CONVERGENCE OF THE CR YAMABE FLOW

PAK TUNG HO, WEIMIN SHENG, AND KUNBO WANG

Abstract. We consider the CR Yamabe flow on a compact strictly pseudoconvex CR manifold $M$ of real dimension $2n+1$. We prove convergence of the CR Yamabe flow when $n = 1$ or $M$ is spherical.

1. Introduction

Suppose $(M, g_0)$ is a compact $n$-dimensional manifold without boundary where $n \geq 3$. As a generalization of Uniformization theorem, the Yamabe problem \cite{39} is to find a metric $g$ conformal to $g_0$ such that its scalar curvature $R_g$ is constant. This problem was solved by Yamabe, Trudinger, Aubin and Schoen \cite{39,36,1,31}. See the survey article \cite{30} by Lee and Parker for more details.

A different approach has been introduced to solve the Yamabe problem. Hamilton \cite{22} introduced the Yamabe flow, which is defined by

$$\frac{\partial}{\partial t} g(t) = -(R_{g(t)} - r_{g(t)}) g(t),$$

where $r_{g(t)}$ is the average of the scalar curvature $R_{g(t)}$ of $g(t)$. The Yamabe flow was considered by Chow \cite{12}, Ye \cite{40}, Schwetlick and Struwe \cite{32}. Finally Brendle \cite{5,6} showed that the Yamabe flow exists for all time and converges to a metric of constant scalar curvature by using the positive mass theorem.

The Yamabe problem can also be formulated in the context of CR manifold. Suppose that $(M, \theta_0)$ is a compact strongly pseudoconvex CR manifold of real dimension $2n+1$ with a given contact form $\theta_0$. The CR Yamabe problem is to find a contact form $\theta$ conformal to $\theta_0$ such that its Webster scalar curvature $R_\theta$ is constant. This was introduced by Jerison and Lee in \cite{28}, and was solved by Jerison and Lee for the case when $n \geq 2$ and $M$ is not spherical in \cite{26,27,28}. We say that $M$ is spherical if and only if $M$ is locally CR equivalent to the CR sphere $S^{2n+1}$. The remaining case, namely, when $n = 1$ or $M$ is spherical, was solved by Gamara and Yacoub in \cite{18,19} by using critical point at infinity. See also the recent work by Cheng-Chiu-Yang \cite{10} and Cheng-Malchiodi-Yang \cite{11} for these cases.

As an analogue to Yamabe flow, one can consider the CR Yamabe flow defined by

$$\frac{\partial}{\partial t} \theta(t) = -(R_{\theta(t)} - r_{\theta(t)}) \theta(t),$$

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Then we divide the proof into two cases, namely, $u \times$ such that $s$ is spherical and $Y$. Here we denote $(z,t)$ such that $M$ is not CR equivalent to the CR sphere $S^{2n+1}$. When $n = 1$, we also assume that the CR Panetiz operator of $M$ is nonnegative. When $n = 2$, we also assume that the minimum exponent of the integrability of the Green function satisfies $s(M) < 1$. Then the Yamabe flow exists for all time and converges to a contact form with constant Webster scalar curvature.

We refer the readers to (3.6) in [10] for the precise definition of $s(M)$. When $n = 1$, the definition of the CR Panetiz operator was included in section 2, but we also refer the readers to [11] for more properties of the CR Panetiz operator. Similar to the Yamabe flow, we need to use the CR positive mass theorem to prove the convergence of the CR Yamabe flow. The CR positive mass theorem when $M$ is spherical was obtained by Cheng-Chiu-Yang in [10] for $n \geq 2$, with further assumption that $s(M) < 1$ for $n = 2$. On the other hand, the CR positive mass theorem for the case when $n = 1$ was obtained by Cheng-Malchiodi-Yang in [11], under the assumption that the CR Panetiz operator of $M$ is nonnegative. Therefore, we have included the assumptions in Theorem 1.1 so that we can apply the CR positive mass theorem.

Our proof of Theorem 1.1 basically follows [5]. We sketch the proof here. Using the concentration-compactness result (see Theorem 5.1), which is the CR version of the results of Struwe [34] and Bahri-Coron [2], we can show that the solution of the CR Yamabe flow either converges, or else concentrates in finite number of bubbles. Concisely, if we write $\theta(t) = u(t)\|\|\theta_0$ for some positive function $u(t)$, then the CR Yamabe flow can be written as follows: Let $\{t_\nu : \nu \in \mathbb{N}\}$ be a sequence of times such that $t_\nu \to \infty$ as $\nu \to \infty$ and $u_\nu = u(t_\nu)$. Then there exists an integer $m \geq 0$, a collection of positive numbers $\{\varepsilon_{i,\nu} : 1 \leq i \leq m, \nu \in \mathbb{N}\}$, and a collection of points $\{p_{i,\nu} : 1 \leq i \leq m, \nu \in \mathbb{N}\} \subset M$, such that

$$u_\nu - \sum_{k=1}^{m} \left( \frac{n(2n+2)}{r_{\infty}} \right) \cdot \left( \frac{\varepsilon_{i,\nu}^2}{(t_\nu^2 + (\varepsilon_{i,\nu}^2 + |z|^2)^2)} \right)^{\frac{n}{2}} \to u_\infty.$$ 

Here we denote $(z,t)$ the CR normal coordinate at $p_{i,\nu}$, and $r_{\infty} = \lim_{\nu \to \infty} r_\theta(t)$. Then we divide the proof into two cases, namely, $u_\infty \equiv 0$ and $u_\infty > 0$. Following the estimates in [5], we can rule out the formation of bubbles and show that $u(t)$ is uniformly bounded from above and below. By using the estimate of Bramanti and Brandolini [3], we can obtain the uniform bounds for the higher-order derivatives of $u(t)$, which implies the convergence of the CR Yamabe flow.
This paper is organized as follows. In section 2, we recall some basic concepts in CR geometry. In section 3, we recall some basic properties of the CR Yamabe flow. In section 4, by assuming Proposition 4.1, we give the proof of Theorem 1.1. We give the blow-up analysis in section 5 and then prove Proposition 4.1 in section 6. In Appendix A, we give some basic estimates regarding the test functions. In Appendix B, we give the proof of the concentration-compactness theorem, i.e. the proof of Theorem 5.1.

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2. Preliminaries and Notations

In this section, we include some basic concepts in the CR geometry. Most of them can be found in [14] or [29]. Let \( M \) be an orientable smooth manifold of real dimension \( 2n + 1 \). A CR structure on \( M \) is given by a complex \( n \)-dimensional subbundle \( T^{1,0} \) of the complexified tangent bundle \( \mathbb{C}TM \) of \( M \), satisfying \( T^{1,0} \cap T^{0,1} = \{0\} \), where \( T^{0,1} = \overline{T^{1,0}} \). We assume the CR structure is integrable, that is, \( T^{1,0} \) satisfies the formal Frobenius condition \([T^{1,0}, T^{1,0}] \subset T^{1,0}\). We set \( G = \text{Re}(T^{1,0} \oplus T^{0,1}) \), so that \( G \) is a real \( 2n \)-dimensional subbundle of \( TM \). Then \( G \) carries a natural complex structure map: \( J : G \to G \) given by \( J(V + \overline{W}) = \sqrt{-1}(V - \overline{W}) \) for \( V \in T^{1,0} \).

Let \( E \subset T^*M \) denote the real line bundle \( G^\bot \). Because we assume \( M \) is orientable and the complex structure \( J \) induces an orientation on \( G, E \) has a global nonvanishing section. A choice of such a 1-form \( \theta \) is called a pseudohermitian structure on \( M \). Associated with each such \( \theta \) is the real symmetric bilinear form \( \mathcal{L}_\theta \) on \( G \):

\[
\mathcal{L}_\theta(V, W) = d\theta(V, JW), \quad V, W \in G
\]

called the Levi-form of \( \theta \). \( \mathcal{L}_\theta \) extends by complex linearity to \( \mathbb{C}G \), and induces a Hermitian form on \( T^{1,0} \), which we write

\[
\mathcal{L}_\theta(V, \overline{W}) = -\sqrt{-1}d\overline{\theta}(V, \overline{W}), \quad V, W \in T^{1,0}
\]

If \( \theta \) is replaced by \( \tilde{\theta} = f\theta \), \( \mathcal{L}_\theta \) changes conformally by \( \mathcal{L}_{\tilde{\theta}} = f\mathcal{L}_\theta \). We will assume that \( M \) is strictly pseudoconvex, that is, \( \mathcal{L}_\theta \) is positive definite for a suitable \( \theta \). In this case, \( \theta \) defines a contact structure on \( M \), and we call \( \theta \) a contact form. Then we define the volume form on \( M \) as \( dV_\theta = \theta \wedge d\theta^n \). The volume of \( M \) is denoted by \( \text{Vol}(M, \theta) \), i.e. \( \text{Vol}(M, \theta) = \int_M dV_\theta \).

We can choose a unique \( T \), which is called the characteristic direction, such that \( \theta(T) = 1, d\theta(T, \cdot) = 0 \), and \( TM = G \oplus \mathbb{R}T \). Then we can define a coframe \( \{\theta, \theta^1, \theta^2, \ldots, \theta^n\} \) that satisfies \( \theta^n(T) = 0 \), which is called admissible coframe. Its dual frame \( \{T, Z_1, Z_2, \ldots, Z_n\} \) is called admissible frame. In the coframe, we have \( d\theta = \sqrt{-1}h_{\alpha\overline{\beta}}\theta^n \wedge \theta^\beta \), where \( h_{\alpha\overline{\beta}} \) is a Hermitian matrix.
The sub-Laplacian operator \( \Delta_b \) is defined by
\[
\int_M (\Delta_b u) f dV_\theta = - \int_M \langle du, df \rangle_\theta dV_\theta,
\]
for all smooth function \( f \). Here \( \langle \cdot , \cdot \rangle_\theta \) is the inner product induced by \( \mathcal{L}_\theta \). Tanaka [35] and Webster [35] showed that there is a natural connection on the bundle \( T^{1,0} \) adapted to a pseudohermitian structure, which is called the Tanaka-Webster connection. To define this connection, we choose an admissible coframe \( \{ \theta^\alpha \} \) and dual frame \( \{ Z_\alpha \} \) for \( T^{1,0} \). Then there are uniquely determined 1-forms \( \omega^\alpha_\beta, \tau_\alpha \) on \( M \), satisfying
\[
d\theta^\alpha = \theta^\beta \wedge \omega^\alpha_\beta + \theta \wedge \tau^\alpha, \\
dh_{\alpha\beta} = \omega^\alpha_\beta + \omega^\alpha_{\overline{\beta}}, \\
\tau_\alpha \wedge \theta^\alpha = 0.
\]
From the third equation, we can find \( A_{\alpha\gamma} \) such that \( \tau_\alpha = A_{\alpha\gamma} \theta^\gamma \) and \( A_{\alpha\gamma} = A_{\gamma\alpha} \). Here \( A_{\alpha\gamma} \) is called the pseudohermitian torsion. With this connection, the covariant differentiation is defined by
\[
DZ_\alpha = \omega^\beta_\alpha \otimes Z_\beta, \quad DZ_\bar{\alpha} = \omega^\beta_{\bar{\alpha}} \otimes Z_\beta, \quad DT = 0.
\]
\( \{ \omega^\alpha_\beta \} \) are called connection 1-forms. For a smooth function \( f \) on \( M \), we write \( f_\alpha = Z_\alpha f, \quad f_\bar{\alpha} = Z_\bar{\alpha} f, \quad f_0 = T f \), so that \( df = f_\alpha \theta^\alpha + f_\bar{\alpha} \overline{\theta}^\bar{\alpha} + f_0 \theta \). The second covariant differential \( D^2 f \) is the 2-tensor with components
\[
f_{\alpha\beta} = \overline{T_{\alpha\beta}} = Z_\beta Z_\alpha f - \omega^\gamma_\beta (Z_\beta) Z_\gamma f, \quad f_{\alpha\bar{\beta}} = \overline{T_{\alpha\bar{\beta}}} = Z_{\bar{\beta}} Z_\alpha f - \omega^\gamma_{\bar{\beta}} (Z_{\bar{\beta}}) Z_\gamma f,
\]
\( f_{0\alpha} = \overline{T_{0\alpha}} = Z_\alpha T f, \quad f_{0\bar{\alpha}} = \overline{T_{0\bar{\alpha}}} = T Z_\alpha f - \omega^\gamma_\alpha (T) Z_\gamma f, \quad f_{00} = T^2 f. \)
\( h_\alpha\beta \) and \( h^{\alpha\beta} \) are used to lower and raise the indices. We have
\[
d\omega^\alpha_\beta - \omega^\beta_\gamma \wedge \omega^\alpha_\gamma = \frac{1}{2} R^\alpha_{\beta \rho \sigma} \theta^\rho \wedge \theta^\sigma + \frac{1}{2} R^\alpha_{\beta \overline{\rho} \overline{\sigma}} \overline{\theta}^\rho \wedge \overline{\theta}^\sigma + R^\alpha_{\beta \rho \sigma} \theta^\rho \wedge \theta^\sigma + R^\alpha_{\beta \overline{\rho} \overline{\sigma}} \overline{\theta}^\rho \wedge \overline{\theta}^\sigma - R^\alpha_{\beta \rho \sigma} \sigma_\rho \theta^\sigma \wedge \theta.
\]
We call \( R^\alpha_{\beta \rho \sigma} \) the pseudohermitian curvature. Contractions of the pseudohermitian curvature yield the pseudohermitian Ricci curvature \( R^\alpha_{\rho \sigma} = R^\alpha_{\alpha \rho \sigma} \), or \( R^\alpha_{\rho \sigma} = h^{\alpha\beta} R_{\beta \rho \sigma} \), and the pseudohermitian scalar curvature \( R = h^{\alpha\beta} R_{\beta \rho \sigma} \).

The sub-Laplacian operator in this connection can be expressed by
\[
\Delta_b u = u^{\alpha \alpha} + u^{\bar{\alpha} \bar{\alpha}}
\]
If we define \( \bar{\theta} = u^{\bar{\alpha}} \theta \), then we have
\[
\bar{\Delta}_b f = u^{-\left(1 + \frac{\bar{n}}{n}\right)} (u \Delta_b f + 2 \langle du, df \rangle_\theta),
\]
where \( \bar{\Delta}_b \) is the sub-Laplacian operator with respect to the contact form \( \bar{\theta} \) (see (2.4) in [24] for example). If we set \( \bar{u} = r^{-1} u \), then we have the following CR transformation law
\[
-(2 + \frac{2}{n}) \bar{\Delta}_b + \bar{R} \bar{u} = r^{-1} \bar{u}^{-\frac{9}{n}} (-(2 + \frac{2}{n}) \Delta_b + R) u.
\]
In particular, if \( r = u \), then we get the CR Yamabe equation
\[
(1.1) \quad -(2 + \frac{2}{n}) \Delta_b u + Ru = \bar{R} u^{1 + \frac{2}{n}},
\]
The Heisenberg group $\mathbb{H}^n$ is a Lie group whose underlying manifold is $\mathbb{C}^n \times \mathbb{R}$ with coordinates $(z, t) = (z_1, z_2, \cdots, z_n, t)$ and $z = x + \sqrt{-1}y$. The group law of $\mathbb{H}^n$ is given by

$$(z, t)(z', t') = (z + z', t + t' + 2 \text{Im}(z \cdot \bar{z}')).$$

The norm on $\mathbb{H}^n$ is given by $\|\cdot\| = (|z|^4 + t^2)^{\frac{1}{4}}$ and $\delta_\lambda : \mathbb{H}^n \to \mathbb{H}^n$ is the dilation on $\mathbb{H}^n$ given by

$$\delta_\lambda(z, t) = (\lambda z, \lambda^2 t),$$

and $\tau_\xi : \mathbb{H}^n \to \mathbb{H}^n$ is the translation on $\mathbb{H}^n$ given by

$$\tau_\xi(x, y, t) = (x + x', y + y', t + t' + 2(xy' - x'y)), \quad \xi = (x', y', t') \in \mathbb{H}^n.$$

The vector fields $Z_j = \frac{\partial}{\partial z_j} + \sqrt{-1}\frac{\partial}{\partial t_j}$, $j = 1, 2, \ldots, n$, are invariant with respect to the group multiplication on the left. And $T^{1,0} = \text{span}\{Z_1, \ldots, Z_n\}$ gives a left-invariant CR structure on $\mathbb{H}^n$. The contact form of $\mathbb{H}^n$ is

$$\theta_{\mathbb{H}^n} = dt + \sqrt{-1} \sum_{j=1}^{n} (z_j dz_j - \bar{z}_j dz_j).$$

If $\{W_1, \ldots, W_n\}$ is a frame for $T^{1,0}$ over some open set $U \subset M$ which is orthonormal with respect to the given pseudohermitian structure on $M$, we call $\{W_1, \ldots, W_n\}$ a pseudohermitian frame. And $\{W_1, \ldots, W_n, \bar{W}_1, \ldots, \bar{W}_n, T\}$ forms a local frame for $\mathbb{C}TM$. Now let $U$ be a relatively compact open subset of a normal coordinate neighborhood, with contact form $\theta$ and pseudo-hermitian frame $\{W_1, \ldots, W_n\}$. Let $X_j = \text{Re}W_j$ and $X_{j+n} = \text{Im}W_j$. Denote $X^\alpha = X_{\alpha_1} \cdots X_{\alpha_k}$, where $\alpha = (\alpha_1, \cdots, \alpha_k)$, and denote $l(\alpha) = k$. Define the norm

$$\|f\|_{\mathcal{S}_r(U)} = \sup_{l(\alpha) \leq k} \|X^\alpha f\|_{L^p(U)}.$$

The Folland-Stein space $\mathcal{S}_r^p(U)$ is defined as the completion of $C_0^\infty$ with respect to the norm $\|\cdot\|_{\mathcal{S}_r(U)}$. (c.f. [17]). We have the following Folland-Stein embedding theorem, which is the CR version of Sobolev embedding theorem.

**Proposition 2.1.** ([25]) For $\frac{1}{s} = \frac{1}{r} - \frac{k}{2n+2}$, where $1 < r < s < \infty$. Then we have

$$\mathcal{S}_r^p(M) \subset L^s(M).$$

When $n = 1$, we define the CR Paneitz operator $P$ by

$$P\varphi = 4(\varphi_1 \bar{1} + \sqrt{-1}A_{11} \varphi_1) \cdot$$

We say $P$ is non-negative if

$$\int_M \varphi P\varphi dV_0 \geq 0$$

for all real smooth functions $\varphi$. The positivity of the CR Paneitz operator and the conformal sub-Laplacian guarantees that a compact three-dimensional CR manifold can be embedded into $\mathbb{C}^n$ for some integer $n$ (see [8]). At the end of this section, we introduce the concept of Carnot-Carathédory distance $d(\cdot, \cdot)$ between any two points $p, q \in M$. A piecewise smooth curve $\gamma : [0, 1] \to M$ is said to be a Legendrian curve if $\gamma'(t) \in G$ whenever $\gamma'(t)$ exists. And

$$l(\gamma) = \int_0^1 h(\gamma'(t), \gamma'(t)) dt,$$
where \( h(X, Y) = d\theta(X, JY) \). We denote \( C_{p,q} \) the set of all Legendrian curves which join \( p \) and \( q \). Then the Carnot-Carathéodory distance is defined as
\[
d(p, q) = \inf\{l(\gamma) : \gamma \in C_{p,q}\}.
\]

3. The CR Yamabe flow

In this section, we recall the definition and some basic facts of the CR Yamabe flow. Throughout this paper, we assume that \((M, \theta_0)\) is a compact strictly pseudo-convex CR manifold of real dimension \( 2n + 1 \) with a contact form \( \theta_0 \). Hereafter, we denote \( R_{\theta_0} \) the Webster scalar curvature with respect to the contact form \( \theta_0 \). The CR Yamabe constant of \( \theta_0 \) is defined as
\[
Y(M, \theta_0) = \inf_{u \in C^\infty(M), u > 0} \frac{\int_M ((2 + \frac{2}{n})|\nabla_{\theta_0} u|^2 + R_{\theta_0} u^2) dV_{\theta_0}}{\left(\int_M u^{\frac{2n}{n+2}} dV_{\theta_0}\right)^{\frac{n+2}{n}}}.
\]

The CR Yamabe flow is defined by
\[
\frac{\partial}{\partial t} \theta(t) = -(R_{\theta(t)} - r_{\theta(t)}) \theta(t) \quad \text{for} \quad t \geq 0, \quad \theta(t)|_{t=0} = \theta_0,
\]
where \( R_{\theta(t)} \) is the Webster scalar curvature with respect to the contact form \( \theta(t) \), and \( r_{\theta(t)} \) is the average value of the Webster scalar curvature:
\[
r_{\theta(t)} = \frac{\int_M R_{\theta(t)} dV_{\theta(t)}}{\int_M dV_{\theta(t)}}.
\]

If we write \( \theta(t) = u(t)^{\frac{n}{2n+2}} \theta_0 \) for some function \( u(t) \), then the CR Yamabe flow in (3.2) can be written as
\[
\frac{\partial}{\partial t} u(t) = -\frac{n}{2}(R_{\theta(t)} - r_{\theta(t)}) u(t) \quad \text{for} \quad t \geq 0, \quad u(t)|_{t=0} = 1.
\]
Since \( \theta(t) = u(t)^{\frac{n}{2n+2}} \theta_0 \), it follows from (3.1) that we have the CR Yamabe equation:
\[
(2 + \frac{2}{n})\Delta_{\theta_0} u(t) + R_{\theta_0} u(t) = R_{\theta(t)} u(t)^{1+\frac{2}{n}}.
\]

By (3.4) and (3.5), the CR Yamabe flow in (3.2) is equivalent to
\[
\frac{\partial}{\partial t} \left(u(t)^{\frac{n+2}{n}}\right) = \frac{n+2}{2} \left((2 + \frac{2}{n})\Delta_{\theta_0} u(t) - R_{\theta_0} u(t) + r_{\theta(t)} u(t)^{1+\frac{2}{n}}\right),
\]
which is a weakly parabolic partial differential equation. The short time existence of the CR Yamabe flow was proved by Chang and Cheng in [7].

As we have mentioned above, the long time existence and convergence of the CR Yamabe flow were proved in [41] when \( Y(M, \theta_0) < 0 \). In this paper, we consider the case when the CR Yamabe constant is positive, i.e. \( Y(M, \theta_0) > 0 \). In this case, the long time existence has already been proved by the first author in [24]. Therefore, to prove Theorem 1.1 we will focus on proving the convergence part. Since \( Y(M, \theta_0) > 0 \), by choosing another contact form in the same conformal class if necessary, we may assume that \( \theta_0 \) has positive Webster scalar curvature, i.e.
\[
R_{\theta_0} > 0.
\]
Without loss of generality, we can choose the initial contact form \( \theta_0 \) such that
\[
\text{Vol}(M, \theta_0) = \int_M dV_{\theta_0} = 1.
\]
Since the CR Yamabe flow preserves volume (see Proposition 3.1 in [24]), it follows from (3.8) that
\[
\text{Vol}(M, \theta(t)) = \int_M u(t)^{2 + \frac{4}{n}} dV_{\theta_0} = 1 \quad \text{for all } t \geq 0.
\]
By Proposition 3.3 in [24], the function \( t \mapsto r_{\theta(t)} \) is non-increasing. Indeed, we have (see (3.5) in [24])
\[
\frac{d}{dt} r_{\theta(t)} = -n \int_M (R_{\theta(t)} - r_{\theta(t)})^2 dV_{\theta(t)}.
\]
It follows from (3.1), (3.3), and (3.5) that \( r_{\theta(t)} \geq Y(M, \theta_0) \). Hence, the following limit exists and satisfies:
\[
\lim_{t \to \infty} r_{\theta(t)} = r_\infty \geq Y(M, \theta_0) > 0.
\]
It was proved in [24] that (see Proposition 4.1 and Corollary 4.1 in [24])
\[
\lim_{t \to \infty} \int_M |R_{\theta(t)} - r_{\theta(t)}|^p dV_{\theta(t)} = 0.
\]
for all \( 1 < p < n + 2 \), and
\[
\lim_{t \to \infty} \int_M |R_{\theta(t)} - r_\infty|^p dV_{\theta(t)} = 0.
\]
for all \( 1 < p < n + 2 \).

4. Proof of the main result assuming Proposition 4.1

The proof of Theorem 1.1 will be based on the following proposition.

**Proposition 4.1.** Let \( \{t_\nu : \nu \in \mathbb{N}\} \) be a sequence of times such that \( t_\nu \to \infty \) as \( \nu \to \infty \). Then we can find a real number \( 0 < \gamma < 1 \) and a constant \( C \) such that, after passing to a subsequence, we have
\[
r_{\theta(t_\nu)} - r_\infty \leq C \left( \int_M u(t_\nu)^{2 + \frac{4}{n}} |R_{\theta(t_\nu)} - r_\infty|^{\frac{2n + 2}{n + 2}} dV_{\theta_0} \right)^{\frac{n + 2}{2n + 2}(1 + \gamma)}
\]
for all integer \( \nu \) in that subsequence. Note that \( \gamma \) and \( C \) may depend on the sequence \( \{t_\nu : \nu \in \mathbb{N}\} \).

By assuming Proposition 4.1, we are going to prove Theorem 1.1. The following result is an immediate consequence of Proposition 4.1.

**Proposition 4.2.** There exists \( 0 < \gamma < 1 \) and \( t_0 > 0 \) such that
\[
r_{\theta(t)} - r_\infty \leq C \left( \int_M u(t)^{2 + \frac{4}{n}} |R_{\theta(t)} - r_\infty|^{\frac{2n + 2}{n + 2}} dV_{\theta_0} \right)^{\frac{n + 2}{2n + 2}(1 + \gamma)}
\]
for all \( t \geq t_0 \).

**Proof.** Suppose this is not true. Then there exists a sequence of times \( \{t_\nu : \nu \in \mathbb{N}\} \) and \( \{C_\nu : \nu \in \mathbb{N}\} \) such that \( t_\nu \geq \nu \), \( C_\nu \to \infty \) as \( \nu \to \infty \) and
\[
r_{\theta(t_\nu)} - r_\infty \geq C_\nu \left( \int_M u(t_\nu)^{2 + \frac{4}{n}} |R_{\theta(t_\nu)} - r_\infty|^{\frac{2n + 2}{n + 2}} dV_{\theta_0} \right)^{\frac{n + 2}{2n + 2}(1 + \frac{\nu}{\nu + 1})}
\]
for all \( 1 < p < n + 2 \).
for all $\nu \in \mathbb{N}$. We now apply Proposition 4.1 to this sequence $\{t_{\nu} : \nu \in \mathbb{N}\}$. Hence, there exists an infinite subset $I \subset \mathbb{N}$, a real number $0 < \gamma < 1$ and a positive constant $C$ such that

$$r_{\theta(t_{\nu})} - r_{\infty} \leq C \left( \int_M u(t_{\nu})^{2 + \frac{2}{n}} |R_{\theta(t_{\nu})} - r_{\infty}|^{\frac{2n+2}{n+2}} dV_{\theta_0} \right)^\frac{n+2}{2n+2} (1+\gamma)$$

for all $\nu \in I$. Thus we conclude that

$$C_{\nu} \leq C \left( \int_M u(t_{\nu})^{2 + \frac{2}{n}} |R_{\theta(t_{\nu})} - r_{\infty}|^{\frac{2n+2}{n+2}} dV_{\theta_0} \right)^\frac{n+2}{2n+2} (1+\gamma)$$

for all $\nu \in I$, which is a contradiction in view of (3.13) and the fact that $C_{\nu} \to \infty$ as $\nu \to \infty$. This proves the assertion. \hfill \Box

**Proposition 4.3.** We have

$$\int_0^\infty \left( \int_M u(t)^{2 + \frac{2}{n}} (R_{\theta(t)} - r_{\theta(t)})^2 dV_{\theta_0} \right)^\frac{1}{2} dt \leq C.$$

**Proof.** It follows from (3.9) and Proposition 4.2 that

$$r_{\theta(t)} - r_{\infty} \leq C \left( \int_M u(t)^{2 + \frac{2}{n}} |R_{\theta(t)} - r_{\infty}|^{\frac{2n+2}{n+2}} dV_{\theta_0} \right)^\frac{n+2}{2n+2} (1+\gamma)$$

$$\leq C \left( \int_M u(t)^{2 + \frac{2}{n}} |R_{\theta(t)} - r_{\theta(t)}|^{\frac{2n+2}{n+2}} dV_{\theta_0} \right)^\frac{n+2}{2n+2} (1+\gamma) + C(r_{\theta(t)} - r_{\infty})^{1+\gamma},$$

hence,

$$r_{\theta(t)} - r_{\infty} \leq C \left( \int_M u(t)^{2 + \frac{2}{n}} |R_{\theta(t)} - r_{\theta(t)}|^{\frac{2n+2}{n+2}} dV_{\theta_0} \right)^\frac{n+2}{2n+2} (1+\gamma)$$

if $t$ is sufficiently large in view of (3.13). Therefore, we obtain

$$\frac{d}{dt}(r_{\theta(t)} - r_{\infty}) \leq -n \left( \int_M u(t)^{2 + \frac{2}{n}} |R_{\theta(t)} - r_{\theta(t)}|^{\frac{2n+2}{n+2}} dV_{\theta_0} \right)^\frac{n+2}{n+2} \leq -C(r_{\theta(t)} - r_{\infty})^{\gamma}$$

where the first equality follows from (3.10), and the first inequality follows from (3.9) and Hölder’s inequality, and the last inequality follows from (1.1). This implies that

$$\frac{d}{dt}(r_{\theta(t)} - r_{\infty})^{1+\gamma} \geq C$$

where $C$ is a positive constant independent of $t$. From this, it follows that

$$r_{\theta(t)} - r_{\infty} \leq C t^{-\frac{1+\gamma}{1+\gamma}}$$

if $t$ is sufficiently large. Therefore, we have

$$\int_T^{2T} \left( \int_M u(t)^{2 + \frac{2}{n}} (R_{\theta(t)} - r_{\theta(t)})^2 dV_{\theta_0} \right)^\frac{1}{2} dt \leq \left( T \int_T^{2T} \int_M u(t)^{2 + \frac{2}{n}} (R_{\theta(t)} - r_{\theta(t)})^2 dV_{\theta_0} dt \right)^\frac{1}{2} \leq \left( \frac{T}{n} (r_{\theta(T)} - r_{\infty}) \right)^\frac{1}{2} \leq CT^{-\frac{1+\gamma}{2n+2}}$$
where the first inequality follows from Hölder’s inequality, the last inequality follows from (3.10) and (4.2). Since $0 < \gamma < 1$, we conclude that
\[
\int_0^\infty \left( \int_M u(t)^{2+\frac{2}{n}} (R_{\theta(t)} - r_{\theta(t)})^2 dV_{\theta_0} \right)^{\frac{1}{2}} dt \\
= \int_0^1 \left( \int_M u(t)^{2+\frac{2}{n}} (R_{\theta(t)} - r_{\theta(t)})^2 dV_{\theta_0} \right)^{\frac{1}{2}} dt \\
+ \sum_{k=0}^{\infty} \int_{2^k}^{2^{k+1}} \left( \int_M u(t)^{2+\frac{2}{n}} (R_{\theta(t)} - r_{\theta(t)})^2 dV_{\theta_0} \right)^{\frac{1}{2}} dt \\
\leq C \sum_{k=0}^{\infty} 2^{-\frac{k}{1-\gamma}} \leq C.
\]
This proves the assertion. □

Proposition 4.4. Given any $\eta_0 > 0$, we can find a real number $r > 0$ such that
\[
\int_{B_r(x)} u(t)^{2+\frac{2}{n}} dV_{\theta_0} \leq \eta_0
\]
for all $x \in M$ and $t \geq 0$. Here $B_r(x) = \{y \in M : d(x, y) < r\}$ and $d$ is the Carnot-Carathéodory distance on $M$ with respect to the contact form $\theta_0$.

Proof. Given any $\eta_0 > 0$, it follows from Proposition 4.3 that there exists a real number $T > 0$ such that
\[
\int_T^\infty \left( \int_M u(t)^{2+\frac{2}{n}} (R_{\theta(t)} - r_{\theta(t)})^2 dV_{\theta_0} \right)^{\frac{1}{2}} dt \leq \frac{\eta_0}{2(n+1)}.
\]
On the other hand, by (3.9), we can choose a real number $r > 0$ such that
\[
\int_{B_r(x)} u(t)^{2+\frac{2}{n}} dV_{\theta_0} \leq \frac{\eta_0}{2}
\]
for all $x \in M$ and $0 \leq t \leq T$. Combining these with (3.4), we have
\[
\int_{B_r(x)} u(t)^{2+\frac{2}{n}} dV_{\theta_0} \leq \int_{B_r(x)} u(T)^{2+\frac{2}{n}} dV_{\theta_0} \\
+ (n+1) \int_T^\infty \left( \int_M u(t)^{2+\frac{2}{n}} (R_{\theta(t)} - r_{\theta(t)})^2 dV_{\theta_0} \right)^{\frac{1}{2}} dt \\
\leq \frac{\eta_0}{2} + (n+1) \cdot \frac{\eta_0}{2(n+1)} = \eta_0
\]
for all all $x \in M$ and $t \geq T$. This proves the assertion. □

Proposition 4.5. The function $u(t)$ satisfies
\[
\text{sup}_M u(t) \leq C
\]
and
\[
\text{inf}_M u(t) \geq c
\]
for all $t \geq 0$. Here, $C$ and $c$ are positive constants independent of $t$.

Proof. We follow the proof of Proposition 5.5 in [24]. It follows from (3.9) and (3.13) that for $n + 1 < p < n + 2$
\[
\int_M |R_{\theta(t)}|^p dV_{\theta(t)} \leq C
\]
for some constant $C$ independent of $t$. Using (4.5), Proposition 4.4, and Hölder’s inequality, we obtain that for $p > q > n + 1$

\[
\left( \int_{B_r(x)} |R_{\theta(t)}|^q dV_{\theta(t)} \right)^{\frac{p-q}{p}} \left( \int_M |R_{\theta(t)}|^p dV_{\theta(t)} \right)^{\frac{q}{p}} \leq C \eta_0.
\]

Since $u(t)$ is smooth by the long time existence of the CR Yamabe flow, we can choose $\eta_0$ sufficiently small in (4.6) so that we can apply Proposition A.2 in [24] to conclude that $u(t)$ is uniformly bounded from above. This proves (4.3).

Now, if we define

\[
P = R_{\theta_0} + \sigma \left( \sup_{t \geq 0} \sup_M u(t) \right)^{\frac{1}{n}}
\]

where $\sigma$ is given by

\[
\sigma = \max \left\{ \sup (1 - R_{\theta_0}), 1 \right\}.
\]

It was proved in [24] that (see Proposition 3.4 in [24])

\[
R_{\theta(t)} + \sigma \geq 1,
\]

which implies that

\[
-(2 + \frac{2}{n}) \Delta_{\theta_0} u(t) + Pu(t) \geq -(2 + \frac{2}{n}) \Delta_{\theta_0} u(t) + R_{\theta_0} u(t) + \sigma u(t)^{1+\frac{2}{n}}
\]

\[
= (R_{\theta(t)} + \sigma) u(t)^{1+\frac{2}{n}} \geq 0.
\]

Hence, we can apply Proposition A.1 in [24] to conclude that

\[
C \left( \inf_M u(t) \right) \left( \sup_M u(t) \right)^{1+\frac{2}{n}} \geq \int_M u(t)^{2+\frac{2}{n}} dV_{\theta_0}
\]

for some positive constant $C$. Hence, (4.4) follows from (3.9) and (4.3). This proves the assertion. \[\square\]

**Proposition 4.6.** Let $0 < \alpha < \frac{2}{n+2}$. There exists a constant $C$ such that

\[
|u(x_2, t_2) - u(x_1, t_1)| \leq C \left( (t_1 - t_2)^{\frac{2}{n+2}} + d(x_1, x_2)^{\alpha} \right)
\]

for all $x_1, x_2 \in M$ and all $t_1, t_2 \geq 0$ satisfying $0 < t_1 - t_2 < 1$. Here $d$ is the Carnot-Carathéodory distance on $M$ with respect to the contact form $\theta_0$.

