THE DODDS-FREMLIN TYPE THEOREM FOR ABSTRACT URYSON OPERATORS

VLADIMIR ORLOV, MARAT PLIEV AND DMITRY RODE

Abstract. We continue the investigation of abstract Uryson operators in vector lattices. Using the recently proved “Up-and-down” theorem for order bounded, orthogonally additive operators [19], we consider the domination problem for AM-compact abstract Uryson operators. We obtain the Dodds-Fremlin type theorem and prove that for an AM-compact, positive abstract Uryson operator $T$ from a Banach lattice $E$ to an order continuous Banach lattice $F$, every abstract Uryson operator $S : E \to F$, such that $0 \leq S \leq T$ is also AM-compact.

1. Introduction

Today the theory of linear and orthogonally additive operators in vector lattices and lattice-normed spaces is an active area of Functional Analysis [1, 2, 6, 7, 8, 9, 15, 16, 17, 18, 20, 21, 22, 23]. The aim of this article is to continue this line of investigations and to prove the Dodds-Fremlin type theorem for abstract Uryson operators acting between Banach lattices.

2. Preliminaries

The goal of this section is to introduce some basic definitions and facts. General information on vector lattices and the reader can find in the books [3] [10].

Let $E$ be a vector lattice. A net $(x_\alpha)_{\alpha \in \Lambda}$ in $E$ order converges to an element $x \in E$ (notation $x_\alpha \omega \rightarrow x$) if there exists a net $(u_\alpha)_{\alpha \in \Lambda}$ in $E_+$ such that $u_\alpha \downarrow 0$ and $|x_\beta - x| \leq u_\beta$ for all $\beta \in \Lambda$. The equality $x = \bigcup_{i=1}^{n} x_i$ means that $x = \sum_{i=1}^{n} x_i$ and $x_i \perp x_j$ if $i \neq j$. An element $y$ of $E$ is called a fragment (in another terminology, a component) of an element $x \in E$, provided $y \perp (x - y)$.

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The notation \( y \sqsubseteq x \) means that \( y \) is a fragment of \( x \). If \( E \) is a vector lattice and \( e \in E \) then by \( F_e \) we denote the set of all fragments of \( e \).

An element \( e \) of a vector lattice \( E \) is called a **projection element** if the band generated by \( e \) is a projection band. A vector lattice \( E \) is said to have the **principal projection property** if every element of \( E \) is a projection element. For instance, every Dedekind \( \sigma \)-complete vector lattice has the principal projection property.

**Definition 2.1.** Let \( E \) be a vector lattice, and let \( F \) be a real linear space. An operator \( T : E \to F \) is called **orthogonally additive** if \( T(x + y) = T(x) + T(y) \) whenever \( x, y \in E \) are disjoint.

It follows from the definition that \( T(0) = 0 \). It is immediate that the set of all orthogonally additive operators is a real vector space with respect to the natural linear operations.

**Definition 2.2.** Let \( E \) and \( F \) be vector lattices. An orthogonally additive operator \( T : E \to F \) is called:

- **positive** if \( T x \geq 0 \) holds in \( F \) for all \( x \in E \);
- **order bounded** if \( T \) maps order bounded sets in \( E \) to order bounded sets in \( F \).

An orthogonally additive, order bounded operator \( T : E \to F \) is called an **abstract Uryson operator**. This class of operators was introduced and studied in 1990 by Mazón and Segura de León [13, 14], and then extended to lattice-normed spaces by Kusraev and the second named author [11, 12, 15].

For example, any linear operator \( T \in L^+(E, F) \) defines a positive abstract Uryson operator by \( G(f) = T|f| \) for each \( f \in E \). Observe that if \( T : E \to F \) is a positive orthogonally additive operator and \( x \in E \) is such that \( T(x) \neq 0 \) then \( T(-x) \neq -T(x) \), because otherwise both \( T(x) \geq 0 \) and \( T(-x) \geq 0 \) imply \( T(x) = 0 \). So, the above notion of positivity is far from the usual positivity of a linear operator: the only linear operator which is positive in the above sense is zero. A positive orthogonally additive operator need not be order bounded. Consider, for example, the real function \( T : \mathbb{R} \to \mathbb{R} \) defined by

\[
T(x) = \begin{cases} \frac{1}{x^2} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}.
\]

The set of all abstract Uryson operators from \( E \) to \( F \) we denote by \( \mathcal{U}(E, F) \). Consider some examples. The most famous one is the nonlinear integral Uryson operator.

