Similarity Analysis of Nonlinear Equations and Bases of Finite Wavelength Solitons

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Abstract

We introduce a generalized similarity analysis which grants a qualitative description of the localised solutions of any nonlinear differential equation. This procedure provides relations between amplitude, width, and velocity of the solutions, and it is shown to be useful in analysing nonlinear structures like solitons, doublets, triplets, compact supported solitons and other patterns. We also introduce kink-antikink compact solutions for a nonlinear-nonlinear dispersion equation, and we construct a basis of finite wavelength functions having self-similar properties.

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1 Introduction

Nonlinear dynamics, which is a relatively new field of study, is a very important frontier for probing natural phenomena. Active research efforts that focus on the mathematics and physics of nonlinear dynamical systems have emerged worldwide in various fields, including fluid dynamics, plasma physics, astrophysics, and even string theory [1]. Among its unique features is the ability to describe a variety of patterns [2] and particle-like traveling solutions [3]. Other notable features of the theory include its description of solitons and breather modes, features in quantum optics [4], molecular and solid state physics phenomena [5], and solitons in nuclear and particle physics [6].
In contrast with linear theories which exhibit smooth regular motion, nonlinear models require nonlinear partial differential equations (NPDE) and show strong couplings between different mechanisms and parts of the system. Also, the nonlinear interactions involve multiple scales [7] and are related with self-similar patterns or fractals. The NPDE solutions of physical interest are mostly localized and demonstrate good stability in time and through scattering with each other. Their shapes are related to the velocity, thus making the nonlinear patterns distinct from linear waves. In the asymptotic domain these solutions consist of isolated traveling pulses that are free of interactions. Close to the scattering domain, the nonlinear solutions obey nonlinear superposition principles.

The main challenge in any nonlinear analysis is the construction of localized or finitely supported analytical solutions for the NPDE of interest. This challenge includes issues regarding the inexistence of a superposition principle for such solutions. Recent examples show that the traditional nonlinear tools (inverse scattering, group symmetry, functional transforms) are not always applicable [8, 9]. On the other hand, from an experimental point of view one knows that patterns which are observed in nature – either stationary, growing, or propagating – generally have finite space-time extension and a multi-scale structure. Since soliton, even when they are localized, have an infinite extent, one needs other appropriate structures, and eventually self-similar bases.

In this paper, wavelet-inspired approaches for localized solutions of NPDE are explored. We propose a new similarity formalism for the qualitative analysis, and classification, of soliton solutions of nonlinear equations. This method provides relations between the characteristics of such solutions (amplitude, width and velocity) without the need of solving the corresponding NPDE. The method uses the multi-resolution analysis [10] where traditional tools like the Fourier integrals or linear harmonic analysis are inadequate for describing the system. Wavelets are functions that have a space-dependent scale which renders them an invaluable tool for analyzing multi-scale phenomena. Wavelets have been used in signal processing, in problems involving singular potentials, pattern recognition, image compression, turbulence and even in radar and acoustic problems [10, 11]. Moreover, the introduction of wavelet analysis in a study of NPDE is very natural because they can accommodate everything from strong variations, even singularities, to a smooth behavior.

The second purpose of the paper is to show an example of construction of a nonlinear basis for a NPDE with nonlinear dispersion. There are many physical reasons favoring wavelets in the construction of nonlinear bases. For example, the breakup process of fluid drops has been shown to be self-similar, and in particular, the singularities (necks) look
identical at any scale \cite{12}. Other nonlinear oscillations of liquid drops, shells, bubbles, or even neutron stars involve such types of behavior \cite{2, 9}. We introduce in section 2 a qualitative similarity analysis for NPDE that yields relations between the amplitude, width and velocity of their traveling solutions, with many examples and predictions. In section 3, we introduce a nonlinear wavelet-like frame, associated with localized analytic solutions of a modified KdV equation, with nonlinear dispersion.

2 Qualitative similarity analysis

Finding analytic solutions for the nonlinear equations which describe physical phenomena is rather an exception than the rule. The traditional nonlinear tools like the inverse scattering theory or functional transforms are not always applicable \cite{8, 16}. Also, when phenomena of interest have many space (or time) scales, hundred times less or smaller than the dimension (or time scale) of the whole system, numerical methods may fail. This is the case of sharp propagating perturbations developing discontinuities, or shock waves. A simple option is the expansion of solutions in a basis of appropriate chosen linear modes. The most common choice are the Fourier series which have the advantage of orthogonality, but can hardly discriminate local behavior of phenomena. Moreover, information about the order of magnitude of the Fourier coefficients of a signal or wave $u(x, t)$ is not sufficient for making conclusions about the size or scale of $u$.