**Proof.** Choose $\alpha = 2 - \frac{2n + 2}{p}$ with $n + 1 < p < n + 2$. Then we have

\[
\int_M \left| -(2 + \frac{2}{n}) \Delta_{\theta_0} u(t) + R_{\theta_0} u(t) \right|^p dV_{\theta_0}
\]

\[
\leq C \int_M |R_{\theta(t)}|^p dV_{\theta(t)} \leq C \left( \int_M |R_{\theta(t)} - r_{\infty}|^p dV_{\theta(t)} + \int_M r_{\infty}^p dV_{\theta(t)} \right) \leq C
\]

where the first inequality follows from Proposition 4.5 and the last inequality follows from (3.9) and (4.3). This implies that

\[
|u(x_2, t) - u(x_1, t)| \leq Cd(x_1, x_2)^{\alpha}
\]
for all $x_1, x_2 \in M$ and all $t \geq 0$. On the other hand,

\begin{equation}
\int_M \left| \frac{\partial}{\partial t} u(t) \right|^p dV_{\theta_0} = \left( \frac{n}{2} \right)^p \int_M |(R_{\theta}(t) - r_{\theta(t)})u(t)|^p dV_{\theta_0} \leq C \int_M |R_{\theta}(t) - r_{\theta(t)}|^p dV_{\theta(t)} \leq C,
\end{equation}

where the first equality follows from (4.4), and the second inequality follows from Proposition 1.5, and the last inequality follows from (3.12). Using (4.7), we get

\[
|u(x, t_1) - u(x, t_2)| \leq C(t_1 - t_2)^{-(n+1)} \int_{B_{\sqrt{t_1-t_2}}(x)} |u(x, t_1) - u(x, t_2)| dV_{\theta_0}
\]

\[
\leq C(t_1 - t_2)^{-(n+1)} \int_{B_{\sqrt{t_1-t_2}}(x)} |u(t_1) - u(t_2)| dV_{\theta_0} + C(t_1 - t_2)^{2/n}
\]

\[
\leq C(t_1 - t_2)^{-n} \sup_{t_2 \leq t \leq t_1} \int_{B_{\sqrt{t_1-t_2}}(x)} \left| \frac{\partial}{\partial t} u(t) \right|^p dV_{\theta_0} + C(t_1 - t_2)^{2/n}
\]

\[
\leq C(t_1 - t_2)^{2/n} \sup_{t_2 \leq t \leq t_1} \left( \int_M \left| \frac{\partial}{\partial t} u(t) \right|^p dV_{\theta_0} \right)^{2/n} + C(t_1 - t_2)^{2/n}
\]

for all $x \in M$ and all $t_1, t_2 \geq 0$ satisfying $0 < t_1 - t_2 < 1$. This proves the assertion. \hfill \Box

In view of Proposition 4.6, it is easy to see that all derivatives of $u(x, t)$ are uniformly bounded on $[0, \infty)$. Indeed, we can apply Theorem 1.1 in [3], which says: let $X_1, X_2, \ldots, X_q$ be a system of real smooth vector fields satisfying Hörmander’s condition in a bounded domain $\Omega$ of $\mathbb{R}^n$. Let $A = \{a_{ij}(x, t)\}_{i,j=1}^q$ be a symmetric, uniformly positive-definite matrix of real functions defined in a domain $U \subset \Omega \times \mathbb{R}$. For operator of the form

\[
H = \partial_t - \sum_{i=1}^q a_{ij}(x, t)X_iX_j - \sum_{i=1}^q b_i(x, t)X_i - c(x, t)
\]

we have a priori estimate of Schauder type in parabolic Hörmander Hölder spaces $C^{k,\beta}_p(U)$. Namely, for $a_{ij}, b_i, c \in C^{k,\beta}_p(U)$ and $U' \subset U$, we have

\begin{equation}
\|u\|_{C^{k+2,\beta}_p(U')} \leq C\{\|Hu\|_{C^{k,\beta}_p(U')} + \|u\|_{L^{\infty}(U')}\}.
\end{equation}

Here, (see P.193-194 in [3])

\[
C^{k,\beta}_p(U) = \{ u : U \to \mathbb{R} : \|u\|_{C^{k,\beta}_p(U)} < \infty \},
\]

\[
\|u\|_{C^{k,\beta}_p(U)} = \sum_{|\alpha|+2b \leq k \atop b \leq k} \|\partial^\alpha X^b u\|_{C^0_p(U)};
\]

\[
\|u\|_{C^0_p(U)} = |u|_{C^0_p(U)} + \|u\|_{L^{\infty}(U)},
\]

\[
|u|_{C^0_p(U)} = \sup \left\{ \left| \frac{u(t,x)-u(s,y)}{d_p((x,t),(y,s))^{\beta}} \right| : (x,t),(y,s) \in U, (t,x) \neq (s,y) \right\},
\]

where $d_p$ is the parabolic Carnot-Carathéodory distance (see P. 189 in [3]) which is given by

\[
d_p((x_1, t_1), (x_2, t_2)) = \sqrt{d(x_1, x_2)^2 + |t_1 - t_2|^2},
\]
Here \( d \) is the Carnot-Carathéodory distance in \( \Omega \). Moreover, for any multiindex \( I = (i_1, i_2, \ldots, i_l) \), with \( 1 \leq i_j \leq q \), \( X^I u = X_{i_1} X_{i_2} \cdots X_{i_l} u \).

It follows from Proposition 4.6 that \( u(x, t) \in C_{P, \lambda}^{0, \frac{\lambda}{p}}([0, \infty) \times M) \). Therefore, with the estimate (4.3), Proposition 4.5 and Proposition 4.6 we can now apply the standard regularity theory for the weakly parabolic equation in (3.6) to show that all higher order derivatives of \( u(\cdot, t) \) are uniformly bounded on \([0, \infty)\). Therefore, \( u(\cdot, t) \) converges to a smooth function \( u_\infty \), which is positive in view of Proposition 4.5 or equivalently, \( \theta(\cdot, t) \) converges to a contact form \( \theta_\infty = u_\infty^2 \theta_0 \) as \( t \to \infty \). On the other hand, the contact form \( \theta_\infty \) has constant Webster scalar curvature thanks to (3.13). This proves Theorem 1.1.

5. Blow-up Analysis

The remaining part of this paper will be concerned with the proof of Proposition 4.1. Let \( \{t_\nu : \nu \in \mathbb{N} \} \) be a sequence of times such that \( t_\nu \to \infty \) as \( \nu \to \infty \). For abbreviation, we write \( u_\nu = u(t_\nu) \) and \( \theta_\nu = \theta(t_\nu) = u(t_\nu)^2 \theta_0 = u_\nu^2 \theta_0 \), it follows from (3.9) that

\[
\int_M u_\nu^{2+\frac{2}{n}} dV_{\theta_0} = 1 \quad \text{for all } \nu \in \mathbb{N}.
\]

On the other hand, it follows from (3.8) and (3.13) with \( p = \frac{2n+2}{n+2} \) that

\[
\int_M \left| - \left( 2 + \frac{2}{n} \right) \Delta_{\theta_0} u_\nu + R_{\theta_0} u_\nu - r_\infty u_\nu^{1+\frac{2}{n}} \right|^{\frac{2n+2}{n+2}} dV_{\theta_0} \to 0 \quad \text{as } \nu \to \infty.
\]

At this point, we may apply the following concentration-compactness result, which is the CR version of the results of Struwe [34] and Bahri and Coron [2]. See also [13] and [14].

**Theorem 5.1.** Under the assumptions of Theorem 1.1, suppose that \( \{u_\nu\} \) be a sequence of positive functions satisfying (4.1) and (4.2). After passing to a subsequence if necessary, we can find an integer \( m \), a smooth nonnegative function \( u_\infty \) and a sequence of \( m \)-tuples \((x^*, \varepsilon^*)\) with the following properties:

(i) The function \( u_\infty \) satisfies

\[
(2 + \frac{2}{n}) \Delta_{\theta_0} u_\infty - R_{\theta_0} u_\infty + r_\infty u_\infty^{1+\frac{2}{n}} = 0.
\]

(ii) For all \( i \neq j \), we have

\[
\frac{\varepsilon^*_j u_\nu - \varepsilon^*_i u_\nu}{\varepsilon^*_j u_\nu + \varepsilon^*_i u_\nu + d(x^*_i, x^*_j)^2} \to \infty \quad \text{as } \nu \to \infty.
\]

(iii) We have

\[
\left\| u_\nu - u_\infty - \sum_{k=1}^m \mathcal{P}(x^*_k, \varepsilon^*_k) \right\|_{S^2_\infty(M)} \to 0 \quad \text{as } \nu \to \infty.
\]

Here \( \mathcal{P}(x^*_k, \varepsilon^*_k) \) are the standard test functions constructed in (4.10) in Appendix A and \( d \) is Carnot-Carathéodory distance on \( M \) with respect to the contact form \( \theta_0 \).

The proof of Theorem 5.1 will be included in Appendix B.

**Proposition 5.2.** If \( u_\infty \) vanishes at one point in \( M \), then \( u_\infty \) vanishes everywhere.
Proof. It follows from (5.3) that
\[-(2 + \frac{2}{n})\Delta_{\theta_0} u_\infty + R_{\theta_0} u_\infty = r_\infty u_\infty^{1+\frac{2}{n}} \geq 0.\]
Since \( R_{\theta_0} \geq 0 \) by (3.17), we can apply Proposition A.1 in (24) to conclude that
\[
C \left( \inf_M u_\infty \right) \left( \sup_M u_\infty \right)^{1+\frac{2}{n}} \geq \int_M u_\infty^{2+\frac{2}{n}} dV_{\theta_0}
\]
for some positive constant \( C \). Proposition 5.2 follows from (5.6). \( \square \)

The cases \( u_\infty \equiv 0 \) and \( u_\infty > 0 \) will be discussed separately. The case \( u_\infty \equiv 0 \) will be studied in section 5.1. The case \( u_\infty > 0 \) will be studied in section 5.2.

We define two functionals \( E(u) \) and \( F(u) \) by
\[
E(u) = \frac{\int_M \left( (2 + \frac{2}{n})|\nabla_{\theta_0} u|^2 + R_{\theta_0} u^2 \right) dV_{\theta_0}}{\left( \int_M u^{2+\frac{2}{n}} dV_{\theta_0} \right)^{\frac{2}{n+2}}}
\]
and
\[
F(u) = \frac{\int_M \left( (2 + \frac{2}{n})|\nabla_{\theta_0} u|^2 + R_{\theta_0} u^2 \right) dV_{\theta_0}}{\int_M u^{2+\frac{2}{n}} dV_{\theta_0}}.
\]
Then we have
\[
1 = \lim_{\nu \to \infty} \int_M u_\nu^{2+\frac{2}{n}} dV_{\theta_0}
\]
\[
= \lim_{\nu \to \infty} \left( \int_M 2^{\frac{2}{n}} u_\infty^{2+\frac{2}{n}} dV_{\theta_0} + \sum_{k=1}^{m} \int_M \frac{2^{\frac{2}{n}}}{(\varepsilon_{k,\nu} + \varepsilon_j^2)} dV_{\theta_0} \right)
\]
\[
= \left( \frac{E(u_\infty)}{r_\infty} \right)^{n+1} + m \left( \frac{Y(S^{2n+1})}{r_\infty} \right)^{n+1}
\]
where we have used (5.1), (5.3), (5.5), and (A.21).

5.1. The case \( u_\infty \equiv 0 \). Throughout this subsection, we assume that \( u_\infty \equiv 0 \). For every \( \nu \in \mathbb{N} \), we denote by \( \mathcal{A}_\nu \) the set of all \( m \)-tuples \((x_k, \varepsilon_k, \alpha_k)_{1 \leq k \leq m} \in (M \times \mathbb{R}_+ \times \mathbb{R}_+)^m\) such that
\[
d(x_k, x_{k,\nu}) \leq \varepsilon_{k,\nu}, \quad \frac{1}{2} \leq \frac{\varepsilon_k}{\varepsilon_{k,\nu}} \leq 2, \quad \frac{1}{2} \leq \alpha_k \leq 2
\]
for all \( 1 \leq k \leq m \). Moreover, we can find \((x_k, \varepsilon_k, \alpha_k)_{1 \leq k \leq m} \in \mathcal{A}_\nu\) such that
\[
\int_M \left( (2 + \frac{2}{n})|\nabla_{\theta_0} (u_\nu - \sum_{k=1}^{m} \alpha_k \varepsilon_k \pi_{(x_k, \varepsilon_k, \alpha_k)})|^2 + R_{\theta_0} (u_\nu - \sum_{k=1}^{m} \alpha_k \varepsilon_k \pi_{(x_k, \varepsilon_k, \alpha_k)})^2 \right) dV_{\theta_0}
\]
\[
\leq \int_M \left( (2 + \frac{2}{n})|\nabla_{\theta_0} (u_\nu - \sum_{k=1}^{m} \alpha_k \varepsilon_k \pi_{(x_k, \varepsilon_k)})|^2 + R_{\theta_0} (u_\nu - \sum_{k=1}^{m} \alpha_k \varepsilon_k \pi_{(x_k, \varepsilon_k)})^2 \right) dV_{\theta_0}
\]
for all \((x_k, \varepsilon_k, \alpha_k)_{1 \leq k \leq m} \in \mathcal{A}_\nu\).

**Proposition 5.3.** (i) For all \( i \neq j \), we have
\[
\frac{\varepsilon_i^2}{\varepsilon_j^2} + \frac{\varepsilon_i^2}{\varepsilon_j^2} + \frac{d(x_i, x_j)^2}{\varepsilon_i^2 \varepsilon_j^2} \to \infty \quad \text{as} \quad \nu \to \infty,
\]
(ii) We have
\[ \left\| u_\nu - \sum_{k=1}^{m} \alpha_{k,\nu} \pi_{(x_{k,\nu}, \varepsilon_{k,\nu})} \right\|_{S_{1}^{2}(M)} \rightarrow 0 \quad \text{as} \quad \nu \rightarrow \infty. \]

Proof. (i) In view of (5.10), we have
\[
\begin{align*}
32 \frac{\varepsilon_{i,\nu}}{\varepsilon_{j,\nu}} + 32 \frac{\varepsilon_{j,\nu}}{\varepsilon_{i,\nu}} &+ 8 \frac{d(x_{i,\nu}, x_{j,\nu})^2}{\varepsilon_{i,\nu} \varepsilon_{j,\nu}} \\
&\geq 8 \frac{\varepsilon_{i,\nu}}{\varepsilon_{j,\nu}} + 8 \frac{\varepsilon_{j,\nu}}{\varepsilon_{i,\nu}} + 2 \frac{d(x_{i,\nu}, x_{j,\nu})^2}{\varepsilon_{i,\nu} \varepsilon_{j,\nu}} \\
&\geq 4 \frac{\varepsilon_{i,\nu}}{\varepsilon_{j,\nu}} + 4 \frac{\varepsilon_{j,\nu}}{\varepsilon_{i,\nu}} + \frac{(d(x_{i,\nu}, x_{j,\nu}) + \varepsilon_{i,\nu} + \varepsilon_{j,\nu})^2}{\varepsilon_{i,\nu} \varepsilon_{j,\nu}} \\
&\geq 4 \frac{\varepsilon_{i,\nu}}{\varepsilon_{j,\nu}} + 4 \frac{\varepsilon_{j,\nu}}{\varepsilon_{i,\nu}} + \frac{d(x_{i,\nu}, x_{j,\nu})^2}{\varepsilon_{i,\nu} \varepsilon_{j,\nu}},
\end{align*}
\]
and the last expression tends to infinity as \( \nu \rightarrow \infty \) by Theorem 5.1. Thus we have
\[
\frac{\varepsilon_{i,\nu}}{\varepsilon_{j,\nu}} + \frac{\varepsilon_{j,\nu}}{\varepsilon_{i,\nu}} + \frac{d(x_{i,\nu}, x_{j,\nu})^2}{\varepsilon_{i,\nu} \varepsilon_{j,\nu}} \rightarrow \infty \quad \text{as} \quad \nu \rightarrow \infty.
\]
Now Proposition 5.3(i) follows from this and Cauchy-Schwarz inequality.

(ii) By definition of \((x_{k,\nu}, \varepsilon_{k,\nu}, \alpha_{k,\nu})\) for all \( k \leq m \) in (5.11), we have
\[
\int_{M} \left( 2 + \frac{2}{n} \right) \left| \nabla_{\theta_{0}} \left( u_{\nu} - \sum_{k=1}^{m} \alpha_{k,\nu} \pi_{(x_{k,\nu}, \varepsilon_{k,\nu})} \right) \right|^{2} + R_{\theta_{0}} \left( u_{\nu} - \sum_{k=1}^{m} \alpha_{k,\nu} \pi_{(x_{k,\nu}, \varepsilon_{k,\nu})} \right)^{2} \right) dV_{\theta_{0}} \\
\leq \int_{M} \left( 2 + \frac{2}{n} \right) \left| \nabla_{\theta_{0}} \left( u_{\nu} - \sum_{k=1}^{m} \pi_{(x_{k,\nu}, \varepsilon_{k,\nu})} \right) \right|^{2} + R_{\theta_{0}} \left( u_{\nu} - \sum_{k=1}^{m} \pi_{(x_{k,\nu}, \varepsilon_{k,\nu})} \right)^{2} \right) dV_{\theta_{0}}.
\]
By Theorem 5.1, the expression on the right-hand side tends to 0 as \( \nu \rightarrow \infty \). This proves the assertion. \( \square \)

**Proposition 5.4.** We have
\[
d(x_{k,\nu}, x_{k,\nu}^*) \leq o(1) \varepsilon_{k,\nu}, \quad \frac{\varepsilon_{k,\nu}}{\varepsilon_{k,\nu}^*} = 1 + o(1), \quad \alpha_{k,\nu} = 1 + o(1)
\]
for all \( 1 \leq k \leq m \). In particular, \((x_{k,\nu}, \varepsilon_{k,\nu}, \alpha_{k,\nu})\) is an interior point of \( \mathcal{A}_{\nu} \) if \( \nu \) is sufficiently large.

Proof. Observe that
\[
\left\| \sum_{k=1}^{m} \alpha_{k,\nu} \pi_{(x_{k,\nu}, \varepsilon_{k,\nu})} - \sum_{k=1}^{m} \pi_{(x_{k,\nu}^*, \varepsilon_{k,\nu}^*)} \right\|_{S_{1}^{2}(M)} \\
\leq \left\| u_{\nu} - \sum_{k=1}^{m} \pi_{(x_{k,\nu}^*, \varepsilon_{k,\nu}^*)} \right\|_{S_{1}^{2}(M)} + \left\| u_{\nu} - \sum_{k=1}^{m} \alpha_{k,\nu} \pi_{(x_{k,\nu}, \varepsilon_{k,\nu})} \right\|_{S_{1}^{2}(M)} = o(1)
\]
by Theorem 5.1 and Proposition 5.10. From this, the assertion follows. \( \square \)

Now we decompose the function \( u_{\nu} \) as
\[
u_{\nu} = v_{\nu} + w_{\nu}
\]
where
\begin{equation}
(5.12)\quad v_\nu = \sum_{k=1}^{m} \alpha_{k,\nu} \mathfrak{m}_{(x_k,\nu,\varepsilon_k,\nu)}
\end{equation}
and
\begin{equation}
(5.13)\quad w_\nu = u_\nu - \sum_{k=1}^{m} \alpha_{k,\nu} \mathfrak{m}_{(x_k,\nu,\varepsilon_k,\nu)}.
\end{equation}

By Proposition 5.3, the function $w_\nu$ satisfies
\begin{equation}
(5.14)\quad \int_M \left( 2 + \frac{2}{n} |\nabla \theta_0 w_\nu |^2_{\theta_0} + R_{\theta_0} w_\nu^2 \right) dV_{\theta_0} = o(1).
\end{equation}

**Proposition 5.5.** If $\nu$ is sufficiently large, then
\begin{equation}
(5.15)\quad \int_M \left( 2 + \frac{2}{n} |\nabla \theta_0 \tilde{w}_\nu |^2_{\theta_0} + R_{\theta_0} \tilde{w}_\nu^2 \right) dV_{\theta_0} = 1
\end{equation}
and
\begin{equation}
(5.16)\quad \lim_{\nu \to \infty} \frac{n + 2}{n} \int_M \sum_{k=1}^{m} \mathfrak{m}_{(x_k,\nu,\varepsilon_k,\nu)}^2 \tilde{w}_\nu^2 dV_{\theta_0} \geq 1.
\end{equation}

Note that
\begin{equation}
(5.17)\quad \int_M |\tilde{w}_\nu|^2 + \frac{2}{n} dV_{\theta_0} \leq Y(M, \theta_0)^{-\frac{n+1}{n}}
\end{equation}
by (5.15). In view of Proposition 5.3, we can find a sequence $\{N_{\nu} : \nu \in \mathbb{N}\}$ such that $N_{\nu} \to \infty$, $N_{\nu} \varepsilon_{j,\nu} \to 0$ for all $1 \leq j \leq m$, and
\begin{equation}
(5.18)\quad \frac{1}{N_{\nu}} \varepsilon_{j,\nu} + d(x_{i,\nu}, x_{j,\nu}) \to \infty
\end{equation}
for all $i < j$. Let
\begin{equation}
(5.19)\quad \Omega_{j,\nu} = B_{N_{\nu} \varepsilon_{j,\nu}} \setminus \bigcup_{i=1}^{j-1} B_{N_{\nu} \varepsilon_{i,\nu}}(x_{i,\nu})
\end{equation}
for every $1 \leq j \leq m$. In view of (5.15) and (5.16), we can find an integer $1 \leq j \leq m$ such that
\begin{equation}
(5.20)\quad \lim_{\nu \to \infty} \int_{\Omega_{j,\nu}} \mathfrak{m}_{(x_{j,\nu},\varepsilon_{j,\nu})}^2 \tilde{w}_\nu^2 dV_{\theta_0} > 0
\end{equation}
and
\begin{equation}
(5.21)\quad \lim_{\nu \to \infty} \int_{\Omega_{j,\nu}} \left( 2 + \frac{2}{n} |\nabla \theta_0 \tilde{w}_\nu |^2_{\theta_0} + R_{\theta_0} \tilde{w}_\nu^2 \right) dV_{\theta_0} \leq \lim_{\nu \to \infty} \frac{n + 2}{n} r_{\infty} \int_M \mathfrak{m}_{(x_{j,\nu},\varepsilon_{j,\nu})}^2 \tilde{w}_\nu^2 dV_{\theta_0}.
\end{equation}
We now define a sequence of functions \( \hat{w}_\nu : T_{x_j,\nu} M \rightarrow \mathbb{R} \) by
\[
\hat{w}_\nu(z, t) = \varepsilon_{j, \nu}^n \hat{w}_\nu(\exp_{x_j, \nu}(\varepsilon_{j, \nu} z, \varepsilon_{j, \nu} t))
\]
for \((z, t) \in T_{x_j,\nu} M\). The sequence \( \{ \hat{w}_\nu : \nu \in \mathbb{N} \} \) satisfies
\[
\lim_{\nu \to \infty} \int_{\{(z, t) \in \mathbb{H}^n : |(z, t)| \leq N_\nu \}} (2 + \frac{2}{n})|\nabla_{\theta_0} \hat{w}_\nu(z, t)|^2 \, dV_{\theta_0} \leq 1
\]
and
\[
\lim_{\nu \to \infty} \int_{\{(z, t) \in \mathbb{H}^n : |(z, t)| \leq N_\nu \}} |\hat{w}_\nu(z, t)|^{2 + \frac{p}{2}} \, dV_{\theta_0} \leq Y(M, \theta_0)^{-\frac{p+n}{2}}
\]
in view of (5.19) and (5.17). Hence, we can take the weak limit to obtain a function \( \hat{w} : \mathbb{H}^n \rightarrow \mathbb{R} \) such that
\[
\int_{\mathbb{H}^n} \frac{1}{t^2 + (1 + |z|^2)^2} \hat{w}(z, t)^2 \, dV_{\theta_0} > 0
\]
and
\[
\int_{\mathbb{H}^n} |\nabla_{\theta_0} \hat{w}(z, t)|^2 \, dV_{\theta_0} \leq (n + 2) \int_{\mathbb{H}^n} \frac{1}{t^2 + (1 + |z|^2)^2} \hat{w}(z, t)^2 \, dV_{\theta_0}
\]
by (5.24), (5.21), and the definition of \( \pi_{(x_j, \nu, \varepsilon_{j, \nu})} \). By definition of \( (x_{k, \nu}, \varepsilon_{k, \nu}, \alpha_{k, \nu}) \) for \( k \leq m \) in (5.11), we have
\[
\frac{d}{d \alpha_k} \int_M \left( (2 + \frac{2}{n}) |\nabla_{\theta_0} (u_\nu - \sum_{k=1}^m \alpha_k \pi_{(x_{k, \nu}, \varepsilon_{k, \nu})})| \right)^2 \, dV_{\theta_0} + R_{\theta_0} \left( u_\nu - \sum_{k=1}^m \alpha_k \pi_{(x_{k, \nu}, \varepsilon_{k, \nu})} \right)^2 \, dV_{\theta_0} \bigg|_{\alpha_k = \alpha_{k, \nu}} = 0,
\]
which implies
\[
0 = \int_M \left( (2 + \frac{2}{n}) \Delta_{\theta_0} \pi_{(x_{k, \nu}, \varepsilon_{k, \nu})} - R_{\theta_0} \pi_{(x_{k, \nu}, \varepsilon_{k, \nu})} + r_n \pi_{(x_{k, \nu}, \varepsilon_{k, \nu})} \right) \, w_\nu \, dV_{\theta_0}.
\]
Using the estimate
\[
\| (2 + \frac{2}{n}) \Delta_{\theta_0} \pi_{(x_{k, \nu}, \varepsilon_{k, \nu})} - R_{\theta_0} \pi_{(x_{k, \nu}, \varepsilon_{k, \nu})} + r_n \pi_{(x_{k, \nu}, \varepsilon_{k, \nu})} \|_{L_{\frac{2n+2}{2n+4}}(M)} = o(1)
\]
and Hölder’s inequality, we conclude that
\[
r_\infty \int_M \pi_{(x_{k, \nu}, \varepsilon_{k, \nu})}^{1 + \frac{p}{2}} \, w_\nu \, dV_{\theta_0}
\leq \left( (2 + \frac{2}{n}) \Delta_{\theta_0} \pi_{(x_{k, \nu}, \varepsilon_{k, \nu})} - R_{\theta_0} \pi_{(x_{k, \nu}, \varepsilon_{k, \nu})} + r_n \pi_{(x_{k, \nu}, \varepsilon_{k, \nu})} \right) \left( \int_M |w_\nu|^{2 + \frac{p}{2}} \, dV_{\theta_0} \right)^{\frac{n}{2n+2}}
\leq o(1) \left( \int_M |w_\nu|^{2 + \frac{p}{2}} \, dV_{\theta_0} \right)^{\frac{n}{2n+2}}
\]
for all \( 1 \leq k \leq m \). Since \( r_\infty > 0 \), we have
\[
\left| \int_M \pi_{(x_{k, \nu}, \varepsilon_{k, \nu})}^{1 + \frac{p}{2}} \, w_\nu \, dV_{\theta_0} \right| \leq o(1) \left( \int_M |w_\nu|^{2 + \frac{p}{2}} \, dV_{\theta_0} \right)^{\frac{n}{2n+2}}
for all $1 \leq k \leq m$. Taking the weak limit yields

$$
\int_{\Omega^n} \frac{1}{(t^2 + (1 + |z|^2)^2)^{n+2}} \hat{w}(z, t) dV_{\theta_0^n} = 0
$$

by the definition of $\pi_{(x_{k, \nu}, \epsilon_{k, \nu})}$, $\hat{w}_{\nu}$, and $\hat{w}$. By definition of $(x_{k, \nu}, \epsilon_{k, \nu}, \alpha_{k, \nu})_{1 \leq k \leq m}$ in (5.11), we also have

$$
\frac{d}{d\epsilon_k} \int_M \left( (2 + \frac{2}{n}) |\nabla_{\theta_0} \left( u_\nu - \sum_{k=1}^m \alpha_k \pi_{(x_{k, \nu}, \epsilon_{k, \nu})} \right) |^2_{\theta_0} + R_{\theta_0} \left( u_\nu - \sum_{k=1}^m \alpha_k \pi_{(x_{k, \nu}, \epsilon_{k, \nu})} \right) \right) dV_{\theta_0} = 0,
$$

which implies that

$$
0 = \int_M \left( (2 + \frac{2}{n}) \Delta_{\theta_0} \left( \frac{\pi_{(x_{k, \nu}, \epsilon_{k, \nu})}}{\partial \epsilon_{k, \nu}} \right) - R_{\theta_0} \left( \frac{\pi_{(x_{k, \nu}, \epsilon_{k, \nu})}}{\partial \epsilon_{k, \nu}} \right) \right) w_\nu dV_{\theta_0}.
$$

Using the estimate

$$
\left\| (2 + \frac{2}{n}) \Delta_{\theta_0} \left( \frac{\pi_{(x_{k, \nu}, \epsilon_{k, \nu})}}{\partial \epsilon_{k, \nu}} \right) - R_{\theta_0} \left( \frac{\pi_{(x_{k, \nu}, \epsilon_{k, \nu})}}{\partial \epsilon_{k, \nu}} \right) \right\|_{L^{\frac{2n+2}{n+2}}(M)} = o(\epsilon_{k, \nu}^{-1})
$$

and Hölder’s inequality, we conclude that

$$
(1 + \frac{2}{n}) r_n \int_M \frac{\hat{w}}{\pi_{(x_{k, \nu}, \epsilon_{k, \nu})}} \left( \frac{\pi_{(x_{k, \nu}, \epsilon_{k, \nu})}}{\partial \epsilon_{k, \nu}} \right) w_\nu dV_{\theta_0}
$$

$$
\leq \left\| (2 + \frac{2}{n}) \Delta_{\theta_0} \left( \frac{\pi_{(x_{k, \nu}, \epsilon_{k, \nu})}}{\partial \epsilon_{k, \nu}} \right) - R_{\theta_0} \left( \frac{\pi_{(x_{k, \nu}, \epsilon_{k, \nu})}}{\partial \epsilon_{k, \nu}} \right) \right\|_{L^{\frac{2n+2}{n+2}}(M)}
\cdot \left( \int_M |w_\nu|^{2 + \frac{2}{\nu_{k, \epsilon}}} dV_{\theta_0} \right)^{\frac{\nu_{k, \epsilon}}{2n+2}} = o(\epsilon_{k, \nu}^{-1}) \left( \int_M |w_\nu|^{2 + \frac{2}{\nu_{k, \epsilon}}} dV_{\theta_0} \right)^{\frac{\nu_{k, \epsilon}}{2n+2}}
$$

for all $1 \leq k \leq m$. Since $r_n > 0$, we have

$$
\left\| \int_M \frac{\hat{w}}{\pi_{(x_{k, \nu}, \epsilon_{k, \nu})}} \left( \frac{\pi_{(x_{k, \nu}, \epsilon_{k, \nu})}}{\partial \epsilon_{k, \nu}} \right) w_\nu dV_{\theta_0} \right\| \leq o(\epsilon_{k, \nu}^{-1}) \left( \int_M |w_\nu|^{2 + \frac{2}{\nu_{k, \epsilon}}} dV_{\theta_0} \right)^{\frac{\nu_{k, \epsilon}}{2n+2}}
$$

for all $1 \leq k \leq m$. Taking the weak limit yields

$$
\int_{\Omega^n} \frac{1 - |z|^4 - t^2}{(t^2 + (1 + |z|^2)^2)^{n+2}} \hat{w}(z, t) dV_{\theta_0^n} = 0
$$

by the definition of $\pi_{(x_{k, \nu}, \epsilon_{k, \nu})}$, $\hat{w}_{\nu}$, and $\hat{w}$. Similarly, we have

$$
\left\| \int_M \frac{\hat{T}_{\pi_{(x_{k, \nu}, \epsilon_{k, \nu})}}}{\pi_{(x_{k, \nu}, \epsilon_{k, \nu})}} (\pi_{(x_{k, \nu}, \epsilon_{k, \nu})}) w_\nu dV_{\theta_0} \right\| \leq o(\epsilon_{k, \nu}^{-2}) \left( \int_M |w_\nu|^{2 + \frac{2}{\nu_{k, \epsilon}}} dV_{\theta_0} \right)^{\frac{\nu_{k, \epsilon}}{2n+2}},
$$

$$
\left\| \int_M \frac{\hat{Z}_{\pi_{(x_{k, \nu}, \epsilon_{k, \nu})}}}{\pi_{(x_{k, \nu}, \epsilon_{k, \nu})}} (\pi_{(x_{k, \nu}, \epsilon_{k, \nu})}) w_\nu dV_{\theta_0} \right\| \leq o(\epsilon_{k, \nu}^{-1}) \left( \int_M |w_\nu|^{2 + \frac{2}{\nu_{k, \epsilon}}} dV_{\theta_0} \right)^{\frac{\nu_{k, \epsilon}}{2n+2}},
$$

$$
\left\| \int_M \frac{\hat{Z}_{\pi_{(x_{k, \nu}, \epsilon_{k, \nu})}}}{\pi_{(x_{k, \nu}, \epsilon_{k, \nu})}} (\pi_{(x_{k, \nu}, \epsilon_{k, \nu})}) w_\nu dV_{\theta_0} \right\| \leq o(\epsilon_{k, \nu}^{-1}) \left( \int_M |w_\nu|^{2 + \frac{2}{\nu_{k, \epsilon}}} dV_{\theta_0} \right)^{\frac{\nu_{k, \epsilon}}{2n+2}}.
$$
for all \(1 \leq k, l \leq n\), where \(T = \frac{\partial}{\partial t}, Z_l = \frac{\partial}{\partial z_l} + \sqrt{-1}z_l\frac{\partial}{\partial t}\) and \(\mathbb{Z}_l = \frac{\partial}{\partial \mathbb{Z}_l} - \sqrt{-1}z_l\frac{\partial}{\partial t}\) are tangent vector fields in \(\mathbb{H}^n\). Taking the weak limit yields

\[
\begin{align*}
\int_{\mathbb{H}^n} \frac{t}{(t^2 + (1 + |z|^2)^2)^{n+1/2}} \tilde{w}(z,t) dV_{\delta^n} &= 0, \\
\int_{\mathbb{H}^n} \frac{(1 + |z|^2)z_l + \sqrt{-1}z_l t}{(t^2 + (1 + |z|^2)^2)^{n+1/2}} \tilde{w}(z,t) dV_{\delta^n} &= 0 \quad \text{for } 1 \leq k \leq n, \\
\int_{\mathbb{H}^n} \frac{(1 + |z|^2)z_l - \sqrt{-1}z_l t}{(t^2 + (1 + |z|^2)^2)^{n+1/2}} \tilde{w}(z,t) dV_{\delta^n} &= 0 \quad \text{for } 1 \leq k \leq n.
\end{align*}
\]

(5.26)

Now we define \(\tilde{w}(z,t) = \tilde{w}(\mathbb{Z},t)\) for \((z,t) \in \mathbb{H}^n \subset \mathbb{C}^n \times \mathbb{R}\). Then it follows from (5.25) and (5.29) that

\[
\int_{\mathbb{H}^n} \frac{1}{t^2 + (1 + |z|^2)^2} \tilde{w}(z,t)^2 dV_{\delta^n} > 0,
\]

(5.27)

\[
\int_{\mathbb{H}^n} |\nabla_{\delta^n} \tilde{w}(z,t)|^2 dV_{\delta^n} \leq (n + 2)n \int_{\mathbb{H}^n} \frac{1}{t^2 + (1 + |z|^2)^2} \tilde{w}(z,t)^2 dV_{\delta^n},
\]

(5.28)

and

\[
\begin{align*}
\int_{\mathbb{H}^n} \frac{1}{(t^2 + (1 + |z|^2)^2)^{n+1/2}} \tilde{w}(z,t) dV_{\delta^n} &= 0, \\
\int_{\mathbb{H}^n} \frac{1 - |z|^4 - t^2}{(t^2 + (1 + |z|^2)^2)^{n+1/2}} \tilde{w}(z,t) dV_{\delta^n} &= 0, \\
\int_{\mathbb{H}^n} \frac{t}{(t^2 + (1 + |z|^2)^2)^{n+1/2}} \tilde{w}(z,t) dV_{\delta^n} &= 0, \\
\int_{\mathbb{H}^n} \frac{(1 + |z|^2)z_l - \sqrt{-1}z_l t}{(t^2 + (1 + |z|^2)^2)^{n+1/2}} \tilde{w}(z,t) dV_{\delta^n} &= 0 \quad \text{for } 1 \leq k \leq n, \\
\int_{\mathbb{H}^n} \frac{(1 + |z|^2)z_l + \sqrt{-1}z_l t}{(t^2 + (1 + |z|^2)^2)^{n+1/2}} \tilde{w}(z,t) dV_{\delta^n} &= 0 \quad \text{for } 1 \leq k \leq n.
\end{align*}
\]

(5.29)

We claim that any \(\tilde{w}\) satisfying (5.25) and (5.29) must be zero identically. Indeed, if we define \(\tilde{u}(x) = |1 + x_{n+1}|^{-n} \tilde{w} \circ F(x)\) where \(F : S^{2n+1} \rightarrow \mathbb{H}^n\) is given by

\[
F(x) = \left(\begin{array}{c}
x_1 \\
\vdots \\
x_n \\
\frac{1}{1 + x_{n+1}} \end{array}\right) + \frac{Re(\sqrt{-1} - 1 + x_{n+1})}{1 + x_{n+1}},
\]

then we can rewrite (5.29) as (see p.177 in [28])

\[
\begin{align*}
\int_{S^{2n+1}} \tilde{u} dV_{\delta^{2n+1}} &= 0, \\
\int_{S^{2n+1}} x_k \tilde{u} dV_{\delta^{2n+1}} = \int_{S^{2n+1}} \tau x_k \tilde{u} dV_{\delta^{2n+1}} &= 0 \quad \text{for } 1 \leq k \leq n + 1,
\end{align*}
\]

(5.30)

where \(x = (x_1, \cdots, x_{n+1}) \in S^{2n+1} \subset \mathbb{C}^{n+1}\), because \(F^{-1} : \mathbb{H}^n \rightarrow S^{2n+1}\) is given by

\[
F^{-1}(z,t) = \left(\begin{array}{c}
z \\
\frac{2z}{1 + |z|^2 - \sqrt{-1}t} \\
\frac{1 - |z|^2 + \sqrt{-1}t}{1 + |z|^2 - \sqrt{-1}t}
\end{array}\right)
\]

where \((z,t) \in \mathbb{H}^n \subset \mathbb{C}^n \times \mathbb{R}\). The eigenfunctions and eigenvalues of \(\Delta_{\delta^{2n+1}}\) are well-known—see [16]. Namely, 1 is an eigenfunction with respect to the eigenvalue...
(5.31) \[ \int_{S^{2n+1}} |\nabla_{\theta} S^{2n+1} \tilde{u}|^2 dV_{\theta} \geq \lambda_2 \int_{S^{2n+1}} \tilde{u}^2 dV_{\theta} \]

where \( \lambda_2 > n/2 \) is the second eigenvalue of \( \lambda_{\theta} S^{2n+1} \). On the other hand, (5.28) can be written as (see p.177 in [28])

(5.32) \[ \int_{S^{2n+1}} \left( |\nabla_{\theta} S^{2n+1} \tilde{u}|^2 + \frac{n^2}{4} \tilde{u}^2 \right) dV_{\theta} \leq \frac{(n+2)n}{4} \int_{S^{2n+1}} \tilde{u}^2 dV_{\theta} \]

Combining this with (5.31), we conclude that \( \tilde{u} \) is zero identically, which implies that \( \tilde{w} \) is zero identically. But this contradicts (5.27). This proves the assertion. \( \square \)

**Corollary 5.6.** If \( \nu \) is sufficiently large, then

\[ \frac{n+2}{n} \underset{r_{\infty}}{\int} \frac{1}{M^2} w_{\nu}^2 dV_{0} \leq (1-c) \int_{M} \left( 2 + \frac{2}{n} |\nabla_{\theta} w_{\nu}|_{\theta}^2 + R_{\theta} w_{\nu} \right) dV_{\theta} \]

for some positive constant \( c \) which is independent of \( \nu \).

