ON THE HOMOTOPY OF SYMPLECTOMORPHISM GROUPS
OF HOMOGENEOUS SPACES

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Abstract. Let $O$ be a quantizable coadjoint orbit of a semisimple Lie group $G$. Under certain hypotheses we prove that $\#(\pi_1(\text{Ham}(O))) \geq \#(Z(G))$, where $\text{Ham}(O)$ is the group of Hamiltonian symplectomorphisms of $O$.

MSC 2000: 53D05, 57S05, 57T20

1. Introduction

Let $(M,\omega)$ be a quantizable symplectic manifold [13]. By $\text{Ham}(M,\omega)$ we denote the group of Hamiltonian symplectomorphisms of $(M,\omega)$ [9]. The group $\pi_1(\text{Ham}(M,\omega))$ is completely known in very rare cases (see [11, page 52] [7]). In this note we determine a lower bound for the cardinal of this homotopy group, when $M$ is an homogeneous space. Our approach is based on properties of the representation “symplectic action”. In [12] we proved the existence a representation $\kappa$ of the group $\pi_1(\text{Ham}(M,\omega))$. $\kappa$ associates to $[\psi]$ the symplectic action around the loop $\psi$ in $\text{Ham}(M,\omega)$. We also established some properties of $\kappa([\psi])$, which permit us to calculate it in particular cases. If $G$ is a semisimple Lie group and the quantizable manifold is the coadjoint orbit $O_\eta$ of an element $\eta \in g^*$, each curve $\{g_t \in G\}_{t \in [0,1]}$ in $G$ with $g_0 = e$ and $g_1 \in Z(G)$ determines a loop $\psi$ in $\text{Ham}(O_\eta)$. In this case $\kappa([\psi])$ is the value at $g_1$ of a character of the stabilizer $G_\eta$ of $\eta$ for the coadjoint action of $G$. Under certain hypotheses, we prove that such two curves $\{g_t\}$ and $\{\tilde{g}_t\}$ with different endpoints generate two loops $\psi$ and $\tilde{\psi}$, such that $\kappa([\psi]) \neq \kappa([\tilde{\psi}])$.

Hence $\#(\pi_1(\text{Ham}(O_\eta))) \geq \#(Z(G))$.

In Section 2 the definition of the representation $\kappa$ and its general properties are reviewed. In Section 3 we determine a lower bound to $\#(\pi_1(\text{Ham}(M,\omega)))$ when $(M,\omega)$ is a coadjoint orbit. As particular case coadjoint orbits of $SU(n)$ are considered, and we prove that $\pi_1(\text{Ham}(\mathbb{C}P^{n-1},\omega_{\text{Fubini-Study}}))$ has at least $n$ elements.

2. The representation $\kappa$

Let $(M,\omega)$ be a compact symplectic $2n$-manifold. We assume that $(M,\omega)$ is quantizable; i. e. $[\omega] \in H^2(M,\mathbb{Z})$. Let $(L,D)$ be a prequantum bundle [13] over $M$. We denote by $\{\psi_t\}_{t \in [0,1]}$ a Hamiltonian isotopy in $M$, with $\psi_0 = \text{id}$. This family determines the set $\{X_t\}$ of vector fields by

$$\frac{d\psi_t}{dt} = X_t \circ \psi_t.$$ 

Key words and phrases. Hamiltonian symplectomorphisms, Coadjoint orbits.
By \( f_t \) is denoted the corresponding normalized time-dependent Hamiltonian; that is, \( f_t \) is the function defined on \( M \), such that \( \omega(X_t, \cdot) = -df_t \) and \( \int_M f_t \omega^n = 0 \).