In these special situations, the use of bases of multiresolution analysis and wavelets has become popular \cite{13}. We introduce a qualitative analysis for the localized traveling solutions belonging to any type of NPDE, in terms of Morlet continuous wavelet approach. In our case of interest, traveling localized perturbations, the most important information are provided by the scale of the pulse (or the width) denoted $L$, the amplitude $A$, and the group velocity $V$. The qualitative analysis in this section provides simple relations between these three parameters, without actually solving the equation. It also gives an estimation about the specific scale of the solutions.

The procedure consists in the substitution of all the following terms in the NPDE, according to the rule

$$u_t \rightarrow \pm V u_x, \quad u \rightarrow \pm A, \quad u_x \rightarrow \pm A/L, \quad u_{xx} \rightarrow \pm A/L^2 \ldots,$$

and so forth for higher order of derivatives. Consequently, the NPDE is mapped into an algebraic equation in $A, L$ and $V$. In Table 1 we present several examples of application of this substitution to some well-known and widely used NPDE in physics.
The validity of the method follows from the expansion of the soliton-like solution $u(x)$ in Morlet wavelets \[1\]

\[
\Psi_\alpha(x) = \pi^{-1/4} e^{-i\alpha x - \frac{x^2}{2}},
\]

where $\alpha$ describes the scale of this mother wavelet. The support of any Morlet wavelet is mainly confined in the $(-1,1)$ interval. We have the discrete Morlet wavelet expansion of $u$

\[
u(x) = \sum_j \sum_k C_{j,k} \Psi_\alpha(2^j x - k),
\]
in terms of integer translations and dyadic dilations of the mother wavelet. In order to reduce the number of scales needed, the range of the summation should be chosen with an eye to the underlying physics. We use in the following the asymptotic formula describing the pointwise behavior of the Morlet wavelet series around of a point $x_0$ of interest \[4\]. For a chosen $x_0$ and scale $j$, there is only one $k$ and $|\epsilon| \leq 1$ such that the support of the corresponding $\Psi_{\alpha,j,k}$ contains this point, $k = 2^j x_0 + \epsilon$. We can express the solution and its derivatives in a neighborhood of this point

\[
u(x_0) \approx \Psi(-\epsilon) \sum_j C_{j,2^j x_0 + \epsilon} \equiv \sum_j u_j(x_0),
\]

\[
\frac{du}{dx}(x_0) \approx -i\Psi(-\epsilon) \sum_j 2^j \alpha C_{j,2^j x_0 + \epsilon} = -i\alpha \sum_j 2^j u_j(x_0),
\]

for $\alpha$ chosen enough large compared to $\epsilon$. Since the coefficient $1/\alpha 2^j$ represents the scale for each $\Psi_{\alpha,j,k}$ Morlet wavelet, we can define it as a characteristic half-width $L_j$. And we finally have in $x_0$, from eqs.(3)

\[
\frac{d^nu}{dx^n}(x_0) \approx \sum_j \frac{u_j(x_0)}{L_j^n},
\]

where $u_j(x_0) = \Psi_\alpha(-\epsilon) C_{j,2^j x_0 + \epsilon} \approx \Psi_\alpha(0) C_{j,2^j x_0}$. Eq.(4) is the many scales generalization of the simpler formula in eq(1). With eq.(4) in hand we can investigate the structure of hypothetic soliton solutions, by choosing $x_0$ in the neighborhood of the maximum value of the solution, $u(x_0,0) = A$. Around this maximum, such solutions can be described very well by a unique scale $L$, and hence the solution and its derivatives can be approximated with the corresponding dominant term, by the substitutions in eqs.(1).
In Table 1 we present a series of examples of NPDE, identified in the first column by the name and the form of the equation. In the second column, we write the corresponding traveling localized solution, if an analytical form is known. Such solutions provide special relations between $L, A$ and $V$, which are given in the third column. In the last column we introduce for comparison the results of this Morlet qualitative analysis, that is the relations between the three parameters, provided by eqs.(1). The usefulness of the approach may be checked, by a quick comparison between the fourth and the fifth columns. While the results in the fourth column are possible only when one knows the analytical solutions, the results presented in the last column, obtained by the similarity approach, result directly from the NPDE, without actually solving it.

The first line in the Table presents a linear case, for comparative purposes. The above Morlet wavelet approximation provides a correct expression for the dispersion relation $(V = c \to k^2 = \omega^2/c^2)$ with no constraint on either the amplitude $A$ or on the width $L$.

