Higher-Order Floquet Topological Phases with Corner and Bulk Bound States

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We report the theoretical discovery and characterization of higher-order Floquet topological phases dynamically generated in a periodically driven system with mirror symmetries. We demonstrate numerically and analytically that these phases support lower-dimensional Floquet bound states, such as corner Floquet bound states at the intersection of edges of a two-dimensional system, protected by the nonequilibrium higher-order topology induced by the periodic drive. We characterize higher-order Floquet topologies of the bulk Floquet Hamiltonian using mirror-graded Floquet topological invariants. This allows for the characterization of a new class of higher-order “anomalous” Floquet topological phase, where the corners of the open system host Floquet bound states with the same as well as with double the period of the drive. Moreover, we show that bulk vortex structures can be dynamically generated by a drive that is spatially inhomogeneous. We show these bulk vortices can host multiple Floquet bound states. This “stirring drive protocol” leverages a connection between higher-order topologies and previously studied fractionally charged, bulk topological defects. Our work establishes Floquet engineering of higher-order topological phases and bulk defects beyond equilibrium classification and offers a versatile tool for dynamical generation and control of topologically protected Floquet corner and bulk bound states.

I. INTRODUCTION

Topological phases of matter are characterized by an intimate relationship between the patterns of motion in the bulk and those at the boundaries of the system. While there is no general theory of this bulk-boundary correspondence, it is known to hold in certain classes of topological phases, e.g. those of non-interacting fermions protected by internal and crystalline symmetries.11 The interface between two such phases with the same symmetries and different topological invariants binds gapless modes. For example, a one-dimensional interface between the two-dimensional quantum spin Hall phase and a trivial insulator supports an odd number of gapless helical edge modes.6–8

In recent years, topological classification of phases of matter has been extended to systems that are driven periodically out of equilibrium.9–11 In these systems, bulk-boundary correspondence acquires a new, temporal character: for a drive frequency Ω, the localized boundary modes may now coexist at the same interface at different values of the quasienergy, ε+ = 0 (Floquet zone center) and ε− = Ω/2 (Floquet zone edge).1213 At high frequencies, the dynamics self-averages to equilibrium and the quasienergies asymptotically approach the energies of the average Hamiltonian. Thus, the boundary modes at the quasienergy zone edge disappear and the Floquet topology coincides with the equilibrium topology of the average Hamiltonian. As the frequency is lowered, topological transitions at the Floquet zone edge and center are induced and Floquet topology acquires a richer structure than any equilibrium topology.1421 Apart from their richer topological structure, Floquet topological phases promise practical advantages over their equilibrium counterparts, such as more control. Indeed, the topological phase of the system can be tuned, usually with great precision, by the drive protocol (drive amplitude, frequency, and shape), thus allowing phase transitions in situ.

More recently, the notion of bulk-boundary correspondence has been generalized to higher-order topological phases in equilibrium, whose surfaces at one lower dimension remain gapped, yet support gapless modes localized at their lower-dimensional boundaries, such as hinges and corners.22–24 For example, a two-dimensional electric quadrupole topological insulator binds corner states with fractional charge e/2. Such higher-order topological phases have been predicted to exist in engineered lattices of cold atoms and in natural elemental bismuth.25 and have been observed in a mechanical system of coupled microwave resonator, optical waveguides, topological circuits, and perturbative mechanical metamaterials.26 In this work, we show that higher-order Floquet topological phases can be realized and controlled in a periodically driven system, supporting lower-dimensional Floquet bound states at the Floquet zone center and/or edge.

Specifically, we study a driven model with mirror symmetries that realizes Floquet topological quadrupole phases and supports Floquet corner states. We show that with open boundary conditions this system supports Floquet bound states at the corners. With periodic boundary conditions, we characterize these phases using mirror-graded Floquet topological invariants. In particular, we show these invariants correctly predict the higher-order anomalous Floquet topological phase that supports Floquet corner states at both Floquet zone center and edge.

Furthermore, we study drive protocols that are spatially inhomogeneous. We design specific protocols that can be used to “stir” topological bulk defects, namely vortices, that host lower-dimensional Floquet bound
states in the bulk. Thus, we expand the notion of higher-order topology to systems with spatiotemporal nonuniformities.

The paper is organized as follows. In Section II we review the model, its symmetries, and the characterization of its equilibrium higher-order topology. Here, we also introduce our notation of Floquet topology and the general scheme of defining Floquet topological invariants for a drive protocol with time-reflection symmetry. In Section III we use this scheme to study the driven model and characterize, analytically and numerically, the higher-order Floquet topology as a function of frequency. This will set the stage for describing higher-order symmetries as well as the Floquet theory of periodic dynamics.

II. MODEL AND FLOQUET THEORY

In this section, we introduce the model that exhibits higher-order topological phases with an emphasis on the algebra of its symmetries. For completeness, we also briefly review the method of characterizing its higher-order topology in equilibrium in the presence of certain symmetries as well as the Floquet theory of periodic dynamics. This will set the stage for describing higher-order Floquet topologies in Section III.

