Molecular Impurities as a Realization of Anyons on the Two-Sphere

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Studies on experimental realization of two-dimensional anyons in terms of quasiparticles have been restricted, so far, to only anyons on the plane. It is known, however, that the geometry and topology of space can have significant effects on quantum statistics for particles moving on it. Here, we have undertaken the first step towards realizing the emerging fractional statistics for particles restricted to move on the sphere, instead of on the plane. We show that such a model arises naturally in the context of quantum impurity problems. In particular, we demonstrate a setup in which the lowest-energy spectrum of two linear bosonic/fermionic molecules immersed in a quantum many-particle environment can coincide with the anyonic spectrum on the sphere. This paves the way towards experimental realization of anyons on the sphere using molecular impurities. Furthermore, since a change in the alignment of the molecules corresponds to the exchange of the particles on the sphere, such a realization reveals a novel type of exclusion principle for molecular impurities, which could also be of use as a powerful technique to measure the statistics parameter. Finally, our approach opens up a new numerical route to investigate the spectra of many anyons on the sphere. Accordingly, we present the spectrum of two anyons on the sphere in the presence of a Dirac monopole field.

The study of quasiparticles with fractional statistics, called anyons, has been an active field of research in the past decades. This field has gained a lot of attention, due to the possible usage of these quasiparticles in quantum computation. In contrast to bosons and fermions, anyons acquire a phase $e^{i\alpha}$ under the exchange of two particles, where the statistical parameter $\alpha$ is not necessarily an integer. The integer cases $\alpha = 0$ and $\alpha = 1$ represent bosons and fermions, respectively. For non-integer $\alpha$, the transformation law $\Psi \to e^{i\alpha}\Psi$ under the exchange of two particles, can only be realized by allowing the wave function $\Psi$ to be multivalued. The idea is that the multiple values keep book of the different possible ways the particles could “braid” around each other. Due to the triviality of the braid group in $3+1$ dimensions, these particles are a purely low-dimensional phenomenon.

Although anyons are predicted to be realized in certain fractional quantum Hall systems, they have not yet been unambiguously detected in experiment. Indeed there has been a recent upsurge in interest concerning the realization of anyons as emergent quasiparticles in experimentally feasible systems. For instance, it has been recently shown in Refs. how these quasiparticles emerge from impurities in standard condensed matter systems. Nevertheless, all these works focus on the particles moving on the two-dimensional plane, i.e., on $\mathbb{R}^2$. Since the theory of anyons and their statistical behavior are strongly dependent on the geometry and topology of the underlying space, investigations on curved spaces reveal novel features of quantum statistics. In particular, theoretical discussions of the fractional quantum Hall effect (FQHE) for systems having various geometry and topology have widened our understanding of the FQHE. Interestingly, in a different context, it was also argued that the black hole event horizon modeled by a topological two-sphere with punctures exhibits anyonic statistics.

In the present Letter, we explore the possibility of emerging fractional statistics for particles restricted to move on the sphere, $S^2$, instead of on the plane. We show that such quasiparticles naturally arise from a system of impurities exchanging angular momentum with a many-particle bath. As a prototypical example, we consider two linear bosonic/fermionic molecules immersed in a quantum many-particle environment. In the regime of low energies, we identify the spectrum of this system with that of two anyons. This does not only allow us to realize anyons on the sphere, but also to open up various numerical approaches to investigate the spectrum of $N$ anyons on the sphere. To illustrate this, we present the spectrum of two anyons on the sphere in the presence of a Dirac monopole field, extending the recent result of Refs. Furthermore, the anyonic behaviour of linear molecules suggests that a novel type of exclusion principle holds, which concerns the alignment of the molecules, instead of the exchange of their actual position.

We start by considering a system of $N$ free anyons on the two-sphere. The corresponding Hamiltonian is given by the sum of the Laplacian of the $j$th particle on the sphere: $H_0 = -\sum_{j=1}^N \nabla_j^2$. The free anyon Hamiltonian acts on a multivalued wave function $\Psi$. By performing a suitable singular gauge transformation, $\Psi \to e^{ij\beta}\Psi$ (see Refs. 28,32), one can get rid of the multivaluedness of $\Psi$ and the free anyon Hamiltonian on the sphere $H_0$ be-
comes equivalent to

$$H_{\text{anyon}} = - \sum_{j=1}^{N} \left( \nabla_j - iA_j \right)^2 ,$$  \hspace{1cm} (1)

which now acts on single valued bosonic/fermionic wave functions. Here anyons can be depicted as bosons/fermions interacting with the magnetic gauge field $A$, which explains that the calculation of the anyonic spectra is very hard [33]. Note that $A = \nabla \beta$ is an almost pure gauge, up to the singularities of $\beta$, where the particles meet, and it can be found as the variational solution of the Chern-Simons (CS) Lagrangian $L_{\text{CS}} = \sum_{k \neq j} (A \cdot \dot{q}_k + A_0) - (4\pi\alpha)^{-1} \int_{\Sigma^2} d\Omega A \wedge dA$, where $q_k$ is the position of the nonrelativistic point particle coupled to the CS field, $A_0$ the time component of the gauge field, and $\wedge$ the wedge product. For anyons on the plane, one can always find a single magnetic potential $A$ as a solution. However, due to the non trivial homology of $\mathbb{S}^2$, the CS Lagrangian on the sphere can only be solved in two different stereographic coordinate charts: north and south patches, $A^N$ and $A^S$. As the fields $A^N$ and $A^S$ should be a single object in the overlap patch, we require them to be gauge equivalent. This equivalence is given by the Dirac quantization condition $(N-1)\alpha \in \mathbb{Z}$, see Refs. [30,31] for the details.

