ORBIT REDUCTION OF EXTERIOR DIFFERENTIAL SYSTEMS,
AND GROUP-INVAR IA NT VARIATIONAL PROBLEMS

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ABSTRACT. For a given PDE system (or an exterior differential system) pos-
sessing a Lie group of internal symmetries the orbit reduction procedure is
introduced. It is proved that the solutions of the reduced exterior differential
system are in one-to-one correspondence with the moduli space of regular
solutions of the original system.

The isomorphism between the local characteristic cohomology of the re-
duced unconstrained jet space and the Lie algebra cohomology of the symmetry
group is established.

The group-invariant Euler-Lagrange equations of an invariant variational
problem are described as a composition of the Euler-Lagrange operators on
the reduced jet space and certain other differential operators on the reduced
jet space. The practical algorithm of computing these operators is given.

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1. INTRODUCTION.

In this paper we introduce and begin to study the orbit reduction of exterior
differential systems.

Recall, that an exterior differential system is a pair \((M, \mathcal{I})\) where \(M\) is a
manifold, and \(\mathcal{I} \subset \bigwedge T^* M\) is a graded differentially closed ideal. It is a geometrical
generalization of partial differential equations (in this case \(M\) is a submanifold of the
jet space and \(\mathcal{I}\) is the contact ideal). The category of exterior differential systems
is bigger then the category of partial differential equations. It can be shown \(^1\) that the category of partial differential equations is not closed under the operation of the orbit reduction (to be described below). This gives yet another reason for considering exterior differential systems.

Let \(\mathcal{E} = (\Delta, T)\) be a system of partial differential equations, or more generally, an exterior differential system, invariant under the action of a group \(G\) of internal symmetries. The action of \(G\) on \(\mathcal{E}\) induces a \(G\)-action on the space \(\text{Sol}(\mathcal{E})\) of the solutions of \(\mathcal{E}\). Let \(\mathcal{E}^{(r)} = (\Delta^{(r)}, T^{(r)})\) be the \(r\)-th order prolongation of \(\mathcal{E}\). The orbit space \(\Delta^{(r)}/G\) possesses the structure of an exterior differential system induced by the structure of \(\mathcal{E}^{(r)}\).

It turns out (see Theorem 2) that for high enough order \(r\) of prolongation the solutions of the reduced system are in one-to-one correspondence with the moduli space \(\text{Sol}(\mathcal{E})/G\) of almost all solutions of the original system. This motivates the studying of a group-invariant PDE system through the study of its reduced exterior differential system.

All the results of the present paper are proved for the case of finite-dimensional Lie group actions. However we believe that the same results remain valid for the case of real-analytic actions of infinite-dimensional groups. The infinite-dimensional group action version of Theorem 2 also suggests a new approach of studying moduli spaces of any locally defined geometrical objects. This will be addressed in some other paper.

The other important reason for studying the orbit reduction is the inverse problem of reduction. By inverse reduction we mean the following. Given a certain system of nonlinear PDEs one may ask a question whether it is an orbit reduction of a different system of PDEs that has a simpler structure. The questions about the solutions of the original system translate into questions about the solutions of the "simpler" system. For example it would be interesting to identify the class of PDEs which are the orbit reduction of an unconstrained jet space. In this case knowing the inverse reduction gives the general solution of the original equations.

As the very first step towards the understanding the inverse reduction, we establish the isomorphism between the local characteristic cohomology of the reduced jet space and the Lie algebra cohomology of a Lie group of contact transformations acting on the jet space (see Theorem 3 in this paper). This in particular, implies that in order to realize a PDE system having an infinite-dimensional characteristic cohomology as an orbit reduction of a jet space one needs to consider actions of infinite-dimensional groups.

The other purpose of the present paper is to understand the group-invariant variational problems via the orbit reduction. As first observed by Sophus Lie [13], the Euler-Lagrange equations of every invariant variational problem can be written in terms of the differential invariants of the group action. In other words, the Euler-Lagrange equations of a group-invariant variational problem can be pushed forward to the orbit space. Surprisingly, up to date there was no general understanding of the meaning of the pushed forward equations on the orbit space, nor there was a general algorithm of producing the group-invariant Euler-Lagrange equations.

The reduced jet space has its own calculus of variations (for example Euler-Lagrange operators), that can be interpreted as a calculus of variations with constraints imposed by the syzygies of the differential invariants. It is well-understood

\(^1\)See Example 2.5 in this paper.
that all the basic ingredients of such calculus of variations come from the edge complex of the corresponding Vinogradov spectral sequence \[\text{[16]}\]. We show (see Theorem \[\text{[4]}\] below ) that for every invariant variational problem the push-forward of the invariant Euler-Lagrange equations onto the orbit space is a composition of the Euler-Lagrange operators on the reduced jet space and certain other differential operators. These other differential operators come from the morphism of the two Vinogradov spectral sequences of the original and the reduced jet spaces. We also give an explicit algorithm for computing these differential operators.

Here we would like to note that an alternative approach based on the Cartan’s moving frame method is used by I. Kogan, and P. Olver \[\text{[9]}\] for computing invariant Euler-Lagrange equations.

2. Reduced Exterior Differential Systems

2.1. Preliminaries: EDS and PDEs. All the geometrical objects considered in this paper are of class $C^\infty$ unless stated otherwise. All the considered manifolds are paracompact.

Let $I = \bigcup_{x \in M} I_x$ be a collection of homogeneous ideals

$$I_x = \bigoplus_{n=1}^{\dim(M)} T^n_x M \subset \bigwedge T^*_x M$$

in the graded exterior algebra $\bigwedge T^*_x M$. We shall say that a differential form $\omega \in \Omega^n(M)$ is a section of $I$ ($\omega \in \Gamma(I)$) if for every $x \in M$, $\omega(x) \in I_x$. The sections of $I$ form a differential ideal if $d\Gamma(I) \subset \Gamma(I)$. We shall assume that $\Gamma(I)$ does not contain any functions except zero.

**Definition 2.1.** We shall say that $\mathcal{E} = (M, I)$ is an Exterior Differential System (or EDS for short) if the space of sections of $I$ is a differential ideal, and there exists a closed subset $X \subset M$ of zero measure, such that for every connected component $U \subset (M \setminus X)$ $I|_U = \bigcup_{x \in U} I_x$ is a subbundle of $\bigwedge T^*U$.

In practice it is convenient to define $I$ by the generators of $\Gamma(I)$. We shall say that $\Gamma(I)$ is generated by the forms $\omega_1, \ldots, \omega_N$ (the notation is $I = \langle \omega_1, \ldots, \omega_N \rangle$) if for every $\omega \in \Gamma(I)$ there exist forms $\alpha_i, \beta_i \in \Omega(M)$ such that $\omega = \sum_{i=1}^N (\omega_i \wedge \alpha_i + d\omega_i \wedge \beta_i)$.

**Definition 2.2.** A $k$-dimensional solution of $\mathcal{E} = (M, I)$ is a connected $k$-dimensional submanifold $S \hookrightarrow M$, such that the pullback of $\Gamma(I)$ to $S$ is zero.

**Example 2.1. Jet spaces.** Let $N$ be a manifold. Consider the $r$-th order jet space $J^r_k N \rightarrow N$ of $k$-dimensional submanifolds together with the standard contact ideal $\mathcal{C}^{(r)} \subset \bigwedge T^* J^r_k N$ (see for example \[\text{[3]}\]). For every $k$-dimensional submanifold $S \rightarrow N$ there is a natural lift $j^r i_S : S \rightarrow J^r_k N$ such that $\theta \in \Gamma(\mathcal{C}^{(r)})$ if and only if the pullback of $\theta$ by $j^r i_S$ is zero for every $k$-dimensional submanifold $S$. The lifts $j^r S = j^r i_S(S)$ are the solutions of the EDS ($J^r_k N, \mathcal{C}^{(r)}$).

---

2 By saying that the ideal $I_x$ is homogeneous we mean that in the homogeneous-degree decomposition $\omega_x = \omega_1^x + \ldots + \omega_{\dim(M)}^x$ of $\omega \in I_x$ every homogeneous element $\omega_i^x \in \bigwedge^i T^* M$ belongs to the ideal.
Example 2.2. PDE systems. Let $\Delta \hookrightarrow J^1_k N$ be a subbundle of the jet space $J^1_k N$. This subbundle can be thought of as a system of partial differential equations, whose solutions are $k$-dimensional submanifolds $S \hookrightarrow N$ such that $j^r S \subset \Delta$. The lifts $j^r S$ of the solutions of $\Delta$ are the solutions of the EDS $E = (\Delta, \iota^* C(r))$.

Note that since the contact ideals on the jet spaces are always generated by one-forms, not every EDS is described by the last example. However the prolongation of every EDS is a first-order PDE system.

Recall that a $(k$-dimensional) prolongation $\mathfrak{I}$ of $E = (M, \mathcal{I})$ is an EDS $E^{(1)}(k) = (M^{(1)}, \mathcal{I}^{(1)}_k)$, where $M^{(1)}$ is a set of all $k$-dimensional planes in $T M$ annihilating the ideal $\mathcal{I}$:

$$M^{(1)}_k = \{(P, z) | z \in M, P \subset T_z M, \dim(P) = k, \text{ and } \mathcal{I}_z |_P = 0 \} \overset{i} \hookrightarrow J^1_k M,$$

$$\mathcal{I}^{(1)}_k = i^*_k C(1).$$

We shall always assume that $\pi^1 : M^{(1)}_k \rightarrow M$ is a smooth fiber bundle. Sometimes it will mean that we remove some closed subset from $M^{(1)}_k$ to make it smooth.

For every $k$-dimensional solution $S \hookrightarrow M$ its lift $j^1 S \hookrightarrow J^1_k M$ is a submanifold of $M^{(1)}_k$, and is a solution of the prolonged EDS $E^{(1)}_k$. Conversely, given a solution $S_1 \hookrightarrow M^{(1)}_k$ of the prolonged EDS the natural projection $\pi^1(S_1) \subset M$ is a solution of the original EDS. However this projection may "lose" some of its dimension, and may happen not to be a smooth manifold anymore.

Example 2.3. The prolongation of $(J^1_k N, C(r))$ is $(J^{r+1}_k N, C^{(r+1)})$

Example 2.4. Prolongation of PDE systems. Consider the Example 2.2. Denote by $\pi^r : J^r_k N \rightarrow N$ the natural projection. For each small enough open neighborhood $U \subset J^r_k N$ we may introduce local coordinates $x^1, \ldots, x^k, u^1, \ldots, u^q$ in $\pi^r(U) \simeq \mathbb{R}^{k+q}$ (this actually means that we artificially impose a structure of a fiber bundle $\pi^r(U) \rightarrow \mathbb{R}^k$). This choice of the coordinates on the base $N$ induces the canonical jet coordinates (see for example [3, 2]) $(x^i, u^\alpha, u^\alpha_{\nu})$ (here $J = (J_1, \ldots, J_{|J|})$ is a multiindex of length $|J| \leq r$). The contact ideal $C(r)$ is generated by the following 1-forms:

$$(1) \quad \{ \theta^\nu_{\alpha} = du^\nu_{\alpha} - u^\alpha_{\nu} dx^\nu \}_{|J| \leq r}$$

Any subbundle $\Delta \hookrightarrow J^r_k N$ can be represented as a zero level set of functions $\Delta_{\nu} \in C^\infty(J^r_k N)$. Denote by $\frac{d}{dx^i} : C^\infty(J^r_k N) \rightarrow C^\infty(J^{r+1}_k N)$ the total derivatives w.r.t. $x^i$. The PDE system

$$\Delta^{(1)} \overset{\text{def}}{=} \{ \Delta_{\nu} = 0, \quad \frac{d}{dx^i} \Delta_{\nu} = 0 \} \overset{i} \hookrightarrow J^{r+1}_k N$$

is called a prolongation of $\Delta$ (see for example [10, 12]).

