Some characterisations of $\Sigma$-pure-injective objects in compactly
generated triangulated categories.

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Abstract

We provide various ways to characterise $\Sigma$-pure-injective objects in a compactly generated
triangulated category. These characterisations mimic analogous well-known results from the
model theory of modules. The proof of our main result involves two approaches. In the
first approach we adapt arguments from the module-theoretic setting. Here the one-sorted
language of modules over a fixed ring is replaced with a canonical multi-sorted language.
Throughout we use a restriction of the Yoneda embedding, which associates to each object
a corresponding multi-sorted structure. The second approach is to translate statements between
the domain and codomain of this restriction. In particular, to obtain results about $\Sigma$-pure-
injectives in triangulated categories, we use results about $\Sigma$-injective objects in locally coherent
Grothendieck categories. By combining the two approaches, we highlight a connection between
sorted pp-definable subgroups and annihilator subobjects of generators in the functor category.

1. Introduction.

The model theory of modules refers to the specification of model theory to the module-
theoretic setting. Fundamental work, such as that of Baur [1], placed focus on certain formulas
in the language of modules, known as pp formulas. In particular, module embeddings which
reflect solutions to pp formulas, so-called pure embeddings, became of particular interest. This
served as sufficient motivation to study modules which are pure-injective: that is, injective with
respect to the class of pure embeddings.

In famous work of Ziegler [16], a topological space was defined whose points are inde-
composable pure-injective modules. The introduction of what is now known as the Ziegler
spectrum proved to be a groundbreaking moment in this branch of model-theoretic algebra, and
interest in understanding pure-injectivity has since grown. Specifically, in work such as that
of Huising-Zimmerman [8], functional results appeared in which pure-injective and so-called
$\Sigma$-pure-injective modules were characterised. These characterisations are well documented, for
example, in a book of Jensen and Lenzing [10].

A frequently used tool in these characterisations is the relationship between a module and
its image in a certain functor category. To explicate, the functor is given by the tensor product,
restricted to the full subcategory of finitely presented modules. For example, a module is pure-
injective if and only if the corresponding tensor functor is injective. Subsequently one may
convert statements about pure-injective modules into statements about injective objects in
Grothendieck categories, and translate problems and solutions back and forth.

For example, Garcia and Dung [4] developed the understanding of $\Sigma$-injective objects
in Grothendieck categories by building on work of Harada [8], which generalised a famous
characterisation of $\Sigma$-injective modules going back to Faith [3]. These authors showed that, as
above, such developments helped simplify arguments about $\Sigma$-pure-injective modules and their
characterisations.
In this article we attempt to provide, in a utilitarian manner, some analogues to the previously mentioned characterisations. The difference here is that, instead of working in a category of modules, we work in a triangulated category which, in a particular sense, is compactly generated. Never-the-less, the statements we prove and arguments used to prove them are motivated directly from certain module-theoretic counterparts.

The notion of compactness we refer to comes from work of Neeman [14], where the idea was generalised from algebraic topology. Krause [13] provided the definitions of pure monomorphisms, pure-injective objects and the Ziegler spectrum of a compactly generated triangulated category. Garkusha and Prest [5] subsequently introduced a multi-sorted language for this setting, which mimics the role played by the language of modules. Furthermore these authors gave a correspondence between the pp-formulas in this multi-sorted language and coherent functors.

In Theorem 1.1 we use the following notation and assumptions.

- \( \mathcal{T} \) is a compactly generated triangulated category and \( \mathcal{T} \) has all small coproducts.
- \( \mathcal{T}^c \) is the full subcategory of compact objects of \( \mathcal{T} \) and \( \mathcal{T}^c \) is skeletally small.
- \( \text{Ab} \) is the category of abelian groups.
- \( \text{Mod}^{-\mathcal{T}^c} \) is the category of contravariant additive functors \( \mathcal{T}^c \to \text{Ab} \).
- \( G \) is a set of finitely presented generators of \( \text{Mod}^{-\mathcal{T}^c} \).
- \( Y: \mathcal{T} \to \text{Mod}^{-\mathcal{T}^c} \) is the functor taking an object \( N \) to the restriction of \( \mathcal{T}((-), N) \).

Recall that, by the Brown representability theorem, since the categories we are considering have all small coproducts, they have all small products; see Remark 4.3.

**Theorem 1.1.** For an object \( M \) of \( \mathcal{T} \) the following statements are equivalent.

1. \( M \) is \( \Sigma \)-pure injective, that is, for any set \( I \) the coproduct \( M(I) \) is pure-injective.
2. The countable coproduct \( M^{(\aleph_0)} \) is pure-injective.
3. For any generator \( \mathfrak{G} \in G \) each ascending chain \( \bigcap_{\theta \in \mathbb{K}[1]} \ker(\theta) \subseteq \bigcap_{\theta \in \mathbb{K}[2]} \ker(\theta) \subseteq \ldots \) of \( Y(M) \)-annihilator subobjects of \( \mathfrak{G} \) must stabilise.
4. For any set \( I \) the canonical morphism from \( M(I) \) to the product \( M^I \) is a section.
5. For any object \( X \) of \( \mathcal{T}^c \) each descending chain \( \varphi_1(M) \supseteq \varphi_2(M) \supseteq \ldots \) of pp-definable subgroups of \( M \) of sort \( X \) must eventually stabilise.
6. \( M \) is pure injective, and for any set \( I \) the object \( M^I \) is isomorphic to a coproduct of indecomposable pure-injective objects with local endomorphism rings.

The proof of Theorem 1.1 is at the end of the article. The equivalences of (i), (ii), (iii) and (iv) in Theorem 1.1 follow by directly combining work of Garcia and Dung [4] and work of Krause [13]. The equivalence of (v) and (vi) with the previous conditions is more involved. For (v) we adapt ideas going back to Faith [3], whilst applying results due to Harada [6] and Garcia and Dung [4]. For (vi) we adapt arguments of Huisgen-Zimmerman [8].

The article is organised as follows. In §2 we recall some prerequisite terminology from multi-sorted model theory. In §3 we specify to compactly generated triangulated categories by recalling the canonical multi-sorted language of Garkusha and Prest [5]. In §4 we build up some results about products and coproducts in these categories which mimic ideas of Huisgen-Zimmerman [8]. In §5 we highlight a connection between annihilator subobjects of finitely generated functors and pp-definable subgroups, where the presentation of the functor determines the sort of the subgroup. In §6 we begin combining the results developed in the previous sections with results from Krause [13]. In §7 we complete the proof of Theorem 1.1.
2. Multi-sorted languages, structures and homomorphisms.

There are various module-theoretic characterisations for the purity in terms of positive-primitive formulas in the underlying one-sorted language of modules over a ring. Similarly, purity in compactly generated triangulated categories may be discussed in terms of formulas in a multi-sorted language. Although Definitions 2.1, 2.3, 2.5, 2.6 and 2.7 are well-known, we recall them for completeness. We closely follow [2, §2, §7] consistency.

**Definition 2.1.** [2 Definition 34] For a non-empty set S an S-sorted predicate language \( \mathcal{L} \) is a tuple \((\text{pred}_S, \text{func}_S, \text{ar}_S, \text{sort}_S)\) where:

(i) each \( s \in S \) is called a sort;
(ii) the symbol \( \text{pred}_S \) denotes a non-empty set of sorted predicate symbols;
(iii) the symbol \( \text{func}_S \) denotes a set of sorted function symbols which is disjoint with \( \text{pred}_S \);
(iv) the symbol \( \text{ar}_S \) denotes an arity function \( \text{pred}_S \cup \text{func}_S \to \mathbb{N} \); and
(v) the symbol \( \text{sort}_S \) denotes a sort function, taking any \( n \)-ary \( R \in \text{pred}_S \) (respectively \( F \in \text{func}_S \)) to a sequence in \( S \) of length \( n \) (respectively \( n + 1 \)).

When \( n > 0 \) in condition (v) we often write \( \text{sort}_S(R) = (s_1, \ldots, s_n) \) (respectively \( \text{sort}_S(F) = (s_1, \ldots, s_n, s) \)). Note that functions \( F \) with \( \text{ar}_S(F) = 0 \) have a sort.

For each sort \( s \) we introduce a countable set \( V_s \) of variables of sort \( s \). The terms of \( \mathcal{L} \) each have their own sort, and are defined inductively by stipulating: any variable \( x \) of sort \( s \) will be considered a term of sort \( s \); and for any \( F \in \text{func}_S \) with \( \text{sort}_S(F) = (s_1, \ldots, s_n, s) \) and for any terms \( t_1, \ldots, t_n \) of sort \( s_1, \ldots, s_n \) respectively, \( F(t_1, \ldots, t_n) \) is considered a term of sort \( s \). Note that constant symbols, given by functions \( F \) with \( \text{ar}_S(F) = 0 \), are also terms.

The atomic formulas with which \( \mathcal{L} \) is equipped are built from the equality \( t = t' \) between terms \( t, t' \) of common sort \( s \), together with the formulas \( R(t_1, \ldots, t_n) \) where \( R \in \text{pred}_S \), \( \text{sort}_S(R) = (s_1, \ldots, s_n) \) and where each \( t_i \) is a term of sort \( s_i \). First-order formulas \( \varphi \) in \( \mathcal{L} \) are built from: the variables of each sort; the atomic formulas; binary connectives \( \land, \lor, \land \); negation \( \neg \); and the quantifiers \( \forall \) and \( \exists \).

A positive-primitive or pp formula \( \varphi(x_1, \ldots, x_n) \) with free variables \( x_1 \) has the form

\[
\exists w_{n+1}, \ldots, w_m : \bigwedge_{j=1}^{k} \psi_j(x_1, \ldots, x_n, w_{n+1}, \ldots, w_m)
\]

where each \( \psi_j \) is an atomic formula (see, for example, [7, p.50]).

One may build a theory for a multi-sorted language \( \mathcal{L} \) by specifying a set of axioms. For our purposes these axioms are those characterising objects and morphisms in a fixed category.

**Example 2.2.** [10 §6] Let \( A \) be a unital ring. We recall how the language \( \mathcal{L}_A \) of \( A \)-modules may be considered as a predicate language in the sense of Definition 2.1. In this case there is only one sort, which we ignore, and which uniquely determines the function \( \text{sort}_A \). Let \( \text{pred}_A = \{0\} \). Let \( \text{func}_A = \{+\} \cup \{a \times - | a \in A\} \) where \(+\) is binary and \(a \times - \) is unary.

In Definition 2.3 the notion of a structure is recalled. For the language \( \mathcal{L}_A \) we have that this notion, together with the appropriate axioms, recovers the properties defining \( A \)-modules.

