HYDRODYNAMICS OF CORE-COLLAPSE SUPERNOVAE AT THE TRANSITION TO EXPLOSION. I. SPHERICAL SYMMETRY

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ABSTRACT

We study the transition to runaway expansion of an initially stalled core-collapse supernova shock. The neutrino luminosity, mass accretion rate, and neutrinospheric radius are all treated as free parameters. In spherical symmetry, this transition is mediated by a global non-adiabatic instability that develops on the advection time and reaches nonlinear amplitude. Here, we perform high-resolution, time-dependent hydrodynamic simulations of stalled supernova shocks with realistic microphysics to analyze this transition. We find that radial instability is a sufficient condition for runaway expansion if the neutrinospheric parameters do not vary with time and if heating by the accretion luminosity is neglected. For a given unstable mode, transition to runaway occurs when fluid in the gain region reaches positive specific energy. We find approximate instability criteria that accurately describe the behavior of the system over a wide region of parameter space. The threshold neutrino luminosities are in general different than the limiting value for a steady-state solution. We hypothesize that multidimensional explosions arise from the excitation of unstable large-scale modes of the turbulent background flow, at threshold luminosities that are lower than in the laminar case.

Key words: hydrodynamics – instabilities – shock waves – supernovae: general

Online-only material: color figures

1. INTRODUCTION

Following collapse of the core in a massive star, a hydrodynamic shock is launched when the central region reaches nuclear density. Significant energy losses due to neutrino emission and dissociation of infalling nuclei drain this shock of thermal energy and cause it to stall (e.g., Bethe 1990). For slowly rotating progenitors, the re-absorption of a small fraction of neutrinos carrying away the gravitational binding energy of the protoneutron star is thought to re-energize this shock and trigger an explosion (Bethe & Wilson 1985). However, ab initio radiation-hydrodynamic simulations in spherical symmetry fail to produce this outcome for stars that form iron cores (Liebendörfer et al. 2001; Rampp & Janka 2002; Thompson et al. 2003; Sumiyoshi et al. 2005). Success is only obtained for progenitors at the lowest end of the mass range leading to core-collapse (~8–10 M⊙; Kitaura et al. 2006; Burrows et al. 2007a).

Increasing the dimensionality of simulations improves conditions for explosion by enabling non-spherical hydrodynamic instabilities. In axisymmetry (2D), large-scale shock oscillations combined with neutrino-driven convection lead to late-time—albeit somewhat marginal—explosions in representative stellar progenitors (Marek & Janka 2009 and references therein). A recent three-dimensional (3D) hydrodynamic study found that the extra spatial dimension could make conditions even more favorable, reducing the neutrino luminosity needed to start an explosion by ~20% relative to axisymmetry (Nordhaus et al. 2010). The way in which hydrodynamic processes combine to cause this decrease is not well understood at present, however. In particular, a change in the methods and approximations employed can erase this dimensionality effect, suggesting that it may manifest only in some region of physical and/or numerical parameter space (Hanke et al. 2011). Given the marginality of 2D models, it is important to identify the conditions under which this additional reduction in neutrino luminosity occurs.

The purpose of this paper and its companion is to systematically examine the hydrodynamic processes responsible for the transition to explosion in a stalled core-collapse supernova shock. As a first step, we study here the spherically symmetric case. Taking advantage of the simplicity of the flow geometry and borrowing tools from stellar pulsation theory, we develop a framework for identifying the relevant processes involved. The spherically symmetric case is also related (in a time-averaged sense) to three-dimensional models that experience small-scale convection with no significant large-scale shock deformations (e.g., Nordhaus et al. 2010; Wongwathanarat et al. 2010).

Spherically symmetric explosions, arising from boosted neutrino luminosities, involve a transition from a stalled configuration into runaway expansion on timescales longer than the dynamical time. In a realistic setting, the problem is fully time dependent, and a complete analysis must include the mutual feedback of several effects (e.g., Janka 2001). As a way to facilitate the analysis, Burrows & Goshy (1993) approximated the problem as a steady-state accretion flow and identified a set of control parameters that determine it. They conjectured that the transition from accretion to explosion involved a global instability of the flow, with an associated critical stability surface in the space of control parameters. State-of-the-art radiation-hydrodynamic simulations show that multidimensional explosions take place at times when the background flow changes slowly (Burrows et al. 2007b; Marek & Janka 2009; Suwa et al. 2010), making a steady-state treatment a reasonable starting point to study dimensionality effects.

Global instabilities of the steady-state flow have been identified in several linear stability studies (Yamasaki & Yamada 2005, 2007; Foglizzo et al. 2007). Using a realistic equation of state (EOS) and weak interactions, Yamasaki & Yamada (2007) found that as the neutrino luminosity is increased, both oscillatory and non-oscillatory modes become unstable. The presence
of these modes can be found in spherically symmetric simulations dating back to the early days of the delayed neutrino mechanism (e.g., Wilson et al. 1986; Burrows et al. 1995; Buras et al. 2006b; Ohnishi et al. 2006; Murphy & Burrows 2008; Fernández & Thompson 2009a; Nordhaus et al. 2010; Hanke et al. 2011). The instability mechanism behind these modes, their nonlinear development, and their connection to traditional diagnostics for explosion conditions are not completely understood at present, however.

In particular, Burrows & Goshy (1993) found that sequences of steady-state models that satisfy a constraint on the neutrino optical depth are possible only up to a limiting neutrino luminosity, for a given mass accretion rate. This limiting value was found by Murphy & Burrows (2008) to be close to the point where spherically symmetric explosions occur, and by Yamasaki & Yamada (2005) to lie at the critical stability curve in the neutrino luminosity versus mass accretion rate plane. These results provide support for the assumption that explosion is indeed obtained at the limiting steady-state luminosity (e.g., Pejcha & Thompson 2012). The use of a realistic EOS in the linear analysis yields, however, a flow that becomes unstable for luminosities $\sim$30% lower than the limiting value, for a specific choice of parameters (Yamasaki & Yamada 2007). Fernández & Thompson (2009a) found, in turn, that unstable modes develop into explosions when parametric microphysics and steady-state boundary conditions are used in time-dependent simulations.

A more widely used diagnostic for explosion is the ratio of the advection to heating time in the gain region (Janka & Keil 1998; Thompson 2000), which is known to have predictive power in the spherically symmetric case (Thompson et al. 2005). If an understanding of the hydrodynamic processes leading to explosion—in any number of dimensions—is to be obtained, then the connection between the limiting steady-state luminosity, stability in spherical symmetry, and proven explosion diagnostics needs to be clarified.

We attempt to shed light on this issue here by studying the properties of unstable modes of the stalled supernova accretion shock, their nonlinear development, and their transition to a runaway solution. Our approach involves time-dependent hydrodynamic simulations, using a realistic EOS and weak interactions. The neutrino radiation field is treated parametrically, as is the fluid accreting onto the stalled shock. We focus on the evolution of the flow in between the neutrinosphere and the shock when the parameters describing this flow are systematically varied. Such a parametric approach has been employed previously to study properties of exploding models and the effects of dimensionality (e.g., Janka & Müller 1996; Ohnishi et al. 2006; Murphy & Burrows 2008).

This paper is organized as follows. Section 2 describes the physical assumptions and numerical methods employed. Section 3 revisits the limiting neutrino luminosity of the steady-state solution and its relation to the neutrino optical depth. Section 4 examines the instability affecting the post-shock flow in the linear and nonlinear phases by borrowing analysis tools from stellar pulsation theory. By studying the energetics of the flow, the nature of the instability cycle, approximate stability criteria, and saturation mechanisms are identified. Section 5 probes the validity of the instability criteria over a wide region of parameter space. The influence of numerical resolution, boundary conditions, and other parameters is also investigated. Finally, Section 6 contains a summary of our results and a discussion of the implications for the multidimensional case. Readers not interested in technical details can start in this section, where references to the relevant figures and equations are made.

2. METHODS

2.1. Physical Model and Approximations

We are interested in hydrodynamic instabilities that mediate the onset of explosion in a stalled core-collapse supernova shock, when the quantities that determine the global character of the accretion flow—mass accretion rate, neutrino luminosity, and protoneutron star radius—begin to evolve slowly relative to the instability timescales.$^2$ This phase starts around 100–200 ms after core bounce (e.g., Liebendörfer et al. 2001). Prior to that, the system is still undergoing transient evolution. Except for progenitors at the lighter mass end (Kitaura et al. 2006; Buras et al. 2006a), simulations with neutrino transport obtain explosions for representative progenitors only after this transient phase is over (Burrows et al. 2007b; Marek & Janka 2009; Suwa et al. 2010).

The background accretion flow then evolves on timescales $t_{\text{bkg}} \gtrsim 0.1$–1 s (e.g., Bethe 1990; Liebendörfer et al. 2001; Buras et al. 2006b). In contrast, the dynamical time, advection time, and thermal time of the flow in between the neutrinosphere and the shock are approximately

$$t_{\text{ff}} \sim 2M_{1.3}^{-1/2} r_7^{3/2} \text{ ms}$$

$$t_{\text{adv}} \sim 10\gamma v_{r,9}^{-1} \text{ ms}$$

$$t_{\text{th}} \sim 20M_{1.3} r_7^{-1} Q_{\text{net,21}}^{-1} \text{ ms},$$

where $M_{1.3}$ is the mass enclosed within the shock in units of 1.3 $M_\odot$, $r_7$ is the radial distance from the center of mass in units of 10$^7$ cm, $v_{r,9}$ is the radial velocity in units of 10$^9$ cm s$^{-1}$, and $Q_{\text{net,21}}$ is the net specific neutrino energy source term, in units of 10$^{31}$ erg g$^{-1}$ s$^{-1}$ ($\sim$1 GeV per baryon per second). These timescales are usually shorter than $t_{\text{bkg}}$, by a factor of at least several, making a steady-state model a reasonable background state to study global hydrodynamic instabilities and the transition to explosion (Burrows & Goshy 1993).

In this study we adopt a parametric approach in that we do not compute the neutrino radiation field and boundary conditions starting from a stellar progenitor. Instead, we regard the quantities determining those processes as free parameters and study the behavior of the system as these parameters are varied. This approach has previously been used by a number of authors to study the hydrodynamic response of the stalled accretion shock (e.g., Blondin et al. 2003; Ohnishi et al. 2006; Murphy & Burrows 2008; Iwakami et al. 2008; Fernández & Thompson 2009a; Nordhaus et al. 2010; Hanke et al. 2011).

In our model, we ignore the interior of the protoneutron star, as we are interested in the dynamics of the fluid between the neutrinosphere and the shock. Instead, we establish a spherical inner boundary at the radius of the forming neutron star. A time-independent neutrino flux is imposed at this surface. We consider only electron-type neutrinos and antineutrinos, as these are the only species that exchange energy with matter with a significant cross-section (e.g., Bethe 1990). For simplicity, we establish a single neutrinosphere at a time-independent radius $R_v$ and assume that neutrinos stream isotropically from this surface. In a more realistic treatment, the difference between the energy

$^2$ The mass of the protoneutron star is another fundamental parameter of the flow, but it must vary slowly in neutron-star-forming supernovae.
averaged electron neutrino- and antineutrinosphere radius is of the order of 10% (e.g., Buras et al. 2006b).

