Vector Bin Packing with Multiple-Choice*

EXTENDED ABSTRACT

Boaz Patt-Shamir† Dror Rawitz
boaz@eng.tau.ac.il rawitz@eng.tau.ac.il
School of Electrical Engineering
Tel Aviv University
Tel Aviv 69978
Israel

Abstract

We consider a variant of bin packing called multiple-choice vector bin packing. In this problem we are given a set of items, where each item can be selected in one of several \( D \)-dimensional incarnations. We are also given \( T \) bin types, each with its own cost and \( D \)-dimensional size. Our goal is to pack the items in a set of bins of minimum overall cost. The problem is motivated by scheduling in networks with guaranteed quality of service (QoS), but due to its general formulation it has many other applications as well. We present an approximation algorithm that is guaranteed to produce a solution whose cost is about \( \ln D \) times the optimum. For the running time to be polynomial we require \( D = O(1) \) and \( T = O(\log n) \). This extends previous results for vector bin packing, in which each item has a single incarnation and there is only one bin type. To obtain our result we also present a PTAS for the multiple-choice version of multidimensional knapsack, where we are given only one bin and the goal is to pack a maximum weight set of (incarnations of) items in that bin.

Keywords: Approximation Algorithms, Multiple-Choice Vector Bin Packing, Multiple-Choice Multidimensional Knapsack.

---

*Research supported in part by the Next Generation Video (NeGeV) Consortium, Israel.
†Supported in part by the Israel Science Foundation (grant 664/05).
1 Introduction

Bin packing, where one needs to pack a given set of items using the least number of limited-space containers (called bins), is one of the fundamental problems of combinatorial optimization (see, e.g., [16]). In the multidimensional flavor of bin packing, each item has sizes in several dimensions, and the bins have limited size in each dimension [13]. In this paper we consider a natural generalization of multidimensional bin packing that occurs frequently in practice, namely multiple-choice multidimensional bin packing. In this variant, items and space are multidimensional, and in addition, each item may be selected in one of a few incarnations, each with possibly different sizes in the different dimensions. Similarly, bins can be selected from a set of types, each bin type with its own size cap in each dimension, and possibly different cost. The problem is to select incarnations of the items and to assign them to bins so that the overall cost of bins is minimized.

Multidimensionality models the case where the objects to pack have costs in several incomparable budgets. For example, consider a distribution network (e.g., a cable-TV operator), which needs to decide which data streams to provide. Streams typically have prescribed bandwidth requirements, monetary costs, processing requirements etc., while the system typically has limited available bandwidth, a bound on the amount of funds dedicated to buying content, bounded processing power etc. The multiple-choice variant models, for example, the case where digital objects (such as video streams) may be taken in one of a variety of formats with different characteristics (e.g., bandwidth and processing requirements), and similarly, digital bins (e.g., server racks) may be configured in more than one way. The multiple-choice multidimensional variant is useful in many scheduling applications such as communication under Quality of Service (QoS) constraints, and including workplans for nursing personnel in hospitals [22].

Specifically, in this paper we consider the problem of multiple-choice vector bin packing (abbreviated mvbp, see Section 2 for a formal definition). The input to the problem is a set of \( n \) items and a set of \( T \) bin types. Each item is represented by at most \( m \) incarnations, where each incarnation is characterized by a \( D \)-dimensional vector representing the size of the incarnation in each dimension. Each bin type is also characterized by a \( D \)-dimensional vector representing the capacity of that bin type in each dimension. We are required to pack all items in the minimal possible number of bins, i.e., we need to select an incarnation for each item, select a number of required bins from each type, and give an assignment of item incarnations to bins so that no bin exceeds its capacity in any dimension. In the weighted version of this problem each bin type has an associated cost, and the goal is to pack item incarnations into a set of bins of minimum cost.

Before stating our results, we note that naïve reductions to the single-choice model do not work. For example, consider the case where \( n/2 \) items can be packed together in a single type-1 bin but require \( n/2 \) type-2 bins, while the other \( n/2 \) items fit together in a single type-2 bin but require \( n/2 \) type-1 bins. If one uses only one bin type, the cost is dramatically larger than the optimum—even with one incarnation per item. Regarding the choice of item incarnation, one may try to use only a cost-effective incarnation for each item (using some natural definition). However, it is not difficult to see that this approach results in approximation ratio \( \Omega(D) \) even when there is only one bin type.

1.1 Our Results

In this paper we give a polynomial-time approximation algorithm for the multiple-choice vector bin packing problem, in the case where \( D \) (the number of dimensions) is constant. The approximation ratio for the general weighted version is \( \ln 2D + 3 \), assuming that \( T \) (the number
of bin types) satisfies $T = O(\log n)$. For the unweighted case, the approximation ratio can be improved to $\ln 2D + 1 + \varepsilon$, for any constant $\varepsilon > 0$, if $T = O(1)$ as well. Without any assumption on $T$, we can guarantee, in the unweighted case, cost of $(\ln(2D) + 1)\text{OPT} + T + 1$, where $\text{OPT}$ denotes the optimal cost. To the best of our knowledge, this is the first approximation algorithm for the problem with multiple choice, and it is as good as the best solution for single-choice vector bin packing (see below).