A. Model

We demonstrate our findings in a driven $\pi$-flux dimerized square lattice as a minimal model of a two-dimensional quadrupole Floquet topological insulator. The Hamiltonian is

$$H = \sum_{(rs)} w_{rs} e^{i\phi_{rs}} c_{r}^{\dagger} c_{s}$$

where $c_{r}$ annihilates a spinless fermion at site $r = (x, y)$, $w_{rs} = w_{sr}^{*}$ are hopping amplitudes between nearest neighbors $(rs)$, and $\phi_{rs}$ are Peierls phases implementing the magnetic flux penetrating the lattice. In the Landau gauge $\phi_{r+e_{x}} = \pi y \mathbf{e}_{x}$, where we have used natural units $\hbar = c = e = 1$. The hopping amplitude in the direction $e_{\mu}$ of a nearest neighbor is modulated as

$$w_{rr+e_{\mu}} = w_{rr}[1 - \text{Re}(n_{rr} e^{i\theta_{r}})]$$

where $n_{rr} = e^{i\arg e_{x} e^{i\pi r} e_{y}}$ are directional complex signs. Here, $f_{r}$ is a complex function that specifies the hopping modulation locally. For a uniform $f = f = |f| e^{i\chi}$ we have

$$w_{rr+\pm x} = w_{1} [1 \mp (-1)^{y}|f| \sin \chi]$$

$$w_{rr+\pm y} = w_{2} [1 \mp (-1)^{y}|f| \cos \chi]$$

i.e. a uniform hopping modulation by $Re f$ (Im $f$) in the $x$ ($y$) direction.

In this case, the unit cell has four basis points; thus, for a unit cell at position $R$ with the site $r$ at its corner, $\psi_{R}^{1} = (c_{r}^{1}, c_{r+\pm x}^{1}, c_{r+\pm y}^{1})$ defines a unit-cell spinor. For a system with $L$ sites and periodic boundary conditions, we can write the Hamiltonian in the Bloch basis $\psi_{k}^{1} = \frac{1}{2} e^{-i\pi/2} \sum_{R} e^{ik_{R}R} \psi_{R}^{1}$ with lattice momentum $k = (k_{1}, k_{2})$, as $H = \sum_{k} \psi_{k}^{1} H(k) \psi_{k}^{1}$, with Bloch Hamiltonian

$$H(k) = A_{1} H_{1}(k_{1}) + A_{2} H_{2}(k_{2})$$

where $f_{1} \equiv \text{Re} f$, $f_{2} \equiv \text{Im} f$, $C_{1} = -i \gamma_{1}$, $C_{2} = \gamma_{2}$, and $\gamma_{0}, \gamma_{\alpha} \in \{0, 1, 2, 3\}$, are Dirac matrices satisfying the Clifford algebra $\{\gamma_{0}, \gamma_{\alpha}\} = 2\delta_{\alpha\beta}$ with the metric $g = \text{diag}(1, -1, -1, -1)$, and $\gamma_{5} = -i \gamma_{0} \gamma_{1} \gamma_{2} \gamma_{3}$. We use the Weyl basis $\gamma_{0} = \sigma_{1} \otimes 1$, $\gamma = i\sigma_{2} \otimes \sigma$, and $\gamma_{5} = \sigma_{3} \otimes 1$ in terms of Pauli matrices $\sigma$.

B. Symmetries

The above model is a two-dimensional generalization of the one-dimensional Su-Schrieffer-Heeger (SSH) model, equipped with the proper Clifford algebra. Here, we focus on the algebra of the symmetries that determine its spectral and topological properties regardless of the choice of unit-cell basis or gauge. The Hamiltonians

$$H_{j}(k_{j}) \equiv A_{j} D_{j}(k_{j})$$

represent a SSH model in the $k_{j}$ direction. In each direction, the Hermitian unitary operator $C_{j} = C_{j}^{1} = C_{j}^{\dagger}$ represents a chiral symmetry:

$$\{A_{j}, C_{j}\} = 0 \Rightarrow C_{j} H_{j}(k_{j}) C_{j} = -H_{j}(k_{j})$$

The commutation algebra

$$[C_{1}, C_{2}] = [C_{1}, A_{2}] = [C_{2}, A_{1}] = \{A_{1}, A_{2}\} = 0$$

ensures that these two SSH Hamiltonians anticommute with each other, $[H_{1}, H_{2}] = 0$. Consequently, the operator $C = C_{1} C_{2} = \gamma_{0} \gamma_{5}$, $\{C, A_{1}\}$, is the chiral symmetry of the full Hamiltonian $H = H_{1} + H_{2}$, $[C, H] = 0$.

