HOMOTOPY TYPES OF HOMEOMORPHISM GROUPS OF NONCOMPACT 2-MANIFOLDS

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Abstract. Suppose \(M\) is a noncompact connected PL 2-manifold and let \(\mathcal{H}(M)_0\) denote the identity component of the homeomorphism group of \(M\) with the compact-open topology. In this paper we classify the homotopy type of \(\mathcal{H}(M)_0\) by showing that \(\mathcal{H}(M)_0\) has the homotopy type of the circle if \(M\) is the plane, an open or half open annulus, or the punctured projective plane. In all other cases we show that \(\mathcal{H}(M)_0\) is homotopically trivial.

1. Introduction

Hamstrom [4] classified the homotopy types of the identity components of homeomorphism groups of compact 2-manifolds \(M\). In this paper we treat the case where \(M\) is noncompact. Suppose \(M\) is a PL 2-manifold and \(X\) is a compact subpolyhedron of \(M\). We denote by \(\mathcal{H}_X(M)\) the group of homeomorphisms \(h\) of \(M\) onto itself with \(h|_X = id\), equipped with the compact-open topology, and by \(\mathcal{H}(M)_0\) the identity component of \(\mathcal{H}(M)\). Let \(\mathbb{R}^2\) denote the plane, \(S^1\) the unit circle and \(\mathbb{P}^2\) the projective plane. The following is the main result of this paper.

Theorem 1.1. Suppose \(M\) is a noncompact connected (separable) PL 2-manifold and \(X\) is a compact subpolyhedron of \(M\). Then

(i) \(\mathcal{H}_X(M)_0 \simeq S^1\) if \((M, X) \cong (\mathbb{R}^2, \emptyset), (\mathbb{R}^2, 1\text{pt}), (S^1 \times \mathbb{R}^1, \emptyset), (S^1 \times [0, 1), \emptyset)\) or \((\mathbb{P}^2 \setminus 1\text{pt}, \emptyset)\),

(ii) \(\mathcal{H}_X(M)_0 \simeq *\) in all other cases.

Corollary 1.1. If \(M\) is a connected (separable) 2-manifold and \(X\) is a compact subpolyhedron of \(M\) with respect to some triangulation of \(M\), then \(\mathcal{H}_X(M)_0\) is an \(\ell^2\)-manifold.

In [14] we obtained a natural principal bundle connecting the homeomorphism group and the embedding space (cf. Section 2). In this paper we will seek a condition under which the fiber of this bundle is connected (Section 3). The contractibility and the ANR property of \(\mathcal{H}_X(M)_0\) in the compact case will then imply the similar properties of embedding spaces and in turn the corresponding properties of \(\mathcal{H}_X(M)_0\) in the noncompact case. Corollary 1.1 follows immediately from the characterization of \(\ell^2\)-manifolds and this enables us to determine the topological type itself of \(\mathcal{H}_X(M)_0\) by the homotopy invariance of infinite-dimensional manifolds.

In a succeeding paper we will investigate the subgroups of \(\mathcal{H}_X(M)_0\) consisting of PL and Lipschitz homeomorphisms from the viewpoints of infinite-dimensional topological manifolds.

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2. Preliminaries

Throughout the paper we follow the following conventions: Spaces are assumed to be separable and metrizable, and maps are always continuous. When $A$ is a subset of a space $X$, the notations $\text{Fr}_X A$, $\text{cl}_X A$ and $\text{Int}_X A$ denote the frontier, closure and interior of $A$ relative to $X$ (i.e., $\text{Int}_X A = \{ x \in A \mid A$ contains a neighborhood of $x$ in $X \}$ and $\text{Fr}_X A = \text{cl}_X A \setminus \text{Int}_X A$). On the other hand, when $M$ is a manifold, the notations $\partial M$ and $\text{Int} M$ denote the boundary and interior of $M$ as a manifold. When $N$ is a 2-submanifold of a 2-manifold $M$, we always assume that $N$ is a closed subset of $M$ and $\text{Fr} N = \text{Fr}_M N$ is a 1-manifold transversal to $\partial M$. Therefore we have $\text{Int} N = \text{Int}_M N \cap \text{Int} M$ and $\text{Fr}_M N \subset \partial N$. A metrizable space $X$ is called an ANR (absolute neighborhood retract) if any map $f : B \to X$ from a closed subset of a metrizable space $Y$ has an extension to a neighborhood $U$ of $B$. If we can always take $U = Y$, then $X$ is called an AR (absolute retract). ANRs are locally contractible and ARs are exactly contractible ANRs (cf. [7]). Finally $\ell_2$ denotes the separable Hilbert space $\{(x_n) \in \mathbb{R}^\infty : \sum_n x_n^2 < \infty \}$.

In [14] we investigated some extension property of embeddings of a compact 2-polyhedron into a 2-manifold, based upon the conformal mapping theorem. The result is summarized as follows: Suppose $M$ is a PL 2-manifold and $K \subset X$ are compact subpolyhedra of $M$. Let $\mathcal{E}_K(X, M)$ denote the space of embeddings $f : X \to M$ with $f|_K = \text{id}$, equipped with the compact-open topology. We consider the subspace of proper embeddings $\mathcal{E}_K(X, M)^* = \{ f \in \mathcal{E}_K(X, M) : f(X \cap \partial M) \subset \partial M, f(X \cap \text{Int} M) \subset \text{Int} M \}$. Let $\mathcal{E}_K(X, M)_0$ denote the connected component of the inclusion $i_X : X \subset M$ in $\mathcal{E}_K(X, M)^*$.

**Theorem 2.1.** For every $f \in \mathcal{E}_K(X, M)^*$ and every neighborhood $U$ of $f(X)$ in $M$, there exists a neighborhood $\mathcal{U}$ of $f$ in $\mathcal{E}_K(X, M)^*$ and a map $\varphi : \mathcal{U} \to \mathcal{H}_{K \cup (M \setminus U)}(M)_0$ such that $\varphi(g)f = g$ for each $g \in \mathcal{U}$ and $\varphi(f) = \text{id}_M$.

**Corollary 2.1.** For any open neighborhood $U$ of $X$ in $M$, the restriction map $\pi : \mathcal{H}_{K \cup (M \setminus U)}(M)_0 \to \mathcal{E}_K(X, U)_0$, $\pi(f) = f|_X$, is a principal bundle with fiber $\mathcal{G} \equiv \mathcal{H}_{K \cup (M \setminus U)}(M)_0 \cap \mathcal{H}_X(M)$, where the group $\mathcal{G}$ acts on $\mathcal{H}_{K \cup (M \setminus U)}(M)_0$ by right composition.

**Proposition 2.1.** $\mathcal{E}_K(X, M)$ and $\mathcal{E}_K(X, M)^*$ are ANRs.

Next we recall some fundamental facts on homeomorphism groups of compact 2-manifolds.

**Fact 2.1.** If $N$ is a compact PL 2-manifold and $Y$ is a compact subpolyhedron of $N$, then $\mathcal{H}_Y(N)$ is an ANR. ([8], [9], cf. [14], Lemma 3.2)]

**Lemma 2.1.** ([13], [12], §3) Suppose $N$ is a compact connected PL 2-manifold and $Y$ is a compact subpolyhedron of $N$.

