Weak lensing of a dirty black hole

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In this paper, we calculated the weak deflection angle of a dirty Schwarzschild black hole of mass $m$ by considering finite-distance correction. The astrophysical environment considered in this study is a spherical shell of dark matter, which is only described by its mass $M$ and some physical parameters. We found out that for notable deviations in the weak deflection angle to occur, the effective radius of the dark matter distribution must be in the order of $\sim 6\sqrt{mM}$, which only happens if the dark matter density is very high near a black hole. This makes the weak deflection angle a preferable method in detecting dark matter near a black hole than using the deviations in the shadow radius which only occurs when $\Delta r_s \sim \sqrt{3mm}$.

I. INTRODUCTION

One of the fruitful achievements of the human mind is the general theory of relativity by Albert Einstein, which relates the phenomena of gravitation to the geometry of spacetime. One consequence of the theory is the existence of black holes that have long remained to be a theoretical construct, not until the release of the first image of the black hole shadow at the center of the M87 galaxy on April 10, 2019\footnote{\textsuperscript{1}}.

At present, another theoretical construct that remains so mysterious and elusive is the dark matter. The $\Lambda$CDM model of cosmology suggests that the content of our Universe is made up of 27\% dark matter, which constitutes 85\% of the total mass there is\footnote{\textsuperscript{2}}. Some attempt to detect dark matter by direct means using Earth-based laboratories gave inconsistent results\footnote{\textsuperscript{3} and \textsuperscript{4}}. Even long before these experiments, indirect search through dark matter annihilation revealed null results\footnote{\textsuperscript{5} and \textsuperscript{6}}. It looks like Earth-based detection of dark matter gives more difficulty compared to the gravitational waves produced by two merging black holes, which were detected recently\footnote{\textsuperscript{7}}.

There were emerging theoretical efforts that showed alternative means of dark matter detection using the silhouette of a black hole. The spacetime of pure dark matter and black hole were combined rigorously by Xu in\footnote{\textsuperscript{8}}, and this formalism of the Gauss-Bonnet theorem\footnote{\textsuperscript{15}}. The literature is exhaustive, but there are notable studies that included dark matter, phantom matter, and quintessential energy,\footnote{\textsuperscript{12} \textsuperscript{16} \textsuperscript{18}} on their analysis of the weak deflection angle. Gravitational lensing by exotic matter or energy, which possibly deviate from some well-known equation of state, is also explored in\footnote{\textsuperscript{19} \textsuperscript{21}}.

Our primary motivation in this paper follows from\footnote{\textsuperscript{13}} where we deviate ourselves from highly model-dependent black hole metrics, and extend the study to the analysis of the weak deflection angle. We aim to derive a possible analytical estimate under what condition can dark matter considerably affect the weak deflection angle.

We organize the paper as follows: Sec.\textsuperscript{11} introduces the dirty Schwarzschild metric alongside with the description of the simple dark matter model in consideration. In sec.\textsuperscript{III} we briefly show the incapability of the Gauss-Bonnet theorem in evaluating the Gaussian optical curvature when non-asymptotic spacetime is involved. In sec.\textsuperscript{IV} we proceed to calculate the weak deflection angle by assuming finite-distance correction to GBT and make an estimate for the condition for dark matter to have a notable effect on the weak deflection angle. In sec.\textsuperscript{V} we summarize the results and recommend some future research directions. The metric signature in this study is $+2$, and we use $G = c = 1$.

II. DIRTY SCHWARZSCHILD BLACK HOLE

The term "dirty" black hole has its roots in\footnote{\textsuperscript{22} \textsuperscript{26}} which describes a black hole surrounded by some astrophysical environment. Here, we describe the dirty Schwarzschild black hole used in\footnote{\textsuperscript{13}} where the author studied the photosphere and black hole shadow. The environment we consider is a spherical shell of dark matter described only by its mass $M$, inner radius $r_s$, and thickness $\Delta r_s$. Furthermore, it doesn’t interact with the electromagnetic field and hence, invisible. The physical dimension can be adjusted in such a way that the piecewise function below is realized:

$$ m(r) = \begin{cases} m, & r < r_s; \\ m + MG(r), & r_s \leq r \leq r_s + \Delta r_s; \\ m + M, & r > r_s + \Delta r_s \end{cases} \tag{1} $$

where

$$ G(r) = \left(3 - 2 \frac{r - r_s}{\Delta r_s}\right) \left(\frac{r - r_s}{\Delta r_s}\right)^2. \tag{2} $$

