An Outer Bound for the Vector Gaussian CEO Problem*

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Abstract

We study the vector Gaussian CEO problem, where there are an arbitrary number of agents each having a noisy observation of a vector Gaussian source. The goal of the agents is to describe the source to a central unit, which wants to reconstruct the source within a given distortion. The rate-distortion region of the vector Gaussian CEO problem is unknown in general. Here, we provide an outer bound for the rate-distortion region of the vector Gaussian CEO problem. We obtain our outer bound by evaluating an outer bound for the multi-terminal source coding problem by means of a technique relying on the de Bruijn identity and the properties of the Fisher information. Next, we show that our outer bound strictly improves upon the existing outer bounds for all system parameters. We show this strict improvement by providing a specific example, and showing that there exists a gap between our outer bound and the existing outer bounds. Although our outer bound improves upon the existing outer bounds, we show that our outer bound does not provide the exact rate-distortion region in general. To this end, we provide an example and show that the rate-distortion region is strictly contained in our outer bound for this example.

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1 Introduction

We study the vector Gaussian CEO problem, where there is a vector Gaussian source which is observed through some noisy channels by an arbitrary number of agents. The agents process their observations independently and communicate them to a central unit (the so-called CEO unit) through orthogonal and rate-limited links (see Figure 1). The goal of the agents is to describe their observations to the central unit in a way that the central unit can reconstruct the source within a given distortion. The fundamental trade-off between the rate spent by the agents to describe the source and the distortion attained by the central unit is characterized by the rate-distortion region, which is unknown in general.

The CEO problem is introduced in [1], where the authors consider a discrete memoryless setting where the source and the observations of the agents all come from some discrete alphabet. In the setting of [1], the central unit is interested in estimating the source with the minimum expected error frequency which corresponds to the Hamming distance between the source sequence and the central unit’s estimation of the source sequence. In [1], the authors consider the decay rate of the error frequency with respect to the rate expenditure of the agents, and obtain the best possible decay rate when the number of agents goes to infinity.

The scalar Gaussian CEO problem is studied in [2], where there is a scalar Gaussian source which is observed through some linear Gaussian channels by the agents. The agents describe their observations to the central unit in a way that the central unit can reconstruct the source within a certain minimum mean square error (MMSE). In [2], the decay rate of the MMSE with respect to the rate expenditure of the agents is considered and shown to be inversely proportional with the rate expenditure of the agents, when the number of agents goes to infinity. The scalar Gaussian CEO problem is further studied in [3, 4], where instead of the decay rate of the achievable MMSE, the focus was on the entire rate-distortion region. In [3, 4], the entire rate-distortion region for the scalar Gaussian problem is established. The achievability is shown by using the Berger-Tung inner bound [5], and the converse is established by using the entropy-power inequality. Recently, an alternative proof for the sum-rate of the scalar Gaussian CEO problem is established in [6] without invoking the entropy-power inequality.

As pointed out by several works [7, 8], although entropy-power inequality is a key tool in providing converse proofs for scalar Gaussian problems, it might be restrictive for vector Gaussian problems. For the vector Gaussian CEO problem, this observation is noticed in [9], where the authors provide a lower bound for the sum-rate of the vector Gaussian CEO problem by using the entropy-power inequality. This lower bound is shown to be tight under certain conditions, although it is not tight in general. Recently, [10] provided an outer bound for the rate-distortion region of the vector Gaussian CEO problem when there are only two agents. They obtain their outer bound by using an extremal inequality, which can be viewed as a generalization of the extremal inequality provided in [11].

In this paper, we consider the vector Gaussian CEO problem for an arbitrary number
Figure 1: The vector Gaussian CEO problem.

of agents and provide an outer bound for its rate-distortion region. We first consider the outer bound provided in [12] for the multi-terminal source coding problem, and evaluate it for the vector Gaussian CEO problem at hand. In the evaluation of the outer bound in [12], we use the de Bruijn identity [13], a connection between the differential entropy and the Fisher information, along with the properties of the MMSE and the Fisher information. This evaluation technique which relies on the de Bruijn identity is useful in the sense that it is able to alleviate some shortcomings of the entropy-power inequality in vector Gaussian problems [8,14].

Next, we compare our outer bound with the best known outer bound for the rate-distortion region of the vector Gaussian CEO problem given in [10]. We show that the outer bound in [10] contains our outer bound in general, for all system parameters. We then provide a specific example where the outer bound in [10] strictly contains our outer bound. In other words, our outer bound brings a strict improvement over the outer bound in [10]. However, in spite of this strict improvement, our outer bound falls short of providing the exact rate-distortion region of the vector Gaussian CEO problem in general. We establish this fact by considering the parallel Gaussian model, for which we obtain the entire rate-distortion region explicitly and show that our outer bound strictly includes this rate-distortion region. In other words, for the parallel Gaussian model, our outer bound is not equal to the rate-distortion region, which shows that our outer bound is not tight in general.

2 Problem Statement and the Main Result

In the CEO problem, there are $L$ sensors, each of which getting a noisy observation of a source. The goal of the sensors is to describe their observations to the CEO unit such that the CEO unit can reconstruct the source within a given distortion. In the vector Gaussian CEO problem, there is an i.i.d. vector Gaussian source $\{X_i\}_{i=1}^n$ with zero-mean
and covariance $K_X$. Each sensor gets a noisy version of this Gaussian source

$$Y_{\ell,i} = X_i + N_{\ell,i}, \quad \ell = 1, \ldots, L$$

where $\{N_{\ell,i}\}_{i=1}^n$ is an i.i.d. sequence of Gaussian random vectors with zero-mean and covariance $\Sigma_\ell$. Moreover, noise among the sensors are independent, i.e., $N_{1,i}, \ldots, N_{L,i}$ are independent $\forall i = 1, \ldots, n$. In the vector Gaussian CEO problem, the distortion of the reconstructed vector is measured by its mean square error matrix

$$\frac{1}{n} \sum_{i=1}^n E \left[ (X_i - \hat{X}_i) (X_i - \hat{X}_i)^\top \right]$$

where $\hat{X}_i$ denotes the reconstructed vector.

An $(n, R_1, \ldots, R_L)$ code for the CEO problem consists of an encoding function at each sensor $f^n \colon \mathbb{R}^{M \times n} \to B^n_\ell = \{1, \ldots, 2^{nR_\ell}\}$, i.e., $B^n_\ell = f^n(Y^n_\ell)$ where $B^n_\ell \in B^n_\ell$, $\ell = 1, \ldots, L$, and a decoding function at the CEO unit $g^n : B^n_1 \times \cdots \times B^n_L \to \mathbb{R}^{M \times n}$, i.e., $\hat{X}^n = g^n(B^n_1, \ldots, B^n_L)$, where $M$ denotes the size of the vector Gaussian source $X$.

We note that since the mean square error is minimized by the MMSE estimator, which is the conditional mean, without loss of generality, the decoding function $g^n$ can be chosen as the MMSE estimator. Consequently, we have

$$\hat{X}_i = E[X_i | B^n_1, \ldots, B^n_L]$$

using which in (2), we get

$$\frac{1}{n} \sum_{i=1}^n E \left[ (X_i - \hat{X}_i) (X_i - \hat{X}_i)^\top \right] = \frac{1}{n} \sum_{i=1}^n \text{mmse}(X_i | B^n_1, \ldots, B^n_L)$$

In view of (4), a rate tuple $(R_1, \ldots, R_L)$ is said to achieve the distortion $D$ if there exists an $(n, R_1, \ldots, R_L)$ code such that

$$\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^n \text{mmse}(X_i | B^n_1, \ldots, B^n_L) \preceq D$$

where $D$ is a strictly positive definite matrix. Throughout the paper, we assume that the distortion matrix $D$ satisfies

$$\left( K_X^{-1} + \sum_{\ell=1}^L \Sigma_\ell^{-1} \right)^{-1} \preceq D \preceq K_X$$

where the lower bound on the distortion constraint $D$ corresponds to the MMSE matrix obtained when the CEO unit has direct access to the observations of the agents $\{Y_{\ell,i}\}_{i=1}^L$. 

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The derivation of this lower bound is provided in Appendix A, where we also provide insight on the upper bound in (6). In Appendix A, we also show that imposing the lower bound on \( D \) in (6), i.e., imposing \( (K_x^{-1} + \sum_{\ell=1}^L \Sigma_\ell^{-1})^{-1} \preceq D \), does not incur any loss of generality, while imposing the upper bound on \( D \) in (6), i.e., imposing \( D \preceq K_x \), might incur some loss of generality.

The rate-distortion region \( \mathcal{R}(D) \) of the vector Gaussian CEO problem is defined as the closure of all rate tuples \((R_1, \ldots, R_L)\) that can achieve the distortion \( D \).

The main result of this paper is the following outer bound on the rate-distortion region \( \mathcal{R}(D) \) of the vector Gaussian CEO problem stated in the following theorem.

**Theorem 1** The rate-distortion region of the Gaussian CEO problem \( \mathcal{R}(D) \) is contained in the region \( \mathcal{R}^o(D) \) which is given by the union of rate tuples \((R_1, \ldots, R_L)\) satisfying

\[
\sum_{\ell \in A} R_\ell \geq \frac{1}{2} \log^+ \left( \frac{(K_x^{-1} + \sum_{\ell \in A^c} \Sigma_\ell^{-1} - \sum_{\ell \in A^c} \Sigma_\ell^{-1} D_\ell \Sigma_\ell^{-1})^{-1}}{|D|} \right) + \sum_{\ell \in A} \frac{1}{2} \log \frac{|\Sigma_\ell|}{|D_\ell|} \tag{7}
\]

for all \( A \subseteq \{1, \ldots, L\} \), where the union is over all positive semi-definite matrices \( \{D_\ell\}_{\ell=1}^L \) satisfying the following constraints

\[
\left( K_x^{-1} + \sum_{\ell=1}^L \Sigma_\ell^{-1} - \sum_{\ell=1}^L \Sigma_\ell^{-1} D_\ell \Sigma_\ell^{-1} \right)^{-1} \preceq D \tag{8}
\]

\[
0 \preceq D_\ell \preceq \Sigma_\ell, \quad \ell = 1, \ldots, L \tag{9}
\]

and \( \log^+ x = \max(\log x, 0) \).

We obtain this outer bound by evaluating the outer bound given in [12]. The proof of Theorem 1 is given in Section 6. Next, we provide the following inner bound for the rate-distortion region \( \mathcal{R}(D) \).

**Theorem 2** An inner bound for the rate-distortion region of the vector Gaussian CEO problem is given by the region \( \mathcal{R}^i(D) \) which is described by the union of rate tuples \((R_1, \ldots, R_L)\) satisfying

\[
\sum_{\ell \in A} R_\ell \geq \frac{1}{2} \log \left( \frac{(K_x^{-1} + \sum_{\ell \in A^c} \Sigma_\ell^{-1} - \sum_{\ell \in A^c} \Sigma_\ell^{-1} D_\ell \Sigma_\ell^{-1})^{-1}}{|D|} \right) + \sum_{\ell \in A} \frac{1}{2} \log \frac{|\Sigma_\ell|}{|D_\ell|} \tag{10}
\]

for all \( A \subseteq \{1, \ldots, L\} \), where the union is over all positive semi-definite matrices \( \{D_\ell\}_{\ell=1}^L \).
satisfying

\[
\left( K_X^{-1} + \sum_{\ell=1}^{L} \Sigma_\ell^{-1} - \sum_{\ell=1}^{L} \Sigma_\ell^{-1} D_\ell \Sigma_\ell^{-1} \right)^{-1} \preceq D
\]

(11)

\[
0 \preceq D_\ell \preceq \Sigma_\ell, \quad \ell = 1, \ldots, L
\]

(12)

We obtain this inner bound by evaluating the Berger-Tung inner bound [5] by jointly Gaussian auxiliary random variables. The proof of Theorem 2 is given in Appendix I.

3 Alternative Characterizations of the Bounds

In this section, we provide alternative characterizations for the outer and inner bounds given in Theorem 1 and Theorem 2, respectively. To this end, we note that since the rate-distortion region \( \mathcal{R}(D) \) is convex, it can be characterized by the tangent hyperplanes to it, i.e., by solving the following optimization problem

\[
\min_{(R_1, \ldots, R_L) \in \mathcal{R}(D)} \sum_{\ell=1}^{L} \mu_\ell R_\ell
\]

(13)

for all \( \mu_\ell \geq 0, \ell = 1, \ldots, L \). Hence, the outer and inner bounds in Theorem 1 and 2 provide lower and upper bounds for the optimization problem in (13), respectively. Since both the outer and inner bounds are also convex, they can also be described by the tangent hyperplanes to them. In particular, the outer and inner bounds can be described by the following optimization problems

\[
\min_{(R_1, \ldots, R_L) \in \mathcal{R}(D)} \sum_{\ell=1}^{L} \mu_\ell R_\ell \quad \text{and} \quad \min_{(R_1, \ldots, R_L) \in \mathcal{R}(D)} \sum_{\ell=1}^{L} \mu_\ell R_\ell
\]

(14)

respectively, where \( \mu_\ell \geq 0, \ell = 1, \ldots, L \). We note that the first optimization problem in (14) corresponds to the alternative characterization of the outer bound in Theorem 1 and hence, provides a lower bound for the optimization problem in (13) that characterizes the rate-distortion region of the vector Gaussian CEO problem. Similarly, the second optimization problem in (14) corresponds to the alternative characterization of the inner bound in Theorem 2 and hence, provides an upper bound for the optimization problem in (13). Now, we state the explicit form of the optimization problems in (14) starting with the one for the outer bound.
Theorem 3 Assume $\mu_1 \geq \ldots \geq \mu_L \geq 0$. We have

$$\min_{(R_1,\ldots,R_L)\in\mathcal{R}(\mathbf{D})} \sum_{\ell=1}^{L} \mu_\ell R_\ell \geq \min_{(R_1,\ldots,R_L)\in\mathcal{R}^+(\mathbf{D})} \sum_{\ell=1}^{L} \mu_\ell R_\ell$$

$$= \min_{\{\mathbf{D}_\ell\}_{\ell=1}^{L}} \sum_{\ell=1}^{L-1} \frac{\mu_\ell - \mu_{\ell+1}}{2} \log^+ \left( \left( K^{-1} + \sum_{j=\ell+1}^{L} \Sigma_j^{-1}(\Sigma_j - \mathbf{D}_j)\Sigma_j^{-1} \right)^{-1} \right) + \sum_{\ell=1}^{L} \frac{\mu_\ell}{2} \log \frac{\Sigma_\ell}{|\mathbf{D}_\ell|} + \frac{\mu_L}{2} \log \frac{|K_X|}{|\mathbf{D}|}$$

(15)

where $\{\mathbf{D}_\ell\}_{\ell=1}^{L}$ are subject to the following constraints

$$\left( K^{-1} + \sum_{\ell=1}^{L} \Sigma_\ell^{-1} - \sum_{\ell=1}^{L} \Sigma_\ell^{-1} \mathbf{D}_\ell \Sigma_\ell^{-1} \right)^{-1} \preceq \mathbf{D}$$

(16)

$$0 \preceq \mathbf{D}_\ell \preceq \Sigma_\ell, \quad \ell = 1, \ldots, L$$

(17)

Next, we provide the explicit form of the other optimization problem in (14), i.e., the one for the inner bound, as follows.

Theorem 4 Assume $\mu_1 \geq \ldots \geq \mu_L \geq 0$. We have

$$\min_{(R_1,\ldots,R_L)\in\mathcal{R}(\mathbf{D})} \sum_{\ell=1}^{L} \mu_\ell R_\ell \leq \min_{(R_1,\ldots,R_L)\in\mathcal{R}^+(\mathbf{D})} \sum_{\ell=1}^{L} \mu_\ell R_\ell$$

$$= \min_{\{\mathbf{D}_\ell\}_{\ell=1}^{L}} \sum_{\ell=1}^{L-1} \frac{\mu_\ell - \mu_{\ell+1}}{2} \log^+ \left( \left( K^{-1} + \sum_{j=\ell+1}^{L} \Sigma_j^{-1}(\Sigma_j - \mathbf{D}_j)\Sigma_j^{-1} \right)^{-1} \right) + \sum_{\ell=1}^{L} \frac{\mu_\ell}{2} \log \frac{\Sigma_\ell}{|\mathbf{D}_\ell|} + \frac{\mu_L}{2} \log \frac{|K_X|}{|\mathbf{D}|}$$

(18)

where $\{\mathbf{D}_\ell\}_{\ell=1}^{L}$ are subject to the following constraints

$$\left( K^{-1} + \sum_{\ell=1}^{L} \Sigma_\ell^{-1} - \sum_{\ell=1}^{L} \Sigma_\ell^{-1} \mathbf{D}_\ell \Sigma_\ell^{-1} \right)^{-1} \succeq \mathbf{D}$$

(19)

$$0 \succeq \mathbf{D}_\ell \preceq \Sigma_\ell, \quad \ell = 1, \ldots, L$$

(20)

The proofs of Theorem 3 and Theorem 4 are given in Appendix B.

Next, we provide some remarks about the outer bound given in Theorem 3 and the inner bound given in Theorem 4. First, we note that in both cases, the bounds are to be optimized over the positive semi-definite matrices $\{\mathbf{D}_\ell\}_{\ell=1}^{L}$, and the feasible sets for both
cases are identical as seen through (16)-(17) and (19)-(20). On the other hand, rate bounds differ as seen through (15) and (18). Despite this difference, there are cases where the outer and inner bounds match, providing a complete characterization of the rate-distortion region. Here, we note a general sufficient condition under which the outer and inner bounds coincide. If the minimum in Theorem 3 is achieved by positive semi-definite matrices \( \{D^*_\ell\}_{\ell=1}^L \) which attain the distortion constraint in (16) with equality, then the optimization problems in Theorem 3 and Theorem 4 yield identical results, implying the tightness of the outer bound. One particular example where the outer and inner bounds match is the scalar Gaussian model considered next.

### 3.1 Scalar Gaussian Model

In this section, we consider the case where the source and the observations are scalar:

\[
Y_{\ell,i} = X_i + N_{\ell,i}, \quad \ell = 1, \ldots, L
\]

where \( X_i \) is an i.i.d. Gaussian source with zero-mean and variance \( \sigma_X^2 \). The noise at the \( \ell \)th sensor \( N_{\ell,i} \) is also an i.i.d. Gaussian random variable sequence with variance \( \sigma_{\ell}^2 \). For the scalar model (scalar Gaussian CEO problem), our outer bound in Theorem 1 reduces to the following form.

