Noise in an insect outbreak model

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We study the steady state properties of an insect (spruce budworm) outbreak model in the presence of Gaussian white noise. Based on the corresponding Fokker-Planck equation the steady state solution of the probability distribution function and its extrema have been investigated. It was found that fluctuations of the insect birth rate reduces the population of the insects while fluctuations of predation rate and the noise correlation can prevent the population of the insects from going into extinction. Noise in the model can induce a phase transition.

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I. INTRODUCTION

Recently, Nonlinear dynamical systems with random noise have been paid much attention both theoretically and experimentally. Phenomena such as noise-induced transitions, stochastic resonance, resonant activation, noise-induced spatial patterns are a few examples of extensive investigations [1] [2] [3]. In most of these areas the noise affects the dynamics through system variables, i.e., the noise is multiplicative in nature [4,5]. The focal theme of these types investigation is the steady state properties of systems in which the fluctuations, generally applied from outside, are considered, independent of the system’s characteristic dissipation. On the level of a Langevin-type description of a dynamical system, the presence of noise can change the dynamics of the system [6,7]. Noise processes have found applications in a broad range of studies, such as the steady state properties of a single mode laser [8], bistable kinetics [9], directed motion in spatially symmetric periodic potentials [10], stochastic resonance in linear systems [11], and steady state entropy production [12]. In this paper we study an insect outbreak model in the presence of the correlated noise, and show how noise can dynamically affect the ecosystem.
II. AN INSECT OUTBREAK MODEL: SPRUCE BUDWORM

A practical model which exhibits two positive linearly steady state populations is that for the spruce budworm, which can, with ferocious efficiency, defoliate the balsam fir. They are major problem in Canada [13], Ludwing [14] (1978) considered the budworm population dynamics to be governed by the equation

\[
\frac{dN}{dt} = r_B N(1 - \frac{N}{K_B}) - P(N). \tag{1}
\]

Here \(r_B\) is the linear birth rate of the budworm and \(K_B\) is carrying capacity which is related to the density of the foliage available on trees. The \(P(N)\) term represents predation, generated by birds. We take the form for \(P(N)\) suggested by Ludwing (1978), namely \(BN^2/(A^2 + N^2)\), where \(A\) is a positive constant and \(B\) represents the predation rate of the birds. The dynamics of \(N(t)\) is then governed by

\[
\frac{dN}{dt} = r_B N(1 - \frac{N}{K_B}) - \frac{BN^2}{A^2 + N^2}. \tag{2}
\]

Before analysing the model we will express it in nondimensional terms. Here we introduce nondimensional quantities by

\[
x = \frac{N}{A}, \quad r = Ar_B, \quad q = \frac{K_B}{A}, \quad \tau = \frac{t}{A}, \quad \beta = B. \tag{3}
\]

Upon on substitution (2) becomes

\[
\frac{dx}{d\tau} = rx(1 - \frac{x}{q}) - \frac{\beta x^2}{1 + x^2}. \tag{4}
\]

Now, if some environmental external disturbances make both the birth rate and the predation rate fluctuate, it is likely to affect \(r\) and \(\beta\) in the form of multiplicative noises that are connected through a correlation parameter \(\lambda\). As a result we have

\[
\frac{dx}{d\tau} = rx(1 - \frac{x}{q}) - \frac{\beta x^2}{1 + x^2} + x(1 - \frac{x}{q})\Gamma(t) - \frac{x^2}{1 + x^2}\xi(t). \tag{5}
\]

Where \(\Gamma(t)\) and \(\xi(t)\)are Gaussian white noises with the following properties:

\[
\langle \Gamma(t) \rangle = \langle \xi(t) \rangle = 0, \tag{6}
\]

\[
\langle \Gamma(t)\Gamma(s) \rangle = 2D\delta(t - s), \tag{7}
\]
\[ \langle \xi(t) \xi(s) \rangle = 2\sigma\delta(t - s), \quad (8) \]

\[ \langle \xi(t) \Gamma(s) \rangle = \langle \Gamma(t) \xi(s) \rangle = 2\lambda\sqrt{D}\sigma\delta(t - s). \quad (9) \]

Where \( D \) and \( \sigma \) are the strength of noise \( \Gamma(t) \) and \( \xi(t) \), respectively, and \( \lambda \) denotes the degree of correlation between \( \Gamma(t) \) and \( \xi(t) \) with \( 0 \leq \lambda \leq 1 \).

