A PEDESTRIAN FLOW MODEL WITH STOCHASTIC VELOCITIES: MICROSCOPIC AND MACROSCOPIC APPROACHES

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ABSTRACT. We investigate a stochastic model hierarchy for pedestrian flow. Starting from a microscopic social force model, where the pedestrians switch randomly between the two states stop-or-go, we derive an associated macroscopic model of conservation law type. Therefore we use a kinetic mean-field equation and introduce a new problem-oriented closure function. Numerical experiments are presented to compare the above models and to show their similarities.

1. Introduction. The modeling of crowd dynamics is a current research topic and provides a useful tool for evacuation planning. A good overview of the existing literature can be found in [4, 9], where mainly two classes of modeling approaches (microscopic versus macroscopic) are distinguished. Microscopic pedestrian models typically rely on Newton-type dynamics as proposed e.g. in [8, 16, 24] or cellular automata e.g. in [7, 25] while macroscopic models can be either derived via limiting processes [8, 6, 9, 21] or phenomenologically, see e.g. [8, 15, 17, 24, 23]. Starting from a microscopic level model extensions include for example vision cones [11], shortest-path information [13] and stochastic velocities modeled with diffusion [9, 13, 26, 27].

In this work, we focus on a well-known phenomenon in crowds which is the sudden presence of non-moving people stopping immediately due to external attractions or, more recently, new cell phone messages. This leads to significant changes in the individual walking velocities, rerouting actions and hence to local bottlenecks depending on the crowd density. From reality we observe that people stop directly in front of attractions at a high probability and therefore influence other individuals also to stop or keep on walking. This random effect will be included as a stochastic process which is dependent on the position of the pedestrians, i.e., the rate to change the dynamics is space-dependent. In contrast to [12], where a model for self-alignment coupled to a compound Poisson process with constant rates and position dependent incremental distributions is proposed, we face the challenge that the space-dependent rates cause new problems in the limiting process. The main difference to existing pedestrian models with random effects is the
motivation mentioned above and the mathematical representation of the stochastic dynamics. Similar problems have been considered in e.g. [9, 13, 21], where a formulation of the dynamical system in terms of Brownian motion driven stochastic differential equations (SDEs) has been used to capture intrinsic random decisions in the velocity of pedestrians. However, this approach is not suitable to cover the stop-and-go behavior of pedestrians we have in mind. We will use a continuous-time Markov chain approach [10] to include the random stop-and-go behavior by considering location-dependent switching rates. The overall goal is then to derive a model hierarchy for the microscopic and macroscopic pedestrian flow. We mainly follow the ideas in [9, 13, 21] to derive the kinetic model. However, different to the already existing approaches, we need to derive an appropriate, non-standard closure distribution to fix the macroscopic model. The latter is a coupled system of two equations for the density of non-moving and walking pedestrians with non-local flux. In a first step, we develop and define the stochastic microscopic pedestrian model based on the social force model by Helbing and Molnár [16] and prove its existence, see section 2. Then, the kinetic formulation of the microscopic model under the molecular chaos assumption in terms of measures is considered. The derivation of a reasonable closure distribution to state the associated macroscopic pedestrian model is non-standard and requires some computational effort. Section 3 deals with the numerical treatment of the microscopic and macroscopic equations. We present a stochastic simulation algorithm for the microscopic model and explain how numerical methods for hyperbolic conservation laws can be used to approximate the solution of the macroscopic model. For simulation purposes we present two examples in section 4 and compare qualitatively both approaches by defining reasonable performance measures. An additional experiment with a deterministic situation, i.e., no random velocity drops to zero, will be also discussed.

2. The models. In this section, we derive a stochastic microscopic model and deduce a corresponding macroscopic model. First, we define the stochastic microscopic model based on a deterministic social force model [16] and show its well-posedness. Second, we determine the time-evolution of the microscopic model and use a mean field assumption to get a kinetic model. This model is then simplified by an appropriate closure assumption to develop a macroscopic model, cf. existing literature on particle systems, e.g. [5, 19, 20, 21] and pedestrian flow models in e.g. [9].

2.1. Microscopic model. We use the ideas introduced by Helbing and Molnár [16] to describe the dynamical behavior of pedestrians. First, we consider the deterministic model formulation which will be extended to the stochastic model by including the stop-and-go dynamics. Let $N \in \mathbb{N}$ be the number of pedestrians and let

$$x_i(t) = (x_i^{(1)}(t), x_i^{(2)}(t)) \in \mathbb{R}^2$$

the position of pedestrian $i$ at time $t \in \mathbb{R}_{\geq 0}$. The velocity of pedestrian $i$ at time $t$ is given by

$$\dot{x}_i(t) = V(x_i(t), v_i(t)) \in \mathbb{R}^2,$$

where we interpret $v_i(t)$ as the desired velocity if we neglect obstacles and walls. In the following, we denote by

$$\vec{x}(t) = (x_1(t), \ldots, x_N(t)) \in \mathbb{R}^{2N}$$

and

$$\vec{v}(t) = (v_1(t), \ldots, v_N(t)) \in \mathbb{R}^{2N}$$
the states of the system consisting of \( N \) pedestrians. The deterministic model equations are of Newton-type dynamics

\[
\begin{align*}
\dot{x}_i(t) &= V(x_i(t), v_i(t)), \\
\dot{v}_i(t) &= F^{\text{dest}}(x_i(t), v_i(t)) + F^{\text{int}}(x_i(t)),
\end{align*}
\]

with initial positions \( \vec{x}(0) = \vec{x}_0 \in \mathbb{R}^{2N} \) and desired velocities \( \vec{v}(0) = \vec{v}_0 \in \mathbb{R}^{2N} \). The acting forces are divided into the destination force \( F^{\text{dest}} \) and the interaction force \( F^{\text{int}} \). The determination of boundary and obstacle constraints is done using the function \( V \) adapted from swarming models [1, 2]. We remark that the equations (1) are non-standard in the sense that the second equation does not describe the realized velocity which is \( V(x_i(t), v_i(t)) \). The expression \( V(x_i(t), v_i(t)) \) will be explained more detailed in the paragraph on obstacle forces.

To keep the notation and calculations well-arranged, we keep the basic model simple and focus on the new modeling ideas.

**Interaction and Destination Forces.** As in [16], the interaction force acting on pedestrian \( i \) is

\[
F^{\text{int}}(\vec{x}(t)) := \frac{1}{N-1} \sum_{j=1}^{N} G(x_i(t) - x_j(t)),
\]

where \( G: \mathbb{R}^2 \to \mathbb{R}^2 \) is a given vector field describing the repulsion and attraction. Let \( v^C > 0 \) be the comfort speed which is achieved approximately in the relaxation time \( \tau > 0 \). We define the destination force by

\[
F^{\text{dest}}(x_i(t), v_i(t)) := \frac{1}{\tau} \left( v^C D(x_i(t)) - v_i(t) \right),
\]

where \( D: \mathbb{R}^2 \to \mathbb{R}^2 \) describes the direction to the destination of a pedestrian at position \( x_i(t) \). If \( x^D \in \mathbb{R}^2 \) is some destination point, then \( D \) can be expressed in the simplest case by

\[
D(x) = \frac{x^D - x}{\|x^D - x\|}.
\]

Note that alternative destination directions could be achieved by considering the shortest path between the current position \( x \) and the destination \( x^D \) by solving the Eikonal equation, see e.g. [13].

**Obstacle Forces.** Different to [16], we choose an alternative way to model the obstacles with forces by applying a kind of specular reflection, see [1, 2]. The reason for this choice is that singular forces can occur close to obstacles leading to numerical difficulties during the simulation of (1).

Let \( \Gamma \subset \mathbb{R}^2 \) be a domain with outer normal unit field \( \vec{n} \) which is continuous and defined on \( \Gamma \). The boundary \( \partial \Gamma \) describes walls as well as obstacles and the outer normal field allows to manipulate the desired velocity vector such that pedestrians walk along these boundaries. We assume a pedestrian very close to a wall at position \( x \in \Gamma \) having a desired direction given by the vector \( v \in \mathbb{R}^2 \) resulting the destination and interaction forces. If we neglect the wall, the position at a later time would be \( x + \Delta t v \) and might be outside of the domain, see figure 1.

