Spin degrees and Polarization Observables in Electromagnetic Reactions

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After a brief introduction into the general importance of polarization observables for the analysis of a reaction, the basic density matrix formalism for the description of polarization phenomena is outlined and illustrated by explicit examples for spin one-half and one. Then various electromagnetic processes like photoreactions and electron scattering are discussed in some detail. Finally, the question of how to determine a complete set of polarization observables providing the maximum information on a reaction is considered.

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I. INTRODUCTION AND MOTIVATION

Historically the notion of “polarization” has had various although related meanings [1]:

(i) Quite early the term “polarity” for the description of the two sides of a magnet was used in order to characterize the separation of magnetic poles. It thus denotes a directedness associated with respect to the orientation of a magnetic dipole (compass).

(ii) Probably this term led E.L. MALUS (1811) to introduce the term “polarization” in order to describe the property of light waves of exhibiting a type of vibrations transversal to the direction of propagation like the vibrations of a string. It was first discovered already by Ch. HUYGENS in 1690, although in those times the nature of the vibrations was not clear. Since then this property is called “transverse polarization”, expressing a directedness.

(iii) Also the separation of charges in a dielectric medium or the orientation of randomly oriented permanent electric dipoles by an external electric field is called polarization, which also means the creation of a preferred direction in the medium.

(iv) In more recent times with the advent of quantum mechanics the term “polarization” is used in a much more general sense in order to describe the orientation of particles, including massless ones, possessing a nonvanishing spin. In fact, all the above mentioned phenomena are related to such spin degrees of freedom.

Therefore, in the present lectures I will use “polarization” in this general sense with respect to spin degrees of freedom and associated phenomena.

It is well known that the spin of a particle is an important dynamical property which often plays a significant role in the dynamics of a reaction. Moreover, whether a particle possesses integer or half-integer spin determines its statistical behavior in an ensemble of identical particles, obeying either BOSE-EINSTEIN or FERMI-DIRAC statistics. The central question which I would like to discuss is “How can one exploit such spin degrees of freedom?”.

Let us consider, for example, a reaction induced by two particles of the type

\[ a + b \rightarrow c + d + \ldots \]  

Let us further assume that the initial system is prepared in a pure state \( |\psi_{i,M_a,M_b}\rangle \) with definite spin projections \( M_a \) and \( M_b \) of the initial particles and, furthermore, that the detector selects ideally another pure final state \( |\psi_{f,M_c,M_d,...}\rangle \). Then the cross section is determined by the absolut square of the corresponding reaction matrix \( \hat{T} \), leaving aside kinematical factors,

\[ \sigma(a + b \rightarrow c + d + \ldots) \propto |\langle \psi_{f,M_c,M_d,...} | \hat{T} | \psi_{i,M_a,M_b}\rangle |^2. \]  

Spin degrees could then be investigated by varying the projections \( M_a \) and \( M_b \) of the initial and \( M_c, M_d, \ldots \) of the final states, and thus study how the cross section changes. This ideal situation, however, is almost never fulfilled,

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because (i) the initial preparation results in general in a statistical ensemble of a subset of different states, and (ii) similarly the detector is sensitive to another statistical ensemble of final states. In other words, in general one only has incomplete information about the initial and final states of the system.

In the simplest case, both are an incoherent mixture of possible initial or final states, respectively, with equal probability, and then the cross section is given by averaging over the initial and summing over the final ensemble, e.g. states with different spin projections, yielding for the cross section

$$\sigma(a + b \rightarrow c + d + \ldots) \propto \sum_{M_a,M_b} \sum_{M_c,M_d} |\langle \psi_f,M_c,M_d,\ldots | \hat{T} | \psi_i,M_a,M_b \rangle|^2.$$  \hspace{1cm} (3)

This is, for example, the case in photoreactions with unpolarized photons, which is another way of saying, that the photon source produces photons which do not have a specific direction of polarization. In this case, any specific dependence of the reaction on the photon polarization is lost by the averaging procedure.

In the present lectures I will illustrate how one can employ spin degrees of freedom for studying details of a reaction. A more detailed account can be found in a monograph by B.A. Robson [1]. A particularly interesting feature of exploiting spin d.o.f. is the possibility to obtain access to small but interesting amplitudes which usually are buried under dominant contributions in an unpolarized experiment. In order to illustrate this feature, I would like to mention two examples:

(i) The first measurement of parity violation in $\beta$-decay of oriented $^{60}$Co [2], where one measures the correlation between the orientation axis of $^{60}$Co and the direction of the emitted electrons, which is proportional to the scalar product between the spin of $^{60}$Co and the electron momentum, a pseudoscalar quantity generated by parity violation. It turned out that the electrons are predominantly emitted opposite to the orientation axis (see Fig. 1).

(ii) The determination of the electric form factor of the neutron in quasielastic electron scattering with longitudinally polarized electrons and measuring the polarization of the outgoing neutron $d(\vec{e}, e' \vec{n})p$ or using a vector polarized deuteron target $d(\vec{e}, e' \vec{n})p$. I will discuss this case in some detail later.

![FIG. 1: Preferred electron emission opposite to the orientation axis in the $\beta$-decay of $^{60}$Co.](image)

II. THE DENSITY MATRIX

I will now briefly introduce the concept of the density matrix as a convenient tool for the description of statistical ensembles (for reviews see [3, 4]). Let us consider a quantum mechanical system which is described by a complete set of states $\{|\alpha\rangle; \alpha \in \{\text{index set}\}\}$. Furthermore, an observable is represented by an hermitian operator $\hat{\Omega}$. Ideally, a measurement of an observable determines matrix elements between pure states

$$\Omega_{fi} = \langle \psi_f | \hat{\Omega} | \psi_i \rangle.$$  \hspace{1cm} (4)

However, as already mentioned, this ideal situation is almost never realized because one deals in general with a large number of identical systems with incomplete information on which states they occupy, so-called mixed states.
Therefore, statistical methods are needed. This leads to the introduction of the density matrix as the appropriate tool for representing pure and mixed states on an equal footing.

For illustration I will consider the expectation value of an observable $\hat{\Omega}$ for both cases, pure and mixed states. For a pure state, represented by $|\psi\rangle$ the expectation value simply is given by

$$\langle \Omega | \psi \rangle = \langle \psi | \hat{\Omega} | \psi \rangle ,$$  

and expanding each member of the set according to

$$| \psi_i \rangle = \sum_{\alpha} | \alpha \rangle \langle \alpha | \psi_i \rangle ,$$  

one finds now for the expectation value

$$\langle \Omega | \{ | \psi_1 \rangle , ..., | \psi_n \rangle \} = \sum_{i} p_i \langle \psi_i | \hat{\Omega} | \psi_i \rangle = \sum_{i} p_i \sum_{\alpha \alpha'} \langle \psi_i | \alpha \rangle \langle \alpha | \hat{\Omega} | \alpha' \rangle \langle \alpha' | \psi_i \rangle = \sum_{\alpha \alpha'} \langle \alpha | \hat{\Omega} | \alpha' \rangle \sum_{i} | \psi_i \rangle p_i \langle \psi_i | \alpha \rangle = \text{Tr}(\hat{\Omega} \hat{\rho}) .$$  

In the last step one has introduced as density matrix operator

$$\hat{\rho} = \sum_{i} | \psi_i \rangle p_i \langle \psi_i | .$$  

Obviously, the case of a pure state is contained in this definition for $p_i = \delta_{i n_0}$ with $n_0 \in \{1, ..., n\}$. 

To give an example, let me consider the case of unpolarized electrons represented by the two possible spin projections on a given quantization axis $\{ | \psi_{1/2} \rangle = | \pm \frac{1}{2} \rangle \}$ with equal probabilities $p_1 = p_2 = \frac{1}{2}$. Then the expectation value is given by

$$\langle \Omega | \{| \psi_{1/2} \rangle , | -1/2 \rangle \} = \sum_{m = -\frac{1}{2} , \frac{1}{2}} \frac{1}{2} \langle m | \hat{\Omega} | m \rangle .$$  

Instead of always distinguishing between these two types of states, the density matrix formalism offers a unified description for both cases. In order to introduce the density matrix, I will again consider the two cases of pure and mixed states. For a pure state $| \psi \rangle$, which can be expanded in terms of a complete set of basis states

$$| \psi \rangle = \sum_{\alpha} | \alpha \rangle \langle \alpha | \psi \rangle ,$$  

the expectation value of $\hat{\Omega}$ is given by

$$\langle \Omega | \psi \rangle = \sum_{\alpha \alpha'} \langle \psi | \alpha \rangle \langle \alpha | \hat{\Omega} | \alpha' \rangle \langle \alpha' | \psi \rangle = \sum_{\alpha \alpha'} \langle \alpha | \hat{\Omega} | \alpha' \rangle \langle \alpha' | \psi \rangle \langle \psi | \alpha \rangle .$$  

Introducing now the density matrix operator

$$\hat{\rho} = | \psi \rangle \langle \psi | ,$$  

one obtains

$$\langle \Omega | \psi \rangle = \sum_{\alpha} \langle \alpha | \hat{\Omega} \hat{\rho} | \alpha \rangle = \text{Tr}(\hat{\Omega} \hat{\rho}) .$$  

Turning to a mixed state described by the set $\{ | \psi_1 \rangle , ..., | \psi_n \rangle \}$ with probabilities $\{ p_1 , ..., p_n \}$ and expanding each member of the set according to

$$| \psi_i \rangle = \sum_{\alpha} | \alpha \rangle \langle \alpha | \psi_i \rangle ,$$  

one finds now for the expectation value

$$\langle \Omega | \{ | \psi_1 \rangle , ..., | \psi_n \rangle \} = \sum_{i} p_i \langle \psi_i | \hat{\Omega} | \psi_i \rangle = \sum_{i} p_i \sum_{\alpha \alpha'} \langle \psi_i | \alpha \rangle \langle \alpha | \hat{\Omega} | \alpha' \rangle \langle \alpha' | \psi_i \rangle = \sum_{\alpha \alpha'} \langle \alpha | \hat{\Omega} | \alpha' \rangle \sum_{i} | \psi_i \rangle p_i \langle \psi_i | \alpha \rangle = \text{Tr}(\hat{\Omega} \hat{\rho}) .$$  

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Finally, I will turn to a general mixed state.$$
A. Properties of the Density Matrix

Here I will list the most important properties of the density matrix operator.

