Exact Solution of an electronic model of superconductivity in $1 + 1$ dimensions I

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ABSTRACT

We study a superconducting integrable model of strongly correlated electrons in $1 + 1$ dimensions. We construct all six Bethe Ansätze for the model and give explicit expressions for lowest conservation laws. We also prove a lowest weight theorem for the Bethe-Ansatz states.

*This work was supported in part by the National Science Foundation under research grants PHY91-07261 and NSF91-08054.

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1. Introduction

The Hubbard model has been studied as the prime example of a microscopic theory of strongly correlated electrons for several decades. More recently it was proposed as a model for superconductivity by P.W. Anderson. It seems to be clear that in order to make close contact to experimental results, the Hubbard hamiltonian has to be modified by adding competing interactions. Despite the simple form of its hamiltonian, the mathematical structure of the Hubbard model is rather complex (as can be seen in the Bethe Ansatz solution in one spatial dimension) and exact results about its physical features are quite difficult to obtain. The ground states in more than one dimension are still unknown, and the question under what conditions the model exhibits superconductivity is yet unanswered. Therefore it would be useful to have a model similar to the Hubbard model, where the physics is more transparent. In a recent publication we have presented a new model, describing strongly correlated electrons on a general $d$-dimensional lattice, which differs from the Hubbard model by moderate interactions of nearest neighbours. The nature of these interactions is very similar to the ones of Hirsch's model of hole superconductivity. In one dimension we the model is integrable. We gave an exact expression for the ground state in the attractive regime in 1, 2 and 3 dimensions and showed that the ground states of the model for attractive and moderately repulsive on-site interaction exhibit off-diagonal long range order (ODLRO) and are thus superconducting. In order to gain further insights into the physical features of the model it is reasonable to first study its one dimensional version, where integrability permits us to obtain exact results. In this paper we construct the nested Algebraic Bethe Ansatz (NABA) for the model and use it to determine higher conservation laws. We also prove a lowest weight theorem for the Bethe Ansatz states.

In two forthcoming publications we will determine the structure of the low lying excitations over the ground states and prove completeness of the $u(2\mid 2)$ extended Bethe Ansatz.

Electrons on a lattice are described by operators $c_{j,\sigma}$, $j = 1, \ldots, L$, $\sigma = \pm 1$, where $L$ is the total number of lattice sites. These are canonical Fermi operators with anti-commutation relations given by $\{c_{i,\sigma}^+, c_{j,\tau}\} = \delta_{i,j}\delta_{\sigma,\tau}$. The state $|0\rangle$ (the Fock vacuum) satisfies $c_{i,\sigma}|0\rangle = 0$. There are four possible electronic states at a given lattice site $i$

$$
|0\rangle, \quad \uparrow_i = c_{i,1}^{|0\rangle}, \quad \downarrow_i = c_{i,-1}^{|0\rangle}, \quad \uparrow\downarrow_i = c_{i,-1}^+ c_{i,1}^{|0\rangle} .
$$

(1.1)

By $n_{i,\sigma} = c_{i,\sigma}^+ c_{i,\sigma}$ we denote the number operator for electrons with spin $\sigma$ on site $i$ and we write $n_i = n_{i,1} + n_{i,-1}$.

The hamiltonian on a lattice of $L$ sites in the grand canonical ensemble is

$$
H = H^0 + U \sum_{j=1}^{L} (n_{j,1} - \frac{1}{2})(n_{j,-1} - \frac{1}{2}) - \mu \sum_{j=1}^{L} n_j + h \sum_{j=1}^{L} (n_{j,1} - n_{j,-1})
$$

(1.2)
where \( H^0 = -\sum_{j=1}^{L} H_{j,j+1}^0 \) is given by

\[
H_{j,j+1}^0 = c_{j+1}^\dagger c_{j,1}(1 - n_{j,-1} - n_{j+1,-1}) + c_{j,1}^\dagger c_{j+1,1}(1 - n_{j,-1} - n_{j+1,-1}) \\
+ c_{j+1,-1} c_{j,-1}(1 - n_{j,1} - n_{j+1,1}) + c_{j,-1}^\dagger c_{j+1,-1}(1 - n_{j,1} - n_{j+1,1}) \\
+ \frac{1}{2}(n_j - 1)(n_{j+1} - 1) + c_{j,1}^\dagger c_{j,-1}^\dagger c_{j+1,1} - c_{j,-1} c_{j+1,1} c_{j,1}^\dagger \\
- \frac{1}{2}(n_j - n_{j+1})(n_{j+1,1} - n_{j,-1,1}) - c_{j,-1}^\dagger c_{j,1}^\dagger c_{j+1,1} - c_{j,1}^\dagger c_{j,-1} c_{j+1,1} \\
+ (n_{j+1} - \frac{1}{2})(n_{j,-1} - \frac{1}{2}) + (n_{j+1,1} - \frac{1}{2})(n_{j+1,-1} - \frac{1}{2}) .
\]

(1.3)

Here \( U \) is the Hubbard model coupling constant, \( h \) is an external magnetic field, and \( \mu \) is the chemical potential.

It was shown in [1] that \( H^0 \) is invariant under a \( u(2|2) \) symmetry algebra. The three other terms in \( H \) are elements of the Cartan subalgebra of \( u(2|2) \) (and commute with \( H^0 \) and each other), and thus \( H^0 \) and \( H \) have a complete set of simultaneous eigenstates. In what follows we will construct a complete set of eigenstates of \( H^0 \) (and of \( H \)) by using a nested Algebraic Bethe Ansatz and the \( u(2|2) \) symmetry of \( H^0 \). The algebra \( u(2|2) \) has a total of 16 generators, 8 of which are fermionic and bosonic, respectively. The eight bosonic operators fall into two \( su(2) \) and two \( u(1) \) subalgebras. The two \( su(2) \) algebras are generated by the spin-operators [We shall always give local expressions \( O_j \) for symmetry generators, implying that the global ones are obtained as \( O = \sum_{j=1}^{L} O_j \).]

\[
S_j = c_{j,1}^\dagger c_{j,-1} , \quad S_j^\dagger = c_{j,-1}^\dagger c_{j,1} , \quad S_j^z = \frac{1}{2}(n_{j,1} - n_{j,-1}) , \quad (1.4)
\]

and by the \( \eta \)-pairing like operators[5]

\[
\eta_j = c_{j,1}^\dagger c_{j,-1} , \quad \eta_j^\dagger = c_{j,-1}^\dagger c_{j,1} , \quad \eta_j^z = -\frac{1}{2}n_j + \frac{1}{2} . \quad (1.5)
\]

Note that unlike for the case of the Hubbard model as treated in [5] there is no factor of \((-1)^j\) in the definition of \( \eta_j \). The two \( u(1) \) charges are given by the identity operator and the operator

\[
X = \sum_{j=1}^{L} X_j , \quad X_j = (n_{j,1} - \frac{1}{2})(n_{j,-1} - \frac{1}{2}) .
\]

(1.6)

As all eigenvalues of \( S^z + \eta^z = -\hat{N}_{-1} + \frac{L}{2} \) (where \( \hat{N}_{-1} = \sum_{j=1}^{L} n_{j,-1} \) is the number operator for spin down electrons) are either integer (for even \( L \)) or half-odd integer (for odd \( L \)), the two \( su(2) \) algebras are not quite independent. Although all six generators of the two \( su(2) \) algebras commute with \( H^0 \), the Hilbert space of eigenstates of \( H^0 \) will only carry representations of \( su(2) \times su(2) \) with integer (for even \( L \)) or half-odd integer (for odd \( L \)) quantum numbers. The same kind of restriction carries through to the complete \( u(2|2) \) symmetry. The eight fermionic generators are given by

\[
Q_{j,\sigma} = (1 - n_{j,-\sigma})c_{j,\sigma} , \quad \tilde{Q}_{j,\sigma} = n_{j,-\sigma}c_{j,\sigma} , \quad \sigma = \pm 1
\]

(1.7)
and their hermitean conjugates.

We have shown in [1] that $H^0$ can be written as minus the sum over graded permutation operators

$$H^0 = - \sum_{j=1}^{L} \Pi^{j+1}_{j,j+1},$$

(1.8)

where the operator $\Pi^{j+1}_{j,j+1}$ permutes the four possible configurations (1.1) between the sites $j$ and $j+1$, picking up a minus sign if both of the permuted configurations are fermionic, i.e.

$$\Pi^{j+1}_{j,j+1} |0\rangle_j \times |0\rangle_{j+1} = |0\rangle_j \times |0\rangle_{j+1}$$

$$\Pi^{j+1}_{j,j+1} |\uparrow\downarrow\rangle_j \times |\downarrow\uparrow\rangle_{j+1} = |\downarrow\uparrow\rangle_j \times |\downarrow\uparrow\rangle_{j+1}$$

$$\Pi^{j+1}_{j,j+1} |\sigma\rangle_j \times |\sigma\rangle_{j+1} = |\sigma\rangle_j \times |\sigma\rangle_{j+1}$$

(1.9)

$$\Pi^{j+1}_{j,j+1} |\tau\rangle_j \times |\sigma\rangle_{j+1} = - |\sigma\rangle_j \times |\tau\rangle_{j+1}, \quad \sigma, \tau = \uparrow, \downarrow$$

etc.

It is clear that this form of interaction conserves the individual numbers $N^\uparrow$ and $N^\downarrow$ of electrons with spin up and spin down, and the numbers $N_l$ and $N_h$ of doubly occupied (“local electron pairs”) and empty sites (“holes”). We will choose the following conventions throughout this paper

- $N^\uparrow = \text{number of single electrons with spin up}$
- $N^\downarrow = \text{number of single electrons with spin down}$
- $N_e = N^\uparrow + N^\downarrow = \text{number of single electrons}$
- $N_l = \text{number of local electron pairs}$
- $N_h = \text{number of holes}$
- $N_b = N_h + N_l = \text{number of “bosons”}$.

The outline of this paper is as follows:

In section 2 we perform a detailed construction of the Algebraic Bethe Ansatz of the model. We derive six different forms for the Bethe Ansatz Equations (BAE) and the eigenvalues of the transfer matrix. The graded Quantum Inverse Scattering Method (QISM), discussed in section 2, enables us to obtain expressions for the hamiltonian and (an infinite number of) higher conservation laws at the quantum level. These conserved charges are of interest, because physical interactions, although of short range, are not generally well approximated by interactions involving only nearest neighbours. The charges under consideration involve interactions of longer range (next nearest neighbours, next next nearest etc.) and can be added to the hamiltonian without spoiling the integrability of the model. Thus it is possible to construct integrable models with longer range interactions by using higher conservation laws[6]. The first nontrivial higher integral of motion is for example given by the expression

$$H_{(3)} = i \sum_{k=1}^{L} [H^0_{k,k+1}, H^0_{k-1,k}]$$

(1.10)
where \( H_{k,k+1}^0 \) is the density of the hamiltonian defined in (1.2). Section 3 is devoted to the derivation of explicit formulas for higher conservation laws.

2. Graded Quantum Inverse Scattering Method

In this section we discuss how to embed the hamiltonian (1.3) into the framework of the Quantum Inverse Scattering Method (QISM). For supersymmetric theories it is necessary to modify the QISM along the lines discussed by Kulish and Sklyanin in [7,8]. In [9] we gave a summary of the “graded” version of the QISM, so that we will constrain ourselves to a very brief discussion below and refer to [9] for more details. Starting point of the QISM is an \( R \)-matrix, which obeys a graded Yang Baxter equation. From this \( R \)-matrix we then construct a family of commuting transfer matrices, describing a “fundamental” spin model. The transfer matrix is the generating functional of an infinite number of conservation laws, the second of which we show to be the hamiltonian (1.3). Finally we construct a set of simultaneous eigenstates of the transfer matrix and the hamiltonian, using a nested Algebraic Bethe Ansatz (NABA)\(^7\). Due to the grading there exist \( a \) priori \( 4! = 24 \) choices for the \( R \)-matrix, all of them describing the same physical system, but leading to different (yet equivalent as shown in Appendix B) forms of the NABA. As the model has the two trivial discrete symmetries of “spin reflection” (interchange of spin up and spin down electrons) and of interchange of empty sites and doubly occupied sites, there are essentially only six different solutions\(^2\). One of the six solutions has been previously obtained in [7]. By the same kind of reasoning one can show that there are \( \frac{(n+m)!}{n! \, m!} \) Bethe Ansatz solutions for a permutation-type model with \( u(m|n) \) symmetry.

2.1. Yang Baxter Equation

Consider the graded linear space \( V^{(m|n)} \), where \( m \) and \( n \) denote the dimensions of the “even” and “odd” parts respectively. A basis \( \{e_1, \ldots, e_{m+n}\} \) of \( V^{(m+n)} \) consists of \( m+n \) vectors \( e_a \) with Grassmann parities \( \epsilon_a \) such that for \( m \) vectors \( \epsilon_a = 0 \) and for \( n \) vectors \( \epsilon_a = 1 \). The grading of the matrix elements \( M_{ab} \) of linear operators \( M \) on \( V^{(m|n)} \) (in the basis \( \{e_1, \ldots, e_{m+n}\} \)) is given by \( \epsilon_{M_{ab}} = \epsilon_a + \epsilon_b \). The supertrace of \( M \) is defined as

\[
str(M) = \sum_{a=1}^{m+n} (-1)^{\epsilon_a} M_{aa} . \tag{2.1}
\]

The graded tensor product space \( V^{(m|n)} \otimes V^{(m|n)} \) is given in terms of its basis vectors \( \{e_a \otimes e_b | a, b = 1, \ldots, m+n\} \) as follows

\[
v \otimes w = (e_a v_a) \otimes (e_b w_b) = (e_a \otimes e_b) v_a w_b (-1)^{\epsilon_a \epsilon_b} . \tag{2.2}
\]

\(^2\)Each of these solutions generates a “multiplet” of four NABA via the substitutions \( N_t \leftrightarrow N_{\bar{t}} \) and \( N_l \leftrightarrow N_{\bar{t}} \).
The action of the linear operator \( F \otimes G \) on the vector \( v \otimes w \) in \( V^{(m|n)} \otimes V^{(m|n)} \) is defined by

\[
(F \otimes G)(v \otimes w) = F(v) \otimes G(w) ,
\]

which results in matrix elements of the form

\[
(F \otimes G)^{ab}_{cd} = F_{ab}G_{cd} \ (-1)^{\epsilon(a+b)} .
\]

The identity operator \( I \) in \( V^{(m|n)} \otimes V^{(m|n)} \) and the matrix \( \Pi \) that permutes the individual linear spaces in the tensor product space are given by

\[
I(v \otimes w) = (v \otimes w), \quad (I)^{a_1b_1}_{a_2b_2} = \delta_{a_1a_2} \delta_{b_1b_2} \quad (\Pi)^{a_1b_1}_{a_2b_2} = \delta_{a_1a_2} \delta_{b_1b_2} \ (-1)^{\epsilon(a_1)\epsilon(b_2)} .
\]

A matrix \( R(\lambda) \) (depending on a spectral parameter \( \lambda \)) is said to fulfill a graded Yang-Baxter-equation, if the following identity on \( V^{(m|n)} \otimes V^{(m|n)} \otimes V^{(m|n)} \) holds

\[
(I \otimes R(\lambda - \mu)) (R(\lambda) \otimes I) (I \otimes R(\mu)) = (R(\mu) \otimes I) (I \otimes R(\lambda)) (R(\lambda - \mu) \otimes I) .
\]

In components this identity reads

\[
R(\lambda - \mu)^{a_2c_2}_{a_3c_3} R(\lambda)^{a_1b_1}_{c_2c_2} R(\mu)^{d_2b_2}_{c_2c_2} = R(\mu)^{a_1c_1}_{a_2c_2} R(\lambda)^{c_2d_2}_{c_2c_2} R(\lambda - \mu)^{c_1b_1}_{d_2d_2} .
\]

Note that despite the fact that the tensor product in (2.6) carries a grading, there are no extra signs in (2.7) compared to the nongraded case. It is easily checked that the R-matrix

\[
R(\lambda) = b(\lambda)I + a(\lambda)\Pi
\]

\[
a(\lambda) = \frac{\lambda}{\lambda + i} , \quad b(\lambda) = \frac{i}{\lambda + i} .
\]

fulfills equation (2.7).

