A Theorem of the Alternative for Personalized Federated Learning

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Abstract

A widely recognized difficulty in federated learning arises from the statistical heterogeneity among clients: local datasets often come from different but not entirely unrelated distributions, and personalization is, therefore, necessary to achieve optimal results from each individual’s perspective. In this paper, we show how the excess risks of personalized federated learning with a smooth, strongly convex loss depend on data heterogeneity from a minimax point of view. Our analysis reveals a surprising theorem of the alternative for personalized federated learning: there exists a threshold such that (a) if a certain measure of data heterogeneity is below this threshold, the FedAvg algorithm [McMahan et al., 2017] is minimax optimal; (b) when the measure of heterogeneity is above this threshold, then doing pure local training (i.e., clients solve empirical risk minimization problems on their local datasets without any communication) is minimax optimal. As an implication, our results show that the presumably difficult (infinite-dimensional) problem of adapting to client-wise heterogeneity can be reduced to a simple binary decision problem of choosing between the two baseline algorithms. Our analysis relies on a new notion of algorithmic stability that takes into account the nature of federated learning.

1 Introduction

As one of the most important ingredients driving the success of machine learning, data have been being generated and subsequently stored in an increasingly decentralized fashion in many real-world applications. For example, mobile devices will in a single day collect an unprecedented amount of data from users. These data commonly contain sensitive information such as web search histories, online shopping records, and health information, and thus are often not available to service providers [Poushter, 2016]. This decentralized nature of (sensitive) data poses substantial challenges to many machine learning tasks.

To address this issue, McMahan et al. [2017] proposed a new learning paradigm, which they termed federated learning, for collaboratively training machine learning models on data that are locally possessed by multiple clients with the coordination of the central server (e.g., service provider), without having direct access to the local datasets. In its simplest form, federated learning considers a pool of m clients, where the i-th client has a local dataset $S_i$ of size $n_i$, consisting of i.i.d. samples.
To better understand this point, consider the extreme heterogeneous scenario, the objective function \(\text{FedAvg}\) in effect learns a shared global model using gradients from each client and outputs a single model as an estimate of \(w^*_i\) for all clients. When the distributions \(\{\mathcal{D}_i\}\) coincide with each other, \(\text{FedAvg}\) with a strongly convex loss achieves a weighted average excess risk of \(O(1/N)\), which is minimax optimal up to a constant factor [Shalev-Shwartz et al., 2009, Agarwal et al., 2012], see the formal statement in Theorem 3.2.

However, it is an entirely different story in the presence of data heterogeneity. \(\text{FedAvg}\) has been recognized to give inferior performance when there is a significant departure from complete homogeneity (see, e.g., Bonawitz et al. 2019). To better understand this point, consider the extreme case where the data distributions \(\{\mathcal{D}_i\}\) are entirely unrelated. This roughly amounts to saying that the model parameters \(\{w^*_i\}\) can be arbitrarily different from each other. In such a “completely heterogeneous” scenario, the objective function (1.2) simply has no clear interpretation, and any single global model—for example, the output of \(\text{FedAvg}\)—would lead to unbounded risks for most,
if not all, clients. As a matter of fact, it is not difficult to see that the optimal training strategy for federated learning in this regime is arguably PureLocalTraining, which lets each client separately run SGD to minimize its own local ERM objective

$$\min_{w^{(i)}} L_i(w^{(i)}, S_i)$$

(1.3)

without any communication. Indeed, PureLocalTraining is minimax rate optimal in the completely heterogeneous regime, just as FedAvg in the completely homogeneous regime (see Theorem 3.2).

The level of data heterogeneity in practical federated learning problems is apparently neither complete homogeneity nor complete heterogeneity. Thus, the foregoing discussion raises a pressing question of what would happen if we are in the wide middle ground of the two extremes. This underlines the essence of personalized federated learning, which seeks to develop algorithms that perform well over a wide spectrum of data heterogeneity. Despite a venerable line of work on personalized federated learning (see, e.g., Kulkarni et al. 2020), the literature remains relatively silent on how the fundamental limits of personalized federated learning depend on data heterogeneity, as opposed to two extreme cases where both the minimax optimal rates and algorithms are known.

1.1 Main Contributions

The present paper takes a step toward understanding the statistical limits of personalized federated learning by establishing the minimax rates of convergence for both individualized excess risks and their weighted average with smooth strongly convex losses. We briefly summarize our main contributions below.

1. We prove that if the client-wise sample sizes are relatively balanced, then there exists a problem instance on which the IER_i’s of any algorithm are lower bounded by

\[
\begin{cases}
\Omega(1/N + R^2) & \text{if } R^2 = O(m/N) \\
\Omega(m/N) & \text{if } R^2 = \Omega(m/N),
\end{cases}
\]

(1.4)

where \( R^2 := \min_{w \in W} \sum_{i \in [m]} n_i \| w^{(i)} - w \|^2 / N \) measures the level of heterogeneity among clients (here \( \| \cdot \| \) throughout the paper denotes the Euclidean distance). Meanwhile, we show that the IER_i’s of FedAvg are upper bounded by \( O(1/N + R^2) \), whereas the guarantee for PureLocalTraining is \( O(m/N) \), regardless of the specific value of \( R \). Moreover, we also establish similar upper and lower bounds for a weighted average of the IER_i’s under a weaker condition.

2. A closer look at the above-mentioned bounds reveals a surprising theorem of the alternative, which states that given a problem instance with a specified level of heterogeneity, either FedAvg is minimax optimal, or PureLocalTraining is minimax optimal. Such a statement is reminiscent of the celebrated Fredholm alternative in functional analysis [Fredholm, 1903] and Farkas’ lemma in linear programming [Farkas, 1902], both of which give two assertions and state that exactly one of them must hold.

3. With the established theorem of the alternative, the originally infinite-dimensional problem of adapting to client-wise heterogeneity is reduced to a binary decision problem of making a choice

\footnote{Technically, our result is slightly weaker than a theorem of the alternative, as there are scenarios (i.e., when \( R^2 \approx m/N \)) where the two algorithms are simultaneously minimax optimal. See Remark 3.1 for more details.}
between the two algorithms. Indeed, the foregoing results suggest that the naïve dichotomous strategy of (1) running FedAVG when $R^2 = O(m/N)$, and (2) running PureLocalTraining when $R^2 = \Omega(m/N)$, attains the lower bound (14). Moreover, for supervised problems, this dichotomous strategy can be implemented without knowing $R$ by (1) running both FedAVG and PureLocalTraining, (2) evaluating the test errors of the two algorithms, and (3) deploying the algorithm with a lower test error. We emphasize that the notion of optimality under our consideration overlooks constant factors. In practice, a better personalization result could be achieved by more sophisticated algorithms.

4. As a side product, we provide a novel analysis of FedProx, a popular algorithm for personalized federated learning that constrains the learned local models to be close via $\ell_2$ regularization [Li et al., 2018]. In particular, we show that its IERs are of order $O\left(\frac{1}{N/m} \land \frac{R}{\sqrt{N/m}} + \frac{\sqrt{m}}{N}\right)$, and a weighted average of the IERs satisfies a tighter $O\left(\frac{1}{N/m} \land \frac{R}{\sqrt{N/m}} + \frac{1}{N}\right)$ bound, where $a \land b = \min\{a, b\}$ for two real numbers $a$ and $b$.

5. On the technical side, our upper bound analysis is based on a generalized notion of algorithmic stability [Bousquet and Elisseeff, 2002], which we term federated stability and can be of independent interest. Briefly speaking, an algorithm $A(S) = \{\hat{w}^{(i)}(S)\}$ has federated stability $\{\gamma_i\}$ if for any $i \in [m]$, the loss function evaluated at $\hat{w}^{(i)}(S)$ has federated stability $\gamma_i$ if we perturb $S_i$ a little bit, while keeping the rest of datasets $\{S_i' : i' \neq i\}$ fixed. Similar ideas have appeared in Maurer [2005] and have been recently applied to multi-task learning [Wang et al., 2018]. However, their notion of perturbation is based on the deletion of the whole client-wise dataset, whereas our notion of federated stability operates at the “record-level” and is more fine-grained. On the other hand, our construction of the lower bound is based on a generalization of Assouad’s lemma [Assouad, 1983] (see also Yu 1997), which enables us to handle multiple heterogeneous datasets.

1.2 Related Work

Ever since the proposal of federated learning by McMahan et al. [2017], recent years have witnessed a rapidly growing line of work that is concerned with various aspects of FedAVG and its variants (see, e.g., Khaled et al. 2019, Haddadpour and Mahdavi 2019, Li et al. 2020b, Bayoumi et al. 2020, Malinovsky et al. 2020, Li and Richtárik 2020, Woodworth et al. 2020, Yuan and Ma 2020, Zheng et al. 2021).

In the context of personalized federated learning, there have been significant algorithmic developments in recent years. While the idea of using $\ell_2$ regularization to constrain the learned models to be similar has appeared in early works on multi-task learning [Evgeniou and Pontil, 2004], its applicability to personalized federated learning was only recently demonstrated by Li et al. [2018], where the FedProx algorithm was introduced. Similar regularization-based methods have been proposed and analyzed from the scope of convex optimization in Hanzely and Richtárik [2020], Dinh et al. [2020], and Hanzely et al. [2020]. In particular, Hanzely et al. [2020] showed that an accelerated variant of FedProx is optimal in terms of communication complexity and the local oracle complexity. There is also a line of work using model-agnostic meta learning [Finn et al., 2017] to achieve personalization [Jiang et al., 2019, Fallah et al., 2020]. Other strategies have been proposed (see, e.g., Arivazhagan et al. 2019, Li and Wang 2019, Mansour et al. 2020, Yu et al. 2020), and we refer readers to Kulkarni et al. [2020] for a comprehensive survey. We briefly remark here that all the papers mentioned above only consider the optimization properties of their proposed algorithms, while we focus on statistical properties of personalized federated learning.
Compared to the optimization understanding, our statistical understanding (in terms of sample complexity) of federated learning is still limited. Deng et al. [2020] proposed an algorithm for personalized federated learning with learning-theoretic guarantees. However, it is unclear how their bound scales with the heterogeneity among clients.

More generally, exploiting the information “shared among multiple learners” is a theme that constantly appears in other fields of machine learning such as multi-task learning [Caruana, 1997], meta learning [Baxter, 2000], and transfer learning [Pan and Yang, 2009], from which we borrow a lot of intuitions (see, e.g., Ben-David et al. 2006, Ben-David and Borbely 2008, Ben-David et al. 2010, Maurer et al. 2016, Cai and Wei 2019, Hanneke and Kpotufe 2019, 2020, Du et al. 2020, Tripuraneni et al. 2020a,b, Kalan et al. 2020, Shui et al. 2020, Li et al. 2020a, Zhang et al. 2020, Shui et al. 2020, Bai et al., 2019, and Caruana 2019). We conclude this paper with a discussion of open problems in Section 5. For brevity, detailed proofs are deferred to the appendix.

1.3 Paper Organization

The rest of this paper is organized as follows. In Section 2, we give an exposition of the problem setup and main assumptions. Section 3 presents our main results. In Section 4, we present the general statements of our main results with relaxed assumptions and give an overview of our proof strategies. We conclude this paper with a discussion of open problems in Section 5. For brevity, detailed proofs are deferred to the appendix.

2 Problem Setup

In this section, we detail some preliminaries to prepare the readers for our main results.

Notation. We introduce the notation we are going to use throughout this paper. For two real numbers $a, b$, we let $a \lor b = \max\{a, b\}$ and $a \land b = \min\{a, b\}$. For two non-negative sequences $a_n, b_n$, we denote $a_n \lesssim b_n$ (resp. $a_n \gtrsim b_n$) if $a_n \leq C b_n$ (resp. $a_n \geq C b_n$) for some constant $C > 0$ when $n$ is sufficiently large. We use $a_n \asymp b_n$ to indicate that $a_n \gtrsim b_n, a_n \lesssim b_n$ hold simultaneously. We also use $a_n = O(b_n)$, whose meaning is the same as $a_n \lesssim b_n$, and $a_n = \Omega(b_n)$, whose meaning is the same as $a_n \gtrsim b_n$. We use $\mathcal{W}$ to denote the parameter space and $\mathcal{Z}$ to denote the sample space. Finally, we let $\mathcal{P}_{\mathcal{W}}(x) := \arg\min_{y \in \mathcal{W}} \|x - y\|$ denote the operator that projects $x$ onto $\mathcal{W}$ in Euclidean distance.

Evaluation Metrics. The presentation of our main results relies on how to evaluate the performance of a federated learning algorithm. To this end, we consider the following two evaluation metrics.

**Definition 2.1.** Consider an algorithm $\mathcal{A}$ that outputs $\mathcal{A}(S) = \{\hat{w}^{(i)}(S)\}_{i=1}^m$ for the $m$ clients. For the $i$-th client, its *individualized excess risk* (IER) is defined as

$$\text{IER}_i(\mathcal{A}) := \mathbb{E}_{Z_i \sim \mathcal{D}_i}[\ell(\hat{w}^{(i)}(S), Z_i) - \ell(w^*_i, Z_i)], \quad (2.1)$$
where \(Z_i \sim D_i\) is a fresh data point independent of \(S\). In addition, the average excess risk (AER) of \(A\) is defined as

\[
\text{AER}(A) := \frac{1}{N} \sum_{i \in [m]} n_i \cdot \text{IER}_i(A). 
\] (2.2)

Recall that \(N = \sum_{i \in [m]} n_i\) is the total sample size.

In words, the IER measures the performance of the algorithm from the client-wise perspective, whereas the AER evaluates the performance of the algorithm from the system-wide perspective.

Notably, while a uniform upper bound on all the IER\(_i\)'s can be carried over to the same bound on the AER, a bound on the AER alone in general does not imply a tight bound on each IER\(_i\), other than the trivial bound \(\text{IER}_i \leq \text{AER} \cdot N/n_i\). Such a subtlety is a distinguishing feature of personalized federated learning in the following sense: under homogeneity, it suffices to estimate a single shared global model, and thus the AER and all of the IER\(_i\)'s are mathematically equivalent.

In the definition of the AER (2.2), the weight we put on each client is \(n_i/N\). In some situations, one may want to use other weighting schemes (e.g., use the uniform weight \(1/m\) to ensure “fairness” among clients). While our main results in Section 3 are stated for this choice of weights, our upper bounds can be readily generalized to other weighting schemes. See Section 4 for the general statements.

**Regularity Conditions.** In this paper, we restrict ourselves to bounded, smooth, and strongly convex loss functions. Such assumptions are common in the federated learning literature (see, e.g., Li et al. 2020b, Hanzely et al. 2020) and cover many unsupervised learning problems such as mean estimation in exponential families and supervised learning problems such as generalized linear models.

**Assumption A (Regularity conditions).** Suppose the following conditions hold:

(a) **Compact and convex domain.** The parameter space \(W\) is a compact convex subset of \(\mathbb{R}^d\) with diameter \(D := \sup_{w, w' \in W} \|w - w'\| < \infty\);

(b) **Smoothness and strong convexity.** For any \(i \in [m]\), the loss function \(\ell(\cdot, z)\) is \(\beta\)-smooth for almost every \(z\) in the support of \(D_i\), and the \(i\)-th ERM objective \(L_i(\cdot, S)\) is almost surely \(\mu\)-strongly convex on the convex domain \(W \subseteq \mathbb{R}^d\). We also assume that there exists a universal constant \(\|\ell\|_\infty\) such that \(0 \leq \ell(\cdot, z) \leq \|\ell\|_\infty\) for almost every \(z\) in the support of \(D_i\);

(c) **Bounded gradient variance at optimum.** There exists a positive constant \(\sigma\) such that for any \(i \in [m]\), we have \(\mathbb{E}_{Z_i \sim D_i} \|\nabla \ell(w^*_i, Z_i)\|^2 \leq \sigma^2\).

**Heterogeneity Conditions.** To quantify the level of heterogeneity among clients, we start by introducing the notion of an average global model. Assuming a strongly convex loss, the optimal local models (1.1) are uniquely defined. Thus, we can define the average global model as

\[
\mathbf{w}_{\text{avg}}^{(\text{global})} = \frac{1}{N} \sum_{i \in [m]} n_i \mathbf{w}_i^{(i)}.
\]

We remark that the average global model \(\mathbf{w}_{\text{avg}}^{(\text{global})}\) defined above should not be interpreted as the “optimal global model”. Rather, it is more suitable to think of \(\mathbf{w}_{\text{avg}}^{(\text{global})}\) as a point in the parameter space, from which every local model is close to. Indeed, a bit of analysis shows that the average global model is the minimizer of \(\frac{1}{N} \sum_{i \in [m]} n_i \|\mathbf{w}_i^{(i)} - \mathbf{w}_{\text{avg}}^{(\text{global})}\|^2\).

We are now ready to quantify the level of client-wise heterogeneity as follows.

**Assumption B (Level of heterogeneity).** There exists a positive constant \(R\) such that
either \( \frac{1}{N} \sum_{i \in [m]} n_i \| \mathbf{w}_i - \mathbf{w}_{\text{avg}} \|_2^2 \leq R^2 \),

(b) or \( \| \mathbf{w}_i - \mathbf{w}_{\text{avg}} \|_2^2 \leq R^2 \) for all \( i \in [m] \).

Our study of the AER and IER will be based on Part (a) and (b) of Assumption B, respectively. Note that this assumption is slightly more general than what we have imposed in Section 1.1, where we considered exact equalities.

3 Main Results

This section presents our main results. For ease of exposition, many results are stated in their special forms, and we refer readers to Section 4 for the general forms.

3.1 Fundamental Limits and Costs of Heterogeneity

The main result in this subsection is the following minimax lower bound, which characterizes the information-theoretic limits of personalized federated learning.

**Theorem 3.1** (Minimax lower bound). Assume there exist constants \( C, C' > 0, c \geq 0 \) such that

\[ n_i \geq C \beta \quad \forall i \in [m] \quad \text{and} \quad m \leq C' \left( \frac{N}{m} \right)^c. \]

Moreover, assume \( n_i \approx n_{i'} \) for any \( i \neq i' \in [m] \). Then there exists an absolute constant \( c' \) such that the following two statements hold:

1. There exists a problem instance such that Assumptions A and B(a) are satisfied with probability at least \( 1 - e^{-c' \sqrt{N/m}} \). Call this high probability event \( \mathcal{E} \). On this problem instance, any randomized algorithm \( \mathcal{A} \) must necessarily suffer

\[
E_{\mathcal{A},S}[\text{AER}(\mathcal{A}) \cdot 1_{\mathcal{E}}] \gtrsim \mu \cdot \left( \frac{\beta}{N/m} \wedge R^2 + \frac{\beta}{N} \right). \tag{3.1}
\]

2. For any \( i \in [m] \), there exists a problem instance such that Assumptions A and B(b) are satisfied with probability at least \( 1 - e^{-c' \sqrt{N/m}} \). Call this high probability event \( \mathcal{E}_i \). On this problem instance, any randomized algorithm \( \mathcal{A} \) must necessarily suffer

\[
E_{\mathcal{A},S}[\text{IER}_i(\mathcal{A}) \cdot 1_{\mathcal{E}_i}] \gtrsim \mu \cdot \left( \frac{\beta}{n_i} \wedge R^2 + \frac{\beta}{N} \right). \tag{3.2}
\]

In the two displays above, the expectation is taken over the randomness in both the algorithm \( \mathcal{A} \) and the sample \( S \).

**Proof.** See Section 4.1. \( \square \)

Focusing on the dependence on the sample sizes, the above theorem tells that the heterogeneity measure \( R \) enters the lower bounds in a dichotomous fashion:

- If \( R^2 \lesssim m/N \), then both lower bounds become \( \Omega(1/N) \), which agree with the minimax rate as if we were under complete homogeneity;

- If \( R^2 \gtrsim m/N \), then both lower bounds become \( \Omega(m/N) \). When sample sizes are balanced, they agree with the minimax rate as if we were under complete heterogeneity.

The proof is based on a generalization of Assouad’s lemma [Assouad, 1983], which enables us to handle multiple heterogeneous datasets. Curious readers may wonder why our working assumptions are only satisfied with high probability in the statement of the above theorem. This is because we construct this problem instance as a logistic regression problem with random design, for which Assumption A(b) holds with high probability.
3.2 Analysis of the Two Baseline Algorithms

In this subsection, we characterize the performance of two baseline algorithms, namely FedAvg and PureLocalTraining, under the heterogeneity conditions imposed by Assumption B. Under our working assumptions, FedAvg is guaranteed to converge to the global optimum under a proper hyperparameter choice (see, e.g., Khaled et al. 2019, Li et al. 2020b, Bayoumi et al. 2020), and so does PureLocalTraining (see, e.g., Rakhlin et al. 2011). Hence, our analyses are to be conducted for the exact minimizers of (1.2) and (1.3), respectively. This is without loss of generality, as the bounds for the approximate minimizers only involve an extra additive term representing the optimization error, and this term will be negligible since our focus is sample complexity.

**Theorem 3.2 (Performance of two baseline algorithms).** Let assumptions A hold, and assume \( n_i \geq 4\beta/\mu \) for any \( i \in [m] \). Suppose the FedAvg algorithm \( A_{FA} \) and the PureLocalTraining algorithm \( A_{PLT} \) output the exact minimizers of (1.2) and (1.3), respectively. Then, under Assumption B(a), we have

\[
\mathbb{E}_S[\text{AER}(A_{FA})] \lesssim \frac{\beta \| \ell \|_\infty}{\mu N} + \beta R^2, \tag{3.3}
\]

\[
\mathbb{E}_S[\text{AER}(A_{PLT})] \lesssim \frac{\beta \| \ell \|_\infty}{\mu N/m}. \tag{3.4}
\]

Meanwhile, under Assumption B(b), for any \( i \in [m] \) we have

\[
\mathbb{E}_S[\text{IER}_i(A_{FA})] \lesssim \frac{\beta \sigma^2}{\mu^2 N} + \frac{\beta^3 R^2}{\mu^2}, \tag{3.5}
\]

\[
\mathbb{E}_S[\text{IER}_i(A_{PLT})] \lesssim \frac{\beta \| \ell \|_\infty}{\mu n_i}. \tag{3.6}
\]

**Proof.** This is a special case of Theorems 4.2 and 4.3 in Section 4.1.

We remark that while the lower bounds in Theorem 3.1 require an extra assumption of relatively balanced sample sizes, such an assumption is not needed for the upper bounds in (3.5) and (3.6). This is favorable, as in practice, client-wise sample sizes can be extremely imbalanced.

While the bounds for PureLocalTraining follow from standard stability arguments, the bounds for FedAvg rely on the notion of federated stability, which we will introduce in Section 4.2. We also remark that for FedAvg, the bound on its IER is slightly worse than the corresponding bound on the AER in terms of the dependence on \( \beta \) and \( \mu \). This is expected, as the IER is a more stringent criterion.

3.3 A Theorem of the Alternative

Inspecting the lower bounds in Theorem 3.1 and the upper bounds in Theorem 3.2, we see an interesting phase transition phenomenon. If \( R^2 \lesssim m/N \), then the upper bound on the AER of FedAvg (3.3) matches the corresponding lower bound (3.1); if \( R^2 \gtrsim m/N \), then the upper bound on the AER of PureLocalTraining matches the corresponding lower bound (3.1), provided the client-wise sample sizes are relatively balanced and \( (\beta, \mu, \| \ell \|_\infty) \) are all of the constant order. The same phase transition phenomenon also holds for the IER\(_i\)'s.

The foregoing observation allows us to establish a theorem of the alternative for personalized federated learning. If FedAvg is not minimax optimal in terms of its AER, then we are certain that \( R^2 \gg m/N \), from which we can conclude that PureLocalTraining is minimax optimal in terms of its AER. Again, such a statement holds for all the IER\(_i\)'s. This theorem of the alternative is formally stated as follows.

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Theorem 3.3 (Theorem of the alternative for personalized federated learning). Under the setup of Theorems 3.1 and 3.2, if \((\beta, \mu, \|\ell\|_\infty)\) are all of constant order, then one of the following two assertions must hold:

1. either \(\text{FedAvg}\) is minimax optimal in terms of its AER;

2. or \(\text{PureLocalTraining}\) is minimax optimal in terms of its AER.

Moreover, the same conclusion holds for the IER\(_i\)'s under the additional assumption that \(\sigma \asymp 1\).

