Infrared Divergent Coulomb Self-Energy in Yang-Mills Theory

Jeff Greensite†, Štefan Olejník§, Daniel Zwanziger¶

†Physics and Astronomy Dept., San Francisco State University, San Francisco, CA 94132 USA
§Institute of Physics, Slovak Acad. of Sciences, SK-845 11 Bratislava, Slovakia
¶Physics Dept., New York University, New York, NY 10003 USA

It is shown numerically that the Coulomb self-energy of an isolated, color non-singlet source diverges in an infinite volume. This is in accord with the Gribov Horizon scenario of confinement advocated by Gribov and Zwanziger. It is also shown that this divergence can be attributed to the presence of center vortices in thermalized lattice configurations.

The energy of an isolated color non-singlet source, in a confining theory, must be infinite in an infinite volume, even when the usual ultraviolet divergence is regulated. We would like to understand this fact a little better. Color confinement is usually attributed to center vortices or abelian monopoles (it is now understood that these objects are related\(^1\)), but there is also another approach to confinement – the Gribov Horizon scenario, formulated in Coulomb gauge – which has been advocated by Gribov and Zwanziger.\(^2\) In this talk I would like to briefly explain the Gribov Horizon scenario, show numerically that it accounts for the infrared divergent self-energy of color sources, and show also that this success is associated with the presence of center vortices in gauge field configurations. A more detailed exposition, on which this talk is based, can be found in a recent article.\(^3\)

As a warm-up exercise, let us consider a familiar abelian example: the electrostatic energy of a static electric point charge, in a cubic volume of length \(L\), with appropriate boundary conditions. In Coulomb gauge the Hamiltonian has the form

\[
H = \frac{1}{2} \int d^3x \left( E^2 + B^2 \right) + H_{\text{coul}}, \quad H_{\text{coul}} = \frac{1}{2} \int d^3x d^3y \rho(x)K(x,y)\rho(y)
\]

\[
K(x,y) = \left[ M^{-1}(-\nabla^2)M^{-1} \right]_{xy}, \quad \mathcal{E} = e^2K(x,x)
\]  

where \(\mathcal{E}\) is the Coulomb self-energy of a charge at point \(x\). \(M\) is the Faddeev-Popov (FP) operator of the abelian theory. Let \(A^\theta_\mu = A^\mu - \partial_\mu \theta\) be a gauge transformation of a configuration \(A^\mu\) fixed to Coulomb gauge. Then

\[
M_{xy} = \frac{\delta}{\delta \theta(x)} \nabla \cdot A^\theta(y) = -\nabla^2 \delta(x - y)
\]  

The eigenstates of the FP eigenvalue equation \(M\phi^{(n)} = \lambda_n \phi^{(n)}\) are of course just the plane wave states, with eigenvalues equal to squared momenta. In a finite volume these states are discrete, with a lattice regularization their number is finite, and we can express the Green’s function corresponding to \(M\) as

\[
G_{xy} = \left[ M^{-1} \right]_{xy} = \sum_n \frac{\phi_x^{(n)} \phi_y^{(n)*}}{\lambda_n}
\]  

After some simple manipulations, one finds that

\[
\mathcal{E} = e^2 \left[ M^{-1}(-\nabla^2)M^{-1} \right]_{xx} = \frac{e^2}{L^3} \sum_n \frac{F(\lambda_n)}{\lambda_n^2}
\]  

where \(F(\lambda_n) = (\phi^{(n)}|(-\nabla^2)|\phi^{(n)})\), and \(L^3\) is the number of lattice sites in the cubic volume. Let \(\rho(\lambda)\) denote the normalized density of eigenvalues. Then at large volumes we can approximate

\(\text{Talk presented by J. Greensite at Rencontres du Vietnam V, Hanoi, Vietnam, August 5-10, 2004.}\)
the sum over eigenstates by an integral

$$\mathcal{E} = e^2 \int d\lambda \frac{\rho(\lambda) F(\lambda)}{\lambda^2}$$

(5)

