Conformal invariance and quantum integrability of sigma models on symmetric superspaces.

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November 2006

Abstract

We consider two dimensional non linear sigma models on few symmetric superspaces, which are supergroup manifolds of coset type. For those spaces where one loop beta function vanishes, two loop beta function is calculated and is shown to be zero. Vanishing of beta function in all orders of perturbation theory is shown for the principal chiral models on group supermanifolds with zero Killing form. Sigma models on symmetric (super) spaces on supergroup manifold $G/H$ are known to be classically integrable. We investigate a possibility to extend an argument of absence of quantum anomalies in non local current conservation from non super case to the case of supergroup manifolds which are asymptotically free in one loop.

1 Introduction

Two dimensional (2d) non linear sigma models (NLSM) on supermanifolds, with and without WZ term, seemed to be exotic objects, when they appeared in condensed matter physics twenty years ago as an elegant calculational tool in problems of self avoiding walks [1] and disordered metals [2]. Later on they appeared in string theory context [3], [4], [5]. A progress in their understanding might be especially important for the theory of integer quantum Hall plateau transition [6], [7] and disordered systems [8], but this progress is very slow. Many difficulties prevent a usage of standard technique in investigation of 2d NLSM. One of them is unavoidable non compactness of relevant target space supermanifolds. Another – a complicated representation theory of the supergroups (their superalgebras), where so called atypical representations play important, if not the main, role [9]. An interest to 2d NLSM on supermanifolds was renewed recently in string theory in the context of ADS/CFT correspondence, when it was understood that some ADS backgrounds can be described in terms of supercosets [10], [11]. For example, $ADS_5 \times S^5$ is nothing but (bosonic part of) the super coset $PSU(2,2|4)/SO(1,4) \times SO(5)$. Hyperactivity in attempts to exploit integrability and methods of Bethe ansatz as a calculational tool in checks of ADS/CFT correspondence (see e.g. reviews [12] and references therein), also supports this interest, since both spin chains (on the gauge theory side) and 2d NLSM (on the ADS side) appearing there, usually have a supergroup symmetry. Some more examples of this kind appear in the context of non critical strings ADS/CFT correspondence [13], [14], [15].
In this paper we try to investigate aspects of conformal invariance and quantum integrability of 2d NLSM (without WZ terms) on some symmetric supergroup manifolds. List of the models we are interested in is the following. It starts from the principal chiral models (PCM) on the basic supergroups Lie: \( G = A(m|n), B(m|n), D(m|n), D(2,1;\alpha), G(3), F(4) \). In addition we consider the following coset superspaces:

\[
\begin{align*}
B(m|n) \\ B(k|l) \times B(i|j) \\ D(m|n) \\ D(k|l) \times D(i|j) \\ D(2,1;\alpha) \\ G(3) \\ F(4) \\ A(m|n) \\ A(1) \times A(1) \times A(1) \times D(2,1;3) \times C(3)
\end{align*}
\]

(1)

where \( m = k+i, n = l+j \). In all these cosets the factor algebra \( H \) is a maximal regular subalgebra of \( G \). Regular subalgebras of the basic Lie superalgebras were classified in [16]. For details of these cosets embedding see Appendix. All the superspaces in (1) are symmetric. We hope that these toy models will serve as a laboratory in investigation of more realistic ones, appearing both in condensed matter physics, and in string theory.

One of the most interesting observations in this subject was done in the paper [17], where it was shown that 2d NLSM without WZ term (a PCM model) on the supermanifold with \( PSL(n|n) \) symmetry is conformal in all orders of perturbation theory. In [11] this result was obtained for \( PSU(2|2) \) PCM. The authors pointed out the existence of a Casimir like chiral algebra of the model, but a principal difficulties did not allow to investigate the full spectrum of its representations. All the machinery of CFT is hardly applicable for these non standard 2d CFTs, although in some cases CFT methods were successfully applied [18], [19], [9].

As it is well known, any 2d NLSM on a symmetric space is classically integrable (see, e.g. [20] and references therein). Classical integrability expresses itself, in particular, in the presence of conserved non local charges, or, in a more rigorous way, in the presence of Backlund transform and spectral parameter dependent Lax pairs. Generalization of the standard procedure of non local current construction to the symmetric superspace case seems straightforward. It was shown for ordinary symmetric (non super) spaces that on the quantum level, absence of anomaly in these non local current conservation is guaranteed only if the factor group of a coset is either simple [21], [22] or consists of a product of identical simple group by itself [23]. One can expect that the same feature will remain in the case of symmetric superspaces. In this sense, the list of cosets above represents a good candidates for quantum integrable models. (one should consider the first and the third cosets with \( k = i \) and \( l = j \)).

Since the argument about presence/absence of anomaly in non local currents is based on the dimensions of operators calculated as engineering dimensions, one should be sure these dimensions are correct in the UV limit. In the ordinary (non super-) case this is guaranteed by asymptotical freedom (positiveness of the beta function, at least in one loop) of 2d NLSM on symmetric spaces. As we will see below, in general it is not the case for symmetric superspaces. Requirement of asymptotic freedom which we are going to impose in order to preserve an ability to talk about naively calculated dimensions of the operators, will restrict possible values of \( m \) in the list above to be grater then \( n \). So we start from calculation of one loop beta functions for the above cosets.

As we will see, part of them

\[
D(n+1|n), D(2,1;\alpha), \frac{D(2n+1|2n)}{D(n+1|n) \times D(n|n)}, \frac{D(n+1|n)}{A(n+1|n)}, \frac{D(2,1;\alpha)}{A(1) \times A(1) \times A(1)}
\]

(2)

have zero one loop beta function. We extended our calculations to two loops and got zero. As we will show, the beautiful proof of [17] that beta function is zero in all orders of perturbation theory, works in the same way for the superspaces on the manifolds \( D(n+1|n) \) and \( D(2,1;\alpha) \). In [24] one loop background field calculations of beta function were done for some supercosets. Extension of all loops proof [17] to the supercosets seems problematical, but two loops beta function calculation we made confirms that it is equal to zero. We calculate the central charges
of these cosets. Calculation of one loop beta functions for the rest of the superspaces selects asymptotically free ones. For them we analyze the quantum anomaly in the first non trivial non local current conservation, and conclude that there is no anomaly with a proper choice of regularization. So the 2d NLSM on the superspaces (1) are quantum integrable, and moreover, those from the list (2) are conformal invariant.