The case of the KdV equation is described in the second row of the Table 1. The method gives a general expression for $L = L(A, V)$. For $L$ to be related to $A$ only, from the fourth column it results that the velocity $V$ must be proportional to $A$. In this case we obtain exactly the well-known relation (column three) among the parameters in the exact solution. A prediction of the method is if we allow $V$ to depend on a power of $A$. This means solutions with a higher nonlinear coupling between the shape and kinematics. A side effect would be a lower limit for $A$. Smaller solitons than this limit can move with velocity proportional to the amplitude only.

The same result is obtained for the MKdV equation (third row), except that in this case $V$ needs to be proportional to the square of $A$ in order to have $L$ a simple function of $A$ only. This prediction is again identical with that in the exact solution (third column). Moreover, the same relations remain valid even for the new exotic solutions of the MKdV equation of compacton type, [13]

$$u(x, t) = \frac{\sqrt{32k \cos[k(x - 4k^2t)]^2}}{3(1 - \frac{2}{3} \cos[k(x - 4k^2t)]^2)},$$

which has $L = 5\pi/6k$ or $\pi/6k$, that is $L \sim 1/A$ like in the Table 1.

Next example (4-th row) is provided by a generalised KdV equation, in which the dispersion term is quadratic

$$\eta_t + (\eta^2)_x + (\eta^2)_{xxx} = 0. \quad (5)$$
Eq. (5), known as K(2,2) from the two quadratic terms, admits compact supported traveling solutions, named compactons [8,16]. The compactons are powers of trigonometric functions defined on a half-period, and zero otherwise. In general, they have the form $A\cos^2 d(x - ct)$, and different from solitons, their width is independent of the amplitude. This is the fact that provides a connection with wavelet bases. They are characterized by a unique scale, and it is this feature that makes it possible to introduce a nonlinear basis starting from this “mother” function. For eq.(5) the compacton solution is given by

$$\eta_c(x - Vt) = \frac{4V}{3} \cos^2 \left[ \frac{x - Vt}{4} \right],$$

if $|x - Vt| < 2\pi$ and zero otherwise. Here the velocity is a function of the amplitude. Notice that the width $L = 4$ of the wave is independent of the amplitude. The quadratic dispersion term is characteristic for the nonlinear coupling in a chain.

The general compacton solution for eq.(5) is actually a ”dilated” version of eq.(6). That is, a combination of the first rising half of the squared cos in eq.(6), followed by a flat domain of arbitrary length ($\lambda$), and finally followed by the second, descending part of eq.(6). Actually, this combination is just a kink compacton joined smoothly with an antikink one

$$\eta_{kak}(x - Vt; \lambda) =
\begin{cases}
0 & \text{if } -2\pi \leq x - Vt \leq 0 \\
\frac{4V}{3} \cos^2 \left[ \frac{x-Vt}{4} \right], & \text{if } 0 \leq x - Vt \leq \lambda \\
\frac{4V}{3} \cos^2 \left[ \frac{x-Vt-\lambda}{4} \right], & \text{if } \lambda \leq x - Vt \leq \lambda + 2\pi \\
0 & \text{otherwise}
\end{cases}$$

In Fig. 1 we present a compacton, eq.(6), a kink-antikink pair (KAK) described by eq.(7), both with the same amplitude and velocity. Although the second derivative of this generalized compacton is discontinuous at its edges, the KAK, eq.(7), is still a solution of eq.(5) because the third derivative acts on $u^2$, which is a function of class $C_3$. Finally, we can construct solutions by placing a compacton on the top of a KAK, like in the third solution in Fig. 1. Such a solution exists only for a short interval of time, since the two structures have different velocities. The solution is given by

$$\eta(x, t) = \eta_{kak}(x - Vt; \lambda) + \left( \eta_c(x - V't - 2\pi) + \frac{4V}{3} \right) \chi\left(\frac{x - V't - 2\pi}{2\pi}\right),$$
for $0 < t < (\lambda - 4\pi)/(V' - V)$. Here $\chi(x)$ is the support function, equal with 1 for $|x| \leq 1$ and 0 in the rest, and $V' = 3\text{max}\{\eta_c\}/4 + 2V$.

