Periodicities of T-systems and Y-systems, Dilogarithm Identities, and Cluster Algebras II: Types $C_r$, $F_4$, and $G_2$

by

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Abstract

We prove the periodicities of the restricted T-systems and Y-systems associated with the quantum affine algebra of type $C_r$, $F_4$, and $G_2$ at any level. We also prove the dilogarithm identities for these Y-systems at any level. Our proof is based on the tropical Y-systems and the categorification of the cluster algebra associated with any skew-symmetric matrix by Plamondon.

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§1. Introduction

This is a continuation of the paper [IIKKN]. In [IIKKN], we proved the periodicities of the restricted T-systems and Y-systems associated with the quantum affine algebra of type $B_r$ at any level. We also proved the dilogarithm identities for these Y-systems at any level. Our proof was based on the tropical Y-systems and the categorification of the cluster algebra associated with any skew-symmetric matrix by Plamondon [P1, P2]. In this paper, using the same method, we prove the corresponding statements for types $C_r$, $F_4$, and $G_2$, thereby completing all the non-simply laced types.

The results are basically parallel to type $B_r$. Since the common method and the proofs of the statements for type $B_r$ were described in [IIKKN] in detail, in this paper, we skip the proofs of most statements, and concentrate on presenting the results with emphasis on the special features of each case. Notably, the tropical Y-system at level 2, which is the core part in the entire method, is quite specific to each case.

While we try to make the paper as self-contained as possible, we also try to minimize duplication with [IIKKN]. Therefore, we have to ask the reader for tolerating numerous references to the companion paper [IIKKN] for the things which are omitted. In particular, basic definitions for cluster algebras are summarized in [IIKKN, Section 2.1].

The organization of the paper is as follows. In Section 2 we present the main results as well as the T-systems and Y-systems for each type. In Section 3 the results for type $C_r$ are established. The key tropical Y-system at level 2 is described in detail in Section 3.6. In Section 4 the results for type $F_4$ are proved. In Section 5 we give the results for type $G_2$. In Section 6 we list the known mutation equivalences of quivers corresponding to the T-systems and Y-systems.

§2. Main results

§2.1. Restricted T-systems and Y-systems of types $C_r$, $F_4$, and $G_2$

Let $X_r$ be the Dynkin diagram of type $C_r$, $F_4$, or $G_2$ with rank $r$, and $I = \{1, \ldots, r\}$ be the enumeration of the vertices of $X_r$:

\[
\begin{align*}
C_r & \quad \cdots \quad F_4 & \quad G_2 \\
1 & \quad 2 & \quad r-1 & \quad r & \quad 1 & \quad 2 & \quad 3 & \quad 4 & \quad 1 & \quad 2
\end{align*}
\]

Let $h$ and $h^\vee$ be the Coxeter number and the dual Coxeter number of $X_r$, respectively. Then
For a given integer \( \ell \) (2.3)

We define numbers \( t \) and \( t_a \) (\( a \in I \)) by

\[
t = \begin{cases} 
2, & X_r = C_r, F_4, \\
3, & X_r = G_2,
\end{cases} \quad t_a = \begin{cases} 
1, & a_a \text{ long root,} \\
t, & a_a \text{ short root.}
\end{cases}
\]

For a given integer \( \ell \geq 2 \), we introduce a set of triplets

\[
\mathcal{I}_{\ell} = \mathcal{I}_{\ell}(X_r) := \{(a, m, u) \mid a \in I; m = 1, \ldots, t_a \ell - 1; u \in (1/t)\mathbb{Z}\}. \tag{2.3}
\]

**Definition 2.1** ([KNS]). Fix an integer \( \ell \geq 2 \). The level \( \ell \) restricted \( T \)-system \( T_\ell(X_r) \) of type \( X_r \) (with the unit boundary condition) is the following system of relations for a family of variables \( T_r = \{T_m^{a}(u) \mid (a, m, u) \in \mathcal{I}_{\ell}\} \), where \( T_m^{a}(0) = T_m^{a}(u) = 1 \), and furthermore, \( T_m^{a}(u) = T_m^{a}(u) = 1 \) (the unit boundary condition) if they occur in the right hand sides of the relations.

(Here and throughout the paper, \( 2m \) (resp. \( 2m + 1 \)) on the left hand sides, for example, represents elements \( 2, 4, \ldots \) (resp. \( 1, 3, \ldots \)).

For \( X_r = C_r \),

\[
T_m^{(a)}(u) - \frac{1}{2} T_m^{(a)}(u + \frac{1}{2}) = T_{m-1}^{(a)}(u) T_{m+1}^{(a)}(u) + T_m^{(a-1)}(u) T_m^{(a+1)}(u) \quad (1 \leq a \leq \ell - 2),
\]

\[
T_{2m-1}^{(r-1)}(u) - \frac{1}{2} T_{2m-1}^{(r-1)}(u + \frac{1}{2}) = T_{2m-2}^{(r-1)}(u) T_{2m}^{(r-1)(u)} + T_{2m}^{(r-2)}(u) T_{m}^{(r)}(u + \frac{1}{2}),
\]

\[
T_{2m+1}^{(r-1)}(u) - \frac{1}{2} T_{2m+1}^{(r-1)}(u + \frac{1}{2}) = T_{2m+2}^{(r-1)}(u) T_{m+1}^{(r-1)(u)} + T_{2m+1}^{(r-2)}(u) T_{m}^{(r)}(u),
\]

\[
T_m^{(r)}(u) - 1 T_m^{(r)}(u + 1) = T_{m-1}^{(r)}(u) T_{m+1}^{(r)}(u) + T_{2m-1}^{(r)}(u).
\]

For \( X_r = F_4 \),

\[
T_m^{(1)}(u) + 1 T_m^{(1)}(u + 1) = T_{m-1}^{(1)}(u) T_{m+1}^{(1)}(u) + T_m^{(2)}(u),
\]

\[
T_m^{(2)}(u) + 1 T_m^{(2)}(u + 1) = T_{m-1}^{(2)}(u) T_{m+1}^{(2)}(u) + T_m^{(1)}(u) T_{2m}^{(3)}(u),
\]

\[
T_m^{(3)}(u) - \frac{1}{2} T_{2m}^{(3)}(u + \frac{1}{2}) = T_{m-1}^{(3)}(u) T_{m+1}^{(3)}(u) + T_{2m-1}^{(3)}(u) T_{2m+1}^{(3)}(u),
\]

\[
T_{2m+1}^{(3)}(u - \frac{1}{2}) T_{2m+1}^{(3)}(u + \frac{1}{2}) = T_{2m}^{(3)}(u) T_{2m-1}^{(3)}(u) + T_{m}^{(3)}(u) T_{m+1}^{(3)}(u) + T_{2m+1}^{(3)}(u).
\]

\[
X_r & \quad C_r & F_4 & G_2 \\
\hline
h & 2r & 12 & 6 \\
\hbar' & r + 1 & 9 & 4
\]
For $X_r = G_2$,

\[
T_m^{(1)}(u-1)T_m^{(1)}(u+1) = T_{m-1}^{(1)}(u)T_{m+1}^{(1)}(u) + T_{3m}^{(2)}(u),
\]

\[
T_{3m}^{(2)}(u - \frac{1}{3})T_{3m}^{(2)}(u + \frac{1}{3}) = T_{3m-1}^{(2)}(u)T_{3m+1}^{(2)}(u)
+ T_m^{(1)}(u - \frac{1}{2})T_m^{(1)}(u)T_m^{(1)}(u + \frac{2}{3}),
\]

\[
T_{3m+1}^{(2)}(u - \frac{1}{3})T_{3m+1}^{(2)}(u + \frac{1}{3}) = T_{3m}^{(2)}(u)T_{3m+2}^{(2)}(u)
+ T_m^{(1)}(u - \frac{1}{2})T_m^{(1)}(u)T_m^{(1)}(u + \frac{1}{3}),
\]

\[
T_{3m+2}^{(2)}(u - \frac{1}{3})T_{3m+2}^{(2)}(u + \frac{1}{3}) = T_{3m+1}^{(2)}(u)T_{3m+3}^{(2)}(u)
+ T_m^{(1)}(u)T_{m+1}^{(1)}(u - \frac{1}{3})T_{m+1}^{(1)}(u + \frac{1}{3}).
\]

**Definition 2.2 ([KN]).** Fix an integer $\ell \geq 2$. The level $\ell$ restricted $Y$-system $Y_\ell(X_r)$ of type $X_r$ is the following system of relations for a family of variables $Y = \{Y_m^{(a)}(u) \mid (a, m, u) \in I_\ell\}$, where $Y_0^{(0)}(u) = Y_0^{(a)}(u)^{-1} = Y_0^{(a)}(u)^{-1} = 0$ if they occur in the right hand sides in the relations:

For $X_r = C_r$,

\[
Y_m^{(a)}(u - \frac{1}{2})Y_m^{(a)}(u + \frac{1}{2}) = \frac{(1 + Y_m^{(a-1)}(u))(1 + Y_m^{(a+1)}(u))}{(1 + Y_m^{(a)}(u-1))(1 + Y_m^{(a)}(u-1))}
(1 \leq a \leq r - 2),
\]

\[
Y_{2m}^{(r-1)}(u - \frac{1}{2})Y_{2m}^{(r-1)}(u + \frac{1}{2}) = \frac{(1 + Y_{2m}^{(r-2)}(u))(1 + Y_{2m}^{(r)}(u))}{(1 + Y_{2m-1}^{(r-1)}(u-1))(1 + Y_{2m+1}^{(r-1)}(u-1))},
\]

\[
Y_{2m+1}^{(r-1)}(u - \frac{1}{2})Y_{2m+1}^{(r-1)}(u + \frac{1}{2}) = \frac{1 + Y_{2m+1}^{(r-2)}(u)}{(1 + Y_{2m}^{(r-1)}(u-1))(1 + Y_{2m+2}^{(r-1)}(u-1))}
\times(1 + Y_{2m-1}^{(r-1)}(u)Y_{2m+1}^{(r-1)}(1 + Y_{2m-1}^{(r-1)}(u))
\times(1 + Y_{2m}^{(r-1)}(u - \frac{1}{2}))(1 + Y_{2m}^{(r-1)}(u + \frac{1}{2}))
\times(1 + Y_{m-1}^{(r)}(u-1))(1 + Y_{m+1}^{(r)}(u-1)).
\]

For $X_r = F_4$,

\[
Y_m^{(1)}(u - 1)Y_m^{(1)}(u + 1) = \frac{1 + Y_m^{(2)}(u)}{(1 + Y_m^{(1)}(u-1))(1 + Y_{m+1}^{(1)}(u-1))}
\times(1 + Y_{m-1}^{(1)}(u)Y_{m+1}^{(1)}(u))(1 + Y_{m+1}^{(3)}(u))
\times(1 + Y_{2m-1}^{(3)}(u)Y_{2m+1}^{(3)}(u + \frac{1}{2}))
\times(1 + Y_{2m+1}^{(3)}(u-1))(1 + Y_{2m+1}^{(3)}(u-1))
\times(1 + Y_{2m}^{(4)}(u))(1 + Y_{2m}^{(4)}(u))
\times(1 + Y_{2m+1}^{(3)}(u-1))(1 + Y_{2m+1}^{(3)}(u-1)).
\]
\begin{align*}
Y_{2m+1}^{(3)}(u - \frac{1}{2})Y_{2m+1}^{(3)}(u + \frac{1}{2}) &= \frac{1 + Y_{2m+1}^{(4)}(u)}{(1 + Y_{2m+1}^{(3)}(u)^{-1})(1 + Y_{2m+2}^{(3)}(u)^{-1})}, \\
Y_{m}^{(4)}(u - \frac{1}{2})Y_{m}^{(4)}(u + \frac{1}{2}) &= \frac{1 + Y_{m}^{(3)}(u)}{(1 + Y_{m-1}^{(4)}(u)^{-1})(1 + Y_{m+1}^{(4)}(u)^{-1})}.
\end{align*}

For \( X_r = G_2 \),
\begin{align*}
Y_{m}^{(1)}(u - 1)Y_{m}^{(1)}(u + 1) &= \frac{1 + Y_{m}^{(3)}(u)}{(1 + Y_{m-1}^{(1)}(u)^{-1})(1 + Y_{m+1}^{(1)}(u)^{-1})}, \\
Y_{3m}^{(2)}(u - \frac{1}{3})Y_{3m}^{(2)}(u + \frac{1}{3}) &= \frac{1 + Y_{3m}^{(1)}(u)}{(1 + Y_{3m-1}^{(2)}(u)^{-1})(1 + Y_{3m+1}^{(2)}(u)^{-1})}, \\
Y_{3m+1}^{(2)}(u - \frac{1}{3})Y_{3m+1}^{(2)}(u + \frac{1}{3}) &= \frac{1}{(1 + Y_{3m+1}^{(2)}(u)^{-1})(1 + Y_{3m+3}^{(2)}(u)^{-1})}, \\
Y_{3m+2}^{(2)}(u - \frac{1}{3})Y_{3m+2}^{(2)}(u + \frac{1}{3}) &= \frac{1}{(1 + Y_{3m+2}^{(2)}(u)^{-1})(1 + Y_{3m+3}^{(2)}(u)^{-1})}.
\end{align*}

Let us write (2.4)–(2.6) in a unified manner
\begin{align*}
T_{m}^{(a)}(u - \frac{1}{t_r})T_{m}^{(a)}(u + \frac{1}{t_r}) &= T_{m-1}^{(a)}(u)T_{m+1}^{(a)}(u) + \prod_{(b,k,v) \in \mathcal{I}_r} T_{k}^{(b)}(v)^{G(b,k,v;a,m,u)}.
\end{align*}

Define the transposition \( ^t G(b,k,v;a,m,u) = G(a,m,u;b,k,v) \). Then (2.7)–(2.9) can be written as
\begin{align*}
Y_{m}^{(a)}(u - \frac{1}{t_r})Y_{m}^{(a)}(u + \frac{1}{t_r}) &= \prod_{(b,k,v) \in \mathcal{I}_r} \frac{1 + Y_{k}^{(b)}(v)^{G(b,k,v;a,m,u)}}{(1 + Y_{m-1}^{(a)}(u)^{-1})(1 + Y_{m+1}^{(a)}(u)^{-1})}.
\end{align*}

\section{2.2. Periodicities}

\textbf{Definition 2.3.} Let \( \mathcal{T}_r(X_r) \) be the commutative ring over \( \mathbb{Z} \) with identity element, with generators \( T_{m}^{(a)}(u)^{\pm 1} \) \((a,m,u) \in \mathcal{I}_r\) and relations \( \mathcal{T}_r(X_r) \) together with \( T_{m}^{(a)}(u)T_{m}^{(a)}(u)^{-1} = 1 \). Let \( \mathcal{T}_r(X_r) \) be the subring of \( \mathcal{T}(X_r) \) generated by \( T_{m}^{(a)}(u) \) \((a,m,u) \in \mathcal{I}_r\).

