Progressive Identification of True Labels for Partial-Label Learning

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Abstract

Partial-label learning is one of the important weakly supervised learning problems, where each training example is equipped with a set of candidate labels that contains the true label. Most existing methods elaborately designed learning objectives as constrained optimizations that must be solved in specific manners, making their computational complexity a bottleneck for scaling up to big data. The goal of this paper is to propose a novel framework of partial-label learning without implicit assumptions on the model or optimization algorithm. More specifically, we propose a general estimator of the classification risk, theoretically analyze the classifier-consistency, and establish an estimation error bound. We then explore a progressive identification method for approximately minimizing the proposed risk estimator, where the update of the model and identification of true labels are conducted in a seamless manner. The resulting algorithm is model-independent and loss-independent, and compatible with stochastic optimization. Thorough experiments demonstrate it sets the new state of the art.

1 Introduction

In practice, the increasing demand for massive data makes it inevitable that not all examples are equipped with high-quality labels. Low efficiency to weakly supervision \(^1\) becomes a critical bottleneck for supervised learning methods. We focus on an important type in weakly supervised learning problems called partial-label learning (PLL) \(^2\)\(^3\)\(^4\), where each training example is associated with multiple candidate labels among which exactly one is true. This problem arises in many real-world tasks such as automatic image annotation \(^5\)\(^6\), web mining \(^7\), ecoinformatics \(^8\), etc.

Related research on PLL was pioneered by a multiple-label learning (MLL) approach \(^9\). Despite the same form of supervision information that each training example is assigned with a set of candidate labels, a vital difference between MLL and PLL is that the goal of PLL is identifying the only one true label among candidate labels whereas for MLL identifying an arbitrary label in the candidate label set is acceptable. In

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practical implementation, [9] formulated the learning objective by minimizing the KL divergence between the prior probability and the model-based conditional distribution, and solved the optimization by Expectation Maximization (EM) algorithm, resulting in a procedure iterates between estimating the prior probability and training the model. The true labels of training examples in PLL are obscured by candidate labels that hinder the learning, the key to success is, therefore, identifying the true labels. We review that [9] attempts to recover the prior information as the fitting target of the model, and then the label with maximum prior probability is naturally identified as the true label. Thus, the foundational work inspires the successors to propose EM-based methods for recovering non-uniform prior information on the labels reasonably. Various effort to design learning objectives from the perspective of modifying learning models and loss functions, or adding proper regularizations and constraints. [10, 11] proposed a method to train linear dictionaries with sparse constraints, and then [12] extended the linear dictionary by the kernel trick. [13] adapted the boosting techniques for maintaining the prior information in each iteration. In [14, 15], the optimization of prior probability is formulated as a quadratic programming (QP) problem. In addition, maximum margin methods [2, 16] identified the true labels by defining a multi-class maximum margin problem [17] to be solved by some off-the-shelf implementation on SVM [18]. Some non-parametric methods have also been proposed. [19] proposed a kNN method, and [4] constructed a QP problem for better leveraging the underlying distribution of the data.

Most existing methods elaborately designed tricky constrained learning objectives that are coupled to some specific optimization algorithms. A non-linear time complexity in the total volume of data becomes a major limiting factor when these methods envision large amounts of training data. Furthermore, [20] proved stochastic optimization algorithm, which few of the existing methods are compatible with, is the best choice for large-scale learning problems considering the estimation–optimization tradeoff, so the PLL problem is still practically challenging.

In this paper, we would like to propose a novel framework of PLL without implicit assumptions on the model and optimization algorithm for scaling to big data. For this purpose, we rethink critically whether the unconstrained method in [9] is worth exploring?

Our answer is in two folds. First, the decade-old method used a simple linear model fed by the handcrafted features, which is incompetent to represent and discriminate, especially when encountering large-scale learning problems, and we also observe that almost all existing methods exploited non-deep models. Fortunately, the learning objective in [9] is easy to be instantiated by deep neural networks with an enormous benefit in practice [21]. Second, the question we explore is essentially whether we can advance the pioneering work not depending solely on the modification of the learning model. We first derive a general risk estimator leading to classifier-consistency, and then propose a new method bringing a step further towards this goal. We also prove that the proposed method is a generalization of [9]. We further experimentally verify its superiority on benchmark datasets corrupted by manual candidate labels, controlled UCI datasets, and real-world partial-label datasets. Our contributions can be summarized as follows:

- **Theoretically**, we propose a new risk estimator for PLL, and analyze the classifier-consistency, i.e., the classifier learned from partially labeled data converges to the optimal one learned from ordinary supervised data under mild conditions. Then, we establish an estimation error bound for it.
- **Practically**, we propose a *progressive identification* method for approximately optimizing the above risk estimator. The proposed method operates in a mini-batched training manner where the update of the model and the identification of true labels are accomplished seamlessly. The resulting method is model-independent and loss-independent, and is compatible with stochastic optimization (e.g., [22, 23, 24]).
2 Basic Risk Estimator for PLL

In this section, we formulate PLL, propose a general risk estimator, and establish the estimation error bound.

2.1 Preliminaries

In ordinary multi-class classification, let $\mathcal{X} \in \mathbb{R}^d$ be the instance space and $\mathcal{Y} = [c]$ be the label space, where $d$ is the feature space dimension, $[c] := \{1, 2, \ldots, c\}$ and $c > 2$ is the number of classes. Let $P[X,Y]$ be the underlying joint distribution of random variables $(X,Y) \in \mathcal{X} \times \mathcal{Y}$. The goal is to learn a decision function $g : \mathcal{X} \rightarrow \mathbb{R}^c$ that minimizes the estimator of the classification risk:

$$\hat{R}(g) = \mathbb{E}_{X,Y}[\ell\{g_1(X), \ldots, g_c(X)\}, Y],$$

(1)

where $\mathbb{E}_{X,Y}$ is short for $\mathbb{E}_{(X,Y) \sim P[X,Y]}$, $\ell : \mathbb{R}^c \times \mathbb{R} \rightarrow \mathbb{R}_+$ is the loss function, and $g_i(X) : \mathcal{X} \rightarrow \mathbb{R}$ is an estimate of $P[Y = i|X]$. Typically, the classifier $f$ is assumed to take the following form:

$$f(X) = \arg \max_{i \in [c]} g_i(X).$$

The hypothesis in $G$ with minimal error $g^* = \arg \min_{g \in G} \hat{R}(g)$ is the optimal decision function. And in this case, $f^*(X) = \arg \max_{i \in [c]} g^*_i(X)$.