For the K(2,2) compacton, eq.(6), the exact relations between the parameters are $A = 4V/3$ and $L = 4$. The relation provided by the similarity method, in the last column of the forth row, predicts the existence of the compacton. That is, for a linear dependence between the amplitude and the speed, the half-width is constant and does not depend on $A$ or $V$. This fact ($L \equiv L_0 = \text{const.}$) is a typical feature of K(2,2) compactons. Moreover, in literature there was found numerically that for any compact supported initial data, wider than $L_0$, the solution decomposes in time into a series of $L_0$ compactons, Fig. 2. For narrower initial data the numeric solution blows up. There is no exact or analytic explanation of this effect, so far. The similarity method can give a hint in this situation, too, by using the graphic of the relation $L = L(V, A)$ provided by this qualitative method. In Fig. 3 $L$ is plotted versus $V$, for several values of $A$ (larger values of $A$ translate the curves to the right). The half-width of a stable compacton was chosen $L_0 = 0.707$. Above this value, Fig. 3a, wider compact pulses produce an intersection for each curve (each $A$) with the axis $L_0$ providing series of compactons of different heights, like in the numerical experiments, [9, 17]. Below this $L_0$ line, all the curves approach infinite amplitude, providing instability of narrower shapes.

Another good example of prediction of the method is exemplified in the case of a general convection-nonlinear dispersion equations, denoted K(n,m)

$$\eta_t + (\eta^n)_x + (\eta^m)_{xxx} = 0.$$  \hspace{1cm} (9)

Compacton solution for any $n \neq m$ are not known in general, except some particular cases. In this case we find a general relation among the parameters, for any $n, m$, shown in the 5th and 6th rows. These general relations $L(A, V)$ approach the known relations for the exact solutions, in the particular cases like $n = m$ (5th row), $n = m = 2$ (4th row), $n = m = 3$. And $n = 3, m = 2$; $n = 2, m = 3$ in the 6th row. These results can be used to predict the behavior of solutions for all values of $n, m$.

Similar analysis can be done in the case of sine-Gordon equation, if we ask that velocity be proportional with $L^2$ (7th row). In this case we obtaining a transcendental equation in $A$, which is just the case of the sine-Gordon soliton. In the 8th row, we present the cubic nonlinear Schrödinger equation (NLS) which has a soliton solution, too [4]. This equation arises, for example, in nonlinear optics or in the polaron model in solid state physics, [4, 5]. In the general case of a NLS of order $n$ (9th row), when the general analytical solution is unknown, the method predicts a special $L = L(A, V)$ dependence shown in
the fourth column and in Fig. 4. Contrary to third order NLS, where the dependence of \(L\) with \(A\) is monotonous for \(V \approx \pm A\) (\(n = 3\) in Fig. 4), at higher order, the \(L(A)\) function has a discontinuity in the first derivative. This wigle of the function (Fig. 4 for \(n = 4\)) yields at a critical width, producing bifurcations in the solutions and scales. As a consequence, initial data close to this width can split into doublet (or even triplet for higher order NLS) solutions, with different amplitudes. Such phenomena have been put into evidence in several numerical experiments for quintic nonlinear equations [17, 18].

In the following, we present another example of applications of this qualitative approach, related to a new type of behavior of nonlinear systems. Traditional solitons move with constant speed on a rectilinear path (except for the roton, [9] which has a circular trajectory with constant angular velocity). The speed is usually equal with the amplitude scaled with a constant. Higher solitons travel faster and there are no solitons at rest (zero speed asks for zero amplitude). They can travel in both directions with opposite signs for the amplitude. The situation is different in the case of compactons, which allow also stationary solutions. When linear and nonlinear dissperion occur simultaneously, like in the so called K(2,1,2) equation

\[
 u_t + (u^2)_x + (u)_{xxx} + \epsilon (u^2)_{xxx} = 0,
\]

where \(\epsilon\) is a control parameter, the similarity approach yields a dependence of the form

\[
 L = \sqrt{(\pm A + \epsilon)/(V \pm A)},
\]

which still provides a constant width if \(V = \pm A + 2\epsilon\). In this case the speed is proportional with the amplitude, but can change its sign even at non-zero amplitude. Solutions with larger amplitude than a critical one (\(A_{\text{crit}} = \mp 2\epsilon\)) move to the right, solutions having the critical amplitude are at rest, and solutions smaller than the critical amplitude move to the left. This behavior was explored in [17]. However, such a switching of the speed is not necessarily a feature of the nonlinear dispersion. A compacton of amplitude \(A\) on the top of a KAK solution of amplitude \(\delta\)

\[
 u(x, t) = A \cos^2\left(\frac{x - V t}{4}\right) + \delta,
\]

is still a solution of the K(2,2) equation, \(u_t + (u^2)_x + (u^2)_{xxx} = 0\), with the velocity given by \(V = \frac{3}{4} (2\delta + A)\). For \(A = -2\delta\) the compacton becomes a stationary anticom pacton,
embedded in the moving, supporting KAK. Such an example is presented in Fig. 5 for a slow-scale time-dependent amplitude compacton. The induced oscillations in the amplitude transform into oscillations in the velocity. While not the topic of this paper, such a dynamic system has been analysed and it will be published soon. The key to such a conversion of oscillations is the coupling between the traditional nonlinear picture (convection-dispersion-diffusion) and the typical Schrödinger terms.

