The Busemann–Petty problem in the complex hyperbolic space

BY SUSANNA DANN

Department of Mathematics,
University of Missouri, Columbia, MO 65211, U.S.A.
e-mail: danns@missouri.edu

(Received 13 December 2011; revised 7 November 2012)

Abstract

The Busemann–Petty problem asks whether origin-symmetric convex bodies in $\mathbb{R}^n$ with smaller central hyperplane sections necessarily have smaller volume. The answer is affirmative if $n \leq 4$ and negative if $n \geq 5$. We study this problem in the complex hyperbolic $n$-space $\mathbb{H}_C^n$ and prove that the answer is affirmative for $n \leq 2$ and negative for $n \geq 3$.

1. Introduction

The Busemann-Petty problem asks the following question. Given two origin symmetric convex bodies $K$ and $L$ in $\mathbb{R}^n$ such that

$$\text{Vol}_{n-1}(K \cap H) \leq \text{Vol}_{n-1}(L \cap H)$$

for every hyperplane $H$ in $\mathbb{R}^n$ containing the origin, does it follow that

$$\text{Vol}_n(K) \leq \text{Vol}_n(L)?$$

The answer is affirmative for $n \leq 4$ and negative for $n \geq 5$. This problem, posed in 1956 in [7], was solved in the late 90's as a result of a sequence of papers [3, 5, 9, 10, 12, 14, 16, 17, 27, 28, 29, 34, 35], see [22, p. 3–5], for the history of the solution.

Since then the Busemann–Petty problem was studied on other spaces as were its numerous generalizations. V. Yaskin studied the Busemann–Petty problem in real hyperbolic and spherical spaces, [33]. He showed that for the spherical space the answer is the same as for $\mathbb{R}^n$, but not so for the real hyperbolic space, where the answer is affirmative for $n \leq 2$ and negative for $n \geq 3$. A. Koldobsky, H. König and M. Zymonopoulou demonstrated in [24] that the answer to the complex version of the Busemann–Petty problem is affirmative for the complex dimension $n \leq 3$ and negative for $n \geq 4$. Other results on the complex Busemann-Petty problem include [23, 30, 37, 38].

In this article we consider the Busemann–Petty problem in the complex hyperbolic $n$-space. For $\xi \in \mathbb{C}^n$ with $|\xi| = 1$, denote by

$$H_\xi := \left\{ z \in \mathbb{C}^n : (z, \xi) = \sum_{k=1}^n z_k \xi_k = 0 \right\}$$

the complex hyperplane through the origin perpendicular to $\xi$. We identify $\mathbb{C}^n$ with $\mathbb{R}^{2n}$ via the mapping

$$(\xi_1 + i\xi_2, \ldots, \xi_{n1} + i\xi_{n2}) \mapsto (\xi_{11}, \xi_{12}, \ldots, \xi_{n1}, \xi_{n2}).$$

(1.1)
Under this mapping the hyperplane $H_ξ$ turns into a $(2n - 2)$-dimensional subspace of $\mathbb{R}^{2n}$ orthogonal to the vectors

$$ξ = (ξ_1, ξ_2, \ldots, ξ_n),$$

and $ξ_⊥ = (-ξ_1, ξ_2, \ldots, -ξ_n)$. A convex body $K$ in $\mathbb{R}^{2n}$ is called $R_θ$-invariant, if for every $θ \in [0, 2π]$ and every $ξ = (ξ_1, ξ_2, \ldots, ξ_n) \in \mathbb{R}^{2n}$

$$∥ξ∥_K = ∥R_θ(ξ_1, ξ_2, \ldots, ξ_n)∥_K,$$

where $R_θ$ stands for the counterclockwise rotation by an angle $θ$ around the origin in $\mathbb{R}^2$.

An origin symmetric body $K$ in $\mathbb{H}^n_C$ is called convex if under the mapping (1.1) it corresponds to an $R_θ$-invariant body in $\mathbb{R}^{2n}$ contained in the unit ball such that for any pair of points in $K ⊂ \mathbb{R}^{2n}$ the geodesic segment with respect to the Bergman metric on $\mathbb{H}^n_C$ joining them also belongs to $K$, see next section for a motivation of this definition. Bodies in $\mathbb{R}^{2n}$ contained in the unit ball and satisfying the latter condition will be called $h$-convex. We denote the volume element on $\mathbb{H}^n_C$ by $dμ_n$ and the volume of a body $K$ in $\mathbb{R}^{2n}$ with respect to this volume element by $HV ol_2n(K)$ to distinguish from the Euclidean volume of $K$.

Now the Busemann–Petty problem in $\mathbb{H}^n_C$ can be posed as follows. Given two $R_θ$-invariant $h$-convex bodies $K$ and $L$ in $\mathbb{R}^{2n}$ such that

$$HV ol_2n−2(K ∩ H_ξ) ≤ HV ol_2n−2(L ∩ H_ξ)$$

for any element $ξ$ of the unit sphere $S^{2n−1}$ of $\mathbb{R}^{2n}$, does it follow that

$$HV ol_2n(K) ≤ HV ol_2n(L)?$$

Analytic solutions of the Busemann–Petty problem in different settings are based on establishing a connection between a certain distribution and the problem. E. Lutwak introduced a class of intersection bodies in [28] and established a connection between this class and the Busemann–Petty problem on $\mathbb{R}^n$. Recall that an origin symmetric star body $K$ in $\mathbb{R}^n$ is an intersection body if and only if $∥·∥_K$ is a positive definite distribution on $\mathbb{R}^n$, see [22, theorem 4.1]. Later, A. Zvavitch solved the Busemann–Petty problem on $\mathbb{R}^n$ for arbitrary measures [36]. He linked the problem to the distribution $∥x∥_K^{−1}(f_n(x∥x∥_K^{−1})/f_{n−1}(x∥x∥_K^{−1}))$, where $f_n$, a locally integrable function on $\mathbb{R}^n$, is the density function for a measure $μ_n$ on $\mathbb{R}^n$ and $f_{n−1}$, a function on $\mathbb{R}^n$ integrable on central hyperplanes, is the density function for a measure $μ_{n−1}$ on $\mathbb{R}^{n−1}$. In [33], V. Yaskin established a connection between the Busemann–Petty problem in hyperbolic and spherical spaces and the distribution $∥x∥_K^{−1}/(1 ± (∥x∥_K^{−1}))^2$ as a special case of Zvavitch’s theorem. Recently, A. Koldobsky at al. found a connection between the 2-intersection bodies and the Busemann-Petty problem on $\mathbb{C}^n$ [24]. The classes of $k$-intersection bodies were introduced in [18, 19]. Recall that an origin symmetric star body $K$ in $\mathbb{R}^n$ is a $k$-intersection body, $0 < k < n$, if and only if $∥·∥^{−k}_K$ is a positive definite distribution on $\mathbb{R}^n$, see [19].

We prove in Propositions 3.2 and 3.3 a similar connection between the Busemann–Petty problem in $\mathbb{H}^n_C$ and the distribution $∥x∥_K^{2k}/(1 − (∥x∥_K^{−1}))^2$. The solution to the problem is provided in Theorem 4.3, namely the answer is affirmative in complex dimensions one and two only. In our proof we use methods from [24, 33] as well as recently obtained results for complex star bodies from [25].

A few generalizations of the complex Busemann–Petty problem have been considered so far, see [37, 38]. For another result in the complex hyperbolic space related to convex geometry see [2]. Some other extensions of results from convex geometry to non-Euclidean
The Busemann–Petty problem in the complex hyperbolic space

settings include [1, 4, 8, 11]. For other generalizations of the Busemann–Petty problem see [6, 18, 19, 20, 21, 26, 36].

2. Preliminaries

2.1. Complex hyperbolic space

The material of sections 2.1.1 and 2.1.2 is taken from the book by Goldman [15]. We refer the interested reader to this book for more information.

