Attractors with Non-Invariant Interior and Pinheiro’s Theorem A

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Abstract

This is a provisional version of an article, intended to be devoted to properties of attractor’s intetrior for smooth maps (not diffeomorphisms). We were originally motivated for this research by Pinhero’s Theorem A from his preprint [P], and in Section 3 we give a simple and straightforward proof of this result.

1 Pinheiro’s Theorem A

In this section we quote the statement of Theorem A from Pinheiro’s preprint [P]. We will give a simple and straightforward proof of this result in Section 3 below.

Let $X$ be a compact metric space. Given a compact set $A$ such that $f(A) = A$, define the basin of attraction of $A$ as

$$\beta_f(A) = \{ x \in X; \omega_f(x) \subset A \}.$$

Following Milnor’s definition of topological attractor (indeed, minimal topological attractor [M]), a compact forward invariant set $A$ is called a topological attractor if $\beta_f(A)$ is not a meager set and $\beta_f(A) \setminus \beta_f(A')$ is not a meager set for every compact forward invariant set $A' \subsetneq A$. According to Guckenheimer [G], a set $\Lambda \subset X$ has sensitive dependence on initial condition if exists $r > 0$ such that $\sup_n \text{diameter}(f^n(\Lambda \cap B_\varepsilon(x))) \geq r$ for every $x \in \Lambda$ and $\varepsilon > 0$.

If $\bigcup_{n \geq 0} f^n(V) = X$ for every open set $V \subset X$, then $f$ is called strongly transitive on $X$. **Theorem A** 1. Let $f : X \to X$ be a continuous open map defined on a compact metric space $X$. If there exists $\delta > 0$ such that $\bigcup_{n \geq 0} f^n(U)$ contains some open ball of radius $\delta$, for every nonempty open set $U \subset X$, then there exists a finite collection of topological attractors $A_1, \ldots, A_\ell$ satisfying the following properties.

1. $\beta_f(A_1) \cup \cdots \cup \beta_f(A_\ell)$ contains an open and dense subset of $X$.

2. Each $A_j$ contains an open ball of radius $\delta$ and $A_j = \text{interior}(A_j)$.

3. Each $A_j$ is transitive and $\omega_f(x) = A_j$ generically on $\beta_f(A_j)$.

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1In [P] sets with this property are called forward invariant sets, but we prefer another convention: in the text below «$A$ is forward invariant» means $f(A) \subset A$. 
4. $\Omega(f) \setminus \bigcup_{j=0}^f A_j$ is a compact set with empty interior.

Furthermore, if $\bigcup_{n \geq 0} f^n(U)$ contains some open ball of radius $\delta$, for every nonempty open set $U \subset X$, then the following statements are true.

5. For each $A_j$ there is a set $A_j \subset A_j$ containing an open and dense subset of $A_j$ such that $f(A_j) = A_j$ and $f|_{A_j}$ is strongly transitive.

6. Either $\omega_f(x) = A_j$ for every $x \in A_j$ or $A_j$ has sensitive dependence on initial conditions.

2 A simple continuous map whose attractor has non-invariant interior

The original statement by Pinheiro [P, Theorem A] did not require the map $f$ to be open. But the proof implicitly used the invariance of the attractor interior under the dynamics, which is not true in general.

Here is a counterexample to the original Pinheiro theorem A. Let $X$ be $[-1, 1] \cup 2$, and let $f([-1,1]) = \{2\}$ and $f(2) = 0$. Clearly, $f$ is continuous. Every orbit of $f$ contains the point $2$, and therefore contains a ball of radius $0.5$ centered at $2$ (this ball coincides with $\{2\}$). The attractor $A_1$ of $f$ is $\{0,2\}$, with $2$ being an interior point of $A_1$ and $0$ being a boundary point. Therefore $A_1 \neq \text{interior}(A_1)$, and the statement (2) of Theorem A is false. Moreover, the interior point $2$ of $A_1$ is taken into a boundary point $0$.

Counterexamples for path-connected manifolds are also possible.

In this paper we slightly alter the statement of the theorem by requiring $f$ to be open. We will only use this assumption in the proofs of claims (2) and (5).

