Lattice Study of $U_A(1)$ Anomaly:
The Role of QCD-Monopoles

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Abstract

We investigate the role of QCD-monopoles for the $U_A(1)$ anomaly in the maximally abelian gauge within the SU(2) lattice gauge theory. The existence of the strong correlation between instantons and QCD-monopoles in the abelian gauge was already shown by both analytic and numerical works including the Monte Carlo simulation. Their interrelation brings us a conjecture that the presence of QCD-monopoles plays a considerable role on the $U_A(1)$ anomaly. We find an evidence for our conjecture on a determination of the fermionic zero modes of the Dirac operator in both the “monopole removed” gauge configuration and the “photon removed” gauge configuration.

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I. INTRODUCTION

In non-abelian SU($N_c$) gauge theory, any finite-action configuration is classified by the topological charge $Q$, which is associated with the homotopy group $\pi_3(SU(N_c)) = Z_\infty$, in the four-dimensional Euclidean space $\mathbb{R}^4$. Instanton configurations are well known as classical and non-trivial gauge configurations, which satisfy the condition that the action is minimized in each sector $Q$. We are reminded that the topological charge is equal to the index of the massless Dirac operator $D$:

$$Q = \text{Index}[D] \equiv n_+ - n_-,$$

where $n_+(n_-)$ is the number of zero-modes with $+(-)$ chirality. This simple relation is known as the Atiyah-Singer index theorem. The existence of the fermionic zero modes has the consequence that the global $U_A(1)$ symmetry is regarded as explicitly broken at the quantum level owing to the $U_A(1)$ anomaly. Thus, such a topological feature implies that instantons are important topological objects in QCD related to the resolution of the $U_A(1)$ problem.

Recently, some interesting results turn our attention to the non-trivial relation between instantons and magnetic monopoles. As for the appearance of magnetic monopoles (QCD-monopoles) in SU($N_c$) gauge theory, 't Hooft proposed a stimulating idea of the abelian gauge fixing. Such a partial gauge fixing is defined by the gauge transformation in the coset space of the gauge group to fix the gauge degrees of freedom up to the maximally abelian subgroup; $U(1)^{N_c-1}$. In the abelian gauge, point-like singularities in the three-dimensional space $\mathbb{R}^3$ under the maximally abelian subgroup can be identified as magnetic monopoles related to the homotopy group $\pi_2(SU(N_c)/U(1)^{N_c-1}) = Z_{\infty}^{N_c-1}$. The lattice simulations show that QCD-monopoles play a crucial role on color confinement in the QCD vacuum, which can be characterized by their condensation (see, e.g. a recent review article).

It is generally believed that instantons and QCD-monopoles are hardly thought to be associated with each other because these topological objects are originated from different
non-trivial homotopy groups. However, the recent analytical works have demonstrated the QCD-monopole as a classically stable solution in the background fields of the instanton configuration using the abelian gauge fixing \[3,4\]. Furthermore, the several lattice simulations have shown the strong correlation between instantons and QCD-monopoles in the highly quantum vacuum \[4–8\] as well as the semi-classical vacuum \[8–10\]. Their interrelation brings us a conjecture that the presence of QCD-monopoles plays a considerable role on the \(U_A(1)\) anomaly.

The main purpose of this paper is to find an evidence for such a conjecture through the Monte Carlo simulation within the SU(2) lattice gauge theory. We examine the low-lying eigenvalue spectra of the Dirac operator to study the existence of the fermionic zero modes in both the “monopole removed” gauge configuration and the “photon removed” gauge configuration, which are defined subsequently. Although the Atiyah-Singer index theorem is not well-defined on the lattice in the strict sense, it might be approximately inherited at the finite lattice spacing. We then expect that the relation between instantons and QCD-monopoles can be reexamined through an investigation of the \(U_A(1)\) anomaly. Here, it is worth mentioning that whereas the measurement of the topological charge on the lattice usually needs some method to smooth Monte Carlo configurations, the fermionic zero modes can be determined without any cooling method.

