A Note on Symmetries of WDVV Equations

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Abstract

We investigate symmetries of Witten-Dijkgraaf-E.Verlinde-H.Verlinde (WDVV) equations proposed by Dubrovin from bi-hamiltonian point of view. These symmetries can be viewed as canonical Miura transformations between genus-zero bi-hamiltonian systems of hydrodynamic type. In particular, we show that the moduli space of two-primary models under symmetries of WDVV can be characterized by the polytropic exponent $h$. Furthermore, we also discuss the transformation properties of free energy at genus-one level.

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1 Introduction

A Frobenius manifold is a kind of complex manifold $\mathcal{M}$ whose tangent space locally defines a commutative and associative algebra with a unit element. The notion of Frobenius manifolds is related to many subjects in mathematics and theoretical physics, such as integrable systems, quantum cohomology, symplectic geometry, singularity theory, topological field theories and mirror symmetry etc. (see e.g. [18, 8, 20, 16, 15, 17] and references therein). One way to define an $N$-dimensional Frobenius manifold $\mathcal{M}$ is to construct a quasi-homogeneous function $F(t)$ of $N$ variables $t = (t^1, t^2, \ldots, t^N)$ such that the associated functions,

$$c_{\alpha\beta\gamma}(t) = \frac{\partial^3 F(t)}{\partial t^\alpha \partial t^\beta \partial t^\gamma}, \quad \alpha, \beta, \gamma = 1, \ldots, N$$

(1)

satisfy the following conditions:

- The metric $\eta_{\alpha\beta} = c_{1\alpha\beta}$ is constant and non-degenerate (for the discussion of degenerate cases, see [23]).
- The functions $c^\beta_{\alpha\gamma} = \eta^{\alpha\sigma} c_{\sigma\beta\gamma}$ with $\eta^{\alpha\beta} = (\eta_{\alpha\beta})^{-1}$ define the structure constants of an associative and commutative algebra $\mathbf{A}_t$ of dimension $N$ so that the basis $\{e_1, \ldots, e_N\}$: $e_\alpha \cdot e_\beta = c_{\alpha\beta}^\gamma e_\gamma$, with a unity element $e_1$ for all algebra $\mathbf{A}_t$ and $c^{\beta}_{\alpha\beta}(t) = \delta^\beta_\alpha$. The associativity condition, $(e_\alpha \cdot e_\beta) \cdot e_\gamma = e_\alpha \cdot (e_\beta \cdot e_\gamma)$ yields $c^\mu_{\alpha\beta} c^\gamma_{\mu\gamma} = c^\mu_{\alpha\gamma} c^\gamma_{\mu\beta}$ or, by virtue of (1), a system of non-linear partial differential equations for $F(t)$

$$\frac{\partial^3 F(t)}{\partial t^\alpha \partial t^\beta \partial t^\gamma} \eta^{\lambda\mu} \frac{\partial^3 F(t)}{\partial t^\mu \partial t^\gamma \partial t^\sigma} = \frac{\partial^3 F(t)}{\partial t^\mu \partial t^\gamma \partial t^\lambda} \eta^{\lambda\mu} \frac{\partial^3 F(t)}{\partial t^\alpha \partial t^\beta \partial t^\sigma}.$$  

(2)

- The function $F(t)$ (also called primary free energy) satisfies a quasi-homogeneity condition,

$$\mathcal{L}_E F = d_F F + \text{(quadratic terms)},$$

(3)

where $E(t) = \sum_{\alpha} (d_\alpha t^\alpha + r^\alpha) \partial_\alpha = \sum_{\alpha} E^\alpha \partial_\alpha$ is known as the Euler vector field.

The associativity equation (2) together with quasi-homogeneity condition (3) constitute the so-called Witten-Dijkgraaf-E.Verlinde-H.Verlinde (WDVV) equations [27, 6]. It is convenient to set $d_1 = 1$ and introduce new parameters $q_1 = 0, q_2, \ldots, q_N = d$ with $q_\alpha + q_{N-\alpha+1} = d$ in such a way that $q_\alpha = 1 - d_\alpha$ and $d = 3 - d_F$.

Given any solution (or primary free energy) of the WDVV equations, one can construct a Frobenius manifold $\mathcal{M}$ associated with it and define a unique structure of a Frobenius algebra $(\mathbf{A}_t, \langle, \rangle)$ on the tangent plane $T_t \mathcal{M}$ such that $\langle \partial_\alpha \cdot \partial_\beta, \partial_\gamma \rangle = \partial_\alpha \partial_\beta \partial_\gamma F(t)$, $\langle \partial_\alpha, \partial_\beta \rangle = \eta_{\alpha\beta}$ with
\[ \partial_\alpha \cdot \partial_\beta = c^\gamma_{\alpha \beta}(t) \partial_\gamma; \]
\[ c^\gamma_{\alpha \beta} = \eta^\delta_{\alpha \beta} \partial_\delta \partial_\gamma F(t) \]
where we set \( \partial_1 \) the unity of the algebra. On such a manifold one may interpret \( \eta^{\alpha \beta} = (\eta_{\alpha \beta})^{-1} \) as a flat metric and \( t^\alpha \) the flat coordinates.

The solution space of the WDVV equations is quite rich. It can be shown \([7]\) that if \( \eta_{11} = 0 \) and all \( d_\alpha \) are distinct, then by a properly change of variables \( t^\alpha \), the matrix \( \eta_{\alpha \beta} \) has an anti-diagonal form \( \eta_{\alpha \beta} = \delta_{\alpha+\beta,N+1} \) and

\[ F(t) = \frac{1}{2}(t^1)^2 t^N + \frac{1}{2} t^1 \sum_{\alpha=2}^{N-1} t^\alpha t^{N-\alpha+1} + f(t^2, \ldots, t^N). \]

The first nontrivial example is the case for \( N = 2 \), where \( F(t) = \frac{1}{2}(t^1)^2 t^2 + f(t^2) \) and \( E = t^1 \partial_1 + (1 - d) t^2 \partial_2 \). The associativity equation \([2]\) are empty, while the quasi-homogeneity conditions \([3]\) yields the equation, \( t^2 f'(t^2) = s f(t^2) + k_1 + k_2 t^2 + k_3(t^2)^2, s = (3 - d)/(1 - d). \)

