A Nonsingular Two Dimensional Black Hole

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\textbf{Abstract}

We construct a model of gravity in 1+1 spacetime dimensions in which the solutions approach the Schwarzschild metric at large $r$ and de Sitter space far inside the horizon. Our model may be viewed as a two dimensional application of the ‘Limiting Curvature Construction’ of reference\textsuperscript{[6]}.
1 Introduction

It has long been realised that the well known singularity theorems of Penrose and Hawking\[1\] imply that general relativity is an incomplete description of the behaviour of spacetime at high curvatures. It is commonly believed that the successful quantization of gravity will provide us with the modifications to the theory that are necessary to avoid the prediction of a geodesically incomplete spacetime manifold. However, as yet such a quantum theory of gravity cannot be said to exist and even if and when it does we will undoubtedly need to look to effective theories in order to understand its implications\[2\]. Since a generic prediction of all attempts to construct a consistent theory of quantum gravity is that the Einstein-Hilbert action be modified at high curvatures by extra, usually higher derivative, invariant terms\[3\]-\[5\] we may expect that this structure will be represented in any effective theory.

We may therefore ask whether it is possible to construct an effective theory of gravity that is intermediate between General Relativity and Quantum Gravity, ie. one that (a) gives the correct (Einstein) behaviour in the low curvature limit and (b) encompasses the nonsingular nature that we hope to be contained in our full theory.

A class of such theories have been studied in references\[6\]-\[9\] with regard to a variety of spacetimes in 3+1 dimensions. The resulting field equations obey the requirements above and in particular satisfy (b) by forcing the spacetime to become de Sitter (and thus singularity free) at high curvatures. This method is known as the ‘Limiting Curvature Construction’\[10\]. However, whilst the theories are successful the calculations can be formidable and numerical solutions are required since the structure of the equations is highly complex.

As a toy model, therefore, it may be instructive to consider the same type of theory in 1+1 dimensions. Our motivations are threefold: (i) to make clear the structure of the 3+1 dimensional models by analogy, (ii) to investigate the relationship between our work and string theory motivated black hole studies\[11\]-\[13\] and (iii) to explore new ideas in a simple and more hospitable setting. Of course, success in two dimensions does not imply success in four but we may hope to shed some new light on the more complicated models with this method.

In this paper we present a modified action for 1+1 dimensional gravity
which we use to describe a black hole spacetime. We show that away from
regions of high curvature we recover the Schwarzschild spacetime and that
as we approach the high curvature regions the spacetime becomes de Sitter.
All our solutions are analytic and the structure of the equations is easy to
see at all times. As in the four dimensional models we achieve the nonsin-
gular behaviour by parametrizing the higher derivative action by means of
a nondynamical scalar field and choosing its potential energy such that the
curvature remains bounded.

As we mentioned above, one motivation for this study is the current in-
terest in two dimensional black holes in string theory\cite{12\textasciitilde15}. There has
recently been substantial progress towards nonperturbative solutions of two
dimensional string theory. Of particular interest have been solutions corre-
sponding to black holes\cite{14}. One approach to solving this problem\cite{13} is to
consider an effective action for the massless modes of the string, in particular
the graviton and the dilaton (the tachyon is usually set to zero), and to study
black hole solutions in this theory.

A further motivation for studying dilatonic gravity in two dimensions is
the hope that such a study might reveal the answer to the quantum mechanical
information loss problem of black holes. What happens to the information
which has entered the Schwarzschild radius before the black hole evaporates
completely by Hawking radiation\cite{15}? Does quantum mechanics break down
in the sense that a pure state evolves into a mixed state, is information con-
tained in the final stages of Hawking radiation, or do stable black hole
remnants remain? In four dimensions, the calculational techniques break
down during the final stages of Hawking radiation, and we are unable to an-
swer the above questions. However, in a simple two dimensional toy model
we might be able to determine the answer using analytical means.

2 The Field Equations

We will consider a Lagrangian consisting of a potential for our non-dynamical
scalar field and a general coupling to the Ricci scalar: (for convenience we
adopt the notation of Banks and O’Loughlin\cite{16})

$$\mathcal{L} = \sqrt{-g} (V(\phi) + D(\phi) R)$$ (1)
In two spacetime dimensions, the most general renormalizable Lagrangian for a graviton and dilaton field theory can be put into this form via a conformal transformation.

