Geometry of the Lagrangian Grassmannian $\text{Sp}(3)/\text{U}(3)$ with applications to Brill-Noether Loci

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January 30, 2022

Dedicated to Andrei Nikolaevich Tyurin

Abstract

The geometry of $\text{Sp}(3)/\text{U}(3)$ as a subvariety of $\text{Gr}(3,6)$ is explored to explain several examples given by Mukai of non-abelian Brill-Noether loci, and to give some new examples. These examples identify Brill-Noether loci of vector bundles on linear sections of the Lagrangian Grassmannian $\text{Sp}(3)/\text{U}(3)$ with orthogonal linear sections of the dual variety and vice versa. A main technical result of independent interest is the fact that any nodal hyperplane section of the Lagrangian Grassmannian projected from the node is a linear section of the Grassmannian $\text{Gr}(2,6)$.

1 Introduction

Mukai’s linear section theorem for canonical curves of genus 8 and 9 says that every smooth canonical curve of genus 8 with no $g_7^1$ is a complete intersection of the Grassmannian $\text{Gr}(2,6) \subset \mathbb{P}^{14}$ with a linear subspace $\mathbb{P}^7$, and that every smooth canonical curve of genus 9 with no $g_6^1$ is a complete intersection of the Lagrangian Grassmannian $\text{LG}(3,6) = \text{Sp}(3)/\text{U}(3) \subset \mathbb{P}^{13}$ and a linear subspace $\mathbb{P}^8$ [31]. Another beautiful theorem of Mukai is his interpretation of the general complete intersection of $\text{LG}(3,6)$ with a linear subspace $\mathbb{P}^{10}$ as a non-abelian Brill-Noether locus of vector bundles on a plane quartic curve [32]. This quartic curve can in a natural way be interpreted as the orthogonal plane section of the dual variety to $\text{LG}(3,6)$ in $\mathbb{P}^{13}$. In this paper we consider, very much in the spirit of [32], general linear sections of the Lagrangian Grassmannian $\text{LG}(3,6) \subset \mathbb{P}^{13}$ of various dimensions, and show that the orthogonal linear section of the dual variety $\tilde{F}$ of $\text{LG}(3,6)$ has an interpretation as a moduli space of vector bundles on the original linear section, and vice versa. A similar study of linear sections of the 10-dimensional spinor variety or orthogonal Grassmannian $\text{OG}(5,10) \subset \mathbb{P}^{15}$ is taken up by the first author and Markusewich in [16].

The moduli spaces of stable vector bundles on curves is by now a classical subject dating back to 1960’s and the fundamental work of Narasimhan, Seshadri and Tyurin (cf. [34], [11]). More recently the subvarieties of these moduli spaces representing bundles with many sections has attracted attention from many authors (cf. [1], [5], [13], [35], [39], [32]). The corresponding theory for sheaves on $K3$ surfaces becomes particularly nice as explained in Mukais fundamental paper [28].

*Partially supported by Grant MM-1106/2001 of the Bulgarian Foundation for Scientific Research and by the Norwegian Research Council
The purpose of this paper is to present examples in this theory where these moduli spaces are all complete linear sections of either $\text{LG}(3, 6) \subset \mathbb{P}^{13}$ or its dual variety of singular hyperplane sections in the dual space.

A general tangent hyperplane section of $\text{LG}(3, 6)$ is nodal, i.e. has a unique tangency point with a quadratic singularity. The main technical result is (3.3.4 and 3.3.10)

**Theorem.** The projection of a nodal hyperplane section of $\text{LG}(3, 6) = \text{Sp}(3)/U(3)$ from the node is a complete 5-dimensional linear section of a Grassmannian variety $\text{Gr}(2, 6)$. This linear section contains a 4-dimensional quadric, and the general 5-dimensional linear section of $\text{Gr}(2, 6)$ that contains a 4-dimensional quadric appears this way.

We expect similar results to hold for the homogeneous varieties whose general curve section are canonical curves of smaller genus. In particular we expect the projection from the node of the general nodal hyperplane section of $\text{Gr}(2, 6)$ to be a 7-fold linear section of a spinor variety $S_{10}$, and the projection from the node of the general nodal 5-fold linear section of $S_{10}$ to be the complete intersection of a Grassmannian variety $\text{Gr}(2, 5)$ and a quadric. These lower genus cases will not be treated here.

Given a smooth linear section $X$ of $\text{LG}(3, 6)$ of dimension at most 4, each nodal hyperplane section that contains $X$ gives rise to an embedding of $X$ into $\text{Gr}(2, 6)$. In particular it gives rise to a rank 2 vector bundle on $X$ with 6 global sections and determinant equal to the restriction of the Plücker divisor on $\text{LG}(3, 6)$. Furthermore, this vector bundle is stable, and when $X$ is at least 2-dimensional, we show that the set $\mathcal{F}(X)$ of nodal hyperplane sections that contain $X$ form a component of the corresponding moduli space of stable rank 2 vector bundles on $X$ (cf. 3.5.1, 3.5.3). When $X$ is a curve, $\mathcal{F}(X)$ form a component of the corresponding Brill-Noether locus in the moduli space of stable rank 2 vector bundles on $X$ (cf. 3.4.7).

In the opposite direction the general 3-fold linear section $X$ of $\text{LG}(3, 6)$ define a Brill-Noether locus of type II for the plane quartic curve $\mathcal{F}(X)$. More precisely, associated to $X$ there is a $\mathbb{P}^1$-bundle over $\mathcal{F}(X)$ naturally embedded in $\mathbb{P}^5$ as a conic bundle of degree 16. It is isomorphic to a $\mathbb{P}(\mathcal{F})$ where $\mathcal{F}$ is a rank 2 vector bundle on $\mathcal{F}(X)$ of degree 3 such that no twist of $\mathcal{F}$ by line bundles of negative degree have sections. Then $X$ is isomorphic to the moduli space of rank two vector bundles $\mathcal{E}$ on $\mathcal{F}(X)$ such that $\text{rank} \text{Hom}_C(\mathcal{F}, \mathcal{E}) \geq 3$ and $\det \mathcal{E} - \det \mathcal{F} = K_{\mathcal{F}(X)}$ (cf. 2.8).

The paper is organized as follows. The first part is devoted to the geometry of $\text{LG}(3, 6)$. It is the minimal orbit of an irreducible representation of the symplectic group $\text{Sp}(3)$. We describe the four orbits of this group and analyse the singular hyperplane sections corresponding to the corresponding 4 orbits of the group in the dual space. $\text{LG}(3, 6) \subset \mathbb{P}^{13}$ parameterizes the 6-fold of Lagrangian planes in $\mathbb{P}^5$ with respect to a given nondegenerate 2-form. A hyperplane section that is singular at a point $p \in \text{LG}(3, 6)$ define naturally a conic section in the Lagrangian plane represented by $p$. The correspondence between singular hyperplane sections and conic sections in Lagrangian planes manifests itself in various ways and is the crucial key to the main results of this paper. In particular, the first part ends with a description of conic bundles in the incidences between linear sections $X$, the set $\mathcal{F}(X)$ of singular hyperplane sections that contain $X$ and the so-called vertex variety of points on conic sections in Lagrangian planes corresponding to the singular hyperplane sections.

In the second part the conic section corresponding to a nodal hyperplane section is again the crucial ingredient in the construction of a rank 2 vector bundle with 6 global sections.
on the nodal hyperplane section blown up in the node. The application to moduli spaces of vector bundles and Brill-Noether loci occupies the rest of the second part and relies on some general results, to ensure that our examples form components of the corresponding moduli spaces.

Notation. We will denote by $\text{Gr}(k,n)$ the Grassmannian of rank $k$ subspaces of a $n$-dimensional vector space when $k < n$, and the Grassmanian of rank $n$ quotient spaces of a $k$-dimensional vector space when $n < k$.

## 2 Geometry of the Lagrangian Grassmannian

### 2.1 The groups $Sp(3, \mathbb{C})$ and $Sp(3)$

Let $V = \mathbb{C}^6$ be a 6-dimensional complex vector space, and let $\alpha : V \times V \to \mathbb{C}$, $\alpha : (v, v') \mapsto \alpha(v, v')$ be a symplectic form on $V$. Thus $\alpha$ is bilinear, skew-symmetric and non-degenerate (i.e. $\alpha(v \times V) = 0$ implies $v = 0$).

The canonical transformations of $V$ are those diffeomorphisms $f : V \to V$ which leave the form $\alpha$ invariant, i.e. $f^* \alpha = \alpha$. The canonical transformations of $V$ form a group (with a multiplication – the composition of maps), and the symplectic group $Sp(3, \mathbb{C})$ is the subgroup of these canonical transformations which are complex-linear.

For the fixed $\alpha$, it is always possible to find a base $\{e_1, ..., e_6\}$ of $V$ in which the Gramm matrix

$$J = (\alpha(e_i, e_j)) = \begin{pmatrix} 0 & I_3 \\ -I_3 & 0 \end{pmatrix},$$

where $I_3$ is the unit $3 \times 3$ matrix. The complex symplectic group $Sp(3, \mathbb{C})$ has a natural embedding in $GL(6, \mathbb{C})$ as the subgroup of all the complex rank 6 matrices which leave the matrix $J$ invariant, i.e.

$$Sp(3, \mathbb{C}) = \{Z \in GL(6, \mathbb{C}) : {^t}ZJZ = J\}.$$

The group $Sp(3, \mathbb{C})$ is a non-compact Lie group of real dimension 42. Its Lie algebra $sp(3, \mathbb{C})$ consists of all the complex $6 \times 6$ matrices of the form $\left(\begin{array}{cc} A & B \\ C & -{^t}A \end{array}\right)$, where $A, B, C$ are $3 \times 3$ matrices such that $^tB = B$ and $^tC = C$. The group

$$Sp(3, \mathbb{C}) = \left\{ \left(\begin{array}{cc} A & B \\ C & D \end{array}\right) | B \cdot {^t}A = A \cdot {^t}B, \quad B \cdot {^t}C - A \cdot {^t}D = -I_3, \quad D \cdot {^t}C = C \cdot {^t}D \right\}.$$  

Along with the non-compact group $Sp(3, \mathbb{C})$ there exists also the compact group $Sp(3)$, which is defined below.

Let $\mathbb{R}$ be the field of real numbers, and let $\mathbb{H} = \mathbb{R} + \mathbb{R}i + \mathbb{R}j + \mathbb{R}k$ be the quaternionic algebra with products $i^2 = j^2 = k^2 = -1$, $ij = k = -ji$, $jk = i = -kj$, $ki = j = -ik$.

For the quaternion $h = a + bi + cj + dk$, the coordinate $a = Re(h)$ is the real part of $h$, and $\bar{h} = 2Re(h) - h = a - bi - cj - dk$ is the conjugate quaternion to $h$. The non-negative number $|h| = (a^2 + b^2 + c^2 + d^2)^{1/2}$ is called the norm of $h = a + bi + cj + dk$; and it is easy to see that $\bar{h}h = |h|^2$ for any $h \in \mathbb{H}$.

Let $e_1, e_2, e_3$ be a base of the 3-dimensional quaternionic vector space $\mathbb{H}^3 = \mathbb{H}e_1 + \mathbb{H}e_2 + \mathbb{H}e_3$. In this base, any vector $u \in \mathbb{H}^3$ can be written uniquely in the form

$$u = \Sigma_{1 \leq m \leq 3} u_me_m = \Sigma_{1 \leq m \leq 3} (a_m + b_mi + c_mj + d_mk)e_m.$$
where \( u_m = a_m + b_m i + c_m j + d_m k \), \( a_m, b_m, c_m, d_m \in \mathbb{R} \), are the quaternionic coordinates of \( u, m = 1, 2, 3 \).

If \( e_{3+m} = je_m, m = 1, 2, 3 \), then the above identity yields a natural identification of \( H^3 \) and the complex 6-space \( V = \mathbb{C}^6 = \Sigma_{1 \leq m \leq 3} Ce_m + C e_{3+m} \).

For any two vectors \( u = u_1 e_1 + u_2 e_2 + u_3 e_3 \) and \( v = v_1 e_1 + v_2 e_2 + v_3 e_3 \) in \( H^3 \), define their quaternionic product \( [u, v] = Re(u_1 \bar{v}_1 + u_2 \bar{v}_2 + u_3 \bar{v}_3) \). By definition

\[
Sp(3) = \{ A \in GL(3, H) | [Au, Av] = [u, v] \text{ for any } u, v \in H^3 \}.
\]

Clearly \( Sp(3) \) is a subgroup of \( GL(3, H) \); and it is well-known that \( Sp(3) \) is a compact connected group of real dimension 21. In the above identification of \( H^3 \) and \( \mathbb{C}^6 \), any \( A \in Sp(3) \) becomes unitary – as an element of \( GL(6, \mathbb{C}) \). If \( \alpha \) is the symplectic form on \( V = \mathbb{C}^6 \) with Gramm matrix \( J \) as above, then any \( A \in Sp(n) \) becomes an element of \( Sp(3, C) \). Moreover \( Sp(3) = U(6) \cap Sp(3, C) \), where \( U(6) \subset GL(6, \mathbb{C}) \) is the unitary subgroup.

The Lie algebra \( sp(3) \) of \( Sp(3) \subset GL(6, \mathbb{C}) \) consists of all the complex \( 6 \times 6 \) matrices of the form

\[
\begin{pmatrix}
  A & B \\
  -\bar{B} & \bar{A}
\end{pmatrix},
\]

where \( A, B, C \) are \( n \times n \) matrices such that \( tB = B \) and \( t\bar{A} = -A \).

The group

\[
Sp(3) = \{ \begin{pmatrix}
  A & B \\
  -\bar{B} & \bar{A}
\end{pmatrix} | A \cdot t\bar{A} + B \cdot t\bar{B} = I_3, \quad B \cdot tA = A \cdot tB \}
\]

2.2 The Lagrangian Grassmannian \( LG(3, 6) = Sp(3)/U(3) \)

The subspace \( U \subset V \) is called isotropic if \( \alpha(U, U) = 0 \). The maximal dimension of an isotropic subspace in \( V \) is 3, and in this case it is called Lagrangian. Any isotropic subspace is contained in some Lagrangian subspace.

By definition the complex Lagrangian Grassmannian (or the complex Symplectic Grassmannian) \( LG(3, V) \) is the set of all the Lagrangian subspaces of \( V = \mathbb{C}^6 \).

The group \( Sp(3, C) \) acts on the set of Lagrangian subspaces by \( U \mapsto A \cdot U \), and it is easy to check that this action is transitive.

The Lagrangian Grassmannian \( LG(3, V) \) is a smooth complex 6-fold which admits a representation as a homogeneous space \( Sp(3, C)/St \) where \( St \) is isomorphic to the stabiliser group \( St_U \) of any Lagrangian subspace \( U \subset V \). In our fixed base the subspace \( U_0 = < e_1, e_2, e_3 > \subset V \) is Lagrangian, and then \( St = St_{U_0} \subset Sp(3, C) \) consists of all the matrices of the form

\[
\begin{pmatrix}
  A & 0 \\
  0 & tA^{-1}
\end{pmatrix},
\]

where \( A \) and \( B \) are complex \( 3 \times 3 \) matrices such that \( A \cdot tB = B \cdot tA \).

The Lagrangian Grassmannian has an alternative representation as a quotient for the compact group \( Sp(3) = Sp(3, C) \cap U(6) \subset Sp(3, C) \) by the subgroup \( St_{U_0} \cap U(6) \subset Sp(3) \).

It is easy to see that \( St_{U_0} \cap U(6) \) consists of all the \( 6 \times 6 \) matrices of the form

\[
\begin{pmatrix}
  A & 0 \\
  0 & tA^{-1}
\end{pmatrix},
\]

where \( A \in U(3) \) is a unitary \( 3 \times 3 \) matrix. Now one can see that the Lagrangian Grassmannian \( LG(3, V) \) is diffeomorphic to \( Sp(3)/U(3) \) – see §17 in [10]; from now on we shall use the former notation or rather \( \Sigma \) for its Plücker embedding that we describe next.
2.3 Representations and Plücker embedding

From here on we fix the form $\alpha$, the basis $\{e_1, ..., e_6\}$ for $V$ and a dual basis $\{x_1, ..., x_6\}$ for $V^*$. With respect to this basis the matrix of the form $\alpha$ is

$$J = (\alpha(e_i, e_j)) = \begin{pmatrix} 0 & I_3 \\ -I_3 & 0 \end{pmatrix},$$

where $I_3$ is the unit $3 \times 3$ matrix. In the $\{x_i \land x_j | 1 \leq i < j \leq 6\}$ for $\wedge^2 V^*$, the symplectic form $\alpha$ has the following expression

$$\alpha = x_1 \land x_4 + x_2 \land x_5 + x_3 \land x_6.$$

It defines a correlation

$$L_\alpha : V \to V^* \quad v \mapsto \alpha(v, -).$$

This is natural up to sign and nonsingular since $\alpha$ is nondegenerate. It is called a correlation since $v \in \ker L_\alpha(v)$ for every $v \in V$. This correlation induces isomorphisms that we also denote by $L_\alpha$:

$$L_\alpha : \wedge^k V \cong \wedge^k V^*$$

for $k = 1 \ldots 6$. Consider the representations $\wedge^k V$ and $\wedge^k V^*$ of $GL(6, \mathbb{C})$, and the induced representation of $Sp(3, \mathbb{C}) \subset GL(6, \mathbb{C})$. The isomorphisms $\wedge^k V \cong \wedge^k V^*$ induced by $\alpha$ are clearly isomorphisms of these $Sp(3, \mathbb{C})$-representations.

For the Plücker embedding of the Lagrangian Grassmannian we consider the $Sp(3, \mathbb{C})$ representation $\wedge^3 V$. The form $\alpha$ defines a contraction which we denote by $\alpha$ itself: $\alpha : \wedge^3 V \to V$. The representation then decomposes

$$\wedge^3 V = V(14) \oplus V(6),$$

where

$$V(14) = \{w \in \wedge^3 V | \alpha(w) = 0\}$$

and

$$V(6) = \{w \in \wedge^3 V | L_\alpha(w) \in \alpha \wedge V^*\}$$

has rank 14 and 6 respectively (see e.g. [10], p.258). Furthermore

$$V(14)^* := L_\alpha(V(14)) = \{\omega \in \wedge^3 V^* | \omega \wedge \alpha = 0\} \subset \wedge^3 V^*$$

while

$$V(6)^* = \alpha \wedge V^*.$$

We now consider the decomposition of $V$ in two Lagrangian subspaces $U_0 = \langle e_1, e_2, e_3 \rangle$ and $U_1 = \langle e_4, e_5, e_6 \rangle$. We denote by $U_0^\perp$ the Lagrangian subspace $L_\alpha(U_0) = \langle x_4, x_5, x_6 \rangle \subset V^*$, and likewise $U_1^\perp = L_\alpha(U_1) = \langle x_1, x_2, x_3 \rangle$. The decomposition

$$V^* = U_1^\perp \oplus U_0^\perp,$$

induces a decomposition of $\wedge^3 V^*$:

$$\wedge^3 V^* = \wedge^3 U_1^\perp \oplus (\wedge^2 U_1^\perp \otimes U_0^\perp) \oplus (U_1^\perp \otimes \wedge^2 U_0^\perp) \oplus \wedge^3 U_0^\perp.$$
Consider the subspace 

\[ U_1^\perp \otimes \wedge^2 U_0^\perp \subset \wedge^3 V^*. \]

The composition of the map \( L_\alpha^{-1}: U_1^\perp \to U_0 \), the restriction \( U_1^\perp \to U_0^* \) and the natural isomorphism (up to scalar) \( \wedge^2 U_0 \to U_0^* \) define a natural map

\[ U_1^\perp \otimes \wedge^2 U_0^\perp \to U_0^* \otimes \wedge^2 U_0 \to U_0^* \otimes U_0^* \]

Thus the component of a 3-form in \( U_1^\perp \otimes \wedge^2 U_0^\perp \) defines a bilinear form on \( U_0 \subset V^* \). Since the restriction \( U_1^\perp \to U_0^* \) is an isomorphism, the inverse map is a section of the restriction \( V^* \to U_0^* \). Therefore the bilinear form is independent of the choice of subspace \( U_1 \).