2.2. Construction of the Transfer Matrix

From (2.7) one can derive the equation

\[
R_{12}(\lambda - \mu) (\Pi_{13}R_{13}(\lambda)) \otimes (\Pi_{23}R_{23}(\mu)) = (\Pi_{13}R_{13}(\mu)) \otimes (\Pi_{23}R_{23}(\lambda)) R_{12}(\lambda - \mu) ,
\]

where the indices 1, 2, 3 indicate in which of the spaces \( V^{(m|n)} \) in the tensor product space \( V^{(m|n)} \otimes V^{(m|n)} \otimes V^{(m|n)} \) the matrices act nontrivially. The tensor product in (2.9) is between the spaces 1 and 2. We now call the third space “quantum space” and the first two spaces “matrix spaces”. The physical interpretation of the quantum space is as the Hilbert space over a single site of a one-dimensional lattice. We now consider the situation, where intertwining relations of the type (2.9) hold for all sites of a lattice of length \( L \). The quantum
space index “3” now gets replaced by an index labelling the number of the site. We define
the $L$-operator (on site $n$) as a linear operator on $\mathcal{H}_n \otimes V_{\text{matrix}}^{(m|n)}$ (where $\mathcal{H}_n \simeq V^{(m|n)}$ is the
Hilbert space over the $n^{th}$ site, and $V_{\text{matrix}}^{(m|n)}$ is a matrix space)

$$L_n(\lambda)^{ab}_{\alpha\beta} = \prod_{\alpha_1}^{\alpha} R(\lambda)^{c^{\beta}}_{\gamma\beta} = (b(\lambda)\Pi + a(\lambda)I)^{ab}_{\alpha\beta}$$ \quad (2.10)

$L_n$ is a quantum operator valued $(m+n) \times (m+n)$ matrix, with quantum operators acting
nontrivially in the $n^{th}$ quantum space (of the direct product Hilbert space over the complete
lattice $\otimes_{j=1}^{L} \mathcal{H}_j$). The greek indices are the “quantum indices” and the roman indices are
the “matrix indices”. Equation (2.9) for the $n^{th}$ quantum space can now be rewritten as the operator equation

$$R(\lambda - \mu) \ (L_n(\lambda) \otimes L_n(\mu)) = (L_n(\mu) \otimes L_n(\lambda)) \ R(\lambda - \mu)$$ \quad (2.11)

Here the graded tensor product is between the two matrix spaces and $R$ only acts in the
matrix spaces. The intertwining relation (2.11) enables us to construct an integrable spin
model as follows.

We first define the monodromy matrix $T_L(\lambda)$ as the matrix product over the $L$-operators on
all sites of the lattice, i.e.

$$T_L(\lambda) = L_L(\lambda) L_{L-1}(\lambda) \ldots L_1(\lambda)$$

$((T_L(\lambda))^{ab}_{\alpha_1 \ldots \alpha_L})^{\beta_1 \ldots \beta_L} = L_L(\lambda)^{ac_L}_{\alpha_1 \alpha_L} L_{L-1}(\lambda)^{c_L c_{L-1}}_{\alpha_L -1 \beta_{L-1}} \ldots L_1(\lambda)^{c_{L^2}}_{\alpha_1 \beta_1} \times \quad (2.12)$

$$\times (-1)^{\sum_{j=2}^{L} (\epsilon_{\alpha_j} + \epsilon_{\beta_j}) \sum_{i=1}^{j-1} \epsilon_{\alpha_i}} .$$

$T_L(\lambda)$ is a quantum operator valued $(m+n) \times (m+n)$ matrix that acts nontrivially in the
graded tensor product of all quantum spaces of the lattice and by construction fulfills the
same intertwining relation as the $L$-operators (as can be proven by induction over the length
of the lattice)

$$R(\lambda - \mu) \ (T_L(\lambda) \otimes T_L(\mu)) = (T_L(\mu) \otimes T_L(\lambda)) \ R(\lambda - \mu)$$ \quad (2.13)

The transfer matrix $\tau(\lambda)$ of the integrable spin model is now given as the matrix supertrace
of the monodromy matrix

$$\tau(\lambda) = \text{str}(T_L(\lambda)) = \sum_{a=1}^{m+n} (-1)^{s_a} (T_L(\lambda))^{aa}_{a}$$ \quad (2.14)

As a consequence of (2.13) transfer matrices with different spectral parameters commute.
This condition implies that the transfer matrix is the generating functional of the hamiltonian
and of an infinite number of “higher” conservation laws of the model.
2.3. Trace Identities

Taking logarithmic derivatives of the transfer matrix at a special value of the spectral parameter, one can generate higher conservation laws \[^{[13]}\]. For the transfer matrix constructed from the \(R\)-matrix (2.8) along the lines discussed in section 2.2., the corresponding hamiltonian is obtained by taking the first logarithmic derivative at zero spectral parameter

\[
H^{(2)} = -i \left. \frac{\partial \log(\tau(\lambda))}{\partial \lambda} \right|_{\lambda=0} = -\sum_{k=1}^{L} (\Pi^{k,k+1} - 1). \tag{2.15}
\]

The proof of this identity can be carried out in the same way as for the ungraded case, the main difference being the grading of the tensor product of the quantum spaces (see (2.12)). By shifting the energy eigenvalues by a constant we obtain the expression (1.8) for the hamiltonian (1.3)

\[
H^0 = -i \left. \frac{\partial \log(\tau(\lambda))}{\partial \lambda} \right|_{\lambda=0} - L = H^{(2)} - L, \tag{2.16}
\]

if we choose our underlying graded vector space to have signature \((2,2)\), i.e. to have a basis with two fermionic and two bosonic states. This shows that the transfer matrix constructed from the \(L\)-operator (2.10) and \(R\)-matrix (2.8) is indeed the correct transfer matrix for the hamiltonian (1.3). Higher conservation laws are obtained as the coefficients of the power series

\[
\log \left(\frac{\tau(\lambda)(\tau(0))^{-1}}{}\right) = \sum_{k=1}^{\infty} \frac{i \lambda^k}{k!} H^{(k+1)}. \tag{2.17}
\]

There exists however a simpler method for the construction of higher integrals of motion than taking logarithmic derivatives, which we will discuss in section 3. For fundamental spin models (and thus for the model at hand) the transfer matrix at the special value of the spectral parameter (which is zero for our case) is equal to the translation operator. Therefore we can construct the momentum operator from the logarithm of the transfer matrix at zero spectral parameter

\[
P = -i \log (\tau(0)) \tag{2.18}
\]

By construction the momentum operator commutes with the hamiltonian \(H^0\) and all higher conservation laws \(H^{(k+1)}\), i.e. \([P, H^0] = 0 = [P, H^{(k+1)}]\).

2.4. Algebraic Bethe Ansatz with a bosonic background (FFBB grading)

Due to the fact that there are four different configurations per site for the model defined by (1.2), the Hilbert space at the \(k^{th}\) site of the lattice is isomorphic to \(\mathbb{C}^4\) and is spanned by the four vectors \(e_1 = (1 \ 0 \ 0 \ 0)^T\), \(e_2 = (0 \ 1 \ 0 \ 0)^T\), \(e_3 = (0 \ 0 \ 1 \ 0)^T\), and \(e_4 = (0 \ 0 \ 0 \ 1)^T\). In this section we consider a grading such that \(e_1\) and \(e_2\) are fermionic.
Using (2.12) and (2.23) we determine the action of the monodromy matrix on the reference state (representing spin down/spin up electrons respectively) and $e_3$ and $e_4$ are bosonic (doubly occupied/empty site). In terms of the Grassmann parities this means that $\epsilon_1 = \epsilon_2 = 1$ and $\epsilon_3 = \epsilon_4 = 0$. We pick the reference state in the $k^{th}$ quantum space $|0\rangle_k$, and the vacuum $|0\rangle$ of the complete lattice of $L$ sites, to be the bosonic completely unoccupied state, i.e.

$$
|0\rangle_n = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad |0\rangle = \otimes_{n=1}^{L} |0\rangle_n .
$$

(2.19)

This choice of grading implies that the $L$-operator (2.10) is given by the following expression

$$
L_n(\lambda) = \begin{pmatrix}
a(\lambda) - b(\lambda)e_{n}^{11} & -b(\lambda)e_{n}^{21} & b(\lambda)e_{n}^{31} & b(\lambda)e_{n}^{41} \\
-b(\lambda)e_{n}^{12} & a(\lambda) - b(\lambda)e_{n}^{22} & b(\lambda)e_{n}^{32} & b(\lambda)e_{n}^{42} \\
b(\lambda)e_{n}^{13} & b(\lambda)e_{n}^{23} & a(\lambda) + b(\lambda)e_{n}^{33} & b(\lambda)e_{n}^{43} \\
b(\lambda)e_{n}^{14} & b(\lambda)e_{n}^{24} & b(\lambda)e_{n}^{34} & a(\lambda) + b(\lambda)e_{n}^{44}
\end{pmatrix},
$$

(2.20)

where $e_{n}^{ab}$ are quantum operators in the $n^{th}$ quantum space with matrix representation $(e_{n}^{ab})_{\alpha\beta} = \delta_{\alpha\alpha}\delta_{\beta\beta}$. The monodromy matrix (2.12) is a quantum operator valued $4 \times 4$ matrix, which we represent as

$$
T_L(\lambda) = L_L(\lambda)L_{L-1}(\lambda)\ldots L_1(\lambda) = 
\begin{pmatrix}
A_{11}(\lambda) & A_{12}(\lambda) & A_{13}(\lambda) & B_1(\lambda) \\
A_{21}(\lambda) & A_{22}(\lambda) & A_{23}(\lambda) & B_2(\lambda) \\
A_{31}(\lambda) & A_{32}(\lambda) & A_{33}(\lambda) & B_3(\lambda) \\
C_1(\lambda) & C_2(\lambda) & C_3(\lambda) & D(\lambda)
\end{pmatrix}.
$$

(2.21)

The transfer matrix is then given by

$$
\tau(\mu) = \text{str}(T_L(\mu)) = -A_{11}(\mu) - A_{22}(\mu) + A_{33}(\mu) + D(\mu).
$$

(2.22)

The action of $L_k(\lambda)$ on the reference state on the $k^{th}$ site can be easily determined by using the matrix representation of the $e_{k}^{ab}$ in (2.20) and the vectorial representation of $|0\rangle_k$ (2.19)

$$
L_k(\lambda)|0\rangle_k = \begin{pmatrix} a(\lambda) & 0 & 0 & 0 \\
0 & a(\lambda) & 0 & 0 \\
0 & 0 & a(\lambda) & 0 \\
b(\lambda)e_{k}^{14} & b(\lambda)e_{k}^{24} & b(\lambda)e_{k}^{34} & 1
\end{pmatrix} |0\rangle_k .
$$

(2.23)

Using (2.12) and (2.23) we determine the action of the monodromy matrix on the reference state to be

$$
T_L(\lambda)|0\rangle = \begin{pmatrix}
(a(\lambda))^L & 0 & 0 & 0 \\
0 & (a(\lambda))^L & 0 & 0 \\
0 & 0 & (a(\lambda))^L & 0 \\
C_1(\lambda) & C_2(\lambda) & C_3(\lambda) & 1
\end{pmatrix} |0\rangle .
$$

(2.24)
We will now construct a set of eigenstates of the transfer matrix using the technique of
the NABA. Inspection of (2.24) reveals that \( C_i(\lambda) \) are creation operators with respect to our
choice of reference state. This observation leads us to the following Ansatz for the eigenstates
of \( \tau(\mu) \)

\[
|\lambda_1, \ldots, \lambda_n|F\rangle = C_{a_1}(\lambda_1) C_{a_2}(\lambda_2) \ldots C_{a_n}(\lambda_n) |0\rangle F^{a_n \ldots a_1}, \tag{2.25}
\]

where the indices \( a_j \) run over the values 1, 2, 3, and \( F^{a_n \ldots a_1} \) is a function of the spectral
parameters \( \lambda_j \). The action of the transfer matrix on states of the form (2.25) is determined
by (2.24) and the intertwining relations (2.13). The components of the intertwining relations
(2.13) needed for the construction of the NABA are

\[
A_{ab}(\mu) C_c(\lambda) = (-1)^{\epsilon_a \epsilon_b} \frac{r(\mu - \lambda)_{cd}^{bc}}{a(\mu - \lambda)} C_d(\lambda) A_{ad}(\mu) \\
- (-1)^{\epsilon_a \epsilon_b} \frac{b(\mu - \lambda)}{a(\mu - \lambda)} C_b(\mu) A_{ac}(\lambda) \tag{2.26}
\]

\[
D(\mu) C_c(\lambda) = \frac{1}{a(\lambda - \mu)} C_c(\lambda) D(\mu) - \frac{b(\lambda - \mu)}{a(\lambda - \mu)} C_\mu(\mu)D(\lambda)
\]

\[
C_{a_1}(\lambda_1) C_{a_2}(\lambda_2) = r(\lambda_1 - \lambda_2)_{b_2 a_1}^{b_1 a_2} C_{b_2}(\lambda_2) C_{b_1}(\lambda_1), \tag{2.27}
\]

where

\[
\begin{align*}
 r(\mu)_{cd}^{ab} &= b(\mu)\delta_{ab}\delta_{cd} + a(\mu)\delta_{ac}\delta_{bd}(-1)^{\epsilon_a \epsilon_c} \\
 &= b(\mu)I^{(1)ab}_{cd} + a(\mu)\Pi^{(1)ab}_{cd}.
\end{align*}
\]

Here \( \Pi^{(1)ab}_{cd} \) is the 9 \( \times \) 9 permutation matrix corresponding to the grading \( \epsilon_1 = \epsilon_2 = 1, \epsilon_3 = 0 \). \( r(\mu) \) can be seen to fulfill a (graded) Yang-Baxter equation on its own

\[
r(\lambda - \mu)_{a_3 c_3}^{a_2 c_2} r(\lambda)_{c_2 d_2}^{a_1 b_1} r(\mu)_{d_2 b_2}^{c_3 b_3} = r(\mu)_{a_2 c_2}^{a_1 c_1} r(\lambda)_{a_3 b_3}^{c_2 d_2} r(\lambda - \mu)_{d_2 b_2}^{c_1 b_1}, \tag{2.28}
\]

and can be identified with the \( R \)-matrix of a fundamental spin model describing two species
of fermions and one species of bosons. Using (2.26) we find that the diagonal elements of
the monodromy matrix act on the states (2.25) as follows

\[
D(\mu)|\lambda_1, \ldots, \lambda_n|F\rangle = \prod_{j=1}^{n} \frac{1}{a(\lambda_j - \mu)}|\lambda_1, \ldots, \lambda_n|F\rangle
\]

\[
+ \sum_{k=1}^{n} \left( \frac{\Lambda_k}{\lambda_k} \right)_{a_1 \ldots a_n}^{b_1 \ldots b_n} C_{b_k}(\mu) \prod_{j=1}^{n} C_{b_j}(\lambda_j)|0\rangle F^{a_n \ldots a_1}, \tag{2.29}
\]

\[
(-A_{11}(\mu) - A_{22}(\mu) + A_{33}(\mu))|\lambda_1, \ldots, \lambda_n|F\rangle =
\]

\[
= (a(\mu))^L \prod_{j=1}^{n} \frac{1}{a(\mu - \lambda_j)} \prod_{l=1}^{n} C_{b_l}(\lambda_l)|0\rangle \tau^{(1)}(\mu)_{b_1 \ldots b_n}^{a_1 \ldots a_n} F^{a_n \ldots a_1}
\]

\[
+ \sum_{k=1}^{n} \left( \frac{\Lambda_k}{\lambda_k} \right)_{a_1 \ldots a_n}^{b_1 \ldots b_n} C_{b_k}(\mu) \prod_{j=1}^{n} C_{b_j}(\lambda_j)|0\rangle F^{a_n \ldots a_1}, \tag{2.30}
\]

9
\( \tau^{(1)}(\mu)^{b_1 \ldots b_n}_{a_1 \ldots a_n} = \text{str}(T^{(1)}_n(\mu)) \)

\[ = \text{str}(L^{(1)}_n(\mu - \lambda_n)L^{(1)}_{n-1}(\mu - \lambda_{n-1}) \ldots L^{(1)}_2(\mu - \lambda_2)L^{(1)}_1(\mu - \lambda_1)) , \]

and

\[ L^{(1)}_k(\lambda) = b(\lambda)\Pi^{(1)} + a(\lambda)I^{(1)} \]

\[ = \Pi^{(1)} r(\lambda) = \begin{pmatrix} a(\lambda) - b(\lambda)e^{11}_n & -b(\lambda)e^{21}_n & b(\lambda)e^{31}_n \\ -b(\lambda)e^{12}_n & a(\lambda) - b(\lambda)e^{22}_n & b(\lambda)e^{32}_n \\ b(\lambda)e^{13}_n & b(\lambda)e^{23}_n & a(\lambda) + b(\lambda)e^{33}_n \end{pmatrix} . \] (2.31)