2.1.1. The ball model

Let \( V \) be a complex vector space. The projective space associated to \( V \) is the space \( \mathbb{P}(V) \) of all lines in \( V \): one dimensional complex linear subspaces through the origin. Let \( \mathbb{C}^{n+1} \) be the \((n+1)\)-dimensional complex vector space consisting of \((n+1)\)-tuples

\[
Z = \begin{bmatrix} Z' \\ Z_{n+1} \end{bmatrix} \in \mathbb{C}^{n+1}
\]

and equipped with the indefinite Hermitian form

\[
(Z, W) := (Z', W') - Z_{n+1} \overline{W}_{n+1} = Z_1 \overline{W}_1 + \cdots + Z_n \overline{W}_n - Z_{n+1} \overline{W}_{n+1},
\]

where \( Z' \) is a vector in \( \mathbb{C}^n \) and \( Z_{n+1} \in \mathbb{C} \). Consider the subset of negative vectors of \( \mathbb{C}^{n+1} \)

\[
N := \{ Z \in \mathbb{C}^{n+1} : \langle Z, Z \rangle < 0 \}.
\]

The complex hyperbolic \( n \)-space \( \mathbb{H}_C^n \) is defined to be \( \mathbb{P}(N) \): the subset of \( \mathbb{P}(\mathbb{C}^{n+1}) \) consisting of negative lines in \( \mathbb{C}^{n+1} \). We identify \( \mathbb{H}_C^n \) with the open unit ball

\[
B^n := \{ z \in \mathbb{C}^n : (z, z) < 1 \}
\]

as follows. Define a mapping \( A \) by

\[
A : \mathbb{C}^n \longrightarrow \mathbb{P}(\mathbb{C}^{n+1})
\]

\[
z' \longmapsto \begin{bmatrix} z' \\ 1 \end{bmatrix}.
\]

Since for negative vectors of \( \mathbb{C}^{n+1} \) the \((n+1)\)-coordinate is necessarily different from zero, \( \mathbb{H}_C^n \subset A(\mathbb{C}^n) \). The mapping \( A \) identifies \( B^n \) with \( \mathbb{H}_C^n \) and \( \partial B^n = S^{2n-1} \subset \mathbb{C}^n \) with \( \partial \mathbb{H}_C^n \).

Theorem 2.1 ([15, theorem 3.1-10]). Let \( F \subset \mathbb{P}(\mathbb{C}^{n+1}) \) be a complex \( m \)-dimensional projective subspace which intersects \( \mathbb{H}_C^n \). Then \( F \cap \mathbb{H}_C^n \) is a totally geodesic holomorphic submanifold biholomorphically isometric to \( \mathbb{H}_C^m \).

The intersection of \( \mathbb{H}_C^n \) with a complex hyperplane is a totally geodesic holomorphic complex hypersurface, called a complex hyperplane in \( \mathbb{H}_C^n \). Its boundary is a smoothly embedded \((2n-3)\)-sphere in \( \partial \mathbb{H}_C^n \).

2.1.2. The Bergman metric and the volume element

We normalize the Bergman metric, a Hermitian metric on \( \mathbb{H}_C^n \), to have constant holomorphic sectional curvature \(-1\). It can be described as follows. Let \( x, y \) be a pair of distinct points in \( B^n \) and let \( \overrightarrow{xy} \) denote the unique complex line they span. The Bergman metric
restricts on \( \mathbb{H}^n \) to the Poincaré metric of constant curvature \(-1\) given by:

\[
\frac{4R^2 dz d\overline{z}}{(R^2 - r^2)^2},
\]

where \( R \) is the radius of the disc \( \mathbb{xy} \cap B^n \) and \( r = r(z) \) is the distance to the center of the disc \( \mathbb{xy} \cap B^n \). As \( \mathbb{xy} \) is totally geodesic, the distance between \( x \) and \( y \) in \( \mathbb{H}^n \) equals the distance between \( x \) and \( y \) in \( \mathbb{xy} \cap B^n \) with respect to the above Poincaré metric. Moreover, the geodesic from \( x \) to \( y \) in \( \mathbb{H}^n \) is the Poincaré geodesic in \( \mathbb{xy} \cap B^n \) joining \( x \) and \( y \). The Poincaré geodesics are circular arcs orthogonal to the boundary and straight lines through the center.

The volume element on \( \mathbb{H}^n \) is

\[
d\mu_n = 8^n r^{2n-1} dr d\sigma, \quad (1 - r^2)^{n+1}
\]

where \( d\sigma \) is the volume element on the unit sphere \( S^{2n-1} \).

2.1.3. Origin symmetric convex sets in \( \mathbb{H}^n \)

Now we give a motivation for the definition of origin symmetric convex bodies in \( \mathbb{H}^n \) given in the introduction. We start by recalling the corresponding concept in \( \mathbb{C}^n \).

A set \( K \subset \mathbb{C}^n \) is called origin symmetric if \( x \in K \) implies that \( -x \in K \). Recall that origin symmetric convex bodies in \( \mathbb{C}^n \) are unit balls of norms on \( \mathbb{C}^n \) and therefore, under the mapping \((1 \cdot 1)\), they are \( R_\theta \)-invariant convex bodies in \( \mathbb{R}^{2n} \), see [24] for details. In other words, an origin symmetric set \( K \subset \mathbb{C}^n \) is convex if it is \( R_\theta \)-invariant and for any two points in \( K \) the straight line segment joining them also belongs to \( K \).

Since the condition of \( R_\theta \)-invariance comes from the complex structure it is natural to keep it. The second condition requires sets to be geodesically convex. In our model geodesics are Poincaré geodesics of affine sections of \( \mathbb{B}^n \) by complex lines. It is clear that we only consider sets contained in the unit ball. Combining all these conditions we arrive at the definition given in the introduction, namely origin symmetric convex bodies in \( \mathbb{H}^n \) are those that under the mapping \((1 \cdot 1)\) correspond to \( R_\theta \)-invariant \( h \)-convex bodies in \( \mathbb{R}^{2n} \).

In the following we shall often consider origin symmetric convex bodies in \( \mathbb{C}^n \) contained in the open unit ball \( \mathbb{B}^n \): sets that are convex in Euclidean sense. They will be referred to as complex convex bodies or as \( R_\theta \)-invariant convex bodies. To avoid confusion between the different notions of convexity, the term \( h \)-convex will be used each time when dealing with origin symmetric convex bodies in \( \mathbb{H}^n \).

While it is not true in general that a convex body contained in the unit ball is \( h \)-convex - a counterexample is given below - one can dilate a convex body with strictly positive curvature to make it \( h \)-convex. By strictly positive curvature we mean that the boundary of \( K \) does not contain any straight line segments. More precisely, the normal curvature of \( K \) at any point \( p \) on the boundary of \( K \) is strictly positive in any direction \( v \), where \( v \in TK_p \) is any element in the tangent space to \( K \) at \( p \), see [32].

**Lemma 2.2.** Let \( D \) be an origin symmetric convex body in \( \mathbb{R}^{2n} \) with strictly positive curvature. Then there is \( \alpha > 0 \) so that the dilated body \( \alpha D \) is \( h \)-convex.
Proof. Denote by $d$ the smallest normal curvature among the points on the boundary of $D$. $d > 0$ since $D$ has strictly positive curvature. The smallest normal curvature among the points on the boundary of $\alpha D$ is $d/\alpha$. Now we consider the boundary curves of sections of the body $\alpha D$ by affine 2-dimensional planes. The curvature of these boundary curves is at least $d/\alpha$ at any point.

Next consider a small neighbourhood of the origin in $\mathbb{H}^n_{\mathbb{C}}$, say a ball of radius $r$. The geodesics in this neighbourhood are contained in the affine 2-dimensional planes corresponding to affine complex lines that have a non-empty intersection with the neighbourhood. The Euclidean curvature of all geodesics in this neighbourhood is less than $\frac{2r}{1-r^2}$. Indeed, let $t$ be the distance from the origin to an affine 2-dimensional plane corresponding to an arbitrary affine complex line. For $t < r$, this plane intersects $B^n$ in a big disc of radius $\sqrt{1 - t^2}$ and it intersects our neighbourhood in a small disc of radius $\sqrt{r^2 - t^2}$. The geodesics in this small disc are straight lines through the center of the disc and circular arcs orthogonal to the boundary of the big disc that intersect the small disc. The radius $S$ of a geodesic tangent to the small disc must satisfy the equation $(S + \sqrt{r^2 - t^2})^2 = 1 - t^2 + S^2$, hence $S = (1 - r^2)/2r$. Thus all geodesics in the small disc are circular arcs with a radius greater than $(1 - r^2)/2r$ and hence their Euclidean curvature is less than $2r/(1 - r^2)$.

Choose $\alpha$ so small that $\alpha D \subset B^n$ and
\[
\frac{d}{\alpha} > \frac{2r}{1-r^2}.
\]
This guarantees that a section of the body $\alpha D$ by an affine 2-dimensional plane corresponding to an arbitrary affine complex line is geodesically convex and hence $\alpha D$ is $h$-convex.

Now we construct an $R_{\theta}$-invariant convex body in $\mathbb{R}^{2n}$ that is not $h$-convex. This counterexample is a modified version of an example due to Andreas Bernig.

