1. Introduction

In this paper, we generalise the results of [5] on the reduction theory of binary forms, which describe positive zero-cycles in $\mathbb{P}^1$, to positive zero-cycles (or point clusters) in projective spaces of arbitrary dimension. This should have applications to more general projective varieties in $\mathbb{P}^n$, by associating a suitable positive zero-cycle to them in an $\text{PGL}(n+1)$-invariant way. We discuss this in the case of (smooth) plane curves.

The basic problem motivating this work is as follows. Consider projective varieties over $\mathbb{Q}$ in some $\mathbb{P}^n$, with fixed discrete invariants. On this set, there is an action of $\text{SL}(n+1, \mathbb{Z})$ by linear substitution of the coordinates. We would like to be able to select a specific representative of each orbit, which we will call reduced, in a way that is as canonical as possible. Hopefully, this representative will then also allow a description as the zero set of polynomials with fairly small integer coefficients.

Recall the main ingredients of the approach taken in [5]. The key role is played by a map $z$ from binary forms of degree $d$ into the symmetric space of $\text{SL}(2, \mathbb{R})$ (which is the hyperbolic plane $\mathcal{H}$ in this case) that is equivariant with respect to the action of $\text{SL}(2, \mathbb{Z})$. We then define a form $F$ to be reduced if $z(F)$ is in the standard fundamental domain for $\text{SL}(2, \mathbb{Z})$ in $\mathcal{H}$. In order to make the map $z$ as canonical as possible, we use a larger group than $\text{SL}(2, \mathbb{Z})$, namely $\text{SL}(2, \mathbb{C})$; we then look for a map $z$ from binary forms with complex coefficients into the symmetric space $\mathcal{H}_C$ for $\text{SL}(2, \mathbb{C})$ that is $\text{SL}(2, \mathbb{C})$-equivariant and commutes with complex conjugation. This map restricted to real forms will have image contained in $\mathcal{H}$ and satisfy our initial requirement.

Now there are in general many possible such maps $z$ (for exceptions, see below). We therefore need to pick one of them. In [5] this is achieved by a geometric property: we define a function on $\mathcal{H}_C$ that measures how far a point is from the roots of $F$ (up to an arbitrary additive constant); the covariant $z(F)$ is then the unique point in $\mathcal{H}_C$ minimising this distance.
In our more general situation, we work with the space $H_{n,R}$ of positive definite quadratic forms in $n + 1$ variables, modulo scaling, and the space $H_{n,C}$ of positive definite Hermitian forms in $n + 1$ variables, modulo scaling (by positive real factors). There is a natural action of complex conjugation on $H_{n,C}$; the subset fixed by it can be identified with $H_{n,R}$.

We use the formula for the distance function mentioned above to obtain a similar function on $H_{n,C}$, depending on a collection of points in $\mathbb{P}^n(\mathbb{C})$. Under a suitable condition on the point cluster or zero-cycle $Z$, this distance function has a unique critical point, which provides a global minimum. We assign this point to $Z$ as its covariant $\tilde{z}(Z)$, thus solving our problem.

2. Basics

In all of the paper, we fix $n \geq 0$.

We consider the group $G = \text{SL}(n + 1, \mathbb{C})$ and its natural action on forms (homogeneous polynomials) in $n + 1$ variables $X_0, \ldots, X_n$ by linear substitutions; this action will be on the right:

$$F(X_0, X_1, \ldots, X_n) \cdot (a_{ij})_{0 \leq i, j \leq n} = F \left( \sum_{j=0}^{n} a_{0j} X_j, \ldots, \sum_{j=0}^{n} a_{nj} X_j \right).$$

The same action is used for Hermitian forms in $X_0, \ldots, X_n$. A Hermitian form can be considered as a bihomogeneous polynomial of bidegree $(1, 1)$ in two sets of variables $X_0, \ldots, X_n$ and $\bar{X}_0, \ldots, \bar{X}_n$, where the action on the second set is through the complex conjugate of the matrix. The form $Q$ is Hermitian if $Q(\bar{X}; X) = \bar{Q}(X; \bar{X})$, where $\bar{Q}$ denotes the form obtained from $Q$ by replacing the coefficients with their complex conjugates. Hermitian forms can also be identified with Hermitian matrices, i.e., matrices $A$ such that $A^\top = \bar{A}$, where $A$ corresponds to $Q$ if $Q(x) = \bar{x} A x^\top$; then the action of $G$ is given by $A \cdot \gamma = \bar{\gamma}^\top A \gamma$.

