Effects of bosonic and fermionic $q$-deformation on the entropic gravity

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(Dated: September 24, 2019)

In this paper, we study thermodynamical contributions to the theory of gravity under the $q$-deformed boson and fermion gas models. According to the Verlinde’s proposal, the law of gravity is not based on a fundamental interaction but it emerges as an entropic force from the changes of entropy associated with the information on the holographic screen. In addition, Strominger shows that the extremal quantum black holes obey neither boson nor fermion statistics, but they obey deformed statistic. Using these notions, we find $q$-deformed entropy and temperature functions. We also present the contributions that come from the $q$-deformed model to the Poisson equation, Newton’s law of gravity and Einstein’s field equations.

PACS numbers: 04.20.-q, 02.20.Uw, 04.70.Dy, 05.30.-d
Keywords: Entropic gravity, extended theory of gravity, quantum algebras, quantum black holes

I. INTRODUCTION

The mechanism of gravity is described by many methods such as the General theory of relativity, the Gauge theory of gravity and the Quantum gravity. These theories try to explain the gravity with different methods by using geometrical description of spacetime, particles (graviton) or fields. There is an interesting study published by Jacobson $^1$ who develop a gravity theory using the first law of thermodynamics on a local Rindler horizon. This approximation is based on Hawking and Bekenstein’s studies $^2$-4 which are about the relation between the black hole physics and thermodynamics. In 2011 Verlinde $^5$ put forward a theory, named as entropic gravity, which describes gravity as an effect of entropy. In this theory, Verlinde obtains Newton’s second law of gravity and certain components of the Einstein field equations by using Tolman-Komar mass and the equipartition rule. The detailed discussion can be found in $^6$.

The discovery of the relation between the theory of gravity and thermodynamics brings a new point of view for understanding the nature of gravity. Thus we have an alternative theory to get a more general theory of gravity by deforming thermodynamical quantities such as entropy, temperature and energy functions. After Verlinde’s work this idea become so popular. Using this notion, the modified Newtonian dynamics (MOND) and generalized Einstein equation is obtained by deforming energy function with Debye model $^7$. The entropic correction to Coulomb’s law is obtained $^8$. Some developments on modified dark matter (MDM) is studied in $^4$11. The $(n+1)$ dimensional Einstein field equation is found by $^12$. This theory is also applied to quantum black holes $^{13}$16.

On the other hand, according to Strominger’s proposal in Ref $^{17}$, extremal quantum black holes, which have minimum mass and behave like a particle, obey neither standard Bose nor Fermi statistics. However, they can be assumed as deformed bosons or fermions and so their thermostatistical properties can be investigated with the help of deformed Bose or Fermi statistics. In recent decades, thermodynamical and statistical properties of deformed boson and fermion systems have been extensively examined $^{18}$20. Moreover, these deformed boson and fermion systems have a variety of applications such as understanding of higher-order effects in the many body interactions $^27$, quantum mechanics in discontinuous spacetime $^28$, thermosize effects $^29$, phonon spectrum in $^4He$ $^{30}$, vortices in superfluid films $^{31}$, and Landau diamagnetism problem embedded in D-dimensions $^{32}$.

With the above motivations, we consider a $q$-deformed boson and fermion systems, which previously was studied in $^{33}$, at high temperature regime and obtain generalized entropy and temperature functions in terms of deformation parameter $q$. Then, we get generalized forms of the Poisson equation, Newton’s second law of gravity and Einstein’s field equations by using $q$-deformed temperature function based on Verlinde’s idea. In the studies $^{13}$16, $q$-deformed Einstein equations were obtained by considering different types $q$-deformed oscillators algebras. The quantum black holes were assumed as $q$-deformed boson particles in Ref. $^{13}$ and as $q$-deformed fermion particles in Ref $^{16}$. However, in this study, we think different $q$-deformed oscillators algebra than in the Ref. $^{13}$16 and assume that the quantum black holes can be described by the $q$-deformed boson particles or fermion particles. There are several differences

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between $q$-deformed model in this study and given in [13, 16]. For instance, the mean occupation number and the deformed entropy function in our model have different properties from the Ref. [13, 16] because of having different quantum oscillators algebra.