Variation of the action with respect to the scalar field yields

\[- \sqrt{-g} \frac{\partial V}{\partial \phi}(\phi) = \frac{\partial D}{\partial \phi}(\phi) \sqrt{-g} R\]  

(2)

Now consider variations with respect to the metric

\[\delta S = \int d^2x \left[ (V(\phi) + D(\phi)R) \frac{1}{2} g_{\alpha\beta} \sqrt{-g} \delta g^{\alpha\beta} + \sqrt{-g} (-D(\phi)R_{\alpha\beta} \delta g^{\alpha\beta} - D(\phi)\nabla^2 g_{\alpha\beta} \delta g^{\alpha\beta} + D(\phi)\nabla_\alpha \nabla_\beta \delta g^{\alpha\beta}) \right] \]  

(3)

Integrating by parts twice, ignoring surface terms and requiring \(\delta S = 0\) gives

\[- D(\phi)R_{\alpha\beta} + (\nabla_\alpha \nabla_\beta - g_{\alpha\beta} \nabla^2)D(\phi) + \frac{1}{2} V(\phi)g_{\alpha\beta} + \frac{1}{2} D(\phi)Rg_{\alpha\beta} = 0\]  

(4)

Finally we note that in two dimensions \(R_{\alpha\beta} = g_{\alpha\beta} R/2\) which puts our field equations in the form

\[V(\phi)g_{\alpha\beta} = 2(\nabla^2 g_{\alpha\beta} - \nabla_\alpha \nabla_\beta)D(\phi)\]  

(5)

Equations (2) and (5) are our vacuum field equations. Together they form three equations, only two of which are independent.

### 3 Two Dimensional Black Hole

By redefining \(\phi\) we can set \(D(\phi) = 1/\phi\). In two spacetime dimensions we can for any ‘static’ metric chose a gauge in which

\[g_{\mu\nu} = \text{diag}(-f(r), f(r)^{-1})\]  

(6)

Our field equations now become

\[\frac{\partial V}{\partial \phi}(\phi) = \frac{1}{\phi^2} R\]  

(7)
\[ g_{\alpha\beta}V(\phi) = 2(g_{\alpha\beta}\nabla^2 - \nabla_\alpha \nabla_\beta) \frac{1}{\phi} \]  \hspace{1cm} (8)

Expanding the covariant derivative in the second expression we obtain

\[ g_{\alpha\beta}V(\phi) = 2g_{\alpha\beta}g^{\gamma\lambda}(\partial_\gamma \partial_\lambda - \Gamma^\sigma_{\gamma\lambda} \partial_\sigma) \frac{1}{\phi} - 2(\partial_\alpha \partial_\beta - \Gamma^\gamma_{\alpha\beta} \partial_\gamma) \frac{1}{\phi} \]  \hspace{1cm} (9)

First we consider the time-time component. This gives us

\[ \dot{\phi}^2 V(\phi) + 2f\phi\dot{\phi}'' - 4f\phi'^2 + f'\phi\dot{\phi}' = 0 \]  \hspace{1cm} (10)

where we have written \((\cdot)' \equiv \partial(\cdot)/\partial r\). Similarly the space-space equation becomes

\[ \dot{\phi}^2 V(\phi) + f'\phi' = 0 \]  \hspace{1cm} (11)

and the variational equation for the scalar field is

\[ \frac{\partial V}{\partial \phi}(\phi) = \frac{1}{\phi^2}(-f'') \]  \hspace{1cm} (12)

Writing \(\Phi \equiv \phi'/\phi\) and combining equations (11) and (12) we obtain the following differential equation

\[ \frac{\Phi'}{\Phi} - \frac{\phi'}{\phi} = 0 \]  \hspace{1cm} (13)

which yields, finally

\[ \frac{\phi'}{\phi^2} = -A, \]  \hspace{1cm} (14)

where \(A\) is a constant of integration. So we may write, formally

\[ \phi = \frac{1}{Ar + B} \]  \hspace{1cm} (15)

Note that by shifting the \(r\) coordinate we can without loss of generality assume \(B = 0\).

Now that we have obtained the dynamics of the scalar field as a function of the metric our logic will be as follows:
1. Choose \( f(r) \) to be of the Schwarzschild form in the asymptotic region \( r \to \infty \) \( (\phi \to 0) \) and solve exactly for the field \( \phi \) and the potential \( V(\phi) \).

2. Choose the potential \( V(\phi) \) such that at large \( \phi \) (small \( r \)) the curvature \( R \) is bounded.

3. Construct an interpolating potential \( V(\phi) \) with the correct behaviour in both asymptotic regions.

4. Use the interpolating potential to solve exactly for the general form of \( f(r) \).

Note that in two dimensions, \( R \) is the only curvature invariant. Hence, after bounding \( R \), all invariants will be bounded. We will show that this procedure singles out de Sitter space as the unique solution. This is a substantial simplification compared to the four dimensional case, where it is necessary to introduce higher derivative curvature invariants in the Lagrangian in order to obtain de Sitter space as a solution.

In the asymptotic region \( r \to \infty \) we have

\[
 f(r) = 1 - \frac{2m}{r} \quad (16)
\]

We then obtain, from equation (12) with \( \phi \to 0 \)

\[
 V(\phi) \sim 2mA^3 \phi^2(r) \quad (17)
\]

where we have ignored a constant of integration.

