GLOBAL ATTRACTOR FOR A SUSPENSION BRIDGE PROBLEM WITH A NONLINEAR DELAY TERM IN THE INTERNAL FEEDBACK

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Abstract. In the present paper, we consider a suspension bridge problem with a nonlinear delay term in the internal feedback. Namely, we investigate the following equation:

$$u_{tt} + \Delta^2 u + \delta_1 g_1(u_t(x,y,t)) + \delta_2 g_2(u_t(x,y,t-\tau)) + h(u(x,y,t)) = f(x,y),$$

together with some suitable initial data and boundary conditions. We prove the global existence of solutions by means of the energy method combined with the Faedo-Galerkin procedure under a certain relation between the weight of the delay term in the feedback and the weight of the nonlinear frictional damping term without delay. Moreover, we establish the existence of a global attractor for the above-mentioned system by proving the existence of an absorbing set and the asymptotic smoothness of the semigroup $S(t)$.

1. Introduction. In recent years, the research of the suspension bridge model has attracted considerable attention. At the very beginning, suspension bridges were modeled as one-dimensional simply supported beams suspended by hangers ([29]). However, such models do not adequately describe torsional oscillations in suspension bridges, and this is a major drawback ([28]). When people studied the collapse of Tacoma Narrows Bridge in 1940 (for the report about the Tacoma Narrows Bridge collapse, we refer readers to [3, 36]), they came to realize that a reliable model for suspension bridges should be nonlinear and it should have enough degrees of freedom to display torsional oscillations.

In [16], Ferrero and Gazzola suggested a rectangular plate $\Omega = (0, \pi) \times (-l, l)$ as a model to describe the statics and dynamics of a suspension bridge. The equation reads

$$u_{tt}(x,y,t) + \Delta^2 u(x,y,t) + \delta u_t(x,y,t) + h(x,y,u(x,y,t)) = f(x,y,t), \quad (1.1)$$

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in $\Omega \times (0,T)$, where $\delta > 0$ is a frictional constant, $u(x,y,t)$ is the vertical displacement of the plate in the downward direction, $h(x,y,u(x,y,t))$ is the hangers’ restoring force, $f(x,y,t)$ is an external force which includes gravity. The plate is assumed to be hinged on its vertical edges

$$u(0, y, t) = u(\pi, y, t) = u_{xx}(\pi, y, t) = 0, \quad (y, t) \in (-l, l) \times (0, T),$$

and the horizontal edges are free

$$u_{yy}(x, \pm l, t) + \sigma u_{xx}(x, \pm l, t) = 0, \quad u_{yyy}(x, \pm l, t) + (2 - \sigma)u_{xxy}(x, \pm l, t) = 0,$$

for $(x,t) \in (0,\pi) \times [0,T)$, where $0 < \sigma < \frac{1}{2}$ is the Poisson ratio. They established the well-posedness of solutions and discussed several other stationary problems for (1.1).

Due to the fact that the action of any external force $f$ on the plate $\Omega$ is transmitted through hangers to the sustaining cables and this may yield some delay, Messaoudi et al. in [26] considered a variant of equation (1.1) in the presence of delay and damping together with the above-mentioned boundary conditions. Namely, they considered the following equation:

$$u_{tt} + \Delta^2 u + \mu_1 u_t(x, y, t) + \mu_2 u(x, y, t - \tau) + h(u(x, y, t)) = f(x, y),$$

in $\Omega \times (0, +\infty)$. They established a well-posedness result and the existence of a finite-dimensional global attractor. For more recent results on suspension bridge models, we refer readers to [2, 4, 8, 10, 17, 18, 22, 23, 35, 38, 39] and the references therein.

Global attractor is a basic concept in the study of the asymptotic behavior of solutions for nonlinear evolution equations with various dissipation. There are many profound researches on the existence of global attractors for different systems. In [24], Messaoudi et al. considered a plate equation in the presence of memory, namely

$$u_{tt} + \mu \Delta^2 u - \int_{-\infty}^{t} g(t-s)\Delta^2 u(s)ds + \delta u_t + h(u) = f, \quad \text{in} \ \Omega \times (0, +\infty).$$

They gave a rigorous well-posedness result and established the existence of a global attractor. In [33], Park et al. considered the following suspension bridge equation with nonlinear damping:

$$u_{tt} + \Delta^2 u + ku^+ + a(x)g(u_t) + f(u) = h(x), \quad \text{in} \ \Omega \times \mathbb{R}^+.$$ 

They proved the existence of an absorbing set and a global attractor. In [27], Ma et al. considered a model of extensible beam with nonlinear damping and source terms

$$u_{tt} + \Delta^2 u - M \left( \int_{\Omega} |\nabla u|^2 dx \right) \Delta u + f(u) + g(u_t) = h, \quad \text{in} \ \Omega \times \mathbb{R}^+.$$ 

By proving the existence of an absorbing set and the asymptotic smoothness of the semigroup $S(t)$, they proved the existence of a global attractor. For more results concerning the existence of a global attractor in different systems, we refer readers to [15, 20, 21, 32, 33, 34, 40, 42, 43] and the references therein.

In recent years, PDEs with time delay effects have become an active area of research and arise in many practical problems. The presence of delay may be a source of instability. For instance, in [14], Datko et al. proved that an arbitrarily small delay may destabilize a system which is uniformly asymptotically stable in the absence of delay. To stabilize $s$ hyperbolic system involving input delay terms, additional control terms are necessary (see [19, 30]). In [30], the authors studied the
wave equation with a linear internal damping term with constant delay and determined suitable relations between $\mu_1$ and $\mu_2$, for which the stability or alternatively instability takes place. More precisely, they showed that the energy is exponentially stable if $\mu_2 < \mu_1$ and they also found a sequence of delays for which the corresponding solution will be instable if $\mu_2 \geq \mu_1$. For nonlinear time-delay damping case, there is a large number of publications concerning it (see [5, 6, 7, 25]). In [5], Benaissa et al. considered the Timoshenko system in bounded domain with a delay term in the nonlinear internal feedback

\begin{align*}
\rho_1 \varphi_{tt}(x, t) - K(\varphi_x + \psi)_x(x, t) &= 0 \quad \text{in } (0, 1) \times (0, +\infty), \\
\rho_2 \psi_{tt}(x, t) - b\psi_{xx}(x, t) + K(\varphi_x + \psi)(x, t) + \mu_1 g_1(\psi(x, t)) + \mu_2 g_2(\psi(x, t - \tau)) &= 0, \quad \text{in } (0, 1) \times (0, +\infty). \tag{1.6}
\end{align*}

They proved the global existence of solutions by means of the energy method combined with the Faedo-Galerkin procedure under a condition between the weight of the delay term in the feedback and the weight of the term without delay. Moreover, they established a decay rate estimate for the energy by introducing suitable Lyapunov functionals.

However, to our best knowledge, there is no result concerning the suspension bridge model with both nonlinear frictional damping and nonlinear time-delay damping. In this paper, we are concerned with the following problem:

\begin{align*}
\rho_1 \varphi_{tt}(x, t) - K(\varphi_x + \psi)_x(x, t) &= 0 \quad \text{in } (0, 1) \times (0, +\infty), \\
\rho_2 \psi_{tt}(x, t) - b\psi_{xx}(x, t) + K(\varphi_x + \psi)(x, t) &= 0, \quad \text{in } (0, 1) \times (0, +\infty). \tag{1.7}
\end{align*}

where $\Omega = (0, \pi) \times (-l, l) \subset \mathbb{R}^2$, $\delta_1, \delta_2$ are positive real constants, $f \in L^2(\Omega)$ and $\tau > 0$ represents the time delay. Our purpose is to investigate the existence of global weak solutions as well as a global attractor for problem (1.7).

Inspired by [5], we prove the existence of solutions by means of the energy method combined with the Faedo-Galerkin procedure. A different point is that we need to address the source term $h(u(x, y, t))$. To overcome this difficulty, we use a method relying on the idea present in [41] (see also [42, 43]). To prove the existence of a global attractor, we divide our proof into two sections: (i) the existence of an absorbing set; (ii) the asymptotic smoothness of the semigroup $S(t)$. For the existence of an absorbing set, our strategy is to construct an inequality for $\bar{L}(t)$, which is equivalent to the modified energy $\bar{E}(t)$. The proof is inspired by previous work in [5, 27]. For the asymptotic smoothness, motivated by [33], we are devote to constructing an inequality of the energy $E_\phi(T)$.

The paper is organized as follows. In Section 2, we present some fundamental and basic results, and give the energy functional of problem (1.1). In Section 3, we use the argument combining the Galerkin approximation scheme with the energy estimate method to prove the existence of solutions. In Section 4, we prove the
existence of an absorbing set. In Section 5, we prove the asymptotic smoothness of the semigroup $S(t)$. Finally, in Section 6, combining the results of Section 4 and Section 5, we prove the existence of a global attractor.

## 2. Preliminaries

In order to state and prove our results, we need some assumptions, as well as, some lemmas. We first make the following assumptions as in [5]:

**Assumption 1.** $g_1 : \mathbb{R} \to \mathbb{R}$ is a non-decreasing function of the class $C(\mathbb{R})$ such that there exist $\epsilon_1, C_1, C_2 > 0$ and a convex increasing function $\Psi : \mathbb{R}_+ \to \mathbb{R}_+$ of the class $C^1(\mathbb{R}_+) \cap C^2((0, \infty))$ satisfying $\Psi(0) = 0$, and $\Psi$ is linear on $[0, \epsilon_1]$ or $\Psi'(0) = 0$ and $\Psi'' > 0$ on $(0, \epsilon_1]$, such that

$$
C_1|s| \leq |g_1(s)| \leq C_2|s|, \quad \text{if } |s| \geq \epsilon_1, \quad (2.1)
$$

$$
s^2 + g_1^2(s) \leq \Psi^{-1}(sg_1(s)), \quad \text{if } |s| \leq \epsilon_1. \quad (2.2)
$$

$g_2 : \mathbb{R} \to \mathbb{R}$ is an odd non-decreasing function of the class $C^1(\mathbb{R})$ such that there exist $C_3, \alpha_1, \alpha_2 > 0$

$$
|g_2'(s)| \leq C_3, \quad (2.3)
$$

$$
\alpha_1 sg_2(s) \leq G_2(s) \leq \alpha_2 sg_1(s), \quad (2.4)
$$

where

$$
G_2(s) = \int_0^s g_2(r)dr,
$$

and

$$
\alpha_2 \delta_2 < \alpha_1 \delta_1. \quad (2.5)
$$

**Assumption 2.** The function $h \in C^1(\mathbb{R})$, and there exist positive constants $C_4, C_5$ and $C_6$ such that

$$
\begin{aligned}
|h(s_1) - h(s_2)| &\leq C_4 \left(|s_1|^\theta + |s_2|^\theta\right)|s_1 - s_2|, &\forall s_1, s_2 \in \mathbb{R}, \theta > 0, \\
-C_5 &\leq H(s) \leq sh(s), &\forall s \in \mathbb{R}, \\
-C_6 &\leq h'(s), &\forall s \in \mathbb{R},
\end{aligned}
$$

(2.6)

where $H(s) = \int_0^s h(r)dr$.

As in [16], we define the space

$$
H^2_+(\Omega) = \left\{ u \in H^2(\Omega), u = 0 \text{ on } \{0, \pi\} \times (-l, l) \right\},
$$

dowered with the inner product

$$
(u, v)_{H^2_+} = \int_{\Omega} [\Delta u \Delta v + (1 - \sigma)(2u_{xy}v_{xy} - u_{xx}v_{yy} - u_{yy}v_{xx})]dxdy.
$$

For simplicity, we denote

$$
\|u\|_{L^2(\Omega)} = \|u\|_2.
$$

The following result was proved by Ferrero and Gazzola [16]:

**Lemma 2.1.** [16] Assume $0 < \sigma < \frac{1}{2}$. Then, the norm $\| \cdot \|_{H^2_+(\Omega)}$ given by

$$
\|u\|^2_{H^2_+(\Omega)} = (u, u)_{H^2_+(\Omega)}
$$

is equivalent to the usual $H^2(\Omega)$-norm. Moreover, $H^2_+(\Omega)$ endowed with the scalar product $(\cdot, \cdot)_{H^2_+(\Omega)}$ is a Hilbert space.

Moreover, we need the following Sobolev embedding inequality:
Lemma 2.2. [39] Assume that $1 \leq q \leq \infty$. Then for any $u \in H^2_s$, the inequality
\[ \|u\|_q \leq S_q \|u\|_{H^2_s} \]
holds, where $S_q = \left( \frac{\pi}{2l} + \sqrt{2} \right) \left( 2\pi \right)^{\frac{q+2}{2}} \left( \frac{1}{1-\sigma} \right)^{\frac{1}{2}}$.

Considering the following eigenvalue problem:
\[
\begin{cases}
\Delta^2 u = \Lambda u, & (x, y) \in \Omega, \\
u(0, y) = u_{xx}(0, y) = u(\pi, y) = u_{xx}(\pi, y) = 0, & y \in (-l, l), \\
u_{yy}(x, \pm l) + \sigma u_{xx}(x, \pm l) = 0, & x \in (0, \pi),
\end{cases}
\tag{2.7}
\]
we learn from [39] that the set of eigenvalues of (2.7) may be ordered in an increasing sequence $\{\Lambda_i\}_{i=1}^{\infty}$ of strictly positive numbers diverging to $+\infty$, and the least eigenvalue $\Lambda_1$ satisfies $0 < \Lambda_1 < 1$.

As in [30, 31], we set
\[ z(\rho, x, y, t) = u_t(x, y, t - \tau \rho), \quad \rho \in (0, 1), \quad (x, y) \in \Omega, \quad t > 0. \tag{2.8} \]
Then differentiation, with respect to $t$, gives
\[ \tau z_t(\rho, x, y, t) + z_p(\rho, x, y, t) = 0, \quad \rho \in (0, 1), \quad (x, y) \in \Omega, \quad t > 0. \]
Thus, we can rewrite problem (1.7) as
\[
\begin{align*}
&\begin{cases}
u_{tt}(x, y, t) + \Delta^2 u(x, y, t) + \delta_1 g_1(u_t(x, y, t)) + \delta_2 g_2(z(1, x, y, t)) \\
\quad + h(u(x, y, t)) = f(x, y), & \text{in } \Omega \times (0, +\infty),
\end{cases} \\
&\tau z_t(\rho, x, y, t) + z_x(\rho, x, y, t) = 0, \quad \text{in } (0, 1) \times \Omega \times (0, +\infty),
\end{align*}
\tag{2.9}
\]
with boundary conditions
\[
\begin{align*}
u(0, y, t) = u_{xx}(0, y, t) = u(\pi, y, t) = u_{xx}(\pi, y, t) = 0, & \text{for } (y, t) \in (-l, l) \times (0, +\infty), \\
u_{yy}(x, \pm l, t) + \sigma u_{xx}(x, \pm l, t) = u_{yy}(x, \pm l, t) & \quad + (2 - \sigma) u_{xx}(x, \pm l, t) = 0, \quad \text{for } (x, t) \in (0, \pi) \times (0, +\infty),
\end{align*}
\tag{2.10}
\]
and initial conditions
\[
\begin{align*}
u(x, y, 0) = u_0(x, y), & \quad u_t(x, y, 0) = u_1(x, y), \quad \text{in } \Omega, \\
z(\rho, x, y, 0) = f_0(x, y, -\rho \tau), & \quad \text{for } (\rho, x, y) \in (0, 1) \times (0, \pi) \times (-l, l).
\end{align*}
\tag{2.11}
\]

Remark 1. By the mean value theorem for integrals and the monotonicity of $g_2$, we find that
\[ G_2(s) = \int_0^s g_2(r)dr \leq sg_2(s). \]
Thus, $a_1 \leq 1$.

