Schur-Weyl duality for the unitary groups of II$_1$-factors

Nessonov N. I. *

Abstract

We obtain the analogue of Schur-Weyl duality for the unitary group of an arbitrary II$_1$-factor.

1 Preliminaries.

Let $\mathcal{M}$ be a separable II$_1$-factor, let $U(\mathcal{M})$ be its unitary group and let $\text{tr}$ be a unique normalized normal trace on $\mathcal{M}$. Assume that $\mathcal{M}$ acts on $L^2(\mathcal{M}, \text{tr})$ by left multiplication: $L(a)\eta = a\eta$, where $a \in \mathcal{M}$, $\eta \in L^2(\mathcal{M}, \text{tr})$. Then $\mathcal{M}'$ coincides with the set of the operators that act on $L^2(\mathcal{M}, \text{tr})$ by right multiplication: $R(a)\eta = \eta a$, where $\eta \in L^2(\mathcal{M}, \text{tr})$, $a \in \mathcal{M}$. Let $S_p$ be the symmetric group of the $n$ symbols $1, 2, \ldots, p$. Take $u \in U(\mathcal{M})$ and define the operators $L^\otimes_p(u)$ and $R^\otimes_p(u)$ on $L^2(\mathcal{M}, \text{tr})^\otimes_p$ as follows

$L^\otimes_p(u) (x_1 \otimes x_2 \otimes \cdots \otimes x_p) = ux_1 \otimes ux_2 \otimes \cdots \otimes ux_p,$

$R^\otimes_p(u) (x_1 \otimes x_2 \otimes \cdots \otimes x_p) = x_1 u^* \otimes x_2 u^* \otimes \cdots \otimes x_p u^*,$

where $x_1, x_2, \ldots, x_p \in L^2(\mathcal{M}, \text{tr}).$

Obviously the operators $L^\otimes_p(u)$ and $R^\otimes_p(u)$, where $u \in U(\mathcal{M})$, form the unitary representations of the group $U(\mathcal{M})$. Also, we define the representation $P_p$ of $S_p$ that acts on $L^2(\mathcal{M}, \text{tr})^\otimes_p$ by

$P_p(s) (x_1 \otimes x_2 \otimes \cdots \otimes x_p) = x_{s^{-1}(1)} \otimes x_{s^{-1}(2)} \otimes \cdots \otimes x_{s^{-1}(p)}, s \in S_p. \quad (1.1)$

Denote by $\text{Aut} \mathcal{M}$ the automorphism group of factor $\mathcal{M}$. Let $\theta^s_p$ be the automorphism of factor $\mathcal{M}^\otimes_p$ that acts as follows

$\theta^s_p(a) = P_p(s)aP_p(s^{-1}),$ where $s \in S_p, a \in \mathcal{M}^\otimes_p \cup \mathcal{M}^\otimes_p. \quad (1.2)$

Let $\mathcal{A}$ be the set of the operators on Hilbert space $H$, let $\mathcal{N}_\mathcal{A}$ be the smallest von Neumann algebra containing $\mathcal{A}$, and let $\mathcal{A}'$ be a commutant of $\mathcal{A}$. By von Neumann’s bicommutant theorem $\mathcal{N}_\mathcal{A} = \{ \mathcal{A}' \}' = \mathcal{A}''$. *This research was supported in part by the grant Network of Mathematical Research 2013–2015
Set \((\mathcal{M}^\otimes p)^{\mathfrak{S}_p} = \{a \in \mathcal{M}^\otimes p : \theta^s_p(a) = a\ \text{for all}\ s \in \mathfrak{S}_p\}\).

The irreducible representations of \(\mathfrak{S}_p\) are indexed by the partitions of \(p\). Let \(\lambda\) be a partition of \(p\), and let \(\chi^\lambda\) be the character of the corresponding irreducible representation \(T^\lambda\). If \(\dim \lambda\) is the dimension of \(T^\lambda\), then operator \(P^\lambda_p = \frac{\dim \lambda}{p!} \sum_{s \in \mathfrak{S}} \chi^\lambda(s) P_p(s)\) is the orthogonal projection on \(L^2(\mathcal{M}, \text{tr})^\otimes p\). Denote by \(\Upsilon_p\) the set of all partitions of \(p\). The following statement is an analogue of the Schur-Weil duality.

**Theorem 1.** Fix the nonnegative integer numbers \(p\) and \(q\). Let \(\lambda\) and \(\mu\) be the partitions from \(\Upsilon_p\) and \(\Upsilon_q\), respectively, and let \(\Pi_{\lambda\mu}\) be the representation of \(L^\otimes p \otimes R^\otimes q\) to the subspace \(\mathcal{H}_{\lambda\mu} = P^\lambda_p \otimes P^\mu_q \left( L^2(\mathcal{M}, \text{tr})^\otimes p \otimes L^2(\mathcal{M}, \text{tr})^\otimes q \right)\).

