A CONSTRUCTION OF COMPLETE RICCI-FLAT KÄHLER MANIFOLDS

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ABSTRACT. We consider an extension of the results of S. Bando, R. Kobyashi, G. Tian, and S. T. Yau on the existence of Ricci-flat Kähler metrics on quasi-projective varieties $Y = X \setminus D$ with $\alpha[D] = c_1(X), \alpha > 1$. The requirement that $D$ admit a Kähler-Einstein metric is generalized to the condition that the link $S \subset N_D$ in the normal bundle of $D$ admits a Sasaki-Einstein structure in the Sasaki-cone of the usual Sasaki structure provided the embedding $D \subset X$ satisfies an additional holomorphic condition. If $D$ is a toric variety, then $S$ always admits a Sasaki-Einstein metric. As an application we prove that every small smooth deformation of a toric Gorenstein singularity admits a complete Ricci-flat Kähler metric asymptotic to a Calabi ansatz metric. Some examples are given which were not previously known.

1. INTRODUCTION

The purpose of this article is to extend the theorem due to G. Tian and S.-T. Yau, and independently by S. Bando and R. Kobyashi, which gives the existence of a complete Ricci-flat Kähler metric on a quasi-projective manifold $Y = X \setminus D$ under the assumption that $D$ admits a Kähler-Einstein metric. Here $X$ is a projective variety, and $D$ is irreducible and supports the anti-canonical divisor. Further, $X$ and $D$ are required to be Kähler orbifolds. An assumption that $X$ and $D$ are manifolds would restrict the examples of smooth $Y = X \setminus D$ with a Ricci-flat metric that this technique produces. For the definition of a Kähler orbifold and the notions of divisors and line bundles on orbifolds see [5]. We first review the original result.

Let $X$ be a compact Kähler orbifold, with $\dim \mathbb{C} X = n$, and with $\dim \mathbb{C} (\text{Sing } X) \leq n - 2$. Suppose there is a divisor $D \subset X$ such that $\alpha[D] = -K_X$, with $\alpha > 1$. We will need the following.

Definition 1.1. Let $D$ be a divisor on a compact Kähler orbifold. Then

(i) $D$ is admissible if $\text{Sing } X \subset D$ and for any local uniformizing chart $\pi : \tilde{U} \rightarrow U$ at $x \in D$, $\pi^{-1}(D)$ is smooth in $\tilde{U}$.

(ii) $D$ is almost ample if there is an integer $k \gg 0$ such that the divisor $kD$ defines a morphism $i_{kD} : X \rightarrow \mathbb{CP}^N$ which is biholomorphic in a neighborhood of $D$ onto its image.

We will call $D$ good if it is admissible and almost ample.

In [42] the following is proved. See also [7, 8] and [41] for similar results.

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Theorem 1.2. Let $X$ be a Kähler orbifold, and let $D$ be a good divisor with $\alpha [D] = -K_X$, $\alpha > 1$. Suppose that $D$ admits a Kähler-Einstein metric, then there exists a complete Ricci-flat Kähler metric $g$ on $Y = X \setminus D$ in every Kähler class in $H^2_c(Y, \mathbb{R})$.

Furthermore, if $\rho$ denotes the distance function on $Y$ from a fixed point and $R_g$ denote the curvature tensor of $g$, then $\|\nabla^k R_g\|_g = O(\rho^{-2-k})$.

We denote by $H^2_c(Y, \mathbb{R})$ cohomology with compact support. The metrics in the theorem have Euclidean volume growth. It follows from the results of [6] that if $\|R_g\|_g = O(\rho^{-j})$ for $j > 2$, then $Y$ is asymptotically locally Euclidean (ALE).

Recall the idea behind theorem 1.2. Choose a hermitian metric on $[D]$ with curvature $\omega_0$, whose restriction $\omega_D = \omega_0|_D$ defines a Kähler-Einstein metric on $D$ with $\text{Ricci}(\omega_D) = (\alpha - 1)\omega_D$. Let $\sigma$ be a section of $[D]$ vanishing on $D$, and let $t = \log \|\sigma\|^{-2}$. Then define the Kähler metric on $Y = X \setminus D$ by

$$\omega = \frac{n}{\alpha - 1} i \partial \bar{\partial} \|\sigma\|^{-2(\alpha - 1)/n} + \omega_0 + \frac{(\alpha - 1)}{n} \|\sigma\|^{-2(\alpha - 1)/n} i \partial t \wedge \bar{\partial} t.$$ 

Then $\omega^n$ has a pole of order $2\alpha$ along $D$. There exists an holomorphic n-form $\Omega$ on $X$ with a pole of order $\alpha$ along $D$. The Kähler-Einstein condition implies that the function $f = \log \frac{(\alpha - 1)\Omega}{\omega_0}$ extends to a smooth function on $X$ constant on $D$, and can be made to vanish reasonably rapidly along $D$. Then the existence of the Ricci-flat metric on $Y = X \setminus D$ is proved by solving a Monge-Ampère equation similar to the compact case.

Of course, in general there is no guarantee that $D$ admits a Kähler-Einstein metric, as there are well known obstructions to the existence of positive scalar curvature Kähler-Einstein metrics. See [35, 21, 22] for obstructions involving the automorphism group, and [40] for further obstructions.

This article is concerned with extending theorem 1.2 to examples where $D$ does not admit a Kähler-Einstein metric. The approach is to consider the link in the conormal bundle $S \subset L := N_D^* = [-D]|_D$. There is a standard Sasaki structure on $S$ that is Sasaki-Einstein if and only if $D$ is Kähler-Einstein. But if $D$ does not admit a Kähler-Einstein metric, then $S$ may still admit a Sasaki-Einstein structure. This structure can arise by varying the Reeb vector field. This is made precise in the notion of the Sasaki cone, in analogy with the Kähler cone of a Kähler manifold.

In the theorem we will need to assume that

$$H^1(D, \Theta_X \otimes \mathcal{O}(-kD)|_D) = 0, \text{ for all } k \geq 2.$$

Let $L = N_D^* = [-D]|_D$ be the conormal bundle of $D \subset X$. In the following theorem we can weaken condition (1) in Definition 1.1 to just the normal bundle $N_D = [D]|_D$ being positive.

Theorem 1.3. Suppose $X$ is a Kähler orbifold and $D \subset X$ is a good divisor with $\alpha [D] = -K_X$, $\alpha > 1$. Suppose (1) is satisfied, and suppose the link $S \subset L$ admits a Sasaki-Einstein structure in the Sasaki-cone of the standard Sasaki structure on $S \subset L$. Then $Y = X \setminus D$ admits a complete Ricci-flat Kähler metric $g$, in every Kähler class in $H^2_c(Y, \mathbb{R})$, which is asymptotic to a Calabi ansatz metric in the following sense. For fixed $k \in \mathbb{N}$ and $\delta > 0$ there is a neighborhood $U \subset Y$ of infinity and a diffeomorphism $\phi : U \to V \subset L$ to a neighborhood of infinity. There is a Ricci-flat Calabi ansatz metric $g_0$ on $V$ so that if we set $\bar{g} = \phi^* g_0$, then on $U$
we have
\[ \nabla^j (g - \bar{g}) = O \left( r^{-2n+j} \right) \quad \text{for } j \leq k, \]
where \( \nabla \) is the covariant derivative of \( \bar{g} \).

**Remark 1.4** One can eliminate the "\( \delta \)" in (2) by applying the proof of a non-compact version of the Calabi Conjecture in [48, Theorem 3.10] where analysis of the Laplacian is used to get the sharp convergence.

One can probably eliminate the restriction to Kähler classes in \( H^2_c(Y, \mathbb{R}) \). R. Goto was able to do this in the special case of a crepant resolution of a Sasaki cone. One motivation for doing this is that many of the examples \( Y \), such as a smoothing of a singularity, will be Stein. And in this case \( H^2_c(Y, \mathbb{R}) = 0 \), since \( H^k(Y, \mathbb{R}) = 0 \) for \( k > n \). So Theorem 1.3 just produces a Ricci-flat metric in the trivial Kähler class.

Note that if \( Y \) is Stein, e.g. \( Y \) is an affine variety, then there is a cohomologically trivial Kähler metric. So there is no difficulty finding a Kähler class in \( H^2(Y, \mathbb{R}) \).

Of course it is desirable to remove the condition (1). But the author does not know how to construct the approximating metric in the proof without it. This work was inspired by interesting recent results on irregular Sasaki-Einstein manifolds such as the solution of the problem of the existence of Sasaki-Einstein structures on toric Sasaki manifolds by A. Futaki, H. Ono, and G. Wang [20].

For a source of examples we consider deformations of isolated toric Gorenstein singularities. It is known [20] that such a singularity itself has a nice Ricci-flat Kähler metric which is a metric cone \( C(S) \) over a Sasaki-Einstein manifold \( S \). The versal deformation space of an isolated toric Gorenstein singularity was constructed by K. Altmann [3]. These singularities are rigid in dimensions \( n \geq 4 \), but there are many interesting examples in dimension 3.

**Corollary 1.5.** Let \( Y \) be a small deformation of an isolated toric Gorenstein singularity. If \( Y \) is smooth, then it admits a complete Ricci-flat Kähler metric as in Theorem 1.3 which is invariant under a \( T^n-1 \)-action.

Note the deformed space \( Y \) is no longer toric but admits a \( (\mathbb{C}^*)^{n-1} \)-action. Corollary 1.5 parallels that of the author in [46] and [45] where it is proved that certain crepant resolutions of isolated toric Gorenstein singularities admit Ricci-flat Kähler metrics which are asymptotic to the cone metric on \( C(S) \). Because of the \( T^n-1 \)-action, the moment map construction of M. Gross [27] produces special Lagrangian fibrations on the examples produced in Corollary 1.3. These fibrations were studied in [27]; the contribution here is in proving the existence of the Calabi-Yau metric.

One motivation for studying this problem is the conjecture, due to S.-T. Yau, that if \( Y \) is a complete Ricci-flat Kähler manifold with finite topology, then \( Y = X \setminus D \) where \( X \) is a compact Kähler orbifold and \( D \) supports \(-K_X\). Another motivation is the construction of complete Ricci-flat Kähler metrics which are asymptotic to the Kähler cone \( C(S) \) of a Sasaki-Einstein manifold \( S \). Some explicit examples are given in [31] of Ricci-flat Kähler metrics on resolutions of Ricci-flat Kähler cones \( C(S) \). The author has considered the general existence problem in [46], and given further families of examples in [45]. This case is somewhat easier than that considered in this article. In particular, the technical assumption (1) is not needed.
These asymptotically conical Calabi-Yau metrics are of interest in the AdS/CFT correspondence [33, 32]. Here one considers a stack of D-branes at the singularity of the Calabi-Yau cone, and it is useful to work with Calabi-Yau resolutions of this singularity.

In fact, one of the motivations of this work is that the problem of finding a Ricci-flat Kähler metric on an affine variety is complimentary to the problem of finding such a metric on a resolution of a singularity. There are two ways of smoothing a Ricci-flat Kähler cone \( X = C(S) \). One can possibly find a crepant resolution \( \pi : \hat{Y} \rightarrow X \). Or one may be able to analytically deform \( X \) to a smooth variety \( \hat{Y} \).

When both of these exist, this is an example of a geometric transition which are of interest to physicists and mathematicians studying Calabi-Yau manifolds [37]. The familiar example is that of the quadric cone \( X = \{(U,V,Y,Z) \in \mathbb{C}^4 : UV - XY = 0\} \) which is known to admit a small resolution \( \hat{Y} \) and a smoothing \( \hat{Y} \) where all three of these manifolds are known to admit Ricci-flat Kähler metrics [15].

In Section 6 an example is given where \( X \) is the cone over \( \mathbb{C}P^2(2) \), the two-points blow-up. The total space of the canonical bundle of \( \mathbb{C}P^2(2) \) gives a resolution \( \hat{Y} \) of \( X \), and \( X \) can also be deformed to a smooth affine variety \( \hat{Y} \equiv \mathbb{C}P^3(1) \setminus D \) where \( D \cong \mathbb{C}P^2_0 \). Theorem [33] shows that both \( \hat{Y} \) and \( \hat{Y} \) have Ricci-flat Kähler metrics. This example is simplest case in Corollary [3] which gives a new Ricci-flat manifold.

We then consider an infinite class of three dimensional toric Kähler cones possessing a symmetry which were previously considered by the author in a different context [44, 47]. They admit large families of smooth deformations and therefore provide an infinite family of Ricci-flat manifolds which are smoothings of toric Gorenstein singularities. One can construct many more examples of three dimensional toric cones with non-trivial deformations by taking Minkowski sums of simple polygons. Many of these, via Corollary [3] should give new examples of affine varieties with Ricci-flat Kähler metrics.

**Notation.** We will denote line bundles in boldface, \( \mathbf{L}, \mathbf{K} \), etc. While the corresponding divisor classes are denoted, \( L, K \), etc. If \( D \) is a divisor then \([D]\) denotes, depending on the context, either the corresponding line bundle or the poincaré dual of the homology class of \( D \). The same notation will be used for the analogous notions of V-bundles and Baily divisors on orbifolds (cf. [5]). The total space of the line bundle \( L \) minus the zero section will be denoted by \( L^\times \).

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2. **Sasaki manifolds**

2.1. **Introduction.** We review here some results from Sasakian geometry. For more details see [10], [20], or the comprehensive monograph [10].

**Definition 2.1.** A Riemannian manifold \((S, g)\) is Sasakian if the metric cone \((C(S), \bar{g}), C(S) = S \times \mathbb{R}_+ \) and \( \bar{g} = dr^2 + r^2 g \), is a Kähler manifold.

Thus \( \dim \mathbb{C} = 2m + 1 \), where \( n = \dim \mathbb{C} C(S) = m + 1 \). Set \( \xi = J(r \frac{\partial}{\partial r}) \), then \( \xi - i \bar{\xi} \) is a holomorphic vector field on \( C(S) \). The restriction \( \xi \) of \( \xi \) to \( S = \{ r = 1 \} \subset C(S) \) is the Reeb vector field of \( S \), which is a Killing vector field. If
the orbits of $\xi$ close, then it defines a locally free $U(1)$-action on $S$ and the Sasaki structure is said to be quasi-regular. If the $U(1)$-action is free, then the Sasaki structure is regular. If the orbits of $\xi$ do not close, then the Sasaki structure is irregular. In this case the closure of the one parameter subgroup of the isometry group generated by $\xi$ is a torus $T^k$ and $T^k \subseteq \text{Aut}(S)$, the automorphism group of the Sasaki structure. We say that the Sasaki structure has rank $k$, rank$(S) = k$.

Let $\eta$ be the dual 1-form to $\xi$ with respect to $g$. Then

$$\eta = (2d^c \log r)|_{r=1},$$

where $d^c = \frac{1}{2}i(\bar{\partial} - \partial)$. Let $D = \text{ker} \eta$. Then $d\eta$ is in non-degenerate on $D$ and $\eta$ is a contact form on $S$. Furthermore, we have

$$d\eta(X,Y) = 2g(\Phi X,Y), \quad \text{for } X,Y \in D_x, x \in S,$$

where $\Phi$ is given by the restriction of the complex structure $J$ on $C(S)$ to $D_x$. We will denote the Sasaki structure on $S$ by $(g, \xi, \eta, \Phi)$. It follows from (3) that the Kähler form of $(C(S), \bar{g})$ is

$$\omega = \frac{1}{2}dd^c r^2.$$

Thus $\frac{1}{2}r^2$ is a Kähler potential for $\omega$.

There is a 1-dimensional foliation $\mathcal{F}_\xi$ generated by the Reed vector field $\xi$. Since the leaf space is identical with that generated by $\tilde{\xi} - i J \tilde{\xi}$ on $C(S)$, $\mathcal{F}_\xi$ has a natural transverse holomorphic structure. And $\omega^T = \frac{i}{2}d\eta$ defines a Kähler form on the leaf space.

