OPE in planar QCD from integrability

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Abstract

We consider the operator product expansion of local single-trace operators composed of the self-dual components of the field strength tensor in planar QCD. Using the integrability of the one-loop matrix of anomalous dimensions of such operators, we obtain a determinant expression for certain tree-level structure constants in the OPE.
1 Introduction

The problem of determining the anomalous dimensions of single-trace operators in planar $\mathcal{N} = 4$ super Yang-Mills (SYM) theory in 3+1 dimensions is integrable. (For a recent review, see [1] and references therein.) This integrability is starting to be exploited for determining also 3-point functions of such operators [2, 3, 4, 5, 6, 7]. In this note, we show that similar techniques can be used to gain some information about operator product expansions (OPEs) in planar QCD, which in this limit ($N_c \to \infty$ and $g \to 0$ with $\lambda = g^2 N_c$ constant) has zero one-loop beta function.

It was noted by Ferretti, Heise and Zarembo [8] that operators composed of the self-dual components of the Yang-Mills field strength tensor mix only among themselves at one-loop order, and that the linear combinations with definite anomalous dimensions correspond to eigenvectors of an integrable spin-1 XXX Hamiltonian [9], which are given by Bethe ansatz [10, 11, 12, 13]. (In recent work on 3-point functions in $\mathcal{N} = 4$ SYM [3, 4, 6, 7], it is instead the spin-1/2 XXX Hamiltonian that appears.) We show that certain tree-level structure constants for such operators can be expressed in terms of a scalar product of Bethe states, which in turn can be expressed as a determinant of the type investigated in [14, 15, 16].

The outline of this paper is as follows. In Section 2, we review the composite operators and their one-loop mixing matrix. In Section 3, we introduce the OPE and structure constants. In Section 4, we briefly review the algebraic Bethe ansatz solution for the eigenvectors and eigenvalues of the mixing matrix. In Section 5, we present our results for the structure constants in terms of solutions of the Bethe equations. Section 6 contains a brief discussion of these results. In Appendix A, we briefly discuss the coordinate Bethe ansatz and the $\mathcal{F}$-conjugation of [6]. In Appendix B, we present the scalar products that enter into the expression for the structure constants.

2 Composite operators

Following [8], we decompose the Yang-Mills field strength tensor $F_{\mu \nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + ig [A_\mu, A_\nu]$ into its self-dual ($f_{\alpha \beta}$) and anti-self-dual ($\bar{f}_{\dot{\alpha} \dot{\beta}}$) components,

$$F_{\mu \nu} = \sigma_{\mu \nu}^{\alpha \beta} f_{\alpha \beta} + \bar{\sigma}_{\mu \nu}^{\dot{\alpha} \dot{\beta}} \bar{f}_{\dot{\alpha} \dot{\beta}},$$

(2.1)

where

$$\sigma_{\mu \nu} = \frac{i}{4} \sigma_2 (\sigma_\mu \sigma_\nu - \sigma_\nu \sigma_\mu), \quad \bar{\sigma}_{\mu \nu} = -\frac{i}{4} (\bar{\sigma}_\mu \sigma_\nu - \bar{\sigma}_\nu \sigma_\mu) \sigma_2,$$

(2.2)
\[ \sigma_\mu = (1, \vec{\sigma}), \quad \bar{\sigma}_\mu = (1, -\vec{\sigma}). \]  

(2.3)

We further define

\[ f_A = (\sigma_2 \sigma_A)^{\alpha \beta} f_{\alpha \beta}, \quad \bar{f}_{\bar{A}} = (\sigma_2 \bar{\sigma})^{\dot{\alpha} \dot{\beta}} \bar{f}_{\dot{\alpha} \dot{\beta}}, \]  

(2.4)

where \( A, \dot{A} = 1, 2, 3 \). We expect that, by Lorentz symmetry, the two-point function of the field strength tensor has the structure

\[ \langle F_{\mu \nu}^a b(x) F_{\rho \sigma}^c d(0) \rangle = \phi(x) (\eta_{\mu \rho} \eta_{\nu \sigma} - \eta_{\mu \sigma} \eta_{\nu \rho}) \delta_a^b \delta_c^d, \]  

(2.5)

where \( a, b, c, d = 1, \ldots, N_c \) are color indices, and \( \phi(x) \) is a scalar function. Hence,

\[ \langle f_a^A b(x) f_c^B d(0) \rangle = \phi(x) \delta_{AB} \delta_a^d \delta_c^b, \quad \langle f_a^A b(x) \bar{f}_c^\dot{B} d(0) \rangle = 0. \]  

(2.6)

In terms of \[17\]

\[ f_+ = f_{11} = \frac{1}{2} (f_2 + i f_1), \quad f_0 = \frac{1}{\sqrt{2}} (f_{12} + f_{21}) = - \frac{i}{\sqrt{2}} f_3, \quad f_- = f_{22} = \frac{1}{2} (f_2 - i f_1), \]  

(2.7)

we have \( \langle f_\pm(x) f_\pm(0) \rangle = \langle f_\pm(x) f_0(0) \rangle = 0 \); the only nonzero two-point functions are between \( f_+ \) and \( f_- \), and between two \( f_0 \) operators.

We focus on local gauge-invariant single-trace operators of length \( L \) composed of only the self-dual components

\[ O(x) = \text{tr} f_{A_1}(x) \cdots f_{A_L}(x). \]  

(2.8)

At one loop in the 't Hooft planar limit, such operators mix only among themselves \[2.6\], and their mixing matrix is given by \[8\]

\[ \Gamma = \frac{\lambda}{48 \pi^2} \sum_{l=1}^L \left[ 7 + 3 \vec{S}_l \cdot \vec{S}_{l+1} - 3(\vec{S}_l \cdot \vec{S}_{l+1})^2 \right], \]  

(2.9)

where \( \lambda = g^2 N_c \), and \( \vec{S}_l \) are \( SU(2) \) spin-1 generators,

\[ (S^i f)_{A} = -i \epsilon_{j A B} f_{B}. \]  

(2.10)

Note that \( f_+, f_0, f_- \) are eigenstates of \( S^3 \) with eigenvalues \(+1, 0, -1\), respectively. Since \( \Gamma \) commutes with \( \vec{S}^2 \) and \( S^3 \) (where \( \vec{S} = \sum_{l=1}^L \vec{S}_l \) is the total spin), all three operators can be diagonalized simultaneously. An eigenvector of \( \Gamma \) corresponds to an operator of definite conformal dimension \( \Delta = 2L + \gamma \), where \( \gamma \) is the corresponding eigenvalue.
3 Operator product expansion