As an aside, to facilitate our algorithm we also improve on the best results for multiple-choice multidimensional knapsack problem (abbreviated MMK), where we are given a single bin and the goal is to load it with the maximum weight set of (incarnations of) items. Specifically, we present a polynomial-time approximation scheme (PTAS) for MMK for the case where the dimension $D$ is constant. The PTAS for MMK is used as a subroutine in our algorithm for MVBP.

### 1.2 Related Work

Classical bin packing (BP) (single dimension, single choice) admits an asymptotic PTAS [6] and an asymptotic fully polynomial-time approximation scheme (asymptotic FPTAS) [12]. Friesen and Langston [7] presented constant factor approximation algorithms for a more general version of BP in which a fixed collection of bin sizes is allowed, and the cost of a solution is the sum of sizes of used bins. For more details about this version of BP see [20] and references therein. Correa and Epstein [5] considered BP with controllable item sizes. In this version of BP each item has a list of pairs associated with it. Each pair consists of an allowed size for this item, and a nonnegative penalty. The goal is to select a pair for each item so that the number of bins needed to pack the sizes plus the sum of penalties is minimized. Correa and Epstein [5] presented an asymptotic PTAS that uses bins of size slightly larger than 1.

Regarding multidimensionality, it has been long known that vector bin packing (VBP, for short) can be approximated to within a factor of $O(D)$ [9, 6]. More recently, Chekuri and Khanna [4] presented an $O(\log D)$-approximation algorithm for VBP, for the case where $D$ is constant. They also showed that approximating VBP for arbitrary dimension is as hard as graph coloring, implying that it is unlikely that VBP admits approximation factor smaller than $\sqrt{D}$. The best known approximation ratio for VBP is due to Bansal, Caprara and Sviridenko [2], who gave a polynomial-time approximation algorithm for constant dimension $D$ with approximation ratio arbitrarily close to $\ln D + 1$. Our algorithm for MVBP is based on their ideas.

For the knapsack problem, Frieze and Clarke [8] presented a PTAS for the (single-choice) multidimensional variant, but obtaining an FPTAS for multidimensional knapsack is NP-hard [13]. Shachnai and Tamir [21] use the approach of [8] to obtain a PTAS for a special case of 2-dimensional multiple-choice knapsack. Our algorithm for MMK extends their technique to the general case. MMK was studied extensively by practitioners. Heuristics for MMK abound, see, e.g., [14, 11, 17, 1, 19]. From the algorithmic viewpoint, the first relevant result is by Chandra et al. [3], who present a PTAS for single-dimension, multiple-choice knapsack.

### 1.3 Paper Organization

The remainder of this paper is organized as follows. In Section 2 we formalize the problems. In Section 3 we present our solution to the MMK problem, which is used in our solution to the MVBP problem that is presented in Section 4.
2 Problem Statements

We now formalize the optimization problems we deal with. For a natural number $n$, let $[n] \defeq \{1, 2, \ldots, n\}$ (we use this notation throughout the paper).

**Multiple-Choice Multidimensional Knapsack problem** (MMK).

**Instance:** A set of $n$ items, where each item is a set of $m$ or fewer $D$-dimensional incarnations.
   Incarnation $j$ of item $i$ has size $a_{ijd} \in (\mathbb{R}^+)^D$, in which the $d$th dimension is a real number $a_{ijd} \geq 0$.
   In the weighted version, each incarnation $j$ of item $i$ has weight $w_{ij} \geq 0$.

**Solution:** A set of incarnations, at most one of each item, such that the total size of the incarnations in each dimension $d$ is at most 1.

**Goal:** Maximize the number (weighted version: total weight) of incarnations in a solution.

When $D = m = 1$, this is the classical Knapsack problem (KNAPSACK).

**Multiple-Choice Vector Bin Packing** (MVBP).

**Instance:** Same as for unweighted MMK, with the addition of $T$ bin types, where each bin type $t$ is characterized by a vector $b_t \in (\mathbb{R}^+)^D$. The $d$th coordinate of $b_t$ is called the capacity of type $t$ in dimension $d$, and denoted by $b_{td}$.
   In the weighted version, each bin type $t$ has a weight $w_t \geq 0$.

**Solution:** A set of bins, each assigned a bin type and a set of item incarnations, such that exactly one incarnation of each item is assigned to any bin, and such that the total size of incarnations assigned to a bin does not exceed its capacity in any dimension

**Goal:** Minimize number of (weighted version: total weight of) assigned bins.

When $m = 1$ we get VBP, and the special case where $D = m = 1$ is the classical bin packing problem (BP).

