Aeroelastic and sloshing stability of slender hypersonic vehicle

Haiyan Qiao¹,², Ya Yang³, Hao Meng¹, Wei Ke³ and Jiatao Xu²

¹College of Automation, Harbin Engineering University, Harbin 150001, China
²Department of Aerospace Engineering, Harbin Engineering University, Harbin 150001, China
³Hebei Hanguang Industry Co., Ltd., Handan 056000, China

⁴E-mail: yangya@hrbeu.edu.cn

Abstract. This paper focuses on the problem of the coupling between the rigid body mode and elastic mode caused by the light structure design and slender geometry of the hypersonic vehicle, and the influence of the liquid sloshing on the aircraft. In order to analyze the complex dynamics of hypersonic vehicle, the nonlinear dynamic model of a flexible vehicle is established in the case of the vehicle's elastic deformation. The yields terms that produce nonlinear coupling between the elastic deflections, liquid sloshing and the rigid-body motion parameters are obtained by Lagrangian approach. Then, the equations are reduced to a linear set for stability analysis. Finally, some numerical simulations are given that reveal a destabilizing influence of elastic deflections and liquid sloshing at stability for a particular vehicle configuration.

1. Introduction

Compared with conventional aircraft, the structure of the hypersonic vehicle is generally used as light flexible material, and its aerodynamic shape is generally slender. The special structure and aerodynamic configuration make the natural vibration frequency of aircraft structure reduced, and hypersonic velocity makes the coupling problem of rigid body mode and elastic mode is more prominent. At the same time, under the condition of high speed flight, the liquid sloshing has a more significant impact on the aircraft. This is required in the hypersonic vehicle dynamics modeling process, the aircraft as an elastic body, rather than a rigid body, but also consider the liquid sloshing.

In view of the above problems, the kinetic energy and potential energy of the aircraft are derived by Doman, Bolender et al. [1,2], and the motion equation of the elastic vehicle is established by using analytical mechanics theory. But it does not consider the inertia force of the liquid sloshing and the engine swing. Clark et al. [3-5] presented a longitudinal dynamics model of hypersonic vehicle with a CFD calculation considering the thrust / attitude / structure coupling. But the research on the source of vibration and the effect of vibration on flight control system is less. Raney and Schmidt, [6] who also use the Lagrange mechanics, established a dynamic model of the aircraft, and the introduction of the concept of "mean axis", simplified the model, more convenient to simulate and analyze. Waszak and Schmidt et al. [7-9] in the process of establishing the dynamic model, use the strip theory to calculate the aerodynamic force to simplify the calculation process of aerodynamic forces. The study of David K. Schmidt [10-15] shows that the strong aero-propulsive/aero-elastic of hypersonic vehicle coupling requires a holistic approach to the development of an integrated airframe-engine control system.

In the rest of paper, based on the characteristics of hypersonic vehicle, the nonlinear dynamic model of the vehicle is established, considering the elastic deformation, the swing of the engine and
the liquid sloshing. On the basis of this, the simulation analysis is carried out to study the flight characteristics of hypersonic vehicle.

2. General equations of motion for an unrestrained elastic body
We will define a coordinate system, \( O-XYZ \), before we start modeling. This reference system, whose origin at the location of launch, called inertial reference.

A body coordinate system, represented by the right-hand unit vector triad \((i, j, k)\) with its origin at the mass center of the body is assumed fixed to the vehicle.

Via the laws for rate of change of linear and angular momentum, we have

\[
m \frac{d\vec{V}}{dt} = \vec{F}
\]

\[
\frac{d\vec{H}}{dt} = \vec{M}
\]

where \(\vec{V}\) represents the velocity vector of origin of body axis system relative to inertial reference, \(\vec{F}\) denotes the all forces vector act on the vehicle, \(\vec{H}\) and \(\vec{M}\) are, respectively, angular momentum and moment of external forces in body coordinate system.