**Proof.** Note that

\[ \int_{M} \left| v_{\nu}^2 - \sum_{k=1}^{m} u_{(x_{k,\nu},\varepsilon_{k,\nu})} \right|^2 dV_{\theta} = o(1) \]

by (5.12) and Proposition 5.4. Thus, Corollary 5.6 follows from Proposition 5.5. \( \square \)

**Proposition 5.7.** If \( \nu \) is sufficiently large, the energy of \( v_{\nu} \) satisfies the estimate

\[ E(v_{\nu}) \leq \left( \sum_{k=1}^{m} E(u_{(x_{k,\nu},\varepsilon_{k,\nu})})^{n+1} \right)^{\frac{1}{n+1}}. \]

**Proof.** By Hölder’s inequality, we have

(5.32) \[ \left( \sum_{k=1}^{m} E(u_{(x_{k,\nu},\varepsilon_{k,\nu})})^{n+1} \right)^{\frac{1}{n+1}} \left( \int_{M} v_{\nu}^{2+\frac{2}{n}} dV_{\theta} \right)^{\frac{n}{n+1}} \]

\[ \geq \int_{M} \left( \sum_{k=1}^{m} F(u_{(x_{k,\nu},\varepsilon_{k,\nu})})^{n+1} \right)^{\frac{2}{n+1}} dV_{\theta} \]

\[ \geq \int_{M} \sum_{k=1}^{m} \alpha_{k,\nu} F(u_{(x_{k,\nu},\varepsilon_{k,\nu})})^{n+1} dV_{\theta} \]

\[ + 2 \int_{M} \sum_{1 \leq i < j \leq m} \alpha_{i,\nu} \alpha_{j,\nu} \left( F(u_{(x_{i,\nu},\varepsilon_{i,\nu})})^{n+1} \right) dV_{\theta} \]

\[ + F(u_{(x_{i,\nu},\varepsilon_{i,\nu})})^{n+1} \]

\[ + \frac{2}{n+1} \left( \frac{2}{n+1} \right)^{\frac{1}{n+1}} \]

\[ \leq \frac{1}{n+1} \sum_{k=1}^{m} E(u_{(x_{k,\nu},\varepsilon_{k,\nu})})^{n+1} dV_{\theta}. \]
Consider a pair $i < j$. We can find positive constants $c$ and $C$ independent of $\nu$ such that

\[
\mathcal{V}_{(x_i, \nu, \varepsilon_i, \nu)}(x)^{1 + \frac{2}{p}} \mathcal{V}_{(x_j, \nu, \varepsilon_j, \nu)}(x) \geq c \left( \frac{\varepsilon_i^2}{\varepsilon_j^2 + d(x_i, x_j)^4} \right)^{\frac{m}{2}} \varepsilon_i^{-2n-2} \quad \text{and}
\]

\[
\mathcal{V}_{(x_i, \nu, \varepsilon_i, \nu)}(x) \mathcal{V}_{(x_j, \nu, \varepsilon_j, \nu)}(x)^{1 + \frac{2}{p}} \leq C \left( \frac{\varepsilon_i^2}{\varepsilon_j^2 + d(x_i, x_j)^4} \right)^{\frac{m}{2}} \varepsilon_i^{-2n-2}
\]

if $d(x_i, x_j) \leq \varepsilon_i^2$ and $\nu$ is sufficiently large. From this, it follows that

\[
\left( F(\mathcal{V}_{(x_i, \nu, \varepsilon_i, \nu)})^n + F(\mathcal{V}_{(x_j, \nu, \varepsilon_j, \nu)})^n + F(\mathcal{V}_{(x_i, \nu, \varepsilon_i, \nu)}) F(\mathcal{V}_{(x_j, \nu, \varepsilon_j, \nu)}) \right) \mathcal{V}_{(x_i, \nu, \varepsilon_i, \nu)}^{1 + \frac{2}{m}} \mathcal{V}_{(x_j, \nu, \varepsilon_j, \nu)}^{1 + \frac{2}{m}}
\]

if $\nu$ is sufficiently large. Integrating it over $M$, we obtain

\[
\int_M \left( F(\mathcal{V}_{(x_i, \nu, \varepsilon_i, \nu)})^n + F(\mathcal{V}_{(x_j, \nu, \varepsilon_j, \nu)})^n + F(\mathcal{V}_{(x_i, \nu, \varepsilon_i, \nu)}) F(\mathcal{V}_{(x_j, \nu, \varepsilon_j, \nu)}) \right) \mathcal{V}_{(x_i, \nu, \varepsilon_i, \nu)}^{1 + \frac{2}{m}} \mathcal{V}_{(x_j, \nu, \varepsilon_j, \nu)}^{1 + \frac{2}{m}} dV_0
\]

(5.33)

\[
\geq C \left( \frac{\varepsilon_i^2}{\varepsilon_j^2 + d(x_i, x_j)^4} \right)^{\frac{m}{2}} + \int_M F(\mathcal{V}_{(x_i, \nu, \varepsilon_i, \nu)}) \mathcal{V}_{(x_i, \nu, \varepsilon_i, \nu)}^{1 + \frac{2}{m}} dV_0.
\]

if $\nu$ is sufficiently large. Combining this with (5.32), we get

\[
\left( \sum_{k=1}^m E(\mathcal{V}_{(x_k, \nu, \varepsilon_k, \nu)})^n \right)^{\frac{1}{n+1}} \left( \int_M v_{\nu}^{2 + \frac{2}{p}} dV_0 \right)^{\frac{m}{n+1}}
\]

(5.34)

\[
\geq \int_M \sum_{k=1}^m \alpha_{k, \nu}^2 F(\mathcal{V}_{(x_k, \nu, \varepsilon_k, \nu)}) \mathcal{V}_{(x_k, \nu, \varepsilon_k, \nu)}^{2 + \frac{2}{p}} dV_0
\]

\[
+ 2 \int_M \sum_{1 \leq i < j \leq m} \alpha_{i, \nu} \alpha_{j, \nu} F(\mathcal{V}_{(x_{i, \nu}, \varepsilon_{i, \nu})}) \mathcal{V}_{(x_{i, \nu}, \varepsilon_{i, \nu})}^{1 + \frac{2}{p}} dV_0
\]

\[
+ C \sum_{1 \leq i < j \leq m} \left( \frac{\varepsilon_i^2 + d(x_i, x_j)^4}{\varepsilon_j^2 + d(x_i, x_j)^4} \right)^{\frac{m}{2}}.
\]

By the definition of $v_{\nu}$ in (5.32), we have

(5.35)

\[
E(v_{\nu}) \left( \int_M v_{\nu}^{2 + \frac{2}{p}} dV_0 \right)^{\frac{m}{n+1}} = \int_M \sum_{k=1}^m \alpha_{k, \nu}^2 F(\mathcal{V}_{(x_k, \nu, \varepsilon_k, \nu)}) \mathcal{V}_{(x_k, \nu, \varepsilon_k, \nu)}^{2 + \frac{2}{p}} dV_0
\]

\[
- 2 \int_M \sum_{1 \leq i < j \leq m} \alpha_{i, \nu} \alpha_{j, \nu} \mathcal{V}_{(x_{i, \nu}, \varepsilon_{i, \nu})} \left( \left( 2 + \frac{2}{n} \right) \Delta_{\theta_0} \mathcal{V}_{(x_{i, \nu}, \varepsilon_{i, \nu})} - \frac{\theta_0}{\theta_0 \mathcal{V}_{(x_{i, \nu}, \varepsilon_{i, \nu})}} \right) dV_0.
\]
Substituting (5.34) into (5.35), we obtain

\[ E(\nu) \left( \int_M \nu^{2+\frac{2}{n}} \, dV_{\theta_0} \right)^{\frac{n}{n+1}} \]

\[ \leq \left( \sum_{k=1}^{m} E(\bar{u}_{(x_0,0,\ldots,\epsilon_k,\nu)}) \right)^{\frac{n}{n+1}} \left( \int_M \nu^{2+\frac{2}{n}} \, dV_{\theta_0} \right)^{\frac{n}{n+1}} \]

\[ - 2 \int_M \sum_{1 \leq i < j \leq m} \alpha_{i,\nu} \alpha_{j,\nu} \bar{u}_{(x_0,0,\ldots,\epsilon_i,\epsilon_j,\nu)} \]

\[ \cdot \left( (2 + \frac{2}{n}) \Delta_{\theta_0} \bar{u}_{(x_0,0,\ldots,\epsilon_i,\epsilon_j,\nu)} - R_{\theta_0} \bar{u}_{(x_0,0,\ldots,\epsilon_i,\epsilon_j,\nu)} + F(\bar{u}_{(x_0,0,\ldots,\epsilon_i,\epsilon_j,\nu)}) \bar{u}_{(x_0,0,\ldots,\epsilon_i,\epsilon_j,\nu)}^{1+\frac{2}{n}} \right) \, dV_{\theta_0} \]

\[ - C \sum_{1 \leq i < j \leq m} \left( \frac{\varepsilon_i^2 \varepsilon_j^2 \nu^2}{\varepsilon_{j,\nu}^4 + d(x_{i,\nu}, x_{j,\nu})^4} \right)^{\frac{\nu}{2}}. \]

Since \( F(\bar{u}_{(x_0,0,\ldots,\epsilon_i,\epsilon_j,\nu)}) = r_{\infty} + o(1) \) by Lemma A.5, it follows from Lemma A.6 and A.7 that

\[ \int_M \bar{u}_{(x_0,0,\ldots,\epsilon_i,\epsilon_j,\nu)} \left( (2 + \frac{2}{n}) \Delta_{\theta_0} \bar{u}_{(x_0,0,\ldots,\epsilon_i,\epsilon_j,\nu)} - R_{\theta_0} \bar{u}_{(x_0,0,\ldots,\epsilon_i,\epsilon_j,\nu)} + r_{\infty} \bar{u}_{(x_0,0,\ldots,\epsilon_i,\epsilon_j,\nu)}^{1+\frac{2}{n}} \right) \, dV_{\theta_0} \]

\[ \leq \int_M \bar{u}_{(x_0,0,\ldots,\epsilon_i,\epsilon_j,\nu)} \left( (2 + \frac{2}{n}) \Delta_{\theta_0} \bar{u}_{(x_0,0,\ldots,\epsilon_i,\epsilon_j,\nu)} - R_{\theta_0} \bar{u}_{(x_0,0,\ldots,\epsilon_i,\epsilon_j,\nu)} + r_{\infty} \bar{u}_{(x_0,0,\ldots,\epsilon_i,\epsilon_j,\nu)}^{1+\frac{2}{n}} \right) \, dV_{\theta_0} \]

\[ + |F(\bar{u}_{(x_0,0,\ldots,\epsilon_i,\epsilon_j,\nu)}) - r_{\infty}| \int_M \bar{u}_{(x_0,0,\ldots,\epsilon_i,\epsilon_j,\nu)} \bar{u}_{(x_0,0,\ldots,\epsilon_i,\epsilon_j,\nu)}^{1+\frac{2}{n}} \, dV_{\theta_0} \]

\[ \leq C(\delta^4 + \delta^{2n} + \frac{\varepsilon_i^2 \varepsilon_j^2 \nu^2}{\delta^4} \left( \frac{\varepsilon_i^2 \varepsilon_j^2 \nu^2}{\varepsilon_{j,\nu}^4 + d(x_{i,\nu}, x_{j,\nu})^4} \right)^{\frac{\nu}{2}} + o(1) \left( \frac{\varepsilon_i^2 \varepsilon_j^2 \nu^2}{\varepsilon_{j,\nu}^4 + d(x_{i,\nu}, x_{j,\nu})^4} \right)^{\frac{\nu}{2}} \]

for \( i < j \). Hence, if we choose \( \delta \) sufficiently small, the assertion follows. \( \square \)

**Corollary 5.8.** The energy of \( \nu_{\nu} \) satisfies the estimate

\[ E(\nu_{\nu}) \leq \left( mY(S^{2n+1}) \right)^{\frac{1}{n+1}} \]

if \( \nu \) is sufficiently large.

**Proof.** Using Proposition A.3, we obtain

\[ E(\bar{u}_{(x_0,0,\ldots,\epsilon_k,\nu)}) \leq Y(S^{2n+1}) \]

for all \( 1 \leq k \leq m \). Now Corollary 5.8 follows from Proposition 5.7. \( \square \)

### 5.2. The case \( u_{\infty} > 0 \)

Next we discuss the case \( u_{\infty} > 0 \).

**Proposition 5.9.** There exists a sequence of smooth functions \( \{ \psi_a : a \in \mathbb{N} \} \) and a sequence of real numbers \( \{ \lambda_a : a \in \mathbb{N} \} \) with the following properties:

(i) For every \( a \in \mathbb{N} \), the function \( \psi_a \) satisfies

\[ \tag{5.36} (2 + \frac{2}{n}) \Delta_{\theta_0} \psi_a - R_{\theta_0} \psi_a + \lambda_a u_{\infty} \psi_a = 0. \]
(ii) For all \(a, b \in \mathbb{N}\), we have
\[
\int_M u_\infty^2 \psi_a \psi_b \, dV_{\theta_0} = \begin{cases} 1, & \text{if } a = b, \\ 0, & \text{if } a \neq b. \end{cases}
\]

(iii) The span of \(\{\psi_a : a \in \mathbb{N}\}\) is dense in \(L^2(M)\).

(iv) \(\lambda_a \to \infty\) as \(a \to \infty\).

Proof. Consider the linear operator
\[
\psi \mapsto u_\infty^2 \left( (2 + \frac{2}{n}) \Delta_{\theta_0} \psi - R_{\theta_0} \psi \right).
\]
The operator is symmetric with respect to the inner product
\[
(\psi_1, \psi_2) \mapsto \int_M u_\infty^2 \psi_1 \psi_2 \, dV_{\theta_0}
\]
on \(L^2(M)\). Hence, the assertion follows from the spectral theorem.

Let
\[
A = \{a \in \mathbb{N} : \lambda_a \leq \frac{n + 2}{n} r_{\infty} \}
\]
which is a finite subset of \(\mathbb{N}\) by Proposition 5.9(iv). We denote \(\Pi\) the projection operator
\[
\Pi f = \sum_{a \in A} \left( \int_M \psi_a f \, dV_{\theta_0} \right) u_\infty^2 \psi_a = f - \sum_{a \in A} \left( \int_M \psi_a f \, dV_{\theta_0} \right) u_\infty^2 \psi_a.
\]

Lemma 5.10. For every \(1 \leq p < \infty\), we can find a constant \(C\) such that
\[
\|f\|_{L^p(M)} \leq C \left\| (2 + \frac{2}{n}) \Delta_{\theta_0} f - R_{\theta_0} f + \frac{n + 2}{n} r_{\infty} u_\infty^2 f \right\|_{L^p(M)} + C \sup_{a \in A} \left| \int_M u_\infty^2 \psi_a f \, dV_{\theta_0} \right|.
\]

Proof. Suppose it is not true. By compactness, we can find a function \(f \in L^p(M)\) satisfying \(\|f\|_{L^p(M)} = 1\),
\[
\int_M u_\infty^2 \psi_a f \, dV_{\theta_0} = 0 \quad \text{for all } a \in A,
\]
and
\[
(2 + \frac{2}{n}) \Delta_{\theta_0} f - R_{\theta_0} f + \frac{n + 2}{n} r_{\infty} u_\infty^2 f = 0
\]
in the sense of distributions. Multiply (5.36) by \(\psi_a\) and integrate over \(M\), we obtain for all \(a \in \mathbb{N}\)
\[
0 = \int_M \psi_a \left( (2 + \frac{2}{n}) \Delta_{\theta_0} f - R_{\theta_0} f + \frac{n + 2}{n} r_{\infty} u_\infty^2 f \right) \, dV_{\theta_0}
\]
(5.41)
\[
= \left( \frac{n + 2}{n} r_{\infty} - \lambda_a \right) \int_M u_\infty^2 \psi_a f \, dV_{\theta_0}
\]
by (5.36). In particular, for \(a \notin A\), we have \(\lambda_a > \frac{n + 2}{n} r_{\infty}\), which implies that
\[
\int_M u_\infty^2 \psi_a f \, dV_{\theta_0} = 0 \quad \text{for all } a \notin A.
\]
Combining this with (5.39), we have \(f \equiv 0\) which contradicts \(\|f\|_{L^p(M)} = 1\). This proves the assertion. \(\square\)
Lemma 5.11. (i) There exists a constant $C$ such that

$$
\|f\|_{L^{1+\frac{2}{n}}(M)} \leq C \left( \| \Pi \left( (2 + \frac{2}{n})\Delta_{th} f - R_{th} f + \frac{n + 2}{n} r_{\infty} u_{\infty}^2 f \right) \|_{L^{1+\frac{2}{n}}(M)} + C \sup_{a \in A} \left| \int_M u_{\infty}^2 \psi_a f \, dV_{th} \right| \right).
$$

(ii) There exists a constant $C$ such that

$$
\|f\|_{L^p(M)} \leq C \left( \| \Pi \left( (2 + \frac{2}{n})\Delta_{th} f - R_{th} f + \frac{n + 2}{n} r_{\infty} u_{\infty}^2 f \right) \|_{L^{1}(M)} + C \sup_{a \in A} \left| \int_M u_{\infty}^2 \psi_a f \, dV_{th} \right| \right).
$$

Proof: To prove (i), we apply the result of Bramanti-Braudolini in \[4\] to conclude

$$
\begin{align*}
\Pi \left( (2 + \frac{2}{n})\Delta_{th} f - R_{th} f + \frac{n + 2}{n} r_{\infty} u_{\infty}^2 f \right) &\leq (2 + 2\theta) \sum_{a \in A} (\frac{n + 2}{n} r_{\infty} u_{\infty}^2 f - l_a) \left( \int_M u_{\infty}^2 \psi_a f \, dV_{th} \right) u_{\infty}^2 \psi_a, \\
&= \left( \Pi \left( (2 + 2\theta) \sum_{a \in A} (\frac{n + 2}{n} r_{\infty} u_{\infty}^2 f - l_a) \left( \int_M u_{\infty}^2 \psi_a f \, dV_{th} \right) u_{\infty}^2 \psi_a \right) \right).
\end{align*}
$$

By (5.42) and Lemma 5.10 with $p = \frac{(n + 1)(n + 2)}{n^2 + 2n + 2}$, we get

$$
\|f\|_{L^{1+\frac{2}{n}}(M)} \leq \left( \| \Pi \left( (2 + \frac{2}{n})\Delta_{th} f - R_{th} f + \frac{n + 2}{n} r_{\infty} u_{\infty}^2 f \right) \|_{L^{1+\frac{2}{n}}(M)} + C \|f\|_{L^{p}(M)} \right).
$$

By (5.43) and the definition of $\Pi$, we have

$$
\Pi \left( (2 + \frac{2}{n})\Delta_{th} f - R_{th} f + \frac{n + 2}{n} r_{\infty} u_{\infty}^2 f \right) = (2 + \frac{2}{n})\Delta_{th} f - R_{th} f + \frac{n + 2}{n} r_{\infty} u_{\infty}^2 f
$$

which implies that

$$
\begin{align*}
\|\Pi \left( (2 + \frac{2}{n})\Delta_{th} f - R_{th} f + \frac{n + 2}{n} r_{\infty} u_{\infty}^2 f \right)\|_{L^p(M)} &\leq \|\Pi \left( (2 + \frac{2}{n})\Delta_{th} f - R_{th} f + \frac{n + 2}{n} r_{\infty} u_{\infty}^2 f \right)\|_{L^p(M)} + C \sup_{a \in A} \left| \int_M u_{\infty}^2 \psi_a f \, dV_{th} \right|.
\end{align*}
$$
for $1 \leq q < \infty$. Now combining (5.43) and (5.44) with $q = \frac{(n+1)(n+2)}{n^2 + 2n + 2}$, we prove (i). To prove (ii), we note that by Lemma 5.10 with $p = 1$ we have

$$\|f\|_{L^1(M)} \leq C\left\| \left(2 + \frac{2}{n}\right)\Delta_{\theta_0}f - R_{\theta_0}f + \frac{n+2}{n}r_\infty u_\infty^2 f \right\|_{L^1(M)} + C \sup_{a \in A} \left| \int_M \tilde{\psi}_a f \right|.$$ Combining this with (5.44) with $q = 1$, we prove (ii).

**Lemma 5.12.** There exists a positive real number $\zeta$ with the following property: for every vector $z = (z_1, z_2, \ldots, z_{|A|}) \in \mathbb{R}^{|A|}$ with $|z| \leq \zeta$, there exists a smooth function $\pi_z$ such that

$$\int_M u_\infty^2(\pi_z - u_\infty) \psi_a dV_{\theta_0} = z_a$$

for all $a \in A$ and

$$\Pi \left( (2 + \frac{2}{n})\Delta_{\theta_0} \pi_z - R_{\theta_0} \pi_z + \frac{n+2}{n}r_\infty u_\infty^2 \pi_z \right) = 0.$$

Furthermore, the map $z \mapsto \pi_z$ defined in (i) is real analytic.

**Proof.** This is the consequence of the implicit function theorem. \qed

**Lemma 5.13.** There exists a real number $0 < \gamma < 1$ such that

$$E(\pi_z) - E(u_\infty) \leq C \sup_{a \in A} \left| \frac{\partial}{\partial \pi_z} E(\pi_z) \right|^{1+\gamma}$$

if $z$ is sufficiently small.

**Proof.** Note that the function $z \mapsto E(\pi_z)$ is real analytic by Lemma 5.12. According to the results of Lojasiewicz (see [33]), there exists a real number $0 < \gamma < 1$ such that

$$|E(\pi_z) - E(u_\infty)| \leq C \sup_{a \in A} \left| \frac{\partial}{\partial \pi_z} E(\pi_z) \right|^{1+\gamma}$$

if $z$ is sufficiently small. The partial derivatives of the function $z \mapsto E(\pi_z)$ are given by

$$\frac{\partial}{\partial \pi_z} E(\pi_z) = -2 \frac{\int_M \tilde{\psi}_{a,z} \left( (2 + \frac{2}{n})\Delta_{\theta_0} \pi_z - R_{\theta_0} \pi_z + r_\infty u_\infty^2 \pi_z + \frac{4}{n}r_\infty u_\infty^2 \right) dV_{\theta_0}}{\left( \int_M u_\infty^2 \pi_z + \frac{4}{n}r_\infty u_\infty^2 \right)^{\frac{1+\gamma}{\gamma}}},$$

$$-2(F(\pi_z) - r_\infty) \frac{\int_M \pi_z^{1+\gamma} \tilde{\psi}_{a,z} dV_{\theta_0}}{\left( \int_M u_\infty^2 + \frac{4}{n}r_\infty u_\infty^2 \right)^{\frac{1+\gamma}{\gamma}}},$$

where $\tilde{\psi}_{a,z} = \frac{\partial}{\partial \pi_z} \pi_z$ for $a \in A$. Since $\int_M u_\infty^2(\pi_z - u_\infty) \psi_b dV_{\theta_0} = z_b$ for all $b \in A$ by Lemma 5.12, by differentiating it with respect to $\pi_z$, we obtain

$$\int_M u_\infty^2 \tilde{\psi}_{a,z} \psi_b dV_{\theta_0} = \begin{cases} 1, & \text{if } a = b, \\ 0, & \text{if } a \neq b. \end{cases}$$

On the other hand, by Lemma 5.12 we have

$$\Pi \left( (2 + \frac{2}{n})\Delta_{\theta_0} \pi_z - R_{\theta_0} \pi_z + \frac{n+2}{n}r_\infty u_\infty^2 \pi_z \right) = 0.$$
Differentiate it with respect to \( z \), we get

\[
\Pi \left( (2 + \frac{2}{n}) \Delta_{\theta_0} \psi_{a,z} - R_{\theta_0} \psi_{a,z} + \frac{n + 2}{n} r_\infty u_\infty \psi_{a,z} \right) = 0.
\]

By definition of \( \Pi \) in (5.38) and (5.48), we have

\[
(2 + \frac{2}{n}) \Delta_{\theta_0} \psi_{a,z} - R_{\theta_0} \psi_{a,z} + \frac{n + 2}{n} r_\infty u_\infty \psi_{a,z} = 0.
\]

By definition of \( F \), we also have

\[
F \left( v z \right) - r_\infty = - \int_M \psi_a \left( (2 + \frac{2}{n}) \Delta_{\theta_0} \psi_{a,z} - R_{\theta_0} \psi_{a,z} + r_\infty u_\infty \psi_{a,z} \right) dV_{\theta_0}.
\]

Substituting (5.51) into (5.46), we obtain

\[
\frac{\partial}{\partial z_a} E (u z) = - \int_M \psi_a \left( (2 + \frac{2}{n}) \Delta_{\theta_0} \psi_{a,z} - R_{\theta_0} \psi_{a,z} + \frac{n + 2}{n} r_\infty u_\infty \psi_{a,z} \right) dV_{\theta_0}.
\]

where the second equality follows from (5.50), and the last equality follows from (5.47). Thus we conclude that

\[
\sup_{a \in A} \left| \frac{\partial}{\partial z_a} E (u z) \right| \leq C \sup_{a \in A} \left| \int_M \psi_a \left( (2 + \frac{2}{n}) \Delta_{\theta_0} \psi_{a,z} - R_{\theta_0} \psi_{a,z} + r_\infty u_\infty \psi_{a,z} \right) dV_{\theta_0} \right|.
\]

Combining this with (5.45), we prove Lemma 5.13. \( \square \)

Now, for every \( \nu \in \mathbb{N} \), we denote by \( A_\nu \) the set of all pairs \((z, (x_k, \epsilon_k, \alpha_k))_{1 \leq k \leq m} \) \( \in \mathbb{R}^{|A|} \times (M \times \mathbb{R}_+ \times \mathbb{R}_+)^m \) such that

\[
|z| \leq \zeta
\]

and

\[
d(x_k, x_{k,\nu}^*) \leq \epsilon_{k,\nu}^*, \quad \frac{1}{2} \leq \frac{\epsilon_k}{\epsilon_{k,\nu}^*} \leq 2, \quad \frac{1}{2} \leq \alpha_k \leq 2
\]
for all $1 \leq k \leq m$. Moreover, we can find $(z, (x_k, \varepsilon_k, \alpha_k))_{1 \leq k \leq m} \in A_\nu$ such that

\[
\int_M \left( (2 + \frac{2}{n}) \left| \nabla_{\theta_0} \left( u_\nu - \overline{v}_z - \sum_{k=1}^{m} \alpha_{k, \nu} \overline{u}_{(x_k, \varepsilon_k, \alpha_k)} \right) \right|^2_{\theta_0} \right) + R_{\theta_0} \left( u_\nu - \overline{v}_z - \sum_{k=1}^{m} \alpha_{k, \nu} \overline{u}_{(x_k, \varepsilon_k, \alpha_k)} \right)^2_{\theta_0} \ dV_{\theta_0}
\]

(5.54)

for all $(z, (x_k, \varepsilon_k, \alpha_k))_{1 \leq k \leq m} \in A_\nu$. Then we can obtain the following two propositions, which are similar to Proposition 5.3 and Proposition 5.4.

**Proposition 5.14.** (i) For all $i \neq j$, we have

\[
\frac{\varepsilon_{i, \nu}^2}{\varepsilon_{j, \nu}^2} + \frac{\varepsilon_{j, \nu}^2}{\varepsilon_{i, \nu}^2} + \frac{d(x_{i, \nu}, x_{j, \nu})^4}{\varepsilon_{i, \nu}^2 \varepsilon_{j, \nu}^2} \to \infty \quad \text{as} \quad \nu \to \infty,
\]

(ii) We have

\[
\left\| u_\nu - \overline{v}_z - \sum_{k=1}^{m} \alpha_{k, \nu} \overline{u}_{(x_k, \varepsilon_k, \alpha_k)} \right\|_{S^2(M)} \to 0 \quad \text{as} \quad \nu \to \infty.
\]

**Proposition 5.15.** We have

\[
|z_\nu| = o(1)
\]

and

\[
d(x_{k, \nu}, x_{k^*, \nu}) \leq o(1) \varepsilon_{k, \mu} \varepsilon_{k, \mu} \varepsilon_{k, \nu} = 1 + o(1), \quad \alpha_{k, \nu} = 1 + o(1)
\]

for all $1 \leq k \leq m$. In particular, $(z_\nu, (x_k, \varepsilon_k, \alpha_k))_{1 \leq k \leq m}$ is an interior point of $A_\nu$ if $\nu$ is sufficiently large.

Now we decompose the function $u_\nu$ as

\[
u_\nu = v_\nu + w_\nu
\]

where

\[
v_\nu = \overline{v}_z + \sum_{k=1}^{m} \alpha_{k, \nu} \overline{u}_{(x_k, \varepsilon_k, \alpha_k)}
\]

and

\[
w_\nu = u_\nu - \overline{v}_z - \sum_{k=1}^{m} \alpha_{k, \nu} \overline{u}_{(x_k, \varepsilon_k, \alpha_k)}.
\]

By Proposition 5.14(ii), the function $w_\nu$ satisfies

\[
\int_M \left( (2 + \frac{2}{n}) \left| \nabla_{\theta_0} w_\nu \right|^2_{\theta_0} + R_{\theta_0} w_\nu^2 \right) dV_{\theta_0} = o(1).
\]
Proposition 5.16. For every $a \in A$, we have

\begin{equation}
\left| \int_M u_0^2 \psi_a w_\nu \, dV_{\theta_0} \right| \leq o(1) \int_M |w_\nu| \, dV_{\theta_0}.
\end{equation}

Proof. As above, let $\tilde{\psi}_{a,z} = \frac{\partial}{\partial z_a} \bar{z}$z. By definition of $(z_\nu, (x_{k,\nu}, \varepsilon_{k,\nu}, \alpha_{k,\nu})_{1 \leq k \leq m})$ in (5.61), we have

\begin{align*}
0 &= \frac{1}{2} \frac{d}{ds_a} \int_M \left( (2 + \frac{2}{n}) \nabla_{\theta_0} (u_\nu - \bar{z}_z - \sum_{k=1}^m \alpha_{k,\nu} \bar{u}_{(x_{k,\nu}, \varepsilon_{k,\nu})}) \right)^2 dV_{\theta_0} \\
&\quad + R_{\theta_0} (u_\nu - \bar{z}_z - \sum_{k=1}^m \alpha_{k,\nu} \bar{u}_{(x_{k,\nu}, \varepsilon_{k,\nu})})^2 \bigg|_{z = z_\nu} \\
&= \int_M \left( (2 + \frac{2}{n}) \Delta_{\theta_0} \tilde{\psi}_{a,z} - R_{\theta_0} \tilde{\psi}_{a,z} \right) w_\nu \, dV_{\theta_0}.
\end{align*}

This implies together with Proposition 5.9(1) that

$$
\lambda_a \int_M u_0^2 \psi_a w_\nu \, dV_{\theta_0} = \int_M \left( (2 + \frac{2}{n}) \Delta_{\theta_0} (\tilde{\psi}_{a,z} - \psi_a) - R_{\theta_0} (\tilde{\psi}_{a,z} - \psi_a) \right) w_\nu \, dV_{\theta_0}
$$

Since $\lambda_a > 0$, we conclude that

$$
\left| \int_M u_0^2 \psi_a w_\nu \, dV_{\theta_0} \right| \leq o(1) \int_M |w_\nu| \, dV_{\theta_0}
$$

for all $a \in A$. This proves the assertion. \hfill \Box

Proposition 5.17. If $\nu$ is sufficiently large, then

$$
\frac{n + 2}{n} \int_M \left( u_\nu^2 + \sum_{k=1}^m \bar{u}_{(x_{k,\nu}, \varepsilon_{k,\nu})}^2 \right) \, dV_{\theta_0} \leq (1 - c) \int_M \left( (2 + \frac{2}{n}) |\nabla_{\theta_0} w_\nu|^2 + R_{\theta_0} w_\nu^2 \right) \, dV_{\theta_0}
$$

for some positive constant $c$ which is independent of $\nu$.

