DYNAMICAL ANALYSIS OF A DIFFUSIVE SIRS MODEL WITH GENERAL INCIDENCE RATE

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Abstract. In this paper, we propose a diffusive SIRS model with general incidence rate and spatial heterogeneity. The formula of the basic reproduction number $R_0$ is given. Then the threshold dynamics, including globally attractive of the disease-free equilibrium and uniform persistence, are established in terms of $R_0$. Special cases and numerical simulations are presented to support our main results.

1. Introduction. In order to study two directly transmitted diseases of mice, Anderson and May [3] proposed the following classical deterministic Susceptible-Infected-Removed-Susceptible (SIRS) model, accounted for the fact that infected individuals may recover and return to the susceptible class [11] (see also [26]):

$$
\begin{align*}
\begin{cases}
\dot{S}(t) &= \Lambda - \mu S(t) - \beta S(t)I(t) + \delta R(t), \\
\dot{I}(t) &= \beta S(t)I(t) - (\mu + \gamma_2 + \alpha)I(t), \\
\dot{R}(t) &= \gamma_2 I(t) - (\mu + \delta)R(t).
\end{cases}
\end{align*}
$$

Here $S(t)$, $I(t)$ and $R(t)$ represent the numbers of susceptible, infective and recovered individuals at time $t$, respectively. $\Lambda > 0$ denotes the recruitment rate, $\mu > 0$ is the natural death rate of the population, $\beta$ denotes the infection rate, $\gamma_2 > 0$ is

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the transfer rate from the infected class to the recovered class, $\alpha \geq 0$ stands for the death rate induced by the disease, $\delta \geq 0$ is the immunity loss rate.

Several researchers have shown that the nonlinear incidence rate plays an important role in epidemic models [1, 5, 10, 12–15, 20, 40, 45]. Motivated by the above discussions, Li et al. [17] extended model (1) to a SIRS model with a general incidence function which takes the following form:

$$\begin{cases}
\dot{S}(t) = \Lambda - \mu S(t) - S(t)f(I(t)) + \gamma_1 I(t) + \delta R(t), \\
\dot{I}(t) = S(t)f(I(t)) - (\mu + \gamma_1 + \gamma_2 + \alpha)I(t), \\
\dot{R}(t) = \gamma_2 I(t) - (\mu + \delta)R(t),
\end{cases}$$

(2)

where $\gamma_1 \geq 0$ denotes the transfer rate from the infected class to the susceptible class, and $\Lambda$, $\mu$, $\delta$, $\gamma_2$ and $\alpha$ have the same descriptions as those parameters in system (1). Moreover, $S(t)f(I(t))$ is the infectious incidence, i.e., the new infectious individuals of the disease per unit time. Then, authors took the basic reproduction number as the threshold parameter and established the complete global dynamics by constructing Lyapunov functions.

A key assumption in system (2) is that all individuals are well mixed. However, individuals may move randomly between regions, and recently various reaction-diffusion models have been used to model such random movements in epidemics. For example, models have been constructed for the spatial transmission of rabies [29], influenza [48], cholera [32, 33, 42–44, 49], malaria [21, 41], dengue [34] and virus dynamics [16, 25, 35, 36]. Individuals may also be living in different regions while traveling from one location to another. The transmission rate, birth rate, death rate, and even the recovery rate might be different in different regions. So, spatial heterogeneity is inevitable and can affect the transmission dynamics of diseases. It also can bring some interesting mathematical problems. In 2008, Allen et al. [2] discussed the effect of spatial heterogeneity on an SIS epidemic reaction-diffusion model. Then, the dynamical behaviors of reaction-diffusion epidemic model with spatial heterogeneity have attracted more attentions (see, for example, [4, 18, 22, 39, 46]). Motivated by the works of [2], we incorporate the diffusion terms into system (2). Suppose that the diffusive coefficients for $S$, $I$ and $R$ are the same. Thus, we modify system (2) to a reaction-diffusion model with model parameters to be spatially dependent as follows:

$$\begin{cases}
\frac{\partial S(x, t)}{\partial t} = D\Delta S(x, t) + \Lambda(x) - \mu(x)S(x, t) - \beta(x)f(S(x, t), I(x, t)) \\
+ \gamma_1(x)I(x, t) + \delta(x)R(x, t), \\
\frac{\partial I(x, t)}{\partial t} = D\Delta I(x, t) + \beta(x)f(S(x, t), I(x, t)) - (\mu(x) + \gamma_1(x) + \gamma_2(x)) \\
+ \alpha(x))I(x, t), \\
\frac{\partial R(x, t)}{\partial t} = D\Delta R(x, t) + \gamma_2(x)I(x, t) - (\mu(x) + \delta(x))R(x, t),
\end{cases}$$

(3)

for $x \in \Omega$, $t > 0$. Here $S(x, t)$, $I(x, t)$ and $R(x, t)$ denote the densities of susceptible, infective and recovered individuals at position $x$ and time $t$, respectively. $\Delta$ is the Laplacian operator and $D > 0$ is the diffusion coefficient. $\gamma_1(x)$, $\delta(x)$ and $\alpha(x)$ are nonnegative and continuous. The other model parameters are positive and continuous. $\Omega \in \mathbb{R}^n$ is a bounded domain with smooth boundary $\partial\Omega$. Function $f$ satisfies
\( f : \mathbb{R}^2_+ \rightarrow \mathbb{R}_+ \) is differentiable, \( f(0, I) = f(S, 0) = 0 \) for all \( S, I \geq 0 \);
\[
\lim_{I \rightarrow 0^+} \frac{f(S, I)}{I} \text{ exists and is positive for all } S > 0;
\]
\[
f_S(S, I) > 0 \text{ and } \frac{\partial f(S, I)}{\partial I} \leq 0 \text{ for all } S, I \geq 0, \text{ where } \hat{f}(S, I) = \frac{f(S, I)}{I}.
\]
Assumption (H1) guarantees that no transmission occurs in the absence of either susceptible individuals or infectious individuals. We also need the following assumption:

(H2) \( f(S, I) \leq SI \) for all \( S, I \geq 0 \).

Assumption (H2) is also reasonable biologically. In fact, the proportion of susceptible individuals in the total population \( N \) can be described as \( S/N \). If an infectious individual contacts others \( N \) times per unit time, the incidence rate can be expressed as the bilinear incidence \( SI \). In general, the contact number will not be greater than \( N \). Clearly, incidence rates such as \( f(S, I) = \frac{SI}{1 + k_1 S + k_2 I} \) (Beddington-DeAngelis incidence [6]), \( f(S, I) = \frac{SI}{(1 + k_1 S)(1 + k_2 I)} \) (Crowley-Martin incidence [7]) and \( f(S, I) = \frac{SI}{1 + k_1 I^2} \) (nonmonotone incidence [40]) satisfy assumptions (H1) and (H2).