Here

\[
\frac{d\vec{V}}{dt} = \frac{\delta \vec{V}}{\delta t} + \dot{\vec{\omega}} \times \vec{V}
\]

\[
\vec{V} = V_{x_i}i + V_{y_j}j + V_{z_k}k
\]

\[
\vec{\omega} = \omega_{x_i}i + \omega_{y_j}j + \omega_{z_k}k
\]

\[
\vec{F} = F_{x_i}i + F_{y_j}j + F_{z_k}k
\]

\(\vec{V}_i, \vec{V}_j, \vec{V}_k\) are the components of velocity in the body coordinate system. And \(F_{x_i}, F_{y_j}, F_{z_k}\) are the three components of the force resolved in the body coordinate system. After substituting equation 3-6 in equation 1-2, we obtain

\[
m \left( \dot{V}_{x_i} + \omega_{y_j} V_{z_k} - \omega_{z_k} V_{y_j} \right) = F_{x_i}
\]

\[
m \left( \dot{V}_{y_j} + \omega_{z_k} V_{x_i} - \omega_{x_i} V_{z_k} \right) = F_{y_j}
\]

\[
m \left( \dot{V}_{z_k} + \omega_{x_i} V_{y_j} - \omega_{y_j} V_{x_i} \right) = F_{z_k}
\]

We can use similar manner to obtain the equation of angular momentum,

\[
J_{x_i} \dot{\omega}_{x_i} + (J_{y_j} - J_{z_k}) \omega_{y_j} \omega_{z_k} = M_{x_i}
\]

\[
J_{y_j} \dot{\omega}_{y_j} + (J_{z_k} - J_{x_i}) \omega_{z_k} \omega_{x_i} = M_{y_j}
\]

\[
J_{z_k} \dot{\omega}_{z_k} + (J_{x_i} - J_{y_j}) \omega_{x_i} \omega_{y_j} = M_{z_k}
\]

where \(M_{x_i}, M_{y_j}, M_{z_k}\) are the three components of the moment resolved in the body coordinate system; \(\omega_{x_i}, \omega_{y_j}, \omega_{z_k}\) are angular velocities in body axis system.

3. Equations of elastic vibration
A hypersonic vehicle is essentially a long slender shape. Therefore, in order to study elastic vibration, the hypersonic vehicle is usually simplified to a one-dimensional Euler Bernoulli beam model. For the purpose of study the effect of elastic deformation of the vehicle, the structure reference frame in pitch plane is defined. The origin \(O\) is the theory cusp of the vehicle, \(X_e - \text{axis}\) along the opposite
direction of $X_1$-axis and $Z_e$-axis perpendicular to the $X_e$-axis. We consider only the torsional vibration, pitch and yaw direction bending vibration. The elastic displacement at any point of the vehicle is given by

$$ z(x,t) = \sum_{i=1}^{n} q_{iz}(t) u_{iz}(x) $$  \hspace{1cm} (11)

Here $u_{iz}(x)$ represents the normalized mode shape of the $i_{th}$ mode in the pitch plane, which is only a function of the elastic structural characteristics and mass distribution along the axis. $q_{iz}(t)$ is the generalized coordinate. It satisfies the equation

$$ \ddot{q}_{iz} + 2\xi_{iz}\omega_{iz}\dot{q}_{iz} + \omega_{iz}^2 q_{iz} = \frac{Q_{iz}}{m_{iz}} \hspace{1cm} i = 1, 2, \cdots $$  \hspace{1cm} (12)

where $Q_{iz}$ and $m_{iz}$ are the generalized force and mass, respectively, and are given by

$$ Q_{iz} = \int_{0}^{l} f_z(x,t) u_{iz}(x) \, dx $$ \hspace{1cm} (13)

$$ m_{iz} = \int_{0}^{l} m(x) \dot{u}_{iz}^2(x) \, dx $$ \hspace{1cm} (14)

and $\omega_{iz}$ denotes the natural frequency of the $i_{th}$ mode.