Following [6,7], we assume that operators of definite conformal dimension are normalized according to

$$\langle O_i(x_i) \bar{O}_j(x_j) \rangle \sim (\mathcal{N}_i \mathcal{N}_j)^{\frac{1}{2}} \frac{\delta_{ij}}{|x_{ij}|^{\Delta_i + \Delta_j}},$$  \hspace{1cm} (3.1)$$

for $x_{ij} \equiv x_i - x_j \to 0$, where $\mathcal{N}_i$ will be specified below in (B.4). The OPE of a pair of such operators $O_1(x)$ and $O_3(x)$ is given by

$$O_1(x_1) O_3(x_3) \sim \sum_{O_2} \left( \frac{\mathcal{N}_1 \mathcal{N}_3}{\mathcal{N}_2} \right)^{\frac{1}{2}} \frac{C_{132}}{|x_{13}|^{\Delta_1 + \Delta_3 - \Delta_2}} O_2(x) + \ldots, \hspace{1cm} x = \frac{1}{2}(x_1 + x_3),$$  \hspace{1cm} (3.2)$$

for $x_{13} \to 0$, where the ellipsis denotes subleading corrections involving conformal descendants of $O_2$ [18]. The structure constants $C_{132}$ have a perturbative expansion in $\lambda$,

$$N_c C_{132} = c_{132}^{(0)} + \lambda c_{132}^{(1)} + \ldots,$$  \hspace{1cm} (3.3)$$

and we focus here on just the leading (tree-level) contribution $c_{132}^{(0)}$.

We shall obtain a result for the special case that $O_3$ on the LHS of (3.2) is a “BPS-like” operator

$$O_3 = \text{tr} \underbrace{f_+ \cdots f_+}_{L_3},$$  \hspace{1cm} (3.4)$$

which corresponds to an eigenvector of $\Gamma$ with highest conformal dimension. Following [6], we associate the operators $O_i$ to states $|O_i\rangle$ of spin chains with lengths $L_i$; we then “split” the spin chains into left and right subchains of lengths

$$L_{i,l} = \frac{1}{2} (L_i + L_j - L_k),$$

$$L_{i,r} = \frac{1}{2} (L_i + L_k - L_j),$$  \hspace{1cm} (3.5)$$

respectively, with $(i,j,k)$ in cyclic order; and we then perform a corresponding split of the states,

$$|O_i\rangle = \sum_a |O_i\rangle_l \otimes |O_i\rangle_r,$$  \hspace{1cm} (3.6)$$

where, roughly speaking, the sum is over all possible ways of distributing the component fields into the left and right subchains. (A more precise definition of this splitting, as well
as a more accurate version of (3.6), will be given below after introducing the Bethe ansatz.)

Note that $|O_{i\alpha l}\rangle$ and $|O_{i\alpha r}\rangle$ are states of subchains with lengths $L_{i\alpha l}$ and $L_{i\alpha r}$, respectively. We then “flip” ($F$-conjugate) the right kets into right bras

$$|O_i\rangle = \sum_a |O_{i\alpha l}\rangle \otimes |O_{i\alpha r}\rangle ightarrow \sum_a |O_{i\alpha l}\rangle \otimes r \langle O_{i\alpha}|. \quad (3.7)$$

Given a pair of elementary fields $A$ and $B$ that are associated with the kets $|\Psi_i\rangle_r$ and $|\Psi_{i+1}\rangle_l$, respectively, the flipped state $r \langle \Psi_i|\Psi_{i+1}\rangle_l$ is defined such that

$$\langle AB \rangle \propto r \langle \Psi_i|\Psi_{i+1}\rangle_l. \quad (3.8)$$

In view of the fact that the only nonzero two-point functions are between $f_+$ and $f_-$, and between two $f_0$ fields, the prescription (3.8) implies that

$$|f_\pm\rangle_r \rightarrow r \langle f_\mp|, \quad |f_0\rangle_r \rightarrow r \langle f_0|. \quad (3.9)$$

(Our convention is that $\langle f_\pm|f_\pm\rangle = 1, \langle f_\pm|f_\mp\rangle = 0.$) The structure constants are therefore given by [6, 7]

$$c^{(0)}_{132} = N_{132} \sum_{a,b,c} r \langle O_{2b}|O_{1a}\rangle_l \langle O_{1a}|O_{3c}\rangle_l r \langle O_{3c}|O_{2b}\rangle_l, \quad (3.10)$$

where

$$N_{132} = \left( \frac{L_1L_2L_3}{\langle O_1|O_1\rangle\langle O_2|O_2\rangle\langle O_3|O_3\rangle} \right)^{\frac{1}{2}}. \quad (3.11)$$

This is represented graphically in Fig. 1. In order to further evaluate the expression (3.10) for the structure constants, it is necessary to have a more explicit construction of the states with definite conformal dimensions. To this end, we now turn to the Bethe ansatz.

4 Algebraic Bethe ansatz

The mixing matrix (2.9) is the Hamiltonian of an integrable (antiferromagnetic) closed spin-1 chain [9], and can be diagonalized by algebraic Bethe ansatz [10, 11, 12, 13]. The basic strategy is to diagonalize a transfer matrix $t^{(1)}(u)$ that is constructed from a monodromy matrix with a 2-dimensional (i.e., spin-1/2) auxiliary space. Although this transfer matrix does not generate the Hamiltonian (2.9), it is related by the fusion procedure to another transfer matrix $t^{(1)}(u)$ that is constructed from a monodromy matrix with a 3-dimensional
(i.e., spin-1) auxiliary space and that does contain the Hamiltonian \[11, 12, 13\]. Hence, by diagonalizing \( t^{(1)}(u) \), one also diagonalizes \( t^{(1)}(u) \), and therefore also the Hamiltonian.

The transfer matrix \( t^{(1)}(u) \) can be constructed using the 6 \( \times \) 6 R-matrix

\[
R^{(\frac{1}{2},1)}(u, v) = \frac{1}{(u - v - \eta)} \begin{pmatrix}
    u - v + \eta & u - v & \sqrt{2}\eta \\
    u - v & u - v - \eta & \sqrt{2}\eta \\
    \sqrt{2}\eta & \sqrt{2}\eta & u - v - \eta \\
\end{pmatrix}
\]

where eventually we shall set \( \eta = i \), and the matrix elements that are zero are left empty.