II. MAXIMALLY ABELIAN PROJECTION

The Maximally Abelian (MA) gauge fixing \[11\] was advocated by ’t Hooft in order to define magnetic monopoles in the renormalizable and the Lorentz invariant way in the continuum: \((\partial_\mu \pm igA_\mu^3)A_\mu^\pm = 0\) where \(A_\mu^\pm = A_\mu^1 \pm iA_\mu^2\). In the lattice formulation \[13\], this gauge fixing corresponds to the maximization of the gauge dependent variable \(R\):

\[
R[\Omega] = \sum_{n, \mu} \text{tr} \left\{ \sigma_3 U_\mu^{-\Omega}(n) \sigma_3 U_\mu^{\Omega\dagger}(n) \right\},
\]

through the gauge transformation; \(U_\mu(n) \rightarrow U_\mu^{\Omega}(n) = \Omega(n)U_\mu(n)\Omega^\dagger(n+\hat{\mu})\) where \(U_\mu\) denotes the SU(2) link variable. Once the gauge transformation is carried out to satisfy the above
condition, the resulting SU(2) link variable $U_{\mu}^\Omega(n)$ is factorized into an abelian link variable $u_{\mu}(n) \equiv \exp\{i\sigma_3\theta_{\mu}(n)\}$ and an adjoint “matter” field $M_{\mu}$ \[^{[13]}\] as

$$U_{\mu}^\Omega(n) = M_{\mu}(n) \cdot u_{\mu}(n) \ ,$$

where

$$M_{\mu}(n) \equiv \begin{pmatrix} \sqrt{1 - |\xi_{\mu}(n)|^2} & \xi_{\mu}(n) \\ -\xi_{\mu}^*(n) & \sqrt{1 - |\xi_{\mu}(n)|^2} \end{pmatrix} .$$

Performing the residual U(1) gauge transformation, $\theta_{\mu}$ and $\xi_{\mu}$ respectively transform like an abelian gauge field and a charged “matter” field \[^{[13]}\].

Our next task is to look for the magnetic monopole in terms of the U(1) variables. We consider the product of U(1) link variables around an elementary plaquette \[^{[13]}\],

$$u_{\mu\nu}(n) = u_{\mu}(n)u_{\nu}(n + \hat{\mu})u_{\nu}^\dagger(n + \hat{\nu})u_{\nu}^\dagger(n) = e^{i\sigma_3\Theta_{\mu\nu}(n)} .$$

It is worth mentioning that the U(1) plaquette $u_{\mu\nu}$ is a multiple valued function as the U(1) plaquette angle \[^{[13]}\]: $\Theta_{\mu\nu}(n) \equiv \partial_{\mu}\theta_{\nu}(n) - \partial_{\nu}\theta_{\mu}(n) \in [-4\pi, 4\pi)$ where $\partial$ denotes the nearest-neighbor forward difference operator. We then divide $\Theta_{\mu\nu}$ into two parts as

$$\Theta_{\mu\nu}(n) = \bar{\Theta}_{\mu\nu}(n) + 2\pi n_{\mu\nu}(n) ,$$

where $\bar{\Theta}_{\mu\nu}$ is defined in the principal domain $[-\pi, \pi)$, which corresponds to the U(1) field strength in the continuum limit. The integer-valued $n_{\mu\nu}$ is restricted as $n_{\mu\nu} = 0, \pm 1, \pm 2$ \[^{[14]}\]. In terms of $\bar{\Theta}_{\mu\nu}$, the electric current $j_{\mu}$ and the magnetic current $k_{\mu}$ are defined as

$$j_{\mu}(n) = \partial_{\nu}\bar{\Theta}_{\mu\nu}(n) \ ,$$

$$k_{\mu}(n) = \frac{1}{4\pi} \varepsilon_{\mu\nu\rho\sigma} \partial_{\nu}\bar{\Theta}_{\rho\sigma}(n + \hat{\mu}) ,$$

where $\partial'$ denotes the nearest-neighbor backward difference operator \[^{[14]}\]. Because of the Bianchi identity on the U(1) plaquette angle; $\varepsilon_{\mu\nu\rho\sigma} \partial_{\nu}\bar{\Theta}_{\rho\sigma} = 0$, the magnetic current is rewritten as
Eq. (9) implies that the magnetic current, which carries some integer values, can be identified to the monopole trajectory located on the boundary of the Dirac sheet; \( *n_{\mu\nu} \equiv \frac{1}{2} \varepsilon_{\mu\nu\rho\sigma} n_{\rho\sigma} \) \[14\]. The magnetic current is topologically conserved; \( \partial_{\mu} k_{\mu}(n) = 0 \) so that the monopole trajectory forms a closed loop in the four-dimensional Euclidean space.

