Neutrino Mass Matrices with Vanishing Determinant and $\theta_{13}$

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Abstract

We investigate the prospects for scenarios with vanishing determinant neutrino mass matrices and vanishing $\theta_{13}$ mixing angle. Normal and inverse mass hierarchies are considered separately. For normal hierarchy it is found that neutrinoless double beta decay cannot be observed by any of the present or next generation experiments. For inverse hierarchy the neutrinoless double beta decay is, on the contrary, accessible to experiments. We also analyse for both hierarchies the case for texture zeros and equalities between mass matrix elements. No texture zeros are found to be possible nor any such equalities, apart from the obvious ones.

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1 Introduction

It is a well known fact that the neutrino mass matrix contains nine parameters while feasible experiments can hope to determine only seven of them in the foreseeable future. This situation can however be overcome, with the number of free parameters being reduced, if physically motivated assumptions are made to restrict the form of the matrix. Among the most common such assumptions and as an incomplete list one may refer the texture zeros [1], hybrid textures [2], traceless condition [3], [4], [5] and vanishing determinant [6]. The former assumptions can be basis independent under certain conditions as shall be seen for the traceless condition. The latter being equivalent to one vanishing neutrino mass.

In this paper we perform the investigation of vanishing determinant neutrino masses with vanishing $\theta_{13}$. We will assume that neutrinos are Majorana [9], as favoured by some experimental evidence [10], and study the neutrino mass matrix $M$ in the weak basis where all charge leptons are already diagonalized. This is related to the diagonal mass matrix $D$ through the unitary transformation

$$D = U_{MNS}^T M U_{MNS}$$

(1)

where we use the standard parametrization [11]

$$U_{MNS} = \begin{pmatrix}
  c_{12} c_{13} & s_{12} c_{13} & s_{13} e^{-i\delta} \\
  -s_{12} c_{23} - c_{12} s_{23} s_{13} e^{i\delta} & c_{12} c_{23} - s_{12} s_{23} s_{13} e^{i\delta} & s_{23} c_{13} \\
  s_{12} s_{23} - c_{12} c_{23} s_{13} e^{i\delta} & -c_{12} s_{23} - c_{12} c_{23} s_{13} e^{i\delta} & c_{23} c_{13}
\end{pmatrix},$$

(2)

where $\delta$ is a Dirac CP violating phase. Equation (1) is equivalent to

$$M = U^* \text{diag}(m_1, m_2 e^{i\phi_1}, m_3 e^{i\phi_2}) U^\dagger$$

(3)

where $\phi_1$, $\phi_2$ are two extra CP violating Majorana phases and $D = \text{diag}(m_1, m_2 e^{i\phi_1}, m_3 e^{i\phi_2})$. Applying determinants properties

$$\det M = \det (U^* D U^\dagger)$$
$$= \det (U^* U^\dagger D)$$
$$= \det U^* \det U^\dagger \det D$$
$$= \det D \text{ (U real)}$$
$$\neq \det D \text{ (U complex)}$$

(4)

because if matrix $U$ is real, $U^* U^\dagger = U U^T = 1$, which is satisfied provided $\delta = 0$ or $\theta_{13} = 0$ (see eq.(2)). Thus the determinant is not in general basis independent. In

\footnote{The 2$\sigma$ range recently obtained for this quantity is [7] $\sin^2 \theta_{13} = 0.9 \pm 0.3 \times 10^{-2} eV^2$, the lower uncertainty being purely formal, corresponding to the positivity constraint $\sin^2 \theta_{13} \geq 0$.}
order that \( \det D = \det M \) it is necessary and sufficient that there is either no Dirac CP violation or that it is unobservable. The same arguments hold for the condition \( TrD = TrM \) [4].