This is the case if the vector \( v \) points out of the domain, i.e., \( v \cdot \vec{n} \geq 0 \). In the other case \( v \cdot \vec{n} < 0 \), the pedestrian walks into the domain and the velocity vector is admissible. Let \( d(x, \partial \Gamma) \) be the distance to the boundary and \( U_\epsilon = \{ x \in \mathbb{R}^2 : d(x, \partial \Gamma) \leq \epsilon \} \) the zone, where pedestrians feel uncomfortable close to a wall or
obstacle. We avoid the collision with the boundary by an orthogonal projection of the point $x + \Delta t v$ onto the tangent space at $x$ spanned by the orthogonal vector

$$\vec{n}^\perp(x) = (-n_2(x), n_1(x))^T$$

to the outer unit normal vector. To project the desired direction, we set

$$\tilde{v}(v, x) = \frac{|v|}{|v \cdot \vec{n}^\perp(x)|} \vec{n}^\perp(x) = |v| \text{sgn} (v \cdot \vec{n}^\perp(x)) \vec{n}^\perp(x)$$

in the case $v \neq 0$ such that $||\tilde{v}(v, x)|| = ||v||$. Due to the assumption that a pedestrian will not change the direction immediately at the obstacle, we take a sufficiently smooth increasing function $J: [0, \infty) \to [0, 1]$ with $J(0) = 0$ and $J(1) = 1$ and define

$$\hat{v}(v, x) = \tilde{v}(v, x) + J \left( \frac{d(x, \partial \Gamma)}{\epsilon} \right) (v - \tilde{v}(v, x)).$$

Summarizing, the velocity vector is given by

$$V(x, v) := \begin{cases} v & \text{if } v \cdot \vec{n}(x) < 0, \\ \hat{v}(v, x) \frac{|v|}{|v(v, x)|} & \text{if } v \cdot \vec{n}(x) \geq 0 \text{ and } v \neq 0, \\ 0 & \text{if } v = 0 \end{cases}$$

and satisfying $||V(x, v)|| = ||v||$. To guarantee that a pedestrian stays inside the domain during one time step, we have to require for the time step size $\Delta t$ that $\Delta t ||v|| < d(x, \partial \Gamma)$ holds. However, in most cases the velocity will be comparable or lower than the comfort velocity $v^C$ and the choice $\Delta t \ll \frac{d}{v^C}$ is sufficient to prevent a pedestrian at $x \in \Gamma \setminus U_\epsilon$ to exit the domain. The latter condition is used for all computational examples presented in section 4.

Random Velocity Effects. All forces (interaction, destination) and boundary constraints considered so far are purely deterministic and include no random effects. Different to pedestrian models modeled by SDEs, we choose an alternative way to include random effects. From phenomenological observations we deduce the intrinsic random effect that pedestrians can decide to walk or to stop independently of each other. This can be done by switching the velocity and acceleration to zero and back again. More precisely, an individual that stops immediately will start walking again after a random time.

To include this phenomenon mathematically we describe the positions, velocities and the status, i.e., stop or go, as a time discrete stochastic process. Let $\Delta t > 0$ be
a small fixed time stepsize and let $x_i^n \in \mathbb{R}^2$ the position, $v_i^n \in \mathbb{R}^2$ the velocity and $r_i^n \in \{0, 1\}$ the status of pedestrian $i$ at time $t_n = n\Delta t$ for $n \in \mathbb{N}_0$.

To ease the notation, we denote by $F_i = F_{\text{dest}} + F_{\text{int}}$ the sum of forces acting on pedestrian $i$.

We consider a stochastic process $X = (X^n, n \in \mathbb{N}_0)$ with $X^n = (\tilde{x}_i^n, \tilde{v}_i^n, r_i^n) \in E^{(N)} = \mathbb{R}^{2N} \times \mathbb{R}^{2N} \times \{0, 1\}^N$ on some probability space $(\Omega, \mathcal{A}, P)$ such that

\[ P(x_i^{n+1} = x_i^n + r_i^n \Delta t \tilde{V}(x_i^n, v_i^n)|X^n) = 1, \quad (2) \]
\[ P(v_i^{n+1} = v_i^n + \Delta t F_i(\tilde{x}_i^n, \tilde{v}_i^n)|X^n) = r_i^n, \quad (3) \]
\[ P(v_i^{n+1} = 0|X^n) = 1 - r_i^n, \quad (4) \]
\[ P(r_i^{n+1} = z|X^n) = \mathbb{1}_z(r_i^n)(1 - \Delta t \lambda(r_i^n, x_i^n)) + \mathbb{1}_{1-z}(r_i^n)\Delta t \lambda(r_i^n, x_i^n) \quad (5) \]

for all $i = 1, \ldots, N, n \in \mathbb{N}_0, z \in \{0, 1\}$ and some rate function $\lambda: \{0, 1\} \times \mathbb{R}^2 \to \mathbb{R}_0^+$. The rate function $\lambda$ describes the expected number of switches per unit time. To avoid doubling effects by the interaction forces and the effects arising from the rate function $\lambda$, the rate function is assumed to be only dependent on the position of the pedestrian $i$ itself. Equations (2)–(4) can be seen as an Euler approximation of the stochastic extension of (1) while equation (5) describes the probability to walk ($z = 1$) or to stop ($z = 0$) during the next time step $t_{n+1}$ given the values of the process at time $t_n$. This idea is adapted from [10], where a continuous-time Markov Chains is used to motivate a production model with random breakdowns.

Note that for the approach (2)–(5), we can only expect a well-defined model if $\Delta t \lambda(\cdot, \cdot) \in [0, 1]$. Therefore we assume $\lambda$ to be uniformly bounded and $\Delta t \leq \|\lambda\|_{-1}$. Furthermore, we have to fix an initial probability measure $\mu_0^{(N)}$ on the measurable space $(E^{(N)}, \mathcal{E}^{(N)})$, where we denote by $\mathcal{E}^{(N)} = \sigma(E^{(N)})$ the smallest $\sigma$-algebra containing $E^{(N)}$. Summarizing, we state the following definition.

**Definition 2.1.** A time discrete stochastic process $X$ on some probability space $(\Omega, \mathcal{A}, P_0^{(N)})$ with values in $E^{(N)} = \mathbb{R}^{2N} \times \mathbb{R}^{2N} \times \{0, 1\}^N$ and initial measure $\mu_0^{(N)}$ satisfying equations (2)–(5) is called a stochastic microscopic pedestrian model.

Now, we will show the well-posedness of the model. We define the mapping $U: E^{(N)} \times \mathcal{E}^{(N)} \to \mathbb{R}_0^+$ by

\[
U((\bar{x}, \bar{v}, \bar{r}), B) := \sum_{\bar{x} \in \{0, 1\}^N} \epsilon_{h(\bar{x}, \bar{v}, \bar{r}, \bar{z})}(B) \prod_{i=1}^N \left( \mathbb{1}_{z_i}(r_i)(1 - \Delta t \lambda(r_i, x_i)) + \mathbb{1}_{1-z_i}(r_i)\Delta t \lambda(r_i, x_i) \right)
\]

with vector valued function

\[
h(\bar{x}, \bar{v}, \bar{r}, \bar{z}) = \begin{pmatrix}
x_1 + r_1 \Delta t V(x_1, v_1) \\
\vdots \\
x_N + r_N \Delta t V(x_N, v_N) \\
r_1(v_1 + \Delta t F_1(\bar{x}, \bar{v})) \\
\vdots \\
r_N(v_N + \Delta t F_N(\bar{x}, \bar{v})) \\
\end{pmatrix}
\]
Here, we denote by $\delta_a$ the Dirac measure with unit mass centered on some fixed point $a$, cf. [3]. Then, well-posedness can be obtained using the Markovian kernel property.

**Theorem 2.2.** Let $\lambda, V, F_{\text{dest}}, F_{\text{int}}, i = 1, \ldots, N$ be measurable mappings, $\lambda$ uniformly bounded and $\Delta t \leq \|\lambda\|^{-1}_\infty$. Then, the mapping $U$ defined by (6) is a Markovian kernel on $(E^{(N)}, E^{(N)})$.

**Proof.** For simplicity, we identify $x$ as $(\vec{x}, \vec{v}, \vec{r})$ in the following. The first step of the proof is to show that for every set $B \in E^{(N)}$ the mapping $x \mapsto U(x, B)$ is measurable. In a second step, we have to prove that for every $x \in E^{(N)}$ the mapping $B \mapsto U(x, B)$ is a probability measure on $(E^{(N)}, E^{(N)})$. From the assumptions we get that $h$ is a measurable mapping and hence $x \mapsto \epsilon_h(x)(B)$ is measurable. The function $\lambda$ is also measurable and the sum as well as the product of (6) are finite such that $U(x, B)$ is measurable in $x$.

The mapping $U$ consists of a finite sum of weighted Dirac measures on $(E^{(N)}, E^{(N)})$ and the assumption $0 < \Delta t \leq \|\lambda\|^{-1}_\infty$ implies that weights must be non-negative. So $U(x, \cdot)$ is a measure on $(E^{(N)}, E^{(N)})$ and it remains to show that $U(x, E^{(N)}) = 1$ to complete the proof. We calculate

$$U(x, E^{(N)}) = \sum_{z \in \{0, 1\}^N} \left[ \prod_{i=1}^N (\mathbb{1}_{z_i} (r_i)(1 - \Delta t \lambda(r_i, x_i)) + \mathbb{1}_{1-z_i} (r_i)\Delta t \lambda(r_i, x_i)) \right]$$

$$= \sum_{z_1 = 0}^1 \cdots \sum_{z_{N-1} = 0}^1 \prod_{i=1}^{N-1} (\mathbb{1}_{z_i} (r_i)(1 - \Delta t \lambda(r_i, x_i)) + \mathbb{1}_{1-z_i} (r_i)\Delta t \lambda(r_i, x_i))$$

$$\cdot \sum_{z_N = 0}^1 (\mathbb{1}_{z_N} (r_N)(1 - \Delta t \lambda(r_N, x_N)) + \mathbb{1}_{1-z_N} (r_N)\Delta t \lambda(r_N, x_N))$$

$$= \sum_{z_1 = 0}^1 \cdots \sum_{z_{N-1} = 0}^1 \prod_{i=1}^{N-1} (\mathbb{1}_{z_i} (r_i)(1 - \Delta t \lambda(r_i, x_i)) + \mathbb{1}_{1-z_i} (r_i)\Delta t \lambda(r_i, x_i)) \cdot 1$$

$$= \cdots$$

$$= 1$$

and conclude that $U$ is a probability measure in the second component. \qed

The mapping $U$ describes the one step transition probability of the system and allows to define a Markovian semigroup of kernels. We choose the typical composition of Markovian kernels, see e.g. [3],

$$U \circ U(x, B) := \int_{E^{(N)}} U(x, dy)U(y, B)$$

and define

$$U^0 := \text{Id},$$

$$U^n := \bigcirc_{k=1}^n U \text{ for } n \in \mathbb{N}.$$  

The family $(U^n, n \in \mathbb{N}_0)$ is then a normal semigroup of Markovian kernels and we get the following well-posedness result.