The group $G$ also acts on coordinates $(\xi_0, \ldots, \xi_n)$ on the right via the contragredient representation,

$$(\xi_0, \ldots, \xi_n) \cdot \gamma = (\xi_0, \ldots, \xi_n) \gamma^{-\top}.$$

These actions are compatible in the sense that

$$(Q \cdot \gamma)(x \cdot \gamma) = Q(x)$$

for Hermitian forms $Q$ and coordinate vectors $x$.

3. Point Clusters

The actions described above induce actions of $\text{PSL}(n + 1, \mathbb{C}) = \text{PGL}(n + 1, \mathbb{C})$ on projective schemes over $\mathbb{C}$ and points in projective space $\mathbb{P}^n(\mathbb{C})$. The first specialises and the second generalises to an action on positive zero-cycles.
Definition 1. A positive zero-cycle or point cluster is a formal sum \( Z = \sum_{j=1}^{m} P_j \) of points \( P_j \in \mathbb{P}^n \). The number \( m \) of points is the degree of \( Z \), written \( \deg Z \). If \( L \subset \mathbb{P}^n \) is a linear subspace, we let \( Z|_L \) be the sum of those points in \( Z \) that lie in \( L \).

Definition 2. Let \( Z \) be a point cluster in \( \mathbb{P}^n \).

1. \( Z \) is split if there are two disjoint and nonempty linear subspaces \( L_1, L_2 \) of \( \mathbb{P}^n \) such that \( Z = Z|_{L_1} + Z|_{L_2} \). Otherwise, \( Z \) is non-split.
2. \( Z \) is semi-stable if for every linear subspace \( L \subset \mathbb{P}^n \), we have
   \[(n+1) \deg Z|_L \leq (\dim L + 1) \deg Z .\]
3. \( Z \) is stable if for every linear subspace \( \emptyset \neq L \subset \mathbb{P}^n \), we have
   \[(n+1) \deg Z|_L < (\dim L + 1) \deg Z .\]

Remark 3. Note that a split point cluster cannot be stable.

If we identify the cluster \( Z = \sum_{j=1}^{m} P_j \), where \( P_j = (a_{j0} : a_{j1} : \ldots : a_{jn}) \), with the form \( F(Z) = \prod_{j=1}^{m} (a_{j0}x_0 + a_{j1}x_1 + \ldots + a_{jn}x_n) \) (up to scaling), then \( Z \) is (semi-)stable if and only if \( F(Z) \) is (semi-)stable in the sense of Geometric Invariant Theory, see [4].

If \( n = 1 \), then the notions of stable and semi-stable defined here coincide with those defined in [5] (in Def. 4.1 and before Prop. 5.2) for binary forms.

Definition 4. Let \( \mathcal{Z}_m \) denote the set of point clusters of degree \( m \) in \( \mathbb{P}^n(\mathbb{C}) \), \( \mathcal{Z}_m^{sst} \) the subset of semi-stable and \( \mathcal{Z}_m^{st} \) the subset of stable point clusters. We denote by \( \mathcal{Z}_m(\mathbb{R}) \) etc. the subset of point clusters fixed by complex conjugation, which acts via \( \sum_j P_j \mapsto \sum_j \bar{P}_j \).

For notational convenience, we define for a point cluster \( Z \) and \(-1 \leq k \leq n\)
\[ \varphi_Z(k) = \max \{ \deg Z|_L : L \subset \mathbb{P}^n \text{ a } k\text{-dimensional linear subspace} \} .\]
Then \( Z \) is semi-stable if and only if \( \varphi_Z(k) \leq \frac{k+1}{n+1} \deg Z \) and stable if and only if the inequality is strict for \( 0 \leq k < n \).

We let \( \langle P, P' \rangle = \bar{P}(P')^\top \) denote the standard Hermitian inner product on row vectors and \( \|P\|^2 = \langle P, P \rangle \) the corresponding norm. The next lemma is the basis for most of what follows.

Lemma 5. Let \( Z \in \mathcal{Z}_m \). Fix row vectors \( P_j, j \in \{1, \ldots, m\} \), representing the points in \( Z \), such that \( \|P_j\|^2 = 1 \). Then there is a constant \( c > 0 \) such that for every positive definite Hermitian matrix \( Q \) with eigenvalues \( 0 < \lambda_0 \leq \lambda_1 \leq \ldots \leq \lambda_n \), we have
\[ \prod_{j=1}^{m} (\bar{P}_j Q P_j^\top) \geq c \prod_{k=0}^{n} \lambda_k^\varphi_Z(k) - \varphi_Z(k-1) .\]
Proof. Let $B = b_0, \ldots, b_n$ be a unitary basis of $\mathbb{C}^{n+1}$. Let $E_k = \langle b_0, \ldots, b_k \rangle$ the subspace generated by the first $k + 1$ basis vectors. By definition of $\varphi_Z$, the set $\Sigma(B) \subset S_m$ of permutations $\sigma$ with the following property is nonempty:

$$\mathcal{P}_{\sigma(j)} \notin E_k \quad \text{if} \quad j > \varphi_Z(k).$$

Define $k_\sigma(j) = \min\{k : \sigma(j) \leq \varphi_Z(k)\}$; then $\mathcal{P}_{\sigma(j)} \notin E_{k_\sigma(j)-1}$ if $\sigma \in \Sigma(B)$. Write $P_j = \sum_{i=0}^n \xi_{ji} b_i$ and define

$$f_\sigma(B) = \prod_{j=1}^m \left( \sum_{i=k_\sigma(j)}^n |\xi_{\sigma(j),i}|^2 \right) = \prod_{j=1}^m \left( \sum_{i=k_\sigma(j)}^n |\langle \mathcal{P}_{\sigma(j)}, b_i \rangle|^2 \right)$$

and

$$f(B) = \max\{f_\sigma(B) : \sigma \in S_m\}.$$  

It is clear that $f_\sigma$ is continuous on the set of unitary bases and that $f_\sigma(B) > 0$ if $\sigma \in \Sigma(B)$. This implies that $f$ is continuous and positive. Since the set of all unitary bases (i.e., $U(n+1)$) is compact, there is some $c > 0$ such that $f(B) \geq c$ for all $B$.

Now let $Q$ be a positive definite Hermitian matrix as in the statement of the Lemma. Let $B = b_0, \ldots, b_n$ be a unitary basis of eigenvectors such that $b_j Q = \lambda_j b_j$. We then have for $\sigma \in S_m$ and using notation introduced above

$$\prod_{j=1}^m (\bar{P}_j Q P_j^T) = \prod_{j=1}^m (\bar{P}_{\sigma(j)} Q P_{\sigma(j)}^T) = \prod_{j=1}^m \left( \sum_{i=0}^n \lambda_i |\xi_{\sigma(j),i}|^2 \right)$$

$$\geq \prod_{j=1}^m \left( \lambda_{k_\sigma(j)} \sum_{i=k_\sigma(j)}^n |\xi_{\sigma(j),i}|^2 \right)$$

$$= f_\sigma(B) \prod_{j=1}^m \lambda_{k_\sigma(j)} = f_\sigma(B) \prod_{k=0}^n \lambda_k^{\varphi_Z(k) - \varphi_Z(k-1)}.$$ 

Taking the maximum over all $\sigma \in S_m$ now shows that

$$\prod_{j=1}^m (\bar{P}_j Q P_j^T) \geq f(B) \prod_{k=0}^n \lambda_k^{\varphi_Z(k) - \varphi_Z(k-1)} \geq c \prod_{k=0}^n \lambda_k^{\varphi_Z(k) - \varphi_Z(k-1)}.$$  

\[ \square \]

4. The Covariant

Definition 6. Let $\tilde{Z}_m$ etc. denote the set of point clusters of degree $m$ with a choice of coordinates for the points, up to scaling the coordinates of the points with factors whose product is 1. We will call $\tilde{Z} \in \tilde{Z}_m$ a point cluster with scaling. For $\lambda \in \mathbb{C}^\times$ and $\tilde{Z} \in \tilde{Z}_m$, we write $\lambda \tilde{Z}$ for the cluster with scaling that we obtain by scaling one of the points in $\tilde{Z}$ by $\lambda$. This defines an action of $\mathbb{C}^\times$ on $\tilde{Z}_m$ such
that the quotient $\mathbb{C}^x \setminus \tilde{Z}_m$ is $\mathcal{Z}_m$. If $\tilde{Z} \in \tilde{Z}_m$, then we write $Z$ for the image of $\tilde{Z}$ in $\mathcal{Z}_m$.

**Definition 7.** For a point cluster with scaling $\tilde{Z} \in \tilde{Z}_m$, pick a representative $\sum_{j=1}^{m} P_j$ with row vectors $P_j$. Then, for $Q \in \mathcal{H}_{n,\mathbb{C}}$, represented by a Hermitian matrix, we define

$$D_{\tilde{Z}}(Q) = D(\tilde{Z}, Q) = \sum_{j=1}^{m} \log(\bar{P}_j Q P_j^\dagger) - \frac{m}{n+1} \log \det Q.$$ 

$D(\tilde{Z}, Q)$ is clearly invariant under scaling of $Q$, and it does not depend on the choice of representative for $\tilde{Z}$. Note also that for $\gamma \in G$,

$$D(\tilde{Z} \cdot \gamma, Q \cdot \gamma) = D(\tilde{Z}, Q).$$

Furthermore, we have $D(\tilde{Z}, \bar{Q}) = D(\tilde{Z}, Q)$ and $D(\lambda \tilde{Z}, Q) = \log |\lambda|^2 + D(\tilde{Z}, Q)$.