Lemma 2.3. Let $\Delta \hookrightarrow J^r_k N$ be a subbundle of $J^r_k N \rightarrow N$, then the prolongation of $E = (\Delta, \iota^* C(r))$ is $E^{(1)}_k = (\Delta^{(1)}, \iota^* C^{(r+1)})$.

The proof of this lemma is analogous to the proof for the case $r = 1$, given in [3] (Example 6.3, pages 153-154).

The prolongation of an EDS can be iterated thus giving a prolongation tower

$$M \leftarrow M^{(1)}_k \leftarrow M^{(2)}_k \leftarrow \cdots \leftarrow M^{(\infty)}_k,$$
where \( M^{(\infty)}_k = \lim_{r \to \infty} M^{(r)}_k \) is the inverse limit. The last lemma furnishes the following

**Corollary 2.4.** Every prolongation tower of an EDS \( \mathcal{E} = (M, \mathcal{I}) \) can be viewed as a prolongation tower of a first-order PDE system \( M^{(1)}_k \to J^1_k M \). In particular, we have the natural embeddings \( \iota_r : M^{(r)}_k \to J^r_k M \), such that \( \iota_r^* \mathcal{C}^{(r)} = \mathcal{I}^{(r)}_k \).

### 2.2. The reduced EDS

Let \( \mathcal{G} \) be some pseudogroup of local diffeomorphisms acting on a manifold \( M \). We shall say that an EDS \( \mathcal{E} = (M, \mathcal{I}) \) is \( \mathcal{G} \)-invariant if for every \( x \in M \), and \( g \in \mathcal{G} \)

\[
(2) \quad g^* \mathcal{I}_x = \mathcal{I}_x
\]

**Remark 2.5.** If \( \mathcal{E} = (\Delta, \iota^* \mathcal{C}^{(r)}) \) as in Example 2.3 then the symmetry group \( \mathcal{G} \) is usually called a group of internal symmetries \([17, 18]\).

We shall always assume that the orbit space \( \overline{M} \overset{def}{=} M/\mathcal{G} \) is again a differentiable manifold (in what follows we shall always denote the orbit spaces by barred symbols). The local coordinates on \( \overline{M} \) may be identified with the \( G \)-invariant functions on \( M \). The local coordinates on \( M^{(r)}_k \) are usually called the differential invariants of order \( r \) of the \( G \)-action.

**Proposition 2.6.** Let \( \mathcal{E} = (M, \mathcal{I}) \) be a \( \mathcal{G} \)-invariant exterior differential system, then there exists an exterior differential system \( \overline{\mathcal{E}} = (\overline{M}, \overline{\mathcal{I}}) \), such that \( \overline{\mathcal{I}} \) is the maximal ideal satisfying

\[
(3) \quad p^* \overline{\mathcal{I}}_{p(x)} \subset \mathcal{I}_x \quad \forall x \in M,
\]

where \( p : M \to \overline{M} = M/\mathcal{G} \) is the natural projection.

**Definition 2.7.** We shall call \( \overline{\mathcal{E}} = (\overline{M}, \overline{\mathcal{I}}) \) the reduced EDS.

We will use in the proof the following simple fact

**Lemma 2.8.** Let \( \mathcal{E} \to B \) be a vector bundle over a manifold \( B \), and \( \mathcal{E}_i \subset \mathcal{E} \), \( (i = 1, 2) \) be two subbundles of \( \mathcal{E} \). Then there exists a closed subset \( X \subset B \) of zero measure such that for every connected component \( U \subset (B \setminus X) \), \( \mathcal{E}_1 \cap \mathcal{E}_2 \to U \) is a subbundle of \( \mathcal{E} \to U \).

**Proof of Proposition 2.6.** For every \( x \in M \) define \( \overline{\mathcal{I}}_{p(x)} \subset \bigwedge T^*_{p(x)} \overline{M} \) as the preimage of

\[
(4) \quad \mathcal{J}_x \overset{def}{=} \mathcal{I}_x \cap \text{Im}^* p_x
\]

under \( p^*_x : \bigwedge T^*_{p(x)} \overline{M} \to \bigwedge T^*_x M \).

Let us show that this definition does not depend on the choice of a particular \( x \in p^{-1}(p(x)) \). Assume \( p(x_1) = p(x) \), then there exists a local diffeomorphism \( g \in \mathcal{G} \), such that \( gx_1 = x \), and \( g^* \mathcal{I}_x = \mathcal{I}_{x_1} \). Since \( p(g(x_1)) = p(x_1) \),

\[
g^* p^*_x \overline{\mathcal{I}}_{p(x)} = p^*_x \overline{\mathcal{I}}_{p(x_1)},
\]

therefore \( p^*_x \overline{\mathcal{I}}_{p(x)} \subset g^* \mathcal{I}_x = \mathcal{I}_{x_1} \), and \( \overline{\mathcal{I}} \) is well-defined. It is straightforward to check that \( \Gamma(\overline{\mathcal{I}}) \) is a differential ideal in \( \Omega(\overline{M}) \). To show that \( \overline{\mathcal{I}} \) is a subbundle of \( \bigwedge T^* \overline{M} \) over each connected component of the complement to a closed subset of zero measure, consider \( \mathcal{J} = \bigcup_{x \in M} \mathcal{J}_x \) (where \( \mathcal{J}_x \) is defined in (3)).
By lemma \((2.8)\), \(J\) is again a subbundle outside a closed subset \(X_1 \subset M\) of zero measure. Moreover \(X_1\) is \(G\)-invariant. The latter implies that \(\mathcal{X} \equiv p(X_1)\) also has Borel measure zero. Therefore \(\mathcal{I}\) is a subbundle over each connected component of \(\mathcal{M} \setminus \mathcal{X}\), where \(\mathcal{X}\) is a closed subset of zero measure.

**Example 2.5.** Consider the action of the abelian group \(G = \mathbb{R}^3\) on itself \((M = \mathbb{R}^3)\) by translations. Define \(\mathcal{E} = (\mathbb{R}^3, <0>)\). The two-dimensional \((k = 2)\) \(r\)-th prolongation of \(\mathcal{E}\) is the jet space of two-dimensional submanifolds:

\[
\mathcal{E}_2^{(r)} = (J_2^r \mathbb{R}^3, C^{(r)}).
\]

In order to coordinatize the orbit spaces \(\overline{J}_2^r \mathbb{R}^3\), we introduce the coordinates \((x^1, x^2, u)\) in \(\mathbb{R}^3\) as well as the standard jet coordinates \(u_j\) in the fibers of \(J_2^r \mathbb{R}^3\) (here \(J\) is a multiindex). Note that in fact we restricted our attention to the coordinate chart \(U_r \subset J_2^r \mathbb{R}^3\) that has a complement of zero Borel measure in \(J_2^r \mathbb{R}^3\). The orbit space \(\overline{\mathcal{U}}_r = U_r / \mathbb{R}^3\) is a Euclidean space with coordinates \((u_j)_1 \leq |j| \leq r\).

Denote by

\[
y^i \equiv u_i, \quad i = 1, 2
\]

the coordinates on the orbit space \(\overline{J}_2^{(1)} \mathbb{R}^3 \simeq \mathbb{R}P^2\). It is obvious that the reduced ideal \(\overline{C}^{(1)}\) is trivial, thus

\[
\overline{\mathcal{E}}_2^{(1)} = (\mathbb{R}P^2, <0>).
\]

The contact ideal on \(J_2^3 \mathbb{R}^3\) is generated by the three \(\mathbb{R}^3\)-invariant 1-forms

\[
\eta^1 = du - u_1 dx^1 - u_2 dx^2,
\]

\[
\eta^2 = du_1 - u_{11} dx^1 - u_{12} dx^2,
\]

\[
\eta^3 = du_2 - u_{12} dx^1 - u_{22} dx^2.
\]

Let us introduce the coordinates on the fiber of \(J_2^3 \mathbb{R}^3 \to \overline{J}_2^1 \mathbb{R}^3\):

\[
v^1 = u_{11}, \ v^2 = u_{22}, \ v^3 = u_{12}.
\]

Direct calculations show that the reduced ideal \(\overline{C}^{(2)} \subset \bigwedge^2(T^*J_2^3 \mathbb{R}^3)\) has no 1-form component, however it does have a nontrivial 2-form component, generated by the 2-forms \(\bar{\omega}_1, \bar{\omega}_2 \in \Omega^2(J_2^3 \mathbb{R}^3)\),

\[
\bar{\omega}_1 = (v^2 dy^3 - v^3 dy^2) \wedge dv^1 + (v^3 dy^2 - v^3 dy^1) \wedge dv^3 = (u_{11} u_{22} - u_{12}^2) dy^1 + (u_{22} u_{12}^2 - u_{12} u_{11}^3) \wedge du_{11} + (u_{11} u_{12}^3 - u_{12} u_{12}^3) \wedge du_{12},
\]

\[
\bar{\omega}_2 = (v^2 dy^1 - v^3 dy^2) \wedge dv^3 + (v^3 dy^2 - v^3 dy^1) \wedge dv^3 = (u_{11} u_{22} - u_{12}^2) dy^3 + (u_{22} u_{12}^2 - u_{12} u_{11}^3) \wedge du_{11} + (u_{11} u_{12}^3 - u_{12} u_{12}^3) \wedge du_{12},
\]

(in fact \(\overline{C}^{(2)}\) is generated by its 2-form component). Therefore

\[
\overline{E}_2^{(2)} \simeq (\mathbb{R}P^1 \times \mathbb{R}^3, <\bar{\omega}_1, \bar{\omega}_2>).
\]

The last example shows that although the original EDS is generated by 1-forms, the reduced EDS does not necessarily have the same property. In particular, it may not be a prolongation of anything. This raises the natural question of whether the reduction procedure commutes with the prolongation. We address this question in Theorem 3 below.
Definition 2.9. We shall say that an EDS $\mathcal{E}$ is of infinite type if for every $r > 1$ $M_k^{(r)} \to M_k^{(r-1)}$ is a differentiable fiber bundle, and $\dim M_k^{(r)} - \dim M_k^{(r-1)} > 0$.