In what follows, in order to simplify our expressions significantly, we represent the stereographic coordinates $(x,y)$ as a complex number, $z = x + iy$. In these coordinates, we define our singular gauge transformation $F = e^{i\beta}$, with the exponent $\beta(z_1,..,z_N) = -i\alpha \sum_{j<k} \log \left( \frac{z_j - z_k}{z_j - \bar{z}_k} \right)$. The corresponding connections (gauge fields) can be written as $A_{z_j} = iD_{z_j} \beta = \frac{-\alpha(1+|z|^2)}{2} \sum_{k \neq j} (\bar{z}_j - z_k)^{-1}$ and $A_{\bar{z}_j} = iD_{\bar{z}_j} \beta = \frac{-\alpha(1+|z|^2)}{2} \sum_{k \neq j} (z_j - \bar{z}_k)^{-1}$, where we encode the contribution from the metric on $\mathbb{S}^2$ in the following differential operators: $D_z = (1 + |z|^2)\partial_z$ and $D_{\bar{z}} = (1 + |z|^2)\partial_{\bar{z}}$; see Ref. [34]. In the language of connections, $F$ represents the holonomy of $A$, and it is discontinuous along the lines which connect the particles with the north (south) pole, usually called the Dirac lines. Without loss of generality, we consider the north pole, which corresponds to the choice of $z = \cot(\theta/2) \exp(i\varphi)$, with spherical coordinates $\theta$ and $\varphi$. These lines represent the magnetic potential in the singular gauge, by assigning the particle an additional phase factor whenever it crosses them. The Dirac quantization condition makes sure that the Dirac lines are invisible, in the sense that one cannot distinguish between the theory where the lines run to the north pole and theories where they run to any other point. This means that our system is rotational invariant, up to gauge equivalences.

The anyon Hamiltonian in our complex stereographic coordinate system can be written as

$$H_{\text{anyon}} = - \sum_{j=1}^{N} \left( D_{z_j} - \bar{z}_j - A_{z_j} \right) \left( D_{\bar{z}_j} - A_{\bar{z}_j} \right) .$$  \hspace{1cm} (2)

Direct calculations to investigate the spectra of $H_{\text{anyon}}$ turn out to be problematic, when the spectrum is calculated from the bosonic end. This is due to the fact that the matrix elements of $A_{z_j}, A_{\bar{z}_j}$ for certain bosonic states are singular, which is similar to the case of anyons on the plane [20]. To overcome this difficulty we will use a different representation of the free anyon Hamiltonian, with the help of a singular pseudo-gauge transformation (also called holomorphic gauge). This transformation is given by $\Psi \rightarrow e^{\alpha \sum_{j<k} \log(z_j - z_k)} \Psi$. We call this gauge “pseudoc”, as the transformation is not unitary. The advantage of this transformation is that one of the two corresponding magnetic potentials $A'_z = \alpha D_z \log(z)$ is zero, since $\log(z)$ is a holomorphic function. Therefore, the Hamiltonian simplifies to

$$H'_{\text{anyon}} = - \sum_{j=1}^{N} \left( D_{z_j} - \bar{z}_j - A'_{z_j} \right) D_{z_j} ,$$  \hspace{1cm} (3)

in this singular pseudo-gauge. The non-zero magnetic potential is given by $A'_{z_j} = 2A_{z_j}$. Note that $H'_{\text{anyon}}$ is a similarity transformation of $H_{\text{anyon}}$, i.e., $H_{\text{anyon}} = e^{\alpha \sum_{j<k} \log(z_j - z_k)} H'_{\text{anyon}} e^{-\alpha \sum_{j<k} \log(z_j - z_k)}$, therefore these two operators have the same eigenvalues. The cost for the simplification in $H'_{\text{anyon}}$ is that it is self-adjoint in a weighted $L^2$ space. As we discuss below, while the first form of the anyon Hamiltonian [2] allows us to realize anyons in natural quantum impurity setups, the Hamiltonian [3] provides new numerical techniques to calculate the spectra of anyons on the sphere within the simplified impurity models.

We will now consider a general impurity problem $H_{\text{imp}}$ of $N$ bosonic/fermionic impurities on $\mathbb{S}^2$ interacting with some Fock space $\mathcal{F}$. Within the Bogoliubov-Fröhlich theory [35,37], the impurity Hamiltonian is given by

$$H_{\text{imp}} = - \sum_{j=1}^{N} \left( D_{z_j} - \bar{z}_j \right) D_{\bar{z}_j} + \sum_{v} \omega_v b_v^\dagger b_v$$  \hspace{1cm} (4)

$$+ \sum_{v} \lambda_v (z_1,..,z_N) \left( e^{-i\beta_v(z_1,..,z_N)} b_v^\dagger + e^{i\beta_v(z_1,..,z_N)} b_v \right) ,$$

where $b_v^\dagger, b_v$ are the bosonic creation and annihilation operators in $\mathcal{F}$, $\omega_v$ is the energy of the corresponding mode $v$, and the coefficients $\lambda_v(z_1,..,z_N)$ and $\beta_v(z_1,..,z_N)$ describe the interaction of the impurities with the Fock space, depending on their coordinates $z_1,..,z_N$. In the limit of $\omega_v \to \infty$ (the adiabatic limit), one can justify that the lowest spectrum of $H_{\text{imp}}$ is described by the Born-Oppenheimer (BO) approximation; see Ref [20] for an analysis of this assumption. The projection of the
Hamiltonian to the smaller Hilbert space manifests itself as a minimal coupling of the otherwise free particles with effective magnetic potentials \( A_{z_1}, \ldots, A_{z_N} \) and a scalar potential \( \Phi \) (see Supplemental Material). Therefore, it is sufficient to understand how \( H_{\text{imp}} \) acts on the vacuum sector.

