Quantitative non-divergence and lower bounds for points with algebraic coordinates near manifolds

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Abstract

Point counting estimates are a key stepping stone to various results in metric Diophantine approximation. In this paper we use the quantitative non-divergence estimates originally developed by Kleinbock and Margulis to improve lower bounds by Bernik, Götz et al. for the number of points with algebraic conjugate coordinates close to a given manifold. In the process, we also improve on a Khinchin-Groshev-type theorem for a problem of constrained approximation by polynomials.

1 Introduction

In the course of developing his classification of real numbers, Mahler conjectured that for every \( \varepsilon > 0 \) and Lebesgue almost every \( x \in \mathbb{R} \) the inequality

\[
|P(x)| < H(P)^{-n-\varepsilon}
\]

has at most finitely many solutions \( P \in \mathbb{Z}[X] \) with \( \deg(P) \leq n \), where \( H(P) \) denotes the (naive) height of \( P \), i.e. the maximum of its coefficients in absolute value. This was later proved by Sprindžuk [44], and it marked the beginning of the theory of Diophantine Approximation of dependent quantities, i.e. the study of the Diophantine properties of points bound to a given manifold.

It is then natural to wonder about the Diophantine properties of the solutions to a system of simultaneous equations of type (1) in multiple independent variables \( x_0, \ldots, x_m \in \mathbb{R} \), i.e.

\[
\max_{0 \leq k \leq m} |P(x_k)| < \psi(H(P))
\]

for some function \( \psi: \mathbb{R}^+ \to \mathbb{R}^+ \), with solutions in integer polynomials \( P \) of degree between \( m+1 \) and \( n \). Indeed, in [43, Problem C] Sprindžuk conjectured that the maximum \( v > 0 \) for which (2) with \( \psi(Q) = Q^{-v} \) has infinitely many solutions for all \( x \) in a set of positive measure is

\[
v = \frac{n + 1}{m + 1} - 1,
\]

and this was later proved by Bernik in [10].

The problem (2) was then considered for arbitrary \( \psi \) and \( m = 1 \) in [10], as well as for the case where the variables \( x_k \) can also take complex or \( p \)-adic values.

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values in \([11, 15]\) \((m = 2)\) and \([20, 19]\) (arbitrary \(m\)). In particular, the following result is contained in the preprint \([5]\), which deals with the more general case of systems of linear forms in dependent variables, i.e.

\[
\max_{0 \leq k \leq m} |a \cdot f_k(x_k)| < \psi(\|a\|)
\]

with solutions in \(a \in \mathbb{Z}^{n+1}\), where \(f_k: U_k \to \mathbb{R}^n\) are sufficiently regular maps defined on open balls \(U_k \subset \mathbb{R}^{d_k}\) and \(\psi: \mathbb{R}^+ \to \mathbb{R}^+\) as before. Here and throughout this paper \(\|\cdot\|\) will denote the sup norm \(\mathbb{R}^n\) unless otherwise specified, although note that most of the results presented here still hold with minor modification for any other choice of norm.

**Theorem 1.1** \(([5, \text{Theorem 1}])\). Consider integers \(n > m \geq 0\), a function \(\psi: \mathbb{R}^+ \to \mathbb{R}^+\), and a ball \(B \subset \mathbb{R}^{m+1}\). Let \(\|\cdot\|\) be the Lebesgue measure on \(\mathbb{R}^{m+1}\). Then

\[
|\mathcal{L}_{n,m+1}(\psi) \cap B| = \begin{cases} 
0 & \text{if } S_{n,m+1}(\psi) < \infty \\
|B| & \text{if } \psi \text{ is monotonic and } S_{n,m+1}(\psi) = \infty
\end{cases}
\]

where \(\mathcal{L}_{n,m+1}(\psi)\) denotes the set of \((x_0, \ldots, x_m) \in \mathbb{R}^{m+1}\) which satisfy \((2)\) for infinitely many polynomials \(P\) of degree up to \(n\), and where

\[
S_{n,m+1}(\psi) := \sum_{Q=1}^{\infty} Q^{n-m-1} \psi^{m+1}(Q).
\]

**Note.** Like many other Khinchin-Groshev-type theorems, this kind of result has already found applications in communication engineering, specifically in the field of interference alignment; see for example \([39, \text{Appendix B}]\) and \([33, \text{Section IV}]\), or \([26, 38]\) for examples which require results of approximation on manifolds. The interested reader may also find a more accessible description of how Khinchin-like theorems come into play in the theory of interference alignment in \([1, \text{Appendix A}]\).

Finally, one might consider what changes after introducing a dependency among the variables \(x_0, \ldots, x_m\) of \((2)\) (or \(x_0, \ldots, x_m\) in \([1]\)), i.e. when they are parametrised by a sufficiently regular map \(f: \mathcal{B} \subset \mathbb{R}^d \to \mathbb{R}^{m+1}\), and this is the subject of the present paper.

Clearly, if \(P(x)\) is small, then \(x\) must be close to at least one of the roots of \(P\). In particular, this means that if \(P\) is irreducible and \(|P(f_0(x))|, \ldots, |P(f_m(x))|\) are all small, then there must be a point \(\alpha \in \mathbb{R}^{m+1}\) close to \(f(x)\), where the coordinates of \(\alpha\) are algebraic and conjugate. Note, however, that there are subtle differences among these two types of approximation, as evinced by the difference between the classifications of numbers of Mahler and Koksma \([21, \text{Section 3.4}]\).

Nonetheless, a good first step towards establishing a result like Theorem \([1]\) is to provide an estimate for the number of such points \(\alpha\) which are sufficiently close to the manifold \(M\) parametrised by \(f\) (see e.g. \([45, \text{Section 2.6}]\)). Furthermore, the techniques used to derive such estimates can be of interest in and of themselves; for example, in the case of rational points they have been
adapted to derive an efficient algorithm to compute the rational points with bounded denominator on a given manifold, see [12] or [34, Section 11] for a nice overview. This problem was first considered for planar curves by Bernik, Götze and Kulik in [16]. In other words, let $\mathcal{R} \subset \mathbb{R}$ be a bounded open interval and let $f_1 : \mathcal{R} \to \mathbb{R}$ be a $C^1$ function; also, define the sets

$$
\mathcal{K}_n(Q) := \{ \alpha \in \mathbb{R} : \alpha \text{ is algebraic, } \deg(\alpha) \leq n, H(\alpha) \leq Q \}
$$

$$
M_{f_1}^n(Q, \gamma, \mathcal{R}) := \{ (\alpha_0, \alpha_1) \in \mathcal{K}_n(Q) : \alpha_0 \in \mathcal{R}, |f_1(\alpha_0) - \alpha_1| < c_0 Q^{-\gamma} \},
$$

where $c_0 > 0$ is fixed. Here by $\alpha \in \mathbb{R}^{m+1}$ algebraic we mean that its coordinates are algebraic conjugate real numbers, and by $H(\alpha)$ we denote the height of their minimal polynomial.

A lower bound for $\#M_{f_1}^n(Q, \gamma, \mathcal{R})$ was provided in [16] for $0 < \gamma < \frac{1}{2}$. This was soon extended in [13], where Bernik, Götze and Gusakova also provided an upper bound. We also note that recently Bernik, Budarina and Dickinson provided an analogous lower bound for surfaces in $\mathbb{R}^3$ [12].

**Theorem 1.2** ([13, Theorem 1]). Suppose that both $\sup_{x \in [0,1]} |f_1(x)|$ and $\# \{ x \in \mathcal{R} : f_1(x) = x \}$ are bounded. If $c_0$ is sufficiently large, then

$$
\#M_{f_1}^n(Q, \gamma, \mathcal{R}) \geq Q^{n+1-\gamma}
$$

for every $Q$ large enough and $0 < \gamma < 1$.

**Note.** Here and throughout the paper we will make extended use of Vinogradov’s notation. Namely, we will write $a \ll b$ if there is a constant $c > 0$ such that $a < cb$, as well as $b \gg a$ if $a \ll b$, and $a \asymp b$ when $a \ll b$ and $b \ll a$ simultaneously, in which case we say that $a$ is *comparable* to $b$. Occasionally we will make dependencies of the implied constant explicit via a subscript, e.g. $a \ll c b$, and $c$ is generally assumed to be independent of the variables that $a$ and $b$ depend on, although it could depend on the other parameters involved.

We also extend this notation to vectors in the natural way: if $a = (a_1, \ldots, a_r)$ and $b = (b_1, \ldots, b_s)$, then $a \ll b$ means that $a_i \ll b_i$ for every $1 \leq i \leq r$, and similarly for $\gg$ and $\asymp$.

In the present paper we will extend the lower bound in Theorem 1.2 to sufficiently regular manifolds in arbitrary dimension. While the characterisation of these manifolds is quite technical, as a special case our results hold true when $\mathcal{R}$ is analytic with algebraically independent components. In particular, the following is a special case of Theorem 2.13 which extends the range of $\gamma$ to the best possible — here $M_\gamma^n$ is the higher dimensional analogue of $M_{f_1}^n$, see [13] for details.

**Theorem 1.3.** Let $\mathcal{R} \subset \mathbb{R}^d$ be a bounded open set, and let $f(x) = (x_0, \ldots, x_{d-1}, f_d(x), \ldots, f_m(x))$ be an analytic function $\mathcal{R} \to \mathbb{R}^{m+1}$ with algebraically independent components. Then for $c_0 > 0$ fixed and for every

$$
0 < \gamma \leq \frac{n+1}{m+1}
$$

we have

$$
\#M_{f}^n(Q, \gamma, \mathcal{R}) \gg Q^{n+1-\gamma(m+1-d)}
$$

for every $Q$ sufficiently large, where the implied constant does not depend on $Q$. 

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In the process of proving Theorem 1.3, we will also be able to extend the divergence part of Theorem 1.1 as follows; here $\mathcal{H}^s$ denotes the usual Hausdorff $s$-measure (see Definition 7.2).

**Theorem 1.4.** Let $B, f$ be as in Theorem 1.3, and let $\psi: \mathbb{R}^+ \to \mathbb{R}^+$ be a decreasing function such that $\psi(Q) \gg Q^{n-m+1}$. Further, denote by $\mathcal{L}_{n,m+1}(\psi)$ the set of $x \in B$ such that $f(x) \in \mathcal{L}_{n,m+1}(\psi)$. Then for any $0 < s \leq d$ we have

$$\mathcal{H}^s(\mathcal{L}_{n,m+1}(\psi)) = \mathcal{H}^s(B) \quad \text{if} \quad \sum_{Q=1}^{\infty} Q^{n-m-1}\psi(Q)^{m+1} \left(\frac{\psi(Q)}{Q}\right)^{s-d} = \infty.$$ 

**Corollary 1.5** (Cfr. [5, Corollary 1]). Let $\dim_H$ denote Hausdorff dimension. In the same setting of Theorem 1.4, we have that

$$\dim_H(\mathcal{L}_{n,m+1}(\psi)) \geq \min \left\{ d, \frac{n+1}{\tau\psi} + d - m - 1 \right\},$$

where

$$\tau_{\psi} := \liminf_{Q \to \infty} -\frac{\log \psi(Q)}{\log Q}$$

is the lower order of $\psi^{-1}$ at infinity.

**Remark 1.6.** The condition $\psi(Q) \gg Q^{n-m+1}$ of Theorem 1.4 implies that $\tau_{\psi} \leq \frac{n-m}{m+1}$, which is precisely the situation where

$$\frac{n+1}{\tau_{\psi}} + d - m - 1 \geq d.$$ 

Therefore in this setting we actually have $\dim_H(\mathcal{L}_{n,m+1}(\psi)) = d$. On the other hand, Theorem 2.16 shows that the picture is more interesting with two separate approximation functions on $\mathbb{R}^d$ and $\mathbb{R}^{m+1-d}$.

Our proof exploits the powerful quantitative non-divergence bounds first introduced by Kleinbock and Margulis in [27], and it has a similar flavour to [5]. This paper is structured as follows:

- in the next section we will describe our setting and main results;
- in sections 3 and 4 we will discuss the regularity conditions that these depend on and provide some examples of functions that satisfy them;
- the next sections are devoted to the proofs, in this order: sections 5 and 6 for the extension of Theorem 1.2, section 7 for the extension of Theorem 1.1 and section 8 for the proof of Theorem 2.8, which underpins the whole argument; and
- the last section contains some final remarks about possible directions in which this work could be extended.

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2 The main result

Let $X$ be a metric space. If $\kappa > 0$ and $B \subset X$ is a ball centred at $x$ and with radius $r$, throughout this paper $\kappa B$ will denote the dilation of $B$ by $\kappa$, i.e. the ball with centre $x$ and radius $\kappa r$.

**Definition 2.1.** Let $N > 0$. Following [29], a metric space $X$ is called $N$-Besicovitch if, for any bounded set $A \subset X$ and any collection of balls $B$ such that every $x \in A$ is in the centre of a ball in $B$, there is a countable collection $\Omega \subseteq B$ which covers $A$ and such that every point $x \in A$ lies in at most finitely many balls in $\Omega$. We will also say that $X$ is Besicovitch if it is $N$-Besicovitch for some $N > 0$.

**Example 2.2.** It is well known that $\mathbb{R}^n$ with the Euclidean metric is Besicovitch, see e.g. [34, Theorem 2.7].

**Definition 2.3.** Let $U \subset X$ be an open subset and let $\nu$ be a Radon measure on $U$. Following [29], we will say that $\nu$ is:

- **D-Federer (or doubling) on** $U$ **for some** $D > 0$ **if**
  \[ \nu(3^{-1}B) > \frac{\nu(B)}{D} \]
  for any ball $B \subset U$ centred on $\text{supp} \, \nu$.

- **Federer** if for $\nu$ almost every $x \in X$ there are a neighbourhood $U$ of $x$ and a $D > 0$ such that $\nu$ is $D$-Federer on $U$.

- **(Rationally) non-planar** if $X = \mathbb{R}^d$ and $\nu(L) = 0$ for every (rational) affine hyperplane of $\mathbb{R}^d$.