From eq. (4) we get that \( \det M = 0 \) if and only if \( \det D = 0 \), because \( \det U^\dagger \) and \( \det U^* \) are not zero. The vanishing determinant condition is basis independent, corresponding to a zero eigenvalue of the mass matrix. So requiring \( \det M = 0 \) is equivalent to assuming one of the neutrinos to be massless. This is realized for instance in the Affleck-Dine scenario for leptogenesis [12],[13],[14] which requires the lightest neutrino to be practically massless \( (m \simeq 10^{-10} eV) \) [15],[16]. Furthermore, since in this paper we consider \( \theta_{13} = 0 \), the Dirac phase is unobservable and the usual definition \( U_{MNS} = U_{23}U_{13}U_{12} \) [17] simplifies to \( U_{MNS} = U_{23}U_{12} \) with

\[
U_{23} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \alpha_{22} & \alpha_{23} \\ 0 & \alpha_{32} & \alpha_{33} \end{pmatrix}, \quad U_{12} = \begin{pmatrix} \beta_{11} & \beta_{12} & 0 \\ \beta_{21} & \beta_{22} & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{5}
\]

where the unitarity condition \(|\alpha_{22}\alpha_{33} - \alpha_{32}\alpha_{23}| = |\beta_{11}\beta_{22} - \beta_{12}\beta_{21}| = 1\) implies \( \alpha_{22}\alpha_{33}\alpha_{32}\alpha_{23} < 0 \) and \( \beta_{11}\beta_{22}\beta_{12}\beta_{21} < 0 \) with \( \alpha_{22} = \pm \cos \theta_{\odot}, \beta_{11} = \pm \cos \theta_{\odot} \), the remaining matrix elements being evident. For neutrino masses and mixings we refer to the following 2\( \sigma \) ranges [7, 8]

\[
\Delta m^2_{\odot} = m_2^2 - m_1^2 = 7.92 \times 10^{-5}(1 \pm 0.09) eV^2, \tag{6}
\]

\[
\Delta m^2_{\odot} = m_3^2 - m_2^2 = \pm 2.4 \times 10^{-3}(1 \pm 0.21) eV^2, \tag{7}
\]

\[
\sin^2 \theta_{\odot} = 0.314(1\pm0.18), \tag{8}
\]

\[
\sin^2 \theta_{\odot} = 0.44(1\pm0.41) \tag{9}
\]

obtained from a 3 flavour analysis of all solar and atmospheric data. This favours the widely used form of the \( U_{MNS} \) matrix [18] (all entries taken in their moduli)

\[
U_{MNS} = \begin{pmatrix} \sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \end{pmatrix} \tag{10}
\]

The paper is organized as follows: in section 2 we derive all possible forms of the mass matrix \( M \) in this scenario for both normal and inverse hierarchies and investigate their consequences for \( 0\nu\beta\beta \) decay. Since one of the neutrinos is massless, there is only one Majorana phase to be considered. In section 3 we investigate the prospects for texture zeros and equalities among matrix elements in both hierarchies and in section 4 we briefly expound our main conclusions.
2 Mass matrices with \( \text{det}M = 0 \) and \( \theta_{13} = 0 \)

2.1 Normal hierarchy (NH)

This is the case where the two mass eigenstates involved in the solar oscillations are assumed to be the lightest so that \( \Delta m^2_{\odot} = \Delta m^2_{32} > 0 \). We will consider this case as a departure from the degenerate one with \( \Delta m^2_{\odot} = \Delta m^2_{21} = 0 \) and break the degeneracy with a real parameter \( \epsilon \). Matrix \( D \) with \( m \) and \( \epsilon \) both real is therefore

\[
D = \text{diag}(0, 3\epsilon e^{i\phi}, m)
\]

where \( \phi \) is the Majorana relative phase between the second and third diagonal matrix elements (\( \phi = \phi_1 - \phi_2 \) in the notation of section 2) and \( \Delta m^2_{\odot} = 9\epsilon^2 \). Using eqs.(5) the matrix \( M \) is

