Bifurcation and chaos in zero-Prandtl-number convection

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Abstract – We present a detailed bifurcation structure and associated flow patterns near the onset of zero-Prandtl-number Rayleigh-Bénard convection. We employ both direct numerical simulation and a low-dimensional model ensuring qualitative agreement between the two. Various flow patterns originate from a stationary square observed at a higher Rayleigh number through a series of bifurcations starting from a pitchfork followed by a Hopf and finally a homoclinic bifurcation as the Rayleigh number is reduced to the critical value. Homoclinic chaos, intermittency, and crises are observed near the onset.

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Thermal convection is observed almost everywhere in the universe: industrial appliances, liquid metals, atmosphere, oceans, interiors of planets and stars, galaxies etc. An idealized version of convection called Rayleigh-Bénard convection (RBC) has been studied for almost a century and it is still an area of intense research \cite{1}. The two most important parameters characterizing convection in RBC are the Rayleigh number, describing the vigour of buoyancy, and the Prandtl number, being the ratio of kinetic viscosity and thermal diffusivity. Solar \cite{2} and geological flows \cite{3} are considered to have very low Prandtl numbers, as do flows of liquid metals \cite{4}. RBC exhibits a wide range of phenomena including instabilities, patterns, chaos, spatio-temporal chaos, and turbulence for different ranges of Rayleigh number and Prandtl number \cite{1}. The origin of instabilities, chaos, and turbulence in convection is one of the major research topics of convection.

Direct numerical simulation (DNS), due to its high dimensionality, generates realistic but excessively voluminous numerical outputs which obscure the underlying dynamics. Lower-dimensional projections lead to models which, if done improperly, lose the overall physics. In this letter, our aim is to unfold and discover the underlying physics of low-Prandtl-number flows \cite{5} by examining the natural limit of zero Prandtl number (zero-P) \cite{6–12}. This offers a dramatic simplification without sacrificing significant physics, as well as displays a fascinatingly rich dynamic behaviour. In particular, since zero-P flows are chaotic immediately upon initiation of convection, we adopt a nonstandard strategy of approaching this system from the post-bifurcation direction. Moreover, we attack the problem simultaneously with DNS (to ensure accuracy) as well as a low-dimensional model (to aid physical interpretation); and we stringently refine both the model and DNS until satisfactory agreement is obtained at all levels of observed behaviour. Our results show a diverse variety of both new and previously observed flow patterns. These flow patterns emerge as a consequence of various bifurcations ranging from pitchfork, Hopf, and homoclinic bifurcations to bifurcations involving double zero eigenvalues.

Convection in an arbitrary geometry is quite complex, so researchers have focused on Rayleigh-Bénard convection wherein the convective flow is between two conducting parallel plates \cite{1}. The fluid has kinematic viscosity $\nu$, thermal diffusivity $\kappa$, and coefficient of volume expansion $\alpha$. The top and bottom plates are separated by distance $d$, and they are maintained at temperatures $T_2$ and $T_1$, respectively, with $T_1 > T_2$. The convective flow in RBC is characterized by the Rayleigh number $R = \alpha(T_1 - T_2)gd^3/\nu\kappa$, where $g$ is the acceleration due to gravity, and the Prandtl number $P = \nu/\kappa$. Various instabilities, patterns, and chaos are observed for different ranges of $R$ and $P$ \cite{1,6,13}. Transition to chaotic states through various routes have been reported in convection \cite{14,15}.
In this letter, we focus on zero-P convection. The governing zero-P Boussinesq equations [7] are nondimensionalized using $d$ as length scale, $d^2/\nu$ as time scale, and $\nu(T_1 - T_2)/\kappa$ as temperature scale which yields

$$
\partial_t (\nabla^2 v_3) = \nabla^4 v_3 + R \nabla^2 \theta,
$$

$$
-\hat{e}_3 \cdot \nabla \times [ (\omega \cdot \nabla) v - (v \cdot \nabla) \omega],
$$