Due to the enlarged dimension of the unit cell, each direction now has a continuum of discrete symmetries. For example, mirror symmetries $M_{j} H_{j}(k_{j}) M_{j}^{-1} = H(-k_{j})$ are given by $M_{j} = A_{j} U_{j}$, where $U_{j}$ is a unitary that commutes with $H_{j}$. This is the $U(2)$ group generated by $\{1, C_{j}, A_{j} C_{j}, iA_{j} C_{j}\}$, where $j \neq j$ is the complement of $j$. Imposing the condition $[M_{j}, H_{j}] = 0$ then chooses
\[ M_j = A_j C_j \] as the mirror symmetry of the full Hamiltonian. Thus, \[ M_1 = i\gamma_3 \] and \[ M_2 = \gamma_3 \gamma_5. \] We have

\[ \{M_1, M_2\} = \{M_j, C_j\} = [M_j, C_j] = [M_j, A_j] = 0. \tag{10} \]

Since \( M_1 \) and \( M_2 \) anticommute, \( I = -iM_1 M_2 = \gamma_5 \) is a Hermitian unitary representing the inversion symmetry.

We note the action of two diagonal mirror symmetries, \( M_1 : (k_1, k_2) \rightarrow (-k_2, -k_1) \) and \( M_2 : (k_1, k_2) \rightarrow (k_2, k_1). \) Supplied with proper Hermitian units, \( M_1' = e^{i(\pi/4)A_1} e^{i(\pi/4)A_2}, \)

\[ M_1^2 = [M_j, C_j] = [M_j, A_j] = 0. \tag{10} \]

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For \( w_1 = w_2 \) and \( f_1 = f_2 \), the model has diagonal symmetries \( M_1' \) as well as a four-fold rotational symmetry generated by \( R_4 = M_2 M_1' \).

This model also has antiunitary particle-hole symmetry, \( P H(k) P^{-1} = -H^*(-k) \) with \( P = P^{-1} = \gamma_1 \gamma_2 K \), and time-reversal symmetry, \( T H(k) T^{-1} = H^*(-k) \) with \( T = T^{-1} = iCP = \gamma_2 K \), where \( K \) is the complex conjugation operator. Therefore, the model belongs to the BDI Altland-Zirnbauer class. These symmetries satisfy

\[ \{P, M_j\} = [T, M_j] = 0, \tag{15} \]

\[ \{P, T\} = [P, C] = [T, C] = 0. \tag{16} \]

C. Higher-Order Topological Phases

In equilibrium, when all parameters are time-independent, and for uniform modulation \( f = |f|e^{i\chi} \), the model exhibits four phases: a trivial insulating phase for \( 0 < \chi < \pi/2 \), a \( x-\) or \( y-\)edge-polarized insulating phase for \( 0 < \chi \neq \pi/2 < \pi/2 \), and a second-order topological insulating phase for \( -\pi < \chi < -\pi/2 \). One can tune between the trivial and second-order topological phases by closing the bulk gap at \( m = 0 \). However, one may also keep the bulk gap open, \( m \neq 0 \), and cross between the trivial and edge-polarized, or edge-polarized and topological quadrupole phases. With open boundary conditions, the gap closes in the edge spectrum, and the second-order topological phase show corner bound states.

With periodic boundary conditions, these phases have been characterized in terms of nested Wilson loops of Wannier bands. A simpler characterization of these phases becomes possible when the model admits diagonal mirror symmetries \( M_j' \), i.e., when \( f_1 = f_2 \) and \( w_1 = w_2 \). In this case, a bulk mirror-graded topological invariant can be defined as

\[ \nu_j = \frac{\nu_{j+} - \nu_{j-}}{2}. \tag{17} \]

where \( \nu_{j+} \) and \( \nu_{j-} \) are the winding numbers of \( H_{j\pm}(k_{mj}) \), the Hamiltonian projected on the mirror eigenspace of \( M_j' \) with eigenvalues \( \pm 1 \), along the symmetric lines of \( \theta_j \) in the Brillioun zone, \( k_{mj} = (k, -k) \) and \( k_{mj}' = (k, k) \). For an open system, the invariant line under diagonal mirror symmetry keeps two corners intact. Thus, the bulk-boundary correspondence relates the mirror-graded bulk invariant to topologically protected bound states at these corners.

Since the diagonal mirror symmetries commute with the chiral symmetry, \( H(k_{mj}) \), \( M_j' \) and \( C \) can all be block-diagonalized simultaneously. In each block \( C_{j+} \) are chiral symmetries of \( H_j(k_{mj}') \), for which one may define, in the usual way, the winding number \( W \) of a chirally symmetric Hamiltonian. In the chiral basis, such a Hamiltonian \( H(\zeta) \) parametrized by a compact variable \( \zeta \), is off-diagonal,

\[ H = \begin{pmatrix} 0 & h \hbar \hbar^0 \end{pmatrix} \Rightarrow W[H] = \frac{1}{2\pi i} \oint \frac{\partial\log\det h(\zeta)}{\partial \zeta} d\zeta. \tag{18} \]

Therefore,

\[ \nu_{j\pm} = W[H_j(\pm k_{mj})]. \tag{19} \]

We note that the presence of two anticommuting diagonal mirror symmetries dictates \( \nu_{j+} = -\nu_{j-} \), thus \( \nu_j = \nu_{j+}. \)

Indeed, in Ref. [38] it was shown that this mirror-graded invariant captures the higher-order topology of the topological quadrupole phase. In Section III, we shall use this characterization to demonstrate the higher-order Floquet topological phases.