Accordingly, we first apply the transformation \( S(z_1, \ldots, z_N) = e^{i \sum_v \beta_v b_v^\dagger b_v} \) to Eq. (4), and then project the transformed Hamilton onto the coherent state \( |\phi(z_1, \ldots, z_N)\rangle = e^{-\frac{1}{2} \sum_v \frac{1}{v^2} (b_v^\dagger - b_v)} |0\rangle \). The emerging magnetic potential in complex coordinates is then given by

\[
A_{z_j}^{\text{imp}} = i \sum_v \left( \frac{\lambda_v}{\omega_v} \right)^2 D_{z_j} \beta_v; \tag{5}
\]

see Ref. [20] for the details concerning the derivation of the emerging gauge field in the analogous planar case. Let us consider the specific choice \( \beta_v(z_1, \ldots, z_N) = -ip_v \sum_{j<k} \log \left( \frac{z_j - z_k}{z_j - z_l} \right) \), which results in

\[
A_{z_j}^{\text{imp}} = \frac{\alpha(1 + |z_j|^2)}{2} \sum_{k \neq j} (z_j - z_k)^{-1} \text{ with } \alpha(z_1, \ldots, z_N) = \sum_v p_v \left( \frac{\lambda_v}{\omega_v} \right)^2. \tag{6}
\]

We thus see that \( A_{z_j}^{\text{imp}} \) is the sought CS gauge field and obeys the Dirac quantization condition if \( \alpha(z_1, \ldots, z_N) \) is a constant and satisfies \((N-1)\alpha \in \mathbb{Z}\).

In general, the impurity Hamiltonian \( \mathcal{H}_{\text{imp}} \) corresponds to interacting anyons due the presence of the scalar potential \( \Phi \). An impurity Hamiltonian whose lowest-energy spectrum is governed by the anyon Hamiltonian in the pseudo-gauge \( \mathcal{H}_{\text{imp}} \), on the other hand, describes free anyons, as the scalar potential vanishes with \( A_{z} = 0 \). Although such an impurity Hamiltonian is not Hermitian, its non-Hermiticity is harmless for our purposes and, as we discuss below, it also opens up new numerical approaches to calculate the spectra of anyons on the sphere.

Our numerical tools work for arbitrary many particles. Nevertheless, we will here study only the two-anyon case, since the computational effort strongly scales with the number of particles. Furthermore, we investigate a configuration where the impurities are subjected to a Dirac monopole field \( B \). This allows us to investigate the spectrum for all values of \( \alpha \), as the Dirac quantization condition in the presence of a Dirac monopole field is given by \( 2B - (N - 1)\alpha \in \mathbb{Z} \). Accordingly, we consider the following one of the simplest models possible

\[
H_{\text{imp}}^{B} = H_{B} + \omega \left( b^\dagger b + \frac{\alpha}{p} \right) + \sqrt{\frac{\alpha}{p}} \omega (e^{-p \log(z_1 - z_2)} b^\dagger + e^{p \log(z_1 - z_2)} b), \tag{7}
\]

where the Hamiltonian \( H_{B} = H_{0} + \sum_{j=1}^{2} A_{z_{j}}^{B} D_{z_{j}} \) governs the bosonic/fermionic particles interacting with the Dirac monopole field \( B \) generated by the gauge field \( A_{z_{j}}^{B} = 2B_{z_{j}} \), \( p \) is an integer and we subtracted the vacuum energy, \( -\omega \alpha/p \), of the pure Fock space part of the Hamiltonian.

For a direct calculation, one could use, for instance, the orthonormal basis \( \{|S(A);n\rangle \} \), where \( |S(A)\rangle = |Y_{1,m_1} \otimes S(A) Y_{2,m_2}\rangle \) are the impurity basis with \( Y_{1,m} \) being the spherical harmonics, \( \otimes S(A) \) the (anti-)symmetric tensor product, and \( |n\rangle \) the \( n \)-particle state in the Fock space. Then, one could calculate the lowest spectrum of \( H_{\text{imp}}^{B} \) by diagonalizing the matrix \( \langle S(A); n \mid H_{\text{imp}}^{B} \mid S'(A'); n' \rangle \). Instead of this direct diagonalization technique, we first diagonalize the Fock space part of the Hamiltonian with the displacement operator. The anyon Hamiltonian \( \mathcal{H}_{\text{anyon}} \) in the presence of a Dirac monopole field, which emerges in the limit of \( \omega \rightarrow \infty \), is, then, given by

\[
\mathcal{H}_{\text{anyon}}^{B} = H_{B} + \frac{\alpha}{p} \left( e^{p \log(z_1 - z_2)} H_{0} e^{-p \log(z_1 - z_2)} - H_{0} \right). \tag{8}
\]

