AN $L_q(L_p)$-THEORY FOR PARABOLIC PSEUDO-DIFFERENTIAL EQUATIONS: CALDERÓN-ZYGMUND APPROACH

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Abstract. In this paper we present a Calderón-Zygmund approach for a large class of parabolic equations with pseudo-differential operators $A(t)$ of arbitrary order $\gamma \in (0, \infty)$. It is assumed that $A(t)$ is merely measurable with respect to the time variable. The unique solvability of the equation
\[ \frac{\partial u}{\partial t} = Au - \lambda u + f, \quad (t, x) \in \mathbb{R}^{d+1} \]
and the $L_q(\mathbb{R}, L_p)$-estimate
\[ \|u\|_{L_q(\mathbb{R}, L_p)} + \|(-\Delta)^{\gamma/2} u\|_{L_q(\mathbb{R}, L_p)} + \lambda \|u\|_{L_q(\mathbb{R}, L_p)} \leq N \|f\|_{L_q(\mathbb{R}, L_p)} \]
are obtained for any $\lambda > 0$ and $p, q \in (1, \infty)$.

1. Introduction

Calderón-Zygmund theorem has been a powerful tool in the theory of both elliptic and parabolic differential equations. See, for instance, [1, 3, 6] (elliptic equations) and [2, 7, 9, 10] (2nd order parabolic equations). In particular, Krylov [9, 10] introduced a Calderón-Zygmund approach to obtain $L_q(\mathbb{R}, L_p)$ and $L_p(\mathbb{R}, C^{2+\alpha})$-estimates for the second-order parabolic equations with merely measurable coefficients with respect to the time variable.

In this article we use a Calderón-Zygmund approach to study the parabolic equation
\[ \frac{\partial u}{\partial t} = Au - \lambda u + f, \quad (t, x) \in \mathbb{R}^{d+1}. \] (1.1)
It is assumed that the pseudo-differential operator $A(t)$ is merely measurable in $t$ and its symbol $\psi(t, \xi)$ satisfies
\[ \Re[-\psi(t, \xi)] \geq \kappa |\xi|^{\gamma}, \quad \forall \xi \in \mathbb{R}^d \] (1.2)
and
\[ |D_\xi^{\alpha} \psi(t, \xi)| \leq \kappa^{-1} |\xi|^{\gamma-|\alpha|}, \quad \forall \xi \in \mathbb{R}^d \setminus \{0\}, \ |\alpha| \leq \left\lfloor \frac{d}{2} \right\rfloor + 1 \] (1.3)
for some $\gamma, \kappa > 0$. No regularity condition of $A(t)$ in the time variable is assumed and the differentiability condition of $\psi(t, \xi)$ with respect to $\xi$ is only up to order $\lfloor \frac{d}{2} \rfloor + 1$. Conditions [1.2] and [1.3] are satisfied by a large class of pseudo-differential operators including $2m$-order differential operators and integro-differential operators. See Section 2 for some examples. We only mention that if $A_1(t)$ and $A_2(t)$ satisfy the conditions with $\gamma_1$ and $\gamma_2$ respectively then for any constants $a, b > 0$...
the operator $A_{a,b} = (-A_1)^a(-A_2)^b$ satisfies the conditions with $\gamma = a\gamma_1 + b\gamma_2$ if for instance the symbols of $A_i$ are real-valued.

Our approach aims to prove the $L_q(R, L_p)$-estimate

$$\|u_t\|_{L_q(R, L_p)} + \|(-\Delta)^{\gamma/2} u\|_{L_q(R, L_p)} + \lambda\|u\|_{L_q(R, L_p)} \leq N\|f\|_{L_q(R, L_p)}$$

(1.4)

for any $u \in C^\infty_0(R^{d+1})$ and $f := u_t - Au + \lambda u$. We remark that the classical multiplier theorem is not applicable to derive estimates like (1.4) because $A(t)$ is only measurable in $t$.

We first prove

$$\lambda\|u\|_{L_q(R, L_p)} \leq N\|f\|_{L_q(R, L_p)}$$

based on the representation formula of solutions and a few direct calculations. Next we introduce a kernel $K(t,x,s,y)$ so that for any $u \in C^\infty_0(R^{d+1})$ and $f := u_t - Au + \lambda u$ we have

$$(-\Delta)^{\gamma/2} u(t,x) = \int_{R^{d+1}} K(t,x,s,y)f(s,y)dsdy =: Gf(t,x).$$

(1.5)

Then, we prove

$$\|Gf\|_{L_q(R, L_p)} \leq N\|f\|_{L_q(R, L_p)}.$$  

(1.6)

The major step to prove (1.6) is to construct $(Q_m, m \in Z)$, a filtration of partitions of $R^{d+1}$ (see Definition 3.7), and show that for any $Q \in \bigcup_{m \in Z} Q_m$ the following Hörmander condition (cf. [4, 5, 10]) holds:

$$\sup_{(s,y),(r,z) \in Q} \int_{R^{d+1}\setminus Q^*} |K(t,x,s,y) - K(t,x,r,z)|dxdt < \infty,$$

(1.7)

where $Q^*$ is an appropriate dilation of $Q$. The Hörmander condition and the Calderón-Zygmund theorem easily yield (1.6).

It is well known that for the elliptic operators Hörmander condition is fulfilled if the related kernel $K(x,y)$ is a standard kernel i.e. $K(x,y)$ defined on $R^{2d} \setminus \{(x,x) : x \in R^d\}$ satisfies

$$|K(x,y)| \leq \frac{N}{|x-y|^d}$$

and

$$|K(x,y) - K(z,y)| \leq N\frac{|x-z|^\alpha}{|x-y|^{d+\alpha}}$$

whenever $|x-z| \leq \frac{1}{2} \max\{|x-y|, |z-y|\}$ and

$$|K(x,y) - K(x,z)| \leq N\frac{|y-z|^\alpha}{|x-y|^{d+\alpha}}$$

whenever $|y-z| \leq \frac{1}{2} \max\{|x-y|, |x-z|\}$ (see [3] for details).

In this article we study a parabolic version of this result and investigate a sufficient condition on kernel $K(t,x,s,y)$ so that (1.7) holds for any $Q \in \bigcup_{m \in Z} Q_m$. The filtration of partition $(Q_m, m \in Z)$ is constructed only according to the order of the operator $A(t)$. It turns out that if the order of $A(t)$ is not rational then constructing appropriate filtration of partitions by itself is a quite challenging work.
To the best of our knowledge, only few studies have been made on Calderón-Zygmund approach for non second-order parabolic equations. If \( p = q \), a result on integro-differential operators of the type
\[
A^{(\alpha)} f := \int_{\mathbb{R}^d} \left( f(x + y) - f(x) - 1_{|y| \leq 1}(y \cdot \nabla f(x)) \right) \frac{m(t, y)}{|y|^{d+\alpha}}, \quad \alpha \in (0, 2)
\]
was introduced in [12] under certain assumptions on \( m(t, y) \). The version of Calderón-Zygmund decomposition of \( \mathbb{R}^{d+1} \) introduced in [12] uses non-congruent rectangles to construct \( Q_n \) for each \( n \) and the non-congruency of such rectangles depends also on the given function \( u \). We believe that the constants in the \( L_p \)-estimates of [12] are not controllable due to such non-congruency and the proof of [12] is incomplete. In this article we use congruent cubes to construct \( Q_n \) and our construction depends only on the order of the operator \( A(t) \). Our results certainly cover that of [12] (see Example 2.6).

Below are some related \( L_p \)-estimates on non-local parabolic equations based on different approaches. Recently in [3] the authors proved a priori estimate [14] for the case \( p = q \) and \( \lambda = 0 \) using a \( \text{BMO-L}^\infty \) type estimate. However this approach by itself is not enough to treat the case \( p \neq q \). Moreover the unique solvability of equation [13] is not obtained in [3]. In [13], [1.0] is proved for the symbol of order \( \gamma \in (0, 2) \) which can be represented by the Lévy-Khintchine’s formula
\[
\psi(\xi) = \int_{\mathbb{R}^d} \left( 1 + i(\xi \cdot y)1_{|y| \leq 1} - \exp\{i\xi \cdot y\} \right) \nu(dy),
\]
where \( \nu \) is a Lévy measure controlled from the below and the above by the Lévy measures of two \( \alpha \)-stable processes. This result is based on a probabilistic method regarding Lévy processes which is legitimate only if the symbol \( \psi(\xi) \) is independent of \( t \) and its order is in \( (0, 2) \). In this article we do not use any probabilistic method and no restriction on the order and time regularity of \( A(t) \) is assumed.

The article is organized as follows. Our main results are formulated in Section 2. In Section 3, we illustrate the division-merger procedure to construct the filtration of partitions we need. The proofs of main theorems and some auxiliary results are given in Sections 4, 5, and 6.

