LONG-TIME DYNAMICS FOR A NON-AUTONOMOUS NAVIER-STOKES-VOIGT EQUATION IN LIPSCHITZ DOMAINS

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ABSTRACT. This article focuses on the optimal regularity and long-time dynamics of solutions of a Navier-Stokes-Voigt equation with non-autonomous body forces in non-smooth domains. Optimal regularity is considered, since the regularity $H^1_0 \cap H^2$ cannot be achieved. Given the initial data in certain spaces, it can be shown that the problem generates a well-defined evolutionary process. Then we prove the existence of a uniform attractor consisting of complete trajectories.

1. Introduction. Navier-Stokes equations (NS for short) are well known and effectively useful in modeling turbulence in fluid phenomena. The mathematical theory on this model, for instance, the well-posedness of strong solutions, has been developed from theoretical and numerical aspects for over 80 years. The analysis of the infinite dimensional dynamical systems from the incompressible Navier-Stokes equations attracts attention as well within physical and mathematical theory. We refer readers to [9, 19, 20, 21, 22].

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An approximating version, the Navier-Stokes-Voigt (or Navier-Stokes-Voight, NSV for short) model, is a modification of the Navier-Stokes equation by adding pseudoparabolic regularization \(-\alpha^2 \Delta u_t\) for velocity field \(u\). This approximation methodology modifies the parabolic property of the equation, and is a type of inviscid regularization. The semigroups generated by NSV are asymptotically compact, hence the model is close to damped hyperbolic type. This regularization is successful in numerical approximations for ocean models, NS, etc. (see, e.g., [2, 10]). This approximation strategy was firstly proposed by Voigt [23] for the NS equations of the Kelvin-Voigt viscoelastic incompressible fluid, and later was studied by Oskolkov [17]. The well-posedness result of solutions for 3D autonomous NSV was provided in his work as well. The semigroups generated by solutions, as well as their global finite dimensional attractors, were obtained (see [5, 6]). The fractional and Hausdorff dimensions of the global attractors were estimated in [8]. We refer readers to [3, 7, 8, 13, 18] for more discussion on autonomous/non-autonomous versions of models considered in bounded/unbounded domains. All these results for NSV were obtained under homogeneous boundary value conditions, such as the Dirichlet problem; however, in this paper, we will consider the 2D NSV prescribed with the non-homogeneous boundary condition, particularly in non-smooth domains.

The well-posedness of global solutions on non-smooth domains with corners or points of singularities were discussed by Brown, Perry, and Shen [1]. They defined a background function in Lipschitz domains for an auxiliary 2D Stokes problem (known as the background flow), and extended the result of non-homogeneous boundary case by Miranville and Wang [15]. [1] established the existence and estimated the finite Hausdorff and fractional dimension of the global attractor of the 2D autonomous incompressible NS equation; Wu and Zhong used this technique to seek the uniform attractors for 2D non-autonomous NS equation (see [24]).

To prove the existence of attractors, the Condition-(C) framework (which is also called Condition-(MWZ)) is used in our paper. The strategy is to find a mode \(m\) such that the solution trajectory gets close to a subspace of an infinite dimensional Banach space \(X\) with basis \(w_1, w_2, \ldots, \text{span}\{w_1, w_2, \ldots, w_m\} = X_1\), thus it proves the asymptotic compactness of semigroup \(S(t)\). That is, for any \(\varepsilon > 0\), \(\| (I - P) S(t) u_0 \|_X \leq \varepsilon\) as the time \(t\) is large enough for any initial data \(u_0\) (\(P\) is the projector operator: \(X \mapsto X_1\)). This strategy is derived by Ma, Wang, and Zhong [14].

The nonhomogeneous boundary value problem for 2D non-autonomous incompressible Navier-Stokes-Voigt equation, in a bounded Lipschitz domain \(\Omega \subset \mathbb{R}^2\), reads

\[
\begin{aligned}
  u_t - \alpha^2 \Delta u_t - \nu \Delta u + (u \cdot \nabla) u + \nabla p &= f(x, t), \quad (x, t) \in \Omega_T, \\
  \text{div} u &= 0, \quad (x, t) \in \Omega_T, \\
  u(t, x)|_{\partial \Omega} &= \varphi, \quad \varphi \cdot n = 0, \quad (x, t) \in \partial \Omega_T, \\
  u(x, t) &= u_t(x), \quad x \in \Omega, 
\end{aligned}
\]  

(1.1)

where \(n\) is the 2D outward unit normal vector of \(\partial \Omega\), \(\Omega_T := \Omega \times (\tau, +\infty) := \Omega \times \mathbb{R}_+, \partial \Omega_T := \partial \Omega \times (\tau, +\infty), \tau \in \mathbb{R}\) is the initial time, \(\nu\) is the kinematics viscosity of the fluid, \(u = (u_1(t, x), u_2(t, x))\) is the unknown velocity vector, \(p\) is the pressure, the regularizing constant \(\alpha > 0\) is a length scale parameter characterizing the elasticity of the fluid, the function \(\varphi \in L^\infty(\partial \Omega)\) is a time independent function, and the external force \(f(x, t) \in L^2_0(\mathbb{R}; H) \subset L^2_{loc}(\mathbb{R}; E)\), where \(E = D(A^{\frac{\alpha}{2}}) (\alpha = -1 \text{ or } -2 \text{ or } 0)\), which satisfies
\[ \|f\|_{L^2_t} = \|f\|_{L^2_t(\mathbb{R}; E)} = \sup_{t \in \mathbb{R}} \int_t^{t+1} \|f(s)\|^2_E ds < +\infty. \tag{1.2} \]

The space \( L^2_t(\mathbb{R}; E) \) denotes the space of the translation bounded functions.

In this paper, we will focus on the existence of the uniform attractor of (1.1) in light of [1, 24]. The main features of our work are summarized as follows.

1. Using the background function for the Stokes problem in [1], if the external force \( f(t, x) \) is only translation bounded, we prove the existence of global unique solution and its continuous dependence on initial data in \( V \) (a function space specified in Preliminaries) for the non-autonomous Navier-Stokes-Voigt equations with nonhomogeneous boundary values. [24] investigated the 2D non-autonomous Navier-Stokes equation but didn’t provide the detail proof here, especially the continuous dependence on the initial data of the solution.

2. For the problem (1.1) in smooth domains, the regularity can be improved to \( W = H^1_0 \cap H^2 \); however, this regularity argument is impossible for Lipschitz domains, in that higher regularity estimates for the terms \( (B(\psi, \psi), A^\alpha v) \) and \( (\nu F, A^\alpha v) \) can not be acquired in function spaces with higher regularity than \( D(A^{\sigma+\frac{1}{2}}) \) (\( A \) is the Stokes operator, \( F \) will be specified later in (3.11)). If \( \sigma \in [0, \frac{1}{2}] \), we are able to obtain the optimal regularity of solutions in \( D(A^{\sigma+\frac{1}{2}}) \) by using the Hardy’s inequality. This space is definitely less smooth than \( W \).

3. Under the continuity of the processes generated by solutions, the dissipation and the Uniform Condition-(C) are invoked to achieve uniformly asymptotic compactness of a process. By using the property of the Stokes operator, we establish the existence of uniform attractors in \( V \) and \( D(A^{\sigma+\frac{1}{2}}) \).

This paper is organized as follows: In Section 2, notations and function spaces are stipulated; we also define the background function and provide its related estimates in this part; In Section 3, the global existence of unique solution and the continuous dependence on initial data of our problem are developed; moreover, the optimal regularity in \( D(A^{\sigma+\frac{1}{2}}) \) for \( \sigma \in [0, \frac{1}{2}] \) is discussed as well. The existence of the uniform attractor for the process in \( V \) and \( D(A^{\sigma+\frac{1}{2}}) \) are concluded in Section 4.

2. Preliminaries. We shall consider the function space corresponding to our problem with an abstract setting

\[ E := \{u|u \in (C_0^\infty(\Omega))^2, \text{div}u = 0\}. \]

Function space \( H \) is the closure of \( E \) in \( (L^2(\Omega))^2 \) norm, \( \|\cdot\| \) and \( (\cdot, \cdot) \) denote the norm and inner product in \( H \) respectively, i.e.,

\[ (u, v) = \sum_{j=1}^2 \int_{\Omega} u_j(x)v_j(x)dx, \forall \ u, v \in (L^2(\Omega))^2, \]

\[ |u|^2 = (u, u), \forall \ u \in (L^2(\Omega))^2. \]

Function space \( V \) is the closure of \( E \) in \( (H^1(\Omega))^2 \) topology, and \( \|\cdot\| \) and \( (\cdot, \cdot) \) denote, respectively, the norm and inner product in \( V \), i.e.,

\[ ((u, v)) = \sum_{i, j=1}^2 \int_{\Omega} \frac{\partial u_j}{\partial x_i} \frac{\partial v_j}{\partial x_i} dx, \forall \ u, v \in (H^1_0(\Omega))^2, \]

\[ |u|^2 = ((u, u)), \forall \ u \in (H^1_0(\Omega))^2. \]

\( H' \) and \( V' \) are dual spaces of \( H \) and \( V \) respectively, the injections \( V \hookrightarrow H \equiv H' \hookrightarrow V' \) are dense and continuous. The notations \( \|\cdot\|_* \) and \( (\cdot, \cdot) \) denote the norm in \( V' \) and
and the dual product between $V$ and $V'$, respectively. In later sections, we might use $\cdot$ or $|\cdot|$ denoting $|\cdot|_{L^2}$ and $\|\cdot\|_H$ denoting the norm of space $H$. When the integral is outside the integrand, $|\cdot|$ exerted on the integrand denotes the absolute value.

$P$ is the Helmholtz-Leray orthogonal projection in $(L^2(\Omega))^2$ onto the divergence-free space $H$. We define

$$A := -P\Delta$$

as the Stokes operator, the sequence $\{\omega_j\}_{j=1}^\infty$ is a series of orthonormal eigenfunctions of $A$, and $\{\lambda_j\}_{j=1}^\infty$ ($0 < \lambda_1 \leq \lambda_2 \leq \cdots$) are eigenvalues of $A$ corresponding to the eigenfunctions $\{\omega_j\}_{j=1}^\infty$.

We define $A^s$ ($s \in \mathbb{C}$) as

$$A^s f = \sum_j \lambda_j^s a_j \omega_j, \quad s \in \mathbb{C}, \quad j \in \mathbb{R},$$

$$V_s = D(A^s) = \{g : A^s g \in H\} = \{g = \sum_j a_j \omega_j : \sum_j \lambda_j^{2ReZ} |a_j|^2 < +\infty\},$$

$D(A^s)$ denotes the domain of $A^s$ and the norm of $D(A^s)$ can be written as $\|\cdot\|_{V_s}$ with the inner product

$$(u, v)_{V_s} = (A^s u, A^s v), \quad \|u\|^2_{V_s} = (u, u)_{V_s},$$

especially, $V = V_1$.