(i) \( \hat{\rho}^\dagger = \hat{\rho} \) i.e. \( \hat{\rho} \) is hermitean, because \( |\psi\rangle^\dagger = \langle \psi | \) and thus \( (|\psi\rangle\langle |\psi|)^\dagger = \langle \psi | |\psi\rangle^\dagger = |\psi\rangle\langle |\psi| \).

(ii) \( \text{Tr} \hat{\rho} = 1 \) because in terms of a complete set one has

\[
\text{Tr} \hat{\rho} = \sum_{\alpha} \langle \alpha | \hat{\rho} | \alpha \rangle \\
= \sum_{\alpha} \sum_{i} \langle \alpha | \psi_i \rangle p_i \langle \psi_i | \alpha \rangle = \sum_{i} p_i \sum_{\alpha} \langle \psi_i | \alpha \rangle \langle \alpha | \psi_i \rangle \\
= \sum_{i} p_i \langle \psi_i | \psi_i \rangle = \sum_{i} p_i = 1. \tag{15}
\]

(iii) \( \rho_{\alpha \alpha} \geq 0 \) i.e. all diagonal elements are real and non-negative, because

\[
\rho_{\alpha \alpha} = \sum_{i} \langle \alpha | \psi_i \rangle p_i \langle \psi_i | \alpha \rangle = \sum_{i} p_i |\langle \alpha | \psi_i \rangle|^2 \geq 0. \tag{16}
\]

Obviously, \( \rho_{\alpha \alpha} \) is the probability for finding the state \( |\alpha \rangle \) in the ensemble. Furthermore, according to (ii) one has for the diagonal elements as upper and lower limits \( 1 \geq \rho_{\alpha \alpha} \geq 0 \).

(iv) \( \text{Tr} \hat{\rho}^2 \leq 1 \) because one finds

\[
\text{Tr} \hat{\rho}^2 = \sum_{\alpha} \langle \alpha | \left( \sum_{i} |\psi_i \rangle p_i \langle \psi_i | \right) \left( \sum_{j} |\psi_j \rangle p_j \langle \psi_j | \right) | \alpha \rangle \\
= \sum_{i,j} p_i p_j \langle \psi_i | \psi_j \rangle \sum_{\alpha} \langle \psi_j | \alpha \rangle \langle \alpha | \psi_i \rangle \\
= \sum_{i,j} p_i p_j \langle \psi_i | \psi_j \rangle^2. \tag{17}
\]

Using now Schwartz’ inequality

\[
|\langle \psi_i | \psi_j \rangle|^2 \leq \langle \psi_i | \psi_i \rangle \langle \psi_j | \psi_j \rangle = 1, \tag{18}
\]

one finds finally

\[
\text{Tr} \hat{\rho}^2 \leq \left( \sum_{i} p_i \right)^2 = (\text{Tr} \hat{\rho})^2 = 1. \tag{19}
\]

Equality is fulfilled if and only if the state is pure, because for a pure state one has \( \hat{\rho} = |\psi\rangle\langle \psi | \) where \( |\psi \rangle \) is normalized to one, and thus \( (\hat{\rho})^2 = \hat{\rho} \). The inverse follows from the next property (v).

(v) \( \hat{\rho}^2 = \hat{\rho} \) if and only if the state is pure. We only have to show that, if this relation holds, then the density matrix has the form \( \hat{\rho} = |\psi\rangle\langle \psi | \). To this end we note that since \( \hat{\rho} \) is hermitean one can diagonalize the density matrix. This means one can find a complete set of states \( \{ |\phi_1 \rangle, \ldots, |\phi_n \rangle, \ldots \} \) with \( \langle \phi_i | \hat{\rho} | \phi_j \rangle = \delta_{ij} \rho_{ii} \). From this feature follows, using the above relation,

\[
\langle \phi_i | \hat{\rho}^2 | \phi_j \rangle = \delta_{ij} \rho_{ii}^2 = \langle \phi_i | \hat{\rho} | \phi_j \rangle = \delta_{ij} \rho_{ii}. \tag{20}
\]

Thus for all \( i \) one has the equation \( \rho_{ii}^2 = \rho_{ii} \) which has the only solutions \( \rho_{ii} = 0 \) or \( \rho_{ii} = 1 \). In view of the properties (ii) and (iii) only for one index \( n_0 \) the solution can be non-vanishing, i.e. \( \rho_{ii} = \delta_{ii n_0} \). This means that \( \hat{\rho} = |\phi_{n_0}\rangle\langle \phi_{n_0} | \), q.e.d. In this case \( \hat{\rho} \) is idempotent or a projection operator onto the pure state \( |\phi_{n_0}\rangle \).
\( \langle \hat{\Omega} \rangle_\rho = \text{Tr}(\hat{\Omega} \rho) \); i.e. the expectation value of any operator \( \hat{\Omega} \) is given by the trace of \( \hat{\Omega} \rho \). This property may also serve as a definition for the density matrix in an operational approach [3].

(vii) For a \( N \)-dimensional density matrix (\( N \) basis states) the matrix is characterized in general by \( N^2 - 1 \) real parameters. Only for a pure state this number reduces to \( 2N - 2 \), because then the pure state is a linear combination of the \( N \) basis states with \( N \) complex coefficients for which an overall phase remains arbitrary.

(viii) The density matrix fulfills the equation of motion
\[
\frac{\partial \hat{\rho}}{\partial t} = i \frac{\hbar}{\hbar} [\hat{\rho}, \hat{H}],
\]
where \( \hat{H} \) denotes the hamiltonian of the system.

As next I will discuss briefly the density matrices for the description of spin degrees of freedom.

B. Density Matrix for Spin 1/2

For a spin-one-half particle, one has only two basis states, namely \( \{|\frac{1}{2} m_s\}; m_s = \pm \frac{1}{2} \} \). Therefore, the corresponding density matrix \( \rho^{(1/2)} \) is a \( 2 \times 2 \)-matrix which can be represented as a linear combination of the \( 2 \times 2 \) unit matrix \( 1_2 \) and the PAULI spin matrices \( \sigma_i \)
\[
\rho = \frac{1}{2} (1_2 + a_1 \sigma_1 + a_2 \sigma_2 + a_3 \sigma_3),
\]
where the coefficients \( a_i \) are real, because both \( \rho \) and \( \sigma_i \) are hermitean. For the interpretation of these coefficients one considers the probability for finding a spin component in the direction of the \( i \)-axis
\[
P_i = \text{Tr}(\rho^{(1/2)} \sigma_i) = \frac{1}{2} \sum_j a_j \text{Tr}(\sigma_i \sigma_j) = a_i,
\]
because \( \text{Tr}(\sigma_i \sigma_j) = 2 \delta_{ij} \). Therefore, the density matrix of a \( (s = 1/2) \)-particle has the form
\[
\rho^{(1/2)} = \frac{1}{2} (1_2 + P \cdot \sigma),
\]
where \( P_i \) is the probability for finding a particle with spin component along the \( i \)-axis in the state described by \( \rho^{(1/2)} \). For a pure state one has \(|P| = 1\), while for a partially polarized particle \(|P| < 1\).

C. Density Matrix for a Photon

Since a photon has only two independent transverse polarizations, one has again only two basis states for which two types of representations are used:

(i) States of linear polarization (cartesian basis) \( \{|e_x\}, |e_y\} \) with respect to a coordinate system with \( z \)-axis parallel to the photon momentum.

(ii) States of circular polarization (spherical basis) \( \{|e_{-1}\}, |e_1\} \), which are related to the cartesian basis by
\[
|e_{\pm 1}\rangle = \pm \frac{1}{\sqrt{2}} (|e_x\rangle \pm i |e_y\rangle).
\]

Since there are again only two basis states, formally the density matrix \( \rho^{(\gamma)} \) has the same structure as for a spin one-half particle
\[
\rho^{(\gamma)} = \frac{1}{2} (1_2 + P \cdot \sigma).
\]
However, the meaning of $P$ is different and depends on the chosen representation, either cartesian or spherical.

At this point I would like to make a digression to classical radiation physics, where most of the concepts for the description of photon polarization have been developed. To this end I will consider an electromagnetic plane wave travelling along the $z$-axis. Its electric field has the form

$$E = \Re\{A e^{i(kz-\omega t)}\},$$

with a complex transverse amplitude in the $x$-$y$-plane $A = a e_x + b e^i e_y$. For a convenient description of all possible polarizations G.G. Stokes has introduced a set of four parameters, called Stokes parameters, $\{I, P_1, P_2, P_3\}$ which are defined by

$$I = a^2 + b^2, \quad P_1 = a^2 - b^2, \quad P_2 = 2ab \cos \delta, \quad P_3 = 2ab \sin \delta,$$

of which only three are independent, because of the condition

$$I^2 = P_1^2 + P_2^2 + P_3^2.$$

Here the intensity is denoted by $I$ and the three parameters $P_i$ characterize the type of polarization, namely linear, circular or elliptic. In detail one has

(i) linear polarization for $\delta = 0$, i.e. $P_3 = 0$. If $P_2 = 0$ one has linear polarization along the $x$-axis if $P_1 > 0$ or along the $y$-axis if $P_1 < 0$. For all other cases, i.e. $P_2 \neq 0$, one has linear polarization at an angle $\tan \alpha = b/a$ in the $x$-$y$-plane (see left panel of Fig. 2).