**Theorem 2.3.** Let $\lambda, V, F_{\text{dest}}, F_{\text{int}}, i = 1, \ldots, N$ be measurable mappings, $\lambda$ uniformly bounded, $\Delta t \leq \|\lambda\|^{-1}_\infty$ and $\mu_0^{(N)}$ a probability measure on $(E^{(N)}, E^{(N)})$. Then,
there exists a stochastic microscopic pedestrian model. Furthermore, the distribution of \( N \) pedestrians at time step \( t_n \) is given by

\[
\mu^n_N(B) = \int_{E^{(N)}} U^n(x, B)\mu_0^0(dx)
\]

for every \( B \in \mathcal{E}^{(N)} \).

**Proof.** The state space \( E^{(N)} \) is a polish space and we hence can use the Daniell-Kolmogorov theorem [3] which guarantees the existence of a canonical coordinate process \( X = (X^n, n \in \mathbb{N}_0) \) on some probability space \((\Omega, \mathcal{A}, P^{n_{(N)}})\) defined by the semigroup of Markovian kernels \((U^n, n \in \mathbb{N}_0)\), cf. (9). The stochastic process \( X \) is a Markov process and therefore \( (10) \) and

\[
P^{n_{(N)}}(X^{n+1} \in B | X^n = x) = U(x, B)
\]

are satisfied. By setting \( x = (\vec{x}, \vec{v}, \vec{r}) \) and

\[
B = \{x_1 + r_1 \Delta t V(x_1, v_1) \times \mathbb{R}^{2(N-1)} \times \mathbb{R}^{2N} \times \{0, 1\}^N
\]

we obtain equation (2) in the case \( i = 1 \). The remaining equations (3)–(5) can be deduced analogously.

2.2. **Kinetic model.** We now analyze the evolution of measures given by the microscopic pedestrian flow model. The principal idea is to derive the Kolmogorov forward equation, see e.g. [14], given by the semigroup of Markovian kernels \((U^n, n \in \mathbb{N}_0)\). To do so, we introduce the following spaces of test-functions \( C^{(N)} \) and \( C \) to capture the discrete states of the pedestrians:

\[
C^{(N)} := \{ \phi: E^{(N)} \to \mathbb{R}: \phi(\vec{x}, \vec{v}, \cdot) \text{ is } \mathcal{P}(\{0, 1\}^N)/\mathcal{B}(\mathbb{R})\text{-measurable,}
\]

\[
\phi(\cdot, \cdot, \vec{r}) \in C^1_b(\mathbb{R}^{2N} \times \mathbb{R}^{2N}; \mathbb{R})
\]

and

\[
C := \{ \psi: E \to \mathbb{R}: \psi(x, v, \cdot) \text{ is } \mathcal{P}(\{0, 1\})/\mathcal{B}(\mathbb{R})\text{-measurable,}
\]

\[
\psi(\cdot, \cdot, r) \in C^1_b(\mathbb{R}^2 \times \mathbb{R}^2; \mathbb{R})
\}

where \( E = \mathbb{R}^2 \times \mathbb{R}^2 \times \{0, 1\} \). Here, \( \mathcal{P} \) denotes the power set, \( \mathcal{B}(\mathbb{R}) \) the Borel \( \sigma \)-algebra on \( \mathbb{R} \) and \( \mathcal{E} = \mathcal{P}(E) \) the smallest \( \sigma \)-algebra containing \( E \). We make the following assumption: For every \( n \in \mathbb{N}_0 \) there exists a probability measure \( \mu^n_{(1)} \) on \((E, \mathcal{E})\) such that for all \( B = B_1^x \times \cdots \times B_N^x \times B_1^v \times \cdots \times B_N^v \times B_1^r \times \cdots \times B_N^r \in \mathcal{E}^{(N)} \) with \( B_i^x \times B_i^v \times B_i^r \in \mathcal{E}, i = 1, \ldots, N \), we have

\[
\mu^n_{(N)}(B) = \prod_{i=1}^N \mu^n_{(1)}(B_i^x \times B_i^v \times B_i^r)
\]

This is the so-called molecular chaos assumption which forces the \( N \)-particle distribution being the product measure of a one particle distribution. In fact, for existing models it is shown that this is true as \( N \) tends to infinity, see e.g. [6]. To keep the derivation of the kinetic equation clearly arranged, we state the following lemma, where we use the common notation

\[
U\phi(\vec{x}, \vec{v}, \vec{r}) := \int_{E^{(N)}} \phi(\vec{y}, \vec{w}, \vec{s})U((\vec{x}, \vec{v}, \vec{r}), d(\vec{y}, \vec{w}, \vec{s}))
\]
Lemma 2.4.
1. We have the representation
\[
\prod_{i=1}^{N} [1 - \Delta t \lambda(r_i, x_i)] + \mathbb{1}_{1-z_i}(r_i) \Delta t \lambda(r_i, x_i)]
\]
\[
= \left[ 1 - \Delta t \sum_{j=1}^{N} \lambda(r_j, x_j) \right] \prod_{i=1}^{N} [1 - \Delta t \lambda(r_i, x_i)] + o(\Delta t)
\]
as \(\Delta t \to 0\).
2. For every \(\phi \in C^{(n)}\) it holds
\[
U\phi(\vec{x}, \vec{v}, \vec{r}) = \phi(\vec{x}, \vec{v}, \vec{r}) + \Delta t \sum_{j=1}^{N} \lambda(r_j, x_j)(\phi(\vec{x}, \vec{v}, \theta_j(\vec{r})) - \phi(\vec{x}, \vec{v}, \vec{r}))
\]
\[
+ \Delta t \sum_{j=1}^{N} r_j(\nabla_x \phi(\vec{x}, \vec{v}, \vec{r}) \cdot V(x_j, v_j) + \nabla_{v_j} \phi(\vec{x}, \vec{v}, \vec{r}) \cdot F_j(\vec{x}, \vec{v}))
+ o(\Delta t),
\]
where we set
\[
\vec{v} \equiv (v_1 r_1, \ldots, v_N r_N) \text{ and } \theta_j(\vec{r}) = (r_1, \ldots, r_{j-1}, 1 - r_j, r_{j+1}, \ldots, r_N).
\]

Proof. The first part of the lemma directly follows by rearranging and collecting the \(o(\Delta t)\) terms. Thus, it remains to show the second part. We have
\[
U\phi(\vec{x}, \vec{v}, \vec{r}) = \sum_{\vec{z} \in \{0, 1\}^N} \phi(h(\vec{x}, \vec{v}, \vec{r}, \vec{z})) \prod_{i=1}^{N} [1 - \Delta t \lambda(r_i, x_i)]
\]
\[
+ \mathbb{1}_{1-z_i}(r_i) \Delta t \lambda(r_i, x_i))
\]
(12)
If we apply a Taylor expansion to \(\phi(h(\vec{x}, \vec{v}, \vec{r}, \vec{z}))\), this reads
\[
\phi(h(\vec{x}, \vec{v}, \vec{r}, \vec{z})) = \phi(\vec{x}, \vec{v}, \vec{z})
\]
\[
+ \Delta t \sum_{j=1}^{N} r_j(\nabla_x \phi(\vec{x}, \vec{v}, \vec{z}) \cdot V(x_j, v_j) + \nabla_{v_j} \phi(\vec{x}, \vec{v}, \vec{z}) \cdot F_j(\vec{x}, \vec{v}))
+ o(\Delta t).
\]
(13)
Inserting (13) into (12) and using the first part of the lemma we end up with
\[
U\phi(\vec{x}, \vec{v}, \vec{r}) = \phi(\vec{x}, \vec{v}, \vec{r})
\]
\[
- \Delta t \sum_{j=1}^{N} \lambda(r_j, x_j) \phi(\vec{x}, \vec{v}, \vec{r})
\]
\[
+ \Delta t \sum_{j=1}^{N} \lambda(r_j, x_j) \phi(\vec{x}, \vec{v}, \vec{r}) \theta_j(\vec{r})
\]
\[
+ \Delta t \sum_{j=1}^{N} r_j(\nabla_x \phi(\vec{x}, \vec{v}, \vec{r}) \cdot V(x_j, v_j) + \nabla_{v_j} \phi(\vec{x}, \vec{v}, \vec{r}) \cdot F_j(\vec{x}, \vec{v}))
+ o(\Delta t).
\]
\]
Furthermore, to state the discrete time evolution of measures, we have to define a consistency relation.