\textbf{Definition 2.4.} Let \( \mathcal{Y}_r(X_r) \) be the semifield with generators \( Y_{m}^{(a)}(u) \) \((a,m,u) \in \mathcal{I}_r\) and relations \( \mathcal{Y}_r(X_r) \). Let \( \mathcal{Y}_r^*(X_r) \) be the multiplicative subgroup of \( \mathcal{Y}_r(X_r) \).
generated by $Y_m^{(a)}(u), 1 + Y_m^{(a)}(u)$ ($(a, m, u) \in \mathcal{I}_\ell$). (Here we use the symbol $+$ instead of $\oplus$ for simplicity.)

The first main result of the paper concerns the periodicities of the T-systems and Y-systems.

**Theorem 2.5** (Conjectured in [IIKNS]). The following relations hold in $\mathcal{T}^c_\ell(X_r)$:

(i) Half periodicity: $T_m^{(a)}(u + h^\vee + \ell) = T_{t_\ell m}^{(a)}(u)$.

(ii) Full periodicity: $T_m^{(a)}(u + 2(h^\vee + \ell)) = T_m^{(a)}(u)$.

**Theorem 2.6** (Conjectured in [KNS]). The following relations hold in $\mathcal{Y}_\ell^c(X_r)$:

(i) Half periodicity: $Y_m^{(a)}(u + h^\vee + \ell) = Y_{t_\ell m}^{(a)}(u)$.

(ii) Full periodicity: $Y_m^{(a)}(u + 2(h^\vee + \ell)) = Y_m^{(a)}(u)$.

§2.3. Dilogarithm identities

Let $L(x)$ be the Rogers dilogarithm function

$$L(x) = -\frac{1}{2} \int_0^x \left\{ \frac{\log(1 - y)}{y} + \frac{\log y}{1 - y} \right\} dy \quad (0 \leq x \leq 1).$$

We introduce the constant version of the Y-system.

**Definition 2.7.** Fix an integer $\ell \geq 2$. The level $\ell$ restricted constant Y-system $\mathcal{Y}_\ell^c(X_r)$ of type $X_r$ is the following system of relations for a family of variables $Y_m^c = \{Y_m^{(a)} \mid a \in I; m = 1, \ldots, t_\ell m - 1\}$, where $Y_m^{(0)} = Y_m^{(a) - 1} = Y_m^{(a) - 1} = 0$ if they occur on the right hand sides of the relations.

For $X_r = C_r$,

\begin{align*}
(Y_m^{(a)})^2 &= \frac{(1 + Y_{m}^{(a-1)})(1 + Y_{m}^{(a+1)})}{(1 + Y_{m-1}^{(a)}) (1 + Y_{m+1}^{(a)})} \quad (1 \leq a \leq r - 2), \\
(Y_{2m}^{(r-1)})^2 &= \frac{(1 + Y_{2m}^{(r-2)})(1 + Y_{2m}^{(r)})(1 + Y_{2m-1}^{(r-1)})(1 + Y_{2m+1}^{(r-1)}),}{(1 + Y_{2m-1}^{(r-1)})(1 + Y_{2m+1}^{(r-1)})}, \\
(Y_{2m+1}^{(r-1)})^2 &= \frac{1 + Y_{2m+1}^{(r-2)}}{(1 + Y_{2m}^{(r-1)})(1 + Y_{2m+1}^{(r-1)})}, \\
(Y_m^{(r)})^2 &= \frac{(1 + Y_{2m-1}^{(r-1)})(1 + Y_{2m}^{(r-1)})(1 + Y_{2m+1}^{(r-1)})(1 + Y_{m+1}^{(r-1)})}{(1 + Y_{m-1}^{(r-1)})(1 + Y_{m+1}^{(r-1)})}.
\end{align*}

(2.13)
For $X_r = F_4$, 
\[
\begin{align*}
(Y_m^{(1)})^2 &= \frac{1 + Y_m^{(2)}}{(1 + Y_m^{(1)} - 1)(1 + Y_m^{(1)} + 1)}.
(Y_m^{(2)})^2 &= \frac{(1 + Y_m^{(1)})(1 + Y_m^{(3)})(1 + Y_m^{(3)} - 1)(1 + Y_m^{(3)} + 1)}{(1 + Y_m^{(2)} - 1)(1 + Y_m^{(2)} + 1)}.
(Y_m^{(3)})^2 &= \frac{(1 + Y_m^{(2)})(1 + Y_m^{(4)})}{(1 + Y_m^{(2)} - 1)(1 + Y_m^{(2)} - 1)}.
(Y_{2m+1}^{(3)})^2 &= \frac{1 + Y_{2m+1}^{(4)}}{(1 + Y_{2m+1}^{(3)} - 1)(1 + Y_{2m+1}^{(3)} - 1)}.
(Y_{2m+1}^{(4)})^2 &= \frac{1 + Y_{2m+1}^{(4)}}{(1 + Y_{2m+1}^{(4)} - 1)(1 + Y_{2m+1}^{(4)} - 1)}.
\end{align*}
\]

(2.14) 

For $X_r = G_2$, 
\[
\begin{align*}
(Y_m^{(1)})^2 &= \frac{(1 + Y_m^{(2)})(1 + Y_m^{(3)})(1 + Y_m^{(3)} + 1)}{(1 + Y_m^{(1)} - 1)(1 + Y_m^{(1)} + 1)}.
(Y_m^{(2)})^2 &= \frac{1 + Y_m^{(1)}}{(1 + Y_m^{(1)} - 1)(1 + Y_m^{(1)} + 1)}.
(Y_m^{(3)})^2 &= \frac{1 + Y_m^{(1)}}{(1 + Y_m^{(1)} - 1)(1 + Y_m^{(1)} - 1)}.
(Y_m^{(3)} + 1)^2 &= \frac{1}{(1 + Y_m^{(3)} - 1)(1 + Y_m^{(3)} + 1)}.
(Y_m^{(3)} + 2)^2 &= \frac{1}{(1 + Y_m^{(3)} - 1)(1 + Y_m^{(3)} + 1)}.
\end{align*}
\]

Proposition 2.8. There exists a unique positive real solution of $Y_c^c(X_r)$. 

Proof. The proof of [IIKKN, Proposition 1.8] is applicable. \qed

The second main result of the paper is the dilogarithm identities conjectured by Kirillov [Ki, Eq. (7)], and properly corrected by Kuniba [Ku, Eqs. (A.1a), (A.1c)]. 

Theorem 2.9 (Dilogarithm identities). Suppose that a family of positive real numbers $\{Y_m^{(a)} | a \in I; m = 1, \ldots, t_a \ell - 1\}$ satisfies $Y_c^c(X_r)$. Then 
\[
6 \sum_{a \in I} \sum_{m=1}^{t_a \ell - 1} L(\frac{Y_m^{(a)}}{1 + Y_m^{(a)}}) = \frac{f \dim \mathfrak{g}}{h^\nu + \ell} - r,
\]

where $\mathfrak{g}$ is the simple Lie algebra of type $X_r$. 

The right hand side of (2.16) is equal to

\begin{equation}
\frac{r(\ell h - h^\vee)}{h^\vee + \ell}.
\end{equation}

In fact, we prove a functional generalization of Theorem 2.9.

**Theorem 2.10** (Functional dilogarithm identities). Suppose that a family of positive real numbers \( \{Y_m^{(a)}(u) \mid (a,m,u) \in \mathcal{I}_\ell \} \) satisfies \( \mathcal{Y}(X) \). Then

\begin{equation}
\sum_{\substack{(a,m,u) \in \mathcal{I}_\ell \\atop 0 \leq u < 2(h^\vee + \ell)}} L \left( \frac{Y_m^{(a)}(u)}{1 + Y_m^{(a)}(u)} \right) = 2tr(\ell h - h^\vee)
\end{equation}

\begin{equation}
= \begin{cases}
4r(2r\ell - r - 1), & C_r, \\
48(4\ell - 3), & F_4, \\
24(3\ell - 2), & G_2,
\end{cases}
\end{equation}

\begin{equation}
\sum_{\substack{(a,m,u) \in \mathcal{I}_\ell \\atop 0 \leq u < 2(h^\vee + \ell)}} L \left( \frac{1}{1 + Y_m^{(a)}(u)} \right) = \begin{cases}
4\ell(2r\ell - \ell - 1), & C_r, \\
8\ell(3\ell + 1), & F_4, \\
12\ell(2\ell + 1), & G_2.
\end{cases}
\end{equation}

The two identities (2.18) and (2.19) are equivalent to each other, since the sum of the right hand sides is equal to \( 2t(h^\vee + \ell)((a_{\sum a \in I} t_a)\ell - r) \), which is the total number of \((a,m,u) \in \mathcal{I}_\ell \) in the region \( 0 \leq u < 2(h^\vee + \ell) \).

It is clear that Theorem 2.9 follows from Theorem 2.10.

**§3. Type \( C_r \)**

The \( C_r \) case is quite parallel to the \( B_r \) case. For the reader’s convenience, we repeat most of the basic definitions and results from [IIKKN]. Most propositions are proved in a parallel manner to the \( B_r \) case, so that proofs are omitted. The properties of the tropical Y-system at level 2 (Proposition 3.10) are crucial and specific to \( C_r \). Since its derivation is a little more complicated than in the \( B_r \) case, an outline of the proof is provided.

**§3.1. Parity decompositions of T-systems and Y-systems**

For a triplet \((a,m,u) \in \mathcal{I}_\ell \), we define the ‘parity conditions’ \( P_+ \) and \( P_- \) by

\begin{align}
P_+ : \ & r + a + m + 2u \text{ is odd if } a \neq r; \ 2u \text{ is even if } a = r, \\
P_- : \ & r + a + m + 2u \text{ is even if } a \neq r; \ 2u \text{ is odd if } a = r.
\end{align}
Figure 1. The quiver $Q_\ell(C_r)$ for $\ell$ even (top) and for $\ell$ odd (bottom), where we identify the rightmost column in the left quiver with the middle column in the right quiver.

We write, for example, $(a,m,u) : P_+$ if $(a,m,u)$ satisfies $P_+$. We have $\mathcal{I}_\ell = \mathcal{I}_{\ell+} \cup \mathcal{I}_{\ell-}$, where $\mathcal{I}_{\ell\varepsilon}$ is the set of all $(a,m,u) : P_{\varepsilon}$.

Define $T_\ell^\varepsilon(C_r)$ $(\varepsilon = \pm)$ to be the subring of $T_\ell^\varepsilon(C_r)$ generated by $T_m^{(a)}(u)$ $((a,m,u) \in \mathcal{I}_{\ell\varepsilon})$. Then we have $T_\ell^\varepsilon(C_r) = T_\ell^\varepsilon(C_r)$ via $T_m^{(a)}(u) \mapsto T_m^{(a)}(u + \frac{1}{2})$

(3.3) $T_\ell^\varepsilon(C_r) = T_\ell^\varepsilon(C_r)_+ \otimes \mathbb{Z} T_\ell^\varepsilon(C_r)_-$.

For a triplet $(a,m,u) \in \mathcal{I}_\ell$, we set other ‘parity conditions’ $P'_+$ and $P'$ by

(3.4) $P'_+: r + a + m + 2u$ is even if $a \neq r$; $2u$ is even if $a = r$,

(3.5) $P': r + a + m + 2u$ is odd if $a \neq r$; $2u$ is odd if $a = r$. 


We have $\mathcal{I}_\ell = \mathcal{I}_{\ell+} \sqcup \mathcal{I}_{\ell-}$, where $\mathcal{I}_{\ell+}$ is the set of all $(a, m, u) : \mathbf{P}'_\ell$. We also have

$$
(a, m, u) : \mathbf{P}'_\ell \leftrightarrow (a, m, u \pm \frac{1}{2}) : \mathbf{P}_\ell.
$$

Define $Y^r_\ell(C_r) (\varepsilon = \pm)$ to be the subgroup of $Y_\ell(C_r)$ generated by $Y^r_\ell(u)$, $1 + Y^r_\ell(u) ((a, m, u) \in \mathcal{I}_{\ell\varepsilon})$. Then $Y^r_\ell(C_r)_+ \simeq Y^r_\ell(C_r)_-$ via $Y^r_\ell(u) \mapsto Y^r_\ell(u + \frac{1}{2})$, and

$$
Y^r_\ell(C_r) \simeq Y^r_\ell(C_r)_+ \times Y^r_\ell(C_r)_-.
$$

\section*{3.2. Quiver $Q_\ell(C_r)$}

With type $C_r$ and $\ell \geq 2$ we associate the quiver $Q_\ell(C_r)$ by Figure 1, where the rightmost column in the left quiver and the middle column in the right quiver are identified. Also, we assign an empty or filled circle $\circ$ or $\bullet$ and a sign $+/-$ to each vertex.