We can derive the corresponding Fokker-planck equation for evolution of Steady Probability Distribution (SPDF) based on Eq. (5)-Eq.(9). The equation is as follows [15]

\[ \frac{\partial P(x, t)}{\partial t} = -\frac{\partial A(x)P(x, t)}{\partial x} + \frac{\partial^2 B(x)P(x, t)}{\partial x^2}, \quad (10) \]

where

\[ A(x) = h(x) + Dg_1(x)g_1'(x) + \lambda\sqrt{D}\sigma g_1(x)g_2(x) + \lambda\sqrt{D}\sigma g_1'(x)g_2(x) + \sigma g_2(x)g_2'(x), \quad (11) \]

\[ B(x) = Dg_1^2(x) + 2\lambda\sqrt{D}\sigma g_1(x)g_2(x) + \sigma g_2^2(x). \quad (12) \]

Here \( h(x) = rx(1 - \frac{x}{q}) - \frac{\beta x^2}{1 + x^2}, \quad g_1(x) = x(1 - x/q), \quad g_2(x) = -\frac{x^2}{1 + x^2} \). The steady probability distribution of the Fokker-Planck equation is given by [15]

\[ \text{P}_{st}(x) = \frac{N_0}{B(x)} \exp\left[\int x A(x') B(x') dx'\right]. \quad (13) \]

Where \( N_0 \) is the normalization constant.

### III. THE FLUCTUATION OF BIRTH RATE IN THE MODEL

If we only consider the fluctuation of birth rate on the model, namely \( \sigma = 0 \) and \( \lambda = 0 \), we can get

\[ A(x) = rx(1 - \frac{x}{q}) - \frac{\beta x^2}{1 + x^2} + Dx(1 - \frac{x}{q})(1 - \frac{2x}{q}), \quad (14) \]

\[ B(x) = D[x(1 - \frac{x}{q})]^2. \quad (15) \]

From Eq. (13), using the forms of \( A(x) \) and \( B(x) \), we get the following integral forms of the SPDF [16].

\[ \text{P}_{st}(x) = \frac{N_0}{|g_1(x)|} \exp\left[\frac{f(x)}{D}\right]. \quad (16) \]

Here
\[ g_1(x) = x(1 - \frac{x}{q}), \]  
\[ (17) \]

\[ f(x) = r \ln \left| \frac{x}{x-q} \right| + \frac{\beta q^2}{1+q^2} (\arctan x + \frac{1}{x-q}) + \frac{2\beta q^3}{(1+q^2)^2} (\ln |\frac{x-q}{\sqrt{1+x^2}}| - q \arctan x). \]
\[ (18) \]

The extrema of the SPDF are calculated using the condition \( A(x) - B'(x) = 0 \)

\[ r(1 - \frac{x}{q}) - \frac{\beta}{1+x^2} - D(1 - \frac{x}{q})(1 - \frac{2x}{q}) = 0. \]
\[ (19) \]

As for the parameters of the above equations we adopt \( r = 1.0, q = 10.0, \beta = 2.0 \), The results are represented in Fig.1-Fig.2

![Plot of the extrema of SPDF as a function of x for different noise strength values: D = 0.0, 0.1, 0.3, 0.7, using \( \beta = 2.0 \), and \( q = 10.0 \).](image)

**FIG. 1.** Plot of the extrema of SPDF as a function of \( x \) for different noise strength values: \( D = 0.0, 0.1, 0.3, 0.7 \), using \( \beta = 2.0 \), and \( q = 10.0 \).

In Fig.1, for zero noise strength, the curve at \( r = 1.0 \) gives only one \( x \) value, but as the value of \( D \) increases, the curve changes. The curve shows three \( x \) values at \( r = 1.0 \). From the figure we can see that the noise strength can change the state of the system.