- **($C,\alpha$)-decaying on $U$** for some $C, \alpha > 0$ **if** $X = \mathbb{R}^d$ **and for any ball** $B$ **centred on** $\text{supp} \, \nu$, **any affine hyperplane** $L \subset \mathbb{R}^d$, **and any** $\varepsilon > 0$ **we have**
  \[ \nu(B \cap L^{(\varepsilon)}) \leq C \left( \frac{\varepsilon}{\|d_L\|_{\nu,B}} \right)^\alpha \nu(B), \]
  where $L^{(\varepsilon)}$ is the $\varepsilon$-neighbourhood of $L$, $d_L$ is the Euclidean distance from $L$, and $\|d_L\|_{\nu,B} = \sup_{x \in B \cap \text{supp} \nu} d_L(x)$. C.f. Definition 4.1.

- **Absolutely ($C,\alpha$)-decaying on** $U$ **if** (6) **holds with the radius of** $B$ **in place of** $\|d_L\|_{\nu,B}$.

- **(Absolutely) decaying** if for $\nu$ almost every $x \in X$ there are a neighbourhood $U$ of $x$ and constants $C, \alpha > 0$ such that $\nu$ is (absolutely) ($C,\alpha$)-decaying on $U$.

**Remark 2.4.** Both classes of Federer and absolutely decaying measures are closed under restriction to open subsets $U \subset X$. Furthermore, they are also closed with respect to taking finite products [29, Theorem 2.4].
Example 2.5. Examples of measures that are Federerer and absolutely decaying on $\mathbb{R}^d$ include the Lebesgue measure and measures supported on certain self-similar sets (see e.g. [29] and [35]).

Now consider a $d$-dimensional manifold $M$ in $\mathbb{R}^{m+1}$, parametrised over a bounded open subset $\mathcal{R} \subset X$ by a continuous map $f(x) = (f_0(x), \ldots, f_m(x))$. Without loss of generality, if $X = \mathbb{R}^d$ we will assume that $f_i(x) = x_i$ for $0 \leq i < d$ and write $f$ for $(f_0, \ldots, f_m)$. Then, for $n \geq m + 1$ fixed, define the vectors in $\mathbb{R}^n$ for the submatrix of $A$ indexed by $\{ \tau \} \cap J$.

$$v_i = v_i(x) := (\begin{array}{c} f_0(x) \\ f_i(x) \\ f_i(x)^2 \\ \vdots \\ f_i(x)^n \end{array}) \quad 0 \leq i \leq m$$

and the $(n+1) \times (n+1)$ matrices

$$M_f := \begin{pmatrix} v_0 \\ \vdots \\ v_m \\ 0 \mid I_{n-m} \end{pmatrix} \quad (7)$$

$$U^h_f := \begin{pmatrix} v_0 \\ \vdots \\ v_m \\ v'_h \mid I_{n-m-1} \end{pmatrix} \quad (8)$$

Remark 2.6. The determinant of $U^h_f$ is the same as the determinant of the submatrix $\tilde{U}^h_f$ formed by its first $n+1$ rows and columns. The latter is an example of what in the literature is known as a confluent Vandermonde matrix, and a theorem of Schendel’s (see e.g. [31, Theorem 20]) shows that

$$|\det \tilde{U}^h_f| = \prod_{0 \leq i < j \leq m} |f_i(x) - f_j(x)|^{e_i e_j},$$

where $e_i$ is 2 if $i = h$ and 1 otherwise. In particular, $\det U^h_f \neq 0$ if and only if the Vandermonde polynomial $V(f)$ is non-zero (see [19] below).

For ease of notation, we will write $[n]$ instead of $\{0, \ldots, n\}$, as well as $[n]_<$ for the set of $I = (i_1, \ldots, i_r) \in [n]^r$ such that $i_1 < i_2 < \cdots < i_r$, where $1 \leq r \leq n+1$. Given an $(n+1) \times (n+1)$ matrix $A$, we will also write $A_{I,J}$ for the submatrix of $A$ with rows indexed by $I \subseteq [n]_<$ and columns indexed by $J \subseteq [n]_<$, and $|A|_{I,J}$ for its determinant.

Then, for every $1 \leq \tau \leq n+1$ and for every $I \in [n]_<^\tau$, define the map from the set $M_{n+1,n+1}$ to $\mathbb{R}^{n+1} \simeq \mathbb{R}^{(n+1)\tau}$ given by

$$G_I: A \mapsto (|A|_{I,J})_{J \in [n]_<^\tau}. \quad (9)$$

In other words, $G_I(A)$ is the image under the Plücker embedding of the linear subspace of $\mathbb{R}^{n+1}$ spanned by the rows of $A$ indexed by $I$. Furthermore, in
Section 3 we will see that for \( I \in \llbracket m \rrbracket \), \( 1 \leq \tau \leq m + 1 \) we have
\[
G_I(M_f) = \left( V(f_I) s_{\lambda} f \right)_{|\lambda| \leq n+1-\tau, \ell(\lambda) \leq \tau}
\]
where \( V(f_I) \) is the Vandermonde polynomial of \( f_I = (f_i)_{i \in I} \), i.e.
\[
V(f_I) := \prod_{i,j \in I, i < j} (f_j - f_i),
\]}

and \( s_{\lambda} \) is the Schur polynomial in \( \tau \) indeterminates corresponding to the partition \( \lambda \) of the integer \( |\lambda| \) with \( \ell(\lambda) \) parts (see Definition 3.3). Therefore, we also define
\[
S_{n,\tau}(T) = \left( s_{\lambda}(T) \right)_{|\lambda| \leq n+1-\tau, \ell(\lambda) \leq \tau}
\]
and with a slight abuse of notation we will write \( S_I(T) \) or \( S(f) \) instead of \( S_{n,\tau}(T) \) whenever \( n \) or \( \tau \) are clear from the context. Finally, observe that if \( V(f_I) \) is bounded on \( \mathcal{B} \), then \( G_I(M_f) \approx S(f) \).

Definition 2.7. Let \( X \) be a measure space and \( \nu \) a measure on \( X \). Fix \( \tau \geq 1 \) and let \( \Lambda^k_\tau \) be the space of rational symmetric polynomials of degree up to \( k \). Given a map \( f: \mathcal{B} \subseteq X \to \mathbb{R}^\tau \), the pair \((f, \nu)\) is called:

- **Non-symmetric (of degree \( k \)) at \( x \)** if for every neighbourhood \( B \ni x \) and \( s \in \Lambda^k_\tau \) we have that \( f(B \cap \text{supp}\ \nu) \) is not contained in the zero locus of \( s \) in \( \mathbb{R}^\tau \). Cf. the definitions of non-planarity from [28, 30].
- **Non-symmetric (of degree \( k \)) on \( \mathcal{B} \)** if it is non-symmetric of degree \( k \) at every \( x \in \mathcal{B} \cap \text{supp}\ \nu \).
- **Symmetrically good (of degree \( k \)) on \( \mathcal{B} \)** if it is non-symmetric (of degree \( k \)) on \( \mathcal{B} \) and there are constants \( C, \alpha > 0 \) such that \( s(f) \) is \((C, \alpha)\)-good on \( \mathcal{B} \) with respect to \( \nu \) for every \( s \in \Lambda^k_\tau \) (see Definition 4.1).

**Note.** See Corollary 3.5 for an equivalent characterisation of \((f, \nu)\) being non-symmetric of degree \( k \) in terms of the components of an appropriate \( S(f) \).

Now take functions \( \psi_0, \ldots, \psi_m, \varphi_{m+1}, \ldots, \varphi_n: \mathbb{R}^+ \to \mathbb{R}^+ \), and for \( Q > 0 \) consider the system of inequalities
\[
\begin{align*}
|P(f_k(x))| &< \psi_k(Q) & \text{for } 0 \leq k \leq m \\
\max_i |P'(f_i(x))| &\leq \varphi_{m+1}(Q) \\
|a_k| &\leq \varphi_k(Q) & \text{for } m + 1 < k \leq n
\end{align*}
\]}

with solutions in integer polynomials \( P = a_n X^n + \ldots + a_0 \) of degree at most \( n \). Our main result concerns the set
\[
D^p_f(Q, \mathcal{B}) = D^p_f(Q, \mathcal{B}; \psi_0, \ldots, \psi_m, \varphi_{m+1}, \ldots, \varphi_n)
\]}

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of points \( x \in B \) for which \([12]\) admits a solution. For ease of notation, given \( I \in [0, m]_c \) and \( J \in [m + 1, n]_c \), let

\[
\psi_I := \prod_{i \in I} \psi_i, \quad \varphi_J := \prod_{j \in J} \varphi_j,
\]
as well as \( \psi := \psi_{[0,m]} \) and \( \varphi := \varphi_{[m+1,n]} \).

**Theorem 2.8.** Let \( X \) be an \( N \)-Besicovitch space, \( B \subset X \) a bounded open subset, and let \( \nu \) be a \( D \)-Federer measure on \( B \). Let \( f : B \to \mathbb{R}^{m+1} \) be a continuous function such that \( c_8 \geq V(f) \geq c_9 \) on \( B \cap \supp \nu \), where \( V(f) \) is the Vandermonde polynomial of \( f \). Furthermore, let \( \psi, \varphi \) be as above, and suppose that for some \( n > 0 \)

\[
\varphi_{m+1}(Q)^{n+1} \gg \psi(Q)\varphi(Q)
\]

and that for every \( 1 \leq \tau \leq m + 1 \) there is a choice of \( I \in \mathbb{N}^n \) such that

\[
\psi(Q)\varphi(Q) \gg \psi_I(Q)^{n+1}, \quad \text{and}
\]

(\( f_1, \nu \)) is symmetrically (\( \tilde{C}, \alpha \))-good of degree \( n + 1 - \tau \) on \( B \)

for some \( \tilde{C}, \alpha > 0 \). Then for any \( 0 < \theta < 1 \) and for \( Q \) large enough we may find a subset \( B_0 \subset B \) with measure \( \nu(B_0) > \theta \nu(B) \), as well as \( C = C(\tilde{C}, c_8, c_9, n, D) > 0 \) and \( \rho = \rho(f, n, B_0) > 0 \), such that

\[
\nu(D^\rho_{Q}(B_0)) \leq C \left( \frac{\psi(Q)\varphi(Q)}{\rho^{n+1}} \right)^{\frac{n}{\rho}} \nu(B_0).
\]

**Remark 2.9.** The set \( B_0 \) can be chosen to be either compact or a union of finitely many open balls (which form a cover for this compact set).

**Note.** Corollaries 3.10 and 4.6 below show that it is relatively straighforward to check condition (16) when \( X = \mathbb{R}^d \) and \( f \) is analytic.

**Corollary 2.10.** Let \( D^\rho_{Q}(B) = \limsup_Q D^\rho_{Q}(B, B) \). Under the hypothesis of Theorem 2.8, further assume that \( X = \mathbb{R}^d \), \( f \) is analytic and \( \nu \) is the Lebesgue measure on \( B \). Then \( \alpha \) can be chosen to be one of:

- \( 1/d(N-1) \), or
- \( 1/\deg(f) \) if \( f \) is a polynomial map.

Moreover,

\[
\sum_{Q=1}^{\infty} (\psi(Q)\varphi(Q))^{\frac{n}{\rho}} < \infty \quad \text{implies} \quad \nu(D^\rho_{Q}(B)) = 0.
\]

**Corollary 2.11.** Consider \( \psi_0, \ldots, \psi_m, f \) as in Theorem 2.8 such that condition \([14]\) holds with \( \varphi_{m+1}(Q) = \cdots = \varphi_n(Q) = Q \). Furthermore, assume that for every \( 0 \leq i \leq m \) and for \( Q \) large enough \( \psi_i(Q)^{-1}Q^{-1} \) is decreasing, and that there are constants \( c_4, c_3 > 0 \) such that

\[
c_3 \leq \psi(Q)Q^{n-m} \leq c_4.
\]
Then for every $0 < \theta < 1$ there are a constant $c > 0$ and a subset $B_\theta$ of $B$, independent of $Q$, such that $\nu(B_\theta) > \theta \nu(B)$ and every $x \in B_\theta$ admits $n + 1$ distinct points $(\alpha_0, \ldots, \alpha_m) \in \mathbb{R}^{m+1}$ with algebraic conjugate coordinates of height $H(\alpha_k) \ll Q$ which satisfy

$$|f_k(x) - \alpha_k| < \frac{c \psi_k(Q)}{Q} \quad \text{for } 0 \leq k \leq m$$

(17)

whenever $Q > 0$ is sufficiently large.

Remark 2.12. Here $c$ can be chosen to be $\frac{c_1}{c_2}$, where the constants $c_1, c_2$ are the same as in Corollary 5.1. In particular, it depends on $c_3$ and $c_4$ but not on the functions $\psi_i$ themselves.

Now suppose that $B \subset \mathbb{R}^d$ and without loss of generality assume that $f_i(x) = x_i$ for each $0 \leq i < d$. Then for a given $c_0 > 0$, let (c.f. (15))

$$K_{n+1}^m(Q) := \{ \alpha \in \mathbb{R}^{m+1} : \alpha \text{ is algebraic, } \deg(\alpha) \leq n, H(\alpha) \leq Q \}$$

$$M_f^\gamma(Q, \gamma, \mathcal{B}) := \{(\alpha_0, \ldots, \alpha_m) \in K_{n+1}^m(Q) : \alpha = (\alpha_0, \ldots, \alpha_{d-1}) \in \mathcal{B}, \max_{d \leq j \leq m} |f_j(\alpha) - \alpha_j| < c_0 Q^{-\gamma} \}$$

(18)

where and $\gamma > 0$. Then we are able to extend the lower bound from Theorem 1.2 as follows:

Theorem 2.13. Let $f : \mathcal{B} \to \mathbb{R}^{m+1}$ be a $C^1$ map as above such that $V(f) \neq 0$, and assume that, up to reordering $f_d, \ldots, f_m$,

$$(x, f_d, \ldots, f_\tau) \text{ is symmetrically good of degree } n - \tau \text{ on } \mathcal{B}$$

(19)

for every $d \leq \tau \leq m$. Then for $c_0 > 0$ fixed and for every

$$0 < \gamma \leq \frac{n + 1}{m + 1}$$

(20)

we have

$$\#M_f^\gamma(Q, \gamma, \mathcal{B}) \gg Q^{n+1-\gamma(m+1-d)}$$

for every $Q$ sufficiently large, where the implied constant does not depend on $Q$.