Pick any affine complex line. Its intersection with the unit ball is a disc. Choose points $x$ and $y$ in this disc that don’t lie on a straight line segment through the center of the disc. Define a body $K$ to be the convex hull of the set
\[
\{(R_{\theta}(x_{11}, x_{12}), \ldots, R_{\theta}(x_{n1}, x_{n2})), (R_{\theta}(y_{11}, y_{12}), \ldots, R_{\theta}(y_{n1}, y_{n2})) : \theta \in [0, 2\pi]\}.
\]
The body $K$ so constructed is an $R_{\theta}$-invariant convex body contained in the unit ball. But the geodesic with respect to the Bergman metric between $x$ and $y$ does not belong to $K$, because the intersection of $K$ with the affine complex line containing $x$ and $y$ is the straight line segment between $x$ and $y$. Hence $K$ is not $h$-convex.

2.2. Convex geometry

2.2.1. Basic definitions

The main tool used in this paper is the Fourier transform of distributions, see [13] as the classical reference for this topic. As usual, denote by $\mathcal{S}(\mathbb{R}^n)$ the Schwartz space of rapidly decreasing infinitely differentiable functions on $\mathbb{R}^n$, also referred to as test functions, and by $\mathcal{S}'(\mathbb{R}^n)$ the space of distributions on $\mathbb{R}^n$, the continuous dual of $\mathcal{S}(\mathbb{R}^n)$. The Fourier transform $\hat{f}$ of a distribution $f$ is defined by $\langle \hat{f}, \varphi \rangle = \langle f, \hat{\varphi} \rangle$ for every test function $\varphi$. A distribution $f$ on $\mathbb{R}^n$ is even homogeneous of degree $p \in \mathbb{R}$, if
\[
\left( f(x), \varphi \left( \frac{x}{\alpha} \right) \right) = |\alpha|^{n+p} \langle f, \varphi \rangle
\]
for every test function $\varphi$ and every $\alpha \in \mathbb{R}$, $\alpha \neq 0$. The Fourier transform of an even homogeneous distribution of degree $p$ is an even homogeneous distribution of degree $-n - p$. We
call a distribution \( f \) positive definite, if for every test function \( \varphi \)
\[
\langle f(x), \varphi \ast \overline{\varphi}(-x) \rangle \geq 0.
\]
This is equivalent to \( \hat{f} \) being a positive distribution, i.e. \( \langle \hat{f}, \varphi \rangle \geq 0 \) for every non-negative test function \( \varphi \).

A compact subset \( K \) of \( \mathbb{R}^n \) containing the origin as an interior point is called a star body if every line through the origin crosses the boundary in exactly two points different from the origin, and its Minkowski functional is defined by
\[
\|x\|_K := \min \{a \geq 0 : x \in aK\}.
\]
The boundary of \( K \) is continuous if \( \| \cdot \|_K \) is a continuous function on \( \mathbb{R}^n \). If in addition \( K \) is origin symmetric and convex, then the Minkowski functional is a norm on \( \mathbb{R}^n \). A star body \( K \) is said to be \( k \)-smooth, \( k \in \mathbb{N} \cup \{0\} \), if the restriction of \( \| \cdot \|_K \) to the unit sphere \( S^{n-1} \) belongs to the class \( C^k(S^{n-1}) \) of \( k \) times continuously differentiable functions on the unit sphere. If \( \| \cdot \|_K \in C^k(S^{n-1}) \) for any \( k \in \mathbb{N} \), then a star body \( K \) is said to be infinitely smooth. For \( x \in S^{n-1} \), the radial function of \( K \), \( \rho_K(x) = \|x\|_K^{-1} \), is the Euclidean distance from the origin to the boundary of \( K \) in the direction \( x \). All bodies considered in the sequel contain the origin as an interior point.

2.2.2. Fourier approach to sections

It was shown in [22, lemma 3.16], that for an infinitely smooth origin symmetric star body \( K \) in \( \mathbb{R}^n \) and \( 0 < p < n \), the Fourier transform of the distribution \( \|x\|^{-p}_K \) is an infinitely smooth function on \( \mathbb{R}^n \setminus \{0\} \), homogeneous of degree \( -n + p \). We shall use a version of the following Parseval’s formula on the sphere:

**Lemma 2.3 ([22, lemma 3.22]).** Let \( K \) and \( L \) be infinitely smooth origin symmetric star bodies in \( \mathbb{R}^n \), and let \( 0 < p < n \). Then
\[
\int_{S^{n-1}} (\| \cdot \|^{-p}_K) \wedge (\| \cdot \|^{-n+p}_L) \wedge (\theta) d\theta = (2\pi)^n \int_{S^{n-1}} \| \cdot \|^{-p}_K \| \cdot \|^{-n+p}_L d\theta.
\]

The classes of \( k \)-intersection bodies were introduced by A. Koldobsky in [18, 19] as follows. Let \( 1 \leq k < n \) and let \( D \) and \( L \) be origin symmetric star bodies in \( \mathbb{R}^n \). The body \( D \) is called a \( k \)-intersection body of \( L \) if for every \((n-k)\)-dimensional subspace \( H \) of \( \mathbb{R}^n \)
\[
\text{Vol}_k(D \cap H^\perp) = \text{Vol}_{n-k}(L \cap H).
\]

Since an origin symmetric star body \( K \) in \( \mathbb{R}^n \) is a \( k \)-intersection body if and only if \( \| \cdot \|^{-k}_K \) is a positive definite distribution on \( \mathbb{R}^n \), by [22, corollary 2.26] to each \( k \)-intersection body is associated a finite Borel measure \( \mu_0 \) on the sphere so that for every even test function \( \phi \),
\[
\int_{\mathbb{R}^n} \|x\|^{-k}_K \phi(x) dx = \int_{S^{n-1}} \left( \int_0^\infty t^{p-1} \hat{\phi}(t \xi) dt \right) d\mu_0(\xi).
\]

The smoothness assumptions on the bodies \( K, L \) in Lemma 2.3 can be weakened.

**Corollary 2.4 ([22, corollary 3.23]).** Let \( k \in \mathbb{N} \) with \( 0 < k < n \), and let \( K \) and \( D \) be origin symmetric star bodies in \( \mathbb{R}^n \). Suppose that \( D \) is \((k-1)\)-smooth if \( k \) is odd and that \( D \) is \( k \)-smooth if \( k \) is even. Also suppose that \( \| \cdot \|^{-k}_K \) is a positive definite distribution, and let \( \mu_0 \) be the finite Borel measure on \( S^{n-1} \) that corresponds to \( \| \cdot \|^{-k}_K \) by [22, corollary 2.26].
Then
\[ \int_{S^{n-1}} (\| \cdot \|_D^{-n+k})^k (\theta) d\mu_0(\theta) = \int_{S^{n-1}} \|\theta\|_K^{-k} \|\theta\|_D^{-n+k} d\theta. \]

We will use the following version of the above corollary.

**Corollary 2.5.** Let \( k \in \mathbb{N} \) with \( 0 < k < n \). Let \( f \) and \( g \) be two even functions on \( \mathbb{R}^n \), continuous on \( S^{n-1} \) and homogeneous of degree \( -k \) and \( -n+k \), respectively. Suppose that the restriction of \( g \) to \( S^{n-1} \) is \((k-1)\)-smooth if \( k \) is odd and \( k \)-smooth if \( k \) is even. Also suppose that \( f \) represents a positive definite distribution on \( \mathbb{R}^n \) and let \( \mu_0 \) be the finite Borel measure on \( S^{n-1} \) that corresponds to \( f \) by [22, corollary 2.26]. Then
\[ \int_{S^{n-1}} \hat{g}(\theta) d\mu_0 = \int_{S^{n-1}} g(\theta) f(\theta) d\theta. \]

Let \( 0 < k < n \) and let \( H \) be an \((n-k)\)-dimensional subspace of \( \mathbb{R}^n \). Fix an orthonormal basis \( e_1, \ldots, e_k \) in the orthogonal subspace \( H^\perp \). For a star body \( K \) in \( \mathbb{R}^n \), define the \((n-k)\)-dimensional parallel section function \( A_{K,H}(u) \) as a function on \( \mathbb{R}^k \) such that for \( u \in \mathbb{R}^k \)
\[
A_{K,H}(u) = \text{Vol}_{n-k}(K \cap \{ H + u_1 e_1 + \cdots + u_k e_k \})
= \int_{\{ x \in \mathbb{R}^n : (x,e_i)=u_i, \ldots, (x,e_i)=u_k \}} \chi(\|x\|_K) dx,
\]
where \( \chi \) is the indicator function of the interval \([0,1]\). If \( K \) is infinitely smooth, the function \( A_{K,H} \) is infinitely differentiable at the origin. This fact is true in a more general sense. A family of functions is said to be **uniformly differentiable** if convergence in the limits defining the derivatives is uniform with respect to this family of functions.

