Uniform CSP Parameterized by Solution Size is in $W[1]$

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We show that the uniform Constraint Satisfaction Problem (CSP) parameterized by the size of the solution is in $W[1]$ (the problem is $W[1]$-hard and it is easy to place it in $W[3]$). We study the problem on the Boolean domain, that is $\{0, 1\}$. The size of a solution is the number of variables that are mapped to 1. Named by Kolaitis and Vardi (2000), uniform CSP means that the input contains the domain and the list of tuples of each relation in the instance. Uniform CSP is polynomial time equivalent to homomorphism problem and also to evaluation of conjunctive queries on relational databases. It also has applications in artificial intelligence.

We do not restrict the problem to any (finite or infinite) family of relations. Uniform CSP restricted to some finite family of Boolean relations (thus with a bound on the arity of relations) can easily be placed in $W[1]$. Marx (2005) proved that any such problem is either $W[1]$-complete or fixed parameter tractable. Together with Bulatov (2014) they extended this result to any finite domain.

Our proof gives a nondeterministic RAM program with special properties deciding the problem. First defined by Chen et al. (2005), such programs characterize $W[1]$.

Our work builds upon the work of Cesati (2002), which, answering a longstanding open question, shows that parameterized Exact Weighted CNF is in $W[1]$. This problem is equivalent to uniform CSP restricted to a specific (infinite) family of Boolean relations, where a tuple is in a relation, if and only if the tuple has exactly one 1. Thus, our result generalizes that of Cesati in at least two ways: The size of the tuples in the relation are not bounded, and the relations do not need to be symmetric.

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1 Introduction

An instance \( I \) of Constraint Satisfaction Problem (CSP) is specified by a domain \( D \), a set of relations over domain \( D \), a set \( V \) of variables, and a set of constraints \( C \) of the form \( R(x_1, \ldots, x_r) \), where \( R \) is one of the relations with arity \( r \geq 1 \) and \( x_1, \ldots, x_r \in V \).

An assignment to a set of variables \( S \) is a mapping from \( S \) to \( D \). An assignment to the set of variables of a constraint satisfies the constraint if evaluating the tuple of variables of the constraint according to the assignment, gives a tuple in the corresponding relation. An assignment to \( V \) is a satisfying assignment of \( I \), if it satisfies all the constraints in \( I \).

When seen as a decision problem, the question in CSP is whether the given instance has a satisfying assignment. On Boolean domains, that is \( \{0, 1\} \), the size of an assignment is the number of variables that are mapped to 1.

Kolaitis and Vardi [7] made a distinction between nonuniform CSP, where the domain and the family of relations are fixed, and uniform CSP, where the input contains the domain and the list of tuples of each relation in the instance. They showed that uniform CSP is polynomial time equivalent to evaluation of conjunctive queries on relational databases. Feder and Vardi [5] observed that uniform CSP (which had already applications in artificial intelligence) and the homomorphism problem are polynomial time equivalent.

We study uniform CSP on Boolean domain in the settings of parameterized complexity. Parameterized Weighted CSP is defined as follows:

\[ p\text{-WCSP} \]

**Instance:** A set of variables \( V \), a set of Boolean relations each with all the tuples in it, a set of constraints, and \( k \geq 0 \).

**Parameter:** \( k \).

**Problem:** Decide whether there is a satisfying assignment of size \( k \).

Many parameterized problems ask, given a structure \( A \) (on universe \( A \)), if there is a set \( S \subset A \) of a given cardinality (the parameter) such that the substructure induced by \( S \) has a special property. Many of these problems can be readily reduced to \( p\text{-WCSP} \), such that the size parameter in the resulting \( p\text{-WCSP} \) instance has the same value as the cardinality of the set looked for. A good example is \( p\text{-Clique} \), which, given an instance \((G, k)\), asks if Graph \( G \) has a clique of size \( k \). \( p\text{-Vertex-Cover} \) is another example. These problems, however, can be expressed with \( p\text{-WCSP} \) restricted to a finite family of relations. To capture the full expressiveness of \( p\text{-WCSP} \) on Boolean domain, we introduce the following problem which is fixed parameter equivalent to \( p\text{-WCSP} \):
p-W-Hypergraphs-Hitting-Set

Instance: A set \( W \), hypergraphs \( (V_1, E_1), \ldots, (V_m, E_m) \) where \( V_i \subseteq W \), and \( k \geq 0 \).