Let us choose the index set $\mathbf{I}$ of the vertices of $Q_\ell(C_r)$ so that $i = (i, i') \in \mathbf{I}$ represents the vertex in the $i'$th row (from the bottom) and the $i$th column (from the left) of the left quiver for $i = 1, \ldots, r - 1$, in the right column of the right quiver for $i = r$, and in the left column of the right quiver for $i = r + 1$. Thus, $i = 1, \ldots, r + 1$, and $i' = 1, \ldots, \ell - 1$ if $i \neq r, r + 1$, while $i' = 1, \ldots, 2\ell - 1$ if $i = r, r + 1$. We use the natural notation $\mathbf{I}^\circ$ (resp. $\mathbf{I}_{\ell+}^\circ$) for the set of vertices $i$ with property $\circ$ (resp. $\circ$ and $\bullet$), and so on. We have $\mathbf{I} = \mathbf{I}^\circ \sqcup \mathbf{I}^\bullet = \mathbf{I}_{\ell+}^\circ \sqcup \mathbf{I}_{\ell-}^\circ \sqcup \mathbf{I}_{\ell+}^\bullet \sqcup \mathbf{I}_{\ell-}^\bullet$.

We define composite mutations,

$$
\mu^\circ = \prod_{i \in \mathbf{I}_{\ell+}^\circ} \mu_i, \quad \mu^\bullet = \prod_{i \in \mathbf{I}_{\ell-}^\bullet} \mu_i, \quad \mu^\bullet = \prod_{i \in \mathbf{I}_{\ell+}^\bullet} \mu_i, \quad \mu^\circ = \prod_{i \in \mathbf{I}_{\ell-}^\circ} \mu_i.
$$

Note that they do not depend on the order of the product.

Let $r$ be the involution acting on $\mathbf{I}$ by left-right reflection of the right quiver. Let $\omega$ be the involution acting on $\mathbf{I}$ defined by, for even $r$, up-down reflection of the left quiver and 180° rotation of the right quiver; and for odd $r$, up-down reflection of the left and right quivers. Let $r(Q_\ell(C_r))$ and $\omega(Q_\ell(C_r))$ denote the quivers induced from $Q_\ell(C_r)$ by $r$ and $\omega$, respectively. For example, if there is an arrow $i \rightarrow j$ in $Q_\ell(C_r)$, then there is an arrow $r(i) \rightarrow r(j)$ in $r(Q_\ell(C_r))$. For a quiver $Q$, $Q^{op}$ denotes the opposite quiver.

\begin{lemma}
Let $Q = Q_\ell(C_r)$.
\begin{enumerate}
\item We have a periodic sequence of mutations of quivers

$$
Q \xrightarrow{\mu^\circ} Q^{op} \xleftarrow{\mu^\bullet} r(Q) \xrightarrow{\mu^\circ} Q^{op} \xleftarrow{\mu^\bullet} Q.
$$

\item $\omega(Q) = Q$ if $h^\vee + \ell$ is even, and $\omega(Q) = r(Q)$ if $h^\vee + \ell$ is odd.
\end{enumerate}
\end{lemma}
§3.3. Cluster algebra and alternative labels

It is standard to identify a quiver $Q$ with no loop and no 2-cycle with a skew-
symmetric matrix $B$. We use the following convention for the direction of arrows:

\begin{equation}
  i \to j \iff B_{ij} = 1.
\end{equation}

(In this paper we only encounter the situation where $B_{ij} = -1, 0, 1$.) Let $B_t(C_r)$
be the skew-symmetric matrix corresponding to the quiver $Q_t(C_r)$. In the rest of
the section, we set $B = (B_{ij})_{i,j \in I} = B_t(C_r)$ unless otherwise mentioned.

Let $A(B, x, y)$ be the cluster algebra with coefficients in the universal semifield
$\mathbb{Q}_{sf}(y)$, where $(B,x,y)$ is the initial seed [FZ2]. See also [IKKN, Section 2.1] for
the conventions and notations on cluster algebras we employ. (Here we use the
symbol $+$ instead of $\oplus$ in $\mathbb{Q}_{sf}(y)$, since it is the ordinary addition of subtraction-
free expressions of rational functions of $y$.)

\textbf{Definition 3.2.} The coefficient group $\mathcal{G}(B,y)$ associated with
$A(B,x,y)$ is the multiplicative subgroup of the semifield $\mathbb{Q}_{sf}(y)$ generated by all the coefficients
$y_i'$ of $A(B,x,y)$ together with $1 + y_i'$.

In view of Lemma 3.1 we set $x(0) = x, y(0) = y$ and define clusters $x(u) = (x_i(u))_{i \in I}$
and coefficient tuples $y(u) = (y_i(u))_{i \in I}$ by the sequence of mutations

\begin{equation}
  \ldots \xleftarrow{\mu_i^+} (B, x(0), y(0)) \xrightarrow{\mu_i^+} (-B, x(\frac{1}{2}), y(\frac{1}{2})) \xleftarrow{\mu_i^+} (r(B), x(1), y(1)) \xrightarrow{\mu_i^+} (-r(B), x(\frac{3}{2}), y(\frac{3}{2})) \xleftarrow{\mu_i^+} \ldots,
\end{equation}

where $r(B) = B'$ is defined by $B'_{ij} = B_{r(i)r(j)}$.

For a pair $(i, u) \in I \times \frac{1}{2}\mathbb{Z}$, we define the parity conditions $p_+$ and $p_-$ by

\begin{equation}
  p_+: \left\{ \begin{array}{ll}
  i \in I^+ \cup I^*_+, & u \equiv 0, \\
  i \in I^*, & u \equiv \frac{1}{2}, \\
  i \in I^- \cup I^*_-, & u \equiv \frac{3}{2},
  \end{array} \right.
  \quad p_-: \left\{ \begin{array}{ll}
  i \in I^+ \cup I^*_+, & u \equiv \frac{1}{2}, \\
  i \in I^*, & u \equiv 0,1, \\
  i \in I^- \cup I^*_-, & u \equiv \frac{3}{2},
  \end{array} \right.
\end{equation}

where $\equiv$ is equivalence modulo $2\mathbb{Z}$. We have

\begin{equation}
  (i, u) : p_+ \iff (i, u + \frac{1}{2}) : p_-.
\end{equation}

Each $(i, u) : p_+$ is a mutation point of (3.11) in the forward direction of $u$, and
each $(i, u) : p_-$ is one in the backward direction of $u$. Notice that there are also
some \((i, u)\) which satisfy neither \(p_+\) nor \(p_-\), and are not mutation points of (3.11); explicitly, they are \((i, u)\) with \(i \in I_\ell^\circ, u \equiv 1, \frac{3}{2} \mod 2\mathbb{Z}\), or with \(i \in I_\ell^\circ, u \equiv 0, \frac{1}{2} \mod 2\mathbb{Z}\).

There is a correspondence between the parity condition \(p_\pm\) here and \(P_\pm, P'_\pm\) in (3.1) and (3.4).
Lemma 3.3. Below \( \equiv \) means equivalence modulo \( 2\mathbb{Z} \).

(i) The map \( g : \mathcal{I}_+ \to \{(a, u) : p_+ \} \) given by

\[
(a, m, u - \frac{1}{\ln r}) \mapsto \begin{cases} 
(a, m, u), & a \neq r, \\
(r + 1, m, u), & a = r; m + u \equiv 0, \\
(r, m, u), & a = r; m + u \equiv 1,
\end{cases}
\]

is a bijection.
The result in this subsection is completely parallel to the B\textsuperscript{T}dence Theorem 3.6. The ring 
\[(a, m, u) \in I_{\ell+} \to (i, u) : p_{\pm} \text{ or } p_{\mp} \} \text{ given by}
\[
(a, m, u) \mapsto \begin{cases} 
(a, m, u), & a \neq r, \\
(r + 1, m, u), & a = r; m + u \equiv 0, \\
(r, m, u), & a = r; m + u \equiv 1, 
\end{cases}
\]
is a bijection.

We introduce alternative labels \(x_{1}(u) = x_{m}^{(a)}(u - 1/t_{a}) ((a, m, u - 1/t_{a}) \in I_{\ell+})\) for \((i, u) = g((a, m, u - 1/t_{a}))\) and \(y_{1}(u) = y_{m}^{(a)}(u) ((a, m, u) \in I_{\ell+})\) for \((i, u) = g'(a, m, u))\), respectively. See Figures 2–3.

\section*{3.4. T-system and cluster algebra}

The result in this subsection is completely parallel to the \(B_{r}\) case [IKKN].

Let \(A(B, x)\) be the cluster algebra with trivial coefficients, where \((B, x)\) is the initial seed [FZ2]. Let \(I = \{1\}\) be the trivial semifield and \(\pi : \mathbb{Q}(y) \to 1\), \(y_{1} \mapsto 1\), be the projection. Let \([x_{1}(u)]_{1}\) denote the image of \(x_{1}(u)\) under the algebra homomorphism \(A(B, x, y) \to A(B, x)\) induced by \(\pi_{1}\). It is called the trivial evaluation.

Recall that \(G(b, k, v; a, m, u)\) is defined in (2.10).

**Lemma 3.4.** The family \(\{x_{m}^{(a)}(u) \mid (a, m, u) \in I_{\ell+}\}\) satisfies a system of relations
\[
x_{m}^{(a)}(u - 1/t_{a})x_{m}^{(a)}(u + 1/t_{a}) = \frac{y_{m}^{(a)}(u)}{1 + y_{m}^{(a)}(u)} \prod_{(b, k, v) \in I_{\ell+}} x_{k}^{(b)(v) G(b, k, v; a, m, u)} + \frac{1}{1 + y_{m}^{(a)}(u)} x_{m-1}^{(a)}(u)x_{m+1}^{(a)}(u),
\]
where \((a, m, u) \in I_{\ell+}\). In particular, the family \(\{[x_{m}^{(a)}(u)]_{1} \mid (a, m, u) \in I_{\ell+}\}\) satisfies the T-system \(T_{\ell}(C_{r})\) in \(A(B, x)\) after replacing \([x_{m}^{(a)}(u)]_{1}\) with \([x_{m}^{(a)}(u)]_{1}\).

**Definition 3.5.** The \(T\)-subalgebra \(A_{T}(B, x)\) of \(A(B, x, y)\) associated with the sequence (3.11) is the subalgebra of \(A(B, x)\) generated by \([x_{1}(u)]_{1} ((i, u) \in I \times \frac{1}{2}\mathbb{Z})\).

**Theorem 3.6.** The ring \(T_{\ell}^{(C_{r})+}\) is isomorphic to \(A_{T}(B, x)\) via the correspondence \(T_{m}^{(a)}(u) \mapsto [x_{m}^{(a)}(u)]_{1}\).

\section*{3.5. Y-system and cluster algebra}

The result in this subsection is completely parallel to the \(B_{r}\) case [IKKN].
Lemma 3.7. The family \( \{ y_m^{(a)}(u) \mid (a, m, u) \in I'_+ \} \) satisfies the Y-system \( Y^{(a)}_m(C_r) \) after replacing \( Y^{(a)}_m(u) \) with \( y_m^{(a)}(u) \).

Definition 3.8. The Y-subgroup \( \mathcal{G}_Y(B, y) \) of \( \mathcal{G}(B, y) \) associated with the sequence (3.11) is the subgroup of \( \mathcal{G}(B, y) \) generated by \( y_1(u) \) for \((i, u) \in I \times \mathbb{Z} / 2\mathbb{Z}\) and \( 1 + y_1(u) \) for \((i, u) : p_+ \) or \( p_- \).

Theorem 3.9. The group \( Y^+_m(C_r) \) is isomorphic to \( \mathcal{G}_Y(B, y) \) via the correspondence \( Y^+_m(u) \mapsto y_m^{(a)}(u) \) and \( 1 + Y^+_m(u) \mapsto 1 + y_m^{(a)}(u) \).

§3.6. Tropical Y-system at level 2

The tropical semifield \( \text{Trop}(y) \) is an abelian multiplicative group freely generated by the elements \( y_i \) \((i \in I)\) with addition

\[
\prod_{i \in I} y_i^{a_i} \oplus \prod_{i \in I} y_i^{b_i} = \prod_{i \in I} y_i^{\min(a_i, b_i)}.
\]

Let \( \pi_T : \mathbb{Q}_s(y) \to \text{Trop}(y) \), \( y_i \mapsto y_i \), be the projection. Let \( [y_1(u)]_T \) and \( [\mathcal{G}_Y(B, y)]_T \) denote the images of \( y_1(u) \) and \( \mathcal{G}_Y(B, y) \) under the multiplicative group homomorphism induced by \( \pi_T \), respectively. They are called the tropical evaluations, and the resulting relations in the group \( [\mathcal{G}_Y(B, y)]_T \) form the tropical Y-system.

We say a (Laurent) monomial \( m = \prod_{i \in I} y_i^{a_i} \) is positive (resp. negative) if \( m \neq 1 \) and \( k_i \geq 0 \) (resp. \( k_i \leq 0 \)) for any \( i \).

The following properties of the tropical Y-system at level 2 will be the key in the entire method.

Proposition 3.10. For \( [\mathcal{G}_Y(B, y)]_T \) with \( B = B_2(C_r) \), the following facts hold:

(i) Let \( 0 \leq u < 2 \). For any \((i, u) : p_+\), the monomial \( [y_1(u)]_T \) is positive.

(ii) Let \( -h^\vee \leq u < 0 \).

(a) Let \( i = (i, 2) \) \((i \leq r - 1)\), \((r, 1)\), or \((r + 1, 1)\). For any \((i, u) : p_+\), the monomial \( [y_1(u)]_T \) is negative.