D. Floquet Theory of Periodically Driven Model

Here, we review Floquet theory and fix our notation to describe the periodic dynamics of the system. The driven model has a periodic Hamiltonian \( H(t) \), with period \( \tau = 2\pi/\Omega \), via a time-periodic hopping modulation \( f(t) \). The dynamics is given by the time-ordered evolution operator

\[ U(t', t) = \exp \left[ -i \int_t^{t'} H(s) ds \right]. \tag{20} \]

We shall use Floquet theory to separate the motion within a drive cycle and the stroboscopic evolution of successive cycles.

According to Floquet theorem, the solutions of the Schrödinger equation take the form \( e^{-i\epsilon_\alpha t} |u_\alpha(t)\rangle \), where quasienergies \( \epsilon_\alpha \in [-\Omega/2, \Omega/2] \) are conserved and the periodic Floquet modes \( |u_\alpha(t)\rangle = |u_\alpha(t + \tau)\rangle \) are eigenstates of the Floquet evolution operator, \( U_F(t)|u_\alpha(t)\rangle = e^{-i\epsilon_\alpha t} |u_\alpha(t)\rangle \). Here,

\[ U_F(t) \equiv U(t + \tau, t) = e^{-iH_F(t)}, \tag{21} \]

defines the Floquet Hamiltonian \( H_F(t) = H_F(t + \tau). \)
E. Floquet Topological Invariants

Consider an instantaneous Hamiltonian, which has a unitary temporal mirror symmetry,

\[ M_t H(t_m + t) M_t^{-1} = H(t_m - t), \]

around reflection-symmetric times \( t_m = 0 \) and \( \tau/2 \). Under this symmetry, the Floquet operator is mapped to \( \tilde{U}_F(t_m) = M_t U_F(t_m) M_t^{-1} \), where \( \tilde{U}_F \) is obtained from \( H = -H \). In the presence of a chiral symmetry, \( \tilde{H} = C^{-1} H C \), and \( \tilde{U}_F = C U_F^\dagger C \). Thus, the Floquet Hamiltonian at reflection-symmetric times is chirally symmetric,

\[ C_i H_F(t_m) C_i^{-1} = -H_F(t_m) \]

with \( C_i = M_i C \), if \( [C, M_i] = 0 \), and \( C_i = i M_i C \) if \( \{C, M_i\} = 0 \). Moreover, any symmetry \( S \) of the instantaneous Hamiltonian, \( [S, H(t)] = 0 \), is also a symmetry of the Floquet Hamiltonian, \( [S, H_F(t)] = 0 \), at any initial time. In the following, we will assume that the temporal and spatial mirror symmetries commute, \( [M_i, M_j] = 0 \). Then, \( C_i \) will have the same commutation algebra as \( C \) with the spatial mirror symmetries.

Consistent with these symmetries, the Floquet Hamiltonians \( H(t_m) \) have their own stable topological indices, \( \nu_F(t_m) \). Accordingly, two topological invariants are defined,\(^{20}\)

\[ \nu_F^\pm = \frac{\nu_F(0) + \eta \nu_F(\tau/2)}{2}, \]

associated with quasienergies \( \epsilon^\eta = 0, \Omega/2 \), where the sign \( \eta = e^{-i\pi \epsilon^\eta} = \pm 1 \). For example, a chirally symmetric Floquet Hamiltonian has a topological invariant defined as its winding number \( \nu_F(t_m) = W[H_F(t_m)] \). Indeed, any other topological invariant of a system with chiral and time-mirror reflection symmetry, such as the invariants defined through nested Wilson loops or mirror-graded eigenspaces, can be converted in this fashion to Floquet topological invariants characterizing the topology of the periodic dynamics.

We will now use this method to study the higher-order Floquet topologies induced in our driven model.