For \( \phi \to \infty \) \( (r \to 0) \) we demand that \( R \) remains bounded. From (7) it follows that \( \frac{\partial V}{\partial \phi} \sim \phi^{-2} \) and hence, introducing a new proportionality constant \( l \),

\[
 V(\phi) = \frac{2}{l^2} \frac{1}{\phi} \quad (18)
\]

By integrating (11) we obtain

\[
 f(r) = \frac{r^2}{l^2} - C \quad (19)
\]
and by rescaling \( r \) we can take \( C = 1 \). In Appendix A it is shown that the metric given by (6) and (19) corresponds to de Sitter space.

As an example, we consider a specific potential which interpolates between (17) and (18), given by

\[
V(\phi) = \frac{2mA^3\phi^2}{1 + mA^3l^2\phi^3}
\]  

(20)

Using the above form of the potential in equation (11) we arrive at an exact form for the metric component \( f(r) \). Using \( \phi(r) = 1/Ar \), equation (11) becomes

\[
f'(r) = \frac{2mr}{r^3 + ml^2}
\]  

(21)

which can be integrated exactly between \( r_0 \) and \( r \) to give

\[
f(r) = \frac{1}{3} \left( \frac{m}{l} \right)^{2/3} \ln \left[ \frac{r^2 - \left( ml^2 \right)^{1/3} r + \left( ml^2 \right)^{2/3}}{r_0^2 - \left( ml^2 \right)^{1/3} r_0 + \left( ml^2 \right)^{2/3}} \left( \frac{r_0 + \left( ml^2 \right)^{1/3}}{r + \left( ml^2 \right)^{1/3}} \right)^2 \right] +
\]

\[
\frac{2}{\sqrt{3}} \left( \frac{m}{l} \right)^{2/3} \left[ \arctan \left( \frac{2r - \left( ml^2 \right)^{1/3}}{\sqrt{3} \left( ml^2 \right)^{1/3}} \right) - \arctan \left( \frac{2r_0 - \left( ml^2 \right)^{1/3}}{\sqrt{3} \left( ml^2 \right)^{1/3}} \right) \right]
\]  

(22)

This function is plotted in figure 1.

We have now completed our calculation. The two dimensional spacetime that we have constructed is defined by the metric (6) with \( f(r) \) given by the above expression. At macroscopic distances from \( r = 0 \) the spacetime is indistinguishable from the Schwarzschild spacetime which would be predicted by general relativity. As we approach the region of high curvature non-Einstein effects become important and the transition is smoothly made to a de Sitter spacetime. This model is singularity free while possessing all the relevant features of the usual Schwarzschild black hole.

4 Conclusions

We have constructed a class of 1 + 1 dimensional gravity theories which have nonsingular black hole solutions. The models are obtained using the
‘Limiting Curvature Construction’ of reference [3]. Starting from the general higher derivative gravity action parametrized by means of a nondynamical scalar field $\phi$, our subclass of models is obtained by choosing the potential $V(\phi)$ such that the Ricci curvature is bounded. This leads to the asymptotic condition (18) for $V(\phi)$ in the large $\phi$ limit.

Any potential satisfying (18) for large $\phi$ leads to a theory in which space-time approaches de Sitter space at large curvatures. Choosing an appropriate form of $V(\phi)$ for small $\phi$ (see (17)) leads to a model in which the ‘static’ solutions approach the ‘usual’ Schwarschild metric at large distances.

Note that the singularity-free nature of our solutions emerges as a consequence of the ‘Limiting Curvature Principle’ and is not put in by hand.

The principles of our construction differ from those upon which the recent construction of nonsingular black holes by Banks and O’Loughlin [17] are based. These authors also obtain solutions which have a horizon and approach de Sitter space inside the horizon, but their solutions are different outside. Our black holes have a dilaton field outside of the Schwarschild horizon which rapidly approaches zero, and thus they for large $r$ correspond to the dimensionally reduced four dimensional black hole. Note that the black hole solutions of Ref. [12] have a nonvanishing dilaton at large $r$.

Our theory should be seen as a two dimensional effective theory of gravity which contains the appropriate Schwarzschild solution as a unique low curvature approximation while at the same time encapsulating the singularity free nature which we hope will be a feature of a more complete, quantum, theory of gravity.

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A The 2-D De Sitter Spacetime

Here we will show that the two dimensional de Sitter spacetime may be represented in the form (6). Consider the metric (6) with

$$f(r) = \left(\frac{r}{l}\right)^2 - 1$$

(23)
Let us introduce a new coordinate $\tau$ defined by

$$\frac{d\tau}{dr} = \left(\left(\frac{r}{l}\right)^2 - 1\right)^{-1/2}$$

Solving for $\tau(r)$ we obtain

$$\frac{\tau(r)}{l} = \text{arccosh}\left(\frac{r}{l}\right)$$

which implies that our line element in $(t, \tau)$ coordinates is

$$ds^2 = d\tau^2 - \sinh^2(\tau/l) dt^2$$

which is of the form of a de Sitter spacetime with a de Sitter bounce as required.

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Figure Caption

Figure 1: The metric coefficient $f(r)$ as a function of $r$. 

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