Parameter: \( k \).

Problem: Decide whether there is a set \( S \subset W \) of cardinality \( k \) such that \( S \cap V_i \in E_i \) for \( 1 \leq i \leq m \).

Our main contribution is the following containment theorem:

**Theorem 1.1.** \( p\text{-WCSP} \in \text{W}[1] \).

**Corollary 1.2.** \( p\text{-W-Hypergraphs-Hitting-Set} \in \text{W}[1] \).

We prove the theorem by giving a tail-nondeterministic \( \kappa \)-restricted NRAM program (explained in the next section) deciding the problem. The significance of our containment result is that it is for the general problem, without restricting it to any (finite or infinite) family of relations.

Our work builds upon the work of Cesati [2], which, answering a longstanding open problem, proved that \( p\text{-Exact-Weighted-CNF} \) is in \( \text{W}[1] \). Downey and Fellow [4] had already shown that this problem is \( \text{W}[1] \)-hard and had conjectured that it either represents a natural degree intermediate between \( \text{W}[1] \) and \( \text{W}[2] \), or is complete for \( \text{W}[2] \). This problem is to decide, given a CNF without negation symbols and a natural number \( k \), whether there is an assignment of size \( k \), such that exactly one variable in each clause is mapped to 1. This can be seen as \( p\text{-WCSP} \) restricted to a specific (infinite) family of Boolean relations, where a tuple is in a relation, if and only if the tuple has exactly one 1 (this implies that \( p\text{-WCSP} \) is \( \text{W}[1] \)-hard). Notice that because we do not restrict the problem to any family of relations, our result generalizes that of Cesati in at least two ways: The size of the tuples in the relation are not bounded, and the relations do not need to be symmetric (symmetric means that a tuple being in the relation depends only on the number of 1s in the tuple).

In fact, \( p\text{-Exact-Weighted-CNF} \) is an example of an interesting special case of our containment result: \( p\text{-WCSP} \) restricted to any (infinite) family of symmetric relations, provided that there is a bound on the size of the tuples of any relation in the family. Notice that the bound implies that the number of tuples of each relation is bounded by a polynomial in the arity of the relation. Thus, listing all the tuples in the input makes the size of input at most polynomially bigger, and uniform and nonuniform CSP in this case have the same complexity.

It is not hard to see that \( p\text{-WCSP} \) is in \( \text{W}[3] \), by reducing it to the problem of \( p\text{-Weighted-Circuit-Satisfiability} \) for a class of circuits with bounded depth and weft 3: One for the conjunction of all constraints in the instance, one for disjunction of all satisfying assignments of each constraint, and one to specify each satisfying assignment of a constraint. So what is the significance of placing a problem from \( \text{W}[3] \) down to \( \text{W}[1] \)? First, although it is a fundamental conjecture that \( \text{W}[1] \)-complete problems are not fixed-parameter tractable, many of them can still be solved substantially faster than exhaustive search over all \( \binom{n}{k} \) subsets. For example, [9] gives an \( O(n^{799k}) \) time algorithm.
for $p$-Clique. In contrast, the $W[2]$-complete problem $p$-Dominating Set, was shown by [11] not to have such algorithms, unless CNF Satisfiability has an $O(2^{\delta n})$ time algorithm for some $\delta < 1$, which is an important open problem. Second, we can express the problems in $W[1]$ by a logic that is (conjectured to be) a proper subclass of that of $W[3]$ problems. This means that putting a problem in $W[1]$ decreases the descriptive complexity of the problem.

It is easy to see that $p$-WCSP restricted to some finite family of Boolean relations (implying the arity of relations is bounded) is in $W[1]$. Notice that listing the tuples of all relations in the input adds just a constant to the size of input. Thus, uniform and nonuniform CSP in this case have the same complexity. These problems are studied by Marx [8], where he provides a dichotomy: If the family of relations has a property that he calls weak-separability, then the problem is fixed-parameter tractable (like $p$-Vertex-Cover), otherwise it is $W[1]$-complete (like $p$-Clique). This result is extended by Bulatov and Marx [1] to any finite domain.

For the basic concepts, definitions and notation of the parameterized complexity theory, we refer the reader to [6].