\[
M = U_{23}U_{12}D U_{12}^T U_{23}^T = \begin{pmatrix}
3\epsilon e^{i\phi}\beta_{12}^2 & 3\epsilon e^{i\phi}\alpha_{22}\beta_{22} & 3\epsilon e^{i\phi}\alpha_{32}\beta_{22} \\
3\epsilon e^{i\phi}\alpha_{22}\beta_{12}\beta_{22} & 3\epsilon e^{i\phi}\alpha_{22}\beta_{22}^2 + m\alpha_{23}^2 & 3\epsilon e^{i\phi}\alpha_{22}\alpha_{23}\beta_{22} \\
3\epsilon e^{i\phi}\alpha_{32}\beta_{12}\beta_{22} & 3\epsilon e^{i\phi}\alpha_{22}\alpha_{32}\beta_{22}^2 + m\alpha_{23}\alpha_{33} & 3\epsilon e^{i\phi}\alpha_{32}\beta_{22}^2 + m\alpha_{33}^2
\end{pmatrix}.
\]

(12)

Owing to the sign ambiguities of parameters \( \alpha \) and \( \beta \), four possibilities for matrix \( M \) arise. Suppose entries 12 and 13 in this matrix have \((+) (+)\) signs. Then \( \alpha_{22}, \alpha_{32} \) have the same sign as \( \beta_{12}\beta_{22} \), that is \( \alpha_{22}\alpha_{32} \) in the \((23)\) entry is \((+), \) implying the opposite sign for the coefficient of \( m \) \((\alpha_{23}\alpha_{33})\). So eq.(12) has the form

\[
M = \begin{pmatrix}
\epsilon e^{i\phi} & \epsilon e^{i\phi} & \epsilon e^{i\phi} \\
\epsilon e^{i\phi} & (m/2) + \epsilon e^{i\phi} & -(m/2) + \epsilon e^{i\phi} \\
\epsilon e^{i\phi} & -(m/2) + \epsilon e^{i\phi} & (m/2) + \epsilon e^{i\phi}
\end{pmatrix}
\]

(13)

Suppose entries 12 and 13 in the matrix have \((-) (-)\) signs. Then \( \alpha_{22}, \alpha_{32} \) have opposite sign to \( \beta_{12}\beta_{22} \), that is they have the same sign, so \( \alpha_{22}\alpha_{32} \) is \((+) \) and \( \alpha_{23}\alpha_{33} \) is \((-) \) so

\[
M = \begin{pmatrix}
\epsilon e^{i\phi} & -\epsilon e^{i\phi} & -\epsilon e^{i\phi} \\
-\epsilon e^{i\phi} & (m/2) + \epsilon e^{i\phi} & -(m/2) + \epsilon e^{i\phi} \\
-\epsilon e^{i\phi} & -(m/2) + \epsilon e^{i\phi} & (m/2) + \epsilon e^{i\phi}
\end{pmatrix}
\]

(14)

Suppose entries 12 and 13 in the matrix have \((+) (-)\) signs. Then \( \alpha_{22}, \alpha_{32} \) have opposite signs to each other, so \( \alpha_{22}\alpha_{32} \) is \((-) \) and \( \alpha_{23}\alpha_{33} \) is \((+) \). Hence

\[
M = \begin{pmatrix}
\epsilon e^{i\phi} & \epsilon e^{i\phi} & -\epsilon e^{i\phi} \\
\epsilon e^{i\phi} & (m/2) + \epsilon e^{i\phi} & (m/2) - \epsilon e^{i\phi} \\
-\epsilon e^{i\phi} & (m/2) - \epsilon e^{i\phi} & (m/2) + \epsilon e^{i\phi}
\end{pmatrix}
\]

(15)
Suppose entries 12 and 13 in the matrix have (-) (+) signs. Then \( \alpha_{22}, \alpha_{32} \) have opposite signs to each other, so \( \alpha_{22} \alpha_{32} \) is (-) and \( \alpha_{23} \alpha_{33} \) is (+). Hence the matrix is

\[
M = \begin{pmatrix}
\epsilon e^{i\phi} & -\epsilon e^{i\phi} & \epsilon e^{i\phi} \\
-\epsilon e^{i\phi} & (m/2) + \epsilon e^{i\phi} & (m/2) - \epsilon e^{i\phi} \\
\epsilon e^{i\phi} & (m/2) - \epsilon e^{i\phi} & (m/2) + \epsilon e^{i\phi}
\end{pmatrix}
\]