We will consider deformations of the transverse Kähler structure. Let $\phi \in C^\infty_B(S)$ be a smooth basic function, meaning $\xi \ll d\phi = 0$. Then set

$$\tilde{\eta} = \eta + 2d^c \phi.$$

Then

$$d\tilde{\eta} = d\eta + 2d_B d^c_B \phi = d\eta + 2i\partial_B \bar{\partial}_B \phi.$$

For sufficiently small $\phi$, $\tilde{\eta}$ is a non-degenerate contact form in that $\tilde{\eta} \wedge d\tilde{\eta}^m$ is nowhere zero. Then we have a new Sasaki structure on $S$ with the same Reeb vector field $\xi$, transverse holomorphic structure on $\mathcal{F}_\xi$, and holomorphic structure on $C(S)$. This Sasaki structure has transverse Kähler form $\tilde{\omega}^T = \omega^T + i\partial_B \bar{\partial}_B \phi$.

One can show [20] that if $\tilde{r} = r \exp \phi$,

then $\frac{1}{4} \tilde{r}^2$ is the Kähler potential of the new Kähler structure on $C(S)$.

**Proposition 2.2.** Let $(S,g)$ be a $2m + 1$-dimensional Sasaki manifold. Then the following are equivalent.

(i) $(S,g)$ is Sasaki-Einstein with the Einstein constant being necessarily $2m$.
(ii) $(C(S), \bar{g})$ is a Ricci-flat Kähler.
(iii) The Kähler structure on the leaf space of $\mathcal{F}_\xi$ is Kähler-Einstein with Einstein constant $2m + 2$.

This follows from elementary computations. In particular, the equivalence of (i) and (iii) follows from

$$\text{Ricci}_g(X,\bar{Y}) = (\text{Ric}_T - 2g_T)(X,Y),$$

where $\text{Ric}_T$ is the transverse Ricci form of $(C(S), \bar{g})$. If $(S,g)$ is Sasaki-Einstein, then $(C(S), \bar{g})$ is a Ricci-flat Kähler manifold. If $(C(S), \bar{g})$ is a Ricci-flat Kähler manifold and the leaf space of $\mathcal{F}_\xi$ is Kähler-Einstein, then $(S,g)$ is Sasaki-Einstein with the Einstein constant $2m$.
where $\tilde{X}, \tilde{Y} \in D$ are lifts of $X, Y$ in the local leaf space.

We will make use of a slight generalization of the Sasaki-Einstein condition.

**Definition 2.3.** A Sasaki manifold $(S, g)$ is $\eta$-Einstein if there are constants $\lambda$ and $\nu$ with
\[
\text{Ric} = \lambda g + \nu \eta \otimes \eta.
\]

We have $\lambda + \nu = 2m$ as $\text{Ric}(\xi, \xi) = 2m$. In fact, this condition is equivalent to the transverse Kähler-Einstein condition $\text{Ric}^T = \kappa \omega^T$, since this implies by the same argument that proves Proposition 2.2 that
\[
(8) \quad \text{Ric} = (\kappa - 2)g + (2m + 2 - \kappa)\eta \otimes \eta,
\]
and conversely.

Given a Sasaki structure we can perform a $D$-homothetic transformation to get a new Sasaki structure. For $a > 0$ set
\[
(9) \quad \eta' = a\eta, \quad \xi' = \frac{1}{a} \xi,
\]
\[
(10) \quad g' = ag^T + \frac{a^2}{2} \eta \otimes \eta = ag + (a^2 - a)\eta \otimes \eta.
\]

Then $(g', \xi', \eta', \Phi)$ is a Sasaki structure with the same holomorphic structure on $C(S)$, and with $r' = r^a$.

Suppose that $g$ is $\eta$-Einstein with $\text{Ric}_g = \lambda g + \nu \eta \otimes \eta$. A simple computation involving (7), $\text{Ric}^T = \text{Ric}^T$ and $\text{Ric}_g(\xi', \xi') = 2m$ shows that the $D$-homothetic transformation gives an $\eta$-Einstein Sasaki structure with
\[
(12) \quad \text{Ric}_{g'} = \lambda' g' + (2m - \lambda')\eta \otimes \eta, \quad \text{with} \quad \lambda' = \frac{\lambda + 2 - 2a}{a}.
\]

If $g$ is $\eta$-Einstein with $\lambda > -2$, then a $D$-homothetic transformation with $a = \frac{\lambda + 2}{2m + 2}$ gives a Sasaki-Einstein metric $g'$. Thus any Sasaki structure which is transversely Kähler-Einstein $\text{Ric}^T = \kappa \omega^T$ with $\kappa > 0$ has a $D$-homothetic transformation to a Sasaki-Einstein structure.

**Proposition 2.4.** The following necessary conditions for $S$ to admit a deformation of the transverse Kähler structure to a Sasaki-Einstein metric are equivalent.

(i) $c_1^B = a[\text{d}\eta]$ for some positive constant $a$.

(ii) $c_1^B > 0$, i.e. represented by a positive $(1,1)$-form, and $c_3(D) = 0$.

(iii) For some positive integer $\ell > 0$, the $\ell$-th power of the canonical line bundle $K_{C(S)}$ admits a nowhere vanishing section $\Omega$ with $L_\xi \Omega = i(m + 1)\Omega$.

If (iii) is satisfied then the singularity $X = C(S) \cup \{a\}$ is $\ell$-Gorenstein, meaning that $K_{C(S)}^\ell$ is trivial. Though, the condition of Proposition 2.4 is stronger. In fact the isolated singularity of $X$ is rational (cf. [13]). If $\ell = 1$ we will say that $X$ is Gorenstein.

**Proof.** Let $\rho$ denote the Ricci form of $(C(S), \bar{g})$, then easy computation shows that
\[
(13) \quad \rho = \rho^T - (2m + 2)\frac{1}{2} d\eta.
\]

If (i) is satisfied, there is a $D$-homothety so that $[\rho^T] = (2m + 2)[\frac{1}{2} d\eta]$ as basic classes. Thus there exists a smooth function $h$ with $\xi h = 0 = r^\frac{a}{2} h$ and
\[
(14) \quad \rho = i \partial \bar{\partial} h.
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This implies that $e^h \frac{m+1}{m+1}$, where $\omega$ is the Kähler form of $\bar{g}$, defines a flat metric $| \cdot |$ on $K_{C(S)}$. Parallel translation defines a multi-valued section which defines a holomorphic section $\Omega$ of $K'_{C(S)}$ for some integer $l > 0$ with $|\Omega| = 1$. Then we have

$$(15) \quad \left( \frac{i}{2} \right)^{m+1} \frac{-1}{(m+1)!} \Omega \wedge \bar{\Omega} = e^h \frac{1}{(m+1)!} \omega^{m+1}.$$  

From the invariance of $h$ and the fact that $\omega$ is homogeneous of degree 2, we see that $L_a \Omega = (m+1)\Omega$.

The equivalence of (i) and (ii) is easy (cf. [20] Proposition 4.3). □

Example 2.1 This is the most elementary construction of Sasaki manifolds. Let $L$ be a negative line bundle over a complex manifold, or orbifold, $M$. Then $L$ has a hermitian metric $h$ with $\omega_M = i\partial \bar{\partial} \log h$ a Kähler form. Set $r_0^2 = h|z|^2$, then $L^{r_0}$ defines the Kähler potential of a Kähler cone structure on $C(S) = L^\times$, $L$ minus the zero section, with Kähler form $\omega = \frac{i}{2} \partial \bar{\partial} r_0^2$. Thus $S = \{ r_0 = 1 \} \subset L$ has a natural Sasaki structure. Note that $\eta$ is a real valued, connection on the $S^1$ bundle $S$ and $\omega^T = \frac{i}{2} \partial \bar{\partial} \eta = \omega_M$. We call this Sasaki structure on $S = \{ r_0 = 1 \} \subset L$ the standard Sasaki structure. This Sasaki structure is well defined up to deformations of the transverse Kähler structure and $D$-homothetic transformations.

In particular if $L = K_M$, where $K_M$ denotes the orbifold canonical bundle of $M$ if $M$ is an orbifold, then this Sasaki structure on $S \subset L^\times$ satisfies the conditions of Proposition 2.4. ◊

We will make use of the notion of the Sasaki cone of a Sasaki structure [12, 11]. Given a Sasaki structure $(g, \xi, \eta, \Phi)$ with $D = \ker \eta$ and $\Phi|_D = J$, we consider the following set.

$$(16) \quad S(D,J) = \left\{ S = (g, \xi, \eta, \Phi) : S \text{ a Sasaki structure with } \ker \eta = \ker \Phi \right\}$$

Thus $S(D,J)$ is the set of Sasaki structures with underlying strictly pseudo-convex CR-structure $(D,J)$.

Let $\mathfrak{cr}(D,J)$ be the Lie algebra of the group $\mathfrak{CR}(D,J)$ of CR automorphisms $(D,J)$. We say a vector field $X \in \mathfrak{cr}(D,J)$ is positive if $\eta(X) > 0$ for any $(g, \xi, \eta, \Phi) \in S(D,J)$. If $\mathfrak{cr}^+(D,J)$ denotes the subset of positive vector fields, then it is easy to see that

$$(17) \quad S(D,J) \to \mathfrak{cr}^+(D,J) \quad (g, \xi, \eta, \Phi) \mapsto \xi$$

is a bijection. This leads to the following.

Definition 2.5. Let $(D,J)$ be a CR structure associated to a Sasaki structure, then we have the Sasaki cone

$$\kappa(D,J) = S(D,J)/\mathfrak{CR}(D,J)$$

The isotropy subgroup of an element $S$ is precisely the automorphism group of the Sasaki structure, $\text{Aut}(S) \subseteq \mathfrak{CR}(D,J)$.

Theorem 2.6 ([11]). Let $(D,J)$ be the CR structure associated to a Sasaki structure on a compact manifold. Then the Lie algebra $\mathfrak{cr}(D,J)$ decomposes as $\mathfrak{cr}(D,J) = \mathfrak{t} \oplus \mathfrak{p}$ where $\mathfrak{t}$ is the Lie algebra of a maximal torus $T$ of dimension $k$, $1 \leq k \leq m+1$, 

and \( p \) is a completely reducible \( T \)-module. Furthermore, every \( X \in \mathfrak{cr}^+(D,J) \) is conjugate to a positive element of \( \mathfrak{t} \).

Considering Theorem 2.6 let us fix a maximal torus \( T \) of a maximal compact subgroup \( G \subseteq CR(D,J) \) with Weyl group \( \mathcal{W} \). Let \( \mathfrak{t}^+ = \mathfrak{t} \cap \mathfrak{cr}^+(D,J) \) denote the subset of positive elements. Then we have

\[
\kappa(D,J) = \mathfrak{t}^+/\mathcal{W}.
\]

In practice we will alter a Sasaki structure \( S = (g, \xi, \eta, \Phi) \) by varying in \( \kappa(D,J) \) to \( S' = (g', \xi', \eta', \Phi) \) and then making a transverse Kähler deformation as in (6) to \( \tilde{S} = (\tilde{g}, \tilde{\xi}', \tilde{\eta}, \tilde{\Phi}) \). This is awkward to formalize since if \( \phi \in C^\infty_T \) is not \( T \) invariant, then the Sasaki cone of \( \tilde{S} \) will have dimension less than \( k \). So one generally fixes a torus \( T^k \) and considers \( T^k \)-invariant Sasaki structures on \( S \). The case \( k = m + 1 \) is well understood and is considered in the next section.

The proof of the following is easy.

**Proposition 2.7.** Suppose \((g, \xi, \eta, \Phi)\) is a Sasaki structure on \( S \) satisfying Proposition 2.4, then every Sasaki structure in \( \kappa(D,J) \) satisfies Proposition 2.4.

2.2. Toric Sasaki-Einstein manifolds. In this section we recall the basics of toric Sasaki manifolds. Much of what follows can be found in [34] or [20].

**Definition 2.8.** A Sasaki manifold \((S, g, \xi, \eta, \Phi)\) of dimension \( 2m + 1 \) is toric if there is an effective action of an \( m + 1 \)-dimensional torus \( T = T^{m+1} \) preserving the Sasaki structure such that \( \xi \) is an element of the Lie algebra \( \mathfrak{t} \) of \( T \). Equivalently, a toric Sasaki manifold is a Sasaki manifold \( S \) whose Kähler cone \( C(S) \) is a toric Kähler manifold.

We have an effective holomorphic action of \( T_C \cong (\mathbb{C}^*)^{m+1} \) on \( C(S) \) whose restriction to \( T \subset T_C \) preserves the Kähler form \( \omega = d(\frac{1}{2}\pi^2 \eta) \). So there is a moment map

\[
\mu : C(S) \longrightarrow \mathfrak{t}^*,
\]

\[
\langle \mu(x), X \rangle = \frac{1}{2}\pi^2 \eta(X_S(x)),
\]

where \( X_S \) denotes the vector field on \( C(S) \) induced by \( X \in \mathfrak{t} \). We have the moment cone defined by

\[
C(\mu) := \mu(C(S)) \cup \{0\},
\]

which from [30] is a strictly convex rational polyhedral cone. Recall that this means that there are vectors \( \lambda_i, i = 1, \ldots, d \) in the integral lattice \( \mathbb{Z}_T = \ker\{\exp : \mathfrak{t} \rightarrow T\} \) such that

\[
C(\mu) = \bigcap_{j=1}^d \{y \in \mathfrak{t}^* : \langle \lambda_j, y \rangle \geq 0\}.
\]

The condition that \( C(\mu) \) is strictly convex means that it is not contained in any linear subspace of \( \mathfrak{t}^* \). It is cone over a finite polytope. We assume that the set of vectors \( \{\lambda_j\} \) is minimal in that removing one changes the set defined by (21). And we furthermore assume that the vectors \( \lambda_j \) are primitive, meaning that \( \lambda_j \) cannot be written as \( p\lambda_j \) for \( p \in \mathbb{Z} \) and \( \lambda_j \in \mathbb{Z}_T \).

Let \( \text{Int} C(\mu) \) denote the interior of \( C(\mu) \). Then the action of \( T \) on \( \mu^{-1}(\text{Int} C(\mu)) \) is free and is a Lagrangian torus fibration over \( \text{Int} C(\mu) \). There is a condition on
the \( \{ \lambda_j \} \) for \( S \) to be a smooth manifold. Each face \( F \subset C(\mu) \) is the intersection of a number of facets \( \{ y \in t^* : l_j(y) = \lambda_j \cdot y = 0 \} \). For \( S \) to be smooth each face \( F \) must be the intersection of codim \( F \) facets. Let \( \lambda_{j_1}, \ldots, \lambda_{j_a} \) be the corresponding collection of normal vectors in \( \{ \lambda_j \} \), where \( a \) is the codimension of \( F \). Then the cone \( C(\mu) \) is smooth if and only if

\[
\left\{ \sum_{k=1}^a \nu_k \lambda_{j_k} : \nu_k \in \mathbb{R} \right\} \cap \mathbb{Z}_T = \left\{ \sum_{k=1}^a \nu_k \lambda_{j_k} : \nu_k \in \mathbb{Z} \right\}
\]

for all faces \( F \).

Note that \( \mu(S) = \{ y \in C(\mu) : y(\xi) = \frac{1}{2} \} \). The hyperplane \( \{ y \in t^* : y(\xi) = \frac{1}{2} \} \) is the characteristic hyperplane of the Sasaki structure. Consider the dual cone to \( C(\mu) \)

\[
C(\mu)^* = \{ \tilde{x} \in t : \tilde{x} \cdot y \geq 0 \text{ for all } y \in C(\mu) \},
\]

which is also a strictly convex rational polyhedral cone by Farkas’ theorem. Then \( \xi \) is in the interior of \( C(\mu)^* \). Let \( \frac{\partial}{\partial \phi_i}, i = 1, \ldots, m + 1, \) be a basis of \( t \) in \( \mathbb{Z}_T \). Then we have the identification \( t^* \cong t \cong \mathbb{R}^{m+1} \) and write

\[
\lambda_j = (\lambda_j^1, \ldots, \lambda_j^{m+1}), \quad \xi = (\xi^1, \ldots, \xi^{m+1}).
\]

If we set

\[
y_i = \mu(x) \left( \frac{\partial}{\partial \phi_i} \right), \quad i = 1, \ldots, m + 1,
\]

then we have symplectic coordinates \( (y, \phi) \) on \( \mu^{-1}(\text{Int} C(\mu)) \cong \text{Int} C(\mu) \times T^{m+1} \). In these coordinates the symplectic form is

\[
\omega = \sum_{i=1}^{m+1} dy_i \wedge d\phi_i.
\]

The Kähler metric can be seen as in [2] to be of the form

\[
g = \sum_{ij} G_{ij} dy_i dy_j + G^{ij} d\phi_i d\phi_j,
\]

where \( G^{ij} \) is the inverse matrix to \( G_{ij}(y) \), and the complex structure is

\[
\mathcal{I} = \begin{pmatrix} 0 & -G^{ij} \\ G_{ij} & 0 \end{pmatrix}
\]

in the coordinates \( (y, \phi) \). The integrability of \( \mathcal{I} \) is \( G_{ij;k} = G_{ik;j} \). Thus

\[
G_{ij} = G_{ij} := \frac{\partial^2 G}{\partial y_i \partial y_j},
\]

for some strictly convex function \( G(y) \) on \( \text{Int} C(\mu) \). We call \( G \) the symplectic potential of the Kähler metric.

