CATEGORICAL $F$-QUOTIENTS AND MODEL-THEORETIC ISOMORPHISM THEOREMS

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Abstract. The standard categorical notion of a categorical quotient is insufficient, in that it fails to recover the expected quotients in certain categories. We present an alternative formulation of $F$-quotients in a category $C$, which are relativized to a faithful functor $F : C \to D$. The isomorphism theorems of universal algebras generalize to this setting, and we additionally find important links between $F$-quotients in the concrete category of first-order structures, and quotients defined for model-theoretic equivalence classes.

1. Introduction

The notions of quotients in hyperalgebras have already been studied extensively, and isomorphism theorems related to them have been researched in the past. Related research on this area includes studies on hyperrings [10, 20], polygroups [11], hypermodules [21] and general hyperalgebras [12]. Discussion on isomorphism theorems in other structures can be found in [3, 13, 16]. Inspired by these advances, this paper aims to study these notions and obtain standard results from universal algebra [6, 9] in the context of model theory and category theory.

Relationships between category theory and universal algebra have also been discussed in [5]. Using a categorical approach, Mousavi recently studied free hypermodules [19]. Here, our motivation also lies in approaching the problem of defining a suitable notion for algebraic quotients from a categorical point of view.

Let us fix the categorical notations and conventions that we will use in this paper. Abstract categories will generally be denoted by boldface, upright, capital letters — $C$, $D$, etc. Special named categories (e.g., Set, Ring) are referred to in the same style. As per usual, a category $C$ consists of a class of objects Obj($C$), whose members are denoted by italic capital letters $A, B, D$, etc, and a class of morphisms Hom($A, B$) for every $A, B \in$ Obj($C$), whose members are denoted by lowercase letters $f, g, h$, etc. We use the notation Hom($A, -$) to denote the class of all morphisms with source $A$. Functors are denoted by the calligraphic capital letter $F$.

We first recall the definition of a quotient in a category $C$ as the dual notion of subobjects. Let $A \in$ Obj($C$). Let $\text{Epi}(A, -) = \{ f \in \text{Hom}(A, -) : f \text{ is an epimorphism} \}$. We define a relation $\leq$ on $\text{Epi}(A, -)$ such that $(f : A \to B) \leq (g : A \to C)$ if and only if

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if $g$ factors through $f$ (i.e., there exists a morphism $h : B \to C$ such that $g = h \circ f$). The relation $\leq$ is easily seen to be a preorder, and thus it generates an equivalence relation on $\text{Epi}(A, -)$ in the usual manner.

1.1. Definition. [17, p. 126] The categorical quotients of $A$ are the equivalence classes of epimorphisms in $\text{Epi}(A, -)$ under the equivalence relation $f \sim g \iff f \leq g$ and $g \leq f$.

Here, categorical quotients are defined as (equivalence classes of) morphisms with source $A$, rather than objects obtained from $A$ via some quotient map. The object formulation of quotients is obtained by considering the target $B$ of an epimorphism $f : A \to B$. If $f : A \to B$ and $g : A \to C$ are such that $f \sim g$, then it follows directly from the definition of $\sim$ that $B$ and $C$ are isomorphic. Therefore, for any $\sim$-equivalence class $[f]_{\sim}$, the class $\{B \in \text{Obj}(C) : (g : A \to B) \in [f]_{\sim}\}$ is an isomorphism class of objects, which we identify with a quotient $[f]_{\sim}$ of $A$.

One problem with this definition is that it does not give a representative view of what quotients are in some common categories. For example, in $\text{Ring}$, the inclusion map $i : \mathbb{Z} \to \mathbb{Q}$ is an epimorphism [1, p. 133], despite not being surjective on the underlying sets. As a consequence, its equivalence class forms a categorical quotient, even though it is absurd to consider $\mathbb{Q}$ as a quotient ring of $\mathbb{Z}$.

In this paper, we define $\mathcal{F}$-quotients, an alternative categorical definition of quotients which better captures the notion in algebraic categories such as $\text{Ring}$. We devote Section 2 to developing the concept of an $\mathcal{F}$-quotient, and giving connections to free objects and the correspondence theorem of universal algebra. In Section 3, we shall see that the natural model-theoretic definition of quotients are essentially $\mathcal{F}$-quotients in the concrete category of first-order structures. Finally, using the theory developed in Section 2 and 3, we give relatively easy proofs for the isomorphism theorems in the realm of first-order structures.

2. $\mathcal{F}$-quotients

Let $\mathcal{F} : C \to D$ be a faithful functor between categories. A morphism $f$ in $C$ is an $\mathcal{F}$-epimorphism if $\mathcal{F}(f)$ is an epimorphism in $D$. Since $\mathcal{F}$ is faithful, any $\mathcal{F}$-epimorphism is necessarily an epimorphism.