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We can then incorporate eq. (1) to the Schwarzschild metric in which the line element takes the form

\[ ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \]

where the metric function \( f(r) \) is now given by

\[ f(r) = 1 - \frac{2m(r)}{r}. \]

The piecewise function then ensures that there are conditions where the Schwarzschild black hole alone is restored, or dark matter acts as an effective mass. Non-trivial is the second relation that complicates the metric due to \( G(r) \). However, it can be easily seen that this really describes a spherical shell of dark matter that envelopes the black hole. Although eq. (2) allows \( r_s \) values smaller than \( r = 2m \), we shall only consider the case where \( r_s \) coincides with the event horizon of the black hole. This is acceptable because as far as the remote observer is concerned, anything that falls into the black hole stops \( r = 2m \).

The idea of a spherical shell of some astrophysical environment that envelopes a black hole is common in several studies for the following reasons: (1) verify whether the deviation observed is due to the new physics happening near the horizon, or due to some effect of astrophysical environment; and (2) directly examine and study the influence of astrophysical environment to the geometry of black hole. Only recently the mass function in eq. (1) has been used to make some estimates of dark matter effects to the radius of the black hole shadow. In this study, we will use the same mass function and explore dark matter effects on the weak deflection angle.

### III. DEFLECTION ANGLE USING GAUSS-BONNET THEOREM

In this section, we show that we cannot simply use the Gauss-Bonnet theorem to obtain the weak deflection angle of a dirty Schwarzschild black hole. The weak deflection angle \( \hat{\alpha} \) in GBT formalism is defined as follows:

\[ \hat{\alpha} = -\int dS \int_{R}^{\infty} K d\ell + \int_{S}^{R} dS \int_{S}^{R} \kappa_{g} d\ell \]

where \( K \) is the Gaussian optical curvature (related to the Riemann tensor) which is integrated over the quadrilateral \( \pi \subset \mathbb{R}^{2} \), \( dS \) is the surface area element, \( \kappa_{g} \) is the geodesic curvature, and \( d\ell \) is the arc length of photon's path from the source to the receiver. The optical and geodesic curvature are given by the following expressions:

\[ K = \frac{R_{\rho\phi\phi}}{\text{det} \gamma}, \quad \kappa_{g} = -\sqrt{\frac{1}{\text{det} \gamma\gamma_{\theta\theta}}} \beta_{\phi,r} \]

and

\[ dS = \sqrt{\text{det} \gamma} dr d\phi, \quad d\ell = b(1 + \tan^2 \varphi). \]

In the equatorial plane, the optical metric for the null geodesic reads

\[ dt^2 = \frac{dr^2}{f(r)^2} + \frac{r^2}{f(r)} d\phi^2 \]

and the Gaussian curvature is then calculated as

\[ K = \frac{1}{2} f(r) f''(r) + \frac{1}{4} f'(r)^2. \]

As usually done in celestial mechanics, we introduce \( u = 1/r \) and for convenience, \( q = 1/\Delta r_s \). After applying series expansion in \( q \) and then in \( m \) in eq. (9), we find that

\[ \int_{0}^{\pi} \int_{0}^{\sin(\phi)} -\frac{K}{u^2} \sqrt{\text{det} \gamma} du d\phi = \left[ 2m + 6Mq^2 r_s^2 \right. \]

\[ - 12mMq^2 r_s^2 + 6mMq^2 r_s^2 u + \left. \frac{12mMq^2}{u} \right] du d\phi \]

which shows divergence in the last term. Hence, this impedes us to proceed with the calculation of the weak deflection angle using the GBT. By inspecting eq. (10), the dark matter distribution based in its physical dimensions is somewhat similar to the effect of the cosmological constant where it doesn’t allow one to calculate \( \hat{\alpha} \) using the GBT. The root cause is the non-asymptotic flatness of spacetime where the assumption that the source and receiver are located at \( r \to \infty \) no longer applies. With this, it is pointed out in [29] that there is a need to generalize the use of GBT in order to allow one to calculate \( \hat{\alpha} \) in non-asymptotic spacetime such as the Kottler spacetime and Schwarzschild-like solutions in the Weyl conformal gravity [31].