The forced vibration equation of the yaw plane is similar to the pitching plane in form. Therefore, the following form of the equation of the yaw plane elastic vibration can be obtained.

$$ y(x,t) = \sum_{i=1}^{n} q_{iy}(t) u_{iy}(x) $$ \hspace{1cm} (15)

$$ \ddot{q}_{iy} + 2\xi_{iy}\omega_{iy}\dot{q}_{iy} + \omega_{iy}^2 q_{iy} = \frac{Q_{iy}}{m_{iy}} \hspace{1cm} i = 1, 2, \cdots $$ \hspace{1cm} (16)

$$ Q_{iy} = \int_{0}^{l} f_y(x,t) u_{iy}(x) \, dx $$ \hspace{1cm} (17)

$$ m_{iy} = \int_{0}^{l} m(x) \dot{u}_{iy}^2(x) \, dx $$ \hspace{1cm} (18)

Here, $y(x,t)$ is the elastic displacement in yaw plane. $f_y(x,t)$ and $m(x)$ are axial distributions of forces and masses, respectively. $u_{iy}(x)$ represents the normalized mode shape of the $i_{th}$ mode in the yaw plane.

Through a method similar to the derivation process of the pitch plane vibration equations, the torsional elastic vibration equations can be obtained.
\[ r(x,t) = \sum_{i=1}^{n} q_i(t) \gamma_i(x) \]  

\[ \ddot{q}_i + 2\xi_i \omega_i \dot{q}_i + \omega_i^2 q_i = \frac{Q_i}{m_i} \quad i = 1,2,\ldots \]  

\[ Q_i = \int_0^l M(x,t) \gamma_i(x) \, dx \]  

\[ m_i = \int_0^l \rho(x) J(x) \gamma_i^2(x) \, dx \]  

Here, \( r(x,t) \) is the elastic angular displacement in roll plane. \( r_i(x) \) represents the normalized mode shape of the \( i_{th} \) mode in the roll plane. \( M(x,t) \) and \( J(x) \) are axial distributions of moment and moment of inertia, respectively. \( \rho(x) \) is density.

The elastic angle deformation are defined in the following manner

\[ \frac{\partial \sigma(x,t)}{\partial x} = \sum_i q_i(t) \frac{\partial u_i(x)}{\partial x} = \sum_i q_i(t) \sigma_i(x) = \phi_z(x,t) \]  

\( \sigma(x,t) \) and \( \phi_z(x,t) \) is elastic angle deformation in pitch plane due to the \( i_{th} \) mode.

4. Sloshing equation

As is known to all, the dynamic characteristics of liquid sloshing can be described by a space pendulum, as shown in Figure 3. The parameters of pendulum are functions of tank shape, liquid level, etc. For the purposes of establish model, these pendulum parameters are considered to be known. Then, the dynamic equation of the \( i_{th} \) pendulum can be derived. Following this, the forces and moments caused by liquid sloshing, which act on the vehicle will be obtained. The velocity of this pendulum relative to inertial coordinate system is given by
\[ \vec{\mu}_{pi} = \vec{\mu} + \frac{d\vec{p}_{pi}}{dt} \]

\[ = \vec{\mu} + \frac{\delta\vec{p}_{pi}}{\delta t} + \omega_{\vec{r}} \times \vec{p}_{pi} \]  

(24)

\( \mu \) is velocity vector of body coordinate system relative to inertial coordinate system. \( \vec{\mu}_{pi} \) is velocity vector of \( i_{th} \) pendulum in pitch plane relative to inertial reference. \( \vec{p}_{pi} \) is radius vector of \( i_{th} \) pendulum in body axis system.

Here

\[ \omega_{\vec{r}} = \omega_{\vec{r}'_{i}} \]

(25)

\[ \mu = V_{x_{i}} \vec{i}_{i} + V_{z_{i}} \vec{k}_{i} \]

(26)

And

\[ \vec{p}_{pi} = \left( \ell_{pi} - L_{pi} \cos \Gamma_{pi} \right) \vec{i}_{i} + \left( L_{pi} \sin \Gamma_{pi} + z_{pi} \right) \vec{k}_{i} \]

(27)

where \( z_{pi} \) represents \( z_{pi}(\ell_{pi},t) \). \( \ell_{pi} \) is the distance from hinge point of \( i_{th} \) pendulum to origin of body axis system; \( L_{pi} \) is the length of \( i_{th} \) pendulum; \( \Gamma_{pi} \) is the pendulum angle in pitch plane.