Proof. Suppose this is not true. Upon rescaling, we obtain a sequence of functions \{\tilde{w}_\nu : \nu \in \mathbb{N}\} such that

$$
\int_M \left( (2 + \frac{2}{n}) |\nabla_{\theta_0} \tilde{w}_\nu|^2 + R_{\theta_0} \tilde{w}_\nu^2 \right) \, dV_{\theta_0} = 1
$$

and

$$
\lim_{\nu \to \infty} \frac{n + 2}{n} \int_M \left( u_\nu^2 + \sum_{k=1}^m \bar{u}_{(x_{k,\nu}, \varepsilon_{k,\nu})}^2 \right) \, dV_{\theta_0} \geq 1.
$$

Note that

$$
\int_M |\tilde{w}_\nu|^2 \, dV_{\theta_0} \leq Y(M, \theta_0)^{-\frac{1}{n+4}}
$$

by (5.60). In view of Proposition 5.14, we can find a sequence \{N_\nu : \nu \in \mathbb{N}\} such that $N_\nu \to \infty$, $N_\nu \varepsilon_{j,\nu} \to 0$ for all $1 \leq j \leq m$, and

$$
\frac{1}{N_\nu} \varepsilon_{j,\nu} + d(x_{i,\nu}, x_{j,\nu}) \to \infty
$$
for all $i < j$. Let

\begin{equation}
\Omega_{j,\nu} = B_{N_{\nu} \varepsilon_{j,\nu}} \setminus \bigcup_{i=1}^{j-1} B_{N_{\nu} \varepsilon_{i,\nu}}(x_{i,\nu})
\end{equation}

for every $1 \leq j \leq m$. In view of (5.60) and (5.61), there are two cases to be considered:

**Case 1.** Suppose that

\begin{equation}
\lim_{\nu \to \infty} \int_M u_{\infty}^2 \tilde{w}^2 \, dV_{\theta_0} > 0
\end{equation}

and

\begin{equation}
\lim_{\nu \to \infty} \int_{M \setminus \bigcup_{j=1}^m \Omega_{j,\nu}} \left(2 + \frac{2}{n}\right)|\nabla_{\theta_0} \tilde{w}_\nu|_{\theta_0}^2 + R_{\theta_0} \tilde{w}_\nu^2 \right) \, dV_{\theta_0} \leq \lim_{\nu \to \infty} \frac{n+2}{2n} r_\infty \int_M u_{\infty}^2 \tilde{w}^2 \, dV_{\theta_0}.
\end{equation}

Let $\tilde{w}$ be the weak limit of the sequence \{$\tilde{w}_\nu : \nu \in \mathbb{N}$\}. Then, by (5.65) and (5.66), the function $\tilde{w}$ satisfies

\begin{equation}
\int_M u_{\infty}^2 \tilde{w}^2 \, dV_{\theta_0} > 0
\end{equation}

and

\begin{equation}
\int_M \left(2 + \frac{2}{n}\right)|\nabla_{\theta_0} \tilde{w}|_{\theta_0}^2 + R_{\theta_0} \tilde{w}^2 \right) \, dV_{\theta_0} \leq \lim_{\nu \to \infty} \frac{n+2}{2n} r_\infty \int_M u_{\infty}^2 \tilde{w}^2 \, dV_{\theta_0}.
\end{equation}

This implies

\begin{equation}
\sum_{a \in \mathbb{N}} \lambda_a \left(\int_M u_{\infty}^2 \psi_a \tilde{w} \, dV_{\theta_0}\right)^2 \leq \sum_{a \in \mathbb{N}} \frac{n+2}{2n} r_\infty \left(\int_M u_{\infty}^2 \psi_a \tilde{w} \, dV_{\theta_0}\right)^2.
\end{equation}

Using Proposition 5.10, we obtain

\begin{equation}
\int_M u_{\infty}^2 \psi_a \tilde{w} \, dV_{\theta_0} = 0
\end{equation}

for all $a \in A$. Thus, we conclude that $\tilde{w}(x) = 0$ for all $x \in M$, which contradicts to (5.67).

**Case 2.** Suppose that there exists an integer $1 \leq j \leq m$ such that

\begin{equation}
\lim_{\nu \to \infty} \int_{\Omega_{j,\nu}} u_{\infty}^2 \tilde{w}^2 \, dV_{\theta_0} > 0
\end{equation}

and

\begin{equation}
\lim_{\nu \to \infty} \int_{\Omega_{j,\nu}} \left(2 + \frac{2}{n}\right)|\nabla_{\theta_0} \tilde{w}_\nu|_{\theta_0}^2 + R_{\theta_0} \tilde{w}_\nu^2 \right) \, dV_{\theta_0} \leq \lim_{\nu \to \infty} \frac{n+2}{2n} r_\infty \int_{\pi_{\nu}(\omega_{j,\nu})} u_{\infty}^2 \tilde{w}_\nu^2 \, dV_{\theta_0}.
\end{equation}

Now we can follow the same arguments in the proof of Proposition 5.5 to finish the proof. More precisely, we can construct a function $\tilde{w} : \mathbb{H}^n \to \mathbb{R}$ which satisfies (5.22)-(5.23), as in the proof of Proposition 5.5 which is a contraction. This proves the assertion. \hfill \Box

**Corollary 5.18.** If $\nu$ is sufficiently large, then

\begin{equation}
\frac{n+2}{2n} r_\infty \int_M u_{\infty}^2 \tilde{w}^2 \, dV_{\theta_0} \leq (1 - c) \int_M \left(2 + \frac{2}{n}\right)|\nabla_{\theta_0} w_{\nu}|_{\theta_0}^2 + R_{\theta_0} w_{\nu}^2 \right) \, dV_{\theta_0}
\end{equation}

for some positive constant $c$ which is independent of $\nu$. 

Proof. By (5.66) and Proposition 5.13 we have

$$\int_M \left| u_\nu^\frac{2}{n} - u_\infty^\frac{2}{n} - \sum_{k=1}^m \tilde{u}_k^\frac{2}{n} \right|^2 \delta_{\varphi_k} \, dV_\theta = o(1).$$

Therefore, Corollary 5.18 follows from Proposition 5.17. 

\[ \square \]

**Lemma 5.19.** The difference $u_\nu - \overline{u}_z$ satisfies the estimate

$$\parallel u_\nu - \overline{u}_z \parallel_{L^{1+\frac{2}{n}}(M)} \leq C \parallel u_\nu^{1+\frac{2}{n}} (R_{\theta_\nu} - r_\infty) \parallel_{L^{n+2} (M)} + C \sum_{k=1}^m \varepsilon_k \parallel u_\nu \parallel_{L^{1+\frac{2}{n}}(M)}$$

if $\nu$ is sufficiently large.

**Proof.** It follows from Lemma 5.12 that

$$\Pi \left( (2 + \frac{2}{n}) \Delta_{\theta_\nu} \overline{u}_z - R_{\theta_\nu} \overline{u}_z + \frac{n + 2}{n} r_\infty u_\nu^{\frac{2}{n}} \overline{u}_z \right) = 0.$$ 

Then we have

$$\Pi \left( (2 + \frac{2}{n}) \Delta (u_\nu - \overline{u}_z) - R_{\theta_\nu} (u_\nu - \overline{u}_z) + \frac{n + 2}{n} r_\infty u_\nu^{\frac{2}{n}} (u_\nu - \overline{u}_z) \right)$$

$$= \Pi \left( - u_\nu^{1+\frac{2}{n}} (R_{\theta_\nu} - r_\infty) + r_\infty (u_\nu^{1+\frac{2}{n}} + \frac{n + 2}{n} u_\nu^{\frac{2}{n}} (u_\nu - \overline{u}_z) - u_\nu^{1+\frac{2}{n}}) \right)$$

where the second equality follows from (5.69). Applying Lemma 5.11(i) with $f = u_\nu - \overline{u}_z$, we get

$$\parallel u_\nu - \overline{u}_z \parallel_{L^{1+\frac{2}{n}}(M)} \leq C \parallel u_\nu^{1+\frac{2}{n}} (R_{\theta_\nu} - r_\infty) \parallel_{L^{n+2} (M)} + C \left( \parallel (\overline{u}_z u_\nu^{\frac{2}{n}}) (u_\nu - \overline{u}_z) \parallel_{L^{n+2} (M)} + \frac{2}{n} \parallel u_\nu \parallel_{L^{1+\frac{2}{n}}(M)} \right)$$

$$+ C \sup_{a \in A} \left| \int_M u_\nu^{\frac{2}{n}} \psi_\alpha (u_\nu - \overline{u}_z) \, dV_\theta \right|,$$ 

where we have used (5.70). Using the pointwise estimate

$$\parallel u_\nu^{1+\frac{2}{n}} + \frac{n + 2}{n} u_\nu^{\frac{2}{n}} (u_\nu - \overline{u}_z) - u_\nu^{1+\frac{2}{n}} \parallel_{L^{1+\frac{2}{n}}(M)} \leq C \parallel u_\nu \parallel_{L^{1+\frac{2}{n}}(M)} \parallel u_\nu \parallel_{L^{1+\frac{2}{n}}(M)} \parallel \psi \parallel_{L^{1+\frac{2}{n}}(M)},$$

we obtain

$$\parallel u_\nu^{1+\frac{2}{n}} + \frac{n + 2}{n} u_\nu^{\frac{2}{n}} (u_\nu - \overline{u}_z) - u_\nu^{1+\frac{2}{n}} \parallel_{L^{1+\frac{2}{n}}(M)} \leq C \parallel u_\nu \parallel_{L^{1+\frac{2}{n}}(M)} \parallel u_\nu \parallel_{L^{1+\frac{2}{n}}(M)} \parallel \psi \parallel_{L^{1+\frac{2}{n}}(M)},$$

(5.72)
By Hölder’s inequality, we have

\[
(5.73) \quad \left\| u_\nu - \overline{u}_{z_\nu} \right\|_{\min \left(1 + \frac{\alpha}{n}, 2 \right)} + \left\| u_\nu - \overline{u}_{z_\nu} \right\|_{1 + \frac{\alpha}{n}} \leq C \sum_{k=1}^{m} (N \in_{k, \nu}) \frac{\alpha^2}{\nu^2} \left( u_\nu \right) \left( \sum_{k=1}^{m} \frac{\alpha^2}{\nu^2} \right) \left( M \setminus \bigcup_{k=1}^{m} B_{N z_{k, \nu}}(x_{k, \nu}) \right) \quad \text{for } \nu \geq \frac{1}{n+2}.
\]

On the other hand, we have

\[
(5.74) \quad \left\| u_\nu - \overline{u}_{z_\nu} \right\|_{L^{2 + \frac{\alpha}{n}}} (M \setminus \bigcup_{k=1}^{m} B_{N z_{k, \nu}}(x_{k, \nu})) \leq \sum_{k=1}^{m} \alpha_k, \nu_{\nu} \left( x_{k, \nu}, \nu \right) \left\| u_\nu \right\|_{L^{2 + \frac{\alpha}{n}}} (M \setminus \bigcup_{k=1}^{m} B_{N z_{k, \nu}}(x_{k, \nu})) + \left\| w_\nu \right\|_{L^{2 + \frac{\alpha}{n}}} (M) \leq CN^{-n} + o(1),
\]

where the first equality follows from (5.57), and the last inequality follows from (5.68). Therefore, using (5.72) and (5.74), we get

\[
(5.75) \quad \left\| u_\nu \right\|_{L^{1 + \frac{\alpha}{n}}} (M) \leq C \sum_{k=1}^{m} (N \in_{k, \nu}) \frac{\alpha^2}{\nu^2} + C \left( N^{-n} + o(1) \right) \left\| u_\nu - \overline{u}_{z_\nu} \right\|_{L^{1 + \frac{\alpha}{n}}} (M)
\]

Moreover, we have

\[
(5.76) \quad \sup_{a \in A} \left( \frac{1}{\frac{2}{\alpha} - \frac{2}{\nu}} \int_{M} u_\nu^2 \psi_a(u_\nu - \overline{u}_{z_\nu}) dV_0 \right) = \sup_{a \in A} \left( \frac{1}{\frac{2}{\alpha} - \frac{2}{\nu}} \sum_{k=1}^{m} \alpha_k, \nu_{\nu} \int_{M} u_\nu^2 \psi_a u_{\nu_{\nu}} \overline{u}_{z_{\nu}} dV_0 + \int_{M} u_{\nu_{\nu}}^2 \psi_a u_\nu dV_0 \right) \leq C \sum_{k=1}^{m} \frac{\alpha^2}{\nu^2} \left( u_\nu \right) \left( \sum_{k=1}^{m} \frac{\alpha^2}{\nu^2} \right) \left( M \setminus \bigcup_{k=1}^{m} B_{N z_{k, \nu}}(x_{k, \nu}) \right) \leq C \sum_{k=1}^{m} \frac{\alpha^2}{\nu^2} \left( u_\nu \right) \left( \sum_{k=1}^{m} \frac{\alpha^2}{\nu^2} \right) \left( M \setminus \bigcup_{k=1}^{m} B_{N z_{k, \nu}}(x_{k, \nu}) \right) \quad \text{for } \nu \geq \frac{1}{n+2},
\]

and

\[
(5.77) \quad \sup_{a \in A} \left( \frac{1}{\frac{2}{\alpha} - \frac{2}{\nu}} \int_{M} u_\nu^2 \psi_a \left( u_\nu - \overline{u}_{z_\nu} - \sum_{k=1}^{m} \alpha_k, \nu_{\nu} u_{\nu_{\nu}} \overline{u}_{z_{\nu}} \right) dV_0 \right) \leq C \sum_{k=1}^{m} \frac{\alpha^2}{\nu^2} \left( u_\nu \right) \left( \sum_{k=1}^{m} \frac{\alpha^2}{\nu^2} \right) \left( M \setminus \bigcup_{k=1}^{m} B_{N z_{k, \nu}}(x_{k, \nu}) \right) \quad \text{for } \nu \geq \frac{1}{n+2}.
\]
Lemma 5.20. The difference $u_\nu - \overline{u}_{z_\nu}$ satisfies the estimate

$$
\|u_\nu - \overline{u}_{z_\nu}\|_{L^1(M)} \leq C \left\| u_\nu^{1+\frac{2}{n}} (R_{\theta_0} - r_\infty) \right\|_{L^\frac{n+2}{n+2} (M)} + C \left\| \nu \overline{z_\nu} - u_\nu \overline{z_\nu} \right\|_{L^\frac{(n+1)(n+2)}{n^2+2n+2} (M)}
$$

if $\nu$ is sufficiently large.

Now, we can prove the following lemma.

Lemma 5.21. We have

$$
\sup_{\alpha \in \mathcal{A}} \left| \int_M \psi_\alpha \left( (2 + \frac{2}{n}) \Delta_0 \overline{u}_{z_\nu} - R_{\theta_0} \overline{u}_{z_\nu} + r_\infty \overline{u}_{z_\nu}^{1+\frac{2}{n}} \right) dV_{\theta_0} \right|
\leq C \left( \int_M u_\nu^{2+\frac{2}{n}} |R_{\theta_0} - r_\infty| \overline{u}_{z_\nu}^{1+\frac{2}{n}} dV_{\theta_0} \right)^{\frac{n+2}{n+2}} + C \sum_{k=1}^m \overline{\varepsilon}_{k,\nu}^n
$$

if $\nu$ is sufficiently large.

Proof. Note that

$$
\int_M \psi_\alpha \left( (2 + \frac{2}{n}) \Delta_0 \nu \overline{u}_{z_\nu} - R_{\theta_0} \nu \overline{u}_{z_\nu} + r_\infty \overline{u}_{z_\nu}^{1+\frac{2}{n}} \right) dV_{\theta_0}

(5.77)

= \int_M \psi_\alpha \left( (2 + \frac{2}{n}) \Delta_0 u_\nu - R_{\theta_0} u_\nu + r_\infty u_\nu^{1+\frac{2}{n}} \right) dV_{\theta_0}

- \lambda_\alpha \int_M \overline{u}_{z_\nu} \psi_\alpha (\overline{u}_{z_\nu} - u_\nu) dV_{\theta_0} + r_\infty \int_M \psi_\alpha (\overline{u}_{z_\nu}^{1+\frac{2}{n}} - u_\nu^{1+\frac{2}{n}}) dV_{\theta_0}
$$
where we have used integration by parts and (5.30). Using the identity
\[
(2 + \frac{2}{n})\Delta u_{\nu} - R_{\theta_{\nu}} u_{\nu} + u_{\nu}^{1+\frac{2}{n}} = -u_{\nu}^{1+\frac{2}{n}} (R_{\theta_{\nu}} - r_{\infty}),
\]
we can rewrite (5.77) as
\[
\int_{M} \psi_{a} \left( (2 + \frac{2}{n})\Delta u_{\nu} - R_{\theta_{\nu}} u_{\nu} + u_{\nu}^{1+\frac{2}{n}} \right) dV_{\theta_{0}} = -\int_{M} \psi_{a} u_{\nu}^{1+\frac{2}{n}} (R_{\theta_{\nu}} - r_{\infty}) dV_{\theta_{0}} - \lambda_{a} \int_{M} u_{\nu}^{\frac{2}{n}} \psi_{a} (\nu_{z_{\nu}} - u_{\nu}) dV_{\theta_{0}} + r_{\infty} \int_{M} \psi_{a} (1+\frac{2}{n}) u_{\nu}^{1+\frac{2}{n}} - u_{\nu}^{1+\frac{2}{n}} dV_{\theta_{0}}.
\]
From this, together with the pointwise estimate
\[
|u_{\nu}^{1+\frac{2}{n}} - u_{\nu}^{1+\frac{2}{n}}| \leq C |\nu_{z_{\nu}} - u_{\nu}| + C |\nu_{z_{\nu}} - u_{\nu}|^{1+\frac{2}{n}},
\]
we conclude
\[
\sup_{a \in A} \left| \int_{M} \psi_{a} \left( (2 + \frac{2}{n})\Delta u_{\nu} - R_{\theta_{\nu}} u_{\nu} + u_{\nu}^{1+\frac{2}{n}} \right) dV_{\theta_{0}} \right| \leq C \| u_{\nu}^{1+\frac{2}{n}} (R_{\theta_{\nu}} - r_{\infty}) \|_{L^{\frac{2n+2}{n+2}}(M)} + C \| \nu_{z_{\nu}} - u_{\nu} \|_{L^{1}(M)} + C \| \nu_{z_{\nu}} - u_{\nu} \|^{1+\frac{2}{n}}_{L^{1+\frac{2}{n}}(M)}.
\]
Hence, it follows from Lemma 5.19 and 5.20 that
\[
(5.78)
\sup_{a \in A} \left| \int_{M} \psi_{a} \left( (2 + \frac{2}{n})\Delta u_{\nu} - R_{\theta_{\nu}} u_{\nu} + u_{\nu}^{1+\frac{2}{n}} \right) dV_{\theta_{0}} \right| \leq C \| u_{\nu}^{1+\frac{2}{n}} (R_{\theta_{\nu}} - r_{\infty}) \|_{L^{\frac{2n+2}{n+2}}(M)} + C \| u_{\nu}^{1+\frac{2}{n}} (R_{\theta_{\nu}} - r_{\infty}) \|^{1+\frac{2}{n}}_{L^{1+\frac{2}{n}}(M)} + C \sum_{k=1}^{m} \epsilon_{k,\nu}^{n(1+\gamma)}.
\]
Now Lemma 5.24 follows from (5.78) because
\[
\| u_{\nu}^{1+\frac{2}{n}} (R_{\theta_{\nu}} - r_{\infty}) \|_{L^{\frac{2n+2}{n+2}}(M)} = \int_{M} |R_{\theta_{\nu}} - r_{\infty}|^{\frac{2n}{n+2}} dV_{\theta_{0}} \rightarrow 0
\]
as \nu \rightarrow \infty by (5.13). \[Q.E.D.\]

**Proposition 5.22.** The energy of $\nu_{z_{\nu}}$ satisfies the estimate
\[
E(\nu_{z_{\nu}}) - E(u_{\infty}) \leq C \left( \int_{M} u_{\nu}^{\frac{2}{n}} \left| R_{\theta_{\nu}} - r_{\infty} \right|^{\frac{2n+2}{n+2}} dV_{\theta_{0}} \right)^{\frac{1+\gamma}{2(n+1)}} + C \sum_{k=1}^{m} \epsilon_{k,\nu}^{n(1+\gamma)}
\]
if \nu is sufficiently large.

**Proof.** It follows immediately from Lemma 5.13 and 5.21. \[Q.E.D.\]

**Proposition 5.23.** The energy of $\nu_{\nu}$ satisfies the estimate
\[
E(\nu_{\nu}) \leq \left( E(\nu_{z_{\nu}}) \right)^{n+1} + \sum_{k=1}^{m} E(\nu_{z_{k,\nu}}) \right)^{n+1} \right)^{\frac{1+\gamma}{m}} - C \sum_{k=1}^{m} \epsilon_{k,\nu}^{n(1+\gamma)}
\]
if \nu is sufficiently large.
Proof. By Hölder’s inequality, we have

\[
\left( E(\pi_{z_0})^{n+1} + \sum_{k=1}^{m} E(\pi(x_{k,v},\tau_{k,v}))^{n+1} \right)^{\frac{1}{n+1}} \left( \int_M 2^{\frac{2}{\alpha}} v^{n+\frac{1}{\alpha}} dV_0 \right)^{\frac{1}{n+1}}
\]

\[
> \int_M \left( F(\pi_{z_0})^{n+1} \pi_{z_0}^{\frac{2}{\alpha}} + \sum_{k=1}^{m} F(\pi(x_{k,v},\tau_{k,v}))^{n+1} \pi(x_{k,v},\tau_{k,v})^{\frac{2}{\alpha}} \right) v^{n+\frac{1}{\alpha}} dV_0
\]

\[
> \int_M F(\pi_{z_0})^{\frac{2}{\alpha}} v^{\frac{1}{\alpha}} dV_0 + \int_M \sum_{k=1}^{m} \alpha_{k,v} F(\pi(x_{k,v},\tau_{k,v}))^{n+1} \pi(x_{k,v},\tau_{k,v})^{\frac{2}{\alpha}} v^{n+\frac{1}{\alpha}} dV_0
\]

\[
+ 2 \int_M \sum_{1 \leq i < j \leq m} \alpha_{i,v} \alpha_{j,v} \left( F(\pi(x_{i,v},\tau_{i,v}))^{n+1} \pi(x_{i,v},\tau_{i,v})^{\frac{2}{\alpha}} + F(\pi(x_{j,v},\tau_{j,v}))^{n+1} \pi(x_{j,v},\tau_{j,v})^{\frac{2}{\alpha}} \right) v^{n+\frac{1}{\alpha}} dV_0
\]

Using the inequality

\[
\left( F(\pi_{z_0})^{n+1} \pi_{z_0}^{\frac{2}{\alpha}} + F(\pi(x_{k,v},\tau_{k,v}))^{n+1} \pi(x_{k,v},\tau_{k,v})^{\frac{2}{\alpha}} \right) v^{n+\frac{1}{\alpha}} dV_0
\]

\[
> F(\pi_{z_0})^{\frac{1}{\alpha}} \pi_{z_0} v + C \varepsilon^{-n-1} \left( d(x_{k,v},x) \leq \varepsilon_{k,v} \right)
\]

we obtain

\[
(5.81)
\]

\[
\int_M \left( F(\pi_{z_0})^{n+1} \pi_{z_0}^{\frac{2}{\alpha}} + F(\pi(x_{k,v},\tau_{k,v}))^{n+1} \pi(x_{k,v},\tau_{k,v})^{\frac{2}{\alpha}} \right) v^{n+\frac{1}{\alpha}} dV_0
\]

\[
> \int_M F(\pi_{z_0})^{\frac{1}{\alpha}} \pi_{z_0} v + C \varepsilon^n_{k,v}
\]

if \( \nu \) is sufficiently large. We next consider \( i < j \). Again we get (5.33). Substituting (5.33) and (5.81) into (5.80), we get

\[
\left( E(\pi_{z_0})^{n+1} + \sum_{k=1}^{m} E(\pi(x_{k,v},\tau_{k,v}))^{n+1} \right)^{\frac{1}{n+1}} \left( \int_M 2^{\frac{2}{\alpha}} v^{n+\frac{1}{\alpha}} dV_0 \right)^{\frac{1}{n+1}}
\]

\[
> \int_M F(\pi_{z_0})^{\frac{2}{\alpha}} v^{\frac{1}{\alpha}} dV_0 + \int_M \sum_{k=1}^{m} \alpha_{k,v} F(\pi(x_{k,v},\tau_{k,v}))^{n+1} \pi(x_{k,v},\tau_{k,v})^{\frac{2}{\alpha}} v^{n+\frac{1}{\alpha}} dV_0
\]

\[
+ 2 \int_M \sum_{1 \leq i < j \leq m} \alpha_{i,v} \alpha_{j,v} F(\pi(x_{i,v},\tau_{i,v}))^{n+1} \pi(x_{i,v},\tau_{i,v})^{\frac{2}{\alpha}} v^{n+\frac{1}{\alpha}} dV_0
\]

\[
+ C \sum_{k=1}^{m} \alpha_{k,v} \varepsilon_{k,v}^{n} + C \sum_{1 \leq i < j \leq m} \left( \varepsilon_{i,v}^{\frac{2}{\alpha}} + d(x_{i,v},x_{j,v})^{\frac{2}{\alpha}} \right)^{\frac{1}{\alpha}}
\]
By the definition of \( v_\nu \) in (5.56), we have

\[
E(v_\nu) \left( \int_M v_\nu^{2+\frac{2}{n}} dV_0 \right)^{\frac{n}{n+1}}
\]

\[
= \int_M F(\pi_{z_\nu})^2 dV_0 + \int_M \sum_{k=1}^m \alpha_{k,\nu}^2 F(\pi_{(x_{k,\nu}, e_{k,\nu})})^2 dV_0
\]

\[
- 2 \int_M \sum_{k=1}^m \alpha_{k,\nu} F(\pi_{(x_{k,\nu}, e_{k,\nu})}) \left( (2 + \frac{2}{n}) \Delta_{\theta_0} \pi_{z_\nu} - R_{\theta_0} \pi_{z_\nu} + F(\pi_{z_\nu}) \right) dV_0
\]

\[
- 2 \sum_{1 \leq i < j \leq m} \alpha_{i,\nu} \alpha_{j,\nu} \left( (2 + \frac{2}{n}) \Delta_{\theta_0} \pi_{(x_{i,\nu}, e_{i,\nu})} - R_{\theta_0} \pi_{(x_{i,\nu}, e_{i,\nu})} + F(\pi_{(x_{i,\nu}, e_{i,\nu})}) \right) dV_0
\]

Substituting (5.82) into (5.83), we obtain

\[
E(v_\nu) \left( \int_M v_\nu^{2+\frac{2}{n}} dV_0 \right)^{\frac{n}{n+1}} \leq \left( E(\pi_{z_\nu})^{n+1} + \sum_{k=1}^m E(\pi_{(x_{k,\nu}, e_{k,\nu})})^{n+1} \right)^{\frac{n}{n+1}} \left( \int_M v_\nu^{2+\frac{2}{n}} dV_0 \right)^{\frac{n}{n+1}}
\]

\[
- 2 \int_M \sum_{k=1}^m \alpha_{k,\nu} F(\pi_{(x_{k,\nu}, e_{k,\nu})}) \left( (2 + \frac{2}{n}) \Delta_{\theta_0} \pi_{z_\nu} - R_{\theta_0} \pi_{z_\nu} + F(\pi_{z_\nu}) \right) dV_0
\]

\[
- 2 \sum_{1 \leq i < j \leq m} \alpha_{i,\nu} \alpha_{j,\nu} \left( (2 + \frac{2}{n}) \Delta_{\theta_0} \pi_{(x_{i,\nu}, e_{i,\nu})} - R_{\theta_0} \pi_{(x_{i,\nu}, e_{i,\nu})} + F(\pi_{(x_{i,\nu}, e_{i,\nu})}) \right) dV_0
\]

\[
- C \sum_{k=1}^m \alpha_{k,\nu} e_{k,\nu}^n - C \sum_{1 \leq i < j \leq m} \left( \frac{\varepsilon_{x_{i,\nu}}^2 \varepsilon_{x_{j,\nu}}^2}{\varepsilon_{j,\nu}^4 + d(x_{i,\nu}, x_{j,\nu})^4} \right)^{\frac{n}{2}}
\]

Note that

(5.85) \[
\int_M \pi_{(x_{i,\nu}, e_{i,\nu})} \left| (2 + \frac{2}{n}) \Delta_{\theta_0} \pi_{(x_{i,\nu}, e_{i,\nu})} - R_{\theta_0} \pi_{(x_{i,\nu}, e_{i,\nu})} + F(\pi_{(x_{i,\nu}, e_{i,\nu})}) \right|^2 dV_0 \leq o(1) e_{x_{i,\nu}}^n
\]

Moreover, since \( F(\pi_{(x_{j,\nu}, e_{j,\nu})}) = r_\infty + o(1) \) by Lemma A.5, it follows from Lemma A.6 and A.7 that

(5.86) \[
\int_M \pi_{(x_{j,\nu}, e_{j,\nu})} \left| (2 + \frac{2}{n}) \Delta_{\theta_0} \pi_{(x_{j,\nu}, e_{j,\nu})} - R_{\theta_0} \pi_{(x_{j,\nu}, e_{j,\nu})} + F(\pi_{(x_{j,\nu}, e_{j,\nu})}) \right|^2 dV_0 \]

\[
\leq \int_M \pi_{(x_{j,\nu}, e_{j,\nu})} \left| (2 + \frac{2}{n}) \Delta_{\theta_0} \pi_{(x_{j,\nu}, e_{j,\nu})} - R_{\theta_0} \pi_{(x_{j,\nu}, e_{j,\nu})} + r_\infty \pi_{(x_{j,\nu}, e_{j,\nu})} \right|^2 + r_\infty \pi_{(x_{j,\nu}, e_{j,\nu})} dV_0
\]

\[
+ \left| F(\pi_{(x_{j,\nu}, e_{j,\nu})}) - r_\infty \right| \int_M \pi_{(x_{j,\nu}, e_{j,\nu})} \pi_{(x_{j,\nu}, e_{j,\nu})} dV_0
\]

\[
\leq C(\delta^4 + \delta^{2n} + \frac{\varepsilon_{x_{j,\nu}}^2 \varepsilon_{x_{j,\nu}}^2}{\varepsilon_{j,\nu}^4 + d(x_{i,\nu}, x_{j,\nu})^4} \left( \frac{\varepsilon_{x_{j,\nu}}^2 \varepsilon_{x_{j,\nu}}^2}{\varepsilon_{j,\nu}^4 + d(x_{i,\nu}, x_{j,\nu})^4} \right)^{\frac{n}{2}}
\]

\[+ o(1) \left( \frac{\varepsilon_{x_{j,\nu}}^2 \varepsilon_{x_{j,\nu}}^2}{\varepsilon_{j,\nu}^4 + d(x_{i,\nu}, x_{j,\nu})^4} \right)^{\frac{n}{2}}\]
for $i < j$. Therefore, the assertion follows from (5.84), (5.85) and (5.86) by choosing $\delta$ sufficiently small. \qed

**Corollary 5.24.** We have

$$E(v_\nu) \leq \left( E(u_\infty)^{n+1} + m Y(S^{2n+1})^{n+1} \right)^{\frac{2}{n+1}} + C \left( \int_M u_\nu^{2+\frac{2}{n}} |\nabla v_\nu|^2 + |\nabla Y|_{g}\right)^{\frac{2}{n+1}} dV_{\theta_0}$$

if $\nu$ is sufficiently large.