(b) Let \( i = (i, 1) \), \((i, 3) \) \((i \leq r - 1)\). For any \((i, u) : p_+\), the monomial \( [y_1(u)]_T \) is positive for \( u = -\frac{1}{2} h^\vee, -\frac{1}{2} h^\vee - \frac{1}{2} \) and negative otherwise.

(iii) \( y_i^{(2)} \) equals \( y_{i-1}^{-1} \) if \( i \leq r - 1 \), and \( y_i^{-1} \) if \( i = r, r + 1 \).

(iv) For even \( r \), \( y_i^{(2)} \) equals \( y_i^{-1} \) if \( i \leq r - 1 \), and \( y_{2r+1-i}^{-1} \) if \( i = r, r + 1 \). For odd \( r \), \( y_i^{(2)} \) equals \( y_{i-1}^{-1} \).

One can directly verify (i) and (iii) in the same way as in the \( B_r \) case \([\text{IIKKN}, \text{Proposition } 3.2]\). In the rest of this subsection we give the outline of the proof.
of (ii) and (iv). Note that (ii) and (iv) can be proved independently for each variable $y_i$. (To be precise, we also need to ensure that no monomial is 1. However, this can be easily guaranteed, so that we do not describe the details here.) Below we separate the variables into two parts. Here is a brief summary of the results.

(1) The $D$ part. The powers of $[y_i(u)]_T$ in the variables $y_{i,2}$ ($i \leq r - 1$) and $y_{r,1}$, $y_{r+1,1}$ are described by the root system of type $D_{r+1}$ with a Coxeter-like transformation. It turns out that they are further described by (a subset of) the root system of type $A_{2r+1}$ with the Coxeter transformation.

(2) The $A$ part. The powers of $[y_i(u)]_T$ in the variables $y_{i,1}$ and $y_{i,3}$ ($i \leq r - 1$) are mainly described by the root system of type $A_{r-1}$ with the Coxeter transformation.

3.6.1. D part. Let us consider the $D$ part first. Let $D_{r+1}$ be the Dynkin diagram of type $D$ with index set $J = \{1, \ldots, r+1\}$. We assign a sign $+/-$ to vertices of $D_{r+1}$ (no sign for $r$ and $r+1$) as inherited from $Q_2(C_r)$.

Let $\Pi = \{\alpha_1, \ldots, \alpha_{r+1}\}$, $-\Pi$, $\Phi_+$ be the sets of simple roots, of negative simple roots, and of positive roots, respectively, of type $D_{r+1}$. Following [FZ1], we introduce the piecewise-linear analogue $\sigma_i$ of the simple reflection $s_i$, acting on the set $\Phi_{\geq -1} = \Phi_+ \sqcup (-\Pi)$ of almost positive roots by

\[
\sigma_i(\alpha) = s_i(\alpha), \quad \alpha \in \Phi_+,
\]

\[
\sigma_i(-\alpha_j) = \begin{cases} 
\alpha_j, & j = i, \\
-\alpha_j, & \text{otherwise}.
\end{cases}
\]

(3.18)

Let

\[
\sigma_+ = \prod_{i \in J_+} \sigma_i, \quad \sigma_- = \prod_{i \in J_-} \sigma_i,
\]

(3.19)

where $J_{\pm}$ is the set of vertices of $D_{r+1}$ with the respective sign. We define

\[
\sigma = \sigma_- \sigma_+ \sigma_{r+1} \sigma_- \sigma_+ \sigma_r.
\]

(3.20)
Lemma 3.11. The following facts hold:

(I) Let \( r \) be even.

(i) For \( i \leq r - 1 \), \( \sigma^k(-\alpha_i) \in \Phi_+ \) \( (1 \leq k \leq r/2) \), \( \sigma^{r/2+1}(-\alpha_i) = -\alpha_i \).

(ii) For \( i \leq r - 1 \), \( \sigma^k(\alpha_i) \in \Phi_+ \) \( (0 \leq k \leq r/2) \), \( \sigma^{r/2+1}(\alpha_i) = \alpha_i \).

(iii) \( \sigma^k(-\alpha_r) \in \Phi_+ \) \( (1 \leq k \leq r/2) \), \( \sigma^{r/2+1}(-\alpha_r) = -\alpha_{r+1} \).

(iv) \( \sigma^k(-\alpha_{r+1}) \in \Phi_+ \) \( (1 \leq k \leq r/2 + 1) \), \( \sigma^{r/2+2}(-\alpha_{r+1}) = -\alpha_r \).

(v) The elements of \( \Phi_+ \) appearing in (i)–(iv) exhaust the set \( \Phi_+ \), thereby providing the orbit decomposition of \( \Phi_+ \) under \( \sigma \).

(II) Let \( r \) be odd.

(i) For \( i \in J_+ \), \( \sigma^k(-\alpha_i) \in \Phi_+ \) \( (1 \leq k \leq r + 1) \), \( \sigma^{(r+1)/2}(-\alpha_i) = -\alpha_i \).

(ii) For \( i \in J_- \), \( \sigma^k(-\alpha_i) \in \Phi_+ \) \( (1 \leq k \leq r + 1) \), \( \sigma^{(r+1)/2}(-\alpha_i) = -\alpha_i \).

(iii) \( \sigma^k(-\alpha_r) \in \Phi_+ \) \( (1 \leq k \leq (r + 1)/2) \), \( \sigma^{(r+3)/2}(-\alpha_r) = -\alpha_r \).

(iv) \( \sigma^k(-\alpha_{r+1}) \in \Phi_+ \) \( (1 \leq k \leq (r + 1)/2) \), \( \sigma^{(r+3)/2}(-\alpha_{r+1}) = -\alpha_{r+1} \).

(v) The elements of \( \Phi_+ \) appearing in (i)–(iv) exhaust the set \( \Phi_+ \), thereby providing the orbit decomposition of \( \Phi_+ \) under \( \sigma \).

Proof. The statements are verified by explicitly calculating \( \sigma^k(-\alpha_i) \) and \( \sigma^k(\alpha_i) \).

The examples for \( r = 10 \) (for even \( r \)) and \( 9 \) (for odd \( r \)) are given in Tables 1 and 2, respectively, where we use the notations

\[
(i, j) = \alpha_i + \cdots + \alpha_j \quad (1 \leq i < j \leq r), \quad [i] = \alpha_i \quad (1 \leq i \leq r),
\]

\[
\{i, j\} = (\alpha_i + \cdots + \alpha_{i-1}) + (\alpha_j + \cdots + \alpha_{r+1}) \quad (1 \leq i < j \leq r + 1, i \leq r - 1),
\]

and \( \{r + 1\} = \alpha_{r+1} \). In fact, it is not difficult to read off the general rule from these examples.

The orbits \( \sigma(-\alpha_i) \) and \( \sigma(\alpha_i) \) are further described by (a subset of) the root system of type \( A_{2r+1} \). Let \( \Pi' = \{\alpha'_1, \ldots, \alpha'_{2r+1}\} \) and \( \Phi'_+ \) be the sets of simple roots and of positive roots of type \( A_{2r+1} \), respectively, with standard index set \( J' = \{1, \ldots, 2r + 1\} \). Define \( J'_+ = \{i \in J' \mid i - r \text{ is even}\} \) and \( J'_- = \{i \in J' \mid i - r \text{ is odd}\} \). We introduce the notations \([i, j]' = \alpha'_i + \cdots + \alpha'_j \) \( (1 \leq i < j \leq 2r + 1) \) and \([i]' = \alpha'_i \), parallel to \( \alpha^k \). Let \( O'_k = \{(\sigma')^k(-\alpha'_i) \mid 1 \leq k \leq r + 1\} \) be the orbit of \(-\alpha'_i \) in \( \Phi'_+ \) under \( \sigma' = \sigma_+ \), \( \sigma'_k = \prod_{i \in J'_+} \sigma'_i \), where \( \sigma'_i \) is the piecewise-linear analogue of the simple reflection \( s'_i \) acting as in (3.18).
Table 1. The orbits $\sigma^k(-\alpha_i)$ and $\sigma^k(\alpha_i)$ in $\Phi_+^r$ under $\sigma$ of (3.20) for $r = 10$. The orbits of $-\alpha_i$ and $\alpha_i$ ($i \leq 8$), for example, $\alpha_1 \rightarrow [2, 3] \rightarrow [6, 7] \rightarrow \cdots \rightarrow -\alpha_1$ and $\alpha_1 \rightarrow [4, 5] \rightarrow [8, 9] \rightarrow \cdots \rightarrow \alpha_1$, are aligned alternatingly. The orbits of $-\alpha_{10}$ and $-\alpha_{11}$, namely, $-\alpha_{10} \rightarrow \{7, 9\} \rightarrow \{3, 5\} \rightarrow \cdots \rightarrow -\alpha_{11}$, and $-\alpha_{11} \rightarrow \{9, 11\} \rightarrow \{5, 7\} \rightarrow \cdots \rightarrow -\alpha_{10}$, are aligned alternatingly. The numbers $-1, -2, \ldots$ in the head line will be identified with the parameter $u$ in (3.24).

|   | $-\alpha_1$ | $\alpha_2$ | $-\alpha_2$ | $\alpha_3$ | $-\alpha_3$ | $\alpha_4$ | $-\alpha_4$ | $\alpha_5$ | $-\alpha_5$ | $\alpha_6$ | $-\alpha_6$ | $\alpha_7$ | $-\alpha_7$ | $\alpha_8$ | $-\alpha_8$ | $\alpha_9$ | $-\alpha_9$ | $\alpha_{10}$ | $-\alpha_{10}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | $\alpha_1$ | $[2, 3]$ | $[4, 5]$ | $[6, 7]$ | $[8, 9]$ | $[11]$ | $[9, 10]$ | $[7, 8]$ | $[5, 6]$ | $[3, 4]$ | $[1, 2]$ | $-\alpha_1$ |
| 2 | $\alpha_2$ | $-\alpha_2$ | $[1, 3]$ | $[4, 7]$ | $[6, 9]$ | $[8, 11]$ | $[9, 10]$ | $[7, 10]$ | $[5, 8]$ | $[3, 6]$ | $[1, 4]$ | $[2]$ | $-\alpha_2$ |
| 3 | $\alpha_3$ | $-\alpha_3$ | $[3]$ | $[1, 5]$ | $[2, 7]$ | $[4, 9]$ | $[6, 11]$ | $[8, 9]$ | $[7, 10]$ | $[5, 10]$ | $[3, 8]$ | $[1, 6]$ | $[2, 4]$ | $-\alpha_3$ | $\alpha_3$ |
| 4 | $\alpha_4$ | $-\alpha_4$ | $[3, 5]$ | $[1, 7]$ | $[2, 9]$ | $[4, 11]$ | $[6, 9]$ | $[7, 8]$ | $[5, 10]$ | $[3, 10]$ | $[1, 8]$ | $[2, 6]$ | $[4]$ | $-\alpha_4$ |
| 5 | $\alpha_5$ | $-\alpha_5$ | $[5]$ | $[3, 7]$ | $[1, 9]$ | $[2, 11]$ | $[4, 9]$ | $[6, 7]$ | $[5, 8]$ | $[3, 10]$ | $[1, 10]$ | $[2, 8]$ | $[4, 6]$ | $-\alpha_5$ | $\alpha_5$ |
| 6 | $\alpha_6$ | $-\alpha_6$ | $[5, 7]$ | $[3, 9]$ | $[1, 11]$ | $[2, 9]$ | $[4, 7]$ | $[5, 6]$ | $[3, 8]$ | $[1, 10]$ | $[2, 10]$ | $[4, 8]$ | $[6]$ | $-\alpha_6$ |
| 7 | $\alpha_7$ | $-\alpha_7$ | $[7]$ | $[5, 9]$ | $[3, 11]$ | $[1, 9]$ | $[2, 7]$ | $[4, 5]$ | $[3, 6]$ | $[1, 8]$ | $[2, 10]$ | $[4, 10]$ | $[6, 8]$ | $-\alpha_7$ | $\alpha_7$ |
| 8 | $\alpha_8$ | $-\alpha_8$ | $[7, 9]$ | $[5, 11]$ | $[3, 9]$ | $[1, 7]$ | $[2, 5]$ | $[3, 4]$ | $[1, 6]$ | $[2, 8]$ | $[4, 10]$ | $[6, 10]$ | $[8]$ | $-\alpha_8$ |
| 9 | $\alpha_9$ | $-\alpha_9$ | $[9]$ | $[7, 11]$ | $[5, 9]$ | $[3, 7]$ | $[1, 5]$ | $[2, 3]$ | $[1, 4]$ | $[2, 6]$ | $[4, 8]$ | $[6, 10]$ | $[8, 10]$ | $-\alpha_9$ | \(\alpha_9\) |
| $\alpha_{10}$ | $-\alpha_{10}$ | $[9, 11]$ | $[7, 9]$ | $[5, 7]$ | $[3, 5]$ | $[1, 3]$ | $[1, 2]$ | $[2, 4]$ | $[4, 6]$ | $[6, 8]$ | $[8, 10]$ | $[10]$ | $-\alpha_{10}$ | $\alpha_{10}$ |
Table 2. The orbit of $\sigma^k(-\alpha_i)$ in $\Phi_+$ under $\sigma$ of (3.20) for $r = 9$. The orbit of $-\alpha_i$ ($i \leq 8$), for example, $-\alpha_1 \rightarrow [3, 4] \rightarrow [7, 8] \rightarrow \cdots \rightarrow \alpha_1 \rightarrow [1, 2] \rightarrow [5, 6] \rightarrow \cdots \rightarrow -\alpha_1$, is aligned in a cyclic and alternating way. The orbits of $-\alpha_9$ and $-\alpha_{10}$, namely, $-\alpha_9 \rightarrow \{6, 8\} \rightarrow \{2, 4\} \rightarrow \cdots \rightarrow -\alpha_9$, and $-\alpha_{10} \rightarrow \{8, 10\} \rightarrow \{4, 6\} \rightarrow \cdots \rightarrow -\alpha_{10}$, are aligned alternatingly. The numbers $-1, -2, \ldots$ in the head line will be identified with the parameter $u$ in (3.24).