One can construct a canonical Kähler structure on the cone \( X = C(S) \), with a fixed holomorphic structure, via a simple Kähler reduction of \( C^d \) (cf. [28] and [24]). The symplectic potential of the canonical Kähler metric is

\[
G^{\text{can}} = \frac{1}{2} \sum_{i=1}^{d} l_i(y) \log l_i(y).
\]
Let
\[ G_\xi = \frac{1}{2} l_\xi(y) \log l_\xi - \frac{1}{2} l_\infty(y) \log l_\infty, \]
where
\[ l_\xi(y) = \xi \cdot y, \quad \text{and} \quad l_\infty(y) = \sum_{i=1}^{d} \lambda_i \cdot y. \]

Then
\[ (30) \quad G_\xi^{can} = G^{can} + G_\xi, \]
defines a symplectic potential of a Kähler metric on \( C(S) \) with induced Reeb vector field \( \xi \). To see this write
\[ (31) \quad \xi = \sum_{i=1}^{m+1} \xi^i \frac{\partial}{\partial \phi_i}, \]
and note that the Euler vector field is
\[ (32) \quad r \frac{\partial}{\partial r} = 2 \sum_{i=1}^{m+1} y_i \frac{\partial}{\partial y_i}. \]

Thus we have
\[ (33) \quad \xi^i = \sum_{j=1}^{m+1} 2G_{ij}y_j. \]

Computing from (30),
\[ (34) \quad (G_\xi^{can})_{ij} = \frac{1}{2} \sum_{k=1}^{d} \lambda_k^i \lambda_k^j + \frac{1}{2} \frac{\xi^i \xi^j}{l_\xi} - \frac{1}{2} \frac{\sum_{k=1}^{d} \lambda_k^i \sum_{k=1}^{d} \lambda_k^j}{l_\infty}, \]
and (33) follows by direct computation.

The general symplectic potential is of the form
\[ (35) \quad G = G^{can} + G_\xi + g, \]
where \( g \) is a smooth homogeneous degree one function on \( C \) such that \( G \) is strictly convex.

Note that the complex structure on \( X = C(S) \) is determined up to biholomorphism by the associated moment polyhedral cone \( C(\mu) \) (cf. [2] Proposition A.1). The following follows easily from this discussion.

**Proposition 2.9.** Let \( S \) be a compact toric Sasaki manifold and \( C(S) \) its Kähler cone. For any \( \xi \in \text{Int}C(\mu)^* \) there exists a toric Kähler cone metric, and associated Sasaki structure on \( S \), with Reeb vector field \( \xi \). And any other such structure is a transverse Kähler deformation, i.e. \( \tilde{\eta} = \eta + 2d\phi \), for a basic function \( \phi \).

Consider now the holomorphic picture of \( C(S) \). Since \( C(S) \) is a toric variety \( (\mathbb{C}^*)^{m+1} \cong \mu^{-1}(\text{Int}C) \subset C(S) \) is an dense orbit. We introduce logarithmic coordinates \((z_1, \ldots, z_{m+1}) = (x_1 + i\phi_1, \ldots, x_{m+1} + i\phi_{m+1})\) on \( \mathbb{C}^{m+1}/2\pi i\mathbb{Z}^{m+1} \cong (\mathbb{C}^*)^{m+1} \cong \mu^{-1}(\text{Int}C) \subset C(S) \), i.e. \( x_j + i\phi_j = \log w_j \) if \( w_j, j = 1, \ldots, m+1 \), are the usual coordinates on \( (\mathbb{C}^*)^{m+1} \). The Kähler form can be written as
\[ (36) \quad \omega = i\partial\bar{\partial}F, \]
where $F$ is a strictly convex function of $(x_1, \ldots, x_{m+1})$. One can check that
\begin{equation}
F_{ij}(x) = G_{ij}(y),
\end{equation}
where $\mu = y = \nabla F$ is the moment map. Furthermore, one can show $x = \nabla G$, and the Kähler and symplectic potentials are related by the Legendre transform
\begin{equation}
F(x) = \sum_{i=1}^{m+1} x_i \cdot y_i - G(y).
\end{equation}
It follows from equation (24) defining symplectic coordinates that
\begin{equation}
F(x) = l_\xi(y) = \frac{r^2}{2}.
\end{equation}
We now consider the conditions in Proposition 2.4 more closely in the toric case. So suppose the Sasaki structure satisfies Proposition 2.4, thus we may assume $c_B^F = (2m+2)[\omega^T]$. Then equation (33) implies that
\begin{equation}
\rho = -i\partial\bar{\partial} \log \det(F_{ij}) = i\partial\bar{\partial} h,
\end{equation}
with $\xi h = 0 = r \frac{\partial}{\partial r} h$, and we may assume $h$ is $T^{m+1}$-invariant. Since a $T^{m+1}$-invariant pluriharmonic function is an affine function, we have constants $\gamma_1, \ldots, \gamma_{m+1} \in \mathbb{R}$ so that
\begin{equation}
\log \det(F_{ij}) = -2 \sum_{i=1}^{m+1} \gamma_i x_i - h.
\end{equation}
In symplectic coordinates we have
\begin{equation}
\det(G_{ij}) = \exp(2 \sum_{i=1}^{m+1} \gamma_i G_i + h).
\end{equation}
Then from (30) one computes the right hand side to get
\begin{equation}
\det(G_{ij}) = \prod_{k=1}^{d} \left( \frac{l_k(y)}{l_\infty(y)} \right)^{(\gamma, \lambda_k)} (l_\xi(y))^{-(m+1)} \exp(h),
\end{equation}
And from (34) we compute the left hand side of (42)
\begin{equation}
\det(G_{ij}) = \prod_{k=1}^{d} (l_k(y))^{-1} f(y),
\end{equation}
where $f$ is a smooth function on $C(\mu)$. Thus $(\gamma, \lambda_k) = -1$, for $k = 1, \ldots, d$. Since $C(\mu)^*$ is strictly convex, $\gamma$ is a uniquely determined element of $t^*$. Applying $\sum_{j=1}^{m+1} y_j \frac{\partial}{\partial y_j}$ to (42) and noting that $\det(G_{ij})$ is homogeneous of degree $-(m+1)$ we get
\begin{equation}
(\gamma, \xi) = -(m+1).
\end{equation}
As in Proposition 2.4 $e^h \det(F_{ij})$ defines a flat metric $\| \cdot \|$ on $K_{C(S)}$. Consider the $(m+1,0)$-form
\begin{equation}
\Omega = e^{i\theta} e^{\frac{h}{2}} \det(F_{ij})^\frac{1}{2} dz_1 \wedge \cdots \wedge dz_{m+1}.
\end{equation}
From equation (41) we have
\[ \Omega = e^{i\theta} \exp(-\sum_{j=1}^{m+1} \gamma_j x_j) dz_1 \wedge \cdots \wedge dz_{m+1}. \]

If we set \( \theta = -\sum_{j=1}^{m+1} \gamma_j \phi_j \), then
\[ (46) \quad \Omega = e^{-\sum_{j=1}^{m+1} \gamma_j x_j} dz_1 \wedge \cdots \wedge dz_{m+1} \]
is clearly holomorphic on \( U = \mu^{-1}(\text{Int } \mathbb{C}) \). When \( \gamma \) is not integral, then we take \( \ell \in \mathbb{Z}^+ \) such that \( \ell \gamma \) is a primitive element of \( \mathbb{Z}^*_T \cong \mathbb{Z}^{m+1} \). Then \( \Omega \otimes \ell \) is a holomorphic section of \( K^\ell_c(S) \) which extends to a holomorphic section of \( K^\ell_c(S) \) as \( ||\Omega|| = 1 \).

It follows from (46) that
\[ (47) \quad L_\xi \Omega = -i(\gamma, \xi) \Omega = i(m+1) \Omega. \]

And note that we have equation (15) from (41) and (46).

**Proposition 2.10.** Let \( S \) be a compact toric Sasaki manifold. Then the conditions of Proposition 2.4 are equivalent to the existence of \( \gamma \in \mathfrak{t}^* \) such that
\begin{enumerate}[(i)]  
  \item \( (\gamma, \lambda_k) = -1 \), for \( k = 1, \ldots, d \),  
  \item \( (\gamma, \xi) = -(m+1) \), and  
  \item there exists \( \ell \in \mathbb{Z}^+ \) such that \( \ell \gamma \in \mathbb{Z}^*_T \cong \mathbb{Z}^{m+1} \)
\end{enumerate}
Then (46) defines a nowhere vanishing section of \( K^\ell_c(S) \).

In the toric case the conditions of Proposition 2.10 are equivalent to \( X = C(S) \cup \{o\} \) being an \( \ell \)-Gorenstein singularity.

We will need the beautiful results of A. Futaki, H. Ono, and G. Wang on the existence of Sasaki-Einstein metrics on toric Sasaki manifolds.

**Theorem 2.11 (20, 18).** Suppose \( S \) is a toric Sasaki manifold satisfying Proposition 2.10. Then we can deform the Sasaki structure by varying the Reeb vector field and then performing a transverse Kähler deformation to a Sasaki-Einstein metric. The Reeb vector field and transverse Kähler deformation are unique up to isomorphism.

In [20] a more general result is proved. It is proved that a compact toric Sasaki manifold satisfying Proposition 2.10 has a transverse Kähler deformation to a Sasaki structure satisfying the transverse Kähler Ricci soliton equation:
\[ \rho^T - (2m+2) \omega^T = L_X \omega^T \]
for some Hamiltonian holomorphic vector field \( X \). The analogous result for toric Fano manifolds was proved in [49]. A transverse Kähler Ricci soliton becomes a transverse Kähler-Einstein metric, i.e. \( X = 0 \), if the Futaki invariant \( f_1 \) of the transverse Kähler structure vanishes. The invariant \( f_1 \) depends only on the Reeb vector field \( \xi \). The next step is to use a volume minimization argument due to Martelli-Sparks-Yau [34] to show there is a unique \( \xi \) satisfying (45) for which \( f_1 \) vanishes.

**Example 2.2** Let \( M = \mathbb{C}P^2(2) \) be the two-points blow up. And Let \( S \subset K_M \) be the \( U(1) \)-subbundle of the canonical bundle. Then the standard Sasaki structure on \( S \) satisfies (1) of Proposition 2.4 and it is not difficult to show that \( S \) is simply connected and is toric. See Example 2.1. But the automorphism group of \( M \)
is not reductive, thus \( M \) does not admit a Kähler-Einstein metric due to Y. Matsushima [35]. Thus there is no Sasaki-Einstein structure with the usual Reeb vector field. But by Theorem 2.11 there is a Sasaki-Einstein structure with a different Reeb vector field.

The vectors defining the facets of \( \mathcal{C}(\mu) \) are

\[
\lambda_1 = (1, 0, 0), \quad \lambda_2 = (1, 0, 1), \quad \lambda_3 = (1, 1, 2), \quad \lambda_4 = (1, 2, 1), \quad \lambda_5 = (1, 1, 0).
\]

The Reeb vector field of the toric Sasaki-Einstein metric on \( S \) was calculated in [34] to be

\[
\xi = \left(3, \frac{9}{16}(-1 + \sqrt{33}), \frac{9}{16}(-1 + \sqrt{33})\right).
\]

One sees that the Sasaki structure is irregular with the closure of the generic orbit being a two torus.

\[\Box\]

3. APPROXIMATING METRIC

3.1. The Calabi Ansatz. The Calabi ansatz constructs a complete Ricci-flat Kähler metric on the total space of the canonical bundle \( K_M \) of a Kähler manifold \((M, \omega)\), provided \( M \) admits a Kähler-Einstein metric. This condition is equivalent, up to homothety, to the standard Sasaki structure on \( S \subset K_M \) being Einstein, where \( S = \{r = 1\} \) with the Sasaki structure \((g, \xi, \eta, \Phi)\) with \( \frac{1}{2}d\eta = \omega \) and \( \xi \) generating the \( S^1 \) action on \( K_M \). We describe an extension of the Calabi ansatz, due to A. Futaki [19], to the case where \( S \) admits a Sasaki-Einstein structure for a possibly different Reeb vector field \( \tilde{\xi} \), with the same Kähler cone.

Suppose \( M \) is a Fano manifold and \( L^p = K_M \) for a positive integer \( p \). Suppose there is an \( \eta \)-Einstein Sasaki structure \((g, \xi, \eta, \Phi)\) on the \( U(1)\)-bundle \( S \) associated to \( L \), where \( \xi \) is a possibly different Reeb vector field to that in the standard Sasaki structure of Example 2.1. Thus

\[
\rho^T = \kappa \omega^T,
\]

where we set \( \kappa = 2p \). Set \( t = \log r \). The Calabi ansatz searches for a Kähler form on \( L \) of the form

\[
\omega_\phi = \omega^T + i\partial\bar{\partial}F(t),
\]

where \( F(t) \) is a smooth function on \((t_1, t_2) \subset (-\infty, \infty)\). Define a new variable and function

\[
\tau = F'(t) \quad \text{and} \quad \phi(\tau) = F''(t).
\]

We must require \( \phi(\tau) > 0 \) for \( \omega \) to be positive. Also assume that \( F' \) maps \((t_1, t_2) \) onto \((0, b)\). Then the Calabi ansatz is

\[
\omega_\phi = \omega^T + d\bar{d}F(t)
\]

\[
= (1 + \tau)\omega^T + \phi(\tau)i\partial \wedge \bar{\partial}t
\]

\[
= (1 + \tau)\omega^T + \phi(\tau)^{-1}i\partial \tau \wedge \bar{\partial}r
\]

which defines a Kähler metric on

\[
C(S)_{(t_1, t_2)} = \{e^{\ell_1} < r < e^{\ell_2}\} \subseteq C(S) \subset L.
\]
Direct computation gives the equations

\begin{align}
\omega^{m+1}_\phi &= (1 + \tau)^m(m + 1)\phi(\tau)dt \wedge d^2t \wedge (\omega^T)^m, \\
\rho_\phi &= \rho^T - i\bar{\phi}\log((1 + \tau)^m\phi(\tau)) \\
&= \kappa\omega^T - i\bar{\phi}\log((1 + \tau)^m\phi(\tau)), \\
\sigma_\phi &= \frac{\sigma^T}{1 + \tau} - i\Delta_\phi\log((1 + \tau)^m\phi(\tau)) \\
&= \frac{m\kappa}{1 + \tau} - i\Delta_\phi\log((1 + \tau)^m\phi(\tau)).
\end{align}

It will be useful to know the relation between the curvature tensors of \(\omega_\phi\) and \(\omega^T\). Denote them respectively by \(R_\phi\) and \(R^T\). Denote by \(\zeta = r\frac{\partial}{\partial r} - \bar{\xi}\) the holomorphic vector field given by the Sasaki structure. Let \(U, V, X, Y\) be complex vector fields which are horizontal with respect to the 1-form \(\frac{dt}{r} + i\eta\) dual to \(\zeta\). Then we have

\begin{align}
R_\phi(U, \bar{V}, X, Y) &= (1 + \tau)R^T(U, \bar{V}, X, Y) + \phi(\omega^T(U, \bar{V})\omega^T(X, \bar{Y}) - \omega^T(U, \bar{Y})\omega^T(X, \bar{V})), \\
R^\phi(U, \bar{V}, \zeta, \bar{\zeta}) &= (\phi - (1 + \tau)^{-1}\phi^2)i\omega^T(U, \bar{V}), \\
R^\phi(\zeta, \bar{\zeta}, \zeta, \bar{\zeta}) &= -\frac{\phi}{1 + \tau} + \phi^{-1}\phi^2,
\end{align}

where dots in the last line denote the derivative with respect to \(t\).

