Radial geodesics as a microscopic origin of black hole entropy.

II: Features of Reissner–Nordstrøm black hole

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Abstract. The entropy of charged black hole is calculated by using the partition function evaluated at radial geodesics confined under horizons. We establish two quantum phase states inside the black hole and a transition between them.

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1. Preface (instead of Introduction)

The second chapter of present paper is devoted to consequent launching the constructed vehicle to a charged black hole. The Reissner–Nordstrøm black hole has got two horizons, that constitutes a source of specific properties: We will show that the thermal quantization leads to i) quantizing a ratio of horizon areas, ii) two states of aggregation with different temperatures and entropies, iii) a possibility of phase transition between two states. We derive the entropy of Reissner-Nordstrøm black hole in terms of partition function evaluated by the action for causal radial geodesics confined under the horizons. The interior structure of charged black hole mathematically differs from that of Schwarzschild one: instead of conic geometry with a single map on the manifold it has got a sphere, which should be covered by two coordinated maps.

In section 2 we describe the mapping of causal radial geodesics confined under the horizon and thermally quantize them. Section 3 is devoted to the evaluation of entropy. Results are summarized in section 4.

2. Radial geodesics

The Reissner–Nordstrøm metric for a static spherically symmetric charged black hole has the form

$$ds^2 = g_{tt}(r) dt^2 - \frac{1}{g_{tt}(r)} dr^2 - r^2 [d\theta^2 + \sin^2 \theta d\phi^2],$$

(1)
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with
\[ g_{tt}(r) = \frac{1}{r^2} (r - r_+) (r - r_-), \]  
whereas the horizon points determine a black hole mass \( M \) and its charge \( Q \) by
\[ M = \frac{1}{2} (r_+ + r_-), \quad Q^2 = r_+ r_- , \]
in units \( G = c = \hbar = 1 \). In a standard manner, an introduction of
\[ r_* = \int \frac{dr}{g_{tt}(r)} \]
transforms the metric of (1) to the form
\[ ds^2 = g_{tt}(r) \left( dt^2 - dr_*^2 - r^2 [d\theta^2 + \sin^2 \theta \, d\phi^2] \right), \]
where
\[ r_* = r + \frac{r_+^2}{r_+ - r_-} \ln \left[ \frac{r}{r_+} - 1 \right] - \frac{r_-^2}{r_+ - r_-} \ln \left[ \frac{r}{r_-} - 1 \right]. \]

Following the Hamilton–Jacobi method for radial geodesics of a particle with mass \( m \), we get
\[ \left( \frac{dr_*}{dt} \right)^2 = 1 - A g_{tt}(r), \]
where \( A = m^2 / \mathcal{E}^2 \) is the only integral of radial motion, so that the Hamilton–Jacobi action \( S_{HJ} \)
\[ S_{HJ} = -\mathcal{E} \, t + S_{HJ}(r_*) \]
satisfies the equation
\[ \frac{1}{m^2} \left( \frac{\partial S_{HJ}}{\partial r_*} \right) = \mathcal{E}_A - U(r), \]
with a scaled energy \( \mathcal{E}_A = 1 / A \), and the potential \( U(r) = g_{tt}(r) \) as shown in Fig. 1.

For a massive particle, the interval on a radial geodesic curve is given by
\[ ds^2 = \frac{A \, dr^2}{1 - A g_{tt}(r)}, \]
and we call it ‘causal’ if it is time-like
\[ ds^2 > 0, \]
which takes place at
\[ \{ A > 0 \} \cup \left\{ A < \frac{4r_+ r_-}{(r_+ - r_-)^2} < 0 \right\}, \]
and
\[ \{ r_1 < r < r_2, A > 1 \cup A < 0 \} \cup \{ r_1 < r \leq +\infty, 0 < A < 1 \} \]
where the return points have to be given by real roots of quadratic equation
\[ (1 - A) r^2 + A(r_+ + r_-) r - A r_+ r_- = 0 \]
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Figure 1. The gravitational potential $U(r)$ for the Reissner–Nordstrøm black hole.

with discriminant

$$D = A[A(r_+ - r_-)^2 + 4r_+r_-] \geq 0.$$  

For geodesics confined under the horizons $A < -4r_+r_-/(r_+ - r_-)^2$, the roots are arranged as

$$r_- \leq r_1 \leq r_2 \leq r_+.$$  

Particles at such the geodesics compose a thermal ensemble, since the condition of causality in terms of time

$$ds^2 = A g_{tt}^2(r) \, dt^2 > 0$$  \hspace{1cm} (12)

implies the imaginary $dt$.