(i) If $(N, Y) \not\cong (\mathbb{D}^2, \emptyset), (\mathbb{D}^2, 0), ([1 \times [0, 1], 0), ([1, 0], 0, 0), ([S^2, 0), ([S^2, 1pt), ([S^2, 2pts), ([T^2, 0), ([K^2, 0), ([E^2, 0), ([E^2, 1pt), then $\mathcal{H}_Y(N)_0 \simeq *$.

(ii) If $A$ is a nonempty compact subset of $\partial N$, then $\mathcal{H}_{Y \cup A}(N)_0 \simeq *$. 


(iii) If \((N, Y) \cong (\mathbb{D}^2, \emptyset), (\mathbb{D}^2, 0), (\mathbb{S}^1 \times [0, 1], \emptyset), (M, \emptyset), (\mathbb{S}^2, 1pt), (\mathbb{S}^2, 2pts), (\mathbb{P}^2, 1pt), (\mathbb{K}^2, \emptyset),\) then \(\mathcal{H}_Y(N)_0 \cong \mathbb{S}^1.\)

**Proof.** In \([12]\) the PL-homeomorphism groups of compact 2-manifolds was studied in the context of semisimplicial complex. However, using Corollary 2.1 and the results in \([4]\), we can apply the arguments and results in \([12, \S 3]\) to our setting.

(i) Let \(L\) be a small regular neighborhood of the union \(Y_1\) of the nondegenerate components of \(Y\) and let \(\partial Y_0 = Y \setminus Y_1.\) Since \(\mathcal{H}_Y(N)_0\) deforms into \(\mathcal{H}_{L \cup Y_0}(N)_0 \cong \mathcal{H}_{\text{Fr} L \cup Y_0}(\text{cl} (N \setminus L))_0,\) we may assume that \(Y_1 \subset \partial N.\) This case follows from \([4]\) and \([12, \S 3].\)

(ii) Let \(\partial_{+} N\) denote the union of the components of \(\partial N\) which meet \(A.\) Then \(\mathcal{H}_{Y \cup A}(N)_0\) strongly deformation retracts onto \(\mathcal{H}_{Y \cup \partial_{+} N}(N)_0,\) and the latter is contractible by the case (i).

## 3. Relative isotopies on 2-manifolds

In Corollary 2.1 we have a principal bundle with a fiber \(G = \mathcal{H}_X(M) \cap \mathcal{H}_K(M)_0.\) In this section we will seek a sufficient condition which implies \(G = \mathcal{H}_X(M)_0.\) Suppose \(M\) is a 2-manifold and \(N\) is a 2-submanifold of \(M.\) In \([4]\) it is shown that (i) two homotopic essential simple closed curves in \(\text{Int} M\) and two proper arcs homotopic rel ends in \(M\) are ambient isotopic rel \(\partial M,\) (ii) every homeomorphism \(h : M \to M\) homotopic to \(id_M\) is ambient isotopic to \(id_M.\) Using these results or arguments we will show that if, in addition, \(h|_N = id_N\) then \(h\) is isotopic to \(id_M\) rel \(N\) under some restrictions on disks, annuli and Möbius bands components (i.e. the pieces which admit global rotations). We denote the Möbius band, the torus and the Klein bottle by \(\mathbb{M}, \mathbb{T}^2\) and \(\mathbb{K}^2\) respectively. The symbol \#\(X\) denotes the number of elements (or cardinal) of a set \(X.\)

**Theorem 3.1.** Suppose \(M\) is a connected 2-manifold, \(N\) is a compact 2-submanifold of \(M\) and \(X\) is a subset of \(N\) such that

(i) \(M \neq \mathbb{T}^2, \mathbb{P}^2, \mathbb{K}^2\) or \(X \neq \emptyset.\)

(ii)

(a) if \(H\) is a disk component of \(N,\) then \(#(H \cap X) \geq 2,\)

(b) if \(H\) is an annulus or Möbius band component of \(N,\) then \(H \cap X \neq \emptyset,\)

(iii)

(a) if \(L\) is a disk component of \(\text{cl}(M \setminus N),\) then \(\text{Fr} L\) is a disjoint union of arcs or \(#(L \cap X) \geq 2,\)

(b) if \(L\) is a Möbius band component of \(\text{cl}(M \setminus N),\) then \(\text{Fr} L\) is a disjoint union of arcs or \(L \cap X \neq \emptyset.\)

If \(h_t : M \to M\) is an isotopy rel \(X\) such that \(h_0|_N = h_1|_N,\) then there exists an isotopy \(h'_t : M \to M\) rel \(N\) such that \(h'_0 = h_0, h'_1 = h_1\) and \(h'_t = h_t (0 \leq t \leq 1)\) on \(M \setminus K\) for some compact subset \(K\) of \(M.\)

**Corollary 3.1.** Under the same condition as in Theorem 3.1, we have \(\mathcal{H}_N(M) \cap \mathcal{H}_X(M)_0 = \mathcal{H}_N(M)_0.\)

First we explain the meaning of the conditions (ii) and (iii) in Theorem 3.1. Suppose \(h \in \mathcal{H}_N(M)\) and \(h\) is isotopic to \(id_M\) rel \(X.\) In order that \(h\) is isotopic to \(id_M\) rel \(N,\) it is necessary that \(h\) does not Dehn twist along the boundary circle of any disk, Möbius band and annulus component of \(N.\) This is ensured by the condition (ii) in Theorem 3.1 (Figure 1.a: \(N = N_1 \cup N_2, \text{cl}(M \setminus N) = L\) and...
$X = \{a, b\}$). However, this is not sufficient because a union of some components of $N$ and $cl(M \setminus N)$ may form a disk, a M"obius band or an annulus. The condition (iii) in Theorem 3.1 is imposed to prevent Dehn twists around these pieces (Figure 1.1: $N = N_1 \cup N_2$, $cl(M \setminus N) = L \cup L_1 \cup L_2$ and $X = \{a_1, a_2, b_1, b_2\}$). This condition is too strong (we can replace $X$ by $Y = \{a_1, b_1\}$), but it is simple and sufficient for our purpose.

Figure 1. $h$ can not Dehn twist $L$.

![Figure 1.a](image1.png) ![Figure 1.b](image2.png)

We proceed to the verification of Theorem 3.1. We need some preliminary lemmas. Throughout this section we assume that $M$ is a connected 2-manifold and $N$ is a 2-submanifold of $M$. When $G$ is a group and $S \subset G$, $\langle S \rangle$ denotes the subgroup of $G$ generated by $S$.

We will use the following facts from [2].

**Fact 3.1.**

(0) ([2, Theorem 1.7]) If a simple closed curve $C$ in $M$ is null-homotopic, then it bounds a disk.

(1) ([2, Theorem 3.1]) Suppose $\alpha$ and $\beta$ are proper arcs in $M$. If they are homotopic relative to end points, then they are ambient isotopic relative to $\partial M$.

(2) ([2, Theorem 4.2]) Let $C$ be a simple closed curve in $M$, which does not bound a disk or a M"obius band. Let $\alpha \in \pi_1(M, \ast)$ be represented by a single circuit of $C$ and let $\alpha = \beta^k$, $k \geq 0$. Then $\alpha = \beta$.

