A new extended matrix KP hierarchy and its solutions

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Abstract

With the square eigenfunctions symmetry constraint, we introduce a new extended matrix
KP hierarchy and its Lax representation from the matrix KP hierarchy by adding a new \( \tau_B \) flow. The extended KP hierarchy contains two time series \( t_A \) and \( \tau_B \) and eigenfunctions and adjoint


eigenfunctions as components. The extended matrix KP hierarchy and its \( t_A \)-reduction and \( \tau_B \)




reduction include two types of matrix KP hierarchy with self-consistent sources and two types of



(1+1)-dimensional reduced matrix KP hierarchy with self-consistent sources. In particular, the



first type and second type of the 2+1 AKNS equation and the Davey-Stewartson equation with



self-consistent sources are deduced from the extended matrix KP hierarchy. The generalized
dressing approach for solving the extended matrix KP hierarchy is proposed and some solutions
are presented. The soliton solutions of two types of 2+1-dimensional AKNS equation with self-



consistent sources and two types of Davey-Stewartson equation with self-consistent sources are



studied.



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method; 2+1 AKNS equation with self-consistent sources; DS equation with self-consistent sources



1 Introduction

Generalizations of Kadomtsev-Petviashvili (KP) hierarchy attracts lots of interests from physical

and mathematical points of view 1−9. One kind of generalization is the so called multi-component

KP (mcKP) hierarchy or matrix KP hierarchy 2−4, which contains many physical relevant non-

linear integrable systems, such as (2+1)-dimensional AKNS hierarchy and Davey-Stewartson (DS)
equation. An extended DS equation can be derived from matrix KP hierarchy 4. The explicit solutions
of the matrix KP equation were studied in 10−11. The relation between the 1+1 dimensional

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C-integrable Bürgers hierarchy and the matrix KP hierarchy is discussed in \textsuperscript{12}. The constrained matrix KP flows was obtained from the matrix KP hierarchy \textsuperscript{3}. Another kind of generalization of KP equation is the so called KP equation with self-consistent sources initiated by Mel’nikov \textsuperscript{5–7}. The first type and second type of KP equation with self-consistent sources were studied in \textsuperscript{5}. The KP equation with self-consistent sources describes the interaction of a long wave with a short-wave packet propagating on the $x, y$ plane at an angle to each other. Recently, a systematic approach inspired by squared eigenfunction symmetry constraint was proposed to construct a new extended KP hierarchy \textsuperscript{13}. The extended KP hierarchy extended the KP hierarchy by containing two times series $t_n$ and $\tau_k$ and more components given by eigenfunctions and adjoint eigenfunctions. This extended KP hierarchy and its $t_n$- and $\tau_k$-reduction provide an unified way to find the two types of KP equation with self-consistent sources and some (1+1)-dimensional soliton equations with self-consistent sources. The solutions of the extended KP hierarchy and extended mKP hierarchy can be derived under a generalized dressing approach \textsuperscript{14}.

In this paper, we will construct the extension of the matrix KP hierarchy. Inspired by the square eigenfunction symmetry constraint of matrix KP hierarchy, we introduce a new $\tau_B$ flow by "extending" a specific $t_A$-flow of matrix KP hierarchy. The extended matrix KP hierarchy consists of $t_A$-flow, $\tau_B$-flow and the $t_A$-evolutions of eigenfunctions and adjoint eigenfunctions. We get the zero curvature representations for the extended matrix KP hierarchy from the commutativity of $t_A$-flow and $\tau_B$-flow and its Lax representation. The extended matrix KP hierarchy contains two time series $t_A$ and $\tau_B$ and more components by adding eigenfunctions and adjoint eigenfunctions, and admits $t_A$-reduction and $\tau_B$-reduction. The extended matrix KP hierarchy and its two reduction provide an unified way to find two types of matrix KP hierarchy with self-consistent sources, and two types of (1+1)-dimensional reduced matrix KP hierarchy with self-consistent sources.

By restricting the elements of the matrix KP hierarchy in $2 \times 2$ matrices, we deduce two types of 2+1-dimensional AKNS equation with self-consistent sources and two types of DS equation with self-consistent sources from the extended matrix KP hierarchy as examples. We can deduce two types of 1+1 AKNS equation with self-consistent sources under two types of reductions. The DS equation with self-consistent sources were also studied in \textsuperscript{15} and \textsuperscript{16}. We would like to emphasize that our DS equation with self-consistent sources is different from those in \textsuperscript{15} and \textsuperscript{16} as they have different types of sources and the sources satisfy different conditions.

The dressing method is an important tool for solving Gelfand Dickey and KP hierarchy \textsuperscript{17}. However the dressing method for the matrix KP hierarchy can not be directly applied to the extended matrix KP hierarchy. With the combination of a dressing approach and the method of variation of constants, we propose a generalized dressing method for extended matrix KP hierarchy. By using this method, we can solve the entire hierarchy of extended matrix KP hierarchy. We solve two types of 2+1 AKNS equation with self-consistent sources and two types of DS equation with self-consistent sources as examples.

This paper is organized as follows. In section 2, we construct the extended matrix KP hierarchy and derive its Lax pair, including two types of the 2+1 AKNS equation with self-consistent sources and DS equation with self-consistent sources as examples. In section 3, $t_A$-reduction and $\tau_B$-reduction of the extended matrix KP hierarchy are given. In section 4, a generalized dressing method for the extended matrix KP hierarchy is discussed. In section 5, we give the N-soliton solutions for the extended matrix KP hierarchy. The soliton solution of 2+1 AKNS equation with self-consistent sources and DS equation with self-consistent sources are studied. In section 6, we
present the conclusion.

2 The extended matrix KP hierarchy

First we review the well known matrix KP hierarchy. Let \( g_0 \) be a finite dimensional matrix algebra, which means that the elements in \( g_0 \) are \( N \times N \) matrices. The matrix KP hierarchy can be formulated by a pseudodifferential operator which is called the dressing operator:

\[
W = 1 + w_1 \partial^{-1} + w_2 \partial^{-2} + \cdots 
\]

where \( \partial = \frac{\partial}{\partial x} \), and the matrix valued coefficients \( w_i \in g_0 \) are functions of \( x \). \( w = (w_1, w_2, \cdots) \) will be the dynamical fields of the matrix KP hierarchy. Define

\[
\mathcal{A} = \{ c_n \partial^n, \quad n \in \mathbb{N}, \quad c_n \in g_0, \quad [c_n, c_m] = 0 \}.
\]

For each element \( A \in \mathcal{A} \), the evolution of \( W \) is given by

\[
W_t A = -P_{<0}(WAW^{-1})W = -WA + M_A W,
\]

where

\[
M_A = P_{\geq 0}(WAW^{-1}),
\]

\( P_{\geq 0}(L) \) and \( P_{<0}(L) \) denote the nonnegative parts and negative parts of pseudodifferential operator \( L \). If we fix an arbitrary element \( C \in \mathcal{A} \) and define a Lax operator

\[
L = WCW^{-1},
\]

the Lax equation of the matrix KP hierarchy is given by

\[
L_t A = [M_A, L].
\]

