THE CONSTRAINT FOR THE LOWEST LANDAU LEVEL AND THE CHERN-SIMONS FIELD THEORY APPROACH FOR THE FRACTIONAL QUANTUM HALL EFFECT: INFINITE AND FINITE SYSTEMS

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ABSTRACT

We build the constraint that all electrons are in the lowest Landau level into the Chern-Simons field theory approach for the fractional quantum Hall system. We show that the constraint can be transmitted from one hierarchical state to the next. As a result, we derive in generic the equations of the fractionally charged vortices (quasi-particles) for arbitrary hierarchy filling. For a finite system, we show that the action for each hierarchical state can be divided into two parts: the surface part provides the action for the edge excitations while the remaining bulk part is exactly the action for the next hierarchical states. In particular, we not only show that the surface action for the edge excitations would be decoupled from the bulk at each hierarchy filling, but also derive the explicit expressions analytically for the drift velocities of the hierarchical edge excitations.

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I Introduction

The discovery of the fractional quantum Hall effect (FQHE) [1] has stimulated extensive studies on the two dimensional quantum many-electron system in a strong magnetic field. A considerable progress [2] has been made in understanding for the FQHE following the seminal paper of Laughlin’s [3]. The description of incompressible fluid states of two dimensional electron system in a magnetic field has provided a key element for such understandings [2,3]. The analogue of electrons and holes with the fractional charge in a new type of many-body condensates leads to a natural interpretation for the hierarchy scheme of the FQHE [4]. On the other hand, motivated by the analogies between the FQHE and the super fluidity [5] as well as the existence of large ring exchanges on a large length scale [6], Girvin and MacDonald [7] raised a subtle question whether there is an off-diagonal long range order (ODLRO) in the FQHE ground state. They also notice that such a ODLRO might not have the same physics in the usual sense. By introducing a 2+1 dimensional bosonization transformation, they did find a sort of the ODLRO for the bosonized Laughlin wave functions [7,8]. Such an observation gives rise to an interesting quasi-particle picture that of a charged electron in the presence of a point ‘vortex-tube’ [9]. Since then on a vast number of works appeared for the eld theoretical realization of the fractional quantum statistics and the effective eld theory description for the FQH system. Among others, the Ginzburg-Landau Chern-Simons approach (GLCS) [10,11,12] successfully interpretes a variety of the properties for the FQH system from an ab initio point of view. The chiral Luttinger liquid approach [13,14,15] for the edge excitations [16] exhibits a deep insight for such an interesting system. And the topological order approach for the long wave length behavior of the quantum Hall fluid [17] interpretes a novel sort of the order
which is not associated with broken symmetries but topological in nature, and it can be characterized by a series of quantum numbers. Furthermore, the C-S field theory approach for the FQHE can be also formulated in the fermionic picture which also interprets various properties for the FQH system [18].

Despite the successes for the various effective field theory approaches, we still have the following questions: (i) whether one should build in the constraints that all the electrons are in the lowest Landau level (LLL) from the very beginning of these approaches. As we have seen in [10,11], the "trivial Gaussian fluctuation" in the GLCS approach arises actually from the inter-Landau level degrees of freedom. From a more basic point of view, it is known that the FQH system is essentially a 1+1 dimensional system. The one dimensional nature of the FQH system should be a direct consequence of the LLL constraint. (ii) Moreover, different from those "conventional" vortices, which have their effective mass depending on the mass of the constituting particles, we expect that the explicitly built LLL constraint may play a crucial role for introducing a proper description for the massless vortices in the hierarchical FQH system in the context of C-S field-theoretical approach. (iii) A complete C-S field-theoretical approach for the FQH should not apply only to an infinite FQH system but also to a finite system. Since the propagation of the "rippling wave" along the boundary for a finite FQH system is essentially induced by the vortices on the boundary, therefore, if we could have a correct as well as unified description for the vortices in the FQH system, it is natural to raise the question whether we could have a description for a finite FQH system in which the action for the edge excitations could be derived branch by branch from the bulk actions for the corresponding hierarchical states successively. And whether the constraint for the LLL would play a non-trivial role again in such a "unified" description.
Motivated by the above arguments, in this paper, we succeed in building explicitly the LLL constraint into the C-S extended-theoretical description for the FQH system and show that both the action and the constraint can be transmitted from one hierarchical state to the next. As its primary consequence, besides the quantization conditions for the FQHE states as well as the corresponding hierarchy scheme [4] can be deduced as usual, the equations for the fractionally charged vortices for any of these hierarchical levels can be derived in generic without any mass scale dependent coefficient. It also does not depend on whether the FQHE has a BCS type of the symmetry breaking [12]. We can calculate accordingly the quasi-particle energy without difficulty. For a finite FQH system, by applying a careful treatment of the partial integrations to the actions, we show that the action for each hierarchical state can be split into two parts: a surface part provides the action of the edge excitations and the remaining bulk part is exactly the action for the next hierarchical states. In particular, the surface action for the edge excitations could be decoupled from the bulk only at each hierarchy filling. Moreover, for the n-th FQH hierarchical states, we derive analytically the expressions for the drift velocities for all the n branches of edge excitations which are different with each other and might be checked in certain properly designed experiments. To our knowledge, this might be a first time derivation for the hierarchical expressions for such drift velocities of the edge excitations. We thus provide a full dynamical description for both infinite and finite hierarchical FQH systems. This approach provides also a field theoretical background for the description of the vortices in the FQH system (quasi-particles) which can have only zero effective mass [19].

Our treatment, in certain sense, is based upon the Dirac quantization procedure [20] proceeded in the first quantization representation. It provides a sound back-
ground for the treatment for systems with constraints i.e., what we have here is the constraint for the LLL. If we restrict ourselves only for the first hierarchical level: the C-S field theory for the bosonized electrons, we may have almost the same results as those we derived in the following without the application of the Dirac quantization method. But it turns out that such a quantization procedure provides a unified highlight as well as a practically applicable method for the massless vortices of all the hierarchical states, which are, in fact, produced as the singular world lines of the phase variables of the wave fields hierarchically.

We would try to present our discussions as transparent as possible with all those detail derivations being properly included. On the meanwhile, we would like to expose all the details of our approaches if there is anything inappropriate even mistaken.

In section II we would treat the constraint for the LLL along the Dirac algorithm [20] and build it [21,22] into the dynamical description for the FQH system. Then we apply the bosonization to the fermion field which makes the bosonized electrons behave as the singular vortices controlled by the C-S gauge field. We obtain a complete path-integral description of the FQH system in the context of 2+1 dimensional C-S field theory, in which the projection to the LLL being carefully considered. In section III, by introducing the generalized (particle density)-(phase variable conjugate to the particle density) representation [10, 11, 21] for the Z-generating functional, we show that the constraint for the LLL plays a crucial role in the description for the quasi-particles and, as a result, we provide a generic description for the quasi-particles of the FQH system which applies to all hierarchical states.

Section IV is devoted specially to the finite FQH system which in fact constitutes
one of the main chapters of this paper while sections II and III might be understood, in certain sense, as the stepping stones for this and the following sections. In this section, after introducing certain proper description for the boundary of a finite two dimensional FQH system, we present a unified treatment for the surface as well as the bulk degrees of freedom and derive the action for the edge excitations from the bulk with both actions being fixed dynamically. It is interesting to realize that the constraint equation once again plays an essential role even in the derivation for the surface actions.

Section V actually completes our approach by showing that it really works for one hierarchical level to the next. We derive successively the bulk actions, the equations for the vortices and edge excitations for the next hierarchical level in detail. Right on the filling of the second hierarchical level, we show there are two coexisting branches of edge excitations which couple to each other but decouple from the bulk system. We distinguish further two limiting cases: the "strong coupling" limit at which the two branches of edge waves couple to each other strongly and the "weak coupling" limit at which these two branches are further decoupled. Base upon these discussions, we might conclude that this form again really provides a hierarchical description for the finite FQH system. In particular, we derive the explicit expressions for the propagation velocities of the edge excitations hierarchically, which should satisfy a sum rule with interesting physical consequence.

The Appendix A concerns the crucial gauge invariant properties for a finite FQH system in the context of C-S gauge field approach, while the Appendix B deals with the decoupling of branches of edge waves in the weak coupling limit.

All our calculations are given in the nonrelativistic framework.
The FQH System As a Dynamical System With The Second Class Constraint

We consider a two dimensional N-electron system subjected to a strong perpendicular magnetic field $B$ while all the electrons being in the lowest Landau level. The Lagrangian for the system has the expression as [6]

$$L = \frac{e}{c} \sum_i r_i \cdot A(r_i(t)) - \sum_{i \neq j} V(r_i, r_j)$$  \hspace{1cm} (2.1)$$

where $r_i(t)$ is the two dimensional coordinate for the $i$-th electron with $i = 1, \ldots, N$, $r_i(t) = \frac{dr_i(t)}{dt}$, $A(r_i(t))$ is the vector potential for the uniform applied magnetic field $A = B$ and $V(r_i, r_j)$ is the interaction between electrons. Throughout this paper, we shall take the axial gauge as $A = (B_y = 2, B_x = 2, 0)$ and the convention that electron's charge equals to $e$ for convenience. Different from those ordinary system, the kinetic energy term, which usually has a bilinear form of the $r_i(t)$'s, is absent in eq. (2.1). Consequently, the canonical momentum $p_i$ conjugating to $r_i$: $L = \frac{\partial L}{\partial \dot{r}_i} = (e/c)A$, would be independent of $r_i(t)$'s. Following the Dirac's algorithm [20], it can be shown that we now have the second class constraint as

$$i p_i + \frac{e}{c} A = 0$$  \hspace{1cm} (2.2)$$

where $i$ indicates Dirac's weak equality [20], and then the N-electron Hamiltonian for the system takes the form as

$$H = \sum_{i < j} V(r_i, r_j)$$  \hspace{1cm} (2.3)$$

Moreover, the canonical quantization for a system with constraints could be accomplished by the correspondence principle as: to replace the Dirac bracket $f;g_0$ of any couple of dynamical variables $f$ and $g$, i.e., $ff;gg_0$, by a quantum commutator $[f;g]=ih$, where $[f;g]$ $fg - gf$ and the canonically invariant Dirac bracket is
defined as

\[ f; g = f f; g g \]