|   | $-1$ | $-2$ | $-3$ | $-4$ | $-5$ | $-6$ | $-7$ | $-8$ | $-9$ | $-10$ |
|---|------|------|------|------|------|------|------|------|------|------|
| 1 + | $\alpha_1$ | $-\alpha_1$ | [1, 2] | [3, 4] | [5, 6] | [7, 8] | (10) | [8, 9] | [6, 7] | [4, 5] | [2, 3] | [1] |
| 2 − | $-\alpha_2$ | [2] | [1, 4] | [3, 6] | [5, 8] | (7, 10) | [8, 9] | [6, 9] | [4, 7] | [2, 5] | [1, 3] |
| 3 + | $\alpha_3$ | $-\alpha_3$ | [2, 4] | [1, 6] | [3, 8] | (5, 10) | [7, 8] | [6, 9] | [4, 9] | [2, 7] | [1, 5] | [3] |
| 4 − | $-\alpha_4$ | [4] | [2, 6] | [1, 8] | (3, 10) | [5, 8] | [6, 7] | [4, 9] | [2, 9] | [1, 7] | [3, 5] |
| 5 + | $\alpha_5$ | $-\alpha_5$ | [4, 6] | [2, 8] | (1, 10) | [3, 8] | [5, 6] | [4, 7] | [2, 9] | [1, 9] | [3, 7] | [5] |
| 6 − | $-\alpha_6$ | [6] | [4, 8] | (2, 10) | [1, 8] | [3, 6] | [4, 5] | [2, 7] | [1, 9] | [3, 9] | [5, 7] |
| 7 + | $\alpha_7$ | $-\alpha_7$ | [6, 8] | (4, 10) | [2, 8] | [1, 6] | [3, 4] | [2, 5] | [1, 7] | [3, 9] | [5, 9] | [7] |
| 8 − | $-\alpha_8$ | [8] | [6, 10] | [4, 8] | [2, 6] | [1, 4] | [2, 3] | [1, 5] | [3, 7] | [5, 9] | [7, 9] |
| $-\alpha_{10}$ | $-\alpha_{10}$ | [8, 10] | [6, 8] | [4, 6] | [2, 4] | (1, 2) | [1, 3] | [3, 5] | [5, 7] | [7, 9] | [9] | $-\alpha_{10}$ | $-\alpha_9$ |
Lemma 3.12. Let

\begin{equation}
\rho : \Phi_+ \to \bigoplus_{i=1}^{r} O'_i
\end{equation}

be the map defined by

\begin{align*}
[i,j] &\mapsto \begin{cases} 
[i,j]', & j - r \text{ odd}, \\
[2r + 2 - j, 2r + 2 - i]', & j - r \text{ even},
\end{cases} \\
\{i,j\} &\mapsto \begin{cases} 
\{i, 2r + 2 - j\}', & j - r \text{ odd}, \\
\{j, 2r + 2 - i\}', & j - r \text{ even},
\end{cases} \\
\{r+1\} &\mapsto [r, r + 1]',
\end{align*}

where \([i] = [i, i]\). Then \(\rho\) is a bijection. Furthermore, under the bijection \(\rho\), the action of \(\sigma\) is translated into the one of the square of the Coxeter element \(s' = s'_r s'_{r-1}\) of type \(A_{2r+1}\) acting on \(\Phi'_+\), where \(s'_\pm = \prod_{i \in J'_\pm} s'_i\).

For \(-h^\vee \leq u < 0\), define

\begin{align*}
\alpha_i(u) &= \begin{cases} 
\sigma^{-u/2}(-\alpha_i), & i \in J_+, \ u \equiv 0, \\
\sigma^{-(u-1)/2}(\alpha_i), & i \in J_+, \ u \equiv -1, \\
\sigma^{-(2u-1)/4}(-\alpha_i), & i \in J_-, \ u \equiv -\frac{3}{2}, \\
\sigma^{-(2u+1)/4}(\alpha_i), & i \in J_-, \ u \equiv -\frac{1}{2}, \\
\sigma^{-u/2}(-\alpha_r), & i = r, \ u \equiv 0, \\
\sigma^{-(u-1)/2}(-\alpha_{r+1}), & i = r + 1, \ u \equiv -1,
\end{cases}
\end{align*}

where \(\equiv\) is equivalence modulo \(2\mathbb{Z}\). Note that they correspond to the positive roots in Tables 1 and 2 with \(u\) being the parameter in the head lines. By Lemma 3.11 they are all the positive roots of \(D_{r+1}\).

Lemma 3.13. The family in (3.24) satisfies the recurrence relations

\begin{align*}
\alpha_i(u - \frac{1}{2}) + \alpha_i(u + \frac{1}{2}) &= \alpha_{i-1}(u) + \alpha_{i+1}(u) \quad (1 \leq i \leq r - 2), \\
\alpha_{r-1}(u - \frac{1}{2}) + \alpha_{r-1}(u + \frac{1}{2}) &= \begin{cases} 
\alpha_{r-2}(u) + \alpha_{r}(u) & \text{ (u even)}, \\
\alpha_{r-2}(u) + \alpha_{r+1}(u) & \text{ (u odd)},
\end{cases} \\
\alpha_r(u - 1) + \alpha_r(u + 1) &= \alpha_{r-1}(u - \frac{1}{2}) + \alpha_{r-1}(u + \frac{1}{2}) \quad (u \text{ odd}), \\
\alpha_{r+1}(u - 1) + \alpha_{r+1}(u + 1) &= \alpha_{r-1}(u - \frac{1}{2}) + \alpha_{r-1}(u + \frac{1}{2}) \quad (u \text{ even}),
\end{align*}

where \(\alpha_0(u) = 0\).
Proof. These relations are easily verified by using the explicit expressions of $\alpha_i(u)$. See Tables 1 and 2. The first two relations can also be obtained from Lemma 3.12 and [FZ2, Eq. (10.9)].

Let us return to the proof of (ii) of Proposition 3.10 for the $D$ part. For a monomial $m$ in $y = (y_i)_{i \in I}$, let $\pi_D(m)$ denote the specialization with $y_{i,1} = y_{i,3} = 1$ ($i \leq r - 1$). For simplicity, we set $y_{i,2} = y_i$ ($i \leq r - 1$), $y_r = y_r$, $y_{r+1,1} = y_{r+1}$, and also $y_{2}(u) = y_{3}(u)$ ($i \leq r - 1$), $y_{r+1}(u) = y_{r+1}(u)$, $y_{r+1,1}(u) = y_{r}(u)$. We define the vectors $t_i(u) = (t_i(u_k))_{k=1}^{r+1}$ by

\[
\pi_D([y_i(u)]_T) = \prod_{k=1}^{r+1} y_k^{t_i(u)_k}.
\]

We also identify each vector $t_i(u)$ with $\alpha = \sum_{k=1}^{r+1} t_i(u_k)\alpha_k \in \mathbb{Z}$.}

**Proposition 3.14.** Let $-h^\vee \leq u < 0$. Then

\[
t_i(u) = -\alpha_i(u)
\]

for $(i,u)$ as in (3.24), and

\[
\pi_D([y_i(u)]_T) = \pi_D([y_{i,3}(u)]_T) = 1 \quad (i \leq r - 1, r + i + 2u \text{ even}).
\]

Note that these formulas determine $\pi_D([y_i(u)]_T)$ for any $(i, u) : p_+$. 

Proof. We can verify the claim for $-2 \leq u \leq -\frac{1}{2}$ by direct computation. Then, by backward induction on $u$, one can establish the claim, together with the recurrence relations among $t_i(u)$’s with $(i, u)$ in (3.24),

\[
t_i(u - \frac{1}{2}) + t_i(u + \frac{1}{2}) = t_{i-1}(u) + t_{i+1}(u) \quad (1 \leq i \leq r - 2),
\]

\[
t_r(u - \frac{1}{2}) + t_r(u + \frac{1}{2}) = \begin{cases} t_{r-2}(u) + t_r(u) & (u \text{ even}), \\ t_{r-2}(u) + t_{r+1}(u) & (u \text{ odd}), \end{cases}
\]

\[
t_r(u - 1) + t_r(u + 1) = t_{r-1}(u - \frac{1}{2}) + t_{r-1}(u + \frac{1}{2}) \quad (u \text{ odd}),
\]

\[
t_{r+1}(u - 1) + t_{r+1}(u + 1) = t_{r-1}(u - \frac{1}{2}) + t_{r-1}(u + \frac{1}{2}) \quad (u \text{ even}).
\]

Note that (3.29) coincides with (3.25) under (3.27). To derive (3.29), one uses the mutations as in [HIKN, Figure 6] (or the tropical version of the Y-system $\mathcal{Y}_2(C_r)$ directly) and the positivity/negativity of $\pi_D([y_i(u)]_T)$ resulting from (3.27) and (3.28) by induction hypothesis.

Now (ii) and (iv) of Proposition 3.10 for the $D$ part follow from Proposition 3.14.

### 3.6.2. A part.

The $A$ part can be studied in a similar way to the $D$ part. Therefore, we only present the result.

---

**Remark.**
First we note that the quiver $Q_2(C_r)$ is symmetric under the exchange $y_i,1 \leftrightarrow y_i,3$ ($i \leq r - 1$). Thus, one can concentrate on the powers of $[y_i(u)]^T$ in the variables $y_i,1$ ($i \leq r - 1$).

Let $A_{r-1}$ be the Dynkin diagram of type $A$ with index set $J = \{1, \ldots, r-1\}$. We assign a sign $+/-$ to vertices (except for $r$) of $A_{r-1}$ as inherited from $Q_2(C_r)$:

\[
\begin{array}{cccccc}
1 & 2 & r-2 & r-1 & & \\
+ & - & + & - & + & \\
\end{array}
\]

$r$ even

\[
\begin{array}{cccccc}
1 & 2 & r-2 & r-1 & & \\
- & + & + & - & + & \\
\end{array}
\]

$r$ odd

Let $\Pi = \{\alpha_1, \ldots, \alpha_{r-1}\}$, $-\Pi$, $\Phi_+$ be the sets of simple roots, of negative simple roots, and of positive roots, respectively, of type $A_{r-1}$. Again, we introduce the piecewise-linear analogue $\sigma_i$ of the simple reflection $s_i$, acting on $\Phi_\geq -1 = \Phi_+ \cup (-\Pi)$ as in (3.18). Let

\[
\begin{align*}
\sigma_+ &= \prod_{i \in J_+} \sigma_i, \\
\sigma_- &= \prod_{i \in J_-} \sigma_i,
\end{align*}
\]

where $J_\pm$ is the set of vertices of $A_{r-1}$ with the respective sign. We define

\[
\sigma = \sigma_- \sigma_+.
\]

For a monomial $m$ in $y = (y_i)_{i \in I}$, let $\pi_A(m)$ denote the specialization with $y_{i,2} = y_{i,3} = 1$ ($i \leq r - 1$) and $y_{r,1} = y_{r+1,1} = 1$. We set $y_{i,1} = y_i$ ($i \leq r - 1$).

We define the vectors $t_i(u) = (t_i(u)_k)_{k=1}^{r-1}$ by

\[
\pi_A([y_i(u)]^T) = \prod_{k=1}^{r-1} y_k^{t_i(u)_k}.
\]

We also identify each vector $t_i(u)$ with $\alpha = \sum_{k=1}^{r-1} t_i(u)_k \alpha_k \in \mathbb{Z}$. With these notations, the result for the $A$ part is summarized as follows.

**Proposition 3.15.** Let $-h^\vee \leq u < 0$. For $(i, u) : p_+, t_i(u)$ is given by, for $i \leq r - 1$,

\[
\begin{align*}
t_{i1}(u) &= \begin{cases} 
-\sigma^{-u}(-\alpha_i), & i \in J_+ \\
-\sigma^{-(2u+1)/2}(-\alpha_i), & i \in J_-
\end{cases}, \\
t_{i2}(u) &= \begin{cases} 
-\lfloor 2r + 2 - i + 2u, r - 1 \rfloor, & -\frac{1}{2}h^\vee \leq u < 0, \\
-\lfloor -1 - i - 2u, r - 1 \rfloor, & -h^\vee \leq u < -\frac{1}{2}h^\vee,
\end{cases} \\
t_{i3}(u) &= 0,
\end{align*}
\]
and

\[ t_{r+1}(u) = \begin{cases} 
-\frac{1}{2}h^y \leq u < 0, \\
-\frac{1}{2}h^y \leq u < -\frac{1}{2}h^y, \\
-1 - r - 2u, r - 1], \\
-1 - r - 2u, r - 1] 
\end{cases} \]

(3.34)

where \([i, j]\) equals \(\alpha_i + \cdots + \alpha_j\) if \(i \leq j\) and 0 if \(i > j\).

Note that \(t_{11}(-h^y/2) = \alpha_{r-1}\) (\(i \in J_+\) for \(r\) even and \(i \in J_-\) for \(r\) odd) and \(t_{21}(-h^y/2 - 1/2) = \alpha_{r-i}\) (\(i \in J_+\) for \(r\) even and \(i \in J_-\) for \(r\) odd), and that they are the only positive monomials in (3.33) and (3.34). Now (ii) and (iv) of Proposition 3.10 for the \(A\) part follow from Proposition 3.15.

This completes the proof of Proposition 3.10.

§3.7. Tropical Y-systems at higher levels

By the same method as for the \(B_\ell\) case [IJKKN, Proposition 4.1], one can establish the ‘factorization property’ of the tropical Y-system at higher levels. As a result, we obtain a generalization of Proposition 3.10.