(ii) For the case $\delta \neq 0$ one has in general right ($\delta > 0$) or left ($\delta < 0$) elliptic polarization with the large axis at an angle $\tan 2\alpha = 2ab \cos \delta/(a^2 - b^2)$ in the $x$-$y$-plane (see right panel of Fig. 2). Circular polarization is a special case if $\delta = \pm \pi/2$ and $a = b$. In this case the Stokes parameters have the values $P_1 = P_2 = 0$ and $P_3 = \pm I$.

\[ \text{FIG. 2: Left panel: linear polarization; Right panel: elliptic polarization of a classical electromagnetic plane wave.} \]

H. Poincaré has introduced a three-dimensional representation for all possible polarization states, the so-called Poincaré vector $P$, defined by

$$P_1 = I \cos 2\chi \cos 2\psi, \quad P_2 = I \cos 2\chi \sin 2\psi, \quad P_3 = I \sin 2\chi,$$

where

$$\tan 2\psi = \frac{2ab \cos \delta}{a^2 - b^2} \quad \text{and} \quad \tan 2\zeta = \tan \delta \sin 2\psi.$$

The Poincaré vector has the length $I$ and the spherical angles $(2\psi, \pi/2 - 2\chi)$ and thus covers a sphere with radius $I$ (see Fig. 3). In other words, each point on the Poincaré sphere characterizes a specific polarization of an electromagnetic plane wave.
Now I will return to the quantum description of photon polarization. Analogous to classical radiation, the most general photon polarization state is given by

$$|e\rangle = a|e_x\rangle + b e^{i\delta} |e_y\rangle,$$

with $a, b$ real, and $a^2 + b^2 = 1$, \hfill (32)

and $\delta$ denotes a phase. It describes the same types of polarization, linear, circular and elliptic, as in the classical case. The only difference is that now the Poincaré vector has length one. With respect to the cartesian basis, the corresponding density matrix has the form

$$\rho^{(\gamma)}_{\text{cart}} = \frac{1}{2} \begin{pmatrix} a^2 & ab e^{-i\delta} \\ ab e^{i\delta} & b^2 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 2ab(\cos \delta - i \sin \delta) & 2ab(\cos \delta + i \sin \delta) \\ 1 - a^2 - 2ab \cos \delta & 1 - a^2 + 2ab \cos \delta \end{pmatrix}.$$ \hfill (33)

Thus one finds with $(P_x, P_y, P_z) = (P_2, P_3, P_1)$ for the photon polarization density matrix in the cartesian basis

$$\rho^{(\gamma)}_{\text{cart}} = \frac{1}{2} (I + 2ab \sin \delta \sigma_x + 2ab \cos \delta \sigma_y + 2\sigma_z).$$ \hfill (34)

For a pure polarization state one has $|\mathbf{P}| = 1$, whereas for partially polarized photons $|\mathbf{P}| < 1$.

As next I will consider the transformation to the spherical basis. The transformation of the density matrix to this basis is given by

$$\rho^{(\gamma)}_{\text{sph}} = \mathcal{U}^* \rho^{(\gamma)}_{\text{cart}} \mathcal{U}^T$$ with $\mathcal{U} = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 & -i \\ 1 & -i \end{pmatrix}$, \hfill (35)

yielding

$$\rho^{(\gamma)}_{\text{sph}} = \frac{1}{2} \begin{pmatrix} 1 + 2ab \sin \delta & b^2 - a^2 + 2iab \cos \delta \\ b^2 - a^2 - 2iab \cos \delta & 1 - 2ab \sin \delta \end{pmatrix}.$$ \hfill (36)

Thus, it results in the same formal expression as for the cartesian basis but with a different polarization vector $(P_x, P_y, P_z) = (-P_1, -P_2, P_3)$ and a different interpretation. Again $P = |\mathbf{P}|$ denotes the fraction of total polarization, $P_\gamma = P_z$ describes right ($P_\gamma > 0$) or left ($P_\gamma < 0$) circular polarization. With respect to the other components $P_x$ and $P_y$ it is useful to consider a rotation in the $x$-$y$-plane by $\phi$ which results in a transformation of the density matrix $\rho^{(\gamma)}_{\text{sph}} \rightarrow \rho^{(\gamma)'}_{\text{sph}}$ with

$$\rho^{(\gamma)'}_{\text{sph}, \lambda \lambda'} = e^{i(\lambda' - \lambda)\phi} \rho^{(\gamma)}_{\text{sph}, \lambda \lambda'}.$$ \hfill (37)

Thus, by a proper rotation in the $x$-$y$-plane one can reach $P_y = 0$ and then $P_\gamma' = P_x$ denotes linear polarization along the $x$-axis ($P_\gamma' < 0$) or along the $y$-axis ($P_\gamma' > 0$).
D. Density Matrix for a Massive Spin-1 Particle

In contrast to the photon, one has three basis states, namely \{|1 m\rangle; m = 0, \pm 1\}. Thus the density matrix \(\rho^{(1)}_{m'm}\) is a 3 \times 3-matrix in spin-1 space which can be decomposed into 9 independent 3 \times 3-matrices, the unit matrix \(\mathbb{1}_3\) and 8 traceless hermitean matrices. In view of angular momentum algebra it is convenient to chosen them as irreducible tensors under rotations of rank one and two, i.e. three vector components \(\Omega^{[1]}_M\) (transforming like a spin 1) and five tensor components \(\Omega^{[2]}_M\) (transforming like a spin 2), defined by their reduced matrix elements

\[
\langle 1|\Omega^{[I]}_M|1\rangle = \sqrt{3} \mathbb{I},
\]

with the notation \(\mathbb{I} = \sqrt{2I+1}\). Accordingly, the density matrix \(\rho^{(1)}\) is governed by eight independent parameters: 3 for vector polarization \((P_{1M})\) and 5 for tensor polarization \((P_{2M})\)

\[
\hat{\rho}^{(1)} = \frac{1}{3} \left( \mathbb{1}_3 + \sum_{I=1,2} \sum_M (-)^M \Omega^{[I]}_M P_{IM} \right).
\]

Under spatial rotations \(P_{1M}\) transforms like a vector and \(P_{2M}\) like a tensor of rank two with respect to a spherical representation. The relation to the cartesian vector and tensor components are

\[
P_{1M} = \frac{\sqrt{3}}{2} (\delta_{M0} P_x - \frac{M}{2} (P_x + iM P_y))
\]

\[
P_{20} = \frac{1}{\sqrt{2}} P_{zz}, \quad P_{2\pm1} = \mp \frac{1}{\sqrt{3}} (P_{xz} \pm iP_{zy}) , \quad P_{2\pm2} = \frac{1}{2\sqrt{3}} (P_{xx} - P_{yy}) \mp \frac{2}{3\sqrt{3}} P_{xy}.
\]

For a diagonal density matrix \(\rho^{(1)}_{m'm} = \delta_{m'm} p_m\), with \(p_m\) denoting the probability for finding the particle in a state \(|1m\rangle\) with respect to an orientation axis parallel to the \(z\)-axis, the orientation parameters have a particularly simple form

\[
P_{1M} = \delta_{M0} P^0_1, \quad P_{2M} = \delta_{M0} P^0_2,
\]

where

\[
P^0_1 = \sqrt{\frac{3}{2}} (p_1 - p_-), \quad P^0_2 = \sqrt{\frac{1}{2}} (p_1 + p_- - 2p_0) = \sqrt{\frac{1}{2}} (1 - 3p_0).
\]

For an arbitrary direction of the orientation axis the density matrix can be obtained from the foregoing by a simple rotation. Then the spin-1 density matrix is characterized by the spherical angles \(\Omega = (\theta, \phi)\) of the orientation axis and the two parameters, \(P^0_1\) for the vector and \(P^0_2\) for the tensor polarization. Furthermore, the general orientation parameters are determined by these parameters \(\Omega = (\theta, \phi)\) and \(P^0_1\) and \(P^0_2\), i.e.

\[
P_{IM}(\Omega) = P^0_1 D^I_{1M}(0, -\theta, -\phi) = P^0_1 e^{iM\phi} d^I_{M0}(\theta) \quad (I = 1, 2),
\]

where \(D^I_{1M}\) denotes a rotation matrix \([3, 4]\) describing the transformation of irreducible tensors under a spatial rotation. Correspondingly, the density matrix with \(|1m\rangle\) as basis states becomes

\[
\rho^{(1)}_{m'm} = \frac{(-)^{1-m}}{\sqrt{3}} \sum_{l=0}^{2} P^0_l \sum_{M=-l}^{l} \mathbb{I} \left( \begin{array}{cc} 1 & I \\ m & -m' & -M \end{array} \right) e^{iM\phi} d^l_{M0}(\theta),
\]

where for convenience I have introduced \(P^0_0 = 1\). The symbol in round brackets denotes a 3\(j\)-symbol \([3, 4]\).

E. Generalization to Massive Particles with Arbitrary Spin

The foregoing density for a massive spin-1 particle can easily be generalized to massive particles with arbitrary spins

\[
\rho^{J} = \frac{1}{J^2} \left( \mathbb{1}_{2J+1} + \sum_{l=1}^{2J} \sum_{M=-l}^{l} (-)^M \Omega^{[l]}_{-M} P_{lM} \right),
\]

where
with the reduced matrix elements of the irreducible operators \( \Omega^{[l]} \)

\[
(J||\Omega^{[l]}||J) = \hat{J} \hat{T}.
\]

This then gives the density matrix for a massive spin-\( J \) particle in the form

\[
\rho^{(J)} = \frac{(-)^{J-m}}{J} \sum_{I=0}^{2J} \sum_{M=-I}^{I} \left( \begin{array}{ccc} J & J & I \\ m & -m' & -M \end{array} \right) P_{IM}.
\]

The associated orientation parameters \( P_{IM} \) are obtained by inversion

\[
P_{IM} = \hat{J} \hat{T} \sum_{m',m} (-)^{J-m} \left( \begin{array}{ccc} J & J & I \\ m & -m' & -M \end{array} \right) \rho^{(J)}.
\]

I will now assume again an orientation axis at \( \Omega = (\theta, \phi) \), for which \( \rho^{(J)} \) becomes diagonal \( \rho^{(J)}_{m'm} = \delta_{m'm} p_m \) with 2\( J \) orientation parameters \( P^0_I (I = 1, \ldots, 2J) \). These are related to the diagonal elements \( p_m \) by

\[
P^0_I = \hat{J} \hat{T} \sum_{m} (-)^{J-m} \left( \begin{array}{ccc} J & J & I \\ m & -m & 0 \end{array} \right) p_m.
\]

Then, using the relation

\[
P_{IM}(\Omega) = P^0_I D^I_{0M}(0, -\theta, -\phi) = \rho^0_I e^{iM\phi} d^I_{M0}(\theta),
\]

the density matrix becomes finally as a generalization of eq. (45)

\[
\rho^0_{m'm'} = \frac{(-)^{J-m}}{J} \sum_{I=0}^{2J} \sum_{M=-I}^{I} \hat{I} \left( \begin{array}{ccc} J & J & I \\ m & -m' & -M \end{array} \right) e^{iM\phi} d^I_{M0}(\theta).
\]

### III. ELECTROMAGNETIC REACTIONS

I will now turn to illustrate the use of polarization observables in photo reactions and electron scattering on nuclei. An extensive discussion of polarization effects in electron scattering can be found in Ref. [7].