In eq. \( (2-4) \) the script brackets without the subscript \( D \) are the usual Poisson brackets and \( , \) are the scripts for the 2-dimensional vector components. The matrix elements of \( C \) are given by \( C ; [i; j] \ f i; g j \) and \( C \). We notice further that

\[ f i; j g = 2 \frac{h}{\hbar^2} \]

where the second rank antisymmetric tensor is defined as \( 2_{12} = 2_{21} = 1 \) and the magnetic length \( \lambda = (\hbar c = e B)^{\frac{1}{2}} \). As a result, \( C [i; j] \) is a non-singular matrix. We may then work out all the Dirac brackets of the canonical variables and further quantize them. The only nontrivial commutation relation is found as

\[ [x_i; x_j] = i2 \lambda_i \lambda_j \]

i.e., the application of the Dirac quantization procedure to the system that all electrons are in the LLL makes the electrons' coordinates acquire the physics of their guiding center coordinates while the canonical momentum being consistently eliminated via the Dirac brackets. We may verify without difficulty that the constraint for the LLL can be equivalently described by the following constraint for the \( N \)-electron wave function defined in the conventional 2-dimensional space as

\[ (r_i; \ i decorated \ n \ i = 0 \]

together with the understanding that, not only the real processes, but also all the virtual processes beyond the subspace of eq. \( (2-7) \) are prohibited at all, where

\[ i = (x_i \ i \ y_i) = \frac{p}{\hbar^2} \]. A detailed account for the application of the Dirac's quantization on such a constraint system is presented in literature [22].
Base upon the above treatment which is accomplished in the first quantization representation, we may introduce the corresponding description in the second quantization representation accordingly. Following eqs. (2-3) and (2-7), the second quantized Hamiltonian now has the form as

\[ H = V \left[ \hat{\psi}^\dagger(x) \hat{\psi}(x) \right] \]

\[ = \frac{1}{2} \int d^2 r_1 d^2 r_2 \left[ \hat{\psi}^\dagger(r_1) \hat{\psi}(r_1) \right] V(r_1 r_2) \left[ \hat{\psi}^\dagger(r_2) \hat{\psi}(r_2) \right] \] (2-8)

while the electron wave field operator \( \hat{\psi}(r) \) satisfying the fermion statistics is subjected to a LLL constraint that

\[ \hat{\psi}(r) = 0 \] (2-9)

where \( \int d^2 r^\dagger(r) \) with \( S \) being the total area of the system and should be equal to the average charge density contributed by the positive background. One can easily verify that the projection to the LLL, even for the virtual processes, is rigorously guaranteed by the constraint (2-9) in the second quantization representation.

By applying the standard procedure, now we introduce further the bosonized representation \( \langle x \rangle \) for the electron field \( \langle x \rangle \) \([7,10,11]\) as

\[ \langle x \rangle = e^{i \langle x \rangle} \langle x \rangle \] (2-10)

with the definition

\[ \langle x \rangle = \frac{1}{Z} \int d^2 z^0 \text{Im} \ln(z z^0) \] (2-11)

and the C-S gauge field can be defined as

\[ a(x) = 5 \langle x \rangle \] (2-12)
In eq. (2-11) and the following, it is often convenient to introduce the complex notations as

\[ z = \frac{1}{2} (x + iy); \quad \bar{z} = \frac{1}{2} (x - iy) \]

\[ \partial = \frac{\partial}{\partial z} = \frac{1}{2} \left( \frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right); \quad \bar{\partial} = \frac{\partial}{\partial \bar{z}} = \frac{1}{2} \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right) \tag{2.13} \]

and

\[ A = \frac{1}{2} (A_x + iA_y) = \frac{B}{2} z; \quad \bar{A} = \frac{1}{2} (A_x - iA_y) = \frac{-B}{2} \bar{z} \tag{2.14} \]

Substituting eqs. (2-10), (2-11) and (2-12) into eqs. (2-8) and (2-9), and noticing eqs. (2-13) and (2-14), we have

\[ H = \frac{1}{2} z \int d^2r_1 d^2r_2 \left( \hat{r}_1 \right) V \left( r_1, r_2 \right) \left( \hat{r}_2 \right) \tag{2.15} \]

and the LLL constraint becomes

\[ \bar{\partial} \left( \frac{\partial}{\partial z} + i \frac{1}{z} A + i a \right) \hat{r} = 0 \tag{2.16} \]

In eq. (2-15), \( \hat{r}(z) = \hat{r}^+(x) \hat{r}(x) \) and \( \hat{r}(z) = S \int d^2r \hat{r}^+(r) \hat{r}(r) \). Due to the singular behavior of function \( \text{Im} \ln(z - z^0) \), following from eq. (2-11), we may derive

\[ 2 \frac{\partial}{\partial z} a = i \frac{\partial}{\partial a} \frac{\partial}{\partial a} = 2m \hat{r} \tag{2.17} \]

which relates the magnetic field of C-S gauge potential to the particle density and has the physical intuition as: attaching magnetic flux of the C-S field to an electron [10,11]. If we impose further the equation of continuity,

\[ \frac{\partial}{\partial t} \hat{r} + \bar{\partial} j(r) = 0, \]

then, the time derivative of C-S gauge potential should relate to the matter current as

\[ 2 \frac{\partial}{\partial z} a = 2m j(r) + 2 \frac{\partial}{\partial a} a \tag{2.18} \]

up to a trivial divergence free term.
Taking into account of all the above considerations as well as the fact that the constraint for the LLL should be imposed on all the time slices in the dynamical evolution, the path integral representation for the Z-generating functional would have the following form

\[ Z[A] = \int D^{} \mathcal{D}^{} a \left( e^{i \int dx L_0} \right) \]  \hspace{1cm} (2.19)

with

\[ L_0 = (i \partial_0 a_0) V \left( \begin{array}{c} \mathcal{B} \mathcal{G} \\ \mathcal{B} \mathcal{G} \end{array} \right) \left( \begin{array}{c} \frac{1}{2} a_0 \\ \frac{1}{4} \end{array} \right) + \frac{1}{2} a_0 a_0 + a_0 a_0 \]  \hspace{1cm} (2.20)

where the gauge fixing condition is understood involved implicitly and \[ \mathcal{B} \mathcal{G} \] is the -functional. Comparing to the conventional 2+1 dimensional C-S field theory, we have not only two second class constraints for the LLL being explicitly built in but also an action in which the kinetic energy is absent. In fact this is a sort of the non-relativistic C-S field theory with its interacting matter field being massless.

III Description For The Vortices (Quasi-particles) In The FQH System

Since now we are in the boson representation, we prefer to introduce the phase \[ \phi(x) \] and the electron density \[ \rho(x) \] for the wave field as the dynamical variables by taking

\[ \phi(x) = q \frac{\rho(x) e^{i \phi(x)}}{\rho(x)} \]  \hspace{1cm} (3.1)

The phase variable \[ \phi(x) \] bears the description for the vortices and can be further decomposed into a regular part \[ \pi \] and a singular part \[ s \] as \[ [10,11] \]

\[ \phi(x) = \pi(x) + s(x) \]  \hspace{1cm} (3.2)
in which \( r \) and \( s \) satisfy

\[
2 \Theta(r) = 0 \quad (3.3)
\]

and

\[
2 \Theta(s) = 2s(x) \quad (3.4)
\]

respectively. We notice that \( s \) has the physical intuition as the density for the vortices. We then substitute eq. (3.1) into eq. (2.16) and its conjugate, the constraint for the LLL can then be expressed in terms of \( - \) variables as

\[
f(\); \( 1 / 2 \Theta_l + i \Theta_l / 2 \Theta_l + i \Theta_s / 2 \Theta_l + i \frac{1}{2} - 1 \Theta_l + ia = 0
\]

\[
f(\); \( 1 / 2 \Theta_l + i \Theta_l / 2 \Theta_l + i \Theta_s / 2 \Theta_l + i \frac{1}{2} - 1 \Theta_l + ia = 0 \quad (3.5)
\]

The \( Z \)-generating functional (2.19) becomes

\[
Z[A] = \int \int \int \int f(\); \( f(\); \( f(\); \( \frac{1}{2} \Theta_l^2 + i \Theta_l^2 + i \Theta_s^2 + i \frac{1}{2} - 1 \Theta_l + ia = 0
\]

\[
\exp \left( \frac{i}{2} \int d^2 x f(\); \( a_0 + e') V[\] \( \frac{1}{2} \Theta_l^2 + i \Theta_l^2 + i \Theta_s^2 + i \frac{1}{2} - 1 \Theta_l + a = 0 \]
\]

\[
\exp \left( \frac{i}{2} \int d^2 x f(\); \( a_0 + e') V[\] \( \frac{1}{2} \Theta_l^2 + i \Theta_l^2 + i \Theta_s^2 + i \frac{1}{2} - 1 \Theta_l + a = 0 \]
\]

where we included an applied electric field with \( e' \) (x) being its scalar potential. It is quite clear from eqs. (3.5) and (3.6) that, as a result of introducing the \( - \) representation, the C-S field acquires a gauge term: \( a \). \( a + \Theta_l = 0; 1; 2 \), not only in the matter part of the action but also in the constraints. It is known that the action for the C-S term of the gauge field itself is invariant respect to the local gauge transformation up to a surface term. Therefore, we may eliminate the regular part of the phase variables \( r \) by performing a gauge transformation \( a \). \( a + \Theta_l \) for the \( Z \)-generating functional expression eq. (3.6) and forget about the induced surface term \( K[a; r] \) tentatively. We will come back to this induced surface term.
in the next section. Moreover, by taking a linear combination of $\partial f = \partial z$ and $\partial f = \partial z$ in which the $\partial$ has been eliminated as just mentioned, the constraints eq. (3-5) can be transformed into the following equivalent form as

$$\frac{1}{2} 5^2 \ln + \frac{1}{2} 2^2 2^2 a = 0 \quad (3 \text{ 7})$$

and

$$5 \quad a = 0 \quad (3 \text{ 8})$$

We then carry out the integration over the zero-component C-S field $a_0$ in eq. (3-6) and recover the C-S constraint (2-17) rst. By solving eqs. (3-8) and (2-17), we may integrate further $D a_1 D a_2$ in eq. (3-6). Finally we derive

$$Z[A] = \int D D s F[; s; B] \exp i d^2 x f_{-s} + e' V[ ] + \frac{1}{4} m^2 2 \quad a a g$$

$$\quad (3 \text{ 9})$$

with

$$F[; s; B] = \frac{1}{2} 5^2 \ln + \frac{1}{2} 2^2 m^2 2^2 s = 0 \quad (3 \text{ 10})$$

and $a$ being now the solution of eq. (2-17) in consistency with the gauge xing condition eq. (3-8). In this equation, the term $s^2$ could be understood as $(e = h c) 5 \quad A$. We would like to emphasize here that apart from surface term $K[a; r]$ contributed by the C-S term due to the gauge transformation $a \quad a \quad a \quad s$, we have not done any partial integration in the above derivations.