We finish the introduction with some notation used in this article. As usual \( \mathbb{R}^d \) stands for the Euclidean space of points \( x = (x^1, \ldots, x^d) \), \( B_r(x) := \{ y \in \mathbb{R}^d : |x - y| < r \} \) and \( B_r := B_r(0) \). For multi-indices \( \alpha = (\alpha_1, \ldots, \alpha_d) \), \( \alpha_i \in \{0, 1, 2, \ldots \} \), \( x \in \mathbb{R}^d \), and functions \( u(x) \) we set
\[
\begin{align*}
 u_{x^i} &= \frac{\partial u}{\partial x^i} = D_{x^i} u, \\
 D^\alpha u &= D_{x^1}^{\alpha_1} \cdots D_{x^d}^{\alpha_d} u, \\
x^\alpha &= (x^1)^{\alpha_1} (x^2)^{\alpha_2} \cdots (x^d)^{\alpha_d}, \quad |\alpha| = \alpha_1 + \cdots + \alpha_d.
\end{align*}
\]
We also use \( D^m_x \) to denote a partial derivative of order \( m \) with respect to \( x \). For an open set \( \Omega \subset \mathbb{R}^d \) by \( C_0^\infty(\Omega) \) we denote the set of infinitely differentiable functions with compact support in \( U \). For a Banach space \( F \) and \( p > 1 \) by \( L_p(U, F) \) we denote the set of \( F \)-valued measurable functions \( u \) on \( \Omega \) satisfying
\[
||u||_{L_p(\Omega, F)} = \left( \int_\Omega ||u(x)||_F^p dx \right)^{1/p} < \infty.
\]
We write \( f \in L_{p, loc}(U, F) \) if \( \zeta f \in L_p(U, F) \) for any real-valued \( \zeta \in C_0^\infty(U) \). Also \( L_p(\Omega) = L_p(\Omega, \mathbb{R}) \) and \( L_p = L_p(\mathbb{R}^d) \). We use \( \lceil a \rceil \) to denote a definition. \( \lfloor a \rfloor \) is the biggest integer which is less than or equal to \( a \). By \( \mathcal{F} \) and \( \mathcal{F}^{-1} \) we denote the
d-dimensional Fourier transform and the inverse Fourier transform, respectively. That is, 
\[ \mathcal{F}(f)(\xi) := \int_{\mathbb{R}^d} \frac{1}{(2\pi)^d} e^{-ix \cdot \xi} f(x) \, dx \]
and 
\[ \mathcal{F}^{-1}(f)(x) := \int_{\mathbb{R}^d} e^{ix \cdot \xi} f(\xi) \, d\xi. \]
For a Borel set \( A \subset \mathbb{R}^d \), we use \(|A|\) to denote its Lebesgue measure and by \( 1_A(x) \) we denote the indicator of \( A \). \( \text{diam} \, A \) := sup\(_{x, y \in A} |x - y| \). For a complex number \( z \), \( \Re(z) \) is the real part of \( z \). Finally if we write \( N = N(a, b, \ldots) \), this means that the constant \( N \) depends only on \( a, b, \ldots \).

2. Main results

Let \( K(t, x, s, y) \) be a complex-valued measurable function on \( \mathbb{R}^{2d+2} \) satisfying 
\[ K(t, x, s, y) = K(t, x, s, y)1_{r>s}. \]
For \( f \in C_0^\infty(\mathbb{R}^{d+1}) \) denote
\[ \mathcal{G}f(t, x) = \int K(t, x, s, y)f(s, y) \, ds \, dy. \]
In this section we provide a sufficient condition on \( K \) so that \( \mathcal{G} \) admits a weak type \((1, 1)\) estimate, and using this result we obtain a \( L_q(\mathbb{R}, L_p) \) estimate for pseudodifferential operators \( A(t) \).

Here is our assumption on the kernel \( K \).

**Assumption 2.1.** There exist a constant \( \gamma > 0 \) and a nonnegative nondecreasing function \( \varphi \) on \( \mathbb{R}_+ \) such that
(i) for all \( a > s \) and \( y, z \in \mathbb{R}^d \),
\[ \int_a^\infty \int_{\mathbb{R}^d} |K(t, x, s, y) - K(t, x, s, z)| \, dx \, dt \leq \varphi\left( \frac{|y - z|}{(a - s)^{1/\gamma}} \right); \]
(ii) for all \( a > b \geq (s \lor r) \) and \( y \in \mathbb{R}^d \)
\[ \int_a^\infty \int_{\mathbb{R}^d} |K(t, x, s, y) - K(t, x, r, y)| \, dx \, dt \leq \varphi\left( \frac{|s - r|}{a - b} \right); \]
(iii) for all \( b > s \) and \( \rho > 0 \),
\[ \int_s^b \int_{|x - y| \geq \rho} |K(t, x, s, y)| \, dx \, dt \leq \varphi\left( \frac{(b - s)^{1/\gamma}}{\rho} \right). \]

The proof of following results are given in Section 4.

**Theorem 2.2.** Let \( 1 < p \leq p_0 \) and Assumption 2.1 hold. Assume that \( \mathcal{G}f \) is well defined for any \( f \in C_0^\infty(\mathbb{R}^{d+1}) \) and the inequality
\[ \|\mathcal{G}f\|_{L_{p_0}(\mathbb{R}^{d+1})} \leq N_0 \|f\|_{L_{p_0}(\mathbb{R}^{d+1})} \]
holds with some constant \( N_0 \) independent of \( f \). Then the operator \( \mathcal{G} \) is uniquely extendable to a bounded operator on \( L_p(\mathbb{R}^{d+1}) \) and satisfies the weak type \((1, 1)\) estimate, (i.e.) for any \( f \in C_0^\infty(\mathbb{R}^{d+1}) \) and \( \alpha > 0 \)
\[ \alpha |\{(t, x) : \mathcal{G}f(t, x) > \alpha\}| \leq N \|f\|_{L_{\alpha}(\mathbb{R}^{d+1})}, \]
where \( N \) depends only on \( d, p_0, \gamma, N_0, \) and the function \( \varphi \).

**Theorem 2.3.** In addition to assumptions of Theorem 2.2, suppose \( K(t, s, x, y) \) depends only on \( (t, s, x - y) \), and for all \( t > s \) and \( f \in C_0^\infty \)
\[ \left\| \int_{\mathbb{R}^d} K(t, x, s, y)f(y) \, dy \right\|_{L_{p_0}} \leq \varphi(t - s) \|f\|_{L_{p_0}(\mathbb{R}^d)} \].
Then it holds that
\[\|Gf\|_{L^p(D,\mu)} \leq N\|f\|_{L^p(D,\mu)}, \quad \forall f \in C_0^\infty(D,\mu),\]
where \(N\) depends only on \(d, p, p_0, \gamma, N_0,\) and the function \(\varphi.\)

For any \(p, q \in (1, \infty),\) by \(H_{p,q}^{1,\gamma} = H_{p,q}^{1,\gamma}(D)\) we denote the space of distributions \(u\) such that
\[\|u\|_{L^q(D,\mu)} < \infty, \quad \|u_t\|_{L^q(D,\mu)} < \infty, \quad \|(-\Delta)^{\gamma/2} u\|_{L^q(D,\mu)} < \infty.\]
The norm of \(u \in H_{p,q}^{1,\gamma}\) is defined by
\[\|u\|_{H_{p,q}^{1,\gamma}} := \|u\|_{L^q(D,\mu)} + \|u_t\|_{L^q(D,\mu)} + \|(-\Delta)^{\gamma/2} u\|_{L^q(D,\mu)}.\]
One can easily check that \(H_{p,q}^{1,\gamma}\) is a Banach space.

Recall that the operator \(\mathcal{A}(t)\) has the symbol \(\psi(t, \xi),\) that is for \(f \in C_0^\infty(D)\)
\[\mathcal{F}(\mathcal{A}f) = \psi(t, \xi)\mathcal{F}(u)(t, \xi).\]
The following result is an application of Theorem 2.3. In the proof of Theorem 2.3 we will take
\[K(t, x, s, y) = 1_{s < t} \mathcal{F}^{-1}\left\{ |\xi|^{\gamma} \exp\left(\int_s^t \psi(r, \xi)dr\right) \right\} (x - y)\]
so that (1.5) holds for \(\lambda = 0.\)

**Theorem 2.4.** Let \(p, q \in (1, \infty), \lambda > 0.\) Suppose that there exist constants \(\gamma, \kappa > 0\) so that
\[\Re[\psi(t, \xi)] \leq -\kappa|\xi|^{\gamma}, \quad \xi \in \mathbb{R}^d\]
and for all multi-index \(\alpha, \ |\alpha| \leq \lfloor \frac{\gamma}{2} \rfloor + 1,\)
\[|D^\alpha \psi(t, \xi)| \leq \kappa^{-1}|\xi|^{-|\alpha|}, \quad \xi \in \mathbb{R}^d \setminus \{0\}.\]
Then for any \(f \in L^q(D,\mu)\), there exists a unique solution \(u \in H_{p,q}^{1,\gamma}\) to equation (1.1). Furthermore, for this solution we have
\[\|u_t\|_{L^q(D,\mu)} + \|(-\Delta)^{\gamma/2} u\|_{L^q(D,\mu)} + \lambda\|u\|_{L^q(D,\mu)} \leq \|f\|_{L^q(D,\mu)},\]
where \(N = N(d, p, q, \kappa, \gamma).\)

Below we introduce some examples related to conditions (2.6) and (2.7).

**Example 2.5.** The symbol of the \(2m\)-order operator
\[A_1(t)u := (-1)^{m-1} \sum_{|\alpha| = |\beta| = m} a^{\alpha \beta}(t)D^{\alpha + \beta} u,\]
is \(\psi(t, \xi) = -a^{\alpha \beta}(t)\xi^\alpha \xi^\beta.\) Hence (2.6) and (2.7) are satisfied if \(a^{\alpha \beta}(t)\) are bounded complex-valued measurable functions satisfying
\[\kappa|\xi|^{2m} \leq \sum_{|\alpha| = |\beta| = m} \xi^\alpha \xi^\beta \Re [a^{\alpha \beta}(t)], \quad \forall \xi \in \mathbb{R}^d.\]

(ii) Similarly the \(\gamma\)-order nonlocal operator
\[A_2(t) := -a(t)(-\Delta)^{\gamma/2}, \quad \gamma \in (0, \infty)\]
has symbol \(\psi(t, \xi) = -a(t)|\xi|^{\gamma}\) and therefore for the above conditions it is sufficient to have
\[\kappa < \Re[a(t)], \quad |a(t)| \leq \kappa^{-1}.\]
The operator in Example 2.6 below is considered in [12].

