Uncertain Fuzzy Hermite-Hadamard Type Inequalities for MT\((m,\varphi)\)-Preinvex Functions

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Abstract In this paper, a new class of MT\((m,\varphi)\)-preinvex functions is introduced and some uncertain fuzzy Hermite-Hadamard type inequalities for MT\((m,\varphi)\)-preinvex functions via Riemann-Liouville fractional integrals are established. At the end, some applications to special means are given.

Keywords MT-convex Function, Hölder’s Inequality, Power Mean Inequality, Fuzzy Hermite-Hadamard Inequality, Fuzzy Fractional Riemann-Liouville Operator, \(m\)-invex

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1 Introduction and Preliminaries

In this paper, we denote \(\mathbb{R}\)\(_F\) the set of all fuzzy numbers on \(\mathbb{R}\). Denote \(L_F[\alpha, \beta]\) the space of fuzzy Lebesgue integrable functions on \([\alpha, \beta]\) and \(C_F[\alpha, \beta]\) the space of fuzzy continuous functions on \([\alpha, \beta]\). Also, we use \(I_{\alpha+}^\alpha f\) and \(I_{\beta-}^\beta f\) for fuzzy fractional left and right Riemann-Liouville operators, where \(0 < \alpha \leq 1\).

The following inequality, named Hermite-Hadamard inequality, is one of the most famous inequalities in the literature for convex functions.

**Theorem 1.1.** Suppose \(f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}\) be convex function, \(b, a \in I\), where \(a < b\). Then

\[
f\left(\frac{b + a}{2}\right) \leq \frac{1}{b - a} \int_a^b f(x)dx \leq \frac{f(b) + f(a)}{2}.
\]

Please, (see \([2]\), \([3]\)) and the references cited therein for other recent results. In (see \([4]\)) and the references cited therein, Yidirim and Tunç defined MT-convex function:

**Definition 1.2.** \(f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}\) belong to the class MT\((I)\), if \(f \geq 0\) and \(\forall x, y \in I\) and \(\xi \in (0, 1)\), we have

\[
f(\xi x + (1 - \xi)y) \leq \frac{\sqrt{1 - \xi}}{2\sqrt{\xi}} f(y) + \frac{\sqrt{\xi}}{2\sqrt{1 - \xi}} f(x).
\]

**Definition 1.3.** (see \([1]\)) A set \(\Omega \subseteq \mathbb{R}^n\) is called \(m\)-invex related to \(\eta : \Omega \times \Omega \times (0, 1] \rightarrow \mathbb{R}^n\) for any fixed \(m \in (0, 1]\) if \(m x + \xi\eta(y, x, m) \in \Omega\) holds \(\forall x, y \in \Omega\) and \(\xi \in [0, 1]\).

**Remark 1.4.** In Definition 1.3, under certain conditions, the mapping \(\eta(y, x)\) could reduce to \(\eta(y, x)\).

We next give new definition, to be referred as MT\((m,\varphi)\)-preinvex function.
Definition 1.5. Let $\Omega \subseteq \mathbb{R}^n$ be open $m$-invex set related to $\gamma : \Omega \times \Omega \times (0,1] \longrightarrow \mathbb{R}^n$ and $\varphi : I \longrightarrow \Omega$ a continuous increasing function. For $f : \Omega \longrightarrow \mathbb{R}$ and any fixed $m \in (0,1]$, if
\[
f(m\varphi(y) + \xi\gamma(f(x), \varphi(y), m)) \leq \frac{m\sqrt{r}}{2\sqrt{r-1}} f(f(x)) + \frac{m\sqrt{r-1}}{2\sqrt{r}} f(f(y)),
\]
\[\forall x, y \in I \text{ and } \xi \in (0, 1), \text{ then } f \in MT_{(m,\varphi)}(\Omega) \text{ related to } \gamma.
\]

Remark 1.6. In Definition 1.5, the class $MT_{(m,\varphi)}(\Omega)$ is a generalization of the class MT(I) given in Definition 1.2 on $\Omega = I$ related to $\gamma(f(x), \varphi(y), m) = f(x) - m\varphi(y)$, $\varphi(x) = x$, $\forall x \in \Omega$ and $m = 1$.