Proposition 3.16. For \([G_y(B, y)]_T\) with \(B = B_\ell(C_r)\), the following facts hold:

(i) Let \(0 \leq u < \ell\). For any \((i, u) : p_+\), the monomial \([y_i(u)]_T\) is positive.

(ii) Let \(-h^y \leq u < 0\).

(a) Let \(i \in I^0\) or \(i = (i, i') (i \leq r - 1, i' \in 2N)\). For any \((i, u) : p_+\), the monomial \([y_i(u)]_T\) is negative.

(b) Let \(i = (i, i') (i \leq r - 1, i' \notin 2N)\). For any \((i, u) : p_+\), the monomial \([y_i(u)]_T\) is positive for \(u = -\frac{1}{2}h^y, -\frac{1}{2}h^y - \frac{1}{2}\) and negative otherwise.

(iii) \(y_i(i') (i \leq r - 1, i' \geq 2N)\) if \(i = r, r + 1\).

(iv) For even \(r\), \(y_{i,i'}(-h^y)\) equals \(y_{i,i'}^{-1}\) if \(i \leq r - 1\), and \(y_{i,i'}^{-1}\) if \(i = r, r + 1\). For odd \(r\), \(y_{i,i'}(-h^y) = y_{i,i'}^{-1}\).

We obtain two important corollaries of Propositions 3.10 and 3.16.

Theorem 3.17. For \([G_y(B, y)]_T\) the following relations hold:

(i) Half periodicity: \([y_i(u + h^y + \ell)]_T = [y_i(u)]_T\).

(ii) Full periodicity: \([y_i(u + 2(h^y + \ell))]_T = [y_i(u)]_T\).
Theorem 3.18. For $[\mathcal{G}(B, y)]_T$ let $N_+$ and $N_-$ denote the total numbers of the positive and negative monomials, respectively, among $[y_i(u)]_T$ for $(i, u) : p_+$ in the region $0 \leq u < 2(h^\vee + \ell)$. Then

\begin{equation}
N_+ = 2\ell(2r\ell - \ell - 1), \quad N_- = 2r(2r\ell - r - 1).
\end{equation}

We observe the symmetry (the level-rank duality) for the numbers $N_+$ and $N_-$ under the exchange of $r$ and $\ell$.

§3.8. Periodicities and dilogarithm identities

Applying [IIKKN, Theorem 5.1] to Theorem 3.17, we obtain the periodicities:

Theorem 3.19. For $A(B, x, y)$, the following relations hold:

(i) Half periodicity: $x_i(u + h^\vee + \ell) = x_{\omega(i)}(u)$.

(ii) Full periodicity: $x_i(u + 2(h^\vee + \ell)) = x_i(u)$.

Theorem 3.20. For $\mathcal{G}(B, y)$, the following relations hold:

(i) Half periodicity: $y_i(u + h^\vee + \ell) = y_{\omega(i)}(u)$.

(ii) Full periodicity: $y_i(u + 2(h^\vee + \ell)) = y_i(u)$.

Theorems 2.5 and 2.6 for $C_r$ follow from Theorems 3.6, 3.9, 3.19, and 3.20. Furthermore, Theorem 2.10 for $C_r$ is obtained from the above periodicities and Theorem 3.18 as in the $B_r$ case [IIKKN, Section 6].

§4. Type $F_4$

The $F_4$ case is quite parallel to the $B_r$ and $C_r$ cases. We do not repeat the same definitions unless otherwise mentioned. Again, the properties of the tropical Y-system at level 2 (Proposition 4.7) are crucial and specific to $F_4$.

§4.1. Parity decompositions of T-systems and Y-systems

For a triplet $(a, m, u) \in T_r$, we reset the ‘parity conditions’ $P_+$ and $P_-$ to be

\begin{align}
P_+ &: \quad 2u \text{ is even if } a = 1, 2; \quad a + m + 2u \text{ is odd if } a = 3, 4, \\
P_- &: \quad 2u \text{ is odd if } a = 1, 2; \quad a + m + 2u \text{ is even if } a = 3, 4.
\end{align}

Then we have $\mathcal{T}_i(F_4)_+ \simeq \mathcal{T}_i(F_4)_-$ via $T_i^{(a)}(u) \mapsto T_i^{(a)}(u + \frac{1}{2})$ and

\begin{equation}
\mathcal{T}_i(F_4) \simeq \mathcal{T}_i(F_4)_+ \otimes \mathcal{T}_i(F_4)_-.
\end{equation}
For a triplet \((a, m, u) \in I\), we reset the parity conditions \(P'_+\) and \(P'_-\) to be

\[
\begin{align*}
\text{P}'_+: \; & 2u \text{ is even if } a = 1, 2; \\
& a + m + 2u \text{ is even if } a = 3, 4,
\end{align*}
\]

\[
\begin{align*}
\text{P}'_-: \; & 2u \text{ is odd if } a = 1, 2; \\
& a + m + 2u \text{ is odd if } a = 3, 4.
\end{align*}
\]

We have

\[
(a, m, u) : P'_+ \Leftrightarrow (a, m, u \pm \frac{1}{2}) : P_+.
\]

Also, we have

\[
Y^o_\ell (F_4)_+ \simeq Y^o_\ell (F_4)_- \quad \text{via} \quad Y^a_m (u) \mapsto Y^a_m (u + \frac{1}{2}), \quad 1 + Y^a_m (u) \mapsto 1 + Y^a_m (u + \frac{1}{2}),
\]

\[
Y^o_\ell (F_4) \simeq Y^o_\ell (F_4)_+ \times Y^o_\ell (F_4)_-.
\]

§4.2. Quiver \(Q_\ell (F_4)\)

With type \(F_4\) and \(\ell \geq 2\) we associate the quiver \(Q_\ell (F_4)\) by Figure 4, where the right column in the left quiver and the middle column in the right quiver are

Figure 4. The quiver \(Q_\ell (F_4)\) for \(\ell\) even (top) and for \(\ell\) odd (bottom), where we identify the right column in the left quiver with the middle column in the right quiver.
identified. Also, we assign an empty or filled circle $\circ$ and a sign $+/-$ to each vertex.

Let us choose the index set $I$ of the vertices of $Q_\ell(F_4)$ so that $i = (i,i') \in I$ represents the vertex in the $i'$th row (from the bottom) and the $i$th column (from the left) of the right quiver for $i = 1,2,3$, in the $(i-1)$th column of the right quiver for $i = 5,6$, and in the left column of the left quiver for $i = 4$. Thus, $i = 1,\ldots,6$, and $i' = 1,\ldots,\ell-1$ if $i = 1,2,5,6$, while $i' = 1,\ldots,2\ell-1$ if $i = 3,4$.

Let $r$ be the involution acting on $I$ by left-right reflection of the right quiver.

Let $\omega$ be the involution acting on $I$ by up-down reflection of the left quiver and $180^\circ$ rotation of the right quiver.

**Lemma 4.1.** Let $Q = Q_\ell(F_4)$.

(i) We have the same periodic sequence of mutations of quivers as in (3.9).

(ii) $\omega(Q) = Q$ if $h^{\vee} + \ell$ is even, and $\omega(Q) = r(Q)$ if $h^{\vee} + \ell$ is odd.

§4.3. Cluster algebra and alternative labels

Let $B_\ell(F_4)$ be the skew-symmetric matrix corresponding to the quiver $Q_\ell(F_4)$. In the rest of this section, we set $B = (B_{ij})_{i,j \in I} = B_\ell(F_4)$ unless otherwise mentioned.

Let $\mathcal{A}(B,x,y)$ be the cluster algebra with coefficients in the universal semifield $Q_{sf}(y)$, and $\mathcal{G}(B,y)$ be the coefficient group associated with $\mathcal{A}(B,x,y)$.

In view of Lemma 4.1 we set $x(0) = x$, $y(0) = y$ and define clusters $x(u) = (x_i(u))_{i \in I}$ ($u \in \frac{1}{2}\mathbb{Z}$) and coefficient tuples $y(u) = (y_i(u))_{i \in I}$ ($u \in \frac{1}{2}\mathbb{Z}$) by the sequence of mutations (3.11).

For a pair $(i,u) \in I \times \frac{1}{2}\mathbb{Z}$, we set the same parity condition $p_+$ and $p_-$ as in (3.12). We have (3.13), and each $(i,u) : p_+$ is a mutation point of (3.11) in the forward direction of $u$, while each $(i,u) : p_-$ is one in the backward direction of $u$ as before.

**Lemma 4.2.** Below $\equiv$ means equivalence modulo $2\mathbb{Z}$.

(i) The map $g : \mathcal{I}_\ell(\mathbb{Z}/4\mathbb{Z}) \to \{(i,u) : p_+\}$ defined by

\[
(a,m,u - \frac{1}{16}) \mapsto \begin{cases}
((a,m),u), & a = 1,2; a + m + u \equiv 0, \\
((7-a,m),u), & a = 1,2; a + m + u \equiv 1, \\
((a,m),u), & a = 3,4,
\end{cases}
\]

is a bijection.
(ii) The map \( g' : I_{\ell+} \to \{(i,u) : p_+\} \) defined by

\[
(a,m,u) \mapsto \begin{cases} 
((a,m),u), & a = 1, 2; a + m + u \equiv 0, \\
((7 - a, m), u), & a = 1, 2; a + m + u \equiv 1, \\
((a,m), u), & a = 3, 4,
\end{cases}
\]

is a bijection.
We introduce alternative labels $x_i(u) = x_m^{(a)}(u - 1/t_a)$ ($(a, m, u - 1/t_a) \in I_{\ell+}$) for $(i, u) = g((a, m, u - 1/t_a))$ and $y_i(u) = y_m^{(a)}(u)$ ($(a, m, u) \in I_{\ell+}'$) for $(i, u) = g'((a, m, u))$, respectively. See Figures 5–6.

§4.4. T-system and cluster algebra

The result in this subsection is completely parallel to the $B_r$ and $C_r$ cases.

**Lemma 4.3.** The family $\{x_m^{(a)}(u) \mid (a, m, u) \in I_{\ell+}\}$ satisfies the system of relations (3.16) with $G(b, k, v; a, m, u)$ for $T_\ell(F_4)$. In particular, the family $\{x_m^{(a)}(u)_1 \mid (a, m, u) \in I_{\ell+}\}$ satisfies the T-system $T_\ell(F_4)$ in $A(B, x)$ after replacing $T_m^{(a)}(u)$ with $[x_m^{(a)}(u)]_1$. 
The T-subalgebra \( \mathcal{A}_T(B, x) \) is defined as in Definition 3.5.

**Theorem 4.4.** The ring \( \mathcal{T}_0^2(F_4) \) is isomorphic to \( \mathcal{A}_T(B, x) \) via the correspondence \( T_m(a)(u) \mapsto [y_m(a)(u)]_1 \).

### §4.5. Y-system and cluster algebra

The result in this subsection is completely parallel to the \( B_t \) and \( C_t \) cases.

**Lemma 4.5.** The family \( \{ y_m(a)(u) \mid (a, m, u) \in \mathcal{T}_t^2 \} \) satisfies the Y-system \( \mathcal{Y}_t(F_4) \) after replacing \( Y_m(a)(u) \) with \( y_m(a)(u) \).

The Y-subgroup \( \mathcal{G}_Y(B, y) \) is defined as in Definition 3.8.

**Theorem 4.6.** The group \( \mathcal{Y}_t^2(F_4) \) is isomorphic to \( \mathcal{G}_Y(B, y) \) via the correspondence \( Y_m(a)(u) \mapsto y_m(a)(u) \) and \( 1 + Y_m(a)(u) \mapsto 1 + y_m(a)(u) \).

### §4.6. Tropical Y-system at level 2

By direct computations, the following properties are verified.

**Proposition 4.7.** For \( \mathcal{G}_Y(B, y) \) with \( B = B_2(F_4) \), the following facts hold:

(i) Let \( 0 \leq u < 2 \). For any \( (i, u) : p_+ \), the monomial \( [y_i(u)]_T \) is positive.

(ii) Let \( -h^i \leq u < 0 \).

(a) Let \( i = (1, 1), (2, 1), (5, 1), (6, 1), (3, 2), \) or \( (4, 2) \). For any \( (i, u) : p_+ \), the monomial \( [y_i(u)]_T \) is negative.

(b) Let \( i = (3, 1), (3, 3), (4, 1), \) or \( (4, 3) \). For any \( (i, u) : p_+ \), the monomial \( [y_i(u)]_T \) is negative for \( u = -\frac{1}{2}, -1, -\frac{3}{2}, -3, -\frac{5}{2}, -5, -\frac{11}{2}, -6, -\frac{13}{2}, -8, -\frac{17}{2}, -9 \) and positive for \( u = -2, -\frac{5}{2}, -\frac{9}{2}, -5, -7, -\frac{15}{2} \).

(iii) \( y_{ii'}(-2) \) equals \( y_{ii'}^{-1} \) if \( i = 1, 2, 5, 6 \), and \( y_{ii'}^{-1} \) if \( i = 3, 4 \).

(iv) \( y_{ii'}(-h^i) \) equals \( y_{ii'}^{-1} \) if \( i = 1, 2, 5, 6 \), and \( y_{ii'} \) if \( i = 3, 4 \).

Also we have a description of the ‘core part’ of \( [y_i(u)]_T \) for \( -h^i \leq u < 0 \), corresponding to the \( D \) part for \( C_t \), in terms of the root system of \( E_6 \). We use the following indexing of the Dynkin diagram \( E_6 \):

\[
\begin{array}{cccccc}
1 & 2 & 3 & 4 & 5 & 6
\end{array}
\]
Let $\Pi = \{\alpha_1, \ldots, \alpha_6\}$, $-\Pi$, $\Phi_+$ be the sets of simple roots, of negative simple roots, and of positive roots, respectively, of type $E_6$. Let $\sigma_i$ be the piecewise-linear analogue of the simple reflection $s_i$, acting on the set $\Phi_{\geq -1} = \Phi_+ \sqcup (-\Pi)$ of almost positive roots. We write $\sum m_i \alpha_i \in \Phi_+$ as $[1^{m_1}, 2^{m_2}, \ldots, 6^{m_6}]$; furthermore, $[1^0, 2^1, 3^1, 4^1, 5^0, 6^0]$, for example, is abbreviated as $[2, 3, 4]$.