We assume there is an initially non-negative population resulting the following initial condition
\[
S(x, 0) = S_0(x) \geq 0, \quad I(x, 0) = I_0(x) \geq 0, \quad R(x, 0) = R_0(x) \geq 0, \quad x \in \Omega, \quad (4)
\]
and the Neumann boundary condition
\[
\frac{\partial S(x, t)}{\partial \nu} = \frac{\partial I(x, t)}{\partial \nu} = \frac{\partial R(x, t)}{\partial \nu} = 0, \quad t > 0, \quad x \in \partial \Omega,
\]
where \( \frac{\partial}{\partial \nu} \) denotes the outward normal derivative on the boundary \( \partial \Omega \) as in [50–52]. Condition (5) implies that there is no population net flux across the boundary.

The remainder of this paper is organized as follows. In Section 2, we give the well-posedness of this model. Section 3 is devoted to the threshold dynamics in terms of \( R_0 \). In Section 4, special case and numerical simulations are performed as a supplementary to our theoretical results. A brief conclusion is presented in Section 5.

2. The well-posedness. Let \( X := C(\Omega, \mathbb{R}^3) \). Then \( X \) is a Banach space with the supremum \( \| \cdot \|_X \). Define \( X^+ := C(\Omega, \mathbb{R}^+_3) \). It is easy to see that \( (X, X^+) \) is a strongly ordered Banach space. Let
\[
\mathcal{T}_1(t), \mathcal{T}_2(t), \mathcal{T}_3(t) : C(\Omega, \mathbb{R}) \rightarrow C(\Omega, \mathbb{R})
\]
be the \( C_0 \) semigroups associated with \( D\Delta - \mu(x) \), \( D\Delta - (\mu(x) + \gamma_1(x) + \gamma_2(x) + \alpha(x)) \) and \( D\Delta - (\mu(x) + \delta(x)) \), respectively, subject to the Neumann boundary condition. Let \( \tilde{\Gamma}_1, \tilde{\Gamma}_2 \) and \( \tilde{\Gamma}_3 \) be the Green functions associated with \( D\Delta - \mu(x) \), \( D\Delta - (\mu(x) + \gamma_1(x) + \gamma_2(x) + \alpha(x)) \) and \( D\Delta - (\mu(x) + \delta(x)) \), respectively, subject to the Neumann boundary condition. Then, for any \( \phi \in C(\Omega, \mathbb{R}) \) and \( t \geq 0 \), we have
\[
(\mathcal{T}_i(t)\phi)(x) = \int_{\Omega} \tilde{\Gamma}_i(x, y, t)\phi(y)dy, \quad i = 1, 2, 3.
\]
Applying [30, Section 7.1 and Corollary 7.2.3], we know that for each \( t > 0 \) and \( i = 1, 2, 3 \), \( \mathcal{T}_i(t) : C(\Omega, \mathbb{R}) \rightarrow C(\Omega, \mathbb{R}) \) is compact and strongly positive.
Let $A_i : D(A_i) \to C(\overline{\Omega}, \mathbb{R})$ be the generator of $\mathcal{T}_i$, $i = 1, 2, 3$. Then $\mathcal{T}(t) = (\mathcal{T}_1(t), \mathcal{T}_2(t), \mathcal{T}_3(t)) : X \to X$ is a semigroup generated by the operator $\mathcal{A} := (A_1, A_2, A_3)$ defined on $D(\mathcal{A}) := D(A_1) \times D(A_2) \times D(A_3)$.

For all $x \in \overline{\Omega}$ and $\phi = (\phi_1, \phi_2, \phi_3) \in X^+$, let $\mathcal{F} = (\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3) : X^+ \to X$ be

$$
\begin{align*}
\mathcal{F}_1(\phi)(x) &= \Lambda(x) - \beta(x)f(\phi_1(x), \phi_2(x)) + \gamma_1(x)\phi_2(x) + \delta(x)\phi_3(x), \\
\mathcal{F}_2(\phi)(x) &= \beta(x)f(\phi_1(x), \phi_2(x)), \\
\mathcal{F}_3(\phi)(x) &= \gamma_2(x)\phi_2(x).
\end{align*}
$$

We can rewrite system (3)-(5) as follows:

$$
\begin{cases}
\frac{d\phi}{dt} = A\phi + \mathcal{F}(\phi), & t > 0, \\
\phi(0) = \phi \in X^+.
\end{cases}
$$

From assumption (H2), we get

$$
\phi(x) + h\mathcal{F}(\phi)(x) = \begin{pmatrix}
\phi_1(x) + h(\Lambda(x) - \beta(x)f(\phi_1(x), \phi_2(x)) + \gamma_1(x)\phi_2(x) + \delta(x)\phi_3(x)) \\
\phi_2(x) + h\beta(x)f(\phi_1(x), \phi_2(x)) \\
\phi_3(x) + h\gamma_2(x)\phi_2(x)
\end{pmatrix} \\
\geq \begin{pmatrix}
\phi_1(x)(1 - h\beta(x)\phi_2(x)) \\
\phi_2(x) \\
\phi_3(x)
\end{pmatrix},
$$

for any $\phi \in X^+$ and small $h \geq 0$. This yields that

$$
\lim_{h \to 0^+} \frac{1}{h} \text{dist} (\phi + h\mathcal{F}(\phi), X^+) = 0, \ \forall \phi \in X^+.
$$

From [24, Corollary 4], we have the following result.

**Theorem 2.1.** For each initial value $\phi = (\phi_1, \phi_2, \phi_3) \in X^+$, the system (3)-(5) has a unique mild solution $u(\cdot, t, \phi)$ on $[0, \tau_{\phi})$ with $u(\cdot, 0, \phi) = \phi$, where $\tau_{\phi} \leq \infty$. Moreover, $u(\cdot, t, \phi) \in X^+$ for all $t \in [0, \tau_{\phi})$.

Then we give the existence of solutions of system (3)-(5).

**Theorem 2.2.** The system (3)-(5) has a unique solution $u(\cdot, t, \phi)$ on $[0, \infty)$ with $\phi \in X^+$ and $u(\cdot, 0, \phi) = \phi$. Furthermore, the solution semiflow $\Phi(t) = u(\cdot, t) : X^+ \to X^+$ of (3) is defined by

$$
\Phi(t)\phi = u(\cdot, t, \phi), \ t \geq 0
$$

admits a global compact attractor.