We now consider the case of constant scalar curvature. Calculation gives

\begin{equation}
\sigma_\phi = \frac{m\kappa}{1 + \tau} - \frac{1}{(1 + \tau)^m} \frac{d^2}{dt^2}((1 + \tau)^m\phi).
\end{equation}

Setting \(\sigma_\phi = c\) we get the differential equation

\begin{equation}
(\phi(1 + \tau)^m)'' = \left(\frac{m\kappa}{1 + \tau} - c\right)(1 + \tau)^m,
\end{equation}

with the solutions

\begin{equation}
\phi(\tau) = \frac{\kappa}{m + 1}(1 + \tau) - \frac{c}{(m + 1)(m + 2)}(1 + \tau)^2 + \frac{c_1\tau + c_2}{(1 + \tau)^m},
\end{equation}

with constants \(c_1\) and \(c_2\).

The function

\begin{equation}
s(t) = \int_{t_0}^{t(t)} \frac{dx}{\sqrt{\phi(x)}}
\end{equation}

gives the geodesic length along the \(t\)-direction. We are interested in metrics with a complete end at infinity. The following is a consequence of \(\Box\).

**Proposition 3.1.** Let \(\omega_\phi\) be the Kähler form of the Calabi ansatz on \(L^\times\) for an \(\eta\)-Einstein Sasaki manifold. Suppose \(\phi\) is defined on \((c, \infty)\) and for some \(c \geq 0\). Then \(\omega_\phi\) defines a metric with a complete non-compact end toward \(\tau = \infty\) on \(L\) if and only if \(\phi\) grows at most quadratically as \(\tau \to \infty\).

We now construct Ricci-flat metrics on a neighborhood of infinity on \(L\) with \(L^p = K_M\), where \(p = \alpha - 1\). Thus \(\kappa = 2(\alpha - 1)\). The desired metric must be
complete and have a pole of order $2\alpha$ at infinity. Calculation gives
\[
\rho_\phi = \kappa \omega^T - i \partial \bar{\partial} \log((1 + \tau)^m \phi(\tau))
\]
(62)
\[
= \left( \kappa - \frac{m \phi + (1 + \tau) \phi'}{1 + \tau} \right) \omega^T - \left( \frac{m \phi}{1 + \tau} \right)' + \bar{\partial} \partial \log((1 + \tau)^m \phi(\tau))). \phi \, dt \wedge d^c t.
\]
Thus $\kappa - \frac{m \phi + (1 + \tau) \phi'}{1 + \tau} = 0$ and $\left( \frac{m \phi}{1 + \tau} \right)' = 0$. Thus
(63)
\[
\frac{m \phi}{1 + \tau} + \phi' = \kappa.
\]
And solving this equation gives
(64)
\[
\phi(\tau) = \frac{\kappa}{m + 1}(1 + \tau) + \frac{a}{(1 + \tau)^m}, \text{ for } a \in \mathbb{R}.
\]
Therefore $\omega_\phi$ is Ricci-flat and is complete toward infinity as $\phi(\tau)$ has less than quadratic growth.

Now solve \( \frac{d\tau}{dt} = \phi(\tau) \) to get
(65)
\[
\tau = \left( \frac{m + 1}{\kappa} e^{\kappa t} + \frac{m + 1}{\kappa} \right)^{\frac{1}{m + 1}} - 1, \text{ for } c > 0.
\]
After changing the constants $a, c$ we have
(66)
\[
\tau = F'(t) = (ce^{\kappa t} + a)^{\frac{1}{m + 1}} - 1
\]
(67)
\[
\phi = F''(t) = \frac{ce^{\kappa t} + a}{m + 1} (ce^{\kappa t} + a)^{-\frac{m}{m + 1}} e^{\kappa t}.
\]
It follows that equation (62) becomes
(68)
\[
\omega_\phi^{m+1} = c e^{\kappa t} dt \wedge d^c t \wedge (\omega^T)^m.
\]
Notice that
(69)
\[
G = G(t) = \int_{t_0}^t (ce^{\kappa s} + a)^{\frac{1}{m + 1}} ds
\]
is a Kähler potential for $\omega_\phi$, i.e. $\omega_\phi = i \partial \bar{\partial} G$.

Thus (62) defines a Ricci-flat Kähler metric on $L^\times$ for the profile function (67) depending on constants $a \geq 0$ and $c > 0$. This two parameter family includes homotheties. Note that for $a = 0$ this is just a rescaling of the Sasaki cone metric as is seen by changing $\omega_\phi$ to the coordinate $r' = r^{\frac{m}{m + 1}}$.

3.2. approximating metric. Let $L = K_M$ be the canonical bundle of a Fano manifold $M$ with $\pi : L \rightarrow M$. Then as in Example 2.1 there is a standard Sasaki structure on $S = \{r_0 = 1\} \subset L$ with Kähler potential \( r_0^2 = h|z|^2 \) on $C(S) = L^\times$ for $h$ a hermitian metric on $L$. Let $\Psi \in \Omega^{m,0}(L)$ be the tautologically defined holomorphic $m$-form on the total space of $L$, i.e. for $u \in L$, $\Psi(u) = \pi^* u$. Define a $(m + 1, 0)$-form
(70)
\[
\Omega = \left( \frac{dr}{r} + i \eta \right) \wedge \Psi.
\]
If $dz_1 \wedge \cdots \wedge dz_m$ is a local section giving fiber coordinate $w$, then
(71)
\[
\Omega = w \left( \frac{dr}{r} + i \eta \right) \wedge dz_1 \wedge \cdots \wedge dz_m.
\]
One easily checks that $d\Omega = 0$, thus $\Omega$ is holomorphic. Also, $\Omega$ has a pole of order 2 at $\infty$, and $\mathcal{L}_\xi \Omega = i\Omega$.

We assume now that $M$ is a Fano orbifold. And let $\mathbf{L}$ be a line bundle on $M$ with $\mathbf{L}^p = K_M$. Suppose there exists an $\eta$-Einstein Sasaki structure on the link $S$ with holomorphic cone $C(S) = \mathbb{L}^\times$ with Kähler potential $\frac{1}{2}r^2$, Reeb vector field $\xi$, and contact form $\eta$, with $\rhoT = \kappa\omegaT$, $\kappa = 2p$, and which is compatible with $\Omega$ in that $\mathcal{L}_\xi \Omega = i\rho\Omega$.

Consider the holomorphic map $\varphi : \mathbf{L} \overset{\phi}{\to} K_M$. Let $\Omega' \in \Omega^{m+1,0}(K_M)$ be the holomorphic form defined above. Define $\Omega = \varphi^*\Omega'$. Then $\Omega$ has a pole of order $p + 1$ along $\infty$. We have $\mathcal{L}_\xi \Omega = i\rho\Omega$, and it is clear that this is the holomorphic form in Proposition 2.4.

Write $\bar{\omega}$ for $\omega_\phi$ defined in (52) using the profile $\phi$ defined in equation (60). Then $\bar{\omega}$ defines a Ricci-flat Kähler metric in a neighborhood of $\infty$ on $\mathbf{L}$. From (68) we have

$$\bar{\omega}^m + 1 = \frac{1}{2} cr^{m-1} dr \wedge \eta \wedge (\omega^T)^m.$$ 

Let $\omega = rdr \wedge \eta + r^2 \omega^T$ be the Kähler form of the $\eta$-Einstein Sasaki structure. We make a $D$-homothety as in Section 2.1 with $a = \frac{p}{m+1}$ to a Sasaki-Einstein structure on $\mathbb{L}^\times$ with $r' = r\frac{m+1}{p}$, $\eta' = \frac{p}{m+1}\eta$, and $\xi' = \frac{m+1}{p}\xi$. Let $\omega' = rdr' \wedge \eta' + r'^2 \omega'^T$ be the Kähler form. Then an easy computation gives

$$\omega^m + 1 = a^{m+1} (a-1)(2m+2) \omega^m + 1 = \left( \frac{-p}{m+1} \right)^{m+2} r^{2(p-m-1)} \omega^m + 1.$$ 

Now since $\omega'$ is a Ricci-flat Kähler form we have

$$\left( \frac{i}{2} \right)^{m+1} (-1)^{\frac{m(m+1)}{2}} \Omega \wedge \bar{\Omega} = e^h \frac{1}{(m+1)!} (\omega')^m + 1,$$

with $\partial \bar{\partial} h = 0$. Since $\mathcal{L}_\xi \Omega = i(m+1)\Omega$, we have $\Omega' = r' \frac{\partial}{\partial r'} = 0$, i.e. $h$ is basic. Thus $h$ is constant. And from (72), (73), and (74) we have

$$\bar{\omega}^m + 1 = \frac{1}{2} c \kappa r^{p-1} dr \wedge \eta \wedge (\omega^T)^m$$

$$= \frac{1}{2(m+1)} c \kappa r^{2(p-m-1)} \omega^m + 1$$

$$= \bar{\omega}^m \wedge \bar{\Omega},$$

where $\bar{\omega}$ is a non-zero constant.

We summarize the properties of the Kähler metric $\bar{\omega}$ which first appeared in [19].

**Proposition 3.2.** Let $M$ be a Fano orbifold, and let $\mathbf{L}$ be a line bundle on $M$ with $\mathbf{L}^\times$ non-singular with $\mathbf{L}^p = K_M$ where $p = \alpha - 1$. Suppose there is an $\eta$-Einstein Sasaki structure on $S \subset \mathbf{L}^\times$ compatible with the holomorphic structure, with Kähler potential $\frac{1}{2}r^2$, $\rhoT = \kappa\omegaT$, $\kappa = 2p$, and which is compatible with the natural $m + 1$-form $\Omega$ on $\mathbf{L}^\times$ in that $\mathcal{L}_\xi \Omega = i\rho\Omega$. Then the metric $\bar{\omega} = \omega_\phi = i\partial \bar{\partial} G$ with $\phi$ as in (60) and $G$ is defined by (64) defines a Ricci-flat metric $g$ on $\mathbf{L}^\times$. Furthermore, $\bar{\omega}$ is complete at infinity and has Euclidean volume growth. The curvature tensor $R_g$ of $g$ satisfies $\|\nabla^k R_g\|_g = O(\rho^{-2-k})$, where $\rho$ denotes the distance from a fixed point. And $\bar{\omega}^m + 1 = \bar{\omega} \wedge \bar{\Omega}$, so it has a pole of order $2\alpha$ along $\infty$. 
4. proof of main theorem

4.1. normal bundle of a divisor. Let $D \subset X$ be a divisor with $\alpha[D] = -K_X$, $\alpha > 1$. Let $N$ be the total space of the normal bundle $N_D \cong [D]|_D$ to $D$, with $D \subset N$ the zero section. Let $\mathcal{p} \subset \mathcal{O}(X)$ and $\mathcal{p} \subset \mathcal{O}(N)$ be the ideal sheaves of $D \subset X$ and $D \subset N$ respectively. Denote by $D_{(\nu)} = (D, \mathcal{O}_\nu)$, where $\mathcal{O}_\nu = \mathcal{O}(X)/\mathcal{p}^\nu|_D$, the $\nu$-th infinitesimal neighborhood of $D$ in $X$. Let $\tilde{D}_\nu$ be the $\nu$-th infinitesimal neighborhood of $D$ in $N$. We have $D_{(\nu)} \cong \tilde{D}_{(\nu)}$. If $\phi_k : D_{(k)} \cong \tilde{D}_{(k)}$ for $k \geq 2$, is an isomorphism, then the obstruction to lifting to an isomorphism $\phi_{k+1} : D_{(k+1)} \cong \tilde{D}_{(k+1)}$ is in $H^1(D, \Theta_X \otimes \mathcal{O}(-kD)|_D)$ (cf. [24] or [26]). Thus we have condition (2) as necessary in order to approximate a holomorphic neighborhood of $D \subset X$ with a neighborhood in the normal bundle $N_D$ by a map whose jet is holomorphic to high order along $D$.

Since we assume condition (2) holds, we have an isomorphism $\phi_\nu : D_{(\nu)} \cong \tilde{D}_{(\nu)}$ for arbitrary large $\nu \geq 2$. Then $\phi_\nu$ defines a jet

$$J^\nu_D \phi_\nu \in J^\nu_D \text{Diff}_D(V, U),$$

along $D$, where $\text{Diff}_D(V, U)$ denotes diffeomorphisms fixing $D$ where $V$ and $U$ are small tubular neighborhoods of $D$ in $X$ and $N$. Provided $V$ and $U$ are sufficiently small, there is a diffeomorphism $\psi \in \text{Diff}_D(U, V)$ with $J^\nu \psi = J^\nu \phi_\nu$ (cf. [18], Ch. II).

We collect some vanishing results which can be used in applying Theorem 1.3. Let $X$, $\dim X \geq 3$, be a Fano manifold and $D \subset X$ be a smooth divisor with $\alpha[D] = c_1(X) > 0$ with $\alpha > 1$. Then $c_1(D) = (\alpha - 1)[D]|_D > 0$, so $D$ is Fano as well. Suppose that $D$ is toric. We have the following:

**Proposition 4.1.** Suppose either $\alpha \leq 2$, or $X$ is toric and $\dim X \geq 4$. Then $H^1(D, \Theta_X \otimes \mathcal{O}(-kD)|_D) = 0$ for all $k \geq 2$.

**Proof.** Suppose $\alpha \leq 2$. We have the exact sequence on $D$,

$$0 \to \Theta_D \to \Theta_X \to \mathcal{N}_{X/D} \to 0.$$

Using $\mathcal{N}_{X/D} = \mathcal{O}([D]|_D)$ we have

$$\cdots \to H^1(D, \Theta_D(-kD)) \to H^1(D, \Theta_X(-kD)|_D) \to H^1(D, \mathcal{O}((1-k)D)) \to \cdots.$$

By Kodaira-Serre duality, $H^1(D, \Theta_D(-kD)) \cong H^{n-2}(D, \Omega^1((k+1-\alpha)D))$ which is zero for $k+1-\alpha > 0$ by the Bott vanishing theorem (cf. p.130 of [30]). We have $H^1(D, \mathcal{O}((1-k)D)) = 0$ by Kodaira vanishing and the negativity of $[(1-k)D]$, for $k \geq 2$.

The proof for the case with $X$ toric is a similar application of the Bott vanishing theorem. Note that we are not assuming that $D \subset X$ is an invariant embedding. \qed

Note that one can make use of some theorems on the existence of smooth divisors (cf. [1]). Suppose $X$ is a Fano manifold with $\text{Ind} X = r$, so $K^{-1}_X = rH$. Then if either $n = 3$ or $r \geq n - 1$, the linear system $|H|$ contains a smooth irreducible divisor.
4.2. The Approximating metric. We will construct an approximate metric, which will be used in the proof of Theorem [13] in each Kähler class in $H^2(Y, \mathbb{R})$.

We have a diffeomorphism $\psi$ of $V \subset X$ with a neighborhood of infinity $U$ of $L = N^{-1}_D = [D]^{-1}_D$, where $L^p = K_D$, $p = a - 1$, whose $\nu$-jet is holomorphic along $D$ for any large $\nu$. Let $G$ be a Kähler potential away from the zero section given in (69) of the Ricci-flat metric from section [3.4]. Define $g = \psi^* G$. Then $\omega = i\partial \bar{\partial} g$ is a Kähler form in a neighborhood of $D$ on $Y = X \setminus D$. By shrinking $V$ we may assume $\omega$ is positive definite on $V$. Let $V_r$ be the subset of $V$ defined by $V_r = \{ x \in V : g(x) > r \}$. Let $0 < a < b$ be such that $V_a \subset V$. Define a smooth function $\lambda : \mathbb{R} \to \mathbb{R}$ so that $\lambda(x) = x$ for $x \geq b$, $\lambda(x) = \frac{b-a}{a-b}$ for $x \leq a$, and in the interval $(a, b)$ $\lambda' > 0$ and $\lambda'' > 0$. Then $h = \lambda \circ g$ extends to a smooth function on $Y = X \setminus D$. Simple calculation shows that $i\partial \bar{\partial} h \geq 0$ on $Y$, and $i\partial \bar{\partial} h > 0$ on $V_a$.