We regard \( R^{(\frac{1}{2},1)}(u, v) \) as an operator acting on \( \mathbb{C}^2 \otimes \mathbb{C}^3 \). This R-matrix can be obtained by fusion \([10, 11]\) from \( R^{(\frac{1}{2},\frac{1}{2})}(u, v) = u - v + \eta P \) (where \( P \) is the permutation matrix on \( \mathbb{C}^2 \otimes \mathbb{C}^2 \) together with a “gauge” transformation that makes the matrix symmetric.

The (inhomogeneous) monodromy matrix is given by

\[
T^{(\frac{1}{2})}_0(u; \{ z \}_L) = R^{(\frac{1}{2},1)}_{01}(u, z_1) \cdots R^{(\frac{1}{2},1)}_{0L}(u, z_L),
\]

where we have introduced the inhomogeneities \( \{ z \}_L = \{ z_1, \ldots, z_L \} \) for later convenience. The auxiliary space (labeled 0) is two-dimensional, while each of the quantum spaces (labeled 1, \ldots, L) are three-dimensional. By tracing over the auxiliary space, we arrive at the (inhomogeneous) transfer matrix

\[
t^{(\frac{1}{2})}(u; \{ z \}_L) = \text{tr}_0 T^{(\frac{1}{2})}_0(u; \{ z \}_L).
\]
It has the commutativity property
\[
[t^{(1/2)}(u; \{z\}_L), t^{(1/2)}(v; \{z\}_L)] = 0
\]
by virtue of the fact that the R-matrix obeys the Yang-Baxter equation.

The eigenvectors of this transfer matrix can be readily obtained by algebraic Bethe ansatz:
we define the operators \( A, B, C, D \) by
\[
T^{(1/2)}_0(u; \{z\}_L) = \begin{pmatrix} A(u; \{z\}_L) & B(u; \{z\}_L) \\ C(u; \{z\}_L) & D(u; \{z\}_L) \end{pmatrix}.
\]
(4.5)

We also introduce the ferromagnetic vacua with all spins up or down,
\[
|0\rangle_\pm = |f_\pm\rangle^\otimes L \equiv |f^L_\pm\rangle.
\]
(4.6)

These states are eigenstates of both \( A(u; \{z\}_L) \) and \( D(u; \{z\}_L) \),
\[
A(u; \{z\}_L)|0\rangle_+ = \prod_{l=1}^L \frac{u - z_l + \eta}{u - z_l - \eta}|0\rangle_+, \quad D(u; \{z\}_L)|0\rangle_+ = \prod_{l=1}^L \frac{u - z_l + \eta}{u - z_l - \eta}|0\rangle_+.
\]

(4.7)

We note that
\[
B(u; \{z\}_L)^\dagger = -\prod_{l=1}^L \frac{u^* - z^*_l - \eta}{u^* - z^*_l + \eta} C(u^*; \{z^*\}_L),
\]
(4.8)

where we have used \( \eta = i \), and \( * \) denotes complex conjugation. Choosing \( |0\rangle_+ \) as the reference state, one finds that the states
\[
|\{u\}_N\rangle_+ = \prod_{j=1}^N B(u_j; \{z\}_L)|0\rangle_+
\]
(4.9)

are eigenstates of the transfer matrix \( t^{(1/2)}(u; \{z\}_L) \) provided that \( \{u\}_N = \{u_1, \ldots, u_N\} \) are distinct and satisfy the spin-1 Bethe equations
\[
\prod_{l=1}^L \frac{u_j - z_l + \eta}{u_j - z_l - \eta} = \prod_{k=1}^N \frac{u_j - u_k + \eta}{u_j - u_k - \eta}.
\]
(4.10)

In particular, in the homogeneous limit \( z_l = 0 \), these states are eigenstates of the Hamiltonian
\[
H = \frac{\lambda}{48\pi^2} \left( 7L - \sum_{k=1}^N \frac{12}{u_k^2 + 1} \right).
\]
(4.11)
The conformal dimensions are therefore given by $\Delta = 2L + \gamma$. The Bethe states (4.9) are also $SU(2)$ highest-weight states, with spin

$$s = s^3 = L - N,$$

(4.12)

and therefore $N \leq L$. If we instead choose $|0\rangle_-$ as the reference state, then the Bethe states are given by

$$|\{u\}_N\rangle_- = \prod_{j=1}^{N} C(u_j; \{z\}_L)|0\rangle_-, $$

(4.13)

which are lowest-weight states, with spin $s = -s^3 = L - N$, so again $N \leq L$.

In order to properly define the splitting of states (3.6), we follow [6] and split the monodromy matrix (4.2),

$$T_0^{L/2}(u; \{z\}_L) = T_0^{L/2}(u; \{z\}_{L_i}) T_0^{L/2}(u; \{z\}_{L_r}), $$

(4.14)

where

$$T_0^{L/2}(u; \{z\}_L) = R_0^{L/2}(u, z_1) \ldots R_0^{L/2}(u, z_{L_i}), $$

$$T_0^{L/2}(u; \{z\}_{L_r}) = R_0^{L/2}(u, z_{L_i+1}) \ldots R_0^{L/2}(u, z_{L}), $$

(4.15)

and $\{z\}_L = \{z_1, \ldots, z_{L_i}\}$, $\{z\}_{L_r} = \{z_{L_i+1}, \ldots, z_L\}$. Correspondingly,

$$\begin{pmatrix} A(u; \{z\}_L) & B(u; \{z\}_L) \\ C(u; \{z\}_L) & D(u; \{z\}_L) \end{pmatrix} = \begin{pmatrix} A_i(u; \{z\}_{L_i}) & B_i(u; \{z\}_{L_i}) \\ C_i(u; \{z\}_{L_i}) & D_i(u; \{z\}_{L_i}) \end{pmatrix} \begin{pmatrix} A_r(u; \{z\}_{L_r}) & B_r(u; \{z\}_{L_r}) \\ C_r(u; \{z\}_{L_r}) & D_r(u; \{z\}_{L_r}) \end{pmatrix}. $$

(4.16)