A last application of this method, occurs if the KdV equation has an additional term depending on the square of the curvature

\[ u_t + uu_x + u_{xxx} + \epsilon(u_{xx}^2)_x = 0. \]  

(11)

This is the case for extremely sharp surfaces (surface waves in solids or granular materials) when the hydrodynamic surface pressure cannot be linearized in curvature. Such a new term yields a new type of localized solution fulfilling the relations

\[ L = \sqrt{\frac{4\epsilon A}{\pm \sqrt{1 - 8\epsilon A(A \pm V)} - 1}}. \]

If we look for a constant half-width solution (compacton of \(1/L = \alpha\)) we need a dependence of velocity of the form \(V = (1 + \alpha^2\epsilon/8)A + 1/8\epsilon A + \alpha/4\). There are many new effects in this situation. The non-monoton dependence of the speed on \(A\) introduces again bifurcations of a unique pulse in doublets and triplets. Also, there is an upper bound for the amplitude at some critical values of the width. Pulses narrower than this critical width drop to zero. Such bumps can exist in pairs of identical amplitude at different widths. They may be related with the recent observed "oscillations" in granular materials, [17, 19].

As the examples presented in Table 1 proved the above method provides a reliable criterium for finding compact supported solutions. The reason this simple prescription works in so many cases follows from the advantages of wavelet analysis on localized solutions. We stress that this method has little to do with the traditional similarity (dimensional) analysis [8, 12, 16, 17, 18, 20]. In the latter case one obtains relations among powers of \(A, L\) and \(V\), not relations with numeric coefficients like those found in our method.
3 Compacton kink-antikink pairs and the multiresolution frame

A common feature of all NPDE and of the finite differences equations is the existence of compact supported solutions. Compactons and discrete wavelets are typical examples. An interesting general conclusion can be obtained if we look at a one-dimensional model described by the most general NPDE dynamical equation

$$\partial_t u = \mathcal{O}(x, \partial_x) u,$$

where $\mathcal{O}$ is a nonlinear differential operator. By taking into account only traveling solutions, this NPDE reduces to a NODE in the coordinate $\xi = x - Vt$ for an arbitrary velocity $V$. If $u(\xi)$ is a compact solution it results that it is not unique for given initial compact data. If one chooses zero initial value for the solution and its derivatives up to the requested, in a certain point $\xi_0$ of the $\xi$ axis, these conditions can be fulfilled by any linear combination of disjoint translated versions of one particular solution, placed everywhere on the axis except $\xi_0$. Consequently, for such initial data, the solution is not unique. This result shows that the compact supported property of the initial data and the solution, implies its non-uniqueness.

Since we can transform the NODE into a nonlinear differential system of order one

$$\frac{d\vec{U}}{dx} = \vec{F}(\xi, \vec{U}), \quad \vec{U} = (u, \partial_x u, ...),$$

we can apply the fundamental theorem of existence and uniqueness to solutions of eq.(13), for given initial data $\vec{U}(\xi_0) = \vec{U}_0$. If the function $\vec{F}$ in eq.(13) fulfills the Lipschitz condition (its relative variation is bounded) than, for any initial condition, the solution is unique, [21]. Since any linear function is analytic and hence Lipschitz, we conclude that only nonlinear functions $\vec{F}$ allow the existence of compact supported solutions. Thus, a compact soliton implies non-uniqueness in the underlying NPDE, which implies non-Lipschitzian structure of the NPDE and hence the existence of nonlinear terms.

In the following we investigate some compact solutions of the K(2,2) equation. The high stability against scattering of the K(2,2) compactons, or compacton generation from compact initial data, suggest they may play the role of a nonlinear local basis. We know form many numerical experiments, [8, 16, 17, 18], that any positive compact initial data decomposes into finite series of compactons and anticompactons. This suggests that an
intrinsic ingredient for a nonlinear basis could be the multiresolution structure of the solutions, similar with the structure of scaling functions in wavelet theory.