Consider a Lie group $G$, acting on $M$, and $G$-invariant EDS $\mathcal{E} = (M, I)$. The action of $G$ on $M$ prolongs to the action on $M_k^{(r)}$. It is well-known [14, 13] that if $\mathcal{E} = (M, I)$ is an infinite-type EDS and the action is effective on open subsets then the $G$-action is locally free (i.e. the stabilizers are discrete) almost everywhere on $M_k^{(r)}$ for big enough $r$. The author is not aware of any example when the action does not eventually become free on high enough prolongation. Moreover, in the real-analytic category there are strong indications that every effective action becomes free on high enough prolongation [1]. Throughout this paper we shall adopt the following hypothesis:

The Main Assumptions.
1. $G$ is a Lie group, and the considered EDS $\mathcal{E}$ is of infinite type.
2. There exists an integer $r_s$, and a closed subset $X \subset M_k^{(r_s)}$ of zero Borel measure such that the action of $G$ is free on $M_k^{(r_s)} \setminus X$.
3. The quotient space $M_k^{(r_s)} = (M_k^{(r_s)} \setminus X)/G$ is a differentiable manifold.

Theorem 1. Assume that the main assumptions hold. Then there exist an integer $r_o \geq 0$ such that for every $r \geq r_o$ the procedure of reduction of $\mathcal{E}_k^{(r)}$ commutes with the procedure of prolongation, i.e.

$$\left(\mathcal{E}_k^{(r)}\right)_k^{(1)} = \mathcal{E}_k^{(r+1)}$$

It will be shown below (see the proof in the section [4]) that $r_o \leq \max(r_s, r_{cl}) + 2$, where $r_{cl}$ is the order of prolongation at which a closed horizontal $G$-invariant coframe appears (see Lemma 3.1 below).

2.3. Moduli space of solutions and the reduced EDS. Let $G$ be a Lie group acting on $M$. Let $\mathcal{E} = (M, I)$ be a $G$-invariant EDS of infinite type. Denote by $\text{Sol}_k(\mathcal{E})$ the space of $k$-dimensional solutions of $\mathcal{E}$. We shall say that a solution $S_r \in \text{Sol}_k(\mathcal{E}_k^{(r)})$ is regular (the notation is $S_r \in \text{Sol}_k^{\text{reg}}(\mathcal{E}_k^{(r)}, G)$ ) if $S_r$ is transversal to the orbits of the $G$-action on $M_k^{(r)}$ (clearly then the lifts of $S_r$ to the higher prolongations are also regular).

For every $r > 0$, and every solution $S_r \in \text{Sol}_k(\mathcal{E}_k^{(r)})$ we may consider the projection $\bar{S}_r = p(S_r) \subset M_k^{(r)}$ of $S$ onto the orbit space. If the solution $S_r$ is regular then $\bar{S}_r$ is a $k$-dimensional submanifold, and is a solution of the reduced EDS $\mathcal{E}_k^{(r)}$. It turns out that on “high enough” prolongation we can also lift a solution of $\mathcal{E}_k^{(r)}$ to a regular solution of $\mathcal{E}_k^{(r)}$.

Theorem 2. There exists $r_o > 0$ (same as in Theorem [4]) such that for every $r \geq r_o$ the moduli space of regular solutions of the prolonged EDS is isomorphic to the solutions of the reduced EDS:

$$\underbrace{\text{Sol}_k^{\text{reg}}(\mathcal{E}_k^{(r)}, G)}_G \cong \text{Sol}_k \mathcal{E}_k^{(r)}$$

The proof is given in section [5].
2.4. Characteristic cohomology of the reduced jet spaces. Let a Lie group $G$ act on a manifold $M$. Assume that the main assumptions hold with regard to the trivial EDS $\mathcal{E} = (M, < 0>)$. By virtue of Theorem 1 we may regard $(J^\infty_k M, \mathcal{C}(\infty))$ as an infinite prolongation of $\tilde{\mathcal{E}}_0 = (J^\infty_k M, \mathcal{C}(\infty))$.

The fact that $\tilde{\mathcal{E}}_0$ is a reduction of an unconstrained jet space allows us to know everything about the solutions of $\tilde{\mathcal{E}}_0$, since every solution of $\tilde{\mathcal{E}}_0$ is an image of a solution of $(J^\infty_k M, \mathcal{C}(\infty))$ under the mapping $p : J^\infty_k M \to J^\infty_k M$. Therefore it is important to investigate the conditions under which a given EDS $\tilde{\mathcal{E}}_0$ can be a reduction of an unconstrained jet space.

It turns out that the local characteristic cohomology of the reduced EDS $\tilde{\mathcal{E}}_0$ is isomorphic to the Lie algebra cohomology of the Lie group $G$.

Denote by $\tilde{\pi}^\infty_r : \tilde{J}^\infty_k M \to J^\infty_k M$ the natural projection.

**Theorem 3.** For every open subset $\hat{U} \subset \tilde{J}^\infty_k M$, such that $\tilde{\pi}^\infty_r \hat{U}$ is contractible for every $r \geq r_0$, the characteristic cohomology of $\tilde{\mathcal{E}}_0$ over $\hat{U}$ is isomorphic to the Lie algebra cohomology of $G$ in dimensions less than $k$:

\[
H^t(\Omega_{hor}(\hat{U}), \tilde{d}_0) \simeq H^t(\mathfrak{g}) \quad \forall t < k,
\]

where $\Omega_{hor}(\hat{U}) \cong \Omega(\hat{U})/\Gamma(\mathcal{C}(\infty))$, $\tilde{d}_0$ is the horizontal differential induced on the horizontal forms $\Omega_{hor}(\hat{U})$, and $H^t(\mathfrak{g})$ is the Lie algebra cohomology of the Lie group $G$.

The proof as well as the practical algorithm of computing the basis of nontrivial conservation laws of $\tilde{\mathcal{E}}_0$ is given in section 3.

2.5. Invariant variational problems. Consider an unconstrained infinite jet space $J^\infty_k M$ of $k$-dimensional submanifolds of a manifold $M$. Denote by $(E^r, d^r)$ the Vinogradov spectral sequence corresponding to the decreasing filtration

$\mathcal{F}_s \Omega(J^\infty_k M) = \Omega(J^\infty_k M) \cap \wedge^s \Gamma(\mathcal{C}(\infty))$.

It is well-known that the $k$-dimensional variational problems on $M$ can be identified with the space

$\tilde{E}^{0,k}_1 = \Omega^k(J^\infty_k M)/\left(\Gamma(\mathcal{C}(\infty)) + d\Omega^{k-1}(J^\infty_k M)\right)$,

and the Euler-Lagrange operator is $d^1_1 : \tilde{E}^{0,k}_1 \to \tilde{E}^{1,k}_1$, where the quotient

$\tilde{E}^{1,k}_1 = \Gamma(\mathcal{C}(\infty)) \cap \Omega^{k+1}(J^\infty_k M)/\left(\wedge^2 \Gamma(\mathcal{C}(\infty)) + d\Gamma(\mathcal{C}(\infty))\right)$

is a free module over the ring of functions on the infinite jet $J^\infty_k M$.

For a given $\lambda \in \Omega^k(J^\infty_k M)$ one may consider the Euler-Lagrange system $EL(\lambda) \in J^{2r}_k M$ defined as the zero locus of $d^1_1(\lambda)$ (we denote by $[\lambda]$ the equivalence class in $\tilde{E}^{-1}_1$). If $(x^i, u^a, u^a_\beta)$ are the standard jet coordinates in some open neighborhood of $J^\infty_k M$, $dx = dx^1 \wedge \cdots \wedge dx^k$, and $\lambda = Ldx + \Gamma(\mathcal{C}(\infty))$ is the variational problem then the Euler-Lagrange system has the form

\[
EL(\lambda) = \{ E_\alpha(L) = 0, \quad \alpha = 1, \ldots, q = \dim M - k \}, \quad \text{where}
\]

$E_\alpha(L) = \sum_{|I| = 0}^r (-1)^{|I|} \frac{d^{|I|}}{dx^I} \frac{\partial}{\partial u^I} L,$
(13) \[ d\lambda = \sum_{\alpha=1}^{q} E_\alpha(L)\theta^\alpha \wedge dx + d\Gamma(C(\infty)) + \wedge^2 \Gamma(C(\infty)), \]

and the forms \([\theta^\alpha \wedge dx]_1\) give the basis in \(E_1^{1,k}\).

(Here \(\frac{d}{dx}\) are the total derivatives w.r.t. multiindex \(I\).)

Let a Lie group \(G\) act on the manifold \(M\). Since the \(G\)-action on \(\Omega(J^\infty_k M)\) preserves the contact ideal, it induces the action on \(E_1^{x,t}\).

**Definition 2.10.** We shall say that \(\lambda \in \Omega^k(J^\infty_k M)\) represents an invariant variational problem if \([\lambda]_1\) is \(G\)-invariant.

It can be shown (see Lemma 3.3 in section 3) that for every invariant variational problem \([\lambda]_1\) there exists a differential form \(\lambda = Ldy^1 \wedge \cdots \wedge dy^k \in \Omega^k(J^\infty_k M)\) such that \([\lambda]_1 = [\pi^*\lambda]_1\). The form \(\lambda\) in its turn defines a variational problem on \(\bar{J}^\infty_k M\) (that is a class in \(E_1^{0,k}\) of the Vinogradov spectral sequence of \(J^\infty_k M\)). Therefore it is desirable to understand \(EL(\lambda)\) in terms of the calculus of variations on the reduced jet space \(\bar{J}^\infty_k M\).

It is well-known [3, 4] that in every small neighborhood of \(\bar{J}^\infty_k M\) there exist functions \((y^1, \ldots, y^k, v^1, \ldots, v^q)\) such that any other differential invariant is a function of the \(y^i\), and the total derivatives \(v^a_i = \frac{d^{1,i}v^a}{dy^i}\) of \(v^a\) w.r.t. \(y^i\) (see Lemma 4.1 in section 4). Moreover as a consequence of Theorem 4 we have \((\bar{J}^\infty_k M, C(\infty)) = (J^\infty_k M, C(\infty))\), where \(\bar{\Delta}(\infty) \hookrightarrow J^\infty_k \mathbb{R}^{k+\hat{q}}\) is an infinite prolongation of a certain PDE system \(\Delta \hookrightarrow J^\infty_k \mathbb{R}^{k+\hat{q}}\) (see section 4 for more details).