Since the joint distribution is usually unknown, the expectation in Eq. (1) is typically approximated by the average over the training examples $\{(x_i, y_i)\}^n_{i=1} \overset{i.i.d.}{\sim} P[X,Y]$: $\hat{R}(f) = \frac{1}{n} \sum_{i=1}^n [\ell\{g_1(x_i), \ldots, g_c(x_i)\}, y_i]$, and $\hat{g} = \arg \min_{g \in G} \hat{R}(g)$ is returned by the empirical risk minimization (ERM) principle [25].

2.2 Risk Estimator of PLL

Next, we formulate the problem of PLL. The candidate label set $\mathcal{S}$ is the powerset of $\mathcal{Y}$ and its cardinality is $|\mathcal{S}| = 2^c - 2$ (the empty set $\emptyset$ and complete set are not included). Let $(X,Y,S)$ be the variables defined on $\mathcal{X} \times \mathcal{Y} \times \mathcal{S}$ with the joint distribution $P[X,Y,S]$, which may be decomposed into ordinary distribution $P[X,Y]$ and label set conditional distribution $P[S|X,Y]$. Note that in PLL, the true label $Y$ is invisible to the learner, so we can only have $P[X,S]$. In this way, we minimize the following risk for PLL:

$$R_{PLL}(g) = \mathbb{E}_{X,S}[\ell_{PLL}(\{g_1(X), \ldots, g_c(X)\}, S)],$$

where $\ell_{PLL} : \mathbb{R}^c \times \mathcal{P}([c]) \rightarrow \mathbb{R}$ and $\mathcal{P}([c])$ is the powerset of $[c]$. $\ell_{PLL}$ is a loss function defined specially for PLL and we will specify its form later. In the same way as ordinary multi-class case, we can obtain the optimal solution

$$g_{PLL}^* = \arg \min_{g \in G} R_{PLL}(g),$$

and we can further learn a classifier $f_{PLL}^*$ by

$$f_{PLL}^*(X) = \arg \max_{i \in [c]} g_{PLL}^*_i(X).$$

Our task now becomes to derive a new loss function $\ell_{PLL}$ such that the learned $f_{PLL}^*$ is equivalent to $f^*$ learned in ordinary multi-class learning if the training sample size is sufficiently large.

Note that in $S$ there is only one true label. Image the case when the learned $g$ can be good enough to approximate $P(Y|X)$. Then the $g$ associated with the true label will incur the minimal loss among all candidate labels. Motivated by this, we write our loss function as

$$\ell_{PLL} = \min_{Y \in S} \ell(\{g_1(X), \ldots, g_c(X)\}, Y'),$$

(2)
which immediately leads to a new risk estimator, namely,
\[
\mathcal{R}_{PLL}(g) = E_{X,S} \min_{Y' \in S} \ell \left( \{ g_1(X), \ldots, g_c(X) \}, Y' \right).
\] (3)

So what is the effect of optimizing the new defined risk for PLL? To show the effect, we will prove that the proposed risk estimator ensures \( f^*_{PLL} \) converges to \( f^* \) under reasonable assumptions.

Our first assumption is that our learning is conducted under the deterministic case.

**Assumption 1.** Consider the general deterministic learning scenario that label \( Y \) of \( X \) is uniquely determined by some measurable function. When \( Y \) is uniquely determined, \( P[Y \in S] = 1 \). (4)

This is the basic assumption in PLL made by previous PLL works (e.g., [2, 3, 26, 4, 14]) that the true label must be included in the candidate label set.

By definition, the Bayes error \( R^* = \inf_g \mathcal{R}(g) \) is defined over all measurable functions and the hypothesis with a Bayes error is a Bayes optimal decision function and denoted by \( g^{**}(X) \). Under Assumption [1] the Bayes error can reach the value 0. In this way, if we use flexible models such as deep neural networks, our hypothesis space will be large enough to have one classifier reach the Bayes error, which can be as low as zero in this case.

Clearly, the true label is determined by \( g^{**}(X) \) and defined in terms of the conditional probabilities as \( Y = \arg \max_{Y' \in [c]} g_i^{**}(X) = \arg \max_{Y' \in [c]} P[Y' \mid X] \). Then, according to Eq. (4), we can also have
\[
Y = \arg \max_{Y' \in [c]} P[Y' \mid X] = \arg \max_{Y' \in S} P[Y' \mid X].
\] (5)

We make another assumption on the classifier learned in ordinary multi-class learning.

**Assumption 2.** By minimizing the expected risk \( \mathcal{R}(g) \), we can obtain \( g^*_i(X) = P[Y = i \mid X] \).

Note that such an assumption can be satisfied when using the cross-entropy loss (Chapter 5 in [21]) or the mean squared error loss [27].

**Lemma 1.** ([28]) When \( \ell(\cdot, \cdot) \) is the cross-entropy loss or mean squared error loss, Assumption [3] satisfies.

With these above assumptions, we can have our main theorem, which specifies that the classifier learned in ordinary multi-class learning, and the classifier learned in PLL by minimizing \( \mathcal{R}_{PLL} \) are equivalent.

**Theorem 1.** (Classifier-consistency) Suppose Assumption [1] and [3] are satisfied, then when the hypothesis space is flexible enough, the optimal one \( f^*_{PLL} \) learned in PLL is equivalent to the optimal classifier \( f^* \) learned in ordinary multi-class learning, i.e., \( f^* = f^*_{PLL} \).

Theorem [1] is proved by substituting \( g^* \) into \( \mathcal{R}_{PLL}(g) \), and shows that \( \mathcal{R}_{PLL}(g^*) = \mathcal{R}(g^*) \). Given that the hypothesis space is flexible enough, the Bayes error can be achieved which means \( \mathcal{R}_{PLL} \) is minimized by \( g^* \), i.e., \( g^*_PLL = g^* \). This further ensures \( f^*_{PLL} = f^* \). A complete proof can be found in Appendix A.

### 2.3 Estimation Error Bound

In this section we establish the estimation error bound for the proposed estimator.