The paper is organised as follows. In Section 2, $q$-deformed boson and fermion gas models are studied at high temperature limit and deformed entropy function is found. In Section 3, we find generalized Poisson equation and Newton’s second law of gravity by using Verlinde’s entropic gravity proposal and deformed temperature. In Section 4, $q$-deformed Einstein equation is obtained. The last section concludes the paper and contains a discussion about possible future application of our results.

II. $q$-DEFORMED BOSON AND FERMION GAS MODELS AT HIGH TEMPERATURE

In this section, we briefly introduce the basic algebraic properties of $q$-deformed boson and fermion gas models. Then, we review some of the high-temperature thermostatistical properties of such $q$-deformed boson and fermion models investigated in [33].

The quantum algebraic structure of the $q$-deformed boson and fermion is defined by the following relations [33–37]

\[ cc^* - \kappa q^* c^* c = q^{-N}, \]
\[ [\hat{N}, c^*] = c^*, \quad [\hat{N}, c] = -c, \]

where $c$ and $c^*$ are, respectively, deformed annihilation and creation operator, $\hat{N}$ is the total number operator, $q$ is the real positive deformation parameter and $\kappa = 1$ for $q$-boson and $\kappa = -1$ for $q$-fermion. Moreover, the operators obey the relations

\[ c^* = \left[ \hat{N} \right], \quad c^* c = \left[ 1 + \kappa \hat{N} \right], \]

where the $q$-basic number is defined as

\[ [x] = \frac{q^x - q^{-x}}{q - q^{-1}}. \]

Also, Jackson derivative (JD) operator for the above mentioned algebra is given as

\[ D^{(q)} f(x) = \frac{1}{x} \left[ f(qx) - f(q^{-1}x) \right], \]

for any function $f(x)$. This JD operator reduces to the ordinary derivative operator in the limit $q \to 1$ [33].

Now, we consider the $q$-deformed boson and fermion gas models constructed by Eqs. (1)-(3). In the grand canonical ensemble, the Hamiltonian $H$ of the such a $q$-deformed gas model have the following form

\[ H = \sum_i (\varepsilon_i - \mu) N_i, \]

where $\varepsilon_i$ is the kinetic energy of a particle in the $i$. state and $\mu$ is the chemical potential. Then, the logarithm of the grand partition function of the model is given by

\[ \ln Z = -\kappa \sum \ln(1 - \kappa ze^{-\beta \varepsilon_i}), \]

where $z = \exp(\mu/kT)$ is the fugacity. Also, this $q$-deformed gas model has the following mean occupation number

\[ N = \sum_i n_i = \sum_i \frac{1}{q - q^{-1}} \ln \left( \frac{z^{-1} e^{\beta \varepsilon_i} - \kappa q^{-\kappa}}{z^{-1} e^{\beta \varepsilon_i} - \kappa q^{\kappa}} \right). \]
In order to obtain the high-temperature thermostatistical properties of the model, the sum over states can be replaced with the integral for a large volume and a large number of particles. Therefore, the equation of state \( P V / kT = \ln Z \) and the particle density can be written, respectively, as

\[
\frac{P}{kT} = \frac{1}{\lambda^3} \frac{4}{3 \sqrt{\pi}} \int_0^\infty x^{3/2} dx \frac{1}{q-q^{-1}} \ln \left( \frac{1 - \kappa q^{-\kappa} ze^{-x}}{1 - \kappa q^\kappa ze^{-x}} \right), \tag{8}
\]