**Example 2.6.** Fix \( \gamma \in (0, 2) \) and denote
\[
Au := \int_{\mathbb{R}^d \setminus \{0\}} \left( u(t, x + y) - u(t, x) - \chi(y)(\nabla u(t, x), y) \right) \frac{m(t, y)}{|y|^{d+\gamma}} dy
\]
where \( \chi(y) = I_{\gamma>1} + I_{|y| \leq 1}I_{\gamma=1} \). Then \( A \) satisfies (2.6) and (2.7) if \( m(t, y) \geq 0 \) is a measurable function satisfying the following (see [8] for details):

1. If \( \gamma = 1 \) then
\[
\int_{\partial B_1} \omega_m(t, w) S_1(dw) = 0, \quad \forall t > 0,
\]
where \( \partial B_1 \) is the unit sphere in \( \mathbb{R}^d \) and \( S_1(dw) \) is the surface measure on it.

2. The function \( m = m(t, y) \) is zero-order homogeneous and \( \left\lfloor \frac{d}{2} \right\rfloor + 1 \)-times differentiable in \( y \).

3. There is a constant \( K \) such that for each \( t \in \mathbb{R} \)
\[
\sup_{|\alpha| \leq d_0, |y|=1} |D^\alpha m^{(\alpha)}(t, y)| \leq K.
\]

4. There exists a constant \( c > 0 \) so that \( m(t, y) > c \) on a set \( E \subset \partial B_1 \) of positive \( S_1(dw) \)-measure.

Next we discuss the issue regarding the compositions and powers of operators. Let \( A_1(t) \) and \( A_2(t) \) be linear operators with symbols \( \psi_1(t) \) and \( \psi_2(t) \) satisfying the above prescribed conditions, that is there exist constants \( \gamma_1, \gamma_2, \kappa_1, \kappa_2 > 0 \) so that
\[
\Re[-\psi_i(t, \xi)] \geq \kappa_i|\xi|^{\gamma_i}, \quad |D^\alpha \psi_i(t, \xi)| \leq \kappa_i^{-1}|\xi|^{\gamma_i-|\alpha|}, \quad (i = 1, 2),
\]
for any multi-index \( \alpha, |\alpha| \leq \left\lfloor \frac{d}{2} \right\rfloor + 1 \). Fix \( a, b > 0 \), and denote \( \gamma := a\gamma_1 + b\gamma_2 \). Consider \( \gamma \)-order operator
\[
A_{a,b}(t) = -(-A_1(t))^a(-A_2(t))^b
\]
with the symbol \( \psi = -(-\psi_1)^a(-\psi_2)^b \). It is easy to check that there exists a constant \( N > 0 \) so that for any multi-index \( \alpha, |\alpha| \leq \left\lfloor \frac{d}{2} \right\rfloor + 1 \),
\[
|D^\alpha \psi(t, \xi)| \leq N|\xi|^{\gamma-a}, \quad \xi \in \mathbb{R}^d \setminus \{0\}.
\]
Therefore, (2.7) is satisfied, and Theorem 2.4 is applicable to \( A_{a,b}(t) \) if
\[
\Re[-\psi(t, \xi)] = \Re[(-\psi_1)^a(-\psi_2)^b] \geq N^{-1}|\xi|^\gamma, \quad \forall \xi \in \mathbb{R}^d.
\]
Obviously (2.8) is satisfied if, for instance, the symbols \( \psi_i(t, \xi) \) are real-valued.

3. Filtration of Partitions

In this section we introduce a version of Calderón-Zygmund theorem we need. We also construct a filtration of partitions suitable for our pseudo-differential operators. Denote \( \mathbb{N} = \{1, 2, \cdots \} \) and \( \mathbb{Z} = \{0, \pm 1, \pm 2, \cdots \} \).

**Definition 3.1.** Let \( n \in \mathbb{N} \) and \( (Q_m, m \in \mathbb{Z}) \) be a sequence of partitions of \( \mathbb{R}^n \) each consisting of disjoint bounded Borel subsets \( Q \in Q_m \). We call \( (Q_m, m \in \mathbb{Z}) \) a **filtration of partitions** if

1. the partitions become finer as \( m \) increases:
\[
\inf_{Q \in Q_m} |Q| \to \infty \text{ as } m \to -\infty, \quad \sup_{Q \in Q_m} \text{diam } |Q| \to 0 \text{ as } m \to \infty;
\]
Example 3.2. For the second-order parabolic equations, $Q_m$ on $\mathbb{R}^{d+1}$ is typically defined by

$$Q_m = \{ (i_0 4^{-m}, (i_0 + 1)4^{-m}) \times Q_m(i_1, \ldots, i_d, i_0, i_1, \ldots, i_d) : i_0, i_1, \ldots, i_d \in \mathbb{Z} \},$$

where

$$Q_m(i_1, \ldots, i_d) := [i_1 2^{-m}, (i_1 + 1)2^{-m}) \times \cdots \times [i_d 2^{-m}, (i_d + 1)2^{-m}).$$

For Banach spaces $F$ and $G$, $L(F, G)$ is the space of bounded linear operators from $F$ to $G$, and $L(F) := L(F, F)$. Define $B^p_c(x) := \{ y \in \mathbb{R}^n : |x - y| \geq r \}$.

Definition 3.3. Let $(Q_m, m \in \mathbb{Z})$ be a filtration of partitions, and for each $x, y \in \mathbb{R}^n$, $x \neq y$, let $K(x, y)$ be a bounded operator from $F$ into $G$. We say that $K$ is a Calderón-Zygmund kernel relative to $(Q_m, m \in \mathbb{Z})$ if

(i) there is a number $p_0 \in (1, \infty)$ such that, for any $x$ and any $r > 0$, $K(x, \cdot) \in L_{p_0, \text{loc}}(B^p_c(x), L(F, G))$;

(ii) for every $y \in \mathbb{R}^n$ the function $|K(x, y) - K(x, z)|$ is measurable as a function of $(x, z)$ on the set $\mathbb{R}^{2n} \cap \{ (x, z) : x \neq z, x \neq y \}$;

(iii) there is a constant $N_0 \geq 1$ and, for each $Q \in \bigcup_{m \in \mathbb{Z}} Q_m$, there is a closed set $Q^*$ with the properties $Q \subset Q^*$, $|Q^*| \leq N_0 |Q|$, and

$$\int_{\mathbb{R}^n \setminus Q^*} |K(x, y) - K(x, z)| dx \leq N_0$$

whenever $y, z \in Q$.

The following version of the Calderón-Zygmund theorem is taken from [9].

Theorem 3.4. Let $p > 1$ and $A : L_p(\mathbb{R}^n, F) \to L_p(\mathbb{R}^n, G)$ be a bounded linear operator. Assume that if $f \in C_0^\infty(\mathbb{R}^n, F)$ then for almost any $x$ outside the support of $f$ we have

$$Af(x) = \int_{\mathbb{R}^n} K(x, y)f(y)dy$$

where $K(x, y)$ is a Calderón-Zygmund kernel relative to a filtration of partitions. Then the operator $A$ is uniquely extendable to a bounded operator from $L_p(\mathbb{R}^n, F)$ to $L_q(\mathbb{R}^n, G)$ for any $q \in (1, p]$, and $A$ is of weak type $(1, 1)$ on smooth functions with compact support.

The filtration of partitions in Example 3.2 is not appropriate for pseudo-differential operators since the kernels corresponding such operators do not satisfy (3.1) in the setting of Example 3.2. Finding an appropriate filtration of partitions requires delicate procedures unless the given order $\gamma$ is rational. The remaining of this section is devoted to construct a filtration of partitions for pseudo-differential operators of arbitrary order.

We fix $\gamma > 0$ and denote

$$Q_0^{(\gamma)} = \{ Q_0 \subset \mathbb{R}^{d+1} : Q_0 = [i_0, i_0 + 1) \times \prod_{j=1}^{d} [i_j, i_j + 1), i_0, i_1, \ldots, i_d \in \mathbb{Z} \}. $$
To construct $Q_m^{(\gamma)}$ we consider the cases $m \geq 0$ and $m < 0$ separately.