In particular,

$$B(u; \{z\}_L) = A_i(u; \{z\}_{L_i}) B_r(u; \{z\}_{L_r}) + B_i(u; \{z\}_{L_i}) D_r(u; \{z\}_{L_r}), $$

$$C(u; \{z\}_L) = C_i(u; \{z\}_{L_i}) A_r(u; \{z\}_{L_r}) + D_i(u; \{z\}_{L_i}) C_r(u; \{z\}_{L_r}). $$

(4.17)

The $F$-conjugation (A.13) implies that

$$B_r(u; \{z\}_L)|f^{L_r+}_+\rangle_r \rightarrow r|f^{L_r+}_+\rangle_r B_r(u; \{z\}_L), $$

$$C_r(u; \{z\}_L)|f^{L_r-}_+\rangle_r \rightarrow r|f^{L_r-}_+\rangle_r C_r(u; \{z\}_L). $$

(4.18)

## 5 Structure constants

We now return to the task of evaluating the expression (3.10) for the structure constants. By assumption, operator $O_3$ is made up only of $f_+$; hence, the the corresponding state is given by

$$|O_3\rangle = |f^{L_3}_+\rangle = |f^{L_3+}_+\rangle_r \otimes |f^{L_3+}_+\rangle_l \rightarrow |f^{L_3+}_+\rangle_l \otimes r|f^{L_3+}_-\rangle_r, $$

(5.1)
where $L_{3,l}$ and $L_{3,r}$ are given by \( (3.5) \). Evidently, since there is only one way to split this state, no summation is necessary. The algebraic Bethe state for operator $O_1$ can be written as

$$|O_1\rangle = \prod_{j=1}^{N_1} [A_l(u_j)B_r(u_j) + D_r(u_j)B_l(u_j)] |f_{+}^{L_1,l}\rangle_t \otimes |f_{+}^{L_1,r}\rangle_r$$

$$= \sum_{\alpha \cup \bar{\alpha} = \{u\}_{N_1}} H_1(\alpha, \bar{\alpha}) \left[ \prod_{j \in \alpha} B_l(u_j) \right] |f_{+}^{L_1,l}\rangle_t \otimes \left[ \prod_{j \in \bar{\alpha}} B_r(u_j) \right] |f_{+}^{L_1,r}\rangle_r, \quad (5.2)$$

where $\{u\}_{N_1}$ is a solution of the Bethe equations \( (4.10) \) with $L = L_1$. Under $\mathcal{F}$-conjugation, this operator becomes

$$|O_1\rangle \rightarrow \sum_{\alpha \cup \bar{\alpha} = \{u\}_{N_1}} H_1(\alpha, \bar{\alpha})|O_{1,\alpha}\rangle_t \otimes r\langle O_{1,\bar{\alpha}}|, \quad (5.3)$$

where

$$|O_{1,\alpha}\rangle_t = \left[ \prod_{j \in \alpha} B_l(u_j) \right] |f_{+}^{L_1,l}\rangle_t, \quad r\langle O_{1,\bar{\alpha}}| = r\langle f_{-}^{L_1,r}| \left[ \prod_{j \in \bar{\alpha}} B_r(u_j) \right]. \quad (5.4)$$

Similarly, the operator $O_2$ can be written as

$$|O_2\rangle \rightarrow \sum_{\beta \cup \bar{\beta} = \{v\}_{N_2}} H_2(\beta, \bar{\beta})|O_{2,\beta}\rangle_t \otimes r\langle O_{2,\bar{\beta}}|, \quad (5.5)$$

with

$$|O_{2,\beta}\rangle_t = \left[ \prod_{j \in \beta} C_l(v_j) \right] |f_{-}^{L_2,l}\rangle_t, \quad r\langle O_{2,\bar{\beta}}| = r\langle f_{+}^{L_2,r}| \left[ \prod_{j \in \bar{\beta}} C_r(v_j) \right]. \quad (5.6)$$

where $\{v\}_{N_2}$ is a solution of the Bethe equations \( (4.10) \) with $L = L_2$. Having chosen to construct the Bethe states for $O_1$ with the reference state $|0\rangle_-$, it is necessary to construct the Bethe states for $O_2$ with the reference state $|0\rangle_-$.

We now insert these results into Eq. \( (3.10) \) to get

$$c_{132} = \lim_{z_l \to 0} N_{132} \sum_{\beta \cup \bar{\beta} = \{v\}_{N_2}} \sum_{\alpha \cup \bar{\alpha} = \{u\}_{N_1}} H_1(\alpha, \bar{\alpha}) H_2(\beta, \bar{\beta}) r\langle O_{2,\beta}|O_{1,\alpha}\rangle_t r\langle O_{1,\bar{\alpha}}|f_{+}^{L_1,l}\rangle_t r\langle f_{-}^{L_2,r}|O_{2,\bar{\beta}}\rangle_t$$

$$= \lim_{z_l \to 0} N_{132} H_2(\{\}, \{v\}_{N_2}) \sum_{\alpha \cup \bar{\alpha} = \{u\}_{N_1}} H_1(\alpha, \bar{\alpha}) r\langle O_{2}|O_{1,\alpha}\rangle_t r\langle O_{1,\bar{\alpha}}|f_{+}^{L_1,l}\rangle_t r\langle f_{-}^{L_2,r}|f_{-}^{L_2,l}\rangle_t. \quad (5.7)$$
In passing to the second line, we have made use of the fact that the expression vanishes unless \( \beta = \{ \} \) (i.e., the set \( \beta \) contains no Bethe roots), and we defined

\[
\mathcal{r}\langle \mathcal{O}_2 \rangle \equiv \langle f^{L_2, r}_+ \left| \prod_{j=1}^{N_2} C_r(v_j) \right. \rangle . \tag{5.8}
\]

With the help of (4.7), we see that

\[
H_2(\{ \}, \{ v \}_{N_2}) = \prod_{j=1}^{N_2} \prod_{l=1}^{L_2, l} \frac{v_j - z_l + \eta}{v_j - z_l - \eta} . \tag{5.9}
\]