Note that the tensor products between the \( n \) quantum spaces in the expression for the operator \( \tau^{(1)}(\mu) \) are again graded, which results in minus signs given by (2.12). \( L^{(1)} \) and \( r(\mu) \) can be interpreted as \( L \)-operator and \( R \)-matrix of a fundamental spin model (\( r \) fulfills the Yang-Baxter equation (2.28)), describing two species of fermions and one species of bosons. Hence \( T^{(1)}_n(\mu) \) and \( \tau^{(1)}(\mu) \) are the monodromy matrix and transfer matrix of the corresponding inhomogeneous model. This model can be identified with the inhomogeneous supersymmetric \( t-J \) model on a lattice with \( n \) sites. The Algebraic Bethe Ansätze for the supersymmetric \( t-J \) model were constructed in detail in [9], so that we will restrict ourselves to an abbreviated discussion and refer to [9] for derivations of some of the results. Inspection of (2.29) and (2.30) together with (2.22) shows that the eigenvalue condition

\[ \tau(\mu)|\lambda_1, \ldots, \lambda_n\rangle F) = \nu(\mu) |\lambda_1, \ldots, \lambda_n\rangle F) \] (2.32)

leads to the requirements that \( F \) ought to be an eigenvector of the “nested” transfer matrix \( \tau^{(1)}(\mu) \), and that the “unwanted terms” cancel, i.e.

\[ \left( (\Lambda_k)^{b_1 \ldots b_n}_{a_1 \ldots a_n} + (\Lambda_k)^{b_1 \ldots b_n}_{a_1 \ldots a_n} \right) F^{a_n \ldots a_1} = 0 . \] (2.33)

The notation \( \nu(\mu) \) for the eigenvalues of the transfer matrix \( \tau(\mu) \) is to be considered as a shorthand as the eigenvalues depend also on the spectral parameters \( \lambda_j \) and coefficients \( F \). The quantities \( \Lambda_k \) and \( \Lambda_k \) are computed in Appendix A. Using their explicit expressions in (2.33) we obtain the following conditions on the spectral parameters \( \lambda_j \) and coefficients \( F \), which are necessary for (2.32) to hold

\[ (a(\lambda_k))^{-L} \prod_{l=1}^{n} \frac{a(\lambda_k - \lambda_l)}{a(\lambda_l - \lambda_k)} F^{b_1 \ldots b_n}_{a_1 \ldots a_n} = \tau^{(1)}(\lambda_k)^{b_1 \ldots b_n}_{a_1 \ldots a_n} F^{a_n \ldots a_1} , \quad k = 1, \ldots, n . \] (2.34)

This completes the first step of the NABA. In the next step we will now solve the first nesting. The condition that \( F \) ought to be an eigenvector of \( \tau^{(1)}(\mu) \) requires the diagonalisation of \( \tau^{(1)}(\mu) \), which can be carried out by a second, “nested” Bethe Ansatz. From (2.28) and (2.31) the following intertwining relation is easily derived

\[ r(\lambda - \mu) (T^{(1)}_n(\lambda) \otimes T^{(1)}_n(\mu)) = (T^{(1)}_n(\mu) \otimes T^{(1)}_n(\lambda)) r(\lambda - \mu) . \] (2.35)
If we write
\[ T_n^{(1)}(\mu) = \begin{pmatrix} A_{11}^{(1)}(\mu) & A_{12}^{(1)}(\mu) & B_{1}^{(1)}(\mu) \\ A_{21}^{(1)}(\mu) & A_{22}^{(1)}(\mu) & B_{2}^{(1)}(\mu) \\ C_{1}^{(1)}(\mu) & C_{2}^{(1)}(\mu) & D^{(1)}(\mu) \end{pmatrix}, \]
then (2.35) and (2.27) imply that
\[ \tau^{(1)}(\mu) = -A_{11}^{(1)}(\mu) - A_{22}^{(1)}(\mu) + D^{(1)}(\mu), \] (2.36)

Here the roman indices take the values 1, 2, both of which are fermionic ($\epsilon_1 = 1 = \epsilon_2$). The $R$-matrix $r^{(1)}(\mu)$ can again be proven to fulfill a Yang-Baxter equation of the form (2.28).

As the reference state for the first nesting we pick
\[ |0\rangle_{k}^{(1)} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad |0\rangle^{(1)} = \otimes_{k=1}^{n} |0\rangle_{k}^{(1)}. \] (2.38)

The action of the nested monodromy matrix $T_n^{(1)}(\mu)$ on the reference state $|0\rangle^{(1)}$ is determined by (2.31) and we find
\[ A_{11}^{(1)}(\mu)|0\rangle^{(1)} = A_{22}^{(1)}(\mu)|0\rangle^{(1)} = \prod_{j=1}^{n} a(\mu - \lambda_j)|0\rangle^{(1)}, \] (2.39)
\[ D^{(1)}(\mu)|0\rangle^{(1)} = |0\rangle^{(1)}. \]

We now make the following Ansatz for the eigenstates of $\tau^{(1)}(\mu)$
\[ |\lambda_{1}^{(1)}, \ldots, \lambda_{n_1}^{(1)} \rangle = C_{b_{1}}^{(1)}(\lambda_{1}^{(1)}) C_{b_{2}}^{(1)}(\lambda_{2}^{(1)}) \ldots C_{b_{n_1}}^{(1)}(\lambda_{n_1}^{(1)}) |0\rangle^{(1)} G^{b_{a_1} \ldots b_{1}}. \] (2.40)

Here the indizes $b_i$ can take the two values 1 and 2. These states can be related to the coefficients $F_{a_{1} \ldots a_{1}}$ in the following way: The state $|\lambda_{1}^{(1)}, \ldots, \lambda_{n_1}^{(1)} \rangle$ “lives” on a lattice of $n$ sites and is thus an element of a direct product over $n$ Hilbert spaces. In components it reads $|\lambda_{1}^{(1)} \ldots \lambda_{n_1}^{(1)}\rangle_{a_{1} \ldots a_{1}}$, which can be directly identified with $F_{a_{1} \ldots a_{1}}$.

The action of $\tau^{(1)}(\mu)$ on the states (2.40) can be evaluated with the help of (2.37) and (2.39)
\[ D^{(1)}(\mu)|\lambda_{1}^{(1)}, \ldots, \lambda_{n_1}^{(1)} \rangle = \prod_{j=1}^{n_1} \frac{1}{a(\lambda_{j}^{(1)} - \mu)} |\lambda_{1}^{(1)}, \ldots, \lambda_{n_1}^{(1)} \rangle \]
\[ + \sum_{k=1}^{n_1} \left( \tilde{A}_{k}^{(1)} \right)_{a_{1} \ldots a_{n_1}} C_{b_{k}}^{(1)}(\mu) \prod_{j=1}^{n_1} C_{b_{j}}^{(1)}(\lambda_{j}^{(1)}) |0\rangle^{(1)} G^{a_{n_1} \ldots a_{1}}, \] (2.41)
\(-A_{11}^{(1)}(\mu) - A_{22}^{(1)}(\mu))|\lambda_1^{(1)}, \ldots, \lambda_{n_1}^{(1)}|G\) =
\[
= \prod_{i=1}^{n} a(\mu - \lambda_i) \prod_{j=1}^{n_1} \frac{1}{a(\mu - \lambda_i^{(1)})} \prod_{l=1}^{n_1} C_{b_{1l}}^{(1)}(\lambda_i^{(1)}) |0\rangle^{(1)} \tau^{(2)}(\mu)_{a_{1a_{n_1}}}^{b_{1a_{n_1}}} G_{a_{1a_{n_1}}}^{a_{1a_{n_1}}} (2.42)
\]
\[= \sum_{k=1}^{n_1} \left( \lambda_k^{(1)} \right)^{b_{1a_{n_1}}} C_{b_{k}^{(1)}}(\mu) \prod_{j=1}^{n_1} C_{b_{j}}^{(1)}(\lambda_j^{(1)}) |0\rangle^{(1)} G_{a_{1a_{n_1}}}^{a_{1a_{n_1}}} .
\]

Here \(\tau^{(2)}(\mu)\) is the transfer matrix of the second nesting
\[\tau^{(2)}(\mu)_{a_{1a_{n_1}}}^{b_{1a_{n_1}}} = \text{str}(T^{(2)}(\mu)) = \text{str}(L^{(2)}(\mu - \lambda_1^{(1)})T^{(2)}(\mu - \lambda_{1n_1}^{(1)}) \cdots L^{(2)}(\mu - \lambda_2^{(1)})L^{(2)}(\mu - \lambda_{1n_1}^{(1)}))\),
\]
where
\[L_k^{(2)}(\lambda) = b(\lambda)\Pi^{(2)} + a(\lambda)I^{(2)} = \Pi^{(2)}r^{(1)}(\lambda) = \begin{pmatrix} a(\lambda) - b(\lambda)e_{k1}^{11} & -b(\lambda)e_{k1}^{21} \\ -b(\lambda)e_{k1}^{12} & a(\lambda) - b(\lambda)e_{k1}^{22} \end{pmatrix} .
\]

The operator \(\tau^{(2)}(\mu)\) can be interpreted as the transfer matrix of an inhomogeneous model on \(n_1\) sites describing two species of fermions. It is easily seen from (2.42) that for the states (2.40) to be eigenstates of the \(\tau^{(1)}(\mu)\), \(G\) must be an eigenvector of \(\tau^{(2)}(\mu)\) with eigenvalue \(\nu^{(2)}(\mu)\). From (2.41) and (2.42) one can read off the eigenvalues of \(\tau^{(1)}(\mu)\)
\[\tau^{(1)}(\mu)|\lambda_1^{(1)}, \ldots, \lambda_{n_1}^{(1)}|G\rangle = \nu^{(1)}(\mu)|\lambda_1^{(1)}, \ldots, \lambda_{n_1}^{(1)}|G\rangle =
\[
\left( \prod_{i=1}^{n} \frac{1}{a(\mu - \lambda_i^{(1)})} \prod_{j=1}^{n_1} a(\mu - \lambda_j)\nu^{(2)}(\mu) + \prod_{i=1}^{n_1} \frac{1}{a(\lambda_i^{(1)} - \mu)} \right) |\lambda_1^{(1)}, \ldots, \lambda_{n_1}^{(1)}|G\rangle .
\]

The unwanted terms \(\tilde{\Lambda}_k^{(1)}\) and \(\tilde{\Lambda}_k^{(1)}\) are computed in Appendix A and their cancellation (which ensures that the states (2.40) are eigenstates of the transfer matrix \(\tau^{(1)}(\mu)\) ) leads to the following set of Bethe Ansatz equations (BAE) for the first nesting
\[\prod_{j=1}^{n} \frac{1}{a(\lambda_j^{(1)} - \mu)} \prod_{l=1}^{n_1} a(\lambda_l^{(1)} - \lambda_j^{(1)}) G_{a_{1a_{n_1}}}^{b_{1a_{n_1}}} = \tau^{(2)}(\lambda_k^{(1)})_{a_{1a_{n_1}}}^{b_{1a_{n_1}}} G_{a_{1a_{n_1}}}^{a_{1a_{n_1}}} , \ k = 1, \ldots, n_1 .
\]

The diagonalisation of \(\tau^{(2)}(\mu)\) can be performed by a third Bethe Ansatz. From the Yang-Baxter relation for \(r^{(2)}(\mu)\) one can derive the following intertwining relation
\[r^{(1)}(\lambda - \mu) \left( T^{(2)}(\lambda) \otimes T^{(2)}(\mu) \right) = \left( T^{(2)}(\mu) \otimes T^{(2)}(\lambda) \right) r^{(1)}(\lambda - \mu) .
\]

The components of (2.46) needed for the construction of an Algebraic Bethe Ansatz are
\[D^{(2)}(\mu)C^{(2)}(\lambda) = \frac{1}{a(\mu - \lambda)} C^{(2)}(\lambda)D^{(2)}(\mu) + \frac{b(\mu - \lambda)}{a(\lambda - \mu)} C^{(2)}(\mu)D^{(2)}(\lambda)
\]
\[A^{(2)}(\mu)C^{(2)}(\lambda) = \frac{1}{a(\mu - \lambda)} C^{(2)}(\lambda)A^{(2)}(\mu) + \frac{b(\mu - \lambda)}{a(\mu - \lambda)} C^{(2)}(\mu)A^{(2)}(\lambda)
\]
\[C^{(2)}(\lambda)C^{(2)}(\mu) = C^{(2)}(\mu)C^{(2)}(\lambda) .
\]
As the reference state we pick
\[ |0\rangle_k^{(2)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad |0\rangle^{(2)} = \otimes_{k=1}^{n_1} |0\rangle_k^{(2)}. \] (2.48)

The action of the nested monodromy matrix \( T_{n_1}^{(2)}(\mu) \) on the reference state \( |0\rangle^{(2)} \) is determined by (2.43) and we find
\[
A^{(2)}(\mu)|0\rangle^{(2)} = \prod_{j=1}^{n_1} a(\mu - \lambda_j)|0\rangle^{(2)}
\]
\[
D^{(2)}(\mu)|0\rangle^{(2)} = \prod_{j=1}^{n_1} \left( a(\mu - \lambda_j) - b(\mu - \lambda_j) \right) |0\rangle^{(2)} = \prod_{j=1}^{n_1} \frac{a(\mu - \lambda_j)}{a(\lambda_j - \mu)} |0\rangle^{(2)}. \tag{2.49}
\]

We now make the following Ansatz for the eigenstates of \( \tau^{(2)}(\mu) \)
\[
|\lambda_1^{(2)}, \ldots, \lambda_{n_2}^{(2)}\rangle = C^{(2)}(\lambda_1^{(2)}) C^{(2)}(\lambda_2^{(2)}) \ldots C^{(2)}(\lambda_{n_2}^{(2)}) |0\rangle^{(2)}. \tag{2.50}
\]

The analysis now proceeds analogous to the one for the first nesting. The eigenvalues of \( \tau^{(2)}(\mu) \) on the states (2.50) are found to be
\[
\tau^{(2)}(\mu)|\lambda_1^{(2)}, \ldots, \lambda_{n_2}^{(2)}\rangle = \nu^{(2)}(\mu)|\lambda_1^{(2)}, \ldots, \lambda_{n_2}^{(2)}\rangle =
- \left( \prod_{i=1}^{n_2} \frac{1}{a(\mu - \lambda_i^{(2)})} \prod_{j=1}^{n_1} \frac{a(\mu - \lambda_j^{(1)})}{a(\lambda_j^{(1)} - \mu)} + \prod_{i=1}^{n_2} \frac{1}{a(\lambda_i^{(2)} - \mu)} \prod_{j=1}^{n_1} \frac{a(\mu - \lambda_j^{(1)})}{a(\lambda_j^{(1)} - \mu)} \right) |\lambda_1^{(2)}, \ldots, \lambda_{n_2}^{(2)}\rangle,
\]
under the condition that the spectral parameters \( \lambda_p^{(2)} \) are solutions to the BAE
\[
\prod_{i=1}^{n_1} a(\lambda_i^{(1)} - \lambda_p^{(2)}) = \prod_{j=1}^{n_2} \frac{a(\lambda_j^{(2)} - \lambda_p^{(2)})}{a(\lambda_j^{(2)} - \lambda_j^{(1)})}, \quad p = 1, \ldots, n_2. \tag{2.52}
\]