**Lemma 2.6 ([22, lemma 2.4]).** Let \( K \) be an \( m \)-smooth origin-symmetric convex body in \( \mathbb{R}^n \), where \( m \in \mathbb{N} \cup \{0\} \). Then for all \( \xi \in S^{n-1} \) the parallel section functions \( A_{K,H_\xi} \) are uniformly with respect to \( \xi \) \( m \)-times continuously differentiable in some neighbourhood \( U \) of zero. Moreover, for every fixed \( t \in U \), the derivative \( A_{K,H_\xi}^{(m)} \) is a continuous function of the variable \( \xi \in S^{n-1} \).

We shall make use of the following fact:

**Lemma 2.7 ([19, theorem 2]).** Let \( K \) be an infinitely smooth origin symmetric star body in \( \mathbb{R}^n \) and \( 0 < k < n \). Then for every \((n-k)\)-dimensional subspace \( H \) of \( \mathbb{R}^n \) and for every \( m \in \mathbb{N} \cup \{0\}, m < (n-k)/2 \),
\[
\Delta^m A_{K,H}(0) = \frac{(-1)^m}{(2\pi)^k (n - 2m - k)} \int_{S^{n-1} \cap H^\perp} \chi(\|x\|_K^{-n+2m+k})^k(\xi) d\xi,
\]
where \( \Delta \) denotes the Laplacian on \( \mathbb{R}^k \).

---

1 The parallel section function in this generality was first introduced in [19], where the author gives a characterization of several classes of generalized intersection bodies in the language of functional analysis. The parallel section function and its further generalizations proved useful for the solution of virtually every generalization of the Busemann–Petty problem, e.g. [26, 37]. In particular, the parallel section function on \( \mathbb{R}^n \) was used in the Fourier analytic proof of the original Busemann–Petty problem [12].
2.2.3. Approximation results

One can approximate any convex body $K$ in $\mathbb{R}^n$ from inside or from outside in the radial metric

$$\rho(K, L) := \max_{x \in S^{n-1}} |\rho_K(x) - \rho_L(x)|$$

by a sequence of infinitely smooth convex bodies with the same symmetries as $K$, see [31, theorem 3.3.1]. In particular, any $R_\theta$-invariant convex body in $\mathbb{R}^{2n}$ can be approximated by infinitely smooth $R_\theta$-invariant convex bodies. Any $k$-smooth star body $K$ can be approximated by a sequence of infinitely smooth star bodies $K_m$ so that the radial functions $\rho_{K_m}$ converge to $\rho_K$ in the metric of the space $C^k(S^{n-1})$, see [22, p. 27], preserving the symmetries of $K$ as well.

A convex body can also be approximated in the radial metric by convex bodies with strictly positive curvature. We shall use the following lemma from [22]:

**Lemma 2.8** ([22, lemma 4.10]). Let $1 \leq k < n$. Suppose that $D$ is an origin symmetric convex body in $\mathbb{R}^n$ that is not a $k$-intersection body. Then there exists a sequence $D_m$ of origin symmetric convex bodies so that $D_m$ converges to $D$ in the radial metric, each $D_m$ is infinitely smooth, has strictly positive curvature and each $D_m$ is not a $k$-intersection body.

Moreover, if $D$ is $R_\theta$-invariant, one can choose $D_m$ with the same property. The proof of the above lemma is based on the following fact, which allows for more general approximation results.

**Lemma 2.9** ([22, lemma 3.11, (i)]). Suppose that $p > -n$ and let $f_k$, $k \in \mathbb{N}$, and $f$ be even continuous functions on the sphere $S^{n-1}$ so that $f_k \rightarrow f$ in $C(S^{n-1})$. Then for every even test function $\phi$

$$\lim_{k \rightarrow \infty} \langle (f_k(\theta)r^p)\wedge, \phi \rangle = \langle (f(\theta)r^p)\wedge, \phi \rangle.$$

2.2.4. The Fourier and Radon transforms of $R_\theta$-invariant functions

The following simple observation is crucial for the application of Fourier methods to sections of convex bodies in the complex case. It translates the $R_\theta$-invariance of a body $K$ into a certain invariance of the Fourier transform of its Minkowski functional raised to some power. We reproduce the proof of this observation here for its simple and insightful nature.

**Lemma 2.10** ([24, lemma 3]). Suppose that $K$ is an infinitely smooth $R_\theta$-invariant star body in $\mathbb{R}^{2n}$. Then for every $0 < p < 2n$ and $\xi \in S^{2n-1}$ the Fourier transform of the distribution $\|x\|_K^{-p}$ is a constant function on $S^{2n-1} \cap H_{\xi}^\perp$.

**Proof.** As mentioned above the Fourier transform of $\|x\|_K^{-p}$ is an infinitely smooth function on $\mathbb{R}^n \setminus \{0\}$. Since the function $\|x\|_K$ is $R_\theta$-invariant, by the connection between the Fourier transform of distributions and linear transformations, the Fourier transform of $\|x\|_K^{-p}$ is also $R_\theta$-invariant. Recall, from the introduction, that the two-dimensional space $H_{\xi}^\perp$ is spanned by two vectors $\xi$ and $\xi_\perp$. Every vector in $S^{2n-1} \cap H_{\xi}^\perp$ is the image of $\xi$ under one of the coordinate-wise rotations $R_\theta$, so the Fourier transform of $\|x\|_K^{-p}$ is a constant function on $S^{2n-1} \cap H_{\xi}^\perp$.

Denote by $C_\theta(S^{2n-1})$ the space of $R_\theta$-invariant continuous functions on the unit sphere $S^{2n-1}$: continuous real-valued functions $f$ satisfying $f(\xi) = f(R_\theta \xi)$ for any $\xi \in S^{2n-1}$ and
The Busemann–Petty problem in the complex hyperbolic space  163

any \( \theta \in [0, 2\pi] \). The \textit{complex spherical Radon transform}, introduced in [25], is an operator \( \mathcal{R}_c : C_\phi(S^{2n-1}) \to C_\phi(S^{2n-1}) \) defined by

\[
\mathcal{R}_c f(\xi) = \int_{S^{2n-1} \cap H_\xi} f(x) \, dx.
\]

To derive a formula for the volume of the \((2n - 2)\)-dimensional section of an \( R_\phi \)-invariant star body in \( \mathbb{R}^{2n} \) contained in the unit ball by a hyperplane \( H_\xi \) with respect to the volume element in \( \mathbb{H}_n^0 \), we use the following recently established connection between the complex spherical Radon transform and the Fourier transform.

\textbf{Lemma 2.11 ([25, lemma 4])}. Let \( f \in C_\phi(S^{2n-1}) \) be an even function. Extend \( f \) to a homogeneous function of degree \(-2n+2\), \( f \cdot r^{-2n+2} \), then the Fourier transform of this extension is a homogeneous function of degree \(-2\) on \( \mathbb{R}^{2n} \), whose restriction to the unit sphere is continuous. Moreover, for every \( \xi \in S^{2n-1} \)

\[
\mathcal{R}_c f(\xi) = \frac{1}{2\pi} (f \cdot r^{-2n+2})^\wedge(\xi).
\]

\textbf{Lemma 2.12}. Let \( K \) be a continuous \( R_\phi \)-invariant star body in \( \mathbb{R}^{2n} \) with \( n \geq 2 \) contained in the unit ball. For \( \xi \in S^{2n-1} \), we have

\[
\text{HVol}_{2n-2}(K \cap H_\xi) = \frac{8^{n-1}}{2\pi} \left( |x|^{-2n+2} \int_0^{\|x\|^2/k} \frac{r^{2n-3}}{(1 - r^2)^n} \, dr \right)^\wedge(\xi).
\]

**Proof.** We compute:

\[
\text{HVol}_{2n-2}(K \cap H_\xi) = \int_{K \cap H_\xi} d\mu_{n-1}
\]

\[= 8^{n-1} \int_{S^{2n-1} \cap H_\xi} |x|^{-2n+2} \int_0^{\|x\|^2/k} \frac{r^{2n-3}}{(1 - r^2)^n} \, dr \, dx.
\]

Using that \( |x| = 1 \), we rewrite the above integral as:

\[
\text{HVol}_{2n-2}(K \cap H_\xi) = 8^{n-1} \int_{S^{2n-1} \cap H_\xi} |x|^{-2n+2} \int_0^{\|x\|^2/k} \frac{r^{2n-3}}{(1 - r^2)^n} \, dr \, dx.
\]

The function under the first integral sign is a homogeneous function of degree \(-2n+2\) and thus by Lemma 2.11 we obtain:

\[
\text{HVol}_{2n-2}(K \cap H_\xi) = \frac{8^{n-1}}{2\pi} \left( |x|^{-2n+2} \int_0^{\|x\|^2/k} \frac{r^{2n-3}}{(1 - r^2)^n} \, dr \right)^\wedge(\xi),
\]

as claimed.