In local coordinates the Euler-Lagrange equations on \(\bar{J}^\infty_k M\) may be written in the same fashion as on the unconstrained jet space \(J^\infty_k \mathbb{R}^{k+\hat{q}}\). More precisely, given a variational problem

\[ \bar{\lambda} = Ldy^1 \wedge \cdots \wedge dy^k \in \Omega^k(\bar{J}^\infty_k M), \]

one can find a function \(L_1(y^1, v^a_i) \in C(\infty)(J^\infty_k \mathbb{R}^{k+\hat{q}})\) such that its restriction to \(\bar{\Delta}(\infty)\) is equal to \(\bar{L}\) (\(\bar{\iota}_* L_1 = \bar{L}\)), then

(14) \[ d\bar{\lambda} = \sum_{a=1}^{\hat{q}} E_\alpha(\bar{L})\bar{\theta}^\alpha \wedge dy + d\Gamma(C(\infty)) + \wedge^2 \Gamma(C(\infty)), \]

where \(\bar{\theta}^\alpha \stackrel{def}{=} dv^a - v^a_i dy^i\), \(d\bar{\lambda} = dy^1 \wedge \cdots \wedge dy^k\), and the expression staying in the place of the Euler-Lagrange operator is defined as

(15) \[ E_\alpha(\bar{L}) \stackrel{def}{=} i^*_{\infty} \left( \sum_{l=1}^{q} (-1)^{l+1} \frac{d^{1,i}}{dy^i} \frac{\partial L_1}{\partial v^a_i} \right), \]

and depends on the particular choice of the function \(L_1\). \(\ast\)This happens because \(E_1^{1,k}\) is not necessary a free module over the ring of functions on the reduced jet space \(\bar{J}^\infty_k M\).
Theorem 4. Suppose that the main assumptions hold with regard to the trivial EDS \((M, < 0 >)\). Then there exist total differential operators on the reduced jet space \(A^\alpha_a : C^\infty(J^\infty_k M) \to C^\infty(J^\infty_k M)\),

\[
A^\alpha_a = \sum_{0 \leq |I| \leq n-1} A^{aI}_{\alpha} \frac{d^{|I|}}{dy^I},
\]

(here \(A^{aI}_{\alpha} \in C^\infty(J^\infty_k M)\), \(\alpha = 1, \ldots, q = \dim M - k\), \(a = 1, \ldots, \tilde{q}\)) such that every invariant variational problem \([\lambda]_1 = [p^* Ldy^1 \wedge \cdots \wedge dy^k]_1\) has its Euler-Lagrange system as

\[
\text{EL}(\lambda) = p^{-1}\left\{ \left\{ \sum_{a=1}^{\tilde{q}} \hat{A}^a_{\alpha} \hat{E}_a(\hat{L}) = 0, \ \alpha = 1, \ldots, q \right\} \right\},
\]

where \(\hat{E}_a\) are the Euler-Lagrange operators \([13]\) on the reduced jet space.

Remark 2.11. Despite the fact that the Euler-Lagrange expressions defined in the formula \([13]\) depend on the choice of the Lagrangian \(L_1\) the expression \(\sum_{a=1}^{\tilde{q}} \hat{A}^a_{\alpha} \hat{E}_a(\hat{L})\) does not depend on this freedom.

The proof as well as the practical algorithm of computing the operators \(\hat{A}^a_{\alpha}\) is given in section 3.

3. IN Variant CONTACT FORMS.

Let \(E = (M, J)\) be a \(G\)-invariant EDS of infinite type. Denote by \(E^{(\infty)}_k = (M^{(\infty)}_k, J^{(\infty)}_k)\) its infinite prolongation, and by \(p : M^{(r)}(\infty) \to M^{(r)}(\infty)\) denote the natural projection onto the orbit space.

Lemma 3.1. There exist \(r_{cf} \geq 0\), and differential invariants of the \(G\)-acton \(y^1, \ldots, y^k \in C^\infty(M^{(r_{cf})})\) such that \(\{p^* dy^1, \ldots, p^* dy^k\}\) form a basis of \(\Omega^1(M^{(\infty)}_k)/\Gamma(J^{(\infty)}_k)\), and \(\Omega^r(M^{(\infty)}_k)/\Gamma(J^{(\infty)}_k)\) is isomorphic to \(\Lambda^r \text{Span} \{p^* dy^1, \ldots, p^* dy^k\}\) as a \(C^\infty(M^{(r_{cf})})\)-module.

The proof is completely analogous to the proof of the same fact \([13]\) about the unconstrained jet spaces \((M, < 0 >)^{(r)} = (J^r_k M, C^{(r)})\) and therefore omitted.

Choosing the differential invariants \(y^i\) allows us to introduce the total differential operators \(\frac{d}{dy^i} : C^\infty(M^{(r)}) \to C^\infty(M^{(r+1)})\) (here \(r \geq r_{cf}\)) in the following way. Denote by \([\frac{d}{dy^i}]_0 : \Omega^1(M^{(r)}) \to \Omega^1(M^{(\infty)}_k)/\Gamma(J^{(\infty)}_k)\) the natural projection to the quotient. Then these operators are defined by the equality

\[
[p^* dF]_0 = \sum_{i=1}^k (p^* \frac{dF}{dy^i})[p^* dy^i]_0, \quad F \in C^\infty(M^{(r)}).
\]

It is easy to show that the functions \(\frac{dF}{dy^i} \in C^\infty(M^{(\infty)}_k)\) actually belong to \(C^\infty(M^{(r+1)})\). Note also that the operators \(\frac{d}{dy^i}\) commute with each other.

Lemma 3.2. Let \((M^{(1)}_k, J^{(1)}_k)\) be the prolongation of an Exterior Differential System \(E = (M, J)\). Assume that \(\pi : M^{(1)}_k \to M\) is a surjection. Then at every point...
$z \in M_k^{(1)}$ the 1-form component $\mathcal{I}^{(1)}_z = \mathcal{I}_k^{(1)} \cap T^*M_k^{(1)}$ of the prolonged ideal lies in the pullback of $T^*M$ by $\pi$:

\begin{equation}
\mathcal{I}^{(1)}_z \subset \pi^*(T^*_\pi(z)M), \quad \text{and}
\end{equation}

\begin{equation}
dim(\mathcal{I}^{(1)}_z) = \dim M - k
\end{equation}

**Proof.** Let us introduce local coordinates $x^1, \ldots, x^k, u^1, \ldots, u^q$ in some open neighborhood of $M$, and the standard jet coordinates $u^k_1$ in the fibers of $J^1_kM$. The ideal $J^1_k = \iota^*\mathcal{C}(1)$ on $M = \iota^*J^1_kM$ is generated by the forms

$$\theta^\alpha = \iota^*(du^\alpha - u^q dx^i), \quad \alpha = 1, \ldots, q = \dim M - k.$$

This proves (17). To see that the forms $\theta^\alpha$ are linearly independent observe that the forms $\iota^*du^1, \ldots, \iota^*du^q, \iota^*dx^1, \ldots, \iota^*dx^k$ are linearly independent due to the surjectivity of $\pi$. \hfill \square

**Corollary 3.3.** For every $r > r_\text{cd}$, $\alpha \in \Omega^1(M_k^{(r-1)}(r))$, and $\beta \in \Omega^2(M_k^{(r-1)}(r))$

\begin{equation}
(\pi_{r-1}^r)^*\alpha = \theta + \sum_{i=1}^{k} f_i p^*dy^i,
\end{equation}

\begin{equation}
(\pi_{r-1}^r)^*\beta = \sum_\alpha \theta^\alpha \wedge \tau_\alpha + \sum_{1 \leq i_1 < i_2 \leq k} f_{i_1i_2}^* (dy^{i_1} \wedge dy^{i_2}),
\end{equation}

where $\theta, \theta^\alpha \in \Gamma(T^r_k) \cap \Omega^1(M_k^{(r)})$, $\tau_\alpha \in \Omega^1(M_k^{(r)})$, and $f_{i_1i_2} \in C^\infty(M_k^{(r)})$.

Since the action of $G$ on the fiber bundle $M_k^{(r-1)}(r) \xrightarrow{\pi_{r-1}} M_k^{(r)}$ is projectible, there is a surjection $\pi_{r-1}$ defined by the following commutative diagram

\begin{equation}
\begin{array}{ccc}
M_k^{(r-1)} & \xrightarrow{\pi_{r-1}} & M_k^{(r)} \\
p \downarrow & & \downarrow p \\
M_k^{(r-1)} & \xrightarrow{\pi_{r-1}} & M_k^{(r)}
\end{array}
\end{equation}

\begin{lemma}
Assume that $r \geq r_\text{cd} + 1$. In an open neighborhood of $M_k^{(r-1)}$, introduce local coordinates $y^1, \ldots, y^k, v^1, \ldots, v^{q_r}$ (here the functions $y^i$ are taken from Lemma 3.2). Then the 1-form component of $\mathcal{I}_k^{(r)}$ is generated by the forms

\begin{equation}
\bar{\theta}^a = dv^a - v^a dy^i, \quad a = 1, \ldots, q_r = \dim M_k^{(r-1)} - k,
\end{equation}

where $v^a = \frac{dv^a}{dy^i} \in C^\infty(M_k^{(r)})$, and the dimension of this 1-form component is equal to $q_r = \dim M_k^{(r-1)} - k$.

**Proof.** Due to the definition of the operators $\frac{d}{dy^i}$, the forms (20) belong to the reduced ideal. Lemma 3.2 implies that any form $\bar{\theta} \in \Omega^1(M_k^{(r)})$ that has nonzero projection into the quotient $\Omega^1(M_k^{(r)})/(\pi_{r-1})^*\Omega^1(M_k^{(r-1)})$ may not belong to the reduced ideal, thus the forms (20) exhaust the list of the generators. These forms are linearly independent because of the surjectivity of $\pi_{r-1}$.

\hfill \square
Lemma 3.5. Assume that a Lie Group $G$ acts freely on a manifold $B$, and has a projective action on a vector bundle $E \to B$. Then every point on the base has an open neighborhood $U \subset B$ having a basis of $G$-invariant sections in $\Gamma(E|_U)$.

Proof. For a given point on the base $B$ choose a small neighborhood $U \subset B$, such that there exists a right moving frame $\rho : U \to G$ (i.e. $\rho(gx) = \rho(x)g^{-1}$, $\forall(x, g) \in U \times G$). Denote by $I(x) = \rho(x)x$ the invaraintization map $\tilde{\rho}$. Its image $L = I(U)$ is a submanifold of $U$, transversal to the orbits of the $G$-action on $B$. We may assume that the restriction vector bundle $E|_L \to L$ is trivial. Consider a basis of sections $\tilde{s}_1, \ldots, \tilde{s}_l \in \Gamma(E|_L)$ ($l = \dim(E_x)$) over $L$. Then it is easy to show that the sections

$$s_i(x) = \rho(x)^{-1} \tilde{s}_i(\rho(x)x), \quad i = 1, \ldots, \dim(E_x), \quad x \in U,$$

are $G$-invariant and constitute a basis in $\Gamma(E|_U)$.

Let $p : M_k^{(r)} \to M_k^{(s)}$ denote the natural projection onto the orbit space. Denote by $\Omega^1(M_k^{(r)}|G)$ the space of $G$-invariant differential 1-forms on $M_k^{(r)}$.

Applying the Lemma 3.3 to the first-degree component of the prolonged ideal $(\mathcal{I}_k^{(r)})^{1} \subset T^*M_k^{(r)}$ gives the following corollary.

Proposition 3.6. Suppose that the main assumptions hold.