The commutativity of \( \partial_t A \) and \( \partial_t B \) flows gives rise to the zero-curvature equations of matrix KP hierarchy.

\[
M_{A,tB} - M_{B,tA} + [M_A, M_B] = 0.
\]

It is known that the evolution of \( W \) given by

\[
W_z = -\sum_{i=1}^{N} \Phi_i \partial^{-1} \Psi_i^T W_{i},
\]

\[
\Phi_{i,tA} = M_A(\Phi_i),
\]

\[
\Psi_{i,tA} = -M_A^*(\Psi_i)
\]

is compatible with the matrix KP hierarchy (2.5) and reduces the matrix KP hierarchy to the constrained matrix KP hierarchy. If \( M_A = \sum_{i=0}^{N} u_i \partial^i \), the adjoint operator \( M_A^* \) is defined by

\[
M_A^* = \sum_{i=0}^{N} (-1)^i \partial^i u_i^T.
\]
Based on this observation, we now introduce a new $\tau_B$ flow given by

$$W_{\tau_B} = -P_{<0}(WBW^{-1})W + \sum_{i=1}^{N} \Phi_i \partial^{-1}\Psi_i^TW,$$

(2.9a)

$$\Phi_{i,t_A} = M_A(\Phi_i),$$

(2.9b)

$$\Psi_{i,t_A} = -M_A^*(\Psi_i).$$

(2.9c)

We have the following lemma.

Lemma 1. $P_{<0}[M_A, \Phi \partial^{-1}\Psi^T] = M_A(\Phi)\partial^{-1}\Psi^T - \Phi \partial^{-1}(M_A^*(\Psi))^T$.

Proof. Without loss of generality, we consider a monomial $Q = u\partial^k$, $u \in g_0$. Using (2.8), we have

$$P_{<0}(\partial^{-1}\Psi^TQ) = P_{<0}(\partial^{-1}\Psi^Tu\partial^k)$$

$$= P_{<0}(\partial^{-1}\partial\Psi^Tu\partial^k - \partial^{-1}\partial(\Psi^Tu)\partial^k)$$

$$= -P_{<0}(\partial^{-1}(\Psi^Tu)\partial^k) = \ldots$$

$$= (-1)^k\partial^{-1}(\Psi^Tu)^{(k)} = (-1)^k\partial^{-1}((u^T\Psi)^{(k)})^T$$

$$= \partial^{-1}(Q^*(\Psi))^T.$$

So we have

$$P_{<0}[M_A, \Phi \partial^{-1}\Psi^T] = P_{<0}(M_A\Phi \partial^{-1}\Psi^T) - P_{<0}(\Phi \partial^{-1}\Psi^TM_A)$$

$$= M_A(\Phi)\partial^{-1}\Psi^T - \Phi \partial^{-1}(M_A^*(\Psi))^T. \quad \square$$

Further more, we find that

$$L_{\tau_B} = W_{\tau_B}CW^{-1} - WCW^{-1}W_{\tau_B}W^{-1}$$

$$= M_BL - LM_B + \Phi \partial^{-1}\Psi^TL - L\Phi \partial^{-1}\Psi^T$$

(2.10)

$$= [M_B + \Phi \partial^{-1}\Psi^T, L].$$

Lemma 2. The $\tau_B$ flow given by (2.9) and (2.10) is compatible with the matrix KP hierarchy (2.5), namely, $(W_{\tau_B})_{t_B} = (W_{\tau_B})_{t_A}$, $(L_{t_A})_{t_B} = (L_{t_B})_{t_A}$.

Proof. For convenience, we omit $\sum$. Notice that $W_{\tau_B} = W_{t_B} + \Phi \partial^{-1}\Psi^TW$, we obtain

$$(W_{\tau_B})_{t_A} = (W_{t_B})_{t_A} + (M_A(\Phi)\partial^{-1}\Psi^T - \Phi \partial^{-1}(M_A^*(\Psi)^T - \Phi \partial^{-1}\Psi^TP_{<0}(WAW^{-1}))W,$$

$$(W_{t_A})_{t_B} = -P_{<0}(W_{t_B}AW^{-1} + \Phi \partial^{-1}\Psi^TWAW^{-1} - WAW^{-1}W_{t_B}W^{-1}$$

$$-WAW^{-1}\Phi \partial^{-1}\Psi^T)W = P_{<0}(WAW^{-1})W_{t_B} - P_{<0}(WAW^{-1})\Phi \partial^{-1}\Psi^TW,$$

so

$$(W_{\tau_B})_{t_A} - (W_{t_A})_{t_B} = (W_{t_B})_{t_A} + (M_A(\Phi)\partial^{-1}\Psi^T - \Phi \partial^{-1}(M_A^*(\Psi)^T)$$

$$-\Phi \partial^{-1}\Psi^T, P_{<0}(WAW^{-1})] + P_{<0}((\Phi \partial^{-1}\Psi^T, WAW^{-1}))W.$$
Using Lemma 1, we find that
\[
P_{<0}([\Phi \partial^{-1} \Psi^T, WAW^{-1}]) = P_{<0}(\Phi \partial^{-1} \Psi^T M_A + \Phi \partial^{-1} \Psi^T P_{<0}(WAW^{-1})
- M_A \Phi \partial^{-1} \Psi^T - P_{<0}(WAW^{-1}) \Phi \partial^{-1} \Psi^T
= [\Phi \partial^{-1} \Psi^T, P_{<0}(WAW^{-1})] + \Phi \partial^{-1} (M_A'(\Psi))^T - M_A(\Phi) \partial^{-1} \Psi^T,
\]
so we have that \((W_{\tau_B})_{t_A} - (W_{\tau_A})_{t_B} = 0\). As \((L_{t_A})_{\tau_B} = ((W_{C}W^{-1})_{t_A})_{\tau_B}, (L_{\tau_B})_{t_A} = ((W_{C}W^{-1})_{\tau_B})_{t_A}\), we find \((L_{t_A})_{\tau_B} = (L_{\tau_B})_{t_A}\) from \((W_{\tau_A})_{\tau_B} = (W_{\tau_B})_{t_A}\).