By now we derive the Z-generating functional for the FQH system in the representation. We see that the LLL constraint not only makes the electrons' kinetic energy disappear, but also manifests itself as a functional relation among $s$ and $B : F[; s; B] = 0$, which plays a crucial role in the understanding of the properties for the FQHE states. The contributions from the C-S field which had
been introduced non-trivially for the bosonization procedure now transfer partly their effect to the statistics index $\hbar^m$ appearing in the constraint functional $F [s; B]$ while the remaining effect is still born by the term $(4m)^{1/2} a$. If we imagine the functional integral $D_s$ in eq. (3-9) being carried out, we may understand that the eq. (3-9) describes a system with $s$ as its only independent dynamical variable. Since $2 \Theta \Theta$ can be nonzero only at certain singular 2+1 dimensional world lines, so $s$ is a smooth functional in space except those singular points (at vortex positions). We interpret these propagating singular points as point particle-like vortex cores. Then the vortex density should have the expression as $s(x) = \sum_j q_j^2 (x - x_j(t))$ with $q_j = 1$ being the vortex charge and $x_j(t)$'s being the world line for the $j$th vortex. The vortex current $j_s(x) = \sum_j q_j x_j(t)^2 (x - x_j(t))$ can also be equivalently expressed as

$$j_s(x) = \frac{1}{2} 2^2 \Theta \Theta \Theta \Theta \Theta \sum_j q_j x_j(t)^2 (x - x_j(t))$$

We can easily verify that the expressions (3-4) and (3-11) are consistent with the conservation of the vortex current: $\omega_s + \Theta j_s = 0$. Keeping with the above understandings, it is obvious that in the expression for the $Z$-generating functional eq. (3-9), the path integral over $D_s$ is essentially an evolution in the 1st quantization representation for the vortices.

It is straightforward to derive from the $Z$-generating functional eq. (3-9) the following equation

$$\frac{1}{2} 2^2 \langle \ln \rangle > 2 m < \rangle + \frac{1}{2} 2 < A_s > = 0$$

where $\langle \rangle$ is the path integral average over the normalized $Z$-generating functional, i.e., average over the physical ground state. This equation in fact had been
rst time derived directly from the constraint equations for the LLL by applying the collective field theory approach [21,23]. What we have here more is to make its connection to the dynamics being explicit. For a homogeneous system with zero vortex, we derive the quantization condition from eq. (3-12) for the FQHE states, 
\[ \sigma = (2m^2)^{1/4} \text{ immediately.} \]
For a single vortex, we can draw the conclusion easily from this equation that it carries a fractional charge of \( qe = m \) where \( q > 0 \) corresponds a quasi-hole. So this equation can be interpreted as the equation for the vortices (quasi-particle) of the rst hierarchy. Its mean field solution can be solved numerically without difficulty and then the energy for the quasiparticles can be calculated subsequently. We notice that different from the usual G-L type description, there is no mass-scale dependent parameter appearing in eq. (3-12). It also does not depend on whether there is a \( \mathrm{BCS} \) type symmetry breaking [12] in the FQHE state.

In the constraint equation (3-10), \( \sigma \) has the \( \psi \) function like singularities at the location of each vortex. While the main role played by the \( 5^2 \ln \) is to cancel such singularities since the \( (r) \) should have certain drastic variations close to the vortex centers. If we further introduce the second quantization representation for the vortices, such singularities would be smeared out in the wave field description. Hence, the \( 5^2 \ln \) term would be no more interesting as the main physics are usually controlled by the long wavelength behaviors. Therefore, for sake of convenience, we would ignore the \( 5^2 \ln \) term in the following with the understanding that there is always a term \( 2 \theta \ln \psi \) associated with \( \theta \), implicitly in the rst quantization representation of the vortices, while such a term could be reasonably ignored in its second quantization representation.
IV Intimate Relation Between Edge Excitations And Hierarchical Structure For A Finite FQH System

Now we shall treat the finite FQH system, i.e., to separate the surface part of the action properly from the bulk part for a finite FQH system. Before going into the details we would like to introduce certain descriptions for the boundary of a finite FQH system. We imagine that the two dimensional system is enclosed by a (spatially) one dimensional boundary. The continuity equation \( \partial_j \Theta = 0 \) can then be written in the integral form as

\[
\int d^2x \Theta(t) = \int dl \, n \wedge v
\]

where \( dl \) is the linear integral along the boundary and \( n \) is the unit normal vector of the boundary being defined always oriented outward from the system. If we imagine a finite period of time \( t \), it becomes \( \int d^2x = \int dl \, n \wedge r \) in which we have introduced a displacement vector \( r \) defined formally along the boundary. We may express as \( \Theta(\sim) = \sim \), where \( \sim \) is certain initial distribution of the electrons in the system. Then, we have

\[
\int d^2x (\sim) = \int dl \, n \wedge r
\]

If we take \( \sim = \) with \( n \) being the average electron density, the lefthand side of the equation should be zero, so that we should have

\[
\int dl \, n \wedge r = 0
\]

Consequently \( r \) can be interpreted either as the displacement for the particles (electrons) passing back and forth through the boundary or as the "rippling" displacement for the boundary [15] deviating out-or inward along the boundary. Obviously, it is understood that these equations are valid up to the first order of \( r \). If
we split $s$ into two parts: $s = \frac{\text{bulk}}{s} + \frac{\text{surf}}{s}$, correspondingly,

$$s = \frac{\text{bulk}}{s} + \frac{\text{surf}}{s}$$

(4.4)

we then have

$$\frac{\text{bulk}}{s} = \frac{1}{2} \ln 2 \ \& \ \frac{\text{surf}}{s}$$

(4.5)

which contributes to the average vortex density of the system $s$ and

$$\frac{\text{surf}}{s} = \frac{1}{2} \ln 2 \ \& \ \frac{\text{bulk}}{s}$$

(4.6)

which is nonzero only at the boundary, and has zero contribution to the $s$ so that $s = \frac{\text{bulk}}{s}$. Making use of the constraint eq. (3-10), we can have both

$$= \frac{1}{2} m^2 \frac{1}{m} \left( \frac{\text{surf}}{s} + \frac{\text{bulk}}{s} \right)$$

(4.7)

and

$$= \frac{1}{2} m^2 \frac{1}{m} \frac{\text{bulk}}{s}$$

(4.8)

where the $5^{\frac{1}{2}} \ln$ terms are ignored with the previously mentioned understanding.

By taking $\sim$ and then substituting eqs. (4-7) and (4-8) into eq. (4-2), we may draw the expression for $r$ from eq. (4-2) as

$$r = \frac{1}{2} m^2 \frac{1}{m} \frac{\text{surf}}{s}$$

(4.9)

up to an arbitrary gauge transformation $\frac{\text{surf}}{s} \# \frac{\text{surf}}{s} + 0$ where $0$ is a regular function defined along the $: H \ln 2 \ \& \ 0_r = 0$ but not determined yet.

Moreover, since a finite two-dimensional FQH system is always confined by some potential, its chemical potential, $\mu$, is determined in such a way that the Gibbs free energy is minimized consistently with the spatial distribution of the electrons. Therefore, the local deviation of the applied electric potential, $e'$, from the chemical
potential at the boundary is equal to the work done by those electrons that passed through the boundary, or in another words, due to the local displacement of the boundary from its equilibrium configuration. Again in the sense of the first order deviation, we should then have

\[(e')j = e'(\nu_0)j = eE\quad \text{r}j \quad (4\ 10)\]

where \(E\) is the applied electric field and can be expressed as \(E = 5'\).

Intuitively, the boundary is an "ininitely thin layer" with a "thickness" of order of the "rippling" displacement \(r\). Such a boundary layer is a layer of \(\text{surf}_{s}\), i.e., in which and only in which \(\text{surf}_{s}\) has nonzero value locally. It has further the following properties

\[\int_{x}^{x} d^{2}x_{s} \text{surf} = 0 \quad (4\ 11)\]

and

\[\int_{b}^{b} n_{s}j = 0 \quad (4\ 12)\]

where \(d^{2}x_{s}\) means a 2D integration carrying over only this surface layer region.

We may also verify without difficulty that eqs. (4-11) and (4-12) are consistent with eqs. (4-4) to (4-8). Eq. (4-11) has the physical meaning similar to those of \(r\) in eq. (4-3) that \(\text{surf}_{s}\) describes the local accumulation or dissipation of the particles in the surface layer with its total accumulation (dissipation) being kept equal to zero. Moreover, since the description for the displacement of the particles (electrons) passing back and forth through the boundary (which results the local accumulation and dissipation of the particle density) has been taken care by eq. (4-11), as a result, we should have eq. (4-12) for consistency. We notice also that eq. (4-12) is valid only up to the leading order where the unit vector \(n\) is defined as the normal of the outer boundary of the layer. If we view the boundary as a surface layer in sense of
eqs. (4-11) and (4-12), then we can show that eq. (4-9) applies locally to the whole boundary layer region. In fact, we may divide imaginary the surface layer further into many sub-layers with the requirement that each of them having eq. (4-11) being satisfied. But for now, instead over the whole boundary region, we should have the 2D integration in eq. (4-11) carrying over only those sub-layers under consideration. Therefore each intersurface between two successive sub-layers encloses an area with its interior bulk part coinciding exactly with that of the original system but its surface layer being only an inner part of that of the original system. Obviously we then can apply the same arguments to derive eq. (4-9) like equation on each intersurface in the interior of the boundary layer, so that, eq. (4-9) is indeed valid within the boundary layer locally. Furthermore, following the similar spirit, it is not difficult to verify that eq. (4-10) is also valid within the boundary layer.

For the term \( \int_{R}^{d^2x} (e') \) in the action of eq. (3-9), by utilizing the constraint eq. (3-10) or eq. (4-7), we have

\[
\int_{R}^{d^2x} (e') = \int_{R}^{d^2x} \left[ \frac{1}{2m} \frac{1}{z} \left( \frac{1}{z} \right) \right](e')
\]

We notice that the term \( \int_{R}^{d^2x}(2m^2)^{-1}(e') \) in the r.h.s. of the above equation will not contribute to the dynamics of the system since \( e' \) is due to the applied electric potential and is a constant determined by the envelope potential. We would like further to keep the bulk term in the r.h.s. of eq. (4-13) to be retained. Moreover, by applying eq. (4-6) to the surf which is nonzero only in the boundary layer, the remaining term in the r.h.s. of eq. (4-13) can be rewritten as

\[
\frac{1}{2m} \int_{R}^{d^2x}(2 \Theta \Theta_{surf})(e')
\]

Taking into account of eqs. (4-9) and (4-10) with the understanding that both of
the two being valid in the whole boundary layer, eq. (4-14) becomes

\[ \frac{1}{(2\,m)^2} \int_x^Z \text{d}^2x \text{d}t (2 @ \text{surf} (E @ \text{surf})) (4-15) \]

We now introduce the following identity for the integrand of the expression eq. (4-15) as

\[ @ M \text{EM} @ (M \text{EM}) \frac{1}{2} @ E @ (M \text{EM}) (2 M \text{E}) (2 @ M) \]

with \( M \) being identified as \( 2 @ \text{surf} \). Since \( 2 @ M = @ @ \text{surf} \), we may choose the gauge for \( \text{surf} \) and make the last term on the r.h.s. of the above identity becomes zero. Substituting the identity into expression (4-15) and then \( \int_x^Z \text{d}^2x \) can be transformed into a "surface" integral \( \int \text{d}l \) which encloses the boundary layer by two line integral one for the outer boundary and the other for the inner boundary, i.e.,

\[ \frac{1}{(2\,m)^2} \int_x^Z \text{d}t \int \text{d}l @ \text{surf} (E @ \text{surf}) (4-16) \]

Without loss of generality, we may assume reasonably that up to the leading order of \( r, n 2 @ \text{surf} \) being zero at the inner boundary line while \( n @ \text{surf} \) taking the same value locally at both boundary lines. Noticing further that \((@ \text{surf})^2 = (n @ \text{surf})^2 + (n 2 @ \text{surf})^2\), then expression (4-14), i.e., eq. (4-16) can be transformed into the following form

\[ \frac{1}{2(2\,m)^2} \int_x^Z \text{d}t \int \text{d}l (E 2 @ \text{surf}) (4-17) \]

Taking into all the above considerations, we derive from eq. (4-13) that

\[ \int_x^Z \text{d}^2x \text{d}t (e') \]

20
\[ = \frac{eE}{2(2m)^2} \int d^2x dt (n(2 \theta_s^\text{surf})^2 \frac{1}{m} \int d^2x dt \theta_s^\text{bulk}(e') \) \] (4 18)

where we have assumed the electric field always parallel to the normal on the boundary.

