EXISTENCE OF TWO SYMMETRIC SOLUTIONS FOR
NEUMANN PROBLEMS

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Abstract. In this paper, we investigate the existence of at least two distinct cylindrically symmetric weak solutions for some elliptic problems involving a $p$-Laplace operator, subject to Neumann boundary conditions in a strip-like domain of the Euclidean space.

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1. INTRODUCTION

Let $\Omega \subset \mathbb{R}^m$ be a bounded domain with smooth boundary and $\Omega := \Theta \times \mathbb{R}^n$ be a strip-like domain. Define the space of cylindrically symmetric functions by

$W^{1,p}_c(\Omega) := \{ u \in W^{1,p}(\Omega) : u(x, \cdot) \text{ is radially symmetric for all } x \in \Theta \}$.

In this space, Molica Bisci and Rădulescu in [7, Theorem 2.1] studied the existence of at least three cylindrically symmetric solutions for the following elliptic Neumann problem

\[
\begin{align*}
-\Delta_p u + |u|^{p-2} u &= \lambda \alpha(x, y) f(u) \quad \text{in } \Omega, \\
\frac{\partial u}{\partial v} &= 0, \quad \text{on } \partial \Omega,
\end{align*}
\]

where $v$ denotes the outward unit normal to $\partial \Omega$, $p > m + n$ is a real number, $\lambda$ is a positive real parameter and $\Delta_p u := \text{div}(\nabla u |^{p-2} \nabla u)$. Moreover, $\alpha \in L^1(\Omega)$ is a non-negative cylindrically symmetric function and $f : \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function.

In this paper, our goal is to obtain the existence of at least two distinct cylindrically symmetric weak solutions for problem (1.1) under suitable conditions on $\alpha$ and $f$.

We denote by $c_p$ the best embedding constant of $W^{1,p}_c(\Omega)$ into $L^\infty(\Omega)$, i.e.,

\[
c_p := \sup_{u \in W^{1,p}(\Omega)} \frac{\|u\|_{L^\infty(\Omega)}}{\|u\|_{W^{1,p}(\Omega)}},
\]

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where
\[ \|u\|_{L^\infty} := \text{esssup}_{(x,y) \in \Omega} |u(x, y)|; \]
see [4, Theorem 2.2]. Further, let \( \alpha \in L^1(\Omega) \) is a non-negative cylindrically symmetric function such that
\[ \alpha_0 := \inf_{(x,y) \in \Omega} \alpha(x,y) > 0, \]
and \( f : \mathbb{R} \to \mathbb{R} \) be a continuous function satisfying the following condition:
\[ |f(t)| \leq a_1 + a_2|t|^{s-1}, \quad \forall t \in \mathbb{R}, \]
for some non-negative constants \( a_1, a_2 \) and \( s > p \). We put \( F(\xi) := \int_0^\xi f(t)dt \), for every \( \xi \in \mathbb{R} \). Moreover, we introduce the functional \( I_\lambda : W^{1,p}(\Omega) \to \mathbb{R} \) associated with problem (1.1),
\[ I_\lambda(u) := \frac{1}{p} \left( \int_{\Omega} |\nabla u(x,y)|^p dxdy + \int_{\Omega} |u(x,y)|^p dxdy \right) - \lambda \int_{\Omega} \alpha(x,y) F(u(x,y)) dxdy. \]
Fixing the real parameter \( \lambda \), a function \( u \in W^{1,p}(\Omega) \) is said to be a weak solution of (1.1) if for all \( v \in W^{1,p}(\Omega) \),
\[ \int_{\Omega} |\nabla u(x,y)|^{p-2} \nabla u(x,y) \cdot \nabla v(x,y) dxdy + \int_{\Omega} |u(x,y)|^{p-2} u(x,y)v(x,y) dxdy = \lambda \int_{\Omega} \alpha(x,y) f(u(x,y))v(x,y) dxdy. \]
Hence, the critical points of \( I_\lambda \) are exactly the weak solutions of problem (1.1).

**Definition 1.** A Gâteaux differentiable function \( I \) satisfies the Palais-Smale condition (in short (PS)-condition) if any sequence \( \{u_n\} \) such that
(a) \( \{I_\lambda(u_n)\} \) is bounded,
(b) \( \|I'_\lambda(u_n)\|_{X^*} \to 0 \), as \( n \to \infty \),
has a convergent subsequence.

We shall prove our results applying the following critical point theorem, which is a more precise version of Ricceri’s variational principle [12, Theorem 2.5]. We point out that Ricceri’s variational principle generalizes the celebrated three critical point theorem of Pucci and Serrin [9, 10] and is an useful result that gives alternatives for the multiplicity of critical points of certain functions depending on a parameter.