**Proof.** By Proposition 5.22, we have

$$E(\pi_{x_0}) - E(u_\infty) \leq C \left( \int_M u_\nu^{2+\frac{2}{n}} |\nabla v_\nu|^2 + |\nabla Y|_{g}\right)^{\frac{2}{n+1}} + C \sum_{k=1}^m \varepsilon_k^{n(1+\gamma)}$$

if $\nu$ is sufficiently large. By Proposition A.4, we have

$$E(\pi_{(x,k,\varepsilon,k,\nu)}) \leq Y(S^{2n+1}).$$

Substituting these into (5.79), we get the result. \qed

6. PROOF OF PROPOSITION 4.1

In this section, we give the proof of Proposition 4.1. Note that

$$r_{\theta_\nu} = \int_M \left( (2 + \frac{2}{n}) |\nabla v_\nu|^2 + R_{\theta_\nu} v_\nu^2 \right) dV_{\theta_0} + 2 \int_M R_{\theta_\nu} u_\nu^{1+\frac{2}{n}} v_\nu dV_{\theta_0}$$

$$- \int_M \left( (2 + \frac{2}{n}) |\nabla v_\nu|^2 + R_{\theta_\nu} v_\nu^2 \right) dV_{\theta_0},$$

where the first equality follows from (3.33), (3.55), (3.9), and integration by parts, and the second equality follows from (5.9). This implies that

$$r_{\theta_\nu} = E(v_\nu) \left( \int_M v_\nu^{2+\frac{2}{n}} dV_{\theta_0} \right)^{\frac{n}{n+1}} + 2 \int_M u_\nu^{1+\frac{2}{n}} (R_{\theta_\nu} - r_{\infty}) v_\nu dV_{\theta_0}$$

$$- \int_M \left( (2 + \frac{2}{n}) |\nabla v_\nu|^2 + R_{\theta_\nu} v_\nu^2 - \frac{n+2}{n} r_{\infty} v_\nu^2 \right) dV_{\theta_0}$$

$$+ r_{\infty} \int_M \left( - \frac{n+2}{n} r_{\infty} v_\nu^2 + 2(v_\nu + w_\nu)^{1+\frac{2}{n}} v_\nu \right) dV_{\theta_0}.$$ 

(6.1)

Note that $\frac{x^{n+1}}{n+1} - 1 \leq \frac{n+1}{n} x - \frac{n}{n+1}$ for $x \geq 0$. Therefore,

$$\left( \int_M v_\nu^{2+\frac{2}{n}} dV_{\theta_0} \right)^{\frac{n}{n+1}} - 1 \leq \int_M \left( \frac{n+1}{n+1} v_\nu^{2+\frac{2}{n}} - \frac{n}{n+1} (v_\nu + w_\nu)^{2+\frac{2}{n}} \right) dV_{\theta_0}$$

where we have used (3.9). Multiplying this by $r_{\infty}$ and adding it to (6.1), we obtain

$$r_{\theta_\nu} \leq r_{\infty} + (E(v_\nu) - r_{\infty}) \left( \int_M v_\nu^{2+\frac{2}{n}} dV_{\theta_0} \right)^{\frac{n}{n+1}} + 2 \int_M u_\nu^{1+\frac{2}{n}} (R_{\theta_\nu} - r_{\infty}) v_\nu dV_{\theta_0}$$

$$- \int_M \left( (2 + \frac{2}{n}) |\nabla v_\nu|^2 + R_{\theta_\nu} v_\nu^2 - \frac{n+2}{n} r_{\infty} v_\nu^2 \right) dV_{\theta_0}$$

$$+ r_{\infty} \int_M \left( \frac{n}{n+1} v_\nu^{2+\frac{2}{n}} - \frac{n+2}{n} r_{\infty} v_\nu^2 \right) dV_{\theta_0}$$

$$+ 2(v_\nu + w_\nu)^{1+\frac{2}{n}} v_\nu - \frac{n}{n+1} (v_\nu + w_\nu)^{2+\frac{2}{n}} dV_{\theta_0}.$$
Using Hölder’s inequality, we obtain
\[
\int_{M} u^{1 + \frac{2}{n}} (R_{\theta_{\nu}} - r_{\infty}) w_{\nu} \, dV_{\theta_{0}} \\
\leq \left( \int_{M} u^{2 + \frac{2}{n}} |R_{\theta_{\nu}} - r_{\infty}|^{\frac{2(n+2)}{n-2}} \, dV_{\theta_{0}} \right)^{\frac{n-2}{n+2}} \left( \int_{M} |w_{\nu}|^{2 + \frac{2}{n}} \, dV_{\theta_{0}} \right)^{\frac{n}{n+2}}.
\]
(6.3)

Moreover, it follows from Corollary 5.8 and Corollary 5.24 that
\[
\int_{M} \left( (2 + \frac{2}{n}) |\nabla_{\theta_{0}} w_{\nu}|^{2} + R_{\theta_{0}}w_{\nu}^{2} - \frac{n + 2}{n} r_{\infty} v_{\nu}^{2} w_{\nu}^{2} \right) \, dV_{\theta_{0}} \\
\geq c \int_{M} \left( (2 + \frac{2}{n}) |\nabla_{\theta_{0}} w_{\nu}|^{2} + R_{\theta_{0}}w_{\nu}^{2} \right) \, dV_{\theta_{0}} \\
\geq c Y(M, \theta_{0}) \left( \int_{M} |w_{\nu}|^{2 + \frac{2}{n}} \, dV_{\theta_{0}} \right)^{\frac{n}{n+2}}.
\]
(6.4)

Finally it follows from the pointwise estimate
\[
\left| \frac{n}{n + 1} v_{\nu}^{2 + \frac{2}{n}} - \frac{n + 2}{n} r_{\infty} v_{\nu}^{2} w_{\nu}^{2} + 2(v_{\nu} + w_{\nu})^{1 + \frac{2}{n}} w_{\nu} - \frac{n}{n + 1} (v_{\nu} + w_{\nu})^{2 + \frac{2}{n}} \right| \\
\leq C v_{\nu}^{\max(0, \frac{2}{n} - 3)} |w_{\nu}|^{\min(3, \frac{2}{n} + 1)} + C |w_{\nu}|^{2 + \frac{2}{n}}
\]
that
\[
\int_{M} \left( \frac{n}{n + 1} v_{\nu}^{2 + \frac{2}{n}} - \frac{n + 2}{n} r_{\infty} v_{\nu}^{2} w_{\nu}^{2} + 2(v_{\nu} + w_{\nu})^{1 + \frac{2}{n}} w_{\nu} - \frac{n}{n + 1} (v_{\nu} + w_{\nu})^{2 + \frac{2}{n}} \right) \, dV_{\theta_{0}} \\
\leq C \int_{M} v_{\nu}^{\max(0, \frac{2}{n} - 3)} |w_{\nu}|^{\min(3, \frac{2}{n} + 1)} \, dV_{\theta_{0}} + C \int_{M} |w_{\nu}|^{2 + \frac{2}{n}} \, dV_{\theta_{0}} \\
\leq C \left( \int_{M} |w_{\nu}|^{2 + \frac{2}{n}} \, dV_{\theta_{0}} \right)^{\frac{n}{n+2} \min(\frac{2}{n}, \frac{n+1}{n})}.
\]
(6.5)

Substituting (6.3), (6.4), and (6.5) into (7.2), we obtain
\[
r_{\theta_{\nu}} \leq r_{\infty} + (E(v_{\nu}) - r_{\infty}) \left( \int_{M} u^{2 + \frac{2}{n}} \, dV_{\theta_{0}} \right)^{\frac{n}{n+2}} \\
+ 2 \left( \int_{M} u^{2 + \frac{2}{n}} |R_{\theta_{\nu}} - r_{\infty}|^{\frac{2(n+2)}{n-2}} \, dV_{\theta_{0}} \right)^{\frac{n}{n+2}} \left( \int_{M} |w_{\nu}|^{2 + \frac{2}{n}} \, dV_{\theta_{0}} \right)^{\frac{n}{n+2}} \\
- c Y(M, \theta_{0}) \left( \int_{M} |w_{\nu}|^{2 + \frac{2}{n}} \, dV_{\theta_{0}} \right)^{\frac{n}{n+2}} + C \left( \int_{M} |w_{\nu}|^{2 + \frac{2}{n}} \, dV_{\theta_{0}} \right)^{\frac{n}{n+2} \min(\frac{2}{n}, \frac{n+1}{n})} \\
\leq r_{\infty} + (E(v_{\nu}) - r_{\infty}) \left( \int_{M} u^{2 + \frac{2}{n}} \, dV_{\theta_{0}} \right)^{\frac{n}{n+2}} + \left( \int_{M} u^{2 + \frac{2}{n}} |R_{\theta_{\nu}} - r_{\infty}|^{\frac{2(n+2)}{n-2}} \, dV_{\theta_{0}} \right)^{\frac{n+1}{n+2}}
\]
where we have used Young’s inequality. By (6.9), Corollary 5.13, and Corollary 5.24, we get
\[
E(v_{\nu}) \leq r_{\infty} + C \left( \int_{M} u^{2 + \frac{2}{n}} |R_{\theta_{\nu}} - r_{\infty}|^{\frac{2(n+2)}{n-2}} \, dV_{\theta_{0}} \right)^{\frac{n+1}{n+2} (1+\gamma)}.
\]
Substituting this into (6.6), we obtain
\[ r_{\Theta_\nu} \leq r_\infty + C \left( \int_M u_\nu^{2 + \frac{2}{n}} |R_{\Theta_\nu} - r_\infty| \frac{2\nu + 2}{2\nu + 2} dV_{\Theta_0} \right)^{\frac{n+2}{2(n+2)(1+\gamma)}} \]
since \( \int_M u_\nu^{2 + \frac{2}{n}} |R_{\Theta_\nu} - r_\infty| \frac{2\nu + 2}{2\nu + 2} dV_{\Theta_0} \to 0 \) as \( \nu \to \infty \) by (3.13). This completes the proof of Proposition 4.1.

**APPENDIX A.**

First we consider the case when \( n = 1 \). We review the definition of CR normal coordinates and recall some of its properties. Give any \( x \in M \), we can find a contact form \( \tilde{\varphi}_x = \varphi_x^* \theta_0 \) conformal to \( \theta_0 \), where \( \tilde{\varphi}_x \) is a contact form defined in \((z, t)\) which is the CR normal coordinates centered at \( x \). On the other hand, \( \tilde{\varphi}_x \) satisfies the following properties: (see Theorem 3.7 in P.172 of [15] and Proposition 6.5 in [11])

\[ \tilde{\varphi}_x = (1 + O(\rho_x(y)))\theta_{\mathbb{H}^n}, \]

\[ dV_{\tilde{\varphi}_x} = (1 + O(\rho_x(y)))dV_{\theta_{\mathbb{H}^n}}, \]

\[ W_k = (1 + O(\rho_x(y)^4))Z_k + O(\rho_x(y)^4)\overline{Z}_k + O(\rho_x(y)^5)\frac{\partial}{\partial t} \]

for \( 1 \leq k \leq n \), in \( \{(z, t): (t^2 + |z|^4)^{\frac{1}{2}} < \tilde{\rho}\} \) for some \( \tilde{\rho} > 0 \). Here \((z, t)\) represents the point \( y \in M \) in the CR normal coordinates centered at \( x \), \( \rho_x(y) \) is the distance in the CR normal coordinates at \( x \), which implies that

\[ \rho_{z_k}(y) = (t^2 + |z|^4)^{\frac{1}{2}}. \]

Also, \( \theta_{\mathbb{H}^n} = dt + \sqrt{-1}\sum_{j=1}^n(z_jd\overline{z}_j - \overline{z}_jdz_j) \) is the standard contact form of the Heisenberg group \( \mathbb{H}^n = \{(z, t) = (z_1, ..., z_n, t) \in \mathbb{C}^n \times \mathbb{R}\} \). Moreover, \( \{W_k\} \) is a pseudo-Hermitian frame, i.e. \( \{W_k\} \) is a local frame of \( \tilde{\varphi}_x \) satisfying \(-\sqrt{-1}\partial\overline{\partial} \tilde{\varphi}_x(W_k, \overline{W}_t) = \delta_{kt} \) (see P.165 in [15]), and \( Z_k = \frac{\partial}{\partial z_k} + \sqrt{-1} \frac{\partial}{\partial t} \). We also have the following expression for the CR conformal sub-Laplacian: (see P.114 in [18])

\[ (2 + \frac{2}{n})\Delta_{\tilde{\varphi}_x} + R_{\tilde{\varphi}_x} = -(2 + \frac{2}{n})\Delta_{\theta_{\mathbb{H}^n}} + O(\rho_x(y)^2). \]

On the other hand, the Webster scalar curvature of \( \tilde{\varphi}_x \) satisfies (see P.38 in [11])

\[ |R_{\tilde{\varphi}_x}(y)| \leq C\rho_x(y)^2, \]

where \( \rho_x(y) \) is the distance in the CR normal coordinates at \( x \). Let \( G_x \) be the Green’s function with pole at \( x \). Then we have (see (105) in [11])

\[ -(2 + \frac{2}{n})\Delta_{\tilde{\varphi}_x} G_x(y) + R_{\tilde{\varphi}_x} G_x(y) = 0 \]

for \( y \neq x \).

Moreover, the Green’s function satisfies the estimates (see Proposition 5.2 and Proposition 5.3 in [11])

\[ |G_x(y) - \rho_x(y)^{-2n} - A_x| \leq C\rho_x(y) \]

and

\[ |\nabla_{\tilde{\varphi}_x}(G_x(y) - \rho_x(y)^{-2n})|^2_{\tilde{\varphi}_x} \leq C, \]

where \( A_x \) is the CR mass. Note that \( \rho_x(y) \), the distance in the CR normal coordinates at \( x \), and \( d(x, y) \), the Carnot-Carathéodory distance on \( M \) with respect to
the contact form $\theta_0$ are equivalent, i.e. there exists a uniform constant $C_0$ such that

$$\frac{1}{C_0} d(x, y) \leq \rho x(y) \leq C_0 d(x, y) \quad \text{for all } x, y \in M.$$  

When $M$ is spherical, give any $x \in M$, we can find a smooth function $\varphi_x$ such that $\vartheta_{x^n} = \varphi_x^2 \theta_0$ in a neighborhood of $x$. That is,

$$\vartheta_x = \vartheta_{x^n}$$

in the above notation. Note also that $R_{\vartheta_x} = R_{\vartheta_{x^n}} \equiv 0$ in this case. Therefore, (A.9) and (A.1) are still true. On the other hand, the Green’s function $G(x)(y)$ defined as in (A.6) satisfies (A.7) and (A.8) by Lemma 5.1 in [10].

Under the assumptions of Theorem 1.1, the CR mass is positive, i.e. $A_x > 0$ for all $x \in M$ by the CR positive mass theorem (see Theorem 1.1 in [11] and Corollary C in [10]). Note that the function $x \mapsto A_x$ is continuous. See Proposition 3.3 in [37] for the proof when $M$ is spherical. See also Remark I.1.2 in [21] and Proposition 3.5 in [20] for the proof of the corresponding statement in the Riemannian case. Hence, we have

$$\inf_{x \in M} A_x > 0.$$

Suppose that we are given a set of pairs $(x_k, \varepsilon_k)_{1 \leq k \leq m}$. For every 1 \leq k \leq m, we define $\varpi_{(x_k, \varepsilon_k)}$ by

$$\varpi_{(x_k, \varepsilon_k)}(y) = \varphi_{x_k}(y) \varpi_{(x_k, \varepsilon_k)}(y).$$

Here

$$\varpi_{(x_k, \varepsilon_k)}(y) = \left(\frac{n(2n + 2)}{r_\infty}\right) \varepsilon_k^n \left[\frac{\chi \delta(x)}{(2 + (\varepsilon_k^2 + |x|^2)^2)^{\frac{n}{2}}} + \left(1 - \chi \delta(x)\right) G(x)(y)\right],$$

where $\chi(s) = \chi(s)$ and $\chi : \mathbb{R} \to [0, 1]$ is a cut-off function satisfying $\chi(s) = 1$ for $s \leq 1$ and $\chi(s) = 0$ for $s \geq 2$. On the other hand, $\delta$ is a positive real number such that $\varepsilon_k \ll \delta$ for all $1 \leq k \leq m$.

**Proposition A.1.** For $n = 1$, we have

$$\left|2 + \frac{2}{n} \Delta_{\vartheta_{x_k}} \varpi_{(x_k, \varepsilon_k)}(y) - R_{\vartheta_{x_k}} \varpi_{(x_k, \varepsilon_k)}(y) + r_\infty \varpi_{(x_k, \varepsilon_k)}(y)\right|^{\frac{1}{1 + \frac{2}{n}}}$$

$$+ \left(\frac{n(2n + 2)}{r_\infty}\right) \varepsilon_k^n \left[\frac{\chi \delta(x)}{(2 + (\varepsilon_k^2 + |x|^2)^2)^{\frac{n}{2}}} + \left(1 - \chi \delta(x)\right) G(x)(y)\right]$$

$$\leq C \left(\frac{\varepsilon_k^2}{(2 + (\varepsilon_k^2 + |x|^2)^2)^{\frac{n}{2}}} \rho_{x_k}(y)^2 1_{\rho_{x_k}(y) \leq 2\delta} + C \frac{\varepsilon_k^n}{\delta} 1_{\delta \leq \rho_{x_k}(y) \leq 2\delta}\right)$$

$$+ C \left(\frac{\varepsilon_k^2}{(2 + (\varepsilon_k^2 + |x|^2)^2)^{\frac{n}{2}}} \rho_{x_k}(y) 1_{\rho_{x_k}(y) \geq \delta}\right).$$
Proof. By definition of \( U_{(x_k, \varepsilon_k)} \), we have

\[
\begin{aligned}
(2 + \frac{2}{n}) \Delta \hat{\theta}_{x_k} \overline{U}_{(x_k, \varepsilon_k)}(y) - R_{\hat{\theta}_{x_k}} \overline{U}_{(x_k, \varepsilon_k)}(y) + r_\infty \overline{U}_{(x_k, \varepsilon_k)}(y)^{1 + \frac{1}{2n}} \\
+ \left( \frac{n(2n + 2)}{r_\infty} \right)^{\frac{1}{2n}} \varepsilon_k^n A_{x_k} \cdot (2 + \frac{2}{n}) \Delta \hat{\theta}_{x_k} \chi \delta(\rho_{x_k}(y)) \\
= \left( \frac{n(2n + 2)}{r_\infty} \right)^{\frac{1}{2n}} \varepsilon_k^n \left( I_1 + I_2 + I_3 + I_4 + I_5 + I_6 + I_7 + I_8 \right).
\end{aligned}
\]

Here,

\[
I_1 = \chi \delta(\rho_{x_k}(y)) \left( 2 + \frac{2}{n} \right) \Delta \hat{\theta}_{x_k} \left( \frac{1}{t^2 + (\varepsilon_k^2 + |z|^2)^2} \right)^{1 + \frac{1}{2n}} \\
+ (2n + 2) n \varepsilon_k^2 \left( \frac{1}{(t^2 + (\varepsilon_k^2 + |z|^2)^2)^{2n}} \right)^{1 + \frac{1}{2n}}.
\]

\[
I_2 = -\chi \delta(\rho_{x_k}(y)) R_{\hat{\theta}_{x_k}} \left( \frac{1}{t^2 + (\varepsilon_k^2 + |z|^2)^2} \right)^{1 + \frac{1}{2n}}.
\]

\[
I_3 = -(2 + \frac{2}{n}) \Delta \hat{\theta}_{x_k} \chi \delta(\rho_{x_k}(y)) \left( G_{x_k}(y) - \rho_{x_k}(y)^{-2n} - A_{x_k} \right),
\]

\[
I_4 = (2 + \frac{2}{n}) \Delta \hat{\theta}_{x_k} \chi \delta(\rho_{x_k}(y)) \left( \frac{1}{(t^2 + (\varepsilon_k^2 + |z|^2)^2)^{2n}} - \rho_{x_k}(y)^{-2n} \right),
\]

\[
I_5 = -2(2 + \frac{2}{n}) \langle \nabla \hat{\theta}_{x_k} \chi \delta(\rho_{x_k}(y)), \nabla \hat{\theta}_{x_k} \left( G_{x_k}(y) - \rho_{x_k}(y)^{-2n} \right) \rangle_{\hat{\theta}_{x_k}},
\]

\[
I_6 = 2(2 + \frac{2}{n}) \left( \chi \delta(\rho_{x_k}(y)), \nabla \hat{\theta}_{x_k} \left( \frac{1}{(t^2 + (\varepsilon_k^2 + |z|^2)^2)^{2n}} - \rho_{x_k}(y)^{-2n} \right) \right) \rho_{x_k}(y)^{-2n},
\]

\[
I_7 = (2n + 2) n \varepsilon_k^2 \left( \frac{\chi \delta(\rho_{x_k}(y))}{(t^2 + (\varepsilon_k^2 + |z|^2)^2)^{2n}} + \left( 1 - \chi \delta(\rho_{x_k}(y)) \right) G_{x_k}(y) \right)^{1 + \frac{1}{2n}}
\]

\[-\chi \delta(\rho_{x_k}(y)) \left( \frac{1}{(t^2 + (\varepsilon_k^2 + |z|^2)^2)^{2n}} \right)^{1 + \frac{1}{2n}}.\]

\[
I_8 = \left( 1 - \chi \delta(\rho_{x_k}(y)) \right) \left[ 2 + \frac{2}{n} \right) \Delta \hat{\theta}_{x_k} G_{x_k}(y) - R_{\hat{\theta}_{x_k}} G_{x_k}(y) \right].
\]

Since

\[
(A.12) \quad \Delta \hat{\theta}_{x_k} = \left( \frac{1}{(t^2 + (\varepsilon_k^2 + |z|^2)^2)^{2n}} \right) = -\left( \frac{n^2 \varepsilon_k^2}{(t^2 + (\varepsilon_k^2 + |z|^2)^2)^{1 + \frac{1}{2n}}} \right)
\]

we have

\[
|I_1| \leq \chi \delta(\rho_{x_k}(y)) \left( 2 + \frac{2}{n} \right) \left( \Delta \hat{\theta}_{x_k} + O(\rho_{x_k}(y)^2) \right) \left( \frac{1}{(t^2 + (\varepsilon_k^2 + |z|^2)^2)^{2n}} \right)
\]

\[
+ (2n + 2) n \varepsilon_k^2 \left( \frac{1}{(t^2 + (\varepsilon_k^2 + |z|^2)^2)^{2n}} \right)^{1 + \frac{1}{2n}}
\]

\[
\leq 1(\rho_{x_k}(y) \leq 2\delta) \frac{C \rho_{x_k}(y)^2}{(t^2 + (\varepsilon_k^2 + |z|^2)^2)^{2n}}
\]

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by (A.1) and (A.2). By (A.4), we have
\[ |I_2| \leq C \rho_{x_k}(y)^2 \frac{1}{(t^2 + (\varepsilon_k^2 + |z|^2)^2)^\frac{1}{2}} \mathbb{1}_{\{\rho_{x_k}(y) \leq 2\delta\}}. \]

By (A.6), we have
\[ |I_3| \leq (2 + \frac{2}{n})|\Delta \hat{g}_{x_k} \chi \delta(\rho_{x_k}(y))| \cdot |G_{x_k}(y) - \rho_{x_k}(y)^{-2n} - A_{x_k}| \]
\[ \leq C \frac{1}{\delta^2} \mathbb{1}_{\{\delta \leq \rho_{x_k}(y) \leq 2\delta\}} \rho_{x_k}(y) \leq C \frac{1}{\delta} \mathbb{1}_{\{\delta \leq \rho_{x_k}(y) \leq 2\delta\}}. \]

Note that
\[ |I_4| = (2 + \frac{2}{n})|\Delta \hat{g}_{x_k} \chi \delta(\rho_{x_k}(y))| \cdot \frac{1}{(t^2 + (\varepsilon_k^2 + |z|^2)^2)^\frac{1}{2}} \rho_{x_k}(y)^{-2n} \]
\[ \leq C \frac{1}{\delta^2} \mathbb{1}_{\{\delta \leq \rho_{x_k}(y) \leq 2\delta\}} \frac{1}{(t^2 + (\varepsilon_k^2 + |z|^2)^2)^\frac{1}{2}} \rho_{x_k}(y)^{-2n} \]
\[ \leq C \frac{1}{\delta^2} \mathbb{1}_{\{\delta \leq \rho_{x_k}(y) \leq 2\delta\}} \frac{\varepsilon_k^2}{\delta^{2n}} \]
\[ \leq C \frac{1}{\delta} \mathbb{1}_{\{\delta \leq \rho_{x_k}(y) \leq 2\delta\}}. \]

by (A.2) and the assumption that \( \varepsilon_k \ll \delta \). Note also that
\[ |I_6| \leq C |\nabla_{\hat{g}_{x_k}} \chi \delta(\rho_{x_k}(y))| \hat{g}_{x_k} \cdot \nabla_{\hat{g}_{x_k}} \left( \frac{1}{(t^2 + (\varepsilon_k^2 + |z|^2)^2)^\frac{1}{2}} \rho_{x_k}(y)^{-2n} \right) \]
\[ \leq C \frac{1}{\delta^2} \mathbb{1}_{\{\delta \leq \rho_{x_k}(y) \leq 2\delta\}} \frac{|z|^2}{(t^2 + |z|^4)^{\frac{1}{2}} + 1} \]
\[ \leq C \frac{\varepsilon_k^2}{\delta^{2n+3}} \mathbb{1}_{\{\delta \leq \rho_{x_k}(y) \leq 2\delta\}} \leq C \frac{1}{\delta} \mathbb{1}_{\{\delta \leq \rho_{x_k}(y) \leq 2\delta\}}. \]

by (A.1), (A.2), and the assumption that \( \varepsilon_k \ll \delta \). By (A.7), we have
\[ |I_5| \leq C |\nabla_{\hat{g}_{x_k}} \chi \delta(\rho_{x_k}(y))| \hat{g}_{x_k} \cdot \nabla_{\hat{g}_{x_k}} \left( G_{x_k}(y) - \rho_{x_k}(y)^{-2n} \right) \hat{g}_{x_k} \leq C \frac{1}{\delta} \mathbb{1}_{\{\delta \leq \rho_{x_k}(y) \leq 2\delta\}}. \]

We also have
\[ |I_7| = (2n + 2)n \varepsilon_k \left( \frac{\chi \delta(\rho_{x_k}(y))}{(t^2 + (\varepsilon_k^2 + |z|^2)^2)^\frac{1}{2}} + \left( 1 - \chi \delta(\rho_{x_k}(y)) \right) G_{x_k}(y)^{1+\frac{2}{n}} \right) \]
\[ - \chi \delta(\rho_{x_k}(y)) \left( \frac{1}{(t^2 + (\varepsilon_k^2 + |z|^2)^2)^\frac{1}{2}} \right)^{1+\frac{2}{n}} \]
\[ \leq C \frac{\varepsilon_k}{(t^2 + (\varepsilon_k^2 + |z|^2)^2)^{\frac{1}{2}}} \mathbb{1}_{\{\rho_{x_k}(y) \geq \delta\}} \]
\[ \leq C \mathbb{1}_{\{\rho_{x_k}(y) \geq \delta\}} \]

by (A.6), and
\[ I_8 = 0 \]
by (A.6). From this, the assertion follows. \( \square \)
Proposition A.2. When $M$ is spherical, we have
\[
\left| 2 + \frac{2}{n} \Delta_{\tilde{g}_{x_k}} \overline{U}_{(x_k, \varepsilon_k)}(y) - R_{\tilde{g}_{x_k}} \overline{U}_{(x_k, \varepsilon_k)}(y) + r_{\infty} \overline{U}_{(x_k, \varepsilon_k)}(y)^{1+\frac{2}{n}} \right|
\]
\[+ \left( \frac{n(2n+2)}{r_{\infty}} \right)^{\frac{2}{n}} \varepsilon_k^2 A_{x_k} (2 + \frac{2}{n}) \Delta_{\tilde{g}_{x_k}} \chi_{\delta}(\rho_{x_k}(y)) \right| \]
\[\leq C \frac{\varepsilon_k^2}{\delta} 1\{\delta \leq \rho_{x_k}(y) \leq 2\} + C \left( \frac{\varepsilon_k^2}{t^2 + (\varepsilon_k^2 + |z|^2)^2} \right)^{\frac{n+2}{2}} 1\{\rho_{x_k}(y) \geq \delta\}. \]

**Proof.** The proof is the same as the proof of Proposition A.1 except we need to prove that $I_1 = 0$. But this follows from (A.9) and (A.12). \(\square\)

Corollary A.3. We have
\[
\left| 2 + \frac{2}{n} \Delta_{\tilde{g}_{x_k}} \overline{U}_{(x_k, \varepsilon_k)}(y) - R_{\tilde{g}_{x_k}} \overline{U}_{(x_k, \varepsilon_k)}(y) + r_{\infty} \overline{U}_{(x_k, \varepsilon_k)}(y)^{1+\frac{2}{n}} \right|
\]
\[\leq C \left( \frac{\varepsilon_k^2}{t^2 + (\varepsilon_k^2 + |z|^2)^2} \right)^{\frac{2}{n}} \rho_{x_k}(y)^2 1\{\rho_{x_k}(y) \leq \delta\} + C \frac{\varepsilon_k^2}{\delta} 1\{\delta \leq \rho_{x_k}(y) \leq 2\} \]
\[+ C \left( \frac{\varepsilon_k^2}{t^2 + (\varepsilon_k^2 + |z|^2)^2} \right)^{\frac{n+2}{2}} 1\{\rho_{x_k}(y) \geq \delta\}. \]

**Proof.** Combining Proposition A.1 with Proposition A.2 and the following estimate:
\[
\left( \frac{n(2n+2)}{r_{\infty}} \right)^{\frac{2}{n}} \varepsilon_k^2 A_{x_k} (2 + \frac{2}{n}) \Delta_{\tilde{g}_{x_k}} \chi_{\delta}(\rho_{x_k}(y)) \right| \leq C \frac{\varepsilon_k^2}{\delta} 1\{\delta \leq \rho_{x_k}(y) \leq 2\}, \]
Corollary A.3 follows. \(\square\)

**Proposition A.4.** If $\delta$ is sufficiently small, then
\[
E(\overline{U}_{(x_k, \varepsilon_k)}) \leq Y(S^{2n+1}) - c A_{x_k} \varepsilon_k^{2n} + C \delta^2 \varepsilon_k^{2n} + C \delta^{2n+1} \varepsilon_k^{2n+1} + C \delta^{-2n-2} \varepsilon_k^{2n+2}
\]
for some $c > 0$.

**Proof.** When $n = 1$, it follows from Proposition A.1 and integration by parts that
\[
\int_M \left( 2 + \frac{2}{n} \right) \nabla_{\tilde{g}_{x_k}} \overline{U}_{(x_k, \varepsilon_k)} \partial_{\tilde{g}_{x_k}} \overline{U}_{(x_k, \varepsilon_k)} \partial_{\tilde{g}_{x_k}} \right) dV_{\tilde{g}_{x_k}}
\]
\[= - \int_M \left( 2 + \frac{2}{n} \right) \Delta_{\tilde{g}_{x_k}} \overline{U}_{(x_k, \varepsilon_k)} \partial_{\tilde{g}_{x_k}} \overline{U}_{(x_k, \varepsilon_k)} + r_{\infty} \overline{U}_{(x_k, \varepsilon_k)} \partial_{\tilde{g}_{x_k}} \right) dV_{\tilde{g}_{x_k}}
\]
\[\leq \left( \frac{n(2n+2)}{r_{\infty}} \right)^{\frac{2}{n}} \varepsilon_k^2 A_{x_k} (2 + \frac{2}{n}) \int_M A_{x_k} \chi_{\delta}(\rho_{x_k}(y))^2 dV_{\tilde{g}_{x_k}}
\]
\[+ C \int_{\{ \rho_{x_k}(y) \leq \delta \}} \left( \frac{\varepsilon_k^2}{t^2 + (\varepsilon_k^2 + |z|^2)^2} \right)^{\frac{2}{n}} \rho_{x_k}(y)^2 dV_{\tilde{g}_{x_k}}
\]
\[+ \frac{\varepsilon_k^2}{\delta} \int_{\{ \rho_{x_k}(y) \leq \delta \}} \left( \frac{\varepsilon_k^2}{t^2 + (\varepsilon_k^2 + |z|^2)^2} \right)^{\frac{n+2}{2}} dV_{\tilde{g}_{x_k}}
\]
\[+ C \int_{\{ \rho_{x_k}(y) \geq \delta \}} \left( \frac{\varepsilon_k^2}{t^2 + (\varepsilon_k^2 + |z|^2)^2} \right)^{\frac{n+2}{2}} \overline{U}_{(x_k, \varepsilon_k)} dV_{\tilde{g}_{x_k}}. \]
We are going to estimate each of the terms on the right hand side of (A.13). Since
\[
\left| \int_M \Delta \hat{\theta}_{x_k} \chi \left( \rho_{x_k}(y) \right) \left( U_{(x_k, \epsilon_k)} - \rho_{x_k}(y)^{-2n} \right) dV_{\hat{\theta}_{x_k}} \right|
\leq C \frac{\delta^2}{\delta} \int_{\{\delta \leq \rho_{x_k}(y) \leq 2\delta\}} \left| \left( t^2 + \left( \frac{\varepsilon_k^2}{\delta} + |z|^2 \right)^2 \frac{1}{2} \right) dV_{\hat{\theta}_{x_k}} \right|
+ C \frac{\delta^2}{\delta} \int_{\{\delta \leq \rho_{x_k}(y) \leq 2\delta\}} \left| G_{x_k}(y) - \rho_{x_k}(y)^{-2n} \right| dV_{\hat{\theta}_{x_k}}
\leq C \frac{\delta^2}{\delta} \int_\delta^{2\delta} \frac{\varepsilon_k^{2n-2} r^{2n-1} dr}{r^{4n}} \leq C \delta^{-2n-2} \varepsilon_k^n
\]
by (A.6) and (A.2), we have
\[
(A.14)
\int_M \Delta \hat{\theta}_{x_k} \chi \left( \rho_{x_k}(y) \right) U_{(x_k, \epsilon_k)} dV_{\hat{\theta}_{x_k}}
= \int_M (\nabla \hat{\theta}_{x_k} \rho_{x_k}(y)^{-2n}, \nabla \hat{\theta}_{x_k} \chi \left( \rho_{x_k}(y) \right)) dV_{\hat{\theta}_{x_k}} + C \delta^{-2n-2} \varepsilon_k^n
= 2n \int_{\{\delta \leq \rho_{x_k}(y) \leq 2\delta\}} \chi \left( \rho_{x_k}(y) \right) \rho_{x_k}(y)^{-2n-1} \left| \nabla \hat{\theta}_{x_k} \rho_{x_k}(y) \right| dV_{\hat{\theta}_{x_k}} + C \delta^{-2n-2} \varepsilon_k^n
= -\frac{C}{\delta} \int_\delta^{2\delta} dr + C \delta^{-2n-2} \varepsilon_k^n = -C_0 + C \delta^{-2n-2} \varepsilon_k^n
\]
by integration by parts. Here \(C_0\) is a positive constant depending only on \(n\). On the other hand, it follows from the definition of \(U_{(x_k, \epsilon_k)}\) that
\[
(A.15)
C \int_{\{\rho_{x_k}(y) \leq 2\delta\}} \left( t^2 + \left( \frac{\varepsilon_k^2}{\delta} + |z|^2 \right)^2 \right) \rho_{x_k}(y)^2 U_{(x_k, \epsilon_k)} dV_{\hat{\theta}_{x_k}}
\leq C \varepsilon_k^{2n} \int_{\{\rho_{x_k}(y) \leq 2\delta\}} \frac{\rho_{x_k}(y)^2}{(t^2 + |z|^4)^n} dV_{\hat{\theta}_{x_k}} = C \delta^{4-2n} \varepsilon_k^{2n}
\]
where we have used the assumption that \(n = 1\). Note also that
\[
(A.16)
C \varepsilon_k^{2n} \delta \int_{\{\delta \leq \rho_{x_k}(y) \leq 2\delta\}} U_{(x_k, \epsilon_k)} dV_{\hat{\theta}_{x_k}}
\leq C \varepsilon_k^{2n} \delta \int_{\{\delta \leq \rho_{x_k}(y) \leq 2\delta\}} \left( \rho_{x_k}(y)^{-2n} + A_{x_k} + C \rho_{x_k}(y) \right) dV_{\hat{\theta}_{x_k}}
\leq C \varepsilon_k^{2n} \delta \int_\delta^{2\delta} \frac{r^{2n+1} dr}{r^{2n}} = C \delta \varepsilon_k^{2n}
\]
and
\[
(A.17)
C \int_{\{\rho_{x_k}(y) \geq 4\}} \left( t^2 + \left( \frac{\varepsilon_k}{\delta} + |z|^2 \right)^2 \right) \frac{\varepsilon_k^2}{(t^2 + |z|^4)^n} U_{(x_k, \epsilon_k)} dV_{\hat{\theta}_{x_k}}
\leq C \varepsilon_k^{2n+2} \int_\delta^{\infty} \frac{z^{2n+1} dr}{r^{4(2n+2) - 2}} = C \varepsilon_k^{2n+2} \int_\delta^{\infty} \frac{dr}{r^{2n+3}} = C \delta^{-2n-2} \varepsilon_k^{2n+2}
\]
where we have used (A.6) and the definition of \(U_{(x_k,\varepsilon_k)}\). Combining (A.13)-(A.17), we have

\[
\int_M \left( 2 + \frac{2}{n} \right) \left| \nabla_{\hat{\theta}_{x_k}} U_{(x_k,\varepsilon_k)} \right|^2_{\theta_{x_k}} + R_{\hat{\theta}_{x_k}} U^2_{(x_k,\varepsilon_k)} \, dV_{\hat{\theta}_{x_k}} \\
\leq r_{\infty} \int_M \frac{2^{2+\frac{2}{n}}}{\varepsilon_k^2} \chi(\hat{\theta}_{x_k}) \left( \frac{2^{2+\frac{2}{n}}}{r^2 + (\varepsilon_k^2 + |z|^2)^2} \right) \, dV_{\hat{\theta}_{x_k}} \\
+ C \delta^2 \varepsilon_k^{2n - 2} \varepsilon_k^{2n + 2} + C \delta \varepsilon_k^{2n} + C \delta^{-2n - 2} \varepsilon_k^{2n + 2}
\]

where the last inequality follows from Hölder’s inequality. It follows from the definition of \(U_{(x_k,\varepsilon_k)}\) that

\[
\leq \left( \int_M \chi(\hat{\theta}_{x_k}) (\hat{\theta}_{x_k})^{2+\frac{2}{n}} \, dV_{\hat{\theta}_{x_k}} \right) \varepsilon_k^{2n + 2} G_{(x_k,y)}^{2+\frac{2}{n}} \, dV_{\hat{\theta}_{x_k}}
\]

Note that

\[
\leq \varepsilon_k^{2n + 2} \int_{\rho_x(y) \geq \delta} \left( \rho_x(y)^2 - 2n + A x_k + C (\rho_x(y) - 2n + A x_k) \right)^{2+\frac{2}{n}} \, dV_{\hat{\theta}_{x_k}}
\]

by (A.6). Note also that

\[
\leq \left( \frac{Y(S^{2n+1})}{n(2n+2)} \right)^{n+1} + O(\varepsilon_k^{2n + 2})
\]

where we have used the change of variables \( (\frac{4}{\varepsilon_k}, \frac{4}{\varepsilon_k}) \rightarrow (t, z) \). Combining (A.19)-(A.21), we get

\[
\int_M \frac{U^2_{(x_k,\varepsilon_k)}}{r_{\infty}} \leq \left( \frac{Y(S^{2n+1})}{r_{\infty}} \right)^{n+1} + C \delta^{-2n - 2} \varepsilon_k^{2n + 2}.
\]
Substituting this into \((A.18)\), we obtain
\[
\int_M \left( (2 + \frac{2}{n}) |\nabla_{\hat{\theta}_k} \overline{U}_{(x_k, \varepsilon_k)}|^2 + R_{\hat{\theta}_k} \overline{U}_{(x_k, \varepsilon_k)}^2 \right) d\hat{V}_{\hat{\theta}_k} \\
\leq Y(S^{2n+1}) \left( \int_M \overline{U}_{(x_k, \varepsilon_k)}^2 dV_{\hat{\theta}_k} \right) \frac{n(2n+2)}{r_{\infty}} - \left( \frac{n(2n+2)}{r_{\infty}} \right)^n \varepsilon_k^{2n} A_{x_k} (2 + \frac{2}{n}) C_0 \\
+ C\delta^2 \varepsilon_k^{2n} + C\delta \varepsilon_k^{2n} + C\delta^{-2n-2}\varepsilon_k^{2n+2}.
\]

This proves Proposition \(A.4\) for the case when \(n = 1\).