2 Beta function in one and two loops

We start from a geometrical approach to background field perturbation theory calculations of beta function for 2d NLSM on a Riemannian supermanifold. We are going to discuss the action

$$S = \frac{1}{4\pi} \frac{1}{\lambda^2} \int d^2 x \ Str[(G^{-1} \partial_\mu G)^2]$$

(3)

where $G$ is an element of supergroup (supercoset) manifold, and $Str$ is the supertrace. A review of the method and main results for non super case one can find in [25]. Recall that usual QFT background field methods should be modified being applied to 2d NLSM, if we wish to preserve target manifold Riemannian covariance of calculations. One should expand the action around the classical geodesic trajectory $\rho^a$ on the manifold. Then a result of calculations is expressed in terms of the basic covariant object – curvature tensor $R^a_{bcd}$, their covariant derivatives, and products with different kind of indices contructions. In particular, the one loop beta function is proportional to the Ricci tensor

$$\beta_{ab}^{(1)} = \frac{1}{2\pi} \lambda^2 R^c_{acb} = \frac{1}{2\pi} \lambda^2 R_{ab}$$

(4)

In general, only the one loop result is regularization scheme independent, higher loops depend on regularization. In dimensional regularization there exists the choice, for which the two loop result looks in the simplest way:

$$\beta_{ab}^{(2)} = -\frac{2}{3(2\pi)^2} \lambda^4 R_{a(cd)e} R^{e(cd)}_{b}$$

(5)

where the parenthesis means the symmetrization over the indices, and lowering/raising of indices is made by the manifold metric/its inverse. In principle, all this technology of beta function calculation may be extended to the supermanifolds. For definitions of the main objects of Riemannian geometry on supermanifolds see for example [26]. On the mathematical level of rigorosity, there are some principal difficulties in basic definitions of supermanifolds (even on the level of charts self consistency [27]). But there is a way to overcome these difficulties in such a way that usual objects of Riemannian geometry will be well defined [28], [29]. The only difference in these objects definitions from the non super case is some extra minus signes related to the grade of corresponding supermanifold coordinate. Carefully following all the steps of covariant background field calculations described, for instance, in [25], we got the same result for beta function (4), (5), where $R$ now is the Riemannian supercurvature tensor (particular expression in terms of structure constants for supergroup manifold will be written below), and additional minus signs, appearing in Feynmann diagrams as a result of grade 1 fields loops propagators, are encoded in the supermanifold metric used for indices contructions:

$$\tilde{\beta}_{ab} = \frac{1}{2\pi} \lambda^2 R_{ab} - \frac{2}{3(2\pi)^2} \lambda^4 R_{a(cd)e} R^{e(cd)}_{b}$$

(6)

Here $[\ ]$ means symmetrization if one of $c$ and $d$ is Grassmann even, and it means anti symmetrization if both of them are Grassmann odd. We hope to describe this technical calculational details elsewhere.
We are going to apply this result to the supergroup manifold. The basic supergroup structure is defined by its superalgebra generators \( Q_A \) with structure constants \( f_{BC}^A \):

\[
[Q_A, Q_B] = f_{AB}^C Q_C
\]

where the commutator is graded \([A, B] = AB - (-1)^{\text{deg} A \text{deg} B} BA\). In what follows we denote the Grassmannian grade of the coordinate \( \text{deg} A \) as \([A]\). Since we are dealing with regular subalgebras \( H \), the root lattice of \( H \) is a sublattice of the root lattice of \( G \). In other words, the whole set of generators of \( G \) can be divided into two subsets \( \{Q_A\} = \{Q_A\} \cup \{Q_i\} - \) generators of \( H \), \( \{Q_i\} \), following [26] (see also [30], but in another setting) one can derive the curvature on the coset superspace in terms of the structure constants:

\[
R_{bed}^a = \frac{1}{2} f_{be}^a f_{cd}^e + \frac{1}{4} (-1)^{|b|(|c|+|d|)} f_{ce}^a f_{db}^e + \frac{1}{4} (-1)^{|d|(|b|+|c|)} f_{de}^a f_{bc}^e + f_{ba}^a f_{cd}^i
\]  

(7)

Due to the main property of symmetric superspaces \( \{Q_A, Q_B\} \subset H \), all terms here but the last one, vanish. The Ricci tensor defined as \( R_{ab} = (-1)^{|c|(|b|+1)} R_{c ab}^c \) is proportional to the Killing form. For the symmetric superspace case this property takes the form

\[
R_{ab} = (-1)^{|c|(|a|+|b|+1)} f_{ai}^c f_{bc} = (-1)^{|c|} f_{ai}^c f_{bc} = -K_{ab}
\]  

(8)

An obvious but important for the future observation is that the summation over \( i \) in the last formula may be extended on all the superalgebra \( G \): \( K_{ab} = (-1)^{|i|} f_{ai}^D f_{ib}^D \). On the other hand, it is well known that the last expression, considered as a relation on the whole superalgebra \( G \), is nothing but \( C_2 \delta_{ab} \), where \( C_2 \) is the value of the second Casimir operator evaluated on the adjoint representation of the superalgebra \( G \), i.e. the dual Coxeter number of \( G \): \( R_{ab} \sim C_2 \delta_{ab} \). The latter can be calculated purely algebraically, since it is the value of the second Casimir operator on the adjoint representation – the rep. with highest weight with the highest root of \( G \). So we have an important simplifying statement: the one loop beta function on symmetric (super)space \( G/H \)

\[
\beta_{ab}^{(1)} = \frac{C_2}{2\pi} \lambda^2 \delta_{ab}
\]

is proportional to the dual Coxeter number of \( G \) itself. This statement is valid for non super symmetric spaces as well. On the other hand, the value of the second Casimir operator on any representation of quotient of a (super)algebra \( G \) by its regular subalgebra \( H \) with highest weight \( \Lambda \), can be calculated using the formula

\[
C = (\Lambda, \Lambda + 2\rho(G) - 2\rho(H))
\]  