Note that, if the series expansion in \( q \) didn’t happen, the third condition in eq. (11) will apply, and we can freely use the GBT that gives the simple result of

\[ \hat{\alpha} = \frac{4(m + M)}{b}. \]

However, this is not the realistic situation we are after.

### IV. DEFLECTION ANGLE WITH FINITE-DISTANCE CORRECTION

In this section, we follow [29] to calculate the weak deflection angle of a black hole under the influence of dark matter. From the GBT, the generalized correspondence between the deflection angle and the surface integral of the Gaussian curvature reads

\[ \hat{\alpha} = \phi_{RS} + \Psi_{R} - \Psi_{S} \]

\[ = \int_{u_R}^{u_{\infty}} \frac{1}{\sqrt{F(u)}} du + \int_{u_{\infty}}^{u_S} \frac{1}{\sqrt{F(u)}} du + \Psi_{R} - \Psi_{S} \]

where \( u_{\infty} = r_{s}^{-1} \) denotes the closest approach and could be also interpreted as the impact parameter \( b \). Here, in the non-rotating case, the function \( F(u) \) is given and defined by

\[ \left( \frac{du}{d\phi} \right)^2 = F(u) = \frac{1}{b^2} - f(u)u^2. \]
unit vector with respect to the lensing object. The unit basis vector \( e^i \), along the equatorial plane, is given by
\[
e^i = \left( \frac{dr}{dt}, 0, \frac{d\phi}{dt} \right) = \frac{d\phi}{dt} \left( \frac{dr}{d\phi}, 0, 1 \right)
\] (14)
while the unit radial vector, which is along the radial direction from the lens is
\[
R^i = \left( \frac{1}{\sqrt{rr^2}}, 0, 0 \right).
\] (15)
Hence, the inner product suggests the definition
\[
\cos \Psi \equiv \gamma_{ij} e^i R^j
\]
cos \( \Psi = \sqrt{\gamma_{i\phi}} f(u) bu^2 \frac{dr}{d\phi}.
\] (16)

Using \( F(u) \) and the basic identity \( \cos^2 \Psi + \sin^2 \Psi = 1, \)
\[
\sin \Psi = \sqrt{f(u) bu}
\] (17)
where it clear that it is favorable to use \( \sin \Psi \) than using \( \cos \Psi. \)
If the dark matter distribution also envelopes the source and receiver, then we must first apply a series expansion of \( q. \) The result for calculating \( \Psi \) is
\[
\Psi = \arcsin(bu \sqrt{1 - 2mu}) + \frac{3bM \sqrt{1 - 2mu} (r_u - 1)^2}{u(2mu - 1) \sqrt{2b^2 u^3} - b^2 u^2 + 1} q^2
\]
\[
+ \frac{2bM \sqrt{1 - 2mu} (r_u - 1)^3}{u(2mu - 1) \sqrt{2b^2 u^3} - b^2 u^2 + 1} q^3 + O(q^4). \] (18)
Continuing, we found that
\[
\Psi_R - \Psi_S = (\Psi_{R}^{\text{Schw}} - \Psi_{S}^{\text{Schw}}) - 3bM \left[ \frac{(r_u R - 1)^2}{\sqrt{1 - b^2 u_R^2}} + \frac{(r_u S - 1)^2}{\sqrt{1 - b^2 u_S^2}} \right] - \frac{2bM}{\Delta r_s^2} \left[ \frac{(r_u S - 1)^3}{u_S \sqrt{1 - b^2 u_S^2}} + \frac{(r_u S - 1)^3}{u_S \sqrt{1 - b^2 u_S^2}} \right]
\]
\[
+ 3bm M \left[ \frac{u_R (2b^2 u_R^2 - 1) (r_u R - 1)^2}{(1 - b^2 u_R^2)^{3/2}} + \frac{u_S (2b^2 u_S^2 - 1) (r_u S - 1)^2}{(1 - b^2 u_S^2)^{3/2}} \right]
\]
\[
+ \frac{2bm M}{\Delta r_s^3} \left[ \frac{(2b^2 u_R^2 - 1) (r_u R - 1)^3}{(1 - b^2 u_R^2)^{3/2}} + \frac{(2b^2 u_S^2 - 1) (r_u S - 1)^3}{(1 - b^2 u_S^2)^{3/2}} \right]
\] (19)
where
\[
(\Psi_{R}^{\text{Schw}} - \Psi_{S}^{\text{Schw}}) = [\arcsin(u_R b) + \arcsin(u_S b) - \pi]
\]
\[-bm \left[ \frac{u_R^2}{\sqrt{1 - b^2 u_R^2}} + \frac{u_S^2}{\sqrt{1 - b^2 u_S^2}} \right]. \] (20)
We now proceed to calculate the \( \phi_{RS} \) part by evaluating the
\[
\int_{u_s}^{u_o} \frac{1}{\sqrt{F(u)}} du = \arcsin bu + \frac{m (b^2 u^2 - 2)}{b \sqrt{1 - b^2 u^2}} + 9Mq^2 \left[ \left( m - \frac{2r_s}{3} \right) + \frac{5mr_s^2}{2b^2} \right] \arcsin(bu)
\]
integrals in eq. (12). The function \( F(u) \) takes the form
\[
\frac{1}{b^2} u^2 - 2 + 6Mu^2 (su - 1)^2 + \mathcal{O}(q^3) \] (21)
and so we have
\[
\frac{1}{\sqrt{F(u)}} = \frac{b}{\sqrt{1 - b^2 u^2}} - \frac{b^3 u^3 m}{(1 - b^2 u^2)^{3/2}}
\]
\[
+ \frac{3bMwq^2 (su - 1)^2}{(1 - b^2 u^2)^{3/2}} + \frac{9b^5 M u^4 (su - 1)^2}{(1 - b^2 u^2)^{5/2}} + \mathcal{O}(q^3). \] (22)
Hence, for the source we found
We can finally calculate the deflection angle \( \hat{\alpha} \) where we introduced \( \hat{\alpha} \approx \alpha \approx \frac{\alpha}{6} \). However, there’s a divergence in the fourth term. We argue that this is not problematic because physically, \( \phi_{RS} \) and \( \phi_{RS} \approx \phi \). However, there’s a divergence in the fourth term. We argue that this is not problematic because physically, \( \phi_{RS} \) is the result of the term. We argue that this is not problematic because physically, \( \phi_{RS} \) and \( \phi_{RS} \approx \phi \).