For the last term of the action (see eq. (3-9)), \( -m (4 \mathbf{m})^2 \frac{1}{2} \) a \( \mathbf{a} \), we notice \( \mathbf{a} \) is the solution of eq. (2-17) which can be expressed in terms of \( \mathbf{s} \) by making use of eqs. (4-7) and (3-4) as

\[ a = \theta_s \frac{1}{2B} A^\text{em} \] (4 19)

Therefore, by applying further eqs. (4-7) and (4-19)

\[ \int d^2x dt ( -m + \frac{1}{4m} \frac{1}{2} \mathbf{a} \mathbf{a} ) \]

\[ = \int d^2x dt \left[ \frac{1}{2m} (2 \theta_s^\text{surf})^2 + \frac{1}{4m} \frac{1}{2} \theta_s \theta_s \right] \] (4 20)

where \( \mathbf{a} \) (and afterward) we have ignored \( \theta_s \) (would ignore) all those integrands of a total time derivative. Taking a partial integration with respect to the \( \partial \) " in the rst term, expression (4-20) becomes

\[ = \int d^2x dt \left[ \frac{1}{2m} (2 \theta_s^\text{surf})^2 + \frac{1}{4m} \frac{1}{2} \theta_s \theta_s \right] \] (4 21)

For the purpose of separating the "surface" and "bulk" degrees of freedom, we express \( \mathbf{s} \) further as \( \mathbf{s} = \mathbf{s}^\text{surf} + \mathbf{s}^\text{bulk} \) in eq. (4-21). Utilizing the following equalities

\[ \int d^2x dt \theta_s^\text{surf} \theta_s^\text{bulk} = \int d^2x dt \theta_s^\text{bulk} \theta_s^\text{surf} = 0 \]

\[ \theta_s^\text{surf} + \theta_s^\text{bulk} = 0 \]

\[ \theta_s^\text{surf} + \theta_s^\text{bulk} = 0 \]
and eq. (4-12), we can derive the following expression from eq. (4-21) by straightforward calculations

\[
\frac{1}{m} \int_{m}^{Z} d\xi \partial_{s}^{2} \varphi_{\text{bulk}}^{s} + \frac{1}{4m} \int_{m}^{Z} d\xi \partial_{s}^{2} \varphi_{\text{surf}}^{s} + \frac{1}{4m} \int_{m}^{Z} \partial_{s}^{2} \partial_{s}^{2} \varphi_{\text{surf}}^{s}
\]

where we have also utilized the expression for \( \varphi_{\text{bulk}}^{s} \) and \( \varphi_{\text{surf}}^{s} \) given by eqs. (4-5), (4-6) and (3-11).

Now we introduce a dual gauge \( \mathbf{A} \) for the bulk system as

\[
\mathbf{A}^{0} = \frac{1}{m} \partial_{s}^{2} \varphi_{\text{bulk}}^{s} \mathbf{A}^{\text{em}}
\]

Making use further of eqs. (4-5) and (4-7), it satisfies

\[
2 \odot \mathbf{A}^{0} = 2 \odot \varphi_{\text{bulk}}^{s}
\]

Substituting eq. (4-23) into the first two terms of expression (4-22), we derive step by step the following expression as

\[
\frac{1}{4m} \int_{m}^{Z} \partial_{s}^{2} \partial_{s}^{2} \varphi_{\text{surf}}^{s} + \frac{1}{4m} \int_{m}^{Z} \partial_{s}^{2} \partial_{s}^{2} \varphi_{\text{surf}}^{s} + \frac{1}{4m} \int_{m}^{Z} \partial_{s}^{2} \partial_{s}^{2} \varphi_{\text{surf}}^{s}
\]

Finally, take into account of all the above considerations, and substitute eqs. (4-25) and (4-18) into the corresponding terms of eq. (3-9) in which \( \mathbf{e} \) being replaced by \( \mathbf{e}(\mathbf{g}) \) as for a finite system, we obtain an interesting form of the \( Z \)-generating functional for the finite FQH system

\[
\mathbb{J} = \frac{Z}{D_{\text{surf}}} \mathbb{D}_{\text{surf}} \left[ \frac{Z}{D_{\text{bulk}}} \mathbb{D}_{\text{bulk}} \right]
\]
The surface action in eq. (4-26) has the form as

\[
I_{[s]} = \frac{1}{4m} \int d^3x \rho_{s} A^0_0 \left( \frac{1}{m} \rho_{\text{bulk}} (e') \right) d\text{Iface} \left( \begin{array}{c} \text{surf} \\ \end{array} \right) \text{d}r \left[ \text{surf} \right] \left( \begin{array}{c} \text{surf} \\ \end{array} \right)
\]

where we have assumed the applied electric field \( E \) is parallel to the normal on the boundary and \( \rho_{\text{surf}} \) can be derived from eq. (4-18) by applying eq. (4-8) as

\[
\rho_{\text{surf}} = \rho_{\text{surf}} = (1 \ 2 \ 2_s)
\]

with \( \rho_{\text{surf}} = eE \).

In eq.(4-26), \( F \left( [s]; B \right) \) is in fact a product of functions

\[
F \left( [s]; B \right) = \prod_x F \left( x; \rho_{s} (x); B \right)
\]

where \( Q_x \) is the product over all the 2D spatial position \( x \)'s and \( F \left( x; \rho_{s} (x); B \right) \) has exactly the same expression as that of eq.(3-10) but picks its value at the spatial points \( x \). Since the hardcore vortices can never coincide at the same spatial point, we may regroup \( Q_x \) into two products as the following. The first product, \( Q_x \), picks up those singular points (attached with its nearest neighbouring regular points) at which only the surface vortices locate. Obviously, these "mini-islands" (may or may not overlap) exist only in the boundary layer region. The second product, \( Q_{\text{bulk}} \), picks up all the other spatial points in both the bulk interior and the remaining points in the boundary layer region in which only the bulk vortices may locate. Therefore, we may identify \( s (x) = \text{surf} (x) \) for those functions in the first product, and
\( s(x) = \text{bulk}_s(x) \) for those functions in the second product. We then have the following expression

\[
\left[ F \left[ ; \text{bulk}_s B \right] \right]^{Y}_{x^2} \left[ F \left[ \text{surf}_s(x); \text{surf}_s B \right] \right] \left[ F \left[ \text{bulk}_s B \right] \right] \tag{4.30}
\]

in which

\[
\left[ F \left[ ; \text{bulk}_s B \right] \right]^{Y}_{x^2} \left[ F \left[ \text{surf}_s(x); \text{bulk}_s B \right] \right] \tag{4.31}
\]

Keeping with the similar understanding, we may further separate the integral measure of \( R^Q \) into two corresponding parts as

\[
Z \text{surf}_s D \text{surf}_s Z \text{bulk}_s D \text{bulk}_s = Z \text{surf}_s D \text{bulk}_s \text{surf}_s \text{bulk}_s \tag{4.32}
\]

Now we introduce the notation

\[
D^T \text{surf}_s \text{surf}_s Z \text{bulk}_s \text{bulk}_s \left[ F \left[ \text{surf}_s(x); \text{bulk}_s B \right] \right] \tag{4.33}
\]

where \( R^Q \) \( x^2 \left[ F \left[ \text{surf}_s(x); \text{bulk}_s B \right] \right] \) means to solve \( x \) as the functional of \( \text{surf}_s(x) \) from eq.(3-10) in the boundary region. Taking into consideration of eqs.(4-30) (4-33), the generating functional (4-26) can be put into the following form as

\[
Z = D^T \text{surf}_s \text{surf}_s Z \text{bulk}_s \text{bulk}_s \left[ F \left[ \text{surf}_s(x); \text{bulk}_s B \right] \right] \exp i \left[ \frac{1}{4} \int A^0 \nabla \cdot A^0 \left( e' \right) \cdot V \left[ s g + I \left[ s \right] \right] \right] \tag{4.34}
\]

For a finite system, if the integration over \( D \left[ F \left[ ; \text{bulk}_s B \right] \right] \) has been taken into account, eq. (4-34) means that the \( Z \)-generating functional for the \( \text{FQH} \) many electron system can be equivalently described in terms of its vortex degrees of freedom while the electrons can be understood as a background condensate. The corresponding action can be divided into two parts: a bulk part and a surface part. The bulk
and carry the fractional statistics \((m)\) with fractional charge \(q = m\). It can be interpreted as the action for the next hierarchy. In particular, when the system is exactly in a FQHE state of the first hierarchical level, i.e., \(b^\text{bulk}_s = b^\text{bulk}_g = 0\), we then have \(s = \text{surf}\), so that the surface action \(I'_s\) will decouple from its bulk and describe an ensemble of independent edge excitations with its propagation velocity \(v^\text{surf}_b = v^\text{surf}_o\). This is one of the interesting results drawn from our approach with its description mainly based upon the constraint condition eq. (3-10). We notice that if we solve \(A^0_s\) in terms of \(b^\text{bulk}_s\), and apply further eq. (3-10) for the \(j^\text{bulk}_s\) term, we may find easily that the bulk action is formally rather similar to that of [10, 11]. The action \(I'_s\) in the FQH state has the form known as a chiral boson action which is consistent also with those proposed in [14, 15]. What we have here is a unified description for a finite FQH system derived from ab initio analytically.

We stress further that if we perform a gauge transformation to the whole action (4-26), it would also produce a surface term which may cancel the surface term left previously in section III. We will show the details in Appendix A.