In Fig.2 we show the effect of the noise strength \( D \) on the SPDF. For a small value of \( D \), the SPDF shows a single peak region, which changes for larger values of \( D \) (see Fig.2). As the value of \( D \) increases, the peak for the larger values of \( x \) decreases. At the same time, for small values of \( x \), a new peak appears. It can be said that the noise causes the system become from two states to one state, namely noise can induce a phase transition. On the other hand, since \( x, D \) give the relative budworm population and the fluctuation of the budworm’s birth rate, respectively, we can see from the figure the effect...
the fluctuation of the birth rate makes on the growth of budworms, they can even make
the budworms go into extinction.

FIG. 2. Plot of $P_{st}(x)$ (denotes the probability) against $x$ for different noise strength values:
$D = 0.04, 0.1, 0.3, 0.6$, using $r = 1.0, \beta = 2.0$, and $q = 10.0$.

IV. THE FLUCTUATIONS OF THE PREDATION RATE IN THE MODEL

On the other hand, if only the fluctuation of predation rate is investigated, namely
$D = 0$ and $\lambda = 0$ we can get the stationary probability distribution function similar to
Eq.(14)

$$P_{st}(x) = \frac{N_0}{|g_2(x)|} \exp\left[\frac{f(x)}{\sigma}\right],$$

(20)

Where

$$g_2(x) = \frac{x^2}{1 + x^2},$$

(21)

$$f(x) = -\frac{r}{3q}x^3 + \frac{r}{2}x^2 - \left(\beta + \frac{2r}{q}\right)x + 2r \ln x + \left(\beta + \frac{r}{q}\right)x^{-1} - \frac{r}{2}x^{-2}.$$  

(22)

In order to discuss the effect of the fluctuation of the predation rate on the system we
adopt $r = 1.0, q = 10.0, \beta = 2.26$, the results are shown in Fig.3.

Fig. 3 shows the effect of the noise strength $\sigma$ on the SPDF. For a small value of $\sigma$,
the SPDF shows the typical bistable region (see Fig.3) which vanishes for large values of
σ. As the value of σ increases the peak at the small x value decreases drastically, while for a large σ value the curve has a single peak at the large x value. Since x denotes the budworm relative population, it is clear from Fig. 3 that, with an increase in the fluctuations of the predation rate, the budworm recover from going into extinction.

FIG. 3. Plot of $P_{st}(x)$ (denotes the probability) against x for different noise strength values: $\sigma = 0.04, 0.10, 0.70$ using $r = 1.0, \beta = 2.26, \text{and } q = 10.0$.

V. EFFECT OF NOISE CORRELATION IN THE SYSTEM

Since the two type of fluctuations have the common origin, (environmental external disturbance), we will consider the effect of the correlation between the birth rate fluctuation and the predation rate fluctuation. Based on Eq. (10)-Eq. (13) the stationary probabilities distribution function is represented by Fig. 4 with $r = 1.0, \beta = 2.0, D = 0.3, \sigma = 0.3$.

Fig. 4 gives the effect of noise correlation parameter $\lambda$ on SPDF. For a small $\lambda$ value the SPDF shows a typical bistable region which will vanish for the large value of $\lambda$. As the correlation parameter increases the height of the peak on the small x value decreases, while the height of the peak on the large x value increases, namely probabilities flow from the small x value to the large x value. Since x denotes the budworm relative population, it is clear from Fig. 4 that the correlation of the noises is a advantageous factor for the growth of the budworm.
FIG. 4. Plot of $P_{st}(x)$ (denotes the probability) against $x$ for different correlation parameter values: $\lambda = 0, 0.4, 0.70, 1.0$ using $r = 1.0, \beta = 2.0, q = 10.0, D = 0.3$, and $\sigma = 0.3$.

VI. SUMMARY

In this paper, we studied the steady state properties of an insect outbreak model in the presence of the correlated noise. Birth rate fluctuation, predation rate fluctuation and the correlation of the noises are investigated. Birth rate fluctuation work against the growth of the budworm population, while both fluctuation of the predation rate and the correlation of the noises can prevent the budworm population from going into extinction. On the other hand, noise can dynamically affect the ecosystem. The noise from the birth rate fluctuation can make the system change from a single steady state to a bistable state, while noise from the predation rate fluctuation induces the system change from a bistable state to a single steady state. Thus in the insect outbreak model, noise from the fluctuation of the parameters can induce a phase transition. This viewpoint is completely novel in the traditional viewpoint it is the stochastic force that disturbs the phase transition.

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