ON THE RIGIDITY OF INTEGRABLE DEFORMATIONS OF
TWIST MAPS AND HAMILTONIAN FLOWS

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Abstract. In this article we investigate rigidity properties of integrable area-
preserving twist maps of the cylinder. More specifically, we prove that if a
deformation of the standard integrable map preserves rotational invariant cir-
cles (i.e., homotopically non-trivial invariant curves of the cylinder on which
the dynamics is conjugate to a rotation) for any rational rotation number in an
open interval (without any further assumption on its size), then the deforma-
tion must be trivial. Analogue phenomena for integrable Tonelli Hamiltonian
flows in higher dimensions are discussed.

1. Introduction and main results

In the study of Hamiltonian systems, an important role is played by the so-called
integrable systems. These systems – whose dynamics is quite simple to describe due
to the presence of a large number of symmetries – arise quite naturally in many
physical and geometric problems. Due to the polyedrical nature of this subject,
several notions of integrability of a Hamiltonian system have been introduced in
various contexts, such as integrability à la Arnol’d-Liouville (i.e., the existence of
a maximal set of independent and Poisson commuting integrals of motion, see for
example [6,9] and [14,35] for non-commutative generalizations) or C\textsuperscript{0}-integrability
(i.e., the existence of a continuous foliation of the phase space by invariant La-
grangian graphs, see for example [12,27]).

Investigating which of the properties of these systems break or are preserved in
the passage from the integrable regime to non-integrable one is a very natural and
enthralling question, which is part of what Henri Poincaré recognised as “the general
problem of dynamics” (see [33 Sec. 13]). Among the many results in this direction,
a place of honor goes undoubtedly to Kolmogorov’s breakthrough in 1954 [25] and
the subsequent works by Arnol’d [7,8] and Moser [31], that paved the ground
to what is nowadays known as KAM theory and to its subsequent developments
(see [16] for a detailed historical account and, for example, [13,17,18,28,29] and
references therein, for more recent advances).

In this article we are interested in a different, but related, question: instead of
investigating which aspects of integrable systems are preserved or which are de-
stroyed, we would like to understand which classes of deformations of the system
do preserve integrability and possibly characterize them. In other words, we aim to
shed more light on the rigidity of these systems. This question appears in various
contexts, where it translates into very difficult problems and conjectures, among
which we mention Birkhoff’s conjecture in billiard dynamics (namely, to prove that only billiard maps corresponding to elliptic tables are integrable, see \cite{22,24} and references therein), the geometric problem of characterizing integrable geodesic flows on a Riemannian manifold (see for example \cite{11,12,26} and references therein), or in the study of the N-vortex problem in bounded domains of the plane, the question of which deformations preserve integrability (a first step in the case of the 2-vortex problem on the disc appears in \cite{19}).

1.1. Deformations of Twist maps. We start by considering area-preserving deformations of an integrable standard twist map of the cylinder $T \times \mathbb{R}$, namely a map of the form

$$f_\varepsilon(\theta, r) = (\theta + r + \varepsilon g(\theta), r + \varepsilon g(\theta)), \quad \varepsilon > 0,$$

where $g$ is assumed to be continuously differentiable; without loss of generality, we can assume that $g$ has zero average (otherwise, we consider the change of coordinates $(\theta, r) \mapsto (\theta, r - \varepsilon \mu)$, where $\mu$ denotes the average of $g$).

For our purposes, we will adopt a very weak notion of integrability in the following sense (compare with \cite{24}): we assume that the system admits invariant curves, homotopic to the zero section $\{r = 0\}$, on which the dynamics is conjugate to a rotation $\theta \in \mathbb{T} \mapsto \theta + \alpha$, for all $\alpha \in \mathbb{Q}$ in a given open interval (without any assumption on its size).

More precisely, we will prove the following result.

**Theorem 1.** Suppose that there exist $\varepsilon_0 > 0$ and some interval $(\alpha_1, \alpha_2) \subset (0,1)$ such that the twist maps $f_\varepsilon(\theta, r) = (\theta + r + \varepsilon g(\theta), r + \varepsilon g(\theta))$ as in (1.1) are integrable for any $0 < \varepsilon \leq \varepsilon_0$, namely for any $0 < \varepsilon \leq \varepsilon_0$ and any $\alpha \in \mathbb{Q} \cap (\alpha_1, \alpha_2)$, $f_\varepsilon$ has an invariant circle, homotopic to the zero section $\{r = 0\}$, on which the dynamics is conjugate to the rotation $\theta \mapsto \theta + \alpha$. Then the perturbation is trivial, i.e., $g \equiv 0$.

**Remark 1.1.** We remark that a-priori we are not asking that the original foliation by invariant curves for $f_0$ is completely preserved, but only that a subset is; moreover, we do not impose any condition on the location of this subset (compare with \cite{24}, in the billiard case, where a similar local integrability result was shown, but with the assumption that the region of integrability accumulates to the boundary of the billiard).

1.2. Hamiltonian flows. In Section 3 we focus on the rigidity of some integrable Hamiltonian flows. In particular, we address the question of what are the implications on the form of the deformation, if we know that all invariant tori with rotation vectors in a given set are preserved. We discuss in Theorem 2 how this condition reflects on which Fourier coefficients of the deformation do vanish.

Let us consider the following class of deformations $H_\varepsilon : \mathbb{T}^n \times \mathbb{R}^n \to \mathbb{R}$, where

$$H_\varepsilon(q, p) := H_0(p) + \varepsilon G(q), \quad 0 < \varepsilon < \varepsilon_0$$

where we assume that $H_0$ (and hence $H_\varepsilon$) is of Tonelli type, namely:
- $H_0 \in C^2(\mathbb{R}^n)$,
- $H_0$ is strictly convex in $p$ in the $C^2$-sense, i.e., $\partial^2_{pp} H_0(p)$ is positive definite for every $p \in \mathbb{R}^n$,
- $H_0$ is superlinear in $p$, i.e., for every $A > 0$ there exists $B = B(A)$ such that $H_0(p) \geq A\|p\| - B$ for every $p \in \mathbb{R}^n$.

**Remark 1.2.** Actually we do not need $H_0$ to be defined on the whole $\mathbb{R}^n$. It suffices that it is defined on an open neighborhood of the origin $U \subset \mathbb{R}^n$ and that it can be extended globally to a Tonelli Hamiltonian (in other words, it is the restriction of a Tonelli Hamiltonian to an open neighborhood of the origin).

Given $\rho \in \mathbb{R}^n$, we define the corresponding module of resonances (it is a module over $\mathbb{Z}$) by

$$R(\rho) := \{k \in \mathbb{Z}^n : k \cdot \rho = 0\};$$

each $k \neq (0, \ldots, 0) \in R(\rho)$ is called a resonance.