**Corollary 1** The rate-distortion region of the scalar Gaussian CEO problem \( \mathcal{R}(D) \) is contained in the region \( \mathcal{R}^o(D) \) which is given by the union of rate tuples \( (R_1, \ldots, R_L) \) satisfying

\[
\sum_{\ell \in A} R_{\ell} \geq \frac{1}{2} \log \left( \frac{1}{\sigma_X^2} + \sum_{\ell \in A^c} \frac{\sigma_{\ell}^2 - D_{\ell}}{\sigma_{\ell}^4} \right)^{-1} + \sum_{\ell \in A} \frac{1}{2} \log \frac{\sigma_{\ell}^2}{D_{\ell}} \tag{22}
\]

for all \( A \subseteq \{1, \ldots, L\} \), where the union is over all \( \{D_{\ell}\}_{\ell=1}^L \) satisfying the following constraints

\[
\left( \frac{1}{\sigma_X^2} + \sum_{\ell=1}^L \frac{\sigma_{\ell}^2 - D_{\ell}}{\sigma_{\ell}^4} \right)^{-1} \leq D \tag{23}
\]

\[
0 \leq D_{\ell} \leq \sigma_{\ell}^2, \quad \ell = 1, \ldots, L \tag{24}
\]
Using Theorem 3, our outer bound for the scalar Gaussian model can be expressed in the following alternative form

\[
\min_{(R_1,\ldots,R_L) \in \mathcal{R}(D)} \sum_{\ell=1}^{L} \mu_{\ell} R_\ell \geq \min_{(R_1,\ldots,R_L) \in \mathcal{R}(D)} \sum_{\ell=1}^{L} \mu_{\ell} R_\ell
\]

\[
= \min_{\{D_\ell\}_{\ell=1}^{L}} \sum_{\ell=1}^{L-1} \frac{\mu_\ell - \mu_{\ell+1}}{2} \log \left( \frac{1}{\sigma_X^2} + \sum_{j=\ell+1}^{L} \frac{\sigma_j^2 - D_\ell}{\sigma_\ell^4} \right)^{-1} + \sum_{\ell=1}^{L} \mu_\ell \log \frac{\sigma_\ell^2}{D_\ell} + \frac{\mu_L}{2} \log \frac{\sigma_X^2}{D}
\]

(25)

where \(\{D_\ell\}_{\ell=1}^{L}\) are subject to the constraints in (23)-(24), and we assume \(\mu_1 \geq \ldots \geq \mu_L \geq 0\). In [3], it is shown that the optimal \(\{D_\ell^*\}_{\ell=1}^{L}\) that minimizes (25) satisfies the constraint in (23) with equality, i.e., for this optimal \(\{D_\ell^*\}_{\ell=1}^{L}\), we have

\[
\left( \frac{1}{\sigma_X^2} + \sum_{\ell=1}^{L} \frac{\sigma_\ell^2 - D_\ell^*}{\sigma_\ell^4} \right)^{-1} = D
\]

(26)

As we pointed out in the previous section, when, for the outer bound, the distortion constraint is satisfied with equality, then the outer bound in Theorem 1 and the inner bound in Theorem 2 match; yielding the rate-distortion region. Hence, in view of (26), we have the entire rate-distortion region for the scalar Gaussian CEO problem.

**Theorem 5 (3,4)** The rate-distortion region of the scalar Gaussian CEO problem \(\mathcal{R}(D)\) is given by the union of rate tuples \((R_1,\ldots,R_L)\) satisfying

\[
\sum_{\ell \in A} R_\ell \geq \frac{1}{2} \log \frac{1}{D} \left( \frac{1}{\sigma_X^2} + \sum_{\ell \in A^c} \frac{\sigma_\ell^2 - D_\ell}{\sigma_\ell^4} \right)^{-1} + \sum_{\ell \in A} \frac{1}{2} \log \frac{\sigma_\ell^2}{D_\ell} \tag{27}
\]

for all \(A \subseteq \{1, \ldots, L\}\), where the union is over all \(\{D_\ell\}_{\ell=1}^{L}\) satisfying the following constraints

\[
\left( \frac{1}{\sigma_X^2} + \sum_{\ell=1}^{L} \frac{\sigma_\ell^2 - D_\ell}{\sigma_\ell^4} \right)^{-1} = D \tag{28}
\]

\[
0 \leq D_\ell \leq \sigma_\ell^2, \quad \ell = 1, \ldots, L \tag{29}
\]

We note that since the distortion constraint in (28) is satisfied with equality, we do not need the positivity operator in (27).

**4 Chen-Wang Outer Bound**

In [10, Theorem 2], the authors provide an outer bound for the rate-distortion region of the vector Gaussian CEO problem when \(L = 2\). In this section, we compare our outer
bound given in Theorem 1. First, we note that the outer bound in [10, Theorem 2] always contains our outer bound for all system parameters. Next, we provide an example and show that the outer bound in [10, Theorem 2] strictly contains our outer bound. In other words, we show that there are rate pairs \((R_1, R_2)\) that are contained in the outer bound given in [10, Theorem 2] and are strictly outside of our outer bound given in Theorem 1. To this end, we specialize our outer bound in Theorem 3 to the case \(L = 2\) as follows.

**Corollary 2** When \(\mu_1 \geq \mu_2 \geq 0\), we have

\[
\mu_1 R_1 + \mu_2 R_2 \geq T^+ = \min_{(D_1, D_2) \in \mathcal{D}^+(D_1, D_2)} \frac{\mu_1}{2} \log \frac{\Sigma_1}{|D_1|} + \frac{\mu_2}{2} \log \frac{\Sigma_2}{|D_2|} + \frac{\mu_2}{2} \log \frac{|K X|}{|D|} + \frac{\mu_1 - \mu_2}{2} \log \left(\frac{(K^{-1}_X + \Sigma_{2}^{-1} - \Sigma_{2}^{-1} D_2 \Sigma_{2}^{-1})^{-1}}{|D|}\right)
\]

where the feasible set \(\mathcal{D}^+(D_1, D_2)\) is given by the union of \((D_1, D_2)\) satisfying

\[
\left(K^{-1}_X + \sum_{\ell=1}^{2} \Sigma_{\ell}^{-1} - \sum_{\ell=1}^{2} \Sigma_{\ell}^{-1} D_{\ell} \Sigma_{\ell}^{-1}\right)^{-1} \preceq D
\]

\[
0 \preceq D_{\ell} \preceq \Sigma_{\ell}, \quad \ell = 1, 2
\]

When \(0 \leq \mu_1 \leq \mu_2\), a lower bound for \(\mu_1 R_1 + \mu_2 R_2\) can be obtained from (30)-(32) by swapping the indices 1 and 2.

Now, we present the outer bound in [10, Theorem 2].

**Theorem 6 ([10, Theorem 2])** When \(\mu_1 \geq \mu_2 \geq 0\), we have

\[
\mu_1 R_1 + \mu_2 R_2 \geq T^- = \min_{(D_1, D_2) \in \mathcal{D}^-(D_1, D_2)} \frac{\mu_1}{2} \log \frac{\Sigma_1}{|D_1|} + \frac{\mu_2}{2} \log \frac{\Sigma_2}{|D_2|} + \frac{\mu_2}{2} \log \frac{|K X|}{|D|} + \frac{\mu_1 - \mu_2}{2} \log \left(\frac{(K^{-1}_X + \Sigma_{2}^{-1} - \Sigma_{2}^{-1} D_2 \Sigma_{2}^{-1})^{-1}}{|D|}\right)
\]

where the feasible set \(\mathcal{D}^-(D_1, D_2)\) is given by the union of \((D_1, D_2)\) satisfying

\[
\left(K^{-1}_X + \sum_{\ell=1}^{2} \Sigma_{\ell}^{-1} - \sum_{\ell=1}^{2} \Sigma_{\ell}^{-1} D_{\ell} \Sigma_{\ell}^{-1}\right)^{-1} \preceq D
\]

\[
0 \preceq D_{\ell} \preceq \Sigma_{\ell}, \quad \ell = 1, 2
\]

When \(0 \leq \mu_1 \leq \mu_2\), a lower bound for \(\mu_1 R_1 + \mu_2 R_2\) can be obtained from (33)-(35) by swapping the indices 1 and 2.
We note that the only difference between the outer bounds in Corollary 2 and Theorem 6 is the positivity operator involved in (30) (compare (30) with (33)). Besides that, the two outer bounds are identical. In the sequel, we first provide an outline for both approaches that explains how the difference between these two outer bounds arises. We note that because of the positivity operator in our outer bound, we always have $T^+ \geq T^-$ in general, and our outer bound is at least as tight as the outer bound in [10, Theorem 2] or tighter, for all instances of the vector Gaussian CEO problem. Next, we provide an example where $T^+ > T^-$, which implies that our outer bound is strictly contained in the outer bound given in [10, Theorem 2].

In [10], the lower bound $T^-$ is obtained by minimizing the following cost function

$$C^- = \frac{\mu_1}{n} I(B_1^n; Y_1^n | X^n) + \frac{\mu_2}{n} I(X^n; B_1^n, B_2^n) + \frac{\mu_1 - \mu_2}{n} I(X^n; B_1^n | B_2^n) + \frac{\mu_2}{n} I(Y_2^n; B_2^n | X^n)$$

(36)

where the authors consider the first and the second terms separately, which leads to the following terms

$$\frac{\mu_1}{2} \log \frac{|\Sigma_1|}{|D_1|} \quad \text{and} \quad \frac{\mu_2}{2} \log \frac{|K_X|}{|D|}$$

(37)

in (33), respectively. The third and fourth terms in (36) are considered jointly. In particular, in [10, Theorem 2], the authors rewrite the third and fourth terms as

$$C^-_{3,4} = \left[ \frac{\mu_1 - \mu_2}{2n} h(X^n | B_2^n) - \frac{\mu_2}{2n} h(Y_2^n | B_2^n, X^n) \right] - \frac{\mu_1 - \mu_2}{2n} h(X^n | B_1^n, B_2^n) + \frac{\mu_2}{2n} h(Y_2^n | X^n)$$

(38)

and minimize $C^-_{3,4}$. In particular, the difference term in the bracket is minimized jointly, which is the reason why there is no positivity operator in the outer bound given by Theorem 6.

On the other hand, we consider the following cost function

$$C^+ = \mu_1 I(U_1; Y_1 | X, W) + \mu_2 I(X; U_1, U_2) + (\mu_1 - \mu_2) I(X; U_1 | U_2) + \mu_2 I(Y_2; U_2 | X, W)$$

(39)

which can be obtained by using the outer bound provided in [12]. (More details about the cost function $C^+$ can be found in Section 6 where we prove Theorem 4.) We note that the cost function $C^+$ can be viewed as a single-letter form of the cost function $C^-$. As opposed to [10] where the mutual information terms involved in the cost function $C^-$ are decomposed into differential entropies and some cross terms are minimized jointly (see $C^-_{3,4}$), we consider each mutual information term in the cost function $C^+$ separately, and find a lower bound for each term. Hence, we find a lower bound for the third term in $C^+$ which, being a mutual information, is non-negative. This is the reason why we have a positivity operator in}
outer bound given in Corollary 2 (and also in Theorem 1 and Theorem 3).

Next, we provide an example where we have $T^+ > T^-$, which implies that our outer bound in Corollary 2 (and, hence in Theorem 1) is strictly contained in the outer bound Theorem 2] in Theorem 6. In other words, there are rate pairs $(R_1, R_2)$ that lie inside the outer bound given by Theorem 6 and lie strictly outside of our outer bound. To show this, we consider the case where the following assumptions hold:

$$\frac{\mu_2}{\mu_1} \Sigma_1^{-1} \prec K_X^{-1} + \Sigma_2^{-1} - D^{-1} \tag{40}$$

$$\frac{\mu_2}{\mu_1 - \mu_2} K_X^{-1} \prec \Sigma_2^{-1} \tag{41}$$

$$\frac{\mu_1}{\mu_1 - \mu_2} D^{-1} \prec K_X^{-1} + \Sigma_2^{-1} \tag{42}$$

Under the assumptions given by (40)-(42), we can obtain our outer bound given in Corollary 2 explicitly in terms of $K_X, D, \Sigma_\ell$ and $\mu_{\ell}, \ell = 1, 2$, as stated in the following corollary.

**Corollary 3** When the assumptions given by (40)-(42) hold, we have

$$T^+ = \frac{\mu_2}{2} \log \frac{|K_X|}{|D|} + \frac{\mu_2}{2} \log \frac{|\Sigma_2^{-1}|}{|K_X^{-1} + \Sigma_2^{-1} - D^{-1}|} \tag{43}$$

Next, we obtain an upper bound for the lower bound given in Theorem 6. In other words, we obtain an upper bound for $T^-$ as stated in the following corollary.

**Corollary 4** When the assumptions given by (40)-(42) hold, we have

$$T^- \leq T^+ + \frac{\mu_2}{2} \log \frac{|K_X^{-1} + \Sigma_2^{-1} - D^{-1}|}{\left| \frac{\mu_2}{\mu_1} (K_X^{-1} + \Sigma_2^{-1}) \right|} + \frac{\mu_1 - \mu_2}{2} \log \frac{\mu_1}{\mu_1 - \mu_2} \frac{(K_X^{-1} + \Sigma_2^{-1})^{-1}}{|D|} \tag{44}$$

The proofs of Corollaries 3 and 4 are given in Appendix C.1 and Appendix C.2 respectively.

\footnote{An example where these conditions hold is $K_X^{-1} = \Sigma_1^{-1} = \Sigma_2^{-1}$ and $\mu_1/\mu_2 = 4$. For this case, one can find $D$ matrices satisfying these constraints in addition to the original constraints on $D$ stated in 6.}
Now, we are ready to compare $T^+$ and $T^-$ as follows

$$T^- - T^+ \leq \frac{\mu_2}{2} \log \left| \frac{K_X^{-1} + \Sigma_2^{-1} - D}{\frac{\mu_2}{\mu_1} (K_X^{-1} + \Sigma_2^{-1})} \right| + \frac{\mu_1 - \mu_2}{2} \log \left| \frac{\mu_1}{\mu_1 - \mu_2} (K_X^{-1} + \Sigma_2^{-1})^{-1} \right|$$  \hspace{1cm} (45)

$$= \frac{\mu_2}{2} \log \left| \frac{\mu_1}{\mu_2} \left( I - (K_X^{-1} + \Sigma_2^{-1})^{-1/2} D^{-1} (K_X^{-1} + \Sigma_2^{-1})^{-1/2} \right) \right|$$  \hspace{1cm} (46)

$$+ \frac{\mu_1 - \mu_2}{2} \log \left| \frac{\mu_1}{\mu_1 - \mu_2} (K_X^{-1} + \Sigma_2^{-1})^{-1/2} D^{-1} (K_X^{-1} + \Sigma_2^{-1})^{-1/2} \right|$$

$$< \frac{\mu_1}{2} \log |I|$$  \hspace{1cm} (47)

$$= 0$$  \hspace{1cm} (48)

where (47) follows from the facts that the function $\log |\cdot|$ is strictly concave over strictly positive definite matrices [15, Theorem 7.6.7], and the two matrices inside the log $|\cdot|$ functions in (46) are not identical, which is due to the assumption in (42).

5 Parallel Gaussian Model and a Counter-Example

In this section, first, we consider the parallel Gaussian model, and obtain its rate-distortion region. Next, we consider a specific parallel Gaussian model and show that our outer bound in Theorem 1 is not tight. In other words, we show that, in general, there are rate tuples $(R_1, \ldots, R_L)$ that lie inside our outer bound and are not contained in the rate-distortion region, i.e., in general, our outer bound strictly contains the rate-distortion region.

In the parallel Gaussian model, the Gaussian source $X_i$ has a diagonal covariance matrix. In particular, we have $X_i = [X_{1,i}, \ldots, X_{M,i}]$ where $\{X_{m,i}\}_{m=1}^M$ are independent Gaussian random variables with zero-mean and variance $\{\sigma_m^2\}_{m=1}^M$, respectively. Moreover, the noise at the $\ell$th sensor $N_{\ell,i}$ also has a diagonal covariance matrix. In particular, we have $N_{\ell,i} = [N_{\ell,1,i}, \ldots, N_{\ell,M,i}]$, where $\{N_{\ell,m,i}\}_{m=1}^M$ are independent Gaussian random variables with zero-mean with variance $\{\sigma_{\ell,m}^2\}_{m=1}^M$, respectively. In the parallel Gaussian model, there is a separate-distortion constraint on each component of the source as follows

$$\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^n \text{mmse}(X_{m,i}|B_1^n, \ldots, B_L^n) \leq D_m, \quad m = 1, \ldots, M$$  \hspace{1cm} (49)

where we have the following constraints on $\{D_m\}_{m=1}^M$

$$\left( \frac{1}{\sigma_m^2} + \sum_{\ell=1}^L \frac{1}{\sigma_{\ell,m}^2} \right)^{-1} \leq D_m \leq \sigma_m^2, \quad m = 1, \ldots, M$$  \hspace{1cm} (50)

We note that the constraints on $D_m$ in (50) are the scalar versions of the constraints in (6)
that we impose for the vector Gaussian model. For the parallel Gaussian model, we establish
the rate-distortion region $R^p(\{D_m\}_{m=1}^M)$ as stated in the following theorem.

**Theorem 7** The rate-distortion region $R^p(\{D_m\}_{m=1}^M)$ of the parallel Gaussian CEO problem
is given by the union of rate tuples $(R_1, \ldots, R_L)$ satisfying

$$
\sum_{\ell \in A} R_\ell \geq \sum_{m=1}^M \frac{1}{2} \log \frac{1}{D_m} \left( \frac{1}{\sigma_m^2} + \sum_{\ell \in A} \frac{\sigma_{\ell m}^2 - D_{\ell m}}{\sigma_{\ell m}^4} \right)^{-1} \right) + \sum_{m=1}^M \sum_{\ell \in A} \frac{1}{2} \log \frac{\sigma_{\ell m}^2}{D_{\ell m}}
$$

for all $A \subseteq \{1, \ldots, L\}$, where the union is over all $\{D_{\ell m}\}_{\ell, m}$ satisfying the following con-
straints

$$
\left( \frac{1}{\sigma_m^2} + \sum_{\ell=1}^L \frac{\sigma_{\ell m}^2 - D_{\ell m}}{\sigma_{\ell m}^4} \right)^{-1} = D_m, \quad m = 1, \ldots, M
$$

$$
0 \leq D_{\ell m} \leq \sigma_{\ell m}^2, \quad \ell = 1, \ldots, L, \quad m = 1, \ldots, M
$$

We note that since the distortion constraints in (52) are met with equality, the first $\log(\cdot)$
in (51) is always positive, and hence, we do not need a positivity operator. We obtain the
rate-distortion region of the parallel Gaussian CEO problem in two steps. In the first step,
we specialize the outer bound in [12] to the parallel model. In the second step, we evaluate
the outer bound we obtain in the first step, and show that it matches the inner bound given
in Theorem 2. The details of the proof are given in Appendix D.