Remark 3.1. Strictly speaking, a theorem of the alternative would give two assertions and state that exactly one of them must hold. In this sense, the above result is slightly weaker than a theorem of the alternative, as there are scenarios (i.e., when \(R \asymp m/N\)) where FedAvg and PureLocalTraining are simultaneously minimax optimal.

The implications of this theorem are two-fold. From the technical side, it effectively reduces the problem of adapting to client-wise heterogeneity to a binary decision problem of making a choice between the two baseline algorithms, as detailed in the following corollary.

Corollary 3.1 (Performance of a dichotomous strategy). Under the setup of Theorem 3.2, consider the following naïve dichotomous strategy: if \(R^2 \leq \frac{\|\ell\|_\infty}{\mu N/m}\), then output \(A = A_{\text{FA}}\); otherwise, output \(A = A_{\text{PLT}}\). Then under Assumption \(B(a)\), we have

\[
\mathbb{E}_S[\text{AER}(A)] \lesssim \beta \left( \frac{\|\ell\|_\infty}{\mu N/m} \land R^2 \right) + \frac{\beta\|\ell\|_\infty}{\mu N}.
\]

If in addition, \(n_i \asymp n_i'\) for any \(i \neq i' \in [m]\), then under Assumption \(B(b)\), for any \(i \in [m]\), we have

\[
\mathbb{E}_S[\text{IER}_i(A)] \lesssim \frac{\beta^3}{\mu^2} \left( \frac{\|\ell\|_\infty}{\mu n_i} \land R^2 \right) + \frac{\beta \sigma^2}{\mu^2 N}.
\]

Proof. This is a direct consequence of Theorem 3.2. \(\square\)

Comparing the above result to the lower bounds in Theorem 3.1, we see that such a dichotomous strategy is minimax optimal, provided the problem-dependent parameters \((\mu, \beta, \|\ell\|_\infty, \sigma)\) are all of constant order.

From the practical side, we see that for supervised learning problems, such a dichotomous strategy can be implemented without prior knowledge of \(R\). Indeed, given a problem instance, we can first run both FedAvg and PureLocalTraining separately, evaluate their test errors, and deploy the one with a lower test error. Due to the theorem of the alternative, such a strategy is guaranteed to be minimax optimal. As a caveat, however, one should refrain from interpreting our results as saying either of the two baseline algorithms is sufficient for practical problems. From a practical viewpoint, constants that are omitted in the minimax analysis are crucial. Nevertheless, our results suggest that the two baseline algorithms can at least serve as a good starting point in the search for efficient personalized algorithms.

For unsupervised problems where the quality of a model is hard to evaluate, implementing the dichotomous strategy in Corollary 3.1 requires estimating the level of heterogeneity \(R\). This is an important open problem, which we leave for future work. Meanwhile, we would like to emphasize that the notion of optimality under our consideration is worst-case in nature and overlooks constant factors. Thus, even for supervised problems, a better personalization result could be achieved by more sophisticated algorithms in practice.
Dinh et al. 2020

1.2

Hanzely et al. 2020

Li et al. 2018

Zhang et al.

Compared to (1.2), we find that a direct application of existing

3.7

obtains a “soft constraint” version of

FedAvg

constraint.

The idea of local SGD can be seamlessly applied to solve (3.7). Note that on the one hand, if the local models \{w^{(i)}\} are fixed, then solving for the global model \(w^{(global)}\) of (3.7) reduces to taking the weighted average of local models. On the other hand, if a global model is fixed, then solving for the local models of (3.7) “decouples” into \(m\) sub-problems, and each one can be solved by running SGD on its local data with an \(\ell_2\) regularized objective function. Thus, one naturally obtains a “soft constraint” version of FedAvg, which is exactly the FedProx algorithm proposed by Li et al. [2018] (see also Hanzely and Richtárik 2020, Dinh et al. 2020, Hanzely et al. 2020 for related ideas).

A detailed description of FedProx is given in Algorithm 2. Note that similar to FedAvg described in Algorithm 1, we introduce an “aggregation step size” \(\eta_t\) and use a generalized notion of weighted average, sometimes known as the “elastic average” in the literature [Zhang et al., 2015]. As is the case with FedAvg, in the aggregation step, only intermediate local models (namely \(w^{(i)}_t\)’s), but not the raw data (namely \(S_i\)’s), are synchronized with the central server, thus satisfying the computational constraint of federated learning.

While the optimization convergence of FedProx has been established in the literature (see, e.g., Li et al. 2018, Dinh et al. 2020, Hanzely et al. 2020), we find that a direct application of existing

\[ \text{Algorithm 2: FedProx [Li et al., 2018]} \]

\[
\text{Initialize } w^{(global)}_0, \{w^{(i)}_0\}, \text{ number of communication rounds } T, \text{ step sizes } \{\eta_t\}_{t=0}^{T-1} \text{ for } t = 0, 1, \ldots, T - 1 \text{ do} \\
\text{Randomly sample a batch } C_t \subseteq [m] \text{ of clients;} \\
\text{for } i \in [m] \text{ do} \\
\begin{array}{l}
\text{if } i \in C_t \text{ then } w^{(i)}_{t+1} \leftarrow w^{(i)}_t \\
\text{else} \\
\text{Obtain } w^{(i)}_{t+1} \text{ by running several steps of SGD on } S_i \text{ with the regularized loss function } \\
L_i(\cdot, S_i) + \frac{\lambda}{2}\|w^{(global)}_t - \cdot\|^2 \\
\text{w}^{(i)}_{t+1} \leftarrow w^{(i)}_t - \frac{\eta_t}{N[C_t]} \sum_{i \in C_t} n_i (w^{(i)}_t - w^{(i)}_{t+1}) \\
\end{array}
\text{return } \hat{w}^{(i)} = w^{(i)}_T, i \in [m]
\]

3.4 Performance of FedProx

In this subsection, we provide an analysis of the FedProx algorithm Li et al. [2018]. FedProx considers the following optimization problem:

\[ \min_{w^{(global)}_1, \{w^{(i)}_1\}_{i=1}^{m}} \frac{1}{N} \sum_{i \in [m]} n_i \left( L_i(w^{(i)}, S_i) + \frac{\lambda}{2}\|w^{(global)} - w^{(i)}\|^2 \right), \]  

(3.7)

where we recall that \(L_i(w, S_i) := \sum_{j \in [n_i]} \ell(w, z^{(i)}_j)/n_i\) is the ERM objective for the \(i\)-th client. Compared to (1.2), which imposes a “hard” constraint \(w^{(i)} \equiv w^{(global)}\), and compared to (1.3), where there is no constraint at all, the above formulation imposes a “soft” constraint that the norm of \((w^{(global)} - w^{(i)})\) should be small, with a hyperparameter \(\lambda\) controlling the strength of this constraint.

The rationale behind the optimization formulation (3.7) of FedProx is clear: by setting \(\lambda = 0\), the optimization formulation of PURELOCALTRAINING (1.3) is recovered, and as \(\lambda \to \infty\), the optimization formulation of FedAvg (1.2) is recovered. The hope is that by varying \(\lambda \in (0, \infty)\), one can interpolate between the two extremes.

The idea of local SGD can be seamlessly applied to solve (3.7). Note that on the one hand, if the local models \{w^{(i)}\} are fixed, then solving for the global model \(w^{(global)}\) of (3.7) reduces to taking the weighted average of local models. On the other hand, if a global model is fixed, then solving for the local models of (3.7) “decouples” into \(m\) sub-problems, and each one can be solved by running SGD on its local data with an \(\ell_2\) regularized objective function. Thus, one naturally obtains a “soft constraint” version of FedAvg, which is exactly the FedProx algorithm proposed by Li et al. [2018] (see also Hanzely and Richtárik 2020, Dinh et al. 2020, Hanzely et al. 2020 for related ideas).

A detailed description of FedProx is given in Algorithm 2. Note that similar to FedAvg described in Algorithm 1, we introduce an “aggregation step size” \(\eta_t\) and use a generalized notion of weighted average, sometimes known as the “elastic average” in the literature [Zhang et al., 2015]. As is the case with FedAvg, in the aggregation step, only intermediate local models (namely \(w^{(i)}_t\)’s), but not the raw data (namely \(S_i\)’s), are synchronized with the central server, thus satisfying the computational constraint of federated learning.

While the optimization convergence of FedProx has been established in the literature (see, e.g., Li et al. 2018, Dinh et al. 2020, Hanzely et al. 2020), we find that a direct application of existing
results is not sufficient to obtain meaningful excess risk guarantees in our case. Hence, in contrast to Theorem 3.2 where the analysis is done for the global minimizers, we will do an algorithm-dependent analysis.

Next, we present the performance guarantees for FedProx.

**Theorem 3.4 (Performance of FedProx).** Let Assumption A hold. Moreover, assume that \( n_i \sim n_{i'} \) for any \( i \neq i' \in [m] \) and \( n_i \geq 4\beta/\mu \) for any \( i \in [m] \). Consider the FedProx algorithm defined in Algorithm 2, denoted as \( \mathcal{A}_{FP} \). With a proper hyperparameter choice (see Theorem 4.4 and 4.5 for details), under Assumption \( B(a) \), we have

\[
\mathbb{E}_{\mathcal{A}_{FP},\mathcal{S}}[\text{AER}(\mathcal{A}_{FP})] \lesssim \left( \mu + \frac{\beta \|\ell\|\infty}{\mu} \right) \cdot \left( \frac{1}{N/m} \wedge \frac{R}{\sqrt{N/m}} + \frac{1}{N} \right),
\]

and under Assumption \( B(b) \), for any \( i \in [m] \) we have

\[
\mathbb{E}_{\mathcal{A}_{FP},\mathcal{S}}[\text{IER}_i(\mathcal{A}_{FP})] \lesssim (\mu + \mu^{-1}) \left( \beta \|\ell\|\infty + \frac{\sigma^2 \beta^2 + \beta^2 + \sigma^2}{\mu^2} + \mu D^2 \right) \left( \frac{1}{n_i} \wedge \frac{R}{\sqrt{n_i}} + \frac{\sqrt{n_i}}{N} \right).
\]

**Proof.** This is a special case of Theorem 4.4 and 4.5 in Section 4.3. \qed

Both the AER and IER guarantees in the above theorem need the assumption that client-wise sample sizes are relatively balanced. This assumption arises because of the empirical success reported in the literature.

The bounds in Theorem 3.4 in general do not attain the lower bounds in Theorem 3.1.

- For the AER, we have the following three cases. If \( R^2 \gtrsim m/N \), then (3.8) becomes \( \mathcal{O}(m/N) \), which matches the lower bound. Meanwhile, if \( 1/mN \lesssim R^2 \lesssim m/N \), then (3.8) becomes \( \mathcal{O}(m/N) \), whereas the lower bound reads \( \Omega(R^2 + 1/N) \), and thus (3.8) is suboptimal unless \( R^2 \asymp m/N \). Moreover, if \( R^2 \lesssim 1/mN \), then (3.8) becomes \( \mathcal{O}(1/N) \), and is minimax optimal again.

- For the IER, we still have three cases as follows. If \( R^2 \gtrsim m/N \), then (3.9) is \( \mathcal{O}(1/n_i) \), which agrees with the lower bound. Meanwhile, if \( 1/N \lesssim R^2 \lesssim m/N \), then (3.9) is \( \mathcal{O}(R/\sqrt{n_i}) \), and is suboptimal compared to the \( \Omega(R^2 + 1/N) \) lower bound unless \( R^2 \asymp m/N \). Moreover, if \( R^2 \lesssim 1/N \), then (3.9) is \( \mathcal{O}(\sqrt{m}/N) \), and is off by a factor of order \( \sqrt{m} \) compared to the \( \Omega(1/N) \) lower bound.

Despite their suboptimality, the bounds in Theorem 3.4 are still non-trivial in the sense that they scale with the heterogeneity measure \( R \). While there are some recent works establishing the AER guarantees for an objective similar to (3.7) under the online learning setup (see, e.g., Denevi et al. 2019, Balcan et al. 2019, Khodak et al. 2019), to the best of our knowledge, this is the first result that establishes both the AER and IER guarantees for (3.7) under the federated learning setup.

The proof is based on federated stability, which we will introduce in Section 4.2 and detail in Section 4.3. The suboptimality arises partly because in our current proof, we cannot choose \( \lambda \) to be arbitrarily large: if we were able to do so, then the objective function for FedAvg (1.2) can be recovered, and we would have been able to get an \( \mathcal{O}(1/N + R^2) \) bound whenever \( R^2 \lesssim m/N \). We suspect that this is an artifact of our technical approaches, and we conjecture that FedProx is indeed minimax optimal, especially in view of its empirical success reported in the literature.
4 Proofs

4.1 Construction of Lower Bounds

In this subsection, we present our construction of the lower bounds in Theorem 3.1, which characterizes the information-theoretic limit of personalized federated learning.

Our construction starts by considering a special class of problem instances: \textit{logistic regression}. In logistic regression, given the collection of regression coefficients \( \{w_i^{(i)}\} \subseteq W \) where \( W \) has a diameter \( D \), the data distributions \( D_i \)'s are supported on \( \mathbb{R}^d \times \{\pm 1\} \) and specified by a two-step procedure as follows:

1. Generate a \textit{feature vector} \( x \), whose coordinates are i.i.d. copies from some distribution \( \mathbb{P}_X \) on \( \mathbb{R} \), which is assumed to have mean zero and is almost surely bounded by some absolute constant \( c_X \);

2. Generate the binary \textit{label} \( y \in \{\pm 1\} \), which is a biased Rademacher random variable with head probability \( 1 + \exp(-x^\top w_i^{(i)}) \)^{-1}.

The loss function is naturally chosen to be the negative log-likelihood function, which takes the following form:

\[
\ell(w, z) = \ell(w, x, y) = \log(1 + e^{-yx^\top w}).
\]

The following lemma says that Assumption A holds for the aforementioned logistic regression models.

\textbf{Lemma 4.1} (Logistic regressions are valid problem instances). The logistic regression problem described above is a class of problem instances that satisfies Assumption A with \( \|\ell\|_\infty = c_X D \sqrt{d} \) and \( \sigma^2 = \beta = c_X^2 d/4 \). Moreover, if \( m \lesssim (N/m)^c \) for some \( c \geq 0 \) and \( N/m \geq Cd \) for some \( C > 1 \), then there exists some event \( E \) which only depends on the features \( \{x_i^{(i)} : i \in [m], j \in [n_i]\} \) and happens with probability at least \( 1 - e^{-O(\sqrt{N/m})} \), such that on this event, the strongly convex constant in Assumption A satisfies

\[
\mu \asymp \mu_0 = \left( \exp\{c_X D \sqrt{d}/2\} + \exp\{-c_X D \sqrt{d}\} \right)^{-2}.
\]  

\textit{Proof.} The compactness of the domain and the boundedness of the loss function hold by construction. To verify the rest parts of Assumption A, with some algebra one finds that

\[
\nabla^2 \ell(w, x, y) = \frac{xx^\top \exp\{yx^\top w\}}{(1 + \exp\{yx^\top w\})^2} \lesssim \frac{1}{4} xx^\top,
\]

where \( \preceq \) is the Loewner order and the inequality holds because \( x/(1+x)^2 = 1/(x^{-1/2} + x^{1/2})^2 \leq 1/4 \) for \( x > 0 \). Since the population gradient has mean zero at optimum, the gradient variance at optimum can be upper bounded by the trace of the expected Hessian matrix, which, by the above display, is further upper bounded by \( c_X^2 d/4 \). Thus, we can take \( \sigma^2 = c_X^2 d/4 \) in Part (c). Another message of the above display is that we can set the smoothness constant in Part (b) to be \( \beta = c_X^2 d/4 \).

The only subtlety that remains is to ensure each local loss function is \( \mu \)-strongly convex. Note that since \( x/(1+x)^2 \) is decreasing from \( (0, 1) \) and is increasing from \( (1, \infty) \), the right-hand side of (4.2) dominates \( \mu_0 xx^\top \) in Loewner order, where \( \mu_0 \) is the right-hand side of (4.1). Thus, the local population losses \( \mathbb{E}_{(x,y) \sim D_i}[\ell(\cdot, x, y)] \) are all \( \mu_0 \)-strongly convex.
Now, note that
\[
\nabla^2 L_i(w^{(i)}, S_i) = \frac{1}{n_i} \sum_{j \in [n_i]} x_j^{(i)} (x_j^{(i)})^\top \exp\{y_j^{(i)}(x_j^{(i)}, w^{(i)})\} \geq \mu_0 \cdot \frac{1}{n_i} \sum_{j \in [n_i]} x_j^{(i)} (x_j^{(i)})^\top.
\]

Invoking Theorem 5.39 of Vershynin [2010] along with a union bound over all clients, we conclude that for any \( i \in [m] \), the minimum eigenvalue of \( \sum_{j \in [n_i]} x_j^{(i)} (x_j^{(i)})^\top \) is lower bounded by a constant multiple of \( n_i - p \gtrsim n_i \) (this is the definition of the event \( E \)) with probability at least \( 1 - \exp^{-\Omega(n_i)} \gtrsim 1 - e^{-\tilde{O}(\sqrt{N/m})} \), and the proof is concluded.

Note that in the proof of the above lemma, we have established the \( \mu_0 \asymp \mu \)-strong convexity of the client-wise population losses. Hence, lower bounding the excess risks reduces to lower bounding the \( \ell_2 \) estimation errors \( \|\hat{w}^{(i)} - w_*^{(i)}\|^2 \) of the estimators \( \hat{w}^{(i)} \) for \( w_*^{(i)} \). Such a reduction allows us to use powerful tools from information theory.

Before we rigorously state the main result of this subsection, from which Theorem 3.1 follows, we pause to introduce two parameter spaces, corresponding to Part (a) and (b) of Assumption B. Recalling that \( w_{\text{avg}}^{(\text{global})} = \frac{1}{N} \sum_{i \in [m]} n_i w_*^{(i)} \), we define
\[
\mathcal{P}_1 := \left\{ \{w_*^{(i)}\}_{i=1}^m \subseteq \mathcal{W} : \frac{1}{N} \sum_{i \in [m]} n_i \|w_*^{(i)} - w_{\text{avg}}^{(\text{global})}\|^2 \leq R^2 \right\},
\]
\[
\mathcal{P}_2 := \left\{ \{w_*^{(i)}\}_{i=1}^m \subseteq \mathcal{W} : \|w_*^{(i)} - w_{\text{avg}}^{(\text{global})}\|^2 \leq R^2 \forall i \in [m] \right\}.
\]

Note that \( \mathcal{P}_1 \) and \( \mathcal{P}_2 \) index all possible values of \( \{w_*^{(i)}\} \) that can arise in the logistic regression models under Assumption B (a) and (b), respectively.

With the notations introduced so far, we are ready to state the main result of this subsection.

**Theorem 4.1** (Minimax lower bounds for estimation errors). Consider the logistic regression model described above. Suppose \( n_i \asymp n_{i'} \) for any \( i \neq i' \in [m] \). Then we have
\[
\inf_{\{\hat{w}^{(i)}\}_{i \in [m]} \subseteq \mathcal{P}_1} \sup_{\{w_*^{(i)}\}_{i \in [m]} \subseteq \mathcal{P}_1} \frac{1}{N} \sum_{i \in [m]} n_i \mathbb{E}_S \|\hat{w}^{(i)} - w_*^{(i)}\|^2 \geq \frac{d}{N/m} \wedge R^2 + \frac{d}{N}, \tag{4.3}
\]
\[
\inf_{\hat{w}^{(i)} \in \mathcal{P}_2} \sup_{\{w_*^{(i)}\}_{i \in [m]} \subseteq \mathcal{P}_2} \mathbb{E}_S \|\hat{w}^{(i)} - w_*^{(i)}\|^2 \geq \frac{d}{n_i} \wedge R^2 + \frac{d}{N}. \tag{4.4}
\]

for all \( i \in [m] \), where the infimum is taken over all possible \( \hat{w}^{(i)} \)'s that are measurable functions of the data \( S \).

**Proof.** See Appendix A.

From Lemma 4.1 and Theorem 4.1, Theorem 3.1 follows by the fact that the smoothness constant \( \beta \) is of the same order as \( d \) and the population losses are all \( \mu_0 \asymp \mu \)-strongly convex. Note that both lower bounds in Theorem 4.1 are a superposition of two terms, and they correspond to two distinct steps in the proof.

The first step in our proof is to argue that the lower bound under complete homogeneity is in fact a valid lower bound under our working assumptions, which gives the \( \Omega(d/N) \) term. This is
reasonable, since estimation under complete homogeneity is, in many senses, an “easier” problem. The proof of the $\Omega(d/N)$ term is based on the classical Assouad’s method [Assouad, 1983].

The second step is to use a generalized version of Assouad’s method that allows us to deal with multiple heterogeneous datasets. In particular, we need to carefully choose the prior distributions over the parameter space based on the level of heterogeneity, which ultimately leads to the $\Omega(\frac{d}{N/m} \wedge R^2)$ term. Recall that in the vanilla version of Assouad’s method where there is only one parameter, say $w_*$, one can lower bounds the minimax risk by the Bayes risk, and the prior distribution is usually chosen to be $w_* = \delta v$, where $v$ follows a uniform distribution over all $d$-dimensional binary vectors and $\delta$ is chosen so that the resulting hypothesis testing problem has large type-I plus type-II error. In our case where there are $m$ parameters $\{w_*(i)\}$, we need to consider a different prior of the following form:

$$w_*(i) = \delta_i v(i),$$

where $v(i)$ are i.i.d. samples from the uniform distribution over all $d$-dimensional binary vectors, and $\delta_i$’s are scalers that need to be carefully chosen to make the resulting hypothesis testing problem hard.

4.2 Analysis of Baseline Algorithms

In this subsection, we present our analyses of the two baseline algorithms, FedAvg and PureLocalTraining, as well as the naive dichotomous strategy that combines the two baseline algorithms.

Analysis of PureLocalTraining. The analysis of PureLocalTraining is based on the classical notion of uniform stability, proposed by Bousquet and Elisseeff [2002].

Definition 4.1 (Uniform stability). Consider an algorithm $A$ that takes a single dataset $S = \{z_j\}_{j=1}^n$ of size $n$ as input and outputs a single model: $A(S) = \hat{w}(S)$. We say $A$ is $\gamma$-uniformly stable if for any dataset $S$, any $j \in [n]$, and any $z_j' \in \mathcal{Z}$, we have

$$\|\ell(\hat{w}(S), \cdot) - \ell(\hat{w}(S \setminus j), \cdot)\|_{\infty} \leq \gamma,$$

where $S \setminus j$ is the dataset formed by replacing $z_j$ with $z_j'$:

$$S \setminus j = \{z_1, \ldots, z_{j-1}, z_j', z_{j+1}, \ldots, z_n\}.$$  

The main implication of uniformly stable algorithms is that “stable algorithms do not overfit”: if $A$ is $\gamma$-uniformly stable, then its generalization error is upper bounded by a constant multiple of $\gamma$. Thus, one can dissect the analysis of $A$ into two separate parts: (1) bounding its optimization error; (2) bounding its stability term.

As pointed out in Section 3.2, under our working assumptions, SGD with properly chosen step sizes is guaranteed to converge to the global minimum of (1.3) (see, e.g., Rakhlin et al. [2011]). Thus, we consider without loss of generality the global minimizer of (1.3), whose performance is given by the following theorem.

Theorem 4.2 (Performance of PureLocalTraining). Let Assumption $A(b)$ hold and assume $n_i \geq 4\beta/\mu \forall i \in [m]$. Then the algorithm $A_{PLT}$ which outputs the minimizer of (1.3) satisfies

$$\mathbb{E}_S[IER_i(A_{PLT})] \lesssim \frac{\beta\|\ell\|_{\infty}}{\mu n_i}$$

for all $i = 1, \ldots, m$. 

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Proof. The proof is a direct consequence of standard results on uniform stability of strongly convex ERM (see, e.g., Section 5 of Shalev-Shwartz et al. [2009] and Section 13 of Shalev-Shwartz and Ben-David [2014]), which assert that under the current assumptions, the minimizer of (1.3) is $O\left(\frac{\beta \|\ell\|_\infty}{\mu n}\right)$-uniformly stable. We omit the details.

Analysis of FedAvg. As mentioned in Section 2, sometimes we are interested in the following more general weighted version of the AER.