After integrating over \( t^2 \), we obtain

\[ f(t^2) = (t^2)^s \left( k_1 \int t^2 \frac{dz}{z^{s+1}} + k_2 \int t^2 \frac{dz}{z^s} + k_3 \int t^2 \frac{dz}{z^{s-1}} + k_4 \right). \]

For \( d \neq 3, \pm 1, f \sim (t^2)^s \) (modulo the quadratic part). For \( d = 3, f \sim \log t^2. \) For \( d = -1, f \sim (t^2)^2 \log t^2. \) For \( d = 1, \) the Euler vector field become \( E = t^1 \partial_1 + 2 \partial_2 \) and \( f \sim e^{t^2}. \) (see Table 1). They correspond to two-primary solutions of WDVV equations, in which two of them have logarithmic-type primary free energy. It turns out that the associated integrable structures behind these two-primary models are polytropic gas dynamics \([26, 21]\), dispersionless Harry Dym (dDym) hierarchy \([3]\), Benney hierarchy \([2]\), and dispersionless Toda (dToda) hierarchy \([13]\), respectively.

| model         | \( F(t) \)                           | \( E(t) \)                           | \( d \)   |
|---------------|--------------------------------------|--------------------------------------|----------|
| polytropic gas| \( \frac{1}{2}(t^1)^2 t^2 + c_h(t^2)^{h+1} \) | \( t^1 \frac{\partial}{\partial t^1} + \frac{2}{h} t^2 \frac{\partial}{\partial t^2} \) | \( 1 - \frac{2}{h}(h \neq -1, 0, 1) \) |
| dDym          | \( \frac{1}{2}(t^1)^2 t^2 - \frac{1}{2} \log t^2 \) | \( t^1 \frac{\partial}{\partial t^1} - 2 t^2 \frac{\partial}{\partial t^2} \) | 3        |
| Benney        | \( \frac{1}{2}(t^1)^2 t^2 + \frac{1}{2}(t^2)^2 \log(t^2 - \frac{1}{2}) \) | \( t^1 \frac{\partial}{\partial t^1} + 2 t^2 \frac{\partial}{\partial t^2} \) | -1       |
| dToda         | \( \frac{1}{2}(t^1)^2 t^2 + e^{t^2} \) | \( t^1 \frac{\partial}{\partial t^1} + 2 \frac{\partial}{\partial t^1} \) | 1        |

For \( N = 3 \), solutions of WDVV can be reducible to a particular case of the Painlevé-VI equation \([8]\). For \( N > 3 \), since the over-determined system \([2]\) are more complicated, it is not an easy task to classify its associated integrable systems completely \([12]\).
In [8], Dubrovin introduced two kinds of nontrivial symmetries of WDVV equations, which seems to provide another point of view to study the integrability associated with Frobenius manifolds. A symmetry of the WDVV equations is the transformation

\[
\begin{align*}
t^\alpha & \mapsto \hat{t}^\alpha, \\
\eta_{\alpha\beta} & \mapsto \hat{\eta}_{\alpha\beta}, \\
F(t) & \mapsto \hat{F}(\hat{t})
\end{align*}
\]

preserving the equations. Some examples have been given to demonstrate symmetries of WDVV in [8]. However, even for the simplest case \((N = 2)\), it still lacks a general discussion to the solution space of WDVV equations using this symmetries. In this work we try to explore the solution space of two-primary models from symmetry point of view, since symmetry is of central importance to the classification programme for the integrable structures of Frobenius manifolds [12]. We shall show that symmetries of WDVV provide Miura transformation between bi-hamiltonian systems of hydrodynamic type associated with Frobenius manifolds.

In particular, we verify the canonical property of symmetries of WDVV and figure out the moduli space of the solution space for two-primary models. More recently, a dual formulation of Frobenius manifolds was introduced by Dubrovin [9]. Although it would be interesting to study symmetries between dual Frobenius manifolds [22], however, we shall not cover this part in the present work.

This paper is organized as follows. In section 2, we recall some basic concepts for constructing bi-hamiltonian systems of hydrodynamic type from Frobenius manifolds. The integrable systems associated with two-primary models are presented. In section 3, we show that symmetries of WDVV can be viewed as canonical Miura transformations for two-primary models. Moreover, we find that, under symmetries of WDVV, the moduli space of two-primary models can be parameterized by a polytropic exponent \(h\). Finally, in section 4, we explore the possibility for promoting symmetries of WDVV to genus-one level.

2 Bi-hamiltonian Structure

Dispersionless integrable hierarchies (see e.g. [25, 1, 19]) with finite number of variables are closely related to Frobenius manifolds. For those bi-hamiltonian hydrodynamic equations [10], a differential-geometric interpretation to bi-hamiltonian structures can be achieved. More precisely, writing the Poisson brackets of a bi-hamiltonian hydrodynamic system as

\[
\{t^\alpha(x), t^\beta(y)\}_i = J^{\alpha\beta}_i \delta(x - y), \quad i = 1, 2; \alpha, \beta = 1, \cdots, N,
\]
where
\[ J_1^{\alpha\beta}(t) = \eta^{\alpha\beta}(t) \partial_x + \gamma^{\alpha\beta}(t) t^\sigma, \quad J_2^{\alpha\beta}(t) = g^{\alpha\beta}(t) \partial_x + \Gamma^{\alpha\beta}(t) t^\sigma, \]
then \( \gamma^{\alpha\beta}(t) \) and \( \Gamma^{\alpha\beta}(t) \) are the contravariant Levi-Civita connections of the contravariant flat metrics \( \eta^{\alpha\beta}(t) \) and \( g^{\alpha\beta}(t) \), respectively. When \( \eta^{\alpha\beta}(t) \) is a constant flat metric (i.e. \( \gamma^{\alpha\beta}(t) = 0 \)) we call \( t^\alpha \) the flat coordinates. Given a primary free energy \( F(t) \) the associated bi-hamiltonian structure \( 4 \) can be constructed in the context of Frobenius manifolds \([7, 8, 11]\). Let us briefly recall the construction. Denoting the multiplication of the algebra as \( u \cdot v \), then another flat metric on \( \mathcal{M} \) can be defined by