Denoting by \( \Gamma(L) \) the space of \( C^\infty \) sections of \( L \), for each \( t \) one defines the operator \( \mathcal{P}_t \in \text{End}(\Gamma(L)) \) by

\[
\mathcal{P}_t(\sigma) = -D_{X_t} \sigma - 2\pi i f_t \sigma.
\]

The differential equation

\[
\frac{d\sigma_t}{dt} = \mathcal{P}_t(\sigma_t), \quad \sigma_0 = \sigma,
\]

determines a family \( \sigma_t \) of sections of \( L \). In [12, Corollary 5] we have proved the following property: If \( \psi_1 = \text{id} \); i.e. \( \psi \) is a loop in \( \text{Ham}(M, \omega) \), then \( \sigma_1 = \kappa(\psi) \sigma \), where the constant \( k(\psi) \) is given by the symplectic action around the curve \( \{\psi_t(x)\}_t \)

\[
\kappa(\psi) = \exp \left( 2\pi i \int_S \omega - 2\pi i \int_0^1 f_t(\psi_t(x)) dt \right),
\]

\( x \) being any point of \( M \) and \( S \) being any 2-chain in \( M \) whose boundary is the nullhomologous curve \( \{\psi_t(x)\}_t \). Moreover \( \kappa(\psi) \) depends only on the homotopy class \([\psi]\) of the loop \( \psi \). That is, \( \kappa \) is a representation \( \kappa : \pi_1(\text{Ham}(M, \omega)) \to U(1) \) [12, Proposition 7].

Let \( G \) be a compact connected Lie group. We denote by \( \mathcal{O} := \mathcal{O}_\eta \) the coadjoint orbit of \( \eta \in \mathfrak{g}^* \). \( \mathcal{O} \) can be identified with \( G/G_\eta \), where \( G_\eta \) is the stabilizer of \( \eta \) for the coadjoint action. Given \( A \in \mathfrak{g} \), by \( X_A \) is denoted the vector field on \( \mathcal{O} \) generated by \( A \). The manifold \( \mathcal{O} \) is equipped with the symplectic structure \( \omega \) defined by \( \omega(X_A(\nu), X_B(\nu)) = \nu([A, B]) \) [5]. The map \( \nu \in \mathcal{O} \mapsto -\nu \in \mathfrak{g}^* \) is a moment map for the action of \( G \) on \( \mathcal{O} \); that is, \( \iota_{X_A} \omega = -df_A \), with \( f_A \in C^\infty(\mathcal{O}) \) given by \( f_A(\nu) = -\nu(A) \). If \( A \) is a vector of \( \mathfrak{z} \), the center of \( \mathfrak{g} \), then \( f_A \) is constant: \( f_A(\nu) = \eta(A) \). So \( f_A \) is not normalized unless \( \eta(A) = 0 \). Henceforth we assume that \( \mathfrak{z} = 0 \).

If the linear functional

\[
\lambda : C \in \mathfrak{g}_0 = \{ A \in \mathfrak{g} \mid \eta([A, \cdot]) \equiv 0 \} \mapsto 2\pi i \eta(C) \in i\mathbb{R}
\]

is integral; that is, if there is a character \( \Lambda : G_\eta \to U(1) \) whose derivative is \( \lambda \), then the orbit \( \mathcal{O} \) is quantizable see [6] [12]. A prequantum bundle \( L \) over \( \mathcal{O} = G/G_\eta \) is defined by \( L = G \times_{X_A} \mathbb{C} = (G \times \mathbb{C})/\sim \), with \( (g, z) \sim (gb^{-1}, \Lambda(b)z) \), for \( b \in G_\eta \).

In this case each section \( \sigma \) of \( L \) determines a \( \Lambda \)-equivariant function \( s : G \to \mathbb{C} \) by the relation

\[
\sigma(gG_\eta) = [g, s(g)].
\]

And given \( A \in \mathfrak{g} \) we denote by \( \mathcal{P}_A \) the operator

\[
-D_{X_A} - 2\pi if_A \in \text{End}(\Gamma(L)).
\]

In [12] is proved that the \( \Lambda \)-equivariant function associated to \( \mathcal{P}_A(\sigma) \) is \(-R_A(s)\), where \( R_A \) is the right invariant vector field on \( G \) determined by \( A \) and \( s \) is the \( \Lambda \)-equivariant function associated to \( \sigma \).