(16)

All matrices (13), (14), (15), (16) have vanishing determinant as can be easily verified. For 0\(\nu\nu\beta\beta\) decay

\[
<m_{ee} > = |U_{e1}^2 m_1 + U_{e2}^2 m_2 e^{i\phi_1} + U_{e3}^2 m_3 e^{i\phi_2}| 
\]

(17)

hence, for vanishing \( m_1 \) and \( U_{e3} = s_{13} e^{-i\delta} \),

\[
<m_{ee} > = |U_{e2}^2 m_2 e^{i\phi_1}| = \frac{1}{3} 3\epsilon = \frac{1}{3} \sqrt{\Delta m^2_{\odot}} \simeq 3 \times 10^{-3} eV
\]

(18)

where we used \( \epsilon = \frac{1}{3} \sqrt{\Delta m^2_{\odot}} \). So the Majorana phase is not an observable.

There is no commonly accepted evidence in favour of 0\(\nu\nu\beta\beta\) decay but there exist reliable upper limits on \(< m_{ee} >\)

\[
<m_{ee} > \leq (0.3 - 1.2) eV \ [10], \quad < m_{ee} > \leq (0.2 - 1.1) eV \ [19]
\]

(19)

where the uncertainties follow from the uncertainties in the nuclear matrix elements.

The future CUORE experiment [20], of which CUORICINO is a test version [19], is expected to improve this upper bound to \( 3 \times 10^{-2} eV \). Other experiments are also proposed (MAJORANA [21], GENIUS [22], GEM [23] and others) in which the sensitivity of a few \( 10^{-2} eV \) is planned to be reached.

Conclusion: vanishing determinant with vanishing \( \theta_{13} \) and NH implies that 0\(\nu\nu\beta\beta\) decay cannot be detected even in the next generation of experiments. This remains unchanged even if \( \theta_{13} \neq 0 \), since the largest mass \( (m_3) \) multiplies \( s_{13}^2 \) in eq.(17).

### 2.2 Inverse Hierarchy (IH)

We start with matrix \( D \) in the form \( D = \text{diag}\{m, (m + \epsilon)e^{i\phi}, 0\} \) where \( m, \epsilon \) are complex, \( |m| \simeq \sqrt{\Delta m^2_{\odot}}, |\epsilon| \simeq \sqrt{\Delta m^2_{\odot}} \) and chosen in such a way that \( m + \epsilon = \tilde{m} \) is real (\( \epsilon = 0 \) corresponds to the degenerate case). Alternatively \( D = \text{diag}\{\tilde{m} - \epsilon, \tilde{m}e^{i\phi}, 0\} \) with, of course, \( \tilde{m} - \epsilon \) complex. Multiplying the whole matrix by the inverse phase of \( \tilde{m} - \epsilon \), it can be redefined as

\[
D = \text{diag}\{\tilde{m} - \lambda, \tilde{m}e^{i(\phi - \psi)}, 0\}
\]

(20)
with \( \lambda \) real and defined by \((\tilde{m} - \epsilon)e^{-i\psi} = \tilde{m} - \lambda \) (notice that \( \tilde{m} - \epsilon = |\tilde{m} - \epsilon|e^{i\psi} \) and \( \tilde{m} = \sqrt{\Delta m^2_{\odot}} \)). There are two solutions for \( \lambda \). In fact, imposing the solar mass square difference

\[
\Delta m^2_{\odot} = |d_{22}|^2 - |d_{11}|^2 = \tilde{m}^2 - 2\lambda\tilde{m} - \lambda^2
\]

and solving the quadratic equation \( \lambda^2 - 2\lambda\tilde{m} + \Delta m^2_{\odot} = 0 \) one gets

\[
\lambda = \tilde{m} \pm \sqrt{\tilde{m}^2 - \Delta m^2_{\odot}} = \lambda_{\pm}.
\]