From (2.5), we can choose a positive constant $\xi$ such that
\[ \tau - \frac{\delta_2(1 - a_1)}{a_1} < \xi < \tau - \frac{\delta_2 - a_2\delta_2}{a_2}. \tag{2.12} \]
We define the energy associated to the solution of problem (2.9)-(2.11) by
\[
E(t) = \frac{1}{2} \|u(t)\|_{H^2_s(\Omega)}^2 + \frac{1}{2} \|u(t)\|_{H^2_s(\Omega)}^2 + \int_{\Omega} H(u(t))dxdy - \int_{\Omega} f(u(t))dxdy
\]
\[ + \xi \int_0^1 \int_\Omega G_2(z(\rho, x, y, t)) d\rho dx dy. \] (2.13)

**Lemma 2.3.** Let \((u, u_1, z)\) be a solution of problem (2.9)-(2.11). Then, the energy functional defined by (2.13) satisfies

\[ E'(t) \leq - \left( \delta_1 - \frac{\xi \alpha_2}{\tau} - \delta_2 \alpha_2 \right) \int_\Omega u_1 g_1(u_1) dx dy - \left( \frac{\xi}{\tau} \alpha_1 - \delta_2 (1 - \alpha_1) \right) \int_\Omega z(1, x, y, t) g_2(z(1, x, y, t)) dx dy \leq 0. \] (2.14)

**Proof.** Our strategy draws substantially from ideas in \([5, 6]\). Multiplying the first equation in (2.9) by \(u_1\) and integrating the result over \(\Omega\), we have

\[
\frac{d}{dt} \left\{ \frac{1}{2} \|u_1\|_{H^2(\Omega)}^2 + \frac{1}{2} \|u\|_{L^2(\Omega)}^2 + \int_\Omega H(u) dx dy - \int_\Omega f u dx dy \right\} + \delta_1 \int_\Omega g_1(u_1) u_1 dx dy + \delta_2 \int_\Omega g_2(z(1, x, y, t)) u_1 dx dy = 0. \] (2.15)

We multiply the second equation in (2.9) by \(\xi g_2(z)\) and integrate over \((0, 1) \times \Omega\) to obtain

\[
\xi \int_0^1 \int_\Omega g_2(z) dx dy = - \frac{\xi}{\tau} \int_\Omega \int_0^1 z \rho g_2(z) dx dy = \frac{\xi}{\tau} \int_\Omega G_2(z(1, x, y, t)) - G_2(z(0, x, y, t)) dx dy. \] (2.16)

Hence,

\[
\frac{d}{dt} \left\{ \xi \int_\Omega g_2(z) dx dy \right\} = - \frac{\xi}{\tau} \int_\Omega G_2(z(1, x, y, t)) dx dy + \frac{\xi}{\tau} \int_\Omega G_2(u_1) dx dy. \] (2.17)

From (2.15) and (2.17), we have

\[
E'(t) = - \delta_1 \int_\Omega g_1(u_1) u_1 dx dy - \delta_2 \int_\Omega g_2(z(1, x, y, t)) u_1 dx dy - \frac{\xi}{\tau} \int_\Omega G_2(z(1, x, y, t)) dx dy + \frac{\xi}{\tau} \int_\Omega G_2(u_1) dx dy. \] (2.18)

Making use of (2.4), we have

\[
E'(t) \leq - \left( \delta_1 - \frac{\xi \alpha_2}{\tau} \right) \int_\Omega g_1(u_1) u_1 dx dy - \delta_2 \int_\Omega g_2(z(1, x, y, t)) u_1 dx dy - \frac{\xi}{\tau} \int_\Omega G_2(z(1, x, y, t)) dx dy. \] (2.19)

Now, we need to deal with \(\delta_2 \int_\Omega g_2(z(1, x, y, t)) u_1 dx dy\). Let us denote the conjugate function \(G_2^*\) by \(G_2^*\), then we have

\[ G_2^*(s) = \sup_{t \in \mathbb{R}^+} (st - G_2(t)) . \]

According to \([1, p.61-62]\), we deduce that \(G_2^*\) is the Legendre transform of \(G_2\), which takes the form

\[ G_2^*(s) = s(G_2^1)^{-1}(s) - G_2[(G_2^1)^{-1}(s)], \quad \forall \ s \geq 0, \] (2.20)
and satisfies the following inequality:

\[ st \leq G_2^s(s) + G_2^t(t), \quad \forall \ s, t \geq 0. \]  

(2.21)

Thus, we have

\[ \delta_2 \int_{\Omega} g_2(z(1, x, y, t))u_t \, dx \, dy \leq \delta_2 \left( G_2(u_t) + G_2^s(g_2(z(1, x, y, t))) \right) \, dx \, dy. \]  

(2.22)

From the definition of \( G_2 \) and noting that \( G_2' = g_2 \), we get

\[ G_2^s(s) = s g_2^{-1}(s) - G_2(g_2^{-1}(s)). \]

Hence,

\[ G_2^s(g_2(z(1, x, y, t))) = z(1, x, y, t) g_2(z(1, x, y, t)) - G_2(z(1, x, y, t)). \]  

(2.23)

Substituting (2.22) and (2.23) into (2.19), and using (2.4), we arrive at

\[
E'(t) \leq - \left( \delta_1 - \frac{\xi \alpha_2}{\tau} \right) \int_{\Omega} g_1(u_t)u_t \, dx \, dy + \delta_2 \int_{\Omega} G_2(u_t) \, dx \, dy \\
+ \delta_2 \int_{\Omega} z(1, x, y, t) g_2(z(1, x, y, t)) \, dx \, dy - \delta_2 \int_{\Omega} G_2(z(1, x, y, t)) \, dx \, dy \\
- \frac{\xi}{\tau} \int_{\Omega} G_2(z(1, x, y, t)) \, dx \, dy \\
\leq - \left( \delta_1 - \frac{\xi \alpha_2}{\tau} - \delta_2 \alpha_2 \right) \int_{\Omega} g_1(u_t)u_t \, dx \, dy \\
- \left( \frac{\xi \alpha_1}{\tau} - \delta_2(1 - \alpha_1) \right) \int_{\Omega} z(1, x, y, t) g_2(z(1, x, y, t)) \, dx \, dy.
\]  

(2.24)

Then, by using (2.12), our conclusion follows.

\( \square \)

3. Global existence. In this section, we prove the global existence of solutions in suitable Sobolev space by means of the energy method combined with the Faedo-Galerkin method.

**Theorem 3.1.** Let \((u_0, u_1, f_0) \in (H^2_s(\Omega) \cap L^2(\Omega)) \times L^2(\Omega) \times L^2(0, 1)\) satisfies the compatibility condition \(f_0(\cdot, 0) = u_1\). Assume that the Assumptions 1 and 2 hold and \( f \in L^2(\Omega) \). Then problem (2.9)-(2.11) admits a unique weak global solution

\[ u \in L^\infty(0, T; H^2_s(\Omega) \cap L^2(\Omega)), \quad u_t \in L^\infty(0, T; L^2(\Omega)), \quad u_{tt} \in L^\infty(0, T; L^2(\Omega)). \]

Moreover, we have \( z \in L^\infty(0, T; L^2((0, 1) \times \Omega)). \)

**Proof.** Let \( T > 0 \) be fixed and denote by \( V_k \) the space generated by \( \{w_1, w_2, \cdots, w_k\} \), where the set \( \{w^k, k \in \mathbb{N}\} \) is a basis of \( H^2_s(\Omega) \cap L^2(\Omega) \).

Now, we define, for \( 1 \leq j \leq k \), the sequence \( \phi^j(x, y, \rho) \) as

\[ \phi^j(x, y, 0) = w^j. \]

Then, we may extend \( \phi^j(x, y, 0) \) to \( \phi^j(x, y, \rho) \) over \( L^2(\Omega \times (0, 1)) \) such that \( \{\phi^j\}_j \) form a basis of \( L^2(\Omega; L^2(0, 1)) \) and denote \( z^k \) the space generated by \( \{\phi^1, \phi^2, \cdots, \phi^k\} \).

We construct approximate solutions \((u^k, z^k), k = 1, 2, 3, \cdots\), in the form

\[ u^k(t) = \sum_{j=1}^{k} c^j(t) w^j, \quad z^k(t) = \sum_{j=1}^{k} d^j(t) \phi^j, \]
where $c^j$ and $d^j$, $j = 1, 2, \cdots, k$, are determined by the following ordinary differential equations:

\[
\begin{aligned}
\begin{cases}
\langle u_v^k(t), w^j \rangle + \langle u^k(t), w^j \rangle_{H^2(\Omega)} + \langle h(u^k(t)), w^j \rangle + \delta_1(g_1(u^k(t)), w^j) \\
+ \delta_2(g_2(z^k(1, x, y, t)), w^j) = (f, w^j), \quad 1 \leq j \leq k,
\end{cases}
\end{aligned}
\]

(3.1)

For sake of convenience, we denote $u^k(0) = u^0_0 = \sum_{j=1}^k (u_0, w^j) w^j \rightarrow u_0$ in $H^2(\Omega) \cap L^2(\Omega)$ as $k \rightarrow +\infty$, (3.2)

\[
\begin{aligned}
\begin{cases}
u^k(0) = u^k_1 = \sum_{j=1}^k (u_1, w^j) w^j \rightarrow u_1 \text{ in } L^2(\Omega) \text{ as } k \rightarrow +\infty,
\end{cases}
\end{aligned}
\]

(3.3)

and

\[
(\tau z^k + z^k, \phi^j) = 0, \quad 1 \leq j \leq k,
\]

(3.4)

\[
z^k(\rho, x, y, 0) = z^k_0 = \sum_{j=1}^k (f_0, \phi^j) \phi^j \rightarrow f_0 \text{ in } L^2(\Omega; L^2(0, 1)) \text{ as } k \rightarrow +\infty.
\]

(3.5)

Standard theory of ordinary differential equations guarantees that system (3.1)-(3.5) has a unique local solution on a maximal interval $[0, T_k)$ (with $0 < T_k < T$) by Zorn lemma since the nonlinear terms in (3.1) are locally Lipschitz continuous. Note that $u^k(t)$ is of class $C^2$. Next we present some estimates that allow us to extend the local solutions to the interval $[0, T]$, for any given $T > 0$.

A. The first estimate.

Since $(u_0, u_1, f_0) \in (H^2(\Omega) \cap L^2(\Omega)) \times L^2(\Omega) \times L^2(\Omega; L^2(0, 1))$, we can deduce from (3.2), (3.3) and (3.5) that

\[
\left\{ \|u_0\|_{H^2(\Omega)}, \|u_1\|_{L^2(\Omega)}, \|z_0\|_{L^2(\Omega; L^2(0, 1))} \right\} \leq C.
\]

(3.6)

For sake of convenience, we denote $a_1 = \delta_1 - \frac{\xi \alpha_2}{\tau} - \delta_2 \alpha_2 > 0$ and $a_2 = \frac{\xi \alpha_1}{\tau} - \delta_2 (1 - \alpha_1) > 0$.

From (2.24), we can deduce that for $\forall \ 0 < t < T_k$,

\[
E^k(t) + a_1 \int_0^t \int_{\Omega} u^k_1(s) g_1(u^k_1(s)) dx dy ds \\
+ a_2 \int_0^t \int_{\Omega} z^k(1, x, y, s) g_2(z^k(1, x, y, s)) dx dy ds \leq E^k(0),
\]

(3.7)

where

\[
E^k(t) = \frac{1}{2} \|u^k(t)\|_2^2 + \frac{1}{2} \|u^k(t)\|_{H^2(\Omega)}^2 + \int_{\Omega} H(u^k(t)) dx dy - \int_{\Omega} f(u^k(t)) dx dy \\
+ \xi \int_{\Omega} \int_0^1 G_2(z^k(\rho, x, y, t)) d\rho dx dy.
\]

(3.8)

Since

\[
\int_{\Omega} f(u^k(t)) dx dy \leq C_\eta \|f\|_2^2 + \eta S^2 \|u^k(t)\|_{H^2(\Omega)}^2,
\]

(3.9)

we have

\[
- \int_{\Omega} f(u^k(t)) dx dy \geq -C_\eta \|f\|_2^2 - \eta S^2 \|u^k(t)\|_{H^2(\Omega)}^2.
\]

(3.10)
B. The second estimate.

Submitting (3.10) into (3.8), using (2.6), we can deduce that
\[
\frac{1}{2} \| u^k_t (t) \|^2 + \left( \frac{1}{2} - \eta S^2 \right) \| u^k (t) \|^2_{H^2_\Omega} + \xi \int_0^1 \int_\Omega G_2 \left( z^k (\rho, x, y, t) \right) \, d\rho \, dx \, dy \\
+ a_1 \int_0^t \int_\Omega u^k_t (s) g_1 (u^k (s)) \, dx \, dy \, ds + a_2 \int_0^t \int_\Omega z^k (1, x, y, s) g_2 \left( z^k (1, x, y, s) \right) \, dx \, dy \, ds \\
\leq C_3 | \Omega | + C_9 \| f \|^2_2 + E^k (0).
\] (3.11)

From (2.6)_1, we have
\[
\left| \int_\Omega H \left( u^k_0 \right) \, dx \, dy \right| \leq \left| \int_\Omega h \left( u^k_0 \right) u^k_0 \, dx \, dy \right| \leq \frac{1}{2} \eta S^2 \| u^k_0 \|^2_{L^2_\Omega} + C \| u^k \|^2_{L^2_\Omega} \| u^k_0 \|^2_{H^2_\Omega} \\
\leq \frac{1}{2} \eta S^2 \| u^k_0 \|^2_{H^2_\Omega} + C \| u^k \|^2_{L^2_\Omega} \| u^k_0 \|^2_{H^2_\Omega}.
\] (3.12)

Choosing \( \eta \) small enough such that \( \frac{1}{2} - \eta S^2 = \frac{1}{4} \) from (3.6), we can deduce that for \( \forall \ 0 \leq t < T_k \),
\[
\frac{1}{2} \| u^k_t (t) \|^2 + \frac{1}{4} \| u^k (t) \|^2_{H^2_\Omega} + \xi \int_0^1 \int_\Omega G_2 \left( z^k (\rho, x, y, t) \right) \, d\rho \, dx \, dy \\
+ a_1 \int_0^t \int_\Omega u^k_t (s) g_1 (u^k (s)) \, dx \, dy \, ds + a_2 \int_0^t \int_\Omega z^k (1, x, y, s) g_2 \left( z^k (1, x, y, s) \right) \, dx \, dy \, ds \\
\leq C,
\] (3.13)