When performing the specified operation, we found

\[ \vec{p}_{pi} = \left[ V_{x_{i}} L_{pi} \Gamma_{pi} + \omega_{r_{i}} \left( L_{pi} \sin \Gamma_{pi} + z_{pi} \right) \right] \vec{i}_{i} + \left[ V_{z_{i}} + \left( L_{pi} \Gamma_{pi} \sin \Gamma_{pi} + \dot{z}_{pi} \right) - \omega_{r_{i}} \left( \ell_{pi} - L_{pi} \cos \Gamma_{pi} \right) \right] \vec{k}_{i} \]

(28)

The kinetic energy is

\[ T = \frac{1}{2} M_{pi} \vec{\mu}_{pi} \cdot \vec{\mu}_{pi} \]

(29)

Since the system is in free fall, it has no potential energy. Therefore, the Lagrange function can be expressed as \( L = T \). Then, the dynamic equation in Lagrangian form can be expressed as follows

\[ \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\Gamma}_{pi}} \right) - \frac{\partial L}{\partial \Gamma_{pi}} = 0 \]

(30)

After \( \dot{\Gamma}_{pi} \) and \( \Gamma_{pi} \) in the above formula are considered as small quantities, we find

\[ \dot{\Gamma}_{pi} + \frac{\dot{V}_{x_{i}}}{L_{pi}} \Gamma_{pi} = -\frac{1}{L_{pi}} \left[ \dot{V}_{z_{i}} - V_{x_{i}} \omega_{r_{i}} + \dot{\omega}_{r_{i}} \left( \ell_{pi} - L_{pi} \right) + \dot{z}_{pi} \right] \]

(31)

Now put

\[ \omega_{r_{i}}^{2} = \frac{\dot{V}_{x_{i}}}{L_{pi}} \]

(32)

We obtain

\[ \dot{\Gamma}_{pi} + \omega_{r_{i}}^{2} \Gamma_{pi} = \frac{1}{L_{pi}} \left[ V_{x_{i}} \dot{\theta} + \ddot{\theta} \left( \ell_{pi} - L_{pi} \right) - V_{z_{i}} + \sum_{i=1}^{\infty} \ddot{q}_{z_{i}} \left( t \right) u_{z_{i}} \left( x_{pi} \right) \right] \]

(33)

The dynamic equation of the \( i_{th} \) pendulum in the yaw plane can be obtained in a similar way. The final result is

\[ \dot{\Gamma}_{yi} + \omega_{\psi}^{2} \Gamma_{yi} = \frac{1}{L_{pi}} \left[ -V_{x_{i}} \dot{\psi} - \psi \left( \ell_{pi} - L_{yi} \right) - V_{z_{i}} + \sum_{i=1}^{\infty} \ddot{q}_{y_{i}} \left( t \right) u_{y_{i}} \left( x_{pi} \right) \right] \]

(34)
$L_{yi}$ is the length of $i_{th}$ pendulum in yaw plane. $\Gamma_{yi}$ is the pendulum angle in yaw plane. The sloshing forces and moments now appear as

$$
F_{xi} = 0
$$

$$
F_{yi} = \sum_i M_{pi} \dot{X}_i \Gamma_{pi}
$$

$$
F_{zi} = \sum_i M_{pi} \dot{Y}_i \Gamma_{yi}
$$

$$
M_{xi} = 0
$$

$$
M_{yi} = -\sum_i M_{pi} \ell_{pi} \dot{X}_i \Gamma_{pi}
$$

$$
M_{zi} = \sum_i M_{pi} \ell_{pi} \dot{Y}_i \Gamma_{yi}
$$

(35)

5. Forces, moment and generalized forces

The forces and moments acting on the vehicle stem from three sources: gravity, thrust, sloshing, engine inertia, and aerodynamics. We consider these in turn.

(1) Gravity

Let $S_1$ represent the body coordinate system, and the orientation of $S_1$ is related to inertial reference, $S$, by three Euler angles, $\psi$, $\theta$, $\varphi$, defined as follows.

$\psi$, angle that rotate $S_1$ about the $Z$ axis in the positive direction.

$\theta$, angle that rotate $S_1$ about the $Y$ axis in the positive direction.

$\varphi$, angle that rotate $S_1$ about the $X$ axis in the positive direction.

