Research Article

On the Vector Degree Matrix of a Connected Graph

Nasr A. Zeyada,1,2 Anwar Saleh,3 Majed Albaity,3 and Amr K. Amin4,5

1Department of Mathematics, College of Science, University of Jeddah, Jeddah 23218, Saudi Arabia
2Department of Mathematics, Faculty of Science, Cairo University, Giza 12613, Egypt
3Department of Mathematics Faculty of Science, King Abdulaziz University, P.O.Box 80348, Jeddah 22254, Saudi Arabia
4Department of Mathematics and Computer Science, Faculty of Science, Beni-Suef University, Beni-Suef, Egypt
5Department of Basic Science, Adham University College, Umm Al-Qura University, Mecca, Saudi Arabia

Correspondence should be addressed to Nasr A. Zeyada; nzeyada@gmail.com

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A matrix representation of the graph is one of the tools to study the algebraic structure and properties of a graph. In this paper, by defining the vector degree matrix of graph G, we provide a new matrix representation of the graph. For some standard graphs, VD-eigenvalues, VD-spectrum, and VD-energy values are defined and calculated. Moreover, we calculate the VD-matrix and calculate the VD-eigenvalues for graphs representing the chemical composition of paracetamol and tramadol.

1. Introduction

The relationship between structural and spectral properties of a network could be a key question within the field of Network Science. Spectral graph strategies have become an elementary tool in the analysis of huge advanced networks and connected disciplines, with a broad variety of applications in machine learning, data processing, Internet search and ranking, scientific computing, and PC vision. The central topic of pure mathematics graph theory is learning the relationship between a graph’s structure and its eigenvalues. Specifically, the eigenvalues of matrices representing the graph structure, like the contiguity or the Laplacian matrices, have an on-the-spot associated with the behavior of many networked slashing processes, like spreading processes, synchronization of oscillators, random walks, accord dynamics, and a large type of distributed algorithms. Details on spectra and the theory of graph energy can be found in [1, 2, 4–6, 10–15], whereas details on its chemical applications are found in the book [9] and the review [7, 8].

Given a graph, we can associate several matrices that record information about vertices and how they are interconnected. In this paper, we introduced a new matrix based on the distances between the vertices and their degrees. By calculating the new eigenvalues for some standard graphs, we study some properties of this new matrix.

Lemma 1 (see [4]). For the standard graphs Kn, Km,n, and Cn, we have

(i) Spec(Kn) = \{n - 1, -1\}

(ii) Spec(Km,n) = \{\sqrt{mn} - \sqrt{mn}, 0\}

(iii) Spec(Cn) = \{2 \cos \frac{2\pi}{n}, \cdots, 2 \cos \frac{(n - 1)\pi}{n}\}

if n is odd;

\{2 \cos \frac{2\pi}{n}, \cdots, 2 \cos \frac{(n - 2)\pi}{n}\}

1 \quad 2 \quad \cdots \quad 0 \quad m + n - 2

if n is even.

2. The Vector Degree Spectrum of a Graph

Definition 1. Let G = (V, E) be an undirected graph with vertex set V = \{v1, v2, \ldots, vn\}. A vector degree function of G denoted by VD(G) is a function VD(G): V(G) → \{Z ∪ {0}}^d where d is the diameter of the graph G such that VD(vj) = \vec{v_j} = (\Gamma_0(v_j), \Gamma_1(v_j), \ldots, \Gamma_d(v_j)), where \Gamma_i(v_j) is the number of vertices of distance i from the vertex v_j in G. The vector \vec{v_j} is called the vector degree vector corresponding to the vertex v_j.
The vector degree product of two-degree vectors \( \overrightarrow{v}_i \) and \( \overrightarrow{v}_j \) in a graph \( G \) is denoted by \( \text{VD}(\overrightarrow{v}_i, \overrightarrow{v}_j) \) and defined as
\[
\text{VD}(\overrightarrow{v}_i, \overrightarrow{v}_j) = \begin{cases} 
\overrightarrow{v}_i \cdot \overrightarrow{v}_j, & \text{if } i \neq j; \\
0, & \text{otherwise},
\end{cases}
\]  
where \( \overrightarrow{v}_i \cdot \overrightarrow{v}_j \) is the dot product of the vectors \( \overrightarrow{v}_i \) and \( \overrightarrow{v}_j \) in the Euclidean space \( \mathbb{R}^d \).