**Proof.** Let

$$
\overline{m} := \max_{x \in \Omega} m(x) \quad \text{and} \quad \underline{m} := \min_{x \in \Omega} m(x),
$$

for any function $m(x)$ defined in $\Omega$. Define

$$
\mathcal{N}(x,t) := S(x,t) + I(x,t) + R(x,t).
$$

Then $\mathcal{N}(x,t)$ satisfies

$$
\begin{cases}
\frac{\partial \mathcal{N}(x,t)}{\partial t} \leq D\Delta \mathcal{N}(x,t) + \Lambda(x) - \mu(x)\mathcal{N}(x,t), & x \in \Omega, \ t > 0, \\
\frac{\partial \mathcal{N}(x,t)}{\partial \nu} = 0, & x \in \partial \Omega, \ t > 0.
\end{cases}
$$
The standard parabolic comparison theorem [30, Theorem 7.3.4] implies that \( \mathcal{N}(x, t) \) is uniformly bounded, and so are \( S(x, t) \), \( I(x, t) \) and \( R(x, t) \). Furthermore, it follows from [9, Lemma 2.2] that

\[
\limsup_{t \to \infty} \mathcal{N}(x, t) \leq \frac{\bar{X}}{\mu}, \quad \text{uniformly for } x \in \bar{\Omega}.
\]

Then there exists a \( t_1 > 0 \) and \( 0 < \hat{\eta} < 1 \) such that

\[
\mathcal{N}(x, t) \leq (1 + \hat{\eta}) \frac{\bar{X}}{\mu}, \quad \forall t \geq t_1.
\]

Thus, we get

\[
S(x, t) \leq (1 + \hat{\eta}) \frac{\bar{X}}{\mu}, \quad I(x, t) \leq (1 + \hat{\eta}) \frac{\bar{X}}{\mu}, \quad R(x, t) \leq (1 + \hat{\eta}) \frac{\bar{X}}{\mu}, \quad \forall t \geq t_1,
\]

which implies that \( S(x, t) \), \( I(x, t) \) and \( R(x, t) \) are ultimately bounded. Therefore, the solution semiflow \( \Phi(t) : \mathbb{X}^+ \to \mathbb{X}^+ \) is point dissipative. Using [38, Theorem 2.1.8], we conclude that \( \Phi(t) : \mathbb{X}^+ \to \mathbb{X}^+ \) is compact for each \( t \geq 0 \). According to [23, Theorem 2.6], we know that \( \Phi(t) : \mathbb{X}^+ \to \mathbb{X}^+, \ t \geq 0 \), admits a global compact attractor. \( \square \)

3. Threshold dynamics. From [9, Lemma 2.2], system (3) always has a disease-free equilibrium \( E_0(S^0(x), 0, 0) \). For convenience, we denote

\[
f_S(S, I) := \frac{\partial f(S, I)}{\partial S} \quad \text{and} \quad f_I(S, I) := \frac{\partial f(S, I)}{\partial I}.
\]

Linearizing system (3) at the disease-free equilibrium \( E_0 \), we have

\[
\begin{aligned}
\frac{\partial I(x, t)}{\partial t} &= D\Delta I(x, t) + \beta(x)f_I(S^0(x), 0)I(x, t) - (\mu(x) + \gamma_1(x) + \gamma_2(x))I(x, t) + \alpha(x)I(x, t), \quad x \in \Omega, \ t > 0, \\
\frac{\partial I(x, t)}{\partial \nu} &= 0, \quad x \in \partial \Omega, \ t > 0.
\end{aligned}
\]

Substituting \( I(x, t) = e^{\lambda t}\psi(x) \) into (7), we obtain

\[
\begin{aligned}
\lambda \psi(x) &= D\Delta \psi(x) + \beta(x)f_I(S^0(x), 0)\psi(x) - (\mu(x) + \gamma_1(x) + \gamma_2(x))\psi(x) + \alpha(x)\psi(x), \quad x \in \Omega, \\
\frac{\partial \psi(x)}{\partial \nu} &= 0, \quad x \in \partial \Omega.
\end{aligned}
\]

It follows from [9, Lemma 2.4] that (8) has a principal eigenvalue \( \lambda(S^0(x)) \) with a strongly positive eigenvector.

We now define a positive linear operator by

\[
C(\varphi)(x) := \beta(x)f_I(S^0(x), 0)\varphi(x), \quad \forall \varphi \in C(\bar{\Omega}, \mathbb{R}), \ x \in \bar{\Omega}.
\]

Thus, the total distribution of new infective individuals is

\[
\int_0^\infty C((T_2(t)\varphi))(x)dt.
\]

Therefore, the next generation operator \( L \) is given by

\[
L(\varphi) := \int_0^\infty C(T_2(t)\varphi)dt = C \left( \int_0^\infty T_2(t)\varphi dt \right).
\]
Motivated by [9, 21, 34], we can define the spectral radius of \( L \) as the basic reproduction number
\[
R_0 := r(L).
\]
According to [34, Lemma 2.2] and [37, Theorem 3.1], we get the following result.

**Lemma 3.1.** \( \lambda(S^0(x)) \) has the same sign as \( R_0 - 1 \).

On the other hand, \( R_0 \) can also be expressed by the following variational formula (see [2, 4, 39]):
\[
R_0 = \sup_{\phi \in H^1(\Omega), \phi \neq 0} \left\{ \frac{\int_{\Omega} \beta(x)f_1(S^0(x), 0)\phi^2 \, dx}{\int_{\Omega}(\mu(x) + \gamma_1(x) + \gamma_2(x) + \alpha(x))\phi^2 \, dx} \right\}. \tag{9}
\]
Then, from [2, Theorem 2.2] and [39, Theorem 3.3], we can obtain the following results on the impact of \( D \) on \( R_0 \).

**Theorem 3.2.** The following statements hold.

(i): \( R_0 \) is a monotone decreasing function of \( D \) with
\[
\lim_{D \to 0} R_0 = \max \left\{ \frac{\beta(x)f_1(S^0(x), 0)}{\mu(x) + \gamma_1(x) + \gamma_2(x) + \alpha(x)} : x \in \bar{\Omega} \right\}
\]
and
\[
\lim_{D \to \infty} R_0 = \frac{\int_{\Omega} \beta(x)f_1(S^0(x), 0) \, dx}{\int_{\Omega}(\mu(x) + \gamma_1(x) + \gamma_2(x) + \alpha(x)) \, dx}.
\]

(ii): If \( \int_{\Omega} \beta(x)f_1(S^0(x), 0) \, dx > \int_{\Omega}(\mu(x) + \gamma_1(x) + \gamma_2(x) + \alpha(x)) \, dx \), then \( R_0 > 1 \) for all \( D > 0 \).

(iii): If \( \int_{\Omega} \beta(x)f_1(S^0(x), 0) \, dx < \int_{\Omega}(\mu(x) + \gamma_1(x) + \gamma_2(x) + \alpha(x)) \, dx \), then there exists \( D^* \in (0, +\infty) \) such that \( R_0 > 1 \) for \( D < D^* \) and \( R_0 < 1 \) for \( D > D^* \).