As we have mentioned before, because \(s(x)\) has only the isolated singularities in the two dimensional plane, \(D^s\) integrates over only the space-time propagation of those singularities: the coordinates of vortices. Therefore, it is not difficult to show that

\[
Z \left( \exp \int d^3 x f^\text{bulk}_s A^0 - \frac{m}{4} A^0 \partial_0 A^0 g \right)^2 = \prod_{j=1}^{N} (r^0_j(t))^{X} D^0_j \left( \exp \int d^3 x f^\text{bulk}_s A^0 (r^0_j(t)) - \frac{m}{4} A^0 \partial_0 A^0 g \right)
\]

where \(r^0_j(t)\) is the coordinate for the \(j\)th bulk vortex. We notice that, following...
from eq. (3-10), we always take the convention that the vortices are counted as quasi-holes. This identity makes the following fact become explicit. The bulk action for the vortices in eq. (4-34) is essentially in a first quantization representation. Moreover, it becomes clear that such an action again involves only terms linear in the first order time derivative of the vortex coordinates but no bilinear term. We may learn from the Dirac's algorithm immediately that once again we have a system of vortices with "zero kinetic energy" which should be described by the second class constraint. In fact, comparing eqs. (4-34) and (4-35) with eq. (2-1), keeping again the understanding that the functional integration over \( D \) being carried through, we can realize that the bulk action for the vortices has a form almost the same as the original action for the electrons in the LLL. Now it becomes also quite clear that the application of the Dirac's quantization theory for the constrained system to the overall space-time propagation of the vortices in the form of eq. (4-35) provides a old-theoretical background for treating these hierarchical vortices (quasi-particles) in FQHE which have only zero effective mass while the "conventional" vortices often have finite effective mass contributed by the massive constituting particles.

V Schematic Outline For The Higher Hierarchical States And The Corresponding Branches of Edge Excitations

Based on the above observations, we may apply the same procedure as those for the electrons to introduce the second quantization representation for the bulk vortices (of the first hierarchy). But there are certain delicate differences which should be carefully treated as the following: (i) Instead of the vector potential \( A \) which couples to the electron velocity and has a constant curl, \( 5 \quad A = B \) as the applied magnetic field, we have now a vector potential \( A^0 \) for the bulk action of the
vortices which plays a similar role but has a curl, $A^0 = 2$, depending on the dynamical variable via the constraint equation $F [ ; B_{\text{bulk}} ] = 0$; (ii) In the application of the Dirac quantization to the vortices in the first quantization representation, we need the condition $[ 0^i ; 0^j ] = 2 2 2 i j \neq 0$ to be satisfied, where $0^i$ has the same form as $i$ with the corresponding quantities substituted by those for the vortices. Since $0^i$ could be zero (or singular) only at the isolated locations for the vortices, in the spirit of long wavelength approximation, we may reasonably take the approximation as $0 > 0$ (finite). In fact, these singular behaviors at the vortex locations will disappear after its second quantization procedure being completed; (iii) Corresponding to the bosonization procedure for the electrons in which we introduced a C-S gauge field with the statistical index being odd integers $m$, we now introduce a C-S gauge field $0^0$ with the statistical index being even integers $2p$.

This is because that the world lines for the vortex "particles" are originated from the singularities of the phase field $s$ of the bosonized electrons, so that they have to have a periodic boundary condition at the $1$ and $+1$ of the time axis [24]. By such a "bosonization" of the vortices, the newly introduced "C-S" gauge field satisfies the gauge constraint as

$$2 \theta 0^0 = 4 p B_{\text{bulk}} \quad (5.1)$$

Comparing eq. (5.1) with eq. (3.4), we have

$$0^0 = 2p \theta B_{\text{bulk}} \quad (5.2)$$

which in fact has the same physics as eq. (2.12). But eq. (2.12) is for the electrons while eq. (5.2) is for the vortices with one hierarchical level in succession.

Substituting eq. (5.2) into eq. (4.23), we have

$$A^0 = \frac{1}{m} A^m + \frac{1}{2p} 0^0 \quad (5.3)$$
This is a relation between the dual field and the new C-S field.

Taking into account all the above considerations, introducing the bosonized "wave field" for the bulk vortices, and running over almost exactly the same procedure as those for the electron case given in the section II, we may introduce the second quantization representation for the vortex part of the the Z-generating functional (4-34). Consequently, it can be transformed into the following form as

\[
Z = D_s^{\text{surf}} D_s^0 D_s^0 \exp \left[ \frac{i}{\hbar} \int_{\mathcal{L}} \frac{1}{m} (e')^0 a_s^0 V_s^0 \right]
\]

where we have also substituted eq. (5-3) into eq. (4-34) and notice that the first term on the r.h.s. of eq. (5-3) would not contribute to the C-S term in eq. (4-34). Since \( s = s_{\text{bulk}} + s_{\text{surf}} \), we understand that the dynamical variable \( s \) for the surface action has its bulk part being now defined in the second quantization representation while its surface part being not. We notice further that in eq. (5-4) and the following, except \( s_{\text{surf}} \), all the second quantized dynamical variables as well as their functional integration measure, such as \( a_0^{0^*} \) and \( V_0^0 \) etc. are of bulk degrees of freedom, and we would keep such understanding but ignore the "bulk" sup- or subscripts for convenience. Separating the modulus part of \( a_0^{0^*} \) from its phase part by writing

\[ a_0^{0^*} = P_s e^{i \theta} \]

with \( \theta = \phi_s + \phi_s^{0^*} \), absorbing the regular part of phase variable \( \phi_s \) into \( a_s^0 \) (see Appendix A) and then integrating over \( a_s^0, a_0^0, a_0^{0^*} \) and \( V_0^0 \) in the Z-generating functional (5-4) as what we did for the electrons in the section III, it becomes

\[
Z = D_s^{\text{surf}} D_s^0 D_s^0 \exp \left[ \frac{i}{\hbar} \int_{\mathcal{L}} \frac{1}{m} (e')^0 a_s^0 V_s^0 \right] \exp \left[ \frac{1}{16 \hbar^2} \left( \frac{1}{m} + 2p \right) 2 a_0^0 a_0^{0^*} + I_s^0 \right]
\]
with \( V^0_s = V \left[ \left( \begin{array}{c} s \\ s \end{array} \right) \right] = m \) and

\[
F^0_s(0; \overline{s}; B) = \frac{1}{2} 5^2 \ln \frac{1}{m^2} + 2 \left( \frac{1}{m^2} + 2p \right) + 2 \overline{s} = 0 \tag{5.6}
\]

where \( a^0 \) is the solution of eq. (5-1) associated with an appropriate gauge fixing condition which is determined again by the constraint \( a^0 \overline{a} = 0 \) and its complex conjugate. (see the corresponding eqs. (3-7) and (3-8), especially (3-8)). And \( s \), the density of the vortices, is the modulus of the vortex wave field which now is in the second quantization representation, while \( \overline{s} \) is the singular part for the conjugated phase field which describes the isolated 'vortices' for the next (higher) hierarchical level with its density having the expression as

\[
\overline{s} = \frac{1}{2} - \left( \begin{array}{c} 0 \\ 0 \end{array} \right) \tag{5.7}
\]

These 'vortices' have the intuition as 'new quasiholes' on the 'old quasihole' condensate so that they are essentially electron-like excitations in nature. We may further solve \( s \) from eq. (5-6) with the consideration of eq. (5-7) as

\[
s = \frac{1}{2} \left( 1 + 2p \right)^2 \left( \frac{1}{m^2 + 2p} \right) \tag{5.8}
\]

In the above derivations, we have carried out the path integral for \( D \) so that the constraint equation (3-10) \( F \left[ \left( \begin{array}{c} s \\ s \end{array} \right) ; B \right] = 0 \) is understood being always satisfied and the ingredient of the constraint (3-10) has been now transmitted into eq. (5-6). If we divide eq. (3-10) by \( 2m \), eq. (5-6) by \( 2 \left( m^{-1} + 2p \right) \) and then compare themselves each other, we may find that instead of \( m^{-1} \) for the vortices of the first hierarchical level, the charge unit of the vortices of the second hierarchical level becomes \( (1 + 2p)^{-1} \). Correspondingly, the statistics index also changes from \( m^{-1} \) to \( \left( m^{-1} + 2p \right)^{-1} \).
We may further separate one more surface part of the action in eq. (5-5) from the bulk in sense of the next hierarchical level. We may work along exactly the same line as those of the electrons from eqs. (4-1) to (4-28). Since we have also the current conservation of the vortices: $\theta_s^0 \partial_s \mathbf{v}_s^0 = 0$, and especially instead of eq. (3-10), we have now the constraint equation eq. (5-6), therefore, by noticing the correspondence as $s \in \text{surf}$, $e = m \in \text{surf}$, and the C-S factor

$$m! \left( \frac{1}{m} + 2p \right)$$

we may split $s \in \text{surf}$ into $s \in \text{surf} + s \in \text{bulk}$, and follow the same line as those of eqs. (4-1) to (4-10), and derive

$$\frac{e^m \partial_s}{m} = \frac{e^m E}{m} \partial_s$$

with

$$r = \frac{1}{2 (m + 2p)} \left[ \partial_s \theta_s^\text{surf} \right]$$

in the boundary layer $x = 0$. In repeating such a processing, we have an interesting question that whether the "boundary" for the second hierarchical level $0$ coincides the boundary of the first hierarchical level: Formally, the FQH system should have only one unique boundary on which all the surface integrals for the system should be defined, i.e., $0 = \cdot$. But intuitively, as it has been already carefully discussed in the previous section, the boundary carries a sort of ripple-like edge waves with an amplitude of order of $r$. It can be equivalently described in terms of the surface vortices in sense of the first hierarchical level which are spreaded over a surface region of depth $r$ and form a boundary layer. We separated the surface degrees of freedom from those of the bulk in such a way that the latter covers not only the whole region of the bulk interior of the 2D FQH system but
also the boundary layer in sense of those surface vortices with its nearest regular
eighbourhood being excluded. This is the basic physics of the boundary, based
upon which we introduced further the surface measure $\mathcal{D}_s^{\text{surf}}$ of eq.(4-33) and the
bulk measure in eq.(4-32). Following the same intuition, the boundary $0$ is in
fact the boundary of the bulk region of the first hierarchical level. It should be a
rippling region with a depth of $r^0$ but accommodates inside the bulk region in a
rather complicated way. In other words, we could imagine that these two successive
boundary layers permeate into each other heavily, and we would like to say that it is
of the 'strong coupling limit'. We may imagine an opposite limiting case: all the
surface vortices of the second hierarchical level, which is essentially the origin of the
surface rippling of the boundary $0$, distribute inside the boundary layer $0$ and form
a layer as $0$. We may have consequently the boundary layer $0$ accommodates inside
the boundary layer with a sharp separation, i.e., up to the second hierarchical level,
the FQH system has two successive boundary regions with the outer boundary being
while the inner one being $0$. We say that is of the 'weak coupling limit'. After
the physics of the two coexisting boundaries being clarified as above, corresponding
to eqs. (4-11) and (4-12), we have that, in the boundary layer $0$

$$
\int_x \mathcal{D} x \mathcal{D}_s^{\text{surf}} = 0 \quad (5\, 12)
$$

and

$$
\mathcal{J}_s^{\text{bulk}} \mathcal{F}_k = 0 \quad (5\, 13)
$$

where $n^0$ is the normal of the boundary $0$. Keeping with such an understanding,
we may process further as follows.

