A q-difference Baxter operator for the Ablowitz–Ladik chain

Federico Zullo

Dipartimento di Matematica e Fisica, Università di Roma Tre, Via della Vasca Navale 84, I-00146 Roma, Italy
INFN, Sezione di Roma Tre, Via della Vasca Navale 84, I-00146 Roma, Italy
E-mail: zullo@fis.uniroma3.it

Received 5 August 2014, revised 3 February 2015
Accepted for publication 6 February 2015
Published 4 March 2015

Abstract

We construct the Baxter operator and the corresponding Baxter equation for a quantum version of the Ablowitz–Ladik model. The result is achieved in two different ways: by using the well-known Bethe ansatz technique and by looking at the quantum analogue of the classical Bäcklund transformations. General results about integrable models governed by the same r-matrix algebra will be given. Baxter’s equation comes out to be a q-difference equation involving both the trace and the quantum determinant of the monodromy matrix. The spectrality property of the classical Bäcklund transformations gives a trace formula representing the classical analogue of Baxter’s equation. A q-integral representation of the Baxter operator is discussed.

Keywords: Baxter’s operator, Baxter’s equation, Ablowitz–Ladik model, Bäcklund transformations

1. Introduction

The Ablowitz–Ladik model [1, 2] is the integrable differential difference version of the nonlinear Schrödinger equation. The dynamical variables lie on a (periodic or infinite) lattice: we shall denote by a subscript ‘k’ the points of the lattice and by \( r_k, q_k \) the dynamical variables. The classical equations of motion are given by

\[
\dot{q}_k = q_{k+1} + q_{k-1} - 2q_k - q_k r_k (q_{k+1} + q_{k-1}),
\]

\[
\dot{r}_k = -q_{k+1} - q_{k-1} + 2r_k + q_k r_k (q_{k+1} + q_{k-1}).
\]

This model possesses a Lax matrix representation. The Lax matrix is given by the product [6, 13, 28]
\[ L(\lambda) = \prod_{k=1}^{N} L_k(\lambda), \quad \text{with} \quad L_k(\lambda) = \begin{pmatrix} \lambda & q_k \\ \eta_k & \lambda^{-1} \end{pmatrix}. \]  

where \( \lambda \) is the spectral parameter. The determinant of \( L(\lambda) \), given by \( \prod_{k=1}^{N} (1 - q_k \eta_k) \), is a conserved quantity. Other \( N - 1 \) conserved quantities appear in the Laurent expansion of the trace of \( L(\lambda) \)

\[ \text{Tr} \left( L(\lambda) \right) = \sum_{i=0}^{N} H_i \lambda^{N-2i}, \quad H_0 = H_N = 1. \]  

The involutivity of these conserved quantities follows from the Poisson structure underlying the model; it can be defined by an \( r \)-matrix relation satisfied by the Lax matrix

\[ \left[ L(\lambda) \otimes L(\nu) \right] = \left[ r, L(\lambda) \otimes L(\nu) \right], \]  

where the \( r \) matrix is defined by [23]

\[
\begin{pmatrix}
\frac{\nu^2 + \lambda^2}{2 \nu^2 - \lambda^2} & 0 & 0 & 0 \\
0 & \frac{1}{2} \frac{\nu^2 - \lambda^2}{\nu^2 - \lambda^2} & \frac{\lambda \nu}{1} & 0 \\
0 & \frac{\lambda \nu}{\nu^2 - \lambda^2} & \frac{1}{2} & 0 \\
0 & 0 & 0 & \frac{1}{2} \frac{\nu^2 + \lambda^2}{\nu^2 - \lambda^2}
\end{pmatrix}
\]

It is easy to check that the above relations are equivalent to the following Poisson brackets for the dynamical variables of the model

\[ \{ q_k, r_j \} = \left( 1 - q_k \eta_j \right) \delta_{kj}, \quad \{ q_k, q_j \} = \{ \eta_k, r_j \} = 0. \]  

The exact discretization of this model through the technique of Bäcklund transformations has been considered recently in [28], where it was also shown how it is possible to get explicit maps preserving exactly the continuous trajectories. Those results were obtained using the general scheme described in [27], that, in turn, was stimulated by earlier results on the application of the theory of Bäcklund transformations to finite dimensional systems [14, 20, 21, 26].

In this work we would like to consider the quantum version of the Ablowitz–Ladik model and the corresponding quantum version of its Bäcklund transformations. The quantum version of the Ablowitz–Ladik model, with the Lax and \( r \)-matrix structures, were introduced by Kulish [13]. In this work the author also presented the Bethe equations obtained by diagonalizing the trace of the Lax matrix (i.e. the generating function of the integrals of motion). Indeed most of the results of section 2 parallel those of Kulish (see also [6]); they are presented here for the sake of completeness. Indeed, as we will show, the quantum analogue of the classical Bäcklund transformations lead to Baxter’s equations as the algebraic Bethe ansatz does. This is not a coincidence: similar results have been found, for example, for the discrete self-trapping model [15] and for the Toda lattice [17, 23]. The quantum Ablowitz–Ladik model corresponds to the \( q \)-boson model [5, 6]: the algebra \( (11) \) defined by the \( r \)-matrix structure \((8)-(10)\) can be put in correspondence with the \( q \)-boson algebra \([5, 7, 12]\) (also called \( q \)-oscillator algebra [11]). The operators \( r_k \) and \( q_k \) then can be interpreted as boson creation/annihilation operators. It is worth mentioning that the eigenvalue problem for the \( q \)-
boson model with periodic boundary conditions is equivalent to the eigenvalue problem of an integrable discretization [24] of the Lieb–Liniger model [16] (see [12]). For more details about the Bethe wave functions, the completeness of the Bethe ansatz, the correlation functions of the model, the properties of the model in the thermodynamic limit, and results about the infinite chain we refer the reader to the papers [5–7, 12, 24, 25] and references therein. In this paper, as said, a particular emphasis will be put on the application of the Bäcklund transformation theory to the quantum model.

In general, for a given integrable system, the quantum Bäcklund transformations can be represented by a unitary operator $Q_\mu$, where $\mu$ is a (set of) parameter(s) of the transformations realized as an integral operator on the space of eigenfunctions [14, 17]. If we denote by $(P, Q)$ a set of canonically conjugated variables and by $(\hat{P}, \hat{Q})$ the new variables transformed according to the Bäcklund maps, we have

$$Q_\mu : \psi (Q) \rightarrow \int f(\hat{Q}, Q) \psi (Q) dQ. \quad (5)$$

The similarity transformations induced by $Q_\mu$ are the equivalent of the classical canonical transformations and the kernel $f(\hat{Q}, Q)$ is given, in the semiclassical approximation, by

$$f(\hat{Q}, Q) \sim \exp \left( -\frac{i}{\hbar} F(\hat{Q}, Q) \right), \quad \hbar \rightarrow 0, \quad (6)$$

where $F(\hat{Q}, Q)$ is the generating function of the classical Bäcklund transformations, that indeed are canonical transformations of the phase space. The $Q$ operator is called the Baxter operator. The explicit construction of a Baxter operator for the model under consideration has been given recently by Christian Korff [12]. The reader can find in this paper (see proposition 3.11) the explicit expression of the Baxter operator in terms of polynomials in the generators of the $q$-boson algebra. The commutativity properties of the $Q$ are described in detail as well. Here, in the spirit of the above description, we obtain an integral formula for Baxter’s operator.

$$Q_\mu \text{ Tr } L(\mu) = \Delta_+ Q_{\mu+\eta} + \Delta_- Q_{\mu-\eta} \quad (7)$$

where $\Delta_\pm$ are scalar functions [3, 14, 17]. Equation (7) is considered as a fundamental attribute of Baxter’s operator and the starting point for further investigations. Here we would like to emphasize also the significant relationships between this operator and the Bäcklund transformation theory. Indeed we will see that in our case, the form of equation (7) is changed in a $q$-difference equation, whether the relationship with the theory of Bäcklund transformations remains unchanged.