**Definition 2.3.** [2 Definition 35] Fix a set \( S \neq \emptyset \) and an S-sorted predicate language \( \mathcal{L} \). An \( \mathcal{L} \)-structure is a tuple \( M = (\text{S}(M), (R(M) | R \in \text{pred}_S), (F(M) | F \in \text{func}_S)) \) such that:

(i) the symbol \( \text{S}(M) \) denotes a family of sets \( \{s(M) | s \in S\} \);
(ii) if \( \text{sort}_S(R) = (s_1, \ldots, s_n) \) then \( R(M) \) is a subset of \( s_1(M) \times \cdots \times s_n(M) \); and
(iii) if \( \text{sort}_S(F) = (s_1, \ldots, s_n, s) \) then \( F(M) : s_1(M) \times \cdots \times s_n(M) \to s(M) \) is a function.
Denoting the cardinality of any set \( X \) by \( |X| \), let \( |\mathcal{L}| = |\text{pred}_s \cup \text{func}_s| \) and, for \( M \) as above, let \( |M| \) be the sum of the cardinalities \( |s(M)| \) as \( s \) runs through \( \mathcal{S} \).

The so-called one-sorted language from Example 2.2 is trivial in the sense that there is only one possibility for the sort function. In this sense, Example 2.4 is a non-trivial example of the multi-sorted languages we recalled in Definition 2.1.

**Example 2.4.** Here we recall an example of an \( \{r, m\} \)-sorted predicate language which is in contrast to Example 2.2. The predicates in this language will be the unary symbols \( 0_r \) and \( 1_r \) of sort \( r \) and \( 0_m \) of sort \( m \). The functions in this language will be the ternary symbols \( + \) and \( \times \) where \( \text{sort}_{r}(m) = (m, m, m) \) and \( \text{sort}_{r, m}(x) = (r, m, m) \). After specifying the appropriate axioms, structures \( A \in \mathcal{M} \) are tuples \( (A, M) \) where \( A \) is a unital ring and \( M \) is an \( A \)-module.

In this way one interprets the symbols \( 0_r \) and \( 1_r \) as the additive and multiplicative identities in \( A \). Similarly the symbol \( 0_m \) is interpreted as the additive identity in \( M \). In Definition 2.3 the notion of a homomorphism between structures is recalled. In this sense, a homomorphism \( (A, M) \rightarrow (B, N) \) is given by a pair \((f, l)\) where \( f : A \rightarrow B \) is a homomorphism of rings and \( l : M \rightarrow N \) is a homomorphism of \( A \)-modules where the action of \( A \) on \( N \) is given by \( f \).

**Definition 2.5.** [Definition 3] Fix a set \( \mathcal{S} \neq \emptyset \), an \( \mathcal{S} \)-sorted predicate language \( \mathcal{L} \) and \( \mathcal{L} \)-structures \( L \) and \( M \). By an \( \mathcal{L} \)-homomorphism \( h : L \rightarrow M \) we mean a family \( \{h_s : s \in \mathcal{S}\} \) of functions \( h_s : s(L) \rightarrow s(M) \) such that:

i) if \( \text{sort}_s(R) = (s_1, \ldots, s_n) \) then \( R(M) = \{(h_{s_1}(a_1), \ldots, h_{s_n}(a_n)) \mid (a_1, \ldots, a_n) \in R(L)\} \);

ii) and if \( \text{sort}_s(F) = (s_1, \ldots, s_n, s) \) then for all \( (a_1, \ldots, a_n) \in s_1(L) \times \cdots \times s_n(L) \) we have \( h_s(F(L)(a_1, \ldots, a_n)) = F(M)(h_{s_1}(a_1), \ldots, h_{s_n}(a_n)) \).

Note that, in the notation of Definition 2.5 [Theorem 17] says that a collection of functions \( h_s : s(L) \rightarrow s(M) \) defines an \( \mathcal{L} \)-homomorphism if and only if, whenever \( \varphi(x_1, \ldots, x_n) \) is an atomic formula with \( \text{sort}_s(x_i) = s_i \), then for all \( (a_1, \ldots, a_n) \in s_1(L) \times \cdots \times s_n(L) \), if \( L \models \varphi(a_1, \ldots, a_n) \) then \( M \models \varphi(h_{s_1}(a_1), \ldots, h_{s_n}(a_n)) \).

**Definition 2.6.** [Definition 36] Fix a set \( \mathcal{S} \neq \emptyset \) and an \( \mathcal{S} \)-sorted predicate language \( \mathcal{L} \). By an \( \mathcal{L} \)-embedding we mean an \( \mathcal{L} \)-homomorphism \( h : L \rightarrow M \) such that:

i) if \( \varphi(x_1, \ldots, x_n) \) is an atomic formula with \( \text{sort}_s(x_i) = s_i \), then for all \( (a_1, \ldots, a_n) \in s_1(L) \times \cdots \times s_n(L) \), \( L \models \varphi(a_1, \ldots, a_n) \) and if only if \( M \models \varphi(h_{s_1}(a_1), \ldots, h_{s_n}(a_n)) \).

By an \( \mathcal{L} \)-pure embedding we mean an \( \mathcal{L} \)-homomorphism \( h : L \rightarrow M \) such that:

ii) if \( \varphi(x_1, \ldots, x_n) \) is a pp formula with \( \text{sort}_s(x_i) = s_i \), then for all \( (a_1, \ldots, a_n) \in s_1(L) \times \cdots \times s_n(L) \), if \( M \models \varphi(h_{s_1}(a_1), \ldots, h_{s_n}(a_n)) \) then \( L \models \varphi(a_1, \ldots, a_n) \).

Note that \( \mathcal{L} \)-pure embeddings are \( \mathcal{L} \)-embeddings. Note also that the statement of Definition 2.5 ii) is the contrapositive of the definition in [7] p.50, so in this sense, over a ring \( A \) and in the notation from Example 2.2, an injective \( A \)-module homomorphism is pure if and only if it is an \( \mathcal{L}_A \)-pure embedding.

**Definition 2.7.** Fix a non-empty set \( \mathcal{S} \), an \( \mathcal{S} \)-sorted predicate language \( \mathcal{L} \) and \( \mathcal{L} \)-structures \( L \) and \( M \). We say \( L \) is an \( \mathcal{L} \)-substructure of \( M \) if \( s(L) \subseteq s(M) \) for each \( s \in \mathcal{S} \) and, labelling these inclusions \( i_s \), the family \( \{i_s : s \in \mathcal{S}\} \) defines an \( \mathcal{L} \)-homomorphism \( i : L \rightarrow M \). If, additionally, \( i : L \rightarrow M \) is a \( \mathcal{L} \)-pure embedding, we say \( L \) is a \( \mathcal{L} \)-pure substructure of \( M \).
3. Purity in the canonical language of a compactly generated triangulated category.

We now specify the setting of multi-sorted model theory outlined in §2. Throughout the sequel we consider a fixed compactly generated triangulated category; see Assumption 3.3. Before recalling their definition (Definition 3.2) we fix some notation.

**Notation 3.1.** Let \( \mathcal{A} \) be an additive category. Denote the hom-sets \( \mathcal{A}(X,Y) \) and the identity maps \( 1_X \). For any set \( I \) and any collection \( B = \{ B_i \mid i \in I \} \) of objects in \( \mathcal{A} \), if the categorical product \( \prod_i B_i \) exists in \( \mathcal{A} \), we write \( p_{i,B} : \prod_i B_i \to B_j \) for the natural morphisms equipping it, in which case the universal property gives unique morphisms \( v_{j,B} : B_j \to \prod_i B_i \) such that \( p_{j,B} v_{j,B} \) is the identity \( 1_B \) on \( B_j \) for each \( j \). Similarly \( u_{i,B} : B_j \to \bigoplus_i B_i \) will denote the morphisms equipping the coproduct \( \bigoplus_i B_i \) if it exists, in which case there exist unique morphisms \( q_{j,B} : \bigoplus_i B_i \to B_j \) such that \( q_{j,B} u_{j,B} = 1 \) for each \( j \).

Fix an object \( A \) in \( \mathcal{A} \) and consider the covariant functor \( \mathcal{A}(A,-) \). Note that both the product and coproduct of the collection \( \mathcal{A}(A,B) = \{ \mathcal{A}(A,B_i) \mid i \in I \} \) exist in the category \( \text{Ab} \) of abelian groups. We identify \( \bigoplus_{i \in I} \mathcal{A}(A,B_i) \) with the subgroup of \( \prod_{i \in I} \mathcal{A}(A,B_i) \) consisting of tuples \( (g_i \mid i \in I) \) such that \( g_i = 0 \) for all but finitely many \( i \in I \).

Consequently, if \( \prod_i B_i \) exists in \( \mathcal{A} \) then map \( \lambda_{A,B} : \mathcal{A}(A,\prod_i B_i) \to \prod_i \mathcal{A}(A,B_i) \) from the universal property is given by \( f \mapsto (p_{i,B} f \mid i \in I) \) for each \( f \in \mathcal{A}(A,\prod_i B_i) \). Similarly if \( \bigoplus_i B_i \) exists in \( \mathcal{A} \) then map \( \gamma_{A,B} : \bigoplus_i \mathcal{A}(A,B_i) \to \mathcal{A}(A,\bigoplus_i B_i) \) from the universal property is given by \( \gamma_{A,B}(g_i \mid i \in I) = \sum_i u_{i,B} g_i \). In general each of the morphisms \( \lambda_{A,B} \) are isomorphisms.

**Definition 3.2.** [14] Definition 1.1] Let \( \mathcal{T} \) be a triangulated category with suspension functor \( \Sigma \), and assume all small coproducts exist in \( \mathcal{T} \). An object \( X \) of \( \mathcal{T} \) is said to be compact if, for any set \( I \) and collection \( M = \{ M_i \mid i \in I \} \) of objects in \( \mathcal{T} \), the morphism \( \gamma_{X,M} \) is an isomorphism. Let \( \mathcal{T}^c \) be the full triangulated subcategory of \( \mathcal{T} \) consisting of compact objects.

Given a set \( \mathcal{G} \) of compact objects in \( \mathcal{T} \), we say that \( \mathcal{T} \) is compactly generated by \( \mathcal{G} \) if there are no non-zero objects \( M \) in \( \mathcal{T} \) satisfying \( T(X,M) = 0 \) for all \( X \in \mathcal{G} \) (or, said another way, any non-zero object \( M \) gives rise to a non-zero morphism \( X \to M \) for some \( X \in \mathcal{G} \)).

If \( \mathcal{T} \) is compactly generated by \( \mathcal{G} \) we call \( \mathcal{G} \) a generating set provided \( \Sigma X \in \mathcal{G} \) for all \( X \in \mathcal{G} \).

**Assumption 3.3.** In the remainder of §2 fix a triangulated category \( \mathcal{T} \) with suspension functor \( \Sigma \), and we assume that \( \mathcal{T} \) has all small coproducts, that \( \mathcal{T} \) is compactly generated by a generating set \( \mathcal{G} \), and that \( \mathcal{T}^c \) is skeletally small.