(3) ([2, Lemma 4.3]) (i) If $M \neq \mathbb{P}$, then $\pi_1(M)$ has no torsion elements.

(ii) Suppose $M \neq \mathbb{T}^2, \mathbb{K}^2$. If $\alpha, \beta \in \pi_1(M)$ and $\alpha \beta = \beta \alpha$, then $\alpha, \beta \in \langle \gamma \rangle$ for some $\gamma \in \pi_1(M)$.

(4) ([2, p101, lines 5 -10]) If $M \neq \mathbb{P}^2$ and every circle component of $\text{Fr} N$ is essential in $M$, then the inclusion induces a monomorphism $\pi_1(N, x) \to \pi_1(M, x)$ for every $x \in N$.

(5) Suppose $M$ is compact, $X$ is a closed subset of $\partial M$, $X \neq \emptyset$ and $h : M \to M$ is a homeomorphism with $h|_X = id_X$.

(i) ([2, Theorem 3.4]) If $M = \mathbb{D}^2$ or $M$ and $h|_{\partial M} : \partial M \to \partial M$ is orientation preserving, then $h$ is isotopic to $id_M$ rel $X$.

(ii) ([2, Proof of Theorem 6.3]) If $M \neq \mathbb{D}^2$ and $h$ satisfies the following condition ($\ast$), then $h$ is isotopic to $id_M$ rel $X$:

(*): $h \ell \simeq \ell$ rel end points for every proper arc $\ell : [0,1] \to M$ with $\ell(0), \ell(1) \in X$ (we allow that $\ell(0) = \ell(1)$ when $X$ is a single point).

**Comments.** (4) Consider the universal covering $\pi : \tilde{M} \to M$. By Fact 3.1.(3-i) $\pi^{-1}(\text{Fr}_M N)$ is a union of real lines, half rays and proper arcs. If $M = \mathbb{P}^2$, then $N = \mathbb{P}^2$.

(5-ii) $M$ is a disk with $k$ holes, $\ell$ handles (a handle = a torus with a hole) and $m$ M"obius bands. The assertion is easily verified by the induction on $n = k + \ell + m$, using Fact 3.1.(1) and (5-i), together with the following remarks:
(a) When \( \#X \geq 2 \), we have \( h\ell \simeq \ell \) rel. end points even if \( \ell(0) = \ell(1) \).
(b) If \( h \) is (ambient) isotopic to \( h_1 \) rel. \( X \), then \( h_1 \) also satisfies the condition (*).
(c) Since \( M \neq \mathbb{D}^2 \), from the condition (*) it follows that for every component \( C \) of \( \partial M \), we have \( h(C) = C \) and \( h \) preserves the orientation of \( C \).
(d) Let \( C_1, \cdots , C_p \) be the components of \( \partial M \) which meet \( X \). Then \( h \) is isotopic rel. \( X \) to \( h_1 \) such that \( h_1 = id \) on each \( C_i \). Furthermore, \( h_1 \) satisfies (*) for \( \cup_iC_i \).

We also need the following remarks.

**Fact 3.2.** Suppose \( M \) is a connected 2-manifold and \( C \) is a circle component of \( \partial M \). If either (i) \( C \neq \partial M \) or (ii) \( M \) is noncompact, then \( C \) is a retract of \( M \).

**Comments.** (ii) Take a half lay \( \ell \) connecting \( C \) and \( \infty \), and consider the regular neighborhood \( N \) of \( C \cup \ell \). Since \( \partial N \) is a real line we can retract \( M \) onto \( N \) and then onto \( C \).

**Fact 3.3.** Suppose \( M \) is a compact 2-manifold, \( \{M_i\} \) is a finite collection of compact connected 2-manifolds such that \( M = \cup_iM_i \) and \( \text{Int}M_i \cap \text{Int}M_j = \emptyset \) \((i \neq j)\).

(i) If \( M \) is a disk, then some \( M_i \) is a disk.
(ii) If \( M \) is a Möbius band, then some \( M_i \) is a disk or a Möbius band.
(iii) If \( M \) is an annulus, then some \( M_i \) is a disk or an essential annulus in \( M \). (If \( N \) is a disk with \( r \) holes in \( M \) \((r \geq 2)\), then there exists a disk \( D \subset \text{Int}M \) such that \( D \cap N = \partial D \subset \partial N \).)

**Lemma 3.1.** Suppose \( M \neq \mathbb{K}^2 \), \( C \) is a simple closed curve in \( M \) which does not bound a disk or a Möbius band in \( M \), \( x \in C \) and \( \alpha \in \pi_1(M,x) \) is represented by \( C \). If \( \beta \in \pi_1(M,x) \) and \( \beta^k = \alpha^\ell \) for some \( k \), \( \ell \in \mathbb{Z} \setminus \{0\} \), then \( \beta \in \langle \alpha \rangle \).

**Proof.** Attaching \( \partial M \times [0,1) \) to \( \partial M \subset M \), we may assume that \( \partial M = \emptyset \). Take a covering \( p : (\hat{M},\hat{x}) \rightarrow (M,x) \) such that \( p_*\pi_1(\hat{M},\hat{x}) = \langle \alpha, \beta \rangle \subset \pi_1(M,x) \).

If \( \hat{M} \) is noncompact, then by [3, Lemma 2.2] there exists a compact connected 2-submanifold \( N \) of \( \hat{M} \) such that \( \hat{x} \in N \) and the inclusion induces an isomorphism \( \pi_1(N,\hat{x}) \rightarrow \pi_1(\hat{M},\hat{x}) \). Since \( \partial N \neq \emptyset \), it follows that \( \pi_1(N,\hat{x}) \cong \langle \alpha, \beta \rangle \) is a free group, so it is an infinite cyclic group \( \langle \gamma \rangle \). By Fact 3.1.(2) \( \gamma = \alpha^{\pm 1} \), so \( \beta \in \langle \alpha \rangle \).

Suppose \( \hat{M} \) is compact. Since \( \text{rank}H_1(\hat{M}) = 0 \text{ or } 1 \) and \( \pi_1(\hat{M}) \neq 1 \), it follows that \( \hat{M} \cong \mathbb{P}^2 \) or \( \mathbb{K}^2 \) and \( M \) is closed and nonorientable. If \( \hat{M} \cong \mathbb{K}^2 \) so \( \chi(\hat{M}) = 0 \), then \( \chi(M) = 0 \) and \( M \cong \mathbb{K}^2 \), a contradiction. Therefore, \( \hat{M} \cong \mathbb{P}^2 \) and \( \chi(M) = 1 \), so \( \chi(M) = 1 \) and \( M \cong \mathbb{P}^2 \). We have \( \pi_1(M) = \langle \alpha \rangle \).

Note that if \( M = \mathbb{K}^2 \) and \( \alpha, \beta \) are represented by the center circles of two Möbius bands, then \( \alpha^2 = \beta^2 \), but \( \beta \notin \langle \alpha \rangle \).