Next, we consider the case $L = M = 2$, and provide an example where our outer bound
strictly contains the rate-distortion region, i.e., our outer bound includes rate pairs which
are outside of the rate-distortion region. In the example we provide, we assume that the
following conditions hold:

$$
\frac{\mu_2}{\mu_1} \frac{1}{\sigma_{12}^2} \frac{1}{\sigma_2^2} < \frac{1}{\sigma_2^2} + \frac{1}{\sigma_{22}^2} - \frac{1}{D_2}
$$

$$
\frac{\mu_2}{\mu_1 - \mu_2} \frac{1}{\sigma_{12}^2} \frac{1}{\sigma_2^2} < \frac{1}{\sigma_{22}^2}
$$

$$
\frac{\mu_1}{\mu_1 - \mu_2} \frac{1}{D_2} < \frac{1}{\sigma_2^2} + \frac{1}{\sigma_{22}^2}
$$

$$
\frac{1}{D_1} \left( \frac{1}{\sigma_1^2} + \frac{1}{\sigma_{21}^2} \right)^{-1} > \frac{\mu_1 - \mu_2}{\mu_1} \frac{1}{D_2} \left( \frac{1}{\sigma_2^2} + \frac{1}{\sigma_{22}^2} \right)
$$

where the first three constraints are analogous the constraints in (10)-(12), which were used
to provide an example that the Chen-Wang outer bound [10] strictly contains our outer
bound. Under the constraints in (54)-(57), the rate-distortion region $R^p(D_1, D_2)$ can be
characterized as follows.

---

\footnote{We note that if one selects $\sigma_m^2 = \sigma_{\ell m}^2 = \sigma^2, D_1 = 2/5\sigma^2, D_2 = 4/5\sigma^2$ and $\mu_1/\mu_2 = 4$, the four
assumptions in (54)-(57) hold in addition to the original constraints on $(D_1, D_2)$ given in (50).}
Corollary 5 Assume that (54)-(57) hold. Then, we have

\[ T_p = \min_{(R_1, R_2) \in \mathbb{R}_p(D_1, D_2)} \mu_1 R_1 + \mu_2 R_2 \]

\[ = \min_{(D_{11}, D_{21}) \in D_1} f_1(D_{11}, D_{21}) + \frac{\mu_2}{2} \log \frac{\sigma^2_2}{D_2} + \frac{\mu_2}{2} \log \frac{1}{\sigma^2_2} \left( \frac{1}{\sigma^2_2} + \frac{1}{\sigma^2_2} - 1 \right)^{-1} \]

(58)

(59)

where \( f_1(D_{11}, D_{21}) \) is given by

\[ f_1(D_{11}, D_{21}) = \sum_{\ell=1}^{2} \frac{\mu_1}{2} \log \frac{\sigma^2_1 D_{11}}{\sigma^4_{\ell 1}} + \frac{\mu_2}{2} \log \frac{\sigma^2_1}{D_1} + \frac{\mu_1 - \mu_2}{2} \log \frac{1}{D_1} \left( \frac{1}{\sigma^2_1} + \frac{\sigma^2_1 - D_{21}}{\sigma^2_2} \right)^{-1} \]

(60)

and the set \( D_1 \) consists of \( (D_{11}, D_{21}) \) pairs satisfying

\[ \frac{1}{\sigma^2_1} + \sum_{\ell=1}^{2} \frac{\sigma^2_{\ell 1} - D_{\ell 1}}{\sigma^4_{\ell 1}} = \frac{1}{D_1} \]

\[ 0 \leq D_{\ell 1} \leq \sigma^2_{\ell 1}, \quad \ell = 1, 2 \]

(61)

(62)

The proof of Corollary 5 is given in Appendix E. Next, we find an upper bound for our outer bound in Theorem 1 as follows.

Corollary 6 Assume that (54)-(57) hold. Then, we have

\[ T^+ = \min_{(R_1, R_2) \in \mathbb{R}_p(D_1, D_2)} \mu_1 R_1 + \mu_2 R_2 \]

\[ \leq \min_{(D_{11}, D_{21}) \in D_1} f_1(D_{11}, D_{21}) + \frac{\mu_2}{2} \log \frac{\mu_1}{\mu_2} \left( \frac{1}{\sigma^2_2} + \frac{1}{\sigma^2_2} \right)^{-1} + \frac{\mu_2}{2} \log \frac{\sigma^2_2}{D_2} \]

\[ + \frac{\mu_1 - \mu_2}{2} \log \frac{\mu_1}{\mu_1 - \mu_2} \left( \frac{1}{\sigma^2_2} + \frac{1}{\sigma^2_2} \right)^{-1} \]

(63)

(64)

where the function \( f_1(D_{11}, D_{21}) \) is given by (60) and the set \( D_1 \) is given by the union of \( (D_{11}, D_{21}) \) satisfying the constraints in (61)-(62).

The proof of Corollary 6 is given in Appendix F.

Now, we are ready to compare our outer bound with the rate-distortion region for the
parallel Gaussian model. Using Corollary 5 and Corollary 6, we have

$$T^+ - T^p \leq \frac{\mu_2}{2} \log \frac{\mu_1}{\mu_2} \left(\frac{1}{\sigma_2^2} + \frac{1}{\sigma_{22}^2}\right) - \frac{\mu_2}{2} \log \frac{1}{\sigma_{22}^2} \left(\frac{1}{\sigma_2^2} + \frac{1}{\sigma_{22}^2} - \frac{1}{D_2}\right)^{-1}$$

$$= \frac{\mu_2}{2} \log \frac{\mu_1}{\mu_2} \left(1 - \frac{1}{D_2} \left(\frac{1}{\sigma_2^2} + \frac{1}{\sigma_{22}^2}\right)^{-1}\right) + \frac{\mu_1 - \mu_2}{2} \log \frac{\mu_1}{\mu_1 - \mu_2} D_2 \left(\frac{1}{\sigma_2^2} + \frac{1}{\sigma_{22}^2}\right)^{-1}$$

$$< \frac{\mu_1}{2} \log 1$$

$$= 0$$

where (67) follows from the facts that $\log(\cdot)$ is strictly concave, and we have

$$\frac{\mu_1}{\mu_2} \left(1 - \frac{1}{D_2} \left(\frac{1}{\sigma_2^2} + \frac{1}{\sigma_{22}^2}\right)^{-1}\right) \neq \frac{\mu_1}{\mu_1 - \mu_2} D_2 \left(\frac{1}{\sigma_2^2} + \frac{1}{\sigma_{22}^2}\right)^{-1}$$

which is due to the assumption in (56). Equation (68) implies that there are some rate pairs $(R_1, R_2)$ in our outer bound which are outside of the rate-distortion region of the parallel Gaussian model. Hence, our outer bound strictly contains the rate-distortion region of the vector Gaussian CEO problem. In other words, our outer bound is not tight in general.

6 Proof of Theorem 1

The following theorem provides an outer bound for the rate-distortion region of the CEO problem.

Theorem 8 ([12, Theorem 1]) The rate region of the CEO problem $\mathcal{R}(D)$ is contained in the union of rate tuples $(R_1, \ldots, R_L)$ satisfying

$$\sum_{\ell \in A} R_\ell \geq I(X; \{U_{\ell}\}_{\ell \in A} | \{U_{\ell}\}_{\ell \in A^c}) + \sum_{\ell \in A} I(Y_{\ell}; U_{\ell} | X, W), \quad \forall A \subseteq \{1, \ldots, L\}$$

where the union is over all joint distributions $p(x, \{y_{\ell}, u_{\ell}\}_{\ell = 1}^L, w)$ that can be factorized as

$$p(x, \{y_{\ell}, u_{\ell}\}_{\ell = 1}^L, w) = p(x)p(w) \prod_{\ell = 1}^L p(y_{\ell} | x)p(u_{\ell} | y_{\ell}, w)$$

and satisfies

$$\text{mmse}(X | U_1, \ldots, U_L) \leq D$$
In [12], the outer bound is stated in a slightly different form, where there is a time-sharing random variable $T$ involved in the description of the outer bound. However, as pointed out by [12], this time-sharing random variable $T$ can be combined with other auxiliary random variables $(W, U_1, \ldots, U_L)$ to obtain the form of the outer bound we stated here.

We now evaluate this outer bound for the vector Gaussian CEO problem. To this end, we first provide some background information which will be used in the proof.

### 6.1 Background

**Lemma 1 ([8])** Let $(U, X)$ be an arbitrarily correlated random vector with well-defined densities. We assume that $\text{mmse}(X|U) > 0$. Then, we have

$$J(X|U) \succeq \text{mmse}^{-1}(X|U)$$

which is satisfied with equality if $(U, X)$ is jointly Gaussian.

Next, we note the following lemma which will be used subsequently.

**Lemma 2 ([16,17])** Let $(U, X)$ be an arbitrary random vector, where the conditional Fisher information of $X$, conditioned on $U$, exists. Then, we have

$$\frac{1}{2} \log |(2\pi e)^{-1} J^{-1}(X|U)| \leq h(X|U)$$

We also need the following lemma in the upcoming proof.

**Lemma 3 ([13])** Let $(V_1, V_2)$ be an arbitrary random vector with finite second moments, and $N$ be a zero-mean Gaussian random vector with covariance $\Sigma_N$. Assume $(V_1, V_2)$ and $N$ are independent. We have

$$\text{mmse}(V_2|V_1, V_2 + N) = \Sigma_N - \Sigma_N J(V_2 + N|V_1) \Sigma_N$$

### 6.2 Proof

Here, we consider the rate bounds in (70) and obtain a lower bound for them for a given $(W, U_1, \ldots, U_L)$. First, we consider the following mutual information terms

$$I(Y_\ell; U_\ell|X, W) = h(Y_\ell|X, W) - h(Y_\ell|X, W, U_\ell)$$

$$= h(Y_\ell|X) - h(Y_\ell|X, W, U_\ell)$$

$$= \frac{1}{2} \log |(2\pi e)^{-1} \Sigma_\ell| - h(Y_\ell|X, W, U_\ell)$$

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Using Lemma 2 and the fact that jointly Gaussian \((X, W, U, Y, U)\) maximizes \(h(Y|X, W, U)\), we have the following bounds for the second term in (78)

\[
\frac{1}{2} \log \left| (2\pi e)^{-1} \right| \leq h(Y|X, W, U) \leq \frac{1}{2} \log \left| \text{mmse}(Y|X, W, U) \right| \quad (79)
\]

Next, we define the function \(D_\ell(\alpha_\ell)\) as follows

\[
D_\ell(\alpha_\ell) = \alpha_\ell J^{-1}(Y|X, W, U) + \bar{\alpha}_\ell \text{mmse}(Y|X, W, U) \quad (80)
\]

where \(\alpha_\ell = 1 - \bar{\alpha}_\ell \in [0, 1]\). Using the function in (80), the bounds in (79) can be expressed as follows

\[
\frac{1}{2} \log \left| (2\pi e) D_\ell(1) \right| \leq h(Y|X, W, U) \leq \frac{1}{2} \log \left| (2\pi e) D_\ell(0) \right| \quad (81)
\]

Since \(\log \left| (2\pi e) D_\ell(\alpha) \right|\) is continuous in \(\alpha\), due to the intermediate value theorem, there exists an \(\alpha^*_\ell = 1 - \bar{\alpha}^*_\ell \in [0, 1]\) such that

\[
h(Y|X, W, U) = \frac{1}{2} \log \left| (2\pi e) D_\ell(\alpha^*_\ell) \right| \quad (82)
\]

\[
= \frac{1}{2} \log \left| (2\pi e) \left( \alpha^*_\ell J^{-1}(Y|X, W, U) + \bar{\alpha}^*_\ell \text{mmse}(Y|X, W, U) \right) \right| \quad (83)
\]

Hence, using (83) in (78), we have

\[
I(Y_\ell; U_\ell|X, W) = \frac{1}{2} \log \left| \frac{\Sigma_\ell}{|D_\ell(\alpha^*_\ell)|} \right|, \quad \ell = 1, \ldots, L \quad (84)
\]

We note the following bounds on \(D_\ell(\alpha^*_\ell)\)

\[
J^{-1}(Y|X, W, U) \leq D_\ell(\alpha^*_\ell) \leq \text{mmse}(Y|X, W, U) \quad (85)
\]

\[
\leq \text{mmse}(Y|X) \quad (86)
\]

\[
= \Sigma_\ell \quad (87)
\]

where (85) is due to Lemma 1 and (86) comes from the fact that conditioning reduces the MMSE matrix in the positive semi-definite ordering sense.

Next, we consider the following mutual information term

\[
I(X; \{U_\ell\}_{\ell \in A}|\{U_\ell\}_{\ell \in A^c}) = h(X|\{U_\ell\}_{\ell \in A^c}) - h(X|U_1, \ldots, U_L) \quad (88)
\]

\[
\geq h(X|\{U_\ell\}_{\ell \in A^c}) - \frac{1}{2} \log \left| (2\pi e) \text{mmse}(X|U_1, \ldots, U_L) \right| \quad (89)
\]

\[
\geq h(X|\{U_\ell\}_{\ell \in A^c}) - \frac{1}{2} \log \left| (2\pi e) D \right| \quad (90)
\]

\[
\geq h(X|\{U_\ell\}_{\ell \in A^c}, W) - \frac{1}{2} \log \left| (2\pi e) D \right| \quad (91)
\]
where (89) comes from the fact that \( h(X|U_1, \ldots, U_L) \) is maximized by jointly Gaussian \((X, U_1, \ldots, U_L)\), (90) follows from the monotonicity of \( \log |\cdot| \) function in positive semi-definite matrices in conjunction with the distortion constraint in (72), and (91) comes from the fact that conditioning cannot increase entropy.

Next, we obtain a lower bound for \( h(X|\{U_\ell\}_{\ell \in A^c}, W) \). To this end, in view of Lemma 2, we note the following lower bound on \( h(X|\{U_\ell\}_{\ell \in A^c}, W) \)

\[
\begin{align*}
\log |J^{-1}(X|\{U_\ell\}_{\ell \in A^c}, W)| \\
\geq \frac{1}{2} \log \left((2\pi e)^{-1} \det \left( \mathbf{K}^{-1} + \sum_{\ell \in A^c} \Sigma_\ell^{-1} \right) \right)
\end{align*}
\]

which implies that a lower bound on \( J^{-1}(X|\{U_\ell\}_{\ell \in A^c}, W) \) will yield a lower bound for \( h(X|\{U_\ell\}_{\ell \in A^c}, W) \). To obtain a lower bound for \( J^{-1}(X|\{U_\ell\}_{\ell \in A^c}, W) \), we will use the connection between the Fisher information and the MMSE given in Lemma 3. To this end, we note that \( X \) can be decomposed as (see (125) in Appendix A.1)

\[
X = \sum_{\ell \in A^c} A_\ell Y_\ell + N_{A^c}
\]

where the matrices \( \{A_\ell\}_{\ell \in A^c} \) are given by (see (127) in Appendix A.1)

\[
A_\ell = \Sigma_{A^c} \Sigma_\ell^{-1}, \quad \ell \in A^c
\]

In (93), \( N_{A^c} \) is a zero-mean Gaussian vector with covariance matrix (see (126) in Appendix A.1)

\[
\Sigma_{A^c} = \left( K_X^{-1} + \sum_{\ell \in A^c} \Sigma_\ell^{-1} \right)^{-1}
\]

We also note that \( N_{A^c} \) is independent of \((\{Y_\ell, U_\ell\}_{\ell \in A^c}, W)\) which implies the following Markov chain

\[
\{U_\ell\}_{\ell \in A^c}, W \rightarrow \sum_{\ell \in A^c} A_\ell Y_\ell \rightarrow X = \sum_{\ell \in A^c} A_\ell Y_\ell + N_{A^c}
\]

In view of this Markov chain, due to Lemma 3, we have

\[
\mmse(S_{A^c}|X, \{U_\ell\}_{\ell \in A^c}, W) = \Sigma_{A^c} - \Sigma_{A^c} J(X|\{U_\ell\}_{\ell \in A^c}, W) \Sigma_{A^c}
\]

where we define \( S_{A^c} \) as follows

\[
S_{A^c} = \sum_{\ell \in A^c} A_\ell Y_\ell
\]

Next, we obtain the MMSE matrix in (97) in terms of the individual MMSE matrices
\{\text{mmse}(Y_\ell|X, U_\ell, W)\}_{\ell \in \mathcal{A}^c} \text{ as given in the following lemma.}

**Lemma 4** Under the current conditions, we have

\[
\text{mmse}(S_{\mathcal{A}^c}|X, \{U_\ell\}_{\ell \in \mathcal{A}^c}, W) = \sum_{\ell \in \mathcal{A}^c} A_\ell \text{mmse}(Y_\ell|X, W, U_\ell) A_\ell^T \tag{99}
\]

The proof of this lemma is given in Appendix G.