2 A Machine Characterization of $W[1]$

We use a nondeterministic random access machine model. It is based on a standard deterministic random access machine (RAM) model. Registers store nonnegative integers. Register 0 is the accumulator. The arithmetic operations are addition, subtraction (cut off at 0), and division by two (rounded off), and we use a uniform cost measure. For more details see [6]. We define a nondeterministic RAM, or NRAM, to be a RAM with an additional instruction “GUESS” whose semantics is:

Guess a natural number less that or equal to the number stored in the accumulator and store it in the accumulator.

Acceptance of an input by an NRAM program is defined as usually for non-deterministic machines. Steps of a computation of an NRAM that execute a GUESS instruction are called nondeterministic steps.

Definition 2.1. Let $\kappa : \Sigma^* \to \mathbb{N}$ be a parameterization. An NRAM program $P$ is $\kappa$-restricted if there are computable functions $f$ and $g$ and a polynomial $p(X)$ such that on every run with input $x \in \Sigma^*$ the program $P$

- performs at most $f(k) \cdot p(n)$ steps, at most $g(k)$ of them being nondeterministic;
- uses at most the first $f(k) \cdot p(n)$ registers;
- contains numbers $\leq f(k) \cdot p(n)$ in any register at any time.

Here $n := |x|$, and $k := \kappa(x)$.

Definition 2.2. A $\kappa$-restricted NRAM program $P$ is tail-nondeterministic if there is a computable function $h$ such that for every run of $P$ on any input $x$ all nondeterministic steps are among the last $h(\kappa(x))$ steps of the computation.

The machine characterization of $W[1]$ reads as follows:
Theorem 2.3 ([3]). Let \((Q, \kappa)\) be a parameterized problem. Then \((Q, \kappa) \in W[1]\) if and only if there is a tail-nondeterministic \(\kappa\)-restricted NRAM program deciding \((Q, \kappa)\).

3 Containment in \(W[1]\)

First, we describe the following transformation, taken from Papadimitriv and Yannakakis [10]. Given an instance \(I\) of \(p\)-WCSP with the set of variables \(V\) and parameter value \(k\), we construct a second instance \(I'\) with the same set of variables and the same parameter value, such that \(I\) and \(I'\) have the same set of satisfying assignments of size \(\leq k\). Furthermore, each variable appears in each constraint at most once, and for each subset \(S \subseteq V\), there is at most one constraint with this set of variables, thus each constraint is characterized by its set of variables.

Because our domain is Boolean, henceforth we characterize an assignment with its support set, that is the set of variables that are mapped to 1.

Fix an order on \(V\). For each subset \(S \subseteq V\), if \(I\) has a constraint such that the set of variables of the constraint are exactly \(S\) (possibly with repetitions), then \(I'\) has Relation \(R_S\) of arity \(|S|\) and Constraint \(R_S(S)\) defined as follows. Each \(A \subseteq S\) is a satisfying assignment of \(R_S(S)\) if and only if \(|A| \leq k\) and \(A\) is a satisfying assignment of every constraint \(C\) in \(I\), such that the set of variables of \(C\) are exactly \(S\) (possibly with repetitions). The order of variables in \(R_S(S)\) is determined by the order on \(V\). Relation \(R_S\) is defined accordingly. Notice that there is a natural bijective mapping of the tuples in \(R_S\) to the satisfying assignments of \(R_S(S)\).

For \(U \subseteq S\), we write \(U \in R_S(S)\), to mean that \(U\) is (the support set of) a satisfying assignment of \(R_S(S)\).

For \(R_S(S)\) a constraint in \(I'\), define \(\tilde{R}_S(S) \subseteq 2^S\) as a set with exactly the following elements:

| \(\emptyset\) if \(\emptyset \notin R_S(S)\), and |
| \(W \cup \{v\}\) for all \(W \in R_S(S)\), for all \(v \in S\), such that \(W \cup \{v\} \notin R_S(S)\). |

Fact 3.1. \(|\tilde{R}_S(S)| \leq |S|(|R_S(S)| + 1)|.

For \(T \in \tilde{R}_S(S)\) and \(T \subset W \subseteq S\), define

\[ R_S(S|T, W) := \{ U \in R_S(S) | T \subset U \subseteq W \}. \] (1)

The main idea of the proof of Theorem 3.4 is the following lemma.

Lemma 3.2. Let \(W \subseteq S\). \(W \in R_S(S)\) if and only if \(R_S(S|T, W) \neq \emptyset\), for all \(T \in \tilde{R}_S(S)\) such that \(T \subseteq W\).