Note that since \( h \)-convex bodies are star bodies, Lemma 2.12 gives a Fourier analytic expression for the volumes of sections of \( R_\phi \)-invariant \( h \)-convex bodies.

3. **Connection with the distribution \( \frac{\|x\|^2}{1 - \left(\frac{|x|}{k}\right)^2} \)**

We now turn to our problem. First we construct counterexamples to the Busemann–Petty problem in \( \mathbb{H}_n^c \) for \( n \geq 4 \). We use the same idea as in [33], namely that any Riemannian space is locally close to being Euclidean.
THEOREM 3.1. There exist $R_\theta$-invariant $h$-convex bodies $K$ and $L$ in $\mathbb{R}^{2n}$ with $n \geq 4$ satisfying
\[ \text{HVol}_{2n-2}(K \cap H_\xi) < \text{HVol}_{2n-2}(L \cap H_\xi) \]
for every $\xi \in S^{2n-1}$, but
\[ \text{HVol}_{2n}(K) > \text{HVol}_{2n}(L). \]

Proof. Let $K$ and $L$ be $R_\theta$-invariant convex bodies in $\mathbb{R}^{2n}$ with $n \geq 4$ that provide a counterexample to the complex version of the Busemann-Petty problem, see [24], i.e.
\[ \text{Vol}_{2n-2}(K \cap H_\xi) \leq \text{Vol}_{2n-2}(L \cap H_\xi) \quad (3\cdot1) \]
for every $\xi \in S^{2n-1}$, but
\[ \text{Vol}_{2n}(K) > \text{Vol}_{2n}(L). \quad (3\cdot2) \]

Recall that in this construction the body $L$ is infinitely smooth and has strictly positive curvature, and the body $K$ is obtained from the body $L$ by small perturbations. The fact that the body $L$ has strictly positive curvature guarantees that for a small enough perturbation the body $K$ is convex. In order to construct $h$-convex bodies from these bodies $K$ and $L$, we need both bodies to have strictly positive curvature. This can be achieved as follows. Note that a dilation by a positive factor is an automorphism of $R_\theta$-invariant convex bodies in $\mathbb{R}^{2n}$. If the body $K$, obtained from $L$, does not have strictly positive curvature, then one can dilate the body $L$ by a positive factor $\beta$ less than one to ensure that small perturbations of $\beta L$ have strictly positive curvature. Thus we can assume that both bodies $L$ and $K$ are infinitely smooth $R_\theta$-invariant bodies with strictly positive curvature.

Since inequality (3\cdot2) is strict, we can dilate the body $L$ by a positive factor greater than one, to make inequality (3\cdot1) strict as well. Furthermore, there is an $\epsilon > 0$ so that
\[ (1 + \epsilon)\text{Vol}_{2n-2}(K \cap H_\xi) < \text{Vol}_{2n-2}(L \cap H_\xi) \]
for every $\xi \in S^{2n-1}$, but
\[ \text{Vol}_{2n}(K) > (1 + \epsilon)\text{Vol}_{2n}(L). \]

Here we are using the fact that $\xi \mapsto A_{K,H_\xi}(0)$ is a continuous function on a compact set.

Any dilation of bodies $K$ and $L$ by a factor $\alpha > 0$ will also provide a counterexample. Choose $\alpha$ so small that both bodies $\alpha K$ and $\alpha L$ are $h$-convex, see Lemma 2\cdot2, and lie in the ball of radius $s$ that satisfies the inequality
\[ 1 < \frac{1}{(1 - s^2)^{n+1}} \leq 1 + \epsilon. \]

We use the same letters for the dilated bodies. Then for the volumes of bodies $K$ and $L$ with respect to the volume element in $\mathbb{H}_C^n$ we obtain:
\[ \text{HVol}_{2n}(L) = \int_L d\mu_n = \int_L \frac{dx_{2n}}{(1 - |x|^2)^{n+1}} \leq 8^n (1 + \epsilon) \int_L dx_{2n} = 8^n (1 + \epsilon) \text{Vol}_{2n}(L) \]
The Busemann–Petty problem in the complex hyperbolic space

\[
\begin{align*}
&< 8^n \text{Vol}_{2n}(K) \\
&= 8^n \int_K dx_{2n} \\
&\leq 8^n \int_K \frac{dx_{2n}}{(1 - |x|^2)^{n+1}} \\
&= \int_K d\mu_n \\
&= \text{HVol}_{2n}(K).
\end{align*}
\]

Similarly, for the volumes of sections with respect to the volume element in \( \mathbb{H}^n_C \), we have:

\[
\begin{align*}
\text{HVol}_{2n-2}(K \cap H_\xi) &= 8^{n-1} \int_{K \cap H_\xi} \frac{dx_{2n-2}}{(1 - |x|^2)^n} \\
&\leq 8^{n-1} (1 + \epsilon) \int_{K \cap H_\xi} dx_{2n-2} \\
&= 8^{n-1} (1 + \epsilon) \text{Vol}_{2n-2}(K \cap H_\xi) \\
&< 8^{n-1} \text{Vol}_{2n-2}(L \cap H_\xi) \\
&= 8^{n-1} \int_{L \cap H_\xi} dx_{2n-2} \\
&\leq 8^{n-1} \int_{L \cap H_\xi} \frac{dx_{2n-2}}{(1 - |x|^2)^n} \\
&= \text{HVol}_{2n-2}(L \cap H_\xi).
\end{align*}
\]

Thus the dilated bodies \( K \) and \( L \) provide the claimed counterexample.

In the following two propositions we describe the connection between the Busemann–Petty problem in \( \mathbb{H}^n_C \) and the distribution \( \|x\|^{-2}_K/(1 - (|x|/\|x\|_K)^2) \).

**Proposition 3.2.** Let \( K \) be an \( R_\theta \)-invariant star body in \( \mathbb{R}^{2n} \), contained in the open unit ball, such that \( \|x\|^{-2}_K/(1 - (|x|/\|x\|_K)^2) \) is a positive definite distribution on \( \mathbb{R}^{2n} \). And let \( L \) be an \( R_\theta \)-invariant star body in \( \mathbb{R}^{2n} \), contained in the open unit ball, so that

\[
\text{HVol}_{2n-2}(K \cap H_\xi) \leq \text{HVol}_{2n-2}(L \cap H_\xi)
\]

for every \( \xi \in S^{2n-1} \). Then

\[
\text{HVol}_{2n}(K) \leq \text{HVol}_{2n}(L).
\]

**Proof.** We may assume that the star body \( K \) is infinitely smooth. Indeed, one can argue that an \( R_\theta \)-invariant star body \( K \), whose distribution \( \|x\|^{-2}_K/(1 - (|x|/\|x\|_K)^2) \) is positive definite (or not positive definite), can be approximated by a sequence \( K_m \) of infinitely smooth \( R_\theta \)-invariant star bodies, whose distributions \( \|x\|^{-2}_{K_m}/(1 - (|x|/\|x\|_{K_m})^2) \) are all positive definite (or all not positive definite). This follows by essentially the same argument as found in the proofs of Lemmas 2.8 and 2.9.

Observe that the function \( r^2/(1 - r^2) \) is an increasing function on the interval \((0, 1)\). We use this observation to estimate the following expression:
\[
\frac{a^2}{1-a^2} \int_a^b \frac{r^{2n-3}}{(1-r^2)^n} dr = \frac{a^2}{1-a^2} \int_a^b \frac{r^{2n-1}}{(1-r^2)^{n+1}} \frac{1-r^2}{r^2} dr \\
= \int_a^b \frac{r^{2n-1}}{(1-r^2)^{n+1}} \frac{a^2}{1-a^2} \frac{1-r^2}{r^2} dr \\
\leq \int_a^b \frac{r^{2n-1}}{(1-r^2)^{n+1}} dr,
\]

where \(a, b\) are in \((0, 1)\). Observe that the above inequality is true in case \(a \leq b\) as well as in case \(b \leq a\). Integrating both sides in the above inequality over the unit sphere \(S^{2n-1}\) with \(a = \|x\|^{-1}_K\) and \(b = \|x\|^{-1}_L\) we obtain:

\[
\int_{S^{2n-1}} \frac{\|x\|^{-2}_K}{1-\|x\|^{-2}_K} \int_0^{\|x\|^{-1}_K} \frac{r^{2n-3}}{(1-r^2)^n} dr dx \leq \int_{S^{2n-1}} \frac{\|x\|^{-2}_K}{1-\|x\|^{-2}_K} \int_0^{\|x\|^{-1}_K} \frac{r^{2n-3}}{(1-r^2)^n} dr dx. \tag{3.3}
\]