Definition 3.4 and Remark 3.6 closely follow [5] §3], in which a multi-sorted language is associated to the category \( \mathcal{T} \).

**Definition 3.4.** In what follows let \( \mathcal{S} \) denote a fixed set of objects in \( \mathcal{T}^c \) given by choosing exactly one representative of each isomorphism class. Such a set \( \mathcal{S} \) exists because we are assuming that \( \mathcal{T}^c \) is skeletally small.

[5] §3] The canonical language \( \mathcal{L}^\Sigma \) of \( \mathcal{T} \) is given by a \( \mathcal{S} \)-sorted predicate language \( \text{pred}_{\mathcal{S}}, \text{func}_{\mathcal{S}}, \text{ars}_{\mathcal{S}}, \text{sort}_{\mathcal{S}} \), defined as follows. The set \( \text{pred}_{\mathcal{S}} \) consists of a symbol \( 0_G \) with \( \text{sort}_{\mathcal{S}}(0_G) = G \) for each \( G \in \mathcal{S} \). The set \( \text{func}_{\mathcal{S}} \) consists of a ternary symbol \( +_G \) with \( \text{sort}_{\mathcal{S}}(+_G) = (G,G,G) \) for each \( G \in \mathcal{S} \); and a unary operation \( \circ a \) with \( \text{sort}_{\mathcal{S}}(\circ a) = (H,G) \) for each morphism \( a : G \to H \) with \( G,H \in \mathcal{S} \). Variables of sort \( G \in \mathcal{S} \) will be denoted \( v_G \).
Equation 3.5. Suppose \( A \) is any additive category. We write \( A\text{-}\mathbf{Mod} \) (respectively \( \mathbf{Mod}\text{-}A \)) for the category of additive covariant (respectively contravariant) functors \( A \to \mathbf{Ab} \) where \( \mathbf{Ab} \) is the category of abelian groups.

For any object \( M \) of \( \mathcal{T} \) we let \( \mathcal{T}(-, M) \) denote the object of \( \mathbf{Mod}\text{-}\mathcal{T}^e \) defined by restriction of the object \( \mathcal{T}(-, M) \) of \( \mathbf{Mod}\text{-}\mathcal{T} \) to \( \mathcal{T}^e \). We write \( Y : \mathbf{Mod}\text{-}\mathcal{T}^e \to \mathbf{Ab} \) to denote the restricted Yoneda functor. That is, \( Y \) takes an object \( M \) to \( Y(M) = \mathcal{T}(-, M) \), and takes a morphism \( h : L \to M \) to the natural transformation \( Y(h) : \mathcal{T}(-, L) \to \mathcal{T}(-, M) \) given by defining, for each compact object \( X \), the map \( Y(h)_X : \mathcal{T}(X, L) \to \mathcal{T}(X, M) \) by \( g \mapsto hg \).

Remark 3.6. \([5, §3]\) Consider the theory given from the set of axioms expressing the positive atomic diagram of the objects in \( \mathcal{T}^e \), including the specification that all functions are additive. In this way the category of models for the above theory is equivalent to the category \( \mathbf{Mod}\text{-}\mathcal{T}^e \). The objects \( M \) of \( \mathcal{T} \) are regarded as structures \( M \) for this language via \( Y \).

That is, in the notation of Definition 2.3 we let \( G(M) = \mathcal{T}(G, M) \), we interpret the predicate symbol \( 0_G \) as the identity element of \( G(M) \), we interpret \( +_G \) as the additive group operation on \( G(M) \), and we interpret \(- \circ a \) as the map \( G(M) \to H(M) \) given by \( f \mapsto fa \).

Lemma 3.7. Let \( L \) and \( M \) be objects in \( \mathcal{T} \) with corresponding \( \Sigma\text{-}\mathcal{T}\text{-}structures \( L \) and \( M \). Then the choice of \( \mathcal{S} \) made in Definition 3.4 defines a bijection between \( \Sigma\text{-}\mathcal{T}\text{-}homomorphisms \( L \to M \) and morphisms \( Y(L) \to Y(M) \) (that is, natural transformations) in \( \mathbf{Mod}\text{-}\mathcal{T}^e \).

Proof. Any object \( X \) of \( \mathcal{T}^e \) lies in the same isoclass as some unique \( c(X) \in \mathcal{S} \), in which case we choose an isomorphism \( \phi_X : c(X) \to X \). In this way, any object \( N \) defines an isomorphism \( - \circ \phi_{X,N} : \mathcal{T}(X, N) \to \mathcal{T}(c(X), N) \) by precomposition with \( \phi_X \). Recall that here the \( \Sigma\text{-}\mathcal{T}\text{-}structure \( N \) is defined by setting \( G(N) = \mathcal{T}(G, N) \) for each sort \( G \in \mathcal{S} \). In case \( X \in \mathcal{S} \) we assume, without loss of generality, that \( \phi_X = 1_X \).

Define the required bijection as follows. Fix an \( \Sigma\text{-}\mathcal{T}\text{-}homomorphism \( h : L \to M \). For any object \( X \) of \( \mathcal{T}^e \) define the function \( \mathcal{H}(h)_X : \mathcal{T}(X, L) \to \mathcal{T}(X, M) \) by \( l \mapsto (h_{c(X)}(l \phi_X)) \phi_{X,L}^{-1} \). Conversely, fixing a natural transformation \( \mathcal{H} : Y(L) \to Y(M) \), let \( h(\mathcal{H})_G = \mathcal{H}_G \) for each \( G \in \mathcal{S} \). It suffices to explain why these assignments swap between \( \Sigma\text{-}\mathcal{T}\text{-}homomorphisms and morphisms in \( \mathbf{Mod}\text{-}\mathcal{T}^e \). To do so, we explain why the compatibility conditions which define these morphisms are in correspondence.

To this end, note firstly that the preservation of (the predicate symbol \( 0_G \) and the function symbols \( +_G \)) is equivalent to saying that each function \( \mathcal{H}_X \) is a homomorphism of abelian groups. Letting \( b : X \to Y \) be a morphism in \( \mathcal{T}^e \) and \( a = \phi_Y^{-1} b \phi_X \), for any object \( N \) of \( \mathcal{T} \) the sorted function symbol \( - \circ a \) is interpreted in \( N \) by the equation \( (- \circ a)(N) = - \circ (\phi_{N,Y}^{-1})b \phi_{N,X} \).

Thus, by construction, saying that the function symbols \( - \circ a \) are preserved is equivalent to saying that the collection of \( \mathcal{H}_X \) (for \( X \) compact) defines a natural transformation.

In what follows we discuss the notion of purity in the context of triangulated categories.

Definition 3.8. \([13, Definition 1.1]\) A morphism \( h : L \to M \) in \( \mathcal{T} \) is called a pure monomorphism if \( Y(h)_X : \mathcal{T}(X, L) \to \mathcal{T}(X, M) \) is injective for each object \( X \) of \( \mathcal{T}^e \).

Now we may begin to build results which mimic well-known ideas from the model theory of modules. To consistently compare and contrast our work with the module-theoretic setting, we use the book of Jensen and Lenzing \([10]\). In this spirit, Lemma 3.9 is analogous to \([10, Theorem 6.4(\text{i,ii})]\), and Lemma 3.10 is analogous to \([10, Proposition 6.6]\).
Lemma 3.9. A morphism $h : L \to M$ in $\mathcal{T}$ is a pure monomorphism if and only if the image $h : L \to M$ of $Y(h)$ under the bijection from Lemma 3.7 is an $\mathcal{L}^T$-pure embedding.

Proof. By [5] Proposition 3.1 any pp-formula $\varphi(v_G)$ is equivalent to a divisibility formula $\exists u_H : v_G = u_H a$ where $a : G \to H$ is morphism and $G, H \in \mathcal{S}$. By Definition 2.6, $h$ is an $\mathcal{L}^T$-pure embedding if and only if, for any morphism $a : G \to H$ with $G, H \in \mathcal{S}$ and any pair $(f, g) \in G(L) \times H(L)$, if $h g = h f a$ then $g = f a$. Since any compact object is isomorphic to an object in $\mathcal{S}$, this is equivalent to the condition which says that, for each compact object $X$, the morphism $T(X, L) \to T(X, M)$ given by $g \mapsto h g$ is injective.

Lemma 3.10. Let $G \in \mathcal{S}$ and let $\varphi(v_G)$ be a pp-formula in one free variable of sort $G$. If $h : L \to M$ is a pure monomorphism in $\mathcal{T}$ then $\varphi(L) = \{ g \in T(G, L) \mid h g \in \varphi(M) \}$.

Proof. The claim follows from Lemma 3.9 together with the definition of an $\mathcal{L}^T$-pure embedding, which states that solutions to the negation of a pp-formula are preserved.

4. Products, coproducts, coherent functors and pp-formulas.

Recall, from Definition 2.1, that pp-formulas in $\mathcal{L}^T$ are those lying in the closure of the set of equations under conjunction and existential quantification.

Definition 4.1. [5] §2 Given $G \in \mathcal{S}$ and an object $M$ of $\mathcal{T}$ with $\mathcal{L}^T$-structure $\mathbf{M}$, a pp-definable subgroup of $M$ of sort $G$ is the set $\varphi(M) = \{ f \in G(M) \mid \mathbf{M} \models \varphi(f) \}$ of solutions (in $M$) to some pp-formula $\varphi(v_G)$ in one free variable of sort $G \in \mathcal{S}$.

For any morphism $b : X \to Y$ in $\mathcal{T}^c$ and any object $M$ in $\mathcal{T}$ recall the map $T(b, M) : T(Y, M) \to T(X, M)$ is defined by precomposition with $b$. In this case we let

$$Mb = \text{im}(T(b, M)) = \{ f b \in T(X, M) \mid f \in T(Y, M) \}$$

If $G, H \in \mathcal{S}$ and $\phi_X : G \to X$ and $\phi_Y : H \to Y$ are isomorphisms in $\mathcal{T}$ (as in the proof of Lemma 3.7), then $f b \mapsto f \phi_Y^{-1} b \phi_X$ defines a isomorphism $Mb \to \varphi(M)$ in $\mathbf{Ab}$ where $\varphi(v_G)$ is the pp-formula $\exists u_H : v_G = u_H a$ where $a = \phi_Y^{-1} b \phi_X$.

We continue, slightly abusing terminology, by referring to any set of the form $Mb$ (for some $b \in T(X, Y)$) as a pp-definable subgroup of $M$ of sort $X$.

