Time-Evolution of the Coherent and the Squeezed States of Many-Body Systems Based on the Basic Idea of the Boson Mapping and the TDHF Method

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Various works performed by the present authors in the 1990s are reviewed. The topics discussed in this paper are mainly related to the time-evolution of the coherent and the squeezed states of the systems obeying the $su(2)$- and the $su(1,1)$-algebra. The formulations are based on the basic idea of the boson mapping and the TDHF theory in canonical form. Under the time-dependent variational procedure for trial states appropriately chosen, the time-evolution of the systems under investigation is described. Further, it is shown that this method enables us to obtain the classical counterparts of the original quantal systems.

§1. Introduction and preliminaries

For the microscopic studies of nuclear collective motions, the year 1960 is unforgettable. In this year, a theory was proposed by Marumori, also independently by Arvieu & Veneroni and Baranger. This theory is called the quasi-particle random phase approximation (QRPA) and, with the help of this theory, microscopic structures of collective vibrational states, especially, the first excited states of spherical even nuclei, were well described. Under the success of the QRPA theory for the first excited states, the next was a problem how to describe the higher excited states, for example, the well-known $(0^+, 2^+, 4^+)$ triplet observed in the second excited states. In response to the above-mentioned situation, in 1962, Belyaev & Zelevinsky proposed so-called boson expansion theory and slightly later, in 1964, Marumori proposed also boson expansion theory, together with Yamamura (one of the present authors) and Tokunaga. The former and the latter are called the BZ and the MYT method. Both are suitable for the case of fermion-pairs. The case of particle-hole pairs was formulated by da Providência (one of the present authors) & Weneser and Marshalek. These are called the boson expansion theory all together. In Ref., the relation between the BZ and the MYT expansion was investigated from the side of the former form. We can find various further studies concerning the boson expansion in the review by Klein & Marshalek in 1991.

However, we should draw a distinction between the BZ and the MYT method. The former is based on the boson realization of fermion-pair algebra (Lie algebra) and the latter based on the mapping from fermion- to boson-space, which is called the boson mapping. The boson mapping is independent from the Lie algebra and,
if the Lie algebra is treated in the framework of the boson mapping, we can arrive at its boson realization. In this sense, it should be noted that the boson mapping is conceptually wider than the boson realization of the Lie algebra. However, it is not necessary for the boson mapping to restrict ourselves to the fermion- and the boson-space. For example, let us consider a quantal system which can be described in the framework of the orthogonal set \{ |n \rangle; n = 0, 1, 2, \ldots, N \}. It is not always many-fermion system. Next, we take up another set \{ |n \rangle \}, which is not always composed of the boson operators, but, here, following the boson mapping, composed of a kind of boson operator (\hat{a}, \hat{a}^*):

\begin{equation}
|n\rangle = \frac{1}{\sqrt{n!}}(\hat{a}^*)^n|0\rangle . \tag{1.1}
\end{equation}

Then, we set up the one-to-one correspondence between the two sets:

\begin{equation}
|n\rangle \sim |n\rangle . \quad (n = 0, 1, 2, \ldots, N) \tag{1.2}
\end{equation}

In the boson mapping, the following operator \hat{U}_N plays a central role:

\begin{equation}
\hat{U}_N = \sum_{n=0}^{N} |n\rangle\langle n| . \tag{1.3}
\end{equation}

With the use of \hat{U}_N, we have

\begin{align}
\hat{U}_N|n\rangle &= |n\rangle , \quad (n = 0, 1, 2, \ldots, N) \tag{1.4} \\
\hat{U}_N^\dagger|n\rangle &= |n\rangle , \quad (n = 0, 1, 2, \ldots, N) \tag{1.5} \\
\hat{U}_N^\dagger|N+k\rangle &= 0 . \quad (k = 1, 2, 3, \ldots) \tag{1.6}
\end{align}

Usually, the space spanned by the set \{ |n\rangle; n = 0, 1, 2, \ldots, N \} is called the physical space. The operators \hat{U}_N and \hat{U}_N^\dagger obey

\begin{align}
\hat{U}_N^\dagger \hat{U}_N &= 1 , \tag{1.7} \\
\hat{U}_N \hat{U}_N^\dagger &= P_N , \quad P_N = \sum_{n=0}^{N} |n\rangle\langle n| . \tag{1.8}
\end{align}

Here, \(P_N\) is a projection operator to the physical space which satisfies \(P_N^\dagger = P_N\) and \(P_N^2 = P_N\). Next, we consider the operator \(\hat{O}_\nu\) in the original space which satisfies

\begin{equation}
\langle n + \nu | \hat{O}_\nu | n \rangle = \sqrt{\frac{n!}{(n + \nu)!}}O_\nu(n) . \quad (\nu \geq 0, n = 0, 1, 2, \cdots, N - \nu) \tag{1.9}
\end{equation}

Here, \(O_\nu(n)\) is a function of \(n\) which is obtained by calculating the matrix element \(\langle n + \nu | \hat{O}_\nu | n \rangle\). Then, in the boson space, \(\hat{O}_\nu\) is transformed to \(\hat{U}_N \hat{O}_\nu \hat{U}_N^\dagger\) and it is expressed as

\begin{align}
\hat{U}_N \hat{O}_\nu \hat{U}_N^\dagger &= P_N \hat{O}_\nu P_N , \tag{1.10} \\
\hat{O}_\nu &= (\hat{a}^*)^\nu O_\nu(\hat{n}) , \quad \hat{n} = \hat{a}^* \hat{a} . \tag{1.11}
\end{align}
The above is an outline of the boson mapping and in later section, we will use the above-mentioned scheme.

On the other hand, there exists another powerful approach to the microscopic theory of collective motion, which has been called the time-dependent Hartree-Fock (TDHF) theory. At the early stage, the idea of the TDHF theory was proposed by Nogami and slightly later by Marumori for the case of the small amplitude vibrational motion. The frequency of the small fluctuation around a static HF field is calculated in the frames of the TDHF theory. At the middle of the 1970s, the TDHF theory was revived not only for the study of anharmonic vibration but also for the studies of heavy-ion reaction and nuclear fissions. Many ideas or forms were proposed for various problems by many authors. Especially, in 1980, a canonical form for the TDHF theory was proposed by Marumori, together with Maskawa, Sakata and Kuriyama (one of the present authors). The TDHF theory is a possible method for obtaining an approximate solution of the time-dependent Schrödinger equation. First, as a trial state for the time-dependent variation, we prepare a Slater determinant under a definite form containing parameters. Through the variational procedure, we obtain certain differential equations with respect to $t$ and, then, by solving under an appropriate initial condition, the time-dependence of the parameters is determined. Thus, we have an approximate solution of the original time-dependent Schrödinger equation. In the canonical form of the TDHF theory, the parameters are required to obey certain relation which the present authors call the canonicity condition. Then, the parameters can be regarded as canonical variables in classical mechanics and the certain differential equation with respect to $t$ is reduced to the Hamilton equation of motion in classical mechanics, the Hamiltonian of which is given as the expectation value of the starting quantal Hamiltonian for the trial function.

Introduction of the canonicity condition into the TDHF theory suggests us to formulate the variation for the time-dependent Schrödinger equation in relation to classical mechanics for any quantal system. Let $\hat{H}$ denote the Hamiltonian under investigation, which is not always many-fermion system. For $\hat{H}$, we set up the time-dependent Schrödinger equation

$$
\hat{H}\langle \text{exact} \rangle = \frac{\partial}{\partial t}\langle \text{exact} \rangle .
$$

As for an approximate solution of Eq. (1.12), we prepare a trial state $|p,q\rangle$. Here, $p$ and $q$ denote two real parameters obeying the relation

$$
\langle p,q|i\hbar\frac{\partial}{\partial q}|p,q\rangle = p + \frac{\partial S}{\partial q},
$$

$$
\langle p,q|i\hbar\frac{\partial}{\partial p}|p,q\rangle = \frac{\partial S}{\partial p}.
$$

(1.13)

Here, $S$ is arbitrary real function of $p$ and $q$. The present authors call the relation (1.13) the canonicity condition. Then, we have the relation

$$
\langle p,q|i\hbar\frac{\partial}{\partial t}|p,q\rangle = p\dot{q} - H(p,q) + \frac{dS}{dt},
$$

$$
H(p,q) = \langle p,q|\hat{H}|p,q\rangle .
$$

(1.14)

(1.15)
The relations (1.14) and (1.15) can be regarded as the Lagrangian and Hamiltonian, respectively, and the variation gives us

$$\delta \int \langle p, q | i\hbar \frac{\partial}{\partial t} - \hat{H} | p, q \rangle dt = \delta \int (p\dot{q} - H(p, q)) dt = 0. \quad (1.16)$$

Thus, we have

$$\dot{q} = \frac{\partial H}{\partial p}, \quad \dot{p} = -\frac{\partial H}{\partial q}. \quad (1.17)$$

The differential equation is nothing but the Hamilton equation of motion in classical mechanics. Therefore, $(p, q)$ can be regarded as canonical variables. By solving Eq. (1.17) under an appropriate initial condition, $p$ and $q$ can be determined as functions of $t$. The substitution of the solution of Eq. (1.17) into $| p, q \rangle$ gives us an approximate solution of Eq. (1.12). If $S$ is chosen as the form $S = pq/2$, the canonicity condition is written as

$$\langle p, q | \partial X | p, q \rangle = X^*, \quad \langle p, q | \partial X^* | p, q \rangle = -\frac{X}{2}, \quad (1.18)$$

$$p = i\sqrt{\frac{\hbar}{2}}(X^* - X), \quad q = \sqrt{\frac{\hbar}{2}}(X^* + X). \quad (1.19)$$

Of course, $(X, X^*)$ denote the complex parameters. The above is an outline of the variational method for the time-dependent Schrödinger equation which is suggested from the canonical form of the TDHF theory. In later sections, we use the above-mentioned scheme.

The above-mentioned time-dependent variational approach to quantal systems has two characteristic aspects. One is to give an approximate description of time-evolution of the quantum state. The other is to give a possible classical counterpart for an original quantal system under a suitable choice of the trial state. Then, it is expected that the original system is reproduced in a disguised form under an appropriate requantization procedure. This viewpoint was originally proposed by Marshalek and Holzwarth for the case of the time-dependent Hartree-Bogoliubov theory.\[11\] They obtained the Holstein-Primakoff boson representation for the fermion-pair algebra, which is of the same form as that given by the boson mapping.\[6\]

With the aim of formulating the TDHF theory and its extension in the canonical form, the present authors (A. K. & M. Y.) have investigated the above-mentioned idea extensively for the cases of various many-fermion systems. For example, with the aid of the Grassmann variables, odd fermion system can be treated in the framework of the above-mentioned scheme. These works have been reviewed in the paper by the present authors (A. K. & M. Y.).\[12\] Further, the above idea was applied to the investigation of the behavior of the systems in highly excited states also by the present authors (A. K., J. da P. & M. Y.).\[13\] The basic idea for this case is in the description of many-fermion systems under the phase space doubling, which is fundamental in the thermo field dynamics formalism.\[14\] This description gave us the so-called thermal boson expansion and the time-evolution of the thermal states. The above all papers are related with many-fermion systems.
Focussing on the $su(2)$- and $su(1,1)$-algebra, in this paper, we will review some investigations performed mainly by the present authors after 1991. Of course, the topics discussed in this paper are related to the title of this paper. It is well known that the $su(2)$-algebra has played an important and interesting role in the study of many-body physics. Even if restricted to nuclear theory, this algebra has served us not only for schematic understandings of collective dynamics such as the Lipkin model but also for studies of realistic phenomena such as superconducting phase and rotational motion. We can learn various investigations related to the $su(2)$-algebra until 1990 in Refs.\[7,8\]. A possible boson representation of the $su(2)$-algebra was given by Schwinger\[9\] and this form will play a central role in this paper. On the other hand, as was originally discussed by Celeghini, Rasetti, Tarlini and Vitiello\[10\], the $su(1,1)$-algebra is also quite interesting. With the aid of this algebra in the Schwinger boson representation,\[9\] we can describe the “damped and amplified oscillator” quantum mechanically in the conservative form. The investigation of the behavior of systems in highly excited states, such as nuclear phenomena observed in highly excited states, is one of the interesting problems in quantum many-body physics. In the theoretical treatment, it is necessary to deal with the systems at finite temperature. The above means that the $su(1,1)$-algebra presents us a possible entrance to the problems related to finite temperature. The interest of the present authors in the 1990s has been concerned mainly with the time-evolution of many-body systems obeying the $su(2)$- and $su(1,1)$-algebra. Of course, as is clear from the title of this paper, the interest is related with the classical descriptions of quantum systems, and the technique of the boson mapping and the time-dependent variational method are fully used. These investigations are divided into two groups. One is related mainly with time-evolution of the coherent states in many-boson systems\[11-14\] and the other with the squeezed states in many-body systems\[15-18\], some of which were published together with Fujiwara\[19-20\]. Our interest is concerned not only with the coherent and the squeezed state for a simple system but also with those for the systems obeying the $su(2)$- and the $su(1,1)$-algebra. Starting from the conventional boson coherent state, we formulate various forms which are suitable for treating the $su(2)$- and the $su(1,1)$-boson models. Especially, one of the present authors (Y. T.) investigated the formalism, which is partially presented in this paper, in relation to the WKB result and the Maslov phase in the limit of $\hbar \to 0$.\[21\] Modified forms of the conventional Holstein-Primakoff boson representation of the $su(1,1)$-algebra and $q$-boson realizations of the $su_q(2)$- and the $su_q(1,1)$-algebra, which we will not contact with in this paper, are discussed in Refs.\[22,23\] respectively.

In §§2 and 3, starting from the Schwinger boson representation for the $su(2)$- and the $su(1,1)$-algebra, the Holstein-Primakoff boson representations are derived under the use of the basic idea of the boson mapping. Further, with the aid of the coherent state, the classical counterparts of the both algebras are obtained. Section 4 and 5 are devoted to investigating the “damped and amplified oscillation” and the $su(2)$-algebraic model in the framework of the $su(1,1)$-algebra and its coherent state. The idea is found in the phase space doubling appearing in the thermo field dynamics formalism. In particular, the thermal effects observed in their systems are discussed. In §§6 ~ 9, a boson system interacting with the external harmonic
oscillator is described. Under a certain condition for the interaction, the results are almost the same as those obtained in §§4 and 5. In §10, the time-dependent variational approach with the squeezed state is presented and, in §§11 and 12, the squeezed state for the \( su(2) \)-algebraic models of both many-boson and many-fermion systems are discussed. Finally, the concluding remarks are mentioned in §13.

§2. The \( su(2) \)-algebra—The boson realization and its classical counterpart

First, we will recapitulate the boson representation of the \( su(2) \)-spin system in a form suitable for later discussion. For the sake of completeness, various familiar relations will be listed. Let us introduce a boson space constructed in terms of two kinds of boson operators \( \hat{a}, \hat{a}^\dagger \) and \( \hat{b}, \hat{b}^\dagger \). In this space, we define the following three operators:

\[
\hat{S}_+ = \hbar \hat{a}^\dagger \hat{b}, \quad \hat{S}_- = \hbar \hat{b}^\dagger \hat{a}, \quad \hat{S}_0 = \frac{\hbar}{2} (\hat{a}^\dagger \hat{a} - \hat{b}^\dagger \hat{b}).
\]  

(2.1)

Here, \( \hbar \) denotes the Planck constant. The above three operators \( \hat{S}_\pm, \hat{S}_0 \) form the \( su(2) \)-algebra, which obeys

\[
[\hat{S}_+, \hat{S}_-] = 2\hbar \hat{S}_0, \quad [\hat{S}_0, \hat{S}_\pm] = \pm \hbar \hat{S}_\pm.
\]  

(2.2) The form (2.1) is called the Schwinger boson representation of the \( su(2) \)-algebra.\(^{15}\)

The Casimir operator \( \hat{\Gamma}_{su(2)} \) can be written as

\[
\hat{\Gamma}_{su(2)} = \hat{S}_0^2 + \frac{1}{2} (\hat{S}_- \hat{S}_+ + \hat{S}_+ \hat{S}_-) = \hat{S} (\hat{S} + \hbar).
\]  

(2.4)

Here, \( \hat{S} \) and its property are given as

\[
\hat{S} = \frac{\hbar}{2} (\hat{a}^\dagger \hat{a} + \hat{b}^\dagger \hat{b}),
\]  

(2.5)

\[
[\hat{S}, \hat{S}_\pm] = 0.
\]  

(2.6)

We can understand that \( \hat{S} \) is an operator denoting the magnitude of the \( su(2) \)-spin.

The eigenstate of \( \hat{S} \) and \( \hat{S}_0 \) with the eigenvalues \( S \) and \( S_0 \), respectively, is obtained in the form

\[
|S, S_0 \rangle = \frac{1}{\sqrt{(s + s_0)! (s - s_0)!}} (\hat{a}^\dagger)^{s + s_0} (\hat{b}^\dagger)^{s - s_0} |0 \rangle,
\]  

(2.7)

\[
\hat{a} |0 \rangle = \hat{b} |0 \rangle = 0.
\]  

(2.8)

The eigenvalues \( S \) and \( S_0 \) are expressed as

\[
S = \hbar s, \quad S_0 = \hbar s_0,
\]  

(2.9)

\[
s = 0, \frac{1}{2}, \frac{3}{2}, \ldots, \quad s_0 = -s, -s + 1, \ldots, s - 1, s.
\]  

(2.10)
We call the space spanned by the set \{\ket{S,S_0}\} for fixed value of $S$ the $su(2)$-space with the magnitude $S$ and at several occasions, we will use the notation $|s,s_0\rangle$ for $|S,S_0\rangle$.

Following Refs.23) ∼ 25), let us derive the Holstein-Primakoff representation from the Schwinger representation. Basic idea comes from the MYT boson mapping method proposed by Marumori, Yamamura and Tokunaga. For the above-mentioned aim, we prepare a new boson space spanned by the set \{\ket{n} = \sqrt{n!} \cdot (\hat{X}^\ast)^n \ket{0}; n = 0,1,2,\ldots\}. Here, $(\hat{X},\hat{X}^\ast)$ denotes boson operator and $\ket{0}$ is the vacuum for $(\hat{X},\hat{X}^\ast)$.

First, following the MYT boson mapping, we set up the following correspondence between the original and the new boson space:

$$
\ket{s,s_0} \sim \ket{s + s_0} . \quad (S = \hbar s , S_0 = \hbar s_0) \tag{2.11}
$$

The spaces spanned by the set \{\ket{s + s_0}; s_0 = -s,-s+1,\ldots,s-1,s\} and the set \{\ket{2s + k}; k = 1,2,3,\ldots\} are physical and unphysical space, respectively. Under the correspondence (2.11), the following mapping operator $\hat{U}$ is introduced:

$$
\hat{U} = \sum_{s_0 = -s}^{s} \ket{s + s_0}\bra{s,s_0} . \tag{2.12}
$$

The properties of $\hat{U}$ are shown in the form

$$
\hat{U}^\dagger \hat{U} = \sum_{s_0 = -s}^{s} \ket{s,s_0}\bra{s,s_0} = 1 ,
$$

$$
\hat{U} \hat{U}^\dagger = \sum_{s_0 = -s}^{s} \ket{s + s_0}\bra{s + s_0} = P_{2s} . \tag{2.13}
$$

Here, $P_{2s}$ denotes a projection operator to the physical space:

$$
P_{2s}^2 = P_{2s} , \quad P_{2s}^* = P_{2s} . \tag{2.14}
$$

From the above conditions, the following relations are derived:

$$
\hat{U}\ket{s,s_0} = \ket{s + s_0} , \quad (2.15)
$$

$$
\hat{U}^\dagger\ket{s + s_0} = \ket{s,s_0} ,
$$

$$
\hat{U}^\dagger\ket{2s + k} = 0 . \quad (k = 1,2,3,\ldots) \tag{2.16}
$$

With the use of $\hat{U}$, the operators $\hat{S}_{\pm,0}$ are transformed to the form

$$
\hat{U} \hat{S}_{\pm,0} \hat{U}^\dagger = P_{2s} \cdot \hat{S}_{\pm,0} \cdot P_{2s} ,
$$

$$
\hat{S}_+ = \sqrt{\hbar} \hat{X}^\ast \cdot \sqrt{2S - \hbar \hat{X}^\ast \hat{X}} ,
$$

$$
\hat{S}_- = \sqrt{2S - \hbar \hat{X}^\ast \hat{X}} \cdot \sqrt{\hbar} \hat{X} ,
$$

$$
\hat{S}_0 = \hbar \hat{X}^\ast \hat{X} - S . \tag{2.17}
$$
The above operators \( \tilde{S}_{\pm,0} \) satisfy
\[
\begin{align*}
\tilde{S}_{-}^* &= \tilde{S}_{+} , \quad \tilde{S}_{0}^* = \tilde{S}_{0} , \quad (2.18) \\
[\tilde{S}_{+}, \tilde{S}_{-}] &= +2\hbar \tilde{S}_{0} , \quad [\tilde{S}_{0}, \tilde{S}_{\pm}] = \pm \hbar \tilde{S}_{\pm} , \quad (2.19) \\
\tilde{S}_{0}^2 + \frac{1}{2}(\tilde{S}_{-} \tilde{S}_{+} + \tilde{S}_{+} \tilde{S}_{-}) &= S(S + \hbar) . \quad (2.20)
\end{align*}
\]

The relations (2.18) \sim (2.20) correspond to Eqs. (2.2) \sim (2.4). The above is nothing but a refined form given by Marshalek \cite{6} and it is called the Holstein-Primakoff boson representation.

Our third problem is concerned with the coherent state \( |\gamma\rangle \) for the \( su(2) \)-spin system. For the construction of the state \( |\gamma\rangle \), we note the form
\[
|s, s_0\rangle = \frac{1}{(2s)!} \sqrt{\frac{(s - s_0)!}{(s + s_0)!}} \left( \frac{\hat{S}_+}{\hbar} \right)^{s + s_0} \cdot (\hat{b}^*)^{2s} |0\rangle . \quad (2.21)
\]

Then, we set up the following correspondence :
\[
\frac{1}{(2s)!} \sqrt{\frac{(s - s_0)!}{(s + s_0)!}} \left( \frac{\hat{S}_+}{\hbar} \right)^{s + s_0} \sim \exp \left( \frac{v}{u} \cdot \frac{\hat{S}_+}{\hbar} \right) = \exp \left( \frac{v}{u} \cdot \hat{a}^* \hat{b} \right) , \quad (2.22)
\]
\[
(\hat{b}^*)^{2s} |0\rangle \sim \exp \left( \sqrt{\frac{2}{\hbar} wv} \hat{b}^* \right) |0\rangle . \quad (2.22)
\]

Here, \( w \) and \( v \) denote complex parameters and \( u \) is given as
\[
u = \sqrt{1 - |v|^2} . \quad (2.23)
\]

Under the correspondence (2.22), we define the state \( |\gamma\rangle \) in the form
\[
|\gamma\rangle = M_c \cdot \exp \left( \frac{v}{u} \cdot \frac{\hat{S}_+}{\hbar} \right) \cdot \exp \left( \sqrt{\frac{2}{\hbar} wv} \hat{b}^* \right) |0\rangle . \quad (2.24)
\]

Here, \( M_c \) denotes the normalization constant given by
\[
M_c = \exp \left( -\frac{|w|^2}{\hbar} \right) . \quad (2.25)
\]

We can see that in the state \( |\gamma\rangle \) the part \( \exp (\sqrt{2/\hbar} wv \hat{b}^*) |0\rangle \) is a superposition of the states with the maximum weight (the operation of \( \hat{S}_- \) on this part makes it vanish) and the part \( \exp [(v/u) \cdot \hat{S}_+ / \hbar] \) denotes a superposition of the raising operation. The state \( |\gamma\rangle \) can be rewritten in the form
\[
|\gamma\rangle = \exp \left( -\frac{|w|^2}{\hbar} \right) \cdot \exp \left[ \sqrt{\frac{2}{\hbar}} w(v \hat{a}^* + u \hat{b}^*) \right] |0\rangle . \quad (2.26)
\]

It may be clear that \( |\gamma\rangle \) is a boson coherent state, which satisfies
\[
\hat{a} |\gamma\rangle = \sqrt{\frac{2}{\hbar}} wv |\gamma\rangle , \quad \hat{b} |\gamma\rangle = \sqrt{\frac{2}{\hbar}} wu |\gamma\rangle . \quad (2.27)
\]
In other representation, we have

\[
\langle v^* \hat{a} + u \hat{b} \rangle |\gamma\rangle = \sqrt{\frac{\gamma}{\hbar}} w |\gamma\rangle ,
\]

\[
\langle u \hat{a} - v \hat{b} \rangle |\gamma\rangle = 0 .
\]

(2.28)

Here, \( v^* \hat{a} + u \hat{b} \) and \( u \hat{a} - v \hat{b} \) represent annihilations of independent boson operators.

The expectation values of \( \hat{S} \) and \( \hat{S}_{\pm,0} \) are calculated in the following form :

\[
\langle \gamma | \hat{S} | \gamma \rangle = (S)_c = |w|^2 ,
\]

(2.29)

\[
\langle \gamma | \hat{S}_+ | \gamma \rangle = (S_+)_c = 2|w|^2 v^* u ,
\]

\[
\langle \gamma | \hat{S}_- | \gamma \rangle = (S_-)_c = 2|w|^2 u v ,
\]

\[
\langle \gamma | \hat{S}_0 | \gamma \rangle = (S_0)_c = |w|^2 (|v|^2 - u^2) .
\]

(2.30)

Instead of the parameters \((w, w^*)\) and \((v, v^*)\), we introduce the parameters \((\psi, S)\) and \((X, X^*)\), which satisfy the conditions

\[
\langle \gamma | i\hbar \partial_{\psi} | \gamma \rangle = S ,
\]

\[
\langle \gamma | i\hbar \partial S | \gamma \rangle = 0 ,
\]

(2.31)

\[
\langle \gamma | \partial X | \gamma \rangle = + \frac{X^*}{2} ,
\]

\[
\langle \gamma | \partial X^* | \gamma \rangle = - \frac{X}{2} .
\]

(2.32)

We can learn in the TDHF theory that the parameters \((\psi, S)\) and \((X, X^*)\) are regarded as canonical variables in classical mechanics.\[11\].\[12\] The variables \((\psi, S)\) are called the phase angle and the action variable and \((X, X^*)\) are classical variables in boson type. For \(S\), we set up the relation

\[
(S)_c = |w|^2 = S .
\]

(2.33)

The conditions (2.31) and (2.32) give us the following relations :

\[
w = \sqrt{S} \exp \left( - i \frac{\psi}{2} \right) ,
\]

\[
v = \sqrt{\frac{\hbar}{2S}} X ,
\]

\[
u = \sqrt{1 - \frac{\hbar}{2S}} X^* X .
\]

(2.34)

Substituting the relation (2.34) into the expectation values \((S_{\pm,0})_c\) in Eq. (2.30), we have

\[
(S_+)_c = \sqrt{\hbar} X^* \cdot \sqrt{2S - \hbar X^* X} ,
\]

\[
(S_-)_c = \sqrt{2S - \hbar X^* X} \cdot \sqrt{\hbar} X ,
\]

\[
(S_0)_c = \hbar X^* X - S .
\]

(2.35)

The expectation values \((S_{\pm,0})_c\) shown in the relation (2.35) satisfy

\[
(S_-)_c^* = (S_+)_c ,
\]

\[
(S_0)_c^* = (S_0)_c ,
\]

(2.36)

\[
[(S_+)_c, (S_-)_c]_P = -i [2(S_0)_c] ,
\]

\[
[(S_0)_c, (S_{\pm})_c]_P = -i [\pm (S_{\pm})_c] ,
\]

(2.37)

\[
(S_0)_c^2 + \frac{1}{2} [(S_-)_c (S_+)_c + (S_+)_c (S_-)_c] = S^2 .
\]

(2.38)
Here, \([A,B]_P\) denotes the Poisson bracket defined by
\[
[A,B]_P = \left( \frac{\partial A}{\partial \psi} \frac{\partial B}{\partial S} - \frac{\partial B}{\partial \psi} \frac{\partial A}{\partial S} \right) + \frac{1}{i\hbar} \left( \frac{\partial A}{\partial X} \frac{\partial B}{\partial X^*} - \frac{\partial B}{\partial X} \frac{\partial A}{\partial X^*} \right). \tag{2.39}
\]
The relations (2.36) \(\sim\) (2.38) correspond to Eqs. (2.18) \(\sim\) (2.20) and, in this sense, the set \(\{(S_{\pm,0})_c\}\) can be called the classical counterpart of the set \(\{S_{\pm,0}\}\) and, further, \(\{\hat{S}_{\pm,0}\}\). Under the replacements \((X,X^*) \rightarrow (\hat{X},\hat{X}^*)\) and \(S \rightarrow S\), the classical counterpart becomes the quantal form.