**Remark 1.** If the functions \( \Lambda(x), \mu(x), \beta(x), \gamma_1(x), \gamma_2(x), \delta(x) \) and \( \alpha(x) \) are constants, the basic reproduction number \( R_0 \) does not change with the diffusion coefficient \( D \). We will show the details with an example in Section 4.

The following lemma is useful in proving the main results of this section.

**Lemma 3.3.** Suppose that \( u(\cdot, t, \phi) \) is the solution of system (3)-(5) with \( u(\cdot, 0, \phi) = \phi \in X^+ \).

(i): For any \( \phi \in X^+ \), we always have \( S(\cdot, t, \phi) > 0, \forall t > 0 \), and there is a constant \( \eta_1 > 0 \) such that
\[
\liminf_{t \to \infty} S(x, t, \phi) \geq \eta_1, \quad \text{uniformly for } x \in \bar{\Omega};
\]

(ii): If there exists \( \hat{t} \geq 0 \) such that \( I(\cdot, \hat{t}, \phi) \neq 0 \) (resp. \( R(\cdot, \hat{t}, \phi) = 0 \)), then, for \( t > \hat{t} \), we have \( I(\cdot, t, \phi) > 0 \) (resp. \( R(\cdot, t, \phi) > 0 \)).

*Proof.* It follows from (6) that there exists a \( t_2 > 0 \) such that
\[
I(x, t) \leq M, \quad \forall t \geq t_2.
\]

Then by the first equation of system (3), we have
\[
\begin{aligned}
\frac{\partial S(x, t)}{\partial t} & \geq D\Delta S(x, t) + \Lambda(x) - (\mu(x) + \beta(x)M)S(x, t), \quad x \in \Omega, \; t \geq t_2, \\
\frac{\partial S(x, t)}{\partial \nu} & = 0, \quad x \in \partial \Omega, \; t \geq t_2.
\end{aligned}
\]
By the standard parabolic comparison theorem and [9, Lemma 2.2], we obtain that
\[ \liminf_{t \to \infty} S(x, t, \phi) \geq \frac{A}{\mu + \beta M} := \eta_1, \]
uniformly for \( x \in \bar{\Omega} \). Thus, part (i) is proved.

Then, from system (3), we can get the following inequalities, respectively:
\[
\begin{cases}
\frac{\partial I(x, t)}{\partial t} \geq D \Delta I(x, t) - (\mu(x) + \gamma_1(x) + \gamma_2(x) + \alpha(x))I(x, t), \ x \in \Omega, \ t > 0, \\
\frac{\partial I(x, t)}{\partial \nu} = 0, \ x \in \partial \Omega, \ t > 0,
\end{cases}
\]
and
\[
\begin{cases}
\frac{\partial R(x, t)}{\partial t} \geq D \Delta R(x, t) - (\mu(x) + \delta(x))R(x, t), \ x \in \Omega, \ t > 0, \\
\frac{\partial R(x, t)}{\partial \nu} = 0, \ x \in \partial \Omega, \ t > 0.
\end{cases}
\]
Using the strong maximum principle (see, e.g., [28, p. 172, Theorem 4]) and the Hopf boundary lemma (see, e.g., [28, p. 170, Theorem 3]), part (ii) is valid. \[ \square \]

Now we are ready to give the main results of this section.

**Theorem 3.4.** Assume that \( u(\cdot, t, \phi) \) is the solution of system (3)-(5) with \( u(\cdot, 0, \phi) = \phi \in X^+ \). Then the following statements are valid.

(i): If \( R_0 < 1 \), then the disease-free equilibrium \( E_0(S^0(x), 0, 0) \) is globally attractive.

(ii): If \( R_0 > 1 \), then there exists a constant \( \rho > 0 \) such that for any \( \phi \in X^+ \) with \( \phi_2(\cdot) \neq 0 \), we have
\[ \liminf_{t \to \infty} S(x, t) \geq \rho, \ \liminf_{t \to \infty} I(x, t) \geq \rho, \ \liminf_{t \to \infty} R(x, t) \geq \rho, \quad \text{uniformly for all } x \in \Omega. \]

**Proof.** (i) By Lemma 3.1, we have \( \lambda(S^0(x)) < 0 \) when \( R_0 < 1 \). Thus, there exists a sufficiently small \( \varepsilon > 0 \) such that \( \lambda(S^0(x) + \varepsilon) < 0 \). According to Theorem 2.2, there exists a \( t_3 > 0 \) such that
\[ S(x, t) \leq S^0(x) + \varepsilon, \ \forall t \geq t_3, \ x \in \Omega. \]
It follows from assumption (H1) that
\[ \frac{f(\cdot, I)}{I} \leq \lim_{t \to 0^+} \frac{f(\cdot, I)}{I} \leq f_I(\cdot, 0). \]
Thus, from the second equation of system (3), we have
\[
\begin{cases}
\frac{\partial I(x, t)}{\partial t} \leq D \Delta I(x, t) + \beta(x)f_I(S^0(x) + \varepsilon, 0)I - (\mu(x) + \gamma_1(x) + \gamma_2(x) + \alpha(x))I(x, t), \ x \in \Omega, \ t \geq t_3, \\
\frac{\partial I(x, t)}{\partial \nu} = 0, \ x \in \partial \Omega, \ t \geq t_3.
\end{cases}
\]
Let \( \psi(x) \) be the eigenfunction corresponding to the principal eigenvalue \( \lambda(S^0(x) + \varepsilon) < 0 \). Suppose \( \xi_1 > 0 \) such that \( I(x, t_3) \leq \xi_1 \psi(x) \). By the comparison principle, we get
\[ I(x, t) \leq \xi_1 \psi(x)e^{\lambda(S^0(x) + \varepsilon)(t-t_3)}, \ \forall t \geq t_3. \]
This yields that \( \lim_{t \to \infty} I(x, t) = 0 \) uniformly for \( x \in \bar{\Omega} \). Then, the third equation of system (3) is asymptotic to
\[
\frac{\partial R(x,t)}{\partial t} = D\Delta R(x,t) - (\mu(x) + \delta(x))R(x,t).
\]
Then we have \( \lim_{t \to \infty} R(x,t) = 0 \) uniformly for \( x \in \bar{\Omega} \). Moreover, the first equation of system (3) is asymptotic to
\[
\frac{\partial S(x,t)}{\partial t} = D\Delta S(x,t) + \Lambda(x) - \mu(x)S(x,t),
\]
which implies that \( \lim_{t \to \infty} S(x,t) = S^0(x) \) uniformly for \( x \in \bar{\Omega} \).

