Observation of the perturbed eigenvalues of PT-symmetric LC resonator systems

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Abstract

We address both theoretically and experimentally the influence of asymmetric perturbation on the eigenvalues of parity-time (PT) symmetric resonator systems under the symmetric gain-loss arrangement, based on an inductively coupled inductor-capacitor-resistor (LCR) pair. The perturbed eigenvalues have been theoretically presented, numerically simulated, and experimentally measured. It shows that the asymmetric perturbation breaks PT-symmetry, leading to complex eigenvalues, which is different from the broken PT-symmetric phase with complex-conjugate eigenvalues. We have analyzed the perturbed frequency responses in all phases. At the exceptional points (EP), the resulting eigenvalues splitting is proportional to the square root of perturbation, showing the advantage of being highly sensitive to asymmetric perturbation. Meanwhile, the smaller the perturbation, the higher the sensitivity. The perturbation effect of PT-symmetric systems may be utilized to detect small signal changes in LC passive wireless sensors.

1. Introduction

Parity-time (PT) symmetric systems are invariant under a combination of parity and time-reversal operations [11]. They have two distinguished phases, an exact PT-symmetric phase with real eigenvalues and a broken PT-symmetric phase with complex-conjugate eigenvalues. Exceptional points (EP), where both eigenvalues and their corresponding eigenvectors simultaneously coalesce, separate the unbroken and broken phases. The introduction of PT symmetry to classical wave dynamics has enhanced the evolution of optics, photonics, acoustics, and electronics [2].

A pair of inductively coupled inductor-capacitor-resistor (LCR) resonators, one with amplification (gain) and the other with an equivalent amount of attenuation (loss), provide easily accessible experimental configurations for exploring PT-symmetry and building PT-symmetric electronic devices [3–8]. Inspired by the realization of highly sensitive sensors through PT-symmetric photonic systems [9], it has been recently proposed and demonstrated that PT-symmetric LCR resonators can be utilized for enhanced sensing [10–14]. They were operated in PT-symmetric phase [10], Eps [11–13], and broken PT-symmetric phase [14]. An external perturbation to the PT-symmetric LCR resonators, which usually origins from the capacitance variation of one resonator in response to parameters of interest to be sensed, was balanced by adjusting the corresponding capacitance value of the other resonator so as to remain PT symmetry during the operation. Obviously, the external perturbation to the PT-symmetric systems is asymmetric. The asymmetric perturbation have been extensively studied for general PT-symmetric systems [15]. We were motivated by the studies to construct its electrical analog by using a pair of coupled LC resonators in order to explore the possibility of detecting small signal changes in LC passive wireless sensors.

Based on an inductively coupled LCR pair [3], one with gain and the other with an equivalent amount of loss, we address both theoretically and experimentally the influence of asymmetric perturbation on the eigenvalues of PT-symmetric resonator systems under the matched gain and loss. The external perturbation is realized by a humidity sensitive capacitor [16]. The perturbed eigenvalues of the PT-symmetric resonator systems are
theoretically obtained by the coupled-mode theory, simulated by Agilent Software ADS (advanced design system), and measured by Vector Network Analyzer (VNA). We shows that the asymmetric perturbation leads to complex-conjugate eigenvalues. The eigenvalues splitting near the EP follows the square root dependence on the perturbation strength.

2. PT-symmetric system

A PT-symmetric LCR resonator system in this study is shown in figure 1 (a). It consists of a pair of inductively coupled LCR resonators, one with gain (−R) and the other with an equivalent amount of loss (R). In the serial LCR dimer, the gain side with subscript r (reader) is indicated by −R and the loss side with subscript s (sensor) is indicated by R, resulting in the gain/loss parameter γ = R^2LC. Two uncoupled identical resonators have the same resonance frequency \( \omega_0 = 1/\sqrt{LC} \). They are coupled through the mutual inductance M between the coils, describing by the coupling coefficient \( \mu = M/L \).

By applying Kirchhoff laws to the PT-symmetric LCR resonator system in figure 1 (a), we can write the coupled mode equations [see the appendix for more details]

\[
\frac{d}{dt} \begin{pmatrix} I_r \\ I_s \end{pmatrix} = \begin{pmatrix} -i\omega_0 + \frac{\omega_0\gamma}{2} & \frac{i\omega_0\mu}{2} \\ \frac{i\omega_0\mu}{2} & -i\omega_0 - \frac{\omega_0\gamma}{2} \end{pmatrix} \begin{pmatrix} I_r \\ I_s \end{pmatrix}
\]

where \( I_r \) and \( I_s \) are the currents in the gain (reader) and loss (sensor) tank, respectively. To find the eigenvalues \( \omega_{0n}^H \), the coupled mode equations are described through the frequency domain (\( \omega_{r,s} \propto e^{-\omega t} \))
\[ \hat{H}_0 = \begin{pmatrix} 1 + \frac{i\gamma}{2} & -\frac{\mu}{2} \\ -\frac{\mu}{2} & 1 - \frac{i\gamma}{2} \end{pmatrix} \]

where \( \hat{H}_0 \) is the Hamiltonian, which meets PT-symmetry, and \( \omega_{0}^{\mu} \) is the unperturbed eigenvalue scaled by \( \omega_0 \).