**Theorem 1** (see [2, Theorem 3.2]). Let \( X \) be a real Banach space and let \( \Phi, \Psi : X \to \mathbb{R} \) be two continuously Gâteaux differentiable functionals such that \( \Phi \) is bounded from below and \( \Phi(0) = \Psi(0) = 0 \). Fix \( r > 0 \) such that \( \sup_{u \in \Phi^{-1}(0, r]} \Psi(u) < +\infty \) and assume that, for each
\[ \lambda \in \left[ 0, \frac{r}{\sup_{u \in \Phi^{-1}(0, r]}} \right], \]
the functional \( I_\lambda := \Phi - \lambda \Psi \) satisfies (PS)-condition and it is unbounded from below. Then, for each \( \lambda \in \left] 0, \sup_{u \in X \setminus \{0\}} \frac{r}{\|u\|_{W^{1,p}} \Psi(u)} \right] \), the functional \( I_\lambda \) admits two distinct critical points.

For completeness, we refer the interested reader to the recent papers [3, 6] where Ricceri’s variational principle has been developed on studying nonlinear Neumann problems. See also [1, 5].

2. MAIN RESULTS

In this section we establish the main abstract result of this paper. We recall that \( c_p \) is the constant of the continuous embedding \( W^{1,p}_c(\Omega) \hookrightarrow L^{\infty}(\Omega) \); see (1.2).

Theorem 2. Let \( f : \mathbb{R} \to \mathbb{R} \) be a continuous function satisfying condition \((f_1)\). Moreover, assume that

\[(f_2)\] there exist two constants \( \eta > p \) and \( L > 0 \) such that

\( 0 < \eta F(t) \leq tf(t), \quad |t| \geq L. \)

Then, for each \( \lambda \in [0, \lambda^*[ \), problem (1.1) admits at least two distinct cylindrically symmetric weak solutions, where

\[ \lambda^* := \frac{s}{(sa_1 c_p \gamma \zeta_{1/p} + a_2 c_p \zeta_{s/p}) \|\alpha\|_{L^1}}. \]

Proof. Our aim is to apply Theorem 1 to problem (1.1) in the case \( r = 1 \) to the Banach space \( X := W^{1,p}_c(\Omega) \) endowed with the norm

\[ \|u\|_{W^{1,p}} := \left( \int_\Omega |\nabla u(x, y)|^p dxdy + \int_\Omega |u(x, y)|^p dxdy \right)^{1/p}. \]

For every \( u \in X \) we set

\[ \Phi(u) := \frac{\|u\|_{W^{1,p}}^p}{p}, \quad \Psi(u) := \int_\Omega \alpha(x, y) F(u(x, y)) dxdy. \]

Clearly \( \Phi \) and \( \Psi \) are continuously Gâteaux differentiable and

\[ \Phi'(u)(v) := \int_\Omega |\nabla u(x, y)|^{p-2} \nabla u(x, y) \cdot \nabla v(x, y) dxdy + \int_\Omega |u(x, y)|^{p-2} u(x, y) v(x, y) dxdy, \]

and

\[ \Psi'(u)(v) := \int_\Omega \alpha(x, y) f(u(x, y)) v(x, y) dxdy, \]

for every \( v \in X \). Moreover, \( \Phi' \) admits a continuous inverse on \( X^* \) and \( \Psi' \) is a compact operator.
Now we prove that $I_\lambda := \Phi - \lambda \Psi$ satisfies $(PS)$-condition for every $\lambda > 0$. Namely, we will prove that any sequence $\{u_n\} \subset X$ satisfying

$$m := \sup_n I_\lambda(u_n) < +\infty, \quad \lim_{n \to +\infty} \|I_\lambda'(u_n)\|_{X^*} = 0,$$

contains a convergent subsequence. From above, we can actually assume that

$$\frac{1}{\eta}(I_\lambda'(u_n), u_n) \leq \|u_n\|_{W^{1,p}}.$$

For $n$ large enough, we have

$$m \geq I_\lambda(u_n) = \frac{1}{p} \left( \int_\Omega |\nabla u_n(x, y)|^p\,dx\,dy + \int_\Omega |u_n(x, y)|^p\,dx\,dy \right)$$

$$-\frac{\lambda}{\eta} \int_\Omega \alpha(x, y) F(u_n(x, y))\,dx\,dy,$$

then

$$I_\lambda(u_n) \geq \frac{1}{p} \left( \int_\Omega |\nabla u_n(x, y)|^p\,dx\,dy + \int_\Omega |u_n(x, y)|^p\,dx\,dy \right)$$

$$-\frac{\lambda}{\eta} \int_\Omega \alpha(x, y) f(u_n(x, y))u_n(x, y)\,dx\,dy$$

$$= \left( \frac{1}{p} - \frac{1}{\eta} \right) \left( \int_\Omega |\nabla u_n(x, y)|^p\,dx\,dy + \int_\Omega |u_n(x, y)|^p\,dx\,dy \right)$$

$$\geq \frac{1}{p} \int_\Omega |\nabla u_n(x, y)|^p\,dx\,dy + \int_\Omega |u_n(x, y)|^p\,dx\,dy$$

$$-\frac{\lambda}{\eta} \int_\Omega \alpha(x, y) f(u_n(x, y))u_n(x, y)\,dx\,dy$$

$$= \left( \frac{1}{p} - \frac{1}{\eta} \right) \|u_n\|_{W^{1,p}}^p + \frac{1}{\eta} (I_\lambda'(u_n), u_n).$$