\[ g^{\alpha\beta}(t) = E(t) \partial_\alpha \partial_\beta = E^\gamma(t) c^{\alpha\beta\gamma}(t), \]

where \( dt^\alpha \cdot dt^\beta = c^{\alpha\beta\gamma} dt^\gamma = \eta^{\alpha\sigma} c^{\beta\gamma\sigma} dt^\gamma \). This metric together with the original metric \( \eta^{\alpha\beta}(t) \) define a flat pencil, i.e., \( \eta^{\alpha\beta} + \lambda g^{\alpha\beta} \) is flat as well, and the associated Levi-Civita connection is given by \( \gamma^{\alpha\beta} + \lambda \Gamma^{\alpha\beta} \) for any value of \( \lambda \), which corresponds to the compatible condition in integrable systems. We remark that, in terms of flat coordinates, the contravariant components of the Levi-Civita connection associated with the metric \( g^{\alpha\beta}(t) \) can be expressed as

\[ \Gamma^{\alpha\beta}(t) = c^{\alpha\beta\gamma}(t) \left( \frac{1}{2} - \mu \right)^\gamma, \]

where the matrix \( \mu = \text{diag}(\mu_1, \cdots, \mu_N) \) with \( \mu_\alpha = q_\alpha - d/2 \). The associated genus-zero hierarchy flows can be written as

\[ \frac{\partial t^\alpha}{\partial T^{\beta,n}} = \{ t^\alpha(x), H_{\beta,n} \}_1, \]

where the Hamiltonians

\[ H_{\beta,n} = \int h^{(n+1)}_{\beta}(t(x)) dx \]

are defined by the recursive relations \([7, 8]\)

\[ \frac{\partial^2 h^{(n+1)}_{\alpha}}{\partial t^\beta \partial t^\gamma} = c^{\sigma\gamma}_{\beta} \frac{\partial h^{(n)}_{\alpha}}{\partial t^\sigma}, \quad h^{(0)}_{\alpha} = \eta_{\alpha\beta} t^\beta, \]

and

\[ \partial_E h^{(n)} = (n + 1 - d/2 + \mu_\beta) h^{(n)}_{\beta} + \sum_{k=1}^{n} (R_k)_{\beta}^{\gamma} h^{(n-k)}_{\gamma}, \]

with the constant matrices \( R_k \) satisfying

\[ [\mu, R_k] = k R_k, \quad (R_k)_{\gamma}^{\alpha} \eta_{\gamma\beta} = (-1)^{k+1} (R_k)_{\beta}^{\gamma} \eta_{\gamma\alpha}. \]

It is straightforward to show that \( \partial t^\alpha / \partial T^{1,0} = \partial t^\alpha / \partial x \) and thus we identify \( T^{1,0} = x \). In fact, one can define the second hamiltonian structure by the recursive relation

\[ \{ t^\alpha(x), H_{\beta,n-1} \}_2 = (n + \mu_\beta + 1/2) \{ t^\alpha(x), H_{\beta,n} \}_1 + \sum_{k=1}^{n} (R_k)_{\beta}^{\gamma} \{ t^\alpha(x), H_{\gamma,n-k} \}_1, \]
so that the hierarchy flows can be expressed in a bi-hamiltonian form

\[
\frac{\partial t^\alpha}{\partial T^\beta,n} = \{t^\alpha(x), H_{\beta,n}\}_1 = \{t^\alpha(x), \tilde{H}_{\beta,n-1}\}_2 \tag{7}
\]

with

\[
\tilde{H}_{\beta,n-1} = \sum_{k,l} (-1)^k(R_{n-l,k})^\gamma_{\beta}(\frac{H_{\gamma,l-1}}{(n + \mu_\beta + 1/2)k+1}),
\]

where \(R_{0,0} = 1, R_{n>0,0} = 0\), and \(R_{n,l>0} = \sum_{i_1+\cdots+i_l=n} R_{i_1} \cdots R_{i_l}\). It may be noticed \cite{7} that the expression (7) of commuting flows fails to satisfy the second structure for the pair \((\beta, n)\) provided that \(n + \mu_\beta + 1/2 = 0\). A Frobenius manifold is resonant if it has such a pair \((\beta, n)\).

Before closing this subsection, we remark that the time parameters \(\{T^\alpha,n\}\) of hierarchy flows correspond to coupling constants with respect to the operators \(\sigma_{n,\alpha}\) in topological field theory coupled to gravity. (see e.g. \cite{5} for a review) As usual, we call the space spanned by \(\{T^\alpha,n\}, n = 0, 1, 2, \ldots\) the full phase space and the subspace parameterized by \(\{T^\alpha,0\}\) the small phase space where the indices \(\alpha(= 1, 2, \ldots, N)\) and \(n \geq 0\) label the primary fields and the level of gravitational descendants, respectively. For a topological field theory, the most important quantities are correlation functions that describe topological properties of the associated manifold. The generating function of correlation functions is the full free energy defined by \(F(T) = \sum_{g=0}^{\infty} F^{(g)}(T) = \sum_{g=0}^{\infty} \langle e^{\sum_{n,\alpha} T^\alpha,n \sigma_{n,\alpha}} \rangle_g\) where \(\langle \cdots \rangle_g\) denotes the expectation value on a Riemann surface of genus \(g\) with respect to a classical action. The so-called primary free energy \(F(t)\) is just the genus-zero free energy restricted on the small phase space, namely, \(F^{(0)}|_{T^\alpha,0=0, T^\alpha,1=0} = F(t)\). After identifying the flat coordinates \(t^\alpha\) with the genus-zero two-point functions \(\eta^{\alpha\beta} \partial^2 F^{(0)}/\partial T^{1,0} \partial T^{\beta,0}\), it turns out that the hierarchy flows (7) coincide with the genus-zero topological recursion relation\cite{27}.