\[
\frac{N}{V} = \frac{1}{\lambda^3} \frac{2}{\sqrt{\pi}} \int_0^\infty x^{1/2} dx \frac{1}{q-q^{-1}} \ln \left( \frac{1 - \kappa q^{-\kappa} ze^{-x}}{1 - \kappa q^\kappa ze^{-x}} \right), \tag{9}
\]

where \( \lambda = h/\sqrt{2\pi mkT} \) is the thermal wavelength, \( x = \beta \varepsilon \) and \( \varepsilon = p^2/2m \). For high-temperature limits \( z \ll 1 \), these integrals can be expanded with Taylor series and defined as

\[
\frac{P}{kT} = \frac{1}{\lambda^3} h_{5/2}^\kappa(z, q), \tag{10}
\]

\[
\frac{N}{V} = \frac{1}{\lambda^3} h_{3/2}^\kappa(z, q), \tag{11}
\]

where \( q \)-deformed \( h_n^\kappa(z, q) \) is defined as

\[
h_n^\kappa(z, q) = \frac{1}{\Gamma(n)} \int_0^\infty x^{n-1} dx \frac{1}{q-q^{-1}} \ln \left( \frac{1 - \kappa q^{-\kappa} ze^{-x}}{1 - \kappa q^\kappa ze^{-x}} \right)
= \frac{1}{q-q^{-1}} \left[ \sum_{l=1}^\infty \frac{(\kappa z q^\kappa)^l}{ln+1} - \sum_{l=1}^\infty \frac{(\kappa z q^{-\kappa})^l}{ln+1} \right]. \tag{12}
\]

In the limit \( q \to 1 \), the deformed \( h_n^\kappa(z, q) \) functions reduce to the standard Bose-Einstein functions \( g_n(z) \) for bosons and Fermi-Dirac functions \( f_n(z) \) for fermions \[38\]. The internal energy of the model can be obtained by using the thermodynamical relation \( U = -(\partial \ln Z / \partial \beta)_{z,V} \) as

\[
U = \frac{3 kT V}{2 \lambda^3} h_{5/2}^\kappa(z, q). \tag{13}
\]

Now, we want to determine the Helmholtz free energy of the model from the thermodynamical relation \( A = \mu N - PV \). From Eqs. (10) and (11), the Helmholtz free energy of the model can be derived as

\[
A = \frac{kT V}{\lambda^3} \left[ h_{5/2}^\kappa(z, q) \ln z - h_{5/2}^\kappa(z, q) \right]. \tag{14}
\]

Then, the \( q \)-deformed entropy function of the model can be found from the thermodynamical relation \( S = (U - A)/T \). From Eqs. (13) and (14), the entropy of the model becomes

\[
S = \frac{kV}{\lambda^3} \left[ \frac{5}{2} h_{5/2}^\kappa(z, q) - h_{3/2}^\kappa(z, q) \ln z \right]. \tag{15}
\]

This deformed entropy function of the model can also be written in the following form

\[
S = \frac{(2\pi m)^{3/2} V}{Th^3} E_5^{5/2} H^\kappa(z, q), \tag{16}
\]

where \( E = kT \) is the one-particle average kinetic energy and \( H^\kappa(z, q) \) is defined as

\[
H^\kappa(z, q) = \frac{5}{2} h_{5/2}^\kappa(z, q) - h_{3/2}^\kappa(z, q) \ln z. \tag{17}
\]

In the next section, we will particularly focus on the derivation of the Poisson equation for gravity from the \( q \)-deformed entropy function defined in Eq. (16) by taking account of the Verlinde’s entropic gravity approach.
According to the Verlinde’s approach [5], the theory of gravity (including Newton’s law of gravity and General theory of relativity) can be obtained by using the holographic principle when the mass is distributed over a holographic screen. In this section, we investigate the effects of the \(q\)-deformed model on the Poisson equation and Newton’s law of gravitation.