Thus, approximate solutions are defined on the whole range \( [0, T] \), and it follows that
\[
u^k \text{ is bounded in } L^\infty (0, T; H^2_\Omega),
\] (3.14)
\[
u^k_t \text{ is bounded in } L^\infty (0, T; L^2_\Omega),
\] (3.15)
\[
u^k_t (t) g_1 (u^k (t)) \text{ is bounded in } L^1 (\Omega \times (0, T)),
\] (3.16)
\[G_2 \left( z^k (\rho, x, y, t) \right) \text{ is bounded in } L^\infty (0, T; L^1 (\Omega \times (0, 1))),
\] (3.17)
\[z^k (1, x, y, t) g_2 \left( z^k (1, x, y, t) \right) \text{ is bounded in } L^1 (\Omega \times (0, T)).
\] (3.18)

B. The second estimate.

First, we estimate \( u^k_t (0) \). Testing (3.1) by \( c^k_\Omega (t) \) and choosing \( t = 0 \), we obtain
\[
\| u^k_t (0) \|_2 \leq \left\{ \| u^k \|_{H^1_\Omega} + \| h \left( u^k_0 \right) \|_2 + \| g_1 \|_2 + \| g_2 \|_2 \right\}.
\]

Since \( h \left( u^k_0 \right) \), \( g_1 \), \( g_2 \), and \( f \) are bounded in \( L^2_\Omega \), (3.6) yields
\[
\| u^k_t (0) \|_2 \leq C.
\]

Differenting (3.1) with respect to \( t \), then multiplying the resulting identity by \( c^k_\Omega (t) \) and summing over \( j \) from 1 to \( k \), we get
\[
\frac{1}{2} \frac{d}{dt} \left\{ \| u^k_t (t) \|^2 + \| u^k (t) \|^2_{H^2_\Omega} \right\} + \left( h' \left( u^k (t) \right) u^k_t (t), u^k_t (t) \right) \\
+ \| \int_\Omega \left( u^k_t (t) \right)^2 g_1 \left( u^k (t) \right) \, dx \, dy + \| \int_\Omega u^k_t (t) z^k (1, x, y, t) g_2 \left( z^k (1, x, y, t) \right) \, dx \, dy \right| = 0.
\] (3.19)
Differentiating (3.4) with respect to $t$, then multiplying the resulting identity by $d_t^k u(t)$ and summing over $j$ from 1 to $k$ leads
\[
\frac{1}{2} \tau \frac{d}{dt} \left\| z_t^k (\rho, x, y, t) \right\|_2^2 + \frac{1}{2} \frac{d}{d\rho} \left\| z_t^k (\rho, x, y, t) \right\|_2^2 = 0. \tag{3.20}
\]
Integrating (3.20) over $(0, 1)$ with respect to $\rho$, we obtain
\[
\frac{\tau}{2} \frac{d}{dt} \int_0^1 \left\| z_t^k (\rho, x, y, t) \right\|_2^2 d\rho + \frac{1}{2} \left\| z_t^k (1, x, y, t) \right\|_2^2 - \frac{1}{2} \left\| u_t^k (t) \right\|_2^2 = 0. \tag{3.21}
\]
Summing (3.19) and (3.21), we obtain
\[
\frac{1}{2} \frac{d}{dt} \left\{ \left\| u_t^k (t) \right\|_2^2 + \left\| u_{tt}^k (t) \right\|_{H^2_\rho (\Omega)} + \tau \left\| z_t^k (\rho, x, y, t) \right\|_{L^2 (\Omega \times (0,1))} \right\}
+ \delta_1 \int_\Omega \left( u_{tt}^k (t) \right)^2 g_t^l \left( u_t^k (t) \right) dxdy + \frac{1}{2} \left\| z_t^k (1, x, y, t) \right\|_2^2
= - \langle h' (u^k (t)) u_t^k (t), u_{tt}^k (t) \rangle - \delta_2 \int_\Omega u_{tt}^k (t) z_t^k (1, x, y, t) g_2^l \left( z^k (1, x, y, t) \right) dxdy
+ \frac{1}{2} \left\| u_t^k (t) \right\|_2^2. \tag{3.22}
\]
Now, we estimate the right-side of (3.22). By using (2.6), we have
\[
\left\| - \langle h' (u^k (t)) u_t^k (t), u_{tt}^k (t) \rangle \right\| \leq \frac{C_6}{2} \left( \left\| u_t^k (t) \right\|_{H^2_\rho (\Omega)} + \left\| u_{tt}^k (t) \right\|_2^2 \right).
\]
Using (2.3) and Young’s inequality, we conclude
\[
\left\| - \delta_2 \int_\Omega u_{tt}^k (t) z_t^k (1, x, y, t) g_2^l \left( z^k (1, x, y, t) \right) dxdy \right\|
\leq \frac{C_3 \delta_2}{2} \left( C_\eta \left\| u_{tt}^k (t) \right\|_2^2 + \eta \left\| z_t^k (1, x, y, t) \right\|_2^2 \right).
\]
Then, we have
\[
\frac{1}{2} \frac{d}{dt} \left\{ \left\| u_t^k (t) \right\|_2^2 + \left\| u_{tt}^k (t) \right\|_{H^2_\rho (\Omega)} + \tau \left\| z_t^k (\rho, x, y, t) \right\|_{L^2 (\Omega \times (0,1))} \right\}
+ \delta_1 \int_\Omega \left( u_{tt}^k (t) \right)^2 g_t^l \left( u_t^k (t) \right) dxdy + \left( \frac{1}{2} - \frac{C_3 \delta_2}{2} \eta \right) \left\| z_t^k (1, x, y, t) \right\|_2^2
\leq \left( \frac{1}{2} + \frac{C_6 \delta_2}{2} + \frac{C_3 \delta_2}{2} C_\eta \right) \left\| u_{tt}^k (t) \right\|_2^2 + \frac{C_6}{2} S_2 \left\| u_t^k (t) \right\|_{H^2_\rho (\Omega)}.
\]
Since $g_t^l \geq 0$, taking $\eta$ small enough such that $c = \frac{1}{2} - \frac{C_3 \delta_2}{2} \eta > 0$, we have
\[
\frac{1}{2} \frac{d}{dt} \left\{ \left\| u_t^k (t) \right\|_2^2 + \left\| u_{tt}^k (t) \right\|_{H^2_\rho (\Omega)} + \tau \left\| z_t^k (\rho, x, y, t) \right\|_{L^2 (\Omega \times (0,1))} \right\}
+ \delta_1 \int_\Omega \left( u_{tt}^k (t) \right)^2 g_t^l \left( u_t^k (t) \right) dxdy + c \left\| z_t^k (1, x, y, t) \right\|_2^2
\leq C \left\| u_t^k (t) \right\|_2^2 + C \left\| u_{tt}^k (t) \right\|_{H^2_\rho (\Omega)}.
\]
Integrating the last inequality over $(0, t)$ and using Gronwall Lemma, we deduce that
\[
\left\| u_t^k (t) \right\|_2^2 + \left\| u_{tt}^k (t) \right\|_{H^2_\rho (\Omega)} + \tau \left\| z_t^k (\rho, x, y, t) \right\|_{L^2 (\Omega \times (0,1))}^2
\]
\[ e^{CT} \left\{ \left\| u_t^k(0) \right\|_{L^2}^2 + \left\| u_t^k(0) \right\|_{H^2(\Omega)}^2 + \tau \left\| z_t^k(\rho, x, y, 0) \right\|_{L^2(\Omega \times (0, 1))}^2 \right\}, \quad \forall \ t \in [0, T]. \]

Therefore, we conclude that
\[ u_t^k \text{ is bounded in } L^\infty (0, T; L^2(\Omega)), \quad (3.23) \]
\[ u_t^k \text{ is bounded in } L^\infty (0, T; H^2(\Omega)), \quad (3.24) \]
\[ z_t^k \text{ is bounded in } L^\infty (0, T; L^2(\Omega \times (0, 1))). \quad (3.25) \]

Since \( H^2(\Omega) \subset L^2(\Omega), \) we conclude that
\[ u_t^k \text{ is bounded in } L^\infty (0, T; L^2(\Omega)). \quad (3.26) \]

**C. The third estimate.**

Multiplying (3.1) by \( c^k(t), \) summing over \( j \) from 1 to \( k, \) implies
\[ \frac{1}{2} \frac{d}{dt} \left\{ \left\| u_t^k(t) \right\|_{L^2}^2 + \left\| u_t^k(t) \right\|_{H^2(\Omega)}^2 \right\} + (h (u^k(t)), u_t^k(t)) + \delta_1 \int_\Omega g_1 (u_t^k(t)) u_t^k(t) dxdy \\
+ \delta_2 \int_\Omega g_2 (z^k(1, x, y, t)) u_t^k(t) dxdy = \int_\Omega f u_t^k(t) dxdy. \quad (3.27) \]

Multiplying (3.4) by \( d^k(t), \) summing over \( j \) from 1 to \( k, \) leads to
\[ \frac{1}{2} \tau \frac{d}{dt} \left\| z^k(\rho, x, y, t) \right\|_{L^2(\Omega \times (0, 1))}^2 + \frac{1}{2} \left\| z^k(1, x, y, t) \right\|_{L^2(\Omega \times (0, 1))}^2 = \frac{1}{2} \left\| u_t^k(t) \right\|_{L^2}^2. \quad (3.28) \]

From (3.27) and (3.28), it follows
\[ \frac{1}{2} \frac{d}{dt} \left\{ \left\| u_t^k(t) \right\|_{L^2}^2 + \left\| u_t^k(t) \right\|_{H^2(\Omega)}^2 + \tau \left\| z^k(\rho, x, y, t) \right\|_{L^2(\Omega \times (0, 1))}^2 \right\} + \frac{1}{2} \left\| z^k(1, x, y, t) \right\|_{L^2(\Omega \times (0, 1))}^2 \\
= - \delta_1 \int_\Omega g_1 (u_t^k(t)) u_t^k(t) dxdy - \delta_2 \int_\Omega g_2 (z^k(1, x, y, t)) u_t^k(t) dxdy \\
- (h (u^k(t)), u_t^k(t)) + \int_\Omega f u_t^k(t) dxdy + \frac{1}{2} \left\| u_t^k(t) \right\|_{L^2}^2. \quad (3.29) \]

Now, we estimate the right-side of (3.29).
\[ \left| - \delta_1 \int_\Omega g_1 (u_t^k(t)) u_t^k(t) dxdy \right| \leq C_{\eta} \| g_1 (u_t^k(t)) \|_{L^2}^2 + \eta \| u_t^k(t) \|_{L^2}^2. \quad (3.30) \]
\[ \left| - \delta_2 \int_\Omega g_2 (z^k(1, x, y, t)) u_t^k(t) dxdy \right| \leq C_{\eta} \| g_2 (z^k(1, x, y, t)) \|_{L^2}^2 + \eta \| u_t^k(t) \|_{L^2}^2. \quad (3.31) \]
\[ \left| - (h (u^k(t)), u_t^k(t)) \right| \leq \frac{1}{2} \| h(u^k(t)) \|_{L^2}^2 + \frac{1}{2} \| u_t^k(t) \|_{L^2}^2 \\
\leq C \| u^k(t) \|_{H^2(\Omega)}^2 \| u_t^k(t) \|_{H^2(\Omega)}^2 + \frac{1}{2} \| u_t^k(t) \|_{L^2}^2 \\
\leq C + \frac{1}{2} \left\| u_t^k(t) \right\|_{L^2}^2. \quad (3.32) \]
\[ \left| \int_\Omega f u_t^k(t) dxdy \right| \leq \frac{1}{2} \| f \|_{L^2}^2 + \frac{1}{2} \left\| u_t^k(t) \right\|_{L^2}^2. \quad (3.33) \]
Therefore, we have
\[
\frac{d}{dt} \left\{ \|u^k(t)\|_2^2 + \|u^k(t)\|_{H^2_\rho(\Omega)}^2 + \|z^k(\rho, x, y, t)\|_{L^2(\Omega \times (0,1))}^2 \right\} + \|z^k(1, x, y, t)\|_2^2 \\
\leq C \|u^k(t)\|_2^2 + C \int_\Omega |g_1(u^k(t))|^2 \, dx \, dy + C \int_\Omega |g_2(z^k(1, x, y, t))|^2 \, dx \, dy + C.
\]

Using (2.1)-(2.2), Jensen’s inequality and the concavity of $\Psi^{-1}$, we obtain
\[
\int_\Omega |g_1(u^k(t))|^2 \, dx \, dy \\
\leq C_2 \int_{|u^k(t)| \geq c_1} u^k(t) g_1(u^k(t)) \, dx \, dy + \int_{|u^k(t)| \leq c_1} |g_1(u^k(t))|^2 \, dx \, dy \\
\leq C_2 \int_{|u^k(t)| \geq c_1} u^k(t) g_1(u^k(t)) \, dx \, dy + \int_{|u^k(t)| \leq c_1} \Psi^{-1}(u^k(t) g_1(u^k(t))) \, dx \, dy \\
\leq C_2 \int_{|u^k(t)| \geq c_1} u^k(t) g_1(u^k(t)) \, dx \, dy + C' \Psi^*(1) + C \int_{|u^k(t)| \geq c_1} u^k(t) g_1(u^k(t)) \, dx \, dy \\
\leq C' \Psi^*(1) + C \int_{\Omega} u^k(t) g_1(u^k(t)) \, dx \, dy \\
\leq C' \Psi^*(1) + C \int_{\Omega} u^k(t) g_1(u^k(t)) \, dx \, dy + C \int_{\Omega} u^k(t) g_1(u^k(t)) \, dx \, dy + C.
\]
and from (2.3) (that is $|g_2(s)| \leq C|s|$, $\forall s \in \mathbb{R}$)
\[
\int_\Omega |g_2(z^k(1, x, y, t))|^2 \, dx \, dy \leq C \int_\Omega z^k(1, x, y, t) g_2(z^k(1, x, y, t)) \, dx \, dy \leq C (-E'(t)).
\]

Submitting (3.35), (3.36) into (3.34), we have
\[
\frac{d}{dt} \left\{ \|u^k(t)\|_2^2 + \|u^k(t)\|_{H^2_\rho(\Omega)}^2 + \|z^k(\rho, x, y, t)\|_{L^2(\Omega \times (0,1))}^2 \right\} + \|z^k(1, x, y, t)\|_2^2 \\
\leq C \|u^k(t)\|_2^2 + C \Psi^*(1) + C (-E'(t)) + C.
\]

Integrating the last inequality over $(0, t)$ and using Gronwall Lemma, we deduce that
\[
\left\{ \frac{d}{dt} \left\{ \|u^k(t)\|_2^2 + \|u^k(t)\|_{H^2_\rho(\Omega)}^2 + \|z^k(\rho, x, y, t)\|_{L^2(\Omega \times (0,1))}^2 \right\} + \|z^k(1, x, y, t)\|_2^2 \right\} \\
\leq e^{CT} \left\{ \|u^k(0)\|_2^2 + \|u^k(0)\|_{H^2_\rho(\Omega)}^2 + \|z^k(\rho, x, y, 0)\|_{L^2(\Omega \times (0,1))}^2 \right\} \\
+ C \int_0^t \left\{ \|u^k(0)\|_2^2 + \|u^k(0)\|_{H^2_\rho(\Omega)}^2 + \|z^k(\rho, x, y, 0)\|_{L^2(\Omega \times (0,1))}^2 \right\} \, ds + C \int_0^t \left| E(0) - E(T) \right| \, ds \\
\leq e^{CT} \left\{ \|u^k(0)\|_2^2 + \|u^k(0)\|_{H^2_\rho(\Omega)}^2 + \|z^k(\rho, x, y, 0)\|_{L^2(\Omega \times (0,1))}^2 \right\} + CT + C |E(0)|,
\]
for all $t \in [0, T]$. Therefore, we conclude that
\[
z^k \text{ is bounded in } L^\infty (0, T; L^2(\Omega \times (0,1))).
\]

Applying Dunford-Pettis theorem, we conclude from (3.14)-(3.18), (3.23), (3.25), (3.26) and (3.39), after replacing the sequence $u^k$ and $z^k$ by subsequence if necessary, that
\[
u^k \to u \text{ weak-star in } L^\infty (0, T; H^2_\rho(\Omega) \cap L^2(\Omega)).
\]
Lemma 3.2. For each $u$, using Aubin-Lion’s theorem, we have $u^k 	o u$ weak-star in $L^\infty (0, T; L^2(\Omega))$, \( u^\prime_t \to u_t \) weak-star in $L^\infty (0, T; L^2(\Omega))$, $g_1 (u^k_t) \to \chi$ weak in $L^2 (\Omega \times (0, T))$, $z^k \to z$ weak-star in $L^\infty (0, T; L^2(\Omega \times (0, 1)))$, $z^k_t \to z_t$ weak-star in $L^\infty (0, T; L^2(\Omega \times (0, 1)))$, and $g_2 (z^k(1, x, y, t)) \to \psi$ weak in $L^2 (\Omega \times (0, T))$, for suitable $u \in L^\infty (0, T; H^2_0(\Omega) \cap L^2(\Omega))$, $z \in L^\infty (0, T; L^2(\Omega \times (0, 1)))$, $\chi \in L^2 (\Omega \times (0, 1))$, $\psi \in L^2 (\Omega \times (0, 1))$ for all $T \geq 0$.