Remark 2.14. Theorem 2.13 is proved by taking $\psi_k = Q^{1-\gamma}$ for $d \leq k \leq m$ in Corollary 2.11. In particular, Sprindžuk’s conjecture (3) shows that the upper bound $\gamma \leq \frac{n + 1}{m + 1}$ is in general the best possible.

Remark 2.15. If we could show that the lower bound for $Q$ and the implied constant in Theorem 2.13 can be chosen independently of translations of $\mathcal{M}$, then we would also have an upper bound for $\#M_f^\gamma(Q, \gamma, \mathcal{B})$. Indeed, without loss of generality we may assume that $f$ is bounded on $\mathcal{B}$, hence $\mathcal{M}$ is contained in an open set $K$ of volume comparable to $\text{vol}(\mathcal{B})$. Now let $\mathcal{M}_\gamma$ be the $\gamma$-neighbourhood of $\mathcal{M}$, i.e. the set

$$\mathcal{M}_\gamma := \{(x, y) \in \mathcal{B} \times \mathbb{R}^{m+1-d} : \|f(x) - y\| < Q^{-\gamma} \}$$
and note that \( \text{vol}(\mathcal{M}_r) \approx \text{vol}(\mathcal{B}) Q^{-\gamma(m+1-d)} \). In particular, up to replacing \( K \) with a slightly bigger open set, we may assume that \( K \) contains a union of disjoint translated copies \( \{\mathcal{M}_j\}_{j \in J} \) of \( \mathcal{M}_r \), with

\[
\#J \approx \text{vol}(K)/\text{vol}(\mathcal{M}_r) \approx Q^{\gamma(m+1-d)}.
\]

If the implied constant in Theorem 2.13 can be chosen to be in a translation invariant way, then we may find \( c, Q_0 > 0 \) such that for every \( Q > Q_0 \) and for every \( j \in J \) we have

\[
\#(\mathcal{M}_j \cap A_n^{m+1}(Q)) > cQ^{n+1-\gamma(m+1-d)}.
\]

It follows that

\[
\#(K \cap A_n^{m+1}(Q)) \gg Q^{n+1-\gamma(m+1-d)}\#J \gg Q^{n+1}.
\]

However, since there are only \( Q^{n+1} \) polynomials of degree at most \( n \) and height at most \( Q \), we can conclude that

\[
\#M_f(Q, r, \mathcal{B}) \ll Q^{n+1-\gamma(m+1-d)}
\]

as well, matching the lower bound.

We conclude this section by stating our extension of Theorem 1.1. Define \( \mathcal{L}_{n,m+1}(\psi; \Psi; d) \) to be the set of \( x \in \mathbb{R}^{n+1} \) such that

\[
\max_{0 \leq k < d} |P(x_k)| < \psi(H(P)) \quad \text{and} \quad \max_{d \leq k \leq m} |P(x_k)| < \Psi(H(P))
\]

for infinitely many \( P \in \mathbb{Z}[X] \) with \( \text{deg}(P) \leq n \), and note that \( \mathcal{L}_{n,m+1}(\psi; \Psi; d) = \mathcal{L}_{n,m+1}(\psi) \) is the set defined in Theorem 1.1. When \( f \) parametrises a \( d \)-dimensional manifold \( M \) and \( f_k(x) = x_k \) for every \( 0 \leq k < d \), will also write \( \mathcal{L}_{n,f}(\psi, \Psi) \) for the set of \( x \in \mathcal{B} \) such that \( f(x) \in \mathcal{L}_{n,m+1}(\psi; \Psi; d) \).

**Theorem 2.16.** Let \( \psi, \Psi : \mathbb{R}^+ \to \mathbb{R}^+ \) be decreasing functions such that

\[
\Psi(Q) \gg \max \{\frac{Q^{m+1}}{r}, \psi(Q)\}, \tag{21}
\]

and let \( g \) be a dimension function such that \( r^{-d}g(r) \) is non-increasing. Also assume that \( r^{-\gamma}g(r) \) is increasing for some \( \gamma > 0 \), and that there are constants \( r_0, c_{12}, c_{13} \in (0, 1) \) such that

\[
r^{-\gamma}g(c_{12}r) \leq c_{13}g(r)(c_{12}r)^\gamma \quad \text{for any} \ r \in (0, r_0). \tag{22}
\]

Further suppose that \( f \) is Lipschitz continuous, that \( V(f) \neq 0 \), and that \( f \) is symmetrically good of degree \( n+1-d \) on \( \mathcal{B} \). Then

\[
\mathcal{H}^d(\mathcal{L}_{n,f}(\psi, \Psi)) = \mathcal{H}^d(\mathcal{B}) \quad \text{if} \quad \sum_{Q=1}^{\infty} Q^{n-m-1+d}\psi(Q)^{m+1-d}g\left(\frac{\psi(Q)}{Q}\right) = \infty.
\]

Moreover,

\[
|\mathcal{L}_{n,f}(\psi, \Psi)| = 0 \quad \text{if} \quad \sum_{Q=1}^{\infty} Q^{n-m-1}\psi(Q)^d\Psi(Q)^{m+1-d} < \infty.
\]
The generalised Hausdorff measure $\mathcal{H}^s$ will be introduced in Definition \[\text{[24]}\]. For the moment observe that when $g(r) = r^d$ we have that $\mathcal{H}^s$ is a constant multiple of the Lebesgue measure on $\mathbb{R}^d$, thus we recover a version of Theorem \[\text{[11]}\] for symmetrically good manifolds.

**Note.** Condition \[\text{[22]}\] is not particularly restrictive, and in particular it is trivially satisfied for the usual Hausdorff $s$-measures, i.e. when $g(r) = r^s$ for some real $s > 0$.

## 3 Schur polynomials

Throughout this section, we will denote by $\Lambda_\tau = \mathbb{Q}[T_0, \ldots, T_{\tau-1}]^{S_\tau}$ the space of symmetric polynomials in $\tau$ variables, and define

$$\Lambda_\tau^k := \{ s \in \Lambda_\tau : \deg(s) \leq k \}.$$  

**Definition 3.1.** Let $f_0, \ldots, f_{\tau-1}$ be a collection of $\tau$ real valued functions. The order of symmetric independence of $f_0, \ldots, f_{\tau-1}$, denoted by $s(f_0, \ldots, f_{\tau-1})$, is either

$$\min \{ k : s(f_0, \ldots, f_{\tau-1}) \neq 0 \text{ for every } s \in \Lambda_\tau^k \},$$

or $\infty$ when $f_0, \ldots, f_{\tau-1}$ are algebraically independent over $\mathbb{Q}$.

**Note.** The functions $f_0, \ldots, f_{\tau-1}$ are algebraically independent over $\mathbb{Q}$ if and only if there is no symmetric polynomial $S \in \Lambda_\tau$ such that $S(f_0, \ldots, f_{\tau-1}) = 0$. Indeed, observe that if $P(f_0, \ldots, f_{\tau-1}) = 0$ for some rational polynomial $P$ in $\tau$ variables, then

$$S(T_0, \ldots, T_{\tau-1}) := P^{S_\tau} = \prod_{\sigma \in S_\tau} P(T_{\sigma(0)}, \ldots, T_{\sigma(\tau-1)})$$

is a symmetric polynomial such that $S(f_0, \ldots, f_{\tau-1}) = 0$.

**Note.** Comparing with Definition \[\text{[24]}\] we see that $(f, \nu)$ is non-symmetric of degree $k$ at $\mathbf{x}$ if and only if for every ball $B$ containing $\mathbf{x}$ we have $s\left( f_{\mid B \cap \text{supp} \nu} \right) \geq k$.

Now consider $\lambda = (\lambda_0, \ldots, \lambda_{\tau-1})$ with $\lambda_0 \geq \lambda_1 \geq \cdots \geq \lambda_{\tau-1} \geq 0$, i.e. a partition of the integer $|\lambda| := \lambda_0 + \cdots + \lambda_{\tau-1}$ with $\ell(\lambda) \leq \tau$ parts. Given two such partitions $\lambda^1$ and $\lambda^2$, we will define their sum component by component, i.e. $\lambda^1 + \lambda^2 = (\lambda_0^1 + \lambda_0^2, \ldots, \lambda_{\tau-1}^1 + \lambda_{\tau-1}^2)$. Also, let $\mu := (\tau-2, \tau-3, \ldots, 0)$ be the minimal such partition with distinct parts. Then, the alternating polynomial corresponding to $\lambda$ is

$$a_{\lambda + \mu}(T_0, \ldots, T_{\tau-1}) := \det(T_1^{\lambda_i + \mu_0}) = \begin{vmatrix} T_0^{\lambda_0 + \mu_0} & \cdots & T_0^{\lambda_{\tau-1} + \mu_{\tau-1}} \\ \vdots & \ddots & \vdots \\ T_{\tau-1}^{\lambda_0 + \mu_0} & \cdots & T_{\tau-1}^{\lambda_{\tau-1} + \mu_{\tau-1}} \end{vmatrix}.$$
Example 3.2. The alternating polynomial corresponding to \((0,\ldots,0)\) is the Vandermonde polynomial on \(T\), i.e. \(a_\mu = V(T)\).

Definition 3.3. By Cauchy’s Bi-Alternant Formula we know that \(a_\mu\) divides \(a_{\lambda+\mu}\) for every partition \(\lambda\) (see \([16]\) for a concise proof). Further, the quotient is a symmetric polynomial, and we define the Schur polynomial in \(r\) variables corresponding to \(\lambda\) as
\[
s_\lambda := \frac{a_{\lambda+\mu}}{a_\mu}.
\]
We can also extend this to \(\ell(\lambda) > r\) by setting \(s_\lambda = 0\), and we will denote by \(S^k_r(T)\) the collection of all the \(s_\lambda(T)\) with \(|\lambda| \leq k\) and \(\ell(\lambda) \leq r\) (c.f. the definition of \(S_n,r(T)\) at \((11)\)). Note that \(\#S^k_r(T) = \binom{k+r-1}{r}\).

One can show that \(s_\lambda\) is symmetric and homogeneous of degree \(|\lambda|\), which makes it straighforward to see that
\[
s_\lambda(T_0,\ldots,T_{\ell(\lambda)},0,\ldots,0) = s_\lambda(T_0,\ldots,T_{\ell(\lambda)})
\]
when \(\ell(\lambda) < r\). There is a wealth of literature about Schur polynomials, and the interested reader is invited to consult either I. G. Macdonald’s book \([32]\), \([24]\) for a more gentle introduction. In particular, we will need the following result.

Proposition 3.4 (\([32\), (3.3), p. 41]). The Schur polynomials in \(S^k_r(T)\) form a basis for \(\Lambda^k_r\) as a module over \(\mathbb{Q}\).

Corollary 3.5. The following are equivalent:

- \((f,\nu)\) is non-symmetric of degree \(k\) on \(\mathcal{B}\);
- for every \(x \in \mathcal{B} \cap \text{supp} \nu\), every neighbourhood \(B \ni x\), and every partition \(\lambda\) with \(|\lambda| \leq k\) and \(\ell(\lambda) \leq r\), the restrictions of \(s_\lambda(f)\) to \(B \cap \text{supp} \nu\) are linearly independent over \(\mathbb{Q}\).
- \((S^k \circ f)_*\nu\) is rationally non-planar.

Remark 3.6. It follows that \((f,\nu)\) is symmetrically good of degree \(k\) if and only if \((S^k \circ f)_*\nu\) is decaying and rationally non-planar. C.f. the notion of friendly measure from \([29]\), i.e. a measure that is Federer, decaying and non-planar.

We conclude this section with some criteria to estimate \(s(f)\).

Proposition 3.7. Let \(f = (f_0,\ldots,f_{r-1})\). Then for every \(2 \leq t \leq r\) and for every \(I \in [r]_t\) we have
\[
s(f) < \frac{n!}{t!}(s(f_I) + 1).
\]
Proof. Fix \(t, I,\) and let \(s \in \Lambda_t\) be a polynomial of degree \(s(f_I) + 1\) such that \(s(f_I) = 0\). Since \(s\) is symmetric we know that \(S_t\) fixes \(s\), thus there is a well defined action of \(G_t := S_t/S_t\) on the image of \(s\) under the inclusion \(\mathbb{Q}[T_0,\ldots,T_{r-1}] \subset \mathbb{Q}[T_0,\ldots,T_{r-1}]\). It follows that \(s^G\) is a symmetric polynomial in \(r\) variables of degree \(\text{deg}(s)\#G\) such that \(s^G(f) = 0\).
Now let \( d, N \) be positive integers, and for every \( 0 \leq s \leq N - 1 \) let \( \Delta_s \) be a differential operator of the form

\[
\Delta_s = \left( \frac{\partial}{\partial x_0} \right)^{j_0} \cdots \left( \frac{\partial}{\partial x_{d-1}} \right)^{j_{d-1}} \text{ where } j_0 + \cdots + j_{d-1} \leq s. \tag{23}
\]

Given a \( C^{N-1} \) map \( g(x_0, \ldots, x_{d-1}) \) with \( N - 1 \) components, define the generalised Wronskian of \( g \) associated with \( \Delta_0, \ldots, \Delta_{N-2} \) to be the determinant

\[
det(\Delta_s(g_j)) = \begin{vmatrix}
\Delta_0(g_0) & \cdots & \Delta_0(g_{N-1}) \\
\vdots & & \vdots \\
\Delta_{N-1}(g_0) & \cdots & \Delta_{N-1}(g_{N-1})
\end{vmatrix}.
\]

This definition can also be extended to the case where \( g_0, \ldots, g_{N-1} \) are formal power series with coefficients in a field \( K \). Furthermore, note that if the components of \( g \) are linearly dependent, then all of its generalised Wronskians vanish.

**Theorem 3.8** ([17, Theorem 3]). Let \( g_0, \ldots, g_{N-1} \) be formal power series with coefficients in a field \( K \) of characteristic 0. If they are linearly independent over \( K \), then at least one of their generalised Wronskians is non-zero.