#### A. Nuclear photo reactions

The cross section of photo absorption on a nucleus \( N \)

\[
\gamma(k) + N(p_i) \rightarrow N'(p_f),
\]

with an initial density matrix \( \rho^{(N)}_i \), characterizing the target preparation, and a final density matrix \( \rho^{(N)}_f \), characterizing the detector, is in first order perturbation given by

\[
d\sigma = \frac{(2\pi)^2 \alpha}{q} \text{Tr} \left( d\mathcal{W}(\rho^{(N)}_f, \rho^{(N)}_i, \rho^{(\gamma)}) \right),
\]

where \( \alpha \) denotes the fine structure constant, \( q \) the photon momentum and \( \rho^{(\gamma)} \) the photon density matrix [8]. The trace refers here to the photon polarization indices. Furthermore, \( d\mathcal{W} \) denotes the hadron electromagnetic current tensor which depends on the density matrices of the nuclear initial and final states, i.e.

\[
d\mathcal{W}(\rho^{(N)}_f, \rho^{(N)}_i) = \int d^3p_f \delta^{(4)}(p_f - p_i - q) \text{Tr}(\mathcal{J}^\dagger_{fi} \mathcal{J}^\dagger_{fi} \rho^{(N)}_f \rho^{(N)}_i),
\]

\[
\rho^{(N)}_f = \left( \begin{array}{cc} \rho^{(N)}_{f1} & 0 \\ 0 & \rho^{(N)}_{f2} \end{array} \right),
\]

\[
\rho^{(N)}_i = \left( \begin{array}{cc} \rho^{(N)}_{i1} & 0 \\ 0 & \rho^{(N)}_{i2} \end{array} \right),
\]

and

\[
\mathcal{J}_{fi} = \left( \begin{array}{cc} \mathcal{J}_{f1} & 0 \\ 0 & \mathcal{J}_{f2} \end{array} \right).
\]
with \( \mathcal{J}_{fi} = (j_{fi,1}, j_{fi,-1}) \) as the transverse nuclear current in a spherical basis with z-axis along \( q \) for \( \lambda = \pm 1 \)

\[
j_{fi,\lambda} = \langle \mathbf{p}_f | j_{\lambda}(0) | \mathbf{p}_i \rangle = \frac{1}{(2\pi)^3} \langle \mathcal{J}_{\lambda}(\mathbf{q}, \mathbf{P}) | i \rangle .
\] (56)

Here \( \mathcal{J}_{\lambda}(\mathbf{q}, \mathbf{P}) \) denotes the Fourier transform of the nuclear current operator with \( \mathbf{q} = \mathbf{p}_f - \mathbf{p}_i \) and \( \mathbf{P} = \mathbf{p}_f + \mathbf{p}_i \), and \( | i \rangle \) and \( | f \rangle \) the initial and final intrinsic nuclear states, respectively. Evaluation of the trace with respect to the photon indices leads to the well-known form

\[
d\sigma = \frac{4\pi^2\alpha}{q} (dF_T + P_i^\gamma dF_{TT} + P_i^\gamma dF_T^\prime),
\] (57)

where the nuclear current structure functions are defined in terms of the hadron current tensor by

\[
d\mathcal{W}_\lambda = \frac{1}{2} \sum_{\lambda = \pm 1} d\mathcal{W}_{\lambda\lambda}, \quad dF_{TT} = \Re(d\mathcal{W}_{-11}), \quad dF_T = \frac{1}{2} \sum_{\lambda = \pm 1} \lambda d\mathcal{W}_{\lambda\lambda} .
\] (58)

As noted above, the tensor \( d\mathcal{W} = d\mathcal{W}(\rho_f^{(N)}, \rho_i^{(N)}) \) still depends on the initial and final nuclear density matrices

\[
d\mathcal{W}_{\lambda\lambda} = \sum_{\lambda = \pm 1} f d^3p_f \delta(4)(p_f - p_i - q) \text{Tr}(\mathcal{J}_{fi,\lambda} \rho_f^{(N)} \mathcal{J}_{fi,\lambda} \rho_i^{(N)}) ,
\] (59)

where here the trace refers to the nuclear initial and final state spin degrees of freedom.

For unpolarized nuclei (\( \rho_i^{(N)} = \mathbb{1}_{2J_i+1} / J_i^2 \)) and no polarization analysis of the final states (\( \rho_f^{(N)} = \mathbb{1}_{2J_f+1} \)), there cannot be any dependence on circular photon polarization and thus the corresponding structure function vanishes in this case (\( dF_T^0 = 0 \)). The remaining structure functions \( dF_T^0 \) and \( dF_{TT}^0 \) determine the cross section

\[
d\sigma = d\sigma^0 (1 + P_i^\gamma \Sigma) ,
\] (60)

with the unpolarized cross section \( d\sigma^0 \) and the linear photon asymmetry \( \Sigma \)

\[
d\sigma^0 = \frac{4\pi^2\alpha}{q} dF_T^0, \quad \Sigma = \frac{dF_{TT}^0}{dF_T^0} .
\] (61)

For the further analysis it useful to apply a multipole decomposition of the current in order to exploit the conservation of angular momentum in conjunction with angular momentum algebra

\[
\mathcal{J}(\mathbf{q}) = -\sqrt{2\pi} \sum_{LM} i^L \hat{L} \left( T_{LM}^{(e)} + \lambda T_{LM}^{(m)} \right) D_{LM}^\lambda(-\phi_q, -\theta_q, \phi_q),
\] (62)

with electric \( T_{LM}^{(e)} \) and magnetic \( T_{LM}^{(m)} \) multipole operators, and \( D_{LM}^\lambda \) denotes a rotation matrix \([5]\).

### B. Total Photoabsorption by an oriented nucleus

As a specific application I will now consider the total photobserption cross section of an oriented nucleus \([10]\). In this case one sums over all possible final spin states (\( \rho_f^{(N)} = \mathbb{1}_{2J_f+1} \)) and integrates over all possible final continuum states, like e.g. emitted particles. Furthermore, according to the optical theorem the total cross section can be related to the imaginary part of the forward elastic photon scattering amplitude \( T^C(q) \). In particular, for an oriented nucleus with spin \( J \) and density matrix \( \rho_i^{(N)} = \rho^{(J)} \) one finds

\[
\sigma_{\text{tot}}(q, \rho^{(\gamma)}, \rho^{(J)}) = \frac{4\pi}{q} \text{Im} \left[ \text{Tr} \left( \rho^{(\gamma)} \rho^{(J)} T^C(q) \right) \right] .
\] (63)

The forward elastic scattering amplitude can be parametrized as \([11, 12]\)

\[
T_{N,M',\lambda M}(q) = (-)^{J-M} \sum_{i=0}^{2J} i \left( \begin{array}{ccc} J & I & J \\ -M' & \lambda & M \end{array} \right) T_{N,\lambda}(q) ,
\] (64)
where the amplitude $T_{L,\lambda}(q)$ describes the contribution of the total angular momentum transfer $I$ to the ground state in the scattering process. It is given in terms of generalized polarizabilities $P_{L'}^{L'\lambda \lambda'}(\gamma, \nu)$ [11, 12]

$$T_{L,\lambda}(q) = \frac{i}{J} \sum_{L', L} (-1)^{L'+L} \left( \begin{array}{ccc} L & L' & I \\ \lambda & -\lambda' & \lambda \end{array} \right) P_{L'}^{L'\lambda \lambda'}(q),$$

which contain the contributions of the multipoles $T_{L}^{\gamma}$ and $T_{L'}^{\nu}$ in the two-step scattering process, and are defined by

$$P_{L'}^{L'\lambda \lambda'}(q) = \sum_{\nu', \nu=0,1} \chi_{\nu'} \chi_\nu P_{I}(T_{L'}^{\nu}, T_{L'}^{\nu}; k),$$

with the notation: $T_{L}^{\gamma} = E_{L}$ for the electric and $T_{L}^{\nu} = M_{L}$ for the magnetic multipoles. The generalized polarizabilities describe the total angular momentum transfer $I$ in the two-step photon scattering process induced by the two multipoles $T_{L}^{\nu}$ and $T_{L'}^{\nu'}$, which in turn transfer angular momenta $L$ and $L'$, respectively (see Fig. 4). Using the explicit expressions for $\rho^{(\gamma)}$ and $\rho^{(J)}$ one obtains applying angular momentum algebra

$$\sigma_{tot}(q, \rho^{(\gamma)}, \rho^{(J)}) = \sigma_{0}^{11}(q) + \frac{1}{2} \sum_{J=1}^{2J} P_{J}^{0} \left[ (1 + (-1)^{J}) \sigma_{J}^{11}(q) + (1 - (-1)^{J}) P_{J}^{0} \sigma_{J}^{11}(q) P_{J}(\cos \theta) ight] + (1 + (-1)^{J}) P_{J}^{0} \sigma_{J}^{11}(q) d_{20}(\theta) \cos(2\phi),$$

where $|P_{J}^{0}|$ and $|P_{J}^{0}|$ denote the degree of linear and circular photon polarization, respectively, $P_{J}^{0}$ the nuclear orientation parameters, and $P_{J}(\cos \theta)$ a LEGENDRE polynomial. The direction of the orientation axis is determined by the angles $\theta$ and $\phi$. Furthermore, the various contributions to the total cross section are related to the $T_{L,\lambda}(q)$ (see eq. (65)) by

$$\sigma_{L,\lambda}^{11}(q) = \frac{4\pi}{q} \Im m \left( T_{L,\lambda}(q) \right).$$

Variation of the photon polarization parameters $P_{J}^{\gamma}$, $P_{J}^{\nu}$, $\theta$ and $\phi$ and the nuclear polarization parameters $P_{J}^{0}$ allows one to determine all contributions separately.