3 Proof of Theorem A

Lemmas on the $\omega$-limit sets

Consider a map that takes a point of the phase space into the closure of its positive semi-orbit under $f$. Denote this map by $\Psi$ and observe that it is lower semicontinuous (w.r.t. the Hausdorff distance between compact subsets of $X$), since finite parts of the forward semi-orbit depend continuously on the initial point. By the semicontinuity lemma [S] the set $R$ of continuity points of $\Psi$ is residual in the phase space.

Lemma 2. In the assumptions of Theorem A, for any point $x \in R$ the $\omega$-limit set of $x$ contains an open ball of radius $\delta$.

Proof. Let $U_n = B_{1/n}(x)$ be an open ball of radius $1/n$ centered at $x \in R$. By assumption, the closure of the forward orbit of $U_n$ contains an open ball of radius $\delta$. Denote the center of this ball by $y_n$. Let $y_0$ be an accumulation point for the sequence $\{y_n\}$. Then a $\delta$-ball centered at $y_0$ is contained in the set $\Psi(x) = \text{Orb}^+(x)$. Indeed, if this is not the case and there is a point $z$ of this ball outside $\Psi(x)$, then a ball of small radius $\varepsilon$ centered at $z$ is disjoint from $\Psi(x)$. But then for every point $\hat{x}$ in a sufficiently small neighborhood of $x$ the set $\Psi(\hat{x})$ is $\varepsilon/2$-close to $\Psi(x)$ (since $x$ is a continuity point for the map $\Psi$) and hence does not contain $z$. But this is in contradiction with the point $z$ being in $\bigcup_{j \geq 0} f^j(U_{n_k})$ for a
sequence \( n_k \to +\infty \). This yields that the set \( \Psi(x) \) contains the \( \delta \)-ball centered at \( y_0 \). Since the continuity set \( R \) is invariant, an analogous statement is true for \( f^n(x) \), \( n \in \mathbb{N} \). Recall that \( \omega(x) = \cap_{n \geq 0} \text{Orb}^+(f^n(x)) \). Each set \( O_n = \overline{\text{Orb}^+(f^n(x))} \) contains a \( \delta \)-ball. As above, we take a subsequence of centers of the balls that converges to some point \( z_0 \) and observe that for a \((\delta - \varepsilon)\)-ball centered at \( z_0 \) we can find arbitrarily large \( n \) such that this ball is contained in \( O_n \). Since \( O_n \) form a nested sequence, \( B_{\delta-\varepsilon}(z_0) \) is contained in the intersection \( \cap_n O_n = \omega(x) \). Hence, there is an open \( \delta \)-ball in \( \omega(x) \).

\[ \text{Lemma 3.} \text{ If the interiors of } \omega(a) \text{ and } \omega(b) \text{ have nonempty intersection for } a, b \in R, \text{ then } \omega(a) = \omega(b). \]

\[ \text{Proof.} \text{ If the two interiors intersect, then the set } \omega(a) \cap \omega(b) \text{ contains an open ball } B. \text{ Since the orbit of } b \text{ must approach every point of this ball, there is } N \text{ such that } f^N(b) \in B \subset \omega(a), \text{ and hence for any } n > N \text{ we have } f^n(b) \in \omega(a), \text{ by the invariance of } \omega(a), \text{ which yields } \omega(b) \subset \omega(a). \text{ Analogously, } \omega(a) \subset \omega(b). \]

**Proof of claims (1)–(4)**

Let us say that the two points in \( R \) are equivalent if the interiors of their \( \omega \)-limit sets intersect (this relation is transitive by Lemma 3). The set \( R \) then splits into a finite number of equivalence classes: indeed, for each class the \( \omega \)-limit set of its point contains an open \( \delta \)-ball and those balls for different classes are disjoint, but one can fit only a finite number of disjoint \( \delta \)-balls into a compact metric space.

The attractors \( A_j \) are exactly the \( \omega \)-limit sets that correspond to these equivalence classes. The basin of each \( A_j \) is open. Indeed, let \( A_j = \omega(x) \). The set \( A_j \) contains a ball, so any point \( y \) close to \( x \) will get inside this ball under the iterates of \( f \), by continuity, and so it will be attracted to \( A_j \): \( \omega(y) \subset A_j \). Hence, the basin of \( A_j \) is open. Also, since for any \( x \) in the residual set \( R \) the limit set \( \omega(x) \) coincides with some \( A_j \), a proper subset of any \( A_j \) contains \( \omega(y) \) only for a meager set of \( y \)-s, so each \( A_j \) is a topological Milnor attractor.