Next we show that the left-hand side in the above expression is positive. This amounts to showing that

\[
\int_{S^{2n-1}} \frac{\|x\|^{-2}_K}{1-\|x\|^{-2}_K} \int_0^{\|x\|^{-1}_K} \frac{r^{2n-3}}{(1-r^2)^n} dr dx \leq \int_{S^{2n-1}} \frac{\|x\|^{-2}_K}{1-\|x\|^{-2}_K} \int_0^{\|x\|^{-1}_K} \frac{r^{2n-3}}{(1-r^2)^n} dr dx.
\]

Indeed, let \(\mu_0\) be the measure corresponding to the Fourier transform of the positive definite distribution \(\|x\|^{-2}_K/(1 - (\|x\|/\|x\|_K)^2)\), then using Corollary 2.5 we obtain:

\[
\int_{S^{2n-1}} \frac{\|x\|^{-2}_K}{1-\|x\|^{-2}_K} \int_0^{\|x\|^{-1}_K} \frac{r^{2n-3}}{(1-r^2)^n} dr dx \\
= \int_{S^{2n-1}} \left( \frac{\|x\|^{-2}_K}{1 - (\|x\|/\|x\|_K)^2} \right) \left( |x|^{-2n+2} \int_0^{\|x\|^{-1}_K} \frac{r^{2n-3}}{(1-r^2)^n} dr \right) dx \\
= \int_{S^{2n-1}} |x|^{-2n+2} \int_0^{\|x\|^{-1}_K} \frac{r^{2n-3}}{(1-r^2)^n} dr \left( \xi \right) d\mu_0(\xi) \\
= \frac{2\pi}{8^{n-1}} \int_{S^{2n-1}} \text{HVol}_{2n-2}(K \cap H_\xi) d\mu_0(\xi),
\]

and the inequality for sections gives:

\[
\int_{S^{2n-1}} \frac{\|x\|^{-2}_K}{1-\|x\|^{-2}_K} \int_0^{\|x\|^{-1}_K} \frac{r^{2n-3}}{(1-r^2)^n} dr dx \\
\leq \frac{2\pi}{8^{n-1}} \int_{S^{2n-1}} \text{HVol}_{2n-2}(L \cap H_\xi) d\mu_0(\xi) \\
= \int_{S^{2n-1}} \left( |x|^{-2n+2} \int_0^{\|x\|^{-1}_L} \frac{r^{2n-3}}{(1-r^2)^n} dr \right) \left( \xi \right) d\mu_0(\xi) \\
= \int_{S^{2n-1}} \frac{\|x\|^{2}_K}{1-\|x\|^{-2}_K} \int_0^{\|x\|^{-1}_L} \frac{r^{2n-3}}{(1-r^2)^n} dr dx,
\]
This implies that the right-hand side in equation (3.3) is positive as well, which, in turn, shows
\[
\int_{S^{2n-1}}\int_0^\|x\|^1_k \frac{r^{2n-1}}{(1-r^2)^n+1} dr dx \leq \int_{S^{2n-1}}\int_0^\|x\|^1_L \frac{r^{2n-1}}{(1-r^2)^n+1} dr dx.
\]

That is,
\[
HV ol_{2n}(K) \leq HV ol_{2n}(L).
\]

This completes the proof.

The proof of the next result is based on a standard perturbation argument, e.g. see [18, theorem 2].

**PROPOSITION 3.3.** Suppose there is an infinitely smooth complex convex body \(K\) in \(B^n\) with strictly positive curvature so that \(\|x\|^2_k/(1-\|x\|^2_k)^2\) is not a positive definite distribution on \(\mathbb{R}^{2n}\). Then one can perturb the body \(K\) to construct another infinitely smooth complex convex body \(L\) with strictly positive curvature, so that for every \(\xi \in S^{2n-1}\)
\[
HV ol_{2n-2}(L \cap H_\xi) \leq HV ol_{2n-2}(K \cap H_\xi),
\]
but
\[
HV ol_{2n}(L) > HV ol_{2n}(K).
\]

**Proof.** It follows from our assumptions that the Fourier transform of \(\|x\|^2_k/(1-\|x\|^2_k)^2\) is negative on some open subset \(\Omega\) of the sphere \(S^{2n-1}\). \(R_\theta\)-invariance of the body \(K\) implies \(R_\theta\)-invariance of the set \(\Omega\). Choose a smooth non-negative \(R_\theta\)-invariant function \(f\) on \(S^{2n-1}\) whose support is contained in \(\Omega\) and extend \(f\) to an \(R_\theta\)-invariant homogeneous function \(f(x/|x|)|x|^{-2}\) of degree \(-2\) on \(\mathbb{R}^{2n}\). The Fourier transform of this extension is an \(R_\theta\)-invariant infinitely smooth function on \(\mathbb{R}^{2n} \setminus \{0\}\), homogeneous of degree \(-2n+2\):
\[
( f \left( \frac{x}{|x|} \right) |x|^{-2} ) (y) = g \left( \frac{y}{|y|} \right) |y|^{-2n+2}
\]
with \(g \in C^\infty(S^{2n-1})\). Since \(f\) is \(R_\theta\)-invariant, so is \(g\). Define an origin symmetric body \(L\) in \(B^n\) by
\[
|x|^{-2n+2} \int_0^{\|x\|^1_L} r^{2n-3} (1-r^2)^n dr = |x|^{-2n+2} \int_0^{\|x\|^1_K} r^{2n-3} (1-r^2)^n dr - \epsilon g \left( \frac{x}{|x|} \right) |x|^{-2n+2}
\]
for some \(\epsilon > 0\). Since the body \(K\) has strictly positive curvature, by a similar argument as in [36, proposition 2], for small enough \(\epsilon\), the body \(L\) is convex. Moreover, we can assume that the body \(L\) has strictly positive curvature as well. Indeed, let \(\bar{\epsilon}\) be so that the body \(L\) is convex, then for the choice of \(\bar{\epsilon}/2\) the body \(L\) will be strictly convex. From the \(R_\theta\)-invariance of \(K\) and \(g\) follows the \(R_\theta\)-invariance of \(L\). Thus \(L\) is a complex convex body in \(B^n\). Using Lemma 2.12, we compute:
\[
HV ol_{2n-2}(L \cap H_\xi) = \frac{8^{n-1}}{2\pi} \left( |x|^{-2n+2} \int_0^{\|x\|^1_L} r^{2n-3} (1-r^2)^n dr \right)^\wedge (\xi) + \epsilon (2\pi)^{2n} f \left( \frac{x}{|x|} \right) |x|^{-2}
\]
\[
\begin{align*}
\leq & \frac{8^{n-1}}{2\pi} \left( |x|^{-2n+2} \int_0^{\|x\|_K} r^{2n-3} \frac{dr}{(1-r^2)^n} \right)^{\wedge} \\
= & \text{HVol}_{2n-2}(K \cap H_\xi).
\end{align*}
\]

As in the proof of the previous lemma, to complete the proof it is enough to show the following:

\[
(2\pi)^{2n} \int_{S^{2n-1}} \frac{\|x\|_K^{-2}}{1-\|x\|_K^2} \left( \frac{\|x\|_K^{-2}}{1-\|x\|_K^2} \right)^{\wedge} (\xi) \left( |x|^{-2n+2} \int_0^{\|x\|_K} r^{2n-3} \frac{dr}{(1-r^2)^n} \right)^{\wedge} (\xi) d\xi
\]

\[
= \left( \frac{\|x\|_K^{-2}}{1-\|x\|_K^2} \right)^{\wedge} (\xi) \left( |x|^{-2n+2} \int_0^{\|x\|_K} r^{2n-3} \frac{dr}{(1-r^2)^n} \right)^{\wedge} (\xi) d\xi
\]

\[
- \epsilon (2\pi)^{2n} \int_{S^{2n-1}} \left( \frac{\|x\|_K^{-2}}{1-\left( |x|/\|x\|_K \right)^2} \right)^{\wedge} (\xi) f(\xi) d\xi
\]

\[
> \left( \frac{\|x\|_K^{-2}}{1-\left( |x|/\|x\|_K \right)^2} \right)^{\wedge} (\xi) \left( |x|^{-2n+2} \int_0^{\|x\|_K} r^{2n-3} \frac{dr}{(1-r^2)^n} \right)^{\wedge} (\xi) d\xi
\]

\[
= (2\pi)^{2n} \int_{S^{2n-1}} \frac{\|x\|_K^{-2}}{1-\|x\|_K^2} \left( \frac{\|x\|_K^{-1}}{1-\|x\|_K^2} \right)^{\wedge} r^{2n-3} \frac{dr}{(1-r^2)^n} dx.
\]

This inequality, by means of equation (3-3), results in

\[
\text{HVol}_{2n}(L) > \text{HVol}_{2n}(K),
\]

proving the claim.

