Probing the eigenfunction fractality with a stop watch

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We study numerically the distribution of scattering phases \( P(\Phi) \) and of Wigner delay times \( P(\tau_W) \) for the power-law banded random matrix (PBRM) model at criticality with one channel attached to it. We find that \( P(\Phi) \) is insensitive to the position of the channel and undergoes a transition towards uniformity as the bandwidth \( b \) of the PBRM model increases. The inverse moments of Wigner delay times scale as \( \langle \tau_W^{-q} \rangle \sim L^{-qD_{q+1}} \), where \( D_q \) are the multifractal dimensions of the eigenfunctions of the corresponding closed system and \( L \) is the system size. The latter scaling law is sensitive to the position of the channel.

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I. INTRODUCTION

The transport properties of random media have been a subject of intensive research activity for more than fifty years.\(^1,2,3,4,5,6,7,8,9,10,11,12,13\) The most fascinating aspect of this study is the appearance (in high-dimensions) of a metal-insulator phase transition (MIT) as an external parameter changes. In the metallic phase, the eigenstates are extended\(^1,2,3,4,7,8,9,10,11,12\) and the statistical properties of the spectrum are described well by random matrix theory\(^2,3,4,7,8,9,10,11,12\). In the localized phase, the levels become uncorrelated leading to a Poissonian level spacing distribution\(^1,2,3,4,7,8,9,10,11,12\), and the eigenfunctions are exponentially localized\(^1,2,3,4,5,7\).

The MIT where the phase transition from localized to extended states occurs, is characterized by remarkably rich critical properties\(^1,2,3,4,7,8,9,10,11,12\). In particular, the eigenfunctions show strong fluctuations on all length scales and represent multifractal distributions\(^7,8,9\). This is one of the most important features at MIT and has considerably helped our understanding of different phenomena in mesoscopic systems\(^2,3\). The multifractal structure of the eigenfunctions is usually quantified by studying the size dependence of the so-called participation numbers (PN)

\[
N_q = \left( \int |\psi(\mathbf{r})|^2 \, d\mathbf{r} \right)^{-1} \propto L^{(q-1)D_q} \tag{1}
\]

where \( L \) is the linear size of the system and \( D_q \) are the multifractal dimensions of the eigenfunction \( \psi(\mathbf{r}) \). Among all the dimensions, the correlation dimension \( D_2 \) plays the most prominent role. The corresponding PN is roughly equal to the number of non-zero eigenfunction components, and therefore is a good and widely accepted measure of the extension of the states. At the same time, \( D_2 \) manifest itself in a variety of other physical observables. As examples we mention the conductance distribution in metals\(^1,2,3,4,7,8,9,12,13\), the statistical properties of the spectrum\(^1,2,3,4,7,8,9,12,13\), the anomalous spreading of a wave-packet, and the spatial dispersion of the diffusion coefficient\(^1,2,3,4,7,8,9,12,13\).

It is therefore highly desirable to have the possibility of measuring the multifractal dimensions \( D_q \), and specifically \( D_2 \), directly from an experiment. Unfortunately, to our knowledge, there is no experimental study of eigenfunction fluctuations at the transition regime\(^2,3\), although in recent years a MIT was experimentally accessible in microwave\(^26\) and optical wave\(^27\) setups. The main difficulty is the lack of eigenfunction accessibility in all these experiments. On the other hand, new techniques were developed that allow accessibility to various quantities associated with the scattering process. Among them is the Wigner delay time\(^18,19,20\), which captures the time-dependent aspects of quantum scattering. It can be interpreted as the typical time an almost monochromatic wave packet remains in the interaction region. It is related to the energy derivative of the total phase shift \( \Phi(E) = -i \ln \det S(E) \) of the scattering matrix \( S(E) \), i.e., \( \tau_W(E) = \frac{\ln |S(E)|}{\ln |S(E)|} \), where \( M \) is the number of channels.

These experimental efforts have promoted the study of Wigner delay times to quantities of interest in their own right and attracted a lot of theoretical work. For chaotic systems many results are known concerning the distribution of Wigner delay times \( P(\tau_W) \).\(^21\) Recently, the interest has extended to systems showing diffusion\(^22,23\) and localization\(^24\). At the same time, an intensive activity to understand \( P(\tau_W) \) for systems at critical conditions was undertaken in\(^25,26,27\). A possibility of anomalous scaling of inverse moments of \( \tau_W \) with the system size \( L \) was suggested in\(^25\). Specific predictions linking the scaling exponents and the multifractal properties of eigenfunctions of the corresponding closed system were made in\(^25,26\). While in both cases an anomalous scaling of the inverse moments of delay times was reported, there was a discrepancy in the exact power dictating these scaling laws.