Since each \( A_j \) contains an open ball, it contains a point \( y \in R \). But then \( A_j = \omega(y) \), and so the forward orbit of \( y \) is dense in \( A_j \). As the point \( y \) is recurrent, this makes the attractor transitive. This, together with the genericity of \( R \) in \( \beta f(A_j) \) yields claim 3. Now, we are assuming that \( f \) is an open map. This implies that \( f(\text{int}(A_j)) \subset \text{int}(A_j) \), and so any \( \omega \)-limit point for a point \( z \in \text{int}(A_j) \) is in \( \text{int}(A_j) \). Since we can take \( z \in R \) with \( \omega(z) = A_j \), this yields that \( A_j \) coincides with \( \overline{\text{int}(A_j)} \). This finishes the proof of claim (2).

Observe that if a non-wandering point is in \( \beta f(A_j) \), it belongs to \( A_j \). Indeed, the orbit of this point visits a \( \delta \)-ball inside \( A_j \), and hence a small neighborhood of this point is taken inside \( A_j \) by some iterate of \( f \), say \( f^N \). But since the point is non-wandering, there is \( K > N \) such that the image of this neighborhood under \( f^K \) (the image is contained in \( A_j \)) intersects the neighborhood itself. This implies that our non-wandering point is accumulated by the points of \( A_j \), and so it belongs to \( A_j = \overline{A_j} \). Hence, if a non-wandering point does not belong to the union of the attractors \( A_j \), it does not belong to the union of \( \beta f(A_j) \), but the complement of the latter union is contained in a closed set with empty interior, by claim (1), and this implies claim (4).
Proof of claims (5)--(6)

Note that it is not very important that the sets $A_j$ in claims 5 and 6 of the theorem coincide: we can construct two different forward-invariant subsets $A_j^5, A_j^6$ that contain open and dense subsets of $A_j$ and have the properties from claims 5 and 6 respectively\footnote{One can check that sensitive dependence on initial conditions on a set $S$ is inherited by any residual subset of $S$.} and then take their intersection as $A_j$.

In the rest of the proof, we will call the assumption that for any nonempty open $U$ the union of its images contains a $\delta$-ball \textit{the main assumption}.

Fix some $j$ and consider a finite set $\{x_i\}$ such that the union of $\delta/3$-balls centered at $x_i$ covers $A_j$. Let us denote by $B_i$ the intersections of these balls with $A_j$. We will refer to $\{x_i\}$ as the $\delta/3$-covering for $A_j$. Let $C_i = \bigcup_{n \in \mathbb{N}_0} f^n(B_i)$. Each $C_j$ is open with respect to the subset topology on $A_j$: the balls $B_i$ are open in this topology and the restriction $f|_{A_j}$ is open since $f$ is. The attractor $A_j$ is transitive, its dense forward orbit visits each $B_i$, so each $C_i = \bigcup_{n \geq 0} f^n(B_i)$ must be dense in $A_j$.

Let $E = \cap_i(C_i)$ and $A_j^5 = \cap_{n \in \mathbb{N}} f^n(E)$. It is clear that $f(A_j^5) = A_j^5$ The set $E$ is open in $A_j$ as a finite intersection of open sets $C_i$. By claim (2) this set has nonempty intersection with $\text{int}(A_j)$, so it contains a small open ball (i.e., the ball is open in $\mathbb{X}$). Since $E$ is forward invariant ($f(E) \subset E$), it contains the union of the images of this ball, and this union contains a $\delta$-ball of the phase space, by the main assumption. The same argument applies to $f(E)$ (recall that $f|_{A_j}$ is open) and every $f^n(E)$. Since the sets $f^n(E)$ form a nested sequence, their intersection also contains a $\delta$-ball (one can argue as in the proof of Lemma 2). So, $A_j^5$ contains a $\delta$-ball and, by forward-invariance, all of its images. But there is a point in this ball whose forward orbit is dense in $A_j$, so the union of open $f^n$-images of the ball is open and dense in $A_j$, so $A_j^5$ contains an open (w.r.t. the topology of $\mathbb{X}$) subset which is dense in $A_j$.