Thus $Y$ is a 1-convex space, meaning that $Y$ carries an exhaustion function $h : Y \to [0, \infty)$ which is strictly plurisubharmonic, i.e. $i\partial \bar{\partial} h > 0$, outside a compact set. A 1-convex space is holomorphically convex. We consequently have the Remmert reduction $\pi : Y \to W$ where $W$ is a Stein space (cf. [24]), and $\pi$ has the following properties.

(i) $\pi$ is proper, surjective, and with connected fibers.
(ii) $\pi_* \mathcal{O}_Y = \mathcal{O}_W$.
(iii) The map $\pi^* : \mathcal{O}_W(W) \to \mathcal{O}_Y(Y)$ is an isomorphism.
(iv) The exceptional set

$$A = \{ y \in Y : \dim_{\mathbb{C}} \pi^{-1}(\pi(y)) > 0 \}$$

is the maximal compact analytic set of $Y$.

We will need the following vanishing results.

**Proposition 4.2.** Suppose $Y$ is a 1-convex manifold with $K_Y^p$ trivial for some $p \in \mathbb{N}$. Then $H^j(Y, \mathcal{O}_Y) = 0$ for $j \geq 1$. If $B = \{ h < c \}$ is strongly pseudoconvex, then $H^j(Y \setminus B, \mathcal{O}_Y) = 0$ for $1 \leq j \leq n - 2$.

**Proof.** Suppose first that $K_Y$ is trivial. By the Grauert-Riemenschneider vanishing theorem [25]

$$H^j(Y, \mathcal{O}_Y) = H^j(Y, i_* \mathcal{O}(K_Y)) = 0, \quad \text{for } j \geq 1. \quad (77)$$

We have a nowhere vanishing section $\sigma \in \Gamma(K_Y)$. Since the singularity set of $W$ has codimension $n \geq 2$, the dualizing sheaf satisfies $\omega_W = i_* \mathcal{O}(K_Y)$, where $i : U \to W$ is the inclusion of the non-singular set, and $\sigma \in \Gamma(\omega_W)$. The singularity set of $W$ is discrete, and any singular point $p \in W$ has a relatively compact neighborhood $V$ with $V \setminus \{ p \}$ smooth. Recall that a point $p \in W$ is a rational singularity if $R^i \pi_* \mathcal{O}_V|_x = 0$ for $i > 0$ where $\pi : Y \to W$ is any resolution of singularities. It is easy to see that

$$\int_V \sigma \wedge \bar{\sigma} < \infty. \quad (78)$$

It is a result of [13] that an isolated singularity for which there is a non-vanishing holomorphic n-form on a deleted neighborhood is rational if and only if (78) holds. Thus $W$ contains only rational singularities.

If $c > 0$ is chosen large enough that $B = \{ h < c \}$ is strongly pseudoconvex and $h$ is strictly plurisubharmonic on $Y \setminus B$, then the exceptional set $A \subset B$. Thus $\pi(B)$ is a strongly pseudoconvex neighborhood in $W$, which we also denote by $B$. 

It is well known that rational singularities are Cohen-Macaulay \[4\]. The vanishing theorem in \([9, I.\S 3, \text{Theorem } 3.1]\) implies that $H^k_B(W, \mathcal{O}_W) = 0$ for $k < n - 1$, where $H^k_B$ denotes cohomology with supports in $\bar{B}$. Recall the exact sequence with cohomology with supports

$$H^{k-1}(W, \mathcal{O}) \to H^{k-1}(W \setminus \bar{B}, \mathcal{O}) \to H^k_B(W, \mathcal{O}) \to H^k(W, \mathcal{O}) \to H^k_B(W, \mathcal{O}) \to \cdots$$

Since $W$ is Stein, we have

$$H^j(Y \setminus \bar{B}, \mathcal{O}) = H^j(W \setminus \bar{B}, \mathcal{O}) = H^{j+1}_B(W, \mathcal{O}) = 0, \text{ for } 1 \leq j \leq n - 2.$$

Now suppose that merely $K^q_Y$ is trivial for some $q \in \mathbb{N}$. Then there is a finite cover $\pi : Y \to Y$ with $K_Y$ trivial and $\pi^* h$ is strictly plurisubharmonic outside a compact subset of $\tilde{Y}$. Clearly, the above proof works on $\tilde{Y}$. Suppose $\beta \in \Omega^0(Y \setminus \bar{B})$ satisfies $\bar{\partial} \beta = 0$ for $1 \leq j \leq n - 2$. Then $\pi^* \beta = \bar{\partial} \gamma$ for some $\gamma \in \Omega^0(Y \setminus \bar{B})$. Let $G$ be the group of deck transformations of $\varphi$, and define $\bar{\gamma} = \frac{1}{|G|} \sum_{g \in G} g^* \beta$. Then it is easy to check that $\bar{\gamma} = \pi^* \theta$ for some $\theta \in \Omega^0(Y \setminus \bar{B})$, and $\bar{\partial} \theta = \beta$. Thus $H^1(Y \setminus \bar{B}, \mathcal{O}) = 0$ for $1 \leq j \leq n - 2$ in this case as well, and the exact argument shows $H^1(Y, \mathcal{O}) = 0$ for $j \geq 1$ as well. \qed

**Lemma 4.3.** The $\bar{\partial} \bar{\partial}$-lemma holds on $Y$, and for $n > 2$ on $Y \setminus \bar{B}$ where $B$ is as in Proposition 4.2. That is if $\beta$ is an exact real $(1, 1)$-form on $Y$, then $\beta = i\bar{\partial} \partial f$ for $f \in C^\infty(Y)$. And the analogous result holds on $Y \setminus \bar{B}$ for $n > 2$.

**Proof.** There exists an $\alpha \in \Omega^1$ with $d\alpha = \beta$. Splitting into types we have $\alpha = \alpha^0 + \alpha^0$ with $\alpha^0 = \bar{\alpha}^1$, and $\bar{\partial} \alpha^0 = 0$. Since $H^1(Y, \mathcal{O}) = H^1(Y \setminus \bar{B}, \mathcal{O}) = 0$, there exists a $\gamma \in C^\infty$ with $\bar{\partial} \gamma = \alpha^0$. Then one easily checks that $\beta = i\bar{\partial} \partial f \Im \gamma$. \qed

Suppose $\omega$ is a Kähler form on $Y$ with $[\omega] \in H^2_Y(\mathbb{R})$. Since $B_\nu := \{x \in Y : h(x) < \nu\}$ is strongly pseudoconvex, we have $\omega|_{Y \setminus \bar{B}} = i\bar{\partial} \partial f$ with $f \in C^\infty(Y \setminus \bar{B})$. We define a cut-off function $\eta : Y \to [0, 1]$ as follows. Choose $c, d$ so that $b < c < d$ and define $\eta(x) = 1$ for $x \leq c$, $\eta(x) = 0$ for $x \geq d$, and define $\eta$ to be decreasing with values in $(0, 1)$ on $(c, d)$. We define

$$\omega_0 = \begin{cases} \omega + C i\bar{\partial} \partial h, & \text{on } B \\ i\bar{\partial} \partial ((\eta \circ h)f) + C i\bar{\partial} \partial h, & \text{on } Y \setminus B. \end{cases}$$

(79)

For $C > 0$ sufficiently large $\omega_0$ is a Kähler form, and clearly $[\omega_0] = [\omega]$.

**4.3. Monge-Ampère equation.** Let $\sigma \in \Gamma(K_X)$ be a section with a pole along $D$, of order $\alpha$. Thus $\sigma \wedge \bar{\sigma}$ has a pole of order $2\alpha$. Also, $\omega^0_\nu$ has a pole of order $2\alpha$ along $D$. Thus

$$f = \log \left( \frac{\sigma \wedge \bar{\sigma}}{\omega^0_\nu} \right)$$

(80)

extends to a smooth function on $X$. We have $i\bar{\partial} \partial f = \text{Ricci}(\omega_0)$ which is zero along $D$. Thus $f$ is constant on $D$, and we may assume $f$ vanishes on $D$. Furthermore, $\partial f \in H^0(D, \mathcal{O}(N^*))$. And since $N^*$ is negative, $\partial f$ vanishes along $D$. Using the negativity of of $N^{-k}, k \geq 1$, and that $i\bar{\partial} \partial f = \text{Ricci}(\omega_0)$ vanishes to order $\nu - 4$ on $D$, one can show that the derivatives of $f$ up to order $\nu - 2$ vanish along $D$.

Then we have the following properties of the approximating metric $\omega_0$. 

Proposition 4.4. The form $\omega_0$ defines a complete Kähler metric $g_0$ on $Y$ such that

$$Ricci(\omega_0) = i\partial \bar{\partial} f,$$

where $f$ is a smooth function on $X$ vanishing along $D$, and whose derivatives up to order $n - 2$ vanish along $D$. Furthermore, the curvature tensor satisfies $\|\nabla^k R(g_0)\|_{g_0} = O(\rho^{-2-k})$, where $\rho$ is the distance from a fixed point.

The following theorem is due to G. Tian and S.-T. Yau. The final statement on the curvature decay follows from [6].

Proposition 4.5 ([12]). Let $\omega_0$ be the Kähler metric on $Y = X \setminus D$ constructed above. And let $f$ be as above with $Ricci(\omega_0) = i\partial \bar{\partial} f$. Then the Monge-Ampère equation

$$\omega_0 + i\partial \bar{\partial} \phi)^n = e^f \omega_0^n,$$

has a smooth solution $\phi \in C^\infty(Y)$ where $\phi$ converges uniformly to zero at infinity, is bounded in $C^2$, and thus $\omega = \omega_0 + i\partial \bar{\partial} \phi$ satisfies $e^{-1} \omega_0 \leq \omega \leq c \omega_0$, for some $c > 0$.

Thus $\omega$ is the Kähler form of a complete Ricci-flat Kähler metric $g$ on $Y$. Furthermore, $g$ has Euclidean volume growth, and $\|R_g\|_g = O(\rho^{-2})$ where $\rho(x) = \text{dist}(o,x)$. If $\|R_g\|_g = O(\rho^{-k})$ for $k > 2$, then $(Y, g)$ is Kähler ALE of order $2n$. In which case $\|R_g\|_g = O(\rho^{-2n-2})$.

By ALE of order $m$ we mean the following. There exists a compact subset $K \subset Y$, a finite group $\Gamma \subset O(2n)$ acting freely on $\mathbb{R}^{2n} \setminus \{0\}$, and a ball $B_R(0) \subset \mathbb{R}^{2n}$ of radius $R > 0$. So that there is a diffeomorphism $\chi : (\mathbb{R}^{2n} \setminus B_R(0)) / \Gamma \rightarrow Y \setminus K$ and

$$\|\nabla^k \chi^* g - \nabla^k h\|_h = O(r^{-m-k}),$$

where $h$ is the flat metric and $\nabla$ its covariant derivative.

We say that is Kähler ALE if in addition we have $\mathbb{R}^{2n} = \mathbb{C}^n$ with the standard complex structure $J_0$ and $\Gamma \subset U(n)$. And if $J$ is the complex structure on $Y$ we have

$$\|\nabla^k \chi^* J - \nabla^k J_0\|_h = O(r^{-m-k}),$$

and Ricci-flatness implies that $\Gamma \subset SU(n)$. It is known that if an ALE manifold $Y$ is Kähler one can choose coordinates at infinity $\chi : (\mathbb{C}^n \setminus B_R(0)) / \Gamma \rightarrow Y \setminus K$ so that (83) holds.

4.4. asymptotic properties of the metric. We want to improve on the asymptotic behavior of $\phi$ in Proposition 4.4. First we need coordinates on $Y$ for which the metric has good bounds. Then we will apply Schauder estimates to the solution to Proposition 4.5. In the following $\rho$ will denote the distance from a fixed point $o \in Y$ outside a compact set $K \subset Y$ containing $o \in Y$ and extended to a continuous function on all of $Y$.

Definition 4.6. A holomorphic coordinate chart $(U, z_1, \ldots, z_n)$ centered at $x \in Y$ in a Kähler manifold is bounded of order $\ell$ with bound $(R, c, c_1, \ldots, c_\ell)$ if

(i) using the Euclidean coordinate distance the ball $B_R(x) \subset U$,

(ii) $\delta_{ij} \leq g_{ij} \leq c_\delta_{ij}$ if $g_{ij}$ denotes the metric tensor in $(U, z_1, \ldots, z_n)$,
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(iii) and for any multi-indices $\alpha, \beta$ with $|\alpha| + |\beta| \leq \ell$

$$\left| \frac{\partial^{\alpha+\beta} g(z)}{\partial z^\alpha \partial \bar{z}^\beta} \right| \leq c_{|\alpha|+|\beta|}.$$ 

We say that $(Y, g)$ has bounded geometry of order $\ell$ if there are constants $(R, c, c_1, \ldots, c_\ell)$ so that each $x_0 \in Y$ has a coordinate neighborhood $(U, z_1, \ldots, z_n)$ bounded of order $\ell$ with bound $(R, c, c_1, \ldots, c_\ell)$ for $\rho(x_0)^{-2} g$.

**Proposition 4.7.** Let $(Y, \omega)$ be the Kähler metric in Proposition 4.4. Then for any positive integer $\ell$, $(Y, \omega)$ has bounded geometry of order $\ell$ if $\nu$ in Proposition 4.3 is chosen sufficiently large.

**Proof.** We first show that the metric $\bar{\omega} = \omega_{\phi}$ of Proposition 3.2 has bounded geometry of any order $\ell$. Let $r_0 > 0$ be sufficiently large so that the set $Y_{r_0} := \{ x \in L : r(x) \geq r_0 \} \subset U$. We denote the metric on $U$ of (52) by $\omega_{\phi,a}$. We compute that $\omega_{\phi,a} = e^{-\phi} \omega_{\phi,a}$. The distance $\rho$ is equivalent to $e^{\phi/a} - \phi$, i.e. $c_1 \rho \leq e^{\phi/|a|} \leq c_2 \rho$ for $c_1, c_2 > 0$. Then $U_x \subset U$ defines the required coordinate chart centered at $bx$.

Now for $b > 1$ let $R_b : U \to U$ be the map generated by the action of $r \frac{\partial}{\partial r} - i \xi$. Thus $R_b(Y_{r_0}) = Y_{b r_0}$. Simple calculation shows that $R_b^* \omega_{\phi,a} = e^{\phi/b \xi/a} \omega_{\phi,a}$. The distance $\rho$ is equivalent to $e^{\phi/b + \xi/b}$, i.e. $c_1 \rho \leq e^{\phi/b + \xi/b} \leq c_2 \rho$ for $c_1, c_2 > 0$. Then $U_x \subset U$ defines the required coordinate chart centered at $bx$.

Recall we are considering $L^\times$ with a Sasaki-Einstein structure on the link $S \subset L^\times$ which we denote $(g_1, \xi_1, \eta_1, \Phi_1)$, with Kähler potential $\frac{\xi_1^2}{2}$. Denote by $(g_0, \xi_0, \eta_0, \Phi_0)$ the standard Sasaki structure on $L^\times$ satisfying Proposition 2.3 with Kähler potential $\frac{\xi_0^2}{2}$. Let $T^k$ be the torus acting on $L^\times$ whose Lie algebra $t^k$ contains both $\xi_0$ and $\xi_1$. Define $Z = \mathbb{P}(L \oplus \mathbb{C})$. Let $H$ be the tautological orbifold bundle on $Z$ restricting to the hyperplane bundle on each fiber of $\pi : Z \to D$. Then the bundle $E = H \otimes \pi^* K_D^j$ is positive for $j >> 0$. Clearly, $T^k$ and its complexification $(\mathbb{C}^*)^k$ acts on $E$. By the Baily embedding theorem [5] we have an embedding $\nu_{E^j} : Z \to \mathbb{C}P^N$ for $\ell >> 0$. It follows that $\psi : (\mathbb{C}^*)^k \times Z \to Z$ is an algebraic braic. In particular, $\psi$ extends to a rational map $\bar{\psi} : (\mathbb{C}P^1)^k \times Z \to Z$. If $z_1 \frac{\partial}{\partial z_1}, \ldots, z_k \frac{\partial}{\partial z_k}$ is the basis of the Lie algebra $t^k$ of $(\mathbb{C}^*)^k$, then $\frac{\partial}{\partial z_j} = \sum_{j} b_j z_j \frac{\partial}{\partial z_j}$ for $(b_1, \ldots, b_k) \in \mathbb{R}^k$ defines an embedding $\mathbb{R}^k \to (\mathbb{C}^*)^k$. And the restriction $\bar{\psi} : \mathbb{R}^k \times S \to Z$ is a diffeomorphism onto $L^\times$. Thus if $P_1 : \mathbb{R}^k \times S \to \mathbb{R}^k$ denotes the projection, then $r_0 = P_1 \circ \bar{\psi}^{-1}$.