The same procedure can be done for the receiver. The expression for \( \phi_{RS} \) includes the sum of these two integrals:

\[
\phi_{RS} = \phi_{RS}^{\text{Schw}} - 9M \frac{\Delta r_s^3}{\sqrt{1 - b^2 u_R^2}} \left[ \left( m - 2r_s \right) + \frac{5m^2}{2b^2} \right] \left[ \arcsin(bu_R) + \arcsin(bu_S) \right]
\]

\[
+ 3M \frac{\Delta r_s^3}{b} \left\{ \frac{b^2 \left( -r_s^2 u_R^2 - 2r_s u_R + 1 \right) + 2r_s^2}{\sqrt{1 - b^2 u_R^2}} + \frac{b^2 \left( -r_s^2 u_S^2 - 2r_s u_S + 1 \right) + 2r_s^2}{\sqrt{1 - b^2 u_S^2}} \right\}
\]

\[
+ \frac{3M r_s}{2b \Delta r_s^3} \left[ \frac{(48b^2 u_R^2 + 15r_s u_R - 32)}{(1 - b^2 u_R^2)^{3/2}} + \frac{(48b^2 u_S^2 + 15r_s u_S - 32)}{(1 - b^2 u_S^2)^{3/2}} \right]
\]

where we introduced

\[
\phi_{RS}^{\text{Schw}} = \pi - \arcsin(u_R b) - \arcsin(u_S b) - \frac{m}{b} \left[ \frac{(b^2 u_R^2 - 2)}{\sqrt{1 - b^2 u_R^2}} + \frac{(b^2 u_S^2 - 2)}{\sqrt{1 - b^2 u_S^2}} \right].
\]