We make this construction explicit in coordinates. The exterior products \( e_{ijk} = e_i \wedge e_j \wedge e_k, 1 \leq i < j < k \leq 6 \) form a basis for \( \wedge^3 V \). Let \( (x_{ijk} = x_i \wedge x_j \wedge x_k)_{1 \leq i,j,k \leq 6} \) be the dual basis. We interpret the basis for \( \wedge^3 V^* \) as the coordinate functions of an element \( w \in \wedge^3 V \), and dually \( \wedge^3 V \) as coordinates on \( \wedge^3 V^* \). Thus a 3-form \( \omega \in \wedge^3 V^* \) has coordinates:

\[ u^* = e_{123}, X^* = (x_{ab}) = \begin{pmatrix} e_{423} & e_{143} & e_{124} \\ e_{523} & e_{153} & e_{125} \\ e_{623} & e_{163} & e_{126} \end{pmatrix}, \]

\[ Y^* = (y_{ab}) = \begin{pmatrix} e_{156} & e_{146} & e_{451} \\ e_{256} & e_{142} & e_{452} \\ e_{356} & e_{136} & e_{453} \end{pmatrix}, z^* = e_{456}. \]

The component of \( \omega \) given by the matrix \( X^* \) defines a bilinear form on \( U_0 \). If \( (a_1, a_2, a_3), (b_1, b_2, b_3) \in U_0 \), then the form defined by \( X \) becomes:

\[ (a_1, a_2, a_3) \cdot \begin{pmatrix} e_{423} & e_{143} & e_{124} \\ e_{523} & e_{153} & e_{125} \\ e_{623} & e_{163} & e_{126} \end{pmatrix} \cdot (b_1, b_2, b_3)^t. \]

Similarly the component of \( w \) given by the matrix \( Y \) defines a bilinear form on \( U_1 \). The subspace \( V(14)^* \subset \wedge^3 V^* \) now has a simple interpretation in these coordinates:

\[ V(14)^* = \{ \omega \in \wedge^3 V^*: e_{i14} + e_{i25} + e_{i36} = 0, 1 \leq i \leq 6 \}. \]

In the decomposition \( \omega = [u^*, X^*, Y^*, z^*] \) we get:

\[ \omega \in V(14)^* \iff X^* = (x_{ab}^*) \text{ and } Y^* = (y_{ab}^*) \text{ are symmetric } 3 \times 3 \text{ matrices} \]

In particular, the bilinear form defined by the component \( X^* \) is the symmetric form

\[ (x_1, x_2, x_3) \cdot \begin{pmatrix} e_{123} & e_{143} & e_{124} \\ e_{523} & e_{153} & e_{125} \\ e_{623} & e_{163} & e_{126} \end{pmatrix} \cdot (x_1, x_2, x_3)^t = x_1^2 + x_2^2 + x_3^2 \]

on \( U_0 \). Thus we have described in coordinates a natural map

\[ q(U_0): \wedge^3 U_0^\perp \oplus U_1^\perp \otimes \wedge^2 U_0^\perp \to Sym^2 U_0^* \]

For \( \omega \in \wedge^3 U_0^\perp \oplus U_1^\perp \otimes \wedge^2 U_0^\perp \) we denote the associated quadratic form by \( q_\omega(U_0) \) or just \( q_\omega \) if \( U_0 \) is understood from the context, and by abuse we sometimes use the same notation for the conic section in \( \mathbf{P}(U_0) \) that the quadratic form defines.
The Plücker embedding $\Sigma := \text{LG}(3, V) \subset \text{Gr}(3, V) \subset \mathbb{P}(\wedge^3 V)$ of the Lagrangian Grassmannian $\text{LG}(3, V)$ is the intersection of $\text{Gr}(3, V)$ with $\mathbb{P}(V(14))$, i.e.

$$\Sigma = \mathbb{P}(V(14)) \cap \text{Gr}(3, V) \subset \mathbb{P}(\wedge^3 V).$$

In coordinates the graph of a linear map $X \in \text{Hom}(U_0, U_1)$ is given by a matrix

$$\begin{pmatrix} I & X \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & x_{11} & x_{12} & x_{13} \\ 0 & 1 & 0 & x_{21} & x_{22} & x_{23} \\ 0 & 0 & 1 & x_{31} & x_{32} & x_{33} \end{pmatrix}.$$

Thus

$$X \mapsto [1, X, \wedge^2 X, \wedge^3 X]$$

is an open immersion of $\mathbb{C}^9$ into $\text{Gr}(3, V)$ in its Plücker embedding defining an affine neighborhood of $[U_0]$. This is the affine representation of coordinates $[u, X, Y, z]$ dual to the coordinates $[u^*, X^*, Y^*, z^*]$ above. In particular, the subspace $U_X$ is Lagrangian if and only if $X$ is symmetric. Thus the intersection of the above coordinate chart of $\text{Gr}(3, V)$ with $\Sigma$ is defined by 6 symmetrizing linear equations of $X$ and $\wedge^2 X$. The closure of the affine chart around $[U_0]$ is clearly $\Sigma$ so $\Sigma = \mathbb{P}(V(14)) \cap \text{Gr}(3, V)$.

Equations for $\Sigma$ in the affine coordinates

$$[1, X, \wedge^2 X, \wedge^3 X] = [1, x_{ij}, y_{ij}, z]$$

homogenized with first coordinate $u$, are given by standard determinantal identities:

$$
\begin{align*}
uy_{11} - x_{22}x_{33} + x_{23}^2 & \quad uy_{12} + x_{12}x_{33} - x_{13}x_{23} \\
u_{22} - x_{11}x_{33} + x_{13}^2 & \quad uy_{13} - x_{12}x_{23} + x_{13}x_{22} \\
u_{33} - x_{11}x_{22} + x_{12}^2 & \quad uy_{23} + x_{11}x_{23} - x_{12}x_{13} \\
u z - x_{11}y_{11} - x_{12}y_{12} - x_{13}y_{13} & \quad zx_{11} - y_{22}y_{33} + y_{23}^2 \\
u z - x_{12}y_{12} - x_{22}y_{22} - x_{23}y_{23} & \quad zx_{22} - y_{11}y_{33} + y_{13}^2 \\
u z - x_{13}y_{13} - x_{23}y_{23} - x_{33}y_{33} & \quad zx_{33} - y_{11}y_{22} + y_{12}^2 \\
x_{11}y_{12} + x_{12}y_{22} + x_{13}y_{23} & \quad zx_{12} + y_{12}y_{33} - y_{13}y_{23} \\
x_{11}y_{13} + x_{13}y_{23} + x_{13}y_{33} & \quad zx_{13} - y_{12}y_{23} + y_{13}y_{22} \\
x_{12}y_{11} + x_{22}y_{12} + x_{23}y_{13} & \quad zx_{23} + y_{11}y_{23} - y_{12}y_{13} \\
x_{12}y_{13} + x_{22}y_{23} + x_{23}y_{33} & \quad x_{13}y_{11} + x_{23}y_{12} + x_{33}y_{13} \\
x_{13}y_{12} + x_{23}y_{22} + x_{33}y_{23} &
\end{align*}
$$

or, in short:

$$uY = \wedge^2 X, uzI = XY, zX = \wedge^2 Y,$$

where $I = I_3$ is the unit $3 \times 3$ matrix. In other words, the equations of $\Sigma \subset \mathbb{P}^{13}$ can be regarded as projectivized Cramer equations for a symmetric $3 \times 3$ matrix, its adjoint matrix and its determinant.

When $u = 1$, the equations define the coordinates $y_i$ and $z$ in terms of $x_i$, so set-theoretically the coordinate chart $\{u = 1\}$ is defined by these equations. So the quadratic equations define a set-theoretic closure of this $U_0$. An easy computation with MACAULAY shows that the resolution of the ideal $I$ generated by these quadrics have betti numbers:
in the notation standard notation in the program (cf.\[2\]). Therefore the ideal \( I \) is arithmetically Cohen-Macaulay and \( Z(I) \) has no embedded components. Thus \( Z(I) = \Sigma \). In fact, one may check that the resolution is symmetric, which means that \( I \) is arithmetically Gorenstein.

2.4 Orbits, pivots and tangent spaces

Recall from \[20\] §9 that the action \( \rho \) of the group \( Sp(3, \mathbb{C}) \) on \( P^{13} = P(V(14)) \) has precisely four orbits:

\( P^{13} \setminus F, F \setminus \Omega, \Omega \setminus \Sigma, \Sigma. \)

The dual action \( \rho \) is equivalent to \( \rho \) induced by \( L_\alpha \), so it has four corresponding orbits

\( \mathcal{P}^{13} \setminus \mathcal{F}, \mathcal{F} \setminus \mathcal{\Omega}, \mathcal{\Omega} \setminus \mathcal{\Sigma}, \mathcal{\Sigma} \)

in \( \mathcal{P}^{13} = P(V(14)^*) \). In this section we give a geometric characterisation of these orbits.

The smallest orbit, and the only closed one, is the Lagrangian Grassmannian itself. The closure of the orbit \( F \setminus \Omega \) is the union of the projective tangent spaces to \( \Sigma \). They form a hypersurface \( F \subset P^{13} \). Similarly, the dual variety to \( \Sigma \), i.e. the set of hyperplanes \( H \subset P^{13} \) containing some projective tangent space to \( \Sigma \) form a hypersurface \( \mathcal{F} \subset \mathcal{P}^{13} \) isomorphic to \( F \). By \[37\] p. 108 the equation defining \( F \) is

\[
f(w) = (uz - tr XY)^2 + 4u \det Y + 4z \det X - 4\Sigma_{ij} \det(X_{ij})\det(Y_{ij}),
\]

where \( X_{ij} \) and \( Y_{ij} \) are the complimentary matrices to the elements \( x_{ij} \) and \( y_{ij} \) (see also \[37\] p. 83). \( \Omega \) is the singular locus of \( F \), defined by vanishing all the partials of \( f \). A simple computation in MACAULAY \[4\] shows that the ideal of \( \Omega \) has a resolution with Betti numbers

\[
\begin{array}{cccccccc}
1 & - & - & - & - & - & - & - \\
- & 14 & 21 & - & - & - & - & - \\
- & - & - & - & - & - & - & - \\
- & - & - & - & - & - & 6 & - \\
\end{array}
\]

Furthermore \( \dim \Omega = 9 \) and \( \deg \Omega = 21 \) and sectional genus 22. We shall see this in a different description of \( \Omega \) later.

These orbits of \( \rho \) in \( P^{13} = P(V(14)) \) are restrictions to \( P^{13} = P(V(14)) \) of the orbits of \( SL(V, \mathbb{C}) \) in \( P^{19} = P(\wedge^3 V) = P(V(14) \oplus V(6)) \). In \[3\], Donagi proves the following:

**Theorem 2.4.1 (Theorem of Segre for \( Gr(3, 6) \))** The natural representation of \( GL(6, \mathbb{C}) \) on \( P^{19} = P(\wedge^3 V) \) has four orbits:

\[
P^{19} = G \cup (W \setminus G) \cup (D \setminus W) \cup (P^{19} \setminus D), \text{ where:}
\]

(i) \( G = Gr(3, 6) \) is the Grassmannian 9-fold of 3-vectors in \( V = \mathbb{C}^6 \), or \( G = \{ \text{3-spaces } U : U \subset V \} \);
(ii) $D \subset \mathbb{P}^{19}$ is a quartic hypersurface which is the union of all the tangent lines to $G$, i.e. $D = \cup_{u \in G} \mathbb{P}_u^0$, where $\mathbb{P}_u^0$ is the tangent projective 9-space to $G$ at $u \in G$;

(iii) $x \in \mathbb{P}^{19} \setminus W \iff$ there exists a unique secant or tangent line $L_x$ to $G$ through $x$, and then $x \in D \setminus W$ iff $L_x$ is tangent to $G$;

(iv) $W$ is the singular locus of $D$, and $x \in W \iff$ there exist infinitely many secant lines to $G$ passing through $x$.

Proof. cf. [8], Ch. 3.

By restriction to $\mathbb{P}(V(14)) \subset \mathbb{P}(\wedge^3 V)$ we get the analogous result for the orbits of $\rho$.

**Corollary 2.4.2 (Theorem of Segre for $Sp(3)/U(3)$)**

The representation $\rho$ of $Sp(3, \mathbb{C})$ on $\mathbb{P}^{13} = \mathbb{P}(V(14))$ has four orbits:

$$
\mathbb{P}^{14} = \Sigma \cup (\Omega \setminus \Sigma) \cup (F \setminus \Omega) \cup (\mathbb{P}^{13} \setminus F), \quad \text{where:}
$$

(i) $\Sigma$ is the 6-fold Lagrangian Grassmannian

$$
Sp(3)/U(3) = \{ \text{Lagrangian 3-spaces } U : U \subset V = \mathbb{C}^6 \}
$$

(ii) $F \subset \mathbb{P}^{13}$ is a quartic hypersurface which is the union of all the tangent lines to $\Sigma$, i.e. $F = \cup_{u \in \Sigma} \mathbb{P}_u^0$, where $\mathbb{P}_u^0 = T_u \Sigma$ is the tangent projective 6-space at $u \in \Sigma$;

(iii) $w \in \mathbb{P}^{13} \setminus \Omega \iff$ there exists a unique secant or tangent line $L_w$ to $\Sigma$ through $w$, and then $w \in F \setminus \Omega$ iff $L_w$ is tangent to $\Sigma$;

(iv) $\Omega$ is the singular locus of $F$, and $w \in \Omega \iff$ there exist infinitely many tangent lines to $\Sigma$ passing through $w$.

**Remark 2.4.3** Decker, Manolache and Schreyer (cf [8] and 2.5.3 below) showed that the partial derivatives of $f$ define a cubo-cubic Cremona transformation on $\mathbb{P}(V(14))$, with base locus $\Omega$ and exceptional divisor $F$.

Furthermore

**Proposition 2.4.4** $\dim F = 12$, $\deg F = 4$, $K_F = \mathcal{O}_F(-10)$.

$\dim \Omega = 9$, $\deg \Omega = 21$. $F$ has quadratic singularities along $\Omega \setminus \Sigma$.

$\dim \Sigma = 6$, $\deg \Sigma = 16$, $K_\Sigma = \mathcal{O}_\Sigma(-4)$.

Proof. For $F$ it remains to check the statements on singularities. Let $f$ be the polynomial defining $F$. The singularities of $f$ along the subscheme defined by the partials are quadratic if and only if the subscheme is smooth. But the subscheme defined by the partials of $f$ is exactly $\Omega$, which is smooth outside $\Sigma$. To compute the invariants of $\Sigma$ we consider the universal exact sequence of vector bundles on $G = \text{Gr}(3, 6)$:

$$0 \to U \to V \otimes \mathcal{O}_G \to Q \to 0,$$

where $U$ is the universal subbundle. The 2-form $\alpha$ restricts naturally to $U$, i.e. is a section $\alpha_U$ of $(\wedge^2 U)^* \cong \wedge^2 U^*$. The variety $\Sigma$ is therefore nothing but the zero-locus $Z(\alpha_U)$ of this section, so the class $[\Sigma] = c_3(\wedge^2 U^*) \cap G = (c_1(U^*)c_2(U^*) - c_3(U^*)) \cap G$. From this description we immediately get that $\deg \Sigma = c_1^0(U^*) \cap \Sigma = 16$, and that the canonical divisor $K_\Sigma = K_G|_{\Sigma} + c_1(\wedge^2 U^*) \cap \Sigma = -4c_1(U^*) \cap \Sigma$. In particular $\Sigma$ is a Fano 6-fold of index 4.

Q.E.D.
We adopt Donagi’s notation and let the *pivots* of \( w \in \mathbb{P}^{13} \setminus \Omega \) be the intersection points \( \{a, b\} = L_w \cap \Sigma \) where \( L_w \) is the unique secant line through \( w \). Similarly if \( w \in F \setminus \Omega \) (= the case when \( a = b \)), call \( a \) the *pivot* of \( w \). When \( w \in \Omega \), a *pivot* of \( w \) is a point \( u \in \Sigma \) such that \( w \) lies on a tangent line through \( u \).

From the universal exact sequence it follows that the tangent bundle \( T_\Sigma \) is a subbundle of \( \text{Hom}(U, U^*) = U^* \otimes U^* \). In fact it is the subbundle consisting of symmetric tensors, i.e.

\[
T_\Sigma = \text{Sym}^2 U^*.
\]

Using the coordinates \([u : X : Y : z]\) above in the point \( u = [1,0,0,0] \) on \( \Sigma \), the tangent space \( T_u \Sigma \) at \( u \) to \( \Sigma \) is the span of \( \{[U_X] | \text{rk} X \leq 1\} \), i.e.

\[
T_u \Sigma = \mathbb{P}^6_u = \langle \{[u : X : 0 : 0] : \text{rk} X = 1\} \rangle
\]

The conic in \( \mathbb{P}(U_0^*) \) defined by the symmetric matrix \( X \) is of course the conic section \( q_w(U_0^*) \) defined in 2.3.

The following is the \( Sp(3) \)-analog of Lemma 3.4 in [9]:

**Proposition 2.4.5** Let \( u \in \Sigma \), and let \( \mathbb{P}^6_u \) be, as above, the tangent projective space to \( \Sigma \subset \mathbb{P}^{13} \) at \( u \). If \( w \in \mathbb{P}^6_u \), then \( q_w \) has rank 0, 1, 2 or 3, when \( w = u \), \( w \in \Sigma \setminus u \), \( w \in \Omega \setminus \Sigma \) or \( w \in \mathbb{P}^6_u \setminus \Omega \) respectively.

(i) \( C_u := \Sigma \cap \mathbb{P}^6_u \) is a cone over the Veronese surface with a vertex \( u \);

(ii) \( \Omega \cap \mathbb{P}^6_u \) is a cubic hypersurface.

**Proof.** The tangent cone \( C_u = \Sigma \cap \mathbb{P}^6_u = \{[U_X] | \text{rk} X \leq 1\} \). But \( \text{rk} X, \text{rk} X = 1 \) are the equations of a Veronese surface, hence \( C_u = \Sigma \cap \mathbb{P}^6_u \) is a cone over a Veronese surface centered at \( u \). The secant lines to this Veronese surface fill the cubic hypersurface in \( \mathbb{P}^6_u \) defined by \( \text{det} X = 0 \), and every point on this hypersurface lies on infinitely many secant lines, so \( \Omega \cap \mathbb{P}^6_u \) must coincide with this cubic hypersurface. Q.E.D.

### 2.5 Special cycles

We describe some special cycles on \( \Sigma \) and on \( \tilde{\Sigma} \). Again we restrict cycles on \( G = \text{Gr}(3, V) \) to \( \Sigma \).

The restriction of the universal exact sequence on \( G \) to \( \Sigma \) becomes

\[
0 \to U \to V \otimes \mathcal{O}_\Sigma \to Q \to 0.
\]

The correlation \( L_\alpha : V \to V^* \) \( (v \mapsto \alpha(v,-)) \) sets up a natural isomorphism: \( Q \cong U^* \), so the universal sequence becomes:

\[
0 \to U \to V \otimes \mathcal{O}_\Sigma \to U^* \to 0.
\]

If \( \tau_i = c_i(U^*) \), then

\[
H^*(\Sigma) \cong \mathbb{Z}[\tau_1, \tau_2, \tau_3]/(\tau_1^2 - 2\tau_2, \tau_2^2 - 2\tau_1\tau_3, \tau_3^2).
\]

Thus the Betti numbers are:

\[
1, 1, 1, 2, 1, 1, 1.
\]
The classes $\tau_i \cap \Sigma$ are represented by cycles that are restriction of special Schubert cycles on $G$ to $\Sigma$. In the notation $\tau_{ijk} = \sigma_{ijk} \cap \Sigma$, we get that $\tau_{i00} = \tau_i \cap \Sigma$.

In projective notation we describe geometrically the cycle of Lagrangian planes that contain a given point, intersect a given line or a given Lagrangian plane. These belong to the classes $\tau_{300}$, $\tau_{200}$ and $\tau_{100}$, respectively.

As before, the basic tool will be the correlation $L_\alpha$. It is a basic simple fact that a line in a Grassmannian $Gr(k,n)$ corresponds to the pencil of $k$-spaces through a $(k - 1)$-space inside a $(k + 1)$-space. Thus a line in $\Sigma$ always correspond to the pencil of 3-spaces with a common 2-subspace in some 4-space. From this it follows easily that a plane in $\Sigma$ would correspond either to the net of 3-spaces with a common 1-dimensional subspace in some 4-space or to the net of 3-spaces with a common 2-subspace in a 5-space.

**Lemma 2.5.1** $\Sigma$ contains no planes.

**Proof.** Consider the restriction of $\alpha$ to a 4-space $W$. The rank of a skew-symmetric matrix is always even, so the rank of $\alpha_W$ is 0, 2, or 4. Let $W_\alpha = \cap_{v \in W} \ker L_\alpha(v)$. Since $L_\alpha$ is nonsingular, $W_\alpha$ is 2-dimensional. Therefore $\dim W \cap W_\alpha = 4 - \rank \alpha_W$ is 0, 2, or 4. Let $W_\alpha = \cap_{v \in W} \ker L_\alpha(v)$ be a Lagrangian 3-subspace of $W$, and let $w \in W \setminus U$. Then $\ker L_\alpha(v) \cap U = W_\alpha$ so in particular $W_\alpha$ is contained in $U$. Thus $\dim W \cap W_\alpha = 0$ means that there are no Lagrangian 3-spaces in $W$, while in case $\dim W \cap W_\alpha = 2$ there is precisely a pencil.

Similarly, consider the restriction of $\alpha$ to a 5-space $W$. The subspace $W_\alpha = \cap_{v \in W} \ker L_\alpha(v)$ is 1-dimensional. But if it contains a net of Lagrangian 3-space with a common 2-subspace, then these Lagrangian subspaces fill all of $W$, and hence $W_\alpha$ has dimension at least 2, a contradiction. Q.E.D.

For $v \in V$ let $V_v = \ker L_\alpha(v)$.

**Lemma 2.5.2** For every point $p = <v> \in \mathbb{P}(V)$ the variety $Q_p$ of Lagrangian planes that contain $p$ is a 3-dimensional smooth quadric in $\Sigma$. It is isomorphic to the Lagrangian Grassmannian $L\mathbf{G}(2,4)$ of Lagrangian subspaces of $V_v/ <v>$ with respect to the restriction of $\alpha$ to $V_v/ <v>$. 

**Proof.** The cycle representing $Q_p$ on $\Sigma$ is $\tau_3$ and has degree 2. Any Lagrangian 3-space that contains $v$ is itself contained in the 5-space $V_v$. The restriction of $\alpha$ to $V_v$ has kernel $v$, so we may identify $Q_p$ with the Lagrangian Grassmannian with respect to the nondegenerate 2-form $\alpha_v$ induced by $\alpha$ on $V_v/ <v>$. This is nothing but a smooth hyperplane section of a $\mathbf{Gr}(2,4)$, i.e. a smooth quadric 3-fold. Q.E.D.