First let $m = 1, 2, \cdots$. We construct $Q_m^{(\gamma)}$ inductively as follows. A similar division procedure when $\gamma \in (0, 2)$ can be found in [11]. Suppose for a given $Q_{m-1} \in Q_{m-1}^{(\gamma)}$, we can write

$$Q_{m-1} = Q_{m-1}^{\text{time}} \times Q_{m-1}^{\text{space}}$$  \hspace{1cm} (3.2)

where

$$Q_{m-1}^{\text{time}} = [i_0 2^{-(m-1)\gamma} \tau_{m-1}, (i_0 + 1)2^{-(m-1)\gamma} \tau_{m-1}),$$

$$Q_{m-1}^{\text{space}} = \prod_{j=1}^{d} [i_j 2^{-(m-1)}, (i_j + 1)2^{-(m-1)}],$$

for integers $i_0, i_1, \ldots, i_d$ and $\tau_{m-1} \in [1, 2)$ (remember $\tau_0 = 1$). Put

$$2^{-(m-1)\gamma} \tau_{m-1} = 2^{-m\gamma} \rho_m.$$  

Then

$$\rho_m = 2^\gamma \tau_{m-1} \in [2^\gamma, 2^{\gamma+1}) \subset [2^{(\gamma)}, 2^{(\gamma)+2}).$$

Put

$$k_m = \begin{cases} \lfloor \gamma \rfloor & \text{if } \rho_m \in [2^{\lfloor \gamma \rfloor}, 2^{\lfloor \gamma \rfloor+1}) \\ \lfloor \gamma \rfloor + 1 & \text{if } \rho_m \in [2^{\lfloor \gamma \rfloor+1}, 2^{\lfloor \gamma \rfloor+2}). \end{cases}$$

We split $Q_{m-1}$ into $2^d$ congruent cubes and subdivide $Q_{m-1}^{\text{time}}$ into $2^{k_m}$ congruent intervals. Taking all possible products of subcubes and subintervals, we obtain the set of offsprings of $Q_{m-1}$ (i.e.) $\{Q_m : Q_m \subset Q_{m-1}\}$ of the form

$$Q_m = Q_m(i, l) = Q_m^{\text{time}}(i) \times Q_m^{\text{space}}(l)$$  \hspace{1cm} (3.3)

where

$$Q_m^{\text{time}}(i) = [i_0 2^{-m\gamma} \rho_m + (i - 1)2^{-m\gamma} \frac{\rho_m}{2^{k_m}}, i_0 2^{-m\gamma} \rho_m + i2^{-m\gamma} \frac{\rho_m}{2^{k_m}})$$

for some $1 \leq i \leq 2^{k_m}$ and

$$Q_m^{\text{space}}(l) = \prod_{j=1}^{d} [i_j 2^{-(m-1)} + (l_j - 1)2^{-m}, i_j 2^{-(m-1)} + l_j 2^{-m})$$

for some $l = (l_1, \ldots, l_d)$, $l_j \in \{1, 2\}$. Denoting

$$\tau_m = \frac{\rho_m}{2^{k_m}} \in [1, 2),$$

we can rewrite (3.3) as

$$Q_m = [\tilde{i}_0 2^{-m\gamma} \tau_m, (\tilde{i}_0 + 1)2^{-m\gamma} \tau_m) \times \prod_{j=1}^{d} [\tilde{i}_j 2^{-m}, (\tilde{i}_j + 1)2^{-m}),$$

where

$$\tilde{i}_0 = 2^{k_m}i_0 + i - 1, \quad \tilde{i}_j = 2i_j + l_j - 1$$

are integers for each $j = 1, \ldots, d$. Hence collecting all such $Q_m \subset Q_{m-1}$ for every $Q_{m-1} \in Q_{m-1}^{(\gamma)}$, we finally obtain the partition $Q_m^{(\gamma)}$. Moreover, since we choose $k_m$ such that $\tau_m \in [1, 2)$, by going back to (3.2), we can repeat the division procedure for $Q_m^{(\gamma)}$ and generates $Q_{m+1}^{(\gamma)}$. 
Now we illustrate a merger procedure. We define the collections of cubes \( Q_m^{(γ)} \) for \( m = -1, -2, \ldots \) inductively. Suppose that \( Q_{m+1} \) is a partition and every cubes \( Q_m \) in \( Q_m^{(γ)} \) is of the form
\[
Q_m = Q_{m+1}^{\text{time}} \times Q_{m+1}^{\text{space}}
\]  
(3.4)
where
\[
Q_{m+1}^{\text{time}} = [i_0 2^{-(m+1)γ}, (i_0 + 1) 2^{-(m+1)γ}],
\]
\[
Q_{m+1}^{\text{space}} = \prod_{j=1}^d [i_j 2^{-(m+1)}, (i_j + 1) 2^{-(m+1)}],
\]
\( \tau_m+1 \in [1, 2] \) and \( i_0, i_1, \ldots, i_d \) are integers. Obviously, \( Q_0^{(γ)} \) satisfies (3.4) with \( τ_0 = 1 \). We put
\[
2^{-(m+1)γ} \tau_{m+1} = 2^{-mγ}ρ_m.
\]
Then
\[
ρ_m = 2^{-γ}τ_{m+1} ∈ [2^{-γ}, 2^{-γ}+1) ⊂ [2^{-|γ|}−1, 2^{-|γ|}+1).
\]
Put
\[
k_m = \begin{cases} 
\lceil γ \rceil & \text{if } ρ_m ∈ [2^{-|γ|}, 2^{-|γ|}+1) \\
\lceil γ \rceil + 1 & \text{if } ρ_m ∈ [2^{-|γ|}−1, 2^{-|γ|})
\end{cases}
\]
(3.5)
By combining cubes \( Q_{m+1} \) of \( Q_m^{(γ)} \), we compose the partition \( Q_m^{(γ)} \) with \( Q_m \) of the form
\[
Q_m = Q_m(i, l) = Q_m^{\text{time}}(i) \times Q_m^{\text{space}}(l)
\]
where
\[
Q_m^{\text{time}}(i) = [i 2^{-mγ}ρ_m, (i + 1) 2^{-mγ}ρ_m],
\]
for \( i ∈ 2^{k_m}Z \) and
\[
Q_m^{\text{space}}(l) = \prod_{j=1}^d [l_j 2^{-(m+1)}, (l_j + 1) 2^{-(m+1)}]
\]
for \( l_j ∈ 2Z, j = 1, \ldots, d \). Denote \( τ_m = 2^{k_m}ρ_m \). Then we can rewrite \( Q_m(i, l) \) as
\[
Q_m = [i_0 2^{-mγ}τ_m, (i_0 + 1) 2^{-mγ}τ_m] \times \prod_{j=1}^d [l_j 2^{-m}, (l_j + 1) 2^{-m}]
\]
where
\[
\tilde{i}_0 = \frac{i}{2^{k_m}}, \quad \tilde{i}_j = \frac{l_j}{2}
\]
are integers for each \( j = 1, \ldots, d \). Furthermore, due to the choice of \( k_m \) and \( ρ_m \), we have
\[
τ_m = 2^{k_m}ρ_m ∈ [1, 2).
\]
Hence \( Q_m^{(γ)} \) satisfies (3.4) with \( m \) in place of \( m + 1 \). By repeating this merger procedure, we construct \( Q_m^{(γ)} \) for all \( m = -1, -2, \ldots \).

**Remark 3.5.** Due to the above procedure, one can write \( Q ∈ Q_m^{(γ)} \) for \( m ∈ Z \) as follows
\[
Q = Q(i_0, \ldots, i_d) = [i_0 2^{-mγ}τ_m, (i_0 + 1) 2^{-mγ}τ_m] \times \prod_{j=1}^d [l_j 2^{-m}, (l_j + 1) 2^{-m})
\]
(3.5)
where \((i_0, \ldots, i_d) \in \mathbb{Z}^{d+1}\) and \(\tau_m \in [1, 2)\). In fact, \(2^{-m\gamma \tau_m}\) is a dyadic number. Indeed, \(\tau_0 = 1\) and recall that 

\[
2^{-m\gamma \tau_m} = 2^{-k_m}2^{-(m-1)\gamma \tau_{m-1}}
\]

for \(m = 1, 2, \ldots\). Therefore, 

\[
2^{-m\gamma \tau_m} = 2^{-k_m}2^{-(m-1)\gamma \tau_{m-1}} = \ldots = 2^{-(k_m+k_{m-1}+\cdots+k_1)}2^{-\gamma \tau_1} = 2^{-(k_m+k_{m-1}+\cdots+k_1)}.
\]

Similarly, for \(m = -1, -2, \ldots\), we have 

\[
2^{-m\gamma \tau_m} = 2^{k_m}2^{-(m+1)\gamma \tau_{m+1}}
\]

so that 

\[
2^{-m\gamma \tau_m} = 2^{k_m+k_{m+1}+\cdots+k_{-1}}.
\]

Therefore, \((Q^{(\gamma)}_m, m \in \mathbb{Z})\) is constituted of a class of dyadic cubes. If \(\gamma = 2\), then obviously the above procedure generates the filtration of partitions in Example 3.2.

**Theorem 3.6.** \((Q^{(\gamma)}_m, m \in \mathbb{Z})\) is a filtration of partitions.

_Proof._ Due to the above division-merger procedures, (i) and (ii) of Definition 3.1 are obvious. Hence it suffices to show the regularity condition (iii). For \(m \in \mathbb{Z}\), take \(Q\) and \(Q'\) such that \(Q \subset Q'\), \(Q' \in Q^{(\gamma)}_m\), and \(Q \in Q^{(\gamma)}_{m+1}\). From (3.5), we can write 

\[
Q'(t', x') = [t', t' + 2^{-m\gamma \tau_m}) \times \prod_{j=1}^{d} [x'_j, x'_j + 2^{-m}),
\]

and 

\[
Q(t, x) = [t, t + 2^{-(m+1)\gamma \tau_{m+1}}) \times \prod_{j=1}^{d} [x_j, x_j + 2^{-m-1}).
\]

Then by Remark 3.5, 

\[
\frac{|Q'(t', x')|}{|Q(t, x)|} = \frac{2^{-m\gamma - md \tau_m}}{2^{-(m+1)\gamma -(m+1)d \tau_{m+1}}} = 2^{d+k_{m+1}} \leq 2^{d+|\gamma|+1}.
\]

The theorem is proved. \(\Box\)

4. **Proof of Theorem 2.2 and 2.3**

We first check the Hörmander condition under Assumption 2.1.