Recall that a covariant functor $\mathcal{F} : \mathcal{T} \to \mathbf{Ab}$ is said to be coherent provided there is an exact sequence in $\mathcal{T}$-$\text{Mod}$ of the form

$$T(A, -) \to T(B, -) \to \mathcal{F} \to 0$$

Remark 4.2. If $t : M \to N$ and $a : G \to H$ are morphisms in $\mathcal{T}$ with $G, H \in \mathcal{S}$, then $tv \in Na$ for any $v \in Ma$. Hence, for the pp-formula $\exists u_H : v_G = u_H a$ in $\mathcal{L}^T$, the assignment of objects $M \mapsto \varphi(M)$ from defines a functor $\varphi : \mathcal{T} \to \mathbf{Ab}$. Furthermore, by [5] Lemma 4.3 these functors are coherent, and any such coherent functor arises this way.

We now recall that the categories we are considering must have all small products.
REMARK 4.3. As a result of Assumption 3.3 by the Brown representability theorem we have that \( \mathcal{T} \) has all small products. See [12] Lemma 1.5 for details.

Lemma 4.4 is analogous to [10] Proposition 6.7(i,ii). Recall the notation from Notation 3.1.

LEMMMA 4.4. Let \( G \in \mathcal{S} \) and let \( \varphi(v_G) \) be a pp-formula in one free variable of sort \( G \). For any set \( I \) and any collection \( M = \{ M_i \mid i \in I \} \) of objects in \( \mathcal{T} \) the restrictions of \( \gamma_{G,M} \) and \( \lambda_{G,M} \) define isomorphisms \( \bigoplus_i \varphi(M_i) \to \varphi(\bigoplus_i M_i) \) and \( \varphi(\prod_i M_i) \to \prod_i \varphi(M_i) \).

Proof. By the existence of small products and coproducts in \( \mathcal{T} \) and the functorality of \( \varphi \), the universal properties give morphisms \( \delta : \bigoplus_i \varphi(M_i) \to \varphi(\bigoplus_i M_i) \) and \( \mu : \varphi(\prod_i M_i) \to \prod_i \varphi(M_i) \).

By [5] Lemma 4.3 the functor \( \varphi \) is coherent, so by the equivalence of statements (1) and (3) from [13] Theorem A the morphisms \( \delta \) and \( \mu \) are isomorphisms. It is straightforward to check that \( \delta \) and \( \mu \) are the restrictions of \( \gamma_{G,M} \) and \( \lambda_{G,M} \) respectively.

We now adapt some technical results from work of Huisgen-Zimmerman [8], in which a (now well-known) characterization of \( \Sigma \)-pure-injective modules was given. Our adaptations, namely Lemmas 4.6 and 7.7, are used in the sequel. Recall Notation 3.1.

COROLLARY 4.5. Fix collections \( M = \{ M_i \mid i \in \mathbb{N} \} \) and \( L = \{ L_j \mid j \in J \} \) of objects in \( \mathcal{T} \) and let \( M = \prod_i M_i \) and \( L = \bigoplus_j L_j \). Suppose that \( M = L \), and let \( a : G \to H \) be a morphism in \( \mathcal{T} \) with \( G, H \in \mathcal{S} \), and let \( \varphi(v_G) = (\exists u_H : v_G = u_H a) \). Then there is an isomorphism

\[
\kappa(\varphi) : \prod_{i \in \mathbb{N}} \varphi(M_i) \to \bigoplus_{j \in J} \varphi(L_j), \quad (f_i \mid i \in \mathbb{N}) \mapsto (q_j L f \mid j \in J)
\]

where \( f : G \to \prod_i M_i \) is given by the universal property, and whose inverse is

\[
\kappa^{-1}(\varphi) : \bigoplus_{j \in J} \varphi(L_j) \to \prod_{i \in \mathbb{N}} \varphi(M_i), \quad (g_j \mid j \in J) \mapsto (\sum_{j \in J} p_i M u_j g_j \mid i \in \mathbb{N}).
\]

Proof. By Lemma 4.4 the restriction of \( \gamma_{G,L} \) and \( \lambda_{G,M} \) define isomorphisms \( \delta : \bigoplus_i \varphi(M_i) \to \varphi(\bigoplus_i M_i) \) and \( \mu : \varphi(\prod_i M_i) \to \prod_i \varphi(M_i) \). It is straightforward to show that the map \( \kappa(\varphi) \), as written in the statement of the claim, is just \( \delta^{-1} \mu^{-1} \).

The proof of Lemma 4.6 follows the proof of the cited result of Huisgen-Zimmerman, who used the language of so-called \( p \)-functors, which commute with small products and coproducts.

LEMMA 4.6. [8] Lemma 4] Suppose \( \prod_{i \in \mathbb{N}} M_i = \bigoplus_{j \in J} L_j \) in the notation from Corollary 4.5 Let \( \varphi_1(M) \supseteq \varphi_2(M) \supseteq \ldots \) be a descending chain of pp-definable subgroups of \( M \) of some sort \( G \in \mathcal{S} \). For each \( n \in \mathbb{N} \) let \( \Psi(n) = \{ \varphi_n(L_j) \mid j \in J \} \), \( \Pi(n) = \{ \prod_{i \in \mathbb{N}} \varphi_n(M_i), \prod_{i \geq n} \varphi_n(M_i) \} \) and for each \( j \in J \) consider the morphisms \( \rho_{n,j} = q_j, \Psi(n) \kappa(\varphi_n) u_{\geq, \Pi(n)} \), given by the composition

\[
\prod_{i \geq n} \varphi_n(M_i) \xrightarrow{u_{\geq, \Pi(n)}} \prod_{i \in \mathbb{N}} \varphi_n(M_i) \xrightarrow{\kappa(\varphi_n)} \bigoplus_{j \in J} \varphi_n(L_j) \xrightarrow{q_j, \Psi(n)} \varphi_n(L_j)
\]

Then there exists \( r \in \mathbb{N} \) and \( J' \subseteq J \) finite where \( \text{im}(\rho_{n,j}) \subseteq \varphi_n(L_j) \) for all \( n \geq r \) and \( j \notin J' \).

Proof. Without loss of generality we assume, for each \( n \in \mathbb{N} \), that \( u_{\geq, \Pi(n)} \) is (the inclusion) given by sending \( (\theta_{n+1} \mid i \in \mathbb{N}) = (\theta_n, \theta_{n+1}, \ldots) \) (where \( \theta_i \in \varphi_n(M_i) \)) to the sequence \((0, \ldots, 0, \theta_n, \theta_{n+1}, \ldots)\), the initial \( n \) terms of which are 0. Similarly, we can assume \( q_j, \Psi(n) \) is the restriction of \( q_j, \mathcal{T}(G,L) \) for each \( j \) and each \( n \).
Assume for a contradiction that for any $r \in \mathbb{N}$ and any finite subset $J'$ of $J$ there exists $n \in \mathbb{N}$ with $n \geq r$ and there exists $j \in J \setminus J'$ such that $\text{im}(\rho_{r,j}) \not\subseteq \varphi_n(L_j)$. We may inductively define: a strictly increasing sequence $(r_t \mid t \in \mathbb{N})$ of positive integers; a sequence $(i_t \mid t \in \mathbb{N})$ of pairwise different elements in $J$; and, for each $t \in \mathbb{N}$, an element $(\theta_{r_t}^t, \theta_{r_t+1}^t, \ldots) \in \prod_{i \geq r_t} \varphi_{r_t}(M_i)$ whose image under $\rho_{r_t,j_t}$ lies outside $\varphi_{r_t+1}(L_j)$. For the moment fix $t \in \mathbb{N}$. By our initial assumptions, we have that

$$u_{\geq i}(\rho_{r_t}^t)(\theta_{r_t}^t, \theta_{r_t+1}^t, \ldots) = (0, \ldots, 0, \theta_{r_t}^t, \theta_{r_t+1}^t, \theta_{r_t+2}^t, \ldots) \in \prod_{i \in \mathbb{N}} \varphi_{r_t}(M_i).$$

The universal property of the product gives unique morphism $\vartheta^t : G \to \prod_{i \in \mathbb{N}} M_i$ such that $p_iM \vartheta^t = \delta_i^t$ for each $i \in \mathbb{N}$, where we set $\delta_i^t = 0$ when $i < r_t$. Define $\sigma^t$ by

$$\sigma^t = \kappa(\varphi_{r_t})(u_{\geq i}(\rho_{r_t}^t)(\theta_{r_t}^t, \theta_{r_t+1}^t, \ldots)) \in \bigoplus_{j \in J} \varphi_{r_t}(L_j) \subseteq \bigoplus_{j \in J} T(G, L_j),$$

and so $\sigma^t = (\omega_{j_1}^t, \ldots, \omega_{j_d}^t)$ for some $d^t \in \mathbb{N}$, where we let

$$\omega_{j_k}^t = u_{i,j_k}(\rho_{r_t}^t(\vartheta_{r_t}^t, \vartheta_{r_t+1}^t, \ldots)).$$

For each $h$ let $\kappa^{-1}(\varphi_{r_t})(\sigma_h^t) = (\theta_{i_1}^t, \theta_{i_2}^t, \ldots)$. This means $\sum_{i=1}^{d^t} \theta_{i_k}^t = \theta_{j_k}^t$ for each $i \in \mathbb{N}$, and so without loss of generality we may assume that $j_k = j_k(h)$ for some $h$ and that $\theta_{i_k}^t = 0$ when $i < r_t$. Now define $\omega^t = (\omega_{j_1}^t, \omega_{j_2}^t, \ldots)$ by

$$\omega_{j_k}^t = \kappa^{-1}(\varphi_{r_t})(u_{i,j_k}(\rho_{r_t}^t(\vartheta_{r_t}^t, \vartheta_{r_t+1}^t, \ldots))) \in \prod_{i \in \mathbb{N}} \varphi_{r_t}(M_i),$$

and so $\omega_{j_k}^t = 0$ whenever $i < r_t$. Now let $t$ vary. Since $(r_t \mid t \in \mathbb{N})$ is strictly increasing, for each $i \in \mathbb{N}$ we have $\omega_{j_k}^t = 0$ for all but finitely many $t$. Let $\omega = (\sum_{t \in \mathbb{N}} \omega_{j_1}^t, \sum_{t \in \mathbb{N}} \omega_{j_2}^t, \ldots)$, considered as an element of $\prod_{i \in \mathbb{N}} T(G, M_i)$. Fix $l \in \mathbb{N}$. Altogether we have

$$q_{j_l}(\rho_{r_t}^t(\vartheta_{r_t}^t, \vartheta_{r_t+1}^t, \ldots)) = \sum_{t \geq l} q_{j_l}(\rho_{r_t}^t(\vartheta_{r_t}^t, \vartheta_{r_t+1}^t, \ldots)) = \rho_{r_t,j_l}(\theta_{r_t}^t, \ldots),$$

Let $\chi = \gamma^{-1}(\lambda_{G,M}(\omega))$. Since $\rho_{r_t,j_l}(\theta_{r_t}^t, \theta_{r_t+1}^t, \ldots) \in \varphi_{r_t+1}(L_{j_l})$ and $\rho_{r_t,j_l}(\theta_{r_t}^t, \theta_{r_t+1}^t, \ldots) \in \varphi_{r_t+1}(L_{j_l})$ for all $t > l$, the above shows $q_{j_l}(\rho_{r_t}^t(\vartheta_{r_t}^t, \vartheta_{r_t+1}^t, \ldots)) \neq 0$. We now have the contradiction that there is some $\chi \in \bigoplus_{j \in J} T(G, L_j)$ with $q_{j_l}(\rho_{r_t}^t(\vartheta_{r_t}^t, \vartheta_{r_t+1}^t, \ldots)) \neq 0$ for infinitely many $j \in J$. \hfill \Box

5. Annihilator subobjects and pp-definable subgroups.

Recall that, in a category with all small coproducts, a set $\mathcal{S}_\alpha \mid \alpha \in \Omega$ of objects is called a set of generators provided, for each object $Q$, there is an epimorphism $\bigoplus_{\alpha} \mathcal{S}_\alpha \twoheadrightarrow Q$. In case $\Omega$ is a singleton we say the category has a generator. Recall an additive category $\mathcal{A}$ is Grothendieck provided: $\mathcal{A}$ is abelian; $\mathcal{A}$ has all small coproducts; $\mathcal{A}$ has a generator; and the direct limit of any short exact sequence in $\mathcal{A}$ is again exact.