**Example 2.3.** Let \( (A, \Sigma, \mu) \) and \( (B, \Xi, \nu) \) be \( \sigma \)-finite complete measure spaces, and let \( (A \times B, \mu \times \nu) \) denote the completion of their product measure space. Let \( K : A \times B \times \mathbb{R} \to \mathbb{R} \) be a function satisfying the following conditions:

\( (C_0) \) \( K(s, t, 0) = 0 \) for \( \mu \times \nu \)-almost all \( (s, t) \in A \times B \);

\( (C_1) \) and \( (C_2) \) are called the Carathéodory conditions.
Given $f \in L_0(B, \Xi, \nu)$, the function $|K(\cdot, f(\cdot))|$ is $\nu$-measurable for $\mu$-almost all $s \in A$ and $h_f(s) := \int_B |K(s, t, f(t))| \, d\nu(t)$ is a well defined and $\mu$-measurable function. Since the function $h_f$ can be infinite on a set of positive measure, we define

$$\text{Dom}_B(K) := \{f \in L_0(\nu) : h_f \in L_0(\mu)\}.$$ 

Then we define an operator $T : \text{Dom}_B(K) \to L_0(\mu)$ by setting

$$(Tf)(s) := \int_B K(s, t, f(t)) \, d\nu(t) \mu - \text{a.e.} \quad (*)$$

Let $E$ and $F$ be order ideals in $L_0(\nu)$ and $L_0(\mu)$ respectively, $K$ a function satisfying $(C_0)$-$(C_2)$. Then $(*)$ defines an orthogonally additive integral operator acting from $E$ to $F$ if $E \subseteq \text{Dom}_B(K)$ and $T(E) \subseteq F$.

Consider the following order in $\mathcal{U}(E, F) : S \leq T$ whenever $T - S$ is a positive operator. Then $\mathcal{U}(E, F)$ becomes an ordered vector space. If a vector lattice $F$ is Dedekind complete we have the following theorem.

**Theorem 2.4.** ([13], Theorem 3.2). Let $E$ and $F$ be a vector lattices, $F$ Dedekind complete. Then $\mathcal{U}(E, F)$ is a Dedekind complete vector lattice. Moreover for $S, T \in \mathcal{U}(E, F)$ and for $f \in E$ following hold

1. $(T \vee S)(f) := \sup\{Tg_1 + Sg_2 : f = g_1 \sqcup g_2\}$.
2. $(T \wedge S)(f) := \inf\{Tg_1 + Sg_2 : f = g_1 \sqcup g_2\}$.
3. $(T)^+(f) := \sup\{Tg : g \sqsubseteq f\}$.
4. $(T)^-(f) := -\inf\{Tg : g \sqsupseteq f\}$.
5. $|Tf| \leq |T|(f)$.

Let $E, F$ be vector lattices with $F$ Dedekind complete and $T \in \mathcal{U}_+(E, F)$. By definition

$$\mathcal{F}_T = \{S \in \mathcal{U}_+(E, F) : S \wedge (T - S) = 0\}.$$ 

For a subset $\mathcal{A}$ of a vector lattice $W$ we employ the following notation:

$$\mathcal{A}^\uparrow = \{x \in W : \exists \text{ a sequence } (x_n) \subset \mathcal{A} \text{ with } x_n \uparrow x\};$$

$$\mathcal{A}^\downarrow = \{x \in W : \exists \text{ a net } (x_\alpha) \subset \mathcal{A} \text{ with } x_\alpha \uparrow x\}.$$ 

The meanings of $\mathcal{A}^\uparrow$ and $\mathcal{A}^\downarrow$ are analogous. As usual, we also write

$$\mathcal{A}^{\downarrow \uparrow} = (\mathcal{A}^\downarrow)^\uparrow; \mathcal{A}^{\uparrow \downarrow} = (\mathcal{A}^\uparrow)^\downarrow.$$ 