§3. The \(su(1,1)\)-algebra—The boson realization and its classical counterpart

Main aim of this section is to recapitulate the boson representation of the \(su(1,1)\)-spin system in a form suitable for later discussion. In this case, various relations are not so familiar to those in the \(su(2)\)-spin system. In a manner similar to the case of the \(su(2)\)-spin, we define the following three operators:
\[
\hat{T}_+ = \hbar \hat{a}^* \hat{b}^* , \quad \hat{T}_- = \hbar \hat{b} \hat{a} , \quad \hat{T}_0 = \frac{\hbar}{2}(\hat{a}^* \hat{a} + \hat{b}^* \hat{b}^*) . \tag{3.1}
\]
The operators \(\hat{T}_{\pm,0}\) form the \(su(1,1)\)-algebra, which obeys
\[
\hat{T}_* = \hat{T}_+ \, , \quad \hat{T}_0^* = \hat{T}_0 \, , \quad [\hat{T}_+, \hat{T}_-] = -2\hbar \hat{T}_0 \, , \quad [\hat{T}_0, \hat{T}_{\pm}] = \pm \hbar \hat{T}_{\pm} . \tag{3.2}
\]
The form (3.1) is called the Schwinger boson representation of the \(su(1,1)\)-algebra. The Casimir operator \(\hat{\Gamma}_{su(1,1)}\) can be written as
\[
\hat{\Gamma}_{su(1,1)} = \hat{T}_0^2 - \frac{1}{2}(\hat{T}_- \hat{T}_+ + \hat{T}_+ \hat{T}_-) = \hat{T}(\hat{T} - \hbar) . \tag{3.4}
\]
Here, \(\hat{T}\) and its property are given as
\[
\hat{T} = -\frac{\hbar}{2}(\hat{a}^* \hat{a} + \hat{b}^* \hat{b}^*) , \tag{3.5}
\]
\[
[\hat{T}, \hat{T}_{\pm,0}] = 0 . \tag{3.6}
\]
The operator \(\hat{T}\) is not positive-definite, but, in a sense analogous to the case of the \(su(2)\)-spin, \(\hat{T}\) may be called the magnitude of the \(su(1,1)\)-spin.

The operator \(\hat{T}\) is not positive-definite and the eigenvalue of \(\hat{T}_0\) is larger than \(\hbar/2\). Then, compared with the case of the \(su(2)\)-spin system in the relations (2.7) \(\sim\) (2.10), the treatment of the eigenstate of \(\hat{T}\) and \(\hat{T}_0\) is rather complicated. The eigenstate of \(\hat{T}\) and \(\hat{T}_0\) with the eigenvalues \(T\) and \(T_0\), respectively, is formally obtained in the form
\[
|T, T_0 \rangle = \frac{1}{\sqrt{(t_0 - t)! (t_0 + t - 1)!}} (\hat{a}^*)^{t_0 - t} (\hat{b}^*)^{t_0 + t - 1} |0 \rangle , \tag{3.7}
\]
\[
T = \hbar t \, , \quad T_0 = \hbar t_0 . \tag{3.8}
\]
Since \( t_0 \geq t \), \( t_0 \geq -t + 1 \) and \( t_0 \geq 1/2 \), the state \( |T, T_0 \rangle \) is divided into two cases:

(i) \( t = \frac{1}{2}, 1, \frac{3}{2}, 2, \ldots \), \( t_0 = t, t + 1, t + 2, \ldots \), \( (3.9) \)

(ii) \( t = 0, -\frac{1}{2}, -1, -\frac{3}{2}, \ldots \), \( t_0 = -t + 1, -t + 2, -t + 3, \ldots \). \( (3.10) \)

In this paper, we restrict ourselves to case (i), i.e., the case \( t \geq 1/2 \). We call the space spanned by the set \( \{|T, T_0 \rangle\} \) for fixed value of \( T \) (\( \geq \bar{\hbar}/2 \)) the \( su(1, 1) \)-space with the magnitude \( T \) and at several occasions, we will use the notation \( |t, t_0 \rangle \) for \( |T, T_0 \rangle \).

In the same manner as that in the case of the \( su(2) \)-spin, let us derive the Holstein-Primakoff representation. \( [3] \) We set up the following correspondence between the original and the new boson space:

\[
|t, t_0 \rangle \sim |t_0 - t \rangle . \quad (T = \hbar t \ , \ T_0 = \hbar t_0) \quad (3.11)
\]

Since \( t_0 = t, t + 1, t + 2, \ldots \), the whole space in the new space is physical. Under the correspondence \( (3.11) \), the following mapping operator \( \hat{V} \) is introduced:

\[
\hat{V} = \sum_{t_0 = t} |t_0 - t \rangle \langle t, t_0 | . \quad (3.12)
\]

The properties of \( \hat{V} \) are shown in the form

\[
\hat{V}^\dagger \hat{V} = \sum_{t_0 = t} |t_0 - t \rangle \langle t_0 - t | = 1 ,
\]

\[
\hat{V} \hat{V}^\dagger = \sum_{t_0 = t} |t_0 - t \rangle \langle t_0 - t | = 1 . \quad (3.13)
\]

In contrast with the relation \( (2.13) \), there does not exist the projection operator such as \( P_{2s} \) in the relation \( (3.13) \). From the above conditions, the following relations are derived:

\[
\hat{V} |t, t_0 \rangle = |t_0 - t \rangle , \quad (3.14)
\]

\[
\hat{V}^\dagger |t_0 - t \rangle = |t, t_0 \rangle . \quad (3.15)
\]

With the use of \( \hat{V} \), the operators \( \hat{T}_{\pm, 0} \) are transformed to the form

\[
\hat{V} \hat{T}_{\pm, 0} \hat{V}^\dagger = \hat{T}_{\pm, 0} ,
\]

\[
\hat{T}_+ = \sqrt{\hbar \hat{X}^* \cdot \sqrt{2T + \hbar \hat{X}^* \hat{X}} } ,
\]

\[
\hat{T}_- = \sqrt{\sqrt{2T + \hbar \hat{X}^* \hat{X}} \cdot \sqrt{\hbar \hat{X}} } ,
\]

\[
\hat{T}_0 = \hbar \hat{X}^* \hat{X} + T . \quad (3.16)
\]

The above operators \( \hat{T}_{\pm, 0} \) satisfy

\[
\hat{T}^*_+ = \hat{T}_+ , \quad \hat{T}^*_0 = \hat{T}_0 , \quad (3.17)
\]

\[
[\hat{T}_+, \hat{T}_-] = -2\hbar \hat{T}_0 , \quad [\hat{T}_0, \hat{T}_{\pm}] = \pm i\hbar \hat{T}_{\pm} , \quad (3.18)
\]

\[
\hat{T}_0^2 - \frac{1}{2} (\hat{T}_- \hat{T}_+ + \hat{T}_+ \hat{T}_-) = T(T - \hbar) . \quad (3.19)
\]
The relations (3.17) ∼ (3.19) correspond to Eqs. (3.2) ∼ (3.4). Judging from the form, it can be called the Holstein-Primakoff boson representation.

In parallel with the case of the su(2)-spin system, our third problem is concerned with the coherent state $|c\rangle$ for the su(1,1)-spin system. For the construction of $|c\rangle$, we note the form

$$|t, t_0\rangle = \frac{1}{\sqrt{(t_0-t)!(t_0+t-1)!}} \left(\frac{T_+}{\hbar}\right)^{t_0-t} \cdot (\hat{b}^*)^{2t-1} |0\rangle.$$  \hspace{1cm} (3.20)

Then, we set up the following correspondence:

$$\frac{1}{\sqrt{(t_0-t)!(t_0+t-1)!}} \left(\frac{T_+}{\hbar}\right)^{t_0-t} \sim \exp\left(\frac{V}{U} \cdot \frac{T_+}{\hbar}\right) = \exp\left( \frac{V}{U} \cdot \hat{a}^* \hat{b}^* \right),$$

$$\exp\left(\sqrt{\frac{2W}{\hbar U}} \hat{b}^* \right) |0\rangle \sim \exp\left(\sqrt{\frac{2W}{\hbar U}} \hat{b}^* \right) |0\rangle.$$  \hspace{1cm} (3.21)

Here, $W$ and $V$ denote complex parameters and $U$ is given as

$$U = \sqrt{1 + |V|^2}.$$  \hspace{1cm} (3.22)

Under the correspondence (3.21), we define the state $|c\rangle$ in the form

$$|c\rangle = N_c \cdot \exp\left(\frac{V}{U} \cdot \frac{T_+}{\hbar}\right) \cdot \exp\left(\sqrt{\frac{2W}{\hbar U}} \hat{b}^* \right) |0\rangle.$$  \hspace{1cm} (3.23)

Here, $N_c$ denotes the normalization constant given by

$$N_c = \frac{1}{U} \exp\left(-\frac{|W|^2}{\hbar}\right).$$  \hspace{1cm} (3.24)

In this case, also, we can see that in the state $|c\rangle$ the part $\exp(\sqrt{2W/\hbar U} \hat{b}^* |0\rangle)$ is a superposition of the states with the maximum weight (the operation of $\hat{T}_+$ on this part makes it vanish) and the part $\exp[(V/U) \cdot \hat{T}_+/\hbar]$ denotes a superposition of the raising operation. The state $|c\rangle$ satisfies

$$(U\hat{b} - V\hat{a}^*) |c\rangle = \sqrt{\frac{2}{\hbar}} W |c\rangle,$$

$$(U\hat{a} - V\hat{b}^*) |c\rangle = 0.$$  \hspace{1cm} (3.25)

The operators $U\hat{b} - V\hat{a}^*$ and $U\hat{a} - V\hat{b}^*$ represent annihilation of independent boson operators and $|c\rangle$ is a coherent state in the sense given in the relation (3.25). As is clear from the form (2.23), the state $|\gamma\rangle$ is one-mode coherent state and, in the present case, $|c\rangle$ is mixed-mode coherent state.

The expectation values of $\hat{T}$ and $\hat{T}_{\pm,0}$ are calculated in the following form:

$$\langle c | \hat{T} | c \rangle = (T)_c = |W|^2 + \frac{\hbar}{2},$$  \hspace{1cm} (3.26)
\[
\langle c| \hat{T}_+ |c \rangle = (T_+)_c = 2 \left( |W|^2 + \frac{\hbar}{2} \right) V^* U ,
\]
\[
\langle c| \hat{T}_- |c \rangle = (T_-)_c = 2 \left( |W|^2 + \frac{\hbar}{2} \right) UV ,
\]
\[
\langle c| \hat{T}_0 |c \rangle = (T_0)_c = \left( |W|^2 + \frac{\hbar}{2} \right) (|V|^2 + U^2) .
\]

(3.27)

In the same manner as in the case of the su(2)-spin system, we introduce the parameters \((\phi, T)\) and \((X, X^*)\) for the old \((W, W^*)\) and \((V, V^*)\). For these variables, we set up the conditions
\[
\langle c| i\hbar \partial \phi |c \rangle = T - \frac{\hbar}{2} , \quad \langle c| i\hbar \partial T |c \rangle = 0 ,
\]
\[
(3.28)
\]
\[
\langle c| \partial X |c \rangle = + \frac{X^*}{2} , \quad \langle c| \partial X^* |c \rangle = - \frac{X}{2} .
\]

(3.29)

The set of the relations (3.28) and (3.29) is called the canonicity condition.

For \(T\), we use the relation
\[
(T)_c = |W|^2 + \frac{\hbar}{2} = T .
\]

(3.30)

The conditions (3.28) and (3.29) give us the form
\[
W = \sqrt{T - \frac{\hbar}{2}} \exp \left( -i \frac{\phi}{2} \right) ,
\]
\[
V = \sqrt{\frac{\hbar}{2T}} X , \quad U = \sqrt{1 + \frac{\hbar}{2T}} X^* X .
\]

(3.31)

Substituting the relation (3.31) into the expectation values \((T_{\pm,0})_c\) in Eq. (3.27), we obtain
\[
(T_+)_c = \sqrt{\hbar} X^* \cdot \sqrt{2T + \hbar X^* X} ,
\]
\[
(T_-)_c = \sqrt{2T + \hbar X^* X} \cdot \sqrt{\hbar} X ,
\]
\[
(T_0)_c = \hbar X^* X + T .
\]

(3.32)

The expectation values \((T_{\pm,0})_c\) shown in the relation (3.32) satisfy
\[
(T_-)_c^* = (T_+)_c , \quad (T_0)_c^* = (T_0)_c ,
\]
\[
[(T_+)_c, (T_-)_c]^P = (-i)[-2(T_0)_c] ,
\]
\[
[(T_0)_c, (T_{\pm})_c]^P = (-i)[\pm (T_{\pm})_c] ,
\]
\[
(T_0)_c^2 - \frac{1}{2} ((T_-)_c(T_+)_c + (T_+)_c(T_-)_c) = T^2 .
\]

(3.33)

Here, \([A, B]^P\) is the same as that shown in Eq. (2.33) under the replacement \((\psi \rightarrow \phi\) and \(S \rightarrow T\)). The relations (3.33) \(\sim\) (3.35) correspond to Eqs. (3.17) \(\sim\) (3.19) and, in this sense, the set \\{(T_{\pm,0})_c\} is the classical counterpart of the set \\{\(T_{\pm,0}\)\}. This is in the same situation as the case of the su(2)-spin.
§4. The \textit{su}(1,1)-algebra—One-dimensional “damped and amplified oscillator” and the \textit{su}(2)-algebraic model

Following Ref.\textsuperscript{[16]}, let us consider two-dimensional space, in which we define the following Lagrangian:
\begin{equation}
L = m\dot{X}\dot{Y} - kXY + m\gamma(X\dot{Y} - \dot{X}Y - \gamma XY) .
\end{equation}
Here, \((X,Y)\) denote two kinds of real variables and \(m, k\) and \(\gamma\) are positive constants. The Lagrange equations of motion lead us to the equations of motion
\begin{align}
    m\ddot{X} &= -(k + m\gamma^2)X - 2m\gamma\dot{X} , \\
    m\ddot{Y} &= -(k + m\gamma^2)Y + 2m\gamma\dot{Y} .
\end{align}

The solutions of Eqs. (4.2) and (4.3) are expressed in the form
\begin{align}
    X &= A_0\exp\left(-\gamma t\right)\cos\left(\omega t + \alpha_0\right) , \\
    Y &= B_0\exp\left(+\gamma t\right)\cos\left(\omega t + \beta_0\right) , \\
    \omega &= \sqrt{\frac{k}{m}} .
\end{align}

The solutions (4.4) and (4.5) show the “damped and amplified oscillation.” The Lagrangian (4.1) can be rewritten as
\begin{equation}
L = \left(\frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2\right) - \left(\frac{1}{2}y^2 - \frac{1}{2}ky^2\right) - m\left[\gamma(xy - \dot{x}y) + \frac{\gamma^2}{2}(x^2 - y^2)\right] .
\end{equation}
Here, \(x\) and \(y\) are defined as
\begin{align}
    x &= \frac{1}{\sqrt{2}}(X + Y) , \\
    y &= \frac{1}{\sqrt{2}}(X - Y) .
\end{align}

The Lagrangian (4.7) gives us the following Hamiltonian:
\begin{equation}
H = \left(\frac{p_x^2}{2m} + \frac{1}{2}m\omega^2x^2\right) - \left(\frac{p_y^2}{2m} + \frac{1}{2}m\omega^2y^2\right) - \gamma(xy^2 + y^2x) , \\
p_x = \frac{\partial L}{\partial \dot{x}} = mx + m\gamma y , \\
p_y = \frac{\partial L}{\partial \dot{y}} = -m\dot{y} - m\gamma x .
\end{equation}

After canonical quantization, i.e., \(x \rightarrow \hat{x}, y \rightarrow \hat{y}, p_x \rightarrow \hat{p}_x\) and \(p_y \rightarrow \hat{p}_y\): \([\hat{x}, \hat{p}_x] = [\hat{y}, \hat{p}_y] = i\hbar\) and \([\text{other combinations}] = 0\), we obtain the quantized Hamiltonian:
\begin{equation}
\hat{H} = \hbar\omega\hat{b}^\dagger\hat{b} - \hbar\omega\hat{a}^\dagger\hat{a} - i\hbar\gamma(\hat{a}^\dagger\hat{b}^* - \hat{b}\hat{a}) \\
= 2\hbar\left(\hat{T} - \frac{\hbar}{2}\right) - i\gamma(\hat{T}_+ - \hat{T}_-) .
\end{equation}
Here, \((\hat{a}, \hat{a}^\dagger)\) and \((\hat{b}, \hat{b}^\dagger)\) are defined as
\begin{align}
    \hat{a} &= \frac{1}{\sqrt{2\hbar}}\left(\sqrt{m\omega}\hat{y} + i\frac{1}{\sqrt{m\omega}}\hat{p}_y\right) , \\
    \hat{a}^\dagger &= \frac{1}{\sqrt{2\hbar}}\left(\sqrt{m\omega}\hat{y} - i\frac{1}{\sqrt{m\omega}}\hat{p}_y\right) , \\
    \hat{b} &= \frac{1}{\sqrt{2\hbar}}\left(\sqrt{m\omega}\hat{x} + i\frac{1}{\sqrt{m\omega}}\hat{p}_x\right) , \\
    \hat{b}^\dagger &= \frac{1}{\sqrt{2\hbar}}\left(\sqrt{m\omega}\hat{x} - i\frac{1}{\sqrt{m\omega}}\hat{p}_x\right) .
\end{align}
We can see that the Hamiltonian (4.11) is expressed in terms of \( \hat{T} \) and \( \hat{T}_\pm \). This means that the quantized “damped and amplified oscillator” can be described in the framework of the \( su(1, 1) \)-algebra, in other word, it is a kind of the \( su(1, 1) \)-spin system.

The above treatment is interpreted as follows: In order to treat the system such as the “damped and amplified oscillator” as an isolated system, so-called phase space doubling is required. Then, the original intrinsic oscillator expressed in terms of the boson \((\hat{b}, \hat{b}^\dagger)\) and the “external environment” expressed in terms of the boson \((\hat{a}, \hat{a}^\dagger)\) appear and the interaction between both system is naturally introduced. This is our interpretation, which permits us to generalize the Hamiltonian (4.11).

Let the Hamiltonian of the intrinsic system be \( \hat{K} \) given in the form

\[
\hat{K} = F(\hbar \hat{b}^\dagger \hat{b}) .
\]  

(4.13)

Here, \( F \) is a function of \( \hbar \hat{b}^\dagger \hat{b} \). Then, a possible generalization is performed by setting the following Hamiltonian:

\[
\hat{H} = F(\hbar \hat{b}^\dagger \hat{b}) - F(\hbar \hat{a}^\dagger \hat{a}) - i\hbar \gamma (\hat{a}^\dagger \hat{b}^\dagger - \hat{b} \hat{a}) .
\]  

(4.14)

If \( F(\hbar \hat{b}^\dagger \hat{b}) = \hbar \omega \hat{b}^\dagger \hat{b} \), the Hamiltonian (4.13) is reduced to the Hamiltonian (4.11). The Hamiltonian can be expressed in terms of the operators \((\hat{T}, \hat{T}_\pm, 0)\):

\[
\hat{H} = F(\hat{T}^0 + \hat{T} - \hbar) - F(\hat{T}_0 - \hat{T}) - i\hbar \gamma (\hat{T}_+ - \hat{T}_- ) .
\]  

(4.15)

For the convenience of late discussion, we rewrite the Hamiltonian (4.14) in the form

\[
\hat{H} = F(\hat{T}_z + \hat{T} - \hbar) - F(\hat{T}_z - \hat{T}) + 2\gamma \hat{T}_y .
\]  

(4.16)

Here, we adopt the following form:

\[
\hat{T}_x = \frac{1}{2}(\hat{T}_+ + \hat{T}_-), \quad \hat{T}_y = -\frac{i}{2}(\hat{T}_+ - \hat{T}_-), \quad \hat{T}_z = \hat{T}_0 .
\]  

(4.17)

We can find two \( su(2) \)-algebraic models in nuclear physics. One is a model which enables us to describe pairing correlation in single-orbit shell model and another is for describing particle-hole correlation in two single-particle levels. In the first case, the intrinsic Hamiltonian \( \hat{K} \) is given by

\[
\hat{K} = 2\epsilon (\hat{S}_0 + S) - G\hat{S}_+ \hat{S}_- . \quad (S = \hbar (2j + 1)/4)
\]  

(4.18)

The quantities \( \epsilon \) and \( G \) denote the energy of the single-particle state and the strength of the pairing interaction, respectively. Further, \( j \) denotes the magnitude of the angular momentum of the orbit. Of course, the set \((\hat{S}_\pm, 0)\) forms the \( su(2) \)-algebra and \( S \) denotes the magnitude of the \( su(2) \)-spin. Then, if \( \hat{S}_\pm, 0 \) are replaced with \( \hat{S}_\pm, 0 \) shown in Eq. (2.17), \( \hat{K} \) becomes of the form

\[
\hat{K} = 2 \left[ \epsilon - G \left( S + \frac{\hbar}{2} \right) \right] \cdot (\hbar \hat{b}^\dagger \hat{b}) + G \cdot (\hbar \hat{b}^\dagger \hat{b})^2 . \]  

(4.19)
Here, \((\hat{X}, \hat{X}^\ast)\) in Eq. (2.17) is replaced with \((\hat{b}, \hat{b}^\ast)\). Then, the Hamiltonian is given in the form

\[
\hat{H} = 4[\epsilon - G(S + \hbar)] \cdot \left(\hat{T} - \frac{\hbar}{2}\right) + 4G \cdot \left(\hat{T} - \frac{\hbar}{2}\right) \hat{T}_z + 2\gamma \hat{T}_y .
\]

(4.20)

The other is called the Lipkin model and the intrinsic Hamiltonian \(\hat{K}\) is expressed as

\[
\hat{K} = 2\epsilon(\hat{S} + S) - \frac{G}{2}(\hat{S}_+^2 + \hat{S}_-^2) .
\]

(4.21)

Here, \(S\) denotes the maximum magnitude of the \(su(2)\)-spin that is equal to \(\hbar\Omega/2\), where \(\Omega\) is the degeneracy of the single-particle level. And \(\epsilon\) and \(G\) \((> 0)\) represent the single-particle energy-spacing and the strength of the interaction, respectively. The set \((\hat{S}_{\pm, 0})\) denotes the \(su(2)\)-spin. In this case, together with the form (2.17), we adopt the basic assumption of the ATDHF theory and the semi-classical approximation. Then, in the \(T\)-conserving framework, the Hamiltonian is given in the following form:

\[
\hat{H} = \left[2\omega_0 - \frac{6\hbar G}{\omega_0} \left(\epsilon + G \left(\hat{T} - \frac{\hbar}{2}\right)\right)^2\right] \cdot \left(\hat{T} - \frac{\hbar}{2}\right)
+ \frac{6G}{\omega_0} \left(\epsilon + G \left(\hat{T} - \frac{\hbar}{2}\right)\right)^2 \cdot \left(\hat{T} - \frac{\hbar}{2}\right) \hat{T}_z + 2\gamma \hat{T}_y .
\]

(4.22)

Here, \(\omega_0\) is defined by

\[
\omega_0 = 2 \left(\epsilon + G \left(\hat{T} - \frac{\hbar}{2}\right)\right) \nu_0 .
\]

(4.23)

The quantity \(\nu_0\) is a solution of the cubic equation

\[
4 \left(\epsilon + G \left(\hat{T} - \frac{\hbar}{2}\right)\right) \nu_0^3 - 4 \left(\epsilon - G \left(\hat{T} - \frac{\hbar}{2}\right)\right) \nu_0 - 3\hbar G = 0 .
\]

(4.24)

The detail of the above derivation has been given in Ref.20. The Hamiltonians (4.20) and (4.22) depend on \((\hat{T} - \hbar/2), (\hat{T} - \hbar/2)\hat{T}_z\) and \(\hat{T}_y\) in the same form as each other and if the coefficient of the term \((\hat{T} - \hbar/2)\hat{T}_z\) vanishes, both Hamiltonians are reduced to the Hamiltonian (4.11). Hereafter, we will treat both systems in a unified way:

\[
\hat{H} = 2\omega_T \cdot \left(\hat{T} - \frac{\hbar}{2}\right) + 2f_T \cdot \left(\hat{T} - \frac{\hbar}{2}\right) \hat{T}_z + 2\gamma \hat{T}_y .
\]

(4.25)

The meanings of \(\omega_T\) and \(f_T\) may be self-evident. Hereafter, we will omit the subscript \(^T\) in \(\omega_T\) and \(f_T\).