### 2.1 Models with Two Primary Fields

#### 2.1.1 dToda hierarchy

The two-dimensional Frobenius manifold associated to the dToda is described by the primary free energy\cite{13} \cite{8}

\[
F_T(t) = \frac{1}{2}(t^1)^2 t^2 + e^{t^2}, \quad t = (t^1, t^2),
\]

which satisfies \(L_E F = 2F\) with \(E = t^1 \partial_t + 2\partial_x\). The corresponding bi-hamiltonian structure can be deduced from the primary free energy as

\[
J_1^{\alpha\beta} = \begin{pmatrix}
0 & \partial_x \\
\partial_x & 0
\end{pmatrix}, \quad J_2^{\alpha\beta} = \begin{pmatrix}
2e^{t^2} \partial_x + e^{t^2} t_x^2 & t^1 \partial_x \\
t^1 \partial_x + t_x^1 & 2\partial_x
\end{pmatrix},
\]
and the commuting hamiltonian flows of the dToda hierarchy are defined as (7) with

\[ \tilde{H}_{\beta,n-1} = \frac{1}{n + \mu_\beta + 1/2} \left( H_{\beta,n-1} - 2\delta_{1\beta} \frac{H_{2,n-2}}{(n + \mu_\beta + 1/2)} \right) , \]

where \( \mu_1 = -1/2, \mu_2 = 1/2 \) and the pair \((\beta, n) = (1, 0)\) is resonant.

### 2.1.2 Benney hierarchy

The two-dimensional Frobenius manifold corresponding to the Benney hierarchy is described by the primary free energy [8, 2]

\[ F_B(t) = \frac{1}{2} (t_1)^2 t_2 + \frac{1}{2} (t_2)^2 \left( \log t_2^2 - \frac{3}{2} \right) , \]

which satisfies \( \mathcal{L}_E F = 4F \) with \( E = t_1 \partial_1 + 2t_2 \partial_2 \). Just like the dToda hierarchy, the associated bi-hamiltonian structure can be constructed as

\[ J_1^{\alpha\beta} = \begin{pmatrix} 0 & \partial_x \\ \partial_x & 0 \end{pmatrix} , \quad J_2^{\alpha\beta} = \begin{pmatrix} 2\partial_x & t_1^2 \partial_x + t_1^2 \\ t_{1}^2 \partial_x + 2t_1^2 & 2t^2 \partial_x + t_2^2 \end{pmatrix} . \]

The hamiltonian flows of the Benney hierarchy are defined as (7) with

\[ \tilde{H}_{\beta,n-1} = \frac{1}{n + \mu_\beta + 1/2} \left( H_{\beta,n-1} - 2\delta_{2\beta} \frac{H_{1,n-2}}{(n + \mu_\beta + 1/2)} \right) , \]

where \( \mu_1 = 1/2, \mu_2 = -1/2 \) and the pair \((\beta, n) = (2, 0)\) is resonant.

### 2.1.3 dDym hierarchy

For the dDym system, the associated two-dimensional Frobenius manifold is described by the primary free energy [8, 3]

\[ F_D(t) = \frac{1}{2} (t_1)^2 t_2 - \frac{1}{2} \log t_2^2 , \]

which satisfies \( \mathcal{L}_E F = 0 \) with \( E = t_1 \partial_1 - 2t_2^2 \partial_2 \). The corresponding bi-hamiltonian structure can be deduced from \( F_D \) as

\[ J_1^{\alpha\beta} = \begin{pmatrix} 0 & \partial_x \\ \partial_x & 0 \end{pmatrix} , \quad J_2^{\alpha\beta} = \begin{pmatrix} \frac{2}{(t_2)^2} \partial_x - \frac{2}{(t_2)^2} t_2^2 \\ t_1^2 \partial_x + 2t_1^2 \\ t_1^2 \partial_x - t_1^2 \\ -2t_2^2 \partial_x - t_2^2 \end{pmatrix} . \]

and the commuting hamiltonian flows of the dDym hierarchy are defined as (7) with

\[ \tilde{H}_{\beta,n-1} = \frac{1}{n + \mu_\beta + 1/2} \left( H_{\beta,n-1} + 2\delta_{1\beta} \frac{H_{2,n-4}}{(n + \mu_\beta + 1/2)} \right) , \]

where \( \mu_1 = -3/2, \mu_2 = 3/2 \) and the resonance occurs at the pair \((\beta, n) = (1, 1)\).
2.1.4 Polytropic gas dynamics

Finally, we come to the Polytropic gas dynamics. The associated two-dimensional Frobenius manifold is described by the primary free energy

\[ F_h(t) = \frac{1}{2}(t^1)^2t^2 + c_h(t^2)^{h+1}, \quad h \neq -1, 0, 1, \]

which is characterized by a polytropic exponent \( h \) and satisfies \( \mathcal{L}_EF_h = (2 + 2/h)F_h \) with \( E = t^1\partial_1 + 2h^{-1}t^2\partial_2 \). The corresponding bi-hamiltonian structure can be obtained as

\[
J_1^{\alpha\beta} = \begin{pmatrix} 0 & \partial_x \\ \partial_x & 0 \end{pmatrix},
\]

\[
J_2^{\alpha\beta} = \begin{pmatrix} 2c_h(h^2 - 1)(t^2)^{h-1}\partial_x + c_h(h - 1)(h^2 - 1)(t^2)^{h-2}t^2_x & t^1\partial_x + h^{-1}t^1_x \\ t^1\partial_x + (1 - h^{-1})t^1_x & 2h^{-1}t^2\partial_x + h^{-1}t^2_x \end{pmatrix},
\]

and the commuting hamiltonian flows of the Polytropic gas dynamics are defined as [7] with

\[
\bar{H}_{\beta,n-1} = \frac{1}{n + \mu_\beta + 1/2} \left( H_{\beta,n-1} - \sum_{k,\alpha} (R_k)^\alpha_\beta \frac{H_{\alpha,n-k-1}}{(n + \mu_\beta + 1/2)} \right),
\]

where \( \mu_1 = 1/h - 1/2, \mu_2 = 1/2 - 1/h \) and the matrices \( R_k \) are given by [3]

\[
(R_k)^\alpha_\beta = \begin{cases} 2(-1)^{m+1} \delta_{k,2m-1}\delta_{\alpha,1}\delta_{\beta,2}, & h = \frac{1}{m}, \ m \geq 2, \\ 2(-1)^m \delta_{k,2m+1}\delta_{\alpha,2}\delta_{\beta,1}, & h = -\frac{1}{m}, \ m \geq 2, \\ 0, & \text{otherwise}. \end{cases}
\]

There are two cases to be discussed. (i) when \( |h|^{-1} \notin \mathbb{N} \) the bi-hamiltonian recursive relation is just Lenard’s relation and the corresponding Frobenius manifolds are nonresonant. (ii) when \( h = 1/m, \ |m| \in \mathbb{N}_{\geq 2} \) resonance occurs at the pair \( (2, m - 1) \) for \( m \geq 2 \), whereas \( (1, -m) \) for \( m \leq -2 \).