**Corollary 3.9** (Wronskian Criterion). Let \( g = (g_1, \ldots, g_{N-1}) \) be a \( C^{N-1} \) real valued map. If at least one of the generalised Wronskians of \( g \) is non-zero, then \( g_1, \ldots, g_N \) are linearly independent over \( \mathbb{R} \), and the converse holds when \( g \) is analytic.

**Corollary 3.10.** Let \( k, \tau \) be positive integers and let \( N = \binom{k+\tau-1}{\tau} \). If \( f \) is a \( C^{N-1} \) real valued map with \( \tau \) components and at least one of the generalised Wronskians of \( \Delta_k^\tau(f) \) is non-zero, then \( s(f) \geq k \).

To simplify the proof of the final result we will rely on another special kind of symmetric polynomial, the monomial symmetric polynomials \( m_\lambda \). Let \( \lambda \) be a partition of integers with at most \( \tau \) parts; then \( m_\lambda \) is defined as

\[
m_\lambda := \sum_\sigma T_0^{\sigma(\lambda_0)} \cdots T_{\tau-1}^{\sigma(\lambda_{\tau-1})}
\]

where \( \sigma \) runs over the distinct permutations of \( \lambda_0, \ldots, \lambda_{\tau-1} \). Again, it can be shown that the collection of monomial symmetric polynomials corresponding to \( \lambda \) with \( |\lambda| \leq k \) and \( \ell(\lambda) = \tau \) forms a basis for \( \Lambda^k_\tau \) as a module over \( \mathbb{Q} \).

**Proposition 3.11.** Let \( p = (p_0, p_1) \) be a polynomial map with \( \deg p_0 > \deg p_1 \). Then \( s(p) \geq \frac{\deg p_0}{\deg p_1} \).

**Proof.** Let \( d_i := \deg p_i \), and note that if \( \lambda = (\lambda_0, \lambda_1) \) is a partition with \( k \geq \lambda_0 \geq \lambda_1 \geq 0 \), then \( \deg m_\lambda(p) = d_0 \lambda_0 + d_1 \lambda_1 \). We will show that the map \( \lambda \mapsto \deg m_\lambda(p) \) is injective for \( k \leq \frac{d_0}{d_1} \), which immediately gives a lower bound for \( s(p) \).
Suppose that $d_0 \lambda_0^2 + d_1 \lambda_1^1 = d_0 \lambda_0^2 + d_1 \lambda_1^1$ for some $\lambda_1^1 \neq \lambda_2^1$, and without loss of generality assume $\lambda_1^1 > \lambda_2^1$. Then $d_1(\lambda_1^1 - \lambda_2^1) = d_0(\lambda_2^1 - \lambda_1^1)$, which results in

$$k \geq \lambda_1^2 = \lambda_1^1 + \frac{d_1}{d_0}(\lambda_1^1 - \lambda_2^1) \geq 1 + \frac{d_1}{d_0} \square$$

**Example 3.12.** At least when $p(x) = (p_0(x), \ldots, p_{\tau-1}(x))$ is a polynomial map with rational coefficients, we can compute $s(p)$ with relative efficiency using variable elimination via Gröbner bases. Even more, it is possible to describe all the symmetric polynomials that vanish on $p$. Indeed, let $e_1, \ldots, e_\tau$ be the elementary symmetric polynomials in $\tau$ variables, that is

$$e_k(T_0, \ldots, T_{\tau-1}) := \sum_{I \in \llbracket \tau-1 \rrbracket^k} T_{i_1} \cdots T_{i_k}.$$  

Then it is well known that $\Lambda_\tau = \mathbb{Q}[e_1(T), \ldots, e_\tau(T)]$, in other words, every symmetric polynomial in $\mathbb{T}$ can be written as a polynomial in $e_1, \ldots, e_\tau$. Now let $Y = (Y_1, \ldots, Y_\tau)$ and consider the ideal $I \subseteq \mathbb{Q}[x, Y]$ generated by the polynomials

$$Y_k - e_k(p) \quad \text{for } 1 \leq k \leq \tau.$$  

It is possible to compute a Gröbner basis $G$ for the ideal $\tilde{I} := I \cap \mathbb{Q}[Y]$ through standard algorithms, and we can see that every symmetric polynomial in $\Lambda_\tau$ which vanishes on $p$ is of the form $h(e_1, \ldots, e_\tau)$ for some $h \in \tilde{I}$. In particular,

$$s(p) = \min_{g \in G} \deg(g(e_1, \ldots, e_\tau)).$$

As an example, these are the orders of symmetric independence for the Veronese curves of degree $\tau$ between 2 and 10, i.e. for $p(x) = (x, x^2, \ldots, x^\tau)$:

| $\tau$ | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|--------|---|---|---|---|---|---|---|---|----|
| $s(p)$| 4 | 5 | 5 | 6 | 6 | 7 | 7 | 7 | 7 |

**4 Good functions**

**Definition 4.1**. Let $X$ be a metric space and $\nu$ a Radon measure on $X$. Also consider an open subset $U \subseteq X$ and a $\nu$-measurable function $f : U \to \mathbb{R}$. For any open ball $B \subseteq U$ and $\varepsilon > 0$, define

$$B^{f, \varepsilon} := \{ x \in B : |f(x)| < \varepsilon \}.$$  

Then we say that $f$ is $(C, \alpha)$-good on $U$ with respect to $\nu$ if there are constants $C, \alpha > 0$ such that for any open ball $B \subseteq U$ centred on $\text{supp} \, \nu$ we have

$$\nu(B^{f, \varepsilon}) \leq C \left( \frac{\varepsilon}{\|f\|_{\nu, B}} \right)^\alpha \nu(B) \quad \text{for all } \varepsilon > 0, \quad (24)$$
where \( \|f\|_{\nu,B} := \sup_{x \in B \cap \text{supp} \nu} |f(x)| \). Also, when \( X = \mathbb{R}^d \) and \( \nu \) is the corresponding Lebesgue measure we will write \( \|f\|_B \) for \( \|f\|_{\nu,B} \), and we say that \( f \) is absolutely \((C, \alpha)\)-good on \( U \) with respect to \( \nu \) if \((24)\) holds with \( \|f\|_B \) in place of \( \|f\|_{\nu,B} \).

Note that absolute \((C, \alpha)\)-goodness implies \((C, \alpha)\)-goodness, while the converse holds for measures with full support. The following properties are a direct consequence of the definition.

**Lemma 4.2** ([31, Lemma 3.1], [14, Lemma 3.1]).

1. If \( f \) is \((C, \alpha)\)-good on \( U \) wrt \( \nu \) if and only if so is \( |f| \).
2. If \( f \) is \((C, \alpha)\)-good on \( U \) wrt \( \nu \), then so is \( \lambda f \) for every \( \lambda \in \mathbb{R} \).
3. If \( f \) is \((C, \alpha)\)-good on \( U \) wrt \( \nu \), then it is also \((C', \alpha')\)-good on \( U' \) wrt \( \nu \) for every \( C' \geq C \), \( \alpha' \leq \alpha \) and \( U' \subseteq U \).
4. If \( \{f_i\}_{i \in I} \) is a collection of \((C, \alpha)\)-good functions on \( U \) wrt \( \nu \) and the function \( f := \sup_{i \in I} |f_i| \) is Borel measurable, then \( f \) is also \((C, \alpha)\)-good on \( U \) wrt \( \nu \).
5. If \( f \) is \((C, \alpha)\)-good on \( U \) wrt \( \nu \) and \( c_6 \leq \frac{|f(x)|}{|g(x)|} \leq c_7 \) for every \( x \in U \cap \text{supp} \nu \), then \( g \) is \((C, c_7/c_6)\)-good on \( U \) wrt \( \nu \).

**Proof of 4.2** Note that if \( \varepsilon > |g(x)| \geq \frac{|f(x)|}{|g(x)|} \) on \( U \cap \text{supp} \nu \), then \( B^{g, \varepsilon} \cap \text{supp} \nu \subseteq B^{|f|, \varepsilon} \cap \text{supp} \nu \) for every ball \( B \subseteq U \). Furthermore,

\[
c_6 \|g(x)\|_{\nu,B} = \sup_{x \in B \cap \text{supp} \nu} c_6 |g(x)| \leq \sup_{x \in B \cap \text{supp} \nu} |f(x)| = \|f(x)\|_{\nu,B}.
\]

Therefore

\[
\nu(B^{g, \varepsilon}) \leq \nu(B^{|f|, \varepsilon}) \\
\leq C \left( \frac{c_7 \varepsilon}{\|f\|_{\nu,B}} \right)^\alpha \nu(B) \\
\leq C \left( \frac{c_7}{c_6} \right)^\alpha \left( \frac{\varepsilon}{\|g\|_{\nu,B}} \right)^\alpha \nu(B).
\]

The papers [27] and [14] include various examples of real valued functions which are \((C, \alpha)\)-good with respect to Lebesgue measure. Moreover, [31] extends those examples to functions with values in non-Archimedean fields which satisfy a condition equivalent to \((24)\). For the purposes of the present paper we are mainly interested in the following propositions.

**Proposition 4.3** ([3, Proposition 2.8]). Fix \( d, m, k \in \mathbb{Z}_{>0} \) and let \( g = (g_1, \ldots, g_N) : \mathbb{R}^d \to \mathbb{R}^N \) be a polynomial map of degree at most \( k \). Then for any convex subset \( B \subseteq \mathbb{R}^d \) we have

\[
|\{x \in B : \|g(x)\| < \varepsilon\}| \leq 4d \left( \frac{\varepsilon}{\|g\|_B} \right)^{\frac{k}{2}} |B|,
\]

where \( \|g\|_B = \sup_B \|g(x)\| \) and \( \|g(x)\| = \max_i |g_i(x)| \).
Now, suppose that \( U \subset \mathbb{R}^d \) is open and that \( g = (g_1, \ldots, g_N) : U \rightarrow \mathbb{R}^N \) is a \( C^\ell \) map. For a given \( x \in U \), we say that \( g \) is \( \ell \)-nondegenerate at \( x \) if the partial derivatives of \( g \) at \( x \) of order up to \( \ell \) span \( \mathbb{R}^N \). In [27], Kleinbock and Margulis proved the following result on the \((C, \alpha)\)-goodness with respect to Lebesgue measure of \( \ell \)-nondegenerate functions, which was later extended in [29] to a wider class of measures.

**Proposition 4.4** ([27, Proposition 3.4]). Let \( g = (g_1, \ldots, g_N) : U \subset \mathbb{R}^d \rightarrow \mathbb{R}^N \) be a \( C^\ell \) map, \( U \) open. If \( g \) is \( \ell \)-nondegenerate at \( x \in U \), then there are a neighbourhood \( V \subset U \) of \( x \) and \( C > 0 \) such that any linear combination of \( 1, g_1, \ldots, g_N \) is \((C, 1/d\ell)\)-good on \( V \) with respect to Lebesgue measure.

Recall from Definition 2.3 that a measure \( \nu \) on \( X \) is called Federer if for \( \nu \) almost every \( x \in X \) there is a neighbourhood \( U \) of \( x \) and a constant \( D > 0 \) such that \( \nu(3B) > \nu(B)/D \) for any ball \( B \subset U \) centred on \( \text{supp} \nu \). Furthermore, observe that \( \nu \) is absolutely \((C, \alpha)\)-decaying according to (6) precisely when every linear function is absolutely \((C, \alpha)\)-good with respect to \( \nu \).

**Proposition 4.5** ([29, Proposition 7.3]). Let \( g = (g_1, \ldots, g_N) : U \subset \mathbb{R}^d \rightarrow \mathbb{R}^N \) be a \( C^{\ell+1} \) map, \( U \) open. Further, let \( \nu \) be a measure which is Federer and absolutely \((\tilde{C}, \tilde{\alpha})\)-decaying on \( U \) for some \( \tilde{C}, \tilde{\alpha} > 0 \). If \( g \) is \( \ell \)-nondegenerate at \( x \in U \), then there are a neighbourhood \( V \subset U \) of \( x \) and \( C > 0 \) such that any linear combination of \( 1, g_1, \ldots, g_N \) is absolutely \((C, \alpha)/(2^{\ell+1} - 2)\)-good on \( V \) with respect to \( \nu \).

**Note.** Consider the Lebesgue measure as \( \nu \). Then Proposition 4.3 shows that for \( d = 1 \) the exponent \( 1/\ell \) in Proposition 4.4 is likely to be optimal, while \( 1/(2^{\ell+1} - 2) \) is much worse. However, the latter is independent of \( d \). Unfortunately, according to [29], finding the optimal exponent seems to be a challenging open problem.

**Corollary 4.6.** Let \( k \) be a positive integer, \( f = (f_0, \ldots, f_{r-1}) \) be an analytic map on \( U \subset \mathbb{R}^d \), and let \( \nu \) be a measure on \( U \) which is Federer and absolutely \((\tilde{C}, \tilde{\alpha})\)-decaying on \( U \). Then for every \( x \in U \setminus Z_f \) there are a neighbourhood \( V \ni x \) and constants \( C_x, \alpha > 0 \) such that \( s(f) \) is \((C_x, \alpha)\)-good on \( V \) for every symmetric polynomial \( s \) of degree up to \( k \), where \( Z_f \) is the zero set of a real analytic function. Furthermore, if \( N = (k+\tau-1) \), then \( \alpha \) can be chosen to be:

- \( \tilde{\alpha}/(2^N - 2) \);
- \( 1/d(N - 1) \) if \( \nu \) is the Lebesgue measure;
- \( 1/k \deg(f) \) if \( f \) is a polynomial map and \( \nu \) is the Lebesgue measure.

**Proof.** Let \( \Sigma \) be a basis for the linear span \( \langle S^k_k(f) \rangle_{\mathbb{R}} \), and note that we may always assume that \( 1 \in \Sigma \), since \( 1 \in S^k_k(f) \) for every \( k > 0 \). Furthermore, \( \#\Sigma \leq N \) and all the elements of \( \Sigma \) are analytic because so is \( f \).

Therefore by the Wronskian Criterion (Corollary 3.3) we know that at least one of the generalised Wronskians of \( \Sigma \), say \( W \), is not identically zero. Hence
Σ is non-degenerate outside of the zero set $Z_f$ of $W$, and the statement follows from Propositions 4.3, 4.4, and 4.5.