It becomes 1 in the homogeneous limit, by virtue of the zero-momentum constraint

\[
\prod_{j=1}^{N_2} \frac{v_j + \eta}{v_j - \eta} = 1 , \tag{5.10}
\]

which arises from the cyclicity of the trace in \( \mathcal{O}_2 \). The remaining sum over partitions in (5.7) can be performed by using \( \mathcal{r}\langle \mathcal{O}_1, \bar{\alpha} \left| f^{L_3, l}_- \right. \rangle t = \mathcal{t}\langle f^{L_3, l}_- \left| \mathcal{O}_1, \bar{\alpha} \right. \rangle r \). Noting also that \( \mathcal{r}\langle f^{L_3, r}_- \left| f^{L_2, l}_- \right. \rangle t = 1 \), we obtain

\[
c_{132} = \lim_{z_l \to 0} \mathcal{N}_{132} \sum_{\alpha, \bar{\alpha} = \{ u \}_{N_1}} H_1(\alpha, \bar{\alpha}) \mathcal{r}\langle \mathcal{O}_2 \left| \mathcal{O}_1, \bar{\alpha} \right. \rangle t \langle f^{L_3, l}_- \left| \mathcal{O}_1, \bar{\alpha} \right. \rangle r \tag{5.11}
\]

\[
= \lim_{z_l \to 0} \mathcal{N}_{132} \mathcal{t}\langle f^{L_3, l}_- \left| \otimes \mathcal{r}\langle \mathcal{O}_2 \left| \mathcal{O}_1 \right. \rangle \right. \rangle \tag{5.12}
\]

We observe that this expression vanishes unless

\[
L_2 - N_2 = L_1 + L_3 - N_1 \geq 0 . \tag{5.13}
\]

Indeed, the factor \( \mathcal{r}\langle \mathcal{O}_2 \left| \mathcal{O}_1, \alpha \right. \rangle t \) in (5.11) vanishes unless \( |\alpha| \) (the number of Bethe roots in \( \alpha \)) is given by \( |\alpha| = N_2 \). It follows that \( |\bar{\alpha}| = N_1 - N_2 \). Moreover, the two states in the factor \( \mathcal{t}\langle f^{L_3, l}_- \left| \mathcal{O}_1, \bar{\alpha} \right. \rangle r \) should have the same \( S^3 \) eigenvalue; hence,

\[
L_{1, r} - |\bar{\alpha}| = -L_{3, l} , \tag{5.14}
\]

which then implies (5.13). The sum over \( \mathcal{O}_2 \) in (3.2) can therefore be understood as the sum over all \( L_2 \) and \( N_2 \) satisfying the constraint (5.13).

The scalar product in (5.12) is a restricted Slavnov scalar product

\[
c^{(0)}_{132} = \mathcal{N}_{132}^{\text{hom}} S^{\text{hom}}(\{ u \}_{N_1}, \{ v \}_{N_2}) . \tag{5.15}
\]
where \( \{u\}_{N_1}, \{v\}_{N_2} \) are the Bethe roots corresponding to operators \( \mathcal{O}_1, \mathcal{O}_2 \), respectively. In Appendix B we obtain an expression (B.19) for the restricted Slavnov scalar product, which in the homogeneous limit \( z_l \to 0 \) becomes

\[
S^{\text{hom}}(\{u\}_{N_1}, \{v\}_{N_2}) = \prod_{k=1}^{N_2} \left( \frac{v_k + \eta}{v_k - \eta} \right)^{(2L_1 - N_1 + N_2)/2} N_1 \prod_{j>k}^{N_1} \frac{1}{v_j - v_k} \prod_{j=1}^{N_2} \frac{1}{(v_k - \eta)v_k} \prod_{k=1}^{N_2} \frac{1}{(v_k - \eta)v_k} \ 
\]

(5.16)

\[
\times \prod_{k=1}^{N_2} \frac{1}{[(v_k - \eta)v_k]^{(N_1 - N_2)/2}} \det \left( \begin{array}{cccc} M_{ij} & 1 \leq i \leq N_2, & 1 \leq j \leq N_1 \\ \Psi^{(i-1)}(u_j, \eta) & 1 \leq i \leq (N_1 - N_2)/2, & 1 \leq j \leq N_1 \\ \Psi^{(i-1)}(u_j + \eta, 0) & 1 \leq i \leq (N_1 - N_2)/2, & 1 \leq j \leq N_1 \end{array} \right),
\]

\[
\text{where}
\]

\[
M_{ij} = \frac{\eta}{(u_j - v_i)} \left[ \prod_{m=1}^{N_1} (v_i - u_m - \eta) - \left( \frac{v_i - \eta}{v_i + \eta} \right) \prod_{m=1}^{N_1} (v_i - u_m + \eta) \right],
\]

\[
\Psi(u, z) = -\frac{1}{(u - z)(u - z - \eta)} \prod_{j=1}^{N_1} \prod_{m \neq j}^{N_1} (z - u_j), \quad \Psi^{(j)}(u, z) = \frac{1}{j!} \frac{\partial^j}{\partial z^j} \Psi(u, z).
\]

Moreover, \( \mathcal{N}_{132} \) (3.11) is given by

\[
\mathcal{N}_{132} = \sqrt{\frac{L_1 L_2 L_3}{N_1 N_2 N_3}},
\]

(5.18)

where \( \mathcal{N}_i \) are given by (B.4). Indeed,

\[
\langle \mathcal{O}_1 | \mathcal{O}_1 \rangle = \langle 0 | \prod_{j=1}^{N_1} B(u_j, \{z\}_{L_1})^\dagger \prod_{j=1}^{N_1} B(u_j, \{z\}_{L_1}) | 0 \rangle + \prod_{j=1}^{N_1} C(u_j^*, \{z^*\}_{L_1}) \prod_{j=1}^{N_1} B(u_j, \{z\}_{L_1}) | 0 \rangle_+,
\]

(5.19)

where we have used (4.8). The prefactor becomes 1 in the homogeneous limit due to the zero-momentum constraint. Furthermore, the set of all Bethe roots \( \{u\}_{N_1} \) transforms into itself under complex conjugation. Hence,

\[
\langle \mathcal{O}_1 | \mathcal{O}_1 \rangle^{\text{hom}} = \lim_{z_l \to 0} \langle 0 | \prod_{j=1}^{N_1} C(u_j, \{z\}_{L_1}) \prod_{j=1}^{N_1} B(u_j, \{z\}_{L_1}) | 0 \rangle_+ = \mathcal{N}_1^{\text{hom}}.
\]

(5.20)

Similar considerations apply to \( \langle \mathcal{O}_2 | \mathcal{O}_2 \rangle \). Finally, we note that \( \mathcal{N}_3 = 1 \).
6 Discussion

We have obtained a determinant expression for the tree-level OPE structure constants in planar QCD for operators of the type \((2.8)\), where one of them is BPS-like \((3.4)\). Indeed, given \((L_1,N_1)\) and \(L_3\), the possible values of \((L_2,N_2)\) are determined by \((5.13)\); then the corresponding Bethe equations \((4.10)\) can be solved, and the structure constants \(c_{132}^{(0)}\) can be efficiently computed using \((5.15)\).