Using (2.51) in (2.45) \( (G \) is an eigenvector of \( \tau^{(2)}(\lambda_k^{(1)}) \) with eigenvalue \( \nu^{(2)}(\lambda_k^{(1)}) \)) we obtain the BAE for the first nesting. If we then insert (2.51) in (2.44) and use the resulting expression for the eigenvalues of \( \tau^{(1)}(\mu) \) in (2.34), we obtain the complete set of three BAE. If we shift the spectral parameters according to
\[
\bar{\lambda}_k = \lambda_k + \frac{i}{2}, \quad \bar{\lambda}_j^{(1)} = \lambda_j^{(1)} + i, \quad \bar{\lambda}_m^{(2)} = \lambda_m^{(2)} + \frac{i}{2},
\]
these equations read
\[
\left( \frac{\bar{\lambda}_k - \frac{i}{2}}{\bar{\lambda}_k + \frac{i}{2}} \right)^L = \prod_{j=1}^{N_e} \frac{\bar{\lambda}_k - \bar{\lambda}_j^{(1)} + \frac{i}{2}}{\bar{\lambda}_k - \bar{\lambda}_j^{(1)} - \frac{i}{2}} \prod_{l=1}^{N_e+N_i} \frac{\bar{\lambda}_l - \bar{\lambda}_k + \frac{i}{2}}{\bar{\lambda}_l - \bar{\lambda}_k - \frac{i}{2}}, \quad k = 1, \ldots, N_e + N_i
\]
\[
\prod_{k=1}^{N_e+N_i} \frac{\bar{\lambda}_k - \bar{\lambda}_j^{(1)} + \frac{i}{2}}{\bar{\lambda}_k - \bar{\lambda}_j^{(1)} - \frac{i}{2}} = \prod_{m=1}^{N_j} \frac{\bar{\lambda}_m - \bar{\lambda}_j^{(1)} + \frac{i}{2}}{\bar{\lambda}_m - \bar{\lambda}_j^{(1)} - \frac{i}{2}}, \quad j = 1, \ldots, N_e \tag{2.54}
\]
\[
\prod_{l=1}^{N_e} \frac{\bar{\lambda}_l^{(2)} - \bar{\lambda}_m^{(2)} + i}{\bar{\lambda}_l^{(2)} - \bar{\lambda}_m^{(2)} - i} = \prod_{j=1}^{N_i} \frac{\bar{\lambda}_j^{(1)} - \bar{\lambda}_j^{(1)} + \frac{i}{2}}{\bar{\lambda}_j^{(1)} - \bar{\lambda}_j^{(1)} - \frac{i}{2}}, \quad m = 1, \ldots, N_i.
\]
Here we have used that \( n = N_e + N_f = N_\uparrow + N_\downarrow, \) \( n_1 = N_e, \) \( n_2 = N_f, \) which follow from our identification of the four \( \Phi^4 \) basis vectors with the four possible configurations per site (1.1). The eigenvalues of the transfer matrix \( \tau(\mu) \) are computed by using only the wanted terms in (2.30) and (2.29), and inserting (2.44) and (2.51) in the resulting expression

\[
\nu(\mu) = \left( a(\mu) \right)^{N_e + N_f} \left( \prod_{j=1}^{N_e + N_f} \frac{1}{a(\mu - \lambda_j)} \right) \nu^{(1)}(\mu) + \left( \prod_{j=1}^{N_e + N_f} \frac{1}{a(\lambda_j - \mu)} \right) \nu^{(2)}(\mu) + \left( \prod_{i=1}^{N_e} \frac{1}{a(\mu - \lambda_i^{(1)})} \right) \nu^{(3)}(\mu) + \left( \prod_{i=1}^{N_f} \frac{1}{a(\mu - \lambda_i^{(2)})} \right) \nu^{(4)}(\mu) .
\]

(2.55)

Note that the condition of cancellation of the residues of the poles in (2.55) implies the BAE (2.54). The states (2.25) are simultaneous eigenstates of \( H^0 \) and \( \tau(\mu) \), and the energy eigenvalues can be determined from the eigenvalues of the transfer matrix (2.55) by using the trace identity (2.16). We find

\[
E^0(\tilde{\lambda}_1, \ldots, \tilde{\lambda}_n) = -i \left. \frac{\partial \log(\tau(\mu))}{\partial \mu} \right|_{\mu = 0} = \sum_{j=1}^{N_e + N_f} \frac{1}{\lambda_j^2 + \frac{1}{4}} - L = -2 \sum_{j=1}^{N_e + N_f} \cos(k_j) + 2(N_e + N_f) - L ,
\]

(2.56)

where we have reparametrised \( \lambda_j = \frac{1}{2} \cot(k_j) \). The eigenvalues of the momentum operator (2.18) can be obtained by setting \( \mu = 0 \) in the expression (2.55) for the eigenvalues of the transfer matrix, and then taking the logarithm. We find

\[
p(\tilde{\lambda}_1, \ldots, \tilde{\lambda}_n) = -i \sum_{j=1}^{N_e + N_f} \log \left( \frac{\tilde{\lambda}_j + \frac{i}{2}}{\tilde{\lambda}_j - \frac{i}{2}} \right) = \sum_{j=1}^{N_e + N_f} k_j .
\]

(2.57)

2.5. Algebraic Bethe Ansatz for the BBFF grading

The construction of the NABA for this choice of grading follows roughly the same lines as for the FFBB case. Due to the fact that we are “truncating” fermionic lines, there are additional complications on the first two steps of the NABA. The situation is analogous to the one for the BFF grading in the supersymmetric \( t-J \) model, which was discussed in [9]. The necessary steps are all explained in that paper, so that we will constrain ourselves to only a brief sketch of the derivations, and mainly just quote the results.

The grading is chosen in such a way that \( e_3 \) and \( e_4 \) are fermionic (representing spin down/spin up electrons, respectively) and \( e_1 \) and \( e_2 \) are bosonic (doubly occupied/empty
site). In terms of the Grassmann parities this means that $\epsilon_3 = \epsilon_4 = 1$ and $\epsilon_1 = \epsilon_2 = 0$. The reference state in the $k^{th}$ quantum space $|0\rangle_k$ and the vacuum $|0\rangle$ of the complete lattice of $L$ sites are fermionic with all spins up, i.e.

$$|0\rangle_n = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad |0\rangle = \otimes_{n=1}^{L} |0\rangle_n . \quad (2.58)$$

The choice of grading leads to the L-operator

$$L_n(\lambda) = \begin{pmatrix} a(\lambda) + b(\lambda)e_{n1}^{11} & b(\lambda)e_{n2}^{11} & b(\lambda)e_{n3}^{11} & b(\lambda)e_{n4}^{11} \\ b(\lambda)e_{n1}^{12} & a(\lambda) + b(\lambda)e_{n2}^{22} & b(\lambda)e_{n3}^{22} & b(\lambda)e_{n4}^{22} \\ b(\lambda)e_{n1}^{13} & b(\lambda)e_{n2}^{23} & a(\lambda) - b(\lambda)e_{n3}^{33} & -b(\lambda)e_{n4}^{33} \\ b(\lambda)e_{n1}^{14} & b(\lambda)e_{n2}^{24} & -b(\lambda)e_{n3}^{44} & a(\lambda) - b(\lambda)e_{n4}^{44} \end{pmatrix}. \quad (2.59)$$

We again partition the monodromy matrix like in (2.21), which implies the following expression for the transfer matrix

$$\tau(\mu) = str(T_L(\mu)) = A_{11}(\mu) + A_{22}(\mu) - A_{33}(\mu) - D(\mu) . \quad (2.60)$$

The action of the monodromy matrix on the vacuum can be computed using (2.58) and (2.59)

$$T_L(\lambda)|0\rangle = \begin{pmatrix} (a(\lambda))^L & 0 & 0 & 0 \\ 0 & (a(\lambda))^L & 0 & 0 \\ 0 & 0 & (a(\lambda))^L & 0 \\ C_1(\lambda) & C_2(\lambda) & C_3(\lambda) & \left(\frac{a(\lambda)}{a(-\lambda)}\right) \end{pmatrix} |0\rangle . \quad (2.61)$$

As before we identify $C_a$ as creation operators with respect to our choice of reference state and make an Ansatz of the type (2.25) for the eigenstates of the transfer matrix (2.60). The relevant intertwining relations are

$$A_{ab}(\mu) C_c(\lambda) = (-1)^{\epsilon_a \epsilon_b + \epsilon_a + \epsilon_b} r_{BBF}(\mu - \lambda)^{de}_{cd} C_p(\lambda) A_{ad}(\mu) + (-1)^{(\epsilon_a + 1)(\epsilon_b + 1)} \frac{b(\mu - \lambda)}{a(\mu - \lambda)} C_b(\mu) A_{ac}(\lambda) \quad (2.62)$$

$$D(\mu) C_c(\lambda) = \frac{1}{a(\mu - \lambda)} C_c(\lambda) D(\mu) + \frac{b(\mu - \lambda)}{a(\mu - \lambda)} C_c(\mu) D(\lambda) \quad (2.63)$$

$$C_{a1}(\lambda_1) C_{a2}(\lambda_2) = r_{FFB}(\lambda_1 - \lambda_2)^{b_1 b_2}_{b_1 b_2} C_{b_1}(\lambda_2) C_{b_1}(\lambda_1) .$$

where

$$r_{BF}(\mu)_{cd}^{ab} = b(\mu) I_{cd}^{ab} + a(\mu) \Pi_{BF}^{ab} . \quad (2.63)$$

Here $\Pi_{FFB}$ and $\Pi_{BBF}$ are the permutation matrices for the gradings $\epsilon_1 = \epsilon_2 = 1$, $\epsilon_3 = 0$ and $\epsilon_1 = \epsilon_2 = 0$, $\epsilon_3 = 1$ respectively. The intertwining relations (2.62) imply the following
action of the diagonal elements of the monodromy matrix on the states $|\lambda_1, \ldots, \lambda_n,F\rangle = C_{a_1}(\lambda_1) C_{a_2}(\lambda_2) \ldots C_{a_n}(\lambda_n) \ |0\rangle \ F^{a_n \ldots a_1}$

$$D(\mu)|\lambda_1, \ldots, \lambda_n|F\rangle = \prod_{j=1}^{n} \frac{1}{a(\mu - \lambda_j)} \left( \frac{a(\mu)}{a(-\mu)} \right)^L |\lambda_1, \ldots, \lambda_n|F\rangle$$
$$+ \sum_{k=1}^{n} (\bar{A}_k)^{b_1 \ldots b_n}_{a_1 \ldots a_n} C_{b_k}(\mu) \prod_{j=1}^{n} C_{b_j}(\lambda_j) |0\rangle F^{a_n \ldots a_1} \ ,$$

(2.64)

$$(A_{11}(\mu) + A_{22}(\mu) - A_{33}(\mu))|\lambda_1, \ldots, \lambda_n|F\rangle =$$

$$= (a(\mu))^L \prod_{j=1}^{n} \frac{1}{a(\mu - \lambda_j)} \prod_{l=1}^{n} C_{b_l}(\lambda_l) |0\rangle \ \tau^{(1)}(\mu)^{b_1 \ldots b_n}_{a_1 \ldots a_n} F^{a_n \ldots a_1}$$
$$+ \sum_{k=1}^{n} (\bar{A}_k)^{b_1 \ldots b_n}_{a_1 \ldots a_n} C_{b_k}(\mu) \prod_{j=1}^{n} C_{b_j}(\lambda_j) |0\rangle F^{a_n \ldots a_1} \ ,$$

(2.65)

where

$$\tau^{(1)}(\mu)^{b_1 \ldots b_n}_{a_1 \ldots a_n} = (-1)^{c_i} L^{(1)}_n(\mu - \lambda_n)^{c_{i-1}} \cdots L^{(1)}_{n-1}(\mu - \lambda_{n-1})^{c_{n-2}} \cdots L^{(1)}_1(\mu - \lambda_1)^{c_1} \times$$
$$\times (-1)^{c_i} \sum_{i=1}^{n} (\epsilon_{i+1}) + \sum_{i=1}^{n-1} \epsilon_i (\epsilon_{i+1} - 1) \ .$$

(2.66)

Here all indices $c_i$ and $c$ are summed over. The $L$-operators $L^{(1)}_n(\lambda)$ are constructed by multiplying the R-matrix $r_{BBF}(\lambda)$ by the permutation matrix $\Pi_{BBF}$ and are found to be

$$L^{(1)}_k(\lambda) = \begin{pmatrix}
    a(\lambda) + b(\lambda) c^{11}_k & b(\lambda) c^{21}_k & b(\lambda) c^{31}_k \\
    b(\lambda) c^{12}_k & a(\lambda) + b(\lambda) c^{22}_k & b(\lambda) c^{32}_k \\
    b(\lambda) c^{13}_k & b(\lambda) c^{23}_k & a(\lambda) - b(\lambda) c^{33}_k
\end{pmatrix} \ .$$

(2.67)

The expression for $\tau^{(1)}(\mu)$ is significantly different from the one in the $FFBBB$ case, but by using an appropriate definition for the graded tensor product $\tau^{(1)}(\mu)$ can again be interpreted as the transfer matrix of an inhomogeneous spin model on a lattice of $n$ sites. Our reference state $|0\rangle$ is now of fermionic nature and we have to define a new graded tensor product reflecting this fact

$$(F \otimes G)^{ab}_{cd} = F_{ab} G_{cd} (-1)^{(\epsilon_a+1)(\epsilon_c+1)} \ .$$

(2.68)

Effectively the new graded tensor product switches even and odd Grassmann parities, i.e. $\epsilon_a \rightarrow \epsilon_a + 1$. In terms of this tensor product the transfer matrix $\tau^{(1)}(\mu)$ given by (2.66) can be rewritten as

$$\tau^{(1)}(\mu)^{b_1 \ldots b_n}_{a_1 \ldots a_n} = str(T^{(1)}_n(\mu)) = str(L^{(1)}_n(\mu - \lambda_n) \otimes L^{(1)}_{n-1}(\mu - \lambda_{n-1}) \otimes \ldots \otimes L^{(1)}_1(\mu - \lambda_1)) \ .$$

(2.69)

In the expression after the second equality of (2.69) we have explicitly written the tensor product $\otimes$ between the quantum spaces over the sites of the inhomogeneous model (the
L-operatrs are of course again multiplied as matrices). As before \( F^{a_n\ldots a_1} \) must be an eigenvector of \( \tau^{(1)}(\mu) \), if \(|\lambda_1\ldots\lambda_n|F\rangle \) is to be an eigenstate of \( \tau(\mu) \). The unwanted terms are computed in Appendix B, and the condition of their cancellation

\[
\left( (\Lambda_k)^{b_1\ldots b_n}_{a_1\ldots a_n} - (\hat{\Lambda}_k)^{b_1\ldots b_n}_{a_1\ldots a_n} \right) F^{a_n\ldots a_1} = 0
\]

leads to the conditions

\[
F^{a_n\ldots a_1} = (a(-\lambda_k))^L \left( \tau^{(1)}(\lambda_k)F \right)^{a_n\ldots a_1}, \quad k = 1, \ldots, n.
\]

In the solution of the first nesting there is an additional complication compared to the FFBB case. Because of our change of tensor product, the \( L^{(1)}(\lambda) \) are not intertwined by the \( R \)-matrix \( r(\mu) \) defined in (2.63), but by the \( R \)-matrix

\[
\hat{r}(\mu)_{ab} = b(\mu)\delta_{ab} - \delta_{cd} + a(\mu)\delta_{ad} \delta_{bc} (-1)^{\epsilon_a+\epsilon_c+\epsilon_a \epsilon_c}.
\]

The following intertwining relation holds

\[
\hat{r}(\lambda - \mu) T_n^{(1)}(\lambda) \otimes T_n^{(1)}(\mu) = T_n^{(1)}(\mu) \otimes T_n^{(1)}(\lambda) \hat{r}(\lambda - \mu).
\]

If we make the following choice of reference state for the first nesting

\[
|0\rangle^{(1)}_k = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad |0\rangle^{(1)} = \bigotimes_{k=1}^{n}|0\rangle^{(1)}_k,
\]

one can show that the model defined by (2.73), (2.74) and the graded tensor product \( \otimes \) is isomorphic to a model of the permutation type with \( BBF \) grading (describing two species of bosons and one species of fermions) and the graded tensor product \( \otimes \). Effectively the first nesting thus describes the inhomogeneous model obtained by just “truncating a fermionic line” from the original model. This holds generally for \( u(m|n) \) symmetric models[14].