We can finally calculate the deflection angle \( \hat{\alpha} \) using eq. (23). Combining eq. (19) and eq. (24), we have

\[
\hat{\alpha} \approx \frac{2m}{b} \left[ \sqrt{1 - b^2 u_R^2} + \sqrt{1 - b^2 u_S^2} \right] - 9M \frac{\Delta r_s^3}{\sqrt{1 - b^2 u_R^2}} \left[ \left( m - 2r_s \right) + \frac{5m^2}{2b^2} \right] \left[ \arcsin(bu_R) + \arcsin(bu_S) \right]
\]

\[
+ 6 \frac{M r_s}{b \Delta r_s^3} \left[ \sqrt{1 - b^2 u_R^2} + \sqrt{1 - b^2 u_S^2} \right] - \frac{2b M}{\Delta r_s^3} \left[ \frac{(r_s u_R - 1)^3}{u_R \sqrt{1 - b^2 u_R^2}} + \frac{(r_s u_S - 1)^3}{u_S \sqrt{1 - b^2 u_S^2}} \right] - \frac{6m M}{\Delta r_s^3} \left\{ \frac{-b^2 \left( 11r_s u_R^2 + \frac{u_R}{2} \right) - 15r_s^2 u_R^2 + 8r_s}{b \left( 1 - b^2 u_R^2 \right)^{3/2}} + \frac{-b^2 \left( 11r_s u_S^2 + \frac{u_S}{2} \right) - 15r_s^2 u_S^2 + 8r_s}{b \left( 1 - b^2 u_S^2 \right)^{3/2}} \right\}
\]

\[
+ \frac{2bm M}{\Delta r_s^3} \left[ \frac{(2b^2 u_R^2 - 1) (r_s u_R - 1)^3}{(1 - b^2 u_R^2)^{3/2}} + \frac{(2b^2 u_S^2 - 1) (r_s u_S - 1)^3}{(1 - b^2 u_S^2)^{3/2}} \right]
\]

The expression in Eq. (26) must be taken to the limit as \( bu_R \to 0 \) and \( bu_S \to 0 \) in order to obtain the simplified expression for \( \hat{\alpha} \). However, there’s a divergence in the fourth term. We argue that this is not problematic because physically, the location of the source and receiver are finite from the black hole. We also know that this divergence is the result of the non-asymptotic spacetime caused by dark matter. Including only a certain finite distance, eq. (26) is further simplified to

\[
\hat{\alpha} \approx \frac{4m}{b} + \frac{12M r_s^2}{b \Delta r_s^2} + \frac{96m M r_s}{b \Delta r_s^2}
\]
For the weak deflection angle, it is difficult to obtain a formula to prove the same conclusion that we can safely ignore the dark matter outside the photon’s path if dark matter density is very low. However, we can follow such conclusion without doing such rigor because both dealt with the approximation $\Delta r_s \to \infty$. Hence, eq. (28) can also be used as a robust estimate for very small deviation in the Schwarzschild case.

To find the condition where dark matter can have noticeable effect on the weak deflection angle, we use eq. (28) again and

$$\Delta r_s \approx 6 \sqrt{mM}. \quad (29)$$

Like the requirement using the deviation in the shadow radius, this condition is also not met even in our galaxy. In particular, an estimate for dark matter mass in our galaxy of $M \approx 1.0 \times 10^{12} M_\odot$ while the mass of the central black hole is around $m \approx 4.3 \times 10^6 M_\odot$. This gives a required dark matter thickness of $\Delta r_s \approx 1.84 \times 10^{13} m \approx 5.96 \times 10^{-4}$ pc. This is lower in many orders of magnitude even if we compare it to the core radius of the dark matter halo present in our galaxy ($r_s \approx 15.7 - 17.46$ kpc) [33]. Looking again at fig. 1 the weak deflection angle decreases when dark matter density increases. This is the effect when the dark matter outside the photon’s path is not ignored.

V. CONCLUSION

In this paper, we found an analytic formula for the weak deflection angle using a simple dark matter model that only incorporates its basic features such as mass and physical dimensions. In the presence of dark matter, the spacetime is not asymptotically flat and a divergent term arises assuming finite-distance corrections. However, this poses no problems because of the realistic finite distance of the source and the receiver. Moreover, the effective radius of dark matter cancels out the apparent divergence. We also made estimates to show that if one uses the weak deflection angle as a means to detect dark matter in the vicinity of a black hole, it provides increased capability compared when using the shadow radius. However, the estimate is valid only if the dark matter density is abnormally high. For very low dark matter density, which is most likely a scenario to every galaxy in our Universe, the result in this study provides some serious challenges if future civilization is interested in using a black hole to detect dark matter. Our work in progress includes examining dark matter effects on a quantum black hole. We are also interested to see if we could make analytical estimates on a dirty rotating black hole.
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