**Remark 4.7.** As a consequence of the Implicit Function Theorem, one can show that the Hausdorff dimension of $Z_f$ is at most $d - 1$. In particular, if $\nu$ is either the Lebesgue measure on $\mathbb{R}^d$ or the natural measure supported on a sufficiently regular IFS of dimension $s > d - 1$ (see Example 2.5), then $\nu$ satisfies the hypotheses of Corollary 4.6 and $\nu(Z_f) = 0$.

5 Points with conjugate coordinates

**Proof of Theorem 2.13 assuming Corollary 2.11.**

First, we are now going to check that the hypotheses of Corollary 2.11 are satisfied for $\psi$ and $\Psi$ defined by

$$
\begin{cases}
\psi_k(Q) = \psi^d(Q) := Q^{d-n-1+\gamma(m+1-d)} & \text{for } 0 \leq k < d \\
\psi_k(Q) = \Psi(Q) := Q^{1-\gamma} & \text{for } d \leq k \leq m.
\end{cases}
$$

Clearly $\gamma > 0$ implies that $\psi_k(Q)Q^{-1} = Q^{-\gamma}$ is decreasing, and observe that

$$\psi(Q)Q^{n-m} = \psi(Q)^d\Psi(Q)^{m+1-d}Q^{n-m} = 1,$$

hence condition (14) of Theorem 2.8 is satisfied for every $1 \leq \tau \leq m+1$ by choosing $I_\tau = (0, \ldots, \tau-1)$, since our choice of $\psi$ and $\Psi$, together with the fact that $\psi$ is decreasing, implies that $\psi_{I_\tau}$ is decreasing as well.

Furthermore, observe that $s(x) = \infty$ because the coordinate functions $x_0, \ldots, x_{d-1}$ are algebraically independent over $\mathbb{R}$. Therefore (19) is enough to guarantee that condition (15) is satisfied as well, since the ordering of $\psi_k$, hence of $f_k$, is irrelevant for $d \leq k \leq m$.

Thus we can apply Corollary 2.11 taking the Lebesgue measure $\text{vol}_d$ on $\mathbb{R}^d$ as $\nu$, and through it we find $\mathcal{B}_\theta \subseteq \mathcal{B}$ with $\text{vol}_d(\mathcal{B}_\theta) \gg \text{vol}_d(\mathcal{B})$. Moreover, for every $x \in \mathcal{B}_\theta$ we have points $(\alpha_0, \ldots, \alpha_m)$ with algebraic conjugate coordinates and $H(\alpha_k) \ll Q$ such that

$$
\begin{cases}
|x_k - \alpha_k| \ll \frac{\psi(Q)}{Q} & \text{for } 0 \leq k < d \\
|f_k(x) - \alpha_k| \ll \frac{\psi(Q)}{Q} & \text{for } d \leq k \leq m.
\end{cases}
$$

However, given that $\psi$ and $\Psi$ are multiplicative, we may assume that $H(\alpha_k) \leq Q$ by rescaling $Q$ and changing the implied constants in (25) accordingly.

Now choose a compact subset $K \subseteq \mathcal{B}_\theta$ such that $\text{vol}_d(K) \gg \text{vol}_d(\mathcal{B})$, which we can always do since $\mathcal{B}$ is assumed to be bounded. Then note that the partial derivatives of $f$ are all bounded on $K$, therefore the Mean Value Theorem implies that for each $d \leq k \leq m$ we have

$$
|f_k(\alpha) - f_k(x)| \ll_{K,d,f_k} \max_{0 \leq i < d} |x_i - \alpha_i| \ll \frac{\psi(Q)}{Q}.
$$

It follows that

$$
|f_k(\alpha) - \alpha_k| \leq |f_k(\alpha) - f_k(x)| + |f_k(x) - \alpha_k| \ll \frac{\psi(Q)}{Q} + \frac{\Psi(Q)}{Q}.
$$
Since $\Psi(Q) \geq \psi(Q)$ precisely when $\gamma \leq \frac{m+1}{m+1}$, we have that

$$|f_k(\alpha) - \alpha_k| \ll \frac{\Psi(Q)}{Q} = Q^{-\gamma},$$

hence if $c_0$ is greater than the implied constant, then $(\alpha_0, \ldots, \alpha_m) \in M_f^\alpha(Q, \gamma, \mathcal{B})$. Therefore we conclude that

$$\# M_f^\alpha(Q, \gamma, \mathcal{B}) \gg \text{vol}_d(K) \left(\frac{Q}{\psi(Q)}\right)^d \gg \text{vol}_d(\mathcal{B}) Q^{n+1-\gamma(m+1-d)}. \tag{26}$$

In section 6 we shall prove the following Corollary of Theorem 2.8, from which Corollary 2.11 follows immediately.

**Corollary 5.1.** Consider $\psi_0, \ldots, \psi_m$, $\varphi_{m+2}, \ldots, \varphi_n$, $f$ as in Theorem 2.8 such that condition (14) holds with

$$\varphi(Q) = \varphi_{m+1}(Q) = \max\{Q, \varphi_{m+2}(Q), \ldots, \varphi_n(Q)\}.$$  

Furthermore, assume that there are constants $c_3, c_4 > 0$ such that

$$c_3 \leq \psi(Q) \varphi(Q) \leq c_4. \tag{27}$$

Then for every $0 < \theta < 1$ there are constants $c_1, c_2 > 0$ and a subset $\mathcal{B}_0$ of $\mathcal{B}$, independent of $Q$, such that $\nu(\mathcal{B}_0) > \theta \nu(\mathcal{B})$ and every $x \in \mathcal{B}_0$ admits $n + 1$ linearly independent irreducible polynomials $P = a_0 + a_1 x + \cdots + a_n x^n \in \mathbb{Z}[X]$ of degree bounded by $n$ such that

$$\begin{cases} |P(f_k(x))| < c_1 \psi_k(Q) & \text{for } 0 \leq k \leq m \\ |P'(f_k(x))| > c_2 \varphi(Q) & \text{for } m < k \leq n \end{cases} \tag{28}$$

whenever $Q$ is sufficiently large. In particular $H(P) \ll \varphi(Q)$.

**Proof of Corollary 2.11** Let $P$ be as in the statement of Corollary 5.1 let $\varphi_{m+1}(Q) = \cdots = \varphi_n(Q) = Q$, so that $\varphi(Q) = Q$ as well, and note that by remark 2.9 we may choose $\mathcal{B}_0$ to be compact. To simplify the notation, let $y_k = f_k(x)$ for $0 \leq k \leq m$. Then observe that, since $P'$ is continuous, $\mathcal{B}_0$ is compact, and $\psi_k(Q)Q^{-1}$ is decreasing, we may choose an open set $U$ with $\mathcal{B}_0 \subset U \subset \mathcal{B}$ and a constant $Q_0 > 0$ such that for every $Q > Q_0$ every interval of the form

$$I_{y_k} := \left[y_k - \kappa \frac{\psi_k(Q)}{Q}, y_k + \kappa \frac{\psi_k(Q)}{Q}\right]$$

is contained in $U$, where $\kappa := \frac{c_2}{2}$, and such that $|P'(z)| > c_2 Q$ for every $z \in U$. Furthermore, by the Mean Value Theorem we know that for every $y_k \in I_{y_k}$ there is a $z_k \in I_{y_k}$ such that

$$P(y_k) = P(y_k) + P'(z_k)(y_k - y_k).$$

Now note that $H(P) \ll Q$, again because $\psi_k(Q)Q^{-1}$ is decreasing for every $0 \leq k \leq m$. As $\mathcal{B}$ is bounded, it follows that $|P'(z_k)|$ is bounded above by $Q$.  

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up to a constant that depends on \( n, f \) and \( B \). Furthermore, \(|P'(z_k)| > c_2 Q\) implies that for \( \bar{y}_k = y_k \pm \kappa \psi_k(Q) \) we have

\[ |P'(z_k)(\bar{y}_k - y_k)| > c_2 \kappa \psi_k(Q) = c_1 \psi_k(Q), \]

therefore

\[ P \left( y_k - \frac{\kappa \psi_k(Q)}{Q} \right) P \left( y_k + \frac{\kappa \psi_k(Q)}{Q} \right) < 0. \]

Applying once more the Mean Value Theorem we obtain, for every \( 0 \leq k \leq m \), a root \( \alpha_k \) of \( P \) such that

\[ |y_k - \alpha_k| < \kappa \psi_k(Q). \]

Finally, note that Corollary 5.1 gives us \( n + 1 \) distinct irreducible polynomials, from which we obtain \( n + 1 \) distinct points \((\alpha_0, \ldots, \alpha_m)\).

**Note.** The numbers \( y_k \) are pairwise distinct on \( B \cap \text{supp } \nu \), since \( \det U_h f \) is non-zero by remark 2.6. By taking \( Q \) large enough if necessary, it follows that we can guarantee that the sets \( I_{y_k} \) are pairwise disjoint, hence the roots \( \alpha_k \) are pairwise distinct. In particular, observe that the constant \( \kappa \) does not depend on \( Q \) and may be chosen uniformly on \( B \).

### 6 Tailored polynomials

Similarly to what Beresnevich, Bernik and Götze did in [6], we call a tailored polynomial an irreducible polynomial which satisfies (12). Our construction follows closely the argument of [6, Section 3], and it is based on Theorem 2.8, which we will then prove in Section 8 using the quantitative non-divergence method of Kleinbock and Margulis.

Now fix \( x \in B \) and observe that solving for \( P \in \mathbb{Z}[X] \) the system of inequalities

\[
\begin{align*}
&|P(f_k(x))| < \psi_k(Q) \quad \text{for } 0 \leq k \leq m, \\
&|a_k| \leq \varphi_k(Q) \quad \text{for } m < k \leq n
\end{align*}
\]

is equivalent to looking for points of the lattice \( L := M \mathbb{Z}^{n+1} \) which lie in the convex body \( \mathcal{C} \), where \( M = M_f(x) \) is the matrix defined in (7) and where

\[ \mathcal{C} := \left\{ \mathbf{y} \in \mathbb{R}^{n+1} : \begin{cases} |y_k| < \psi_k(Q) & \text{for } 0 \leq k \leq m, \\
|y_k| \leq \varphi_k(Q) & \text{for } m < k \leq n \end{cases} \right\}. \]

Note that \( \det M \neq 0 \) on \( B \cap \text{supp } \nu \), since \( V(f) \neq 0 \) implies \( \det U_f \neq 0 \) by remark 2.6. Furthermore, since \( \det M \) is continuous in \( x \) we may assume without loss of generality that it is bounded away from 0 on \( B \), up to replacing \( B \) with the interior of a compact subset with measure arbitrarily close to \( \nu(B) \) (which we can always find since \( \nu \) is Radon). Then Minkowski’s second convex body theorem tells us that the successive minima \( \lambda_0 \leq \ldots \leq \lambda_n \) of \( \mathcal{C} \) with respect to \( L \) satisfy

\[ \frac{2^{n+1}}{(n+1)!} \det M \leq \lambda_0 \cdots \lambda_n \text{ vol}(\mathcal{C}) \leq 2^{n+1} \det M. \]
where $\operatorname{vol}(\mathcal{C}) = 2^{n+1} \psi(Q) \varphi(Q)$ is the volume of $\mathcal{C}$. Therefore we have

$$\lambda_n \leq \frac{\det(M)}{c_3 \lambda_0^n},$$

since $\psi(Q) \varphi(Q) \geq c_3$ by condition (24).

Now note that if $P = a_0 + a_1 X + \cdots + a_n X^n$ is such that $M a \in \lambda_0 \mathcal{C}$ where $a = (a_0, \ldots, a_n)^T \neq (0, \ldots, 0)^T$, then $H(P) \ll \lambda_0 \varphi(Q)$ as long as $\det(M)$ is uniformly bounded away from 0. Indeed, there is a $b \in \lambda_0 \mathcal{C}$ such that $M a = b$, thus for $Q$ large enough

$$H(P) = \|a\|_\infty \leq \|M^{-1}\|_\infty \|b\|_\infty \leq \lambda_0 \varphi(Q) \frac{\|\text{adj}(M)\|_\infty}{\det(M)}$$

where $\text{adj}(M)$ is the adjugate matrix of $M$, whose norm depends only on $n$, $x$ and $f(x)$, and thus can be bounded above by a constant depending on $n$, $f$, and $\mathcal{B}$. Since $\mathcal{B}$ is bounded, it follows that there is a constant $c_m > 0$ such that

$$\max_{0 \leq i \leq m} |P'(f_i(x))| \leq c_m \lambda_0 \varphi(Q).$$

Therefore Theorem (23) implies that for any given $\delta_0 > 0$ the set of $x \in \mathcal{B}$ for which $\lambda_0 = \lambda_0(x) \leq \delta_0$ is bounded above by

$$\delta_0^n \nu(\mathcal{B})$$

up to a constant, since condition (27) implies that $\operatorname{vol}(\lambda_0 \mathcal{C}) \leq 2^{n+1} c_4 \lambda_0^{n+1}$. In particular, we may choose $\delta_0$ depending only on $\theta$, $n$, $f$ and $\mathcal{B}$ such that for every $x$ in a subset $B(\delta_0)$ of measure at least $\sqrt{\theta} \nu(\mathcal{B})$ we have $\lambda_0 > \delta_0$.

Now, let $\delta_n := \frac{\det(M)}{c_3 \lambda_0^n}$. Then for any $x \in B(\delta_0)$ we may find $n + 1$ linearly independent polynomials $P_i$ whose vectors of coefficients $a_i$ satisfy $M a_i \in \delta_n \mathcal{C}$. If $A$ is the matrix with columns $a_i$, $0 \leq i \leq n$, then

$$1 \leq |\det(A)| \leq \operatorname{vol}(\delta_n \mathcal{C}) \leq 2^{n+1} c_4 \delta_n^{n+1} : = c'$$

and by Bertrand’s postulate we may find a prime $p$ such that

$$c' < p < 2c'.$$

In particular, this implies that $\det(A) \equiv 0 \pmod{p}$, hence the system

$$A t \equiv b$$

has a unique solution $t \in \mathbb{F}_p^{n+1}$, where $b = (0, \ldots, 0, 1)^T$. Now, for $\ell = 0, \ldots, n$ define $r_{\ell} = (1, \ldots, 1, 0, \ldots, 0)^T \in \mathbb{F}_p^{n+1}$, where $\ell$ denotes the number of zeroes. Then write $A t - b = p w$ after choosing representatives for $t$ in $\{0, \ldots, p - 1\}$, let $\gamma_{\ell} \in \mathbb{F}_p^{n+1}$ be the unique solution to

$$A \gamma_{\ell} \equiv -w + r_{\ell}$$

modulo $p$, and define $\eta_{\ell} = t + p \gamma_{\ell}$. For each $\ell = 0, \ldots, n$ let

$$\tilde{P}_\ell := \sum_{i=0}^{n} \eta_{\ell} P_i$$

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and note that the linear independence of the vectors \( r_\ell \) implies the linear independence of the polynomials \( \tilde{P}_\ell \).