Thus,

$$m + \|u_n\|_{W^{1,p}} \geq I_\lambda(u_n) - \frac{1}{\eta} (I_\lambda'(u_n), u_n) \geq \left( \frac{1}{p} - \frac{1}{\eta} \right) \|u_n\|_{W^{1,p}}^p.$$

Consequently, $\{\|u_n\|\}$ is bounded. By the Eberlian-Smulyan theorem, without loss of generality, we assume that $u_n \rightharpoonup u$. Then $\Psi'(u_n) \to \Psi'(u)$ because of compactness. Since $I_\lambda'(u_n) = \Phi'(u_n) - \lambda \Psi'(u_n) \to 0$, then $\Phi'(u_n) \to \lambda \Psi'(u)$. Since $\Phi'$ has a continuous inverse, then $u_n \to u$ and so $I_\lambda$ satisfies $(PS)$-condition.

From $(f_2)$, there is a positive constant $C$ such that

$$F(t) \geq C |t|^\eta$$

for all $|t| > L$. In fact, setting $b := \min_{|\xi| = L} F(\xi)$ and

$$\varphi_t(\beta) := F(\beta t), \quad \forall \beta > 0,$$

(2.1) holds. \end{proof}
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by \((f_2)\), for every \(|t| > L\) one has

\[
0 < \eta \varphi_t(\beta) = \eta F(\beta t) \leq \beta t \cdot f(\beta t) = \beta \varphi_t'(\beta), \quad \forall \beta > \frac{L}{|t|}.
\]

Therefore,

\[
\int_{L/|t|}^{1} \frac{\varphi_t'(\beta)}{\varphi_t(\beta)} d\beta \geq \int_{L/|t|}^{1} \frac{\eta}{\beta} d\beta.
\]

Then

\[
\varphi_t(1) \geq \varphi_t\left( \frac{L}{|t|} \right) \frac{|t|^\eta}{L^\eta}.
\]

Taking into account of \((2.2)\), we obtain

\[
F(t) \geq F\left( \frac{L}{|t|} \right) \frac{|t|^\eta}{L^\eta} \geq C |t|^\eta,
\]

where \(C > 0\) is a constant. Thus, \((2.1)\) is proved.

Fixed \(u_0 \in X \setminus \{0\}\), for each \(t > 1\) one has

\[
I_\lambda(tu_0) \leq \frac{1}{p} t^p \|u_0\|_{W^{1,p}}^p - \lambda\alpha_0 C t^\eta \int_{\Omega} |u_0(x, y)|^\eta dxdy.
\]

Since \(\eta > p\), this condition guarantees that \(I_\lambda\) is unbounded from below. Fixed \(\lambda \in ]0, \lambda^*[\), from definition of \(\Phi\) it follows that

\[
\|u\|_{W^{1,p}} < p^{1/p}, \tag{2.3}
\]

for each \(u \in X\) such that \(u \in \Phi^{-1}([-\infty, 1])\). Moreover, \((f_1)\), the compact embedding \(X \hookrightarrow L^\infty(\Omega)\) and \((2.3)\) imply that, for each \(u \in \Phi^{-1}([-\infty, 1])\), we have

\[
\Psi(u) \leq \int_{\Omega} \alpha(x, y)(a_1 |u(x, y)| + \frac{a_2}{s} |u(x, y)|^s) dxdy
\]

\[
\leq (a_1 \|u\|_{L^\infty} + \frac{a_2}{s} \|u\|_{L^\infty}^s)\|\alpha\|_{L^1}
\]

\[
\leq (a_1 c_p \|u\|_{W^{1,p}} + \frac{a_2 c_p^s}{s} \|u\|_{W^{1,p}}^s)\|\alpha\|_{L^1}
\]

\[
< (a_1 c_p p^{1/p} + \frac{a_2 c_p^s p^s/p}{s})\|\alpha\|_{L^1},
\]

and so,

\[
\sup_{u \in \Phi^{-1}([0, 1])} \Psi(u) \leq (a_1 c_p p^{1/p} + \frac{a_2 c_p^s p^s/p}{s})\|\alpha\|_{L^1} = \frac{1}{\lambda^*} < \frac{1}{\lambda} \tag{2.4}
\]

From \((2.4)\) one has

\[
\lambda \in ]0, \lambda^*[ \leq \frac{1}{\sup_{u \in \Phi^{-1}([-\infty, 1])} \Psi(u)\lambda^*}.\]
Hence, Theorem 1.2 assures the existence of at least two distinct critical points for problem (1.1). Also, it is proved in [7, proof of Theorem 2.1] that $I_\lambda$ is an invariant functional with respect to the action of the compact group of linear isometries of $\mathbb{R}^n$. Thus, we can apply the principle of symmetric criticality (see [8]) to the smooth and isometric invariant functional $I_\lambda$ and deduce that problem (1.1) admits at least two distinct cylindrically symmetric weak solutions. The proof is complete. □

Remark 1. We observe that, if $f$ is non-negative and $f(0) \neq 0$, then Theorem 2 ensures the existence of two positive cylindrically symmetric weak solutions for problem (1.1) (see, e.g., [11, Theorem 11.1]).

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