In Section 3 we will prove the following result.

**Theorem 2.** Let

$$\mathcal{W} := \{\rho := (\rho_1, \ldots, \rho_n) \in \mathbb{Q}^n : \text{rank} R(\rho) = n - 1 \quad \text{and} \quad \rho_i \neq 0 \quad \text{for all} \ i\}$$

and let $\Omega = A \cap \mathcal{W}$ where $A$ is an open neighborhood of the origin $\mathbb{R}^n$.

(i) Assume that for every $\rho \in \Omega$ and for every $0 < \varepsilon < \varepsilon_0$, the perturbed Hamiltonian system associated to $H_\varepsilon$ admits an invariant Lagrangian torus, hamiltonianly isotopic to the zero section, on which the motion is conjugate to a translation by $\rho$ on $\mathbb{T}^n$. Then:

$$G(q) = \sum_{i=1}^n G_i(q_i),$$

where $G_i : \mathbb{T} \to \mathbb{R}$ for every $i = 1, \ldots, n$.

(ii) If in addition for every $0 < \varepsilon < \varepsilon_0$ the perturbed Hamiltonian system associated to $H_\varepsilon$ admits an invariant Lagrangian torus, hamiltonianly isotopic to the zero section, on which the motion is conjugate to a translation by $\rho = (\rho_1, \ldots, \rho_n)$, with $\rho_i = 0$ for some $1 \leq i \leq n$, then $G_i(q_i) \equiv \text{constant}$.

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2. Families of integrable exact twist maps

Let $T := \mathbb{R}/\mathbb{Z}$ be the 1-dimensional torus. Let

\begin{equation}
    f_\varepsilon(\theta, r) := (\theta + r + \varepsilon g(\theta), r + \varepsilon g(\theta)) \quad \varepsilon \in \mathbb{R}
\end{equation}

be a one parameter family of exact twist maps as in (1.1); note that these maps are orientation preserving.

In order to prove Theorem 1, we start by some preliminary facts. Let $f_\varepsilon : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \times \mathbb{R}$ be a lift of $f_\varepsilon : T \times \mathbb{R} \to T \times \mathbb{R}$ to the universal cover $\mathbb{R} \times \mathbb{R}$, namely a map satisfying $\pi \circ f_\varepsilon = f_\varepsilon \circ \pi$, where $\pi : \mathbb{R} \times \mathbb{R} \to T \times \mathbb{R}$ is the projection $(\bar{\theta}, r) \mapsto (\bar{\theta} \pmod{\mathbb{Z}}, r)$. Since $f_\varepsilon$ is an orientation preserving diffeomorphism, then $f_\varepsilon(\bar{\theta} + 1, r) = f(\bar{\theta}, r) + (1, 0)$, where we denoted with $(\bar{\theta}, r)$ the variables in the covering space. We will indicate $(\bar{\theta}', r') := f_\varepsilon(\bar{\theta}, r)$.

Let $n \geq 1$ and $m \in \mathbb{Z}$, such that $\gcd(n, m) = 1$. We say that a curve $\Gamma$ in $T \times \mathbb{R}$ is $(n, m)$-completely periodic for a twist map $f$, if it is invariant, homotopic to the zero section $\{r = 0\}$ and every point $(\bar{\theta}_0, r_0) \in \Gamma$ (where $\Gamma$ denotes a lift of $\Gamma$ to the universal cover) satisfies

\begin{equation}
    \tilde{f}_n(\bar{\theta}_0, r_0) = (\bar{\theta}_0 + m, r_0).
\end{equation}

In particular, it follows that $\Gamma$ has rotation number $\frac{m}{n}$ and it must be a Lipschitz graph, as it follows from Birkhoff’s theorem (see for example, [20, Theorem 35.4]) namely

$$
\Gamma := \{(\theta, \gamma(\theta)) : \theta \in T\}
$$

where $\gamma : T \to \mathbb{R}$ is a Lipschitz function.

Note that in general, the twist property of a map is not closed under composition. However any power of a twist map is locally twist over its invariant resonant curve. See [32, Lemma 2.1].

The map $f_\varepsilon(\theta, r)$ is twist and exact and let $h_\varepsilon : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be its generating function, given by:

\begin{equation}
    h_\varepsilon(\bar{\theta}, \bar{\theta}') = \frac{1}{2}(\bar{\theta}' - \bar{\theta})^2 + \varepsilon G(\bar{\theta}) := h_0(\bar{\theta}, \bar{\theta}') + \varepsilon G(\bar{\theta}),
\end{equation}

where $G(\bar{\theta})$ is a 1-periodic primitive of $g(\bar{\theta})$ (recall that we are assuming that $g$ has zero average). Take now $\frac{m}{n} \in (\alpha_1, \alpha_2)$, with $\gcd(m, n) = 1$; By hypothesis, for any $\alpha \in (\alpha_1, \alpha_2)$ and $\varepsilon$ small enough $f_\varepsilon$ has an invariant rotational circle $\Gamma_\varepsilon^\gamma$ on which the dynamics is conjugate to a rotation by $\alpha$, then the circle

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1 A map $f$ is twist when the twist property $\partial_r \pi_1 \tilde{f}(\bar{\theta}, r) \neq 0$ holds, where $\pi_1 : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ the projection onto the first component. If the twist is positive (resp. negative), then any vertical line is deviated to the right (resp. left) by its tangent map.

2 Namely, there exists a generating function, i.e., a function $h : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ such that $h(\bar{\theta} + 1, \bar{\theta}' + 1) = h(\bar{\theta}, \bar{\theta}')$ (periodicity) and $r' \, d\bar{\theta}' - r \, d\bar{\theta} = dh(\bar{\theta}, \bar{\theta}')$ (exactness condition); in other words:

$$
\begin{cases}
    r' = \partial_{\bar{\theta}'} h(\bar{\theta}, \bar{\theta}') \\
    r = -\partial_{\bar{\theta}} h(\bar{\theta}, \bar{\theta}').
\end{cases}
$$
\( \Gamma_\mathbb{R}^\varepsilon \) is foliated by minimizing periodic orbits of rotation number \( \frac{m}{n} \); in particular, all these periodic orbits have the same Lagrangian action (see for example \[20\] Theorem 35.2): to each \( \theta_0 \in \mathbb{T} \) there corresponds a periodic orbit on \( \Gamma_\mathbb{R}^\varepsilon \), \( \theta_0 \mapsto (\theta_j^\varepsilon, r_j^\varepsilon) \), with \( \theta_j^\varepsilon = \theta_j^0 + \varepsilon G(\theta_j^0) \) and \( \theta_j^0 := \theta_0 \).