This function generalises the distance function used in Prop. 5.3 of [5]. We will now proceed to show that for stable clusters, there is a unique form $Q \in \mathcal{H}_{n,\mathbb{C}}$ that minimises this distance.

To that end, we now identify $\mathcal{H}_{n,\mathbb{C}}$ with the set of positive definite Hermitian matrices of determinant 1. This is a real $n(n+2)$-dimensional submanifold of the space of all complex $(n+1) \times (n+1)$-matrices. $\text{SL}(n+1, \mathbb{C})$ acts transitively on this space, and the tangent space $T$ at the identity matrix $I$ consists of the Hermitian matrices of trace zero. We say that a twice continuously differentiable function on $\mathcal{H}_{n,\mathbb{C}}$ is convex if its second derivative is positive semidefinite, and strictly convex if its second derivative is positive definite. Then the usual conclusions on convex functions apply.

**Lemma 8.** Let $\tilde{Z} \in \tilde{Z}_m$ be a point cluster with scaling.

1. The function $D_{\tilde{Z}}$ is convex.
2. If $Z$ is non-split, then $D_{\tilde{Z}}$ is strictly convex.
3. If $Z$ is semi-stable, then $D_{\tilde{Z}}$ is bounded from below.
4. If $Z$ is stable, then the sets $\{Q \in \mathcal{H}_{n,\mathbb{C}} : D_{\tilde{Z}}(Q) \leq B\}$ are compact for all $B \in \mathbb{R}$.

**Proof.** Since scaling $\tilde{Z}$ only changes $D_{\tilde{Z}}$ by an additive constant, we can assume that $\tilde{Z} = P_1 + \ldots + P_m$ with row vectors $P_j$ satisfying $\|P_j\|^2 = 1$.

(1) Since $D_{\tilde{Z}}(Q \cdot \gamma) = D_{\tilde{Z} \cdot \gamma^{-1}}(Q)$, we can assume that $Q = I$. We compute the second derivative at $\lambda = 0$ of $\lambda \mapsto f(\lambda) = D_{\tilde{Z}}(\exp(\lambda B))$, where $B$ is a Hermitian matrix.
trace-zero matrix (i.e., \( B \in T \)). We have

\[
D \tilde{Z}(\exp(\lambda B)) = \sum_j \log(1 + \bar{P}_j B P_j^\top \cdot \lambda + \bar{P}_j B^2 P_j^\top \cdot \lambda^2/2 + \ldots )
\]
\[
= \sum_j (\bar{P}_j B P_j^\top \cdot \lambda + (\bar{P}_j B^2 P_j^\top - (\bar{P}_j B P_j^\top)^2) \cdot \lambda^2/2 + \ldots )
\]

The second derivative therefore is

\[
\sum_j (\bar{P}_j B^2 P_j^\top - (\bar{P}_j B P_j^\top)^2) = \sum_j (\|P_j \bar{B}\|^2 \|P_j\|^2 - |\langle P_j \bar{B}, P_j \rangle|^2) \geq 0
\]

by the Cauchy-Schwarz inequality. This shows that the second derivative is positive semidefinite, whence the first claim.

(2) The second derivative vanishes exactly when every \( P_j \) is an eigenvector of \( B \) for all \( j \). Since \( B \) has trace zero, there are at least two distinct eigenvalues (unless \( B \) is the zero matrix). The ‘non-split’ condition therefore excludes this possibility, implying that the second derivative at \( I \) is positive definite. Since the condition is invariant under the action of \( \text{SL}(n+1, \mathbb{C}) \), the second derivative is positive definite everywhere.

(3) By Lemma 5, we find some \( c > 0 \) such that for \( Q \in \mathcal{H}_{n,\mathbb{C}} \) with eigenvalues \( \lambda_0 \leq \ldots \leq \lambda_n \), we have

\[
\prod_{j=1}^m (\bar{P}_j Q P_j^\top) \geq c \prod_{k=0}^n \lambda_k^{\varphi_Z(k) - \varphi_Z(k-1)}.
\]

With \( \varphi_Z(k) \leq (k + 1) \frac{m}{n+1} \), we obtain

\[
D \tilde{Z}(Q) \geq \log c + \sum_{k=0}^n (\varphi_Z(k) - \varphi_Z(k-1)) \log \lambda_k
\]
\[
= \log c + m \log \lambda_n - \sum_{k=1}^n \varphi_Z(k-1)(\log \lambda_k - \log \lambda_{k-1})
\]
\[
\geq \log c + m \log \lambda_n - \frac{m}{n+1} \sum_{k=1}^n k(\log \lambda_k - \log \lambda_{k-1})
\]
\[
= \log c + \frac{m}{n+1} \sum_{k=0}^n \log \lambda_k
\]
\[
= \log c
\]