Assume that \( \hat{\mathcal{R}}_{PLL} \) is the empirical counterpart of \( \mathcal{R}_{PLL} \), and we denote by \( \hat{g}_{PLL} \) the optimal solution obtained by minimizing \( \hat{\mathcal{R}}_{PLL}(g) \). We will upper bound the difference between \( g^*_PLL \) and \( \hat{g}_{PLL} \) by upper bounding \( \mathcal{R}_{PLL}(g^*_PLL) - \mathcal{R}_{PLL}(\hat{g}_{PLL}) \). We have the following estimation error bound.
Theorem 2. (Estimation Error Bound) Assume $g \in \mathcal{G}$ and $\ell(g(X), Y)$ is Lipschitz continuous with respect to $g(X)$ with a Lipschitz constant $L_\ell$. Let $\mathcal{R}_n(\mathcal{G})$ be the Rademacher Complexity on $\mathcal{G}$ when given sample sized $n$, and the loss function is upper bounded by $M$, then with probability at least $1 - \delta$ we have

$$R_{PLL}(\hat{g}_{PLL}) - R_{PLL}(g^*_{PLL}) \leq 4cL_\ell\mathcal{R}(\mathcal{G}) + M\sqrt{\frac{\log(1/\delta)}{2n}}.$$ 

To prove the estimation error bound, we first need the following conclusion between the estimation error bound and the generalization error bound.

Lemma 2. The estimation error $R_{PLL}(\hat{g}) - R_{PLL}(g^*)$ can be bounded by

$$R_{PLL}(\hat{g}_{PLL}) - R_{PLL}(g^*_{PLL}) \leq 2\sup_{g \in \mathcal{G}}|R_{PLL}(g) - \hat{R}_{PLL}(g)|.$$ 

That is, the generalization error $R_{PLL}(g) - \hat{R}_{PLL}(g)$ can be used to bound the estimation error of the ERM algorithm. Then we use the following generalization error bound.

Theorem 3. (29) Let $g \in \mathcal{G}$ and $\mathcal{R}_n(\ell_{PLL} \circ \mathcal{G})$ be the Rademacher complexity of $\ell_{PLL} \circ \mathcal{G}$. If the loss function be upper bounded by $M$, then for any $\delta > 0$, with the probability $1 - \delta$, we have

$$\sup_{g \in \mathcal{G}}|R_{PLL}(g) - \hat{R}_{PLL}(g)| \leq 2\mathcal{R}_n(\ell_{PLL} \circ \mathcal{G}) + M\sqrt{\frac{\log(1/\delta)}{2n}}.$$ 

We further bound the relationship between $\mathcal{R}_n(\ell_{PLL} \circ \mathcal{G})$ and $\mathcal{R}_n(\mathcal{G})$.

Lemma 3. Let $g \in \mathcal{G}$. $\ell_{PLL}$ is defined in Eq. (2), $\ell(g(X), Y)$ is Lipschitz continuous with respect to $g(X)$ with a Lipschitz constant $L_\ell$. Then we have

$$\mathcal{R}_n(\ell_{PLL} \circ \mathcal{G}) \leq c\mathcal{R}_n(\ell \circ \mathcal{G}) \leq cL_\ell\mathcal{R}_n(\mathcal{G}).$$ 

Combining Lemma 2, Theorem 3 and Lemma 3 we can prove Theorem 2. Please find in Appendix A the detailed proof.

Note that if $|S| \leq s$ for all $X$, we can have the following corollary.

Corollary 4. Assume $|S| \leq s$ for all $X$ and all other conditions same as Theorem 2, we can have

$$R_{PLL}(\hat{g}_{PLL}) - R_{PLL}(g^*_{PLL}) \leq 4sL_\ell\mathcal{R}(\mathcal{G}) + M\sqrt{\frac{\log(1/\delta)}{2n}}.$$ 

Corollary 4 implies that the smaller the set of $S$, the better the learned classifier, given that the true label $Y$ lies in the set $S$. This agrees with our intuition on PLL. This section discusses an expected risk estimator for PLL. In the next section, we will discuss how to approximately optimizing the proposed risk.

3 Progressive Identification of True Labels

Obviously, it is not easy to directly do stochastic gradient descend on Eq. (3), due to the non-differentiabilty of the min operator. However, we still would like to train deep networks to obtain $\hat{g}_{PLL}$ by stochastic optimization with great benefit. This motivates our efforts to propose a novel progressive identification
**Algorithm 1 PRODEN Algorithm**

**Input:** $\mathcal{D}$: the partial-label training set $\{(\mathbf{x}_i, s_i)\}_{i=1}^n$, $T$: epoch number

**Output:** model parameter for $g(\mathbf{x}; \Theta)$

1. Let $\mathcal{A}$ be an SGD-like stochastic optimization algorithm
2. Initialize uniform confidences $\mathbf{w}^0$ according to Eq. (7);
3. for $t = 1$ to $T$ do
   4. Shuffle training set $\mathcal{D}$ into $B$ mini-batches;
   5. for $k = 1$ to $B$ do
      6. Compute $L$ according to Eq. (6);
      7. Set gradient $-\nabla_\Theta L$;
      8. Update $\mathbf{w}^t$ according to Eq. (8);
      9. Update $\Theta$ by $\mathcal{A}$;

Method for approximately minimizing Eq. (3). To this end, we first assume that the loss function can be decomposed onto each label, i.e.,

$$
\ell_{PLL}(\{g_1(X), \ldots, g_c(X)\}, S) = \sum_{i=1}^c \tilde{\ell}(g_i(X), I(i \in S)),
$$

where $\tilde{\ell} : \mathbb{R} \times \mathbb{R} \to \mathbb{R}_+$ is the label-wise loss. In this way, with appropriate $\mathbf{w}_i \in \mathbb{R}^c$ associated with the training example $(\mathbf{x}_i, s_i)$, we can have the relaxed empirical loss

$$
\hat{R}_{PLL} = \frac{1}{n} \sum_{i=1}^n \sum_{j=1}^c w_{ij} \tilde{\ell}(g_j(\mathbf{x}_i), I(j \in s_i)),
$$

(6)

where

$$
\begin{align*}
\mathbf{w}_i &\in \Delta^{c-1} \quad \forall i \in [n], \\
w_{ij} &= 0 \quad \forall i \in [n], \forall j \notin s_i.
\end{align*}
$$

$\Delta^{c-1}$ refers to a standard simplex in $\mathbb{R}^c$, i.e., $\sum_{j=1}^c w_{ij} = 1, 0 \leq w_{ij} \leq 1, \forall j \in [c]$. $w_{ij}$ can be interpreted as the corresponding confidence, i.e., the confidence of the $j$-th label being the true label of the $i$-th example. Ideally, the confidence of a label would trend to 1 progressively, which means that we finally have full confidence on which label is a true label. We will explain how to eventually achieve such an ideal situation.