III. HIGHER-ORDER FLOQUET TOPOLOGICAL INSULATOR

For simplicity of our presentation, we will focus on two-step drive protocols, in which the hopping modulation vector \( \mathbf{f} \) periodically switches between two values \( \mathbf{f}_1 \) with a duration \( \tau_1 \), and \( \mathbf{f}_2 \) with a duration \( \tau_2 = \tau - \tau_1 \). This is simple enough to allow analytical and exact numerical calculations, yet rich enough to demonstrate the physics of interest. This two-step protocol is time-mirror symmetric with \( M_t = 1 \) and reflection-symmetric times in the middle of each step, which we set at 0 and \( \tau/2 \), respectively. The Floquet evolution operator at these times

\[ U_F(0) = U_{f_1}^0 U_{t_2}^2 U_{t_1}, \quad U_F(\tau/2) = U_{t_2} U_{t_1}^2 U_{t_2}, \]

where \( U_{f_j} = e^{-i(\pi/2) H_{f_j}} \), with \( H_{f_j} = H(\mathbf{w}_{f_j}, \mathbf{f}_{j\pi}) \), for drive step \( j = 1, 2 \). The two Floquet evolution operators are related by the half-cycle micromotion operator \( \Phi = U_{t_2} U_{t_1} \), as \( U_F(0) = \Phi U_F(\tau/2) \Phi \).

A. Floquet corner states

We first present the numerical evidence for higher-order topological phases in open boundary conditions. In Fig. 1 we show the Floquet spectrum of the driven model as the frequency of the drive is lowered. The hopping modulations \( \mathbf{f}_{1\pi} \) and \( \mathbf{f}_{2\pi} \) are both chosen to be in the trivial phase of the instantaneous Hamiltonian and respect diagonal mirror symmetries. The frequency is shown in units of inter-unit-cell hopping \( \lambda = w_1/(1 - m) \), taken to be the same for both drive steps. Floquet bound states are seen in different ranges of frequency at \( \epsilon^+ = 0 \) and \( \epsilon^- = \Omega/2 \). The probability density of the four degenerate Floquet bound states, at \( \epsilon^+ \) (b) and \( \epsilon^- \) (c) are shown. The drive frequency here is \( \Omega/\lambda = 4.1 \), marked with a dashed line in (a).
As the frequency is lowered, the coupling between Floquet modes in different Floquet zones increases and the Floquet topology can change when the quasienergy gap at the Floquet zone edge ($\epsilon^- = \Omega/2$) and/or center ($\epsilon^+ = 0$) close, either in the bulk or at the edges. With our choice of drive parameters, the changes of topology are accompanied by quasienergy gap closings in the bulk as seen in Fig. 1(a). These nontrivial topologies host Floquet bound states at the corners of the system, see Fig. 1(b), which signal the higher-order nature of the Floquet topological phase.

At reflection-symmetric times, the Floquet Hamiltonian has all the same symmetries as the instantaneous Hamiltonian. Thus, when $H_t$ have diagonal mirror symmetries, so do the Floquet Hamiltonians $H_F(t_m)$. Thus, the topological phase transitions can only happen when the bulk gap closes.

B. High-Frequency Approximation

The algebraic form of the Floquet Hamiltonians is dictated by the symmetries to be,

$$H_F(t_m) = A_1 D F_{1,1}(k) + A_2 D F_{2,2}(k).$$

The main difference with $H(t)$ here is that the operators $D F_{j} = \{d F_{j} e^{i\varphi F_{j}} C_{j}\}$ now depend on both components of the lattice momentum $k$.

In the high-frequency limit, we can use the Baker-Campbell-Hausdorff formula,

$$e^{X} e^{Y} = e^{X+Y + \frac{1}{2} [X,Y] + \frac{1}{12} ([X,[X,Y]] + [Y,[Y,X]] + \cdots)},$$

to find

$$H_F(t_m) = \overline{H} + e^{i\Omega t_m} \frac{\tau_1 \tau_2}{24} \overline{H}, [H_{t_1}, H_{t_2}]) + O(\tau^3),$$

where $\overline{H} = (\tau_1 H_{t_1} + \tau_2 H_{t_2})/\tau$ is the average Hamiltonian, and $e^{i\Omega t_m} = \pm 1$ for the two reflection-symmetric times. For example, taking $w_x = w_y = w$ to be time-independent, we find $\{d F_{j} e^{i\varphi F_{j}}\}$ given by Eq. (6) with $w$ and $f_j$ replaced with $w_F$ and $f_{F_j}$, where

$$\frac{w_F}{w} = 1 - e^{i\Omega t_m} \frac{w^2 \tau_1 \tau_2}{12} \sum_{j=1,2} f_j \Delta f_j (1 - \cos k_j),$$

and

$$\frac{w_F}{w} f_{F_j} = \overline{f}_j - e^{i\Omega t_m} \frac{w^2 \tau_1 \tau_2}{12} \left[ \Delta f_j \sum_{j=1,2} (1 + \cos k_j) \right. + \left. (-1)^{j} f_{21} (1 - \cos k_j) \right].$$

Here, the average $\overline{f}_j = (\tau_1 f_{11} + \tau_2 f_{21})/\tau$, the difference $\Delta f = f_{21} - f_{11}$, and $f_{21} = \text{Im}(f_{12} f_{11})$. Details of this calculation are presented in the Appendix.