The vector degree matrix of \( G \) is then \( \text{VD} = \text{VD}(G) = [a_{ij}] \), where
\[
a_{ij} = \text{VD}(\overrightarrow{v}_i, \overrightarrow{v}_j).
\]  

The characteristic polynomial \( \det(\gamma I - \text{VD}(G)) \) of \( \text{VD}(G) \) is called the \( \text{VD} \)-characteristic polynomial of \( G \) and is denoted by \( P_{\text{VD}}(G) = \sum_{i=0}^n a_i \gamma^{n-i} \). The eigenvalues of the matrix \( \text{VD}(G) \) which are the zeros of \( |\gamma I - \text{VD}(G)| \) are called the \( \text{VD} \)-eigenvalues of \( G \) and form its Spectrum denoted by \( \text{Spec}_{\text{VD}}(G) \).

By the above definition, the degree matrix is a real symmetric \( n \times n \) matrix. Therefore, its eigenvalues \( y_1, y_2, \ldots, y_n \) are real numbers. Since the trace of \( \text{VD}(G) \) is zero, the sum of its eigenvalues is also equal to zero.

**Lemma 2.** Let \( G \) be a connected graph with \( n \) vertices, and let \( y_1, y_2, \ldots, y_n \) be its \( \text{VD} \)-eigenvalues. Then,
\[
\begin{align*}
(1) & \quad \sum_{i=1}^n y_i = 0, \\
(2) & \quad \sum_{i=1}^n y_i^2 = 2 \sum_{1 \leq i < j \leq n} (\overrightarrow{y}_i \cdot \overrightarrow{y}_j)^2.
\end{align*}
\]

**Definition 2.** The \( \text{VD} \)-degree energy of the graph \( G \) is
\[
E_{\text{VD}} = E_{\text{VD}}(G) = \sum_{i=1}^n |y_i|.
\]

**Theorem 1.** For the complete graph \( K_n \) of order \( n \geq 2 \),
\[
\text{Spec}_{\text{VD}}(K_n) = \left( \frac{(-n^2 + 2n - 2)}{(n-1)} \right),
\]
\[
E_{\text{VD}}(K_n) = 2(n-1)(n^2 - 2n + 2).
\]  

**Proof.** Let \( G = K_n \) be a complete graph with vertices \( v_1, v_2, \ldots, v_n \). Then, for every vertex \( v_i, i = 1, 2, \ldots, n \), the vector of a degree corresponding to the vertex \( \overrightarrow{v}_i = (1, n-1) \). Thus, for any two vectors \( \overrightarrow{v}_i \) and \( \overrightarrow{v}_j \),
\[
\overrightarrow{v}_i \cdot \overrightarrow{v}_j = n^2 - 2n + 2.
\]

Hence, \( \text{VD}(K_n) = (n^2 - 2n + 2)A(K_n) \), where \( A(K_n) \) is the adjacency matrix of \( K_n \). Therefore, Lemma 1 gives that
\[
\text{Spec}_{\text{VD}}(K_n) = \left( \frac{(-n^2 + 2n - 2)}{(n-1)} \right),
\]
\[
E_{\text{VD}}(K_n) = 2(n-1)(n^2 - 2n + 2).
\]

We now determine the \( \text{VD} \)-spectrum and \( \text{VD} \)-energy of any cycle \( C_n \).

**Theorem 2.** Let \( n \geq 4 \) be an even integer. Then, for a cycle \( C_n \), we have
\[
\text{Spec}_{\text{VD}}(C_n) = \left( \frac{(-2n - 2)}{(n-1)} \right),
\]
\[
E_{\text{VD}}(C_n) = 2(n-1)E(K_n),
\]
where \( E(K_n) \) is the energy of \( K_n \).