If \( \{g_t\}_{t\in[0,1]} \) is a smooth curve in \( G \) with \( g_0 = e \), then

\[
\{\psi_t : g'G_\eta \in G/G_\eta \mapsto g_tg'G_\eta \in G/G_\eta \},
\]

is a Hamiltonian isotopy. This isotopy is generated by the vector fields \( X_{A_t} \), with \( A_t = \dot{g}_t g_t^{-1} \) (see [12]). The stabilizer \( G_\eta \) contains a maximal torus of \( G \) (see [4]).
Since the center \(Z(G)\) of \(G\) is the intersection of all maximal tori of \(G\), \(\{\psi_t\}\) defines a loop in \(\text{Ham}(\mathcal{O})\) if \(g_1 \in Z(G)\). From now on we assume that \(g_1 \in Z(G)\).

One can consider the differential equation

\[
\frac{d\sigma_t}{dt} = \mathcal{P}_{A_t}\sigma_t, \quad \sigma_t = \sigma.
\]

If \(s_t\) is the equivariant function associated to \(\sigma_t\), by the above remark \(s_t\) satisfies

\[
\dot{s}(g_t) = -R_{A_t}(g_t)(s_t).
\]

But \(\dot{g}_t = R_{A_t}(g_t) \in T^\mu(G)\). So \(\dot{s}_t(g_t) + \dot{g}_t(s_t) = 0\); that is, the function \(h : [0, 1] \to \mathbb{C}\) defined by \(h(t) = s_t(g_t)\) is constant. Hence \(s_1(g_1) = s_0(c)\).

\[
\sigma_1(eG_\eta) = [eG_\eta, s_1(c)] = [eG_\eta, \Lambda(g_1)s_1(g_1)] = \Lambda(g_1)[eG_\eta, s_0(c)] = \Lambda(g_1)s_0(eG_\eta),
\]

in other words, \(\Lambda(g_1) = \kappa([\psi])\). We have the following Theorem

**Theorem 1.** Let \(\{\psi_t\}_{t \in [0, 1]}\) be the closed isotopy on \(\mathcal{O}_\eta\) defined by \(\psi_t(x) = g_t \cdot x\), where \(g_t \in G, \ g_0 = c\) and \(g_1 \in Z(G)\), if the functional \(2\pi i \eta\) is integral, then \(\kappa([\psi]) = \Lambda(g_1)\), where \(\Lambda\) is the character of \(G_\eta\) whose derivative is \(2\pi i \eta\).

If \(G_\eta\) is a maximal torus of \(G\) there is another description of the action \(\kappa([\psi])\) based on the Borel-Weil theorem [1]. Here we quote the result of [12].

**Proposition 2.** If \(2\pi i \eta\) an integral character of \(g_\eta\) and \(G_\eta\) is a maximal torus of \(G\), then the symplectic action around the loop (2.3) is

\[
\kappa([\psi]) = \frac{\chi(\pi^*)^\alpha(g_1)}{\dim \pi}, \tag{2.4}
\]

where \(\pi\) is the irreducible representation of \(G\) whose highest weight is \(-2\pi i \eta\).

3. Symplectic action in flag manifolds

Let \(G\) be a compact connected semisimple Lie group. As we said \(G_\eta\) contains a maximal torus \(T\). Let us consider the decomposition of \(g_c\) as direct sum of root spaces

\[
g_c = t_c \oplus \bigoplus_{\alpha \in \mathcal{R}} g_\alpha.
\]

As \(T \subset G_\eta\), \(\eta\) vanishes over \(g_\alpha\). If \(\tilde{\alpha} \in [g_\alpha, g_{-\alpha}]\) is the coroot to \(\alpha\), and \(s_0 := g_\alpha \oplus g_{-\alpha} \oplus [g_\alpha, g_{-\alpha}]\), it turns out that the complexification \((g_\eta)_c\) of \(g_\eta\) is generated by \(t_c\) and the \(s_0\)'s for which \(\eta(\tilde{\alpha}) = 0\). If we define

\[
p := t_c \oplus \bigoplus_{\eta(\tilde{\alpha}) \geq 0} g_\alpha,
\]

\(p\) is a subalgebra which generates a parabolic subgroup \(P\) of the complexification \(G_c\) of \(G\) [3], furthermore \(G_c/P\) and \(G/G_\eta\) can be identified as differential manifolds.