2. The Bethe ansatz and Baxter’s equation

In this section we shall construct Baxter’s equation by diagonalizing the operator $\text{ Tr } L(\mu)$. We will use the standard algebraic Bethe ansatz technique (see e.g. [8]).

Let us first define the quantum version of the Ablowitz–Ladik model. We take the same Lax matrix as above (1), where now the elements of the matrices are operators. The commutation relations are defined by the following quantum $r$-matrix (see also [13])
where, for simplicity of notation, we set \( c = \frac{i^2}{\lambda - i^2} \) and \( b = \frac{\lambda + i^2}{\lambda - i^2} \). This quantum r-matrix is related to the classical one by the relation \( R = (1 + \eta/2)I - \eta r \). The r-matrix (8) solves the Yang–Baxter equation:

\[
R_{ic,jb}(\lambda/\nu)R_{cm,kb}(\lambda)R_{an,hr}(\nu) = R_{ja,kb}(\nu)R_{ic,hr}(\lambda)R_{cm,an}(\lambda/\nu). \tag{9}
\]

In equation (9) there is a sum over repeated indices and we use the usual convention for the Kronecker product of two matrices to label the elements \( R_{ij,kl} \) (i.e. if \( T = A \otimes B \) then \( T_{ijkl} = A_{ij}B_{kl} \)).

The Poisson brackets (quantum commutators) among phase space variables are defined by the relation

\[
R(\lambda/\nu)L(\nu)L(\lambda)R(\lambda/\nu) = \frac{1}{L}L(\lambda)L(\nu)L(\lambda/\nu), \tag{10}
\]

where \( L \) and \( \bar{L} \) mean the tensor products \( L = L \otimes 1 \) and \( \bar{L} = 1 \otimes L \). The commutation relations (10) are equivalent to

\[
[q_j, r_k] = \eta(1 - q_j r_j)\delta_{j,k}, \tag{11}
\]

that are the quantum analogue of the Poisson brackets (4). The relations (10) define the commutators among the matrix elements of the monodromy matrix \( L(\lambda) \). If we set

\[
L(\lambda) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix}, \tag{12}
\]

it is not difficult to show, for example, that the following relations hold:

\[
A(\nu)C(\lambda) = f(\nu, \lambda)C(\lambda)A(\nu) + g(\lambda, \nu)C(\nu)A(\lambda),
\]

\[
D(\nu)C(\lambda) = \tilde{f}(\nu, \lambda)C(\lambda)D(\nu) + \tilde{g}(\lambda, \nu)C(\nu)D(\lambda), \tag{13}
\]

where \( f(\nu, \lambda) = \frac{1}{1 + \eta} \left( 1 - \eta \frac{\lambda^2}{\lambda^2 - \nu^2} \right) \), \( g(\lambda, \nu) = \frac{\eta}{1 + \eta} \frac{\nu}{\lambda - \nu} \), \( \tilde{f}(\nu, \lambda) = f(\lambda, \nu) \) and \( \tilde{g}(\lambda, \nu) = -g(\lambda, \nu) \). The relations (13) are the most relevant for what we are going to say, so we will not write the commutation relations among the other matrix elements of the monodromy matrix. If we define a pseudo-vacuum state \( \left| 0 \right> \) by

\[
B(\lambda)|0\rangle = 0
\]

then, if it is unique, it is also an eigenvector of \( A(\lambda) \) and \( D(\lambda) \)

\[
A(\lambda)|0\rangle = a(\lambda)|0\rangle, \quad D(\lambda)|0\rangle = d(\lambda)|0\rangle.
\]

Also, because of the factorization (1), if we define a vacuum state \( |0\rangle_j \) for each lattice site \( j \) by \( q_j|0\rangle_j = 0 \), then \( |0\rangle = \prod |0\rangle_j \) and

\[
a(\lambda) = \lambda^N, \quad d(\lambda) = \lambda^{-N}.
\]
Let us look for a set of eigenfunctions in the form
\[ \phi(\lambda) = \prod_{k=1}^{m} C(\lambda_k)|0\rangle. \] (14)

By using recursively the relations (13) to move the operator \( A(\nu) \) on the right, it is possible to show that
\[ A(\nu) \prod_{k=1}^{m} C(\lambda_k) = \prod_{k=1}^{m} f(\nu, \lambda_k)C(\lambda_k)A(\nu) + \sum_{k=1}^{m} \Lambda_k \prod_{j \neq k} C(\lambda_j)C(\nu)A(\lambda_k), \]
where \( \Lambda_k = g(\lambda_k, \nu) \prod_{j \neq k} f(\lambda_k, \lambda_j). \) Analogously, for \( D(\nu) \) we have
\[ D(\nu) \prod_{k=1}^{m} C(\lambda_k) = \prod_{k=1}^{m} \tilde{f}(\nu, \lambda_k)C(\lambda_k)D(\nu) + \sum_{k=1}^{m} \tilde{\Lambda}_k \prod_{j \neq k} C(\lambda_j)C(\nu)D(\lambda_k), \]
with \( \tilde{\Lambda}_k = \tilde{g}(\lambda_k, \nu) \prod_{j \neq k} \tilde{f}(\lambda_k, \lambda_j). \) So, for the trace of \( L(\nu) \) we have
\[ \text{Tr} (L(\nu))\phi = \left(a(\nu) \prod_{j} f(\nu, \lambda_j) + d(\nu) \prod_{j} \tilde{f}(\nu, \lambda_j)\right)\phi + \sum_{k} \left(a(\lambda_k)\Lambda_k + d(\lambda_k)\tilde{\Lambda}_k\right) \prod_{j \neq k} C(\lambda_j)C(\nu)|0\rangle, \]
and, if the Bethe equations
\[ a(\lambda_k)\Lambda_k + d(\lambda_k)\tilde{\Lambda}_k = 0 \implies \prod_{j \neq k} \left(\lambda_j^2(1 + \eta) - \lambda_k^2\right) = \lambda_k^{2N}, \quad k = 1 \ldots m, \] (15)
are satisfied, then \( \phi \) is an eigenvector of \( \text{Tr} (L(\nu)) \) with the eigenvalue equation
\[ \text{Tr} (L(\nu))\phi = \frac{\nu^N}{(1 + \eta)^m} \prod_{j=1}^{m} \left(1 - \eta - \frac{\nu^2}{\lambda_j^2 - \nu^2}\right)\phi + \frac{\nu^{-N}}{(1 + \eta)^m} \prod_{j=1}^{m} \left(1 + \eta - \frac{\lambda_j^2}{\lambda_j^2 - \nu^2}\right)\phi. \] (16)