For details concerning the derivation, see Supplemental Material. For an even integer \( p \), the second term of the right hand side can be written in terms of the composite bosons (fermions) \( [20] \). The equation can be further simplified for the choice of \( p = 1 \). In this case Eq. (8) can be written as the following matrix equation

\[
\mathcal{F}_{\text{anyon}}^{B} = E_{\text{bos}} + 2B W_{S} + \alpha (Z^{-1} E_{\text{fer}} Z - E_{\text{bos}}), \tag{9}
\]

where the elements of the matrices are given by \( E_{\text{bos}} = \langle S|H_{0}|S\rangle \), \( E_{\text{fer}} = \langle A|H_{0}|A\rangle \), \( W_{S} = \langle S| \sum_{j=1}^{2} z_{j} D_{z_{j}} |S\rangle \), and \( Z^{-1} = \langle S|z_{1} - z_{2}|A\rangle \). As the latter two terms are straightforward to calculate numerically, and the matrix \( Z \) can be obtained by taking the (pseudo)inverse of \( Z^{-1} \), Eq. (9) opens up a new powerful route to calculate the anyonic spectrum. The spectrum from the fermionic end in terms of the relative statistics...
parameter can be calculated with the replacement of the basis $|S(A)\rangle \rightarrow |A(S)\rangle$ in Eq. (8).

As an example, we compute the eigenvalues for values of $\alpha$ ranging from 0 to 1. For an easier comparison with the result existing in Ref. [25], we calculate the spectrum from the fermionic end and present the result in terms of the relative statistics parameter such that $\alpha = 0$ corresponds to fermions and $\alpha = 1$ to bosons. The result presented in Fig. 1 is consistent with the one shown in Ref. [25], where the spectrum was calculated only for the energy levels with unit total angular momentum.

Thus, the general form of the impurity Hamiltonian [4], whose lowest spectra show anyonic behavior in the adiabatic limit, allow us to physically realize anyons on the sphere in terms of quantum impurities. First of all, the kinetic energy of the particles on the sphere, which is given by the Laplacian $-\Delta \hat{z}$, can be realized as the angular momentum operator $\mathbf{L}^2$. This enable us to consider linear molecules as the impurities. Such a realization exposes a novel correlation between molecules when they are immersed in a bath. Specifically, the exchange of the particles on the sphere corresponds to a change in the alignment of the molecules, but not the exchange of the molecules themselves, see Fig. 2 (Top). Therefore, the emerging statistical interaction manifests itself in the alignment of molecules.

To illustrate this transparent way, we consider again the simple impurity Hamiltonian [4], but in the absence of the Dirac monopole field. We note that in this case the Hamiltonian is still well-defined for all values of $\alpha$ which do not satisfy the Dirac quantization condition. The only difference for these values is that the theory is no longer fully rotational invariant, but, instead, it is invariant under rotation around the $z$ axis. In other words, the Dirac lines are not visible and they puncture the sphere. Accordingly, we investigate the alignment $\langle (\cos \theta_1 - \cos \theta_2)^2 \rangle$ as a function of the statistics parameter for two molecules. In Fig. 2 (Bottom) we present the alignment for the ground state, which is obtained from Eq. (8) for the case of $B = 0$. This could be of a new use as an experimental measure of the statistics parameter. Such a measurement can be performed, for instance, within the technique of laser-induced molecular alignment [39–40]. Further discussion of the alignment of molecules as a consequence of the statistical interaction will be the subject of future work.

A physical realization of the interaction between the molecules and a many-particle bath is also natural in the context of quantum impurity problems. Indeed, it was shown that the molecular impurities rotating in superfluid helium can be described within an impurity problem [41–43]. The resulting quasiparticle, which is called the angulon, represents a quantum impurity exchanging orbital angular momentum with a many-particle bath, and serves as a reliable model for the rotation of molecules in superfluids [44]. Therefore, we consider the following angulon Hamiltonian [45–46]

\begin{equation}
H_{\text{angulon}} = \sum_{j=1}^{2} \mathbf{L}_j^2 + V(q_1, q_2) + \sum_{k,l,m} \omega_{k,l,m} b_{k,l,m}^\dagger b_{k,l,m} + \sum_{k,l,m} \lambda_{k,l,m}(q_1, q_2) e^{-i\beta_{k,l,m}(q_1, q_2)} b_{k,l,m}^\dagger b_{k,l,m} + \text{H.c.}
\end{equation}

where $\hat{b}_{k,l,m}^\dagger$ and $\hat{b}_{k,l,m}$ are the bosonic creation and annihilation operators written in the spherical basis [41], $q_i = (\theta_i, \varphi_i)$ is the angular coordinates representing the molecular rotation of the $i$-th molecule, $V$ is a confining potential, and H.c. stands for Hermitian conjugate. Note that the coupling terms might depend on the inter-molecular distance, as well. For small values of the rotational constant, which corresponds to heavy molecules, the BO approximation can be justified with a gapped dispersion $\omega_{k,l,m}$. Furthermore, following our previous reasoning and Eq. (6), if the impurity-bath coupling satisfies the relation $i \sum_{k,l,m} \left( \frac{\lambda_{k,l,m}}{\omega_{k,l,m}} \right)^2 D_{zj} \hat{b}_{k,l,m} = A_{zj}$, then the