Of particular interest is the so-called spin asymmetry of the total cross section defined as the difference of cross sections for right and left circularly polarized photons on a target with maximal spin projection on the photon momentum, i.e. with orientation parameters (see eq. (51) with $p_{m} = \delta_{m,J}$)

$$P_{J}^{0} = \hat{J} \hat{T} \left( \begin{array}{ccc} J & J & I \\ J & -J & 0 \end{array} \right).$$

Then one finds for the spin asymmetry

$$\sigma_{tot}^{p}(q) - \sigma_{tot}^{A}(q) = \frac{2\sqrt{J}}{\sqrt{J+1}} \sigma_{J}^{11}(q) + \cdots$$

FIG. 4: Total angular momentum transfer $I$ in the two-step scattering process induced by the multipoles $T_{L}^{\nu}$ and $T_{L'}^{\nu'}$. 

\[
\begin{align*}
T_{L,\lambda}(q) &= \frac{i}{J} \sum_{L', L} (-1)^{L'+L} \left( \begin{array}{ccc} L & L' & I \\ \lambda & -\lambda' & \lambda \end{array} \right) P_{L'}^{L'\lambda \lambda'}(q), \\
T_{L}^{\gamma} &= E_{L}, \\
T_{L}^{\nu} &= M_{L}, \\
\sigma_{tot}(q, \rho^{(\gamma)}, \rho^{(J)}) &= \sigma_{0}^{11}(q) + \frac{1}{2} \sum_{J=1}^{2J} P_{J}^{0} \left[ (1 + (-1)^{J}) \sigma_{J}^{11}(q) + (1 - (-1)^{J}) P_{J}^{0} \sigma_{J}^{11}(q) P_{J}(\cos \theta) ight] + (1 + (-1)^{J}) P_{J}^{0} \sigma_{J}^{11}(q) d_{20}(\theta) \cos(2\phi), \\
\sigma_{L,\lambda}^{11}(q) &= \frac{4\pi}{q} \Im m \left( T_{L,\lambda}(q) \right). \\
P_{J}^{0} &= \hat{J} \hat{T} \left( \begin{array}{ccc} J & J & I \\ J & -J & 0 \end{array} \right). \\
\sigma_{tot}^{p}(q) - \sigma_{tot}^{A}(q) &= \frac{2\sqrt{J}}{\sqrt{J+1}} \sigma_{J}^{11}(q) + \cdots
\end{align*}
\]
This asymmetry determines the Gerasimov-Drell-Hearn sum rule \[13, 14\]
\[\int_0^\infty \frac{dq}{q} \left( \sigma_P(q) - \sigma_A(q) \right) = 4 \pi^2 \kappa^2 \frac{e^2}{M_N} J, \tag{71}\]
where $M_N$ denotes the nuclear mass and $\kappa$ the anomalous magnetic moment of the nuclear ground state. This sum rule thus links a ground state property, $\kappa$, to the whole internal excitation spectrum.

C. Deuteron photodisintegration

As next application I will discuss the photodisintegration of the deuteron. The differential cross section for partially polarized photons and oriented deuterons is \[15\]
\[\frac{d\sigma}{d\Omega}(\theta) = T \left( (T_\gamma^\dagger) T_\gamma \rho(\gamma) \rho(d) \right) \]
\[= \frac{d\sigma_0}{d\Omega} \left( 1 + P_I^\gamma \Sigma^I(\theta) \cos 2\phi \right) + \sum_{I=1,2} P_I^d \left( \sum_{M=0}^I \left( T_{IM}(\theta) \cos (M(\phi_d - \phi) - \delta_I M_1) + P_I^\gamma T_{IM}(\theta) \sin (M(\phi_d - \phi) + \delta_I M_1) \right) d_M(\theta_d) \right) + \sum_{M=-I}^I T_{IM}(\theta) \cos (M(\phi_d - \phi) + 2\phi - \delta_I M_1) d_M(\theta_d) \right) \tag{72}\]
This expression defines the various photon, target and photon-target asymmetries $\Sigma^I$, $T_{IM}$, and $T_{IM}'$. Of particular interest is the vector target asymmetry for circularly polarized photons $T_{10}$, because it determines the GDH sum rule. A few examples should serve for illustrating the importance of polarization observables.

1. Linear photon asymmetry for an unoriented deuteron target

For an unoriented deuteron, i.e. $P_1^d = P_2^d = 0$, and photons linearly polarized perpendicular and parallel to the reaction plane, the corresponding cross sections are

\[\frac{d\sigma_{\perp}}{d\Omega} = \frac{d\sigma_0}{d\Omega} (1 - P_I^\gamma \Sigma(\theta)) \quad \text{for} \quad \phi = \pi/2 \quad \text{(perpendicular polarization)}, \tag{73}\]

\[\frac{d\sigma_{\parallel}}{d\Omega} = \frac{d\sigma_0}{d\Omega} (1 + P_I^\gamma \Sigma(\theta)) \quad \text{for} \quad \phi = 0 \quad \text{(parallel polarization)}. \tag{74}\]

Thus measuring the cross sections for parallel and perpendicular linear polarization allows one to determine the photon asymmetry

\[\Sigma(\theta) = \frac{1}{P_I^\gamma} \frac{d\sigma_{\parallel} - d\sigma_{\perp}}{d\sigma_{\parallel} + d\sigma_{\perp}}. \tag{75}\]
As is demonstrated by the example in Fig. 5, this asymmetry $\Sigma$ is quite sensitive to the interference of various multipoles and thus sensitive to various current contributions from meson exchange, isobar configurations and relativistic effects, which enter differently into the multipoles.

2. Target asymmetries for unpolarized photons and oriented deuterons

By proper choices of the deuteron orientation axis one can extract the target asymmetries $T_{IM}$ for unpolarized photons, namely one has

\[T_{11} = \frac{1}{\sqrt{2} P_1^d} \frac{d\sigma^\perp - d\sigma^\parallel}{d\sigma_0}, \quad T_{20} = \frac{1}{P_2^d} \left( 2 - \frac{d\sigma^\parallel + 1/2 (d\sigma^\perp + d\sigma^\parallel)}{d\sigma_0} \right), \quad T_{22} = \sqrt{3} \frac{d\sigma^\parallel - 1/2 (d\sigma^\perp + d\sigma^\parallel)}{d\sigma_0}. \tag{76}\]
where the symbols characterize the various choices for the deuteron orientation angles, i.e. perpendicular to the photon momentum ($\theta_d = 90^\circ$), either in the reaction plane ($\odot$) or perpendicular to the reaction plane ($\uparrow, \downarrow$):

$$\odot : \phi_d = 0^\circ, \quad \uparrow : \phi_d = 90^\circ, \quad \downarrow : \phi_d = -90^\circ.$$ (77)

As an example, Fig. 6 shows the vector target asymmetry $T_{11}$ at various energies with contributions from meson exchange currents (MEC), isobar configurations (IC), relativistic effects (RC), which again shows strong sensitivity to the different dynamical ingredients.

3. Spin asymmetry of the total cross section of deuteron photodisintegration

As already mentioned above, the vector beam-target $T_{10}$ asymmetry of the total cross section of deuteron photodisintegration for circularly polarized photons determines the spin asymmetry which enters the GDH sum rule

$$\frac{d\sigma}{d\Omega}|_{\theta_d=0, \rho^z=1} - \frac{d\sigma}{d\Omega}|_{\theta_d=0, \rho^z=-1} = 2T_{10} \frac{d\sigma_0}{d\Omega}.$$ (78)

Fig. 7 shows this spin asymmetry from threshold up to 1 GeV. It starts negatively near threshold. The reason for this feature is easily understood. Near threshold the dominant contribution to the total cross section arises from the $M1$ transition to the final $^1S_0$-state. Since the total spin is zero it can only be reached when photon and deuteron spins have opposite direction, which means a negative $T_{10}$. Finally I show the spin asymmetry for pion photoproduction on deuteron and nucleon in Fig. 8.

D. Electron scattering and the electric form factor of the neutron

The cross section of electron scattering on a nucleon $N$

$$e(k_i) + N(p_i) \to e(k_f) + N'(p_f),$$ (79)
FIG. 6: Vector target asymmetry $T_{11}$ for deuteron photo disintegration with various contributions (see caption of Fig. 5) from [12].

FIG. 7: Spin asymmetry for deuteron photodisintegration using the realistic Bonn r-space $NN$-interaction and a potential which takes into account retardation of the exchanged pion [16].

with an initial density matrix $\rho_i^{(N)}$ and a final density matrix $\rho_f^{(N)}$ has the formal appearance of the absorption of a virtual photon [17]

$$\frac{d^2\sigma}{d\epsilon_f d\Omega_e} = \frac{2\alpha^2 k_f}{q_i^4} \sum_{f} d^3 p_f \delta^{(4)}(p_f - p_i - q) \text{Tr} \left( d\mathcal{W}(\rho_f^{(N)}, \rho_i^{(N)}) \rho^{(\gamma)} \right),$$

(80)

where the nuclear current matrix elements comprise, besides the two transverse contributions, the nuclear charge as zeroth component which has been introduced in favour of the longitudinal one via current conservation, i.e. $J_{fi} = (j_{fi,0}, j_{fi,1}, j_{fi,-1})$. Furthermore, the virtual photon density matrix for unpolarized electrons is given as a
\[ (\rho^{(\gamma^*)})_{\lambda'\lambda} = -\frac{1}{2} q_{\mu}^2 \begin{pmatrix} \frac{\xi^2}{\eta} & \frac{\xi}{\sqrt{2\eta}} & -\frac{\xi}{\sqrt{2\eta}} \\ \frac{\xi}{\sqrt{2\eta}} & 1 + \frac{\xi}{\eta} & -\frac{\xi}{\sqrt{2\eta}} \\ -\frac{\xi}{\sqrt{2\eta}} & -\frac{\xi}{\sqrt{2\eta}} & 1 + \frac{\xi}{2\eta} \end{pmatrix}, \]  \hspace{1cm} (81)

with \( \lambda = 0, 1, -1, \xi = -q_{\mu}/q^2, \eta = \tan^2(\theta_e/2), \) \( q \) as momentum transfer, and \( \theta_e \) as the electron scattering angle. For longitudinally polarized electrons of degree \( h \) the virtual photon density matrix has to be supplemented according to

\[ \rho^{(\gamma^*)} = \rho^{(\gamma^*)0} + h \rho^{(\gamma^*)'}, \]  \hspace{1cm} (82)

where

\[ (\rho^{(\gamma^*)'})_{\lambda'\lambda} = \frac{1}{2} q_{\mu}^2 \begin{pmatrix} 0 & \frac{\xi}{\sqrt{2\eta}} & \frac{\xi}{\sqrt{2\eta}} \\ \frac{\xi}{\sqrt{2\eta}} & \sqrt{1 + \frac{\xi}{\eta}} & 0 \\ \frac{\xi}{\sqrt{2\eta}} & 0 & -\sqrt{1 + \frac{\xi}{\eta}} \end{pmatrix}. \]  \hspace{1cm} (83)