3 Multiple-Choice Multidimensional Knapsack

In this section we present a PTAS for weighted MMK for the case where $D$ is a constant. Our construction extends the algorithms of Frieze and Clarke [8] and of Shachnai and Tamir [21].

We first present a linear program of MMK, where the variables $x_{ij}$ indicate whether the $j$th incarnation of the $i$th item is selected.

\[
\text{max} \quad \sum_{i=1}^{n} \sum_{j=1}^{m} w_{ij}x_{ij} \\
\text{s.t.} \quad \sum_{i=1}^{n} \sum_{j=1}^{m} a_{ijd}x_{ij} \leq 1 \quad \forall d \in [D] \\
\sum_{j=1}^{m} x_{ij} \leq 1 \quad \forall i \in [n] \\
x_{ij} \geq 0 \quad \forall i \in [n], j \in [m]
\]  

(MMK)

In the program, the first type of constraints make sure that the load on the knapsack in each dimension is bounded by 1; the second type of constraints ensures that at most one copy of
each element is taken into the solution. Constraints of the third type indicate the relaxation: the integer program for MMK requires that \( x_{ij} \in \{0,1\} \).

Our PTAS for MMK is based on the linear program (MMK). Let \( \varepsilon > 0 \). Suppose we somehow guess the heaviest \( q \) incarnations that are packed in the knapsack by some optimal solution, for \( q = \min \{ n, \lceil D/\varepsilon \rceil \} \). Formally, assume we are given a set \( G \subseteq [n] \) of at most \( q \) items and a function \( g : G \to [m] \) that selects incarnations of items in \( G \). In this case we can assign values to some variables of (MMK) as follows:

\[
    x_{ij} = \begin{cases} 
        1, & \text{if } i \in G \text{ and } j = g(i) \\
        0, & \text{if } i \in G \text{ and } j \neq g(i) \\
        0, & \text{if } i \notin G \text{ and } w_{ij} > \min \{ w_{\ell g(\ell)} | \ell \in G \} 
    \end{cases} 
\]

That is, if we guess that incarnation \( j \) of item \( i \) is in the optimal solution, then \( x_{ij} = 1 \) and \( x_{ij'} = 0 \) for \( j' \neq j \); also, if the \( j \)th incarnation of item \( i \) weighs more than some incarnation in our guess, then \( x_{ij} = 0 \). Denote the resulting linear program (MMK) \((G, g)\).

Let \( x^*(G, g) \) be an optimal (fractional) solution of (MMK) \((G, g)\). The idea of Algorithm 1 below is to simply round down the values of \( x^*(G, g) \). We show that if \( G \) and \( g \) are indeed the heaviest incarnations in the knapsack, then the rounded-down solution is very close to the optimum. Therefore, in the algorithm we loop over all possible assignments of \( G \) and \( g \) and output the best solution.

**Algorithm 1**

1. for all \( G \subseteq [n] \) such that \( |G| \leq q \) and \( g : G \to [m] \) do
2. \hspace{1em} \( b_d(G, g) \leftarrow 1 - \sum_{i \in G} a_{g(i)d} \) for every \( d \in [D] \)
3. \hspace{1em} if \( b_d(G, g) \geq 0 \) for every \( d \) then
4. \hspace{2em} Compute an optimal basic solution \( x^*(G, g) \) of (MMK) \((G, g)\)
5. \hspace{1em} \( x_{ij}(G, g) \leftarrow \lfloor x^*_{ij}(G, g) \rfloor \) for every \( i \) and \( j \)
6. \hspace{1em} end if
7. \hspace{1em} \( x \leftarrow \arg\max_{x(G, g)} w \cdot x(G, g) \)
8. end for
9. return \( x \)

**Theorem 1.** If \( D = O(1) \), then Algorithm 1 is a PTAS for MMK.

**Proof.** Regarding running time, note that there are \( O(n^q) \) choices of \( G \), and \( O(m^q) \) choices of \( g \) for each choice of \( G \), and hence, the algorithm runs for \( O((nm)^q) = O((nm)^{D/\varepsilon}) \) iterations, i.e., time polynomial in the input length, for constant \( D \) and \( \varepsilon \).