Take any $U$ open in $A_j$. Denote by $V$ the union of the $f^n$-images of $U$. By claim (2) and the main assumption, there is a ball of radius $\delta$ in $V$. This ball contains one of the sets $B_i$ (because those have diameter $2\delta/3$ and cover $A_j$), and hence $V$ contains the set $C_j$, and so $V$ contains the whole $A_j^5$, which means that $f|_{A_j^5}$ is strongly transitive. Also note that, as we showed, $A_j^5$ coincides with some $C_i$.

Now let us prove claim (6) of the theorem. We fix some attractor $A_j$ and consider a point in its basin. We will call such a point $r$-punctured if its $\omega$-limit set has empty intersection with some open $r$-ball in $A_j$.

Either there exists an $r$ such that $A_j$ admits a $\delta$-covering (with elements contained in $A_j$) that consists of $3r$-punctured points, or not.

In the first case let us take an arbitrary ball $G$ in $A_j$ (i.e., we regard $A_j$ as a metric space and take a ball in it). The union of its images contains a $\delta$-ball (as $G$ contains an open subset of interior $A_j$ by claim (2)). Then there is a point of our $\delta$-covering in this $\delta$-ball, and so the union has a $3r$-punctured point. Since any preimage of a $3r$-punctured point is also $3r$-punctured, there is a $3r$-punctured point $b$ in $G$. Denote by $B$ a $3r$-ball in $A_j$ such that $B \cap \omega(b) = \emptyset$. There exists an integer $M$ such that for $m > M$ we have that $f^m(b)$ is at a distance at least $2r$ to a center $y$ of $B$ (otherwise $B$ would intersect $\omega(b)$; note that the ball can have more than one center). Since the attractor is transitive, there is a point $x \in G$ whose forward orbit is dense in $A_j$. Let $N > M$ be a moment of time such that $f^N(x)$ is $r$-close to $y$. Then $\text{dist}(f^N(x), f^N(b)) > r$, and we have sensitive dependence on initial
condition, with this $r$.

Now suppose that for any $r$ there is no $\delta$-covering of $3r$-punctured points for $A_j$. For $r = 1/n$ there must be a $\delta$-ball in $A_j$ that has no $3/n$-punctured points: otherwise we would construct a $\delta$-covering of punctured points. Denote a center of this ball by $w_n$. Let $w_0$ be an accumulation point for $\{w_n\}$. In a $\delta/2$-ball cantered at $w_0$ there are no $r$-punctured points, for arbitrary $r$; denote this ball by $P$. For every point in $P$ its forward orbit is dense in $A_j$ — otherwise there would be $r$-punctured points. Let $A^6_j = \bigcup_{k \in \mathbb{Z}} f^k(P) \cap A_j$. This set satisfies $f(A^6_j) = A^6_j$ and consists of points whose forward orbits are dense in $A_j$.

The subset $\hat{A}^6_j = \bigcup_{n \in \mathbb{N}_0} f^{-n}(P) \cap A_j \subset A^6_j$ is open and dense in $A_j$. Indeed, it is open as a union of preimages of open subset $P$ of $A_j$ under continuous $f|_{A_j}$. Suppose that this set is not dense in $A_j$, that is, there is a ball $V$ disjoint from it. But any point of $P$ is taken into $V$ by some iterate of $f$, and then gets back to $P$ (recall that for $z \in P$ the forward orbit is dense), so some points of $V$ are preimages of points in $P$, i.e., they belong to $\hat{A}^6_j$. This contradiction shows that $\hat{A}^6_j$ is dense in $A_j$.

Thus, $A^6_j$ has the properties from claim (6).

4 A smooth map with non-invariant attractor interior on a solid torus

In this section we present an example of a smooth map from a path-connected manifold to itself that has a Milnor topological attractor whose interior is not forward invariant.

**Theorem 4.** There exists a smooth map $F$ on a solid torus with a structure of a skew product over the octupling map on a circle $S^1$: $\varphi \mapsto 8\varphi$ with a Milnor topological attractor whose interior is not forward invariant under $F$.

**Proof.** The fiber $M$ of our skew product is a two-dimensional disk with radius 3. The skew product itself has the form

$$S^1 \times M \ni (\varphi, x) \mapsto (8\varphi, f_\varphi(x)),$$

and we refer to $f_\varphi$ as fiber maps. The octupling map on $S^1$ is semi-conjugate via the so-called symbolic coding with the left shift $\sigma$ on the space $\Sigma^+_8$ of one-sided infinite sequences over the alphabet $\{0, \ldots, 7\}$. This allows us to encode the fiber maps using these sequences and write $f_\omega$ instead of $f_\varphi$ if $\omega$ is taken to $\varphi$ by the semiconjugacy map.