The commutativity of \(t_A\) flow and \(\tau_B\) flow enables us to obtain a new extended matrix KP hierarchy as
\[
\begin{align*}
\partial_{t_A} L &= [M_A, L], \quad (2.11a) \\
\partial_{\tau_B} L &= [M_B + \sum_{i=1}^{N} \Phi_i \partial^{-1} \Psi_i^T, L] \quad (2.11b) \\
\Phi_{i,t_A} &= M_A(\Phi_i), \quad i = 1, \ldots, N, \quad (2.11c) \\
\Psi_{i,t_A} &= -M_A'(\Psi_i), \quad i = 1, \ldots, N. \quad (2.11d)
\end{align*}
\]

**Proposition 1.** The commutativity of (2.11a) and (2.11b) under (2.11c) and (2.11d) gives rise to the zero-curvature equation for extended matrix KP hierarchy (2.11)
\[
M_{A,\tau_B} - (M_B + \sum_{i=1}^{N} \Phi_i \partial^{-1} \Psi_i^T)_{t_A} + [M_A, M_B + \sum_{i=1}^{N} \Phi_i \partial^{-1} \Psi_i^T] = 0, \quad (2.12a)
\]
\[
\Phi_{i,t_A} = M_A(\Phi_i), \quad (2.12b)
\]
\[
\Psi_{i,t_A} = -M_A'(\Psi_i), \quad i = 1, \ldots, N, \quad (2.12c)
\]
or equivalently
\[
M_{A,B} - M_{B,t_A} + [M_A, M_B] - \sum_{i=1}^{N} P_{\geq 0}([\Phi_i \partial^{-1} \Psi_i^T, M_A]) = 0, \quad (2.13a)
\]
\[
\Phi_{i,t_A} = M_A(\Phi_i), \quad (2.13b)
\]
\[
\Psi_{i,t_A} = -M_A'(\Psi_i), \quad i = 1, \ldots, N, \quad (2.13c)
\]
with the Lax representation given by
\[
\psi_{t_A} = M_A(\psi), \quad (2.14a)
\]
\[
\psi_{\tau_B} = (M_B + \sum_{i=1}^{N} \Phi_i \partial^{-1} \Psi_i^T)(\psi). \quad (2.14b)
\]

**Proof.** The commutativity of (2.11a) and (2.11b) under (2.11c) and (2.11d) immediately gives rise to (2.12). Now we prove (2.13). Using Lemma 1, we have
\[
-(\Phi \partial^{-1} \Psi^T)_{t_A} + [M_A, \Phi \partial^{-1} \Psi^T] = -M_A(\Phi) \partial^{-1} \Psi^T + \Phi \partial^{-1} M_A'(\Psi)^T + [M_A, \Phi \partial^{-1} \Psi^T]
= -P_{<0}(M_A, \Phi \partial^{-1} \Psi^T) + [M_A, \Phi \partial^{-1} \Psi^T]
= P_{\geq 0}(M_A, \Phi \partial^{-1} \Psi^T). \quad \Box
\]
**Remark**. The extended matrix KP hierarchy extends the matrix KP hierarchy by containing two time series $t_A$ and $\tau_B$ and more components $\Phi_i$ and $\Psi_i$, $i = 1, \ldots, N$.

In the following we restrict $w_i$ to $2 \times 2$ matrices and consider the extended matrix KP hierarchy with $A = \sigma_3 \partial$, $B = \sigma_3 \partial^2$ and $C = \sigma_3 \partial$, where $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. The associated times are $t_A = y$, $t_B = t$. We introduce $U = \begin{pmatrix} 0 & r \\ q & 0 \end{pmatrix} = [\omega_1, \sigma_3] = -2\sigma_3 \omega_1^{off}$, $D = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} = -2\omega_1^{diag}$. The zero-curvature equation (2.6) leads to the evolution equation

\begin{align}
U_t &= \frac{1}{2} \sigma_3 (U_{xx} + U_{yy}) + \sigma_3 U^3 + [D_y, U], \\
\sigma_3 D_y - D_x + U^2 &= 0,
\end{align}

i.e.

\begin{align}
rt &= \frac{1}{2} (r_{xx} + r_{yy}) + r^2 q + (a_y - b_y)r, \quad a_y - a_x + rq = 0, \\
qt &= -\frac{1}{2} (q_{xx} + q_{yy}) - q^2 r - (a_y - b_y)q, \quad b_y + b_x - rq = 0,
\end{align}

As $a_x - a_y = b_y + b_x$, we may assume $a + b = \phi_x$, $a - b = \phi_y$, which leads to $\phi_{xx} - \phi_{yy} = 2rq$. Denote $v = \phi_y$, then we have

\begin{align}
rt &= \frac{1}{2} (r_{xx} + r_{yy}) + r^2 q + v yr, \\
qt &= -\frac{1}{2} (q_{xx} + q_{yy}) - q^2 r - v y q, \\
v_{xx} - v_{yy} &= 2(rq)_y,
\end{align}

which is $2 + 1$ dimensional AKNS equation \(^3\). If we replace $U_t$ by $iU_t$ and assume that $q = \tilde{r}$, $\phi = \tilde{\phi}$, then the above system reduces to the Davey-Stewartson I (DSI) equation

\begin{align}
ir_t &= \frac{1}{2} (r_{xx} + r_{yy}) + |r|^2 r + v yr, \quad v_{xx} - v_{yy} = 2(|r|^2)_y.
\end{align}

Now we derive the $(2+1)$ dimensional AKNS equation with self-consistent sources and Davey-Stewartson equation with self-consistent sources from (2.13).

**Example 1.** If we take $t_A = y$ and $\tau_B = t$, we obtain the first type of $2 + 1$ dimensional AKNS equation with self-consistent sources from (2.13)

\begin{align}
U_t &= \frac{1}{2} \sigma_3 (U_{xx} + U_{yy}) + \sigma_3 U^3 + [D_y, U] + \sum_{i=1}^N \Phi_i \Psi_i^T, \sigma_3], \\
\sigma_3 D_y - D_x + U^2 &= 0, \\
\Phi_{i,y} &= \sigma_3 \Phi_{i,x} + U \Phi_i, \\
\Psi_{i,y} &= \sigma_3 \Psi_{i,x} - U^T \Psi_i,
\end{align}
or we can rewrite it as

\[ r_t = \frac{1}{2}(r_{xx} + r_{yy}) + r^2q + v_y r - \sum_{i=1}^{N} (2(\phi_{i11}\psi_{i21} + \phi_{i12}\psi_{i22})), \]  

(2.21a)

\[ q_t = -\frac{1}{2}(q_{xx} + q_{yy}) - q^2 r - v_y q + \sum_{i=1}^{N} (2(\phi_{i21}\psi_{i11} + \phi_{i22}\psi_{i12})), \]  

(2.21b)

\[ v_{xx} - v_{yy} = 2(qr)_y, \]  

(2.21c)

\[ \Phi_{i,y} = \sigma_3 \Phi_{i,x} + U \Phi_i, \]  

(2.21d)

\[ \Psi_{i,y} = \sigma_3 \Psi_{i,x} - U^T \Psi_i. \]  

(2.21e)

Here and afterward \( \Phi_i = \begin{pmatrix} \phi_{i11} & \phi_{i12} \\ \phi_{i21} & \phi_{i22} \end{pmatrix}, \Psi_i = \begin{pmatrix} \psi_{i11} & \psi_{i12} \\ \psi_{i21} & \psi_{i22} \end{pmatrix}, \) for \( i = 1, \ldots, N \) are \( 2 \times 2 \) matrices.