Unlike the spin-1/2 case \([7]\), here there is no domain-wall contribution, since one of the operators is BPS-like. It should be possible to obtain the structure constants with operators other than \((3.4)\), but the expressions will be more complicated, as they will involve more than one determinant.

In the QCD literature, operators of the form \((2.8)\) would be classified as “chiral odd.” While chiral-odd operators involving quark fields play an important role in certain hadronic scattering processes \([19]\), the purely gluonic chiral-odd operators that we have considered here (with no covariant derivatives) do not seem to have direct relevance to QCD phenomenology.

It would be interesting to generalize this work to operators with covariant derivatives, which are more relevant to phenomenology. Such operators comprise the largest sector of QCD that is known to be integrable at one loop \([17, 20, 21, 22, 23]\). Another challenge is to go to higher loops (see e.g. \([24, 25]\)).

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A Coordinate Bethe ansatz and \(F\)-conjugation

In order to properly formulate \(F\)-conjugation in the algebraic Bethe ansatz formalism, it is necessary to first formulate it in the coordinate Bethe ansatz formalism.
We begin by reviewing the coordinate Bethe ansatz for spin-1, which has been discussed in [26, 27]. For simplicity, we consider the homogeneous case $z_l = 0$, and restrict to states with just two excitations, which are given by

$$|\{u_1, u_2\}\rangle^{co} = \sum_{1 \leq n_1 \leq n_2 \leq L} \left[ e^{i(p_1 n_1 + p_2 n_2)} + S(p_2, p_1) e^{i(p_2 n_1 + p_1 n_2)} \right] |n_1, n_2\rangle. \quad (A.1)$$

Here $|n_1, n_2\rangle$ is given by [27]

$$|n_1, n_2\rangle = e^{-n_1} e^{-n_2} |f_L\rangle, \quad e^- = \begin{pmatrix} 0 & 0 & 0 \\ 2^{1/2} & 0 & 0 \\ 0 & 2^{-1/2} & 0 \end{pmatrix}, \quad (A.2)$$

and

$$S(p_2, p_1) = \frac{u_2 - u_1 + i}{u_2 - u_1 - i}, \quad e^{ip_j} = \frac{u_j + i}{u_j - i}. \quad (A.3)$$

The expression (A.1) is almost the same as for the spin-1/2 case [6], the main difference being that now the summation includes $n_1 = n_2$.

We define $\mathcal{F}$-conjugation by

$$\mathcal{F} \circ |n_1, n_2\rangle = \langle L + 1 - n_2, L + 1 - n_1|\hat{\mathcal{C}} \otimes L \rangle,$$  

where

$$\hat{\mathcal{C}} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} = \hat{\mathcal{C}}^\dagger, \quad (A.5)$$

which has the properties

$$\hat{\mathcal{C}} |f_\pm\rangle = |f_\pm\rangle, \quad \hat{\mathcal{C}} |f_0\rangle = |f_0\rangle, \quad \hat{\mathcal{C}}^2 = 1,$$

$$\hat{\mathcal{C}} \otimes L B(u) \hat{\mathcal{C}} \otimes L = C(u). \quad (A.6)$$

The definition (A.4) is consistent with (3.9), and is a generalization of the definition for the spin-1/2 case [6]. It follows, as in the spin-1/2 case, that $\mathcal{F}$-conjugation of the coordinate Bethe ansatz state (A.1) is given by

$$\mathcal{F} \circ |\{u_1, u_2\}\rangle^{co} = e^{i(L+1)(p_1+p_2)} S(p_2, p_1)^{co} |\{u_1^*, u_2^*\}\rangle \hat{\mathcal{C}} \otimes L,$$  

where $^{co} \langle\{u_1, u_2\}| \equiv (|\{u_1, u_2\}\rangle^{co})^\dagger.$
We now proceed to translate this result to the algebraic Bethe ansatz. One can show that the algebraic and coordinate Bethe ansatz states are related (in our normalization) by
\[ |\{u_1, u_2\}\rangle_{al} = -\frac{(u_1 - u_2 + i)}{(u_1 + i)(u_2 - u_1)} |\{u_1, u_2\}\rangle_{co}, \]
(A.8)
generalizing the known spin-1/2 result [28, 6]. The corresponding hermitian-conjugate result is
\[ a^\dagger |\{u_1, u_2\}\rangle_{al} = -\frac{(u_1^* - u_2^* - i)}{(u_1^* - i)(u_2^* - i)(u_1 - u_2)} a^\dagger |\{u_1, u_2\}\rangle_{co}, \]
(A.9)
and therefore
\[ a^\dagger \langle\{u_1^*, u_2^*\}|_{al} = -\frac{(u_1 - i)(u_2 - i)(u_1 - u_2)}{(u_1 - u_2 - i)} a^\dagger \langle\{u_1, u_2\}|_{co}. \]
(A.10)

Using (A.8), (A.7), (A.10) and (A.3), we obtain
\[ \mathcal{F} \circ |\{u_1, u_2\}\rangle_{al} = \prod_{j=1}^{2} \left( \frac{(u_j + i)}{(u_j - i)} \right)^L a^\dagger \langle\{u_1^*, u_2^*\}|_{\tilde{C}^{\otimes L}}. \]
(A.11)

Since \[ |\{u_1, u_2\}\rangle_{al} = B(1)B(z_2)|f^L_+\rangle, \]
with the help of (4.8) we see that
\[ a^\dagger \langle\{u_1^*, u_2^*\}|_{al} = \prod_{j=1}^{2} \left( \frac{(u_j - i)}{(u_j + i)} \right)^L \langle f^L_+|C(u_1)C(u_2)\rangle. \]
(A.12)

We conclude that \(\mathcal{F}\)-conjugation of an algebraic Bethe ansatz state is given by
\[ \mathcal{F} \circ \left[ B(1)B(z_2)|f^L_+\rangle \right] = \langle f^L_+|C(u_1)C(u_2)\tilde{C}^{\otimes L} = \langle f^L_-|B(1)B(z_2). \]
(A.13)

### B Scalar products

We present here results for scalar products that enter into the expression for the structure constants.