**Proof.** By labeling the vertices of the cycle \( C_n \) in the anti-clockwise direction as \( \{v_1, v_2, \ldots, v_n\} \), we observe that for any vertex \( v_i \), we have
\[
\overrightarrow{v}_i = \left( \frac{1, 2, 2, \ldots, 2, 2, 1}{(n/2)\text{-times}} \right).
\]

Then, clearly for any two vertices \( v_i \) and \( v_j \),
\[
\overrightarrow{v}_i \cdot \overrightarrow{v}_j = ((1)(1) + (2)(2) + (2)(2) + \cdots + (2)(2) + (1)(1)) = 2(n-1).
\]

Therefore, \( \text{VD}(C_n) = 2(n-1)A(K_n) \).

Hence, by Lemma 1, we get
\[
\text{Spec}_{\text{VD}}(C_n) = \left( \frac{(-2n - 2)}{(n-1)} \right),
\]
and it is easy to see that
\[
E_{\text{VD}}(C_n) = 2(n-1)E(K_n),
\]
where \( E(K_n) \) is the energy of \( K_n \).

**Theorem 3.** Let \( n \geq 3 \) be an odd integer. Then, for a cycle \( C_n \), we have
\[
\text{Spec}_{\text{VD}}(C_n) = \left( \frac{(-2n - 1)}{(n-1)} \right),
\]
\[
E_{\text{VD}}(C_n) = 2(n-1)E(K_n),
\]
where \( E(K_n) \) is the energy of \( E(K_n) \).

**Proof.** By labeling the vertices of the cycle \( C_n \) in the anti-clockwise direction as \( \{v_1, v_2, \ldots, v_n\} \), we observe that for any vertex \( v_i \), we have
Then, clearly for every two vertices \( v_i \) and \( v_j \) with \( i \neq j \),
\[
\overrightarrow{v_i} \cdot \overrightarrow{v_j} = (1) (1) + (2) (2) + (2) (2) + \cdots + (2) (2) + (2) (2) \\
= (2n - 1).
\]
Therefore, \( V(D(C_n)) = (2n - 1)A(K_n) \).
Hence, by Lemma 1, we get,
\[
\text{Spec}_{VD}(C_n) = \left( -(2n - 1) \quad (2n - 1)(n - 1) \right).
\]
Further, \( E_{VD}(C_n) = (2n - 1)E(K_n) \). \( \square \)

**Theorem 4.** Let \( G \) be a complete bipartite graph \( K_{n,m} \), where \( 1 \leq n \leq m \). Then,
\[
\text{Spec}_{VD}(K_{n,m}) = \begin{pmatrix} -\alpha & -\gamma & r_1 & r_2 \\ n-1 & m-1 & 1 & 1 \end{pmatrix},
\]
where \( \alpha = 1 + m^2 + (n-1)^2 \), \( \beta = 1 + nm + (n-1)(m-1) \),
\( \gamma = 1 + n^2 + (m-1)^2 \), and \( r_1 \) and \( r_2 \) are the zeros of the polynomial.
\[
[(n-1)\alpha - x][x - (m-1)\gamma] + nm\beta^2.
\]
Furthermore,
\[
E_{VD}(K_{n,m}) = (n-1)\alpha + (m-1)\gamma + \sqrt{[(n-1)\alpha + (m-1)\gamma]^2 - 4 [(n-1)\alpha (m-1)\gamma - nm\beta^2]}.
\]
Thus, \( -\alpha \) is an eigen value of \( VD(K_{n,m}) \), and the corresponding basis of the eigen space \( E_{\alpha} \leq \mathbb{R}^{n+m} \) is
\[
\begin{pmatrix} x_1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} x_2 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \ldots , \begin{pmatrix} x_{n-1} \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} x_n \\ 0 \\ \vdots \\ 0 \end{pmatrix}.
\]
Similarly, \( -\gamma \) is an eigen value of \( VD(K_{n,m}) \) with multiplicity \( m-1 \). Consequently, the characteristic polynomial can be written as \((x + \alpha)^{n-1}(x + \gamma)^{m-1}p(x)\) where \( p(x) \) is a polynomial of degree 2 and by a routine calculations, \( p(x) = [(n-1)\alpha - x][x - (m-1)\gamma] + nm\beta^2 \). Hence,
\[
\text{Spec}_{VD}(K_{n,m}) = \begin{pmatrix} -\alpha & -\gamma & r_1 & r_2 \\ n-1 & m-1 & 1 & 1 \end{pmatrix},
\]
where \( r_1 \) and \( r_2 \) are the zeros of the polynomial.
\[
[(n-1)\alpha - x][x - (m-1)\gamma] + nm\beta^2.
\]
Furthermore,
\[
E_{VD}(K_{n,m}) = (n-1)\alpha + (m-1)\gamma + \sqrt{[(n-1)\alpha + (m-1)\gamma]^2 - 4 [(n-1)\alpha (m-1)\gamma - nm\beta^2]}.
\]
Thus, the following results are obtained from Theorem 4. \( \square \)