Next, we have to show that $(u, z)$ is a solution of (2.9)-(2.11).

From (3.14) and (3.15), we have that $(u^k)$ is bounded in $L^2 (0, T; H^2_0(\Omega))$, $(u^\prime_t)$ is bounded in $L^2 (0, T; L^2(\Omega))$. Thanks to the compact embedding $H^2_0(\Omega) \hookrightarrow L^2(\Omega)$, using Aubin-Lion’s theorem, we have

$$u^k \to u \text{ strongly in } L^2(\Omega \times (0, T)).$$

Therefore,

$$u^k \to u \text{ a.e. in } \Omega \times (0, T). \quad (3.47)$$

Similarly, we obtain

$$u^\prime_t \to u_t \text{ a.e. in } \Omega \times (0, T), \quad (3.48)$$

$$z^k \to z \text{ a.e. in } \Omega \times (0, 1) \times (0, T). \quad (3.49)$$

Lemma 3.2. For each $T > 0$, one has $g_1(u_t), g_2(z(1, x, y, t)) \in L^1(\Omega \times (0, T))$ and $\|g_1 (u_t)\|_{L^1(\Omega \times (0, T))} \leq K_1, \|g_2 (z(1, x, y, t))\|_{L^1(\Omega \times (0, T))} \leq K_2$, where $K_1, K_2$ are constants independent of $t$.

Proof. By Assumption 1 and (3.47), we have

$$g_1 (u^k_t(x, y, t)) \to g_1(u_t(x, y, t)) \text{ a.e. in } Q,$$

$$0 \leq g_1 (u^\prime_t(x, y, t)) u^k_t(x, y, t) \to g_1(u_t(x, y, t))u_t(x, y, t) \text{ a.e. in } Q.$$

Hence, by (3.16) and Fatou’s Lemma, we have

$$\int_0^T \int_{\Omega} u_t(x, y, t)g_1(u_t(x, y, t))dxdydt \leq K \text{ for } T > 0. \quad (3.50)$$

By using (3.50) and (2.1), we have

$$\int_0^T \int_{\Omega} |g_1(u_t(x, y, t))|dxdydt$$

$$\leq \int_0^T \int_{|u_t| \leq \varepsilon_1} |g_1(u_t(x, y, t))|dxdydt + \int_0^T \int_{|u_t| \geq \varepsilon_1} |g_1(u_t(x, y, t))|dxdydt$$

$$\leq \int_0^T \int_{\Omega} |g_1(\varepsilon_1)|dxdydt + (|\Omega \times (0, T)|)^{\frac{1}{2}} \left( \int_0^T \int_{|u_t| \geq \varepsilon_1} |g_1(u_t(x, y, t))|^2dxdydt \right)^{\frac{1}{2}}$$

$$\leq C + C \left( \int_0^T \int_{\Omega} u_t(x, y, t)g_1(u_t(x, y, t))dxdydt \right)^{\frac{1}{2}} \leq K_1.$$

Similarly, from (2.3) (that is $|g_2(s)| \leq C|s|, \forall s \in \mathbb{R}$), we can conclude that

$$\int_0^T \int_{\Omega} |g_2(z(1, x, y, t))|dxdydt.$$
Applying (3.16), we deduce that
\[
\sup_{k} \frac{g_{1}(u_{k}^{k})}{g_{2}(z(1, x, y, t))} \leq K_{2}.
\]
The proof is completed. \hfill \square

**Lemma 3.3.** \( g_{1}(u_{k}^{k}) \rightarrow g_{1}(u_{i}) \) in \( L^{1}(\Omega \times (0, T)) \) and \( g_{2}(z^{k}(1, x, y, t)) \rightarrow g_{2}(z(1, x, y, t)) \) in \( L^{1}(\Omega \times (0, T)) \).

**Proof.** Let \( E \subset \Omega \times [0, T] \), and set
\[
E_{1} = \left\{ (x, y, t) \in E; |g_{1}(u_{k}^{k}(x, y, t))| \leq \frac{1}{\sqrt{|E|}} \right\}, \quad E_{2} = E \setminus E_{1},
\]
where \( |E| \) is the measure of \( E \). If \( M(r) := \inf\{|s|; s \in \mathbb{R} \text{ and } |g_{1}(s)| \geq r\}, \)
\[
\int_{E} |g_{1}(u_{k}^{k})| dx \leq |E| + \left( M\left( \frac{1}{\sqrt{|E|}} \right) \right)^{-1} \int_{E_{2}} |u_{k}^{k}| dx.
\]
Applying (3.16), we deduce that \( \sup_{k} \int_{E} |g_{1}(u_{k}^{k})| \rightarrow 0 \) as \( |E| \rightarrow 0 \). From Vital’s convergence theorem, we deduce that
\[
g_{1}(u_{k}^{k}) \rightarrow g_{1}(u_{i}) \text{ in } L^{1}(\Omega \times (0, T)).
\]
Similarly, we have
\[
g_{2}(z^{k}(1, x, y, t)) \rightarrow g_{2}(z(1, x, y, t)) \text{ weak in } L^{1}(\Omega \times (0, T)).
\]
The proof is completed. \hfill \square

From Lemma 3.2 and 3.3, we have
\[
g_{1}(u_{k}^{k}) \rightarrow g_{1}(u_{i}) \text{ weak in } L^{2}(\Omega \times (0, T)),
\]
\[
g_{2}(z^{k}) \rightarrow g_{2}(z) \text{ weak in } L^{2}(\Omega \times (0, T)),
\]
and this implies that for all \( v \in L^{2}(0, T; L^{2}(\Omega)) \)
\[
\int_{0}^{T} \int_{\Omega} g_{1}(u_{k}^{k}) v dx dt \rightarrow \int_{0}^{T} \int_{\Omega} g_{1}(u_{i}) v dx dt, \quad (3.51)
\]
\[
\int_{0}^{T} \int_{\Omega} g_{2}(z^{k}(1, x, y, t)) v dx dt \rightarrow \int_{0}^{T} \int_{\Omega} g_{2}(z(1, x, y, t)) v dx dt. \quad (3.52)
\]
as \( k \rightarrow +\infty \). By using (2.6)$_{1}$, for any \( \phi \in C_{0}^{\infty}(0, T) \), we have
\[
\left| \int_{0}^{T} \int_{\Omega} \left( h(u_{k}^{k}) - h(u(t)) \right) w_{j} \phi(t) dx dt \right|
\leq C \int_{0}^{T} \int_{\Omega} (|u_{k}^{k}(t)|^{a} + |u(t)|^{a}) |u_{k}^{k}(t) - u(t)| w_{j} dx dt
\leq C \int_{0}^{T} \left( |u_{k}^{k}(t)|_{L^{\infty}(\Omega)}^{a} + |u(t)|_{L^{\infty}(\Omega)}^{a} \right) \int_{\Omega} |u_{k}^{k}(t) - u(t)| w_{j} dx dt
\leq C \int_{0}^{T} \|u(t) - u(t)\|_{2} dt.
Global Attractor for a Suspension Bridge Problem

Since we have proved that $u^k \to u$ strongly in $L^2(\Omega \times (0, T))$, by using the embedding $L^2(\Omega) \hookrightarrow L^1(\Omega)$, we have

$$
\int_0^T \|u^k(t) - u(t)\|_2 dt \to 0, \quad \text{as } k \to 0,
$$

which implies that

$$
h(u^k(t)) \to h(u(t)) \text{ weak in } L^2(\Omega \times (0, T)).
$$

Then, we have

$$
\int_0^T \int_\Omega h(u^k) v dx dy dt \to \int_0^T \int_\Omega h(u) v dx dy dt \quad \text{for all } v \in L^2(0, T; L^2(\Omega)),
$$

as $k \to +\infty$. It follows at once from (3.40), (3.42), (3.44), (3.51)-(3.53) that, for each $v \in L^2(0, T; L^2(\Omega))$ and $w \in L^2(0, T; L^2(\Omega \times (0, 1)))$,

$$
\int_0^T \int_\Omega (u_{tt}^k + \Delta^2 u^k + h(u^k) + \delta_1 g_1(u^k) + \delta_2 g_2(z^k(1, x, y, t)) - f) v dx dy dt
$$

and

$$
\int_0^T \int_\Omega (u_{tt} + \Delta^2 u + h(u) + \delta_1 g_1(u) + \delta_2 g_2(z(1, x, y, t)) - f) v dx dy dt = 0,
$$

as $k \to +\infty$. Hence,

$$
\int_0^T \int_\Omega (u_{tt} + \Delta^2 u + h(u) + \delta_1 g_1(u) + \delta_2 g_2(z(1, x, y, t)) - f) v dx dy dt = 0,
$$

for all $v \in L^2(0, T; L^2(\Omega))$.

Thus, problem (2.9)-(2.11) admits a global weak solution $(u, z)$.

E. Uniqueness. Let $(u_1, z_1)$ and $(u_2, z_2)$ be two solutions of problem (2.9)-(2.11). Then $(w, \bar{w}) = (u_1, z_1) - (u_2, z_2)$ satisfies

$$
\begin{cases}
    w_{tt} + \Delta^2 w + h(u_1(x, y, t)) - h(u_2(x, y, t)) + \delta_1 g_1(u_{1tt}(x, y, t)) - \delta_1 g_1(u_{2tt}(x, y, t)) + \delta_2 g_2(z_1(1, x, y, t)) - \delta_2 g_2(z_2(1, x, y, t)) = 0, &\text{in } \Omega \times (0, +\infty),
    \\
    \tau \bar{w}_t(\rho, x, y, t) + \frac{\partial}{\partial \rho} \bar{w}(\rho, x, y, t) = 0, &\text{in } (0, 1) \times \Omega \times (0, +\infty),
    \\
    \bar{w}(0, x, y, t) = u_{1tt}(x, y, t) - u_{2tt}(x, y, t), &\text{in } \Omega \times (0, +\infty),
    \\
    w(x, y, 0) = 0, \quad w_t(x, y, 0) = 0, &\text{in } \Omega,
    \\
    \bar{w}(\rho, x, y, 0) = 0, &\text{in } \Omega \times (0, 1),
    \\
    w(0, y, t) = w_{xx}(0, y, t) = w(\pi, y, t) = w_{xx}(\pi, y, t) = 0, &\text{for } (y, t) \in (-l, l) \times (0, +\infty),
    \\
    w_{yy}(x, \pm l, t) + \sigma w_{xx}(x, \pm l, t) = w_{yy}(x, \pm l, t) \quad &\text{for } (x, t) \in (0, \pi) \times (0, +\infty),
    \\
    + (2 - \sigma) w_{xy}(x, \pm l, t) = 0, &\text{for } (x, t) \in (0, \pi) \times (0, +\infty).
\end{cases}
$$

(3.54)
Multiplying the first equation in (3.54) by \( w_t \), integrating over \( \Omega \) and using integration by parts, we get
\[
\frac{1}{2} \frac{d}{dt} \left( \|w_t\|_2^2 + \|w\|_{L^2(\Omega)}^2 \right) + (h(u_1) - h(u_2), w_t) + \delta_1 (g_1(u_1) - g_1(u_2), w_t) + \delta_2 (g_2(z_1(1, x, y, t)) - g_2(z_2(1, x, y, t)), w_t) = 0. \tag{3.55}
\]

Multiplying the second equation in (3.54) by \( \tilde{w} \), integrating over \( \Omega \times (0, 1) \) and using integration by parts, we get
\[
\frac{\tau}{2} \frac{d}{dt} \|\tilde{w}\|_2^2 + \frac{1}{2} \frac{d}{dt} \|\tilde{w}\|_2^2 = 0. \tag{3.56}
\]

Then,
\[
\frac{\tau}{2} \frac{d}{dt} \|\tilde{w}\|_{L^2(\Omega \times (0, 1))}^2 + \frac{1}{2} (\|\tilde{\omega}(1, x, y, t)\|_2^2 - \|w_1\|_2^2) = 0. \tag{3.57}
\]

From (3.55) and (3.57), we get
\[
\frac{1}{2} \frac{d}{dt} \left( \|w_t\|_2^2 + \|w\|_{L^2(\Omega)}^2 + \tau \|\tilde{w}\|_{L^2(\Omega \times (0, 1))}^2 \right) + \delta_1 (g_1(u_1) - g_1(u_2), w_t) + \frac{1}{2} \|\tilde{w}(1, x, y, t)\|_2^2
\]
\[
= - (h(u_1) - h(u_2), w_t) - \delta_2 (g_2(z_1(1, x, y, t)) - g_2(z_2(1, x, y, t)), w_t) + \frac{1}{2} \|w_t\|_2^2
\]
\[
\leq \|w_t\|_2^2 + \frac{1}{2} \|h(u_1) - h(u_2)\|_2^2 + \eta \|\tilde{w}(1, x, y, t)\|_2^2 + C_\eta \|w_t\|_2^2 + C_\eta \|w\|_{L^2(\Omega)}^2 + \eta \|\tilde{w}(1, x, y, t)\|_{L^2(\Omega \times (0, 1))}^2.
\tag{3.58}
\]

Since \( \delta_1 (g_1(u_1) - g_1(u_2), w_t) \geq 0 \), we choose \( \eta \) small enough such that \( \frac{1}{2} - \eta > 0 \), we deduce that
\[
\frac{1}{2} \frac{d}{dt} \left( \|w_t\|_2^2 + \|w\|_{L^2(\Omega)}^2 + \tau \|\tilde{w}\|_{L^2(\Omega \times (0, 1))}^2 \right) \leq C \left( \|w_t\|_2^2 + \|w\|_{L^2(\Omega)}^2 \right),
\]
where \( C \) is a positive constant. Using Gronwall’s Lemma, we conclude
\[
\|w_t\|_2^2 + \|w\|_{L^2(\Omega)}^2 + \tau \|\tilde{w}\|_{L^2(\Omega \times (0, 1))}^2 = 0.
\]

Hence, uniqueness follows. \( \square \)

4. Absorbing set. This section is devoted to establish an absorbing set to system (2.9)-(2.11). Thanks to Theorem 3.1, we can define the semigroup \( S(t) \)
\[
S(t) : \mathcal{H} \to \mathcal{H}, (u_0, u_1, f_0) \mapsto (u(t), u_t(t), z(\rho, x, y, t)),
\]
where \( \mathcal{H} := L^2_0(\Omega) \times L^2(\Omega) \times L^2_{G_2}(0, 1) \times \Omega \), which is equipped with the norm
\[
\| (u(t), u_t(t), z(\rho, x, y, t)) \|_{\mathcal{H}}^2 = \|u(t)\|_{L^2(\Omega)}^2 + \|u_t(t)\|_2^2 + \int_0^1 \int_{\Omega} G_2(z(\rho, x, y, t)) \, d\rho \, dx \, dy,
\]
and \( (u(t), u_t(t), z(\rho, x, y, t)) \) is the solution to problem (2.9)-(2.11) at time \( t \). Here the definition of \( L^2_{G_2}(0, 1) \times \Omega \) is as follows
\[
L^2_{G_2}(0, 1) \times \Omega = \left\{ z(\rho, x, y) \mid z \in L^2((0, 1) \times \Omega) \text{ and } \int_0^1 \int_{\Omega} G_2(z(\rho, x, y)) \, d\rho \, dx \, dy < \infty \right\}.
\]
Definition 4.1. Let $X$ be a Banach space. A set $B \subset X$ is an absorbing set for the semigroup $S(t) : X \to X$ if, given any bounded set $B \subset X$, there exists a time $t_0(B)$ such that $S(t)B \subset B$, for every $t \geq t_0(B)$.

