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Monodromy of Fano Problems

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A Fano problem \([d], n, r\) is the problem of enumerating all \(r\)-dimensional linear subspaces on a general complete intersection \(X[d] = X_{d_1} \cap \cdots \cap X_{d_s} \subset \mathbb{P}^n_K\) whenever this is finite.

**Example**

- \(N((3), 3, 1) = 27\)
- \(N((3), 18, 6) = 38406501359372282063949\) (H.-Kadets)
Fix a Fano problem \(([d], n, r)\).

- \(M_{[d]} := \prod_{i=1}^{s} \mathbb{P}^{(d_i + n) - 1}\) moduli space of tuples of hypersurfaces of degrees \([d]\).
- Incidence scheme \(I := \{(X_{[d]}, \Lambda) \mid \Lambda \subset X_{[d]}\} \subset M_{[d]} \times \mathbb{G}(r, n)\)
- \(\pi : I \to M_{[d]}\) is generically étale: let \(G_{([d], n, r)}\) be the Fano monodromy group.
Motivational question

Question
What can we say about the Fano monodromy group?

Basic principle: we need geometry to compute the monodromy group, which in turn reveals information about the geometry.
Example (Harris, 1979)

The Fano monodromy group of the cubic surface is the Weyl group $G_{((3),3,1)} = W(E_6) \subset S_{27}$. The Fano monodromy groups of lines in hypersurfaces $G_{((d),n,1)}$ are symmetric groups $S_N((d),n,1)$ when $(d) \neq (3)$. 
Main Theorems

Theorem (H.-Kadets)

Suppose we are not in the case of a cubic surface in $\mathbb{P}^3$ or the intersection of two quadrics in $\mathbb{P}^n$. Then the Fano monodromy group is the alternating group or the symmetric group, $A_N([d],n,r)$ or $S_N([d],n,r)$.

Theorem (H.-Kadets)

Suppose $\text{char } K \neq 2$. The Fano monodromy group of $k$-planes on the complete intersection of two quadrics in $\mathbb{P}^{2k+2}$ is the Weyl group $W(D_{2k+3})$. 
There is a good classification of multiply transitive groups: if $G$ is a 4-transitive permutation group which acts on a set of size $n > 24$, it must be $A_n$ or $S_n$.

A group $G$ acting on a set $S$ is $(m + 1)$-transitive if the stabilizer of a set of $m$ points $T \subset S$ acts transitively on $S \setminus T$. 
An étale covering $f : X \to Y$ of a normal irreducible scheme $Y$ has transitive monodromy iff $X$ is irreducible.

Our situation:

$$
\begin{array}{c}
I \xrightarrow{\pi_2} \mathbb{G}(r, n) \\
\downarrow \pi \\
M[d]
\end{array}
$$
Double Transitivity

Fix an $r$-plane $\Lambda$. Consider $M_\Lambda$, parametrizing $X[d]$ containing $\Lambda$.

$\pi : I_\Lambda \to M_\Lambda$ has monodromy group $G_\Lambda$ contained in a 1-point stabilizer. Show it acts transitively on a smooth fiber of $I' := I_\Lambda \setminus \pi_2^{-1}(\Lambda)$. 
Equidimensionality

Optimistic approach: $\pi_2 : I' \to \mathbb{G}(r, n)$, hope this is proper with irreducible equidimensional fibers over an open set of $\mathbb{G}(r, n)$.

Modify $I'$ to make this true: $U := \{\Sigma|\Sigma \cap \Lambda = \emptyset\} \subset \mathbb{G}(r, n)$, and $I'' := I' \cap \pi_2^{-1}(U)$ has equidimensional fibers over $U$.

For a general $X_{[d]}$, a dimension count shows $\pi^{-1}(X_{[d]})$ is contained in $\pi_2^{-1}(U)$, so suffices to show $I''$ irreducible.
Remark

Double transitivity of the Fano monodromy group $G([d],n,r)$ implies that any two distinct $r$-planes $\Lambda_1$ and $\Lambda_2$ on a general complete intersection $X[d]$ are linearly independent.
Inductive approach

As long as there is room for $m$ linearly independent $r$-planes, we can go from $m$-transitive to $(m + 1)$-transitive by considering $\pi : I_\Lambda \to M_\Lambda$ fixing $\Lambda_1, \ldots, \Lambda_m$ contained in $X_{[d]}$. Monodromy group $G_\Lambda$ contained in $m$-point stabilizer.
Sometimes the induction stops short: 6-planes on $X(3) \subset \mathbb{P}^{18}$, $18 + 1 < 3(6 + 1)$, can’t fit 3 linearly independent planes to prove 4-transitivity.

Group theory fact: the only 3-transitive groups that act on sets with cardinality $n > 24$ and $n$ not a power 2 or one more than a power of a prime are $A_n$ and $S_n$. So we must compute $N((3), 18, 6) = \deg(F_6(X(3)))$.

There is a finite list of “small Fano problems”, $[d] \neq (2, 2, 2)$. 
Small Fano problems

By Debarre-Manivel, \( N((3), 18, 6) \) is the coefficient of \( x_0^{18} x_1^{17} \ldots x_6^{12} \) in

\[
\left( \prod_{a_0+a_1+\cdots+a_6=3} a_0 x_0 + \cdots + a_6 x_6 \right) \prod_{0 \leq i < j \leq 6} (x_i - x_j).
\]

Has 105 linear factors, after multiplying only 50 together, have roughly 1 million monomials...

\( N((3), 22, 7) \) is an even bigger small problem!