FIG. 2. (Top) Realization of anyons on the sphere in terms of linear molecules immersed in a quantum many-particle environment. A change in the alignment of the molecules (dumbbells), which is depicted by the white arrows, corresponds to the exchange of the particles on the sphere (dots), shown by the curvy black arrows. (Bottom) The alignment $\langle (\cos \theta_1 - \cos \theta_2)^2 \rangle$ as a function of the absolute statistics parameter for the ground state. The curve follows the bosonic state $|Y_{0,0} \otimes S Y_{0,0}\rangle$ at $\alpha = 0$ to the fermionic state $|Y_{1,0} \otimes A Y_{0,0}\rangle$ at $\alpha = 1$. We consider spherical harmonics with the angular momentum up to $l_{\text{max}} = 8$ for the numerics.
more, we consider a gapped dispersion

\[ \theta \]

concrete choice of model-parameters: \( c_i = 12(l + 1)^{-1}, \omega_i = 13, M = 1, m_{\text{Max}} = 6 \) and \( l_{\text{Max}} = 50 \).

lowest-energy spectrum of the two linear molecules immersed in the bath coincide with the spectrum of two anyons on the sphere. In principle, such an interaction is feasible with the state-of-art techniques in ultracold atomic physics.

In order to present a simple and intuitive realization, we first neglect the intermolecular distance. In this case, the coupling is defined by

\[ \lambda_{k,l,m}(q_1, q_2) = c_{k,l} \sum_{j=1}^{m} Y_{l,m}(q_j) \]

with some constants \( c_{k,l} \). Furthermore, we consider a gapped dispersion

\[ \omega_{k,l,m} = \omega_{k,l} + \Omega m \]

with \( m \) being the magnetic quantum number. Note that

\[ \sum_{k,l,m} \omega_{k,l,m} b_{k,l,m}^\dagger b_{k,l,m} \]

is not fully rotational invariant, hence the reference axis for the magnetic quantum number \( m \) matters. In the model which we investigate, the reference axis is aligned in the direction

\[ \eta_{\text{center}} = (q_1 + q_2)/(q_1 + q_2) \]

(see Supplemental Material for details). Concerning the emergence of anyons in the plane, an experimental realization of a dispersion

\[ \omega_{k,l,m} = \omega_{k,l} + \Omega m \]

was proposed in Ref. [20] by coupling \( H_{\text{angulon}} \) to an additional constant magnetic field of strength and rotating the whole system at the cyclotron frequency \( \Omega \). Note that this magnetic field breaks time reversal symmetry so that anyons can emerge on the lowest-energy spectrum; cf. Ref. [20]. A priori, the emerging statistics parameter

\[ \alpha = \alpha(\theta_r) \]

depends on the relative angle \( \theta_r \) between the points \( q_1 \) and \( q_2 \) (see Supplemental Material for the derivation of the statistics parameter). However, with a careful choice of the model parameters, \( \alpha \) becomes approximately constant. The \( \theta_r \) dependence of \( \alpha \) is demonstrated in Fig. 3 for a suitable choice of parameters. In general, the statistics parameter does not satisfy the Dirac quantization condition. Therefore, the molecular impurities correspond to anyons interacting with the magnetic potential depicted by the Dirac lines, with broken rotational symmetry.

Thus, we see that a system of two linear molecules exchanging angular momentum with a many-particle bath can give rise to a system of quasiparticles with the correct anyonic transformation law. It would be interesting to continue this approach and investigate, whether one can generalize the results above e.g. to non-Abelian Chern-Simons particles with the help of a higher order Born-Oppenheimer approximation.

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![Graph](image_url)

FIG. 3. The dependence of the statistical parameter \( \alpha \) on the relative angle \( \theta_r \). The computation is performed for the concrete choice of model-parameters: \( c_i = 12(l + 1)^{-1}, \omega_i = 13, M = 1, m_{\text{Max}} = 6 \) and \( l_{\text{Max}} = 50 \).
(1985)

[22] T. Einarsson, Phys. Rev. Lett. 64, 1995 (1990).
[23] T. Einarsson, Modern Physics Letters B 5, 675 (1991).
[24] S. Ouvry and A. P. Polychronakos, Nuclear Physics B 949, 114797 (2019).
[25] A. P. Polychronakos and S. Ouvry, Nuclear Physics B 951, 114906 (2020).
[26] P. D. M. Haldane, Phys. Rev. Lett. 51, 605 (1983).
[27] A. G. A. Pithis and H.-C. Ruiz Euler, Phys. Rev. D 91, 064053 (2015).
[28] Y.-S. Wu, Phys. Rev. Lett. 53, 111 (1984).
[29] J. Mund and R. Schrader, arXiv preprint hep-th/9310054 (1993).
[30] T. Lee and P. Oh, Physical Review Letters 72, 1141–1144 (1994).
[31] P. Oh, Nuclear Physics B 462, 551–570 (1996).
[32] S. Rao, "Introduction to abelian and non-abelian anyons," (2016), arXiv:1610.09260 [cond-mat.mes-hall].
[33] D. Lundhholm, Phys. Rev. A 96, 012116 (2017).
[34] A. Comtet, J. McCabe, and S. Ouvry, Phys. Rev. D 45, 709 (1992).
[35] H. Frohlich, Advances in Physics 3, 325 (1954).
[36] N. N. Bogolyubov, Izv. Akad. Nauk Ser. Fiz. 11, 23 (1947).
[37] L. P. Pitaevskii and S. Stringari, *Bose-Einstein Condensation and Superfluidity* (Oxford University Press, 2016).
[38] N. Park, C. Rim, and D. Soh, Physical Review D 50, 5241–5251 (1994).
[39] B. Friedrich and D. Herschbach, Phys. Rev. Lett. 74, 4623 (1995).
[40] M. Lemeshko, R. V. Krems, J. M. Doyle, and S. Kais, Molecular Physics 111, 1648 (2013).
[41] R. Schmidt and M. Lemeshko, Phys. Rev. Lett. 114, 203001 (2015).
[42] R. Schmidt and M. Lemeshko, Phys. Rev. X 6, 011012 (2016).
[43] M. Lemeshko and R. Schmidt, *Molecular impurities interacting with a many-particle environment: from ultracold gases to helium nanodroplets*, book chapter in "Low Energy and Low Temperature Molecular Scattering" edited by A. Osterwalder and O. Dulieu (Royal Society of Chemistry, 2016).
[44] M. Lemeshko, Phys. Rev. Lett. 118, 095301 (2017).
[45] E. Yakaboylu, B. Midya, A. Deuchert, N. Leopold, and M. Lemeshko, Phys. Rev. B 98, 224506 (2018).
[46] X. Li, E. Yakaboylu, G. Bighin, R. Schmidt, M. Lemeshko, and A. Deuchert, arXiv preprint arXiv:1912.02658 (2019).
Here we provide some further technical details for the Letter.