The method begins with training an initial model based on the uniform confidences:

$$
\mathbf{w}_i^0 = \begin{cases} 
1/|s_i| & \text{if } j \in s_i \\
0 & \text{otherwise} 
\end{cases}.
$$

(7)

According to the memorization effects [30, 31] of deep networks, the deep networks will remember “frequent patterns” in the first few iterations. If the given partially labeled data has a reasonable ambiguity degree [31, 20], the deep networks tend to remember the true labels in the initial epochs, and guides us towards a classifier giving relatively low prediction losses for more possible true labels. In this way, the initial informative predictions are used to update the confidences for further training:

$$
\mathbf{w}_i^t = \begin{cases} 
g_j(\mathbf{x}_i) / \sum_{k \in s_i} g_k(\mathbf{x}_i) & \text{if } j \in s_i \\
0 & \text{otherwise} 
\end{cases}.
$$

(8)
Table 1: Summary of benchmark datasets and models.

| Dataset             | # Train | # Test | # Feature | # Class | Model g(x; Θ)                  |
|---------------------|---------|--------|-----------|---------|--------------------------------|
| MNIST               | 60,000  | 10,000 | 784       | 10      | Linear Model, MLP (depth 5)    |
| Fashion-MNIST       | 60,000  | 10,000 | 784       | 10      | Linear Model, MLP (depth 5)    |
| Kuzushiji-MNIST     | 60,000  | 10,000 | 784       | 10      | Linear Model, MLP (depth 5)    |
| CIFAR-10            | 50,000  | 10,000 | 3,072     | 10      | ConvNet (depth 10), ResNet (depth 32) |

In summary, we begin with training a neural network to optimize the risk $\hat{R}_{PLL}$ using the uniform confidences given in Eq. (7). Then we update the confidences by Eq. (8) after each iteration, and continue to train the neural network using the new updated confidences. The procedure of our algorithm is summarized in Algorithm 3 where we call our proposal PRODEN (PRogressive IDEntification).

At first glance, the proposed method shares some similarities with the iterative EM method proposed in [9]. In fact, the method in [9] has a tendency to overfit in the M-step, and has the limitation of using only one specific loss function. Conversely, PRODEN makes use of a more effective learning framework to get rid of overfitting, and besides, it is also flexible enough to use other loss functions. We provide more detailed arguments on the superiority and generalization of the proposed method in the following.

First, the iterative EM method in previous works trains the model until convergence in the M-step, but overemphasizing the convergence may result in redundant computation and overfitting issue, as the model will eventually fit the initial inexact prior knowledge and make a less informative estimate in the E-step on which the subsequent learning is based. To mitigate the overfitting issue, our method advances the procedure by merging the E-step and M-step. While the model is trained in a seamless manner without clear separation of E-step and M-step, the confidences can be updated at any iteration such that the convergence is not necessary in our training procedure.

Second, in the deep learning era, loss functions are one of the key elements and many useful loss functions are proposed, such as the mean squared error loss [27] and the mean absolute error loss [32]. In this way, models are welcomed to be loss-independent that allow usage of any loss function [33]. However, existing EM methods are restricted to some specific loss function, e.g., [9] limits the loss function to the KL divergence which is equivalent to the cross-entropy loss. Such restriction on loss functions is not suitable for practical use. Thus, our proposal is flexible enough to be compatible with a large group of loss functions. Moreover, we will show in the appendix that the proposal of [9] is a special case of ours.

4 Experiments

In this section, we verify the effectiveness of the proposed method PRODEN. First we analyze different strategies of PRODEN in training deep neural networks on benchmark datasets corrupted by manual candidate labels, and then compare it with non-deep state-of-the-art PLL methods on controlled UCI datasets and real-world datasets.

4.1 Experiments with deep networks

Datasets Experiments are conducted on four widely adopted benchmarks MNIST, Fashion-MNIST, Kuzushiji-MNIST and CIFAR-10 which are summarized in Table 1. We manually corrupted these datasets into partially labeled versions. Firstly, we probabilistically add any negative labels into the candidate label sets with a flipping probability $p$ where $p = P[\tilde{y} = 1|y = 0]$. Secondly, for the training examples only have
Figure 1: Inductive accuracy for various models and datasets with $p=0.1$ and $p=0.7$. Dark colors show the mean accuracy of 5 trials and light colors show standard deviation. Fashion is short for Fashion-MNIST, Kuzushiji is short of Kuzushiji-MNIST, CIFAR is short of CIFAR-10.

one candidate label (the true label), we randomly flip a negative label to positive label in order to ensure all the training examples are partially labeled.

**Baselines** In order to analyze the proposed method, we compare it with six baselines:
• *PN-oracle* means supervised learning from ordinary supervised data. It is merely for a proof of concept.

• *PN-transform* means decomposing multiple candidate labels into many single labels, so that we could use any ordinary multi-class classification methods.

• *PRODEN-iterative* means updating the label confidences in the iterative EM manner.

• *PRODEN-deterministic* means updating the label confidences in a hard manner, i.e., the confidence $w_{ij}$ of label $y_j$ with maximum modeling output over $x_i$ equals 1 and 0 otherwise.

• *PRODEN-naive* means never updating the uniform confidences.

• *GA* [33] means complementary label learning with gradient ascent.

PLL may be tackled by the method learning from complementary labels, which specifies a class that an example does not belong to. A set of candidate labels can be regarded as an inverse case of multi-complementary labels. We compare the proposed method with a state-of-the-art complementary-label learning method.

**Experimental setup** The robustness to partially labeled data is tested by conducting the proposed method and comparing methods under low-level partial circumstances $p = 0.1$ and under extremely partial circumstances $p = 0.7$. Table 1 describes the models on each dataset, where MLP refers to multi-layer perceptron, ConvNet follows the architecture in [31] and ResNet refers to residual networks [34]. The optimizer is stochastic gradient descent (SGD) [22] with momentum 0.9. We train these models 500 epochs with cross-entropy loss function for all the experiments. Because the M-step of iterative EM method is time-consuming, *PRODEN-iterative* updates label confidences every 100 epochs, which suffice to demonstrate the overfitting issue. We implement all methods by PyTorch, and conduct all the experiments on an NVIDIA Tesla V100 GPU. Please find the details in Appendix B.

