Combined Subset Simulation and Comprehensive Learning Particle Swarm Optimization in Reliability-Based Structural Optimization

Arnut Sutha, Thu Huynh Van, Sawekchai Tangaramvong

1 Applied Mechanics and Structures Research Unit, Department of Civil Engineering, Chulalongkorn University, 10330 Bangkok, Thailand

Abstract. Reliability-based design optimization (RBDO) addresses the cost-effective integrity design of structures in the presence of inherent uncertain parameters. Processing this class of problem is challenging from the computational burden to determine the failure probability of structures violating the limit-state function. This paper proposes an efficient decoupling RBDO method that advantageously couples a comprehensive learning particle swarm optimization (CLPSO) algorithm with a subset simulation (SS), termed as SS-CLPSO approach. In essence, the proposed method iteratively performs the CLPSO assuming deterministic parameters based on the most probable point underpinning limit-state functions updated within the reliability evaluation process. Based on the CLPSO design data, the SS approximates the spectrum of limit-state functions under uncertain parameters, and hence enables the significant reduction of Monte-Carlo simulations for the failure probability prediction. The SS map outs the failure probability from the conditional samples constructed at each intermediate event. The proposed SS-CLPSO terminates the optimal solution to the RBDO problem as when the resulting failure probability converges to the permissible threshold. The applications of the present approach are illustrated through the steel truss design under probabilistic uncertain parameters and constraints.

1. Introduction

Deterministic optimization has been extensively applied in engineering structures to improve the design performance with minimum resources. The design solution computed by the deterministic optimization becomes unreliable in some cases, especially when the influences of uncertainties inheriting structural dimensions, material properties, loading and operating conditions are significant and cannot be eliminated. By addressing the performance and reliability of the structure together, the structural reliability-based design optimization (RBDO) has been considered as the alternative approach in recent years. More explicitly, the RBDO problem minimizes the cost function, denoted as C, and satisfies the certain deterministic and probabilistic constraints, as state by the following generic mathematical formulations [1]:

$$\min C(s)$$

s.t. $$\mathbb{P}[G(s,x) - y \leq 0] - P_a \leq 0$$

$$s \in [s_L, s_U]$$ (1)
where \( \mathbf{s} \) and \( \mathbf{x} \) are the vectors of deterministic design variables and random parameters, respectively. The vector \( \mathbf{x} \) is characterized by the joint probability density function (PDF) \( f(\mathbf{x}) \) in the space \( \Omega \). The two \( \mathbf{s} \) and \( \mathbf{x} \) denote the lower and upper bounds on the variables \( \mathbf{s} \). The functions \( G(\mathbf{s}, \mathbf{x}) \) respectively the performance and limit-state design expressions considered, where \( F \) is a constant threshold of the failure domain specified for the design variables (i.e. \( F = \{ \mathbf{x} \in \Omega | G(\mathbf{s}, \mathbf{x}) - F \leq 0 \} \)). The probability of failure reads \( P_f = \prod [G(\mathbf{s}, \mathbf{x}) - F] = \int_{-\infty}^{\infty} f(x) dx \), where \( P_a \) is the allowable threshold of \( P_f \).

2. CLPSO Algorithm

Whilst the general PSO algorithm provides the good convergence rate to the optimal solutions, the method is often trapped into the local optimal. To enable the diversity of design particles and overcome the premature convergence, Liang and Qin [2] developed the CLPSO algorithm using the comprehensive learning (CL) strategy. The position \( \mathbf{x}_{\text{ext}}^d,\mathbf{d} \) and velocity \( \mathbf{v}_{\text{ext}}^d,\mathbf{d} \) functions of CLPSO are written as follows:

\[
\mathbf{x}_{\text{ext}}^{d,\text{next}} = \mathbf{x}_i^d + \mathbf{v}_{\text{ext}}^{d,\text{next}} \quad (2)
\]

\[
\mathbf{v}_{\text{ext}}^{d,\text{next}} = w \mathbf{v}_i^d + c r_1 \left( p_{\text{best}, f(d)} - \mathbf{x}_i^d \right) \quad (3)
\]