#### B.1 Slavnov scalar product

Let us first consider the matrix element
\[ S_N(u_N, v_N, z_L) = \langle 0| \prod_{j=1}^{N} C(v_j; z_L) \prod_{k=1}^{N} B(u_k; z_L) |0 \rangle, \]
(B.1)
Figure 2: 2D lattice configuration for the Slavnov determinant. Double vertical lines denote spin-1 quantum spaces with double up-arrows for the \( f_+ \) state. Horizontal lines with incoming spin-1/2 arrows denote \( B \) operators, while those with outgoing arrows denote \( C \) operators. If we impose \( u_i = v_i \) (\( i = 1, \ldots, N_1 \)), then this configuration depicts the Gaudin norm.

where \( \{u\}_N = \{u_1, \ldots, u_N\} \) (but not necessarily \( \{v\}_N = \{v_1, \ldots, v_N\} \)) satisfy the Bethe equations (4.10), and \( |0\rangle \equiv |0\rangle_+ \). In the two-dimensional vertex-model description, this scalar product is represented by Fig. 2. It follows from Slavnov [14] that this matrix element is given by

\[
S_N(\{u\}_N, \{v\}_N, \{z\}_L) = \prod_{j>i}^N \left( \frac{1}{v_j - v_i} \frac{1}{u_i - u_j} \right) \det M_{lk}, \quad (B.2)
\]

where the \( N \times N \) matrix \( M_{lk} \) is given by

\[
M_{lk} = \frac{\eta}{u_k - v_l} \left[ \prod_{m=1}^N (v_l - u_m - \eta) \prod_{j=1}^L (v_l - z_j + \eta) - \prod_{m=1}^N (v_l - u_m + \eta) \prod_{j=1}^L (v_l - z_j - \eta) \right]. \quad (B.3)
\]

### B.2 Gaudin norm

For the special case that \( \{v_i\} \) coincide with \( \{u_i\} \), the scalar product (B.1) reduces to the Gaudin norm [29, 30, 31]

\[
\mathcal{N}(\{u\}_N, \{z\}_L) = \langle 0 | \prod_{j=1}^N C(u_j; \{z\}_L) \prod_{k=1}^N B(u_k; \{z\}_L) | 0 \rangle = \eta^N \prod_{j \neq k} \frac{u_j - u_k - \eta}{u_j - u_k} \det \Phi', \quad (B.4)
\]

---

We identify \(-ic\) in [14] with \( \eta \).
where $\Phi'$ is an $N \times N$ matrix given by
\[
\Phi'_{jk} = \frac{\partial}{\partial u_k} \log \left( \prod_{l=1}^{L} \frac{u_j - z_l + \eta}{u_j - z_l - \eta} \prod_{m \neq j}^{L} \frac{u_j - u_m - \eta}{u_j - u_m + \eta} \right). \tag{B.5}
\]

### B.3 Restricted Slavnov scalar product

We now show how to restrict the Slavnov scalar product (B.1)-(B.3) (with $N = N_1$ and $L = L_1$) to obtain (B.19). The basic trick [7, 15, 16] is to set the “extra” $v$’s equal to inhomogeneities:
\[
v_{N_1 - 2j + 1} = z_j, \quad v_{N_1 - 2j + 2} = z_j + \eta, \quad j = 1, \ldots, \frac{1}{2}(N_1 - N_2), \quad N_2 < N_1. \tag{B.6}
\]

However, since the expression (B.3) for $M_{lk}$ then becomes singular, it is convenient to first change normalization. Using a tilde to denote quantities in the new normalization, we see that
\[
\tilde{R}^{(\frac{1}{2},1)}(u, v) = \alpha(u, v) R^{(\frac{1}{2},1)}(u, v) \tag{B.7}
\]
implies that
\[
\tilde{B}(u; \{z\}_L) = \prod_{l=1}^{L} \alpha(u, z_l) B(u; \{z\}_L), \quad \tilde{C}(u; \{z\}_L) = \prod_{l=1}^{L} \alpha(u, z_l) C(u; \{z\}_L). \tag{B.8}
\]

Hence,
\[
\tilde{S}_{N_1} = \langle 0 | \prod_{j=1}^{N_1} \tilde{C}(v_j; \{z\}_L) \prod_{k=1}^{N_1} \tilde{B}(u_k; \{z\}_L) | 0 \rangle
\]
\[
= \prod_{j=1}^{N_1} \prod_{l=1}^{L_1} \alpha(u_j, z_l) \alpha(v_j, z_l) S_{N_1}. \tag{B.9}
\]

We choose the normalization factor
\[
\alpha(u, v) = \frac{u - v - \eta}{u - v + \eta}, \tag{B.10}
\]
which will avoid the singularity. Then
\[
\tilde{S}_{N_1} = \prod_{j=1}^{N_1} \prod_{l=1}^{L_1} \frac{u_j - z_l + \eta}{u_j - z_l - \eta} \prod_{j>i}^{N_1} \left( \frac{1}{v_j - v_i} \frac{1}{u_i - u_j} \right) \det \tilde{M}_{lk}, \tag{B.11}
\]
Figure 3: On the left, we “freeze” the two bottom rows of Fig. 2 by imposing (B.6). These two frozen rows are then eliminated. By repeating this procedure, we freeze out and eliminate the $N_1 - N_2$ bottom rows, thereby obtaining the figure on the right. The spins in the bottom-left part of the remaining 2D lattice are in fact completely fixed. After also removing this part, we obtain the restricted Slavnov determinant depicted in Fig. 4.

where

\[
\tilde{M}_{lk} = \frac{\eta}{(u_k - v_l)} \left[ \prod_{m=1}^{N_1} (v_l - u_m - \eta) - \prod_{l=1}^{L_1} \frac{v_l - z_j - \eta}{v_l - z_j + \eta} \prod_{m=1}^{N_1} (v_l - u_m + \eta) \right]. \tag{B.12}
\]