Fractional calculus (see [3]) and the references cited therein, was introduced by Riemann and Liouville. Fuzzy Riemann integrals were introduced by Wu (see [5]). Let $r, s \in \mathbb{R}_+$ and $\lambda \in \mathbb{R}$. Define
\[|r \oplus s|^\alpha = [r]^{\alpha} + |s|^{\alpha}, \quad \lambda \ominus r]^\alpha = \lambda [r]^{\alpha}, \quad \forall \alpha \in [0,1],
\]
and $D : \mathbb{R}_+ \times \mathbb{R}_+ \longrightarrow \mathbb{R}_0$ by
\[D(r, s) := \sup_{\alpha \in [0,1]} \max \{|r|^{\alpha} - s^{-\alpha}, |r|^{\alpha} - s^{+\alpha} - \}
\]
where
\[|r|^{\alpha} = [r^{-\alpha}, r^{+\alpha}], \quad r \in \mathbb{R}_+.
\]

It is easy to show that $D$ is a metric on $\mathbb{R}_+$ and ($\mathbb{R}_+, D$) is a complete metric space with the following properties:

1. $D(u \oplus v, w \oplus e) = D(u, v), \quad \forall u, v, w \in \mathbb{R}_+$;
2. $D(k \ominus u, k \ominus v) = |k|D(u, v), \quad \forall u, v \in \mathbb{R}_+, \forall k \in \mathbb{R}$;
3. $D(u \ominus v, w \ominus e) = D(u, w) + D(v, e), \quad \forall u, v, w, e \in \mathbb{R}_+$;
4. $D(u \ominus v, 0) \leq D(u, 0) + D(v, 0), \quad \forall u, v \in \mathbb{R}_+$;
5. $D(u \ominus v, w \ominus 0) \leq D(u, w) + D(v, 0), \quad \forall u, v, w \in \mathbb{R}_+$.

The aim of this paper is to applied the notion of $MT_{(m,\varphi)}$-preinvex function to establish some uncertain fuzzy Hermite-Hadamard type inequalities via Riemann-Liouville fractional integrals. Holder’s inequality and the well-known power mean inequality will be used to find new bounds for uncertain fuzzy Hermite-Hadamard inequalities. At the end, some applications to special means are given.

2 Fractional uncertain fuzzy Hermite-Hadamard for $MT_{(m,\varphi)}$-preinvex functions

In order to prove in this section our main results regarding uncertain fuzzy Hermite-Hadamard type inequalities for $MT_{(m,\varphi)}$-preinvex functions we need the following new lemma:

Lemma 2.1. Let $\varphi : I \longrightarrow \Omega$ be a continuous increasing function. Suppose $\Omega \subseteq \mathbb{R}$ an open $m$-invex subset related to $\eta : \Omega \times \Omega \times (0,1] \longrightarrow \mathbb{R}$ for any fixed $m \in (0,1]$, $\varphi(a), \varphi(b) \in \Omega$, $a < b$, where $m\varphi(a) < m\varphi(a) + \gamma(\varphi(b), \varphi(a), m)$. Let $f : \Omega \longrightarrow \mathbb{R}_+$ be a differentiable function on $\Omega$ and $f' \in \mathbb{C}_f[m\varphi(a), m\varphi(a) + \gamma(\varphi(b), \varphi(a), m)] \cap L_f[m\varphi(a), m\varphi(a) + \gamma(\varphi(b), \varphi(a), m)]$. Then for all $0 < \alpha \leq 1$, we have
\[
\gamma(\varphi(x), \varphi(a), m)^\alpha f(m\varphi(a) - \gamma(\varphi(x), \varphi(b), m)^\alpha f(m\varphi(b))
\]
\[= \frac{\Gamma(\alpha + 1)}{\gamma(\varphi(b), \varphi(a), m)} \left[ I_{m\varphi(a) + \gamma(\varphi(x), \varphi(a), m)} - I_{m\varphi(b) + \gamma(\varphi(x), \varphi(b), m)} - I_{m\varphi(b)} \right]
\]
\[+ \frac{\gamma(\varphi(x), \varphi(a), m)^\alpha + 1}{\gamma(\varphi(b), \varphi(a), m)} (FR) \int_0^1 (t^n - 1) f'(m\varphi(a) + t\gamma(\varphi(x), \varphi(a), m)) dt
\]
\[+ \frac{\gamma(\varphi(x), \varphi(b), m)^\alpha + 1}{\gamma(\varphi(b), \varphi(a), m)} (FR) \int_0^1 (1 - t^n) f'(m\varphi(b) + t\gamma(\varphi(x), \varphi(b), m)) dt,
\]  
(2.1)