2.1. Definition. Let $\mathcal{F} : C \to D$ be a faithful functor. Let $A \in \text{Obj}(C)$ and let

\[ \text{Epi}_{\mathcal{F}}(A, -) = \{ f \in \text{Hom}(A, -) : f \text{ is an } \mathcal{F}\text{-epimorphism}\}. \]

We define a relation $\leq$ on $\text{Epi}_{\mathcal{F}}(A, -)$ by

\[ (f : A \to B) \leq (g : A \to C) \iff \text{there exists } h : B \to C \text{ such that } g = h \circ f. \]

Then, $\mathcal{F}$-quotients of $A$ are equivalence classes under the equivalence relation $f \sim g \iff f \leq g$ and $g \leq f$. The class of all $\mathcal{F}$-quotients of $A$ is denoted by $\text{Quo}_{\mathcal{F}}(A)$. 
Since $\mathcal{F}$-epimorphisms are epimorphisms, we have $\text{Epi}_\mathcal{F}(A, -) \subseteq \text{Epi}(A, -)$. Therefore, the preceding definition allows us to restrict the set $\text{Epi}(A, -)$ to a certain extent. We note that Definition 2.1 generalizes the usual categorical definition of a quotient given in Definition 1.1.

2.2. Example. Let $D = C$, and $\mathcal{F} : C \to C$ be the identity functor. Then, the $\mathcal{F}$-quotients of $A \in \text{Obj}(C)$ are exactly the categorical quotients of $A$.

2.3. Example. Let $R$ be a unital ring and let $\mathcal{F} : \text{Ring} \to \text{Set}$ be the forgetful functor. Then $\text{Epi}_\mathcal{F}(R, -)$ precisely contains surjective ring homomorphisms with $R$ as the source, in contrast to the set $\text{Epi}(R, -)$ of all ring epimorphisms from $R$, which contains the troublesome inclusion map $i : \mathbb{Z} \to \mathbb{Q}$.

Note that the class $\text{Quo}_\mathcal{F}(A)$ might be proper for large categories, although in the next section, we show that it is a set in the concrete category of first-order structures. The relation $\leq$ forms a well-defined partial order on $\text{Quo}_\mathcal{F}(A)$, where for every $[f]_\sim, [g]_\sim \in \text{Quo}_\mathcal{F}(A)$, we define $[f]_\sim \leq [g]_\sim \iff f \leq g$.

2.4. Proposition. Let $\mathcal{F} : C \to D$ be a faithful functor and let $A \in \text{Obj}(C)$. If $f, g \in \text{Epi}_\mathcal{F}(A, -)$ and $f \leq g$, then the induced morphism $h$ such that $g = h \circ f$ is unique. Moreover, $h$ is an $\mathcal{F}$-epimorphism.

Proof. Suppose $h_1, h_2$ are such that $h_1 \circ f = g = h_2 \circ f$, then clearly $\mathcal{F}(h_1) \circ \mathcal{F}(f) = \mathcal{F}(h_2) \circ \mathcal{F}(f)$, and so since $\mathcal{F}(f)$ is an epimorphism, we have $\mathcal{F}(h_1) = \mathcal{F}(h_2)$. Since $\mathcal{F}$ is faithful, we have $h_1 = h_2$. To prove that $\mathcal{F}(h)$ is an epimorphism, let $j_1, j_2$ be such that $j_1 \circ h = j_2 \circ h$. We thus have $j_1 \circ g = j_2 \circ g$, and so $j_1 = j_2$ since $g$ is an epimorphism.

Let $f, g \in \text{Epi}_\mathcal{F}(A, -)$ and $f \leq g$. We will denote the unique induced morphism $h$ from Definition 2.1 by the suggestive notation $g/f$. Moreover, if $f : A \to B$, we will denote $B$ as $A/f$. By Proposition 2.4, we have $g/f \in \text{Epi}_\mathcal{F}(A/f, -)$. We note that $f/f$ is the identity morphism and that if $g \leq h$, then $(h/g) \circ (g/f) = (h/f)$.

2.5. Proposition. Let $\mathcal{F} : C \to D$ be a faithful functor and let $A \in \text{Obj}(C)$. If $f, g \in \text{Epi}_\mathcal{F}(A, -)$ and $f \leq g$, then

$$\frac{(A/f)}{(g/f)} = A/g.$$

Proof.

$$A \xymatrix{ \ar[r]^g & A/g = \frac{(A/f)}{(g/f)} \ar[d]_{g/f} \ar[dl] \cr A/f }$$
2.6. Lemma. Let $F: C \to D$ be a faithful functor and let $A \in \text{Obj}(C)$. If $f, g \in \text{Epi}_F(A, -)$, then $f \sim g$ implies that $A/f \cong A/g$.