**Lemma 2.1.** Suppose that there exist \( \varepsilon_0 > 0 \) and some rational number \( \frac{m}{n} \in (\alpha_1, \alpha_2) \), with \( m, n \) coprime, such that the exact twist maps \( f_\varepsilon \) are integrable for every \( 0 < \varepsilon \leq \varepsilon_0 \). Then, for every \( k \in n\mathbb{Z} \setminus \{0\} \) the \( k \)-th Fourier coefficient of \( G \) vanishes.

**Proof.** Let

\[
L_\varepsilon^\ast (\tilde{\theta}_0) := \sum_{j=0}^{n-1} h_\varepsilon(\tilde{\theta}_j^\varepsilon, \tilde{\theta}_{j+1}^\varepsilon) = \sum_{j=0}^{n-1} h_0(\tilde{\theta}_j^\varepsilon, \tilde{\theta}_{j+1}^\varepsilon) + \varepsilon G(\tilde{\theta}_j^\varepsilon)
\]

be the Lagrangian action functional associated to the orbit of \( \tilde{f}_\varepsilon \) of rotation number \( \frac{m}{n} \) starting at \( \tilde{\theta}_0 = \tilde{\theta}_0 \), which necessarily lies on \( \Gamma_\mathbb{R}^\varepsilon \). Observe that \( L_\varepsilon^\ast \) is 1-periodic. Since the circle \( \Gamma_\mathbb{R}^\varepsilon \) is invariant, by Mather’s theory it follows that \( L_\varepsilon^\ast \) must be constant (see for example \[20\] Theorem 35.2)). By following a perturbative approach, we are going to get information on its first-order expansion in power of \( \varepsilon \), namely:

\[
L_\varepsilon^\ast (\tilde{\theta}_0) = \ell_\varepsilon^{(0)}(\tilde{\theta}_0) + \varepsilon \ell_\varepsilon^{(1)}(\tilde{\theta}_0) + O(\varepsilon^2).
\]

Let \( \tilde{\theta}_j^0(\tilde{\theta}_0) := \tilde{\theta}_0 + j \frac{m}{n} \) be the zero order term in \( \varepsilon \) of \( \tilde{\theta}_j^\varepsilon = \tilde{\pi}_1 \tilde{f}_\varepsilon(\tilde{\theta}_0, \gamma_0(\tilde{\theta}_0)) \), where \( \tilde{\pi}_1 : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R} \) denotes the projection on the first component. Then, \( \tilde{\theta}_j^\varepsilon = \tilde{\theta}_j^0 + O(\varepsilon) \), for \( j = 1, \ldots, n-1 \) and by expanding \( L_\varepsilon^\ast \) with respect to \( \varepsilon \) we get:

\[
L_\varepsilon^\ast (\tilde{\theta}_0) = \sum_{j=0}^{n-1} h_0(\tilde{\theta}_j^0, \tilde{\theta}_{j+1}^0) + \varepsilon G(\tilde{\theta}_j^0)
\]

\[
= \sum_{j=0}^{n-1} \frac{1}{2} [\tilde{\theta}_{j+1}^0 - \tilde{\theta}_j^\varepsilon + (\tilde{\theta}_{j+1}^0 - \tilde{\theta}_j^0) - (\tilde{\theta}_{j+1}^0 - \tilde{\theta}_j^0)]^2 + \varepsilon G(\tilde{\theta}_j^0) + O(\varepsilon^2).
\]

\[\text{In the literature, } L_\varepsilon^\ast \text{ is often referred to as the subharmonic potential of } \Gamma_\mathbb{R}^\varepsilon \text{ under the twist perturbation } f_\varepsilon, \text{ while its first order term as its (subharmonic) Melnikov potential (see, for instance, } [32] \text{ p. 1882)}).\]
Let us now observe that the term
\[
\mathcal{J}(\hat{\theta}_0, \varepsilon) := \frac{1}{2} \sum_{j=0}^{n-1} (\hat{\theta}_{j+1}^0 - \hat{\theta}_j^0)^2 + \left[ (\hat{\theta}_{j+1}^\varepsilon - \hat{\theta}_j^\varepsilon) - (\hat{\theta}_{j+1}^0 - \hat{\theta}_j^0) \right]^2 + 2(\hat{\theta}_{j+1}^0 - \hat{\theta}_j^0) \left[ (\hat{\theta}_{j+1}^\varepsilon - \hat{\theta}_j^\varepsilon) - (\hat{\theta}_{j+1}^0 - \hat{\theta}_j^0) \right] = \frac{1}{2} \sum_{j=0}^{n-1} (\hat{\theta}_{j+1}^0 - \hat{\theta}_j^0)^2 + O(\varepsilon^2) + \frac{m}{n} (\hat{\theta}_n^\varepsilon - \hat{\theta}_0^\varepsilon - m)
\]
where we used that \( \hat{\theta}_{j+1}^0 - \hat{\theta}_j^0 = \frac{m}{n} \), that the \( \sum_{j=0}^{n-1} (\hat{\theta}_{j+1}^\varepsilon - \hat{\theta}_j^\varepsilon) \) is telescopic, and that \( \hat{\theta}_n^\varepsilon = \hat{\theta}_0^\varepsilon + m \).
Hence, we get
\[
L_{\varepsilon \pi}^\varepsilon(\hat{\theta}_0) = \sum_{j=0}^{n-1} h_0(\hat{\theta}_j^0, \hat{\theta}_{j+1}^0) + \varepsilon \sum_{j=0}^{n-1} G(\hat{\theta}_j^0) + O(\varepsilon^2).
\]
By identifying terms in (2.6) with those in equation (2.5) we conclude:
\[
\ell_\pi^{(0)}(\hat{\theta}_0) := \sum_{j=0}^{n-1} h_0(\hat{\theta}_j^0, \hat{\theta}_{j+1}^0) = \frac{1}{2} \sum_{j=0}^{n-1} (\hat{\theta}_{j+1}^0 - \hat{\theta}_j^0)^2 \equiv \frac{1}{2} \frac{m^2}{n},
\]
and
\[
\ell_\pi^{(1)}(\hat{\theta}_0) := \sum_{j=0}^{n-1} G(\hat{\theta}_j^0).
\]
Since \( \ell_\pi^{(0)}(\hat{\theta}_0) \) is constant, then we conclude that in order to have \( L_{\varepsilon \pi}^\varepsilon(\hat{\theta}_0) \) constant, we necessarily need
\[
\ell_\pi^{(1)}(\hat{\theta}_0) = \sum_{j=0}^{n-1} G(\hat{\theta}_j^0) \equiv \text{const}.
\]
In particular, let \( k \neq 0 \) be a multiple of \( n \) and multiply the above relation by \( e^{-i2\pi k\theta_0} \); integrating and applying the change of variables \( \varphi := \theta_0 - \frac{im}{n} \), we get:
\[
0 = \int_{\pi} G(\theta_0 + \frac{im}{n}) e^{-i2\pi k\theta_0} d\theta_0 = \sum_{j=0}^{n-1} \int_{\pi} G(\varphi) e^{-i2\pi k(\varphi - \frac{jm}{n})} d\varphi.
\]
Being \( e^{i2\pi k\frac{jm}{n}} = 1 \), equation (2.7) reduces to
\[
0 = \int_{\pi} G(\varphi) e^{-i2\pi k\varphi} d\varphi = \int_{\pi} G(\varphi) e^{-i2\pi k\varphi} d\varphi \equiv \hat{G}(k),
\]
where \( \hat{G}(k) \) denotes the \( k \)-th Fourier’s coefficient of the function \( G \).
Lemma 2.2. Suppose that the hypotheses of Lemma 2.1 hold for all \( r \in (\alpha_1, \alpha_2) \cap \mathbb{Q} \), then \( G \) is a trigonometric polynomial of order \( N > (\alpha_2 - \alpha_1)^{-1} \).