4. Solution of the problem

In view of Theorem 3.1, we only have to find out what happens in dimensions one, two and three. The case of the complex dimension one is trivial. Indeed, \( R_\theta \)-invariant \( h \)-convex bodies in \( \mathbb{C} = \mathbb{R}^2 \) are closed discs around the origin. Moreover, we see that the class of origin symmetric convex bodies in the complex hyperbolic 1-space, \( \mathbb{H}^1_\mathbb{C} \), corresponds to the class of origin symmetric convex bodies in \( \mathbb{C} \) contained in the open unit ball \( B^1 \). The parallel section function, defined earlier, will help us to find solutions in dimensions two and three.

In dimension \( 2n \), Lemma 2.7 with \( H = H_\xi, \xi \in S^{2n-1} \), in which case \( k = 2 \), reads as follows: Let \( K \) be an infinitely smooth origin-symmetric star body in \( \mathbb{R}^{2n} \), then for
\[ m \in \mathbb{N} \cup \{0\}, m < n - 1 \]
\[
\Delta^m A_{K, H_t}(0) = \frac{(-1)^m}{(2\pi)^2(2n - 2m - 2)} \int_{S^{2n-1} \cap H_t^+} (\|x\|^2_{K} - 2m + 2)^\wedge (v) dv.
\]

Since the above integral is taken over the region \( S^{2n-1} \cap H_t^+ \), by Lemma 2.10, it follows that
\[
\Delta^m A_{K, H_t}(0) = \frac{(-1)^m}{2\pi(2n - 2m - 2)} (\|x\|^2_{K} - 2m + 2)^\wedge (\xi). \tag{4.1}
\]

In complex dimension two there is only one choice for \( m \), namely \( m = 0 \), and equation (4.1) becomes
\[
A_{K, H_t}(0) = \frac{1}{4\pi} (\|x\|^2_{K})^\wedge (\xi).
\]

In complex dimension three, evaluating equation (4.1) for \( m = 1 \), we obtain
\[
\Delta A_{K, H_t}(0) = -\frac{1}{4\pi} (\|x\|^2_{K})^\wedge (\xi).
\]

**Lemma 4.1.** For any \( R_\theta \)-invariant star body \( K \) in \( \mathbb{R}^4 \), contained in the unit ball, the distribution \( \|x\|^2_{K}/(1 - (|x|/\|x\|)^2) \) is positive definite.

**Proof.** As in the proof of Proposition 3.2, we may assume that the star body \( K \) is infinitely smooth. Define another body \( M \) by
\[
\|x\|^2_{M} = \frac{\|x\|^2_{K}}{1 - \left(\frac{|x|}{\|x\|_K}\right)^2}.
\]

Since \( K \) is an \( R_\theta \)-invariant star body, so is \( M \). Note also that the denominator in the defining expression for \( M \) is never zero, since the body \( K \) is contained in the open unit ball. This implies that the body \( M \) is also infinitely smooth. Hence
\[
A_{M, H_t}(0) = \frac{1}{4\pi} (\|x\|^2_{M})^\wedge (\xi),
\]
and consequently \( \|x\|^2_{M} \) is positive definite. Thus for any infinitely smooth \( R_\theta \)-invariant star body \( K \) the distribution \( \|x\|^2_{K}/(1 - (|x|/\|x\|_K)^2) \) is positive definite. And therefore, this is true for any \( R_\theta \)-invariant star body.

**Lemma 4.2.** There is an infinitely smooth \( R_\theta \)-invariant convex body \( K \) in \( \mathbb{R}^6 \) contained in the unit ball for which the distribution \( \|x\|^2_{K}/1 - (|x|/\|x\|_K)^2 \) is not positive definite.

**Proof.** For an element \( \xi \) of \( \mathbb{R}^6 \), \( \xi = (\xi_{11}, \xi_{12}, \xi_{21}, \xi_{22}, \xi_{31}, \xi_{32}) \), denote by \( \xi_3 = (\xi_{31}, \xi_{32}) \) and by \( \xi = (\xi_{11}, \xi_{12}, \xi_{21}, \xi_{22}) \), then \( \xi = (\xi, \xi_3) \). We will work with the following map, written in polar coordinates as
\[
(r, \theta) \mapsto \left(\sqrt{\frac{r^2}{1 - r^2}}, \theta\right). \tag{4.2}
\]

Note that this map, restricted to the two-dimensional plane \( xy \), takes the line \( x = 1/a \) to the hyperbola \( (a^2 - 1)x^2 - y^2 = 1 \), provided that \( a^2 - 1 > 0 \), and it takes the ellipse \( x^2 + (1 + b^2)y^2 = 1 \) to the line \( y = 1/b \). Indeed, writing the equation of the line \( x = 1/a \) in polar coordinates, we obtain \( r^2/(1 - r^2) = 1/(a^2 \cos^2 \theta - 1) \), and so the image of the line
is \( r^2 = 1/(a^2 \cos^2 \theta - 1) \). Similarly to find the image of the ellipse, we write its equation in polar coordinates: \( r^2 = 1/(1 + b^2 \sin^2 \theta) \). Then \( r^2/(1 - r^2) = 1/(b^2 \sin^2 \theta) \) and hence the image of the ellipse is \( r = 1/b \sin \theta \). Denote the equation of the elliptic arc above the \( x \)-axis by \( e(x) = \sqrt{(1 - x^2)/(1 + b^2)} \), and the equation of the hyperbolic arc to the right of the \( y \)-axis by \( h(y) = \sqrt{(1 + y^2)/(a^2 - 1)} \). Now define a convex body \( K \) in \( \mathbb{R}^6 \) by

\[
K = \left\{ \xi \in \mathbb{R}^6 : |\tilde{\xi}| \leq 1/a \text{ and } |\xi_3| \leq e(|\tilde{\xi}|) \right\}.
\]

We restrict the values of \( b \) to be strictly greater then one, this ensures that the body \( K \) is contained in the unit ball. As before, define a star body \( M \) by

\[
\|x\|_M^2 = \frac{\|x\|_K^2}{1 - \left( \frac{|x|}{\|x\|_K} \right)^2}.
\]

The body \( M \) is an image of the body \( K \) under the map (4.2) and hence it can be described as

\[
M = \left\{ \xi \in \mathbb{R}^6 : |\tilde{\xi}| \leq h(|\tilde{\xi}|) \text{ with } |\xi_3| \leq 1/b \right\}.
\]

The bodies \( K \) and \( M \) we constructed are not infinitely smooth. However, since we can approximate the body \( K \) in the radial metric by a sequence of infinitely smooth convex bodies, there is an infinitely smooth convex body \( K' \) that differs from \( K \) only in an arbitrary small neighborhood of the boundary of \( K \). This modification of \( K \) will make the body \( M \) infinitely smooth as well. We use the same letters \( K \) and \( M \) for these modified bodies. By construction both bodies are \( R_\theta \)-invariant.

Let \( x = (\tilde{x}, x_3) \in M \) with \( x_3 \neq (0, 0) \). Choose \( \xi \in S^5 \) in the direction of \( x_3 \). Fix an orthonormal basis \( \{e_1, e_2\} \) for \( H_\xi \). For \( u \in \mathbb{R}^2 \) with \( |u| > 1/b \), \( A_{M,H_\xi}(u) = 0 \), and otherwise

\[
A_{M,H_\xi}(u) = \text{Vol}_4(M \cap \{H_\xi + u_1 e_1 + u_2 e_2\})
\]

\[
= \int_{\{x \in \mathbb{R}^2 : (x,e_1) = u_1, (x,e_2) = u_2\}} \chi(\|x\|_M) dx
\]

\[
= \int_{S^3} \int_0^{r_0} r^3 dr d\theta
\]

\[
= |S^3| \frac{h(|u|)^4}{4}
\]

\[
= \frac{\pi^2}{2} h(|u|)^4,
\]

where \( |S^{n-1}| \) stands for the surface area of the unit sphere \( S^{n-1} \) in \( \mathbb{R}^n : |S^{n-1}| = 2\pi^{\frac{n}{2}}/\Gamma(n/2) \). Setting \( a = 2 \), we get

\[
A_{M,H_\xi}(u) = \frac{\pi^2}{2} \left( 1 + |u|^2 \right)^2 / 3, \text{ and consequently } \Delta A_{M,H_\xi}(u) = \frac{4\pi^2}{9} (1 + 2|u|^2).
\]

Since \( M \) is infinitely smooth we have

\[
\left( \|x\|_K^2/(1 - (|x|/\|x\|_K)^2) \right)^\wedge(\xi) = -4\pi \Delta A_{M,H_\xi}(0) = -16\pi^3/9.
\]

This shows that \( (\|x\|_K^2/(1 - (|x|/\|x\|_K)^2)) \wedge(\xi) = (\|x\|_M^2)^\wedge(\xi) \) is negative for some direction \( \xi \).
Now we are ready to prove the main result of this paper:

**Theorem 4.3.** The answer to the Busemann–Petty problem in the complex hyperbolic $n$-space, $\mathbb{H}^n_C$, is affirmative for $n \leq 2$ and negative for $n \geq 3$.