Since \( A \eta_\ell = s \) is the vector of coefficients of \( \tilde{P}_\ell \) and since \( \eta_\ell \equiv t \pmod{p} \), it follows that \( s_n \equiv 1 \pmod{p} \) and \( s_i \equiv 0 \pmod{p} \) for \( 0 \leq i \leq n-1 \). Furthermore, the definition of \( \gamma_\ell \) implies that

\[
A \eta_\ell = b + pr_\ell,
\]

thus \( s_0 \equiv p \pmod{p^2} \). Therefore, by Eisenstein’s criterion it follows that \( \tilde{P}_\ell \) is irreducible. Finally, observe that taking representatives for \( t \) and \( \gamma_\ell \) in \( \{0, \ldots, p-1\} \) we have \( |\eta|_\ell \leq p^2 \), thus \( \tilde{P}_\ell \) satisfies

\[
\begin{align*}
|P(f_k(x))| &< c_1 \psi_k(Q) \quad \text{for } 0 \leq k \leq m \\
|a_k| &\leq c_1 \varphi_k(Q) \quad \text{for } m < k \leq n,
\end{align*}
\]

where

\[
c_1 = 4(n+1)\delta_n c^2 = 2^{2n+4}(n+1)c_1^2 \delta_n^{2n+3} = 2^{2n+4}(n+1)c_2^2 \left( \frac{\det(M)}{c_3 \delta_0^n} \right)^{2n+3}.
\]

Then, Theorem 2.8 implies that the measure of the set of \( x \in \mathcal{B} \) which admit a solution \( P \) to (30) such that \( \max |P'(f_k(x))| \leq c_2 \varphi(Q) \) is bounded above by

\[
c_2 \frac{\nu(\mathcal{B})}{c_2^{2n+3}}
\]

up to a constant. In particular, we may choose \( c_2 > 0 \), depending only on \( \theta, n, f \) and \( \mathcal{B} \), such that for every \( x \) in a subset \( \mathcal{B}_\theta = B(\delta_0, c_2) \subseteq B(\delta_0) \) of measure at least \( \sqrt{\theta} \nu(B(\delta_0)) \geq \theta \nu(\mathcal{B}) \) we have \( \min |P'(f_k(x))| > c_2 \varphi(Q) \).

7 Ubiquity

**Definition 7.1.** A dimension function \( g: \mathbb{R}^+ \to \mathbb{R}^+ \) is a continuous increasing function such that \( g(r) \to 0 \) as \( r \to 0 \). Now suppose that \( F \) is a non-empty subset of a metric space \( \Omega \). For \( \rho > 0 \), a \( \rho \)-cover of \( F \) is a countable collection \( \{B_i\} \) of balls in \( \Omega \) of radii \( r(B_i) \leq \rho \) whose union contains \( F \). Define

\[
\mathcal{H}^\rho(g)(F) := \inf \left\{ \sum_i g(r(B_i)) : \{B_i\} \text{ is a } \rho \text{-cover of } F \right\}.
\]

The (generalised) Hausdorff measure \( \mathcal{H}^\rho(g)(F) \) of \( F \) with respect to the dimension function \( g \) is defined as

\[
\mathcal{H}^\rho(g)(F) := \lim_{\rho \to 0} \mathcal{H}^\rho(g)(F) = \sup_{\rho > 0} \mathcal{H}^\rho(g)(F).
\]

See [34, Chapter 4] for more details.

**Example 7.2.** Given \( s > 0 \), the usual Hausdorff \( s \)-measure \( \mathcal{H}^s \) coincides with \( \mathcal{H}^\rho \) where \( g(r) = r^s \). In particular, when \( s \) is an integer \( \mathcal{H}^s \) is a constant multiple of the \( s \)-dimensional Lebesgue measure.
Proposition 7.3. Let $r$ of the following Proposition, of which Theorem 2.16 is a direct consequence. Also assume that $x$ satisfies condition $B$ for infinitely many $\alpha$. Then $x \in \mathcal{B}$ and let $f(x) \in \mathcal{L}_{n,m}^*(\psi, \Psi; d)$. This section is devoted to the proof of the following Proposition, of which Theorem 2.16 is a direct consequence.

Proposition 7.3. Let $\psi, \Psi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ be decreasing functions which satisfy \([21]\), and let $g$ be a dimension function such that $r^d g(r)$ is non-increasing. Also assume that $r^{-d} g(r)$ is increasing for some $\gamma > 0$, and that it satisfies \([22]\). Further suppose that $f$ is Lipschitz continuous, that $V(f) \neq 0$, and that $f$ satisfies condition \([13]\) on $\mathcal{B}$. Then

$$\mathcal{H}^d(\mathcal{L}_{n,f}^*(\psi)) = \mathcal{H}^d(\mathcal{B}) \text{ if } \sum_{Q=1}^{\infty} Q^{n-m+1-d} \psi(Q)^{m+1-d} \left(\frac{\psi(Q)}{Q}\right) = \infty.$$ 

Remark 7.4. There is a constant $c_{10} > 0$, depending only on $n$ and $\mathcal{M}$, such that

$$\mathcal{L}_{n,f}^*(\psi, \Psi) \subseteq \mathcal{L}_n(f(c_{10} \psi, c_{10} \Psi)).$$

Indeed, suppose that $y \in \mathcal{L}_{n,f}^*(\psi)$, and let $\alpha \in \mathbb{A}_n^{m+1}$ be such that $\|y - \alpha\| < \frac{\psi(H(\alpha))}{H(\alpha)}$. If $P$ is the minimum polynomial of $\alpha_0$ (and hence of $\alpha_k$ for every $0 \leq k \leq m + 1$), then by the Mean Value Theorem we have

$$|P(y_k)| = |P(y_k) - P(\alpha_k)|$$

$$\leq |y_k - \alpha_k| \sup_{z \in \mathcal{M}} |P'(z_k)|$$

$$< C_{10} \frac{\psi_k(H(\alpha_k))}{H(\alpha_k)} - H(P)$$

$$= C_{10} \psi_k(H(\alpha_k))$$

since $\mathcal{B}$ bounded implies that $P'$ is bounded above on $\mathcal{M}$, and of course $H(P) = H(\alpha)$. Thus it follows that $y \in \mathcal{L}_n(f(c_{10} \psi, c_{10} \Psi))$, as required.

Therefore the convergence part of Theorem \([11]\) immediately gives the following partial counterpart of Proposition 7.3. Here, as before, $|U|$ denotes the Lebesgue measure of a measurable set $U \subset \mathbb{R}^{m+1}$.

Lemma 7.5. For any function $\psi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ we have

$$|\mathcal{L}_{n,f}^*(\psi, \Psi)| = 0 \text{ if } \sum_{Q=1}^{\infty} Q^{n-m+1} \psi(Q)^{m+1-d} < \infty.$$ 

Our proof relies on a powerful tool of Diophantine Approximation, ubiquitous systems, adapted to the case of approximation of dependent quantities like in \([1]\). Consider the following setting:

- $\Omega$, a compact subset of $\mathbb{R}^d$;
• $J$, a countable set;
• $\mathcal{R} = (R_\alpha)_{\alpha \in J}$ a family of points in $\Omega$ indexed by $J$, referred to as resonant points;
• a function $\beta: J \rightarrow \mathbb{R}^+, \alpha \mapsto \beta_\alpha$, which assigns a weight to each $R_\alpha$ in $\mathcal{R}$;
• a function $\rho: \mathbb{R}^+ \rightarrow \mathbb{R}^+$ such that $\lim_{r \rightarrow \infty} \rho(r) = 0$, referred to as a ubiquitous function; and
• $J(t) = J_\kappa(t) := \{\alpha \in J: \beta_\alpha \leq \kappa t\}$, assumed to be finite for every $t \in \mathbb{N}$, where $\kappa > 1$ is fixed.

Furthermore, $B(x, r)$ will denote a ball in $\Omega$ with respect to the sup norm, and for a given function $\hat{\psi}: \mathbb{R}^+ \rightarrow \mathbb{R}^+$ we will also consider the limsup set
$$\Lambda_{\mathcal{R}}(\hat{\psi}) := \{x \in \Omega: \|x - R_\alpha\| < \hat{\psi}(\beta_\alpha) \text{ for infinitely many } \alpha \in J\}.$$ 

**Definition 7.6.** The pair $(\mathcal{R}, \beta)$ is a locally ubiquitous system in $\Omega$ with respect to $\rho$ if for any ball $B \subseteq \Omega$
$$\left| \bigcup_{\alpha \in J(t)} B(\alpha, \rho(\kappa t)) \cap B \right| \gg |B|$$
for every $t$ large enough, where the implied constant is absolute.

Like with [8, Theorem 1], the following statement can be readily obtained by combining Corollaries 2 and 3 from [7]:

**Theorem 7.7.** In the above setting, suppose that $(\mathcal{R}, \beta)$ is a locally ubiquitous system in $\Omega$ with respect to $\rho$, and let $g$ be a dimension function such that $r^{-d} g(r)$ is non-increasing. Furthermore, suppose that $r^{-\gamma} g(r)$ is increasing for some $\gamma > 0$, and that there are constants $r_0, c_{12}, c_{13} \in (0, 1)$ such that
$$r^{-\gamma} g(c_{12} r) \leq c_{13} g(r) c_{12} r^{-\gamma} \text{ for any } r \in (0, r_0).$$

Also assume that $\hat{\psi}$ is decreasing and that
$$\limsup_{t \rightarrow \infty} \frac{\hat{\psi}(\kappa t+1)}{\psi(\kappa t)} < 1.$$

Then
$$\mathcal{H}(\Lambda_{\mathcal{R}}(\hat{\psi})) = \mathcal{H}^\beta(\Omega) \text{ if } \sum_{t=0}^{\infty} g(\hat{\psi}(\kappa t)) \rho(\kappa t)^d = \infty.$$ 

Now let $f, \psi$ and $\Psi$ be as in Proposition 7.3. Since $\mathcal{B}$ is assumed to be bounded, for every integer $q \geq 2$ we may find a compact subset $\mathcal{B}_q \subset \mathcal{B}$ such that $|\mathcal{B}_q| \geq (1 - \frac{1}{q})|\mathcal{B}|$. It follows that $\mathcal{H}^\beta(\mathcal{B}) = \lim_{q \rightarrow \infty} \mathcal{H}^\beta(\mathcal{B}_q)$, so it suffices to prove the proposition with $\mathcal{B}_q$ in place of $\mathcal{B}$ for any fixed $q$. For ease of notation, given $y = (y_0, \ldots, y_m) \in \mathbb{R}^{m+1}$ we will write $\tilde{y}$ for $(y_0, \ldots, y_{d-1})$. Then let $\Omega := \mathcal{B}_q$ and define

$$J := \left\{ \alpha \in \mathbb{R}_m^{m+1}: \tilde{\alpha} \in \Omega \text{ and } \max_{d \leq k \leq m} |f_k(\tilde{\alpha}) - \alpha_k| < \frac{1}{2} \frac{\Psi(H(\alpha))}{H(\alpha)} \right\}$$
$$\mathcal{R} := (\tilde{\alpha})_{\alpha \in J} \quad \beta_\alpha := H(\alpha).$$

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Also let
\[ \rho(Q) = \rho_0 (Q^{n-d} \Psi(Q)^{m+1-d})^{-\frac{1}{2}} \]
for some constant \( \rho_0 > 0 \) to be determined later, and observe that (21) implies
\[ \rho(Q) \ll (Q^{n-m+d} \Psi^{m+1-d})^{-\frac{1}{2}} = (Q^{n-m+d} Q^{m-n-d})^{-\frac{1}{2}} = Q^{-1+\frac{d}{m+1}}, \]
which shows that \( \rho(Q) \to 0 \) as \( Q \to \infty \).

\textbf{Lemma 7.8.} Suppose that \( f_d, \ldots, f_m \) are Lipschitz continuous with constant bounded above by \( c_f \). If \( y \in \mathbb{R}^{m+1} \) is such that \( \hat{y} \in B \),
\[ \max_{0 \leq k < d} |x_k - y_k| < \Theta_x \quad \text{and} \quad \max_{d \leq k \leq m} |f_k(x) - y_k| < \Theta_f \]
for some \( \Theta_x, \Theta_f > 0 \), then
\[ \max_{d \leq k \leq m} |f_k(\hat{y}) - y_k| < \Theta_f \left(1 + c_f \frac{\Theta_x}{\Theta_f}\right). \]

\textit{Proof.} Simply observe that, by the triangle inequality,
\[ |f_k(\hat{y}) - y_k| \leq |f_k(\hat{y}) - f_k(x)| + |f_k(x) - y_k| \]
\[ < c_f \|x - \hat{y}\| + |f_k(x) - y_k| \]
\[ < c_f \Theta_x + \Theta_f \]
\[ = \Theta_f \left(1 + c_f \frac{\Theta_x}{\Theta_f}\right). \]

\textbf{Lemma 7.9.} Let \( J, R, \beta, \rho \) be as above, and suppose that \( f \) is Lipschitz continuous. Then there is a choice of \( \rho_0 > 0 \) such that \((R, \beta)\) is a locally ubiquitous system in \( \Omega \) with respect to \( \rho \).