Proof. The proof is an easy consequence of the following fact. Let us start by observing that the open interval \( I := (\alpha_1, \alpha_2) \) is homeomorphic to an open arc \( A_I \) of \( S^1 \subset \mathbb{C} \) via the identification \( x \mapsto e^{2\pi i x} \). By uniform distribution of the roots of unity, for \( N > (\alpha_2 - \alpha_1)^{-1} \), there exists a primitive root of unity \( \lambda' \in A_I \) of order \( N \) such that \( \arg \lambda' \in (\alpha_1, \alpha_2) \) and a neighborhood \( U_N \subset A_I \) of \( \lambda' \) containing, for any natural \( n \geq N \), a primitive root of unity \( \lambda_n \) of order \( n \). For any \( n \geq N \), let \( m/n := \arg \lambda_n \in (\alpha_1, \alpha_2) \). Either \( m \) and \( n \) are coprime, in which case, by hypothesis, there corresponds an \( n \)-periodic circle \( \Gamma_{\vec{m}} \) of rotation number \( m/n \), or \( m/n \equiv m'/n' \) with \( m'/n' \) coprime and \( n = kn' \) for some \( k \in \mathbb{Z} \). In both cases, it follows from Lemma 2.1 that \( \hat{G}(\pm n) = 0 \) for every \( n \geq N \) and the thesis follows. \( \square \)

Remark 2.1. One could actually prove that eventually, the interval \( (\alpha_1, \alpha_2) \) contains rationals \( m/n \) with \( m \) and \( n \) coprime, for any \( n \geq N \), to which Lemma 2.1 applies. In fact, in [23] Iwaniec proved that the Jacobsthal function \( J : \mathbb{N} \to \mathbb{N}, n \mapsto J(n) \) has a sublinear growth of order \( O(\log^2 n) \). As a byproduct, there exists \( N = N((\alpha_1 - \alpha_2)) \in \mathbb{N} \) such that for any \( n \geq N \) we have that \( (J(n) + 1)/n < (\alpha_2 - \alpha_1) \), which guarantees that there exists a sequence of \( J(n) \) rational numbers of the form \( \frac{\pm 1}{n} \) with \( k \in \mathbb{N} \) and \( j = 1, \ldots, J(n) \) which actually belong to \( (\alpha_1, \alpha_2) \), among which at least one is irreducible. Then, by hypothesis of Lemma 2.1, for any of such \( n \geq N \) the \( \pm n \)-th Fourier’s coefficient of \( G \) vanish, this implying that \( G \) is a trigonometric polynomial of order \( N \). This concludes the proof.

2.1. Proof of Theorem 1 In this section we want to prove Theorem 1. First, let us observe that it follows from Lemma 2.2 that

\[
(2.9) \quad f_\varepsilon(\theta, r) = (\theta + r + \varepsilon P_N(\theta), r + \varepsilon P_N(\theta)),
\]

where \( P_N(\theta) \) is a (non constant) trigonometric polynomial of degree \( N \); in other words, \( g \) and hence \( G \) are trigonometric polynomials.

Let us prove the following proposition, which is a (slight) generalization of a result which has been pointed out to us by Marie-Claude Arnaud (see unpublished notes 5).

Proposition 2.1. Let \( n \geq 1 \) and \( m \in \mathbb{Z} \) with \( \gcd(n, m) = 1 \) and let \( P_N \) be a non-constant trigonometric polynomial. The set \( \mathcal{I}_{n,m} = \mathcal{I}_{n,m}(P_N) \) of \( \varepsilon \in \mathbb{R} \) such that the map \( f_\varepsilon \) as in (2.9) admits an \( (n,m) \)-completely periodic curve, is closed with empty interior.

4The Jacobsthal function of \( n \in \mathbb{N} \) is defined as the smallest \( m \in \mathbb{N} \) such that every sequence of \( m \) consecutive integers contains at least one which is co-prime with \( n \).

5Since \( J \) is logarithmic, one easy (non sharp) estimate on \( N \) can be given as \( N > N_0 = C(b - a)^{-2} \), for some positive \( C \) big enough.
Proof. The proof is essentially the same as the one provided by M.-C. Arnaud in the case where \( f \) reduces to the standard map. We repeat it here for the reader’s convenience. It consists of two main steps:

- **The set \( \mathcal{I}_{n,m} \) is closed.** The fact that this set is closed follows from Birkhoff’s theory of periodic orbits (see [21], for instance). Indeed, let a sequence \( \varepsilon_k \to \varepsilon, \ k \to \infty \), be such that for each \( \varepsilon_k \) there exists a Lipschitz continuous \( \gamma_k : T \to \mathbb{R} \) whose graph is \((n,m)\)-completely periodic, see (2.2). The rotation number of any such a curve is \( \frac{m}{n} \) and the lift of any periodic orbit

\[
(\tilde{\theta}_0, \gamma_k(\theta_0)), \ldots, (\tilde{\theta}_n = \tilde{\theta}_0 + m, \gamma_k(\theta_0))
\]

is well ordered (the graphs \( \Gamma^{(k)}_{\frac{m}{n}} := \text{Graph}(\gamma_k) \) correspond to the Mather set of rotation number \( \frac{m}{n} \) of the maps \( f_{\varepsilon_k} \); see, for example, [20,34] for more details):

\[(2.10)\]

\[
\tilde{\theta}_j \leq \tilde{\theta}_\ell + p \implies \tilde{\theta}_{j+1} \leq \tilde{\theta}_{\ell+1} + p \quad \forall \ j, \ell, p \in \mathbb{Z}.
\]