**Lemma 4.1.** Under Assumption 2.1, the kernel \(K(t, x, y)\) satisfies Hörmander condition (6.1) with respect to \((Q^{(\gamma)}_m, m \in \mathbb{Z})\), the filtration of partitions in Theorem 3.6.

_Proof._ Let 

\[
Q = [t_0, t_0 + 2^{-m\gamma \tau_m}) \times \prod_{j=1}^{d} [0, 2^{-m}), \quad m \in \mathbb{Z}
\]

and 

\[
Q^* = [t_0, t_0 + 4 \cdot 2^{-m\gamma}) \times \prod_{j=1}^{d} [-2 \cdot 2^{-m}, 2 \cdot 2^{-m}), \quad m \in \mathbb{Z},
\]

where \(1 \leq \tau_m \leq 2\).
It suffices to show
\[
\sup_{(s,y),(r,z) \in Q} \int_{\mathbb{R}^{d+1} \setminus Q^*} |K(t, x, s, y) - K(t, x, r, z)| \, dx \, dt < \infty. \tag{4.1}
\]
Put
\[
\Gamma_1 = \{ t \geq t_0 + 4 \cdot 2^{-m \gamma} \} \times \mathbb{R}^d,
\]
and
\[
\Gamma_2 = \{ t_0 < t < t_0 + 4 \cdot 2^{-m \gamma} \} \cap (\mathbb{R}^{d+1} \setminus Q^*).
\]
Recall that \( K(t, x, s, y) \) vanishes if \( t \leq s \). Then obviously for any \( (s,y),(r,z) \in Q \),
\[
\int_{\mathbb{R}^{d+1} \setminus Q^*} |K(t, x, s, y) - K(t, x, r, z)| \, dx \, dt
\]
\[
\leq \int_{\Gamma_1} |K(t, x, s, y) - K(t, x, r, z)| \, dx \, dt + \int_{\Gamma_2} |K(t, x, s, y) - K(t, x, r, z)| \, dx \, dt
\]
\[
\leq \int_{\Gamma_1} |K(t, x, s, y) - K(t, x, s, z)| \, dx \, dt + \int_{\Gamma_1} |K(t, x, s, z) - K(t, x, r, z)| \, dx \, dt
\]
\[
+ 2 \sup_{(s,y) \in Q} \int_{\Gamma_2} |K(t, x, s, y)| \, dx \, dt =: I_1 + I_2 + I_3.
\]
First we estimate \( I_1 \). Observe that
\[
t_0 + 4 \cdot 2^{-m \gamma} - s \geq 2^{-m \gamma},
\]
and \( |z - y| \leq 2 \cdot 2^{-m} \). So by (2.1),
\[
I_1 \leq \int_{t_0 + 4 \cdot 2^{-m \gamma}}^{\infty} \int_{\mathbb{R}^d} |K(t, x, s, y) - K(t, x, s, z)| \, dx \, dt
\]
\[
\leq \varphi \left( (t_0 + 4 \cdot 2^{-m \gamma} - s)^{-1/\gamma} |z - y| \right) \leq \varphi(2).
\]
For \( I_2 \) we use (2.2). Since \( s, r \in [t_0, t_0 + 2 \cdot 2^{-m \gamma}] \), we have
\[
I_2 \leq \int_{t_0 + 4 \cdot 2^{-m \gamma}}^{\infty} \int_{\mathbb{R}^d} K(t, x, s, z) - K(t, x, r, z) \, dx \, dt
\]
\[
\leq \varphi \left( (2 \cdot 2^{-m \gamma})^{-1} |s - r| \right) \leq \varphi(1).
\]
Finally we estimate \( I_3 \). Note that for \( (t, x) \in \Gamma_2 \) and \( (s, y) \in Q \)
\[
|x - y| \geq 2^{-m}.
\]
Hence from (2.3),
\[
I_3 \leq 2 \sup_{(s,y) \in Q} \int_{t_0}^{t_0 + 4 \cdot 2^{-m \gamma}} \int_{|x-y| \geq 2^{-m}} K(t, x, s, y) \, dx \, dt
\]
\[
\leq 2 \sup_{(s,y) \in Q} \int_{t_0}^{t_0 + 4 \cdot 2^{-m \gamma}} \int_{|x-y| \geq 2^{-m}} K(t, x, s, y) \, dx \, dt
\]
\[
\leq 2 \varphi \left( (4 \cdot 2^{-m \gamma})^{1/\gamma} (2^{-m})^{-1} \right) \leq 2 \varphi(4^{1/\gamma}),
\]
where the second inequality is because \( K \) vanishes if \( t \leq s \). Therefore (4.1) is proved. \( \square \)
We define the operator $K(t, s)$ as follows:

$$K(t, s)f(x) = \int_{\mathbb{R}^d} K(t, x, y)f(y)dy, \quad f \in C_0^\infty.$$ 

Suppose that (2.5) holds, that is for any $t > s$ and $f \in C_0^\infty$

$$\left\| \int_{\mathbb{R}^d} K(t, x, y)f(y)dy \right\|_{L_{p_0}} \leq \varphi(t - s)\|f\|_{L_{p_0}}. \tag{4.2}$$

Since $K(t, s)$ is linear and (1.2) holds, the operator $K(t, x)$ is uniquely extendible to $L_{p_0}$. Hence we can consider $K(t, s)$ as a bounded operator on $L_{p_0}$. Denote $Q_{m}^{time} := \{[4^{-m}i, 4^{-m}(i + 1)) : i \in \mathbb{Z}\}, \quad m \in \mathbb{Z}$.

**Lemma 4.2.** Suppose that Assumption 2.1 (ii) and (2.3) hold, and $K(t, s, x, y) = K(t, s, x - y)$. Then $K(t, s)$ satisfies the Hörmander condition 6.1 with $n = 1$ and $(Q_{m}^{time}, m \in \mathbb{Z})$

**Proof.** Let

$$Q = [t_0, t_0 + \delta], \quad Q^* = [t_0 - 2\delta, t_0 + 2\delta].$$

Note that for $t \notin Q^*$ and $s, r \in Q$, we have

$$|s - r| \leq \delta, \quad |t - (t_0 + \delta)| \geq \delta,$$

and recall $K(t, s, x - y) = 0$ if $t \leq s$. Note

$$\|K(t, s) - K(t, r)\|_{L_{p_0}} = \sup_{\|f\|_{L_{p_0}} = 1} \left\| \int_{\mathbb{R}^d} (K(t, s, x - y) - K(t, r, x - y))f(y)dy \right\|_{L_{p_0}}$$

$$\leq \sup_{\|f\|_{L_{p_0}} = 1} \|f\|_{L_{p_0}} \int_{\mathbb{R}^d} |K(t, s, x) - K(t, r, x)|dx$$

$$= \int_{\mathbb{R}^d} |K(t, s, x) - K(t, r, x)|dx. \tag{4.3}$$

Therefore, by Assumption 2.1 (ii) and (4.3),

$$\int_{\mathbb{R}^d} |K(t, s) - K(t, r)|_{L_{p_0}} dt \leq \int_{\mathbb{R}\backslash Q^*} \int_{\mathbb{R}^d} |K(t, s, x) - K(t, r, x)|dxdt$$

$$\leq \int_{t \geq t_0 + 2\delta} \int_{\mathbb{R}^d} |K(t, s, x) - K(t, r, x)|dxdt$$

$$\leq N\varphi\left(\frac{|s - r|}{\delta}\right) \leq N\varphi(1) \leq N.$$

The lemma is proved. \qed

**Proof of Theorem 2.2 and 2.3.** Due to Lemma 4.1 and Lemma 4.2, these are easy consequences of Theorem 3.3. We only mention that in the proof of Theorem 2.3 following the proof of Theorem 1.1 of [9], one can easily check that for almost any $x$ outside of the closed support of $f \in C_0^\infty(\mathbb{R}, L_{p_0})$,

$$Gf(t, x) = \int_{-\infty}^{\infty} K(t, s)f(s, x) ds,$$

where $G$ denote the unique extension on $L_{p_0}(\mathbb{R}^{d+1})$ stated in Theorem 2.2. The theorems are proved. \qed
5. Auxiliary results

In this section we study a kernel $p_λ$ and an operator $R_λ$ which are related to $A(t) - λ$.

**Lemma 5.1.** Let $σ ≥ 0$, $δ > σ - \frac{d}{2}$ and $h ∈ C^{1+|σ|}(\mathbb{R}^d \setminus \{0\})$. Suppose that there exists constant $c > 0$ such that

$$|D^m_x (−Δ)^{σ/2} h(x)| ≤ c|x|^{δ−2|σ|} \exp\{-c|x|^γ\}, \quad ∀x ∈ \mathbb{R}^d \setminus \{0\} \quad (5.1)$$

for $m = 0, 1$. Also assume (5.1) holds for $m = 2$ if $1 ≤ σ - 2|\frac{d}{2}| < 2$. Then

$$\| (−Δ)^{σ/2} h \|_{L^2(\mathbb{R}^d)} ≤ N$$

where $N = N(c, d, γ, δ, σ)$.