Definition 4.2 (p-average excess risk). Consider an algorithm $A$ that outputs $A(S) = \{\hat{w}^{(i)}(S)\}$. For a vector $p = (p_1, \ldots, p_m)$ lying in the $m$-dimensional probability simplex (i.e., all $p_i$'s are non-negative and they sum to one), we define the p-average excess risk (AER$_p$) of $A$ to be

$$AER_p(A) := \sum_{i \in [m]} p_i \cdot IER_i(A).$$

Intuitively speaking, the weight vector $p$ can be regarded as the importance weight on each client and controls “how many resources are allocated to each client”. For example, setting $p_i = 1/m$ enforces “fair allocation”, so that each client is treated uniformly, regardless of sample sizes. As another example, setting $p_i = n_i/N$ means that the central server pays more attention to clients with larger sample sizes, which, to a certain extend, incentivize the clients to contribute more data.

In view of Definition 4.2, it is natural to consider the following generalization of Assumption B.

Assumption C (p-heterogeneity). For a vector $p = (p_1, \ldots, p_m)$ lying in the $m$-dimensional probability simplex, define the p-average global model as

$$w^{(\text{global}, p)}_{\text{avg}} = \sum_{i \in [m]} p_i w^{(i)}_*. $$

We assume that there exists a positive constant $R$ such that

(a) either $\sum_{i \in [m]} p_i \|w^{(\text{global}, p)}_{\text{avg}} - w^{(i)}_*\|^2 \leq R^2$,

(b) or $\|w^{(i)}_* - w^{(\text{global}, p)}_{\text{avg}}\|^2 \leq R^2 \forall i \in [m]$.

As is the case with Assumption B, we assume Part (a) when we deal with AER$_p$, and we assume Part (b) when we deal with the IER$_i$'s.

We consider the following weighted version of (1.2):

$$\min_{w \in \mathcal{W}} \sum_{i \in [m]} p_i L_i(w, S_i), \quad (4.5)$$

and the FedAvg algorithm also seamlessly generalizes. The above optimization formulation is in fact covered by the general theory of Li et al. [2020b], where they showed that FedAvg is guaranteed to converge to the global optimum under a suitable hyperparameter choice, even in the presence of heterogeneity (but the convergence is slower). Thus, in the following discussion, we only consider the global minimizer of (4.5).

It turns out that a tight analysis of FedAvg requires a more “fine-grained” notion of uniform stability, which we present below.
Definition 4.3 (Federated stability). An algorithm $\mathcal{A}$ that outputs $\mathcal{A}(S) = \{\hat{w}^{(i)}(S)\}$ has federated stability $\{\gamma_i\}_{i=1}^m$ if for every $S \sim \otimes_i^m D_i^{\otimes n_i}$ and for any $i \in [m], j_i \in [n_i], z_{i,j_i} \in Z$, we have

$$\|\ell(\hat{w}^{(i)}(S), \cdot) - \ell(\hat{w}^{(i)}(S^{\backslash(i,j_i)}), \cdot)\|_\infty \leq \gamma_i.$$ 

Above, $S^{\backslash(i,j_i)}$ is the dataset formed by replacing $z_{j_i}^{(i)}$ in the $i$-the dataset with $z_{i,j_i}^{(i)}$:

$$S^{\backslash(i,j_i)} = \{S_1, \ldots, S_{i-1}, S_i^{\backslash j_i}, S_{i+1}, \ldots, S_m\},$$

$$S_i^{\backslash j_i} = \{z_1^{(i)}, \ldots, z_{j_i-1}^{(i)}, z_{i,j_i}^{(i)}, z_{i,j_i+1}^{(i)}, \ldots, z_{n_i}^{(i)}\}.$$

Compared to the conventional uniform stability in Definition 4.1, federated stability provides a finer control by allowing distinct stability measures $\{\gamma_i\}$ for different clients. Moreover, the classical statement that “stable algorithms do not overfit” still holds, in the sense that the average (resp. individualized) generalization error can be upper bounded by $O(\sum_{i\in[m]} n_i \gamma_i / N)$ (resp. $O(\gamma_i)$), plus a term scaling with the level of heterogeneity $R$. And this again enables us to separate the analysis of $A$ into two parts (namely bounding the optimization error and bounding the stability), as is the case with the conventional uniform stability. The notion of federated stability has other implications when restricted to the FedProx algorithm, and we refer the readers to Section 4.3 for details.

We are now ready to state the theorem that characterizes the performance of FedAvg.

Theorem 4.3 (Performance of FedAvg). Let Assumption A(b, c) hold and assume $n_i \geq 4 \beta p_i / \mu \forall i \in [m]$. Suppose the FedAvg algorithm $\mathcal{A}_{FA}$ outputs the minimizer of (4.5). Then under Assumption C(a), we have

$$\mathbb{E}_S[\text{AER}_p(\mathcal{A}_{FA})] \lesssim \frac{\beta \|\cdot\|_\infty}{\mu} \sum_{i \in [m]} \frac{p_i^2}{n_i} + \beta R^2,$$  \hfill (4.6)$$

and under Assumption C(b), we have

$$\mathbb{E}_S[\text{IER}_i(\mathcal{A}_{FA})] \lesssim \frac{\beta \sigma^2}{\mu_2} \sum_{i' \in [m]} \frac{p_{i'}^2}{n_{i'}} + \frac{\beta^3}{\mu^2} R^2.$$  \hfill (4.7)$$

Proof. The proof of (4.6) is, roughly speaking, based on the fact that the global minimizer of (4.5) has federated stability $\gamma_i \lesssim \frac{\beta \|\cdot\|_\infty p_i}{\mu n_i}$, and thus the first term in the right-hand side of (4.6) corresponds to the average federated stability $\sum_{i \in [m]} p_i \gamma_i$. The second term $\beta R^2$ in the right-hand side of (4.6) reflects the presence of heterogeneity. For Equation (4.7), we were not able to obtain a federated stability based proof, and our current proof is based on an adaptation of the arguments in Theorem 7 of Foster et al. [2019], which explains why the dependence on $(\sigma, \beta, \mu)$ are different (and slightly worse) compared to Equation (4.6). We refer the readers to Appendix C.1 for details.

Note that both bounds in the above theorem are minimized by choosing $p_i = n_i / N$. This makes sense, since this choice of weight corresponds to the ERM objective under complete homogeneity. This observation also suggests that ensuring “fair resource allocation” (i.e., setting $p_i = 1/m$) can lead to statistical inefficiency, especially when the sample sizes are imbalanced.

Analysis of the dichotomous strategy. With Theorem 4.2 and 4.3 at hand, we are ready to state the following result, from which Corollary 3.1 follows as a special case.
Corollary 4.1 (Performance of a dichotomous strategy). Let Assumptions A(b, c) hold and assume \( n_i \geq 4\beta p_i / \mu \) \( \forall i \in [m] \). Consider the following naïve dichotomous strategy: if output \( R^2 \leq \frac{\|\epsilon\|_{\infty}}{\mu N/m} \), then output \( A = A_{FA} \); otherwise, output \( A = A_{PLT} \). Then under Assumption C(a), we have

\[
\mathbb{E}_S[AER_p(A)] \lesssim \beta \left( \frac{\|\epsilon\|_{\infty}}{\mu N/m} \land R^2 \right) + \frac{\beta\|\epsilon\|_{\infty}}{\mu} \sum_{i \in [m]} \frac{p_i^2}{n_i}
\]

If in addition, \( n_i \sim n_i' \) for any \( i \neq i' \in [m] \), then under Assumption C(b), for any \( i \in [m] \), we have

\[
\mathbb{E}_S[IER_i(A)] \lesssim \frac{\beta^3}{\mu^2} \left( \frac{\|\epsilon\|_{\infty}}{\mu n_i} \land R^2 \right) + \frac{\beta\sigma^2}{\mu^2} \sum_{i' \in [m]} \frac{p_{i'}^2}{n_{i'}}
\]

Proof. This is a direct consequence of Theorem 4.2 and 4.3. \( \square \)

We conclude this subsection by noting that though the compactness assumption (Assumption A(a)) is not needed in the above result, it is usually needed in the analysis of the optimization error of FedAvg and PureLocalTraining (see, e.g., Rakhlin et al. [2011], Li et al. [2020b]).

4.3 Analysis of FedProx

In this subsection, we are concerned with the performance guarantees for FedProx. As with our earlier analysis of FedAvg, we will consider a p-weighted version of FedProx, and Algorithm 2 is a special case with \( p_i = n_i / N \). Specifically, we consider the following p-weighted generalization of (3.7):

\[
\min_{\{w^{(i)}\}_{i=1}^m \subseteq W, \{w^{(i)}\}_{i=1}^m \subseteq W} \sum_{i=1}^m p_i \left( L_i(w^{(i)}, S_i) + \frac{\lambda}{2} \| w^{(global)} - w^{(i)} \|^2 \right).
\]

In this subsection, we let \( (\hat{w}^{(global)}, \{\hat{w}^{(i)}\}) \) be the global minimizer of the above problem.

The FedProx algorithm generalizes in a straightforward fashion (see Algorithm 3 for a detailed description), where the only change to Algorithm 2 is the aggregation step: at the \( t \)-th communication round, we aggregate by

\[
w_{t+1}^{(global)} = w_t^{(global)} - \frac{\lambda m \eta^{(global)}}{|C_t|} \sum_{i \in [C_t]} p_i (w_t^{(global)} - w_t^{(i)}).
\]

Curious readers may wonder why we deliberately separate the training into two stages in the algorithm description. This is because, in order for FedProx to generalize well, we need both global and local models output by this algorithm to approach the optimum: \( w_{T}^{(global)} \approx \hat{w}^{(global)} \), \( w_{T+1}^{(i)} \approx \hat{w}^{(i)} \) \( \forall i \in [m] \). Thus, the two stages have distinct interpretations: in Stage I, the central server aims to learn a good global model with the help of local clients, whereas in Stage II, each local client takes advantage of the global model to personalize. Alternatively, one can also interpret FedProx as an instance of the general framework of model-agnostic meta learning [Finn et al., 2017], where Stage I learns a good initialization, and Stage II trains the local models starting from this initialization.

In contrast to our analyses for FedAvg and PureLocalTraining, where we largely focus on global minimizers, the analysis for FedProx will be carried out for the approximate minimizer output by Algorithm 3. The reason for this, again, is rooted in the fact that a rigorous statement on
4.2 Dinh et al. [72x72]

Let us first define the optimization error of a generic algorithm $O$ by its average generalization error (resp. individualized generalization error). Can be upper bounded to the generalization ability of FedProx in Section 4.2. Rigorous statement on the performance of FedProx.

**Algorithm 3: FedProx, general version**

**Input:** Initial global model $w^{(global)}_0$, initial local models $\{w^{(i)}_0\}_{i=1}^m \equiv \{w^{(i)}_0\}_{i=1}^m$, global rounds $T$, global batch size $B^{(global)}$, global step sizes $\{\eta^{(global)}_{t}\}_{t=0}^{T-1}$, local rounds $\{K_{t}\}_{t=0}^{T}$, local batch sizes $\{B^{(i)}\}_{i=0}^{m}$, local step sizes $\{\eta^{(i)}_{t,k}\}_{0 \leq t \leq T, 0 \leq k \leq K_{t} - 1}$.

**Output:** Local models $\{w^{(i)}_{T+1}\}_{i=1}^m$.

# Stage I: joint training

for $t = 0, 1, \ldots, T - 1$ do

Randomly sample a batch $\mathcal{C}_t \subseteq [m]$ of size $B$.

for $i \in [m]$ do

if $i \in \mathcal{C}_t$ then $w^{(i)}_{t+1} \leftarrow w^{(i)}_t$.

else

Pull $w^{(global)}_t$ from the server.

$w^{(i)}_{t+1} \leftarrow \text{SoftLocalSGD}(i, w^{(i)}_t, w^{(global)}_t, K_t, B^{(i)}, \{\eta^{(i)}_{t,k}\}_{k=0}^{K_t-1})$.

Push $w^{(i)}_{t+1}$ to the server.

end if

end for

$w^{(global)}_{t+1} \leftarrow w^{(global)}_t - \frac{\lambda m^{(global)}}{B^{(global)}} \sum_{i \in \mathcal{C}_t} p_i (w^{(global)}_t - w^{(i)}_{t+1})$.

# Stage II: final training before deployment

for $i \in [m]$ do

Pull $w^{(global)}_T$ from the server.

$w^{(i)}_{T+1} \leftarrow \text{SoftLocalSGD}(i, w^{(i)}_T, w^{(global)}_T, K_T, B^{(i)}, \{\eta^{(i)}_{T,k}\}_{k=0}^{K_T-1})$.

end for

return $\{w^{(i)}_{T+1}\}_{i=1}^m$.

# Local SGD subroutine

**Function SoftLocalSGD**($i, w^{(i)}_t, w^{(global)}_t, K_t, B^{(i)}; \{\eta^{(i)}_{t,k}\}_{k=0}^{K_t-1}$)

for $k = 0, 1, \ldots, K_t - 1$ do

Randomly sample a batch $I \subseteq [n_i]$ of size $|I| = B$.

$w^{(i)} \leftarrow \mathcal{P}_W \left[w^{(i)} - \frac{\eta^{(i)}}{B} \sum_{j \in I} \left( \nabla \ell(w^{(i)}, z^{(i)}_j) + \lambda (w^{(i)} - w^{(global)}) \right) \right]$.

return $w^{(i)}$.

the generalization ability of FedProx calls for both global and local models to approach the optimum. While the convergence of $w^{(global)}_T$ to $\tilde{w}^{(global)}$ has appeared in Li et al. [2018] and Dinh et al. [2020], to the best of our knowledge, the convergence of $w^{(i)}_T$ to $\tilde{w}^{(i)}$ has been largely overlooked in the literature, and we need to address this additional technical challenge in order to have a fully rigorous statement on the performance of FedProx.

**Implications of federated stability for FedProx.** We have briefly mentioned the main implications of federated stability in Section 4.2: for an algorithm $\mathcal{A} = \{\tilde{w}^{(i)}\}$ with federated stability $\{\gamma_i\}$, its average generalization error (resp. individualized generalization error) can be upper bounded by $O(\sum_{i \in [m]} p_i \gamma_i)$ (resp. $O(\gamma_i)$), plus a term scaling with the level of heterogeneity $R$. We make such a statement precise here. Let us first define the optimization error of a generic algorithm.

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The main implications of federated stability, when applied to the specifics of FedProx, can then be summarized in the following proposition.

**Proposition 4.1 (Implications of federated stability restricted to FedProx).** Consider an algorithm \( \mathcal{A} = (\mathbf{w}^{(\text{global})}, \{\mathbf{w}^{(i)}\}) \) with federated uniform stability \( \{\gamma_i\}_i \). Then we have

\[
\mathbb{E}_{\mathcal{A}, \mathcal{S}}[\operatorname{AER}_{\mathbf{p}}(\mathcal{A})] \leq \mathbb{E}_{\mathcal{A}, \mathcal{S}}[\mathcal{E}_{\text{OPT}}] + 2 \sum_{i \in [m]} p_i \mathbb{E}_{\mathcal{A}, \mathcal{S}}[\gamma_i] + \frac{\lambda}{2} \sum_{i \in [m]} p_i \|\mathbf{w}^{(\text{global})} - \mathbf{w}^{(i)}\|^2, \tag{4.9}
\]

\[
\mathbb{E}_{\mathcal{A}, \mathcal{S}}[\operatorname{IER}_s(\mathcal{A})] \leq \frac{\mathbb{E}_{\mathcal{A}, \mathcal{S}}[\mathcal{E}_{\text{OPT}}]}{p_i} + 2 \mathbb{E}_{\mathcal{A}, \mathcal{S}}[\gamma_i] + \frac{\lambda}{2} \mathbb{E}_{\mathcal{A}, \mathcal{S}}[\|\mathbf{w}^{(\text{global})} - \mathbf{w}^{(i)}\|^2 \forall i \in [m]]. \tag{4.10}
\]

**Proof.** The proof of (4.9) is based on the following basic inequality for the AER:

\[
\sum_{i \in [m]} p_i \left( L_i(\mathbf{w}^{(i)}, S_i) + \frac{\lambda}{2} \|\mathbf{w}^{(\text{global})} - \mathbf{w}^{(i)}\|^2 \right) \leq \sum_{i \in [m]} p_i \left( L_i(\mathbf{w}^{(i)}, S_i) + \frac{\lambda}{2} \|\mathbf{w}^{(\text{global})} - \mathbf{w}^{(i)}\|^2 \right), \tag{4.11}
\]

whereas the proof of (4.10) is based on the following basic inequality for the IER: for any \( s \in [m] \), we have

\[
\sum_{i \in [m]} p_i \left( L_i(\mathbf{w}^{(i)}, S_i) + \frac{\lambda}{2} \|\mathbf{w}^{(\text{global})} - \mathbf{w}^{(i)}\|^2 \right) \\
\leq p_s L_s(\mathbf{w}^{(s)}, S_s) + \frac{\lambda}{2} \|\mathbf{w}^{(\text{global})} - \mathbf{w}^{(s)}\|^2 + \sum_{i \neq s} p_i \left( L_i(\mathbf{w}^{(i)}, S_i) + \frac{\lambda}{2} \|\mathbf{w}^{(\text{global})} - \mathbf{w}^{(i)}\|^2 \right). \tag{4.12}
\]

We refer the readers to Appendix C.2 for details. \( \square \)

Note that both bounds in Proposition 4.1 involve a term that scales linearly with both \( \lambda \) and the heterogeneity measure. In general, we expect the stability measures to scale inversely with \( \lambda \), and thus opening the possibility of carefully choosing \( \lambda \) to balance the stability term and the heterogeneity term.

Let us observe that the heterogeneity term of (4.10) is slightly different than that of (4.9), in that it involves the estimated global model \( \mathbf{w}^{(\text{global})} \). This suggests that achieving the IER guarantees might be intrinsically more difficult than achieving the AER guarantees, which partially explains why the IER guarantees are largely missing in the literature, whereas the AER guarantees are already available in recent works such as Denevi et al. [2019], Balcan et al. [2019], and Khodak et al. [2019]. It also explains why our bound for the IER in Theorem 3.4 is worse compared to the bound for the AER.

In view of Proposition 4.1, we are left to bound the optimization error and the federated stability of FedProx. As discussed above, achieving the AER and IER guarantees requires somewhat different assumptions, as the latter involves characterizing the performance of the global model. So we split our discussion into two parts below.

**Bounding the average excess error.** The following theorem characterize the performance of FedProx in terms of the AER.
**Theorem 4.4** (AER guarantees for FedProx). Let Assumptions A and C(a) hold, and assume $n_i \geq 4\beta/\mu$ for all $i \in [m]$. Choose the weight vector $p$ such that

$$
\frac{p_{\max} \sum_{i \in [m]} p_i / n_i}{\sum_{i \in [m]} p_i^2 / n_i} \leq C_p \tag{4.13}
$$

for some constant $C_p$, where $p_{\max} = \max_i p_i$. Consider the FedProx algorithm, $A_{FP}$, with the following hyperparameter configuration:

1. In the joint training stage (i.e., $0 \leq t \leq T - 1$), set

$$
\eta_{t,k}^{(i)} = \frac{1}{(\mu + \lambda)(k + 1)}, \quad \eta^{(\text{global})} = \frac{2(\mu + \lambda)}{\lambda \mu(t + 1)}, \quad K_t + 1 \geq C_1(\lambda^2 \lor 1)t, \quad T \geq C_2 \lambda(\lambda \lor 1)m\|p\|^2 \cdot \left(\sum_{i \in [m]} p_i / n_i\right)^{-1} \lor \left[\lambda(\lambda \lor 1)n_{\max}^2\right]; \tag{4.14}
$$

2. In the final training stage (i.e., $t = T$), set

$$
\eta_{T,k}^{(i)} = \frac{1}{(\mu + \lambda)(k + 1)}, \quad K_T \geq C_3(\lambda + 1)^2 \cdot \left(\sum_{i \in [m]} p_i / n_i\right)^{-1} \lor \left[\lambda^2 \max_{i \in [m]} (p_i n_i)^2\right], \tag{4.15}
$$

where $C_1, C_2, C_3$ are constants depending only on $(\mu, \beta, \|\ell\|_\infty, D)$. Then, there exists a choice of $\lambda$ such that

$$
\mathbb{E}_{A_{FP}, S}[\text{AER}_p(A_{FP})] \lesssim \left(\frac{\mu}{1 \lor C_p} + \frac{(1 \lor C_p)\beta\|\ell\|_\infty}{\mu}\right) \left(\sum_{i \in [m]} p_i / n_i\right)^{-1} \lor \left(\sum_{i \in [m]} p_i / n_i\right) + \sum_{i \in [m]} p_i^2 / n_i. \tag{4.16}
$$

**Proof.** See Appendix C.3.

A few remarks are in order. First, (4.13) essentially says that the weight $p$ cannot be too imbalanced, and too much imbalance in $p$ can hurt the performance in view of the multiplicative factor of $C_p$ in our bound (4.16). If we set $p_i = 1/m$, then $C_p$ is naturally of constant order; whereas if we set $p_i = n_i/N$, we have $C_p \propto mn_{\max}/N$, where $n_{\max} = \max_i n_i$, which calls for relative balance of the sample sizes.

We then briefly comment on the hyperparameter choice in the above theorem. The step sizes are of the form $1/((\mu + \lambda)(k + 1))$, and such a choice is common in strongly convex stochastic optimization problems (see, e.g., Rakhlin et al. 2011, Shamir and Zhang 2013). Such a choice, along with the smoothness of the problem, is also the key for us to by-pass the need of doing any time-averaging operation, as is done by, for example, Dinh et al. [2020].

In Theorem 4.4, the choice of the communication rounds $T$ and the final local training round $K_T$ both scale polynomially with $\lambda$, which means that the optimization convergence of FedProx is slower when the data are less heterogeneous. This phenomenon happens more generally. For example, in Hanzely and Richtárik [2020], they proposed a variant of SGD that optimizes (4.8) with $p_i = 1/m$ in $O\left(\frac{L + \lambda}{\mu} \log 1/\varepsilon\right)$-many iterations, where $L$ is the Lipschitz constant of the loss function and $\varepsilon$ is the desired accuracy level.
The constants $C_1, C_2, C_3$ in the statement of Theorem 4.4 can be explicitly traced in our proof. We remark that the dependence on problem-specific constants $(\mu, \beta, \|\ell\|_\infty, D)$ in our hyperparameter choice and on $\lambda$ may not be tight. A tight analysis of the optimization error is interesting, but less relevant for our purpose of understanding the sample complexity. So we defer such an analysis to future work\footnote{The theories developed by Hanzely et al. [2020] can be useful for such an analysis.}.

**Bounding individualized excess errors.** The following theorem gives the IER guarantees for FedProx.

**Theorem 4.5** (IER guarantees for FedProx). Let Assumptions A and C(b) hold. Moreover, assume that $n_i \asymp n_{i'}$ for any $i \neq i' \in [m]$ and $n_i \geq 4\beta/\mu \forall i \in [m]$. Let the weight vector be chosen as $p_i \asymp 1/m \forall i \in [m]$. Consider the FedProx algorithm, $A_{FP}$, with the following hyperparameter configuration:

1. In the joint training stage (i.e., $0 \leq t \leq T-1$), set $\eta_{t,k}^{(i)}, \eta_t^{(\text{global})}, K_t$ as in (4.14), and set
   \[
   T \geq C'_2\lambda(1 + \lambda) \max_{i \in [m]} n_i \cdot \left(p_i^{-1} \lor [\lambda(1 + \lambda)n_i]\right);
   \]

2. In the final training stage (i.e., $t = T$), set $\eta_{T,k}^{(i)}$ as in (4.15), and set
   \[
   K_T \geq C'_3\lambda^2 \max_{i \in [m]} n_i \left(p_i^{-1} \lor \lambda^2 p_i^2 n_i\right),
   \]

where $C'_2, C'_3$ are constants only depending on $(\mu, \beta, \|\ell\|_\infty, D)$. Then, there exists a choice of $\lambda$ such that for any $i \in [m]$, we have

\[
\mathbb{E}_{A_{FP}, S}[\text{IER}_i(A_{FP})] \lesssim \left((\mu + \mu^{-1})(\beta\|\ell\|_\infty + \frac{\sigma^2 + \beta^2 + \sigma^2}{\mu^2}) + \mu D^2\right) \cdot \left(\frac{R}{\sqrt{n_i}} \lor \frac{1}{n_i} + \frac{\sqrt{m}}{N}\right).
\]

**Proof.** See Appendix C.4.\qed

Compared to Theorem 4.4, the above theorem imposes extra assumptions that the sample sizes are relative balanced and that $p_i \asymp 1/m$, both of which are due to the fact that we need to additionally take care of the estimation error of the global model. The hyperparameter choice slightly differs from that in Theorem 4.4 for the same reason. Note that Theorem 3.4 is then a direct consequence of Theorems 4.4 and 4.5.