Let us call \( t(\nu) \) the eigenvalue of \( \text{Tr} L(\nu) \). If we set
\[ \psi(\nu, \{\lambda\}) \equiv \prod_{j=1}^{m} \left(\nu^2 - \lambda_j^2\right) \]
then equation (16) is equivalent to
\[ t(\nu)\psi(\nu, \{\lambda\}) = \frac{\nu^N}{(1 + \eta)^m}\psi(\nu, \sqrt{1 + \eta}, \{\lambda\}) + \frac{1}{\nu}\psi\left(\frac{\nu}{\sqrt{1 + \eta}}, \{\lambda\}\right). \] (17)

To better understand the structure of this equation we need to introduce the quantum determinant of the monodromy matrix (12) (see also [13]). From the commutation relations (10) it is possible to show that
\[ \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix} \begin{pmatrix} D(\lambda\sqrt{\alpha}) \\ \sqrt{\alpha} \end{pmatrix} = \begin{pmatrix} A(\lambda\sqrt{\alpha}) \\ C(\lambda\sqrt{\alpha}) \end{pmatrix} \begin{pmatrix} D(\lambda) & -B(\lambda) \\ \sqrt{\alpha} & A(\lambda) \end{pmatrix} = (\sqrt{\alpha})^{N-1} \Delta \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \] (18)
and that
\[
\begin{pmatrix}
\frac{D(\lambda \sqrt{\alpha})}{\sqrt{\alpha}} & \frac{B(\lambda \sqrt{\alpha})}{\alpha} \\
-C(\lambda \sqrt{\alpha}) & \frac{A(\lambda \sqrt{\alpha})}{\sqrt{\alpha}}
\end{pmatrix}
\begin{pmatrix}
A(\lambda) & B(\lambda) \\
C(\lambda) & D(\lambda)
\end{pmatrix}
= \left( \sqrt{\alpha} \right)^{-1} \Delta \left( \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right),
\]  
(19)

where we pose
\[
\alpha \doteq \frac{1}{1 + \eta} = \frac{1}{1 + i \hbar}
\]
and we define
\[
\left( \sqrt{\alpha} \right)^{-1} \Delta \doteq \frac{A(\lambda)D(\lambda \sqrt{\alpha})}{\sqrt{\alpha}} - \frac{B(\lambda)C(\lambda \sqrt{\alpha})}{\alpha} = \frac{D(\lambda)A(\lambda \sqrt{\alpha})}{\sqrt{\alpha}} - C(\lambda)B(\lambda \sqrt{\alpha})
\]
\[
= \frac{A(\lambda \sqrt{\alpha})D(\lambda)}{\sqrt{\alpha}} - C(\lambda)B(\lambda) = \frac{D(\lambda \sqrt{\alpha})A(\lambda)}{\sqrt{\alpha}} - \frac{B(\lambda \sqrt{\alpha})C(\lambda)}{\alpha}.
\]
(21)

The previous four expressions are equal by virtue of the commutation relations (10). The factor \(\left( \sqrt{\alpha} \right)^{-1} \) is not essential but is just a normalization constant (i.e. when all the fields \(q_k\) and \(r_k\) are zero then \(\Delta = 1\)). It is a useful exercise to prove the following remark explicitly using the commutation relations defined by (10).

**Remark 2.1.** The operators \(\Delta(\lambda)\) and \(\text{Tr}(L(\mu))\) commute.

We stress that all the previous formulae and proposition are independent of the particular structure of the monodromy matrix since they follows from the Poisson structure defined by (10) for any monodromy matrix (12). If we specialize to our monodromy matrix, defined by (1), \(\Delta\) is also independent of \(\lambda\). Indeed, let us define
\[
T_n(\lambda) = \prod_{k=1}^{n} L_k(\lambda),
\]
where the matrices \(L_k(\lambda)\) are defined by (1). Writing down explicitly the equation \(T_{n+1} = L_{n+1} T_n\), using the commutativity of the elements of \(L_{n+1}\) with those of \(T_n\) and the expressions (11) we get
\[
\Delta_{n+1} = (1 - n_{n+1} q_{n+1}) \Delta_n \implies \Delta_n = \prod_k \left( 1 - n_k q_k \right)
\]
where with \(\Delta_n\) we denoted the quantum determinant of the matrix \(T_n\). So we get
\[
\Delta = \prod_{k=1}^{N} \left( 1 - n_k q_k \right).
\]
(22)

At this point we could ask about the eigenvalues of \(\Delta\) on the eigenfunctions (14). Again, from the commutation relations (10) it follows that
\[
\Delta(\lambda) C(\mu) = \alpha C(\mu) \Delta(\lambda)
\]
entailing the following
Corollary 2.2. The eigenvalues of the quantum determinant $\Delta$ (21) on the eigenfunctions (14) are given by the relation

$$\Delta(\lambda) \prod_{j=1}^{m} C(\lambda_j)|0\rangle = a^m \hat{d}(\lambda) a(\sqrt{\alpha}) \prod_{j=1}^{m} C(\lambda_j)|0\rangle.$$ 

In our case, for the monodromy matrix defined by (1), the eigenvalues are independent of $\lambda$ and are given by $a^m$.

The meaning of equation (17) is now clearer: the trace decomposes in the sum of two terms that classically correspond to the two eigenvalues of the monodromy matrix. If we denote by $\delta$ the eigenvalue of the quantum determinant $\Delta$, we have

$$t(\nu)\psi(\nu, \{\lambda\}) = \delta \nu^N \psi\left(\frac{\nu}{\sqrt{\alpha}}, \{\lambda\}\right) + \frac{1}{\nu^N} \psi\left(\nu \sqrt{\alpha}, \{\lambda\}\right).$$ \tag{24}

By choosing a different normalization for the function $\psi(\nu)$, that is $\psi(\nu) = \hat{\psi}(\nu) \nu^{2m}$, one gets

$$t(\nu)\hat{\psi}(\nu, \{\lambda\}) = \nu^N \hat{\psi}\left(\frac{\nu}{\sqrt{\alpha}}, \{\lambda\}\right) + \delta \nu^N \hat{\psi}\left(\nu \sqrt{\alpha}, \{\lambda\}\right)$$ \tag{25}

that is the factor $\delta$ moved from one addend to the other.