3 Miura Transformations

3.1 WDVV Symmetries as Canonical Miura Maps

Dubrovin [5] introduced two types of symmetries for the WDVV equations as follows.

Type 1. Legendre-type transformation \( S_\kappa \):

\[
\dot{\eta}_{\beta} = \eta_{\beta}, \quad \dot{F}(t) = \frac{\partial^2 F(t)}{\partial t^\alpha \partial t^\beta}, \quad \dot{t} = \partial_t F(t),
\]

\[
\dot{t}^\alpha = \partial_{t^\alpha} F(t), \quad (8)
\]
where $\kappa = 2, \cdots, N$ since $S_1$ is the identity transformation. The Euler vector field transforms as $\hat{E}(\hat{t}) = E(t)$ with $\hat{d} = d - 2 + 2\kappa$ and $\hat{\mu} = \mu$. Based on above, it can be shown that the vector field $\partial/\partial \hat{t}^\kappa$ should be identified as the identity $e$, and we have to interchange the indices 1 and $\kappa$ after the transformation $S_\kappa$.

Type 2. Inversion transformation $I$:

\[
\hat{t}^1 = \frac{1}{2} t^1 t^\sigma, \quad \hat{t}^\alpha = \frac{t^\alpha}{t^\sigma}, (\alpha \neq 1, N), \quad \hat{t}^N = -\frac{1}{t^N},
\]

\[
\hat{F}(\hat{t}) = (\hat{t}^N)^2 F(t) + \frac{1}{2} t^1 t^\sigma t^\sigma, \quad (9)
\]

\[
\hat{\eta}_{\alpha\beta} = \eta_{\alpha\beta}.
\]

From above it can be shown that

\[
\hat{c}_{\alpha\beta\gamma}(\hat{t}) = (t^N)^{-1} \frac{\partial t^\lambda}{\partial t^\alpha} \frac{\partial t^\mu}{\partial t^\beta} \frac{\partial t^\nu}{\partial t^\gamma} c_{\lambda\mu\nu}(t),
\]

and the Euler vector field transforms as $\hat{E}(\hat{t}) = E(t)$ with $\hat{d} = 2 - d$ and $\hat{\mu} = (d - 1)(E_{11} - E_{NN}) + \mu$. Note that the Legendre-type transformation $S_\kappa$ is an involution transformations, i.e., $(S_\kappa)^2 = id$, while $I$ is an involution transformations up to an equivalence

\[
I^2 : (t^1, t^2, \cdots, t^{N-1}, t^N) \mapsto (t^1, -t^2, \cdots, -t^{N-1}, t^N).
\]

Below, we would like to investigate the canonical property of the WDVV symmetries generated by $S_\kappa$ and $I$. In terms of flat coordinate $t^\alpha$, $\gamma_{\alpha\beta}^\gamma(t) = 0$ and the WDVV transformations $S_\kappa$ and $I$, by virtue of (8) and (9), preserve the first hamiltonian structure, i.e.,

\[
J_1^{\alpha\beta}(t) \rightarrow \hat{J}_1^{\alpha\beta}(\hat{t}) = \eta^{\alpha\beta} \partial_x.
\]

For the second structure, since the flat metric $g^{\alpha\beta}(t)$ depends on $t$ nontrivially, it would be interesting to work out the transformations of second hamiltonian structures under $S_\kappa$ and $I$.

Type 1. Legendre-type transformation $S_\kappa$.

From (5) and (6), we have

\[
g^{\alpha\beta}(t) \xrightarrow{S_\kappa} \hat{g}^{\alpha\beta}(\hat{t}) = \hat{E}^\gamma(\hat{t}) c_{\alpha\beta}^\gamma(\hat{t}) = g^{\alpha\beta}(t), \quad (11)
\]

\[
\Gamma_{\gamma}^{\alpha\beta}(t) t_x^\gamma \xrightarrow{S_\kappa} \hat{\Gamma}_{\gamma}^{\alpha\beta}(\hat{t}) \hat{t}_x^\gamma = c_{\alpha\beta}^\gamma(\hat{t}) \hat{t}_x^\gamma \left(\frac{1}{2} - \hat{\mu}\right), \quad (12)
\]

where $\hat{\Gamma}_{\gamma}^{\alpha\beta}(\hat{t})$ is the connection with respect to the metric $\hat{g}^{\alpha\beta}(\hat{t})$. Therefore,

\[
J_2^{\alpha\beta}(t) \xrightarrow{S_\kappa} \hat{J}_2^{\alpha\beta}(\hat{t}) = J_2^{\alpha\beta}(t),
\]

which is just a coordinate transformation.
Type 2. Inversion transformation I.

In this case, using (5), (6) and (10), we have

\[ g^{αβ}(t) \xrightarrow{L} \hat{g}^{αβ}(\hat{t}) = (t^N)^{-2} \frac{∂t_λ}{∂t_α} \frac{∂t_μ}{∂t_β} g^{μν}(t), \]

and hence

\[ \Gamma^{αβ}_γ(t) t^γ_2 \xrightarrow{L} \hat{Γ}^{αβ}_γ(\hat{t}) \hat{t}^γ_2 = (t^N)^{-2} \frac{∂t_λ}{∂t_α} \frac{∂t_μ}{∂t_β} c^{μν}_γ(t) t^ν_2 \left( \frac{1}{2} - \mu + (1 - d)(E_{11} - E_{NN}) \right)^β, \]

and hence symmetries of WDVV can be viewed as canonical Miura transformations between bi-hamiltonian system of hydrodynamic type.