Regarding approximation, fix an optimal integral solution \( x^I \) to (MMK). If \( x^I \) assigns at most \( q \) incarnations of items to the knapsack, then we are done. Otherwise, let \( G^I \) be the set of items that correspond to the \( q \) heaviest incarnations selected to the knapsack by \( x^I \). For \( i \in G^I \), let \( g^I(i) \) denote the incarnation of \( i \) that was put in the knapsack by \( x^I \). Consider the iteration of Algorithm 1 in which \( G = G^I \) and \( g = g^I \). Clearly, \( w \cdot x^I \leq w \cdot x^*(G^I, g^I) \). Let \( n' \) denote the number of items that were not chosen by \( (G^I, g^I) \) or were eliminated because their incarnations weigh too much. Let \( k \) be the number of variables of the form \( x_{ij} \) in (MMK) \((G^I, g^I)\). When using slack form, we add \( D + n' \) slack variables: each constraint of the first type is written as \( \sum_{i=1}^n \sum_{j=1}^m a_{ijd}x_{ij} + s_d = 1 \), for \( 1 \leq d \leq D \), and each constraint of the second type is written as \( \sum_{j=1}^m x_{ij} + s_i' = 1 \) for \( 1 \leq i \leq n' \). Thus, the total number of variables in the program to \( k + D + n' \), and since (MMK) \((G^I, g^I)\) has \( D + n' \) constraints (excluding positivity constraints \( x_{ij} \geq 0 \)), it follows any basic solution of (MMK) \((G^I, g^I)\) has at most \( D + n' \) positive variables. Since by constraints of the second type in (MMK) \((G^I, g^I)\) there is at least one positive variable
for each item, it follows that \( x^*(G^I, g^I) \) has at most \( D \) non-integral entries, and therefore, by rounding down \( x^*(G^I, g^I) \) we lose at most \( D \) incarnations of items. Let \( W^I = \sum_{i \in G^I} w_i g^I(i) \). Then each incarnation lost due to rounding weighs at most \( W^I/q \) (because it is not one of the \( q \) heaviest). We conclude that
\[
w \cdot (G^I, g^I) \geq w \cdot x^*(G^I, g^I) - D \cdot \frac{W^I}{q} \geq w \cdot x^*(G^I, g^I)(1 - \frac{D}{q}) \geq w \cdot x^I(1 - \frac{D}{q}) = \frac{OPT}{1+\epsilon},
\]
and we are done. \( \square \)

4 Multiple-Choice Vector Bin Packing

In this section we present our main result, namely, an \( O(\log D) \)-approximation algorithm for \( \text{mvbp} \), assuming that \( D \) and \( T \) (number of dimensions and bin types, respectively) are constants.

Our algorithm is based on and extends the work of \cite{2}.

The general idea is as follows. We first encode \( \text{mvbp} \) using a covering linear programming formulation with exponentially many variables, but polynomially many constraints. We find a near optimal fractional solution to this (implicit) program using a separation oracle of the dual program. (The oracle is implemented by the MMK algorithm from Section \( \ref{sec:mmk} \).) We assign some incarnations to bins using a greedy rule based on some “well behaved” dual solution (the number of greedy assignments depends on the value of the solution to the primal program). Then we are left with a set of unassigned items, but due to our greedy rule we can assign these remaining items to a relatively small number of bins.

4.1 Covering Formulation

We start with the transformation of \( \text{mvbp} \) to weighted Set Cover (sc). An instance of sc is a family of sets \( \mathcal{C} = \{C_1, C_2, \ldots\} \) and a cost \( w_C \geq 0 \) for each \( C \in \mathcal{C} \). We call \( \bigcup_{C \in \mathcal{C}} C \) the ground set of the instance, and usually denote it by \( I \). The goal in sc is to choose sets from \( \mathcal{C} \) whose union is \( I \) and whose overall cost is minimal. Clearly, sc is equivalent to the following integer program:

\[
\begin{align*}
\min & \sum_{C \in \mathcal{C}} w_C \cdot x_C \\
\text{s.t.} & \sum_{C \ni i} x_C \geq 1 \quad \forall i \in I \\
& x_C \in \{0, 1\} \quad \forall C \in \mathcal{C}
\end{align*}
\tag{P}
\]

where \( x_C \) indicates whether the set \( C \) is in the cover. A linear program relaxation is obtained by replacing the integrality constraints of (P) by positivity constraints \( x_C \geq 0 \) for every \( C \in \mathcal{C} \). The above formulation is very general. We shall henceforth call problems whose instances can be formulated as in (P) for some \( \mathcal{C} \) and \( w_C \) values, (P)-problems.

In particular, \( \text{mvbp} \) is a (P)-problem, as the following reduction shows. Let \( \mathcal{I} \) be an instance of \( \text{mvbp} \). Construct an instance \( \mathcal{C} \) of sc as follows. The ground set of \( \mathcal{C} \) is the set of items in \( \mathcal{I} \), and sets in \( \mathcal{C} \) are the subsets of items that can be assigned to some bin. Formally, a set \( C \) of items is called compatible if and only if there exists a bin type \( t \) and an incarnation mapping \( f : C \to [m] \) such that \( \sum_{i \in C} a_{if(i)d} \leq b_{td} \) for every dimension \( d \), i.e., if there is a way to accommodate all members of \( C \) is the same bin. In the instance of sc, we let \( \mathcal{C} \) be the collection of all compatible item sets. Note that a solution to set cover does not immediately solve \( \text{mvbp} \), because selecting incarnations and bin-types is an NP-hard problem in its own right. To deal with this issue we have one variable for each possible assignment of incarnations and bin type. Namely, we may have more than one variable for a compatible item subset.
4.2 Dual Oblivious Algorithms