Proof. The induced morphisms $f/g$ and $g/f$ are inverses of each other. To see this, notice that since $(g/f) \circ f = g$ and $(f/g) \circ g = f$, we have

$$
\frac{f}{g} \circ \frac{g}{f} = \frac{f}{g} \circ g = f.
$$

Thus, $(f/g) \circ (g/f) = \text{id}_{A/f}$ since $f$ is an epimorphism. Similarly, we can show that $(g/f) \circ (f/g) = \text{id}_{A/g}$, proving that $A/f \cong A/g$.

2.7. Lemma. Let $F: C \to D$ be a faithful functor and let $A \in \text{Obj}(C)$. Suppose that $f, g_1, g_2 \in \text{Epi}_F(A, -)$ are such that $f \leq g_1$ and $f \leq g_2$. Then:

1. $g_1 \leq g_2$ if and only if $g_1/f \leq g_2/f$;

2. $g_1 \sim g_2$ if and only if $g_1/f \sim g_2/f$.

Proof. The statement in (ii) obviously follows from (i), so we only prove (i). Suppose that $g_1 \leq g_2$. To show that $g_1/f \leq g_2/f$, we just need to verify that

$$
\begin{array}{ccc}
A/f & \xrightarrow{g_2/f} & A/g_2 \\
\downarrow{g_1/f} & & \downarrow{g_2/g_1} \\
A/g_1 & & \\
\end{array}
$$

commutes. However we have

$$
\frac{g_2}{g_1} \circ \frac{g_1}{f} \circ f = \frac{g_2}{g_1} \circ g_1 = g_2 = \frac{g_2}{f} \circ f,
$$

and since $f$ is an epimorphism, our claim follows.

Conversely, suppose that $g_1/f \leq g_2/f$. The diagram

$$
\begin{array}{ccc}
A & \xrightarrow{g_2} & A/g_2 \\
\downarrow{g_1} & & \downarrow{(g_2/f)/(g_1/f)} \\
A/g_1 & & \\
\end{array}
$$

commutes since

$$
\frac{(g_2/f)}{(g_1/f)} \circ g_1 = \frac{(g_2/f)}{(g_1/f)} \circ (g_1/f) \circ f = \frac{g_2}{f} \circ f = g_2.
$$

Thus, we have $g_1 \leq g_2$.
We can now provide a categorical analogue for the universal algebraic correspondence theorem.

2.8. Theorem. [Categorical Correspondence Theorem] Let $\mathcal{F} : \mathcal{C} \to \mathcal{D}$ be a faithful functor. Let $A \in \text{Obj}(\mathcal{C})$ and let $\uparrow [f]_\sim = \{[g]_\sim : f \leq g\}$ be the principal filter of $\text{Quo}_\mathcal{F}(A)$ generated by the quotient $[f]_\sim$. Then the partially ordered classes $(\uparrow [f]_\sim, \leq)$ and $(\text{Quo}_\mathcal{F}(A/f), \leq)$ are isomorphic.

Proof. Consider the map $[g]_\sim \mapsto [g/f]_\sim$. By Lemma 2.7, this map is a strongly order preserving, well-defined injection. To prove that it is surjective, suppose that $h : A/f \to B$ is an $\mathcal{F}$-epimorphism, and then show that $h \sim g/f$ for some $g \in \uparrow [f]_\sim$. Setting $g = h \circ f$ gives us the desired result.

To end this section, we present a direct connection between the concept of $\mathcal{F}$-free objects and $\mathcal{F}$-quotients. The following definition for $\mathcal{F}$-free objects generalizes the definition found in [15, p. 55] which requires that $\mathcal{F} : \mathcal{C} \to \text{Set}$.

2.9. Definition. Let $\mathcal{F} : \mathcal{C} \to \mathcal{D}$ be a faithful functor, $X$ a $\mathcal{D}$-object, $A$ a $\mathcal{C}$-object and $i : X \to \mathcal{F}(A)$ a monomorphism. We say the pair $(A, i)$ is $\mathcal{F}$-free over $X$ if, for any object $B \in \text{Obj}(\mathcal{C})$ and morphism $f : X \to \mathcal{F}(B)$, there exists a unique morphism $\varphi : A \to B$ in $\mathcal{C}$ such that the following commutes:

$$
\begin{array}{ccc}
X & \xleftarrow{i} & \mathcal{F}(A) \\
\downarrow{f} & & \downarrow{\mathcal{F}(\varphi)} \\
\mathcal{F}(B) & \end{array}
$$

2.10. Definition. With respect to a faithful functor $\mathcal{F} : \mathcal{C} \to \mathcal{D}$, a category $\mathcal{C}$ has $\mathcal{F}$-free objects if for every $X \in \text{Obj}(\mathcal{D})$, there is a pair $(A, i)$ which is $\mathcal{F}$-free over $X$.

Certain types of concrete categories $\mathcal{C}$ arising from algebra always have free objects with respect to their forgetful functors $\mathcal{F} : \mathcal{C} \to \text{Set}$. Famously, any non-trivial variety of algebras always has free objects [9, p. 170].