Denote $r_\eta = \max(r_s, r_d) + 1$. Then every point $z \in M_k^{(r_\eta)} \setminus (\pi_{r_\eta}^r)^{-1} X$ has an open neighborhood $U \subset M_k^{(r_\eta)}$, and the $G$-invariant contact forms

$$(21) \quad \eta_1, \eta^2, \ldots, \eta^{\dim G} \in \Omega^1(U)^G \cap \Gamma(\mathcal{I}_k^{(r_\eta)}),$$

such that

$$\text{Span}\{\eta_1, \eta^2, \ldots, \eta^{\dim G}\} \simeq \Omega^1(U)^G / p^*\Omega^1(p(U)).$$

Proof. Apply Lemma 3.3 to the subbundle $\mathcal{I}_k^{(r_\eta)} \cap T^*M_k^{(r_\eta)}$. Then over a certain neighborhood $U \subset M_k^{(r_\eta)}$ we have a $G$-invariant basis of contact forms $\zeta^1, \ldots, \zeta^{q_{r_\eta}} \in \Omega^1(U)^G \cap \Gamma(\mathcal{I}_k^{(r_\eta)})$, where $q_{r_\eta} = \dim M_{k}^{(r_\eta) - 1} - k$ (see Lemma 3.3). Since $G$ acts freely on $\pi_{r_\eta-1}(U)$, the difference between the dimensions of the 1-form component of $\mathcal{I}_k^{(r_\eta)}$, and the 1-form component of $\mathcal{I}_k^{(r_\eta)}$ is equal to the dimension of $G$:

$$\dim(\mathcal{I}_k^{(r_\eta)}) - \dim(\mathcal{I}_k^{(r_\eta)}) = q_{r_\eta} - q_{r_\eta} = \dim M_{k}^{(r_\eta) - 1} - \dim M_{k}^{(r_\eta) - 1} = \dim G$$

(here we used Lemma 3.4). Therefore the projection of $\text{Span}\{\zeta^1, \ldots, \zeta^{q_{r_\eta}}\}$ into the quotient $\Omega^1(U)^G / p^*\Omega^1(p(U))$ is surjective. Thus we can find the forms (21) as a linear combination of the forms $\tilde{\zeta}^1$ with the lifts of some functions on $M_k^{(r_\eta)}$.

Corollary 3.7. For every $r \geq r_\eta$, and $z \in M_k^{(r_\eta)} \setminus (\pi_{r_\eta}^r)^{-1} X$ each $z' \in (\pi_{r_\eta}^r)^{-1}(z)$ has an open neighborhood $U' \subset (\pi_{r_\eta}^r)^{-1}(U) \subset M_k^{(r)}$, and contact forms on the orbit space

$$\overline{\theta}^1, \ldots, \overline{\theta}^{q_{r}} \in \Omega^1(p(U')),$$
such that the forms $p^*\bar{\theta}^a$, and $(\pi_{T_e}^r)^*\eta^i$ generate $\Gamma(\mathcal{I}_k^{(r)})^G$, i.e. every $G$-invariant differential form $\omega \in \Gamma(\mathcal{I}_k^{(r)}) \cap (\Omega^n(U'))^G$ satisfies

$$\omega = \sum_{i=1}^{\dim G} ((\pi_{T_e}^r)^*\eta^i \wedge \alpha_i + (\pi_{T_e}^r)^*d\eta^i \wedge \beta_i) + \sum_{a=1}^{\tilde{q}_r} (p^*\bar{\theta}^a \wedge \gamma_a + p^*d\bar{\theta}^a \wedge \delta_a),$$

where the differential forms $\alpha_i, \gamma_a \in (\Omega^{n-1}(U'))^G$, $\beta_i, \delta_a \in (\Omega^{n-2}(U'))^G$ are $G$-invariant.

Moreover, if $r > r_\eta$, then

$$\sum_{i=1}^{\dim G} (\pi_{T_e}^r)^*d\eta^i = \sum_{j=1}^{\dim G} (\pi_{T_e}^r)^*\eta^i \wedge \alpha_i + \sum_{a=1}^{\tilde{q}_r} p^*\bar{\theta}^a \wedge \zeta_a,$$

where $i = 1, \ldots, \dim G$, $\alpha_i, \zeta_a \in \Omega^1(M_k^{(r)})^G$.

**Proof.** The forms $\bar{\theta}^a$ are obtained by applying Lemma 3.3 to the subbundle $\mathcal{I}_k^{(r)1} = \mathcal{I}_k^{(r)} \cap T^*M_k^{(r)}$, and using the forms $(\pi_{T_e}^r)^*\eta^i$ to project the $G$-invariant basis of $\Gamma(\mathcal{I}_k^{(r)1})$ to $p^*(\Omega^1(M_k^{(r)}))$ thus getting the forms on the orbit space. The formula (22) holds because $\Gamma(\mathcal{I}_k^{(r)})^G$ is generated by its 1-form component.

If $r > r_\eta$, we can use formula (13), and write

$$(\pi_{T_e}^r)^*d\eta^i = \sum_{\alpha} \theta^\alpha \wedge \tau_\alpha + \sum_{1 \leq i_1 < i_2 \leq k} f_{i_1i_2}p^* (dy^{i_1} \wedge dy^{i_2}).$$

Taking into account that the right-hand side of the last equation belongs to the ideal $\Gamma(\mathcal{I}_k^{(r)1})$, and using Lemma 3.1 we conclude that $f_{i_1i_2} = 0$. Rewriting $\theta^\alpha$ as linear combination of $(\pi_{T_e}^r)^*\eta^i$, and $\bar{\theta}^a$ gives the formula (23).

**Example 3.1.** Consider the example [23]. Here $r_x = r_{cf} = 1$, thus $r_\eta = 2$. The generators $\mathfrak{f}$ of the contact ideal $C^{(r)}$ are already invariant. However none of them lives on the orbit space. The forms $\eta^1, \eta^2, \eta^3$ are defined in [1]-[8]. Consider $M_2^{(3)} = \mathbb{R}^3$ with the coordinates $(y^i, v^a, v^a_0)$, where $(y^i, v^a)$ are defined in [1],[2], and $v^a_0 = \frac{dv^a}{dy^i}$ (here $i = 1, 2$; $a = 1, 2, 3$). (Note that there are functional dependencies among $v^a_0$.) Then $\tilde{q}_3 = 3$, and the forms $\bar{\theta}^a$ are given by the formula

$$\bar{\theta}^a = dv^a - v^a_0 dy^i.$$

**Proposition 3.8.** For every $r \geq \max(r_x, r_{cf}) + 2$, $\mathcal{I}_k^{(r)}$ is generated by 1-forms, i.e. there exist $\bar{\eta}^1, \ldots, \bar{\eta}^r \in \Gamma(\mathcal{I}_k^{(r)}) \cap \Omega^1(M_k^{(r)})$ such that for every $\varpi \in \Gamma(\mathcal{I}_k^{(r)})$

$$\varpi = \sum_{a=1}^{\tilde{q}_r} (\bar{\eta}^a \wedge \varpi_a + d\bar{\theta}^a \wedge \bar{\beta}_a) \quad \text{where} \quad \varpi_a, \bar{\beta}_a \in \Omega(M_k^{(r)}).$$

**Proof.** It suffices to prove this proposition in a small neighborhood in $M_k^{(r)}$. Given the forms (24), define a $G$-invariant subbundle

$$\mathcal{P} = \text{Span}\{ (\pi_{T_e}^r)^*\eta^i \}_{i=1}^{\dim G} \subset T^*M_k^{(r)}.$$
There are $G$-invariant decompositions
\[ T^*M_k^{(r)} = \mathcal{P} \oplus p^*T^*M_k^{(r)}, \]
\[ \wedge^n T^*M_k^{(r)} = \bigoplus_{l=0}^{n} \wedge^l \mathcal{P} \otimes \wedge^{n-l} p^*T^*M_k^{(r)}. \]

The corresponding $G$-invariant projectors $p_n : \wedge^n T^*M_k^{(r)} \to \wedge^n p^*\overline{T^*M_k^{(r)}}$ have the following properties:
\[ \omega \in p^*(\Omega^n(M_k^{(r)})) \iff p_n \omega = \omega, \text{ and } \omega \text{ is } G\text{-invariant}; \]
\[ \omega \text{ is } G\text{-invariant} \implies p_n \omega = p^*\varpi \text{ for some } \varpi \in \Omega^n(M_k^{(r)}); \]
\[ p_{n_1+n_2}(\omega_1 \wedge \omega_2) = (p_{n_1}\omega_1) \wedge (p_{n_2}\omega_2) \quad \omega_i \in \Omega^{n_i}(M_k^{(r)}); \]
\[ p_1((\pi^r_{G})^* \eta^i) = 0 \quad \forall i = 1, \ldots, \dim G. \]

Now assume that $\varpi \in \Gamma(\mathcal{I}_k^{(r)}) \cap \Omega^n(M_k^{(r)})$. Using Corollary [3.7] and the properties (28)-(29) we conclude that
\[ p^*\varpi = p_n p^*\varpi = \]
\[ p_n \left( \sum_{i=1}^{\dim G} \left( (\pi^r_{G})^* \eta^i \wedge \alpha_i + (\pi^r_{G})^* d\eta^i \wedge \beta_i \right) + \sum_{a=1}^{\tilde{q}_r} (p^* \overline{\eta}^j \wedge \gamma_a + p^* d\overline{\theta}^a \wedge \delta_a) \right) = \]
\[ = \sum_{i=1}^{\dim G} p_2(\pi^r_{G})^* d\eta^j \wedge p_{n-2} \beta_l + \sum_{a=1}^{\tilde{q}_r} (p^* \overline{\eta}^j \wedge p_{n-1} \gamma_a + p^* d\overline{\theta}^a \wedge p_{n-2} \delta_a) = \]
\[ = \sum_{i=1}^{\dim G} p_2 \left( \sum_{j=1}^{\dim G} (\pi^r_{G})^* \eta^j \wedge \alpha^j + \sum_{a=1}^{\tilde{q}_r} p^* \overline{\eta}^j \wedge \beta^j \right) \wedge p_{n-2} \beta_l + \]
\[ + \sum_{a=1}^{\tilde{q}_r} (p^* \overline{\eta}^j \wedge p_{n-1} \gamma_a + p^* d\overline{\theta}^a \wedge p_{n-2} \delta_a) = \]
\[ = \sum_{a=1}^{\tilde{q}_r} \left( p^* \overline{\eta}^a \wedge p_{n-1} \left( \sum_{i=1}^{\dim G} \zeta^i_\alpha \wedge \beta^i + \gamma_a \right) + p^* d\overline{\theta}^a \wedge p_{n-2} \delta_a \right). \]

This together with the property (27) proves formula (24).

4. SYZYGIES OF DIFFERENTIAL INVARIANTS AND THE PROOF OF THEOREM [1]

The following lemma originally appeared in the work of A. Tresse [13] in the context of what was later called the jet spaces (See more recent treatment in [13]). It says that the differential invariants of any order are generated by taking total derivatives of finitely many differential invariants. The proof for the case of jet bundles is given in [13] (Theorem 5.49 page 171). The proof for the case of an EDS of infinite type is completely analogous, and therefore omitted.
Lemma 4.1. Assume that the EDS $\mathcal{E}$ is of infinite type, then for every $r \geq \max(r_s, r_{cf}) + 2$ the differential invariants of order $r$ are obtained by taking the total derivatives of the invariants of order $r - 1$:

$$\forall f \in C^\infty(M^{(r)_k}) \quad f = f(y^i, v^a, \frac{dv^a}{dy^i})$$

where $(y^i, v^a)$ are local coordinates on $M^{(r-1)}_k$, $i = 1, ..., k$, $a = 1, ..., \hat{q}$.

We may think of $(y^i, v^a, v^a_q)$ as standard jet coordinates on $J^1_k M^{(r-1)}_k$. These coordinate functions give the mapping $\iota_1 : M^{(r)_k} \to J^1_k M^{(r-1)}_k$. The last lemma implies that $\iota_1$ is an injective immersion of an open conull subset of $M^{(r)}_k$. The image $\Delta$ of $\iota_1$ is a PDE system that can be described locally as a zero locus of functions $\Delta(\pi_1^{-1}(\hat{U})) \subset C^\infty(J^1_k M^{(r-1)}_k)$. These functions are sometimes called syzygies of differential invariants [13].