The closure of $E$ in the topology $D(A)$ denote as $W$. Moreover, $A^s$ satisfies (see [1])

$$\int_\Omega \frac{|A^s u|^2}{\text{dist}(x, \partial \Omega)} \, dx \leq C_0 \int_\Omega |A^{s+\frac{1}{4}} u|^2 \, dx, \quad \forall u \in D(A^{s+\frac{1}{4}}),$$

$$\|u\|_{L^4} \leq C_1 |A^{\frac{1}{4}} u|, \quad \forall u \in D(A^{\frac{1}{4}}).$$

We define the bilinear and trilinear forms as follows (see [19, 21])

$$B(u, v) := P((u \cdot \nabla) v), \quad \forall u, v \in E,$n

$$b(u, v, w) = (B(u, v), w) = \sum_{i,j=1}^2 \int_\Omega u_i \frac{\partial v_j}{\partial x_i} w_j \, dx,$$

where $B(u, v)$ is a linear continuous form and $b(u, v, w)$ satisfies

$$\begin{cases}
    b(u, v, v) = 0, & \forall u, v \in V; \\
    b(u, v, w) = -b(u, w, v), & \forall u, v, w \in V; \\
    |b(u, v, w)| \leq C |u|^\frac{1}{2} \|v\|^\frac{1}{2} \|w\|^\frac{1}{2}, & \forall u, v, w \in V.
\end{cases}$$

We also need to use some useful inequalities: the Gagliardo-Nirenberg interpolation inequality

$$|A^{1/2} u|^2 \leq C_2 A^{1/4} u \|A^{3/4} u\|, \quad \forall u \in D(A^{3/4});$$

the Hardy’s inequality

$$\int_\Omega \frac{|u(x)|^2}{\text{dist}(x, \partial \Omega)^2} \, dx \leq C_3 \int_\Omega |\nabla u(x)|^2 \, dx, \quad \forall u \in V.$$
3. Existence of solutions, uniqueness, and continuity results. We transform (1.1) to a homogeneous boundary problem by $v = u - \psi$, here the background function $\psi$ need to satisfies

$$ \begin{cases} \text{div}\psi = 0, & \text{in } \Omega, \\ \psi = \varphi, & \text{on } \partial\Omega, \end{cases} \quad (3.1) $$

This transform was proposed by Miranville and Wang [15, 16]. The main idea is to localize the solution of the Stokes problem with boundary data $\psi$ to an $\varepsilon$-neighborhood of boundary $\partial\Omega$. It aims to reduce the problem into a homogeneous boundary one. Brown, Perry and Shen [1] extended this nonhomogeneous boundary problem for Lipschitz domains.

3.1. The background function for Stokes problem. Let $\varepsilon \in (0, c \cdot \text{diam}(\Omega))$ be a constant to be determined later, $\eta_\varepsilon \in C_0^\infty(\mathbb{R}^2)$ satisfies

$$ \begin{cases} \eta_\varepsilon = 1, & \text{in } \{ x \in \mathbb{R}^2; \text{dist}(x, \partial\Omega) \leq C_1'\varepsilon \}, \\ \eta_\varepsilon = 0, & \text{in } \{ x \in \mathbb{R}^2; \text{dist}(x, \partial\Omega) \geq C_2'\varepsilon \}, \\ 0 \leq \eta_\varepsilon \leq 1, & \text{otherwise}, \end{cases} \quad (3.2) $$

and

$$ |\nabla^\alpha \eta_\varepsilon| \leq C'_\varepsilon |\alpha|, \quad (3.3) $$

where $\eta_\varepsilon$ is defined as the form $h(\rho(x)/\varepsilon)$, $h$ is a standard bump function and $\rho \in C^\infty$ is a regularized distance bump function to $\partial\Omega$.

The background function $\psi := \psi_\varepsilon = (\partial_2 (g_\eta_\varepsilon), -\partial_1 (g_\eta_\varepsilon))$ which is obtained by solving the Stokes system (See [1])

$$ \begin{cases} -\Delta u + \nabla q = 0, & \text{in } \Omega, \\ \text{div } u = 0, & \text{in } \Omega, \\ u = \varphi, & \text{a.e. on } \partial\Omega \text{ in the sense of nontangential convergence.} \end{cases} \quad (3.4) $$

The background function $\psi$ satisfies

$$ \sup_{x \in \Omega} |\psi(x)| + \sup_{x \in \Omega} |\nabla \psi(x)| \text{dist}(x, \partial\Omega) \leq C_4 ||\varphi||_{L^\infty(\partial\Omega)}, \quad (3.5) $$

$$ \|\nabla \psi|\text{dist}(\cdot, \partial\Omega)|^{1-\frac{1}{p}}\|_{L^p(\Omega)} \leq C_5 ||\varphi||_{L^p(\partial\Omega)}, \quad 2 \leq p \leq \infty, \quad (3.6) $$

here $g := \int_{\mathbb{R}^2} (-u_2, u_1) \cdot dx$ for fixed $x \in \partial\Omega$ and $\varphi \cdot n = 0$, where $n$ is the outer unit normal vector of $\Omega$.

Lemma 3.1. (See e.g. [1]) (1) From the above properties, we have

$$ \|\psi\|_{L^\infty(\Omega)} \leq C_4 ||\varphi||_{L^\infty(\partial\Omega)}. \quad (3.7) $$

(2) If $\psi$ is defined as

$$ \begin{cases} \text{div}\psi = 0, & x \in \Omega, \\ \psi = u, & \text{if } x \in \{ x \in \Omega; \text{dist}(x, \partial\Omega) < \varepsilon \}, \\ \psi = \varphi, & \text{on } \partial\Omega \text{ in the sense of nontangential convergence.} \end{cases} \quad (3.8) $$

Then

$$ \Delta \psi = \nabla (q_\eta_\varepsilon) + F, \quad (3.11) $$
where
\[ \text{supp}\psi \subset \{ x \in \Omega; \text{dist}(x, \partial\Omega) < C_2'\varepsilon \}, \quad (3.12) \]
\[ \| F \|_{L^2(\Omega)} \leq \frac{C}{\varepsilon^2} \| \varphi \|_{L^2(\partial\Omega)}, \quad \nabla q = \Delta u, \quad (3.13) \]
\[ F = 0, \quad x \in \{ x | \text{dist}(x, \partial\Omega) < C_1'\varepsilon \text{ or } \text{dist}(x, \partial\Omega) > C_2'\varepsilon \}. \quad (3.14) \]

3.2. Equivalent homogeneous boundary value problem. Let \( v = u - \psi \), then (1.1) is transformed into the following homogeneous boundary value problem
\[
\begin{aligned}
v_t - \alpha^2 \Delta v_t - \nu \Delta v + (v \cdot \nabla)v + (v \cdot \nabla)\psi + (\psi \cdot \nabla)v + \nabla(p - \nu q_\varepsilon) \\
v_t = f + \nu F, \quad \text{div}v = 0, \\
v|_{\partial\Omega} = 0, \\
v(\tau, x) = v_\tau(x),
\end{aligned}
\]
where \( \bar{f} = f + \nu F \).

Let \( v_\tau \in \mathcal{V} \), the abstract equivalent form of (3.15) reads
\[
\begin{aligned}
v_t + \nu Av + \alpha^2 Av_t + B(v) + B(v, \psi) + B(\psi, v) = P\bar{f} - B(\psi), \\
\text{div}v = 0.
\end{aligned}
\]

Similarly as the definition of the bilinear continuous operator \( B(u, v), \quad R(v) := B(v, \psi) + B(\psi, v) \) is a continuous linear operator from \( \mathcal{V} \) onto \( \mathcal{V}' \) as well.

3.3. Existence of solutions and uniqueness. Firstly, we define the global weak solution of (1.1).

**Definition 3.2.** Let \( u_\tau \in \mathcal{V}, \quad f(x, t) \in L^2_t(\mathbb{R}; H), \varphi \in L^\infty(\partial\Omega) \) and \( \varphi \cdot n = 0 \) on \( \partial\Omega \), \( u \) is called a weak solution of the problem (1.1) provided that
(i) \( u \in C(\mathbb{R}; \mathcal{V}), u(\cdot, 0) = u_0 \), and \( du/dt \in L^2_{\text{loc}}(\mathbb{R}; \mathcal{V}') \);
(ii) for all \( v \in C^{\infty}_0(\Omega) \) with \( \text{div}v = 0 \), we get
\[ \frac{d}{dt} < u, v > - \nu < u, \Delta v > - \alpha^2 \frac{d}{dt} < u, \Delta v > - \int_{\Omega} \sum_{i,j=1}^2 u^i \frac{\partial v^j}{\partial x_j} dx = < f, v >; \]
(iii) \( \exists \psi \in C^2(\Omega) \cap L^\infty(\partial\Omega), q \in C^1(\Omega) \) and \( g \in L^2(\Omega) \) such that
\[
\begin{aligned}
\Delta \psi = \nabla q + g, & \quad \text{in } \Omega, \\
\text{div} \psi = 0, & \quad \text{in } \Omega, \\
\psi = \varphi & \quad \text{on } \partial\Omega,
\end{aligned}
\]
where \( \psi \) can reach its boundary values in the sense of non-tangential convergence and \( u - \psi \in L^2_{\text{loc}}(\mathbb{R}; \mathcal{V}) \).

Then, the existence of global weak solutions of (3.16) (or (3.15)) can be stated in following theorem.

**Theorem 3.3.** Let \( v_\tau \in \mathcal{V}, \quad f(x, t) \in L^2_t(\mathbb{R}; H) \text{ or } L^2_t(\mathbb{R}; \mathcal{V}'), \quad \text{then there exists a unique solution } v(t) \text{ of (3.16) (or (3.15)) satisfying}
\[ v(t) \in L^\infty(\mathbb{R}; \mathcal{V}) \bigcap L^4_{\text{loc}}(\mathbb{R}; \mathcal{V}), \]
and \( \frac{du}{dt} \) is uniformly bounded in \( L^4_{\text{loc}}(\mathbb{R}; \mathcal{V}') \).
Proof. Step 1. We aim to use the standard Faedo-Galerkin procedure to establish the existence of solution to (3.16).