where $\Gamma(\alpha) = \int_0^{+\infty} e^{-u} u^{\alpha-1} du$ is the Euler gamma function.
Proof. Denote
\[ I = \frac{\gamma(\varphi(x), \varphi(a), m)^{\alpha+1}}{\gamma(\varphi(b), \varphi(a), m)^{\alpha+1}} \int_0^1 (t^\alpha - 1)f'(m\varphi(a) + t\gamma(\varphi(x), \varphi(a), m)) dt \]
\[ + \frac{\gamma(\varphi(x), \varphi(b), m)^{\alpha+1}}{\gamma(\varphi(b), \varphi(a), m)^{\alpha+1}} \int_0^1 (1 - t^\alpha)f'(m\varphi(b) + t\gamma(\varphi(x), \varphi(b), m)) dt. \]
By integration by parts and using properties of fuzzy numbers the lemma follows.

By using Lemma 2.1, we have the following interesting results.

**Theorem 2.2.** Let \( \varphi : I \rightarrow S \) be a continuous increasing function. Suppose \( S \subseteq \mathbb{R} \) an open \( m \)-invex subset related to \( \eta : S \times S \times [0,1] \rightarrow \mathbb{R} \) for any fixed \( m \in (0,1) \), \( \varphi(a), \varphi(b) \in S \), \( a < b \), where \( m\varphi(a) + \gamma(\varphi(b), \varphi(a), m) \) and \( f : S \rightarrow \mathbb{R} \) be a differentiable function on \( S \), and \( f' \) \( C_F[\varphi(a), \varphi(a) + \gamma(\varphi(b), \varphi(a), m)] \bigcap L_F[\varphi(a), \varphi(a) + \gamma(\varphi(b), \varphi(a), m)] \). If \( D(f'(x), \tilde{0}) \) is a \( MT_{(m, \varphi)} \)-preinvex function on \([m\varphi(a), m\varphi(a) + \gamma(\varphi(b), \varphi(a), m)], \) then for all \( 0 < \alpha \leq 1 \), we have
\[ D \left( \frac{\gamma(\varphi(x), \varphi(a), m)\alpha f(m\varphi(a)) - \gamma(\varphi(x), \varphi(b), m)\alpha f(m\varphi(b))}{\gamma(\varphi(b), \varphi(a), m)} \right) \]
\[ \leq \frac{\Gamma(\alpha + 1)}{\gamma(\varphi(b), \varphi(a), m)} \left[ I_0^{(m\varphi(a) + \gamma(\varphi(x), \varphi(a), m))} - f(m\varphi(a)) - I_0^{(m\varphi(b) + \gamma(\varphi(x), \varphi(b), m))} - f(m\varphi(b)) \right] \]
\[ \leq \frac{1}{\gamma(\varphi(b), \varphi(a), m)} D \left( f'(\varphi(x)), \tilde{0} \right) \]
\[ \times \left[ \frac{\pi}{2} - \frac{\Gamma(\alpha + \frac{1}{2})}{2\Gamma(\alpha + 2)} \right] \left[ \gamma(\varphi(x), \varphi(a), m) \right]^{\alpha+1} + \gamma(\varphi(x), \varphi(b), m) \right]^{\alpha+1} \]
\[ \times \left[ \gamma(\varphi(x), \varphi(a), m) + \gamma(\varphi(x), \varphi(b), m) \right] \right]. \] (2.2)

Proof. Using Lemma 2.1, \( MT_{(m, \varphi)} \)-preinvexity of \( D \left( f'(\varphi(x)), \tilde{0} \right) \),
the fact that \( f' \in C_F[m\varphi(a), m\varphi(a) + \gamma(\varphi(b), \varphi(a), m)] \bigcap L_F[m\varphi(a), m\varphi(a) + \gamma(\varphi(b), \varphi(a), m)] \) and \( 0 < \alpha \leq 1 \), we have
\[ D \left( \frac{\gamma(\varphi(x), \varphi(a), m)\alpha f(m\varphi(a)) - \gamma(\varphi(x), \varphi(b), m)\alpha f(m\varphi(b))}{\gamma(\varphi(b), \varphi(a), m)} \right) \]
\[ \leq \frac{\Gamma(\alpha + 1)}{\gamma(\varphi(b), \varphi(a), m)} \left[ I_0^{(m\varphi(a) + \gamma(\varphi(x), \varphi(a), m))} - f(m\varphi(a)) - I_0^{(m\varphi(b) + \gamma(\varphi(x), \varphi(b), m))} - f(m\varphi(b)) \right] \]
\[ \leq \frac{1}{\gamma(\varphi(b), \varphi(a), m)} D \left( f'(m\varphi(a) + t\gamma(\varphi(x), \varphi(a), m)), \tilde{0} \right) dt \]
\[ + \left| \gamma(\varphi(x), \varphi(b), m) \right|^{\alpha+1} \]
\[ \times \left[ \gamma(\varphi(x), \varphi(a), m) \right] \right]. \]
Corollary 2.3. If we choose \( m = 1 \) and \( \gamma(\varphi(x), \varphi(y)) = \varphi(x) - m\varphi(y) \) in Theorem 2.2, then inequality (2.2) reduces to