We conclude this subsection by a remark on practical implementations of FedProx. In practice, when one is to use FedProx to optimize highly non-convex functions like the loss function of deep neural networks, instead of sticking to the choices made in Theorems 4.4 and 4.5, the hyperparameters are usually tuned by trial-and-error for best test performance.

## 5 Discussion

This paper studies the statistical properties of personalized federated learning. Focusing on strongly-convex, smooth, and bounded empirical risk minimization problems, we have established a theorem of the alternative, stating that given a specific level of heterogeneity, either FedAvg is minimax optimal, or PureLocalTraining is minimax optimal. In the course of proving this theorem of
the alternative for personalized federated learning, we obtained a novel analysis of FedProx and introduced a new notion of algorithmic stability termed federated stability, which is possibly of independent interest for analyzing generalization properties in the context of federated learning.

We close this paper by mentioning several open problems.

- **Dependence on problem-specific parameters.** This paper focuses on the dependence on the sample sizes, and in our bounds, the dependence on problem-specific parameters (e.g., the smoothness and strong convexity constants) may not be optimal. This can be problematic if those parameters are not of constant order, and it will be interesting to give a refined analysis that gives optimal dependence on those parameters.

- **A refined analysis of FedProx.** The upper bounds we develop for FedProx, as we have mentioned, do not match our minimax lower bounds. We suspect that this is an artifact of our analysis and a refined analysis of FedProx would be a welcome advance.

- **Estimation of the level of heterogeneity.** For unsupervised problems where evaluation of a model is difficult, implementation of the dichotomous strategy described in Corollaries 3.1 and 4.1 would require estimating the level of heterogeneity $R$. Even for supervised problems, estimation of $R$ would be interesting, as it allows one to decide which algorithm to choose without model training.

- **Beyond convexity.** Our analysis is heavily contingent upon the strong convexity of the loss function, which, to the best of our knowledge, is not easily generalizable to the non-convex case. Meanwhile, our notion of heterogeneity, which is based on the distance of optimal local models to the convex combination of them, may not be natural for non-convex problems. It is of interest, albeit difficult, to have a theoretical investigation of personalized federated learning for non-convex problems.

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Appendices for “A Theorem of the Alternative for Personalized Federated Learning”

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Appendix A Proof of Theorem 4.1: Lower Bounds

We start by presenting a lower bound when all \( w^{(i)}_* \)'s are the same.

**Lemma A.1** (Lower bound under homogeneity). Consider the logistic regression model with \( w^{(i)}_* = w^{(global)}_{avg} \) for any \( i \in [m] \). Then

\[
\inf_{\tilde{w}^{(global)}} \sup_{w^{(global)}_{avg}} \mathbb{E}_S \| \tilde{w}^{(global)} - w^{(global)}_{avg} \|^2 \geq \frac{d}{N}.
\]

**Proof.** This is a classical result. See, e.g., Example 8.4 of Duchi [2019].

**Proof of (4.3).** We first give a lower bound based on the observation that the homogeneous case is in fact included in the parameter space \( P_1 \). More explicitly, let us define \( P_0 = \{ \{ w^{(i)}_* \} \in P_1 : w^{(i)}_* = w^{(global)}_{avg} \ \forall i \in [m] \} \). By Lemma A.1, we have

\[
\inf_{\{ \tilde{w}^{(i)} \}} \sup_{\{ w^{(i)}_* \}} \sum_{i \in [m]} p_i \mathbb{E}_S \| \tilde{w}^{(i)} - w^{(i)}_* \|^2 \geq \inf_{\{ \tilde{w}^{(i)} \}} \sup_{\{ w^{(i)}_* \}} \sum_{i \in [m]} p_i \mathbb{E}_S \| \tilde{w}^{(i)} - w^{(i)}_* \|^2
\]

\[
= \inf_{\tilde{w}^{(global)}} \sup_{w^{(global)}_{avg}} \mathbb{E}_S \| \tilde{w}^{(global)} - w^{(global)}_{avg} \|^2 \gtrsim \frac{d}{N}.
\]

We now use a variant of Assouad’s method [Assouad, 1983] that allows us to tackle multiple datasets. Consider the following data generating process: nature generates \( V = \{ v^{(i)} : i \in [m] \} \) i.i.d. from the uniform distribution on \( V = \{ \pm 1 \}^d \) and sets \( w^{(i)}_* = \delta_i v^{(i)} \) for some \( \delta_i \) such that the following constraint is satisfied:

\[
\sum_{i \in [m]} p_i \| w^{(i)}_* - w^{(global)}_{avg} \|^2 = \sum_{i \in [m]} p_i \| \delta_i v^{(i)} - \sum_{s \in [m]} p_s \delta_s v^{(s)} \|^2 \leq R^2.
\]

We will specify the choice of \( \delta_i \)'s later. Denoting \( \mathbb{E}_X \) as the marginal expectation operator with respect to all the features \( \{ x^{(i)}_j \} \) and \( \mathbb{E}_{Y|X} \) as the conditional expectation operator with respect to \( \{ y^{(i)}_j \}|\{ x^{(i)}_j \} \), we
can lower bound the minimax risk by the Bayes risk as follows:

\[
\inf_{\{\hat{\omega}^{(i)}\}} \sup_{\{w^{(i)}\} \in P_i \in \mathcal{M}} \sum_{i \in [m]} p_i E_S \|\hat{\omega}^{(i)} - w^{(i)}\|^2 \\
\geq \inf_{\{\hat{\omega}^{(i)}\}} \sum_{i \in [m]} p_i E_S \|\hat{\omega}^{(i)} - \delta_i v^{(i)}\|^2 \\
= \inf_{\{\hat{\omega}^{(i)}\} \subseteq \mathcal{V}} \sum_{i \in [m]} p_i E_V,S \|\delta_i \hat{\omega}^{(i)} - \delta_i v^{(i)}\|^2 \\
\geq E_X \sum_{i \in [m]} p_i \delta_i^2 \inf_{\{v^{(i)}\} \subseteq \mathcal{V}} E_{V,Y \mid X} \|\hat{\omega}^{(i)} - v^{(i)}\|^2 \\
\geq E_X \sum_{i \in [m]} p_i \delta_i^2 \sum_{k \in [d]} \inf_{\{v^{(i)}\} \subseteq \mathcal{V}} \mathbb{P}_{V,Y \mid X} (\hat{\omega}_k^{(i)} \neq v_k^{(i)}) \\
= \frac{1}{2} E_X \sum_{i \in [m]} p_i \delta_i^2 \sum_{k \in [d]} \inf_{\{v^{(i)}\} \subseteq \mathcal{V}} \mathbb{P}_{v \mid k} (\hat{\omega}_k^{(i)} = -1) + \mathbb{P}_{v \mid k} (\hat{\omega}_k^{(i)} = +1),
\]

where in the last line, we have let \( \mathbb{P}_{i,\pm k} (\cdot) = \mathbb{P}_{V,Y \mid X} (\cdot | v_k^{(i)} = \pm 1) \) to denote the probability measure with respect to the randomness in \((V, S)\) conditional on the features \( \{x_j^{(i)}\} \) as well as the realization of \( v_k^{(i)} = \pm 1 \). More explicitly, we can write

\[
\mathbb{P}_{i,\pm k} = \left( \bigotimes_{s \neq i} \mathbb{P}_{v^{(s)}} \otimes \mathbb{P}_{\{y^{(s)}\}_{j \neq 1} | v^{(s)}, \{x_j^{(s)}\}_{j \neq 1}} \right) \otimes \left( \mathbb{P}_{v^{(i)} | v_k^{(i)} = \pm 1} \otimes \mathbb{P}_{\{y_j^{(i)}\}_{j \neq 1} | v^{(i)}, v_k^{(i)} = \pm 1, \{x_j^{(i)}\}_{j \neq 1}} \right) = \frac{1}{2(m-1)d+d-1} \sum_{V \setminus \{v_k^{(i)}\}} \mathbb{P}_{V,i,\pm k},
\]

where the \( \otimes \) symbol stands for taking the product of two measures and \( \mathbb{P}_{V,i,\pm k} \) corresponds to the law of all the labels \( Y \) conditional on a specific realization of \( \{V : v_k^{(i)} = \pm 1\} \) and the features \( X \). With the current notations and letting \( \|\mathbb{P} - \mathbb{Q}\|_{TV} \) be the total variation distance between two probability measures \( \mathbb{P} \) and \( \mathbb{Q} \), we can invoke Neyman-Pearson lemma to get

\[
\inf_{\{\hat{\omega}^{(i)}\}} \sup_{\{w^{(i)}\} \in \mathcal{M}} \sum_{i \in [m]} p_i E_S \|\hat{\omega}^{(i)} - w^{(i)}\|^2 \geq E_X \sum_{i \in [m]} p_i \delta_i^2 \sum_{k \in [d]} \left( 1 - \|\mathbb{P}_{i,+,k} - \mathbb{P}_{i,-,k}\|_{TV} \right) \\
= d \sum_{i \in [m]} p_i \delta_i^2 - E_X \sum_{i \in [m]} p_i \delta_i^2 \sum_{k \in [d]} \|\mathbb{P}_{i,+,k} - \mathbb{P}_{i,-,k}\|_{TV}. \tag{A.3}
\]

We then proceed by

\[
\sum_{i \in [m]} p_i \delta_i^2 \sum_{k \in [d]} \|\mathbb{P}_{i,+,k} - \mathbb{P}_{i,-,k}\|_{TV} \\
\leq \sum_{i \in [m]} p_i \delta_i^2 \sqrt{d} \left( \sum_{k \in [d]} \|\mathbb{P}_{i,+,k} - \mathbb{P}_{i,-,k}\|^2_{TV} \right)^{1/2} \\
= \sum_{i \in [m]} p_i \delta_i^2 \sqrt{d} \left( \sum_{k \in [d]} \frac{1}{2(m-1)d+d-1} \sum_{V \setminus \{v_k^{(i)}\}} \|\mathbb{P}_{V,i,+,k} - \mathbb{P}_{V,i,-,k}\|^2_{TV} \right)^{1/2} \\
= \sum_{i \in [m]} p_i \delta_i^2 \sqrt{d} \left( \sum_{k \in [d]} \frac{1}{2(m-1)d+d-1} \sum_{V \setminus \{v_k^{(i)}\}} \|\mathbb{P}_{V,i,+,k} - \mathbb{P}_{V,i,-,k}\|^2_{TV} \right)^{1/2},
\]
where the last inequality is by convexity of the total variation distance. Note that \( F_{V,i,±k} \) is the product of biased Rademacher random variables: if we let \( \text{Rad}(p) \) be the \( \pm 1 \)-valued random variable with positive probability \( p \), we can write

\[
P_{V,i,±k} = \bigotimes_{s \in [m]} \bigotimes_{j \in [n_s]} \text{Rad}\left(\frac{1}{1 + \exp\{-\delta_s \langle \mathbf{v}(s), \mathbf{x}_j^{(s)} \rangle\}\right), \quad \mathbf{v}_k^{(i)} = \pm 1.
\]

Thus, by Pinsker’s inequality, we have

\[
\|P_{V,i,+k} - P_{V,i,-k}\|_{TV}^2 \\
\leq \frac{1}{2} D_{JS}(P_{V,i,+k} \| P_{V,i,-k}) \\
= \frac{1}{2} \sum_{s \neq i} \sum_{j \in [n_i]} D_{JS}\left[\text{Rad}\left(\frac{1}{1 + \exp\{-\delta_i \langle \mathbf{v}(i), \mathbf{x}_j^{(i)} \rangle\}\right) \bigg| \text{Rad}\left(\frac{1}{1 + \exp\{-\delta_i \langle \tilde{\mathbf{v}}(i), \mathbf{x}_j^{(i)} \rangle\}\right)\right],
\]

where \( D_{JS}(\| \cdot \|_Q) = D_{KL}(\P \| \Q) + D_{KL}(\Q \| \P) \) is the Jensen–Shannon divergence between \( \P \) and \( \Q \), and \( \mathbf{v}(s), \tilde{\mathbf{v}}(s) \) are two \( V \)-valued vectors that only differs in the \( k \)-th coordinate. By a standard calculation, one finds that

\[
D_{JS}\left[\text{Rad}\left(\frac{1}{1 + \exp\{-\delta_i \langle \mathbf{v}(i), \mathbf{x}_j^{(i)} \rangle\}\right) \bigg| \text{Rad}\left(\frac{1}{1 + \exp\{-\delta_i \langle \tilde{\mathbf{v}}(i), \mathbf{x}_j^{(i)} \rangle\}\right)\right] \\
\leq \delta_i^2 (\mathbf{v}_k^{(i)} - \tilde{\mathbf{v}}_k^{(i)})^2 (\mathbf{x}_{j,k}^{(i)})^2 \\
= 4\delta_i^2 (\mathbf{x}_{j,k}^{(i)})^2.
\]

This gives

\[
\|P_{V,i,+k} - P_{V,i,-k}\|_{TV}^2 \leq 2\delta_i^2 \sum_{j \in [n_i]} (\mathbf{x}_{j,k}^{(i)})^2 \leq 2\delta_i^2 c_X^2 n_i.
\]

and hence

\[
\sum_{i \in [m]} \sum_{k \in [d]} p_i \delta_i^2 \|P_{i,+k} - P_{i,-k}\|_{TV} \leq \sqrt{2c_X} \sum_{i \in [m]} p_i \delta_i^3 d n_i^{1/2}.
\]

Plugging the above display to (A.3) gives

\[
\inf_{\{\mathbf{w}^{(i)}\}} \sup_{\{\mathbf{w}^{(i)}\} \in \mathcal{P}} \sum_{i \in [m]} p_i \mathbb{E}_S \|\mathbf{w}^{(i)} - \mathbf{w}_*^{(i)}\|^2 \\
\geq d \left( \sum_{i \in [m]} p_i \delta_i^2 - \sqrt{2c_X} \sum_{i \in [m]} p_i \delta_i^3 \sqrt{n_i} \right).
\]

To this end, all that is left is to choose \( \delta_i \) appropriately so that (1) the above display is as tight as possible; (2) (A.2) is satisfied. We consider the following two cases:

1. Assume \( R^2 \geq d \sum_{i \in [m]} p_i/n_i = dm/N \). Note that we can re-write the requirement (A.2) to be

\[
d \sum_{i \in [m]} p_i \delta_i^2 - \| \sum_{i \in [m]} p_i \delta_i \mathbf{v}^{(i)} \|^2 \leq R^2.
\]

Under the current assumption, this requirement will be satisfied if we choose \( \delta_i = c/\sqrt{n_i} \) for any \( c \leq 1 \). Under such a choice, the right-hand side of (A.4) becomes \( \frac{c}{N} \sum_{i \in [m]} p_i \delta_i^3 \sqrt{n_i} \). Thus, by setting \( c = 2\sqrt{2c_X} \), we get the following lower bound:

\[
\inf_{\{\mathbf{w}^{(i)}\}} \sup_{\{\mathbf{w}^{(i)}\} \in \mathcal{P}} \sum_{i \in [m]} p_i \mathbb{E}_S \|\mathbf{w}^{(i)} - \mathbf{w}_*^{(i)}\|^2 \geq \frac{d}{N/m}.
\]

2. Assume \( R^2 \leq d \sum_{i \in [m]} p_i/n_i = dm/N \). Note that if we set \( \delta_i \equiv \delta = cR/\sqrt{d} \) where \( c \leq 1 \), (A.2) reads

\[
c^2 R^2 - \| \sum_{i \in [m]} p_i \delta_i \mathbf{v}^{(i)} \|^2 \leq R^2,
\]

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which trivially holds. Now, the right-hand side of (A.4) becomes
\[ c^2 R^2 (1 - \sqrt{2cc_X} \sum_{i \in [m]} p_i R \sqrt{n_i} / \sqrt{d}). \]

Since \( p_i = n_i / N \) and \( n_i \approx N / m \), our assumption on \( R \) gives
\[ \sqrt{2cc_X} \sum_{i \in [m]} p_i R \sqrt{n_i} / \sqrt{d} \lesssim \sum_{i} n_i \cdot \sqrt{\frac{mn_i}{N}} = 1. \]

This means that we can choose \( c \) to be a small constant such that the following lower bound holds:
\[ \inf_{\{\hat{w}^{(i)}\}} \sup_{\{w^{(i)}\} \in P} \sum_{i \in [m]} p_i \mathbb{E}_S \| \hat{w}^{(i)} - w^{(i)}_* \|^2 \gtrsim R^2. \]

Summarizing the above two cases, we arrive at
\[ \inf_{\{\hat{w}^{(i)}\}} \sup_{\{w^{(i)}\} \in P} \sum_{i \in [m]} p_i \mathbb{E}_S \| \hat{w}^{(i)} - w^{(i)}_* \|^2 \gtrsim \frac{d}{N/m} \land R^2. \]

Combining the above bound with (A.1), we get
\[ \inf_{\{\hat{w}^{(i)}\}} \sup_{\{w^{(i)}\} \in P} \sum_{i \in [m]} p_i \mathbb{E}_S \| \hat{w}^{(i)} - w^{(i)}_* \|^2 \gtrsim \frac{d}{N/m} \land R^2 + \frac{d}{N}, \]
which is the desired result.

\[ \square \]

Proof of (4.4). The proof is similar to the proof of (3.1), and we only provide a sketch here. Without loss of generality we consider the first client. By the same arguments as in the proof of (3.1), the left-hand side of (4.4) is lower bounded by a constant multiple of \( d / N \). Now, by considering the same prior distribution on \( P \) as in the proof of (3.1), we get
\[ \inf_{\{\hat{w}^{(i)}\}} \sup_{\{w^{(i)}\} \in P} \mathbb{E}_S \| \hat{w}^{(1)} - w^{(1)}_* \|^2 \gtrsim d \delta_1^2 (1 - \delta_1 \sqrt{n_1}), \]
where the \( \delta_i \)'s should obey the following inequality:
\[ \| \delta_i v^{(i)} - \sum_{s \in [m]} p_s \delta_s v^{(s)} \|^2 \leq R^2. \]

Choosing \( \delta_i \approx 1 / \sqrt{n_i} \) when \( R \geq dm/N \) and \( \delta_i \approx R/\sqrt{d} \) otherwise, we arrive at
\[ \inf_{\{\hat{w}^{(i)}\}} \sup_{\{w^{(i)}\} \in P} \mathbb{E}_S \| \hat{w}^{(1)} - w^{(1)}_* \|^2 \gtrsim \frac{d}{n_1} \land R^2, \]
and the proof is concluded.

\[ \square \]

Appendix B  Optimization Convergence of FedProx

This section concerns the optimization convergence of FedProx. We first introduce some notations. Let \( w^{(i)}_{t,t,k} \) be the output of \( k \)-th step of Algorithm 3 when the initial local model is given by \( w^{(i)}_t \equiv w^{(i)}_{t,0} \equiv w^{(i)}_{t-1,k} \); let \( I^{(i)}_{t,k} \) be the corresponding minibatch taken, and denote the initial global model by \( w^{(global)}_t \). Let \( \mathcal{F}_{t,k} \) be the sigma algebra generated by the randomness by Algorithm 3 up to \( w^{(i)}_{t,k} \), namely the randomness in
\[ \{ C_t, \{ I^{(i)}_{t,l} : i \in C_t, 0 \leq l \leq K_t - 1 \} \}_{t=0}^{l-1}, C_l, \text{ and } \{ I^{(i)}_{t,l} : i \in C_t, 0 \leq l \leq k - 1 \}. \]

For notational convenience we
let $C_T = [m]$ (i.e., all clients are involved in local training in Stage II of Algorithm 3). Then the sequence \{w^{(i)}_{t,k}\} is adapted to the following filtration:

$$F_{0,0} \subseteq F_{0,1} \subseteq \cdots \subseteq F_{0,K} \subseteq F_{1,0} \subseteq F_{1,1} \subseteq \cdots \subseteq F_{1,K} \subseteq \cdots \subseteq F_{T,K}.$$  

We write the optimization problem \ref{eq:inner_loop} as

$$\min_{w^{(\text{global})} \in W} \sum_{i \in [m]} p_i F_i(w^{(\text{global})}, S_i),$$  \hspace{1cm} \text{(B.1)}

where

$$F_i(w^{(\text{global})}, S_i) = \min_{w^{(i)} \in W} \left\{ L_i(w^{(i)}, S_i) + \frac{\lambda}{2} \| w^{(\text{global})} - w^{(i)} \|^2 \right\}. \hspace{1cm} \text{(B.2)}$$

To simplify notations, we introduce the proximal operator

$$\text{Prox}_{L_i/\lambda}(w^{(\text{global})}) = \text{Prox}_{L_i/\lambda}(w^{(\text{global})}, S_i) = \arg\min_{w^{(i)} \in W} \left\{ L_i(w^{(i)}, S_i) + \frac{\lambda}{2} \| w^{(\text{global})} - w^{(i)} \|^2 \right\}. \hspace{1cm} \text{(B.3)}$$

The high-level idea of this proof is to regard $\lambda \sum_{i \in C_t} (w^{(i)}_t - w^{(i)}_{t+1})/B^{(\text{global})}$ as a biased stochastic gradient of $\frac{1}{m} \sum_{i \in [m]} F_i(w^{(i)}_t, S_i)$. This idea has appeared in various places (see, e.g., the proof of Proposition 5 in Denevi et al. \cite{denevi2019local} and the proof of Theorem 1 in Dinh et al. \cite{dinh2020localization}). However, the implementation of this idea in our case is more complicated than the above mentioned works in that (1) we are not in an online learning setup (compared to Denevi et al. \cite{denevi2019local}); (2) we don’t need to assume all clients are training at every round (compared to Dinh et al. \cite{dinh2020localization}); and (3) we use local SGD for the inner loop (instead of assuming the inner loop can be solved with arbitrary precision as assumed in Dinh et al. \cite{dinh2020localization}), so the gradient norm depends on $\lambda$, and could in principle be arbitrarily large, which causes extra complications.

**Lemma B.1** (Convergence of the inner loop). Let Assumption A(a, b) holds. Choose $\eta^{(i)}_{t,k} = \frac{1}{\mu + \lambda(k+1)}$. Then for any $k \geq 0$, we have

$$\mathbb{E} \left[ \| w^{(i)}_{t,k} - \text{Prox}_{L_i/\lambda}(w^{(i)}_t) \|^2 \bigg| F_{t,0}, i \in C_t \right] \leq \frac{8 \beta^2 D^2}{\mu^2(k+1)}.$$  \hspace{1cm} \text{(B.4)}

**Proof.** See Appendix B.1. \hfill \Box

**Lemma B.2** (Convergence of the outer loop). Let the assumptions in Lemma B.1 hold. Choose $\eta^{(\text{global})} = \frac{2(\mu + \lambda)}{\mu(t+1)}$ and assume

$$K_T + 1 \geq \frac{(4\tau + 20)\lambda^2 \beta^2 D^2}{\mu^2(\beta^2 D^2 \wedge 2\lambda\|\ell\|_\infty \wedge \lambda^2 D^2)} \quad \forall 0 \leq \tau \leq t - 1.$$  \hspace{1cm} \text{(B.5)}

Then for any $t \geq 0$, we have

$$\mathbb{E}_{A_{FP}} \| w^{(\text{global})}_t - \bar{w}^{(\text{global})} \|^2 \leq \frac{12(\lambda + \mu)^2 m \| p \|^2 (\beta^2 D^2 \wedge 2\lambda\|\ell\|_\infty \wedge \lambda^2 D^2)}{\lambda^2 \mu^2(t+1)}.$$  \hspace{1cm} \text{(B.6)}

where the expectation is taken over the randomness in Algorithm 3.