The span of the quadric $Q_p$ is a $\mathbb{P}^4 \subset \mathbb{P}(V(14))$ which we denote by $\mathbb{P}^4_p$. Consider the incidence variety

$$I_Q = \{([P(U)], p) | p \in \mathbb{P}(U)\} \subset \Sigma \times \mathbb{P}(V)$$

(2)

By 2.5.2 this is a quadric bundle over $\mathbb{P}(V)$. Now $\Sigma$ spans $\mathbb{P}(V(14))$, and each $\mathbb{P}^4_p$ is contained in this span so we may also consider the incidence

$$I_P = \{(q, p) | q \in \mathbb{P}^4_p\} \subset \mathbb{P}(V(14)) \times \mathbb{P}(V).$$

(3)
This is a $\mathbb{P}^4$-bundle over $\mathbb{P}(V)$, which has been studied by Decker, Manolache and Schreyer. Its associated rank 5-bundle is selfdual, so we describe a construction which is dual to their: Consider the third exterior power of the Euler sequence on $\mathbb{P}(V)$ twisted by $\mathcal{O}_{\mathbb{P}(V)}(2)$

$$0 \rightarrow \wedge^2 T_{\mathbb{P}(V)}(-1) \rightarrow \wedge^3 V \otimes \mathcal{O}_{\mathbb{P}^5}(2) \rightarrow \wedge^3 T_{\mathbb{P}(V)}(-1) \rightarrow 0$$

and restrict the contraction $\alpha : \wedge^3 V \rightarrow V$ defined by

$$u \wedge v \wedge w \mapsto \alpha(u \wedge v)w + \alpha(v \wedge w)u + \alpha(w \wedge u)v$$

over $\mathbb{P}(V)$ to the subbundle $\wedge^2 T_{\mathbb{P}(V)}(-1)$. This restriction, denote it by $\alpha_U$, is nothing but the restriction of $\alpha$ over each point $p = < v >$ to 3–dimensional subspaces in $V$ which contain $v$. On the other hand $L_\alpha$ defines a map $L_{\alpha_U} : V \otimes \mathcal{O}_{\mathbb{P}(V)}(2) \rightarrow \mathcal{O}_{\mathbb{P}(V)}(3)$ by $L_{\alpha_U}(u)(v) = \alpha(u \wedge v)$. The composition

$$L_{\alpha_U} \circ \alpha_U : \wedge^2 T_{\mathbb{P}(V)}(-1) \rightarrow V \otimes \mathcal{O}_{\mathbb{P}(V)}(2) \rightarrow \mathcal{O}_{\mathbb{P}(V)}(3)$$

is zero since

$$\alpha(u \wedge v)\alpha(w \wedge u) + \alpha(v \wedge w)\alpha(u \wedge u) + \alpha(w \wedge u)\alpha(v \wedge u) = 0.$$ 

The kernel $\ker L_{\alpha_U}$ is the rank 5 bundle $\Omega_{\mathbb{P}(V)}(3)$, so $\alpha_U$ defines a bundle map

$$\alpha_U : \wedge^2 T_{\mathbb{P}(V)}(-1) \rightarrow \Omega_{\mathbb{P}(V)}(3).$$

It is easy to check that this map is surjective as soon as $\alpha$ is nondegenerate. Denote by $E$ the rank 5 kernel bundle $\ker \alpha_U$. If $U$ is a Lagrangian 3-space that contains $v$, then $\wedge^3 U$ clearly is contained in $E_p$ over the point $p = < v >$. Thus $\mathbb{P}(E_p)$ coincides with the fiber of the incidence $I_P$ over $p$.

**Proposition 2.5.3 (8 Propositions (1.2) and (1.3))** Let $E$ be the rank 5 kernel bundle of the natural surjective map

$$\alpha_U : \wedge^2 T_{\mathbb{P}(V)}(-1) \rightarrow \Omega_{\mathbb{P}(V)}(3),$$

as above.

Then $E$ has Chern polynomial is $c_t(E) = 1 + 5t + 12t^2 + 16t^3 + 8t^4$, and $H^0(\mathbb{P}(V), E) \cong V(14)^*$. Furthermore $E$ is the rank 5 bundle associated to the $\mathbb{P}^4$-bundle $I_P$ over $\mathbb{P}(V)$, and the projection of $I_P$ into the first factor $\mathbb{P}(V(14))$ is $\Omega$.

**Proof.** It remains to compute the Chern polynomial, but this is straightforward from the construction. The twisted bundle $E(-1)$ coincides with the dual of the bundle $B$ defined in [8]. They show that $B$ is selfdual, so the invariants of the bundle also follows from their results. Q.E.D.
The lines \( l \subset \mathbb{P}^5 \) fall into two cases w.r.t. \( \alpha \). The cycle \( \tau_{330} \) representing planes containing a line \( l \) is empty for the general line, while it is a line for each line \( l \) in a Lagrangian plane. The latter lines are of course precisely the isotropic lines.

**Lemma 2.5.4** If \( l \) is a non-isotropic line, then the variety \( \Sigma_l \) of Lagrangian planes that intersect \( l \) is a smooth 4-dimensional variety of degree 8 which is a quadric bundle inside a rational normal scroll of degree 5, the Segre embedding of \( \mathbb{P}^1 \times \mathbb{P}^4 \) in a \( \mathbb{P}^9 \). If \( l \) is an isotropic line the variety \( \Sigma_l \) is a singular 4-dimensional variety of degree 8 which is a quadric bundle inside a rational normal scroll of degree 5 with vertex a line that spans a \( \mathbb{P}^9 \).

**Proof.** The variety \( \Sigma_l \) is represented by the cycle \( \tau_{200} \), the restriction \( \sigma_{200} \cap \Sigma \). Consider the subvariety \( T_l \subset \Sigma_l \) parameterizing Lagrangian planes that contain \( l \). The subvariety \( T_l \) is represented by the class \( \tau_{330} = \tau_3^2 \) which is 0 in \( H^*(\Sigma) \). So there are two cases:

- the case when \( T_l \) is positive dimensional (and nonempty), and
- the case when \( T_l \) is empty.

In the above coordinates the first case occurs when \( x_{11}, x_{12}, y_{33}, y_{23} \) vanish. The equations then reduces to the \( 2 \times 2 \)-minors of the matrix

\[
\begin{pmatrix}
  u & x_{13} & x_{23} & -x_{22} & y_{13} \\
  x_{13} & -y_{22} & y_{12} & y_{13} & z
\end{pmatrix}
\]

which define a rational normal 5-fold scroll with vertex a line \( L(l) \), and

\[
x_{22}x_{33} - x_{23}^2 - uy_{11}, \quad x_{13}x_{33} - y_{11}y_{22} + y_{12}^2, \quad x_{13}y_{11} + x_{23}y_{12} + x_{33}y_{13}.
\]

These quadrics also vanish on \( L(l) \). This is a 4-dimensional variety of degree 8 in a \( \mathbb{P}^9 \). The line \( L(l) \) clearly parameterizes the Lagrangian planes that contain \( l \).

The second case occurs when the coordinates \( x_{12}, x_{13}, y_{12}, y_{13} \) vanish. The equations then reduces to the \( 2 \times 2 \) minors of the matrix

\[
\begin{pmatrix}
  u & x_{33} & x_{22} & x_{23} & y_{11} \\
  x_{11} & y_{22} & y_{33} & -y_{23} & z
\end{pmatrix}
\]

which define a smooth rational normal 5-fold scroll, the Segre embedding of \( \mathbb{P}^1 \times \mathbb{P}^4 \), and

\[
x_{22}x_{33} - x_{23}^2 - uy_{11}, \quad x_{11} - y_{22}y_{33} + y_{23}^2, \quad x_{11}y_{11} - x_{22}y_{22} - x_{23}y_{23}.
\]

This is a 4-dimensional variety of degree 8 in a \( \mathbb{P}^9 \). A bundle over \( \mathbb{P}^1 \) of smooth 3-dimensional quadrics. Q.E.D.

**Corollary 2.5.5** The restriction \( E_l \) of the vector bundle \( E \) to a line \( l \) is \( E_l = 5 \mathcal{O}_l(1) \) when \( l \) is non-Lagrangian; and \( E_l = 2 \mathcal{O}_l \oplus \mathcal{O}_l(1) \oplus 2 \mathcal{O}_l(2) \) when \( l \) is Lagrangian.

**Proof.** The restriction is determined by the \( 2 \times 5 \) matrices above. Q.E.D.

We turn to planes. The isomorphism \( L_\alpha : \wedge^3 V \cong \wedge^3 V^* \) induces an isomorphism \( \text{Gr}(3,V) \to \text{Gr}(V,3) \), which composed with the natural isomorphism \( \text{Gr}(V,3) \to \text{Gr}(3,V) \), \( \omega \mapsto \text{ker} \omega \), defines an involution

\[
i_\alpha : \text{Gr}(3,V) \to \text{Gr}(3,V).
\]
The fixed point locus of this involution is of course $\Sigma$, the set of Lagrangian planes. On the other hand every involutive pair $\{P, \iota_\alpha(P)\}$ of planes have a common point $P \cap \iota_\alpha(P) = \ker(\alpha|_P)$ and is of course contained in the corresponding correlated hyperplane.

Let $P \subset \mathbb{P}(V)$ be any plane. Then the restriction $\tau_1(P)$ of the Schubert cycle $\sigma_1(P)$ in $\text{Gr}(3, V)$ is clearly a hyperplane section of $\Sigma$. Thus we have shown

**Proposition 2.5.6** If $P \subset \mathbb{P}(V)$ is a non-Lagrangian plane, then the variety $\Sigma_P$ of Lagrangian planes that intersect $P$ coincides with the variety $\Sigma_{\iota_\alpha(P)}$ of Lagrangian planes that intersect $\iota_\alpha(P)$ and they define a hyperplane section of $\Sigma$.

**Proposition 2.5.7** For a Lagrangian plane $P(U) \subset \mathbb{P}(V)$ corresponding to $u \in \Sigma$ the variety $\Sigma_u$ of Lagrangian planes that intersect $P(U)$ is the hyperplane section defined by the hyperplane $H_{L_{\alpha}(u)}$, while the variety of Lagrangian planes that intersect $P(U)$ in a cone $C_u$ over a Veronese surface of degree 4 with vertex at $u$ in a $\mathbb{P}^6 = \mathbb{P}_u^6$, the projective tangent space to $\Sigma$ at $u$.

**Proof.** $\Sigma_u$ is a hyperplane section with $P(U)$ invariant under the involution $\iota_\alpha$. The lines in $\Sigma$ through $u$ represent precisely the pencils of Lagrangian planes that contain $P(U)$. On the other hand these lines generate precisely the tangent cone $C_u$ to $\Sigma$ at $u$, so the lemma follows from 2.4.3. Q.E.D.

Recall the decomposition $V = U_0 \oplus U_1$ from 2.3. The restriction of the global sections $V(14)^*$ of $E$ to the Lagrangian plane $P = P(U_0)$ decomposes as the restriction of the decomposition

$$\wedge^3 V^* = \wedge^3 U_1^\perp \oplus (\wedge^2 U_1^\perp \otimes U_0^\perp) \oplus (U_1^\perp \otimes \wedge^2 U_0^\perp) \oplus \wedge^3 U_0^\perp$$

to $V(14)^*$. By the natural map $q(U_0)$ this decomposition becomes

$$H^0(P, E|_P) \cong \wedge^3 U_1^\perp \oplus U^\prime \oplus \text{Sym}^2 U_0^\perp,$$

where the first summand consists of the constant forms, the second is the restriction of the summand $(\wedge^2 U_1^\perp \otimes U_0^\perp)$, while the last summand is the quadratic forms on $P$. The restriction of the vector bundle $E$ to $P$ therefore decomposes as a sum of two line bundles and a rank 3 bundle. Thus

**Proposition 2.5.8** For a Lagrangian plane $P = P(U) \subset \mathbb{P}(V)$ the restriction of the vector bundle $E$ to $P$ is $E|_P = \mathcal{O}_P \oplus \mathcal{O}_P(2) \oplus E_P$, where $E_P$ is a rank 3 vector bundle with Chern polynomial $c_t(E_P) = 1 + 3t + 6t^2$.

**Proof.** The Chern polynomial follows by a direct calculation. Q.E.D.

### 2.6 Singular hyperplane sections

In this section we describe the hyperplane sections of $\Sigma$ corresponding to the different $Sp(3, \mathbb{C})$-orbits in $\mathbb{P}(V(14)^*)$. In particular we describe their singular locus. Four maps are crucial. First there is the basic correlation

$$L : \Sigma \to \overline{\Sigma}, \quad P(U) \mapsto P(L_{\alpha}(U)).$$

The second map is the involution

$$\iota_\alpha : \text{Gr}(3, V) \to \text{Gr}(3, V).$$
The third map, which we call the vertex map, is
\[ v : \Omega \setminus \Sigma \to P(V) \quad q \mapsto \pi_2 \pi_1^{-1}(q), \]
where \( \pi_1 \) and \( \pi_2 \) are the projections from the incidence \( I_P \). The point \( v(w) \in P(V) \), is
called the vertex of \( w \in \Omega \). The third map is the pivot map:
\[ piv : \hat{F} \setminus \hat{\Omega} \to \Sigma \quad p \mapsto \text{the (unique) pivot of } p. \]
The corresponding maps on the dual space are marked with a \(*\). Notice that we have the
following relations:
\[ L^{-1} \circ piv^* = piv \circ L^{-1} : \hat{F} \setminus \hat{\Omega} \to \Sigma \]
and
\[ L^{-1} \circ v^* = v \circ L^{-1} : \hat{\Omega} \setminus \Sigma \to P(V). \]
The first of these will be denoted by \( u \):
\[ u = L^{-1} \circ piv^* : \hat{F} \setminus \hat{\Omega} \to \Sigma \]
while, by abuse of notation, the second one will be denoted by \( v \):
\[ v = L^{-1} \circ v^* : \hat{\Omega} \setminus \Sigma \to P(V). \]
For the point \( \omega \in P(V(14)^* \setminus \hat{F} \), denote by \( P_{\omega}^{12} \subset P(V(14)) \) the hyperplane defined by \( \omega \), and
denote by \( H_\omega = P_{\omega}^{12} \cap \Sigma \) the corresponding hyperplane section of \( \Sigma \).

**Lemma 2.6.1** The hyperplane section \( H_\omega = P_{\omega}^{12} \cap \Sigma \) is singular precisely in the points \( u \in \Sigma \) such that the tangent space \( P_{\omega}^6 \) to \( \Sigma \) contains \( \omega \).

**Proof.** The hyperplane section \( H_\omega \) is singular at \( u \) if and only if it contains the tangent
space \( P_{u}^6 \), which is equivalent by \( L_\alpha \) to \( \omega \in P_{\omega}^6 \). Q.E.D.

**Proposition 2.6.2** (i) When \( \omega \in P(V(14)^*) \setminus \hat{F} \), then \( H_\omega = P_{\omega}^{12} \cap \Sigma \) is smooth.

(ii) When \( \omega \in \hat{F} \setminus \hat{\Omega} \), then \( H_\omega \) is singular precisely in the point \( u(\omega) \).

(iii) When \( \omega \in \hat{\Omega} \setminus \Sigma \), then \( H_\omega = \Sigma_{P_1} = \Sigma_{P_2} \) for an involutive pair of planes \( P_1(\omega) \) and \( P_2(\omega) = \iota_\alpha(P_1(\omega)) \). Furthermore \( H_\omega \) is singular along a smooth quadric surface \( Q_\omega \) in \( P_{v(\omega)} \) which parameterizes the set of Lagrangian planes passing through \( v(\omega) \) and intersecting \( P_1 \) and \( P_2 \) in a line.

(iv) When \( \omega \in \hat{\Sigma} \), then \( H_\omega \) is singular along \( C_u \) the cone over a Veronese surface with
vertex at \( u = L^{-1}(\omega) \).

**Proof.** We check one point \( \omega \) in each orbit and use 2.6.1 to find the singular points of \( H_\omega \)
as the points \( u \in \Sigma \) such that the tangent spaces \( P_{\omega}^6 \) contain \( \omega \). Clearly \( H_\omega \) is smooth iff \( w \in P_{\omega}^{13} \setminus \hat{F} \), while \( H_\omega \) is singular precisely at \( u(\omega) \). On the other hand 2.4.3 implies that \( H_\omega \) is singular along the cone \( C_u \) with vertex at \( u = L^{-1}(\omega) \) when \( \omega \in \hat{\Sigma} \).

It remains to check a point \( \omega \in \hat{\Omega} \setminus \hat{\Sigma} \) and we start with a point \( p \) such that \( p = v(\omega) \). In the usual coordinates we consider the point \( p = < e_1 > \in P(V) \). Any Lagrangian plane through \( p \) lies in the hyperplane \( x_4 = 0 \). We construct the singular locus of a hyperplane
section \( H_\omega \) with \( \omega \in P_{\alpha}(p) \setminus Q_{\alpha}(p) \). Consider the two planes
\[ P_1 = P(< e_1, e_2, e_5 >) \quad \text{and} \quad P_2 = \iota_\alpha(P_1) = P(< e_1, e_3, e_6 >) \]
passing through \( p \) and spanning this hyperplane. For any \((t) = (t_2 : t_5)\) and for any \((s) = (s_3 : s_6)\) the plane \( \mathbf{P}^2(s; t) = \mathbf{P}( < e_1, t_2 e_2 + t_5 e_5, s_3 e_3 + s_6 e_6 > ) \) is Lagrangian and intersect both \( P_1 \) and \( P_2 \) in a line. So the corresponding point

\[
u(s; t) = < e_1 \wedge (t_2 e_2 + t_5 e_5) \wedge (s_3 e_3 + s_6 e_6) > \in \Sigma.
\]

Consider \( P_u(s; t) \), the projective tangent space to \( \Sigma \) at \((s; t) \in \mathbf{P}^1 \times \mathbf{P}^1\). For a 3-vector \( u = x \wedge y \wedge z \in \Sigma \) the projective tangent space \( P_u \) is spanned by

\[
\{ w \in \wedge^2 U \wedge V | \alpha(w) = 0 \}
\]

where \( U = < x, y, z > \). Applied to the tangent spaces \( \mathbf{P}^6(s; t) \), we find that the point \( w = e_{125} + e_{163} \) is a common point. For this point only the coordinate \( x_{23} \) is nonzero, so by the equations, it does not lie on \( \Sigma \). On the other hand all the \((u; t)\) are “pivots” of the point \( w \), i.e. for any \((u; t)\) the line \( < u; t, w > \) is tangent to \( \Sigma \). Thus the point \( w \) will belong to \( \Omega \setminus \Sigma \). In fact \( w \in \mathbf{P}_p^4 \) and every pivot of \( w \) is a Lagrangian plane through \( p \).

Since \( Q_p = \Sigma \cap \mathbf{P}_p^4 \) is a quadric 3-fold, the set of pivots of \( w \) is the intersection of \( Q_p \) and the polar of \( w \) w.r.t. \( Q_p \), i.e. a quadric surface. Therefore \( S_w = \{ u(s; t) : (s; t) \in \mathbf{P}^1 \times \mathbf{P}^1 \} \) is precisely the set of pivots of \( w \).

Let \( \omega = L(w) = x_{452} + x_{436} \in \Omega \setminus \Sigma \), then by \([2.6.1]\) the hyperplane section \( H_\omega \) is singular precisely along \( S_{L^{-1}(\omega)} = S_w \). Finally, any Lagrangian subspace \( U' \subset V \) that intersects \( < e_1, e_2, e_5 > \), intersects also \( < e_1, e_3, e_6 > \), so it is of the form

\[
< a e_1 + b e_2 + c e_5, a' e_1 + b' e_3 + e'_6, v >
\]

so \( \omega(\wedge^3 U') = 0 \), i.e. the corresponding point \( u' \in \Sigma_{P_3} \). Q.E.D.

We give a more precise description of the type of singularities of hypersections.

**Proposition 2.6.3** (i) If \( \omega \in \tilde{\Omega} \setminus \tilde{\Sigma} \) then the hyperplane section \( H_\omega \subset \Sigma \) has a quadratic singularity at the pivot \( u = u(\omega) = L(\text{pivot}(\omega)) \in \Sigma \).

(ii) If \( \omega \in \tilde{\Omega} \setminus \tilde{\Sigma} \) then the hyperplanes section \( H_\omega \subset \Sigma \) has a quadratic singularity along the quadric surface \( Q_\omega \subset \Sigma \).

(iii) Let \( \omega \in \tilde{\Sigma} \) and let \( w = L^{-1}(\omega) \in \Sigma \). Then the projective tangent cone is a cone \( C_w \) over a Veronese surface with vertex \( w \). The hyperplane section \( H_\omega \subset \Sigma \) has a double singularity of rank 3 along the punctured cone \( C_w \setminus \{ w \} \).

**Proof.** We choose one representative for each orbit. Let \( u = [1, 0, 0, 0] \). By \([2.3]\) symmetric 3×3-matrices \( X \) and \( Y \) such that \([1, X, Y, z] \in \mathbf{C}^{13} \subset \mathbf{P}(V(14)) = < \Sigma > \) form local coordinates at the point \( u = [1, 0, 0, 0] \) in which a neighborhood \( \Sigma^o \subset \Sigma \) has a local parameterization \([1, X, \wedge^2 X, \det(X)]\). The tangent space \( T\Sigma|_u \) is parameterized by the linear matrix \( X = (x_{ij}) \), and the projection \([1, X, Y, z] \rightarrow X \) sends the neighborhood \( \Sigma^o \) to \( \mathbf{C}^6(x_{ij}) \iso T\Sigma|_u \).