The compactons given in eqs. (6,7) have constant half-width and hence describe a unique scale, which can cover all the space by integer translations. From the point of view of multi-resolution analysis, the K(2,2) equations acts like a $L$-band filter, allowing only a particular scale to emerge for any given set of initial condition. To each scale, from zero to infinity, we can associate a K(2,2) equation with different coefficients. However, the compacton solution is not the unique one with this property. For a given K(2,2) equation, we can thus extend the scale from $L$ to any larger scale. These more general compact supported solutions are still $C^2(R)$ and are combinations of piece-wise constant and piece-wise $\cos^2$ functions. The simplest shape is given by a half-compacton prolonged with a constant level, that is a kink solution. The basis solution is a kink-antikink (KAK) compact supported combination, Fig. 1. Such kink-antikink pairs of different length, can be associated with other compactons, or KAK pairs, one on the top of the other

$$\eta_{\text{comp+KAK}}(x - Vt; \lambda) = \begin{cases} 
0... & 

\frac{4V}{3} \cos^2\left[\frac{x-Vt}{4}\right], \quad -2\pi \leq x - Vt \leq 0 \\
\frac{4V}{3}, & 0 \leq x - Vt \leq \delta \\
\frac{4V}{3} + \frac{4}{3}(V' - 2V) \cos^2\left[\frac{x-V't}{4}\right], \quad \delta \leq x - Vt \leq \delta + 4\pi \\
\frac{4V}{3}, & \delta + 4\pi \leq x - Vt \leq \lambda \\
\frac{4V}{3} \cos^2\left[\frac{x-Vt-\lambda}{4}\right], & \lambda \leq x - Vt \leq \lambda + 2\pi \\
0... & 
\end{cases}$$

where $\delta < \lambda$ characterizes the initial position (at $t = 0$) of the top compacton, with respect to the flat part of the KAK solution. The amplitude $4(V' - 2V)/3$ of the compacton, and the amplitude $4V/3$ of the KAK, are related to their velocities $V'$ and $V$, respectively. The length of the flat part, $\lambda$, is arbitrary. A compound solution is not stable in time since the different elements travel with different velocities. The total height of the compacton is $4(V' - V)/3$. Since the higher the amplitude is, the faster the structure travels, the top compacton moves faster than the KAK, and at a certain moment it passes the KAK. Because the area of the solution is conserving, such a compound structure decomposes into compactons and KAK pairs. Similar and even more complicated constructions can be imagined, with indefinite number of compactons and KAK’s, if one just fulfills the $C_3$ continuity condition for the square of the total structure. Such structures, defined at the
initial moment can interpolate any function, playing a similar role with wavelets or spline bases. It has been also proved that the KAK solutions are stable, by using both a linear stability analysis and Lyapunov stability criteria. [10, 20].

For a given K(2,2) equation, the compacton solution, eq.(6) and in addition the family of KAK solutions, eq.(7) can be organized as a scaling functions system. They act like a low-pass filter in terms of space-time scales and give the opportunity to construct frames of functions from the wavelet model, [10, 11, 13].

For the sake of simplicity we will renormalize the coefficients of the K(2,2) equation such that the support of the simple compacton is one. That is, we take \( \eta_c(x, t) = \eta_{kak}(\pi(x - V t), 0) \) on the interval \( |x - V t| \) in \([-1/2, 1/2]\). We construct a multiresolution approximation of \( L^2(\mathbb{R}) \), that is an increasing sequence of closed subspaces \( V_j \), \( j \in \mathbb{Z} \), of \( L^2(\mathbb{R}) \) with the following properties, [11, 13]

1. The \( V_j \) subspaces are all disjoint and their union is dense in \( L^2(\mathbb{R}) \).
2. For any function \( f \in L^2(\mathbb{R}) \) and for any integer \( j \) we have \( f(x) \in V_j \) if and only if \( D^{-1}f(x) \in V_{j-1} \) where \( D^{-1} \) is an operator that will be defined later.
3. For any function \( f \in L^2(\mathbb{R}) \) and for any integer \( k \), we have \( f(x) \in V_0 \) is equivalent with \( f(x - k) \in V_0 \).
4. There is a function \( g(x) \in V_0 \) such that the sequence \( g(x - k) \) with \( k \in \mathbb{Z} \) is a Riesz basis of \( V_0 \).