\[
D \left( \frac{(\varphi(x) - \varphi(a))^\alpha f(\varphi(a)) - (\varphi(x) - \varphi(b))^\alpha f(\varphi(b))}{\varphi(b) - \varphi(a)} \right) + \frac{\Gamma(\alpha + 1)}{\varphi(b) - \varphi(a)} \left[ I_{\varphi(x)}^\alpha f(\varphi(a)) - I_{\varphi(x)}^\alpha f(\varphi(b)) \right] 
\]

\[
\leq D \left( f'(\varphi(x)), \tilde{0} \right) + \frac{1}{2(\varphi(b) - \varphi(a))} \left[ \frac{\pi}{2} - \frac{\Gamma(\alpha + \frac{1}{2}) \Gamma(\frac{1}{2})}{\Gamma(\alpha + 2)} \left[ (\varphi(x) - \varphi(a))^{\alpha + 1} + (\varphi(b) - \varphi(x))^{\alpha + 1} \right] \right] 
\]

\[
\times \left[ (\varphi(x) - \varphi(a))^{\alpha + 1} D \left( f'(\varphi(a)), \tilde{0} \right) + (\varphi(b) - \varphi(x))^{\alpha + 1} D \left( f'(\varphi(b)), \tilde{0} \right) \right].
\]

Theorem 2.4. Let \( \varphi : I \to S \) be a continuous increasing function. Suppose \( S \subseteq \mathbb{R} \) be an open \( m \)-invex subset related to \( \eta : S \times S \times [0, 1] \to \mathbb{R} \) for any fixed \( m \in (0, 1) \), \( \varphi(a), \varphi(b) \in S, a < b \), where \( m\varphi(a) < m\varphi(a) + \gamma(\varphi(b), (a), m) \). Let \( f : S \to \mathbb{R} \) be a differentiable function on \( S^* \), and \( f' \in C_\\mathcal{F}[m\varphi(a), m\varphi(a) + \gamma(\varphi(b), (a), m)] \cap L_\\mathcal{F}[m\varphi(a), m\varphi(a) + \gamma(\varphi(b), (a), m)] \). If \( D \left( f'(\varphi(x)), \tilde{0} \right)^q \) is a \( MT_{(m, \varphi)} \)-preinvex function on \( [m\varphi(a), m\varphi(a) + \gamma(\varphi(b), (a), m)] \), \( q > 1, p^{-1} + q^{-1} = 1, \) then for all \( 0 < \alpha \leq 1 \), we have

\[
D \left( \frac{(\varphi(x) - \varphi(a))^\alpha f(m\varphi(a)) - (\varphi(x) - \varphi(b))^\alpha f(m\varphi(b))}{\gamma(\varphi(b), \varphi(a), m)} \right) 
\]

\[
\leq \left( \frac{\pi}{4} \right)^{\frac{1}{q}} \left[ \frac{\Gamma(\alpha + 1)}{\gamma(\varphi(b), \varphi(a), m)} \right]^{\frac{1}{q}} \gamma(\varphi(x), \varphi(a), m)^{\alpha + 1} \left[ D \left( f'(\varphi(x)), \tilde{0} \right)^q + D \left( f'(\varphi(a)), \tilde{0} \right)^q \right]^{\frac{1}{q}} 
\]

\[
+ \frac{\Gamma(\alpha + 1)}{\gamma(\varphi(b), \varphi(a), m)} \gamma(\varphi(x), \varphi(b), m)^{\alpha + 1} \left[ D \left( f'(\varphi(x)), \tilde{0} \right)^q + D \left( f'(\varphi(b)), \tilde{0} \right)^q \right]^{\frac{1}{q}} 
\]

(2.3)