We then have

$$
S_1 = AS
$$

(36)

where $A$ is the transformation matrix given by

$$
A = \begin{bmatrix}
\cos \theta \cos \psi & \cos \theta \sin \psi & -\sin \theta \\
\sin \varphi \sin \theta \cos \psi - \cos \theta \sin \psi & \sin \varphi \sin \theta \sin \psi - \cos \theta \cos \psi & \sin \varphi \cos \theta \\
\cos \varphi \sin \theta \cos \psi + \sin \varphi \sin \psi & \cos \varphi \sin \theta \sin \psi + \sin \varphi \cos \psi & \cos \varphi \cos \theta
\end{bmatrix}
$$

(37)

The force of gravity resolved in inertial reference system can write

$$
\begin{bmatrix}
G_X \\
G_Y \\
G_Z
\end{bmatrix} =
\begin{bmatrix}
0 \\
0 \\
mg
\end{bmatrix}
$$

(38)

The component of gravity vector in the body coordinate system can be obtained by using equation 24-26.

$$
\begin{bmatrix}
G_{X_i} \\
G_{Y_i} \\
G_{Z_i}
\end{bmatrix} =
\begin{bmatrix}
-mg \sin \theta \\
mg \sin \varphi \cos \theta \\
mg \cos \varphi \cos \theta
\end{bmatrix}
$$

(39)

There is one hypothesis that the location of mass center of the vehicle is unchanged. So the generalized forces of gravity are zero.
(2) Thrust

We assume that the angular quantities which we used in the derivation of thrust components are small, and the positive direction of engine swing angle is anticlockwise. Now, assume that the quantities of angle are small. So in the body coordinate system

\[ P_{x_1} = (P \cos(\delta_1) + P \cos(\delta_2) + P \cos(\delta_3) + P \cos(\delta_4)) \cos(\phi_1(x,t)) \]
\[ = 4P(\phi_1(x,t)) \]
\[ = 4P \]

\[ P_{y_1} = P\sin(\delta_1) - P\sin(\delta_2) + P\cos(\delta_3) + P\cos(\delta_4) \sin(\phi_1(x,t)) \]
\[ = P(\delta_3 - \delta_2) + 4P\phi_1(x,t) \]
\[ = P(\delta_3 - \delta_2) + 4P \sum_i q_{yi}(t) \sigma_{yi}(x_R) \]  \( (41) \)

\[ P_{z_1} = P\sin(\delta_1) - P\sin(\delta_2) + P\cos(\delta_3) + P\cos(\delta_4) \sin(\phi_1(x,t)) \]
\[ = P(\delta_3 - \delta_2) + 4P\phi_1(x,t) \]
\[ = P(\delta_3 - \delta_2) + 4P \sum_i q_{zi}(t) \sigma_{zi}(x_R) \]  \( (42) \)

\( \delta_1, \delta_2, \delta_3 \) and \( \delta_4 \) are swing angles of the engine, respectively.

**Thrust moment:**

\[ M_{Px_1} = P(\sin(\delta_1) + \sin(\delta_2) + \sin(\delta_3) + \sin(\delta_4))r_0 \]
\[ = P(\delta_1 + \delta_2 + \delta_3 + \delta_4)r_0 \]  \( (43) \)

\[ M_{Py_1} = -F_{Pz1}z_M + F_{Pz1}x_M \]
\[ = \left[ P(\delta_3 - \delta_1) + 4P\phi_1(x_R,t) \right](x_R - x_T) - 4Pz(x_R,t) \]  \( (44) \)

\[ M_{Pz_1} = F_{Pz1}y_M + F_{Pz1}x_M \]
\[ = 4Py(x_R,t) - \left[ P(\delta_3 - \delta_1) + 4P\phi_1(x_R,t) \right](x_R - x_T) \]
\[ = 4P \sum_i q_{yi}(t)u_y(x_R) - \left[ P(\delta_3 - \delta_1) + 4P \sum_i q_{zi}(t)\sigma_{zi}(x_R) \right](x_R - x_T) \]  \( (45) \)

where \( x_R \) and \( x_T \) is the distance from the engine swing point and mass center to the theory cusp of the vehicle.