I. THE NEW ANYON HAMILTONIAN

We start by considering Eq. (6) in the Letter

\[ H'_{\text{imp}} = H_B + \omega \left( b^\dagger b + \frac{\alpha}{p} \right) + \sqrt{\frac{\alpha}{p}} \omega \left( e^{-p \log(z_1-z_2)} b^\dagger + e^{p \log(z_1-z_2)} b \right). \]  

(1)

The Fock space part of the Hamiltonian can be diagonalized with the following displacement operator

\[ T = \exp \left[ -\sqrt{\frac{\alpha}{p}} \left( e^{-p \log(z_1-z_2)} b^\dagger - e^{p \log(z_1-z_2)} b \right) \right]. \]  

(2)

The transformed Hamiltonian is written as

\[ T^{-1} H'_{\text{imp}} T = T^{-1} H_B T + \omega b^\dagger b. \]  

(3)

In the limit of \( \omega \to \infty \) the vacuum sector decouples from the rest of the spectrum, and the lowest energy levels are given by the anyon Hamiltonian in the pseudo gauge (see [20] for further details)

\[ H_{\text{anyon}}^{IB} = \langle 0 | T^{-1} H'_{\text{imp}} T | 0 \rangle = \langle 0 | T^{-1} H_B T | 0 \rangle = \sum_{n=0}^{\infty} \frac{e^{-\alpha/p}}{n!} \left( \frac{\alpha}{p} \right)^n e^{np \log(z_1-z_2)} H_0 e^{-np \log(z_1-z_2)}, \]  

(4)

where we made use of the coherent state. After using the series expansion for \( e^{-\alpha/p} \) and the double sum identity, Eq. (4) can be written as

\[ H_{\text{anyon}}^{IB} = \sum_{n=0}^{\infty} \frac{1}{n!} \left( \frac{\alpha}{p} \right)^n K_n, \text{ with } K_n = \sum_{j=0}^{n} \binom{n}{j} (-1)^{n-j} e^{jp \log(z_1-z_2)} H_0 e^{-jp \log(z_1-z_2)}. \]  

(5)

As \( \log(z_1 - z_2) \) is a holomorphic function, i.e., \( D_z \log(x) = 0 \), and \( H_B = H_0 + \sum_{j=1}^{2} A_{z_j} D_{z_j} \), \( K_n = 0 \) for all \( n \geq 2 \), the anyon Hamiltonian is further simplified to

\[ H_{\text{anyon}}^{IB} = H_B + \frac{\alpha}{p} \left( e^{p \log(z_1-z_2)} H_0 e^{-p \log(z_1-z_2)} - H_0 \right). \]  

(6)

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II. EMERGING ANYONS FROM THE ANGULON HAMILTONIAN

Let us first consider the angulon Hamiltonian

\[
H_{\text{angulon}} = - \sum_{j=1}^{2} (D_{zj} - \bar{z}_j) D_{zj} + \sum_{k,l,m} \omega_{k,l} b^\dagger_{k,l,m} b_{k,l,m} + \sum_{j=1}^{2} \sum_{k,l,m} c_{k,l} \left( Y_{l,m}(\theta_j, \phi_j) b^\dagger_{k,l,m} + Y^*_{l,m}(\theta_j, \phi_j) b_{k,l,m} \right).
\]