In order to show the existence of an absorbing set to problem (2.9)-(2.11), we need to establish several lemmas.

Lemma 4.2. The functional
$$I_1(t) = \int_{\Omega} u_t \, dx \, dy$$
satisfies, along the solution to (2.9)-(2.11),
$$\frac{dI_1(t)}{dt} \leq \|u_t(t)\|^2 - [1 - \eta S_2^2(\delta_1 + \delta_2 + 1)] \|u(t)\|_{H_0^2(\Omega)}^2 + \frac{\delta_1}{4\eta} \int_{\Omega} |g_1(u_t)|^2 \, dx \, dy$$
$$+ \frac{\delta_2}{4\eta} \int_{\Omega} |g_2(z(1, x, y, t))|^2 \, dx \, dy - \int_{\Omega} H(u(t)) \, dx \, dy + \frac{1}{4\eta} \|f\|^2_2. \quad (4.1)$$

Proof. Direct differentiation, using (2.9) gives
$$\frac{dI_1(t)}{dt} = \|u_t(t)\|^2 - \|u(t)\|_{H_0^2(\Omega)}^2 - \delta_1 \int_{\Omega} g_1(u_t) u' dx \, dy - \delta_2 \int_{\Omega} g_2(z(1, x, y, t)) u' dx \, dy$$
$$- \int_{\Omega} h(u) u' dx \, dy + \int_{\Omega} f u' dx \, dy. \quad (4.2)$$

Using Assumption 2 and Lemma 2.2, we have, for any $\eta > 0$,
$$\begin{cases}
- \int_{\Omega} g_1(u_t) u' dx \, dy \leq \eta S_2^2 \|u(t)\|^2_{H_0^2(\Omega)} + \frac{1}{4\eta} \int_{\Omega} |g_1(u_t)|^2 \, dx \, dy, \\
- \int_{\Omega} g_2(z(1, x, y, t)) u' dx \, dy \leq \eta S_2^2 \|u(t)\|^2_{H_0^2(\Omega)} + \frac{1}{4\eta} \int_{\Omega} |g_2(z(1, x, y, t))|^2 \, dx \, dy, \\
- \int_{\Omega} h(u) u' dx \, dy \leq - \int_{\Omega} H(u(t)) \, dx \, dy, \\
\int_{\Omega} f u' dx \, dy \leq \eta S_2^2 \|u(t)\|^2_{H_0^2(\Omega)} + \frac{1}{4\eta} \|f\|^2_2.
\end{cases} \quad (4.3)$$

Substituting (4.3) into (4.2), we obtain the result. The proof is completed.

Lemma 4.3. The functional
$$I_2(t) = \int_{\Omega} \int_{0}^{1} e^{-2\tau \rho} G_2(z(\rho, x, y, t)) \, d\rho \, dx \, dy$$
satisfies, along the solution to problem (2.9)-(2.11),
$$\frac{dI_2(t)}{dt} \leq - 2I_2(t) + \frac{\alpha_2}{\tau} \int_{\Omega} u_t(t) g_1(u_t(t)) dx \, dy$$
$$- \frac{\alpha_1 e^{-2\tau}}{\tau} \int_{\Omega} z(1, x, y, t) g_2(z(1, x, y, t)) dx \, dy. \quad (4.4)$$

Proof. Direct differentiation, using (2.9) gives
$$\frac{dI_2(t)}{dt} = \int_{\Omega} \int_{0}^{1} e^{-2\tau \rho} z_2(\rho, x, y, t) g_2(z(\rho, x, y, t)) \, d\rho \, dx \, dy$$
Therefore,\[ \frac{1}{\tau} \int_{\Omega} f(u(t))dxdy \leq \frac{1}{\lambda_1} \|f\|_2^2 + \frac{\lambda_1}{4} \|u(t)\|_2^2 \leq \frac{1}{\lambda_1} \|f\|_2^2 + \frac{1}{4} \|u(t)\|_2^2, \]

and because \( H(u(t)) \geq -C_5 \), we get
\[ \bar{E}(t) \geq \frac{1}{2} \|u(t)\|_2^2 + \frac{1}{4} \|u(t)\|_2^2_H^2 - C_5 |\Omega| - \frac{1}{\lambda_1} \|f\|_2^2 - \frac{1}{4} \|u(t)\|_2^2_H^2 + \xi \int_0^1 G_2(z(\rho, x, y, t)) d\rho dxdy + C_5 |\Omega| + \frac{1}{\lambda_1} \|f\|_2^2 \]
\[ \geq a_4 \|u(t)\|_2^2 + \frac{1}{4} \|u(t)\|_2^2_H^2 + \xi \int_0^1 G_2(z(\rho, x, y, t)) d\rho dxdy \]

where \( a_4 = \min \left\{ \frac{1}{4}, \xi \right\} \) is a positive constant.

We define the functional
\[ L(t) = M \bar{E}(t) + N I_1(t) + I_2(t), \]
where \( M, N > 0 \) are constants to be specified later.

**Lemma 4.4.** There exist two positive constants \( B_1 \) and \( B_2 \) depending on \( M \) and \( N \) such that for all \( t > 0 \),
\[ B_1 \bar{E}(t) \leq L(t) \leq B_2 \bar{E}(t). \]
Proof. We directly estimate the latter two terms of (4.9)

\[
|N I_1(t) + I_2(t)|
\]

\[
= |N \int_\Omega uu_t dx dy + \int_\Omega \int_0^1 e^{-2\tau_p} G_2(z(\rho, x, y, t)) d\rho dx dy|
\]

\[
\leq \frac{N}{2} \|u_t(t)\|_2^2 + \frac{N}{2} S_2^2 \|u(t)\|_{H_2(\Omega)}^2 + \int_\Omega \int_0^1 G_2(z(\rho, x, y, t)) d\rho dx dy
\]

\[
\leq a_4 \left\{ \|u_t(t)\|_2^2 + \|u(t)\|_{H_2(\Omega)}^2 + \int_\Omega \int_0^1 G_2(z(\rho, x, y, t)) d\rho dx dy \right\}
\]

\[
\leq a_4 \frac{a_4}{a_3} E(t),
\]

(4.11)

where \(a_4 = \max \left\{ \frac{N}{2}, \frac{N}{2} S_2^2, 1 \right\} \) is a positive constant. Thus

\[
|L(t) - M \bar{E}(t)| \leq a_4 \frac{a_4}{a_3} E(t).
\]

(4.12)

Choosing \(M\) large enough such that \(M - a_4 \frac{a_4}{a_3} > 0\), we obtain the result. \(\Box\)

Lemma 4.5. For \(N\) small enough and \(M\) large enough, there exist positive constants \(\beta_1, \beta_2\) such that

\[
\beta_1 \| (u(t), u_t(t), z(\rho, x, y, t)) \|_{H_2}^2 - C_\tau \|f\|_{L^2}^2
\]

\[
\leq L(t)
\]

\[
\leq \beta_2 \| (u(t), u_t(t), z(\rho, x, y, t)) \|_{H_2}^2 + C_8 \|f\|_{L^2}^2 + C_9.
\]

(4.13)

Proof. Since

\[
\frac{d}{dt} \bar{E}(t) \leq \frac{d}{dt} E(t)
\]

\[
\leq -a_1 \int_\Omega g_1(u_t(t))u_t(t) dx dy - a_2 \int_\Omega g_2(z(1, x, y, t)) z(1, x, y, t) dx dy \leq 0,
\]

(4.14)

one has

\[
\| (u(t), u_t(t), z(\rho, x, y, t)) \|_{H_2}^2 \leq \frac{1}{a_3} \bar{E}(t) \leq \frac{1}{a_3} \bar{E}(0) \leq C.
\]

(4.15)

Using (2.6), we have

\[
\left| \int_\Omega H(u(t)) dx dy \right| \leq \left| \int_\Omega h(u(t))u_t(t) dx dy \right| \leq \eta S_2^2 \|u(t)\|_{H_2}^2 + C_7 \|h(u(t))\|_{L^\infty(\Omega)}^2.
\]

(4.16)

Thus, by (4.16) and Young’s inequality, we have, on the one hand,

\[
L(t) \leq M \bar{E}(t) + \frac{N}{2} \|u_t(t)\|_2^2 + \frac{N}{2} S_2^2 \|u(t)\|_{H_2(\Omega)}^2 + \int_\Omega \int_0^1 G_2(z(\rho, x, y, t)) d\rho dx dy
\]

\[
\leq \frac{M + N}{2} \|u_t(t)\|_2^2 + \frac{M + 2M \eta S_2^2 + MS_2^2 + NS_2^2}{2} \|u(t)\|_{H_2(\Omega)}^2
\]

\[
+ (M \xi + 1) \int_\Omega \int_0^1 G_2(z(\rho, x, y, t)) d\rho dx dy
\]

\[
+ \left( \frac{M}{2} + \frac{M}{M_1} \right) \|f\|_{L^2}^2 + (C + MC_9(\Omega))
\]

\[
\leq \beta_2 \| (u(t), u_t(t), z(\rho, x, y, t)) \|_{H_2}^2 + C_8 \|f\|_{L^2}^2 + C_9.
\]

(4.17)
and on the other hand, we have, for any $\epsilon > 0$,
\[
L(t) \geq \left( \frac{M(1 - 2\epsilon S_2^2) - NS_2^2}{2} \right) \|u(t)\|^2_{H_2^2(\Omega)} + \frac{M - N}{2} \|u_t(t)\|^2_2 \\
+ (\xi M + e^{-2\tau}) \int_0^1 G_2(z(\rho, x, y, t))d\rho d\sigma - \left( \frac{M}{4\epsilon} - \frac{M}{\Lambda_1} \right) \|f\|^2_2. \tag{4.18}
\]
We first choose $\epsilon$ small enough such that $1 - 2\epsilon S_2^2 > 0$ and $\frac{M}{4\epsilon} - \frac{M}{\Lambda_1} > 0$. Then we choose $M$ large enough and $N$ small enough such that
\[
\frac{M(1 - 2\epsilon S_2^2) - NS_2^2}{2} > 0, \quad \frac{M - N}{2} > 0. \tag{4.19}
\]
Thus, we have
\[
L(t) \geq \beta_1 \| (u(t), u_t(t), z(\rho, x, y, t)) \|^2_{H_2^2} - C_7 \|f\|^2_2.
\]
The proof is completed. \hfill \Box

**Theorem 4.6.** Under the hypotheses of Theorem 3.1, the semigroup $S(t)$ of problem (2.9)-(2.11) possesses a bounded absorbing set $B_1$ in $\mathcal{H}$.

**Proof.** Before we make proof, we need to introduce two functions as follows:
\[
H_1(t) = \int_t^1 \frac{1}{H_2(s)} ds, \tag{4.20}
\]
and
\[
H_2(t) = \begin{cases} \quad t & \text{if } \Psi \text{ is linear on } [0, \epsilon_1], \\ t\Psi'(\epsilon_0 t) & \text{if } \Psi'(0) = 0 \text{ and } \Psi'' > 0 \text{ on } (0, \epsilon_1]. \tag{4.21}
\end{cases}
\]
Using (2.14), Lemma 4.2 and Lemma 4.3, direct computations, we have
\[
\frac{dL(t)}{dt} = M\frac{dE(t)}{dt} + N\frac{dI_1(t)}{dt} + \frac{dI_2(t)}{dt} \\
\leq - \left( Ma_1 - \frac{\alpha_2}{\tau} \right) \int_\Omega u_t(t)g_1(u_t(t))d\rho d\sigma \\
- \left( Ma_2 + \frac{\alpha_1 e^{-2\tau}}{\tau} \right) \int_\Omega z(1, x, y, t)g_2(z(1, x, y, t))d\rho d\sigma \\
- [1 - \eta S_2^2(\delta_1 + \delta_2 + 1)]N\|u(t)\|^2_{H_2^2(\Omega)} - N\|u_t(t)\|^2_2 \\
- 2I_2(t) - N \int_\Omega H(u(t))d\rho d\sigma + 2N\|u_t(t)\|^2_2 + \frac{\delta_1 N}{4\eta} \int_\Omega |g_1(u_t(t))|^2 d\rho d\sigma \\
+ \frac{\delta_2 N}{4\eta} \int_\Omega |g_2(z(1, x, y, t))|^2 d\rho d\sigma + \frac{N}{4\eta} \|f\|^2_2. \tag{4.22}
\]
From (2.3), we have $|g_2(s)| \leq C_3|s|$, then
\[
\int_\Omega |g_2(z(1, x, y, t))|^2 d\rho d\sigma \leq C_3 \int_\Omega g_2(z(1, x, y, t))z(1, x, y, t)d\rho d\sigma. \tag{4.23}
\]
Therefore, we have
\[
\frac{dL(t)}{dt}
\]
Then, we have
\[
\text{We choose } \eta > 0 \text{ large enough such that } 1 - \eta S_2^2 (\delta_1 + \delta_2 + 2) > 0.
\]

Then, we choose \( M \) large enough and \( N \) small enough such that (4.22) remains valid and, further,
\[
Ma_1 - \alpha_2 \frac{\tau}{\tau} > 0, \quad Ma_2 + \alpha_1 e^{-2\tau} - \frac{\delta_2 N}{4\eta} C_3 > 0.
\]