**Proof.** By Lemma 4-1, the distribution $\|x\|^2/(1 - (\|x\|/\|x\|_K)^2)$ is positive definite for any $R_0$-invariant $h$-convex body $K$ in $\mathbb{R}^4$, as any such body is a star body. The affirmative answer for $n = 2$ now follows from Proposition 3-2.

For $n = 3$, by Lemma 4-2, there is an infinitely smooth complex convex body $K$ in $\mathbb{R}^6$ contained in the unit ball for which the distribution $\|x\|^2/(1 - (\|x\|/\|x\|_K)^2)$ is not positive definite. Observe that we can assume that the body $K$ has strictly positive curvature by setting $\|x\|^{-1}_K = \|x\|^{-1}_K + \epsilon |x|$ for some $\epsilon > 0$ small. Proposition 3-3 now gives another infinitely smooth complex convex body $L$ with strictly positive curvature that along with the body $K$ satisfies the required inequalities. Bodies $K$ and $L$ might not be $h$-convex. However, since both bodies have strictly positive curvature, we can scale them by a small factor $\alpha$ so that the scaled bodies $\alpha K$ and $\alpha L$ are $h$-convex, see Lemma 2-2. Moreover, scaled bodies $\alpha K$ and $\alpha L$ satisfy same volume inequalities as bodies $K$ and $L$, providing a counterexample in dimension three.

For $n \geq 4$ the negative answer was provided in Theorem 3-1.

**Acknowledgments.** The author wishes to thank Alexander Koldobsky for proposing to work on this problem as well as for encouragement and many suggestions, to Vladislav Yaskin for many useful discussions, and to the anonymous referee for the thorough review of the paper and many constructive suggestions.

**References**

[1] J. Abardia and A. Bernig. Projection bodies in complex vector spaces. *Adv. Math.* **227** 2 (2011), 830–846.

[2] J. Abardia and E. Gallego. Convexity on complex hyperbolic space. arXiv:1003.4667.

[3] K. Ball. Some remarks on the geometry of convex sets. In *Geometric aspects of functional analysis* (1986/87). Lecture Notes in Math. vol. 1317 (Springer, Berlin, 1988), pp. 224–231.

[4] K. Bezdek and R. Schneider. Covering large balls with convex sets in spherical space. *Beiträge Algebra Geom.* **51** 1 (2010), 229–235.

[5] J. Bourgain. On the Busemann–Petty problem for perturbations of the ball. *Geom. Funct. Anal.* **1** 1 (1991), 1–13.

[6] J. Bourgain and G. Zhang. On a generalization of the Busemann–Petty problem. In *Convex geometric analysis (Berkeley, CA, 1996)*, Math. Sci. Res. Inst. Publ. vol. 34 (Cambridge University Press, Cambridge, 1999), pp. 65–76.

[7] H. Busemann and C. M. Petty. Problems on convex bodies. *Math. Scand.* **4** (1956), 88–94.

[8] F. Gao, D. Hug and R. Schneider. Intrinsic volumes and polar sets in spherical space. *Math. Notae.* **41** (2001/02), 159–176 (2003). Homage to Luis Santaló. vol. 1 (Spanish).

[9] R. J. Gardner. Intersection bodies and the Busemann–Petty problem. *Trans. Amer. Math. Soc.* **342** 1 (1994), 435–445.

[10] R. J. Gardner. A positive answer to the Busemann-Petty problem in three dimensions. *Ann. of Math.* (2) **140** 2 (1994), 435–447.

[11] R. J. Gardner. The Brunn–Minkowski inequality. *Bull. Amer. Math. Soc. (N.S.)* **39** 3 (2002), 355–405.

[12] R. J. Gardner, A. Koldobsky and T. Schlumprecht. An analytic solution to the Busemann–Petty problem on sections of convex bodies. *Ann. of Math.* (2) **149** 2 (1999), 691–703.

[13] I. M. Gel‘fand and G. E. Shilov. *Generalized functions*. Vol. 1. (Academic Press [Harcourt Brace Jovanovich Publishers], New York, 1964 [1977]). Properties and operations, Translated from the Russian by Eugene Saletan.
[14] A. A. Giannopoulos. A note on a problem of H. Busemann and C. M. Petty concerning sections of symmetric convex bodies. *Mathematika* **37** 2 (1990), 239–244.

[15] W. M. Goldman. *Complex hyperbolic geometry*. Oxford Mathematical Monographs. (The Clarendon Press Oxford University Press, New York, 1999). Oxford Science Publications.

[16] A. Koldobsky. An application of the Fourier transform to sections of star bodies. *Israel J. Math.* **106** (1998), 157–164.

[17] A. Koldobsky. Intersection bodies, positive definite distributions, and the Busemann–Petty problem. *Amer. J. Math.* **120** 4 (1998), 827–840.

[18] A. Koldobsky. A generalization of the Busemann–Petty problem on sections of convex bodies. *Israel J. Math.* **110** (1999), 75–91.

[19] A. Koldobsky. A functional analytic approach to intersection bodies. *Geom. Funct. Anal.* **10** 6 (2000), 1507–1526.

[20] A. Koldobsky. On the derivatives of X-ray functions. *Arch. Math. (Basel)* **79** 3 (2002), 216–222.

[21] A. Koldobsky. The Busemann-Petty problem via spherical harmonics. *Adv. Math.* **177** 1 (2003), 105–114.

[22] A. Koldobsky. *Fourier Analysis in Convex Geometry*. Math. Surv. Monogr. vol. 116 (American Mathematical Society, Providence, RI, 2005).

[23] A. Koldobsky. Stability of volume comparison for complex convex bodies. *Arch. Math. (Basel)* **97** 1 (2011), 91–98.

[24] A. Koldobsky, H. König and M. Zymonopoulou. The complex Busemann–Petty problem on sections of convex bodies. *Adv. Math.* **218** 2 (2008), 352–367.

[25] A. Koldobsky, G. Paouris and M. Zymonopoulou. Complex intersection bodies. preprint.

[26] A. Koldobsky, V. Yaskin and M. Yaskina. Modified Busemann–Petty problem on sections of convex bodies. *Israel J. Math.* **154** (2006), 191–207.

[27] D. G. Larman and C. A. Rogers. The existence of a centrally symmetric convex body with central sections that are unexpectedly small. *Mathematika* **22** 2 (1975), 164–175.

[28] E. Lutwak. Intersection bodies and dual mixed volumes. *Adv. in Math.* **71** 2 (1988), 232–261.

[29] M. Papadimitrakis. On the Busemann–Petty problem about convex, centrally symmetric bodies in $\mathbb{R}^n$. *Mathematika* **39** 2 (1992), 258–266.

[30] B. Rubin. Comparison of volumes of convex bodies in real, complex, and quaternionic spaces. *Adv. Math.* **225** 3 (2010), 1461–1498.

[31] R. Schneider. *Convex bodies: the Brunn-Minkowski theory*. Encyclopedia Math. Appl. vol. 44 (Cambridge University Press, 1993).

[32] J. A. Thorpe. *Elementary Topics in Differential Geometry*. Undergraduate Texts in Mathematics. (Springer-Verlag, New York, 1979).

[33] V. Yaskin. The Busemann-Petty problem in hyperbolic and spherical spaces. *Adv. Math.* **203** 2 (2006), 537–553.

[34] G. Y. Zhang. Intersection bodies and the Busemann–Petty inequalities in $\mathbb{R}^4$. *Ann. of Math. (2)* **140** 2 (1994), 331–346.

[35] G. Y. Zhang. A positive solution to the Busemann–Petty problem in $\mathbb{R}^3$. *Ann. of Math. (2)* **149** 2 (1999), 535–543.

[36] A. Zvavitch. The Busemann–Petty problem for arbitrary measures. *Math. Ann.* **331** 4 (2005), 867–887.

[37] M. Zymonopoulou. The complex Busemann–Petty problem for arbitrary measures. *Arch. Math. (Basel)* **91** 5 (2008), 436–449.

[38] M. Zymonopoulou. The modified complex Busemann-Petty problem on sections of convex bodies. *Positivity* **13** 4 (2009), 717–733.