We are now at stage to discuss the time-evolution of the coherent state (3.23) for the Hamiltonian (4.25) under the variational procedure. The expectation value of the Hamiltonian (4.25) for the state (3.23), which we denote \(H\), is expressed in the form

\[
H = 2\omega \left(\hat{T} - \frac{\hbar}{2}\right) + 2\lambda \hat{T}_z + 2\gamma \hat{T}_y ,
\]

\[
\lambda = \lambda(\hat{T}) = f \cdot \left(1 + \frac{\hbar}{2\hat{T}}\right) \left(\hat{T} - \frac{\hbar}{2}\right) .
\]
The expectation values of \( \hat{T}_x, \hat{T}_y \) and \( \hat{T}_z \) for the state \((3.23)\) are given as follows:

\[
\begin{align*}
\hat{T}_x &= \sqrt{\frac{\hbar}{2}} (\hat{X}^* + \hat{X}) \cdot \sqrt{2T + \hbar \hat{X}^* \hat{X}} , \\
\hat{T}_y &= -i \sqrt{\frac{\hbar}{2}} (\hat{X}^* - \hat{X}) \cdot \sqrt{2T + \hbar \hat{X}^* \hat{X}} , \\
\hat{T}_z &= \hbar \hat{X}^* \hat{X} + T .
\end{align*}
\] (4.28)

Here, we used the definition \((4.17)\) and the form \((3.32)\) under the omission of the symbol \(( )_c\). The set \((4.28)\) satisfies the relations

\[
\begin{align*}
[T_y, T_z]_P &= T_x , \\
[T_z, T_x]_P &= -T_y , \\
[T_x, T_y]_P &= [T, T_z]_P = [T, T_x]_P = 0 , \\
T_z^2 - T_x^2 - T_y^2 &= T^2 .
\end{align*}
\] (4.29) (4.30) (4.31)

Since \(T\) and \(\phi\) are mutually canonical, the equations of motion are obtained as

\[
\begin{align*}
\dot{T} &= [T, H]_P = 0 , \\
\dot{\phi} &= [\phi, H]_P = 2\omega + \frac{\partial}{\partial T}(2\lambda T_z + 2\gamma T_y) .
\end{align*}
\] (4.32) (4.33)

Since \(T\) is a constant of motion, we set up

\[
T = I .
\] (4.34)

The quantity \(I\) is determined at the initial time. Instead of examining the Hamilton equations of motion for \((X, X^*)\), let us consider the equations of motion for \(T_x, T_y\) and \(T_z\):

\[
\begin{align*}
\dot{T}_x &= [T_x, H]_P = -2\lambda T_y - 2\gamma T_z , \\
\dot{T}_y &= [T_y, H]_P = +2\lambda T_x , \\
\dot{T}_z &= [T_z, H]_P = -2\gamma T_x .
\end{align*}
\] (4.35) (4.36) (4.37)

From these equations, we have

\[
\dot{T}_x = -4(\lambda^2 - \gamma^2)T_x . \quad (\lambda = \lambda(T)) \] (4.38)

Clearly, the solutions of Eq. \((4.38)\) are classified into four cases:

(i) \(\lambda^2 > \gamma^2\), (ii) \(\lambda^2 < \gamma^2\), (iii) \(\lambda = -\gamma\), (iv) \(\lambda = +\gamma\) . (4.39)

The results of the solutions of Eq. \((4.38)\) and the related points were given in Ref.21. In this paper, only the results are summarized. Including Eqs. \((4.32)\) and \((4.33)\), we adopt the canonical transformation from \((\phi, T; X, X^*)\) to \((\theta, I; \chi, J)\).

Case (i) : \(\lambda^2 > \gamma^2\) \((\lambda_0 = \sqrt{\lambda^2 - \gamma^2})\)

\[
T_x = -\sqrt{J^2 - T^2} \sin \chi ,
\]
\[ T_y = \frac{\lambda}{\lambda_0} \sqrt{J^2 - \bar{I}^2} \tan \chi - \frac{\gamma}{\lambda_0} J, \]
\[ T_z = -\frac{\gamma}{\lambda_0} \sqrt{J^2 - \bar{I}^2} \tan \chi + \frac{\lambda}{\lambda_0} J, \]  
\[ T = I, \]
\[ \phi = \theta + 2 \tan^{-1} \left[ \frac{\lambda_0 \bar{I} + \lambda J + \gamma \sqrt{J^2 - \bar{I}^2}}{\lambda \bar{I} + \lambda_0 J} \right] \tan \left( \frac{\chi}{2} \right) + \frac{\gamma}{\lambda_0} \frac{d\lambda}{dI} \sin \chi, \]  
\[ H = 2\omega \left( I - \frac{\chi}{2} \right) + 2\lambda_0 J, \]
\[ \chi = 2\lambda_0 t + \chi_0 , \quad \theta = 2\omega t + \theta_0 . \]

Case (ii) \( \lambda^2 < \gamma^2 \) (\( \gamma_0 = \sqrt{\gamma^2 - \lambda^2} \))
\[ T_x = -\sqrt{J^2 + \bar{I}^2} \sinh \chi, \]
\[ T_y = -\frac{\lambda}{\gamma_0} \sqrt{J^2 + \bar{I}^2} \cosh \chi + \frac{\gamma}{\gamma_0} J, \]
\[ T_z = \frac{\gamma}{\gamma_0} \sqrt{J^2 + \bar{I}^2} \cosh \chi - \frac{\lambda}{\gamma_0} J, \]
\[ T = I, \]
\[ \phi = \theta + 2 \tan^{-1} \left[ \frac{\gamma_0 \bar{I} - \lambda J - \gamma \sqrt{J^2 + \bar{I}^2}}{\lambda \bar{I} + \gamma_0 J} \tanh \left( \frac{\chi}{2} \right) \right] + \frac{\gamma}{\gamma_0} \frac{d\lambda}{dI} \sinh \chi, \]  
\[ H = 2\omega \left( I - \frac{\chi}{2} \right) + 2\gamma_0 J, \]
\[ \chi = 2\gamma_0 t + \chi_0 , \quad \theta = 2\omega t + \theta_0 . \]

Case (iii) \( \lambda = -\gamma \)
\[ T_x = \chi J, \]
\[ T_y = \frac{\chi^2}{2} J - \frac{1}{2} \left( J - \frac{\bar{I}^2}{J} \right), \]
\[ T_z = \frac{\chi^2}{2} J + \frac{1}{2} \left( J + \frac{\bar{I}^2}{J} \right), \]
\[ T = I, \]
\[ \phi = \theta + 2 \tan^{-1} \left[ \frac{\chi J}{I + J} \right], \]
\[ H = 2\omega \left( I - \frac{\chi}{2} \right) + 2\gamma J, \]
\[ \chi = 2\gamma t + \chi_0 , \quad \theta = 2\omega t + \theta_0 . \]

Case (iv) \( \lambda = +\gamma \)
\[ T_x = -\chi J, \]
Time-evolution of the coherent and the squeezed states

\[
T_y = -\frac{\chi}{2} J + \frac{1}{2} \left( J - I^2 J \right),
\]

\[
T_z = \frac{\chi}{2} J + \frac{1}{2} \left( J + I^2 J \right),
\]

\[
T = I,
\]

\[
\phi = \theta + 2 \tan^{-1} \left( \frac{JX}{I+J} \right),
\]

\[
H = 2\omega (I - \hbar) + 2\gamma J,
\]

\[
\chi = 2\gamma t + \chi_0,
\]

\[
\theta = 2\omega t + \theta_0.
\]

We can see that \( J \) is also a constant of motion. The variables \((X, X^\ast)\) are given by

\[
\sqrt{\hbar} X = \frac{T_x - iT_y}{\sqrt{T + T_z}}, \quad \sqrt{\hbar} X^\ast = \frac{T_x + iT_y}{\sqrt{T + T_z}}.
\]

Then, the time-dependence of various quantities is determined as functions of \( t \) for four cases. For example, \(|V|^2\) given in Eq. (3.31) is expressed in the form

\[
|V|^2 = \frac{\hbar X^* X}{2T} = \frac{T_x^2 + T_y^2}{2T(T + T_z)} = \frac{T_z - I}{2I}.
\]

Here, the relation (4.31) was used. It may be interesting to see that there exist four phases.

§5. Description of thermal effects in terms of the coherent state for the \( su(1,1) \)-algebra

In §4, we discussed the \( su(1,1) \)-algebraic model, which permits us to describe influences of the “external environment” on the intrinsic system under investigation such as the harmonic oscillator. As the result of the interaction between two systems, the intrinsic system becomes the “damped and amplified oscillator.” Therefore, it may be interesting to investigate such influences in the language of the thermal effects. We know that, in the thermo field dynamics formalism, \( 14) \) statistically mixed state can be described by the conventional Schrödinger equation with a modified Hamiltonian in a space with phase space doubling. In order to specify mixture of the pure states, the auxiliary variables are introduced. The formalism of the present \( su(1,1) \)-algebraic model resembles the thermo field dynamics formalism. The boson operators \((\hat{b}, \hat{b}^*)\) describe the intrinsic system under investigation and \((\hat{a}, \hat{a}^*)\) are introduced for the auxiliary variables. The coherent state \(|c\rangle\) shown in Eq. (3.23) contains both boson operators and, in the sense of the variational procedure, satisfies the Schrödinger equation derived from the intrinsic Hamiltonian (4.13) through the form (4.14). Therefore, \(|c\rangle\) can be regarded as the state which describes the mixed state for the present system. \( 18), 19), 21), 22) \)

First, let us expand partially the exponential function in the mixed-mode coherent state \(|c\rangle\) in Eq. (8.23). Since the vacuum state \(|0\rangle\), which satisfies the relations
\[ |c\rangle = \sum_{n=0}^{\infty} \Gamma(n;c) |n(c)\rangle \otimes |n\rangle \]  \hspace{1cm} (5.1)

Here, \(|n(c)\rangle\) and \(|n\rangle\rangle denote normalized states for \(b\)- and \(a\)-boson, respectively. They are defined together with \(\Gamma(n;c)\) as follows:

\[ |n(c)\rangle = e^{i\sigma/2} N(n;|\delta|^2)^{-\frac{1}{2}} \exp \left(-\frac{|\delta|^2}{2}\right) \frac{1}{\sqrt{n!}} (\hat{b}^*)^n \exp(\hat{b}^*)|0\rangle_b, \]  \hspace{1cm} (5.2)

\[ |n\rangle = \frac{1}{\sqrt{n!}} (\hat{a}^*)^n |0\rangle_a, \]  \hspace{1cm} (5.3)

\[ \Gamma(n;c) = |V|^n U^{-(n+1)} \exp \left(-|V|^2 \frac{|\delta|^2}{2}\right) N(n;|\delta|^2)^{\frac{1}{2}}. \]  \hspace{1cm} (5.4)

We should note that the set of the states \(|\{n\}\rangle\) in Eq. (5.3) is an orthonormal one satisfying (\(|m|n\rangle\rangle = \delta_{mn\rangle}). However, \(|n(c)\rangle\) is not orthogonalized but normalized, that is, \(|m(c)|n(c)\rangle\rangle \neq \delta_{mn\rangle} for m \neq n but \langle n(c)|n(c)\rangle = 1. Since \langle c|c\rangle = 1 and \langle n\rangle\rangle and \(|n(c)\rangle\) are the normalized states, we have the relation

\[ \sum_{n=0}^{\infty} \Gamma(n;c)^2 = 1. \]  \hspace{1cm} (5.7)

Since our intrinsic system which we are interested in is given only in terms of the \(b\)-boson, we need to evaluate an expectation value for any operator \(\hat{O}_b\) consisting of only \((\hat{b}, \hat{b}^*)\) with respect to \(|c\rangle\) :

\[ (O_b)_c = \langle c|\hat{O}_b|c\rangle = \sum_{n=0}^{\infty} \Gamma(n;c)^2 \langle n(c)|\hat{O}_b|n(c)\rangle. \]  \hspace{1cm} (5.8)

This situation is strongly similar to that in the case of thermo field dynamics formalism. The relation (5.8) tells us that the factor \(\Gamma(n;c)^2\) denotes the weight of the mixture of the pure state \(|n(c)\rangle\rangle in the statistically mixed state under consideration and the quantum mechanical average gives us the statistical average for the physical quantity \(\hat{O}_b\) automatically. In §9, the above will be discussed again.

From the above argument, we learned that \(\Gamma(n;c)^2\) denotes the statistical weight. Then, it is possible to introduce the entropy in the present formalism. First, we note
the following relation:

\[ |c\rangle = \Gamma(\hat{n}_a; c) \sum_{n=0}^{\infty} |n(c)\rangle \otimes |n\rangle, \]  
\[ \hat{n}_a = \hat{a}^* \hat{a}. \]  

Here, \( \Gamma(\hat{n}_a; c) \) is obtained by replacing \( n \) in \( \Gamma(n; c) \) with the \( a \)-boson number operator. With use of the operator \( \Gamma(\hat{n}_a; c) \), we define the operator \( \hat{S} \) in the form

\[ \hat{S} = -\ln \Gamma(\hat{n}_a; c)^2. \]

Then, the expectation value of \( \hat{S} \) for the state \( |c\rangle \) is calculated as follows:

\[ (S)_c = \langle c| \hat{S} |c\rangle = -\sum_{n=0}^{\infty} \Gamma(n; c)^2 \ln \Gamma(n; c)^2. \]  

The relation (5.12) tells that \( (S)_c \) is the entropy and, then, \( \hat{S} \) denotes the entropy operator. With the help of \( \Gamma(n; c) \) shown in Eq. (5.4), \( \hat{S} \) is given as

\[ \hat{S} = (\hat{n}_a + 1) \ln U^2 - \hat{n}_a \ln |V|^2 + |V|^2 |\delta|^2 - \ln N(\hat{n}_a; |\delta|^2). \]  

The expectation value of \( \hat{S} \) for \( |c\rangle \) becomes

\[ (S)_c = U^2 \ln U^2 - |V|^2 \ln |V|^2 + \left( \frac{2}{\hbar} |W|^2 \right) (|V|^2 \ln U^2 - |V|^2 \ln |V|^2) \]
\[ - \left( \frac{2}{\hbar} |W|^2 \right)^2 \cdot \frac{1}{2} \frac{|V|^2}{U^2} + \cdots. \]

Here, we used the relation

\[ \ln N(\hat{n}_a; |\delta|^2) = |\delta|^2 \cdot \hat{n}_a - \frac{1}{4} |\delta|^4 (\hat{n}_a + \hat{n}_a^2) + \cdots. \]

We can show that, compared with the second term on the right-hand side of Eq. (5.14), the third one is expected to be small in a rather wide region of \( |V|^2 \). Then, hereafter, we treat the following form for the entropy:

\[ (S)_c = U^2 \ln U^2 - |V|^2 \ln |V|^2 + \left( \frac{2}{\hbar} |W|^2 \right) (|V|^2 \ln U^2 - |V|^2 \ln |V|^2). \]

With the use of the entropy \( \hat{S} \) and the intrinsic Hamiltonian \( \hat{K} \), we define the free energy \( F \) in the form

\[ F = \langle c| \hat{K} - \frac{\hat{S}}{\beta} |c\rangle. \]

Here, \( \beta \) denotes the inverse of the temperature, i.e., \( \beta = (k_B T)^{-1} \). The intrinsic Hamiltonian \( \hat{K} \) in the present case is expressed as

\[ \hat{K} = K_0 + \hbar \hat{b}^* \hat{b} + \hbar^2 \hat{G} \hat{a}^* \hat{a}^2. \]
(i) harmonic oscillator
\[ K_0 = 0 , \quad e = \omega , \quad g = 0 . \] (5.19)

(ii) pairing model
\[ K_0 = 0 , \quad e = 2(\epsilon - GS) , \quad g = G . \] (5.20)

(iii) Lipkin model
\[ K_0 = -\hbar \epsilon + \frac{\hbar}{2} \left( \omega_0 - \frac{3\hbar g}{2\omega_0} \right) \left( \epsilon + G \left( S + \frac{\hbar}{2} \right) \right)^2 , \quad e = \omega_0 , \quad g = \frac{3\hbar G^2}{2\omega_0} \left( \epsilon + G \left( S + \frac{\hbar}{2} \right) \right)^2 . \] (5.21)

The expectation value of \( \hat{K} \) for \( |c\rangle \) is calculated in the form
\[ \langle K \rangle_c = K_0 + \frac{\hbar}{2} \left( \omega_0 - \frac{3\hbar g}{2\omega_0} \right) \left( \epsilon + G \left( S + \frac{\hbar}{2} \right) \right)^2 + \hbar^2 g \left[ 2|V|^4 + \left( \frac{2}{\hbar} |W|^2 \right) \cdot 4U^2|V|^2 \right] . \] (5.22)

Here, we neglect the term with the order \( (2|W|^2/\hbar)^2 \). Then, combining \( \langle K \rangle_c \) with \( \langle S \rangle_c \) shown in Eq. (5.16), we can express \( F \) as a function of \( |V|^2 \). The variation of \( F \) with respect to \( |V|^2 \) leads us to the following relation :
\[ |V|^2 \exp\left[ \beta \mathcal{E}(|V|^2) \right] = U^2 \exp \left[ -\frac{2}{\hbar} |W|^2 \cdot \frac{1}{1 + \frac{2}{\hbar} |W|^2} \cdot \frac{1}{U^2} \right] . \] (5.23)

Here, \( \mathcal{E}(|V|^2) \) is defined as
\[ \mathcal{E}(|V|^2) = \hbar e + 4\hbar^2 g \left[ |V|^2 + \frac{\frac{2}{\hbar} |W|^2}{1 + \frac{2}{\hbar} |W|^2} \cdot \frac{1}{U^2} \right] . \] (5.24)

Under the approximation \( e^{-X} = 1 - X \) for \( X = (2|W|^2/\hbar)/(1 + 2|W|^2/\hbar) \cdot (1/U^2) \), we have
\[ |V|^2 = \frac{1}{1 + \frac{2}{\hbar} |W|^2} \cdot \frac{1}{e^{\mathcal{E}(|V|^2)\beta}} - 1 . \] (5.25)

The relation (5.23) gives us
\[ \beta = \frac{1}{\mathcal{E}(|V|^2)} \ln \left( \frac{U^2 + |V|^2 \cdot \frac{\frac{2}{\hbar} |W|^2}{|V|^2 + |V|^2 \cdot \frac{\frac{2}{\hbar} |W|^2}{}}}{U^2} \right) . \] (5.26)

The quantities \( |W|^2 \) and \( |V|^2 \) are given in Eqs. (3.30) and (4.57), respectively, and \( T_z \) is obtained as a function of \( t \). Therefore, with the aid of the relation (5.26), \( \beta \), i.e., the temperature of the system can be determined as a function of \( t \).

In order to make further discussion simpler, let us treat the case of \( \lambda = 0 \), where \( \lambda \) is given in Eq. (4.27), i.e., the case of harmonic oscillator. As is shown in Eq. (5.19), \( K_0 = 0, e = \omega \) and \( g = 0 \), and \( \mathcal{E}(|V|^2) \) in Eq. (5.24) becomes
\[ \mathcal{E}(|V|^2) = \hbar \omega . \] (5.27)
Then, the relation (5.25) is reduced to
\[ |V|^2 = \frac{1}{1 + \frac{1}{\hbar} |W|^2} \cdot \frac{1}{e^{\epsilon \beta} - 1} \cdot (\epsilon = \hbar \omega) \] (5.28)

With the relations (5.22) and (5.28), we have
\[ (K)_c = \epsilon \left( \frac{2}{\hbar} |W|^2 + \left( 1 + \frac{2}{\hbar} |W|^2 \right) |V|^2 \right) \]
\[ = \epsilon \cdot \frac{2}{\hbar} |W|^2 + \epsilon \cdot \frac{1}{e^{\epsilon \beta} - 1} \cdot (5.29) \]

The quantity \( 2 |W|^2 / \hbar \) is a constant of motion and at the low temperature limit, the second term on the right-hand side of Eq. (5.29) vanishes. Therefore, the first term represents the energy at the low temperature limit and the second the energy coming from the thermal fluctuations in the boson distribution.

Next, let us investigate the time-evolution of the system under a special initial condition \((I = I, J = 0, \chi_0 = 0)\). Then, \( |V|^2 \) in Eq. (4.57) can be expressed as
\[ |V|^2 = [\sinh(\gamma t)]^2 \] (5.30)

Here, we used the relation (4.44) with \( \gamma_0 = \gamma \). At the initial time \( t = 0 \), \( |V|^2 = 0 \) and in this case \( \Gamma(n; c)^2 \) becomes \( \Gamma(n = 0; c)^2 = 1 \) and \( \Gamma(n \neq 0; c)^2 = 0 \). This means that the system is in a pure state with \( n = 0 \). At the time \( t \to \infty \), \( |V|^2 \to \infty \) and in this case, all \( \Gamma(n; c)^2 \) becomes equal to one another. This means that the present system is in the equal weight mixing. On the basis of the above fact, let us investigate the time evolution of the system. The energy \( (K)_c \) in Eq. (5.29) is expressed as
\[ (K)_c = (2I - \hbar)\omega + \omega \cdot 2I |V|^2 \] (5.31)

Since \( |V|^2 = 0 \) at \( t = 0 \), \( E(t) \) is expressed as
\[ E(0) = (2I - \hbar)\omega \] (5.32)

On the other hand, \( E(t) \) can be also expressed in the following form:
\[ E(t) = \sum_{n=0}^{\infty} \Gamma(n; c)^2 \langle n(c) | \hat{K}_b | n(c) \rangle \] (5.33)
\[ \langle n(c) | \hat{K}_b | n(c) \rangle = \hbar \omega n - \hbar \omega |\delta|^2 \left[ \frac{dL_{n+1}(x)}{dx} \right]_{x=\delta^2} \cdot \frac{1}{L_n(-|\delta|^2)} \] (5.34)

Then, we have the relations
\[(\text{Contribution of the 1st term in Eq.(5.34)})\]
\[ = \sum_{n=0}^{\infty} \Gamma(n; c)^2 \cdot \hbar \omega n = \omega \cdot 2I |V|^2 \]
\[ = \epsilon \cdot \frac{1}{e^{\epsilon \beta} - 1} \] (5.35)
\begin{equation}
\sum_{n=0}^{\infty} \Gamma(n; c)^2 \left\{ -\hbar \omega |\delta|^2 \left[ \frac{dL_{n+1}(x)}{dx} \right]_{x=-|\delta|^2} \cdot \frac{1}{L_n(-|\delta|^2)} \right\} = \omega(2I - \hbar) . \tag{5.36}
\end{equation}

It is interesting to compare the above results with the initial value of the energy given in Eq. (5.32). The ensemble average of the second term shown in Eq. (5.36) coincides with the initial energy (5.32). The initial state is in the pure state with \( n = 0 \) and the energy of the system concentrates on this state. At the time \( t > 0 \), the states for \( n \neq 0 \) exist in \( |c\rangle \) and the energy is distributed to each state. This fact tells us that the energy in the pure state with \( n = 0 \) at \( t = 0 \) dissipates to other pure states at \( t \) under the mixing weight \( \Gamma(n; c)^2 \). However, totally, it conserves.

Next, let us investigate the contribution of the first term. It is seen from the result in Eq. (5.35) that this energy originates in the thermal effect. Therefore, the energy is supplied from the “external environment” and by this energy the temperature of the system increases. More detail informations are obtained in Refs.18), 21) and 22).

\section{The pseudo \( su(1,1) \)-algebra in the \( su(2) \)-spin system}

As was discussed in \S 5, the \( su(1,1) \)-algebraic approach to the description of thermal effect gives us various quite interesting features. However, this approach contains an undesirable feature. The Hamiltonian for the \( su(1,1) \)-spin system does not represent the energy of the entire system. Thus, it may not be justified to assume that the additional degree of freedom expressed in terms of the boson \( (\hat{a}, \hat{a}^*) \) can be specified as a physical object characterizing the external environment. Rather, it may be natural to regard the additional degree of freedom as a theoretical tool for describing dynamical motion such as the damped oscillation in the frame of isolated system. In this sense, a fictitious degree of freedom is introduced in the \( su(1,1) \)-algebraic approach, in spite of giving us interesting results. Then, it may be interesting to present, without introducing the fictitious degree of freedom, a possible approach which gives us the same results as those obtained in the \( su(1,1) \)-algebraic approach. In this section, we will present the basic theoretical framework for this problem\textsuperscript{23}.

Let us prepare a boson space constructed in terms of the boson operators \( (\hat{c}, \hat{c}^*) \) and \( (\hat{d}, \hat{d}^*) \). In this space, we introduce the following operators :

\begin{align*}
\tau_+ & = \sqrt{\hbar} \hat{c}^* \cdot \sqrt{2T + \hbar \overset{\text{c}}{\cdots}} \cdot \frac{1}{\sqrt{\hbar + \hbar \overset{\text{d}}{\cdots}}} \cdot \sqrt{\hbar} \hat{d} , \\
\tau_- & = \sqrt{\hbar} \hat{d}^* \cdot \frac{1}{\sqrt{\hbar + \hbar \overset{\text{d}}{\cdots}}} \cdot \sqrt{2T + \hbar \overset{\text{c}}{\cdots}} \cdot \sqrt{\hbar} \hat{c} , \\
\tau_0 & = T + \hbar \overset{\text{c}}{\cdots} \hat{c} . \tag{6.1}
\end{align*}
Here, $T$ is a real parameter which is assumed to be
\[ T = \hbar t, \quad t = \frac{1}{2}, 1, \frac{3}{2}, 2, \cdots. \tag{6.2} \]

The above three operators are rewritten in the form
\[
\tau_+ = \frac{1}{\sqrt{h + \hat{S} - \hat{S}_0}} \hat{S}_+ \sqrt{2T + \hat{S} + \hat{S}_0}, \quad \tau_- = \sqrt{2T + \hat{S} + \hat{S}_0} \cdot \hat{S}_- \frac{1}{\sqrt{h + \hat{S} - \hat{S}_0}}, \quad \tau_0 = T + \hat{S} + \hat{S}_0. \tag{6.3}
\]

The operators $(\hat{S}_{\pm,0})$ compose the $su(2)$-algebra and are defined as
\[
\hat{S}_+ = \frac{\hbar}{2} (\hat{c}^* \hat{d} - \hat{d}^* \hat{c}) , \quad \hat{S}_- = \frac{\hbar}{2} (\hat{c}^* \hat{d} + \hat{d}^* \hat{c}) , \quad \hat{S}_0 = \frac{\hbar}{2} (\hat{c}^* \hat{c} - \hat{d}^* \hat{d}) , \quad \hat{S} = \frac{\hbar}{2} (\hat{c}^* \hat{c} + \hat{d}^* \hat{d}). \tag{6.4}
\]

It is interesting to see that $(\hat{\tau}_{\pm,0})$ can be expressed in terms of $(\hat{S}_{\pm,0}, \hat{S})$.

Further, we can prove the following relations:
\[
[\tau_+, \tau_-] = -2h \tau_0 + ((2T + \hbar \hat{c}^* \hat{d})(\hbar + \hbar \hat{c}^* \hat{d} \hat{D}) + (1 - \epsilon) \hbar \cdot \hbar \hat{c}^* \hat{d} \hat{D}),
\]
\[
[\tau_0, \tau_{\pm}] = \pm h \tau_{\pm}, \quad \tau_0^2 = \frac{1}{2} (\tau_- \tau_+ + \tau_+ \tau_-),
\]
\[
= T(T - \epsilon h) + \frac{1}{2} (2T + \hbar \hat{c}^* \hat{d})(\hbar + \hbar \hat{c}^* \hat{d}) - (1 - \epsilon)(2T + \hbar \hat{c}^* \hat{d})(\hbar - \hbar \hat{c}^* \hat{d}) \hat{D}. \tag{6.8}
\]

Here, $\epsilon$ and $\hat{\circ} \hat{D}$ are given as
\[
\epsilon = 1, \quad \hat{\circ} \hat{D} = 1 - \hat{d}^* \frac{1}{1 + \hat{d}^* \hat{d}} \hat{d}, \quad \hat{\circ}^2 \hat{D} = \hat{D}, \quad \hat{\circ}^* \hat{D} = \hat{D}. \tag{6.9}
\]

The relations (6.6) $\sim$ (6.8) should be compared with the relations (3.2) $\sim$ (3.4).