**Corollary 1.** Let \( G \) be a complete bipartite graph, where \( n \geq 1 \). Then,
Spec\(_{VD}(K_{n,n}) = \begin{pmatrix} (n^2 - 2n + 2) & (n - 1)(n^2 - 2n + 2) \\ (2n - 1) & 1 \end{pmatrix}, \) (27)

\[ E_D \left( K_{n,n} \right) = (3n - 2)(n^2 - 2n + 2). \]

**Corollary 2.** Let \( G \) be any star graph \( K_{1,n} \). Then,

\[ E_D(K_{1,n}) = (n - 1)(n^2 - 2n + 3) + \sqrt{(n-1)^2(n^2 - 2n + 3)^2 + 4n(1+n)^2}. \] (29)

### 3. Vector Degree Spectrum and Energy of Regular and Strongly Regular Graphs

One of the most important families of regular graphs is strongly regular graphs (abbreviated SRG), which has so many beautiful properties. There are many SRGs which arise from combinatorial concepts such as orthogonal arrays, Latin squares, conference matrices, designs, and geometric graphs.

A strongly regular graph (SRG) with parameters \((n,k,\lambda,\mu)\) is a graph on \( n \) vertices which is regular with valency \( k \) and has the following properties:

(i) Any two adjacent vertices have exactly \( \lambda \) common neighbors

(ii) Any two nonadjacent vertices have exactly \( \mu \) common neighbors [3]

**Theorem 5** (see [5]). Let \( G \) be a strongly regular graph with parameters \((n,k,\lambda,\mu)\). Then, the eigenvalues of \( G \) satisfy the following properties:

1. \( \lambda \) has exactly three distinct eigenvalues which are \( k, \theta \) and \( \tau \), where
   \[ \theta = \frac{1}{2} \left( \lambda - \mu + \sqrt{\left(\lambda - \mu\right)^2 + 4(k - \mu)} \right), \] (30)
   \[ \tau = \frac{1}{2} \left( \lambda - \mu - \sqrt{\left(\lambda - \mu\right)^2 + 4(k - \mu)} \right). \] (31)

2. The multiplicity of the eigenvalue \( k \) is 1, and the multiplicities of \( \theta \) and \( \tau \) are \( f \) and \( g \), respectively, where
   \[ f = n - 1 + \frac{(n - 1)(\mu - \lambda) - 2k}{\sqrt{\left(\lambda - \mu\right)^2 + 4(k - \mu)}}, \] (32)
   \[ g = n - 1 - \frac{(n - 1)(\mu - \lambda) - 2k}{\sqrt{\left(\lambda - \mu\right)^2 + 4(k - \mu)}}. \]

3. If \((n - 1)(\mu - \lambda) - 2k \neq 0\), then the eigenvalues \( \theta \) and \( \tau \) are integers. On the other hand, if \((n - 1)(\mu - \lambda) - 2k = 0\), then \( f = g \) and \( \theta \) and \( \tau \) need not be integers.