In general we expect Baxter’s equation defined by monodromy matrices satisfying the commutation relation (10) to be

$$t(\nu)\psi(\nu, \{\lambda\}) = \Lambda_+ \psi\left(\frac{\nu}{\sqrt{\alpha}}, \{\lambda\}\right) + \Lambda_- \psi\left(\nu \sqrt{\alpha}, \{\lambda\}\right),$$ \tag{26}

where $\Lambda_+$ and $\Lambda_-$ are two scalar factors whose product gives $\delta$, the eigenvalue of the corresponding quantum determinant. A similar functional equation has been obtained by Korff for the $Q$ operator (see [12], equation 3.50).

3. Baxter’s equation and Bäcklund transformations

In this section we would like to find an expression for Baxter’s $Q$ operator investigating the relationships with a set of Bäcklund transformations for the model. We will divide the section in three parts: the first one will describe a set of classical Bäcklund transformations for the model, mainly emphasizing the aspects having a clear correspondence with the quantum case, that will be described in the second part of the section. In the last subsection we represent the Baxter’s operator as a q-integral operator, proving its main properties (i.e. Baxter’s equation and the commutativity with the conserved quantities of the model).

3.1. Classical Bäcklund transformations

Classical Bäcklund transformations for the Ablowitz–Ladik model have been considered in different works (see e.g. [23, 28]). It is possible to compose elementary parametric transformations to get more complex multi-parametric maps. Here, for what concerns our purposes, it is sufficient to take just an elementary transformation. We shall use the dressing matrix technique adapting the results of [28]. We take a dressing matrix of the form
\[ D_k(\lambda) = \begin{pmatrix} \lambda^2 + a_k & \lambda b_k \\ \lambda c_k & 1 \end{pmatrix} \]

and require that, for a given value of \( \lambda \), say \( \lambda = \mu \), the matrix is degenerate. Then we can parametrize \( D_k \) as
\[ D_k(\lambda) = \begin{pmatrix} \lambda^2 - \mu^2(1 - b_k c_k) & \lambda b_k \\ \lambda c_k & 1 \end{pmatrix} \] (27)

Now we must look at the matrix equation
\[ \tilde{L}_k D_k - D_{k+1} L_k = 0 \] (28)

that has to be satisfied for every value of \( \lambda \). The elements (12) and (21) of the previous expression then give
\[ b_k = q_k, \quad c_k = \tilde{r}_{k-1}. \]

By choosing the set \( \{ n_k, \tilde{r}_k \}_{k=1}^N \) as independent variables, from (28) we get
\[ 1 - q_k n_k = \frac{(\tilde{r}_{k-1} - n_k)(\tilde{r}_k \mu^2 + n_k)}{\mu^2 \tilde{r}_{k-1}}, \quad k = 1 \ldots N \] (29)

defining \( \tilde{q}_k \) and \( q_k \) in terms of the independent variables. We are assuming periodic boundary conditions, i.e. \( q_{k+N} = q_k \) and \( \tilde{r}_{k+N} = \tilde{r}_k \). Let us show that the expressions (29) do define Bäcklund transformations for the Ablowitz–Ladik model. To do so, we must show that [14, 27]

1. the conserved quantities are invariant under the action of the map (29),
2. the transformations are canonical.

Point 1 is trivial: equation (28) implies, for the monodromy matrix \( L(\lambda) \) (1), the equivalence \( \tilde{L} D = D L \). From this isospectral equation we get the preservation of the conserved quantities under the action of the map. Indeed the generating functions of the integrals of motion are given by \( \text{Tr} L \) and \( \text{det}(L) \) (see equation (2)): since \( L \) and \( \tilde{L} \) are related by a similarity transformations we get \( \text{Tr}(L) = \text{Tr}(\tilde{L}) \) and \( \text{det}(L) = \text{det}(\tilde{L}) \).

For point 2 it is possible to write down explicitly the generating function of the transformations. Indeed, from the Poisson brackets (4), we get that \( \frac{\ln(1 - q_k n_k)}{n_k} \) is a canonical one-form. The generating function of the transformations (29) then satisfy
\[ dF(r, \tilde{r}) = \sum_k \ln \left( 1 - \tilde{q}_k \tilde{r}_k \right) d\tilde{r}_k - \ln \left( 1 - q_k n_k \right) dn_k. \] (30)

It is possible to check that the function \( F(r, \tilde{r}) \) is given by
\[ F = \sum_k \int_{n_k}^{\tilde{r}_k} \ln \left( z - n_{k+1} \right) dz + \int_{\mu^2}^{\tilde{r}_k} \ln \left( \mu^2 z + n_k \right) dz - \ln \left( \tilde{r}_k \right) \ln \left( \mu^2 \tilde{r}_{k-1} \right) - 2 \ln (\mu) z^2. \]

Now let us show how it is possible to obtain the classical version of Baxter’s equation (26) by looking at the action of the dressing matrix on the matrices \( L_k(\lambda) \). The result will be trivial but it will help to understand what we get in the quantum case. The idea is a mix
of the observations due to Pasquier and Gaudin \cite{17} and Kuznetsov and Sklyanin \cite{14}. Baxter’s equation (26) involves the trace of the monodromy matrix so we would like to obtain an expression for this trace. The observation in \cite{17} is that \( \text{Tr} \ L(\mu) \) does not change if we perform a sort of similarity transformation on the matrices \( L_k \) like \( L_k = M_k^{-1} L_k M_k \). To choose the matrix \( M_k \) we look again at equation (28). The matrix \( D_k(\mu) \) is singular; its kernel is given by
\[
|w_k\rangle = \left( \begin{array}{c} 1 \\ -\mu \bar{r}_{k-1} \end{array} \right).
\]
From equation (28) we readily get
\[
L_k(\mu)|w_k\rangle = \gamma_k |w_{k+1}\rangle
\]
for some function \( \gamma_k \). We notice that \( |w_k\rangle \) is an eigenvector of the monodromy matrix \( L(\mu) \) and \( \gamma = \prod_{k=1}^{N} \gamma_k \) is the corresponding eigenvalue. The other eigenvalue is given by
\[
\frac{\det(L(\mu))}{\gamma} = \prod_{k=1}^{N} \frac{1 - q_k \bar{r}_k}{\gamma_k},
\]
since the product of this eigenvalue with the other is equal to the determinant of the monodromy matrix. Now let us build a matrix \( M_k \) with the column \( |w_k\rangle \) and another arbitrary column \( |x_k\rangle \): \( M_k = (|x_k\rangle, |w_k\rangle) \). With the help of relation (31) it is simple to show that \( \tilde{L}_k \) is lower triangular
\[
\tilde{L}_k = M_k^{-1} L_k M_k = \begin{pmatrix} \frac{\det(M_k)}{\gamma_k} & 0 \\ \frac{1 - q_k \bar{r}_k}{\gamma_k} & \gamma_k \end{pmatrix}
\]
immediately implying
\[
\text{Tr} \ \tilde{L}(\mu) = \text{Tr} \ L(\mu) = \prod_{k=1}^{N} \frac{1 - q_k \bar{r}_k}{\gamma_k} + \prod_{k=1}^{N} \gamma_k = \frac{\det(L(\mu))}{\gamma} + \gamma.
\]
Although the result is trivial (the trace of the monodromy matrix is the sum of its eigenvalues), there is a strong correspondence with the quantum case. Indeed (33) is the semiclassical limit of Baxter’s equation, as we will show in the next subsection. Now we will just make some further considerations, postponing more comments to the end of section 3.3.