In our example, the fiber maps $f_\omega$ will depend only on the element at the leftmost position in the sequence $\omega$, provided that this element is even ($\{0, 2, 4, 6\}$). We start indexing with 0 and write $f_\omega = f_{\omega_0}$; $\omega_0$ is the element at position 0 in the sequence $\omega$. The fiber maps for the sequences that start with an odd element in general depend on the whole sequence and are used primarily to make the transition between the “even” fiber maps smooth.

Now we describe the properties of the four fiber maps $f_0, f_2, f_4, f_6$. For convenience, we choose a rectangular coordinate system on the disk $M$, such that the point $(0,0)$ is at the center of $M$. Denote a semi-disk defined by condition $x < 0$ by $D$ and a subset defined by $x < 1$ by $Z$. Now, the three maps $f_2, f_4, f_6$ must have the following properties:

1) they are smooth on $M$ and contracting on $Z$ (i.e., there is $\lambda < 1$ such that for any $x, y$

$$\text{dist}(f_i(x), f_i(y)) < \lambda \cdot \text{dist}(x, y), \text{ for } i = 2, 4, 6);$$
2) \( D \subset f_2(D) \cup f_4(D) \cup f^6_2(D) \subset Z \);

3) \( f_i(M \setminus Z) \subset D \) for \( i = 2, 4, 6 \);

4) \( f_i(Z) \subset Z \) for \( i = 2, 4, 6 \).

Let the last map \( f_0 \) just take the whole fiber into a point \( q \) with coordinates \((2, 0)\).

Now we are prepared to use the Hutchinson lemma for maps \( f_2, f_4, f^6_2 \).

**Lemma 5.** (Hutchinson [H], in the form from [VI]) Consider a metric space \((M, \rho)\) and maps \( f_n : M \to M \). Suppose that there exist compact sets \( D \subset Z \subset M \) such that \( f_n(Z) \subset Z \) for all \( f_n \), all \( f_n \) are contracting on \( Z \) and \( D \subset \cup f_n(D) \). Then for any open \( U, U \cap K \neq \emptyset \), there exists a finite word \( w = \omega_1 \ldots \omega_m \) such that \( f^{[m]}_w(Z) \subset U \), where \( f^{[m]}_w = f_{\omega_1} \circ \cdots \circ f_{\omega_m} \).

The smoothing maps must belong to one of the following two types:

(i) either they take the whole \( M \) into a point at the axis \( \{ y = 0 \} \),

(ii) or they are smooth and take \( M \) into a subset of \( Z \).

For convenience we describe a tuple of maps with these properties.

**Example.** Let \( f_s \) take \( M \) into the point \( s = (-1, 0) \) and let \( a_{[l,r]} \) be a smooth monotonically increasing map from \([l, r]\) to \([0, 1]\) whose every derivative vanishes at the endpoints: \( a^{(m)}(l) = a^{(m)}(r) = 0 \).

Now put

\[
\begin{align*}
f_\varphi &= \left(1 - a_{\left[\frac{3\pi}{4}, \frac{4\pi}{4}\right]}(\varphi)\right) \cdot f_2 + a_{\left[\frac{3\pi}{4}, \frac{4\pi}{4}\right]}(\varphi) \cdot f_4 \quad \text{for } \varphi \in \left[\frac{3\pi}{4}, \frac{4\pi}{4}\right]; \\
f_\varphi &= \left(1 - a_{\left[\frac{5\pi}{4}, \frac{6\pi}{4}\right]}(\varphi)\right) \cdot f_4 + a_{\left[\frac{5\pi}{4}, \frac{6\pi}{4}\right]}(\varphi) \cdot f^6_2 \quad \text{for } \varphi \in \left[\frac{5\pi}{4}, \frac{6\pi}{4}\right].
\end{align*}
\]

Every \( f_\varphi \) takes \( M \) to itself, because \( M \) is convex (and therefore if, for example, \( f_2(p) \in M \) and \( f_4(p) \in M \), then \((1-a)f_2(p) + af_4(p) \in M \)). Moreover, \( f_\varphi(M) \subset Z \), because for any \( p \) \( x(f_2(p)) < 1, x(f_4(p)) < 1 \) and then \( x((1-a)f_2(p) + af_4(p)) < 1 \).