**Theorem 6.** Let \( G \) be a strongly regular graph with parameters \((n,k,\lambda,\mu)\). Then,

\[ \text{Spec}_{VD}(G) = \begin{pmatrix} -\delta & (n-1)\delta \\ n-1 & 1 \end{pmatrix}, \] (33)

and further \( E_{VD}(G) = \delta E(K_n) \), where \( \delta = 1 + k^2 + (n - k - 1)^2 \) and \( E(K_n) \) is the energy of \( K_n \).

**Proof.** Let \( G \) be a strongly regular graph with the parameters \((n,k,\lambda,\mu)\). For any vertex \( v \) clearly \( \overline{v} = (1,k,n-k-1) \), then the degree product of any two-degree vectors is equal to \( 1 + k^2 + (n - k - 1)^2 \). Hence,

\[ VD(G) = \left( 1 + k^2 + (n - k - 1)^2 \right) A(K_n). \] (34)

If we put \( \delta = 1 + k^2 + (n - k - 1)^2 \), then

\[ \text{Spec}_{VD}(G) = \begin{pmatrix} -\delta & (n-1)\delta \\ n-1 & 1 \end{pmatrix}, \] (35)

and \( E_{VD}(G) = \delta E(K_n) \).

We can generalize Theorem 6 as the follows. \( \Box \)

**Theorem 7.** Let \( G = (V,E) \) be a \( k \)-regular graph of diameter two. Then,

\[ \text{Spec}_{VD}(G) = \begin{pmatrix} -\delta & (n-1)\delta \\ n-1 & 1 \end{pmatrix}, \] (36)

and \( E_{VD}(G) = \delta E(K_n) \). Further, \( E_{VD}(G) = \delta E(K_n) \), where \( \delta = 1 + k^2 + (n - k - 1)^2 \) and \( E(K_n) \) is the energy of \( K_n \).

In the following, we calculate \( VD \)-matrix and calculate \( VD \)-eigenvalues for graphs representing the chemical composition of paracetamol and tramadol. Also, we note that the closeness of eigen values between paracetamol and tramadol might indicate pharmacological and chemical characteristics convergence.
Scheme 1: The graph of Paracetamol.
**Example 1.** Paracetamol

\[
\begin{align*}
V_1 &= (1, 1, 3, 2, 2, 2, 4, 3, 1, 1), \\
V_2 &= (1, 1, 3, 2, 2, 2, 4, 3, 1, 1), \\
V_3 &= (1, 4, 2, 2, 2, 4, 3, 1, 1, 0), \\
V_4 &= (1, 1, 3, 2, 2, 2, 4, 3, 1, 1), \\
V_5 &= (1, 3, 5, 2, 4, 3, 1, 1, 0, 0), \\
V_6 &= (1, 1, 2, 5, 2, 4, 3, 1, 1, 0), \\
V_7 &= (1, 3, 4, 7, 3, 1, 1, 0, 0, 0), \\
V_8 &= (1, 1, 2, 4, 7, 3, 1, 1, 0, 0), \\
V_9 &= (1, 3, 6, 5, 4, 1, 0, 0, 0, 0), \\
V_{10} &= (1, 3, 4, 5, 4, 3, 0, 0, 0, 0), \\
V_{11} &= (1, 1, 2, 4, 5, 4, 3, 0, 0, 0), \\
V_{12} &= (1, 3, 4, 4, 3, 3, 2, 3, 0, 0), \\
V_{13} &= (1, 1, 2, 4, 4, 3, 2, 3, 0, 0), \\
V_{14} &= (1, 3, 5, 3, 1, 2, 2, 3, 0, 0), \\
V_{15} &= (1, 2, 2, 4, 3, 1, 2, 2, 3, 0), \\
V_{16} &= (1, 1, 1, 2, 4, 3, 1, 2, 2, 3), \\
V_{17} &= (1, 1, 2, 4, 4, 3, 2, 3, 0, 0), \\
V_{18} &= (1, 3, 4, 4, 3, 2, 3, 0, 0, 0), \\
V_{19} &= (1, 3, 4, 5, 4, 3, 0, 0, 0, 0), \\
V_{20} &= (1, 1, 2, 4, 5, 4, 3, 0, 0, 0).
\end{align*}
\]
Example 2. Tramadol