![FIG. 2. Floquet topological invariants for the driven model with diagonal mirror symmetries. The drive parameters are the same as in Fig. 1(a) The mirror-graded Floquet topological invariants vs. drive frequency change at topological phase transitions and identify the presence of Floquet corner states. (b,c,d) The variation of mirror-graded topological invariant of the Floquet Hamiltonian, $\nu_F(t)$, as a function of the initial time in the cycle, $t$ (see text for definitions). The stable mirror-graded invariants, $\nu_F(t_m)$, at reflection-symmetric times $t_m = 0$ and $\tau/2$ are marked with triangles. The mirror-graded Floquet topological invariant in panel (a) is obtained as $\nu_F^\pm = \frac{1}{2} [\nu F(0) + \eta \nu F(\tau/2)]$ for $\eta = \pm 1$.]

While the algebraic structure of the Floquet Hamiltonian $H_F(t_m)$ is the same as the instantaneous Hamiltonian, its dependence on lattice momentum $k$ and parameters $w$ and $f$ can be quite different and complicated. This can be seen explicitly in our high-frequency expansion above. In this way, the periodic drive generates a whole family of different Hamiltonians consistent with the algebra of symmetries.

C. Mirror-Graded Floquet Topological Invariants

Choosing the instantaneous Hamiltonian to have diagonal mirror symmetries, i.e. $w_1 = w_2$ and $f_1 = f_2$ for both steps of the drive, we obtain Floquet Hamiltonians at reflection symmetric times, which also have the diagonal mirror symmetries. Thus, we can define stable mirror-graded topological invariants for these Floquet Hamiltonians. Following the definition of the Floquet topological invariant, Eq. (24), we can thus compute the mirror-graded Floquet topological invariants of the periodic drive.

In Fig. 2 we plot the mirror-graded Floquet topological invariants of a driven model with periodic boundary conditions and the same parameters as in Fig. 1. In Fig. 2(a), the Floquet invariants are shown as a function of frequency in the same range as in Fig. 1(a). They correctly show topological phase transitions when bulk gap closes. They also correctly predict the presence of
Floquet corner states in both quasienergy gaps around $\epsilon^+ = 0$ and $\epsilon^- = \Omega/2$ for the system with open boundary conditions.

In order to visualize how mirror-graded Floquet topological invariants arise in the periodic dynamics, we show in Fig. 3(b,c,d) for three representative cases, the evolution of the winding number $\nu(t)$ of $H_F(t)$ as the initial time is varied through a cycle. This winding number is calculated as $\nu(t) = \frac{1}{2}[\nu^+(t) - \nu^-(t)]$, with

$$\nu_{\pm}(t) = \frac{1}{2\pi i} \oint \frac{\partial}{\partial k} \log \det[h_{\pm}(k,t)]dk,$$

where $h_{\pm}(t)$ is the off-diagonal element of $H_F(t)$ along $k_{2m} = (k,k)$ projected on the eigenspaces of $M_2^\ast$ with eigenvalues $\pm 1$. Note that these projections are chirally symmetric at reflection symmetric times $t_m$ only. Thus, $\nu(t)$ is a stable topological invariant only at $t = t_m$. This is why it changes as $t$ is varied through the cycle. However, by plotting $\nu(t)$ for all times in the cycle, we can track its changes more easily and obtain the mirror-graded Floquet topological invariants with confidence.

In particular, in certain ranges of frequency, we find $\nu_F^+ = \pm 1$. This corresponds to two Floquet corner states at each corner for the open system as shown in Fig. 1(b). We call this, in accord with previous literature, a higher-order anomalous Floquet topological phase.

### IV. FLOQUET TOPOLOGICAL BULK DEFECTS

#### A. Vortices in the Static Model

In previous studies, the equilibrium $\pi$-flux dimerized square lattice model was shown to host topologically protected bulk bound states with support at the core of vortex defects in the hopping modulation $f_r$. Given a pattern of hopping modulations, we may define a defect order parameter

$$m(r) = -\sum_{j} \frac{1}{w_{\mu}} \eta_{r\mu} w_{r\mu+e_j},$$

where $w_{r\mu+e_j}$ and $\eta_{r\mu}$ are the same as in Eq. (2). The low-energy theory of excitations at half-filling is given by Dirac Hamiltonian on the background of this defect order parameter

$$\mathcal{H} = \sum_{j=1,2} (p_j A_j + m_j B_j),$$

where $B_j = iA_j C_j$, we have identified $m = m_1 + im_2$ again, and $p_j = -i\partial_j$ is the momentum operator of the excitations.

Due to this Dirac form, vortex configurations of $m(r)$ bind localized excitations. A vortex defect is realized for $f_r = |f(r)|e^{imq_\ast} \arg r$, where $n \in \mathbb{Z}$ is the quantized vorticity. It supports $n$ mid-gap bound states at zero energy, which are protected by the chiral symmetry $C$ and whose number is a topological invariant related to the index of the Dirac Hamiltonian. The presence of bound states endows a vortex with fractional quantum numbers.