(1)

$$
\partial_t \omega_3 = \nabla^2 \omega_3 + [ (\omega \cdot \nabla) v_3 - (v \cdot \nabla) \omega_3],
$$

(2)

$$
\nabla^2 \theta = -v_3,
$$

(3)

$$
\nabla \cdot v = 0,
$$

(4)

where $v \equiv (v_1, v_2, v_3)$ is the velocity field, $\theta$ is the deviation in the temperature field from the steady conduction profile, $\omega = \nabla \times v$ is the vorticity field, $\hat{e}_3$ is the vertically directed unit vector, and $\nabla^2_H = \partial_{xx} + \partial_{yy}$ is the horizontal Laplacian. We consider perfectly conducting and free-slip boundary conditions at the top and bottom plates, and periodic boundary conditions along the horizontal directions [6,15]. In the following discussion we also use the reduced Rayleigh-number $r = R/R_c$, where $R_c$ is the critical Rayleigh number.

Straight two-dimensional (2D) rolls that have zero vertical vorticity are neutrally stable solution of zero-P convection at $r = 1$. However they become unstable for $r > 1$. Busse [9], Thual [6] and Kumar et al. [8] showed that these 2D rolls saturate through generation of vertical vorticity (wavy rolls) for $r > 1$ both for low-Prandtl-number and zero-P fluids. Thus vorticity plays a critical role in zero-P convection.

Herring [10] was first to simulate these equations under the free-slip boundary conditions. However he observed divergence of the solutions possibly due to the instabilities described above. The first successful simulation of zero-P equations with free-slip boundary conditions was done by Thual [6]. He reported many interesting flow patterns including relaxation oscillation of square patterns (SQOR) and stationary square patterns (SQ). Later Knobloch [11] studied the stability of the SQ patterns using amplitude equations. Pal and Kumar [12] explained the mechanism of selection of the square patterns using a 15-dimensional
Flow patterns | $r$ (Model) | $r$ (DNS)
--- | --- | ---
Chaotic | 1–1.0045 | 1–1.0048
SQOR | 1.0045–1.0175 | 1.0048–1.0708
OASQ | 1.0175–1.0703 | 1.0709–1.1315
ASQ | 1.0703–1.2201 | 1.1316–1.2005
SQ | 1.2201–1.4373 | 1.2006–1.4297

Table 1: Range of reduced Rayleigh number $r$ corresponding to various flow patterns observed in the model and the DNS. Here SQ, ASQ, OASQ, and SQOR represent stationary squares, stationary asymmetric squares, oscillatory asymmetric squares, and relaxation oscillation of squares, respectively.

The origin of the above flow patterns can be nicely understood using the bifurcation diagram of the low-dimensional model. To generate the bifurcation diagram, we first evaluate a fixed point numerically using the Newton-Raphson method for a given $r$. The branch of the fixed points is subsequently obtained using a fixed arc-length based continuation scheme [17]. Stability of the fixed points is ascertained through an eigenvalue analysis of the Jacobian and accordingly the bifurcation points are located. New branches of fixed points are born when the eigenvalue(s) become zero (pitchfork), and branches of periodic solutions appear when the eigenvalue(s) become purely imaginary (Hopf). Subsequent branches are generated by calculating and continuing the new steady solutions close to the bifurcation points.

Fig. 2: (Colour on-line) Bifurcation diagram of the model for $0.95 \leq r \leq 1.4$. The stable branches corresponding to stationary squares (SQ) and stationary asymmetric squares (ASQ) are represented by solid black and solid blue lines, respectively. Red, green, and brown curves represent the extrema of oscillatory asymmetric squares (OASQ), relaxation oscillation of squares (SQOR), and chaotic solutions, respectively. A zoomed view of the bifurcation diagram for the chaotic regime is shown in the inset. In the inset, the $x$-axis is chosen as $\log(r−1)$ to highlight the behaviour near $r = 1$. Branches corresponding to the unstable fixed points are represented by dashed lines. Cyan line represents the conduction state.
Fig. 3: (Colour on-line) Phase space projections of fixed points on the $W_{101}$-$W_{011}$ plane for (a) $r = 1.4$ and (b) $r = 1.15$. The cyan square, black filled circles, and blue triangles represent the conduction fixed point, SQ fixed points, and ASQ fixed points, respectively.