Its Lax representation is

\[ \psi_y = (\sigma_3 \partial + U)\psi, \]  

(2.22a)

\[ \psi_t = (\sigma_3 \partial^2 + U \partial + \frac{1}{2}U_x + \frac{1}{2}\sigma_3 U^2 + \frac{1}{2}\sigma_3 U_y + D_y + \sum_{i=1}^{N} \Phi_i \partial^{-1} \Psi_i^T)\psi. \]  

(2.22b)

**Example 2.** When we take \( \tau_A = y \) and \( \tau_B = t \), we get the second type of \( 2 + 1 \) dimensional AKNS equation with self-consistent sources from (2.13)

\[ U_t = \frac{1}{2}\sigma_3(U_{xx} + U_{yy}) + \sigma_3 U^3 + [D_y, U] - \sum_{i=1}^{N} \left( [U, (\Phi_i \Psi_i^T)_{\text{diag}}] + 2\sigma_3 (\Phi_i \Psi_i^T)_{\text{off}} \right) \]  

(2.23a)

\[ (\sigma_3 D_y - D_x + U^2)_y - \sum_{i=1}^{N} \left( [U, (\Phi_i \Psi_i^T)_{\text{off}}] + 2\sigma_3 (\Phi_i \Psi_i^T)_{\text{diag}} \right) = 0, \]  

(2.23b)

\[ \Phi_{i,t} = \sigma_3 \Phi_{i,xx} + U \Phi_{i,x} + \frac{1}{2}(U_x + \sigma_3 U^2 + \sigma_3 U_y + D_y) \Phi_i, \quad i = 1, \ldots, N, \]  

(2.23c)

\[ \Psi_{i,t} = -\sigma_3 \Psi_{i,xx} + U^T \Psi_{i,x} - \frac{1}{2}(U_x + \sigma_3 U^2 + \sigma_3 U_y + D_y)^T \Psi_i, \quad i = 1, \ldots, N, \]  

(2.23d)
or

\[
    r_t = \frac{1}{2}(r_{xx} + r_{yy}) + r^2q + v_yr + \sum_{i=1}^{N}(r(\phi_{i11}\psi_{i11} + \phi_{i12}\psi_{i12} - \phi_{i21}\psi_{i21} - \phi_{i22}\psi_{i22})
    - 2(\phi_{i11,x}\psi_{i21} + \phi_{i12,x}\psi_{i22}))
    \]

\[
    q_t = -\frac{1}{2}(q_{xx} + q_{yy}) - q^2 r - v_y q - \sum_{i=1}^{N}(q(\phi_{i11}\psi_{i11} + \phi_{i12}\psi_{i12} - \phi_{i21}\psi_{i21} - \phi_{i22}\psi_{i22})
    - 2(\phi_{i21,x}\psi_{i11} + \phi_{i22,x}\psi_{i12}))
    \]

\[
    v_{yy} - v_{xx} = 2(\sum_{i=1}^{N}(r(\phi_{i21}\psi_{i11} + \phi_{i22}\psi_{i12} - q(\phi_{i11}\psi_{i21} + \phi_{i12}\psi_{i22})
    + 2(\phi_{i11}\psi_{i11} + \phi_{i12}\psi_{i12})_{x,i}))
    \]

\[
    \Phi_{i,t} = \sigma_3\Phi_{i,x} + U\Phi_{i,x} + \frac{1}{2}(U_x + \sigma_3U^2 + \sigma_3U_y + D_y)\Phi_{i}, \ i = 1, \ldots, N
    \]

\[
    \Psi_{i,t} = -\sigma_3\Psi_{i,x} + U^T\Psi_{i,x} - \frac{1}{2}(U_x + \sigma_3U^2 + \sigma_3U_y + D_y)^T\Psi_{i}, \ i = 1, \ldots, N
    \]

Its Lax representation is

\[
    \psi_y = (\sigma_3\partial + U + \sum_{i=1}^{N}\Phi_{i}\partial^{-1}\Psi_{i}^T)\psi,
    \]

\[
    \psi_t = (\sigma_3\partial^2 + U\partial + \frac{1}{2}U_x + \frac{1}{2}\sigma_3U^2 + \frac{1}{2}\sigma_3U_y + D_y)\psi.
    \]

**Example 3.** Define \( \tilde{U} = \begin{pmatrix} 0 & r \\ \bar{r} & 0 \end{pmatrix} \), the first type of DSI equation with self-consistent sources is

\[
    ir_t = \frac{1}{2}(r_{xx} + r_{yy}) + |r|^2 r + v_y r - 2\sum_{j=1}^{N}(\phi_{j11}\psi_{j21} + \phi_{j12}\psi_{j22}),
    \]

\[
    v_{xx} - v_{yy} = 2(|r|^2)_{y},
    \]

\[
    \Phi_{j,x} = \sigma_3\Phi_{j,x} + \tilde{U}\Phi_{j}, \ j = 1, \ldots, N,
    \]

\[
    \Psi_{j,x} = \sigma_3\Psi_{j,x} - \tilde{U}^T\Psi_{j}, \ j = 1, \ldots, N.
    \]

Its Lax representation is

\[
    \psi_y = (\sigma_3\partial + \tilde{U})\psi,
    \]

\[
    \psi_t = -i(\sigma_3\partial^2 + \tilde{U}\partial + \frac{1}{2}\tilde{U}_x + \frac{1}{2}\sigma_3\tilde{U}^2 + \frac{1}{2}\sigma_3\tilde{U}_y + D_y + \sum_{j=1}^{N}\Phi_{j}\partial^{-1}\Psi_{j}^T)\psi.
    \]
Example 4. The second type of DSI equation with self-consistent sources is
\[
ir_t = \frac{1}{2}(r_{xx} + r_{yy}) + |r|^2 r + v_y r + \sum_{j=1}^{N}(r(\phi_{j11} \psi_{j11} + \phi_{j12} \psi_{j12} - \phi_{j21} \psi_{j21} - \phi_{j22} \psi_{j22})
-2(\phi_{j11,x} \psi_{j21} + \phi_{j12,x} \psi_{j22}))
\]
\[= 2(\sum_{j=1}^{N}(r(\phi_{j21} \psi_{j11} + \phi_{j22} \psi_{j12} - \bar{r}(\phi_{j11} \psi_{j21} + \phi_{j12} \psi_{j22})
+2(\phi_{j11} \psi_{j11} + \phi_{j12} \psi_{j12}))), \tag{2.28a}
\]
\[v_{yy} - v_{xx} = 2(\sum_{j=1}^{N}(r(\phi_{j21} \psi_{j11} + \phi_{j22} \psi_{j12} - \bar{r}(\phi_{j11} \psi_{j21} + \phi_{j12} \psi_{j22})
+2(\phi_{j11} \psi_{j11} + \phi_{j12} \psi_{j12}))), \tag{2.28b}
\]
\[\Phi_{j,t} = \sigma_3 \Phi_{j,xx} + \bar{U} \Phi_{j,x} + \frac{1}{2}(\bar{U}_x + \sigma_3 \bar{U}^2 + \sigma_3 \bar{U}_y + D_y) \Phi_{j}, \; j = 1, \ldots, N, \tag{2.28c}
\]
\[\Psi_{j,t} = -\sigma_3 \Psi_{j,xx} + \bar{U}^T \Psi_{j,x} - \frac{1}{2}(\bar{U}_x + \sigma_3 \bar{U}^2 + \sigma_3 \bar{U}_y + D_y)^T \Psi_{j}, \; j = 1, \ldots, N. \tag{2.28d}
\]