It follows from (2.10) and (2.9) that

\[
|m| \geq |\tilde{\theta}_{j+1} - \tilde{\theta}_j| = |\gamma_k(\theta_j) + \varepsilon_k P_N(\theta_j)|.
\]

Since \( \Gamma^{(k)}_{\frac{m}{n}} \) is completely foliated by periodic points, we obtain:

\[(2.11)\]

\[
-|m| - \|P_N\|_{\infty}|\varepsilon_k| \leq -|m| - \varepsilon_k P_N(\theta) \leq \gamma_k(\theta) \leq |m| + \varepsilon_k P_N(\theta) \leq |m| + \|P_N\|_{\infty}|\varepsilon_k|.
\]

Moreover, since \( f_\varepsilon \) is a (right) twist, the vertical line \((0,1)\) is deviated on the right by the differential

\[
Df_\varepsilon(\theta, r) = \begin{pmatrix}
1 + \varepsilon P_N'(\theta) & 1 \\
\varepsilon P_N'(\theta) & 1
\end{pmatrix},
\]

while on the left by

\[
Df^{-1}_\varepsilon(\theta, r) = \begin{pmatrix}
1 & -1 \\
-\varepsilon P_N'(\theta) & 1 + \varepsilon P_N'(\theta)
\end{pmatrix},
\]

respectively. Thus one can deduce \textit{a priori} the following estimate on the Lipschitz constant \( \lambda_k \) of \( \gamma_k \) (see [3, Proposition 6])

\[(2.12)\]

\[
0 \leq \lambda_k \leq \max \{1, \max \{1 + \varepsilon_k P_N'(\theta)\}\} \leq 1 + |\varepsilon_k|\|P_N\|_{\infty}.
\]

Finally, by Ascoli-Arzelà theorem, there exists a subsequence converging to some \( \gamma \) in the \( C^0 \)-topology, whose graph is then invariant by \( f_\varepsilon \) and \((n,m)\)-completely periodic. We conclude that \( \varepsilon \in \mathcal{I}_{n,m} \), hence \( \mathcal{I}_{n,m} \) is closed.

- **The set of \( \mathcal{I}_{n,m} \) has empty interior.** We are going to prove that the closed set \( \mathcal{I}_{n,m} \) does not contain any non-trivial interval. Since \( f_\varepsilon \) is real analytic, its lift admits a complex holomorphic extension in \( \theta, r \) and \( \varepsilon \); for the sake of simplicity, we use the same notation for \( f_\varepsilon \) and its lift to the universal cover. In [3, Proposition 6], Arnaud points out that, if \( \Gamma \) is the invariant graph of an exact twist map \( f \), then for any \( x \in \Gamma \) and any \( n \in \mathbb{N} \setminus \{0\} \), the first (or "horizontal") coordinate
of \( Df^n(f^{-n}(x))(0,1) \) is different from 0, which results in the fact that

\[
Df^n(\theta, \gamma(\theta)) = \begin{pmatrix} a_n(\theta, \gamma(\theta)) & b_n(\theta, \gamma(\theta)) \\ c_n(\theta, \gamma(\theta)) & d_n(\theta, \gamma(\theta)) \end{pmatrix}
\]

has the property that \( b_n(\theta, \gamma(\theta)) \neq 0 \).

So, let now

\[
Df^n(\theta, r) = \begin{pmatrix} a_n,\varepsilon(\theta, r) & b_n,\varepsilon(\theta, r) \\ c_n,\varepsilon(\theta, r) & d_n,\varepsilon(\theta, r) \end{pmatrix}
\]

and define the following set

\[
(2.13) \quad \mathcal{R}_\varepsilon = \{ (\theta, r) \in \mathbb{C}^2 : \pi_1 f^n_\varepsilon(\theta, r) = \theta + m, \quad b_n,\varepsilon(\theta) \neq 0 \}.
\]

where \( \pi_1 \) denotes the canonical projection on \( \theta \)-component. The set

\[
(2.14) \quad \mathcal{R} := \bigcup \{ \varepsilon \times \mathcal{R}_\varepsilon \}
\]

is a 2-dimensional complex submanifold of \( \mathbb{C}^3 \). As a matter of fact, since the map

\[
g : \mathbb{C}^3 \longrightarrow \mathbb{C} \\
(\varepsilon, \theta, r) \mapsto \pi_1 f^n_\varepsilon(\theta, r) - \theta - m
\]

satisfies \( \partial_\varepsilon g(\theta, r) = b_n,\varepsilon(\theta, r) \neq 0 \), is then a holomorphic submersion. Note that \( b_n(\theta, r) \neq 0 \) is an open condition. Thus

\[
(2.15) \quad \mathcal{V} := g^{-1}(0) \cap (\{ \varepsilon \} \times \mathcal{R}_\varepsilon)
\]

is a submanifold of (complex) dimension 2.

Let us now suppose by contradiction that for \( \varepsilon \in [a, b] \) with \( a < b \), \( f_\varepsilon \) has an invariant \((n,m)\)-completely periodic curve, which is the graph of \( \theta \mapsto \gamma_\varepsilon(\theta) \).

Then

\[
(2.16) \quad \mathcal{E} = \{ (\varepsilon, \theta, \gamma_\varepsilon(\theta)) : \varepsilon \in [a, b], \theta \in \mathbb{R} \} \subset \mathcal{V}.
\]

Let now \( \mathcal{C} \) be the connected component of \( \mathcal{V} \) containing \( \mathcal{E} \) and let us define the map

\[
\Delta : \mathcal{V} \longrightarrow \mathbb{C} \\
(\varepsilon, \theta, r) \mapsto \pi_2 f^n_\varepsilon(\theta, r) - r,
\]

where \( \pi_2 \) denotes the projection on the \( r \)-component. Since the map \( \Delta \) is holomorphic and vanishes on \( \mathcal{E} \), it follows from the Cauchy-Riemann equations that it also vanishes on \( \mathcal{C} \). So, all \((\varepsilon, \theta, r) \in \mathcal{C}\) are such that

\[
(2.17) \quad f^n_\varepsilon(\theta, r) = (\theta + m, r).
\]

In particular \( \{ (b, \theta, \gamma_b(\theta)) : \theta \in \mathbb{R} \} \in \mathcal{V} \) and, by the implicit function theorem, we can find a neighborhood of \( b \), hence some \( b' > b \) such that for all \( c \in [b, b'] \), \( f_c \) has a curve, graph of \( \gamma_c \), such that \( \pi_1 f^n_c(\theta, \gamma_c(\theta)) = \theta + m \), this providing an open condition.

\[\text{In particular, since the twist maps is positive, then } b_n(\theta, \gamma(\theta)) > 0.\]
extension $E_{b'}$ of $E$. Then the set $G_{b'} = \{ (\varepsilon, \theta, \gamma_{\varepsilon}(\theta) : \varepsilon \in [b, b'], \theta \in \mathbb{R}) \}$ is still in $C$; the map $\Delta$ vanishing on $E_{b'}$, it also vanishes on $G_{b'}$, this proving the existence of an $(n, m)$-completely invariant curve for $\varepsilon \in [b, b']$ and hence in $[a, b']$. Since the set $I_{n,m}$ is closed, then the maximal extension of the interval $[a, b]$ must be infinite, namely $[a, +\infty)$, which would contradict Lemma 2.3 (see below). In fact, it follows from the closedness of $I_{n,m}$ that the maximal extension must be closed too (otherwise $I_{n,m}$ would not contain one of its accumulation points) and, by repeating the same argument as above, any closed finite interval $[a, b_{\infty}]$ can be further extended, thus contradicting maximality.