The following properties are true.

1. (1) \(\{L^\otimes p \otimes R^\otimes q (U(M))\}''' = (\mathcal{M}^\otimes p)^{\mathfrak{S}_p} \otimes (\mathcal{M}^\otimes q)^{\mathfrak{S}_q}\). In particular, the algebra \((\mathcal{M}^\otimes p)^{\mathfrak{S}_p} \otimes (\mathcal{M}^\otimes q)^{\mathfrak{S}_q}\) is the finite factor.

2. (2) For any \(\lambda\) and \(\mu\) the representation \(\Pi_{\lambda\mu}\) is quasi-equivalent to \(L^\otimes p \otimes R^\otimes q\).

3. (3) Let \(\gamma \vdash p\) and \(\delta \vdash q\). The representations \(\Pi_{\lambda\mu}\) and \(\Pi_{\gamma\delta}\) are unitary equivalent if and only if \(\dim \lambda \cdot \dim \mu = \dim \gamma \cdot \dim \delta\).

## 2 The proof of property (1)

In this section we give three auxiliary lemmas and the proof of property (1) in theorem 1.

**Lemma 1.** Let \(A\) be a self-adjoint operator from \(\Pi_1\)-factor \(\mathcal{M}\). Then for any number \(\epsilon > 0\), there exist a hyperfinite \(\Pi_1\)-subfactor \(\mathcal{R}_0 \subset \mathcal{M}\) and self-adjoint operator \(A_\epsilon \in \mathcal{R}_0\) such that \(\|A - A_\epsilon\| < \epsilon\). Here \(\|\|\) is the ordinary operator norm.

**Proof.** Let \(A = \int_a^b t \, dE_t\) be the spectral decomposition of \(A\). Fix the increasing finite sequence of the real numbers \(a = a_1 < a_2 < \ldots < a_m > b\) such that \(|a_i - a_{i+1}| < \epsilon\). Hence, choosing \(t_i \in [a_i, a_{i+1})\), we have

\[
\left\| A - \sum_{i=1}^m t_i E_{[a_i, a_{i+1})} \right\| < \epsilon. \tag{2.3}
\]

It is obvious that \(\mathcal{M}\) contains the sequence of the pairwise commuting \(I_2\)-factors \(M_i\), where \(i \in \mathbb{N}\). Notice that exist the pairwise orthogonal projections \(F_i\) from the hyperfinite \(\Pi_1\)-factor \(\left( \bigcup_i M_i \right)''\) and unitary \(u \in \mathcal{M}\) such that

\[
E_{[a_i, a_{i+1})} = u F_i u^* \quad \text{for } i = 1, 2, \ldots, m - 1.
\]

\(^1\)A partition \(\lambda = (\lambda_1, \lambda_2, \ldots)\) is a weakly decreasing sequence of non-negative integers \(\lambda_j\), such that \(\sum \lambda_j = p\). As usual, we write \(\lambda \vdash p\).
It follows from (2.3) that \( R_0 = u \left( \bigcup_{i} M_i \right)'' \) and \( A_e = \sum_{i=1}^{m} t_i E_{(a_i, a_{i+1})} \in R_0 \)
satisfy the conditions as in the lemma. \( \square \)

Consider the operators \( l(a) \) and \( r(a) \), where \( a \in \mathcal{M} \), acting in Hilbert space \( L^2(\mathcal{M}, tr) \) by

\[
l(a\eta) = a\eta, r(a) = \eta a, \quad \eta \in L^2(\mathcal{M}, tr).
\]

Let us denote by \( k \) the operator \( I \otimes \cdots \otimes I \otimes A \otimes I \otimes \cdots \) in \( \mathcal{M}^{\otimes p} \otimes \mathcal{M}^{\otimes q} \), where \( p+q \) satisfy the conditions as in the lemma.

**Proof.** At first we will prove that \( \text{hyperfinite } \text{II}_1 \)-factor \( \mathcal{R}_0 \) and \( a_e \in \mathcal{R}_0 \) such that \( \| a - a_e \| < \epsilon \).

Let \( M_i, i \in \mathbb{N} \) be the sequence of pairwise commuting \( I_2 \)-subfactors from \( \mathcal{R}_0 \) such that \( \left\{ \bigcup_{j \in \mathbb{N}} M_j \right\}'' = \mathcal{R}_0 \) and \( \mathcal{N}_0 \) be the relative commutant of \( \mathcal{R}_0 \) in \( \mathcal{M} \).