Remark 5.1. Let $\mathcal{A}$ be a Grothendieck category. Recall that an object $Q$ of $\mathcal{A}$ is finitely presented provided the functor $\mathcal{A}(Q, -) : \mathcal{A} \to \text{Ab}$ commutes with direct limits. Recall that an object $S$ of $\mathcal{A}$ is finitely generated provided there is an exact sequence $R \to Q \to 0$ in $\mathcal{A}$. The categories considered in work of Garcia and Dung and Harada were Grothendieck categories with a set of finitely generated generators.

Following Krause, a category $\mathcal{A}$ is said to be locally coherent provided: $\mathcal{A}$ is a Grothendieck category; $\mathcal{A}$ has a set $\{\mathcal{S}_\alpha \mid \alpha \in \Omega\}$ of generators such that each $\mathcal{S}_\alpha$ is finitely presented; and the full subcategory of $\mathcal{A}$ consisting of finitely presented objects is abelian. As noted at the top of [5] p.3] the category $\text{Mod-}\mathcal{T}^c$ is locally coherent. Thus, we may specify various definitions and results from [4] and [6] to the category $\text{Mod-}\mathcal{T}^c$. This is done in Definition [5.2] and Lemmas 5.3 and 5.4.
We now recall a notion introduced by Harada.

**Definition 5.2.** [6] §1] Let \( \mathcal{A} \) be a Grothendieck category with a set \( \{ S_\alpha | \alpha \in \Omega \} \) of finitely generated generators. Let \( \mathcal{Q} \) and \( \mathcal{R} \) be objects in \( \mathcal{A} \). A subobject \( \mathcal{P} \) of \( \mathcal{Q} \) is said to be an \( \mathcal{R} \)-annihilator subobject of \( \mathcal{Q} \) provided \( \mathcal{P} = \bigcap \ker(\theta) \) where the intersection runs over all \( \theta \) from some subset \( \mathcal{K} \) of \( \mathcal{A}(\mathcal{Q}, \mathcal{R}) \).

Lemma 5.3 focuses on a particular context of the specification of Definition 5.2 to the locally coherent category \( \text{Mod} \cdot \mathcal{T}^c \), and was written only to simplify the proof of Lemma 5.4.

**Lemma 5.3.** Let \( M \) and \( X \) be objects in \( \mathcal{T} \) and \( \mathcal{T}^c \) respectively, and let \( \pi : \mathcal{T}(\mathcal{X}) \to \Omega \) be an epimorphism in \( \text{Mod} \cdot \mathcal{T}^c \). For each object \( W \) of \( \mathcal{T}^c \) and any \( Y(M) \)-annihilator subobject of \( \mathcal{Q} \), say of the form \( \mathcal{P} = \bigcap_{\theta \in \mathcal{K}} \ker(\theta) \), we have

\[
\mathcal{P}(W) = \{ \pi_W(g) \mid g \in \mathcal{T}(W, X) \text{ and } \theta_X(\pi_X(1_X))g = 0 \text{ for all } \theta \in \mathcal{K} \}.
\]

**Proof.** It suffices to assume \( \mathcal{K} \neq \emptyset \), and we fix \( \theta \in \mathcal{K} \). The morphisms \( \pi \) and \( \theta \) of \( \text{Mod} \cdot \mathcal{T}^c \) are, by definition, given by homomorphisms \( \pi_U : \mathcal{T}(U, X) \to \Omega(U) \) and \( \theta_U : \Omega(U) \to \mathcal{T}(U, M) \) such that the following diagram commutes for any morphism \( f : U \to V \) in \( \mathcal{T}^c \).

\[
\begin{array}{ccc}
\mathcal{T}(V, X) & \xrightarrow{\pi_V} & \Omega(V) \\
\mathcal{T}(f, X) \downarrow & & \downarrow \theta_V \\
\mathcal{T}(U, X) & \xrightarrow{\pi_U} & \Omega(U)
\end{array}
\]

Now, for the compact object \( W \), let \( z \in \Omega(W) \). In what follows we use the commutativity of the above diagram in different cases. Since \( \pi_W \) is onto, \( z = \pi_W(g) \) for some \( g \in \mathcal{T}(W, X) \).

Take the case \( f = g \), so that \( U = W \) and \( V = X \). By evaluating the commutativity of the left hand square at the identity \( 1_X \) on \( X \), we have that \( \pi_W(g) = \Omega(g)(\pi_X(1_X)) \). Together with the commutativity of the left hand square, this shows \( \theta_X(z) = \theta_X(\pi_X(1_X))g\). Hence, whenever \( z = \pi_W(g) \), we have \( z \in \ker(\theta_W) \) if and only if \( \theta_X(\pi_X(1_X))g = 0 \). So far we have shown \( \ker(\theta_W) \) is the set of elements of the form \( \pi_W(g) \) such that \( g : W \to X \) satisfies \( \theta_X(\pi_X(1_X))g = 0 \).

Now suppose \( \theta, \theta' \in \mathcal{K} \) and \( z \in \ker(\theta_W) \cap \ker(\theta'_W) \). By the above we have \( \pi_W(g) = z = \pi_W(g') \) where \( g, g' \notin \mathcal{T}(W, X) \) satisfy \( \theta_X(\pi_X(1_X))g = 0 \) and \( \theta'_X(\pi_X(1_X))g' = 0 \). Take the case \( f = g - g' \). The commutativity of the outer rectangle shows that

\[
\theta_X(\pi_X(1_X))g' = \theta_X(\pi_X(1_X))(g' - g) = \theta_W(\pi_W(g' - g)) = 0.
\]

We now have that \( \mathcal{P}(W) \) is contained in the right hand side of the required equality. The reverse inclusion is straightforward. \( \square \)

Lemma 5.4 is based on a proof of a given by Huisgen-Zimmerman [9] Corollary 7] of a well-known characterisation of \( \Sigma \)-injective modules due to Faith [3] Proposition 3]. We use Lemma 5.4 to simplify the proof of Lemma 5.6, a key result employed in the sequel.

**Lemma 5.4.** Let \( M \) and \( X \) be objects in \( \mathcal{T} \) and \( \mathcal{T}^c \) respectively, and let \( \mathcal{T}(\mathcal{X}) \to \mathcal{Q} \to 0 \) be an exact sequence in \( \text{Mod} \cdot \mathcal{T}^c \). Any strictly ascending chain of \( \mathcal{Y}(M) \)-annihilator subobjects of \( \mathcal{Q} \) gives a strictly descending chain of \( \alpha \)-injective modules due to Faith [3] Proposition 3].
**Proof.** Suppose \( P_1 \subseteq P_2 \subseteq \ldots \) is a strictly ascending chain of \( Y(M) \)-annihilator subobjects of \( Q \), say where, for each integer \( n > 0 \), we have \( P_n = \bigcap_{\theta \in K[n]} \ker(\theta) \) for some subset \( K[n] \) of morphisms \( \theta : Q \to Y(M) \) in \( \text{Mod-}\mathcal{T}^c \).

We assume \( K[1] \supseteq K[2] \supseteq \ldots \) without loss of generality. For each \( n \) there is an object \( W_n \) of \( \mathcal{T}^c \) for which \( P_n(W_n) \subseteq P_{n+1}(W_n) \), and we choose \( h_n \in P_{n+1}(W_n) \setminus P_n(W_n) \). Let \( \pi : \mathcal{T}(X, -) \to Q \) be the epimorphism in \( \text{Mod-}\mathcal{T}^c \) giving the exact sequence \( \mathcal{T}(\cdot, X) \to Q \to 0 \).

Since \( \pi_{W_n} \) is onto and \( h_n \in \mathcal{T}(W_n) \) we have \( h_n = \pi_{W_n}(g_n) \) for some morphism \( g_n : W_n \to X \) in \( \mathcal{T}^c \). By Lemma 5.3 since \( h_n \in P_{n+1}(W_n) \) we have that \( \theta_X(\pi_X(1_X))g_n = 0 \) for all \( \theta \in K[n + 1] \). Similarly, since \( h_n \notin P_n(W_n) \) there exists \( \tau(n) \in K[n] \) such that \( \tau(n)_X(\pi_X(1_X))g_n \neq 0 \).

We now follow the proof of [5 Proposition 3.1]. Using the axioms of triangulated categories there exists a morphism \( b_n : X \to Y_n \) which completes \( g_n : W_n \to X \) to a triangle

\[
W_n \xrightarrow{g_n} X \xrightarrow{b_n} Y_n \xrightarrow{\Sigma} \Sigma W_n
\]

Applying the covariant functor \( \mathcal{T}(W_n, -) : \mathcal{T} \to \text{Ab} \) to this triangle yields an exact sequence of abelian groups, and so \( b_n \) is a pseudocokernel of \( g_n \) in \( \mathcal{T}^c \). That is, for any morphism \( t : X \to Z \) in \( \mathcal{T}^c \) with \( tg_n = 0 \), there exists a morphism \( s : Y_n \to Z \) for which \( ts = gb_n \).