For this purpose, we firstly derive deformed temperature by using deformed entropy function given in Eq.(16) with the methods presented in [16]. The total entropy \(S\) remains constant when the entropic force is equal to the force increasing the entropy. In this case, the system reaches the statistical equilibrium and the variation of the entropy goes to zero, such that

\[
\frac{d}{dx^a} S(E, x^a) = 0,
\]

(18)

and it can be written as following relation

\[
\frac{\partial S}{\partial E} \frac{\partial E}{\partial x^a} + \frac{\partial S}{\partial x^a} = 0.
\]

(19)

To obtain deformed temperature, we now assume that the mass enclosed by the surface is formed by the \(q\)-deformed bosons or fermions. Accordingly, the \(q\)-deformed entropy function in Eq.(16) can be used to find the deformed temperature. So that the Eq.(19) gives

\[
- \frac{5V(2\pi mE)^{3/2}}{2\hbar^3} H^\kappa(z, q) F_a + T \nabla_a S = 0,
\]

(20)

where \(\partial E/\partial x_a = -F_a\) and \(\partial S/\partial x_a = \nabla_a S\). By applying the relations \(F = ma = -m\nabla \Phi\) and \(\nabla_a S = (-2\pi m N_a)/\hbar\) on the Eq.(20). Also, the deformed temperature on the holographic screen \(S\) can be obtained as

\[
T = \frac{5V(2mE)^{3/2}}{8\sqrt{\pi} \hbar^2} H^\kappa(z, q) N^a \nabla_a \Phi,
\]

\[
= \tilde{\alpha}(z, q) T_U,
\]

(21)

where \(T_U = \frac{\hbar}{2\pi} N^a \nabla_a \Phi\) represents the standard Unruh temperature and \(N_a\) is the unit outward pointing vector which is normal to the screen. The parameter \(\tilde{\alpha}(z, q)\) is defined as follows,

\[
\tilde{\alpha}(z, q) := \frac{5V(2\pi mE)^{3/2}}{2\hbar^3} H^\kappa(z, q).
\]

(22)

Let us now derive a modified version of the Poisson equation based on the entropic gravity approach by using the \(q\)-deformed temperature obtained in Eq. (21). According to Bekenstein [4], if we suppose that there is a test particle near the black hole horizon which is distant from one Compton wavelength, it increases the black hole mass and horizon area. This process is identified as one bit of information. According to the holographic principle, the total number of bits \(N\) is proportional to the area \(A\),

\[
N = \frac{A}{G\hbar},
\]

(23)

where the speed of light \(c = 1\) and \(G\) is a constant. The constant \(G\) will be identified with Newton’s gravitational constant when we consider gravity. Now, we suppose that the total energy of the system \(E\) is associated with the total mass distributed over all the bits. By taking into account the equipartition law of energy, the total mass can be defined [5] as

\[
M = \frac{1}{2} \int_{S} T dN.
\]

(24)
where the integration over the holographic screen $S$. Substituting Eq. (21) in Eq. (24), then using Eq. (23) for the number of bits on the holographic screen Eq. (24), we obtain

$$M = \frac{\tilde{\alpha}(z,q)}{4\pi G} \int_S \nabla \Phi dA. \quad (25)$$

Using the divergence theorem, the Eq. (25) can be written as

$$M = \frac{\tilde{\alpha}(z,q)}{4\pi G} \int_V \nabla \cdot (\nabla \Phi) dV, \quad (26)$$

where $V$ represents three dimensional volume element. On the other hand, the mass distribution in the closed surface can be given as

$$M = \int_V \rho(r) dV, \quad (27)$$

where $\rho(r)$ is the mass density. Comparing Eq. (26) with Eq. (27), we get the Poisson equation for gravity for $q$-deformed boson and fermion models as follows,

$$\nabla^2 \Phi(r) = 4\pi G_{eff} \rho(r), \quad (28)$$

where $G_{eff}$ is effective or $q$-deformed Newton’s gravitational constant and defined as,