It is clear that $\mathcal{A}^{\downarrow \uparrow} = \mathcal{A}^\downarrow$, $\mathcal{A}^{\uparrow \downarrow} = \mathcal{A}^\uparrow$. Consider a positive abstract Uryson operator $T : E \to F$, where $F$ is Dedekind complete. Since $\mathcal{F}_T$ is a Boolean algebra, it is closed under finite suprema and infima. In particular, all “ups and downs” of $\mathcal{F}_T$ are likewise closed under finite suprema and infima, and therefore they are also directed upward and, respectively, downward.

**Definition 2.5.** A subset $D$ of a vector lattice $E$ is called a lateral ideal if

$$(C_1) \ K(\cdot, r) \text{ is } \mu \times \nu\text{-measurable for all } r \in \mathbb{R};$$

$$(C_2) \ K(s, t, \cdot) \text{ is continuous on } \mathbb{R} \text{ for } \mu \times \nu\text{-almost all } (s, t) \in A \times B.$$
(1) if \( x \in D \) then \( y \in D \) for every \( y \in F_x \);
(2) if \( x, y \in D \), \( x \perp y \) then \( x + y \in D \).

Consider some examples.

**Example 2.6.** Let \( E \) be a vector lattice. Every order ideal in \( E \) is a lateral ideal.

**Example 2.7.** Let \( E, F \) be a vector lattices and \( T \in U_+(E, F) \). Then \( \mathcal{N}_T := \{ e \in E : T(e) = 0 \} \) is a lateral ideal.

The following example is important for further considerations.

**Lemma 2.8.** ([4], Lemma 3.5). Let \( E \) be a vector lattice and \( x \in E \). Then \( F_x \) is a lateral ideal.

Let \( T \in U_+(E, F) \) and \( D \subset E \) be a lateral ideal. Then for every \( x \in E \), we define a map \( \pi^D T : E \rightarrow F_+ \) by the following formula

\[
\pi^D T(x) = \sup \{ Ty : y \in F_x \cap D \}.
\]

(2.1)

**Lemma 2.9.** ([4], Lemma 3.6). Let \( E, F \) be vector lattices with \( F \) Dedekind complete, \( \rho \in B(F) \), \( T \in U_+(E, F) \) and \( D \) be a lateral ideal. Then \( \pi^D T \) is a positive abstract Uryson operator and \( \rho \pi^D T \in F_T \).

If \( D = F_x \) then the operator \( \pi^D T \) is denoted by \( \pi^x T \). Let \( F \) be a vector lattice. Recall that a family of mutually disjoint order projections \( (\rho_\xi)_{\xi \in \Xi} \) on \( F \) is said to be partition of unity if \( \bigvee_{\xi \in \Xi} (\rho_\xi)_{\xi \in \Xi} = \text{Id}_F \). Any fragment of the form \( \sum_{i=1}^{n} \rho_i \pi^{x_i} T \), \( n \in \mathbb{N} \), where \( \rho_1, \ldots, \rho_n \) is a finite family of mutually disjoint order projections in \( F \), like in the linear case is called an elementary fragment \( T \). The set of all elementary fragments of \( T \) we denote by \( \mathcal{A}_T \).

The following theorem ([19], Theor. 3.14) is described the structure of Boolean algebras of fragments of a positive abstract Uryson operator.

**Theorem 2.10.** ([19], Theor. 3.14). Let \( E, F \) be vector lattices, \( F \) Dedekind complete, \( T \in U_+(E, F) \) and \( S \in F_T \). Then \( S \in \mathcal{A}_T^{1\uparrow} \).

3. Result

In this section we consider a domination problem for AM-compact abstract Uryson operators. In the classical sense, the domination problem can be stated as follows. Let \( E, F \) be vector lattices, \( S, T : E \rightarrow F \) linear operators with \( 0 \leq S \leq T \). Let \( \mathcal{P} \) be some property of linear operators \( R : E \rightarrow F \), so that \( \mathcal{P}(R) \) means that \( R \) possesses \( \mathcal{P} \). Does \( \mathcal{P}(T) \) imply \( \mathcal{P}(S) \)?