3.2 Miura Transformations between Two-Primary Models

Now let us apply WDVV transformations on those two-primary models \((N = 2)\) listed in Table 1. Here we have \((S_2)^2 = I^2 = id\). Note that we have to interchange the indices 1 and 2 after the transformation \(S_2\).

3.2.1 Benney ↔ dToda

Under the Legendre-type transformation \(S_2\), the Benney system transforms as

\[
\begin{align*}
F_B(t) & = \frac{1}{2}(t^1)^2 t^2 + \frac{1}{2}(t^2)^2 (\text{log } t^2 - \frac{3}{2}) \\
E_B(t) & = \left(t^1 \frac{∂}{∂t^2} + 2t^2 \frac{∂}{∂t^1}\right) t^2 \\
(J_1, J_2)_B & \xrightarrow{S_2} \left(\hat{J}_1, \hat{J}_2\right)_\hat{T}
\end{align*}
\]

where

\[ t^1 = \hat{t}^2, \quad t^2 = e^{\hat{t}^1}. \]

Let us verify the canonical property of \(S_2\) for \(J_2\). From (11) and (12) we have

\[
\begin{align*}
g(t) & \xrightarrow{S_2} \hat{g}(\hat{t}) = g(t) = \begin{pmatrix} 2 & \hat{t}^2 \\ \hat{t}^2 & 2e^{\hat{t}^1} \end{pmatrix} \\
\Gamma^{αβ}_γ(t) t^γ_2 & \xrightarrow{S_2} \hat{Γ}^{αβ}_γ(\hat{t}) \hat{t}^γ_2 = \Gamma^{αβ}_γ(t) t^γ_2 = \begin{pmatrix} 0 & \hat{t}^2_x \\ 0 & 0 \end{pmatrix} \left(\begin{pmatrix} 2e^{\hat{t}^2} & 0 \\ \hat{t}^1_x e^{\hat{t}^1} & 0 \end{pmatrix}\right).
\end{align*}
\]

Therefore, the bi-hamiltonian structures of the dToda and Benney hierarchies are canonically related under the transformation \(S_2\).
3.2.2 Benney ↔ dDym

Under the inversion transformation $I$ the Benney system transforms as

\[
\begin{cases}
F_B(t) &= \frac{1}{2}(t^1)^2 t^2 + \frac{1}{2}(t^2)^2 \log t^2 \\
E_B(t) &= t^1 \frac{\partial}{\partial t^1} + 2t^2 \frac{\partial}{\partial t^2} \\
(J_1, J_2)_B
\end{cases}
\]

where

\[
t^1 = \hat{t}^1, \quad t^2 = -\frac{1}{t^2}.
\]

To verify the canonical property under the inversion $I$, let us define the matrix

\[
S^\alpha_\lambda = \left( \frac{\partial t_\lambda}{\partial t^\alpha} \right) = \left( \begin{array}{cc} \frac{1}{(t^2)^2} & 0 \\ 0 & 1 \end{array} \right),
\]

then from (13) and (14) we have

\[
g^{\alpha\beta}(t) \xrightarrow{I} \hat{g}^{\alpha\beta}(\hat{t}) = \frac{1}{(t^2)^2} S^\alpha_\lambda S^\lambda_\mu (S^t)^\beta_{\mu} = \left( \begin{array}{cc} \frac{2}{(t^2)^2} & \hat{t}^1 \\ -\hat{t}^1 & -2t^2 \end{array} \right),
\]

\[
\Gamma^\alpha_\gamma(\hat{t}) \xrightarrow{I} \hat{\Gamma}^\alpha_\gamma(\hat{t}) \hat{t}^\gamma_x = (t^2)^{-2} S^\alpha_\lambda (c^\mu_\nu(t)(S^t)^\nu_\mu)(S^t)^\epsilon_{\epsilon} \left( \frac{1}{2} + 2(E_{11} - E_{22}) - \mu \right) \epsilon_x
\]

\[
= \left( \begin{array}{cc} -\frac{2t^2}{(t^2)^3} & -\hat{t}^1_x \\ 2\hat{t}^1_x & -\hat{t}^2_x \end{array} \right).
\]

Hence, the bi-hamiltonian structure of the Benney hierarchy is mapped to that of the dDym hierarchy.

3.2.3 polytropic gas ↔ dDym

Under the Legendre-type transformation $S_2$ the Polytropic gas dynamics with $h = 1/2$ transforms as

\[
\begin{cases}
F_2(t) &= \frac{1}{2}(t^1)^2 t^2 + c_{1/2}(t^2)^{3/2} \\
E_2(t) &= t^1 \frac{\partial}{\partial t^1} + 4t^2 \frac{\partial}{\partial t^2} \\
(J_1, J_2)_2
\end{cases}
\]

where

\[
t^1 = \hat{t}^2, \quad t^2 = \frac{1}{2(t^1)^2}, \quad c_{1/2} = \frac{2^{3/2}}{3}.
\]
3.2.4 polytropic gas ↔ polytropic gas

Under the Legendre-type transformation $S_2$, the polytropic gas system with $h \neq 1/2$ transforms as

\[
\begin{align*}
F_{h \neq 1/2}(t) &= \frac{1}{2} (t^1)^2 t^2 + c_h (t^2)^{1+h} \\
E_{h \neq 1/2}(t) &= t^1 \frac{\partial}{\partial t^1} + \frac{2}{h} t^2 \frac{\partial}{\partial t^2}
\end{align*}
\]

\[
\begin{align*}
\left( J_1, J_2 \right)_{h \neq 1/2}
\end{align*}
\]

where

\[
t^1 = \hat{t}^2, \quad t^2 = \left( \frac{\hat{t}^1}{c_h h(1+h)} \right) \frac{1}{\hat{t}^2}, \quad \hat{c}_h = \frac{(h-1)^2}{h(2h-1)} (c_h h(h+1)) \frac{1}{\hat{t}^2}.
\]