(i) We choose \( \omega \in T\Sigma|_u \) such that \( H_\omega = \{ y_{11} + y_{22} + y_{33} = 0 \} \). Clearly this hyperplane contains the projective tangent space \( \mathbf{P}^6_u = \mathbf{P}(T\Sigma|_u) \) to \( \Sigma \) at \( u \), in particular \( u \in H_\omega \). In matrix coordinates \( \omega \) is represented by the matrix

\[
Y = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.
\]
Since \( \text{rank}(Y) = 3 \), form \( \omega \in \mathcal{F} \setminus \mathcal{S} \). Since on \( \Sigma \) the parameters \( y_{ij} \) are in fact the minors of the matrix \( X \), the isomorphic projection of the neighborhood \( H_\omega = \Sigma \cap P^1_\omega \subseteq H_\omega \) of \( u \) on \( T\Sigma|_u = \mathbb{C}^6(x_{ij}) \) is a hypersurface with a local equation \( y_{11} + y_{22} + y_{33} = Q(X) = x_{22}x_{33} - x_{23}^2 + x_{11}x_{33} - x_{12}^2 = 0 \). Therefore \( \text{mult}_u H_\omega = \deg Q(X) = 2 \). Moreover \( u \) is a quadratic singularity on \( H_\omega \) since the quadric \( \{Q(X) = 0\} \subseteq P(T\Sigma|_u) \) has maximal rank \( = 6 = \dim H_\omega + 1 \).

(ii) Next, we choose \( \omega \in \mathcal{S} \setminus \mathcal{S} \) such that \( H_\omega = (y_{11} + y_{22} = 0) \). Thus \( u \in Q_\omega \), the singular locus of \( H_\omega \). Now, the equation of the projectivized tangent cone to \( H_\omega \) at \( u \) becomes:

\[
y_{11} + y_{22} = Q(X) = x_{22}x_{33} - x_{23}^2 + x_{11}x_{33} - x_{12}^2 = 0.
\]

Therefore \( \text{mult}_u H_\omega = \deg Q(X) = 2 \), but in this case \( \text{rank}(Q) = 4 < 6 \). On the other hand \( H_\omega \) is singular along the surface \( Q_\omega \), so the singularity has maximal rank, in particular it is quadratic along \( Q_\omega \).

(iii) Let \( \omega \in \mathcal{S} \) such that \( H_\omega = (y_{11} = 0) \), and let \( w = L^{-1}(\omega) \). The tangent cone at \( w \) of \( H_\omega \) is a cone \( C_w \) over a Veronese surface with vertex \( w \). The point \( u = [1, 0, 0, 0] \) lies on this cone and is different from \( w \). The local equation of \( H_\omega \) at \( u \) is:

\[
y_{11} = Q(X) = x_{22}x_{33} - x_{23}^2 = 0.
\]

Since \( \text{rank}(Q) = 3 \) the hyperplane section \( H_\omega \) has a quadratic singularity along the 3-dimensional punctured cone \( C_w^* = C_w \setminus \{w\} \).

Q.E.D.

Now, we turn to the involutive pair of planes \( P_1(\omega), P_2(\omega) \) that appear in \( \ref{sec:lagrangian} \). It follows from \( \ref{sec:lagrangian} \) (iii) that the union of the planes \( P_1(\omega) \cup P_2(\omega) \) is the set of points \( q \) in \( P(V) \), such that \( P_q^4 \subseteq P^\omega \) (or, equivalently, \( Q_q \subseteq H_\omega \)). This fact fits in the description of the incidence

\[
J = \{(p, w) \in P(V) \times P(V(14)^*) | P_p^4 \subseteq P^\omega \} \subseteq P(V) \times P(V(14)^*)
\]  

Let \( \pi : J \to P^5 = P(V) \) and \( \psi : J \to P^{13} = P(V(14)^*) \) be the two projections of \( J \).

**Proposition 2.6.4** The image of the second projection is precisely the quartic \( \mathcal{F} \subseteq \mathcal{P}^{13} \).

The fiber of the second projection over a point \( \omega \in \mathcal{F} \setminus \mathcal{S} \) is the smooth conic section \( q_\omega(U) \subseteq P(U) \), where \( P(U) \) is the plane of the pivot \( u(\omega) \). The fiber of \( \psi \) over a point \( \omega \) on \( \mathcal{S} \setminus \mathcal{S} \) is the union of the two planes \( P_1(\omega) \cup P_2(\omega) \). The fiber of \( \psi \) over a point \( \omega \in \mathcal{S} \) is the Lagrangian plane \( P(U) \) of \( u = L^{-1}(\omega) \).

**Proof.** The last statement follows from \( \ref{sec:lagrangian} \) (vii), while the case \( \omega \in \mathcal{S} \setminus \mathcal{S} \) follows as explained above from \( \ref{sec:lagrangian} \) (iii). The remainder of the proposition we reformulate:

**Proposition 2.6.5** Let \( \omega \in P(V(14)^*) \), and let \( p \in P(V) \). Then \( P_p^4 \subseteq P^\omega \Leftrightarrow \omega \in T^1_{[L]} \) for some Lagrangian subspace \( U \) with \( p \in P(U) \) and \( q_\omega(U)(p) = 0 \).

**Proof.** By abuse of notation we do not distinguish between \( \omega \in P(V(14)^*) \) and any nonzero vector in \( V(14)^* \) representing it. Thus we consider \( \omega \) as a section of the vector bundle \( E \), and analyse its restriction to Lagrangian planes.

Let \( P = P(U) \subseteq P(V) \) be a Lagrangian plane and \( u \in \mathcal{S} \) the corresponding point. According to \( \ref{sec:lagrangian} \) the restriction of the vector bundle \( E \) to \( P \) decomposes into three direct summands:

\[
E|_P = O_P \oplus O_P(2) \oplus E_P
\]
where \( E_P \) is a rank 3 vector bundle with Chern polynomial \( c_t(E_P) = 1 + 3t + 6t^2 \). Therefore the restriction \( \omega_P \) of \( \omega \) to \( P \) decomposes into \( \omega_P = a \oplus b \oplus c \) where \( a \) is a constant, while \( b \) is a quadratic form and \( c \) is a section of the rank 3 bundle \( E_P \).

**Lemma 2.6.6** (i) If \( \omega \in \tilde{\Sigma} \), such that the dual pivot \( u(\omega) = u \), then \( a(\omega) = b(\omega) = c(\omega) = 0 \).

(ii) \( a(\omega) = c(\omega) = 0 \) if and only if \( \omega \in \tilde{F} \) and the dual pivot \( u(\omega) = u \).

(iii) \( u \in \mathbb{P}^1_{\omega} \) if and only if \( a = 0 \).

**Proof.** Let \( u \in \Sigma \) with corresponding Lagrangian plane \( P = \mathbb{P}(U) \). The hyperplane \( \mathbb{P}^1_{\omega} \) contains \( \Sigma_u \) (cf. [2.5.7]) if and only if \( a = b = c = 0 \), so (i) follows. The hyperplane \( \mathbb{P}^1_{\omega} \) contains the tangent cone \( C_u \) at \( u \) (the cone over a Veronese surface) if and only if the restriction \( \omega_P \in \mathcal{S} \text{ym}^2(U)^* \), i.e. \( a(\omega_P) = c(\omega_P) = 0 \), so (ii) follows. Finally \( a = 0 \) if and only if the \( \mathbb{P}^1_{\omega} \) passes through the vertex \( u \) of \( \Sigma_u \). Q.E.D.

If \( \omega \in \tilde{F} \) and the dual pivot \( u(\omega) = u \), then the quadratic form \( b(\omega_P) \) is nothing but the quadratic form \( q(\omega(U)) \). Let \( p \in P = \mathbb{P}(U) \). Then \( \omega(p) = 0 \) if and only if \( \omega \in \tilde{F} \) with dual pivot \( u(\omega) = u \) and \( q(\omega(U))(p) = 0 \) by (2.6.6) (ii). On the other hand \( \omega(p) = 0 \) if and only if \( \mathbb{P}^1_{P} \subset \mathbb{P}^1_{\omega} \) so the proposition follows. Q.E.D.

### 2.7 Linear sections, \( Sp(3) \)-dual sections and vertex varieties

In this section we explore the incidence correspondences \( I_P, I_Q \) and \( J \) of the previous two sections to study the relations between linear sections of \( \Sigma \), the dual linear sections of \( \tilde{F} \) and certain vertex varieties.

For \( 2 \leq k \leq 5 \), let \( \mathbb{P}^{13-k} \subset \mathbb{P}(V(14)) \) be a general linear subspace of codimension \( k \), and let \( \Pi^{k-1} = (\mathbb{P}^{13-k})^\perp \subset \mathbb{P}(V(14)^*) \) be the orthogonal subspace of hyperplanes which pass through \( \mathbb{P}^{13-k} \). Let

\[
X^{6-k} = \Sigma \cap \mathbb{P}^{13-k}, \quad \tilde{F}(X^{6-k}) = \Pi^{k-1} \cap \tilde{F},
\]

and let

\[
\tilde{\Omega}(X^{6-k}) = \tilde{F}(X^{6-k}) \cap \tilde{\Omega} = \Pi^{k-1} \cap \tilde{\Omega}.
\]

Thus the superscript always indicates the dimension of the linear section \( X \). We call \( \tilde{F}(X^{6-k}) \) the \( Sp(3) \)-dual section to \( X^{6-k} \).

We restrict our attention to general linear subspaces, more precisely we will assume that \( X^{6-k} \) is smooth. For singular sections \( X^{6-k} \), there are obviously similar results.

**Lemma 2.7.1** Let \( P \subset \mathbb{P}(V(14)) \) be a linear subspace and let \( \omega \in P^\perp \cap \tilde{F} \) such that a dual pivot \( u \in \Sigma \) of \( \omega \) lies in \( P \). Then \( u \in \operatorname{Sing} P \cap \Sigma \).

**Proof.** Let \( H_\omega = \Sigma \cap \mathbb{P}^1_{\omega} \) be the hyperplane section of \( \Sigma \) defined by \( \omega \). By 2.6.1, \( H_\omega \) is singular at the pivot \( u \). Since \( \omega \in P^\perp \), the variety \( P \cap \Sigma \subset H_\omega \) is a complete intersection of \( H_\omega \) and hyperplanes which pass through the singular point \( u \). Therefore \( P \cap \Sigma \) is singular at \( u \). Q.E.D.

It follows from 2.7.1 that if \( X^{6-k} = \Sigma \cap \mathbb{P}^{13-k} \) is smooth then \( u(\omega) \notin \mathbb{P}^{13-k} \) for any \( \omega \in \tilde{F}(X^{6-k}) \setminus \tilde{\Omega}(X^{6-k}) \). Combined with 2.4.4 this yields:
Proposition 2.7.2 Let $X^{6-k}$ be a smooth $(6-k)$-dimensional linear section of $\Sigma$.

If $2 \leq k \leq 4$, then $\hat{\Omega}(X^{6-k}) = \emptyset$ and $\hat{F}(X^{6-k}) \subset \Pi^{k-1}$ is a smooth quartic $(k-2)$-fold (e.g. when $k=2$ then $\hat{F}(X^{6})$ is a set of 4 points each with multiplicity 1); and if $k=5$ then $\text{Sing} \hat{F}(X^{1}) = \hat{\Omega}(X^{1}) = \{\omega_{1},...,\omega_{21}\}$ is a set of 21 ordinary double points (nodes) of the quartic threefold $\hat{F}(X^{1})$.

The linear section $X^{6-k}$ is subcanonical, more precisely $-K_{X} = (-4 + k)H$ where $H$ is the class of the hyperplane section. When $k \leq 4$ and $X^{6-k}$ is general, then $\text{Pic}(X^{6-k}) = \mathbb{Z}[H]$

Proof. Only the last statement remains to be shown, but this follows from [25] (cf. also [30]). Q.E.D.

Two quadric bundle fibrations Let $1 \leq k \leq 6$, let $P = P^{13-k}$ be a general linear space in $P(V(14))$ as above and consider the incidence variety

$$I_{P}(k) = \{(x, y) \in X^{6-k} \times P(V) | \dim(P_{y}^{4} \cap P^{13-k}) \geq k - 1, x \in Q_{y}\} \subset I_{P}$$

We denote the image of the second projection in $P(V)$ by $Y(k)$ and call it the vertex variety of $X^{6-k}$. The linear space $P = P^{13-k}$ is defined by $k$ linear forms. These linear forms may be pulled back to global sections of the vector bundle $E$ (cf. 2.5.3), therefore they define a map $\Phi_{P} : k\mathcal{O}_{P(V)} \to E$ and $Y(k)$ is just the $(k-1)$th degeneracy locus

$$Y(k) = \{y | \text{rank} \Phi_{P}(y) \leq k - 1\} \subset P(V).$$

Thus $Y(k)$ has dimension $k - 1$ and degree given by Porteous’ formula [11]:

$$\text{deg} Y(k) = c_{6-k}(E), \quad k = 1, \ldots, 5$$

The fibers of the map $I_{P}(k) \to Y(k)$ are all quadrics. In fact

Proposition 2.7.3 When $2 \leq k \leq 4$ and $P^{13-k}$ is a general linear subspace in $P(V(14))$, then the map $I_{P}(k) \to Y(k)$ is a quadric bundle map, i.e. the fibers are all quadrics of dimension $k - 2$.

Proof. When the $k$ sections of $E$ are general, $\text{rank} \Phi_{P}(y) \geq k - 1$ for every $y$. Therefore $\dim(P_{y}^{4} \cap P^{13-k}) = k - 1$ for every $y \in Y(k)$ and the proposition follows. Q.E.D.

The fibers of the other projection

$I_{P}(k) \to X^{6-k}$

are the intersections of a Lagrangian plane of $X^{6-k}$ with the subvariety $Y(k)$. This intersection is the rank $k - 1$ degeneracy locus of the vector bundle map $\Phi$ restricted to these Lagrangian planes.
Consider now $k$ sections of $E$, and let $\mathbb{P}^{k-1}$ be the linear subspace of $\mathbb{P}^{13}$ spanned by these sections. Let $M(k)$ denote the (locally) $k \times 5$ matrix with rows defined by the sections as above.

**Lemma 2.7.4** Let $u \in (\mathbb{P}^{k-1})^\perp \cap \Sigma$ and let $P$ be the corresponding Lagrangian plane in $\mathbb{P}(V)$, then $Y(k)$ contains $P$ if $k = 5$, it intersects $P$ in a quintic curve if $k = 4$ and in a finite scheme of length $12$ if $k = 3$. Let $\omega \in \tilde{F} \cap \mathbb{P}^{k-1}$, and let $P_u$ be the Lagrangian plane of $u = u(\omega)$ the dual pivot. If $u \notin (\mathbb{P}^{k-1})^\perp \cap \Sigma$, then $P_u \cap Y(k) = \{q\omega = 0\}$, the conic section defined by $\omega$.

**Proof.** The restriction to $P$ of $M(k) = (a(\omega); b(\omega); c(\omega))$ has, by 2.6.6 (iii), $a(\omega) = 0$. Therefore the rank of the matrix is at most $(k - 1) \leq 3$ along a subvariety of class $c_{5-k}(E) \cap P$. If $\omega \in \tilde{F} \cap \mathbb{P}^{k-1}$, then the rank of the matrix is at most $(k - 1)$ along the conic section $q\omega$.

Q.E.D.

**Proposition 2.7.5** The fibers of the projection 

$$I_P(k) \rightarrow X^{6-k}$$

are Lagrangian planes if $k = 5$, plane quintic curves if $k = 4$, and finite planar subschemes of length $12$ if $k = 3$.

We turn to the second incidence variety $J$ of $\Omega$. It involves the $Sp(3)$--dual $\tilde{F}(X^{6-k}) = \tilde{F} \cap (\mathbb{P}^{13-k})^\perp$. Consider first

$$I_{J_P}(k) = \{(x, y, \omega) \in X^{6-k} \times P^5 \times \tilde{F}(X^{6-k}) | \dim(P^4_y \cap \mathbb{P}^{13-k}) \geq k - 1, x \in Q_y, P^4_y \subset P^{12}_\omega\}$$

and its projection onto the second and third factor

$$J_P(k) = \{(y, \omega) \in P^5 \times \tilde{F}(X^{6-k}) | \dim(P^4_y \cap \mathbb{P}^{13-k}) \geq k - 1, P^4_y \subset P^{12}_\omega\}.$$

**Lemma 2.7.6** When $k \leq 4$ and $(\mathbb{P}^{13-k})^\perp$ does not intersect $\tilde{\Omega}$, then the projections $I_{J_P}(k) \rightarrow I_P(k)$ and $J_P(k) \rightarrow Y(k)$ are isomorphisms.

**Proof.** When $(\mathbb{P}^{13-k})^\perp$ does not intersect $\tilde{\Omega}$, then for every $y \in Y(k)$ there is a unique $\omega \in \tilde{F}(X^{6-k})$ such that $P^4_y \subset P^{12}_\omega$.

Q.E.D.

**Proposition 2.7.7** When $\tilde{F}(X^{6-k}) \subset \tilde{F} \setminus \tilde{\Omega}$, then $Y(k)$ has a conic bundle structure over $\tilde{F}(X^{6-k})$, where every conic is smooth.

**Proof.** Since $Y(k) \cong J_P(k)$ by 2.7.6, it suffices to consider the map $J_P(k) \rightarrow \tilde{F}(X^{6-k})$. Let $\omega \in \tilde{F}(X^{6-k})$. Since $\omega \in \tilde{F} \setminus \tilde{\Omega}$, there is a double pivot $u = u(\omega) \in \Sigma$. According to 2.6.4 (or 2.7.4) the Lagrangian plane $P(U)$ corresponding to $u$ intersects $Y(k)$ in a smooth conic section $\{q\omega(U) = 0\}$.

Q.E.D.

The invariants of the vertex variety $Y(k)$ is easily computed from the degree and the conic bundle structure. The only involved case is $k = 4$, i.e. when $Y(4)$ is a threefold conic bundle over the $K3$-surface $\tilde{F}(X^2)$. These threefolds form one of only two families of smooth threefold conic bundles in $\mathbb{P}^5$ and is extensively studied in [3].
Corollary 2.7.8 ([6]) For a general linear surface section $X^2 \subset \Sigma$ the vertex variety $Y(4)$ is a threefold smooth conic bundle over a quartic surface. It has degree 12 and intersection numbers:

$$H^{3-i} \cdot (H + K)^i = c_{2+i}(I_{Y^*}(5h)).$$

Corollary 2.7.9 For a general 3-fold linear section $X^3 \subset \Sigma$ the vertex variety $Y(3)$ is a minimal smooth conic bundle of degree 16 over a quartic plane curve.

Corollary 2.7.10 For a general 4-fold linear section $X^4 \subset \Sigma$ the vertex variety $Y(2)$ is 4 disjoint conic sections.

We summarize these results in the following table:

| $k$ | $X^{6-k}$ | $Y(k)$ | $F(X^{6-k})$ |
|-----|-----------|--------|--------------|
| 1   | Fano 5-fold of index 3 | $\emptyset$ | $\emptyset$ |
| 2   | Fano 4-fold of index 2 | 4 disjoint conics | 4 points |
| 3   | Fano 3-fold of index 1 | surface of degree 16 | plane quartic curve |
| 4   | K3-surface | 3-fold of degree 12 | quartic surface |
| 5   | Canonical curve of genus 9 | quintic hypersurface | 21-nodal quartic 3-fold |

2.8 The Fano 3-fold linear sections

The general 3-dimensional linear section $X = X^3$ of $\Sigma$ is a prime Fano 3-fold of genus 9 (see [26, 29]). The vertex variety $Y = Y(3)$ of $X$ is a conic bundle over the $Sp(3)$-dual plane quartic curve $\tilde{F}(X)$ by [2.7.7]. Below we give several relations between the Fano 3-fold $X$ and its vertex surface $Y$. For a different approach to these relations see [15].

Proposition 2.8.1 For the general smooth 3-fold linear section $X$ of $\Sigma$ there is a one to one correspondence between the set of lines in $X$, the 5-secant lines to its vertex variety $Y$ and sections $C$ with minimal self intersection $C^2 = 3$ of $Y$ as a conic bundle.

Proof. A line $L$ in $\Sigma$ parameterizes the pencil of Lagrangian planes through an isotropic line $l$, in the notation of 2.5.4 $L = L(l)$. The union of these Lagrangian planes is a 3-space $\mathbb{P}_l^3$. We will show that $l$ is a 5-secant line to $Y$ while $\mathbb{P}_l^3 \cap Y$ contains a section $C_L$ of the conic bundle $Y \to \tilde{F}(X)$. First we consider the restriction of the vector bundle map $\Phi$ to $l$. It is defined by a matrix with rows

$$M_l = (a_1 \quad a_2 \quad b_1 \quad c_1 \quad c_2).$$

Then

$$\Phi_l : \mathcal{O}_l \to 2\mathcal{O}_l \oplus \mathcal{O}_l(1) \oplus 2\mathcal{O}_l(2).$$

Thus the $a_i$ are constants, while $b_i$ and $c_i$ are linear and quadratic respectively. A point $\omega \in \mathbb{P}^{13}$ define a section of $E$, i.e. a matrix $M_l(\omega)$. In the notation of 2.5.4 we have...
Lemma 2.8.2 If \( l \) is an isotropic line, then \( \Sigma_l \) is contained in \( \mathbf{P}^{12}_\omega \) if and only if \( M_1(\omega) = 0 \). The line \( L(l) \) parameterizing the pencil of Lagrangian planes through \( l \) is contained in the hyperplane \( \mathbf{P}^{12}_\omega \) defined by \( \omega \), if and only if \( a(\omega) = 0 \).

Proof. The first statement is obvious, while \( L(l) \) is the vertex of \( \Sigma_l \) (cf. 2.5.4), so clearly, \( \mathbf{P}^{12}_\omega \) contains the vertex \( L(l) \) of \( \Sigma_l \) precisely when \( a = 0 \). Q.E.D.

When \( l \) is a non-isotropic line, then the row matrix

\[
M_l = (a_1 \ a_2 \ a_3 \ a_4 \ a_5)
\]

has only linear entries.

Lemma 2.8.3 If \( l \) is a non-isotropic line, then \( \Sigma_l \) is contained in \( \mathbf{P}^{12}_\omega \) if and only if \( M_1(\omega) = 0 \). Furthermore \( \mathbf{P}^{12}_\omega \) contains a quadric \( Q_p \) for a point \( p \in l \) if and only if \( M_1(p) = 0 \).