(ii) Define
\[
W = \{ \phi \in X^+ : \phi_2(\cdot) \neq 0 \},
\]
and
\[
\partial W := X^+ \setminus W = \{ \phi \in X^+ : \phi_2(\cdot) \equiv 0 \}.
\]
If \( \phi_2(\cdot) \neq 0 \), then \( I(x,t) > 0, \forall x \in \bar{\Omega} \) and \( t > 0 \) by Lemma 3.3. From the third equation of system (3), a straightforward contradiction argument shows that \( R(x,t) > 0, \forall x \in \Omega \) and \( t > 0 \). By Lemma 3.3, for all \( \phi \in W \), we get \( I(x,t,\phi) > 0 \) and \( R(x,t,\phi) > 0, \forall x \in \bar{\Omega} \) and \( t > 0 \). This implies that \( \Phi(t)W \subseteq W, \forall t \geq 0 \). Define
\[
\mathcal{M}_\partial := \{ \phi \in \partial W : \Phi(t)\phi \in \partial W, \forall t \geq 0 \},
\]
and the orbit
\[
\mathcal{O}^+ (\phi) := \{ \Phi(t)\phi : t \geq 0 \}.
\]
\( \omega(\phi) \) is the omega limit set of \( \mathcal{O}^+ (\phi) \).

We first show that \( \omega(\phi) = \{ (S^0(x),0,0) \}, \forall \phi \in \mathcal{M}_\partial \). By \( \phi \in \mathcal{M}_\partial \) and \( \forall t \geq 0 \), we have \( \Phi(t)\phi \in \partial W \). Thus, \( I(\cdot,t,\phi) \equiv 0 \). When \( I(\cdot,t,\phi) \equiv 0, \forall t \geq 0 \), according to the last equation of system (3), it is easy to see that \( \lim_{t \to \infty} R(x,t,\phi) = 0 \) uniformly for \( x \in \bar{\Omega} \). Further, we can show that \( \lim_{t \to \infty} S(x,t,\phi) = S^0(x) \) uniformly for \( x \in \bar{\Omega} \). Therefore, \( \omega(\phi) = \{ (S^0(x),0,0) \}, \forall \phi \in \mathcal{M}_\partial \).

From Lemma 3.1, we have \( \lambda(S^0(x)) > 0 \). Since \( \lambda(S^0(x)) \) is continuous on \( S^0(x) \), there exists a sufficiently small \( \xi_0 > 0 \) such that \( \lambda(S^0(x) - \xi_0) > 0 \). Next we prove
\[
\lim_{t \to \infty} \sup \| \Phi(t)\phi - (S^0(x),0,0) \| \geq \xi_0, \forall \phi \in W.
\]
Assume that there exists \( \phi_0 \in W \) such that
\[
\lim_{t \to \infty} \sup \| \Phi(t)\phi_0 - (S^0(x),0,0) \| < \xi_0.
\]
Thus, there exists \( t_4 > 0 \) such that \( S(x,t,\phi_0) > S^0(x) - \xi_0 \) and \( 0 < I(x,t,\phi_0) < \xi_0 \), for all \( x \in \bar{\Omega} \) and \( t \geq t_4 \). Therefore, from assumption (H1), we know that \( I(x,t,\phi_0) \) satisfies
\[
\begin{cases}
\frac{\partial I(x,t)}{\partial t} \geq D\Delta I(x,t) + \beta(x)f_I(S^0(x) - \xi_0,\xi_0)I(x,t) - (\mu(x) + \gamma_1(x) + \gamma_2(x) + \alpha(x))I(x,t), & x \in \bar{\Omega}, \ t \geq t_4, \\
\frac{\partial I(x,t)}{\partial \nu} = 0, & x \in \partial \Omega, \ t \geq t_4.
\end{cases}
\]
From (8), we can also obtain
\[
\begin{align*}
\lambda \psi(x) &= D \Delta \psi(x) + \beta(x)f_1(S^0(x) - \xi_0, \xi_0)\psi(x) - (\mu(x) + \gamma_1(x) + \gamma_2(x) \\
&\quad + \alpha(x))\psi(x), \ x \in \Omega, \\
\frac{\partial \psi(x)}{\partial \nu} &= 0, \ x \in \partial \Omega.
\end{align*}
\] (10)

Let \(\hat{\psi}\) be the strongly positive eigenfunction of problem (10) with principal eigen-value \(\lambda(S^0(x) - \xi_0)\). Since \(I(x, t, \phi_0) > 0\) for all \(x \in \Omega\) and \(t > 0\), there exists \(\varepsilon > 0\) such that \(I(x, t, \phi_0) \geq \varepsilon \hat{\psi}\). It is clear that \(u_1(x, t) = \varepsilon e^{\lambda(S^0(x) - \xi_0)(t-t_4)}\hat{\psi}\) is a solution of the following linear system
\[
\begin{align*}
\frac{\partial u_1(x, t)}{\partial t} &= D \Delta u_1(x, t) + \beta(x)f_1(S^0(x) - \xi_0, \xi_0)u_1(x, t) - (\mu(x) + \gamma_1(x) \\
&\quad + \gamma_2(x) + \alpha(x))u_1(x, t), \ x \in \Omega, \ t \geq t_4, \\
\frac{\partial u_1(x, t)}{\partial \nu} &= 0, \ x \in \partial \Omega, \ t \geq t_4.
\end{align*}
\]

Applying the comparison principle, we get
\[
I(x, t, \phi_0) \geq \varepsilon e^{\lambda(S^0(x) - \xi_0)(t-t_4)}\hat{\psi}, \ \forall x \in \bar{\Omega}, \ t \geq t_4.
\]

Due to \(\lambda(S^0(x) - \xi_0) > 0\), we conclude that \(I(x, t, \phi_0)\) is unbounded. This is a contradiction.