(9)

where \( 2\rho(G) \) (\( 2\rho(H) \)) is the sum of positive roots of \( G \) (\( H \)). Let’s emphasis here that \( 2\rho(G) \) (\( 2\rho(H) \)) depend on the particular form of usually non distinguished Dynkin diagrams one should use for the proper embedding of \( H \) into \( G \). Change of the Dynkin diagram for a given superalgebra changes also the order of its roots, and hence, the sum of positive roots. Explicit case by case analysis confirms the statement we did above: the contribution of the term \( (\Lambda, 2\rho(H)) \) with \( \Lambda \) the highest root of \( G \), vanishes for all the cosets from the list [1]. The values of dual Coxeter number for the basic Lie superalgebras, which one can extract from, e.g. [31], we list in the Table 1.

| \( G \) | \( A(m|n) \) (\( m \neq n \)) | \( A(m|m) \) | \( B(m|n) \) | \( C(m+1) \) |
|---|---|---|---|---|
| \( C_2 \) | \( 2(m-n) \) | 0 | 2(m - n - \frac{1}{2}) | -2m |
| \( C_2 \) | 2(m - n - 1) | 0 | 0 | 6 | 2 |

\(^1\)Note that almost in all the supercosets from the list [1] one should use non distinguished Dynkin diagrams of \( G \) for realization of proper embedding of the factor subalgebra \( H \) (see Appendix).
Table 1.

From this table one can see that if we are interested only in asymptotically free 2d NLSM on supergroups and their maximal regular supercosets with a non positive beta function (non negative $C_2$), one should reject the supergroup $C(m+1)$. $A(m|n)$ ($m \neq n$) and $B(m|n)$ can be taken only with $m > n$, and $D(m|n)$ ($m \neq n+1$) is acceptable if $m > n + 1$. In addition $A(m|m), D(n+1|n), D(2,1;\alpha)$ and their cosets from (1) are candidates for conformal field theories. Moreover the case $A(m|m)$, which is the most popular in string oriented literature, was proven to be really conformal field theory in all loops of perturbation theory. We are going to concentrate on other cases. As we said, the values of $C_2$ listed above are at the same time the values for one loop beta functions for the cosets of these supergroups (1). The cases of PCM models on $D(n+1|n), D(2,1;\alpha)$, and 2d NLSM on (2) are good candidates to be CFTs exactly as it happened to $A(m|m)$ [17]. In order to check this statement one can try to calculate the two loop beta function, using (5), for these supergroup manifolds and their cosets from list (2).

With this goal, it is useful to work in matrix representations of the superalgebras and their cosets (see Appendix). It is convenient to chose the defining representation of $D(n+1|n)$, since in addition to the minimal dimension, it gives an invariant non degenerate bilinear form on the algebra by $g_{AB} = Str(E_A E_B)$, where $E_A$ are the supermatrices of the algebra generators. They can be chosen as follows (see [31]):

$$E_{IJ} = G_{IK} e_{KJ} + (-1)^{(1+\deg I)(1+\deg J)} G_{JK} e_{KI}$$

(10)

Here $I, J, K = 1, \ldots, 4n+2, (e_{IJ})_{KL} = \delta_{IK} \delta_{JL}$, and the ortosymplectic form in the supermatrix block form is

$$G = \begin{pmatrix}
    \tilde{G} & 0 \\
    0 & \overline{G}
\end{pmatrix}$$

(11)

The Grassmann even $2n \times 2n$ matrix $\overline{G}$ will be chosen as

$$\overline{G} = \begin{pmatrix}
    0 & 1_n \\
    -1_n & 0
\end{pmatrix}$$

(12)

and, for the moment, the $(2n+2) \times (2n+2)$ matrix $\tilde{G}$ we fix as

$$\tilde{G} = \begin{pmatrix}
    0 & 1_{n+1} \\
    1_{n+1} & 0
\end{pmatrix}$$

(13)

One can easily write down the (anti)commutation relations for the generators (10) and read off from them the structure constants. It is a straightforward (but not trivial) exercise (using Mathematica program) to check that not only (8), but also the second term in (6) calculated from these structure constants through (7), vanishes. For raising/lowering the indices necessary in (6), one should use the metric $g_{AB} = Str(E_A E_B)$ and its inverse defined on the generators (10).

The same (actually much easier) calculation may be done for the $D(2,1;\alpha)$ Lie superalgebra. Here one can extract the structure constants for the generators $T_i^{(a)}, i = 1, 2, 3, a = 1, 2, 3$ (Grassmann even), and $F_{\alpha\beta\gamma}, \alpha, \beta, \gamma = 1, 2$ (Grassmann odd), from the following (anti)commutation relations (see e.g. [32])

$$[T_i^{(a)}, T_j^{(b)}] = i \delta_{ab} e_{ijk} T_k^{(a)},$$

$$[T_i^{(1)}, F_{\alpha\beta\gamma}] = \frac{1}{2} \sigma_i^{\mu\alpha} F_{\mu\beta\gamma},$$

$$[T_i^{(2)}, F_{\alpha\beta\gamma}] = \frac{1}{2} \sigma_i^{\mu\beta} F_{\alpha\mu\gamma},$$

$$[T_i^{(3)}, F_{\alpha\beta\gamma}] = \frac{1}{2} \sigma_i^{\mu\gamma} F_{\alpha\beta\mu},$$