Proof. Clear. Q.E.D.

For the first part of the proposition we let \( l \) be a 5-secant line to \( Y \), and let \( M(l) \) be the matrix of \( \Phi \) restricted to \( l \). Assume first that \( L \) is a non-isotropic 5-secant line to \( Y(3) \). Then \( M(l) = (a_1, \ldots, a_5) \) has rank 2 in at 5 points. But this is possible only if \( M(l) \) has rank 2 on all of \( l \). For a general \( \Phi \) this is not the case of any non-isotropic line. If \( l \) is an isotropic 5-secant line to \( Y^s \) then again \( M(l) \) must have rank 2 in 5 points on the line. This is precisely the case when \( l \) is contained in \( Y(3) \) or when \( a(l) = 0 \).

For the last part of the proposition, consider first the set of Lagrangian planes that pass through a point \( p \in Y \). In \( X \), these planes are parameterized by a conic section, this is 2.7.3. In \( \mathbf{P}(V) \), these planes form a pencil on a 3-dimensional quadric \( Q \) in the hyperplane \( H_p \) defined by \( L_a(p) \). The conic section \( F_p \) in \( Y \) through \( p \) lies in a Lagrangian plane whose point in \( \Sigma \) is not in \( X \). In particular its plane is contained in \( H_p \), but not in the quadric \( Q \). The intersection \( Q \cap Y \) is therefore a curve of degree at most 14. But every Lagrangian plane is 12-secant, unless the curve is rational it has degree at least 14. Therefore we have equality. When the conic in \( X \) degenerates into two lines, the quadric degenerates into two 3-spaces, which each intersects \( Y \) in a curve of degree 7 that obviously is a section of the conic bundle. Let \( C \) be such a curve on \( Y \), let \( F \) be conic, and let \( H \) be a hyperplane section. Then \( C \) has genus 3, so by adjunction \( C^2 + C \cdot K = C^2 + C \cdot (\cdot H + 8F) = C^2 - 7 + 8 = 4 \), i.e. \( C^2 = 3 \). If \( D \subset Y \) is a section with \( D^2 \leq 2 \), then \( D \cdot K = D \cdot (\cdot H + 8F) = -D \cdot H + 8 \geq 2 \), i.e. \( D \cdot H \leq 6 \). Since \( Y \) is a conic bundle the degree of the sections all have the same parity. So the degree of \( D \) is 5 or 3. But \( D \) is a smooth curve of genus 3, so this is impossible. Q.E.D.

By 2.7.3, 2.7.3, it is clear that the 3-fold \( X \) parameterizes 12-secant planes to \( Y \), while \( Y \) parameterizes conic sections contained in \( X \).

Proposition 2.8.4 For the general smooth 3-dimensional linear section \( X \) of \( \Sigma \) the vertex variety \( Y \) coincides with the Hilbert scheme \( \text{Hilb}_2(X) \) of conic sections on \( X \).

Proof. It remains to show that any conic section on \( X \) is a plane section of the 3-dimensional quadric \( Q_y \subset \Sigma \) of a point \( y \in Y \). But any conic section in \( \Sigma \) corresponds to one of the two pencils of planes on a 3-dimensional quadric of rank 4. But these planes all have a common point \( y \), so \( Q_y \cap X \) is a conic, and \( y \in Y \). Q.E.D.
Proposition 2.8.5 The general smooth 3-fold linear section $X$ of $\Sigma$ coincides with the set of planes in $\mathbf{P}(V)$ that intersect the vertex variety $Y$ in a subscheme of length 12.

Proof. It remains to prove that a plane $P$ that intersects $Y$ in a finite subscheme of length at least 12 is Lagrangian and belong to $X$. Notice first that the 4-space $\mathbf{P}^4_y$ intersects $<X> = \mathbf{P}^{10}$ in a line when $y \in \mathbf{P}(V) \setminus Y$, while the intersection is a plane when $y \in Y$. Since $Q_y = \mathbf{P}^4_y \cap \Sigma$ is a quadric threefold that contains lines but not planes, the intersection $Q_y \cap X$ may be of three different kinds, namely

i) $Q_y \cap X$ is a scheme of length 2,

ii) $Q_y \cap X$ is a line or

iii) $Q_y \cap X$ is a (possibly singular) conic section.

Each plane in $\mathbf{P}(V)$ defines a hyperplane section of $X$ consisting of those Lagrangian planes of $X$ that intersect $P$. Similarly to the points $y \in \mathbf{P}(V)$ are the following three possibilities for a plane $P \subset \mathbf{P}(V)$:

I) There is a curve of points in $P$ through which passes infinitely planes from $X$.

II) Through only finitely many points in $P$ pass infinitely many planes from $X$, but through every $y \in P$ there is a plane of $X$ that intersects $P$ along a line through $y$.

III) Through only finitely many points in $P$ pass infinitely many planes from $X$, and only finitely many planes of $X$ intersect $P$ in a line.

Recall that $Y$ is a minimal conic bundle over a plane quartic curve, whose sections of minimal self-intersection are curves $C$ of degree 7. In particular the only plane curves on $Y$ are the smooth conics. Each minimal section $C$ lies in a $\mathbf{P}^3_l$ of a 5-secant line $l$ to $Y$. In fact the scheme of intersection $l \cap Y$ is the residual intersection to $C$ in $\mathbf{P}^3_l \cap Y$. Assume that $P$ is a plane through a 5-secant line $l$ that intersects $Y$ in a scheme of length 12. If $P$ is not contained in $\mathbf{P}^3_l$, i.e. it is not a plane of $X$, then the span $<P \cup \mathbf{P}^3_l>$ is a hyperplane that intersects $Y$ in $C \cup C'$, where $C'$ is a section of degree 9. Since $P \cap C'$ contains a scheme of length 12, we already have a contradiction. Thus we may give a more precise description of when the different cases above occur.

For points case iii) occurs, of course, when $y \in Y$, while case ii) occurs when $y$ lies outside $Y$ but on a 5-secant line $l$ to $Y$. For planes case I) occur precisely when $P$ intersects $Y$ in a conic or passes through a 5-secant line $l$. In the former case $P$ is Lagrangian but does not belong to $X$. The latter case was argued above: $P$ intersects $Y$ in a scheme of length 12 only if $P$ is in $X$. For the case II) assume that infinitely many planes of $X$ meet $P$ in a line. Since the planes of $X$ that intersects $P$ form a hyperplane section of $X$, there still is a 2-dimensional family of planes of $X$ that meet $P$. Therefore there is a most one plane of $X$ through a general point of $P$ that intersects $P$ in a line, so the family of such lines of intersection form a line in the dual space and have some point $p$ in $Y$ in common. This point $p$ is obviously of type ii) or iii). In case it is of type ii), the plane $P$ and a 5-secant line $l$ meet in $p$, so they span a $\mathbf{P}^3$. On the other hand this $\mathbf{P}^3$ intersects $Y$ in a scheme of length at least 17, so it intersects $Y$ is a curve. Any curve on $Y$ that span a $\mathbf{P}^3$ is a minimal section, so $P$ must contain some 5-secant line, and belong to case I). In case $p$ is of type iii) the planes of $X$ that meet $P$ in a line through $p$ form a conic in $X$, and therefore forms one family of a quadric 3-fold in $\mathbf{P}(V)$. The plane $P$ is then a member of the other family, so $P$ and any plane of the conic intersects $Y$ in at most a 0-dimensional $\mathbf{P}^3$ section of $Y$, i.e. in a scheme of length 16. Since the plane of the conic meet $Y$ in a scheme of length 12, and no line intersects $Y$ in a scheme of length 5, the plane $P$ meets $Y$ in a scheme of length at most 9, contrary to our assumption. The remaining case III) follows from the following lemma.
**Lemma 2.8.6** Let $P \subset \mathbb{P}(V)$ be a plane that intersects $Y$ in a scheme of length 12. Assume that only finitely many planes of $X$ meet $P$ in a line. Then $P$ belongs to $X$.

**Proof.** Assume that $P$ does not belong to $X$. If $P$ intersects a point of ii) we are done by the above argument. So the points of $P \cap Y$ are precisely the points of $P$ through which there pass infinitely many planes from $X$, i.e. a conic section of planes from $X$. Since finitely many planes of $X$ meet $P$ in a line, there are (with multiplicity) through the general point in $P$ precisely two planes of $X$, and they intersect $P$ precisely in the point. Let $C \subset P$ be a curve and consider the curve $D$ of planes in $X$ that meet $C$. If $C$ intersects $Y$, there are conic sections in $X$ that are components of $D$. We define $D_C$ to be the complement of the conics in $D$. Since the curve $D$ for a general line $C$ has degree 8, the degree of $D$ for a curve $C$ of degree $d$ is $8d$. The curve $D_C$ has degree $8d - 2e$, where $e$ is the length of the subscheme $C \cap Y$. Clearly, for any curve $C$ in $P$, the curve $D_C$ has positive degree. Furthermore if the degree of $D_C$ is 2, then $D_C$ is a conic or two disjoint lines. In the first case the planes of $D_C$ fill a quadric that intersects $P$ in a conic. But since there are two planes in $D_C$ through a general point on $C$, this is impossible unless $C$ is a line. In the second case the plane $P$ meets two 5-secant lines $l$, so this is ruled out above. Therefore if a curve $C$ of degree $d \geq 2$ in $P$ passes through a subscheme of length $a$ of $Y$, then $8d - 2e \geq 4$, or $e \leq 4d - 2$.

Consider now the linear system of quartic curves in $P$ that pass through $P \cap Y$. If $C$ is a line resp. conic or cubic curve in $P$ intersecting $Y$ in a scheme of length $a$, then $a \leq 3$ resp. $a \leq 6$ or $a \leq 10$. Therefore the linear system of quartic curves through $P \cap Y$ can have no fixed component. Furthermore the general member of this system has at most one singular point (supported in $P \cap Y$), and this point is at worst an ordinary double point. Let $C$ be a general such curve. Then $D_C$ is a curve of degree 8 in $X$. Since $P$ is not in $X$, there are two planes form $X$ distinct from $P$ through a general point of $C$, so $D_C$ is a double covering of $C$. If $D_C$ is nonreduced, its reduction is a curve of genus at least 2 and degree 4, i.e. $D_C$ is a plane quartic curve. But $X$ is cut out by quadrics so this is impossible. If $D_C$ is reduced its genus is at least 3, by Hurwitz’ formula. A curve of degree 8 and genus at least 3 spans at most a $\mathbb{P}^5$. If the genus is 3, the residual to a canonical divisor in a hyperplane section is a divisor of degree 4 that spans a plane. This divisor moves in a pencil, so $D_C$ has a pencil of 4-secant planes. If the genus is bigger, the span of $D_C$ is smaller, so $D_C$ has even more 4-secant planes. But any plane that meet $X$ in a scheme of length at least 4 intersects $X$ in a curve. Therefore the span of $D_C$ intersects $X$ in a surface. On the other hand the Picard group of $X$ is generated by a hyperplane section, so we get a contradiction. Q.E.D.

**Remark 2.8.7** Mukai shows in [31] Theorem 9.1 that $X$ is the Brill-Noether locus of Type II: Let $F$ be a rank 2 vector bundle on a plane quartic curve of odd degree, such that any section $C$ on the associated $\mathbb{P}^1$ bundle has self intersection $C^2 \geq 3$, then the moduli space $M$ of rank 2 vector bundles $E$, such that $\det E - \det F = K$ and $\text{rankHom}(F, E) \geq 3$, is a Fano 3-fold of index 1 and genus 9. More precisely, $M$ is isomorphic to a 3-fold linear section $X$ of $\Sigma$. The plane quartic curve is the Sp(3)-dual curve $\tilde{F}(X)$. The $\mathbb{P}^1$ bundle associated to $F$ is nothing but the vertex surface $Y$. It would be interesting to see the identification of the bundles $E$ directly in the above picture.

In our setup the result of Mukai has the following corollary:

**Corollary 2.8.8** Fix a smooth plane quartic curve $C$. There is a one-one correspondence between the isomorphism classes of ruled surfaces $S$ over $C$ with minimal selfintersection of a section equal to 3 and the 3-dimensional linear sections $X \subset \Sigma$ such that the Sp(3)-dual section $\tilde{F}(X) \cong C$. The ruled surface $S$ is isomorphic to the vertex variety $Y$ of $X$. 

Iskovskikh [13] described the 3-folds $X$, and in particular their rationality via the double projection from a line:

**Theorem 2.8.9** (Iskovskikh) Let $X$ be a smooth Fano 3-fold of genus $g = 9$ of rank $\text{Pic} = 1$ and of index 1, and let $L \subset X$ be a line. Then the double projection $\pi_{2L}$ from $L$, defined by the non-complete linear system $|h - 2L|$ on $X$, sends $X$ birationally to $\mathbb{P}^3$. Moreover, on $X$ there exists a unique cubic section $M \subset |\mathcal{O}_X(3h - 7L)|$ which intersects the line $L$, and this family of conics is contracted to a smooth curve $\Gamma = \Gamma_3^2 \subset \mathbb{P}^3$ of genus 3 and of degree 7 which lies on a unique cubic surface $S_3 \subset \mathbb{P}^3$.

The inverse birational map $\psi = \pi_{2L}^{-1} : \mathbb{P}^3 \rightarrow X$ is given by the non-complete linear system $|7H - 2\Gamma|$ on $\mathbb{P}^3$. Moreover, the cubic surface $S_3$ is swept out by the 1-dimensional family $\mathcal{C}_L$ of conics $q \subset \mathbb{P}^3$ which are 7-secant to $\Gamma$, and this family of conics is contracted to the line $L \subset X$.

In addition, for any smooth curve $\Gamma = \Gamma_3^2 \subset \mathbb{P}^3$ which lies in a unique cubic surface $S_3 \subset \mathbb{P}^3$, the rational map $\psi$ defined by the non-complete linear system $|7H - 2\Gamma|$ on $\mathbb{P}^3$ defines a birational map from $\mathbb{P}^3$ to a smooth Fano 3-fold $X = X_{16}$ of rank $\text{Pic} = 1$ and of index 1, and on this $X$ there exists a line $L$ such that $\psi = \pi_{2L}^{-1}$ where $\pi_{2L}$ is the double projection from $L$.

The birationality between $X_{16}$ and the projective 3-space $\mathbb{P}^3$ defined by double projection from a line $L \subset X_{16}$

\[
\phi = |h - 2L| \quad \quad \quad \quad \quad \psi = |7H - 2\Gamma| \quad \quad \quad \quad \quad \mathbb{P}^3 \supset \Gamma_3^2
\]

extremal curves $\mathcal{C}_L$

extremal divisor $M_t \equiv 3H - 7L$

extremal curves $\mathcal{C}_\Gamma$

extremal divisor $N_\Gamma \equiv 3h - \Gamma$

We may use 2.8.9 to construct an isomorphism $f : \tilde{F}(X) \rightarrow \Gamma_3^2$.

Consider a line $L \subset X$. By 2.8.1 the pencil of Lagrangian planes in $\mathbb{P}^5$ corresponding to this line each intersects the vertex surface $Y$ in a finite scheme of length 12, and the common line is a 5-secant. The $\mathbb{P}^3$ union of these planes cut $Y$ in a section $C_L$ of $Y(3)$ of degree 7. On the other hand $Y$ parameterizes conic sections on $X$. The curve $C_L$ parameterizes a family of conic sections on $X$ which intersects $L$. Since $Y$ is a fiber bundle over the plane quartic curve $\tilde{F}(X)$, we get an induced map from a family of conics on $X$ that meet $L$ onto $\tilde{F}(X)$. On the other hand this family of conics is also fibered over $\Gamma$, so we get an induced map from $\Gamma$ to $\tilde{F}(X)$. Since they are both of genus 3, this must be an isomorphism.

**Corollary 2.8.10** The principally polarized intermediate jacobian $J_X$ of $X = X_{16}$ is isomorphic to the jacobian $J_{\tilde{F}(X)}$ of the plane quartic curve $\tilde{F}(X)$, the $\text{Sp}(3)$-dual to $X$.

**Proof.** Indeed $J_X \cong J_\Gamma$ since $\pi_{2L}^{-1} : \mathbb{P}^3 \rightarrow X$ is a composition of a blow-up of $\Gamma$, a flop and a blow-down of a divisor to $L \cong \mathbb{P}^1$ (see [19] and Corollary 9.7 of [3]). Since $\Gamma \cong \tilde{F}(X)$ then $J_X \cong J_\Gamma \cong J_{\tilde{F}(X)}$. Q.E.D.
3 Rank two vector bundles on linear sections

We turn to the main application of our study of \( \Sigma \). On each nodal hyperplane section of \( \Sigma \) we will construct a rank two vector bundle with 6 global sections.

As before we use the following notation: For a point \( \omega \in \mathbb{P}^{13} \) we consider the hyperplane \( \mathbb{P}^{12}_\omega \) and the hyperplane section \( H_\omega = \mathbb{P}^{12}_\omega \cap \Sigma \). When \( \omega \in \mathbb{F} \setminus \Omega \subset \mathbb{P}^{13} \) then \( u(\omega) \in \Sigma \) is the pivot of \( \omega \) on \( \Sigma \), and when \( \omega \in \Omega \setminus \Sigma \subset \mathbb{P}^{13} \), then \( Q_\omega \subset \Sigma \) is the smooth quadric surface of pivots of \( \omega \) on \( \Sigma \) and also the singular locus of \( H_\omega \).

3.1 The projection of \( H_\omega \) from the pivot \( u(\omega) \)

Let \( \mathbb{P}^{12}_\omega \subset \mathbb{P}^{13} \) be the hyperplane defined by \( \omega \in \mathbb{F} \subset \mathbb{P}^{13} \), let \( \pi_u : \mathbb{P}^{12}_\omega \rightarrow \mathbb{P}^{11}_\omega \) be the projection from \( u = u(\omega) \in \Sigma \), and let the variety \( \mathbb{P}_\omega \subset \mathbb{P}^{11}_\omega \) be the proper \( \pi_u \)-image of the hyperplane section \( H_\omega = \Sigma \cap \mathbb{P}^{12}_\omega \subset \mathbb{P}^{12}_\omega \).

Let \( \sigma : H'_\omega \rightarrow H_\omega \) be the blow-up of \( u \in H_\omega \), and \( \psi : H'_\omega \rightarrow \mathbb{P}_\omega \) the projection into \( \mathbb{P}^{11}_\omega \). By 2.6.3(i), \( u = u(\omega) \) is an ordinary double point of \( H_\omega \), therefore the exceptional divisor \( Q' = \sigma^{-1}(u) \subset H'_\omega \) of \( \sigma \) is isomorphic to a smooth 4-dimensional quadric, i.e. \( Q' \cong G(2, \mathbb{C}^4) \).

The projection \( \pi_u \) contracts the tangent cone \( C_u \subset \mathbb{P}^6_u = T_u \Sigma \) at \( u \) to a Veronese surface \( S_u \) (cf 2.4.5).

Since the exceptional divisor \( Q' \subset H'_\omega \) is isomorphic to the projectivized tangent cone to \( H_\omega \) at \( u = u(\omega) \), the strict transform \( C'_u \) of \( C_u \) in \( H'_\omega \) intersects \( Q' \) in a Veronese surface. By 2.6.3(i), \( Q' \) is isomorphic to a smooth 4-dimensional quadric; therefore the isomorphic image \( Q = \psi(Q') \subset \mathbb{P}_u \) is a smooth 4-dimensional quadric containing the surface \( S_u \).

Since \( \mathbb{P}_\omega \) is a birational projection of \( H_\omega \) from its double point \( u = u(c) \), the degree \( \deg \mathbb{P}_\omega = \deg H_\omega - 2 = 14 \). Let \( L \) be the hyperplane divisor on \( H_\omega \). We denote by \( L \) also the pullback \( \sigma^*L \) and let \( L' \) be the strict transform of the general hyperplane divisor which passes through the point \( u \), i.e. \( L' \equiv L - Q' \). We summarize some further properties of the morphism \( \psi \).

**Lemma 3.1.1** \( \mathbb{P}_\omega \subset \mathbb{P}^{11}_\omega \) is a 5-fold with singularities at most on the surface \( S_u \subset \overline{Q} \). \( H'_\omega \) is a smooth 5-fold, the morphism \( \psi : H'_\omega \rightarrow \mathbb{P}_\omega \) contracts the codimension 2 subvariety \( C'_u \) to the surface \( S_u \subset \overline{Q} \) and is an isomorphism outside \( C'_u \). In particular, \( \mathbb{P}_\omega \) has singularities at most on the surface \( S_u \). The canonical divisor on \( H'_\omega \) is \( K_{H'_\omega} = -3L' \), where \( L' \) is the pullback of a hyperplane divisor on \( \mathbb{P}_\omega \).

**Proof.** Since \( \psi \) induces the projection from \( u \), the divisor \( L' = L - Q' \) is the pullback of a hyperplane divisor on \( \mathbb{P}_\omega \). It remains only to compute the canonical divisor. The canonical
divisor on $\Sigma$ is $K_{\Sigma} \equiv -4H$, so by adjunction the canonical divisor $K_{H_\omega} = -3L$. Since $\sigma : H'_\omega \to H_\omega$ blows up the double point $u \in H_\omega$, the canonical divisor

$$K_{H_\omega} = \sigma^* K_{H_\omega} + (\dim H_\omega \setminus 2)Q' \equiv \sigma^*(-3L) + 3Q' \equiv -3L'. $$

Q.E.D.

We shall see in [3, 4] below that the 5-fold $\overline{\Pi}_\omega$ is in fact a linear section of the Grassmannian $G(2, 6) \subset \mathbb{P}^{14}$ with a special codimension 3 subspace in $\mathbb{P}^{14}$.