Solving equation (2) yields

\[ \omega_{1,2}^{0} = 1 \pm \frac{1}{2} \sqrt{\mu^2 - \gamma^2}. \]

In the exact PT-symmetric phase \( \gamma < \mu \), the coupling between the gain and loss elements is capable of exactly communicating a balanced flow of energy, and the modes have real eigenvalues and identical steady-state oscillatory magnitudes. In the broken phase \( \gamma > \mu \), the flow of energy overwhelms the coupling and effectively decouples the two resonators into modes with a single real frequency and conjugate imaginary parts. The exact and broken PT-symmetry phases are separated by the exceptional point (EP) at \( \gamma = \mu \) (black dot) where the eigenvalues merge. Figures 1(c) and (d) plot the real parts, \( \text{Re}(\omega) \), and the imaginary parts, \( \text{Im}(\omega) \), of the eigenvalues as a function of the normalized coupling coefficient \( \mu/\gamma \) at \( \delta = 0 \) (red line), respectively.

3. Perturbation effect

When the external perturbation, as shown in figure 1(b), is exerted on the PT-symmetric resonator systems in the sensor side. The capacitance in the loss side is now equal to \( C + \Delta C = C\left(1 + \frac{\Delta C}{C}\right) = C\frac{1}{(1 + \delta)} \). The coupled mode equations and the effective Hamiltonian of the perturbed resonator systems are given by [see the appendix for more details],

\[ \hat{H} = \begin{pmatrix} 1 + \frac{i\gamma}{2} & -\frac{\mu}{2} \\ -\frac{\mu}{2} & 1 + \delta - \frac{i\gamma}{2} \end{pmatrix} \]

where \( \omega_{0}^{\mu} \) denotes the perturbed eigenvalues scaled by \( \omega_0 \). It is clear from equation (6) that the perturbation is asymmetric. Solving equation (5) yields

\[ \omega_{1,2} = 1 + \frac{1}{4} \delta \pm \frac{1}{4} \sqrt{\left(4\mu^2 - 4\gamma^2 + \delta^2\right)^2 + 16\gamma^2\delta^2 + (4\mu^2 - 4\gamma^2 + \delta^2)} \]

\[ \mp \frac{1}{4} \sqrt{\left(4\mu^2 - 4\gamma^2 + \delta^2\right)^2 + 16\gamma^2\delta^2 - (4\mu^2 - 4\gamma^2 + \delta^2)} \]

 Obviously, the asymmetric perturbation breaks PT-symmetry, leading to complex eigenvalues. Figures 1(d) and (e) plot the eigenvalues as a function of the normalized coupling coefficient \( \gamma/\mu \) and the perturbation \( \delta/\mu \). The response is highlighted in the two purple cross-sections (the orange and green curved surfaces). The real parts of the eigenvalues bifurcate according to

\[ \text{Re}(\Delta \omega) = \text{Re}(\omega_2 - \omega_1) = \frac{1}{2} \sqrt{\frac{\left(4\mu^2 - 4\gamma^2 + \delta^2\right)^2 + 16\gamma^2\delta^2 + (4\mu^2 - 4\gamma^2 + \delta^2)}{2}} \]

At the EP (i.e., \( \mu = \gamma \)), the eigenvalues reduce to

\[ \omega_{1,2} = 1 + \frac{1}{4 0.5(504,642),(529,672)} \pm \frac{1}{4} \sqrt{\frac{\delta^4 + 16\gamma^2\delta^2 + \delta^2}{2}} \]

\[ \mp \frac{1}{4} \sqrt{\frac{\delta^4 + 16\gamma^2\delta^2 - \delta^2}{2}} \]

It shows that at the EPs the unperturbed eigenvalues (\( \omega_{1,2}^{0} = 1 \)) in equation (4) are now split into two complex ones in equation (9). The real parts of the eigenvalues at the EPs bifurcate according to

\[ \text{Re}(\Delta \omega) = \text{Re}(\omega_2 - \omega_1) = \frac{1}{2} \sqrt{\frac{\delta^4 + 16\gamma^2\delta^2 + \delta^2}{2}} \approx \sqrt{\frac{\gamma^2}{2}} \]
These expressions indicate that for the PT-symmetric LCR resonator systems biased around the EPs the real parts of the eigenvalues bifurcate with a square-root dependence on the applied asymmetric perturbation. This is in qualitative agreement with a pair of coupled optical PT-symmetric resonators [9].

To verify the results, the PT-symmetric LCR resonator systems have been simulated by Agilent Software ADS. In the closed-loop analysis, a reader with impedance normalized to 50 Ω can be modelled as a negative resistance $-50 \, \Omega$, and the eigenvalues as a function of asymmetric perturbation (the capacitance variation in the loss side) are extracted from $S_{11}$ parameters [see the appendix for more details]. Figure 2(a) plots the real parts, $\text{Re}(\omega)$, of the eigenvalues as a function of perturbation for both theoretical and simulated results at the exceptional points. Figure 2(b) clearly demonstrates the eigenvalues bifurcate with a square-root dependence on the perturbation in response to changes in the sensitive capacitance in the sensor, in accordance with theoretical expectations.