\[ C = \begin{pmatrix}
0 & 50 & 43 & 50 & 44 & 46 & 42 & 43 & 42 & 40 & 46 & 46 & 47 & 48 & 42 & 38 & 47 & 46 & 40 & 46 \\
50 & 0 & 43 & 50 & 44 & 46 & 42 & 43 & 42 & 40 & 46 & 46 & 47 & 48 & 42 & 38 & 47 & 46 & 40 & 46 \\
43 & 43 & 0 & 43 & 55 & 50 & 48 & 47 & 47 & 51 & 52 & 52 & 52 & 46 & 48 & 42 & 38 & 46 & 52 & 51 & 52 \\
50 & 50 & 43 & 0 & 44 & 46 & 42 & 43 & 42 & 40 & 46 & 46 & 47 & 48 & 42 & 38 & 47 & 46 & 40 & 46 \\
44 & 44 & 55 & 44 & 0 & 49 & 60 & 61 & 69 & 65 & 57 & 59 & 52 & 56 & 44 & 41 & 52 & 59 & 65 & 57 & 57 \\
46 & 46 & 50 & 46 & 49 & 0 & 60 & 56 & 53 & 57 & 61 & 55 & 55 & 49 & 48 & 42 & 55 & 55 & 57 & 61 & 61 \\
42 & 42 & 48 & 42 & 60 & 60 & 0 & 65 & 82 & 76 & 62 & 68 & 66 & 58 & 55 & 38 & 66 & 68 & 76 & 62 & 62 \\
43 & 43 & 47 & 43 & 61 & 56 & 65 & 0 & 67 & 69 & 72 & 58 & 64 & 44 & 51 & 52 & 64 & 58 & 69 & 72 & 72 \\
42 & 42 & 47 & 42 & 69 & 53 & 82 & 67 & 0 & 78 & 60 & 68 & 55 & 61 & 52 & 39 & 55 & 68 & 78 & 60 & 60 \\
40 & 40 & 51 & 40 & 65 & 57 & 76 & 69 & 78 & 0 & 64 & 64 & 57 & 55 & 50 & 43 & 57 & 64 & 76 & 64 & 64 \\
46 & 46 & 52 & 46 & 57 & 61 & 62 & 72 & 60 & 64 & 0 & 60 & 60 & 45 & 48 & 47 & 60 & 60 & 64 & 72 & 72 \\
46 & 46 & 52 & 46 & 59 & 55 & 68 & 58 & 68 & 64 & 60 & 0 & 52 & 55 & 48 & 37 & 52 & 64 & 64 & 60 & 60 \\
47 & 47 & 46 & 47 & 59 & 55 & 66 & 64 & 55 & 57 & 60 & 52 & 0 & 49 & 48 & 45 & 60 & 52 & 57 & 60 & 60 \\
48 & 48 & 48 & 48 & 56 & 49 & 58 & 44 & 61 & 55 & 45 & 55 & 49 & 0 & 44 & 33 & 49 & 55 & 55 & 45 & 45 \\
42 & 42 & 42 & 42 & 44 & 48 & 55 & 51 & 52 & 50 & 48 & 48 & 48 & 44 & 0 & 40 & 48 & 48 & 50 & 48 & 48 \\
38 & 38 & 38 & 38 & 41 & 42 & 38 & 52 & 39 & 43 & 47 & 37 & 45 & 33 & 40 & 0 & 45 & 37 & 43 & 47 & 47 \\
47 & 47 & 46 & 47 & 59 & 55 & 66 & 64 & 55 & 57 & 60 & 52 & 60 & 49 & 48 & 45 & 0 & 52 & 57 & 60 & 60 \\
45 & 46 & 52 & 46 & 59 & 55 & 68 & 58 & 68 & 64 & 60 & 64 & 52 & 55 & 48 & 37 & 52 & 0 & 64 & 60 & 60 \\
40 & 40 & 51 & 40 & 65 & 57 & 76 & 69 & 78 & 78 & 64 & 64 & 57 & 55 & 50 & 43 & 57 & 64 & 0 & 64 & 64 \\
46 & 46 & 52 & 46 & 57 & 61 & 62 & 72 & 60 & 64 & 72 & 60 & 60 & 45 & 48 & 47 & 60 & 60 & 64 & 64 & 64
\end{pmatrix}\]