The following result follows from the proof of the main Theorem of [30], in section 6.

**Lemma 2.3.** The set

$I_{PN} := \{ \varepsilon_{0} \geq 0 : f_{\varepsilon} \text{ admits homotopically non trivial invariant curves for every } \varepsilon \in (0, \varepsilon_{0}) \}$

is bounded from above.

**Proof.** From [30, formula (10)] (compare our notation with the one in [30, formula (1)]) we have $h(x) = \varepsilon P'_{N}(x)$) a necessary condition for $\varepsilon_{0}$ to belong to $I_{PN}$ is

\[
2 + \frac{\varepsilon_{0}m}{2} + \frac{2 + \varepsilon_{0}m}{2} \leq 2 + \varepsilon_{0}M,
\]

where $m, M$ are, respectively, the minimum and the maximum of $P'_{N}$ (observe that $m < 0 < M$). Hence, we must have $\varepsilon_{0} < -2/m$, Otherwise we would get a contradiction since the limit of the left-hand side in (2.19), as $\varepsilon_{0} \to (2/m)^{-}$ goes to $+\infty$, while the right-hand side remains bounded. □

We can finally prove Theorem 1.

**Proof of Theorem 1.** From Lemma 2.2 it follows that $G$, hence $g$ (see 2.1 and 2.3), must be a trigonometric polynomial. Since we are assuming that for any $0 \leq \varepsilon \leq \varepsilon_{0}$, $f_{\varepsilon}$ admits invariant curves for any rotation number in $(\alpha_{1}, \alpha_{2})$, then $g$ must be constant otherwise it would contradict Proposition 2.1 since, by hypothesis, for every $\frac{m}{n} \in (\alpha_{1}, \alpha_{2})$ the set $I_{n,m}$ contains the interval $(0, \varepsilon_{0})$. Since by hypothesis $g$ has zero average, then $g \equiv 0$. □

3. HAMILTONIAN FLOWS POSSESSING INVARIANT RESONANT TORI

Let us consider an integrable $C^{1}$ Hamiltonian in action-angle coordinates

$$H_{0} : \mathbb{T}^{n} \times \mathcal{U} \longrightarrow \mathbb{R}$$

$$(q, p) \longmapsto H_{0}(p)$$

where $\mathbb{T}^{n} := \mathbb{R}^{n}/\mathbb{Z}^{n}$ and $\mathcal{U} \subseteq \mathbb{R}^{n}$ is an open set. Let us recall some basic facts. The equations of motion are extremely simple

\[
\begin{align*}
\dot{q} &= \partial_{p}H_{0}(p) =: \rho_{H_{0}}(p) \\
\dot{p} &= 0
\end{align*}
\]
and the flow is given by
\[ \Phi_{tH_0}(q_0, p_0) = (q_0 + t\rho_{H_0}(p_0) \mod \mathbb{Z}^n, p_0). \]

Hence, for every \( p_0 \in \mathcal{U} \) the Lagrangian torus \( T_{p_0} := \mathbb{T}^n \times \{p_0\} \) is invariant and the motion on it corresponds to a translation by \( \rho_{H_0}(p_0) \).

We denote by \( d(\rho) = \text{rank} R(\rho) \), as defined in (1.2); clearly, \( 0 \leq d(\rho) \leq n \). In particular, the nature of the dynamics on \( T_{p_0} \) depends on the arithmetic properties of \( \rho_{H_0}(p_0) \), or more specifically by \( d(\rho) \) (the number of its independent resonances):

- If \( d(\rho_{H_0}(p_0)) = n \), then \( \rho_{H_0}(p_0) = (0, \ldots, 0) \) and \( T_{p_0} \) consists of fixed points of the flow.
- If \( d(\rho_{H_0}(p_0)) = 0 \), then \( \rho_{H_0}(p_0) \) is said non-resonant and every orbit on \( T_{p_0} \) is dense and the flow on it is uniquely ergodic.
- If \( 1 \leq d(\rho_{H_0}(p_0)) \leq n - 1 \), then \( \rho_{H_0}(p_0) \) is said \( d(\rho_{H_0}(p_0)) \)-resonant; more specifically, the closure of every orbit is a \( n - d(\rho_{H_0}(p_0)) \) dimensional invariant torus; hence, \( T_{p_0} \) decomposes into a \( d(\rho_{H_0}(p_0)) \)-dimensional family of invariant tori of dimension \( n - d(\rho_{H_0}(p_0)) \). In particular, when \( d(\rho_{H_0}(p_0)) = n - 1 \), \( T_{p_0} \) is foliated by periodic orbits.

Given a point \((q_0, p_0) \in T_{p_0}\), we denote by \( \mu_{(q_0, p_0)} \) the invariant probability measure uniformly distributed on the orbit starting at \((q_0, p_0)\); namely, the unique probability measure such that for every \( f \in C(T_{p_0}) \) we have

\[
\int_{T_{p_0}} f \, d\mu_{(q_0, p_0)} = \lim_{T \to +\infty} \frac{1}{T} \int_0^T f(\Phi_{H_0}^t(q_0, p_0)) \, dt.
\]

Clearly, each of these measures is invariant and ergodic; moreover, \( \mu_{(q_0, p_0)} = \mu_{\Phi_{H_0}^t(q_0, p_0)} \) for every \( t \in \mathbb{R} \). In particular:

- if \( d(\rho_{H_0}(p_0)) = n \), then \( \mu_{(q_0, p_0)} = \delta_{(q_0, p_0)} \) i.e., the Dirac measure concentrated at \((q_0, p_0)\);
- if \( d(\rho_{H_0}(p_0)) = 0 \), then \( \mu_{(q_0, p_0)} \) is the \( n \)-dimensional Lebesgue measure on \( T_{p_0} \);
- if \( 1 \leq d(\rho_{H_0}(p_0)) \leq n - 1 \), then \( \mu_{(q_0, p_0)} \) is supported on an invariant torus of dimension \( n - d(\rho_{H_0}(p_0)) \). Observe that the \( n \)-dimensional Lebesgue measure on \( T_{p_0} \) is still invariant, but not ergodic anymore.