Notice that \( \lambda_+ \) is large and \( \lambda_- \) is small. To first order in \( \frac{\Delta m^2_{\odot}}{m^2} = \frac{\Delta m^2_{\odot}}{\Delta m^2_{\odot}} \simeq 0.30 \) one has

\[
\lambda_+ = \tilde{m}(2 - \frac{1}{2} \frac{\Delta m^2_{\odot}}{\Delta m^2_{\odot}}) \simeq 1.85\tilde{m}
\]

\[
\lambda_- = \frac{\tilde{m} \Delta m^2_{\odot}}{2\Delta m^2_{\odot}} \simeq \frac{\tilde{m}}{60}
\]

It is straightforward to see that \( D(\lambda_-, \phi + \pi) = -D(\lambda_+, \phi) \) and the same property holds for matrix \( M \), namely \( M(\lambda_-, \phi + \pi) = -M(\lambda_+, \phi) \) because \( U_{MNS} \) is invariant under the transformations \( \lambda_+ \rightarrow \lambda_- \) and \( \phi \rightarrow \phi + \pi \). So the two solutions for \( \lambda \) are equivalent: one may take either

\[
\lambda_+ \ , \ \psi = 0
\]

or

\[
\lambda_- \ , \ \psi = \pi.
\]

Using \( M = U_{23}U_{12}DU_{12}^TU_{23}^T \) with eqs.(5), (20), the matrix \( M \) has now the form

\[
M = \begin{pmatrix}
\tilde{m}(1 - \frac{1}{3}) - \frac{2}{3} \lambda & (\text{sign})\frac{1}{3}(\tilde{m}t - \lambda) & (\text{sign})\frac{1}{3}(\tilde{m}t - \lambda)\\
(\text{sign})\frac{1}{3}(\tilde{m}t - \lambda) & \tilde{m}(\frac{1}{2} - \frac{1}{3}) - \frac{1}{6} \lambda & (\text{sign})[\tilde{m}(\frac{1}{2} - \frac{1}{3}) - \frac{1}{6} \lambda]\\
(\text{sign})\frac{1}{3}(\tilde{m}t - \lambda) & (\text{sign})[\tilde{m}(\frac{1}{2} - \frac{1}{3}) - \frac{1}{6} \lambda] & \tilde{m}(\frac{1}{2} - \frac{1}{3}) - \frac{1}{6} \lambda
\end{pmatrix}
\]

which also verifies \( det \ M = 0 \) as expected. Equation (27) is formally the same for \( \lambda = \lambda_+ \) and \( \lambda = \lambda_- \) with the definition \( t = 1 - e^{i\phi} \) for \( \lambda = \lambda_+ \) and \( t = 1 + e^{i\phi} \) for \( \lambda = \lambda_- \), the sign affecting the exponential being related to the \( \psi \) phase. The structure of (+) and (-) in eq.(27) is the same as before ((13), (14), (15), (16)): equal signs in entries \( M_{12} \), \( M_{13} \) correspond to (+) in both entries \( M_{23} \), \( M_{32} \) while different signs in \( M_{12} \), \( M_{13} \) correspond to (-) in both entries \( M_{23} \), \( M_{32} \). Eq. (27) is the equivalent for IH of (13), (14), (15), (16) for NH.

For \( \beta\beta_{0\nu} \) decay we have

\[
< m_{ee} >= |U_{e1}^2m_1 + U_{e2}^2m_2e^{i\phi_1} + U_{e3}^2m_3e^{i\phi_2}| = \frac{2}{3}(|\tilde{m} - \lambda_-| \pm \frac{1}{3}\tilde{m}e^{i(\phi)}).
\]
The quantity $m_{ee}$ is displayed in fig.1 as a function of the phase difference $\phi$. The shaded areas correspond to the $1\sigma$ uncertainties in the solar angle $\theta_\odot$. It is seen from eq.(28) and fig.1 that for inverse hierarchy (vanishing $\theta_{13}$ and mass matrix determinant) $\beta\beta_{0\nu\nu}$ decay is phase dependent and within observational limits of forthcoming experiments. So:

Conclusion: models with vanishing determinant mass matrix and vanishing $\theta_{13}$ provide, in inverse hierarchy, a Majorana phase dependent $\beta\beta_{0\nu\nu}$ decay which is physically observable for most values of the phase in the next generation of experiments.