(14)
\[ \{ F_{\alpha\beta\gamma}, F_{\mu\nu\rho} \} = C_{\beta\gamma}(C_{\sigma\tau})_{\alpha\mu}T^{(1)}_i + \alpha C_{\gamma\rho}C_{\alpha\mu}(C_{\sigma\tau})_{\beta\nu}T^{(2)}_i - (1 + \alpha)C_{\alpha\mu}C_{\beta\nu}(C_{\sigma\tau})_{\gamma\rho}T^{(3)}_i \]

where \( \varepsilon \) is totally antisymmetric tensor, \( \sigma^i \) are the Pauli matrices, and \( C = i\sigma^2 \) is the 2 \times 2 "charge conjugation" matrix. The invariant metric \( g \) on the algebra can be chosen as \([32]\)

\[
\begin{align*}
    g(T^{(a)}_i, F_{\alpha\beta\gamma}) &= 0, \\
    g(F_{\alpha\beta\gamma}, F_{\mu\nu\rho}) &= C_{\alpha\mu}C_{\beta\nu}C_{\gamma\rho}, \\
    g(T^{(a)}_i, T^{(b)}_j) &= -\frac{1}{2\kappa^2} \delta_{ij} \delta^{ab}
\end{align*}
\]

with \( \kappa^1 = 1, \kappa^2 = \alpha, \kappa^3 = -1 - \alpha \). Again, the calculation of the beta function in one and two loops with these structure constants and supersymmetric bilinear non degenerate form gives zero.

If we now want to do the same calculations for the coset superalgebras from the list \([2]\), we have to find matrix representation which will give possibility to divide all the generators of \( H \) into two parts – those which are generators of \( G \) and any element is represented as \( \text{in the case of the coset} D \) of the coset. It can be easily done for all the coset cases \([2]\). The situation is the most simple in the case of the coset \( \frac{D(2,1;\alpha)}{A(1) \times A(1) \times A(1)} \). Here we factor out all the Grassmann even part of the algebra. It means that the coset contains only the generators \( F \) from \([14]\). Simple calculation of one and two loop beta function \([6]\) through the structure constants gives zero.

Block structure of matrix realization of other two cosets embedding from the list \([2]\) are explained in Appendix. Again the calculation of one and two loop beta function using the Mathematica gives zero.

3 Conformal invariance in all orders of perturbation theory.

The proof of conformal invariance \([17]\) in all loops of perturbation theory can be repeated for the non coset supermanifolds from the list \([2]\). Here we recall this proof. In the background field method of beta function calculation one starts from the action

\[
S[G] = \frac{1}{4\pi \lambda^2} \int d^2 x \text{Str}[(G^{-1}\partial_\mu G)^2]
\]

and any element is represented as \( G(x) = \bar{g}(x)G_0(x) \) with a classical background field \( G_0(x) \) and quantum fluctuations \( \bar{g}(x) \). Then the current

\[
J_\mu = G^{-1}\partial_\mu G = G_0^{-1}\partial_\mu G_0 + \bar{g}^{-1}\partial_\mu \bar{g} = J_\mu^0 + \bar{g}^{-1}\partial_\mu \bar{g} = J_\mu^0 + \bar{J}_\mu
\]

and the action, after passing to the Lie superalgebra fields \( \bar{g} = e^{\lambda \bar{g}} \), separates into three pieces

\[
S = S[\bar{g}] + S[G_0] + \frac{1}{2\pi \lambda^2} \int d^2 x \text{Str}[(\bar{J}_\mu (\partial_\mu G_0)G_0^{-1})]
\]

When passing to the Lie superalgebra from supergroup \( \bar{g}(x) = e^{\lambda \bar{g}(x)} \), the fluctuation current \( \bar{J}_\mu = e^{-\lambda \bar{g}} \partial_\mu e^{\lambda \bar{g}} \) can be expanded in series in \( \lambda \)

\[
\bar{J}_\mu = \lambda \partial_\mu \bar{g} + \lambda^2 \frac{2}{2} [\partial_\mu \bar{g}, \bar{g}] + \lambda^3 \frac{2}{3} \llbracket [\partial_\mu \bar{g}, \bar{g}], \bar{g} \rrbracket + ...
\]

In the same way, Lagrangian for self interacting part \( S[\bar{g}] \) takes the form

\[
L[e^{\lambda \bar{g}}] = \frac{1}{4\pi} \left\{ \text{Str}(\partial_\mu \bar{g} \partial^\mu \bar{g}) + \frac{2\lambda^2}{4!} \text{Str}([\partial_\mu \bar{g}, \bar{g}] \partial^\mu \bar{g}) + ... \right\}
\]
Figure 1: Structure of divergent Feynmann diagrams for back ground field beta function calculation

Insertion of these relations into the (17) gives the action for $g$ with the kinetic term $\text{Str}(\partial_{\mu} g \partial^{\mu} g)$ and two types of interaction terms. The first type is powers of $\partial_{\mu} g$ each coming with the external field $G_{0}$, and the second – powers of $\partial_{\mu} g \partial^{\mu} g$ coming without external background field. Each interaction vertex is proportional to the structure constant coming from (graded) commutators.

In calculation of beta function we have to renormalize all 1PI UV divergent diagrams. (Of course there are IR divergent diagrams, but suppose they were regularized by inclusion of small mass terms.) Standard power counting arguments lead to the naive divergence formula $D = 2 - V_{1}$, where $V_{1}$ is the number of vertices with the single derivative $\partial_{\mu} g$. Since each single derivative $\partial_{\mu} g$ is coming together with an external line of the background field, the naive divergence formula takes the form $D = 2 - E$, where $E$ is the number of external lines. Therefore, we should analyze all 1PI diagrams with one or two external lines. Diagrams without external background lines will renormalize the action $S[e^{b g}]$. It leads to the vertex and wave function renormalization which will not be important for us since we are interested in diagrams with no external $g$-lines. Moreover, the renormalization procedure can be chosen in such a way that the group structure of renormalized vertices does not change and they remain proportional to the structure constants also after the renormalization. So we should consider 1PI diagrams with one external background line (Fig.1.a) and two such external lines (Fig.1.b). The key observation is that the blob parts of both diagrams should be supergroup invariant tensors, whereas the pulled off vertex parts of diagrams are proportional to the structure constants. Recall here that we call a tensor $t_{i_{1}i_{2}...i_{k}}$ supergroup $G$-invariant if

$$\sum_{a=1}^{k} \sum_{a \in G} (-1)^{|a|} f_{i_{a}a}^{a} t_{i_{1}i_{2}...i_{s-1}a_{s+1}...i_{k}} = 0, \forall b \in G,$$