When \(M\) is spherical, it follows from Proposition \(A.2\) and integration by parts that
\[
\int_M \left( (2 + \frac{2}{n}) |\nabla_{\hat{\theta}_k} \overline{U}_{(x_k, \varepsilon_k)}|^2 + R_{\hat{\theta}_k} \overline{U}_{(x_k, \varepsilon_k)}^2 - r_{\infty} \overline{U}_{(x_k, \varepsilon_k)}^{2 + \frac{2}{n}} \right) dV_{\hat{\theta}_k} \\
\leq \left( \frac{n(2n+2)}{r_{\infty}} \right)^n \varepsilon_k^{2n} A_{x_k} (2 + \frac{2}{n}) \int_M \Delta_{\hat{\theta}_k} \chi(\rho_{x_k}(y)) \overline{U}_{(x_k, \varepsilon_k)} dV_{\hat{\theta}_k} \\
+ C\varepsilon_k^n \int_{\{\delta \leq \rho_{x_k}(y) \leq 2\delta\}} \overline{U}_{(x_k, \varepsilon_k)} dV_{\hat{\theta}_k} \\
+ C \int_{\{\rho_{x_k}(y) \geq 4\}} \left( \frac{\varepsilon_k^2}{I^2 + (\varepsilon_k^2 + |z|^2)^2} \right) \overline{U}_{(x_k, \varepsilon_k)} dV_{\hat{\theta}_k}.
\]

Combining \((A.14), (A.16), (A.17),\) and \((A.22)\), we also obtain \((A.18)\). Now we can follow the same argument as above to finish the proof. 

\[\square\]

**Lemma A.5.** We have
\[
F(\overline{u}_{(x_k, \varepsilon_k)}) = r_{\infty} + o(1).
\]

**Proof.** It follows from \((A.18)\) that
\[
\int_M \left( (2 + \frac{2}{n}) |\nabla_{\theta_0} \overline{U}_{(x_k, \varepsilon_k)}|^2 + R_{\theta_0} \overline{U}_{(x_k, \varepsilon_k)}^2 \right) dV_{\theta_0} \\
\leq r_{\infty} \int_M \overline{U}_{(x_k, \varepsilon_k)}^{2 + \frac{2}{n}} dV_{\hat{\theta}_k} - \left( \frac{n(2n+2)}{r_{\infty}} \right)^n \varepsilon_k^{2n} A_{x_k} (2 + \frac{2}{n}) C_0 \\
+ C\delta^2 \varepsilon_k^{2n} + C\delta \varepsilon_k^{2n} + C\delta^{-2n-2}\varepsilon_k^{2n+2},
\]

which implies that
\[
\int_M \left( (2 + \frac{2}{n}) |\nabla_{\theta_0} \overline{u}_{(x_k, \varepsilon_k)}|^2 + R_{\theta_0} \overline{u}_{(x_k, \varepsilon_k)}^2 \right) dV_{\theta_0} \leq r_{\infty} \int_M \overline{u}_{(x_k, \varepsilon_k)}^{2 + \frac{2}{n}} dV_{\theta_0} + o(1).
\]

This implies that \(F(\overline{u}_{(x_k, \varepsilon_k)}) \leq r_{\infty} + o(1)\). On the other hand, following the proof of Proposition \(A.4\), one can prove that
\[
\int_M \left( (2 + \frac{2}{n}) |\nabla_{\theta_0} \overline{u}_{(x_k, \varepsilon_k)}|^2 + R_{\theta_0} \overline{u}_{(x_k, \varepsilon_k)}^2 \right) dV_{\theta_0} \\
\geq r_{\infty} \int_M \overline{u}_{(x_k, \varepsilon_k)}^{2 + \frac{2}{n}} dV_{\hat{\theta}_k} - \left( \frac{n(2n+2)}{r_{\infty}} \right)^n \varepsilon_k^{2n} A_{x_k} (2 + \frac{2}{n}) C_0 \\
- C\delta^2 \varepsilon_k^{2n} - C\delta \varepsilon_k^{2n} - C\delta^{-2n-2}\varepsilon_k^{2n+2},
\]

by using the same arguments to obtain \((A.18)\). This implies that
\[
\int_M \left( (2 + \frac{2}{n}) |\nabla_{\theta_0} \overline{u}_{(x_k, \varepsilon_k)}|^2 + R_{\theta_0} \overline{u}_{(x_k, \varepsilon_k)}^2 \right) dV_{\theta_0} \geq r_{\infty} \int_M \overline{u}_{(x_k, \varepsilon_k)}^{2 + \frac{2}{n}} dV_{\theta_0} + o(1),
\]
which gives \( F(\overline{\nu}_{(x_1, \varepsilon_k)}) \geq r_{\infty} + o(1) \). This proves the assertion. \(\square\)

**Lemma A.6.** We have the estimate

\[
\int_M \overline{\nu}_{(x, \varepsilon_i)}^{1+\frac{\delta}{2}} dV_{\theta_0} \leq C \left( \frac{\varepsilon_i^2 \varepsilon_j^2}{\varepsilon_j^4 + d(x_i, x_j)^4} \right)^{\frac{\delta}{2}}.
\]

**Proof.** It follows from the definition of \( \overline{\nu}_{(x, \varepsilon_i)} \) that \( \overline{\nu}_{(x, \varepsilon_i)}(y) = \varphi_x(y) \overline{U}_{(x, \varepsilon_i)}(y) \) and

\[
\overline{U}_{(x, \varepsilon_i)}(y) \leq \left( \frac{n(2n + 2)}{r_{\infty}} \right)^{\frac{n}{2}} \varepsilon_i^n \left[ \frac{\chi_{\delta}(\varphi_x(y))}{(\varepsilon_i^4 + d(x_i, y)^4)^{\frac{n}{2}}} + \left( 1 - \chi_{\delta}(\varphi_x(y)) \right) G_{x_i}(y) \right].
\]

On the set \( U = \{2d(x_i, y) \leq \varepsilon_j + d(x_i, x_j)\} \), we have

\[
\varepsilon_j + d(y, x_j) \geq \varepsilon_j + d(x_i, x_j) - d(x_i, y) \geq \frac{1}{2}(\varepsilon_j + d(x_i, x_j)).
\]

Therefore, we have

\[
\int_M \left( \frac{\varepsilon_i^2}{\varepsilon_i^4 + d(x_i, y)^4} \right)^{\frac{n}{2}} \left( \frac{\varepsilon_j^2}{\varepsilon_j^4 + d(y, x_j)^4} \right)^{\frac{n}{2}} dV_{\theta_0}
\]

\[
\leq \left( \int_U + \int_{M \setminus U} \right) \left( \frac{\varepsilon_i^4}{\varepsilon_i^4 + d(x_i, y)^4} \right)^{\frac{n}{2}} \left( \frac{\varepsilon_j^4}{\varepsilon_j^4 + d(y, x_j)^4} \right)^{\frac{n}{2}} dV_{\theta_0}
\]

\[
\leq C \frac{\varepsilon_i^n \varepsilon_j^n}{(\varepsilon_i^4 + d(x_i, x_j)^4)^{\frac{n}{2}}} \int_0^{\varepsilon_j + d(x_i, x_j)} \frac{r^{2n+1} dr}{(\varepsilon_i^4 + r^4)^{\frac{n}{2}}}
\]

\[
+ C \frac{\varepsilon_i^n \varepsilon_j^{n+2}}{(\varepsilon_j^4 + d(x_i, x_j)^4)^{\frac{n}{2}}} \int_0^{\infty} \frac{r^{2n+1} dr}{(\varepsilon_j^4 + r^4)^{\frac{n}{2}}}
\]

\[
\leq C \left( \frac{\varepsilon_i^2 \varepsilon_j^2}{\varepsilon_j^4 + d(x_i, x_j)^4} \right)^{\frac{n}{2}}
\]

because

\[
\int_0^{\varepsilon_j + d(x_i, x_j)} \frac{r^{2n+1} dr}{(\varepsilon_i^4 + r^4)^{\frac{n}{2}}} \leq \int_0^{(\varepsilon_j + d(x_i, x_j))^2} r dr = \frac{(\varepsilon_j + d(x_i, x_j))^2}{8} \leq (\varepsilon_j^4 + d(x_i, x_j)^4)^{\frac{\delta}{2}}
\]

and

\[
\int_0^{\infty} \frac{r^{2n+1} dr}{(\varepsilon_j^4 + r^4)^{\frac{n}{2}}} \leq \int_0^{\varepsilon_j} \frac{r^{2n+1} dr}{\varepsilon_j^4 r^4} + \int_\varepsilon^{\infty} \frac{r^{2n+1} dr}{r^4(\varepsilon_j^4 + r^4)} = C \frac{\varepsilon_j^{2n+\frac{n}{2}}}{\varepsilon_j^{2n+\frac{n}{2}} + \varepsilon_j^4}
\]

This proves the assertion. \(\square\)

**Lemma A.7.** We have the estimate

\[
\int_M \overline{\nu}_{(x, \varepsilon_i)} \left( 2 + \frac{2}{n} \right) \Delta_{\theta_0} \overline{\nu}_{(x, \varepsilon_i)} - R_{\theta_0} \overline{\nu}_{(x, \varepsilon_i)} + r_{\infty} \overline{\nu}_{(x, \varepsilon_i)}^{1+\frac{\delta}{2}} dV_{\theta_0}
\]

\[
\leq C(\delta^4 + \delta^{2n} + \frac{\varepsilon_i^2}{\delta^2}) \left( \frac{\varepsilon_i^2 \varepsilon_j^2}{\varepsilon_j^4 + d(x_i, x_j)^4} \right)^{\frac{\delta}{2}}
\]
Proof. Recall \( \hat{\theta}_z = \varphi \theta_0 \) for some smooth function \( \varphi \) on \( M \). Therefore, we can find a positive constant \( c_0 \) such that \( c_0^{-n} \leq \varphi \leq c_0^{n} \), which implies \( \frac{1}{c_0} d(x, y) \leq \rho_M(x, y) \leq c_0 d(x, y) \) for all \( x, y \in M \). Hence, it follows from Corollary A.3 that
\[
\left| (2 + \frac{2}{n}) \Delta_{\theta_0} \psi(x, \varepsilon_j) - R_{\theta_0} \psi(x, \varepsilon_j) + r_0 \psi(x, \varepsilon_j) \right|
\leq C (\delta^2 + \delta^{2n-2}) \left( \frac{\varepsilon_j^2}{\varepsilon_j^4 + d(x, y)^4} \right)^{\frac{n}{2}} 1_{\{d(x, y) \leq 2c_0\delta\}} + C \left( \frac{\varepsilon_j^2}{\varepsilon_j^4 + d(x, y)^4} \right)^{\frac{n+2}{2}} 1_{\{d(x, y) \geq \frac{c_0}{c_0}\}}.
\]

On the set \( U := \{2d(x, y) \leq \varepsilon_j + d(x, x_j)\} \cap \{d(y, x_j) \leq 2c_0\delta\} \), we have
\[
d(x, y) \leq \frac{1}{2} (\varepsilon_j + d(x, x_j)) \leq \varepsilon_j + d(y, x_j) \leq 4c_0\delta.
\]
From this, it follows that
\[
\int_{\{d(y, x_j) \leq 2c_0\delta\}} \left( \frac{\varepsilon_j^2}{\varepsilon_j^4 + d(x, y)^4} \right)^{\frac{n}{2}} \left( \frac{\varepsilon_j^2}{\varepsilon_j^4 + d(y, x_j)^4} \right)^{\frac{n}{2}} dV_{\theta_0}
\leq \left( \int_{U} \int_{\{d(y, x_j) \leq 2c_0\delta\}} \left( \frac{\varepsilon_j^2}{\varepsilon_j^4 + d(x, y)^4} \right)^{\frac{n}{2}} \left( \frac{\varepsilon_j^2}{\varepsilon_j^4 + d(y, x_j)^4} \right)^{\frac{n}{2}} dV_{\theta_0}
\leq C \int_{\{d(x, y) \leq 4c_0\delta\}} \left( \frac{\varepsilon_j^2}{\varepsilon_j^4 + d(x, x_j)^4} \right)^{\frac{n}{2}} \left( \frac{\varepsilon_j^2}{\varepsilon_j^4 + d(y, x_j)^4} \right)^{\frac{n}{2}} dV_{\theta_0}
\leq C \delta^2 \left( \frac{\varepsilon_j^2 \varepsilon_j^2}{\varepsilon_j^4 + d(x, x_j)^4} \right)^{\frac{n}{2}}.
\]
Similarly, on the set \( V := \{2d(x, y) \leq \varepsilon_j + d(x, x_j)\} \cap \{d(y, x_j) \geq \frac{c_0}{c_0}\} \), we have
\[
\varepsilon_j + d(y, x_j) \geq \varepsilon_j + d(x, x_j) - d(x, y) \geq \frac{1}{2} (\varepsilon_j + d(x, x_j)).
\]
This implies
\[
\int_{\{d(y, x_j) \geq \frac{c_0}{c_0}\}} \left( \frac{\varepsilon_j^2}{\varepsilon_j^4 + d(x, y)^4} \right)^{\frac{n}{2}} \left( \frac{\varepsilon_j^2}{\varepsilon_j^4 + d(y, x_j)^4} \right)^{\frac{n+2}{2}} dV_{\theta_0}
\leq \left( \int_{V} \int_{\{d(y, x_j) \geq \frac{c_0}{c_0}\}} \left( \frac{\varepsilon_j^2}{\varepsilon_j^4 + d(x, y)^4} \right)^{\frac{n}{2}} \left( \frac{\varepsilon_j^2}{\varepsilon_j^4 + d(y, x_j)^4} \right)^{\frac{n+2}{2}} dV_{\theta_0}
\leq C \int_{\{2d(x, y) \leq \varepsilon_j + d(x, x_j)\}} \left( \frac{\varepsilon_j^2}{\varepsilon_j^4 + d(x, y)^4} \right)^{\frac{n}{2}} \left( \frac{\varepsilon_j^2}{\varepsilon_j^4 + d(y, x_j)^4} \right)^{\frac{n+2}{2}} \delta^2 (\varepsilon_j^4 + d(x, x_j)^4)^{\frac{n}{2}} dV_{\theta_0}
\leq C \int_{\{d(y, x_j) \geq \frac{c_0}{c_0}\}} \left( \frac{\varepsilon_j^2 \varepsilon_j^2}{\varepsilon_j^4 + d(x, x_j)^4} \right)^{\frac{n}{2}} dV_{\theta_0}
\leq C \frac{\varepsilon_j^2 \varepsilon_j^2}{\varepsilon_j^4 + d(x, x_j)^4}.
\]
because
\[
\int_0^{\varepsilon_j + d(x_i, x_j)} \frac{r^{2n+1} dr}{(\varepsilon_j^4 + r^4)^{\frac{n}{2}}} \leq \int_0^{\varepsilon_j + d(x_i, x_j)} \frac{r^{2n+1} dr}{r^4} \leq (\varepsilon_j^4 + d(x_i, x_j)^4)^{\frac{1}{2}}
\]
and
\[
\int_{\varepsilon_j}^{\infty} \frac{r^{2n+1} dr}{(\varepsilon_j^4 + r^4)^{1+\frac{n}{2}}} \leq \int_{\varepsilon_j}^{\infty} \frac{r^{2n+1} dr}{r^{4(1+\frac{n}{2})}} = \frac{C}{\delta^2}.
\]
This proves the assertion. \(\square\)

### APPENDIX B.

Here we prove Theorem 5.1. We discuss both of the cases when \(n = 1\) or \(M\) is spherical. For convenience, we denote the CR conformal sub-Laplacian of a contact form \(\theta\) by \(L_\theta\), i.e.
\[
L_\theta = -(2 + \frac{2}{n})\Delta_\theta + R_\theta.
\]
Let \(\{u_\nu\}\) be a sequence of positive functions satisfying (5.1) and (5.2). Note that \(u_\nu\) is uniformly bounded in \(S^2_1(M)\). To see this, it follows from integration by parts and Hölder’s inequality that
\[
\int_M \left(2 + \frac{2}{n}\right)|\nabla_\theta u_\nu|^2_{\theta_0} + R_\theta u_\nu^2 \, dV_{\theta_0}
\]
\[
\leq \left(\int_M |L_\theta u_\nu - r_\infty u_\nu^{1+\frac{2}{n}}|^{\frac{2n+2}{n}} \, dV_{\theta_0}\right)^{\frac{n+2}{2n+2}} \left(\int_M u_\nu^{2+\frac{2}{n}} \, dV_{\theta_0}\right)^{\frac{n}{2n+2}} + r_\infty \int_M u_\nu^{2+\frac{2}{n}} \, dV_{\theta_0},
\]
which is uniformly bounded by (5.1) and (5.2). Thus, by passing to a subsequence if necessary, we assume that \(u_\nu\) converges to \(u_\infty\) weakly in \(S^2_1(M)\). Since the Folland-Stein embedding \(S^2_1(M) \hookrightarrow L^s(M)\) is compact for \(1 < s < 2 + \frac{2}{n}\) (see Proposition 5.6 in [28]), \(u_\nu\) converges to \(u_\infty\) in \(L^s(M)\) for \(1 < s < 2 + \frac{2}{n}\).

**Proposition B.1.** The function \(u_\infty\) is a smooth nonnegative function satisfying (5.3).

**Proof.** Note that for any \(\varphi \in C^\infty(M)\)
\[
(\text{B.1) } \int_M \varphi u_\nu^{1+\frac{2}{n}} \, dV_{\theta_0} \to \int_M \varphi u_\infty^{1+\frac{2}{n}} \, dV_{\theta_0} \quad \text{as } \nu \to \infty.
\]
Indeed, there exists a constant \(c\) such that
\[
|u_\nu^{1+\frac{2}{n}} - u_\infty^{1+\frac{2}{n}}| \leq c|u_\nu - u_\infty| \left(|u_\nu|^{\frac{2}{n}} + |u_\infty|^{\frac{2}{n}}\right)
\]
Hence, it follows from Hölder’s inequality that
\[
\left|\int_M \varphi u_\nu^{1+\frac{2}{n}} \, dV_{\theta_0} - \int_M \varphi u_\infty^{1+\frac{2}{n}} \, dV_{\theta_0}\right|
\]
\[
\leq \int_M |u_\nu^{1+\frac{2}{n}} - u_\infty^{1+\frac{2}{n}}| \|\varphi\|dV_{\theta_0}
\]
\[
\leq c\|\varphi\|_{L^\infty(M)} \|u_\nu - u_\infty\|_{L^s(M)} \left(|u_\nu|^{\frac{2}{n}} + |u_\infty|^{\frac{2}{n}}\right)\|_{L^s(M)}
\]

where $\beta$ are chosen such that $1 + \frac{2}{n} < \beta < 2 + \frac{2}{n}$ and $\frac{1}{\beta} + \frac{1}{\beta'} = 1$. Therefore, $\frac{2}{\beta'} < 2 + \frac{n}{2}$. Combining all these, we can conclude that the right hand side of (B.2) tends to 0 as $\nu \to \infty$. This proves (B.1). On the other hand, for any $\varphi \in C^\infty(M)$, we have

$$
(B.3) \quad \int_M \varphi L_{\theta_0} u_\nu dV_{\theta_0} = \int_M u_\nu L_{\theta_0} \varphi dV_{\theta_0} \to \int_M u_\infty L_{\theta_0} \varphi dV_{\theta_0} = \int_M \varphi L_{\theta_0} u_\infty dV_{\theta_0}
$$
as $\nu \to \infty$. Therefore, it follows from (B.1) and (B.3) that for any $\varphi \in C^\infty(M)$

$$
\int_M \left( L_{\theta_0} u_\nu - r_\infty |u_\nu|^\frac{1+\beta}{n} \right) \varphi dV_{\theta_0} \to \int_M \left( L_{\theta_0} u_\infty - r_\infty |u_\infty|^\frac{1+\beta}{n} \right) \varphi dV_{\theta_0}
$$
as $\nu \to \infty$. Combining this with (B.2), we can conclude that $u_\infty$ satisfies (B.3). Since $u_\nu$ is nonnegative, $u_\infty$ is also nonnegative. It follows from Theorem 3.22 in [15] that $u_\infty$ is smooth. This proves the assertion.\[ □ \]

**Proposition B.2.** There holds

$$
(B.4) \quad u_\nu^{\frac{1}{n}} - u_\infty^{\frac{1}{n}} - |u_\nu - u_\infty|^\frac{\beta}{n} (u_\nu - u_\infty) \to 0 \quad \text{in} \quad L^{\frac{2n}{n+\beta}}(M) \quad \text{as} \quad \nu \to \infty.
$$

**Proof.** We use the following inequality:

$$
(B.5) \quad |a + b|^\frac{\beta}{n} - |a|^\frac{\beta}{n} \leq c \left( |b|^\frac{\beta}{n} + \max\{|a|, |b|\}^{\frac{\beta}{n} - 1} |b| \right) \quad \text{for all} \quad a, b.
$$

Applying (B.5), we get

$$
(B.6) \quad u_\nu^{\frac{2}{n}} = u_\nu^{\frac{2}{n}} u_\infty + (u_\nu - u_\infty + u_\infty)^{\frac{2}{n}} (u_\nu - u_\infty) \quad = u_\nu^{\frac{2}{n}} u_\infty + |u_\nu - u_\infty|^\frac{\beta}{n} (u_\nu - u_\infty) + O \left( u_\infty^{\frac{2}{n}} |u_\nu - u_\infty| + \max\{|u_\nu - u_\infty|, |u_\infty|\}^{\frac{\beta}{n} - 1} |u_\nu - u_\infty| u_\infty \right) 
$$

$$
\quad \quad = u_\nu^{\frac{2}{n}} u_\infty + |u_\nu - u_\infty|^\frac{\beta}{n} (u_\nu - u_\infty) + O \left( u_\infty^{\frac{2}{n}} |u_\nu - u_\infty| + |u_\nu - u_\infty|^\frac{\beta}{n} u_\infty \right).
$$

Applying (B.5) again, we obtain

$$
(B.7) \quad u_\nu^{\frac{2}{n}} u_\infty = (u_\nu - u_\infty + u_\infty)^{\frac{2}{n}} u_\infty \quad = u_\nu^{\frac{2}{n}} u_\infty + O \left( u_\infty^{\frac{2}{n}} |u_\nu - u_\infty|^{\frac{\beta}{n}} + \max\{|u_\nu - u_\infty|, |u_\infty|\}^{\frac{\beta}{n} - 1} |u_\nu - u_\infty| u_\infty \right) 
$$

$$
\quad \quad = u_\nu^{\frac{2}{n}} u_\infty + O \left( u_\infty^{\frac{2}{n}} |u_\nu - u_\infty| + |u_\nu - u_\infty|^\frac{\beta}{n} u_\infty \right).
$$

Combining (B.6) and (B.7), we get

$$
(B.8) \quad u_\nu^{1+\frac{\beta}{n}} - u_\infty^{1+\frac{\beta}{n}} - (u_\nu - u_\infty)^{\frac{\beta}{n}} (u_\nu - u_\infty) = O \left( u_\infty^{\frac{2}{n}} |u_\nu - u_\infty| + |u_\nu - u_\infty|^\frac{\beta}{n} u_\infty \right).
$$

Since $u_\infty$ is smooth and $u_\nu$ converges to $u_\infty$ in $L^s(M)$ for $1 < s < 2 + \frac{n}{2}$, (B.4) follows from (B.8).\[ □ \]
If $u_\nu$ converges to $u_\infty$ strongly in $S^2(M)$, then Theorem 5.1 follows. Therefore, we assume that $u_\nu$ converges to $u_\infty$ weakly in $S^2(M)$, but not strongly in $S^2(M)$. On the other hand, it follows from \([5.2], [19]\) and \([14]\) that

\[
\int_M |L_{\theta_0}(u_\nu - u_\infty) - r_\infty|u_\nu - u_\infty|^2 \theta_0 dV_{\theta_0}
\leq C \int_M |L_{\theta_0}u_\nu - r_\infty u_\nu|^{1+\frac{2}{n}} \frac{2n+2}{n+2} dV_{\theta_0}
+ C r_\infty \int_M \left| u_\nu - u_\infty \right|^{1+\frac{2}{n}} - \left| u_\nu - u_\infty \right|^2 \left( u_\nu - u_\infty \right)^\theta \frac{2n+2}{n+2} dV_{\theta_0} \to 0
\]

as $\nu \to \infty$. That is to say, if we let $v_\nu := u_\nu - u_\infty$, then $\{v_\nu\}$ is a sequence of functions such that $v_\nu$ converges to 0 weakly in $S^2(M)$, but not strongly in $S^2(M)$, and satisfies

\[
(B.9) \quad \int_M |L_{\theta_0}v_\nu - r_\infty v_\nu|^\frac{2}{n} v_\nu \frac{2n+2}{n+2} dV_{\theta_0} \to 0 \quad \text{as} \quad \nu \to \infty.
\]

We are going to extract bubbles from $v_\nu$. To do this, we mainly follow the proof of Proposition 8 in \([19]\). The argument is almost the same except with some small modifications. However, there are some parts in the argument of \([19]\) which are not very precise. So we are going to provide all the details of the proof.

As before, we denote $B_r(x) = \{y \in M : d(x, y) < r\}$, where $d$ is the Carnot-Carathéodory distance on $M$ with respect to the contact form $\theta_0$.

**Lemma B.3.** There exists $x \in M$ such that for every $\rho > 0$, there exists $\delta(\rho) > 0$ such that

\[
(B.10) \quad \liminf_{\nu \to \infty} \int_{B_\rho(x)} \left( (2 + \frac{2}{n}) |\nabla_{\theta_0} v_\nu|^2_{\theta_0} + R_{\theta_0} v_\nu^2 \right) dV_{\theta_0} \geq \delta(\rho).
\]

**Proof.** Suppose that for all $x \in M$, there is $\rho(x) > 0$ such that

\[
\liminf_{\nu \to \infty} \int_{B_{\rho(x)}(x)} \left( (2 + \frac{2}{n}) |\nabla_{\theta_0} v_\nu|^2_{\theta_0} + R_{\theta_0} v_\nu^2 \right) dV_{\theta_0} = 0.
\]

Since $M$ is compact, we can cover $M$ with finite number of $B_{\rho(x)}(x)$ with $i = 1, \ldots, L$. As a result, there exists a subsequence of $\{v_\nu\}$, which is still denoted by $\{v_\nu\}$, such that

\[
\int_M \left( (2 + \frac{2}{n}) |\nabla_{\theta_0} v_\nu|^2_{\theta_0} + R_{\theta_0} v_\nu^2 \right) dV_{\theta_0} \to 0
\]

as $\nu \to \infty$. This contradicts to the assumption that $v_\nu$ does not converge to 0 strongly in $S^2(M)$. This proves the assertion. \(\square\)

**Lemma B.4.** The constant $\delta(\rho)$ in Lemma B.3 can be taken to be $a_0 r_\infty^{-n} Y(M, \theta_0)^{n+1}$, where $a_0$ is any positive real number strictly less than 1.

Before we give the proof of Lemma B.4, we remark that our constant in Lemma B.4 is different from that of \([19]\). But one can see from the following arguments that a uniform constant will be sufficient for our purpose.
Proof of Lemma B.4. The proof is similar to the proof of Lemma 10 in [19]. Let \( \psi_\nu : \mathbb{R} \to [0, 1] \) be a \( C^1 \) cut-off function such that
\[
\psi_\nu(s) = \begin{cases} 
1, & \text{for } s \leq \rho_\nu; \\
0, & \text{for } s \geq \rho_\nu + \delta_\nu,
\end{cases}
\]
where \( \rho < \rho_\nu < 2\rho \) and \( 0 < \delta_\nu < \rho \). Fix a point \( x \in M \) where (B.10) holds for \( x \). Define the function \( \phi_\nu \) by \( \phi_\nu(y) = \psi_\nu(d(x, y)) \) where \( d \) is the Carnot-Carathéodory distance on \( M \) with respect to the contact form \( \theta_0 \). Then we have
\begin{equation}
|\nabla \theta_0 \phi_\nu|_{\theta_0} \leq \frac{C}{\delta_\nu}.
\end{equation}

By Hölder’s inequality, we have
\begin{equation}
\int_M \left( L_{\theta_0} \nu_\nu - r_\infty |\nu_\nu |^2 \nu_\nu \right) \nu_\nu \phi_\nu dV_{\theta_0} \leq \| \alpha_\nu \|_{L^{\frac{n+2}{n}}(M)} \left( \int_M (\nu_\nu \phi_\nu)^{2+\frac{2}{n}} dV_{\theta_0} \right)^{\frac{n}{n+2}}
\end{equation}
where \( \alpha_\nu = L_{\theta_0} \nu_\nu - r_\infty |\nu_\nu |^2 \nu_\nu \). By integration by parts, the left hand side of (B.12) can be written as
\begin{equation}
\int_M \left( L_{\theta_0} \nu_\nu - r_\infty |\nu_\nu |^2 \nu_\nu \right) \nu_\nu \phi_\nu dV_{\theta_0} = (2 + \frac{2}{n}) \int_M \nabla_{\theta_0} \nu_\nu \phi_\nu dV_{\theta_0} + (2 + \frac{2}{n}) \int_M \nu_\nu \nabla_{\theta_0} \phi_\nu \nabla_{\theta_0} \phi_\nu dV_{\theta_0} + \int_M R_{\theta_0} \nu_\nu^2 \phi_\nu dV_{\theta_0} - r_\infty \int_M |\nu_\nu |^{2+\frac{2}{n}} \phi_\nu dV_{\theta_0}.
\end{equation}

It follows from (B.12) and (B.13) that
\begin{equation}
(2 + \frac{2}{n}) \int_M |\nabla_{\theta_0} \nu_\nu |^2 \phi_\nu dV_{\theta_0} + (2 + \frac{2}{n}) \int_M \nu_\nu \nabla_{\theta_0} \phi_\nu \nabla_{\theta_0} \phi_\nu dV_{\theta_0} + \int_M R_{\theta_0} \nu_\nu^2 \phi_\nu dV_{\theta_0} \leq r_\infty \int_M |\nu_\nu |^{2+\frac{2}{n}} \phi_\nu dV_{\theta_0} + \| \alpha_\nu \|_{L^{\frac{n+2}{n}}(M)} \left( \int_M (\nu_\nu \phi_\nu)^{2+\frac{2}{n}} dV_{\theta_0} \right)^{\frac{n}{n+2}}.
\end{equation}

Also, by Folland-Stein embedding theorem in Proposition 2.1 we have
\begin{equation}
\left( \int_M (|\nabla \theta_0 (\nu_\nu \phi_\nu) |_{\theta_0}^2 + R_{\theta_0} (\nu_\nu \phi_\nu)^2) dV_{\theta_0} \right)^{\frac{1}{2}} \leq C \left( \int_M (|\nabla \theta_0 (\nu_\nu \phi_\nu) |_{\theta_0}^2 + R_{\theta_0} (\nu_\nu \phi_\nu)^2) dV_{\theta_0} \right)^{\frac{1}{2}}.
\end{equation}

By Cauchy-Schwarz inequality, we have
\begin{equation}
\left( \int_M |\nabla \theta_0 (\nu_\nu \phi_\nu) |_{\theta_0}^2 dV_{\theta_0} \right)^{\frac{1}{2}} \leq \left( \int_M |\nabla \theta_0 \nu_\nu |^2 \phi_\nu dV_{\theta_0} \right)^{\frac{1}{2}} + \left( \int_M |\nabla \theta_0 \phi_\nu |^2 \nu_\nu dV_{\theta_0} \right)^{\frac{1}{2}}.
\end{equation}
Since \( \{v_\nu\} \) is uniformly bounded in \( S^2(M) \), it follows from (B.9) that

\[
\|\alpha_\nu\|_{L^{\frac{n+2}{n-2}}(M)} \left( \left( \int_M |\nabla_{\theta_0} v_\nu|^2 |\phi_\nu|^2 dV_{\theta_0} \right)^{\frac{1}{2}} + \left( \int_M R_{\theta_0} (v_\nu \phi_\nu)^2 dV_{\theta_0} \right)^{\frac{1}{2}} \right) \to 0
\]
as \( \nu \to \infty \). By (B.18) and (B.19), we can rewrite (B.17) as follows:

\[
(1 + o(1)) \int_M \left( 2 + \frac{2}{n} \right) |\nabla_{\theta_0} v_\nu|^2 |\phi_\nu|^2 + R_{\theta_0} v_\nu^2 \phi_\nu \right) dV_{\theta_0}
\]
\[
+ (2 + \frac{2}{n}) \int_M v_\nu (\nabla_{\theta_0} v_\nu, \nabla_{\theta_0} \phi_\nu)_{\theta_0} dV_{\theta_0}
\]
\[
\leq r_\infty \int_M |v_\nu|^2 + \frac{2}{n} \phi_\nu dV_{\theta_0} + C \|\alpha_\nu\|_{L^{\frac{n+2}{n-2}}(M)} \left( \int_M |\nabla_{\theta_0} \phi_\nu|^2 |\phi_\nu|^2 dV_{\theta_0} \right)^{\frac{1}{2}}.
\]

Note that

\[
\int_M |\nabla_{\theta_0} \phi_\nu|^2 |\phi_\nu|^2 dV_{\theta_0} \leq C \frac{1}{\delta_\nu^2} \int_M v_\nu^2 dV_{\theta_0}
\]

by (B.11). Hence,

\[
\left| \int_M v_\nu (\nabla_{\theta_0} v_\nu, \nabla_{\theta_0} \phi_\nu)_{\theta_0} dV_{\theta_0} \right|
\]
\[
\leq \left( \int_M |\nabla_{\theta_0} v_\nu|^2 |\phi_\nu|^2 dV_{\theta_0} \right)^{\frac{1}{2}} \left( \int_M |\nabla_{\theta_0} \phi_\nu|^2 |\phi_\nu|^2 dV_{\theta_0} \right)^{\frac{1}{2}} \leq C \frac{1}{\delta_\nu} \left( \int_M v_\nu^2 dV_{\theta_0} \right)^{\frac{1}{2}}
\]
by Hölder’s inequality and the fact that \( \{v_\nu\} \) is uniformly bounded in \( S^2(M) \).