Then, we have
\[
\frac{d L(t)}{dt} \leq -\beta_3 \left\{ \frac{1}{2} \| u_t(t) \|_2^2 + \frac{1}{2} \| u(t) \|_{H^2(\Omega)}^2 + \xi \int_\Omega \int_0^1 G_2 (z(\rho, x, y, t)) d\rho dx dy + \int_\Omega H(u(t)) dx dy - \int_\Omega f(u(t)) dx dy + C_5 |\Omega | + \frac{1}{\Lambda_1} \| f \|_2^2 \right\}
\]
\[
+ C_{10} \left( \| u_t(t) \|_2^2 + \| g_1 (u_t(t)) \|_2^2 \right) + \left( \frac{N}{4\eta} + \frac{1}{4\eta} \right) \| f \|_2^2 + C_5 |\Omega | N
\]
\[
= -\beta_3 \tilde{E}(t) + C_{10} \left( \| u_t(t) \|_2^2 + \| g_1 (u_t(t)) \|_2^2 \right) + C_{11}. \tag{4.24}
\]

We define
\[
\Omega^+ = \{(x, y) \in \Omega; \ |u_t(t)| \geq \epsilon_1 \}, \quad \Omega^- = \{(x, y) \in \Omega; \ |u_t(t)| \leq \epsilon_1 \}.
\]

From (2.1), we can deduce that
\[
\int_{\Omega^+} \left( \| u_t(t) \|_2^2 + \| g_1 (u_t(t)) \|_2^2 \right) dx dy \leq \beta_5 \int_{\Omega^+} u_t(t) g_1 (u_t(t)) dx dy \leq -\beta_5 \tilde{E}'(t). \tag{4.25}
\]

**Case 1:** \( \Psi \) is linear on \([0, \epsilon_1]\).

In this case, one can easily check that there exists \( \beta_5 > 0 \) such that \( |g_1(s)| \leq \beta_5 |s| \) for all \( |s| \leq \epsilon_1 \), and thus
\[
\int_{\Omega^-} \left( \| u_t(t) \|_2^2 + \| g_1 (u_t(t)) \|_2^2 \right) dx dy \leq \beta_5 \int_{\Omega^-} u_t(t) g_1 (u_t(t)) dx dy \leq -\beta_5 \tilde{E}'(t). \tag{4.26}
\]

Substitution of (4.25) and (4.26) into (4.24) gives
\[
\frac{d}{dt} \left( L(t) + \mu_1 \tilde{E}(t) \right) \leq -\beta_3 \tilde{E}(t) + C_{11}, \tag{4.27}
\]
where \( \mu_1 = C_{10} (\beta_4 + \beta_5) \). Thus, we have
\[
\frac{d}{dt} \left( L(t) + \mu_1 \tilde{E}(t) \right) \leq -\beta_3 H_2 \left( \tilde{E}(t) \right) + C_{11}. \tag{4.28}
\]

**Case 2:** \( \Psi'(0) = 0 \) and \( \Psi'' > 0 \) on \((0, \epsilon_1]\).
Since $\Psi$ is convex and increasing, $\Psi^{-1}$ is concave and increasing. By the virtue of (2.2), the reversed Jensen’s inequality for concave function, and (2.14), it follows that
\[
\int_{\Omega^-} \left( |u_t(t)|^2 + |g_1(u_t(t))|^2 \right) \, dx \, dy \leq \int_{\Omega^-} \Psi^{-1}(u_t(t)g_1(u_t(t))) \, dx \, dy
\]
\[
\leq |\Omega| \Psi^{-1} \left( \frac{1}{|\Omega|} \int_{\Omega^-} u_t(t)g_1(u_t(t)) \, dx \, dy \right) \leq C_{12} \Psi^{-1} \left( -C_{13} \bar{E}'(t) \right). \tag{4.29}
\]
A combination of (4.24), (4.25) and (4.29) yields
\[
\frac{d}{dt} \left( L(t) + \beta_4 \bar{E}(t) \right) \leq -\beta_3 \bar{E}(t) + C_{12} \Psi^{-1} \left( -C_{13} \bar{E}'(t) \right) + C_{11}. \tag{4.30}
\]
Let us denote by $\Psi^*$ the conjugate function of the convex function $\Psi$, i.e.,
\[
\Psi^*(s) = \sup_{t \in \mathbb{R}^+} (st - \Psi(t)).
\]
Then $\Psi^*$ is the Legendre transform of $\Psi$, which is given by
\[
\Psi^*(s) = s (\Psi')^{-1}(s) - \Psi \left( (\Psi')^{-1}(s) \right), \quad \forall s \geq 0, \tag{4.31}
\]
and satisfies the following inequality:
\[
st \leq \Psi^*(s) + \Psi(t), \quad \forall s, t \geq 0. \tag{4.32}
\]
The relation (4.32) and the fact that $\Psi'(0) = 0$, and $(\Psi')^{-1}, \Psi$ are increasing functions yield
\[
\Psi^*(s) \leq s (\Psi')^{-1}(s), \quad \forall s \geq 0. \tag{4.33}
\]
Making use of $\bar{E}'(t) \leq 0$, $\Psi''(t) \geq 0$, (4.28) and (4.31), we derive for $\epsilon_0 > 0$ small enough
\[
\frac{d}{dt} \left[ \Psi' \left( \epsilon_0 \bar{E}(t) \right) \left\{ L(t) + \beta_4 \bar{E}(t) \right\} + C_{12} C_{13} \bar{E}(t) \right]
\]
\[
\leq \Psi' \left( \epsilon_0 \bar{E}(t) \right) \left\{ -\beta_3 \bar{E}(t) + C_{12} \Psi^{-1} \left( -C_{13} \bar{E}'(t) \right) + C_{11} \right\} + C_{12} C_{13} \bar{E}'(t)
\]
\[
\leq -\beta_3 \Psi' \left( \epsilon_0 \bar{E}(t) \right) \bar{E}(t) + C_{12} C_{13} \Psi' \left( \psi' \left( \epsilon_0 \bar{E}(t) \right) \right) + C_{11} \Psi' \left( \epsilon_0 \bar{E}(0) \right)
\]
\[
\leq -\beta_3 \Psi' \left( \epsilon_0 \bar{E}(t) \right) \bar{E}(t) + C_{12} C_{13} \psi' \left( \epsilon_0 \bar{E}(t) \right) \epsilon_0 \bar{E}(t) + C_{11} \psi' \left( \epsilon_0 \bar{E}(0) \right)
\]
\[
\leq -\beta_7 \Psi' \left( \epsilon_0 \bar{E}(t) \right) \bar{E}(t) + C_{11} \Psi' \left( \epsilon_0 \bar{E}(0) \right)
\]
\[
= -\beta_7 H_2 \left( \bar{E}(t) \right) + C_{11} \Psi' \left( \epsilon_0 \bar{E}(0) \right). \tag{4.34}
\]
Let
\[
\bar{L}(t) = \begin{cases} L(t) + \mu_1 \bar{E}(t), & \text{if } \Psi \text{ is linear on } [0, \epsilon_1], \\ \Psi' \left( \epsilon_0 \bar{E}(t) \right) \left\{ L(t) + \beta_4 \bar{E}(t) \right\} + C_{12} C_{13} \bar{E}(t), & \text{if } \Psi'(0) = 0 \text{ and } \Psi'' > 0 \text{ on } (0, \epsilon_1]. \end{cases} \tag{4.35}
\]
From (4.28) and (4.34), it follows
\[
\frac{d}{dt} \bar{L}(t) \leq -\beta_8 H_2 \left( \bar{E}(t) \right) + C_{14}, \quad \forall t \geq 0, \tag{4.36}
\]
where \( C_{14} = \max \left\{ C_{11}, C_{11}\Psi' \left( \epsilon_0 \tilde{E}(0) \right) \right\} \). On the other hand, after choosing \( M > 0 \) larger if needed, we can observe that \( L(t) \) is equivalent to \( \tilde{E}(t) \). So, \( \tilde{L}(t) \) is also equivalent to \( \tilde{E}(t) \). Thus, we have

\[
\frac{d}{dt} \tilde{L}(t) \leq -\beta_3 H_2 \left( \tilde{L}(t) \right) + C_{14}, \quad \forall t \geq 0. \tag{4.37}
\]

Applying Lemma 4.7 below, we infer from (4.37)

\[
\tilde{L}(t) \leq H_1^{-1}(\beta_3 t) + H_2^{-1} \left( \frac{C_{14}}{\beta_3} \right), \quad \forall t > 0.
\]

Consequently, the equivalence of \( L(t), \tilde{L}(t) \) and \( E(t) \) yields the estimate

\[
L(t) \leq \omega_1 H_1^{-1}(\beta_3 t) + \omega_2 H_2^{-1} \left( \frac{C_{14}}{\beta_3} \right),
\]

for all \( t > 0 \). From (4.13), we have

\[
\| (u(t), u_t(t), z(\rho, x, y, t)) \|_{\mathcal{H}}^2 \leq \frac{\omega_1}{\beta_1} H_1^{-1}(\beta_3 t) + \frac{1}{\beta_1} \left( \omega_1 H_2^{-1} \left( \frac{C_{14}}{\beta_9} \right) + C_7 \| f \|_2^2 \right),
\]

for all \( t > 0 \). Thus, for \( R > \frac{1}{\beta_1} \left( \omega_1 H_2^{-1} \left( \frac{C_{14}}{\beta_9} \right) + C_7 \| f \|_2^2 \right) \), the ball \( B_1 = B(0, R) \) is a bounded absorbing set of \((\mathcal{H}, S(t))\). The proof is completed. \( \square \)

Now, we give a detailed proof of solving inequality (4.37).

**Lemma 4.7.** Let \( y(t) \) be a positive absolutely continuous function on \((0, +\infty)\) satisfying

\[
\frac{d}{dt} y(t) \leq -\beta H_2(y(t)) + \gamma, \tag{4.38}
\]

where \( \beta > 0, \gamma > 0 \), \( H_2(s) \) is defined as (4.21). Then for any \( t > 0 \),

\[
y(t) \leq H_1^{-1}(\beta t) + H_2^{-1} \left( \frac{\gamma}{\beta} \right), \tag{4.39}
\]

where \( H_1(s) \) is defined as (4.20), \( H_1^{-1}, H_2^{-1} \) represent the inverse function of \( H_1, H_2 \) respectively.

**Proof.** If \( y(0) \leq H_2^{-1} \left( \frac{\gamma}{\beta} \right) \), then for all \( t \geq 0 \),

\[
y(t) \leq H_2^{-1} \left( \frac{\gamma}{\beta} \right). \tag{4.40}
\]

If \( y(0) \geq H_2^{-1} \left( \frac{\gamma}{\beta} \right) \), then there exists \( t_0 \in (0, +\infty) \) such that

\[
\begin{cases}
y(t) \geq H_2^{-1} \left( \frac{\gamma}{\beta} \right), & \text{for all } 0 \leq t < t_0, \\
y(t) \leq H_2^{-1} \left( \frac{\gamma}{\beta} \right), & \text{for all } t \geq t_0.
\end{cases}
\]

For all \( t \in [0, t_0) \), we write \( z(t) = y(t) - H_2^{-1} \left( \frac{\gamma}{\beta} \right) \geq 0 \). From (4.38), we can deduce that

\[
\frac{d}{dt} z(t) + \beta H_2 \left[ z(t) + H_2^{-1} \left( \frac{\gamma}{\beta} \right) \right] \leq \gamma. \tag{4.41}
\]

Now, we prove that, for all \( a \geq 0, b \geq 0 \),

\[
H_2(a + b) \geq H_2(a) + H_2(b). \tag{4.42}
\]
Indeed, if \( H_2(t) = t \), it is easy to deduce the conclusion. If \( H_2(t) = t\Psi'(\epsilon_0 t) \), since \( \Psi'' > 0, \Psi'(0) = 0 \), we can deduce that \( \Psi' > 0 \) and \( \Psi \) is increasing on \((0, \epsilon_1)\). Then, we have

\[
H_2(a + b) - [H_2(a) + H_2(b)] = (a + b)\Psi'(\epsilon_0 (a + b)) - [a\Psi'(\epsilon_0 a) + b\Psi'(\epsilon_0 b)] \\
= a\{\Psi'[\epsilon_0 (a + b)] - \Psi'(\epsilon_0 a)\} + b\{\Psi'[\epsilon_0 (a + b)] - \Psi'(\epsilon_0 b)\} \geq 0,
\]

which means \( H_2(a + b) \geq H_2(a) + H_2(b) \). Then, from (4.41), we can deduce that

\[
\frac{d}{dt} z(t) + \beta H_2[z(t)] \leq 0. \tag{4.43}
\]

Recalling that \( H_1' = -\frac{1}{\beta} \), we infer from (4.43)

\[
z'(t)H_1'(z(t)) \geq \beta. \tag{4.44}
\]

A simple integration over \((0, t)\) yields

\[
H_1(z(t)) \geq H_1(z(0)) + \beta t.
\]

Then, exploiting the fact that \( H_1^{-1} \) is decreasing, we infer

\[
z(t) \leq H_1^{-1}(H_1(z(0)) + \beta t) \leq H_1^{-1}(\beta t),
\]

which implies (4.39) for all \( t \in [0, t_0] \) and, since this inequality is obvious for all \( t \geq t_0 \), the lemma is proved.

\[\square\]

**Remark 2.** If \( H_2(t) = t \), then we obtain in a neighborhood of 0, \( H_1(t) = -\ln t \). Thus, in a neighborhood of \(+\infty\), \( H^{-1}_1(t) = e^{-t} \), \( H^{-1}_2(t) = t \). By using (4.39), we have

\[
y(t) \leq e^{-\beta t} + \frac{\gamma}{\beta},
\]

which has the same decay rate as the one in [26].

**Remark 3.** If \( H_2(t) = t^p, p > 1 \), then we obtain in a neighborhood of 0, \( H_1(t) = (1 - p) - \frac{1 - p}{p} \). Thus, in a neighborhood of \(+\infty\), \( H_1^{-1}(t) = ct^{-\frac{1}{p-1}}, H_2^{-1}(t) = t^\frac{1}{p} \). By using (4.39), we have

\[
y(t) \leq c(\beta t)^{-\frac{1}{p-1}} + \left(\frac{\gamma}{\beta}\right)^{\frac{1}{p}},
\]

which is the Ghidaglia inequality, see [37, Lemma 5.1].

5. **Asymptotic smooth.** Here we show that the semigroup \( S(t) \) is asymptotically smooth in \( \mathcal{H} \) (cf. [13, 21]).

First, we recall the following result:

**Lemma 5.1.** [13] Let \( H \) be a Banach space. Assume that for any \( B \subset H \) bounded and positively invariant and for any \( \epsilon > 0 \), there exists \( T = T(\epsilon, B) \) such that

\[
\|S(T)y_1 - S(T)y_2\|_H \leq \epsilon + \Phi_T(y_1, y_2), \quad \forall y_1, y_2 \in B,
\]

where \( \Phi_T : H \times H \to \mathbb{R} \) satisfies, for any sequence \( \{y_n\} \subset B \),

\[
\lim_{j \to \infty} \lim_{k \to \infty} \Phi_T(y_{n_j}, y_{n_k}) = 0.
\]

Then \( S(t) \) is asymptotically smooth.