(recall that \( \sum_k \log \lambda_k = \log \det Q = 0 \)).
We now use that 
\[ \psi_Z(k) \leq (k+1) \frac{m}{n+1} - \frac{1}{n+1} \text{ for } 0 \leq k \leq n-1. \]

The computation in the proof of (3) above then yields

\[
D_{\tilde{Z}}(Q) \geq \log c + m \log \lambda - \sum_{k=1}^{n} \psi_Z(k-1)(\log \lambda_k - \log \lambda_{k-1})
\]

\[
\geq \log c + m \log \lambda
\]

\[
- \frac{m}{n+1} \sum_{k=1}^{n} k(\log \lambda_k - \log \lambda_{k-1}) + \frac{1}{n+1} \sum_{k=1}^{n} (\log \lambda_k - \log \lambda_{k-1})
\]

\[
= \log c + \frac{1}{n+1}(\log \lambda_n - \log \lambda_0).
\]

So \( D_{\tilde{Z}}(Q) \leq B \) implies that \( \lambda_n/\lambda_0 \) is bounded, but this implies that the subset of \( Q \in \mathcal{H}_{n,\mathbb{C}} \) satisfying \( D_{\tilde{Z}}(Q) \leq B \) is also bounded. Since it is obviously closed, it must be compact. \( \square \)

**Remark 9.** Note that if \( Z \) is not stable, then there are sets \( \{Q : D_{\tilde{Z}}(Q) \leq B\} \) that are not compact. Indeed, there is a linear subspace \( L_0 \subset \mathbb{C}^{n+1} \) of some dimension \( 0 < k+1 < n+1 \) containing at least \( (k+1)m/(n+1) \) points of \( Z \). Let \( L_1 \) be its orthogonal complement. Let \( Q_\lambda \) be the Hermitian matrix with eigenvalue \( \lambda^{-(n-k)} \) on \( L_0 \) and eigenvalue \( \lambda^{k+1} \) on \( L_1 \). Then we have for \( \lambda \geq 1 \) that

\[
D_{\tilde{Z}}(Q_\lambda) \leq \text{const.} + (k+1) \frac{m}{n+1} \log \lambda^{-(n-k)} + (n-k) \frac{m}{n+1} \log \lambda^{k+1} = \text{const.};
\]

but the set \( \{Q_\lambda : \lambda \geq 1\} \) is not relatively compact.

We also see that \( D_{\tilde{Z}} \) is not bounded from below when \( Z \) is not semi-stable, since using the corresponding strict inequality, we find with a similar argument that

\[
D_{\tilde{Z}}(Q_\lambda) \leq \text{const.} - \varepsilon \log \lambda
\]

for some \( \varepsilon > 0 \).

**Corollary 10.** If \( \tilde{Z} \in \tilde{Z}_{m+} \), then the function \( D_{\tilde{Z}} \) has a unique critical point \( z(Z) \) on \( \mathcal{H}_{n,\mathbb{C}} \), and at this point \( D_{\tilde{Z}} \) achieves its global minimum \( \log \theta(\tilde{Z}) \) (for some \( \theta(\tilde{Z}) \in \mathbb{R}_{>0} \)).

**Proof:** By Lemma 8, we know that \( D_{\tilde{Z}} \) is strictly convex and that the set \( \{Q \in \mathcal{H}_{n,\mathbb{C}} : D_{\tilde{Z}}(Q) \leq B\} \) is always compact. The first property implies that every critical point must be a local minimum. By the second property, there exists a global minimum. If there were two distinct local minima, then on a path joining the two, there would have to be a local maximum, but then the second derivative would not be positive definite in this point, a contradiction. Hence there is a unique local minimum, which must then also be the global minimum and the unique critical point.
Since $D\lambda\tilde{Z} = \log|\lambda|^2 + D\tilde{Z}$, the minimising point in $\mathcal{H}_{n,\mathbb{C}}$ does not depend on the scaling, so it only depends on $Z$, and the notation $z(Z)$ is justified.

Note that we have $\theta(\lambda\tilde{Z}) = |\lambda|^2\theta(\tilde{Z})$.