Let \( \tau_n = \tau(n)_X(\pi_X(1_X)) \) for each \( n \). Combining what we have so far, for each \( n \) we have \( \tau_n(1 + 1) \in K[n + 1] \), so \( \tau_{n+1}g_n = 0 \), and so \( \tau_{n+1} = s_nb_n \) for some morphism \( s_n : Y_n \to M \), and so \( \tau_{n+1} \in M_{b_n} \). On the other hand, if \( \tau_{n+1} \in M_{b_n+1} \) then \( \tau_{n+1}g_n = 0 \) which contradicts that \( \tau(n+1)_X(\pi_X(1_X))g_n \neq 0 \), and so \( \tau_{n+1} \notin M_{b_n+1} \). This gives a strict descending chain

\[
M_{b_1} \supseteq M_{b_1} \cap M_{b_2} \supseteq M_{b_1} \cap M_{b_2} \cap M_{b_3} \supseteq \cdots \supseteq \bigcap_{i=1}^d M_{b_i} \supseteq \cdots
\]

A direct application of [5 Proposition 3.1] shows that each finite intersection \( \bigcap_{i=1}^d M_{b_i} \) has the form \( M_{a_d} \) for some morphism \( a_d : X \to Z_d \) in \( \mathcal{T}^c \), and so the chain above is, as required, a strictly descending chain of pp-definable subgroups of \( M \) of sort \( X \).

**Definition 5.5.** [6 §1] Let \( A \) be a Grothendieck category with a set of finitely generated generators. Fix an object \( M \) of \( A \). We say that \( M \) is \( \Sigma \)-injective if, for any set \( I \), the coproduct \( M^{(I)} = \bigoplus_{i \in I} M \) is injective. We say that \( M \) is fp-injective if, whenever \( 0 \to P \to Q \to \Omega \to 0 \) is an exact sequence in \( A \) where \( Q \) is finitely presented, any morphism \( P \to M \) extends to a morphism \( Q \to M \); see for example [4 §1].

For the proof of Corollary 5.8 we recall two results: Proposition 5.6, due to Garcia and Dung, characterises \( \Sigma \)-injectivity in the fp-injective setting; and Lemma 5.7, due to Krause, shows that it is sufficient to consider the fp-injective setting.

**Proposition 5.6.** [4 Proposition 1.3] Let \( M \) be an fp-injective object in a Grothendieck category \( A \) which has a set \( G \) of finitely presented generators \( G \). Then \( M \) is \( \Sigma \)-injective if and only if, for each \( \mathcal{J} \subseteq G \), every ascending chain of \( M \)-annihilator subobjects of \( \mathcal{J} \) must stabilise.

**Lemma 5.7.** [12 Lemma 1.6] For any \( M \) in \( \mathcal{T} \) the object \( Y(M) \) of \( \text{Mod-}\mathcal{T}^c \) is fp-injective.

**Corollary 5.8.** Let \( M \) be an object in \( \mathcal{T} \). Suppose, for any compact object \( X \) of \( \mathcal{T} \), that every descending chain \( Ma_1 \supseteq Ma_2 \supseteq \ldots \) of pp-definable subgroups of \( M \) of sort \( X \) must stabilise. Then the image \( Y(M) \) of \( M \) in \( \text{Mod-}\mathcal{T}^c \) is \( \Sigma \)-injective.
Proof. We prove the contrapositive, so we assume $Y(M)$ is not $\Sigma$-injective. Recall, from Remark 5.1, that $\text{Mod}^{-T^c}$ is locally coherent, and so it is a Grothendieck category with a set $\mathcal{G}$ of finitely presented generators. Note $M = Y(M)$ is fp-injective by Lemma 5.7, and combining our initial assumption with Proposition 5.6 shows that, for some $G \in \mathcal{G}$, there exists a strictly ascending chain of $M$-annihilator subobjects of $G$. Since $G$ is finitely presented, there is an exact sequence of the form $T(-, Y) \to T(-, X) \to G \to 0$ in $\text{Mod}^{-T^c}$ where $X$ and $Y$ lie in $T^c$. By Lemma 5.4 the aforementioned ascending chain strict ascending chain gives rise to a strictly descending chain of pp-definable subgroups of $M$ of sort $X$.

6. $\Sigma$-pure-injective objects and canonical morphisms.

**Definition 6.1.** Recall Notation 3.1. Let $I$ be a set and let $M$ be an object of $\mathcal{T}$. By the universal properties of the product and coproduct of the collection $M = \{M \mid i \in I\}$, there exists a unique summation morphism $\sigma_{I,M} : \bigoplus_i M \to M$ and a unique canonical morphism $\iota_{I,M} : \bigoplus_i M \to \prod_i M$ for which $\sigma_{I,M}u_{i,M} = 1_M$ and $\iota_{I,M}u_{i,M} = v_{i,M}$ for each $i$.

**Proposition 6.2.** Let $M$ be an object of $\mathcal{T}$ and let $I$ be a set. Then the canonical morphism $\iota_{I,M}$ is a pure monomorphism.

Proof. Let $X$ be any compact object in $\mathcal{T}$. In general: the morphism $\lambda_{X,M}$ is an isomorphism; the canonical morphism $\iota_{I,T(X,M)} : \bigoplus_i M \to \prod_i M$ is injective; and $\lambda_{X,M}T(X, \iota_{I,M}) = \iota_{I,T(X,M)}$. Since $X$ is compact the morphism $\gamma_{X,M}$ is an isomorphism. This shows $T(X, \iota_{I,M})$ is injective if $X$ is compact, and so $\iota_{I,M}$ is a pure monomorphism.

**Definition 6.3.** [12, Definition 1.1] An object $M$ of $\mathcal{T}$ is called pure-injective if each pure monomorphism $M \to N$ is a section, and $M$ is called $\Sigma$-pure-injective if, for any set $I$, the coproduct $\bigoplus_{i \in I} M = M^{(I)}$ is pure-injective.

At this point it is worth recalling some characterisations of purity due to Krause. Theorem 6.4 is analogous to [10, Theorem 7.1 (ii,v,vi)].

**Theorem 6.4.** [13, Theorem 1.8, (1,3,5)] Let $M$ be an object of $\mathcal{T}$. Then the following statements are equivalent.

(i) $M$ is pure-injective.
(ii) The image $Y(M)$ of $M$ is an injective object of $\text{Mod}^{-T}$.
(iii) For any set $I$ the morphism $\sigma_{I,M}$ factors through the morphism $\iota_{I,M}$.

Proposition 6.5 is analogous to the equivalence of (i) and (ii) in [10, Theorem 8.1].

**Proposition 6.6.** Let $M$ be an object of $\mathcal{T}$. Then $M$ is $\Sigma$-pure-injective if and only if, for each set $I$, the canonical morphism $\iota_{I,M}$ is a section.

Proof. Assume that $M$ is $\Sigma$-pure-injective and that $I$ is a set. By assumption the domain of $\iota_{I,M}$ is pure-injective. Since $\iota_{I,M}$ is a pure monomorphism, this means it is a section. Supposing conversely that $\iota_{I,M}$ is a section for each set $I$, it remains to show that $M$ is $\Sigma$-pure-injective. Choose a set $T$ and let $N = \bigoplus_{t \in T} M$. It suffices to prove $N$ is pure-injective.
Let $S$ be any set and consider the collection $N = \{N \mid s \in S\}$. By Theorem 6.4 it suffices to find a map $\theta_{SN}: \prod_{\omega} N \rightarrow N$ such that $\sigma_{SN} = \theta_{SN \times T SN}$. Let $M = \{M \mid t \in T\}$. For each $(s,t) \in S \times T$ the morphisms $u_{s,t}u_{t,M}$ satisfy the universal property of the coproduct $\bigoplus_{s,t} M$, and so we assume $u_{s,t,M} = u_{s,N}u_{t,M}$ without loss of generality.

Consider the morphisms $\varphi_{s,t,M} = q_{t,M}p_{s,N}$ for each $(s,t) \in S \times T$. Since $u_{s,t,M} = u_{s,N}u_{t,M}$ we have $q_{s,t,M} = q_{t,M}q_{s,N}$ by uniqueness. Consequently $\varphi_{s,t,M}u_{s,N}q_{s,N}u_{t,M}$ is the identity on $M$. By the universal property of the product, there is a morphism $\omega: \prod_{\omega} N \rightarrow \prod_{s,t} M$ such that $p_{s,t,M}\omega = \varphi_{s,t,M}$ for each $(s,t)$. It suffices to let $\theta_{SN} = \sigma_{SN}p_{SN \times T,M}\omega$. By the uniqueness of the involved morphisms, it is straightforward to see that $\sigma_{SN} = \theta_{SN \times T SN}$.

Lemma 6.6 is analogous to 10 Theorem 8.1(ii,iii).

**Lemma 6.6.** Let $G \in S$ and let $M$ be a $\Sigma$-pure-injective object in $T$. Then every descending chain of pp-definable subgroups of $M$ of sort $G$ stabilises.

**Proof.** For a contradiction we assume the existence of a strictly descending chain $M_0 \supseteq M_1 \supseteq M_2 \supseteq \ldots$ of (pp-definable subgroups of $M$ of sort $G$) for some $G \in S$. Hence there is a collection of compact objects $H_n \in S$ such that $a_n \in T(G, H_n)$ for each $n \in \mathbb{N}$. By our assumption we may choose elements $f_n \in T(H_n, M)$ such that $f_na_n \not\in M_{n+1}$.

By Proposition 6.10 the canonical morphism $\Gamma_{G,M} : \bigoplus_{\omega} M \rightarrow \prod_{\omega} M$ is a section, and so there is some morphism $\pi_{G,M} : \prod_{\omega} M \rightarrow \bigoplus_{\omega} M$ such that $\pi_{G,M}\Gamma_{G,M}$ is the identity on $\bigoplus_{\omega} M$. Let $(f) = (f_n a_n | n \in \mathbb{N})$, considered as an element of $\prod_{\omega} T(G, M)$. Fix $n \in \mathbb{N}$ and let $\varphi_n(v_G)$ be the formula $\exists u_{H_n} : v_G = u_{H_n}a_n$. Let $M = \{M | n \in \mathbb{N}\}$. Recall $\lambda_{G,M}$ is always an isomorphism, and since $G$ is compact, $\gamma_{G,M}$ is an isomorphism. Define the map $\omega$ by

$$\omega = (\gamma_{G,M})^{-1}\Gamma_{G,M}(\lambda_{G,M})^{-1} : \prod_{\omega} T(G, M) \rightarrow \bigoplus_{\omega} T(G, M).$$

Let $\omega((f)) = (w_n | n \in \mathbb{N})$. The contradiction we will find is that $w_l \neq 0$ for all $l \in \mathbb{N}$, which contradicts $\omega$ has codomain $\bigoplus_{\omega} T(G, M)$. Fix $l \in \mathbb{N}$. Define $\langle f \rangle_{\leq l}$ and $\langle f \rangle_{>l}$ by

$$\langle f \rangle_{\leq l} = (f_0 a_0, \ldots, f_l a_l, 0, 0, \ldots)$$

and

$$\langle f \rangle_{>l} = (0, \ldots, 0, f_{l+1}a_{l+1}, f_{l+2}a_{l+2}, \ldots),$$

where the first $l$ entries of $\langle f \rangle_{>l}$ are 0. Note that $\langle f \rangle_{\leq l} \in \bigoplus_{\omega} T(G, M)$. Furthermore, since the chain $M_0 \supseteq M_1 \supseteq M_2 \supseteq \ldots$ is descending, we have $f_{n} a_n \in \varphi_l (M)$ for all $n > l$ and so $f_{n} a_n \in \bigoplus_{\omega} T(G, M).$

By Lemma 3.10(ii), the restrictions of $(\lambda_{G,M})^{-1}$ and $(\gamma_{G,M})^{-1}$ respectively define isomorphisms $\prod_{\omega} \varphi_{l+1}(M) \rightarrow \varphi_{l+1}(\prod_{\omega} M)$ and $\varphi_{l+1}(\prod_{\omega} M) \rightarrow \prod_{\omega} \varphi_{l+1}(M)$. Similarly $T(G, \pi_{G,M})$ restricts to a morphism $\prod_{\omega} \varphi_{l+1}(M) \rightarrow \prod_{\omega} \varphi_{l+1}(M)$. Altogether we have that $\omega$ restricts to a morphism $\prod_{\omega} \varphi_{l+1}(M) \rightarrow \prod_{\omega} \varphi_{l+1}(M)$.