Now, we introduce a lemma, which will be used in the following paper.
\textbf{Lemma 5.2.} Let $g_1(\cdot)$ satisfy Assumption 1. Then for any $\delta > 0$, there exists $C(\delta) > 0$ such that

$$
|u - v|^2 \leq \delta + C(\delta)(g_1(u) - g_1(v))(u - v) \quad \text{for } u, v \in \mathbb{R}.
$$

(5.1)

\textit{Proof.} From (2.1), we have $\lim_{|s| \to \infty} \inf |g_1'(s)| > 0$. Indeed, since $g_1 : \mathbb{R} \to \mathbb{R}$ is a non-decreasing function, we have $\lim_{|s| \to +\infty} \inf |g_1'(s)| \geq 0$. If $\lim_{|s| \to +\infty} \inf |g_1'(s)| = 0$, then we have $g_1(s)$ monotonically increases and tends to a positive constant $K_1$, which contradicts (2.1). Thus

$$
\lim_{|s| \to +\infty} \inf |g_1'(s)| > 0.
$$

(5.2)

Assume (5.1) does not hold. Then there exist $\delta_0 > 0$, $C_n \to +\infty$, and $u_n \in \mathbb{R}$, $v_n \in \mathbb{R}$ such that

$$
|u_n - v_n|^2 > \delta_0 + C_n(g_1(u_n) - g_1(v_n))(u_n - v_n),
$$

from which we obtain

$$
|u_n - v_n|^2 > \delta_0, \quad \text{and} \quad \frac{1}{u_n - v_n} \int_{v_n}^{u_n} g_1'(s)ds \to 0,
$$

which contradicts (5.2). The proof is completed. \qed

\textbf{Theorem 5.3.} Let the assumptions of Theorem 3.1 hold. Then, $S(t)$ is asymptotically smooth in $\mathcal{H}$.

\textit{Proof.} Without loss of generality, we deal only with the strong solutions in the following, the generalized solution case then follows easily by a density argument. The following process is derived from the standard energy method given in [11, 12, 20, 40].

Let $B_1$ be bounded and positively invariant for $S(t)$ and $(u_1^1, u_1^2, f_1^0), (u_0^1, u_0^2, f_0^0) \in B_1$. Let $(u^1(t), u_1^2(t), z^1(\rho, \cdot, t))$ and $(u^2(t), u_2^2(t), z^2(\rho, \cdot, t))$ be the solutions of problem (2.9)-(2.11) with the initial conditions $(u_0^1, u_0^2, f_0^0)$ and $(u_0^1, u_0^2, f_0^0)$ respectively. We set $\phi = u^1 - u^2$ and $\varphi = z^1 - z^2$, so the pair $(\phi, \varphi)$ satisfies the problem

$$
\begin{align*}
\phi_{tt} + \Delta^2 \phi + h(u^1(x, y, t)) - h(u^2(x, y, t)) + \delta_1 g_1(u_1^1(x, y, t)) \\
- \delta_1 g_1(u_1^2(x, y, t)) + \delta_2 g_2(z_1^1(x, y, t)) - \delta_2 g_2(z_1^2(x, y, t)) = 0,
\end{align*}
$$

in $\Omega \times (0, +\infty)$,

(5.3)

with boundary conditions

$$
\begin{align*}
\phi(0, y, t) = \phi_{xx}(0, y, t) = \phi_{xy}(\pi, y, t) = \phi_{xx}(\pi, y, t) = 0, \\
& \text{for } (y, t) \in (-l, l) \times (0, +\infty), \\
\phi_{yy}(x, \pm l, t) + \sigma \phi_{xx}(x, \pm l, t) = w_{yy}(x, \pm l, t) \\
+ (2 - \sigma)\phi_{xy}(x, \pm l, t) = 0, \quad \text{for } (x, t) \in (0, \pi) \times (0, +\infty),
\end{align*}
$$

(5.4)

and initial conditions

$$
\begin{align*}
\varphi(0, x, y, t) := \phi_t(x, y, t) = u_1^1(x, y, t) - u_1^2(x, y, t), \quad \text{in } \Omega, \\
\varphi(x, y, 0) := \phi_0 = u_0^1 - u_0^2, \quad \text{in } \Omega, \\
\varphi_t(x, y, 0) := \phi_0 = u_1^1 - u_1^2, \quad \text{in } \Omega, \\
\varphi(\rho, x, y, 0) := \varphi_0 = f_0^1 - f_0^2, \quad \text{for } (\rho, x, y) \in (0, 1) \times (0, \pi) \times (-l, l).
\end{align*}
$$

(5.5)
The energy functional for (5.3)-(5.5) is

\[ E_\phi(t) = \frac{1}{2} \|\phi(t)\|^2_{H^2(\Omega)} + \frac{1}{2} \|\phi_t(t)\|^2_2 + \xi \int_\Omega \int_0^1 G_2 (\varphi(\rho, x, y, t)) \, dpdx. \] (5.6)

Since \( B_1 \) is positively invariant for \( S(t) \), we have

\[ (u^i, u^t, z^i) = S(t) (u^i_0, u^t_0, f^i_0) \in B_1, \quad i = 1, 2. \] (5.7)

At first, multiplying (5.3) by \( \phi \) and integrating over \([0, T] \times \Omega \), we get

\[
\int_0^T \|\phi(s)\|^2_{H^2(\Omega)} \, ds
= \int_\Omega \phi(0)\phi(0) \, dxdy - \int_\Omega \phi_t(T)\phi(T) \, dxdy + \int_0^T \int_\Omega \phi_t(s)^2 \, dxdyds
- \frac{\delta_1}{2} \int_0^T \int_\Omega (g_1 (u^1_t(s)) - g_1 (u^2_t(s))) \phi(s) \, dxdyds
- \frac{1}{2} \int_0^T \int_\Omega (h (u^1(s)) - h (u^2(s))) \phi(s) \, dxdyds
- \frac{\delta_2\int_0^T \int_\Omega (g_2 (z^1(1, x, y, s)) - g_2 (z^2(1, x, y, s))) \phi(s) \, dxdyds. \] (5.8)

Then we have

\[
\int_0^T E_\phi(s) \, ds
= \xi \int_0^T \int_\Omega \int_0^1 G_2 (\varphi(\rho, x, y, s)) \, dpdxds + \frac{1}{2} \int_\Omega \phi(0)\phi(0) \, dxdy
- \frac{1}{2} \int_\Omega \phi_t(T)\phi(T) \, dxdy + \int_0^T \int_\Omega \phi_t(s)^2 \, dxdyds
- \frac{1}{2} \int_0^T \int_\Omega (g_1 (u^1_t(s)) - g_1 (u^2_t(s))) \phi(s) \, dxdyds
- \frac{1}{2} \int_0^T \int_\Omega (h (u^1(s)) - h (u^2(s))) \phi(s) \, dxdyds
- \frac{1}{2} \int_0^T \int_\Omega (g_2 (z^1(1, x, y, s)) - g_2 (z^2(1, x, y, s))) \phi(s) \, dxdyds. \] (5.9)

Secondly, multiplying (5.3) by \( \phi_t \) and integrating over \([s, T] \times \Omega \), multiplying (5.3) by \( \xi g_2(\varphi) \) and integrating over \([s, T] \times \Omega \times [0, 1] \), taking the sum, we get

\[
E_\phi(T) + \delta_1 \int_s^T \int_\Omega (g_1 (u^1(\tau)) - g_1 (u^2(\tau))) \phi_t(\tau) \, dxdydt
+ \frac{\xi}{\tau} \int_s^T \int_\Omega G_2 (\varphi(1, x, y, \tau)) \, dxdydt
= E_\phi(s) - \delta_1 \int_s^T \int_\Omega (h (u^1(\tau)) - h (u^2(\tau))) \phi_t(\tau) \, dxdydt
- \frac{1}{\tau} \int_s^T \int_\Omega (g_2 (z^1(1, x, y, \tau)) - g_2 (z^2(1, x, y, \tau))) \phi_t(\tau) \, dxdydt.
\]
Integrating (5.14) over \([0, T]\) with respect to \(t\), we get that, for any \(\delta > 0\),

\[
\int_0^T \int_\Omega G_2(\phi_t(\tau)) \, dx \, dy \, d\tau,
\]

where \(0 \leq s \leq T\). Integrating (5.10) over \([0, T]\) with respect to \(s\), we obtain that

\[
TE_\phi(T) \leq \int_0^T \int_\Omega \left( h(u^1(\tau)) - h(u^2(\tau)) \right) \phi_t(\tau) \, dx \, dy \, d\tau - \delta_2 \int_0^T \int_\Omega \left( g_2(z^1(1, x, y, \tau)) - g_2(z^2(1, x, y, \tau)) \right) \phi_t(\tau) \, dx \, dy \, d\tau 
\]

+ \frac{\xi}{\tau} \int_0^T \int_\Omega G_2(\phi_t(\tau)) \, dx \, dy \, d\tau.
\]

From (5.10), we also have

\[
\int_0^T \int_\Omega (g_1(u^1(\tau)) - g_1(u^2(\tau))) \phi_t(\tau) \, dx \, dy \, d\tau = E_\phi(0) - \int_0^T \int_\Omega \left( h(u^1(\tau)) - h(u^2(\tau)) \right) \phi_t(\tau) \, dx \, dy \, d\tau
\]

- \delta_2 \int_0^T \int_\Omega \left( g_2(z^1(1, x, y, \tau)) - g_2(z^2(1, x, y, \tau)) \right) \phi_t(\tau) \, dx \, dy \, d\tau
\]

+ \frac{\xi}{\tau} \int_0^T \int_\Omega G_2(\phi_t(\tau)) \, dx \, dy \, d\tau.
\]

Combining with Lemma 5.2, we get that, for any \(\delta > 0\),

\[
\int_0^T \int_\Omega |\phi_t(\tau)| \, dx \, dy \, d\tau \leq \delta T \text{mes}(\Omega) + C_\delta E_\phi(0) - C\delta \int_0^T \int_\Omega \left( h(u^1(\tau)) - h(u^2(\tau)) \right) \phi_t(\tau) \, dx \, dy \, d\tau
\]

- \delta_2 C_\delta \int_0^T \int_\Omega \left( g_2(z^1(1, x, y, \tau)) - g_2(z^2(1, x, y, \tau)) \right) \phi_t(\tau) \, dx \, dy \, d\tau
\]

+ \frac{\xi}{\tau} C_\delta \int_0^T \int_\Omega G_2(\phi_t(\tau)) \, dx \, dy \, d\tau.
\]

Thirdly, multiplying (5.3) by \(\xi G_2(\varphi)\) and integrating over \(\Omega \times (0, 1)\), we have

\[
\frac{d}{dt} \left\{ \xi \int_\Omega \int_0^1 G_2(\varphi(\rho, x, y, t)) \, dp \, dx \, dy \right\} + \frac{\xi}{\tau} \int_\Omega G_2(\varphi(1, x, y, t)) \, dx \, dy
\]

= \frac{\xi}{\tau} \int_\Omega G_2(\phi_t(t)) \, dx \, dy.
\]

Integrating (5.14) over \([0, s]\) with respect to \(t\), where \(0 \leq s \leq T\), we have

\[
\xi \int_\Omega \int_0^1 G_2(\varphi(\rho, x, y, s)) \, dp \, dx \, dy + \frac{\xi}{\tau} \int_0^s \int_\Omega G_2(\varphi(1, x, y, \tau)) \, dx \, dy \, d\tau
\]

= \frac{\xi}{\tau} \int_0^s \int_\Omega G_2(\phi_t(\tau)) \, dx \, dy \, d\tau + \xi \int_\Omega \int_0^1 G_2(\varphi(\rho, x, y, 0)) \, dp \, dx \, dy.
\]

Integrating (5.13) over \([0, T]\) with respect to \(s\), we deduce that

\[
\xi \int_0^T \int_\Omega \int_0^1 G_2(\varphi(\rho, x, y, s)) \, dp \, dx \, dy \, ds
\]
Thus, we have
\[
\leq \frac{\xi}{T} \int_0^T \int_0^s \int_\Omega G_2(\phi_i(\tau))dxdydrds + \xi \int_0^T \int_\Omega \int_0^1 G_2(\varphi(\rho, x, y, 0))dpdxdydt.
\]

(5.16)

Thus, we have
\[
TE_\phi(T) \leq \int_0^T E_\phi(s)ds - \int_0^T \int_\Omega \left( h (u^1(\tau)) - h (u^2(\tau)) \right) \phi_i(\tau)dxdydrds
- \delta_2 \int_0^T \int_\Omega \left( g_2 \left( z^1(1, x, y, \tau) \right) - g_2 \left( z^2(1, x, y, \tau) \right) \right) \phi_i(\tau)dxdydrds
+ \frac{\xi}{T} \int_0^T \int_\Omega G_2(\phi_i(\tau))dxdydrds + \xi \int_0^T \int_\Omega \int_0^1 G_2(\varphi(\rho, x, y, 0))dpdxdydt
\]

\[
\leq \frac{\xi}{T} \int_0^T \int_0^s \int_\Omega G_2(\phi_i(\tau))dxdydrds + \frac{1}{2} \int_\Omega \phi_i(0)\phi(0)dxy - \frac{1}{2} \int_\Omega \phi_i(T)\phi(T)dxy
+ \delta Tmes(\Omega) + C_\delta E_\phi(0) - C_\delta \int_0^T \int_\Omega \left( h (u^1(\tau)) - h (u^2(\tau)) \right) \phi_i(\tau)dxdydr
\]

(5.17)

Now, we need to tackle with \[ \int_0^T \int_\Omega \left( g_1 \left( u^i_1(s) \right) - g_1 \left( u^2_1(s) \right) \right) \phi(s)dxdyds \]

Multiplying (2.9) by \( u^i_1(t) \), (2.9) by \( z^i(t)g_2 \left( \varphi(t) \right) \), and integrating over \( \Omega \) and \( (0, 1) \times \Omega \) respectively, we obtain
\[
\frac{d}{dt} \left\{ \frac{1}{2}\|u^i_1(t)\|^2 + \frac{1}{2}\|u^i(t)\|^2_{H_2(\Omega)} + \int_\Omega H \left( u^i(t) \right) dxy \right\}
+ \xi \int_\Omega \int_0^1 G_2 \left( z^i(\rho, x, y, t) \right) dpdxdy
+ a_1 \int_\Omega g_1 \left( u^i_1(t) \right) u^i_1(t)dxy + a_2 \int_\Omega g_2 \left( z^i(1, x, y, t) \right) z^i(1, x, y, t)dxy
\]
\[ \leq \int_{\Omega} f u_i^t(t) \, dx \, dy, \]

which combining with the existence of bounded absorbing set \( B_1 \), implies that

\[ \int_0^T \int_{\Omega} g_2 \left( z^i(t, x, y, t) \right) z^i(t, x, y, t) \, dx \, dy \, dt \leq C_R, \quad \text{(5.19)} \]

where \( C_R \) is a constant which depends on the size of \( B_1 \), but is independent of \( T \). Therefore,

\[ \left| \int_0^T \int_{\Omega} g_1 \left( u_i^t(s) \right) \phi(s) \, dx \, dy \, ds \right| \]

\[ \leq \int_0^T \int_{\Omega \left( \{ u_i^t \leq \epsilon_1 \} \right)} \left| g_1 \left( u_i^t(s) \right) \phi(s) \right| \, dx \, dy \, ds + \left( \int_0^T \int_{\Omega \left( \{ u_i^t \geq \epsilon_1 \} \right)} \left| g_1 \left( u_i^t(s) \right) \right|^2 \, dx \, dy \, ds \right)^{\frac{1}{2}} \]

\[ \times \left( \int_0^T \int_{\Omega \left( \{ u_i^t \geq \epsilon_1 \} \right)} \left| \phi(s) \right|^2 \, dx \, dy \, ds \right)^{\frac{1}{2}} \]

\[ \leq C \int_0^T \int_{\Omega} \left| \phi(s) \right| \, dx \, dy \, ds + \left( C_2 \int_0^T \int_{\Omega \left( \{ u_i^t \geq \epsilon_1 \} \right)} g_1 \left( u_i^t(s) \right) u_i^t(s) \, dx \, dy \, ds \right)^{\frac{1}{2}} \]

\[ \times \left( 2 S^2 \int_0^T \left( \left\| u^1(t) \right\|_{H_x^2(\Omega)}^2 + \left\| u^2(t) \right\|_{H_x^2(\Omega)}^2 \right) \, ds \right)^{\frac{1}{2}} \]

\[ \leq C \int_0^T \int_{\Omega} \left| \phi(s) \right| \, dx \, dy \, ds + CC_R^2 T^{\frac{1}{2}}. \quad \text{(5.20)} \]