Where \( i \) and \( \mathbf{d} \) are the array indices of the particles for \( i \in \{1, \ldots, n_p\} \) and dimensions for \( \mathbf{d} \in \{1, \ldots, nd\} \), respectively; \( \mathbf{x}_i^d \) and \( \mathbf{v}_i^d \) the position and velocity of the \( i \)-th particle at the \( d \)-th dimension, respectively; \( w \) the inertia weight; \( c \) the acceleration coefficient (equal to 1.5) attracting \( \mathbf{x}_i^d \) to \( \mathbf{x}_{\text{ext}}^{d,\text{next}} \); \( r_1 \) a random number in the range of [0,1]; \( f \) the exemplar index the \( i \)-th particle follows; \( p_{\text{best}, f(d)} \) the best position of the \( f_i \)-th particle for the \( d \)-th dimension. The updates for \( \mathbf{x}_i^d \) and \( \mathbf{v}_i^d \) across swarm population comply with the learning probability function [3]:

\[
P_{C_i} = 0.05 + 0.45 \left( \exp \left( \frac{10(i-1)}{N_p - 1} \right) - 1 \right) f(\exp(10) - 1) \quad (4)
\]

3. Subset Simulation

Subset simulation (SS) is a well-known, efficient Monte Carlo technique for variance reduction in the structural reliability community. It exploits the concept of conditional probability and advanced Markov Chain Monte Carlo technique [4]. The formulation for SS optimization is given by

\[
P_f = P(F) = P(F_1) \prod_{k=1}^{m-1} P(F_{k+1} | F_k) \quad (5)
\]

where \( F_k \) for \( \forall k \in \{1, \ldots, m\} \) are the intermediate failure events nested (satisfying \( F_1 \supseteq F_2 \supseteq \ldots \supseteq F_m = F \)). The nesting feature of all intermediate events gives the decomposition of small probability. Then, searching for \( \mathbf{C}(\mathbf{s}) \) in Eq. (1) is converted to the failure region in the reliability problem. A key step for the success of SS is the computation of conditional samples for each intermediate event to estimate its corresponding failure probability. Because the probability density functions (PDFs) for intermediate events are implicit functions, it is not practical to generate samples directly from their PDFs. This can be achieved using Markov Chain Monte Carlo (MCMC). A modified Metropolis–Hastings (MMH) algorithm has been developed for SS and summarized.

Modified Metropolis-Hastings algorithm

Let \( \mathbf{s} \sim \pi(\cdot | F^1) \) be a sample from the conditional distribution \( \pi(\cdot | F^1) \). The MMH algorithm generates another sample \( \mathbf{\bar{s}} \) from \( \pi(\cdot | F^1) \) as follows [5]:

\[
\mathbf{\bar{s}} \sim \pi(\cdot | F^1) \text{ with probability } P = \frac{\pi(\cdot | F^1) \cdot \pi(F^1 | \mathbf{s})}{\pi(\mathbf{\bar{s}} | F^1) \cdot \pi(F^1 | \mathbf{\bar{s}})}
\]
1. Generate a “candidate” sample $\xi$: for each coordinate $j = 1, \ldots, nd$

   a) Sample $\eta_j - q_j(\cdot | s_j)$, where $q_j(\cdot | s_j)$ called the proposal distribution, is a univariate PDF for $\eta_j$ centered at $s_j$ with the symmetry property $q_j(\eta_j | s_j) = q_j(s_j | \eta_j)$. The proposal distribution adopts a Gaussian PDF with mean $s_j$ and variance $\sigma_j^2$:

   $$q_j(\eta_j | s_j) = \frac{1}{\sqrt{2\pi\sigma_j}} \exp \left( -\frac{(\eta_j - s_j)^2}{2\sigma_j^2} \right)$$  \hspace{1cm} (6)

   b) Define the $j^{th}$ coordinate of the candidate sample by accepting or rejecting $\eta_j$:

   $$\xi_j = \begin{cases} 
   \eta_j, & \text{with probability } \min \left\{ 1, \frac{\phi(\eta_j)}{\phi(s_j)} \right\}, \text{ where } \phi(\cdot) \text{ denotes the PDF.} \end{cases} \hspace{1cm} (7)$$