We shall be concerned with approximation algorithms for (P)-problems which have a special property with respect to the dual program. First, we define the dual to the LP-relaxation of (P):

\[
\begin{align*}
\max & \sum_{i \in I} y_i \\
\text{s.t.} & \sum_{i \in C} y_i \leq w_C \forall C \in \mathcal{C} \\
& y_i \geq 0 \forall i \in I
\end{align*}
\]

(D)

Next, for an instance \(C\) of set cover and a set \(S\), we define the restriction of \(C\) to \(S\) by \(C|_S \defeq \{C \cap S \mid C \in \mathcal{C}\}\), namely we project out all elements not in \(S\). Note that for any \(S\), a solution to \(C\) is also a solution to \(C|_S\): we may only discard some of the constraints in (P). We now arrive at our central concept.

**Definition 1 (Dual Obliviousness).** Let \(\Pi\) be a (P)-problem. An algorithm \(A\) for \(\Pi\) is called \(\rho\)-dual oblivious if there exists a constant \(\delta\) such that for every instance \(C \in \Pi\) there exists a dual solution \(y \in \mathbb{R}^n\) to (D) satisfying, for all \(S \subseteq I\), that

\[
A(C|_S) \leq \rho \sum_{i \in S} y_i + \delta.
\]

Let us show that the First-Fit (FF) heuristic for \(bp\) is dual oblivious (we use this property later). In FF, the algorithm scans the items in arbitrary order and places each item in the leftmost bin which has enough space to accommodate it, possibly opening a new bin if necessary. A newly open bin is placed to the right of rightmost open bin.

**Observation 1.** First-Fit is a 2-dual oblivious algorithm for bin packing.

**Proof.** In any solution produced by FF, all non-empty bins except perhaps one are more than half-full. Furthermore, this property holds throughout the execution of FF, and regardless of the order in which items are scanned. It follows that if we let \(y_i = a_i\), where \(a_i\) is the size of the \(i\)th item, then for every \(S \subseteq I\) we have \(FF(S) \leq \max\{2 \sum_{i \in S} y_i, 1\} \leq 2 \sum_{i \in S} y_i + 1\), and hence FF is dual oblivious for \(bp\) with \(\rho = 2\) and \(\delta = 1\). \[\square\]

The usefulness of dual obliviousness is expressed in the following result. Let \(\Pi\) be a (P)-problem, and suppose that APPR is a \(\rho\)-dual oblivious algorithm for \(\Pi\). Suppose further that we can efficiently find the dual solution \(y\) promised by dual obliviousness. Under these assumptions, Algorithm 2 below solves any instance \(C\) of \(\Pi\).

**Theorem 2.** Let \(\Pi\) be a (P)-problem. Then for any instance of \(\Pi\) with optimal fractional solution \(OPT^*\), Algorithm 2 outputs \(G \cup A\) satisfying

\[
w(G \cup A) \leq (\ln \rho + 1)OPT^* + \delta + w_{\max},
\]

where \(w_{\max} = \max_t w_t\).

**Proof.** Clearly, \(w(G) < \ln \rho \cdot OPT^* + w_{\max}\). It remains to bound the weight of \(A\). Let \(S'\) be the set of items not covered by \(G\). We prove that \(\sum_{i \in S'} y_i \leq \frac{1}{\rho} \sum_{i \in I} y_i\), which implies

\[
w(A) \leq \rho \sum_{i \in S'} y_i + \delta \leq \rho e^{-\ln \rho} \sum_{i=1}^n y_i + \delta \leq OPT^* + \delta,
\]
Algorithm 2

1: (Linear Programming) Find an optimal solution $x^*$ to $[P]$. Let $\text{OPT}^*$ denote its value.
2: (greedy phase) Let $C^+ = \{ C : x^*_C > 0 \}$. Let $G \leftarrow \emptyset$, $S \leftarrow I$.
3: while $\sum_{C \in G} w_C < \ln \rho \cdot \text{OPT}^*$ do
   4: Find $C \in C^+$ for which $\frac{1}{w_C} \sum_{i \in S \cap C} y_i$ is maximized;
   5: $G \leftarrow G \cup \{ C \}$, $S \leftarrow S \setminus C$.
4: end while
7: (residual solution) Apply APPR to the residual instance $S$, obtaining solution $A$.
8: return $G \cup A$.

proving the theorem.