### 4. Frequency response under humidity perturbation near EP

In order to demonstrate our results, we have built a prototype as shown in figure 1(b) using onboard circuit technology [14]. The experimental setup is shown in the figure 3. The sensor and reader tanks are directly coupled through two identical planar spiral inductors ($L = 5 \, \mu\text{H}$). A variable capacitor in the sensor side consists of capacitive humidity sensors, which is connected in series to the planar spiral inductor. A resistor ($R = 0.5\, \Omega$) accounts for the effective parasitic resistance of the sensor. The information provided by the sensor is then read by a VNA reader connected to the Vector Network Analyzer (VNA) for measuring the reflection spectrum, with setting the operation frequency range from 12 MHz to 18 MHz. When constructing PT symmetric LRC resonator, the characteristic impedance of RF signal source should be considered. The external VNA source with characteristic impedance 50 Ω is usually modelled as a negative resistance ($-50 \, \Omega$) because it provides energy to the system in the closed loop circuit, and the equivalent resistance in the reader side is $-0.5 \, \Omega$ [see the appendix for more details].

A capacitive humidity sensor was fabricated with 3 μm standard CMOS technology and a post-processing step. It consists of the interdigital electrodes covered with graphene oxide (GO) as the humidity sensing material. The capacitance changes in response to ambient humidity since its relative dielectric constant is dependent of humidity levels. Here, the small change in capacitance is applied as a perturbation. A variable capacitor was used in the reader so that its capacitance can be adjusted to the initial capacitance of the capacitive humidity sensor. The reader coil and sensor were fixed on a linear translation stage used to precisely control the coupling distance $d$. The sensor and reader were placed in a humidity test chamber OMEGA205, which is used to control the relative humidity. In the humidity experiments, the relative humidity level is adjusted from 40%RH to 80%RH with a step of 10%RH, which corresponds to negative ($\delta < 0$) and positive ($\delta > 0$) perturbations [see the appendix for more details].

The reflection coefficient, $S_{11}$, is saved at different relative humidity levels with negative and positive perturbations, as shown in figures 4(a) and (b), respectively. The capacitance value of the capacitive humidity sensors varies with the change of ambient humidity levels, which causes the frequency (eigenvalues) to change with the ambient humidity levels. By extracting the frequency from the measured reflection spectra as a function of relative humidity levels in figures 4(a)–(c) shows the measured frequency ($f = \omega / 2\pi$) as a function of negative
and positive (right part) perturbations $\delta$ corresponding to humidity changes, taking 40%RH and 80% RH as the initial humidity respectively, at $d = 1$ cm. Figure 4 shows the frequency responses at the $PT$-symmetric exceptional point $d = 1.36$ cm, when ambient humidity is 75%RH (black line), as well as with the different perturbation $\delta$ due to changes in humidity. By extracting the frequency from the measured reflection spectra as a function of relative humidity levels in figures 4, 5 shows the measured frequency $f = \omega / 2\pi$ as a function of perturbation $\delta$, at the $PT$-symmetric exceptional point with $d = 1.36$ cm. The orange and green lines denote high frequency $f_2$ and low frequency $f_1$ respectively. At relative humidity 80%RH, the double frequencies cannot be observed in the experiment, because the negative perturbation makes the system enter the weakly coupled region and the larger perturbation means the serious detuning. The inset shows frequency splitting $\Delta f = f_2 - f_1$ corresponding to figure 4 as a function of positive perturbations $\delta$. Sensitivity is defined as the ratio of the frequency change to the perturbation change, $\delta = \partial f / \partial \delta$. At the exceptional point, it can be directly observed that the high frequency sensitivity is significantly higher than the low frequency within the positive perturbation range ($\delta > 0$) in figure 4(e). By comparing the sensitivity $\delta$ of high frequency $f_2$ and frequency difference $\Delta f$ in figure 4(f), it is clear that the latter is superior to the former for very small perturbations. In addition, the smaller the perturbation, the higher the sensitivity, which is a great advantage in measuring small changes, compensating for the insensitivity of other detection methods to tiny perturbation.

The reflection coefficient $S_{11}$ at the terminals of the readout circuit, as a function of the input impedance $Z_{in}$ with respect to the characteristic impedance $Z_0$ of the measurement system, is written as

$$S_{11} = \frac{Z_{in} - Z_0}{Z_{in} + Z_0} \bigg|_{Z_0 = 50\Omega}$$

where $Z_0$ is the characteristic impedance of the signal source. It is 50 $\Omega$ for the VNA Network Analyzer (Agilent N5224A). The input impedance $Z_{in}$ at the terminal of the reader is written as

$$Z_{in} = j\omega L + \frac{1}{j\omega C_s} + Z_0 - R + \frac{\mu^2 L^2 \omega^2}{j\omega L + \frac{1}{j\omega C_s}} + R$$

where $\omega$ is the angular frequency, $L$, $C_s$, and $R$ are the inductance, capacitance and resistance of the sensor, respectively. If the two resonators are identical without coupling, $C_s = C_t = C$, with the same natural frequency $\omega_0 = 1/\sqrt{LC}$. 