We are now ready to “freeze” (restrict) by choosing \{v_{N_2+1}, \ldots, v_{N_1}\} according to (B.6). We obtain (see Fig. 3)

\[
\tilde{S}_{\text{restricted}} = \tilde{S}_{N_1} \left| v_{N_1-2j+1} = z_j, \quad v_{N_1-2j+2} = z_j + \eta, \quad j = 1, \ldots, \frac{1}{2}(N_1 - N_2), \quad N_2 < N_1 \right.
\]

\[
= \prod_{j=1}^{N_1} \prod_{l=1}^{L_1} \frac{u_j - z_l - \eta}{u_j - z_l + \eta} \prod_{l=1}^{L_1} \frac{1}{u_j - u_k} \prod_{N_2 \geq j > k \geq 1} \frac{1}{v_j - v_k} \times \prod_{j=1}^{\frac{1}{2}(N_1-N_2)\geq k \geq 1} \frac{1}{(z_j - z_k)^2(z_j - z_k - \eta)(z_j - z_k + \eta)}
\]

\[
\times \prod_{j=1}^{\frac{1}{2}(N_1-N_2)} \prod_{k=1}^{N_2} \frac{1}{(z_j - v_k + \eta)(z_j - v_k)} \det \mathcal{M}_{lk}, \tag{B.13}
\]
where $\mathcal{M}_{lk}$ is an $N_1 \times N_1$ matrix, which for $l \leq N_2$ is given by (B.12)

$$
\mathcal{M}_{lk} = \frac{\eta}{(u_k - v_l)} \left[ \prod_{m=1, m \neq k}^{N_1} (v_l - u_m - \eta) - \prod_{m=1, m \neq k}^{N_1} (v_l - u_m + \eta) \prod_{j=1}^{L_1} \frac{v_l - z_j - \eta}{v_l - z_j + \eta} \right],
$$

$$
l \leq N_2; \tag{B.14}
$$

and for $l > N_2$,

$$
\mathcal{M}_{N_2+2j-1,k} = \frac{1}{(u_k - z_j)} \left[ \prod_{n=1, n \neq k}^{N_1} (z_j - u_n - \eta) - \prod_{l=1}^{L_1} \frac{z_j - z_l - \eta}{z_j - z_l + \eta} \prod_{n=1, n \neq k}^{N_1} (z_j - u_n + \eta) \right],
$$

$$
\mathcal{M}_{N_2+2j,k} = \frac{1}{(u_k - z_j - \eta)} \prod_{n=1, n \neq k}^{N_1} (z_j - u_n), \quad j = 1, \ldots, \frac{1}{2}(N_1 - N_2). \tag{B.15}
$$

We now observe that $\det \mathcal{M}_{lk}$ does not change if we add to $\mathcal{M}_{N_2+2j-1,k}$ any $k$-independent factor times $\mathcal{M}_{N_2+2j,k}$. The second term of $\mathcal{M}_{N_2+2j-1,k}$ can therefore be dropped, since it can be written as

$$
-\frac{1}{(u_k - z_j)(z_j - u_k + \eta)} \prod_{l=1}^{L_1} \frac{z_j - z_l - \eta}{z_j - z_l + \eta} \prod_{n=1, n \neq k}^{N_1} (z_j - u_n + \eta),
$$

which is a $k$-independent factor times $\mathcal{M}_{N_2+2j,k}$. In short, for $l > N_2$, $\mathcal{M}_{lk}$ is given by

$$
\mathcal{M}_{N_2+2j-1,k} = \frac{1}{(u_k - z_j)} \prod_{n=1, n \neq k}^{N_1} (z_j - u_n - \eta), \tag{B.16}
$$

$$
\mathcal{M}_{N_2+2j,k} = \frac{1}{(u_k - z_j - \eta)} \prod_{n=1, n \neq k}^{N_1} (z_j - u_n), \quad j = 1, \ldots, \frac{1}{2}(N_1 - N_2). \tag{B.17}
$$

With the help of the vertex-model correspondence, we can make the identification

$$
\tilde{S}_{\text{restricted}} = \langle 1, \ldots, \frac{1}{2}(N_1 - N_2) | \prod_{j=1}^{N_1} \tilde{C}(v_j; \{z\}_{L_1}) \prod_{k=1}^{N_1} \tilde{B}(u_k; \{z\}_{L_1}) | 0 \rangle, \tag{B.18}
$$

where $|1, \ldots, \frac{1}{2}(N_1 - N_2)\rangle$ is the state with down-spins at the sites $1, \ldots, \frac{1}{2}(N_1 - N_2)$ and up-spins at the remaining $L_1 - \frac{1}{2}(N_1 - N_2)$ sites. See Fig. 4. Finally, returning to the original
Figure 4: 2D lattice representation of the restricted Slavnov determinant, which is a scalar product between $|O_1\rangle$ (where the $B$ operators with arguments $\{u\}_{N_1}$ act on all the quantum spaces $1, \ldots, L_1$) and $r\langle O_2|$ (where the $C$ operators with arguments $\{v\}_{N_2}$ act only on the quantum spaces $1 + L_{1,r}, \ldots, L_1$).

normalization using (B.8), we obtain

$$S(\{u\}_{N_1}, \{v\}_{N_2}, \{z\}_{L_1}) = \langle 1, \ldots, 1, \frac{1}{2}(N_1 - N_2)\rangle \prod_{j=1}^{N_2} C(v_j; \{z\}_{L_1}) \prod_{k=1}^{N_1} B(u_k; \{z\}_{L_1}) |0\rangle$$



$$= \prod_{l=\frac{1}{2}(N_1 - N_2) + 1}^{L_1} \prod_{k=1}^{N_2} \frac{v_k - z_l + \eta}{v_k - z_l - \eta} \prod_{N_1 \geq j > k \geq 1}^{1} \frac{1}{u_j - u_k} \prod_{N_2 \geq j > k \geq 1}^{1} \frac{1}{v_j - v_k}$$

$$\times \prod_{\frac{1}{2}(N_1 - N_2) \geq j > k \geq 1}^{1} \frac{1}{(z_j - z_k)^2(z_j - z_k - \eta)(z_j - z_k + \eta)}$$

$$\times \prod_{j=1}^{\frac{1}{2}(N_1 - N_2)} \prod_{k=1}^{N_2} \frac{1}{(z_j - v_k + \eta)(z_j - v_k)} \det M_{lk}. \quad (B.19)$$
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