Fix $n \geq 1$, $w_j$ ($j \geq 1$) be the normalized eigenfunction basis of the Stokes operator: $V \rightarrow H$ with the corresponding eigenvalue $\lambda_j$ ($j \geq 0$) being $0 < \lambda_1 \leq \lambda_2 \leq \cdots$ and $\lim_{j \rightarrow \infty} \lambda_j = \infty$. We define an approximate solution $v_n$ to (3.16) as $v_n(t) = \sum_{j=1}^{n} a_{nj}(t)w_j \in V_n = \text{span}\{w_1, w_2, \cdots, w_n\}$ which satisfies the following initial data problem of ordinary differential equations with respect to unknown variables $\{a_{nj}\}_{j=1}^{n}$

\[
\begin{aligned}
\frac{d}{dt}(v_n, w_j) + \nu \langle Av_n, w_j \rangle + \alpha^2 \langle Av_{nt}, w_j \rangle + b(v_n, v_n, w_j) + b(\psi, w_j) \\
+ \mu (\nabla(p - \mu q_n), w_j) + b(\psi, v_n, w_j) = (P_n \tilde{f}, w_j) - b(\psi, w_j),
\end{aligned}
\]

(3.18)

By the local existence theory of solutions of ordinary differential equations, there exists a solution in a local interval for problem (3.18).

Step 2. We will get the uniform priori estimate.

Multiplying (3.18) by $a_{nj}$, integrating by parts, noting $b(v_n, v_n, v_n) = 0$ and $b(\psi, v_n, v_n) = 0$ from (2.8), using the incompressible condition, we have

\[
\begin{aligned}
\frac{1}{2} \frac{d}{dt} \left( |v_n|^2 + \alpha^2 |v_n|^2 \right) + \nu |v_n|^2 &\leq |b(v_n, v_n)| + |\langle P_n \tilde{f}, v_n \rangle| + |b(\psi, v_n)|,
\end{aligned}
\]

and we shall estimate the terms of (3.20) on the right hand side.

Using the Hardy’s inequality (2.10), and (2.8) and (3.5), choosing suitable $\varepsilon$ such that

\[
C_2 C_3 C_4 \varepsilon \| \varphi \|_{L^\infty(\partial \Omega)} \leq \frac{\nu}{4},
\]

hence

\[
\begin{aligned}
|b(v_n, \psi, v_n)| &\leq \int_{\Omega} |v_n| |\nabla \psi| |v_n| dx \\
&\leq C_4 \| \varphi \|_{L^\infty(\partial \Omega)} \int_{\text{dist}(x, \partial \Omega) \leq C_2 \varepsilon} \frac{|v_n|^2}{|\text{dist}(x, \partial \Omega)|^2} dx \\
&\leq C_2 C_3 C_4 \varepsilon \| \varphi \|_{L^\infty(\partial \Omega)} \| v_n \|^2 \leq \frac{\nu}{4} \| v_n \|^2.
\end{aligned}
\]

Similarly using the Hölder inequality and Young’s inequality for (3.22), we derive

\[
\begin{aligned}
|b(\psi, \psi, v_n)| &\leq \int_{\Omega} |\psi| |\nabla \psi| |v_n| dx \\
&\leq C_4 \| \varphi \|_{L^\infty(\partial \Omega)} \int_{\text{dist}(x, \partial \Omega) \leq C_2 \varepsilon} \frac{|v_n|}{|\text{dist}(x, \partial \Omega)|} |\psi| dx \\
&\leq C \| \varphi \|_{L^\infty(\partial \Omega)}^2 C' \sqrt{\varepsilon} |\partial \Omega|^{1/2} \| v_n \| \\
&\leq \frac{\nu}{4} \| v_n \|^2 + \frac{C \varepsilon |\partial \Omega|}{\nu} \| \varphi \|_{L^\infty(\partial \Omega)}^4.
\end{aligned}
\]

(3.23)
By the Cauchy inequality, Hardy’s inequality and Young’s inequality, via Lemma 3.1, we deduce that

\[
\begin{align*}
| < P_n f, v_n > | & \\
\leq | < f, v_n > | + \nu | < F, v_n > | & \\
\leq | f || v_n || + \nu \int_{C^1_{1, \varepsilon} \leq \text{dist}(x, \partial \Omega) \leq C^1_{1, \varepsilon}} \text{dist}(x, \partial \Omega) || F || \frac{|v_n|}{\text{dist}(x, \partial \Omega)} dx & \\
\leq | f || v_n || \sqrt{\lambda_1} + C \varepsilon \nu || F ||_{L^2(\Omega)} \{ \int_{\Omega} \frac{|v_n|^2}{\text{dist}(x, \partial \Omega)^2} dx \}^{1/2} & \\
\leq || v_n || \{ \frac{| f |}{\sqrt{\lambda_1}} + \frac{C \nu}{\varepsilon} || \varphi ||_{L^2(\partial \Omega)} \} & \\
\leq \frac{\nu}{4} || v_n ||^2 + C \left( \frac{| f |^2}{\nu \lambda_1} + \frac{C \nu}{\varepsilon} || \varphi ||_{L^2(\partial \Omega)}^2 \right). & (3.24)
\end{align*}
\]

Combining (3.22)–(3.24), we get

\[
\begin{align*}
\frac{1}{2} \frac{d}{dt} \left( | v_n |^2 + \alpha^2 || v_n ||^2 \right) + \frac{\nu}{4} || v_n ||^2 & \\
\leq \frac{C}{\nu} \left[ \frac{| f |^2}{\lambda_1} + \frac{C \nu}{\varepsilon} || \varphi ||_{L^2(\partial \Omega)}^2 + \frac{C \varepsilon | \partial \Omega |}{\nu} || \varphi ||_{L^\infty(\partial \Omega)}^4 \right] := K_0^2. & (3.25)
\end{align*}
\]

By the Poincaré inequality, we have

\[
\frac{1}{2} \frac{d}{dt} \left( | v_n |^2 + \alpha^2 || v_n ||^2 \right) + \frac{\nu}{8} (\lambda_1 | v_n |^2 + || v_n ||^2) \leq K_0^2. & (3.26)
\]

Choosing \( C' = \lambda_1 \) if \( \frac{1}{\lambda_1} \geq \alpha^2 \) and \( C' = \min \{ 1, \frac{1}{\lambda_1^2} \} \) if \( \frac{1}{\lambda_1} < \alpha^2 \), (3.26) reduces to

\[
\frac{d}{dt} \left( | v_n |^2 + \alpha^2 || v_n ||^2 \right) + \frac{C' \nu}{4} (| v_n |^2 + \alpha^2 || v_n ||^2) \leq 2K_0^2. & (3.27)
\]

The Gronwall inequality leads to

\[
| v_n |^2 + \alpha^2 || v_n ||^2 \leq (| v_r |^2 + \alpha^2 || v_r ||^2) e^{\int_{t}^{s} \left( \frac{C' \nu}{4} \right) ds} + \int_{t}^{s} e^{-\frac{C' \nu}{4} (s-t)} 2K_0^2 ds. & (3.28)
\]

Moreover, since the external force \( f \) is translation bounded and ||\( \varphi ||_{L^2(\partial \Omega)} \leq || \partial \Omega || \|| \varphi ||_{L^2(\partial \Omega)}^2 \), it holds

\[
\begin{align*}
\int_{t}^{s} e^{-\frac{C' \nu}{4} (s-t)} 2K_0^2 ds & \\
\leq \int_{t}^{s} e^{-\frac{C' \nu}{4} (s-t)} \frac{C}{\nu} \left[ \frac{| f |^2}{\lambda_1} + \frac{\nu^2}{\varepsilon} || \varphi ||_{L^2(\partial \Omega)}^2 + \varepsilon | \partial \Omega || \varphi ||_{L^\infty(\partial \Omega)}^4 \right] ds & \\
\leq \int_{t}^{s} e^{-\frac{C' \nu}{4} (s-t)} \frac{C | f |^2}{\nu \lambda_1} ds & \\
\quad + \frac{C}{\nu} \left[ \frac{\nu^2}{\varepsilon} | \partial \Omega || \varphi ||_{L^\infty(\partial \Omega)}^2 + \varepsilon | \partial \Omega || \varphi ||_{L^\infty(\partial \Omega)}^4 \right] \int_{t}^{s} e^{-\frac{C' \nu}{4} (s-t)} ds & \\
\text{and} & \\
\frac{C}{\nu \lambda_1} \int_{t}^{s} e^{-\frac{C' \nu}{4} (s-t)} ds \leq C & (3.29)
\end{align*}
\]

\[
\frac{C}{\nu \lambda_1} \int_{t}^{s} e^{-\frac{C' \nu}{4} (s-t)} ds \leq C \|| f_0 ||_{L^2(\mathbb{R}, H)}^2 & (3.30)
\]
From (3.32) and (3.33), we have the following convergence

Theorem 3.4. Let \( f \) for an arbitrary fixed \( f_0 \in L^2(\mathbb{R}; H) \) which is uniformly bounded in a symbol space \( \Sigma \) defined later.

Integrating (3.28) over \([s, t]\), we have

\[
\int_s^t \frac{C^2 \nu}{4} \left( |v_n'(r)|^2 + \alpha^2 |v_n(r)|^2 \right) dr \\
\leq |v_n(s)|^2 + C \|f_0\|_{L^2(\mathbb{R}; H)}^2 + \frac{C}{\nu} \left( \frac{\nu^2}{\varepsilon} |\partial \Omega| \|\varphi\|_{L^\infty(\partial \Omega)}^2 \\
+ C \varepsilon |\partial \Omega| |\varphi|^4_{L^\infty(\partial \Omega)} \right) (t-s).
\]

(3.31)

Combining (3.28)-(3.31), we conclude that

\[
v_n(t) \text{ is bounded in } L^\infty(\mathbb{R}; V) \cap L^2_{\text{loc}}(\mathbb{R}; V).
\]

(3.32)

**Step 3.** We will establish the global weak solution by compact argument.

Using the technique in [22], considering the equation in distribution sense, we can derive that

\[
\frac{dv_n}{dt} \text{ is bounded in } L^4(\mathbb{T}; V).
\]

(3.33)

From (3.32) and (3.33), we have the following convergence

\[
v_n \to v \text{ strongly in } V;
\]

(3.34)

\[
v_n \to v \text{ weakly }* \text{ in } L^\infty(\mathbb{R}; V);
\]

(3.35)

\[
v_n \to v \text{ weakly in } L^2_{\text{loc}}(\mathbb{R}; V);
\]

(3.36)

\[
\frac{dv_n}{dt} \to \frac{dv}{dt} \text{ weakly in } L^4(\mathbb{T}; V').
\]

(3.37)

Using the Lions-Aubin Lemma, i.e., the compact argument, passing the limit as \( n \to +\infty \), we can derive the global weak solution \( v(t) \in L^\infty(\mathbb{R}; V) \cap L^2_{\text{loc}}(\mathbb{R}; V) \) and \( L^4_{\text{loc}}(\mathbb{R}; V') \).

\[\square\]

**Theorem 3.4.** Let \( u_\tau \in V, f \in L^2(\mathbb{R}; H), \varphi \in L^\infty(\partial \Omega), \) and \( \varphi \cdot n = 0 \) on \( \partial \Omega \). Then the problem (1.1) possesses a unique weak solution which continuously depends on the initial data, i.e., \( u(t, x) \in C(\mathbb{R}_\tau; V) \).