2. Accept or reject the candidate sample $\xi$ by setting

   $$\bar{s} = \begin{cases} 
   \xi, & \text{if } \xi \in F_1 \\
   s_j, & \text{otherwise}
   \end{cases}$$ \hspace{1cm} (8)

4. Failure Probability Estimation Using MCS

Monte-Carlo simulations can be performed to approximate the failure probability in Eq. (1) by

$$P_f = \int_{\Omega} f(x) dx = \frac{1}{N_{MCS}} \sum_{i=1}^{N_{MCS}} \left[ G_j(s_j, x_j) - \bar{z} \right]$$ \hspace{1cm} (9)

where $N_{MCS}$ is the number of random samples generated within the space $\Omega$, and $[.]$ denotes the indicator function, namely

$$\left[ G_j(s_j, x_j) - \bar{z} \right] = \begin{cases} 
1 & \text{if } G_j(s_j, x_j) - \bar{z} \leq 0 \\
0 & \text{otherwise}
\end{cases} \hspace{1cm} (10)$$

5. Combined SS-CLPSO Algorithm

The proposed SS-CLPSO method for the optimal solution of the RBDO in Eq. (1) is summarized:

Step 1: Initialize the random variables in the original RBDO Eq. (1).

Step 2: Perform the deterministic optimization using the CLPSO algorithm to obtain the optimal design solutions.

Step 3: Import $s_j$ into SS using $s_j + \mathcal{N}(0,1)$, number of $n_s$ samples and $n_s$ cycles, and then sort the results $G_0(s, x)$ in descending order first. Store the value $G_0^{(nc)}(s, x)$, then select $n_s$ sample to create the next level.

Step 4: The new sample group must be less than $G_0^{(nc)}(s, x)$, the number of $n_s$ samples and $n_s$ cycles, and then sort the results $G_1(s, x)$ in ascending order first. Collect the value $G_1^{(nc)}(s, x)$, then select $n_s$ for example and calculate $P_f$. If $P_f > P_o$, repeat this Step 4 until $P_{MMP}$ is achieved.

Step 5: Bring $x_{MMP}$ repeat Steps 2 to 5. If the estimated failure probability $P_f$ converges, terminate, where $n_s = np$ is the number of Markov chains, $n_s = n/n_c$ is the number of states in each chain, $n$ is the number of samples per level, and $p = [0.1, 0.2]$ is the level probability.
6. Illustrative Example

Table 1. Probabilistic properties of random parameters.

| Variable | Distribution | Mean value | COV |
|----------|--------------|------------|-----|
| $P_1$ [kN] | Normal | 60 | 0.20 |
| $P_2$ [kN] | Normal | 40 | 0.20 |
| $P_3$ [kN] | Normal | 10 | 0.20 |
| $E$ [GPa] | Normal | 200 | 0.10 |
| $L$ [m] | Normal | 1 | 0.05 |

A 10-bar truss in Fig. 1 was considered [1]. The RBDO problem in Eq. (1) was formulated. The cross-sectional areas were employed as the design variables, namely $s = [s_1, \ldots, s_{10}]^T$, where each variable is bounded within the intervals, i.e., $s_i \in [1,20] \times 10^{-4} m^2$. The intrinsic (indeterministic probabilistic) uncertainties inherited the external loads ($P_1$, $P_2$, and $P_3$), Young’s modulus ($E$) and dimension length ($L$), simultaneously. The random variables were $x = [P_1, P_2, P_3, E, L]^T$ with the probabilistic properties listed in Table 1. The vertical displacement at node 3, denoted as $\Delta_3$, was considered as the response performance of interest ($y = -\Delta_3$), whose probability of exceeding the allowable value of $\zeta = 4 \times 10^{-3}$ m. was less than or equal to $P_d = 6.21 \times 10^{-3}$. Thus, the specific RBDO Eq. (1) was written as follows:

$$
\min C(s) = \sum_{i=1}^{10} s_i
$$

s.t. \quad \mathbb{P}[G(s,x) + 4 \times 10^{-3} \leq 0] - 6.21 \times 10^{-3} \leq 0 \quad (11)