**Results** We record inductive results which indicate the classification accuracy in the test set. The experimental results are shown in Figure 1. And transductive results which reflect the ability in identifying the true labels in the training set can be found in Appendix C.

Firstly, we observe the performance under $p = 0.1$ (cf. the left two columns). The proposed method *PRODEN* is always the best method and comparable to *PN-oracle* with all the models. *PRODEN-iterative* is comparable to *PRODEN* with linear model, but its performance deteriorates drastically with complex models because the overfitting issue is severe. These results are consistent with the discussions in Section 3. And the residual blocks can remedy this problem. The superiority always stands out for *PRODEN* compared with *GA*. Besides, *PRODEN* is much more stable than *GA* without learning rate decay strategy because the progressively identified true labels make the training examples much cleaner. The awful performance of *PRODEN-deterministic* proves the hard labels discard the helpful learning history. *PN-transform* and *PRODEN-naive* suffer from overfitting with deep models, where the performance even drops behind their linear counterparts. The reason for this phenomenon is more complex architectures have stronger capacities to fit all labels.

Secondly, we investigate these methods under $p = 0.7$ (cf. the right two columns). In the easier learning scenarios (MNIST, Fashion-MNIST), *PRODEN* is comparable to *PN-oracle* even when the flipping probability is considerably large. In the hardest learning scenario (CIFAR-10), the progressive identification method can also alleviate overfitting. The generalization degeneration of *PRODEN-iterative* is more serious, however, *GA* which is designed for extremely partial circumstances (complementary labels) performs better (surpasses *PRODEN-iterative*) instead. It is impressive that *PRODEN* still achieves the superior performance against all the comparing methods.

**4.2 Comparison with non-deep PLL approaches**

We further verify the performance of the proposed method by comparing it with six state-of-the-art PLL approaches which cannot be generalized to deep networks:
- SURE \[14\]: an iterative EM approach [suggested configuration: \(\lambda, \beta \in \{0.001, 0.01, 0.05, 0.1, 0.3, 0.5, 1\}\].
- CLPL \[3\]: a parametric approach transforming the PLL problem to the binary learning problem [suggested configuration: SVM with squared hinge loss].
- ECOC \[35\]: a disambiguation-free approach adapting the binary decomposition strategy to PLL [suggested configuration: \(L = \log_2(l)\)].
- PLSVM \[2\]: a maximum margin approach [suggested configuration: \(\lambda \in \{10^{-3}, \ldots, 10^3\}\)].
- PL\(k\)NN \[19\]: a non-parametric approach [suggested configuration: \(k \in \{5, 6, \ldots, 10\}\)].
- IPAL \[4\]: a non-parametric approach [suggested configuration: \(\alpha = 0.95, k = 10, T = 100\)].

The original comparing approaches are implemented in Matlab. For a fair comparison, all the parametric approaches exploit a linear model. We train PRODEN 500 epochs for all the datasets and average the classification accuracy over the last 10 epochs as the final results. For each dataset, ten-fold cross-validation is performed.

**Controlled UCI datasets** The characteristics of UCI datasets are reported in Appendix B. Following the widely-used controlling protocol, we can generate several artificial partial-label datasets from each UCI dataset by adjusting the controlling parameters \(p, r\) and \(\epsilon\). Here, \(p\) controls the proportion of partially labeled examples, \(r\) controls the number of false positive labels in each candidate label sets, and \(\epsilon\) controls the occurring probability between the true label and a specific false positive label. There are 4 configurations of controlling parameters corresponding to \(28(4 \times 7)\) results for each UCI dataset.

Figure 2 shows the classification accuracy of the comparing approaches as \(p\) ranges from 0.1 to 0.7 when \(r = 2\) (Configuration 2). Figure 3 illustrates the classification accuracy as \(\epsilon\) varies from 0.1 to 0.7 when
Table 2: Win/tie/loss (t-test at the significance level 5%) on the classification accuracy of **PRODEN** against each comparing approach.

| Configuration | PRODEN-Linear | SURE | CLPL | ECOC | PLSVM | PLkNN | IPAL |
|---------------|---------------|------|------|------|-------|-------|------|
| 1             | 22/4/2        | 18/7/3 | 15/2/11 | 27/1/0 | 17/3/8 | 20/1/7 |
| 2             | 22/6/0        | 20/7/1 | 17/2/9 | 26/2/0 | 20/1/7 | 18/2/8 |
| 3             | 21/6/1        | 19/8/1 | 20/0/8 | 28/0/0 | 19/1/8 | 21/0/7 |
| 4             | 24/4/0        | 22/5/1 | 16/4/8 | 28/0/0 | 19/2/7 | 12/2/14 |
| **Total**     | **89/20/3**   | **79/27/6** | **68/8/36** | **109/3/0** | **75/7/30** | **71/5/36** |

Table 3: Mean±standard deviations of the classification accuracy in percentage with linear model in real-world datasets. Paired t-test at 5% significance level is conducted, and ◦/○ represents whether the proposed method is significantly better/worse than the comparing approach. Besides, the best results are highlighted in boldface.

| Dataset       | PRODEN-Linear | SURE | CLPL | ECOC | PLSVM | PLkNN | IPAL |
|---------------|---------------|------|------|------|-------|-------|------|
| Lost          | 81.59±3.46    | 71.33±3.57 ● | 74.87±4.30 ● | 49.03±8.36 ● | 75.31±3.81 ● | 36.73±2.99 ● | 72.12±4.48 ● |
| MSRCv2        | 43.44±3.28    | 46.88±4.67 ● | 36.53±4.59 ● | 41.53±3.25 ● | 35.85±4.41 ● | 41.36±2.89 ● | **50.80±4.46** ◦ |
| BirdSong      | 68.90±0.72    | 58.92±1.28 ● | 63.56±1.40 ● | 71.58±1.81 ○ | 49.90±2.07 ● | 64.94±1.42 ● | **72.06±1.55** ◦ |
| Soccer Player | **55.32±0.56** | 49.41±0.86 ● | 36.82±1.04 ● | 53.70±2.02 ○ | 46.29±0.96 ● | 49.62±0.67 ● | 55.03±0.77 |
| Yahoo!News    | **67.48±0.65** | 45.49±1.15 ● | 46.21±0.90 ● | 66.22±1.01 ● | 56.85±0.91 ● | 41.07±1.02 ○ | 66.79±1.22 ● |

To further statistically compare the proposed method with other algorithms, Table 2 summarizes the win/tie/loss counts between **PRODEN-Linear** and the comparing approaches. We can find that the accuracy of **PRODEN-Linear** is highly competitive to all the parametric approaches. And compared with the best non-parametric method **IPAL**, **PRODEN-Linear** still significantly outperforms it in 63.4% cases, but it is easy to use other networks for further advancing the classification ability of **PRODEN-Linear**.