**Proof.** The case $σ ∈ [0, 2)$ is proved in [8, Lemma 5.1]. For $σ ≥ 2$, denote

$$\hat{σ} := σ - 2\left\lfloor \frac{σ}{2} \right\rfloor, \quad v := (−Δ)^{\frac{σ}{2}} h ∈ C^{1+|σ|−2|\frac{σ}{2}|}(\mathbb{R}^d).$$

Then

$$\| (−Δ)^{σ/2} h \|_{L^2(\mathbb{R}^d)} = ∥ (−Δ)^{\hat{σ}/2} (−Δ)^{\frac{σ}{2}} h \|_{L^2(\mathbb{R}^d)} = ∥ (−Δ)^{\hat{σ}/2} v \|_{L^2(\mathbb{R}^d)}.$$

Thus it is enough to apply the result for $σ ∈ [0, 2)$. The lemma is proved. □

Recall that $ψ(t, ξ)$ is the symbol of $A(t)$ satisfying

$$\Re[ψ(t, ξ)] ≤ −κ|ξ|^γ, \quad (5.2)$$

$$|D^α ψ(t, ξ)| ≤ κ^{-1}|ξ|^{−|α|}, \quad |α| ≤ \left\lfloor \frac{d}{2} \right\rfloor + 1.$$  

Note that due to (5.2),

$$p_λ(t, s, x) := 1_{s< t} F^{-1}\left\{ \exp\left( \int_s^t ψ(r, ξ) − λ dr \right) \right\}$$

is well defined for any $λ ≥ 0$. Similarly one can check that

$$R_λ f(t, x) := F^{-1}\left( \int_{−∞}^∞ 1_{s< t} \exp\left( \int_s^t (ψ(r, ξ) − λ) dr \right) Ff(s, ξ) ds \right)(x)$$

is well defined for any $λ > 0$ and $f ∈ L^2(\mathbb{R}^{d+1})$. Obviously

$$R_λ f(t, x) = \int_{−∞}^∞ \int_{\mathbb{R}^d} p_λ(t, s, x− y) f(s, y) dy ds, \quad ∀f ∈ C_0^∞(\mathbb{R}^{d+1}).$$

In the following lemma we show that the operator $R_λ$ is continuously extensible to $L^q(\mathbb{R}, L^p)$ for any $p, q > 1$.

**Lemma 5.2.** Let $λ > 0$ and $p, q > 1$. Then

$$\| p_λ(t, s, ·) \|_{L^1} ≤ N 1_{s< t} e^{-λ(t−s)}$$

$$\| R_λ f(·, ·) \|_{L^p} ≤ N λ^{−(p−1)/p} \| f \|_{L^p(\mathbb{R}^{d+1})}, \quad ∀f ∈ C_0^∞(\mathbb{R}^{d+1}),$$

and

$$\| R_λ f \|_{L^q(\mathbb{R}, L^p)} ≤ \frac{N}{λ} \| f \|_{L^q(\mathbb{R}, L^p)}, \quad ∀f ∈ C_0^∞(\mathbb{R}^{d+1}), \quad (5.3)$$

where $N = N(d, p, q, κ, γ)$.  

Proof: (i) Using $xe^{-x} \leq 1$ for $x \geq 0$, one can check that for any multi-index $|\alpha| \leq \lfloor \frac{d}{2} \rfloor + 1$

$$\left| D_\xi^\alpha \exp \left( \int_s^t \psi(r, (t-s)^{-\frac{s}{2}})dr \right) \right| \leq N(\kappa) |\xi|^{1-|\alpha|} e^{-\kappa|\xi|^2}.$$  

Denote

$$q(t, s, x) := 1_{\lambda < t} \left\{ \exp \left( \int_s^t (\psi(r, (t-s)^{-\frac{s}{2}})dr \right) \right\} = (t-s)^{\frac{d}{2}} e^{\lambda(t-s)} p_\lambda(t, s, (t-s)^{\frac{1}{2}}).$$

By using Hölder inequality and Parseval’s identity, for $\varepsilon < (\gamma/4 \wedge 1/4)$,

$$\|p_\lambda(t, s, \cdot)\|_{L_1} = \|(1 + |x|^{\frac{d+\gamma}{2}})^{-1}(1 + |x|^{\frac{d+\gamma}{2}}) p_\lambda(t, s, \cdot)\|_{L_1} \leq N 1_{\lambda < t} e^{-\lambda(t-s)} \left( \int_{\mathbb{R}^d} \|F(q(t, s, \cdot))\|_{L_2} \right)^{1/2} \leq N 1_{\lambda < t} e^{-\lambda(t-s)} \left( \|F(q(t, s, \cdot))\|_{L_2} + \|F(q(t, s, \cdot))\|_{L_2} \right).$$

Then by using Lemma 5.1 with $\sigma = (d + \varepsilon)/2$ and $h(\cdot) = F(q(t, s, \cdot))$, we have

$$\|p_\lambda(t, s, \cdot)\|_{L_1} \leq N(\delta, \gamma, \kappa) 1_{\lambda < t} e^{-\lambda(t-s)}.$$

To apply Lemma 5.1, $h$ should be $m + 2\lfloor \frac{d}{2} \rfloor$-times differentiable, and this is possible since $m + 2\lfloor \frac{d}{2} \rfloor \leq \lfloor \frac{d}{2} \rfloor + 1$ for $m = 0, 1$ and $m = 2$ if $1 \leq \sigma - 2\lfloor \frac{d}{2} \rfloor < 2$.

(ii) By Minkowski’s inequality, Young’s inequality, and (i),

$$\|R_\lambda f(t, \cdot)\|_{L_p} \leq \int_{-\infty}^{\infty} 1_{\lambda < t} \|p_\lambda(s, t, \cdot)\|_{L_1} \|f(s, \cdot)\|_{L_p} ds \leq N \int_{-\infty}^{\infty} 1_{\lambda < t} e^{-\lambda(t-s)} \|f(s, \cdot)\|_{L_p} ds (5.4) \leq N \lambda^{-(p-1)/2} \|f\|_{L_p(\mathbb{R}^{d+1}).}$$

(iii) By (5.4),

$$\|R f\|_{L_q(\mathbb{R}, L_p)} \leq \left\| \int_0^\infty e^{-\lambda s} \|f(t-s, \cdot)\|_{L_p} ds \right\|_{L_q(\mathbb{R})} \leq \frac{1}{\lambda} \|f\|_{L_q(\mathbb{R}, L_p)).}$$

The lemma is proved. 

Remark 5.3. Due to the above lemma, we can consider the continuous extension of $R_\lambda$ on $L_q(\mathbb{R}, L_p)$ for any $p, q > 1$. From now on, we regard the operator $R_\lambda$ as this extension on $L_q(\mathbb{R}, L_p)$. Actually $R_\lambda$ was already defined on $L_q(\mathbb{R}^{d+1})$, but two different definitions coincide on $L_2(\mathbb{R}^{d+1}) \cap L_q(\mathbb{R}, L_p)$ due to Riesz-Fischer theorem.
6. Proof of Theorem 2.4

Define $K(t, s, x, y) = K(t, s, x - y)$ by

$$K(t, s, x) = 1_{s < t} F^{-1} \left\{ |\xi|^\gamma \exp \left( \int_s^t \psi(r, \xi) dr \right) \right\}. \quad (6.1)$$

Also define

$$\mathcal{G} f(t, x) = F^{-1} \left\{ \int_{-\infty}^\infty 1_{s < t} |\xi|^\gamma \exp \left( \int_s^t \psi(r, \xi) dr \right) F f(s, \xi) ds \right\}.$$ 

It is easy to check that $\mathcal{G} f$ is well defined if $f \in C_0^\infty(\mathbb{R}^{d+1})$, and furthermore

$$\mathcal{G} f(t, x) = \int_{-\infty}^\infty \int_{\mathbb{R}^d} K(t, s, x - y) f(s, y) dy ds.$$ 

**Theorem 6.1.** Let $p, q \in (1, \infty)$. Under the assumptions in Theorem 2.4, the kernel $K$ satisfies Assumption 2.1 and (2.5), and it holds that

$$\|\mathcal{G} f\|_{L_p(\mathbb{R}^d)} \leq N \|f\|_{L_q(\mathbb{R}^d)} \quad \forall f \in C_0^\infty(\mathbb{R}^{d+1}),$$

where $N = N(d, p, q, \gamma, \kappa)$.

**Proof.** **Part 1.** We show that the kernel $K$ defined in (6.1) satisfies Assumption 2.1 and (2.5). Observe that

$$(t - s)^{1 + \frac{d}{2}} K(t, s, (t - s)^{\frac{1}{2}} x)$$

$$= N 1_{s < t} (t - s)^{1 + \frac{d}{2}} \int_{\mathbb{R}^d} e^{i(t - s)^{\frac{1}{2}} x \cdot \xi} |\xi|^\gamma \exp \left\{ \int_s^t \psi(r, \xi) dr \right\} d\xi$$

$$= N 1_{s < t} (t - s)^{1 + \frac{d}{2}} \int_{\mathbb{R}^d} e^{i(x \cdot (t - s)^{\frac{1}{2}} \xi)} |\xi|^\gamma \exp \left\{ \int_s^t \psi(r, \xi) dr \right\} d\xi$$

$$= N 1_{s < t} \int_{\mathbb{R}^d} e^{i(x \cdot \xi)} |\xi|^\gamma \exp \left\{ \int_s^t \psi(r, (t - s)^{-\frac{1}{2}} \xi) dr \right\} d\xi.$$ 

Denote

$$F(t, s, \xi) = 1_{s < t} |\xi|^\gamma \exp \left\{ \int_s^t \psi(r, (t - s)^{-\frac{1}{2}} \xi) dr \right\}.$$ 

Due to the assumptions on $\psi(t, \xi)$,

$$|D^\alpha \xi F(t, s, \xi)| \leq N |\xi|^{|\gamma| - |\alpha|} \exp\{-\kappa |\xi|^\gamma\}, \quad \forall \xi \in \mathbb{R}^d \setminus \{0\}$$

for every multi-index $\alpha$ with $|\alpha| \leq \lfloor \frac{d}{2} \rfloor + 1$. Therefore, by Lemma 5.1

$$\|(-\Delta)^{\sigma/2} F(t, s, \xi)\|_{L_2(\mathbb{R}^d)} \leq N(d, \kappa, \gamma)$$

for all $\sigma \in \left[ 0, \frac{d}{2} + \gamma \right) \cap \left[ 0, \lfloor \frac{d}{2} \rfloor + 1 \right]$.