Let $X^{6-k} = \mathbb{P}^{13-k} \cap \Sigma$ be a smooth $(6-k)$-dimensional linear section of $\Sigma$ as in [2, 7]. Let $\omega \in \hat{F}(X^{6-k}) - \hat{\Omega}(X^{6-k})$. Then by [2, 7, 1] the pivot $u(\omega)$ is not contained in $\mathbb{P}^{13-k}$. Let $P^{14-k}_\omega$ be the subspace of $\mathbb{P}^{13} = \mathbb{P}(V(14))$ spanned by $P^{13-k}$ and $u(\omega)$, and let $\Pi^{k-2}_\omega = (P^{14-k}_\omega)^\perp$. Clearly $\Pi^{k-2}_\omega$ is a linear subspace of $\Pi^{k-1}$ of codimension one. Denote by $W^{7-k}_\omega$ the intersection

$$W^{7-k}_\omega = \Sigma \cap \mathbb{P}^{14-k}_\omega. $$

Since the $(6-k)$-fold $X^{6-k}$ is a proper linear section of $\Sigma$ and a linear section of $W^{7-k}_\omega$ with the codimension 1 subspace $\mathbb{P}^{13-k} \subset \mathbb{P}^{14-k}_\omega$, the dimension $\dim W^{7-k}_\omega = 7 - k$. Furthermore, by [2, 7, 1], the pivot $u(\omega)$ is a singular point of $W^{7-k}_\omega$.

Consider now the projection $\pi_{u(\omega)} : P^{12}_\omega \to P^{11}_\omega$ from the pivot $u(\omega)$. Since $u(\omega) \not\in \mathbb{P}^{13-k}$, the restriction of $\pi_{u(\omega)}$ to $\mathbb{P}^{13-k}$ is an projective-linear isomorphism onto $P^{13-k}_\omega := \pi_{u(b)}(\mathbb{P}^{13-k})$, in particular $\pi_{u(b)} : X^{6-k} \to X^{6-k}_\omega = \pi_{u(b)}(X^{6-k}) \subset P^{13-k}_b$ is a projective-linear isomorphism.

Since $P^{13-k} \subset \mathbb{P}^{14-k}_\omega$ is a hyperplane, and since $u(\omega) \in \mathbb{P}^{14-k}_\omega$, the projection $\pi_{u(\omega)}$ maps $P^{14-k}_\omega$ onto $P^{13-k}_\omega$. The pivot point $u(\omega)$ is a quadratic singularity of $W^{7-k}_\omega$, therefore the proper $\pi_{u(\omega)}$-image $\overline{W}^{7-k}_\omega$ of $W^{7-k}_\omega$ will contain a quadric $Q^{6-k}_\omega \subset Q$ of dimension $6 - k$; under the condition $2 \leq k \leq 5$.

We will show in [3, 4] that the projection $\pi_{u(\omega)}$ sends the hyperplane section $H_\omega = \Sigma \cap P^{12}_\omega$ to a codimension 3 linear section $\overline{\Pi}_\omega = P^{11}_\omega \cap G(2, C^6)$ containing a smooth 4-fold quadric $Q = G(2, C^4)$ for some $C^4 \subset C^6$.

Thus $\overline{X}^{6-k}_\omega$ is a subvariety of the linear section $\overline{W}^{7-k}_\omega$ of $G(2, C^6)$, a linear section that contains a $6 - k$-dimensional quadric.

In the rest of this section these observations lie behind a description of embeddings of linear sections of $\Sigma$ into $G(2, C^6)$, or what amounts to the same, a description of a family of rank two vector bundles on linear sections of $\Sigma$ with 6 global sections.

### 3.2 Del Pezzo and Segre threefolds

We define special Del Pezzo threefolds and identify them with projections of Segre threefolds. We shall show later that these are subcanonical varieties on $\overline{\Pi}_\omega$, and thus they are zero loci of sections of a rank 2 vector bundle (via the Serre construction).

Let $\text{Gr}(2, 5) \subset \mathbb{P}^9 = \mathbb{P}(\wedge^2 C^5)$ be the Grassmannian of lines in $\mathbb{P}^4 = \mathbb{P}(C^5)$, and let $\text{Gr}(5, 2) \subset \mathbb{P}^9 = \mathbb{P}(\wedge^2 \hat{C}^5)$ be the Grassmannian of lines in the dual space $\mathbb{P}^4 = \mathbb{P}(C^5)$. Any plane $\Pi \subset \mathbb{P}^9$ is the plane of linear equations of its zero-space $P^9_\Pi \subset \mathbb{P}^9$. The group $GL(5, C)$ acts by $\wedge^2$ on the space $\mathbb{P}^9$. Therefore $GL(5, C)$ acts on $\text{Gr}(3, \wedge^2 \hat{C}^5)$, the family of planes in $\mathbb{P}^9$, and denote this action by $\rho_5$. 
Let $U_0 \subset \text{Gr}(3, \mathbb{C}^5)$ be the open set of these planes $\Pi \subset \mathbb{P}^9$ such that $\Pi \cap \text{Gr}(5, 2) = \emptyset$. The linear section $V_\Pi = \mathbb{P}^6_\Pi \cap \text{Gr}(2, 5)$ is singular if and only if it is contained in a tangent hyperplane since the contact locus of a tangent hyperplane is a plane. Therefore $\Pi \in U_0$ iff $V_\Pi = \mathbb{P}^6_\Pi \cap \text{Gr}(2, 5)$ is a smooth Fano threefold of degree 5 and of index 2, (i.e. $K_{V_\Pi} \equiv \mathcal{O}_{V_\Pi}(-2)$). The action $\rho_5$ is transitive on $U_0$, i.e. $U_0$ is an orbit of $\rho_5$ (Proposition 14 §5 in [37]). Therefore all the $V_\Pi, \Pi \in U_0$ are conjugate to each other by the action $\wedge^2$ of $GL(5, \mathbb{C})$ on $\mathbb{P}^9$ (see also [7] §(6.5)).

Let $U_{xxx} \subset \text{Gr}(3, \wedge^2 \mathbb{C}^5)$ be the subset of these planes $\Pi \subset \mathbb{P}^9$ such that $\Pi$ intersects $\text{Gr}(5, 2)$ transversally in exactly three points. These three points of intersection cannot be collinear, since $\text{Gr}(5, 2)$ is an intersection of quadrics.

As above, $U_{xxx}$ is an orbit of $\rho_5$, and all the $V_\Pi, \Pi \in U_{xxx}$ are conjugate to each other by the action $\wedge^2$ of $GL(5, \mathbb{C})$ on $\mathbb{P}^9$.

We call the unique threefold $V_\Pi, \Pi \in U_{xxx}$ the Del Pezzo threefold of type $xxx$.

Next, we turn to Segre threefolds. For this we first make a slight detour to 2–forms on even dimensional spaces and prove:

**Proposition 3.2.1** Let $V = \mathbb{C}^{2n}$ and let $\alpha, \alpha' \in \wedge^2 V$ be two general 2–vectors. Then there exists a unique (up to scalars) $n$– tuple $\gamma_1, \ldots, \gamma_n$ of 2–vectors of rank 2 such that both $\alpha$ and $\alpha'$ are linear combinations of the $\gamma_i$.

**Remark 3.2.2** The proposition may be reformulated in terms of multisecant spaces to the Grassmannian $G(2, 2n)$ of lines in $\mathbb{P}^{2n-1}$ embedded in Plücker space: A general line in the Plücker space is contained in a unique $n$–secant $(n-1)$–space to $G(2, 2n)$.

**Proof of 3.2.1.** First we prove uniqueness. Since the pair $\alpha, \alpha'$ of 2–vectors is general we may suppose that they both have rank 2n and that

$$\alpha = \sum_{i=1}^{n} \gamma_i, \quad \text{and} \quad \alpha' = \sum_{i=1}^{n} \lambda_i \gamma_i,$$

where the $\lambda_i$ are pairwise distinct coefficients. Let

$$\beta_i = \lambda_i \alpha - \alpha', \quad i = 1, \ldots, n.$$

Then the $\beta_i$ are the precisely the 2–vectors of the pencil generated by $\alpha$ and $\alpha'$ that have rank less than 2n. Furthermore their rank is exactly $2n-2$ since $\lambda_i \neq \lambda_j$ for $i \neq j$. Therefore each $\beta_i \in \wedge^2 V_i$ for a unique rank $2n-2$ subspace $V_i \subset V$. Let $U_j = \cap_{i \neq j} V_i$. Then $U_j$ is 2-dimensional and $\gamma_j$ is a nonzero 2–vector that generates the subspace $\wedge^2 U_j \subset \wedge^2 V$, so the 2–vectors $\gamma_i$ are determined uniquely by the pencil generated by $\alpha$ and $\alpha'$.

For existence we use a dimension argument. On the one hand, the Grassmannian of lines in the Plücker space $\mathbb{P}(\wedge^2 V)$ has dimension $2(n(2n-1)-2) = 4n^2-2n-4$. On the other hand the Grassmannian $G(2, 2n)$ has dimension $4n-4$, so the family of lines contained in $n$–secant $(n-1)$–spaces to $G(2, 2n)$ has dimension at most $(4n-4)n+2(n-2) = 4n^2-2n-4$. By the uniqueness argument, a general such line lies in a unique $n$–secant $(n-1)$–space, so the two dimensions actually coincide. Since there is an obvious inclusion of the latter into the former, and the former is irreducible, we may conclude. Q.E.D.
This result leads to a simple description of Segre $n$-folds

$$X_n = \mathbb{P}^1 \times \mathbb{P}^1 \times \cdots \times \mathbb{P}^1$$

in the Grassmannian $G(n, 2n)$.

**Proposition 3.2.3** Let $V = \mathbb{C}^{2n}$ and let $\alpha, \alpha' \in \wedge^2 V^*$ be two general 2-forms on $V$. Then the set of common Lagrangian $n$-spaces of $V$ with respect to the forms $\alpha$ and $\alpha'$ form a Segre $n$-fold $\mathbb{P}^1 \times \mathbb{P}^1 \times \cdots \times \mathbb{P}^1$ in the Grassmannian $G(n, 2n)$. Let $\gamma_1, \ldots, \gamma_n$ be the unique $n$-tuple of 2-forms of rank 2 such that both $\alpha$ and $\alpha'$ are linear combinations of the $\gamma_i$, and let $U_i \subset V$ be the $2n - 2$-dimensional kernel of $\gamma_i$. Then the common Lagrangian $n$-spaces $U$ with respect to the forms $\alpha$ and $\alpha'$ are precisely the $n$-spaces that intersects each $U_i$ in a $n - 1$-space. Equivalently, if $W_i = \cap_{i \neq j} U_j$, then $W_i$ is 2-dimensional, and a $n$-space is Lagrangian with respect to $\alpha$ and $\alpha'$ if and only if it has a nontrivial intersection with each $W_i$.

**Proof.** First, the two characterizations of Lagrangian $n$-spaces are clearly equivalent. Furthermore, the last one describe a family of $n$-spaces that clearly form a Segre $n$-fold, since every $n$-space intersects each line $\mathbb{P}(W_i)$ in a unique point.

A $n$-space $U$ that intersects each $U_i$ in a $n - 1$-space is clearly isotropic with respect to each 2-form $\gamma_i$, therefore also Lagrangian for $\alpha$ and $\alpha'$.

On the other hand it is a straightforward exercise in Schubert calculus to show that the set of common Lagrangian $n$-spaces for $\alpha$ and $\alpha'$ is $n$-dimensional of degree $n!$ i.e. the degree of the Segre $n$-fold. Since we have an inclusion the conclusion follows. Q.E.D.

We turn back to the case $n = 3$:

**Lemma 3.2.4** Let $X = X_3 \subset \mathbb{P}^7$ be the Segre threefold, and let $u \in X$. Then the projection $\overline{V}$ of $X$ from $u$ is a Del Pezzo threefold of type $xxx$.

Conversely, the Del Pezzo threefold $\overline{V}$ of type $xxx$ is a projection of the Segre threefold $X$ from a point $u \in X$.

**Proof.** Consider the blowup $X' \to X$ centered at $u$, and let $E$ denote the exceptional divisor. Let $L_i$ be the pullback to $X$ of $O_{\mathbb{P}^1}(1)$ on each factor of $X$, and let $F_X = L_1 \oplus L_2 \oplus L_3$. Let $s_u$ be the unique global section of $F_X$ whose zero locus is $u$, and let $F_{X'}$ be the pullback of $F_X$ to $X'$. The pullback of $s_u$ to $F_{X'}$ vanishes on $E$, and correspond to the unique nonvanishing section $s'$ of $F_{X'}(-E)$. The exterior multiplication with $s'$ defines a surjective map $\wedge^2 F_{X'}(-E) \to \wedge^3 F_{X'}(-2E)$ that fits into an exact sequence

$$0 \to F_0 \to \wedge^2 F_{X'}(-E) \to \wedge^3 F_{X'}(-2E) \to 0,$$

where $F_0$ is a rank 2 vector bundle on $X'$. Notice $\wedge^3 F_{X'}(-2E)$ is the line bundle $O_{X'}(H_X - 2E)$ where $H_X$ is the pullback to $X'$ of the hyperplane class on $X$ in the Segre embedding. Furthermore $c_1(\wedge^2 F_{X'}(-E)) = 2H_X - 3E$ so $c_1(F_0) = H_X - E$. Similarly one computes $c_2(F_0) = (H_X - E)^2$. On the other hand, $h^0(O_{X'}(H_X - 2E)) = 4$, while there $h^0(\wedge^2 F_{X'}(-E)) = 9$ so $h^0(F_0) \geq 5$. Therefore the morphism defined by $H_X - E$, i.e. the projection of $X$ from $u$, maps $X'$ into $Gr(2, 5)$. It is a threefold of degree 5 that spans a $\mathbb{P}^6$, so it is a linear section of $Gr(2, 5)$.

$X$ contains three quadric surfaces that meet pairwise along a line through $u$. Thus the projection $\overline{X}$, i.e. the image of $X'$ contains three planes that meet pairwise in three points.
Evidently these points are precisely the singularities of $X$. Thus $X$ is a Del Pezzo threefold of type $xxx$.

The converse is clear by the transitivity of $\rho_5$. Q.E.D.

**Lemma 3.2.5** Let $X = X_3 \subset \Sigma$ be a Segre threefold, and let $H$ be a hyperplane section of $\Sigma$ that contains $X$. Then $H$ is singular, i.e. a tangent hyperplane section to $\Sigma$, at some point of $X$.

**Proof.** For each point $u$ on $X$ there is a $P^2$ of hyperplanes tangent to $\Sigma$ at $u$ that contain $X$. Since the general such hyperplane is tangent to $\Sigma$ at $u$ only, there is altogether a 5 dimensional family of tangent hyperplanes that contain $X$. But $X$ spans a $P^7$ so there is a $P^5$ of hyperplanes that contain $X$. Therefore the two sets must coincide and the lemma follows. Q.E.D.

### 3.3 A rank 2 vector bundle on singular hyperplane sections

Recall the universal sequence of vector bundles on $\Sigma$, the restriction of the universal sequence on $G = \text{Gr}(3, V)$:

$$0 \to U \to V \otimes \mathcal{O}_\Sigma \to Q \to 0,$$

where $U$ is the universal subbundle, and $U^* \cong Q$, by the natural map induced by $\alpha$. Any global section of the rank 3 bundle $\wedge^2 U^*$ comes from a 2-form $\alpha' \in \wedge^2 V^*$. In the previous section we saw that if the 2-form $\alpha'$ is general, then the zero-locus $X = Z(\alpha')$ is a Segre 3-fold. In fact the characterization of 3.2.3 yields a straightforward argument that any Segre 3-fold in $\Sigma$ is the zero-locus a section of $\wedge^2 U^*$. Since it is not essential we leave this out here.

From 3.2.5 we know that any hyperplane that contains the Segre 3-fold $X$ is tangent to $\Sigma$. So we fix a hyperplane $P^1_\omega$ that contains $X$, and assume that it is tangent only at $u \in X$. Then $X$ has codimension 2 in this hyperplane section, and still it is the zero-locus of a 2-form restricted to the hyperplane. Consider the blowup $H'_\omega$ of the hyperplane section $H_\omega = \Sigma \cap P^1_\omega$ in the singular point $u$. Let $Q'_\omega$ be the exceptional divisor on $H'_\omega$. It is isomorphic to a 3-dimensional quadric (when $\omega$ is general).

In the notation of 3.1.1 the canonical bundle $K_X = \mathcal{O}_X(-2L)$. On the strict transform $X'_\omega$ of $X$, the canonical bundle is therefore

$$K_{X'_\omega} = \mathcal{O}_{X'_\omega}(-2L + 2Q'_\omega) = \mathcal{O}_{X'_\omega}(-2L'),$$

so $X'_\omega$ is subcanonical with respect to the hyperplane line bundle $L'$ induced by the projection from $u$. Therefore, by the Serre construction, $X'_\omega$ is the zero-locus of a rank 2 vector bundle on $H'_\omega$. The aim of this section is to identify this vector bundle. To construct it one may apply the Serre construction starting with $X'_\omega$. We will use a different, and for our purposes, more direct argument, similar to the one used in the proof of 3.2.4.

Let

$$\wedge^2 U'_\omega^* = \wedge^2 U^* \otimes \mathcal{O}_{H'_\omega},$$

and consider the twisted bundle

$$\wedge^2 U'_\omega^*(-Q'_\omega).$$

Notice that the section of $\wedge^2 U'_\omega^*$, given by the restriction and pullback of the section $\alpha'$ vanishes on $Q'_\omega$. Therefore it corresponds to a section $\alpha'_\omega$ of $\wedge^2 U'_\omega^*(-Q'_\omega)$. The vector bundle
\( \wedge^2 U^*_\omega(-Q'_\omega) \) has rank 3, while the zero-locus \( X_\omega \) of the section \( \alpha'_\omega \) has codimension 2. We will show that \( \alpha'_\omega \) is a section of a rank 2 subbundle of \( \wedge^2 U^*_\omega(-Q'_\omega) \). To do this we consider maps of vector bundles:

\[
\wedge^2 U^*_\omega(-Q'_\omega) \to O_{H'_\omega}(L - 2Q'),
\]

where \( L \) is the pullback of the general hyperplane divisor on \( H_\omega \). Let \( U \subset V \) be the Lagrangian 3-space represented by \( u \in \Sigma \) and let \( U^\perp = L_\alpha(U) \subset V^* \). Then any element \( x \in U^\perp \) induces by exterior multiplication such a map:

\[
m_x : \wedge^2 U^*_\omega(-Q'_\omega) \to O_{H'_\omega}(L - 2Q'_\omega).
\]

The kernel of this map, we denote it by \( E'_x \), is of course a torsion free sheaf. If the map \( m_x \) is surjective, the kernel is even a vector bundle of rank 2. Therefore \( E'_x \) is our candidate for a rank 2 vector bundle. If we look at the stalks, we see that the multiplication by \( x \) is surjective outside the zero locus of \( x \), i.e. outside the strict transform \( C(x) \) on \( H'_\omega \) of the quadric cone \( Q_x \cap H_\omega \) with vertex at \( u \). Since \( Q_x \) is 3-dimensional, \( C(x) \) is a (rational) surface scroll whose image in \( \overline{\mathcal{H}}_\omega \) is a conic section on the Veronese surface \( S_u \). Thus we have an exact sequence

\[
0 \to E'_x \to \wedge^2 U^*_\omega(-Q'_\omega) \to O_{H'_\omega}(L - 2Q'_\omega) \to O_{C(x)}(L - 2Q'_\omega) \to 0.
\]

Outside \( C(x) \) the kernel sheaf \( E'_x \) is a rank 2 vector bundle. This will be enough for our purposes at this point, but eventually we will show that \( E'_x \) is a subsheaf of a bundle \( E_x \) that coincides with \( E'_x \) outside \( C_x \).

The problem is to get \( h^0(E'_x) = 6 \). By restriction and pullback from \( \Sigma \) there is a natural surjection of sections

\[
r_\omega : V^*(14) \cap (\wedge^2 U^\perp \otimes U^\perp_1 \oplus \wedge^3 U^\perp) = \text{Sym}^2 U^* \oplus \wedge^3 U^\perp \to H^0(O_{H'_\omega}(L - 2Q'_\omega)).
\]

The kernel of this map is generated by \( \omega \in \text{Sym}^2 U^* \), so \( h^0(O_{H'_\omega}(L - 2Q'_\omega)) = 6 \).

Similarly there is a natural surjection

\[
U^\perp \otimes U^\perp_1 \oplus \wedge^2 U^\perp \to H^0(\wedge^2 U^*_\omega(-Q'_\omega)).
\]

Here the kernel is generated by \( \alpha \), while \( U^\perp \otimes U^\perp_1 \oplus \wedge^2 U^\perp \) is 12-dimensional, so \( h^0(\wedge^2 U^*_\omega(-Q'_\omega)) = 11 \). Thus \( h^0(E'_x) = 6 \) only if the map \( m_x \) is not surjective on global sections.

We consider more carefully the image of the map \( m_x \) on global sections. Notice that \( r_\omega(\eta) \) for a form

\[
\eta \in V^*(14) \cap (\wedge^2 U^\perp \otimes U^\perp_1 \oplus \wedge^3 U^\perp)
\]

is in the image of \( m_x \) if and only if there exists a 2-form \( \beta \in U^\perp \otimes U^\perp_1 \oplus \wedge^2 U^\perp \) and a 1-form \( y \in U^\perp \) such that

\[
\eta = \alpha \wedge y + \beta \wedge x.
\]

The subspace of 3-forms of this kind in \( \wedge^2 U^\perp \otimes U^\perp_1 \oplus \wedge^3 U^\perp \) has dimension 9, i.e. codimension 1: The 3-forms of the kind \( \alpha \wedge y \) form a 3-dimensional space, while the 3-forms of the kind \( \beta \wedge x \), where \( \beta \) varies, form a subspace of dimension 7. Since these two subspaces intersect in \( < \alpha \wedge x > \), the dimension of their sum is 9. The intersection with \( V^*(14) \) has codimension 3, it is the symmetrizer relations in \( V^*(14) \cap \wedge^2 U^\perp \otimes U^\perp_1 \), so the subspace

\[
U_x = \{ \eta = \alpha \wedge y + \beta \wedge x | \eta \wedge \alpha = 0 \} \subset V^*(14) \cap (\wedge^2 U^\perp \otimes U^\perp_1 \oplus \wedge^3 U^\perp)
\]

has dimension 6. The image of the map \( m_x \) on global sections is nothing but the projection of \( U_x \) from the form \( \omega \). Therefore we have shown
**Lemma 3.3.1** The exterior multiplication

\[ m_x : \wedge^2 U_x^*(-Q'_\omega) \to O_{H'_\omega}(L - 2Q'_\omega) \]

is not surjective on global sections if and only if \( \omega \) is an element of \( U_x \).