(18)

In other words, the blob on the Fig.1.a is rank two supergroup invariant tensor, and the blob on the Fig.1.b – is rank three invariant tensor. An obvious fact is that the only rank two invariant tensor is the non degenerate bilinear invariant form of the Lie superalgebra. Recall that a natural way to chose this form for the Lie superalgebras with the vanishing Killing form, as the nominators in (2), is $g_{ab} \sim \text{Str}(X_{a} X_{b})$, where $X$ are the generators in the defining representation for the $D(n+1|n)$ case, or the metric [15] for the $D(1,2;\alpha)$ case. In both cases it is a feature of these superalgebras that the contraction of these metrics with the structure constant, as it appears on the Fig.1.a, gives zero. In order to calculate the contribution of diagrams Fig.1.b, one should classify all the rank three superalgebra invariant tensors. The full set of rank 3 tensors with indices $a, b, c$ can be chosen as $\text{Str}X_{a} \text{Str}X_{b} \text{Str}X_{c}$, $\text{Str}(X_{a} X_{b}) \text{Str}X_{c}$, $\text{Str}(X_{a} X_{c}) \text{Str}X_{b}$, $\text{Str}(X_{b} X_{c}) \text{Str}X_{a}$, $\text{Str}(X_{[a}, X_{b} X_{c]}),$ $\text{Str}(X_{[a}, X_{b} X_{c]}),$ where $(\ ) (\ )$ means total (anti) symmetrization over indices $a,b,c$. Our basis of generators is traceless, hence only last two tensors may be non zero. Using the relation $G X = -X^{s t} G$ valid for $D(n+1|n)$, one can easily show that the last tensor vanishes, as well as all other odd rank totally symmetric invariant tensors. For $D(2,1;\alpha)$ one can check this fact using explicit form of the generators [14]. Hence we are left with only one invariant tensor of rank 3: $\text{Str}(X_{[a}, X_{b} X_{c]})$ and it is proportional to the structure constants. (Invariance of the structure constant as a tensor is nothing but the Jacobi identity). It means that also all diagrams of kind Fig.1.b vanish, since the contraction of two structure constants they represent is proportional to
the value of the second Casimir in the adjoint representation (the Killing form), which vanishes both for $OSP(2n + 2|2n)$ and $D(2, 1; \alpha)$. This complete the proof of vanishing of beta function in all loops in perturbation theory in $\lambda$.

Unfortunately we didn’t succeed to generalize this proof of all loops conformal invariance to the supercoset case. But we consider one and two loops conformal invariance discussed above as a good evidence for all loops conformal invariance of the supercosets from the list \[2\].

4 Central charges

A two dimensional quantum conformal field theory contains the symmetry generator – the energy momentum tensor $T(z, \bar{z})$. The first question one asks, what is the central charge of the CFT.

We start from the cases of supergroup manifolds. Here we follow the arguments of [17]. We are interested in exact non perturbative calculation of the correlation function $\langle TT \rangle$. First of all, as it was shown in [17] using the Zamolodchikov’s equations for different components of $T$, the coefficient $\alpha$ in the Sugawara constructed component $T_{z\bar{z}} = \alpha \text{Str}(J_z J_{\bar{z}})$, should be equal to zero, and hence the energy-momentum tensor is holomorphic. Then with a normalization $T_{z\bar{z}} = -\frac{1}{2\lambda^2} \text{Str}(J_z J_{\bar{z}})$ we are interested in extraction of the central charge from a non perturbative calculation of

$$
\langle T(z)T(w) \rangle = \frac{1}{4\lambda^4}\langle \text{Str}(J_z(z)J_z(z))\text{Str}(J_z(w)J_z(w)) \rangle = \frac{c/2}{(z-w)^4} \tag{19}
$$

The crucial observation which enables a non perturbative calculation is that in the expansion of $J_z$ all higher loops come proportional to the contractions of the structure constants with themselves, which give zero because of vanishing of the Killing form (the dual Coxeter number) of the algebras $OSP(2n + 2|2n)$ and $D(2, 1; \alpha)$. It means one can use just the free fields $A$ instead of currents:

$$
J_z \rightarrow \partial A, J_{\bar{z}} \rightarrow \overline{\partial} A \tag{20}
$$

with the action

$$
S = \frac{1}{4\pi\lambda^2} \int d^2x \text{Str}(\partial_\mu A\partial^\mu A) \tag{21}
$$

The propagator for the field $A$ should respect the supergroup symmetry. For example, for the $OSP(2n + 2|2n)$ case we can chose

$$
A_{I\bar{J}} = \begin{pmatrix}
a_{ij} & b_{ij} \\
c_{ij} & d_{ij}
\end{pmatrix}
$$

with Grassmann even matrices $a, d$ and odd $b, c$, such that it preserves the form

$$
G = \begin{pmatrix}
1_{2n+2} & 0 \\
0 & 0_{2n}
\end{pmatrix}, \quad \overline{G} = \begin{pmatrix}
0 & 1_n \\
-1_n & 0
\end{pmatrix}: \quad A^T G = -GA,
$$

which leads to

$$
a^T = -a, \quad d^T = \overline{G}d\overline{G}, \quad b^T = -\overline{G}c, \quad c^T = -b\overline{G}.
$$

The following index structure of propagators respects this symmetry:

$$
\langle a_{ij}(z)a_{kl}(w) \rangle = \frac{1}{2} (\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk})(-\ln(z-w))
$$

$$
\langle d_{ij}(z)d_{kl}(w) \rangle = \frac{1}{2} (\delta_{il}\delta_{kj} - \delta_{ik}\delta_{jl})(-\ln(z-w))
$$

$$
\langle c_{ij}(z)c_{kl}(w) \rangle = \frac{1}{2} \delta_{ij}\overline{G}_{kl}(-\ln(z-w))
$$
Using these propagators, Wick theorem applied to (19) with substitution of (20) gives the result
\[ \langle T(z)T(w) \rangle = \frac{1/2}{(z-w)^4} \]
meaning that the central charge is \( c = 1 \).