If the terms related with the projection operator $\hat{D}$ are omitted, all the relations are reduced to those governed by the $su(1,1)$-algebra in the $su(1,1)$-space with the magnitude $T$. The operator $\hat{\circ} \hat{D}$ plays a role of the projection operator; i.e.,
\[
\hat{\circ} \hat{D} \cdot (1/\sqrt{n}) \cdot (\hat{d}^* \circ [0])_d = \delta_n [0]_d \text{ for } \hat{d} [0]_d = 0. \tag{6.10}
\]

This means that the behavior of the set $(\hat{\tau}_{\pm,0})$ is quite similar to that of the $su(1,1)$-spin with the magnitude $T$. 
In this sense, it may be possible to regard the set \((\tau_{\pm,0})\) as composing the pseudo \(su(1,1)\)-algebra. It is interesting to see that it is constructed in the \(su(2)\)-spin.

Our next problem is to introduce a possible form of coherent state which leads us to the classical counterpart of the set \((\tau_{\pm,0})\). For the preparation, let us consider a state \(|\alpha\rangle\) which satisfies

\[
\hat{\alpha}|\alpha\rangle = \alpha|\alpha\rangle \quad .
\]  

Here, the operators \(\hat{\alpha}\) and \(\hat{\alpha}^\ast\) are defined by

\[
\hat{\alpha} = f(\hat{a}^\ast \hat{a})^{-1} \hat{a} \quad , \quad \hat{\alpha}^\ast = \hat{a}^\ast f(\hat{a}^\ast \hat{a})^{-1} \quad .
\]  

The operators \(\hat{\alpha}\) and \(\hat{\alpha}^\ast\) denote boson annihilation and creation operators, respectively. Of course, \(f(\hat{a}^\ast \hat{a})\) and its inverse \(f(\hat{a}^\ast \hat{a})^{-1}\) should be defined. The eigenvalue \(\alpha\) is, in general, complex. We can see that if \(f(\hat{a}^\ast \hat{a}) = 1\), the solution of Eq. (6.11) is the well-known boson coherent state. In association with \(\hat{\alpha}\) and \(\hat{\alpha}^\ast\), we introduce the operators \(\hat{A}\) and \(\hat{A}^\ast\) in the form

\[
\hat{A} = f(\hat{a}^\ast \hat{a})\hat{a} \quad , \quad \hat{A}^\ast = \hat{a}^\ast f(\hat{a}^\ast \hat{a}) \quad .
\]  

The commutation relation between \(\hat{\alpha}\) and \(\hat{A}^\ast\) is given as

\[
[\hat{\alpha}, \hat{A}^\ast] = 1 \quad .
\]  

Then, the state \(|\alpha\rangle\) is obtained in the following form :

\[
|\alpha\rangle = M_c \exp(\alpha \hat{A}^\ast)|0\rangle \quad , \quad \hat{a}|0\rangle = 0 \quad , \quad (\alpha|\alpha\rangle = 1 \quad .
\]  

(6.15)

\[(M_c)^{-2} = 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \prod_{k=0}^{n-1} f(k)^2 \cdot (|\alpha|^2)^n \quad .
\]  

(6.16)

For the complex variable \(z\), we can derive the relation

\[
(\alpha|\partial_z|\alpha) = \frac{1}{2} (a^\ast \partial_z a - a \partial_z a^\ast)(\alpha|f(\hat{a}^\ast \hat{a})|\alpha) = \frac{1}{2} (a^\ast \partial_z a - a \partial_z a^\ast) \quad .
\]  

(6.17)

Here, \(a\) is defined by

\[
a = \alpha \sqrt{(\alpha|f(\hat{a}^\ast \hat{a})|\alpha)} \quad .
\]  

(6.18)

The relation (6.17) supports that the parameters \(a\) and \(a^\ast\) can be regarded as canonical variables in classical mechanics. The expectation value of \(\hat{a}^\ast \hat{a}\) for \(|\alpha\rangle\) is given as

\[
(\alpha|\hat{a}^\ast \hat{a}|\alpha) = (\alpha|\hat{a}^\ast f(\hat{a}^\ast \hat{a})^2 \hat{a}|\alpha) = |\alpha|^2 (\alpha|f(\hat{a}^\ast \hat{a})^2|\alpha) = a^\ast a \quad .
\]  

(6.19)

Under the above preparation, let us investigate a possible form of coherent state which gives us the classical counterpart of the set \((\tau_{\pm,0})\). For this aim, we set up the
following wave packet:

\[ |c\rangle = M_c \exp \left( \gamma \cdot \frac{\tau}{\hbar} \right) |\delta\rangle, \]
\[ |\delta\rangle = \sqrt{1 - |\delta|^2} \exp \left( \delta \cdot \frac{\Delta^*}{\hbar} \right) |0\rangle, \]
\[ \Delta^* = \delta^* \sqrt{1 + \delta^* \delta}. \]  

Here, \( \gamma \) and \( \delta \) denote the complex parameters. Clearly, the state \( |\delta\rangle \) is nothing but a simple example of \( |\alpha\rangle \) shown in Eq. (6.15) and satisfies

\[ \delta |\delta\rangle = |\delta\rangle, \]
\[ \delta^* = \delta^* \left( \sqrt{1 + \delta^* \delta} \right)^{-1}, \]
\[ \tau_+ |\delta\rangle = 0. \]

Therefore, we can see that the state \( |c\rangle \) consists of a superposition of successive operation of \( \tau_+ \) on the state \( |\delta\rangle \) satisfying the relation (6.24). With the use of the relation (6.23), together with the definition of \( \tau_+ \) shown in Eq. (6.1), the state \( |c\rangle \) can be rewritten in the form

\[ |c\rangle = (1 - |\gamma|^2)^t (1 - |\delta|^2)^\frac{1}{2} \cdot \exp \left( \gamma \cdot \sqrt{2t} \cdot \hat{c}^* \hat{c} \right) \exp \left( \delta \cdot \sqrt{1 + \delta^* \delta} |0\rangle \right). \]  

The relation (6.18) with \( f(\hat{c}^* \hat{c}) = 2t + \hat{c}^* \hat{c} \) and \( f(\hat{d}^* \hat{d}) = 1 + \hat{d}^* \hat{d} \) tells us that \( \gamma \) and \( \delta \) can be expressed in terms of the canonical variables \( c \) and \( d \):

\[ \gamma = \frac{c}{\sqrt{2t + \hat{c}^* \hat{c}}}, \quad \delta = \frac{d}{\sqrt{1 + \hat{d}^* \hat{d}}}. \]  

The relation (6.19) gives us

\[ \langle c| \hat{c}^* \hat{c} |c\rangle = c^* c, \quad \langle c| \hat{d}^* \hat{d} |c\rangle = d^* d. \]  

Then, the expectation value of \( \hat{S} \) in Eq. (6.5) is calculated as

\[ \langle \hat{S} |c\rangle = \frac{1}{2} (c^* c + d^* d) = S. \]  

Instead of the variables \( (c, c^*) \) and \( (d, d^*) \), we use the variables \( (\psi, S) \) and \( (X, X^*) \) used in §2:

\[ c = X \exp \left( -i \frac{\psi}{2} \right), \quad c^* = X^* \exp \left( +i \frac{\psi}{2} \right), \]
\[ d = \frac{1}{\sqrt{\hbar}} \sqrt{2S - \hbar X^* X} \exp \left( -i \frac{\psi}{2} \right), \quad d^* = \frac{1}{\sqrt{\hbar}} \sqrt{2S - \hbar X^* X} \exp \left( +i \frac{\psi}{2} \right). \]
With the use of the relation \([6.11]\) for \(f(c^\dagger c) = \sqrt{2t + c^\dagger c}\) and \(f(d^\dagger d) = \sqrt{1 + d^\dagger d}\), together with the relation \([6.20]\) and (6-29), the expectation values of \(\langle \tau_{\pm,0} \rangle\) are given in the following form:

\[
\begin{align*}
(\tau_+)_c &= \frac{1}{\sqrt{h + 2S - h\hat{X}^*\hat{X}}} \cdot \sqrt{h}\hat{X}^*\sqrt{2T + h\hat{X}^*\hat{X}} \cdot \sqrt{2S - h\hat{X}^*\hat{X}} , \\
(\tau_-)_c &= \sqrt{2S - h\hat{X}^*\hat{X}} \cdot \sqrt{2T + h\hat{X}^*\hat{X}} \cdot \sqrt{h}\frac{1}{\sqrt{h + 2S - h\hat{X}^*\hat{X}}} , \\
(\tau_0)_c &= h\hat{X}^*\hat{X} + T .
\end{align*}
\]

\(\text{(6.30)}\)

We can prove that \((\tau_{\pm,0})_c\) obey the relations

\[
\begin{align*}
(\tau_-)_c^* &= (\tau_+)_c , \\
(\tau_0)_c^* &= (\tau_0)_c , \\
[[(\tau_+)_c, (\tau_-)_c]_p &= -(i) \{ -2h(\tau_0)_c + [(2T + hc^c)(h + hc^cD) + (1 - \epsilon)h \cdot hc^cD] \} , \\
[[(\tau_0)_c, (\tau_{\pm})_c]_p &= -(i)(\pm h(\tau_{\pm})_c) , \\
(\tau_0)_c^2 &= \frac{1}{2}[(\tau_-)_c(\tau_+)_c + (\tau_+)_c(\tau_-)_c] \\
&= T(T - \epsilon h) + \frac{1}{2}[(2T + hc^c)(h + hc^c) - (1 - \epsilon)(2T + hc^c)(h - hc^c)]D .
\end{align*}
\]

\(\text{(6.31) - (6.33)}\)

Here, \(\epsilon\) and \(D\) are given as

\[
\begin{align*}
\epsilon &= 0 , \\
D &= (1 + d^*d)^{-1} = 1 - d^*\frac{1}{1 + d^*d}d .
\end{align*}
\]

\(\text{(6.34) - (6.35)}\)

It is interesting to see that the relations (6.31) \(\sim\) (6.35) correspond to the relations (6-6) \(\sim\) (6-10) completely. The quantity \(\epsilon\) appears from the non-commutability of various operators, i.e., it is a kind of quantum effect. From the above correspondence, we obtain a conclusion that the expectation values \((\tau_{\pm,0})_c\) are the classical counterparts of \((\tau_{\pm,0})\).

With the help of the relation (6.3), together with the form (2-17), the Holstein-Primakoff boson representation for the set \((\tau_{\pm,0})\) is obtained easily:

\[
\begin{align*}
\hat{\tau}_+ &= \frac{1}{\sqrt{h + 2S - h\hat{X}^*\hat{X}}} \cdot \sqrt{h}\hat{X}^*\sqrt{2T + h\hat{X}^*\hat{X}} \cdot \sqrt{2S - h\hat{X}^*\hat{X}} , \\
\hat{\tau}_- &= \sqrt{2S - h\hat{X}^*\hat{X}} \cdot \sqrt{2T + h\hat{X}^*\hat{X}} \cdot \sqrt{h}\frac{1}{\sqrt{h + 2S - h\hat{X}^*\hat{X}}} , \\
\hat{\tau}_0 &= h\hat{X}^*\hat{X} + T .
\end{align*}
\]

\(\text{(6.36)}\)

Here, \(\hat{S}_{\pm,0}\) and \(\hat{S}\) in Eq. (6.3) are replaced with \(\hat{S}_{\pm,0}\) in Eq. (2-17) and \(S\), respectively. We can see that under the replacements \((\hat{X}, \hat{X}^*) \to (\hat{X}, \hat{X}^*)\) and \(S \to S\), the form (6.36) is reduced to the form (6.30).
§7. Description of a many-boson system interacting with an external harmonic oscillator

The aim of this section was already mentioned in the beginning part of the previous section. Corresponding the Hamiltonian (5.18), we treat the following Hamiltonian for a many-boson system:

\[ \hat{K}_c = \hbar e^0 \cdot \hat{c}^* \hat{c} + \hbar^2 g^0 \cdot \hat{c}^* \hat{c}^2. \]  

(7.1)

It may be self-evident why we treat the Hamiltonian (7.1) in the present case. The above boson system interacts with a harmonic oscillator, the Hamiltonian of which is given by

\[ \hat{K}_d = \hbar \omega^0 \cdot \hat{d}^* \hat{d}. \]  

(7.2)

As the interaction between the two systems, we are interested in the following form:

\[ \hat{V}_{cd} = \hat{V}_{cd} (1) + \hat{V}_{cd} (2) , \]  

(7.3)

\[ \hat{V}_{cd} (1) = \hbar^2 f^0 \cdot (\hat{c}^* \hat{d} \cdot \hat{d}^* \hat{c} + \hat{d}^* \hat{c} \cdot \hat{c}^* \hat{d}) , \]  

(7.4)

\[ \hat{V}_{cd} (2) = -\hbar \gamma^0 \cdot i \left[ \left( \frac{\sqrt{\hbar + \hbar \hat{d}^* \hat{d}}}{\sqrt{2T + \hbar \hat{c}^* \hat{c}}} \right)^{-1} \cdot \hat{c}^* \hat{d} \cdot \sqrt{2T + \hbar \hat{c}^* \hat{c}} 
\right. \]

\[ \left. -\sqrt{2T + \hbar \hat{c}^* \hat{c}} \cdot \hat{d}^* \hat{c} \cdot \left( \frac{\sqrt{\hbar + \hbar \hat{d}^* \hat{d}}}{\sqrt{2T + \hbar \hat{c}^* \hat{c}}} \right)^{-1} \right] . \]  

(7.5)

Of course, \( e^0, g^0, \omega^0, f^0, \gamma^0 \) and \( T \) denote real parameters. In particular, we regard \( T \) as the parameter obeying the condition \( T = \frac{\hbar}{2}, (t = 1/2, 1, 3/2, 2, \cdots) \). Thus, the Hamiltonian of the entire system is expressed as

\[ \hat{H}_{su(2)} = \hat{K}_c + \hat{K}_d + \hat{V}_{cd} . \]  

(7.6)

The meaning of the “\( su(2) \)” in \( \hat{H}_{su(2)} \) is interpreted as follows: We are now describing the system obeying the \( su(2) \)-algebra and \( \hat{H}_{su(2)} \) can be expressed in terms of the operators \( (\hat{S}_{\pm,0}) \) and \( \hat{S} \) shown in Eqs. (6.4) and (6.5). The Hamiltonian (7.6) can be rewritten in the form

\[ \hat{H}_{su(2)} = \hat{U}_{su(2)} + \hat{K}_{su(2)} + \hat{V}_{su(2)} , \]  

(7.7)

\[ \hat{U}_{su(2)} = 2 \left[ \omega^0 - 2 f^0 \cdot \left( T - \frac{\hbar}{2} \right) \right] \cdot \hat{S} 
\right. \]

\[ \left. - [(e^0 - \omega^0) - \hbar g^0] \cdot T + (g^0 - 2 f^0) \cdot T^2 , \right] \]  

(7.8)

\[ \hat{K}_{su(2)} = \left[ (e^0 - \omega^0) - \hbar g^0 + 4 f^0 \hat{S} - 2(g^0 - 2 f^0)T \right] \frac{\hbar}{\tau_0} + (g^0 - 2 f^0) \cdot \frac{\hbar}{\tau_0} , \]  

(7.9)

\[ \hat{V}_{su(2)} = -\gamma^0 \cdot i \left( \frac{\hbar}{\tau_+} - \frac{\hbar}{\tau_-} \right) . \]  

(7.10)
The operator $\hat{S}$ commutes with $\hat{H}_{su(2)}$, and $\hat{U}_{su(2)}$ contains only $\hat{S}$. Therefore, the dynamics of our present system is described in the framework of $(\hat{K}_{su(2)} + \hat{V}_{su(2)})$ under a given eigenvalue $S$ of $\hat{S}$.

In associating with the $su(2)$-spin system introduced in the above, we will recapitulate a possible form of the $su(1,1)$-spin system discussed in §4. Our interest is in the following Hamiltonian:

$$\hat{H}_{su(1,1)} = \hat{K}_b - \hat{K}_a + \hat{V}_{ba}, \quad (7.11)$$

$$\hat{K}_b = \hbar e \cdot \hat{b}^* \hat{b} + \hbar^2 g \cdot \hat{b}^* \hat{b}^2, \quad (7.12)$$

$$\hat{K}_a = \hbar e \cdot \hat{a}^* \hat{a} + \hbar^2 g \cdot \hat{a}^* \hat{a}^2, \quad (7.13)$$

$$\hat{V}_{ba} = -\hbar \gamma \cdot i(\hat{a}^* \hat{b}^* - \hat{b} \hat{a}). \quad (7.14)$$

The Hamiltonian $\hat{K}_b$ is of the same form as that given in Eq. (7.18). The Hamiltonian (7.11) can be rewritten in the form

$$\hat{H}_{su(1,1)} = \hat{U}_{su(1,1)} + \hat{K}_{su(1,1)} + \hat{V}_{su(1,1)}, \quad (7.15)$$

$$\hat{U}_{su(1,1)} = 2(e - \hbar g) \left( \hat{T} - \frac{\hbar}{2} \right), \quad (7.16)$$

$$\hat{K}_{su(1,1)} = 4g \cdot \left( \hat{T} - \frac{\hbar}{2} \right) \hat{T}_0, \quad (7.17)$$

$$\hat{V}_{su(1,1)} = -\gamma \cdot i(\hat{T}_+ - \hat{T}_-). \quad (7.18)$$

In a sense similar to the case of the $su(2)$-spin system, the operator $\hat{T}$ commutes with $\hat{H}_{su(1,1)}$, and $\hat{U}_{su(1,1)}$ contains only $\hat{T}$. Therefore, the dynamics of the present system is described in the framework of $(\hat{K}_{su(1,1)} + \hat{V}_{su(1,1)})$ under a given eigenvalue $T$ of $\hat{T}$. As was already mentioned, the Hamiltonian $\hat{H}_{su(1,1)}$ gives us various interesting aspects of thermal properties and dissipation for boson systems. In this sense, the form (7.15) is quite interesting. However, it does not represent the total energy of the system. On the other hand, $\hat{H}_{su(2)}$ in Eq. (7.6) represents the total energy of the system. Therefore, it is interesting to investigate the relation between both systems.

At the present stage, we have two spin systems. One is the $su(2)$- and the other is the $su(1,1)$-spin system. For a given value of $S (= \hbar s; s = 0, 1/2, 1, 3/2, \cdots)$, the $su(2)$-spin system is described by the complete set $\{|n; 2s - n\}\rangle$, which is expressed as

$$|n; 2s - n\rangle = \frac{1}{\sqrt{n!(2s-n)!}} (\hat{c}^*)^n (\hat{d}^*)^{2s-n}|0\rangle. \quad (7.19)$$

Here, $n$ runs in the region

$$n = 0, 1, 2, \cdots, 2s. \quad (7.20)$$

For a given value of $T (= \hbar t, t = 1/2, 1, 3/2, 2, \cdots)$, the $su(1, 1)$-spin system can be treated by the complete set $\{|n; 2t - 1 + n\}\rangle$, which is given by

$$|n; 2t - 1 + n\rangle = \frac{1}{\sqrt{n!(2t-1+n)!}} (\hat{a}^*)^n (\hat{b}^*)^{2t-1+n}|0\rangle. \quad (7.21)$$
Here, $n$ runs in the region
\[ n = 0, 1, 2, \ldots, 2s, 2s + 1, \ldots. \]  
(7.22)

We refer to the spaces spanned by the complete set (7.19) and (7.21) as the $su(2)$-space with $s$ and the $su(1, 1)$-space with $t$, respectively.

Let us transcribe the $su(2)$-spin system of the $su(2)$-space with $s$ into that in the $su(1, 1)$-space with $t$. For this aim, first, we set up the following correspondence:
\[ |n; 2s - n\rangle \sim |n; 2t - 1 + n\rangle, \quad n = 0, 1, 2, \ldots, 2s. \]
(7.23)

Following the basic idea of the boson mapping, we call the space spanned by the set \{\(|n; 2t - 1 + n\rangle; n = 0, 1, 2, \cdots, 2s\}\} as the physical space and define the operator $\hat{U}_{st}$
\[ \hat{U}_{st} = \sum_{n=0}^{2s} |n; 2t - 1 + n\rangle\langle n| = 1. \]
(7.24)

The operator $\hat{U}_{st}$ satisfies
\[ \hat{U}_{st}^\dagger \hat{U}_{st} = \sum_{n=0}^{2s} |n; 2s - n\rangle\langle n; 2s - n| = 1, \]
(7.25)
\[ \hat{U}_{st} \hat{U}_{st}^\dagger = \sum_{n=0}^{2s} |n; 2t - 1 + n\rangle\langle n| = P_{st}. \]
(7.26)

Here, $P_{st}$ plays the role of the projection operator to the physical space ($P_{st}^* = P_{st}$, $P_{st}^2 = P_{st}$). The operators $\hat{U}_{st}$ and $\hat{U}_{st}^\dagger$ give us
\[ \hat{U}_{st}|n; 2s - n\rangle = |n; 2t - 1 + n\rangle, \quad (n = 0, 1, 2, \cdots, 2s) \]
\[ \hat{U}_{st}^\dagger|n; 2t - 1 + n\rangle = \begin{cases} |n; 2s - n\rangle, & (n = 0, 1, 2, \cdots, 2s) \\ 0, & (n = 2s + 1, 2s + 2, \cdots) \end{cases} \]
(7.27)

The operator $\hat{U}_{st}$ permits us to transform $\hat{\tau}_{\pm, 0}$, $\hat{S}$ and $T$ as
\[ \hat{U}_{st} \cdot \hat{\tau}_{\pm, 0} \cdot \hat{U}_{st}^\dagger = P_{st} \cdot \hat{T}_{\pm, 0} \cdot P_{st}, \]
\[ \hat{U}_{st} \cdot \hat{S} \cdot \hat{U}_{st}^\dagger = \hat{U}_{st} \cdot \hat{S} \cdot \hat{U}_{st} = P_{st} \cdot \hat{S} \cdot P_{st}, \]
\[ \hat{U}_{st} \cdot \hat{T} \cdot \hat{U}_{st}^\dagger = P_{st} \cdot \hat{T} \cdot P_{st}. \]
(7.28)

Here, it should be noted that $S$ and $T$ are $c$-numbers. With the aid of $U_{st}$ and the relation (7.30), we are able to transcribe $\hat{H}_{su(2)}$ in the $su(1, 1)$-space with $t$:
\[ \hat{U}_{st} \cdot \hat{H}_{su(2)} \cdot \hat{U}_{st}^\dagger = P_{st} \cdot \hat{H}_{su(2)} \cdot P_{st}, \]
\[ \hat{H}_{su(2)} = \hat{U}_{su(2)} + \hat{K}_{su(2)} + \hat{V}_{su(2)}. \]  
(7.29)
Under the condition (7.36), we obtain
\[
\hat{U}_{su(2)} = 2 \left[ \omega^0 - 2f^0 \cdot \left( \hat{T} - \frac{\hbar}{2} \right) \right] \cdot S
\]
\[
- \left[ (e^0 - \omega^0) - h^0 \right] \cdot \hat{T} + (g^0 - 2f^0) \cdot \hat{T}^2 ,
\]
(7.33)
\[
\hat{K}_{su(2)} = [(e^0 - \omega^0) - h^0 + 4f^0 S - 2(g^0 - 2f^0) T] \hat{T}_0 + (g^0 - 2f^0) \cdot \hat{T}_0^2 ,
\]
(7.34)
\[
\hat{V}_{su(2)} = - \gamma^0 \cdot i(\hat{T}_+ - \hat{T}_-) .
\]
(7.35)

We can regard \(\hat{H}_{su(2)}\) as the Hamiltonian in the \(su(1,1)\)-space which has a connection with the original \(\hat{H}_{su(2)}\) in the \(su(2)\)-space. In the same meaning as that mentioned for the Hamiltonian \(\hat{H}_{su(1,1)}\), we are interested in the part \((\hat{K}_{su(2)} + \hat{V}_{su(2)})\). Then, we require the following condition for the parameters \((e^0, \omega^0, g^0, f^0, \gamma^0, \gamma)\) :
\[
e^0 - \omega^0 + 2g^0 \left( S - \frac{\hbar}{2} \right) = 4g \left( T - \frac{\hbar}{2} \right) ,
\]
\[
g^0 = 2f^0 , \quad \gamma^0 = \gamma .
\]
(7.36)

Under the condition (7.36), we obtain
\[
P_{st} \cdot (\hat{K}_{su(2)} + \hat{V}_{su(2)}) \cdot P_{st} = P_{st} \cdot (\hat{K}_{su(1,1)} + \hat{V}_{su(1,1)}) \cdot P_{st} .
\]
(7.37)

This means that, under the condition (7.36), \(\hat{H}_{su(2)}\) is expected to display various results similar to those given by \(\hat{H}_{su(1,1)}\). A possible reexpression of the condition (7.36) is given as follows :
\[
e^0 = \omega^0 , \quad f^0 = \frac{g_0}{2} \cdot \left( T - \frac{\hbar}{2} \right) , \quad g^0 = g_0 \cdot \left( T - \frac{\hbar}{2} \right) ,
\]
\[
\gamma^0 = \gamma , \quad g = \frac{g_0}{2} \cdot \left( S - \frac{\hbar}{2} \right) . \quad (g_0 : \text{parameter})
\]
(7.38)

Under the condition (7.38), \((\hat{K}_{su(2)}, \hat{V}_{su(2)})\) and \((\hat{K}_{su(1,1)}, \hat{V}_{su(1,1)})\) can be expressed in the form
\[
\hat{K}_{su(2)} = 2g_0 \left( T - \frac{\hbar}{2} \right) \left( \hat{S} - \frac{\hbar}{2} \right) \cdot \hat{\tau}_0 ,
\]
\[
\hat{V}_{su(2)} = - \gamma \cdot i(\hat{\tau}_+ - \hat{\tau}_-) ,
\]
(7.39)
\[
\hat{K}_{su(1,1)} = 2g_0 \left( \hat{T} - \frac{\hbar}{2} \right) \left( S - \frac{\hbar}{2} \right) \cdot \hat{T}_0 ,
\]
\[
\hat{V}_{su(1,1)} = - \gamma \cdot i(\hat{T}_+ - \hat{T}_-) .
\]
(7.40)

Thus, in the \(su(2)\)-space with \(S\) and the \(su(1,1)\)-space with \(T\), we have the following correspondence
\[
\hat{K}_{su(2)} \sim \hat{K}_{su(1,1)} , \quad \hat{V}_{su(2)} \sim \hat{V}_{su(1,1)} .
\]
(7.41)

The above correspondence results from
\[
\hat{\tau}_{\pm,0} \sim \hat{T}_{\pm,0} , \quad T \sim \hat{T} , \quad \hat{S} \sim S .
\]
(7.42)
Of course, $\hat{U}_{su(2)}$ and $\hat{U}_{su(1,1)}$ do not correspond in forms. However, it should be noted that there exists a difference between $\hat{H}_{su(2)}$ and $\hat{H}_{su(1,1)}$ in addition to $\hat{U}_{su(2)}$ and $\hat{U}_{su(1,1)}$. The Holstein-Primakoff representations shown in Eqs. (3.16) and (6.36) teach us the difference. In the case of $(\bar{T}_{\pm,0})$ in Eq. (3.16), the integer $n$ in the boson state $(1/\sqrt{n!}) \cdot (\hat{X}^*)^n |0\rangle$ is permissible up to the infinity. But, in the case of $(\bar{T}_{\pm,0})$ in Eq. (6.36), $n$ is permissible up to $2s$. This can be seen in the factor $\sqrt{2s - \hbar \hat{X}^* \hat{X}}$. It is quite natural, because the $su(2)$-algebra is compact, and the $su(1,1)$-algebra noncompact. If $s$ is sufficiently large compared with $n$, the two algebras give almost the same results.