Its Lax representation is
\[
\psi_y = (\sigma_3 \partial + \bar{U} + \sum_{j=1}^{N} \Phi_{j,\partial^{-1}} \Psi_j^T) \psi, \tag{2.29a}
\]
\[
\psi_t = -i(\sigma_3 \partial^2 + \bar{U} \partial + \frac{1}{2} \bar{U}_x + \frac{1}{2} \sigma_3 \bar{U}^2 + \frac{1}{2} \sigma_3 \bar{U}_y + D_y) \psi. \tag{2.29b}
\]

Remark. The extended matrix KP hierarchy (2.11) provides a unified way to construct the first type and second type of the (2+1)-dimensional AKNS equation (and DSI equation) with self-consistent sources and their Lax representation.

3 Reductions of the extended matrix KP hierarchy

The extended KP hierarchy depends on two time series \( t_A \) and \( \tau_B \). It is natural to consider its \( t_A \)-reduction and \( \tau_B \)-reduction.

3.1 The \( t_A \)-reduction

The \( t_A \)-reduction of the extended matrix KP hierarchy is given by
\[
WAW^{-1} = M_A, \tag{3.1}
\]
where \( A = C_k \partial^k \). The wave function and the adjoint wave function are given by
\[
\Phi(t, z) = W \exp(\xi(t, z)), \; \Phi^*(t, z) = (W^*)^{-1} \exp(-\xi(t, z)), \tag{3.2}
\]
where \( \xi(t, z) = \sum_{i>0} t_i z^i, \; t_1 = x \). Then we have
\[
M_A(\Phi) = WAW^{-1} \Phi = z^k \Phi C_k, \tag{3.3}
\]
\[
M_A^*(\Phi^*) = (WAW^{-1})^* \Phi^* = -z^k \Phi^* C_k. \tag{3.4}
\]
\[
L_{t_A} = [M_A, \; L] = [WAW^{-1}, \; W CW^{-1}] = W[A, \; C]W^{-1} = 0. \tag{3.5}
\]
So $L$ is independent of $t_A$ and we can drop $t_A$ dependence from (2.12)

$$(M_A)_B = [M_B + \sum_{i=1}^{N} \Phi_i \partial^{-1}\Psi_i^T, M_A], \quad (3.6a)$$

$M_A(\Phi_i) = \lambda_i^k \Phi_i C_k,$ \hspace{1cm} (3.6b)

$M_A^*(\Psi_i) = \lambda_i^k \Psi_i C_k.$ \hspace{1cm} (3.6c)

When $A = \sigma_3 \partial$ and $B = \sigma_3 \partial^2$ we have $M_A(\Phi_i) = W A W^{-1}(\Phi_i) = \lambda_i \Phi_i \sigma_3$. Then the first type of the $2 + 1$-dimensional AKNS equation with self-consistent sources reduces to the first type of $1 + 1$-dimensional AKNS equation with self-consistent sources

$U_t = \frac{1}{2} \sigma_3 U_{xx} + \sigma_3 U^3 + \sum_{i=1}^{N} \Phi_i \Psi_i^T, \quad (3.7a)$

$U^2 = D_x,$ \hspace{1cm} (3.7b)

$\sigma_3 \Phi_{i,x} + U \Phi_i = \lambda_i \Phi_i \sigma_3,$ \hspace{1cm} (3.7c)

$\sigma_3 \Psi_{i,x} - U^T \Psi_i = \lambda_i \Psi_i \sigma_3.$ \hspace{1cm} (3.7d)

### 3.2 The $\tau_B$-reduction

The $\tau_B$-reduction is given by \(^3\)

$$WBW^{-1} = M_B + \sum_{i=1}^{N} \Phi_i \partial^{-1}\Psi_i^T, \quad (3.8)$$

Then we can drop $\tau_B$ dependence from (2.12)

$$(M_B + \sum_{i=1}^{N} \Phi_i \partial^{-1}\Psi_i^T)_{t_A} = [M_A, M_B + \sum_{i=1}^{N} \Phi_i \partial^{-1}\Psi_i^T], \quad (3.9a)$$

$$\Phi_{i,t_A} = M_A(\Phi_i), \quad (3.9b)$$

$$\Psi_{i,t_A} = -M_A^*(\Psi_i). \quad (3.9c)$$

which is just the constrained matrix KP hierarchy given in \(^3\).

When $B = \sigma_3 \partial$ and $A = \sigma_3 \partial^2$, we get the constrained (2+1)-dimensional AKNS equation or second type of AKNS equation with self-consistent sources.

$$U_t = \frac{1}{2} \sigma_3 U_{xx} + \sigma_3 U^3 + 2 \sum_{j=1}^{m} (\Phi_j \Psi_j^T)_{\text{diag}}, \quad (3.10a)$$

$$\Phi_{it} = \sigma_3 \Phi_{i,xx} + U \Phi_{ix} + \frac{1}{2} U_{x} \Phi_i + \frac{1}{2} \sigma_3 U^2 \Phi_i \sum_{j=1}^{m} \Phi_{j} \Psi_j \Phi_i + \sum_{j=1}^{m} (\Phi_j \Psi_j^T)_{\text{diag}} \Phi_i, \quad (3.10b)$$

$$\Psi_{it} = -\sigma_3 \Psi_{i,xx} + U^T \Psi_{ix} + \frac{1}{2} U_{x} \Psi_i - \frac{1}{2} \sigma_3 U^2 \Psi_i - \sum_{j=1}^{m} \Psi_j \Phi_j^T \Psi_i - \sum_{j=1}^{m} (\Psi_j \Phi_j^T)_{\text{diag}} \Psi_i. \quad (3.10c)$$

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**Remark.** The extended matrix KP hierarchy and its $t_A$-reduction and $\tau_B$-reduction provide a simple and unified way to obtain the two types of (2+1)-dimensional and (1+1)-dimensional AKNS equation with self-consistent sources.