**Proof.** **Step 1.** The existence of weak solution is established.

From the property of background flows class \( \psi_\tau = \psi \in C^\infty(\Omega) \) and \( v = u - \psi \) and the solution \( v \) for (3.15) is obtained in Theorem 3.1, it can be shown easily that \( u \) satisfies the conditions (i) (ii) and (iii) in Definition 3.1. Hence \( u \) is a weak solution.

**Step 2.** The uniqueness of the solution and its continuous dependence of the initial data.

Consider the two solutions \( u_1(\cdot) \) and \( u_2(\cdot) \) to problem (1.1) with corresponding initial data \( u_\tau \) and \( u_\tau' \), respectively, and flows \( \psi_1 \) and \( \psi_2 \), respectively. Denote \( w = u_1 - u_2 \), then \( w \) satisfies the problem:

\[
\begin{aligned}
\frac{dw}{dt} - \nu \Delta w - \alpha^2 \Delta w_1 + (u_1 \cdot \nabla) u_1 - (u_2 \cdot \nabla) u_2 &= 0, \\
d \text{div} w &= 0, \ (x, t) \in \Omega_\tau, \\
w(t, x)|_{\partial \Omega} &= 0, \ (x, t) \in \partial \Omega_\tau, \\
w(t, x) &= u_\tau(x) - u_\tau'(x).
\end{aligned}
\]

(3.38)
The system is rewritten as:
\[
\begin{align*}
\frac{dw}{dt} + \nu Aw + \alpha^2 A\omega_t + B(u_1, u_1) - B(u_2, u_2) &= 0, \\
\text{div} w &= 0,
\end{align*}
\] (3.39)

Let \( \omega \in C_0^\infty(\Omega) \), \( \text{div} \omega = 0 \), from the condition (ii) in Definition 3.1, we can derive
\[
\frac{d}{dt} < u_1 - u_2, \omega > -\nu < u_1 - u_2, \Delta \omega > -\alpha^2 \frac{d}{dt} < u_1 - u_2, \Delta \omega > = \int_\Omega \sum_{i,j=1}^2 (u_1^i u_1^j - u_2^i u_2^j) \frac{\partial \omega^j}{\partial x_i} dx.
\] (3.40)

 Apparently, (3.40) holds for any \( \omega \in V \). In fact, from the condition (ii) we have
\[
< u_1 - u_2, \Delta \omega > = -((u_1 - u_2, \omega)),
\]
\[
\frac{d}{dt} < u_1 - u_2, \omega > = 0 \text{ for } v \in V,
\]
hence (3.40) holds for any \( v \in V \).

Let \( \omega = w = u_1 - u_2 \), we have
\[
\frac{1}{2} \frac{d}{dt} (|\omega|^2 + \alpha^2 |\omega|^2) + \nu |\omega|^2 \leq C \int_\Omega |u_1||\omega||\nabla \omega| dx
\]
\[
\leq \|u_1\|_{L^4(\Omega)}|\nabla \omega|^{1/2}\|\omega\|^{1/2}|\nabla \omega|
\]
\[
\leq \nu |\omega|^2 + C \nu \|u_1\|_{L^4(\Omega)}^4 |\omega|^2,
\]
that is,
\[
\frac{d}{dt} (|\omega|^2 + \alpha^2 |\omega|^2) \leq 2C \nu \|u_1\|_{L^4(\Omega)}^4 |\omega|^2.
\]

Since
\[
\|u_j\|_{L^4(\Omega)} \leq \|u_j - \psi\|_{L^4(\Omega)} + \|\psi\|_{L^4(\Omega)}
\]
\[
\leq C \|
abla (u_j - \psi)\|_{L^2(\Omega)} \|u_j - \psi\|_{L^2(\Omega)}^{1/2} + \|\psi\|_{L^4(\Omega)}
\]
(3.41)

\( u_j \in L^4(\Omega) \) \((j = 1, 2)\). Note \( \omega(\cdot, \tau) = 0 \), hence \( \omega \equiv 0 \), i.e., the solution is unique on \( \mathbb{R}_\tau \).

Multiplying (3.39) with \( w \), since \( B(u_1, u_1) - B(u_2, u_2) = B(w, u_1) - B(u_2, w) \) and \( (B(u_2, w)) = b(u_2, w, w) = 0 \), we obtain
\[
\frac{1}{2} \frac{d}{dt} (|w|^2 + \alpha^2 |w|^2) + \nu |w|^2 \leq |b(w, u_1, w)|.
\] (3.42)

Since \( u_\tau, u'_\tau \in V \), using (2.8), we have
\[
|b(w, u_1, w)| \leq C |w| \|w\| |\nabla u_1|.
\] (3.43)

Integrating (3.42) over \([\tau, t]\), using (3.43), we obtain
\[
|w|^2 + \alpha^2 |w|^2 + 2\nu \int_\tau^t |w|^2 ds
\]
\[
|w|^2 + \alpha^2\|w\|^2 + 2C \int_\tau^t |w|\|\nabla u_1|ds \\
\leq 2|w_\tau|^2 + 2\alpha^2\|w_\tau\|^2 + \nu \int_\tau^t \|w\|^2ds + \frac{2C}{\nu} \int_\tau^t |u_1|^2|w|ds, \tag{3.44}
\]
i.e.,
\[
|w|^2 + \alpha^2\|w\|^2 + \nu \int_\tau^t \|w\|^2ds \leq |w_\tau|^2 + \alpha^2\|w_\tau\|^2 + \frac{2C}{\nu} \int_\tau^t |u_1|^2|w|ds. \tag{3.45}
\]

Since \(u_j(t) \in L^\infty([\tau,T];V) \cap L^2([\tau,T];V)\) \((j = 1,2)\), neglecting the integrating term on left side of (3.45), we have
\[
|w|^2 \leq |w_\tau|^2 + \alpha^2\|w_\tau\|^2 + \frac{C'}{\nu} \int_\tau^t |w|ds. \tag{3.46}
\]

Using the Gronwall inequality to (3.46), we see
\[
|u_1(t) - u_2(t)|^2 \leq (\|u_\tau - u_\tau'\|^2_{H} + \|u_\tau - u_\tau'\|^2_{V}) \times e^{\frac{C'}{\nu}(t-\tau)}. \tag{3.47}
\]

Similarly, using the Poincaré inequality, we can derive
\[
\alpha^2\|w\|^2 \leq |w_\tau|^2 + \alpha^2\|w_\tau\|^2 + \frac{C'}{\nu} \int_\tau^t \|w\|^2ds. \tag{3.48}
\]
and
\[
\|u_1(t) - u_2(t)\|^2 \leq \frac{1}{\alpha^2}(\|u_\tau - u_\tau'\|^2_{H} + \|u_\tau - u_\tau'\|^2_{V}) \times e^{\frac{C'}{\nu}(t-\tau)}. \tag{3.49}
\]
Moreover,
\[
\int_\tau^t \|u_1(t) - u_2(t)\|^2ds \leq \frac{\lambda_1}{C}(\|u_\tau - u_\tau'\|^2_{H} + \|u_\tau - u_\tau'\|^2_{V}) \times \left( e^{\frac{C'}{\nu}(t-\tau)} + 1 \right). \tag{3.50}
\]

Therefore, (3.47), (3.49) and (3.50) imply solution’s continuous dependence on the initial data for the solution which generates a continuous process \(U_f(t,\tau)u_\tau = u(x, t) : V \rightarrow V\).

3.4. Difficulty of achieving regularity in \(W\) in Lipschitz domains. In this section, we aim to state why the estimates for the norms such as \(\|u\|_{L^2([\tau,T];W)}\) and \(\|u\|_{L^\infty([\tau,T];W)}\) (more regularity in \(W\)) cannot be obtained. Note this difficulty exists in proving the boundedness of same norms of \(v\). This situation is also similar, as in study on the 2D Navier-Stokes equations with nonhomogeneous boundary in non-smooth domains, the regularity of \(u\) can not reach \(D(A^2)\) (see [1]).

Multiplying the equation (3.16) with \(Av\), integrating by parts, we have
\[
\frac{1}{2} \frac{d}{dt} \left( \|v\|^2 + \alpha^2|Av|^2 \right) + \nu|Av|^2 = (P_f, Av) - (B(\psi), Av) - (B(v), Av) - (B(v, \psi), Av). \tag{3.51}
\]

Next, we will estimate every term on the right side of (3.51). Using (3.24) and the Cauchy inequality, the property of trilinear operator, the Sobolev embedding theorem, the Poincaré inequality, choosing \(\frac{C_{\psi}||\psi||_{L^\infty}}{2} \leq \frac{\nu}{8}\), and by the Hardy’s
inequality, we obtain estimates:

\[ |(P^f, Av)| \leq \frac{\nu}{8} |Av|^2 + \frac{C}{\nu} \int_{\Omega} |P|^2 dx, \]  
\[ |(B(v, v), Av)| \leq \frac{\nu}{8} |Av|^2 + \frac{C}{\lambda^2 \nu} \|v\|^6, \]  
\[ |(B(v, \psi), Av)| \leq \frac{C_4 \|\varphi\|_{L^\infty(\partial\Omega)}}{\nu \lambda_1} \|v\|^2 + \frac{\nu}{8} |Av|^2, \]  
\[ |(B(\psi, v), Av)| \leq \frac{C \|\varphi\|_{L^\infty(\partial\Omega)}}{\nu \lambda_1} \|v\|^2 + \frac{\nu}{8} |Av|^2, \]  

but for the last one, we cannot give the appropriate estimate

\[ |(B(\psi, v), Av)| \leq \int_{\Omega} |\psi| |\nabla\psi| |Av| dx \]
\[ = \int_{\Omega} |\psi| |\nabla\psi| |\text{dist}(x, \partial\Omega)|^{\frac{1}{2}} |Av| \left(\frac{\text{dist}(x, \partial\Omega)}{|\text{dist}(x, \partial\Omega)|^2}\right) dx, \]  

in that we do not have more regularity of \( L^2([\tau, T]; D(A^{1+\beta})) \) such as \( \beta = \frac{1}{4} \). This implies that \( \|v\|_{L^2([\tau, T]; W)}^p \) and \( \|v\|_{L^\infty([\tau, T]; W)}^p \) are not necessarily bounded, which means the higher regularity for \( v \), and also \( u \) in \( W \) can not be obtained. This is the motivation that we will discuss the optimal regularity in \( D(A^{\frac{2}{4}}) \) in the next subsection.