Figure 1: $\beta\beta_{0\nu\nu}$ decay effective mass parameter $<m_{ee}>$ as a function of the Majorana phase $\phi$ showing its accessibility for forthcoming experiments.
3  Texture zeros and equalities between $M$ matrix elements

3.1  Texture zeros

Here we analyze the possibility of vanishing entries in the mass matrix $M$. Taking first NH and recalling eqs. (13)-(16), it is seen that this implies either $\tilde{m}/2 = \pm \epsilon e^{i\phi}$ or $\epsilon = 0$, both situations being impossible. For IH three cases need to be considered:

(a) $M_{11} = 0$
We have in this case $\tilde{m}(1 - \frac{t}{3}) - \frac{2}{3} \lambda = 0$ implying

$$\tilde{m}(3 - t) = 2\lambda. \quad (29)$$

Replacing $t \rightarrow 1 - e^{i\phi}$ and $\lambda \rightarrow \lambda_\pm$ this leads to

$$e^{i\phi} = 2 \sqrt{1 - \frac{\Delta m^2_{\odot}}{\Delta m^2_{\odot}}} \quad (30)$$

which is experimentally excluded.

(b) $M_{12} = 0$
This gives $\tilde{m}t - \lambda = 0$, hence using the same replacement

$$e^{i\phi} = -\sqrt{1 - \frac{\Delta m^2_{\odot}}{\Delta m^2_{\odot}}} \quad (31)$$

which is also impossible since $\Delta m^2_{\odot} = 0$ is strictly excluded experimentally.

(c) $M_{22} = 0$
This gives $\tilde{m}(\frac{1}{2} - \frac{t}{3}) - \frac{\lambda}{6} = 0$, hence using the same replacement

$$e^{i\phi} = \frac{1}{2} \sqrt{1 - \frac{\Delta m^2_{\odot}}{\Delta m^2_{\odot}}} \quad (32)$$

which is also experimentally excluded. In the former cases (a), (b), (c) the same results are of course obtained with the replacement $t \rightarrow 1 + e^{i\phi}$ and $\lambda \rightarrow \lambda_-$, as can be easily verified. So zero mass textures are not possible in the present scenario.

The same conclusion can be obtained using the results from the literature. In fact the analytical study of various structures of the neutrino mass matrix was presented systematically by Frigerio and Smirnov [24] who also discussed the case of equalities of matrix elements. Here we use a result from [25] where specific relations
among the mixing angles were derived for one texture zero and one vanishing eigenvalue. We refer to table I of [25] and first to NH. Using their definition of parameter \( \chi = \left| \frac{m_2}{m_1} \right| \) we have in our model \( \chi = \sqrt{\frac{\Delta m^2_{21}}{\Delta m^2_{31}}} = 0.182 \) and so for cases A, B, C, D, E, F respectively in their notation
\[
\chi = 0, \chi = 0, \chi = 0, \chi = 1.50, \chi = 1.50, \chi = 1.50^2
\]
For inverse hierarchy, defining \( \eta = \frac{m_1}{m_2} = \frac{\vert m - \lambda \vert}{\sqrt{\Delta m^2}} = \sqrt{1 - \frac{\Delta m^2_{31}}{\Delta m^2_{12}}} = 0.983 \) we have for cases A, B, C, D, E, F respectively
\[
\eta = 0.50, \eta = 1, \eta = 1, \eta = 2.0, \eta = 2.0, \eta = 2.0
\]
Notice that 0.953 < \( \eta \) < 0.988 (using 1\( \sigma \) upper and lower values for the solar and atmospheric mass square differences). So one can draw the following

**Conclusion:** both NH and IH cannot work with \( \det D = \det M = 0 \), vanishing \( \theta_{13} \) and one texture zero. In other words, vanishing determinant scenarios with \( \theta_{13} = 0 \) are experimentally excluded, unless they have no texture zeros.