$$
\quad \quad \quad \quad \quad s_i \in [1,20] \times 10^{-4} (m^2), \ i = 1,\ldots,10
$$

$$
\quad \quad \quad \quad \quad G(s,x) = -\Delta_3
$$

In the first instance, the initial random variables were assigned to take the mean values given in Table 1. The deterministic counterpart of the problem in Eq. (11) was solved using the CLPSO algorithm. The parameters adopted: the total number of particle populations of $p_s = 30$; $c = 1.5$; the inertial weight of $w$ linearly declining from 0.9 to 0.4; and the maximum number of iterations of 1,000 as per each particle set. The parameters employed in SS were samples per level $n = 1,000$ and level probability $p = 0.1$ set.

The coupling CLPSO and SS procedures were iterated until the estimated probability of failure (viz., $P_f = 6.07 \times 10^{-3}$) was converged and well complied with the limit of $P_d = 6.21 \times 10^{-3}$. The proposed SS-CLPSO method was encoded in Python, and run using the computer hardware with RYZEN 7 4800 HS CPU @ 2.9 GHz and 16 GB RAM. The optimal solutions reported in Table 2 were successfully computed and agreed very well with the benchmark references [1, 6, 7].

Table 2. Comparisons of optimization results for various analysis approaches.

| Design variables | FEA-FORM [6] | RSM2-FORM-MCS [6] | PSA-ISAP [7] | MGP-SA [1] | GRP-CLPSO [1] | SS-CLPSO (This Study) |
|-----------------|-------------|------------------|-------------|------------|--------------|---------------------|
| $s_1$ | 10.493 | 10.705 | 10.482 | 10.333 | 10.635 | 9.924 |
| $s_2$ | 5.772 | 5.914 | 4.421 | 5.371 | 5.589 | 5.750 |
| $s_3$ | 14.098 | 14.424 | 15.685 | 13.579 | 13.481 | 13.845 |
| $s_4$ | 1.000 | 1.000 | 1.089 | 1.000 | 1.000 | 1.082 |
$s_5$ 1.000 1.000 1.000 1.000 1.000 1.030
$s_6$ 1.000 1.000 1.000 1.000 1.000 1.006
$s_7$ 5.460 5.531 7.851 6.418 5.883 5.913
$s_8$ 11.586 11.853 10.048 11.273 11.149 11.165
$s_9$ 1.000 1.000 1.121 1.000 1.000 1.042
$s_{10}$ 10.958 11.223 9.650 10.508 10.555 10.186

$C(s)$ [× 10$^{-4}$ m$^2$] 62.367 63.649 62.347 61.482 61.293 60.943

No. of FEA iterates 2,240 1,904 524 20,000 225 +15 4 Cycles

Computational time [s] - - - 3,208 2,213 -

$P[G(s, x)-z \leq 0]$ $8.51 \times 10^{-3}$ $6.19 \times 10^{-3}$ $6.19 \times 10^{-3}$ $4.34 \times 10^{-3}$ $5.43 \times 10^{-3}$ $2.98 \times 10^{-3}$

$P_f$(MCS 10$^6$ samples) $3.59 \times 10^{-3}$ $5.86 \times 10^{-3}$ $5.72 \times 10^{-3}$ $5.25 \times 10^{-3}$ $5.22 \times 10^{-3}$ $6.07 \times 10^{-3}$

Note: 4 cycles = 10,000$^SS$ samples. +4,000$^{CLPSO}$ iterates.

Figure 2. Normalized cross-sectional areas of optimally designed members, where blue and red-colour bars indicate compression and tension members, respectively.

Figure 3. Sample generated by SS, where red sample are MCMC generated at conditional level convergence to $P_f$ and red line is the most probable point underpinning the limit-state function.
7. Conclusion
The combined SS-CLPSO method determine the optimal solution of the RBDO problem. It is based the cooperative procedures between CLPSO (the sizing design of members) and SS (the efficient failure probability approximations). A number of steel truss design examples and benchmarks (one of which has been provided herein) illustrate the robustness and accuracy of the proposed RBDO solution approach.

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