Fig. 4: (Colour on-line) Phase space projections of limit cycles on the $W_{101}$-$W_{011}$ plane for (a) $r = 1.0494$ and (b) $r = 1.0099$. The limit cycles in (a) merge to form a single limit cycle in (b). Black dots indicate the symmetric square saddle.

We start our analysis at $r = 1.4$ where we observe stable symmetric squares (SQ) corresponding to the black curve in fig. 1. The state-space projection on $W_{101}$-$W_{011}$ plane for $r = 1.4$ is shown in fig. 3(a). In this figure SQ fixed points are represented by the filled circles. In fig. 2 we represent only the $W_{101} = W_{011}$ solution. As $r$ is reduced from $1.4$, the SQ branch of fixed points loses stability via a supercritical pitchfork bifurcation at $r \approx 1.2201$, after which we observe stationary solutions with $W_{101} \neq W_{011}$ (blue curves of figs. 1 and 2). These solutions correspond to asymmetric square patterns (ASQ), either dominant along the $x$ axis ($|W_{101}| > |W_{011}|$), or dominant along the $y$ axis ($|W_{101}| < |W_{011}|$). These states are represented by filled triangles in fig. 3(b) for $r = 1.15$. The SQ solution $|W_{101}| = |W_{011}|$ continues as a saddle. With a further reduction of $r$, ASQ branches lose stability through a supercritical Hopf bifurcation at $r \approx 1.0703$ and limit cycles are born. These limit cycles are represented by red curves in fig. 2. Physically they represent oscillatory asymmetric square patterns (OASQ). Figure 4(a) illustrates the projection of two of these stable limit cycles (for $r = 1.0494$) on the $W_{101}$-$W_{011}$ plane.

The limit cycles grow in size as $r$ is lowered. A homoclinic orbit is formed at $r \approx 1.0175$. Afterwards, homoclinic chaos is observed in a narrow window. At lower $r$ the attractor becomes regular resulting in a larger limit cycle that corresponds to the relaxation oscillations with an intermediate square pattern (SQOR). Figure 4(b) illustrates the projection of this limit cycle at $r = 1.0099$. The flow pattern in this regime changes in time from

Fig. 5: (Colour on-line) The three different chaotic solutions observed near $r = 1$: Ch1 at $r = 1.0041$ for the model (a) and at $r = 1.0045$ in DNS (b); Ch2 at $r = 1.0038$ for the model (c) and at $r = 1.0030$ in DNS (d); Ch3 at $r = 1.0030$ for the model (e) and at $r = 1.0023$ in DNS (f). These solutions belong to (i), (ii), and (iii) regimes in the bifurcation diagram (fig. 2).

Fig. 6: (Colour on-line) (a) The three-dimensional projection of the Ch3 orbit on the ($W_{101}$, $W_{011}$, $W_{013}$) space for $r = 1.0030$. This solution belongs to the (ii) regime in the bifurcation diagram (fig. 2). (b) The first return map of the variable $W_{011}$ on the Poincaré plane $W_{011} = 25$ shown in (a).
an approximate pure roll in one direction to a symmetric square, and then to an approximate pure roll in the perpendicular direction. The SQOR solution is represented by the green curve in fig. 2.