In the following, we want to derive the emerging magnetic fields as \( \omega \to \infty \). Since the anyon gauge field only depends on the relative position of the particles to each other, we start our analysis by fixing the center of mass coordinate \( q_{\text{center}} = (q_1 + q_2)/|q_1 + q_2| \). In order to do so, let \( T = T(q_{\text{center}}) \) be a rotation which maps the \( q_{\text{center}} \) into the south pole and \( T_{\mathcal{F}} = T_{\mathcal{F}}(q_{\text{center}}) \) its promotion to a rotation on the whole Fock space. Furthermore, let \((\theta, \phi)\) be the spherical coordinates of the rotated point \( \bar{q}_1 = T(q_{\text{center}}) q_1 \). The transformed Hamilton operator \( H'_{\text{angulon}} = T_{\mathcal{F}} H_{\text{angulon}} T_{\mathcal{F}}^{-1} \) then reads

\[
H'_{\text{angulon}} = - \sum_{j=1}^{2} (D_{zj} - \bar{z}_j - T_{\mathcal{F}} D_{zj} T_{\mathcal{F}}^{-1}) (D_{zj} - T_{\mathcal{F}} D_{zj} T_{\mathcal{F}}^{-1})
+ \sum_{k,l,m} \omega_{k,l} b^\dagger_{k,l,m} b_{k,l,m} + \sum_{k,l,m} \lambda_{k,l,m} \left( e^{im\phi_1} b^\dagger_{k,l,m} + e^{-im\phi_1} b_{k,l,m} \right),
\]

with \( \lambda_{k,l,m}(\theta) = c_{k,l} (Y_{l,m}(\theta,0) + Y^*_{l,m}(\theta, \pi)) \). We define the function \( \beta(q_1, q_2) = \phi \) as the polar angle of the rotated point \( \bar{q}_1 = T(q_{\text{center}}) q_1 \), which is indeed dependent on \( q_1, q_2 \). In the following, we want to think of \( T_{\mathcal{F}} D_{zj} T_{\mathcal{F}}^{-1} \) as a non-singular error term. To achieve this we have to confine the particles \( q_1, q_2 \) to a subregion \( O \subset S^2 \) of the sphere, since \( q_{\text{center}} = q_{\text{center}}(q_1, q_2) \) is singular for \( q_1 = -q_2 \). A suitable choice would be an open set \( O \) with \( O \subset \{ q = (x, y, z) \in S^2 : z < 0 \} \).

As a consequence, the map \((q_1, q_2) \mapsto T^{-1}(q_{\text{center}})\) is a smooth \( SO(3) \) valued map, and therefore the complex derivative \( T D_{zj} T^{-1} \) is a smooth \( \mathbb{C} \otimes \mathfrak{so}(3) \) valued map. With this at hand, we can write its Fock space promotion as \( T_{\mathcal{F}} D_{zj} T_{\mathcal{F}}^{-1} = f_j \Lambda_{\nu_j} + ig_j \Lambda_{\eta_j} \), where the \( \Lambda_{\nu_j}, \Lambda_{\eta_j} \) are the angular momentum operators oriented along the \( q_1, q_2 \) dependent unit vectors \( \nu_j = \nu_j(q_1, q_2) \in S^2 \), \( \eta_j = \eta_j(q_1, q_2) \in S^2 \) and \( f_j, g_j \) are smooth functions of \( q_1, q_2 \).

Similar to the planar case, we observe that the emerging theory for \( H_{\text{angulon}} \) in its initial symmetric form leads to anyons with the statistical parameter \( \alpha = 0 \), i.e. non interacting bosons, since symmetry breaking is required. A more interesting theory emerges from the modified Hamilton
operator

\[ H'_{\text{angulon}, \Omega} = -\sum_{j=1}^{2} (D_{z_j} - T_F D_{z_j} T_F^{-1}) \left( D_{\bar{z}_j} - T_F D_{\bar{z}_j} T_F^{-1} \right) + \Omega^2 V(q_1, q_2) \]

\[ + \sum_{k,l,m} (\omega_{k,l} + \Omega m) b_{k,l,m}^\dagger b_{k,l,m} + \sum_{k,l,m} \lambda_{k,l,m} \left( e^{im\phi} b_{k,l,m}^\dagger + e^{-im\phi} b_{k,l,m} \right), \]

where \( V \) is a confining potential. Note that \( H'_{\text{angulon}, \Omega} \) is written in the \( T_F \) transformed picture. This means, that in the original picture the magnetic quantum number \( m \) is oriented in the direction \( q_{\text{center}} \). For anyons in the plane, a realization of \( H'_{\text{angulon}, \Omega} \) can be achieved by coupling \( H_{\text{angulon}} \) to an additional constant magnetic field and rotating the whole system at the cyclotron frequency \( \Omega \).

We now define \( H_{\text{emerg}} = \langle \Psi | H'_{\text{angulon}, \Omega} | \Psi \rangle \) as the 0-th order Born-Oppenheimer approximation, where \( \Psi \) is the vacuum sector corresponding to \( H'_{\text{angulon}, \Omega} \). Note that we can write \( \Psi = S \Psi_0 \), with

\[ S = e^{i \sum_{k,l,m} m \phi b_{k,l,m}^\dagger b_{k,l,m}}, \]

\[ \Psi_0 = e^{-\frac{i}{\sqrt{2}} \sum_{k,l,m} \frac{\lambda_{k,l,m}}{\omega_{k,l} + \Omega m} \left( b_{k,l,m}^\dagger - b_{k,l,m} \right)} |0\rangle. \]