Let $(W, z_1, \ldots, z_m)$ be a coordinate neighborhood on $D$ trivializing $L$ with fiber coordinate $w$, where $m = n - 1$ as above. Then $(z_1, \ldots, z_{n-1}, z_n)$ with $z_n = \frac{1}{w}$ defines a coordinate $U$ chart intersecting $D \subset Y$, as the section at infinity, with $D \cap U = \{ z \in V : z_n = 0 \}$. And $D$ can be covered by finitely many such coordinate charts. Since $\psi$ is a rational map, in $U$ we have

$$r_1 = \frac{1}{|z_n|^q} f(z),$$

with $q \in \mathbb{R}_+$ and $f(z)$ a smooth function.
As before denote \( t_0 = \log r_0 \) and \( t_1 = \log r_1 \). Then from (83) we have

\[
\frac{\partial t_1}{\partial t_0} = \frac{r_0}{r_1} \frac{\partial r_1}{\partial r_0} = 1/2 \eta_t(\xi_0) > c > 0,
\]

for a constant \( c \). Since \( \left[ \frac{\partial}{\partial r_1}, \frac{\partial}{\partial r_0} \right] = 0 \), it follows from integrating (85) and a similar bound for \( \frac{\partial t_0}{\partial t_1} \) that

\[
C_1 r_0^{q_1} \leq r_1 \leq C_2 r_0^{q_2},
\]

for \( q_1, q_2 \in \mathbb{R}_+, q_1 > q_2 \). We denote by \( \rho_0 \) and \( \rho_1 \) the distance from a fixed point \( x \in S \subset \mathbb{L}^x \) with respect to the metrics \( g_0 \) and \( g_1 \). Then we have

\[
C_1 \rho_0(y)^{q_1} \leq \rho_1(y) \leq C_2 \rho_0(y)^{q_2}, \quad \text{for} \quad y \in \mathbb{L}^x.
\]

And we have a bound as in (87) for the distance \( \rho \) of \( \bar{\omega} \).

We have the diffeomorphism \( \psi : V \to U \), where \( V \subset X \) is a neighborhood of \( D \subset X \), whose jet is holomorphic to arbitrarily high order along \( D \). Let \( \bar{g} \) denote the metric on \( U \subset \mathbb{L}^x \) with Kähler form \( \bar{\omega} \). For notational simplicity denote the metric \( \psi^* \bar{g} \) on \( V \subset X \) also by \( \bar{g} \). We also have the metric on \( V \) with Kähler form \( d\bar{\omega}^n \psi^* G \) which we denote \( g \). It follows from (84) and (87) that for \( k_0 > 0 \) if \( \psi : V \to U \) is chosen to have holomorphic jet along \( D \) of sufficiently high order \( \mu \), then

\[
\| \nabla^j (g - \bar{g}) \| = O(\rho^{-\alpha-2-j}), \quad \text{for} \quad 0 \leq j \leq k_0,
\]

where \( \alpha > 2n \) and \( \nabla \), the norm, and the distance \( \rho \) is with respect to \( \bar{g} \).

Let \( (U'_x, z_1, \ldots, z_n), x \in U' \) be the family of neighborhoods on \( U' \subset \mathbb{L}^x \) satisfying Definition 4.6. Then the \( w_j = \phi^* z_j \) form non-holomorphic coordinates on \( U_x = \phi^{-1}(U'_x), \phi(v) = x \) satisfying the conditions in 4.6 for the metric \( \rho(v)^{-2} \psi^* \bar{g} \), and thus also for \( \rho(v)^{-2} g \) by (88). In the following we will consider each \( U_x, x \in V \), with the metric \( \rho(x)^{-2} g \), and \( \rho \) is the distance with respect to this metric.

Consider the real coordinates \( w_j = x_j + iy_j, j = 1, \ldots, n, \) on \( U_x, v \in V \), and denote \( y_j \) by \( x_{n+j} \). First make a linear change of coordinates so that each \( dw_j |_v \in \Lambda^{1,0} \), i.e. \( \bar{\partial}w_j = 0 \) at \( v \). Then make the change of coordinates \( x_k \mapsto x_k + \frac{1}{2} \Gamma^k_{ij} x_i x_j \), so that \( \Gamma^k_{ij} |_v = 0 \). Thus \( \nabla dv_j = 0 \) at \( v \).

Let \( \rho \) be the distance from \( x \in U_x \). Since \( \sup\| R(g) \|_g \leq C \), by the Hessian comparison theorem (cf. (83)) there are positive constants \( r \) and \( C_1 \) so that restricting to \( B_r = \{ y : \rho(y) < r \} \subset U_x \) the functions \( \rho^2 \) and \( \log(\rho^2) \) are plurisubharmonic and

\[
i\partial \bar{\partial} \rho^2 \geq C_1 > 0, \quad \text{on} \quad B_r.
\]

Here \( r, C_1 \) can be chosen uniformly for every chart \( U_x \). Replace each \( U_x, x \in V \) with its restriction to \( B_r \). Thus, in particular, each \( U_x \) is strictly pseudoconvex.

Let \( P_{0,1} \) be the projection of \( \Lambda^1 \) onto its \((0,1)\) component, then \( \bar{\partial}w_j = P_{0,1} dw_j \). Since \( \nabla \) preserves types, \( \nabla \bar{\partial}w_j = 0 \) at \( x \). Let \( t \) denote the radial distance from \( x \in U_x \) along a geodesic. Then \( \frac{\partial}{\partial t} g(\bar{\partial}w_j, \bar{\partial}w_j) = 0 \) at \( x \in U_x \) for \( 0 \leq i \leq 4 \).
Expanding gives
\[ \frac{d^4}{dt^4} g(\partial w_j, \partial w_j) = g(P_{0,1} \nabla^4 dw_j, P_{0,1} dw_j) + g(P_{0,1} \nabla^3 dw_j, \partial w_j) + g(P_{0,1} \nabla^2 dw_j, P_{0,1} \nabla^2 dw_j) \]
\[ + g(P_{0,1} \nabla^3 dw_j, P_{0,1} \nabla_\ell dw_j) + g(P_{0,1} dw_j, P_{0,1} \nabla^4 dw_j) \]
\[ \leq g(\nabla^4 dw_j, dw_j) + g(\nabla^3 dw_j, \nabla_\ell dw_j) + g(\nabla^2 dw_j, \nabla^2 dw_j) \]
\[ + g(\nabla dw_j, \nabla^3 dw_j) + g(dw_j, \nabla^4 dw_j) \]
\[ \leq C, \]
where \( C \) depends on the uniform bounds of the metric and its derivatives in the coordinates \((w_1, \ldots, w_n)\). Integrating \( (90) \) four times gives
\[ g(\partial w_j, \partial w_j) \leq C \rho^4. \]

By Hörmander’s \(L^2\)-estimate with weight function \( \phi = (2n+3) \log \rho + 2 \log(1 + \rho^2) \), which is strictly plurisubharmonic after possibly shrinking \( R \), there is a function \( u_j \in C^\infty(U_x) \) so that \( \partial u_j = \partial w_j \) and
\[ \int_{U_x} |u_j|^2 e^{-\phi} d\mu \leq C_1 \int_{U_x} |\partial w_j|^2 e^{-\phi} d\mu \leq C_2, \]
where \( C_2 > 0 \) follows from \( (91) \) and \( C_1, C_2 > 0 \) are independent of the neighborhood \( U_x \). Because of the singularity of \( \phi, u_j = du_j = 0 \) at \( x \). Then the functions \( z_j = w_j - u_j, j = 1, \ldots, n \), give a holomorphic coordinate system in a neighborhood of \( x \in U_x \) which, after shrinking \( R \) if necessary, we denote by \( U_x \) again.

In each \( U_x \) we consider the Laplacian \( \Delta \) with respect to the Kähler metric \( \rho(x)^{-2} g \). Let \( U'_x \subset U_x \) be the ball of radius \( R' < R \) with respect to this metric. Then for \( \Delta u_j = \Delta w_j = f \) we have the Sobolev estimate
\[ \|u_j\|_{L^2(L^2(U'_x))} \leq C \left( \|f\|_{L^2(U_x)} + \|u\|_{L^2(U_x)} \right), \]
where \( C \) is independent of \( U_x \). Also the Sobolev inequality gives the Hölder bound
\[ \|u_j\|_{C^{\gamma}(U'_x)} \leq C' \|u_j\|_{L^2(U'_x)}, \]
for \( 0 < \gamma < 1 \), where \( C' \) depends only on \( s \) and \( \gamma \). By choosing \( \phi : V \to U \) to have jet along \( D \) which is holomorphic of sufficiently high order \( \mu \) and shrinking \( R' > 0 \) if necessary we may bound \( u_j \) in \( C^{s, \gamma} \) by an arbitrarily small constant in each \( U'_x \). So in each \( U'_x \) the \( z_j = w_j - u_j, j = 1, \ldots, n \), define a system of coordinates satisfying the conditions of Definition 4.6.

In the following we will denote the metric of Proposition 4.4 by \( g_0 \) with Kähler form \( \omega_0 \). We will prove that the Ricci-flat metric \( g \) with Kähler form \( \omega \) of Proposition 4.5 has nice asymptotic properties.

**Proposition 4.8.** If \( \phi \) is the solution to \((37)\) of Proposition 4.5 for any \( \delta > 0 \) there is a compact \( K \subset Y \) with
\[ -C_1 \rho(y)^{-2n+2+\delta} \leq \phi \leq C_2 \rho(y)^{-2n+2+\delta}, \]
for \( y \in Y \setminus K \), where \( C_1, C_2 > 0 \).
Proof. Outside a compact set the metric $g_0$ is close according to the estimate (88) to the Calabi ansatz $\omega_{\phi}$ with profile function (67). The distance from a fixed point $\rho(y)$ is equivalent to $r' = r_{\phi}$. Then a complicated but straightforward calculation shows that with $\tau = C_1 r^\alpha$ we have

$$\omega_{\phi} + i \partial \bar{\partial} \tau)^n < \omega_{\phi}^n,$$

for $\alpha > -\frac{(a-1)\kappa}{n}$. And similarly, if $\tau = -C_2 r^\alpha$

$$\omega_{\phi} + i \partial \bar{\partial} \tau)^n > \omega_{\phi}^n,$$

for $\alpha > -\frac{(a-1)\kappa}{n}$ (cf. [16] Lemma 5.2). Then for $C_1, C_2 > 0$ chosen sufficiently large we can apply the maximum principle. But first observe that $f$ vanishes to arbitrarily high order along $D$. Then from (87) and (88) if we denote $\psi^* \tau$ by $\tau$ also, we have in a neighborhood $V$ of $D$

$$(\omega_{\phi} + i \partial \bar{\partial} \tau)^n \leq \omega_{\phi}^n e^f.$$ And for $\tau = -C_2 \psi^*(r^\alpha)$ we have on $V$

$$(\omega_{\phi} + i \partial \bar{\partial} \tau)^n \geq \omega_{\phi}^n e^f.$$ Thus if $C_1, C_2 > 0$ are chosen sufficiently large, since $\phi$ converges uniformly to zero at infinity, we may apply the maximum principle as in [17] to get the inequalities.

Alternatively, one can apply the non-linear version of the maximum principle as follows. We have (97) in $V$. Choose $C_1 > 0$ large enough that $\tau = C_1 r^\alpha \geq \phi$ on $\partial V$. Since $\lim_{\rho \to \infty} \phi - \tau = 0$ and $\phi - \tau \leq 0$ on $\partial V$, either $\phi - \tau \leq 0$ on $V$ or $\phi - \tau$ attains its maximum at $x \in V$. If the latter, then [23] Theorem 17.1 applied in a coordinate neighborhood $(U, z_1, \ldots, z_n)$ centered at $x \in V$ implies that $\phi - \tau$ is constant on $U$. And by taking other coordinate neighborhood one sees that $\phi - \tau$ is constant, and therefore $\phi = \tau$. The lower bound is obtained similarly. \qed

For the following proposition we will need weighted H"older spaces. For $\beta \in \mathbb{R}$ and $k$ a nonnegative integer we define $C^k_\beta(Y)$ to be the space of continuous functions $f$ with $k$ continuous derivatives for which the norm

$$\|f\|_{C^k_\beta} := \sum_{j=0}^k \sup_Y |\rho^{j-\beta} \nabla_j f|$$

is finite. Then $C^k_\beta(Y)$ is a Banach space with this norm.

Let $\text{inj}(x)$ be the injectivity radius at $x \in Y$, and $d(x, y)$ the distance between $x, y \in Y$. Then for $\alpha, \beta \in \mathbb{R}$ and $T$ a tensor field define

$$[T]_{\alpha, \beta} := \sup_{x \neq y, d(x, y) < \text{inj}(x)} \left[ \min_{\rho(x), \rho(y)} |T(x) - T(y)| \right],$$

where $|T(x) - T(y)|$ is defined by parallel translation along the unique geodesic between $x$ and $y$.

For $\alpha \in (0, 1)$ define the weighted H"older space $C^{k, \alpha}_\beta(Y)$ to be the set of $f \in C^k_\beta(Y)$ for which the norm

$$\|f\|_{C^{k, \alpha}_\beta} := \|f\|_{C^k_\beta} + |\nabla^k f|_{\alpha, \beta - k - \alpha}$$

is finite. Then $C^{k, \alpha}_\beta(Y)$ is a Banach space with this norm.
In the case of \((Y, g_0)\) with bounded geometry of order \(\ell > k\) it is not difficult to see that the norm \(\|\cdot\|_{C^{k,\alpha}}\) is equivalent to taking the norm \(\rho(x)\|\cdot\|_{C^{k,\alpha}}\) in each coordinate system \((U_x, z_1, \ldots, z_n)\), where \(\|\cdot\|_{C^{k,\alpha}}\) is the Hölder norm with respect to the usual Euclidean metric.

**Proposition 4.9.** For any \(\delta > 0\) the solution to \((104)\) of Proposition 4.7 satisfies
\[
\|\nabla^j \phi\|_{g_0} = O(\rho^{-2n+2-j+\delta}), \quad \text{for} \quad j \leq k.
\]

**Proof.** Let \(\{(U_x, \varphi_x, z_1, \ldots, z_n)\}_{x \in Y}\) be a system of holomorphic charts giving \((Y, g_0)\) bounded geometry of order \(\ell\) as in Definition 4.0 where \(\varphi_x : B_R \to U_x \subset Y\) is the chart centered at \(x \in Y\). We define a chart centered at \(x \in Y\) depending on \(r > 0\), \(\psi_{x,r} : B_R \to U_x\). Let \(R_a : B_R \to B_R\) be scalar multiplication by \(a\). Then define \(\psi_{x,r} := \varphi_x \circ R_{r \rho(x)^{-1}}\). Note that for \(r = 1\) the charts \(\psi_{x,r}\) are bounded of order \(\ell\) for \(g_0\), and for \(r = \rho(x)\) the charts \(\psi_{x,r} = \varphi_x\) are bounded of order \(\ell\) for \(\rho^{-2}(x)g_0\).

We first must show that for each \(x \in Y\) we have \(\|\psi_{x,1}^* \phi\|_{C^{\ell,\alpha}} \leq C\) for some \(C > 0\) where the Hölder norm is with respect to the Euclidean metric \(h\) on \(B_R \subset \mathbb{C}^n\). Let \(\Delta'\) denotes the Laplacian with respect to the metric \(g\) associated to \(\omega = \omega_0 + i\partial\bar{\partial}\phi\); let \(\Delta, \nabla, \text{ and } R\) denote the Laplacian, connection, and curvature of \(g\). Then the Hölder bound follows from a bootstrapping argument involving the equation
\[
\Delta' \Delta \phi = g^{\alpha \bar{\mu}} \gamma^\beta \left(\nabla^\nu \nabla^\mu \phi - \Delta f + g^{\lambda \bar{\nu}} R^\alpha_{\beta \lambda \bar{\nu}} \nabla^\alpha \phi - g^{\nu \bar{\nu}} R^\lambda_{\mu \nu} \nabla^\lambda \phi\right).
\]

The complete proof is given in [48, Theorem 3.21]. The arguments are also explained in [29].