Precise reasons for these discrepancies are still not clear (see though the discussion in\(^26\)) and deserve investigation. Here we undertake this task by studying numerically the distribution of Wigner delay times at MIT for the prototype power-law banded random matrix (PBRM) model with one channel attached to it. We provide clear evidence showing that the inverse moments of Wigner delay times \( \langle \tau_W^{-q} \rangle \) scale as

\[
\langle \tau_W^{-q} \rangle \propto L^{-f(q)}, \quad f(q) = qD_{q+1} \tag{2}
\]

where \( \langle \cdot \rangle \) stands for an ensemble average in agreement
with the theoretical prediction of\textsuperscript{28}. However we find that this relation is extremely fragile. Namely we show that it holds for channels attached to a typical position in the sample. At the same time we perform a detail analysis of the distribution of phases \( \mathcal{P}(\Phi) \) of the scattering matrix associated with our scattering setup. We show that \( \mathcal{P}(\Phi) \) is insensitive to the position of the channel while it depends drastically on the band-width \( b \) which characterize the criticality. It undergoes a transition towards uniformity as \( b \) increases. We note that the theoretical analysis of\textsuperscript{28} applies only for \( b \gg 1 \) while our numerical investigations extends over the whole \( b \)-range.

The structure of this paper is as follows. In the next section we describe the scattering setup and summarize the properties of the corresponding closed system. Section III discuss the distribution of phases for various bandwidths, coupling strengths, and position of the channel. In Section IV we analyze the statistical properties of delay times. Finally, our conclusions are given in Section V.

II. SCATTERING SETUP

The isolated sample is represented by \( L \times L \) real symmetric matrices whose entries are randomly drawn from a normal distribution with zero mean and a variance depending on the distance of the matrix element from the diagonal

\[
\langle (H_{ij})^2 \rangle = \frac{1}{1 + (|i-j|/b)^{2\alpha}},
\]

where \( b \) and \( \alpha \) are parameters. In a straightforward interpretation, the PBRM model describes a 1D sample with random long-range hopping. Also, the PBRM ensemble arises as an effective description to a variety of physical systems, such as the quantum Fermi accelerator, the scattering by a Coulomb center in an integrable billiard, and in billiard systems with non-analytic boundaries.

Field-theoretical considerations\textsuperscript{7,29,30} and detail numerical investigations\textsuperscript{10,11} verified that the model undergo a transition at \( \alpha = 1 \) from localized states for \( \alpha > 1 \) to delocalized states for \( \alpha < 1 \). This transition shows all the key features of the Anderson MIT, including multifractality of eigenfunctions and non-trivial spectral statistics at the critical point. At the center of the spectral-band a theoretical estimation for the multifractal dimensions \( D_q \) gives\textsuperscript{28}

\[
D_q = \begin{cases} 
4b\Gamma(q-1/2)\sqrt{\pi}(q-1)^{-1} & , b \ll 1 \\
1 - q(2\pi b)^{-1} & , b \gg 1 
\end{cases}
\]

where \( \Gamma \) is the Gamma function. Thus model\textsuperscript{28} possesses a line of critical points \( b \in (0, \infty) \), where the multifractal dimensions \( D_q \) change with \( b \).

We turn the isolated system to a scattering one by attaching one semi-infinite single channel lead to it. The lead is described by a 1D semi-infinite tight-binding Hamiltonian

\[
H_{\text{lead}} = \sum_{n=1}^{\infty} \left( |n+1, n+1 \rangle \langle n+1, n+1| + |n+1, n \rangle \langle n, n+1| \right). \tag{5}
\]

Using standard methods\textsuperscript{28} one can write the scattering matrix in the form\textsuperscript{28}

\[
S(E) = e^{i\Phi(E)} = 1 - 2i\pi W T - \frac{1}{E - \mathcal{H}_{\text{eff}}} W, \tag{6}
\]

where \( \mathcal{H}_{\text{eff}} \) is an effective non-hermitian Hamiltonian given by

\[
\mathcal{H}_{\text{eff}} = H - i\pi WW T. \tag{7}
\]

Here \( W_{nm} = w_0 \delta_{n0} \) is a \( L \times 1 \) vector where \( w_0 \) is the coupling strength and \( n_0 \) is the site at which we attach the lead. All our calculations take place in an energy window close to the band center (\( E = 0 \)). The delay time is given by\textsuperscript{21,27}

\[
\tau_W(E = 0) = \left. \frac{d\Phi(E)}{dE} \right|_{E = 0} = -2i\pi \text{Tr}(E - \mathcal{H}_{\text{eff}})^{-1} \bigg|_{E = 0} \tag{8}
\]

We will see later that on the scaling properties of inverse moments of delay times are independent of \( w_0 \) while they depend drastically on the position \( n_0 \). Specifically, if \( n_0 \) is a representative site (like any site in the bulk of the sample) where ergodicity of wavefunctions can be assumed, then Eq (\textsuperscript{8}) applies. Instead if \( n_0 \) belongs to the boundary (i.e. \( n_0 = 1 \) or \( L \)), then Eq (\textsuperscript{8}) does not valid any more.