On the other hand, under the inversion transformation $I$ the polytropic gas system transforms as

\[
\begin{align*}
F_h(t) &= \frac{1}{2} (t^1)^2 t^2 + c_h (t^2)^{1+h} \\
E_h(t) &= t^1 \frac{\partial}{\partial t^1} + \frac{2}{h} t^2 \frac{\partial}{\partial t^2}
\end{align*}
\]

\[
\begin{align*}
\left( J_1, J_2 \right)_h
\end{align*}
\]

where

\[
t^1 = \hat{t}^1, \quad t^2 = -\frac{1}{\hat{t}^2}, \quad \hat{c}_h = (-1)^{1-h} c_h.
\]

3.3 Moduli Space

Based on the above discussions, we find that the relationship between the four types of two-primary models can be summarized by the following sequences of WDVV transformations:

**Resonant sequence.**

For those two-dimensional Frobenius manifolds including $F_T$, $F_B$, $F_D$ and $F_h$ with $|h|^{-1} \in \mathbb{N}_{\geq 2}$, they are all resonant and the Miura transformations between them constitute a sequence of WDVV transformations:

\[
F_0 \cdots F_{n+1} \xrightarrow{S_{n+1}} F_{1} \xrightarrow{I} F_{1} \cdots F_{n} \xrightarrow{I} F_{1} \xrightarrow{S_{n}} F_D \xrightarrow{I} F_B \xrightarrow{S_{h}} F_T,
\]

where the coefficient $c_{\frac{1}{n}} (n \geq 2)$ of $F_{\frac{1}{n}}$ can be computed as $c_{\frac{1}{n}} = (-1)^{n} \frac{\hat{c}_h}{n} (n!)^{\frac{2}{h}} / (n^2 - 1)$. Since $F_0$ is an invariant function under $S_2$ and $I$ transformations, thus, on the left end, the resonant sequence terminates at the limiting function $F_0$. On the right end, the resonant sequence terminates at $F_T$ because under inversion transformation $I$ the transformed primary free energy no longer satisfies the quasi-homogeneity condition.

**Nonresonant sequences.**

For those polytropic gas systems defined by $F_h$ with $|h|^{-1} \not\in \mathbb{N}$, we have

\[
F_0 \cdots I \xrightarrow{F_{h \neq 1/2} S_{h \neq 1/2}} F_{h \neq 1/2} \xrightarrow{S_{h \neq 1/2}} F_{h \neq 1/2} \cdots F_{h \neq 1/2} \xrightarrow{S_{h \neq 1/2}} F_{h \neq 1/2} \xrightarrow{I} \cdots F_0.
\]
It may be noticed that $F_2$ is also an invariant function of $S_2$, thus the sequence generated by $F_2$ is semi-infinite; namely,

$$F_0 \rightarrow F_2 \rightarrow F_2 \rightarrow F_2 \rightarrow \cdots$$

There are some peculiar properties associated with nonresonant-sequences.

(i) No two polytropic exponents are the same in a sequence of WDVV transformations. It can be seen from the fact that for a nonresonant-sequence (16) containing $F_h$, the possible values of polytropic exponents are $\pm h/(kh \pm 1)$, where $k \geq 0$, $h > 0$ and $h^{-1} \notin \mathbb{N}$. If two exponents in a sequence coincide, it must be one of the cases $h/(kh + 1) = \pm h/(k'h \pm 1)$. However, they all give contradictions.

(ii) Two nonresonant sequences either have no common elements or overlap completely. If two nonresonant sequences associated with $F_h$ and $F_{h'}$ ($h \neq h'$, $h, h' > 0$) have a common element, then we have $\pm h/(kh \pm 1) = \pm h'/(k'h \pm 1)$ where $k, k' \geq 0$, and $h$ and $h'$ belong to two different sequences. However, for example, if $h/(kh + 1) = h'/(k'h + 1)$ which implies that $h' = h/((k - k')h + 1)$ and thus $h$ and $h'$ belong to the same sequence. Other cases can be verified in a similar manner.

The above properties motivate us to introduce a fundamental interval parameterized by the polytropic exponent $h$ so that every sequence has a representative value of $h$ in this interval. It turns out that the moduli space for two-primary models can be chosen as one of the following intervals:

$$D_n = \left\{ h \mid h \in \left[ \frac{2}{2n + 1}, \frac{1}{n} \right] \right\}, \quad n = 0, 1, 2, \ldots$$

with the proviso that an “effective” polytropic exponent was assigned to dToda, Benney, and dDym as $\infty$, 1, and $-1$, respectively. Note that $D_n$ are isomorphic to the quotient space $\mathbb{R}P_1^\times / \{S_2, I\}$ where $\mathbb{R}P_1^\times = \mathbb{R} \cup \{\infty\} \setminus \{0\}$, and are isomorphic to each other via the map $S_2 I : D_n \rightarrow D_{n+1}$. It was pointed out [8] that a Frobenius manifold is called reduced if it satisfies the inequality:

$$0 \leq q_0 \leq d \leq 1,$$  \hspace{1cm} (17)

and one can reduce by the transformations $S_\kappa$ and $I$ any solution of WDVV to a solution with property (17). It is indeed the case for two-primary models since $D_0 = \{ h | h \in [2, \infty] \}$ is just the only fundamental interval satisfying the inequality (17).
4 WDVV Symmetries as Quantum Miura Maps