Both neutrino species are assumed to have a Fermi–Dirac spectrum with zero chemical potential. To remain close to results from more detailed radiation-hydrodynamic simulations, we set the neutrino luminosities of both species to be equal, but allow them to have different neutrinospheric temperatures (e.g., Janka 2001). This is achieved by allowing the normalization of the neutrino luminosity to vary freely relative to the neutrinospheric temperature and radius (Appendix A).

The contribution to neutrino heating from the accretion luminosity is not included explicitly. To first approximation, the amount of heating required to start an explosion depends only on the total flux incident on the gain region, which lies above the cooling region, so both contributions can be absorbed by the core luminosity in steady state. However, instability of the post-shock flow generates changes in the mass accretion rate, hence a non-trivial feedback from the accretion luminosity can alter the stability properties. We discuss the implications of this approximation in Section 4.5.

The mass accretion rate is taken to be constant in time. Only the point-mass gravity of the protoneutron star is considered, as the self-gravity of the envelope is a ≲ 10% correction to the dynamics. This gravitating mass remains constant in time.

Starting from a steady-state accretion flow defined as above, we evolve the system for different values of the control parameters that set the background solution. In particular, we study the evolution of sequences of models with different neutrino luminosities, for fixed values of the mass accretion rate and neutrinospheric radius. Our goal is to identify the hydrodynamic processes by which a quasi-steady-state configuration transitions into an exploding solution and to determine the conditions (quantified by the control parameters) for which this transition occurs.

2.2. Equations and Coordinate System

Spherical symmetry is assumed throughout this paper, with the origin at the center of the protoneutron star. The time-dependent system is described by the equations of mass, momentum, energy, and lepton number conservation in spherical polar coordinates, with source terms due to the gravity of a point mass $M$ and charged-current weak interactions:

$$\frac{\partial \rho}{\partial t} + \frac{1}{r^2} \frac{\partial}{\partial r}(r^2 \rho v_r) = 0$$

(4)

$$\frac{\partial v_r}{\partial t} + v_r \frac{\partial}{\partial r}(v_r \rho + p) + \frac{GM}{r^2} = 0$$

(5)

$$\frac{\partial (\rho e)}{\partial t} + \frac{1}{r^2} \frac{\partial}{\partial r}(r^2 \rho e v_r) + \frac{GM}{r^2} = \mathcal{L}_\text{net}$$

(6)

$$\frac{\partial Y_e}{\partial t} + v_r \frac{\partial Y_e}{\partial r} = \Gamma_\text{net}.$$  

(7)

We denote, respectively, by $\rho$, $v_r$, $p$, $G$, $e$, and $Y_e$, the mass density, radial velocity, total pressure, gravitational constant, specific energy, and electron fraction. Source terms due to weak interactions are calculated in tabular form, with details provided in Appendix A. The relevant contributions are the net rate of heating minus cooling per unit volume $\mathcal{L}_\text{net}$, and the net rate per baryon of electron minus positron generation $\Gamma_\text{net}$.

The system of Equations (4)–(7) is closed with an EOS that yields the relation $p(\rho, e_{\text{int}}, Y_e)$, with $e_{\text{int}} = e - \nu_e^2/2$ the specific internal energy. We use the model of Shen et al. (1998), as implemented by O’Connor & Ott (2010). The high-density part of EOS does not make a significant difference at the densities considered here, $\rho \lesssim 10^{11}$ g cm$^{-3}$. The important components are nucleons, alpha particles, and heavy nuclei in nuclear statistical equilibrium, supplemented by photons and electron–positron pairs with an arbitrary degree of degeneracy and relativity.

2.3. Initial Conditions

The initial condition for time-dependent simulations consists of a steady-state spherical accretion flow with a shock at some radial distance $R_s$ from the center, obtained by solving Equations (4)–(7) without time derivatives. Solutions upstream and downstream of the shock are connected through the Rankine–Hugoniot jump conditions (e.g., Landau & Lifshitz 1987).

Conditions upstream of the shock are determined by requiring that the velocity, entropy, and electron fraction of the supersonic flow have specified values at a fixed radial coordinate $r_{\text{cf}}$. In order to obtain values that are in close agreement with more realistic one-dimensional simulations (e.g., Liebendorfer et al. 2001; Buras et al. 2006b), we set $r_{\text{cf}} = 100$ km for all simulations. At this radius, we set the radial velocity to the local free-fall velocity, the entropy per baryon to $5k_B$, and the electron fraction to 1/2. These parameters together with the mass accretion rate $\dot{M}$, the gravitating mass $M$, and the shock radius $R_s$ yield the mass density, internal energy, electron fraction, and velocity upstream of the shock, by integrating the steady-state equations from $r_{\text{cf}}$ to $R_s$. The sensitivity of our results to these choices is examined in Section 5.2.

For a given set of parameters, the shock radius $R_s$ is obtained by iteratively solving for the downstream flow from a trial shock position to $R_s$, until an additional constraint is satisfied. Following previous studies of steady-state core-collapse supernova flows (e.g., Burrows & Goshy 1993; Yamasaki & Yamada 2005, 2006, 2007; Pejcha & Thompson 2012), this additional closure relation is obtained by requiring that the neutrino optical depth from $R_s$ to the shock

$$\tau_{\nu} = \int_{R_s}^{R_s} \kappa_{\nu} dr$$

(8)

equal is to 2/3. By default, we set $\kappa_{\nu}$ to the effective absorption coefficient of electron-type neutrinos $\kappa_{\text{eff}} = \sqrt{\kappa_{\text{abs}}(\kappa_{\text{sc}} + \kappa_{\text{abs}})}$ (Janka 2001), where

$$\kappa_{\text{abs}} \simeq 1.96 \times 10^{-7} T_{\nu, 4}^2 \rho_{10} Y_e \text{ cm}^{-1}$$

(9)
corresponds to charged-current absorption and

$$\kappa_{\text{sc}} \simeq 0.51 \times 10^{-7} T_{\nu, 4}^2 \rho_{10} (Y_e + Y_p) \text{ cm}^{-1}$$

(10)
to elastic scattering with nucleons, to lowest order in the neutron–proton mass difference over the neutrino energy (e.g., Bruenn 1985). In Equations (9) and (10), $T_{\nu, 4}$ is the electron neutrinospheric temperature in units of 4 MeV, $\rho_{10}$ is the mass density in units of $10^{10}$ g cm$^{-3}$, and $\{Y_e, Y_p\}$ are the number of neutrons and protons per baryon, respectively. To test the

3 Available at http://stellarcollapse.org.
sensitivity of our results to this prescription for the optical depth, we also compute solutions with pure absorption ($\kappa_\nu = \kappa_{abs}$), or absorption plus scattering ($\kappa_\nu = \kappa_{abs} + \kappa_{sc}$), thereby bracketing our default choice (Section 5.2).

Figure 1 shows characteristic radial profiles of density, entropy, electron fraction, and net specific neutrino energy generation rate for a few neutrino luminosities, given $M = 0.3 M_\odot$ s$^{-1}$ and $R_\nu = 30$ km.

2.4. Time-dependent Implementation

We use FLASH3.2 (Dubey et al. 2009) to evolve the system of Equations (4)−(7). The public version of the code has been modified to include the EOS implementation of O'Connor & Ott (2010), weak interaction rates, and a grid of variable spacing. Details about the latter are provided in Appendix B.

We locate the inner simulation boundary at the neutrinospheric radius $R_\nu$. The outer boundary $r_{max}$ is chosen to be 1000 km for most models, corresponding to approximately four times the radius where the nuclear binding energy of alpha particles equals their gravitational binding energy,

$$r_\alpha = 254 \left( \frac{M_{enc}}{1.3 M_\odot} \right) \text{km},$$

where $M_{enc}$ is the mass enclosed at a radius $r_\alpha$. Above this radius, the shock accelerates significantly during an explosion (e.g., Fernández & Thompson 2009a and Section 4.4).

We use two types of boundary condition at $R_\nu$. Our default implementation fixes all variables in the ghost cells to their initial values, obtained by continuing the steady-state solution inside $R_\nu$. This prescription fixes the mass, momentum, and energy fluxes leaving the domain. A truly reflecting boundary condition would cause the shock to drift outward with time due to the accumulation of mass around $R_\nu$, because the latter does not recede (as it would do with a more realistic treatment, e.g., Scheck et al. 2006). Fernández & Thompson (2009b) and Fernández & Thompson (2009a) were able to use a reflecting boundary condition with a fixed neutrinospheric radius because their parametric cooling function was very centrally concentrated, resulting in accumulation of mass in a few cells outside $R_\nu$, and thus eliminating the shock drift effect. The cooling function used here has, in contrast, a shallower radial dependence.

To test the influence of the default inner boundary condition on our results, we also run a few models that allow an arbitrary amount of mass to leave the domain, while still providing pressure support to the accretion flow. The density and radial velocity in the ghost cells inside $R_\nu$ are set to

$$\rho(r) = \rho_0(r) + \rho(r_1) - \rho_0(r_1),$$

$$v_r(r) = \left( \frac{r_1}{r} \right)^2 v_r(r_1),$$

where $r$ is the radial position of a given ghost cell, $r_1$ corresponds to the center of the first active cell outside the inner boundary, and the subscript zero labels the initial steady-state solution. All other variables are set to have zero gradient and are thus copied from the innermost active cell into the ghost cells. The additional factor multiplying the velocity accounts for the radial convergence of the flow (Ohnishi et al. 2006).

The radial cell spacing $\Delta r$ is ratioed (as in, e.g., Stone & Norman 1992): $\Delta r_{i+1}/\Delta r_i = \zeta$, with $i$ denoting cell number and $\zeta$ some positive real number greater than 1. We space our cells logarithmically, with $\zeta = (r_{max}/r_1)^{N_r}$, where $N_r$ is the total number of cells, and the size of the cell closest to the inner boundary $\Delta r_{min} = R_\nu(\zeta - 1)$ (Appendix B). Guided by convergence tests, we choose either 880 or 1760 cells for our runs, although some models are evolved at resolutions up to $N_r = 3200$ (Section 5.2). The corresponding values of $(\zeta - 1)$ are 0.4%, 0.2%, and 0.1%, respectively, for $r_{max} = 1000$ km and $R_\nu = 30$ km.

To avoid problems with the Riemann solver whenever the internal energy becomes negative, we implement the prescription of Buras et al. (2006b), adding the nuclear binding energy of alpha particles, heavy nuclei, and the rest mass energy of electrons to the internal energy during the hydrodynamic step. The mass fractions of alpha particles and heavy nuclei are then advected as passive scalars in between calls to the EOS. In some cases where the Mach number of the upstream flow is $\gtrsim 5$, we add an additional zero point of $7 \times 10^{17}$ erg g$^{-1}$ to prevent the internal energy to be negative at large radii. This constant shift makes a negligible difference in the evolution of the shock.