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**Lemma 3.2.** Suppose $C$ is a circle component of $\partial M$, $x \in C$ and $\alpha \in \pi_1(M, x)$ is represented by $C$. If $M \neq \mathbb{D}^2$, $\mathbb{M}$ or $S^1 \times [0, 1] \setminus A$ ($A$ is a compact subset of $S^1 \times \{1\}$), then there exists a $\gamma \in \pi_1(M, x)$ such that $\gamma \alpha^n \neq \alpha^n \gamma$ for any $n \in \mathbb{Z} \setminus \{0\}$.

**Proof.** By the claim below we have a $\gamma \in \pi_1(M, x) \setminus \langle \alpha \rangle$. If $\gamma \alpha^n = \alpha^n \gamma$ for some $n \neq 0$, then by Fact 3.1.(3-ii) $\alpha^n, \gamma \in \langle \beta \rangle$ for some $\beta \in \pi_1(M, x)$ and $\alpha^n = \beta^k$ for some $k \in \mathbb{Z}$. Since $\alpha \neq 1$ and $M \neq \mathbb{P}^2$, by Fact 3.1.(3-i) $k \neq 0$. Hence by Lemma 3.1 $\beta \in \langle \alpha \rangle$ so $\gamma \in \langle \alpha \rangle$, a contradiction. \qed

**Claim.** Suppose $M$ is a connected 2-manifold, $C$ is a circle component of $\partial M$, $x \in C$ and $\alpha \in \pi_1(M, x)$ is represented by $C$. If $\pi_1(M, x) = \langle \alpha \rangle$, then $M \cong \mathbb{D}^2$ or $S^1 \times [0, 1] \setminus A$ for some compact subset $A$ of $S^1 \times \{1\}$.

**Proof.** First we note that $M$ does not contain any handles or Möbius bands. In fact if $H$ is a handle or a Möbius band in $M$, then we can easily construct a retraction $r : M \to H$ which maps $C$ homeomorphically onto $\partial H$ (Fact 3.2), and we have the contradiction $\pi_1(H) = \langle r_* \alpha \rangle$. In particular, if $M$ is compact then $M$ is a disk or an annulus.

Suppose $M$ is noncompact. It follows that $\partial M$ contains no circle components other than $C$. In fact if $C'$ is a circle in $\partial M \setminus C$, then we can join $C$ and $C'$ by a proper arc $A$ in $M$ and by Fact 3.2 we have a retraction $M \to C \cup A \cup C'$, a contradiction. We can write $M = \bigcup_{i=1}^{\infty} N_i$, where $N_i$ is a compact connected 2-submanifold of $M$, $C \subset \text{Int}_M N_1$, $N_i \subset \text{Int}_M N_{i+1}$ and each component of $\text{cl}(M \setminus N_i)$ is noncompact. We will show that each $N = N_i$ is an annulus. This easily implies the conclusion.

Let $C_1, \ldots, C_m$ be the components of $\partial N \setminus C$. By the above remark $C_j \not\subset \partial M$, so $C_j$ meets a component of $\text{cl}(M \setminus N)$. Let $N'$ be a submanifold of $N$ obtained by removing an open color of each $C_j$ from $N$. It follows that $N' \cong N$, $\text{Fr} N'$ is the union of circles $C_j'$ associated with $C_j$’s, each $C_j'$ is contained in some component $L_j$ of $\text{cl}(M \setminus N')$, $\text{cl}(M \setminus N') = \bigcup_j L_j$, and each $L_j$ is noncompact. Since $M$ contains no handles or Möbius bands (so no one point union of two circles), it follows that $L_j \cap L_j' = \emptyset$ ($j \neq j'$) and $L_j \cap N' = C_j'$. By Fact 3.2 $N'$ is a retract of $M$, so $\pi_1(N', x) = \langle \alpha \rangle$. This implies that $N \cong N'$ is an annulus. \qed

The next lemma is a key point in the proof of Theorem 3.1. In [2, Lemma 6.1] the condition “the loop $h_t(x)$ is null-homotopic in $M$” is achieved by rotating $x$ along $C$. However, this process does not keep the condition “isotopic rel $N$”.

**Lemma 3.3.** Suppose $C$ is a circle component of $\text{Fr} N$ which does not bound a disk or a Möbius band in $M$, $h : M \to M$ is a homeomorphism with $h|_N = \text{id}_N$ and $h_t : M \to M$ ($0 \leq t \leq 1$) is a homotopy with $h_0 = h$, $h_1 = \text{id}_M$. If the following conditions are satisfied, then for any $x \in C$ the loop $m = \{h_t(x) : 0 \leq t \leq 1\}$ is null-homotopic in $M$:

(i) $M \neq \mathbb{T}^2$, $\mathbb{P}^2$, $\mathbb{K}^2$,

(ii) each circle component of $\text{Fr} N$ is essential in $M$,

(iii) each component of $N \not\cong \mathbb{D}^2$, $\mathbb{M}$, $S^1 \times [0, 1] \setminus A$ ($A \subset S^1 \times \{1\}$, compact).
Proof. Let \( \alpha = \{ \ell \} \in \pi_1(M, x) \) be represented by \( C \) and let \( \beta = \{ m \} \in \pi_1(M, x) \). The homotopy \( h_\ell \) implies that \( \alpha \beta = \beta \alpha \). Since \( M \not\cong \mathbb{T}^2, \mathbb{P}^2, \mathbb{K}^2 \), by Fact.3.1.(3-ii) \( (\alpha, \beta) \subset (\delta) \) for some \( \delta \in \pi_1(M, x) \). Since \( C \) does not bound a disk or a Möbius band, by Fact 3.1.(2) \( \delta = \alpha^k \) so \( \beta = \alpha^k \) for some \( k \in \mathbb{Z} \). Let \( \alpha_1 = \{ \ell \} \in \pi_1(N, x) \). By Lemma 3.2 there exists a \( \gamma = \{ n \} \in \pi_1(N, x) \) such that \( \gamma \alpha_1 \neq \alpha_1 \gamma \) for any \( i \in \mathbb{Z} \setminus \{ 0 \} \) (Figure 2). The homotopy \( htn \) implies that \( \gamma \beta = \beta \gamma \) in \( \pi_1(M, x) \). Since \( \pi_1(N, x) \to \pi_1(M, x) \) is monomorphic by Fact.3.1.(4), \( \gamma \alpha^k_1 = \alpha^k_1 \gamma \) in \( \pi_1(N, x) \) so that \( k = 0 \) and \( \beta = 1 \) in \( \pi_1(M, x) \).

\[ \Box \]

**Figure 2.** The loops \( \ell, m \) and \( n \) in Lemma 3.3.

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**Lemma 3.4.** Suppose \( N \not= \emptyset \), \( cl(M \setminus N) \) is compact, each component of \( FrN \) is a circle, \( h : M \to M \) is a homeomorphism such that \( h|_N = id_N \) and \( h \) is homotopic to \( id_M \). If the following conditions are satisfied, then \( h \) is isotopic to \( id_M \) rel \( N \) :

(i) \( M \not\cong \mathbb{T}^2, \mathbb{P}^2, \mathbb{K}^2 \),

(ii) each component \( C \) of \( FrN \) does not bound a disk or a Möbius band,

(iii) each component of \( N \not\cong S^1 \times [0, 1] \setminus A \) (\( A \subset S^1 \times \{ 1 \} \), compact).