### 4 Generalized dressing approach for extended matrix KP hierarchy

In the following, we restrict $g_0$ to be the matrix algebra of dimensional $2 \times 2$. Now we propose a generalized dressing approach for the extended matrix KP hierarchy. For the dressing form of \( L \) given by (2.4)

\[
L = WCW^{-1},
\]

(4.1)

usually the $W$ has finite terms, so it is equivalent to assume that the dressing operator $W$ is a pure differential operator of order $N$ as follows

\[
W = \partial^N + w_1 \partial^{N-1} + w_2 \partial^{N-2} + \cdots + w_N.
\]

(4.2)

Let $2 \times 2$ matrices $f_i$, $g_i$ satisfy

\[
f_{i,t_A} = A(f_i), \quad f_{i,\tau_B} = B(f_i), \quad g_{i,t_A} = A(g_i), \quad g_{i,\tau_B} = B(g_i), \quad i = 1, \cdots, N.
\]

(4.3)

By means of the method of variation of constants, let $2 \times 2$ matrices $h_i$ be the linear combination of $f_i$ and $g_i$ as

\[
h_i = f_i + \alpha_i(\tau_B)g_i, \quad i = 1, \cdots, N
\]

(4.4)

with $\alpha_i$ being a function of $\tau_B$. We assume that $h_i$ and its derivatives are invertible matrices and the $2N \times 2N$ matrix

\[
\begin{pmatrix}
  h_1 & h_2 & \cdots & h_N \\
  h_1^{(1)} & h_2^{(1)} & \cdots & h_N^{(1)} \\
  \vdots & \vdots & \ddots & \vdots \\
  h_1^{(N-1)} & h_2^{(N-1)} & \cdots & h_N^{(N-1)}
\end{pmatrix}
\]

is invertible.

**Theorem 1**: Let $a_0, \ldots, a_{N-1}$ be the $2 \times 2$ matrix functions determined as the solution of the linear algebraic system

\[
\sum_{i=0}^{N} a_i h_j^{(i)} = 0, \quad j = 1, \ldots, N,
\]

(4.5)

with $a_N = 1$. Then $W = \sum_{i=0}^{N} a_i \partial^i$ satisfies (2.2) and $L = WCW^{-1}$ satisfies the matrix KP hierarchy (2.5).

We have

\[
W(h_i) = 0, \quad i = 1, \cdots, N
\]

(4.6)

**Theorem 2**: Let $b_1, \ldots, b_N$ be the $2 \times 2$ matrix functions satisfy

\[
\sum_{j=1}^{N} h_j^{(i)} b_j = \delta_{i,N-1}, \quad i = 0, \ldots, N - 1.
\]

(4.7)
Define $\Phi_i = -\dot{\alpha}_iW(g_i)$, and $\Psi_i^T = b_i$, where $\dot{\alpha}_i = \partial \dot{\alpha}_i/d\tau_B$, then $W = \sum_{i=0}^{N} a_i \partial^i \Phi_i, \Psi_i$ and $L = WCW^{-1}$ satisfy extended matrix KP hierarchy (2.11).

To proof Theorem 2, we need several lemmas under the above assumptions.

**Lemma 3:** $W^{-1} = \sum_{i=1}^{N} h_i \partial^{-1} \Psi_i^T$.

**Proof:** Using (4.6) and (4.7), we have

$$P_{\geq 0}(W \sum_{i=1}^{N} h_i \partial^{-1} \Psi_i^T) = P_{\geq 0}(W \sum_{i=1}^{N} h_i \partial^{-1} b_i) = P_{\geq 0}(W \sum_{k=0}^{\infty} \partial^{-k-1} \sum_{i=1}^{N} h_i^{(k)} b_i)$$

$$= P_{\geq 0}(W \partial^{-N} + \sum_{k=0}^{\infty} \partial^{-k-1} \sum_{i=1}^{N} h_i^{(k)} b_i)$$

$$= P_{\geq 0}(W \partial^{-N} (1 + \sum_{k=0}^{\infty} \partial^{-k-1} \sum_{i=1}^{N} h_i^{(N+k)} b_i)) = 1.$$  

$$P_{\leq 0}(W \sum_{i=1}^{N} h_i \partial^{-1} \Psi_i^T) = \sum_{i=1}^{N} (W(h_i)) \partial^{-1} \Psi_i^T = 0. \quad (4.8)$$

So we know that $W^{-1} = \sum_{i=1}^{N} h_i \partial^{-1} \Psi_i^T$.

**Lemma 4:** $W^*(\Psi_i) = 0$.

**Proof:** From the relation $W^*(W^{-1})^* \partial^j = \partial^j$, we know that

$$0 = \text{Res}_\partial W^* \left( \sum_{i=1}^{N} h_i \partial^{-1} \Psi_i^T \right) \partial^j = -\text{Res}_\partial W^* \sum_{i=1}^{N} \Psi_i \partial^{-1} h_i^T \partial^j = (-1)^{j+1} \sum_{i=1}^{N} W^*(\Psi_i) h_i^{T(j)}.$$  

As the $2N \times 2N$ matrix

$$\begin{pmatrix} h_1 & h_2 & \cdots & h_N \\ h_1^{(1)} & h_2^{(1)} & \cdots & h_N^{(1)} \\ \vdots & \vdots & \ddots & \vdots \\ h_1^{(N-1)} & h_2^{(N-1)} & \cdots & h_N^{(N-1)} \end{pmatrix}$$

is invertible, we find that $W^*(\Psi_i) = 0$.

**Lemma 5:** The operator $\partial^{-1} \Psi_i^T W$ is a pure differential operator for each $i$. Further more, for $1 \leq i, j \leq N$, $(\partial^{-1} \Psi_i^T W)(h_j) = \delta_{ij} I$.

**Proof:** As $P_{\leq 0}(\partial^{-1} \Psi_i^T W) = \partial^{-1}(W^*(\Psi_i))^T = 0$, we know that $\partial^{-1} \Psi_i^T W$ is a pure differential operator. Let $c_{ij} = (\partial^{-1} \Psi_i^T W)(h_j)$. We find that $\partial(c_{ij}) = \Psi_i^T W(h_j) = 0$ and

$$\sum_{i=1}^{N} h_i^{(k)} c_{ij} = \partial^k \left( \sum_{i} h_i c_{ij} \right) = \partial^k \left( \sum_{i} (h_i \partial^{-1} \Psi_i^T W)(h_j) \right) = h_j^{(k)}.$$  

So we have $c_{ij} = \delta_{ij} I$.

**Proposition 2:** $W$ satisfies $W_{tb} = -P_{\leq 0}(W B W^{-1}) W + \sum_{i=1}^{N} \Phi_i \partial^{-1} \Psi_i^T W$.  

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Proof: Taking $\partial_{\tau_B}$ to the identity $W(h_i) = 0$ and using (4.3), Lemma 3 and Lemma 5, we have

$$0 = W_{\tau_B}(h_i) + W B(h_i) + \dot{\alpha}_i W(g_i)$$

$$= (\partial_{\tau_B} W)(h_i) + (W B W^{-1} W)(h_i) - \sum_{j=1}^{N} \Phi_j \delta_{ji}$$

$$= (\partial_{\tau_B} W + P_{<0}(W B W^{-1} W) - \sum_{j=1}^{N} \Phi_j \partial^{-1} \Psi_j^T W)(h_i).$$

(4.9)

Since the non-negative operator acting on $h_i$ above has degree lower than $N$ and $h_i$ are $N$ independent functions, the operator itself must be zero. Hence the proposition is proven.