Figure 1. Initial profiles of (a) density, (b) entropy, (c) electron fraction, and (d) net neutrino energy generation rate $L_{net}/\rho$ as a function of radius, obtained by solving the steady-state version of Equations (4)−(7) as described in Section 2.3. Parameters correspond to our fiducial sequence (Section 2.5). Curves shown correspond to neutrino luminosities at the approximate instability thresholds for oscillatory modes (blue) and non-oscillatory modes (red), plus the limiting luminosity for a steady-state solution (black).

(A color version of this figure is available in the online journal.)
To prevent the system from reaching the lower limit on $Y_e$ allowed by the EOS implementation, and to account for the decrease in neutrino flux around the neutrinosphere in a very crude manner, we multiply all the neutrino source terms by a suppression factor

$$f_{\text{sup}}(\rho) = e^{-\rho/\rho_0},$$

where $\rho_0 = 10^{11}$ g cm$^{-3}$ is a fiducial density, chosen so as to match the characteristic density at the neutrinosphere. Given that the effective neutrino optical depth depends chiefly on density (Equation (8)), the factor in Equation (14) amounts to a reduction $\sim e^{-\tau}$ (e.g., Murphy & Burrows 2008). The choice of $\rho_0$ has some mild influence on the threshold for instability, but it does not significantly affect the relative differences between models (Section 5.2).

A small initial transient is produced in the form of an outgoing sound wave from the inner boundary and downgoing entropy and sound waves from the shock. The former acts as an initial perturbation.

2.5. Models Evolved

We evolve sequences of models with varying neutrino luminosity $L_{\nu_e}$, for a given mass accretion rate $M$ and prescription for the neutrinospheric radius $R_{\nu}$. Our fiducial sequence corresponds to $M = 0.3 M_\odot$ s$^{-1}$ and $R_{\nu} = 30$ km, with other parameters fixed to values as described in Section 2.3. This default sequence closely resembles the flow obtained at late times in the evolution of a $15 M_\odot$ progenitor (e.g., Liebendörfer et al. 2001; Marek & Janka 2009).

We explore a wider region of parameter space with a grid of constant $R_{\nu}$ and $M$ sequences such that $R_{\nu} = [20, 30, 40]$ km and $M = \{0.1, 0.3, 0.5, 0.75, 1\} M_\odot$ s$^{-1}$, with our fiducial sequence as one of its members. We add another sequence, following Burrows & Goshy (1993), that relates the neutrinospheric radius to the neutrino luminosity through the Fermi–Dirac distribution at constant neutrinospheric temperature and zero chemical potential (Equation (A3)) with $N_{\nu_e} = 1$, namely, $R_{\nu} \propto L_{\nu_{e}}^{1/2}$. In this case we use the same grid in $M$ as in the constant $R_{\nu}$ models. All sequences are summarized in Table 1.

3. ON THE LIMITING NEUTRINO LUMINOSITY OF THE STEADY-STATE SOLUTION

Burrows & Goshy (1993) found that steady-state solutions to the accretion shock problem in the supernova context can exist only up to a maximum value of the neutrino luminosity (the Burrows–Goshy limit hereafter). Recently, Pejcha & Thompson (2012) have related this limit to a critical value of the ratio of sound speed to free-fall speed in the postshock region, in analogy with isothermal accretion flows. Here, we provide an alternative (but equivalent) explanation.

The existence of a limiting neutrino luminosity for the steady-state solution is related to the requirement that the neutrino optical depth has a fixed value (Equation (8)). This closure relation is used for self-consistency with the assumed neutrino radiation field, but has no significance from the purely hydrodynamic point of view. By relaxing this constraint, one can examine the behavior of $\tau_{\nu_e}$ as a function of shock radius and neutrino luminosity, in order to gain insight into the processes that determine this limit.

Figure 2 shows the neutrino optical depth $\tau_{\nu_e}$ as a function of shock radius $R_s$ at $t = 0$ for several neutrino luminosities in our fiducial sequence. Each point along a given curve is obtained by integrating the steady-state version of Equations (4)–(7) from an arbitrary shock position down to a constant $R_{\nu}$, keeping the upstream flow fixed, and calculating Equation (8). For small shock radii, the optical depth is a monotonically increasing function of $R_s$. For large enough shock radii, however, $\tau_{\nu_e}(R_s)$ reaches a maximum and then decreases for increasing $R_s$. Larger neutrino luminosities move this maximum toward smaller shock radii and decrease its magnitude. The two points where this curve reaches a value of $2/3$ correspond to the two physical solutions found by Yamasaki & Yamada (2005) and Pejcha & Thompson (2012). Further increases in the neutrino luminosity cause the maximum of $\tau_{\nu_e}$ to fall below $2/3$, making it impossible to satisfy the optical depth constraint.

The function $\partial \tau_{\nu_e}/\partial R_s$ has two contributions. The first comes from the increase in the volume of the postshock region and is always positive for fixed $R_{\nu}$. The second arises from a change in the quantities that determine $\kappa_{\nu_e}$, chiefly the density profile (Equations (9) and (10)), and can have either sign. Figure 3(a)
In the case where the neutrinospheric radius is tied to the neutrino luminosity at constant temperature (e.g., Burrows & Goshy 1993; Yamasaki & Yamada 2007), an additional contribution arises from the change of the lower limit of integration in Equation (8). This decreases the size of the integration volume for increasing neutrino luminosity relative to the case where $R_\nu$ is constant, making $\tau_\nu$ smaller than it would otherwise be.

Pejcha & Thompson (2012) found that the Burrows–Goshy limit can be characterized by a critical value of the ratio of the sound speed to free-fall speed at some point in the flow, above which it is not possible to connect a subsonic settling solution to a supersonic accretion flow via the hydrodynamic shock jump conditions (see also Yamasaki & Yamada 2005).

Implicit in this result, however, is the simultaneous fulfillment of the boundary conditions at the upstream side of the shock and at the neutrinosphere when constructing the two solutions. Here we have adopted a unidirectional approach, keeping only the upstream boundary condition and integrating the fluid equations inward until an additional constraint is met at the base. This allows us to violate the optical depth condition by varying the shock position, which can be placed at any point in the upstream solution. The conclusions of both approaches regarding the nature of the Burrows–Goshy limit are equivalent when viewed in this light, as a higher sound speed relative to the free-fall velocity implies higher entropy (see Murphy & Meakin 2011 for a discussion of the connection between these two quantities through the entropy equation).

4. TRANSITION TO RUNAWAY EXPANSION

In this section we analyze the instabilities that mediate the transition to runaway expansion, using simulations results and adapting analysis techniques from stellar pulsation theory. The term explosion implies the existence of a star whose envelope is to be ejected by a successful shock. Given the parametric character of our study, we employ instead the term runaway explosion to denote the onset of explosion in a medium with constant mass accretion rate, as the asymptotic explosion energy is determined by additional processes not included here (e.g., Marek & Janka 2009).

Evolving any of the model sequences described in Section 2.5 yields a characteristic outcome for the evolution of the shock radius as a function of time. The result for selected models from our fiducial sequence is shown in Figure 4. Models with low neutrino luminosity are stable to small perturbations. For $4.05 < L_\nu / (10^{52} \text{ erg s}^{-1}) < 4.55$, the shock undergoes oscillations of growing amplitude that transition to runaway expansion after some time delay. For $L_\nu > 4.55 \times 10^{52} \text{ erg s}^{-1}$, initial exponential growth in amplitude occurs without any oscillation, transitioning into a nearly constant-velocity expansion phase at later times.

The characteristic timescale associated with both types of modes is the advection time from the shock to the proto-neutron star

$$t_{\text{adv}} = \int_{R_\nu}^{R_s} \frac{dr}{|v_r|} = \frac{M_{\text{env}}}{\dot{M}},$$

where the second equality is valid when $\dot{M}$ is constant, and $M_{\text{env}}$ is the mass enclosed between the neutrinospheric radius and the shock. The fluid velocity is very subsonic, hence the kinetic energy content is small. Changes in the accretion flow involve changes in the heat content of the fluid through modulation of the boundary conditions at the upstream side of the shock and at the neutrinosphere when constructing the two solutions. Here we have adopted a unidirectional approach, keeping only the upstream boundary condition and integrating the fluid equations inward until an additional constraint is met at the base. This allows us to violate the optical depth condition by varying the shock position, which can be placed at any point in the upstream solution. The conclusions of both approaches regarding the nature of the Burrows–Goshy limit are equivalent when viewed in this light, as a higher sound speed relative to the free-fall velocity implies higher entropy (see Murphy & Meakin 2011 for a discussion of the connection between these two quantities through the entropy equation).

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The characteristic timescale associated with both types of modes is the advection time from the shock to the proto-neutron star

$$t_{\text{adv}} = \int_{R_\nu}^{R_s} \frac{dr}{|v_r|} = \frac{M_{\text{env}}}{\dot{M}},$$

where the second equality is valid when $\dot{M}$ is constant, and $M_{\text{env}}$ is the mass enclosed between the neutrinospheric radius and the shock. The fluid velocity is very subsonic, hence the kinetic energy content is small. Changes in the accretion flow involve changes in the heat content of the fluid through modulation

5 The steepening of the density profile with a higher adiabatic index occurs because the temperature varies slowly with radius near the neutrinosphere. The pressure gradient needed to balance gravity thus arises mostly from the density gradient. In contrast, in the $\gamma = 4/3$ region the pressure gradient arises mostly from the changes in the temperature, with the density profile being thus shallower (Janka 2001).
The work integral for stars is also simplified by the pressure falling to very small values at the surface, which is not the case for flow confined by an accretion shock.

The lowest panel in Figure 5 shows that, in fact, the flow adjusts in as close as possible to $R_s$, to eliminate boundary effects (Appendix C).

In the linear phase, the magnitude of the energy generation by shock motion $\dot{E}_s$ peaks when the shock velocity is the largest and is dominated by the gravitational term in $\dot{E}_{\text{tot}}$. Contractions thus inject positive energy into the postshock region. The net energy generation from accretion $\dot{E}_N - \dot{E}_{\text{up}} + \dot{E}_{\text{dn}}$ also becomes non-zero once the system deviates from equilibrium. Its magnitude is much smaller than the individual terms that comprise it, showing that imperfect cancellation between the energy fluxes and neutrino source terms is the additional source of energy. The lowest panel in Figure 5 shows that, in fact, the flow adjusts itself in the linear phase to match the energy generation from shock motion with the excess energy flux entering the domain, $\dot{E}_s(t) \approx \dot{E}_{\text{up}}(t) - \dot{E}_{\text{up}}(0)$. The total energy generation can then be accounted for entirely by the deviation from steady state of the neutrino source terms and the energy flux leaving the domain toward the protoneutron star. In other words, oscillatory instability arises from a mismatch in the dissipation of the accretion energy in a system with a moving boundary.