3.5. Optimal regularity of solutions. In this section, we need to estimate the norms of \( u \) as well as some regular norms of \( v \). Since \( v(t) = U_f(t, \tau) v_\tau \), then we have a decomposition of the process

\[ U_f(t, \tau) v_\tau = D(t, \tau) v_\tau + K_f(t, \tau) v_\tau = v_1(t) + v_2(t). \]  

By the Duhamel’s principle, \( v_1(t) = D(t, \tau) v_\tau \) and \( v_2(t) = K_f(t, \tau) v_\tau \) are solutions satisfying the sub-problems

\[
\begin{align*}
    v_{1t} + \alpha^2 A v_{1t} + \nu Av_1 &= -B(v_1, v_1) - B(\psi, v_1) - B(v_1, \psi), \\
    \text{div} v_1 &= 0, \\
    v_1|_{\partial\Omega} &= 0, \\
    v_1(\tau, x) &= u_\tau(x) + \psi(x),
\end{align*}
\]

and

\[
\begin{align*}
    v_{2t} + \alpha^2 A v_{2t} + \nu Av_2 &= -B(v_2, v_1) - B(v_2, v_1) - B(v_2, v_2) \\
    &\quad - B(v_2, \psi) - B(\psi, v_1) - B(\psi, \psi) + \hat{f}, \\
    \text{div} v_2 &= 0, \\
    v_2|_{\partial\Omega} &= 0, \\
    v_2(\tau, x) &= 0,
\end{align*}
\]

respectively. Here \( \hat{f}(t, x) := P \hat{f} \).

The problem \((3.58)\) is a linear autonomous equation with the process operator \( D(t, \tau) = D(t - \tau, 0) \), which implies it can generate a semigroup in the phase space.
Lemma 3.5. For any \( \nu > \frac{3}{2\lambda_1} + \frac{3C_4\|\psi\|_{L^\infty(\partial\Omega)}}{\nu} + \frac{3C_4^2\|\psi\|^2_{L^\infty(\partial\Omega)}}{\nu} \) and \( v_\tau \in W \), then there exists \( k > 0 \), such that the solution of (3.58) satisfies

\[
\|v_1(t)\|^2_W = \|D(t,\tau)v_\tau\|^2_W \leq Q(\|v_\tau\|_W)e^{-k(t-\tau)},
\]

\[
\int_\tau^t |Av_1(s)|^2 ds \leq C(k, \nu, \|v_\tau\|_W),
\]

here \( Q(\cdot) \) denotes an increasing nonnegative function.

Proof. Multiplying the equation (3.58) with \( Av_1 \), we have

\[
\frac{1}{2}(\|v_1\|^2 + \alpha^2|Av_1|^2) + \nu|Av_1|^2
\]

\[
= |b(v_1, v_1, Av_1)| + |b(\psi, v_1, Av_1)| + |b(v_1, \psi, Av_1)|,
\]

since

\[
|b(v_1, v_1, Av_1)| \leq \|v_1\|^2\|\nabla v_1\|^2|Av_1| \leq \frac{\nu}{6}|Av_1|^2 + \frac{3}{2\nu\lambda_1}\|v_1\|^2,
\]

and

\[
|b(\psi, v_1, Av_1)| \leq C_4\|\psi\|_{L^\infty(\partial\Omega)}\|\nabla v_1\||Av_1|
\]

\[
\leq \frac{\nu}{6}|Av_1|^2 + \frac{3C_4\|\psi\|_{L^\infty(\partial\Omega)}\|v_1\||Av_1|,
\]

and

\[
|b(v_1, \psi, Av_1)| = \int_\Omega \frac{v_1}{\text{dist}(x, \partial\Omega)}|\nabla \psi|\text{dist}(x, \partial\Omega)|Av_1|dx
\]

\[
\leq C_4\|\psi\|_{L^\infty(\partial\Omega)}\left(\int_\Omega \frac{|v_1|^2}{\text{dist}^2(x, \partial\Omega)}dx\right)^\frac{1}{2}|Av_1|
\]

\[
\leq C_4C_4\|\psi\|_{L^\infty(\partial\Omega)}\|v_1\||Av_1|
\]

\[
\leq \frac{\nu}{6}|Av_1|^2 + \frac{3C_4^2\|\psi\|^2_{L^\infty(\partial\Omega)}}{2\nu}\|v_1\|^2.
\]

Hence, combining (3.62)-(3.65), we have

\[
\frac{d}{dt}(\|v_1\|^2 + \alpha^2|Av_1|^2) + \nu|Av_1|^2
\]

\[
\leq \left(\frac{3}{\nu\lambda_1} + \frac{3C_4\|\psi\|_{L^\infty(\partial\Omega)}}{\nu} + \frac{3C_4^2\|\psi\|^2_{L^\infty(\partial\Omega)}}{\nu}\right)\|v_1\|^2.
\]

Via the Gronwall inequality and the Poincaré inequality, the theorem is proved. \( \square \)

Lemma 3.6. For the fractional operator \( A^\alpha \), we have

(a) the generalized Sobolev embedding inequality

\[
\|A^{\frac{\alpha+1}{2}} u\|_H \leq \frac{C}{\lambda_1^{\frac{\alpha}{2}}} \|A^{\frac{\alpha+1}{2}} u\|_H, \quad \forall \ u \in D(A^{\frac{\alpha+1}{2}}),
\]

(b) the generalized Hardy’s inequality

\[
\int_\Omega \frac{|A^\alpha u|^2}{\text{dist}(x, \partial\Omega)}dx \leq C \int_\Omega |A^{\alpha+\frac{1}{2}} u|^2 dx,
\]

\[
|A^{\alpha+\frac{1}{2}} u| \leq \frac{C}{\lambda^{\frac{\alpha}{2}}} |A^{\frac{\alpha+1}{2}} v_2|, \quad 0 \leq \alpha \leq \frac{1}{2}
\]
for all \( u \in D(A^{\sigma+1}) \).

**Proof.** See, e.g., [25]. \( \square \)

Next, we will estimate the nonlinear nonhomogeneous problem (3.59) with the homogeneous boundary condition.

**Lemma 3.7.** For any \( \nu > \frac{3}{\nu \lambda_1^2} + \frac{3C_s||\nu||_{L^\infty(\partial\Omega)}}{\nu} + \frac{3C \gamma_1^2 ||\nu||_{L^\infty(\partial\Omega)}}{\nu} \) and \( f \in L^2_{loc}([0, T); H) \), there exist positive constants

\[
M_1 = M_1(\alpha, \nu, \|f\|_{L^2_{loc}([0, T); H)}, \|\nu\|_W, \|\varphi\|_{L^\infty(\partial\Omega)}^2) > 0
\]

and

\[
M_2 = M_2(T, \alpha, \nu, \|f\|_{L^2_{loc}([0, T); H)}, \|\nu\|_W, \|\varphi\|_{L^\infty(\partial\Omega)}^2) > 0
\]

such that the solution of (3.59) satisfies that for any \( 0 \leq \sigma \leq \frac{1}{2} \)

\[
\|K_f(t, \tau)\|_{D(A^{\sigma+1})}^2 \leq M_1,
\]

\[
\int_0^T \|K_f(s, \tau)\|_{D(A^{\sigma+1})}^2 ds \leq M_2.
\]

**Proof.** Taking inner product of (3.59) with \( A^\sigma v_2(t) \) in \( H \)-norm, we have

\[
(v_{2t}, A^\sigma v_2) + \alpha^2 (Av_{2t}, A^\sigma v_2) + \nu (Av_2, A^\sigma v_2) \]
\[
= -\langle B(v_1, v_2), A^\sigma v_2 \rangle - \langle B(v_2, v_1), A^\sigma v_2 \rangle - (B(v_2, v_2), A^\sigma v_2) \]
\[
-\langle B(\psi, v_2), A^\sigma v_2 \rangle - \langle \hat{f}, A^\sigma v_2 \rangle,
\]

i.e.,

\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega} |A^{\frac{\sigma+1}{2}} v_2|^2 + \alpha^2 |A^{\frac{\sigma+1}{2}} v_2|^2 dx + \nu \int_{\Omega} |A^{\frac{\sigma+1}{2}} v_2|^2 dx \leq
\]
\[
|\langle B(v_1, v_2), A^\sigma v_2 \rangle| + |\langle B(v_2, v_1), A^\sigma v_2 \rangle| + |\langle B(v_2, v_2), A^\sigma v_2 \rangle| + |\langle B(\psi, v_2), A^\sigma v_2 \rangle| + |\langle \hat{f}, A^\sigma v_2 \rangle|.
\]

Next, we shall estimate every terms on the right hand side of (3.73). Using the technique in [25], we have the following estimates.