### 3.2 Equalities between matrix elements

First we consider the case of NH. Equations (13)-(16) can be written in the general form
\[
M = \begin{pmatrix}
\epsilon e^{i\phi} & \text{sign}(\epsilon e^{i\phi}) & \text{sign}(\epsilon e^{i\phi}) \\
\text{sign}(\epsilon e^{i\phi}) & (m/2) + \epsilon e^{i\phi} & \text{sign}[-(m/2) + \epsilon e^{i\phi}] \\
\text{sign}(\epsilon e^{i\phi}) & \text{sign}[-(m/2) + \epsilon e^{i\phi}] & (m/2) + \epsilon e^{i\phi}
\end{pmatrix}
\]
and using the same sign conventions as in eqs.(13)-(16), it is seen that \( |M_{11}| = |M_{12}|, |M_{12}| = |M_{13}|, M_{22} = M_{33} \). Hence the relations to be investigated are \( M_{11} = M_{22}, |M_{11}| = |M_{23}|, |M_{22}| = |M_{23}|. \)

Equation \( M_{11} = M_{22} \) implies \( m = 0 \) which is impossible.

Equation \( |M_{11}| = |M_{23}| \) yields
\[
(a) \quad \epsilon e^{i\phi} = (-m/2) + \epsilon e^{i\phi}
\]
leading to \( m = 0 \), and
\[
(b) \quad \epsilon e^{i\phi} = (+m/2) - \epsilon e^{i\phi}
\]
\footnote{For instance for case D, normal hierarchy and \( \theta_{13} \neq 0 \) we have, with the best fit values given in [7], \( \chi = \frac{0.386 \pm 0.0397 e^{-2}}{0.436} \) (\( \delta \) – Dirac phase).}
leading to \( \frac{4}{3} = \sqrt{1 + \frac{\Delta m^2}{\Delta m^2}} \), which is also experimentally excluded.

Equation \( |M_{22}| = |M_{23}| \) yields

\[
(a) \quad (+m/2) + \epsilon e^{i\phi} = (-m/2) + \epsilon e^{i\phi}
\]

leading to \( m = 0 \), and

\[
(a) \quad (+m/2) + \epsilon e^{i\phi} = (+m/2) - \epsilon e^{i\phi}
\]

leading to \( \epsilon = 0 \), both experimentally excluded.

Next we consider IH. We use eq. (27) and note that the matrix is symmetric, so there are at first sight 6 independent entries. However \( M_{22} = M_{33}, |M_{12}| = |M_{13}|, |M_{22}| = |M_{23}| \), so there remain 3 independent matrix elements and therefore 3 equalities to be investigated. In other words, there are three different moduli only: \( |M_{11}|, |M_{12}|, |M_{22}| \). So the three equalities to be investigated are \( |M_{11}| = |M_{12}|, |M_{11}| = |M_{23}|, |M_{12}| = |M_{23}| \).

Equality \( |M_{11}| = |M_{12}| \) yields two cases

\[
(a) \quad \tilde{m}(1 - \frac{t}{3}) - \frac{2}{3} \lambda = \frac{1}{3} (\tilde{m}t - \lambda)
\]

which upon using \( \lambda = \lambda_{\pm} \) for \( t = 1 \mp e^{i\phi} \) gives

\[
\tilde{m} - \lambda_{\pm} = \mp 2\tilde{m}e^{i\phi}
\]

which is impossible to satisfy, as seen from eq.(22), and

\[
(b) \quad \tilde{m}(1 - \frac{t}{3}) - \frac{2}{3} \lambda = -\frac{1}{3} (\tilde{m}t - \lambda)
\]

leading to

\[
\tilde{m} = \lambda,
\]

also impossible, eq.(22).