We define

$$\sigma = \sigma_3(\sigma_4 \sigma_2 \sigma_6) \sigma_3(\sigma_4 \sigma_1 \sigma_5).$$

**Lemma 4.8.** The orbits under $\sigma$ are:

(4.11)

$$-\alpha_1 \to [1, 2, 3] \to [2, 3, 4, 5, 6] \to [1, 2, 3^2, 4, 5] \to [5, 6] \to -\alpha_6,$$

$$-\alpha_2 \to [2, 3] \to [1, 2^2, 3^2, 4, 5, 6] \to [1, 2^2, 3^3, 4^2, 5^2, 6]$$

$$\to [1, 2, 3^2, 4, 5^2, 6] \to [5] \to -\alpha_5,$$

$$-\alpha_3 \to [2, 3, 4] \to [1, 2, 3^2, 4, 5, 6] \to [2, 3^2, 4^2, 5^2, 6] \to [1, 2, 3, 5] \to -\alpha_3,$$

$$\alpha_3 \to [2, 3, 5, 6] \to [1, 2^2, 3^2, 4, 5] \to [1, 2, 3, 4, 5, 6] \to [3, 4, 5] \to \alpha_3,$$

$$-\alpha_4 \to [2] \to [1, 2, 3, 4] \to [3, 4, 5, 6] \to [3, 5] \to -\alpha_4,$$

$$\alpha_4 \to [3, 4] \to [3, 5, 6] \to [2, 3, 5] \to [1, 2] \to \alpha_4,$$

$$-\alpha_5 \to [2, 3^2, 4, 5, 6] \to [1, 2^2, 3^2, 4, 5^2, 6] \to [1, 2^2, 3^2, 4, 5, 6]$$

$$\to [1, 2, 3, 4, 5] \to -\alpha_2,$$

$$-\alpha_6 \to [6] \to [2, 3^2, 4, 5] \to [1, 2, 3, 5, 6] \to [2, 3, 4, 5] \to [1] \to -\alpha_1.$$ 

In particular, these elements of $\Phi_+$ exhaust the set $\Phi_+$, thereby providing the orbit decomposition of $\Phi_+$ under $\sigma$.

For $-h^\vee \leq u < 0$, define

(4.12)  

$$\alpha_i(u) = \begin{cases} 
\sigma^{-u/2}(-\alpha_i), & i = 1, 4, 5; u \equiv 0, \\
\sigma^{-(u-1)/2}(-\alpha_i), & i = 2, 6; u \equiv -1, \\
\sigma^{-(u-1)/2}(\alpha_4), & i = 4; u \equiv -1, \\
\sigma^{-(2u-1)/4}(-\alpha_3), & i = 3; u \equiv -\frac{3}{2}, \\
\sigma^{-(2u+1)/4}(\alpha_3), & i = 3; u \equiv -\frac{1}{2},
\end{cases}$$

where $\equiv$ is equivalence mod $2\mathbb{Z}$. By Lemma 4.8 they are (all the) positive roots of $E_6$.

For a monomial $m$ in $y = (y_i)_{i \in \mathbb{Z}}$, let $\pi_A(m)$ denote the specialization with $y_{31} = y_{33} = y_{41} = y_{43} = 1$. For simplicity, we set $y_{i1} = y_i$ ($i = 1, 2, 5, 6$), $y_{i2} = y_i$ ($i = 3, 4$), and also $y_{i1}(u) = y_i(u)$ ($i = 1, 2, 5, 6$), $y_{i2}(u) = y_i(u)$ ($i = 3, 4$). We
define the vectors \( t_i(u) = (t_i(u)_k)_{k=1}^6 \) by
\[
(4.13) \quad \pi_A([y_i(u)]_T) = \prod_{k=1}^6 g_k^{t_i(u)_k}.
\]
We also identify each vector \( t_i(u) \) with \( \alpha = \sum_{k=1}^6 t_i(u)_k \alpha_k \in \mathbb{Z}^2 \).

**Proposition 4.9.** Let \(-h^\vee \leq u < 0\). Then
\[
(4.14) \quad t_i(u) = -\alpha_i(u)
\]
for \((i, u)\) as in (4.12).

§4.7. Tropical Y-systems at higher levels

Due to the factorization property, we obtain the following.

**Proposition 4.10.** Let \( \ell > 2 \) be an integer. For \([G_Y(B, y)]_T\) with \( B = B_\ell(F_4) \), the following facts hold.

(i) Let \( 0 \leq u < \ell \). For any \((i, u) : p_+\), the monomial \([y_i(u)]_T\) is positive.

(ii) Let \(-h^\vee \leq u < 0\).

(a) Let \( i \in \mathbf{I}^o \) or \( i = (3, i'), (4, i') \ (i' \in 2\mathbb{N}) \). For any \((i, u) : p_+\), the monomial \([y_i(u)]_T\) is negative.

(b) Let \( i = (3, i'), (4, i') \ (i' \not\in 2\mathbb{N}) \). For any \((i, u) : p_+\), the monomial \([y_i(u)]_T\) is negative for \( u = -\frac{1}{2}, -2, -\frac{9}{2}, -5, -7, -\frac{15}{2}, -\frac{13}{2} \) and positive for \( u = -2, -\frac{5}{2}, -\frac{9}{2}, -5, -7, \frac{15}{2} \).

(iii) \( y_{i'}(\ell) \) equals \( y_{i'}^{1 \cdots 1 \ell-i'} \) if \( i = 1, 2, 5, 6 \), and \( y_{i'}^{1 \cdots 2\ell-i'} \) if \( i = 3, 4 \).

(iv) \( y_{i'}(-h^\vee) \) equals \( y_{i'}^{1 \cdots 1 \ell-i'} \) if \( i = 1, 2, 5, 6 \), and \( y_{i'}^{1 \cdots 1} \) if \( i = 3, 4 \).

We obtain corollaries of Propositions 4.7 and 4.10.

**Theorem 4.11.** For \([G_Y(B, y)]_T\) the following relations hold:

(i) Half periodicity: \([y_i(u + h^\vee + \ell)]_T = [y_{\omega(i)}(u)]_T\).

(ii) Full periodicity: \([y_i(u + 2(h^\vee + \ell))]_T = [y_i(u)]_T\).

**Theorem 4.12.** For \([G_Y(B, y)]_T\), let \( N_+ \) and \( N_- \) denote the total numbers of the positive and negative monomials, respectively, among \([y_i(u)]_T\) for \((i, u) : p_+\) in the region \( 0 \leq u < 2(h^\vee + \ell) \). Then
\[
(4.15) \quad N_+ = 4\ell(3\ell + 1), \quad N_- = 24(4\ell - 3).
\]
§4.8. Periodicities and dilogarithm identities

Applying [IIKKN, Theorem 5.1] to Theorem 4.11, we obtain the periodicities:

**Theorem 4.13.** For \( \mathcal{A}(B,x,y) \), the following relations hold:

(i) **Half periodicity:** \( x_i(u + h^\vee + \ell) = x_{\omega(1)}(u) \).

(ii) **Full periodicity:** \( x_i(u + 2(h^\vee + \ell)) = x_i(u) \).

**Theorem 4.14.** For \( \mathcal{G}(B,y) \), the following relations hold:

(i) **Half periodicity:** \( y_i(u + h^\vee + \ell) = y_{\omega(1)}(u) \).

(ii) **Full periodicity:** \( y_i(u + 2(h^\vee + \ell)) = y_i(u) \).

Theorems 2.5 and 2.6 for \( F_4 \) follow from Theorems 4.4, 4.6, 4.13, and 4.14. Furthermore, Theorem 2.10 for \( F_4 \) is obtained from the above periodicities and Theorem 4.12 as in the \( B_r \) case [IIKKN, Section 6].

§5. Type \( G_2 \)

The \( G_2 \) case is mostly parallel to the former cases, but slightly different because the number \( t \) in (2.2) is three. Again, the properties of the tropical Y-system at level 2 (Proposition 5.9) are crucial and specific to \( G_2 \).

§5.1. Parity decompositions of T-systems and Y-systems

For a triplet \((a,m,u)\) \( \in I_\ell \), we reset the parity conditions \( P_+ \) and \( P_- \) to be

\[
\begin{align*}
P_+ &: a + m + 3u \text{ is even}, \\
P_- &: a + m + 3u \text{ is odd}.
\end{align*}
\]

Then we have \( T^+_\ell(G_2)_+ \simeq T^+_\ell(G_2)_- \) via \( T^{(a)}_m(u) \mapsto T^{(a)}_m(u + \frac{1}{3}) \) and

\[
T^+_\ell(G_2)_+ \simeq T^+_\ell(G_2)_- \otimes \mathbb{Z} T^+_\ell(G_2)_-.
\]

For \((a,m,u)\) \( \in I_\ell \), we reset the parity conditions \( P'_+ \) and \( P'_- \) to be

\[
\begin{align*}
P'_+ &: a + m + 3u \text{ is odd}, \\
P'_- &: a + m + 3u \text{ is even}.
\end{align*}
\]

We have

\[
(a,m,u) : P'_+ \leftrightarrow (a,m,u \pm \frac{1}{3}) : P_+.
\]
Also, we have $Y^\circ(G_2)_+ \simeq Y^\circ(G_2)_-$ via $Y_m^{(a)}(u) \mapsto Y_m^{(a)}(u + \frac{1}{3})$, $1 + Y_m^{(a)}(u) \mapsto 1 + Y_m^{(a)}(u + \frac{1}{3})$, and

$$Y^\circ(G_2) \simeq Y^\circ(G_2)_+ \times Y^\circ(G_2)_-.$$  

§5.2. Quiver $Q_{l}(G_2)$

With type $G_2$ and $\ell \geq 2$ we associate the quiver $Q_{l}(G_2)$ by Figure 7, where the right columns in the three quivers are identified. Also we assign an empty or filled
circle $\circ/\bullet$ to each vertex; furthermore, we assign a sign $+/-$ to each vertex with $\bullet$, and one of the numbers $1, \ldots, 6$ to each vertex with $\circ$.

Let us choose the index set $I$ of the vertices of $Q_2(G_2)$ so that $i = (i, i') \in I$ represents the vertex in the $i$th row (from the bottom) and in the left column in the $i$th quiver (from the left) for $i = 1, 2, 3$, and in the right column in any quiver for $i = 4$. Thus, $i = 1, \ldots, 4$, and $i' = 1, \ldots, \ell - 1$ if $i \neq 4$, while $i' = 1, \ldots, 3\ell - 1$ if $i = 4$.

For a permutation $s$ of $\{1, 2, 3\}$, let $\nu_s$ be the permutation of $I$ such that $\nu_s(i, i')$ equals $(s(i), s(i'))$ for $i \neq 4$, and $(4, i')$ for $i = 4$. Let $\omega$ be the involution acting on $I$ by up-down reflection. Let $\nu_s(Q_2(G_2))$ and $\omega(Q_2(G_2))$ denote the quivers induced from $Q_2(G_2)$ by $\nu_s$ and $\omega$, respectively.

**Lemma 5.1.** Let $Q = Q_2(G_2)$.

(i) We have a periodic sequence of mutations of quivers

\begin{align*}
Q \overset{\nu_{(23)}^\ast\mu_{(23)}}{\longrightarrow} \nu_{(23)}(Q)^{\text{op}} \overset{\mu_{(312)}^\ast\nu_{(312)}}{\longrightarrow} \nu_{(13)}(Q)^{\text{op}} \overset{\nu_{(12)}(312)}{\longrightarrow} \nu_{(12)}(Q)^{\text{op}} \overset{\nu_{(13)}(12)}{\longrightarrow} Q.
\end{align*}

(ii) $\omega(Q) = Q$ if $h^\nu + \ell$ is even, and $\omega(Q) = \nu_{(13)}(Q)^{\text{op}}$ if $h^\nu + \ell$ is odd.

See Figures 8–10 for an example.

§5.3. Cluster algebra and alternative labels

Let $B_2(G_2)$ be the skew-symmetric matrix corresponding to the quiver $Q_2(G_2)$. In the rest of the section, we set $B = (B_i)_{i \in I} = B_2(G_2)$ unless otherwise mentioned.

Let $A(B, x, y)$ be the cluster algebra with coefficients in the universal semifield $\mathbb{Q}(x, y)$, and $G(B, y)$ be the coefficient group associated with $A(B, x, y)$.

In view of Lemma 5.1 we set $x(0) = x$, $y(0) = y$ and define clusters $x(u) = (x_i(u))_{i \in I}$ and coefficient tuples $y(u) = (y_i(u))_{i \in I}$ ($u \in \frac{1}{\ell} \mathbb{Z}$) by the sequence of mutations

\begin{align*}
\cdots \overset{\nu_{(23)}^\ast\mu_{(23)}}{\longrightarrow} (B, x(0), y(0)) \overset{\nu_{(23)}^\ast\mu_{(23)}}{\longrightarrow} (-\nu_{(23)}(B), x(\frac{1}{2}), y(\frac{1}{2})) \overset{\nu_{(312)}^\ast\nu_{(312)}}{\longrightarrow} (-\nu_{(13)}(B), x(1), y(1)) \overset{\nu_{(13)}^\ast\nu_{(13)}}{\longrightarrow} (-\nu_{(12)}(B), x(\frac{5}{2}), y(\frac{5}{2})) \overset{\nu_{(12)}^\ast\nu_{(12)}}{\longrightarrow} \cdots ,
\end{align*}

where $\nu_s(B) = B'$ is defined by $B'_{\nu_s(i)}(y) = B_i(y)$.
Figure 8. (Continued in Figures 9 and 10.) Labeling of cluster variables $x_i(u)$ by $I_\ell$ for $G_2$, $\ell = 4$. The right columns in the middle and right quivers (marked by $\diamond$) are identified with the right column in the left quiver.
Figure 9. Continuation of Figure 8.
Figure 10. Continuation of Figure 9.
Lemma 5.2. Below \(\equiv\) means equivalence modulo \(2\mathbb{Z}\). We have
\[
(5.11) \quad (i, u) : p_+ \iff (i, u + \frac{1}{3}) : p_-.
\]
Each \((i, u) : p_+\) is a mutation point of (5.9) in the forward direction of \(u\), and each \((i, u) : p_-\) is one in the backward direction of \(u\).