\( \mathcal{N}_0 = \mathcal{R}_0' \cap \mathcal{M} \). There exists the unique normal conditional expectation \( \mathcal{E} \) of \( \mathcal{M} \) onto \( \mathcal{N}_0 \) satisfying the next conditions

- **a)** \( \text{tr} (a) = tr (\mathcal{E} (a)) \) for all \( a \in \mathcal{M} \);
- **b)** \( \mathcal{E} (xay) = x\mathcal{E} (a) y \) for all \( a \in \mathcal{M} \) and \( x, y \in \mathcal{N}_0 \);
- **c)** \( \mathcal{E} (a) = \text{tr} (a) \) for all \( a \in \mathcal{R}_0 \).
Denote by $U_k (2^{l-k})$ the unitary subgroup of the $I_{2^{l-k}}$-factor $\left\{ \bigcup_{j=k+1}^{l} M_j \right\}''$. Let $d u$ be Haar measure on $U_k (2^{l-k})$.

Since, by lemma 2,

$$p^T(a u^*) \cdot p^T(u) \quad \text{lies in} \quad \{ \Sigma^p \otimes R^q (U(\mathcal{M})) \}''$$

for all $a, u \in \mathcal{M}$, to prove the theorem, it suffices to show that

$$\lim_{n \to \infty} \int_{U_0(2^n)} p^T(a u^*) \cdot p^T(u) \, d u = p^T(a) - q^T (\mathcal{E}(a)), \quad a \in \mathcal{M} \quad (2.5)$$

with respect to the strong operator topology.

Indeed, then, by property (c), the operator $p^T(a) - q^T (\mathcal{E}(a))$, where $I$ is the identity operator from $M^p \otimes M^q$, lies in $\{ \Sigma^p \otimes R^q (U(\mathcal{M})) \}''$. Hence, using (2.4), we obtain

$$p^T(a) \in \{ \Sigma^p \otimes R^q (U(\mathcal{M})) \}'' .$$

To calculate of the left side in (2.5) we notice that

$$p^T(a u^*) \cdot p^T(u) = \sum_{k=1}^{p} k(l(a) + \sum_{k=p+1}^{p+q} k \tau(u^*) + \Sigma(a, u), \quad \text{where} \quad (2.6)$$

$$\Sigma(a, u) = \sum_{\{k,j=1\} \& \{k \neq j\}}^{p} k(l(a) \cdot (u^*) \cdot j(l(u)) + \sum_{\{k,j=p+1\} \& \{k \neq j\}}^{p+q} k(u^*) \cdot (\tau(u) \cdot k(a)$$

$$- \sum_{k=1}^{p} \sum_{j=p+1}^{p+q} k(l(a) \cdot (u^*) \cdot j(l(u)) - \sum_{k=1}^{p} \sum_{j=p+1}^{p+q} k(u) \cdot j(\tau(u^*) \cdot \tau(a)). \quad (2.7)$$

Let us first prove that

$$\lim_{n \to \infty} \int_{U_0(2^n)} k^T(uau^*) \, d u = k^T(\mathcal{E}(a)) \quad \text{for all} \quad a \in \mathcal{M} \quad (2.8)$$

with respect to the strong operator topology.

For this purpose we notice that the map

$$a \ni L^2(\mathcal{M}, \text{tr}) \xrightarrow{\mathcal{E}_n} \int_{U_0(2^n)} uau^* \, d u \in L^2(\mathcal{M}, \text{tr})$$

is the orthogonal projection. Since $\mathcal{E}_n \geq \mathcal{E}_{n+1}$, then

$$\lim_{n \to \infty} \mathcal{E}_n(a) = \mathcal{E}(a) \quad \text{for all} \quad a \in \mathcal{M}$$
with respect to the norm on $L^2(\mathcal{M}, \text{tr})$. Hence, applying the inequality $\|\mathcal{E}_n(a)\|_{L^2(\mathcal{M}, \text{tr})} \leq \|a\|$, we obtain $\lim_{n \to \infty} \|\mathcal{E}_n(a) - \mathcal{E}(a)\|_{L^2} = 0$ for all $\eta \in L^2(\mathcal{M}, \text{tr})$. This gives (2.8).