For \( \varphi \in \left[\frac{3\pi}{4}, \frac{5\pi}{4}\right) \) let \( f_\varphi \) be \((1-a)[\left[\frac{3\pi}{4}, \frac{4\pi}{4}\right]](\varphi) f_0 + a[\left[\frac{3\pi}{4}, \frac{4\pi}{4}\right]](\varphi) f_s \), and for \( \varphi \in \left[\frac{15\pi}{8}, 2\pi\right] \) let \( f_\varphi \) be \((1-a)[\left[\frac{15\pi}{8}, 2\pi\right]](\varphi) f_s + a[\left[\frac{15\pi}{8}, 2\pi\right]](\varphi) f_0 \). All this maps take \( M \) into a point \((1-a)q + as \) or \((1-a)s + aq \) respectively, and this points all lie on the axis \( y = 0 \).

At last, for \( \varphi \in \left[\frac{3\pi}{4}, \frac{2\pi}{4}\right] \) we put \( f_\varphi = (1-a[\left[\frac{3\pi}{4}, \frac{2\pi}{4}\right]](\varphi) f_s + a[\left[\frac{3\pi}{4}, \frac{2\pi}{4}\right]](\varphi) f_2 \), and for \( \varphi \in \left[\frac{7\pi}{4}, \frac{15\pi}{8}\right) \) we put \( f_\varphi = (1-a[\left[\frac{7\pi}{4}, \frac{15\pi}{8}\right]](\varphi) f_2 + a[\left[\frac{7\pi}{4}, \frac{15\pi}{8}\right]](\varphi) f_s \). Map \( f_\varphi \) here is from \( M \) to \( M \), because \( M \) is convex. Moreover, \( f_\varphi(M) \subset Z \), because for all \( p \) \( x(f_s(p)) < 1 \) and \( x(f_2(p)) < 1 \), and then \( x((1-a)f_2(p) + af_4(p)) < 1 \).

Clearly, all the maps are smooth and we are done.

**Remark 6.** Since any point in \((M \setminus Z) \cap (y \neq 0)\) has an empty preimage, such points are not in the Milnor topological attractor of \( F \).

We will need the following auxiliary statement.

**CLAIM.** A generic sequence \( \omega \in \Sigma^+_g \) contains infinitely many instances of every finite word of \( 0,\ldots,7 \).
Proof. The space $\Sigma_8^+$ is a Baire space. Given any finite word, the set of sequences that contain this word is open and dense in $\Sigma_8^+$. The intersection $\mathcal{R}$ of these sets over all finite words is residual in $\Sigma_8^+$ and has the required property. Indeed, given a word $w$ and a sequence $\omega \in \mathcal{R}$, one can find an instance of $w$ in $\omega$ simply by the definition of $\mathcal{R}$. Now we take another word $w_1$ such that $\omega$ does not contain an instance of the concatenated word $ww_1$ that starts on the left from the right end of the instance of $w$ above. Since $\omega \in \mathcal{R}$ must contain an instance of $ww_1$, this yields another instance of $w$ on the right from the first one, and we get an infinite series of instances by continuing in the same way.

Denote the subset $\Sigma_8^+ \times (D \cup \{q\})$ by $A$.

**Proposition 7.** The set $A$ is a subset of the Milnor topological attractor.\(^3\)

**Proof.** Let $\hat{\mathcal{R}} = \mathcal{R} \times M$, where $\mathcal{R}$ is the residual subset of $\Sigma_8^+$ from the claim. The set $\hat{\mathcal{R}}$ is residual in the phase space. Fix any point $r = (\omega, x) \in \hat{\mathcal{R}}$. Let us prove that the forward $F$-orbit of $r$ is dense in $A$. Given $u = (\hat{\omega}, x_0) \in A$, we consider its neighborhood $U$ that consists of points whose projections to $\Sigma_8^+$ and $M$ are $\varepsilon$-close to the projections of $u$.