**Definition 2.5.** The consistency relation for the initial measure $\mu^0_{(N)}$ is defined by
\[
\int_{E^{(N)}} \phi(\vec{x}, \vec{v}, \vec{r}) \mu^0_{(N)}(d(\vec{x}, \vec{v}, \vec{r})) = \int_{E^{(N)}} \phi(\vec{x}, \vec{v}, \vec{r}) \mu^0_{(N)}(d(\vec{x}, \vec{v}, \vec{r})). \tag{14}
\]

The relation (14) means that non-moving pedestrians at the initial time $t_0$ have zero velocity. The next lemma shows that the consistency relation is also conserved in time.

**Lemma 2.6.** Let the measure $\mu^0_{(N)}$ satisfy the consistency relation (14), then it holds
\[
\int_{E^{(N)}} \phi(\vec{x}, \vec{v}, \vec{r}) \mu^n_{(N)}(d(\vec{x}, \vec{v}, \vec{r})) = \int_{E^{(N)}} \phi(\vec{x}, \vec{v}, \vec{r}) \mu^n_{(N)}(d(\vec{x}, \vec{v}, \vec{r})). \tag{15}
\]
for every $n \in \mathbb{N}_0$.

**Proof.** The case $n = 0$ is true by assumption. Let us further assume that (15) is true for some arbitrary, fixed $n \in \mathbb{N}_0$, then due to (10)
\[
\int_{E^{(N)}} \phi(\vec{x}, \vec{v}, \vec{r}) \mu^{n+1}_{(N)}(d(\vec{x}, \vec{v}, \vec{r})) = \int_{E^{(N)}} U \phi(\vec{x}, \vec{v}, \vec{r}) \mu^n_{(N)}(d(\vec{x}, \vec{v}, \vec{r})).
\]

It remains to show that
\[
U \phi(\vec{x}, \vec{v}, \vec{r}) = U \phi(\vec{x}, \vec{v}, \vec{r}).
\]

We know that
\[
V(x_i, v_i) = r_i V(x_i, v_i)
\]
and hence
\[
h(\vec{x}, \vec{v}, \vec{r}, \vec{z}) = h(\vec{x}, \vec{v}, \vec{r}, \vec{z})
\]
due to $r_i \in \{0, 1\}$, i.e., $r_i^2 = r_i$. Then it is straightforward to see
\[
U((\vec{x}, \vec{v}, \vec{r}), B) = U((\vec{x}, \vec{v}, \vec{r}), B)
\]
and
\[
U \phi(\vec{x}, \vec{v}, \vec{r}) = U \phi(\vec{x}, \vec{v}, \vec{r}).
\]

The discrete time evolution of measures can be finally summarized in the following theorem.

**Theorem 2.7.** Let the consistency relation (14) be satisfied. Then, for all $\phi \in \mathcal{C}^{(N)}$, the $N$-particle distribution $(\mu^n_{(N)}, n \in \mathbb{N}_0)$ satisfies
\[
\frac{1}{\Delta t} \left( \int_{E^{(N)}} \phi(\vec{x}, \vec{v}, \vec{r}) \mu^{n+1}_{(N)}(d(\vec{x}, \vec{v}, \vec{r})) - \int_{E^{(N)}} \phi(\vec{x}, \vec{v}, \vec{r}) \mu^n_{(N)}(d(\vec{x}, \vec{v}, \vec{r})) \right)
\]
\[
= \int_{E^{(N)}} \sum_{j=1}^N \left[ r_j \nabla x_j \phi(\vec{x}, \vec{v}, \vec{r}) \cdot V(x_j, v_j) + r_j \nabla v_j \phi(\vec{x}, \vec{v}, \vec{r}) \cdot F_j(\vec{x}, \vec{v}) + \lambda(r_j, x_j)(\phi(\vec{x}, \vec{v}, \theta_j(\vec{r})) - \phi(\vec{x}, \vec{v}, \vec{r})) \right] \mu^n_{(N)}(d(\vec{x}, \vec{v}, \vec{r})) + o(1).
\]

**Proof.** The result is a direct application of lemma 2.4 and 2.6. \qed
The kinetic model to the stochastic microscopic pedestrian flow model can be derived as follows: If we apply theorem 2.7 on the function \( \phi(x, \bar{v}, \bar{r}) = \psi(x_1, v_1, r_1) \in C \), we get

\[
\frac{1}{\Delta t} \left( \int_E \psi(x_1, v_1, r_1) \mu^{n+1}_{(1)}(d(x_1, v_1, r_1)) - \int_E \psi(x_1, v_1, r_1) \mu^{n}_{(1)}(d(x_1, v_1, r_1)) \right)
\]

\[
= \int_E \left[ r_1 \nabla_x \psi(x_1, v_1, r_1, r_1) \cdot V(x_1, v_1) + r_1 \nabla_v \psi(x_1, v_1, r_1, r_1) \cdot F_{\text{dest}}(x_1, v_1)
+ \lambda(r_1, x_1)(\psi(x_1, v_1 r_1, 1 - r_1) - \psi(x_1, v_1 r_1, r_1)) \right] \mu^{n}_{(1)}(d(x_1, v_1, r_1))
+ \frac{1}{N - 1} \sum_{j=2}^N \int_{E^{(N)}} r_1 \nabla_x \psi(x_1, v_1 r_1, r_1) \cdot G(x_1 - x_j) \mu_{(N)}^n(d(x_1, v_1, r_1)),
\]

(16)

where \( \mu^{n}_{(1)} \) denotes the distribution of the first pedestrian.

Using the molecular chaos assumption (11), we can express the fourth line of (16) as

\[
\int_{E^{(N)}} r_1 \nabla_v \psi(x_1, v_1 r_1, r_1) \cdot G(x_1 - x_j) \mu_{(N)}^n(d(x_1, v_1, r_1))
= \int_E r_1 \nabla_v \psi(x_1, v_1 r_1, r_1) \cdot \int_E G(x_1 - x_j, v_2) \mu_{(1)}^n(d(x_1, v_1, r_1))
\]

and therefore

\[
\frac{1}{N - 1} \sum_{j=2}^N \int_{E^{(N)}} r_1 \nabla_v \psi(x_1, v_1 r_1, r_1) \cdot G(x_1 - x_j) \mu_{(N)}^n(d(x_1, v_1, r_1))
= \int_E r_1 \nabla_v \psi(x_1, v_1 r_1, r_1) \cdot \int_E G(x_1 - x_j, v_2) \mu_{(1)}^n(d(x_1, v_1, r_1)).
\]

We introduce \( (\mu^t, t \geq 0) \) such that \( \mu^{n}_{(1)} = \mu^{n \Delta t} \) and obtain a continuous time equation for \( \mu^t \) by \( \Delta t \to 0 \) reading

\[
\frac{d}{dt} \int_E \psi(x, v, r) \mu^t(d(x, v, r))
= \int_E \left[ r \nabla_x \psi(x, v, r) \cdot V(x, v) + r \nabla_v \psi(x, v, r) \cdot F_{\text{dest}}(x, v)
+ \lambda(r, x)(\psi(x, v, 1 - r) - \psi(x, v, r))
+ r \nabla_v \psi(x, v, r) \cdot \int_E G(x - y) \mu^t(d(y, w, s)) \right] \mu^t(d(x, v, r)).
\]

(17)

Remark 1. We prefer to use integral equations instead of a differential representation of (17). This is due to the fact that non-moving pedestrians have zero velocity and hence for fixed \( B^x \) the measure

\[
B^x \mapsto \mu^t(B^x \times B^y \times \{0\})
\]

is a Dirac distribution \( \sim c_0(B^y) \).

Note that the one-particle distribution \( \mu^t \) is often called kinetic model as the following definition indicates:

Definition 2.8. We call a family of measures \( (\mu^t, t \geq 0) \) satisfying (17) for every \( \psi \in C \) stochastic kinetic pedestrian model.
2.3. Macroscopic model. As we have seen, the kinetic model describes the one particle distribution including detailed information about the velocity of the pedestrians. However, we are mainly interested in the derivation of a simplified deterministic macroscopic model capturing the stochastic dynamics and hence significant less computational costs.

Let

\[ \rho^t_k(B^x) := \int_{B^x \times \mathbb{R}^2 \times \{k\}} \mu^t(d(x, v, r)) \]

be the probability to find a pedestrian in \( B^x \in \mathcal{B}(\mathbb{R}^2) \) with status \( k \in \{0, 1\} \) and \( \rho^t(B^x) := \rho^t_0(B^x) + \rho^t_1(B^x) \). If we choose \( \phi(x, v, r) = \eta(x, r) \in \mathcal{C} \), then

\[
\frac{d}{dt} \int_E \phi(x, v, r) \mu^t(d(x, v, r)) = \frac{d}{dt} \int_{\mathbb{R}^2} \eta(x, 0) \rho^0_0(dx) + \frac{d}{dt} \int_{\mathbb{R}^2} \eta(x, 1) \rho^1_1(dx)
\]

and together with (17) we obtain

\[
\frac{d}{dt} \int_{\mathbb{R}^2} \eta(x, 0) \rho^0_0(dx) + \frac{d}{dt} \int_{\mathbb{R}^2} \eta(x, 1) \rho^1_1(dx)
\]

\[
= \int_E r \nabla_x \eta(x, r) \cdot V(x, v) \mu^t(d(x, v, r))
\]

\[
+ \frac{1}{\delta} \int_{\mathbb{R}^2} \left( r \nabla_x \psi(x, rv, r) \cdot (F_{\text{dest}}(x, v) + \int_{\mathbb{R}^2} G(x-y) \rho^{t, \delta}(dy)) \right) \mu^{t, \delta}(d(x, v, r)),
\]

where we consider no change in the velocity anymore.