\[ \text{eig}(C) = \begin{pmatrix}
1.0112 \\
19.2 \\
-9.0 \\
-87.6 \\
-35.7 \\
-38.6 \\
-42.0 \\
-79.3 \\
-48.9 \\
-51.8 \\
-75.8 \\
-60.9 \\
-65.0 \\
-63.7 \\
-50.0 \\
-50.0 \\
-76.0 \\
-72.0 \\
-64.0 \\
-60.0
\end{pmatrix} \]
\[ V_1 = (1, 1, 1, 2, 2, 3, 3, 2, 2), \]
\[ V_2 = (1, 2, 2, 2, 3, 3, 2, 0), \]
\[ V_3 = (1, 3, 3, 2, 3, 3, 2, 0, 0), \]
\[ V_4 = (1, 2, 3, 5, 4, 2, 0, 0, 0), \]
\[ V_5 = (1, 2, 3, 2, 1, 3, 2, 2, 0), \]
\[ V_6 = (1, 2, 3, 4, 3, 2, 0, 0, 0), \]
\[ V_7 = (1, 3, 5, 3, 2, 0, 0, 0), \]
\[ V_8 = (1, 4, 5, 4, 1, 0, 0, 0, 0), \]
\[ V_9 = (1, 3, 5, 4, 1, 0, 0, 0, 0), \]
\[ V_{10} = (1, 3, 5, 5, 2, 1, 0, 0, 0), \]
\[ V_{11} = (1, 2, 4, 4, 3, 2, 1, 0, 0), \]
\[ V_{12} = (1, 3, 1, 2, 4, 3, 2, 1, 0, 0), \]
\[ V_{13} = (1, 1, 2, 1, 2, 4, 3, 2, 1), \]
\[ V_{14} = (1, 2, 3, 4, 4, 2, 1, 0, 0), \]
\[ V_{15} = (1, 2, 2, 2, 3, 4, 2, 1, 0, 0), \]
\[ V_{16} = (1, 2, 3, 3, 4, 4, 1, 0, 0), \]
\[ V_{17} = (1, 2, 4, 4, 3, 1, 0, 0, 0, 0), \]
\[ V_{18} = (1, 2, 4, 4, 1, 0, 0, 0, 0), \]
\[ V_{19} = (1, 2, 4, 4, 3, 1, 0, 0, 0). \]