Figure 3. (a) Schematic diagram of the experimental setup, where a planar loop inductor coil was used to enquire a capacitive sensor through inductive coupling. (b) The equivalent circuit for the whole experimental setup, where the parameters to be measured can be obtained from the reflection coefficient monitored via the VNA connected to the reader, and a source impedance from the VNA is modelled as $-Z_0$. (c) Simplified circuit model for the whole experimental setup.
For the initial PT-symmetric system, \( C_r = C_s = C \),

\[
Z_{in} = j\omega L + \frac{1}{j\omega C} + Z_0 - R + \frac{\mu^2L^2\omega^2}{j\omega L + \frac{1}{j\omega C} + R}.
\]  \( \tag{13} \)

For the perturbed PT-symmetric system, \( C_r = C \) and \( C_s = C + \Delta C \),

\[
Z_{in} = j\omega L + \frac{1}{j\omega C} + Z_0 - R + \frac{\mu^2L^2\omega^2}{j\omega L + \frac{1}{j\omega C} + R + \frac{-\Delta C}{C + \Delta C}}.
\]  \( \tag{14} \)

where

\[
\delta = \frac{-\Delta C}{C + \Delta C}.
\]

By inserting equations (13) and (14) into equation (10), respectively, and letting \( \partial S_1/\partial \omega = 0 \), the resonant frequency at which the amplitude of \( S_{11} \) gets its minimum. The obtained resonant frequency is the same as equations (4) and (7), respectively [see the appendix for more details].

By extracting the frequency from the measured reflection spectra as a function of relative humidity levels, figure 5(a) shows the measured frequency \( (f = \omega/2\pi) \) as a function of coupling coefficients \( \mu \) (corresponding to

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**Figure 4.** Measured reflection spectra as a function of relative humidity levels. (a) The frequency response under asymmetric negative perturbations \( (\delta < 0) \), with setting the initial PT-symmetric system at 40%RH, at \( d = 1 \) cm. (b) The frequency response under asymmetric positive perturbations \( (\delta > 0) \), with setting the initial PT-symmetric system at 80%RH, at \( d = 1 \) cm. (c) The real parts of eigenfrequencies as a function of the negative perturbations \( \delta \) corresponding to (a) and the positive perturbations \( \delta \) corresponding to (b). (d) The frequency response at the PT-symmetric exceptional point \( d = 1.36 \) cm at 75%RH (black line) with the different perturbation \( \delta \) due to changes in humidity. (e) The real parts of eigenfrequencies as a function of the perturbations \( \delta \) corresponding to (d). The inset (purple line) demonstrates a slope of \( 1/2 \) on a logarithmic scale for \( \text{Re}(\Delta f) = \text{Re}(f_2 - f_1) \). (f) The sensitivity of high frequency \( f_2 \) and frequency splitting \( (f_2 - f_1) = \Delta f \) as a function of the perturbations \( \delta \) due to changes in humidity corresponding to (e).
at the humidity level 75%RH and perturbation $\delta = -0.022$. The theoretical and measured resonance frequency $\omega_{1,2}$ as a function of perturbation $\delta$ with the coupling distance $d = 2.5$ (solid line and diamonds) and $d = 2.7$ (dashed line and stars). Figure 5(c) shows frequency splitting $(\omega_2 - \omega_1) = \Delta f$ corresponding to figure 5(a) as a function of coupling coefficients $\mu$. Figure 5(d) shows frequency splitting $\Delta f$ corresponding to figure 5(b) as a function of perturbations $\delta$. Solid and hollow circles represent the experimental data of $d = 2.5$ cm and $d = 2.7$ cm respectively. The insets (purple interval) in each panel indicate where experiment is performed.

$\mu = 1/\left[1 + 2^{2/3}\left(d/\sqrt{\gamma}\right)^{2/3}\right]. r$ denotes the radius of the inductance coil. The measured frequency is in agreement with the theoretical values except for the large coupling coefficients. The reason for this is that the coupled-mode equation is obtained under the weak coupling. Figures 5(a), (c) indicate the experimental results are obtained in exact phase, Figure 5(d) shows frequency splitting $\Delta f$ as a function of perturbations $\delta$. The insets (purple interval) in figures 5(b), (d) indicate the experimental results are obtained near EP. The purple solid line and the dotted line represent the variation of frequency splitting with perturbations at different distances from the theory. Solid and hollow circles represent the experimental data of $d = 2.5$ cm and $d = 2.7$ cm respectively. As can be seen from the diagram figures 5(c), (d), the experimental results agree well with the theoretical ones.

### 5. Conclusion

In conclusion, based on a $PT$-symmetric LCR resonator pair, we have shown both theoretically and experimentally the influence of asymmetric perturbation on the eigenvalues of $PT$-symmetric systems under the matched gain and loss. The perturbed eigenvalues have been exactly solved through the coupled-mode equation, which are numerically and experimentally confirmed. It shows that the asymmetric perturbation breaks $PT$-symmetry, leading to complex eigenvalues, which is different from the broken $PT$-symmetric phase with complex-conjugate eigenvalues. We have analyzed the perturbed frequency responses under symmetric perturbation in two phases. The eigenvalues splitting near the exceptional points exhibits the square root
dependence on the perturbation strength. The results presented here can be used to design LC passive wireless sensor without adjusting the readout capacitor, and be extended to study the fundamental of PT-symmetry due to asymmetric perturbation from the fabrication imperfections or fluctuating environments. Based on these observations at EP, it is clear that the frequency splitting is exceptionally sensitive to the asymmetric tiny perturbation \( \delta \), compared with the individual frequencies \( f_1, f_2 \), and the smaller the perturbation, the higher the sensitivity.