Solving eq. (5-6) for $s$ and splitting then $\mathcal{D}_s^{0}$ into $\mathcal{D}_s^{\text{surf}} + \mathcal{D}_s^{\text{bulk}}$, we substitute it
into the second term of the action in eq.(5-5). We would like to keep the $\mathcal{D}_s^{\text{bulk}}$ term
to be survived and perform a partial integration for the remaining terms. This is in fact the same procedure as done in eqs. (4-13)-(4-18) but with one hierarchical level higher. As the result, the term involving the applied electric field in action eq. (5-5) can then be transformed into the following expression as

\[ \frac{1}{m} \int_{s}^{z} d^2x \int_{t}^{s} (e') = \frac{1}{1 + 2p} \int_{s}^{z} d^2x \int_{s}^{0} \text{bulk} (e') \]

\[ \frac{eE}{2m (2 \cdot (m + 2p))^2} \int_{s}^{z} dt \int_{o}^{l} dln \left[ 2 \cdot \text{surf} \right]^2 \]

(5-14)

On the meanwhile, we solve \( a^0 \) from eq. (5-1) and then utilize eq. (5-6), we derive

\[ a^0 = \frac{2m p}{1 + 2m p} \frac{1}{m} \int_{s}^{z} d^2x \int_{s}^{0} \text{A}_{em} \left[ ?^0 \text{S}_{bulk} \right] \]

(5-15)

In the above equation we ignore the \( 5^2 \text{ln} s \) term with the same understanding as those for \( 5^2 \text{ln} s \) in the previous sections. And we introduce further a dual field \( A^{\infty} \) for the new bulk system which is the correspondent of \( A^0 \) introduced by eq. (4-23)

\[ A^{\infty} = \frac{1}{m \cdot (m + 2p)} \int_{s}^{z} d^2x \int_{s}^{0} \text{A}_{em} \left[ ?\text{bulk} \right] \]

(5-16)

with

\[ 2 \cdot \text{A}^{\infty} = 2 \cdot \text{bulk} \]

(5-17)

Then applying almost the same procedure as those from eq. (4-19) to eq. (4-25) correspondingly, the first as well as the third term of the action in eq. (5-5) can be transformed into the following form as

\[ \int_{s}^{z} d^2x \int_{t}^{s} \left[ ? \right] \]

\[ = \frac{1}{4 \cdot (m + 2p)} \int_{s}^{z} dt \int_{o}^{l} dln \left[ \left( \text{surf} + \text{bulk} \right) \right] \]

\[ \frac{1}{4} \int_{s}^{z} d^2x \int_{s}^{0} \text{bulk} + \frac{1}{4} \int_{s}^{z} d^2x \int_{s}^{0} \text{bulk} \]

(5-18)

32
In deriving eq. (5-18), we notice that the expression for the \( s_s^0 \), eq. (5-7), has a formal sign difference with that of eq. (3-4), therefore the corresponding expression for \( j_s^0 \) should also have a sign difference with that of eq. (3-11) formally. Substituting eqs. (5-14) and (5-18) into eq. (5-5) and applying further those arguments as well as treatments similar to that of eqs. (4-29) to (4-34), the \( Z \)-generating functional can be put into a new form as

\[
Z = \hat{D}^\text{surf} Z_0 \hat{D}^\text{bulk} D_s^0 \left[ \mathcal{F}^0[s_s^0; \ \hat{D}^\text{bulk} B] \right]
\]

\[
= \exp i \int \text{d}^3x f \hat{A}^\text{bulk} + \frac{1}{1+2pm} \hat{A}^\text{bulk} (e') V^0[s_s^0]
\]

\[
+ \frac{1}{4} \left( \frac{1}{m} + 2p \right) 2 \ A^0 \partial \hat{A}^0 + I[s_s^0] + I^0[s_s^0]
\]

(5-19)

with the additional surface action as

\[
I^0[s_s^0] = \frac{1}{4} \left( \frac{1}{m} + 2p \right) \int_0^1 dt \ dt \ 2 \ \partial \hat{A}^\text{surf} (n \ 2 \ \partial \hat{A}^\text{surf})^2 g
\]

(5-20)

where the drift velocity for the new edge excitations is now

\[
\nu_0^0 = \nu_0 = (1 + 2m \ \hat{A}^0) : \ (5-21)
\]

For now we practiced our scheme once again that the bulk action for the vortices eq. (4-34) can be also divided into two parts: a surface part may describe one more branch of edge excitations, while the remaining bulk part is exactly for the third hierarchical states. Both of them have their forms almost the same as those given in eqs. (4-34) and (4-27), and the only difference is that we have now the statistics parameter changed from \( m \) to \( m + 2p \) and the fractional charge changed from \( e = m \) to \( e = (1 + 2pm) \). Especially, noticing the sign difference between eq. (3-4) and
eq. (5-7), the surface action $I^0_s[\sigma^s]$ of eq.(5-20) has consistently an additional global minus sign compared to $I_s[\sigma_s]$ of eq.(4-27). The fact that these signs change from one to the next reflects the hole-particle nature for the vortices of different hierarchical levels which depends actually on our convention that we keep the vortex particles as quasiholes for each hierarchical level.

For a homogeneous system with $0_s$ being equal to zero, it means that the system is now lying exactly on the second hierarchical FQHE filling, i.e., we have a condensate for both electrons and vortices. Then the constraints eq. (3-10), $F[\sigma_s;B] = 0$, and eq. (5-6), $F^0[\sigma_s;0_s;B] = 0$, will give the expression for filling factor as

$$\frac{1}{m + \frac{1}{2p}} \quad (5.22)$$

For the system having isolated vortices on the condensate of the second hierarchical level, then $F^0[\sigma_s;0_s;B] = 0$ will provide the corresponding vortex equation with each vortex carrying a fractional charge as $(1 + 2p m)^{-1}$. It becomes so obvious that our approach does provide a dynamical description for these massless vortices for whole hierarchical scheme.

One more interesting question is for the surface actions as we derived now two surface actions co-existing in a FQH system at the second hierarchical level. For the second one, eq. (5-20), we have the understanding that $0_s^g = g^g_{s,\text{bulk}} + g^g_{s,\text{surf}}$ where the $g^g_{s,\text{bulk}}$ is contributed by the $g^g_{s,\text{bulk}}$ while $g^g_{s,\text{surf}}$ is contributed by the $g^g_{s,\text{surf}}$. If the FQH system is precisely on the second hierarchical level with $g^g_{s,\text{bulk}} = g^g_{s,\text{bulk}} = 0$ so that we have $0_s^g = g^g_{s,\text{surf}}$ then the surface action eq. (5-20) will be decoupled from the bulk as $I_s[0_s^g]$! $I_s[\sigma^s]$ and on the meanwhile, its drift velocity $v_0^s$ becomes $v_0$. But on the other hand, due to the boundary $0^s$ accommodates inside
the boundary, the \( s_{\text{surf}} \) and \( s_{\text{surf}} \) should contribute in principle to the \( s_{\text{bulk}} \)

variable demand on. Therefore, if we split \( s_{\text{surf}} \) into its bulk and surface part in eq.

(5-6) with the condition \( s_{\text{bulk}} = 0 \), we have

\[
s = \frac{1}{2} \frac{1}{(1 + 2pm)} \frac{1}{m} \frac{1}{1 + 2p} \quad s_{\text{surf}}
\]

(5 23)

In eq.(5-23), \( s \) actually satisfies eq.(4-5) since we had ignore the \( \text{"bulk"-superscript} \)

for the second quantized \( s \) after (including) eq.(5-4). Moreover, \( s_{\text{surf}} \) satisfies an
equation of the same form as that of eq.(5-8) but with \( s_{\text{bulk}} \) and \( s_{\text{surf}} \) substituted by \( s_{\text{bulk}} \)

and \( s_{\text{surf}} \) respectively. Then we may solve \( s_{\text{surf}} \) from eq.(5-23) as

\[
s_{\text{surf}} = \frac{1}{m} \frac{1}{1 + 2p} \quad s_{\text{surf}} = \frac{1}{m} \frac{1}{2B} \quad s_{\text{surf}}
\]

(5 24)

up to a trivial curl free 2-dimensional vector. On the other hand, we may also

express \( s_{\text{bulk}} \) directly in terms of \( s \) which is entirely equivalent to eq.(4-5),

\[
s^{\text{bulk}} = Z \frac{\mathcal{d}^2 x \Im \ln(z \cdot z^0)}{s(z^0)}
\]

subsequently, we have

\[
s^{\text{bulk}} = Z \frac{\mathcal{d}^2 x \Im \ln(z \cdot z^0)}{s(z^0)}
\]

(5 25)

For the \( s_{\text{surf}} \); we should have similar equations followed from eq.(5-7) with the

condition \( s_{\text{surf}} = 0 \); these are

\[
s_{\text{surf}} = Z \frac{\mathcal{d}^2 x \Im \ln(z \cdot z^0)}{s(z^0)}
\]

and

\[
s_{\text{surf}} = Z \frac{\mathcal{d}^2 x \Im \ln(z \cdot z^0)}{s(z^0)}
\]

(5 26)

By utilizing further eq. (5-23) again, we may show that

\[
s_{\text{bulk}} = s_{\text{surf}} = \frac{1}{m} \frac{1}{1 + 2p} \quad s_{\text{surf}}
\]

(5 27)
The underlying physics could be understood as follows: due to the further condensation of the vortices on the first hierarchical level, the singular behavior for the boundary vortices preserves and transmits itself into the singular behavior for the vortices of the next hierarchical level via the constraint equation (5-6) or eq.(5-23). Substituting eqs. (5-24) and (5-27) into the surface action eq. (4-27), it becomes

\[
\mathcal{I}^s \mathcal{I}^s_{\text{surf}} = \frac{1}{4m} \int_0^Z dt \int d\mathbf{l}_f \left[ \mathcal{I}^s_{\text{surf}} \mathcal{I}^s_{\text{surf}} \left( \mathcal{I}^s_{\text{surf}} + \frac{1}{m+2p} \mathcal{I}^s_{\text{surf}} \right) \right] 
\]

Eq.(5-28) contains only surface variables \( \mathcal{I}^s_{\text{surf}} \) and \( \mathcal{I}^s_{\text{surf}} \) so that they decouple also from the bulk as long as the system is on the second FQH hierarchy: \( \mathcal{I}^\text{bulk} = 0 \). But the two branches of edge excitations described by \( \mathcal{I}^s_{\text{surf}} \) and \( \mathcal{I}^s_{\text{surf}} \) will form ally couple to each other as shown by the explicit expressions for actions \( \mathcal{I}^s_{\text{surf}} \) and \( \mathcal{I}^s_{\text{surf}} \) as eqs.(5-20) and (5-28) respectively. In the weak coupling limit, i.e., the two boundary layer and \( \mathcal{I}^s_{\text{surf}} \) being sharply separated, due to \( \mathcal{I}^s_{\text{surf}} \) has its source \( \mathcal{I}^s_{\text{surf}} \) being nonzero only strictly inside the boundary layer, we may show in the Appendix B that \( \mathcal{I}^s_{\text{surf}} \) will not contribute to eq. (5-28). It would be then simplified to the following form as

\[
\mathcal{I}^s \mathcal{I}^s_{\text{surf}} = \frac{1}{4m} \int_0^Z dt \int d\mathbf{l}_f \left[ \mathcal{I}^s_{\text{surf}} \mathcal{I}^s_{\text{surf}} \left( \mathcal{I}^s_{\text{surf}} + \frac{1}{m+2p} \mathcal{I}^s_{\text{surf}} \right) \right] 
\]

so that the two branches of edge excitations will further decouple into two independent edge excitations. Associated with the action eq.(5-20) in which \( \mathcal{I}^\text{bulk} \) being
now set to be zero, $s^\text{surf}_S$ describes one branch of edge excitation propagation along boundary $^0$ with drift velocity $v_D$; while $s^\text{surf}_S$, associated with the action (5-29), describes one another branch of edge wave with the propagation velocity $v_D$.