Kruskal coordinates are introduced by

$$\begin{cases} u = t - r_*, \\ v = t + r_*, \end{cases}$$  \hspace{1cm} (13)

and two maps marked by “+” and “−”

$$\begin{cases} \bar{u}_+ = - \frac{2r_+^2}{r_+ - r_-} \exp \left[- \frac{u}{2r_+^2}(r_+ - r_-) \right], \\ \bar{v}_+ = + \frac{2r_+^2}{r_+ - r_-} \exp \left[+ \frac{v}{2r_+^2}(r_+ - r_-) \right], \end{cases} \hspace{1cm} \text{at } r_- < r < \infty,$$  \hspace{1cm} (14)
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\begin{equation}
\begin{cases}
\bar{u}_- = + \frac{2r_-^2}{r_+ - r_-} \exp \left[ + \frac{u}{2r_-^2} (r_+ - r_-) \right] C_-, \\
\bar{v}_- = - \frac{2r_-^2}{r_+ - r_-} \exp \left[ - \frac{v}{2r_-^2} (r_+ - r_-) \right] C_+, \\
\end{cases}
\end{equation}

where \( C_\pm \) compensate complex phases of \( r_\ast \) at \( r_- < r < r_+ \), for instance: \( C_\pm = \exp[i\pi(r_-^2/r_+^2 + l)] \), \( l \in \mathbb{N} \).

One could compactify the above representations by substitution \( w \to 2\arctan[w] \) for each variable to get a scheme of causal structure [1] for the Reissner–Nordstrøm space-time shown in Fig. 2.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure2}
\caption{The Reissner–Nordstrøm space-time in compactified variables. The singularity \( r = 0 \) is shown by bold dashes. The horizons are borders of black hole interior marked by shaded polygons. A short-dashed curve represents a geodesic line reaching the infinity, while a long-dashed curve does a geodesic line oscillating between maximal and minimal radii. Mirror geodesics are given by reflection with respect to the vertical axis. The top and bottom of tape should be glued (taking into account that the geodesics will not return to regions, wherein they visited in past).}
\end{figure}

Two maps give regular metric near the horizons, but it has the true singularity at \( r = 0 \) and a coordinate singularity at \( r = \infty \), which is removed by mapping in the \( \{t, r\} \) plane.
Appropriate coordinates for geodesics completely confined under the horizons are given by
\[
\begin{align*}
\bar{u}_+ &= \kappa_+ i \rho_+ e^{i \varphi_+}, \\
\bar{v}_+ &= -\frac{i}{\kappa_+} \rho_+ e^{-i \varphi_+}, \\
\bar{u}_- &= \kappa_- i \rho_- e^{i \varphi_-}, \\
\bar{v}_- &= -\frac{i}{\kappa_-} \rho_- e^{-i \varphi_-},
\end{align*}
\]
(16)
or (at ‘initial’ times \(\Delta t^\pm_0 = -\frac{2r^2}{r^+_+ - r^-_+} \ln \kappa^\pm\), which are set to zero)
\[
\begin{align*}
t &= \frac{r^2_+}{r^+_+ - r^-_+} \ln e^{-2i \varphi^\pm} = -\frac{i}{r^+_+ - r^-_+} 2r^2_+ \varphi^\pm, \quad \varphi^\pm \in [0, 2\pi], \\
r^* &= \frac{2r^2_+}{r^+_+ - r^-_+} \ln \left[-\frac{\rho^\pm}{2r^2_+ (r^+_+ - r^-_-)}\right], \quad \rho^\pm \in [0, \infty],
\end{align*}
\]
(17)
while two maps are consistent with each to other if only the ratio of periods in \(\varphi^\pm\) is a natural number \(l\), so that the periods in \(t\) are also consistent,
\[
\frac{r^2_+}{r^2_-} = l \in \mathbb{N}.
\]
(18)
Remember that the periods in the imaginary time give the inverse temperature \(\beta\) of thermal ensemble. Therefore, we deduce that horizons are in general at different temperatures, which ratio is quantized:
\[
\beta_+ = \frac{4\pi r^2_+}{r^+_+ - r^-_+} > \beta_- = \frac{4\pi r^2_-}{r^+_+ - r^-_-}, \quad \frac{\beta_+}{\beta_-} = l.
\]
(19)
An example of consistent mapping at \(l = 3\) is shown in Fig. 3 where we use the convenient plane of \(\{\varphi, r\}\).