To estimate of $\Sigma(a, u)$ fix the matrix unit $\{e_{pq} : 1 \leq p, q \leq 2^n\}$ of the $I_{2n}$-factor $\bigcup_{j=1}^n M_j$. We recall that the operators $e_{pq}$ satisfy the relations

$$e_{pq}^* = e_{qp}, \quad e_{pq} e_{st} = \delta_{qs} e_{pt}, \quad 1 \leq p, q, s, t \leq 2^n.$$

Denote by $\{a_{pq}\}_{p,q=1}^{2n} \subset \mathbb{C}$ the corresponding matrix elements of the operator $a \in \bigcup_{j=1}^n M_j''$:

$$a = \sum_{p,q=1}^{2n} a_{pq} e_{pq}.$$

If $k \neq j$, then, applying Peter-Weyl theorem, we obtain

$$k^j T_l = \int_{U_0(2^n)} k^l (u^*) \cdot \mathcal{I}(u) \, du = 2^{-n} \sum_{p,q=1}^{2n} k^l (e_{pq}) \cdot \mathcal{I}(e_{qp}),$$

$$k^j T_r = \int_{U_0(2^n)} k^r (u^*) \cdot \mathcal{I}(u) \, du = 2^{-n} \sum_{p,q=1}^{2n} k^r (e_{pq}) \cdot \mathcal{I}(e_{qp}),$$

$$k^j P = \int_{U_0(2^n)} k^l (u^*) \cdot \mathcal{I}(u) \, du = 2^{-n} \sum_{p,q=1}^{2n} k^l (e_{pq}) \cdot \mathcal{I}(e_{qp}).$$

A trivial verification shows that

$$(k^j T_l)^* = k^j T_l, \quad (k^j T_l)^* = k^j T_r, \quad k^j T_l^2 = k^j T_r^2 = 2^{-2n} I,$$

$$(k^j P)^* = k^j P^2 = 2^{-n} \cdot k^j P.$$

Hence, using (2.7), we have

$$\lim_{n \to \infty} \int_{U_0(2^n)} \Sigma(a, u) \, du = 0$$

with respect to the operator norm. We thus get (2.9).

The proof above works for the operator $\eta^T(a)$. But we must examine $p^T(u^*a) \cdot p^T(u)$ instead $p^T(u^*a) \cdot p^T(u)$ (see (2.6)).

The proof of Theorem 1(1). By lemma 3, it suffices to show that

$$\{ p^T(a), a \in \mathcal{M} \}'' = (\mathcal{M}^{\otimes p})^{\mathcal{S}_p}. \quad (2.9)$$

Fix the orthonormal bases $\{b_j\}_{j=0}^{\infty}$ in $L^2(\mathcal{M}, \text{tr})$ such that $b_j \in \mathcal{M}$ and $b_0 = 1$. Let $j = (j_1, j_2, \ldots, j_p)$ be the ordered collection of the indexes, and let $b_j =
Let \( i = (i_1, i_2, \ldots, i_p) \) and \( j = (j_1, j_2, \ldots, j_p) \) be equivalent if there exists \( s \in \mathcal{S} \) such that \((i_1, i_2, \ldots, i_p) = (j_{s(1)}, j_{s(2)}, \ldots, j_{s(p)})\). Denote by \( \mathcal{T} \) the equivalence class containing \( i \). Set \( s(j) = (j_{s(1)}, j_{s(2)}, \ldots, j_{s(p)}) \), \( s \in \mathcal{S}_p \).

It is clear that the elements \( b_j = \sum_{s \in \mathcal{S}_p} b_{s(j)} \in (\mathcal{M}^{\otimes p})^{\mathcal{S}_p} \) form the orthogonal bases in \( L^2 \left( (\mathcal{M}^{\otimes p})^{\mathcal{S}_p}, \text{tr}^{\otimes p} \right) \). So to prove (2.9), it suffices to show that

\[
b_j \in \{ pT^+(a), a \in \mathcal{M} \}'' I.
\]

Let us prove this, by induction on \( m \).

If \( m = 1 \) then \( L_1 \subset \{ pT^+(a), a \in \mathcal{M} \}'' I \), by the definition of \( pT^+(a) \) (see lemma 3). Assuming (2.11) to hold for \( m = 1, 2, \ldots, k \), we will prove that

\[
L_{k+1} \subset \{ pT^+(a), a \in \mathcal{M} \}'' I.
\]

Indeed, if \( b_j \) lies in \( L_k \) then without loss of generality we can assume that

\[
j = \left( \underbrace{j_1, j_2, \ldots, j_k}_{k}, 0, \ldots, 0 \right), \text{ where } j_i \neq 0 \text{ for all } i \in \{1, 2, \ldots, k\}.
\]

If \( l \neq 0 \) then \( pT^+(b_l) b_j = b_l^{(k)} + b_l \), where \( b_l^{(k)} \in \bigoplus_{m=0}^{k} L_m \subset \{ pT^+(a), a \in \mathcal{M} \}'' I \)

and \( i = \left( \underbrace{j_1, j_2, \ldots, j_k}_{k+1}, l, 0, \ldots, 0 \right) \). Therefore, \( b_i \) lies in \( \{ pT^+(a), a \in \mathcal{M} \}'' I \).