Equality \( |M_{11}| = |M_{23}| \). The two cases to be considered are

\[
(a) \quad \tilde{m}(1 - \frac{t}{3}) - \frac{2}{3} \lambda = \tilde{m}(\frac{1}{2} - \frac{t}{3}) - \frac{\lambda}{6}
\]

from which

\[
\tilde{m} = \lambda
\]

which cannot be satisfied (eq.(22)) and

\[
(b) \quad \tilde{m}(1 - \frac{t}{3}) - \frac{2}{3} \lambda = -\tilde{m}(\frac{1}{2} - \frac{t}{3}) + \frac{\lambda}{6}
\]
which upon using $\lambda = \lambda_{\pm}$ for $t = 1 \mp e^{i\phi}$ gives
\[ \tilde{m} - \lambda_{\pm} = \mp \frac{4}{5} \tilde{m} e^{i\phi} \] (47)
or equivalently
\[ 5 \sqrt{1 - \frac{\Delta m^2_{\odot}}{\Delta m^2_{\odot}}} = 4e^{i\phi} \] (48)
which is cannot be satisfied even if $\phi = 0$. (Maximizing $\Delta m^2_{\odot}$ and minimizing $\Delta m^2_{\odot}$ (1 $\sigma$) the above square root verifies $0.953 < \sqrt{1 - \frac{\Delta m^2_{\odot}}{\Delta m^2_{\odot}}} < 0.988$).

Equality $|M_{12}| = |M_{23}|$. The two cases are now
\[ (a) \quad \frac{1}{3}(\tilde{m} t - \lambda) = \tilde{m}(\frac{1}{2} - \frac{t}{3}) - \frac{\lambda}{6} \] (49)
which gives $\tilde{m} - \lambda_{\pm} = \pm 4\tilde{m} e^{i\phi}$ or $\pm \sqrt{1 - \frac{\Delta m^2_{\odot}}{\Delta m^2_{\odot}}} = \pm 4e^{i\phi}$, again impossible, and
\[ (b) \quad \frac{1}{3}(\tilde{m} t - \lambda) = -\tilde{m}(\frac{1}{2} - \frac{t}{3}) - \frac{\lambda}{6} \] (50)
or $\tilde{m} = \lambda$, also impossible. All these impossibilities mean experimentally excluded.

Moreover, if $M_{12}$ and $M_{13}$ have opposite signs, since $|M_{12}| = |M_{13}|$, they both vanish, implying two texture zeros which is excluded. The same is true for $M_{22}$ and $M_{23}$. Recall that one texture zero with vanishing determinant cannot work with $\theta_{13} = 0$ (see section 3.1). Hence:

**Conclusion:** equalities between mass matrix elements apart from the obvious ones are experimentally excluded.

### 4 Conclusions

We have investigated the prospects for neutrino mass matrices with vanishing determinant and $\theta_{13}$. The vanishing determinant condition alone is expressed by two real conditions, so the original nine independent parameters in these matrices are reduced to seven. Hence the undesirable situation of existing and planned experiments not being able to determine all these nine quantities is in this case overcome. Furthermore, as shown in the introduction, the vanishing of $\theta_{13}$ implies that the CP violating Dirac phase is unobservable and the mass matrix can be diagonalized by a real and orthogonal matrix. In such case the mass matrix determinant is basis
independent, \( \det M = \det D \), while the vanishing determinant condition is always basis independent. So \( \det M = 0 \) is always equivalent to the lightest neutrino being massless.

We considered both the normal and inverse mass hierarchies. Summarizing our main conclusions for vanishing determinant mass matrices with vanishing \( \theta_{13} \):

In the case of normal hierarchy there can be no observable \( \beta \beta_{0
u} \) decay. For inverse hierarchy \( \beta \beta_{0
u} \) decay depends on the Majorana phase and can be observed in the next generation of experiments for all or most of the possible phase range.

Texture zeros and equalities between mass matrix elements besides the obvious ones are incompatible with experimental evidence.

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