The parameter \( \mu \) can be seen as an evolution parameter for the Bäcklund transformations. Then the maps are the integral curves of a non autonomous Hamiltonian system of equations, the flow being generated by the variable conjugated to \( \mu \) expressed in the variables \((r, q)\) and \(\Phi = \frac{\partial F}{\partial \mu}\bigg|_{\mu=\bar{r}(r,q)}\). Explicitly we have
\[
\Phi = \frac{\partial F}{\partial \mu}\bigg|_{\mu=\bar{r}(r,q)} = \frac{2}{\mu} \sum_k \ln \left( \frac{\mu^2 \bar{r}_k + \bar{q}_k}{\mu^2 \bar{r}_k} \right)\bigg|_{\mu=\bar{r}(r,q)}.
\]
Using the expressions of Bäcklund transformations it is simple to show that
\[
\Phi = \frac{2}{\mu} \sum_k \ln \left( \frac{\mu^2 \bar{r}_k + \bar{q}_k}{\mu^2 \bar{r}_k} \right)\bigg|_{\mu=\bar{r}(r,q)} = \frac{2}{\mu} \ln \left( \prod_k \frac{1 - q_k \bar{r}_k}{\mu \bar{r}_k} \right) = \frac{2}{\mu} \ln \left( \frac{\det(L(\mu))}{\mu^N \gamma} \right).
\]
Comparing with equation (33), we see that it can be rewritten as

$$\text{Tr } L(\mu) = \mu^N e^{z\Phi} + \frac{\det(L(\mu))}{\mu^N} e^{-z\Phi}.$$  \hspace{1cm} (36)

We shall return to this equation at the end of the next section.

3.2. Quantum case

The quantum model is described by the monodromy matrix (1) and the commutation relations (10). Equation (11) suggests that the action of $q_k$ on a function $f(\{r_j\})$ is proportional to the Jackson derivative (or $q$-derivative, but to avoid confusion with the dynamical variables we shall use the symbol $\alpha$ instead of $q$) in the direction of $q_k$ (see also [11], chapter 5)

$$q_k f(\{r_j\}) = (1 - \alpha) D_{\alpha,k} f(\{r_j\}),$$  \hspace{1cm} (37)

where $\alpha$ is defined by (20).

Again we shall follow [17] and [14] (see also [3]). We are looking for an operator $Q_\mu$ satisfying (see equation (26))

$$\text{Tr } L(\mu) Q_\mu = \mu^N Q_\mu + \frac{\Delta}{\mu^N} Q_{\mu,\nu}, \quad \alpha \equiv \frac{1}{1 + \eta} = \frac{1}{1 + i\hbar}$$  \hspace{1cm} (38)

commuting with the trace of $L(\lambda)$, $[Q_\mu, \text{Tr } L(\lambda)] = 0$.

Let us consider the columns of equation (38), say $\rho_k$. If we take $\rho_k$ in the form of a product, i.e. $\rho_k = \prod_j \rho_j(r_j)$, then, because $L(\mu)$ is itself in the form of a product (see equation (1)), the action of $L(\mu)$ on $\rho$ decomposes:

$$\text{Tr } L(\mu) \rho = \text{Tr} \left( \left( L_N \rho_N \right) \cdots \left( L_1 \rho_1 \right) \right)$$  \hspace{1cm} (39)

Exactly as in the classical case, $\text{Tr } L(\mu)$ does not change if we perform the transformation $\hat{L}_k = M_{k+1}^{-1} L_k M_k$. So we take the following matrix $M_k$:

$$M_k = \begin{pmatrix} 0 & 1 \\ -1 & -\mu \tilde{\eta}_{k-1} \end{pmatrix}$$

that is of the same form as in the classical case, but now we specified the vector $|\chi_k\rangle$ to be $(0.1)^T$. We fixed this vector for the sake of simplicity since now the determinant of the matrix $M_k$ is equal to 1. We get

$$\hat{L}_k = \begin{pmatrix} \mu \tilde{\eta}_k q_k + \frac{1}{\mu} -\mu^2 \tilde{\eta}_k \left( 1 - q_k \tilde{\eta}_{k-1} \right) - (\tilde{\eta}_k - \tilde{\eta}_{k-1}) \\ -q_k \mu \left( 1 - q_k \tilde{\eta}_{k-1} \right) \end{pmatrix}.$$  \hspace{1cm} (40)

The functions $\rho_k$ are then defined by requiring (see equation (32))

$$\mu^2 \tilde{\eta}_k \left( 1 - q_k \tilde{\eta}_{k-1} \right) \rho_k = (\tilde{\eta}_{k-1} - \eta_k) \rho_k$$  \hspace{1cm} (41)

Recalling the action of the operator $q_k$ on the functions of $r_k$ (37), equation (41) can be written as
\[ \rho_k(\mu, n_k) = \frac{\mu^2 \tilde{\rho}_{k-1}}{\left(\mu^2 \tilde{\rho}_k + n_k\left(\tilde{\rho}_{k-1} - n_k\right)\right)} \rho_k(\mu, \alpha n_k) \]

that is solved by
\[ \rho_k(\mu, n_k) = G_k \prod_{p=0}^{\infty} \frac{1}{1 + a^{\frac{n_k}{\mu^2 \tilde{\rho}_k}}} \frac{1 - a^{\frac{n_k}{\mu^2 \tilde{\rho}_k}}}{\left(\frac{n_k}{\tilde{\rho}_{k-1}}; a\right)_{\infty} - \frac{n_k}{\mu^2 \tilde{\rho}_k}}. \quad (42) \]

where \( G_k \) is independent of \( r_k \) and we introduced the usual notation for the q-Pochhammer symbol \([9]\)
\[ (x; \alpha)_{\infty} = \prod_{p=0}^{\infty} \left(1 - x a^p\right). \quad (43) \]

We notice that \( \frac{1}{(x; \alpha)_{\infty}} \) is a q-analog of the exponential function \( e^x \) since
\[ \frac{1}{(x; \alpha)_{\infty}} \to e^x \text{ in the limit } \alpha \to 1. \]