Hence, using Lemma 4 in (97), we get

\[
\Sigma_{\mathcal{A}^c} - \Sigma_{\mathcal{A}^c} J(X|\{U_\ell\}_{\ell \in \mathcal{A}^c}, W) \Sigma_{\mathcal{A}^c} = \sum_{\ell \in \mathcal{A}^c} A_\ell \text{mmse}(Y_\ell|X, W, U_\ell) A_\ell^T \tag{100}
\]

\[
\geq \sum_{\ell \in \mathcal{A}^c} A_\ell D_\ell (\alpha^*_\ell) A_\ell^T \tag{101}
\]

\[
= \Sigma_{\mathcal{A}^c} \left( \sum_{\ell \in \mathcal{A}^c} \Sigma^{-1}_\ell D_\ell (\alpha^*_\ell) \Sigma^{-1}_\ell \right) \Sigma_{\mathcal{A}^c} \tag{102}
\]

where (101) is due to (85), and in (102), we use the definition of \(A_\ell\) given in (94). We note that (102) implies

\[
J^{-1}(X|\{U_\ell\}_{\ell \in \mathcal{A}^c}, W) \geq \left( \Sigma^{-1}_{\mathcal{A}^c} - \sum_{\ell \in \mathcal{A}^c} \Sigma^{-1}_\ell D_\ell (\alpha^*_\ell) \Sigma^{-1}_\ell \right)^{-1} \tag{103}
\]

\[
= \left( K^{-1}_X + \sum_{\ell \in \mathcal{A}^c} \Sigma^{-1}_\ell - \sum_{\ell \in \mathcal{A}^c} \Sigma^{-1}_\ell D_\ell (\alpha^*_\ell) \Sigma^{-1}_\ell \right)^{-1} \tag{104}
\]

where (104) comes from the definition of \(\Sigma_{\mathcal{A}^c}\) in (95). In view of (92) and (104), we have the following lower bound for \(h(X|\{U_\ell\}_{\ell \in \mathcal{A}^c}, W)\) as follows

\[
h(X|\{U_\ell\}_{\ell \in \mathcal{A}^c}, W) \geq \frac{1}{2} \log \left( 2\pi e \left( K^{-1}_X + \sum_{\ell \in \mathcal{A}^c} \Sigma^{-1}_\ell - \sum_{\ell \in \mathcal{A}^c} \Sigma^{-1}_\ell D_\ell (\alpha^*_\ell) \Sigma^{-1}_\ell \right)^{-1} \right) \tag{105}
\]

Hence, using (105) in (91), we get

\[
I(X; \{U_\ell\}_{\ell \in \mathcal{A}}|\{U_\ell\}_{\ell \in \mathcal{A}^c}) \geq \frac{1}{2} \log \left( \frac{|K^{-1}_X + \sum_{\ell \in \mathcal{A}^c} \Sigma^{-1}_\ell - \sum_{\ell \in \mathcal{A}^c} \Sigma^{-1}_\ell D_\ell (\alpha^*_\ell) \Sigma^{-1}_\ell |^{-1}}{|D|} \right) \tag{106}
\]

Moreover, using the non-negativity of the mutual information, we can improve this lower bound as follows

\[
I(X; \{U_\ell\}_{\ell \in \mathcal{A}}|\{U_\ell\}_{\ell \in \mathcal{A}^c}) \geq \frac{1}{2} \log^+ \left( \frac{|K^{-1}_X + \sum_{\ell \in \mathcal{A}^c} \Sigma^{-1}_\ell - \sum_{\ell \in \mathcal{A}^c} \Sigma^{-1}_\ell D_\ell (\alpha^*_\ell) \Sigma^{-1}_\ell |^{-1}}{|D|} \right) \tag{107}
\]
where $\log^+ x = \max(\log x, 0)$. Using (84) and (107) in the rate bounds given in (70), we get

$$\sum_{\ell \in A} R_\ell \geq \frac{1}{2} \log^+ \left| \left( K^{-1}_X + \sum_{\ell \in A^c} \Sigma^{-1}_\ell - \sum_{\ell \in A^c} \Sigma^{-1}_\ell D_\ell (\alpha^*_\ell) \Sigma^{-1}_\ell \right)^{-1} \right| + \frac{1}{2} \sum_{\ell \in A} \log \left| \frac{\Sigma_\ell}{|D_\ell (\alpha^*_\ell)|} \right| $$  

(108)

Next, we establish a connection between $D$ and $(D_1(\alpha^*_1), \ldots, D_L(\alpha^*_L))$. To this end, by taking $A^c = \{1, \ldots, L\}$ in (104), we get

$$\left( K^{-1}_X + \sum_{\ell = 1}^L \Sigma^{-1}_\ell - \sum_{\ell = 1}^L \Sigma^{-1}_\ell D_\ell (\alpha^*_\ell) \Sigma^{-1}_\ell \right)^{-1} \preceq \text{J}^{-1}(X|\{U_\ell\}_{\ell = 1}^L, W) $$

(109)

$$\preceq \text{mmse}(X|\{U_\ell\}_{\ell = 1}^L, W) $$

(110)

$$\preceq \text{mmse}(X|\{U_\ell\}_{\ell = 1}^L) $$

(111)

$$\preceq D $$

(112)

where (110) is due to Lemma 1, (111) comes from the fact that conditioning reduces the MMSE matrix in the positive semi-definite ordering sense, and (112) follows from the distortion constraint in (72). Hence, in view of (108) and (112), we show that the rate region of the vector Gaussian CEO problem is included in the union of rate tuples $(R_1, \ldots, R_L)$ satisfying

$$\sum_{\ell \in A} R_\ell \geq \frac{1}{2} \log^+ \left| \left( K^{-1}_X + \sum_{\ell \in A} \Sigma^{-1}_\ell - \sum_{\ell \in A} \Sigma^{-1}_\ell D_\ell (\alpha^*_\ell) \Sigma^{-1}_\ell \right)^{-1} \right| + \frac{1}{2} \sum_{\ell \in A} \log \left| \frac{\Sigma_\ell}{|D_\ell (\alpha^*_\ell)|} \right| $$

(113)

for all $A \subseteq \{1, \ldots, L\}$, where the union is over all positive semi-definite matrices $D_1(\alpha^*_1), \ldots, D_L(\alpha^*_L)$ satisfying the following orders

$$\left( K^{-1}_X + \sum_{\ell = 1}^L \Sigma^{-1}_\ell - \sum_{\ell = 1}^L \Sigma^{-1}_\ell D_\ell (\alpha^*_\ell) \Sigma^{-1}_\ell \right)^{-1} \preceq D $$

(114)

$$0 \preceq D_\ell (\alpha^*_\ell) \preceq \Sigma_\ell, \quad \ell = 1, \ldots, L $$

(115)

The orders in (115) follow from (87). The region given in Theorem 1 can be obtained from the outer bound described in (113)-(115) by setting $D_\ell (\alpha^*_\ell) = D_\ell$, which completes the proof of Theorem 1.
Generalization of the Bounds

In this section, we consider the most general form of the vector Gaussian CEO problem, and generalize the outer and the inner bounds in Theorem 1 and Theorem 2, respectively. In the most general form of the vector Gaussian CEO problem, the observations at the sensors are given by

\[ Y_\ell = H_\ell X + N_\ell, \quad \ell = 1, \ldots, L \]

(116)

where \( \{N_\ell\}_{\ell=1}^L \) are i.i.d. zero-mean Gaussian random vectors with identity covariance matrices. We note that the general form for the observations in (116) cover the model in (1) we studied so far. All definitions we introduced in Section 2 hold for the general model defined by (116) except for the distortion constraints in (6). In the general model, the distortion \( D \) is assumed to satisfy

\[
\left( K_X^{-1} + \sum_{\ell=1}^L H_\ell^\top H_\ell \right)^{-1} \preceq D \preceq K_X
\]

(117)

where the left hand-side is the MMSE matrix obtained when the CEO unit has access to all observations in (116). Similar to the model given by (1), here also, imposing the lower bound constraint on \( D \) in (117) does not incur any loss of generality, while the upper bound constraint on \( D \) in (117) might incur some loss of generality.

Now, we provide an outer bound for the rate-distortion region \( R(D) \) for the general model given by (116), which, in fact, corresponds to the generalization of the outer bound in Theorem 1 to the most general form of the vector Gaussian CEO problem.

**Theorem 9** An outer bound for the rate-distortion region of the general vector Gaussian CEO problem is given by the union of rate tuples \((R_1, \ldots, R_L)\) satisfying

\[
\sum_{\ell \in A} R_\ell \geq \frac{1}{2} \log + \left| \frac{K_X^{-1} + \sum_{\ell \in A^c} H_\ell^\top (I - D_\ell) H_\ell)^{-1}}{|D|} \right| + \sum_{\ell \in A} \frac{1}{2} \log \frac{1}{|D_\ell|}
\]

(118)

for all \( A \subseteq \{1, \ldots, L\} \), where the union is over all positive semi-definite matrices \( \{D_\ell\}_{\ell=1}^L \) satisfying the following constraints

\[
\left( K_X^{-1} + \sum_{\ell=1}^L H_\ell^\top (I - D_\ell) H_\ell \right)^{-1} \preceq D
\]

(119)

\[
0 \preceq D_\ell \preceq I, \quad \ell = 1, \ldots, L
\]

(120)

We prove Theorem 9 in two steps. In the first step, we enhance (improve) the observations at the sensors in a way that the enhanced observations are in a similar form given by (1).
In the next step, we use Theorem 1 to obtain an outer bound for the enhanced model, and from this outer bound, we obtain Theorem 9 by using some limiting arguments. The details of the proof can be found in Appendix H.

Now, we introduce an inner bound for the rate-distortion region $R(D)$ for the general model given by (116), which, in fact, corresponds to the generalization of the inner bound in Theorem 2 to the most general form of the vector Gaussian CEO problem.

**Theorem 10** An inner bound for the rate-distortion region of the general vector Gaussian CEO problem is given by the union of rate tuples $(R_1, \ldots, R_L)$ satisfying

$$\sum_{\ell \in A} R_\ell \geq \frac{1}{2} \log \left( \frac{(K_X^{-1} + \sum_{\ell \in A^c} H_\ell^\top (I - D_\ell) H_\ell)^{-1}}{(K_X^{-1} + \sum_{\ell=1}^L H_\ell^\top (I - D_\ell) H_\ell)^{-1}} \right) + \sum_{\ell \in A} \frac{1}{2} \log \frac{1}{|D_\ell|}$$

for all $A \subseteq \{1, \ldots, L\}$, where the union is over all positive semi-definite matrices $\{D_\ell\}_{\ell=1}^L$ satisfying

$$\left( K_X^{-1} + \sum_{\ell=1}^L H_\ell^\top (I - D_\ell) H_\ell \right)^{-1} \preceq D$$

$$0 \preceq D_\ell \preceq I, \quad \ell = 1, \ldots, L$$

The proof of Theorem 10 is given in Appendix H. We obtain this inner bound by evaluating the Berger-Tung inner bound [5] by jointly Gaussian auxiliary random variables.

We note that since the outer and the inner bounds in Theorem 9 and Theorem 10 correspond to the generalizations of the outer and inner bounds in Theorem 1 and Theorem 2 respectively, our previous comments and remarks about Theorem 1 and Theorem 2 hold for Theorem 9 and Theorem 10 as well. In particular, similar to Theorem 1 and Theorem 2, we can provide alternative characterizations for Theorem 9 and Theorem 10 as well. Moreover, similar to Theorem 1 and Theorem 2, the bounds in Theorem 9 and Theorem 10 match when the boundary of the outer bound in Theorem 9 can be described by the matrices $\{D_\ell\}_{\ell=1}^L$ that satisfy the distortion constraint in (119) with equality.

### 8 Conclusions

In this paper, we study the vector Gaussian CEO problem and provide an outer bound for its rate-distortion region. We obtain our outer bound by evaluating the rather general outer bound in [12]. We accomplish this evaluation by using a technique that relies on the de Bruijn identity along with the properties of the MMSE and Fisher information. We show that our outer bound strictly improves the existing outer bounds by providing an example, in which, our outer bound is strictly contained in the existing outer bounds. However, despite
this improvement, we show that our outer bound does not provide the exact rate-distortion region in general. We show this by providing an example where our outer bound strictly includes the rate-distortion region.

Appendices

A Distortion Limits

In this appendix, we first note some facts about Gaussian random vectors that are used throughout the paper.

A.1 Gaussian Random Vectors

Let $\mathbf{T}$ be a zero-mean Gaussian random vector with covariance matrix $\Sigma_T \succ 0$. We define the Gaussian random vectors $\{\mathbf{T}_\ell\}_{\ell=1}^L$ as

$$\mathbf{T}_\ell = \mathbf{H}_\ell \mathbf{T} + \mathbf{N}_\ell$$

where $\{\mathbf{N}_\ell\}_{\ell=1}^L$ are zero-mean independent Gaussian random vectors with covariance matrices $\{\Sigma_\ell\}_{\ell=1}^L$, which are also independent of $\mathbf{T}$. We assume $\Sigma_\ell \succ 0$, $\ell = 1, \ldots, L$.

For any subset $\mathcal{A} \subseteq \{1, \ldots, L\}$, we have

$$\mathbf{T} = \sum_{\ell \in \mathcal{A}} \mathbf{A}_\ell \mathbf{T}_\ell + \mathbf{N}_\mathcal{A}$$

where $\mathbf{N}_\mathcal{A}$ is a zero-mean Gaussian random vector with covariance matrix $\Sigma_\mathcal{A}$ given by

$$\Sigma_\mathcal{A} = \left( \Sigma_T^{-1} + \sum_{\ell \in \mathcal{A}} \mathbf{H}_\ell^\top \Sigma^{-1}_\ell \mathbf{H}_\ell \right)^{-1}$$

and is independent of $\{\mathbf{T}_\ell\}_{\ell \in \mathcal{A}}$. The matrices $\{\mathbf{A}_\ell\}_{\ell \in \mathcal{A}}$ are given by

$$\mathbf{A}_\ell = \Sigma_\mathcal{A} \mathbf{H}_\ell^\top \Sigma^{-1}_\ell, \quad \ell \in \mathcal{A}$$

The decomposition in (125) follows from the MMSE estimation of Gaussian random vectors, which is equivalent to the linear MMSE estimation. In particular, we have

$$\hat{\mathbf{T}} = E[\mathbf{T}|\{\mathbf{T}_\ell\}_{\ell \in \mathcal{A}}] = \sum_{\ell \in \mathcal{A}} \mathbf{A}_\ell \mathbf{T}_\ell$$

which is the MMSE, equivalently the linear MMSE, estimator of $\mathbf{T}$ from $\{\mathbf{T}_\ell\}_{\ell=1}^L$. The error
in estimation is $N_A$, and the MMSE matrix is

$$\text{mmse}(T|\{T_\ell\}_{\ell \in A}) = \Sigma_A$$  (129)

### A.2 Regarding (6)

We first obtain the lower bound on the distortion constraint $D$ in (6) as follows

$$\text{mmse}(X_i|B_1^n, \ldots, B_L^n) \succeq \text{mmse}(X_i|B_1^n, \ldots, B_L^n, Y_1^n, \ldots, Y_L^n)$$  (130)

$$= \text{mmse}(X_i|Y_1^n, \ldots, Y_L^n)$$  (131)

$$= \text{mmse}(X_i|Y_{1,i}, \ldots, Y_{L,i})$$  (132)

$$= \left( K_X^{-1} + \sum_{\ell=1}^{L} \Sigma_{\ell}^{-1} \right)^{-1}$$  (133)

where (130) follows from the fact that conditioning reduces the MMSE matrix in the positive semi-definite ordering sense, (131) is due to the fact that $B_\ell^n$ is a function of $Y_\ell^n$, (132) comes from the independence of $(X_i, Y_{1,i}, \ldots, Y_{L,i})$ across time, and (133) is due to (126) and (129).

Hence, (133) implies that imposing the constraint $D \succeq K_X$ does not incur any loss of generality.

Next, we consider the upper bound on the distortion constraint in (72). To this end, we note the following order

$$\text{mmse}(X_i|B_1^n, \ldots, B_L^n) \preceq \text{mmse}(X) = K_X$$  (134)

where we use the fact that conditioning reduces the MMSE matrix in the positive semi-definite ordering sense. Equation (134) implies that all $(n, R_1, \ldots, R_L)$ codes achieve a distortion which is smaller than $K_X$. In other words, if $\hat{D}$ is the distortion achieved by a specific code, we always have $\hat{D} \preceq K_X$. In spite of this fact, we still cannot impose the constraint $D \preceq K_X$ without loss of generality. To demonstrate this point, assume that $K_X - D$ is indefinite. Hence, to be able to impose the constraint $D \preceq K_X$, we should find a new distortion constraint $D'$ which satisfies $D' \preceq \{D, K_X\}$ and the rate-distortion regions $R(D)$ and $R(D')$ are identical. In other words, there needs to be a distortion matrix $D' \preceq \{D, K_X\}$, and for any code achieving a distortion $\hat{D} \preceq D$, we also have $\hat{D} \preceq D'$. However, as we will show now, this is not possible in general. Assume that there exist two codes achieving the distortion $\hat{D}_j$, $j = 1, 2$, where $\hat{D}_j \preceq \{D, K_X\}$. Hence, $D'$ needs to satisfy

$$\{\hat{D}_1, \hat{D}_2\} \preceq D' \preceq \{D, K_X\}$$  (135)

However, there are cases where it is impossible to find a matrix $D'$ satisfying the order
in (135) as shown in [18, Appendix I] by a counter-example. Consequently, imposing the constraint \( D \preceq K_X \) might incur some loss of generality.

B Proofs of Theorem 3 and Theorem 4

Here, we prove only Theorem 3. The proof of Theorem 4 is similar to the proof of Theorem 3 and can be concluded from the proof we present here. First, we note that our outer bound in Theorem 1 can be expressed as

\[
\sum_{\ell \in A} \bar{R}_\ell \geq f(A), \quad A \subseteq \{1, \ldots, L\} \tag{136}
\]

where

\[
\bar{R}_\ell = R_\ell - \frac{1}{2} \log \frac{\left| \Sigma_\ell \right|}{\left| D_\ell \right|}, \quad \ell = 1, \ldots, L \tag{137}
\]

\[
f(A) = \frac{1}{2} \log^+ \left| \left( K_X^{-1} + \sum_{\ell \in A} \Sigma_\ell^{-1} (\Sigma_\ell - D_\ell) \Sigma_\ell^{-1} \right)^{-1} \right|, \quad \forall A \subseteq \{1, \ldots, L\} \tag{138}
\]

Next, we show that \( f(A) \) satisfies the following properties.

Lemma 5

\[
f(\emptyset) = 0 \tag{139}
\]

\[
f(A \cup \{t\}) \geq f(A), \quad \forall t \in \{1, \ldots, L\} \tag{140}
\]

\[
f(A \cup B) + f(A \cap B) \geq f(A) + f(B) \tag{141}
\]

The proof of Lemma 5 is given in Appendix B.1.