Proof of Theorem 4. Let $r \geq r_0 = \max(r_s, r_{cf}) + 2$. In some open neighborhood $\hat{U} \subset M^{(r)}_k$ consider the functions $(y^i, v^a, v^a_q)$ as in Lemma 4.1. These coordinates define the embedding $\iota_1 : \hat{U} \to J^1_k(\pi_1^{-1}(\hat{U}))$.

Since $p^{-1}(\hat{U}) \to \hat{U}$ is a principle $G$-bundle we may find another principle $G$-bundle $p_1 : \hat{U} \to J^1_k(\pi_1^{-1}(\hat{U}))$ together with the embedding $i_1 : p^{-1}(\hat{U}) \to \hat{U}$ such that we have the following commutative diagram:

$$\begin{array}{ccc}
p^{-1}(\hat{U}) & \xrightarrow{i_1} & \hat{U} \\
p | & & | \\
\hat{U} & \xrightarrow{\iota_1} & J^1_k(\pi_1^{-1}(\hat{U}))
\end{array}$$

(in fact we find $\hat{U} \to J^1_k(\pi_1^{-1}(\hat{U})$ only over a certain tubular neighborhood of the image $\Delta$ of $\iota_1$). The image of the embedding $i_1$ is the zero locus of the pullbacks of the functions $\Delta_{\nu}$.

Consider the $G$ invariant coframe $(\tilde{\eta}^l, \pi_1^a \frac{\partial}{\partial y^l}, \pi_1^a \frac{\partial}{\partial v^a}, p_1^a \frac{\partial}{\partial v^a_q})$ on $\hat{U}$, where the forms $\tilde{\eta}^l$ are uniquely determined by the condition $\iota_1^* \tilde{\eta}^l = \pi_1^{-1} \eta^l$. Denote the dual basis of vector fields on $\hat{U}$ by $(H_j, \frac{\partial}{\partial y^l}, \frac{\partial}{\partial v^a}, \frac{\partial}{\partial v^a_q})$.

In order to construct the prolongation of $\mathcal{E}_{k}^{(r)}$ we may consider the coordinates $(p_1^a \frac{\partial}{\partial y^l}, p_1^a \frac{\partial}{\partial v^a}, p_1^a \frac{\partial}{\partial v^a_q})$ in an open subset of the fiber of $J^1_k \hat{U} \to \hat{U}$ such that each $k$-dimensional $P \subset T\hat{U}$ is given as

$$P = \text{Span}_{i=1,...,k}\{\frac{\partial}{\partial y^l} + p_1^a \frac{\partial}{\partial y^l} + p_1^a \frac{\partial}{\partial v^a} + p_1^a \frac{\partial}{\partial v^a_q}\}$$

(here we use the standard summation convention).

Due to Corollary 4.7 the ideal $\Gamma(J^1_k)^{(r)}_{\omega}$ is algebraically generated by the forms $(\pi_1^a)^* \eta^l$, $\tilde{\eta}^l$, and $d\theta^a$. The embedding $p^{-1}(\hat{U}) \to \hat{U}$ induces the embedding $J^1_k p^{-1}(\hat{U}) \to J^1_k \hat{U}$, thus we may view $M^{(r-1)}_k$ as a submanifold of $J^1_k \hat{U}$. We may construct the prolongation of $\mathcal{E}_{k}^{(r)}$ as the prolongation of the ideal in $\Omega(\hat{U})$ generated by the forms $\tilde{\eta}^l$, $\tilde{\theta}^a = p_1^a(\frac{dv^a}{dy^l} - v^a_q \frac{dv^a_q}{dy^l})$, $d\theta^a$, and the functions $p_1^a \Delta_{\nu}$. Direct
calculation shows that this prolongation $M_k^{(r+1)} \hookrightarrow J_k^1 \tilde{U}$ is defined by the equations
\begin{equation}
 p_i^1 \Delta \nu = 0, \quad p_i^1 \frac{d}{dy^i} \Delta \nu = 0,
\end{equation}
\begin{equation}
 p_i^j = 0, \quad p_i^a - v_i^a = 0, \quad p_i^{a_{i_2}} - p_{i_{i_2} i_1} = 0.
\end{equation}

The prolonged ideal $\Gamma(\mathcal{I}^{(r+1)}_k)$ is obtained as the restriction of the standard contact ideal on $J_k^1 \tilde{U}$ to the zero locus of these functions, and is generated by the forms $\tilde{\eta}^i$, $\tilde{\eta}^a = dv^a - v_i^a dy^i$, and $\tilde{\eta}_i^a = dv_i^a - p_{i_{i_1}}^a dy^{i_1}$.

Due to their definition, the functions $p_i^1, p_i^a, p_{i_1 i_2}^a \in C^\infty(J_k^1 \tilde{U})$ are $G$-invariant, thus the equations (30), (31) can be pushed forward by $p_1$, and the reduced EDS $\overline{\mathcal{E}}^{(r+1)}_k = (M_k^{(r+1)}, \mathcal{I}^{(r+1)}_k)$ is described by the following data:
\begin{equation}
 M_k^{(r+1)} = \{ \Delta \nu = 0, \quad \frac{d \Delta \nu}{dy^1} = 0, \quad p_i^{a_{i_2}} - p_{i_{i_2} i_1}^a = 0 \} \hookrightarrow \mathbb{R}^{\tilde{\eta}^1 + k}_{(y', v^a, v_i^a, p_{i_1 i_2}^a)},
\end{equation}
\begin{equation}
 \mathcal{I}^{(r+1)}_k = \langle \tilde{\eta}_i^a = dv_i^a - p_{i_{i_1}}^a dy^{i_1} \rangle.
\end{equation}

(here we used Proposition 3.8 to find $\mathcal{I}^{(r+1)}_k$). It is easy to see (Lemma 2.3) that this is exactly the prolongation of PDE system defined by the syzygies $\Delta \nu$, thus
\begin{equation}
 \overline{\mathcal{E}}^{(r+1)}_k = (\tilde{\Delta}_1, \tilde{\nu}^1 C^{(1)})^{(1)}_k = (\tilde{\mathcal{E}}^{(r)}_k)^{(1)}.
\end{equation}

\[ \square \]

**Example 4.1.** Consider the example 2.5. On the space $\tilde{J}_2^2 \mathbb{R}^3$ we introduced the local coordinates $(y^1, y^2, v^1, v^2, v^3)$. Since $r_\eta = 2$, all the higher order differential invariants are generated by the total derivatives of $v^a$. Counting the dimensions shows that there are two functionally independent syzygies, namely
\begin{equation}
 \tilde{\Delta}_1 = v^3(v_2^2 - v_1^2) + v^1 v_2^2 - v^2 v_1^3 = 0,
\end{equation}
\begin{equation}
 \tilde{\Delta}_2 = v^3(v_1^2 - v_2^2) + v^2 v_1^2 - v^1 v_2^3 = 0.
\end{equation}

Theorem 2 implies that for every $r \geq 3$ the reduced EDS $\overline{\mathcal{E}}^{(r)}_k = (\tilde{J}_2^2 \mathbb{R}^3, C^{(r)})$ is isomorphic to the $(r - 3)$-th prolongation $(\tilde{\Delta}^{(r-3)}, \tilde{\nu}_1^{r-2} C^{(r-2)})$ of the PDE system $\tilde{\Delta} = \{ \tilde{\Delta}_1 = \tilde{\Delta}_2 = 0 \} \hookrightarrow J_2^1 \mathbb{R}^5$.

5. **Reconstructing the solutions of the original EDS and the proof of Theorem 2**

**Definition 5.1.** Let $\mathcal{J}^1 \subset T^*M$ be a subbundle of the cotangent bundle. Denote by $\mathcal{J} \subset T^*M$ the ideal generated by $\mathcal{J}^1$. The EDS $(M, \mathcal{J})$ is called Frobenius if $d\Gamma(\mathcal{J}^1) \subset \Gamma(\mathcal{J}) \bigwedge \Omega^1(M)$.

In this case the manifold $M$ is foliated by $k = (\dim M - \dim \mathcal{J}^1_x)$-dimensional solutions of $(M, \mathcal{J})$.

Let $\tilde{S} \hookrightarrow M^{(r)}_k$ be a $k$-dimensional solution of the reduced EDS $\overline{\mathcal{E}}^{(r)}_k$. Consider $p^{-1}(\tilde{S}) \hookrightarrow M^{(r)}_k$. Define $\mathcal{J}(\tilde{S}) \overset{\text{def}}{=} i^* \mathcal{I}^{(r)}_k \subset T^* p^{-1}(\tilde{S})$.

\[ ^4 \text{In order to fit everything into one coordinate chart we actually cut off certain closed subset of zero measure from } J_2^{(r)} \mathbb{R}^3. \]
Proposition 5.2. Let $r \geq \max(r_s, r_{\text{cf}}) + 2$, then the exterior differential system $(p^{-1}(\bar{S}), J(\bar{S}))$ is Frobenius. The solutions of this EDS are transversal to the orbits of the $G$-action, and form a foliation of codimension $\dim G$.

Proof. Since $\bar{S}$ is a solution of the reduced EDS, $i^*p^*\bar{\theta} = 0$. We can apply the mapping $i^*$ to both sides of the equation (23), and conclude that

$$d i^*(\pi_{r^n}^r)^*\eta^j = \sum_{j=1}^{\dim G} i^*(\pi_{r^n}^r)^*\eta^j \wedge i^*\alpha_j^i.$$ 

Therefore the ideal $J(\bar{S})$ is algebraically generated by the 1-forms $\{i^*(\pi_{r^n}^r)^*\eta^j\}$. This proves that the EDS $(p^{-1}(\bar{S}), J(\bar{S}))$ is Frobenius.