This proves (2.11), (2.10) and (2.9).

\[\square\]

### 3 The proof of the properties (2) and (3)

Let \( \text{Aut} \mathcal{N} \) be the group of all automorphisms of von Neumann algebra \( \mathcal{N} \). We recall that automorphism \( \theta \) of factor \( \mathcal{F} \) is inner if there exists unitary \( u \in \mathcal{F} \).
such that $\theta(a) = uau^* = \text{Ad } u(a)$. Let us denote by $\text{Int } \mathcal{F}$ the set of all inner automorphisms of factor $\mathcal{F}$. An automorphism $\theta \in \text{Aut } \mathcal{F}$ is called outer if $\theta \notin \text{Int } \mathcal{F}$.

Consider $\Pi_1$-factor $\mathcal{F} = \mathcal{M}^\otimes p \otimes (\mathcal{M}')^\otimes q$. We emphasize that $\mathcal{F}$ is generated by the operators $A = \{a_1, \ldots, a_p\} \otimes \{a_p+1, \ldots, a_{p+q}\}$, $(1 \leq j \leq p + q)$ which act in $L^2 \left( \mathcal{M}^\otimes (p+q), \text{tr}^\otimes (p+q) \right)$ as follows

$$A (\eta_1 \otimes \cdots \eta_p \otimes \eta_{p+1} \otimes \cdots \eta_{p+q}) = a_1 \eta_1 \otimes \cdots a_p \eta_p \otimes a_{p+1} \eta_{p+1} \otimes \cdots a_{p+q} \eta_{p+q}.$$  

From now on, $\text{tr}^\otimes (p+q)$ denotes the unique normal normalized trace on the factor $\mathcal{F}$:

$$\text{tr}^\otimes (p+q) (A) = \prod_{k=1}^p \text{tr} (a_k) \prod_{k=p+1}^{p+q} \text{tr} (a_k^*). \quad (3.13)$$

If $J$ is the antilinear isometry on $L^2 \left( \mathcal{M}^\otimes (p+q), \text{tr}^\otimes (p+q) \right)$ defined by $L^2 \left( \mathcal{M}^\otimes (p+q), \text{tr}^\otimes (p+q) \right) \ni X \mapsto X^* \in L^2 \left( \mathcal{M}^\otimes (p+q), \text{tr}^\otimes (p+q) \right)$, then

$$J \mathcal{A} \mathcal{J} (\eta_1 \otimes \cdots \eta_p \otimes \eta_{p+1} \otimes \cdots \eta_{p+q}) = \eta_1 a_1^* \otimes \cdots \eta_p a_p^* \otimes a_{p+1} \eta_{p+1} \otimes \cdots a_{p+q} \eta_{p+q}. \quad (3.14)$$

Well-known that $\mathcal{F}^* = J \mathcal{F} J$ (see [7]).

Let $\mathcal{P}_{p+q}(s), (s \in \mathcal{S}_{p+q})$ be the unitary operator on $L^2 \left( \mathcal{M}^\otimes (p+q), \text{tr}^\otimes (p+q) \right)$ defined by [13], and let $\mathcal{S}_p \times \mathcal{S}_q = \{s \in \mathcal{S}_{p+q} : s \{1, 2, \ldots, p\} = \{1, 2, \ldots, p\}\}$. Denote by $e$ the unit in the group $\mathcal{S}_{p+q}$. The next lemma is obvious from the definition of factor $\mathcal{F}$.

**Lemma 4.** For each $s \in \mathcal{S}_p \times \mathcal{S}_q$ the map $\mathcal{F} \ni a \mapsto \mathcal{P}_{p+q}(s)a \mathcal{P}_{p+q}(s^{-1}) = \theta^*_{p+q}(a)$ is the automorphism of factor $\mathcal{F}$.

**Lemma 5.** If $s$ is any non-identical element from $\mathcal{S}_p \times \mathcal{S}_q$ then $\text{Ad } \mathcal{P}_{p+q}$ is the outer automorphism of factor $\mathcal{F}$.

**Proof.** On the contrary, suppose that there exists the unitary operator $U \in \mathcal{F}$ such that

$$\mathcal{P}_{p+q}(s)a \mathcal{P}_{p+q}(s^{-1}) = U a U^* \quad \text{for all } a \in \mathcal{F}. \quad (3.15)$$

Let us prove that $U = 0$. For this, it suffices to show that

$$\text{tr}^\otimes (p+q) (U (u_1 \otimes u_2 \otimes \cdots \otimes u_{p+q})) = 0 \quad \text{for all unitary } u_j \in \mathcal{M}. \quad (3.16)$$