In the case of compact solutions of K(2,2) of unit length, we chose for the space \( V_0 \) that which is generated by all translation of \( \eta_c \) with any integer \( k \). The subspaces \( V_j \) for \( j \geq 0 \) are generated by all integer translations of the compressed version of this function, namely, by \( \eta_{kak}(2^j\pi(x - V t), 0) \). The subspaces \( V_j \) for \( j \leq 0 \) are generated by all integer translations of the KAK solution of length \( 2^j - 1 \). For example, \( V_{-1} \) is generated by \( \eta_{kak}(\pi(x - 2V t), 0) \). The spaces \( V_j, j \geq 0 \) are all solutions of K(2,2); the others are not. The function \( g(x) \) is taken to be \( \eta_{kak}(\pi(x - V t), 0) \). It is not difficult to prove that these definitions fulfill restrictions one, three, and four. As for the second criterion, we define the action of the operator \( D^{-1}f(x) = f(2x) \) if \( f(x) \in V_j \) with a \( j \) positive integer, and \( D^{-1}\eta_{kak}(\pi 2^j(x - 2^jV t), 2^{-j} - 1) = \eta_{kak}(\pi 2^{j+1}(x - 2^{-j+1}V t), 2^{-j+1} - 1) \) for negative \( j \). In conclusion, we construct a frame of functions made of contractions of compactons and sequences of KAK solutions. We can write the corresponding two-scale equation which
connects the subspaces (the equivalent of eq.(15)),
\[ \eta_{kak}(\pi(x - Vt), 1) = \eta_{kak}(\pi(x - Vt), 0) + \eta_{kak}(\pi(x - Vt - 1), 0). \] (15)

We will denote generically by \( \eta_{k,j} \) the elements of this frame, that is
\[ \eta_{k,j}(x) = \eta_{kak}(\pi(x - 2^j Vt - k), 2^j - 1)|_{t=0}, \]
where \( t = 0 \) means that we neglect the time evolution, but the amplitude is still amplified with a factor of \( 2^j \), in virtue of relation \( \eta_{\text{max}} = 4V/3 \). In the following, we can expand any initial data for the K(2,2) equation in this basis.
\[ u_0(x) = \sum_k \sum_j C_{k,j} \eta_{k,j}(x). \] (16)

We notice that the following equality holds for \( j',j \)
\[ \eta_{k,j} \eta_{k',j'} \left\{ \begin{array}{ll} \neq 0 & \text{if } k' = k \cdot 2^{j'-j}, \ldots, (k + 1) \cdot 2^{j'-j} - 1 \\ = 0 & \text{otherwise.} \end{array} \right. \] (17)

After some rather elaborate algebraic calculations and by using eq.(21), we show that the square of this function (since the equations is nonlinear and of order two) will be given by
\[ u^2(x) = \sum_{k,j} \sum_{j' \geq j} \sum_{k' \in I} C_{k,j} C_{k',j'} \left( \sum_{i_1=0}^{1} \sum_{i_2=0}^{1} \ldots \sum_{i_{j'-j}=0}^{1} \eta_{\sigma(i_1,i_2,\ldots,i_{j'-j}),j'} \right) \eta_{k',j'}, \] (18)
where \( I \) is the range of \( k' \) described in the first line of eq. (21), and
\[ \sigma(i_1,i_2,\ldots,i_{j'-j}) = \sum_{l=1}^{j'-j} i_l 2^{j'-j+l+\frac{(j'-j)(j'-j+1)-l(l+1)}{2}} + k 2^{j'-j} j + \frac{(j'-j)(j'-j+1)}{2}. \]

From eq.(21) we notice that in eq.(22)the unique nonzero terms are those for which \( \sigma(i_1,i_2,\ldots,i_{j'-j}) = k' \) with \( k' \in I \). This result express the following simple fact. The initial data is expanded in different scales and different translations. The translations are

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mutually orthogonal so they do not give a contribution to the square. When we have to multiply two different scales in the expression of the square, we reduce the wider scale in terms of linear combination of the narrower one by using the two-scale equation, eq. (19). This is what eq. (22) expresses. Out of all the terms in such a product only approximately \((2^{-j} - 1)/(2^{-j'} - 1) \approx 2^{j'-j}\) give non-zero contributions. In other words this number is given by the number of solutions of equation \(\sigma(i_1, i_2, ..., i_{j'-j}) = k', \) with \(k' \in I.\) This is the primary advantage of treating nonlinear problems with a basis that has a scale criterion. Another advantage is that all the function in the basis are actually contractions or dilations, and translations of only two basic ones.

4 Comments and conclusions

In the present paper we introduce new applications for wavelets, in the field of the study of localized solutions of nonlinear differential equations. The existence of compactons and discrete wavelets underlines a common feature of NPDE and finite differences equations, that is the existence of compact supported solutions. We propose a new similarity formalism for the qualitative analysis and classification of soliton solutions, without the need of solving the corresponding NPDE. Also, we proved that starting from any unique soliton solution of a NPDE, we can construct a frame of solutions organized under a multiresolution criterium. This approach provides the possibility of constructing nonlinear frames for NPDE. We show that frames of self-similar functions are related with solitons with compact support. In addition, we notice the evidence that compactons fulfil both characteristics of solitons and wavelets, suggesting possible new applications. Such unifying direction between nonlinearity and self-similarity, can bring new applications of wavelets in cluster formation, at any scale, from supernovae through fluid dynamics to atomic and nuclear systems. The similarity approach can be applied with success to the physics of droplets, bubbles, traveling patterns, fragmentation, fission and inertial fusion.