First assume that $x_0 \in D$ and denote $p = f_2(q)$, $f_2(q) \in D$ by (3). Due to Hutchinson lemma, there are $n$ and $w_p$ such that $f^{[n]}_w(p)$ is $\varepsilon$-close to $x_0$ (and $w_p$ only contains symbols 2, 4, 6). Furthermore, let $w_0$ be an initial word of the sequence $\hat{\omega}$ such that any sequence that starts with $w_0$ is $\varepsilon$-close to $\hat{\omega}$. Let $w$ be the word that consists of a zero, two and the word $w_p$ followed by the word $w_0$. Since the base component $\omega$ of $r$ is in $\mathcal{R}$, it contains the word $w$; say the $w_0$-part of this word starts at position $k$. Observe that the map $f_0$ takes any point of the fiber to $q$, then $f_2$ takes $q$ into $p$ and $f^{[n]}_w$ takes $p$ into a point that is $\varepsilon$-close to $x_0$. Hence we have $F^k(r) \in U$: we stop iterating when the fiber-component is near $x_0$ and the $\Sigma_8^+$-component begins with $w$ and so is close to $\hat{\omega}$.

If $x_0 = q$, we argue analogously, with $w$ being a word of one 0 followed by $w_0$. This shows that the orbit of $z \in \hat{\mathcal{R}}$ is dense in $A$, and hence $\omega(z) \supset A$ and, by the way, $A$ is a transitive set. Therefore, $A$ is a subset of a Milnor topological attractor.

Now, any point $(\omega, x) \in A$ with $\omega_0 = 0$ and $x \in D$ is in $\text{Int}(A)$, but its $F$-image belongs to a boundary of the attractor (because points from $(M \setminus Z) \cap (y \neq 0)$ are not in the attractor). This finishes the proof of the theorem.

It is worth noting, that from density of $\hat{\mathcal{R}}$ and from previous result one can conclude, that $F$ is a map with the Pinheiro property: for any open $V \bigcup_{n \geq 0} f^n(V) \supset A$ contains a fixed ball.

An analogous proof can be done for the measure Milnor attractor, i.e., the likely limit set.

\(\square\)

5 On generic transitive attractors with nonempty interior

**Theorem 8.** For a generic $C^1$-map $F$ from a manifold $M$ to itself, if $F$ has a transitive closed forward-invariant set $A$ with nonempty interior, then $A$ coincides with the closure of its interior.

\(^3\)Note, that in [M] it is proved, that measure Milnor attractor always exists. One can easily modify this proof to check, that Milnor topological attractor also always exists.
Proof. 1. For a generic $C^1$-smooth map $F: M \to M$ there is an open and dense set of regular points in $M$. Indeed, fix some countable dense subset $C$ of $M$. Given a point in this subset, there is an open and dense set in the space of $C^1$-maps of $M$ such that the point is regular for maps in this set. The intersection of these open and dense sets is a residual set that consists of maps for which every point in $C$ is regular. Clearly, the set of regular points is open by the continuity of the Jacobian.

2. Denote by $R$ the residual set of maps for which regular points form an open and dense set in $M$. For $F \in R$ for any $N > 0$ there is an open and dense set $X_N \subset M$ such that on $X_N \cup F(X_N) \cup \cdots \cup F^N(X_N)$ the map $F$ is regular. Indeed, let $Y$ be the set of regular points of $F$, it is open and dense. The set $F^{-1}(Y)$ is open by continuity, let us show that it is dense. Take some open set $U$; restricting to a small open subset we can assume that $F$ is regular on $U$ and, moreover, that $F : U \mapsto F(U)$ is a diffeomorphism. This implies $F^{-1}(Y)$ is dense in $U$. We have proved that $F^{-1}(Y)$ is open and dense; we can show in the same way that all sets $F^{-k}(Y)$, $k > 0$, are open and dense too.

3. Pick open $B \subset A$ and any $a \in A$. Let $U \ni a$ be some neighborhood. By transitivity for some $N > 0$ the set $F^{-N}(U)$ intersects $B$. Denote $C = X_N \cap F^{-N}(U) \cap B$, it is an open set and $F^N$ is regular on it. Thus $F^N(C)$ contains an open set. We have $F^N(C) \subset A$, as $A$ is forward-invariant. Also, $F^N(C) \subset U$. Thus $U$ intersects $\text{Int}(A)$. As this holds for any $U$, this means $a \in \text{Int}(A)$.

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