A set function \( f(A) \) satisfying the properties in Lemma 5 is called a supermodular function. The region defined by means of a supermodular function as in (136) is called a contra-polymatroid [19]. We denote the contra-polymatroid defined in (136) by \( G(f) \). An important property of contra-polymatroids is that all of their vertices can be found in an explicit form. In particular, when \( \mu_1 \geq \ldots \geq \mu_L \geq 0 \), the vertex corresponding to the tangent hyperplane \( \sum_{\ell=1}^L \mu_\ell \bar{R}_\ell \) is given by [20, Lemma 3.3]

\[
\bar{R}'_\ell = f(\{1, \ldots, \ell\}) - f(\{1, \ldots, \ell - 1\}), \quad \ell = 1, \ldots, L \tag{142}
\]
using which, we have

\[
\min_{(R_1, \ldots, R_L) \in G(f)} \sum_{\ell=1}^{L} \mu_\ell \bar{R}_\ell = \sum_{\ell=1}^{L} \mu_\ell (f(\{1, \ldots, \ell\}) - f(\{1, \ldots, \ell-1\}))
\]

(143)

\[
= \sum_{\ell=1}^{L} \mu_\ell f(\{1, \ldots, \ell\}) - \sum_{\ell=1}^{L-1} \mu_{\ell+1} f(\{1, \ldots, \ell\})
\]

(144)

\[
= \sum_{\ell=1}^{L-1} \frac{\mu_\ell - \mu_{\ell+1}}{2} \log^+ \left| \frac{\left( K_X^{-1} + \sum_{j=\ell+1}^{L} \Sigma_j^{-1} (\Sigma_j - D_j) \Sigma_j^{-1} \right)^{-1}}{|D|} \right| + \frac{\mu_L}{2} \log \left| \frac{K_X}{|D|} \right|
\]

(145)

which, in turn, implies

\[
\min_{(R_1, \ldots, R_L) \in R(D)} \sum_{\ell=1}^{L} \mu_\ell R_\ell = \min_{\{D_\ell\}_{\ell=1}^{L}} \sum_{\ell=1}^{L-1} \frac{\mu_\ell - \mu_{\ell+1}}{2} \log^+ \left| \frac{\left( K_X^{-1} + \sum_{j=\ell+1}^{L} \Sigma_j^{-1} (\Sigma_j - D_j) \Sigma_j^{-1} \right)^{-1}}{|D|} \right| + \frac{\mu_L}{2} \log \left| \frac{K_X}{|D|} \right|
\]

(146)

where \(\{D_\ell\}_{\ell=1}^{L}\) are subject to the constraints in (16)-(17). Since (146) is the desired result in Theorem 3, this completes the proof of Theorem 3.

### B.1 Proof of Lemma 5

Now we prove Lemma 5. The first property of the set function \(f(A)\) given in (139) is immediate by noting from (8) that

\[
\left( K_X^{-1} + \sum_{\ell=1}^{L} \Sigma_\ell^{-1} - \Sigma_i^{-1} D_i \Sigma_\ell^{-1} \right)^{-1} \preceq D
\]

(147)

Next, we prove (140) and (141). To this end, we define the jointly Gaussian random vector tuple \((U_1^*, \ldots, U_L^*)\) which satisfies the Markov chain

\[
U_i^* \rightarrow X \rightarrow \{U_j^*_\}_{j=1, j \neq i} \quad i = 1, \ldots, L
\]

(148)

and

\[
h(X | \{U_\ell^*_\}_{\ell \in A^c}) = \frac{1}{2} \log \left| \frac{\left( K_X^{-1} + \sum_{\ell \in A^c} \Sigma_\ell^{-1} - \sum_{\ell \in A^c} \Sigma_\ell^{-1} D_\ell \Sigma_\ell^{-1} \right)^{-1}}{|D|} \right|
\]

(149)

for all \(A^c \subseteq \{1, \ldots, L\}\). Here, we do not show the existence of the jointly Gaussian random vector tuple \((U_1^*, \ldots, U_L^*)\) satisfying (148)-(149), however the existence of such Gaussian
random vector tuples can be concluded from the analysis in Appendix I where we prove Theorem 2 (the inner bound for the rate-distortion region). Hence, using \((U^*_1, \ldots, U^*_L)\), the set function \(f(A)\) can be written as

\[
f(A) = \max \left( 0, h(X|\{U^*_\ell\}_{\ell \in A^c}) - \frac{1}{2} \log \left| (2\pi e)^D \right| \right), \quad A \subseteq \{1, \ldots, L\}
\]  
(150)

The monotonicity of the set function \(f(A)\) can be shown as follows

\[
f(A \cup \{t\}) = \max \left( 0, h(X|\{U^*_\ell\}_{\ell \in A^c, \ell \neq t}) - \frac{1}{2} \log \left| (2\pi e)^D \right| \right)
\]  
(151)

\[
\geq \max \left( 0, h(X|\{U^*_\ell\}_{\ell \in A_c}) - \frac{1}{2} \log \left| (2\pi e)^D \right| \right)
\]  
(152)

\[
= f(A)
\]  
(153)

where (152) follows from the fact that conditioning cannot increase entropy. Equation (153) proves (140).

Finally, we consider (141) as follows

\[
f(A \cup B) + f(A \cap B)
\]

\[
= \max \left( 0, h(X|\{U^*_\ell\}_{\ell \in A^c \cap B^c}) - \frac{\alpha}{2} \right) + \max \left( 0, h(X|\{U^*_\ell\}_{\ell \in A^c \cup B^c}) - \frac{\alpha}{2} \right)
\]  
(154)

\[
= \max \left( 0, h(X|\{U^*_\ell\}_{\ell \in A^c \cap B^c}) - \frac{\alpha}{2} \right) + \max \left( 0, h(X|\{U^*_\ell\}_{\ell \in A^c \cup B^c}) - \frac{\alpha}{2} \right)
\]  
(155)

\[
\geq \max \left( 0, h(X|\{U^*_\ell\}_{\ell \in A^c \cap B^c}) - \frac{\alpha}{2} \right) \geq \max \left( h(X|\{U^*_\ell\}_{\ell \in A^c}), h(X|\{U^*_\ell\}_{\ell \in B^c}) \right)
\]  
(156)

where \(\alpha = \log \left| (2\pi e)^D \right|\), and (155)-(156) follow from

\[
h(X|\{U^*_\ell\}_{\ell \in A^c \cap B^c}) \geq h(X|\{U^*_\ell\}_{\ell \in A^c \cup B^c})
\]  
(157)

\[
h(X|\{U^*_\ell\}_{\ell \in A^c \cap B^c}) \geq \max \left( h(X|\{U^*_\ell\}_{\ell \in A^c}), h(X|\{U^*_\ell\}_{\ell \in B^c}) \right)
\]  
(158)

respectively, which, in turn, come from the fact that conditioning cannot increase entropy.
Next, we consider the last term in (156) as follows

\[ h(X \{ U^*_\ell \}_{\ell \in \mathcal{A} \cap \mathcal{B}^c} + h(X \{ U^*_\ell \}_{\ell \in \mathcal{A} \cup \mathcal{B}^c}) \]
\[ = h(X, \{ U^*_\ell \}_{\ell \in \mathcal{A} \cap \mathcal{B}^c}) + h(X, \{ U^*_\ell \}_{\ell \in \mathcal{A} \cup \mathcal{B}^c}) - h(\{ U^*_\ell \}_{\ell \in \mathcal{A} \cap \mathcal{B}^c}) - h(\{ U^*_\ell \}_{\ell \in \mathcal{A} \cup \mathcal{B}^c}) \] (159)
\[ = 2h(X) + \sum_{\ell \in \mathcal{A}^c} h(U^*_\ell | X) + \sum_{\ell \in \mathcal{B}^c} h(U^*_\ell | X) - h(\{ U^*_\ell \}_{\ell \in \mathcal{A} \cap \mathcal{B}^c}) - h(\{ U^*_\ell \}_{\ell \in \mathcal{A} \cup \mathcal{B}^c}) \] (160)
\[ = 2h(X) + \sum_{\ell \in \mathcal{A}^c} h(U^*_\ell | X) + \sum_{\ell \in \mathcal{B}^c} h(U^*_\ell | X) - h(\{ U^*_\ell \}_{\ell \in \mathcal{A} \cap \mathcal{B}^c}) - h(\{ U^*_\ell \}_{\ell \in \mathcal{A} \cup \mathcal{B}^c}) - h(\{ U^*_\ell \}_{\ell \in \mathcal{A}^c}) \] (161)
\[ \geq h(X \{ U^*_\ell \}_{\ell \in \mathcal{A}^c}) + h(X) + \sum_{\ell \in \mathcal{B}^c} h(U^*_\ell | X) - h(\{ U^*_\ell \}_{\ell \in \mathcal{A} \cap \mathcal{B}^c}) - h(\{ U^*_\ell \}_{\ell \in \mathcal{A} \cup \mathcal{B}^c}) \] (162)
\[ = h(X \{ U^*_\ell \}_{\ell \in \mathcal{A}^c}) + h(X) + \sum_{\ell \in \mathcal{B}^c} h(U^*_\ell | X) - h(\{ U^*_\ell \}_{\ell \in \mathcal{B}^c}) \] (163)
\[ = h(X \{ U^*_\ell \}_{\ell \in \mathcal{A}^c}) + h(X | \{ U^*_\ell \}_{\ell \in \mathcal{B}^c}) \] (164)
\[ = h(X \{ U^*_\ell \}_{\ell \in \mathcal{A}^c}) + h(X) + \sum_{\ell \in \mathcal{B}^c} h(U^*_\ell | X) - h(\{ U^*_\ell \}_{\ell \in \mathcal{B}^c}) \] (165)

where (160) comes from the Markov chain in (148), and (163) follows from the fact that conditioning cannot increase entropy. Using (165) in (156), we get

\[ f(A \cup B) + f(A \cap B) \]
\[ \geq \max \left( 0, h(X \{ U^*_\ell \}_{\ell \in \mathcal{A}^c}) - \frac{\alpha}{2}, h(X | \{ U^*_\ell \}_{\ell \in \mathcal{B}^c}) - \frac{\alpha}{2} \right) \]
\[ = \max \left( 0, h(X \{ U^*_\ell \}_{\ell \in \mathcal{A}^c}) - \frac{\alpha}{2}, h(X | \{ U^*_\ell \}_{\ell \in \mathcal{B}^c}) - \frac{\alpha}{2} \right) \]
\[ = f(A) + f(B) \]

which proves (141); completing the proof of Lemma 5.
C Proofs of Corollaries 3 and 4

C.1 Proof of Corollary 3

We define the set $\mathcal{D}^{++}(D_1, D_2)$ as the union of $(D_1, D_2)$ satisfying

$$
\left( K_X^{-1} + \sum_{\ell=1}^2 \Sigma_{\ell}^{-1} - \sum_{\ell=1}^2 \Sigma_{\ell}^{-1} D_{\ell} \Sigma_{\ell}^{-1} \right)^{-1} \preceq D
$$

(169)

$$
0 \preceq D_1 \preceq \Sigma_1
$$

(170)

We note that $\mathcal{D}^+(D_1, D_2) \subseteq \mathcal{D}^{++}(D_1, D_2)$. Using this in Corollary 2 we have

$$
T^+ \geq \min_{(D_1, D_2) \in \mathcal{D}^{++}(D_1, D_2)} \frac{\mu_1}{2} \log \frac{\Sigma_1}{|D_1|} + \frac{\mu_2}{2} \log \frac{|D_2|}{|D|} + \frac{\mu_1 - \mu_2}{2} \log^+ \frac{|(K_X^{-1} + \Sigma_1^{-1} - \Sigma_2^{-1} D_2 \Sigma_2^{-1})^{-1}|}{|D|}
$$

(171)

$$
\geq \min_{(D_1, D_2) \in \mathcal{D}^{++}(D_1, D_2)} \frac{\mu_1}{2} \log \frac{\Sigma_1}{|D_1|} + \frac{\mu_2}{2} \log \frac{|D_2|}{|D|} + \frac{\mu_1 - \mu_2}{2} \log^+ \frac{|(K_X^{-1} + \Sigma_1^{-1} - \Sigma_2^{-1} D_2 \Sigma_2^{-1})^{-1}|}{|D|}
$$

(172)

$$
= \min_{(D_1, D_2) \in \mathcal{D}^{++}(D_1, D_2)} \frac{\mu_1}{2} \log \frac{\Sigma_1^{-1}}{|D_1|} - \frac{\mu_1}{2} \log \frac{|D_1|}{|D|} + \frac{\mu_2}{2} \log \frac{|\Sigma_2^{-1}|}{|D_2|} - \frac{\mu_2}{2} \log \frac{|D_2|}{|D|} + \frac{\mu_1 - \mu_2}{2} \log^+ \frac{|(K_X^{-1} + \Sigma_1^{-1} + \Sigma_2^{-1} - \Sigma_1^{-1} D_1 \Sigma_1^{-1} - D^{-1})|}{|D|}
$$

(173)

$$
\geq \min_{a \leq D_1 \leq \Sigma_1} \frac{\mu_1}{2} \log \frac{|\Sigma_1^{-1}|}{|D_1|} - \frac{\mu_1}{2} \log \frac{|D_1|}{|D|} + \frac{\mu_2}{2} \log \frac{|\Sigma_2^{-1}|}{|D_2|} - \frac{\mu_2}{2} \log \frac{|D_2|}{|D|} + \frac{\mu_1 - \mu_2}{2} \log^+ \frac{|(K_X^{-1} + \Sigma_1^{-1} + \Sigma_2^{-1} - \Sigma_1^{-1} D_1 \Sigma_1^{-1} - D^{-1})|}{|D|}
$$

(174)

where we obtain (174) by using the fact that $\log |\Sigma_2^{-1} D_2 \Sigma_2^{-1}|$ is monotonically increasing in positive semi-definite matrices $D_2$ and the order on $D_2$ given in (169). Next, we show that the cost function in (174) is monotonically decreasing in $D_1$, or equivalently in $\Sigma_1^{-1} D_1 \Sigma_1^{-1}$. To this end, we consider the gradient of the cost function in (174) with respect to the matrix $\Sigma_1^{-1} D_1 \Sigma_1^{-1}$, which is equivalent to

$$
-\mu_1 (\Sigma_1^{-1} D_1 \Sigma_1^{-1})^{-1} + \mu_2 (K_X^{-1} + \Sigma_1^{-1} + \Sigma_2^{-1} - \Sigma_1^{-1} D_1 \Sigma_1^{-1} - D^{-1})^{-1}
$$

(175)

Next, we show that (175) is strictly negative definite; implying that the cost function in (174) is monotonically decreasing in $D_1$. To this end, using the assumption in (40), we have

$$
\frac{1}{\mu_2} (K_X^{-1} + \Sigma_1^{-1} + \Sigma_2^{-1} - D^{-1}) \succ \left( \frac{1}{\mu_1} + \frac{1}{\mu_2} \right) \Sigma_1^{-1}
$$

(176)

$$
\preceq \left( \frac{1}{\mu_1} + \frac{1}{\mu_2} \right) \Sigma_1^{-1} D_1 \Sigma_1^{-1}
$$

(177)
where we use the fact that $D_1 \preceq \Sigma_1$. We note that the order in (177) can be written as

$$\frac{1}{\mu_2} \left( K_{X}^{-1} + \Sigma_1^{-1} + \Sigma_2^{-1} - \Sigma_1^{-1} D_1 \Sigma_1^{-1} - D^{-1} \right) \succeq \frac{1}{\mu_1} \Sigma_1^{-1} D_1 \Sigma_1^{-1}$$

(178)

which is equivalent to

$$\mu_2 \left( K_{X}^{-1} + \Sigma_1^{-1} + \Sigma_2^{-1} - \Sigma_1^{-1} D_1 \Sigma_1^{-1} - D^{-1} \right)^{-1} \succeq \mu_1 \left( \Sigma_1^{-1} D_1 \Sigma_1^{-1} \right)^{-1}$$

(179)

which, in turn, implies that the gradient of the cost function in (174) is negative definite, and hence, the cost function in (174) is monotonically decreasing in $D$, which implies that the gradient of the cost function in (174) is negative definite, and hence, the cost function in (174) is monotonically decreasing in $D$. Consequently, this implies that the minimum in (174) is attained when $D_1 = \Sigma_1$, i.e., we have

$$T^+ \geq \frac{\mu_2}{2} \log \frac{\Sigma_2^{-1}}{|K_{X}^{-1} + \Sigma_2^{-1} - D^{-1}|} + \frac{\mu_2}{2} \log \frac{|K_{X}|}{|D|}$$

(180)

Finally, we note that $(D_1 = \Sigma_1, D_2 = \Sigma_2 (K_{X}^{-1} + \Sigma_2^{-1} - D^{-1}) \Sigma_2) \in D^+(D_1, D_2)$ attains the lower bound for $T^+$ in (180); which completes the proof of Corollary 3.

### C.2 Proof of Corollary 4

To obtain an outer bound for $T^-$, we consider the following $(D_1, D_2)$ pair

$$D_1 = \Sigma_1$$

$$D_2 = \frac{\mu_2}{\mu_1} \Sigma_2 \left( K_{X}^{-1} + \Sigma_2^{-1} \right) \Sigma_2$$

(181)

(182)

which is feasible, i.e., $(D_1, D_2) \in D^-(D_1, D_2)$. (To show that $D_2$ is feasible, we use (11).) Consequently, using this pair of matrices in the cost function of $T^-$, we get the following upper bound for $T^-$

$$T^- \leq \frac{\mu_2}{2} \log \frac{|\Sigma_2^{-1}|}{\mu_2 \mu_1 \left( K_{X}^{-1} + \Sigma_2^{-1} \right)} + \frac{\mu_2}{2} \log \frac{|K_{X}|}{|D|} + \frac{\mu_1 - \mu_2}{2} \log \frac{\mu_1}{\mu_1 - \mu_2} \left( K_{X}^{-1} + \Sigma_2^{-1} \right)^{-1}$$

(183)

$$= \frac{\mu_2}{2} \log \frac{|\Sigma_2^{-1}|}{|K_{X}^{-1} + \Sigma_2^{-1} - D^{-1}|} + \frac{\mu_2}{2} \log \frac{|K_{X}^{-1} + \Sigma_2^{-1} - D^{-1}|}{\mu_2 \mu_1 \left( K_{X}^{-1} + \Sigma_2^{-1} \right)} + \frac{\mu_2}{2} \log \frac{|K_{X}|}{|D|}$$

$$+ \frac{\mu_1 - \mu_2}{2} \log \frac{\mu_1}{\mu_1 - \mu_2} \left( K_{X}^{-1} + \Sigma_2^{-1} \right)^{-1}$$

(184)

$$= T^+ + \frac{\mu_2}{2} \log \frac{|K_{X}^{-1} + \Sigma_2^{-1} - D^{-1}|}{\mu_2 \mu_1 \left( K_{X}^{-1} + \Sigma_2^{-1} \right)} + \frac{\mu_1 - \mu_2}{2} \log \frac{\mu_1}{\mu_1 - \mu_2} \left( K_{X}^{-1} + \Sigma_2^{-1} \right)^{-1}$$

(185)

which is the desired end result in Corollary 4, completing the proof.
D Proof of Theorem 7

We prove Theorem 7 in two steps. In the first step, we specialize the outer bound in [12] to the parallel model defined by the following joint distribution

\[
p(x^M, \{y^M_\ell\}_{\ell=1}^L) = \prod_{m=1}^M p(x_m) \prod_{\ell=1}^L p(y_{\ell m}|x_m)
\]  

(186)

Next, we evaluate the outer bound we obtain in the first step, and show that it can be attained by the inner bound provided in Theorem 2.