If we assume that \( h \) is isotopic to \( id_M \), then the condition (iii) is weakened to the condition:

(iii') each component of \( N \not\cong S^1 \times [0, 1], S^1 \times [0, 1) \).

Proof. Let \( h_t : h \simeq id_M \) be any homotopy and let \( L_1, \cdots, L_m \) be the components of \( cl(M \setminus N) \). By Lemma 3.3 the loop \( h_t(x) \simeq * \) in \( M \) for any \( x \in FrN = \bigcup_j FrL_j \). We must find an isotopy \( h|_{L_j} \simeq id_{L_j} \) rel \( FrL_j \).

Let \( f : [0, 1] \to L_j \) be any path with \( f(0), f(1) \in FrL_j \). The homotopy \( h_t f \) yields a contraction of the loop \( h_f \cdot h_t(f(1)) \cdot f^{-1} \cdot (h_t(f(0)))^{-1} \) in \( M \). Since \( h_t(f(0)), h_t(f(1)) \simeq * \), it follows that \( h_f \cdot f^{-1} \simeq * \) in \( M \). Since \( \pi_1(L_j) \to \pi_1(M) \) is monomorphic by Fact.3.1.(4), the loop \( h_f \cdot f^{-1} \simeq * \) in \( L_j \), and the desired isotopy is obtained by Fact.3.1.(5-ii).

Figure 3 illustrates an original idea to prove Lemma 3.4 and Theorem 3.1: Consider the loop \( m = n_1 f n_2 f^{-1} \) (\( f^{-1} \) is the inverse path of \( f \)). Any isotopy \( h_t : id_M \simeq h \) rel \( \{ a, b \} \) induces a homotopy \( h_t m : m \simeq n_1(hf)n_2(hf)^{-1} \) in \( M \setminus \{ a, b \} \). Modify the homotopy \( h_t m \) to simplify the intersection of the image of \( h_t m \) and \( FrN \), and obtain a homotopy \( F : S^1 \times [0, 1] \to M \setminus \{ a, b \} \) shown in Figure 3. The homotopies \( F|_{A_1} \) in \( N_1 \) and \( F|_{A_2} \) in \( N_2 \setminus \{ a, b \} \) imply that \( k_i = 0 \) (\( i = 1, 2 \)), and the homotopy \( F|_{B_1} \) in \( L \) implies that \( f \simeq hf \) rel end points in \( L \) as required.

**Proof of Theorem 3.1.** We can assume that \( X \) is a finite set, since there exists a finite subset \( Y \) of \( X \) such that \( (M, N, Y) \) satisfies the conditions (i) - (iii) in Theorem 3.1. Replacing \( h_t \) by \( h^{-1}_t h_t \), we may assume that \( h_1 = id_M \).
Figure 3.

(I) The case where \( M \) is compact: Let \( N_1, \ldots, N_p \) be the components of \( N \) and \( L = \partial(M \setminus N) \). Let \( K_1, \ldots, K_p \) be the components of \( L \) which are disks or Möbius bands and let \( L_1, \ldots, L_q \) be the remaining components. For each \( j \) we can write

\[
\partial L_j = \left( \bigcup_{i=1}^{k(j)} A_i^j \right) \cup \left( \bigcup_{i=1}^{l(j)} B_i^j \right) \cup \left( \bigcup_{i=1}^{m(j)} C_i^j \right),
\]

where \( A_i^j \)'s are the circle components of \( \text{Fr} L_j \), \( B_i^j \)'s are the components of \( \partial L_j \) which contain some arc components of \( \text{Fr} L_j \) and \( C_i^j \)'s are the remaining components of \( \partial L_j \). We choose disjoint collars \( E_i^j \) of \( A_i^j \) and \( F_i^j \) of \( B_i^j \) in \( L_j \) and set \( \hat{A}_i^j = \partial E_i^j \setminus A_i^j \), \( \hat{B}_i^j = \partial F_i^j \setminus B_i^j \) and

\[
L_j' = \text{cl}(L_j \setminus (\bigcup_{i=1}^{k(j)} E_i^j) \cup (\bigcup_{i=1}^{l(j)} F_i^j)), \quad N' = N \cup (\bigcup_{k=1}^{p} K_k) \cup \left( \bigcup_{j=1}^{q} \left( \bigcup_{i=1}^{k(j)} \hat{A}_i^j \right) \cup \left( \bigcup_{i=1}^{l(j)} \hat{B}_i^j \right) \right).
\]

Note that \( \text{Fr} N' = \bigcup_{j=1}^{q} \left( \bigcup_{i=1}^{k(j)} \hat{A}_i^j \right) \cup \left( \bigcup_{i=1}^{l(j)} \hat{B}_i^j \right) \subset \text{Int} M \). Since \( \mathcal{H}_0(\mathbb{D}) \simeq \mathcal{H}_0(\mathbb{M}) \simeq * \) by Fact 3.1.(5-i), we can isotope \( h_0 \mid \text{rel } N \) to an \( h' \in \mathcal{H}_{N'}(M) \).

By the construction \((M,N',X,h')\) satisfies the following conditions:

1. \( N' \) is a 2-submanifold of \( M \), every component of \( \text{Fr} N' \) is a circle and \( X \subset \text{Int}_M N' \).
2. \( h'|_{N'} = id_{N'} \) and \( h' \) is isotopic to \( id_M \) rel \( X \).
3. Suppose \( C \) is a component of \( \text{Fr} N' \). If \( C \) bounds a disk \( D \) then \( \#(D \cap X) \geq 2 \), and if \( C \) bounds a Möbius band \( E \) then \( E \cap X \neq \emptyset \).
4. If \( H \) is an annulus component of \( N' \) then \( H \cap X \neq \emptyset \).

To see (3) first note that \( M \) is the union of compact 2-manifolds \( N_i \)'s, \( E_i^j \)'s, \( F_i^j \)'s, \( K_k \)'s and \( L_i' \)'s, which have disjoint interiors. Suppose \( G \) is a compact connected 2-manifold in \( M \) with \( \partial G \subset \text{Fr} N' \). Since \( G \subset \text{Int} M \) and each \( F_i^j \) meets \( \partial M \), it follows that \( G \) is the union of \( N_i \)'s, \( E_i^j \)'s, \( L_i' \)'s and \( K_k \)'s contained in \( G \). Since \( E_i^j \) is an annulus and \( L_i' \cong L_j \) is not a disk or a Möbius band, from Fact 3.3 it follows that (i) if \( G \) is a disk, then \( G \) contains a disk which is some \( N_i \) or \( K_k \) with \( K_k \subset G \subset \text{Int} M \), so \( \#(G \cap X) \geq 2 \), (ii) if \( G \) is a Möbius band, then \( G \) contains a disk or a Möbius band which is some \( N_i \) or \( K_k \) with \( K_k \subset G \subset \text{Int} M \), so \( G \cap X \neq \emptyset \).