There is more compact way to do the same calculations. Using the matrix realization basis \( E^m_{IJ} \) for the superalgebra \( osp(2n+2|2n) \) described above, one can write \( A_{IJ} \) of the action (21) as
\[ A_{IJ} = \sum_m a_m E^m_{IJ}. \]
The propagator for \( a_m \) is just free field propagator \( \langle a_m(z)a_l(w) \rangle = -\delta_{ml} \ln(z-w) \).
We have to calculate
\[ \langle \text{Str}(JJ)\text{Str}(JJ) \rangle = g_{ml}g_{pq}(\partial a_m\partial a_l)(z)(\partial a_p\partial a_q)(w) \]
where \( g_{ml} = \text{Str}(E^m E^l) \) is bilinear invariant and non degenerate form on the superalgebra. Explicit calculation of the previous expression reduces just to counting of different generators with the proper weights and gives the same answer. The advantage of this method is obvious when we calculate the supercoset central charge: one can use the last formula, but the summation is running only over generators of \( G \) which are not generators of \( H \). This calculation for the PCM sigma model on the \( D(2,1;\alpha) \) supermanifold gives \( c = 1 \). For the cosets \( D(n+1|n) \) and \( D(2n+1|2n) \) we got \( c = 0 \), and the coset \( D(2,1;\alpha) \) has \( c = -8 \).

Of course, the central charge says almost nothing about the two dimensional CFT – one has to know the full extended algebra of the theory and its representations, i.e. the spectrum of the primary fields. This problem doesn’t seem solvable for today, since almost nothing is known about CFTs where \textit{apriori} there is no explicit factorization into holomorphic and antiholomorphic parts, at least for the representations. As it was shown in [17], one can construct holomorphic algebras, which are believed remain holomorphic anomaly free on the quantum level, i.e they are chiral Casimir like algebras. The question is whether these algebras contain all the symmetry of the theory. Another, more realistic for solvability problem is an investigation of representations of these chiral Casimir algebras at least on the subset of holomorphic representations. Some steps in this direction were done recently in [9].

5 Quantum integrability

Here we repeat the arguments [22], [21] about the absence of quantum anomaly in the conservation of the first non trivial non local current for UV asymptotically free sigma models on the symmetric superspaces from the list (1) which are not conformal, with either simple \( H \), or semisimple consisting from identical simple subalgebras. It means we are going to reproduce the argument [21] of quantum integrability of the sigma models on the following supercosets:

\[ \frac{B(2i|2j)}{B(i|i) \times B(j|j)} \cdot \frac{B(m|n)}{D(m|m) \times D(i|i) \times D(j|j)} \cdot \frac{D(2i|2j)}{D(2,1;3) \times C(3)} \]
with \( i > j \) and \( m > n \).

Recall that sigma models on symmetric spaces are always classically integrable due to a possibility to construct a parameter dependent flat connection. The same is true for symmetric superspaces (see e.g. [33], [34] and references therein). In the same way as in the usual (not super) symmetric spaces one can construct the first non trivial non local conserved current [20]:

\begin{align*}
Q^{(2)} &= \frac{1}{2} \int dy_1 dy_2 \varepsilon(y_1 - y_2) [j_0(t, y_1), j_0(t, y_2)] - \int dy j_1(t, y) \\
\varepsilon(x) &= \begin{cases} 
1, & x > 0 \\
-1, & x < 0 
\end{cases}
\end{align*}
The action of 2d NLSM on symmetric superspace $G/H$ is defined in terms of currents

$$S = \frac{1}{4\pi^2\lambda} \int d^2 x \text{Str}(k_\mu k^\mu) = \frac{1}{4\pi^2\lambda} \int d^2 x \text{Str}(j_\mu j^\mu),$$

$$k_\mu = G^{-1} D_\mu G = G^{-1}(\partial_\mu G - GA_\mu), j_\mu = -G^{-1}k_\mu G$$

where $A_\mu$ is the $H$ current. Equations of motion can be written as

$$D_\mu k^\mu = \partial_\mu j^\mu = 0$$

Potential source of anomaly on the quantum level is the first term in \[23\], since product of operators at the same point requires regularization. One can say that there is no anomaly if there exists a regularization procedure for $Q^{(2)}$ which preserves its conservation. The arguments of \[22\], \[21\] are based on the counting of all possible terms compatible with the symmetries, which can appear in the operator product expansion of two currents. The counting of such terms is possible if one can trust to engineering dimensions of operators in the UV. This is guaranteed in the asymptotically free theories, and that is why we consider NLSM from list \[22\] with the restriction $i > j, m > n$. So the starting point is the splitting point regularization of OPE

$$[j_\mu(t, x + \varepsilon), j_\nu(t, x)] = \sum_k C^{(k)}_{\mu\nu}(\varepsilon) Y^{(k)}(t, x)$$

where $Y^{(k)}$ is a complete set of local operators of dimension not greater than two, such that $C^{(k)}_{\mu\nu}(\varepsilon)$ is divergent or non zero when $\varepsilon \to 0$. All possible operators $Y^{(k)}$ should be consistent with the existing symmetries. The left hand side is globally $G$-covariant and locally $H$-invariant, so the same should be on the right hand side. One can count all possible composite operators of dimension not greater than 2 with these symmetry properties. If one chooses a hermitian matrix realization for $G$: $gg^\dagger = 1$, any operator of this kind can be written as $L_1 g L_2 g^\dagger \ldots L_{2k-1} g L_{2k} g^\dagger$, where $L_i$ is a product of any number (including zero) of covariant derivatives $D_\mu$. Not all of these operators are independent. There are no operators of dimension 0 of this kind, there is one independent operator of spin 1 – it is $g D_\mu g^\dagger \equiv j_\mu$, and one can chose two independent operators of dimension 2. The first is $D_\mu D_\nu g g^\dagger \equiv F_{\mu\nu} g^\dagger$, and the second – $D_\mu g D_\nu g^\dagger + g D_\mu D_\nu g^\dagger \equiv \partial_\mu j_\nu$. Here $F_{\mu\nu}$ is the stress tensor in the subgroup $H$. The irreducible parts of $g F_{\mu\nu} g^\dagger$ in the case of semisimple $H$, are $G^{(i)}_{\mu\nu}$. We also used the fact that symmetric superspace $G/H$ is by itself an irreducible representation of $H$. The proof of this statement \[35\] for the non super case uses, in addition to the symmetry property, the fact that $G$ is simple, and existence of invariant bilinear non degenerate form on $G$. The only subtle point in copying of this proof on symmetric superspace case is the last one: the natural form - the Killing one - is identically zero for few cases of symmetric superspaces. But as we said, there exists another bilinear form, which is invariant and non degenerate. It can be successfully used in this proof instead of the Killing form, and the proof still works. Finally, the most general form of possible singular terms in the Wilson expansion is