D.1 A General Outer Bound

First, we restate the outer bound in [12] for the parallel model satisfying (186) as follows.

**Theorem 11 ([12, Theorem 1])** We have

\[
R_{p}(\{D_m\}_{m=1}^M) \subseteq R_{p-o}(\{D_m\}_{m=1}^M),
\]

where \(R_{p-o}(\{D_m\}_{m=1}^M)\) is given by the union of rate tuples \((R_1, \ldots, R_L)\) satisfying

\[
\sum_{\ell \in A} R_\ell \geq I(X^M; \{U_\ell\}_{\ell \in A}|\{U_\ell\}_{\ell \in A^c}) + \sum_{\ell \in A} I(U_\ell; Y^M|X^M, W)
\]  

(187)

for all \(A \subseteq \{1, \ldots, L\}\), where the union is over all \(\{U_\ell\}_{\ell=1}^L\) satisfying

\[
p(x^M, \{y^M_\ell\}_{\ell=1}^L, \{u_\ell\}_{\ell=1}^L, w) = p(w) \prod_{m=1}^M p(x_m) \prod_{\ell=1}^L p(y_{\ell m}|x_m)p(u_\ell|w, y^M_\ell)
\]  

(188)

and

\[
\text{mmse}(X_m|U_1, \ldots, U_L) \leq D_m, \quad m = 1, \ldots, M
\]  

(189)

Next, we define the following auxiliary random variables

\[
U_{\ell m} = U_\ell X^{m-1}, \quad \ell = 1, \ldots, L, \quad m = 1, \ldots, M \\
W_{m} = W X^{m-1}\{Y^{M}_{\ell, m+1}\}_{\ell=1}^L, \quad m = 1, \ldots, M
\]  

(190) (191)

Using these auxiliary random variables, we will find lower bounds for the rate constraints in (187). We start with the following term

\[
I(X^M; \{U_\ell\}_{\ell \in A}|\{U_\ell\}_{\ell \in A^c}) = \sum_{m=1}^M I(X_m; \{U_\ell\}_{\ell \in A}|\{U_\ell\}_{\ell \in A^c}, X^{m-1})
\]

(192)

\[
= \sum_{m=1}^M I(X_m; \{U_{\ell m}\}_{\ell \in A}|\{U_{\ell m}\}_{\ell \in A^c})
\]  

(193)
Next, we consider the following term

\[
\begin{align*}
I(U_\ell; Y^M \mid X^M, W) &= h(U_\ell; X^M, W) - h(U_\ell; X^M, W, Y^M) \\
&\geq h(U_\ell; X^M, W, \{Y_j^M\}_{j=1,j\neq \ell}^L) - h(U_\ell; X^M, W, Y^M) \\
&= h(U_\ell; X^M, W, \{Y_j^M\}_{j=1,j\neq \ell}^L) - h(U_\ell; X^M, W, \{Y_j^M\}_{j=1}^L) \\
&= I(U_\ell; Y^M \mid X^M, W, \{Y_j^M\}_{j=1,j\neq \ell}^L) \\
&= \sum_{m=1}^M I(U_\ell; Y_{\ell m} \mid X^M, W, \{Y_j^M\}_{j=1,j\neq \ell}^L, Y_{\ell,m+1}^M) \\
&= \sum_{m=1}^M h(Y_{\ell m} \mid X^M, W, \{Y_j^M\}_{j=1,j\neq \ell}^L, Y_{\ell,m+1}^M) - h(Y_{\ell m} \mid X^M, W, \{Y_j^M\}_{j=1,j\neq \ell}^L, Y_{\ell,m+1}^M, U_\ell) \\
&\geq \sum_{m=1}^M h(Y_{\ell m} \mid X^M, W, \{Y_j^M\}_{j=1,j\neq \ell}^L, Y_{\ell,m+1}^M) - h(Y_{\ell m} \mid X^m, W, \{Y_j^M\}_{j=1,j\neq \ell}^L, U_\ell) \\
&= \sum_{m=1}^M I(U_\ell; Y_{\ell m} \mid X^m, W, \{Y_j^M\}_{j=1}^L) \\
&= \sum_{m=1}^M I(U_{\ell m}; Y_{\ell m} \mid X_m, W_m)
\end{align*}
\]

where \((195)\) follows from the fact that conditioning cannot increase entropy, \((196)\) and \((200)\) come from the following Markov chains

\[
\begin{align*}
U_\ell &\rightarrow W, Y^M_\ell \rightarrow X^M, \{Y_j^M\}_{j=1,j\neq \ell}^L \\
Y_{\ell m} &\rightarrow X_m \rightarrow W, X^{m-1}, X_{m+1}^M, \{Y_j^M\}_{j=1,j\neq \ell}^L, Y_{\ell,m+1}^M
\end{align*}
\]

respectively, which are consequences of the joint distribution in \((188)\), and \((201)\) is due to the fact that conditioning cannot increase entropy.

Next, we consider the distortion constraints in \((189)\) as follows

\[
D_m \geq \text{mmse}(X_m \mid \{U_{\ell m}\}_{\ell=1}^L) \\
\geq \text{mmse}(X_m \mid \{U_{\ell m}\}_{\ell=1}^L, X^{m-1}) \\
= \text{mmse}(X_m \mid \{U_{\ell m}\}_{\ell=1}^L)
\]

where we use the fact that conditioning reduces MMSE.
Hence, using (193) and (203), the rate constraints in Theorem 11 can be expressed as
\[ \sum_{\ell \in A} R_{\ell} \geq \sum_{m=1}^{M} I(X_m; \{U_{\ell m}\}_{\ell \in A}\{U_{\ell m}\}_{\ell \in A'}) + \sum_{m=1}^{M} \sum_{\ell \in A} I(U_{\ell m}; Y_{\ell m}|X_m, W_m) \] 
(209)
and the distortion constraints in Theorem 11 are
\[ \text{mmse}(X_m|\{U_{\ell m}\}_{\ell=1}^{L}) \leq D_m \] 
(210)
We note that the random variable tuples
\[ \{(X_m, \{Y_{\ell m}, U_{\ell m}\}_{\ell=1}^{L}, W_m)\}_{m=1}^{M} \] 
(211)
might be correlated over the index \( m \). However, neither the expressions in the rate bounds given by (209) nor the distortion constraints in (210) depend on the entire joint distribution of \( \{(X_m, \{Y_{\ell m}, U_{\ell m}\}_{\ell=1}^{L}, W_m)\}_{m=1}^{M} \). Instead, both the expressions in the rate bounds given by (209) and the distortion constraints in (210) depend only on the distribution of \( (X_m, \{Y_{\ell m}, U_{\ell m}\}_{\ell=1}^{L}, W_m) \) for each \( m \) involved. Hence, without loss of generality, we can assume that
\[ (X_m, \{Y_{\ell m}, U_{\ell m}\}_{\ell=1}^{L}, W_m) \text{ and } \{(X_j, \{Y_{\ell j}, U_{\ell j}\}_{\ell=1}^{L}, W_j)\}_{j=1,j\neq m} \] 
(212)
are independent for all \( m = 1, \ldots, M \). Next, we note that the joint distribution of \( (X_m, \{Y_{\ell m}, U_{\ell m}\}_{\ell=1}^{L}, W_m) \) can be factorized as follows
\[ p(x_m, \{y_{\ell m}, u_{\ell m}\}_{\ell=1}^{L}, w_m) = p(x_m)p(w_m) \prod_{\ell=1}^{L} p(y_{\ell m}|x_m)p(u_{\ell m}|y_{\ell m}, w_m) \] 
(213)
whose proof is given in Appendix D.3. In view of (209)-(210) and (213), we obtain the following outer bound for the parallel model.

**Theorem 12** We have \( \mathcal{R}^{p}(\{D_m\}_{m=1}^{M}) \subseteq \mathcal{R}^{p-\alpha}(\{D_m\}_{m=1}^{M}) \), where \( \mathcal{R}^{p-\alpha}(\{D_m\}_{m=1}^{M}) \) is given by the union of rate tuples \( (R_1, \ldots, R_L) \) satisfying
\[ \sum_{\ell \in A} R_{\ell} \geq \sum_{m=1}^{M} I(X_m; \{U_{\ell m}\}_{\ell \in A}\{U_{\ell m}\}_{\ell \in A'}) + \sum_{m=1}^{M} \sum_{\ell \in A} I(U_{\ell m}; Y_{\ell m}|X_m, W_m) \] 
(214)
for all \( A \subseteq \{1, \ldots, L\} \), where the union is over all \( \{U_{\ell m}\}_{\ell \in A} \) satisfying
\[ p(x^M, \{y_{\ell m}^M\}_{\ell=1}^{L}, \{u_{\ell m}^L\}_{\ell=1}^{L}, w) = \prod_{m=1}^{M} p(x_m)p(w_m) \prod_{\ell=1}^{L} p(y_{\ell m}|x_m)p(u_{\ell m}|w_m, y_{\ell m}) \] 
(215)
D.2 Evaluation of the Outer Bound

Now, we evaluate the outer bound in Theorem 12 for the parallel Gaussian model, and show that it is attainable by the inner bound given in Theorem 2. To this end, we note that following the analysis in Section 6, one can evaluate the outer bound in Theorem 12 yielding the following outer bound for the parallel Gaussian model.

**Theorem 13** An outer bound for the rate-distortion region \( R^p(\{D_m\}_{m=1}^M) \) of the parallel Gaussian model is given by

\[
\tilde{R}^p(\{D_m\}_{m=1}^M) \quad \text{which corresponds to the union of rate tuples} \quad (R_1, \ldots, R_L)
\]

satisfying

\[
\sum_{\ell \in A} R_\ell \geq \sum_{m=1}^M \frac{1}{2} \log + \frac{1}{D_m} \left( \frac{1}{\sigma_m^2} + \sum_{\ell \in A^c} \frac{\sigma_{\ell m}^2 - D_{\ell m}}{\sigma_{\ell m}^4} \right)^{-1} + \sum_{m=1}^M \sum_{\ell \in A} \frac{1}{2} \log \frac{\sigma_{\ell m}^2}{D_{\ell m}}
\]

for all \( A \subseteq \{1, \ldots, L\} \), where the union is over all \( \{D_{\ell m}\}_{\ell, m} \) satisfying the following constraints

\[
\left( \frac{1}{\sigma_m^2} + \sum_{\ell=1}^L \frac{\sigma_{\ell m}^2 - D_{\ell m}}{\sigma_{\ell m}^4} \right)^{-1} \leq D_m, \quad m = 1, \ldots, M
\]

and

\[
0 \leq D_{\ell m} \leq \sigma_{\ell m}^2, \quad \ell = 1, \ldots, L, \quad m = 1, \ldots, M
\]

Next, we show that there is no loss of generality to assume that the constraints in (218) are satisfied with equality. To prove this, we consider an alternative description of the outer bound in Theorem 13 by means of the tangent hyperplanes. In other words, we consider the following optimization problem

\[
\min_{(R_1, \ldots, R_L) \in \tilde{R}^p(\{D_m\}_{m=1}^M)} \sum_{\ell=1}^L \mu_\ell R_\ell
\]
where we assume $\mu_1 \geq \ldots \geq \mu_L \geq 0$. Using the analysis in Appendix B, we can express the optimization problem in (220) as follows

$$
\min_{(R_1, \ldots, R_L) \in \tilde{\mathcal{R}}^P} \sum_{\ell=1}^L \mu_\ell R_\ell \\
= \min_{\{D_{\ell m}\}_{\ell, m}} \sum_{\ell=1}^{L-1} \frac{\mu_\ell - \mu_{\ell+1}}{2} \sum_{m=1}^M \log^+ \frac{1}{D_m} \frac{1}{\sigma_m^2} + \sum_{j=\ell+1}^L \frac{\sigma_{jm}^2 - D_{jm}}{\sigma_{jm}^4} - 1 \\
+ \sum_{\ell=1}^L \frac{\mu_L}{2} \sum_{m=1}^M \log \frac{\sigma_{\ell m}^2}{D_{\ell m}} + \sum_{\ell=1}^L \frac{\mu_L}{2} \sum_{m=1}^M \log \frac{\sigma_m^2}{D_m} \\
(221)
$$

$$
= \min_{\{D_{\ell m}\}_{\ell, m}} \sum_{m=1}^M f_m(\{D_{\ell m}\}_{\ell=1}^L) \\
(222)
$$

$$
= \sum_{m=1}^M \min_{\{D_{\ell m}\}_{\ell=1}^L} f_m(\{D_{\ell m}\}_{\ell=1}^L) \\
(223)
$$

where we define the function $f_m(\{D_{\ell m}\}_{\ell=1}^L)$ as follows

$$
f_m(\{D_{\ell m}\}_{\ell=1}^L) = \sum_{\ell=1}^{L-1} \frac{\mu_\ell - \mu_{\ell+1}}{2} \log^+ \frac{1}{D_m} \frac{1}{\sigma_m^2} + \sum_{j=\ell+1}^L \frac{\sigma_{jm}^2 - D_{jm}}{\sigma_{jm}^4} - 1 \\
+ \sum_{\ell=1}^L \frac{\mu_L}{2} \log \frac{\sigma_m^2}{D_m} \\
(224)
$$

and the feasible set of the minimizations in (221)-(223) are defined by the constraints in (218)-(219). Equation (223) follows from the fact that $f_m(\{D_{\ell m}\}_{\ell=1}^L)$ depends only on $\{D_{\ell m}\}_{\ell=1}^L$ but not on $\{D_{\ell j}\}_{\ell=1}^L, j \neq m$.

Next, we note that each minimization

$$
\min_{\{D_{\ell m}\}_{\ell=1}^L} f_m(\{D_{\ell m}\}_{\ell=1}^L) \\
(225)
$$

is identical to the optimization problem we encounter for the scalar Gaussian model in Section 3.1, and hence, the minimum is attained by those $\{D_{\ell m}\}_{\ell=1}^L$ that satisfy the constraint in (218) with equality. This implies that the outer bound in Theorem 13 is attainable; completing the proof of Theorem 7.
D.3 Proof of (213)

We first note that

\[ p(x_m, \{y_{\ell m}, u_{\ell m}\}_{\ell=1}^L, w_m) = p(x_m)p(w_m) \left( \prod_{\ell=1}^L p(y_{\ell m}|x_m) \right) p(\{u_{\ell m}\}_{\ell=1}^L|x_m, w_m, \{y_{\ell m}\}_{\ell=1}^L) \]

(226)

where we use the fact that \((X_m, \{Y_{\ell m}\}_{\ell=1}^L)\) and \(W_m = W X_{\ell m-1}^m \{Y_{\ell m+1}^M\}_{\ell=1}^L\) are independent, which is a consequence of the joint distribution in (188). Next, we consider the following term

\[ p(\{u_{\ell m}\}_{\ell=1}^L|x_m, w_m, \{y_{\ell m}\}_{\ell=1}^L) = \sum_{\forall \{y_{\ell m}^{-1}\}_{\ell=1}^L} \sum_{\{y_{\ell m}\}_{\ell=1}^L} p(\{y_{\ell m}^{-1}\}_{\ell=1}^L|x_m, w_m, \{y_{\ell m}\}_{\ell=1}^L) p(\{u_{\ell m}\}_{\ell=1}^L|x_m, w_m, \{y_{\ell m}\}_{\ell=1}^L) \]

(227)

\[ = \sum_{\forall \{y_{\ell m}^{-1}\}_{\ell=1}^L} p(\{y_{\ell m}^{-1}\}_{\ell=1}^L|x_m, w_m, \{y_{\ell m}\}_{\ell=1}^L) p(\{u_{\ell m}\}_{\ell=1}^L|x_m, w_m, \{y_{\ell m}\}_{\ell=1}^L) \]

(228)

where the first term in the summation is

\[ p(\{y_{\ell m}^{-1}\}_{\ell=1}^L|x_m, w_m, \{y_{\ell m}\}_{\ell=1}^L) = \prod_{\ell=1}^L p(y_{\ell m}^{-1}|x_m, w_m, \{y_{\ell m}\}_{\ell=1}^L, \{y_{j m}^{-1}\}_{j=1}^L) \]

(229)

\[ = \prod_{\ell=1}^L p(y_{\ell m}^{-1}|w_m) \]

(230)

\[ = \prod_{\ell=1}^L p(y_{\ell m}^{-1}|w_m, y_{\ell m}) \]

(231)

where (230)-(231) come from the following Markov chain

\[ Y_{\ell m}^{-1} \rightarrow W_m \rightarrow X_m, \{Y_{\ell m}\}_{\ell=1}^L, \{Y_{j m}^{-1}\}_{j=1}^L \]

(232)

which is a consequence of the definition of \(W_m\) and the joint distribution in (188).