So far we only focus our discussions on bi-hamiltonian systems at genus-zero level. For higher genus we may consult the work of Dubrovin and Zhang (DZ)\[11\], which provide us a starting point to extend a bi-hamiltonian hydrodynamic system to its dispersive counterpart and obtain the corresponding commuting flows up to genus-one corrections. The DZ approach \[11\] consists two main ingredients: (a) introducing slow spatial and time variables scaling $T^{\alpha, n} \rightarrow \epsilon T^{\alpha, n}$, $n = 0, 1, 2, \ldots$. (b) changing the full free energy as $F \rightarrow \sum_{g=0}^{\infty} \epsilon^{2g-2} F^{(g)}$, where $\epsilon$ is the parameter of genus expansion. Thus all of the corrections become series in $\epsilon$. To get an unambiguous genus-one correction of the hamiltonian flows (7) one may expand the flat coordinates up to the $\epsilon^2$ order as $t^{(0)}_{\alpha} + \epsilon^2 t^{(1)}_{\alpha} + O(\epsilon^4)$, $t^{(0)}_{\alpha}$ are the ordinary flat coordinates, and $t^{(1)}_{\alpha}$ are the genus-one correction defined by $t^{(1)}_{\alpha} = \partial^2 F^{(1)}(T)/\partial T^{\alpha, 0} \partial x$. The genus-one part of the free energy has the form\[11\],

\[ F^{(1)}(T) = \frac{1}{24} \log \det M^{\alpha}_{\beta}(t, \partial_x t) + G(t), \]

where the matrix $M^{\alpha}_{\beta}(t, \partial_x t) = c^{\alpha}_{\beta \gamma}(t) \partial_x \gamma$ and $G(t)$ is the so-called $G$-function which depends on $(t^2, \ldots, t^N)$ and satisfies the Getzler’s equation \[13\]. Using $c^{\alpha}_{\beta \gamma}(t)$ and $F^{(1)}(t)$ and consulting the procedure developed in \[11\](c.f. Theorem 1 and 2 and Proposition 3), one can obtain the genus-one corrections of the Poisson brackets and hierarchy flows. This means that the bi-hamiltonian structure $J_1$ and $J_2$ and the hamiltonians will receive corrections up to $\epsilon^2$ such that the hamiltonian flows still commute with each other. From (8) and (9), we have

\[ M^{\alpha}_{\beta}(t) \overset{S}{\rightarrow} \tilde{M}^{\alpha}_{\beta}(\hat{t}) = M^{\alpha}_{\beta}(t), \]

\[ M^{\alpha}_{\beta}(t) \overset{I}{\rightarrow} \tilde{M}^{\alpha}_{\beta}(\hat{t}) = (t^N)^{-2} \frac{\partial \lambda}{\partial \eta} \frac{\partial \mu}{\partial \eta} M^{\lambda}(t). \]

On the other hand, the transformations of the $G$-function under WDVV symmetries have been obtained by Strachan as\[24\]

\[ G(t) \overset{S}{\rightarrow} \hat{G}(\hat{t}) = G(t) - \frac{1}{24} \log \det \left( \frac{\partial \hat{t}^{\alpha}}{\partial \hat{t}^{\beta}} \right), \]

\[ G(t) \overset{I}{\rightarrow} \hat{G}(\hat{t}) = G(t) + \left( \frac{N}{24} - \frac{1}{2} \right) \log t^N. \]

Combining the above results we obtain the WDVV transformations for the genus-one free energy $F^{(1)}$ as

\[ F^{(1)}(t) \overset{S}{\rightarrow} \hat{F}^{(1)}(\hat{t}) = F^{(1)}(t) - \frac{1}{24} \log \det c^{\alpha}_{\beta \gamma}(t), \]

\[ F^{(1)}(t) \overset{I}{\rightarrow} \hat{F}^{(1)}(\hat{t}) = F^{(1)}(t) + \left( \frac{N}{24} - \frac{1}{2} \right) \log t^N, \]

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where we have used the formula \( \det \left( \frac{\partial \alpha}{\partial y} \right) = (t^N)^{-N} \) for the inversion transformation \( I \).

For the two-primary models (\( N = 2 \)), we list the \( G \)-function and the genus-one free energy \( \mathcal{F}^{(1)} \) in Table 2.

| model         | \( G \)-function                                  | \( \mathcal{F}^{(1)} \)                      |
|---------------|---------------------------------------------------|-----------------------------------------------|
| polytropic gas| \(-\frac{1}{24}\left(\frac{(2-h)(3-h)}{h}\right)\log t^2\) | \(\frac{1}{24}\log \left[\left(\frac{t^1_x}{t^2_x}\right)^2 - \left(\frac{t^1_y}{t^2_y}\right)^2\right] - \frac{1}{24}\left(\frac{(h-2)(h-3)}{h}\right)\log t^2\) |
| dDym          | \(-\frac{1}{12}\log t^2\)                        | \(\frac{1}{24}\log \left[\left(\frac{t^2_1(t^1_x)^2}{t^2_y}\right)^2 + \left(\frac{t^2_2(t^1_x)^2}{t^2_y}\right)^2\right] + \frac{3}{8}\log t^2\) |
| Benney        | \(-\frac{1}{24}\log t^2\)                        | \(\frac{1}{24}\log \left[\left(\frac{t^1_1(t^1_x)^2}{t^1_y}\right)^2 - \left(\frac{t^1_2(t^1_x)^2}{t^1_y}\right)^2\right] - \frac{1}{24}\log t^2\) |
| dToda         | \(-\frac{1}{24}\log t^2\)                        | \(\frac{1}{24}\log \left[\left(\frac{t^1_1(t^2_x)^2}{t^2_y}\right)^2 - \left(\frac{t^1_2(t^2_x)^2}{t^2_y}\right)^2\right] - \frac{1}{24}\log t^2\) |

In fact, both resonant and nonresonant sequences of WDVV transformations for genus-zero free energy can be promoted to genus-one level. For example, consider the genus-one free energy of the Benney system under the transformation \( S_2 \):

\[
\mathcal{F}^{(1)}_B(t) \to \mathcal{F}^{(1)}_T(h) = \frac{1}{24} \log \left[\left(\frac{t^2_1(t^1_x)^2}{t^2_y}\right)^2 - \left(\frac{t^2_2(t^1_x)^2}{t^2_y}\right)^2\right] - \frac{1}{24} t^2^1, \quad t^1_1 = t^2_1, \quad t^1_2 = t^2_2,
\]

which is just the genus-one free energy of the dToda hierarchy, \( \mathcal{F}^{(1)}_T(h) \). In a similar manner, other transformations between genus-zero free energy in the sequences \( \{15\} \) and \( \{18\} \) can be lifted to genus-one free energy without difficulty. Therefore, it seems to suggest that the bi-hamiltonian structures of the two-primary models, up to the genus-one corrections, are canonically related under WDVV transformations. A direct verification of this claim remains to be worked out.

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