**Remark 3.1.**

1. Trivially, if one perturbs by a function \( G = G(p) \) only depending on \( p \), then the Hamiltonian remains integrable.
2. If \( H_0(p) = \sum_{i=1}^n h_i(p_i) \) and we add a potential (split potential)

\[
G(q) = \sum_{i=1}^n G_i(q_i),
\]

then the system remains integrable. In fact, the equations of motion are:

\[
\begin{align*}
\dot{q}_i &= \partial_{p_i} h_i(p_i) \\
\dot{p}_i &= -\varepsilon G_i'(q_i) & i = 1, \ldots, n
\end{align*}
\]
hence the system consists of $n$ uncoupled 1-dimensional system and therefore it is integrable.

(3) One could consider the more difficult problem of perturbations (not necessarily deformations) that preserve integrability. Some results in this direction have appeared in [15].

We start by proving some preliminary results. Let us consider the Fourier expansion of $G$:

$$G(q) = \sum_{k \in \mathbb{Z}^n} \hat{G}_k e^{2\pi i k \cdot q}.$$ 

**Lemma 3.1.** Let $\rho \in \mathbb{R}^n$. Suppose that for every $0 < \varepsilon < \varepsilon_0$ the perturbed system $H_\varepsilon$ admits an invariant Lagrangian torus, hamiltonianly isotopic to the zero section, on which the motion is conjugate to a translation by $\rho$. Then

$$\hat{G}_k = 0 \quad \forall k \in \mathcal{R}(\rho) \setminus \{(0, \ldots, 0)\}.$$ 

**Proof.** If $\rho$ is non-resonant, then the statement is empty. Let us assume that $\rho$ is resonant.

By hypothesis, $H_\varepsilon$ has an invariant torus $\mathcal{T}_\rho^\varepsilon$ on which the motion is conjugate to a rotation by $\rho$, in particular this torus must be a graph by Birkhoff theorem (see [10, Theorem 1.1] and [4, Theorem 1]). For every $q_0 \in \mathbb{T}^n$, there exists a unique $p_0^\varepsilon$ such that $(q_0, p_0^\varepsilon) \in \mathcal{T}_\rho^\varepsilon$. We denote by

$$(q_\varepsilon(q_0, t), p_\varepsilon(q_0, t)) := \Phi^t_{H_\varepsilon}(q_0, p_0^\varepsilon)$$

its evolution under the flow of $H_\varepsilon$ and by $\mu_\varepsilon^{(q_0, p_0^\varepsilon)}$ the unique invariant probability measure uniformly distributed along this orbit, defined in the same way as in (3.1).

The key observation is that – as it follows, for example, from Aubry-Mather theory (see [34, Proposition 2.1.6 and Remark 2.1.7]) – all these measures $\mu_\varepsilon^{(q_0, p_0^\varepsilon)}$ are action-minimizing (or Mather’s measures) with rotation vector $\rho$ and the values of their average actions are all the same, coinciding with the so-called Mather’s $\beta$-function computed at $\rho$, that we denote by $\beta_\varepsilon(\rho)$ (we refer to [34, Section 3.3] for more details about this function). We can express the average action of the
measure $\mu^\varepsilon_{(q_0,p_0)}$ in the following way:

\[
\beta_\varepsilon(\rho) = \int_{T^r} \left[ \partial_p H_\varepsilon(q,p) \cdot p - H_\varepsilon(q,p) \right] d\mu^\varepsilon_{(q_0,p_0)}
\]

\[
= \lim_{T \to +\infty} \frac{1}{T} \int_0^T \left[ \partial_p H_\varepsilon(q_0(t),p_\varepsilon(q_0(t))) \cdot p_\varepsilon(q_0(t)) - H_\varepsilon(q_0(t),p_\varepsilon(q_0(t))) \right] dt
\]

\[
= \lim_{T \to +\infty} \frac{1}{T} \int_0^T \left[ \partial_p H_0(p_\varepsilon(q_0(t))) \cdot p_\varepsilon(q_0(t)) - H_0(p_\varepsilon(q_0(t))) - \varepsilon G(q_\varepsilon(q_0(t))) \right] dt
\]

\[
= \lim_{T \to +\infty} \frac{1}{T} \int_0^T \left[ \partial_p H_0(p_\varepsilon(q_0(t))) \cdot (p_\varepsilon(q_0(t)) - p_0) + \partial_p H_0(p_\varepsilon(q_0(t))) \cdot p_0
\]

\[
- H_0(p_0) - \partial_p H_0(p_0) \cdot (p_\varepsilon(q_0(t)) - p_0) - \varepsilon G(q_0 + \varepsilon p_0) \right] dt + o(\varepsilon)
\]

\[
= \lim_{T \to +\infty} \frac{1}{T} \int_0^T \left[ \partial_p H_0(p_\varepsilon(q_0(t))) \cdot p_0 - H_0(p_0) - \varepsilon G(q_0 + \varepsilon p_0) \right] dt + o(\varepsilon)
\]

\[
= \lim_{T \to +\infty} \frac{1}{T} \int_0^T \left[ \partial_p H_0(p_\varepsilon(q_0(t))) \cdot p_0 - H_0(p_0) - \varepsilon G(q_0 + \varepsilon p_0) \right] dt + o(\varepsilon)
\]

\[
= -H_0(p_0) + \lim_{T \to +\infty} \frac{1}{T} \int_0^T \left( \frac{\tilde{q}_\varepsilon(q_0(t)) - \tilde{q}_\varepsilon(q_0,0)}{T} \right) - \varepsilon \left. \lim_{T \to +\infty} \frac{1}{T} \int_0^T G(q_0 + \varepsilon p_0) \right) dt + o(\varepsilon)
\]

where, in the second-last line, $\tilde{q}(q_0(t))$ denotes a lift of the trajectory $q_\varepsilon(q_0(t))$ to the universal cover $\mathbb{R}^n$; in particular, it follows from the definition of rotation vector of an orbit that $\lim_{T \to +\infty} \frac{\tilde{q}_\varepsilon(q_0(t)) - \tilde{q}_\varepsilon(q_0,0)}{T} = \rho$.