Similarly,

\[ \left| \int_0^T \int_{\Omega} g_2 \left( z^i(t, x, y, s) \right) \phi(s) \, dx \, dy \, ds \right| \]

\[ \leq \int_0^T \int_{\Omega \left( \{ \left| z^i \right| \leq \epsilon_1 \} \right)} \left| g_2 \left( z^i(t, x, y, s) \right) \phi(s) \right| \, dx \, dy \, ds \]

\[ + \left( \int_0^T \int_{\Omega \left( \{ \left| z^i \right| \geq \epsilon_1 \} \right)} \left| g_2 \left( z^i(t, x, y, s) \right) \right|^2 \, dx \, dy \, ds \right)^{\frac{1}{2}} \left( \int_0^T \int_{\Omega \left( \{ \left| z^i \right| \geq \epsilon_1 \} \right)} \left| \phi(s) \right|^2 \, dx \, dy \, ds \right)^{\frac{1}{2}} \]

\[ \leq C \int_0^T \int_{\Omega} \left| \phi(s) \right| \, dx \, dy \, ds \]

\[ + \left( \int_0^T \int_{\Omega \left( \{ \left| z^i \right| \geq \epsilon_1 \} \right)} C_2 \frac{\alpha_2}{\alpha_1} \left| g_2 \left( z^i(t, x, y, s) z^i(t, x, y, s) \right) \right| \, dx \, dy \, ds \right)^{\frac{1}{2}} \]

\[ \times \left( \int_0^T \int_{\Omega} \left| \phi(s) \right| \, dx \, dy \, ds \right)^{\frac{1}{2}} \]

\[ \leq C \int_0^T \int_{\Omega} \left| \phi(s) \right| \, dx \, dy \, ds + CC_R^2 T^{\frac{1}{2}}. \quad \text{(5.21)} \]

\[ \left| \int_0^T \int_{\Omega} g_2 \left( z^i(t, x, y, \tau) \right) \phi(t) \, dx \, dy \, d\tau \right| \]
\[\int_0^T \int_{\Omega} g_2(z^i(1, x, y, \tau)) \phi_i(\tau) \, dxdy d\tau\]

\[+ \int_0^T \int_{\Omega} g_2(z^i(1, x, y, \tau)) \phi_i(\tau) \, dxdy d\tau\]

\[\leq C \int_0^T \int_{\Omega} |\phi_i(\tau)| \, dxdy d\tau + CC_T^2 T^2. \quad (5.22)\]

\[\int_0^T \int_{\Omega} G_2(\phi_i(\tau)) \, dxdy d\tau\]

\[\leq \alpha_2 \int_0^T \int_{\Omega} \phi_i(\tau) g_1(\phi_i(\tau)) \, dxdy d\tau\]

\[\leq \alpha_2 \int_0^T \int_{\Omega(|\phi_i| \leq \epsilon_1)} |\phi_i(\tau) g_1(\epsilon_1)| \, dxdy d\tau + \alpha_2 \int_0^T \int_{\Omega(|\phi_i| \geq \epsilon_1)} |\phi_i(\tau) g_1(\phi_i(\tau))| \, dxdy d\tau\]

\[\leq C \int_0^T \int_{\Omega} |\phi_i(\tau)| \, dxdy d\tau + C \int_0^T \int_{\Omega} |\phi_i(\tau)|^2 \, dxdy d\tau. \quad (5.23)\]

\[\int_0^T \int_{\Omega} g_2(z^i(1, x, y, \tau)) \phi_i(\tau) \, dxdy d\tau\]

\[\leq T \int_0^T \int_{\Omega(|z^i| \leq \epsilon_1)} |g_2(z^i(1, x, y, \tau)) \phi_i(\tau)| \, dxdy d\tau\]

\[+ T \int_0^T \int_{\Omega(|z^i| \geq \epsilon_1)} |g_2(z^i(1, x, y, \tau)) \phi_i(\tau)| \, dxdy d\tau\]

\[\leq C T \int_0^T \int_{\Omega} |\phi_i(\tau)| \, dxdy d\tau + \epsilon T \int_0^T \int_{\Omega(|z^i| \geq \epsilon_1)} |g_2(z^i(1, x, y, \tau))|^2 \, dxdy d\tau\]

\[+ C \epsilon T \int_0^T \int_{\Omega(|z^i| \geq \epsilon_1)} |\phi_i(\tau)|^2 \, dxdy d\tau\]

\[\leq C T \int_0^T \int_{\Omega} |\phi_i(\tau)| \, dxdy d\tau + \epsilon TC_R + C \epsilon T \int_0^T \int_{\Omega} |\phi_i(\tau)|^2 \, dxdy d\tau. \quad (5.25)\]

\[\int_0^T \int_{\Omega} G_2(\varphi(\rho, x, y, 0)) \, dxdy d\tau\]

\[\leq \alpha_2 \int_0^T \int_{\Omega(|\varphi(\rho, x, y, 0)| \leq \epsilon_1)} \int_0^1 |\varphi(\rho, x, y, 0)| g_1(\varphi(\rho, x, y, 0)) \, d\rho \, dxdy d\tau\]

\[+ \alpha_2 \int_0^T \int_{\Omega(|\varphi(\rho, x, y, 0)| \geq \epsilon_1)} \int_0^1 |\varphi(\rho, x, y, 0)| g_1(\varphi(\rho, x, y, 0)) \, d\rho \, dxdy d\tau\]
From these estimates, we deduce from (5.17) that

\[
T E_\phi(T) \lesssim CT \int_0^1 \int_0^1 |\varphi_0| dp dxdy + CT \int_0^1 \int_0^1 |\varphi_0|^2 dp dxdy. \tag{5.26}
\]

Set

\[
C_{B_1} = \frac{\alpha_3}{a_4} \left( CT^\frac{1}{2} + \delta Tmes(\Omega) + 2\delta_2 \varepsilon TC_R + C_\delta E_\phi(0) + \frac{1}{2} \int_{\Omega} \phi_1(0) \phi(0) dxdy \right).
\]
where \( a_4 = \max\left\{ \frac{1}{2}, \xi \right\} \), \( a_4 = \min\left\{ \frac{1}{2}, \xi \right\} \). Then, we have
\[
\frac{\|S(T)(u^1_0, u^1_1, f^1_0) - S(T)(u^2_0, u^2_1, f^2_0)\|_H^2}{T} \lesssim \frac{a_3}{a_4} E_\delta(T) \leq \frac{C_{B_1}}{T} + \frac{1}{T} \Phi_T \left( \left( u^1_0, u^1_1, f^1_0 \right), \left( u^2_0, u^2_1, f^2_0 \right) \right).
\]
(5.30)

Since the semigroup \( \{S(t)\}_{t \geq 0} \) has a bounded absorbing set \( B_1 \), for any fixed \( \epsilon > 0 \), we can choose \( \delta \leq \frac{a_3 \epsilon}{a_4 \nu} \), \( \epsilon \leq \frac{a_3 \epsilon}{8a_4 \delta_2 C_R} \), and then let \( T \) so large that
\[
\frac{C_{B_1}}{T} \leq \epsilon.
\]

For each fixed \( T \), let \( (u^n, u^n_t, z^n) \) be the corresponding solution of \( (u^n_0, u^n_1, f^n_0) \in B_1, n = 1, 2, \cdots \). Then, since \( B_1 \) is a bounded positively invariant set in \( \mathcal{H} \), without loss of generality (at most by passing subsequences), we assume that
\[
u^n \to \nu \text{ weakly star in } L^\infty(0, T; H^2(\Omega) \cap L^2(\Omega)),
\]
(5.31)
\[
u^n_t \to \nu_t \text{ weakly star in } L^\infty(0, T; L^2(\Omega)),
\]
(5.32)
\[
u^n_{tt} \to \nu_{tt} \text{ weakly star in } L^\infty(0, T; \mathcal{H}(\Omega)),
\]
(5.33)
\[f^n_0 \to f_0 \text{ weak in } L^2((0, 1) \times \Omega),
\]
(5.34)
\[
u^n \to \nu \text{ strongly in } L^2(0, T; H^2(\Omega)),
\]
(5.35)
\[
u^n_t \to \nu_t \text{ strongly in } L^2(0, T; L^2(\Omega)),
\]
(5.36)
where we use the embedding \( L^2(\Omega) \hookrightarrow \mathcal{H}(\Omega) \).

Now, we deal with each term corresponding to that in (5.29). At first, since \( T \) is fixed, \( T \) is a constant. From (5.36), we have
\[
\lim \left( \lim \right)_{n \to \infty} \left( \lim \right)_{m \to \infty} T \int_0^T \int_\Omega |u^n_t(\tau) - u^m_t(\tau)| \, dx \, dy \, d\tau = 0,
\]
(5.37)
\[
\lim \left( \lim \right)_{n \to \infty} \left( \lim \right)_{m \to \infty} T \int_0^T \int_\Omega |u^n_t(\tau) - u^m_t(\tau)|^2 \, dx \, dy \, d\tau = 0,
\]
(5.38)
\[
\lim \left( \lim \right)_{n \to \infty} \left( \lim \right)_{m \to \infty} T \int_0^T \int_\Omega |u^n(\tau) - u^m(\tau)| \, dx \, dy \, d\tau = 0,
\]
(5.39)
Applying Lemma 5.1, we conclude that

\[
\lim_{n \to \infty} \lim_{m \to \infty} \int_0^T \int_\Omega (u_i^n(\tau) - u_i^m(\tau))^2 \, dx \, dy \, d\tau = 0. \quad (5.40)
\]

Secondly, since \(H^2_2(\Omega) \hookrightarrow L^2(\Omega), L^2(\Omega) \hookrightarrow L^1(\Omega)\), we have \(H^2_2(\Omega) \hookrightarrow L^1(\Omega)\). From (5.35), we obtain that

\[
\lim_{n \to \infty} \lim_{m \to \infty} \int_0^T \int_\Omega |u^n(\tau) - u^m(\tau)| \, dx \, dy \, d\tau = 0. \quad (5.41)
\]

Thirdly, from (5.34), we have

\[
\lim_{n \to \infty} \lim_{m \to \infty} T \int_\Omega \int_0^1 |f^n_0 - f^m_0| \, dp \, dx \, dy = 0, \quad (5.42)
\]

\[
\lim_{n \to \infty} \lim_{m \to \infty} T \int_\Omega \int_0^1 |f^n_0 - f^m_0|^2 \, dp \, dx \, dy = 0. \quad (5.43)
\]

Finally, from (5.35) and (5.36), noticing Assumption 2, we obtain that

\[
\begin{align*}
&\lim_{n \to \infty} \lim_{m \to \infty} \int_0^T \int_\Omega (h(u^n(\tau)) - h(u^m(\tau))) (u_i^n(\tau) - u_i^m(\tau)) \, dx \, dy \, d\tau \\
&\leq \frac{1}{2} \lim_{n \to \infty} \lim_{m \to \infty} \int_0^T \int_\Omega C \left( |u^n(\tau)|^2 + |u^m(\tau)|^2 \right) |u^n(\tau) - u^m(\tau)|^2 \, dx \, dy \, d\tau \\
&+ \frac{1}{2} \lim_{n \to \infty} \lim_{m \to \infty} \int_0^T \int_\Omega |u_i^n(\tau) - u_i^m(\tau)|^2 \, dx \, dy \, d\tau \\
&\leq C(B_1) \lim_{n \to \infty} \lim_{m \to \infty} \int_0^T \int_\Omega |u^n(\tau) - u^m(\tau)|^2 \, dx \, dy \, d\tau \\
&+ \frac{1}{2} \lim_{n \to \infty} \lim_{m \to \infty} \int_0^T \int_\Omega |u_i^n(\tau) - u_i^m(\tau)|^2 \, dx \, dy \, d\tau = 0, \quad (5.44)
\end{align*}
\]

where we use the embedding of \(H^2_2(\Omega)\) in \(L^\infty(\Omega)\). Similarly,

\[
\begin{align*}
&\lim_{n \to \infty} \lim_{m \to \infty} \int_0^T \int_\Omega |(h(u^n(\tau)) - h(u^m(\tau))) (u^n(\tau) - u^m(\tau))| \, dx \, dy \, d\tau = 0. \quad (5.45)
\end{align*}
\]

\[
\begin{align*}
&\lim_{n \to \infty} \lim_{m \to \infty} \int_0^T \int_s^T \int_\Omega |(h(u^n(\tau)) - h(u^m(\tau))) (u_i^n(\tau) - u_i^m(\tau))| \, dx \, dy \, d\tau \, ds \\
&\leq \lim_{n \to \infty} \lim_{m \to \infty} T \int_0^T \int_\Omega |(h(u^n(\tau)) - h(u^m(\tau))) (u_i^n(\tau) - u_i^m(\tau))| \, dx \, dy \, d\tau = 0. \quad (5.46)
\end{align*}
\]

Hence, combining (5.37)-(5.46), we have

\[
\lim_{n \to \infty} \lim_{m \to \infty} \frac{1}{T} \Phi_T ((u_0^n, u_1^n, f_0^n), (u_0^m, u_1^m, f_0^m)) = 0.
\]

Applying Lemma 5.1, we conclude that \(S(t)\) is asymptotically smooth in \(\mathcal{H}\).

The proof is completed.
6. **Global attractor.** In this section, we establish a global attractor to system (2.9)-(2.11). We first recall some results.

**Definition 6.1.** [26] The global attractor for a semigroup $S(t)$ acting on a Hilbert space $H$ is a compact subset $A$ of $H$ satisfying the following conditions:

(i) $A$ is invariant for $S(t)$; i.e.,

$$S(t)A = A, \ \forall t \geq 0.$$ 

(ii) $A$ attracts bounded sets; this means, for any bounded set $B \subset H$, we have

$$\lim_{t \to \infty} d_H (S(t)B, A) = 0,$$ 

where $d_H$ is the Hausdorff semi-distance defined by

$$d_H (A, B) = \sup_{a \in A} \inf_{b \in B} \|a - b\|_H.$$ 

**Theorem 6.2.** [13] Let $S(t)$ be a dissipative semigroup on a metric space $H$. Then, $S(t)$ has the compact global attractor in $H$ if and only if it is asymptotically smooth in $H$.

Now, we are in position to establish our main result.

**Theorem 6.3.** Let the assumptions of Theorem 3.1 hold. Then, the semigroup $S(t)$ associated to problem (2.9)-(2.11) possess the global attractor in $H$.

**Proof.** Theorem 4.6 and 5.3 imply the existence of a global attractor. The proof is completed.

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