There is indeed a close relationship between these bulk vortices and corner states of the static model. In the bulk, $m(r) = f_r$. However, for an open system, hopping amplitudes in the outward directions are set to zero. So, even when $f_r$ is uniform in the bulk, the defect order parameter may be nontrivial at the edges and corners. Fig. 3 shows plots of $m(r)$ for the case of uniform $f_r = f$ with open boundary conditions. Indeed, all phases of the system and, in particular the corner states of the higher-order topological phase, correspond directly to the domain-wall and vortex defects of $m(r)$.

A vortex defect in $m(r)$ need not have full rotational symmetry to support bound states at its core. For example, a $\mathbb{Z}_4$ vortex defect formed at the intersection of four domains with $f = |f|e^{i\chi}$, $\chi = q\pi/2 + \chi_0$, where $0 < \chi_0 < \pi/2$ and the domain index $q \in \mathbb{Z}$, say, in the clockwise direction, also hosts a bound states at the core.

We now show that a whole family of bulk vortex defects supporting Floquet topological bound states can be realized dynamically in the driven model.
FIG. 4. Stirring protocols and Floquet topological defects. A two-step drive (a) switches between two domain wall configurations with the defect order parameter as shown. The quasienergy spectrum vs. drive frequency for an open (b) shows a bound state at \( \epsilon^+ = 0 \) at arbitrarily high frequencies and, following a bulk gap closing at a lower frequency, a set of 6 degenerate bound states at \( \epsilon^- = \Omega/2 \). The system size here is 63 \times 63 and the hopping parameters are \( w_1 = w_2 = w \) and \( |f|/w = 0.8 \). The probability density of the bound states are shown in (c,d). The high-frequency bound state at \( \epsilon^+ = 0 \) (c) and two of the lower-frequency ones at \( \epsilon^- = \Omega/2 \) (d, first two panels) are localized at the intersection of the two domain wall. The other 4 lower-frequency bound states at \( \epsilon^- = \Omega/2 \) (d, last panel, shown together with different colors) are localized at the intersection of domain walls and the edges of the system.

B. Stirring Drive Protocols and Bulk Floquet Topological Defects

The \( Z_4 \) vortex can be generated dynamically by a two-step “stirring” drive protocol that switches between two domain-wall configurations, one with a vertical domain wall with \( f = \pm |f| \) on each side, and the other with a horizontal domain wall with \( f = \pm |f|e^{2i\chi_0} \) on each side. This is shown in Fig. 4(a) for \( \chi_0 = \pi/4 \). At high frequency, the Floquet Hamiltonian \( H_F = \bar{H} \) to the lowest order, where \( \bar{H} \) is the average Hamiltonian. Keeping \( w_1 = w_2 = w \) fixed in time, we can see easily that the average Hamiltonian will have four domains intersecting at a vortex defect. Thus, at sufficiently high frequency, we expect to see a Floquet topological bound states localized at the intersection of the two domain walls.

As frequency is lowered, this picture is modified as higher-order corrections to Floquet Hamiltonian grow and the topology of quasienergy bands is modified by gap closings. In Fig. 4(b), we plot the quasienergy of the driven model with the two-step stirring protocol as a function of drive frequency. As expected, at high frequency, there is a Floquet bound state at \( \epsilon^+ = 0 \), whose wavefunction is localized at the intersection of the two domain walls, see Fig. 4(c). At lower frequency, a quasi energy gap closing is observed at Floquet zone edge, below which a set of six degenerate Floquet bound states appear at \( \epsilon^- = \Omega/2 \). In Fig. 4(d) we plot the wavefunctions of these Floquet bound states. Two of them are localized at the intersection of the domain walls. The other four are localized at the intersection of domain walls with the edges.

The above structure can be understood as the dynamical generation of two higher-order anomalous Floquet domains and two trivial ones joined along the oscillating domain walls. Indeed, this configuration is similar to that of a static model with a defect order parameter obtained by the superposition of two \( Z_4 \) vortices. This would make a vortex with double vorticity that binds two states at its core. However, in the driven case, the low-frequency driving evidently produces a single vortex structure for \( \epsilon^+ = 0 \) with a single Floquet bound states, and a double-vortex structure for \( \epsilon^- = \Omega/2 \) with two Floquet bound states. These bulk Floquet bound states coexist as steady states of the same model at different quasienergies. This is a novel feature of the stirred Floquet bulk vortices that has no counterpart in the equilibrium model.

V. DISCUSSION AND OUTLOOK

We note that the scheme of dynamically generating bulk vortex defects is not limited to \( Z_4 \) vortices. For example, a rotationally symmetric vortex defect can be realized in the high-frequency regime by a continuously stirred domain wall at an angle \( \theta = \Omega t \) with the hopping modulation \( f(t) = \pm |f|e^{i\theta} \) on each side. Similarly, a multi-step drive stirring a domain wall through \( N \) steps, each rotating the domain wall by an angle \( \pi/N \), would create a vortex structure with \( 2N \) domain walls. As the frequency is lowered in these protocols, we would expect a series of transitions with multiple bulk Floquet bound states appearing at the Floquet zone edge and center.