Define \(p : X^+ \to [0, \infty)\) by
\[
p(\phi) = \min_{x \in \Omega} \phi_2(x), \ \forall \phi \in X^+,
\]
which is a continuous function. Clearly, \(p^{-1}(0, \infty) \subseteq \mathcal{W}\). We can also know that if \(p(\phi) > 0\) or \(p(\phi) = 0\) and \(\phi \in \mathcal{W}\), then \(p(\Phi(t)\phi) > 0\) for all \(t > 0\). Hence, for the semiflow \(\Phi(t) : X^+ \to X^+\), \(p\) is a generalized distance function. From the above discussions, we know that any forward orbit of \(\Phi(t)\) in \(\mathcal{M}_0\) converges to \((S^0(x), 0, 0)\), which is isolated in \(X^+\) and \(W^*(S^0(x), 0, 0) \cap \mathcal{W} = \emptyset\), where \(W^*(S^0(x), 0, 0)\) is the stable set of \((S^0(x), 0, 0)\). Obviously, there is no cycle in \(\mathcal{M}_0\) from \((S^0(x), 0, 0)\) to \((S^0(x), 0, 0)\). By using [31, Theorem 3], there exists a constant \(\tilde{\rho} > 0\) such that
\[
\min_{\psi \in \omega(\phi)} p(\psi) > \tilde{\rho}, \ \forall \phi \in \mathcal{W}.
\]
This implies that \(\liminf_{t \to \infty} I(\cdot, t, \phi) \geq \tilde{\rho}, \ \forall \phi \in \mathcal{W}\). Then for arbitrary \(\varepsilon_1 > 0\), there exists \(t_5 > 0\) such that \(I(\cdot, t, \phi) \geq \tilde{\rho} - \varepsilon_1, \ \forall \phi \in \mathcal{W}, \ t \geq t_5\). From the third equation of system (3), we obtain
\[
\begin{align*}
\frac{\partial R(x, t)}{\partial t} &\geq D \Delta R(x, t) + \gamma_2(\tilde{\rho} - \varepsilon_1) - (\mu + \delta)R(x, t), \ x \in \Omega, \ t \geq t_5, \\
\frac{\partial R(x, t)}{\partial \nu} &= 0, \ x \in \partial \Omega, \ t \geq t_5,
\end{align*}
\]
and then we have \(\liminf_{t \to \infty} R(\cdot, t, \phi) \geq \frac{\gamma_2(\tilde{\rho} - \varepsilon_1)}{\mu + \delta}, \ \forall \phi \in \mathcal{W}\). In fact, \(\liminf_{t \to \infty} R(\cdot, t, \phi) \geq \frac{\gamma_2\tilde{\rho}}{\mu + \delta}, \ \forall \phi \in \mathcal{W}\), since \(\varepsilon_1\) is an arbitrary positive constant. From Lemma 3.3, we can choose \(\rho = \min \left\{ \eta_1, \tilde{\rho}, \frac{\gamma_2\tilde{\rho}}{\mu + \delta} \right\} \) such that
\[
\liminf_{t \to \infty} S(\cdot, t, \phi) \geq \rho, \ \liminf_{t \to \infty} I(\cdot, t, \phi) \geq \rho, \ \liminf_{t \to \infty} R(\cdot, t, \phi) \geq \rho, \ \forall \phi \in \mathcal{W}.
\]
This completes the proof. □

4. Special cases and numerical simulations. In this section, we discuss some special cases. Firstly, we assume that $\Lambda(x)$, $\mu(x)$, $\beta(x)$, $\gamma_1(x)$, $\gamma_2(x)$, $\delta(x)$ and $\alpha(x)$ are constants in system (3). Then system (3) becomes to

\[
\begin{cases}
\frac{\partial S(x,t)}{\partial t} = D\triangle S(x,t) + \Lambda - \mu S(x,t) - \beta f(S(x,t), I(x,t)) + \gamma_1 I(x,t) + \delta R(x,t), \\
\frac{\partial I(x,t)}{\partial t} = D\triangle I(x,t) + \beta f(S(x,t), I(x,t)) - (\mu + \gamma_1 + \gamma_2 + \alpha) I(x,t), \\
\frac{\partial R(x,t)}{\partial t} = D\triangle R(x,t) + \gamma_2 I(x,t) - (\mu + \delta) R(x,t),
\end{cases}
\]  

(11)

with initial data (4) and Neumann boundary condition (5). According to [34, Theorem 2.3] and formula (9), we can obtain

\[ R_0 = \frac{\beta f_1(S^0, 0)}{\mu + \gamma_1 + \gamma_2 + \alpha}, \]  

(12)

where $S^0 := \Lambda/\mu$.

Then, we have the following results.

**Theorem 4.1.** (i) System (11) only has the disease-free equilibrium $E_0$ ($S^0, 0, 0$) when $R_0 < 1$;

(ii) Besides $E_0$, system (11) also has a unique endemic equilibrium $E^*$ ($S^*, I^*, R^*$) when $R_0 > 1$.

**Proof.** We can easily obtain the conclusion (i). From the third equation of system (11), we have $R = \frac{\gamma_2 I}{\mu + \delta}$. Adding the three equations of system (11), we get

\[ S = \frac{\Lambda(\mu + \delta) - ((\mu + \delta)(\mu + \alpha) + \mu \gamma_2) I}{\mu(\mu + \delta)}. \]

This yields that in order to have $S > 0$ and $I > 0$ at an equilibrium, we must have $I \in \left(0, \frac{\Lambda(\mu + \delta)}{\mu(\mu + \alpha) + \mu \gamma_2}\right)$. Substituting the expressions $S$ and $R$ into the second equation of system (11), we obtain

\[ \beta f \left( \frac{\Lambda(\mu + \delta) - ((\mu + \delta)(\mu + \alpha) + \mu \gamma_2) I}{\mu(\mu + \delta)}, I \right) - (\mu + \gamma_1 + \gamma_2 + \alpha) I = 0. \]

Define

\[ \mathcal{H}(I) = \frac{\beta f \left( \frac{\Lambda(\mu + \delta) - ((\mu + \delta)(\mu + \alpha) + \mu \gamma_2) I}{\mu(\mu + \delta)}, I \right)}{I} - (\mu + \gamma_1 + \gamma_2 + \alpha), \]

for $I \in \left(0, \frac{\Lambda(\mu + \delta)}{\mu(\mu + \alpha) + \mu \gamma_2}\right)$. Clearly, $\mathcal{H} \left( \frac{\Lambda(\mu + \delta)}{\mu(\mu + \alpha) + \mu \gamma_2} \right) = -(\mu + \gamma_1 + \gamma_2 + \alpha) < 0$. Therefore, we have

\[ \mathcal{H}'(I) = -\beta \frac{(\mu + \delta)(\mu + \alpha) + \mu \gamma_2}{\mu(\mu + \delta)} - \frac{\beta f_1 \left( \frac{\Lambda(\mu + \delta) - ((\mu + \delta)(\mu + \alpha) + \mu \gamma_2) I}{\mu(\mu + \delta)}, I \right)}{I}. \]
According to assumption (H1), we can get $\mathcal{H}'(I) < 0$ for $I \in \left(0, \frac{\lambda_{\alpha}}{\mu + \delta} \right)$ and
\[
\lim_{I \to 0^+} \mathcal{H}(I) = \beta \lim_{I \to 0^+} \frac{f(S^0, I)}{I} - (\mu + \gamma_1 + \gamma_2 + \alpha) = \beta f_I(S_0, 0) - (\mu + \gamma_1 + \gamma_2 + \alpha) = (\mu + \gamma_1 + \gamma_2 + \alpha)(R_0 - 1) > 0, \text{ when } R_0 > 1.
\]
Thus, we know that system (11) has a unique equilibrium. \hfill \square