In the next step of the NABA we thus face the task of solving the inhomogeneous BBF model. As in the first step we have to truncate a fermionic line, which can be done in a way analogous to the procedure described above. The intertwining relations are of the same form as (2.62), but the \( R \)-matrices are now the ones of the BB and FF models respectively. Taking only the wanted terms into account we obtain the eigenvalues of the transfer matrix \( \tau^{(1)}(\mu) \) as given below. The condition of the cancellation of the unwanted terms now reads

\[
G^{a_n\ldots a_1} = \prod_{j=1}^{n} a(\lambda_j - \lambda_k^{(1)}) \left( \tau^{(2)}(\lambda_k^{(1)})G \right)^{a_n\ldots a_1}, \quad k = 1, \ldots, n_1
\]

where we have made an Ansatz of the form (2.40) for the eigenstates of the transfer matrix \( \tau^{(1)}(\mu) \), and where \( \tau^{(2)}(\mu) \) is the transfer matrix of the second nesting given by the following expression

\[
\tau^{(2)}(\mu)_{a_1\ldots a_n} = L^{(2)}_{n_1}(\mu - \lambda^{(1)}_{n_1})_{b_1\ldots b_{n_1}} L^{(2)}_{n_1-1}(\mu - \lambda^{(1)}_{n_1-1})_{b_{n_1-1}\ldots b_1} \ldots L^{(2)}_1(\mu - \lambda^{(1)}_{1})_{b_1\ldots b_{n_1}}.
\]
Here $L^{(2)}(\mu) = a(\mu)I^{(2)} + b(\mu)\Pi_{BB}$. Note that the roman indices in (2.75) and (2.76) can take the values 1, 2, both of which carry bosonic grading and thus do not generate any minus signs. The second nesting can be shown to be equivalent to the inhomogeneous spin $\frac{1}{2}$ Heisenberg XXX model, or BB model in our terminology. The eigenvalues of $\tau^{(2)}(\mu)$ and the BAE for the inhomogeneous BB model are easily determined, and we can put together the three steps of the NABA like in the case of the FFBB grading. After reparametrising according to

\[ \tilde{\lambda}_k = \lambda_k - \frac{i}{2}, \quad \tilde{\lambda}_j^{(1)} = \lambda_j^{(1)} - i, \quad \tilde{\lambda}_m^{(2)} = \lambda_m^{(2)} - \frac{i}{2}, \]  

(2.77)

the BAE take the form

\[ \left( \frac{\lambda_k - \frac{i}{2}}{\lambda_k + \frac{i}{2}} \right)^L \prod_{j=1}^{N_b} \frac{\tilde{\lambda}_k - \tilde{\lambda}_j^{(1)} + \frac{i}{2}}{\tilde{\lambda}_k - \tilde{\lambda}_j^{(1)} - \frac{i}{2}} \prod_{l \neq k}^{N_b+N_i} \frac{\tilde{\lambda}_l - \tilde{\lambda}_k + i}{\tilde{\lambda}_l - \tilde{\lambda}_k - i}, \quad k = 1, \ldots, N_b + N_i \]

(2.78)

The eigenvalues of the transfer matrix are given by

\[ \nu(\mu) = (a(\mu))^L \left( \prod_{j=1}^{N_b+N_i} \frac{1}{a(\mu - \lambda_j^{(1)})} \right) \left( \nu^{(1)}(\mu) - \left( \frac{1}{a(-\mu)} \right)^L \right) \]

\[ \nu^{(1)}(\mu) = \prod_{i=1}^{N_b} \frac{1}{a(\mu - \lambda_i^{(1)})} \left( \prod_{j=1}^{N_b+N_i} a(\mu - \lambda_j) \right) \left( \nu^{(2)}(\mu) - \prod_{l=1}^{N_i} \frac{1}{a(\lambda_l - \mu)} \right) \]

\[ \nu^{(2)}(\mu) = \prod_{k=1}^{N_b+N_i} \frac{1}{a(\mu - \lambda_k^{(2)})} \prod_{j=1}^{N_b} a(\mu - \lambda_j^{(1)}) + \prod_{j=1}^{N_i} \frac{1}{a(\lambda_j^{(2)} - \mu)}. \]

(2.79)

Energy and momentum eigenvalues are given by

\[ E^0(\tilde{\lambda}_1, \ldots, \tilde{\lambda}_n) = L - \sum_{j=1}^{N_b+N_i} \frac{1}{\lambda_j^2 + \frac{1}{4}} = -2 \sum_{j=1}^{N_b+N_i} \cos(k_j) + L - 2(N_b + N_i) \]

(2.80)

\[ p(\tilde{\lambda}_1, \ldots, \tilde{\lambda}_n) = -i \sum_{j=1}^{N_b+N_i} \log(\frac{\tilde{\lambda}_j - \frac{i}{2}}{\lambda_j + \frac{i}{2}}) = \sum_{j=1}^{N_b+N_i} k_j + (N_f + 1)\pi, \]

where we have defined $\tilde{\lambda}_j = \frac{i}{2} \tan(\frac{k_j}{2})$. Note that after the reparametrisations (2.77) and (2.53) the expressions for the BAE for the BBFF and FFBB gradings coincide up to the
substitutions \( N_\uparrow \leftrightarrow N_l \) and \( N_\downarrow \leftrightarrow N_h \). However there is no such obvious relation between the eigenvalues of the transfer matrix. For the energy eigenvalues it can be easily shown by taking logarithmic derivatives of the eigenvalues \( \nu(\mu) \), that the expressions for BBFF and FFBB are identical up to the transformation \( N_\downarrow \leftrightarrow N_l \) and \( N_\uparrow \leftrightarrow N_h \) apart from an overall minus sign. This is clear as under a particle-hole transformation for spin-up \( H^0 \) transforms into \(-H^0\). There is no such relation for the eigenvalues of the momentum operator.

The BAE (2.78) are particularly useful, because they reduce to the BAE for the Sutherland solution of the supersymmetric \( t-J \) model in the sector without localons \((N_l = 0)\). The classification of ground states and low lying excited states is particularly easy in this case. In our forthcoming publication we will analyse the bound state and ground state structure of this set of BAE, and describe the model by a set of integral equations in the thermodynamic limit\([15]\).

2.6. Algebraic Bethe Ansatz for the FBFB grading

Here the first step of the NABA is completely analogous to the one for the FFBB grading. The R-matrix and monodromy matrix of the nesting can be identified with the ones for FBF grading of the inhomogeneous supersymmetric \( t-J \) model, which was solved in [9]. The eigenvalues of the transfer matrix are found to be

\[
\nu(\mu) = (a(\mu))^L \left( \prod_{j=1}^{N_e+N_l} \frac{1}{a(\mu - \lambda_j)} \right) \nu^{(1)}(\mu) + \left( \prod_{j=1}^{N_e+N_l} \frac{1}{a(\lambda_j - \mu)} \right) \nu^{(2)}(\mu) \\
\nu^{(1)}(\mu) = \prod_{l=1}^{N_\downarrow+N_l} \frac{1}{a(\mu - \lambda_l^{(1)})} \left( \prod_{j=1}^{N_e+N_l} a(\mu - \lambda_j) \right) \left( \nu^{(2)}(\mu) - \prod_{k=1}^{N_e+N_l} \frac{1}{a(\lambda_k - \mu)} \right) \\
\nu^{(2)}(\mu) = \prod_{m=1}^{N_\uparrow} \frac{1}{a(\lambda_m^{(2)} - \mu)} \left( 1 - \prod_{j=1}^{N_e+N_l} a(\mu - \lambda_j^{(1)}) \right)
\]

Using the reparametrisation

\[
\tilde{\lambda}_k = \lambda_k + \frac{i}{2}, \quad \tilde{\lambda}_m^{(2)} = \lambda_m^{(2)} + \frac{i}{2},
\]

the following form of the BAE is derived

\[
\left( \frac{\tilde{\lambda}_k - \frac{i}{2}}{\tilde{\lambda}_k + \frac{i}{2}} \right)^L = \prod_{j=1}^{N_e+N_l} \frac{\tilde{\lambda}_k - \lambda_j^{(1)} - \frac{i}{2}}{\lambda_k - \lambda_j^{(1)} + \frac{i}{2}}, \quad k = 1, \ldots, N_e + N_l \\
\prod_{k=1}^{N_e+N_l} \frac{\tilde{\lambda}_k - \lambda_j^{(1)} - \frac{i}{2}}{\lambda_k - \lambda_j^{(1)} + \frac{i}{2}} = \prod_{m=1}^{N_\uparrow} \frac{\tilde{\lambda}_m^{(2)} - \lambda_j^{(1)} - \frac{i}{2}}{\lambda_m^{(2)} - \lambda_j^{(1)} + \frac{i}{2}}, \quad j = 1, \ldots, N_\downarrow + N_l \\
1 = \prod_{j=1}^{N_e+N_l} \frac{\tilde{\lambda}_m^{(2)} - \lambda_j^{(1)} - \frac{i}{2}}{\lambda_m^{(2)} - \lambda_j^{(1)} + \frac{i}{2}}, \quad m = 1, \ldots, N_\downarrow
\]
2.7. Algebraic Bethe Ansatz for the BFBF grading

In this choice of grading we first truncate a fermionic line analogously to the first step of the BBFF case. In the first nesting we truncate a bosonic line, which is done like in the first step of the FFBB case. The resulting expression for the eigenvalues of the transfer matrix are

\[ \nu(\mu) = (a(\mu))^L \left( \prod_{j=1}^{N_b+N_f} \frac{1}{a(\mu - \lambda_j)} \right) \left( \nu^{(1)}(\mu) - \left( \frac{1}{a(-\mu)} \right)^L \right) \]

\[ \nu^{(1)}(\mu) = \prod_{i=1}^{N_i+N_f} \frac{1}{a(\mu - \lambda^{(1)}_i)} \left( \prod_{j=1}^{N_b+N_f} a(\mu - \lambda_j) \right) \nu^{(2)}(\mu) + \prod_{i=1}^{N_i+N_f} \frac{1}{a(\lambda^{(1)}_i - \mu)} \]

\[ \nu^{(2)}(\mu) = \prod_{k=1}^{N_f} \frac{1}{a(\mu - \lambda^{(2)}_k)} \prod_{m=1}^{N_i+N_f} a(\mu - \lambda^{(2)}_m) \left( 1 - \prod_{j=1}^{N_i+N_f} \frac{1}{a(\lambda^{(1)}_j - \mu)} \right) \] (2.84)

After reparametrising the spectral parameters according to

\[ \tilde{\lambda}_k = \lambda_k - \frac{i}{2}, \quad \tilde{\lambda}^{(2)}_m = \lambda^{(2)}_m - \frac{i}{2} , \] (2.85)

the BAE take the form

\[ \left( \frac{\tilde{\lambda}_k - \frac{i}{2}}{\tilde{\lambda}_k + \frac{i}{2}} \right)^L = \prod_{j=1}^{N_i+N_f} \frac{\tilde{\lambda}_k - \lambda^{(1)}_j - \frac{i}{2}}{\tilde{\lambda}_k - \lambda^{(1)}_j + \frac{i}{2}} , \quad k = 1, \ldots, N_i + N_f 

\[ \prod_{k=1}^{N_i+N_f} \frac{\tilde{\lambda}_k - \lambda^{(1)}_j - \frac{i}{2}}{\tilde{\lambda}_k - \lambda^{(1)}_j + \frac{i}{2}} = \prod_{m=1}^{N_i+N_f} \frac{\tilde{\lambda}^{(2)}_m - \lambda^{(1)}_j - \frac{i}{2}}{\tilde{\lambda}^{(2)}_m - \lambda^{(1)}_j + \frac{i}{2}} , \quad j = 1, \ldots, N_i + N_f \] (2.86)

\[ 1 = \prod_{j=1}^{N_i+N_f} \frac{\tilde{\lambda}^{(2)}_m - \lambda^{(1)}_j - \frac{i}{2}}{\tilde{\lambda}^{(2)}_m - \lambda^{(1)}_j + \frac{i}{2}} , \quad m = 1, \ldots, N_i . \]

The BAE (2.86) can be mapped onto the BAE (2.83) by a boson-fermion interchange like in the FFBB/BBFF case.

2.8. Algebraic Bethe Ansatz for the BFFB grading

The results for the eigenvalues of the transfer matrix and BAE are

\[ \nu(\mu) = (a(\mu))^L \left( \prod_{j=1}^{N_i+N_f} \frac{1}{a(\mu - \lambda_j)} \right) \nu^{(1)}(\mu) + \left( \prod_{j=1}^{N_i+N_f} \frac{1}{a(\lambda_j - \mu)} \right) \nu^{(2)}(\mu) - \prod_{l=1}^{N_i+N_f} \frac{1}{a(\lambda_l - \mu)} \] (2.87)

\[ \nu^{(1)}(\mu) = \prod_{j=1}^{N_i+N_f} \frac{1}{a(\mu - \lambda^{(1)}_j)} \left( \prod_{k=1}^{N_i+N_f} a(\mu - \lambda_k) \right) \nu^{(2)}(\mu) - \prod_{k=1}^{N_i+N_f} \frac{1}{a(\lambda_k - \mu)} \]

\[ \nu^{(2)}(\mu) = \prod_{m=1}^{N_f} \frac{1}{a(\mu - \lambda^{(2)}_m)} \prod_{j=1}^{N_i+N_f} a(\mu - \lambda^{(1)}_j) \left( 1 - \prod_{p=1}^{N_i+N_f} \frac{1}{a(\lambda^{(1)}_p - \mu)} \right) . \]
where we have reparametrised

\[ \left( \frac{\tilde{\lambda}_k - \frac{i}{2}}{\lambda_k + \frac{i}{2}} \right)^L = \prod_{j=1}^{N_i+N_b} \frac{\tilde{\lambda}_k - \lambda_j^{(1)} - \frac{i}{2}}{\lambda_k - \lambda_j^{(1)} + \frac{i}{2}}, \quad k = 1, \ldots, N_e + N_l \]

\[ \prod_{i=1 \atop i \neq j}^{N_i+N_i} \frac{\lambda_j^{(1)} - \lambda_i^{(1)} + i}{\lambda_j^{(1)} - \lambda_i^{(1)} - i} = \prod_{k=1}^{N_e+N_b} \frac{\tilde{\lambda}_k - \lambda_j^{(1)} - \frac{i}{2}}{\lambda_k - \lambda_j^{(1)} + \frac{i}{2}} \prod_{m=1}^{N_i} \frac{\tilde{\lambda}_m^{(2)} - \lambda_j^{(1)} - \frac{i}{2}}{\lambda_m^{(2)} - \lambda_j^{(1)} + \frac{i}{2}}, \quad j = 1, \ldots, N_l + N_b \]

\[ 1 = \prod_{m=1}^{N_e+N_i} \frac{\tilde{\lambda}_m^{(2)} - \lambda_j^{(1)} - \frac{i}{2}}{\lambda_m^{(2)} - \lambda_j^{(1)} + \frac{i}{2}}, \quad m = 1, \ldots, N_l, \quad (2.88) \]

where we have reparametrised

\[ \tilde{\lambda}_k = \lambda_k + \frac{i}{2}, \quad \tilde{\lambda}_m^{(2)} = \lambda_m^{(2)} - \frac{i}{2}. \quad (2.89) \]

2.9. Algebraic Bethe Ansatz for the FBBF grading

This last possible choice of grading leads to the following results

\[ \nu(\mu) = (a(\mu))^L \left( \prod_{j=1}^{N_i+N_b} \frac{1}{a(\mu - \lambda_j^{(1)})} \right) \left( \nu^{(1)}(\mu) - \left( \frac{1}{a(-\mu)} \right)^L \right) \]

\[ \nu^{(1)}(\mu) = \prod_{j=1}^{N_i+N_i} \frac{1}{a(\mu - \lambda_j^{(1)})} \prod_{k=1}^{N_e+N_b} a(\mu - \lambda_k) \nu^{(2)}(\mu) + \prod_{j=1}^{N_i+N_l} \frac{1}{a(\mu - \lambda_j^{(1)})} \quad (2.90) \]

\[ \nu^{(2)}(\mu) = \prod_{k=1}^{N_i} \frac{1}{a(\lambda_k^{(2)} - \mu)} \left( 1 - \prod_{j=1}^{N_i+N_l} a(\mu - \lambda_j^{(1)}) \right) \]

\[ \left( \frac{\tilde{\lambda}_k - \frac{i}{2}}{\lambda_k + \frac{i}{2}} \right)^L = \prod_{j=1}^{N_i+N_b} \frac{\tilde{\lambda}_k - \lambda_j^{(1)} - \frac{i}{2}}{\lambda_k - \lambda_j^{(1)} + \frac{i}{2}}, \quad k = 1, \ldots, N_e + N_b \]

\[ \prod_{i=1 \atop i \neq j}^{N_i+N_i} \frac{\lambda_j^{(1)} - \lambda_i^{(1)} + i}{\lambda_j^{(1)} - \lambda_i^{(1)} - i} = \prod_{k=1}^{N_e+N_b} \frac{\tilde{\lambda}_k - \lambda_j^{(1)} - \frac{i}{2}}{\lambda_k - \lambda_j^{(1)} + \frac{i}{2}} \prod_{m=1}^{N_i} \frac{\tilde{\lambda}_m^{(2)} - \lambda_j^{(1)} - \frac{i}{2}}{\lambda_m^{(2)} - \lambda_j^{(1)} + \frac{i}{2}}, \quad j = 1, \ldots, N_l + N_b \]

\[ 1 = \prod_{m=1}^{N_e+N_i} \frac{\tilde{\lambda}_m^{(2)} - \lambda_j^{(1)} - \frac{i}{2}}{\lambda_m^{(2)} - \lambda_j^{(1)} + \frac{i}{2}}, \quad m = 1, \ldots, N_l, \quad (2.91) \]

where

\[ \tilde{\lambda}_k = \lambda_k - \frac{i}{2}, \quad \tilde{\lambda}_m^{(2)} = \lambda_m^{(2)} + \frac{i}{2}. \quad (2.92) \]

Again there exists a boson-fermion interchange map between (2.88) and (2.91).
3. Lowest Weight properties of the Bethe Ansatz states

In this section we prove that all Bethe Ansatz states with finite spectral parameters are lowest weight states of the global \( u(2|2) \) symmetry algebra. For the case of the FFBB grading the lowest-weight conditions read

\[
0 = \eta |\lambda_1, \ldots, \lambda_n, F\rangle = S |\lambda_1, \ldots, \lambda_n, F\rangle \\
= Q_1 |\lambda_1, \ldots, \lambda_n, F\rangle = Q_1^\dagger |\lambda_1, \ldots, \lambda_n, F\rangle \\
= Q_{-1} |\lambda_1, \ldots, \lambda_n, F\rangle = Q_{-1}^\dagger |\lambda_1, \ldots, \lambda_n, F\rangle .
\]

Lowest weight theorems hold for all choices of the grading (the grading determines which generators annihilate the BA states), but without loss of generality we will perform the computations only for the specific case of the FFBB grading. The lowest weight property is of great importance as it first of all demonstrates that the Bethe Ansatz does not provide a complete set of eigenstates of the hamiltonian \( H^0 \), and because it can be used together with the global symmetry structure to generate a complete set of eigenstates. Similar computations have been performed for other models in [16–19].