Let $\omega(\langle f \rangle_{>l}) = (z_n | n \in \mathbb{N})$, and so $z_n \in \varphi_l (M)$ for all $n$. Recall it suffices to show $w_l \neq 0$ where $\omega(f) = (w_n | n \in \mathbb{N})$. From the above we have

$$w_l = \omega(\langle f \rangle_{\leq l}) + \omega(\langle f \rangle_{>l}) = (f_0 a_0 + z_0, \ldots, f_l a_l + z_l, f_{l+1}a_{l+1} + z_{l+1}, \ldots),$$

and so $w_l \neq 0$ as otherwise $\varphi_{l+1}(M) \ni -z_l = f_l a_l \notin \varphi_{l+1}(M)$.

We now recall a result of Krause which is used heavily in the sequel.

**Corollary 6.7.** 12 Corollary 1.10] The restricted Yoneda functor $Y$ induces an equivalence between the full subcategory of pure-injective objects in $T$ and the full subcategory of injective objects in $\text{Mod}\negthinspace -T^c$. 
Note that, since $Y$ is additive, by Corollary 6.7 a pure-injective object $M$ is indecomposable if and only if $Y(M)$ is an indecomposable (injective) object.

**Remark 6.8.** Fix a collection $M = \{ M_i \mid i \in I \}$ of objects in $T$. Since small coproducts exist in $T$ (by Assumption 3.3) and $\text{Ab}$, the morphisms $\gamma_{X,M}$ combine to define a natural transformation $\bigoplus_i T(-, M_i) \to T(-, \bigoplus_i M_i)$. By Definition 3.2 this transformation is in fact an isomorphism $\bigoplus_i Y(M_i) \cong Y(\bigoplus_i M_i)$ in $\text{Mod-}T^c$, and so $Y$ preserves small coproducts.

**Corollary 6.9.** The restricted Yoneda functor $Y$ induces an equivalence between the full subcategory of $\Sigma$-pure-injectives in $T$ and the full subcategory of $\Sigma$-injectives in $\text{Mod-}T^c$.

**Proof.** By definition, an object $M$ of $T$ is $\Sigma$-pure-injective if and only if, for every set $I$, the coproduct $M(I)$ is pure-injective. By Corollary 6.7 and Remark 6.8 given any set $I$, $M(I)$ is pure-injective if and only if $Y(M(I)) \cong (Y(M))(I)$ is injective. This shows that $Y$ induces a functor from the full subcategory of $\Sigma$-pure-injectives in $T$ and the full subcategory of $\Sigma$-injectives in $\text{Mod-}T^c$. To see that this functor is full, faithful and dense, one uses Corollary 6.7 together with the fact that any $\Sigma$-pure-injective object of $T$ is pure-injective.

Corollary 6.10 is analogous to [10, Corollary 8.2(i,ii)].

**Corollary 6.10.** Let $G \in S$ and let $M$ be a $\Sigma$-pure-injective object of $T$.

(i) For any set $I$, the objects $M(I)$ and $M^I$ are $\Sigma$-pure-injective.

(ii) If $h : L \to M$ is a pure monomorphism in $T$ then $L$ is $\Sigma$-pure-injective and $h$ is a section.

**Proof.** It is worth noting that we now have the equivalence of (i) and (v) of Theorem 1.1. That is, by Lemma 6.6 and Corollaries 5.8 and 6.9 an object $M$ is $\Sigma$-pure-injective if and only if every descending chain of $\text{pp}$-definable subgroups of $M$ of each sort stabilises.

(i) This is now straightforward, recalling that, by Lemma 4.4 we have that $\varphi(M)^I \cong \varphi(M(I))$ and $\varphi(M^I) \cong \varphi(M)(I)$ for any $\text{pp}$-formula $\varphi$ of sort $G$.

(ii) As above, it suffices to recall that, in this setting, by Lemma 3.10(i) we have that $\varphi(L) = \{ g \in T(G, L) \mid hg \in \varphi(M) \}$ for any $\text{pp}$-formula $\varphi$ of sort $G$.

To prove Lemma 6.12 we use Corollary 6.11 a result of Garcia and Dung.

**Corollary 6.11.** [4, Corollary 1.6] Let $A$ be a Grothendieck category with a set of finitely generated generators. Any $\Sigma$-injective object of $A$ is a direct sum of indecomposable objects.

Lemma 6.12 is analogous to the statement that (i) implies (v) in [10, Theorem 8.1].

**Lemma 6.12.** Let $M$ be a $\Sigma$-pure-injective object of $T$. Then for any set $I$ the product $M^I$ is a direct sum of indecomposable $\Sigma$-pure-injective objects of $T$.

**Proof.** Recall $\text{Mod-}T^c$ is a Grothendieck category with a set of finitely generated generators. Let $K = M^I$. By Corollary 6.10(i) $K$ is $\Sigma$-pure-injective, and so $Y(K)$ is $\Sigma$-injective by Corollary 6.9. By Corollary 6.11 this means $Y(K) \cong \bigoplus_{j \in J} \mathcal{L}_j$ where each $\mathcal{L}_j$ is an indecomposable object of $\text{Mod-}T^c$. 

For each $j$, there is a section $u_j : \mathcal{L}_j \to \mathcal{Y}(K)$, which means $\mathcal{L}_j$ is injective, and so by Corollary 6.7 we have $\mathcal{L}_j \simeq \mathcal{Y}(L_j)$ for some pure-injective object $L_j$ of $\mathcal{T}$. Since $\mathcal{L}_j$ is indecomposable, we have that $L_j$ is indecomposable. Again, applying Corollary 6.7 gives a section $h_j : L_j \to K$ in $\mathcal{T}$ with $\mathcal{Y}(h_j) = u_j$. Altogether this means $h_j$ is a pure monomorphism into a $\Sigma$-pure-injective object, and so $L_j$ is $\Sigma$-pure-injective by Corollary 6.10(i). By Remark 6.8 we have that $\mathcal{Y}(\bigoplus_j L_j) \simeq \mathcal{Y}(K)$ is injective, and so $\bigoplus_j L_j \simeq K$ by Corollary 6.7.

7. Completing the proof of the main result.

Before proving Theorem 1.1 we note a consequence of the results gathered so far. Recall, from Definitions 2.3 and 3.2, that the cardinality of the $\mathcal{L}_\mathcal{T}$-structure $\mathcal{M}$ underlying any object $\mathcal{M}$ of $\mathcal{T}$ is defined and denoted $|\mathcal{M}| = |\bigcup_{G \in \mathcal{S}} \mathcal{T}(G, \mathcal{M})|$ where $\mathcal{S}$ is a fixed chosen set of isoclass representatives, one for each class. Corollary 7.1 is analogous to \cite[Corollary 8.2(iii)]{10}.

Corollary 7.1. There exists a cardinal $\kappa$ such that the $\mathcal{L}_\mathcal{T}$-structure underlying any indecomposable pure-injective object of $\mathcal{T}$ has cardinality at most $\kappa$.

We delay the proof of Corollary 7.1 until after Corollary 7.3.

Remark 7.2. We now note a non-trivial complication in our setting of compactly generated triangulated categories, which is absent in the module-theoretic situation. In the spirit of the results presented so far, it is natural to ask if one may adapt the proof of \cite[Corollary 8.2(iii)]{10} to prove Corollary 7.1. This seems straightforward at first glance, since there are multi-sorted versions of the downward Löwenheim-Skolem theorem; see for example \cite[Theorem 37]{2}.

Note that, in the language $\mathcal{L}_\mathcal{A}$ of modules over a fixed ring $\mathcal{A}$ from Example 2.2, any $\mathcal{L}_A$-structure is an object in the category of $\mathcal{A}$-modules, and any elementary embedding of $\mathcal{L}_A$-structures is an elementary (and hence pure) embedding of $\mathcal{A}$-modules. The same correspondence between structures need not be true here. By Remark 3.6, structures over $\mathcal{L}_\mathcal{T}$ correspond to objects of $\mathcal{Mod}\cdot\mathcal{T}^c$, which need not be given by objects of $\mathcal{T}$, since $\mathcal{Y}$ need not be essentially surjective. Fortunately, here we may instead use Corollary 7.3, a remark due to Krause, which shortens the proof.

Corollary 7.3. (See \cite[Corollary 1.10]{12}). There is a set $\mathcal{Sp}$ of isomorphism classes of pure-injective indecomposable objects in $\mathcal{T}$.

Proof of Corollary 7.3. It suffices to let $\kappa = |\bigcup_{G \in \mathcal{S}, M \in \mathcal{Sp}} \mathcal{T}(G, M)|$ as in Corollary 7.3.

We now proceed toward proving Theorem 1.1. For this we require Corollary 7.8 and to this end, in Theorem 7.4 we recall well-known results about decompositions of coproducts in abelian categories. For consistency we follow \cite{15}.

Theorem 7.4. Let $\mathcal{A}$ be an abelian category, and let $L = \{\mathcal{L}_j \mid j \in J\}$ and $N = \{\mathcal{N}_k \mid k \in K\}$ be collections of objects in $\mathcal{A}$. The following statements hold.

(i) \cite[p.82, Theorem 4.A7]{15} Suppose the collections $L$ and $N$ consist of indecomposable objects with local endomorphism rings. If we have $\bigoplus_{j \in J} \mathcal{L}_j \simeq \bigoplus_{k \in K} \mathcal{N}_k$ then there is a bijection $\sigma : J \to K$ such that $\mathcal{L}_j \simeq \mathcal{N}_{\sigma(j)}$ for all $j \in J$.
(ii) [15] p.82, Theorem 4.A11, Exchange Property| Suppose that \( J = K \), and that for all \( j \in J \), \( L_j \) is indecomposable and \( \mathcal{L}_j \) is the injective hull of \( N_j \). Let \( \mathcal{P} \) be a direct summand of \( \bigoplus_{j \in J} L_j \). Then there is a subset \( H \subseteq J \) with \( \bigoplus_{j \in J} L_j \cong \mathcal{P} \oplus \bigoplus_{j \in H} L_j \).