2.11. Proposition. Suppose $\mathcal{C}$ has $\mathcal{F}$-free objects for a faithful functor $\mathcal{F} : \mathcal{C} \to \mathcal{D}$. Then, every $\mathcal{C}$-object $K$ is realizable as an $\mathcal{F}$-quotient of an $\mathcal{F}$-free object. That is, we can find a pair $(A, i)$, which is $\mathcal{F}$-free over some $X \in \text{Obj}(\mathcal{D})$, and an $\mathcal{F}$-epimorphism $\varphi : A \to B$ in $\mathcal{C}$ such that $K = A/\varphi$.

Proof. Let $(A, i)$ be the free object over $\mathcal{F}(K)$. By definition, $A$ satisfies the universal property mentioned in Definition 2.9. Applying this property when $B = K$ and $f = \text{id}_{\mathcal{F}(K)}$, we get the existence of $\varphi : A \to K$ such that

$$
\begin{array}{ccc}
\mathcal{F}(K) & \xleftarrow{i} & \mathcal{F}(A) \\
\downarrow{\text{id}_{\mathcal{F}(K)}} & & \downarrow{\mathcal{F}(\varphi)} \\
\mathcal{F}(K) & \end{array}
$$
The fact that $F(\varphi)$ is an epimorphism follows trivially from the fact that $\text{id}_{F(K)}$ is an epimorphism, and so we have $K = A/\varphi$.

3. Quotients in elementary classes

In this section, we assume some familiarity with elementary model theory, which can be found in any model theory text such as [8, 18]. Let us fix a first-order language $\mathcal{L}$. A particularly important concept that will be used throughout this paper is the notion of a strong homomorphism between $\mathcal{L}$-structures.

3.1. Definition. [18, p. 24] Let $\mathcal{M}$ and $\mathcal{N}$ be $\mathcal{L}$-structures. A strong homomorphism from $\mathcal{M}$ to $\mathcal{N}$ is a map $f : \mathcal{M} \to \mathcal{N}$ such that for all $n \in \mathbb{N}$ and $x_1, \ldots, x_n \in M$, where $\mathcal{M}$ is the universe of $\mathcal{M}$:

1. For every $n$-ary function symbol $F \in \mathcal{L}$, we have
   $$f(F^\mathcal{M}(x_1, \ldots, x_n)) = F^\mathcal{N}(f(x_1), \ldots, f(x_n));$$

2. For every $n$-ary relation symbol $R \in \mathcal{L}$, we have
   $$R^\mathcal{M}(x_1, \ldots, x_n) \iff R^\mathcal{N}(f(x_1), \ldots, f(x_n)).$$

The $\mathcal{L}$-structures and strong homomorphisms form a concrete category $\mathcal{L}\text{Str}$ under the forgetful functor $F : \mathcal{L}\text{Str} \to \text{Set}$ mapping every $\mathcal{L}$-structure to its universe. Setting $\mathcal{L} = \{c\}$ where $c$ is a constant symbol (or equivalently, a 0-ary function symbol) as an example, we obtain a category isomorphic to the category of pointed sets $\text{pSet}$.

If we also have an $\mathcal{L}$-theory $\mathcal{T}$, then the $\mathcal{T}$-models and strong homomorphisms form a full subcategory $\mathcal{T}\text{Mdl} \subseteq \mathcal{L}\text{Str}$, called an elementary class. For example, if $\mathcal{L} = \{+,-,0\}$, where $-$ denotes the unary negation symbol, and $\mathcal{T}$ consists of the abelian group axioms, then clearly $\mathcal{T}\text{Mdl} \cong \text{Ab}$, the category of abelian groups. If $\mathcal{T} = \emptyset$, then trivially we have $\mathcal{T}\text{Mdl} \cong \mathcal{L}\text{Str}$. Observe that in $\mathcal{T}\text{Mdl}$, bijective strong homomorphisms are precisely the isomorphisms.

Throughout this paper, the universe (or underlying set) of an $\mathcal{L}$-structure $\mathcal{M}$ will be denoted as $M$. In addition, the universe of the $F$-quotient $\mathcal{M}/f$ will be denoted as $M/f$.

In [4], Barrett introduces the idea of a logical quotient in the context of first-order model theory. This idea generalizes the notion of quotients in universal algebra.