The origin of this imbalance can be traced back to the steep temperature dependence of the neutrino cooling function,
\( \mathcal{L}_c / \rho \propto T^6 \) (Equation (A13)). Figure 6 shows that the perturbation to the specific neutrino source term closely mirrors the temperature perturbation, with the opposite sign (after removing the density dependence, the heating term depends only on the neutron and proton abundances, linearly). Thus, an increase in the temperature from its steady-state value causes an increase in the rate of cooling and hence excess energy dissipation. This results in contraction and cooling of the postshock volume on a sound crossing time, reversing the cycle when the lower temperature fluid reaches the cooling region by advection.

Figure 6 also shows that the period of the oscillation is related to the advection time from the shock to a radius near the point of maximum cooling. At this radius the flow achieves maximum deceleration, and advected perturbations are efficiently converted into acoustic waves (Scheck et al. 2008; Foglizzo 2009; Sato et al. 2009). The normalized density perturbation is significant only outside of this radius, below which it undergoes a phase shift. The model shown in Figure 6 has a period very close to twice the advection timescale from the shock to this radius. This is very close to the period of an oscillatory mode with no heating (Fernández & Thompson 2009b). As the flow becomes unstable, however, the oscillation period becomes increasingly longer, achieving nearly four advection times at the threshold for oscillatory instability. This lengthening of the oscillation period with neutrino luminosity has been attributed to buoyancy effects by Yamasaki & Yamada (2007).

Non-oscillatory modes grow exponentially in amplitude and transition directly into runaway expansion. The different terms that make up the rate of change of total energy are shown in Figure 7 (the model has \( L_{\nu_e, \nu_x} = 3.5 \)). The evolution of these terms resembles the last expansion phase of oscillatory modes, where the shock motion term acts only as a sink of energy and is not matched by the deviation in \( E_{\nu} \) from steady state. The expansion is driven by the net energy generation from accretion, which is positive throughout.
The approximate instability criterion is then

\[ t_{\text{adv, gain}} > 1.5 t_{\text{max}}, \]

(27)

and where the solution to Equation (27) closest to \( r_e \) is taken. The dependence of these two timescales on neutrino luminosity is shown in Figure 8.

It is worth emphasizing that \( r_e \) was chosen because it yields a coefficient of unity. Another location close to the node in the density perturbation could also have been used, with a slightly different coefficient. For example, taking the radius of maximum specific cooling \( r_{\text{max}} \) yields an instability condition \( t_{\text{adv, gain}} > 1.5 t_{\text{max}} \), where the latter timescale is computed by replacing \( r_e \) with \( r_{\text{max}} \) in Equation (26).

A physical justification for \( r_e \) can be found by looking at the behavior of the relative phase between the pressure and density perturbations. It is a general result from stellar pulsation theory that, for oscillatory instability, an excitation mechanism is required that causes the system to be absorbing heat at the point of maximum compression (e.g., Cox 1974). This way, the pressure maximum is reached during the expansion phase, leading to positive work over an oscillation cycle, and hence to a net increase in the pulsation kinetic energy. In the system under study, such a phase lag of the pressure perturbation relative to the density is satisfied below the node of the density perturbation. The point where both perturbations are 180° apart is approximately \( r_e \) for most of our models, with the pressure leading the density at any given instant. In cases where \( r_e \) does not trace the phase radius correctly, the criterion in Equation (24) loses accuracy. The relation between the phase radius and the cooling profile is most likely a consequence of the way we suppress the neutrino source terms with density (Equation (14)); different implementations of neutrino transport will most likely find slightly different values.

\[ t_{\text{adv, g}} > t_{\text{adv, e}}. \]

(24)

With

\[ t_{\text{adv, g}} = \int_{R_e}^{R_g} \frac{dr}{|v_r|} \]

(25)

and

\[ t_{\text{adv, e}} = \int_{R_e}^{R_c} \frac{dr}{|v_r|} \]

(26)

Figure 7. Same as Figure 5, but for a model that explodes via the non-oscillatory instability (\( L_{\nu_c, 52} = 4.6 \)). Only the two upper panels are shown, as the fluctuation in \( \dot{E}_{\nu} \) does not compensate \( E_n \) as in the last expansion of the oscillatory mode.

(A color version of this figure is available in the online journal.)

4.3. Approximate Instability Criteria

The only way to obtain an exact instability threshold for these non-adiabatic modes is to perform a linear perturbation analysis of Equations (4)–(7) such as that done by Yamasaki & Yamada (2007). In the absence of such a calculation, we have instead searched for physically motivated instability criteria that correctly describe the behavior of our simulations over a wide range of parameter space. We discuss here their definition, and leave for Section 5 their comparison with simulations over an extended region of parameter space.

For oscillatory modes, we have found that instability sets in when the advection time through the gain region is longer than the time required to advect from the gain radius to a point close to the node in the density perturbation (Figure 6) times some coefficient close to unity. This is equivalent to requiring more mass to reside in the gain region than in the part of the cooling region above the node in the density perturbation, at any given instant (Equation (15)). To obtain a coefficient of unity, the relevant location is the radius at which the density and pressure perturbations are 180° out of phase with each other. In the steady-state solution, and for most of our models, this position is approximately the radius closest to \( R_e \) where the specific cooling has decreased in magnitude by one e-folding from its maximum (e.g., Figure 1(d)).

The approximate instability criterion is then

\[ t_{\text{adv, g}} > t_{\text{adv, e}}, \]

(24)

with

\[ t_{\text{adv, g}} = \int_{R_e}^{R_g} \frac{dr}{|v_r|} \]

(25)

and

\[ t_{\text{adv, e}} = \int_{R_e}^{R_c} \frac{dr}{|v_r|} \]

(26)
An interesting question is how the phase lag criterion for stars can be adapted to the problem at hand, given that fluid elements are advected downstream instead of returning to their original position. Taken at face value, the phase lag criterion would dictate that the driving region resides below the node of the density perturbation. However, the phase lags the density in this region for both stable and unstable models, and \( \chi \) hardly changes over a wide range in \( L_\nu \) (Figure 8). Such a question is likely to be at the root of the instability mechanism of long wavelength modes of the standing accretion shock instability (SASI; Blondin et al. 2003) with heating included, and will not be further pursued in this paper.

In the case of non-oscillatory modes, the instability threshold is very close to the point where the advection time through the gain region is longer than the time required to change the total energy through heating:

\[
\tau_{\text{adv}} > \tau_{\text{heat}},
\]

with

\[
\tau_{\text{heat}} = \int \frac{d^3x}{R_f} \frac{\rho \sigma_{\text{tan}}}{G M}.
\]

The dependence of \( \tau_{\text{heat}} \) on neutrino luminosity is also shown in Figure 8. This relation has long been known to provide a predictive criterion for runaway expansion in spherically symmetric core-collapse simulations (Thompson et al. 2005; Buras et al. 2006a; Murphy & Burrows 2008; Marek & Janka 2009). Its interpretation is straightforward: Neutrino heating is effective enough to change the internal energy of the flow during its transit time through the gain region. As a consequence, the pressure changes significantly and the shock re-adjusts to a new position (Janka & Keil 1998).

Note that Equation (28) is a global condition on the gain region and includes the gravitational binding energy, differing from local definitions of this ratio such as those in Thompson et al. (2005) and Pejcha & Thompson (2012). The fluid does not achieve positive energy over a time \( \tau_{\text{adv}} \) (~10 ms, Figure 8), but instead takes a multiple of this timescale to transition into runaway expansion. As we discuss in the next section, onset of runaway occurs when the fluid achieves positive energy for the first time.

We have checked whether any instability threshold can be described by a fixed value of the parameter \( \chi \) that measures the effects of buoyancy in an advective flow,

\[
\chi = \int \frac{d^3x}{|\omega_{\text{bv}|} v_r},
\]

where \( \omega_{\text{bv}} \) is the Brunt–Väisälä frequency,

\[
\omega_{\text{bv}}^2 = \frac{GM}{r^2} \left( \frac{\partial \ln \rho}{\partial r} - \frac{\partial \ln \rho}{\partial r} \right),
\]

with \( \Gamma_r = (\rho/p)^2 \). When \( \chi > 3 \), buoyancy is expected to overcome the stabilizing effect of advection in the multidimensional case (Foglizzo et al. 2006). Even though all of our sequences have \( \chi \) in the range 1–10 near the threshold for instability, in none of them does this parameter have a constant value along the critical stability curves. For example, in our fiducial sequence, the value of this parameter at the non-oscillatory threshold ranges from \( \chi \approx 7 \) for \( M = 0.1 M_\odot \, \text{s}^{-1} \) to \( \chi = 1.7 \) at \( M = 1 M_\odot \, \text{s}^{-1} \). It is worth pointing out that the analysis of Foglizzo et al. (2006) applies to infinitesimal perturbations. The results of Scheck et al. (2008) show that the SASI can trigger convection through finite amplitude density fluctuations in cases where \( \chi < 3 \), so our findings do not necessarily imply that convection will be suppressed at large accretion rates in the quasi-steady-state approximation (Section 2.1).

### 4.4. Nonlinear Phase and Runaway Expansion

Once oscillatory modes become unstable, the amplitude grows steadily until the oscillation cycle is broken and the system transitions into runaway expansion. Figure 5 shows that not only the shock radius but also the different energy generation terms undergo a qualitative change in behavior relative to the linear phase once this point is reached. In particular, the energy generation due to shock motion decouples from the fluctuation in the energy flux entering through the shock. Expansion is driven by the net energy generation from accretion, with damping from the shock motion term.

A physical origin for this transition point can be found by inspecting the evolution of the total specific energy and radial velocity in the system. Figure 9 shows that this time corresponds approximately to the point where the specific energy behind the shock becomes positive for the first time. This time is marked by a vertical black dashed line in Figure 5(a) and the snapshot at 277 ms in Figure 9. At about the same time, the fluid just inside the shock achieves positive radial velocity, reversing the accretion flow. This transition point is equivalent to the escape temperature condition discussed in Burrows et al. (1995).

Physically, the above result is dependent on the definition of the zero points of energy in the context of a Newtonian framework. The EOS used here takes this zero point to be, per baryon, the atomic mass unit (Shen et al. 1998). Our expression for the total specific energy, Equation (17), takes the zero point of gravitational binding energy at an infinite distance from the central mass. Given these definitions, the condition of positive energy leading to runaway expansion is a well-defined mathematical concept. A self-consistent determination of this condition would require inclusion of the rest mass energy...
and gravitational field calculated self-consistently in a general relativistic framework.

The shock accelerates when it approaches the radius where the binding energy of alpha particles equals their gravitational binding energy (Equation (11)). This can be understood from the fact that the dissipation due to shock motion $E_s$ decreases in magnitude with increasing radius, because the total specific energy behind the shock becomes less negative. This can be seen in Figure 5(b), where the blue curve reaches a minimum value. By now most of the gain region has reached positive energy, except for a narrow layer behind the shock, and most of the fluid has positive velocity, effectively becoming a wind-like solution.