Next, we investigate the correspondence between the two systems in the classical counterpart. The expectation values of $\hat{H}_{su(2)}$ and $\hat{H}_{su(1,1)}$ with respect to the coherent states (3.25) and (3.23), respectively, give us the following results:

\begin{align}
H_{su(2)} &= U_{su(2)} + K_{su(2)} + V_{su(2)} , \\
U_{su(2)} &= 2 \left[ \omega^0 - 2f^0 \cdot \left( T - \frac{\hbar}{2} \right) \right] \cdot \mathbf{S} \\
&\quad - [(e^0 - \omega^0) - \hbar g^0] \cdot T + \left[ g^0 \left( 1 - \frac{\hbar}{2T} \right) - 2f^0 \right] \cdot T^2 , \\
K_{su(2)} &= [e^0 - \omega^0 - \hbar g^0 + 4f^0 \mathbf{S} - 2(g^0 - 2f^0)T] (\tau_0)_c \\
&\quad + \left[ g^0 \left( 1 + \frac{\hbar}{2T} \right) - 2f^0 \right] \cdot (\tau_0)_c^2 , \\
V_{su(2)} &= -\gamma^0 \cdot i((\tau_+)_c - (\tau_-)_c) , \\
H_{su(1,1)} &= U_{su(1,1)} + K_{su(1,1)} + V_{su(1,1)} , \\
U_{su(1,1)} &= 2(e - 2\hbar g) \left( T - \frac{\hbar}{2} \right) , \\
K_{su(1,1)} &= 4g \cdot \left( T - \frac{\hbar}{2} \right) \left( 1 + \frac{\hbar}{2T} \right) (T_0)_c , \\
V_{su(1,1)} &= -\gamma \cdot i((T_+)_c - (T_-)_c) .
\end{align}

Here, $(\tau_{\pm,0})_c$ and $(T_{\pm,0})_c$ are given in the relations (6.30) and (3.32), respectively. In the same meaning as that in the quantum version, $U_{su(2)}$ and $U_{su(1,1)}$ do not relate with the dynamics. For the Hamiltonian $(K_{su(2)} + V_{su(2)})$ and $(K_{su(1,1)} + V_{su(1,1)})$, we set up the following condition:

\begin{align}
\omega^0 &= \omega^0 , \\
f^0 &= \frac{g_0}{2} \cdot \left( T - \frac{\hbar}{2} \right) \left( 1 + \frac{\hbar}{2T} \right) , \\
g^0 &= g_0 \cdot \left( T - \frac{\hbar}{2} \right) , \\
\gamma^0 &= \gamma , \\
g &= \frac{g_0}{2} \cdot \mathbf{S} .
\end{align}

The relation (7.51) corresponds to Eq. (7.38). Then, $(K_{su(2)}, V_{su(2)})$ and $(K_{su(1,1)}, V_{su(1,1)})$ can be expressed in the form:

\begin{align}
K_{su(2)} &= 2g_0 \left( T - \frac{\hbar}{2} \right) \left( 1 + \frac{\hbar}{2T} \right) \mathbf{S} (\tau_0)_c ,
\end{align}
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\[ V_{su(2)} = -\gamma \cdot i((\tau_+)_c - (\tau_-)_c), \quad (7.53) \]

\[ K_{su(1,1)} = 2g_0 \left( T - \frac{\hbar}{2} \right) \left( 1 + \frac{\hbar}{2T} \right) S(T_0)_c, \quad (7.54) \]

\[ V_{su(1,1)} = -\gamma \cdot i((T_+)_c - (T_-)_c). \quad (7.55) \]

It should be noted that \((S, T)\) are the parameters and \((S, T)\) denote the dynamical variables, which are constants of motion in our present system. Under the condition \((S = S, T = T)\) and the corresponding \((\tau_{\pm,0})_c \sim (T_{\pm,0})_c\), we have

\[ K_{su(2)} \sim K_{su(1,1)}, \quad V_{su(2)} \sim V_{su(1,1)}. \quad (7.56) \]

However, in the case of the \(su(2)\)-spin system, we have the restriction \(h |X|^2 \leq 2S\) and in the case of the \(su(1,1)\)-spin system, these does not exist such a restriction. Then, if \(S\) is sufficiently large compared with \(hX^*X\), the results of the two systems are almost the same as each other.

Let us investigate the difference between \((\tau_{\pm})_c\) in Eq. (6.30) and \((T_{\pm})_c\) in Eq. (3.32). Under \(T = T\), both are related with each other through the relation

\[ (\tau_{\pm})_c = (T_{\pm})_c \cdot \sqrt{1 - \theta(hX^*X)}, \quad (7.57) \]

\[ \theta(hX^*X) = \frac{\hbar}{2S + \hbar - hX^*X}. \quad (7.58) \]

The behavior of the function \(\theta(hX^*X)\) is summarized as follows:

\[
\theta(hX^*X) = \begin{cases} 
O(h^{-\frac{1}{2}}), & (hX^*X \approx 2S - O(h^{\frac{1}{2}})) \\
O(h^0), & (hX^*X \approx 2S \pm O(h^{\frac{1}{2}})) \\
O(h^{\frac{1}{2}}), & (hX^*X \approx 2S + \hbar - O(h^{\frac{3}{2}})) \\
O(h^{\frac{3}{2}}), & (hX^*X \approx 2S + \hbar - O(h^2)) 
\end{cases} \quad (7.59)
\]

Here, \(O(h^k)\) denotes the term of order \(h^k\). Therefore, under the condition \(h \to 0\), \(\theta(hX^*X)\) can be approximated in the form

\[ \theta(hX^*X) = \begin{cases} 
0, & (hX^*X < 2S) \\
1, & (hX^*X = 2S) \\
\infty, & (hX^*X > 2S)
\end{cases} \quad (7.60) \]

The behavior of \(\theta(hX^*X)\) given in the above tells us that \(\theta(hX^*X)\) is equal to zero in the region \(hX^*X < 2S\) and at the point \(hX^*X = 2S\), \(\theta(hX^*X)\) behaves like an infinite wall. The existence of the wall suggests us that the quantity \(hX^*X\) shows a certain periodic behavior with the period \(\sim \ln S\). Then, if \(S\) is sufficiently large, we can expect that the solution for the pseudo \(su(1,1)\)-algebra is well approximated as that for the \(su(1,1)\)-algebra. The details are discussed in Ref. 24.

\section*{§8. The \(su(2,1)\)-algebra and its modified form in terms of three kinds of boson operators}

In the previous section, we presented a possible description of a boson system interacting with an external harmonic oscillator. Of course, two kinds of boson
operators (\(\hat{c}, \hat{c}^*\)) and (\(\hat{d}, \hat{d}^*\)) are introduced and a coherent state for a possible time-
evolution of the system is expressed in terms of these two kinds of boson operators,
which describe the physical objects. Therefore, these bosons are not related to the
phase space doubling such as mentioned in \(\S 5\). In this sense, with the help of the
cohere coherent state adopted in \(\S 7\), it may be impossible to treat statistically mixed state.
In this section, we will give a theoretical framework which enables us to describe the
mixed state of a boson system interacting with the external harmonic oscillator. \(^{(28)}\)

In addition to the boson operators (\(\hat{c}, \hat{c}^*\)) and (\(\hat{d}, \hat{d}^*\)), we introduce the boson
(\(\hat{a}, \hat{a}^*\)). With the use of these bosons, the following bi-linear form can be defined :

\[
\hat{T}_+ = \hbar \hat{a}^* \hat{c}^* , \quad \hat{T}_- = \hbar \hat{c} \hat{a} , \quad \hat{T}_0 = \frac{\hbar}{2} (\hat{a}^* \hat{a} + \hat{c}^* \hat{c} ) + \frac{\hbar}{2} , \quad (8.1)
\]

\[
\hat{R}_+ = \hbar \hat{a}^* \hat{d}^* , \quad \hat{R}_- = \hbar \hat{d} \hat{a} , \quad \hat{R}_0 = \frac{\hbar}{2} (\hat{a}^* \hat{a} + \hat{d}^* \hat{d} ) + \frac{\hbar}{2} , \quad (8.2)
\]

\[
\hat{\sigma}_+ = \hbar \hat{c} \hat{d}^* , \quad \hat{\sigma}_- = \hbar \hat{d}^* \hat{c} , \quad \hat{\sigma}_0 = \frac{\hbar}{2} (\hat{c}^* \hat{c} - \hat{d}^* \hat{d} ) , \quad (8.3)
\]

The set (\(\hat{T}_{\pm,0}\)) and (\(\hat{R}_{\pm,0}\)) obey the su\((1,1)\)-algebra, respectively, and (\(\hat{\sigma}_{\pm,0}\)) the
su\((2)\)-algebra. The operator \(\hat{\sigma}_0\) is related with

\[
\hat{\sigma}_0 = \hat{T}_0 - \hat{R}_0 . \quad (8.4)
\]

We can prove that the set (\(\hat{T}_{\pm,0}, \hat{R}_{\pm,0}, \hat{\sigma}_{\pm}\)) composes the su\((2,1)\)-algebra. In the
above set, there exists an operator which commutes with \(\hat{T}_{\pm,0}, \hat{R}_{\pm,0}\) and \(\hat{\sigma}_{\pm,0}\) :

\[
\hat{K} = \frac{\hbar}{2} (\hat{c}^* \hat{c} + \hat{d}^* \hat{d} - \hat{a}^* \hat{a} ) + \hbar , \quad (8.5)
\]

\[
[\hat{K}, \hat{T}_{\pm,0}] = [\hat{K}, \hat{R}_{\pm,0}] = [\hat{K}, \hat{\sigma}_{\pm,0}] = 0 . \quad (8.6)
\]

As was shown in Ref.\(^{(28)}\), there exist many cases for constructing the orthogonal
set of the su\((2,1)\)-algebra. In this paper, we show one example. For this aim, we
introduce the operator \(\hat{T}\) in the form

\[
\hat{T} = \frac{\hbar}{2} (\hat{c}^* \hat{c} - \hat{a}^* \hat{a} ) + \frac{\hbar}{2} . \quad (8.7)
\]

The operator \(\hat{T}\) commutes with \(\hat{K}\) and \(\hat{T}_0\) and, then, we set up the eigenvalue equation

\[
\hat{K} |K, T, T_0\rangle = K |K, T, T_0\rangle , \quad K = \hbar k ,
\]

\[
\hat{T} |K, T, T_0\rangle = T |K, T, T_0\rangle , \quad T = \hbar t ,
\]

\[
\hat{T}_0 |K, T, T_0\rangle = T_0 |K, T, T_0\rangle , \quad T_0 = \hbar t_0 . \quad (8.8)
\]

A possible solution of the eigenvalue equation (8.8) is given as follows :

\[
|K, T, T_0\rangle = \frac{1}{\sqrt{(t_0 + t - 1)!(2k - 2t - 1)!(t_0 - t)!}}
\]
\( \times (\hat{a}^\dagger)^{t_0 + t - 1} (\hat{a}^\dagger)^{2k - 2t - 1} (\hat{a}^\dagger)^{t_0 - t} |0\rangle \rangle \\
= \frac{1}{(2k - 2)!} \sqrt{\frac{(2k - 2t - 1)!}{(t_0 + t - 1)!(t_0 - t)!}} \times \left( \frac{\tilde{T}_+}{\hbar} \right)^{t_0 - t} \left( \frac{\tilde{S}_+}{\hbar} \right)^{2t - 1} (\hat{a}^\dagger)^{2k - 2} |0\rangle \rangle . \quad (8.9) \)

The classical counterpart can be derived by the following coherent state:

\[ |c\rangle \rangle = M_c \exp \left( \frac{V}{U} \frac{\tilde{T}_+}{\hbar} \right) \exp \left( \frac{w}{vU} \frac{\tilde{S}_+}{\hbar} \right) \exp(v\hat{a}^\dagger)|0\rangle \rangle . \quad (8.11) \]

Here, \( V, w \) and \( v \) are complex parameters and \( U = \sqrt{1 + |V|^2} \). Instead of the above parameters, \((\theta, K), (X, X^*)\) and \((Y, Y^*)\) are convenient for our discussion. They are required to satisfy the relation

\[ \langle\langle c| i\hbar \tilde{K}_0 |c\rangle\rangle = K - \hbar , \quad \langle\langle c| i\hbar \tilde{K}_1 |c\rangle\rangle = 0 , \quad (8.12) \]

\[ \langle\langle c| \partial X |c\rangle\rangle = + \frac{X^*}{2} , \quad \langle\langle c| \partial X^* |c\rangle\rangle = - \frac{X}{2} , \quad (8.13) \]

\[ \langle\langle c| \partial Y |c\rangle\rangle = + \frac{Y^*}{2} , \quad \langle\langle c| \partial Y^* |c\rangle\rangle = - \frac{Y}{2} . \quad (8.14) \]

Further, \( K \) obeys the condition

\[ \langle\langle c| \tilde{K} |c\rangle\rangle = K . \quad (8.15) \]

The relations \((8.12) \sim (8.15)\) give us the form

\[ w = \exp \left( -i \frac{\theta}{2} \right) X , \]

\[ v = \exp \left( -i \frac{\theta}{2} \right) \sqrt{\frac{2(K - \hbar)}{\hbar} - X^* X} , \]

\[ V = Y \frac{1}{\sqrt{1 + X^* X}} . \quad (8.16) \]

Then, the expectation values of \( \tilde{T}_{\pm,0} \) and \( \tilde{R}_{\pm,0} \) are given by

\[ (T_+)_c = \sqrt{\hbar Y^*} \cdot \sqrt{\hbar X^* X + \hbar + hY^*Y} , \]

\[ (T_-)_c = \sqrt{\hbar X^* X + \hbar + hY^*Y} \cdot \sqrt{\hbar Y} , \]
The time-evolution of the coherent and the squeezed states is given by:

\[
\begin{align*}
(T_0)_c &= \hbar Y^* Y + \frac{1}{2} (\hbar X^* X + \hbar) , \\
(R_+)_c &= \sqrt{\hbar} Y^* \cdot \frac{1}{\sqrt{\hbar X^* X + \hbar}} \cdot \sqrt{\hbar} X , \\
(R_-)_c &= \sqrt{\hbar} X^* \cdot \frac{1}{\sqrt{\hbar X^* X + \hbar}} \cdot \sqrt{\hbar} Y , \\
(R_0)_c &= K - \frac{\hbar}{2} X^* X + \hbar Y^* Y - \frac{\hbar}{2} . \tag{8.17}
\end{align*}
\]

In Ref. [28], it was discussed in detail that the form (8.17) can be regarded as the classical counterpart of the \(su(2,1)\)-algebra in terms of three kinds of boson operators.

In order to describe a many-boson system interacting with an external harmonic oscillator, we modified the \(su(1,1)\)-algebra, which was called the pseudo \(su(1,1)\)-algebra. Then, in the present case, we must modify the \(su(2,1)\)-algebra. For this aim, first, we replace \(\sigma_{\pm 0}\) in Eq. (8.3) with \(\tau_{\pm 0}\) defined in the relation (6.1). Then, by extending the coherent states shown in Eqs. (6.20) and (8.11), respectively, we adopt the following form:

\[
|c\rangle = M_c \exp \left( \frac{V}{U} \cdot \frac{\xi}{h} \right) \exp \left( \frac{\eta}{\xi} \cdot \frac{\tau}{h} \right) \exp \left( \eta \tilde{\Delta}^* \right) |0\rangle . \tag{8.18}
\]

Here, \(V, \xi\) and \(\eta\) are complex parameters and \(U = \sqrt{1 + |V|^2}\). The operator \(\tilde{\Delta}^*\) is defined in Eq. (6.22). In the same way as in the previous case, we introduce \((\theta, K), (X, X^*)\) and \((Y, Y^*)\) which obey the conditions (8.12) \sim (8.15). Then, the parameters \(\xi, \eta\) and \(V\) are expressed in the form

\[
\begin{align*}
\xi &= \exp \left( -i \frac{\theta}{2} \right) \frac{1}{\sqrt{2t + X^* X}} , \\
\eta &= \exp \left( -i \frac{\theta}{2} \right) \sqrt{2(K - \hbar) - hX^* X} \frac{1}{\sqrt{2(K - \hbar) + \hbar - hX^* X}} , \\
V &= Y \sqrt{1 + X^* X} . \tag{8.19}
\end{align*}
\]

On the basis of the form presented in the above, we will treat statistically mixed state of a boson system interacting with an external harmonic oscillator. The state \(|c\rangle\) in Eq. (8.18) can be factorized in the form

\[
|c\rangle = |\Phi_{ca}\rangle \otimes |\Psi_d\rangle , \tag{8.20}
\]

\[
|\Phi_{ca}\rangle = U^{-1} (1 - |\xi|^2)^t \exp \left( \frac{V}{U} \hat{a}^* \hat{c}^* \right) \cdot \exp \left( \frac{\xi}{U} \hat{c}^* \sqrt{2t + \hat{c}^* \hat{c}} \right) |0\rangle_{ca} , \tag{8.21}
\]

\[
|\Psi_d\rangle = (1 - |\eta|^2)^t \exp \left( \eta \hat{d}^* \sqrt{1 + \hat{d}^* \hat{d}} \right) |0\rangle_d . \tag{8.22}
\]

The above factorization will be used in the next section.
§9. Description of statistically mixed state of a boson system interacting with an external harmonic oscillator

Before entering the concrete treatment for the present problem, we will contact with the basic idea of our treatment in rather general form. Let us investigate a many-body system consisting of two parts. We refer to the first and the second part as the \( c \)-part and the \( d \)-part. The Hamiltonian consists of

\[
\hat{H} = \hat{K}_c + \hat{K}_d + \hat{V}_{cd} .
\]  

(9.1)

Here, \( \hat{K}_c \) and \( \hat{K}_d \) denote the Hamiltonian of the \( c \)- and the \( d \)-part, respectively, and the two parts interact with each other through the term \( \hat{V}_{cd} \). Therefore, we must introduce two spaces, which we call the \( c \)- and the \( d \)-space, and the present system is treated in the product of the two spaces. The operators \( \hat{K}_c \) and \( \hat{K}_d \) belong to the \( c \)- and the \( d \)-space, respectively, and \( \hat{V}_{cd} \) connects the two spaces.

For the Hamiltonian (9.1), we set up the time-dependent Schrödinger equation

\[
i\hbar \partial_t |c\rangle = \hat{H} |c\rangle, \quad \langle c | c \rangle = 1 .
\]  

(9.2)

As a possible solution of Eq. (9.2), let us assume the form

\[
|c\rangle = |\Phi_c\rangle \otimes |\Psi_d\rangle, \quad \langle \Phi_c | \Phi_c \rangle = \langle \Psi_d | \Psi_d \rangle = 1 .
\]  

(9.3)

Here, \( |\Phi_c\rangle \) and \( |\Psi_d\rangle \) are given in the \( c \)- and the \( d \)-space, respectively. For the state (9.3), we impose two conditions. The first is the following: For any state \( |k\rangle \) in the \( d \)-space which is orthogonal to \( |\Psi_d\rangle \), \( |\Psi_d\rangle \) is governed by

\[
\langle k | \hat{K}_d + \hat{V}_{cd} - i\hbar \partial_t |\Psi_d\rangle = 0 ,
\]

(9.4)

\[
\langle k |\Psi_d \rangle = 0 .
\]

(9.5)

The second condition is the following: For the state \( |\Psi_d\rangle \) satisfying the condition (9.4), \( |\Phi_c\rangle \) obeys

\[
i\hbar \partial_t |\Phi_c\rangle = \hat{H}_c |\Phi_c\rangle ,
\]

(9.6)

\[
\hat{H}_c = \hat{K}_c + \langle \Psi_d | \hat{K}_d + \hat{V}_{cd} - i\hbar \partial_t |\Psi_d\rangle .
\]

(9.7)

Here, it should be noted that the left-hand side of Eq. (9.4) and the second term on the right-hand side of Eq. (9.7) are calculated only for operators in the \( d \)-space and, then, naturally, they are operators in the \( c \)-space. Under the conditions (9.4) and (9.6), we can prove that the state \( |c\rangle \) given in Eq. (9.3) is an exact solution of the Schrödinger equation (9.2). As is clear from the forms (9.4) \( \sim \) (9.7), the above-mentioned two conditions are too strong to get the exact solution. However, it may be permissible as an approximation. In general, the time-dependent variational procedure gives us a possible approximate solution of the Schrödinger equation.
Therefore, under the procedure with the trial function $|c\rangle$ in the form (9.3), we obtain an approximate solution of Eq. (9.2).

Our chief concern is related to a possible description of statistically mixed state realized in the $c$-part. Then, with an idea similar to that used in thermo field dynamics formalism, we perform the phase space doubling for the $c$-part. This procedure is done through the introduction of a new space which is conjugate to the $c$-space. We call this space the $a$-space. Let a possible solution of the Schrödinger equation in a form analogous to $|c\rangle$ given in Eq. (9.3) be obtained:

$$ |c\rangle = \Phi_{ca} \otimes |\Psi_d\rangle , \quad \langle c|c\rangle = 1 . $$

The part $|\Psi_d\rangle$ obeys the conditions (9.4) and (9.5) and $|\Phi_{ca}\rangle$ is given by solving Eq. (9.6) in the $ca$-space:

$$ i\hbar \partial_t |\Phi_{ca}\rangle = \hat{H}_c |\Phi_{ca}\rangle , \quad \langle \Phi_{ca}|\Phi_{ca}\rangle = 1 . $$

Here, $\hat{H}_c$ is given in Eq.(9.7). With the use of the state $|c\rangle$ obtained in the above process, we are able to describe the mixed state in a manner similar to the standard thermo field dynamics formalism. In our present system, let us introduce the operator $\hat{\rho}_c$ in the form

$$ \hat{\rho}_c = \sum_n (n|\Phi_{ca}\rangle\langle \Phi_{ca}|n) . $$

Here, $\{ |n\rangle \}$ denotes an orthonormal set in the $a$-space. Then, $\hat{\rho}_c$ is an operator given in the $c$-space. We can prove that $\hat{\rho}_c$ satisfies the conditions which the density matrix obeys:

$$ \text{Tr} \ hat{\rho}_c = 1 , \quad i\hbar \partial_t \hat{\rho}_c = [\hat{H}_c, \hat{\rho}_c] . $$

Here, $\hat{H}_c$ is given in Eq.(9.7). The ensemble average of the operator $\hat{O}_c$ in the $c$-space is expressed as

$$ \text{Tr} (\hat{O}_c \hat{\rho}_c) = \langle \Phi_{ca}|\Phi_{ca}\rangle = \langle c|\Phi_{ca}|c\rangle . $$

Further, the entropy $\hat{S}_c$ in the $c$-space can be defined as

$$ \hat{S}_c = -\sum_n \langle \Phi_{ca}|n\rangle (n|\Phi_{ca}\rangle) \cdot \ln \langle \Phi_{ca}|n\rangle(n|\Phi_{ca}\rangle) . $$

We can discriminate the pure state in terms of the index $n$.

On the basis of the basic idea mentioned in the above, let us investigate the Hamiltonian (7.6) in the framework of the modification of the $su(2, 1)$-algebra given in §8. The expectation values of $\hat{K}_c$, $\hat{K}_d$ and $\hat{V}_{cd}$ (1) shown in Eqs. (7.1) $\sim$ (7.3), respectively, are calculated as follows:

$$ (K_c)_c = e^0 \cdot h(X^*X + Y^*Y) $$

$$ + g^0 \cdot \left\{ 2[h(X^*X + Y^*Y)]^2 - \left( 1 - \frac{\hbar}{2T} \right) (hX^*X)^2[1 + \hbar Y^*Y(h + \hbar X^*X)^2] \right\} , $$
\[ (K_d)_c = \omega^0 \cdot [2(K - h) - hX^*X], \]  
\[ (V_{cd}(1))_c = 2f^0 \cdot h(X^*X + Y^*Y)[2(K - h) - hX^*X] + h f^0 \cdot [2(K - h) + hY^*Y]. \]  

The above three are exact. But, in the case of \((V_{cd}(2))_c\), the approximation form is given as

\[ (V_{cd}(2))_c = -\gamma^0 \cdot i\sqrt{h}(X^* - X)\sqrt{2\tau + hX^*X} \sqrt{\frac{2(K - h) - hX^*X}{2(K - h) + h - hX^*X}}. \]  

Here, \(\tau\) is defined by

\[ \tau = T + hY^*Y, \quad T = \hbar t. \]  

Concerning the derivation of the relation (9.17), the following approximate form plays a central role:

\[ \langle \langle c|\sqrt{h}\hat{c}^*\sqrt{2T + h\hat{c}^*\hat{c}}|c\rangle \rangle = \exp \left( i\frac{\theta}{2} \right) \cdot \sqrt{h}X^*\sqrt{2\tau + hX^*X}. \]

The detail is given in Ref. [29]. Further, we introduce the quantities \(\tau_{\pm,0}\) as follows:

\[ \tau_+ = \sqrt{h}X^*\sqrt{2\tau + hX^*X} \sqrt{\frac{2(K - h) - hX^*X}{2(K - h) + h - hX^*X}}, \]  
\[ \tau_- = \sqrt{h}X\sqrt{2\tau + hX^*X} \sqrt{\frac{2(K - h) - hX^*X}{2(K - h) + h - hX^*X}}, \]  
\[ \tau_0 = \tau + hX^*X, \]  
\[ S = K - h + \frac{\hbar}{2}Y^*Y. \]  

The quantity \(S\) is the expectation value of \(\hat{S} = \hbar(\hat{c}^*\hat{c} + \hat{d}^*\hat{d})/2\) for the state \(|c\rangle\).

Then, we have

\[ H_{su(2,1)} = U_{su(2,1)} + K_{su(2,1)} + V_{su(2,1)}, \]  
\[ U_{su(2,1)} = 2 \left[ \omega^0 - 2f^0 \cdot \left( T - \frac{h}{2} \right) \right] S - [(e^0 - \omega^0) - h g^0] \cdot T + \left[ g^0 \left( 1 - \frac{h}{2T} \right) - 2f^0 \right] T^2, \]  
\[ K_{su(2,1)} = [(e^0 - \omega^0) - h g^0 + 4f^0 S - 2(g^0 - 2f^0)T] \tau_0 \]  
\[ + \left[ g^0 \left( 1 + \frac{h}{2T} \right) - 2f^0 \right] \tau_0^2, \]  
\[ V_{su(2,1)} = -\gamma^0 \cdot i(\tau_+ - \tau_-). \]

We can see that the Hamiltonian (9.22) with the forms (9.23) \sim (9.25) is of the same form as that shown in Eq. (7.43) with the forms (7.44) \sim (7.46) if \(\tau\) and \((K - h)\) in
\(\tau_{\pm,0}\) appearing in Eq. (9.20) are replaced with \(T\) and \(S\) in \((\tau_{\pm,0})\) appearing in Eq. (6.30), respectively. However, as was discussed in Ref.29), the meaning of \(\not\!\not\!\not\!c\) is different from that of \(H_{\text{su}(2)}\). Since \(H_{\text{su}(2)}\) is of the same form as \(H_{\text{su}(2)}\), we can treat \(H_{\text{su}(2)}\) under the same method as that adopted in §§4 and 7. Then, we will not contact with this method in detail.