Corollary 10 defines $z : \mathcal{Z}^*_{st} \to \mathcal{H}_{n,\mathbb{C}}$ and $\theta : \mathcal{Z}^*_{st} \to \mathbb{R}_{>0}$. The latter extends to

$$\theta : \tilde{Z}_m \longrightarrow \mathbb{R}_{>0}$$

with the definition $\theta(\tilde{Z}) = \inf_{Q \in \mathcal{H}_{n,\mathbb{C}}} \exp(D(\tilde{Z}, Q))$. By Lemma 8, (3), we have $\theta(\tilde{Z}) > 0$ if $\tilde{Z} \in \mathcal{Z}^*_{st}$, and by the preceding remark, $\theta(\tilde{Z}) = 0$ if $\tilde{Z}$ is not semi-stable.

**Corollary 11.** The function $z : \mathcal{Z}^*_{st} \to \mathcal{H}_{n,\mathbb{C}}$ is $\text{SL}(n + 1, \mathbb{C})$-equivariant. It also satisfies $z(\tilde{Z}) = \bar{z}(\bar{Z})$. In particular, $z$ restricts to $z : \mathcal{Z}^*_{st}(\mathbb{R}) \to \mathcal{H}_{n,\mathbb{R}}$.

The function $\theta : \tilde{Z}_m \to \mathbb{R}_{>0}$ is invariant under $\text{SL}(n + 1, \mathbb{C})$ and under complex conjugation.

**Proof:** The first statement follows from the invariance of $D$ (under the action of both $\text{SL}(n + 1, \mathbb{C})$ and complex conjugation) and the uniqueness of $z(Z)$. The second statement follows from the invariance of $D$. □

In some cases the point $z(Z)$ is uniquely determined by symmetry considerations. Namely if the point cluster $Z \in \mathcal{Z}^*_{st}$ is stabilised by a subgroup of $\text{SL}(n + 1, \mathbb{C})$ that fixes a unique point in $\mathcal{H}_{n,\mathbb{C}}$, then $z(Z)$ must be this point, compare Lemma 3.1 in [5]. This facilitates the numerical computation of $z(Z)$, since it eliminates the need for finding numerically the minimum of the distance function on $\mathcal{H}_{n,\mathbb{C}}$.

**Example 12.** Consider a sum $Z$ of $n + 2$ points in general position in $\mathbb{P}^n(\mathbb{C})$. Then $Z$ is stable. Since $\text{PGL}(n + 1, \mathbb{C})$ acts transitively on $(n + 2)$-tuples of points in general position, we can assume that the points in $Z$ are the coordinate points together with the point $(1 : \ldots : 1)$. Let this specific cluster be $Z_0$. The stabiliser of $Z_0$ in $\text{PGL}(n + 1)$ is isomorphic to the symmetric group $S_{n+2}$; its preimage $\Gamma$ in $\text{SL}(n + 1, \mathbb{C})$ acts irreducibly on $\mathbb{C}^{n+1}$. By Schur’s lemma, there is a unique (up to scaling) $\Gamma$-invariant positive definite Hermitian form. It can be checked that

$$Q_0(x_0, \ldots, x_n) = \sum_{i=0}^n |x_i|^2 + \sum_{0 \leq i < j \leq n} |x_i - x_j|^2 = (n + 2) \sum_{i=0}^n |x_i|^2 - \left| \sum_{i=0}^n x_i \right|^2$$

is invariant under $\Gamma$, hence $z(Z_0) = Q_0$. In general, we just have to find a matrix $\gamma$ such that $Z_0 \cdot \gamma^{-\top} = Z$; then

$$z(Z) = z(Z_0 \cdot \gamma^{-\top}) = Q_0 \cdot \gamma^{-\top}.$$  

Note that $Z_0 \cdot \gamma^{-\top} = \sum_j P_{0,j} \gamma$ if $Z_0 = \sum_j P_{0,j}$ and we think of the $P_{0,j}$ as row vectors. So if $Z = \sum_j P_j$, then the rows of $\gamma$ are coordinate vectors for the first $n + 1$ points in $Z$, scaled in such a way that their sum is a coordinate vector for the last point.
5. Reduction of Point Clusters

We can now define when a point cluster is reduced.

**Definition 13.** Let $Z \in \mathbb{Z}_m^\text{st}(\mathbb{R})$. We say that $Z$ is LLL-reduced, resp., Minkowski-reduced, if the positive definite real quadratic form corresponding to $z(Z)$ is LLL-reduced, resp., Minkowski-reduced.

By definition, there is an essentially unique Minkowski-reduced representative in the $\text{SL}(n+1, \mathbb{Z})$-orbit of a given point cluster $Z \in \mathbb{Z}_m^\text{st}(\mathbb{R})$. On the other hand, for computational purposes, it is usually more convenient to work with LLL-reduced representatives. In order to find an LLL-reduced representative of $Z$’s orbit, we compute the covariant $Q = z(Z)$. Then we use the LLL algorithm [3] to find $\gamma \in \text{SL}(n+1, \mathbb{Z})$ such that $Q \cdot \gamma$ is LLL-reduced. Then $Z \cdot \gamma$ is an LLL-reduced representative of the orbit of $Z$.