One should notice that the trace of \( \rho^{(\gamma^*)} \) is not normalized. Instead one has

\[ \text{Tr}(\rho^{(\gamma^*)}) = -\frac{1}{2} q_{\mu}^2 \left( \frac{\xi^2}{\eta} + \frac{\xi}{\eta} + 2 \right). \]  \hspace{1cm} (84)

The normalized density matrix \( \tilde{\rho}^{(\gamma^*)} \) is given by

\[ (\tilde{\rho}^{(\gamma^*)})_{\lambda'\lambda} = \frac{1}{\xi^2 + \xi + 2\eta} \begin{pmatrix} \xi^2 & \xi \rho_{01} & \xi \rho_{0-1} \\ \xi \rho_{01} & \eta + \xi - h \sqrt{\eta(\xi + \eta)} & -\frac{\xi}{\eta} \\ \xi \rho_{0-1} & -\frac{\xi}{\eta} & \eta + \xi + h \sqrt{\eta(\xi + \eta)} \end{pmatrix}, \]  \hspace{1cm} (85)

where for convenience I have introduced \( \rho_{0\pm 1} = \pm \xi \sqrt{\frac{3}{2}(\xi + \eta)} - h \xi \frac{\eta}{2} \). This leads to the following interpretation of the virtual photon density matrix:
(i) $P_{\text{long}} = \frac{\xi^2}{\xi + 2\eta}$ describes the fraction of longitudinal photons.

(ii) $P_{\text{trans}} = \frac{\xi + 2\eta}{\xi^2 + \xi + 2\eta}$ describes the fraction of transverse photons.

(ii) The ratio longitudinal to transverse photons is $P_{\text{long}}/P_{\text{trans}} = \frac{\xi^2}{\xi + 2\eta}$. For $\theta_e \to 0$ this ratio approaches $P_{\text{long}}/P_{\text{trans}} \to \xi = 1 - \omega^2/q^2$, since then $\eta \to 0$.

Furthermore, the transverse part of $\rho(\gamma^*)$ is

\[
\rho_T^{(\gamma^*)} = \frac{\eta}{\xi^2 + \xi + 2\eta} \left( \frac{1 + \frac{\xi}{2\eta} + \hbar \sqrt{1 + \frac{\xi}{\eta}}}{1 + \frac{\xi}{2\eta} - \hbar \sqrt{1 + \frac{\eta}{\xi}}} \right) \\
= \frac{\eta + \frac{\xi}{2\eta}}{\xi^2 + \xi + 2\eta} (\mathbb{1} - \varepsilon_x \sigma_x + \hbar \varepsilon_y \sigma_y).
\]

This means that for unpolarized electrons the transverse part of the virtual photon density matrix describes linear polarization in the scattering plane of degree $\varepsilon_T = \xi/(\xi + 2\eta)$. Furthermore, longitudinally polarized electrons create in addition circular polarization of the virtual photon of degree $\hbar \varepsilon_T$ with $\varepsilon_T = 2\sqrt{\eta(\xi + \eta)/(\xi + 2\eta)}$.

Evaluation of the trace with respect to the virtual photon polarization indices leads to the well-known form of the differential scattering cross section \cite{3, 9}

\[
\frac{d^2\sigma}{d\varepsilon_f d\Omega_e} = \frac{2\alpha^2}{q_i^2} \frac{k_f}{k_i} (\rho_L dF_L + \rho_T dF_T + \rho_{LT} dF_{LT} + \rho_{TT} dF_{TT}) + \hbar (\rho'_L dF'_{LT} + \rho'_T dF'_{TT}), \tag{87}
\]

with the electron kinematical functions

\[
\rho_L = \rho^{(\gamma^*)}_0, \quad \rho_T = \rho^{(\gamma^*)}_1, \quad \rho_{LT} = \rho^{(\gamma^*)}_{01}, \quad \rho_{TT} = \rho^{(\gamma^*)}_{-11}, \quad \rho'_L = \rho^{(\gamma^*)}_1, \quad \rho'_T = \rho^{(\gamma^*)}_1, \quad \tag{88}
\]

and the nuclear current structure functions

\[
dF_L = dW_0, \quad dF_T = \sum_{\lambda = \pm 1} dW_{\lambda}, \quad dF_{LT} = 2\Re (dW_0 + dW_{10}), \quad dF_{TT} = 2\Re (dW_{-11}), \tag{89}
\]

\[
dF'_{LT} = -2\Im (dW_0 + dW_{10}), \quad dF'_T = \sum_{\lambda = \pm 1} \lambda dW_{\lambda}.
\]

The hadron current tensor $dW_{\lambda \lambda}$ still depend on the initial and final orientation parameters

\[
dW_{\lambda \lambda} = \sum_f d^3 p_f d^4(p_f - p_i - q) \text{Tr}(J_{f_i \lambda}^\dagger \rho_f^{(N)} J_{f_i \lambda} \rho_i^{(N)}), \tag{90}
\]

where here the trace refers to the nuclear spin degrees of freedom.

1. **Deuteron electrodisintegration and the neutron electric form factor**

In the absence of neutron targets of sufficient density one uses light nuclei as effective neutron targets, deuteron or $^3$He, and studies, for example, inelastic electron scattering off the deuteron $d(e, e' n)p$ in quasi-free kinematics, where one observes the outgoing neutron. Quasi-free means that the emitted neutron takes up energy and momentum transfer such that the recoiling proton remains at rest in the laboratory.

For this kinematics the scattering cross section is roughly given by the cross section on the neutron multiplied by the probability of finding a neutron at rest within the deuteron. In this way, the neutron magnetic form factor $G_M$ can be determined. However, since the electric form factor $G_E$ of the neutron is much smaller than its magnetic one, its contribution is buried under competing binding and final state interaction effects. In this situation polarization observables come into play as they often allow one to investigate small contributions by interference with large ones.

In order to illustrate this, I will first consider an exercise with a simplified discussion of elastic electron-neutron scattering. Assuming a nonrelativistic form of charge and transverse current \cite{17}

\[
j_0 = G_E, \quad j_{\pm 1} = \sqrt{\frac{1}{2M}} G_M \sigma_{\pm}, \tag{91}
\]
and taking for the polarized initial neutron the density matrix
\[ \rho^{(n)} = \frac{1}{2}(1 + \mathbf{P} \cdot \mathbf{\sigma}) , \]
one finds for the structure functions
\[ F_L = G_E^2 , \quad F_T = \frac{q^2}{2M^2} G_M^2 , \quad F_{LT} = 0 , \quad F_{TT} = 0 , \]
\[ F'^{LT} = -\sqrt{2} \frac{q}{M} G_E G_M P_x , \quad F'^{TT} = -\frac{q^2}{2M} G_M^2 P_z . \]
Thus one obtains as differential cross section
\[ \frac{d\sigma}{d\Omega_e} = \frac{d\sigma_0}{d\Omega_e} (1 + h \mathbf{A} \cdot \mathbf{P}) , \quad \frac{d\sigma_0}{d\Omega_e} = c \frac{q^2}{2M^2} G_M^2 S_0 , \]
with the neutron target asymmetry
\[ \mathbf{A} = (-2\rho'^L_T R/S_0, 0, -\rho'^T/S_0) , \]
and furthermore
\[ S_0 = \rho_T + \rho_L R^2 , \quad \text{and} \quad R = \sqrt{2} \frac{M}{q} \frac{G_E}{G_M} . \]

Thus one notes that the unpolarized cross section is dominated by the neutron magnetic form factor $G_M$, while the electric form factor $G_E$ gives via $R^2$ a tiny contribution only. But for longitudinally polarized electrons and initially polarized neutrons with spin lying in the scattering plane and perpendicular to $\mathbf{q}$, the corresponding asymmetry $A_x = -2\rho'^L_T R/S_0$ is linear in the ratio $R$ and thus more sensitive to $G_E$. The same holds for an unpolarized initial neutron and the final neutron polarization component in the scattering plane and perpendicular to $\mathbf{q}$, i.e. $P_x = A_x$.

Now I will return to quasi-free scattering on the deuteron. One needs a vector polarized deuteron target for having an initially polarized neutron. The polarization component of the bound neutron in a vector polarized deuteron along the orientation axis is given by
\[ P = (p_1 - p_{-1})(1 - \frac{3}{2} P_D) , \]
with $P_D$ for the $D$-wave probability. In Born approximation, neglecting final state interactions and also the small $D$-component in the deuteron, the cross section in quasi-free kinematics for polarized electrons and a vector-polarized deuteron target has the form
\[ \frac{d\sigma}{d\Omega_e d\Omega_{np}} = \frac{d\sigma_0}{d\Omega_e d\Omega_{np}} (1 + h P'^L_A V_{ed}) , \quad \frac{d\sigma_0}{d\Omega_e} = c' \frac{q^2}{2M^2} G_M^2 S_0 , \]
where the vector asymmetry is given by
\[ A'^V_{ed} = \frac{1}{S_0} \sqrt{\frac{2}{3}} \left( \rho'_T \cos \theta_d + 2\rho'_L \sin \theta_d \cos \phi_d R \right) . \]
Thus indeed, this vector asymmetry depends linearly on the ratio $R$. In particular one finds for $\theta_d = 90^\circ$ and $\phi_d = 0$
\[ A'^V_{ed} = \frac{2}{S_0} \sqrt{\frac{2}{3}} \rho'_L R , \]
FIG. 9: Kinematics of quasi-free electron scattering on the deuteron in the laboratory.
which means, that for this kinematics one has direct access to the ratio containing the electric form factor of the neutron. Instead of using an initially polarized neutron, one can measure the polarization of the outgoing neutron in $d(\vec{e},\vec{e}'\vec{n})p$ for which one obtains

$$P'_x(n) = \frac{2}{S_0} \rho'_L R = \sqrt{\frac{3}{2}} A_{ed}.$$  

(101)

An example is shown in Fig. 10 where one readily notices the strong influence of the electric form factor of the neutron right on the quasi-free peak ($\theta_{np} = 180^\circ$). Naturally, one has to check what the influence is of the neglected effects from final state interaction, relativistic effects and additional current contributions from meson exchange and isobar configurations. Indeed, a thorough calculation has shown that indeed these additional effects have quite small effect at $\theta_{np} = 180^\circ$.