1. By the Hölder inequality, for any \( \sigma \in [0, \frac{1}{2}] \subset [0, 1) \), we have

\[
|\langle B(v_2, v_2), A^\sigma v_2 \rangle| \leq \int_{\Omega} |v_2||\nabla v_2||A^\sigma v_2| dx \leq \frac{\nu}{16} |A^{\frac{\sigma+1}{2}} v_2|^2 + \frac{C}{\nu \lambda_1^2} \|v_2\|^2.
\] (3.74)

2. By the Hardy inequality, since \( \sigma \in [0, \frac{1}{2}] \subset [0, 1) \), we derive

\[
|\langle B(v_2, \psi), A^\sigma v_2 \rangle| \leq \int_{\Omega} |v_2||\nabla \psi||A^\sigma v_2| dx \leq \int_{\Omega} \frac{|v_2|}{\text{dist}(x, \partial\Omega)} |\nabla \psi| \cdot \text{dist}(x, \partial\Omega)|A^\sigma v_2| dx.
\]
\[ \leq C \| \varphi \|_{L^\infty(\partial \Omega)} \left( \int_\Omega \frac{|v_2|^2}{(\text{dist}(x, \partial \Omega))^2} dx \right)^{\frac{1}{2}} \left( \int_\Omega |A^\sigma v_2|^2 dx \right)^{\frac{1}{2}} \]
\[ \leq C \| \varphi \|_{L^\infty(\partial \Omega)} \| v_2 \| \| A^{\frac{\sigma+1}{2}} v_2 \| \]
\[ \leq \frac{\nu}{16} |A^{\frac{\sigma+1}{4}} v_2|^2 + C \| \varphi \|^2_{L^\infty(\partial \Omega)} \| v_2 \|^2. \quad (3.75) \]

(3) By the estimates of background function and the property of trilinear operator, since \( \sigma \in [0, \frac{1}{2}] \subset [0, 1) \), it follows
\[
| - \langle B(\psi, v_2), A^\sigma v_2 \rangle | = \int_\Omega |\nabla \psi| \| A^\sigma v_2 \| dx
\leq C_4 \| \varphi \|_{L^\infty(\partial \Omega)} |\nabla v_2| \| A^\sigma v_2 \|
\leq \frac{\nu}{16} |A^{\frac{\sigma+1}{4}} v_2|^2 + C \| \varphi \|^2_{L^\infty(\partial \Omega)} \| v_2 \|^2. \quad (3.76)
\]

(4) From Lemma 3.1 and the Young’s inequality, for any \( \sigma \in [0, \frac{1}{2}] \), we deduce
\[
| - \langle B(\psi, \psi), A^\sigma v_2 \rangle | = C \| \varphi \|_{L^\infty(\partial \Omega)} \int_\Omega |\nabla \psi| \| A^\sigma v_2 \| dx
\leq C \| \varphi \|_{L^\infty(\partial \Omega)} \left( \int_\Omega |\nabla \psi|^2 \text{dist}(x, \partial \Omega) dx \right)^{\frac{1}{2}} \left( \int_\Omega |A^\sigma v_2|^2 \text{dist}(x, \partial \Omega) dx \right)^{\frac{1}{2}}
\leq \frac{\nu}{16} |A^{\frac{\sigma+1}{4}} v_2|^2 + C |\partial \Omega| \| \varphi \|^2_{L^\infty(\partial \Omega)}. \quad (3.77)
\]

(5) By the Young’s inequality and Lemma 4.6, for any \( \sigma \in [0, \frac{1}{2}] \), we obtain
\[
|\langle f, A^\sigma v_2 \rangle | \leq C |f| \| A^\sigma v_2 \| \leq C |f|^2 + \frac{\nu}{16} |A^{\frac{\sigma+1}{4}} v_2|^2 \quad (3.78)
\]
and
\[
|\langle \nu F, A^\sigma v_2 \rangle | \leq C \nu \sqrt{\varepsilon} \int_\Omega \frac{|A^\sigma v_2|}{|\text{dist}(x, \partial \Omega)|^2} dx
\leq C \nu \sqrt{\varepsilon} \| F \|_{L^2(\partial \Omega)} \| A^{\sigma + \frac{1}{4}} v_2 \|
\leq C \frac{\nu}{\varepsilon} |\partial \Omega| \| \varphi \|^2_{L^\infty(\partial \Omega)} + \frac{\nu}{16} |A^{\frac{\sigma+1}{4}} v_2|^2. \quad (3.79)
\]

(6) Using the property of the trilinear operator, Lemma 3.5, the Cauchy inequality and Poincaré inequality, we obtain
\[
|\langle B(v_1, v_2), A^\sigma v_2 \rangle | \leq |\nabla v_1| \| \nabla v_2 \| |A^\sigma v_2|
\leq \frac{\nu}{16} |A^{\frac{\sigma+1}{4}} v_2|^2 + \frac{Q(\|v_r\|_W) e^{-k(t-\tau)}}{\nu \lambda_1^{\frac{3}{2}}} \| v_2 \|^2
\leq \frac{\nu}{16} |A^{\frac{\sigma+1}{4}} v_2|^2 + \frac{C}{\nu \lambda_1^2} \| v_2 \|^2 \quad (3.80)
\]
and
\[
|\langle B(v_2, v_1), A^\sigma v_2 \rangle | \leq |\nabla v_2| \| \nabla v_1 \| |A^\sigma v_2|
\leq \frac{\nu}{16} |A^{\frac{\sigma+1}{4}} v_2|^2 + \frac{C}{\nu \lambda_1^2} \| v_2 \|^2. \quad (3.81)
\]
Combining (3.73)–(3.81), we have
\[
\frac{d}{dt} \int_{\Omega} \left( |A^{\frac{2}{\alpha}}v_2|^2 + \sigma^2 |A^{\frac{2}{\sigma^2}}v_2|^2 \right) dx + \nu \int_{\Omega} |A^{\frac{2}{\alpha^2}}v_2|^2 dx \\
\leq \frac{CV}{\varepsilon} |\partial \Omega| \|\varphi\|_{L^\infty(\partial \Omega)} + C|f|^2 + \left( \frac{C}{\nu \lambda_1^2} + 4C \|\varphi\|_{L^\infty(\partial \Omega)} \right) \|v_2\|^2 \\
+ C|\partial \Omega| \|\varphi\|_{L^\infty(\partial \Omega)} + \frac{2C}{\nu \lambda_1^3} \|v_2\|^2. \tag{3.82}
\]
Integrating (3.82) over \([\tau, t]\), since \(f \in L^2_{\text{loc}}([\tau, T]; H), v \in L^\infty([\tau, T]; V) \cap L^2([\tau, T]; W)\), noting \(v = v_1 + v_2\) and the Gronwall inequality, we obtain
\[
|A^{\frac{2}{\alpha}}v_2|^2 + \sigma^2 |A^{\frac{2}{\sigma^2}}v_2|^2 \leq M_1, \tag{3.83}
\]
\[
\int_\tau^t |A^{\frac{2}{\alpha^2}}v_2(s)|^2 ds \leq M_2. \tag{3.84}
\]
The proof is finished. \(\square\)

**Theorem 3.8.** Let \(v(t, x)\) be the solution of (3.15) with the initial data \(v_\tau \in W \cap D(A^{\frac{-1}{\alpha}}), \varphi \in L^\infty(\partial \Omega), f \in L^2_{\text{loc}}(\mathbb{R}; H)\), then for any \(\sigma \in [0, \frac{1}{2}]\), \(v(t, x)\) satisfy the optimal regularity
\[
|A^{\frac{-1}{\alpha}}v|^2 \leq M'_1, \tag{3.85}
\]
\[
\int_\tau^t |A^{\frac{-1}{\alpha^2}}v(s)|^2 ds \leq M', \tag{3.86}
\]
here
\[
M'_1 = M'_1(\tau, \lambda_1, \|f\|_{L^2_{\text{loc}}(\mathbb{R}; H)}, \|v_\tau\|_W, \|\varphi\|_{L^\infty(\partial \Omega)}),
\]
\[
M' = M'(T, \tau, \lambda_1, \|f\|_{L^2_{\text{loc}}(\mathbb{R}; H)}, \|v_\tau\|_W, \|\varphi\|_{L^\infty(\partial \Omega)}).
\]

**Proof.** Combining the Lemmas 3.5 and 3.7, we have this theorem immediately. \(\square\)

4. **Existence of the uniform attractor.** Here we shall use the preliminary theory on attractors of PDEs to find our uniform attractors. The theory can be founded in \([4, 11, 12, 24]\), and we omit the detail in this section.

4.1. **Some preliminaries for uniform attractors.** We select \(\Sigma := \mathcal{H}(f_0)\) as our symbol space of the system, and
\[
\mathcal{H}(f_0) = \{f_0(\cdot + h)| h \in \mathbb{R}\}^{L^2_{\text{loc}}(\mathbb{R}; H)}.
\]
We define that \(f \in L^2_{\text{loc}}(\mathbb{R}; H)\) is translation compact in \(L^2_{\text{loc}}(\mathbb{R}; H)\) if \(\mathcal{H}(f)\) is compact in \(L^2_{\text{loc}}(\mathbb{R}; H)\) which is denoted as \(L^2_{\text{loc}}(\mathbb{R}; H)\). Moreover, if we choose \(f_0 \in L^2_{\text{loc}}(\mathbb{R}; H)\) and fixed, then for every \(f \in \mathcal{H}(f_0)\), we have \(\|f\|_{L^2_{\text{loc}}} \leq C\|f_0\|_{L^2_{\text{loc}}}\).

Let \(\{T(\cdot)\}_{t \geq 0}\) be the translation semigroup acting on \(L^2_{\text{loc}}(\mathbb{R}; H)\) which is defined by
\[
T(h)u(t) = u(t + h), \quad \forall \ h \geq 0, \ \forall \ u \in L^2_{\text{loc}}(\mathbb{R}; H).
\]
For the non-autonomous system, the solutions generates a two-parameter operator which is called process \(U_f(t, \tau) : E \rightarrow E\) which satisfies
\[
U_f(t, s)U_f(s, \tau) = U_f(t, \tau), \quad \forall \ t \geq s \geq \tau, \ \tau \in \mathbb{R}, \tag{4.1}
\]
\[
U_f(\tau, \tau) = I \quad \text{(the identity)}. \tag{4.2}
\]
and the translation identity holds

\[ U_f(t + h, \tau + h) = U_{T(h)} f(t, \tau), \quad \forall f \in \Sigma, \quad t \geq \tau, \quad \tau \in \mathbb{R}, \quad h \geq 0. \]  

(4.3)

**Definition 4.1.** A set \( B_0 \) is said to be a uniformly (w.r.t. \( f \in \mathcal{H}(f_0) \)) absorbing set for the process \{\( U_f(t, \tau) \)\} (\( f \in \mathcal{H}(f_0) \)), if for every bounded set \( B \) of \( H \) and any \( \tau \in \mathbb{R} \), there exists some time \( t_0 = t_0(B, \tau) \geq \tau \) such that \( \bigcup_{f \in \mathcal{H}(f_0)} U_f(t, \tau) B \subseteq B_0 \) for all \( t \geq t_0 \).

**Definition 4.2.** The process \{\( U_f(t, \tau) \)\} (\( f \in \mathcal{H}(f_0) \)) is said to be asymptotically compact in \( H \) if \{\( U_{f(n)}(t_n, \tau) u^{(n)}_+ \)\} is precompact in \( H \), whenever \{\( u^{(n)}_+ \)\} is bounded in \( H \), \( f^{(n)} \in \mathcal{H}(f_0) \) and \( \{t_n\} \subseteq \mathbb{R}, \quad t_n \to +\infty \) as \( n \to +\infty \).

**Definition 4.3.** A closed set \( \Lambda \subset H \) is called the uniformly (w.r.t., \( f \in \mathcal{H}(f_0) \)) attracting set of the process \{\( U_f(t, \tau) \)\} (\( f \in \mathcal{H}(f_0) \)) if for any bounded set \( B \) of \( H \) and arbitrarily fixed \( \tau \in \mathbb{R} \),

\[
\lim_{t \to +\infty} \sup_{f \in \mathcal{H}(f_0)} \text{dist}_H(U_f(t, \tau) B, \Lambda) = 0,
\]

where \( \text{dist}_H \) refers to the Hausdorff distance.