Generalized forces will be got with equation 13, 17 and 21.

\[ Q_{tx_1} = \int_0^t M_{x_1}(x,t)\gamma_{x_1}(x)dx \]
\[ = -P\gamma_0(\delta_1 + \delta_2 + \delta_3 + \delta_4)\gamma_{x_1}(x_R) \]  \( (46) \)

\[ Q_{ty_1} = \int_0^t f_{y_1}(x,t)u_y(x)dx \]
\[ = \left[ P(\delta_3 - \delta_1) + 4P\phi_1(x_R,t) \right]u_y(x_R) \]
\[ = P(\delta_3 - \delta_1)u_y(x_R) + 4P \sum_i q_{yi}(t)\sigma_{yi}(x_R)u_y(x_R) \]  \( (47) \)
\[
Q_{\alpha_0} = \int_0^t f_z(x,t)u_z(x)dx \\
= \left[ P(\delta_2 - \delta_3) - 4P\phi_q(x_R, t) \right]u_z(x_R) \\
= P(\delta_2 - \delta_3)u_z(x_R) + 4\sum_j q_{\alpha_j}(t)\sigma_{\alpha_j}(x_R)u_z(x_R)
\]

(48)

(3) Engine swing inertia force (in the body coordinate system)

The inertia force acted on the vehicle of engine swing only exists in the roll plane. So

\[
F_{\delta X_1} = 0
\]

(49)

D'Alembert's principle and virtual work principle has been used to develop the equations of inertia force. The final result is

\[
F_{\delta Y_1} = m_Rl_R\left(\ddot{\delta}_1 - \ddot{\delta}_2\right)
\]

(50)

\[
F_{\delta z_1} = m_Rl_R\left(\ddot{\delta}_2 - \ddot{\delta}_3\right)
\]

(51)

Engine swing moment of inertia

\[
M_{\delta X_1} = -m_Rl_R\left[\ddot{\delta}_1 + \ddot{\delta}_2 + \ddot{\delta}_3 + \ddot{\delta}_4\right]
\]

(52)

\[
M_{\delta Y_1} = F_{\delta z_1}x_{XYZ} - m_Rl_R\ddot{W}_{x_1}\left(\delta_2 - \delta_4\right)
\]

(53)

\[
= -m_Rl_R\left[(x_R - x_T) + l_R\right]\left(\ddot{\delta}_2 - \ddot{\delta}_4\right) - m_Rl_RW_{x_1}\left(\delta_2 - \delta_4\right)
\]

\[
M_{\delta Z_1} = F_{\delta z_1}x_{ZMV} + m_Rl_R\ddot{W}_{x_1}\left(\delta_1 - \delta_3\right)
\]

(54)

\[
= -m_Rl_R\left[(x_R - x_T) + l_R\right]\left(\ddot{\delta}_1 - \ddot{\delta}_3\right) + m_Rl_RW_{x_1}\left(\delta_1 - \delta_3\right)
\]

Generalized forces:

\[
Q_{\alpha_0} = \int_0^t M_{\alpha}(x,t)\gamma_{r\alpha}(x)dx \\
= -m_Rl_R\ddot{w}_{r\alpha}(x_R)\left(\ddot{\delta}_1 + \ddot{\delta}_2 + \ddot{\delta}_3 + \ddot{\delta}_4\right)
\]

(55)

\[
Q_{\gamma_0} = \int_0^t f_j(x,t)u_\gamma(x)dx \\
= m_Rl_R\left(\ddot{\delta}_1 - \ddot{\delta}_3\right)u_\gamma(x_R) + \left[ -m_Rl_R\left[(x_R - x_T) + l_R\right]\left(\ddot{\delta}_2 - \ddot{\delta}_4\right) - m_Rl_RW_{x_1}\left(\delta_2 - \delta_4\right) \right]\sigma_{\gamma}(x_R)
\]

(56)

\[
Q_{\alpha_2} = \int_0^t f_j(x,t)u_\alpha(x)dx \\
= m_Rl_R\left(\ddot{\delta}_2 - \ddot{\delta}_4\right)u_\alpha(x_R) + \left[ -m_Rl_R\left[(x_R - x_T) + l_R\right]\left(\ddot{\delta}_1 - \ddot{\delta}_3\right) + m_Rl_RW_{x_1}\left(\delta_1 - \delta_3\right) \right]\sigma_{\alpha}(x_R)
\]

(57)