The interesting point is that the latter would have a different drift velocity from that of $I_0(s^\text{surf}_S)$. Following from eq. (5-8), we have now $s = (2^2(1 + 2m \rho))^{1/2}$ which is non zero for the second FQHE hierarchical level. By substituting it into eq. (4-28) we derive then

$$v_D = v_0 (1 + \frac{1}{2m \rho})$$

This is a rather interesting result that we derived the analytical expressions for the propagation velocities of the edge excitations which are different for its different branches. We expect it could be checked by certain properly designed experiment.

So far, we derived the corresponding edge excitations for the second hierarchical level and the bulk action for the \textbackslash vortex " of the third hierarchical level in which the \textbackslash vortex current " would couple to a new \textbackslash C-S " gauge field as $\oint_s A^\theta$ with a C-S action $(1)^{1/m} (1 + 2m \rho) \frac{1}{2} A^\theta A^\theta$. Now it is sufficiently convincing that by repeating the procedure developed above, we arrive a complete description for the FQH system that, based upon a careful consideration of the LLL constraint, the action incorporated with the constraint can be transformed from one hierarchical state to the next in an almost universal form, and the n-th hierarchical state can be viewed as n branches of interacting edge excitations coupled to a (n-th) bulk vortices system. In particular, only at the hierarchical dilution of the FQHE, these branches of edge state excitation will decouple from bulk and bear the main physics of the FQHE state.

We would summarize further the analytical expressions for propagation velocities
of the edge excitations hierarchically as the following. The statistics index \( n \) for the \( n \)-th hierarchical level has the expression as

\[
n = \frac{1}{n_1 + 2p_{n_1}}
\]  

(5.31)

where \( n_1 \) is the corresponding index for the \((n-1)\)-th hierarchical level with \( 1 = 1 = m \) and \( p_{n_1} \) is an integer. Then, the fractional charge for the vortices on the \((n-1)\)-th hierarchical states can be expressed as \( e = m_n \) with

\[
m_n = \frac{\nu_0}{\lfloor 1 \rfloor}
\]  

(5.32)

in which we have \( m_1 = m \). And the vortex density for the \((n-1)\)-th hierarchical states can be expressed as

\[
\langle n^{(1)} \rangle = \frac{1}{2 \nu_0 m_n \nu_0^{(n)}}
\]  

(5.33)

with \( \langle n = 0 \rangle = \). If the FQH system is on the \( N \)-th hierarchical filling, we have \( \nu_0^{(N)} = 0 \), and the filling can be expressed as

\[
= \frac{1}{m} \lfloor 1 \ 1 \ 2 \ (1 \ 2 \ 3 \ (1 \ N \ 1 \ N) \ )) \rfloor
\]  

(5.34)

If we substitute eq. (5-31) successively into eq. (5-34), it coincides with the Halperin expression \([4]\) precisely. With the above notations, we can show that the \( n \) branches of edge excitations for the \( n \)-th hierarchical level have the general expressions as

\[
v^{(j)}_D = \frac{v_D}{2 \nu_0 m_j \nu_0^{(j)}}
\]  

(5.35)

with \( j = 1; \ldots; n \) and \( \nu_0 \ 0 \ = \). In case of the FQH system being on the \( N \)-th hierarchical filling, \( \nu_0^{(N)} = 0 \), we have then the hierarchical expression for the drift velocities of the edge excitations as

\[
v^{(1)} = \frac{v_D}{1 \ 2 \ (1 \ 2 \ 3 \ (1 \ N \ 1 \ N) \ )) \ m}
\]  

38
\[ v_D^{(2)} = \frac{v_D}{1^{\frac{3}{1}}(1^{\frac{3}{3}}4^{(1_N^{1}1_N^{1}))}} \]

\[ v_D^{N^{1})} = \frac{v_D}{1_N^{N}} \]

\[ v_D^{N^{1})} = v_D \quad (536) \]

We derive eq. (5-36) by substituting eqs. (5-33), (5-32) and (5-34) into eq. (5-35).

V I S U M M A R Y A N D D I S C U S S I O N S

In summary, our whole discussion is essentially based upon two basic observations as follows. The first is that since the vortices for any hierarchical level (including the bosonized electrons) have all their actions having only term 1 linear in the vortex velocities, therefore, the Dirac algorithm provides a high light guiding line so that we could have a unified treatment for the dynamics of the quasi-particles in the FQH system. The second is that, in association with the constraint for the LLL, a careful treatment of the partial integrations in the actions for the finite FQH system may separate the surface degrees of freedom from the bulk which makes a proper description for the dynamics of the edge excitations being possible. What we have succeeded in this paper is mainly that we derive not only the expressions for the bulk actions as well as the equations for the fractionally charged quasi-particles of each hierarchical state, but also the expressions of the actions, and subsequently the propagation velocities, for the associated branches of edge excitations analytically. (We notify that, since the edge excitations are essentially a sort of rippling wave of the boundary of an incompressible liquid, we, as a primary study, ignored the effect of Coulomb interactions among the surface vortices at the hierarchy filling.) Especially, we show that the branches of edge excitations can be decoupled from the bulk only at the hierarchical fillings in the context of C-S field theory approach.
What we have found is that the constraint equation, which can be transmitted from one hierarchical level to the next, plays a central role in the whole formulation not only for the bulk but also for the boundary. We hope that the calculated expressions for the propagation velocities of the edge excitations could be checked experimentally.

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APPENDIX A:

In the section III we have absorbed the regular functional \( r \) in the C-S gauge field \( a \). This could be realized by performing a gauge transformation \( a \to r \) in eq. (3-6) and it gives

\[
Z \frac{d^2}{dxdt}(-\pi \to e^' \ a_0 \ V \ \frac{1}{2} a_0 2 \ \Theta a + \frac{1}{4} a_0 2 \ \Theta a) = Z \frac{d^2}{dxdt}(-\pi \to e^' \ a_0 \ V \ \frac{1}{2} a_0 \ 2 \ \Theta a + \frac{1}{4} a_0 \ 2 \ \Theta a) \tag{A1}
\]

Utilizing the regular behavior of \( r \); \( 2 \ \Theta a \to r = 0 \), and considering further that a term of total time derivative in the Lagrangian will give a zero contribution since the bosonized system is periodic at \( t = 1 \), the r.h.s. of eq. (A1) can be transformed
into the following form by simple algebraic manipulations,

$$Z \quad \int \frac{d^2x \text{d}t}{2m} \; e^V \; a_0 \frac{1}{2m} a_0 \; 2 \; @ a + \frac{1}{4m} \; 2 \; a \; a \; V + K \quad \left[ \tau; a \right]$$

(A 2)

with $K \quad \left[ \tau; a \right]$ having the expression as

$$K \quad \left[ \tau; a \right] = \frac{1}{2m} \frac{Z}{2} \; \int \text{d}t \; \text{d}ln \; 2 \; a \; \tau - \frac{1}{4m} \frac{Z}{2} \; \int \text{d}t \; \text{d}ln \; 2 \; @ \; \tau \tau$$

(A 3)

In fact, $K \quad \left[ \tau; a \right]$ is the right term which had been forgotten tentatively in section III, especially in eq. (3-9).

On the other hand, the $s$ as well as $A^0$ dependent parts of the action in eq. (4-26) have the following form

$$L \quad \left[ s; A^0 \right] = \frac{1}{4m} \; v_b \int \text{d}t \; (2 \; @ \; s)^2 \; \int \text{d}t \; \text{d}ln \; 2 \; (\text{bulk}_s \quad \text{bulk}_s)$$

$$+ \frac{Z}{2} \; d^2x \text{d}t A^0 \quad \text{bulk}_s$$

(A 4)

where we recovered the term $K \quad \left[ a; \tau \right]$ and introduced a notation $L \quad \left[ s; A^0 \right]$ for convenience. If we perform further a gauge transformation as

$$A^0 \quad \rightarrow \quad A^0 + \frac{1}{m} \; @ \; \tau$$

(A 5)

for the action (A-4), i.e., $L \quad \left[ s; A^0 \right] \rightarrow L \quad \left[ s; \quad \tau; \quad A^0 + m \; @ \; \tau \right]$. The first term of eq. (A-4), $(2 \; m) \; v_b \int \text{d}t \; \text{d}ln \; (2 \; @ \; s)^2$, is invariant under the gauge transformation (A-5). Its second, third and fourth terms would be transformed into

$$\frac{1}{4m} \frac{Z}{2} \; \int \text{d}t \; \text{d}ln \; 2 \; @ \; (s \quad \tau) \; (\tau \quad \tau) + @ \; (\text{bulk}_s \quad \tau) \; (\text{bulk}_s \quad \tau)$$
Substituting the gauge invariant expression for \( \tilde{j}_{\text{bulk}} \) as given in eq. (3-11), and once again considering that \( r \) satisfies 2 \( \Theta \Theta r = 0 \) as well as the fact that a total time derivative term in Lagrangian would give zero contribution, we may transform eq. (A-6) into the following form via step-by-step calculations:

\[
\frac{1}{4} m \int_0^Z \frac{Z}{dt} d\ln 2 (\Theta_{s-r} + \Theta_{s-r}^{\text{bulk}}) + \int_0^Z \frac{Z}{dt} d\ln 2 \Theta_{s-r} - \frac{1}{4} m \int_0^Z \frac{Z}{dt} d\ln 2 \Theta_{s-r}
\]

Moreover, substituting eq. (4-19) into eq. (A-3), the last term of eq. (A-4), \( K_{[r,a]} \) would transform simultaneously into a form as

\[
K_{[r,a]} \rightarrow K_{[r,a+\Theta r]}
\]

Comparing eq. (A-8) with the last two terms of eq. (A-7), we see that the surface terms in eq. (A-7) which is induced by the gauge transformation eq. (A-5) are cancelled by \( K_{[r,a+\Theta r]} \).