We claim that for any $\mu \in \left[ 0, \min\{\gamma, \lfloor \frac{d}{2} \rfloor + 1 - \frac{d}{2} \} \right)$,

$$\int_{\mathbb{R}^d} |x|^{\mu} |K(t, s, x)| dx \leq N(d, \gamma, \kappa, \mu) (t - s)^{\frac{d}{2} - 1}. \quad (6.3)$$

Indeed, fix $\mu \in \left[ 0, \min\{\gamma, \lfloor \frac{d}{2} \rfloor + 1 - \frac{d}{2} \} \right)$ and choose $\sigma > 0$ such that

$$\mu + \frac{d}{2} < \sigma < \min\left\{ \gamma + \frac{d}{2}, \left\lfloor \frac{d}{2} \right\rfloor + 1 \right\}.$$
Therefore, Hölder inequality, Parseval’s identity, and Lemma 5.1
\[
\int_{\mathbb{R}^d} |x|^{\mu} K(t, s, x) dx = (t-s)^{\frac{d+\mu}{d}} \int_{\mathbb{R}^d} |x|^{\mu} K(t, s, (t-s)^{\frac{1}{d}} x) dx \\
\leq N(t-s)^{\frac{d+\mu}{d}} \left( \int_{\mathbb{R}^d} |1+|x|^\sigma|K(t, s, (t-s)^{\frac{1}{d}} x)|^2 dx \right)^{\frac{1}{2}} \\
\leq N(t-s)^{\frac{\mu}{d} - 1} \left( \int_{\mathbb{R}^d} |1+(-\Delta \xi)^{\sigma/2})F(t, s, \xi)|^2 \, d\xi \right)^{\frac{1}{2}} \leq N(t-s)^{\frac{\mu}{d} - 1}.
\]
Hence (3.5) holds for any \(0 \leq \mu < \min\{\gamma, \frac{d}{2}\} + 1 - \frac{d}{2}\). One also can see that
\[
\int_s^b \int_{|x|\geq \rho} |K(t, s, x)| dx dt \leq \int_s^b \int_{\mathbb{R}^d} \frac{|x|^{\mu}}{\rho^{\mu}} K(t, s, x) dx dt \\
\leq N \int_s^b \rho^{-\mu} (t-s)^{\frac{\mu}{d} - 1} dt = N \rho^{-\mu} (b-s)^{\frac{\mu}{d}}.
\]
Therefore, \(K\) satisfies (2.3) with \(\varphi(t) = N^\mu t^\nu\) for some constant \(N = N(d, \gamma, \kappa)\).
Next we prove (2.1) and (2.2). Note that
\[
(t-s)^{1+\frac{d+\mu}{d}} \frac{\partial K}{\partial x}(t, s, (t-s)^{\frac{1}{d}} x) \\
= N 1_{s<t}(t-s)^{1+\frac{d+1}{d}} \int_{\mathbb{R}^d} e^{i((t-s)^{\frac{1}{d}} x, \xi)} i\xi |\gamma| \exp \left\{ \int_s^t \psi(r, \xi) dr \right\} \, d\xi \\
= N 1_{s<t}(t-s)^{1+\frac{d+1}{d}} \int_{\mathbb{R}^d} e^{i(x, (t-s)^{\frac{1}{d}} \xi)} i\xi |\gamma| \exp \left\{ \int_s^t \psi(r, \xi) dr \right\} \, d\xi \\
= N 1_{s<t} \int e^{i(x, \xi)} i\xi |\gamma| \exp \left\{ \int_s^t \psi(r, (t-s)^{-\frac{1}{d}} \xi) \, dr \right\} \, d\xi.
\]
For \(\frac{d}{2} < \sigma \leq \min\{\gamma + 1 + \frac{d}{2}, \frac{d}{2}\} + 1\), by Hölder inequality, Parseval’s identity, and Lemma 5.1
\[
\int_{\mathbb{R}^d} |K(t, s, x+y) - K(t, s, x)| dx \\
= |y| \int_{\mathbb{R}^d} |\nabla K(t, s, x+\theta y)| dx \quad (\theta \in [0, 1]) \\
= |y|(t-s)^{\frac{d}{2}} \int_{\mathbb{R}^d} |\nabla K(t, s, (t-s)^{\frac{1}{d}} x)| dx \\
\leq |y|(t-s)^{\frac{d}{2}} \left( \int_{\mathbb{R}^d} |(1+|x|^\sigma)\nabla K(t, s, (t-s)^{\frac{1}{d}} x)|^2 \, dx \right)^{\frac{1}{2}} \\
\leq N|y|(t-s)^{\frac{\mu}{d} - 1} \left( \int_{\mathbb{R}^d} |1+(-\Delta \xi)^{\sigma/2})F(t, s, \xi)|^2 \, d\xi \right)^{\frac{1}{2}} \leq N|y|(t-s)^{\frac{\mu}{d} - 1}.
\]
Hence
\[
\int_{a}^{\infty} \int_{\mathbb{R}^d} |K(t, s, x+y) - K(t, s, x)| \, dx \, dt \leq N|y|(a-s)^{-\frac{\mu}{d}},
\]
and therefore (2.1) holds. Finally, denote
\[ F(t, s, \xi) = 1_{s < t} \psi(t, (t - s)^{-\frac{d}{2}} \xi) \exp \left\{ \int_s^t \psi(r, (t - s)^{-\frac{d}{2}} \xi) dr \right\} \]
and observe that for \( s \leq \max\{r, s\} \leq b < a < t \) and \( \frac{d}{2} < \sigma < \min\{2\gamma + \frac{d}{2}, \frac{d}{2} + 1\} \),
(\( \tau = \theta s + (1 - \theta) r \))
\[ \int_{\mathbb{R}^d} |K(t, r, x) - K(t, s, x)| dx = |s - r| \int_{\mathbb{R}^d} |\partial_s K(t, \tau, x)| dx \]
\[ = |s - r| |(t - \tau)^{-\frac{d}{2}} \int_{\mathbb{R}^d} |\partial_s K(t, \tau, x)|^2 dx| \]
\[ = \frac{|s - r|}{(t - \tau)^{\frac{d}{2}}} \int_{\mathbb{R}^d} |F(t, \tau, \cdot)|^2 (\int_{\mathbb{R}^d} |(1 - \Delta_{\xi}^{\gamma/2}) F(t, \tau, \xi)|^2 d\xi)^{1/2} \]
\[ \leq N \frac{|s - r|}{(t - b)^2} 1_{r < t}. \]

Therefore,
\[ \int_a^\infty \int_{\mathbb{R}^d} |K(t, r, x) - K(t, s, x)| dxdt \]
\[ \leq N \int_a^\infty \frac{|s - r|}{(t - b)^2} dt \leq N |s - r| \int_a^\infty \frac{1}{t^2} dt = \frac{N |s - r|}{a - r}. \]

This certainly leads to (2.2), and thus Assumption (2.1) holds.

**Part 2.** We prove (2.2) when \( p = q \). First we show \( K \) satisfies (2.4) with \( p_0 = 2 \).

Due to Parseval’s identity, for any \( f \in L_2(\mathbb{R}^{d+1}) \) it holds that
\[ \|Gf\|^2_{L_2(\mathbb{R}^{d+1})} \]
\[ = N \int_{-\infty}^\infty \int_{\mathbb{R}^d} \left| \int_{-\infty}^\infty \xi^\gamma \exp \left\{ \int_s^t \psi(r, \xi) dr \right\} F(f)(s, \xi) ds \right|^2 d\xi dt \]
\[ \leq N \int_{-\infty}^\infty \int_{\mathbb{R}^d} \left| \int_{-\infty}^\infty \xi^\gamma \exp \{-\kappa |\xi^\gamma (t - s)| \} F(f)(s, \xi) ds \right|^2 d\xi dt \]
\[ = N \int_{-\infty}^\infty \int_{\mathbb{R}^d} \left| \int_{-\infty}^\infty e^{it\tau} |\xi^\gamma | \exp \{-\kappa |\xi^\gamma (t - s)| \} F(f)(s, \xi) ds \right| dt^2 d\xi d\tau \]
\[ = N \int_{-\infty}^\infty \int_{\mathbb{R}^d} \left| \int_{-\infty}^\infty e^{it\tau} |\xi^\gamma | \exp \{-\kappa |\xi^\gamma (t - s)| \} F(f)(s, \xi) ds \right| dt^2 d\xi d\tau \]
\[ \leq N \int_{-\infty}^\infty \int_{\mathbb{R}^d} \left| \int_{-\infty}^\infty e^{it\tau} |F(|t, \xi)| dt \right|^2 d\xi d\tau \]
\[ \leq N \int_{-\infty}^\infty \int_{\mathbb{R}^d} |F(f)|^2 d\xi dt = N \|f\|^2_{L_2(\mathbb{R}^{d+1})}. \]