Proof. Let \( q > 1 \). Using Lemma 2.1, \( MT_{(m, \varphi)} \)-preinvexity of \( D \left( f'(\varphi(x)), \tilde{0} \right)^q \), the fact that \( f' \in C_\\mathcal{F}[m\varphi(a), m\varphi(a) + \gamma(\varphi(b), (a), m)] \cap L_\\mathcal{F}[m\varphi(a), m\varphi(a) + \gamma(\varphi(b), (a), m)] \) and Hölder’s inequality, then for all \( 0 < \alpha \leq 1 \), we have

\[
D \left( \frac{(\varphi(x) - \varphi(a))^\alpha f(m\varphi(a)) - (\varphi(x) - \varphi(b))^\alpha f(m\varphi(b))}{\gamma(\varphi(b), \varphi(a), m)} \right) 
\]

\[
\leq \frac{\Gamma(\alpha + 1)}{\gamma(\varphi(b), \varphi(a), m)} \left[ I_{(m\varphi(a) + \gamma(\varphi(x), \varphi(a), m))}^\alpha f(m\varphi(a)) - I_{(m\varphi(b) + \gamma(\varphi(x), \varphi(b), m))}^\alpha f(m\varphi(b)) \right] 
\]

\[
\leq \frac{|\gamma(\varphi(x), \varphi(a), m)|^{\alpha + 1}}{|\gamma(\varphi(b), \varphi(a), m)|} \int_0^1 t^{\alpha} - 1 |D \left( f'(m\varphi(a) + t\gamma(\varphi(x), \varphi(a), m)), \tilde{0} \right) dt 
\]

\[
+ \frac{|\gamma(\varphi(x), \varphi(b), m)|^{\alpha + 1}}{|\gamma(\varphi(b), \varphi(a), m)|} \int_0^1 |1 - t^{\alpha} |D \left( f'(m\varphi(b) + t\gamma(\varphi(x), \varphi(b), m)), \tilde{0} \right) dt 
\]
Theorem 2.6. Let 

\[ \frac{\gamma(\varphi(x), \varphi(a), m)}{|\gamma(\varphi(b), \varphi(a), m)|} \left( \int_0^1 (1 - t^\alpha)^p dt \right)^{\frac{1}{p}} \left( \int_0^1 D \left( f'(m\varphi(a) + t\gamma(\varphi(x), \varphi(a), m)), \tilde{0} \right)^q dt \right)^{\frac{1}{q}} \]

\[ + \frac{|\gamma(\varphi(x), \varphi(b), m)|^{\alpha+1}}{|\gamma(\varphi(b), \varphi(a), m)|} \left( \int_0^1 (1 - t^\alpha)^p dt \right)^{\frac{1}{p}} \left( \int_0^1 D \left( f'(m\varphi(b) + t\gamma(\varphi(x), \varphi(b), m)), \tilde{0} \right)^q dt \right)^{\frac{1}{q}} \]

\[ \leq \frac{|\gamma(\varphi(x), \varphi(a), m)|^{\alpha+1}}{|\gamma(\varphi(b), \varphi(a), m)|} \left( \int_0^1 (1 - t^\alpha)^p dt \right)^{\frac{1}{p}} \]

\[ \times \left[ \int_0^1 \left( \frac{m\sqrt{t}}{2\sqrt{1 - t}} D \left( f'(\varphi(x)), \tilde{0} \right)^q + \frac{m\sqrt{1 - t}}{2\sqrt{t}} D \left( f'(\varphi(a)), \tilde{0} \right)^q dt \right]^{\frac{1}{q}} \]

\[ + \frac{|\gamma(\varphi(x), \varphi(b), m)|^{\alpha+1}}{|\gamma(\varphi(b), \varphi(a), m)|} \left( \int_0^1 (1 - t^\alpha)^p dt \right)^{\frac{1}{p}} \left( \int_0^1 D \left( f'(\varphi(b)), \tilde{0} \right)^q dt \right)^{\frac{1}{q}} \]

\[ \leq \frac{\Gamma(\alpha + 1)}{\varphi(b) - \varphi(a)} \left[ f'_{\varphi(x)} - f(\varphi(a)) - f'_{\varphi(x)} - f(\varphi(b)) \right] \]

\[ \leq \left( \frac{\alpha}{4} \right)^{\frac{1}{4}} \frac{1}{\varphi(b) - \varphi(a)} \left[ \frac{\Gamma(p + 1)\Gamma(\frac{1}{2})}{\Gamma(p + \frac{1}{2})} \right]^{\frac{1}{2}} \]

\[ \times \left\{ (\varphi(x) - \varphi(a))^{\alpha+1} \left[ D \left( f'(\varphi(x)), \tilde{0} \right)^q + D \left( f'(\varphi(a)), \tilde{0} \right)^q \right]^{\frac{1}{q}} \right\} \]

\[ + (\varphi(b) - \varphi(x))^{\alpha+1} \left[ D \left( f'(\varphi(x)), \tilde{0} \right)^q + D \left( f'(\varphi(b)), \tilde{0} \right)^q \right]^{\frac{1}{q}} \} \right\}. \]