In order to make the problem simpler, we consider the case of \(e^{\theta} = \omega^{0}\) and \(g^{0} = f^{0} = 0\), which corresponds to the harmonic oscillator. Further, \((\mathbf{K} - \hbar)\) is sufficiently large, i.e., \(2(\mathbf{K} - \hbar) > \hbar \mathbf{X}^{\ast} \mathbf{X}\). In this case, \(\hbar \mathbf{X}^{\ast} \mathbf{X}\) shows a certain long periodic behavior and, with the help of Eqs. (4.44), (4.47) and (4.56), we have

\[
\hbar \mathbf{X}^{\ast} \mathbf{X} = \sqrt{T^{2} + J^{2}} \cosh(2\gamma t + \chi^{0}) - I ,
\]

i.e.,

\[
\langle \langle \langle c | \hbar \not\!\not\!\not\!c | c \rangle \rangle \rangle = \sqrt{T^{2} + J^{2}} \cosh(2\gamma t + \chi^{0}) - T .
\]

Since \(\theta = 2\omega t + \theta^{0}\), to order \(\gamma^{1}\), the expectation value \(\langle \langle \langle c | \hbar \not\!\not\!\not\!c | c \rangle \rangle \rangle\) can be expressed as

\[
\langle \langle \langle c | \hbar \not\!\not\!\not\!c | c \rangle \rangle \rangle = \langle \langle \langle c | \hbar \not\!\not\!\not\!c | c \rangle \rangle \rangle_{t=0}
- \gamma t \left[ \langle \langle \langle c | \sqrt{\hbar} \not\!\not\!\not\!c | \sqrt{\hbar} \not\!\not\!\not\!c \rangle \rangle \rangle \exp \left( -i\frac{\theta^{0}}{2} \right)
- \langle \langle \langle c | \hbar \not\!\not\!\not\!c | c \rangle \rangle \rangle \exp \left( i\frac{\theta^{0}}{2} \right) \right] .
\]

Here, the symbol \(t\) denotes time.

Now let us investigate our system in the framework mentioned in the beginning part of this section. With the use of \(|\Phi_{\text{eq}}\rangle\rangle\) and \(|\Psi_{d}\rangle\rangle\) in Eqs. (8.21) and (8.22) and the form \(\eta\) in Eq. (8.19), \(\hat{H}_{\text{c}}\) defined in the relation (9.7) can be expressed as

\[
\hat{H}_{c} = \hat{H}_{0} + \hat{H}_{1} + \hat{H}_{2} ,
\]

\[
\hat{H}_{0} = \hbar \omega \not\!\not\!\not\!c ,
\]

\[
\hat{H}_{1} = -\gamma \cdot i[\sqrt{\hbar} \not\!\not\!\not\!c \cdot \sqrt{2T + \hbar \not\!\not\!\not\!c \cdot \eta} - \sqrt{2T + \hbar \not\!\not\!\not\!c \cdot \eta} \cdot \not\!\not\!\not\!c \cdot \eta^{\ast}] ,
\]

\[
\hat{H}_{2} = -\frac{i\hbar}{2} \frac{1}{1 - |\eta|^{2}} [(\dot{\eta} + i\omega \eta)\eta^{\ast} - (\dot{\eta}^{\ast} - i\omega \eta^{\ast})\eta] .
\]

Substituting \(\theta = 2\omega t + \theta^{0}\) into the form (8.19), \(\eta\) can be expressed as

\[
\eta = \exp \left( -i\frac{\theta^{0}}{2} \right) \exp(-i\omega t) \sqrt{1 - \frac{\hbar}{2(\mathbf{K} - \hbar) + \hbar h \mathbf{X}^{\ast} \mathbf{X}}} .
\]

Then, we can prove that, at order \(\hbar^{0}\), the part \(\hat{\not\!\not\!\not\!c}\) vanishes and \(\eta = \exp(-i\theta^{0}/2)\times\exp(-i\omega t)\). Thus, at this order, \(\hat{H}_{c}\) can be approximated as

\[
\hat{H}_{c} = \hat{H}_{0} + \hat{H}_{1}(t) .
\]
Here, $\hat{H}_0$ is given in Eq. (9.30) and $\hat{H}_1(t)$ is written in the form

$$
\hat{H}_1(t) = -\gamma \cdot i \left[ \sqrt{h^*c} \cdot \sqrt{2T + \hat{c}^*\hat{c}} \cdot \exp(-i\theta^0/2) \cdot \exp(-i\omega t) \right.
- \sqrt{2T + h^*\hat{c}} \cdot \sqrt{h^*} \cdot \exp(i\theta^0/2) \cdot \exp(+i\omega t)
\left. \right] .
$$

The Hamiltonian $\hat{H}_c$ indicates that the present system is a harmonic oscillator (or many-boson system) in a classical external field with the frequency $\omega$. The interaction with this is $\hat{H}_1(t)$. We will apply linear response theory of non-equilibrium statistical mechanics. For the Hamiltonian (9.29), this method leads us to the following form in the case of $\hat{c}^*\hat{c}$:

$$
\text{Tr}(\hat{h}\hat{c}^*\hat{c} \hat{\rho}(t)) = \text{Tr}(\hat{h}\hat{c}^*\hat{c} \hat{\rho}(0))
- \gamma t \cdot \text{Tr} \left[ \sqrt{h^*c} \cdot \sqrt{2T + h^*\hat{c}} \cdot \exp \left( -i\frac{\theta^0}{2} \right) 
+ \sqrt{2T + h^*\hat{c}} \cdot \sqrt{h^*} \cdot \exp \left( +i\frac{\theta^0}{2} \right) \right] \hat{\rho}(0) .
$$

Here, $\hat{\rho}(t)$ obeys the von Neumann equation

$$
i\hbar \partial_t \hat{\rho}(t) = [\hat{H}_c, \hat{\rho}(t)] .
$$

A possible solution of Eq. (9.37) is given in the form $\hat{\rho}(t)$ shown in Eq. (9.10) and the relation (9.12) tells us that, for any operator $\hat{O}_c$, we have

$$
\text{Tr}(\hat{O}_c \hat{\rho}(0)) = \langle \langle c | \hat{O}_c | c \rangle \rangle_{t=0} .
$$

Then, the results (9.28) and (9.36) coincide with each other under the correspondence (9.38). In our present method, we can calculate $\text{Tr}(\hat{h}\hat{c}^*\hat{c} \hat{\rho}(t))$ for any order of $\gamma$ in the specific order. Also, it may be interesting to see that our approach is closely related to linear response theory of non-equilibrium statistical mechanics.

In our present system, $\hat{h}Y^*Y$ does not depend on $t$, while $\hat{h}X^*X$ depends on $t$. Therefore, in the present system, the minimum value of $\hat{h}X^*X$ for $t$ can be regarded as the equilibrium value, that is, the equilibrium state appears. This state realizes at $2\gamma t + \chi^0 = 0$ in the relation (9.26). Then, we have

$$
\langle \hat{h}X^*X \rangle_{eq} = \sqrt{I^2 + J^2 - I}
= \sqrt{(T + \hat{h}Y^*Y)^2 + J^2} - (T + \hat{h}Y^*Y) .
$$

In this case, the entropy $S_c$ is derived from the relation (9.13) in the following approximate form:

$$
S_c = (1 + Y^*Y) \ln(1 + Y^*Y) - Y^*Y \ln(Y^*Y) .
$$
At the equilibrium point, we define the free energy $F_c$ in the form

$$F_c = E_c - \frac{S_c}{\beta},$$

(9.41)

$$E_c = \langle\langle c|\hat{h}\omega \hat{c}^* \hat{c}|c\rangle\rangle$$

$$= \omega \left( \sqrt{T^2 + \hbar^2 Y^* Y + J^2} - T \right),$$

(9.42)

$$\beta = (k_B T_{eq})^{-1}.$$  

(9.43)

By minimizing $F_c$ with respect to $Y^* Y$, we have the following relation:

$$I = T + \hbar \frac{1}{\exp \left( \frac{I}{\sqrt{T^2 + J^2}} \right) - 1}. $$

(9.44)

From the above relation, we can calculate the energy $E_c$ as a function of $\hbar \omega \beta$. Two extreme cases are rather easily derived:

1) The low temperature limit:

$$E_c = \frac{\omega J^2}{T + \sqrt{T^2 + J^2}} + \hbar \omega_c \frac{1}{\exp(\hbar \omega_c \beta) - 1}$$

$$- \frac{\hbar}{2} \frac{\hbar \omega J^2}{(T^2 + J^2)} \cdot \left( \frac{1}{\exp(\hbar \omega_c \beta) - 1} \right)^2 + \cdots,$$

(9.45)

$$\omega_c = \omega \sqrt{T^2 + J^2}.$$  

(9.46)

2) The high temperature limit:

$$E_c = k_B T_{eq} + \omega^2 \sqrt{T^2 + J^2} \cdot (k_B T_{eq})^{-1} + \cdots.$$  

(9.47)

We see that in the relation (9.45), the first term denotes the energy of $T_{eq} = 0$, and the second represents the energy coming from the thermal fluctuations and the distribution function of free bose particle. The frequency $\omega$ is changed effectively to $\omega_c$, which may be the result of interaction with the external oscillator. On the other hand, the high temperature limit reveals that $E_c \sim k_B T_{eq}$ and we can see that present system obeys the well-known law of equi-partition of energy. The details can be found in Ref.29).

§10. Time-dependent variational approach with squeezed state to quantum mechanical systems

In this and succeeding sections, we introduce another trial state in the time-dependent variational approach for quantum many-particle systems with the aim of investigating the classical motion including quantum fluctuations in the quantized systems. The new state, that is called the (one-mode) squeezed state, is introduced in
this section, which gives a possible classical motion in quantum mechanical systems including quantum fluctuations beyond the order of $\hbar$. The similar variational approach is found in Ref.42), while our treatment is developed widely in the systematic way. We here present instructively our formalism based on the time-dependent variational principle with the squeezed state in quantum mechanical systems.

The squeezed state is defined as

$$\left| \psi(\alpha, \beta) \right> = (1 - \beta^* \beta) \frac{1}{2} \exp \left( \frac{\beta^* \hat{b}^2}{2} \right) \cdot \exp \left( -\frac{1}{2} \alpha^* \alpha \right) \exp(\alpha \hat{a}^*) |0\rangle .$$  \hspace{1cm} (10.1)$$

Here, $\hat{a}^*$ is a boson creation operator and $|0\rangle$ is a vacuum state for the boson annihilation operator $\hat{a}$: $\hat{a} |0\rangle = 0$. The operators $\hat{b}^*$ and $\hat{b}$ are defined as

$$\hat{b}^* = \hat{a}^* - \alpha^* , \quad \hat{b} = \hat{a} - \alpha .$$  \hspace{1cm} (10.2)$$

The operator $\hat{b}$ is nothing but the annihilation operator for the usual coherent state: $\hat{b} \exp \left( -\frac{1}{2} \alpha^* \alpha \right) \exp(\alpha \hat{a}^*) |0\rangle = 0$. Introducing the operators which correspond to the coordinate and momentum operators, we have another expression of the squeezed state (10.1):

$$\left| \psi(\alpha, \beta) \right> = e^{i\varphi} (2G)^{-1/4} \exp \left( i \frac{\hbar}{2} (p \dot{Q} - q \dot{P}) \right) \exp \left\{ \frac{1}{4 \hbar} \left( 1 - \frac{1}{2G} + i2\Pi \right) \dot{Q}^2 \right\} |0\rangle ,$$  \hspace{1cm} (10.3)$$

$$\dot{Q} = \frac{\hbar}{2} (\hat{a}^* + \hat{a}) , \quad \dot{P} = i \frac{\hbar}{2} (\hat{a}^* - \hat{a}) ,$$  \hspace{1cm} (10.4)$$
q = \sqrt{\frac{\hbar}{2}} (\alpha^* + \alpha) , \quad p = i \sqrt{\frac{\hbar}{2}} (\alpha^* - \alpha) ,$$

$$G = \sqrt{\frac{1}{2} + |y|^2 + y^2} , \quad \Pi = \frac{i}{2} (y^* - y) \sqrt{\frac{1}{2} + |y|^2} G^{-1} ,$$  \hspace{1cm} (10.5)$$

$$e^{-i2\varphi} = \frac{1}{\sqrt{G}} \left( \sqrt{\frac{1}{2} + |y|^2 + y} \right) ,$$

where $y$ is related to $\beta$ as $y = \beta / \sqrt{2(1 - |\beta|^2)}$. The reason why we have introduced the new variables $y$ and $y^*$ instead of $\beta$ and $\beta^*$ is shown later.

The time-evolution of this quantum state is governed by the time-dependent variational principle:

$$\delta \int_{t_0}^{t_1} dt \langle \psi(\alpha, \beta) | i\hbar \partial_t - \hat{H} | \psi(\alpha, \beta) \rangle = 0 .$$  \hspace{1cm} (10.6)$$

In the present method, one can deal with any Hamiltonian which is a function $\hat{Q}$ and $\hat{P}$. However, we restrict ourselves in this section on the Hamiltonian of the form

$$\hat{H} = \frac{1}{2} \hat{P}^2 + V(\hat{Q}) .$$  \hspace{1cm} (10.7)$$
Further, the following canonicity conditions
\[
\langle \psi(\alpha, \beta) | \partial_x | \psi(\alpha, \beta) \rangle = \frac{1}{2} x^* , \quad \langle \psi(\alpha, \beta) | \partial_y | \psi(\alpha, \beta) \rangle = \frac{1}{2} y ,
\]
\[
\langle \psi(\alpha, \beta) | \partial_y | \psi(\alpha, \beta) \rangle = \frac{1}{2} y^* , \quad \langle \psi(\alpha, \beta) | \partial_x | \psi(\alpha, \beta) \rangle = -\frac{1}{2} y \quad (10.8)
\]
give us the sets of canonical variables:
\[
x = \alpha , \quad x^* = \alpha^* ; \quad y = \frac{\beta}{\sqrt{2(1 - \beta^* \beta)}} , \quad y^* = \frac{\beta^*}{\sqrt{2(1 - \beta^* \beta)}} . \quad (10.9)
\]
This is the reason why we have already expressed \( G \) and \( \Pi \) in Eq.(10.5) in terms of \( y \) and \( y^* \) instead of \( \beta \) and \( \beta^* \). Thus, the variational principle (10.6) leads us to the following equations of motion:
\[
\dot{q} = \partial_p \mathcal{H} , \quad \dot{p} = -\partial_q \mathcal{H} , \quad i\hbar \dot{y} = \partial_{y^*} \mathcal{H} , \quad i\hbar \dot{y}^* = -\partial_y \mathcal{H} , \quad (10.10)
\]
where \( \mathcal{H} \) is the expectation value of the Hamiltonian with respect to the squeezed state. Also, the variables \( G \) and \( \Pi \) obey the similar canonical equations of motion instead of the variables \( y \) and \( y^* \): \( \hbar G = \partial_{\Pi} \mathcal{H} \) and \( \hbar \Pi = -\partial_G \mathcal{H} \). Here, we summarize the expectation values for the various operators:
\[
\langle \psi(\alpha, \beta) | \hat{Q} | \psi(\alpha, \beta) \rangle = q , \quad \langle \psi(\alpha, \beta) | \hat{P} | \psi(\alpha, \beta) \rangle = p , \quad (10.11)
\]
\[
\langle \psi(\alpha, \beta) | \hat{Q}^2 | \psi(\alpha, \beta) \rangle = q^2 + \hbar \xi , \quad \langle \psi(\alpha, \beta) | \hat{P}^2 | \psi(\alpha, \beta) \rangle = p^2 + \hbar \eta ,
\]
\[
\langle \psi(\alpha, \beta) | \hat{H} | \psi(\alpha, \beta) \rangle = \frac{1}{2} p^2 + \frac{1}{2} \hbar \eta + \exp \left\{ \frac{1}{2} \hbar \xi \left( \frac{\partial}{\partial q} \right)^2 \right\} V(q) , \quad (10.12)
\]
where the real quantities \( \xi (> 0) \) and \( \eta (> 0) \) are defined as
\[
\xi = \left| \sqrt{\frac{1}{2} + y^* y} \right|^2 = G , \quad \eta = \left| \sqrt{\frac{1}{2} + y^* y - y} \right|^2 = \frac{1}{4G} + G\Pi^2 . \quad (10.13)
\]
The squares of the standard deviations for \( \hat{Q} \) and \( \hat{P} \) are expressed as \( \langle \psi(\alpha, \beta) | (\hat{Q} - q)^2 | \psi(\alpha, \beta) \rangle = \hbar \xi \) and \( \langle \psi(\alpha, \beta) | (\hat{P} - p)^2 | \psi(\alpha, \beta) \rangle = \hbar \eta \). Thus, the uncertainty relation is expressed in terms of \( \xi \) and \( \eta \) as \( \langle \psi(\alpha, \beta) | (\hat{Q} - q)^2 | \psi(\alpha, \beta) \rangle \langle \psi(\alpha, \beta) | (\hat{P} - p)^2 | \psi(\alpha, \beta) \rangle = \hbar^2 \xi \eta = \hbar^2 (1/4 + G^2 \Pi^2) \). It can be seen from this uncertainty relation that, if one direction of the uncertainty is relaxed, the other can be squeezed. Also, one can see from (10.12) that the quantum effects beyond the order of \( \hbar \) are included in this formalism. This novel feature is originated from the degree of freedom of the squeezing.

We have to give the initial conditions for solving the Hamilton equations of motion derived in Eq. (10.11). Among four initial values for \( q_0, p_0, y_0 \) and \( y_0^* \) which lead to a unique set of solutions of Eq.(10.10), those for \( q_0 \) and \( p_0 \) can be selected arbitrary as is usually done in the ordinary TDHF theory. Thus, our task is to determine the initial conditions for \( y \) and \( y^* \). Since we are interested in classical motions in quantal systems, it may be natural to require that, at least, the initial
state has the least quantal effects. Thus, the initial state satisfies the minimal uncertainty relation. Also, this means that the absolute value of the energy due to quantum effect at the very beginning is as small as possible. These two conditions give us the initial values of \( y_0 \) and \( y_0^* \):

\[
\xi_0 y_0 = \left| \frac{1}{2} + y_0^* y_0 - y_0^* \right|^2 = \frac{1}{4}, \quad |\mathcal{H}(q_0, p_0, y_0, y_0^*) - \mathcal{H}_c(q_0, p_0)| : \text{minimal}.
\]

(10.14)

Here, each variable with suffix 0 denotes the initial value of each dynamical variable, and \( \mathcal{H}_c(q, p) \) denotes the classical value of \( \mathcal{H} \) defined as

\[
\mathcal{H}_c(q, p) = \frac{1}{2} p^2 + V(q).
\]

(10.15)

The first condition in Eq. (10.14) gives us \( y_0^* = y_0 = \rho \), which is identical with \( \Pi_0 = 0 \). The second condition in Eq. (10.14) together with \( y_0^* = y_0 \) is reduced to

\[
\left| \frac{1}{8G_0} + \frac{1}{\hbar} \left[ \exp \left( \frac{1}{2} \hbar G_0 \left( \frac{\partial}{\partial q_0} \right)^2 \right) \right] - 1 \right| V(q_0) : \text{minimal}.
\]

(10.16)

Here, \( G_0 \) has already been defined in the relation (10.5).

In the order of \( \hbar^0 \) and \( \hbar^1 \), the condition (10.16) is further reduced to

\[
\left| \frac{1}{G_0} + 4vG_0 + \hbar wG_0^2 \right| : \text{minimal}, \quad v = \frac{\partial^2}{\partial q_0^2}V(q_0), \quad w = \frac{\partial^4}{\partial q_0^4}V(q_0).
\]

(10.17)

Approximate solutions of Eq. (10.17) and the corresponding energies are given as:

(i) \( v > 0 \);

\[
G_0 = \frac{1}{2\sqrt{v}} - \frac{\hbar w}{16v^2}, \quad E = E_{c,0} + \frac{\hbar}{2} \sqrt{v} + \frac{\hbar^2 w}{32v}.
\]

(ii) \( v < 0 \);

\[
G_0 = \frac{1}{2\sqrt{-v}} + \frac{\hbar w}{32v^2}, \quad E = E_{c,0}.
\]

(iii) \( v = 0, w > 0 \);

\[
G_0 = h^{-1/3}(2w)^{-1/3}, \quad E = E_{c,0} + h^{4/3} \frac{3}{16}(2w)^{1/3}.
\]

(iv) \( v = 0, w < 0 \);

\[
G_0 = h^{-1/3}(-w)^{-1/3}, \quad E = E_{c,0}.
\]

(10.18-10.21)

Here, the quantity \( E_{c,0} \) is defined by \( E_{c,0} = \mathcal{H}_c(q_0, p_0) \). The quantity \( \rho (= y_0^* = y_0) \) is obtained from \( G_0 \) with the use of the relation \( G_0 = (\sqrt{1/2} + \rho^2 + \rho^4/2) \).

For various potentials, let us investigate the energy expectation values under the above initial conditions:

(1) The harmonic oscillator potential \( \omega^2 \hat{Q}^2/2 \) : In this case, \( v = \omega^2 \) and \( w = 0 \) are obtained. We then have \( \rho = (1/2\sqrt{\omega})(1/\sqrt{\omega} - \sqrt{\omega}) \). Thus, the total energy \( E \) is expressed as

\[
E = \frac{1}{2} \rho^2 + \frac{1}{2} \omega^2 q_0^2 + \frac{1}{2} \hbar \omega.
\]

(10.22)

(2) The Morse potential \( W_0[\exp(-2\mu \hat{Q}) - 2 \exp(-\mu \hat{Q})] \) with \( W_0 > h^2 \mu^2/8 \) : In this case, \( v = 2\mu^2 W_0[2 \exp(-2\mu q_0) - \exp(-\mu q_0)] \). We will show the results of the order
\[ E = \frac{1}{2} p_0^2 + W_0 [\exp(-2\mu q_0) - 2 \exp(-\mu q_0)] + \hbar \sqrt{\frac{W_0}{2}} [2 \exp(-2\mu q_0) - \exp(-\mu q_0)]^{\frac{3}{2}}. \]  
(10.23)

If \( q_0 = p_0 = 0 \), we have the minimal classical energy
\[ E_0 = -W_0 + \hbar \mu \sqrt{\frac{W_0}{2}}. \]  
(10.24)

This energy should be compared with the exact eigenvalue of this quantum Hamiltonian which has the form
\[ E_{\text{exact}}^n = -W_0 \left[ 1 - \hbar \frac{\mu}{\sqrt{2} W_0} \left( n + \frac{1}{2} \right) \right]^2. \quad (n = 0, 1, 2, \cdots) \]  
(10.25)

If \( n = 0 \), in the order of \( \hbar^1 \), the eigenvalue (10.25) is reduced to the result (10.24).

We can conclude that the conditions in Eq. (10.14) are plausible for describing the classical motion in quantal system in our squeezed state approach. The time-evolution of the dynamical variables in the squeezed state, which governs the time-evolution of the quantum state, is presented in Ref. [32] in the case of \( V(\hat{Q}) = 0 \), \( V(\hat{Q}) = \gamma \hat{Q} \) and the upside-down harmonic oscillator potential.
the order of $\hbar$. If we take the limit of $\hbar \to 0$, the WKB result is reproduced. Further, we can show that the Maslov phase which appears in the Bohr-Sommerfeld quantization rule has the geometrical property such as the Berry phase in the limit of $\hbar \to 0$ in our formalism. Of course, we can take into account of the higher order contributions of $\hbar$. The details can be found in Ref.36. Secondly, we point out that our squeezed state formalism can be easily extended to the case of the scalar field theory. The details can be found in Ref.33).

§11. Coherent and squeezed states in the $su(2)$-boson model

We extend the squeezed state approach developed in §10 to the $su(2)$-algebraic model. The $su(2)$-algebraic models such as the Lipkin model and the pairing model are regarded as the schematic models which describe many-nucleon and/or fermion system. The $su(2)$-algebra can be expressed in terms of two kinds of boson operators, which is known as the Schwinger boson realization. In this sense, we call the $su(2)$-model treated here the $su(2)$-boson model. Focussing on the least quantal effect, in this section, we give an analysis of the $su(2)$-boson model based on the coherent and squeezed state.

We introduce two kinds of boson operators $\hat{a}$ and $\hat{b}$. The generators of the $su(2)$-algebra are expressed by these boson operators, which have already given in Eq. (2.1). The Casimir operator $\hat{\Gamma}_{su(2)}$ has also been defined in Eq. (2.4) with (2.5). The magnitude of the $su(2)$-spin, $S$, as a mean value is given by

$$\langle w | \hat{S} | w \rangle = S, \quad (11.1)$$

where $|w\rangle$ is a certain state. If $|w\rangle$ is an eigenstate for $\hat{S}$, then the expectation value of the Casimir operator is obtained as $\langle w | \hat{\Gamma}_{su(2)} | w \rangle = S^2 + \hbar S$. The second term may be regarded as the quantal effect because the mean value of $\hat{S}$ is $S$. In general, this relation is not satisfied.

First, let us adopt a coherent state as $|w\rangle$. The investigation is expected to be a powerful help to the classical description of the $su(2)$-boson model. As a preparation, we introduce the new sets of boson operators:

$$\hat{c} = U\hat{a} - V\hat{b}, \quad \hat{d} = V^*\hat{a} + U\hat{b}. \quad (11.2)$$

Here, $U$ is real and $V$ is complex and they obey the condition

$$U^2 + V^*V = 1. \quad (11.3)$$

Then, $(\hat{c}, \hat{c}^*)$ and $(\hat{d}, \hat{d}^*)$ are independent boson annihilation and creation operators, respectively. Further, we introduce the boson operators $(\hat{C}, \hat{C}^*)$ and $(\hat{D}, \hat{D}^*)$ which are defined as

$$\hat{C} = \hat{c}, \quad \hat{D} = \hat{d} - \exp(-i\theta/2)\sqrt{\frac{2\sigma}{\hbar}}. \quad (11.4)$$

The boson operator $\hat{D}$ expresses the displacement from the complex value $e^{-i\theta/2}\sqrt{2\sigma/\hbar}$. 