$$G_{eff} := G\tilde{\alpha}^{-1}(z,q). \quad (29)$$

Now if we define the Newton potential and gravitational acceleration as follows respectively,

$$\Phi(r) := -\frac{G_{eff} M}{r}, \quad (30)$$

$$g = -\nabla \Phi(r), \quad (31)$$

then the Eq. (28) takes following form,

$$\nabla \cdot g = -4\pi G_{eff} \rho(r). \quad (32)$$

After these calculations, substituting Eqs. (30) and (31) in Newton’s second law $F = mg$ we find $q$-deformed force formula with effective gravitational constant as,

$$F = -\frac{G_{eff} M m}{r^2}. \quad (33)$$

The Eqs. (28) and (33) show that the gravity can be modified by $q$-deformed temperature function based on Verlinde’s proposal. In other words, the Eq. (33) can be seen as the generalization of the Newton’s law of universal gravitation.

IV. GENERALIZATION OF THE EINSTEIN EQUATIONS

Here, we focus on generalizing the Einstein equations by applying Verlinde’s entropic gravity proposal [5]. For this aim, we will use a deformed temperature. Before calculating the Einstein equations we give some information about the deformed temperature function in Eq. (21). The Unruh temperature on the holographic screen is given as
\[ T_U = \frac{\hbar a}{2\pi}, \]  
(34)

where the constants \( c \) and \( k_B \) equal to one and \( a \) represents the acceleration which is perpendicular to screen \( S \). This temperature is experienced by an observer in an accelerated frame. The acceleration also defined with the Newton potential \( \phi \) in general relativity as

\[ a^b = -\nabla^b \phi. \]  
(35)

Here \( \phi \) can be seen as a generalization of Newton potential in general relativity when we describe it by time like killing vector \( \xi^a \) [39],

\[ \phi = \ln (-\xi^a \xi_a). \]  
(36)

The exponential \( e^\phi \) represents the redshift factor that relates local time coordinate to that at a reference point with \( \phi = 0 \), which we will take to be at infinity [4]. When taking into account the redshift factor \( e^\phi \), the Unruh temperature \( T_U \) on the holographic screen as a equipotential surface with non relativistic case is defined as,

\[ T_U = \frac{\hbar}{2\pi} e^\phi N^a \nabla_a \phi. \]  
(37)

In a similar calculations given in previous section with generalized Newton’s potential, the deformed temperature can be found as,

\[ T = \frac{5V}{8\sqrt{\pi}} \frac{(2mE)^{3/2}}{H^k (z, q)} e^\phi N^a \nabla_a \phi, \]

\[ = \tilde{\alpha} (z, q) T_U. \]  
(38)

Substituting Eq.(38) in Eq.(24) and using Eq.(23), we obtain for the total mass

\[ M = \frac{\tilde{\alpha} (z, q)}{4\pi G} \int_S e^\phi \nabla \phi dA. \]  
(39)

where \( \tilde{\alpha} (z, q) \) is defined in Eq.(22). The Eq.(39) can be seen as the natural generalization of Gauss’s law to the General Relativity for \( q \)-deformed model and the mass \( M \) corresponds to Komar mass. The Komar mass can also be, alternatively, re-expressed in terms of the Killing vector \( \xi^a \) and Eq. (29) as,

\[ M = \frac{1}{4\pi G_{\text{eff}}} \int \Sigma R_{ab} n^a \xi^b dV, \]  
(40)