**Real-world datasets** The characteristics of real-world partial-label datasets are summarized in Appendix B. Table 3 reports the mean classification accuracy as well as the standard deviation of each method. Out of the 30 statistical tests, we can see that **PRODEN-Linear** achieves superior or at least comparable performance against all the comparing methods in 73.3% cases, and outperformed by them in only 10% cases. This confirms that the advantage of **PRODEN** afforded not only by the deep neural networks, more important, the progressive identification process.

### 5 Conclusion

In this paper, we proposed a progressive identification method for partial-label learning, named **PRODEN**. We first proposed a novel risk estimator, which will optimize the minimal loss incurred by candidate labels. We
then proved theoretically that the classifier learned with the proposed risk estimator is equal to the classifier learned in ordinary multi-class learning under mild conditions. We also derived the estimation error bound, which theoretically guarantees the learned classifier’s performance. Furthermore, we proposed a progressive identification method for approximately minimizing the proposed risk estimator. Our key idea is to conduct identifying and classifying seamlessly for mitigating the overfitting problem. At last, we experimentally demonstrated the proposed method could successfully train linear, fully connected, convolutional networks and residual networks, and confirmed the superiority of the progressive identification process.
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A Proofs

A.1 Proof of Lemma 1

According to [36], since the $\ell_{CE}$ is non-negative, minimizing the conditional risk $E_{Y|X}[\ell_{CE}((g_1(X), \ldots, g_c(X)), Y) | X = x]$, $\forall x \in X'$ is an alternative of minimizing $R(g)$. The conditional risk can be written as

$$C(g) = -\sum_{i=1}^{c} P[Y = i | X = x] \log(g_i(x)).$$

Given that $g_i(X) : X' \rightarrow \mathbb{R}$ is an estimate of $P[Y = i|X]$, $g_i(x) \in \Delta^{c-1}$. $\Delta^{c-1}$ is $(c-1)$-dimensional probability simplex, i.e., $\forall x \in \Delta^{c-1}, x_i \geq 0 \forall i \in [c], \sum_{i=1}^{c} x_i = 1$. Add the constraints $g_i(x) \in \Delta^{c-1}$ $\forall i \in [c]$ into the objective function according to the Lagrange Multiplier method [37], we have

$$\mathcal{L} = C(g) - \lambda \left( \sum_{i=1}^{c} g_i(x) - 1 \right).$$

To minimize $\mathcal{L}$, we take the partial derivative of $\mathcal{L}$ with respect to $g_i$ and let it be 0:

$$\frac{\partial \mathcal{L}}{\partial g_i} = -\frac{P[Y = i | X]}{g_i(x)} - \lambda = 0,$$

$$g_i^*(x) = -\frac{1}{\lambda} P[Y = i | x].$$

$\lambda = -1$ can ensure $\sum_{i=1}^{c} g_i^*(x) = \sum_{i=1}^{c} P[Y = i | x] = 1, \forall i \in [c], \forall x \in X$, which concludes the proof. \(\square\)

A.2 Proof of Theorem 1

We substitute the optimal decision function $g^*$ into the PLL risk estimator $R_{PLL}(g)$, we have

$$R_{PLL}(g^*) = E_{X,S}[\min_{Y \in S} \ell((g_1^*(X), \ldots, g_c^*(X)), Y')]$$

$$= \int \sum_{S} \min_{Y \in S} \ell((g_1^*(X), \ldots, g_c^*(X)), Y') P[S|X]P[X]dX$$

$$= \int \sum_{S} \min_{Y \in S} \ell((g_1^*(X), \ldots, g_c^*(X)), Y') \sum_{\tilde{Y}} P[S, \tilde{Y}|X]P[X]dX$$

$$= \int \sum_{\tilde{Y}} \sum_{S} \min_{Y \in S} \ell((g_1^*(X), \ldots, g_c^*(X)), Y') P[S, \tilde{Y}|X]P[X]dX$$

$$= \int \sum_{\tilde{Y}} \sum_{S} \ell((g_1^*(X), \ldots, g_c^*(X)), Y) P[S|X, \tilde{Y}]P[\tilde{Y}|X]P[X]dX$$

$$= \int \sum_{\tilde{Y}} \ell((g_1^*(X), \ldots, g_c^*(X)), Y) P[S|X, \tilde{Y}]P[\tilde{Y}|X]P[X]dX$$

$$= \int \sum_{\tilde{Y}} \ell((g_1^*(X), \ldots, g_c^*(X)), Y) P[X, \tilde{Y}]dX$$

$$= R(g^*).$$
Note that the sixth equality is due to that $g^*_i(X) = P[Y = i|X]$ (Assumption 2) and that $Y = \arg\max_{Y' \in S} P[Y'|X]$ (Assumption 1), we have

$$\arg\min_{Y' \in S} \ell(\{g^*_1(X), \ldots, g^*_n(X)\}, Y') = Y.$$ 

According to the assumption that the hypothesis space is flexible enough, the Bayes error can be achieved, i.e., $R(g^*) = 0$, which means $R_{PLL}(g^*)$ has been minimized, i.e., $g^*_{PLL} = g^*$. Therefore $f^*_{PLL} = f^*$.

### A.3 Proof of Lemma 2

$$R_{PLL}(\hat{g}_{PLL}) - R_{PLL}(g^*_{PLL}) = \left( R_{PLL}(\hat{g}_{PLL}) - \hat{R}_{PLL}(\hat{g}_{PLL}) \right) + \left( \hat{R}_{PLL}(\hat{g}_{PLL}) - \hat{R}_{PLL}(g^*_{PLL}) \right)$$

$$\leq \left( R_{PLL}(\hat{g}_{PLL}) - \hat{R}_{PLL}(\hat{g}_{PLL}) \right) + \left( \hat{R}_{PLL}(\hat{g}_{PLL}) - R_{PLL}(g_{PLL}) \right)$$

$$\leq 2\sup_{g \in G} | R_{PLL}(g) - \hat{R}_{PLL}(g) | .$$

The proof is completed.