We may assume that \(f\) is chosen to vanish to high enough order in Proposition 4.3 that \(f \in C^{k,\alpha}_\beta(Y)\) where \(\beta < -2n\). Suppose now that \(\phi \in C^{k,\alpha}_\beta(Y)\) and \(\|\phi\|_{C^0} = P\) for some \(\beta + 2 \leq \gamma < 0\) and \(P > 0\). In the following we will also assume that \(\ell \geq k + 2\). The proof of the proposition will be based on the following lemma due to D. Joyce [29].

**Lemma 4.10.** Let \(N_1, N_2 > 0\) and \(\lambda \in [0, 1]\). Then there exists an \(N_3 > \) depending on \((X, g, J), \gamma, \alpha, \beta, k, P\) and \(N_1, N_2, \lambda\), so that the following holds.

Suppose \(\|f\|_{C^{k,\alpha}} \leq N_1\) and
\[
\|\nabla^j df \phi\|_{C^{0,\alpha}_{\gamma,\lambda}} \leq N_2, \quad \text{for} \quad j = 0, \ldots, k,
\]
and
\[
\|\nabla^k df \phi\|_{C^{0,\alpha}_{-(k+\alpha)\lambda}} \leq N_2.
\]

Then the following inequalities hold, where the norms exist,
\[
\|\nabla^j \phi\|_{C^{0,\alpha}_{\gamma,\lambda}} \leq N_3, \quad \text{for} \quad j = 0, \ldots, k + 2,
\]
and
\[
\|\nabla^{k+2} \phi\|_{C^{0,\alpha}_{-(k+2+\alpha)\lambda}} \leq N_3.
\]

**Proof.** We define an operator \(P_{x,r} : C^{k+2,\alpha}(B_R) \to C^{k,\alpha}(B_R)\) by
\[
(\psi_{x,r})_*[P_{x,r} \omega_0^n] = -r^2 df \left[(\psi_{x,r})_* u \right] \wedge (\omega_0^{n-1} + \cdots + \omega^{n-1}).
\]
Thus \(P_{x,r}\) is an elliptic operator, and it follows from [81] that
\[
P_{x,r}(\psi_{x,r}^* \phi) = r^2 \left(1 - \psi_{x,r}^*(e^f)\right).
\]
Now set $r = \rho(x)^\lambda$ for $\lambda \in [0, 1]$. By shrinking $R > 0$ if necessary we may suppose that
\begin{equation}
\frac{1}{2} \rho(x) \leq \rho(y) \leq 2 \rho(x), \quad \text{for all } y \in \psi_{x,r}(B_R),
\end{equation}
and this completes the proof of Theorem 1.3.

And it follows from (104) and the Schauder interior estimates [23, Th. 6.2, Th. 6.7] that there are constants $C > 0$, independent of $x \in Y$, so that
\begin{equation}
\| \varphi \|_{C^{k,\alpha}} \leq C (r^2) |1 - \psi^*_x,\varphi|_{C^{k,\alpha}} + \| \psi^*_x,\varphi\|_{C^n}
\end{equation}
for all $x \in Y$ and $\varphi \in \mathcal{O}_{k,\alpha}$. The proposition is now proved by repeatedly applying the lemma. Let $\gamma = -2n + 2 + \delta$. Let $p$ be the smallest integer with $-p \gamma > k + \alpha$ and define $\lambda_i = \frac{\gamma}{p + \alpha}$ for $i = 0, \ldots, n-1$ and $\lambda_p = 1$. We have already shown that (101) holds for $\lambda_0 = 0$. For the inductive step suppose we have (101) for $\lambda = \lambda_i$. Then we have (102), and it is easy to see that this implies (101) for $\lambda_{i+1}$. Thus we get (102) for $\lambda_p = 1$, and this implies the proposition.

In particular, we have
\begin{equation}
\| \nabla^j (g - g_0) \|_{g_0} = O(\rho^{-2n-j+\delta}), \quad \text{for } j \leq k.
\end{equation}
And this completes the proof of Theorem 1.3.

Remark 4.11 One can remove the “$\delta$” in Proposition 4.9 by studying the Laplacian on weighted Hölder spaces on manifolds with a conical end. Thus the Ricci-flat
metric decays to the Calabi Ansatz metric \( g_0 \) as in the ALE case \([29]\). The details are in \([48]\).

5. **Deformations of Gorenstein toric singularities**

In this section \( X_\sigma \) is a Gorenstein affine toric variety associated to the strictly convex rational polyhedral cone \( \sigma \subset \mathbb{R}^n \). As in Section 2.3 \( X_\sigma \) has a Kähler cone structure and as such \( \sigma \) is precisely the dual moment cone \( \mathcal{C}(\mu)^* \) of \([23]\). Thus \( \sigma \) is the span of the \( \lambda_i \in \mathbb{Z}^n, i = 1, \ldots, d \), defining the moment cone in \([21]\). And Gorenstein means that Proposition 2.10 is satisfied with \( \gamma \in (\mathbb{Z}^n)^* = \text{Hom}(\mathbb{Z}^n, \mathbb{Z}) \). It follows that \( \sigma \) is the cone over a lattice polytope \( P \subset \mathbb{R}^{n-1} \).

We will assume that \( X_\sigma \) is toric with an isolated Gorenstein singularity. In this case there is a beautiful description by K. Altmann \([3]\) of the versal deformation space \( \mathcal{M}_0 \) of the versal family is described in Figure 2.

The functions \( t_0, \ldots, t_p \) are invariant under the complex \( n-1 \)-torus \( T_{\mathbb{C}}(L) \subset T_{\mathbb{C}}(N') \). Thus we have a \( (\mathbb{C}^*)^{n-1} \cong T_{\mathbb{C}}(L) \) action on \( X_\sigma \). Note that the cone...
Proposition 2.10. Let \( \hat{\sigma} \) be Gorenstein, since \( \ell' \in M' \) with \( \ell' = e_p + \cdots + e_0 \) evaluates to 1 on all the generators. So there is a nowhere vanishing holomorphic \( p + p \)-form \( \Omega' \) on \( Y_{\hat{\sigma}} \) as in Proposition 2.10.

Let \( \Sigma \) be any fan in \( N = \mathbb{Z}^n \). Recall that there a bijective correspondence between \( p \)-dimensional cones \( \sigma \in \Sigma \) and orbits \( T_{\sigma} \subset X_{\Sigma} \) of codimension \( p \) isomorphic to \( T_{\mathbb{C}}^{n-p} \).

**Definition 5.1.** We say that a codimension \( p \) analytic subvariety \( V \) of a toric variety \( X_{\Sigma} \) is \( \Sigma \)-regular if for every \( \sigma \in \Sigma \) the intersection \( V \cap T_{\sigma} \) is a smooth codimension \( p \) subvariety.

**Proposition 5.2.** A \( \Sigma \)-regular subvariety of a smooth toric variety \( V \subset X_{\Sigma} \) is smooth.

**Proof.** We denote by \( A_{\sigma} \subset X_{\Sigma} \) the \( n \)-dimensional affine toric variety associated to the cone \( \sigma \in \Sigma \). If \( \sigma \in \Sigma(r) \) is \( r \)-dimensional, let \( N_{\sigma} \) be the \( r \)-dimensional sublattice containing \( \sigma \cap N \). Denote by \( A_{\sigma,N(\sigma)} \) the \( r \)-dimensional affine toric variety corresponding to \( \sigma \subset N(\sigma) \). Then \( A_{\sigma} = A_{\sigma,N(\sigma)} \times (\mathbb{C}^*)^{n-r} \). The implicit function theorem implies that \( V \) is smooth at every point of \( V \cap T_{\sigma} = V \cap \{0\} \times (\mathbb{C}^*)^{n-r} \subset A_{\sigma,N(\sigma)} \times (\mathbb{C}^*)^{n-r} \).

**Proposition 5.3.** Suppose \( Y_{\hat{\sigma}} \) is smooth besides the one isolated singularity corresponding to the \( n + p \)-dimensional cone, i.e. \( \hat{\sigma} \) satisfies (\ref{eq:5.2}). Then for generic \( \tilde{c} = (c_1, \ldots, c_p) \in \mathbb{C}^p \), the subvariety \( \pi^{-1}(\tilde{c}) =: Y_{\tilde{\sigma}} \subset Y_{\hat{\sigma}} \) is smooth.

**Proof.** Let \( e_{k}^{\perp} \in \hat{\sigma}^\perp \) be the extension of the standard dual basis element on \( \mathbb{R}^{p+1} \). For \( \tau \in \hat{\sigma} \), \( t_k \) vanishes on \( T_{\tau} \) unless \( e_k^{\perp} \in \tau^\perp \). Suppose \( e_{k_1}^{\perp}, e_{k_2}^{\perp} \notin \tau^\perp \). Then \( t_{k_1} - t_{k_2} \) vanishes on \( T_{\tau} \). So \( Y_{\tilde{\sigma}} \cap T_{\tau} = \emptyset \) if \( c_{k_1} - c_{k_2} \neq 0 \). Thus we may assume that only one \( e_k^{\perp} \notin \tau^\perp \). Then the \( t_1 - t_0 = c_1, \ldots, t_{k-1} - t_0 = c_{k-1}, -t_0 = c_k, t_{k+1} - t_0 = c_{k+1}, \ldots, t_p - t_0 = c_p \) define \( Y_{\tilde{\sigma}} \cap T_{\tau} \) which is a complete intersection for sufficiently general \( c_1, \ldots, c_p \). The result then follows from Proposition 5.2.

Notice that for generic \( \tilde{c} \in \mathbb{C}^p \), as in the proposition, \( Y_{\tilde{\sigma}} \cap T_{\tau} = \emptyset \) if \( \dim T_{\tau} < p \).

Supposing that \( X_{\tilde{\sigma}} \subset Y_{\tilde{\sigma}} \) is smooth, adjunction gives a nowhere vanishing \( n \)-form \( \Omega \) on \( X_{\tilde{\sigma}} \). In order to prove Corollary 1.5 we will compactify \( Y_{\hat{\sigma}} \). Assume...
that $n_0 \in \text{Int}(P) \cap L$, and define $\tau = -n_0 - e_0 - \cdots - e_p$. We define a new
fan $\Sigma$ in $N'$. For each face $\mathcal{F}$ of $\hat{\sigma}$ spanned by $\beta_1, \ldots, \beta_r$, $\Sigma$ contains the cone
$\mathbb{R}_{\geq 0}\beta + \mathbb{R}_{\geq 0}\beta_1 + \cdots + \mathbb{R}_{\geq 0}\beta_r$. If the fan $\hat{\sigma}(n+p-1)$, ignoring the $n+p$-dimensional
cone, is simplicial, then $\Sigma$ is simplicial apart for the $n+p$-cone spanned by $\hat{\sigma}$. Thus
apart from the singular point corresponding to this cone, $\bar{Y}_\sigma := Y_{\Sigma}$ has an orbifold
structure. And $\bar{Y}_\sigma$ is a compactification of $Y_{\sigma}$ by adding a divisor $D_{\sigma}$ at infinity.

Even if $\Sigma$ is not simplicial $D_{\sigma}$ structure. And $\bar{Y}_\sigma$ is simplicial, then $\Sigma$ is simplicial apart for the
$n$-cone of $\sigma$. We have the embedding $\bar{X}_\sigma \hookrightarrow Y_{\sigma}$. And we define $X_{\bar{\varepsilon}} \subset Y_{\sigma}$ to be the closure of $X_{\bar{\varepsilon}} \subset Y_{\sigma}$.

The regular functions $f_1 = t_1 - t_0, \ldots, f_p = t_p - t_0$ on $Y_{\sigma}$ extend to rational functions on
$Y_{\bar{\varepsilon}}$. Suppose that Proposition 5.3 is satisfied for $\bar{\varepsilon} \in \mathbb{C}^P$. Then the
Weil divisor which is the closure of $\{f_i = c_i\}$ in $Y_{\bar{\varepsilon}}$. Since $f_i - c_i$ has a pole along
$D_{\sigma}$ and for generic $\bar{\varepsilon} \{f_i = c_i\}$ contains no other $T_C(N')$-invariant divisors, we have
$D_{\bar{\varepsilon}} = D_\sigma - (f_i - c_i)$. The holomorphic $n+p$-form $\Omega$ on $Y_{\bar{\varepsilon}}$ has a pole of order
$p + 2$ along $D_{\bar{\varepsilon}}$. By applying the adjunction formula $p$ times we get a holomorphic
$n$-form $\Omega$ on $X_{\bar{\varepsilon}}$ with a pole of order $2$ along $D_{\bar{\varepsilon}} \cap X_{\bar{\varepsilon}}$. It is also not difficult to see
that $D := D_{\bar{\varepsilon}} \cap X_{\bar{\varepsilon}} = D_\sigma \cap X_{\bar{\varepsilon}}$.

**Proposition 5.4.** Suppose that $X_{\bar{\varepsilon}}$ is smooth. Then $X_{\bar{\varepsilon}}$ is an orbifold. And we
have $K_{X_{\bar{\varepsilon}}} = [-2D]$. If $n = 3$ then condition (11) holds for $D \subset X_{\bar{\varepsilon}}$.

**Proof.** Proposition 4.1 is not applicable as $X_{\bar{\varepsilon}}$ and $D$ are orbifolds. But condition (11)
does hold, where the sheaves are coherent sheaves of orbifold bundles. By the argument in Proposition 4.1 it is sufficient to prove that $H^1(D, \Omega^1((k-1)D)) = 0$ and $H^1(D, \mathcal{O}((1-k)D)) = 0$ for all $k \geq 2$. The second holds by the negativity of
$[(1-k)D]$. Let $\sum_{i=1}^d C_i$, be the anti-canonical divisor of $D$. Consider the exact sequence

$$0 \rightarrow \Omega^1_D \rightarrow \mathcal{O}_D \oplus \mathcal{O}_D \rightarrow \bigoplus_{i=1}^d \mathcal{O}_{C_i} \rightarrow 0,$$

where $\Omega^1_D$ is the sheaf of sections of the orbifold bundle of holomorphic 1-forms. Then tensor with $E = [(1-k)D]|_D$, to get

$$0 \rightarrow \Omega^1_D(E) \rightarrow \mathcal{O}_D(E) \oplus \mathcal{O}_D(E) \rightarrow \bigoplus_{i=1}^d \mathcal{O}_{C_i}(E) \rightarrow 0.$$

Since $E$ is negative, Kodaira vanishing and the cohomology sequence gives

$$H^1(D, \Omega^1((k-1)D)) = H^1(D, \Omega^1(E)) = 0,$$

where the first equality is Serre duality.

It is proved in [3] that isolated Gorenstein singularities are rigid for $n \geq 4$. We
assumed that there is an $n_0 \in \text{Int}(P) \cap N$. The only isolated Gorenstein singularity
with $n \leq 3$ for which this is not the case is the quadric cone $X = \{(V, W, Y, Z) \in \mathbb{C}^4 : VW - YZ = 0\}$. And $X$ admits a one dimensional versal space of deformations
(cf. [33]), $X_{\bar{\varepsilon}} = \{(V, W, Y, Z) \in \mathbb{C}^4 : VW - YZ = \epsilon\}$. For $\epsilon \neq 0$, $X_{\bar{\varepsilon}}$ is smooth and
diffeomorphic to $T^*S^3$. In fact, with the symplectic structure as a submanifold of
$\mathbb{C}^4$, $X_{\bar{\varepsilon}}$ is symplectomorphic to $T^*S^3$ with the canonical symplectic structure. The
complex structure is that given as the complexification $SL(2, \mathbb{C})$ of $S^3$. In [39] a
Ricci-flat Kähler metric is constructed on $X$, which is asymptotic to an obvious cone metric on $T^*S^3$ as in \textup{(2)} with exponent $-3$ rather than $-2n + \delta$.