To apply Theorem [1.4] we use the following observation of Garkusha and Prest.

**Lemma 7.5.** [5] Lemma 2.2] A pure-injective object \( M \) of \( T \) is indecomposable if and only if the endomorphism ring \( \text{End}_T(M) \) is local.

**Corollary 7.6.** Let \( L = \{ L_j \mid j \in J \} \) be a collection of indecomposable objects in \( T \) such that the coproduct \( \bigoplus_{j \in J} L_j \) is pure injective. If \( P \) is a summand of \( \bigoplus_{j \in J} L_j \) then there exists \( H \subseteq J \) such that \( \bigoplus_{j \in J} L_j \cong P \oplus \bigoplus_{j \in H} L_j \). If additionally \( P \cong \bigoplus_{k \in K} N_k \) where each \( N_k \) is indecomposable, then there is a bijection \( \sigma : J \to K \sqcup H \) with \( L_j \cong N_{\sigma(j)} \) for all \( j \in J \).

**Proof.** Let \( \mathcal{L}_j = \mathcal{Y}(L_j) \) for each \( j \). Let \( L_T = \bigoplus_{j \in T} L_j \) and \( \mathcal{L}_T = \bigoplus_{j \in T} \mathcal{L}_j \) for any subset \( T \subseteq J \). Since \( \mathcal{Y} \) preserves small coproducts by Remark 6.8 and since each \( L_j \) is a summand of \( L_T \), each \( \mathcal{L}_j \) is a summand of \( \mathcal{L}_T \). By Theorem 6.4 and Remark 6.8 the object \( \mathcal{L}_j \) is an injective object in \( \mathbf{Mod} \mathcal{T}^c \). Thus each \( \mathcal{L}_j \) is injective. Since each \( L_j \) is indecomposable, we have that each \( \mathcal{L}_j \) is indecomposable. The same argument shows that, in the second claim, each \( N_k \) is pure-injective. We now use Theorem 7.4 and Lemma 7.6 to complete the proof.

Let \( \mathcal{P} = \mathcal{Y}(P) \). Since \( P \) is a direct summand of \( L_J \) by assumption, \( \mathcal{P} \) is a direct summand of \( \mathcal{L}_J \). Hence by Theorem 7.4(ii) there exists a subset \( H \subseteq J \) such that \( \mathcal{L}_J \cong \mathcal{P} \oplus \mathcal{L}_H \) in \( \mathbf{Mod} \mathcal{T} \). By Corollary 6.7 since we assume \( L_J \) is pure injective, \( \mathcal{L}_J \) is injective, and hence so too are \( \mathcal{P} \) and \( \mathcal{L}_H \). Again, by Corollary 6.7 this gives \( L_J \cong P \oplus L_H \). This gives the first claim.

Now suppose also \( P \cong \bigoplus_{k \in K} N_k \) where each \( N_k \) is indecomposable and, as above, necessarily pure-injective. By Lemma 7.5 each of the objects \( L_j \) and \( N_k \) has a local endomorphism ring. By Corollary 6.7 for any object \( Z \) of \( T \) we have \( \text{End}_T(Z) \cong \text{End}_{\mathbf{Mod} \mathcal{T}^c}(\mathcal{Y}(Z)) \). Since \( \mathcal{L}_J \cong \mathcal{P} \oplus \mathcal{L}_H \), the second claim follows, as above, by Theorem 7.4(i), using again that \( \mathcal{Y} \) preserves small coproducts by Remark 6.8. This gives the second claim. 

Lemma 7.6 is analogous to the cited result of Huisen-Zimmerman.

**Lemma 7.7.** [8] Lemma 5] Suppose, in the notation from Lemma 4.6, that \( M = \prod_{i \in \mathbb{N}} M_i = \bigoplus_{j \in J} L_j \) and \( \varphi_1(M) \supseteq \varphi_2(M) \supseteq \ldots \) is a descending chain of pp-definable subgroups. Suppose additionally that each object \( L_j \) is pure-injective and indecomposable.

Then there exists \( r \in \mathbb{N} \) such that, for each collection \( N = \{ N_i \mid i \in \mathbb{N} \} \) where \( N_i \) is an indecomposable summand of \( M_i \), we have \( \varphi_r(N_i) = \varphi_n(N_i) \) for all \( n \in \mathbb{N} \) with \( n \geq r \) and all but finitely many \( i \in \mathbb{N} \).

**Proof.** By Lemma 4.6 there exists \( r \in \mathbb{N} \) and a finite subset \( J' \) of \( J \) such that we have the containment \( \text{im}(\varphi_{r,j}) \subseteq \varphi_n(L_j) \) for all \( n \in \mathbb{N} \) with \( n \geq r \) and all \( j \in J \setminus J' \). Choose arbitrary \( m \in \mathbb{N} \) with \( m > |J'| \), and choose arbitrary \( i_1, \ldots, i_m \in \mathbb{N} \) with \( i_r \geq r \) for each \( p \). Let \( P = N_{i_1} \oplus \cdots \oplus N_{i_m} \). By construction \( P \) is a summand of \( M = \bigoplus_{j \in J} L_j \). By Corollary 7.6 since \( P \) is a direct sum of \( m \) indecomposable pure-injective objects, we have \( M \cong P \oplus \bigoplus_{j \in K} L_j \) and a bijection \( \sigma : J \to K \sqcup \{ i_1, \ldots, i_m \} \) with \( L_j \cong N_{\sigma(j)} \) for all \( j \in J \).

From this point one may complete the proof by closely following the proof of [8] Lemma 5]. In doing so one applies Lemma 4.6 where Huisen-Zimmerman applied [8] Lemma 4]. Note that, in a similar fashion, our proof of Lemma 4.6 closely followed that of [8] Lemma 4].

\[ \square \]
Corollary 7.8. Let $M$ be a pure-injective object of $T$ such that for any set $I$ the product $M^I$ is a coproduct of indecomposable pure-injective objects. Then $M$ is $\Sigma$-pure-injective.

Proof. Let $K = M^N$ and $M_i = M$ for each $i \in \mathbb{N}$, so that $K = \coprod_{i \in \mathbb{N}} M_i$. By hypothesis we have $K = \bigoplus_{j \in J} L_j$ where each $L_j$ is an indecomposable pure-injective object of $T$. By Lemma [7.7] there exists $r \in \mathbb{N}$ such that, for each collection $N = \{N_i \mid i \in \mathbb{N}\}$ where $N_i$ is an indecomposable summand of $M_i$, we have $\varphi_r(N_i) = \varphi_n(N_i)$ for all $n \in \mathbb{N}$ with $n \geq r$ and all but finitely many $i \in \mathbb{N}$. Fixing $j \in J$, for the collection given by $N_i = L_j$ for all $i$, we have $\varphi_r(L_j) = \varphi_n(L_j)$ for all $n \in \mathbb{N}$, and $\varphi_n(K) \cong \bigoplus_j \varphi_n(L_j)$ for all $n$ by Lemma [1.4]. □

Theorem [7.9] goes back to a characterisation due to Faith [3, Proposition 3].

Theorem 7.9. Let $M$ be an injective object in a Grothendieck category $A$ which has a set $\mathcal{G}$ of finitely generated generators. Then the following statements are equivalent.

(i) $M$ is $\Sigma$-injective.

(ii) The countable coproduct $M^{(\mathbb{N})}$ is injective.

(iii) For any $\mathfrak{F} \in \mathcal{G}$ every ascending chain of $M$-annihilator subobjects of $\mathfrak{F}$ must stabilise.

Proof of Theorem 7.9. Taking $I = \mathbb{N}$ shows (i) implies (ii). That (ii) implies (iii) follows from Remark 6.8 and Theorem 7.9. That (iii) implies (i) follows from Remark 6.1 and Proposition 5.6. The equivalence of (ii) and (iv) follows from Corollary 6.9. That (i) implies (v) follows from Lemma 6.6 and the converse follows from Corollaries 5.6 and 6.9. The equivalence of (i) and (vi) follows from Lemma 6.12 and Corollary 7.8. □

References

1. Walter Baur. Elimination of quantifiers for modules. Israel J. Math., 25(1-2):64–70, 1976.
2. Pilar Dellunde, Ángel García-Cerdaña, and Carles Noguera. Löwenheim-Skolem theorems for non-classical first-order algebrizable logics. Log. J. IGPL, 24(3):321–345, 2016.
3. Carl Faith. Rings with ascending condition on annihilators. Nagoya Math. J., 27:179–191, 1966.
4. Jose Luis Garcia and Nguyen Viet Dung. Some decomposition properties of injective and pure-injective modules. Osaka J. Math., 31(1):95–108, 1994.
5. Grigory Garkusha and Mike Prest. Triangulated categories and the Ziegler spectrum. Algebr. Represent. Theory, 8(4):499–523, 2005.
6. Manabu Harada. Perfect categories. IV. Quasi-Frobenius categories. Osaka Math. J., 10:585–596, 1973.
7. Wilfrid Hodges. A shorter model theory. Cambridge University Press, Cambridge, 1997.
8. Birge Huisgen-Zimmermann. Rings whose right modules are direct sums of indecomposable modules. Proc. Amer. Math. Soc., 77(2):191–197, 1979.
9. Birge Huisgen-Zimmermann. Purity, algebraic compactness, direct sum decompositions, and representation type. In Infinite length modules (Bielefeld, 1998), Trends Math., pages 331–367. Birkhäuser, Basel, 2000.
10. Christian Jensen and Helmut Lenzing. Model-theoretic algebra with particular emphasis on fields, rings, modules, volume 2 of Algebra, Logic and Applications. Gordon and Breach Science Publishers, New York, 1989.
11. Henning Krause. The spectrum of a locally coherent category. J. Pure Appl. Algebra, 114(3):259–271, 1997.
12. Henning Krause. Smashing subcategories and the telescope conjecture—an algebraic approach. Invent. Math., 139(1):99–133, 2000.
13. Henning Krause. Coherent functors in stable homotopy theory. Fund. Math., 173(1):33–56, 2002.
14. Amnon Neeman. The connection between the $K$-theory localization theorem of Thomason, Trobaugh and Yao and the smashing subcategories of Bousfield and Ravenel. Ann. Sci. École Norm. Sup. (4), 25(5):547–566, 1992.
15. Mike Prest. Model theory and modules, volume 130 of London Mathematical Society Lecture Note Series. Cambridge University Press, Cambridge, 1988.
16. Martin Ziegler. Model theory of modules. Ann. Pure Appl. Logic, 26(2):149–213, 1984.