We start by noting that in the basis of the Hilbert space introduced at the beginning of section 2.4, all lowering generators of the \( u(2|2) \) algebra are represented in the form \( \sum_{k=1}^{L} e_{k}^{\alpha \beta} \). By using the correspondence of the four basis vectors \( e_i \) over one site of the lattice with the four possible configurations (1.1), we find

\[
S = \sum_{k=1}^{L} e_{k}^{21} , \quad \eta = \sum_{k=1}^{L} e_{k}^{43} , \quad Q_1 = \sum_{k=1}^{L} e_{k}^{42} \\
Q_{-1} = \sum_{k=1}^{L} e_{k}^{41} , \quad Q_1^\dagger = - \sum_{k=1}^{L} e_{k}^{31} , \quad Q_{-1}^\dagger = \sum_{k=1}^{L} e_{k}^{32} ,
\]

(3.2)

where \( e_{k}^{ij} \) are quantum operators acting nontrivially only on the \( k^{th} \) site of the lattice with matrix representation

\[
(e_{k}^{ij})^{\alpha \beta} = \delta_{i\alpha} \delta_{j\beta} .
\]

(3.3)

Using the representation (3.2) and the expression (2.20) for the \( L \)-operator we are able to rewrite graded quantum commutators as matrix commutators

\[
\left[ (L_k(\mu))^{ab}, e_{k}^{ij} \right]_{\text{quantum}} = (-1)^{(\epsilon_i + \epsilon_j)\epsilon_b} \left( (e_{k}^{ij} L_k(\mu))^{ab} - (L_k(\mu) e_{k}^{ij})^{ab} \right) \\
= (-1)^{(\epsilon_i + \epsilon_j)\epsilon_b} \left[ [e_{k}^{ij}, L_k(\mu)]_{\text{matrix}} \right]^{ab} .
\]

(3.4)

where \( e_{k}^{ij} \) is a matrix of the form

\[
(e_{k}^{ij})^{ab} = \delta_{i\alpha} \delta_{j\beta} .
\]

(3.5)
Note that on the r.h.s of (3.4) $e^{ij}$ and $L_k(\mu)$ are multiplied as matrices. The graded quantum commutator of $e^{ij}$ with $(T_L(\mu))^{ab}$ can now be expressed as a matrix commutator by using (3.4) and (2.21)

$$[\{T_L(\mu)\}^{ab}, e^{ij}]_{\text{quantum}} = (-1)^{(\epsilon_i+\epsilon_j)\epsilon_b} L_L(\mu)^{ac_a} \ldots L_{k+1}(\mu)^{ck+2c_{k+1}} \times (e^{ij} L_k(\mu)^{ck+1c_k} - (L_k(\mu)^{e^{ij}})^{ck+1c_k}) \times L_{k-1}(\mu)^{c_kc_{k-1}} \ldots L_1(\mu)^{c_2c_1}.$$ (3.6)

Summing over all sites of the lattice we arrive at

$$\left[ \sum_{k=1}^{L} e^{ij}, (T_L(\mu))^{ab} \right]_{\text{quantum}} = (-1)^{(\epsilon_i+\epsilon_j)\epsilon_b} \left[ [e^{ij}, T_L(\mu)]_{\text{matrix}} \right]^{ab}. \tag{3.7}$$

Equation (3.7) enables us to compute the graded commutators of the $u(2|2)$-lowering operators with the creation operators $C_a(\lambda)$. We find

$$[\eta, C_a(\lambda)] = \delta_{a3} D(\lambda) - A_{3a}(\lambda)$$
$$[Q_1, C_a(\lambda)] = \delta_{a2} D(\lambda) - A_{2a}(\lambda)$$
$$[Q_{-1}, C_a(\lambda)] = \delta_{a1} D(\lambda) - A_{1a}(\lambda)$$
$$[-\bar{Q}_1^+, C_a(\lambda)] = \delta_{a1} C_3(\lambda)$$
$$[\bar{Q}_1^+, C_a(\lambda)] = \delta_{a2} C_3(\lambda)$$
$$[S, C_a(\lambda)] = \delta_{a1} C_2(\lambda). \tag{3.8}$$

We also note that all lowering operators annihilate the vacuum

$$0 = \eta |0\rangle = \bar{Q}_1^+ |0\rangle = \bar{Q}_{-1}^+ |0\rangle = Q_1 |0\rangle = Q_{-1} |0\rangle = S |0\rangle. \tag{3.9}$$

**Lowest weight property for $Q_1$, $Q_{-1}$ and $\eta$**

In this subsection we show that all Bethe Ansatz eigenstates are are annihilated by the $u(2|2)$ generators $J_1 = Q_{-1}$, $J_2 = Q_1$, and $J_3 = \eta$. Using (3.9) and (3.8) we find that

$$J_a |\lambda_1, \ldots, \lambda_n\rangle = 0$$

$$= \left[ J_a, \prod_{i=1}^{n} C_{a_i}(\lambda_i) \right] |0\rangle F^{a_n \ldots a_1}$$

$$= \sum_{k=1}^{n} \prod_{i=1}^{k-1} C_{a_i}(\lambda_i) \left[ J_a, C_{a_k}(\lambda_k) \right] \prod_{j=k+1}^{n} C_{a_j}(\lambda_j) |0\rangle F^{a_n \ldots a_1} (-1)^{\epsilon_a} \sum_{l=1}^{k-1} \epsilon_{a_l}$$

$$= \sum_{k=1}^{n} \prod_{i=1}^{k-1} C_{a_i}(\lambda_i) \left( \delta_{aa_k} D(\lambda_k) - A_{aa_k}(\lambda_k) \right) \prod_{j=k+1}^{n} C_{a_j}(\lambda_j) |0\rangle F^{a_n \ldots a_1} (-1)^{\epsilon_a} \sum_{l=1}^{k-1} \epsilon_{a_l}$$

$$= \sum_{k=1}^{n} \prod_{i=1}^{k-1} C_{b_i}(\lambda_i) \prod_{j=k+1}^{n} C_{b_j}(\lambda_j) |0\rangle (\Omega_k)^{b_1 \ldots b_{k-1}a b_{k+1} \ldots b_n} (-1)^{\epsilon_a} \sum_{l=1}^{k-1} \epsilon_{a_l}. \tag{3.10}$$
The last equality follows by inspection of the r.h.s. of the previous line, as \( D(\lambda_k) \) and \( A_{aa_k}(\lambda_k) \) can be moved past the string of \( C_b \)'s by using (2.26). Due to the cancellation of the unwanted terms (which are explicitly computed in Appendix A) all of the quantities \( \Omega_k \) vanish as we will demonstrate below. From (3.10) it follows that \( \Omega_1 \) is the coefficient of the term, where the creation operator \( C_b \) with spectral parameter \( \lambda_1 \) is missing. It can be obtained from the term

\[
(\delta_{aa_1} D(\lambda_1) - A_{aa_1}(\lambda_1)) \prod_{j=2}^{n} C_{a_j}(\lambda_j) |0 \rangle F^{a_n \ldots a_1} \tag{3.11}
\]

by moving \((\delta_{aa_1} D(\lambda_1) - A_{aa_1}(\lambda_1))\) past \( \prod_{j=2}^{n} C_{a_j}(\lambda_j) \) by only taking “wanted” terms into account, i.e., by only using the first contribution of the intertwining relations (2.26) when moving \( A_{ab}(\lambda_1) \) or \( D(\lambda_1) \) past \( C_{b_j}(\lambda_j) \). The resulting expression for \( \Omega_1 \) is found to be

\[
(\Omega_1)^{ab_2 \ldots b_n} = \delta_{aa_1} \left[ \prod_{i=2}^{n} \frac{1}{a(\lambda_i - \lambda_1)} \prod_{j=2}^{n} \delta_{a_ib_j} \right] F^{a_n \ldots a_1} \\
- \prod_{i=2}^{n} \frac{1}{a(\lambda_i - \lambda_1)} r(\lambda_1 - \lambda_2)^{d_1 a_2} r(\lambda_1 - \lambda_3)^{d_2 a_3} \ldots r(\lambda_1 - \lambda_n)^{d_n a_n} \prod_{i=2}^{n} (-1)^{\sum_{i=2}^{n} c_i e_i} (a(\lambda_1))^L F^{a_n \ldots a_1} . \tag{3.12}
\]

If we compare (3.12) with the condition of the cancellation of the unwanted terms (\( \bullet \)) and (\( \circ \)), we find that \( \Omega_1 \) vanishes because

\[
(\Lambda_1 F)^{ab_2 \ldots b_n} + (\Lambda_1 F)^{ab_2 \ldots b_n} = 0 .
\]

The expression for \( \Omega_k \) can be obtained in a similar fashion:

We use (2.26) to rewrite \(|\lambda_1 \ldots \lambda_n\rangle\) as

\[
\prod_{i=1}^{n} C_{a_i}(\lambda_i) |0 \rangle F^{a_n \ldots a_1} = C_{c_k}(\lambda_k) \prod_{i=1}^{k-1} C_{c_i}(\lambda_i) \prod_{j=k+1}^{n} C_{a_j}(\lambda_j) S(\lambda_k)^{c_1 \ldots c_k} |0 \rangle F^{a_n \ldots a_1} ,
\]

\[
S(\lambda_k)^{c_1 \ldots c_k} = r(\lambda_{k-1} - \lambda_k)^{d_{k-1} a_k} r(\lambda_{k-2} - \lambda_k)^{d_{k-2} a_{k-1}} \ldots r(\lambda_1 - \lambda_k)^{d_1 a_1} . \tag{3.13}
\]

Now the action of the lowering operators \( J_a \) on the Bethe Ansatz states can be written in the form

\[
J_a |\lambda_1 , \ldots , \lambda_n F \rangle = \\
= \left[ J_a , C_{b_k}(\lambda_k) \prod_{i=1}^{k-1} C_{b_i}(\lambda_i) \prod_{j=k+1}^{n} C_{a_j}(\lambda_j) S(\lambda_k)^{b_1 \ldots b_k} \right] |0 \rangle F^{a_n \ldots a_1}. \tag{3.14}
\]

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Going through the same steps as in (3.10) it follows that $\Omega_k$ can be obtained in essentially the same way as $\Omega_1$ with the result

$$
(\Omega_k)^{b_1 \ldots b_{k-1} a b_{k+1} \ldots b_n} = \left[ \prod_{i=1 \atop i \neq k}^{n} \frac{1}{a(\lambda_i - \lambda_k)} \prod_{j=k+1}^{n} \delta_{a_i b_j} S(\lambda_k)^{b_1 \ldots b_{k-1} a} \right] F^{a_n \ldots a_1} - \prod_{i=1 \atop i \neq k}^{n} \frac{1}{a(\lambda_i - \lambda_k)} S(\lambda_k)^{c_1 \ldots c_k} (-1)^{\sum_{i=1 \atop i \neq k}^{n} e_i e_a} F^{a_n \ldots a_1} \\
\times r(\lambda_k - \lambda_1)^{d_1 c_1} r(\lambda_k - \lambda_2)^{d_2 c_2} \ldots r(\lambda_k - \lambda_{k-1})^{d_{k-1} c_{k-1}} \\
\times r(\lambda_k - \lambda_{k+1})^{d_k a_{k+1}} r(\lambda_k - \lambda_{k+2})^{d_{k+1} a_{k+2}} \ldots r(\lambda_k - \lambda_n)^{a_{a_n} a_{n-2}} \\
\times (a(\lambda_k))^L. 
$$

(3.15)

Inspection of the (●) and (●) in Appendix A shows that the condition of the cancellation of the unwanted terms

$$(\lambda_1 F)^{b_1 \ldots b_{k-1} a b_{k+1} \ldots b_n} + (\lambda_n F)^{b_1 \ldots b_{k-1} a b_{k+1} \ldots b_n} = 0$$

implies the vanishing of $\Omega_k$. This completes the proof of the lowest weight property of the Bethe Ansatz states for the generators $Q_1$, $Q_{-1}$ and $\eta$.

**Lowest weight property for $\tilde{Q}_1^+$ and $\tilde{Q}_{-1}^+$**

In this subsection we show that the lowering generators $\tilde{Q}_{\pm 1}^\dagger$ annihilate all Bethe Ansatz states with finite spectral parameters $\lambda_j$. We first reexpress $\tilde{Q}_{\pm 1}^\dagger$ in terms of matrix elements of the monodromy matrix (2.21)

$$
\tilde{Q}_{-1}^\dagger = \sum_{k=1}^{L} e_k^{32} = -i \lim_{\mu \to \infty} \mu A_{23}(\mu) \\
\tilde{Q}_1^\dagger = -\sum_{k=1}^{L} e_k^{31} = i \lim_{\mu \to \infty} \mu A_{13}(\mu) .
$$

(3.16)

Using the intertwining relation (2.26) we are able to determine the action of $\mu A_{ab}(\mu)$ on the states $|\lambda_1 \ldots \lambda_n\rangle$ in the limit $\mu \to \infty$

$$
-i \lim_{\mu \to \infty} \mu A_{ab}(\mu)|\lambda_1 \ldots \lambda_n\rangle F = -i \prod_{l=1}^{n} C_{b_l}(\lambda_l)|0\rangle \lim_{\mu \to \infty} \mu \left(T_n^{(1)}(\mu)^{ab}\right)_{a_1 \ldots a_n}^{b_1 \ldots b_n} F^{a_1 \ldots a_n} .
$$

(3.17)

This implies that

$$
\tilde{Q}_1^\dagger |\lambda_1 \ldots \lambda_n\rangle F = \prod_{l=1}^{n} C_{b_l}(\lambda_l)|0\rangle \lim_{\mu \to \infty} i \mu \left(B_1^{(1)}(\mu)F\right)^{b_1 \ldots b_n} \\
\tilde{Q}_{-1} |\lambda_1 \ldots \lambda_n\rangle F = \prod_{l=1}^{n} C_{b_l}(\lambda_l)|0\rangle \lim_{\mu \to \infty} -i \mu \left(B_2^{(1)}(\mu)F\right)^{b_1 \ldots b_n} .
$$

(3.18)
We now define new operators $J_a^{(1)}$ according to

$$J_a^{(1)} = \lim_{\mu \to \infty} -i \mu B_a^{(1)}(\mu), \quad a = 1, 2 \quad (3.19)$$

From (3.18) it is clear that the lowest weight property of the Bethe Ansatz states with respect to $\tilde{Q}^\dagger_{\pm 1}$ is equivalent to the conditions

$$J_a^{(1)} F = 0, \quad a = 1, 2 \quad (3.20)$$

Recalling that

$$F = \prod_{j=1}^{n_1} C_{b_j}^{(1)}(\lambda_j^{(1)}) |0^{(1)}\rangle G^{\alpha_1, \ldots, \alpha_{n_1}}$$

we realize that we now face a problem very similar to the one we just solved for the case of $\eta$ and $Q_{\pm 1}$. The graded commutator of $J_a^{(1)}$ with $C_{b}^{(1)}(\lambda^{(1)})$ can be computed by using (2.37) and (3.19)

$$\left[ J_a^{(1)}, C_{b}^{(1)}(\lambda^{(1)}) \right] = \delta_{ab}D^{(1)}(\lambda^{(1)}) - A_{ab}^{(1)}(\lambda^{(1)}) \quad (3.21)$$

The action of $J_a^{(1)}$ on the states $|\lambda_1^{(1)}, \ldots, \lambda_{n_1}^{(1)}\rangle G$ now takes a form very similar to (3.10)

$$J_a^{(1)} |\lambda_1^{(1)}, \ldots, \lambda_{n_1}^{(1)}\rangle G =$$

$$\sum_{k=1}^{n_1} \prod_{i=1}^{k-1} C_{a_i}^{(1)}(\lambda_i^{(1)}) \left( \delta_{a_k} D^{(1)}(\lambda_k^{(1)}) - A_{a_k}^{(1)}(\lambda_k^{(1)}) \right) \prod_{j=k+1}^{n_1} C_{a_j}^{(1)}(\lambda_j^{(1)}) |0^{(1)}\rangle G^{\alpha_1, \ldots, \alpha_{n_1}} (-1)^{\epsilon_a} \sum_{i=1}^{k-1} \epsilon_{a_i}$$

$$= \sum_{k=1}^{n_1} \prod_{i=1}^{k-1} C_{a_i}^{(1)}(\lambda_i^{(1)}) \prod_{j=k+1}^{n_1} C_{a_j}^{(1)}(\lambda_j^{(1)}) (-1)^{\epsilon_a} \sum_{i=1}^{k-1} \epsilon_{a_i}$$

$$= \sum_{k=1}^{n_1} \prod_{i=1}^{k-1} C_{b_i}^{(1)}(\lambda_i^{(1)}) \prod_{j=k+1}^{n_1} C_{b_j}^{(1)}(\lambda_j^{(1)}) |0^{(1)}\rangle \left( \Omega_k^{(1)} \right)^{b_1, \ldots, b_{k-1} a b_{k+1} \ldots b_{n_1}} (-1)^{\epsilon_a} \sum_{i=1}^{k-1} \epsilon_{a_i} \quad (3.22)$$

The computation of the quantities $\Omega_k^{(1)}$ follows the same strategy employed above for the determination of $\Omega_j$, and after some computations we find that all $\Omega_k^{(1)}$ vanish, because the unwanted terms for the nesting cancel, i.e.,

$$\left( \tilde{A}_k^{(1)} G \right)^{b_1, \ldots, b_{k-1} a b_{k+1} \ldots b_n} + \left( \tilde{A}_k^{(1)} G \right)^{b_1, \ldots, b_{k-1} a b_{k+1} \ldots b_n} = 0$$

This completes the proof of the lowest weight property for $\tilde{Q}^\dagger_{\pm 1}$.  