**Definition 4.4.** A closed set \( \mathcal{A} \subset H \) is called the uniform (w.r.t., \( f \in \mathcal{H}(f_0) \)) attractor of the process \{\( U_f(t, \tau) \)\} (\( f \in \mathcal{H}(f_0) \)) acting on \( H \) if \( \mathcal{A} \) is a uniformly attracting set and \( \mathcal{A} \) satisfies the following minimality property: \( \mathcal{A} \) belongs to any closed uniformly attracting set of the process \{\( U_f(t, \tau) \)\}.

**Definition 4.5.** The process \{\( U_f(t, \tau) \)\} (\( f \in \Sigma \)) is called satisfying the uniform (w.r.t. \( f \in \Sigma \)) condition-(C) if for all fixed \( \tau \in \mathbb{R} \), \( B \in \mathcal{B}(E) \) and \( \varepsilon \in \mathbb{R}_+ \), there exist \( t_0 = t_0(\tau, B, \varepsilon) \) and a subspace \( E_1 \), \( \dim(E_1) < \infty \), such that

\[
(1) \quad \|P(\bigcup_{f \in \Sigma} U_f(t, \tau) B)\|_E < \infty,
\]

\[
(2) \quad \|(I - P)(\bigcup_{f \in \Sigma} U_f(t, \tau) x)\|_E \leq \varepsilon, \quad \forall x \in B,
\]

(4.5)

where \( P : E \to E_1 \) is a projection, \( \mathcal{B}(E) \) is the union of all bounded subsets in \( E \).

**Theorem 4.6.** The process \{\( U_f(t, \tau) \)\} (\( f \in \Sigma \)) has a uniformly (w.r.t. \( f \in \Sigma \)) compact attractor \( \mathcal{A}_\Sigma \) (\( K_f(0) \) denotes the nonempty kernel) which satisfies

\[
\mathcal{A}_\Sigma = \omega_{\emptyset, \Sigma}(B_0) = \omega_{\tau, \Sigma}(B_0) = \bigcup_{B \in \mathcal{B}(E)} \omega_{\tau, \Sigma}(B) = \bigcup_{B \in \mathcal{B}(E)} K_f(\tau),
\]

(4.6)

if (i) there exists a bounded uniformly (w.r.t. \( f \in \Sigma \)) absorbing set \( B_0 \) for process \{\( U_f(t, \tau) \)\} (\( f \in \Sigma \)); (ii) the process \{\( U_f(t, \tau) \)\} (\( f \in \Sigma \)) satisfy the uniform condition-(C).

### 4.2. The uniformly absorbing sets in \( V \)

In this section, we shall prove the existence of uniformly absorbing ball for the process generated by the solution in Theorem 3.4. From section 4.1, we need to establish the existence of a uniformly absorbing ball firstly for the process in \( V \). Define the symbol space as

\[
\Sigma = \mathcal{H}(f_0) = \{ f(s) = f_0(s + h) \mid h \in \mathbb{R} \}^{L^2_{loc}(\mathbb{R}; H)}\), \quad \text{here } f_0 \in L^2_b(\mathbb{R}; H) \text{ is an arbitrarily fixed function.} \]
Theorem 4.7. Assuming \( f \in L^2_0(\mathbb{R}; H) \) and \( v_\tau \in V \), the process \( U_f(t, \tau) \) possesses a uniformly absorbing set in \( V \) for the system (3.16) which is equivalent to (1.1).

Proof. Since the external forces \( f \in \Sigma \) and \( f_0 \in \Sigma \) are translation bounded and \( \|f(t)\|^2_{L^2_{\text{loc}}(\mathbb{R}; H)} \leq C\|f_0\|^2_{L^2_{\text{loc}}(\mathbb{R}; H)} \) (See [4]), it holds
\[
\frac{2}{\nu\lambda_1} \int_{\tau}^{t} e^{-\alpha(t-s)} |f(s)|^2 ds \\
\leq \frac{4}{\alpha} \left( \int_{\tau}^{t} e^{-\alpha(t-s)} |f(s)|^2 ds + \int_{t-1}^{t-1} e^{-\alpha(t-s)} |f(s)|^2 ds + \cdots \right) \\
\leq \frac{2}{\nu\lambda_1} \left( \int_{\tau}^{t-1} |f(s)|^2 ds + e^{-\alpha(t-1)} \int_{t-2}^{t-2} |f(s)|^2 ds + \cdots \right) \\
\leq \frac{2}{\nu\lambda_1} \left( 1 + e^{-\alpha(t-1)} + e^{-2\alpha(t-2)} + \cdots \right) \|f(s)\|^2_{L^2_{\text{loc}}(\mathbb{R}; H)} \\
\leq \frac{C}{\nu\lambda_1} \left( 1 + \frac{1}{\nu\lambda_1} \right) \|f_0\|^2_{L^2_{\text{loc}}(\mathbb{R}; H)}.
\] (4.7)

Let \( D \subset V \) be any bounded set, and \( v_\tau \in D \), then there exists a constant \( d > 0 \) such that
\[
(\|v_0\|^2 + \alpha^2\|v_0\|^2) e^{\int_{\tau}^{t} \left(-\frac{\nu\lambda_1}{2}\right) ds} + \int_{\tau}^{t} e^{-\alpha(t-s)} 2K_0^2 ds \leq d^2,
\] (4.8)
where \( \frac{C}{\nu} \left[ \frac{\nu\lambda_1^2}{\alpha} ||\varphi||^2_{L^\infty(\partial \Omega)} + C_\varepsilon \left| \partial \Omega \right| ||\varphi||^2_{L^\infty(\partial \Omega)} \right] := K_0^2. \)

From the proof of Theorem 3.3, we see that
\[
|v|^2 + \alpha^2|v|^2 \leq \left( \|v_0\|^2 + \alpha^2\|v_0\|^2 \right) e^{\int_{\tau}^{t} \left(-\frac{\nu\lambda_1}{2}\right) ds} + \int_{\tau}^{t} e^{-\alpha(t-s)} 2K_0^2 ds \\
\leq \left( \|v_0\|^2 + \alpha^2\|v_0\|^2 \right) e^{\int_{\tau}^{t} \left(-\frac{\nu\lambda_1}{2}\right) ds} + \int_{\tau}^{t} e^{-\alpha(t-s)} 2K_0^2 ds \\
+ \frac{C}{\nu\lambda_1} \left( 1 + \frac{1}{\nu\lambda_1} \right) \|f_0\|^2_{L^2_{\text{loc}}(\mathbb{R}; H)} \\
\leq \left( d^2 + \frac{C}{\nu\lambda_1} \left( 1 + \frac{1}{\nu\lambda_1} \right) \right) \|f_0\|^2_{L^2_{\text{loc}}(\mathbb{R}; H)}.
\] (4.9)

There exists a fixed time \( T_d > \tau \) such that \( d^2 \leq \frac{C}{\nu\lambda_1} \left( 1 + \frac{1}{\nu\lambda_1} \right) \|f_0\|^2_{L^2_{\text{loc}}(\mathbb{R}; H)} \) for each \( f \in \Sigma \). Let \( \rho_{\nu}^2 = \frac{C}{\nu\lambda_1} \left( 1 + \frac{1}{\nu\lambda_1} \right) \|f_0\|^2_{L^2_{\text{loc}}(\mathbb{R}; H)} \) for any \( v_0 \in V \), then for every bounded domain \( D \subset V \), we have \( U_f(t, \tau) D \subset B_{\nu}(0, \rho_{\nu}) \), where \( B_{\nu}(0, \rho_{\nu}) \) is a uniformly absorbing ball centered at 0 with radius \( \rho_{\nu} \) in \( V \). Hence we complete the proof. 

4.3. The uniformly asymptotic compactness for the process. In this section, we shall verify the uniform condition-(C) to achieve the uniformly (w. r. t. the symbol \( f \in \Sigma \)) asymptotic compactness for the process.

Theorem 4.8. Assume \( f \in L^2_0(\mathbb{R}; H) \) and \( v_\tau \in V \), then the process \( U_f(t, \tau) \) is uniformly asymptotical compactness in \( V \) for the system (3.16), which is equivalent to (1.1).
Proof. **Step 1.** Let $D \subset V$ be any bounded set, and $v_\tau \in V$, then there exists a constant $d > 0$ such that (4.8) holds, and there exists a uniformly absorbing ball $B(0, \rho_V)$ with radius $\rho_V$, so for any $v(x, t) \in D$, we have $\|v\| = \|A^{\frac{\alpha}{2}} v\|_H \leq \rho_V^2$.

**Step 2.** Since $V \subset H$ is a Hilbert space, the process is defined as $U_f(t, \tau) : V \mapsto V$. Let $V = V^1 \bigoplus V^2$, here $V^1 = \text{span}\{\omega_1, \omega_2, \cdots, \omega_m\}$. We have the decomposition of $v = v_1 + v_2$, $v_1 \in V^1$, $v_2 \in V^2$. From the existence of global solution and the existence of a uniformly (with respect to $f \in \Sigma$) absorbing ball, we know $\|v_1\|^2 \leq \|v\|^2 \leq \rho_V^2$, then we only need to prove the $V$-norm of $v_2$ is uniformly small enough.

**Step 3.** Taking inner product of (3.16) with $v_2$, using the orthonormal of $v_1$ and $v_2$, we have

\[
(v_1, v_2) + (\alpha^2 Av_1, v_2) + (\nu Av_2) + (B(v, v), v_2) + (B(\psi, v), v_2) = -(B(\psi, \psi), v_2) + |PF, v_2 > - \nu PF, v_2 > , \tag{4.10}
\]

Next, we shall estimate every terms at the right-hand side of (4.11).