Let us abbreviate the operator \( X_{z_j} = -i \sum_{k,l,m} m \left( D_{z_j} \phi \right) b_{k,l,m}^\dagger b_{k,l,m} + ST_F D_{z_j} T_F^{-1} S^{-1} \). A straightforward computation exhibits

\[ SH'_{\text{angulon}, \Omega} S^{-1} = -\sum_{j=1}^{2} (D_{z_j} - \bar{z}_j - X_{z_j}) \left( D_{\bar{z}_j} - X_{\bar{z}_j} \right) + \Omega^2 V \]

\[ + \sum_{k,l,m} (\omega_{k,l} + \Omega m) \left( b_{k,l,m}^\dagger + \frac{\lambda_{k,l,m}}{\omega_{k,l} + \Omega m} \right) \left( b_{k,l,m} + \frac{\lambda_{k,l,m}}{\omega_{k,l} + \Omega m} \right) - \sum_{k,l,m} \frac{\lambda_{k,l,m}^2}{\omega_{k,l} + \Omega m}. \]

It is an important observation that \( \frac{\lambda_{k,l,m}}{\omega_{k,l} + \Omega m} \in \mathbb{R} \). As a consequence, we obtain \( \langle \Psi_0 | D_{z_j} | \Psi_0 \rangle = 0 \). With this at hand, we can express the 0-th order Born-Oppenheimer approximation as

\[ H_{\text{emerg}} = \langle \Psi_0 | S H'_{\text{angulon}, \Omega} S^{-1} | \Psi_0 \rangle = -\sum_{j=1}^{n} \left( D_{z_j} - \bar{z}_j - \tilde{A}_{z_j} \right) \left( D_{\bar{z}_j} - \tilde{A}_{\bar{z}_j} \right) + \Phi \]

\[ - \sum_{j=1}^{n} \left( \delta_{z_j} D_{z_j} + \delta_{\bar{z}_j} D_{\bar{z}_j} \right), \]

where \( \Phi \) is a scalar potential, \( \delta_{z_j} \) an error contribution which can be neglected in the limit \( \Omega \to \infty \) (see below) and where the emerging gauge field \( \tilde{A}_{z_j} \) is given by

\[ \tilde{A}_{z_j} = \alpha(\theta) \ D_{z_j} \phi, \]

with \( \alpha(\theta) = \sum_{k,l,m} m \left( \frac{\lambda_{k,l,m}(\theta)}{\omega_{k,l} + \Omega m} \right)^2 \). We can explicitly write down the additional emerging quanti-
ties $\Phi$ and $\delta z_j$ as

$$\delta \bar{z}_j = \langle \Psi | T_F D \bar{z}_j T_F^{-1} | \Psi \rangle = f_j \langle \Psi | A_{\nu_j} | \Psi \rangle + ig_j \langle \Psi | A_{\eta_j} | \Psi \rangle,$$

$$\Phi = \sum_{j=1}^{2} \left( \bar{A}_{z_j} \bar{A}_{\bar{z}_j} - D_{z_j} \bar{A}_{\bar{z}_j} + \langle \Psi_0 | X_{z_j} D_{\bar{z}_j} + D_{\bar{z}_j} X_{z_j} - D_{z_j} D_{\bar{z}_j} - X_{z_j} X_{\bar{z}_j} | \Psi_0 \rangle \right)$$

$$- \sum_{k,l,m} \frac{\lambda_{k,l,m}^2}{\omega_{k,l} + \Omega m} + \Omega^2 V.$$

In order to neglect the error contribution $\delta \bar{z}_j$, it is an important observation that $\delta \bar{z}_j$, in contrast to $\bar{A}_{\bar{z}_j}$, is a non-singular quantity. In the limit $\Omega \to \infty$, we expect $D_{\bar{z}_j}$ to scale like $\Omega$ and therefore the kinetic energy scales like $\Omega^2$ while $\delta z_j D_{\bar{z}_j} + \delta \bar{z}_j D_{z_j}$ only scales like $\Omega$. Therefore, we are going to neglect the error contribution (compare Ref. [20]).

Note that the statistics parameter $\alpha$ is a priori a $\theta$-dependent function

$$\alpha(\theta) = \sum_{k,l,m} m \left( \frac{\lambda_{k,l,m}(\theta)}{w_{k,l} + \Omega m} \right)^2$$

$$= \sum_{k,l,m} m \left( \frac{c_{k,l}(2\pi)^{-\frac{1}{2}} N_{l,m} (1 + (-1)^m) P_{l,m}(\cos(\theta))}{w_{k,l} + \Omega m} \right)^2.$$

However, by choosing the interaction coefficients carefully, it is possible to make $\alpha$ approximately constant. This is demonstrated in Fig. 3 in the Letter, where we omit the $k$ dependence, since $l, m$ are the essential indices. For constant $\alpha$, the emerging magnetic field $\alpha \bar{A}_{\bar{z}_j}$ is almost pure gauge, up to the singularities of $\bar{A}_{\bar{z}_j}$. In case that $\alpha$ satisfies the Dirac quantization condition $\alpha \bar{A}_{\bar{z}_j}$ is even gauge equivalent to the usual anyon field $\alpha A_{z_j}$. One can verify this, by passing to the singular gauge via the transformation $e^{i \alpha \phi}$, which has no line of discontinuity as long as $\alpha \in \mathbb{Z}$ and the right behavior under the exchange of $z_1$ with $z_2$.

We have seen that in the limit $\Omega \to \infty$ the 0-th order Born-Oppenheimer approximation of the Hamiltonian $H_{\text{angulon},\Omega}^\prime$ corresponds to anyons coupled to a scalar potential $\Phi$, at least if the parameters are chosen suitable as in Fig. 3 in the Letter. Note that for large $\Omega$, the additional confining potential $\Omega^2 V$ also justifies our assumption that the particles are confined to one of the half spheres.