**Example 14.** We can use our results to reduce pencils of quadrics in three variables whose generic member is smooth. These correspond to four points in general position in $\mathbb{P}^2$. We illustrate the method with a concrete example. Let

$$Q_1(x, y, z) = 857211194051x^2 - 10879213981695xy - 1296007209476x^3 + 345181244996y^2 + 8224075847095yz + 489854396055z^2$$

$$Q_2(x, y, z) = 2274418654562x^2 - 28865567091425xy - 3438665984061x^3 + 9158614684213y^2 + 21820750429746yz + 1299719350945z^2$$

be a pair of quadrics. We first determine a good basis of the pencil spanned by $Q_1$ and $Q_2$ by reducing the binary cubic

$$\det(xM_1 + yM_2) = 27348x^3 + 215720x^2y + 567184xy^2 + 497080y^3$$

with the approach described in [5]. Here $M_1$ and $M_2$ are the matrices of second partial derivatives of $Q_1$ and $Q_2$, respectively. This suggests the new basis

$$Q'_1 = -21Q_1 + 8Q_2, \quad Q'_2 = -8Q_1 + 3Q_2$$

with already somewhat smaller coefficients; the new binary cubic is

$$-4x^3 + 88x^2y + 112xy^2 - 24y^3.$$

Now we find the four points of intersection numerically. We obtain

$$P_1 = (0.3038054131 + 0.0003625989i : -0.0712511408 + 0.000571409i : 1)$$

$$P_2 = (0.3038054131 - 0.0003625989i : -0.0712511408 - 0.000571409i : 1)$$

$$P_3 = (0.3038639670 + 0.0003672580i : -0.0712419135 + 0.000578751i : 1)$$

$$P_4 = (0.3038639670 - 0.0003672580i : -0.0712419135 - 0.000578751i : 1)$$
and from this a matrix $\gamma \in \text{SL}(3, \mathbb{C})$ that brings these points in standard position:

$$
\gamma^{-1} = \begin{pmatrix}
-1358.01 - 1762.69i & 3186.66 + 407.04i & -44719.72 - 5748.66i \\
8318.54 + 10882.75i & -1945.84 - 2556.21i & 27338.35 + 35854.08i \\
14176.55 + 2104.80i & -3324.73 - 486.76i & 46662.58 + 6870.37i
\end{pmatrix}
$$

From this, we obtain a matrix representing $z(P_1 + P_2 + P_3 + P_4)$ as

$$
\bar{\gamma} \begin{pmatrix}
3 & -1 & -1 \\
-1 & 3 & -1 \\
-1 & -1 & 3
\end{pmatrix} \gamma^T = \begin{pmatrix}
241474533625 & -1532325529959 & -182541212588 \\
-1532325529959 & 9723681808257.5 & 11583552212636.4 \\
-182541212588 & 11583552212636.4 & 137990925143.2
\end{pmatrix}
$$

(For the actual computation, more precision is needed than indicated by the numbers above.) An LLL computation applied to this Gram matrix suggests the transformation given by

$$
g = \begin{pmatrix}
3780 & 19276 & -12561 \\
-889 & -4515 & 2953 \\
12463 & 63400 & -41405
\end{pmatrix}
$$

and indeed, if we apply the corresponding substitution to $Q'_1$ and $Q'_2$, we obtain the nice and small quadrics

$$2x^2 - xy + xz + 2z^2 \quad \text{and} \quad -2xz + 3y^2 - yz + 2z^2.
$$

6. Reduction of Ternary Forms

In this section, we apply the reduction theory of point clusters to ternary forms. The idea is to associate to a ternary form, or rather, to the plane curve it defines, a stable point cluster in a covariant way. This should be a purely geometric construction working over any base field of characteristic zero.

We will only consider irreducible ternary forms $F$ of degree $d$. Assume that the curve defined by $F$ has $r$ nodes and no other singularities; then its genus is

$$g = \frac{1}{2}(d - 1)(d - 2) - r,
$$

and by [2, Exercise IV.4.6, p. 337], the number of inflection points is

$$6(g - 1) + 3d = 3d(d - 2) - 6r.
$$

We let $Z(F)$ be the sum of the inflection points, counted with multiplicity. When is $Z(F)$ stable? The first condition is that the multiplicity of any point must be less than $d(d - 2) - 2r$. Now the multiplicity is 2 less than the order of tangency of the inflectional tangent, so it is at most $d - 2$. Hence the condition is satisfied if $d - 2 < d(d - 2) - 2r$, i.e., if $0 < (d - 1)(d - 2)/2 - r = g$. The second condition is that the multiplicities of points on a line add up to less than $2d(d - 2) - 4r$. Since there are at most $d$ points on the curve on a line, this sum is at most $d(d - 2)$. Hence the condition is satisfied if $r < d(d - 2)/4.$
In any case, if $F$ defines a nonsingular plane curve of positive genus, then $Z(F)$ is stable, and we can set $z(F) = z(Z(F))$. We then define $F$ to be reduced if $z(F)$ is reduced (i.e., if $Z(F)$ is reduced).