**Proof.** See Appendix B.2. \hfill \Box

**Proposition B.1** (Optimization error of $A_{FP}$). Under the assumptions of Lemma B.1 and B.2, for any dataset $S \sim \otimes_i D_i^\infty$, we have

$$\mathbb{E}_{A_{FP}}[\mathcal{E}_{OPT}] \leq \frac{4(\beta + \lambda)\beta^2 D^2}{\mu^2(K_T + 1)} + \frac{6(\lambda + \mu)^2 m \| p \|^2 (\beta^2 D^2 \wedge 2\lambda\|\ell\|_\infty \wedge \lambda^2 D^2)}{\lambda \mu^2(t+1)}.$$  \hspace{1cm} \text{(B.7)}

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Proof. By definition we have

\[
\mathbb{E}_{A_{FP}}[\mathcal{E}_{OPT}] := \mathbb{E}_{A_{FP}} \left[ \sum_{i \in [m]} p_i \left( L_i(w_{T+1}^{(i)}, S_i) + \lambda \frac{2}{\mu} \|w_{T+1}^{(i)} - w_{T}^{(i)}\|^2 \right) - \sum_{i \in [m]} p_i F_i(\tilde{w}^{(i)}_t, S_i) \right]
\]

\[
\begin{align*}
\leq & \sum_{i \in [m]} \frac{p_i(\beta + \lambda)}{2} \mathbb{E}_{A_{FP}} \|w_{T+1}^{(i)} - \text{Prox}_{L_i/\lambda}(w_{T}^{(i)})\|^2 \\
+ & \mathbb{E}_{A_{FP}} \left[ \sum_{i \in [m]} p_i F_i(\tilde{w}^{(i)}_T, S_i) - \sum_{i \in [m]} p_i F_i(\tilde{w}^{(i)}_T, S_i) \right] \\
\leq & \frac{4(\beta + \lambda)\beta^2 D^2}{\mu^2(K_T + 1)} + \frac{\lambda}{2} \mathbb{E}_{A_{FP}} \|w_{T}^{(i)} - \tilde{u}^{(i)}_T\|^2 \\
\leq & \frac{4(\beta + \lambda)\beta^2 D^2}{\mu^2(K_T + 1)} + 6(\lambda + \mu)^2 m ||p||^2 (\beta^2 D^2 \vee 2\lambda \|f\|_\infty \vee \lambda^2 D^2) \frac{\lambda^2 t + 1}{\mu^2 t + 1},
\end{align*}
\]

where (a) is by smoothness of $L_i$, (b) is by Lemma B.1 and $\lambda$-smoothness of $\sum_{i \in [m]} p_i F_i$ (which holds by Lemma B.3), and (c) is by Lemma B.2. \qed

### B.1 Proof of Lemma B.1: Convergence of the Inner Loop

The proof is an adaptation of the proof of Lemma 1 in Rakhlin et al. [2011]. However, we need to deal with the extra complication that the hyperparameter $\lambda$ can in principle be arbitrarily large. We start by noting that

\[
\begin{align*}
\|w_{t+1}^{(i)} - \text{Prox}_{L_i/\lambda}(w_t^{(i)})\|^2 \\
= & \|P_W \left[ \frac{\eta_{t,k}^{(i)}}{B^{(i)}} \sum_{j \in I_{t,k}^{(i)}} \left( \nabla \ell(w_{t,k}^{(i)}, z_j^{(i)}) + \lambda(w_{t,k}^{(i)} - w_t^{(i)}) \right) \right] - \text{Prox}_{L_i/\lambda}(w_t^{(i)}) \|^2 \\
\leq & \left\| w_{t,k}^{(i)} - \text{Prox}_{L_i/\lambda}(w_t^{(i)}) \right\|^2 + \left\| \frac{\eta_{t,k}^{(i)}}{B^{(i)}} \sum_{j \in I_{t,k}^{(i)}} \left( \nabla \ell(w_{t,k}^{(i)}, z_j^{(i)}) + \lambda(w_{t,k}^{(i)} - w_t^{(i)}) \right) \right\|^2 \\
& - 2 \left\langle w_{t,k}^{(i)} - \text{Prox}_{L_i/\lambda}(w_t^{(i)}), \frac{\eta_{t,k}^{(i)}}{B^{(i)}} \sum_{j \in I_{t,k}^{(i)}} \left( \nabla \ell(w_{t,k}^{(i)}, z_j^{(i)}) + \lambda(w_{t,k}^{(i)} - w_t^{(i)}) \right) \right\rangle,
\end{align*}
\]

where the inequality is because $\text{Prox}_{L_i/\lambda}(w_t^{(i)}) \in W$ and $P_W$ is non-expansive. Now by strong convexity and unbiasedness of the stochastic gradients, we have

\[
\mathbb{E} \left[ \left\langle w_{t,k}^{(i)} - \text{Prox}_{L_i/\lambda}(w_t^{(i)}), \frac{1}{B^{(i)}} \sum_{j \in I_{t,k}^{(i)}} \left( \nabla \ell(w_{t,k}^{(i)}, z_j^{(i)}) + \lambda(w_{t,k}^{(i)} - w_t^{(i)}) \right) \right\rangle \mid F_{t,k}, i \in C_t \right] \\
\geq \left( L_i(w_{t,k}^{(i)}, S_i) + \frac{\lambda}{2} \|w_{t,k}^{(i)} - w_t^{(i)}\|^2 \right) \\
- \left( L_i(\text{Prox}_{L_i/\lambda}(w_t^{(i)}), S_i) + \frac{\lambda}{2} \|\text{Prox}_{L_i/\lambda}(w_t^{(i)}) - w_t^{(i)}\|^2 \right) \\
+ \frac{1}{2} \left( \mu_i + \frac{\lambda n}{m n_i} \right) \|w_{t,k}^{(i)} - \text{Prox}_{L_i/\lambda}(w_t^{(i)})\|^2 \\
\geq (\mu + \lambda) \|w_{t,k}^{(i)} - \text{Prox}_{L_i/\lambda}(w_t^{(i)})\|^2.
\]
On the other hand, applying Lemma B.5 gives

$$E \left[ \frac{\eta_{i,k}}{B(i)} \sum_{j \in F_{i,k}} \left( \nabla \ell(w_{i,k}^{(i)}, z_{j}^{(i)}) + \lambda(w_{i,k}^{(i)} - w_{i}^{\text{global}}) \right) \right|^2 \biggm| F_{i,k}, i \in C_{i}$$

$$= (\eta_{i,k})^2 \cdot \left[ \frac{n_{i}/B(i) - 1}{n_{i}(n_{i} - 1)} \sum_{j \in [n_{i}]} \left( \nabla \ell(w_{i,k}^{(i)}, z_{j}^{(i)}) - \nabla \ell(w_{i,k}^{(i)}, z_{j}^{(i)}) \right)^2 \right. + \left. \frac{1}{n_{i}} \sum_{j \in [n_{i}]} \nabla \ell(w_{i,k}^{(i)}, z_{j}^{(i)}) + \lambda(w_{i,k}^{(i)} - w_{i}^{\text{global}}) \right|^2 \right]$$

$$\leq 2(\eta_{i,k})^2 \beta^2 D^2 \cdot \left( \frac{n_{i}/B(i) - 1}{n_{i} - 1} \right) + \left( \beta + \frac{\lambda n}{mn_{i}} \right)^2 \|w_{i,k}^{(i)} - \text{Prox}_{L_{i}/\lambda}(w_{i}^{\text{global}})\|^2,$$

where in the second line we let $\nabla \ell(w_{i,k}^{(i)}, z_{j}^{(i)}) := \sum_{j \in [n_{i}]} \nabla \ell(w_{i,k}^{(i)}, z_{j}^{(i)})/n_{i}$, and in the last line is by the $\beta$-smoothness of $\ell(\cdot, z)$. Thus, we get

$$E \left[ \|w_{i,k+1}^{(i)} - \text{Prox}_{L_{i}/\lambda}(w_{i}^{\text{global}})\|^2 \biggm| F_{i,k}, i \in C_{i} \right]$$

$$\leq \left[ 1 - 2\eta_{i,k}(\mu + \lambda) + (\eta_{i,k})^2(\beta + \lambda)^2 \right] \|w_{i,k}^{(i)} - \text{Prox}_{L_{i}/\lambda}(w_{i}^{\text{global}})\|^2$$

$$+ 2(\eta_{i,k})^2 \beta^2 D^2 \cdot \left( \frac{n_{i}/B(i) - 1}{n_{i} - 1} \right). \quad \text{(B.6)}$$

We then proceed by induction. Note that if $k + 1 \leq \frac{8\beta^2}{\mu^2}$, then we have the following trivial bound:

$$E \left[ \|w_{i,k}^{(i)} - \text{Prox}_{L_{i}/\lambda}(w_{i}^{\text{global}})\|^2 \biggm| F_{i,0}, i \in C_{i} \right] \leq D^2 \leq \frac{8\beta^2 D^2}{\mu^2(k + 1)}, \quad \text{(B.7)}$$

where the first inequality is by $w_{i,k}^{(i)}, \text{Prox}_{L_{i}/\lambda}(w_{i}^{\text{global}}) \in W$ and the second inequality is by our assumption on $k$. Thus, it suffices to show

$$E \left[ \|w_{i,k+1}^{(i)} - \text{Prox}_{L_{i}/\lambda}(w_{i}^{\text{global}})\|^2 \biggm| F_{i,0}, i \in C_{i} \right] \leq \frac{8\beta^2 D^2}{\mu^2(k + 2)} \quad \text{(B.8)}$$

based on the inductive hypothesis (B.7) and $k + 1 \geq \frac{8\beta^2}{\mu^2}$. By the recursive relationship (B.6) and taking expectation, we have

$$E \left[ \|w_{i,k+1}^{(i)} - \text{Prox}_{L_{i}/\lambda}(w_{i}^{\text{global}})\|^2 \biggm| F_{i,0}, i \in C_{i} \right]$$

$$\leq \left[ 1 - 2\eta_{i,k}(\mu + \lambda) + (\eta_{i,k})^2(\beta + \lambda)^2 \right] \frac{8\beta^2 D^2}{k + 1} + 2(\eta_{i,k})^2 \beta^2 D^2 \cdot \left( \frac{n_{i}/B(i) - 1}{n_{i} - 1} \right).$$

Hence (B.8) is satisfied if

$$8\beta^2 D^2 \cdot \left[ \frac{1}{k + 2} - \frac{1}{k + 1} + \frac{2\eta_{i,k}(\mu + \lambda) - (\eta_{i,k})^2(\beta + \lambda)^2}{k + 1} \right] \geq 2(\eta_{i,k})^2 \beta^2 D^2 \cdot \left( \frac{n_{i}/B(i) - 1}{n_{i} - 1} \right).$$

By our choice of $\eta_{i,k}$, the above display is equivalent to

$$8\beta^2 D^2 \cdot \left[ - \frac{1}{(k + 1)(k + 2)} + \frac{2}{(k + 1)^2} - \frac{1}{(k + 1)^3} \left( \frac{\beta + \lambda}{\mu + \lambda} \right)^2 \right] \geq \frac{2\beta^2 D^2}{(\mu + \lambda)^2(k + 1)^2} \cdot \frac{n_{i}/B(i) - 1}{n_{i} - 1}.$$
which is further equivalent to
\[ 8 \beta^2 D^2 \cdot \left[ - \frac{k + 1}{k + 2} + 2 - \frac{1}{k + 1} \left( \frac{\beta + \lambda}{\mu + \lambda} \right)^2 \right] \geq \frac{2 \beta^2 D^2}{(\mu + \lambda)^2} \cdot \frac{n_i/B(i) - 1}{n_i - 1}. \]
We now claim that
\[ \frac{1}{k + 1} \left( \frac{\beta + \lambda}{\mu + \lambda} \right)^2 \leq \frac{1}{2}. \]
Indeed, since \( k + 1 \geq 8 \beta^2 / \mu^2 \), (1) if \( \lambda \leq \beta \), then the left-hand side above is less than \( \frac{4 \beta^2}{\mu^2} \geq \frac{1}{2} \); and (2) if \( \lambda \geq \beta \), the left-hand side above is less than \( \frac{4 \beta^2}{8 \beta^2} \leq \frac{1}{2} \). By the above claim, (B.8) would hold if
\[ 4 \beta^2 D^2 \geq \frac{2 \beta^2 D^2}{(\mu + \lambda)^2} \cdot \frac{n_i/B(i) - 1}{n_i - 1}. \]
We finish the proof by noting that the right-hand side above is bounded above by \( \frac{2 \beta^2 D^2}{\mu^2} \).

### B.2 Proof of Lemma B.2: Convergence of the Outer Loop

By construction we have
\[
\| \mathbf{u}_t^{\text{(global)}} - \mathbf{w}^{\text{(global)}} \|^2 \\
= \left\| \frac{\lambda m i}{B^{\text{(global)}}} \sum_{i \in C_t} p_i (\mathbf{u}_t^{\text{(global)}} - \mathbf{u}_t^{(i)}) \right\|^2 \\
= \| \mathbf{u}_t^{\text{(global)}} - \mathbf{w}^{\text{(global)}} \|^2 + \left\| \frac{\lambda m i}{B^{\text{(global)}}} \sum_{i \in C_t} p_i (\mathbf{u}_t^{\text{(global)}} - \mathbf{u}_t^{(i)}) \right\|^2 \\
- 2 \langle \mathbf{u}_t^{\text{(global)}} - \mathbf{w}^{\text{(global)}} , \frac{\lambda m i}{B^{\text{(global)}}} \sum_{i \in C_t} p_i (\mathbf{u}_t^{\text{(global)}} - \mathbf{u}_t^{(i)}) \rangle \\
\leq \| \mathbf{u}_t^{\text{(global)}} - \mathbf{w}^{\text{(global)}} \|^2 - 2 \left\langle \mathbf{u}_t^{\text{(global)}} - \mathbf{w}^{\text{(global)}} , \frac{\lambda m i}{B^{\text{(global)}}} \sum_{i \in C_t} p_i (\mathbf{u}_t^{\text{(global)}} - \text{Prox}_{L_i/\lambda}(\mathbf{u}_t^{\text{(global)}})) \right\rangle \\
+ \frac{2 \lambda m i}{B^{\text{(global)}}} \sum_{i \in C_t} p_i \left( \left\| \text{Prox}_{L_i/\lambda}(\mathbf{u}_t^{\text{(global)}}) - \mathbf{u}_t^{(i)} \right\| \right)^2 \\
+ \frac{2 \lambda m i}{B^{\text{(global)}}} \sum_{i \in C_t} p_i \left( \left\| \text{Prox}_{L_i/\lambda}(\mathbf{u}_t^{\text{(global)}}) - \mathbf{u}_t^{(i)} \right\| \right)^2 \\
- 2 \langle \mathbf{u}_t^{\text{(global)}} - \mathbf{w}^{\text{(global)}} , \frac{\lambda m i}{B^{\text{(global)}}} \sum_{i \in C_t} p_i (\text{Prox}_{L_i/\lambda}(\mathbf{u}_t^{\text{(global)}})) - \mathbf{u}_t^{(i)} \rangle.
\]

We first consider Term I. Note that \( \frac{\lambda m i}{\text{Prox}_{L_i/\lambda}} \sum_{i \in C_t} p_i (\mathbf{u}_t^{\text{(global)}} - \text{Prox}_{L_i/\lambda}(\mathbf{u}_t^{\text{(global)}})) \) is an unbiased stochastic gradient of \( \sum_i p_i F_i \), which is \( \mu F = \lambda \mu / (\lambda + \mu) \)-strongly convex. Thus, we have
\[
\mathbb{E}\left[ I \mid \mathcal{F}_{t-1,K_{t-1}} \right] = 2 \eta_i \left\langle \mathbf{u}_t^{\text{(global)}} - \mathbf{w}^{\text{(global)}} , \sum_{i \in [m]} p_i \nabla F_i(\mathbf{u}_t^{\text{(global)}}) , S_i ) \right\rangle \\
\geq 2 \eta_i \mu_F \| \mathbf{u}_t^{\text{(global)}} - \mathbf{w}^{\text{(global)}} \|^2.
\]
Now for Term II, we have
\[
E[\|F_{t-1,K_{t-1}}\|^{2}] \leq 2(\eta_t)_{(global)}^{2} \cdot \mathbb{E}\left[\left(\frac{1}{B_{(global)}} \sum_{i \in C_t} m_{p_i}\right)^{2} \left| F_{t-1,K_{t-1}} \right| \cdot \max_{i \in [m]} \|\nabla F_i(\mathbf{w}_{t}^{(global)}, S_i)\|^{2}\right]
\]
\[
\leq 2(\eta_t)_{(global)}^{2} \cdot \mathbb{E}\left[\left(\frac{1}{B_{(global)}} \sum_{i \in C_t} m_{p_i}\right)^{2} \left| F_{t-1,K_{t-1}} \right| \cdot \max_{i \in [m]} \left(\beta^2 D^2 \wedge 2\lambda \|\ell\|_{\infty} \wedge \lambda^2 D^2\right)\right]
\]
\[
\leq 2(\eta_t)_{(global)}^{2} \cdot \left(\frac{1}{m} \sum_{i \in [m]} (m_{p_i} - 1)^2 + 1\right) \cdot \left(\beta^2 D^2 \wedge 2\lambda \|\ell\|_{\infty} \wedge \lambda^2 D^2\right)
\]
\[
= 2(\eta_t)_{(global)}^{2} m \|p\|^{2} \left(\beta^2 D^2 \wedge 2\lambda \|\ell\|_{\infty} \wedge \lambda^2 D^2\right),
\]
where the second line is by Lemma B.4 and the third line is by Lemma B.5. For Term III, we invoke Lemma B.1 to get
\[
E[\|F_{t-1,K_{t-1}}\|^{2}] \leq 2\lambda^2 (\eta_t)_{(global)}^{2} \cdot \frac{8\beta^2 D^2}{\mu^2(K_t + 1)} \cdot \mathbb{E}\left[\left(\frac{1}{B_{(global)}} \sum_{i \in C_t} m_{p_i}\right)^{2} \left| F_{t-1,K_{t-1}} \right|\right]
\]
\[
\leq \frac{16\lambda^2 (\eta_t)_{(global)}^{2} \beta^2 D^2 m \|p\|^{2}}{\mu^2(K_t + 1)},
\]
where the last line is again by Lemma B.5. For Term IV, we invoke Young’s inequality for products to get
\[
E[\|F_{t-1,K_{t-1}}\|^{2}] \leq (\eta_t)_{(global)} \mu_{F} \|\mathbf{w}_{t}^{(global)} - \tilde{\mathbf{w}}^{(global)}\|^{2} + \frac{8\beta^2 D^2}{\mu^2(K_t + 1)} \cdot \mathbb{E}\left[\|F_{t-1,K_{t-1}}\|^{2}\right]
\]
\[
\leq (\eta_t)_{(global)} \mu_{F} \|\mathbf{w}_{t}^{(global)} - \tilde{\mathbf{w}}^{(global)}\|^{2} + (\eta_t)_{(global)} \mu_{F}^{-1} \cdot \frac{8\lambda^2 (\eta_t)_{(global)}^{2} \beta^2 D^2 m \|p\|^{2}}{\mu^2(K_t + 1)}.
\]
Summarizing the above bounds on the four terms, we arrive at
\[
E\left[\|\mathbf{w}_{t+1}^{(global)} - \tilde{\mathbf{w}}^{(global)}\|^{2} \left| F_{t-1,K_{t-1}} \right|\right]
\]
\[
\leq (1 - \eta_t)_{(global)} \mu_{F} \|\mathbf{w}_{t}^{(global)} - \tilde{\mathbf{w}}^{(global)}\|^{2} + 2(\eta_t)_{(global)} m \|p\|^{2} \left(\beta^2 D^2 \wedge 2\lambda \|\ell\|_{\infty} \wedge \lambda^2 D^2\right)
\]
\[
+ \frac{\lambda^2 (\eta_t)_{(global)}^{2} \beta^2 D^2 m \|p\|^{2}}{\mu^2(K_t + 1)} \cdot \left[\frac{8}{\delta_{(global)}^{(global)}} \mu_{F}\right].
\]
We claim that VI $\leq V$. Indeed, with our choice of $\eta_t^{(global)} = \frac{2}{\mu_{F}(t+1)}$, with some algebra, one recognizes that this claim is equivalent to
\[
\frac{20 + 4t}{\mu^2(K_t + 1)} \leq \left(\frac{1}{\lambda^2} \wedge \frac{2\|\ell\|_{\infty}}{\lambda \beta^2 D^2} \wedge \frac{1}{\beta^2}\right),
\]
which is exactly (B.4). Thus, we have
\[
E\left[\|\mathbf{w}_{t+1}^{(global)} - \tilde{\mathbf{w}}^{(global)}\|^{2} \left| F_{t-1,K_{t-1}} \right|\right]
\]
\[
\leq (1 - \eta_t)_{(global)} \mu_{F} \|\mathbf{w}_{t}^{(global)} - \tilde{\mathbf{w}}^{(global)}\|^{2} + 3 \cdot V
\]
\[
= \left(1 - \frac{2}{t + 1}\right) \|\mathbf{w}_{t}^{(global)} - \tilde{\mathbf{w}}^{(global)}\|^{2} + \frac{12m \|p\|^{2} \left(\beta^2 D^2 \wedge 2\lambda \|\ell\|_{\infty} \wedge \lambda^2 D^2\right)}{\mu_{F}^{2}(t + 1)^2}.
\]
We then proceed by induction. For the base case, we invoke the strong convexity of $\sum_i p_i F_i$ and Lemma B.4 to get
\[
\frac{\mu^2}{4} \| w_0^{(\text{global})} - \bar{w}^{(\text{global})} \|^2 \leq \left\| \sum_{i \in [m]} p_i \nabla F_i(w_0^{(\text{global})}, S_i) \right\|^2 \leq \beta^2 D^2 \wedge 2\lambda \| \ell \|_{\infty} \wedge \lambda^2 D^2.
\]
Along with the fact that $1 = (\sum_{i \in [n]} p_i)^2 \leq m\| p \|^2$, we conclude that (B.2) is true for $t = 0$. Now assume (B.5) hold for any $0 \leq t \leq \tau$. For $t = \tau + 1$, using (B.9) and the inductive hypothesis, we have
\[
\mathbb{E}_{A_{FP}} \| w^{(\text{global})}_{\tau+1} - \bar{w}^{(\text{global})} \|^2 \leq \left( 1 - \frac{2}{\tau + 1} \right) \frac{12m\| p \|^2 (\beta^2 D^2 \wedge 2\lambda \| \ell \|_{\infty} \wedge \lambda^2 D^2)}{(\tau + 1)\mu_F^2} + \frac{12m\| p \|^2 (\beta^2 D^2 \wedge 2\lambda \| \ell \|_{\infty} \wedge \lambda^2 D^2)}{\mu_F^2}
\[
= \left( 1 - \frac{1}{(\tau + 1)^2} \right) \frac{12m\| p \|^2 (\beta^2 D^2 \wedge 2\lambda \| \ell \|_{\infty} \wedge \lambda^2 D^2)}{(\tau + 2)\mu_F^2}
\]
which is the desired result.

### B.3 Auxiliary lemmas

**Lemma B.3** (Convexity and smoothness $F_i$). Under Assumption $A(b)$, each $F_i$ is $\lambda$-smooth and $\frac{\lambda}{\mu_F}$-strongly convex.