Let $C_k \in C^+$ denote the $k$th subset added to $G$ during the greedy phase, and let $S_k \subseteq I$ be the set of items not covered after the $k$th subset was chosen. Define $S_0 = I$. We prove, by induction on $|G|$, that for every $k$,

$$\sum_{i \in S_k} y_i \leq \prod_{q=1}^{k} \left( 1 - \frac{w_{C_q}}{\text{OPT}^*} \right) \cdot \sum_{i \in I} y_i \quad (1)$$

For the base case we have trivially $\sum_{i \in S_0} y_i \leq \sum_{i \in I} y_i$. For the inductive step, assume that

$$\sum_{i \in S_{k-1}} y_i \leq \prod_{q=1}^{k-1} \left( 1 - \frac{w_{C_q}}{\text{OPT}^*} \right) \cdot \sum_{i \in I} y_i \cdot$$

By the greedy rule and the pigeonhole principle, we have that

$$\frac{1}{w_{C_k}} \sum_{i \in S_{k-1} \cap C_k} y_i \geq \frac{1}{\text{OPT}^*} \sum_{i \in S_{k-1}} y_i \cdot$$

It follows that

$$\sum_{i \in S_k} y_i = \sum_{i \in S_{k-1}} y_i - \sum_{i \in S_{k-1} \cap C_k} y_i \leq \left( 1 - \frac{w_{C_k}}{\text{OPT}^*} \right) \sum_{i \in S_{k-1}} y_i \leq \prod_{q=1}^{k} \left( 1 - \frac{w_{C_q}}{\text{OPT}^*} \right) \cdot \sum_{i \in I} y_i \cdot$$

completing the inductive argument. The theorem now follows, since by (1) we have

$$\sum_{i \in S'} y_i \leq \left( 1 - \frac{\ln \rho}{k} \right)^k \cdot \sum_{i \in I} y_i \leq e^{-\ln \rho} \sum_{i \in I} y_i \cdot$$

and we are done.

Note that if $x^*$ can be found in polynomial time, and if APPR is a polynomial-time algorithm, then Algorithm 2 runs in polynomial time. Also observe that Theorem 2 holds even if $x^*$ is not an optimal solution of $[P]$, but rather a $(1 + \varepsilon)$-approximation. We use this fact later.

In this section we defined the notion of dual obliviousness of an algorithm. We note that Bansal et al. [2] defined a more general property of algorithms called subset obliviousness. (For example, a subset oblivious algorithm is associated with several dual solutions.) Furthermore, Bansal et al. showed that the asymptotic PTAS for BP from [6] with minor modifications is subset


oblivious and used it to obtain a subset oblivious \((D + \varepsilon)\)-approximation algorithm for MVBP. This paved the way to an algorithm for VBP, whose approximation guarantee is arbitrarily close to \(\ln D + 1\). However, in the case of MVBP, using the above APTAS for BP (at least in a straightforward manner) would lead to a subset oblivious algorithm whose approximation guarantee is \((DT + \varepsilon)\). In the next section we present a 2D-dual oblivious algorithm for weighted MVBP that is based on First-Fit.

4.3 Algorithm for Multiple-Choice Vector Bin Packing

We now apply the framework of Theorem 2 to derive an approximation algorithm for MVBP. There are several gaps we need to fill.

First, we need to solve (P) for MVBP, which consists of a polynomial number of constraints (one for each item), but an exponential number of variables. We circumvent this difficulty as follows. Consider the dual of (P). The separation problem of the dual program in our case is to find (if it exists) a subset \(C\) with \(\sum_{i \in C} y_i > w_C\) for given item profits \(y_1, \ldots, y_n\). The separation problem can therefore be solved by testing, for each bin type, whether the optimum is greater than \(w_t\), which in turn is simply an MMK instance, for which we have presented a PTAS in Section 3. In other words, the separation problem of the dual program \((D)\) has a PTAS, and hence there exists a PTAS for the LP-relaxation of \((P)\) \([18, 10]\).

Second, we need to construct a dual oblivious algorithm for MVBP. To do that, we introduce the following notation. For every item \(i \in I\), incarnation \(j\), dimension \(d\), and bin type \(t\) we define the load of incarnation \(j\) of \(i\) on the \(d\)th dimension of bins of type by \(\ell_{ijtd} = a_{ijd}/b_{td}\). For every item \(i \in I\) we define the effective load of \(i\) as

\[
\bar{\ell}_i = \min_{1 \leq j \leq m, 1 \leq t \leq T} \left\{ w_t \cdot \max_d \ell_{ijtd} \right\}.
\]

Also, let \(t(i)\) denote the bin type that can contain the most (fractional) copies of some incarnation of item \(i\), where \(j(i)\) and \(d(i)\) are the incarnation and dimension that determine this bound. Formally:

\[
j(i) = \arg\min_j \min_t \{ w_t \cdot \max_d \ell_{ijtd} \}
\]

\[
t(i) = \arg\min_t \{ w_t \cdot \max_d \ell_{ij(i)td} \}
\]

\[
d(i) = \arg\max_d \ell_{ij(i)t(i)d}.
\]

Assume that \(j(i), t(i)\) and \(d(i)\) are the choices of \(j\), \(t\) and \(d\) that are taken in the definition of \(\bar{\ell}_i\). Our dual oblivious algorithm APPR for MVBP is as follows:

1. Divide the item set \(I\) into \(T\) subsets by letting \(I_t = \{ i : t(i) = t \}\).

2. Pack each subset \(I_t\) in bins of type \(t\) using FF, where the size of each item \(i\) is \(a_{ij(i)d(i)}\).

Observe that the size of incarnation \(j(i)\) of item \(i\) in dimension \(d(i)\) is the largest among all other sizes of this incarnation. Hence, the solution computed by FF is feasible for \(I_t\).