3.2. Definition. [4] Let $\mathcal{M}$ be an $\mathcal{L}$-structure. An equivalence relation $\theta$ on $M$ is called a congruence on $\mathcal{M}$ if for all $n \in \mathbb{N}$ and $(x_i, y_i) \in \theta$, $i \leq n$, we have:

1. For every $n$-ary function symbol $F \in \mathcal{L}$,
   $$(F^\mathcal{M}(x_1, \ldots, x_n), F^\mathcal{M}(y_1, \ldots, y_n)) \in \theta;$$

2. For every $n$-ary relation symbol $R \in \mathcal{L}$,
   $$R^\mathcal{M}(x_1, \ldots, x_n) \iff R^\mathcal{M}(y_1, \ldots, y_n).$$

The set of congruences on $\mathcal{M}$ is denoted as $\text{Con}(\mathcal{M})$. 
3.3. Remark. This definition of congruence on \( \mathcal{L} \)-structures is not compatible with the notion of (strong) congruence on hyperalgebras found in [2], where every \( n \)-ary hyperoperation is considered as an \( (n+1) \)-ary relation. This definition is instead motivated by the desire to construct the \( \mathcal{F} \)-quotients in the concrete category \( \mathcal{L} \mathbf{Str} \) in a natural way. Theorem 3.13, for example, gives a direct link between \( \text{Quo}_\mathcal{F}(\mathcal{M}) \) and \( \text{Con}(\mathcal{M}) \), where \( \mathcal{F} : \mathcal{L} \mathbf{Str} \to \mathbf{Set} \) is the forgetful functor.

It is known that the lattice \( \text{Eq}(\mathcal{M}) \) of equivalence relations on \( \mathcal{M} \), ordered by inclusion, is complete. Furthermore, for \( \theta_i \in \text{Eq}(\mathcal{M}) \), \( i \in I \), we have

\[
\bigwedge_{i\in I} \theta_i = \bigcap_{i\in I} \theta_i
\]

and

\[
\bigvee_{i\in I} \theta_i = \bigcup_{\{\theta_{i_1} \circ \ldots \circ \theta_{i_k} : i_1, \ldots, i_k \in I\}}
\]

where \( \circ \) denotes the composition of binary relations.

3.4. Proposition. Let \( \mathcal{M} \) be an \( \mathcal{L} \)-structure. Then \( (\text{Con}(\mathcal{M}), \subseteq) \) is a complete sublattice of \( (\text{Eq}(\mathcal{M}), \subseteq) \).

We can naturally define quotients of first-order structures using congruences:

3.5. Definition. [4] Let \( \theta \) be a congruence on an \( \mathcal{L} \)-structure \( \mathcal{M} \). The quotient of \( \mathcal{M} \) by \( \theta \) is defined as the \( \mathcal{L} \)-structure \( \mathcal{M}/\theta \) such that:

1. The universe of \( \mathcal{M}/\theta \) is \( \mathcal{M}/\theta = \{[x]_\theta : x \in \mathcal{M}\} \);
2. For every \( n \)-ary function symbol \( F \in \mathcal{L} \), we have
   \[
   F^{\mathcal{M}/\theta}(\ldots, [x_n]_\theta) = [F^\mathcal{M}(\ldots, x_n)]_\theta;
   \]
3. For every \( n \)-ary relation symbol \( R \in \mathcal{L} \), we have
   \[
   R^{\mathcal{M}/\theta}(\ldots, [x_n]_\theta) \iff R^\mathcal{M}(\ldots, x_n).
   \]

The notation \( \mathcal{M}/\theta \) for a quotient generated by a congruence and \( \mathcal{M}/f \) for an \( \mathcal{F} \)-quotient can sometimes be in conflict. To avoid confusion, we shall consistently use lowercase Greek letters such as \( \theta \) and \( \psi \) to denote congruences and lowercase letters such as \( f \) and \( g \) to denote \( \mathcal{F} \)-epimorphisms.

Given a non-abelian group \( G \), the quotient \( G/G \) is abelian. This implies that the elementary class of non-abelian groups is not closed under quotients. However, we can give a sufficient condition for the class of \( \mathcal{T} \)-models to be closed under quotients by giving some requirements for \( \mathcal{T} \) to fulfill.

Recall that an atomic formula is an \( \mathcal{L} \)-formula of the form \( R(t_1, \ldots, t_n) \) or \( s = t \), where \( s, t, t_1, \ldots, t_n \) are \( \mathcal{L} \)-terms. A literal is an atomic formula or the negation of one (i.e., of the form \( R(t_1, \ldots, t_n) \), \( \neg R(t_1, \ldots, t_n) \), \( s = t \) or \( \neg(s = t) \)).
3.6. **Definition.** [7] An \( \mathcal{L} \)-formula \( \varphi(x_1, \ldots, x_n) \) is in *prenex conjunctive normal form* (PCNF) if it is of the form

\[
Q_1 y_1 \ldots Q_m y_m \left( \psi_{11} \lor \ldots \lor \psi_{1n_1} \right) \land \ldots \land \left( \psi_{k1} \lor \ldots \lor \psi_{kn_k} \right)
\]

where each \( Q_i \) is either the \( \forall \) or \( \exists \) symbol, and each \( \psi_{ij} \) is a literal with free variables in \( x_1, \ldots, x_n, y_1, \ldots, y_m \).

3.7. **Remark.** It should be noted that every \( \mathcal{L} \)-formula is equivalent to one in PCNF (see [7]). Moreover, Barrett [4] showed that formulae written in PCNF without literals of the form \( \neg(s = t) \) have their truth preserved under quotients. Thus, if an elementary class can be axiomatized by formulae of this form, it is closed under quotients. Varieties of algebras, e.g., groups and rings, provide examples of such classes.