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Figure Captions

- **Fig. 1**
  A compacton and a kink-antikink pair solution (KAK) of the equation K(2,2), both having the same amplitude, and hence velocity V. To the right, there is a smaller compacton on the top of KAK. The upper compacton has higher speed, V'.

- **Fig. 2**
  A finite series of K(2,2) compactons emerging from initial compact data, with the width larger than the compacton width.

- **Figs. 3**
  The half-width L versus velocity V for the K(2,2) equation, for different amplitudes A. Figure 3a shows widths larger than $L_{\text{compacton}} = 3/4$, and Fig. 3b shows narrower widths, $L < 3/4$. Amplitude increases from left to right, in the range 0.01-0.85.

- **Fig. 4**
  The half-width L plot versus amplitude V, for the third (n=3) and forth (n=4) order NLS equation, in two $V = \pm A$ cases. From the figure one notes that the quartic NLS equation yields bifurcations in the solutions.

- **Fig. 5**
  The solution of the mixed linear plus nonlinear-dispersion K(2,2) equation, in the case of a solution with a slow oscillating shape.
Table 1: Nonlinear equations, exact solutions and Morlet similarity analysis.

| Equation                  | Solution                  | Relations             | Wavelet                  |
|---------------------------|---------------------------|-----------------------|--------------------------|
| Linear wave               | $\sum C_k e^{i(kx \pm \omega t)}$ | $k^2 = \omega^2/c^2$  | $V = c$ A,L arbitrary   |
| $u_{xx} - (1/c^2)u_t = 0$ |                           |                       |                          |
| KdV=K(2,1)                | $A \text{sech}^2 \frac{x-Vt}{L}$ | $L = \sqrt{\frac{A}{2}}$ | $L = \frac{1}{\sqrt{|\pm V \pm 6A|}}$ |
| $u_t + 6uu_x + u_{xxx} = 0$ |                           | $V = 2A$              |                          |
| MKdV=K(3,1)               | $A \text{sech}^2 \frac{x-Vt}{L}$ | $L = 1/A$             | $L = \frac{1}{\sqrt{|\pm V \pm 6A^2|}}$ |
| $u_t + u^2u_x + u_{xxx} = 0$ |                           | $A = \sqrt{V}$       |                          |
| K(2,2)                    | $A \cos^2 \frac{x-Vt}{L}$  | $L = 4$               | $L = \sqrt{\frac{8A}{|\pm V \pm 2A|}}$, if  |
| $u_t + (u^2)_x + (u^2)_{xxx} = 0$ |                       | $V = 3A/4$            | $V = -3A/2$, $L = 4$    |
| K(n,n)                    | $A \cos^2 \left( \frac{x-Vt}{L} \right)^{\frac{1}{n-1}}$ | $A = \frac{2Vn}{n+1}$ | $L = \sqrt{\frac{n(n^2+1)}{\pm \alpha \pm n}}$ |
| $u_t + (u^n)_x + (u^n)_{xxx} = 0$ |                       | $L = \frac{4n}{(n-1)}$ | if $V = \alpha A^{n-1}$ |
| K(n,m)                    | unknown                   | if $n \neq m$         | $L = \sqrt{\frac{n(n^2+1)A^{n-1}}{\pm V \pm mA^{n-1}}}$ |
| $u_t + (u^n)_x + (u^m)_{xxx} = 0$ |                       |                       |                          |
| sine-Gordon               | $A \tan^{-1} \gamma e^{\frac{x-Vt}{L}}$ | $A = 4$               | $\pm \frac{VA}{L^2} = \sin A$ |
| $u_{xt} - \sin u = 0$     |                           | $V = L^2$             |                          |
| NLS(3)                    | $A e^{i(\omega t + kx)} \text{sech} \frac{x-Vt}{L}$ | $L = \frac{1}{A}$ | $L = \frac{\pm V \pm \sqrt{V^2-4A^2}}{2A^2}$ |
| $\Psi_t + \Psi_{xx} + \Psi^3 = 0$ |                       | $A \approx V$         |                          |
| NLS(n)                    | unknown                   |                       | $L = \frac{\pm V \pm \sqrt{V^2-4A^n}}{2A^n}$ |