III. PHOTON AND MONOPOLE CONTRIBUTION

Next, we aim to decompose the abelian gauge field into the regular (photon) part and the singular (monopole) part \[15,16\]. We first perform the Hodge decomposition on the U(1) field strength \( \bar{\Theta}_{\mu\nu} \) \[15\] as

\[
\bar{\Theta}_{\mu\nu}(n) = \partial_{\mu} \theta'_{\nu}(n) - \partial_{\nu} \theta'_{\mu}(n) + \varepsilon_{\mu\nu\rho\sigma} \partial'_{\rho} B_{\sigma}(n)
\]

with the dual gauge field \( B_{\mu} \) satisfying the following equation \[13\]:

\[
\left( \partial^2 \delta_{\mu\nu} - \partial'_{\mu} \partial_{\nu} \right) B_{\nu}(n) = -2\pi k_{\mu}(n - \hat{\mu})
\]

where \( \partial^2 = \partial'_{\mu} \partial_{\mu} \). The Gaussian fluctuation \( \theta'_{\mu} \) contributes only the electric current \( j_{\mu} \) and not the magnetic current \( k_{\mu} \) so that it just corresponds to the regular (photon) part of the abelian gauge field. As a result, the singular (monopole) part of the abelian gauge field can be identified by subtracting the Gaussian fluctuation from the abelian gauge field \[15,16\].

In the Landau gauge, the definite identification of the Gaussian fluctuation, \textit{i.e.} the regular (photon) part, is given by convolution of the U(1) field strength \( \bar{\Theta}_{\mu\nu} \) with the lattice Coulomb propagator \( G(n - m) \) \[13,16\] as

\[
\theta_{\mu}^{Ph}(n) \equiv - \sum_{m, \lambda} G(n - m) \partial'_{\lambda} \bar{\Theta}_{\lambda\mu}(m)
\]

Here the lattice Coulomb propagator satisfies the equation; \( \partial^2 G(n - m) = -\delta_{n, m} \). Immediately, we can obtain the singular (monopole) part from the following definition \[13,16\]:

\[
k_{\mu}(n) = -\frac{1}{2} \varepsilon_{\mu\nu\rho\sigma} \partial_{\nu} n_{\rho\sigma}(n + \hat{\mu})
\]
\[ \theta_{\mu}^{Mo}(n) \equiv \theta_{\mu}^{L}(n) - \theta_{\mu}^{Ph}(n) \]
\[ = -2\pi \sum_{m, \lambda} G(n - m) \partial_\lambda n_{\lambda\mu}(m) . \] (13)

Here \( \theta_{\mu}^{L} \) denotes the abelian gauge field in the Landau gauge where \( \partial_\mu \theta_{\mu}^{L}(n) = 0 \). The singular part \( \theta_{\mu}^{Mo} \) actually carries the same amount of the magnetic current as the original abelian gauge field in the infinite volume limit [17]. In addition, one may note that the singular part \( \theta_{\mu}^{Mo} \) keeps essential contributions to confining features of the Polyakov loop [16] and finite quark condensate [17] in the finite temperature phase transition. Also, the SU(2) string tension is almost evaluated from the singular part in the MA gauge (\( \sigma_{Mo} \approx 0.87 \sigma_{SU(2)} \)) [18].