In a next step, we need to approximate the expression

\[
\int_E r \nabla_x \eta(x, r) \cdot V(x, v) \mu^t(d(x, v, r))
\]

with measures \( \rho^0_0 \) and \( \rho^1_1 \) to get a closed representation. This can be achieved by the following rescaling arguments of the kinetic pedestrian flow model: Let \( t^\delta = t / \delta \) and \( x^\delta = x / \delta \) be the rescaled variables satisfying \( 1 \gg \delta > 0 \) and let \( \mu^{t, \delta}, \rho^{t, \delta}, \rho^0_0^{t, \delta} \) and \( \rho^1_1^{t, \delta} \) the corresponding rescaled variables introduced before. We can rewrite (17) as

\[
\frac{d}{dt} \int_E \psi(x, v, r) \mu^{t, \delta}(d(x, v, r))
\]

\[
= \int_{\mathbb{R}^2} \left( r \nabla_x \psi(x, rv, r) \cdot V(x, v) \right)
\]

\[
+ \frac{1}{\delta} \left( r \nabla_x \psi(x, rv, r) \cdot (F_{\text{dest}}(x, v) + \int_{\mathbb{R}^2} G(x-y) \rho^{t, \delta}(dy)) \right) \mu^{t, \delta}(d(x, v, r))
\]

where the index \( \delta \) at \( x \) and \( t \) is neglected. Due to the fact that \( \delta \) can be chosen arbitrarily small, the change in the velocity and the switching from stop-to-go (and vice versa) dominates the right hand side. In detail, if equation (19) is multiplied by \( \delta \) with \( \delta \) tends to zero, we obtain an equation without time and spatial derivatives. One solution \( \mu^t \) to the unscaled equation (17) which additionally satisfies

\[
\frac{d}{dt} \int_{\mathbb{R}^2} \psi(x, v, r) \mu^t(d(x, v, r))
\]

\[
= \int_{\mathbb{R}^2} r \nabla_x \psi(x, rv, r) \cdot V(x, v) \mu^t(d(x, v, r))
\]
for every test function \( \psi \) is a solution to the limit \( \delta \to 0 \) of (19). This exactly
describes the case in which the distribution of the velocities and the switching does
not change in time. By defining
\[
\mathbf{F}(x, \rho^{t,\delta}) := \frac{\nu C}{\tau} D(x) + \int_{\mathbb{R}^2} G(x - y) \rho^{t,\delta}(dy),
\]
we can rewrite
\[
\int_{E} \left[ \tau \nabla_{v} \psi(x, rv, r) \cdot (\mathbf{F}^{dest}(x, v) + \int_{\mathbb{R}^2} G(x - y) \rho^{t,\delta}(dy))
+ \lambda(r, x)(\psi(x, rv, 1 - r) - \psi(x, rv, r))] \mu^{t,\delta}(d(x, v, r)) \right.
\]
\[
= \int_{E} \left[ \tau \nabla_{v} \psi(x, rv, r) \cdot \left( \mathbf{F}(x, \rho^{t,\delta}) - \frac{\nu}{\tau} \right)
+ \lambda(r, x)(\psi(x, rv, 1 - r) - \psi(x, rv, r))] \mu^{t,\delta}(d(x, v, r)). \tag{20}
\]
In [18] it is analytically shown that if \( G, \mathbf{F}^{dest} \in W^{1,\infty} \) and
\[
\frac{1}{\delta} \int_{\mathbb{R}^2 \times \mathbb{R}^2} \left[ \nabla_{v} \psi(x, v) \cdot \left( \mathbf{F}(x, \rho^{t,\delta}) - \frac{\nu}{\tau} \right) \right] \mu^{t,\delta}(d(x, v)) \to 0
\]
for \( \delta \to 0 \), the sequence of measures \( \nu^{t,\delta} \) weakly converges to a measure \( \nu^{t,0} \) satisfying
\[
\nu^{t,0}(B^x \times B^v) = \int_{B^x} \epsilon_{\mathbf{F}(x, \rho^{t,0})} (B^v) \hat{\rho}^{t,0}(dx) \tag{21}
\]
for every \( B^x, B^v \in \mathcal{B}(\mathbb{R}^2) \) and
\[
\hat{\rho}^{t,\delta}(B^x) = \int_{B^x \times \mathbb{R}^2} \nu(d(x, v)).
\]
This is exactly our model if no random stop-and-go behavior is considered, i.e.,
\( \lambda(1, \cdot) \equiv 0 \) and \( \mu^\delta(B \times \{0\}) = 0 \) for every \( B \in \mathcal{B}(\mathbb{R}^2 \times \mathbb{R}^2) \). In this case, the
macroscopic pedestrian dynamics is given by the left column of figure 2.

This motivates the use of a Dirac distribution for the desired velocity to deduce
a simplified macroscopic model. We set
\[
\hat{\mu}^{t,0}(B) := \int_{\mathbb{R}^2} \epsilon_{(x,0,0)}(B) \nu_{0}^{t,0}(dx) + \int_{\mathbb{R}^2} \epsilon_{(x,v^*(x),1)}(B) \nu_{1}^{t,0}(dx) \tag{22}
\]
for every \( B \in \mathcal{E} \) and intend to find a macroscopic velocity \( v^* \) to obtain an appropriate
velocity distribution. The first integral in (22) is related to pedestrians with zero
velocity, i.e. \( r = 0 \). The second integral is related to the moving pedestrians
\( (r = 1) \) and, motivated by (21), we assume a Dirac distribution concentrated on
some closure velocity \( v^*(x) \). Inserting the measure (22) into (20) yields for \( \delta \to 0 \)
\[
\int_{E} \left[ \tau \nabla_{v} \psi(x, rv, r) \cdot \left( \mathbf{F}(x, \rho^{t,\delta}) - \frac{\nu}{\tau} \right)
+ \lambda(r, x)(\psi(x, rv, 1 - r) - \psi(x, rv, r))] \hat{\mu}^{t,0}(d(x, v, r))
\]
\[
= \int_{\mathbb{R}^2} \lambda(0, x)(\psi(x, 0, 1) - \psi(x, 0, 0)) \rho_{0}^{t,0}(dx)
+ \int_{\mathbb{R}^2} \lambda(1, x)(\psi(x, v^*(x), 0) - \psi(x, v^*(x), 1)) \rho_{1}^{t,0}(dx)
\]
If we apply the test function \( \hat{\psi}(x,v,r) = (\psi(x,0,1) - \psi(x,0,0))\mathbb{1}(r) \) to (20), we see that

\[
\begin{align*}
\int_E & \left[ \tau \nabla_v \hat{\psi}(x,rv,r) \cdot \left( F(x,\rho^{t,\delta}) - \frac{v^*(x)}{\tau} \right) \\
& + \lambda(r,x)(\hat{\psi}(x,rv,1-r) - \hat{\psi}(x,rv,r)) \right] \mu^{t,\delta}(dx,v) \bigg| _{r=0} = 0
\end{align*}
\]

for some \( \tau \in [0,1] \) and assume that \( \hat{\psi} \) is twice time differentiable with respect to \( v \).

Inserting the Taylor expansion into (23) leads to

\[
\int_{\mathbb{R}^2} \nabla_v \psi(x,v^*(x),1) \cdot \left( F(x,\rho^{t,\delta}) - \frac{v^*(x)}{\tau} - \lambda(1,x)v^*(x) \right) \rho^{t,0}_1(dx) + R = 0
\]

with

\[
R = \int_{\mathbb{R}^2} \left[ \nabla_v \psi(x,\tau_2 v^*(x),0) \cdot v^*(x) \\
+ \frac{1}{2} v^*(x)^T \nabla_v^2 \psi(x,\tau_1 v^*(x),1)v^*(x) \right] \lambda(1,x) \rho^{t,0}_1(dx).
\]

For the choice

\[
v^*(x) = \frac{\tau F(x,\rho^{t,\delta})}{1 + \tau \lambda(1,x)}
\]

as \( \delta \to 0 \), the first term in (24) vanishes and only \( R \) remains. If we choose \( \psi(x,v,r) = \mathbb{1}(r) \nabla_x \eta(x,v) \cdot v \) and assume (24) is valid for the latter, we get \( R = 0 \) and the closure distribution \( \hat{\mu}^{t,0} \) satisfies (19) in the limit \( \delta \to 0 \). Then, we can deduce \( v^*(x) \) as the expected desired velocity of moving pedestrians of the scaled kinetic model (19) in the limit \( \delta \to 0 \). In the case \( \lambda(1,x) \equiv 0 \), i.e. the stop-and-go effect is neglected, \( R \) is zero and the closure velocity \( v^*(x) \) coincides with the closure velocity in the deterministic case (21) as expected. More precisely, \( v^*(x) \) is close to (21) up to the scaling factor \( (1 + \tau \lambda(1,x))^{-1} \approx 1 - \tau \lambda(1,x) \). This means that the expected velocity of moving pedestrians is reduced by \( \tau \lambda(1,x) \) proportional to the
expected total time a pedestrian is at rest during a time interval of length $\tau$. Summarizing, we approximate (18) using $\hat{\mu}_t^0$, i.e.,

$$\int_E \tau \nabla_x \eta(x, r) \cdot V(x, v) \mu_t^0 (d(x, v, r)) \approx \int_{\mathbb{R}^2} \nabla_x \eta(x, 1) \cdot V(x, v^*(x)) \rho^1_t (dx).$$

Finally, we collect all ideas and computations so far.