**Proof.** The smoothness is a standard fact about the Moreau envelope. The strongly convex constant of $F_i$ follows from Theorem 2.2 of LeMaréchal and Sagastizábal [1997]. \qed

**Lemma B.4** (A priori gradient norm bound). Under Assumption $A(a, b)$, for any $w \in \mathcal{W}$ and $i \in [m]$, we have
\[
\| \nabla F_i(w, S_i) \|^2 \leq \beta^2 D^2 \wedge 2\lambda \| \ell \|_{\infty} \wedge \lambda^2 D^2.
\]

**Proof.** Since $\nabla F_i(w, S_i) = \lambda(w - \text{Prox}_{L_i/\lambda}(w))$, its norm is trivially bounded by $\lambda D$. Now, since $\text{Prox}_{L_i/\lambda}(w)$ achieves a lower objective value than $w$ for the objective function $L_i(\cdot, S_i) + \frac{\lambda}{2}\| w - \cdot \|^2$, we have
\[
\frac{\lambda}{2}\| w - \text{Prox}_{L_i/\lambda}(w) \|^2 \leq L_i(w, S_i) - L_i(\text{Prox}_{L_i/\lambda}(w), S_i) \leq \| \ell \|_{\infty},
\]
and hence $\| \nabla F_i(w, S_i) \|^2 \leq 2\lambda \| \ell \|_{\infty}$. Finally, by the first-order condition, we have
\[
\nabla L_i(\text{Prox}_{L_i/\lambda}(w), S_i) + \lambda(\text{Prox}_{L_i/\lambda}(w) - w) = 0.
\]
Hence, we get $\| \nabla F_i(w, S_i) \| = \| \nabla L_i(\text{Prox}_{L_i/\lambda}(w), S_i) \| \leq \beta D$. \qed

**Lemma B.5** (Variance of minibatch sampling). Let $\mathcal{B} \subseteq [n]$ be a randomly sampled batch with batch size $B$ and let $\{ x_i \}_{i=1}^n \subseteq \mathbb{R}^d$ be an arbitrary set of vectors, then
\[
\mathbb{E}_\mathcal{B} \left\| \frac{1}{B} \sum_{i \in \mathcal{B}} x_i \right\|^2 = \frac{n/B - 1}{n(n-1)} \sum_{i \in [n]} \| x_i - \bar{x} \|^2 + \| \bar{x} \|^2 \leq \frac{1}{n} \sum_{i \in [n]} \| x_i - \bar{x} \|^2 + \| \bar{x} \|^2,
\]
where $\bar{x} := \sum_{i \in [n]} x_i / n$.

**Proof.** Since $\mathbb{E}_\mathcal{B} \sum_{i \in \mathcal{B}} x_i / B = \bar{x}$, we have
\[
\mathbb{E}_\mathcal{B} \left\| \frac{1}{B} \sum_{i \in \mathcal{B}} x_i \right\|^2 = \mathbb{E}_\mathcal{B} \left\| \frac{1}{B} \sum_{i \in \mathcal{B}} x_i - \bar{x} \right\|^2 + \| \bar{x} \|^2
\]

where the last line is by \( \mathbb{P}_B(i \in B) = B/n \) and \( \mathbb{P}_B(i, j \in B) = B(B-1)n^{-1}(n-1)^{-1} \) for any \( i \neq j \). Now, since \( \sum_{i \in [n]} \|x_i - \bar{x}\|^2 + 2 \sum_{i < j} \langle x_i - \bar{x}, x_j - \bar{x} \rangle = 0 \), we arrive at

\[
\mathbb{E}_B \left[ \frac{1}{B} \sum_{i \in B} x_i \right]^2 = \frac{1}{B^2} \left( \frac{B}{n} - \frac{B(B-1)}{n(n-1)} \right) \sum_i \|x_i - \bar{x}\|^2 + \|x\|^2
= \frac{n/B - 1}{n(n-1)} \sum_{i \in [n]} \|x_i - \bar{x}\|^2 + \|\bar{x}\|^2,
\]

which is the desired result. □

### Appendix C Proofs of Upper Bounds

#### C.1 Proof of Theorem 4.3

In this proof, we let \( \hat{w}^{(\text{global})} \) be the global minimizer of (4.5) and we write \( w^{(\text{global,p})} \equiv w^{(\text{global})} \) when there is no ambiguity.

**Proof of (4.6).** We have

\[
0 = -\sum_{i \in [m]} p_i L_i(\hat{w}^{(\text{global})}(\mathcal{S}), S_i) + \sum_{i \in [m]} p_i L_i(\hat{w}^{(\text{global})}(\mathcal{S}), S_i)
\leq -\sum_{i \in [m]} p_i L_i(\hat{w}^{(\text{global})}(\mathcal{S}), S_i) + \sum_{i \in [m]} p_i L_i(w^{(1)}_i, S_i)
= -\sum_{i \in [m]} \frac{p_i}{n_i} \sum_{j \in [n_i]} \left( \ell(\hat{w}^{(\text{global})}(\mathcal{S}^{(i,j)}), z_j^{(i)}) - \ell(w^{(1)}_i, z_j^{(i)}) \right)
+ \sum_{i \in [m]} \frac{p_i}{n_i} \sum_{j \in [n_i]} \left( \ell(\hat{w}^{(\text{global})}(\mathcal{S}^{(i,j)}), z_j^{(i)}) - \ell(\hat{w}^{(\text{global})}(\mathcal{S}), z_j^{(i)}) \right),
\]

where \( S^{(i,j)} \) stands for the dataset formed by replacing \( z_j^{(i)} \) by another \( z'_j \sim \mathcal{D}_i \), which is independent of everything else. Taking expectation in both sides, we get

\[
0 \leq -\sum_{i \in [m]} p_i \cdot \mathbb{E}_{S_i, Z_i \sim \mathcal{D}_i}[\ell(\hat{w}^{(\text{global})}(\mathcal{S}), Z_i) - \ell(w^{(1)}_i, Z_i)]
+ \sum_{i \in [m]} \frac{p_i}{n_i} \sum_{j \in [n_i]} \mathbb{E}_{S_i, Z_i \sim \mathcal{D}_i}[\ell(\hat{w}^{(\text{global})}(\mathcal{S}^{(i,j)}), z_j^{(i)}) - \ell(\hat{w}^{(\text{global})}(\mathcal{S}), z_j^{(i)})]
\]

\[
= -\sum_{i \in [m]} p_i \cdot \mathbb{E}_{S_i, Z_i \sim \mathcal{D}_i}[\ell(\hat{w}^{(\text{global})}(\mathcal{S}), Z_i) - \ell(w^{(1)}_i, Z_i)]
- \sum_{i \in [m]} p_i \cdot \mathbb{E}_{Z_i \sim \mathcal{D}_i}[\ell(w^{(1)}_i, Z_i) - \ell(w^{(1)}_i, Z_i)]
+ \sum_{i \in [m]} \frac{p_i}{n_i} \sum_{j \in [n_i]} \mathbb{E}_{S_i, Z_i \sim \mathcal{D}_i}[\ell(\hat{w}^{(\text{global})}(\mathcal{S}^{(i,j)}), z_j^{(i)}) - \ell(\hat{w}^{(\text{global})}(\mathcal{S}), z_j^{(i)})].
\]

Noting that \( w^{(1)}_i \) is the argmin of \( \mathbb{E}_{Z_i \sim \mathcal{D}_i}[\ell(\cdot, Z_i)] \) and invoking the \( \beta \)-smoothness assumption, we get

\[
\mathbb{E}_S[AER_p(\hat{w}^{(\text{global})})] \geq \ldots
\]

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Without loss of generality we consider the first client. By taking a weighted average, we arrive at

\[
\left(\sum_{i \in [m]} \frac{p_i}{n_i} \sum_{j \in [n_i]} \mathbb{E}_{S, z_{i,j}}[\ell(\tilde{\mathbf{w}}^{(global)}(S^{(i,j)}), z_{j,i}^{(i)}) - \ell(\hat{\mathbf{w}}^{(global)}(S), z_{j,i}^{(i)})]\right)
\]

leading to

\[
\text{taking a weighted average, we arrive at}
\]

\[
\mathbb{E}_{S}[\text{AER}_p(\mathbf{w}^{(global)})]
\]

\[
\leq 4\beta R^2 + \sum_{i \in [m]} \frac{p_i}{n_i} \sum_{j \in [n_i]} \mathbb{E}_{S, z'_{i,j}}[\ell(\tilde{\mathbf{w}}^{(global)}(S^{(i,j)}), z_{j,i}^{(i)}) - \ell(\hat{\mathbf{w}}^{(global)}(S), z_{j,i}^{(i)})]
\]

To bound the second term in the right-hand side above, we bound the federated stability of \(\hat{\mathbf{w}}^{(global)}\). Without loss of generality we consider the first client. By \(\mu\)-strongly convexity of \(L_1\), for any \(j_1 \in [n_1]\) we have

\[
\frac{\mu}{2} ||\mathbf{w}^{(global)}(S) - \hat{\mathbf{w}}(S^{(1,j_1)})||^2
\]

\[
\leq \sum_{i \in [m]} p_i \left(L_i(\tilde{\mathbf{w}}^{(global)}(S^{(1,j_1)}), S_i) - L_i(\hat{\mathbf{w}}^{(global)}(S), S_i)\right)
\]

\[
= \left(\sum_{i \neq 1} p_i L_i(\tilde{\mathbf{w}}^{(global)}(S^{(1,j_1)}), S_i) + p_1 L_i(\tilde{\mathbf{w}}^{(global)}(S^{(1,j_1)}), S_1^{(j_1)})\right)
\]

\[
- \left(\sum_{i \neq 1} p_i L_i(\hat{\mathbf{w}}^{(global)}(S), S_i) + p_1 L_i(\hat{\mathbf{w}}^{(global)}(S), S_1^{(j_1)})\right)
\]

\[
+ p_1 \left(L_i(\hat{\mathbf{w}}^{(global)}(S^{(1,j_1)}), S_1^{(j_1)}) - L_i(\hat{\mathbf{w}}^{(global)}(S^{(1,j_1)}), S_1^{(j_1)})\right)
\]

\[
\leq p_1 \left(L_i(\tilde{\mathbf{w}}^{(global)}(S^{(1,j_1)}), S_1^{(j_1)}) - L_i(\tilde{\mathbf{w}}^{(global)}(S^{(1,j_1)}), S_1^{(j_1)})\right)
\]

\[
+ p_1 \left(L_i(\hat{\mathbf{w}}^{(global)}(S), S_1^{(j_1)}) - L_i(\hat{\mathbf{w}}^{(global)}(S), S_1^{(j_1)})\right)
\]

\[
= \frac{p_1}{n_1} \left(\ell(\tilde{\mathbf{w}}^{(global)}(S^{(1,j_1)}), z_{j_1}^{(1)}) - \ell(\hat{\mathbf{w}}^{(global)}(S), z_{j_1}^{(1)})\right)
\]

\[
+ \frac{p_1}{n_1} \left(\ell(\tilde{\mathbf{w}}^{(global)}(S), z_{1,j_1}^{(1)}) - \ell(\hat{\mathbf{w}}^{(global)}(S^{(1,j_1)}), z_{1,j_1}^{(1)})\right)
\]

where the second inequality is because \(\tilde{\mathbf{w}}^{(global)}(S^{(1,j_1)})\) minimizes \(L_i(\cdot, S_1^{(j_1)}) + \sum_{i \neq 1} n_i L_i(\cdot, S_i)\). By an identical argument as in the proof of Lemma C.3, we have

\[
\ell(\tilde{\mathbf{w}}^{(global)}(S^{(1,j_1)}), z_{j_1}^{(1)}) - \ell(\hat{\mathbf{w}}^{(global)}(S), z_{j_1}^{(1)})
\]

\[
\leq \sqrt{2\beta ||\ell||_\infty \cdot ||\hat{\mathbf{w}}^{(global)}(S) - \tilde{\mathbf{w}}^{(global)}(S^{(1,j_1)})||^2 + \frac{\beta}{2} ||\hat{\mathbf{w}}^{(global)}(S) - \hat{\mathbf{w}}^{(global)}(S^{(1,j_1)})||^2}
\]

The same bound also holds for \(\ell(\tilde{\mathbf{w}}^{(global)}(S), z_{1,j_1}^{(1)}) - \ell(\hat{\mathbf{w}}^{(global)}(S^{(1,j_1)}), z_{1,j_1}^{(1)})\). Plugging these two bounds to (C.2) and rearranging terms, we get

\[
\left(\frac{\mu}{2} - \frac{\beta p_1}{n_1}\right) ||\hat{\mathbf{w}}^{(global)}(S) - \tilde{\mathbf{w}}^{(global)}(S^{(1,j_1)})|| \leq \frac{2\sqrt{2\beta ||\ell||_\infty} \cdot p_1}{n_1}.
\]
Thus, we get
\[
\frac{\mu}{4} \| \hat{w}^{(\text{global})}(S) - \hat{w}^{(\text{global})}(S^{(1,j)}) \| \leq \frac{2\sqrt{2\beta \| \ell \|_{\infty} \cdot p_1}}{n_1}.
\]
Plugging the above display back to (C.3), we arrive at
\[
\ell(\hat{w}^{(\text{global})}(S^{(1,j)}), z_j^{(1)}) - \ell(\hat{w}^{(\text{global})}(S), z_j^{(1)}) \leq \frac{16\beta \| \ell \|_{\infty} p_1}{\mu n_1} \left( 1 + \frac{4\beta p_1}{\mu n_1} \right) \leq \frac{32\beta \| \ell \|_{\infty} p_1}{\mu n_1},
\]
where the last inequality is again by \( n_1 \geq 4\beta p_1/\mu \). The desired result follows by plugging the above inequality back to (C.1) \( \square \)

Proof of (4.7). Without loss of generality we consider the first client. Since \( \hat{w}_*^{(1)} \) is the minimizer of \( \mathbb{E}_{Z_1 \sim D_1} \ell(\cdot, Z_1) \), by \( \beta \)-smoothness we have
\[
\mathbb{E}_{Z_1 \sim D_1} [\ell(\hat{w}^{(\text{global})}, Z_1) - \ell(\hat{w}_*^{(1)}, Z_1)] \leq \beta \cdot \mathbb{E}_{Z_1 \sim D_1} \| \hat{w}^{(\text{global})} - \hat{w}_*^{(1)} \|^2 \]
\[
\leq \beta \cdot \mathbb{E}_{Z_1 \sim D_1} \| \hat{w}^{(\text{global})} - w_\text{avg}^{(\text{global})} \|^2 + \beta R^2, \quad (C.4)
\]
where the last inequality is by Part (b) of Assumption C. By optimality of \( \hat{w}^{(\text{global})} \) and the strong convexity of \( L_i \)'s, we have
\[
\left( \sum_{i \in [m]} p_i \nabla L_i(\hat{w}_\text{avg}^{(\text{global})}, S_i), \hat{w}^{(\text{global})} - w_\text{avg}^{(\text{global})} \right) + \frac{\mu}{2} \| \hat{w}^{(\text{global})} - w_\text{avg}^{(\text{global})} \|^2 \leq 0.
\]
If \( \hat{w}^{(\text{global})} - w_\text{avg}^{(\text{global})} = 0 \) then we are done. Otherwise, the above display gives
\[
\| \hat{w}^{(\text{global})} - w_\text{avg}^{(\text{global})} \|^2 \leq \frac{2}{\mu} \left( \sum_{i \in [m]} p_i \| \nabla L_i(\hat{w}_\text{avg}^{(\text{global})}, S_i) \| \right) + \frac{2}{\mu} \left( \sum_{i \in [m]} p_i \| \nabla L_i(\hat{w}_*^{(i)}, S_i) \| + \beta \sum_{i \in [m]} p_i \| w_\text{avg}^{(\text{global})} - w_*^{(i)} \| \right) \]
\[
\leq \frac{2}{\mu} \left( \sum_{i \in [m]} p_i \| \nabla L_i(\hat{w}_*^{(i)}, S_i) \| + \beta \sum_{i \in [m]} p_i \| w_\text{avg}^{(\text{global})} - w_*^{(i)} \| \right) \]
\[
\leq \frac{2}{\mu} \left( \sum_{i \in [m]} p_i \| \nabla L_i(\hat{w}_*^{(i)}, S_i) \| + \beta R \right).
\]
Thus, we get
\[
\| \hat{w}^{(\text{global})} - w_\text{avg}^{(\text{global})} \|^2 \leq \frac{8}{\mu^2} \left( \sum_{i \in [m]} p_i \| \nabla L_i(\hat{w}_*^{(i)}, S_i) \|^2 + \beta^2 R^2 \right).
\]
Taking expectation with respect to the sample \( S \) at both sides, we have
\[
\mathbb{E}_S \| \hat{w}^{(\text{global})} - w_\text{avg}^{(\text{global})} \|^2 \leq \frac{1}{\mu^2} \mathbb{E}_S \left( \sum_{i \in [m]} p_i \| \nabla L_i(\hat{w}_*^{(i)}, S_i) - \mathbb{E}_S [\nabla L_i(\hat{w}_*^{(i)}, S_i)] \|^2 + \frac{\beta^2 R^2}{\mu^2} \right) \]
\[
\leq \frac{1}{\mu^2} \sum_{i \in [m]} \frac{p_i^2 \sigma^2}{n_i} + \frac{\beta^2 R^2}{\mu^2}.
\]
Plugging the above inequality to (C.4) gives the desired result. \( \square \)
C.2 Proof of Proposition 4.1

Proof of (4.9). By the definitions of the AER and $\mathcal{E}_{\text{OPT}}$, we have

$$
\text{AER}_p = \mathcal{E}_{\text{OPT}} + \frac{\lambda}{2} \sum_{i \in [m]} p_i \left( \| \hat{w}^{(\text{global})}(S) - \hat{w}^{(i)}(S) \|^2 - \| \hat{w}^{(\text{global})}(S) - \hat{w}^{(i)}(S) \|^2 \right)
$$

$$
+ \sum_{i \in [m]} p_i \left( \mathbb{E}_{Z_i \sim \mathcal{D}_i} [\ell(\hat{w}^{(i)}(S), Z_i)] - L_i(\hat{w}^{(i)}(S), S_i) \right)
$$

$$
+ \sum_{i \in [m]} p_i \left( L_i(\hat{w}^{(i)}(S), S_i) - \mathbb{E}_{Z_i \sim \mathcal{D}_i} [\ell(\hat{w}^{(i)}(S), Z_i)] \right).
$$

By the basic inequality (4.11), we can bound the AER by

$$
\text{AER}_p \leq \mathcal{E}_{\text{OPT}} + \frac{\lambda}{2} \sum_{i \in [m]} p_i \| w^{(\text{avg})}_{\text{avg}} - \hat{w}^{(i)} \|^2 + \sum_{i \in [m]} p_i \left( \mathbb{E}_{Z_i \sim \mathcal{D}_i} [\ell(\hat{w}^{(i)}(S), Z_i)] - L_i(\hat{w}^{(i)}(S), S_i) \right)
$$

$$
+ \sum_{i \in [m]} p_i \left( L_i(\hat{w}^{(i)}(S), S_i) - \mathbb{E}_{Z_i \sim \mathcal{D}_i} [\ell(\hat{w}^{(i)}(S), Z_i)] \right).
$$

Now, invoking federated stability, we can further bound the AER by

$$
\text{AER}_p \leq \mathcal{E}_{\text{OPT}} + \frac{\lambda}{2} \sum_{i \in [m]} p_i \| w^{(\text{avg})}_{\text{avg}} - \hat{w}^{(i)} \|^2 + 2 \sum_{i \in [m]} p_i \gamma_i
$$

$$
+ \sum_{i \in [m]} p_i \cdot \frac{1}{n_i} \sum_{j \in [n_i]} \mathbb{E}_{Z_{i,j} \sim \mathcal{D}_i} \left[ \mathbb{E}_{Z_i \sim \mathcal{D}_i} [\ell(\hat{w}^{(i)}(S^{(i,j)}), Z_i)] - \ell(\hat{w}^{(i)}(S^{(i,j)}), z_{i,j}) \right]
$$

$$
+ \sum_{i \in [m]} p_i \left( L_i(\hat{w}^{(i)}(S), S_i) - \mathbb{E}_{Z_i \sim \mathcal{D}_i} [\ell(\hat{w}^{(i)}(S), Z_i)] \right),
$$

where $S^{(i,j)}$ is the dataset formed by replacing $z_{i,j}$ with a new sample $z'_{i,j}$, and here we are choosing $z'_{i,j}$ to be an independent sample from $\mathcal{D}_i$. Note that the last two terms of the above display have mean zero under the randomness of the algorithm $\mathcal{A}$, the dataset $S$, and $\{ z_{i,j} : i \in [m], j \in [n_i] \}$. Thus, the desired result follows by taking expectation in both sides. \(\square\)

Proof of (4.10). Without loss of generality we consider the first client. By definitions of IER_1 and $\mathcal{E}_{\text{OPT}}$, we have

$$
p_1 \cdot \text{IER}_1 = \mathcal{E}_{\text{OPT}} + \sum_{i \in [m]} p_i \left( L_i(\hat{w}^{(i)}(S), S_i) + \frac{\lambda}{2} \| \hat{w}^{(\text{global})}(S) - \hat{w}^{(i)}(S) \|^2 \right)
$$

$$
- \sum_{i \in [m]} p_i \left( L_i(\hat{w}^{(i)}(S), S_i) + \frac{\lambda}{2} \| \hat{w}^{(\text{global})}(S) - \hat{w}^{(i)}(S) \|^2 \right)
$$

$$
+ p_1 \mathbb{E}_{Z_i \sim \mathcal{D}_i} [\ell(\hat{w}^{(1)}(S), Z_1)] - \ell(\hat{w}^{(1)}(S), Z_1)\).$$

Invoking the basic inequality (4.12), with some algebra, we arrive at

$$
p_1 \cdot \text{IER}_1 \leq \mathcal{E}_{\text{OPT}} + \frac{p_1 \lambda}{2} \| \hat{w}^{(\text{global})}(S) - \hat{w}^{(1)} \|^2 + p_1 \left( \mathbb{E}_{Z_i \sim \mathcal{D}_i} [\ell(\hat{w}^{(1)}(S), Z_1)] - L_i(\hat{w}^{(1)}(S), S_1) \right)
$$

$$
+ p_1 \left( L_i(\hat{w}^{(1)}(S), S_1) - \mathbb{E}_{Z_i \sim \mathcal{D}_i} [\ell(\hat{w}^{(1)}(S), Z_1)] \right).$$
Now, invoking federated stability for the first client, we can bound its IER by

\[ p_1 \cdot \text{IER}_1 \leq \mathcal{E}_{\text{OPT}} + \frac{p_1\lambda}{2} \| \hat{w}^{(\text{global})}(S) - \hat{w}^*_1 \|^2 + 2p_1\gamma_1 \]

\[ + \frac{p_1}{n_1} \sum_{j \in [n_1]} \mathbb{E}_{Z_1 \sim D_1} \left[ \mathbb{E}_{Z_1 \sim D_1} \left[ \ell(\hat{w}^{(1)}(S^{(1,j)}), Z_1) \right] \right] - \ell(\hat{w}^{(1)}(S^{(1,j)}), z_1^{(1)}) \]

\[ + p_1 \left( L_1(\hat{w}^{(1)}_1, S_1) - \mathbb{E}_{Z_1 \sim D_1} [\ell(\hat{w}^*_1, Z_1)] \right), \]

where we recall that \( S^{(1,j)} \) is the dataset formed by replacing \( z_j^{(1)} \) with a new sample \( z'_{1,j} \), and here we are choosing \( z'_{1,j} \) to be an independent sample from \( D_1 \). We finish the proof by taking the expectation with respect to \( A, S, \{z'_{1,j} : j \in [n_1] \} \) at both sides. \( \square \)

### C.3 Proof of Theorem 4.4

In this proof, we let \( A = (\hat{w}^{(\text{global})}, \{\hat{w}^{(i)}\}_{i=1}^m) \) be a generic algorithm that tries to minimize (4.8). For notational simplicity, we use \( a_n \leq \beta b_n \) (resp. \( a_n \geq \beta b_n \)) to denote that \( a_n \leq C_\beta b_n \) (resp. \( a_n \geq C_\beta b_n \)) for large \( n \), where \( C_\beta \) has explicit dependence on a parameter \( \beta \).

Recall that \( (\hat{w}^{(\text{global})}, \{\hat{w}^{(i)}\}_{i=1}^m) \) is the global minimizer of (4.8), and recall the notations in (B.1)–(B.3).

We start by bounding the federated stability of approximate minimizers of (4.8). We need the following definition.

**Definition C.1 (Approximate minimizers).** We say an algorithm \( A = (\hat{w}^{(\text{global})}, \{\hat{w}^{(i)}\}_{i=1}^m) \) produces an \( (\varepsilon^{(\text{global})}, \{\varepsilon^{(i)}\}_{i=1}^m) \)-minimizer of the objective function (4.8) on the dataset \( S \) if the following two conditions hold:

1. there exist a positive constant \( \varepsilon^{(\text{global})} \) such that \( \| \hat{w}^{(\text{global})} - \hat{w}^{(\text{global})} \| \leq \varepsilon^{(\text{global})} \);
2. for any \( i \in [m] \), there exist a positive constant \( \varepsilon^{(i)} \) such that \( \| \hat{w}^{(i)} - \text{Prox}_{L_i/\lambda}(\hat{w}^{(\text{global})}) \| \leq \varepsilon^{(i)} \).