As for (4), \( H \) is the union of \( N_i \)'s \( E_i^j \)'s, \( F_i^j \)'s and \( K_k \)'s contained in \( H \), and \( H \) contains at least one \( N_i \), which is a disk with \( r \) holes. If \( r \leq 1 \) then by the assumption \( N_i \cap X \neq \emptyset \). If \( r \geq 2 \) then we can find a disk \( D \) in \( \text{Int} H \) such that \( D \cap N_i = \partial D \subset \partial N_i \) (Fact 3.3.(iii)). Since \( D \subset \text{Int} N' \subset \text{Int} M \), \( D \) is a union of \( N_i \)'s and \( K_k \)'s and we can conclude that it coincides with some \( K_k \subset \text{Int} M \), which meets \( X \). These imply (4).

It remains to show that \( h' \) is isotopic to \( id_M \) rel \( N' \) under the conditions (1) - (4). (i) When \( X \subset \text{Int} N' \), we can apply Lemma 3.4 to the triple \((M \setminus X, N' \setminus X, h'|_{M \setminus X})\). To verify the condition (iii) in Lemma 3.4, note that (a) each component of \( N' \setminus X \) takes of the form \( H \setminus X \) for some component of \( H \) of \( N' \), and, in particular, (b) if \( H \setminus X \cong S^1 \times [0,1) \), then \( H \) is a disk and \( \#H \cap X \geq 2 \), a
and $Y$ then $Fr_H H_{3.1}$ when $id_L$ and consider $(\tilde{M} = M \cup C \times [0,1], \tilde{N} = N' \cup C \times [0,1], X, \tilde{h})$, where $\tilde{h}$ is the extension of $h'$ by $id_{C \times [0,1]}$. Then (a) $X \subset Int \tilde{N}$ and $(\tilde{M}, \tilde{N}, X, \tilde{h})$ satisfies (1) - (4), and (b) an isotopy of $\tilde{h}$ to $id_{\tilde{M}}$ rel $\tilde{N}$ restricts to an isotopy of $h'$ to $id_{M}$ rel $N'$. (Alternatively, we can modify the isotopy of $h'$ to $id_{M}$ rel $X$ to an isotopy rel $X \cup V$, where $V$ is a neighborhood of $X \cap \partial M$ in $M$. We can replace $X$ so that $X \subset Int N'$.) This completes the proof of the case (I).

(II) The case where $M$ is noncompact: Choose a compact connected 2-submanifolds $L_0$ and $L$ of $M$ such that $h_t(N) \subset Int_M L_0$ ($0 \leq t \leq 1$) and $L_0 \subset Int_M L$. Let $N_1 = N \cup cl(L \setminus L_0)$. Since $N_1$ is a subpolyhedron of $L$ with respect to some triangulation of $L$ (cf. [2]), by Corollary 2.1 we have the principal bundle : $\mathcal{H}(L)_0 \to \mathcal{E}(N_1, L)_0^*$. Let $f_t \in \mathcal{H}(L)_0$, $f_1 = id_L$, be any lift (= extension) of the path $e_t \in \mathcal{E}(N_1, L)_0^*$, defined by $e_t|N = h_t|N$ and $e_t = id$ on $cl(L \setminus L_0)$.

We can apply the case (I) to $(L, N_1, X_1, f_1)$, $X_1 = X \cup cl(L \setminus L_0)$. For the condition (iii) in Theorem 3.1, when $E$ is a component of $cl(L \setminus N_1) = cl(L_0 \setminus N)$, (a) if $E \cap Fr L_0 = \emptyset$, then $E$ is a component of $cl(M \setminus N)$ and (b) if $E \cap Fr L_0 \neq \emptyset$, then $E$ contains a component of Fr$L_0$ and Fr$L_0 \subset cl(L \setminus L_0) \subset X_1$ (it also follows that Fr$L E$ is not connected since $E \cap Fr N \neq \emptyset$, so if $E$ is a disk or a Möbius band, then Fr$L E$ is a disjoint union of arcs).

Therefore we have an isotopy $k_t : L \to L$ rel $N_1$ such that $k_0 = f_0$, $k_1 = id_L$. We can extend $f_t$ and $k_t$ to $M$ by $id$. The required isotopy $h'_t$ is defined by $h'_t = k_t f_t^{-1} h_t$. \hfill $\Box$

**Proof of Corollary 3.1.** Let $G_1$ denote the unit path-component of a topological group $G$. Theorem 3.1 implies $\mathcal{H}_N(M) \cap \mathcal{H}_X(M)_1 = \mathcal{H}_N(M)_1$. When $M$ is compact, from Fact 2.1 it follows that $\mathcal{H}_K(M)_0 = \mathcal{H}_K(M)_1$ for any compact subpolyhedron $K$ of $M$. Since $X$ can be replaced by a finite subset $Y$ of $X$ as in the above proof, we have $\mathcal{H}_N(M) \cap \mathcal{H}_X(M)_0 \subset \mathcal{H}_N(M) \cap \mathcal{H}_Y(M)_0 = \mathcal{H}_N(M)_0$. The noncompact case follows from the same argument when we will show that $\mathcal{H}_K(M)_0$ is an ANR (Propositions 4.1,4.2) in the next section.

4. **The homotopy types of the identity components of homeomorphism groups of noncompact 2-manifolds**

In this final section we will prove Theorem 1.1 and Corollary 1.1. Below we assume that $M$ is a noncompact connected PL 2-manifold and $X$ is a compact subpolyhedron of $M$. We set $M_0 = X$ and write as $M = \cup_{i=0}^{\infty} M_i$, where for each $i \geq 1$ (a) $M_i$ is a nonempty compact connected PL 2-submanifold of $M$ and $M_{i-1} \subset Int_M M_i$, (b) for each component $L$ of $cl(M \setminus M_i)$, $L$ is noncompact and $L \cap M_{i+1}$ is connected and (c) $M_1 \cap \partial M \neq \emptyset$ if $\partial M \neq \emptyset$. Taking a subsequence, we have the following cases:

(i) each $M_i$ is a disk, (ii) each $M_i$ is an annulus, (iii) each $M_i$ is a Möbius band, and (iv) each $M_i$ is not a disk, an annulus or a Möbius band.

In (ii) the inclusion $M_i \subset M_{i+1}$ is essential, otherwise a boundary circle of $M_i$ bounds a disk component in $cl(M \setminus M_i)$, and it contradicts the condition (b).
\textbf{Proposition 4.1.} In the cases (i) – (iii) it follows that

(i) \quad (a) \ \partial M = \emptyset \implies M \cong \mathbb{R}^2, \\
(b) \ \partial M \neq \emptyset \implies M \cong \partial \mathbb{D} \setminus A, \text{ where } A \text{ is a nonempty 0-dimensional compact subset of } \partial \mathbb{D};

(ii) \quad (a) \ \partial M = \emptyset \implies M \cong S^1 \times \mathbb{R}, \\
(b) \ \partial M \neq \emptyset \implies \\
(b)_1 \ M \cong S^1 \times [0,1), \\
(b)_2 \ M \cong S^1 \times [0,1) \setminus A, \text{ where } A \text{ is a nonempty 0-dimensional compact subset of } S^1 \times \{0\}, \\
(b)_3 \ M \cong S^1 \times [0,1) \setminus A, \text{ where } A \text{ is a nonempty 0-dimensional compact subset of } S^1 \times \{0,1\},

(iii) \quad (a) \ \partial M = \emptyset \implies M \cong \mathbb{P}^2 \setminus 1pt, \\
(b) \ \partial M \neq \emptyset \implies M \cong M \setminus A, \text{ where } A \text{ is a nonempty 0-dimensional compact subset of } \partial M.