(4) Aerodynamics

Additional angle of attack caused by elastic deformation is

\[
\alpha_1 = \frac{\dot{\varepsilon}z(x,t)}{\dot{\alpha}_X} = \dot{\phi}_z(x,t)
\]

(58)

Additional angle of attack caused by rotation and elastic vibration velocity is

\[
\alpha_2 = -\frac{x_T - x}{V}\omega_{z_1} - \frac{\dot{z}(x,t)}{V}
\]

(59)

So, the total angle of attack is given by

\[
\alpha' = \alpha + \alpha_1 + \alpha_2 = \alpha + \phi_z(x,t) - \frac{x_T - x}{V}\omega_{z_1} - \frac{\dot{z}(x,t)}{V}
\]

(60)
In a similar way, the sideslip angle under the influence of elastic deformation can be expressed as

$$\beta' = \beta + \beta_1 + \beta_2 = \beta + \phi(x, t) - \frac{x_y - x}{V} \omega_z - \frac{\dot{y}(x, t)}{V}$$  \hspace{1cm} (61)$$

The components of the forces and moments resolved in velocity coordinate system are expressed as follows.

$$F_{Ax} = -qSC_x$$  \hspace{1cm} (62)$$

$$F_{Ay} = \int_0^1 \frac{\partial (C_y)}{\partial \beta} qS \left[ \beta + \sum_{i=1}^n \sigma_i (x) q_{iy} - \frac{x_y - x}{V} \omega_z - \frac{1}{V} \sum_{j=1}^n u_{iy} (x) \dot{q}_{iy} \right] dx$$  \hspace{1cm} (63)$$

$$F_{Az} = \int_0^1 \frac{\partial (C_z)}{\partial \alpha} qS \left[ \alpha + \sum_{i=1}^n \sigma_i (x) q_{iz} - \frac{x_y - x}{V} \omega_z - \frac{1}{V} \sum_{j=1}^n u_{iz} (x) \dot{q}_{iz} \right] dx$$  \hspace{1cm} (64)$$

where

$$C_y, C_z, C_x$$  \hspace{1cm} (65)$$

is the aerodynamic coefficients, $q$ is the dynamic pressure, $\rho$, $V$ is the density and velocity of the air flow, and $S$ is aerodynamic reference area.

$$M_{Ax} = -\frac{1}{V} m_x qS \omega_i'^2$$  \hspace{1cm} (66)$$

$$M_{Ay} = -\int_0^1 \frac{\partial (C_y)}{\partial \alpha} qS (x - x_y) \left[ \alpha + \sum_{i=1}^n \sigma_i (x) q_{iy} - \frac{x_y - x}{V} \omega_z - \frac{1}{V} \sum_{j=1}^n u_{iy} (x) \dot{q}_{iy} \right] dx$$  \hspace{1cm} (67)$$

$$M_{Az} = -\int_0^1 \frac{\partial (C_z)}{\partial \beta} qS (x - x_y) \left[ \beta + \sum_{i=1}^n \sigma_i (x) q_{iz} - \frac{x_y - x}{V} \omega_z - \frac{1}{V} \sum_{j=1}^n u_{iz} (x) \dot{q}_{iz} \right] dx$$  \hspace{1cm} (68)$$

Generalized aerodynamic forces and damping force in the pitch direction and yaw direction:

$$Q_{Ay} = qS \alpha \int_0^1 u_{iy} (x) (C_y^\beta)_{ix} dx - qS \omega_y \sum_{j=1}^n \int_0^1 (C_y^\beta)_{iy} (x) \sigma_{ij} (x) dx$$  \hspace{1cm} (69)$$
\[ Q_{az} = qS\alpha \int_0^1 u_{iz} (x)(C_{iz}^n) dx - qS\beta \sum_{j=1}^{n} \int_0^1 \left( C_{iz}^j \right) u_{iz} (x) \sigma_{iz} \left(x\right) dx \]

\[ = -\frac{1}{V} qS\beta \int_0^1 u_{iz} (x)(C_{iz}^n) \left(x_d - x_T\right) dx - \frac{1}{V} qS\beta \sum_{j=1}^{n} \int_0^1 \left( C_{iz}^j \right) u_{iz} (x) u_{iz} (x) dx \]

(70)

When the results obtained so far are collected and summarized, a complete elastic hypersonic vehicle equation will be formed. Then, on the basis of the above model, some simulation will be carried out.