Proof. The forward direction follows from (1). We give a contrapositive proof for the backward direction. Suppose \(W \notin R_S(S)\). If there exists a maximal subset \(Z \subset W\) such that \(Z \in R_S(S)\), then there is \(v \in W\) such that \(Z \cup \{v\} \notin R_S(S)\). Thus, \(Z \cup \{v\} \subseteq W\), \(Z \cup \{v\} \in R_S(S)\) and \(R_S(S|Z \cup \{v\}, W) = \emptyset\). Otherwise, we have \(\emptyset \notin R_S(S)\) (thus \(\emptyset \in \tilde{R}_S(S)\)) and \(R_S(S|\emptyset, W) = \emptyset\). \(\square\)
And a variant of the above:

**Lemma 3.3.** Let $Q \subset W \subseteq S$. If $Q \in R_S(S)$ and $W \not\in R_S(S)$, then $R_S(S|T,W) = \emptyset$ for some $T \in \tilde{R}_S(S)$ such that $Q \subset T \subseteq W$.

**Proof.** Adapt the proof of the backward direction of Lemma 3.2, by adding the requirement that $Z \supseteq Q$. 

Our containment theorem reads as follows:

**Theorem 3.4.** $p$-WCSP $\in W[1]$.

**Proof.** We give a tail-nondeterministic $\kappa$-restricted NRAM program $Q$ deciding the problem. The result follows by Theorem 2.3. Let $I$ be the given instance with the set of variables $V$ and the parameter value $k$. Program $Q$ first constructs Instance $I'$ from Instance $I$ as described above. This can easily be done in polynomial time. Let

$$S := \{S \subseteq V | R_S(S) \text{ is in } I'\},$$

$$T := \{T | T \in \tilde{R}_S(S) \text{ for some } S \in S\}.$$

Next, $Q$ calculates two tries:

- Trie 1 stores the values $d[T] := |D_T|$ for all $T \in T$, where
  $$D_T := \{S \in S | T \in \tilde{R}_S(S)\}.$$

- Trie 2 stores the values

  $$l[T,U] := \sum_{S \in D_T} L_{T,S}(U),$$

  for all $T \in T$, and every $U$ such that

  $$U \in R_S(S|T,S) \text{ for some } S \in D_T,$$

  and using the recursive definition

  $$L_{T,S}(U) := 1 - \sum_{W \in R_S(S|T,S)} L_{T,S}(W) \quad \text{if } U \in R_S(S|T,S),$$

  and $L_{T,S}(U) := 0$ otherwise. The number of keys stored in Trie 1 and Trie 2 is bounded by a fixed polynomial in $n$ (size of $I$) and the stored values are of polynomial size. To verify this, remember that for each relation in $I$, all the tuples in the relation are listed in the input. Thus $R_S(S)$ has at most $n$ satisfying assignments, for all $S \in S$. It follows that the summation in the recursive definition (3) has at most $n$ summands, and moreover, $|T|$ is bounded by a fixed polynomial in $n$.

The tries are arranged such that for all $T,U \subset V$ of size $|T|, |U| \leq k$, the queries for $d[T]$ and $l[T,U]$ is answered in $O(2k)$ time (a general property of the trie data structure).
and for nonexistent keys 0 is returned. The tries can easily be computed in polynomial time.

Now the nondeterministic part of the computation starts: Program Q guesses \( k \) variables in \( V \) and checks they are distinct. Let \( B \) be the set of \( k \) variables. Then \( Q \) iterates over subsets \( T \subseteq B \), and if \( d(T) > 0 \), checks if

\[
\sum_{T \subseteq W \subseteq B} l[T, W] = d[T].
\]  

The summation in Check (4) has at most \( 2^{2k} \) summands, thus Program Q is tail-nondeterministic.

We should show now that Q actually decides \( p \)-WCSP. By construction, \( I \) and \( I' \) have the same set of satisfying assignments of size \( \leq k \). Thus, it is enough to prove that on a computation branch of Program Q, where the set of guessed variables \( B \) is of size \(|B|=k\), the following four statements are equivalent:

(i) \( B \) is a satisfying assignment of \( I' \), that is, \( B \) satisfies all the constraints in \( I' \).
(ii) (By Lemma 3.2) For all \( S \in S \), for all \( T \in R_S(S) \) such that \( T \subseteq B \),

\[
R_S(S|T, B \cap S) \neq \emptyset.
\]

(iii) (By swapping the universal quantifiers) For all \( T \in T \) such that \( T \subseteq B \),

\[
\text{for all } S \in D_T, \text{ we have } R_S(S|T, B \cap S) \neq \emptyset.
\]  

(iv) For all \( T \in T \) such that \( T \subseteq B \), Check (4) is passed.