If we further take into account of the remaining term of the action in eq. (4-26) \( m \int \frac{1}{2} \left( e^{\text{bulk}} \right) \), with \( \frac{1}{2} \left( e^{\text{bulk}} \right) \), it is also invariant with respect to the gauge transformation (A-5). Consequently, the transformation eq. (A-5) indeed cancels the \( K_{[a]} \) term and keeps all the remaining terms have the form as in the text.
Alternatively, we may not cancel the surface term $K_{a; r}$ at this stage and keep it to be remained as we process to the next hierarchical level, i.e., in eq. (5-4) we keep this additional term $K_{a; r}$ with $a$ being defined as eq. (4-19). Furthermore, similar to what we have done for the first hierarchical level, there is one another regular phase variable $0_r$ in eq. (5-4) contributed by the vortex field $0_s$ which arises from the second quantization representation of the $f_{b}^{b}$ $0$ term in eq. (4-34) (or eq. (4-25)).

This $0_r$ should be absorbed into $a^0$ via a transformation $a^0 ! a^0 @ 0_r$ in the same way as those for eq. (3-6) (i.e., eq. (A-1)). Since $a^0 = 2p^0_s$ $b_k$ $b_k$ should transform simultaneously as $b_k ! b_k (2p) \frac{1}{2} 0_r$ for consistency. Therefore, all those terms beside the $f_{b}^{b}$ $0$ in action (4-25)

$$
\frac{1}{4m} \zeta_2 \int \frac{1}{16 p^2 m} \frac{1}{d^2 \gamma 2} a^0 a^0 R
$$

will transform accordingly as

$$
R! \frac{1}{4m} \zeta_2 \int \frac{1}{16 p^2 m} \frac{1}{d^2 \gamma 2} a^0 a^0 R (A-10)
$$

The derivation from eq. (A-9) to eq. (A-10) actually is almost the same as that from eq. (A-4) to eq. (A-7) with the $f_{b}^{b}$ $0$ term being kept away. Correspondingly, noticing eq. (4-19), the additional term $K_{a; r}$ should transform also into

$$
K_{a+ \frac{1}{2p} \theta ^0_r; r} = \frac{1}{4m} \zeta_2 \int \frac{1}{d^2 \gamma 2} f 2 \gamma a + 2 \gamma (\frac{1}{2p} \theta ^0_r) - \theta ^0_r g (A-11)
$$

If we set $r = (2p) \frac{1}{2} 0_r$, the $K_{a+ (2p) \frac{1}{2} \theta ^0_r; 0_r}$ term will be cancelled exactly by the extra terms in eq. (A-10). On the meanwhile, the C-S term for the $a^0$ field with statistics index $(8, p) \frac{1}{2}$ will induce a new $K^0$ term (due to absorbing the $0_r$ variable...
leaving to the next higher hierarchical level. This part of discussion indicates that
the additional surface term $K \{ s; r \}$ really does not contributed to the dynamics
of the next hierarchical level. Therefore, the procedure in sections III and IV as
well as the previous part of this appendix that to cancel $r$ before going to the next
hierarchical level is reasonably correct.

APPENDIX B:

In section V, we derived the surface action of the boundary for the system
precisely on the FQH state of the second hierarchical level as

$$ I \left[ \text{surf}, 0 \text{surf} \right] = \frac{1}{4 m} \int \frac{Z}{2} \frac{1}{I} \text{d}t \text{d}l \text{f} \text{n} \left[ 2 \text{surf} \text{surf} \right] \text{v} \text{b} \left( n 2 \text{surf} \text{surf} \right)^2 \text{g} \left[ \begin{array}{l} \text{surf} \text{surf} \\ \text{surf} \text{surf} \end{array} \right] $$

$$ \left( B-1 \right) $$

It is straightforward to verify that eq. \( B-1 \) is exactly identical to eq. \( 5-28 \). In
eq. \( B-1 \), it is known from the sections IV and V that

$$ \frac{1}{2} \left[ 2 \text{surf} \text{surf} \right] = \text{surf} \text{surf} \left( B-2 \right) $$

$$ \frac{1}{2} \left[ 2 \text{surf} \text{surf} \right] = \text{surf} \text{surf} \left( B-3 \right) $$

where $\text{surf}$ is nonzero only in the boundary layer while $0 \text{surf}$ is nonzero only in
the layer $0$.

In the weak coupling limit, the boundary layer $0$ is enclosed inside the boundary
layer with a sharp separation. It is equivalently to say that the bundle of world
lines for the surface vortex particle (described by $0 \text{surf}$) in $0$ will never penetrate
into the bundle of the world lines of surface vortex particles in (although they
are vortex particles in sense of different hierarchical level). Based upon such an assumption (approximation), we will show in this appendix that the third and fourth terms on the r.h.s. of eq. (B-1) have zero contribution.

Introduce

$$s_{\text{surf}}^j(x) = \frac{X}{q_j^0} x_j^0 (x \cdot x_j^0(t))$$

(B 4)

and

$$s_{\text{surf}}^i(x) = \frac{X}{q_i^0} x_i^0 (x \cdot x_i(t))$$

(B 5)

where $q_i$ and $q_j^0$ are the vortex charge for the vortex particle $i$ and $j$ respectively.

Then we may solve $s_{\text{surf}}^j$ and $s_{\text{surf}}^i$ from eqs. (B-2) and (B-3) as

$$s_{\text{surf}}^j(x) = \frac{X}{q_j^0} \text{Im} \ln (z \cdot z_j^0(t))$$

(B 6)

$$s_{\text{surf}}^i(x) = \frac{X}{q_i^0} \text{Im} \ln (z \cdot z_i(t))$$

(B 7)

and subsequently,

$$s_{\text{surf}}^j(x) = \frac{X}{q_j^0} x_j^0 (t) \cdot \text{Im} \ln (z \cdot z_j^0(t))$$

(B 8)

By applying eqs. (B-6), (B-7) and (B-8), the third and fourth terms in eq. (B-1) can be rewritten as

$$\frac{1}{2m(m + 1 + 2p)^2} \int \frac{X}{q_j^0} x_j^0 \frac{z}{q_j^0} \frac{dt x_j^0}{z_j^0(t)}$$

I \int \text{Im} \ln (z \cdot z_j^0(t)) \cdot \text{Im} \ln (z \cdot z_j^0(t))

$$+ \frac{1}{2m(m + 1 + 2p)} \int \frac{X}{q_j^0} x_j^0 \frac{z}{q_j^0} \frac{dt x_j^0}{z_j^0(t)}$$

I \int \text{Im} \ln (z \cdot z_i(t)) \cdot \text{Im} \ln (z \cdot z_j^0(t))$$

(B 9)
Utilizing
\[
\begin{align*}
\int \ln 2 \theta &= \int \theta = (dz\theta_x + dz\theta_z) \\
\int dz\theta_x &= z\theta_x + z\theta_z;
\end{align*}
\]
and

\[\text{eq. (B-9) becomes}\]
\[
\frac{1}{2m} \left( \frac{m}{1 + 2p} \right)^2 \int \frac{1}{q_i q_j} \frac{z_1}{z} \int dz \theta_x \ln (z) z_j^0(t) z_j^0(t) \theta_x \ln (z) z_j^0(t) \\
+ \int dz \theta_x \ln (z) z_j^0(t) z_j^0(t) \theta_x \ln (z) z_j^0(t) \\
+ \int dz \theta_x \ln (z) z_j^0(t) z_j^0(t) \theta_x \ln (z) z_j^0(t) \\
+ \int dz \theta_x \ln (z) z_j^0(t) z_j^0(t) \theta_x \ln (z) z_j^0(t) \]

\[\text{We would like to discuss the eight group terms of eq. (B-10) term by term. If we take the derivatives to the imaginary part of the ln function, any of the first group terms of eq. (B-10) would be proportional to}\]
\[
\int dz \frac{1}{(z z_j^0(t))} = 0 \quad \text{(B 11)}
\]
where we have utilized the fact that, as what we have assumed, \(x^0_j(t), x^0_{j0}(t)\) always stay inside the \(W\). With the similar arguments, we can show easily that the second, fifth and sixth group terms are also equal to zero. If we take a partial integration with respect to \(dz@z\) for any of the third group term of eq. (B-10), it would transform into a form proportional to

\[
\int_{z} \text{Im} \ln \left( z \right) \frac{z^0_j(t)}{z^0_{j0}(t)} \text{Im} \ln \left( z \right) \frac{z^0_{j0}(t)}{z^0_j(t)} dz \int \text{Im} \ln \left( z \right) \frac{z^0_j(t)}{z^0_{j0}(t)} \left( i \right)^2 (x \cdot x^0_{j0}(t)) \quad (B\ 12)
\]

where we have made use of the identities

\[
\left( \partial_z \partial_{\bar{z}} \right) \text{Im} \ln \left( z \right) \frac{z^0_j(t)}{z^0_{j0}(t)} = 2 i^2 (x \cdot x^0_{j0}(t));
\]

\[
\left( \partial_z \partial_{\bar{z}} + \partial_{\bar{z}} \partial_z \right) \text{Im} \ln \left( z \right) \frac{z^0_j(t)}{z^0_{j0}(t)} = 0:
\]

Since \(x^0_j(t)\)'s stay always inside the \(W\) while \(x\) is in the \(W\), the \(2 (x \cdot x^0_{j0}(t))\) in eq. (B-12) should always take the value zero. As a result, the third group term of eq. (B-10) has only zero contribution. By applying the similar arguments, we may show also that the fourth, seventh and eighth group terms of eq. (B-10) do not contribute too.

Consequently, in the weak coupling limit, we have shown in this appendix that eq. (B-1), i.e., eq. (5-28) can be simplified into a form as eq. (5-29)

\[
I \left[ \text{surf} \right] = \frac{1}{4m} \int_{z} \text{Im} \ln \left( z \right) \frac{z^0_j(t)}{z^0_{j0}(t)} \text{Im} \ln \left( z \right) \frac{z^0_{j0}(t)}{z^0_j(t)} dz \int \text{Im} \ln \left( z \right) \frac{z^0_j(t)}{z^0_{j0}(t)} \left( i \right)^2 (n \cdot \partial_{\text{surf}}) (n \cdot \partial_{\text{surf}})^2 g \quad (B\ 13)
\]

which indeed decoupled form the \(n_{\text{surf}}\) right on the \(W\) of the second hierarchical level.

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