Corollary 2.5. If we choose \( m = 1 \) and \( \gamma(\varphi(x), \varphi(y), m) = \varphi(x) - m\varphi(y) \) in Theorem 2.4, then inequality (2.3) reduces to

\[ \frac{D}{\varphi(b) - \varphi(a)} \left( \frac{(\varphi(x) - \varphi(a))^\alpha f(\varphi(a)) - (\varphi(x) - \varphi(b))^\alpha f(\varphi(b))}{\varphi(b) - \varphi(a)} \right) \]

\[ \leq \left( \frac{\alpha}{4} \right)^{\frac{1}{4}} \frac{1}{\varphi(b) - \varphi(a)} \left[ \frac{\Gamma(p + 1)\Gamma(\frac{1}{2})}{\Gamma(p + \frac{1}{2})} \right]^{\frac{1}{2}} \]

\[ \times \left\{ (\varphi(x) - \varphi(a))^{\alpha+1} \left[ D \left( f'(\varphi(x)), \tilde{0} \right)^q + D \left( f'(\varphi(a)), \tilde{0} \right)^q \right]^{\frac{1}{q}} \right\} \]

\[ + (\varphi(b) - \varphi(x))^{\alpha+1} \left[ D \left( f'(\varphi(x)), \tilde{0} \right)^q + D \left( f'(\varphi(b)), \tilde{0} \right)^q \right]^{\frac{1}{q}} \} \right. \}

Theorem 2.6. Let \( \varphi : I \rightarrow S \) be a continuous increasing function. Suppose \( S \subseteq \mathbb{R} \) be an open m-invex subset related to \( \eta : S \times S \times [0, 1] \rightarrow \mathbb{R} \) for any fixed \( m \in (0, 1] \). Let \( f : S \rightarrow \mathbb{R} \) be a differentiable function on \( S^\circ \), and \( f' \in C_f[m\varphi(a), m\varphi(a) + \gamma(\varphi(b), \varphi(a), m)] \right\} \right. \}

\[ \frac{D}{\gamma(\varphi(b), \varphi(a), m)} \left( \frac{(\varphi(x) - \varphi(a))^\alpha f(\varphi(a)) - (\varphi(x) - \varphi(b))^\alpha f(\varphi(b))}{\gamma(\varphi(b), \varphi(a), m)} \right) \]
\[
\frac{\Gamma(\alpha + 1)}{\gamma(\varphi(b), \varphi(a), m)} \left[ P_{(m\varphi(a) + \gamma(\varphi(x), \varphi(a), m))} - f(m\varphi(a)) \right] - \frac{\alpha}{2} \left[ \frac{\Gamma\left(\frac{\alpha}{2}\right)}{\Gamma(\alpha + 1)} \right] \left[ D\left(f'\left(\varphi(x), \tilde{0}\right)\right) \right]^q \left( \frac{\pi}{2} - \frac{\Gamma\left(\frac{\alpha + 1}{2}\right)}{\Gamma(\alpha + 1)} \right) \\
+ D\left(f'\left(\varphi(a), \tilde{0}\right)\right) \left( \frac{\pi}{2} - \frac{\Gamma\left(\frac{\alpha + 1}{2}\right)}{\Gamma(\alpha + 1)} \right) \right]^{\frac{q}{2}} \\
+ D\left(f'\left(\varphi(b), \tilde{0}\right)\right) \left( \frac{\pi}{2} - \frac{\Gamma\left(\frac{\alpha + 1}{2}\right)}{\Gamma(\alpha + 1)} \right) \right]^{\frac{q}{2}}.
\]
Definition 3.1. (see [6]) A function $3$ Applications to special means

Remark 2.8

If we choose $Corollary 2.7.

1. Homogeneity: $M$
2. Symmetry: $x$
3. Reflexivity: $min$
5. Internality: $\in$

In the following we give certain generalizations of some notions for a positive valued function of a positive variable.