In order to show the explicit contribution of monopoles for the \( U_A(1) \) anomaly, we define two types of gauge configuration; the “monopole removed \((\text{photon-dominating})\)” link variable \( U_{\mu}^{Ph} \) and the “photon removed \((\text{monopole-dominating})\)” link variable \( U_{\mu}^{Mo} \) as the corresponding SU(2) variables [5,19]. \( U_{\mu}^{Ph} \) is defined by removing the monopole contribution from the original gauge configuration as

\[ U_{\mu}^{Ph}(n) \equiv U_{\mu}(n) \cdot u_{\mu}^{\dagger Mo}(n) . \] (14)

On the other hand, \( U_{\mu}^{Mo} \) is defined by removing the photon contribution from the original gauge configuration as

\[ U_{\mu}^{Mo}(n) \equiv U_{\mu}(n) \cdot u_{\mu}^{\dagger Ph}(n) . \] (15)

Here, \( u_{\mu}^{i}(n) \equiv \exp\{i\sigma_3\theta_{\mu}^{i}(n)\} \) \((i = \text{Ph or Mo})\). It is noted that these definitions exactly correspond to the reconstruction of the resulting SU(2) variables from \( u_{\mu}^{i} \) by multiplying the adjoint “matter” field \( \text{in the Landau gauge} \) [19]:

\[ \tilde{U}_{\mu}^{i}(n) \equiv \tilde{M}_{\mu}(n) \exp\{i\sigma_3\theta_{\mu}^{i}(n)\} , \] (16)

where \( \tilde{M}_{\mu}(n) = d(n)M_{\mu}(n)d^{\dagger}(n) \) with \( d(n) = e^{i\varphi(n)\sigma_3} \). One can easily see the relation: \( \tilde{U}_{\mu}^{i}(n) = d(n)U_{\mu}^{i}(n)d^{\dagger}(n + \mu) \). In this sense, we call \( U_{\mu}^{Mo} \) as the monopole-dominating gauge configuration.
configuration (Mo part) and $U^{\text{Ph}}_\mu$ as the photon-dominating gauge configuration (Ph part) respectively. In previous publications, we found the corresponding topological charge, which can be classified by an “integer” value, in the background of $U^{\text{Mo}}_\mu$ \cite{5,19}. On the other hand, the non-zero topological charge was never found in the background of $U^{\text{Ph}}_\mu$ \cite{3,19}.

IV. ZERO MODES OF THE DIRAC OPERATOR

We study the low-lying eigenvalue spectra of the Dirac operator in the background of three types of configuration; the monopole-dominating gauge fields, the photon-dominating gauge fields and the original SU(2) gauge fields \cite{19}. For the Dirac operator on the lattice, we adopt the Wilson fermion \cite{20}. In the background of $U^i_\mu$ ($i = \text{Ph or Mo}$) and the original gauge fields; $U_\mu$, $\mathcal{D}$ is expressed as

$$
\mathcal{D}(n, m) = \delta_{n, m} - \kappa \sum_\mu \left[ (1 - \gamma_\mu) U^{(i)}_\mu(n) \delta_{n+\mu, m} + (1 + \gamma_\mu) U^{\dagger(i)}_\mu(n - \mu) \delta_{n-\mu, m} \right],
$$

(17)

where $\kappa$ is the hopping parameter. Although the Wilson fermion does not have the chiral symmetry in the naive argument, the partial symmetry restoration would be realized near the critical value $\kappa_c$ where the pseudo-scalar mass vanishes \cite{21}.

The operator $\mathcal{D}$ loses a feature as the hermitian operator owing to the discretization of the space-time. However, one can easily find that $\gamma_5 \mathcal{D}^\dagger \gamma_5 = \mathcal{D}$. We then examine the eigenvalue spectrum of the hermitian operator $\gamma_5 \mathcal{D}$ by using the Lanczos algorithm. We can identify the fermionic zero-modes in the following procedure. First, the existence of zero modes could be found by the zero-line crossing in eigenvalue spectra of $\gamma_5 \mathcal{D}$ through the variation of the hopping parameter around $\kappa_c$ \cite{22}. Then, the chirality of zero-modes could be defined by the slope of the eigenvalue spectrum \cite{22}.