Since \(\rho^\pm \to 0\) at \(r \to r^\pm\) correspondingly, the interior space-time of confined geodesics is a sphere with poles at \(r = r^\pm\), wherein the horizons are contracted, as pictured in Fig. 4.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure3}
\caption{The mapping of interior: the phases of aggregation in the charged black hole: a cool “ice” (top), a hot “water” (bottom), and the transition layer of “melting ice” (middle).}
\end{figure}
The ground state of $\mathcal{E}_A \to 0$ is easily calculable. So, the integral increment of interval per cycle is given by
\[
\Delta c_s = 2 \int_{r_-}^{r_+} \frac{r dr}{\sqrt{(r_+ - r)(r - r_-)}} \equiv \pi (r_+ + r_-),
\]
(20)
while the time increment
\[
\Delta c t_E = \int \frac{dt_E}{dr} dr = -i \int \frac{dr}{g_{tt}(r)} \frac{1}{\sqrt{1 - A g_{tt}(r)}},
\]
(21)
can be simply evaluated at $A \to -\infty$, since in this limit the integral over the contour shown in Fig. 5 should be equal to zero, because the integrand does not cross any singularity, and it tends to zero.

Therefore, the increment is given by the residuals in the poles of inverse $g_{tt}(r)$,
\[
\Delta c t_E = -i 2\pi \left[ \frac{1}{g_{tt}(r_+)} + \frac{1}{g_{tt}(r_-)} \right] = 2\pi (r_+ + r_-).
\]
(22)
Winding numbers are given by

\[ n_\pm = \frac{2\pi}{\Delta_c \phi_\pm} = \frac{4\pi r_\pm^2}{(r_+ - r_-) \Delta_c t_E}, \quad (23) \]

so that

\[ n_+ = \frac{2l}{l - 1} \in \mathbb{N}, \quad n_- = \frac{2}{l - 1} \in \mathbb{N}. \quad (24) \]

Then, we can list quantized winding numbers as given in Table 1.

**Table 1.** Admissible values of winding numbers \( n_\pm \) in the ground state versus the quantum ratio of horizon areas \( l \).

| \( l \) | \( n_+ \) | \( n_- \) |
|-------|-------|-------|
| 1     | \( \infty \) | \( \infty \) |
| 2     | 4     | 2     |
| 3     | 3     | 1     |
| \( \infty \) | 2     | 0     |

The limit cases of \( l = 1 \) and \( l = \infty \) correspondingly give an extremal charged black hole \( (r_+ = r_-) \) and Schwarzschild one (uncharged). There are four kinds of quantum charged black holes in the ground state, only.

Generically, at admissible \( A \) under horizons we get

\[ \Delta_c^A s = x^{3/2} \cdot \Delta_c s, \quad (25) \]

where again

\[ x = \frac{-A}{1 - A}, \quad (26) \]

while the winding numbers are given by

\[ n_\pm^A = \frac{2\pi}{\Delta_c^A \phi_\pm}, \quad (27) \]

with

\[ \Delta_c^A \phi_\pm = \frac{\pi r_\pm^2 - r_\mp^2}{2} \left[ 2 - (x + 2) \sqrt{1 - x} \right]. \quad (28) \]

The action of particle is determined by

\[ S_n = -mc \cdot n_\pm^A \Delta_c^A s = -mc \cdot \beta_\pm \frac{x^{3/2}}{2 - (x + 2) \sqrt{1 - x}}, \quad (29) \]

which is the same function of quantized \( x \) as was studied in the case of Schwarzschild black hole. The value at the ground level \( (x \rightarrow 1) \) is equal to

\[ S_{gr} = -mc \cdot \frac{\beta_\pm}{2}. \quad (30) \]
3. Entropy

Following the method invented in the case of Schwarzschild black hole, we evaluate the partition function by making use of the quantum action gap:

$$\ln Z \approx \sum S_{gr} = -\frac{\beta_+}{2} \sigma_+ - \frac{\beta_-}{2} \sigma_-,$$

where we have introduced the sums of masses on geodesics with corresponding maps,

$$\sigma_{\pm} = \sum \pm mc.$$  \(32\)