Since, the above equality must hold for every $q_0 \in T^n$ and for every $0 < \varepsilon < \varepsilon_0$, then we can conclude that

\[
(3.2) \quad \lim_{T \to +\infty} \frac{1}{T} \int_0^T G(q_0 + \varepsilon p_0) \right) dt \equiv \text{constant independent of } q_0.
\]

Substituting the Fourier expansion of $G$, we obtain:

\[
\lim_{T \to +\infty} \frac{1}{T} \int_0^T G(q_0 + \varepsilon p_0) \right) dt = \lim_{T \to +\infty} \frac{1}{T} \int_0^T \left( \sum_{k \in \mathbb{Z}^n} \tilde{G}_k e^{2\pi i k \cdot (q_0 + \varepsilon p_0)} \right) dt
\]

\[
= \sum_{k \in \mathbb{Z}^n} \tilde{G}_k \left( \lim_{T \to +\infty} \frac{1}{T} \int_0^T e^{2\pi i (k \cdot \rho) t} dt \right) e^{2\pi i k \cdot q_0}
\]

\[
= \sum_{k \in \mathcal{R}(\rho)} \tilde{G}_k e^{2\pi i k \cdot q_0},
\]

(3.3)

where we have used that if $k \notin \mathcal{R}(\rho)$ then

\[
\lim_{T \to +\infty} \frac{1}{T} \int_0^T e^{2\pi i (k \cdot \rho) t} dt = 0.
\]

Using that (3.3) is independent of $q_0 \in T^n$ (see (3.2)), then it follows that

\[
(3.4) \quad \tilde{G}_k = 0 \quad \forall k \in \mathcal{R}(\rho) \setminus \{0, \ldots, 0\},
\]

which completes the proof.
Corollary 3.1. Let $\Omega \subset \mathbb{R}^n$ and assume that for every $\rho \in \Omega$ and for every $0 < \varepsilon < \varepsilon_0$, the perturbed Hamiltonian system associated to $H_\varepsilon$ admits an invariant Lagrangian torus, hamiltonianly isotopic to the zero section, on which the motion is conjugate to a translation by $\rho$. Then

$$\hat{G}_k = 0 \quad \forall \ k \in \left( \bigcup_{\rho \in \Omega} \mathcal{R}(\rho) \right) \setminus \{(0, \ldots, 0)\}.$$ 

Let

$$W := \{\rho := (\rho_1, \ldots, \rho_n) \in \mathbb{Q}^n : \text{rank} \mathcal{R}(\rho) = n - 1 \text{ and } \rho_i \neq 0 \text{ for all } i\}.$$ 

Lemma 3.2. Let $\Omega = A \cap W$ where $A$ is an open neighbourhood of the origin $\mathbb{R}^n$. Then

$$(3.5) \quad \mathcal{R}(\Omega) \supseteq \{\nu \in \mathbb{Z}^n : \nu \text{ has more than one non-zero component} \}.$$ 

From these results, Proposition 2 follows.

Proof of Proposition 2. i) Lemma 3.2 and corollary 3.1 imply that if $\Omega$ is as in (3.5) and for every $0 < \varepsilon < \varepsilon_0$ and for every $\rho \in \Omega$ the perturbed Hamiltonian system associated to $H_\varepsilon$ admits an invariant Lagrangian torus, hamiltonianly isotopic to the zero section, on which the motion is conjugate to a translation by $\rho$, then the potential $G$ splits into

$$G(q) = \sum_{i=1}^n G_i(q_i);$$

in fact, it is sufficient to observe that the Fourier coefficients $\hat{G}_k = 0$ for every $k \in \mathbb{Z}^n$ with more than one non-zero component.

ii) Let $\rho_i = 0$, for some $1 \leq i \leq n$. Then every $k \in \mathbb{Z}^n$ such that $k_i \neq 0$ and $k_j = 0$ for $j \neq i$ solves $k \cdot \rho = 0$, then $k \in \mathcal{R}(\rho)$ and (3.4) holds. The thesis follows.  

Proof. (Lemma 3.2) Let $\nu \in \mathbb{Z}^n$ with more than one component different from zero; we aim to find $\rho \in \Omega$ such that $\rho \cdot \nu = 0$.

We observe that:

- If $\rho$ is such that $\rho \cdot \nu = 0$, then $\lambda \rho$ is also a solution for every $\lambda \in \mathbb{R} \setminus \{0\}$. Clearly, if $\rho$ is $(n-1)$-resonant, also $\lambda \rho$ is, for every $\lambda \neq 0$. Being $A$ an open neighbourhood of the origin, it suffices to find a solution in $W$ and then rescale it to find a solution in $\Omega$.

- Let $\sigma$ be a permutation of $\{1, \ldots, n\}$; given a vector $v = (v_1, \ldots, v_n) \in \mathbb{R}^n$, we denote by

$$v_\sigma = (v_{\sigma(1)}, \ldots, v_{\sigma(n)}).$$

If $\rho$ is a solution of $\rho \cdot \nu = 0$, then $\rho_\sigma$ satisfies $\rho_\sigma \cdot \nu_\sigma = 0$. Hence, without loss of generality, we can assume that $\nu_1 \neq 0$ and $\nu_2 \neq 0$. 

Let us first assume that $\nu$ has only two non-zero components $\nu_1, \nu_2$. In this case, it suffices to take $\rho = \left(-\frac{\nu_2}{\nu_1}, 1, \ldots, 1\right)$, which is indeed in $\mathcal{W}$, since its components are non-zero and all its resonances $\alpha \in \mathbb{Z}^n$ are given for example by

$$\left\{ \nu_1(\alpha_1, \frac{\nu_2}{\nu_1} - \sum_{j=3}^{n} \alpha_j, \alpha_3, \ldots, \alpha_n) : \alpha_j \in \mathbb{Z}, 1 \leq j \leq n, j \neq 2 \right\}$$

which is of dimension $n - 1$.

Let us now consider the case in which $\nu$ has $k$ non-zero components (with $3 \leq k \leq n$) that we can assume are $\nu_1, \ldots, \nu_k$.

Consider the vectors

$$\rho_{\pm} = \left(-\frac{1}{\nu_1}(\pm \nu_2 + \sum_{j=3}^{k} \nu_j), \pm 1, 1, \ldots, 1\right).$$

It is easy to check that

$$\rho_{\pm} \cdot \nu = 0.$$ 

Moreover, they are $(n - 1)$-resonant. In fact, as before,

$$\mathcal{R}(\rho_{\pm}) = \left\{ \nu_1(\alpha_1, \frac{\nu_2}{\nu_1}(\mp \nu_2 + \sum_{j=3}^{k} \nu_j) + \sum_{j=3}^{n} \alpha_j, \alpha_3, \ldots, \alpha_n) : \alpha_j \in \mathbb{Z}, 1 \leq j \leq n, j \neq 2 \right\}$$

which is an $(n - 1)$-dimensional subspace. Finally, at least one between $\rho_+$ and $\rho_-$ must have all non-zero components (since $\nu_2 \neq 0$), hence belonging to $\mathcal{W}$.

□

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