3.8. **Example.** An MI-monoid [14] (MI stands for Many Identities) can be defined as the structure \( G \) with universe \( G \) and language \( \mathcal{L} = \{ \circ, E \} \), where \( \circ \) is a binary operation and \( E \) is a unary relation, such that:

1. \( \forall x \forall y \forall z \ (x \circ y) \circ z = x \circ (y \circ z) \);
2. \( \exists e \forall x \ (\neg E(e) \lor x \circ e = x) \land (\neg E(e) \lor e \circ x = x) \);
3. \( \forall a \forall b \ (\neg E(a) \lor \neg E(b) \lor E(a \circ b)) \);
4. \( \forall x \forall a \ (\neg E(a) \lor x \circ a = a \circ x) \).

We say that \( e \) is a *pseudoidentity* of \( G \) if \( E(e) \). Since this axiomatization includes no \( \neg(s = t) \) literals, it follows from the above discussion that the class of MI-monoids is closed under quotients.

3.9. **Definition.** Let \( f : \mathcal{M} \to \mathcal{N} \) be a strong homomorphism between \( \mathcal{L} \)-structures. The *kernel* of \( f \) is defined as

\[
\ker f = \{(x, y) \in M^2 : f(x) = f(y)\}.
\]

3.10. **Definition.** Let \( \mathcal{M} \) be an \( \mathcal{L} \)-structure and let \( \theta \in \text{Con}(\mathcal{M}) \). The quotient map \( \pi_\theta : \mathcal{M} \to \mathcal{M}/\theta \) is defined as \( \pi_\theta(x) = [x]_\theta \).

It is straightforward to show that \( \ker f \) is a congruence and that \( \pi_\theta \) is a surjective strong homomorphism. The existence of \( \pi_\theta \) leads to the fact that every congruence on \( \mathcal{M} \) is the kernel of some surjective strong homomorphism from \( \mathcal{M} \).

3.11. **Proposition.** Let \( \mathcal{M} \) be an \( \mathcal{L} \)-structure and let \( \theta \) be a congruence on \( \mathcal{M} \). Then, \( \theta = \ker \pi_\theta \).

We now connect the notion of \( \mathcal{F} \)-quotients from the previous section and the notion of quotients of \( \mathcal{L} \)-structures from this section.
3.12. **Lemma.** Let $\mathcal{M}$ be an $\mathcal{L}$-structure and let $\mathcal{F} : \mathcal{L}\text{Str} \to \text{Set}$ be the forgetful functor. Suppose that $f, g \in \text{Epi}_\mathcal{F}(\mathcal{M}, -)$. Then:

1. $f \leq g$ if and only if $\ker f \subseteq \ker g$;
2. $f \sim g$ if and only if $\ker f = \ker g$.

**Proof.** Since (ii) is easily obtained from (i), we only give a proof of (i). Suppose that $f \leq g$ and that $(x, y) \in \ker f$. Since $f(x) = f(y)$, we then have

$$g(x) = ((g/f) \circ f)(x) = ((g/f)(y)) = g(y),$$

and thus $(x, y) \in \ker g$. Conversely, suppose that $\ker f \subseteq \ker g$. Set $h : \mathcal{M}/f \to \mathcal{M}/g$ to be the map $h(y) = g(x)$, where $x$ is any element of $\mathcal{M}$ such that $f(x) = y$. From the fact that $f$ is surjective and that $\ker f \subseteq \ker g$, it is clear that $h$ is well-defined. Moreover, we can easily see that $g = h \circ f$. Now let $y_1, \ldots, y_n \in \mathcal{M}/f$ and $F, R \in \mathcal{L}$ be $n$-ary. Suppose that $f(x_i) = y_i$ for $i \leq n$. The map $h$ preserves the function $F$ since

$$h(F^{\mathcal{M}/f}(y_1, \ldots, y_n)) = h(F^{\mathcal{M}/f}(f(x_1), \ldots, f(x_n))) = (h \circ f)(F^{\mathcal{M}}(x_1, \ldots, x_n)) = g(F^{\mathcal{M}}(x_1, \ldots, x_n)) = F^{\mathcal{M}/g}(g(x_1), \ldots, g(x_n)) = F^{\mathcal{M}/g}(h(y_1), \ldots, h(y_n)).$$

Moreover,

$$R^{\mathcal{M}/f}(y_1, \ldots, y_n) \iff R^{\mathcal{M}/f}(f(x_1), \ldots, f(x_n)) \iff R^{\mathcal{M}}(x_1, \ldots, x_n) \iff R^{\mathcal{M}/g}(g(x_1), \ldots, g(x_n)) \iff R^{\mathcal{M}/g}(h(y_1), \ldots, h(y_n))$$

which implies that $h$ also preserves the relation $R$. Thus, $h$ is a strong homomorphism and so we have $f \leq g$. \hfill \blacksquare

3.13. **Theorem.** Let $\mathcal{M}$ be an $\mathcal{L}$-structure and let $\mathcal{F} : \mathcal{L}\text{Str} \to \text{Set}$ be the forgetful functor. The map $(\text{Quo}_\mathcal{F}(\mathcal{M}), \leq) \to (\text{Con}(\mathcal{M}), \subseteq), [f] \sim \mapsto \ker f$ defines a lattice isomorphism.