\textit{Proof.} Fix a ball \( B \subset \Omega \) and let
\[ \psi_k(Q) = \begin{cases} Q \rho(Q) & \text{for } 0 \leq k < d \\ \psi_k(Q) = \Psi(Q) & \text{for } d \leq k \leq m \\ \varphi_k(Q) = Q & \text{for } m < k \leq n. \end{cases} \]

Then observe that \( \psi_k(Q) Q^{-1} \) is decreasing for every \( 0 \leq k \leq m \), and that
\[ \psi(Q) \varphi(Q) = \rho(Q)^d \Psi(Q)^{m+1-d} Q^{n-m+d} = \rho_0^d. \]

Furthermore, by condition (21) we know that \( \Psi(Q) \gg Q^{m-n} \), which implies \( \Psi(Q) \gg Q \rho(Q) \). Therefore for every \( 1 \leq \tau \leq m+1 \) and every choice of \( I \in [m]_\tau \) we have that
\[ \psi_I(Q) \ll \Psi(Q)^\tau \ll 1 \]
for every \( Q \) large enough, thus we may apply Corollary 2.11 with \( \nu(\cdot) = |\cdot| \) and \( B \) in place of \( \mathcal{F} \). Hence for any fixed \( 0 < \theta < 1 \) we find a set \( B_\theta \subseteq B \) with

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Theorem 7.7 once we've shown that Λ ∈ ℜ(Ψ). Since we may assume without loss of generality that Ψ is decreasing and that Ψ > c for some c > 0 such that, for t large enough, every x ∈ B_0 admits n + 1 points α ∈ ℜ_{n+1}(ck^t) with

\[ \max_{0 \leq k < d} |x_k - α_k| < c\rho(n^t) \quad \text{and} \quad \max_{d \leq k \leq m} |f_k(x) - α_k| < c\frac{\Psi(κ^t)}{κ^t}. \]

Now, again because of Ψ(Q) ≫ Qρ(Q), by Lemma 7.8 it follows that

\[ \max_{d \leq k \leq m} |f_k(\hat{α}) - α_k| < cc\frac{\Psi(κ^t)}{κ^t} \]

for some c > 1. Finally, observe that Remark 2.12 and equation (21) show that we can choose c by manipulating the value of ρ_0. In particular, we can ensure that c < \hat{c}^{-1}, thus α ∈ J(t) and

\[ \left| \bigcup_{α ∈ J(t)} B(α, ρ(κ^t)) ∩ B \right| ≥ |B_0| > θ|B|. \]

**Note.** Condition (14) is actually satisfied even in the absence of (21). Indeed, using the fact that Ψ is decreasing and that ψ is constant, one can show that ψ is decreasing for every I = (m + 1 − τ, . . . , m) where 1 ≤ τ ≤ m + 1.

**Proof of Proposition 7.3.** Note that since ψ ∈ O(Ψ), there is a c_{11} > 0 such that Ψ(Q) > c_{11}ψ(Q) for any integer Q > 0. Then let \( \hat{ψ}(Q) = \frac{c_{11} \Psi(Q)}{2c_f Q} \), where c_f is as in Lemma 7.8. It is clear that this choice of \( \hat{ψ} \) satisfies (22); indeed,

\[ \lim_{t → ∞} \hat{ψ}(κ^{t+1}) = \frac{1}{κ} \lim_{t → ∞} \hat{ψ}(κ^t) < \frac{1}{κ} \]

since ψ is assumed to be decreasing. The proposition will follow as an immediate consequence of Theorem 4.4 once we've shown that Λ(Ψ) ≤ ℒ_{n,f}^∗(ψ, Ψ).

If x ∈ Λ(Ψ), then there are infinitely many α ∈ ℜ_{n+1} such that

\[ \max_{0 \leq k < d} |x_k - α_k| < c_{11} \frac{ψ(H(α))}{2c_f H(α)} \quad \text{and} \quad \max_{d \leq k \leq m} |f_k(\hat{α}) - α_k| < \frac{1}{2} \frac{Ψ(H(α))}{H(α)}. \]

Therefore the same argument of Lemma 7.8 gives

\[ \max_{d \leq k \leq m} |f_k(x) - α_k| < \frac{Ψ(H(α))}{H(α)}. \]

It follows that x ∈ ℒ_{n,f}^∗(ψ, Ψ), since we may assume without loss of generality that c_f ≥ \frac{1}{2}. The proof is concluded by observing that by Cauchy’s Condensation Test

\[ \sum_{t=0}^{∞} \frac{g(\hat{ψ}(κ^t))}{ρ(κ^t)^d} = ρ_0^{-d} \sum_{Q=1}^{∞} κ^{t(n-m+d)} \Psi(κ^t)^{m+1-d} g(\frac{c_{11} ψ(κ^t)}{2c_f}) = ∞ \]

if and only if

\[ S_1 := \sum_{Q=1}^{∞} Q^{n-m-1+d} \Psi(Q)^{m+1-d} g(\frac{c_{11} ψ(Q)}{2c_f}) = ∞, \]
and that the same argument of [40, Lemma 3.2] shows that the latter happens if and only if

\[ S_2 := \sum_{Q=1}^{\infty} Q^{n-m-1+d} \psi(Q)^{m+1-d} g \left( \frac{\psi(Q)}{Q} \right) = \infty \]

when \( g \) is increasing and \( \psi \) is decreasing. Indeed, note that without loss of generality we may assume that \( 2c_f > c_{11} \), and let \( c := \frac{2c_f}{c_{11}} > 1 \). Furthermore, for ease of notation let

\[ \sigma(z, Q) := Q^{n-m-1+d} \psi(Q)^{m+1-d} g \left( \frac{\psi(Q)}{Q} \right). \]

Now, on one hand \( g \) increasing immediately implies that \( S_1 \leq S_2 \). On the other hand,

\[
S_2 = \sum_{Q=1}^{c-1} \sigma(1, Q) + \sum_{q=1}^{\infty} \sum_{Q=q+1}^{\infty} \sigma(1, Q) \\
\ll \sum_{q=1}^{\infty} \sigma(1, q) \\
\ll \sum_{q=1}^{\infty} \sigma(c^{-1}, q) \\
= S_1,
\]

where the last inequality is due to the fact that \( \psi \) is decreasing and \( g \) is increasing.

8 Proofs of Theorem 2.8 and Corollary 2.10

We will first prove the following local version of Theorem 2.8 which can then be extended via a compactness argument.

Theorem 8.1. Under the hypotheses of Theorem 2.8, fix a point \( x \in B \cap \supp \nu \) and let \( B \ni x \) be a ball such that \( B = 3^n B \subset B \). Then for \( Q \) large enough we may find constants \( C, \rho > 0 \), the latter dependent on \( B \), such that

\[
\nu \left( D^{\rho}_x(Q, B) \right) \leq C \left( \frac{\psi(Q) \varphi(Q)}{\rho^{n+1}} \right)^{\frac{1}{\alpha}} \nu(B).
\]

Proof of Theorem 2.8 given Theorem 8.1. Let \( B_\theta \subset B \) be a compact subset such that \( \nu(B_\theta) \geq \nu(B) \), which exists because \( B \) is bounded and \( \nu \) is Radon. Then note that, since \( B \cap \supp \nu \) is contained in the interior of \( B \) by hypothesis, for every \( x \in B_\theta \) we may find a ball \( B_{\theta} \ni x \) as in Theorem 8.1 as well as the respective constants \( C_x \) and \( \rho_x \). Hence by compactness there is a finite subset \( \{ x_k \}_{k \in K} \subset B_\theta \) such that \( \{ B_{x_k} \}_{k \in K} \) is an open cover of \( B_\theta \). Therefore the result follows by observing that

\[
\nu \left( D^{\rho}_x(Q, B_\theta) \right) \leq \sum_{k \in K} \nu \left( D^{\rho}_x(Q, B_{x_k}) \right)
\]

and by taking \( C = \max_K C_{x_k} \) and \( \rho = \max_K \rho_{x_k} \).
Through the Dani-Kleinbock-Margulis correspondence between Diophantine Approximation and flows on homogeneous spaces \cite{22, 27}, we will reinterpret the problem of finding points \(x \in \mathcal{B}\) for which (12) has a solution as a shortest vector problem. First, we expand (12) into the \(m+1\) systems of inequalities

\[
\begin{cases}
|P(f_k(x))| < \psi_k(Q) & \text{for } 0 \leq k \leq m \\
|P(x)| < \varphi_{m+1}(Q) & 0 \leq h \leq m, \\
|a_k| \leq \varphi_k(Q) & \text{for } m+1 < k \leq n,
\end{cases}
\]

(33)

and observe that these can be rewritten in matrix form using the matrices \(U_h\).

However, to be able to view this as a smallest vector problem we also need to rescale the inequalities. Consider the scaling matrix

\[
g_t := \text{diag}(e^{t_0}, \ldots, e^{t_m}, e^{-t_{m+1}}, \ldots, e^{-t_n}),
\]

(34)

where \(t = (t_0, \ldots, t_n) \in \mathbb{R}^{n+1}\) is such that

\[
l_{[0,m]} = l_{[m+1,n]},
\]

and where for every \(1 \leq \tau \leq n+1\) and \(I \in \llbracket n \rrbracket^\tau\) we defined

\[
t_I = \sum_{i \in I} t_i.
\]

(35)

Then we need \(\delta = \delta(Q) > 0\) such that

\[
\begin{cases}
\delta = e^{t_k} \psi_k(Q) & \text{for } 0 \leq k \leq m \\
\delta = e^{-t_k} \varphi_k(Q) & \text{for } m < k \leq n,
\end{cases}
\]

(36)

and multiplying those \(n+1\) equations together we see that

\[
\delta^{n+1} = \psi(Q) \varphi(Q).
\]

(37)

Therefore, taking logarithms we may rewrite \(t_k\) in terms of \(\psi_k\) and \(\varphi_k\), as

\[
\begin{cases}
t_k = \log \delta - \log \psi_k(Q) & \text{for } 0 \leq k \leq m \\
t_k = \log \varphi_k(Q) - \log \delta & \text{for } m < k \leq n.
\end{cases}
\]

(38)

We can now see that (12) has a solution for a given \(x \in \mathcal{B}\) if and only if for every \(0 \leq h \leq m\) the lattice \(g_tU^h(x)\mathbb{Z}^{n+1}\) has a non-zero vector with sup-norm at most \(\delta\), thus we have indeed reduced to a shortest vector problem. In other words,

\[
\mathcal{D}_f^p(Q, B) = \left\{ x \in B : \lambda(g_tU^h(x)\mathbb{Z}^{n+1}) < (\psi(Q) \varphi(Q))^{\frac{1}{n+1}} \right\},
\]

where \(\lambda(\Gamma) = \inf_{v \in \Gamma \setminus \{0\}} \|v\|\) denotes the length of the shortest vector in a discrete subgroup \(\Gamma \subset \mathbb{R}^{n+1}\).
Remark 8.2. Condition (14) of Theorem 2.8 is equivalent to asking that for every $1 \leq \tau \leq m+1$ there is a choice of $I \in \llbracket m \rrbracket_{\leq}$ such that $t_I = t_I(Q)$ is bounded below. Indeed, by (38)

\[
t_I = \tau \log \delta - \sum_{i \in I} \log \psi_i(Q) = \frac{\tau}{n+1} \log(\psi(Q)\varphi(Q)) - \log \psi_I(Q) \geq c
\]

precisely when $\psi(Q)\varphi(Q) \geq e^c \psi_I(Q)^{\frac{n-1}{2}}$.

Example 8.3. Let $d = m = 1$, as in the context of Theorem 1.2. Furthermore, let $\psi_0(Q) = \psi_1(Q) = Q^{-\frac{n+1}{2}}$, $\varphi_2(Q) = e^{n+1}Q$ and $\varphi_3(Q) = \cdots = \varphi_n(Q) = Q$. Then by (37) we have $\delta = \varepsilon$. Moreover, the equations (38) become

\[
\begin{cases}
  t_k = \log \varepsilon + \frac{n-1}{2} \log Q & \text{for } 0 \leq k \leq 1 \\
  t_2 = n \log \varepsilon + \log Q \\
  t_k = \log Q - \log \varepsilon & \text{for } 3 \leq k \leq n.
\end{cases}
\]

Therefore $t_{\llbracket 1 \rrbracket} = 2 \log \varepsilon + (n-1) \log Q$, which in particular gives that $t_{\llbracket 1 \rrbracket} \geq c$ for

\[
\log \varepsilon \geq \frac{\log c}{2} - \frac{n-1}{2} \log Q,
\]

i.e. $\varphi_2(Q) \gg Q^{\frac{n-2}{2}}$. Then $\psi(Q)\varphi(Q) \gg Q^{\frac{n-2}{2}}$ and we easily see that condition (14) of Theorem 2.8 is satisfied for all choices of $I$, since

\[
\psi(Q)^{\frac{n-2}{2}} = \psi_0(Q)^{n+1} = \psi_1(Q)^{n+1} = Q^{\frac{n-2}{2}}.
\]

The main tool in our proof will be the following Theorem from [28]. Here $W_{\tau}$ denotes the set of elements

\[
w = w_1 \wedge \cdots \wedge w_{\tau} \in \bigwedge^{\tau} \mathbb{Z}^{n+1}
\]

where $\{w_1, \ldots, w_{\tau}\}$ is a primitive $\tau$-tuple, i.e. it can be completed to a basis of $\mathbb{Z}^{n+1}$. Furthermore, $\|\cdot\|$ will denote both the sup-norm and the norm it induces on $\bigwedge \mathbb{R}^{n+1}$.

Note. The elements of $W_{\tau}$ can be identified with the primitive subgroups of $\mathbb{Z}^{n+1}$ of rank $\tau$, i.e. those non-zero subgroups $\Gamma \subseteq \mathbb{Z}^{n+1}$ of rank $\tau$ such that $\Gamma = \Gamma_{\mathbb{R}} \cap \mathbb{Z}^{n+1}$, where $\Gamma_{\mathbb{R}}$ denotes the linear subspace generated by $\Gamma$ in $\mathbb{R}^{n+1}$. Therefore, up to a sign they can also be identified with the rational $\tau$-dimensional subspaces of $\mathbb{R}^{n+1}$.