The element \(\eta\) is said to be regular if \(G_\eta = T\); that is, \(\eta(\tilde{\alpha}) \neq 0\) for all \(\alpha \in \mathcal{R}\). In this case \(G/G_\eta\) can be identified with \(G_c/B\), where \(B\) is a Borel subgroup of \(G_c\). Therefore the differential structure of \(\mathcal{O}_\eta\) is fixed by the set of roots \(\alpha \in \mathcal{R}\) such that \(\eta(\tilde{\alpha}) \neq 0\). But in the symplectic structure of \(\mathcal{O}_\eta\) are also involved the values of \(\eta\).

If the functional \(\lambda\) considered in (2.1) is integral (so \(\mathcal{O}_\eta\) is quantizable) then

\[2\pi i \eta : t \mapsto i \mathcal{R}\]

is the derivative of \(\lambda_T\), where \(\lambda\) is a character of \(G_\eta\). In this case, if \(T \simeq T^k\), there are integers \(m_j, \ j = 1, \ldots, k\) such that \(\lambda(A_1, \ldots, A_k) = \prod A_j^{m_j}\), for \((A_1, \ldots, A_k) \in T^k\). Given a smooth curve \(\{g_t \in G\}_{t \in [0, 1]}\) the symplectic action around the loop \(\psi\) in \(\text{Ham}(\mathcal{O})\) defined in (2.3) is \(\kappa([\psi]) = \lambda(g_1) = \prod B_j^{m_j}\), assumed
that \( g_1 \in Z(G) \) as element of \( \mathbb{T}^k \) is \( g_1 = (B_1, \ldots, B_k) \). So we have the following Theorem

**Theorem 3.** If \( \eta \in \mathfrak{g}^* \) is integral, and the character \( \Lambda \) of \( G_\eta \) whose derivative is \( 2\pi i\eta \) is injective on \( Z(G) \), then

\[
\sharp((\pi_1(\text{Ham}(O_\eta)))) \geq \sharp(Z(G)).
\]

*Note.* For a general coadjoint orbit \( O \) of \( G \) McDuff and Tolman have proved the following property: If \( G \) acts effectively on \( O \), the inclusion \( G \to \text{Ham}(O) \) induces an injection on \( \pi_1 \) \([8]\).

**Example**

Given

\[
d = (p_1, \ldots, p_k, m_1, \ldots, m_k) \in \mathbb{R}^{n-1},
\]

with \( 0 < n_1 \leq n_2 \leq \cdots \leq n_k \), we define \( \eta \in \mathfrak{su}(n) \) by

\[
\eta(Y) = i \sum_{j=1}^{n-1} d_j Y_{jj},
\]

where \( d = (d_1, \ldots, d_{n-1}) \). The stabilizer of \( \eta \) for the coadjoint action of \( G = SU(n) \) is \( G_\eta = U(n_1) \times \cdots \times U(n_k) \subset SU(n) \). The orbit \( G/G_\eta \) is the flag manifold \( F_\eta \) in \( \mathbb{C}^n \), where \( q \) is the partition \((n_1, \ldots, n_k)\) of \( n - 1 \). So the differential structure of \( O_\eta \) is determined by the partition \( q \).

For

\[
Y = (B_1, \ldots, B_k) \in \bigoplus_{j=1}^k \mathfrak{u}(n_j) = \mathfrak{g}_\eta
\]

one has \( \eta(Y) = i \sum_{j=1}^k p_j \text{tr}(B_j) \). If \(-2\pi p_j =: m_j \in \mathbb{Z} \) for \( j = 1, \ldots, k \), then the character \( \Lambda : \prod U(n_j) \to U(1) \) defined by

\[
\Lambda(A_1, \ldots, A_k) = \prod_{j=1}^k (\det(A_j))^{m_j}
\]

has as derivative \( 2\pi i\eta \). That is, the coadjoint orbit \( O_\eta \) endowed with the natural symplectic structure is a quantizable manifold. Hence, given \( q = (n_1, \ldots, n_k) \) a partition of \( n - 1 \) and \( \vec{m} \in \mathbb{Z}^k \), the pair \((q, \vec{m})\) =: \( \vec{d} \) determines a quantizable symplectic manifold \( O_{\vec{d}} \), which is diffeomorphic to \( F_\eta \).