Let $N$ be any natural number $N$

$$\left| \text{tr}^\otimes (p+q) (U (u_1 \otimes u_2 \otimes \cdots \otimes u_{p+q})) \right| \leq \frac{1}{N}. \quad (3.17)$$

To this purpose we find the pairwise orthogonal projections $p_j \in \mathcal{M}, j = 1, 2, \ldots, N$ with the properties

$$\sum_{j=1}^N p_j = I, \quad \text{tr} (p_j) = \frac{1}{N} \quad \text{for all } j = 1, 2, \ldots, N. \quad (3.18)$$
Remark 1. Without loss generality we can assume that \( s(1) = i \neq 1 \). Then

\[
\left| \text{tr}^{\otimes(p+q)} \left( U \left( u_1 \otimes u_2 \otimes \ldots \otimes u_{p+q} \right) \right) \right|
\]

is the faithful normal trace on \( F \). The involution:

\[
\text{tr}^{\otimes(p+q)} \left( \frac{1}{\lambda} U \left( u_1 \otimes u_2 \otimes \ldots \otimes u_{p+q} \right) \frac{1}{\lambda} \right)
\]

isometry. It follows immediately that

\[
\left| \text{tr}^{\otimes(p+q)} \left( \frac{1}{\lambda} U \left( u_1 \otimes u_2 \otimes \ldots \otimes u_{p+q} \right) \frac{1}{\lambda} \right) \right|
\]

\[
\leq \sum_{j=1}^{N} \text{tr}^{\otimes(p+q)} \left( \frac{1}{\lambda} \theta \left( u_1 p_j u_1^* \right) \right) = \sum_{j=1}^{N} \text{tr} \left( p_j \right) \text{tr} \left( u_1 p_j u_1^* \right) \overset{3.18}{=} \frac{1}{N}.
\]

This establishes (3.17) and (3.18).

To simplify notation, we will write \( \theta_s \) instead \( \theta^s_{p+q} = \text{Ad} \mathcal{P}_p \mathcal{Q}(s) \) (see lemma [4]).

Now we consider the crossed product \( \mathcal{F} \rtimes \theta (\mathcal{S}_p \times \mathcal{S}_q) \) of the factor \( \mathcal{F} \) by the finite group \( \mathcal{S}_p \times \mathcal{S}_q \) acting via \( \theta : s \in \mathcal{S}_p \times \mathcal{S}_q \mapsto \theta_s \in \text{Aut} \mathcal{F} \).

Von Neumann algebra \( \mathcal{F} \rtimes \theta (\mathcal{S}_p \times \mathcal{S}_q) \) is generated in Hilbert space \( L^2 (G, \mathcal{H}) \), where \( G = \mathcal{S}_p \times \mathcal{S}_q, \mathcal{H} = L^2 (M^{\otimes(p+q)}), \text{tr}^{\otimes(p+q)} \), by the operators \( \Pi_\theta (a), a \in \mathcal{F} \) and \( \lambda_g, g \in G \), which act as follows

\[
\Pi_\theta (a) \eta(g) = \theta_{g^{-1}} (a) \eta(g), \eta \in L^2 (G, \mathcal{H}),
\]

\[
(\lambda_g \eta)(g) = \eta(g^{-1} g), g \in G.
\]

(3.19)

Remark 1. Let \( a \in L^2 \left( M^{\otimes(p+q)}, \text{tr}^{\otimes(p+q)} \right) \). Set \( \xi_1 (g) = \begin{cases} a & \text{if } g = e \\ 0 & \text{if } g \neq e \end{cases} \). It is easy to check that \( \xi_1 \) is the cyclic vector for \( \mathcal{F} \rtimes \theta (\mathcal{S}_p \times \mathcal{S}_q) \). Namely, the set of the vectors \( A \xi_1, a \in \mathcal{F} \rtimes \theta (\mathcal{S}_p \times \mathcal{S}_q) \) is dense in \( L^2 (G, \mathcal{H}) \). In addition, the functional \( \hat{\tau} \) defined on \( A = \sum_{s \in \mathcal{S}_p \times \mathcal{S}_q} \Pi_\theta (a_s) \cdot \lambda_s \in \mathcal{F} \rtimes \theta (\mathcal{S}_p \times \mathcal{S}_q) \) by

\[
\hat{\tau}(A) = (A \xi_1, \xi_1) = \text{tr}^{\otimes(p+q)}(a_e),
\]

is the faithful normal trace on \( \mathcal{F} \rtimes \theta (\mathcal{S}_p \times \mathcal{S}_q) \).

Remark 2. The involution: \( \theta \mathcal{F} \xi_1 \ni A \xi_1 \mapsto \bar{A}^* \xi_1 \) extends to the antilinear isometry. It follows immediately that \( \hat{J} \eta(x) = \theta_{x^{-1}} (b_{x^{-1}}^*) \), \( \eta \in \mathcal{L}^2 (G, \mathcal{H}) \).