Let \( p = \langle v \rangle \in P(U) \) and \( x = L_\alpha(v) \in U^\perp \). Let \( q_\omega \) be the quadratic form defined by \( \omega \) on \( U \) (cf. 2.3).

**Lemma 3.3.2** Let \( \omega \in V^*(14) \cap (\wedge^2 U^\perp \otimes U^\perp_1 \oplus \wedge^3 U^\perp) \). Then \( \omega \in U_x \) if and only if \( q_\omega(v) = 0 \).

**Proof.** First we assume that 

\[ \omega = \alpha \wedge y + \beta \wedge x. \]

The common zero-locus of the 2-forms \( \alpha \) and \( \beta \) is then contained in \( H_\omega \). Therefore we may choose coordinates \((e_i, x_i)\) on \( V \) such that \( U = U_0 = \langle e_1, e_2, e_3 \rangle \), \( U_1 = \langle e_4, e_5, e_6 \rangle \) and assume that

\[ \beta = sx_{14} + tx_{25} + ux_{36}. \]

Since \( \alpha \wedge \omega = (x_{14} + x_{25} + x_{36}) \wedge \omega = 0 \), then

\[ \omega = b(x_{415} + x_{356}) - c(x_{416} + x_{256}) - a(x_{452} + x_{436}), \]

for suitable scalar coefficients \( a, b, c \).

The quadratic form \( q_\omega \) on \( U \) is then (cf. 2.3)

\[ q_\omega = bx_1x_3 - cx_1x_2 - ax_2x_3. \]

In the expression

\[ \omega = \beta \wedge x + \alpha \wedge y, \]

it is clear that

\( x, y \in \langle x_4, x_5, x_6 \rangle = L_\alpha(U) \).

Thus we may write \( x = \beta_4x_4 + \beta_5x_5 + \beta_6x_6 \) and \( y = \alpha_4x_4 + \alpha_5x_5 + \alpha_6x_6 \). A straightforward calculation gives the following solutions:

\[ \alpha_4 = \frac{a}{u - t}, \alpha_5 = \frac{b}{u - s}, \alpha_6 = \frac{c}{t - s} \]

and

\[ \beta_4 = \frac{a}{u - t}, \beta_5 = \frac{b}{u - s}, \beta_6 = \frac{c}{t - s}, \]

thus \( v = \frac{a}{u - t}e_1 + \frac{b}{u - s}e_2 + \frac{c}{t - s}e_3 \) and \( q_\omega(v) = 0 \).

Conversely assume \( q_\omega(v) = 0 \). Let \( X \) be a Segre 3-fold through \( u \) contained in \( H_\omega \). Then we may assume that \( X \) is the zero locus of a 2-form \( \beta \). Coordinates may therefore be chosen as above, and \( q_\omega(v) = 0 \) implies that

\[ v = \frac{a}{u - t}e_1 + \frac{b}{u - s}e_2 + \frac{c}{t - s}e_3. \]

With

\[ y = \frac{a}{u - t}x_4 + \frac{b}{u - s}x_5 + \frac{c}{t - s}x_6 \]

we get

\[ \omega = \alpha \wedge y + \beta \wedge x. \]

Q.E.D.
Corollary 3.3.3 Let \( v \in U \), and \( x = L_\alpha(v) \in U^\perp \). Let \( E'_x \) on \( H'_\omega \) be the kernel sheaf of the map \( m_x \) above. Then \( h^0(E'_x) = 6 \) if and only if \( q_\omega(v) = 0 \), where \( H'_\omega \) is tangent at \( u \in \Sigma \) and \( q_\omega \) is the quadratic form defined by \( \omega \) on \( U \).

Proof. Since \( \wedge^2 U'_\omega(-Q'_\omega) \) has 11 sections, \( h^0(E'_x) = 6 \) if and only if \( m_x \) is not surjective on global sections, i.e. \( \omega = \alpha \wedge y + \beta \wedge x \) for some 1-form \( y \) and 2-form \( \beta \). So the corollary follows from 3.3.2. Q.E.D.

For each \( v \in U \) such that \( \{ q_\omega(v) = 0 \} \) we have contracted a sheaf \( E'_x \), where \( x = L_\alpha(v) \), locally free of rank 2 outside \( C(x) \) on \( H'_\omega \) with \( h^0(E'_x) = 6 \). Each of them gives rise to a rational map of \( H'_\omega \) into the Grassmannian \( \text{Gr}(2,6) \). This map is defined by sections of the determinant line bundle of \( E'_x \), whose first Chern class is given by:

\[
c_1(E_x) = c_1(\wedge^2 U'_\omega(-Q'_\omega)) - c_1(\mathcal{O}_{\Sigma}(L - 2Q'_\omega))
= (2L - 3Q'_\omega) - (L - 2Q'_\omega) = L - Q'_\omega = L'.
\]

If the natural map

\[
\wedge^2 H^0(E_x) \to H^0 \mathcal{O}_{H'_\omega}(L')
\]

is surjective, then the map to \( \text{Gr}(2,6) \) is nothing but the projection of \( H'_\omega \) from its singular point \( u = u(\omega) \). If it is not surjective the image in \( \text{Gr}(2,6) \) spans at most a \( \mathbb{P}^{10} \). Since \( H'_\omega \) is not a cone, the image is 5-dimensional, and the intersection of its span with \( \text{Gr}(2,6) \) is not proper. It is now a straightforward but tedious to check that no \( \mathbb{P}^{10} \) section of \( \text{Gr}(2,6) \) can contain the image of \( H'_\omega \). Therefore the image in \( \text{Gr}(2,6) \) is precisely the projection \( \mathcal{P}_\omega \). Furthermore this map is independent of \( x \).

Theorem 3.3.4 The projection of \( H_\omega \) from the singular point is a linear section of the Grassmannian \( \text{Gr}(2,6) \).

Proof. What remains is to show that the image \( \mathcal{P}_\omega \) of \( H'_\omega \) under the projection is a linear section of \( \text{Gr}(2,6) \). Since \( \Sigma \) has degree 16 and sectional genus 9, the projection of \( H_\omega \) must have degree 14 and sectional genus 8. It is 5-dimensional, spans a \( \mathbb{P}^{11} \) and is contained in \( \text{Gr}(2,6) \). Furthermore, it contains a 4-dimensional quadric, the image of the exceptional divisor on \( H'_\omega \). Thus \( \mathcal{P}_\omega \) has the same degree, sectional genus and codimension as \( \text{Gr}(2,6) \). So if the intersection

\[
\text{Gr}(2,6) \cap \langle \mathcal{P}_\omega \rangle
\]

is different from \( \mathcal{P}_\omega \), then this intersection is 6-dimensional and not proper. But the only codimension 2 varieties in \( \text{Gr}(2,6) \) that are contained in a \( \mathbb{P}^{11} \) are those representing special Schubert cycles of codimension 2. One is represented by the subvariety of rank 2 subspaces that intersect a given rank 3 subspace and the other is represented by Grassmannians \( \text{Gr}(2,5) \). The latter does not span a \( \mathbb{P}^{11} \), while the former do. In the former case \( \mathcal{P}_\omega \) is contained in a variety that is a \( \mathbb{P}^4 \) scroll parameterized by a \( \mathbb{P}^2 \). Each \( \mathbb{P}^4 \) must intersect \( \mathcal{P}_\omega \) in a threefold. But \( H_\omega \) is cut out by quadrics and contains only a 1-parameter family of threefold hypersurfaces (in fact the linear 3-spaces that appear as projections of \( Q_p \subset H'_\omega \) of \( 26.4 \)), so we have a contradiction and the theorem follows. Q.E.D.

The restriction and pullback to \( H'_\omega \) of the universal rank 2 quotient bundle on \( \text{Gr}(2,6) \) is clearly a rank 2 vector bundle. We denote it by \( E_\omega \). We immediately get:
Corollary 3.3.5 The sheaf $E'_x$ is a subsheaf of the restriction and pullback $E_\omega$ to $H'_\omega$ of the universal rank 2 quotient bundle on $\text{Gr}(2, 6)$. The bundle $E_\omega$ is independent of $x$, has $h^0(E_\omega) = 6$, $\det E_\omega = O_{H'_\omega}(I')$ and the zero scheme of a general section is isomorphic to the strict transform of a Segre threefold that passes through the singular point.

Proof. Outside $C(x)$ the two sheaves $E'_x$ and $E_\omega$ coincides. The zeros of a general section is precisely the zeros of a 2-form on $\Sigma$, i.e. the strict transform on $H'_\omega$ of a Segre threefold that passes through the singular point $u$. Since $C(x)$ has codimension 3 the corollary follows. Q.E.D.

Clearly $\overline{\Pi}_\omega$ is a special linear section of $\text{Gr}(2, 6)$ since it contains a 4-dimensional quadric, but a natural question arises: Is a general $P^{11}$ section of $\text{Gr}(2, 6)$ that contains a 4-dimensional quadric the projection of a singular section of the Lagrangian Grassmannian $\Sigma$.

We now prove that this is the case, and give another characterization of these linear sections of $\text{Gr}(2, 6)$. We set $Z = \overline{\Pi}_\omega$, and notice that the projection of $H'_\omega$ is an isomorphism on the exceptional quadric and outside the tangent cone at $u$. Thus it is singular at most along the image of the tangent cone, i.e. a Veronese surface (inside the 4-dimensional quadric).

First, since $\tilde{\mathcal{F}} \setminus \Omega$ is an orbit for $\rho$, all the $\overline{\Pi}_\omega$ are projectively equivalent to the same 5-fold $Z$.

To fix notation, let $V \cong \mathbb{C}^6$ and let $P^{14} = P(\wedge^2 V)$. Let $\text{Gr}(2, V)$ be the Grassmannian of 2-dimensional subspaces $U \subset V$, and let

$$\text{Gr}(2, V) \to P^{14}, \quad U \mapsto P(\wedge^2 U)$$

be the Plücker embedding.

Let $P^{14} = P(\wedge^2 V^*)$ be the dual space to $P^{14}$. The space $\wedge^2 V^*$ is isomorphic to the space $\text{Alt}(V, V^*)$ of skew-symmetric linear maps $A : V \to V^*$. Recall that the rank of $A$ is even. The rank stratification is given by the inclusions

$$\text{Gr}(V, 2) = \text{Gr}(2, V^*) \subset \tilde{P}f \subset \tilde{P}f$$

that is by the Grassmannian variety parameterizing $\mathbb{C}^*$-classes of maps $A \not= 0$ s.t. $\text{rank}(A) = 2$, and the Pfaffian cubic hypersurface parameterizing $\mathbb{C}^*$-classes of maps $A$ s.t. $\text{rank}(A) \leq 4$.

Let $\Pi^2 \subset \tilde{P}f$ be the plane of linear equations that define $P^{11} \subset P^{14}$, such that the 5-fold $Z' = P^{11} \cap \text{Gr}(2, V)$ contains a smooth 4-dimensional quadric $Q$.

Obviously, $Q = \text{Gr}(2, W) \subset \text{Gr}(2, V)$ for some 4--dimensional subspace $W \subset V$.

Lemma 3.3.6 $\Pi^2 \subset \tilde{P}f$.

Proof. The forms in $V^*$ that vanish on the 4--dimensional subspace $W \subset V$ is a rank 2 subspace $W^\perp \subset V^*$. Then

$$\wedge^2 W^\perp = V^* \wedge W^\perp \subset \wedge^2 V^*.$$ 

Therefore any $A \in (\wedge^2 W)^\perp$ is of rank at most 4, i.e.

$$P((\wedge^2 W)^\perp) \subset \tilde{P}f.$$
Since \( Q = G(2, W) \subset Z' \), the lemma follows. Q.E.D.

Fix \( W \subset V \) a 4–dimensional subspace, and let \( P^8_W = P(\langle \wedge^2 W \rangle^\perp) \subset \tilde{P}f \). Then \( P^8_W \) intersects the Grassmannian \( \Gr(V, 2) \) along the 5-fold Schubert cycle \( Y_W := \sigma_{30}(W^\perp) \) of 2-dimensional subspaces of \( V^* \) that intersects the rank 2 subspace \( W^\perp \) nontrivially. Therefore the general plane in \( P^8 \) does not intersect \( Y_W \).

When \( A \) has rank 4, the kernel is a rank 2 subspace \( U_A \subset V \). So there is a natural kernel map:

\[
pker : \tilde{P}f \setminus \Gr(V, 2) \to \Gr(2, V) \quad [A] \mapsto [U_A].
\]

This map can also be seen as the map

\[
\wedge^2 V^* \to \wedge^4 V^* \cong \wedge^2 V \quad \alpha \mapsto \alpha \wedge \alpha
\]

so it is quadratic in the coordinates. Now, the hyperplane section \( H_A \cap \Gr(2, V) \) is singular precisely in \( P(\langle \wedge^2 U_A \rangle) \). On the other hand \( P(\langle \wedge^2 U_A \rangle) \in \Gr(2, W') \) for any 4-dimensional subspace \( W' \subset V \) that contains \( U_A \). Clearly \( U_A \subset W \) for any \( A \in \wedge^2 W^\perp \). Therefore

\[
Z' = \cap_{A \in \Pi^2} H_A \cap \Gr(2, V)
\]

contains the image of

\[
s : \Pi^2 \to \Gr(2, V) \quad A \mapsto P(\langle \wedge^2 U_A \rangle).
\]

Since \( P(\langle \wedge^2 U_A \rangle) \) is a singular point in \( H_A \cap \Gr(2, V) \), the image of \( s \) is contained in the singular locus of \( Z' \).

**Lemma 3.3.7** Sing(\( Z' \)) is contained in a Veronese surface if and only if \( \Pi^2 \cap \Gr(V, 2) = \emptyset \).

**Proof.** Indeed, the above map \( s \) is defined everywhere on \( \Pi^2 \) only if \( \Pi^2 \cap \Gr(V, 2) = \emptyset \), and in this case the image is clearly a Veronese surface. On the other hand if \( A \in \Pi^2 \cap \Gr(V, 2) \), and \( U_A \subset V \) is the kernel of \( A \) (regarded as a skew-symmetric map as above), then the hyperplane section \( H_A \subset \Gr(2, V) \) defined by \( A \) is singular along a 4-fold quadric \( Q_A = \Gr(2, U_A) \subset \Gr(2, V) \). Thus

\[
Z' = \cap_{A \in \Pi^2} H_A \cap \Gr(2, V)
\]

is singular at least along a codimension 2 linear section of a 4–fold quadric \( Q_A \), which is clearly not contained in a Veronese surface. Q.E.D.

**Proposition 3.3.8** The linear section \( Z = \Pi_W \) of \( \Gr(2, V) \) is defined by a plane \( \Pi^2 \) of linear equations in \( \tilde{P}f \setminus \Gr(V, 2) \). The singular locus of \( Z \) is a Veronese surface.

**Proof.** We noticed above that the singular locus of \( Z \) is contained in a Veronese surface. It follows from [3.3.6] and [3.3.7] that the singular locus of \( Z \) is a Veronese surface and the orthogonal plane does not intersect \( \Gr(2, V)^* \). Q.E.D.

The following result on the orbits of the \( GL(V) \)-action on \( \tilde{P}^{14} \) fits well with the remark above.

**Proposition 3.3.9** ([Sato and Kimura] [37] p. 94) The action \( \wedge^2 \) of \( GL(V) \) on \( \tilde{P}^{14} \) is transitive on the set of the planes \( \Pi^2 \subset \tilde{P}f \setminus \Gr(V, 2) \).

[3.3.8, 3.3.9] immediately implies the following corollary and the second part of theorem [1].
Corollary 3.3.10 Let $Z \subset \text{Gr}(2,V)$ be a 5-fold linear section. Then the following are equivalent:

i) $Z$ contains a 4-dimensional smooth quadric and $\text{sing}Z$ is a Veronese surface.

ii) The orthogonal complement of the span $<Z>_+ \subset \hat{P}f \setminus \text{Gr}(V,2)$.

iii) $Z$ is the projection of a nodal hyperplane section of $\Sigma = \text{LG}(3,6) \subset \mathbb{P}^{13}$ from its node.

Let $S = Z \cap \mathbb{P}^8$ be a general linear surface section of $Z$. Then, clearly, $S$ is a $K3$ surface with a conic section $C$.

Corollary 3.3.11 The Picard group of a general linear surface section $S$ of $Z$ has rank 2 and is generated by the class of a hyperplane and the class of the conic section on $S$.

Proof. According to the refined versions of Mukai’s linear section theorem [26], [31], the general $K3$ surface $S$ with Picard group generated by a very ample line bundle $O_S(H)$ of degree $H^2 = 14$ and the line bundle $O_S(C)$ of a rational curve $C$ with $C \cdot H = 2$ is a linear section of $\text{Gr}(2,6)$. The conic section $C$ lies then in a unique 4-dimensional quadric $\text{Gr}(2,4)$ inside $\text{Gr}(2,6)$ and $S$ is therefore a linear section of a subvariety $Z$. Q.E.D.

3.4 Stable rank 2 vector bundles on linear sections

Theorem 3.3.4 allows us to construct families of rank two vector bundles on linear sections of $\Sigma$ as promised in the end of section 3.1. Let $\omega < k < H$ be a smooth linear section $X = X^{6-k} = \mathbb{P}^{13-k} \cap \Sigma$ and its $\text{Sp}(3)$-dual linear section $\hat{F}(X)$ of the quartic $\hat{F}$. Then to each point $\omega \in \hat{F}(X) \setminus \hat{\Omega}(X)$ we may associate a rank 2 vector bundle $E_{\omega,X}$ on $X$ with chern classes $c_1(E_{\omega,X}) = H$ and $c_2(E_{\omega,X}) = \sigma_X$, where $\sigma_X$ is the class of a codimension $k - 1$ linear section of a Segre 3-fold. In case $X$ is a curve $c_2(E_{\omega,X}) = 0$, but the vector bundle is special having at least 6 global sections.

Namely, let $\omega \in \hat{F}(X) \setminus \hat{\Omega}(X)$, then by 3.3.4 the projection $\pi_{u(\omega)}$ from the pivot $u(\omega) = L(\text{pivot}(\omega))$ sends the hyperplane section $H_\omega = \Sigma \cap \mathbb{P}^{12}_\omega$ to the linear section $\mathbb{P}^{11}_\omega \subset \text{Gr}(2,6) \cap \mathbb{P}^{11}_\omega$. Now $\omega \in \hat{F}(X) \subset (\mathbb{P}^{13-k})^\perp$, so $H_\omega \supset X$. By assumption $X$ is smooth, so 2.7.1 implies that $u(\omega) \notin \mathbb{P}^{13-k} = <X>$. Therefore the projection $\pi_{u(\omega)} : \mathbb{P}^{12}_\omega \to \mathbb{P}^{11}_\omega$ restricts to a linear isomorphism of $X \subset \mathbb{P}^{13-k}$. If $E_{\omega}$ is the rank 2 vector bundle on $H_\omega$ constructed in 3.3.3, then the restriction $E_{\omega,X} = E_{\omega}|_X$ is a rank 2 vector bundle on $X$. Via the linear isomorphism $X \subset \text{Gr}(2,6)$, and $E_{\omega,X}$ is the pullback of the universal rank 2 quotient bundle. Therefore $h^0(E_{\omega,X}) \geq 6$; and $c_1(E_{\omega,X}) = H_X$. By 3.3.1 the general section of $E_{\omega}$ vanishes on the projection of a Segre 3-fold inside $H_\omega$ through $u(\omega)$. Since $X$ does not pass through $u(\omega)$, the restriction to $X$ is that of a codimension $k - 1$ linear section of this Segre 3-fold, so $c_2(E_{\omega,X}) = \sigma_X$.

The vector bundle $E_{\omega,X}$ is in fact stable with respect to $H_X$. Denote by $\mathcal{M}_X(2,H,\sigma_X)$ the moduli space of stable vector bundles on $X$ with chern classes $c_1(E) = H, c_2(E) = \sigma_X$. It exists as a quasiprojective variety (cf. [29], [23], [24]).

Proposition 3.4.1 Let $1 < k < 6$, and let $X = X^{6-k} = \Sigma \cap \mathbb{P}^{13-k}$ be a smooth linear section of $\Sigma$ without non-trivial automorphisms, and let $\hat{F}(X)$ be its $\text{Sp}(3)$-dual linear section of the quartic $\hat{F}$. Then there is a natural map

$$e_X : \hat{F}(X) \setminus \hat{\Omega}(X) \to \mathcal{M}_X(2,H,\sigma_X) \quad \omega \mapsto [E_{\omega,X}]$$

where $[E_{\omega,X}]$ is the isomorphism class of $E_{\omega,X}$, and $\sigma_X = 0$ if $X$ is a curve. Furthermore, the map is injective on the set where $H^0(E_{\omega,X}) = 6$.
Proof. First we prove stability. Recall that a rank 2 vector bundle \( E \) is \textit{Takemoto-Mumford stable} (resp. \textit{semistable}) with respect to the polarization \( H \), if for each line subbundle \( L \), the inequality

\[
2H^i \cdot L < H^i \cdot c_1(E), \quad \text{(resp. } 2H^i \cdot L \leq H^i \cdot c_1(E)\text{)}
\]

holds, where \( i = \dim X - 1 \).