Next, we consider the second term in the summation given by (228) as follows

\[ p(\{u_{\ell m}\}_{\ell=1}^L|x_m, w_m, \{y_{j m}\}_{j=1}^L) = \prod_{\ell=1}^L p(u_{\ell m}|x_m, w_m, \{y_{j m}\}_{j=1}^L, \{u_{j m}\}_{j=1}^L) \]

(233)

\[ = \prod_{\ell=1}^L p(u_{\ell m}|w_m, y_{\ell m}) \]

(234)

where (234) comes from the following Markov chain

\[ U_{\ell m} \rightarrow W_m, Y_{\ell m}^m \rightarrow X_m, \{Y_{j m}^m\}_{j=1, j \neq \ell}^L, \{U_{j m}\}_{j=1}^{\ell-1} \]

(235)
which is also a consequence of the definition of $W_m$ and the joint distribution in (188).

Using (231) and (234) in (228), we get

\[
p(\{u_{\ell m}\}_{\ell=1}^L | x_m, w_m, \{y_{\ell m}\}_{\ell=1}^L) = \sum_{\forall(y_{m}^{m-1})_{\ell=1}^L} \prod_{\ell=1}^L p(y_{\ell m}^{m-1} | w_m, y_{\ell m})p(u_{\ell m} | w_m, y_{\ell m}^m) \tag{236}\]

\[
= \sum_{\forall(y_{m}^{m-1})_{\ell=1}^L} \prod_{\ell=1}^L p(y_{\ell m}^{m-1}, u_{\ell m} | w_m, y_{\ell m}) \tag{237}\]

\[
= \prod_{\ell=1}^L p(u_{\ell m} | w_m, y_{\ell m}) \tag{238}\]

using which in (226), we get

\[
p(x_m, \{y_{\ell m}, u_{\ell m}\}_{\ell=1}^L, w_m) = p(x_m)p(w_m) \prod_{\ell=1}^L p(y_{\ell m} | x_m)p(u_{\ell m} | w_m, y_{\ell m}) \tag{239}\]

which is the desired result in (213); completing the proof.

## E Proof of Corollary 5

From the analysis in Appendix D.2 when $\mu_1 \geq \mu_2 \geq 0$, we have

\[
\min_{(R_1, R_2) \in \mathcal{R}^p(D_1, D_2)} \mu_1 R_1 + \mu_2 R_2 = \sum_{m=1}^2 \min_{(D_{1m}, D_{2m}) \in \mathcal{D}_m} f_m(D_{1m}, D_{2m}) \tag{240}\]

where the function $f_m(D_{1m}, D_{2m})$ is given by

\[
f_m(D_{1m}, D_{2m}) = \sum_{\ell=1}^2 \frac{\mu_\ell}{2} \log \frac{\sigma_{\ell m}^2}{D_{\ell m}} + \frac{\mu_2}{2} \log \frac{\sigma_m^2}{D_m} + \frac{\mu_1 - \mu_2}{2} \log \frac{1}{D_m} \left( \frac{1}{\sigma_m^2} + \frac{\sigma_{2m}^2 - D_{2m}}{\sigma_{m}^4} \right)^{-1} \tag{241}\]

and the set $\mathcal{D}_m$ consists of $(D_{1m}, D_{2m})$ pairs satisfying

\[
\frac{1}{\sigma_m^2} + \sum_{\ell=1}^2 \frac{\sigma_{\ell m}^2 - D_{\ell m}}{\sigma_{m}^4} = \frac{1}{D_m} \tag{242}\]

\[
0 \leq D_{\ell m} \leq \sigma_{\ell m}^2, \quad \ell = 1, 2 \tag{243}\]
Next, we define the function $\tilde{f}_2(D_{12}, D_{22})$ as

$$\tilde{f}_2(D_{12}, D_{22}) = \sum_{\ell=1}^{2} \frac{\mu_\ell}{2} \log \frac{\sigma_{\ell 2}^2}{D_{\ell 2}} + \frac{\mu_2}{2} \log \frac{\sigma_2^2}{D_2}$$  \hspace{1cm} (244)$$

and the set $\tilde{D}_2$ as the union of $(D_{12}, D_{22})$ satisfying

$$\left(\frac{1}{\sigma_2^2} + \sum_{\ell=1}^{2} \frac{\sigma_{\ell 2}^2 - D_{\ell 2}}{\sigma_{\ell 2}^4}\right)^{-1} \leq D_2$$ \hspace{1cm} (245)

$$0 \leq D_{12} \leq \sigma_{12}^2$$ \hspace{1cm} (246)

We note the following facts

$$f_2(D_{12}, D_{22}) \geq \tilde{f}_2(D_{12}, D_{22}), \quad \forall (D_{12}, D_{22}) \in D_2$$ \hspace{1cm} (247)

$$D_2 \subseteq \tilde{D}_2$$ \hspace{1cm} (248)

Next, we consider the following optimization problem

$$\min_{(D_{12}, D_{22}) \in D_2} f_2(D_{12}, D_{22}) \geq \min_{(D_{12}, D_{22}) \in D_2} \tilde{f}_2(D_{12}, D_{22})$$ \hspace{1cm} (249)

$$\geq \min_{(D_{12}, D_{22}) \in \tilde{D}_2} \tilde{f}_2(D_{12}, D_{22})$$ \hspace{1cm} (250)

where (249)-(250) follow from (247)-(248), respectively. We note that the optimization problem in (250) is the scalar version of the optimization problem we consider in Appendix C.1. Using the result from Appendix C.1, we have

$$\min_{(D_{12}, D_{22}) \in \tilde{D}_2} \tilde{f}_2(D_{12}, D_{22}) = \frac{\mu_2}{2} \log \frac{\sigma_2^2}{D_2} + \frac{\mu_2}{2} \log \frac{1}{\sigma_{22}^2} \left(\frac{1}{\sigma_2^2} + \frac{1}{\sigma_{22}^2} - \frac{1}{D_2}\right)^{-1}$$ \hspace{1cm} (251)

Next, we note that by setting $(D_{12}^* = \sigma_{12}^2, D_{22}^* = \sigma_{22}^4(1/\sigma_2^2 + 1/\sigma_{22}^2 - 1/D_2)) \in D_2$, we get

$$f_2(D_{12}^*, D_{22}^*) = \min_{(D_{12}, D_{22}) \in \tilde{D}_2} \tilde{f}_2(D_{12}, D_{22})$$ \hspace{1cm} (252)

using which, and (251) in (240), we get

$$\min_{(R_1, R_2) \in \mathcal{R}_p(D_1, D_2)} \mu_1 R_1 + \mu_2 R_2 = \min_{(D_{11}, D_{21}) \in D_1} f_1(D_{11}, D_{21})$$

$$+ \frac{\mu_2}{2} \log \frac{\sigma_2^2}{D_2} + \frac{\mu_2}{2} \log \frac{1}{\sigma_{22}^2} \left(\frac{1}{\sigma_2^2} + \frac{1}{\sigma_{22}^2} - \frac{1}{D_2}\right)^{-1}$$ \hspace{1cm} (253)

which is the desired result in Corollary 5, completing the proof.
Proof of Corollary 6

Using Corollary 2, our outer bound for the parallel Gaussian model can be expressed as follows.

\[
T^+ = \min_{(R_1,R_2) \in \mathbb{R}^n(D_1,D_2)} \mu_1 R_1 + \mu_2 R_2
\]

\[
= \min_{(D_1,D_2,D)} \sum_{\ell=1}^2 \frac{\mu_\ell}{2} \log \left| \frac{\Sigma_\ell}{D_\ell} \right| + \frac{\mu_2}{2} \log \left| \frac{K_X}{D} \right|
\]

\[
+ \frac{\mu_1 - \mu_2}{2} \log^+ \left| \frac{(K_X^{-1} + \Sigma_2^{-1} - \Sigma_2^{-1}D_2\Sigma_2^{-1})^{-1}}{D} \right|
\]

where \((D_1,D_2,D)\) are subject to the following constraints

\[
\left( K_X^{-1} + \sum_{\ell=1}^2 \Sigma_\ell^{-1} - \sum_{\ell=1}^2 \Sigma_\ell^{-1}D_\ell\Sigma_\ell^{-1} \right)^{-1} \preceq D
\]

\[
0 \preceq D_\ell \preceq \Sigma_\ell, \quad \ell = 1,2
\]

\[
D_{mm} \preceq D_m, \quad m = 1,2
\]

where \(D_{mm}\) denotes the \(m\)th diagonal element of \(D\). By restricting \((D_1,D_2,D)\) to be diagonal, we have

\[
T^+ \leq \min_{\{D_{\ell m}\}_{\ell \neq \ell', m}} \sum_{m=1}^2 \frac{\mu_1}{2} \log \frac{\sigma_{\ell m}^2}{D_{1m}} + \frac{\mu_2}{2} \log \frac{\sigma_{2m}^2}{D_{2m}} + \frac{\mu_2}{2} \log \frac{\sigma_m^2}{D_m}
\]

\[
+ \frac{\mu_1 - \mu_2}{2} \log^+ \prod_{m=1}^2 \frac{1}{D_m} \left( \frac{1}{\sigma_m^2} + \frac{1}{\sigma_{2m}^2} - \frac{D_{2m}}{\sigma_{2m}^4} \right)^{-1}
\]

where \(\{D_{\ell m}\}_{\ell \neq \ell', m}\) are subject to the following constraints

\[
\left( \frac{1}{\sigma_m^2} + \sum_{\ell=1}^2 \frac{1}{\sigma_{\ell m}^2} - \frac{D_{\ell m}}{\sigma_{\ell m}^4} \right)^{-1} \leq D_m, \quad m = 1,2
\]

\[
0 \leq D_{\ell m} \leq \sigma_{\ell m}^2, \quad \ell = 1,2 \quad m = 1,2
\]

Next, we set

\[
D_{12} = \sigma_{12}^2
\]

\[
D_{22} = \frac{\mu_2}{\mu_1} \sigma_{22}^4 \left( \frac{1}{\sigma_2^2} + \frac{1}{\sigma_{22}^2} \right)
\]
which are feasible, i.e., satisfy the constraints in (260)-(261), due to the assumptions in (55)-(56). Next, we note the following

\[
\prod_{m=1}^{2} D_m \left( \frac{1}{\sigma_m^2} + \frac{1}{\sigma_{2m}^2} - \frac{D_{2m}}{\sigma_{2m}^4} \right)^{-1} = \frac{1}{D_1} \left( \frac{1}{\sigma_1^2} + \frac{1}{\sigma_{21}^2} - \frac{D_{21}}{\sigma_{21}^4} \right)^{-1} \frac{1}{D_2} \frac{\mu_1}{\mu_1 - \mu_2} \left( \frac{1}{\sigma_2^2} + \frac{1}{\sigma_{22}^2} \right)^{-1}
\]

(264)

\[
\geq \frac{1}{D_1} \left( \frac{1}{\sigma_1^2} + \frac{1}{\sigma_{21}^2} \right)^{-1} \frac{1}{D_2} \frac{\mu_1}{\mu_1 - \mu_2} \left( \frac{1}{\sigma_2^2} + \frac{1}{\sigma_{22}^2} \right)^{-1}
\]

(265)

\[
> 1
\]

(266)

where, in (264), we use (262)-(263), (265) follows from the fact that \( D_{21} \geq 0 \), and (266) is due to the assumption in (57). Hence, using (262)-(263) and (265) in (259), we get

\[
T^+ \leq \min_{(D_{11}, D_{21}) \in \hat{D}_1} \left[ f_1(D_{11}, D_{21}) + \frac{\mu_2}{2} \log \frac{\sigma_2^2}{D_2} + \frac{\mu_2}{2} \log \frac{\mu_1}{\mu_2} \frac{1}{\sigma_{22}^2} \left( \frac{1}{\sigma_2^2} + \frac{1}{\sigma_{22}^2} \right)^{-1} \right.
\]

\[
+ \frac{\mu_1 - \mu_2}{2} \log \frac{\mu_1}{\mu_1 - \mu_2} \frac{1}{D_2} \left( \frac{1}{\sigma_2^2} + \frac{1}{\sigma_{22}^2} \right)^{-1}
\]

(267)

where the set \( \hat{D}_1 \) is defined as the union of \((D_{11}, D_{21})\) pairs satisfying

\[
\left( \frac{1}{\sigma_1^2} + \sum_{\ell=1}^{2} \frac{1}{\sigma_{\ell 1}^2} - \frac{D_{11}}{\sigma_{11}^2} \right)^{-1} \leq D_1
\]

(268)

\[
0 \leq D_{11} \leq \sigma_{11}^2
\]

(269)

We note that \( \mathcal{D}_1 \subseteq \hat{D}_1 \), where \( \mathcal{D}_1 \) is the region defined in Corollary 6. Hence, using this fact in (267), we get

\[
T^+ \leq \min_{(D_{11}, D_{21}) \in \mathcal{D}_1} \left[ f_1(D_{11}, D_{21}) + \frac{\mu_2}{2} \log \frac{\sigma_2^2}{D_2} + \frac{\mu_2}{2} \log \frac{\mu_1}{\mu_2} \frac{1}{\sigma_{22}^2} \left( \frac{1}{\sigma_2^2} + \frac{1}{\sigma_{22}^2} \right)^{-1} \right.
\]

\[
+ \frac{\mu_1 - \mu_2}{2} \log \frac{\mu_1}{\mu_1 - \mu_2} \frac{1}{D_2} \left( \frac{1}{\sigma_2^2} + \frac{1}{\sigma_{22}^2} \right)^{-1}
\]

(270)

which is the desired result in Corollary 6 completing the proof.

\section*{G Proof of Lemma 4}

We first note the following Markov chain

\[(U_j, Y_j) \to (W, X) \to (U_{[1:L] \setminus j}, Y_{[1:L] \setminus j})\]

(271)
whose proof is given in Appendix G.1. Next, we note that

\[ E \left[ S_{A^c} | X, \{U_\ell\}_{\ell \in A^c}, W \right] = \sum_{\ell \in A^c} A_\ell E \left[ Y_\ell | X, \{U_\ell\}_{\ell \in A^c}, W \right] \]

(272)

\[ = \sum_{\ell \in A^c} A_\ell E \left[ Y_\ell | U_\ell, W \right] \]

(273)

where (273) follows from the Markov chain in (271). Now, we consider \( \text{mmse}(S_{A^c} | X, \{U_\ell\}_{\ell \in A^c}, W) \) as follows

\[
\text{mmse}(S_{A^c} | X, \{U_\ell\}_{\ell \in A^c}, W) = E \left[ (S_{A^c} - E[S_{A^c} | X, \{U_\ell\}_{\ell \in A^c}, W]) (S_{A^c} - E[S_{A^c} | X, \{U_\ell\}_{\ell \in A^c}, W])^T \right]
\]

(274)

\[
= E \left[ \left( \sum_{\ell \in A^c} A_\ell (Y_\ell - E[Y_\ell | X, U_\ell, W]) \right) \left( \sum_{\ell \in A^c} A_\ell (Y_\ell - E[Y_\ell | X, U_\ell, W]) \right)^T \right]
\]

(275)

\[
= \sum_{\ell \in A^c} A_\ell \text{mmse}(Y_\ell | X, U_\ell, W) A_\ell^T
\]

(276)

where (275) is due to (273). Next, we consider the cross-terms in (276) as follows

\[
E \left[ (Y_i - E[Y_i | X, U_i, W]) (Y_j - E[Y_j | X, U_j, W])^T \right]
\]

(277)

\[
= E \left[ (Y_i - E[Y_i | X, U_i, W]) (Y_j - E[Y_j | X, U_j, W])^T | X, W \right]
\]

(278)

\[
= 0
\]

(279)

where (278) is due to the Markov chain in (271). Using (279) in (276), we get

\[
\text{mmse}(S_{A^c} | X, \{U_\ell\}_{\ell \in A^c}, W) = \sum_{\ell \in A^c} A_\ell \text{mmse}(Y_\ell | X, U_\ell, W) A_\ell^T
\]

(280)

which completes the proof of Lemma 4.
G.1 Proof of (271)

We first consider the joint distribution in (71) as follows

\[ p(x, \{y_\ell, u_\ell\}_{\ell=1}^L, w) = p(x)p(w) \left( \prod_{\ell=1}^L p(y_\ell|x)p(u_\ell|y_\ell, w) \right) p(y_j|x)p(u_j|y_j, w) \]  

(281)

which implies

\[ U_\ell \to (Y_\ell, W) \to X, \quad \ell = 1, \ldots, L \]  

(282)

Next, we note that

\[ p(y_j|x)p(u_j|y_j, w) = p(y_j|x)p(u_j|y_j, w, x) \]  

(283)

\[ = p(y_j|x)\frac{p(u_j, y_j, w, x)}{p(y_j, x)p(w)} \]  

(284)

\[ = \frac{p(u_j, y_j, w, x)}{p(x)p(w)} \]  

(285)

\[ = \frac{p(u_j, y_j, w, x)}{p(w, x)} \]  

(286)

\[ = p(u_j, y_j|w, x) \]  

(287)

where (283) comes from the Markov chain in (282), (284) and (286) follow from the fact that \((X, Y_j)\) and \(W\) are independent which is a consequence of the factorization in (281). Using (287) in (281), we get

\[ p(x, \{y_\ell, u_\ell\}_{\ell=1}^L, w) = p(x)p(w) \left( \prod_{\ell=1}^L p(y_\ell|x)p(u_\ell|y_\ell, w) \right) p(u_j, y_j|x, w) \]  

(288)

which implies the Markov chain in (271); completing the proof.

H Proof of Theorem 9

The singular value decomposition of the matrices \(\{H_\ell\}_{\ell=1}^L\) are given by

\[ H_\ell = U_\ell \Lambda_\ell V_\ell^T, \quad \ell = 1, \ldots, L \]  

(289)

where \(\{U_\ell\}_{\ell=1}^L\) and \(\{V_\ell\}_{\ell=1}^L\) are orthonormal matrices. Next, we show that without loss of generality, we can assume that \(\{H_\ell\}_{\ell=1}^L\) are square matrices. To this end, we define the
following observations

\[ \tilde{Y}_\ell = U_\ell^T Y_\ell \]
\[ = \Lambda_\ell V_\ell X + \tilde{N}_\ell \]

(290)

(291)

where \( \tilde{N}_\ell \) is again a zero-mean Gaussian random vector with an identity covariance matrix.