Actually \( Gf \) was defined only for \( f \in C_0^\infty(\mathbb{R}^{d+1}) \). However, the calculations above show it is also defined on \( L_2(\mathbb{R}^{d+1}) \). Therefore \( K \) satisfies (2.4) with \( p_0 = 2 \). Hence
by Theorem 2.3 holds for $p = q$, $1 < p \leq 2$, and for all $f \in L_p(\mathbb{R}^{d+1})$. For $p \in (2, \infty)$, we apply the standard duality argument. Denote $p' = p/(p-1)$ and

$$P(s, t, x) = K(-t, -s, x) = 1_{t < s}F^{-1} \left\{ |\xi|^\gamma \exp \left( \int_t^s \psi(-r, \xi)dr \right) \right\},$$

and define operator $\mathcal{P} : L_p'(\mathbb{R}^{d+1}) \to L_{p'}(\mathbb{R}^{d+1})$ by

$$\mathcal{P}g(s, y) = \int_{\mathbb{R}^{d+1}} P(s, t, y - x)g(t, x)dxdt.$$

Note that $\psi(-r, \xi)$ also satisfies (2.6) and (2.7). Then for $g \in C_0^\infty(\mathbb{R}^{d+1})$, by change of variable $(t, s, x, y) \to (-t, -s, -x, -y)$ and Fubini’s theorem we have

$$\int_{\mathbb{R}^{d+1}} g(t, x)\mathcal{G}f(t, x)dxdt$$

$$= \int_{\mathbb{R}^{d+1}} g(t, x) \left( \int_{\mathbb{R}^{d+1}} K(t, s, x - y)f(s, y)dyds \right)dxdt$$

$$= \int_{\mathbb{R}^{d+1}} f(s, y) \left( \int_{\mathbb{R}^{d+1}} K(t, s, x - y)g(t, x)dxdt \right)dyds$$

$$= \int_{\mathbb{R}^{d+1}} f(-s, -y) \left( \int_{\mathbb{R}^{d+1}} P(s, t, y - x)g(-t, -x)dxdt \right)dyds$$

$$= \int_{\mathbb{R}^{d+1}} f(-s, -y)\mathcal{P}g(s, y)dyds,$$

where $\bar{g}(t, x) = g(-t, -x)$. By applying Hölder inequality,

$$\int_{\mathbb{R}^{d+1}} g(t, x)\mathcal{G}f(t, x)dxdt \leq N\|f\|_{L_p(\mathbb{R}^{d+1})}\|\mathcal{P}g\|_{L_{p'}(\mathbb{R}^{d+1})}.$$

Since $p' \in (1, 2]$, we have $\|\mathcal{P}g\|_{L_{p'}(\mathbb{R}^{d+1})} \leq N\|g\|_{L_p(\mathbb{R}^{d+1})}$. This implies the desired result since $g \in C_0^\infty(\mathbb{R}^{d+1})$ is arbitrary. Thus (6.2) holds for all $p \in (1, \infty)$.

**Part 3.** Finally we check that $K$ satisfies (2.5) and prove (6.2) for general $p, q > 1$. Recall the operator $\mathcal{K}(t, s)$, that is

$$\mathcal{K}(t, s)f(x) = \int_{\mathbb{R}^d} K(t, s, x - y)f(y)dy$$

for $f \in C_0^\infty$ and $t, s \in \mathbb{R}$. Fix $p \in (1, \infty)$. By (6.3), we have

$$\|\mathcal{K}(t, s)f\|_{L_p} \leq \|f\|_{L_p} \int_{\mathbb{R}^d} |K(t, s, y)|dy \leq N\|f\|_{L_p}(t - s)^{-1}.$$

Hence (2.5) is satisfied with $\varphi(t) = t^{-1}$ and $p_0 = p$. Therefore from Theorem 2.3 we conclude that for any $1 < q \leq p$, (6.2) holds for all $f \in C_0^\infty(\mathbb{R}, L_p)$.

Now let $1 < p < q < \infty$. Define $p' = p/(p-1)$, $q' = q/(q-1)$. Since $1 < q' < p'$, by (6.4) we conclude that

$$\int_{\mathbb{R}^{d+1}} g(t, x)\mathcal{G}f(t, x)dxdt = \int_{\mathbb{R}} \left( \int_{\mathbb{R}^d} f(-s, -y)\mathcal{P}g(s, y)dy \right)ds$$

$$\leq \int_{\mathbb{R}} \|f(-s, \cdot)\|_{L_p}\|\mathcal{P}g(s)\|_{L_{p'}}ds$$

$$\leq N\|f\|_{L_q(\mathbb{R}, L_p)}\|g\|_{L_{q'}(\mathbb{R}, L_{p'})}.$$
for any \( f, g \in C_0^\infty(\mathbb{R}^{d+1}) \). Since \( g \) is arbitrary and
\[
\|Gf\|_{L_q(\mathbb{R}, L_p)} = \sup_{\|g\|_{L_q(\mathbb{R}, L_p)} \leq 1} \left| \int_{\mathbb{R}^{d+1}} g(t, x) Gf(t, x) dx dt \right|
\]
we have
\[
\|Gf\|_{L_q(\mathbb{R}, L_p)} \leq N \|f\|_{L_q(\mathbb{R}, L_p)}.
\]
Therefore for any \( p, q \in (1, \infty) \), we obtain
\[
\|Gf\|_{L_q(\mathbb{R}, L_p)} \leq N \|f\|_{L_q(\mathbb{R}, L_p)}, \quad \forall f \in C_0^\infty(\mathbb{R}^{d+1}), \tag{6.5}
\]
where \( N \) is independent of \( f \). Since \( C_0^\infty(\mathbb{R}^{d+1}) \) is dense in \( L_q(\mathbb{R}, L_p) \), \( G \) is continuously extendible to \( L_q(\mathbb{R}, L_p) \).

Next, we prove a priori estimate.

**Lemma 6.2** (a priori estimate). Let \( \lambda \geq 0 \) and \( p, q \in (1, \infty) \). Suppose that conditions 2.4 and 2.7 are fulfilled. Then for any \( u \in C_0^\infty(\mathbb{R}^{d+1}) \), we have
\[
\|(-\Delta)^{\gamma/2} u\|_{L_q(\mathbb{R}, L_p)} + \lambda \|u\|_{L_q(\mathbb{R}, L_p)} \leq N \|\Delta u + \lambda u\|_{L_q(\mathbb{R}, L_p)}, \tag{6.6}
\]
where \( N \) depends only on \( d, p, \gamma, \) and \( \kappa \).

**Proof.** Put \( f := \frac{\partial}{\partial t} u - \Delta u + \lambda u \). Then obviously \( f \in L_2(\mathbb{R}^{d+1}) \).

**Case 1** \( \lambda = 0 \). By taking the Fourier transform, we can easily check that
\[ (-\Delta)^{\gamma/2} u = Gf \ (a.e). \]

Hence from (6.5), we have
\[ \|(-\Delta)^{\gamma/2} u\|_{L_q(\mathbb{R}, L_p)} \leq N \|\Delta u\|_{L_q(\mathbb{R}, L_p)}. \tag{6.7} \]

**Case 2** \( \lambda > 0 \). Similarly one can also check \( u = \mathcal{R} f \ (a.e) \). Hence (6.6) is a consequence of (6.6) because
\[ \|\Delta u + \lambda u\|_{L_q(\mathbb{R}, L_p)} \leq \|f\|_{L_q(\mathbb{R}, L_p)} + \lambda \|u\|_{L_q(\mathbb{R}, L_p)}. \]

The lemma is proved. \( \square \)

**Proof of Theorem 2.4**

Note that \( C_0^\infty(\mathbb{R}^{d+1}) \) is dense in \( \mathbb{H}_{q, p}^{\gamma} \) and \( \mathcal{A}(t) \) is a continuous operator on \( H_p^\gamma \) due to Mihlin multiplier theorem. Indeed, for \( |\alpha| \leq \lfloor \frac{d}{2} \rfloor + 1, \)
\[
\left| D_\xi^\gamma \left( \frac{\psi(t, \xi) - \psi(t, 0)}{|\xi|^{\gamma}} \right) \right| \leq N \left( \frac{|\xi|^{2|\alpha| - |\alpha|}}{|\xi|^{2\gamma|\alpha|}} \right) \leq N(\kappa)|\xi|^{-|\alpha|}.
\]

Hence from Lemma 6.2
\[
\|u_t\|_{L_q(\mathbb{R}, L_p)} + \|(-\Delta)^{\gamma/2} u\|_{L_q(\mathbb{R}, L_p)} + \lambda \|u\|_{L_q(\mathbb{R}, L_p)}
\]
\[
\leq N(d, p, q, \gamma, \kappa) \|u_t - \Delta u + \lambda u\|_{L_q(\mathbb{R}, L_p)}
\]
for any \( u \in \mathbb{H}_{q, p}^{\gamma} \), and the uniqueness of solutions to (1.1) is proved.

It only remains to prove the existence of solutions. For \( f \in \mathbb{H}_{q, p}^{1, \gamma} \), we consider a sequence \( f_n \in C_0^\infty(\mathbb{R}^{d+1}) \) so that \( \|f_n - f\|_{\mathbb{H}_{q, p}^{1, \gamma}} \to 0 \) as \( n \to \infty \). For each \( n \), we can easily check that \( \mathcal{R} f_n \) is a solution to (1.1). Since \( \mathbb{H}_{q, p}^{1, \gamma} \) is a Banach space, we
can find a solution \( u \) as the limit of \( R f_n \) in \( H^{1,2}_{q,p} \) using the a priori estimate. The theorem is proved. \( \square \)

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