Given \( z \in \mathbb{C} \) such that \( z^n = 1 \), a curve \( g_t \in SU(n) \) with \( g_1 = z I_n \) defines a closed Hamiltonian isotopy \( z \psi \) on \( O_{\vec{d}} \) by \((2.3)\). The corresponding symplectic action around the loop \( z \psi \) in \( \text{Ham}(O_{\vec{d}}) \) can be calculated by Theorem 1

\[
\kappa([z\psi]) = \prod_{j=1}^k z^{m_j n_j} = z^q \vec{m}.
\]

If \( q \cdot \vec{m} := \sum j n_j m_j \) and \( n \) are relatively prime, then \( \kappa([z\psi]) \neq \kappa([y\psi]) \), for \( z \neq y \). We have proved the following Proposition

**Proposition 4.** Let \( O_{\vec{d}} \) the coadjoint orbit of \( SU(n) \) determined by \( \vec{d} := (q, \vec{m}) \). If \( q \cdot \vec{m} \) and \( n \) are relatively prime, then \( \pi_1(\text{Ham}(O_{\vec{d}})) \) has at least \( n \) elements.
When $k = 1$ in (3.1); that is, $n_1 = n - 1$ and $m_1 = m$ then $G/G_\eta = SU(n)/U(n-1) = CP^{n-1}$. The symplectic manifold $O_\eta$ is $CP^{n-1}$ equipped with a multiple of the Fubini-Study symplectic structure. If $g_1 = zI_n$, with $z^n = 1$ then by (3.2)
\[
\kappa([z\eta]) = z^{-m}.
\] (3.3)

In particular, when $m = -1$ the corresponding coadjoint orbit is $CP^{n-1}$ endowed with the Fubini-Study symplectic structure $\omega_{FS}$. Therefore

**Proposition 5.** $\pi_1(\text{Ham}(CP^{n-1}, \omega_{FS}))$ has at least $n$ elements.

**Remark 1.** It is known that Ham($CP^2$, $\omega_{FS}$) has the homotopy type of $PU(3)$ [10]. So $\#\pi_1(\text{Ham}(CP^2, \omega_{FS})) = 3$, and this result is consistent with Proposition 5.

**Remark 2.** If $n = 2$, then $\mathcal{F}_\eta = CP^1$, and the value for $\kappa([z\eta])$ obtained in (3.3) can be also calculated using Proposition 2.4. Now $-2\pi i\eta$ is the weight $(-m_1, 0)$. Assumed that $m < 0$, if $\pi$ is the irreducible representation of $SU(2)$ whose highest weight is $-2\pi i\eta$, then the Weyl character formula [2] gives
\[
\chi_\pi(t_1, t_2) = \frac{\sum_{s \in S_2} \text{sig } s \ e^{s(-2\pi i\eta + \rho)}(t_1, t_2)}{\sum_{t \in S_2} \text{sig } t \ e^{t(\eta)}(t_1, t_2)},
\]
for $(t_1, t_2) \in U(1) \subset SU(2)$, $\rho$ being the weight $(1, 0)$ and $S_2$ the symmetric group of 2 elements. So
\[
\chi_\pi(t_1, t_2) = \frac{t_1^{-m+1} - t_2^{-m+1}}{t_1 - t_2} = \sum_{k=0}^{m} t_1^k t_2^{-m-k}
\]

When $t_1 = t_2 = z$, $\chi_\pi(t_1, t_2) = (-m+1)z^{-m}$. On the other hand, Weyl dimension formula gives $\dim \pi = -m + 1$. So, by Proposition 2.4
\[
\kappa([z\eta]) = \chi_\pi(z^{-1}L_2)/(-m+1) = z^m.
\]
As $z^2 = 1$ this value agrees with (3.3).

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