Theorem 8.4 ([28, Theorem 2.2]). Fix $n, N \in \mathbb{N}$ and $\hat{C}, D, \alpha, \rho > 0$. Given an $N$-Besicovitch metric space $X$, let $B$ be a ball in $X$ and $\nu$ be a measure which is D-Federer on $\hat{C} = 3^{n+1}B$. Suppose that $\eta: \hat{C} \to GL_{n+1}(\mathbb{R})$ is a map such that for every $1 \leq \tau \leq n+1$ and for every $w \in W_{\tau}$:
1. the function \( x \mapsto \| \eta(x)w \| \) is \((\bar{C}, \alpha)\)-good on \(B\) with respect to \(\nu\), and
2. \( \| \eta(\cdot)w \|_{\nu, B} \geq \rho^\tau \).

Then for any \(0 < \delta \leq \rho\) we have

\[
\nu \left( \{ x \in B : \lambda (\eta(x)Z^{n+1}) < \delta \} \right) \leq C \left( \frac{\delta}{\rho} \right)^\alpha \nu(B)
\]

with \( C = (n+1)\bar{C}(ND^2)^{n+1} \).

**Note.** In light of Lemma 4.2 we may extend this to \( \delta > \rho \) as well, since we may always exchange \( \bar{C} \) with \( \max \{ \bar{C}, (n+1)^{-1}(ND^2)^{-n-1} \} \), so that \( C \geq 1 \).

For our purposes we would like to take \( \eta(x) = g_tU^b_J(x) \), and to show that it satisfies hypotheses 1 and 2 we will need the following Lemma. Here, for each \( I \subseteq [n]_\prec \) we will denote by \( e_I \) the standard basis element \( e_{i_1} \wedge \cdots \wedge e_{i_r} \) of \( \Lambda^{- \tau}_{\mathbb{R}^{n+1}} \).

**Lemma 8.5.** Let \( w = w_1 \wedge \cdots \wedge w_\tau \in W_\tau \) and let \( A \) be an \((n+1) \times (n+1)\) matrix. Then, for every \( I \subseteq [n]_\prec \) we have the component of \( Aw \) corresponding to \( e_I \) is an integer linear combination of the minors \( |A|_{I,J} \), where \( J \) runs through \([n]_\prec\). Furthermore, the coefficients are independent from \( I \) and not all zero.

**Proof.** Let \( W = (w_1| \cdots |w_\tau) \) be the matrix obtained by juxtaposition of the vectors \( w_1, \ldots, w_\tau \), and recall the well-known fact that the \( e_I \) component of \( w \) is just the \( \tau \times \tau \) minor \( |W|_{I,J} \) of \( W \) (see e.g. \[42\] Chapter 10, Section 3), where \([\tau]^n = \{1, \ldots, \tau\} \). Now observe that

\[
Aw = (A w_1) \wedge \cdots \wedge (A w_\tau) = AW(e_1 \wedge \cdots \wedge e_\tau)
\]

and that \( (A w_1| \cdots |A w_\tau) = AW \). Finally, the statement follows by the Cauchy-Binet formula (see e.g. \[42\] Example 10.31) or \[13\] Cauchy-Binet Corollary, p. 214), i.e.

\[
|AW|_{I,J} = \sum_{J \subseteq [n]_\prec} |A|_{I,J} |W|_{I,J}. \tag*{\Box}
\]

It follows that the component of \( g_tU^b_J(x)w \) corresponding to \( e_I \) is of the form

\[
e^{tI} \sum_{J \subseteq [n]_\prec} c_J |U^b_J|_{I,J}
\]

with \( c_J \in \mathbb{Z} \) not all zero and independent from \( I \). Therefore we have that

\[
\| g_tU^b_J(x)w \| \gg \sum_{J \subseteq [n]_\prec} c_J |U^b_J|_{I,J}
\]

as long as \( e^{tI} \) is bounded below, which we know from Remark 8.2 to be guaranteed by condition \(13\) of Theorem 2.3. We can now prove that for every \( 1 \leq \tau \leq n+1 \) the norm of \( g_tU^b_J(x)w \) is bounded below uniformly in \( w \in W_\tau \).
This is straightforward for \( \tau = n + 1 \), since in that case
\[
\|c_i U^j(x)w\| = |c_{i\tau}| \det U^j \geq |\det U^j| > 0
\]
by remark 2.6. When \( \tau \leq n \), on the other hand, condition 13 of Theorem 2.8 guarantees that
\[
t_{m+1} = \sum_{i=0}^{m} t_i - \sum_{i=m+2}^{n} t_i
\]
is bounded below, hence we may always find an index set \( I \in [n]_\leq \) such that \( m + 1 \notin I \) and \( e^{t_i} \) is bounded below. But then
\[
\mathcal{G}_I(U^j) = \mathcal{G}_I(M_f) = \mathcal{G}_I(M_f)
\]
for some \( \tilde{I} \in [m]_\leq \) and \( 1 \leq \tilde{\tau} \leq m + 1 \). Therefore it is enough to check that for every such \( \tilde{I} \) and \( \tilde{\tau} \)
\[
\|c \cdot \mathcal{G}_I(M_f(x))\|_{\nu,B} = \sup_{x \in B^{\text{supp} \nu}} |c \cdot \mathcal{G}_I(M_f(x))| \gg 1 \quad (40)
\]
uniformly in non-zero integer vectors \( c \). By Lemma 5.7 this can be guaranteed by requiring that \( f_1 \) is non-symmetric of degree \( n + 1 - \tau \), because when \( V(f) \) is bounded we have
\[
|c \cdot \mathcal{G}_I(M_f(x))| \gg |c \cdot S_{n,\tau}(f_1(x))| \quad (41)
\]
and from Proposition 5.4 we know that the components of \( S_{n,\tau}(T) = S^{n+1-\tau}_T(T) \) form a basis for the module of symmetric polynomials in \( \tau \) variables and degree bounded by \( n + 1 - \tau \).

**Lemma 8.6.** Given a continuous map \( g = (g_0, \ldots, g_{r}) : \mathcal{B} \to \mathbb{R}^r \), let \( \tilde{g} = (\tilde{g}_0, \ldots, \tilde{g}_{r}) \) be a basis for the linear span \( \langle g_0, \ldots, g_{r} \rangle_{\mathbb{R}} \) and let \( R \) be the real matrix such that \( \tilde{g} R = g \). Then \( \ker(R) \cap \mathbb{Q}^{r+1} = \{0\} \) if and only if \( g_0, \ldots, g_{r} \) are linearly independent over \( \mathbb{Q} \).

**Proof.** The components of \( g \) are linearly dependent over \( \mathbb{Q} \) if and only if there is a non-zero \( q \in \mathbb{Q}^{r+1} \) such that
\[
0 = g \cdot q = \tilde{g} R q,
\]
but then it must be that \( q \in \ker(R) \), since by hypothesis the components of \( \tilde{g} \) are linearly independent over \( \mathbb{R} \).

**Lemma 8.7.** Let \( g = (g_1, \ldots, g_{r}) : \mathcal{B} \to \mathbb{R}^r \) be a continuous map with components linearly independent over \( \mathbb{Q} \). Then there is a \( \rho > 0 \) such that \( \|g\|_{\nu,B} \geq \rho \) for every integer linear combination \( g \) of the components of \( g \).

**Proof.** Let \( \tilde{g} \) be a basis for the linear span \( \langle g_0, \ldots, g_{r} \rangle_{\mathbb{R}} \). Further, let \( S' \) be the unit sphere in \( \mathbb{R}^{r+1} \) and note that if \( b \in S' \setminus \{0\} \), then \( \tilde{b} := \frac{b}{|b|} \in S' \cap \mathbb{Q}^{r+1} \).

Therefore
\[
\min_{b \in S' \setminus \{0\}} \|g \cdot b\|_{\nu,B} \geq \min_{b \in S'} \|g \cdot b\|_{\nu,B} = \min_{b \in S'} \|\tilde{g} \cdot (R \tilde{b})\|_{\nu,B} =: \rho,
\]
which is well defined since \( \tilde{g} R \tilde{b} \) is continuous in \( \tilde{b} \) and \( S' \) is compact. Finally, Lemma 8.6 implies that \( \rho > 0 \).
Having shown that \( \eta(x) = g_U h f(x) \) satisfies condition (2) of Theorem 8.4, we note that (41) implies that \( \eta \) satisfies condition 1 as well. Indeed, write \( \varpi \) for \( \| g_U h f(x) \|_{\nu_B} \) and \( \varpi_I \) for the component of \( g_U h f(x) \) corresponding to \( e_I \). Since \( B \) is bounded and \( \varpi, \varpi_I \) are continuous, (40) implies that \( \varpi \leq c \varpi_I \) on \( B \cap \supp \nu \) for some \( c > 0 \). Furthermore, (41) shows that \( \varpi_I \) is \((C, \alpha)-good\) on \( B \) with respect to \( \nu \), since \((S(f_I), \nu)\) is \((C, \alpha)-good\) by hypothesis. Therefore by Lemma 5 we have that \( \varpi \) is \((c \alpha C, \alpha)-good\) on \( B \) with respect to \( \nu \).

This concludes the proof of Theorem 8.1.

Proof of Corollary 2.10. Note that the first part is just a special case of Corollary 4.6. Then for each integer \( k > 1 \) apply Theorem 2.8 with \( \theta_k = 1 - \frac{1}{k} \), resulting in a sequence of subsets \( B_k \subset B \) with \( \nu(B_k) > \theta_k \nu(B) \) and such that

\[
\nu(D^n f(Q, B_k)) \ll_k \left( \psi(Q) \varphi(Q) \right)^{\frac{1}{k} + \alpha} \nu(B_k)
\]

for \( Q \) large enough, where the implied constant is independent of \( Q \). Therefore by condition (16) we have

\[
\sum_{Q=1}^{\infty} \nu(D^n f(Q, B_k)) \ll \nu(B_k) \sum_{Q=1}^{\infty} \left( \psi(Q) \varphi(Q) \right)^{\frac{1}{k} + \alpha} < \infty
\]

and by the Borel-Cantelli Lemma this implies that \( \nu(D^n f(B_k)) = 0 \).

Now observe that for every \( k, Q > 1 \) we have \( D^n f(Q, B_k) \subseteq D^n f(Q, B) \), hence \( D^n f(B_k) \subseteq D^n f(B) \). Thus

\[
\nu(D^n f(B)) \leq \nu(D^n f(B) \setminus D^n f(B_k)) + \nu(D^n f(B_k)) = \nu(D^n f(B) \setminus D^n f(B_k)) \\
= \nu(B) \setminus \nu(B_k) \\
\leq \frac{1}{k} \nu(B) \to 0
\]

as \( k \to \infty \), and we conclude that \( \nu(D^n f(B)) = 0 \), as required.

9 Final remarks

There is a notable gap between the hypotheses of Theorem 1.2 and those of Theorem 2.13. For example, when \( f \) is a polynomial map our theorem only applies to at most finitely many values of \( n \). It would therefore be interesting to explore the limit of the techniques presented in this paper, and a possible approach would be to adapt the work of Aka, Breuillard, Rosenzweig, and de Saxcé [2] to determine the precise obstruction to the applicability of Theorem 8.4 to the present problem.

We also note that Theorem 2.8 suggests that the volume of the approximation targets plays a greater role than the length of their sides in determining whether a certain rate of approximation is achievable or not. In other words, we conjecture the following improvement of Proposition 7.3 for the set.
\( L^*_n f(\psi_0, \ldots, \psi_m) \) of points \( x \in \mathcal{B} \) such that \( f(x) \in L^*_{n,m+1}(\psi_0, \ldots, \psi_m) \), where the latter is the set of \( x \in \mathbb{R}^{m+1} \) such that

\[
|x_k - \alpha_k| < \frac{\psi_k(H(\alpha))}{H(\alpha)}
\]

for infinitely many \( \alpha \in \mathbb{R}_m^{m+1} \).

**Conjecture 9.1.** Let \( \psi_0, \ldots, \psi_m : \mathbb{R}^+ \to \mathbb{R}^+ \) be decreasing functions such that \( \psi_i \in O(\psi_j) \) for every \( 0 \leq i < d \) and \( d \leq j \leq m \), and suppose that there is a \( \kappa > 0 \) such that

\[
\kappa^{n-m+d} > \lim_{t \to \infty} \frac{\psi_d(c^t) \cdots \psi_m(c^t)}{\psi_d(c^{t+1}) \cdots \psi_m(c^{t+1})}.
\]

Further, let \( g \) be a dimension function such that \( r^{-d}g(r) \) is non-increasing, and assume that \( f \) is Lipschitz continuous, that \( V(f) \neq 0 \), and that \( f \) satisfies condition (15) on \( B \). Then

\[
\mathcal{H}^d(L^*_n f(\psi_0, \ldots, \psi_m)) = \begin{cases} 0 & \text{if } S^d_{n,d}(\psi_0, \ldots, \psi_m) < \infty, \\ \mathcal{H}^d(\mathcal{B}) & \text{if } S^d_{n,d}(\psi_0, \ldots, \psi_m) = \infty. \end{cases}
\]

where

\[
S^d_{n,d}(\psi_0, \ldots, \psi_m) := \sum_{Q=1}^{\infty} Q^d \frac{\psi_d(Q) \cdots \psi_m(Q)}{Q^{m+1-d} g(Q^{d+1})}.
\]

Furthermore, observe that a version of [23, Lemma 4.6] for flows \( B \to \text{GL}_{m+1}(\mathbb{R}) \) would allow us to extend Theorem 2.8 to more general measures. In this spirit and motivated by [29, 47, 41], as well as recent work by Khalil and Luethi, we propose the following:

**Conjecture 9.2.** Let \( f : \mathcal{B} \subseteq \mathbb{R}^d \to \mathbb{R}^{m+1} \) be a continuous map, and let \( \nu \) be a measure on \( \mathcal{B} \) such that \( (S^{n-d} \circ f)_* \nu \) is Federer, decaying, and rationally non-planar. Also let \( \psi : \mathbb{R}^+ \to \mathbb{R}^+ \) be a decreasing function. Then for any ball \( B \subseteq \mathcal{B} \)

\[
\nu(L^*_n f(\psi) \cap B) = \nu(B) \text{ if } \sum_{Q=1}^{\infty} Q^{n-m+1} \psi^{-1}(Q) = \infty.
\]

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