Proof of the Theorem 2: The proof of (2.11a) is similar as we have in Theorem 1. By using (2.10), we get (2.11b). By a direct calculation, we get (2.11c) and (2.11d).

5 Solutions for extended matrix KP hierarchy

By using the generalized dressing approach given by Theorem 2, we can construct the explicit solutions to the extended matrix KP hierarchy.

For the (2+1)-dimensional AKNS equation with self-consistent sources, we choose proper $2 \times 2$ matrices $f_i$ and $g_i$. The solution of $f_y = \sigma_3 f_x, f_t = \sigma_3 f_{xx}$ is

$$f = \left(\begin{array}{cc}
\frac{c_{11}}{\lambda_i} + d_{11} e^{\lambda x + \lambda y + \lambda^2 t} & 0 \\
0 & \frac{c_{12}}{\mu_i} + d_{21} e^{\mu x + \mu y + \mu^2 t}
\end{array}\right),$$

where $c_{ij}, d_{ij}$ are arbitrary constants, but they should be chosen such that $f$ is invertible for any $x, y, t$. We can define

$$f_i = \left(\begin{array}{cc}
\frac{1}{\lambda_i} + e^{\lambda_i x + \lambda_i y + \lambda_i^2 t} & 0 \\
0 & \frac{1}{\mu_i} + e^{\mu_i x - \mu_i y - \mu_i^2 t}
\end{array}\right),$$

$$g_i = \left(\begin{array}{cc}
0 & -e^{\mu_i x + \mu_i y + \mu_i^2 t} \\
e^{\lambda_i x - \lambda_i y - \lambda_i^2 t} & 0
\end{array}\right).$$

(5.1)

(5.2)

(5.3)

where $\lambda_i > 0, \mu_i > 0$. In this way we have $a_i$ and $b_i$ and the N-soliton solution of the (2+1)-dimensional AKNS equation with self-consistent sources is

$$U = -2\sigma_3 a_{n-1}^{off}, D = a_{n-1}^{diag}, \Phi_i = -\alpha_i W(g_i), \Psi_i = b_i, i = 1, \ldots, n.$$  

(5.4)

The one-soliton solution of the first type of $(2 + 1)$-AKNS equation with a self-consistent source (2.21) can be constructed from $h = f + \alpha(t)g = \left(\begin{array}{cc}
\frac{1}{\lambda} + e^{\lambda x + \lambda y + \lambda^2 t} & -\alpha(t) e^{\mu x + \mu y + \mu^2 t} \\
\alpha(t) e^{\lambda x - \lambda y - \lambda^2 t} & \frac{1}{\mu} + e^{\mu x - \mu y - \mu^2 t}
\end{array}\right) = \left(\begin{array}{cc}
\frac{1}{\lambda} + e^{\xi_{11}} & -\alpha(t) e^{\xi_{12}} \\
\alpha(t) e^{\xi_{21}} & \frac{1}{\mu} + e^{\xi_{22}}
\end{array}\right)$
From Theorem 1 and Theorem 2 we have \( a_0 = -h_x h^{-1}, a_1 = 1, b_0 = h^{-1} \), which gives rise to

\[
W = \partial - h_x h^{-1}, \quad U = 2\sigma_3(h_x h^{-1})^{off}, \quad D = 2(h_x h^{-1})^{diag}, \quad \Phi = -\dot{\alpha}(t)(g_x - h_x h^{-1}g), \quad \Psi T = h^{-1}. \tag{5.5}
\]

So the one-soliton of the first type of (2+1)-AKNS equation with a self-consistent source (2.21) is

\[
q = -2\alpha(t)\frac{\lambda e^{\xi_{21}} + (\lambda - \mu)e^{\xi_{21} + \xi_{22}}}{(\frac{1}{\lambda} + e^{\xi_{11}})(\frac{1}{\mu} + e^{\xi_{22}}) + \alpha(t)^2 e^{\xi_{12} + \xi_{21}}},
\]

\[
r = 2\alpha(t)\frac{\lambda e^{\xi_{11}} + (\lambda - \mu)e^{\xi_{11} + \xi_{12}} + \alpha(t)^2 e^{\xi_{12} + \xi_{21}}}{(\frac{1}{\lambda} + e^{\xi_{11}})(\frac{1}{\mu} + e^{\xi_{22}}) + \alpha(t)^2 e^{\xi_{12} + \xi_{21}}},
\]

\[
v = 2\alpha(t)\frac{\lambda e^{\xi_{22}} + (\lambda - \mu)(e^{\xi_{11} + \xi_{22}} - \alpha(t)^2 e^{\xi_{12} + \xi_{21}})}{(\frac{1}{\lambda} + e^{\xi_{11}})(\frac{1}{\mu} + e^{\xi_{22}}) + \alpha(t)^2 e^{\xi_{12} + \xi_{21}}},
\]

\[
\Phi = -\dot{\alpha}(t)\left(\frac{\alpha(t)^2 e^{\xi_{12} + \xi_{21}} + (\mu - \lambda)e^{\xi_{11} + \xi_{12} + \xi_{21}}}{(\frac{1}{\lambda} + e^{\xi_{11}})(\frac{1}{\mu} + e^{\xi_{22}}) + \alpha(t)^2 e^{\xi_{12} + \xi_{21}}} - \frac{\alpha(t)^2 e^{\xi_{12} + \xi_{21}} + (\mu - \lambda)e^{\xi_{11} + \xi_{12} + \xi_{21}}}{(\frac{1}{\lambda} + e^{\xi_{11}})(\frac{1}{\mu} + e^{\xi_{22}}) + \alpha(t)^2 e^{\xi_{12} + \xi_{21}}}ight),
\]

\[
\Psi = \begin{pmatrix}
\frac{1}{\lambda} + e^{\xi_{11}} & -\alpha(t)e^{\xi_{21}} \\
\alpha(t)e^{\xi_{21}} & \frac{1}{\lambda} + e^{\xi_{11}} - \alpha(t)e^{\xi_{12}} \\
\frac{1}{\mu} + e^{\xi_{22}} & \frac{1}{\mu} + e^{\xi_{22}} - \alpha(t)e^{\xi_{12}} \\
\frac{1}{\mu} + e^{\xi_{22}} - \alpha(t)e^{\xi_{12}} & \frac{1}{\mu} + e^{\xi_{22}}
\end{pmatrix}.
\tag{5.10}
\]