We generate the gauge configuration by using the Wilson action on an $8^4$ lattice with $\beta = 2.4$. As the hopping parameter, we change within the range; $0.130 \leq \kappa \leq 0.180$. (In ref. \cite{23}, $\kappa_c = 0.175 \pm 0.002$ has been obtained at $\beta = 2.4$ on a $5^3 \times 10$ lattice.) We measure the eigenvalue of the operator $\gamma_5 \mathcal{D}$ in the background of $U^i_\mu$ ($i = \text{Ph or Mo}$) and also in the
original SU(2) gauge field; $U_\mu$ for 32 independent configurations without any cooling \cite{19}. Fig.1(a)-1(d) show the low-lying spectra in 4 independent gauge configurations as typical examples. In each configuration, we can see (a) 1 zero mode of chirality $-$, (b) 2 zero modes of chirality $+$, (c) no zero mode and (d) 1 zero mode of chirality $+$. Shows that Fig.2(a)-2(d) and Fig.3(a)-3(d), we can also observe the low-lying spectra in the background of the monopole-dominating gauge field and the photon-dominating gauge field, which are defined on the basis of the same gauge configurations in Fig.1(a)-1(d). The same number of zero modes as be observed in Fig.1(a)-1(d) can be found in the background of $U_{\mu}^{Mo}$ \cite{19}. This remarkable coincidence for the number of zero modes and its chirality is not well identified in 6 configurations, but is confirmed in all the rest 26 configurations \cite{19}. However, we can not find the corresponding zero modes in the background of $U_{\mu}^{Ph}$ within 32 configurations.

It is worth mentioning that this result is consistent with our previous works in Ref. \cite{3,18,19}, which showed that the non-zero value of the topological charge was never found in any photon-dominating gauge configuration after several cooling sweeps. Thus, we can interpret that instantons can not live in the monopole removed gauge configuration.

V. SUMMARY

We have investigated topological aspects of the QCD vacuum structure by using the Monte Carlo simulation within the SU(2) gauge theory. We defined two types of gauge configuration; the monopole removed gauge configuration and the photon removed gauge configuration after the MA gauge fixing. We measured the fermionic zero modes of the Dirac operator in each gauge field background without any cooling method. In only the background of the monopole-dominating gauge field, the explicit breaking of the $U_A(1)$ symmetry occurs owing to the existence of the fermionic zero modes. On the other hand, we can never find the corresponding zero modes in the background where monopole contributions are completely removed. These results imply that topological features are inherited in the monopole-dominating (photon removed) gauge field, but spoiled in the photon-dominating
(monopole removed) gauge field since the $U_A(1)$ anomaly is related to topological objects, i.e. instantons. Of course, we must not forget that an $8^4$ lattice at $\beta = 2.4$ in our simulation might be considerably small in physical units to produce definitive results. However, it seems reasonable to suppose that our numerical data shows the strong evidence for the topologically close relation between instantons and QCD-monopoles in the quantum vacuum of QCD after the typical abelian gauge fixing. This statement is also strongly supported by our next study [24], which shows that the topological charge can be approximately reconstructed from the monopole current and the abelian component of gauge fields in the MA gauge.

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FIGURE CAPTIONS

FIG.1. Typical examples of low-lying spectra of $\gamma_5 D$ through the variation of $\kappa$ in the background of the original SU(2) gauge field on an $8^4$ lattice at $\beta = 2.4$.

FIG.2. Low-lying spectra of $\gamma_5 D$ through the variation of $\kappa$ in the background of $U_{\mu}^{Mo}$ corresponding to the examples of (a)-(d) in Fig.1.

FIG.3. Low-lying spectra of $\gamma_5 D$ through the variation of $\kappa$ in the background of $U_{\mu}^{Ph}$ corresponding to the examples of (a)-(d) in Fig.1.
Fig. 1 (Phys. Lett. B) Shoichi Sasaki et al.

Fig. 1(a)

Fig. 1(b)

Fig. 1(c)

Fig. 1(d)
FIG. 2 (Phys. Lett. B) Shoichi Sasaki et al.

Fig. 2(a)

Fig. 2(b)

Fig. 2(c)

Fig. 2(d)
FIG. 3 (Phys. Lett. B) Shoichi Sasaki et al.

Fig. 3(a)  
Fig. 3(b)  

Fig. 3(c)  
Fig. 3(d)