Using (41), the matrix \( \hat{L}(\mu) \rho_k \) can be written as
\[ \hat{L}_k \rho_k = \begin{pmatrix} \frac{\mu^2 \tilde{\rho}_k + n_k}{\mu \tilde{\rho}_{k-1}} & 0 \\ -\frac{n_k}{\mu \tilde{\rho}_k} & \frac{n_k}{\tilde{\rho}_{k-1}} - \frac{n_k}{\mu \tilde{\rho}_k} \end{pmatrix} \rho_k \quad (44) \]

or, using the explicit form of the functions \( \rho_k \) (42)
\[ \hat{L}_k \rho_k(\mu, n_k) = \begin{pmatrix} \frac{\mu \tilde{\rho}_k}{\tilde{\rho}_{k-1}} \rho_k \left(\frac{\mu}{\sqrt{\alpha}}, n_k\right) & 0 \\ -\frac{n_k}{\mu \tilde{\rho}_k} \rho_k(\mu, n_k) & \frac{n_k}{\tilde{\rho}_{k-1}} - \frac{n_k}{\mu \tilde{\rho}_k} \rho_k \left(\mu \sqrt{\alpha}, \alpha \rho_k\right) \end{pmatrix}. \]

Then it follows from (39) that the trace of the monodromy matrix \( L(\mu) \) on \( \rho = \prod_k \rho_k \) is given by
\[ \text{Tr} (L(\mu)) \rho(\mu, r) = \mu^N \rho \left(\frac{\mu}{\sqrt{\alpha}}, r\right) + \frac{1}{\mu^N} \rho(\mu \sqrt{\alpha}, \alpha r). \quad (45) \]

We notice that the action of \( \Delta = \prod_k (1 - n_k q_k) \) on \( \rho \) is given by
\[ \prod_k (1 - n_k q_k) \rho(r) = \rho(\alpha r) \]
so that equation (45) can be also written as
\[ \text{Tr} (L(\mu)) \rho(\mu, r) = \mu^N \rho \left(\frac{\mu}{\sqrt{\alpha}}, r\right) + \frac{\Delta}{\mu^N} \rho(\mu \sqrt{\alpha}, r). \quad (46) \]

From the Bäcklund transformations point of view there are clear analogies between the classical and the quantum cases. Baxter’s equation
\[ \text{Tr} (L(\mu)) Q_\alpha = \mu^N Q_\alpha + \frac{\Delta}{\mu^N} Q_{\mu \sqrt{\alpha}} \quad (47) \]
is the quantum analogue of the classical equation (36) for the trace of the monodromy matrix. Indeed in (36) \( \Phi \) is the canonical conjugated variable with respect to \( \mu \). If we substitute
directly for \( \Phi \) the corresponding quantum variable, i.e. \( \Phi = \eta \frac{\partial}{\partial \mu} \), then equation (36) gives

\[
\text{Tr} \left( L(\mu) \rho(\mu, t) \right) = \mu^N e^{\frac{\lambda_1^2}{\mu}} \partial_{\mu} \rho(\mu, t) + \frac{\Delta}{\mu^N} e^{-\frac{\lambda_1^2}{\mu}} \rho(\mu, t)
\]

\[
= \mu^N \rho \left( \mu \left( 1 + \frac{\eta}{2} \right) \right) + \frac{\Delta}{\mu^N} \rho \left( \mu \left( 1 - \frac{\eta}{2} \right) \right).
\]

(48)

that agrees at first order in \( \eta \) with (46) since \( \alpha = \frac{1}{1 + \eta} \). The fact that the semi-classical limit of the quantum Bäcklund transformations is linked with the generating function of the corresponding classical transformations is well known [17, 22]. If we consider the equation (47) for its eigenvalues

\[
t(\mu) q(\mu) = \mu^N q \left( \frac{\mu}{\sqrt{\alpha}} \right) + \delta \mu^N q \left( \mu \sqrt{\alpha} \right)
\]

(49)

and we seek a solution \( q(\mu) \) in the following form

\[
q(\mu) = e^{\frac{\lambda_1^2}{\mu}} \left( \sigma \eta^5 \sigma \eta^5 \eta^5 \right),
\]

we get for \( S_0 \) (the prime indicates derivative with respect to \( \mu \))

\[
t(\mu) = \mu^N e^{\frac{\lambda_1^2}{\mu}} + \frac{\det(L(\mu))}{\mu^N} e^{-\frac{\lambda_1^2}{\mu}}.
\]

(50)

Confronting with equations (36) and (34) we see that \( S_0 = F \), the generating function of the classical transformations.

Let us give a remark. Usually Baxter’s equation involves two operators, that is the trace of the monodromy matrix and the Baxter operator. This because the quantum determinant of the monodromy matrix is a c-number or a Casimir of the system. In this case however it is a true operator, so Baxter’s equation involves three operators commuting with each other and with themselves. However this single equation is still enough to get the eigenvalues of both the trace of the monodromy matrix and of its quantum determinant. Indeed the trace of \( L(\mu) \) is given by

\[
\text{Tr} \left( L(\mu) \right) = \sum_{k=0}^{N} \mu^{N-2k} H_k, \quad H_0 = H_N = 1.
\]

If we expand

\[
q(\mu) = \prod_{j=1}^{m} \left( 1 - \frac{1}{\mu^2} - \frac{1}{\lambda_j^2} \right)
\]

then, in the limit \( \mu \to 0 \) equation (49) gives \( \delta = \alpha^m \). The zeros of the right-hand side of equation (49) gives the Bethe equations (15)

\[
\lambda_k^{2N} = \prod_{j \neq k} \left( \frac{\lambda_j^2 (1 + \eta) - \lambda_k^2}{\lambda_j^2 (1 + \eta) \lambda_k^2} \right)
\]

and the eigenvalues \( H_m \) can be found from the relation

\[
\sum_{k=0}^{N} \mu^{N-2k} H_k = \frac{\mu^N q \left( \frac{\mu}{\sqrt{\alpha}} \right) + \delta \mu^N q \left( \mu \sqrt{\alpha} \right)}{q(\mu)}.
\]
3.3. A q-integral formula for the Baxter operator

For quantum models with commutation relations described by the $r$-matrix given by the permutation operator (see equation (7)), it is possible to give an explicit integral formula for the Baxter operator [23]. The kernel of the integral is the object related to the Bäcklund transformations, since it gives, in the semi-classical limit, the generating function of the classical transformations [14]. Further, the Baxter operator can be expressed as the trace of a monodromy matrix [3, 4, 18]. It is possible to combine these two approaches [15]: the monodromy matrix is again the product of elementary operators $\mu_k$ from the spaces $\mathcal{H}_{rc}[\tilde{t}, k]$ to $\mathcal{H}_{rc}[t, k+1]$, where $\mathcal{H}_{rc}[\cdot]$ are the spaces of the so-called ‘auxiliary variables’—their classical counterparts are given by the variables $c_k$ appearing in the dressing matrix $D_k(\lambda)$ (27) so we maintain the same symbol for these variables. The monodromy operator $R_\mu$ is then a map from $\mathcal{H}[\tilde{r}, c_1]$ to $\mathcal{H}[r, c_1]$. Since in our case Baxter’s equation is a q-difference equation, we expect that the Baxter operator can be represented by a q-integral formula. So we introduce the inverse of the operator $q_k$ defined in (37)

$$ (q_k)^{-1}f(\{r_j\}) = \int d_{aj}\alpha_j f(\{r_j\}) = \sum_{n=0}^\infty a^n r_j f(n,...,\alpha_n,...,\gamma). $$