We note that the rate-distortion region for the observations \( \{\tilde{Y}_\ell\}_{\ell=1}^L \) is identical to the rate-distortion region for the observations \( \{Y_\ell\}_{\ell=1}^L \), since we obtain the observations \( \{\tilde{Y}_\ell\}_{\ell=1}^L \) from \( \{Y_\ell\}_{\ell=1}^L \) by an invertible transformation. Now, we show that there is no loss of generality to assume that the matrices \( \{H_\ell\}_{\ell=1}^L \) are square matrices. Assume that \( H_\ell \) is an \( r_\ell \times M \) matrix. Hence, \( \Lambda_\ell \) is also an \( r_\ell \times M \) matrix. First, consider \( r_\ell > M \). In this case, \( r_\ell - M \) entries of \( \tilde{Y}_\ell \) consists of only noise. Since the noise \( \tilde{N}_\ell \) is i.i.d., we can drop these \( r_\ell - M \) entries of the observation \( \tilde{Y}_\ell \) without altering the rate-distortion region. Hence, when \( r_\ell > M \), there is an equivalent model with the same rate-distortion region and \( r_\ell = M \). Next, assume \( r_\ell < M \). In this case, we can add \( M - r_\ell \) i.i.d. noise entries to the observation \( \tilde{Y}_\ell \) without altering the rate-distortion region. Hence, when \( r_\ell < M \), there is also an equivalent model with the same rate-distortion region and \( r_\ell = M \). Consequently, from now on, we assume that \( r_1 = \ldots = r_L = M \).

Next, we define

\[ H_{\ell,\alpha} = U_\ell (\Lambda_\ell + \alpha I) V_\ell^T, \quad \ell = 1, \ldots, L \]

(292)

where \( \alpha > 0 \). We note that \( \{H_{\ell,\alpha}\}_{\ell=1}^L \) are invertible, i.e., \( \{H_{\ell,\alpha}^{-1}\}_{\ell=1}^L \) exist, and in particular, we have

\[ H_{\ell,\alpha}^{-1} = V_\ell (\Lambda_\ell + \alpha I)^{-1} U_\ell^T, \quad \ell = 1, \ldots, L \]

(293)

Using \( \{H_{\ell,\alpha}\}_{\ell=1}^L \), we define an enhanced model as follows

\[ Y_{\ell,\alpha} = H_{\ell,\alpha} X + N_\ell, \quad \ell = 1, \ldots, L \]

(294)

Using these enhanced observations in (294), we can rewrite the original observations in (116) as follows

\[ Y_\ell = H_\ell H_{\ell,\alpha}^{-1} Y_{\ell,\alpha} + \bar{N}_\ell, \quad \ell = 1, \ldots, L \]

(295)

where \( \bar{N}_\ell \) is a zero-mean Gaussian random vector, and independent of \( \{Y_{\ell,\alpha}\}_{\ell=1}^L \) and \( \{\bar{N}_j\}_{j=1,j\neq \ell}^L \). The decomposition in (295) is possible, since we have

\[ (H_\ell H_{\ell,\alpha}^{-1})(H_\ell H_{\ell,\alpha}^{-1})^T = U_\ell \Lambda_\ell^2 (\Lambda_\ell + \alpha I)^{-2} U_\ell^T \leq I \]

(296)

(297)
Moreover, due to the decomposition in (295), we can assume that the following holds

\[ p(x, \{y_{\ell,\alpha}, y_{\ell}\}_{\ell=1}^L) = p(x) \prod_{\ell=1}^L p(y_{\ell,\alpha} | x)p(y_{\ell} | y_{\ell,\alpha}) \] (298)

which implies that the original observations \( \{Y_{\ell}\}_{\ell=1}^L \) are degraded versions the enhanced observations \( \{Y_{\ell,\alpha}\}_{\ell=1}^L \). Consequently, we have

\[ R(D) \subseteq R_\alpha(D) \] (299)

where \( R_\alpha(D) \) denotes the rate-distortion region for the enhanced model defined by (294).

Next, we note that the enhanced model defined by (294) is equivalent to the following one

\[ \bar{Y}_{\ell,\alpha} = H_{\ell,\alpha}^{-1}Y_{\ell,\alpha} = X + N_{\ell,\alpha}, \quad \ell = 1, \ldots, L \] (300)

where the covariance matrix of \( N_{\ell,\alpha} \) is given by

\[ \Sigma_{\ell,\alpha} = (H_{\ell,\alpha}^\top H_{\ell,\alpha})^{-1}, \quad \ell = 1, \ldots, L \] (302)

Using Theorem 1, we can obtain an outer bound for the rate-distortion region of the model defined by (301), which is equivalent to the enhanced model given by (294). In particular, we have \( R_\alpha(D) \subseteq R_\alpha^o(D) \), where \( R_\alpha^o(D) \) is given by the union of rate tuples \( (R_1, \ldots, R_L) \) satisfying

\[ \sum_{\ell \in A} R_\ell \geq \frac{1}{2} \log \left| \left( K_X^{-1} + \sum_{\ell \in A^c} \Sigma_{\ell,\alpha}^{-1} - \sum_{\ell \in A} \Sigma_{\ell,\alpha}^{-1} \tilde{D}_\ell \Sigma_{\ell,\alpha}^{-1} \right)^{-1} \right| + \sum_{\ell \in A} \frac{1}{2} \log \left| \Sigma_{\ell,\alpha} \right| \] (303)

for all \( A \subseteq \{1, \ldots, L\} \), where the union is over all \( \{\tilde{D}_\ell\}_{\ell=1}^L \) satisfying

\[ \left( K_X^{-1} + \sum_{\ell=1}^L \Sigma_{\ell,\alpha}^{-1} - \sum_{\ell=1}^L \Sigma_{\ell,\alpha}^{-1} \tilde{D}_\ell \Sigma_{\ell,\alpha}^{-1} \right)^{-1} \preceq \bar{D} \] (304)

\[ 0 \preceq \tilde{D}_\ell \preceq \Sigma_{\ell,\alpha}, \quad \ell = 1, \ldots, L \] (305)

Next, we set \( D_\ell = H_{\ell,\alpha} \tilde{D}_\ell H_{\ell,\alpha}^\top, \ell = 1, \ldots, L \), using which in (303)-(305), we can express the outer bound \( R_\alpha^o(D) \) as the union of rate tuples \( (R_1, \ldots, R_L) \) satisfying

\[ \sum_{\ell \in A} R_\ell \geq \frac{1}{2} \log \left| \left( K_X^{-1} + \sum_{\ell \in A^c} H_{\ell,\alpha}^\top (I-D_\ell) H_{\ell,\alpha} \right)^{-1} \right| + \sum_{\ell \in A} \frac{1}{2} \log \left| D_\ell \right| \] (306)
for all $A \subseteq \{1, \ldots, L\}$, where the union is over all $\{D_\ell\}_{\ell=1}^L$ satisfying

$$\left( K_X^{-1} + \sum_{\ell=1}^L H_\ell^T (I - D_\ell) H_\ell \right)^{-1} \preceq D $$

$$0 \preceq D_\ell \preceq I, \quad \ell = 1, \ldots, L$$ (307)

In view of (299), we have the following

$$\mathcal{R}(D) \subseteq \mathcal{R}_\alpha^o(D), \quad \forall \alpha > 0$$ (309)

which implies that

$$\mathcal{R}(D) \subseteq \lim_{\alpha \to 0} \mathcal{R}_\alpha^o(D)$$ (310)

Hence, to obtain an outer bound for the rate-distortion region of the general model defined by (116), it is sufficient to obtain the limiting region $\lim_{\alpha \to 0} \mathcal{R}_\alpha^o(D)$. To this end, we introduce the following lemma.

**Lemma 6** For all $A^c \subseteq \{1, \ldots, L\}$, we have

$$\lim_{\alpha \to 0} \left( K_X^{-1} + \sum_{\ell \in A^c} H_\ell^T (I - D_\ell) H_\ell \right)^{-1} = \left( K_X^{-1} + \sum_{\ell \in A^c} H_\ell^T (I - D_\ell) H_\ell \right)^{-1}$$ (311)

The proof of Lemma 6 is given in Appendix H.1. Using this lemma in (306)-(308), we obtain the region $\lim_{\alpha \to 0} \mathcal{R}_\alpha^o(D)$ as the union of rate tuples $(R_1, \ldots, R_L)$ satisfying

$$\sum_{\ell \in A} R_\ell \geq \frac{1}{2} \log \left| \frac{K_X^{-1} + \sum_{\ell \in A^c} H_\ell^T (I - D_\ell) H_\ell}{|D|} \right| + \sum_{\ell \in A} \frac{1}{2} \log \frac{1}{|D_\ell|}$$ (312)

for all $A \subseteq \{1, \ldots, L\}$, where the union is over all $\{D_\ell\}_{\ell=1}^L$ satisfying

$$\left( K_X^{-1} + \sum_{\ell=1}^L H_\ell^T (I - D_\ell) H_\ell \right)^{-1} \preceq D $$

$$0 \preceq D_\ell \preceq I, \quad \ell = 1, \ldots, L$$ (313)

which is the desired result in Theorem 9; completing the proof.

**H.1 Proof of Lemma 6**

In the proof of Lemma 6, we use the following fact.
Lemma 7 ([15, page 258]) Let $C$ be a matrix satisfying $\lim_{n \to \infty} C^n = 0$. Then, we have

$$(I + C)^{-1} = \sum_{n=0}^{\infty} (-1)^n C^n$$

(315)

where $C^0 = I$.

Next, we note that

$$\sum_{\ell \in A^c} H^\top_{\ell,\alpha} (I - D_\ell) H_{\ell,\alpha} = \sum_{\ell \in A^c} H^\top_{\ell} (I - D_\ell) H_{\ell} + M(\alpha)$$

(316)

where $\lim_{\alpha \to 0} M(\alpha) = 0$. We define

$$M_{A^c} = K^{-1}_X + \sum_{\ell \in A^c} H^\top_{\ell} (I - D_\ell) H_{\ell}$$

(317)

Using (316)-(317), we have

$$\left( K^{-1}_X + \sum_{\ell \in A^c} H^\top_{\ell,\alpha} (I - D_\ell) H_{\ell,\alpha} \right)^{-1} = \left( M_{A^c} + M(\alpha) \right)^{-1}$$

(318)

$$= M^{-1/2}_{A^c} \left( I + M^{-1/2}_{A^c} M(\alpha) M^{-1/2}_{A^c} \right)^{-1} M^{-1/2}_{A^c}$$

(319)

Since we have $\lim_{\alpha \to 0} M(\alpha) = 0$, there exists $\alpha^*$ such that

$$\lim_{n \to \infty} \left( M^{-1/2}_{A^c} M(\alpha) M^{-1/2}_{A^c} \right)^n = 0, \quad \forall \alpha \in (0, \alpha^*)$$

(320)

In view of (320), using Lemma 7 in (319) yields

$$\left( M_{A^c} + M(\alpha) \right)^{-1} = M^{-1/2}_{A^c} \left( \sum_{n=0}^{\infty} \left( M^{-1/2}_{A^c} M(\alpha) M^{-1/2}_{A^c} \right)^n \right) M^{-1/2}_{A^c}, \quad \alpha \in (0, \alpha^*)$$

(321)

using which yields

$$\lim_{\alpha \to 0} \left( K^{-1}_X + \sum_{\ell \in A^c} H^\top_{\ell,\alpha} (I - D_\ell) H_{\ell,\alpha} \right)^{-1} = \lim_{\alpha \to 0} \left( M^{-1/2}_{A^c} \left( \sum_{n=0}^{\infty} \left( M^{-1/2}_{A^c} M(\alpha) M^{-1/2}_{A^c} \right)^n \right) M^{-1/2}_{A^c} \right)$$

(322)

$$= M^{-1}_{A^c}$$

(323)

$$= \left( K^{-1}_X + \sum_{\ell \in A^c} H^\top_{\ell} (I - D_\ell) H_{\ell} \right)^{-1}$$

(324)
where (323) is due to the fact that \( \lim_{\alpha \to 0} M(\alpha) = 0 \), and (324) is due to (317). Equation (324) is the desired end result in Lemma 6 completing the proof.

1  Proofs of Theorem 2 and Theorem 10

We obtain the inner bound for the rate-distortion region \( \mathcal{R}(D) \) by evaluating the Berger-Tung achievable scheme with jointly Gaussian auxiliary random vectors. For that purpose, we consider the most general form of the vector Gaussian CEO model defined by the observations in (116). In other words, we first obtain an inner bound for the most general form given by (116), i.e., we prove Theorem 10, and next, show that Theorem 2 follows from Theorem 10.

Let \( \mathcal{R}^{\text{BT}}(D) \) denote the Berger-Tung inner bound. \( \mathcal{R}^{\text{BT}}(D) \) is given by the union of rate tuples \( (R_1, \ldots, R_L) \) satisfying

\[
\sum_{\ell \in \mathcal{A}} R_\ell \geq I(X; \{U_\ell\}_{\ell \in \mathcal{A}}|\{U_\ell\}_{\ell \in \mathcal{A}^c}) + \sum_{\ell \in \mathcal{A}} I(Y_\ell; U_\ell|X) \tag{325}
\]

for all \( \mathcal{A} \subseteq \{1, \ldots, L\} \), where the union is over all \( (U_1, \ldots, U_L) \) satisfying the Markov chain

\[
U_j \rightarrow Y_j \rightarrow X \rightarrow Y_k \rightarrow U_k, \quad j \neq k \tag{326}
\]

and the distortion constraint

\[
\text{mmse}(X|U_1, \ldots, U_L) \preceq D \tag{327}
\]

We select the auxiliary random variables \( \{U_\ell\}_{\ell=1}^L \) as follows

\[
U_\ell = Y_\ell + \bar{N}_\ell, \quad \ell = 1, \ldots, L \tag{328}
\]

where \( \{\bar{N}_\ell\}_{\ell=1}^L \) are zero-mean independent Gaussian random vectors with covariance matrices \( \{\bar{\Sigma}_\ell\}_{\ell=1}^L \), and are independent of \( \{Y_\ell\}_{\ell=1}^L, X \). We assume that the covariance matrices \( \{\bar{\Sigma}_\ell\}_{\ell=1}^L \) are strictly positive definite, i.e., we have \( \bar{\Sigma}_\ell \succ 0, \forall \ell \in \{1, \ldots, L\} \). This assumption arises from the fact that if one of these matrices is singular, for example, if \( \bar{\Sigma}_\ell \) is singular, then, as we will show soon, the corresponding MMSE matrix \( \text{mmse}(Y_\ell|X,U_\ell) \) will be singular as well, and consequently, \( I(U_\ell; Y_\ell|X) \rightarrow \infty \). When the auxiliary random variables \( \{U_\ell\}_{\ell=1}^L \) are selected to be Gaussian as in (328), the rate bound in (325) becomes

\[
\sum_{\ell \in \mathcal{A}} R_\ell \geq \frac{1}{2} \log \frac{\text{mmse}(X|\{U_\ell\}_{\ell \in \mathcal{A}})}{\text{mmse}(X|\{U_\ell\}_{\ell=1}^L)} + \sum_{\ell \in \mathcal{A}} \frac{1}{2} \log \frac{1}{\text{mmse}(Y_\ell|X,U_\ell)} \tag{329}
\]

where, as it will become clear soon, all MMSE matrices are strictly positive definite; implying that the rate bounds in (329) are finite.

Next, we evaluate the MMSE terms in (329). Using the definition of auxiliary random
variables in (328), we have (see (126) and (129) in Appendix A.1)

\[ D_\ell \triangleq \text{mmse}(Y_\ell | X, U_\ell) \]  
\[ = (I + \Sigma_\ell^{-1})^{-1}, \quad \ell = 1, \ldots, L \]  

where \( D_\ell \) satisfies the following orders

\[ 0 \prec D_\ell \preceq I \]  

where the upper bound on \( D_\ell \) follows from the following fact

\[ \text{mmse}(Y_\ell | X, U_\ell) = \text{mmse}(N_\ell | N_\ell + \bar{N}_\ell) \preceq I \]  

Using (331), we have

\[ \bar{\Sigma}_\ell = (D_\ell^{-1} - I)^{-1}, \quad \ell = 1, \ldots, L \]  

Next, we evaluate the MMSE matrices \( \text{mmse}(X|\{U_\ell\}_{\ell \in A^c}) \) as follows (see (126) and (129) in Appendix A.1)

\[ \text{mmse}(X|\{U_\ell\}_{\ell \in A^c}) = \left( K_X^{-1} + \sum_{\ell \in A^c} H_\ell^T (I + \bar{\Sigma}_\ell)^{-1} H_\ell \right)^{-1} \]  
\[ = \left( K_X^{-1} + \sum_{\ell \in A^c} H_\ell^T (I - D_\ell) H_\ell \right)^{-1} \]  

where we used the following identity

\[ (I + \bar{\Sigma}_\ell)^{-1} = I - D_\ell \]  

which can be shown by using (334). Hence, using (331) and (336) in (329), we obtain the inner bound as the union of rate tuples \( (R_1, \ldots, R_L) \) satisfying

\[ \sum_{\ell \in A} R_\ell \geq \frac{1}{2} \log \frac{\left( K_X^{-1} + \sum_{\ell \in A^c} H_\ell^T (I - D_\ell) H_\ell \right)^{-1}}{\left( K_X^{-1} + \sum_{\ell=1}^L H_\ell^T (I - D_\ell) H_\ell \right)^{-1}} + \sum_{\ell \in A} \frac{1}{2} \log \frac{1}{|D_\ell|} \]  

for all \( A \subseteq \{1, \ldots, L\} \), where the union is over all positive semi-definite matrices \( \{D_\ell\}_{\ell=1}^L \)
satisfying
\[
\left( K_X^{-1} + \sum_{\ell=1}^{L} H_\ell^\top (I - D_\ell) H_\ell \right)^{-1} \preceq D \tag{339}
\]
\[
0 \preceq D_\ell \preceq I, \quad \ell = 1, \ldots, L \tag{340}
\]
where the first constraint in (339) is obtained by using (336) in (327), and the second constraint in (340) comes from (332). Hence, in view of (338)-(340), we obtain the inner bound given in Theorem 10, completing the proof.

Next, we show that Theorem 2 follows from Theorem 10. We note that the observations in (1) are equivalent to the general form of the observations in (116), when one sets \( H_\ell = \Sigma_\ell^{-1/2}, \quad \ell = 1, \ldots, L \). Using this observation in (338)-(340) in conjunction with the definition \( D_\ell = \Sigma_\ell^{-1/2} \tilde{D}_\ell \Sigma_\ell^{-1/2}, \quad \ell = 1, \ldots, L \), one can get the inner bound in Theorem 2, completing the proof.

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