**Definition 2.9.** A family of measures $((\rho^0_t, \rho^1_t), t \geq 0)$ satisfying

$$\frac{d}{dt} \int_{\mathbb{R}^2} \eta(x, 0) \rho^0_t (dx) + \frac{d}{dt} \int_{\mathbb{R}^2} \eta(x, 1) \rho^1_t (dx)$$

$$= \int_{\mathbb{R}^2} \nabla_x \eta(x, 1) \cdot V \left( x, \frac{\tau F(x, \rho^1)}{1 + \tau \lambda(1, x)} \right) \rho^1_t (dx)$$

$$+ \int_{\mathbb{R}^2} \lambda(0, x)(\eta(x, 1) - \eta(x, 0)) \rho^0_t (dx) + \int_{\mathbb{R}^2} \lambda(1, x)(\eta(x, 0) - \eta(x, 1)) \rho^1_t (dx)$$

for every $\eta(\cdot, 0), \eta(\cdot, 1) \in C^1_b(\mathbb{R}^2; \mathbb{R})$ with

$$\rho^0_0 = \rho^1_0, \quad \rho^0_1 = \rho^1_1, \quad \rho^0(\mathbb{R}^2) + \rho^1(\mathbb{R}^2) = 1$$

for some positive measures $\rho^0_t, \rho^1_t$ on $(\mathbb{R}^2, \mathcal{B}(\mathbb{R}^2))$ is called a **stochastic macroscopic pedestrian model**.

We are able to obtain the differential representation of the stochastic macroscopic pedestrian model by applying the closure distribution (25) to eliminate the dynamics of the velocity. To do so, we assume that the measures $\rho^0_t$ and $\rho^1_t$ are absolutely continuous with respect to the Lebesgue measure and possess density functions $u_0$ and $u_1$, respectively. Then,

$$\frac{d}{dt} \int_{\mathbb{R}^2} \eta(x, 0) u_0(t, x) dx + \frac{d}{dt} \int_{\mathbb{R}^2} \eta(x, 1) u_1(t, x) dx$$

$$= \int_{\mathbb{R}^2} \nabla_x \eta(x, 1) \cdot V \left( x, \frac{\tau F(x, \rho^1)}{1 + \tau \lambda(1, x)} \right) u_1(t, x) dx$$

$$+ \int_{\mathbb{R}^2} \lambda(0, x)(\eta(x, 1) - \eta(x, 0)) u_0(t, x) dx$$

$$+ \int_{\mathbb{R}^2} \lambda(1, x)(\eta(x, 0) - \eta(x, 1)) u_1(t, x) dx$$

$$= \int_{\mathbb{R}^2} \eta(x, 0) (\lambda(1, x) u_1(t, x) - \lambda(0, x) u_0(t, x)) dx$$

$$+ \int_{\mathbb{R}^2} \left[ \nabla_x \eta(x, 1) \cdot V \left( x, \frac{\tau F(x, \rho^1)}{1 + \tau \lambda(1, x)} \right) u_1(t, x) \right. \left. + \eta(x, 1) (\lambda(0, x) u_0(t, x) - \lambda(1, x) u_1(t, x)) \right] dx$$

for every $\eta(\cdot, 0), \eta(\cdot, 1) \in C^1_b$. Due to $C^1_b \subset C^\infty$, the macroscopic model implies the strong formulation

$$\partial_t u_0(t, x) = \lambda(1, x) u_1(t, x) - \lambda(0, x) u_0(t, x), \quad (26)$$

Due to $\mu^0_t \subset C^\infty$, the macroscopic model implies the strong formulation

$$\partial_t u_0(t, x) = \lambda(1, x) u_1(t, x) - \lambda(0, x) u_0(t, x), \quad (26)$$

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$$\partial_t u_0(t, x) = \lambda(1, x) u_1(t, x) - \lambda(0, x) u_0(t, x), \quad (26)$$
\[
\partial_t u_1(t, x) = \lambda(0, x) u_0(t, x) - \lambda(1, x) u_1(t, x) \\
- \text{div}_x \left( V \left( x, \frac{\tau F(x, u(t, \cdot))}{1 + \tau \lambda(1, x)} \right) u_1(t, x) \right),
\]

(27)

\[
u(t, x) = u_0(t, x) + u_1(t, x)
\]

(28)
equipped with initial conditions \(u_0(0, x) = g_0(x) \geq 0, u_1(0, x) = g_1(x) \geq 0\) satisfying

\[
\int_{\mathbb{R}^2} g_0(x) + g_1(x) \, dx = 1.
\]

Note that the latter representation of the stochastic macroscopic pedestrian model also allows for comparisons to the deterministic model setting. Figure 2 illustrates our main results starting from the microscopic model to the resulting first order macroscopic model using the proposed closure velocities. The corresponding deterministic results can be directly obtained from [6, 13, 18] and are consistent with the stochastic model (26)-(28) if the stochastic stop-and-go effect is neglected, i.e., \(\lambda(1, \cdot) \equiv 0\) and \(g_0 \equiv 0\). However, the main difference to the deterministic model is the appearance of the reaction term and the model-based closure velocity leading to two dynamical regimes of moving and non-moving pedestrians.

| Deterministic | Stochastic |
|---------------|------------|
| Microscopic | | |
| \(x_i = V(x_i, v_i)\) | \(P(x_i^{n+1} = x_i^n + r_i^n \Delta t V(x_i^n, v_i^n) | X^n) = 1,\) |
| \(v_i = F_i(x, v)\) | \(P(v_i^{n+1} = v_i^n + \Delta t F_i(x^n, \bar{v}^n) | X^n) = r_i^n,\) |
| \(r_i^{n+1} = 1 - r_i^n | X^n) = \Delta t \lambda(r_i^n, x_i^n)\) | \(v_0^* = 0, v_1^* = \frac{\tau}{1 + \tau \lambda(1, x)} F(x, u_0 + u_1)\) |

| Closure Velocity | | |
| v* = \(\tau F(x, u)\) | \(\partial_t u_0 = \lambda(1, \cdot) u_1 - \lambda(0, \cdot) u_0\) |
| | \(\partial_t u_1 = \lambda(0, \cdot) u_0 - \lambda(1, \cdot) u_1 - \text{div}_x(V(x, v^*) u_1)\) |

**Figure 2.** Overview of the deterministic and stochastic model hierarchy equations

### 3. Numerical approximation.

This section deals with the numerical treatment of the presented stochastic microscopic and macroscopic pedestrian model. First, we present a stochastic simulation algorithm for the microscopic model to generate efficiently sample paths. Second, we introduce a numerical approximation scheme for the macroscopic model in its differential form.

#### 3.1. Microscopic model.

The generation of sample paths for the microscopic model can be directly implemented with the transition kernel and the Markov property. In detail, let \(X^n = (\bar{x}^n, \bar{v}^n, \bar{r}^n) \in E^N\) be the current state of the system and \(\Delta t \leq ||\lambda||_\infty^{-1}\). Due to
we compute $x^{n+1}_i = x^n_i + r^n_i \Delta t V(x^n_i, v^n_i)$ and set $v^{n+1}_i = v^n_i + \Delta t F_i(x^n_i, v^n_i)$ if $r^n_i = 1$. The equation

$$P^{0(N)}(X^{n+1}) = z|X^n| = \prod_{i=1}^N (I_{z_i}(r^n_i)(1 - \Delta t \lambda(r^n_i, x^n_i)) + I_{1-z_i}(r^n_i)\Delta t \lambda(r^n_i, x^n_i))$$

implies that the status of each pedestrian $r^n_i$ to walk or to stop can be chosen independently according to a Bernoulli distribution with probabilities $p_i = \Delta t \lambda(r^n_i, x^n_i)$ where $\lambda$ is the given rate function.

Note that we need to sample initial values according to the initial distribution $\mu^0(N)$. Then, trajectories of the microscopic pedestrian model can be computed by following the procedure described above until a finite time horizon is reached.