The stability bound is as follows.

**Proposition C.1 (Federated stability of approximate minimizers).** Let Assumption A(b) holds, and consider an algorithm \( A = (\hat{w}^{(\text{global})}, \{\hat{w}^{(i)}\}_{i=1}^m) \) that produces an \( (\varepsilon^{(\text{global})}, \{\varepsilon^{(i)}\}_{i=1}^m) \)-minimizer of the objective function (4.8) on the dataset \( S \). Assume in addition that

\[ n_i \geq \frac{4\beta}{\mu}, \quad p_i \lambda \leq \frac{\mu}{16} \quad \forall i \in [m]. \]  

(C.5)

Then \( A \) has federated stability

\[ \gamma_i \leq \frac{160\beta \| \varepsilon \|}{n_i (\mu + \lambda)} + \text{Err}_i, \]

where

\[ \text{Err}_i := 2\sqrt{2\beta} \| \varepsilon \| \left[ 4\varepsilon^{(\text{global})} \left( \frac{\beta + \lambda}{\mu + \lambda} + \frac{3\lambda}{\mu} \right)^2 + \varepsilon^{(i)} \left( \frac{\beta + \lambda}{\mu + \lambda} + \frac{16p_i\lambda}{\mu} \right)^2 \right] \]

\[ + 16\varepsilon^{(\text{global})} \| \varepsilon^{(i)} \|^2 \left( \frac{\beta + \lambda}{\mu + \lambda} + \frac{3\lambda}{\mu} \right)^2 \]

\[ + \varepsilon^{(i)} \| \varepsilon^{(i)} \|^2 \left( \frac{\beta + \lambda}{\mu + \lambda} + \frac{16p_i\lambda}{\mu} \right)^2 \]

is the error term due to not exactly minimizing the soft weight sharing objective (3.7).

**Proof.** See Appendix C.3.1. \( \square \)

Taking the optimization error into account, we have the following result.
Proposition C.2 (Federated stability of $A_{FP}$). Let Assumption A(a, b) and (C.5) hold. Run $A_{FP}$ with hyperparameters chosen as in Lemma B.1 and B.2. Then, as long as

$$T \geq C_1 \cdot \lambda^2 (\lambda \vee 1)^2 m \|p\|^2 n_i^2, \quad K_T \geq C_2 \cdot \lambda^2 (\lambda \vee 1)^2 p_i^2 n_i^2 \quad \forall i \in [m],$$

the algorithm $A_{FP}$ have expected federated stability

$$E_{A_{FP}}[\eta_i] \leq C \cdot \frac{\beta \|\ell\|_\infty}{n_i (\mu + \lambda)};$$

where $C_1, C_2$ are two constants only depending on $(\mu, \beta, \|\ell\|_\infty, D)$, and $C$ is an absolute constant.

Proof. By Proposition C.1, it suffices to upper bound the error term $Err_i$ by a constant multiple of $\frac{\beta \|\ell\|_\infty}{n_i (\mu + \lambda)}$. Invoking Lemma B.2, we have

$$\left( \frac{E_{A_{FP}}[\varepsilon^{(\text{global})}]}{E_{A_{FP}}[\varepsilon] \Vdash 12 (\lambda + \mu)^2 m \|p\|^2 (\beta^2 D^2 \wedge 2 \lambda \|\ell\|_\infty \wedge \lambda^2 D^2)} \times \lambda^2 e^2 (T + 1) \right) \leq \sum_{i=1}^m \frac{\|p\|^2}{\lambda^2 (T + 1)} \leq \frac{m \|p\|^2}{T + 1},$$

This gives

$$E_{A_{FP}}[\varepsilon] \leq \frac{E_{A_{FP}}[\varepsilon^{(\text{global})}] + \lambda \varepsilon \varepsilon_{A_{FP}}[\varepsilon^{(i)}] + \lambda^2 E_{A_{FP}}[(\varepsilon^{(\text{global})})^2] + p_i^2 \lambda^2 E_{A_{FP}}[(\varepsilon^{(i)})^2]}{\lambda^2 \varepsilon \varepsilon + p_i^2 \lambda^2} \leq \frac{8 \beta^2 D^2}{\mu^2 (K_T + 1)} \leq \frac{1}{K_T + 1}.$$
for any $i \in [m]$, the algorithm $A_{FP}$ satisfies

$$E_{A_{FP}, S}[AER_p(A_{FP})] \leq C \cdot \frac{\beta \|\ell\|_\infty}{\mu + \lambda} \sum_{i \in [m]} \frac{p_i}{n_i} + \frac{\lambda}{2} \sum_{i \in [m]} p_i \|w_\text{avg}^{(\text{global})} - w_i^{(i)}\|^2,$$

where $C_1, C_2$ are two constants only depending on $(\mu, \beta, \|\ell\|_\infty, D)$, and $C$ is an absolute constant.

**Proof.** In view of Propositions 4.1 and C.2, it suffices to set $T, K_T$ such that (1) (C.6) is satisfied; and (2) $E_{A_{FP}}[\mathcal{E}_{\text{OPT}}]$ is upper bounded by a constant multiple of $\frac{\beta \|\ell\|_\infty}{\mu + \lambda} \sum_{i \in [m]} \frac{p_i}{n_i}$. To achieve the second case, note that by Proposition B.1, the optimization error is bounded by

$$E_{A_{FP}}[\mathcal{E}_{\text{OPT}}] \leq \frac{\lambda \vee 1}{K_T + 1} \frac{\lambda m \|p\|^2}{T + 1}. \quad (C.9)$$

Thus, it suffices to require $T \geq \frac{\lambda(\lambda \vee 1)m\|p\|^2}{\sum_{i \in [m]} p_i/n_i}$ and $K_T \geq \frac{\lambda \vee 1}{\sum_{i \in [m]} p_i/n_i}$. This requirement, combined with (C.6), is exactly (C.8). \qed

With the above proposition at hand, we are ready to give our proof of Theorem 4.4.

**Proof of Theorem 4.4.** We first define the following three events:

$$A := \left\{ R \geq \sqrt{\sum_{i \in [m]} \frac{p_i}{n_i}} \right\}, \quad B := \left\{ \frac{\sum_{i \in [m]} p_i^2 / n_i}{\sqrt{\sum_{i \in [m]} p_i / n_i}} \leq R \leq \sqrt{\sum_{i \in [m]} \frac{p_i}{n_i}} \right\}, \quad C := \left\{ R \leq \frac{\sum_{i \in [m]} p_i^2 / n_i}{\sqrt{\sum_{i \in [m]} p_i / n_i}} \right\}.$$

We then choose $\lambda$ to be

$$\lambda = \frac{\mu}{16R^2} \sum_{i \in [m]} \frac{p_i}{n_i} \cdot 1_A + \frac{\mu}{16C_p R^2} \sqrt{\sum_{i \in [m]} \frac{p_i}{n_i}} \cdot 1_B + \frac{\mu}{16C_p \sum_{i \in [m]} p_i / n_i} \sum_{i \in [m]} p_i / n_i \cdot 1_C.$$

We now consider the three events separately.

1. If $A$ holds, then $p_i \lambda = \frac{p_i \mu}{16n_i R^2} \sum_{i \in [m]} \frac{p_i}{n_i} \leq \frac{p_i \mu}{16} \leq \frac{\mu}{16}$. Thus we can invoke Proposition C.3 to get

$$E_{A_{FP}, S}[AER_p] \leq \left( \frac{C \beta \|\ell\|_\infty}{\mu} + \frac{\mu}{32} \right) \sum_{i \in [m]} \frac{p_i}{n_i} \leq \text{right-hand side of (4.16)}.$$

2. If $B$ holds, then $p_i \lambda = \frac{p_i \mu}{16n_i R \sqrt{\sum_{i \in [m]} \frac{p_i}{n_i}}} \leq \frac{p_i \mu}{16C_p \sum_{i \in [m]} p_i / n_i} \sum_{i \in [m]} \frac{p_i}{n_i} \leq \frac{\mu}{16}$, where the last inequality is by the definition of $C_p$. Hence, by Proposition C.3, we have

$$E_{A_{FP}, S}[AER_p] \leq \left( \frac{16CC_p \beta \|\ell\|_\infty}{\mu} + \frac{\mu}{32C_p} \right) R \sqrt{\sum_{i \in [m]} \frac{p_i}{n_i}} \leq \text{right-hand side of (4.16)}.$$

3. If $C$ holds, then $p_i \lambda = \frac{p_i \mu}{16n_i \sum_{i \in [m]} p_i / n_i} \sum_{i \in [m]} p_i / n_i \leq \frac{\mu}{16}$, and thus Proposition C.3 gives

$$E_{A_{FP}, S}[AER_p] \leq \left( \frac{16CC_p \beta \|\ell\|_\infty}{\mu} + \frac{\mu}{32C_p} \right) \sum_{i \in [m]} \frac{p_i^2}{n_i} \leq \text{right-hand side of (4.16)}.$$

The desired result follows by combining the above three cases together. \qed
C.3.1 Proof of Proposition C.1: Stability of Approximate Minimizers

We first present two lemmas, from which Proposition C.1 will follow.

**Lemma C.1** (Federated stability of approximate minimizers, Part I). Let Assumption $A(b)$ holds, and consider an algorithm $\mathcal{A} = (\hat{\mathbf{w}}^{(\text{global})}, \{\hat{\mathbf{w}}^{(i)}\}_{i=1}^m)$ that satisfies the following conditions:

1. there exist positive constants $\delta^{(\text{global})}, \zeta^{(\text{global})}$ such that
   \[
   \sum_{i \in [m]} p_i F_i(\hat{\mathbf{w}}^{(\text{global})}, S_i) \leq \delta^{(\text{global})} + \sum_{i \in [m]} p_i F_i(\hat{\mathbf{w}}^{(\text{global})}, S_i),
   \]
   \[
   \| \sum_{i \in [m]} p_i \nabla F_i(\hat{\mathbf{w}}^{(\text{global})}, S_i) \| \leq \zeta^{(\text{global})},
   \]

2. for any $i \in [m]$, there exist positive constants $\{\delta^{(i)}, \zeta^{(i)}, \varepsilon^{(i)}\}_{i=1}^m$ such that
   \[
   L_i(\hat{\mathbf{w}}^{(i)}, S_i) + \frac{\lambda}{2} \| \hat{\mathbf{w}}^{(\text{global})} - \hat{\mathbf{w}}^{(i)} \|^2 \leq \delta^{(i)} + F_i(\hat{\mathbf{w}}^{(\text{global})}, S_i),
   \]
   \[
   \| \nabla L_i(\hat{\mathbf{w}}^{(i)}, S_i) + \lambda (\hat{\mathbf{w}}^{(i)} - \hat{\mathbf{w}}^{(\text{global})}) \| \leq \zeta^{(i)},
   \]
   \[
   \| \hat{\mathbf{w}}^{(i)} - \text{Prox}_{L_i/\lambda}(\hat{\mathbf{w}}^{(\text{global})}) \| \leq \varepsilon^{(i)}.
   \]

Assume in addition that (C.5) holds. Then $\mathcal{A}$ has federated stability

\[
\gamma_i \leq \frac{1600\beta \| \ell \|_\infty}{n_i (\mu + \lambda)} + \sqrt{2\beta \| \ell \|_\infty} \cdot \mathcal{E}_{\lambda,i} + \beta \mathcal{E}_{\lambda,i}^2,
\]

where

\[
\mathcal{E}_{\lambda,i} := \frac{8\varepsilon^{(i)}(\mu + \lambda)}{\mu + \lambda} + \sqrt{8\delta^{(i)}(\mu + \lambda)} + 8\mu^{-1} \left( 2\zeta^{(\text{global})} + 4p_i \lambda \varepsilon^{(i)} + \sqrt{2\mu \lambda \delta^{(\text{global})}} \right)
\]

is the error term due to not exactly minimizing (4.8).

**Lemma C.2** (Federated stability of approximate minimizers, Part II). Let Assumption $A(b)$ holds and consider an algorithm $\mathcal{A} = (\hat{\mathbf{w}}^{(\text{global})}, \{\hat{\mathbf{w}}^{(i)}\}_{i=1}^m)$ that produces an $(\varepsilon^{(\text{global})}, \{\varepsilon^{(i)}\}_{i=1}^m)$-minimizer in the sense of Definition C.1. Then $\mathcal{A}$ also satisfies Equations (C.10)–(C.14) with

\[
\delta^{(\text{global})} = \frac{\lambda}{2} \varepsilon^{(\text{global})}, \quad \zeta^{(\text{global})} = \lambda \varepsilon^{(\text{global})}, \quad \delta^{(i)} = \frac{\beta + \lambda}{2} \varepsilon^{(i)}, \quad \zeta^{(i)} = (\beta + \lambda) \varepsilon^{(i)}.
\]

**Proof.** These correspondences are consequences of $\lambda$-smoothness of $F_i$ and $(\beta + \lambda)$-smoothness of $L_i(\cdot, S_i) + \frac{\lambda}{2} \| \hat{\mathbf{w}}^{(\text{global})} - \cdot \|^2$. We omit the details.

With the above two lemmas at hand, the proof of Proposition C.1 is purely computational:

**Proof of Proposition C.1 given Lemma C.1 and C.2.** Invoking C.2, the error term $\mathcal{E}_{\lambda,i}$ defined in Equation C.16 can be bounded above by

\[
\mathcal{E}_{\lambda,i} \leq \frac{(\beta + \lambda)}{\mu + \lambda} \varepsilon^{(\text{global})} + \sqrt{\frac{4(\beta + \lambda)}{\mu + \lambda}} \cdot \varepsilon^{(i)} + \frac{8}{\mu} \left( 2\lambda \varepsilon^{(\text{global})} + 4p_i \lambda \varepsilon^{(i)} + \sqrt{\frac{\mu \lambda^2}{\mu + \lambda}} \right).
\]

With the above two lemmas at hand, the proof of Proposition C.1 is purely computational:

\[
\mathcal{E}_{\lambda,i} \leq \frac{(\beta + \lambda)}{\mu + \lambda} \varepsilon^{(\text{global})} + \sqrt{\frac{4(\beta + \lambda)}{\mu + \lambda}} \cdot \varepsilon^{(i)} + \frac{8}{\mu} \left( 2\lambda \varepsilon^{(\text{global})} + 4p_i \lambda \varepsilon^{(i)} + \sqrt{\frac{\mu \lambda^2}{\mu + \lambda}} \right).
\]

\[
\leq 8\varepsilon^{(\text{global})} \left( \frac{\beta + \lambda}{\mu + \lambda} + \frac{3\lambda}{\mu} \right) + 2\varepsilon^{(i)} \left( \sqrt{\frac{\beta + \lambda}{\mu + \lambda}} + \frac{16p_i \lambda}{\mu} \right).
\]
This gives
\[ \mathcal{E}_{S, i}^2 \leq 128(\varepsilon^{\text{global}})^2 \left( \frac{\beta + \lambda}{\mu + \lambda} + \frac{3\lambda}{\mu} \right)^2 + 8(\varepsilon^{(i)})^2 \left( \sqrt{\frac{\beta + \lambda}{\mu + \lambda} + \frac{16p_i\lambda}{\mu}} \right)^2. \]

Plugging the above two displays to (C.15) gives the desired result. \hfill \Box

We now present our proof of Lemma C.1. We start by stating and proving several useful lemmas.

**Lemma C.3** (From loss stability to parameter stability). Let Assumption A(b) holds. Then the algorithm \( \mathcal{A} = (\hat{w}^{\text{global}}(\cdot), \hat{w}^{(i)}) \) has federated stability
\[ \gamma_i \leq \sqrt{2\beta\|\ell\|_{\infty} \cdot \|\hat{w}^{(i)}(S) - \hat{w}^{(i)}(S^{(i,j_i)})\|} + \frac{\beta}{2} \|\hat{w}^{(i)}(S) - \hat{w}^{(i)}(S^{(i,j_i)})\|^2. \]

**Proof.** This lemma has implicitly appeared in the proofs of many stability-based generalization bounds (see, e.g., Section 13.3.2 of Shalev-Shwartz and Ben-David [2014]), and we provide a proof for completeness. By \( \beta \)-smoothness, for an arbitrary \( z \in Z \) we have
\[
\ell(\hat{w}^{(i)}(S), z) - \ell(\hat{w}^{(i)}(S^{(i,j_i)}), z) \\
\leq \langle \nabla \ell(\hat{w}^{(i)}(S^{(i,j_i)})), z \rangle - \langle \nabla \ell(\hat{w}^{(i)}(S)), z \rangle \\
\leq \|\nabla \ell(\hat{w}^{(i)}(S^{(i,j_i)}), z)\| \cdot \|\hat{w}^{(i)}(S) - \hat{w}^{(i)}(S^{(i,j_i)})\| + \frac{\beta}{2} \|\hat{w}^{(i)}(S) - \hat{w}^{(i)}(S^{(i,j_i)})\|^2 \\
\leq \sqrt{2\beta \left( \ell(\hat{w}^{(i)}(S^{(i,j_i)}), z) - \min_{w^{(i)} \in W} \ell(w^{(i)}, z) \right) \cdot \|\hat{w}^{(i)}(S) - \hat{w}^{(i)}(S^{(i,j_i)})\|} \\
+ \frac{\beta}{2} \|\hat{w}^{(i)}(S) - \hat{w}^{(i)}(S^{(i,j_i)})\|^2 \\
\leq \sqrt{2\beta\|\ell\|_{\infty} \cdot \|\hat{w}^{(i)}(S) - \hat{w}^{(i)}(S^{(i,j_i)})\|} + \frac{\beta}{2} \|\hat{w}^{(i)}(S) - \hat{w}^{(i)}(S^{(i,j_i)})\|^2, 
\]
where the last inequality follows from boundedness of \( \ell \). By a nearly identical argument, the above upper bound also holds for \( -\ell(\hat{w}^{(i)}(S), z) + \ell(\hat{w}^{(i)}(S^{(i,j_i)}), z) \), and the desired result follows. \hfill \Box

**Lemma C.4** (Local stability implies global stability). Assume Assumption A(b) holds and consider an algorithm \( \mathcal{A} = (\hat{w}^{\text{global}}(\cdot), \hat{w}^{(i)}(\cdot)) \) that satisfies Equations (C.10), (C.11) and (C.14). Then for any \( i \in [m], j_i \in [n_i] \), we have
\[
\|\hat{w}^{\text{global}}(S^{(i,j_i)}) - \hat{w}^{\text{global}}(S)\| \\
\leq \frac{\lambda + \mu}{\lambda \mu} \left( 2\varepsilon^{\text{global}} + \frac{2\lambda \mu \delta^{\text{global}}}{\lambda + \mu} + 4p_i \lambda \varepsilon^{(i)} + 2p_i \mu \|\hat{w}^{(i)}(S^{(i,j_i)}) - \hat{w}^{(i)}(S)\| \right). \tag{C.17}
\]

**Proof.** Without loss of generality we consider the first client. Let \( \mu_F \) be the strongly convex constant of \( \sum_i p_i F_i \), which, by Lemma B.3, is equal to \( \sum_i p_i \frac{\mu \lambda}{\mu + \lambda} = \lambda \mu / (\lambda + \mu) \). Now, by strong convexity, we have
\[
\frac{\mu_F}{2} \|\hat{w}^{\text{global}}(S) - \hat{w}^{\text{global}}(S^{(1,j_1)})\|^2 \\
\leq \sum_{i \in [m]} p_i \left( F_i(\hat{w}^{\text{global}}(S^{(1,j_1)}), S_i) - F_i(\hat{w}^{\text{global}}(S), S_i) \right) \\
+ \left( \sum_{i \in [m]} p_i \nabla F_i(\hat{w}^{\text{global}}(S), S_i), \hat{w}^{\text{global}}(S^{(1,j_1)} - \hat{w}^{\text{global}}(S)) \right) \\
\leq \sum_{i \in [m]} p_i \left( F_i(\hat{w}^{\text{global}}(S^{(1,j_1)}), S_i) - F_i(\hat{w}^{\text{global}}(S), S_i) \right)
\]

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\[ \begin{align*}
&+ \zeta^{(\text{global})} \| \hat{\omega}^{(\text{global})}(S^{\setminus(1,j_1)}) - \hat{\omega}^{(\text{global})}(S) \| \\
&= \left( p_1 F_1(\hat{\omega}^{(\text{global})}(S^{\setminus(1,j_1)}), S_{j_1}^{\setminus 1}) + \sum_{i \neq 1} p_i F_i(\hat{\omega}^{(\text{global})}(S^{\setminus(1,j_i)}), S_i) \right) \\
&\quad - \left( p_1 F_1(\hat{\omega}^{(\text{global})}(S, S_{j_1}^{\setminus 1}) + \sum_{i \neq 1} p_i F_i(\hat{\omega}^{(\text{global})}(S, S_i) \right) \\
&\quad + p_1 \left( F_1(\hat{\omega}^{(\text{global})}(S^{\setminus(1,j_1)}), S_1) - F_1(\hat{\omega}^{(\text{global})}(S^{\setminus(1,j_1)}), S_1^{\setminus 1}) \right) \\
&\quad + F_1(\hat{\omega}^{(\text{global})}(S), S_{j_1}^{\setminus 1}) - F_1(\hat{\omega}^{(\text{global})}(S), S_1) \right) \\
&+ \zeta^{(\text{global})} \| \hat{\omega}^{(\text{global})}(S^{\setminus(1,j_1)}) - \hat{\omega}^{(\text{global})}(S) \| \\
&\leq \delta^{(\text{global})} + \zeta^{(\text{global})} \| \hat{\omega}^{(\text{global})}(S^{\setminus(1,j_1)}) - \hat{\omega}^{(\text{global})}(S) \| + p_1 \lambda \| \hat{\omega}^{(\text{global})}(S^{\setminus(1,j_1)}) - \hat{\omega}^{(\text{global})}(S) \| \\
&\quad + p_1 \left( \nabla F_1(\hat{\omega}^{(\text{global})}(S), S_1) - \nabla F_1(\hat{\omega}^{(\text{global})}(S^{\setminus(1,j_1)}), S_{j_1}^{\setminus 1}) \hat{\omega}^{(\text{global})}(S^{\setminus(1,j_1)}) - \hat{\omega}^{(\text{global})}(S) \right). \\
\end{align*} \]

Since \( F_1 \) is \( \lambda \)-smooth by Lemma B.3, we can proceed by

\[ \frac{\mu F}{2} \| \hat{\omega}^{(\text{global})}(S) - \hat{\omega}^{(\text{global})}(S^{\setminus(1,j_1)}) \|^2 \leq \delta^{(\text{global})} + \zeta^{(\text{global})} \| \hat{\omega}^{(\text{global})}(S^{\setminus(1,j_1)}) - \hat{\omega}^{(\text{global})}(S) \| + p_1 \lambda \| \hat{\omega}^{(\text{global})}(S^{\setminus(1,j_1)}) - \hat{\omega}^{(\text{global})}(S) \| \\
+ p_1 \left( \nabla F_1(\hat{\omega}^{(\text{global})}(S), S_1) - \nabla F_1(\hat{\omega}^{(\text{global})}(S^{\setminus(1,j_1)}), S_{j_1}^{\setminus 1}) \hat{\omega}^{(\text{global})}(S^{\setminus(1,j_1)}) - \hat{\omega}^{(\text{global})}(S) \right). \]

Since \( \nabla F_1(\hat{\omega}^{(\text{global})}, S_1) = \lambda \left( \hat{\omega}^{(\text{global})} - \text{Prox}_{L_1/\lambda}(\hat{\omega}^{(\text{global})}, S_1) \right) \), with some algebra, the right-hand side above is in fact equal to

\[ \begin{align*}
&\delta^{(\text{global})} + \zeta^{(\text{global})} \| \hat{\omega}^{(\text{global})}(S^{\setminus(1,j_1)}) - \hat{\omega}^{(\text{global})}(S) \| \\
&\quad + p_1 \lambda \left( \hat{\omega}^{(1)}(S) - \text{Prox}_{L_1/\lambda}(\hat{\omega}^{(\text{global})}(S^{\setminus 1}), S_{j_1}^{\setminus 1}), \hat{\omega}^{(\text{global})}(S^{\setminus(1,j_1)}) - \hat{\omega}^{(\text{global})}(S) \right) \\
&\quad + p_1 \lambda \left( \text{Prox}_{L_1/\lambda}(\hat{\omega}^{(\text{global})}(S^{\setminus(1,j_1)}), S_{j_1}^{\setminus 1}) - \hat{\omega}^{(1)}(S^{\setminus(1,j_1)}), \hat{\omega}^{(\text{global})}(S^{\setminus(1,j_1)}) - \hat{\omega}^{(\text{global})}(S) \right) \\
&\quad + p_1 \lambda \left( \hat{\omega}^{(1)}(S^{\setminus(1,j_1)}), S_{j_1}^{\setminus 1} \hat{\omega}^{(\text{global})}(S^{\setminus(1,j_1)}) - \hat{\omega}^{(\text{global})}(S) \right) \\
&\leq \delta^{(\text{global})} + \zeta^{(\text{global})} + 2p_1 \lambda \| \hat{\omega}^{(\text{global})}(S^{\setminus(1,j_1)}) - \hat{\omega}^{(\text{global})}(S) \| \\
&\quad + p_1 \| \hat{\omega}^{(1)}(S^{\setminus(1,j_1)}) - \hat{\omega}^{(1)}(S) \| \| \hat{\omega}^{(\text{global})}(S^{\setminus(1,j_1)}) - \hat{\omega}^{(\text{global})}(S) \|. \\
\end{align*} \]

The above bound gives a quadratic inequality: if we let \( s_G := \| \hat{\omega}^{(\text{global})}(S^{\setminus(1,j_1)}) - \hat{\omega}^{(\text{global})}(S) \| \) and \( s_1 := \| \hat{\omega}^{(1)}(S^{\setminus(1,j_1)}) - \hat{\omega}^{(1)}(S) \| \), then the above bound can be written as

\[ \frac{\mu F}{2} \cdot s_G^2 = \zeta^{(\text{global})} + 2p_1 \lambda \| \hat{\omega}^{(1)} + p_1 \lambda s_1 \| - \delta^{(\text{global})} \leq 0. \]

Solving this inequality gives

\[ \begin{align*}
s_G \leq \frac{1}{\mu F} \left[ \zeta^{(\text{global})} + 2p_1 \lambda \| \hat{\omega}^{(1)} + p_1 \lambda s_1 \| + \sqrt{\zeta^{(\text{global})} + 2p_1 \lambda \| \hat{\omega}^{(1)} + p_1 \lambda s_1 \|^2 + 2\mu F \delta^{(\text{global})}} \right] \\
\leq \frac{1}{\mu F} \left( \zeta^{(\text{global})} + 4p_1 \lambda \| \hat{\omega}^{(1)} + 2p_1 \lambda s_1 + \sqrt{2\mu F \delta^{(\text{global})}} \right). \]

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which is exactly (C.17).