Let \( E \) be a vector lattice and \( x \in E_+ \). The order ideal generated by \( x \) we denote by \( E_x \). The following theorem is an important tool for further considerations. An \( x \)-step function is any vector \( s \in E \) for which there exist
pairwise disjoint fragments \(x_1, \ldots, x_n\) of \(x\) with \(x = \bigcup_{i=1}^{n} x_i\) and real numbers \(\lambda_1, \ldots, \lambda_n\) satisfying \(s = \sum_{i=1}^{n} \lambda_i x_i\).

**Theorem 3.1.** (Freudenthal Spectral Theorem) ([3], Theorem 2.8). Let \(E\) be a vector lattice with the principal projection property and let \(x \in E_+\). Then for every \(y \in E_x\) there exists a sequence \((u_n)\) of \(x\)-step functions satisfying \(0 \leq y - u_n \leq \frac{1}{n} x\) for each \(n\) and \(u_n \uparrow y\).

**Definition 3.2.** Let \(E\) be a vector lattice and \(F\) a Banach space. An orthogonally additive operator \(T : E \to F\) is called:

1. \(AM\)-compact if for every order bounded set \(M \subset E\) its image \(T(M)\) is a relatively compact set in \(F\);
2. \(C\)-compact if the set \(T(F_x)\) is relatively compact in \(F\) for every \(x \in E\).

**Theorem 3.3.** ([14], Theor. 3.5) Let \(E, F\) be a Banach function spaces, \(F\) having order continuous norm. Then every integral Uryson operator \(T \in \mathcal{U}(E, F)\) is \(AM\)-compact.

The next theorem is the main result of the article.

**Theorem 3.4.** Let \(E, F\) be Banach lattices, \(F\) having order continuous norm, and \(T \in \mathcal{U}_+(E, F)\) be an \(AM\)-compact operator. Then every operator \(S \in \mathcal{U}_+(E, F)\), such that \(0 \leq S \leq T\) is \(AM\)-compact.

Recall that a family of mutually disjoint order projections \((\rho_\xi)_{\xi \in \Xi}\) on \(F\) is said to be partition of unity if \(\bigvee_{\xi \in \Xi} (\rho_\xi)_{\xi \in \Xi} = Id_F\). For the proof we need an some auxiliary result.

**Lemma 3.5.** Let \(E, F\) be vector lattices with \(F\) Dedekind complete, \(M \subset E\) be an order bounded set, \((T_\alpha)_{\alpha \in \Lambda} \subset \mathcal{U}_+(E, F)\) be a decreasing net of positive operators such that \(\inf (T_\alpha)_{\alpha \in \Lambda} = 0\) and \(v_\alpha = \sup \{T_\alpha(M)\}\). Then \((v_\alpha)\) is the decreasing net of positive elements of \(F\) and \(\inf (v_\alpha)_{\alpha \in \Lambda} = 0\).

**Proof.** Let us show that \((v_\alpha)_{\alpha \in \Lambda}\) is the decreasing net in \(F\). Indeed, \(F\) with a vector sublattice of the Dedekind complete vector lattice \(C_\infty(Q)\) of all extended real valued continuous functions with pointwise ordering in some extremally disconnected compact space \(Q\) (more exactly with its image under some vector lattice isomorphism), (see [10], Theorem 1.4.5). Take an arbitrary \(v_\alpha \in F_+\), then there exists the partition of unity \((\rho_\xi)_{\xi \in M}\), such that \(\rho_\xi v_\alpha = \rho_\xi T_\alpha x_\xi\), where \(x_\xi \in M\) and \(M = \{x_\xi : \xi \in M\}\). Since the net \((T_\alpha)_{\alpha \in \Lambda}\) is decreasing, we have \(\rho_\xi T_\beta x_\xi \leq \rho_\xi T_\alpha x_\xi\) for every \(\alpha \leq \beta\), \(\alpha, \beta \in \Lambda\) and every \(\xi \in M\) and therefore \(v_\beta \leq v_\alpha\). The same arguments show that \(\inf (v_\alpha)_{\alpha \in \Lambda} = 0\).