A final transition occurs when the shock begins to mechanically decouple from the protoneutron star atmosphere. This occurs approximately at a time when the sound crossing time from the shock to $R_s$ becomes longer than the shock expansion time $R_s/v_s$, where $v_s$ is the shock velocity. This time is marked as a vertical green dashed line in Figure 5(a) and corresponds to the curve at 314 ms in Figure 9. This mechanical decoupling explains the fact that the net energy generation becomes negative yet the shock continues to expand, as inferred from Figure 5. It is worth noting that velocities below the shock are subsonic throughout the time period considered here.

Non-oscillatory modes transition directly into runaway expansion and their evolution closely resembles that of oscillatory modes after reaching positive energy. The corresponding snapshots of total specific energy and radial velocity at different phases are shown in Figure 10.

In all of the sequences simulated, transition to runaway expansion is achieved after the flow becomes radially unstable. We have found no unstable mode that saturates, nor any model that has achieved positive energy, failed to continue into runaway expansion. Our preliminary conclusion is that for the set of assumptions adopted in this paper, radial instability is a sufficient condition for transitioning into runaway expansion.

4.5. Conditions for Saturation

There are many examples in the literature where one-dimensional stalled shocks start to expand or oscillate, but then fizzle (e.g., Figure 1 of Janka & Müller 1996). It is thus imperative to identify the processes that can lead to saturation, as our preliminary finding linking instability to runaway may break down in some region of parameter space.

Insight on this question can be obtained again by inspecting the different processes that contribute to the change in the total energy in the post-shock region (Equation (19)). What we are interested in are dissipation processes that contribute with negative energy generation.

In the present study, we are ignoring evolution of the neutrinospheric parameters ($R_s$, core luminosities, and spectra) and heating due to the accretion luminosity, hence the dominant dissipation mechanism arises from the change in the post-shock volume, $E_s$, as inferred from Figures 5 and 7. This term is made up of three factors: (1) the specific energy below the shock, (2) the density profile, and (3) the rate of change of the post-shock volume.

1. At the typical radii where supernova shocks stall (100–200 km), the total specific energy $E_{tot}$ is usually negative. Because the temperature decreases outward, nucleons below the shock recombine first into alpha particles and then heavy nuclei as the shock moves out, yielding an increase in the thermal energy that is comparable to the gravitational binding energy (Bethe 1996; Fernández & Thompson 2009a). This causes the total specific energy to become significantly less negative, decreasing the magnitude of $E_s$ as the shock expands. This can be most clearly seen in Figure 5(b), which shows a clear transition in the evolution of $E_s$ when the shock reaches $r_s$. The results of Fernández & Thompson (2009a) show that using a constant dissociation energy indeed quenches runaway expansion, in direct contrast to allowing alpha particles to recombine, because in the first case the dissociation energy becomes an increasingly larger fraction of the local gravitational binding energy as the shock expands. Even then, saturation occurs at a radius that is several times the initial shock radius. Except for unrealistically small shock stagnation radii ($\lesssim 50$ km), this saturation channel is unlikely to be of importance.

2. The evolution of the density profile below the shock and the mass accretion rate are tied to the density profile of the progenitor at the onset of collapse. For an iron core supported by relativistic electrons, the density scales like $r^{-3}$, yielding a mass accretion rate that scales inversely with time at fixed radius when mass conservation and a near free-fall velocity field are assumed (Bethe 1990). A time-independent mass accretion rate corresponds to a progenitor density profile $\propto r^{-3/2}$, which would be given by an adiabatic index of 5/3 in hydrostatic equilibrium. In other words, a time-independent mass accretion rate, as assumed in our sequences, is already unrealistically high and overestimates the dissipation.

3. Because $E_s$ is negative on expansion, it tends to stabilize the shock velocity for a given energy dissipation rate. The dominant energy source for expansion is neutrino heating. As long as the total heating remains larger than $E_s$ before the shock is completely decoupled from the atmosphere, the shock will continue to expand.

We therefore conclude that for the physical assumptions of the present study, radial instability is a sufficient condition for runaway expansion.

In the general case, however, evolution of the neutrinospheric parameters and neutrino heating by the accretion luminosity can provide significant energy dissipation and saturate the
instability. By inspection of Equation (18), one can find three sources of negative energy.

First, evolution of the core neutrino flux affects the energetics via $\dot{E}_N$. If the net effect of decreasing neutrino luminosities and increasing mean neutrino energies is a decrease in the energy deposition in the gain region, the instability can be suppressed in two ways. The linear instability depends on the extent of the gain region through $t_{\text{adv}, \text{gain}}$, hence a rapid decrease of heating can stabilize oscillations or non-oscillatory expansion if the flow has not yet achieved positive energy when the stability criterion is reversed. Second, a decrease in the rate of neutrino energy deposition can lead to quenching of the runaway phase if the total rate of heating fails to keep up with dissipation due to $E_s$. The results of Janka & Müller (1996) are consistent with the operation of this saturation channel.

Second, the contribution of the accretion luminosity to the total neutrino flux in realistic models is $\sim 50\%$ (e.g., Liebendörfer et al. 2001). The important aspect to keep in mind here is that shock expansion relative to its steady-state position leads to a decrease in the magnitude of the mass accretion rate. Perturbing the mass conservation equation yields (Foglizzo et al. 2007)

$$\frac{\delta M}{M} = \left( \frac{1}{v_2} - \frac{1}{v_1} \right) \delta v_s, \quad (32)$$

where $v_1$ and $v_2$ are the (negative) upstream and downstream fluid velocities, respectively, and $\delta v_s$ is the shock velocity perturbation. Taking $\delta v_s$ real and positive (shock expansion) yields $\delta M/M < 0$ or a decrease in the magnitude of the accretion rate. Hence including the heating from accretion neutrinos adds a non-trivial feedback to both oscillatory and non-oscillatory modes, altering the instability criteria. Buras et al. (2006b) find that indeed spherical oscillations are damped by the decrease in the heating due to the dropping mass accretion rate. As noted by a number of previous works, the runaway expansion phase also cuts off accretion in spherical symmetry. A larger core neutrino flux is therefore required to sustain expansion, relative to the light bulb heating case, and the causality relation between radial instability and explosion we have found here will be violated.

Finally, the contraction of the protoneutron star generates in itself a sink of energy in the post-shock region. Including this effect, and keeping everything else constant, would add a term of the form

$$\dot{E}_{\nu, \text{ph}} = -4\pi R_s^2 \dot{R}_s \rho \epsilon_{\text{tot}} R_s, \quad (33)$$

$$\lesssim -10^{50} R_{s,30}^2 \dot{R}_{s,6} \rho_{11} \epsilon_{\text{tot},19} \text{ erg s}^{-1} \quad (34)$$

to Equation (18), with $R_{bc} = R_s$. The notation in the second equality is $R_{s,30} = R_s/(30 \text{ km})$, $R_{s,6} = R_s/(-10^6 \text{ cm s}^{-1})$, $\rho_{11} = \rho/(10^{11} \text{ g cm}^{-3})$, and $\epsilon_{\text{tot},19} = \epsilon_{\text{tot}}/(-10^{19} \text{ erg g}^{-1})$. In contrast to $E_s$, contraction leads to energy dissipation. This quantity is smaller than typical core neutrino luminosities, but approaches the net energy generation terms that regulate instabilities (Section 4.2). A more careful analysis would need to include the additional cooling and heating by accretion neutrinos generated by enlarging the post-shock cavity. The one-dimensional results of Janka & Müller (1996) show that indeed $\sim 10\%$ higher core neutrino luminosities are required to start an explosion in models that experience protoneutron star contraction relative to fixed cores, when all other parameters are similar.

5. INSTABILITY THRESHOLDS AND RELATION TO THE LIMITING STEADY-STATE LUMINOSITY

5.1. Dependence on Mass Accretion Rate and Neutrinospheric Radius

For each of the simulation sequences described in Section 2.5, we have searched for the instability thresholds of oscillatory and non-oscillatory modes. Figure 11 shows the resulting threshold luminosities as a function of mass accretion rate. In all cases, data points correspond to the average luminosity between two models at both sides of the instability threshold. The separation in luminosity, shown as error bars, is less than $10^{51} \text{ erg s}^{-1}$. Figure 11 also shows the limiting luminosity for a steady-state configuration (the Burrows–Goshy limit), calculated as described in Section 3. The resulting curve agrees qualitatively with the results of Burrows & Goshy (1993), Yamasaki & Yamada (2005, 2006), and Pejcha & Thompson (2012). Note that the neutrinospheric temperatures, neutrino opacities, and EOS employed here differ from what was used in those studies, hence numerical values are expected to differ.

The calculation of this limiting luminosity in a consistent manner with the microphysics, initial conditions, and boundary conditions employed in our simulations allows direct testing of the hypothesis that the critical stability threshold for explosion is given by the Burrows–Goshy limit. We find that in all cases, the measured thresholds for both types of mode lie below this limiting value. When the neutrinospheric radius is held constant, the instability thresholds approach (but never equal) the Burrows–Goshy limit for increasing mass accretion rate. In the sequence that relates the neutrinospheric radius to the neutrino luminosity via the blackbody relation (Figure 11(d)), the instability thresholds have a nearly constant separation in luminosity over the entire range of accretion rates investigated.

The sequence with $R_s = 20 \text{ km}$ yielded unexpected results for $M \gtrsim 0.5 M_\odot \text{ s}^{-1}$, however. In this case there is neither instability nor runaway for all luminosities up to and including the limiting value, independent of numerical resolution. Our approximate instability criteria fail to predict this behavior, showing that additional constraints play a role in determining instability. Also, the fact that no explosion is found at the Burrows–Goshy limit shows that this luminosity is not an independent tracer of stability either. Increasing the neutrino luminosity above the limiting value (at constant shock radius) by $\sim 10\%$ causes the shock to readjust to a new equilibrium position, which is also stable. It is worth noting however that the shock radius is $R_s \lesssim 61 \text{ km}$ for the non-exploding segment of this sequence. This is an unrealistically low value, with advection times of the order of $\sim 3 \text{ ms}$, a factor of a few from the dynamical time. It is likely that the fixed value of $r_{sc}$ used to set the upstream velocity (Section 2.3) leads to these extreme conditions.

In all other sequences, the approximate instability criteria found in Section 4.3 provide a good description of the measured values over a wide region of parameter space, with agreement better than $5\%$ in $L_{\nu}$. They correctly capture the disappearance of the oscillatory mode in the $R_s = 30 \text{ km}$ sequence at high accretion rates, where only a non-oscillatory mode is measured. This phenomenon occurs because at large accretion rates, the shock radius becomes increasingly smaller. The size of the resulting gain region is such that the advection time through it never exceeds $t_{\text{adv}, -e}$ before the heating time drops below $t_{\text{adv}, -g}$ due to the increasing luminosity. The deviation of the measured oscillatory threshold from the approximate criterion
at high accretion rate in the $R_e = 40 \text{ km}$ sequence is due to $r_e$ not being a good tracer of the point where pressure and density perturbations are $180^\circ$ out-of-phase (Section 4.3).