The coherent state $|c\rangle$ has the following standard form:

$$|c\rangle = N_c \exp(e^{-ix/2 \cdot (\alpha \hat{a}^* + \beta \hat{b}^*)})|0\rangle,$$

$$N_c = \exp(-(|\alpha|^2 + \beta^2)/2). \quad (11.5)$$

Here, $\alpha$ is complex and $\beta$ is real with the help of the introduction of the phase angle $\chi$. Let $\alpha$, $\beta$ and $\chi$ correspond to

$$\alpha = \sqrt{2\sigma \bar{\hbar}} \cdot V, \quad \beta = \sqrt{2\sigma \bar{\hbar}} \cdot U, \quad \chi = \theta. \quad (11.6)$$

Then, the coherent state $|c\rangle$ is rewritten as

$$|c\rangle = N_c \cdot \exp\left(e^{-i\theta/2 \cdot \sqrt{2\sigma / \bar{\hbar}} \cdot \hat{d}^*}\right)|0\rangle,$$

$$N_c = \exp(-\sigma / \bar{\hbar}). \quad (11.7)$$

The state $|c\rangle$ is also the coherent state of $\hat{c}$ and $\hat{d}$ with the eigenvalues 0 and $\exp(-i\theta/2) \cdot \sqrt{2\sigma / \bar{\hbar}}$, respectively. Therefore, $|c\rangle$ is the vacuum for $\hat{C}$ and $\hat{D}$:

$$\hat{c}|c\rangle = 0, \quad \hat{d}|c\rangle = \exp(-i\theta/2) \cdot \sqrt{2\sigma / \bar{\hbar}}; \quad \hat{C}|c\rangle = \hat{D}|c\rangle = 0. \quad (11.8)$$

It is noted that the coherent state $|c\rangle$ is specified by four variables $\theta$, $\sigma$, $V$ and $V^*$. Let us investigate the quantal fluctuations in the coherent state $|c\rangle$. The squares of standard deviations for the quasi-spin operators are calculated as follows:

$$(\Delta S_x^2)_c = (\Delta S_y^2)_c = (\Delta S_z^2)_c = \frac{\hbar S^2}{2}, \quad (11.9)$$

where $(\Delta S_x^2)_c = \langle c|\hat{S}_x^2|c\rangle - \langle c|\hat{S}_x|c\rangle^2$ etc., and $\hat{S}_x = (\hat{S}_+ + \hat{S}_-)/2$, $\hat{S}_y = (\hat{S}_+ - \hat{S}_-)/(2i)$ and $\hat{S}_z = \hat{S}_0$. It should be noted that the standard deviations are fixed in the coherent state. Further, it is interesting to investigate the quantal fluctuations by means of the expectation value of the Casimir operator $\hat{\Gamma}_{su(2)}$. We can calculate it as

$$\langle c|\hat{\Gamma}_{su(2)}|c\rangle = \langle c|\hat{S}^2 + \hbar \hat{S}|c\rangle = S^2 + \frac{\hbar S}{2} \cdot (1 - W_c), \quad (11.10)$$

where $W_c$ is related to the standard deviation for $\hat{S}$, i.e., $(\Delta S^2)_c = \langle c|\hat{S}^2|c\rangle - \langle c|\hat{S}|c\rangle^2$:

$$W_c = 1 - \frac{2(\Delta S^2)_c}{\hbar S}. \quad (11.11)$$

Here, for the coherent state $|c\rangle$, we obtain $W_c = 0$. In other words, $(\Delta S^2)_c$ is equal to $\hbar S/2$. Namely, it is of the same order as that of $\hbar S$, which is the exact quantal fluctuation. From the above fact, we have to get the conclusion that the coherent state cannot give a correct quantal fluctuation with respect to the Casimir operator.
The fictitious condition $W_c = 1$ gives us the exact relation for the Casimir operator. Therefore, it is inevitable to look for the state $|w\rangle$ which makes the order of $(\Delta S^2)_w$ less than that of $\bar{h}S$.

In order to overcome the above-mentioned shortcoming of the coherent state, we extend the coherent state $|c\rangle$ given in Eq.(11.5) to the squeezed state $|s\rangle$ in the following form:

$$|s\rangle = N_s \exp(u \hat{C}^* + v \hat{D}^*) \cdot \exp\left(e^{-i\theta/2} \sqrt{\frac{2\sigma}{\bar{h}}} \cdot \hat{d}^*\right) |0\rangle, \quad (11.12)$$

$$u = \frac{\xi e^{-i\theta}}{\sqrt{2(1 + 2\xi^* \xi)}}, \quad v = \frac{\eta e^{-i\theta}}{\sqrt{2(1 + 2\eta^* \eta)}}, \quad (11.13)$$

$$N_s = \left[(1 + 2\xi^* \xi)(1 + 2\eta^* \eta)\right]^{-1/4} \cdot \exp\left(-\frac{\sigma}{\bar{h}}\right). \quad (11.14)$$

It should be noted that the term related to the quadratic for the bosons, which is characteristic to the squeezed state, is added. The state $|s\rangle$ contains new complex variables $(\xi^*, \xi)$ and $(\eta^*, \eta)$. Under the condition (11.1) for $|w\rangle = |s\rangle$, $\sigma$ can be related to $S$ in the following form:

$$S = \langle s|\hat{S}|s\rangle = \sigma + \bar{h}(\xi^* \xi + \eta^* \eta). \quad (11.15)$$

First, let us investigate the square of the standard deviation $(\Delta S^2)_s$. The quantity $(\Delta S^2)_s$ can be calculated as

$$(\Delta S^2)_s = \langle s|\hat{S}^2|s\rangle - \langle s|\hat{S}|s\rangle^2$$

$$= \frac{\hbar}{2} \cdot (S - \hbar \xi^* \xi)[1 + \sqrt{2}(\eta^* + \eta)\sqrt{1 + 2\eta^* \eta + 4\eta^* \eta}]$$

$$+ \hbar^2 \cdot \left(2(\xi^* \xi)^2 + \xi^* \xi + \frac{\eta^* \eta}{2} - \frac{1}{\sqrt{2}}(\eta^* + \eta)\eta^* \eta\sqrt{1 + 2\eta^* \eta}\right). \quad (11.16)$$

The differential of $(\Delta S^2)_s$ for $\eta$ and $\eta^*$ leads us to the following relations for the minimum value of $(\Delta S^2)_s$, while we cannot make $(\Delta S^2)_s$ vanish any choice of $\eta$ and $\eta^*$ for given $S$ and $\xi^* \xi$:

$$\eta = \eta^* = -\frac{1}{2\sqrt{2}}(\sqrt{\zeta} - \frac{1}{\sqrt{\zeta}}), \quad (11.17)$$

where $\zeta$ satisfies the following equation:

$$\zeta^3 - 2 + \frac{1}{\zeta} = 8 \left(\frac{S}{\hbar} - \xi^* \xi\right). \quad (11.18)$$

Here, Eq.(11.18) has a real solution if $S/\hbar + (1 - 4\xi^* \xi)/4 \geq 1/2\sqrt{27}$. Substituting the solutions (11.17) into the relation (11.16), the deviation $(\Delta S^2)_s$ is given as

$$\frac{(\Delta S^2)_s}{\hbar S} = \frac{3}{8} \sqrt{\frac{\hbar}{S}} \left[1 + \frac{8}{3} \left(\frac{\hbar}{S}\right)^2 \left(-\frac{1}{8} + \xi^* \xi + 2(\xi^* \xi)^2\right) + \frac{1}{6} \cdot \frac{\hbar}{S} \cdot (1 - 4\xi^* \xi) + \cdots \right], \quad (11.19)$$
where $S$ and $\xi^* \xi$ are regarded as the same order. Thus, up to the order of $\hbar^1$, $(\Delta S^2)_s$ can be neglected: $(\Delta S^2)_s = 0$. From this result, we have

$$\langle s | \dot{I}_{\text{su}(2)} | s \rangle = S^2 + \hbar S .$$ (11.20)

This is the desired result. As was discussed previously, the condition $W_s = 1$ is realized, while $W_c = 0$ in the coherent state. Thus, the squeezed state $|s\rangle$ gives us an appropriate quantal fluctuation for the Casimir operator in contrast to the coherent state $|c\rangle$.

Before investigating the deviations $(\Delta S^2)_s$ etc., let us parametrize the state $|s\rangle$ in terms of the canonical variables. We have new variables $\xi^*$ and $\xi$ adding to the four variables characterizing the coherent state $|c\rangle$. We impose the following canonicity conditions:

$$\langle s | i\hbar \partial_\theta | s \rangle = S , \quad \langle s | i\hbar \partial_S | s \rangle = 0 ,$$ (11.21)

$$\langle s | \partial_X | s \rangle = \frac{X^*}{2} , \quad \langle s | \partial_X^* | s \rangle = - \frac{X}{2} ,$$ (11.22)

$$\langle s | \partial_\xi | s \rangle = \frac{x^*}{2} , \quad \langle s | \partial_\xi^* | s \rangle = - \frac{x}{2} ,$$ (11.23)

where $S$ is nothing but the expectation value of $\hat{S}$ and $\theta$ corresponds to its conjugate angle variable. A possible solutions of Eqs. (11.22) and (11.23) are obtained as

$$V = \sqrt{\frac{\hbar}{2(S - 2\hbar x^* x)}} \cdot X , \quad U = \sqrt{1 - \frac{\hbar X^* X}{2(S - 2\hbar x^* x)}} ; \quad \xi = x , \quad \xi^* = x^* .$$ (11.24)

Then, we can express the expectation values of the generators of the $\text{su}(2)$-algebra in terms of these canonical variables:

$$(S_x)_s = \langle s | \hat{S}_x | s \rangle = Q \sqrt{S - (P^2 + Q^2)/4 - 2\hbar x^* x} ,$$

$$(S_y)_s = \langle s | \hat{S}_y | s \rangle = -P \sqrt{S - (P^2 + Q^2)/4 - 2\hbar x^* x} ,$$

$$(S_z)_s = \langle s | \hat{S}_z | s \rangle = -S + (P^2 + Q^2)/2 + 2\hbar x^* x ,$$ (11.25)

where $Q = \sqrt{\hbar/2 \cdot (X^* + X)}$ and $P = i\sqrt{\hbar/2 \cdot (X^* - X)}$. The expression (11.25) contains the terms related to the quantal fluctuations, i.e., $\hbar x^* x$. The $\text{su}(2)$-generators are expressed in terms of the quadratic forms for the boson operators. In contrast to the case of the coherent state, the forms are not of the normal ordered product for the squeezed state $|s\rangle$. From this reason, the expressions contain the terms of the quantal fluctuations.

Now we are ready to give the deviations $(\Delta S^2)_s$ etc., which are expressed up to the order of $\hbar^1$ in terms of the canonical variables:

$$(\Delta S^2)_s = \frac{\hbar}{8S} \cdot [4S^2 - Q^2(4S - (P^2 + Q^2))W_s]$$

$$+ \frac{\hbar}{4S} \cdot [4S^2 - Q^2(4S + (P^2 - Q^2))] \cdot (x^* + x) \sqrt{\frac{1}{2} + x^* x}$$
\[-\frac{\hbar}{2S} \cdot Q P(2S - Q^2) \cdot i(x^* - x) \sqrt{\frac{1}{2} + x^* x} \]
\[+ \frac{\hbar}{2S} \cdot [4S^2 - Q^2(4S - (P^2 + Q^2))] \cdot x^* x , \]
\[(\Delta S^2_y)_s = \frac{\hbar}{8S} \cdot [4S^2 - P^2(4S - (P^2 + Q^2))] W_s \]
\[\quad - \frac{\hbar}{4S} \cdot [4S^2 - P^2(4S - (P^2 - Q^2))] \cdot (x^* + x) \sqrt{\frac{1}{2} + x^* x} \]
\[+ \frac{\hbar}{2S} \cdot [4S^2 - P^2(4S - (P^2 + Q^2))] \cdot x^* x , \]
\[\quad + \frac{\hbar}{2S} \cdot (P^2 + Q^2)[4S - (P^2 + Q^2)] \cdot x^* x . \quad (11.26)\]

Here, \(W_s\) is defined by
\[W_s = 1 - \frac{2(\Delta S^2)_s}{\hbar S} = 1 , \quad (11.27)\]
where the last equality is valid for the approximation up to the order of \(\hbar^1\) as was seen in Eq. (11.13). In contrast to the case of the coherent state, the above uncertainties do not have fixed value \(\bar{\eta}/2\) and \(W_s\) is equal to unity even if the variables describing the quantal effects, \(x^*\) and \(x\), are equal to zero. This fact comes from the inclusion of the parameters \(\eta\) and \(\eta^*\).

As an example, we sketch our basic idea of a possible application to an \(su(2)\)-model, that is, the Lipkin model whose Hamiltonian is given as
\[\hat{H} = 2\epsilon \hat{S}_z - G(S^2_x - \hat{S}_y^2) . \quad (11.28)\]

Here, \(\epsilon\) and \(G\) denote positive constant. Main interest is to investigate the time-evolution of this quantal system with the least quantal effects. The expectation value of this Hamiltonian with respect to the squeezed state \(|s\rangle\) is expressed as
\[\langle s | \hat{H} | s \rangle + 2\epsilon S = H_s = H_{s,cl} + H_{s,ql} , \quad (11.29)\]
\[H_{s,cl} = \epsilon (P^2 + Q^2) + GS(P^2 - Q^2) - \frac{G}{4} \cdot (P^4 - Q^4) , \quad (11.30)\]
\[H_{s,ql} = \hbar \left[ 4\epsilon x^* x - \frac{G}{8S} \cdot (P^2 - Q^2)(4S - (P^2 + Q^2)) \right. \]
\[\quad - \left. \frac{G}{4S} \cdot [8S^2 - 4S(P^2 + Q^2) + (P^2 - Q^2)^2](x^* + x) \sqrt{\frac{1}{2} + x^* x} \right] . \]
The part $H_{s,\text{cl}}$ ($H_{s,\text{ql}}$) denotes the classical (quantal) part of $H_s$. It is seen from (11.29) $\sim$ (11.31) that the $c$-number Hamiltonian is expressed in terms of the canonical variables $(\theta, S)$, $(Q, P)$ and $(x^*, x)$. Then, the Hamilton equations of motion are written as

\begin{align}
\dot{\theta} &= \partial_S H_s, \\
\dot{S} &= -\partial_\theta H_s = 0, \\
\dot{Q} &= \partial_P H_s, \\
\dot{P} &= -\partial_Q H_s, \\
i\hbar \dot{x} &= \partial_{x^*} H_s, \\
i\hbar \dot{x}^* &= -\partial_x H_s. 
\end{align}

The second equation of Eq.(11.32) shows us that $S$ is a constant of motion. By solving Eqs.(11.33) and (11.34) with appropriate initial conditions, the time-dependence of $Q$, $P$, $x$ and $x^*$ can be obtained. Here, let us analyze the Lipkin model under the adiabatic approximation like the adiabatic time-dependent Hartree-Fock (ATDHF) theory. In the ATDHF theory, the classical Hamiltonian is approximately expressed in terms of the quadratic form for the momentum. Following the same viewpoint as that of the ATDHF theory, $H_{s,\text{cl}}$ can be expressed in terms of the variables $Q$ and $P$ as follows:

\begin{align}
H_{s,\text{cl}} &= \frac{P^2}{2M} + V(Q), \\
M &= \frac{1}{2(\epsilon + GS)}, \quad V(Q) = (\epsilon - GS) \cdot Q^2 + \frac{G}{4} \cdot Q^4. 
\end{align}

Further, we approximate $H_{s,\text{ql}}$ in the frame of the term with $P^0$:

\begin{align}
H_{s,\text{ql}} &= \hbar \left( \frac{G}{8S} \cdot Q^2(4S - Q^2) + \left( 4\epsilon + \frac{G}{2S} \cdot Q^2(8S - Q^2) \right) x^* x \\
&\quad - \frac{G}{4S} \cdot (8S^2 - 4SQ^2 + Q^4)(x^* + x) \sqrt{\frac{1}{2} + x^* x} \right). 
\end{align}

In the same way as that in $H_{s,\text{ql}}$, we have the following approximate uncertainty relations:

\begin{align}
(D_x)_s &= \frac{4}{\hbar^2} \left[ (\Delta S_y)_s^2 \cdot (\Delta S_z)_s^2 - \frac{\hbar^2}{4} \cdot (S_x^2)_{s,\text{cl}} \right] = Q^2(4S - Q^2)[i(x^* - x)]^2 \left( \frac{1}{2} + x^* x \right), \\
(D_y)_s &= \frac{4}{\hbar^2} \left[ (\Delta S_z)_s^2 \cdot (\Delta S_x)_s^2 - \frac{\hbar^2}{4} \cdot (S_y^2)_{s,\text{cl}} \right] \\
&= \frac{1}{4S^2} \cdot Q^2(4S - Q^2)(2S - Q^2)^2 \left( x^* + \sqrt{\frac{1}{2} + x^* x} \right)^2 \left( x + \sqrt{\frac{1}{2} + x^* x} \right)^2, 
\end{align}
(D_2)_s = \frac{4}{\hbar^2} \left[ (\Delta S^2_y)_s \cdot (\Delta S^2_z)_s - \frac{\hbar^2}{4} \cdot (S_z)_{s,cl}^2 \right] = (2S - Q^2)^2 [i(x^*-x)]^2 \left( \frac{1}{2} + x^*x \right). \tag{11.37}

Here, \((S_x)_{s,cl} = Q \sqrt{S - (P^2 + Q^2)/4}\), \((S_y)_{s,cl} = -P \sqrt{S - (P^2 + Q^2)/4}\) and \((S_z)_{s,cl} = -S + (P^2 + Q^2)/2\). It is noted that the variable \(Q\) should run in the region \(Q^2 \leq 4S\). Our task is to give the initial conditions in order to solve the Hamilton equations of motion \((11.33)\) and \((11.34)\). As for the initial conditions for \(Q\) and \(P\), they are taken arbitrary. It is necessary to give the initial conditions for \(x^*\) and \(x\). The basic idea has been already presented in the previous section, that is, the least quantal effects should be realized at the initial time. The idea for investigating the least quantal effects in the classical motion of quantized system is given in Ref. \[34\]. Translating the basic idea into the present case, the following criteria to give the initial conditions should be taken into account for the squeezed state:

| \(|H^0_{s,ql}| \) : minimal , \tag{11.38} \\
| any two of \((D_x)_s^0\), \((D_y)_s^0\) and \((D_z)_s^0 = 0\) , \tag{11.39} |

where the superscript 0 denotes the initial value for each quantity. As can be seen in Eq.(11.37), the criterion \((11.39)\) is satisfied for the case \((D_x)_s^0 = (D_z)_s^0 = 0\) which gives us \(x_0 = x^*_0\), that is, \(x_0\) is real. For convenience, let us introduce the quantities \(g\), \(z\) and \(y\) instead of \(GS/\epsilon\), \(Q_0\) and \(x_0\):

\[
\frac{GS}{\epsilon} = g , \quad \frac{Q_0^2}{4S} = z , \quad x_0 = \frac{1}{2\sqrt{2}} \cdot \left( \sqrt{y} - \frac{1}{\sqrt{y}} \right). \tag{11.40}
\]

Then, the quantities \(g\) and \(z\) run from 0 to \(\infty\) and from 0 to 1, respectively. From the criterion \((11.38)\), the initial condition is obtained together with the initial value of the energy of the quantal part:

1. in the regions \(0 \leq z \leq z_1\) and \(z_2 \leq z \leq 1\),

\[
y = \frac{g \cdot 2z + 1 - \sqrt{g^2 \cdot (8z^4 - 4z^2 - 4z + 1)} + g \cdot 4z(z - 1)}{-g \cdot (4z^2 - 6z + 1) + 1}, \tag{11.41}
\]

\[
H^0_{s,ql} = 0, \tag{11.42}
\]

2. in the region \(z_1 \leq z \leq z_2\),

\[
y = \frac{g \cdot (2z + 1) + 1}{-g \cdot (4z^2 - 6z + 1) + 1}, \tag{11.43}
\]

\[
H^0_{s,ql} = \hbar \epsilon \left[ g \cdot (2z + 1) + 1 \right] - (g \cdot 2z + 1) . \tag{11.44}
\]

Here, \(z_1\) and \(z_2\) are given as

\[
z_1 = \frac{1}{6g} \cdot \left( g - 1 + 2 \sqrt{7g^2 + 4g + 1} \cos(\phi/3 + \pi/3) \right) ,
\]

\[
z_2 = \frac{1}{6g} \cdot \left( g - 1 + 2 \sqrt{7g^2 + 4g + 1} \cos(\phi/3 - \pi/3) \right) ,
\]
\[
\cos \phi = \frac{7g^3 + 6g^2 + 12g + 2}{2(\sqrt{7g^2 + 4g + 1})^3}. \tag{11.45}
\]

The details are found in Ref.\[35\]. It may be interesting to calculate \((D\gamma)_s^0\) which is given by 
\[(D\gamma)_s^0 = S^2 \cdot f(z) \text{ where } f(z) = 4z(1 - 2z)^2(1 - z)^2.\] It can be shown that the factor \(f(z)\) is rather small with some exceptions. The minimal uncertainty for the quantity \((D\gamma)_s^0\) is satisfied at rather high accuracy. The detailed numerical estimation is given in Ref.\[35\].

§12. Quasi-spin squeezed state approach to many-fermion systems

A possible classical description of many-fermion systems such as nucleus is given by the TDHF theory\[12\]. In this theory, the Slater determinant is usually adopted as a trial state for the variational procedure. The Slater determinant is essentially a coherent state which yields a minimal uncertainty relation where each uncertainty is fixed. Therefore, in order to take into account of appropriate quantal effects in many-fermion system in terms of classical mechanics, we need to proceed beyond the usual coherent state approach.

In our formalism developed in the preceding two sections, the boson squeezed state, which is introduced as a trial state in the time-dependent variational approach in the quantum mechanical or many-boson systems, has been originally constructed as an extension of the familiar boson coherent state. In many-fermion systems, our first task is to define an appropriate trial state similar to the boson squeezed state.

We formulate our idea for the \(su(2)\)-algebraic models, namely, a single-level model with pairing interaction and a two-level model with particle-hole interaction. In these models the trial state of the time-dependent Hartree-Fock or Hartree-Fock-Bogoliubov theory is reduced to the \(su(2)\)-coherent state. With the aim of going beyond this theory, we consider a possible extension of the \(su(2)\)-coherent state. This new state, which we call “a quasi-spin squeezed state,” is used as a trial state in the variational method, and we show in this section that the quasi-spin squeezed state approach is workable and successful.

First, we investigate a simple system: \(N\) identical nucleons are moving in a spherical orbit with the pairing interaction. The single-particle state is specified by a set of quantum numbers \((j, m)\), where \(j\) and \(m\) denote the magnitude of angular momentum of the single-particle state (the half-integer) and its projection to the \(z\)-axis, respectively. The annihilation and the creation operators for the nucleon with \(m\), \(\hat{c}_m\) and \(\hat{c}_m^\dagger\), satisfy the usual anti-commutation relations: \(\{\hat{c}_m, \hat{c}_{m'}^\dagger\} = \delta_{mm'}\) and the other combinations are equal to 0. The Hamiltonian is written as

\[
\hat{H} = \sum_{m=-j}^{j} \hbar \epsilon \hat{c}_m^\dagger \hat{c}_m - \frac{1}{4} \hbar^2 G \sum_{m=-j}^{j} (-1)^{j-m} \hat{c}_m^\dagger \hat{c}_{m} \sum_{m'=-j}^{j} (-1)^{j-m'} \hat{c}_{m'}^\dagger \hat{c}_{m'} - 2\epsilon (\hat{S}_0 + \hat{S}_j) - GS_+ \hat{S}_- , \tag{12.1}
\]

where \(\hbar \epsilon\) and \(\hbar^2 G\) denote the single-particle energy and the force strength, respectively. Also, \(S_j = \hbar \Omega/2\) where \(\Omega\) is the half of the degeneracy: \(\Omega = j + 1/2\). This
model has been used in §4 as an example of the \( su(2) \)-algebraic models. Here, we have defined the quasi-spin operators \( \hat{S}_0, \hat{S}_\pm \) as

\[
\hat{S}_+ = \frac{\hbar}{2} \sum_m (-)^j \hat{c}_m^* \hat{c}_m , \quad \hat{S}_- = \frac{\hbar}{2} \sum_m (-)^j \hat{c}_m \hat{c}_m^* , \\
\hat{S}_0 = \frac{\hbar}{2} (\sum_m \hat{c}_m^* \hat{c}_m - \Omega) = \frac{\hbar}{2} (\hat{N} - \Omega) ,
\]

where \( \hat{N} \) is a number operator defined by \( \hat{N} = \sum_m \hat{c}_m^* \hat{c}_m \). The set \( (12.2) \) is governed by the \( su(2) \)-algebra :

\[
[\hat{S}_+, \hat{S}_-] = 2i \hbar \hat{S}_0 , \\
[\hat{S}_0, \hat{S}_\pm] = \pm i \hbar \hat{S}_\pm .
\]

As is well-known, this model can be solved exactly. The exact eigenvalue \( H_{ql} \) in the ground state for this Hamiltonian is given by

\[
H_{ql} = \langle q | \hat{H} | q \rangle = 2 \epsilon \Lambda - G \Lambda (2S_j - \Lambda) - \hbar \Lambda A = H_{cl} - \hbar \Lambda A (= E_{ql}) .
\]

Here, \( \Lambda = \frac{\hbar N}{2} \) in which \( N \) is the particle number. The state \( |q\rangle \) is expressed in the following form :

\[
|q\rangle = \left( \frac{\Omega!(N/2)!}{(\Omega - N/2)!} \right)^{-\frac{1}{2}} \left( \frac{\hat{S}_+ \hbar}{\Omega} \right)^{\hat{N}/2} |0\rangle \cdot \exp \left( - \frac{i \hbar}{\hbar} E_{ql} t \right) .
\]

Let us investigate the state \( |w\rangle \) which gives a classical counterpart for the quasi-spin. The state \( |w\rangle \) should satisfy the exact relation for the Casimir operator \( \hat{\Gamma}_{su(2)} \).

We set up the following conditions :

\[
\Gamma_w = \langle w | \hat{\Gamma}_{su(2)} | w \rangle = \langle w | \hat{S}_0^2 + \frac{1}{2} (\hat{S}_- \hat{S}_+ + \hat{S}_+ \hat{S}_-) | w \rangle \\
= S_j^2 + \hbar S_j , \\
N_w = \langle w | \hat{N} | w \rangle = N .
\]

First, we consider a quasi-spin coherent state \( |c\rangle \), which is generalized from the boson coherent state :

\[
|c\rangle = C_c^{-1/2} \cdot \exp \left( \frac{V}{U} \cdot \frac{\hat{S}_+}{\hbar} \right) |0\rangle , \\
C_c = \sum_{k=0}^{\Omega} \frac{\Omega!}{k!(\Omega - k)!} \cdot \left( \left| \frac{V}{U} \right|^2 \right)^k = (U^{-2})^{\Omega} ,
\]

where \( V \) is complex and \( U \) is real parameter and the relation \( U^2 + |V|^2 = 1 \) is satisfied. This state is known as the \( su(2) \)-coherent state which corresponds to the BCS ground-state. It has been regarded as an approximate state for \( |q\rangle \) in Eq.\((12.5)\).