**Theorem 4.2.** The disease-free equilibrium $E_0$ of system (11) is locally asymptotically stable when $R_0 < 1$.

**Proof.** Linearizing system (11) at $E_0$, we obtain
\[
\frac{\partial U(x,t)}{\partial t} = \bar{D} \Delta U(x,t) + \mathcal{P}_1 U(x,t),
\]
where $\bar{D} = \text{diag}(D, D, D)$ and
\[
\mathcal{P}_1 = \begin{pmatrix} -\mu & -\beta f_I(S^0, 0) + \gamma_1 & \delta \\ -\beta f_I(S^0, 0) - (\mu + \gamma_1 + \gamma_2 + \alpha) & 0 & 0 \\ \gamma_2 & -(\mu + \delta) & \end{pmatrix}.
\]
Hence, we get the characteristic polynomial as follows
\[
(\lambda + l^2 D + \mu)(\lambda + l^2 D + \mu + \delta)(\lambda + l^2 D + \mu + \gamma_1 + \gamma_2 + \alpha - \beta f_I(S^0, 0)) = 0,
\]
where $\lambda$ is the eigenvalue which determines temporal growth and $l$ is the wavenumber (see [27, 51]). It follows from (13) that $\lambda_1 = -(l^2 D + \mu) < 0$, $\lambda_2 = -(l^2 D + \mu + \delta) < 0$ and
\[
\lambda_3 = -(l^2 D + \mu + \gamma_1 + \gamma_2 + \alpha - \beta f_I(S^0, 0)).
\]
Since $R_0 < 1$, we can obtain $\mu + \gamma_1 + \gamma_2 + \alpha > \beta f_I(S^0, 0)$. Therefore, we conclude that $E_0$ is locally asymptotically stable when $R_0 < 1$. \hfill \square

**Theorem 4.3.** The endemic equilibrium $E^*$ of system (11) is locally asymptotically stable when $R_0 > 1$.

**Proof.** By similar calculation as Theorem 4.2, we obtain the characteristic equation at $E^*$ as follows
\[
\lambda^3 + a_3(l^2)\lambda^2 + a_2(l^2)\lambda + a_1(l^2) = 0,
\]
where
\[
a_3(l^2) = l^2 D + \mu + \gamma_1 + \gamma_2 + \alpha + l^2 D + \mu + \beta f_s(S^*, I^*) + l^2 D + \mu + \delta - \beta f_I(S^*, I^*),
\]
\[
a_2(l^2) = (l^2 D + \mu + \beta f_s(S^*, I^*)) + l^2 D + \mu + \beta f_s(S^*, I^*) + (l^2 D + \mu + \gamma_1 + \gamma_2 + \alpha) + (l^2 D + \mu + \beta f_s(S^*, I^*) + l^2 D + \mu + \delta - \beta f_I(S^*, I^*) - \gamma_1 \beta f_s(S^*, I^*) - (l^2 D + \mu + \beta f_s(S^*, I^*) + l^2 D + \mu + \delta - \beta f_I(S^*, I^*) - \gamma_1 \beta f_s(S^*, I^*) - \gamma_2 \delta f_s(S^*, I^*)),
\]
\[
a_1(l^2) = (l^2 D + \mu + \beta f_s(S^*, I^*) + l^2 D + \mu + \beta f_s(S^*, I^*) + (l^2 D + \mu + \gamma_1 + \gamma_2 + \alpha) + (l^2 D + \mu + \beta f_s(S^*, I^*) + l^2 D + \mu + \delta - \beta f_I(S^*, I^*) - (l^2 D + \mu + \beta f_s(S^*, I^*) + l^2 D + \mu + \delta - \beta f_I(S^*, I^*) - \gamma_1 \beta f_s(S^*, I^*) - \gamma_2 \delta f_s(S^*, I^*).
Since
\[\beta f_I(S^*, I^*) \leq \beta \frac{f(S^*, I^*)}{I^*} = \mu + \gamma_1 + \gamma_2 + \alpha,\]
we obtain
\[a_3(l^2) \geq 3l^2 D + 2\mu + \beta f_S(S^*, I^*) + \delta > 0,\]
\[a_2(l^2) \geq (l^2 D + \mu + \beta f_S(S^*, I^*))(l^2 D + \mu + \delta) + \beta f_S(S^*, I^*)(l^2 D + \mu + \gamma_2 + \alpha) > 0,\]
and
\[a_1(l^2) \geq \beta f_S(S^*, I^*)(l^2 D + \mu)(l^2 D + \mu + \gamma_2 + \alpha) > 0.\]

Then, we have
\[a_3(l^2)a_2(l^2) - a_1(l^2) > (l^2 D + \mu + \delta)^2(l^2 D + \mu + \gamma_1 + \gamma_2 + \alpha + l^2 D + \mu + \beta f_S(S^*, I^*) - \beta f_I(S^*, I^*)) + \gamma_2 \delta \beta f_S(S^*, I^*) \]
\[> (l^2 D + \mu + \delta)^2(l^2 D + \mu + \beta f_S(S^*, I^*)) + \gamma_2 \delta \beta f_S(S^*, I^*) > 0.\]

It follows from Routh-Hurwitz criterion that all eigenvalues of (14) have negative real parts. Therefore, the endemic equilibrium \(E^*\) of system (11) is locally asymptotically stable when \(R_0 > 1\).

From Theorems 3.4 (i) and 4.2, we get the following result.

**Theorem 4.4.** If \(R_0 < 1\), then the disease-free equilibrium \(E_0\) of system (11) is globally asymptotically stable.

However, it is difficult to study the global stability of endemic equilibrium \(E^*\) of system (11). Here we only consider the special incidence rate \(f(S, I) = Sg(I)\), where \(g(I)\) satisfies
\[g(0) = 0; \ g(I) \geq 0 \text{ for } I > 0; \ \lim_{I \to 0^+} \frac{g(I)}{I} \text{ exists and } \left(\frac{g(I)}{I}\right)' \leq 0 \text{ for } I > 0.\]

By Theorem 2.3 and Corollary 2.4 of [19], we have the following result.

**Theorem 4.5.** If \(R_0 > 1\) and \(f(S, I) = Sg(I)\), then the endemic equilibrium \(E^*\) of system (11) is globally asymptotically stable.