The equivalence of (i), (ii) and (iii) should be evident. The rest of the proof is to show that (iii) and (iv) are equivalent.

**Claim 1.** Let \( T \in T \) and \( T \subseteq B \). Suppose either \( R_S(S|T, B) \) is empty or it has a maximum with respect to set inclusion, for all \( S \in D_T \). Then (5) holds for \( T \) if and only if Check (4) is passed for \( T \).

**Proof.** We rewrite the summation in (4) as

\[
\sum_{T \subseteq W \subseteq B} l[T, W] = \sum_{T \subseteq W \subseteq B} \sum_{S \in D_T} L_{T,S}(W) \tag{by (2)}
\]

\[
= \sum_{S \in D_T} \sum_{W \in R_S(S|T, B \cap S)} L_{T,S}(W) \tag{by the Fubini’s Principle}.
\]

To complete the proof, we show that the inner summation (as a function of \( S \)) is the **indicator function** for the statement \( R_S(S|T, B \cap S) \neq \emptyset \). That is, for all \( S \in D_T \),

\[
\sum_{W \in R_S(S|T, B \cap S)} L_{T,S}(W) = \begin{cases} 
1, & R_S(S|T, B \cap S) \neq \emptyset, \\
0, & \text{otherwise}.
\end{cases}
\]
In the second case, the summation is empty, thus it sums to 0. In the first case, by hypothesis, the set \( R_S(S|T, B \cap S) \) has a maximum, \( U \), with respect to inclusion. Thus, we have

\[
\sum_{W \in R_S(S|T, B \cap S)} L_{T,S}(W) = L_{T,S}(U) + \sum_{W \subset U \cap R_S(S|T, B \cap S)} L_{T,S}(W).
\]

Then, by plugging (3) for \( L_{T,S}(U) \), it all sums to 1, as desired.

Now, if all \( T \in T \) such that \( T \subseteq B \) satisfy the hypothesis of Claim 1, then (by Claim 1) (iii) and (iv) are equivalent. Otherwise, by Claim 2 below, there is \( T' \in T \) that satisfies the hypothesis of Claim 1, and (5) does not hold for \( T' \) (implying (iii) is false). But then (by Claim 1) Check (4) fails for \( T' \), meaning that (iv) is false. Thus (iii) and (iv) are equivalent. This completes the proof.

Claim 2. Suppose \( R_S(S|T, B \cap S) \) is not empty and it does not have a maximum, for some \( S \in D_T \). Then there is a \( T' \in T \) such that \( T' \subseteq B \) and

- \( R_{S'}(S'|T', B \cap S') = \emptyset \) for some \( S' \in D_{T'} \), and
- either \( R_{S''}(S''|T', B \cap S'') \) is empty or it has a maximum, for all \( S'' \in D_{T'} \).

Proof. We have \( B \cap S \notin R_S(S) \). Otherwise, \( B \cap S \) is the maximum of \( R_S(S|T, B \cap S) \), a contradiction.

Now, let \( U \in R_S(S|T, B \cap S) \). By Lemma 3.3, there is \( T_2 \in R_S(S) \) such that \( U \subseteq T_2 \subseteq B \cap S \) and \( R_S(S|T_2, B \cap S) = \emptyset \). Notice that \( T \subseteq U \subseteq T_2 \), thus we have \(|T_2| > |T|\).

If \( R_{S''}(S''|T_2, B \cap S'') \) is empty or has a maximum, for all \( S'' \in D_{T_2} \), then set \( T' \coloneqq T_2 \) (and \( S' \coloneqq S \)). Otherwise, we recursively apply the above argument to a counterexample \( S_2 \neq S \) (and \( T_2 \)), getting sets \( S_i \) and \( T_{i+1} \). Now, because \(|T_{i+1}| > |T_i|\) and \( T_i \subseteq B \) for all \( i \), this recursion ends in \( j \leq |B| + 1 \) steps. Then, set \( T' \coloneqq T_j \) (and \( S' \coloneqq S_{j-1} \)).

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