Lemma C.5 (Parameter stability). Under the same assumptions as Proposition C.1, for any $i \in [m], j_i \in [n]$, we have

\[
\|\tilde{w}^{(i)}(S^{(i,j_i)}) - \tilde{w}^{(i)}(S)\| \leq \frac{16\sqrt{2}\beta\|\ell\|_{\infty}}{n_1(\mu + \lambda)} + \mathcal{E}_{\lambda,i}.
\]

Proof. Without loss of generality we consider the first client. Since $L_1(\cdot, S_1) + \frac{\lambda}{2}\|\tilde{w}^{(\text{global})}(S) - \cdot\|^2$ is $(\mu + \lambda)$-strongly convex, we have

\[
\begin{align*}
\frac{1}{2}(\mu + \lambda)\|\tilde{w}^{(1)}(S) - \tilde{w}^{(1)}(S^{(1,j_1)})\|^2 & \leq \left( L_1(\tilde{w}^{(1)}(S^{(1,j_1)}), S_1) + \frac{\lambda}{2}\|\tilde{w}^{(\text{global})}(S) - \tilde{w}^{(1)}(S^{(1,j_1)})\|^2 \right) \\
& \quad - \left( L_1(\tilde{w}^{(1)}(S), S_1) + \frac{\lambda}{2}\|\tilde{w}^{(\text{global})}(S) - \tilde{w}^{(1)}(S)\|^2 \right) \\
& \quad + \left\langle \nabla L_1(\tilde{w}^{(1)}(S), S_1) + \lambda(\tilde{w}^{(1)}(S) - \tilde{w}^{(\text{global})}(S)), \tilde{w}^{(1)}(S^{(1,j_1)}) - \tilde{w}^{(1)}(S) \right\rangle \\
& \overset{(C.13)}{\leq} \left( L_1(\tilde{w}^{(1)}(S^{(1,j_1)}), S^{(j_1)}) + \frac{\lambda}{2}\|\tilde{w}^{(\text{global})}(S^{(1,j_1)}) - \tilde{w}^{(1)}(S^{(1,j_1)})\|^2 \right) \\
& \quad - \left( L_1(\tilde{w}^{(1)}(S), S^{(j_1)}) + \frac{\lambda}{2}\|\tilde{w}^{(\text{global})}(S^{(1,j_1)}) - \tilde{w}^{(1)}(S)\|^2 \right) \\
& \quad - \frac{1}{n_1}\ell(\tilde{w}^{(1)}(S^{(1,j_1)}), z_{1,j_1}^{(1)}) + \frac{1}{n_1}\ell(\tilde{w}^{(1)}(S^{(1,j_1)}), \tilde{z}_{1,j_1}^{(1)}) + \frac{1}{n_1}\ell(\tilde{w}^{(1)}(S), \tilde{z}_{1,j_1}^{(1)}) - \frac{1}{n_1}\ell(\tilde{w}(S), \tilde{z}_{1,j_1}^{(1)}) \\
& \quad - \frac{\lambda}{2}\|\tilde{w}^{(\text{global})}(S^{(1,j_1)}) - \tilde{w}^{(1)}(S^{(1,j_1)})\|^2 + \frac{\lambda}{2}\|\tilde{w}^{(\text{global})}(S) - \tilde{w}^{(1)}(S^{(1,j_1)})\|^2 \\
& \quad + \frac{\lambda}{2}\|\tilde{w}^{(\text{global})}(S^{(1,j_1)}) - \tilde{w}^{(1)}(S)\|^2 - \lambda\|\|\tilde{w}^{(\text{global})}(S) - \tilde{w}^{(1)}(S)\|^2 \\
& \overset{(C.12)}{\leq} \delta^{(1)} + \zeta^{(1)}\|\tilde{w}^{(1)}(S^{(1,j_1)}) - \tilde{w}^{(1)}(S)\| \\
& \quad + \lambda\left( \|\tilde{w}^{(\text{global})}(S) - \tilde{w}^{(\text{global})(S^{(1,j_1)})}, \tilde{w}^{(S)} - \tilde{w}^{(1)}(S^{(1,j_1)}) \right) \\
& \quad + \frac{1}{n_1}\left( \ell(\tilde{w}^{(1)}(S), z_{1,j_1}^{(1)}) - \ell(\tilde{w}^{(1)}(S^{(1,j_1)}), \tilde{z}_{1,j_1}^{(1)}) + \ell(\tilde{w}^{(1)}(S^{(1,j_1)}), \tilde{z}_{1,j_1}^{(1)}) - \ell(\tilde{w}^{(1)}(S), \tilde{z}_{1,j_1}^{(1)}) \right) \\
& \leq \delta^{(1)} + \zeta^{(1)}\|\tilde{w}^{(1)}(S^{(1,j_1)}) - \tilde{w}^{(1)}(S)\| \\
& \quad + \frac{2}{n_1}\left( \sqrt{2\beta\|\ell\|_{\infty}}\|\tilde{w}^{(1)}(S) - \tilde{w}^{(1)}(S^{(1,j_1)})\| + \frac{\beta}{2}\|\tilde{w}^{(1)}(S) - \tilde{w}^{(1)}(S^{(1,j_1)})\|^2 \right) \\
& \quad + \frac{\lambda + \mu}{\mu}\left( 2\zeta^{(\text{global})} + \sqrt{\frac{2\lambda\mu\delta^{(\text{global})}}{\lambda + \mu}} + 4p_1\lambda\zeta^{(1)} + 2p_1\lambda\|\tilde{w}^{(i)}(S^{(i,j_1)}) - \tilde{w}^{(i)}(S)\| \right) \\
& \quad \times \|\tilde{w}^{(1)}(S) - \tilde{w}^{(1)}(S^{(1,j_1)})\|,
\end{align*}
\]

where the last inequality is by Lemma C.3 and Lemma C.4. Denoting $s_1 := \|\tilde{w}^{(1)}(S) - \tilde{w}^{(1)}(S^{(1,j_1)})\|$, the above inequality can be written as

\[
C_{\lambda,1}s_1^2 - \left[ \frac{2\sqrt{2}\beta\|\ell\|_{\infty}}{n_1} + \zeta^{(1)} + \frac{\lambda + \mu}{\mu}\left( 2\zeta^{(\text{global})} + 4p_1\lambda\zeta^{(1)} + \sqrt{\frac{2\lambda\mu\delta^{(\text{global})}}{\lambda + \mu}} \right) \right] s_1 - \delta^{(1)} \leq 0,
\]
where

\[
C_{\lambda,1} := \frac{1}{2}(\mu + \lambda) - \frac{\beta}{n_1} - \frac{2p_1\lambda(\lambda + \mu)}{\mu}.
\]
By (C.5), we have
\[ C_{\lambda,1} \geq \frac{\mu + \lambda}{2} - \frac{\mu}{4} - \frac{2\mu_1\lambda + \mu_1}{\mu} = \frac{\lambda + \mu}{4} - \frac{2\mu_1\lambda + \mu_1}{\mu} = \frac{\lambda + \mu}{4} - \frac{8\mu_1\lambda}{\mu} \geq \frac{\lambda + \mu}{8}. \]
In particular, \( C_{\lambda,1} > 0 \), and thus we can solve the quadratic inequality (C.18) (similar to the proof of Lemma C.4) to get
\[ s_1 \leq \frac{2\sqrt{2\|\ell\|_{\infty}}}{C_{\lambda,1} n_1} + \frac{\zeta^{(1)}}{C_{\lambda,1}} + \frac{\delta^{(1)}}{C_{\lambda,1}}. \]
Plugging in \( C_{\lambda,1} \geq (\lambda + \mu)/8 \) to the above inequality gives the desired result. \( \square \)

We are finally ready to present a proof of Lemma C.1:

**Proof of Lemma C.1.** Invoking Lemma C.3, we have
\[ \gamma_i \leq \sqrt{2\|\ell\|_{\infty}} \cdot \left( \frac{16\sqrt{2\|\ell\|_{\infty}}}{n_1(\mu + \lambda)} + \mathcal{E}_{\lambda,i} \right) + \frac{\beta}{2} \left( \frac{16\sqrt{2\|\ell\|_{\infty}}}{n_1(\mu + \lambda)} + \mathcal{E}_{\lambda,i} \right)^2 \]
\[ \leq \frac{32\beta\|\ell\|_{\infty}}{n_1(\mu + \lambda)} \cdot \left( 1 + \frac{\beta}{n_1(\mu + \lambda)} \right) + \sqrt{2\|\ell\|_{\infty}} \cdot \mathcal{E}_{\lambda,i} + \beta \mathcal{E}_{\lambda,i}^2, \]
where in the last line we have used \((a+b)^2 \leq 2a^2 + 2b^2\). We finish the proof by noticing that \( \frac{\beta}{n_1(\mu + \lambda)} \leq \frac{\beta}{n_1 \mu} \leq 4 \), where the last inequality is by (C.5). \( \square \)

### C.4 Proof of Theorem 4.5

Compared to the proof of Theorem 4.4, we need to additionally control the estimation error of the global model.

**Proposition C.4** (Estimation error of the global model). Let Assumptions \( A(b) \) and \( C(b) \) hold. Then
\[ \mathbb{E}_{S} \| \tilde{w}^{(\text{global})} - w^{(\text{global})}_{\text{avg}} \|^2 \leq \frac{48\beta^2 \sigma^2}{\mu^2 \lambda^2} \left( \sum_{i \in [m]} \frac{p_i}{\sqrt{n_i}} \right)^2 + \frac{48\beta^2 \mathbf{R}^2}{\mu^2} + \frac{12(\mu + \lambda)^2 \sigma^2}{\mu^2 \lambda^2} \sum_{i \in [m]} \frac{p_i^2}{n_i}. \]

**Proof.** See Appendix C.4.1 \( \square \)

With the above proposition, the following result is a counterpart of Proposition C.3.

**Proposition C.5** (\( \lambda \)-dependent bound on the IER). Let Assumptions \( A(a, b), C(b) \) and Equation (C.5) hold. Run \( \mathcal{A}_{FP} \) with hyperparameters chosen as in Lemma B.1 and B.2. Then, for any \( i \in [m] \), as long as
\[ T \geq C_1 \lambda(\lambda \vee 1)n_i \left\| p_i^{-1} \vee [\lambda(\lambda \vee 1)n_i] \right\|, \]
\[ K_T \geq C_2(\lambda + 1)^2 n_i \left( \frac{p_i^{-1} + \lambda^2 p_i^2}{n_i} \right), \]
\[ \lambda \Delta \mathcal{E}_{\mathcal{A}_{FP},s}[I_{ER_i}(\mathcal{A}_{FP})] \leq \frac{C}{\lambda n_i} \left[ \left\| \ell \right\|_{\infty} + \sigma^2 \beta \lambda \sigma \left( \sum_{i \in [m]} \frac{p_i}{\sqrt{n_i}} \right)^2 + \sigma^2 n_i \sum_{i \in [m]} \frac{p_i^2}{n_i} \right] + C \lambda \left[ \left( 1 + \frac{\beta^2 \mathbf{R}^2}{\mu^2} \right)^2 + \sigma^2 \sum_{i \in [m]} \frac{p_i^2}{n_i} \right], \]
and
\[ \mathbb{E}_{\mathcal{A}_{FP},s}[I_{ER_i}(\mathcal{A}_{FP})] \leq C \left( \frac{\beta \left\| \ell \right\|_{\infty}}{\mu n_i} + \lambda D^2 \right), \]
where \( C_1, C_2 \) are two constants only depending on \( (\mu, \beta, \left\| \ell \right\|_{\infty}, D) \), and \( C \) is an absolute constant.
We first show that the expected value of $\mathcal{E}_{\text{OPT}}/p_1$ and $\lambda(\varepsilon(\text{global}))^2$ are both bounded above by a constant multiple of $\frac{\beta\|\ell\|_{\infty}}{n_1(\mu + \lambda)}$. Indeed, by the estimates we have established in Equations (C.7) and (C.9), it suffices to require

$$T \gtrsim_{(\mu, \beta, \|\mu\|_{\infty}, D)} \lambda(\lambda \vee 1)m\|p\|_{2}^{2}n_{1},$$

and

$$K_{T} \gtrsim_{(\mu, \beta, \|\mu\|_{\infty}, D)} \frac{(\lambda \vee 1)^{2}n_{1}}{p_{1}}, \quad T \gtrsim_{(\mu, \beta, \|\mu\|_{\infty}, D)} \frac{\lambda(\lambda \vee 1)m\|p\|_{2}^{2}n_{1}}{p_{1}},$$

respectively. And the above two displays, combined with (C.6), is exactly (C.19). (C.21) then follows from the compactness of $\mathcal{W}$. To prove (C.20), we invoke Proposition C.4 to get

$$E_{\text{AFP}, S}[\mathcal{I}_{\text{ER}_1}] \leq \frac{\beta\|\ell\|_{\infty}}{n_1(\mu + \lambda)} + \lambda \left(1 + \frac{\beta^2}{\mu^2}\right)R^2 + \frac{\beta^2\sigma^2}{\mu^2\lambda} \left(\sum_{i \in [m]} \frac{p_i}{\sqrt{n_i}}\right)^2 + \frac{(\mu + \lambda)^2\sigma^2}{\mu^2\lambda} \sum_{i \in [m]} \frac{\sigma^2_i}{n_i},$$

and (C.21) follows by rearranging terms. \qed

We now present our proof of Theorem 4.5.

**Proof of Theorem 4.5.** Without loss of generality we consider the first client. Since all $n_i$’s are of the same order, it suffices to show

$$E[\mathcal{I}_{\text{ER}_1}(A_{\text{FP}})] \leq \left[(\mu + \mu^{-1})\left(\beta\|\ell\|_{\infty} + \frac{\sigma^2\beta^2 + \sigma^2}{\mu^2} + \frac{\sigma^2}{\mu^2}\right) + \mu D^2\right] \cdot \left(\frac{R}{\sqrt{N/m}} \wedge \frac{1}{N/m} + \frac{\sqrt{m}}{N}\right). \quad (C.22)$$

We define the following two events:

$$A := \{R \geq \sqrt{m/N}\}, \quad B := A^c = \{R < \sqrt{m/N}\},$$

and we set

$$\lambda = \frac{c_A m}{D^2 N} \cdot 1_A + c_B \sqrt{\frac{m}{R^2 N + 1}} \cdot 1_B,$$

where $c_A, c_B$ are two constants to be specified later. We consider two cases:

1. If $A$ holds, then from (C.21) we have $E_{\text{AFP}, S}[\mathcal{I}_{\text{ER}_1}] \leq \frac{\beta\|\ell\|_{\infty}}{\mu} + c_A \cdot \frac{1}{N/m}$, provided $\lambda p_{\max} \leq \mu/16$. Note that $\lambda p_{\max} \approx \frac{c_A}{D^2 N} \leq c_A/D^2$. So we can choose $\lambda \approx \mu D^2$, which gives

$$E_{\text{AFP}, S}[\mathcal{I}_{\text{ER}_1}] \leq \left(\frac{\beta\|\ell\|_{\infty}}{\mu} + \mu D^2\right) \cdot \frac{1}{N/m} \leq \text{right-hand side of } (C.22).$$

2. If $B$ holds, and if $\lambda p_{\max} \leq \mu/16$ holds, then from (C.20) we have

$$E_{\text{AFP}, S}[\mathcal{I}_{\text{ER}_1}] \leq \frac{1}{\lambda N/m} \left(\beta\|\ell\|_{\infty} + \frac{\sigma^2\beta^2}{\mu^2} + \frac{\sigma^2}{\mu^2}\right) + \lambda \left(1 + \frac{\sigma^2}{\mu^2}\right)R^2 + \frac{\sigma^2 N}{\mu^2 N}$$

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\[ \sum \left( \beta \| \ell \|_\infty + \frac{\sigma^2 \beta^2 + \beta^2 + \sigma^2}{\mu^2} \right) \cdot \left( \frac{1}{\lambda N/m} + \lambda (R^2 + N^{-1}) \right) \]
\[ = \left( \beta \| \ell \|_\infty + \frac{\sigma^2 \beta^2 + \beta^2 + \sigma^2}{\mu^2} \right) \cdot \left( \frac{\sqrt{R^2 N + 1}}{c_B N/\sqrt{m}} + c_B \sqrt{m} \sqrt{R^2 N + 1} \right) \]
\[ \leq \left( \beta \| \ell \|_\infty + \frac{\sigma^2 \beta^2 + \beta^2 + \sigma^2}{\mu^2} \right) \cdot \left( \frac{1}{c_B \sqrt{N/m}} + \frac{1}{c_B \sqrt{m} \sqrt{N}} + \frac{c_B \sqrt{m}}{N} \right) \]
\[ = \left( \beta \| \ell \|_\infty + \frac{\sigma^2 \beta^2 + \beta^2 + \sigma^2}{\mu^2} \right) \cdot (c_B + c_B^{-1}) \left( \frac{1}{c_B \sqrt{N/m}} + \frac{\sqrt{m}}{N} \right). \]

Note that \( p_{\max} \lambda \leq c_B p_{\max} \sqrt{m} < c_B / \sqrt{m} \leq c_B \). So to satisfy \( p_{\max} \lambda \leq \mu/16 \), we can choose \( c_B \approx \mu \). This gives

\[ E_{\text{FP}, s}[\text{IER}] \lesssim \left( \beta \| \ell \|_\infty + \frac{\sigma^2 \beta^2 + \beta^2 + \sigma^2}{\mu^2} \right) (\mu + \mu^{-1}) \left( \frac{R}{\sqrt{N/m}} + \frac{\sqrt{m}}{N} \right) \]
\[ \leq \text{right-hand side of (C.22)}. \]

The desired result follows by combining the above two cases together. \( \square \)

### C.4.1 Proof of Proposition C.4: Estimation Error of the Global Model

We begin by proving a useful lemma.

**Lemma C.6** (Estimating \( \mathbf{w}_i^{(i)} \), given the knowledge of \( \mathbf{u}_{\text{avg}}^{(\text{global})} \)). Let Assumption A(b) hold. Then for any \( i \in [m] \), we have

\[ \| \mathbf{w}_i^{(i)} - \text{Prox}_{\lambda/\lambda}(\mathbf{w}_{\text{avg}}^{(\text{global})}) \| \leq \frac{2}{\mu + \lambda} \left\| \nabla L_i(\mathbf{w}_i^{(i)}, S_i) + \lambda(\mathbf{w}_i^{(i)} - \mathbf{u}_{\text{avg}}^{(\text{global})}) \right\|. \]

**Proof.** This follows from an adaptation of the arguments in Theorem 7 of Foster et al. [2019]. By strong convexity, we have

\[ \left\langle \nabla L_i(\mathbf{w}_i^{(i)}, S_i) + \lambda(\mathbf{w}_i^{(i)} - \mathbf{w}_{\text{avg}}^{(\text{global})}), \text{Prox}_{\lambda/\lambda}(\mathbf{w}_{\text{avg}}^{(\text{global})}) - \mathbf{w}_i^{(i)} \right\rangle \]
\[ + \frac{\mu + \lambda}{2} \left\| \mathbf{w}_i^{(i)} - \text{Prox}_{\lambda/\lambda}(\mathbf{w}_{\text{avg}}^{(\text{global})}) \right\|^2 \]
\[ \leq L_i(\mathbf{w}_i^{(i)}, S_i) + \lambda \left\| \mathbf{w}_{\text{avg}}^{(\text{global})} - \mathbf{w}_i^{(i)} \right\|^2 - L_i(\text{Prox}_{\lambda/\lambda}(\mathbf{w}_{\text{avg}}^{(\text{global})}), S_i) \]
\[ - \frac{\lambda}{2} \left\| \mathbf{w}_{\text{avg}}^{(\text{global})} - \text{Prox}_{\lambda/\lambda}(\mathbf{w}_{\text{avg}}^{(\text{global})}) \right\|^2 \]
\[ \leq 0. \]

If \( \| \mathbf{w}_i^{(i)} - \text{Prox}_{\lambda/\lambda}(\mathbf{w}_{\text{avg}}^{(\text{global})}) \| = 0 \) we are done. Otherwise, Cauchy-Schwartz inequality applied to the above display gives the desired result. \( \square \)

Now, since \( \sum_{i \in [m]} p_i F_i \) is \( \mu F = \mu/\lambda \)-strongly convex, we have

\[ \left\langle \sum_{i \in [m]} p_i \nabla F_i(\mathbf{w}_{\text{avg}}^{(\text{global})}), \mathbf{w}^{(\text{global})} - \mathbf{w}_{\text{avg}}^{(\text{global})} \right\rangle + \frac{\mu F}{2} \left\| \mathbf{w}^{(\text{global})} - \mathbf{w}_{\text{avg}}^{(\text{global})} \right\|^2 \]
\[ \leq \sum_{i \in [m]} p_i F_i(\mathbf{w}^{(\text{global})}) - \sum_{i \in [m]} p_i F_i(\mathbf{w}_{\text{avg}}^{(\text{global})}) \]
\[ \leq 0. \]
Taking expectation at both sides, we arrive at
\[
\mathbb{E}_S \left[ \left( \sum_{i \in [m]} p_i \| \nabla L_i(w^*_i, S_i) \| \right)^2 \right] \leq \sum_{i \in [m]} \frac{p_i^2 \sigma^2}{n_i} + \sum_{i \neq s} \frac{p_i p_s \sigma_i \sigma_s}{\sqrt{n_i n_s}} = \sigma^2 \left( \sum_{i \in [m]} \frac{p_i}{\sqrt{n_i}} \right)^2.
\]

Meanwhile, we have
\[
\mathbb{E}_S \left\| \sum_{i \in [m]} p_i \nabla L_i(w^*_i, S_i) \right\|^2 \leq \sum_{i \in [m]} \frac{p_i^2 \sigma^2}{n_i}.
\]

The desired result follows by plugging the previous two displays to (C.23).