![Fig. 10: Two examples for the influence of the electric form factor of the neutron in quasi-free electron deuteron scattering at quasi-free kinematics ($q^2 = 12 \text{ fm}^{-2}$) for $G_{En} = 0$ (dotted) and two different models for $G_{En} = 0$ from [18]. Left panel: Polarization component $P'_x$ of the final neutron. Right panel: Vector asymmetry $A_{ed}$ at $\theta_d = 90^\circ, \phi_d = 0$. Dotted $G_{En} = 0$](image)

**IV. COMPLETE SETS OF OBSERVABLES**

The polarization observables of a reaction, i.e. beam and target asymmetries, polarization components of the outgoing particles and combinations thereof, provide a wealth of information. Thus the question naturally arises, does one need all of them in order to obtain complete information on this reaction or is a smaller number carefully chosen already sufficient. In order to illustrate this problem I will consider again a reaction of the type

$$a + b \rightarrow c + d + \cdots,$$  

(102)

where the participating particles have spins $s_a, s_b, \ldots$. Let me denote the corresponding $T$-matrix by

$$T_{\lambda_a, \lambda_b, \lambda_c, \ldots}.$$  

(103)

with helicities (spin projections on the particle momenta) $\lambda_a, \lambda_b, \lambda_c, \ldots$. Angular momentum conservation requires that $s_a + s_b$ as well as $s_c + s_d + \cdots$ are either half-integer or integer. The whole set of these complex $T$-matrix elements contains the maximal dynamic information available for this reaction. But in general not all of them are independent, since there might exist relations between the matrix elements due to some conservation laws. Therefore, it suffices to determine the maximal subset of independent matrix elements. The total number of $T$-matrix elements is given by

$$N = N(s_a)N(s_b)N(s_c)N(s_d)\cdots,$$  

(104)

with

$$N(s_{\alpha}) = \begin{cases} 2s_\alpha + 1 & \text{for massive particles,} \\ 2 & \text{for massless particles.} \end{cases}$$  

(105)
If parity is conserved, the number of independent matrix elements reduces to

\[ n = \begin{cases} \frac{N}{2} & \text{for } N \text{ even}, \\ \frac{(N-1)}{2} & \text{for } N \text{ odd}. \end{cases} \]  

(106)

What one measures are observables \( \mathcal{O} \), like cross sections, asymmetries etc., which are hermitean forms of the \( T \)-matrix elements

\[ \mathcal{O}_\alpha = \sum_{j'j} T^*_j O^\alpha_{j'j} T_j. \]  

(107)

Here, \( j' \) and \( j \) enumerate all independent \( T \)-matrix elements and \( O^\alpha_{j'j} \) denotes a hermitean \( n \times n \) matrix associated with the observable \( \mathcal{O} \). For example, the unpolarized differential cross section is given by

\[ \frac{d\sigma}{d\Omega} \propto \sum_{j} T^*_j T_j, \]  

(108)

or the polarisation component \( P^\xi_c \) of the outgoing particle \( c \)

\[ P^\xi_c \frac{d\sigma}{d\Omega} \propto \sum_{j'j} T^*_j (S^\xi_c)_{j'j} T_j, \]  

(109)

with \( S^\xi_c \) as the corresponding spin operator.

The number of \textit{linearly} independent observables is equal to the number of linearly independent hermitean \( n \times n \) matrices, namely \( n^2 \). On the other hand, the \( n \) independent complex matrix elements are determined by \( 2n - 1 \) real numbers, since a common phase remains unobserved. This seeming contradiction is resolved by the observation that exactly \( (n-1)^2 \) quadratic relations exist between the linearly independent observables (see Appendix B of [20]). Thus one can conclude

\begin{quote}
\textit{A set of} \( 2n - 1 \) \textit{properly chosen observables should allow a determination of all independent } T \text{-matrix elements except for possible discrete ambiguities, whose elimination requires additional observables.}
\end{quote}

For illustration, take the simplest case of the absorption of a scalar particle on a \( s = 1/2 \) particle. One has four matrix elements which reduce to two independent complex matrix elements \( T_1 \) and \( T_2 \) if parity is conserved. In this case, each observable is represented by a \( 2 \times 2 \) matrix for which we may choose the unit matrix \( (\sigma_0 = 1_2) \) and the three \textsc{Pauli} matrices \( (\sigma_i, i = 1, 2, 3) \). Thus we have four independent observables

\[ \mathcal{O}_\alpha = T^\dagger \sigma_\alpha T, \quad (\alpha = 0, \ldots, 3), \]  

(110)

or in detail

\[ \begin{align*}
\mathcal{O}_0 &= |T_1|^2 + |T_2|^2, \\
\mathcal{O}_1 &= 2 \text{Re}(T_1^* T_2), \\
\mathcal{O}_2 &= 2 \text{Im}(T_1^* T_2), \\
\mathcal{O}_3 &= |T_1|^2 - |T_2|^2.
\end{align*} \]  

(111)

One easily finds the following quadratic relation

\[ \mathcal{O}_3^2 = \mathcal{O}_0^2 - 4|T_1|^2|T_2|^2 = \mathcal{O}_0^2 - (\mathcal{O}_1^2 + \mathcal{O}_2^2), \]  

(112)

which leads to the inequality

\[ \mathcal{O}_0^2 \geq \mathcal{O}_1^2 + \mathcal{O}_2^2. \]  

(113)

Without loss of generality we can choose the overall phase such that \( T_1 \) becomes real and non-negative. Then the two matrix elements can be obtained from three observables, for example \( \mathcal{O}_0, \mathcal{O}_1 \) and \( \mathcal{O}_2 \).

\[ T_1^2 = \frac{1}{2} \left( \mathcal{O}_0 \pm \sqrt{\mathcal{O}_0^2 - (\mathcal{O}_1^2 + \mathcal{O}_2^2)} \right), \]  

(114)

\[ T_2 = \frac{1}{2T_1} (\mathcal{O}_1 + i\mathcal{O}_2). \]  

(115)
Obviously, one finds in general two solutions. This discrete ambiguity can be resolved by considering in addition $O_3$, for which one finds, inserting the general solution,
\[ O_3 = \pm \sqrt{O_0^2 - (O_1^2 + O_2^2)}, \]  
(116)
Thus the sign of $O_3$ selects the proper solution.

An alternative method exploits the fact that all bilinear forms $T_i^*T_j$ can be given as linear expressions in the observables
\[ T_1^*T_1 = \frac{1}{2}(O_0 + O_3), \quad T_1^*T_2 = \frac{1}{2}(O_1 + iO_2), \quad T_2^*T_2 = \frac{1}{2}(O_0 - O_3). \]  
(117)
Setting $T_1$ real and positive, one finds
\[ T_1 = \sqrt{\frac{1}{2}(O_0 + O_3)}, \quad T_2 = \frac{O_1 + iO_2}{\sqrt{2(O_0 + O_3)}}. \]  
(118)
The disadvantage of this approach is that the choice of observables is fixed by the bilinear expressions. Thus the essential question is:

**Is there a criterion which allows one to decide whether a chosen set of $2n - 1$ observables is complete, i.e. allows the determination of all independent $T$-matrix elements, except for eventual discrete ambiguities?**

Indeed, such a criterion exists and has been applied successfully. For details see [19].

### A. Explicit inversion

Now I will briefly describe an alternative method based on the explicit inversion of the observables with respect to the bilinear forms in the $T$-matrix elements. As an example, I will consider as observables the longitudinal structure functions $(120)$ of deuteron electrodisintegration (for details see [21]). Here $X = x_{\alpha_1}x_{\alpha_2}$ ($\alpha_i = 0, \ldots, 3$) describes the polarization components $P_{\alpha_1}P_{\alpha_2}$ of the two outgoing nucleons ($\alpha_i = 0$ means no polarization of the $i$-th nucleon). In particular, $X = x_0x_0 = 1$ refers to the differential cross section. Furthermore, $IM$ ($0 \leq M \leq I$) describes the deuteron polarization, i.e.
\[ I = \begin{cases} 
0 & \text{unpolarized,} \\
1 & \text{vector polarized,} \\
2 & \text{tensor polarized.} 
\end{cases} \]  
(119)
The initial state has three spin states for the deuteron and only one for the longitudinal virtual photon, and the final state has for each nucleon two spin states. Thus one has 12 $T$-matrix elements which however are not independent if parity is conserved which we will assume. Then the number reduces to six independent matrix elements which leads to 36 linearly independent observables. On the other hand, a complete set should comprise eleven observables. Because of the fact that the spin operators form a complete basis in spin space, it is possible to represent all bilinear forms $T_i^*T_j$ as linear combinations of structure functions $f_{L,X}^{IM}$
\[ T_j^*T_j = T_{j,j}^{IM} [f_{L,X}^{IM}]. \]  
(120)
For example one finds for $T_{j,1} = T_j^*T_1$ ($j = 2, \ldots, 6$)
\[ T_{2,1} = \frac{1}{6} \left( -f_{L,zz}^{00} + \sqrt{2} f_{L,zz}^{20} + i \left( f_{L,zz}^{00} - \sqrt{2} f_{L,zz}^{20} \right) \right), \]  
(121)
\[ T_{3,1} = \frac{1}{4\sqrt{6}} \left( -f_{L}^{21} - f_{L,y_1}^{21} + i \left( -f_{L}^{11} - f_{L,y_1}^{11} \right) \right), \]  
(122)
\[ T_{4,1} = \frac{1}{4\sqrt{6}} \left( -f_{L,zz}^{11} + f_{L,z_2}^{11} + i \left( -f_{L,zz}^{21} - f_{L,z_2}^{21} \right) \right), \]  
(123)
\[ T_{5,1} = \frac{1}{4\sqrt{6}} \left( -f_{L,zz}^{11} + f_{L,z_1}^{21} + i \left( -f_{L,zz}^{21} - f_{L,z_1}^{21} \right) \right), \]  
(124)
\[ T_{6,1} = \frac{1}{4\sqrt{6}} \left( f_{L,zz}^{11} + f_{L,zz}^{21} + i \left( -f_{L,zz}^{11} + f_{L,zz}^{21} \right) \right), \]  
(125)
which obviously are determined by 20 observables. These expressions can be used to relate all matrix elements to one

\[ T_i = \frac{1}{T_j} T_{ji} \quad \text{for } i \neq j, \quad (126) \]

where the \( T_{ji} \) on the right hand side are given in terms of the observables \( f_{L,X}^{M} \). The missing matrix element \( T_j \) can be determined from a properly chosen additional observable. The disadvantage of this approach is that one needs more than 11 observables, for example, in the above case 21.