Let $\rho = \frac{\alpha}{\alpha + 1}$

Corollary 2.7. If we choose $m = 1$ and $\gamma(x, \varphi(y), m) = \varphi(x) - m \varphi(y)$ in Theorem 2.6, then inequality (2.4) reduces to

$$D \left( \frac{(\varphi(x) - \varphi(a))^\alpha f(\varphi(a)) - (\varphi(x) - \varphi(b))^\alpha f(\varphi(b))}{\varphi(b) - \varphi(a)} \right) \leq \left( \frac{1}{2} \right)^{\frac{1}{q}} \left( \frac{\alpha}{\alpha + 1} \right)^{1 - \frac{1}{q}} \left( \frac{\varphi(x) - \varphi(a))^{\alpha + 1}}{\varphi(b) - \varphi(a)} \right) \left[ D \left( f'(\varphi(x)), \tilde{0} \right) \right]^q \left( \frac{\pi}{2} - \frac{\Gamma \left( \alpha + \frac{1}{2} \right) \Gamma \left( \frac{1}{2} \right)}{2 \Gamma(\alpha + 1)} \right) \right] \frac{1}{q}$$

Remark 2.8. Let $M \in \mathbb{R}, q \geq 1$. For $D(f'(\varphi(x)), \tilde{0}) \leq M$ or $D(f'(\varphi(x)), \tilde{0})^q \leq M$, by our theorems mentioned we get some special kinds of fuzzy Hermite-Hadamard type inequalities.

3 Applications to special means

In the following we give certain generalizations of some notions for a positive valued function of a positive variable.

Definition 3.1. (see [6]) A function $M : \mathbb{R}^2_+ \rightarrow \mathbb{R}_+$, is called a Mean function if it has the following properties:

1. Homogeneity: $M(ax, ay) = aM(x, y)$, for all $a > 0$,
2. Symmetry: $M(x, y) = M(y, x)$,
3. Reflexivity: $M(x, x) = x$,
4. Monotonicity: If $x \leq x'$ and $y \leq y'$, then $M(x, y) \leq M(x', y')$,
5. Internality: $\min \{x, y\} \leq M(x, y) \leq \max \{x, y\}$.

We consider some means for arbitrary positive real numbers $\alpha, \beta$ ($\alpha \neq \beta$).

1. The arithmetic mean:

$$A := A(\alpha, \beta) = \frac{\alpha + \beta}{2}$$
2. The geometric mean:
\[ G := G(\alpha, \beta) = \sqrt{\alpha \beta} \]

3. The harmonic mean:
\[ H := H(\alpha, \beta) = \frac{2}{\frac{1}{\alpha} + \frac{1}{\beta}} \]

4. The power mean:
\[ P_r := P_r(\alpha, \beta) = \left( \frac{\alpha^r + \beta^r}{2} \right)^{\frac{1}{r}}, \quad r \geq 1. \]

5. The identric mean:
\[ I := I(\alpha, \beta) = \left\{ \begin{array}{cl} \frac{1}{e} \left( \frac{\beta^\alpha}{\alpha^\beta} \right), & \alpha \neq \beta; \\ \alpha, & \alpha = \beta. \end{array} \right. \]

6. The logarithmic mean:
\[ L := L(\alpha, \beta) = \frac{\beta - \alpha}{\ln \beta - \ln \alpha}; \quad |\alpha| \neq |\beta|, \quad \alpha \beta \neq 0. \]

7. The generalized log-mean:
\[ L_p := L_p(\alpha, \beta) = \left[ \frac{\beta^{p+1} - \alpha^{p+1}}{(p+1)(\beta - \alpha)} \right]^{\frac{1}{p}}, \quad p \in \mathbb{R} \setminus \{-1, 0\}, \quad \alpha \neq \beta. \]

8. The weighted p-power mean:
\[ M_p \left( \alpha_1, \alpha_2, \ldots, \alpha_n; u_1, u_2, \ldots, u_n \right) = \left( \sum_{i=1}^{n} \alpha_i u_i^p \right)^{\frac{1}{p}} \]

where \( 0 \leq \alpha_i \leq 1, \ u_i > 0 \) (\( i = 1, 2, \ldots, n \)) with \( \sum_{i=1}^{n} \alpha_i = 1 \).