In our numerical experiments bellow we used matrices of size varying from \( L = 50 \) up to \( L = 2000 \). For better statistics a considerable number of different disorder realizations has been used. In all cases we had at least 100000 data for statistical processing.

III. DISTRIBUTION OF PHASES

We start our analysis with the distribution \( \mathcal{P}(\Phi) \) of phases of the \( S \)-matrix. Analytical considerations applied to the large \( b \gg 1 \) limit\textsuperscript{28} led to the following expression

\[
\mathcal{P}(\Phi) = \frac{1}{2\pi(\gamma + \sqrt{\gamma^2 - 1} \cos(\Phi))} \tag{9}
\]

where \( \gamma = (1 + |\langle S \rangle|^2)/(1 - |\langle S \rangle|^2) \). Notice that in the limit of \( |\langle S \rangle| = 0 \) we recover the uniform distribution, where the corresponding setup is associated with perfect coupling of the sample to the leads. Eq. (\textsuperscript{9}) was also derived in the framework of Random Matrix Theory\textsuperscript{28} using the maximum entropy ansatz. It is interesting that the same expression applies (see bellow) in our case as well.
sample. For large responds to one channel attached to the boundary of the sample. Various coupling strengths \( w \) for the case where the lead is attached to the bulk of the sample. Similar results (not reported here) are found for the case where the lead is attached. In this section we analyze the distribution of Wigner delay times (8). Recent investigations \( \tau \) scale in an anomalous way with the system size. In the insets we concentrate on the scaling of the first inverse moment \( \langle \tau_W^{-1} \rangle \sim L^{-\alpha} \). A nice scaling is observed. Using a least square fitting we can extract the scaling exponent \( \alpha \).

Our results are summarized in Fig. 4 where we compare the scaling exponents obtained for the two scattering setups (channel attached at the boundary and to the bulk) together with the theoretical expectation (2) which predicts that in the limit of \( b \gg 1 \) the scaling exponent \( \alpha = D_2 \). For completeness we also include the value of \( D_2 \) numerically extracted from the scaling analysis of eigenvectors of the closed system (11). We observe that Eq. (2) for the \( \langle \tau_W^{-1} \rangle \) is valid as well as long as the channel is connected to the bulk of the sample where the ergodic hypothesis for the statistical properties of the eigenstates of the corresponding closed system holds. Although the theoretical expression (2) was derived under the assumption of \( b \gg 1 \) we see that our numerical results for \( b \ll 1 \) can also be described by Eq. (2) relatively good. Moreover, if we find that the scaling relation (2) persist for any value of \( \langle S \rangle \neq 0 \) in agreement with (26). On the contrary, for the case where a channel is attached to the boundary, we observe strong deviations from Eq. (2). This is because the boundary is not a representative position and the ergodicity hypothesis as far as the eigenfunctions are concerned is not any more applicable.

Let us finally comment on the anomalous scaling behavior of the other moments \( \langle \tau_W^{-q} \rangle \). A simple homogeneous fractal would be completely characterized by two of the moments and the respective \( f(q) \) curve would be a straight line while a non-linearity in \( f(q) \) would signify a multifractal behavior. In Fig. 4a) we report our results for the case with a channel attached to the boundary. Again we see that the numerical data deviates from the theoretical predictions (4) for any value of \( b \). Instead, the agreement is quite good for the case where the channel is attached to the bulk of the sample (see Fig. 1).

**FIG. 1:** Left panels: \( \langle |S| \rangle \) as a function of \( w_0 \) for a channel attached to the first site (boundary) of the sample. Right panels: \( P(\Phi) \) corresponding to \( w_0 \) where \( \langle |S| \rangle \) takes its minimum value (see left panels) and for various system sizes, ranging from \( L = 50 \) up to \( L = 800 \). In the insets of the left panels we show a representative \( P(\Phi) \) for \( \langle |S| \rangle \sim 0.5 \). The dashed line is Eq. (9). From top to bottom: \( b = 0.1, 1, \) and 10.
V. CONCLUSIONS

In summary we have studied the distribution of scattering phases $P(\Phi)$ and delay times for the power law banded matrix model at criticality. For small bandwidths $b \ll 1$ the distribution of phases is non-uniform irrespective of the value of $|\langle S \rangle|$, while for large $b$ the theoretical prediction $26$ applies.

We found that the multifractal nature of the eigenfunctions is reflected in the scaling of the inverse moments of Wigner delay times $\langle \tau_W^{-q} \rangle$ with the system size $L$ provided that the channel is attached to a representative position in the sample. We expect that these results will provide a new method for evaluating $D_2$ (and in general $D_q$) in microwave and light wave experiments where $\tau_W$ can be extracted even in the presence of weak absorption. At the same time the appearance of the anomalous scaling of $\langle \tau_W^{-q} \rangle$ can be used as a criterion for detecting MIT.

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