### A.4 Proof of Lemma 3

By definition of $\ell_{PLL}$, $\ell_{PLL} \circ \mathcal{G}(x_i, S_i) = \min_{g \in S} \ell \circ \mathcal{G}(x_i, y) = \min_{g \in \mathcal{G}} \ell \circ \mathcal{G}(x_i, y)$. Given sample sized $n$, we first prove the result in the case $c = 2$. The min operator can be written as

$$\min\{g_1, g_2\} = \frac{1}{2}[g_1 + g_2 - |g_1 - g_2|].$$

In this way, we can write

$$\mathcal{R}_n(\ell_{PLL} \circ \mathcal{G}) = \mathbb{E}_\sigma \left[ \sup_{g \in \mathcal{G}} \frac{1}{n} \sum_{i=1}^{n} \sigma_i \ell(g(x_i, y)) \right]$$

$$= \mathbb{E}_\sigma \left[ \sup_{g \in \mathcal{G}} \frac{1}{n} \sum_{i=1}^{n} \sigma_i \min\{\ell(g(x_i, y_1)), \ell(g(x_i, y_2))\} \right]$$

$$= \mathbb{E}_\sigma \left[ \sup_{g \in \mathcal{G}} \frac{1}{2n} \sum_{i=1}^{n} \sigma_i |\ell(g(x_i, y_1)) + \ell(g(x_i, y_2)) - |\ell(g(x_i, y_1)) - \ell(g(x_i, y_2))| | \right]$$

$$\leq \mathbb{E}_\sigma \left[ \sup_{g \in \mathcal{G}} \frac{1}{2n} \sum_{i=1}^{n} \sigma_i |\ell(g(x_i, y_1)) + \ell(g(x_i, y_2)) | \right] + \mathbb{E}_\sigma \left[ \sup_{g \in \mathcal{G}} \frac{1}{2n} \sum_{i=1}^{n} \sigma_i |\ell(g(x_i, y_1)) - \ell(g(x_i, y_2)) | \right]$$

$$= \frac{1}{2}(\mathcal{R}_n(\ell \circ \mathcal{G}) + \mathcal{R}_n(\ell \circ \mathcal{G})) + \mathbb{E}_\sigma \left[ \sup_{g \in \mathcal{G}} \frac{1}{2n} \sum_{i=1}^{n} \sigma_i |\ell(g(x_i, y_1)) - \ell(g(x_i, y_2)) | \right].$$
Since $x \mapsto |x|$ is a 1-Lipschitz function, by Talagrand’s lemma [38], the last term can be bounded:

$$\mathbb{E}_{\sigma} \left[ \sup_{y \in \mathcal{Y}} \frac{1}{2n} \sum_{i=1}^{n} \sigma_i(\ell(g(x_i, y_1)) - \ell(g(x_i, y_2))) \right]$$

(10)

$$\leq \mathbb{E}_{\sigma} \left[ \sup_{y \in \mathcal{Y}} \frac{1}{2n} \sum_{i=1}^{n} \sigma_i(\ell(g(x_i, y_1)) - \ell(g(x_i, y_2))) \right]$$

$$= \frac{1}{2} (\mathcal{R}_n(\ell \circ \mathcal{G}) + \mathcal{R}_n(\ell \circ \mathcal{G})).$$

Combining Eq. (9) and Eq. (10), we have

$$\mathcal{R}_n(\ell_{PLL} \circ \mathcal{G}) \leq \mathcal{R}_n(\ell \circ \mathcal{G}) + \mathcal{R}_n(\ell \circ \mathcal{G}).$$

The general case can be derived from the case $c = 2$ using $\min\{g_1, \ldots, g_c\} = \min\{g_1, \min\{g_2, \ldots, g_c\}\}$ and an immediate recurrence.

Again we apply the Talagrand’s contraction lemma,

$$c\mathcal{R}_n(\ell \circ \mathcal{G}) \leq cL\mathcal{R}_n(\mathcal{G}).$$

The proof is completed. \qed

### A.5 Supplementary theorem on Section 3

**Theorem 5.** The learning objective in [9] is a special case of Eq. (6).

**Proof.** First, recall the learning objective in [9] is formulated as:

$$\hat{R}_{PLL} = \frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{c} P[j \mid \mathbf{x}_i] \cdot \mathbb{KL}[\hat{P}[j \mid \mathbf{x}_i] \mid \mid P[j \mid \mathbf{x}_i; \theta]].$$

(11)

Here, $\hat{P}[j \mid \mathbf{x}_i]$ represents the unknown prior probability and $P[j \mid \mathbf{x}_i; \theta]$ is the model-based conditional distribution.

Then in Eq. (11), the loss function can be specified as the cross-entropy loss: $\hat{\ell}_{CE}(g_j(x_i), 1(j \in s_i)) = -\mathbb{KL}(g_j(x_i) \mid \mid P[j \mid \mathbf{x}_i; \theta])$. We can easily find the cross-entropy loss is linear in the second term, i.e.,

$$w_{ij} \hat{\ell}_{CE}(g_j(x_i), 1(j \in s_i)) = \hat{\ell}_{CE}(g_j(x_i), w_{ij} 1(j \in s_i)).$$

Thus, the confidences $\mathbf{w}$ can be moved into the loss function:

$$\hat{R}_{PLL} = \frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{c} \hat{\ell}_{CE}(g_j(x_i), z_{ij}),$$

(12)

where $z_{ij} = w_{ij} 1(j \in s_i)$. $g_j(x_i)$ is the estimate of $P[j \mid \mathbf{x}_i]$.

For optimizing $g$ in Eq. (12), we can add a term which is independent from $g$ to the learning objective:

$$\hat{R}_{PLL} = \frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{c} \left[ \hat{\ell}_{CE}(g_j(x_i), z_{ij}) + \hat{\ell}_{CE}(z_{ij}, z_{ij}) \right]$$

$$= \frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{c} \left[ -z_{ij} \log g_j(x_i) + z_{ij} \log z_{ij} \right]$$

$$= \frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{c} \log g_j(x_i)$$

$$= \frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{c} \mathbb{KL}[z_{ij} \mid \mid g_j(x_i)],$$

which is essentially equivalent to Eq. (11). Therefore, our learning objective is a strict extension of [9]. Instead, the learning objective in [9] is a special case of the proposed method. \qed
B Dataset information and experimental setting

B.1 Benchmark dataset

MNIST  This is a grayscale image dataset of handwritten digits from 0 to 9 where the size of the images is $28 \times 28$.