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Lowest weight property for $S$

Noting that

$$S = \sum_{k=1}^{L} \varepsilon_{k}^{2l} = i \lim_{\mu \rightarrow \infty} \mu A_{12}(\mu)$$

(3.23)

we find

$$S |\lambda_1 \ldots \lambda_n \rangle = i \lim_{\mu \rightarrow \infty} \mu B^{(2)}(\mu) |\lambda_1^{(2)} \ldots \lambda_n^{(2)} \rangle =: J^{(2)} |\lambda_1^{(2)} \ldots \lambda_n^{(2)} \rangle.$$  

(3.24)

Therefore the condition that $S$ annihilates all Bethe Ansatz states with finite spectral parameters is equivalent to

$$0 = i \lim_{\mu \rightarrow \infty} \mu B^{(2)}(\mu) |\lambda_1^{(2)} \ldots \lambda_n^{(2)} \rangle =: J^{(2)} |\lambda_1^{(2)} \ldots \lambda_n^{(2)} \rangle.$$  

(3.25)

From the intertwining relation (2.47) we deduce

$$[J^{(2)}, C^{(2)}(\lambda^{(2)})] = D^{(2)}(\lambda^{(2)}) - A^{(2)}(\lambda^{(2)}),$$

(3.26)

which enables us to show by direct computation that

$$J^{(2)} |\lambda_1^{(2)} \ldots \lambda_n^{(2)} \rangle = 0$$

due to the cancellation of the unwanted terms for the second nesting

$$(\tilde{\Lambda}_k^{(2)}) + (\Lambda_k^{(2)}) = 0.$$  

This completes our proof of (3.1).

4. Higher Conservation Laws

In this section we derive explicit expressions for the conservation laws $H_{(3)}$ and $H_{(4)}$ (which involve interactions between 3 and 4 neighbouring sites respectively). We use a generalisation of Tetel’man’s method\cite{20,21} to the supersymmetric case\cite{9}.

Let us define the “boost”-operator

$$B = \sum_{n} n H^{n,n+1}_{(2)},$$

(4.1)
where \( H_{n,n+1}^{n,n+1} \) is the density of the Hamiltonian given by the right hand side of (2.15). This operator obviously violates periodicity on the finite chain, but if used in commutators (which “differentiate” the linear n-dependence) it yields formally periodic expressions.

The integrals of motion can be successively obtained by commutation with the boost-operator

\[
H_{(k+1)} = i[B, H_{(k)}] = i[\tilde{B}, H_{(k)}],
\]

where

\[
\tilde{B} = \sum_{n} n H_{n,n+1}^{0}.
\]

Using (4.2) we obtain explicit expressions for higher conservation laws as follows: We have shown in [1] that the Hamiltonian \( H^{0} \) can be written in terms of \( u(2|2) \) generators \( \{ J_{j,\alpha}, \alpha = 1 \ldots 16 \} = \{ S_{j}, S_{j}^{\dagger}, S_{j}, \eta_{j}, \eta_{j}^{\dagger}, I_{j}, X_{j}, Q_{j,\sigma}, Q_{j,\sigma}^{\dagger}, \tilde{Q}_{j,\sigma}, \tilde{Q}_{j,\sigma}^{\dagger} | \sigma = \pm 1 \} \) as

\[
H^{0} = - \sum_{j=1}^{L} H_{j,j+1}^{0} = - \sum_{j=1}^{L} \Pi_{j,j+1} = - \sum_{j=1}^{L} K_{\alpha\beta}^{\gamma} J_{j,\alpha} J_{j,\beta}.
\]

The structure functions of \( u(2|2) \) are defined by

\[
[J_{k,\alpha}, J_{k,\beta}] = J_{k,\alpha} J_{k,\beta} - (-1)^{\epsilon_{\alpha}^{\epsilon_{\beta}}} J_{k,\beta} J_{k,\alpha} = f_{\alpha\beta}^{\gamma} J_{k,\gamma},
\]

\( H_{(3)} \) can be obtained by commutation with the boost operator \( \tilde{B} \)

\[
H_{(3)} = i[\tilde{B}, H_{(2)}] = i[\tilde{B}, H^{0}] = i \sum_{k=1}^{L} [H^{0}_{k+1,k+2}, H^{0}_{k,k+1}]
\]

\[
= -i \sum_{k=1}^{L} K_{\alpha\beta}^{\gamma} K_{\gamma\delta}^{\epsilon} f_{\beta\gamma}^{\epsilon} J_{k-1,\alpha} J_{k,\epsilon} J_{k+1,\delta}.
\]

The next highest conservation law can be computed along similar lines and we find

\[
H_{(4)} = i[\tilde{B}, H_{(3)}] = -2 \sum_{k=1}^{L} K_{\mu\nu}^{\alpha\beta} K_{\gamma\delta}^{\epsilon} f_{\beta\gamma}^{\epsilon} f_{\delta\mu}^{\omega} J_{k-1,\alpha} J_{k,\epsilon} J_{k+1,\omega} J_{k+2,\nu}
\]

\[
+ \sum_{k=1}^{L} P^{k-1,k+1} - 2 \sum_{k=1}^{L} \Pi^{k,k+1},
\]

where \( P^{k-1,k+1} \) is a graded permutation operator between the sites \( k - 1 \) and \( k + 1 \) with definition

\[
P^{k-1,k+1} = \Pi^{k-1,k} \Pi^{k,k+1} \Pi^{k-1,k}.
\]

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A. Appendix : Computation of the “unwanted terms” for the FFBB grading

In this section we compute the so-called “unwanted terms” in the expressions (2.29) and (2.30). These unwanted terms are characterized by containing a creation operator $C_a$ with spectral parameter (SP) $\mu$ in place of a creation operator with SP $\lambda_k$ (or $\lambda_k^{(1)}$, $\lambda_k^{(2)}$ for the nestings). The condition of cancellation of the unwanted terms leads to the Bethe equations. In order to obtain the expression for $\tilde{\Lambda}_k$, we first move the creation operator with SP $\lambda_k$ to the first place in (2.25), using (2.26)

$$\prod_{i=1}^{n} C_{a_i} (\lambda_i) = C_{b_{k}} (\lambda_k) \prod_{i=1}^{k-1} C_{b'_i} (\lambda_i) \prod_{j=k+1}^{n} C_{a_j} (\lambda_j) S(\lambda_k)^{b_{k} \ldots b_{k}'} S_{a_{1} \ldots a_{k}} ,$$

(A.1)

To get an unwanted term, we now have to use the second term in (2.26) to move $D$ past $C_{b_{k}} (\lambda_k)$, and then always the first term in (2.26) to move $D$ (which now carries SP $\lambda_k$) to the very right, until it hits the vacuum, on which it acts according to (2.24). This way we obtain

$$\left( \tilde{\Lambda}_k F \right)^{b_{1} \ldots b_{n}} = S(\lambda_k)^{b_{1} \ldots b_{k}} F^{a_{1} \ldots a_{k}} \left( -\frac{b(\lambda_k - \mu)}{a(\lambda_k - \mu)} \right) \prod_{i=1}^{n} \frac{1}{a(\lambda_i - \lambda_k)} .$$

(A.2)

The computation of $\Lambda_k$ is more complicated. We first derive an expression for the contribution of $A_{11}(\mu)$, which we denote by $\Lambda_{k,1}$. Proceeding along the same lines as in the computation of $\tilde{\Lambda}_k$, we find

$$\left( \Lambda_{k,1} F \right)^{b_{1} \ldots b_{n}} = -S(\lambda_k)^{c_{1} \ldots c_{k}} F^{a_{1} \ldots a_{k}} \left( \frac{b(\mu - \lambda_k)}{a(\mu - \lambda_k)} \right) \prod_{i=1}^{n} \frac{1}{a(\lambda_i - \lambda_k)}$$

$$\times r(\lambda_k - \lambda_1)^{d_{1}c_{1}} r(\lambda_k - \lambda_2)^{d_{2}c_{2}} \ldots r(\lambda_k - \lambda_{k-1})^{d_{k-1}c_{k-1}}$$

$$\times r(\lambda_k - \lambda_{k+1})^{d_{k+1}c_{k+1}} r(\lambda_k - \lambda_{k+2})^{d_{k+2}c_{k+2}} \ldots r(\lambda_k - \lambda_n)^{d_{n}c_{n}}$$

$$\times (a(\lambda_k))^L \delta_{d_{n-1},1} (-1)^{\sum_{i=1}^{n} \epsilon_{b_{i}} \epsilon_{c_{i}}} .$$

(A.3)

The $\delta_{d_{n-1},1}$ stems from the action of $A_{11,1}(\lambda_k)$ (which is what we get after moving $A$ past all the $C$’s) on the vacuum. We also had to include a $\delta_{b_{k},1}$ due to the fact that in (2.30) we denoted by $b_k$ the index of the $C$ with SP $\mu$. The contributions of $A_{22}(\mu)$ and $A_{33}(\mu)$ can be obtained in a similar way, and $\Lambda_k = \Lambda_{k,1} + \Lambda_{k,2} + \Lambda_{k,3}$ is found to be

$$\left( \Lambda_{k} F \right)^{b_{1} \ldots b_{n}} = -S(\lambda_k)^{c_{1} \ldots c_{k}} F^{a_{1} \ldots a_{k}} \left( \frac{b(\mu - \lambda_k)}{a(\mu - \lambda_k)} \right) \prod_{i=1}^{n} \frac{1}{a(\lambda_i - \lambda_k)} (a(\lambda_k))^L$$

$$\times r(\lambda_k - \lambda_1)^{d_{1}c_{1}} r(\lambda_k - \lambda_2)^{d_{2}c_{2}} \ldots r(\lambda_k - \lambda_{k-1})^{d_{k-1}c_{k-1}} (-1)^{\sum_{i=1}^{n} \epsilon_{b_{i}} \epsilon_{c_{i}}}$$

$$\times r(\lambda_k - \lambda_{k+1})^{d_{k+1}c_{k+1}} r(\lambda_k - \lambda_{k+2})^{d_{k+2}c_{k+2}} \ldots r(\lambda_k - \lambda_n)^{d_{n}c_{n}} .$$

(A.4)
This expression can be simplified by carrying out the contractions over the summation indices $c_1, \ldots, c_k$. Noting that

$$r(\lambda_1 - \lambda_2)^{b_1 a_1} r(\lambda_2 - \lambda_1)^{c_1 b_2} = f_{a_2 c_2} = \delta_{a_1 c_1} \delta_{a_2 c_2}, \quad (A.5)$$

we are able to perform all $c_i$-summations with the result

$$S(\lambda_k)^{c_k \ldots c_1} r(\lambda_k - \lambda_1)^{d_k c_1} \ldots r(\lambda_k - \lambda_{k-1})^{d_{k-1} c_{k-1}} = \prod_{i=1}^{k-1} \delta_{a_i b_i} \delta_{d_{k-1} a_k}. \quad (A.6)$$

Now we transform the remaining $r$-matrices into $L$-operators, using the identity

$$r(\lambda)^{cd} = (r(\lambda)\Pi^{(1)})^{ad} (-1)^{\epsilon_a \epsilon_c} = (-1)^{\epsilon_a \epsilon_c} L^{(1)}(\lambda)^{ad}. \quad (A.7)$$

Thus we obtain our final form for the unwanted terms due to $-A_{11}(\mu) - A_{22}(\mu) + A_{33}(\mu)$

$$(\Lambda_k F)^{b_1 \ldots b_n} = \frac{b(\mu - \lambda_k)}{a(\mu - \lambda_k)} \prod_{i=1}^{n} \frac{1}{a(\lambda_k - \lambda_i)} (a(\lambda_k))^L \ F^{a_n \ldots a_{b_{k-1}} \ldots b_1}$

$$\times L^{(1)}(\lambda_k - \lambda_{n-1})^{b_{k-1} a_{n-1}} L^{(1)}(\lambda_{n-1} - \lambda_{n-2})^{d_{n-2} a_{n-2}} \ldots L^{(1)}(\lambda_{k+1} - \lambda_k)^{d_k a_k}$

$$\times (-1) \sum_{i \neq k} \epsilon_{b_i f_i} + \sum_{j=1}^{n} \epsilon_{c_j a_j + \epsilon_{b_k f_k}}. \quad (A.8)$$

We now insert (A.8) and (A.2) into the condition (2.33) for the cancellation of the unwanted terms, and multiply the resulting equation by the inverse of $S(\lambda_k)^{b_1 \ldots b_k}$, which satisfies $(S^{-1}(\lambda_k))^{p_1 \ldots p_k}$, $S(\lambda_k)^{b_1 \ldots b_k} = \prod_{i=1}^{k} \delta_{a_i p_i}$, and which is computed via (A.5). After carrying out these manipulations we arrive at (2.34).