It is well known that \( L_p \) is monotonic nondecreasing over \( p \in \mathbb{R} \) with \( L_{-1} := L \) and \( L_0 := I \). In particular, we have the following inequality \( H \leq G \leq L \leq I \leq A \). Now, let \( a \) and \( b \) be positive real numbers such that \( a < b \). Consider the function \( M := M(\varphi(a), \varphi(b)) : [\varphi(a), \varphi(a) + \gamma(\varphi(b), \varphi(a))] \times [\varphi(a), \varphi(a) + \gamma(\varphi(b), \varphi(a))] \to \mathbb{R}_+ \), which is one of the above mentioned means and \( \varphi : I \to \mathbb{R} \) be a continuous increasing function, therefore one can obtain various inequalities using the results of Section 2 for these means as follows:

Replace \( \gamma(\varphi(x), \varphi(y), m) \) with \( \gamma(\varphi(x), \varphi(y)) \) and setting \( \gamma(\varphi(x), \varphi(b)) = M(\varphi(x), \varphi(b)), \gamma(\varphi(x), \varphi(a)) = M(\varphi(x), \varphi(a)), \gamma(\varphi(x), \varphi(b)) = M(\varphi(x), \varphi(b)), \forall x \in I \), for value \( m = 1 \) in (2.2), (2.3) and (2.4), one can obtain the following interesting inequalities involving means:

\[ \frac{\Gamma(\alpha + 1)}{M(\varphi(a), \varphi(b))} \left[ \frac{\Gamma(\alpha) + \Gamma(\varphi(x), \varphi(b))}{2} \right] \left[ M(\varphi(x), \varphi(a))^{\alpha+1} + M(\varphi(x), \varphi(b))^{\alpha+1} \right] \]
\[ \times \left[ M(\varphi(x), \varphi(a))^{\alpha+1} D \left( f'(\varphi(a)) \right) + M(\varphi(x), \varphi(b))^{\alpha+1} D \left( f'(\varphi(b)) \right) \right], \]

(3.1)
\[
\frac{\Gamma(\alpha + 1)}{M(\varphi(a), \varphi(b))} \left[ I_{(\varphi(a) + M(\varphi(x), \varphi(a)))}^\alpha f(\varphi(a)) - I_{(\varphi(b) + M(\varphi(x), \varphi(b)))}^\alpha f(\varphi(b)) \right]
\leq \left( \frac{\pi}{4} \right)^{\frac{1}{2}} M(\varphi(a), \varphi(b)) \frac{1}{\alpha} \left[ \Gamma(p + 1) \Gamma \left( \frac{1}{p} \right) \right]^{\frac{1}{2}}
\times \left\{ M(\varphi(x), \varphi(a))^\alpha + M(\varphi(x), \varphi(b))^{\alpha + 1} \left[ D \left( f'(\varphi(x)), \tilde{0} \right)^q + D \left( f'(\varphi(a)), \tilde{0} \right)^q \right] \right\}^{\frac{1}{2}},
\]
\[
\left(3.2\right)
\]
\[
\frac{\Gamma(\alpha + 1)}{M(\varphi(a), \varphi(b))} \left[ I_{(\varphi(a) + M(\varphi(x), \varphi(a)))}^\alpha f(\varphi(a)) - I_{(\varphi(b) + M(\varphi(x), \varphi(b)))}^\alpha f(\varphi(b)) \right]
\leq \left( \frac{1}{2} \right)^{\frac{1}{2}} \left( \frac{\alpha}{\alpha + 1} \right)^{1 - \frac{1}{2}} M(\varphi(x), \varphi(a))^{\alpha + 1} \left[ D \left( f'(\varphi(x)), \tilde{0} \right)^q \left( \frac{\pi}{2} - \frac{\Gamma \left( \alpha + \frac{3}{2} \right) \Gamma \left( \frac{1}{2} \right)}{\Gamma(\alpha + 2)} \right) \right]^{\frac{1}{2}}
\times \left\{ D \left( f'(\varphi(x)), \tilde{0} \right)^q \left( \frac{\pi}{2} - \frac{\Gamma \left( \alpha + \frac{3}{2} \right) \Gamma \left( \frac{1}{2} \right)}{\Gamma(\alpha + 2)} \right) \right\}^{\frac{1}{2}}.
\]
\[
\left(3.3\right)
\]
Letting \( M(\varphi(x), \varphi(y)) = A, G, H, P_r, I, L, L_p, M_p, \forall x, y \in I \) in (3.1), (3.2) and (3.3), we get the inequalities involving means for a particular choices of a differentiable MT(1,\varphi)-preinvex function \( f \). The details are left to the interested reader.

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