(1) By the property of trilinear operator and (2.5), using (3.5) and the definition of fractional power of operator $A$,

\[
\|v_2\|_{L^4} \leq \|A^{\frac{\alpha}{2}} v_2\|_H , \tag{4.12}
\]

\[
\|A^{\frac{\alpha}{2}} v_2\|_H = \| \sum_{j=m+1} \lambda_j^\frac{\alpha}{2} a_j \omega_j \| H \leq \frac{\rho}{\lambda_{m+1}^\frac{\alpha}{2}} \|A^{\frac{\alpha}{2}} v_2\|_H \tag{4.13}
\]

\[
\|B(v, v)\| \leq \int_{\Omega} |v||\nabla v|| v_2 | dx \leq \|v\|_{L^4} \|\nabla v\|_{L^2} \|v_2\|_{L^4} \leq C \|A^{\frac{\alpha}{2}} v\|_H \|v\|_{L^4} \|A^{\frac{\alpha}{2}} v_2\|_H \leq C \frac{\|v\|^2}{\lambda_{m+1}^\frac{\alpha}{2}} \|A^{\frac{\alpha}{2}} v_2\|_H \leq \frac{\nu}{12} \|v_2\|^2 + \frac{\nu \lambda_{m+1}^\frac{\alpha}{2}}{\lambda_{m+1}^{\frac{\alpha}{2}}} \rho_V^4. \tag{4.14}
\]

(2) By the property of the trilinear operator and (2.5), using the Hardy’s inequality and the definition of fractional power of operator $A$, since $\lambda_j$ is increasing for $j = m + 1, \cdots$, we have

\[
\|v_2\|_H = \| \sum_{j=m+1} \lambda_j^{\frac{\alpha}{2}} a_j \omega_j \| H \leq \| \frac{1}{\lambda_{m+1}^{\frac{\alpha}{2}}} \sum_{j=m+1} \lambda_j^{\frac{\alpha}{2}} a_j \omega_j \| H = \frac{1}{\lambda_{m+1}^\frac{\alpha}{2}} \|A^{\frac{\alpha}{2}} v_2\|_H = \frac{1}{\lambda_{m+1}^{\frac{\alpha}{2}}} \|v_2\|_H . \tag{4.15}
\]
then

\[
|(B(v, \psi), v_2)| \leq \int_{\Omega} |v||\nabla \psi||v_2|dx
\]

\[
\leq \int_{\text{dist}(x, \partial \Omega) \leq C_\varepsilon} |v||\nabla \psi| \cdot \text{dist}(x, \partial \Omega) \frac{|v_2|}{\text{dist}(x, \partial \Omega)}dx
\]

\[
\leq C_4 \|\phi\|_{L^\infty(\partial \Omega)} \int_{\Omega} \frac{|v|}{\text{dist}(x, \partial \Omega)}|v_2|dx
\]

\[
\leq C_4 \|\phi\|_{L^\infty(\partial \Omega)} \left( \left( \int_{\Omega} \frac{|v|^2}{[\text{dist}(x, \partial \Omega)]^2} \right)^{\frac{1}{2}} \left( \int_{\Omega}|v_2|^2dx \right)^{\frac{1}{2}} \right)
\]

\[
\leq C_4 \|\phi\|_{L^\infty(\partial \Omega)} \|A^\frac{\nu}{2}v\|_H \|v_2\|_H
\]

\[
\leq C_4 \|\phi\|_{L^\infty(\partial \Omega)}^2 \|\phi\|_{L^\infty(\partial \Omega)} \frac{1}{\nu \lambda_{m+1}} \|v\|^2 + \frac{\nu}{12} \|v_2\|^2
\]

\[
\leq \frac{\nu}{12} \|v_2\|^2 + \frac{C_4 \|\phi\|_{L^\infty(\partial \Omega)}^2}{\nu \lambda_{m+1}} \rho^2_v. \quad (4.16)
\]

(3) By the similar technique in (2), we derive

\[
|(B(\psi, v), v_2)| \leq \int_{\Omega} |\psi||\nabla v||v_2|dx \leq C_4 \|\phi\|_{L^\infty(\partial \Omega)} \int_{\Omega} |\nabla v||v_2|dx
\]

\[
\leq \frac{\nu}{12} \|v_2\|^2 + \frac{C_4 \|\phi\|_{L^\infty(\partial \Omega)}^2}{\nu \lambda_{m+1}} \rho^2_v. \quad (4.17)
\]

(4) By the Hardy’s inequality and the property of trilinear operator, we obtain

\[
|(B(\psi, \psi), v_2)| \leq \int_{\Omega} |\psi||\nabla \psi||v_2|dx
\]

\[
= \int_{\Omega} |\psi||\nabla \psi| \cdot \text{dist}(x, \partial \Omega) \frac{|v_2|}{\text{dist}(x, \partial \Omega)}dx
\]

\[
\leq C_4 \|\phi\|_{L^\infty(\partial \Omega)} \int_{\text{dist}(x, \partial \Omega) \leq C_\varepsilon} |\psi| \frac{|v_2|}{\text{dist}(x, \partial \Omega)}dx
\]

\[
\leq C_4 \|\phi\|_{L^\infty(\partial \Omega)} \left( \left( \int_{\Omega} \frac{|v_2|^2}{[\text{dist}(x, \partial \Omega)]^2} \right)^{\frac{1}{2}} \left( \int_{\text{dist}(x, \partial \Omega) \leq C_\varepsilon} |\psi|^2dx \right)^{\frac{1}{2}} \right)
\]

\[
\leq \frac{\nu}{12} \|v_2\|^2 + \frac{C_4 \|\phi\|_{L^\infty(\partial \Omega)}^2}{\nu \lambda_{m+1}} \frac{1}{\rho^2_v} \rho^2_v. \quad (4.18)
\]

(5) By the Hardy’s inequality and Lemma 3.1, we derive

\[
\nu|< F, v_2 |
\]

\[
\leq \nu \int_{\Omega} |F||v_2|dx
\]

\[
\leq \nu \int_{\Omega} \frac{|v_2|}{\text{dist}(x, \partial \Omega)} \cdot \text{dist}(x, \partial \Omega)dx
\]

\[
\leq \nu \left( \int_{\Omega} \frac{|v_2|^2}{[\text{dist}(x, \partial \Omega)]^2}dx \right)^{\frac{1}{2}} \left( \int_{C_\varepsilon \| \text{dist}(x, \partial \Omega) \leq C_\varepsilon} |F|^2[\text{dist}(x, \partial \Omega)]^2dx \right)^{\frac{1}{2}}
\]

\[
\leq \nu \|v_2\| \times C_2^\varepsilon \|F\|_{L^2(\Omega)}
\]
Using the Gronwall inequality from $\tau$

In fact, there exists a constant $\tilde{C}$ such that

Combining (4.10)–(4.20), we conclude

Using the Cauchy inequality and Young’s inequality, we have

(6) Using the Cauchy inequality and Young’s inequality, we have

Combining (4.10)–(4.20), we conclude

In fact, there exists a constant $\tilde{C}$ such that $\lambda_1 |v_2|^2 + \|v_2\|^2 \geq \tilde{C}(|v_2|^2 + \alpha^2 \|v_2\|^2)$, and then we have

Using the Gronwall inequality from $\tau$ to $t$, we obtain

Thus, we obtain

By the technique in [4], we have that

Since $\lim_{m\to \infty} \lambda_{m+1} = +\infty$ and $CL_1$, $CL_2$ are bounded, letting $t \to +\infty$ and $m \to +\infty$, so we have

\begin{align}
I_3 &= CL_1 \left( \frac{1}{\lambda_{m+1}^2} + \frac{1}{\lambda_{m+1}} \right) \leq \frac{\varepsilon}{4}, \\
I_4 &= CL_2 \left( \frac{t - \tau}{e^{\frac{\varepsilon}{2}(t-\tau)}} + \frac{1}{\lambda_{m+1}} \right) \leq \frac{\varepsilon}{4}.
\end{align}
Moreover, there exists a time \( t_0 \) such that for \( t > t_0 \), it follows
\[
I_1 = (|v_2(\tau)|^2 + \alpha^2||v_2(\tau)||^2)e^{-\frac{2t}{b_2}}(1-\tau) \leq \frac{\varepsilon}{4}.
\]
Combining (4.23)–(4.26), we conclude that
\[
|v_2|^2 + \alpha^2||v_2||^2 \leq \left( \frac{3}{4} + CL_2 \right)\varepsilon,
\]
which leads to the uniform condition-(C) for the process. This implies the uniformly asymptotic compactness. The proof is complete.

4.4. The uniform attractor and its structure in \( V \). The main result in the paper is stated as follows:

**Theorem 4.9.** Assume \( f \in L_0^2(\mathbb{R}; H) \) and \( v_\tau \in V \), then the continuous process \( U_f(t, \tau) \) possesses a uniformly (with respect to the symbol \( f \in \Sigma \)) compact attractor \( \mathcal{A}_\Sigma = \omega V (B_0) = \bigcup_{f \in \Sigma} \mathcal{K}_f (\tau) \) in \( V \) for the system (3.16), which is equivalent to our original problem (1.1). Here \( B_0 \) is the uniformly absorbing set in \( V \), and \( \mathcal{K}_f (\tau) \) is the nonempty kernel in \( V \) contains almost all bounded completely trajectories.

**Proof.** Given the uniformly absorbing ball and uniformly asymptotical compactness of the process, from the theory of uniform attractor, it is direct to prove the main result.

4.5. The uniform attractor and its structure in \( D(A^{\sigma+\frac{1}{2}}) \) when \( \sigma \in [0, \frac{1}{2}] \). In this final subsection, we reach the existence of the uniform attractor in \( D(A^{\sigma+\frac{1}{2}}) \) (\( \sigma \in [0, \frac{1}{2}] \)). The proof is similar to subsection 3.5, i.e., take inner product for the equation (3.16) with \( A^\sigma v \) or \( A^\sigma v_2 \) to achieve the uniformly absorbing ball and then verify the Uniform Condition-(C) for the process by uniform estimates. We just omit the detail here.

**Theorem 4.10.** Let \( v(t, x) \) be the solution of (3.15) with the initial data \( v_\tau \in \mathcal{W} \cap D(A^{\sigma+\frac{1}{2}}) \), \( \varphi \in L^\infty(\partial \Omega) \), \( f \in L_0^2(\mathbb{R}; H) \), then for any \( \sigma \in [0, \frac{1}{2}] \), the continuous process \( U_f(t, \tau) : D(A^{\sigma+\frac{1}{2}}) \to D(A^{\sigma+\frac{1}{2}}) \) possesses a uniformly (with respect to the symbol \( f \in \Sigma \)) compact attractor \( \mathcal{A}_{\sigma}^f = \omega V (B_0') = \bigcup_{f \in \Sigma} \mathcal{K}_f (\tau) \) in \( D(A^{\sigma+\frac{1}{2}}) \) for the system (3.16), which is equivalent to our original problem (1.1). Here \( B_0' \) is the uniformly absorbing set in \( D(A^{\sigma+\frac{1}{2}}) \), \( \mathcal{K}_f (\tau) \) is the nonempty kernel in \( D(A^{\sigma+\frac{1}{2}}) \) containing almost all bounded completely trajectories.

**Proof.** Using the same technique in Theorem 4.9, we can prove the result, and we omit the detail.

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384 XINGUANG YANG, BAOWEI FENG, THALES MAIER DE SOUZA AND TAIGE WANG
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