The unwanted terms for the nestings can be computed along similar lines, we refer to [9] for more detailed derivations and just give the results. For the first nesting we find

$$\left(\Lambda_k^{(1)} G\right)^{g_1 \ldots g_{n_1}} = \frac{b(\mu - \lambda_k)}{a(\mu - \lambda_k)} \prod_{i=1}^{n_1} \frac{1}{a(\lambda_k - \lambda_i)} \prod_{j=1}^{n} a(\lambda_k - \lambda_j) \ G^{f_{n_1} \ldots f_{k} g_{k-1} \ldots g_1} (-1)^{k+1}$$

$$\times L^{(2)}(\lambda_k - \lambda_{n-1})^{g_{n-1} f_{n-1}} L^{(2)}(\lambda_{n-1} - \lambda_{n-2})^{d_{n-2} g_{n-2}} \ldots L^{(2)}(\lambda_{k+1} - \lambda_k)^{d_k f_k}$$

and

$$\left(\Lambda_k^{(1)} G\right)^{g_1 \ldots g_{n_1}} = S^{(1)}(\lambda_k)^{f_{1} \ldots f_{k}} F^{g_{n_1} \ldots g_{k+1} f_{k} \ldots f_1} \left(\frac{-b(\lambda_k - \mu)}{a(\lambda_k - \mu)}\right) \prod_{i=1}^{n_1} \frac{1}{a(\lambda_i - \lambda_1)} \quad (A.9)$$

where

$$S^{(1)}(\lambda_k)^{f_{1} \ldots f_{k}} =$$

$$r^{(1)}(\lambda_k - \lambda_{k-1})^{f_{k-1} c_{k-1}} r^{(1)}(\lambda_k - \lambda_{k-2})^{f_{k-2} c_{k-2}} \ldots r^{(1)}(\lambda_k - \lambda_1)^{f_1 c_1}. \quad (A.10)$$
The unwanted terms for the second nesting are

\[ \Lambda_k^{(2)} = \frac{b(\mu - \lambda_k^{(2)})}{a(\mu - \lambda_k^{(2)})} \prod_{j=1}^{n_2} \frac{1}{a(\lambda_j^{(2)} - \lambda_k^{(2)})} \prod_{i=1}^{n_1} a(\lambda_k^{(2)} - \lambda_i^{(1)}) \]

\[ \tilde{\Lambda}_k^{(2)} = \frac{b(\lambda_k^{(2)} - \mu)}{a(\lambda_k^{(2)} - \mu)} \prod_{j=1}^{n_2} \frac{1}{a(\lambda_k^{(2)} - \lambda_j^{(2)})} \prod_{i=1}^{n_1} a(\lambda_k^{(2)} - \lambda_i^{(1)}) \]  

(A.12)

In all the above computations we have used that no two spectral parameters coincide, even if they belong to different nestings (e.g. \( \lambda_k \neq \lambda_j^{(1)} \) \( \forall j, k \)). For the Bethe Ansatz without nesting the wave function vanishes if two spectral parameters coincide. This is sometimes called “Pauli principle in rapidity space”. The implications of coinciding spectral parameters in the nested Bethe Ansatz is currently under investigation\(^{[15]}\).

**B. Appendix : Computation of the “unwanted terms” for the BBFF grading**

In this section we compute the unwanted terms in the expressions (2.64) and (2.65).

To obtain the expression for \( \tilde{\Lambda}_k F \) we first move the creation operator with SP \( \lambda_k \) to the first place in (2.25), using (2.62)

\[ \prod_{i=1}^{n} C_{a_i}(\lambda_i) = C_{e_k}(\lambda_k) \prod_{i=1}^{k-1} C_{e_i}(\lambda_i) \prod_{j=k+1}^{n} C_{a_j}(\lambda_j) \ S(\lambda_k)^{e_1...e_k} \]

\[ S(\lambda_k)^{e_1...e_k} = r_{FFB}(\lambda_k - \lambda_{k-1})^{e_{k-1}a_{k-1}} r_{FFB}(\lambda_k - \lambda_{k-2})^{e_{k-2}a_{k-2}} \cdots r_{FFB}(\lambda_k - \lambda_1)^{e_1c_1} . \]

We then use the second term in (2.62) to move \( D \) past \( C_{b_k}(\lambda_k) \), and then always the first term in (2.62) to move \( D \) (which now carries SP \( \lambda_k \)) to the very right, until it hits the vacuum, on which it acts according to (2.61). We find

\[ (\tilde{\Lambda}_k F)^{b_1...b_n} = S(\lambda_k)^{b_1...b_k} F^{b_{n-k+1}...b_n} \left( \frac{b(\lambda_k - \mu)}{a(\lambda_k - \mu)} \right) \prod_{i=1}^{n} \frac{1}{a(\lambda_k - \lambda_i)} \left( \frac{a(\lambda_k)}{a(-\lambda_k)} \right) . \]  

(B.2)

The computation of the unwanted term \( \Lambda_k F \) proceeds along the same lines as in the FFBB case (Appendix A). After some elementary algebra we arrive at

\[ (\Lambda_k F)^{b_1...b_n} = S(\lambda_k)^{e_1...e_k} F^{a_{n-k+1}...a_n} \left( \frac{b(\mu - \lambda_k)}{a(\mu - \lambda_k)} \right) \prod_{i=1}^{n} \frac{1}{a(\lambda_k - \lambda_i)} \left( a(\lambda_k) \right)^L \]

\[ \times r(\lambda_k - \lambda_1)^{d_1} r(\lambda_k - \lambda_2)^{d_2} \cdots r(\lambda_k - \lambda_{k-1})^{d_{k-1}} \]

\[ \times r(\lambda_k - \lambda_{k+1})^{e_k} r(\lambda_k - \lambda_{k+2})^{e_{k+1}} \cdots r(\lambda_k - \lambda_n)^{e_n} \]  

(B.3)

\[ \times (-1)^{\varepsilon_k + \sum_{i=1}^{n-2} \varepsilon_i + \varepsilon_k \sum_{i \neq k} (\varepsilon_i + 1)} (-1)^{\varepsilon_{b_k} + 1}(\varepsilon_{b_k} + 1) . \]

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The summations over $e_1, \ldots, e_k$ can be carried out by using the identity
\[
r_{FBF}(\lambda)_{e_1} e_2 r_{BBF}(\lambda)_{e_1} (-1)^{e_k} = (-1)^{e_1} (b(\lambda) - a(\lambda)) I_{e_1}^{e_2}.
\] (B.4)

Evaluation of the condition of cancellation of the unwanted terms (2.70) then leads to (2.71).

C. Appendix : Equivalence of the BAE

In this appendix we establish the equivalence of the six different sets of BAE by means of particle-hole transformations in rapidity space\textsuperscript{[22]3}. Let us denote the various sets of BAE by their underlying grading, \textit{i.e.} FBBF, BFBF, BBFF, BFFB, FBFB, and FFBB. We first will demonstrate the equivalence of FBBF and BBFF.

(i) \textit{FBBF} $\leftrightarrow$ \textit{BFBF}

We start by expressing the third set of (2.91)
\[
1 = \prod_{j=1}^{N_i+N_i} \frac{\tilde{\lambda}_m^{(2)} - \lambda_j^{(1)} - i \frac{1}{2}}{\tilde{\lambda}_m^{(2)} - \lambda_j^{(1)} + i \frac{1}{2}}, \quad m = 1, \ldots, N_i.
\] (C.1)
as a polynomial equation of degree $N_i + N_i$
\[
p(w) = \prod_{j=1}^{N_i+N_i} (w - \lambda_j^{(1)} - i \frac{1}{2}) - \prod_{j=1}^{N_i+N_i} (w - \lambda_j^{(1)} + i \frac{1}{2}) = 0 \quad .
\] (C.2)

Among the $N_i + N_i$ roots of (C.2) we consider $N_i$ roots $w_1, \ldots, w_N$, which we identify with $\tilde{\lambda}_m^{(2)}, \ldots, \tilde{\lambda}_N^{(2)}$. The $N_i$ other roots of (C.2) we denote by $w_j'$. Using the residue theorem we can derive the following equality
\[
\sum_{j=1}^{N_i} -i \log \left( \frac{\lambda_j^{(1)} - w_j + i \frac{1}{2}}{\lambda_j^{(1)} - w_j - i \frac{1}{2}} \right) = \sum_{j=1}^{N_i} \frac{1}{2\pi i} \oint_{C_j} dz (-i) \log \left( \frac{\lambda_j^{(1)} - z + i \frac{1}{2}}{\lambda_j^{(1)} - z - i \frac{1}{2}} \right) \frac{d}{dz} \log(p(z)) ,
\] (C.3)

where $C_j$ is a small contour around $w_j$. The branch cut of the logarithm extends from $z_n = \lambda_j^{(1)} + i \frac{1}{2}$ to $z_p = \lambda_j^{(1)} - i \frac{1}{2}$. By deforming the contours on the r.h.s. of (C.3) we arrive the the following equality
\[
\sum_{j=1}^{N_i} -i \log \left( \frac{\lambda_j^{(1)} - w_j + i \frac{1}{2}}{\lambda_j^{(1)} - w_j - i \frac{1}{2}} \right) = - \sum_{j=1}^{N_i} -i \log \left( \frac{\lambda_j^{(1)} - w_j + i \frac{1}{2}}{\lambda_j^{(1)} - w_j - i \frac{1}{2}} \right) - i \log \left( \frac{p(z_n)}{p(z_p)} \right) ,
\] (C.4)

\textsuperscript{3}We thank P.A. Bares for explaining this method to us.
where the last term on the r.h.s. comes from integration around the branch cut. The form of the polynomial \( p \) now implies that

\[
p(z_n) = - \prod_{m=1}^{N_i+N_\downarrow} \left( \lambda^{(1)}_l - \lambda^{(1)}_m + i \right)
\]

\[
p(z_p) = \prod_{m=1}^{N_i+N_\downarrow} \left( \lambda^{(1)}_l - \lambda^{(1)}_m - i \right) .
\]

Inserting (C.5) into (C.4) and exponentiating the result we obtain the identity

\[
\prod_{j=1}^{N_i+N_\downarrow} \frac{\bar{\lambda}^{(1)}_j - w_j + \frac{i}{2}}{\lambda^{(1)}_j - w_j - \frac{i}{2}} = \prod_{k=1}^{N_i} \frac{\lambda^{(1)}_l - w'_k - \frac{i}{2}}{\lambda^{(1)}_l - w'_k + \frac{i}{2}} \prod_{m=1}^{N_i+N_\downarrow} \frac{\lambda^{(1)}_l - \lambda^{(1)}_m + i}{\lambda^{(1)}_l - \lambda^{(1)}_m - i} .
\]

(C.6)

Now we recall that \( w_j = \bar{\lambda}^{(2)}_j \) and use (C.6) in the second set of BAE in (2.91) with the result

\[
\prod_{k=1}^{N_i+N_\downarrow} \frac{\bar{\lambda}^{(1)}_k - \lambda^{(1)}_j - \frac{i}{2}}{\lambda^{(1)}_k - \lambda^{(1)}_j + \frac{i}{2}} = \prod_{m=1}^{N_i} \frac{w'_m - \lambda^{(1)}_j - \frac{i}{2}}{w'_m - \lambda^{(1)}_j + \frac{i}{2}} , \quad j = 1, \ldots, N_i+N_\downarrow .
\]

(C.7)

This is precisely the second set of BAE for the BFBF grading (2.86), if we make the identification \( w'_k = \bar{\lambda}^{(2)}_k \). The third set of the BAE (2.86) is also fulfilled by the spectral parameters \( \bar{\lambda}^{(2)}_k \), because they are roots of the polynomial equation (C.2). The first set of BAE for the BFBF and the FBBF gradings are identical anyway, so that we thus have established the equivalence of the BAE (2.91) and (2.86).

(ii) BFBF ↔ BFFB

We start with the first set of the BAE (2.86) for the BFBF grading

\[
\left( \frac{\bar{\lambda}^{(1)}_l - \frac{i}{2}}{\lambda^{(1)}_l + \frac{i}{2}} \right) \prod_{j=1}^{N_i+N_\downarrow} \frac{\bar{\lambda}^{(1)}_k - \lambda^{(1)}_j - \frac{i}{2}}{\lambda^{(1)}_k - \lambda^{(1)}_j + \frac{i}{2}} = \prod_{j=1}^{L-N_h} \left( w - \lambda^{(1)}_j - \frac{i}{2} \right) \prod_{j=1}^{N_i+N_\downarrow} \left( w - \lambda^{(1)}_j + \frac{i}{2} \right) = 0 .
\]

(C.8)

and rewrite it as the polynomial equation

\[
p(w) = \left( w - \frac{i}{2} \right) \prod_{j=1}^{N_i+N_\downarrow} (w - \lambda^{(1)}_j + \frac{i}{2}) - \left( w + \frac{i}{2} \right) \prod_{j=1}^{L-N_h} (w - \lambda^{(1)}_j - \frac{i}{2}) = 0 .
\]

(C.9)

Partitioning the roots of (C.9) in the two sets \( \{ w_k | k = 1 \ldots N_i+N_\downarrow \} = \{ \bar{\lambda}^{(2)}_k | k = 1 \ldots N_i+N_\downarrow \} \) and \( \{ w'_j | j = 1 \ldots L-N_h \} \) and proceeding along the same lines as in step (i), we can derive the following equality

\[
\prod_{k=1}^{N_i+N_\downarrow} \frac{\lambda^{(1)}_l - w_k + \frac{i}{2}}{\lambda^{(1)}_l - w_k - \frac{i}{2}} = \prod_{j=1}^{L-N_h} \frac{\lambda^{(1)}_l - w'_j - \frac{i}{2}}{\lambda^{(1)}_l - w'_j + \frac{i}{2}} \prod_{m=1}^{N_i+N_\downarrow} \frac{\lambda^{(1)}_l - \lambda^{(1)}_m + i}{\lambda^{(1)}_l - \lambda^{(1)}_m - i} .
\]

(C.10)
Recalling that $w_k = \tilde{\lambda}_k$ and inserting (C.10) into the second set of BAE for the BFBF (2.86) grading and using that $L = N_b + N_e$, we obtain

$$
\prod_{m=1}^{N_l} \frac{\tilde{\lambda}_m^{(2)} - \lambda_j^{(1)} - i \frac{\lambda_j^{(1)}}{2}}{\tilde{\lambda}_m^{(2)} - \lambda_j^{(1)} + i \frac{\lambda_j^{(1)}}{2}} = \prod_{l=1}^{N_e+N_l} \frac{\tilde{\lambda}_l^{(1)} - w_j' - i \frac{\lambda_j^{(1)}}{2}}{\tilde{\lambda}_l^{(1)} - w_j' + i \frac{\lambda_j^{(1)}}{2}} \prod_{k=1}^{N_l+N_e} \frac{\tilde{\lambda}_k^{(1)} - \lambda_j^{(1)} + i \frac{\lambda_j^{(1)}}{2}}{\tilde{\lambda}_k^{(1)} - \lambda_j^{(1)} - i \frac{\lambda_j^{(1)}}{2}}, \quad j = 1, \ldots, N_l+N_e. \tag{C.11}
$$

Identifying $w_j'$ with $\tilde{\lambda}_j$ this implies the equivalence of the BAE (2.86) and (2.88).

(iii) $BFFB \leftrightarrow FBFB$

The equivalence of (2.88) and (2.83) can be shown by an analogous computation to the one in step (i).

(iv) $BFBF \leftrightarrow BBFF$

We start by converting the second set of the BFBF BAE (2.86) into the polynomial equation

$$
p(w) = \prod_{k=1}^{N_b+N_l} (\tilde{\lambda}_k - w - i \frac{\lambda_k^{(1)}}{2}) \prod_{j=1}^{N_l} (w - \tilde{\lambda}_j^{(2)} - i \frac{\lambda_j^{(1)}}{2}) - \prod_{k=1}^{N_b+N_l} (\tilde{\lambda}_k - w + i \frac{\lambda_k^{(1)}}{2}) \prod_{j=1}^{N_l} (w - \tilde{\lambda}_j^{(2)} + i \frac{\lambda_j^{(1)}}{2}) = 0. \tag{C.12}
$$

We partition the roots of (C.12) into the sets $\{w_k | k = 1 \ldots N_l+N_l\} = \{\lambda_k^{(1)} | k = 1 \ldots N_l+N_l\}$ and $\{w_j' | j = 1 \ldots N_b\}$. Proceeding along the same lines as in step (i), we can derive the following equalities

$$
\prod_{j=1}^{N_l+N_e} \frac{\tilde{\lambda}_k - w_j' - i \frac{\lambda_k^{(1)}}{2}}{\tilde{\lambda}_k - w_j' + i \frac{\lambda_k^{(1)}}{2}} = \prod_{l=1}^{N_b+N_l} \frac{\tilde{\lambda}_l^{(2)} - \lambda_k + i \frac{\lambda_k^{(1)}}{2}}{\tilde{\lambda}_l^{(2)} - \lambda_k - i \frac{\lambda_k^{(1)}}{2}}, \quad k = 1 \ldots N_b + N_e
$$

and

$$
\prod_{j=1}^{N_l+N_e} \frac{\tilde{\lambda}_m^{(2)} - w_j - i \frac{\lambda_m^{(1)}}{2}}{\tilde{\lambda}_m^{(2)} - w_j + i \frac{\lambda_m^{(1)}}{2}} = \prod_{l=1}^{N_b+N_l} \frac{\tilde{\lambda}_l^{(2)} - \lambda_m^{(2)} + i \frac{\lambda_m^{(1)}}{2}}{\tilde{\lambda}_l^{(2)} - \lambda_m^{(2)} - i \frac{\lambda_m^{(1)}}{2}}, \quad m = 1 \ldots N_l. \tag{C.13}
$$

Inserting (C.13) into the BFFB BAE (2.88) and identifying the $w_j'$ with $\tilde{\lambda}_j^{(1)}$ in the BBFF BAE, we see that (2.88) and (2.78) are equivalent.

(v) $FBFB \leftrightarrow FFBB$

This step is analogous to step (iv).
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