**Proof of Theorem 3.4.** Let \(T \in \mathcal{U}_+(E, F)\) be an \(AM\)-compact operator, and \(x \in E\). Firstly, we prove that operator \(\pi^x T\) is also \(AM\)-compact. Fix an order bounded set \(M \subset E\). By definition of the operator \(\pi^x T\) we have that
\( \pi^xT(M) \subset T(F_x) \). Since, \( F_x \) is an ordered bounded set and operator \( T \) is \( AM \)-compact, we have that \( \pi^xT(M) \) is the relatively compact subset of \( F \) and therefore \( \pi^xT \) is an \( AM \)-compact operator. It is clear that \( \rho \pi^xT \) is also \( AM \)-compact, for every order projection \( \rho \) on \( F \). The finite sum of \( AM \)-compact operators is also \( AM \)-compact operator. Thus, the assertion is proved for every \( S \in A_T \). Now, let \( M \) be an ordered bounded set in \( E \) and \( S \in A_T^{\uparrow \downarrow} \). Then by the definition, there exists a net \((S_\alpha)_{\alpha \in \Lambda} \subset A_T \) and a decreasing net \((G_\alpha)_{\alpha \in \Lambda} \subset U_+(E,F) \), such that \( \inf (G_\alpha)_{\alpha \in \Lambda} = 0 \) and \( |S - S_\alpha| \leq G_\alpha \), \( \alpha \in \Lambda \). Let \( v_\alpha := \sup \{G(x) : x \in M \} \). Then by the Lemma 3.5 we have that \((v_\alpha)_{\alpha \in \Lambda} \) is a decreasing net and \( \inf (v_\alpha)_{\alpha \in \Lambda} = 0 \). Fix \( \varepsilon > 0 \). Since, the norm in \( F \) is order continuous there exists a \( \alpha_0 \in \Lambda \), such that \( \|v_\alpha\| < \frac{\varepsilon}{3} \), for every \( \alpha \geq \alpha_0 \). Let \( S_\alpha(x_1), \ldots, S_\alpha(x_n) \) be a finite \( \frac{\varepsilon}{3} \)-net in \( S_\alpha(M) \). Then we may write

\[
\|S(x) - S(x_i)\| \leq \|S(x) - S_\alpha(x) + S_\alpha(x) - S_\alpha(x_i) + S_\alpha(x_i) - S(x_i)\| \leq \\
\|S(x) - S_\alpha(x)\| + \|S_\alpha(x) - S_\alpha(x_i)\| + \|S(x_i) - S_\alpha(x_i)\| \leq \\
\|G_\alpha(x)\| + \frac{\varepsilon}{3} + \|G_\alpha(x)\| \leq \\
\|v_\alpha\| + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon,
\]

where \( x \in M \). Thus \( S(x_1), \ldots, S(x_n) \) is a finite \( \varepsilon \)-net in \( S(M) \). The same arguments are valid for every operator \( S \in A_T^{\uparrow \downarrow} \). By the Theorem 3.1 an every abstract Uryson operator \( S \) such that \( 0 \leq S \leq T \) is the relatively uniform limit of some net of elements in the linear span of \( A_T^{\uparrow \downarrow} \) and therefore is the \( AM \)-compact operator.

**Corollary 3.6.** Let \( E, F \) be a Banach function spaces lattice, \( F \) having order continuous norm \( F \) be a order continuous Banach lattice and \( T \in U_+(E,F) \) be an integral Uryson operator. Then every abstract Uryson operator \( S \in U(E,F) \), such that \( 0 \leq S \leq T \) is \( AM \)-compact.

Remark that for linear positive operators the similar theorem was proved by Dodds and Fremlin in [5].

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Voronezh State University, Universitetskaya pl., Voronezh, 394006, Russia
Southern Mathematical Institute of the Russian Academy of Sciences, str. Markusa 22, Vladikavkaz, 362027 Russia

Voronezh State University, Universitetskaya pl., Voronezh, 394006, Russia