$$[j_\mu(t, x + \varepsilon), j_\nu(t, x)] = C^0_{\mu\nu}(\varepsilon) j_\rho(x) + D^{\sigma\rho}_{\mu\nu}(\varepsilon) \partial_\sigma j_\rho(x) + \sum_i E^{(i)\sigma\rho}_{\mu\nu}(\varepsilon) G^{(i)}_{\mu\nu}(x) \quad (24)$$

Moreover, because of identity $\sum_i G^{(i)}_{\mu\nu} = \partial_\mu j_\nu - \partial_\nu j_\mu$ one can impose

$$\sum_i E^{(i)\sigma\rho}_{\mu\nu}(\varepsilon) = 0 \quad (25)$$

Lorentz and PT invariance and charge conjugation of \[24\] lead to an ansatz for unknown coefficients $C, D, E$ with some properties of their transformation under inversion of their argument sign. In particular,

$$C^0_{\mu\nu}(\varepsilon) = C_1(\varepsilon^2) g_{\mu\nu} \varepsilon^\rho + C_2(\varepsilon^2) (\varepsilon_\mu \delta_\nu^\rho + \varepsilon_\nu \delta_\mu^\rho) + C_3(\varepsilon^2) \varepsilon_\mu \varepsilon_\nu \varepsilon^\rho$$

$$E^{(i)\sigma\rho}_{\mu\nu}(\varepsilon) = E^{(i)}(\varepsilon^2) \varepsilon_\mu \varepsilon_\nu \varepsilon^\rho$$
Ward identity applied to (24) gives first order differential equations on the scalar functions $C_i, D_i, E^{(i)}$ [21]. One can show that these equations have a solution. The most singular part on the right hand side of (24) is defined by $C_1(\varepsilon^2) \sim 1/\varepsilon^2$, and (24) implies that the regularization

$$Q_\delta^{(2)} = \frac{1}{2} \int_{|y_1 - y_2| > \delta} dy_1 dy_2 \varepsilon(y_1 - y_2) [j_0(t, y_1); j_0(t, y_2)] - Z_\delta \int dy_1(t, y)$$

$$Z_\delta = \int_0^\Lambda dx C_1(-x^2)x$$

has a finite limit when $\delta \to 0$ (\Lambda here is an irrelevant IR cutoff). Explicit calculation of $dQ_\delta^{(2)}/dt$ using (24) and differential equations for $C_1, D_i, E^{(i)}$, gives finally

$$\frac{dQ_\delta^{(2)}}{dt} = \sum_i E^{(i)} \int dy \varepsilon \mu \nu G^{(i)}_{\mu \nu}(t, y)$$

One can see that in the case of simple $H$ we have zero on the right hand side just because of condition (25). Moreover, if we have semisimple $H$ composed as a tensor product power of one simple $H': H = H' \times H' \times ... \times H'$, there is no reason why corresponding $E^{(i)}$ for each $H'$ should be different. Due to the same condition (25) we again get zero. It completes the proof of absence of anomaly in the non local current conservation for the symmetric superspaces from the list (22). Let us emphasis again that the anomaly analysis presented here is a complete copy of the analysis for the purely bosonic case, since the presence of the Grassmann odd variables doesn’t change any of the steps described above.

An important remark is in order here. As it was mentioned in [20], if one computes the potential anomaly terms coefficients $C_i, D_i, E^{(i)}$ explicitly, they (including $C_1$) turn out to be proportional to the dual Coxeter number of the group $G$. Therefore the renormalization procedure scheme of $Q_\delta^{(2)}$ described above doesn’t work for supercosets with $G$ either $A(m|m)$, or $D(m + 1|m)$, or $D(2, 1; \alpha)$, since for them $C_1 = 0$. It shows that conformal invariant cases are different and cannot be considered on the same footing with other quantum integrable sigma models. At least role of non local currents on the quantum level are different in the conformal and integrable symmetric superspaces NLSM.

6 Discussion

We saw that beta function for symmetric superspaces of regular type (11) can be easily calculated in one loop. Those cases where the beta function is negative (the value of the second Casimir operator on the coset is positive), one can expect an asymptotical freedom behavior. We have shown, that in the case of vanishing of one loop beta function, both in the case of supergroup manifold and in the case of supercoset spaces, the two loop beta function vanishes. In the case of the supergroup manifolds, the proof of [17] of all loops vanishing of beta function on the supermanifold $PSL(n|m)$, can be copied to the cases of supermanifolds $D(n + 1|m), D(2, 1; \alpha)$. It would be important to extend the all loops conformal invariance proof to supercosets, and investigate higher orders conformal invariance for other Ricci flat supercosets, in particular, constructed recently in [36].

One of the possible tools of analysis of these new two dimensional CFT may be investigation of the chiral algebra of Casimir operators. Recall, that by conjecture of [17] the algebra of operators

$$W^{(k)} = t_{i_1 i_2 ... i_k} J^{(i_1)}(z) J^{(i_2)}(z) ... J^{(i_k)}(z)$$

where $t$ is invariant tensor of rank $k$, remains chiral also on the quantum level. It seems such a conjecture should be universal for all the conformal NLSM [22]. Of course, there is a question
whether such algebras contain the full symmetry of these models, but in any case, an understanding of their structure and especially of their representations, including the spectrum of primary fields, is an important and challenging problem. Another interesting open question is the relation between these NLSM CFTs and WZW models for the same supergroup (supercoset) manifolds [9].