If we choose \( \delta_\nu = \left( \int_M v_\nu^2 dV_{\theta_0} \right)^{\frac{1}{2}} \), then

\[
\frac{C}{\delta_\nu} \left( \int_M v_\nu^2 dV_{\theta_0} \right)^{\frac{1}{2}} \to 0 \quad \text{as} \quad \nu \to \infty,
\]
since \( v_\nu \to 0 \) in \( L^2(M) \) as \( \nu \to \infty \). Using (B.21), (B.23), we can rewrite (B.20) as (B.24)

\[
(1 + o(1)) \int_M \left( 2 + \frac{2}{n} \right) |\nabla_{\theta_0} v_\nu|^2 + R_{\theta_0} v_\nu^2 \phi_\nu \right) dV_{\theta_0} \leq r_\infty \int_M |v_\nu|^2 + \frac{2}{n} \phi_\nu dV_{\theta_0} + o(1).
\]
In view of (B.18), we deduce from (B.24) that (B.25)

\[(1 + o(1)) \int_M \left(2 + \frac{2}{n}\right) |\nabla_{\theta_0} v_\nu|_{\theta_0}^2 \phi_\nu + R_{\theta_0} v_\nu^2 \phi_\nu \right) dV_{\theta_0} \leq r_\infty \int_M |v_\nu|^{2 + \frac{2}{n}} \phi_\nu dV_{\theta_0}.
\]

We introduce \(\gamma_\nu > 0\) which will be chosen latter. If \(\gamma_\nu\) is sufficiently small, it follows from (B.25) that (B.26)

\[(1 + o(1)) \int_M \left(2 + \frac{2}{n}\right) |\nabla_{\theta_0} v_\nu|_{\theta_0}^2 + R_{\theta_0} w^2 \right) dV_{\theta_0} \leq Y(M, \theta_0) ||w||^2_{L^{\frac{2n}{n-2}}(M)}.
\]

By the definition of the CR Yamabe constant in (3.1), we have (B.27)

\[
Y(M, \theta_0) \left(\int_M |v_\nu|^{2 + \frac{2}{n}} (\phi_\nu + \gamma_\nu) dV_{\theta_0}\right)^{\frac{n}{n-2}} \\
\leq \left(2 + \frac{2}{n}\right) \int_M \left(\left|\nabla_{\theta_0} v_\nu\right|_{\theta_0}^2 (\phi_\nu + \gamma_\nu) + v_\nu^2 \left|\nabla_{\theta_0} ((\phi_\nu + \gamma_\nu) \frac{2}{n-2})\right|_{\theta_0}^2 + 2|v_\nu|(\phi_\nu + \gamma_\nu) \frac{2}{n-2} \nabla_{\theta_0} v_\nu, \nabla_{\theta_0} ((\phi_\nu + \gamma_\nu) \frac{2}{n-2})\right)_{\theta_0} dV_{\theta_0}
\]

\[+ \int_M R_{\theta_0} v_\nu^2 (\phi_\nu + \gamma_\nu) \frac{2}{n-2} dV_{\theta_0}.
\]

By (B.21), (B.23) and the fact that \(\frac{n}{n+1} - 2 < 0\), we can choose \(\gamma_k\) sufficiently small such that

\[\int_M v_\nu^2 \gamma_\nu \frac{2}{n-2} |\nabla_{\theta_0} \phi_\nu|_{\theta_0}^2 = o(1),
\]

which implies that

\[\int_M v_\nu^2 |\nabla_{\theta_0} ((\phi_\nu + \gamma_\nu) \frac{2}{n-2})|_{\theta_0}^2 dV_{\theta_0} = \int_M v_\nu^2 (\phi_\nu + \gamma_\nu) \frac{2}{n-2} |\nabla_{\theta_0} \phi_\nu|_{\theta_0}^2 dV_{\theta_0}
\]

\[\leq \int_M v_\nu^2 \gamma_\nu \frac{2}{n-2} |\nabla_{\theta_0} \phi_\nu|_{\theta_0}^2 dV_{\theta_0} = o(1).
\]

It follows from (B.25), Hölder’s inequality and the fact that \(\{v_\nu\}\) is uniformly bounded in \(S^2_1(M)\) that (B.28)

\[
\left|\int_M |v_\nu|(\phi_\nu + \gamma_\nu) \frac{n}{n-2} \nabla_{\theta_0} v_\nu, \nabla_{\theta_0} ((\phi_\nu + \gamma_\nu) \frac{2}{n-2})\right|_{\theta_0} dV_{\theta_0}
\]

\[\leq \left(\int_M |\nabla_{\theta_0} v_\nu|_{\theta_0}^2 (\phi_\nu + \gamma_\nu) \frac{2}{n-2} dV_{\theta_0}\right)^{\frac{n}{n-2}} \left(\int_M v_\nu^2 ((\phi_\nu + \gamma_\nu) \frac{2}{n-2}) |\nabla_{\theta_0} v_\nu|_{\theta_0}^2 dV_{\theta_0}\right)^{\frac{n}{n-2}} = o(1).
\]

Putting (B.28) and (B.29) into (B.27), we get

\[
Y(M, \theta_0) \left(\int_M |v_\nu|^{2 + \frac{2}{n}} (\phi_\nu + \gamma_\nu) dV_{\theta_0}\right)^{\frac{n}{n-2}} \\
\leq \left(2 + \frac{2}{n}\right) \int_M |\nabla_{\theta_0} v_\nu|_{\theta_0}^2 (\phi_\nu + \gamma_\nu) \frac{2}{n-2} + R_{\theta_0} v_\nu^2 (\phi_\nu + \gamma_\nu) \frac{2}{n-2} dV_{\theta_0} + o(1).
\]
Combining this with (B.26) and using (B.18), we obtain

\( (1 + o(1)) \int_M \left( 2 + \frac{2}{n} \right) |\nabla_{\hat{\theta}_0} v_\nu|_{\hat{\theta}_0}^2 \phi_{\nu} + R_{\hat{\theta}_0} v_\nu^2 \phi_{\nu} \right) dV_{\hat{\theta}_0} \)

\[ \leq r_\infty Y(M, \theta_0)^{-\frac{n+1}{n}} \left( \int_M \left( 2 + \frac{2}{n} \right) |\nabla_{\hat{\theta}_0} v_\nu|_{\hat{\theta}_0}^2 \phi_{\nu} + R_{\hat{\theta}_0} v_\nu^2 \phi_{\nu} \right) dV_{\hat{\theta}_0} \]

It follows from (B.30) and the definition of \( \phi_{\nu} \) that

\( (1 + o(1)) \int_{B_{\rho_\nu}(x)} \left( 2 + \frac{2}{n} \right) |\nabla_{\hat{\theta}_0} v_\nu|_{\hat{\theta}_0}^2 + R_{\hat{\theta}_0} v_\nu^2 \right) dV_{\hat{\theta}_0} \)

\[ \leq r_\infty Y(M, \theta_0)^{-\frac{n+1}{n}} \left( \int_{B_{\rho_\nu}(x)} \left( 2 + \frac{2}{n} \right) |\nabla_{\hat{\theta}_0} v_\nu|_{\hat{\theta}_0}^2 + R_{\hat{\theta}_0} v_\nu^2 \right) dV_{\hat{\theta}_0} \]

\[ + r_\infty Y(M, \theta_0)^{-\frac{n+1}{n}} \left( \int_{B_{\rho_\nu+i_\nu}(x) - B_{\rho_\nu}(x)} \left( 2 + \frac{2}{n} \right) |\nabla_{\hat{\theta}_0} v_\nu|_{\hat{\theta}_0}^2 + R_{\hat{\theta}_0} v_\nu^2 \right) dV_{\hat{\theta}_0} \]

\[ := r_\infty Y(M, \theta_0)^{-\frac{n+1}{n}} (I + II). \]

It follows from (B.18) that \( I \geq C \delta(\rho)^{-\frac{n+1}{n}} \). We have the following two cases:

Case (i). If \( II = o(I) \), then it follows from (B.31) that

\[ (1 + o(1)) \int_{B_{\rho_\nu}(x)} \left( 2 + \frac{2}{n} \right) |\nabla_{\hat{\theta}_0} v_\nu|_{\hat{\theta}_0}^2 + R_{\hat{\theta}_0} v_\nu^2 \right) dV_{\hat{\theta}_0} \]

\[ \leq r_\infty Y(M, \theta_0)^{-\frac{n+1}{n}} \left( \int_{B_{\rho_\nu}(x)} \left( 2 + \frac{2}{n} \right) |\nabla_{\hat{\theta}_0} v_\nu|_{\hat{\theta}_0}^2 + R_{\hat{\theta}_0} v_\nu^2 \right) dV_{\hat{\theta}_0} \]

which implies that

\[ \int_{B_{\rho_\nu}(x)} \left( 2 + \frac{2}{n} \right) |\nabla_{\hat{\theta}_0} v_\nu|_{\hat{\theta}_0}^2 + R_{\hat{\theta}_0} v_\nu^2 \right) dV_{\hat{\theta}_0} \geq a_0 r_\infty^{-n} Y(M, \theta_0)^{n+1} \]

where \( a_0 \) is any positive real number strictly less than 1.

Case (ii). Suppose that there exists a fixed constant \( C \) such that \( II \geq C \delta(\rho)^{-\frac{n+1}{n}} \) for all choices of \( \rho_\nu \in [\rho, 2\rho] \) with \( \delta_\nu = \left( \int_M v_\nu^2 dV_{\hat{\theta}_0} \right)^{\frac{1}{2}} \). In fact, it can not occur since \( \delta_\nu \to 0 \) as \( \nu \to 0 \), and \( \int_M \left( 2 + \frac{2}{n} \right) |\nabla_{\hat{\theta}_0} v_\nu|_{\hat{\theta}_0}^2 + R_{\hat{\theta}_0} v_\nu^2 \right) dV_{\hat{\theta}_0} \) is uniformly bounded. This proves Lemma B.4. \( \square \)

It follows from Lemma B.4 that for any \( x \in M \) satisfying (B.10) and for any given \( a_0 < 1 \), and \( \nu \) sufficiently large, there exists a \( \rho_\nu(x) \) such that

\[ \int_{B_{\rho_\nu}(x)} \left( 2 + \frac{2}{n} \right) |\nabla_{\hat{\theta}_0} v_\nu|_{\hat{\theta}_0}^2 + R_{\hat{\theta}_0} v_\nu^2 \right) dV_{\hat{\theta}_0} = a_0 r_\infty^{-n} Y(M, \theta_0)^{n+1}. \]

Which means for and \( \rho > \rho_\nu(x) \), we have

\[ \int_{B_{\rho_\nu}(x)} \left( 2 + \frac{2}{n} \right) |\nabla_{\hat{\theta}_0} v_\nu|_{\hat{\theta}_0}^2 + R_{\hat{\theta}_0} v_\nu^2 \right) dV_{\hat{\theta}_0} > a_0 r_\infty^{-n} Y(M, \theta_0)^{n+1}. \]
Then for every \( \nu \) sufficiently large, we define:

\[
\rho_{1,\nu} = \inf \rho_\nu(x),
\]

where the infimum is taken among all \( x \in M \) satisfying (B.33), which is a closed set. Thus the infimum is attained. That is, there exists \( x_{1,\nu}^* \in M \) such that

\[
\rho_{1,\nu} = \rho_\nu(x_{1,\nu}^*).
\]

For any \( \rho_0 > 0 \), by the proof of Lemma B.4 we have

\[
\int_{B_{2\nu}(x)} \left( 2 + \frac{2}{n} \right) \nu_0 \nu_\nu \left( \nabla \theta_0 \nu_\nu \right)_0^2 + R_{\theta_0} \nu_\nu^2 \, dV_{\theta_0} \geq a_0 r_\nu^{-n} Y(M, \theta_0)^{n+1}
\]

for all \( \nu \) sufficiently large. Since \( \rho_0 \) is arbitrary, and \( \rho_\nu(x) \leq \rho_0 \), then we have the following lemma:

**Lemma B.5.** The sequence \( \{\rho_{1,\nu}\} \) converges to zero as \( \nu \to \infty \).

As explained in Appendix A, we can find \( \tilde{\theta}_x = \varphi_x^{-1} \theta_0 \) in a neighborhood \( B_{3\nu}(x) \) of \( x \) such that (A.3) and (A.4) hold in the CR normal coordinates \( \{(z, t) : (t^2 + |z|^2)^{1/2} < \tilde{\rho} \} \subset \mathbb{H}^n \), where \( \tilde{\rho} > 0 \) is independent of \( x \). By (B.34) and (B.33), we have

\[
\int_{\{(t^2 + |z|^2)^{1/2} < \tilde{\rho}\}} \left| L_{\tilde{\theta}_x} \tilde{\nu}_\nu - r_\nu \tilde{\nu}_\nu \right|^2 \, dV_{\tilde{\theta}_x} \to 0 \quad \text{as} \quad \nu \to \infty.
\]

By the properties of \( \nu_\nu \), we know that \( \tilde{\nu}_\nu \) is bounded in \( L^{2 + \frac{2}{n}} \) and \( \tilde{\nu}_\nu \to 0 \) in \( L^s \) for all \( s < 2 + \frac{2}{n} \) as \( \nu \to \infty \).

When \( n = 1 \), it follows from (A.1), (A.3), (A.4) and (B.35) that

\[
\int_{\{(t^2 + |z|^2)^{1/2} < \tilde{\rho}\}} \left| L_{\tilde{\theta}_{\nu_0}} \tilde{\nu}_\nu - r_\nu \tilde{\nu}_\nu \right|^2 \, dV_{\tilde{\theta}_{\nu_0}} \leq C \int_{\{(t^2 + |z|^2)^{1/2} < \tilde{\rho}\}} \left| L_{\tilde{\theta}_{\nu_0}} \tilde{\nu}_\nu - L_{\tilde{\theta}_x} \tilde{\nu}_\nu \right|^2 \, dV_{\tilde{\theta}_{\nu_0}} + C \int_{\{(t^2 + |z|^2)^{1/2} < \tilde{\rho}\}} \left| L_{\tilde{\theta}_x} \tilde{\nu}_\nu - r_\nu \tilde{\nu}_\nu \right|^2 \, dV_{\tilde{\theta}_x} \to 0 \quad \text{as} \quad \nu \to \infty.
\]

as \( \nu \to \infty \). When \( M \) is spherical, then it follows from (A.9) and (B.35) that

\[
\int_{\{(t^2 + |z|^2)^{1/2} < \tilde{\rho}\}} \left| L_{\tilde{\theta}_{\nu_0}} \tilde{\nu}_\nu - r_\nu \tilde{\nu}_\nu \right|^2 \, dV_{\tilde{\theta}_{\nu_0}} = 0 \quad \text{as} \quad \nu \to \infty,
\]

since \( R_{\theta_{\nu_0}} = 0 \).
Let \( \tilde{\chi} \) be a cut off function such that \( \tilde{\chi}(s) = 1 \) if \( 0 \leq s \leq \frac{\rho}{2} \) and 0 if \( s \geq \rho \). Let \( \{ \tilde{V}_\nu \} \) be a sequence of functions in \( \mathbb{H}^n \) defined by

\[
\tilde{V}_\nu(z,t) = \begin{cases} 
(\rho_1,\nu)^n \tilde{\chi}(\rho_1,\nu(t^2 + |z|^2)^{1/4}) \overline{\nu}(\rho_1,\nu z, (\rho_1,\nu)^2 t), & \text{for } (t^2 + |z|^2)^{1/4} < \frac{\rho}{\rho_1,\nu}; \\
n_0, & \text{otherwise},
\end{cases}
\]

where \( \rho_1,\nu \) is defined as in (B.32). Then we have the following:

**Proposition B.6.** (i) For any fixed ball \( B \) of \( \mathbb{H}^n \), we have

\[
\int_B \left| L_{\theta_{1,\nu}} \tilde{V}_\nu - r_\infty |\tilde{V}_\nu|^{\frac{2n+2}{n-2}} \tilde{V}_\nu \right| dV_{\theta_{1,\nu}} \to 0 \quad \text{as } \nu \to \infty.
\]

(ii) There exists a constant \( C \) such that

\[
\int_{\mathbb{H}^n} \left( |\nabla_{\theta_{1,\nu}} \tilde{V}_\nu|^2 + |\tilde{V}_\nu|^{2+\frac{2}{n-2}} \right) dV_{\theta_{1,\nu}} \leq C \quad \text{for all } \nu.
\]

**Proof.** To prove (i), we fix \( \rho > 0 \). Since \( \rho_1,\nu \to 0 \) as \( \nu \to \infty \) by Lemma B.5, there exists \( N \) such that \( \rho \leq \frac{\rho}{2\rho_1,\nu} \) when \( \nu \geq N \). Hence, if \( \nu \geq N \), we have

\[
\int_{\{t^2 + |z|^2 \leq \rho \}} \left| L_{\theta_{1,\nu}} \tilde{V}_\nu(z,t) - r_\infty |\tilde{V}_\nu(z,t)|^{\frac{2n+2}{n-2}} \tilde{V}_\nu(z,t) \right| ^{\frac{2n+2}{n-2}} dV_{\theta_{1,\nu}}
\]

\[
= \int_{\{t^2 + |z|^2 \leq \rho \}} \left| L_{\theta_{1,\nu}} \left( (\rho_1,\nu)^n \overline{\nu}(\rho_1,\nu z, (\rho_1,\nu)^2 t) \right) 
- r_\infty \left( (\rho_1,\nu)^n \overline{\nu}(\rho_1,\nu z, (\rho_1,\nu)^2 t) \right) ^{\frac{2n+2}{n-2}} \right| ^{\frac{2n+2}{n-2}} dV_{\theta_{1,\nu}}
\]

\[
= \int_{\{t^2 + |z|^2 \leq \rho \nu \}} \left| L_{\theta_{1,\nu}} \overline{\nu}(\tilde{z},\tilde{t}) - r_\infty |\overline{\nu}(\tilde{z},\tilde{t})|^{\frac{2n+2}{n-2}} \tilde{V}_\nu(\tilde{z},\tilde{t}) \right| ^{\frac{2n+2}{n-2}} dV_{\theta_{1,\nu}} = o(1),
\]

where the first equality follows from (B.33) and the fact that \( \rho \leq \frac{\rho}{2\rho_1,\nu} \), the second equality follows from the change of variables \((\tilde{z},\tilde{t}) = (\rho_1,\nu z, (\rho_1,\nu)^2 t)\), and the last equality follows from (B.36), (B.37) and the fact that \( \rho \rho_1,\nu \leq \frac{\rho}{2} \). This proves (i).
For (ii), we have
\[
\int_{\mathbb{H}^n} \left( 2 + \frac{2}{n} \right) \left\| \nabla_{\theta_{\text{im}}} \tilde{V}_\nu \right\|^2_{\theta_{\text{im}}} + r_\infty |\tilde{V}_\nu|^{2+\frac{2}{n}} dV_{\theta_{\text{im}}}
\]
\[
= \int_{\{t^2 + |z|^4 < \frac{1}{\nu}\}} \left( 2 + \frac{2}{n} \right) \left\| \nabla_{\theta_{\text{im}}} \tilde{V}_\nu \right\|^2_{\theta_{\text{im}}} + r_\infty |\tilde{V}_\nu|^{2+\frac{2}{n}} dV_{\theta_{\text{im}}}
\]
\[
= \int_{\{t^2 + |z|^4 < \nu\}} \left( 2 + \frac{2}{n} \right) \left\| \nabla_{\theta_{\text{im}}} \tilde{V}_\nu \right\|^2_{\theta_{\text{im}}}
\]
\[
+ r_\infty \left| \tilde{\chi} \left( t^2 + |z|^4 \right)^{\frac{2}{n}} \tilde{V}_\nu \tilde{\nu} (\tilde{z}, \tilde{t}) \right|^{2+\frac{2}{n}} dV_{\theta_{\text{im}}}
\]
\[
\leq \int_{\{t^2 + |z|^4 < \nu\}} \left( 2 + \frac{2}{n} \right) \left| \tilde{\chi} \left( t^2 + |z|^4 \right)^{\frac{2}{n}} \tilde{V}_\nu \tilde{\nu} (\tilde{z}, \tilde{t}) \right|^{2+\frac{2}{n}} dV_{\theta_{\text{im}}}
\]
\[
+ \int_{\{t^2 + |z|^4 < \nu\}} \left( 2 + \frac{2}{n} \right) \left( 2 + \frac{2}{n} \right) \left| \nabla_{\theta_{\text{im}}} \tilde{V}_\nu \tilde{\nu} (\tilde{z}, \tilde{t}) \right|^{2+\frac{2}{n}} dV_{\theta_{\text{im}}}
\]
\[
\leq C,
\]
where the first equality follows from (B.38), the second equality follows from the change of variables \((\tilde{z}, \tilde{t}) = (\rho_{1,\nu} \tilde{z}, (\rho_{1,\nu})^2 \tilde{t})\), the third inequality follows from the property of the cut-off function \(\tilde{\chi}\), and the last inequality follows from the fact that \(\tilde{\nu}_\nu\) is uniformly bounded in \(S^2_1(\mathbb{H}^n)\). This proves (ii). \qed

It follows from Proposition B.6(ii) that, by passing to subsequence if necessary, \(\tilde{V}_\nu\) converges to \(\tilde{V}\) weakly in \(S^2_1(B)\) as \(\nu \to \infty\) on each ball \(B\) of \(\mathbb{H}^n\). Since the Folland-Stein embedding \(S^2_1(B) \to L^s(B)\) is compact for \(1 < s < 2 + \frac{2}{n}\) on each ball \(B\) of \(\mathbb{H}^n\), \(\tilde{V}_\nu\) converges to \(\tilde{V}\) in \(L^s(B)\) for \(1 < s < 2 + \frac{2}{n}\). On the other hand, it follows from Proposition B.6(i) that \(\tilde{V}\) satisfies
\[
(2 + \frac{2}{n}) \Delta_{\theta_{\text{im}}} \tilde{V} = r_\infty |\tilde{V}|^{\frac{2}{n}} \tilde{V} \quad \text{in} \quad \mathbb{H}^n.
\]

**Lemma B.7.** (i) For every ball \(B\) in \(\mathbb{H}^n\), there holds
\[
\int_B \left| \nabla_{\theta_{\text{im}}} (\tilde{V}_\nu - \tilde{V}) - r_\infty \tilde{V}_\nu - \tilde{V} \right|^2 (\tilde{V}_\nu - \tilde{V}) \left| \tilde{V}_\nu + \tilde{V} \right|^\frac{2n+2}{n} dV_{\theta_{\text{im}}} \to 0 \quad \text{as} \quad \nu \to \infty.
\]
(ii) There exists a constant \(C\) such that
\[
\int_{\mathbb{H}^n} \left( \left| \nabla_{\theta_{\text{im}}} (\tilde{V}_\nu - \tilde{V}) \right|^2_{\theta_{\text{im}}} + |\tilde{V}_\nu - \tilde{V}|^{2+\frac{2}{n}} \right) dV_{\theta_{\text{im}}} \leq C \quad \text{for all} \quad \nu.
\]

**Proof.** By the same proof of Proposition B.2, we have
\[
\tilde{V}_\nu^{1+\frac{2}{n}} - \tilde{V}^{1+\frac{2}{n}} - |\tilde{V}_\nu - \tilde{V}|^{\frac{2}{n}} (\tilde{V}_\nu - \tilde{V}) \to 0 \quad \text{in} \quad L^{\frac{2n+2}{n}}(B) \quad \text{as} \quad \nu \to \infty.
\]
This together with (B.39) and Proposition [B.6(i)] implies that
\[
\int_B \left| L_{\theta_{\tilde{r}_n}} (\tilde{V}_\nu - \tilde{V}) - r_\infty |\tilde{V}_\nu - \tilde{V}|^\frac{2n+2}{n+2} \right| dV_{\theta_{\tilde{r}_n}} \\
\leq C \int_B \left| L_{\theta_{\tilde{r}_n}} \tilde{V}_\nu - r_\infty |\tilde{V}_\nu|^\frac{2n+2}{n+2} \right| dV_{\theta_{\tilde{r}_n}} \\
+ C r_\infty \int_B \left| \tilde{V}_\nu^{1+\frac{2}{n+2}} - \tilde{V}_\nu^{1+\frac{2}{n+2}} - |\tilde{V}_\nu - \tilde{V}|^\frac{2n+2}{n+2} \right| dV_{\theta_{\tilde{r}_n}} \to 0
\]
as \nu \to \infty. This proves (i). On the other hand, (ii) follows from (B.39) and Proposition [B.6(ii)]. This proves the assertion. \(\square\)

**Lemma B.8.** For every ball \(B\) in \(\mathbb{H}^n\), \(\tilde{V}_\nu\) converges to \(\tilde{V}\) strongly in \(S^2(B)\) as \(\nu \to \infty\).

**Proof.** By contradiction, we assume that \(\tilde{V}_\nu\) does not converge to \(\tilde{V}\) strongly in \(S^2(B)\) as \(\nu \to \infty\) for some \(B\) in \(\mathbb{H}^n\). Therefore, it follows from Lemma [B.7] that the sequence \(\{\tilde{V}_\nu - \tilde{V}\}\) satisfied the same properties of the sequence \(\{v_\nu\}\). In particular, it follows from Lemma [B.4] that there exists a sequence \((\tilde{x}_\nu, \tilde{\rho}_\nu)\) with \(\tilde{x}_\nu \in B_{\rho_1, \nu}(x_1, \nu)\) and \(\tilde{\rho}_\nu \to 0\) as \(\nu \to \infty\) such that
\[
(2 + \frac{2}{n}) \int_{\exp_{\tilde{x}_\nu, \nu}^{-1}} \left| \nabla_{\theta_{\tilde{r}_n}} (\tilde{V}_\nu - \tilde{V}) \right|^2_{\theta_{\tilde{r}_n}} dV_{\theta_{\tilde{r}_n}} \geq \frac{1}{2} + a_0 r_\infty^n Y(B, \theta_{\tilde{r}_n})^{n+1}
\]
for \(\nu\) sufficiently large. It follows from Lemma [B.7] that
\[
\int_{\exp_{\tilde{x}_\nu, \nu}^{-1}} \langle \nabla_{\theta_{\tilde{r}_n}} \tilde{V}, \nabla_{\theta_{\tilde{r}_n}} (\tilde{V}_\nu - \tilde{V}) \rangle_{\theta_{\tilde{r}_n}} dV_{\theta_{\tilde{r}_n}} \to 0
\]
as \(\nu \to \infty\). Combining (B.40) and (B.41), we get
\[
(2 + \frac{2}{n}) \int_{\exp_{\tilde{x}_\nu, \nu}^{-1}} \left| \nabla_{\theta_{\tilde{r}_n}} \tilde{V}_\nu \right|^2_{\theta_{\tilde{r}_n}} dV_{\theta_{\tilde{r}_n}} \\
= (2 + \frac{2}{n}) \int_{\exp_{\tilde{x}_\nu, \nu}^{-1}} \left| \nabla_{\theta_{\tilde{r}_n}} (\tilde{V}_\nu - \tilde{V}) \right|^2_{\theta_{\tilde{r}_n}} dV_{\theta_{\tilde{r}_n}} \\
+ (2 + \frac{2}{n}) \int_{\exp_{\tilde{x}_\nu, \nu}^{-1}} \left| \nabla_{\theta_{\tilde{r}_n}} \tilde{V} \right|^2_{\theta_{\tilde{r}_n}} dV_{\theta_{\tilde{r}_n}} \\
- 2(2 + \frac{2}{n}) \int_{\exp_{\tilde{x}_\nu, \nu}^{-1}} \langle \nabla_{\theta_{\tilde{r}_n}} \tilde{V}, \nabla_{\theta_{\tilde{r}_n}} (\tilde{V}_\nu - \tilde{V}) \rangle_{\theta_{\tilde{r}_n}} dV_{\theta_{\tilde{r}_n}} \\
\geq \frac{1}{2} + a_0 r_\infty^n Y(B, \theta_{\tilde{r}_n})^{n+1} + o(1) \\
\geq a_0 r_\infty^n Y(M, \theta_0)^{n+1}
\]
for \(\nu\) sufficiently large, where the last inequality follows from
\[
Y(B, \theta_{\tilde{r}_n}) = Y(\mathbb{H}^n, \theta_{\tilde{r}_n}) = Y(S^{2n+1}, \theta_{S^{2n+1}}) \geq Y(M, \theta_0).
\]
Proposition B.9. For any \( \chi \) that follows from (B.42) that there exists \( \tilde{\chi}_\nu \in B_{\rho_1,\nu}(x^*,\nu) \) such that
\[
\int_{B_{\rho_1,\nu}(x^*,\nu)} \left( 2 + \frac{2}{n} |\nabla \theta_0 v_\nu|_{\theta_0}^2 + R_{\theta_0} v_\nu^2 \right) dV_{\theta_0} \geq a_0 r^{-n} Y(M, \theta_0)^{n+1}
\]
for \( \nu \) sufficiently large. This implies that
\[
\rho_\nu(\tilde{\chi}_\nu) \leq \tilde{\rho}_\nu \rho_1, \nu < \rho_1, \nu,
\]
where we have used the fact that \( \tilde{\rho}_\nu \to 0 \) as \( \nu \to \infty \). But this contradicts to (B.32). This proves Lemma B.8. \( \square \)

Since \( \tilde{V} \) satisfies (B.39), it follows from the result of Jerison and Lee in [20] that there exists \( (z_0, t_0) \in \mathbb{H}^n \) and \( \gamma_1 > 0 \) such that
\[
\tilde{V}(z, t) = W \circ T_{(z_0, t_0)}(z, t),
\]
where
\[
W(z, t) = \left( \frac{n(2n + 2)}{r_\infty} \right)^{\frac{1}{2}} \left( \frac{\gamma_1^2}{\gamma_1^2 + (\gamma_1^2|z|^2 + 1)^2} \right)^{\frac{1}{2}}
\]
and
\[
T_{(z_0, t_0)}(z, t) = (z + z_0, t + t_0 + 2Im(z \cdot z_0)) \text{ for } (z, t) \in \mathbb{H}^n.
\]
By the optimality of \((x_1, \rho_1, \nu)\), we can conclude that \((z_0, t_0) = (0, 0)\); if \((z_0, t_0) \neq (0, 0)\), we can find \((\tilde{x}_1, \tilde{\rho}_1, \nu)\) with \( \tilde{\rho}_1, \nu < \rho_1, \nu \) such that
\[
\int_{B_{\rho_1,\nu}(\tilde{x}_1, \nu)} \left( 2 + \frac{2}{n} |\nabla \theta_0 v_\nu|_{\theta_0}^2 + R_{\theta_0} v_\nu^2 \right) dV_{\theta_0} \geq a_0 r^{-n} Y(M, \theta_0)^{n+1}.
\]
Therefore, we have
\[
\tilde{V}(z, t) = \left( \frac{n(2n + 2)}{r_\infty} \right)^{\frac{1}{2}} \left( \frac{\gamma_1^2}{\gamma_1^2 + (\gamma_1^2|z|^2 + 1)^2} \right)^{\frac{1}{2}}
\]
We remark that it was claimed in [19] that \( \gamma_1 = 1 \) (see the last line in P.146 in [19]), which does not seem to be true. In fact, we will show that \( \gamma_1 \geq C_1 \). Here \( C_1 \) is a positive constant depending only on \( a_0, r_\infty, \) and \( M \). We need the following:

Proposition B.9. For any \( x \in M \) and for any \( r > 0 \), there holds
\[
\int_{B_r(x)} v_\nu \Delta_0 v_\nu dV_{\theta_0} = \int_{B_r(x)} |\nabla \theta_0 v_\nu|_{\theta_0}^2 dV_{\theta_0} + o(1).
\]
Proof. We consider the following sequence of cut-off functions:
\[
\chi_\nu(y) = \begin{cases} 
1, & \text{if } d(x, y) \leq r; \\
0, & \text{if } d(x, y) \geq r + r_\nu,
\end{cases}
\]
such that
\[
0 \leq \chi_\nu \leq 1, \quad |\nabla_\theta \chi_\nu|_{\theta_0} \leq \frac{C}{r_\nu} \quad \text{and} \quad |\Delta_0 \chi_\nu| \leq \frac{C}{r_\nu^2},
\]
where \( r_\nu \) will be chosen later. Since the function \( \chi_\nu v_\nu \) has support in \( B_{r+2r_\nu}(x) \), it follows from integration by parts that
\[
\int_{B_{r+2r_\nu}(x)} \chi_\nu v_\nu \Delta_0 (\chi_\nu v_\nu) dV_{\theta_0} = \int_{B_{r+2r_\nu}(x)} |\nabla_\theta_0 (\chi_\nu v_\nu)|_{\theta_0}^2 dV_{\theta_0}.
\]
We are going to expand the left and right hand side of (B.46). By (B.44), the left hand side of (B.46) can be written as

\[ \int_{B_r(x)} \chi_\nu v_\nu \Delta \theta_0 (\chi_\nu v_\nu) dV_0 \]

\[ = \int_{B_r(x)} v_\nu \Delta \theta_0 v_\nu dV_0 + \int_{B_{r+2r_\nu}(x)-B_r(x)} \chi_\nu^2 v_\nu^2 \Delta \theta_0 v_\nu dV_0 \]

\[ + \int_{B_{r+2r_\nu}(x)-B_r(x)} \chi_\nu v_\nu^2 \Delta \theta_0 \chi_\nu dV_0 + 2 \int_{B_{r+2r_\nu}(x)-B_r(x)} \chi_\nu v_\nu \langle \nabla \theta_0 \chi_\nu, \nabla \theta_0 v_\nu \rangle dV_0 \]

\[ := \int_{B_r(x)} v_\nu \Delta \theta_0 v_\nu dV_0 + I + II + III. \]

By (B.44) and (B.45), we have

\[ \left| \frac{B.9}{B.9} \right| \left| \frac{B.48}{B.48} \right| \left| \frac{B.49}{B.49} \right| \]

\[ \left| II \right| \leq \int_{B_{r+2r_\nu}(x)-B_r(x)} \left| \chi_\nu v_\nu^2 \Delta \theta_0 \chi_\nu \right| dV_0 \]

\[ \leq \frac{C}{v_\nu^4} \int_M v_\nu^2 dV_0. \]

Since \( v_\nu \) converges to 0 in \( L^2(M) \) as \( \nu \to \infty \), if we choose

\[ (B.47) \quad r_\nu = \left( \frac{1}{M} \int_M v_\nu^2 dV_0 \right)^\frac{1}{2} \to 0 \quad \text{as} \quad \nu \to \infty, \]

then \( |II| \to 0 \) as \( \nu \to \infty \). Since \( v_\nu \) converges to 0 in \( L^2(M) \) as \( \nu \to \infty \), we have

\[ (B.48) \quad \int_M |R_{\theta_0} v_\nu^2 | dV_0 \to 0 \quad \text{as} \quad \nu \to \infty. \]