For the sake of simplicity and spectacular clarity, let us, first, evaluate the partition function for a pure “+”-state ($\sigma_- = 0$), that has a temperature less then a pure “−”-state ($\sigma_+ = 0$), which we respectively call “ice” and “water”. Then, the solution for $\sigma$ is explicit:

$$\sigma_+ = 2M - \frac{1}{2\beta_+} A_+,$$  \(33\)

where $A_+ = 4\pi r_+^2$ is the area of horizon at $r = r_+$. Indeed, since the temperature $T_+ = 1/\beta_+$ is determined by a ‘surface gravity’ \([1, 2]\) as

$$T_+ = 4 \frac{\partial M}{\partial A_+}, \quad \text{at } dQ^2 \equiv 0,$$  \(34\)

for the average energy equal to the mass we get the following identity:

$$M = \langle E \rangle = -\frac{\partial \ln Z_+}{\partial \beta_+} = \frac{\partial}{\partial \beta_+} (\beta_+ \sigma_+)/2$$

$$= M - \frac{1}{4\beta_+} A_+ + \frac{\beta_+}{2} \left[ 2 \frac{\partial M}{\partial \beta_+} + \frac{1}{2\beta_+^2} A_+ - \frac{1}{2\beta_+} \frac{\partial A_+}{\partial \beta_+} \right] = M.$$  \(35\)

Then, the entropy of “ice” is equal to

$$S_+ = \frac{1}{4} A_+.$$  \(36\)

We can repeat the same manipulations with the pure “water” state of aggregation\(\dagger\) and get its entropy

$$S_- = \frac{1}{4} A_-.$$  \(37\)

Thus, the “water” has the entropy less than the “ice”, as well as the greater temperature, that means the layer of “melting ice” between two phases is not stable: the interior is cooling to the “ice”; the quantum transition of aggregation states takes place with a jump of temperature and entropy. We have just found that the space-time outside the external

\(\dagger\) Formally, the Christodoulou–Ruffini formula $M^2 = [(Q^2 + A/4\pi)^2 + 4J^2]^{1/2}$, yielding the black hole mass in terms of (outer/inner) horizon area $A = A_{\pm}$, charge $Q$ and angular momentum $J$, gives negative value of temperature at the inner horizon $T_- = -1/\beta_-$, which corresponds to the black body Hawking radiation confined in the inner space-time at $r < r_-$. The situation is analogous to inverse saturation of levels in lasers. Geometrically, one could also relate the negative sign with the opposite orientation of inner horizon surface with respect to the outer one.
horizon forms the cool thermostats, while the space-time inside the inner horizon forms the hot thermostats. For the extremal black hole both thermostats have the identical temperature (the hole does not radiate), while the temperature of singularity in the Schwarzschild black hole is equal to infinity (the hole radiates, since it is heated by the singularity). To our opinion, thermodynamically the singularity should cool off, as well as the inner horizon does.

As for a mixed stage of “ice” and “water”, one could introduce fractions of aggregate states, $f_{\pm}$, so that

$$\ln Z = f_+ \ln Z_+ + f_- \ln Z_-, \quad f_+ + f_- = 1,$$

and

$$M = -f_+ \frac{\partial \ln Z_+}{\partial \beta_+} - f_- \frac{\partial \ln Z_-}{\partial \beta_-} \equiv M.$$

Thus, the Bekenstein–Hawking entropy [3, 4] gives the entropy of pure “ice”.

As we have already seen in chapter I, the entropy evaluated by the partition function represents the number of microstates with various winding numbers.

4. Discussion and conclusion

In present chapter we have expanded the method based on the consideration of radial geodesics confined under the horizons, to the charged black hole. We have found the quantization of areas of horizons due to the consistency of maps covering the black hole interior. Two states of aggregation have been deduced: the cool “ice” and hot “water”, which can be exposed by the quantum phase transition.

The ground state coherently matches all points $r$ inside the interior of black hole to its external surface at $r_+$. Therefore, we do not expect any straightforward contradiction with estimates based on the Cardy formula [5] in conformal field theories [6–9], in agreement with the principle of holography [10]. The same note concerns for an evaluation of quantum field information eaten by the black hole [11]. Note, that the extremal black hole is described by solitonic BPS states in superstrings [9]. The most general case of extremal black hole involves its rotation. So, we are permanently travelling to an analogous exercise with the Kerr-Newman black hole (a charged and rotating one), which will be considered in chapter III.

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