The flow becomes chaotic as $r \to 1$. The chaotic flow manifests itself in three different forms: Ch1, Ch2, and Ch3 as shown in the inset of fig. 2 as (i), (ii), and (iii), respectively. The phase space projection for these three solutions are depicted in fig. 5 for $r = 1.0041, 1.0038$ and $1.0030$ for the 13-mode model, and for $r = 1.0045, 1.0032$ and 1.0023 in the DNS. A three-dimensional projection of the Ch3 attractor is shown in fig. 6(a). The first return map of $W_{101}$ for the Poincaré plane $W_{011} = 25$ is shown in fig. 6(b). A scatter in this map indicates the chaotic nature of the attractor. We also observe these chaotic attractors in DNS albeit at different $r$ values. The state space projections corresponding to DNS are shown in fig. 5(b), (d), (f).

In fig. 7 we plot the time series for the three types of chaotic attractors for the values of $r$ given in fig. 5 using random initial conditions. In each subplot, the solid and dotted lines represent two time series generated using two initial conditions that differ by $\Delta W_{101} = 10^{-6}$. A clear divergence between the two solutions (sensitivity to initial conditions) shows that the attractors are chaotic. The power spectra of the above time series are broadband as shown in fig. 8. This result corroborates with the earlier conclusions on the chaotic nature of the attractor.

The time series for the $W_{101}$ mode of the Ch1 attractor appears to be periodic visually, but the power spectra and the sensitivity to initial conditions indicate its chaotic nature. The Poincaré first return map of the attractor also shows scatter similar to that in Ch3 shown in fig. 6(b).

In the second time series (for the Ch2 attractor), the modes $W_{101}$ switches sign intermittently, which is due to the transit of the system from one quadrant to another in the state space projection shown in fig. 5. For the Ch3 attractor, the time series of the mode $W_{101}$ has a very similar feature. The time series of the Ch2 and Ch3 attractors differ when we plot $W_{011}$ as evident from state space plots of fig. 5.

We investigate the origin of the attractors Ch1, Ch2, and Ch3. The attractor Ch1 originates from homoclinic
chaos associated with the saddle corresponding to the dashed line originating from SQ. The attractor Ch1 however is quite thin, and its largest Lyapunov exponent is very small but positive. Note that there are four Ch1 attractors due to symmetry. As r is reduced, the four attractors collide simultaneously with their respective basin boundaries and yields a larger chaotic attractor Ch2 of fig. 5(c) [19]. This is the “attractor merging crisis”. The resulting dynamics is intermittent (crisis induced intermittency [19]), as exhibited by the time series (see fig. 7). As r is reduced further, another crisis occurs when the Ch2 attractor breaks into four small attractors Ch3 of fig. 5(e). Note that the nature of these three chaotic attractors are quite different. With a further reduction in the Rayleigh number, the size of these chaotic attractors decreases and they ultimately merge with one of the branches of the unstable ASQ fixed points at $r = 1$. In fig. 2, we exhibit the merger of one of these chaotic attractors with the unstable ASQ fixed point with $W_{101} \approx 26.4$.

In conclusion, we present for the first time a numerically obtained, DNS validated, detailed bifurcation diagram and associated flow structures of zero-P convective flow near the onset of convection. The whole spectrum of phenomena observed in DNS near the onset of convection is replicated by the low-dimensional model. Hence, the bifurcation structure presented here explains the origin and dynamics of various patterns near the onset of convection. Recent analysis of VKS (Von-Karman-Sodium) experimental results indicate a strong role of large-scale modes for the magnetic field reversal [20]. A study of large-scale modes as outlined in this letter may provide useful insights into the mechanism behind the generation and reversal of magnetic field. The dynamics of large-scale modes in other hydrodynamic systems like rotating turbulence, magneto-convection etc. could also be captured by a similar approach.

In this paper we have performed analysis for zero-P convection for $1 \leq r \leq 1.4$. The bifurcation analysis for $r > 1.4$ is reasonably complex, and it will be reported later. In addition, preliminary results show a reasonable amount of similarity between low-Prandtl-number convection and zero-P convection. These issues are under investigation.

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