Our results extend the findings of Yamasaki & Yamada (2007), who obtained $\ell = 0$ instability thresholds below the Burrows–Goshy limit, to a wider region of parameter space. Our results are qualitatively different from theirs, partly due to the different neutrinospheric temperatures that they employed ($T_{\nu_e} = T_{\bar{\nu}_e} = 4.5 \text{ MeV}$), their use of the pure absorption coefficient for computation of the optical depth (see Sections 2.3 and 5.2), and their inclusion of self-gravity. Using the same parameters as Yamasaki & Yamada (2007), we obtain a limiting luminosity of $L_{\nu_e,\nu_{\bar{\nu}}} = 7.36 \times 10^{52} \text{ erg s}^{-1}$ for $M = 1 M_\odot \text{ s}^{-1}$, which differs by a factor of nearly two and is closer to what was found originally by Burrows & Goshy (1993). The corresponding approximate instability criteria for oscillatory and non-oscillatory modes are $6.41 \times 10^{52} \text{ erg s}^{-1}$ and $7.04 \times 10^{52} \text{ erg s}^{-1}$. These values are lower than our limiting luminosity by 13% and 5%, respectively.

5.2. Other Parameter Dependencies

Given the parametric character of this study, certain choices had to be made in order to construct a background flow that is as realistic as possible. Here, we explore how our results depend on numerical resolution, boundary conditions, parameters that determine the upstream flow, and the suppression of source terms near the neutrinosphere.

Figure 12(a) shows the approximate instability criteria and Burrows–Goshy limit for our fiducial sequence, together with instability thresholds measured from simulations at different resolutions. The grid is chosen logarithmically spaced, with a ratio of spacing between adjacent cells $\Delta z = (R_{\min}/R_\max)^{1/N_r}$ and $\Delta r_{\min} = R_{\max}(\zeta - 1)$, with $\Delta r_{\min}$ the cell adjacent to the inner boundary (Appendix B). Oscillatory modes asymptote to the theoretical threshold for increasing resolution, albeit convergence is non-monotonic. Non-oscillatory modes converge monotonically to a slightly different value, indicating that the instability criterion is indeed approximate.

The results shown in Figure 12 also indicate that previous hydrodynamic studies that used similar methods to solve the hydrodynamic equations (Murphy & Burrows 2008; Nordhaus et al. 2010; Hanke et al. 2011) have enough resolution to capture the critical stability thresholds correctly. However, we have found in our models that there are noticeable differences in the growth rates of oscillatory modes as a function of resolution. For the model with $L_{\nu_e,52} = 4.1$ in our fiducial sequence,
Figure 12. Dependence of the approximate instability thresholds and Burrows–Goshy limit on various parameters, taking our fiducial sequence as baseline. Solid lines denote the approximate instability condition for oscillatory (blue, Equation (24)) and non-oscillatory modes (red, Equation (28)), while the black solid line denotes the Burrows–Goshy limit. Panel (a) shows simulation results (as data points) as a function of radial resolution in a logarithmic grid from 30 km to 1000 km. Other panels show (b) the dependence of these critical points on the upstream electron fraction, (c) upstream entropy, (d) radius at which upstream velocity is set to the free-fall speed, and (e) density cutoff for source terms (Equation (14)). Parameters of our fiducial sequence are denoted by a black dashed line.

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Figure 13. Evolution of the shock radius in selected models of our fiducial sequence using different prescriptions for the inner boundary condition. Panel (a) shows models with different neutrino luminosities using the outflow boundary condition (Equations (12) and (13)). The curves shown bracket the instability threshold for oscillatory (black and solid blue) and non-oscillatory (dashed blue and red) modes. Except for the boundary condition, parameters are identical to those in Figure 4. Panel (b) compares two runs with the same parameters except the inner boundary condition. Both models are stable and do not explode (note the vertical scale).

(A color version of this figure is available in the online journal.)

to prevent the shocked envelope from collapsing (Section 2.4). Figure 13(a) shows that the system still goes through the same stability phases as the neutrino luminosity is increased, with a small quantitative difference $\lesssim 5\%$ in the critical stability points relative to our fiducial sequence. We thus conclude that the two types of instability do not depend on the fluxes of mass, momentum, and energy leaving the domain being fixed. The quantitative differences can be explained by the fact that the outflow boundary condition acts as a persistent source of waves, keeping shock oscillations at some non-zero amplitude. This effect is shown in Figure 13(b), which compares the shock radius evolution for two stable runs with identical parameters except for the boundary condition.

This small quantitative difference, caused by a persistent wave source, is interesting as it provides an illustration of what we expect will occur in the multidimensional case. The critical stability thresholds are modified in the presence of this additional wave generation, as comparison of Figures 4 and 13 shows. Taking the time average of two stable runs with differing boundary conditions after they have settled into a steady or a quasi-steady state shows that the mean velocity profiles differ (Figure 14). The magnitude of the change in the threshold luminosities is consistent with the magnitude and direction of these changes in the approximate instability criteria, using the mean advection times rather than the unperturbed ones. A stronger source of wave energy (e.g., convection) will affect the mean advection time in a stronger way, and critical stability points will move accordingly.
steady-state

opacity (Section 2.3) in Equation (8). Again, the three curves
pure absorption, effective, or total (absorption plus scattering)
in Figure 15(b). Here, we compare the effects of using the
rates.

sen seems to matter the least when going to low mass accretion
fixed accretion rate. Interestingly, the exact optical depth cho-
can also be seen from the non-overlapping curves in Figure 2 for
optical depth (Equation (8)) by a factor of two. Neither of these
shows the effect of increasing or decreasing the neutrinospheric
quantities used to calculate the latter are changed. Figure 15(a)
tween instability thresholds and Burrows–Goshy limit when the
thresholds and the Burrows–Goshy limit persist without qual-
itive changes. All other parameters produce changes smaller
than ∼5% over a large range of values.

We have also examined the robustness of the hierarchy be-
tween instability thresholds and Burrows–Goshy limit when the
quantities used to calculate the latter are changed. Figure 15(a)
shows the effect of increasing or decreasing the neutrinospheric
optical depth (Equation (8)) by a factor of two. Neither of these
changes cause the Burrows–Goshy limit or instability thresh-
olds to cross each other, for a given optical depth. This result
can also be seen from the non-overlapping curves in Figure 2 for
fixed accretion rate. Interestingly, the exact optical depth chos-
en seems to matter the least when going to low mass accretion
rates.

The dependence on the choice of neutrino opacity is shown
in Figure 15(b). Here, we compare the effects of using the
pure absorption, effective, or total (absorption plus scattering)
opacity (Section 2.3) in Equation (8). Again, the three curves
maintain their relative hierarchy and do not cross for a given
choice of opacity.

5.3. Relation between The Burrows–Goshy Limit and The
Instability Thresholds

We have found that the difference between the Burrows–
Goshy limit and the radial instability thresholds is finite and
measurable, and is independent of parameter choices. However,
the question remains as to why this limiting luminosity is
always close to the threshold for non-oscillatory instability in
the parameter region relevant to core-collapse supernovae.

In a more extended parameter space study of steady-state
configurations, Pejcha & Thompson (2012) found that at the
Burrows–Goshy limit, the ratio between $t_{\text{adv}}/t_{\gamma}$ and the global
heating timescale (without gravity) is always close to unity.

![Figure 14](image1.png)

**Figure 14.** Time averaged radial velocity as a function of radius for two stable models in our fiducial sequence with the same parameters except the inner boundary condition. The time average is taken after the models have settled into a steady or a quasi-steady state.

(A color version of this figure is available in the online journal.)

Figures 12(b)–(e) illustrate the sensitivity of the approximate
instability thresholds and the Burrows–Goshy limit to the pa-
parameter choices made in Section 2.3. Shown is the dependence
on the upstream electron fraction, upstream entropy, radius at
which the upstream flow has free-fall speed, and cutoff density
for neutrino source terms (Equation (14)). There is a some-
what sensitive dependence on the upstream entropy, which in a
realistic situations will be set by the entropy in the progen-
itor core. Note that larger luminosities are needed to cause
stalled shocks to become unstable when the progenitor entropy
is smaller. Nonetheless, the relation between the approximate
thresholds and the Burrows–Goshy limit persists without qual-
itative changes. All other parameters produce changes smaller
than ∼5% over a large range of values.

Ultimately, this proximity must depend on the characteristic
magnitude of the neutrino opacities, the gravitational binding
energy around a forming neutron star, and the fact that the flow
is supported against gravity by the thermodynamic pressure of
the gas (e.g., not centrifugal forces). In other words, it could
be just a result of dimensionality. A possible way to test this
hypothesis is to explore the behavior of the system at lower
mass accretion rates, where the expected trend is an increasing
deviation.

6. SUMMARY AND DISCUSSION

We have investigated the transition to runaway expansion
of a stalled core-collapse supernova shock in spherical sym-
metry, when the parameters that describe the accretion flow
are varied systematically. A realistic EOS and weak interac-
tions were employed to perform time-dependent simulations
with FLASH3.2. Starting from steady-state solutions to the hy-
drodynamic equations, we evolve sequences of time-dependent
models with increasing neutrino luminosity and analyze the hy-
drodynamic processes that mediate the transition from accretion
to runaway expansion. Our findings can be summarized as fol-

ds.

1. The onset of radial instability is a sufficient condition for
runaway expansion when heating by the accretion lumi-
nosity is ignored and neutrinospheric parameters remain
constant in time. Radial instability can manifest itself via

![Figure 15](image2.png)

**Figure 15.** Approximate instability thresholds and Burrows–Goshy limit as a function of mass accretion rate for different optical depths (top) and different
prescriptions for the neutrino opacity (bottom). The three curves remain
separated.

(A color version of this figure is available in the online journal.)
oscillatory and non-oscillatory modes, as found in the linear stability analysis of Yamasaki & Yamada (2007) and numerous time-dependent hydrodynamic studies. These modes are non-adiabatic, as they involve changes in the heat content of the fluid (Section 4.2).

2. For both types of modes, transition to runaway expansion occurs after a portion of the fluid in the gain region achieves positive energy. This coincides with the fluid just below the shock achieving positive velocity, starting the phase of runaway expansion (Section 4.4; Figures 9 and 10).

3. The only significant source of dissipation in our models is the energy loss from shock motion (Equation (23); Figures 5 and 7), which provides damping on expansion. Nuclear recombination and a steady source of neutrino heating combine to ensure that this term never extinguishes the runaway or saturate the linear instability. In a more realistic context, the dominant dissipative processes are the decrease of the core neutrino luminosity with time and self-consistent heating by the accretion luminosity. The latter is expected to provide a negative feedback during expansion due to the decrease of the mass accretion rate (e.g., Janka & Müller 1996). The contraction of the protoneutron star is a somewhat smaller correction, of the order of $\sim 10\%$. None of these effects is included in this study (Section 4.5).