In QED, it’s easy to show that \(\rho(\lambda) = \sqrt{\lambda/(4\pi^2)}\), \(F(\lambda) = \lambda\). The eigenvalues span a finite range, with \(\lambda_{\text{min}} \sim 1/L^2\) and \(\lambda_{\text{max}} \sim 1/a^2\), where \(L\) is the extension of the lattice, and \(a\) is the lattice spacing. Putting it all together, we find that while \(\mathcal{E}\) has an ultraviolet divergence in the continuum \(a \to 0\) limit, it is finite in the infinite volume \(L \to \infty\) limit. However, finiteness at infinite volume clearly depends on the small \(\lambda\) behavior of \(\rho(\lambda) F(\lambda)\). If instead we had \(\lim_{\lambda \to 0} \rho(\lambda) F(\lambda) / \lambda > 0\), then the Coulomb energy would be divergent in the infinite volume limit.

Figure 1: Data for full, unprojected configurations.

In non-abelian theories, there are many gauge copies – Gribov copies – that satisfy the Coulomb gauge condition. The Gribov Region is the space of all Gribov copies with positive FP eigenvalues. The boundary of the Gribov region is the Gribov Horizon. Configurations lying on the Gribov Horizon have at least one FP eigenvalue \(\lambda = 0\). Typical Coulomb-gauge lattice configurations are expected to approach the Gribov horizon, in the infinite-volume limit, due to entropy considerations.\(^4\) But what counts for confinement is the density of eigenvalues \(\rho(\lambda)\) near \(\lambda = 0\), and the “smoothness” of these near-zero eigenvalues, as measured by \(F(\lambda)\). In non-abelian theories the FP operator is gauge-field dependent, i.e. \(M = -\nabla \cdot D\) where \(D_\mu\) is the covariant derivative, and the self-energy of an isolated static charge in group representation \(r\) is proportional to the quadratic Casimir \(C_r\), and to \(\mathcal{E}\) in eq. (5). So our approach is to calculate \(\rho(\lambda)\) and \(F(\lambda)\) by lattice Monte Carlo simulations of lattice SU(2) gauge theory, and extrapolate the results to infinite volume.

The results are shown in Fig. 1, obtained at \(\beta = 2.1\) for a variety of lattice sizes. We have applied a scaling analysis derived from random matrix theory,\(^5\) based on the scaling of the low-lying eigenvalue distributions with lattice size \(L\), to estimate that in the infinite volume limit we have \(\rho(\lambda) \sim \lambda^{0.25}\), \(F(\lambda) \sim \lambda^{0.4}\) at small \(\lambda\). Substituting these power behaviors into eq. (5), we find that \(\mathcal{E}\) has a divergence in the infrared \(L \to \infty\), \(\lambda_{\text{min}} \to 0\) limit, in addition to the usual ultraviolet divergence in the continuum \(a \to 0\) limit. In other words, the Coulomb self-energy of an isolated color charge is infrared infinite, by the mechanism envisaged in the Gribov Horizon scenario.

We will now display a connection with the center vortex confinement mechanism. Recall that center vortices are surfacelike objects in the (D=4) SU(N) gauge theory vacuum, which can be topologically linked to closed loops. Creation of a center vortex linked to a Wilson
loop multiplies the loop by a center element of the gauge group. The center vortex theory of confinement holds that the area law falloff of Wilson loops is due to vacuum fluctuations in the number of vortices linking the loop.

In 1997, methods were devised for locating center vortices in lattice configurations, and, a little later, for removing them. This was followed by many investigations of the vortex theory in the lattice community, with results discussed in two recent reviews. It was found that (i) center vortices, by themselves, account for the bulk of asymptotic string tension; (ii) the density of these vortices scales according to asymptotic freedom; and (iii) removing center vortices removes the string tension, removes chiral symmetry breaking, and sends the topological charge to zero.