3.2. Macroscopic model. We consider equations (26)–(28) to solve the macroscopic pedestrian model numerically by established approximation schemes. Let $u^n_0(i, j)$ and $u^n_1(i, j)$ be the numerical approximation of $u_0(t_n, (x_i, y_j))$ and $u_1(t_n, (x_i, y_j))$, where $x_i = i \Delta x$, $y_j = j \Delta y$ is the spatial discretization with $\Delta x, \Delta y > 0$ and $t_n$ the $n$-th time-step. We decouple the original model into two separate problems (advection and reaction) in a straightforward way using a fractional-step method, see e.g. [22], and solve the problem in parts:

1. Advection part

$$\partial_t u_1 + \text{div}_x \left( V \left( x, \frac{\tau F(x, u(t, \cdot))}{1 + \tau \lambda(1, x)} \right) u_1(t, x) \right) = 0.$$ 

This is a conservation law (given as one equation) on $\mathbb{R}^2$ with a space dependent, non-local flux function that is solved applying a first order Roe method while the convolution

$$F(x, u(t, \cdot)) = \frac{u^C}{\tau} D(x) + \int_{\mathbb{R}^2} G(x - y) u(t, y) dy$$

is approximated by a rectangular rule. Dimension splitting allows the consideration of one dimensional problems to solve the two dimensional advection equation in each time step. As a numerical scheme this reads

$$\tilde{u}^{n+1}_1(i, j) = u^n_1(i, j) - \frac{\Delta t}{\Delta x} (F_x(i, j, u^n_1, u^n_0) - F_x(i-1, j, u^n_1, u^n_0))$$

$$- \frac{\Delta t}{\Delta y} (F_y(i, j, u^n_1, u^n_0) - F_y(i, j-1, u^n_1, u^n_0)), \quad (29)$$

where

$$F_x(i, j, u^n_1, u^n_0) = F_{\text{num}}(u^n_1(i+1, j), u^n_1(i, j), f_x, L(i, j), f_x, R(i, j))$$

and the numerical flux given by

$$F_{\text{num}}(u_L, u_R, F_L, F_R) = \begin{cases} F_L & \text{if } (F_R - F_L)(u_R - u_L) \geq 0, \\ F_R & \text{else}. \end{cases}$$

Here, the fluxes $f_x, L(i, j)$ and $f_x, R(i, j)$ are computed as follows: Let

$$V^{\text{num}}_x(i, j) = V_1 \left( (x_i, y_j), \frac{\tau F^{\text{num}}(i, j)}{1 + \tau \lambda(1, (x_i, y_j))} \right)$$

$$\partial_t u_0 + \text{div}_x \left( V \left( x, \frac{\tau \mu(x, \cdot)}{1 + \tau \lambda(1, x)} \right) u_0(t, x) \right) = 0.$$
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with

\[ F_{\text{num}}(i,j) = \frac{v}{\tau} D((x_i, y_j)) \]

\[ + \Delta x \Delta y \sum_{k_1, k_2 \in \mathbb{Z}} G((x_i - k_1, y_j - k_2))(u^0_1(k_1, k_2) + u^0_0(k_1, k_2)), \]

then

\[ f_{x,L}(i,j) = u^0_1(i,j) \frac{V^{\text{num}}_x(i+1,j) + V^{\text{num}}_x(i,j)}{2} \]

and

\[ f_{x,R}(i,j) = u^0_1(i+1,j) \frac{V^{\text{num}}_x(i+1,j) + V^{\text{num}}_x(i,j)}{2}. \]

The same holds for the numerical flux \( F_y \) in \( y \)-direction.

2. Reaction part

\[ \partial_t u_0(t,x) = \lambda(1,x)u_1(t,x) - \lambda(0,x)u_0(t,x), \]

\[ \partial_t u_1(t,x) = \lambda(0,x)u_0(t,x) - \lambda(1,x)u_1(t,x). \]

These equations can be explicitly solved for every \( x \in \mathbb{R}^2 \) since the reaction is linear. Let

\[ u^R(t,x) = (u^R_0(t,x), u^R_1(t,x)) \]

denote the solution of the reaction equation, then

\[ u^R(t,x) = e^{t\Lambda(x)} u^R(0,x) \]

\[ = \frac{1}{\overline{\lambda}(x)} \begin{pmatrix} (\lambda(1,x) + \lambda(0,x)e^{-t\overline{\lambda}(x)})u^R_0(0,x) & +\lambda(1,x)(1 - e^{-t\overline{\lambda}(x)})u^R_1(0,x) \\ \lambda(0,x)(1 - e^{-t\overline{\lambda}(x)})u^R_0(0,x) & +\lambda(0,x) + \lambda(1,x)e^{-t\overline{\lambda}(x)})u^R_1(0,x) \end{pmatrix} \]

with \( \overline{\lambda}(x) = \lambda(0,x) + \lambda(1,x) \) and

\[ \Lambda(x) = \begin{pmatrix} -\lambda(0,x) & \lambda(1,x) \\ \lambda(0,x) & -\lambda(1,x) \end{pmatrix}. \]

Using the intermediate approximation (29) from part 1., the complete numerical method is given by

\[ \begin{pmatrix} u^{n+1}_0(i,j) \\ u^{n+1}_1(i,j) \end{pmatrix} = e^{\Delta t \Lambda(x_i,y_j)} \begin{pmatrix} u^n_0(i,j) \\ u^n_1(i,j) \end{pmatrix}. \]

It is obvious to see that the proposed numerical scheme is mass conservative, i.e.,

\[ \sum_{i,j \in \mathbb{Z}} (u^{n+1}_0(i,j) + u^{n+1}_1(i,j)) = \sum_{i,j \in \mathbb{Z}} (u^n_0(i,j) + u^n_1(i,j)) \]

\[ = \sum_{i,j \in \mathbb{Z}} (u^n_0(i,j) + u^n_1(i,j)). \]

Furthermore, we expect first order numerical convergence rates since the Roe approximation and the operator splitting are both first order approximations. We also refer to section 4 and the numerical examples therein.
4. Computational results. For simulation purposes, we focus on new effects arising from the stochastic dynamics and qualitative comparisons of the model hierarchy. We need to define performance measures to evaluate the quality of approximations of the microscopic and macroscopic model. Therefore, we assume that the macroscopic measure $\rho_t$ admits a Lebesgue density $u(x,t)$ which is approximated by cell means $u_{ij}^{Mac,n}$ on rectangles $Q_{ij}$ with lengths $\Delta x$, $\Delta y$ and center $x_{ij}$ at time step $t_n$.

Furthermore, the average density of pedestrians in $Q_{ij}$ for the microscopic model is given by

$$u_{ij}^{Mic,n} := \mathbb{E}_{\mu_0(N)} \left[ \frac{1}{N} \sum_{l=1}^N 1_{Q_{ij}}(x^n_l) \right] \Delta x \Delta y$$

This allows to specify the error between the microscopic and macroscopic model as

$$\epsilon^n := u_{ij}^{Mic,n} - u_{ij}^{Mac,n}$$

measured in terms of the $L^p$-norm

$$||\epsilon^n||_{L^p} := \left( \Delta x \Delta y \sum_{ij} |u_{ij}^{Mic,n} - u_{ij}^{Mac,n}|^p \right)^{\frac{1}{p}}$$

for every $p \geq 1$.

The mass balance

$$MB_x(u(t,\cdot)) := \int_{(-\infty,\hat{x}] \times \mathbb{R}} u(t,x)dx$$

at some vertical cut at position $\hat{x} \in \mathbb{R}$ determines the amount of mass to the left of the cut. This measure extracts information about the time needed for a certain percentage of pedestrians to cross the cut.

In the following examples, we assume that the destination force is a point force

$$D(x) = \frac{x^D - x}{||x^D - x||}$$

with destination point $x^D \in \mathbb{R}^2$ and interaction kernel $G$ as the gradient of a Morse Potential, i.e.

$$G(x) = -2(e^{-||x||-0.9} - e^{-2||x||-0.9}) \frac{x}{||x||}.$$ 

The microscopic model is only comparable to the macroscopic one if we assume that the initial velocities are given by the closure velocities of the macroscopic model, i.e., for randomly generated positions $(x_1,\ldots,x_N)$ and states $(r_1,\ldots,r_N)$, we set

$$v_i = r_i \frac{\tau}{1 + \tau \lambda(1, x_i)} \left( \frac{v_C}{\tau} D(x_i) + \frac{1}{N} \sum_{j=1}^N G(x_i - x_j) \right),$$

where the initial states $r_i$ are computed according to a Bernoulli distribution with initial probability $P(r_i = 0) = p_0 \in [0,1]$.

Example 1: Corridor with Space Dependent Rates. As a first example, we consider a strip without obstacles with boundaries given by bold lines, see figure 3. The comfort velocity $v_C$ and the relaxation time $\tau$ are assumed to be one. The distribution of the initial positions should be
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$$\rho^0(B) = \frac{1}{2} \int_B \mathbb{1}_{[-2,-1] \times [-1,1]}(x,y)d(x,y).$$

Additionally, the initial conditions for the macroscopic model are

$$g_0(x,y) = p_0 \mathbb{1}_{[-2,-1] \times [-1,1]}(x,y)$$

and

$$g_1(x,y) = (1-p_0) \mathbb{1}_{[-2,-1] \times [-1,1]}(x,y)$$

supplemented by the initial probability to stop $$p_0 = \frac{1}{2}$$.