One is lead to a better strategy by the observation that disjunct groups of observables exist, which determine a subgroup of interference terms, e.g.

\[ T_{3,1} = \frac{1}{4\sqrt{6}} \left( - f_{L,zz}^{11} + f_{L,zz}^{21} + i \left( - f_{L,zz}^{11} + f_{L,zz}^{21} \right) \right), \]

\[ T_{6,2} = \frac{1}{4\sqrt{6}} \left( - f_{L,zz}^{11} + f_{L,zz}^{21} + i \left( - f_{L,zz}^{11} + f_{L,zz}^{21} \right) \right), \]

or

\[ T_{6,1} = \frac{1}{4\sqrt{6}} \left( f_{L,zz}^{11} + f_{L,zz}^{21} + i \left( - f_{L,zz}^{11} + f_{L,zz}^{21} \right) \right), \]

\[ T_{3,2} = \frac{1}{4\sqrt{6}} \left( - f_{L,zz}^{11} + f_{L,zz}^{21} + i \left( - f_{L,zz}^{11} + f_{L,zz}^{21} \right) \right). \]

For a survey it is useful to introduce a diagrammatic representation in the following way: Each matrix element \( T_j \) is represented by a point on a circle with equal distance between neighbouring points, and an interference term \( T_{ij} \) is represented by a straight line connecting point “i” with point “j”. Those interference terms determined by the same group of observables are represented by the same type of lines. An example is shown in Fig. 11.

![Diagram](image_url)

**FIG. 11:** Diagramatic representation of groups of longitudinal observables determining the interference terms of \( T \)-matrix elements. The nomenclature for the groups and the corresponding observables are listed in Table 1 (from 20).

| Line type | Observable group        | Line type | Observable group        |
|-----------|-------------------------|-----------|-------------------------|
| (a) solid | \( x_{z}^{10}, x_{x}^{20}, x_{z}^{22}, z_{z}^{22} \) | (b) solid | \( 1^{11}, 1^{21}, y_{1}^{21}, y_{2}^{21} \) |
|           | \( z_{z}^{20}, z_{z}^{22} \)        | dashed    | \( x_{z}^{11}, x_{z}^{21}, z_{z}^{11}, z_{z}^{21} \) |
| (c) solid | \( x_{1}^{10}, x_{1}^{22}, z_{1}^{10}, z_{2}^{22} \) | (d) solid | \( x_{2}^{11}, x_{2}^{21}, z_{2}^{11}, z_{2}^{21} \) |
| dashed    | \( x_{2}^{10}, x_{2}^{22}, z_{2}^{10}, z_{2}^{22} \)        | dashed    | \( x_{2}^{11}, x_{2}^{21}, z_{2}^{11}, z_{2}^{21} \) |
Now, the better strategy consists in finding a set of \( n - 1 \) independent interference terms. Independent means a representation by a diagram with all \( n \) points connected by exactly \( n - 1 \) lines such that each point is connected to all the others, not necessarily directly. Then closed loops cannot appear. Examples are shown in Fig. 12.

In such a set, all matrix elements can be expressed by one, because one finds for a matrix element \( j' \), which is connected via the points \( j_1 \) through \( j_k \) with point \( j \),

\[
T_j' = \begin{cases} 
\frac{T_{j_1j} T_{j_2j}}{T_{j_1j_2} T_{j_2j_3}} \cdots \frac{T_{j_{k-2}j_{k-1}} T_{j_{k-1}j_k}}{T_{j_{k-2}j_{k-1}} T_{j_{k-1}j_k}}, & \text{if } k \text{ odd} \\
\frac{T_{j_1j} T_{j_2j}}{T_{j_1j_2} T_{j_2j_3}} \cdots \frac{T_{j_{k-2}j_{k-1}} T_{j_{k-1}j_k}}{T_{j_{k-2}j_{k-1}} T_{j_{k-1}j_k}}, & \text{if } k \text{ even}
\end{cases}
\]

(131)

The procedure is to select a minimal number of groups of observables which contain an independent set of interference terms. These, however, may in general contain also additional interference terms leading to closed loops which constitute additional conditions. For example, for a loop through an even number of points, one obtains from (131) with \( k \) odd and \( j' = j \)

\[
\frac{T_{j_{11}} T_{j_{32}}}{T_{j_{11}j_{12}} T_{j_{32}j_4}} \cdots \frac{T_{j_{k-2}j_{k-1}} T_{j_{k-1}j_k}}{T_{j_{k-2}j_{k-1}} T_{j_{k-1}j_k}} = 1,
\]

(132)

which constitutes two real quadratic relations between the observables. For a loop with an odd number of points follows from (131) with \( k \) even

\[
|T_j|^2 = \frac{T_{j_{11}} T_{j_{32}}}{T_{j_{11}j_{12}} T_{j_{32}j_4}} \cdots \frac{T_{j_{k-2}j_{k-1}} T_{j_{k-1}j_k}}{T_{j_{k-2}j_{k-1}} T_{j_{k-1}j_k}},
\]

(133)

by which the moduli of all participating matrix elements are fixed. Examples of minimal sets are shown in Fig. 12. In both cases three groups of observables are participating, which each consists of four observables. Because of the loop through an even number of points, one can eliminate two of the observables. Thus all matrix elements can be expressed by one matrix element and 10 observables. For fixing the remaining matrix element one needs only one additional observable, so that indeed a complete set of 11 observables is obtained. An example for an explicit solution is

\[
T_2 = \frac{-f_{L_{,yz}}^2 + f_{L_{,yy}}^2 + i(-f_{L_{,yz}}^1 + f_{L_{,yy}}^1)}{f_{L_{,yz}}^1 + f_{L_{,yy}}^1 - i(f_{L_{,yz}}^1 - f_{L_{,yy}}^1)} T_1,
\]

(134)

\[
T_3 = \left(-f_{L_{,yz}}^2 - f_{L_{,yy}}^2 + i(f_{L_{,yz}}^1 + f_{L_{,yy}}^1)\right) \frac{1}{4\sqrt{6} T_1},
\]

(135)

\[
T_4 = \frac{\sqrt{2} f_{L_{,yz}}^{10} + f_{L_{,zz}}^{10} + i(f_{L_{,yz}}^{22} + \sqrt{2} f_{L_{,zz}}^{10})}{f_{L_{,zz}}^{10} + f_{L_{,zz}}^{10} - i(f_{L_{,zz}}^{22} - f_{L_{,zz}}^{10})} T_1,
\]

(136)

\[
T_5 = \frac{\sqrt{2} f_{L_{,yz}}^{10} + f_{L_{,zz}}^{22} + i(f_{L_{,yz}}^{22} - \sqrt{2} f_{L_{,zz}}^{10})}{-f_{L_{,zz}}^{22} - f_{L_{,zz}}^{10} - i(f_{L_{,zz}}^{22} + f_{L_{,zz}}^{10})} T_1,
\]

(137)

\[
T_6 = \left(f_{L_{,zz}}^{10} + f_{L_{,zz}}^{22} + i(f_{L_{,zz}}^{22} - f_{L_{,zz}}^{10})\right) \frac{1}{4\sqrt{6} T_1}.
\]

(138)
It contains 12 observables but with two conditions
\[
(f_{L,zz}^{21})^2 - (f_{L,xz}^{11})^2 + (f_{L,xz}^{21})^2 = (f_{L}^{21})^2 - (f_{L,y1}^{21})^2 + (f_{L,y1}^{11})^2, \tag{139}
\]
\[
f_{L,zz}^{11}f_{L,xz}^{11} + f_{L,xz}^{21}f_{L,xz}^{21} = f_{L}^{11}f_{L}^{21} - f_{L,y1}^{11}f_{L,y1}^{21}, \tag{140}
\]
which results in a reduction to 10 observables. Finally, $T_1$ can be determined from $f_{L}^{00}$, the eleventh observable.

V. CONCLUSIONS AND OUTLOOK

In these two lectures, serving as a compact survey, I have tried to convey to the reader the vast possibilities, offered by the study of spin degrees of freedom in electromagnetic reactions. The main conclusions are:

- The density matrix formalism is an adequate, convenient and economic framework for the treatment of spin degrees of freedom.
- Polarization observables provide us with much more detailed information on a reaction than just unpolarized total and differential cross sections.
- The study of polarization effects allows one to test theoretical models in much greater detail.
- In particular, polarization observables are in general much more sensitive to small but interesting dynamical effects like, for example, parity violation.
- In principle, the selection and determination of a complete set of observables provides the most detailed analysis of a reaction.

Future experimental and theoretical efforts should be devoted to:

- The development of more intense polarized beams and targets as well as highly efficient polarimeters.
- Experimental studies of various polarization observables over a large range of energy and momentum transfers.
- Theoretical predictions for preferentially complete sets of polarization observables also over a large range of energy and momentum transfers in order to provide sensitive tests for theoretical ingredients and thus guidelines for the planning of polarization experiments.

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quasi-free point
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