**Proof.** This map is surjective by Proposition 3.11. The theorem then clearly follows from Lemma 3.12, which ensures that the map is a strongly order-preserving, well-defined injection. \hfill \blacksquare
3.14. **Proposition.** Let $\mathcal{M}$ be an $\mathcal{L}$-structure and let $F : \mathcal{L}\text{Str} \to \text{Set}$ be the forgetful functor. Let $f \in \text{Epi}_F(\mathcal{M},\mathcal{M})$. Then the map $\mathcal{M}/\ker f \to \mathcal{M}/f$ defined by $[x]_{\ker f} \mapsto f(x)$, where $x \in \mathcal{M}$, is a well-defined isomorphism.

**Proof.** It is easy to see that the map is a well-defined bijection. Let $x_1, \ldots, x_n \in \mathcal{M}$ and $F, R \in \mathcal{L}$ be $n$-ary. We see that the map preserves $F$ by combining the fact that $F^{\mathcal{M}/\ker f}([x_1]_{\ker f}, \ldots, [x_n]_{\ker f}) = [F^{\mathcal{M}}(x_1, \ldots, x_n)]_{\ker f}$ and $f(F^{\mathcal{M}}(x_1, \ldots, x_n)) = F^{\mathcal{M}/f}(f(x_1), \ldots, f(x_n))$.

The map also preserves $R$ since $R^{\mathcal{M}/\ker f}([x_1]_{\ker f}, \ldots, [x_n]_{\ker f}) \iff R^{\mathcal{M}}(x_1, \ldots, x_n) \iff R^{\mathcal{M}/f}(f(x_1), \ldots, f(x_n))$.

Therefore, the map is a strong homomorphism.

4. **Model-theoretic isomorphism theorems**

In this section, the isomorphism theorems for universal algebras (see [6]) are generalized to the setting of $\mathcal{F}$-quotients. Most of the isomorphism theorems presented are easier to see due to the progress made previously in Section 2 and 3. It can be verified that the image $f(\mathcal{M})$ of a strong homomorphism $f : \mathcal{M} \to \mathcal{N}$ forms a substructure of $\mathcal{N}$.

4.1. **Theorem.** [First Isomorphism Theorem] Let $f : \mathcal{M} \to \mathcal{N}$ be a strong homomorphism. Then

$$\frac{\mathcal{M}}{\ker f} \cong f(\mathcal{M}).$$

**Proof.** Set a surjective homomorphism $f' : \mathcal{M} \to f(\mathcal{M})$ as $f'(x) = f(x)$. By Proposition 3.14, we have

$$\frac{\mathcal{M}}{\ker f} \cong \frac{\mathcal{M}}{\ker f'} = f(\mathcal{M})$$

as desired.

Suppose $K$ is an elementary class axiomatized by PCNF formulae without $\neg(s = t)$ literals. By Remark 3.7, for any $\mathcal{M}, \mathcal{N} \in K$ and any strong homomorphism $f : \mathcal{M} \to \mathcal{N}$, $f(\mathcal{M}) \in K$ also, since it is isomorphic to $\mathcal{M}/\ker f$. In other words, the first isomorphism theorem easily shows that $K$ is closed under strong homomorphic images.

Suppose that $\mathcal{N}$ is a substructure of $\mathcal{M}$, and that $\theta$ is a congruence on $\mathcal{M}$. We can define a subset of $\mathcal{M}$ as $N^\theta = \{x \in \mathcal{M} : N \cap [x]_\theta \neq \emptyset\}$. The smallest substructure of $\mathcal{M}$ containing the set $N^\theta$ is denoted as $\mathcal{N}^\theta$.

4.2. **Proposition.** If $\mathcal{N}$ be a substructure of $\mathcal{M}$ and $\theta \in \text{Con}(\mathcal{M})$, then the universe of $\mathcal{N}^\theta$ is $\mathcal{N}^\theta$.