In the case (ii)(b)_3 we may further assume that \( M_1 \) meets both \( S^1 \times \{0\} \) and \( S^1 \times \{1\} \).
We choose a metric \( d \) on \( M \) with \( d \leq 1 \) and metrize \( \mathcal{H}_X(M) \) by the metric \( \rho \) defined by

\[
\rho(f,g) = \sum_{i=1}^{\infty} \frac{1}{2^i} \sup_{x \in M_i} d(f(x), g(x)).
\]

We separate the following two cases:

(I) \quad (M,X) \cong (\mathbb{R}^2, \emptyset), (\mathbb{R}^2, 1pt), (S^1 \times \mathbb{R}, \emptyset), (S^1 \times [0,1), \emptyset), (\mathbb{P}^2 \setminus 1pt, \emptyset).

(II) \quad (M,X) \text{ is not Case (I)}.

\textbf{Case (II):} First we treat Case (II) and prove the following statements:

\textbf{Proposition 4.1.} In Case (II), we have (1) \( \mathcal{H}_X(M)_0 \cong * \) and (2) \( \mathcal{H}_X(M)_0 \) \text{ is an ANR.}

\textbf{Proof.} We use the following notation: For each \( j \geq 1 \) let \( U_j = \text{Int}_M M_j \) and \( L_j = \text{Fr}_M M_j \), and for each \( j > i \geq k \geq 0 \) let \( \mathcal{H}_{k,j} = \mathcal{H}_{M_{k\cup \cup_M U_j}}(M)_0, \mathcal{U}_{k,j} = \mathcal{E}_{M_{k\cup \cup_M U_j}}(M)_0 \) and let \( \pi_{k,j} : \mathcal{H}_{k,j} \rightarrow \mathcal{U}_{k,j} \) denote the restriction map, \( \pi_{k,j}(h) = h|_{M_i} \).

\textbf{Lemma 4.2.} (1) \( \mathcal{H}_{k,j} \cong \mathcal{H}_{M_{k\cup \cup_M L_j}}(M_j)_0 \) \text{ is an AR}.

(2) \ The map \( \pi_{k,j} : \mathcal{H}_{k,j} \rightarrow \mathcal{U}_{k,j} \) is a principal bundle with the structure group \( \mathcal{H}_{k,j} \cap \mathcal{H}_{M_j}(M) \cong \mathcal{G}_{k,j} \equiv \mathcal{H}_{M_{k\cup \cup_M L_j}}(M_j)_0 \cap \mathcal{H}_{M_{k\cup \cup_M L_j}}(M_j) \) \text{ under the restriction map}.

(3) \ \mathcal{U}_{k,j} \text{ is an open subset of } \mathcal{E}_{M_k}(M_i,M)_0, \text{ cl}\mathcal{U}_{k,j} \subset \mathcal{U}_{k,j+1} \text{ and } \mathcal{E}_{M_k}(M_i,M)_0 = \cup_{j>i} \mathcal{U}_{k,j}.

\textbf{Proof.} The statement (1) follows from Fact 2.1 and Lemma 2.1.(ii), and (2) follows from Corollary 2.1. For (3), note that \( \mathcal{E}_{M_k}(M_i,M)_0 \) \text{ is path connected (Proposition 2.1) and each } f \in \mathcal{E}_{M_k}(M_i,M)_0 \text{ is isotopic to the inclusion } M_i \subset M \text{ in a compact subset of } M. \hfill \Box

\textbf{Lemma 4.3.} In Case (II), for each \( j > i > k \geq 0 \), (a) \( \mathcal{G}_{k,j} \) \text{ is an AR}, (b) \text{ the restriction map } \pi_{k,j} : \mathcal{H}_{k,j} \rightarrow \mathcal{U}_{k,j} \text{ is a trivial bundle} \text{ and (c) } \mathcal{U}_{k,j} \text{ is also an AR}.

\textbf{Proof.} Once we show that (a) \( \mathcal{G}_{k,j} = \mathcal{H}_{M_{k\cup \cup_M L_j}}(M_j)_0 \), then (a) the fiber \( \mathcal{G}_{k,j} \) is an AR by Fact 2.1 and Lemma 2.1.(ii), so (b) the principal bundle has a global section and it is trivial and (c) follows from Lemma 4.2.(1). It remains to prove (a).
(1) The cases (i)(a), (ii)(a), (iii)(a), (ii)(b) and (iv) (under the condition (II)): We can apply Theorem 3.1 to \((\widetilde{M}_j = M_j \cup L_j \times [0, 1], \tilde{M}_i = M_i \cup L_i \times [0, 1], \tilde{M}_k = M_k \cup L_k \times [0, 1])\). We can verify the conditions (ii) and (iii) in Theorem 3.1 as follows: (ii) By the assumption \((M_i, X) \not\approx (\mathbb{D}, \emptyset), (\mathbb{D}, \text{1pt}), (\mathbb{S}^1 \times [0, 1], \emptyset), (\mathbb{M}, \emptyset)\) for each \(i \geq 1\). (iii) If \(H\) is a component of \(cl_{\widetilde{M}_j} (\widetilde{M}_j \setminus \widetilde{M}_i) = cl_M (M_j \setminus M_i)\), then \(H\) contains a component of \(L_j\). (Also, \(H\) meets both \(M_i\) and \(L_j\) if \(H \cap L_j = \emptyset\) then \(H\) is a compact component of \(cl(M \setminus M_i)\), a contradiction.). So \(Fr_{\widetilde{M}_j}\) is not connected. Hence if \(H\) is a disk or a Möbius band, then \(Fr_{\widetilde{M}_j}\) is a disjoint union of arcs.) By Corollary 3.1 (Compact case) it follows that \(\mathcal{H}_{\widetilde{M}_k} (\widetilde{M}_j) \cap \mathcal{H}_{\widetilde{M}_l} (\widetilde{M}_j) = \mathcal{H}_{\widetilde{M}_l} (\widetilde{M}_j)\) and this implies (*).

(2) The cases (i)(b), (ii)(b) and (iii)(b): Since \(M_1 \cap \partial M \neq \emptyset\) and \(M_1\) meets both \(\mathbb{S}^1 \times \{0\}\) and \(\mathbb{S}^1 \times \{1\}\) in the case (ii)(b)3, it follows that \(cl(M_j \setminus M_i)\) is a disjoint union of disks, thus \(\mathcal{H}_{\mathcal{M}_i \cup L_j} (M_j) = \mathcal{H}_{\mathcal{M}_i \cup L_j} (M_j)\) by Fact 3.1.(5-i). This implies (*).