The operators \( \Pi_\theta (a) = \bar{J} \Pi_\theta (a) \bar{J}, a \in \mathcal{F} \) and \( \lambda_s' = \bar{J} \lambda_s \bar{J}, s \in \mathcal{S}_p \times \mathcal{S}_q \) act by

\[
(\Pi_\theta (a) \eta)(g) = JaJ \eta(g), \eta \in \mathcal{L}^2 (G, \mathcal{H}),
\]

\[
(\lambda_s' \eta)(g) = \theta_s (\eta(gs)), s \in G.
\]

(3.20)
The equality \( \hat{J}^{\theta} F \hat{J} = \theta F' \) is true. In particular, the vector \( \xi_I \) is cyclic for \( \theta F' \). Set \( \hat{\tau}'(A) = \hat{\tau}(\hat{J} A \hat{J}) \), where \( A \in \theta F' \). Then \( \hat{\tau}' \) is the faithful normal trace on \( \theta F' \).

**Lemma 6.** Von Neumann algebra \( \theta F' \) is \( \Pi_1 \)-factor.

**Proof.** It follows from remarks 1 and 2 that any operator \( A \in \theta F \) has a unique decomposition \( A = \sum_{g \in G} \Pi_{\theta}(a_g) \lambda_g \). Thus, if \( A \) lies in the centrum of \( \theta F \) then

\[
\Pi_{\theta}(a_g) \lambda_g \cdot \Pi_{\theta}(b) = \Pi_{\theta}(b) \cdot \Pi_{\theta}(a_g) \lambda_g \quad \text{for all} \quad g \in G \quad \text{and} \quad b \in F.
\]

Hence, using (3.19), we obtain

\[
a_g \cdot \theta_g(b) = b \cdot a_g \quad \text{for all} \quad g \in G \quad \text{and} \quad b \in F.
\]

Therefore, we have

\[
a_g^* a_g \theta_g(b) = a_g^* b a_g, \quad \theta_g'(b^*) a_g = a_g^* b^* a_g \quad \text{for all} \quad g \in G \quad \text{and} \quad b \in F.
\]

Hence, we conclude

\[
a_g^* a_g \in F \cap F' = \mathbb{C} I \quad \text{for all} \quad g \in G.
\]

We thus get \( a_g = z_g u_g \), where \( z_g \in \mathbb{C} \), \( u_g \) is the unitary operator from \( F \).

Assuming \( g \neq e \), we obtain from (3.21)

\[
\theta_g(b) = u_g^* \cdot b \cdot u_g \quad \text{for all} \quad b \in F. \tag{3.21}
\]

It follows from lemma 5 that \( a_g = 0 \) for all \( g \neq e \). But, by (3.21), \( a_e \in F \cap F' = \mathbb{C} I \). Therefore, \( A \in \mathbb{C} I \).

Let \( P = \frac{1}{|G|} \sum_{g \in G} \lambda_g \). We will identify \( \eta \in \mathcal{H} = L^2(\mathcal{M}^{\otimes(p+q)}; \text{tr}^{\otimes(p+q)}) \) with the function \( \tilde{\eta} \in l^2(\mathcal{H}, G) \) defined by: \( \tilde{\eta}(g) = \eta \) for all \( g \in G \). Define the unitary operator \( U_g \) on \( \mathcal{H} \) by

\[
U_g \eta = \theta_g(\eta), \quad \eta \in \mathcal{M}^{\otimes(p+q)},
\]

where \( \theta_g \) denote the automorphism \( \text{Ad} \mathcal{P}_{p+q}(g) \) (see lemma 4). It is easy to check that

\[
P L^2(\mathcal{H}, G) = \mathcal{H}, \quad P \cdot \theta F \cdot P = \{ a \in F : \theta_g(a) = a \quad \text{for all} \quad g \in G \} = \mathcal{F}^G,
\]

\[
P \lambda_g P = I \quad \text{for all} \quad g \in G, \quad P \Pi_{\theta}(a) P = \frac{1}{|G|} \sum_{g \in G} \theta_g(a) \in F \tag{3.22}
\]

\[
P \lambda_g P \eta = \theta_g(\eta) = U_g \eta, \quad P \Pi_{\theta}(a) P \eta = J a J \eta = \eta a^*, \quad \text{where} \quad \eta \in \mathcal{H}.
\]

We sum up this discussion in the following.

**Lemma 7.** Von Neumann algebra \( (\mathcal{F}^G)' \) is generated by \( F' \) and \( \{ U_g \}_{g \in G} \).

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Proof. According to remark 2, von Neumann algebra \( \theta F' \) is generated by the operators \( \Pi_g(a), \ a \in F \) and \( \lambda_g, \ g \in G \). Hence, using (3.22), we obtain the desired conclusion.