Since \( r_\nu \to 0 \) as \( \nu \to \infty \) by (B.47) and \( v_\nu \) is uniformly bounded in \( S^2(M) \to L^{2+\frac{2}{n}}(M) \)

\[ (B.49) \quad \int_{B_{r+2r_\nu}(x)-B_r(x)} |v_\nu|^{2+\frac{2}{n}} dV_0 \to 0 \quad \text{as} \quad \nu \to \infty. \]

Therefore, we have

\[ \left| 2 + \frac{2}{n} I \right| \leq \left( \int_{B_{r+2r_\nu}(x)-B_r(x)} v_\nu^{2+\frac{2}{n}} dV_0 \right)^\frac{n}{2n+2} \left( \int_M |L_{\theta_0} v_\nu - r_{\infty} |v_\nu|^{\frac{2n+2}{n+2}} dV_0 \right)^\frac{n+2}{2n+2} \]

\[ \leq C \left( \int_M v_\nu^2 dV_0 \right)^\frac{1}{2} \left( \int_M |\nabla \theta_0 v_\nu|_0^2 dV_0 \right)^\frac{1}{2} \to 0 \quad \text{as} \quad \nu \to \infty, \]

where we have used (B.9), (B.48) and (B.49) in the last equality. By (B.47), Cauchy-Schwarz inequality and the fact that \( v_\nu \) is bounded in \( S^2(M) \), we have

\[ |III| \leq 2 \int_{B_{r+2r_\nu}(x)-B_r(x)} |\chi_\nu v_\nu \langle \nabla \theta_0 \chi_\nu, \nabla \theta_0 v_\nu \rangle | dV_0 \]

\[ \leq \frac{C}{r_\nu} \left( \int_M v_\nu^2 dV_0 \right)^\frac{1}{2} \left( \int_M |\nabla \theta_0 v_\nu|_0^2 dV_0 \right)^\frac{1}{2} \to 0 \quad \text{as} \quad \nu \to \infty. \]

Combining all these, we can see that the left hand side of (B.46) is equal to

\[ (B.51) \quad \int_{B_{r+2r_\nu}(x)} \chi_\nu v_\nu \Delta \theta_0 (\chi_\nu v_\nu) dV_0 = \int_{B_r(x)} v_\nu \Delta \theta_0 v_\nu dV_0 + o(1). \]
On the other hand, the right hand side of (B.46) can be written as
\[
\int_{B_{r}(x)} |\nabla \theta_0 (\chi_\nu v_\nu)|^2_{0,0} dV_{\theta_0}
\]
\[
= \int_{B_{r}(x)} |\nabla \theta_0 v_\nu|^2_{0,0} dV_{\theta_0} + \int_{B_{r+2r_\nu}(x)-B_{r}(x)} \chi_\nu^2 |\nabla \theta_0 v_\nu|^2_{0,0} dV_{\theta_0}
\]
\[
+ \int_{B_{r+2r_\nu}(x)-B_{r}(x)} v_\nu^2 |\nabla \theta_0 \chi_\nu|^2_{0,0} dV_{\theta_0} + \int_{B_{r+2r_\nu}(x)-B_{r}(x)} \chi_\nu v_\nu (\nabla \theta_0 \chi_\nu, \nabla \theta_0 v_\nu)_{0,0} dV_{\theta_0}
\]
\[
:= \int_{B_{r}(x)} |\nabla \theta_0 v_\nu|^2_{0,0} dV_{\theta_0} + I' + II' + III'.
\]
Since \( r_\nu \to 0 \) as \( \nu \to \infty \) by (B.47) and \( v_\nu \) is uniformly bounded in \( S^2_1(M) \), we have
\[
|I'| \leq \int_{B_{r+2r_\nu}(x)-B_{r}(x)} |\nabla \theta_0 v_\nu|^2_{0,0} dV_{\theta_0} \to 0 \quad \text{as } \nu \to \infty.
\]
By (B.45) and (B.47), we have
\[
|II'| = \int_{B_{r+2r_\nu}(x)-B_{r}(x)} \chi_\nu^2 |\nabla \theta_0 v_\nu|^2_{0,0} dV_{\theta_0} \leq \frac{C}{r_\nu^2} \int_M v_\nu^2 dV_{\theta_0} \to 0 \quad \text{as } \nu \to \infty.
\]
Since \( III = III' \), it follows from (B.50) that \( |III'| \to 0 \) as \( \nu \to \infty \). Combining all these, we can see that the right hand side of (B.46) is equal to
\[
(B.52) \quad \int_{B_{r+2r_\nu}(x)} |\nabla \theta_0 (\chi_\nu v_\nu)|^2_{0,0} dV_{\theta_0} = \int_{B_{r}(x)} |\nabla \theta_0 v_\nu|^2_{0,0} dV_{\theta_0} + o(1).
\]
The assertion follows from combining (B.46), (B.51) and (B.52). \( \square \)

**Proposition B.10.** There exists a positive constant \( C_1 \) depending only on \( a_0, \gamma_1, \gamma_2 \), and \( M \) such that
\[
\gamma_1 \geq C_1.
\]

**Proof.** Since \( \rho_{1,\nu} \to 0 \) as \( \nu \to \infty \) by Lemma B.5, we can find \( N \) such that
\[
(C.53) \quad C_0 \leq \frac{\rho}{2\rho_{1,\nu}} \quad \text{whenever } \nu \geq N,
\]
where \( C_0 \) is the uniform constant given in (A.8). Note that
\[
\int_{\{(x^1, |z|) \in C_0 \}} |\nabla \tilde{V}_\nu (z, t)|^{2+\frac{2}{\alpha}} dV_{\theta_0}|^n
\]
\[
= \int_{\{(x^1, |z|) \in C_0 \}} \left( \rho_{1,\nu} \alpha \tilde{v}_\nu (\rho_{1,\nu} z, (\rho_{1,\nu})^2 t) \right)^{2+\frac{2}{\alpha}} dV_{\theta_0}|^n
\]
\[
= \int_{\{(x^1, |z|) \in C_0 \}} |\tilde{v}_\nu (y)|^{2+\frac{2}{\alpha}} dV_{\theta_0} + o(1)
\]
\[
\geq \int_{\{(x^1, |z|) \in C_0 \}} |v_\nu (y)|^{2+\frac{2}{\alpha}} dV_{\theta_0} + o(1),
\]
where the first equality follows from (B.53) and the definition of \( \tilde{V}_\nu \) in (B.38), and the second equality follows from (A.1) and the change of variables \((\tilde{z}, t) =\)
\((\rho_1, \rho_2, (\rho_1, \rho_2)^2) \), and the last inequality follows from (A.3) and the definition of \(\tilde{v}_\nu\).

For sufficiently large \(\nu\), we have

\[
\begin{align*}
\lim_{\nu \to 0} r_\infty & \int_{B_{1,\nu} (x_1, \nu)} |v_\nu|^{2+\frac{2}{n}} dV_{\theta_0} \\
& \ge \int_{B_{1,\nu} (x_1, \nu)} v_\nu L_{\theta_0} v_\nu dV_{\theta_0} \\
& \ge \left( \int_M |L_{\theta_0} v_\nu - r_\infty |v_\nu|^\frac{2n}{n-2} v_\nu dV_{\theta_0} \right)^{\frac{n-2}{2n}} \left( \int_M |v_\nu|^{2+\frac{2}{n}} dV_{\theta_0} \right)^{\frac{n}{2n}} \\
& \quad - \left( \int_M |\nabla_\theta v_\nu|_0^2 + R_{\theta_0} v_\nu^{2+\frac{2}{n}} \right) dV_{\theta_0} + o(1) \\
& \ge a_0 r_\infty^{n-1} Y(M, \theta_0)^{n+1} + o(1),
\end{align*}
\]

where the first inequality follows from Hölder’s inequality, the last equality follows from Proposition B.9, and the fact that \(v_\nu\) is bounded in \(L^{2+\frac{2}{n}}(M)\), and the last inequality follows from the definition of \((x_1, \nu, \rho_1, \nu)\) in (B.33).

Combining (B.54) and (B.55), we obtain

\[
\int \int \int_{\{z \in |\tilde{z}|^\frac{1}{n} < C_0\}} |V_\nu(z, t)|^{2+\frac{2}{n}} dV_{\theta_0} \ge a_0 r_\infty^{n-1} Y(M, \theta_0)^{n+1} + o(1).
\]

Combining this with Lemma B.8, we get

\[
\int \int \int_{\{z \in |\tilde{z}|^\frac{1}{n} < C_0\}} V(z, t)^{2+\frac{2}{n}} dV_{\theta_0} \ge a_0 r_\infty^{n-1} Y(M, \theta_0)^{n+1} + o(1).
\]

We compute

\[
\begin{align*}
& \int \int \int_{\{z \in |\tilde{z}|^\frac{1}{n} < C_0\}} V(z, t)^{2+\frac{2}{n}} dV_{\theta_0} \\
& = \left( \frac{n(2n+2)}{r_\infty} \right)^{n+1} \int \int \int_{\{z \in |\tilde{z}|^\frac{1}{n} < C_0\}} \left( \frac{\gamma_1^2}{\gamma_1^2 + (|z|^2 + 1)^2} \right)^{n+1} dV_{\theta_0},
\end{align*}
\]

where the first equality follows from (B.43) and the second equality follows from change of variables \((\tilde{z}, \tilde{t}) = (\gamma_1 z, \gamma_1^2 \tilde{t})\). Note that the last term in (B.57) tends to zero as \(\gamma_1 \to 0^+\). Hence, combining (B.56) and (B.57), we can conclude that \(\gamma_1\) is bounded below by a positive constant \(C_1\) depending only on \(a_0\), \(r_\infty\), and \(M\). This proves the assertion.

For any \((x, \lambda) \in M \times (0, \infty)\), we can find a unique solution \(\hat{\omega}(x, \lambda)\) of the following equation:

\[
L_{\theta_0} \hat{\omega}(x, \lambda) = r_\infty \omega'(x, \lambda)^{1+\frac{2}{n}} \quad \text{in} \ M,
\]

where \(\omega'(x, \lambda)\) is defined as

\[
\omega'(x, \lambda)(y) = \begin{cases} \chi_\delta(\rho_2(y)) \varphi_x(y) \omega(x, \lambda)(y), & \text{for} \ y \in B_{2\delta}(x); \\ 0, & \text{otherwise}. \end{cases}
\]
Here $\chi_\delta$ is the cut-off function defined in (A.11), $\varphi_x$ is the conformal factor such that $\tilde{\theta}_x = \varphi_x \tilde{\theta}_0$ in a neighborhood $B_{3\delta}(x)$ of $x$. Moreover, $\omega(x, \lambda)(y)$ is given by
\begin{equation}
\omega(x, \lambda)(y) = \left( \frac{n(2n + 2)}{\delta r_\infty} \right)^{\frac{2}{n}} \left( \frac{\lambda^2}{\lambda^2 + (\lambda^2 |z|^2 + 2)^2} \right)^{\frac{2}{n}},
\end{equation}
where $(z, t)$ is CR normal coordinates of $y$ centered at $x$. It follows from the definition of $\tilde{\omega}(x, \lambda)$ that
\begin{equation}
\tilde{\omega}(x, \lambda)(y) = \omega'(x, \lambda)(y) = 0 \quad \text{for } y \in M - B_{2\delta}(x).
\end{equation}
When $n = 1$, there holds (see Proposition 1 in [18])
\begin{equation}
|\tilde{\omega}(x, \lambda)(y) - \omega'(x, \lambda)(y)| \leq C\lambda^{-1}(1 + |\log(\lambda^{-2} + \rho_x(y)^2)|) \quad \text{for } y \in B_{2\delta}(x)
\end{equation}
and (see (3.6) in [18])
\begin{equation}
|L_{\theta_0}(\tilde{\omega}(x, \lambda)(y) - \omega'(x, \lambda)(y))| \leq \inf \left\{ 1, \frac{C}{\rho_x(y)^2 + \lambda^{-2}} \right\} \quad \text{for } y \in B_{2\delta}(x).
\end{equation}
It follows from (B.60) and (B.62) that
\begin{equation}
\|\tilde{\omega}(x, \lambda) - \omega'(x, \lambda)\|_{S^2(M)} \to 0 \quad \text{as } \lambda \to \infty.
\end{equation}
Similarly, when $M$ is spherical, it follows from Lemma 3 and Lemma 4 in [19] that
\begin{equation}
|\tilde{\omega}(x, \lambda)(y) - \omega'(x, \lambda)(y)| \leq \frac{C}{\lambda^n} \quad \text{for } y \in B_{2\delta}(x)
\end{equation}
and
\begin{equation}
\|\tilde{\omega}(x, \lambda) - \omega'(x, \lambda)\|_{S^2(M)} = O\left( \frac{1}{\lambda} \right).
\end{equation}

We have the following:

**Proposition B.11.** There holds
\begin{equation}
\int_{B_{\rho_1, \nu}(x_{1,\nu})} \left| \nu_{\nu} - \omega'(x^*_{1,\nu}, \frac{\gamma_1}{\rho_{1,\nu}}) \right|^{2 + \frac{2}{n'}} dV_{\tilde{\theta}_0} \to 0 \quad \text{as } \nu \to \infty.
\end{equation}

**Proof.** By Lemma B.53, we can find $N$ such that
\begin{equation}
C_0 \rho_{1,\nu} \leq \delta \quad \text{for } \nu \geq N,
\end{equation}
where $C_0$ is the constant in (A.8). Therefore, for $\nu \geq N$, we have
\begin{equation}
\int_{B_{\rho_1, \nu}(x_{1,\nu})} \left| \nu_{\nu} - \omega'(x^*_{1,\nu}, \frac{\gamma_1}{\rho_{1,\nu}}) \right|^{2 + \frac{2}{n'}} dV_{\tilde{\theta}_0}
\end{equation}
\begin{equation}
= \int_{B_{\rho_1, \nu}(x_{1,\nu})} \left| \tilde{\nu}_{\nu}(x) - \chi_\delta(\rho_{x^*_{1,\nu}}(x))\omega(x^*_{1,\nu}, \frac{\gamma_1}{\rho_{1,\nu}})(x) \right|^{2 + \frac{2}{n'}} dV_{\tilde{\theta}_0}
\end{equation}
\begin{equation}
\leq \int_{\{|\tilde{z}|^2 + \tilde{t}^2 \leq C_0 \rho_{1,\nu}\}} \left| \tilde{\nu}_{\nu}(\tilde{z}, \tilde{t}) - \omega(x^*_{1,\nu}, \frac{\gamma_1}{\rho_{1,\nu}})(x, t) \right|^{2 + \frac{2}{n'}} dV_{\tilde{\theta}_0}
\end{equation}
\begin{equation}
= \int_{\{|\tilde{z}|^2 + \tilde{t}^2 \leq C_0\}} \left| \tilde{V}_{\nu}(\tilde{z}, \tilde{t}) - \tilde{V}_{\nu}(\tilde{z}, \tilde{t}) \right|^{2 + \frac{2}{n'}} dV_{\tilde{\theta}_0},
\end{equation}
where the first inequality follows from (A.8), the first equality from the definition of $\omega'(x^*_{1,\nu}, \frac{\gamma_1}{\rho_{1,\nu}})$ in (B.58), and the last equality follows from (B.33), (B.59) and the change of variables $(\tilde{z}, \tilde{t}) = (\frac{\tilde{z}}{\rho_{1,\nu}}, \frac{\tilde{t}}{\rho_{1,\nu}})$). Thanks to Lemma B.8, the last expression in (B.66) tends to zero as $\nu \to \infty$. This proves the assertion. \qed
Lemma B.12. (i) The sequence \( \{v^1_{\nu}\} \) satisfies
\[
\int_M \left| L_{\theta_0} v^1_{\nu} - r_\infty |v^1_{\nu}| \hat{\omega}_0 \right|^{\frac{2n+2}{n-2}} dV_{\theta_0} \to 0 \quad \text{as} \; \nu \to \infty
\]

(ii) There exists a constant \( C \) such that
\[
\int_M \left( |\nabla_\theta v^1_{\nu}|^2 + (v^1_{\nu})^2 \right) dV_{\theta_0} \leq C \quad \text{for all} \; \nu.
\]

Proof. We follow the proof of Lemma 14 in [19]. By Lemma B.5 we can choose \( N \) such that \( \rho_1, \omega \leq 2\rho \) for \( \nu \geq N \). For \( \nu \geq N \), it follows from (B.60) that
\[
\int_M \left| L_{\theta_0} v^1_{\nu} - r_\infty |v^1_{\nu}| \hat{\omega}_0 \right|^{\frac{2n+2}{n-2}} dV_{\theta_0}
\]
\[
\leq C \int_M \left| L_{\theta_0} v_{\nu} - r_\infty |v^1_{\nu}| \hat{\omega}_0 \right|^{\frac{2n+2}{n-2}} dV_{\theta_0}
\]
\[
+ C \int_{B_{\rho_1, \omega}(x^1_{\nu})} \left| L_{\theta_0} \hat{\omega}(x^1_{\nu}, \frac{\gamma_1}{\rho_1, \nu}) - r_\infty \hat{\omega}(x^1_{\nu}, \frac{\gamma_1}{\rho_1, \nu}) \right|^{1+\frac{2}{n}} dV_{\theta_0}
\]
\[
+ C r_\infty \int_{B_{\rho_1, \omega}(x^1_{\nu})} \left| v^1_{\nu} \hat{\omega}(x^1_{\nu}, \frac{\gamma_1}{\rho_1, \nu}) \right|^{1+\frac{2}{n}} dV_{\theta_0}.
\]

It follows from (B.60), (B.61), (B.64), Lemma B.5 and the definition of \( \hat{\omega}(x, \lambda) \) that (B.68)
\[
\int_M \left| L_{\theta_0} \hat{\omega}(x^1_{\nu}, \frac{\gamma_1}{\rho_1, \nu}) - r_\infty \hat{\omega}(x^1_{\nu}, \frac{\gamma_1}{\rho_1, \nu}) \right|^{1+\frac{2}{n}} dV_{\theta_0}
\]
\[
= r_\infty \int_M \left| \omega'(x^1_{\nu}, \frac{\gamma_1}{\rho_1, \nu}) \right|^{1+\frac{2}{n}} dV_{\theta_0} = O(\rho_1, \omega) = o(1).
\]

On the other hand,
\[
\left| v^1_{\nu} \hat{\omega}(x^1_{\nu}, \frac{\gamma_1}{\rho_1, \nu}) \right|^{1+\frac{2}{n}} dV_{\theta_0}
\]
\[
\leq \left| v^1_{\nu} \hat{\omega}(x^1_{\nu}, \frac{\gamma_1}{\rho_1, \nu}) \right|^{1+\frac{2}{n}} dV_{\theta_0} + \left| \omega'(x^1_{\nu}, \frac{\gamma_1}{\rho_1, \nu}) \right|^{1+\frac{2}{n}} dV_{\theta_0}
\]
\[
+ \left| \omega'(x^1_{\nu}, \frac{\gamma_1}{\rho_1, \nu}) \right|^{1+\frac{2}{n}} dV_{\theta_0} = O(\rho_1, \omega) = o(1).
\]

We can now extract from \( \{v_{\nu}\} \) the first bubble and consider the following new sequence of functions:
\[
v^1_{\nu}(x) = v_{\nu}(x) - \hat{\omega}(x^1_{\nu}, \frac{\gamma_1}{\rho_1, \nu})(x).
\]
It follows from the proof of (B.8) that
\[ I = O \left( |v_\nu|^\frac{2}{\gamma_1} |v_\nu - \omega'(x_{1,\nu}^*, \frac{\gamma_1}{\rho_{1,\nu}}) | + |v_\nu - \omega'(x_{1,\nu}^*, \frac{\gamma_1}{\rho_{1,\nu}}) |^\frac{\gamma_1}{\gamma_1 - 2} |v_\nu| \right). \]
This together with Hölder’s inequality implies that
\[
\int_{B_{\rho_1, \nu}(x_{1,\nu}^*)} \left| v_\nu \right|^{\frac{2}{\gamma_1}} v_\nu - \omega'(x_{1,\nu}^*, \frac{\gamma_1}{\rho_{1,\nu}})^{1+\frac{2}{\gamma_1}} - |v_\nu - \omega'(x_{1,\nu}^*, \frac{\gamma_1}{\rho_{1,\nu}})|^{\frac{\gamma_1}{\gamma_1 - 2}} \left( v_\nu - \omega'(x_{1,\nu}^*, \frac{\gamma_1}{\rho_{1,\nu}}) \right)^{\frac{2n+2}{n+2}} dV_{\theta_0}
\]
\[
\leq C \left( \int_M |v_\nu|^{2+\frac{2}{\gamma_1}} \right)^{\frac{\gamma_1}{2+\frac{2}{\gamma_1}}} \left( \int_{B_{\rho_1, \nu}(x_{1,\nu}^*)} \left| v_\nu - \omega'(x_{1,\nu}^*, \frac{\gamma_1}{\rho_{1,\nu}}) \right|^{2+\frac{2}{\gamma_1}} dV_{\theta_0} \right)^{\frac{\gamma_1}{2+\frac{2}{\gamma_1}}}
\]
\[
+ C \left( \int_{B_{\rho_1, \nu}(x_{1,\nu}^*)} \left| v_\nu - \omega'(x_{1,\nu}^*, \frac{\gamma_1}{\rho_{1,\nu}}) \right|^{2+\frac{2}{\gamma_1}} dV_{\theta_0} \right)^{\frac{\gamma_1}{2+\frac{2}{\gamma_1}}}
\]
\[
= o(1),
\]
where the last equality follows from Proposition [B.11] and the fact that $v_\nu$ is bounded in $L^{2+\frac{2}{\gamma_1}}(M)$. Also, it follows from \[(B.60), (B.61), (B.64)\] and Hölder’s inequality that
\[
II \leq C \frac{\rho_{1, \nu}}{\gamma_1} \quad \text{and} \quad III \leq C \frac{\rho_{1, \nu}}{\gamma_1}.
\]
It follows from \[(B.60), (B.61), (B.64)\] and Hölder’s inequality that
\[
IV \leq C \frac{\rho_{1, \nu}}{\gamma_1} \left| v_\nu - \omega'(x_{1,\nu}^*, \frac{\gamma_1}{\rho_{1,\nu}}) \right|^{\frac{\gamma_1}{\gamma_1 - 2}} \quad \text{and} \quad V \leq C \left( \frac{\epsilon_{1, \nu}}{\gamma_1} \right)^{\frac{\gamma_1}{\gamma_1 - 2}} \left| v_\nu - \omega'(x_{1,\nu}^*, \frac{\gamma_1}{\rho_{1,\nu}}) \right|.
\]
Combining all these with Lemma [B.5] Proposition [B.10] and Proposition [B.11] we can conclude that
\[
\int_{B_{\rho_1, \nu}(x_{1,\nu}^*)} \left| v_\nu \right|^{\frac{2}{\gamma_1}} v_\nu - \omega'(x_{1,\nu}^*, \frac{\gamma_1}{\rho_{1,\nu}})^{1+\frac{2}{\gamma_1}}
\]
\[
- \left| v_\nu - \omega'(x_{1,\nu}^*, \frac{\gamma_1}{\rho_{1,\nu}}) \right|^{\frac{\gamma_1}{\gamma_1 - 2}} \left( v_\nu - \omega'(x_{1,\nu}^*, \frac{\gamma_1}{\rho_{1,\nu}}) \right)^{\frac{2n+2}{n+2}} dV_{\theta_0} = o(1).
\]
Now (i) follows from \[(B.9), (B.67), (B.68)\] and \[(B.69)\].

For (ii), note that
\[
\int_M \left( (2 + \frac{2}{n}) |\nabla_{\theta_0} \omega(x, \lambda)|^2 + R_{\theta_0} \omega(x, \lambda)^2 \right) dV_{\theta_0}
\]
\[
= r_\infty \int_M \omega'(x, \lambda)^{1+\frac{2}{\gamma_1}} \omega(x, \lambda) dV_{\theta_0}
\]
\[
= r_\infty \int_M \omega'(x, \lambda)^{2+\frac{2}{n}} dV_{\theta_0} + O \left( \frac{1}{\lambda} \right) \int_M \omega'(x, \lambda)^{1+\frac{2}{\gamma_1}} dV_{\theta_0}
\]
\[
\leq r_\infty \int_M \omega'(x, \lambda)^{2+\frac{2}{n}} dV_{\theta_0} + O \left( \frac{1}{\lambda} \right) \left( \int_M \omega'(x, \lambda)^{2+\frac{2}{n}} dV_{\theta_0} \right)^{\frac{n+2}{n+2}}
\]
where the first equality follows from the definition of $\omega(x, \lambda)$, the second equality follows from \[(B.60), (B.61)\] and \[(B.64)\], the last inequality follows from the Hölder’s inequality. Note also that it follows from the definition of $\omega'(x, \lambda)$ that
\[
(B.71) \quad \int_M \omega'(x, \lambda)^{2+\frac{2}{n}} dV_{\theta_0} \leq \int_{B_{2\lambda}(x)} \omega(x, \lambda)^{2+\frac{2}{n}} dV_{\theta_0} \leq \int_M \omega(x, \lambda)^{2+\frac{2}{n}} dV_{\theta_0},
\]
where the last inequality follows from the change of variables \((\tilde{z}, \tilde{t}) = (\lambda z, \lambda^2 t)\) in the CR normal coordinates. We derive from (B.70) and (B.71) that there exists a uniform constant \(C\) such that

\[
(B.72) \quad \int_M \left(2 + \frac{2}{n}\right)\left|\nabla_{\theta_0} \bar{\omega}(x, \lambda)\right|^2_{\theta_0} + R_{\theta_0} \bar{\omega}(x, \lambda)^2 \right) \, dV_{\theta_0} \leq C
\]

when \(\lambda\) is sufficiently large. By Cauchy-Schwarz inequality, we have

\[
(B.73) \quad \int_M \left(\left|\nabla_{\theta_0} v_\nu^1\right|^2 + (v_\nu^1)^2\right) \, dV_{\theta_0} \\
\leq 2 \int_M \left(\left|\nabla_{\theta_0} v_\nu\right|^2_{\theta_0} + v_\nu^2\right) \, dV_{\theta_0} + 2 \int_M \left(\left|\nabla_{\theta_0} \bar{\omega}(x_1^+ \nu, \frac{\gamma_1}{\rho_{1, \nu}})\right|^2_{\theta_0} + \bar{\omega}(x_1^+ \nu, \frac{\gamma_1}{\rho_{1, \nu}})^2\right) \, dV_{\theta_0}.
\]

Now (ii) follows from combining (B.72), (B.73) and the assumption that \(\{v_\nu\}\) is uniformly bounded in \(S^2(M)\). This proves Lemma B.12.

Iterating the above procedure, either \(v_\nu^1\) converges strongly to 0 in \(S^2(M)\) as \(\nu \to \infty\), or we can find a new sequence \((x_2^+ \nu, \rho_{2, \nu})\) and extract another bubble by defining

\[
v_\nu^2(x) = v_\nu^1(x) - \bar{\omega}(x_2^+ \nu, \frac{\gamma_2}{\rho_{2, \nu}})(x)
\]

and show that

\[
\int_M \left|L_{\theta_0} v_\nu^2 - r_\infty |v_{\nu}^2|^2 \hat{\theta} v_\nu^2 \right|^{2+n+2} \, dV_{\theta_0} \to 0 \quad \text{as} \quad \nu \to \infty.
\]

On the other hand, it can be shown that (see Lemma 15 and Lemma 16 in [19])

\[
(B.74) \quad \rho_{2, \nu} \geq \frac{1}{2} \rho_{1, \nu} \quad \text{and} \quad \frac{\rho_{2, \nu}}{\rho_{1, \nu}} + \frac{d(x_{1, \nu}^+, x_{2, \nu}^+)^2}{\rho_{1, \nu} \rho_{2, \nu}} \to \infty
\]

as \(\nu \to \infty\). Here \(d\) is Carnot-Carathéodory distance on \(M\) with respect to the contact form \(\theta_0\). This argument can be iterated as long as the new sequence \(\{v_\nu\}\) does not converge strongly to 0 in \(S^2(M)\). And we claim that the iteration must terminate in finite steps. To see this, note that

\[
\int_M \left(2 + \frac{2}{n}\right)\left|\nabla_{\theta_0} v_\nu^1\right|^2_{\theta_0} + R_{\theta_0} (v_\nu^1)^2 \right) \, dV_{\theta_0} \\
= \int_M \left(2 + \frac{2}{n}\right)\left|\nabla_{\theta_0} \left(v_{\nu}^{l-1} - \bar{\omega}(x_{1, \nu}^+, \frac{\gamma_l}{\rho_{1, \nu}})\right)\right|^2_{\theta_0} + R_{\theta_0} \left(v_{\nu}^{l-1} - \bar{\omega}(x_{1, \nu}^+, \frac{\gamma_l}{\rho_{1, \nu}})\right)^2 \right) \, dV_{\theta_0} \\
= \int_M \left(2 + \frac{2}{n}\right)\left|\nabla_{\theta_0} v_{\nu}^{l-1}\right|^2_{\theta_0} + R_{\theta_0} (v_{\nu}^{l-1})^2 \right) \, dV_{\theta_0} \\
+ r_\infty \int_M \bar{\omega}(x_{1, \nu}^+, \frac{\gamma_l}{\rho_{1, \nu}}) \omega'(x_{1, \nu}^+, \frac{\gamma_l}{\rho_{1, \nu}})^{1+\frac{n}{2}} \, dV_{\theta_0} + o(1),
\]
where the first equality follows from the fact that $v_{\nu}^{l-1}$ converges to 0 weakly in $S^2_\nu(M)$ as $\nu \to \infty$, and the last equality follows from (B.58). We compute

$$
\int_M \hat{\omega}(x^*_l, \frac{\gamma_l}{\rho_l, \nu}) \omega'(x^*_l, \frac{\gamma_l}{\rho_l, \nu}) 1 + \frac{\hat{\epsilon}}{n} dV_{\theta_0}
$$

$$
\geq \int_{B_3(x^*_l)} (\varphi_{x^*_l, \nu}(y) \omega(x^*_l, \frac{\gamma_l}{\rho_l, \nu})(y))^{2 + \frac{\hat{\epsilon}}{n}} dV_{\theta_0} + o(1)
$$

$$
\geq \int_{\{(||z|| + t^2) \leq \frac{r}{2}\}} (\varphi_{x^*_l, \nu}(y) \omega(x^*_l, \frac{\gamma_l}{\rho_l, \nu})(y))^{2 + \frac{\hat{\epsilon}}{n}} dV_{\theta_0} + o(1)
$$

$$
= \left( \frac{n(2n + 2)}{r^2} \right)^{n+1} \int_{\{(||z|| + t^2) \leq \frac{r}{2}\}} \left( \frac{\gamma_l^2}{\gamma_l^4 t^2 + (\gamma_l^2 ||z||^2 + 1)^2} \right)^{n+1} dV_{\theta_0} + o(1)
$$

$$
= \left( \frac{n(2n + 2)}{r^2} \right)^{n+1} \int_{\{(||z|| + t^2) \leq \frac{r}{2}\}} \left( \frac{1}{t^2 + (||z||^2 + 1)^2} \right)^{n+1} dV_{\theta_0} + o(1)
$$

$$
\geq \left( \frac{n(2n + 2)}{r^2} \right)^{n+1} \int_{\{(||z|| + t^2) \leq \frac{r}{2}\}} \left( \frac{1}{t^2 + (||z||^2 + 1)^2} \right)^{n+1} dV_{\theta_0} + o(1)
$$

where the first inequality follows from (B.60), (B.61), (B.64) and the definition of $\omega'(x^*_l, \frac{\gamma_l}{\rho_l, \nu})$, the second inequality follows from (A.11) and (A.8), the first equality follows from the change of variables $(\tilde{z}, \tilde{t}) = (\frac{||z||}{\rho_l, \nu}, \frac{t}{\rho_l, \nu})$, the second equality follows from the change of variables $(\tilde{z}, \tilde{t}) = (\frac{||z||}{\rho_l, \nu}, \frac{t}{\rho_l, \nu})$, and the last inequality follows from Proposition B.10 and Lemma B.5. Hence, if we let

$$
C_2 = \frac{n(2n + 2)}{r^2} \left( \frac{n(2n + 2)}{r^2} \right)^{n+1} \int_{\{(||z|| + t^2) \leq \frac{r}{2}\}} \left( \frac{1}{t^2 + (||z||^2 + 1)^2} \right)^{n+1} dV_{\theta_0},
$$

then it follows from the above computation that

$$
\int_M \left( 2 + \frac{2}{n} \right) |\nabla_{\theta_0} v_{\nu}^l|^2 dV_{\theta_0} + R_{\theta_0}(v_{\nu}^l)^2 dV_{\theta_0} + C_2 + o(1)
$$

That is to say, the quantity $\int_M \left( 2 + \frac{2}{n} \right) |\nabla_{\theta_0} v_{\nu}^l-1|^2 dV_{\theta_0} + R_{\theta_0}(v_{\nu}^{l-1})^2 dV_{\theta_0}$ at the l-th step decreases by at least $C_2$ after extraction of a bubble. Therefore, the iteration must stop after finite steps.

Therefore, there exists an integer $m$ and a sequence of $m$-tuples $(x^*_{k, \nu}, \varepsilon^*_{k, \nu})_{1 \leq k \leq m}$ where $\varepsilon^*_{k, \nu} = \frac{\rho_{k, \nu}}{\gamma_k}$ such that

$$
\varepsilon^*_{k, \nu} \to 0 \quad \text{as} \quad \nu \to \infty \quad \text{for all} \quad 1 \leq k \leq m,
$$

by Lemma B.5 and Proposition B.10. Also, we have

$$
(B.75) \quad \|v_{\nu} - \sum_{k=1}^{m} \hat{\omega}(x^*_{k, \nu}, \frac{1}{\varepsilon^*_{k, \nu}})\|_{S^2_\nu(M)} \to 0 \quad \text{as} \quad \nu \to \infty.
$$
Now (5.4) follows from (B.74) and Proposition B.10. On the other hand, we have

\[
\left\| u_\nu - u_\infty - \sum_{k=1}^{m} \frac{\tilde{u}(x^*_{k,\nu}, \varepsilon^*_{k,\nu})}{\varepsilon^*_{k,\nu}} \right\| _{S^2_1(M)}^2 \\
= \left\| v_\nu - \sum_{k=1}^{m} \varepsilon^*_{k,\nu} \left( \frac{\omega'(x^*_{k,\nu}, \frac{1}{\varepsilon^*_{k,\nu}}) - \tilde{\omega}(x^*_{k,\nu}, \frac{1}{\varepsilon^*_{k,\nu}})}{\varepsilon^*_{k,\nu}} \right) \right\| _{S^2_1(M)}^2 \\
- \sum_{k=1}^{m} \frac{\varphi_{x^*_{k,\nu}}(y)}{\varepsilon^*_{k,\nu}^2} \left( n(2n + 2) \right) \frac{\partial}{\partial \varepsilon^*_{k,\nu}} \left( \varepsilon^*_{k,\nu} \right) \left( 1 - \chi_s(\rho_{x^*_{k,\nu}}(y)) \right) G_{x^*_{k,\nu}}(y) \right\| _{S^2_1(M)}^2 = o(1)
\]

where the first equality follows from (A.10) and (A.11), and the last equality follows from (B.59), (B.63), (B.75), Lemma B.5 and the fact that the Green’s function $G_{x^*_{k,\nu}}(y)$ is bounded in $S^2_1(K)$ for any compact set $K \subset M - \{x^*_{k,\nu}\}$ (see (A.6) and (A.7)). This proves (5.5) and this completes the proof of Theorem 5.1.

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