**Example 15.** If $F$ is a nonsingular cubic, then it defines a smooth curve $C$ of genus 1, with Jacobian elliptic curve $E$. The 3-torsion subgroup $E[3]$ acts on $C$ by linear automorphisms of the ambient $\mathbb{P}^2$. The preimage of $E[3]$ in $\text{SL}(3, \mathbb{C})$ is a nonabelian group $\Gamma$ of order 27 that acts irreducibly on $C^3$. Therefore there is a unique $Q \in \mathcal{H}_{2,\mathbb{C}}$ that is invariant under the action of $E[3]$. This $Q$ is then $z(F)$. If we know explicit matrices $M_T \in \text{SL}(3, \mathbb{C})$ for $T \in E[3]$ that give the action of $E[3]$ on $\mathbb{P}^2$, then we can compute a representative of $Q$ as a Hermitian matrix as

$$Q = \sum_{T \in E[3]} M_T\top M_T,$$

compare [1, § 6].

We get the same result if we consider the cluster of inflection points on $C$, since this cluster (which is a principal homogeneous space for the action of $E[3]$) is invariant under the same group $\Gamma$. Numerically, however, the method using the action of $E[3]$ seems to be more stable. See [1, § 6] for some more discussion and details.

In general, we have to find the inflection points numerically and then find the minimum of $D\tilde{Z}$, also numerically. This can be done by a steepest descent method. We will illustrate this by reducing a ternary quartic.

**Example 16.** Let

$$F(x, y, z) = 3900908548757x^4 - 1083699236751x^3y + 835578482044x^3z$$

$$+ 1126610184312x^2y^2 - 1737329379412x^2yz + 669777678687x^2z^2$$

$$- 520542386163xy^3 + 1204081445939xy^2z - 928398396271xyz^2$$

$$+ 238611653627x^3z + 90192376558y^4 - 278168756247y^3z$$

$$+ 321720059816y^2z^2 - 165373310794yz^3 + 31877479532z^4.$$

We compute the inflection points as the intersection points of $F = 0$ and $H = 0$, where $H$ is the Hessian of $F$. This gives 24 coordinate vectors and defines the point cluster $\tilde{Z}$. We then use a steepest descent method to find (an approximation to) $z(Z)$, represented by the matrix

$$
\begin{pmatrix}
367751.9942 & -254909.8720 & 196557.1210 \\
-254909.8720 & 176692.9800 & -136245.3974 \\
196557.1210 & -136245.3974 & 105056.8935
\end{pmatrix}.
$$
LLL applied to this Gram matrix suggests the transformation
\[
\begin{pmatrix}
-7 & 23 & -89 \\
-34 & 118 & -443 \\
-31 & 110 & -408
\end{pmatrix},
\]
which turns $F$ into
\[
3x^4 - 3x^3y + 3x^3z + x^2y^2 - 2x^2z^2 + xy^2z - xyz^2 - 2xz^3 + 3y^4 - 3y^3z + y^2z^2 - 3z^4.
\]

References

[1] J.E. Cremona, T.A. Fisher and M. Stoll, Minimisation and reduction of 2-, 3- and 4-coverings of elliptic curves, Preprint (2009). arXiv:0908.1741v1 [math.NT]

[2] R. Hartshorne, Algebraic Geometry, Springer GTM 52, corr. 3rd printing (1983).

[3] A.K. Lenstra, H.W. Lenstra and L. Lovász, Factoring polynomials with rational coefficients, Math. Ann. 261, no. 4, 515–534 (1982).

[4] D. Mumford, J. Fogarty and F. Kirwan, Geometric invariant theory, 3rd enl. ed., Erg. Math. Grenzgeb. 3. Folge, vol. 34. Berlin: Springer-Verlag (1993).

[5] M. Stoll and J.E. Cremona, On the reduction theory of binary forms, J. reine angew. Math. 565, 79–99 (2003).

Mathematisches Institut, Universität Bayreuth, 95440 Bayreuth, Germany.

E-mail address: Michael.Stoll@uni-bayreuth.de