According to the Stokes theorem, the surface integral in Eq.(40) can be converted into a volume integral if the surface is two dimensional boundary of the hyper-surface. Using the relation \( \nabla_a \nabla^a \xi^b = -R^b_{\phantom{b}a} \xi^a \), which is implied by the Killing equation for \( \xi^a \), the Komar mass can be written as

\[ M = \frac{1}{4\pi G_{\text{eff}}} \int \Sigma R_{ab} n^a \xi^b dV, \]  
(41)

where \( R_{ab} \) is the Ricci curvature tensor, \( \Sigma \) is the three dimensional volume bounded by the holographic screen and \( n^a \) is its outward normal. Furthermore, the Komar mass can also be written as an integral over the enclosed volume of certain components of the energy-momentum tensor \( T_{ab} \) [38] as

\[ M = 2 \int_{\Sigma} \left( T_{ab} - \frac{1}{2} g_{ab} T \right) n^a \xi^b dV, \]  
(42)
where \( g_{ab} \) is space-time metric tensor. When Eqs. (11) and (12) are equated, we find

\[
R_{ab} = 8\pi G_{\text{eff}} \left( T_{ab} - \frac{1}{2} g_{ab} T \right) \quad (43)
\]

This gives us only a time-time component of the Einstein field equations \([5, 40]\). Taking the trace of Eq. (43) leads to

\[
R_{ab} - \frac{1}{2} g_{ab} R = 8\pi G_{\text{eff}} T_{ab} \quad (44)
\]

The last equation is the \( q \)-deformed Einstein equations resulting from considering the quantum black holes as deformed bosons or fermions. The factor \( \tilde{\alpha}(z, q) \) in the definition of \( G_{\text{eff}} \) Eq.(29) carries the information of the total energy-momentum of the deformed systems.

V. CONCLUSION

In this study, we have derived deformed entropy and temperature functions for \( q \)-deformed boson and fermion gas models at high temperature limit. Using Verlinde’s entropic gravity approach with deformed temperature function, we have obtained \( q \)-deformed Poisson’s equations for gravity, Newton’s law of universal gravitation and the Einstein field equations. According to the Strominger’s idea, quantum black holes obey deformed statistics, neither boson nor fermion. Thus our results provide an alternative framework for describing gravitational effect around the quantum black holes.

In the limit \( q \to 1 \) the deformed boson and fermion gas models transform into its ideal form in which there is no interaction between related particles, in other words, the model goes non-deformed case. Thus, we can not use this model to describe the extremal black holes in this limit because it needs deformed statistic. Furthermore, the deformed temperature functions of the model, given in Eq.(21) and Eq.(38), do not reduce to their standard forms in \( q \to 1 \) limit because the parameter \( \tilde{\alpha}(z, q) \) can not be equal to one. Due to the effective gravitational constant Eq.(29) which depend on \( \tilde{\alpha}(z, q) \), our deformed gravitational equations Eq.(32), Eq.(33) and Eq.(44) remain deformed under this limit.

Here we want to give a remark about possible application. If we consider the Brans-Dicke theory of gravity \([11–42]\), the Lagrangian density which is invariant under the scale transformation can be written as

\[
\mathcal{L} = \frac{1}{16\pi} \int d^4x \sqrt{-g} \varphi^2 R \quad (45)
\]

where \( g \) is determinant of metric tensor, \( R \) is the Ricci scalar and \( \varphi(x) \) is a scalar field. In this case, the Einstein equations take the following form:

\[
R_{ab} - \frac{1}{2} g_{ab} R = 8\pi \varphi^{-1} T_{ab} \quad (46)
\]

and comparing the Eqs. (44) and (46) a close connection emerges between \( \varphi(x) \) and \( \tilde{\alpha}(z, q) \) as,

\[
\varphi(x) \sim \frac{\tilde{\alpha}(z, q)}{G} = \frac{1}{G_{\text{eff}}} \quad (47)
\]

this relation means that the scalar field \( \varphi(x) \) can be described by thermodynamical quantities in the deformed model and thus, if it is correct, one can say that the Brans-Dicke like theory of gravity can be obtained by using Verlinde’s entropic gravity approach under the \( q \)-deformed systems.

[1] T. Jacobson, Phys. Rev. Lett. 75, 1260 (1995).
[2] S. W. Hawking, Phys. Rev. Letters 26, 1344 (1971).
[3] S. W. Hawking, Commun. Math. Phys. 43, 199 (1975).