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**Data availability statement**

All data that support the findings of this study are included within the article (and any supplementary files).

**Appendix A. Hamiltonian \( \hat{H}_0 \) of PT-symmetric LCR systems.**

A PT-symmetric circuit presented in figure 1(a) consists of −RLC and RLC tanks. In the serial LRC dimer, applying Kirchhoff voltage laws yields

\[
-i \omega (L_i + ML_i) - I_i R + \frac{I_i}{-i \omega C} = 0 \\
-i \omega (L_i + ML_i) + I_i R + \frac{I_i}{-i \omega C} = 0
\]

where \( I_i \) and \( I_s \) are the current in the gain and loss tank, respectively. Defining the resonant frequency \( \omega_0 = \frac{1}{\sqrt{LC}} \); coupling coefficient \( \mu = \frac{M}{\sqrt{L}} \), dimensionless non-Hermiticity parameter or gain-loss parameter \( \gamma = R \sqrt{\frac{C}{L}} \), the above equation is written as

\[
\left(-i \left(1 - \frac{\omega_0^2}{\omega^2}\right) - \frac{\omega_0 \gamma}{\omega} + i \mu \right) \left(\frac{I_i}{I_s}\right) = 0. \tag{A3}
\]

In the weak coupling between the dimer, we make the approximation \( \mu \ll 1, \gamma \ll 1 \), \( \left(1 - \frac{\omega_0^2}{\omega^2}\right) \approx \frac{2(\omega - \omega_0)}{\omega_0} \). Equation (A1) (A2) then reduces to

\[
\left(\begin{array}{c}
i(\omega - \omega_0) + \frac{\omega_0 \gamma}{2} \\
i(\omega - \omega_0) - \frac{\omega_0 \gamma}{2}
\end{array}\right) \left(\begin{array}{c}I_i \\ I_s\end{array}\right) = 0. \tag{A4}
\]

Equation (A4) is equivalent to the coupled-mode equation with time harmonic currents \( I_{i,s}(t) \rightarrow I_{i,s} e^{-i \omega t} \).

\[
\frac{d}{dt} \left(\begin{array}{c}I_i \\ I_s\end{array}\right) = \left(\begin{array}{cc}-i \omega_0 + \frac{\omega_0 \gamma}{2} & i \mu \\i \mu & -i \omega_0 - \frac{\omega_0 \gamma}{2}\end{array}\right) \left(\begin{array}{c}I_i \\ I_s\end{array}\right). \tag{A5}
\]

This formulation can be interpreted as a Schrödinger equation with PT-symmetric effective Hamiltonian \( \hat{H}_0 \)

\[
\hat{H}_0 \left(\begin{array}{c}I_i \\ I_s\end{array}\right) = \frac{d}{dt} \left(\begin{array}{c}I_i \\ I_s\end{array}\right) = \left(\begin{array}{cc}i \omega_0 - \frac{\omega_0 \gamma}{2} & -i \mu \\i \mu & -i \omega_0 - \frac{\omega_0 \gamma}{2}\end{array}\right) \left(\begin{array}{c}I_i \\ I_s\end{array}\right) \tag{A6}
\]

Scaling frequency by \( \omega_0 \), the effective Hamiltonian \( \hat{H}_0 \) is obtained

\[
\hat{H}_0 = \left(\begin{array}{cc}1 + \frac{\gamma}{2} - \frac{\mu}{2} \\\frac{\mu}{2} & -1 - \frac{\gamma}{2}\end{array}\right)
\]
\[ \hat{H}_n I_n^0 = \omega_n^0 I_n^0 \]  

where \( \hat{H}_n \) is a non-Hermitian but PT-symmetric operator.

**Appendix B. Effect of asymmetric perturbation on the PT-symmetric Hamiltonian.**

Considering the tiny change \( \Delta C \) of the capacitor in loss (sensor) tank as the perturbation in the serial LRC dimer, as presented in figure 1(b), equation (A1) and equation (A2) are rewritten as

\[
-i\omega (I_{L} + M_{L}) - I_{R} + \frac{I_{L}}{-i\omega C} = 0 \\
-i\omega (I_{L} + M_{L}) + I_{R} + \frac{I_{L}}{-i\omega (C + \Delta C)} = 0
\]

Defining \( \delta = -\frac{\Delta C}{C + \Delta C} \), equations (B1) and (B2) are given by

\[
\begin{pmatrix}
-i \left( 1 - \frac{\omega^2}{\omega^2} \right) \frac{\omega_0 \gamma}{\omega} - i\mu \\
-i\mu - i \left( 1 - \frac{\omega^2}{\omega^2} (1 + \delta) \right) + \frac{\omega_0 \gamma}{\omega}
\end{pmatrix}
\begin{pmatrix}
I_{L} \\
I_{R}
\end{pmatrix} = 0
\]