This state is a vacuum state for the Bogoliubov-transformed fermion operator \( \hat{a}_m \) :

\[
\hat{a}_m = U \hat{c}_m - V (-)^j \hat{c}_m^* , \quad \hat{a}_m |c\rangle = 0 .
\]
By the use of this operator, let us define new operators:

\[
\hat{B}^* = \sqrt{\frac{\hbar}{8S_j}} \sum_m (-)^{j-m} \hat{a}_m^* \hat{a}_m^* , \quad \hat{B} = \sqrt{\frac{\hbar}{8S_j}} \sum_m (-)^{j-m} \hat{a}_m \hat{a}_m , \quad \hat{M} = \sum_m \hat{a}_m^* \hat{a}_m .
\]

(12.10)

These operators satisfy the following commutation relations:

\[
[\hat{B}, \hat{B}^*] = 1 - \frac{\hbar}{2S_j} \hat{M} , \quad [\hat{M}, \hat{B}^*] = 2\hat{B}^* , \quad [\hat{M}, \hat{B}] = -2\hat{B} .
\]

(12.11)

It should be noted that the state \(|c\rangle\) gives the following expectation value for the quasi-particle number operator \(\hat{M} : n = \langle c|\hat{M}|c\rangle = 0\). The expectation value of the Hamiltonian is expressed as

\[
H_{ch} = \langle c|\hat{H}|c\rangle = H_{cl} - \hbar GA + F_{ch} ,
\]

\[
F_{ch} = \frac{\hbar G}{2S_j} \Lambda (2S_j - \Lambda) .
\]

(12.12)

The expectation value is clearly different from the exact energy eigenvalue. Further, it is interesting to investigate the uncertainty relations. Let us introduce another \(su(2)\)-generators, \((\hat{s}_\pm, \hat{s}_0)\), which consist of the quasi-particle operators \(\hat{a}_m\) and \(\hat{a}_m^*\) instead of \(\hat{c}_m\) and \(\hat{c}_m^*\) in Eq.(12.2) : \(\hat{s}_+ = (\hbar/2) \sum_m (-)^{j-m} \hat{a}_m^* \hat{a}_m\), \(\hat{s}_- = (\hbar/2) \sum_m (-)^{j-m} \hat{a}_m \hat{a}_m^*\), \(\hat{s}_0 = (\hbar/2) \cdot (\sum_m \hat{a}_m^* \hat{a}_m - \Omega)\). Then, the following squares of the standard deviations for \(\hat{s}_x,y,z\), which are defined as \(\hat{s}_x = (\hat{s}_+ + \hat{s}_-)/2\), \(\hat{s}_y = (\hat{s}_+ - \hat{s}_-)/2i\), and \(\hat{s}_z = \hat{s}_0\), are obtained:

\[
(\Delta s_x^2)_{ch} = (\Delta s_y^2)_{ch} = \frac{\hbar S_j}{2} ,
\]

\[
(\Delta s_z^2)_{ch} = 0 ,
\]

(12.13)

where \((\Delta s_x^2)_{ch} = \langle c|\hat{s}_x^2|c\rangle - \langle c|\hat{s}_x|c\rangle^2\), etc., and \(\langle c|\hat{s}_z|c\rangle = -S_j\). The above relations mean that the state \(|c\rangle\) is governed by the minimal uncertainty for the quasi-spin operators \(\hat{s}_x\), \(\hat{s}_y\), and \(\hat{s}_z\), namely, \((\Delta s_x^2)_{ch}(\Delta s_y^2)_{ch} = (\hbar/2)(\langle c|\hat{s}_x|c\rangle^2\langle c|\hat{s}_y|c\rangle^2\), etc. Also, each standard deviation has a fixed value. We see that the above-mentioned situation is similar to the case of the boson coherent state.

In order to relax the expectation value, \(n\), of the quasi-particle number operator, \(\hat{M}\), and the fixed standard deviations for the quasi-spin operators, we here introduce a new state \(|s\rangle\), which we call the quasi-spin squeezed state, as an extension of the quasi-spin coherent state \(|c\rangle\). The new state is defined as

\[
|s\rangle = C_s^{-1/2} \cdot \exp \left( \frac{1}{2} \cdot \frac{v}{u} \cdot \hat{B}^2 \right) |c\rangle ,
\]

\[
C_s = \sum_{k=0}^{[\Omega/2]} \frac{(2k-1)!!}{(2k)!!} \cdot \prod_{p=0}^{2k-1} \left( 1 - \frac{p}{\Omega} \right) \cdot z^k .
\]

(12.14)

For the state \(|s\rangle\), we have

\[
n = \langle s|\hat{M}|s\rangle = 4z \frac{C_s'}{C_s} ,
\]

(12.15)
where the prime denotes the first derivative with respect to \( z \). The dynamical variables in the quasi-spin squeezed state are \( V, V^*, v/u \) and \( v^*/u \). For the canonicity conditions for the present variables, the following relations are adopted:

\[
\langle s | i \hbar \partial_\Phi | s \rangle = A, \quad \langle s | i \hbar \partial_\Lambda | s \rangle = 0, \\
\langle s | i \hbar \partial_{\bar{\phi}} | s \rangle = \hbar \lambda, \quad \langle s | i \hbar \partial_{\bar{\Lambda}} | s \rangle = 0. \tag{12.16}
\]

We then get possible solutions for the above canonicity conditions:

\[
V = \sqrt{\frac{A - \hbar \lambda}{2(S_j - \hbar \lambda)}} \cdot \exp(-i\Phi), \quad U = \sqrt{1 - |V|^2} = \sqrt{1 - \frac{A - \hbar \lambda}{2(S_j - \hbar \lambda)}}, \tag{12.17}
\]

\[
v = \sqrt{\lambda} \cdot \exp(-2i(\Phi + \phi)), \\
u = \sqrt{\frac{2C_s'}{C_s}} = \sqrt{\left(\lambda + 1\right) \left(1 - \frac{\lambda}{\Omega}\right) \left(1 - \frac{\lambda + 1}{\Omega}\right) (1 + \Delta)^{-1}}. \tag{12.18}
\]

The explicit form of \( \Delta \) is found in Ref. 37). Here, it is easily shown that \( \lambda \) is related to the quasi-particle number:

\[
\lambda = \frac{1}{2} \langle s | \hat{M} | s \rangle = \frac{n}{2}. \tag{12.19}
\]

If \( \lambda = 0 \), the quasi-spin squeezed state is reduced to the quasi-spin coherent state. Thus, we see that the variables \( (\phi, \lambda) \) describe the quantal fluctuation and the variables \( (\Phi, \Lambda) \) mainly describe the classical motion.

The expression of the factor \( (\lambda + 1)(1 - \lambda/\Omega)(1 - (\lambda + 1)/\Omega) \) in \( u^2 \) is interesting. If \( \Omega \) is even or odd, then the maximum of \( \lambda \) should be \( \Omega \) or \( \Omega - 1 \), respectively. This realizes the Pauli principle. On the other hand, if \( \lambda \ll \Omega \), \( u^2 \) is approximated as \( (\lambda + 1) \). In this case, we have \( u^2 - |v|^2 = 1 \), which leads us to the boson approximation for \( \hat{B} \) and \( \hat{B}^* \).

Hereafter, the discussion is restricted to the first order quantal fluctuation. In this approximation, it is enough to adopt the following form for \( V, U, v \) and \( u \):

\[
V = \sqrt{\frac{A}{2S_j}} e^{-i\Phi}, \quad U = \sqrt{1 - \frac{A}{2S_j}}, \quad v = \sqrt{\lambda} e^{-2i(\Phi + \phi)}, \quad u = \sqrt{1 + \lambda}. \tag{12.20}
\]

Then, the expectation value of the Hamiltonian with respect to the quasi-spin squeezed state is approximately given as

\[
H_{sq} = \langle s | \hat{H} | s \rangle = H_{cl} - \hbar GA + F_{sq}, \tag{12.21}
\]

\[
F_{sq} = F_{ch} \cdot f(\lambda, \phi), \\
f(\lambda, \phi) = 1 + 2\lambda + 2\sqrt{\lambda(\lambda + 1)} \cdot \cos 2\phi. \tag{12.22}
\]

Here, \( F_{ch} \) has been defined in Eq.(12.12). In this approximation, the squares of the standard deviations for the quasi-spin operators \( \hat{s}_{x,y,z} \) are given as follows:

\[
(\Delta s_x^2)_{sq} = \frac{\hbar S_j}{2} \cdot \left[1 + 2\lambda + 2\sqrt{\lambda(\lambda + 1)} \cdot \cos[2(\Phi + \phi)]\right], \\
(\Delta s_y^2)_{sq} = \frac{\hbar S_j}{2} \cdot \left[1 + 2\lambda - 2\sqrt{\lambda(\lambda + 1)} \cdot \cos[2(\Phi + \phi)]\right], \\
(\Delta s_z^2)_{sq} = 0. \tag{12.23}
\]
where \((\Delta s^2_{x})_{sq} = \langle s|\hat{s}^2_x|s\rangle - \langle s|\hat{s}_x|s\rangle^2\), etc., and \(\langle s|\hat{s}_z|s\rangle = -S_j\).

Our problem is to solve the equations of motion derived from the time-dependent variational principle:

\[
\begin{align*}
\dot{\Lambda} &= -\frac{\partial H_{sq}}{\partial \Phi}, \\
\dot{\Phi} &= \frac{\partial H_{sq}}{\partial \Lambda}, \\
\hbar \dot{\lambda} &= -\frac{\partial H_{sq}}{\partial \phi}, \\
\hbar \dot{\phi} &= \frac{\partial H_{sq}}{\partial \lambda}.
\end{align*}
\tag{12.24}
\]

The initial conditions for \(\lambda\) and \(\phi\) which describe the classical image of quantal fluctuation are obtained under the similar criterion to that given in the case of quantum mechanical system in §10 or the \(su(2)\)-boson model in §11. Namely, at the initial time, the minimal uncertainty should be realized. Further, the energy originated from quantum fluctuations is minimal at the initial time.

\[
(\Delta s^2_{x})^0_{sq} \cdot (\Delta s^2_{y})^0_{sq} = \left(\frac{\hbar}{2}\right)^2 \cdot |\langle s|\hat{s}_z|s\rangle^0|^2, \\
f(\lambda_0, \phi_0) : \text{minimal},
\tag{12.25}
\]

where the superscript and subscript “0” denotes the initial value of corresponding variable. These criteria lead us to the following initial conditions:

\[
\phi_0 = \frac{k\pi}{2} - \Phi_0, \quad (k = 0, \pm1, \pm2, \cdots), \\
\lambda_0 = \frac{1}{2} \left(\frac{1}{|\sin 2\Phi_0|} - 1\right). \\
\tag{12.26}
\]

If \(\cos 2\phi_0 > 0\), then \(\lambda_0 = 0\). For the solution \(12.26\), \(H_{sq}\) is given in the form

\[
H_{sq} = H_{cl} - \hbar G\Lambda + \hbar G \frac{S_j}{2S_j} \cdot \Lambda(2S_j - \Lambda) \cdot |\sin 2\Phi_0|.
\tag{12.27}
\]

It may be interesting to see that if \(|\sin 2\Phi_0|\) vanishes, \(F_{sq} = 0\) and if \(|\sin 2\Phi_0|\) is equal to 1, \(F_{sq} = \hbar G/2S_j \cdot \Lambda(2S_j - \Lambda)\). The former and the latter correspond to the situations given by \(|q\rangle\) and \(|c\rangle\), respectively. Therefore, by the parameter \(|\sin 2\Phi_0|\), we can reproduce the intermediate situation. Concerning the minimal uncertainty, we have

\[
(\Delta s^2_{x})^0_{sq} = \frac{\hbar S_j}{2} \cdot \rho_0^2, \\
(\Delta s^2_{y})^0_{sq} = \frac{\hbar S_j}{2} \cdot \rho_0^{-2}, \\
\rho_0 = \frac{\sqrt{1 + |\sin 2\Phi_0|} \pm \sqrt{1 - |\sin 2\Phi_0|}}{\sqrt{2} |\sin 2\Phi_0|}.
\tag{12.28}
\]

The relation \(12.28\) shows us that at the initial time the magnitudes of \((\Delta s^2_{x})^0_{sq}\) and \((\Delta s^2_{y})^0_{sq}\) are controlled by the quantity \(\sin 2\Phi_0\), which is the same situation as that in the case of \(F_{sq}\). Under the above initial conditions, we obtain the following solutions for the Hamilton equations of motion:

\[
\begin{align*}
\Lambda(t) &= \Lambda, \\
\Phi(t) &= 2\omega_{sq} t + \Phi_0, \\
\lambda(t) &= \left(\frac{G}{S_j}\right) \Lambda^2(2S_j - \Lambda)^2 |\sin 2\Phi_0| t^2
\end{align*}
\]
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\[ + \frac{G}{S_j} \Lambda(2S_j - \Lambda)|\cos 2\Phi_0|t + \frac{1}{2} \left( \frac{1}{|\sin 2\Phi_0|} - 1 \right), \]

\[ \phi(t) = \frac{1}{2} \cdot \cos^{-1} \chi(t), \quad (12.29) \]

where \( \omega_{sq} \) and \( \chi(t) \) are defined as

\[ \omega_{sq} = \epsilon - G(S_j - \Lambda) - \frac{\hbar G}{2} \cdot (1 - |\sin 2\Phi_0|) - \frac{\hbar GA}{2S_j} |\sin 2\Phi_0|, \]

\[ \chi(t) = \frac{|\sin 2\Phi_0| - (1 + 2\lambda(t))}{2\sqrt{\lambda(t)(\lambda(t) + 1)}}, \quad (12.30) \]

Next, let us investigate another many-fermion system which consists of two-energy levels with same degeneracy \( 2\Omega = 2j + 1 \). The two-body interaction is active only for a particle-hole pair with coupled momentum being 0. This model is called the Lipkin model whose Hamiltonian is written as

\[ \hat{H} = 2\epsilon \hat{S}_0 - \frac{1}{2} G(\hat{S}_+^2 + \hat{S}_-^2), \quad (12.31) \]

where \( 2\hbar\epsilon \) is the level spacing between lower and upper energy levels labeled by \( g = 1 \) and 2, respectively. We have already used this model in §§4 and 11 as an \( su(2) \)-boson model. However, in this section, we deal with this model as an example of many-fermion system. Thus, the quasi-spin operators consist of the fermion operators, which are defined as

\[ \hat{S}_+ = \hbar \sum_{m=-j}^j \hat{c}_{2m}^* \hat{c}_{1m}, \quad \hat{S}_- = \hbar \sum_{m=-j}^j \hat{c}_{1m}^* \hat{c}_{2m}, \quad \hat{S}_0 = \frac{\hbar}{2} \sum_{m=-j}^j (\hat{c}_{2m}^* \hat{c}_{2m} - \hat{c}_{1m}^* \hat{c}_{1m}), \]

\[ (12.32) \]

where \( \hat{c}_{2m} \) and \( \hat{c}_{1m} \) are fermion creation and annihilation operators. Here, the above quasi-spin operators satisfy the commutation relations of the \( su(2) \)-algebra : \( [\hat{S}_+ , \hat{S}_-] = 2\hbar \hat{S}_0 \) and \( [\hat{S}_0 , \hat{S}_\pm] = \pm \hbar \hat{S}_\pm \). Hereafter, we concentrate on the case \( N = 2\Omega \) where \( N \) represents particle number.

The \( su(2) \)-coherent state is adopted as a trial state in the time-dependent Hartree-Fock theory. This state is given as

\[ |c\rangle \equiv \frac{1}{(1 + |Z|^2)^{1/2}} \exp \left( Z \frac{\hat{S}_+}{\hbar} \right) |0_{ph}\rangle \quad (12.33) \]

with \( \hat{c}_{2m} |0_{ph}\rangle = \hat{c}_{1m}^* |0_{ph}\rangle = 0 \). This coherent state is a vacuum state with respect to the unitary-transformed fermion operators as

\[ \hat{a}_{2m} = \frac{1}{(1 + |Z|^2)^{1/2}} \hat{c}_{2m} - \frac{Z}{(1 + |Z|^2)^{1/2}} \hat{c}_{1m}, \]

\[ \hat{a}_{1m} = \frac{Z^*}{(1 + |Z|^2)^{1/2}} \hat{c}_{2m} + \frac{1}{(1 + |Z|^2)^{1/2}} \hat{c}_{1m}, \]

\[ (12.34) \]

with \( \hat{a}_{2m} |c\rangle = \hat{a}_{1m}^* |c\rangle = 0 \).
Our aim is to extend the coherent state to the squeezed state in order to go beyond the usual Hartree-Fock (HF) approximation. Now, let us introduce a quasi-spin squeezed state from the $su(2)$-coherent state in a manner similar to that for the case of the pairing model discussed previously. The new operators which consist of the unitary-transformed quasi-fermion operators are introduced like (12.10):

$$\hat{B}^* = \sqrt{\frac{\hbar}{2\Omega}} \sum_{m=-j}^{j} \hat{a}_{2m}^* \hat{a}_{1m}, \quad \hat{B} = \sqrt{\frac{\hbar}{2\Omega}} \sum_{m=-j}^{j} \hat{a}_{1m}^* \hat{a}_{2m},$$

$$\hat{M} = \sum_{m=-j}^{j} (\hat{a}_{2m}^* \hat{a}_{2m} + \hat{a}_{1m}^* \hat{a}_{1m}). \quad (12.35)$$

The operator $\hat{M}$ is nothing but the number operator of the quasi-particles and holes. The commutation relations among these operators have the same forms as Eq.(12.11) except for the replacement of $S_j$ to $\Omega$. The quasi-spin squeezed state in this model is defined as

$$|s\rangle = C_s^{-1/2} \cdot \exp \left( \frac{1}{2} \cdot \frac{v}{u} \cdot \hat{B}^2 \right) |c\rangle,$$

$$C_s = 1 + \sum_{k=1}^{[\Omega]} \frac{(2k-1)!!}{(2k)!!} \cdot \prod_{p=0}^{k-1} \left( 1 - \frac{p}{2\Omega} \right) \cdot (v/u)^2 \cdot (z = (|v|/u)^2) \quad (12.36)$$

Let us investigate the classical counterpart given by the quasi-spin squeezed state. We impose the canonicity conditions so as to transform the variables, which are contained in the $|s\rangle$, from $Z$, $Z^*$, $v/u$ and $v^*/u$ to $b$, $b^*$, $\beta$ and $\beta^*$:

$$\langle s| \frac{\partial}{\partial b} |s\rangle = \frac{1}{2} b^*, \quad \langle s| \frac{\partial}{\partial b^*} |s\rangle = -\frac{1}{2} b; \quad \langle s| \frac{\partial}{\partial \beta} |s\rangle = \frac{1}{2} \beta^*, \quad \langle s| \frac{\partial}{\partial \beta^*} |s\rangle = -\frac{1}{2} \beta. \quad (12.37)$$

Under these conditions, the sets of variables $(b, b^*)$ and $(\beta, \beta^*)$ are those of canonical variables, respectively. Solving the above canonicity conditions, we can express the expectation values of the quasi-spin operators in terms of $b$, $b^*$, $\beta$ and $\beta^*$:

$$\langle s| \hat{S}_+ |s\rangle = \sqrt{\hbar} b^* \sqrt{2\Omega - \hbar |b|^2 - 4\hbar |\beta|^2} = \langle s| \hat{S}_- |s\rangle^*, \quad \langle s| \hat{S}_0 |s\rangle = -\Omega + \hbar |b|^2 + 2\hbar |\beta|^2. \quad (12.38)$$

These expressions remind us of the Holstein-Primakoff boson representation for the $su(2)$-algebra. Further, we can calculate the energy expectation value with respect to the quasi-spin squeezed state. The result is almost identical with the exact eigenvalue for the ground state in the wide region of the coupling strength $G$. The details are found numerically in Refs. 33 and 34.

Finally, let us investigate the implication to the RPA. We now set the operator $\hat{B}$ as a pure boson operator $\hat{B}$. Further, under this replacement, the number operator of the quasi-particles and quasi-holes, $\hat{M}$, should be replaced so as to satisfy the original commutation relation (12.11). Namely,

$$\hat{B} \rightarrow \hat{B}, \quad \hat{B}^* \rightarrow \hat{B}^*, \quad [\hat{B}, \hat{B}^*] = 1; \quad \hat{M} \rightarrow \hat{M} \equiv 2\hat{B}^* \hat{B}. \quad (12.39)$$
Also, the quasi-spin squeezed state \(|s\rangle\) has to be replaced into the usual boson squeezed state \(|b\rangle\):

\[
|s\rangle \rightarrow |b\rangle \equiv \Gamma_b^{-1/2} \exp \left( \frac{1}{2} \frac{v}{u} \hat{B}^\dagger \hat{b} \right) |c\rangle , \quad (12.40)
\]

where the relation \(u^2 - |v|^2 = 1\) is satisfied. As is well known, the boson squeezed state is a vacuum with respect to the unitary-transformed boson operator:

\[
\hat{b} |b\rangle = 0 , \quad \hat{b} = u\hat{B} - v\hat{B}^\dagger , \quad [\hat{b} , \hat{b}^\dagger] = 1 . \quad (12.41)
\]

Let us consider to diagonalize the model Hamiltonian with respect to the boson operators \(\hat{b}\) and \(\hat{b}^\dagger\). The Hamiltonian (12.31) is expressed in terms of \(\hat{B}, \hat{B}^\dagger\) and \(\hat{M}\) as

\[
\hat{H} = E_{ch} + E_{11}\hat{M} + E_{20}\hat{B}^\dagger + E_{20}\hat{B} + E_{M}\hat{M}^2 + E_{22}\hat{B}^\dagger \hat{B} + (E_{40}\hat{B}^\dagger + E_{60}\hat{B}\hat{M} + \text{h.c.}) ,
\]

where \(E_{ch} \equiv \langle c|\hat{H}|c\rangle\), \(E_{ij}\) denote \(c\)-number coefficients and \(\text{h.c.}\) means the Hermitian conjugate. In the HF approximation, \(E_{20} = E_{20}^\dagger = 0\) are demanded because \(\hat{B}^\dagger\) is a particle-hole pair-creation operator. Now, let us take into account of the higher order terms, which are neglected in the HF approximation. Following the replacement (12.39), neglecting the terms of third and fourth order with respect to the boson operators \(\hat{B}\) and \(\hat{B}^\dagger\), and transforming \(\hat{B}\) to \(\hat{b}\) in (12.41), we obtain the Hamiltonian as

\[
\hat{H}_b \simeq E_{ch} + \alpha|v|^2 + \frac{\beta}{2}(uv + uv^\dagger) + \left[ \alpha uv + \frac{\beta}{2}(u^2 + v^2) \right] \hat{b}^\dagger \hat{b} + \text{h.c.} \\
+ \left[ \alpha(u^2 + |v|^2) + \beta(uv + uv^\dagger) \right] \hat{b}^\dagger \hat{b} . \quad (12.43)
\]

Here, if we define \(\chi = \hbar^2 G\Omega(1 - 1/2\Omega)/(\hbar\epsilon)\), then \(\alpha\) and \(\beta\) are given as \(\alpha = 2\hbar\epsilon\) for \(\chi \leq 1\) or \(\hbar\epsilon(3\chi^2 - 1)/\chi\) for \(\chi \geq 1\) and \(\beta = -2\hbar\epsilon\chi\) for \(\chi \leq 1\) or \(-\hbar\epsilon(\chi^2 + 1)/\chi\) for \(\chi \geq 1\), respectively. We demand that the terms which is proportional to \(\hat{b}^\dagger \hat{b}\) should be zero. As a result, in the limit of \(2\Omega \gg 1\), \(u\) and \(v\) are determined as

\[
u^2 \simeq \frac{1}{2} \left( 1 + \frac{\alpha}{\sqrt{\alpha^2 - \beta^2}} \right) , \quad |v|^2 \simeq -\frac{1}{2} \left( 1 - \frac{\alpha}{\sqrt{\alpha^2 - \beta^2}} \right) . \quad (12.44)
\]

The Hamiltonian is finally expressed as

\[
\hat{H}_b \simeq E_{RPA} + \hbar\omega_b \hat{b}^\dagger \hat{b} , \quad E_{RPA} \equiv E_{ch} + \frac{1}{2}(\sqrt{\alpha^2 - \beta^2} - \alpha) , \quad \hbar\omega_b = \sqrt{\alpha^2 - \beta^2} . \quad (12.45)
\]

Here, \(E_{RPA}\) is identical to the ground state energy of the RPA calculation. Namely, the ground state correlation is included automatically in our approach. The boson operator \(\hat{b}\) corresponds to the phonon operator. Thus, our quasi-spin squeezed state approach includes the RPA in a certain limit.
§13. Concluding remarks

On the basis of the celebrated boson expansion theory\(^1\) proposed by Marumori, together with Yamamura (one of the present authors) and Tokunaga, we have developed the idea of the boson mapping method, which is contained in the Marumori-Yamamura-Tokunaga boson expansion theory, to describe the thermal states in quantum many-body systems. The original boson mapping method in which the fermion space is mapped to the physical boson-space is not restricted to the many-fermion systems such as nucleus. We have used this method in §§2 and 3 to give the classical counterparts of the \(su(2)\)- and the \(su(1, 1)\)-algebras realized by the Schwinger boson representation. As a result, it has been shown that the classical counterparts of these algebras realized in terms of the Schwinger bosons could be obtained in the same forms as those of the Holstein-Primakoff representations of the \(su(2)\)- and the \(su(1, 1)\)-algebras, respectively. With the aim of describing the thermal behaviors of the quantum many-body systems, we have learned in §§4 and 5 that the thermal properties appear with the \(su(1, 1)\)-algebraic features in the “damped and amplified oscillator” model. With this insight, in §6, we have considered the \(su(2)\)-spin system and our idea has been applied to this model. The \(su(2)\)-spin model is often regarded as a schematic model of nucleus. It has been indicated that the \(su(2)\)-spin model shows the \(su(1, 1)\)-like behavior with the help of a certain kind of the coherent state giving a classical counterpart. Further, we have treated the \(su(2)\)-spin system interacting with an external harmonic oscillator. With the help of the MYT boson mapping method again, we have shown that the \(su(1, 1)\)-like behavior is realized in the physical space spanned by the boson mapping from the original space. Thus, it becomes possible to describe the thermal behavior in the system composed of the \(su(2)\)-spin interacting with the harmonic oscillator. It is seen in §7 ∼ §9 that our formalism is workable to describe the thermal behaviors in this model.

Another powerful method to describe the quantum many-body systems in the microscopic way is the time-dependent Hartree-Fock theory. To investigate the small amplitude nuclear collective vibrational motion, the idea of the TDHF theory was proposed by Marumori.\(^1\) In the TDHF theory, an approximation of the small amplitude fluctuation around a static Hartree-Fock field results in the random phase approximation. This approximation is not enough to investigate the large amplitude nuclear collective motion. To go beyond the RPA, the TDHF theory devised by the canonical form was proposed by Marumori,\(^8\) together with Maskawa, Sakata and Kuriyama (one of the present authors), in which the collective coordinate was extracted self-consistently from many-nucleon degrees of freedom. The time-dependent variational approach with the canonical form developed by Marumori et al. is not restricted to treating the many-nucleon or -fermion systems. Also, it is not restricted to using the Slater determinant, which is essentially a coherent state, as a trial state for the time-dependent variational procedure. We have extended this formulation to quantum mechanical systems and quantum many-boson systems in order to describe the time-evolution of these systems in terms of the classical mechanics. The key idea is to use squeezed states as trial states in the time-dependent variational procedure. As a result, it has been shown in §§10 and 11 that the higher order quantal effects...
than the order of $\hbar^1$ have been automatically included in our squeezed state formalism. Further, our idea is applied to the quantum many-fermion systems governed by the $su(2)$-algebra. We have defined and introduced the quasi-spin squeezed state to go beyond the usual TDHF theory in §12. It has been shown that, in the single orbit shell model with pairing interaction, the BCS result and the exact one for the ground state have been connected by the use of this new state. Also, in the Lipkin mode, we have shown that the quasi-spin squeezed state approach could reproduce the result obtained in the RPA under a certain limit and, further, seemed to give us a possible way to go beyond the RPA.

The boson expansion theory and the TDHF theory in the canonical form are powerful theories to investigate diverse quantum many-body systems in various situations. It is expected that these theories will be developed in and successfully applied to quantum physics in addition to the microscopic studies of nuclear collective motions. For example, we can find such studies in Ref. [13].

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