The linear model was a linear-in-input model: $d$-10 and used softmax activation in the output layer. An $\ell_2$-regularization was added where the regularization parameter was fixed to 1e-5. The model was trained by SGD with a fixed learning rate 1e-3 and a batch size 1000.

The MLP used for training MNIST was FC with ReLU as the activation function: $d$-300-300-300-300-10. The softmax activation function was also used in the output layer. Batch normalization [39] was applied before hidden layers. The regularization parameter was 1e-3 and the learning rate was 1e-2.

Fashion-MNIST  This is also a grayscale image dataset similarly to MNIST, but here each data is associated with a label from 10 fashion item classes.

The models and optimizer were the same as MNIST, except that the regularization parameter was 1e-5 for MLP.

Kuzushiji-MNIST  This is another variant of MNIST dataset, and each example is associated with a label from 10 cursive Japanese (Kuzushiji) characters.

The models and optimizer were the same as MNIST, except that the regularization parameter was 1e-6 for linear model and 1e-4 for MLP.

CIFAR-10  This dataset consists of 60,000 $32 \times 32$ color images in 10 classes.

The detailed architecture of ConvNet [31] was as follows.

0th (input) layer: $(32*32*3)$-
1st to 3rd layers: Max Pooling-\(C(3*3, 128)\)*3-
4th to 6th layers: Max Pooling-\(C(3*3, 256)\)*3-
7th to 9th layers: Max Pooling-\(C(3*3, 512)-C(3*3, 256)-C(3*3, 128)\)-
10th layer: Average Pooling-10

where $C(3*3, 128)$ means 128 channels of $3*3$ convolutions followed by Leaky-ReLU (LReLU) active function [40], $\cdot \cdot \cdot$ means 3 such layers, etc. Again, the softmax activation function was used in the output layer. Besides, dropout (the dropout rate was set to 50%) and batch normalization were also used. The model was trained by SGD with the default momentum parameters. The regularization parameter was 1e-4 and the batch size was 500. In addition, the initial learning rate was 1e-2 and decreased by \(\text{decay}^{\lfloor \text{epoch}/50 \rfloor}\),

where decay was 0.9.

The detailed architecture of ResNet-32 [34] was as follows.

0th (input) layer: $(32*32*3)$-
1st to 11th layers: $C(3*3, 16)-C(3*3, 16), C(3*3, 16)]*5$-
12th to 21st layers: $[C(3*3, 32), C(3*3, 32)]*5$-
22nd to 31st layers: $[C(3*3, 64), C(3*3, 64)]*5$-
32nd layer: Average Pooling-10

where $[ \cdot , \cdot ]$ means a building block [34]. The optimization setup was the same as for MNIST, except that the regularization parameter was 1e-3 and the fixed learning rate was 5e-2.
Table 4: Summary of controlled UCI datasets.

| Dataset | # Examples | # Feature | # Class | Configurations                           |
|---------|------------|-----------|---------|------------------------------------------|
| ecoli   | 336        | 7         | 8       | Configuration 1 | $r = 1, \ p \in \{0.1, 0.2, \ldots, 0.7\}$ |
| deter   | 358        | 23        | 6       | Configuration 2 | $r = 2, \ p \in \{0.1, 0.2, \ldots, 0.7\}$ |
| glass   | 214        | 10        | 29      | Configuration 3 | $r = 3, \ p \in \{0.1, 0.2, \ldots, 0.7\}$ |
| usps    | 9,298      | 256       | 10      | Configuration 4 | $p = 1, \ r = 1, \ \varepsilon \in \{0.1, 0.2, \ldots, 0.7\}$ |

Table 5: Summary of real-world partial label datasets.

| Dataset    | # Examples | # Feature | # Class | # Avg. CLs | Task Domain                  |
|------------|------------|-----------|---------|-----------|-----------------------------|
| Lost       | 1122       | 108       | 16      | 2.23      | automatic face naming [41]  |
| MSRCv2     | 1758       | 48        | 23      | 3.16      | object classification [8]   |
| BirdSong   | 4998       | 38        | 13      | 2.18      | bird song classification [42]|
| Soccer Player | 17472   | 279       | 171     | 2.09      | automatic face naming [43]  |
| Yahoo!News | 22991      | 163       | 219     | 1.91      | automatic face naming [44]  |

B.2 UCI datasets and real-world datasets

The characteristics of UCI datasets and real-world datasets are reported in Table 4 and Table 5 respectively. First we normalized these dataset by the Z-scores by convention. In all these datasets, PRODEN used linear model: d-c trained by SGD with momentum 0.9. The regularization parameter was 1e-3 and the learning rate was 1e-3 in usps, the regularization parameter was 1e-2 and the learning rate was 1e-1 in ecoli, deter, glass, Lost, MSRCv2, and BirdSong, the regularization parameter was 1e-3 and the learning rate was 1e-2 in Soccer Player, the regularization parameter was 1e-4 and the learning rate was 1e-2 in Yahoo!News. In regular-scale dataset (#examples $\leq$ 5000), we used a full batch update, whereas in large-scale dataset (#examples $\geq$ 5000), the batch size was fixed to 1000.

C Additional experimental results

C.1 Transductive results on benchmark datasets

The transductive results which reflect the ability in identifying the true labels in the training sets are shown in Figure 4.

We can find these results are consistent with findings in Figure 1 that the proposed method can successfully train linear, fully connected, convolutional networks and residual networks with stochastic optimization. The progressive identification process can successfully mitigate the overfitting issue of the iterative EM method, and outperforms other comparing methods even when the flipping probability is considerably large.
Figure 4: Transductive accuracy for various models and datasets with $p=0.1$ and $p=0.7$. Dark colors show the mean accuracy of 5 trials and light colors show standard deviation. Fashion is short for Fashion-MNIST, Kuzushiji is short of Kuzushiji-MNIST, CIFAR is short of CIFAR-10.

C.2 Classification accuracy on controlled UCI datasets

Figure 5 and Figure 6 shows the classification accuracy of the comparing approaches as $p$ ranges from $0.1$ to $0.7$ when $r=1$ (C. 1) and $r=3$ (C. 3) respectively.
Similar to Figure 2 and Figure 3, the proposed method PRODEN-Linear is highly comparable to all the parametric methods, and compared with the best non-parametric approach IPAL, PRODEN-Linear still significantly outperforms it in most cases.