6. Simulation

In this part, the simulation of sloshing and elastic vibration is carried out. The geometry of the hypersonic vehicle is given in the Figure 4.

![Figure 4. The geometry of the hypersonic vehicle.](image)

![Figure 5. The first four order vibration modes.](image)

The modal and frequency data of the aircraft are calculated by the finite element method. The first four order vibration modes are shown in Figure 5. We take the first order vibration modal data for simulation, and the natural frequency of the first order vibration mode is 6.9486 Hz.

In the following simulation, we can clearly see the effects of sloshing and elastic vibrations on flight states.

![Figure 6. The change of angle of attack and side slip.](image)
Figure 6 is the simulation results of the attack angle and sideslip angle. We can see that the effects of elastic vibration and sloshing on flight states are different. The elastic vibration has a great effect on the two angles, but the effect of sloshing is very small. However, when we study the speed, the results are different. In Figure 7, at a certain moment, the change of speed is approximately the same. This shows that the influence of elastic vibration and sloshing on flight state is different. From the point of view of elastic vibration, it has more serious influence on the attack and sideslip angle than on the speed. On the other hand, when only the sloshing is considered, the influence of the sloshing angle is larger, however, less impact on the angle of sideslip. Figure 8 and 9 are respectively the changes of the moment and angular velocities. In Figure 9, the elastic vibration has a significant effect on the moment in all of the XYZ direction. However, the effect of sloshing is only at the Y direction. The result of the moment is also reflected in the angular velocity. In Figure 8, at the Y direction, the change of the angular velocities is great when the effect of sloshing is considered. From the above simulation, we can see that in this period of time, sloshing occurs mainly in the direction of Z, and bring great
disturbance to the moment at Y direction. However, the elastic vibration has influence in all directions. The change of all flight parameters will be recorded by the sensor, and then affect the control system work. On the other hand, the flutter is enhanced by the sloshing inertial force, and the amplitude of the flutter is increased by the liquid fuel sloshing. However, in the Z direction, the amplitude of flutter is not significantly affected by sloshing, but the flutter frequency is increased. The simulation results show that the sloshing inertia force has different effects on the dynamic characteristics of each motion direction. Elastic vibrations can increase the amplitude of sloshing and cause instability of sloshing.

![Figure 9. The simulation results of the moment.](image)

![Figure 10. Root locus without considering elasticity and sloshing.](image)
Next, a flight time is chosen to linearize the dynamics equation at this moment. Then, the dynamic stability of the aircraft is analyzed when considering the effects of elasticity and liquid sloshing.

After linearization, we obtain the transfer function of the pitch angle to the engine rudder deflection angle. Next, we get the root locus. Figure 10 is a root locus without considering elasticity and sloshing. Figure 11 is a root locus with only elasticity considered. It can be seen from the diagram that the elastic vibration increases two poles, and one of the poles is positive. At this natural frequency, the divergence of pitch angle is faster.

Figure 12 is root locus considering elasticity and liquid sloshing. Liquid sloshing increases two pairs of poles, and one of them is positive pole. The two pairs of poles are caused by two tanks. This shows that one of the tanks makes the divergence faster. Figure 13 is a comparison of bode diagram in three cases. It can be seen from the amplitude characteristic diagram that the amplitude fluctuates greatly near the sloshing and the elastic frequency. This change also occurs in phase diagrams. Moreover, the elasticity changes the amplitude characteristic and reduces the amplitude. This is because the elastic natural frequency and sloshing frequency are very close to the motion frequency of the vehicle, so that they are coupled each other. So when we design the control system, we need to consider the influence of elasticity and liquid sloshing.
7. Conclusion
This article mainly aims at the study hypersonic vehicle, the six degree of freedom nonlinear dynamic model of a flexible vehicle is established by considering the elastic deformation, the swing of the engine and the liquid sloshing, and the longitudinal channel simulation analysis is conducted according to the model. The results show that the model can fully reflect the influence of the elastic and sloshing effects on the flight characteristics of hypersonic vehicle, and can clearly see the physical meaning of the aircraft's force in the model. Because of its own characteristics, the design of the control system is more difficult and the control scheme is more demanding.

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