Notice that $(1 - \alpha)(q_k)^{-1}$ is the usual Jackson integral, the inverse of the Jackson derivative $D_{aj}f(\{r_j\})$ defined in formula (37). From the definition of the definite Jackson integral (see e.g. [10]) we also have

$$ \int_0^b d_{aj}\alpha_j f(\{r_j\}) = \sum_{n=0}^\infty a^n \beta f(n,...,\alpha_n,...,\gamma) $$

and $\int_a^b d_{aj}\alpha_j f(\{r_j\}) = \int_0^b d_{aj}\alpha_j f(\{r_j\}) - \int_0^a d_{aj}\alpha_j f(\{r_j\})$. The calculus rules for $q_k$ and $(q_k)^{-1}$ then follow directly from the well-known q-calculus rules (see e.g. [10]). In particular we shall need the q-Leibniz rule, given by

$$ q_k(f(\{r_j\})g(\{r_j\})) = f(\{r_j\})q_k g(\{r_j\}) + g(n,...,\alpha_n,...,\gamma)q_k f(\{r_j\}), $$

and of the integration by parts

$$ \int_a^b d_{aj}\alpha_j f(\{r_j\})(q_k g(\{r_j\})) = f_k(b)g_k(b) - f_k(a)g_k(a) $$

$$ - \int_a^b d_{aj}\alpha_j g(\{r,...,\alpha_n,...,\gamma\})(q_k f(\{r_j\})), $$

where we used the shorthand notation $f_k(b) = (n,...,b,...,\gamma)$, that is $f_k(b)$ is the function $f(\{r_j\})$ with the variable $r_k$ replaced by $b$ and the same for $g_k(a)$.

We introduce the following q-integral representation for the operators $R^k_\mu$

$$ R^k_\mu : \psi(c_k, h_\tilde{k}) \to \int d_{aj}c_k \int d_{aj}h_\tilde{k} P_\mu(\alpha c_k, \alpha h_\tilde{k}, c_{k+1}, h_\tilde{k}) \psi(c_k, h_\tilde{k}). $$

where the limits of integration are determined by the range of variations of our variables. For definiteness we can look to integrals between $-1$ and $1$. We also assume that the function $\psi$ vanishes on these boundaries.
The $Q$ operator is defined to be
\[ Q_\mu : \psi (r) \rightarrow \int d_\alpha N_\alpha \ldots \int d_r \hat{Q}_\mu (a_r |r) \psi (r) \] (53)
where the kernel $\hat{Q}_\mu (a_r |r)$ is given by
\[ \hat{Q}_\mu (a_r |r) = \int d_\alpha c_\alpha \ldots \int d_r c_1 \prod_{k=1}^N P_k (ac_k, \alpha |c_{k+1}, n_k). \] (54)

The factor $\alpha$ multiplying the variables $c_k$ and $\tilde{\alpha}$ is just for convention: it will become useful when we need to apply the formula for integration by parts (52). The commutation between $\text{Tr} (L(\lambda))$ and $Q_\mu$ is ensured by the quantum analogue of the relation (28), that is [15]
\[ R^k_q \lambda_L (\lambda, \mu) = D_{k+1}(\lambda, \mu)L_k (\lambda) R^k_q. \] (55)

The fundamental point is that the matrix $D_k (\lambda, \mu)$, given by (27), is another representation of the algebra (10), that is
\[ R(\lambda, \nu)D(\lambda, \mu)D(\nu, \mu) = D(\nu, \mu)D(\lambda, \mu)R(\lambda, \nu), \] (56)
equivalent to the commutation relations for the variables $\{b_j, c_j\}_{j=1}^N$.
\[ \left[ b_j, c_k \right] = \eta (1 - b_j c_j) \delta_{j,k}. \] (57)

The equivalence (55) gives the following set of equations for $R^k_q$:
\[ R^k_q c_k = c_{k+1} R^k_q, \quad R^k_q b_k = q_k R^k_q, \]
\[ R^k_q \tilde{\alpha} = (b_{k+1} - \mu^2 (1 - b_{k+1} c_k + 1) q_k) R^k_q, \]
\[ R^k_q (c_k - \mu^2 \tilde{\alpha} (1 - b_k c_k)) = n_k R^k_q, \]
\[ R^k_q (\tilde{\alpha} c_k - \mu^2 (1 - b_k c_k)) = (b_{k+1} n_k - \mu^2 (1 - b_{k+1} c_k + 1)) R^k_q. \] (58)

The first equation gives a contribution proportional to a q-delta function $\delta (\tilde{\alpha} - c_{k+1})$ to $P_k (ac_k, \alpha |c_{k+1}, n_k)$, that is we can set $P_k (ac_k, \alpha |c_{k+1}, n_k) = \delta_k (\tilde{\alpha} - c_{k+1}) F_k (ac_k, c_{k+1}, n_k)$.

Here the q-delta function is defined by the identity
\[ \int d_x \delta_q (x - y) g(y) = g(x). \] (59)

We do not enter into the discussions about a formal definition of the q-delta function but we shall just use equation (59) as a working identity to find an explicit expression for the kernel $\hat{Q}_\mu$ (54). By using the q-Leibniz rule (51) and the q-integration by parts (52) we get from (58) the following set of equations for $F_k (ac_k, c_{k+1}, n_k)$:
\[ n_k F_k (ac_k, c_{k+1}, n_k) + \mu^2 ac_{k+1} F_k (ac_k, ac_{k+1}, n_k), \]
\[ = (n_k + \mu^2 ac_{k+1}) F_k (ac_k, ac_{k+1}, n_k), \]
\[ (n_k - c_k) F_k (ac_k, c_{k+1}, n_k) = n_k F_k (c_k, c_{k+1}, n_k) - c_k F_k (ac_k, c_{k+1}, n_k), \]
\[ c_k F_k (ac_k, c_{k+1}, n_k) = (n_k + \mu^2 ac_{k+1}) F_k (ac_k, ac_{k+1}, n_k), \]
\[ (c_k - n_k) F_k (ac_k, c_{k+1}, n_k) = \mu^2 c_{k+1} F_k (c_k, c_{k+1}, n_k). \] (60)

It is possible to check that these equations are indeed compatible. The first and the third equation of the set together gives
Since this equation involves only differences in the variable \( r_k \), we expect that it is compatible with equation (42) for the function \( \rho_k \) that we found in the previous section. Indeed, if we set \( \alpha \) and \( \mu \), equations (60) reduce to

\[
G_k(a c_k, a c_{k+1}) = \frac{\mu^2 a c_k c_{k+1}}{c_k} G_k(a c_k, a c_{k+1}), \quad G_k(a c_k, a c_{k+1}) = \frac{\mu^2 a c_k c_{k+1}}{c_k} G_k(a c_k, a c_{k+1}). \tag{61}
\]