In the last section we saw that the standard arguments about the absence of anomalies in non local current conservation known for the symmetric non super cases may be extended to the super case. It would be interesting to extend the all orders perturbation theory analysis of non local currents conservation developed in [37] to the NLSM considered here. Note that as we mentioned, there is an obstacle on the way to treat conformal NLSM [22] on the same footing as integrable ones: the proof of absence of anomaly in the non local current conservation doesn’t work, at least formally. Quantum integrability poses a question about construction of their exact (relativistic) S-matrices. This problem requires an understanding in which representation of $G$ leaves the fundamental massive multiplet of $G/H$ for all the cases. One can also expect serious problems in the bootstrap program realization, since, as usual, it goes in parallel to the construction of a ”fusion ring” of representations of algebra $G$, and situation here is much more involved compared to the case of non super spaces, mainly because of atypical representations, which complicates the S-matrix construction (see e.g. [38]). Of course the permanent problem of fixing of CDD ambiguity is also there. All these interesting and not simple problems are important in applications of 2d NLSM on supermanifolds to problems of condensed matter and string theory.

7 Acknowledgements

I am thankful to D.Gepner, M.Gorelik and B.Noyvert for useful discussions and to V.Kac, S.Ketov and C.Young for communications. I am especially grateful to S.Elitzur for many productive discussions and remarks. I am also thankful to Einstein Center for financial support.

8 Appendix

8.1 Embedding for cosets

On the Fig.2 the supercosets embeddings are shown in terms of superalgebras simple roots systems embeddings. Recall that we are dealing with the regular subalgebras, when the root system of subalgebra $H$ is embedded into the root system of $G$. For all the supercosets from the list [1] the construction may be described in terms of Dynkin diagrams as follows. One starts from (usually not distinguished) extended Dynkin diagram of $G^{(1)}$. The Dynkin diagram should be chosen in such a way that among possible subdiagrams of $G^{(1)}$ one could find a diagram of $H$ by removing one (in the case of proper subalgebras) or more then one (non proper case) nodes from diagram $G^{(1)}$. As we already said, the full classification of such subalgebras (and hence corresponding supercosets) was done in [16]. All cosets of this kind as supergroup manifolds, turn out to be symmetric superspaces. Using the data [31] for full root systems of Lie superalgebras, and the diagrams below for the simple roots, one can list sets of positive and negative roots both for $G$ and $H$. This data is necessary if one wishes to check the equality of the value of second Casimir operator calculated on adjoint representation of $G$ and on $G/H$ by the formula [9]. The positive roots system data is necessary also for much more tedious exercise – check that all the coset $G/H$ root systems are by themselves irreducible representation moduli of $H$. As it was mentioned above, this check is unnecessary, since the theorem of this kind proved in [35] can be extended on the super case.
8.2 Matrix realization for cosets

Here we explain the block structure of embedding for matrix realization of cosets $D(n+1|n)$ and $D(2n+1|2n)$. We start from the former. We chose the standard matrix realization of $D(n+1|n)$ as a $4n+2$ by $4n+2$ even supermatrix built from two even diagonal blocks $((2n+2) \times (2n+2)$ and $2n \times 2n)$, and two non diagonal Grassmann odd blocks. Generators were defined in \( D(n+1|n) \). We divide the matrix into 9 blocks. The rows and columns are divided into 3 intervals: $I : [1, n+1]$, $II : [n+2, 3n+2]$, $III : [3n+3, 4n+2]$. One can see that generators of $D(n+1|n)$ which are non zero in the block $(II,II)$ (up to a linear transform of diagonal Cartan subalgebra generators) form a matrix realization of $A(n+1|n)$. Non zero entries of generators in the blocks $(I,I)$, $(I,III)$, $(III,I)$, $(III,III)$ just "copy" corresponding entries of the same generator located in the block $(II,II)$. All of them should be reduced as elements of factor algebra of the coset. The remaining generators of $D(n+1|n)$ have non zero elements located in the blocks $(I,II)$, $(II,I)$, $(II,III)$, $(III,II)$ and represent the generators of the coset.

Similar but more involved "chessboard" embedding can be constructed for the coset $D(2n+1|2n)$. Consider $n$ even case. We divide rows and columns of the $(8n+2) \times (8n+2)$ even supermatrix realization of $D(2n+1|2n)$ into 9 intervals: $I : [1, n/2]$, $II : [n/2 + 1, 3n/2 + 1]$, $III : [3n/2 + 2, 5n/2 + 1]$, $IV : [5n/2 + 2, 7n/2 + 2]$, $V : [7n/2 +3, 9n/2 +2]$, $VI : [9n/2 + 3, 11n/2 + 2]$, $VII : [11n/2 + 3, 13n/2 + 2]$, $VIII : [13n/2 + 3, 15n/2 + 2]$, $IX : [15n/2 + 3, 18n/2 + 2]$. One can see that generators with non zero elements in the blocks with odd interval coordinate number – $(I,I)$, $(I,III)$,...$(III,I)$,... – form the subalgebra $D(n+1|n)$, whereas the generators with non zero elements in the blocks with even interval coordinate numbers $(II,II)$, $(II,IV)$,... – form subalgebra $D(n|n)$. Remaining generators with non zero elements in the blocks with even-odd and odd-even interval coordinates numbers represent the coset generators.

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Figure 2: Roots system embedding for supercosets. (a) – $B(m|n)/D(m|n)$, (b) – $B(m|n)/B(i|j)B(k|l)$, (c) – $D(m|n)/D(i|j)D(k|l)$, (d) – $D(m|n)/A(m|n)$, (e) – $G(3)/D(2,1;3)$, (f) – $F(4)/C(3)$. 