We now show that this algorithm is 2D-dual oblivious.

Lemma 2. Algorithm APPR above is a polynomial time 2D-dual oblivious algorithm for MVBP.
Proof. Consider an instance of mvbp with item set \( I \), and let the corresponding set cover problem instance be \( C \). We show that there exists a dual solution \( y \in \mathbb{R}^n \) such that for any \( S \subseteq I \),

\[
\text{APPR}(C|S) \leq 2D \cdot \sum_{i \in S} y_i + \sum_{t=1}^T w_t .
\]

Define \( y_i = \bar{\ell}_i / D \) for every \( i \). We claim that \( y \) is a feasible solution to (D). Let \( C \in \mathcal{C} \) be a compatible item set. \( C \) induces some bin type \( t \), and an incarnation \( j'(i) \) for each \( i \in C \). Let \( d'(i) = \arg\max_d \{ a_{ij'(i)d} / b_{td} \} \), i.e., \( d'(i) \) is a dimension of bin type \( t \) that receives maximum load from (incarnation \( j'(i) \) of) item \( i \). Then

\[
\sum_{i \in C} y_i = \sum_{d=1}^D \sum_{i : d'(i) = d} \frac{\bar{\ell}_i}{D} \\
\leq \frac{1}{D} \sum_{d=1}^D \sum_{i : d'(i) = d} w_t \cdot \bar{\ell}_{ij'(i)d} \\
= \frac{w_t}{D} \sum_{d=1}^D \sum_{i : d'(i) = d} \frac{a_{ij'(i)d}}{b_{td}} \\
\leq \frac{w_t}{D} \sum_{d=1}^D \frac{1}{b_{td}} \cdot b_{td} \\
= w_t ,
\]

where the last inequality follows from the compatibility of \( C \).

Now, since FF computes bin assignments that occupy at most twice the sum of bin sizes, we have that

\[
\text{FF}(I_t) \leq w_t \cdot \max \left\{ 2 \sum_{i \in I_t} \bar{\ell}_i / w_t, 1 \right\} \leq 2 \sum_{i \in I_t} \bar{\ell}_i + w_t .
\]

Hence, for every instance \( \mathcal{I} \) of mvbp we have

\[
\text{APPR}(\mathcal{I}) = \sum_{t=1}^T \text{FF}(I_t) \\
\leq \sum_{t=1}^T \left( 2 \sum_{i \in I_t} \bar{\ell}_i + w_t \right) \\
= 2 \sum_{i \in I} \bar{\ell}_i + \sum_{t=1}^T w_t \\
= 2D \sum_{i \in I} y_i + \sum_{t=1}^T w_t \\
\leq 2D \cdot \text{OPT}^* + \sum_{t=1}^T w_t .
\]
Furthermore, for every $S \subseteq I$ we have
\[
\text{APPR}(C|S) = \sum_{t=1}^{T} FF(S \cap I_t) \leq 2 \sum_{i \in S} \bar{\ell}_i + \sum_{t=1}^{T} w_t = 2D \sum_{i \in S} y_i + \sum_{t=1}^{T} w_t ,
\]
and we are done.

Based on Theorem 2 and Lemma 2 we obtain our main result.

**Theorem 3.** If $D = O(1)$, then there exists a polynomial time algorithm for MVBP with $T$ bin types that computes a solution whose size is at most
\[
(\ln 2D + 1)\text{OPT}^* + \sum_{t=1}^{T} w_t + w_{\text{max}}.
\]

This implies the following result for unweighted MVBP:

**Corollary 4.** If $D = O(1)$, then there exists a polynomial time algorithm for unweighted MVBP with $T$ bin types that computes a solution whose size is at most
\[
(\ln 2D + 1)\text{OPT}^* + T + 1.
\]

Furthermore, if $T = O(1)$, then there exists a polynomial time $(\ln 2D + 1 + \varepsilon)$-approximation algorithm for unweighted MVBP, for every $\varepsilon > 0$.

We also have the following for weighted MVBP.

**Corollary 5.** If $D = O(1)$ and $T = O(\log n)$, then there exists a polynomial time $(\ln 2D + 3)$-approximation algorithm for MVBP.