Simplifying equation (B3) by the similar methods in equations (A1), (B3) becomes

\[
\begin{pmatrix}
-i(\omega - \omega_0) + \frac{i\omega_0 \gamma}{2} \\
-i\omega_0 \mu + i \left( \omega - \omega_0 - \frac{\gamma \delta}{2} \right) - \frac{i\omega_0 \gamma}{2}
\end{pmatrix}
\begin{pmatrix}
I_{L} \\
I_{R}
\end{pmatrix} = 0
\]

Scaling frequency by \( \omega_0 \), the effective Hamiltonian \( \hat{H} \) can be obtained

\[
\hat{H} = \begin{pmatrix}
1 + \frac{i\gamma}{2} & -\frac{\mu}{2} \\
-\frac{\mu}{2} & 1 + \frac{\delta}{2} - \frac{i\gamma}{2}
\end{pmatrix}
\]

\[ H I_n = \omega_n I_n \]

where \( \omega_n, I_n \) denote the perturbed eigenvalues and eigenvectors of the system, respectively. Solving equation (B5) yields

\[
\omega_{1,2} = 1 + \frac{1}{4} \delta \mp \frac{1}{4} \sqrt{(4\mu^2 - 4\gamma^2 + 4i\gamma\delta + \delta^2)}
\]

The complex eigenvalue is expanded as

\[
\omega_{1,2} = 1 + \frac{1}{4} \delta \mp \frac{1}{4} \sqrt{(4\mu^2 - 4\gamma^2 + \delta^2)^2 + 16\gamma^2\delta^2 + (4\mu^2 - 4\gamma^2 + \delta^2)}
\]

\[
\mp \frac{1}{4} \sqrt{(4\mu^2 - 4\gamma^2 + \delta^2)^2 + 16\gamma^2\delta^2 - (4\mu^2 - 4\gamma^2 + \delta^2)}.
\]

**Appendix C. Simulation**

Agilent ADS Software was used to conduct simulations. Figure C1 plots a Simulation Circuit diagram. The readout circuit was linked to a term with impedance normalized to 50 \( \Omega \) in the schematic module. The sensor resistance \( R_2 = 5 \Omega \) was connected to an inductance coil \( L_2 = 10 \mu H \), and a variable capacitor \( C_2 \) with initial capacitance 13.4 pF was used. In order to explore the perturbation effect on the resonance responses of the PT-symmetric system, a slight change in sensor capacitance \( C_2 \) is used as a perturbation.

For the PT-symmetric system, the readout circuit has the same inductance coil as the sensor, \( L_1 = 10 \mu H \), and similarly, a capacitor connected to the readout inductor is 13.4 pF. Corresponding to the resistance \( R_2 = 5 \Omega \) in the sensor circuit, the readout circuit has a negative resistance \( -5 \Omega \). In the reader closed-loop analysis, a term with impedance normalized to 50 \( \Omega \) can be modelled as a negative resistance \( -50 \Omega \), as it supplies energy to the system. So a resistance \( R_1 = 45 \Omega \) is selected here. As a result, the whole system is PT-symmetric, one with amplification as the reader and the other with an equivalent amount of attenuation as the sensor. In our simulations, \( C_2 \) was ranged from 13.2 pF to 13.6 pF with a step 0.1 pF, setting \( C_1 = 13.4 \) pF, leading to negative and positive perturbations, as shown in figure C2(a). We also simulated the frequency response with keeping
perturbation $\delta = 0$ for comparison, i.e. the system was maintained to be PT-symmetric, as shown in figure C2(b).

Appendix D. Experimental setup: The reflection spectra $S_{11}$ and eigenfrequencies $\omega_{1,2}$ of the perturbed system

The schematic diagram of the experimental setup is shown in figure 3(a). Using the one-port measurement, the information is encoded in the reflection coefficient $S_{11}$. We can get $\omega_{1,2}$ by finding the minimum of reflection coefficient $S_{11}$ in the lumped parameter circuit. The reflection coefficient $S_{11}$ at the terminals of the readout circuit in Fig. D1(b), as a function of the input impedance $Z_{in}$ (red box) with respect to the measurement system characteristic impedance $Z_0$ is written as

$$S_{11} = \frac{Z_{in} - Z_0}{Z_{in} + Z_0} \bigg|_{Z_{in}=50\Omega} .$$

(D1)

where $Z_0$ is the characteristic impedance of the RF signal source, which is equal to 50 for the PNA Network Analyzer (Agilent N5224A). The input impedance $Z_{in}$ at the terminals of the reader is written as

$$Z_{in} = j\omega L + \frac{1}{j\omega C_s} + Z_0 - R + \frac{\mu^2 L^2 \omega^2}{j\omega L + \frac{1}{j\omega C_s} + R} ,$$