**Lemma 8.** \( (F^G)' \cap F = CI \).

**Proof.** Let \( A \in (F^G)' \cap F = (F^G)' \cap (F')' \). Lemma 7 assures that \( A = \sum_{g \in G} a_g U_g \), where \( a_g \in F' \) for all \( g \in G \). Therefore,

\[
\sum_{g \in G} b a_g U_g = \sum_{g \in G} a_g U_g b \quad \text{for all} \quad b \in F'.
\]

Hence, applying lemma 6 and (3.22), we have

\[
b a_g U_g = a_g U_g b \quad \text{for all} \quad b \in F' \quad \text{and} \quad g \in G.
\]

This means that

\[
b a_g = a_g \theta_g(b) \quad \text{for all} \quad b \in F' \quad \text{and} \quad g \in G. \tag{3.33}
\]

Now we recall that, by remark 2, since the relations \( JFJ = F' \) and \( JU_g = U_g J \) are true, we obtain from lemma 5 that

\[
F' \ni b \mapsto U_g b U_g^* = \theta_g(b) \in F'
\]
is the outer automorphism of the factor \( F' \) for all \( g \neq e \). Now as in the proof of lemma 6 the equality (3.23) gives that \( A \in CI \).

**The proof of Theorem 1(2-3).** We recall that \( F = M \otimes_p (M') \otimes_q G = G_p \times G_q \). By theorem 1(1), \( \{ L \otimes_p (R \otimes_q U(M)) \}' = F^G \). It follows from (3.22) and the lemmas 6 7 that \( F^G \) is a factor, and the map

\[
\theta F' \ni a R_p \mapsto P a P \in (F^G)' 
\]
is an isomorphism. Therefore, the formula

\[
\tau'(a) = (R_p^{-1}(a)\xi_1, \xi_1), \ a \in (F^G)' \quad (\text{see remarks 1 and 2}) \tag{3.24}
\]
defines the normal, normalized trace on \( (F^G)' \).

Since the projection \( P^l = P^l \otimes P^l \) lies in \( (F^G)' \), the map

\[
\{ L \otimes_p \otimes_q (U(M)) \}' = F^G \ni a \mapsto P^l \otimes^\nu P^l \otimes^\nu P^l \in P^l F^G P^l \tag{3.25}
\]
is an isomorphism. In particular, \( \Pi^{l \nu}_{pq} (U(M) \otimes (L \otimes q(u))) = \Pi^{l \nu}_{pq} \) for all \( u \in U(M) \) and \( (\lambda, \mu) \in G_p \times G_q \). This proves the property (2) from the theorem 1.
To prove the property (3), we notice that the projections $P_{p}^{\lambda} \otimes P_{q}^{\mu}$ and $P_{p}^{\gamma} \otimes P_{q}^{\delta}$ are in $(\mathcal{F}G)^{\prime}$. It follows from (3.24) that

$$\tau'(P^{\lambda}_{p} \otimes P^{\mu}_{q}) = \frac{\dim \lambda \cdot \dim \mu}{p!q!}.$$ 

Thus, assuming that $\dim \lambda \cdot \dim \mu = \dim \gamma \cdot \dim \delta$, we obtain

$$\tau'(P^{\lambda}_{p} \otimes P^{\mu}_{q}) = \tau'(P^{\gamma}_{p} \otimes P^{\delta}_{q}).$$

Since $(\mathcal{F}G)^{\prime}$ is a factor, there exist the partial isometry $U \in (\mathcal{F}G)^{\prime}$ such that

$$UU^{\ast} = P_{p}^{\lambda} \otimes P_{q}^{\mu} \quad \text{and} \quad U^{\ast}U = P_{p}^{\gamma} \otimes P_{q}^{\delta}.$$ 

Hence we have

$$\Pi_{\lambda\mu}(u) = P_{p}^{\lambda} \otimes P_{q}^{\mu} \ (\mathcal{L}^{\oplus p} \otimes \mathcal{R}^{\oplus q}(u)) \ P_{p}^{\lambda} \otimes P_{q}^{\mu} = \mathfrak{U}U^{\ast} \ (\mathcal{L}^{\oplus p} \otimes \mathcal{R}^{\oplus q}(u)) \ \mathfrak{U}U^{\ast} = \mathfrak{U} \ U^{\ast} (\mathcal{L}^{\oplus p} \otimes \mathcal{R}^{\oplus q}(u)) \ U^{\ast} = \Pi_{\gamma\delta}(u) \quad \text{for any} \quad u \in U(M).$$

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