The one-soliton solution of the second type of (2+1)-AKNS equation with a self-consistent source (2.24) can be constructed from \( h = f + \alpha(y)g = \begin{pmatrix} (\frac{1}{\lambda} + e^{\lambda x + \lambda y + \lambda^2 t}) & -\alpha(y)e^{\lambda x + \mu y + \mu^2 t} \\
\alpha(y)e^{\lambda x + \mu y + \mu^2 t} & (\frac{1}{\mu} + e^{\mu x - \mu y - \mu^2 t}) \end{pmatrix} :=
\begin{pmatrix} \frac{1}{\lambda} + e^{\xi_{11}} & -\alpha(y)e^{\xi_{12}} \\
\alpha(y)e^{\xi_{12}} & \frac{1}{\lambda} + e^{\xi_{22}} \end{pmatrix}.
\]

From Theorem 1 and Theorem 2 we have \( a_0 = -h_x h^{-1}, a_1 = 1, b_0 = h^{-1} \), which gives rise to

\[
W = \partial - h_x h^{-1}, \quad U = 2\sigma_3(h_x h^{-1})^{off}, \quad D = 2(h_x h^{-1})^{diag}, \quad \Phi = -\dot{\alpha}(y)(g_x - h_x h^{-1}g), \quad \Psi T = h^{-1}. \tag{5.11}
\]

So the one-soliton of the second type of (2+1)-AKNS equation with a self-consistent source
The one-soliton solution of the first type of the (2+1)-DS equation with a self-consistent source (2.26) can be constructed from

\[ h = f + \alpha(t)g = \left( \frac{1}{\lambda} + e^{\lambda x - \lambda y + i\lambda^2 t} - \alpha(t)e^{-\lambda x - \lambda y + i\lambda^2 t} \right) = \left( \frac{1}{\lambda} + e^{\eta_{11}} - \alpha(t)e^{\eta_{12}} \right), \]

where \( \lambda > 0 \).

We find that the one-soliton solution of the first type of the (2+1)-DS equation with a self-consistent source is

\[ r = 2\alpha(t) \frac{e^{\eta_{12}} + 2\lambda e^{\eta_{11} + \eta_{22}}}{(\frac{1}{\lambda} + e^{\eta_{11}})(\frac{1}{\lambda} + e^{\eta_{22}}) + \alpha(t)^2 e^{\eta_{12} + \eta_{21}}}, \]

\[ v = 2 \frac{e^{\eta_{22}} + e^{\eta_{11}} + 2\lambda(e^{\eta_{11} + \eta_{22}} - \alpha(t)^2 e^{\eta_{12} + \eta_{21}})}{(\frac{1}{\lambda} + e^{\eta_{11}})(\frac{1}{\lambda} + e^{\eta_{22}}) + \alpha(t)^2 e^{\eta_{12} + \eta_{21}}}, \]

\[ \Phi = -\dot{\alpha}(t) \left( \frac{-\alpha(t) e^{\eta_{12} + \eta_{21} + 2\lambda e^{\eta_{11} + \eta_{22}} + \alpha(t)^2 e^{\eta_{12} + \eta_{21}}}}{(\frac{1}{\lambda} + e^{\eta_{11}})(\frac{1}{\lambda} + e^{\eta_{22}}) + \alpha(t)^2 e^{\eta_{12} + \eta_{21}}} \right), \]

\[ \Psi = \left( \frac{\frac{1}{\lambda} + e^{\eta_{22}}}{(\frac{1}{\lambda} + e^{\eta_{11}})(\frac{1}{\lambda} + e^{\eta_{22}}) + \alpha(t)^2 e^{\eta_{12} + \eta_{21}}} \right). \]

The one-soliton solution of the second type of the (2+1)-DS equation with a self-consistent source (2.28) can be constructed from

\[ h = f + \alpha(y)g = \left( \frac{1}{\lambda} + e^{\lambda x + \lambda y + \lambda^2 t} - \alpha(y)e^{-\lambda x - \lambda y + i\lambda^2 t} \right) = \left( \frac{1}{\lambda} + e^{\eta_{11}} - \alpha(y)e^{\eta_{12}} \right). \]
where $\lambda > 0$.

We find that the one-soliton solution of the second type of the $(2+1)$-DS equation with a self-consistent source is

$$
\begin{align*}
  r &= 2\alpha(y) \frac{e^{\eta_{12}} + 2\lambda e^{\eta_{11} + \eta_{21}}}{(\frac{2}{\lambda} + e^{\eta_{11}})(\frac{2}{\lambda} + e^{\eta_{22}}) + \alpha(y)^2 e^{\eta_{11} + \eta_{21}}}, \\
v &= 2\frac{e^{\eta_{22}} + e^{\eta_{11}} + 2\lambda(e^{\eta_{11} + \eta_{22}} - \alpha(y)^2 e^{\eta_{11} + \eta_{22}})}{(\frac{2}{\lambda} + e^{\eta_{11}})(\frac{2}{\lambda} + e^{\eta_{22}}) + \alpha(y)^2 e^{\eta_{11} + \eta_{22}}}, \\
\Phi &= -\dot{\alpha}(y) \frac{-\alpha(y) e^{\eta_{12} + \eta_{21} + 2\lambda e^{\eta_{11} + \eta_{12} + \eta_{21}}} + \alpha(y) e^{\eta_{11} + \eta_{22}} + \alpha(y)^2 e^{\eta_{11} + \eta_{22}}}{\alpha(y) e^{\eta_{12} + \eta_{21}} + \alpha(y)^2 e^{\eta_{11} + \eta_{22}}}, \\
\Psi &= \left( \begin{array}{c}
\frac{\lambda}{\lambda + e^{\eta_{11}}} + e^{\eta_{22}} + \alpha(y)^2 e^{\eta_{11} + \eta_{21}} \\
\frac{\lambda}{\lambda + e^{\eta_{11}}} + e^{\eta_{22}} + \alpha(y)^2 e^{\eta_{11} + \eta_{21}} \\
\alpha(y) e^{\eta_{12}} \\
\alpha(y) e^{\eta_{12}} \\
\frac{\lambda}{\lambda + e^{\eta_{11}}} + e^{\eta_{22}} + \alpha(y)^2 e^{\eta_{11} + \eta_{21}} \\
\frac{\lambda}{\lambda + e^{\eta_{11}}} + e^{\eta_{22}} + \alpha(y)^2 e^{\eta_{11} + \eta_{21}}
\end{array} \right). 
\end{align*}
$$

(5.21)  

(5.22)  

(5.23)  

(5.24)

6 Conclusion

We extend the matrix KP hierarchy by introducing a new $\tau_B$ flow and adding eigenfunctions and adjoint eigenfunctions as new components. The zero curvature equation and Lax representation for the extended matrix KP hierarchy and its $t_A$-reduction and $\tau_B$-reduction are presented. The extended matrix KP hierarchy and its two reductions provide an unified way to find two types of $(2+1)$-dimensional and $(1+1)$-dimensional AKNS equation (and DS equation) with self-consistent sources. With the combination of dressing method and the method of variation of constants, we propose a generalized dressing method to solve the extended matrix KP hierarchy and obtain some of its solutions. The soliton solution of two types of $2+1$ AKNS equation with self-consistent sources and two types of DS equation with self-consistent sources are studied.

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