The method of creating vortex defects can be easily
generalized to higher dimensions. In a three-dimensional model with higher-order topological phases, it would offer a practical way of creating monopoles with fractional quantum numbers using a series of pulses at high frequency. Again, at lower frequencies we expect to find an interesting set of intrinsically non-equilibrium bulk Floquet bound states. By combining these pulses, we envision designing additional dynamics for bulk defects in general and, in particular, adiabatic manipulations that can be useful for quantum information processing.

A problem that is opened by our work for future study is to find a unified classification higher-order topological phases in and out of equilibrium. The original study is to find a unified classification higher-order topological phases with second-order spatial symmetries. However, since the interpretation of Wilson loops and their Berry phases in terms of bulk and edge polarization leaves us with only a $\mathbb{Z}_2$ invariant, this approach may not yield the general classification. A more recent study found parallels between the topological structure of Wilson loops and those of Floquet operators, including the presence of anomalous bound states in the Wannier bands. However, this relationship is not in general well understood. A significant step towards the general classification scheme was taken in Ref. 35 by utilizing the classification of topological crystalline phases with second-order spatial symmetries. Nevertheless, this classification has been achieved for models with only a single spatial symmetry. Thus, the model we study in this paper, with two anticommuting mirror symmetries, is not covered by this classification.

In this work, we have presented a periodically driven $\pi$-flux dimerized square-lattice model that realizes the dynamical generation of robust corner states as Floquet bound states. In the presence of four-fold symmetry, these corner states can be classified with two bulk Floquet topological invariants obtained from the winding number of Floquet Hamiltonians graded with a mirror symmetry, along the diagonal lines of the Brillouin zone. Since winding numbers are integers, we conjecture this classification to be given by $\mathbb{Z} \times \mathbb{Z}$, even when the four-fold symmetry is broken, as long as the original mirror symmetries about the unit-cell axes are preserved. We find evidence for this integer classification when we consider simple stirring protocols that dynamically generate vortex defects in the hopping modulations with multiple Floquet bound states in the bulk. Thus, our work paves the way to using simple drive protocols that leverage spatiotemporal inhomogeneities to generate higher-order Floquet topologies.

Note added. In the final stage of preparing this paper, we became aware of two concurrent papers dis-cussing higher-order Floquet topological phases. We note that our stirring drive protocols to generate higher-order bulk Floquet topological bound states is not discussed in these papers. The stacking construction is different from our model. In particular, it does not admit a low-energy Dirac Hamiltonian at high drive frequency, and only supports weak higher-order topological phases.

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Appendix: High-frequency Floquet Hamiltonian

In this appendix, we provide some of the details for deriving the Floquet Hamiltonian at reflection symmetric times and Eqs. (28) and (29). First, we define a four-dimensional vector, $\vec{d} \equiv (|d_1| \cos \phi_1, |d_1| \sin \phi_1, |d_2| \cos \phi_2, |d_2| \sin \phi_2)$, for any Hamiltonian of the form \((1)\). Using the commutation relations of matrices $A_1$, and $C_1$ and their products to calculate the high-frequency Floquet Hamiltonian in Eq. (27), we find

$$d_{F,\alpha} = \frac{e^{i\alpha\phi}}{6} \left[ (\vec{d}, \vec{d}^\alpha) d_{t,\alpha} - \{ t_1 \leftrightarrow t_2 \} \right],$$

\(\text{(A.1)}\)

where $\alpha \in \{0, 1, 2, 3\}$ index the components, $\vec{d}$ denotes the cycle average, and $\vec{d}^\alpha$ indicates a projection on the three-dimensional space normal to the direction $\alpha$.

Now, we observe that for a given $k$ the Hamiltonian can be cast in a new basis by the gauge transformation $H(k) \rightarrow G^\dagger(k) H(k) G(k)$ with

$$G(k) = e^{-i \sum_{j=1,2} (k_j/4) C_j},$$

\(\text{(A.2)}\)

so that in the new basis $|d_j| e^{i\phi} = 2w_j [\cos(k_j/2) + i f_j \sin(k_j/2)]$. Then, for $w_1 = w_2 = w$, we have in the new basis,

$$\vec{d} = w \left( \cos \frac{k_1}{2}, f_1 \sin \frac{k_1}{2}, \cos \frac{k_2}{2}, f_2 \sin \frac{k_2}{2} \right).$$

(A.3)

Replacing $d_{t_1}$, $d_{t_2}$, and $\vec{d}$ in Eq. \((\text{A.1})\) in this form, we obtain Eqs. (28) and (29) in the main text.

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