These two equations imply that \( G_k(a c_k, a c_{k+1}) \) is an homogeneous function of its arguments of degree \(-1\), that is \( G_k(a c_k, a c_{k+1}) = \alpha G_k(a c_k, a c_{k+1}) \). We can use this homogeneity to reduce the two equations (61) to a single equation. Let us set

\[
\hat{G}_k(z_k) = z_k \hat{G}_k(a c_k).
\]

If we introduce \( z_k = \frac{c_k}{c_{k+1}} \), equations (61) give

\[
\hat{G}_k(z_k) = z_k \hat{G}_k(a c_k).
\]

Notice that in formula (54) only the product over \( k \) is involved, so, recalling the periodic conditions \( c_{N+j} = c_j \), we can set the product \( \Pi_k \hat{G}_k \) equal to a constant. Putting all together, formula (54) becomes

\[
\hat{Q}_p(a \hat{r}|\hat{r}) = A \prod_{k=1}^{N} \frac{1}{\hat{r}_k \left( \frac{\mu \hat{r}_k}{\lambda_k}; \alpha \right)_{\infty} \left( -\frac{\mu \hat{r}_k}{\lambda_k}; \alpha \right)_{\infty}}, \tag{62}
\]

where the q-Pochhammer symbol \((x; \alpha)_\infty\) is defined by (43) and \( A \) is a constant of normalization that can depend on \( \alpha \) and \( \mu \).

### 4. Discussion

In this paper we considered a quantum version of the Ablowitz–Ladik model. We built Baxter’s equation in two different ways: by using the algebraic Bethe ansatz technique (section 2) and with the help of the quantum analogue of the classical Bäcklund transformations (sections 3.2 and 3.3). Our main aim was to underline the deep relationships between the Bäcklund transformations and the Baxter operator. Baxter’s equation turns out to be a q-difference equation (see equation (47)) whose semi-classical limit is linked with the classical trace formula for the monodromy matrix and with the spectrality property of the classical Bäcklund transformations (see equation (50)). The quantum determinant of the monodromy matrix is a conserved quantity but not a Casimir of the Poisson algebra defined by the commutation relations (10), and it plays an explicit role in Baxter’s equation. The construction leading to formulae (29) for the classical Bäcklund transformations has a well defined and precise quantum counterpart, described in section 3.2. In section 3.3 we gave a q-integral formula for the Baxter operator and proved the commutativity properties of \( Q_\mu \) with the other conserved quantities of the model encoded into the trace and quantum determinant.
of the monodromy matrix. A detailed study of the analytic properties of the q-integral representation of the $\mathcal{Q}$ operator will be discussed in future works. Also, it would be interesting to investigate the relationships between the same q-integral representation and the Green function of the Schrödinger equation corresponding to the interpolating flow of the Bäcklund transformations, in line with the research developed in [19].

References

[1] Ablowitz M J and Ladik J F 1975 Nonlinear differential-difference equations J. Math. Phys. 16 598–603
[2] Ablowitz M J and Ladik J F 1976 Nonlinear differential-difference equations and Fourier analysis J. Math. Phys. 17 1011–8
[3] Baxter R J 1989 Exactly Solved Models in Statistical Mechanics (London: Academic)
[4] Bazhanov V V, Lukyanov S L and Zamolodchikov A B 1997 Integrable structure of conformal field theory II. Q-operator and DDV equation Commun. Math. Phys. 190 247–78
[5] Bogoliubov N M, Bullough R K and Pang G D 1993 Exact solution of a q-boson hopping model Phys. Rev. B 47 495–8
[6] Bogoliubov N M, Izergin A G and Kitanine N A 1998 Correlation functions for a strongly correlated boson system Nucl. Phys. B 516 501–28
[7] Bullough R K, Bogoliubov N M and Pang G D 1992 The quantum Ablowitz–Ladik equation as a q-boson system Future Directions of Nonlinear Dynamics in Physical and Biological Systems (NATO ASI Series vol 32) ed P L Christiansen, J C Eilbeck and R D Parmentier (New York: Springer) pp 217–22
[8] Faddeev L D 1996 How algebraic Bethe Ansatz works for integrable model Symétries Quantiques Proc. 1995 Les Houches summer school pp 149–219 (arXiv:hep-th/9605187)
[9] Gasper G and Rahman M 2004 Basic Hypergeometric Series vol 96 (Cambridge: Cambridge University Press)
[10] Kac V and Cheung P 2002 Quantum Calculus (New York: Springer)
[11] Kilmyk A and Schmüdgen K 1997 Quantum Groups and Their Representations (Berlin: Springer)
[12] Korff C 2013 Cylindric versions of specialised Macdonald functions and a deformed Verlinde algebra Commun. Math. Phys. 318 73–246
[13] Kulish P P 1981 Quantum difference nonlinear Schrödinger equation Lett. Math. Phys. 5 191–7
[14] Kuznetsov V B and Sklyanin E K 1998 On Bäcklund transformations for many-body systems J. Phys. A: Math. Gen. 31 2241–51
[15] Kuznetsov V B, Salerno M and Sklyanin E K 2000 Quantum Bäcklund transformation for the integrable DST model J. Phys. A: Math. Gen. 33 171
[16] Lieb E H and Liniger W 1963 Exact analysis of an interacting Bose gas: I. The general solution and the ground state Phys. Rev. 130 1605–16
[17] Pasquier V and Gaudin M 1992 The periodic Toda chain and a matrix generalization of Bessel function recursion relations J. Phys. A: Math. Gen. 25 5243–52
[18] Pronko G P 2000 On the Baxter’s Q operator for the XXX spin chain Commun. Math. Phys. 212 678–701
[19] Ragnisco O and Zullo F 2012 Quantum Bäcklund transformation: some ideas and examples Theor. Math. Phys. 172 1159–70
[20] Ragnisco O and Zullo F 2011 Bäcklund transformation for the Kirchhoff top SIGMA 7 13
[21] Ragnisco O and Zullo F 2010 Bäcklund transformations as exact integrable time discretizations for the trigonometric Gaudin model J. Phys. A: Math. Theor. 43 434029
[22] Sklyanin E K 2000 Bäcklund transformation and Baxter’s Q-operator Integrable Systems: From Classical to Quantum (CRM Proc. Lecture Notes vol 26) ed J Harnad, G Sabidussi and P Winternitz (Providence, RI: American Mathematical Society) pp 227–50
[23] Suris Y B 1997 A note on an integrable discretization of the nonlinear Schrödinger equation Inverse Problems 13 1121–36
[24] van Diejen J F and Emsiz E 2014 Diagonalization of the infinite q-boson system J. Funct. Anal. 266 5801–17
[25] van Diejen J F 2006 Diagonalization of an integrable discretization of the repulsive delta Bose gas on the circle Commun. Math. Phys. 267 451–76
[26] Zullo F 2011 Bäcklund transformations for the elliptic Gaudin model and a Clebsch system J. Math. Phys. 52 073507
[27] Zullo F 2013 Bäcklund transformations and Hamiltonian flows J. Phys. A: Math. Theor. 46 145203
[28] Zullo F 2013 On an integrable discretisation of the Ablowitz–Ladik hierarchy J. Math. Phys. 54 053515