For $n = 2$ all toric Gorenstein singularities $X$ are cyclic quotients of $\mathbb{C}^2$ by $\mathbb{Z}_{k+1} \subset SL(2, \mathbb{C})$, where $\mathbb{Z}_{k+1}$ is generated by $(\alpha, \alpha^k)$ with $\alpha$ a primitive $k + 1$-th root of unity. Thus it is the rational double point $A_k$ which we denote $X_k$. Recall $X_k = \{(x, y, z) \in \mathbb{C}^3 : xy + z^{k+1} = 0\}$. The versal deformation space of $X_k$ is easy to construct \textup{[13]}. The versal deformation space of $X_k$ is the subspace $Y_k \subset \mathbb{C}^3 \times \mathbb{C}^k(t_1, \ldots, t_k)$ defined by the equation

$$F = xy + z^{k+1} + t_1 z^{k-1} + \cdots + t_{k-1} z + t_k = 0.$$  

And the projection $\pi : Y_k \to \mathbb{C}^k(t_1, \ldots, t_k)$ is the restriction of the obvious projection $Y_k \subset \mathbb{C}^3 \times \mathbb{C}^k(t_1, \ldots, t_k) \to \mathbb{C}^k(t_1, \ldots, t_k)$. We compactify $X_k(t_1, \ldots, t_k) = \pi^{-1}(t_1, \ldots, t_k)$ as follows. For $k$ even, consider the weighted projective space $\mathbb{C}P^3(k + 1, k + 1, 2, 1)$ with homogeneous coordinates $[x : y : z : s]$ and weights $w = (w_0, w_1, w_2, w_3) = (k + 1, k + 1, 2, 1)$. Then define $\hat{X}_k(t_1, \ldots, t_k) \subset \mathbb{C}P^3(k + 1, k + 1, 2, 1)$ to be the hyperpersurface

$$f(x, y, z, s) = xy + z^{k+1} + t_1 z^{k-1} s^4 + t_2 z^{k-2} s^6 + \cdots + t_{k-1} z^{2k} + t_k s^{2k+2} = 0.$$  

Since $f$ is weighted homogeneous of degree $d = 2k + 2$ and $\sum_i w_i - d = 3 > 0$, we have that $K_X^{-1} > 0$. If we denote $D = \{s = 0\} \cap \hat{X}$, then $-K_{\hat{X}} = 3[D]$. And $D$ is $\mathbb{C}P^1$ with two antipodal orbifold points of degree $k + 1$. Thus $D$ admits a Kähler-Einstein metric, i.e. the $\mathbb{Z}_{k+1}$ quotient of the constant curvature metric. When $X_k(t_1, \ldots, t_k)$ is smooth, Theorem \textup{(1.2)} is applicable and gives the required metric. The case with $k$ odd is similar. Just use the weights $w = (\frac{k+1}{2}, \frac{k+1}{2}, 1, 1)$ and \textup{(110)} with the necessary changes. This completes the proof of Corollary \textup{[13]}

Suppose that a toric Gorenstein singularity $X$, with dim$_{\mathbb{C}} X = 3$, admits a smoothing which we denote $\hat{X}$. It was proved in \textup{[46, 45]} that $X$ also has a resolution $\hat{X}$ which admits a Ricci-flat Kähler metric asymptotic to a cone metric. Thus we have two smooth Calabi-Yau varieties $\hat{X}$ and $\hat{X}$ which admit Ricci-flat Kähler metrics. We can transition $\hat{X} \rightsquigarrow X$ by shrinking the exceptional divisor to a point. And we can deform $X \rightsquigarrow \hat{X}$. This is a \textit{geometric transition} analogous to that given by the small resolution and smoothing of the conifold $\{(U, V, X, Y) \in \mathbb{C}^4 : UV - XY = 0\}$ which has been investigated in \textup{[13]} and elsewhere. See \textup{[37]} for more on geometric transitions in the study of Calabi-Yau manifolds and their relevance to physics. Since the resolved space $\hat{X}$ is toric, and also the smoothing $\hat{Y}$, when it exists, admits a $(\mathbb{C}^*)^{n-1}$ action (cf. \textup{[3]}), it follows from the work of M. Gross \textup{[27]} that both $\hat{X}$ and $\hat{X}$ admit special Lagrangian fibrations. Thus this phenomenon should be of interest in the Strominger-Yau-Zaslow conjecture and mirror symmetry.

Though a toric Gorenstein singularity $X$, with dim$_{\mathbb{C}} X = 3$, always admits a resolution $\hat{X}$ with a Ricci-flat Kähler metric, a smoothing $\hat{X}$ may not exist. For example, if as in Example \textup{6.1} one takes $M = \mathbb{C}P^2_{(1)}$, then the total space $\hat{Y} = K_M$ is a resolution of the cone $X = K_M^* \cup \{0\}$. And $\hat{Y}$ admits a Ricci-flat Kähler metric. But it is proved in \textup{[3]} that the cone $X$ is rigid.

6. Examples

\textbf{Example 6.1} Let $M$ be a toric Fano manifold. Let $X = \mathbb{F}(K_M \oplus \mathbb{C})$. If $D \subset X$ is the $\infty$-section of $K_M$, then $2[D] = -K_X$. Note that $X$ is not a Fano manifold, but
$D$ is a good divisor as in Definition 1.1. The arguments in Proposition 4.1 show that condition (1) holds. But this is immaterial as it is clear that the normal bundle $N_D$ is biholomorphic to a neighborhood of $D$ in $X$ in an obvious way. Theorem 2.11 implies the $U(1)$-subbundle $S \subset K_M$ admits a Sasaki-Einstein structure. And Theorem 1.3 implies that $K_M$ admits a complete Ricci-flat Kähler metric which converges to the Calabi ansatz at infinity.

Of course, if $M$ admits a Kähler-Einstein metric then the Calabi ansatz [14] constructs a complete Ricci-flat Kähler metric on $K_M$ as in section 3.1 which is explicit up to the Kähler-Einstein metric on $M$. The problem of the existence of a Kähler-Einstein metric on a toric Fano manifold was solved in [49], where it was proved that the only obstruction is the Futaki invariant. We saw that the Calabi ansatz always constructs a Ricci-flat metric in a neighborhood of infinity on $K_M$. But the author does not believe that this metric extends smoothly across the zero section in the case when $M$ does not admit a Kähler-Einstein metric.

In particular, suppose $M = \mathbb{CP}^2$ the two-points blow-up. Then the total space $Y = K_M$ admits a complete Ricci-flat Kähler metric which converges to the Calabi ansatz at infinity. Note that $Y$ is also a toric variety. Thus as an analytic variety $Y$ is described by a fan $\Delta$ in $\mathbb{R}^3$ which in this situation is a cone over a triangulated polytope in $P \subset \mathbb{R}^2 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_3 = 1\}$. The polytope $P$ and its triangulation are shown in Figure 3. The cone over $P$ is just the dual cone $C^\vee$ to the moment polytope of $K_M = C(S)$ where $S$ is the Sasaki-Einstein manifold of Example 2.2. Note that collapsing the zero section of $K_M$ gives a morphism $\pi : Y \to C(S) \cup \{o\}$.

![Figure 3. Canonical bundle of $\mathbb{CP}^2_{(2)}$](image)

**Example 6.2** Consider the Fano 3-fold $X = \mathbb{P}(\mathcal{O}_{\mathbb{CP}^3}(1) \oplus \mathcal{O}) = \mathbb{CP}^3_{(1)}$, the one-point blow-up of $\mathbb{CP}^3$. This is $V_7$ in the classification of Fano 3-folds of V. A. Iskovskikh [1]. Then $\text{Ind}(X) = 2$, and there exists a smooth subvariety $D \subset X$ with $2D = -K_X$. Since $-K_X^3 = 56$, one easily sees that $K_D^2 = 7$. But since $D$ is a del Pezzo surface, we must have $D = \mathbb{CP}_{(2)}^2$, the two-points blow-up. Then Proposition 4.1 implies that condition (1) is satisfies and by Theorem 1.3 $Y = X \setminus D$ admits a complete Ricci-flat Kähler metric.

More explicitly, if $\pi : \mathbb{CP}^3_{(1)} \to \mathbb{CP}^3$ is the blow-down collapsing the exceptional divisor $E$ to $p \in \mathbb{CP}^3$, then $-K_X = 4\pi^*H - 2E$. We have $D = 2\pi^*H - E$. Then $D$ can be represented by the strict transform of a smooth quadric surface through $p$. So $D = \mathbb{CP}^1 \times \mathbb{CP}^1_{(1)} = \mathbb{CP}^2_{(2)}$.

Notice that the end of $Y$ is diffeomorphic to a cone over the $U(1)$-bundle $M \subset K_D$, and $M$ is diffeomorphic to $(S^2 \times S^3) \# (S^2 \times S^3)$ by the S. Smale’s classification.
of smooth simply connected spin 5-manifolds. This is the only example of a smooth Fano 3-fold $X$ with smooth divisor $D \subset X$, $\alpha D = -K_X$, for which $D$ does not admit a Kähler-Einstein metric. It is not difficult to show that $Y$ is simply connected, and the Betti numbers are $b_0(Y) = b_2(Y) = b_3(Y) = 1$ with the rest zero.

The topology of $Y$ can be described in more detail. The compliment of a smooth quadric $Q \subset \mathbb{CP}^3$ is diffeomorphic to $T^*S^3/\mathbb{Z}_2$, the quotient of the cotangent bundle of $S^3$ where $\mathbb{Z}_2$ acts by the antipodal map on $S^3$. This is easily seen to be $T^*\mathbb{R}P^3$. Let $W$ be the unit disk bundle in $T^*\mathbb{R}P^3$. This is the trivial bundle and $\partial W = \mathbb{R}P^3 \times S^2$. Take a smooth $S^1$ in $\partial W$ generating $\pi_1(\partial W)$. It has a neighborhood $N \cong S^1 \times B^4$ in $\partial W$. Then $\partial(\partial W \setminus N) = S^1 \times S^3$. Now glue $B^2 \times B^4$ to $W \setminus N$ along $S^1 \times S^3 \subset B^2 \times B^4$. We have $Y \cong (W \setminus N) \cup_{S^1 \times S^3} B^2 \times B^4$. It follows that $Y$ is homotopy equivalent to $\mathbb{R}P^3 \cup_f B^2$, where $f : S^1 \to \mathbb{R}P^3$ generates $\pi_1(\mathbb{R}P^3)$. If $X$ denotes the cone over $\mathbb{CP}^2$, i.e., the toric variety given by a fan which is the cone over the polytope $P$ in Figure 3, then topologically we have replaced the vertex with $\mathbb{R}P^3 \cup_f B^2$.

This example is a smoothing $\hat{X}$ of the toric Gorenstein variety $X$ in Example 6.2 as discussed in Section 5. This can be described geometrically. The divisor $D = 2\pi^*H - E$ on $\mathbb{CP}^3(1)$ is very ample. This follows from the result that an ample divisor on a non-singular toric variety is very ample [36, Theorem 2.22]. And $H^0(\mathbb{CP}^3(1), O(D))$ is spanned by 9 sections. Thus we have an embedding $\iota : \mathbb{CP}^3(1) \to \mathbb{CP}^8$. Let $V \subset \mathbb{C}^9$ be the cone over this variety. If $H \subset \mathbb{C}^9$ is a generic hyperplane through $o \in \mathbb{C}^9$, then $X = H \cap V$. Generically deform $H$ away from $o \in \mathbb{C}^9$, then $H \cap V = \hat{X}$. This deformation of $X$ is given by the Minkowski decomposition of $P$ in Figure 2.

Examples 6.1 and 6.2 provide an example of a geometric transition. Let $X = C(S) \cup \{o\}$ be the Ricci-flat toric Kähler cone which is given by the fan which is the cone over the polytope $P$ in Figure 3. Let $\hat{X}$ denote the toric resolution of Example 6.1 and $\tilde{X}$ the smoothing of $X$ of Example 6.2. We can transition $\tilde{X} \to X$ by shrinking the exceptional $\mathbb{CP}^2(2)$ to a point. And then we can deform $X \to \hat{X}$.

Example 6.3 We consider a series of examples of toric singularities with a symmetry property. The symmetry ensures that the polytope $P$ is a Minkowski sum with a large number of components. These examples arose in the author’s investigation of submanifolds of toric 3-Sasakian manifolds [44, 47].

Let $P \subset \mathbb{R}^2$ be an integral polytope invariant under the antipodal map $(x_1, x_2) \mapsto (-x_1, -x_2)$. Then considering $P \subset \mathbb{R}^2 \times \{(0, 0, 1)\} \subset \mathbb{R}^3$, $\sigma = \text{Cone}(P)$. We require that $X_\sigma$ has only an isolated singularity. The variety $X_\sigma \setminus \{o\}$, minus the singular point, has $\pi_1(X_\sigma \setminus \{o\}) = \mathbb{Z}_2$. If $\pi_1(X_\sigma \setminus \{o\}) = \mathbb{Z}_2$, then take the sublattice $N' \subset \mathbb{Z}^2$ generated by the $\{(x_1, x_2, 1)\}$ where $\langle x_1, x_2 \rangle$ is a vertex of $P$. Denote by $\sigma'$ the cone $\sigma$ as a cone in $N'$. Then $X_{\sigma'} \setminus \{o\}$ is simply connected and $\sigma'$ has the essential property used below. It was shown in [44, 47], by considering submanifolds of toric 3-Sasakian 7-manifolds, how to produce infinite families of $\sigma'$ with this property.

Let $d_0, \ldots, d_k \in \mathbb{Z}^2$ be the edge vectors of $P$ taken counterclockwise around $P$. Thus if $\lambda_0, \ldots, \lambda_k$ span $\sigma'$, then $d_j = \lambda_j - \lambda_{j-1}$. These are primitive, since $X_{\sigma'}$ has only an isolated singularity. Then $k + 1 = 2(p + 1)$ and $d_{j+p+1} = -d_j$, where indices are taken mod $k + 1$. It follows that the elements $d_0, \ldots, d_p$ give
a Minkowski decomposition of $P$ (cf. [3], §2). More precisely, if $R_j \subset \mathbb{R}^2$ is the segment spanned by $(0, 0)$ and $d_j$, then

\begin{equation}
P = R_0 + \cdots + R_p.\end{equation}

Proposition 6.1. With $\sigma$ as above, the tautological cone $\tilde{\sigma}$ satisfies the smoothness condition (22), i.e. $Y_{\tilde{\sigma}}$ has only an isolated singularity.

Proof. Let $\alpha \in \tilde{\sigma}^\vee$. Then $\alpha = \beta + \sum_{j=0}^{p} c_j e_j^*$ where $\beta \in (\mathbb{R}^2)^*$ and $c_j \geq 0, 0 \leq j \leq p$. The cone $\tilde{\sigma}$ is generated by $\tau_j = (0, e_j)$ and $\tau'_j = (d_j, e_j)$ for $0 \leq j \leq p$. Thus $\beta(d_j) + c_j \geq 0$ for $0 \leq j \leq p$.

A facet of $\tilde{\sigma}$ is given by choice of $\alpha$ vanishing on $p + 2$ generators. The only possibilities are \{$\tau'_0, \ldots, \tau'_i, \tau_i, \ldots, \tau_p$\}, where $\beta(d_i) = 0$, $c_j = 0, i \leq j \leq p$, and $\beta(d_j) = -c_j$ for $0 \leq j \leq i$. And it is easy to see that the face spanned by this set satisfies (22). \hfill $\Box$

By Proposition 5.3 for generic $\bar{c} \in \mathbb{C}^p$ the affine variety $X_{\bar{c}} \subset Y_{\tilde{\sigma}}$ is smooth. And by Corollary 1.5 there is a complete Ricci-flat Kähler metric asymptotic to a Calabi ansatz metric.

This provides infinitely many examples of geometric transitions. As shown in [46, 45] the toric cone $X_\sigma$ admits a resolution $\hat{X}$ with Ricci-flat Kähler metric asymptotic to the Kähler cone metric on $X_\sigma$ of Theorem 2.11. And $X$ can also be smoothed to an affine variety $\hat{X}$ with a Ricci-flat metric with similar asymptotic properties. \hfill $\Diamond$

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