In the following, let \(\Omega = [0, 10]\) and choose \(f(S, I) = \frac{SI}{(1 + \alpha_2 S)(1 + \alpha_1 I)}\) with \(\alpha_1, \ \alpha_2 \geq 0\). Clearly, \(f(S, I)\) satisfies the assumptions (H1) and (H2). We will discuss a homogeneous case and a heterogeneous case in the following.
Case I: Homogeneous case. Suppose that every parameter is constant. Then, system (11) becomes to

\[
\begin{align*}
\frac{\partial S(x,t)}{\partial t} &= D \Delta S(x,t) + \Lambda - \mu S(x,t) - \frac{\beta S(x,t)I(x,t)}{(1 + \alpha_2 S(x,t))(1 + \alpha_1 I(x,t))} \\
&\quad + \gamma_1 I(x,t) + \delta R(x,t), \\
\frac{\partial I(x,t)}{\partial t} &= D \Delta I(x,t) + \frac{\beta S(x,t)I(x,t)}{(1 + \alpha_2 S(x,t))(1 + \alpha_1 I(x,t))} \\
&\quad - (\mu + \gamma_1 + \gamma_2 + \alpha) I(x,t), \\
\frac{\partial R(x,t)}{\partial t} &= D \Delta R(x,t) + \gamma_2 I(x,t) - (\mu + \delta) R(x,t), \\
\frac{\partial S(x,t)}{\partial \nu} &= \frac{\partial I(x,t)}{\partial \nu} = \frac{\partial R(x,t)}{\partial \nu} = 0.
\end{align*}
\]  
(15)

We consider system (15) with initial value

\[
S_0(x) = 200, \quad I_0(x) = 20e^{-x^2}, \quad \text{and} \quad R_0(x) = 0.
\]

From (12), the basic reproduction number of system (15) is

\[
R_0 = \frac{\beta S^0}{(\mu + \gamma_1 + \gamma_2 + \alpha)(1 + \alpha_2 S^0)}.
\]

For numerical simulations, we assume \(D = 1\) and choose the following parameters from [3, 40]:

\[
\begin{align*}
\Lambda &= 0.33 \text{ day}^{-1}, \quad \mu = 0.006 \text{ day}^{-1}, \quad \beta = 0.0056, \quad \alpha_1 = 0.5, \\
\gamma_1 &= 0.07 \text{ day}^{-1}, \quad \gamma_2 = 0.04 \text{ day}^{-1}, \quad \alpha = 0.06 \text{ day}^{-1}, \quad \delta = 0.021 \text{ day}^{-1}.
\end{align*}
\]  
(16)

When \(\alpha_2 = 0\), we get \(R_0 = 1.75 > 1\). By Theorem 4.5, we obtain that the endemic equilibrium \(E^* = (44.5655, 0.836, 1.2385)\) of system (15) is globally asymptotically stable (see Fig. 1), and the disease will keep endemic. Then, we set \(\alpha_2 = 0.1\) and keep the other parameters next. We have \(R_0 = 0.2692 < 1\). By Theorem 4.4, the disease-free equilibrium \(E_0 = (55, 0, 0)\) of system (15) is globally asymptotically stable (see Fig. 2), and the disease will be eliminated finally.

![Figure 1. Variation of populations of system (15) with \(D = 1\), \(\alpha_2 = 0\) and other parameters in (16).]
Changing $\gamma_1$ from 0 to 0.07 and fixing the other parameters, we show that the basic reproduction number $R_0$ of system (15) will decrease when $\gamma_1$ increases (see Fig. 3). When $\alpha_2$ increases from 0 to 0.1 and fixing the other parameters, Fig. 4 shows that the basic reproduction number $R_0$ of system (15) will decrease.

**Case II: Spatially heterogeneous case.** We assume that $\alpha_2 = 0$ and the other parameters are chosen as constants in (16) except $\beta$. First, we let

$$\beta(x) = 3.1 \times 10^{-3} \times (1 + c \sin(9\pi x/10)), \quad x \in [0, 10],$$

where $0 \leq c \leq 1$ is the magnitude of spatially heterogeneous transmission rate. Let $D = 10^{-3}$ and $c = 1$. By Theorem 3.2 (i), we obtain that $R_0 \approx 1.9375 > 1$. Fig. 5
shows that the density of infective individuals converges to a positive distribution. Then, we choose \( D = 10^6 \). By Theorem 3.2 (i) again, we can know that \( R_0 \) is an increasing function of \( c \) in \( \beta(x) \). Fig. 6 shows that \( R_0 < 1 \) when \( c < c^* \approx 0.4566 \), and \( R_0 > 1 \) when \( c > c^* \). Further, let \( c = 1 \) and we obtain that \( R_0 \approx 1.0373 \). We find that the larger diffusion rate can eliminate the spatial heterogeneity (see Fig. 7).

We next choose \( c = 0.15 \) in \( \beta(x) \). By simple calculation, we obtain
\[
\int_0^{10} \beta(x) f_1(S^0(x), 0) dx \approx 1.7231 < \int_0^{10} (\mu(x) + \gamma_1(x) + \gamma_2(x) + \alpha(x)) dx = 1.76.
\]

According to Theorem 3.2 (i), we can get \( \lim_{D \to 0} R_0 \approx 1.1141 > 1 \) and \( \lim_{D \to \infty} R_0 \approx 0.979 < 1 \). Since \( R_0 \) is a monotone decreasing function of \( D \), we conclude that there exists a critical value \( D^* > 0 \) such that \( R_0 > 1 \) when \( D < D^* \) and \( R_0 < 1 \) when \( D > D^* \) supporting Theorem 3.2 (iii).

5. Conclusions. In this paper, we have extended the ODE model investigated by Li et al. [17] to a reaction-diffusion one with a general incidence rate by taking into account the spatial heterogeneity. For this model, the basic reproduction number \( R_0 \) is defined by applying the next generation operator. As suggested by (9), we found that the diffusion can effect the value of \( R_0 \) due to the spatial heterogeneity. Then, we have established the results of threshold dynamics in term of \( R_0 \): the disease-free equilibrium is globally attractive if \( R_0 < 1 \) and persists if \( R_0 > 1 \). When model parameters are constants, we have derived the following results: the disease-free equilibrium is globally asymptotically stable if \( R_0 < 1 \), and the endemic equilibrium is globally asymptotically stable if \( R_0 > 1 \) with special incidence rate. However, when model parameters of system (3) are constants, it is difficult to study the global stability of the endemic equilibrium. We leave this as an open
When $\alpha_2 = 0$ and $D = 10^{-5}$, the evolution of the infective individuals $I(x,t)$ with parameters in (16) and $R_0 \approx 1.9375$.

Figure 5. When $\alpha_2 = 0$ and $D = 10^{-5}$, the evolution of the infective individuals $I(x,t)$ with parameters in (16) and $R_0 \approx 1.9375$.

Figure 6. The relation between $R_0$ and $c$ in $\beta(x)$.

problem. From Fig. 5 and Fig. 7, we know that the spatial heterogeneity can be eliminated when the diffusion rate $D$ is large enough. Fig. 6 shows that the spatial heterogeneous can effect the value of $R_0$: it can attain the minimum value with $c = 0$ (spatially homogeneous case), and become greater than one when $c$ passes through the critical value $c^*$. Hence, the disease will break out.

When $\alpha_1 = \alpha_2 = \delta = 0$, the existence of traveling wave solution of system (15) has been established (see [8, 47]). Therefore, it is an interesting problem to study the existence and nonexistence of traveling wave solutions and the minimum wave speed of system (15). We leave this for future study.
When $\alpha_2 = 0$ and $D = 10^6$, the evolution of the infective individuals $I(x,t)$ with parameters in (16) and $R_0 \approx 1.0373$.

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