(D2)

where $\omega$ is the angular frequency, $L$, $C_s$, and $R$ are the inductance, capacitance and resistance of the LC sensor, respectively. If the two resonators are identical without coupling, $C_r = C_s = C$, with the same natural frequency.
$$\omega_0 = \frac{1}{\sqrt{LC}}, \mu = M/L, \text{ where } M \text{ is the mutual inductance between the readout coil and sensor coil, and } \mu \text{ is the geometry-dependent coupling strength with a value between 0 and 1. The amplitude of } S_{11} \text{ will get its minimum when } \partial S_{11}/\partial \omega = 0, \text{ i.e. } \omega = \omega_{\text{dip1,2}}. $$

For the initial PT-symmetric system, \( C_r = C_s = C \),

$$Z_{in} = j\omega L + \frac{1}{j\omega C} + 50 - R + \frac{\mu^2 L^2 \omega^2}{j\omega L + \frac{1}{j\omega C} + R}. \quad \text{(D3)}$$

By inserting equation (D3) into equation (D1), and letting \( \partial S_{11}/\partial \omega = 0 \), the resonant frequency at which the amplitude of \( S_{11} \) has its minimum is given by [14]

$$\omega_{\text{dip1,2}} = \frac{1}{2} \left\{ \sqrt{2 - \gamma^2 + 2\sqrt{1 - \mu^2} \over 1 - \mu^2} \pm \sqrt{2 - \gamma^2 - 2\sqrt{1 - \mu^2} \over 1 - \mu^2} \right\}. \quad \text{(D4)}$$

Under the coupled mode approximation \( \gamma \ll 1, \mu \ll 1 \), equation (S16) can be simplified as

$$\omega_{\text{dip1,2}} \approx \frac{1}{2} \left( 2 + \sqrt{-\gamma^2 + \mu^2} \right) = \omega_{1,2}^0, \quad \text{(D5)}$$

where \( \omega_{1,2}^0 \) is the eigenfrequencies of the PT-symmetric system as introduced in equation (A1).

For the perturbed PT-asymmetric system, \( C_r = C \) and \( C_s = C + \Delta C \),

$$Z_{in} = j\omega L + \frac{1}{j\omega C} + 50 - R + \frac{\mu^2 L^2 \omega^2}{j\omega L + \frac{1}{j\omega C} + R + \delta}. \quad \text{(D6)}$$

where \( \delta = \frac{\Delta C}{C + \Delta C} \). Similarly, by inserting equation (D6) into equation (D1), and letting \( \partial S_{11}/\partial \omega = 0 \), the resonant frequency at which the amplitude of \( S_{11} \) has its minimum is given by

$$\omega_{\text{dip1,2}} = \text{Re}(\omega_{1,2}) = 1 + \frac{1}{4}\delta \mp \frac{1}{4} \sqrt{4(4\mu^2 - 4\gamma^2 + \delta^2)^2 + 16\gamma^2\delta^2 + (4\mu^2 - 4\gamma^2 + \delta^2)^2} / 2. \quad \text{(D7)}$$

Appendix E. Experimental design

E.1. Design of an inductor coil

The main geometrical parameters of planar spiral inductors are the outside diameter \( d_{\text{out}} \), inside diameter \( d_{\text{in}} \), line width \( w \), spacing \( s \) and winding number \( N \). A relatively simple calculation method suitable for the circular plane inductance in DC is the current sheet approximation, which is based on the basic principles of electromagnetics, assuming that each side of the planar inductance is a current sheet with uniform current distribution, and the inductance expression was derived [17]

$$L = \frac{\mu_0 N^2 d_{\text{avg}}}{2} \ln \left( \frac{2.46}{\rho} + 0.2 \rho^2 \right) \quad \text{(E1)}$$

where average diameter \( d_{\text{avg}} = (d_{\text{in}} + d_{\text{out}})/2 \), filling ratio \( \rho = (d_{\text{out}} - d_{\text{in}})/(d_{\text{out}} + d_{\text{in}}) \), and \( \mu_0 \) is the space permeability.

Using equation (E1) to design the reader/sensor coils, the inductance values and parameters used in our system (figure D1) are summarized in table E1.

The parasitic series resistance \( R \) of the inductance is derived from the inductance metal line, taking into account the high-frequency skin effect [18]:

$$R = \frac{\rho l}{\omega_0 (1 - e^{-h/\alpha})} \quad \text{(E2)}$$

where \( \rho \) is the resistivity of the inductor metal line, \( l = 2d_{\text{out}} N \left( 1 + \frac{d_{\text{in}}}{d_{\text{avg}}} \right) \) is the total length of the inductor metal line, \( w \) and \( h \) are the width and height of the metal line, \( \alpha = \sqrt{\rho / \pi \mu_0 f} \) is the frequency-dependent skin depth of the metal, and \( \mu \) is the permeability of the metal wire.

We used the impedance analyzer (Agilent 4294A) to measure our designed inductors, and the frequency response curve is given in figure E1(c).

E.2. Design of capacitive humidity sensor

Figure E2(a) presents the standard CMOS fabrication flow of the interdigital capacitor and the Micro-electromechanical System (MEMS) post-processing of the graphene oxide film as sensing material. The
Figure E1. (a) Configurations and physical parameters of planar spiral coils used in the reader and sensor; (b) A lumped parameter model for the inductor. (c) Frequency characteristic curve of the inductance and parasitic series resistance of reader(s) and sensor(s) coils.