\[
A = \begin{pmatrix}
0 & 38 & 35 & 34 & 37 & 34 & 29 & 28 & 34 & 30 & 34 & 37 & 39 & 39 & 35 & 37 & 38 & 32 \\
38 & 0 & 42 & 41 & 43 & 42 & 42 & 39 & 38 & 40 & 41 & 42 & 42 & 41 & 43 & 44 & 42 \\
35 & 42 & 0 & 48 & 42 & 48 & 49 & 50 & 51 & 49 & 49 & 48 & 46 & 40 & 40 & 48 & 46 & 47 & 50 \\
34 & 41 & 48 & 0 & 40 & 52 & 60 & 63 & 62 & 63 & 61 & 57 & 46 & 36 & 36 & 58 & 52 & 59 \\
37 & 43 & 42 & 40 & 0 & 40 & 41 & 41 & 39 & 41 & 43 & 42 & 39 & 42 & 42 & 40 & 42 & 43 & 43 \\
37 & 42 & 48 & 52 & 40 & 0 & 52 & 50 & 50 & 54 & 48 & 49 & 48 & 40 & 40 & 51 & 47 & 49 & 51 \\
34 & 42 & 49 & 60 & 41 & 52 & 0 & 60 & 59 & 63 & 58 & 55 & 47 & 39 & 39 & 56 & 47 & 52 & 59 \\
29 & 39 & 50 & 63 & 41 & 50 & 60 & 60 & 72 & 64 & 70 & 60 & 43 & 33 & 33 & 38 & 44 & 47 & 64 \\
28 & 38 & 51 & 62 & 39 & 50 & 59 & 72 & 60 & 60 & 64 & 60 & 60 & 61 & 57 & 47 & 40 & 40 & 40 \\
34 & 42 & 49 & 63 & 41 & 54 & 63 & 64 & 60 & 0 & 61 & 57 & 47 & 40 & 40 & 58 & 49 & 52 & 64 \\
30 & 40 & 49 & 61 & 43 & 48 & 58 & 70 & 68 & 61 & 0 & 59 & 41 & 34 & 34 & 56 & 43 & 48 & 62 \\
34 & 41 & 48 & 57 & 42 & 49 & 55 & 60 & 59 & 57 & 59 & 0 & 43 & 37 & 37 & 54 & 42 & 49 & 56 \\
37 & 42 & 46 & 46 & 39 & 48 & 47 & 43 & 45 & 47 & 41 & 41 & 0 & 40 & 40 & 46 & 46 & 46 & 45 \\
39 & 42 & 40 & 36 & 42 & 40 & 39 & 39 & 33 & 31 & 40 & 34 & 37 & 40 & 40 & 45 & 37 & 43 & 42 & 40 \\
39 & 42 & 40 & 36 & 42 & 40 & 39 & 33 & 31 & 40 & 34 & 37 & 40 & 45 & 0 & 37 & 43 & 42 & 40 \\
35 & 41 & 48 & 58 & 40 & 51 & 56 & 58 & 58 & 58 & 56 & 54 & 46 & 37 & 37 & 0 & 45 & 50 & 55 \\
37 & 43 & 46 & 45 & 42 & 47 & 47 & 44 & 43 & 49 & 43 & 42 & 46 & 43 & 43 & 45 & 0 & 46 & 48 \\
38 & 44 & 47 & 52 & 43 & 49 & 52 & 47 & 46 & 52 & 48 & 49 & 46 & 42 & 42 & 50 & 46 & 0 & 50 \\
32 & 42 & 50 & 59 & 43 & 51 & 59 & 64 & 61 & 64 & 62 & 56 & 45 & 40 & 40 & 55 & 48 & 50 & 0
\end{pmatrix}
\]
\[
\text{eig}(A) = \begin{pmatrix}
-72.8695 \\
-68.7152 \\
-65.8056 \\
-61.3030 \\
-60.3018 \\
-57.6119 \\
-54.6774 \\
-54.2252 \\
-50.0643 \\
-47.6270 \\
-45.4837 \\
-42.8595 \\
-39.9573 \\
-34.9652 \\
-32.9403 \\
 39.4311 \\
 840.9000
\end{pmatrix}
\]

4. Conclusions

One of the main tools for studying the algebraic structure and properties of a graph is a matrix representation of the graph. We have introduced a new matrix representation, called the vector degree matrix of graph \( G \). For analogues to this new matrix, we have defined and calculated VD-eigenvalues, VD-eigenvectors, VD-spectrum, and VD-energy for various standard graphs. Furthermore, for graphs illustrating the chemical composition of paracetamol and tramadol, we have constructed and calculated the VD-matrix and VD-eigenvalues.

Data Availability

The datasets used during the current study are available from the corresponding author upon reasonable request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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