4. We have found approximate instability criteria for oscillatory and non-oscillatory modes that correctly describe the behavior of the system over a wide region of parameter space, with a precision better than 5% in neutrino luminosity (Section 4.3; Figure 11). For oscillatory modes, instability arises when the advection time over the gain region becomes longer than the advection time from the gain radius to the point where the pressure and density perturbations are $180^\circ$ out of phase (Figures 6 and 8). In our implementation, this point lies at a position in the background flow where the cooling has decreased by an $e$-folding from its peak value. This equality of advection times is equivalent to an equality of masses of the respective regions when the accretion rate is constant. We have not been able to identify a conclusive physical reason for why this condition on the masses or advection times triggers an oscillatory instability. We surmise that a relation might exist between the fraction of the oscillation cycle that the fluid gains energy and the phase lag criterion in stellar pulsation theory (e.g., Cox 1974).

5. Non-oscillatory modes become unstable when the advection time through the gain region becomes longer than the integrated total energy divided by the integrated net heating in the gain region. This condition means that heating is effective at increasing the thermal energy while the fluid transits the gain region, increasing the pressure and causing the shock to adjust to a new equilibrium position (Janka & Keil 1998). This global criterion has been used by a number of previous studies as an explosion diagnostic (Buras et al. 2006a; Murphy & Burrows 2008; Marek & Janka 2009).

6. The instability thresholds are in general different from the limiting luminosity for the steady-state solutions (Burrows & Goshy 1993, Section 3). For constant neutrinospheric radius and increasing accretion rates, the thresholds asymptote to the limiting luminosity, but never coincide with it (Figure 11). This separation does not depend on how this limiting luminosity is calculated (Figure 15) or on specific parameter choices (Figure 12).

7. The existence of a limiting luminosity for steady-state solutions is a direct consequence of requiring the optical depth between the neutrinosphere and the shock to have a fixed value (Figure 2). For fixed upstream conditions and increasing neutrino luminosity, the entropy increases and hence the radius at which the pressure transitions from being dominated by relativistic particles to being dominated by non-relativistic nucleons moves inward (Figure 3). This causes the density profile to soften at fixed radius, resulting in a lower density at the neutrinosphere. Above a certain limit, there is not enough mass to provide sufficient neutrino optical depth to satisfy the closure relation, and no steady-state solution is possible. This result is equivalent to that of Pejcha & Thompson (2012), differing only in which boundary conditions are assumed to be fulfilled.

8. We find neither instability nor explosion in our sequence with constant $R_s = 20\ km$ for $\dot{M} \geq 0.5 M_\odot\ s^{-1}$ and luminosities up to and including the limit luminosity (Figure 11(b)). At the lower end of these mass accretion rates, the shock radius is $R_s \geq 60\ km$ and decreases for larger accretion rates. We did not find an explanation for this result within our framework. Regardless of the reason, however, it shows that the transition to runaway expansion does not necessarily occur at the limiting luminosity. It also shows that the approximate instability criteria derived here are incomplete, as an additional constraint is likely to determine the stability of the flow.

Our results show that the Burrows & Goshy (1993) conjecture, relating the transition to explosion to a global instability of the shocked envelope, is correct within a restricted set of assumptions. The critical stability surface, however, is not given by the limiting luminosity of the steady-state configuration. This limiting luminosity remains nonetheless close to the instability threshold over a large section of the parameter space relevant to core-collapse supernovae, a result that we have not accounted for and which deserves further investigation.

Murphy & Burrows (2008) and Fernández & Thompson (2009a) have also found oscillations around the transition to explosion. However, the width in neutrino luminosity separating from non-exploding configurations is larger in those studies. In the case of Fernández & Thompson (2009a), this can be attributed to the differences in the employed microphysics relative to the present study. Large amplitude oscillations are excited in the simulations of Murphy & Burrows (2008) when the Si/O composition interface is accreted through the shock. Stable models damp oscillations, and unstable modes explode, with neutrally stable oscillations confined to a range $\Delta L_{\nu}/L_{\nu} \simeq 4\%$. This range is wider than that found here, presumably because Murphy & Burrows (2008) include the contraction of the protoneutron star. Our results are in good agreement with those of Ohnishi et al. (2006), who used essentially the same physical assumptions as we do.

The main result of this paper, point (1) above, does not apply to all modes of higher dimensionality. Non-radial oscillatory instability of the shock (the SASI) does not necessarily lead to explosion when the same set of assumptions adopted in this study is employed (Ohnishi et al. 2006). Given our results and those of Yamasaki & Yamada (2007), the question then arises as to whether a multidimensional explosion can be thought of as the excitation of an unstable radial-oscillatory and/or non-oscillatory mode (radial or otherwise) by the action of turbulent stresses from the SASI and convection. In this case, the excited mode would arise from a background state that differs from...
the laminar steady-state solution by the presence of a turbulent pressure term in the momentum equation and a convective flux term in the energy equation. The excited mode would also become unstable at a lower neutrino luminosity than in the spherically symmetric flow.

According to Yamasaki & Yamada (2007), the non-radial-non-oscillatory modes that have the lowest threshold luminosity for instability have Legendre indices $\ell \sim 6$. These modes are likely to be associated with convection in the gain region, as they have no unstable oscillatory counterpart. The results of Ohnishi et al. (2006) and Iwakami et al. (2008) show that small-scale convection in itself does not lead to runaway expansion. In contrast, an $\ell = 1$ or $\ell = 2$ non-oscillatory mode, such as that envisioned by Thompson (2000), is likely to be behind unipolar or bipolar explosions seen in axisymmetric simulations (e.g., Scheck et al. 2006). The results of Yamasaki & Yamada (2007) show that the threshold luminosities of these large-scale non-oscillatory modes are larger than that of $\ell \geq 3$ modes and very close to that of the spherical oscillatory mode.

Which of these modes is excited first in a multidimensional context will obviously depend on the nature of the new background flow. The exploding three-dimensional models of Nordhaus et al. (2010) and Hanke et al. (2011) lack a dominant oscillatory mode in their transition to explosion, even though oscillations in the average shock position are still visible (particularly for the $L_{\nu_e} = 9 \times 10^{53}$ erg s$^{-1}$ model in the 11.2 $M_\odot$ progenitor sequence of Hanke et al. 2011). In contrast, two-dimensional models have a noticeable radial oscillatory component, with successive dips in the average shock radius before and during runaway expansion. Yet the amplitude of these oscillations is smaller than in the spherically symmetric case, suggesting that more than one mode is likely to be involved.

Support for this interpretation of multidimensional explosions can be found in the results of Buras et al. (2006a) and Marek & Janka (2009), who use the same definition of $t_{\text{heat-g}}$ as we do here. They find that the ratio $t_{\text{adv-g}}/t_{\text{heat-g}}$ exceeding unity corresponds roughly to the time when the shock begins its expansion toward explosion in two-dimensional runs. Obviously, all the saturation mechanisms discussed in Section 4.5 are present in those simulations, so analogies need to be made cautiously. A similar result is obtained by Murphy & Burrows (2008), with a heating time defined without gravity and kinetic energy.

An attempt to include the effects of convection in the steady-state solution was made by Yamasaki & Yamada (2006), setting the convective flux to a value that yielded a flat entropy gradient. With this maximally efficient convection, they found that the Burrows–Goshy limit can decrease by several tens of percent from the radial direction. In Equation (A3), $\sigma_{SB}$ is the Stefan–Boltzmann constant.

To approximate the results of radiation-hydrodynamic simulations with our simplified assumptions, we set $L_{\nu_e} = L_{\nu_\mu}$ but specify different temperatures, $T_{\nu_e} = 4$ MeV and $T_{\nu_\mu} = 6$ MeV, reflecting the fact that the photospheres are at slightly different locations, but with a spectrum such that luminosities are roughly comparable (Janka 1995, 2001). We ignore however the correction due to the spatial difference between the electron neutrino- and antineutrinospheric radius, which is of the order of 10%. This difference is absorbed by the normalization factor in Equation (A3).

The evolution of $Y_e$ above the neutrinosphere is determined by the net production rate of electron and positrons $\Gamma_{e^-}$ and...
\[ \Gamma_{\text{net}} = \Gamma_e - \Gamma_{\nu} \tag{A5} \]
\[ \Gamma_e = \frac{2 \pi m_e c}{(hc)^3 \rho} \int d \cos \theta k \int e^2 d e [\kappa_{\nu e} f_{\nu e} - j_{\nu e} (1 - f_{\nu e})] \tag{A6} \]
\[ \Gamma_{\nu e} = \frac{2 \pi m_e c}{(hc)^3 \rho} \int d \cos \theta k \int e^2 d e [\kappa_{\nu e} f_{\nu e} - j_{\nu e} (1 - f_{\nu e})], \tag{A7} \]

where \( j_{\nu e} \) and \( \kappa_{\nu e} \) are the emissivity and absorption coefficient, respectively, associated with electron-type neutrinos or antineutrinos (as subscripted), and the integrals are performed over all propagation angles and positive energies. Expressions for these coefficients are obtained by Bruenn (1985) assuming detailed balance, matter in nuclear statistical equilibrium, and non-relativistic nucleons, whose recoil is neglected. In addition, we ignore here the nucleon phase space blocking factors, which deviate only slightly from unity at densities \( \lesssim 10^{11} \text{ g cm}^{-3} \) (Bruenn 1985), yielding

\[ j_{\nu e} (\epsilon, T, \mu_e, n_p) = \frac{G_F^2}{\pi} (g^2_{\nu e} + 3 g^2_{\nu e A}) n_p F_{\text{FD}} (\epsilon, T, \mu_e) [\epsilon + \Delta_m]^2 \left[ 1 - \frac{m_e^2 c^4}{(\epsilon + \Delta_m)^2} \right]^{1/2} \tag{A8} \]

\[ j_{\bar{\nu} e} (\epsilon, T, \mu_e, n_p) = \frac{G_F^2}{\pi} (g^2_{\bar{\nu} e} + 3 g^2_{\nu e A}) n_p F_{\text{FD}} (\epsilon - \Delta_m, T, -\mu_e) \times [\epsilon - \Delta_m]^2 \times \left[ 1 - \frac{m_e^2 c^4}{(\epsilon - \Delta_m)^2} \right]^{1/2} \times \Theta(\epsilon - \Delta_m - m_e c^2) \tag{A9} \]

\[ \kappa_{\nu e} (\epsilon, T, \mu_e, n_p) = \frac{G_F^2}{\pi} (g^2_{\nu e} + 3 g^2_{\nu e A}) n_n [1 - F_{\text{FD}}] \times (\epsilon, T, \mu_e) [\epsilon + \Delta_m]^2 \times \left[ 1 - \frac{m_e^2 c^4}{(\epsilon + \Delta_m)^2} \right]^{1/2} \tag{A10} \]

\[ \kappa_{\bar{\nu} e} (\epsilon, T, \mu_e, n_p) = \frac{G_F^2}{\pi} (g^2_{\bar{\nu} e} + 3 g^2_{\nu e A}) \times n_p [1 - F_{\text{FD}} (\epsilon - \Delta_m, T, -\mu_e)] [\epsilon - \Delta_m]^2 \times \left[ 1 - \frac{m_e^2 c^4}{(\epsilon - \Delta_m)^2} \right]^{1/2} \times \Theta(\epsilon - \Delta_m - m_e c^2), \tag{A11} \]

where \( \mu_e \) is the chemical potential of electrons, \( n_n \) and \( n_p \) are the number density of free neutrons and protons, respectively, \( G_F = G_F / (hc)^3 \) is the Fermi constant, \( g_{\nu e} \) and \( g_{\nu e A} \) are the vector and axial coupling constants, respectively, and \( \Delta_m = (m_n - m_p)c^2 \) is the difference between the rest mass energy of neutrons and protons.