In addition, we can define the restriction of $\theta$ to a subuniverse $N$ as $\theta|_N = \theta \cap N^2$. It is not hard to show that $\theta|_N \in \text{Con}(\mathcal{N})$. 
4.3. **Theorem.** [Second Isomorphism Theorem] If \( \mathcal{N} \) is a substructure of \( \mathcal{M} \) and \( \theta \in \text{Con}(\mathcal{M}) \), then

\[
\frac{\mathcal{N}}{\theta|_{\mathcal{N}}} \cong \frac{\mathcal{N}^\theta}{\theta|_{\mathcal{N}^\theta}}.
\]

**Proof.** It is straightforward to show that the map \( [y]_{\theta|_{\mathcal{N}}} \mapsto [y]_{\theta|_{\mathcal{N}^\theta}}, y \in \mathcal{N} \) is well-defined isomorphism. \( \blacksquare \)

4.4. **Definition.** Let \( \theta \subseteq \psi \) be congruences on \( \mathcal{M} \). Then,

\[
\psi/\theta = \{([x]_\theta, [y]_\theta) \in (\mathcal{M}/\theta)^2 : (x, y) \in \psi\}.
\]

It is straightforward to confirm that \( \psi/\theta \in \text{Con}(\mathcal{M}/\theta) \). From the following definition, we find that \( \ker(g/f) \), where \( f \leq g \) are surjective strong homomorphisms, and \( \ker g/\ker f \) are essentially equivalent.

4.5. **Proposition.** Let \( \mathcal{F} : \mathcal{LStr} \to \mathbf{Set} \) be the forgetful functor and let \( f, g \in \text{Epi}_\mathcal{F}(\mathcal{M}, -) \). Suppose that \( \phi : \mathcal{M}/\ker f \to \mathcal{M}/f \) is the isomorphism \( [x]_{\ker f} \mapsto f(x) \). If \( f \leq g \), then

\[
\phi^2(\ker g/\ker f) = \ker(g/f).
\]

**Proof.** For every \( x, y \in \mathcal{M} \), we have the following chain of equivalences:

\[
([x]_{\ker f}, [y]_{\ker f}) \in \ker g/\ker f \iff (x, y) \in \ker g \\
\implies g(x) = g(y) \\
\implies ((g/f) \circ f)(x) = ((g/f) \circ f)(y) \\
\implies (f(x), f(y)) \in \ker(g/f).
\]

This proves that \( \phi^2(\ker g/\ker f) = \ker(g/f) \). \( \blacksquare \)

4.6. **Theorem.** [Third Isomorphism Theorem] Let \( \theta \subseteq \psi \) be congruences on \( \mathcal{M} \). Then,

\[
\frac{(\mathcal{M}/\theta)}{\psi/\theta} \cong \frac{\mathcal{M}}{\psi}.
\]

**Proof.** Let \( \mathcal{F} : \mathcal{LStr} \to \mathbf{Set} \) be the forgetful functor and let \( f, g \in \text{Epi}_\mathcal{F}(\mathcal{M}, -) \), \( f \leq g \), be such that \( \psi = \ker g \) and \( \theta = \ker f \). By Proposition 4.5, we have

\[
\frac{(\mathcal{M}/\theta)}{\psi/\theta} = \frac{(\mathcal{M}/\ker f)}{\ker g/\ker f} \cong \frac{\phi(\mathcal{M}/\ker f)}{\phi(\ker g/\ker f)} = \frac{(\mathcal{M}/f)}{\ker(g/f)}.
\]

Moreover, by Proposition 3.14 and Proposition 2.5, we obtain

\[
\frac{(\mathcal{M}/f)}{\ker(g/f)} \cong \frac{(\mathcal{M}/f)}{(g/f)} = \frac{\mathcal{M}}{\ker g} \cong \frac{\mathcal{M}}{\psi},
\]

which completes the proof. \( \blacksquare \)
4.7. **Theorem.** [Correspondence Theorem] Let $[\theta, M^2] = \{ \psi : \theta \subseteq \psi \}$ be the principal filter of $\text{Con}(\mathcal{M})$ generated by the congruence $\theta$. Then $[\theta, M^2]$ and $\text{Con}(\mathcal{M}/\theta)$ are isomorphic lattices.

**Proof.** Let $\mathcal{F} : \mathcal{LStr} \to \text{Set}$ be the forgetful functor. By invoking Theorem 3.13, Proposition 3.14 and Theorem 2.8 judiciously, we have

$$[\theta, M^2] \cong \uparrow [f] \cong \text{Quo}_{\mathcal{F}}(\mathcal{M}/f) \cong \text{Con}(\mathcal{M}/f) \cong \text{Con}(\mathcal{M}/\theta),$$

where $f \in \text{Epi}_{\mathcal{F}}(\mathcal{M}, -)$ is such that $\theta = \ker f$.

5. **Conclusion**

We hope that this paper brings to light some potential applications of model theory to studying algebraic structures. While we have presented some definitions for quotients and basic isomorphism theorems, one could easily provide generalizations for the Zassenhaus lemma and the Jordan-Hölder theorem, as well as free structures in the realm of model theory. Alternatively, one could attempt to define the notion of “weak” congruences which agrees with the $\mathcal{F}$-quotients on the concrete category of $\mathcal{L}$-structures with weak homomorphisms as morphisms.

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