Figure E2. (a) The CMOS fabrication process of an interdigital capacitor and the MEMS post-processing of GO films as sensing material. (b) Sensing humidity capacitance versus relative humidity levels corresponding to figure 5, measured at 13.914 MHz. (c) Deduced humidity capacitance in the sensor versus perturbation δ corresponding to (b). (d) Variable capacitance in the sensor versus perturbation δ corresponding to figure 5.

Table E1. Physical parameters for sensor and reader.

|       | $L$ ($\mu$H) | $d_{in}$ (cm) | $d_{out}$ (cm) | $N$ |
|-------|--------------|---------------|----------------|-----|
| Sensor| 4.58         | 1.2           | 1.9            | 14  |
| Reader| 4.93         | 1.2           | 1.9            | 14.5|
passivation around the aluminum electrodes was to prevent leakage and ensure the reliability of the humidity sensor. The use of polysilicon heaters improved the hysteresis performance of the sensor. The substrate was grounded to reduce the parasitic capacitance and the external interference. The humidity sensor based on GO film with appropriate dispersion concentration shows a monotonic relationship between the sensing capacitance and relative humidity, with fast response and good repeatability [16].

Figure E2(b) displays the measured capacitance changes with the relative humidity level. In humidity chamber (OMEGA 205), the relative humidity of the ambient chamber was increased from 40%RH to 80%RH. In the whole test range, the capacitance change is about 0.82 pF. The capacitance variation corresponds to perturbation. Figure E2(c) shows the humidity capacitance deduced as a function of negative and positive perturbation, with setting the initial PT-symmetric system at 40%RH and 80%RH respectively. Figure E2(d) demonstrates the variable capacitance in the sensor as a function of perturbation, corresponding to the experiment in figure 5.

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References
[1] Bender C M 2019 PT Symmetry in Quantum and Classical Physics (Singapore: World Scientific Publishing)
[2] Christodoulides D and Yang J 2018 Parity–time Symmetry and Its Applications (Singapore: Springer)
[3] Schindler J, Li A, Zheng M C, Ellis F M and Kottos T 2011 Experimental study of active LRC circuits with PT symmetries Phys. Rev. A 84 040101
[4] Lin Z, Schindler J, Ellis F M and Kottos T 2012 Experimental observation of the dual behavior of PT-symmetric scattering Phys. Rev. A 85 050101(R)
[5] Bender N, Factor S, Bodyfelt J D, Ramezani H, Christodoulides D N, Ellis F M and Kottos T 2013 Observation of asymmetric transport in structures with active nonlinearities Phys. Rev. Lett. 110 234101
[6] Chitsazi M, Li H, Ellis F M and Kottos T 2017 Experimental Realization of Floquet PT-Symmetric Systems Phys. Rev. Lett. 119 093901
[7] Ye Z, Farhat M and Chen P-Y 2020 Tunability and switching of Fano and Lorentz resonances in PTX-symmetric electronic systems Appl. Phys. Lett. 117 031101
[8] Sakhdari M, Hajizadega M, Zhong Q, Christodoulides D N, El-Ganainy R and Chen P-Y 2019 Experimental Observation of PT Symmetry Breaking near Divergent Exceptional Points Phys. Rev. Lett. 123 193901
[9] Wiersig J 2020 Review of exceptional point-based sensors Photonics Research 8 1457
[10] Chen P Y, Sakhdari M, Hajizadegan M, Cui Q, Cheng M M C, El-Ganainy R and Aliu A 2018 Generalized parity–time symmetry condition for enhanced sensor telemetry Nat. Electron. 1 297
[11] Dong Z, Li Z, Yang F, Qiu C-W and Ho J S 2019 Sensitive readout of implantable microsensors using a wireless system locked to an exceptional point Nat. Electron. 2 335
[12] Xiao Z, Li H, Kottos T and Aliu A 2019 Enhanced Sensing and Nondegraded Thermal Noise Performance Based on PT-Symmetric Electronic Circuits with a Sixth-Order Exceptional Point Phys. Rev. Lett. 123 213901
[13] Zeng C, Sun Y, Li G, Li Y, Jiang H, Yang Y and Chen H 2019 Enhanced sensitivity at high-order exceptional points in a passive wireless sensing system Opt. Express 27 27562
[14] Zhou B-B, Deng W-J, Wang L-F, Dong L and Huang Q-A 2020 Enhancing the Remote Distance of LC Passive Wireless Sensors by Parity–Time Symmetry Breaking Phys. Rev. Appl. 13 064022
[15] Hodaei H, Hassan A U, Wittke S, Garcia-Gracia H, El-Ganainy R, Christodoulides D N and Khajavikhan M 2017 Nature 548 187
[16] Zhao C L, Qin M, Li W H and Huang Q A 2011 Enhanced performance of a CMOS interdigital capacitive humidity sensor by graphene oxide in Proc. 16th Int. Conf. Solid-State Sensors, Actuators, Microsystems. 1954
[17] Mohan S S, del Mar Hershenson M, Boyd S P and Lee T H 1999 IEEE J. Solid-State Circuits 34 1419
[18] Chen P, Saati S, Varma R, Humayun M S and Tai Y 2010 J. Microelectromech. Syst. 19 721