The rates of energy exchange between neutrinos and matter are similarly found as a function of \( f_{\nu e}, j_{\nu e}, \) and \( \kappa_{\nu e} \). The heating and cooling rates per unit volume are given, respectively, by

\[ \mathcal{L}_H = \frac{\rho}{m_n} (Q_{\nu e}^+ + Q_{\bar{\nu} e}^+), \tag{A12} \]
\[ \mathcal{L}_C = \frac{\rho}{m_n} (Q_{\nu e}^- + Q_{\bar{\nu} e}^-), \tag{A13} \]

where \( Q_{\nu e}^\pm \) are rates per baryon of heating (+ superscript) and cooling (− superscript) due to electron-type neutrinos \( (\nu_e \text{ subscript}) \) and antineutrinos \( (\bar{\nu}_e \text{ subscript}) \). These rates are given by

\[ Q_{\nu e}^+ = \frac{2 \pi m_e c}{(hc)^3 \rho} \int d \cos \theta k \int e^2 d e [j_{\nu e} + \kappa_{\nu e}] f_{\nu e} \tag{A14} \]
\[ Q_{\nu e}^- = \frac{2 \pi m_e c}{(hc)^3 \rho} \int d \cos \theta k \int e^2 d e [j_{\nu e} + \kappa_{\nu e}] f_{\bar{\nu} e} \tag{A15} \]
\[ Q_{\bar{\nu} e}^+ = \frac{4 \pi m_e c}{(hc)^3 \rho} \int e^3 d e j_{\bar{\nu} e} \tag{A16} \]
\[ Q_{\bar{\nu} e}^- = \frac{4 \pi m_e c}{(hc)^3 \rho} \int e^3 d e j_{\bar{\nu} e}. \tag{A17} \]

Numerical calculation of Equations (A6) and (A7) and (A14)−(A17) involves tabulating Fermi–Dirac integrals as a function of \( T, \mu_e, \) and \( T_{\nu e} \), with other dependencies entering as global scaling factors. The left panel of Figure 16 shows the total cooling per baryon \( Q_{\text{cool}} = Q_{\nu e}^+ + Q_{\bar{\nu} e}^- \) as a function of temperature, for fixed \( Y_e \) and different densities. In the degenerate regime \( kT < |\mu_e| \), the cooling rates are nearly independent of temperature, whereas for higher temperatures they approach the asymptotic expression \( Q_{\text{cool}} \simeq 145 (kT/2 \text{ MeV})^6 \) from Janka (2001), obtained by assuming zero electron chemical potential. The right panel of Figure 16 shows heating by absorption of electron-type neutrinos as a function of \( T_{\nu e} \), for fixed parameters as shown in the figure. For reference, the approximate expression \( Q_{\text{heat}}^+ \simeq 160 L_{\text{SN}}^{1/2} r_{18}^{-2}(1 - Y_e) (kT_{\nu e}/4 \text{ MeV})^2 \text{ MeV s}^{-1} \) (Janka 2001) is also shown, agreeing with our results to within \( \lesssim 10\% \).

APPENDIX B

GRID OF VARIABLE SPACING IN FLASH3.2

Time-dependent modeling of post-bounce core-collapse supernova hydrodynamics demands the ability to resolve a steep density gradient in the proto-neutron star atmosphere (\( \sim 30 \) km) while also allowing for the shock to expand out to at least \( \sim 1000 \) km to track an explosion. For a grid in spherical coordinates, an efficient way to accomplish this is to use variable spacing in radius.

The public version of FLASH3.2 was modified to include this capability, starting from the existing uniform grid mode. The implementation can be decomposed into two parts. First, defining the cell sizes and coordinates appropriately when the computational domain is initialized. The second part involves modifying all the subroutines that assume a uniform grid spacing.

Grid initialization is accomplished by modifying the Grid_init and gr_create_domain subroutines. We define...
the grid points in between \( r_{\min} \) and \( r_{\max} \) such that consecutive cell sizes have a ratio \( \Delta r_{i+1}/\Delta r_i = \zeta > 1 \), where \( i \) is a cell index that increases with increasing radius. For a given number of grid cells, \( N_r \), the ratio can be obtained by solving (e.g., Stone & Norman 1992)

\[
\frac{r_{\max} - r_{\min}}{\Delta r_{\min}} = \zeta \sum_{i=0}^{N_r-1} \zeta^i = \Delta r_{\min} \left[ \frac{\zeta^{N_r} - 1}{\zeta - 1} \right],
\]

where \( \Delta r_{\min} \) is the minimum cell size, whose inner edge is located at \( r_{\min} \). A logarithmic spacing can be achieved by setting \( \zeta = (r_{\max}/r_{\min})^{1/N_r} \), which then determines the minimum cell size: \( \Delta r_{\min} = r_{\min} (\zeta - 1) \). It then follows that for \( 0 \leq q \leq N_r \)

\[
\frac{\Delta r_q}{r_q} = \frac{\zeta^q}{(r_{q}/r_{\min})^{1/N_r}} = \frac{\Delta r_{\min}}{r_{\min}},
\]

with \( r_0 = r_{\min} \) and \( r_{N_r} = r_{\max} \).

There are only three subroutines that make the assumption of a uniform grid when using spherical coordinates. They are \( \text{hy}_\text{ppm}_\text{sweep} \), \( \text{Driver}_\text{computeDt} \), and \( \text{Driver}_\text{verifyInitDt} \). In all three cases, a scalar cell spacing was replaced with a vector in the appropriate locations. The subroutines that compute cell areas and volumes make direct use of coordinate information, so no additional modification is required for spherical coordinates.

As a test of the grid implementation, we have run the self-similar explosion problem of Sedov (1982). An energy \( E_0 \) is initially placed inside some spherical volume with a radius smaller than some characteristic radius \( R_0 \). The medium has uniform density \( \rho_0 \) and has an ideal gas EOS with adiabatic index \( \gamma \). Outside of the injection volume, the pressure is uniform and equal to \( 10^{-5} E_0 R_0^{-5} \). Our benchmark simulation has 500 cells uniformly spaced in radius from \( r = 0 \) to \( r = R_0 \). The energy is placed in the first grid cell, with radius \( r_{\text{inj}} = 2 \times 10^{-3} R_0 \), and we set \( \gamma = 1.4 \).

We then run a sequence of simulations that preserve the minimum cell spacing next to the origin, \( \Delta r_{\min} = r_{\text{inj}} \), but vary the number of cells, resulting in a ratioed (not logarithmic) grid that satisfies Equation (B1). The number of cells are \( N_r = \{466, 400, 300, 180\} \) which results in \( (\zeta - 1) \approx \{3 \times 10^{-4}, 10^{-3}, 3 \times 10^{-3}, 10^{-2}\} \), respectively. The left panel of Figure 17 shows the shock radius (obtained by linear interpolating the location of the surface with pressure \( 10^{-3} E_0 R_0^{-3} \)) as a function of time for the sequence of runs. As the number of cells is decreased relative to the benchmark model, the cells in the upper part of the domain become increasingly coarser, and the shock trajectory gradually diverges from the uniform grid case. The right panel of Figure 17 shows the fractional deviation from the uniform grid result at a time \( 0.75 E_0^{-1/2} \rho_0^{1/2} R_0^{5/2} \), with the error bars indicating the fractional size of the cell at the given shock position, normalized to the shock position in the benchmark model. As the spacing ratio decreases, deviations tend to zero close to linearly in \( (\zeta - 1) \).

APPENDIX C

NUMERICAL CALCULATION OF THE RATE OF CHANGE OF ENERGY

Here, we describe the calculation of the different terms that comprise the rate of change of the total energy with time (Equation (19)) for Figures 5 and 7. In the Piecewise Parabolic Method (Colella & Woodward 1984) used in FLASH, the shock is typically broadened along 2–3 cells. To compute an accurate shock position, we begin by finding the minimum in the radial derivative of the pressure. We establish a reference position by taking a weighted average of the radial coordinate, with weight equal to \( |\partial p/\partial r| \) around the minimum. This reference position, which varies smoothly with time, is a good tracer of the center of the shock. Because we want quantities below the shock, we define our actual shock position to be a fixed distance (of the order of two cell widths) behind the center of the shock, so that the fluid quantities are...
in the post-shock regime while remaining as close to the shock center as possible.

Next, the energy flux entering through the shock \( E_{up} \) (Equation (21)) is computed by interpolating variables (linearly for velocity and internal energy; logarithmically for pressure and density) to the shock position. The energy flux leaving the domain \( E_{dn} \) (Equation (22)) is computed at a radius \( R_{out} \) corresponding to the inner edge of some cell inside the domain. After some experimentation, we found that the innermost location that is not significantly affected by boundary effects is the third active cell from the inner boundary at \( R_{n} \). To compute the energy flux at \( R_{n} \), we average the fluid variables from the cells that this radius separates as an approximation to face-centered values. The integrated neutrino source terms \( E_N \) (Equation (20)) are computed by simple integration of the cells fully contained between \( R_n \) and the shock, and then a small differential term is added by linear interpolation, \( 4\pi R^2 \frac{\rho}{\rho_0} \frac{\eta}{\eta_0} \Delta R \), where \( R_{out} \) is the outer radius of the cell that is closest to and fully contained within the shock surface, and \( \Delta R = R_S - R_{out} \). The shock motion term \( E_S \) is found by first computing the shock velocity through time-centered finite differences and then evaluating fluid quantities at the same position as for \( E_{up} \).

For calibration, we also compute the total energy contained between \( R_n \) and the shock. The total rate of change, computed through time-centered finite differences, and the sum of the separate terms that make it up (Equation (19)) are shown in Figure 18. Quantities are sampled every 2 ms to eliminate high frequency noise. Agreement is very good if a zero point, equal to the sum of all the terms that make up \( E_{tot} \) at \( t = 0 \), is subtracted.

Also shown in Figure 18 is the net energy generation from accretion, \( E_{N} - E_{up} + E_{dn} \), together with what is obtained if the shock motion term \( E_s \) is subtracted from the total rate of change of the energy \( E_{tot} \), computed from finite differencing the total energy. Agreement is again very good if the same zero point is subtracted. This smoother version of the accretion energy generation, together with \( E_{tot} \) from finite differencing, is what is shown in Figures 5 and 7.

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