Let $\bar{x} \in M_{k(\tau)}$ At every point $x \in p^{-1}(\bar{x}) \subset p^{-1}(\bar{S})$ the ideal $J_x(\bar{S})$ is generated by its 1-form component

$$J^1_x(\bar{S}) = \text{Span}\{i^*(\pi_{r^n}^r)^*\eta^j(x)\}_{j=1}^{\dim G}.$$ 

The commutative diagram

\[
\begin{array}{ccc}
\bar{S} & \xrightarrow{p_{\bar{S}}} & p^{-1}(\bar{S}) \\
\downarrow & & \downarrow^i \\
M_{k(\tau)} & \xleftarrow{p} & M_{k(\tau)}
\end{array}
\]

where the horizontal rows are principal $G$-bundles, and the vertical rows are embeddings, gives the following commutative diagram

\[
\begin{array}{ccc}
J^1_x(\bar{S}) & \xrightarrow{J^1_x(\bar{S})} & J^1_x(\bar{S}) \\
\downarrow & & \downarrow \\
0 & \xrightarrow{T_xp_{\bar{S}}} & T_xp^{-1}(\bar{S}) \text{ / } p^*T_xp_{\bar{S}} & \xrightarrow{\text{Im} \ p^*} & 0 \\
0 & \xrightarrow{T_xM_{k(\tau)}} & T_xM_{k(\tau)} \text{ / } p^*T_xM_{k(\tau)} & \xrightarrow{\text{Im} \ p^* \text{ / } T_xM_{k(\tau)}} & 0 \\
\downarrow & & \downarrow & & \downarrow \\
\text{Im} \ p^* & \xleftarrow{p^*} & \text{Im} \ p^* \text{ / } T_xM_{k(\tau)} & \xleftarrow{\text{Im} \ p^* \text{ / } T_xM_{k(\tau)}} & 0
\end{array}
\]

where the horizontal rows are exact, the leftmost vertical arrow is epimorphic, and $\text{Ker} \ i^* \subset \text{Im} \ p^*$. It is easy to see that this implies that the rightmost vertical arrow is a bijection, thus $\dim J^1_x(\bar{S}) = \dim \mathcal{P}_x = \dim G$ (here the subbundle $\mathcal{P} \subset T^*M_{k(\tau)}$ is defined in (24)). The transversality of the solutions and the group orbits follows from the decomposition $T_x^*p^{-1}(\bar{S}) = \text{Im} \ p^*_{\bar{S}} \oplus J^1_x(\bar{S})$. \qed
To prove Theorem 3 we need the following simple

Lemma 5.3. Let \( p : B \to \tilde{B} \) be a principal \( G \)-bundle. Assume that there exists a \( G \)-invariant Frobenius EDS \((B, \mathcal{J})\) such that \( \dim \mathcal{J}^1 = \dim G \), and the leaves of the foliation defined by \((B, \mathcal{J})\) are transversal to the fibers of \( p \). Then for every two connected leaves \( S_1, S_2 \) of this foliation there exists a group element \( g \in G \) such that \( gS_1 = S_2 \).

Proof of Theorem 3. For every \( S_r \in \text{Sol}_k^{\text{reg}}(E^{(r)}_k, G) \) the projection \( \bar{S} = p(S_r) \) depends only on the equivalence class of \( S_r \) in \( \frac{\text{Sol}_k^{\text{reg}}(E^{(r)}_k, G)}{G} \), and is a \( k \)-dimensional solution of the reduced EDS. Given a solution \( \bar{S} \in \text{Sol}_k(M^{(r)}_k) \) consider a \( k \)-dimensional solution \( S_r \to p^{-1}(\bar{S}) \to M^{(r)}_k \) of the Frobenius EDS \((p^{-1}(\bar{S}), \mathcal{J}(\bar{S}))\). Proposition 5.2 implies that \( S_r \) is a regular \( k \)-dimensional solution of \( E^{(r)}_k \). Lemma 5.3 implies that a different choice of a solution of \((p^{-1}(\bar{S}), \mathcal{J}(\bar{S}))\) lies in the same equivalence class of the moduli space. This completes the proof.

Remark 5.4. For every solution \( \bar{S} \) of the reduced EDS the forms \( i^*(\pi_r)^*\eta^j \in \Omega^1(p^{-1}(\bar{S})) \) define a flat connection in the principle bundle \( p^{-1}(\bar{S}) \to \bar{S} \). Thus the reconstruction of a solution \( S_r \to M^{(r)}_k \) constitutes finding the parallel transport of a point in \( p^{-1}(\bar{S}) \) w.r.t. this flat connection. In practical terms this means solving a sequence of \( k \) systems of ODEs.

6. Proof of Theorem 3 and computing the conservation laws of the syzygy equations.

Let \( G \) be a Lie group acting on a manifold \( M \). For every open subset \( U \subset J^\infty_k M \) consider a \( G \)-invariant Vinogradov spectral sequence \((E^{s,t}_r, d_{r}^{s,t})\) corresponding to the differential ideal \( \Gamma(\mathcal{C}(\infty)) \cap \Omega(U)^G \) in \( \Omega(U)^G \). Denote by \((\tilde{E}^{s,t}_r, \tilde{d}_{r}^{s,t})\) the Vinogradov spectral sequence \([10, 6]\) corresponding to the differential ideal \( \Gamma(\mathcal{C}(\infty)) \) in \( \Omega(p(U)) \). The mapping \( p : J^\infty_k M \to J^\infty_k \tilde{M} \) induces the morphism of spectral sequences \( p^* : \tilde{E}^{s,t}_r \to E^{s,t}_r \).

Lemma 6.1. The map \( p^* \) induces an isomorphism of characteristic cohomology:

\[
p^* : \tilde{E}^{0,1}_1 \cong E^{0,1}_1.
\]

Proof. Since \( dp^* - p^*d = 0 \), and \( p^*(\mathcal{C}(\infty)) \subset \mathcal{C}(\infty) \), \( p^* \) induces the mapping \( p^* : \tilde{E}^{0,1}_1 \to E^{0,1}_1 \). Due to Lemma 6.1 the mapping \( p^* : \tilde{E}^{0,1}_1 \to E^{0,1}_1 \) is an isomorphism, thus the induced mapping in cohomology is also an isomorphism.

The following theorem first was announced in the paper \([1]\) by I. Anderson, and J. Pohjanpelto for the special case when \( M = \mathbb{R}^k \times \mathbb{R}^q \), and the action of \( G \) is projectable w.r.t. the fibration \( \mathbb{R}^k \times \mathbb{R}^q \to \mathbb{R}^k \). It turns out that both these assumptions are superfluous.

Theorem 6.2. For every open subset \( U \subset J^\infty_k M \), for every \( s \geq 1 \), and \( t \neq k \) the corresponding \( G \)-invariant Vinogradov spectral sequence \( E^{s,t}_r \) satisfies

\[
E^{s,t}_1 = 0
\]

The proof is done by constructing a \( G \)-invariant variant of Spencer cohomology, and proving that it vanishes for the free complex. The complete proof will be given elsewhere \([8]\).
Corollary 6.3.

\begin{equation}
\label{eq:corollary}
E^{0,t}_1 \simeq H^t(\Omega(U)G, d), \quad 0 < t < k,
\end{equation}

\begin{equation}
E^{s,k}_2 \simeq H^{k+s}(\Omega(U)G, d),
\end{equation}

where \( H^t(\Omega(U)G, d) \) is the G-invariant deRham cohomology of \( U \subset J^\infty_k M \).

**Proof of Theorem 3** For every contractible \( \hat{U} \) as in the theorem consider \( U = p^{-1}(\hat{U}) \subset J^\infty_k M \). Using Lemma 6.1 and Corollary 6.3 we conclude that for every \( t < k \), \( E^{0,t}_1 \simeq H^t(\Omega(U)G, d) \).

To prove (11), observe that for every \( r > r_0 \), \( \pi_r^\infty U \to \pi_r^\infty \hat{U} \) is a principal G-bundle with contractible base, therefore \( H^t(\Omega(\pi_r^\infty U)G, d) \simeq H^t(\pi_r^\infty g) \). This implies that \( H^t(\Omega(U)G, d) \simeq H^t(\pi_r^\infty g) \), thus completing the proof. \( \square \)

Now we would like to describe the practical algorithm for computing the representatives in the characteristic cohomology classes of the syzygy equations.

**The practical algorithm.**

1. We may identify \( \bigwedge g^* \) with right-invariant differential forms on \( G \). Therefore the basis \( \tilde{\omega}_1, \ldots, \tilde{\omega}_N \in \bigoplus_{t=1}^{k-1} H^t(\pi_r^\infty g) \) gives the closed forms \( \tilde{\omega}_i \) in \( \Omega_{\text{right-inv}}(G) \). For a given contractible subset \( \hat{U} \in J^\infty_k M \) choose a right moving frame \( \hat{U} \), i.e. a mapping \( \rho : \hat{U} \to G \), such that \( \rho(gz) = \rho(z)g^{-1} \). The pullbacks \( \omega_i = (\pi_r^\infty)^* \rho^* \tilde{\omega}_i \) represent the basis in \( \bigoplus_{t=1}^{k-1} H^t(\Omega(U)G, d) \).

2. Using the G-invariant coframe \( \{\eta^i, dy^i, \theta^o\} \) in \( (\pi_r^\infty)^* \Omega^1(J^\infty_k M) \) we may rewrite each of the forms \( \omega_i \) as \( \omega_i = \omega_{i_1, \ldots, i_t} dy^{i_1} \wedge \cdots \wedge dy^{i_t} + \Gamma(C^{(t)}) \).

It is easy to see that the function \( \omega_{i_1, \ldots, i_t} \) are G-invariant, thus we may consider the forms on the reduced jet space \( \tilde{\omega}_i = \omega_{i_1, \ldots, i_t} dy^{i_1} \wedge \cdots \wedge dy^{i_t} \in \Omega^t(J^\infty_k M) \).

These forms represent the basis of characteristic cohomology classes of \( \hat{\mathcal{E}}_0 \), or using different terminology, nontrivial conservation laws \( \{12\} \) of the syzygy equations \( \Delta_\nu = 0 \).

**Example 6.1.** Let us compute the nontrivial conservation laws for the syzygy equations \( \Delta = \{\Delta_1 = \Delta_2 = 0\} \subset \mathbb{R}^5 \) in the example \( \{11\} \). The Lie algebra cohomology of \( \mathbb{R}^3 \) is given by the generators \( dx^1, dx^2, du \). The moving frame \( \mathbb{R}^3 \to \mathbb{R}^3 \) is the multiplication by \(-1\), thus the forms \( dx^1, dx^2, du \) represent the basis in \( H^1(\Omega(J^\infty_2\mathbb{R}^3)G, d) \). Using the forms \( \eta^1, \eta^2, \eta^3 \) \( \{11\} \) we can notice that

\begin{align*}
dx^1 &= \frac{1}{v^2} \left( v^2 dy^1 - v^3 dy^2 \right) + \Gamma(C^{(2)}), \\
dx^2 &= \frac{1}{v^2} \left( v^3 dy^2 - v^2 dy^1 \right) + \Gamma(C^{(2)}), \\
du &= y^1 dx^1 + y^2 dx^2 + \Gamma(C^{(2)}).
\end{align*}

Therefore the forms

\begin{align*}
\bar{\omega}_1 &= \frac{1}{v^2} \left( v^2 dy^1 - v^3 dy^2 \right), \\
\bar{\omega}_2 &= \frac{1}{v^2} \left( v^3 dy^2 - v^2 dy^1 \right), \\
\bar{\omega}_3 &= y^1 \bar{\omega}_1 + y^2 \bar{\omega}_2
\end{align*}

give the basis of nontrivial conservation laws for the syzygy equations \( \{12\} \).
7. Invariant Euler-Lagrange equations.

Consider the increasing filtration \( F_1 \subset F_2 \subset \cdots \subset F_r = \Gamma(C^{(r)}) \cap \Omega^1(J^r_k M) \), where

\[
F_r \stackrel{\text{def}}{=} (\pi^*_r)^* \Gamma(C^{(r)}) \cap \Omega^1(J^r_k M).
\]

Outside of a certain set of zero measure \( F_r \) is a space of sections of a certain subbundle of \( T^* J^r_k M \). For each of these subbundles we can apply Lemma 6.3 (in a small neighborhood of every point), and find \( G \)-invariant contact forms \( \eta^\alpha_r \in \Omega^1(J^r_k M)^G \cap \Gamma(C^{(r)}) \) such that for each \( r \leq r_o \) Span\( \{\eta^\alpha_r\}_{1 \leq \alpha \leq q} \) is a basis of \( F_{r+1} \).