We have used the standard technique of maximal center gauge fixing plus center projection to separate each Monte Carlo generated configuration into two components: the vortex-only (or “center-projected”) component, containing only the identified center vortices, and the vortex-removed component, in which those same vortices have been removed from the original lattice configuration. Each component is then transformed to minimal Coulomb gauge. Our data for the vortex-only configurations is shown in Fig. 2. This time a finite-volume scaling analysis indicates that \( \rho(\lambda) \sim \lambda^{0.05} \), \( F(\lambda) \approx 1 \) as \( \lambda \to 0 \), which again implies (from eq. (5)) an infrared divergence of the Coulomb energy, resulting from the vortex configurations alone.

By contrast, when vortices are removed from thermalized lattice configurations, there is a dramatic change in the eigenvalue spectrum, illustrated in Fig. 3 for the \( 20^4 \) lattice volume. Inspection of this data reveals that the number of eigenvalues in each peak of \( \rho(\lambda) \), and each band of \( F(\lambda) \), matches the degeneracy of eigenvalues of \( -\nabla^2 \), the zeroth-order Faddeev-Popov operator, at the given lattice size. We know that \( \rho(\lambda) \) for the \( -\nabla^2 \) operator at finite volume is just a series of \( \delta \)-function peaks. In the vortex-removed configurations, these peaks broaden to finite width, but the qualitative features of \( \rho(\lambda) \), \( F(\lambda) \) at zeroth order, i.e. the absence of confinement, remains.

These numerical results establish that the Coulomb self-energy of a color non-singlet state is infrared divergent, due to the enhanced density \( \rho(\lambda) \) of Faddeev-Popov eigenvalues near \( \lambda = 0 \). This supports the Gribov-Zwanziger picture of confinement. It also appears that the confining property of the FP eigenvalue density can be entirely attributed to center vortices, since (i) enhancement of \( \rho(\lambda) \), \( F(\lambda) \) is found in the vortex-only content of lattice gauge configurations; while (ii) the confining properties of \( \rho(\lambda) \), \( F(\lambda) \) disappear when vortices are removed from the lattice.

We conclude with two further facts about center vortices and the Gribov horizon, stated here without proof: First, vortex-only configurations have non-trivial Faddeev-Popov zero modes,
and therefore lie precisely on the Gribov horizon, which is a convex manifold in the space of gauge fields, both in the continuum and on the lattice. Secondly, vortex-only configurations are conical singularities on the Gribov horizon.\footnote{It appears that center vortices have a special geometrical status in Coulomb gauge, although the physical implications of this fact are not yet fully understood.}

Acknowledgments

Our research is supported in part by the U.S. Department of Energy under Grant No. DE-FG03-92ER40711 (J.G.), the Slovak Grant Agency for Science, Grant No. 2/3106/2003 (Š.O.), and the National Science Foundation, Grant No. PHY-0099393 (D.Z.).

References

1. J. Ambjørn, J. Giedt, and J. Greensite, *Journal of High Energy Phys.* \textbf{2}, 033 (2000), arXiv: hep-lat/9907021;
   P. de Forcrand and M. Pepe, *Nucl. Phys.* B \textbf{598}, 557 (2001);
   A. Kovalenko et al., arXiv: hep-lat/0402017;
   M. Chernodub et al., arXiv: hep-lat/0406015.
2. V. Gribov, *Nucl. Phys.* B \textbf{139}, 1 (1978);
   D. Zwanziger, *Nucl. Phys.* B \textbf{518}, 237 (1998).
3. J. Greensite, Štefan Olejník, and D. Zwanziger, arXiv: hep-lat/0407032.
4. D. Zwanziger, *Nucl. Phys.* B \textbf{378}, 525 (1992).
5. M. Berbenni-Bitsch et al., *Phys. Rev. Lett.* \textbf{80}, 1146 (1998), arXiv: hep-lat/9704018;
   R. Janik, *Nucl. Phys.* B \textbf{635}, 492 (2002), arXiv: hep-th/0201167.
6. J. Greensite, *Prog. Part. Nucl. Phys.* \textbf{51}, 1 (2003); arXiv: hep-lat/0301023;
   M. Engelhardt, arXiv: hep-lat/0409023.
7. J. Greensite, Štefan Olejník, and D. Zwanziger, *Phys. Rev.* D \textbf{69}, 074506 (2004); arXiv: hep-lat/0401003.