**Proof.** The result follows from the fact that as we show, we may assume that $\sum_t w_t \leq \text{OPT}$. In this case, due to Theorem 3 we have that the cost of the computed solution is at most
\[
(\ln 2D + 1)\text{OPT}^* + \sum_{t=1}^{T} w_t + w_{\text{max}} \leq (\ln 2D + 3)\text{OPT}.
\]

The above assumption is fulfilled by the following wrapper for our algorithm: Guess which bin types are used in some optimal solution. Iterate through all $2^T - 1$ guesses, and for each guess, compute a solution for the instance that contains only the bin types in the guess. Output the best solution. Since our algorithm computes a $(\ln 2D + 1)$-approximate solution for the right guess, the best solution is also a $(\ln 2D + 3)$-approximation.

**References**

[1] M. M. Akbara, M. S. Rahman, M. Kaykobadb, E. Manninga, and G. Shojaa. Solving the multidimensional multiple-choice knapsack problem by constructing convex hulls. *Computers & Operations Research*, 33:1259–1273, 2006.

[2] N. Bansal, A. Caprara, and M. Sviridenko. Improved approximation algorithms for multidimensional bin packing problems. In *47th IEEE Annual Symposium on Foundations of Computer Science*, pages 697–708, 2006.
[3] A. K. Chandra, D. S. Hirschberg, and C. K. Wong. Approximate algorithms for some generalized knapsack problems. *Theoretical Computer Science*, 3(3):293–304, 1976.

[4] C. Chekuri and S. Khanna. On multidimensional packing problems. *SIAM Journal on Computing*, 33(4):837–851, 2004.

[5] J. R. Correa and L. Epstein. Bin packing with controllable item sizes. *Information and Computation*, 206(8):1003–1016, 2008.

[6] W. Fernandez de la Vega and G. S. Lueker. Bin packing can be solved within 1+epsilons in linear time. *Combinatorica*, 1(4):349–355, 1981.

[7] D. K. Friesen and M. A. Langston. Variable sized bin packing. *SIAM Journal on Computing*, 15(1):222–230, 1986.

[8] A. M. Frieze and M. R. B. Clarke. Approximation algorithms for the m-dimensional 0–1 knapsack problem: worst-case and probabilistic analyses. *European Journal of Operational Research*, 15:100–109, 1984.

[9] M. R. Garey, R. L. Graham, D. S. Johnson, and A. C. Yao. Resource constrained scheduling as generalized bin packing. *J. Comb. Theory, Ser. A*, 21(3):257–298, 1976.

[10] M. Grötschel, L. Lovasz, and A. Schrijver. *Geometric Algorithms and Combinatorial Optimization*. Springer-Verlag, 1988.

[11] M. Hifi, M. Michrafy, and A. Sbihi. Heuristic algorithms for the multiple-choice multidimensional knapsack problem. *Journal of the Operational Research Society*, 55:1323–1332, 2004.

[12] N. Karmarkar and R. M. Karp. An efficient approximation scheme for the one-dimensional bin-packing problem. In 23rd *IEEE Annual Symposium on Foundations of Computer Science*, pages 312–320, 1982.

[13] H. Kellerer, U. Pferschy, and D. Pisinger. *Knapsack Problems*. Springer, Berlin, 2004.

[14] M. S. Khan. *Quality Adaptation in a Multisession Multimedia System: Model, Algorithms and Architecture*. PhD thesis, Dept. of Electrical and Computer Engineering, 1998.

[15] M. J. Magazine and M.-S. Chern. A note on approximation schemes for multidimensional knapsack problems. *Mathematics of Operations Research*, 9(2):244–247, May 1984.

[16] C. H. Papadimitriou and K. Steiglitz. *Combinatorial optimization : algorithms and complexity*. Prentice-Hall, 1981.

[17] R. Parra-Hernández and N. J. Dimopoulos. A new heuristic for solving the multiochoice multidimensional knapsack problem. *IEEE Trans. on Systems, Man, and Cybernetics—Part A: Systems and Humans*, 35(5):708–717, 2005.

[18] S. A. Plotkin, D. B. Shmoys, and É. Tardos. Fast approximation algorithms for fractional packing and covering problems. *Mathematics of Operations Research*, 20:257–301, 1995.

[19] A. Sbihi. A best first search exact algorithm for the multiple-choice multidimensional knapsack problem. *Journal of Combinatorial Optimization*, 13(4):337–351, May 2007.

[20] S. S. Seiden, R. van Stee, and L. Epstein. New bounds for variable-sized online bin packing. *SIAM Journal on Computing*, 32(2):455–469, 2003.

[21] H. Shachnai and T. Tamir. Approximation schemes for generalized 2-dimensional vector packing with application to data placement. In 6th *International Workshop on Approximation Algorithms for Combinatorial Optimization Problems*, number 2764 in LNCS, pages 129–148, 2003.

[22] D. Warner and J. Prawda. A mathematical programming model for scheduling nursing personnel in a hospital. *Manage. Sci. (Application Series Part 1)*, 19:411–422, Dec. 1972.