Denote by \( dy = dy^1 \wedge \cdots \wedge dy^k \) the horizontal volume on the reduced jet space. The following lemma gives the group-invariant version of the integration by parts used in the deducing the Euler-Lagrange equations.

**Lemma 7.1.** For every \( \eta^\alpha_r, |\alpha| > 0 \) there exist invariant total differential operators \( \hat{T}^\alpha_{\alpha'} = T^\alpha_{\alpha'}(dy) + T^\alpha_{\alpha'}(\text{here } T^\alpha_{\alpha'}, T^\alpha_{\alpha'} \in C^\infty(J^r_k M) \) \), such that for every \( \bar{f} \in C^\infty(J^r_k M) \)

\[
(p^* \bar{f})[\eta^\alpha_r \wedge p^* dy]_0 = \sum_{|\alpha'| < |\alpha|; \alpha' = 1, \ldots, q} (p^* \hat{T}^\alpha_{\alpha'}(\bar{f})[\eta^\alpha_r' \wedge p^* dy]_0 + d_0^{1,k-1}|\chi|_0
\]

for some \( \chi \in (F_{r+1})^G \wedge p^*(C^\infty(J^r_k M))dy \).

The proof is based on the same fact about the (noninvariant) standard contact forms \( \theta^\alpha_r ) \).

**Corollary 7.2.** Let \( \bar{\theta}^a_r \in \Gamma(C^{(r)}) \) be the generating 1-forms of the reduced ideal \( \bar{C}^{(r)} \).

Then there exist total differential operators

\[
\hat{A}^a_r : C^\infty(J^r_k M) \to C^\infty(J^r_k M) + 1
\]

(here \( r \geq r_o \),

\[
\hat{A}^a_r = \sum_{0 \leq |\alpha'| \leq r_o} A^a_{\alpha'} \frac{d|\alpha'|}{dy}, \quad A^a_{\alpha'} \in C^\infty(J^r_k M)
\]

such that for every \( \bar{f} \in C^\infty(J^r_k M) \)

\[
p^* \bar{\theta}^\alpha_r \wedge \bar{f}dy = \sum_{\alpha=1}^q (p^* \hat{A}^a_r(\bar{f})[\eta^\alpha_r \wedge p^* dy]_0 + d_0^{1,k-1}|\chi|_0,
\]

for some \( |\chi|_0 \in E^1_k \), where \( \{\eta^\alpha_r\}_{1 \leq \alpha \leq q} \) are the basis of forms in \( F_1 \).

**Proof of Theorem 4.** Let \( |\lambda|_1 \) be a \( G \)-invariant variational problem, then there exists \( \bar{\lambda} = \bar{L}dy \in \Omega^k(J^r_k M) \) such that \( p^* \bar{\lambda} = [\lambda]_1 \). Using the above corollary we conclude that

\[
d_1^{0,k}[\lambda]_1 = d_1^{0,k}[p^* \bar{L}dy]_1 = p^* d_1^{0,k}[\bar{L}d\lambda]_1 = p^* \sum_{\alpha=1}^q \bar{E}_\alpha(\bar{L})\bar{\theta}^\alpha \wedge dy|_1 =
\]

\[
= \sum_{\alpha=1}^q \left( p^* \sum_{a=1}^q \hat{A}^a_r(E_\alpha(\bar{L})) \right) [\eta^\alpha_r \wedge p^* dy]_1 = \left( p^* \hat{A}^a_r(E_\alpha(\bar{L})) \right) \frac{dy}{dx} c^\alpha_r [\theta^\alpha_r \wedge dx]_1
\]
where $\theta^\alpha \in \Gamma(C^{(1)})$ are the standard contact forms corresponding to the choice of local coordinates $(x^i, u^i)$ on $M$, $\bar{\eta}^\alpha = c_\alpha^p \theta^p$, and $\hat{p}^*dy = \frac{dx}{\alpha}$. Since the matrix $(\frac{dx}{\alpha} c_\alpha^p)$ is nondegenerate the formulas $[\eta]$, and $[\alpha]$ imply $[\beta]$.

**Remark.** Since the functions $(\frac{dx}{\alpha} c_\alpha^p)$ depend only on the choice of the horizontal volumes, and the basis of contact forms, the equality $[\eta]$ implies that for every $\alpha = 1, \ldots, q$ the function $\sum_{\alpha=1}^{\beta} A_\alpha^\alpha(E_\alpha(L))$ does not depend on the choice of the Lagrangian $L_1$ used in the definition (formula $[\alpha]$) of $E_\alpha(L)$.

Now we would like to describe the practical algorithm of computing the operators $A_\alpha^\alpha$.

**The practical algorithm.**

1. We can compute the forms $\bar{\eta}^\alpha$ by applying the moving frame construction (described in the proof of Lemma $[\beta]$) consecutively to each of the subbundles $(\pi_c^s)^*C^{(r)} \subset T^*J^\infty_1 M$.
2. For every $r$, $0 \leq r < r_0$, consider the system of equations

   $$(I', \alpha', i') \quad a_{0,i}^{k-1}[\bar{\eta}^\beta_{I'} \wedge d\nu_i]|_{0} = \sum_{|I'| = r+1} \hat{p}^*(\bar{f}^\alpha_{I'} \wedge d\nu_i)| \bar{\eta}^\beta_{I'} \wedge dy|_{0} + \sum_{|I'| \leq r} \hat{p}^*(\bar{f}^\alpha_{I'} \wedge d\nu_i)| \bar{\eta}^\beta_{I'} \wedge dy|_{0}$$

3. We can rewrite the forms $\bar{\eta}^\alpha \in (F_{\alpha})^G$ as a linear combination (over the ring $C^\infty(J^\infty_1 M)$) of the forms $\bar{\eta}^\alpha$. Consecutively using the formula $[\alpha]$ we obtain the operators $A_\alpha^\alpha$.

**Example 7.1.** Consider the (nonprojectable) action of the group of Euclidean motions $G = SE(2)$ on $M = \mathbb{R}^2$. Introduce the standard jet coordinates $(x, u, u_1, u_2, \ldots)$ on $J_1^\infty \mathbb{R}^2$. The Euclidean curvature $\kappa = u_2(1 + u_1^2)^{-3/2}$, and its derivative with respect to the arclength $\kappa_s = u_3(1 + u_1^2)^{-2} - 3u_1u_2(1 + u_1^2)^{-3}$ provide the local coordinates $y = \kappa, v = \kappa_s$ on the reduced jet space $J_1^\infty \mathbb{R}^2$. Here $r_0 = 4$, and the reduced EDS $E^{(4)} = (J_1^\infty \mathbb{R}^2, C^{(4)})$ is isomorphic to the first jet space of curves: $E^{(4)} = (J_1^\infty \mathbb{R}^2, C^{(4)})$. Thus the reduced infinite jet space is again the infinite jet space of curves $\mathbb{R}^2$. In particular the reduced Euler-Lagrange operators $[\beta]$ coincide with the usual ones in $J_1^\infty \mathbb{R}^2$.

Let $(c_1, c_2, \phi)$ be the coordinates on the group $SE(2)$ such that the action on $\mathbb{R}^2 = C$ is given by the formula

$$ (c_1, c_2, \phi)(x + iu) = e^{i\phi}(x + iu) + c_1 + ic_2. $$

We can use the right moving frame $\rho : J_1^\infty \mathbb{R}^2 \to SE(2)$,

$$ \rho(x, u, u_1) = (c_1, c_2, \phi) = \left( -\frac{x + uu_1}{\sqrt{1 + u_1^2}}, \frac{xu_1 - u}{\sqrt{1 + u_1^2}}, -\tan^{-1} u_1 \right) $$

$^5$In fact it is true for any group action on $\mathbb{R}^2$. 


to pull back the right Maurer-Cartan forms
\[ \mu^1 = d\phi, \quad \mu^2 = dc_1 + c_2 d\phi, \quad \mu^3 = dc_2 - c_1 d\phi, \]
and obtain the basis
\[ \zeta^i \equiv \rho^i \mu^i, \quad i = 1, 2, 3 \]
of invariant 1-forms in \( \Omega^1(J^\infty_1 \mathbb{R}^2)^G / p^* \Omega^1(J^\infty_1 \mathbb{R}^2) \). Using the procedure given in the proof of Lemma 3.5 we obtain the filtered basis in \( \Gamma(C) \cap \Omega^1(J^\infty_1 \mathbb{R}^2)^G : \)
\[ \tilde{\eta}_0 = \zeta^1, \quad \tilde{\eta}_1 = y\zeta^2 - \zeta^1, \quad \tilde{\eta}_2 = dy + v\zeta^2, \quad \tilde{\eta}_3 = p^* \tilde{\theta}_0 \]
(here \( y = \kappa, \ v = \kappa_s \), and \( \tilde{\theta}_0 = dv - v_1 dy \)). The table of horizontal differentiation
\[ \begin{align*}
  d^{1,0}_0[\tilde{\eta}_0]_0 &= \left[ \frac{1}{v} \tilde{\eta}_1 \wedge dy \right]_0, \\
  d^{1,0}_0[\tilde{\eta}_1]_0 &= \left[ \frac{y^2}{v} dy \wedge \tilde{\eta}_0 + \frac{1}{v} dy \wedge \tilde{\eta}_2 \right]_0, \\
  d^{1,0}_0[\tilde{\eta}_2]_0 &= \left[ \frac{v_1}{v} dy \wedge \tilde{\eta}_2 + y dy \wedge \tilde{\eta}_0 + \frac{1}{v} dy \wedge p^* \tilde{\theta}_0 \right]_0
\end{align*} \]
allows us to compute the operators \( \hat{T} \) in the formula (35):
\[ (p^* \hat{f})[p^* (\tilde{\theta}_0 \wedge dy)]_1 = -p^*(2v_1 \hat{f} + v \frac{df}{dy})[\tilde{\eta}_2 \wedge p^* dy]_1 - p^*(vy \hat{f})[\tilde{\eta}_0 \wedge p^* dy]_1, \]
\[ (p^* \hat{f})[\tilde{\eta}_2 \wedge p^* dy]_1 = -p^*(v_1 \hat{f} + v \frac{df}{dy})[\tilde{\eta}_1 \wedge p^* dy]_1 - p^*(y^2 \hat{f})[\tilde{\eta}_0 \wedge p^* dy]_1, \]
\[ (p^* \hat{f})[\tilde{\eta}_1 \wedge p^* dy]_1 = p^*(v_1 \hat{f} + v \frac{df}{dy})[\tilde{\eta}_0 \wedge p^* dy]_1, \]
therefore the operator \( \hat{A} \) is computed by composing the operators \( \hat{T} \):
\[ \hat{A} = \left( (v_1 + v \frac{d}{dy})^2 + y^2 \right) (2v_1 + v \frac{d}{dy}) - vy, \]
and the Euler-Lagrange system of every invariant variational problem \( \lambda = Ldx = \hat{L}dy + \Gamma(C(\infty)) \) is the lift of the equation \( \hat{A}(\sum_{i=0}^\infty (-\frac{d}{dy})^i \frac{d}{dy}) = 0 \) on the reduced jet space.

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