(3) The remaining case (ii)(b)2: It follows that (a) \(cl(M_j \setminus M_i)\) is a disjoint union of disks \(D_k\) and an annulus \(L\) and (b) \(D_k \cap M_i\) is an arc, \(D_k \cap L_j\) is a disjoint union of arcs (\(\neq \emptyset\)) and \(H \cap M_i, H \cap L_j\) are the boundary circles of \(H\). Since \(N = M_i \cup (\cup_k D_k)\) is an annulus and \(N \cap L_j \neq \emptyset\), from Theorem 3.1 it follows that \(\mathcal{H}_{\mathcal{M}_i \cup L_j} (M_j) \cap \mathcal{H}_{N \cup L_j} (M_j) = \mathcal{H}_{N \cup L_j} (M_j)\). Each \(f \in \mathcal{G}_{k,j}^i\) is isotopic rel \(M_i \cup L_j\) to \(f' \in \mathcal{H}_{N \cup L_j} (M_j)\). Since \(f'\) is isotopic to id rel \(M_k \cup L_j\), it follows that \(f' \in \mathcal{H}_{N \cup L_j} (M_j)\) and so \(f \in \mathcal{H}_{\mathcal{M}_i \cup L_j} (M_j)\). This completes the proof.

\[\square\]

**Lemma 4.4.** In Case (II), for each \(i \geq k \geq 0\),

(a) \(\mathcal{E}_{\mathcal{M}_k} (M_i, M)\) is an AR,

(b) the restriction map \(\pi : \mathcal{H}_{\mathcal{M}_k} (M_i) \rightarrow \mathcal{E}_{\mathcal{M}_k} (M_i, M)^\ast\) is a trivial principal bundle with fiber \(\mathcal{H}_{\mathcal{M}_i} (M)_0\),

(c) \(\mathcal{H}_{\mathcal{M}_k} (M)\) strongly deformation retracts onto \(\mathcal{H}_{\mathcal{M}_i} (M)_0\).

**Proof.** By Lemma 4.3(c) each \(U_{k,j}^i\) (\(j > i\)) is an AR. Thus by Fact 4.2(3) \(\mathcal{E}_{\mathcal{M}_k} (M_i, M)_0\) is also an AR and it strongly deformation retracts onto the single point set \(\{M_i \subset M\}\). Hence the principal bundle

\[\mathcal{G}_{k}^i = \mathcal{H}_{\mathcal{M}_k} (M) \cap \mathcal{H}_{\mathcal{M}_i} (M) \subset \mathcal{H}_{\mathcal{M}_k} (M)_0 \rightarrow \mathcal{E}_{\mathcal{M}_k} (M_i, M)^\ast\]

is trivial and \(\mathcal{H}_{\mathcal{M}_k} (M)_0\) strongly deformation retracts onto the fiber \(\mathcal{G}_{k}^i\). In particular, \(\mathcal{G}_{k}^i\) is connected and \(\mathcal{G}_{k}^i = \mathcal{H}_{\mathcal{M}_i} (M)_0\).

\[\square\]

**Proof of Proposition 4.1.(1).** By Lemma 4.4(c), for each \(i \geq 0\) there exists a strong deformation retraction \(h_i^t (0 \leq t \leq 1)\) of \(\mathcal{H}_{\mathcal{M}_i} (M)_0\) onto \(\mathcal{H}_{\mathcal{M}_{i+1}} (M)_0\). A strong deformation retraction \(h_i (0 \leq t \leq \infty)\) of \(\mathcal{H}_{\mathcal{X}} (M)_0\) onto \(\{id_M\}\) is defined as follows:

\[h_i(f) = h_{t-i}^1 \cdots h_0^1(f) \quad (f \in \mathcal{H}_{\mathcal{X}} (M)_0, \ i \geq 0, \ i \leq t \leq i + 1)\]

\[h_\infty(f) = id_M.\]

Since \(\text{diam} \mathcal{H}_{\mathcal{M}_i} (M)_0 \leq 1/2^i \rightarrow 0\), the map \(h : \mathcal{H}_{\mathcal{X}} (M)_0 \times [0, \infty) \rightarrow \mathcal{H}_{\mathcal{X}} (M)_0\) is continuous.
(In the cases (i), (ii) and (iii), the same conclusion follows from Lemma 2.1.(i)(ii) by taking the end compactification of \( M \).

For the proof of Proposition 4.1.(2), we will apply Hanner’s criterion of ANRs:

**Fact 4.1.** ([\( \mathbf{1} \)]) A metric space \( X \) is an ANR iff for any \( \varepsilon > 0 \) there is an ANR \( Y \) and maps \( f : X \to Y \) and \( g : Y \to X \) such that \( gf \) is \( \varepsilon \)-homotopic to \( id_X \).

**Proof of Proposition 4.1.(2).** By Lemma 4.4.(b) and Proposition 4.1.(1) for each \( i \geq 1 \) we have the trivial principal bundle
\[
\mathcal{H}_{M_i}(M)_0 \subset \mathcal{H}_X(M)_0 \xrightarrow{\pi} \mathcal{E}_X(M_i, M)_0^* \quad \text{with} \quad \mathcal{H}_{M_i}(M)_0 \simeq *. 
\]
It follows that \( \pi \) admits a section \( s \), and the map \( s\pi \) is fiber preserving homotopic to \( id_{\mathcal{H}_X(M)_0} \) over \( \mathcal{E}_X(M_i, M)_0^* \). Since each fiber of \( \pi \) has diam \( \leq 1/2^i \), this homotopy is a \( 1/2^i \)-homotopy. Since \( \mathcal{E}_X(M_i, M)_0^* \) is an ANR (Proposition 2.1), by Fact 4.1 \( \mathcal{H}_X(M)_0 \) is also an ANR.

**Case (I):** The next statements follow from Lemma 2.1.(iii) and Fact 2.1 by taking the end compactification of \( M \).

**Proposition 4.2.** In Case (I), we have (1) \( \mathcal{H}_X(M)_0 \simeq \mathbb{S}^1 \) and (2) \( \mathcal{H}_X(M)_0 \) is an ANR.

Theorem 1.1 follows from Propositions 4.1, 4.2, and Corollary 1.1 now follows from the following characterization of \( \ell_2 \)-manifold topological groups.

**Fact 4.2.** ([\( \mathbf{2} \)]) A topological group is an \( \ell_2 \)-manifold iff it is a separable, non locally compact, completely metrizable ANR.

**Proof of Corollary 1.1.** Since \( M \) is locally compact and locally connected, \( \mathcal{H}(M) \) is a topological group and \( \mathcal{H}_X(M) \) is a closed subgroup of \( \mathcal{H}(M) \). Since \( M \) is locally compact and second countable, \( \mathcal{H}(M) \) is also second countable. A complete metric \( \rho \) on \( \mathcal{H}(M) \) is defined by
\[
\rho(f, g) = d_\infty(f, g) + d_\infty(f^{-1}, g^{-1}), \quad d_\infty(f, g) = \sum_{n=1}^{\infty} \frac{1}{2^n} \sup_{x \in M_n} d(f(x), g(x))
\]
for \( f, g \in \mathcal{H}(M) \), where \( d \) is a complete metric on \( M \) with \( d \leq 1 \). Since \( \mathcal{H}_X(M)_0 \simeq \mathcal{H}_X(M)_0 \times s [\mathbf{3}] \), \( \mathcal{H}_X(M)_0 \) is not locally compact. Finally, by Propositions 4.1, 4.2 \( \mathcal{H}_X(M)_0 \) is an ANR.

This completes the proof.

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