We are interested in the dynamic behavior of the model for a space dependent rate function:

$$\lambda(0,x) = \begin{cases} 6 & \text{if } ||x|| \leq 0.5 \\ 10 & \text{else} \end{cases} \quad (33)$$

$$\lambda(1,x) = \begin{cases} 5 & \text{if } ||x|| \leq 0.5 \\ 4 & \text{else} \end{cases} \quad (34)$$

The results of a simulation with destination $$x^D = (100,0)$$, $$\Delta x = \Delta y = \frac{1}{40}$$, $$N = 100$$ pedestrians and $$M = 1000$$ Monte Carlo iterations are shown in figure 3. For our computations, we have used a WIN10 operated PC with 4-core i5-4690 CPU 3.50GHz, 16GB DDR3 RAM. All codes are implemented in MATLAB R2016b. The computation times we observed are 5149 seconds for the microscopic and 434 seconds for the macroscopic model.

In this scenario, the rate $$\lambda$$ describes the number of events per time such that equations (33) and (34) indicate that pedestrians will stop more frequently than intend to walk inside the circle around the origin with radius 0.5. We expect that walking pedestrians circulate around the non-moving crowd and merge behind the bulk until a comfort density is reached again, cf. times $$t = 10$$ and $$t = 15$$ in figure 3. As we can also see, the influence of the space dependent rate on the density is significant since the density is much higher in the region where people stop longer and walk shorter.

Apparently, the results from the microscopic and the macroscopic simulation are quite close, i.e. the density profiles show the same qualitative behavior in time. This can be also verified by the mass balance in figure 4 and the $$L^1$$ and $$L^2$$ error in figure 5. In detail, the error at time $$t = 0$$ is the error caused by sampling and converges to zero as $$N \rightarrow \infty$$ due to the law of large numbers. For the $$L^2$$ error we have 0.048 at $$t = 0$$ and maximally 0.09 at $$t = 2.6$$ which is less than half of the initial error.

We also study the convergence of the numerical scheme for the macroscopic equation experimentally. Therefore, we assume $$u_{i,j}^{\text{Ref}} = v_0^{\text{Ref}} + u_1^{\text{Ref}}$$ is the reference solution computed by $$\Delta x = \Delta y = \frac{1}{40}$$. Let $$u_{i,j}^{\text{Ref}}$$ be the value of the reference solution at grid point $$(x_i,y_j)$$ with $$\Delta x = \Delta y > \frac{1}{40}$$ and $$u_{i,j}$$ the corresponding numerical approximation. We define the error to the reference solution by

$$\text{err} = \max_n \left( \Delta x \Delta y \sum_{i,j} |u_{i,j}(t_n) - u_{i,j}^{\text{Ref}}(t_n)| \right).$$

In the first column of table 1 the error err is shown for different grid sizes. The experimental order of convergence (EOOC) is deduced in the second column. We observe that the distance to the reference solution gets smaller for decreasing $$\Delta x$$ and that the order of convergence is approximately one.
Figure 3. Densities at different times: $u_{ij}^{\text{Mic},n}$ for the microscopic and $u_{ij}^{\text{Mac},n}$ for the macroscopic model.

| $\Delta x$     | $\text{err}$ | EOOC |
|----------------|--------------|------|
| $1/5$          | 0.3251       | -    |
| $1/10$         | 0.1755       | 0.8897|
| $1/20$         | 0.0717       | 1.2919|

Table 1. Numerical error and EOOC for the first example
Example 2: Corridor including Bottlenecks. Different to the first example, we now focus on a scenario where two obstacles are located oppositely, see bold boundaries in figures 6 and 7. We distinguish the rate functions $\lambda_1$ and $\lambda_2$ to basically cover two relevant scenarios:

$$
\lambda_1(0, x) = \begin{cases} 
1 & \text{if } x \in [-1, 1], \\
10 & \text{else,}
\end{cases} \quad \lambda_1(1, x) = \begin{cases} 
1 & \text{if } x \in [-1, 1], \\
\frac{1}{100} & \text{else,}
\end{cases}
$$

$$
\lambda_2(0, x) = 10, \quad \lambda_2(1, x) = \frac{1}{100}.
$$

The rate function $\lambda_1$ is in-homogeneous in space and divided into two cases. In the case $x \in [-1, 1]$ which is exactly between the obstacles (e.g. external attractions) pedestrians will stop more often and stay for a longer period. Otherwise, for $x \in (-\infty, -1) \cup (1, \infty)$, pedestrians behave deterministically since the scaling factor $(1 + \tau \lambda(1, x))^{-1}$ in (25) becomes very close to one and we get a velocity which corresponds to the deterministic model, cf. left column in figure 2. On the contrary, the rate function $\lambda_2$ is homogeneous in space. For $\lambda_2$ the transition rate from walking to stopping is quite small and the opposite rate from stopping to walking quite high which results in a situation where almost everyone is in motion. This scenario is quite close to the deterministic model, where $\lambda(1, x) = 0$, cf. figure 2, and allows for studying and comparing the influence of space-dependent rate functions such as $\lambda_1$ at bottlenecks.

We consider the following setup: The destination is again located at $x^D = (100, 0)$, the comfort speed is $v^C = 1$ and the relaxation time is supposed to be $\tau = 0.2$. The initial distribution is

$$
\rho^0(B) = \frac{2}{3} \int_B \mathbb{1}_{[-2.5, -1] \times [-0.5, 0.5]}(x, y) d(x, y)
$$

with probability $p_0 = 0.01$ to stop in the beginning. Therefore, the initial conditions for the macroscopic model read

$$
g_0(x, y) = p_0 \frac{2}{3} \mathbb{1}_{[-2.5, -1] \times [-0.5, 0.5]}(x, y) \quad \text{and} \quad g_1(x, y) = (1 - p_0) \frac{2}{3} \mathbb{1}_{[-2.5, -1] \times [-0.5, 0.5]}(x, y).
$$

Figure 4. Mass balances at $x = -1$ and $x = 0$
The simulation is performed with step-sizes $\Delta x = \Delta y = \frac{1}{40}$, $N = 100$ pedestrians and $M = 1000$ Monte-Carlo iterations. We have used the same PC as before and the required computation times are 3006 seconds ($\lambda_1$), 3205 seconds ($\lambda_2$) for the microscopic model and 485 seconds ($\lambda_1$), 495 seconds ($\lambda_2$) for the macroscopic model.

In figures 6–7 we plot the densities for different rate functions. The specular reflection type boundaries seem to work well and the pedestrians are rerouted along the obstacles. For the choice of $\lambda_1$ in figure 6, where the pedestrians tend to stop between the obstacles, we observe the expected queuing behavior. In contrast for the deterministic setting $\lambda_2$, higher densities in front of the obstacles result from the spatial conditions only and are not caused by stop-or-go decisions, cf. figure 7.

The influence of different rate functions is also reflected by the comparison of the mass balances in figure 8. The first scenario, the bottleneck situation, reduces significantly the mean velocity of the pedestrians and hence leads to longer travel times, see left picture in figure 8. After approximately 14 time units all pedestrians crossed the cut at $x = 1$ while for the deterministic scenario only 2.5 time units are needed, cf. right picture in figure 8.

Finally, we stress that the error between the microscopic and macroscopic model is again bounded and maximally three times the initial error, as figure 9 illustrates. Furthermore, the error is not strongly affected by the different rate functions. The second increase in the error, see figure 9 (b), is due to the numerical diffusion of the numerical scheme for macroscopic equation. This can be also observed in figure 7 at time point $t = 5$. Then, the error decreases again since the mass flows out of the system.

We also add results for the convergence of the numerical scheme. We observe in table 2 and 3 again that the distance to the reference solution decreases for finer grids. The order of convergence is also close to one again.

**Conclusion.** We have introduced a stochastic microscopic pedestrian model with specular reflection type boundary conditions and proved its existence. Based on
Figure 6. Densities for $\lambda_1$ at different times: $u_{ij}^{\text{Mic},n}$ for the microscopic and $u_{ij}^{\text{Mac},n}$ for the macroscopic model.

| $\Delta x$ | err | EOOC |
|------------|-----|------|
| $1/5$      | 0.4457 | -    |
| $1/10$     | 0.2215 | 1.0085 |
| $1/20$     | 0.0889 | 1.3170 |

Table 2. Numerical error and EOOC for the second example with rate function $\lambda_1$

| $\Delta x$ | err | EOOC |
|------------|-----|------|
| $1/5$      | 0.5203 | -    |
| $1/10$     | 0.2873 | 0.8567 |
| $1/20$     | 0.1153 | 1.3176 |

Table 3. Numerical error and EOOC for the second example with rate function $\lambda_2$.

the microscopic model, we have derived a kinetic formulation under a mean field assumption in terms of measures. A non-standard closure assumption has been
Figure 7. Densities for $\lambda_2$ at different times: $u_{ij}^{\text{Mic},n}$ for the microscopic and $u_{ij}^{\text{Mac},n}$ for the macroscopic model

Established to extract a first order macroscopic model given by a system of conservation laws. Numerical comparisons of the microscopic and macroscopic pedestrian flow model have been performed to demonstrate the same dynamical behavior for both modeling approaches.

Future work might include the investigation of a stochastic model hierarchy, where the Eikonal equation (instead of the destination force) is used to determine the shortest travel time to the desired destination, see [13]. The derivation of a second order macroscopic model seems to be also very appealing in this context. From the numerical point of view, higher order unsplit methods should be studied to reduce the numerical diffusion and consequently the error to the microscopic results.
Figure 8. Mass balances at $x = 1$

Figure 9. $L^1$ and $L^2$ errors

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