Lemma 5.2. Below \(\equiv\) means equivalence modulo \(2\mathbb{Z}\).

(i) The map \(g : \mathcal{I}_f \to \{ (i, u) : p_+ \}\) defined by
\[
(5.12) \quad (a, m, u - \frac{1}{3}) \mapsto \begin{cases} ((1, m), u), & a = 1; m + u \equiv 0, \\ ((2, m), u), & a = 1; m + u \equiv \frac{4}{3}, \\ ((3, m), u), & a = 1; m + u \equiv \frac{2}{3}, \\ ((4, m), u), & a = 2, \end{cases}
\]
is a bijection.

(ii) The map \(g' : \mathcal{I}_f' \to \{ (i, u) : p_+ \}\) defined by
\[
(5.13) \quad (a, m, u) \mapsto \begin{cases} ((1, m), u), & a = 1; m + u \equiv 0, \\ ((2, m), u), & a = 1; m + u \equiv \frac{4}{3}, \\ ((3, m), u), & a = 1; m + u \equiv \frac{2}{3}, \\ ((4, m), u), & a = 2, \end{cases}
\]
is a bijection.

We introduce alternative labels \(x_1(u) = x_{1(a)}^<(u - 1/t_a) \) \(((a, m, u - 1/t_a) \in \mathcal{I}_f)\) for \((i, u) = g'((a, m, u))\) and \(y_1(u) = y_{1(a)}^<(u) \) \(((a, m, u) \in \mathcal{I}_f')\) for \((i, u) = g'((a, m, u))\), respectively. See Figures 8–10 for an example.

§5.4. T-system and cluster algebra

Lemma 5.3. The family \(\{ x_{1(a)}^<(u) \mid (a, m, u) \in \mathcal{I}_f \}\) satisfies the system of relations (3.16) with \(G(b, k, v; a, m, u)\) for \(T_f(G_2)\). In particular, the family
\{ [x_m^{(a)}(u)]_1 \mid (a, m, u) \in I_{t+} \} \) satisfies the T-system \( T_t(G_2) \) in \( A(B, x) \) after replacing \( T_m^{(a)}(u) \) with \( [x_m^{(a)}(u)]_1 \).

**Definition 5.4.** The \( T \)-subalgebra \( A_T(B, x) \) of \( A(B, x) \) associated with the sequence \((5.9)\) is the subring of \( A(B, x) \) generated by \([x_1(u)]_1 \) \((i, u) \in I \times \frac{1}{5} \mathbb{Z})\).

**Theorem 5.5.** The ring \( T_T(G_2) \) is isomorphic to \( A_T(B, x) \) via the correspondence \( T_m^{(a)}(u) \to [x_m^{(a)}(u)]_1 \).

### §5.5. Y-system and cluster algebra

**Lemma 5.6.** The family \( \{ y_m^{(a)}(u) \mid (a, m, u) \in I_{t+} \} \) satisfies the Y-system \( Y_t(G_2) \) after replacing \( Y_m^{(a)}(u) \) with \( y_m^{(a)}(u) \).

**Definition 5.7.** The \( Y \)-subgroup \( \mathcal{G}_Y(B, y) \) of \( \mathcal{G}(B, y) \) associated with the sequence \((5.9)\) is the subgroup of \( \mathcal{G}(B, y) \) generated by \( y_1(u) \) \((i, u) \in I \times \frac{1}{5} \mathbb{Z})\) and \( 1 + y_1(u) \) \((i, u) : p_+ \) or \( p_- \).

**Theorem 5.8.** The group \( \mathcal{Y}_T(G_2) \) is isomorphic to \( \mathcal{G}_Y(B, y) \) via the correspondence \( Y_m^{(a)}(u) \to y_m^{(a)}(u) \) and \( 1 + Y_m^{(a)}(u) \to 1 + y_m^{(a)}(u) \).

### §5.6. Tropical Y-system at level 2

By direct computations, the following properties are verified.

**Proposition 5.9.** For \( \mathcal{G}_T(B, y) \) with \( B = B_2(G_2) \), the following facts hold:

(i) Let \( 0 \leq u < 2 \). For any \((i, u) : p_+ \), the monomial \( y_{1}(u)u \) is positive.

(ii) Let \( -h^r \leq u < 0 \).

(a) Let \( i = (1, 1), (2, 1), (3, 1), \) or \((4, 3)\). For any \((i, u) : p_+ \), the monomial \( y_{1}(u)u \) is negative.

(b) Let \( i = (4, 1), (4, 2), (4, 4), \) or \((4, 5)\). For any \((i, u) : p_+ \), the monomial \( y_{1}(u)u \) is negative for \( u = -1, -\frac{4}{5}, -\frac{2}{5}, -\frac{2}{3}, -2, -\frac{4}{3}, -\frac{11}{4}, -4 \), and positive for \( u = -1, -\frac{4}{5}, -\frac{2}{5}, -\frac{2}{3}, -3, -\frac{10}{7} \).

(iii) \( y_{u^r}(2) \) equals \( y_{u^r}^{-1} \) if \( i \neq 4 \), and \( y_{4,6-i^r}^{-1} \) if \( i = 4 \).

(iv) \( y_{u^r}(-h^r) = y_{u^r}^{-1} \).

Also we have a description of the core part of \( [y_1(u)]_T \) in the region \( -h^r \leq u < 0 \) in terms of the root system of \( D_4 \). We use the following indexing of the
Dynkin diagram $D_4$:

```
1 --- 2
   |   |
   3
```

Let $\Pi = \{\alpha_1, \ldots, \alpha_4\}$, $-\Pi$, $\Phi_+$ be the sets of simple roots, of negative simple roots, and of positive roots, respectively, of type $D_4$. Let $\sigma_i$ be the piecewise-linear analogue of the simple reflection $s_i$, acting on the set $\Phi_{\geq -1} = \Phi_+ \cup (-\Pi)$ of almost positive roots. We define

$$\sigma = \sigma_3 \sigma_4 \sigma_1 \sigma_2 \sigma_4.$$  

**Lemma 5.10.** The orbits under $\sigma$ are:

- $-\alpha_1 \to \alpha_1 + \alpha_3 + \alpha_4 \to \alpha_1 + \alpha_2 + \alpha_4 \to -\alpha_1$,
- $-\alpha_2 \to \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 \to \alpha_2 + \alpha_4 \to -\alpha_2$,
- $-\alpha_3 \to \alpha_3 \to \alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 \to -\alpha_3$,
- $-\alpha_4 \to \alpha_2 + \alpha_3 + \alpha_4 \to \alpha_4 \to \alpha_1 \to \alpha_2 \to \alpha_3 + \alpha_4 \to \alpha_1 + \alpha_4 \to -\alpha_4$.

In particular, these elements of $\Phi_+$ exhaust the set $\Phi_+$, thereby providing the orbit decomposition of $\Phi_+$ under $\sigma$.

For $-h' \leq u < 0$, define

$$\alpha_i(u) = \begin{cases} 
\sigma^{-(u-1)/2}(-\alpha_1), & i = 1; u = -1, -3, \\
\sigma^{-(3u-1)/6}(-\alpha_2), & i = 2; u = \frac{5}{3}, \frac{11}{3}, \\
\sigma^{-(3u-5)/6}(-\alpha_3), & i = 3; u = \frac{1}{3}, -\frac{2}{3}, \\
\sigma^{-(3u+2)/6}(\alpha_3 + \alpha_4), & i = 4; u = \frac{2}{3}, -\frac{8}{3}, \\
\sigma^{-(3u+4)/6}(\alpha_1), & i = 4; u = \frac{4}{3}, -\frac{10}{3}, \\
\sigma^{-u/2}(-\alpha_4), & i = 4; u = -2, -4.
\end{cases}$$

By Lemma 5.10 these are (all the) positive roots of $D_4$.

For a monomial $m$ in $y = (y_i)_{i \in I}$, let $\pi_D(m)$ denote the specialization with $y_{41} = y_{42} = y_{44} = y_{45} = 1$. For simplicity, we set $y_{i1} = y_i$ ($i \neq 4$), $y_{43} = y_4$, and also $y_{i1}(u) = y_i(u)$ ($i \neq 4$), $y_{43}(u) = y_4(u)$. We define the vectors $t_i(u) = (t_i(u)_k)_{k=1}^4$ by

$$\pi_D([y_i(u)]_T) = \prod_{k=1}^4 y_k^{t_i(u)_k}.$$
We also identify each vector $\mathbf{t}_i(u)$ with $\alpha = \sum_{k=1}^{4} t_i(u) \alpha_k \in \mathbb{Z}$.

**Proposition 5.11.** Let $-h^\vee \leq u < 0$. Then

\begin{equation}
(5.18) \quad \mathbf{t}_i(u) = -\alpha_i(u)
\end{equation}

for $(i, u)$ as in (5.16).

§5.7. Tropical Y-systems of higher levels

**Proposition 5.12.** Let $\ell > 2$ be an integer. For $[\mathcal{G}_Y(B, y)]_T$ with $B = B_\ell(G_2)$, the following facts hold:

(i) Let $0 \leq u < \ell$. For any $(i, u) : \mathbf{p}_+$, the monomial $[y_i(u)]_T$ is positive.

(ii) Let $-h^\vee \leq u < 0$.

(a) Let $i \in I$ or $i = (4, i')$ ($i' \in 3\mathbb{N}$). For any $(i, u) : \mathbf{p}_+$, the monomial $[y_i(u)]_T$ is negative.

(b) Let $i = (4, i')$ ($i' \notin 3\mathbb{N}$). For any $(i, u) : \mathbf{p}_+$, the monomial $[y_i(u)]_T$ is negative for $u = -\frac{1}{4}, -\frac{2}{3}, -2, -\frac{7}{3}, -4$ and positive for $u = -1, -\frac{1}{3}, -\frac{5}{3}$.

(iii) $y_{i,i'}(\ell)$ equals $y_{i,i'-i}$ if $i \neq 4$, and $y_{4,3e-i}$ if $i = 4$.

(iv) $y_{i,i'}(-h^\vee) = y_{i,i'}^{-1}$.

The following are corollaries of Propositions 5.9 and 5.12.

**Theorem 5.13.** For $[\mathcal{G}_Y(B, y)]_T$, the following relations hold:

(i) Half periodicity: $[y_i(u + h^\vee + \ell)]_T = [y_{\omega(i)}(u)]_T$.

(ii) Full periodicity: $[y_i(u + 2(h^\vee + \ell))]_T = [y_i(u)]_T$.

**Theorem 5.14.** For $[\mathcal{G}_Y(B, y)]_T$, let $N_+$ and $N_-$ denote the total numbers of the positive and negative monomials, respectively, among $[y_i(u)]_T$ for $(i, u) : \mathbf{p}_+$ in the region $0 \leq u < 2(h^\vee + \ell)$. Then

\begin{equation}
(5.19) \quad N_+ = 6\ell(2\ell + 1), \quad N_- = 12(3\ell - 2).
\end{equation}

§5.8. Periodicities and dilogarithm identities

Applying [IIKKN, Theorem 5.1] to Theorem 5.13, we obtain the periodicities:

**Theorem 5.15.** For $\mathcal{A}(B, x, y)$, the following relations hold:

(i) Half periodicity: $x_i(u + h^\vee + \ell) = x_{\omega(i)}(u)$.

(ii) Full periodicity: $x_i(u + 2(h^\vee + \ell)) = x_i(u)$. 
Theorem 5.16. For \( G(B, y) \), the following relations hold:

(i) **Half periodicity:** \( y_i(u + h^\vee + \ell) = y_{\omega(B)}(u) \).

(ii) **Full periodicity:** \( y_i(u + 2(h^\vee + \ell)) = y_i(u) \).

Theorems 2.5 and 2.6 for \( G_2 \) follow from Theorems 5.5, 5.8, 5.15, and 5.16. Furthermore, Theorem 2.10 for \( G_2 \) is obtained from the above periodicities and Theorem 5.14 as in the \( B_r \) case [IIKKN, Section 6].

§6. Mutation equivalence of quivers

Recall that two quivers \( Q \) and \( Q' \) are said to be **mutation equivalent**, denoted \( Q \sim Q' \) here, if there is a quiver isomorphism from \( Q \) to some quiver obtained from \( Q' \) by successive mutations.

Below we present several mutation equivalent pairs of quivers \( Q_\ell(X_r) \), though the list is not complete at all. For simply laced \( X_r \), \( Q_\ell(X_r) \) is the quiver defined as the square product \( \tilde{X_r} \square \tilde{A_{\ell-1}} \) in [Ke, Section 8].

**Proposition 6.1.** We have the following mutation equivalences of quivers:

\[
\begin{align*}
Q_2(B_r) & \sim Q_2(D_{2r+1}), \\
Q_3(C_3) & \sim Q_3(D_4), \\
Q_2(F_4) & \sim Q_4(D_5), \\
Q_3(C_2) & \sim Q_4(A_3), \\
Q_\ell(G_2) & \sim Q_\ell(C_3).
\end{align*}
\]

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