It is clear from the definition of Takemoto-Mumford that it is enough in our case to check stability in the curves \( X^1 \); the instability in the other cases would imply instability by restriction to a curve section \( X^1 \subset X^{6-k} \).

So let \( C = X^1 \subset \Sigma \) be a smooth linear curve section, and let \( \omega \in \tilde{F}(C) \setminus \tilde{\Omega}(C) \). We may assume that \( C \) has no automorphisms and that \( C \) has no \( g^1_3 \) (see 3.4.3 below). Consider the vector bundle \( E_{\omega,C} \). By assumption it is the pullback of the universal rank 2 quotient bundle on \( \text{Gr}(2,6) \), so the associated map of the \( \mathbf{P}^1 \)-bundle

\[
\mathbf{P}(E_{\omega,C}) \to \mathbf{P}^5
\]

is a morphism. Let \( S_C \subset \mathbf{P}^5 \) be the image ruled surface of this morphism. The curve \( C \) is contained in a 5-dimensional linear section \( Z \subset \text{Gr}(2,6) \) that contains a 4-dimensional quadric. The linear span of \( C \) intersects \( Z \) in a surface, which by 3.3.11 is a \( K3 \)-surface section \( Y \) containing a unique conic section that does not meet \( C \). Assume now that \( E_{\omega,C} \) has a subbundle of degree \( d \geq 6 \). Then \( S_C \) has \( d \) members of the ruling contained in a hyperplane \( H \). In particular \( C \subset \text{Gr}(2,6) \) meets the Grassmannian \( \text{Gr}(2,H) \) in \( d \) points. But \( \text{Gr}(2,H) \) has degree 5, so \( \text{Gr}(2,H) \) must intersect the surface \( Y \) in at least a curve. Since \( Y \) is an irreducible surface, the intersection must be a curve and spans at most a \( \mathbf{P}^4 \). Since \( d \geq 6 \), the corresponding divisor on \( C \) of degree \( d \) spans at least a \( \mathbf{P}^4 \). Therefore this curve has degree 5. But by 3.3.11 the Picard group of \( Y \) is generated class of \( H \) and the class of the unique conic section, so in particular every curve on it has even degree, a contradiction. Therefore \( E_{\omega,C} \) is stable.

For injectivity, notice that for a given embedding of \( X \in \text{Gr}(2,6) \) the linear span of \( X \) cuts the Grassmannian in a variety \( Y \) of dimension \( \dim X + 1 \). Given two elements \( \omega \) and \( \omega' \) in \( \tilde{F}(X) \setminus \tilde{\Omega}(X) \), then the vector bundles \( E_{\omega,X} \) and \( E_{\omega',X} \) are isomorphic only if the two linear sections \( Y_\omega \) and \( Y_{\omega'} \) are projectively equivalent. In fact the global sections of \( E_{\omega,X} \) define the map into \( \text{Gr}(2,6) \) and the linear span of the image defines \( Y_\omega \). Now \( Y_\omega \) is the projection from \( u(\omega) \) of the linear section \( Y_\omega \) of \( \Sigma \) defined by the span of \( X \) and \( u(\omega) \). Therefore \( Y_\omega \) and \( Y_{\omega'} \) are projectively equivalent if and only if \( Y_\omega \) and \( Y_{\omega'} \) are equivalent. Now, Mukai proves in \cite{24} Theorem 0.2, that two smooth linear surface sections of \( \Sigma \) are projectively equivalent if and only if they lie in the same orbit of the group action \( \rho \). It is straightforward to extend his argument to our nodal case. Therefore the linear span of \( X \) and \( u(\omega) \) and the linear span of \( X \) and \( u(\omega') \) are in the same orbit under the action \( \rho \). As soon as \( X \) has no nontrivial automorphisms, this cannot happen.

Q.E.D.

We next see what is the image of the map \( e_X \), and start with the curve case. The general stable rank 2 vector bundle with canonical determinant on \( C \) has no sections. The subset of \( M_C(2;K) \) corresponding to vector bundles with a given number of sections has the structure of a subvariety, which have been studied by several authors (cf. \cite{32}, \cite{33}, \cite{35}, \cite{1}). Following their notation we define the Brill-Noether locus \( M_C(2;K,k) \) to be the subvariety of \( M_C(2;K) \) corresponding to vector bundles with at least \( k+2 \) sections. Let \( E \) be a rank 2 vector bundle on a general linear curve section \( C \subset \Sigma \) with \([E] \in M_C(2;K,4)\).
Assume that $E$ is generated by global sections. Let
\[ \wedge^2 H^0(C, E) \to H^0(C, K) \]
be the natural map. If this map is surjective, then $E$ clearly is in the image of the map $e_C$ if the induced image $C \subset \Gr(2, H^0(C, K))$ is contained in a subvariety isomorphic to $Z$. By 3.3.10 this is satisfied as soon as the orthogonal complement of the span of $C$ in $P(\wedge^2 H^0(C, K))$ does not meet $\Gr(H^0(C, K), 2)$. Thus

**Lemma 3.4.2** The isomorphism class $[E] \in M_C(2; K, 4)$ fails to be in the image of $e_C$ only if
\[ \wedge^2 H^0(C, E) \to H^0(C, K) \]
is not surjective, or the orthogonal complement of the span of $C$ meets $\Gr(4, H^0(C, K))$, or $E$ is not globally generated.

We will not show that the cases of the lemma do not occur, but simply note that they represent closed subvarieties of $M_C(2; K, 4)$. A more general result is:

**Proposition 3.4.3** ([33] Theorem 4.7 or [3] p. 260-261): Let $C$ be a curve of genus 9 with no $g_1^5$, $g_2^7$, $g_3^9$, or $g_4^{11}$. If $M_C(2; K, 5) = \emptyset$, then $M_C(2; K, 4)$ is smooth and of dimension three precisely at the points representing bundles $E$ for which the Petri map $\mu : \Sym^2 H^0(C, E) \to H^0(C, \Sym^2 E)$ is injective.

The injectivity of the Petri map is shown by Bertram and Feinberg for $g(C) \geq 2$ and any stable rank 2 vector bundle with canonical determinant and $h^0(C, E) \leq 5$ in [3]. The same line of argument yields

**Lemma 3.4.4** Let $C$ be a curve of genus 9 as above, then the Petri map
\[ \mu : \Sym^2 H^0(C, E) \to H^0(C, \Sym^2 E) \]
is injective for any stable bundle $E \in M_C(2; K, 4)$.

**Proof.** Let $S$ be the scroll in $P(H^0(C, E)^*)$ defined by mapping $P(E)$ by the global sections of $E$. Then an element in the kernel of the Petri map
\[ \mu : \Sym^2 H^0(C, E) \to H^0(C, \Sym^2 E) \]
defines a quadric hypersurface containing $S$ (see [3] p. 267). So the Petri map is injective if and only if $S$ is contained in any quadric $Q$. Since $\det E$ is the canonical line bundle, the degree of $S$ is 16. Since $C$ is a section of $\Sigma$ it has no $g_1^5$, $g_2^7$, $g_3^9$ or $g_4^{11}$. Furthermore, since $E$ is stable, it has no sub linebundle of degree greater than 7, or equivalently, no section of $E$ vanishes in a divisor of degree 8. In particular no $P^4$ contains 8 lines on $S$, and no plane intersects $S$ in a section. For the latter, it is clear that a plane section of degree at most 7 corresponds to a linear series on $C$ of dimension 2 and degree at most 7, while for a plane section of degree at least 8, the residual net of hyperplanes define a linear series of dimension 2 and degree at most 8. Equality corresponds to a semistable bundle $E$.

Following [3] §4, we regard separately the cases $1 \leq \text{rank } Q \leq 6$:

If $\text{rank } Q = 6$, then the family $F(Q) \subset \Gr(2, 6)$ of lines on the smooth quadric $Q$ is isomorphic to the 5-fold
Thus the hyperplane divisor on $S$ intersection of $C$ contradicting the fact that $H$ semistable.

Assume that one of the two projections maps $C$ to a line. This line will then intersect all lines in $P^3$ that are parameterized by $S$. But this means that $S$ is degenerate, contained in a $P^1$, a contradiction.

Thus the linear series defined by $h$ and $h'$ both have dimension at least 2. Since $C$ has no $g^2_5$, the degree of both $h$ and $h'$ is at least 8. Since they are complimentary in a canonical divisor, this happens only if they both define $g^2_5$'s. This corresponds to a semistable vector bundle $E$.

If rank $Q = 5$, then $Q$ is a cone with vertex a point, and the planes in $Q$ all pass through the vertex and are parameterized by $P^3$. Therefore $F(Q) \subset Gr(2,6)$ is a $P^2$-bundle over a $P^3$. Since $S$ is no cone, only finitely many lines of $S$, say $d$ lines, pass through the vertex of $Q$. Let $P$ be the $P^4$ of lines in $Gr(2,6)$ passing through the vertex, and let $p : C \to Gr(2,5)$, be the projection from $P$. Then $p$ corresponds to the projection of $S$ from the vertex of $Q$, and maps $C$ into the double Veronese embedding of $P^3$ in $Gr(2,5)$. Thus the canonical linear series has a decomposition as a sum $K_C = D + 2L$, where $D = C \cap P$ is a divisor of even degree $d$. Since $D$ spans at most a $P^4$ in the canonical embedding of $C$, the degree $d \leq 6$. If $d \leq 2$, then $p(C)$ spans at least a $P^6$, and $L$ is of degree at most 8 and dimension 3 contrary to the assumption on $C$. If $d = 4$ or $d = 6$, then $L$ is a $g^6_6$ resp. a $g^3_5$, again a contradiction.

If rank $Q = 4$, then $Q$ has two pencils of $P^3$'s. The restriction of these pencils to $S$ define pencils of curves $|D|$ and $|D'|$ on $S$ such that $D + D'$ is a hyperplane section. We may assume that $D$ is a section of $S$, while $D'$ is the pullback of a divisor on $C$. Thus $\deg D' \geq 6$ and $\deg D \leq 10$. Since $C$ has no $g^2_5$ only equality is possible. In this case the decomposition $D + D'$ of a hyperplane section of $S$, correspond to an exact sequence

$$0 \to O_C(D'_C) \to E \to O_C(K_C - D'_C) \to 0.$$

The assumption on $C$ in fact implies that this sequence is exact on global sections, i.e. the connecting homomorphism $\delta_{[E]} : H^0(O_C(K_C - D'_C)) \to H^1(O_C(D'_C))$ is zero. Therefore $E = O_C(D'_C) \oplus O_C(K_C - D'_C)$; and since $\deg(K_C - D'_C) = 10 > 8$ the bundle $E$ is not even semistable.

If rank $Q = 3$, then $Q$ is a cone with vertex a plane $P^2$ over a smooth plane conic $q$. Thus the hyperplane divisor on $S$ decomposes $H = 2D + D_0$, where $D_0 = S \cap P^2$ is the intersection of $S$ with the vertex of $Q$. Note that $D_0$ has to be a curve, otherwise $H = 2D$ contradicting the fact that $H$ is a section of $S$ over $C$. Thus $D_0$ must be a section of $S$, but we saw above that no section of $S$ lie in a plane so this case is impossible.

If rank $Q = 2$ or 1, then the scroll $S$ span at most a $P^4$ contrary to the assumption.

Q.E.D.

One part of Mukai’s famous linear section theorem says:

**Theorem 3.4.5 (Mukai [24]).** Any smooth curve $C$ of genus 9 with no $g^1_3$ is isomorphic to a linear section $C = X^1$ of $\Sigma = LG(3,6)$.
On the other hand

**Lemma 3.4.6** No smooth linear curve section $C$ in a $\Sigma$ has a $g^1_9$.

Proof. Consider the curve $C$ as a subvariety of $\text{Gr}(3,6)$, and let $D$ be a member of a $g^1_9$ on $C$. Then $D$ spans a $\mathbb{P}^3$ and must correspond to five 3-spaces in contained in a $\text{Gr}(3,5)$. The intersection of $\Sigma$ with any $\text{Gr}(3,5)$ is however always a $\mathbb{P}^4$ section of a Grassmannian quadric. Therefore the intersection with the span of $D$ must be a quadric surface, a contradiction with the fact that $C$ is a linear section.

Q.E.D.

Therefore, if we combine 3.4.4 and 3.4.3 and the injectivity of the map $e_C$, we recover Mukai’s result on the Brill-Noether locus:

**Theorem 3.4.7** ([24] p.17) For a smooth linear curve section $C$ of $\Sigma$ the quartic 3-fold $\tilde{F}(C) \setminus \tilde{\Omega}(C)$ is a connected component of the Brill-Noether locus $M_C(2;K,4)$. The 21 double points $\tilde{\Omega}(C) \subset \tilde{F}(C)$ in the boundary correspond to semistable vector bundles that are not stable.

Proof. It only remains to check the semistable boundary. The semistable boundary $\delta_{ss}M_C(2;K)$ of $M_C(2;K)$ is the image of $Pic^{g-1}(C)$ under the map

$$j : Pic^{g-1}(C) \rightarrow M_C(2;K), \quad j : L \mapsto L \oplus K \otimes L^{-1}$$

see [35] §1. The semistable boundary of the locus $M_C(2;K,4) \subset M_C(2;K)$ is the intersection $\delta_{ss}M_C(2;K,4) = \delta_{ss}M_C(2;K) \cap M_C(2;K,4)$. Therefore

$$\delta_{ss}M_C(2;K,4) = \{ L \in Pic^8(C) : L \oplus K \otimes L^{-1} \in M_C(2;K,4) \}$$

$$= \{ L \in Pic^8(C) : h^0(L \oplus K \otimes L^{-1}) \geq 6 \}.$$

Since $C \subset \text{LG}(3,6)$ has no $g^1_9$, it has no $g^3_6$, so any line bundle $L$ and likewise $K \otimes L^{-1}$, such that $h^0(L \oplus K \otimes L^{-1}) \geq 6$ must be a $g^2_8$. Let $W_d^8(C) \subset Pic^d(C)$ be the Brill-Noether locus of all the invertible sheaves $L$ of degree $d$ on $C$ such that $h^0(C,L) \geq r$. Since $C$ is general of genus $g = 9$ then the fundamental class of $W_d^8(C)$ in $Pic^d(C) \cong J(C)$ is

$$[W_d^8] = \frac{r!(r-1)! \ldots 0!}{(g+2r-d)! \ldots (g+r-d)!} \Theta^{(r+1)(g+r-d)},$$

where $(J(C), \Theta)$ is the principally polarized Jacobian of $C$ (see [12], Ch. 2 §7 – Special linear systems IV). In particular $\text{dim} \ W_8^2(C) = 0$; and since $\text{deg}(\Theta^9/9!) = 1$

$$\text{deg} \ W_8^2(C) = \frac{2! \cdot 1! \cdot 0!}{5! \cdot 4! \cdot 3!} \cdot 0! = 42.$$

Therefore on the general curve $C$ of genus $9$ there are exactly 42 line bundles $L$ such that $\text{deg}(L) = 8$ and $h^0(C,L) = 3$. Moreover, since $K \otimes L^{-1}$ also has degree $8$ and $3$ sections the map

$$^- : W_8^2(C) \rightarrow W_8^2(C), \quad ^- : L \mapsto \tilde{L} = K \otimes L^{-1}$$

is an involution of $W_8^2(C)$. The fixed points, if such exist, of the involution $^-$ are these $L$ such that $L \otimes^2 = K_C$ (i.e. $L$ is a theta-characteristic of $C$) for which $h^0(C,L) = 3$. But since $C$ is general $h^0(C,L) \leq 1$ for any theta-characteristic of $C$, i.e. $^-$ has no fixed points.
Therefore on the general curve $C$ of genus 9 there exist exactly 21 (non-ordered) pairs $(L_i, \bar{L}_i), 1 \leq i \leq 21$ of line bundles such that $\deg L_i = \deg \bar{L}_i = 8$, $h^0(C, L_i) = h^0(C, \bar{L}_i) = 3$ and $L_i \otimes \bar{L}_i = K_C$.

Therefore the semistable boundary $\delta_{ss} M_C(2; K, 4)$ of $M_C(2; K, 4)$ is a finite set of 21 points representing the 21 rank 2 vector bundles $E_i = L_i \oplus \bar{L}_i, 1 \leq i \leq 21$.

In our setting, when $C$ is a general linear section of $\Sigma$, the $Sp(3)$-dual $\hat{F}(C)$ is a quartic 3-fold that intersects $\Omega$ in 21 nodes. Since this number fits with the number of semistable vector bundles just computed we try to extend the map $e_C$ above to $\hat{\Omega}(C)$. This is possible: Let $\omega \in \hat{\Omega}$ and let $u$ be a dual pivot of $\omega$ and consider the blowup of $\Sigma$ in $u$. Let $Q'_\omega$ be the strict transform of the exceptional divisor on $H'_\omega$, as above. Consider the exact sequence

$$0 \to E'_x \to \wedge^2 U^*_\omega(-Q'_\omega) \to \mathcal{O}_{H'_\omega}(L') \to N_x \to 0,$$

where the cokernel sheaf $N_x = \mathcal{O}_{C(x)}(L)$ is the restriction of the line bundle $\mathcal{O}_{S_{\omega}}(L)$ to the zero-scheme of $x$. By §3.2 the kernel sheaf $E'_x$ has 6 sections as soon as $\mathbb{P}^4 \subset \mathbb{P}^{12}_\omega$, where $v = L^{-1}(x)$. So $v$ belongs to one of the planes in the involutive pair $P_1$ and $P_2$ belonging to $\omega$. Furthermore the zeros of $x$ have codimension 2 so it does not intersect a general curve section $C \subset H_\omega$. Therefore the restriction of $E'_x$ to $C$ becomes a rank 2 vector bundle $E_\omega$ with canonical determinant and 6 sections. On the other hand for any line $l$ in one of the planes $P_i$, the subvariety $\Sigma_l$ of Lagrangian planes that meet $l$, is a Weil-divisor on $H_\omega$. Thus there are two nets of Weil divisors on $H_\omega$. Restricted to the curve $C$ these divisors become Cartier divisors, and the vector bundle $E_\omega$ splits as the sum of the corresponding line bundles. This yields the desired semistable vector bundle corresponding to the point $\omega \in \hat{\Omega}(C)$.

Q.E.D.

In the next section we study the image of the map $e_X$ of §3.1 in the cases where $X$ is a surface, a threefold or a fourfold. Thus we recover and generalize the results by Mukai that initiated this investigation.

## 3.5 The moduli space $M_S(2; h, 4)$ for a K3-surface $S$ of genus 9

Let $(S, h)$ be a polarized K3 surface of genus $g = 2n + 1$, let $s$ be an integer such that $s \leq n$. By §10 of [28] or §3 of [27], the moduli space of stable rank 2 vector bundles $E$ on $S$:

$$M_S(2, h, s) = \{ E | E \text{ is stable}, c_1(E) = h \text{ and } \chi(S, E) = s + 2 \} / (iso).$$

is a nonsingular symplectic variety of dimension $2(g - 2s)$. In particular if $s = n = (g - 1)/2$ then $\hat{S} := M_S(2, h, s)$ is a K3 surface. Irreducibility has been given several proofs, see [13] for a recent one and including references to other proofs.

Let $S = S_{16}$ be a general K3 surface of genus 9 embedded as a linear section of $\Sigma$ by a codimension 4 subspace $\mathbb{P}^9 \subset \mathbb{P}^{13}$, and let $H$ be the hyperplane class of $S \subset \mathbb{P}^9$. The moduli space $M_S(S, H, \sigma_S)$, then coincides with $M_S(S, H, 4)$ since $\chi(S, E) = 4 + 2 = c_2(E) = \deg \sigma_S$. Therefore the above combine with §3.4.1 to yield

**Theorem 3.5.1** For the general linear surface section $S = X \subset \Sigma$ the K3 surface $\hat{S} = M_S(2, H, 4)$ is isomorphic to the $Sp(3)$-dual quartic surface $\hat{F}(S)$.

**Proof.** The map $e_X$ is injective, so $\hat{F}(S)$ is a subvariety of $M_S(2, H, 4)$. On the other hand $M_S(2, H, 4)$ is a K3 surface so they must coincide.

Q.E.D.
If we compare this with Proposition 3.4.1 we see that in fact the map $e_S$ for linear surface sections $S \subset \Sigma$ is surjective.

**Proposition 3.5.2** Let $X$ be a general smooth threefold or fourfold linear section of $\Sigma$ and let $E$ be a stable rank two vector bundle on $X$ with $h^0(X,E) = 6$ and $\det E = \mathcal{O}_X(H)$. Assume that natural map $\wedge^2 H^0(X,E) \to H^0(\mathcal{O}_X(H))$ is surjective. Then $E$ is in the image of $e_X$.

**Proof.** Assume that $E$ is a rank 2 vector bundle on $X$ that satisfies the conditions of the proposition. Then the surjection $\wedge^2 H^0(X,E) \to H^0(\mathcal{O}_X(H))$ defines an embedding $X \subset \text{Gr}(2,H^0(X,E)\ast)$. Then clearly $E$ is in the image of $e_X$ if and only if there is a $\mathbb{P}^{11}$ such that $X \subset Z_X = \mathbb{P}^{11} \cap \text{Gr}(2,H^0(X,E)\ast)$ for some $Z_X \cong Z$ of 3.3.8.

Assume that $\dim X \geq 3$. Since $e_S$ is surjective for any general surface section each surface section $S$ intersects the Grassmannian in a threefold that is a linear section of a variety $Z_S \cong Z$. Therefore the 4-dimensional quadric $Q \subset Z_S$ intersects the linear span of $S$ in a quadric surface. If two surface sections $S$ and $S'$ of $X$ give rise to subvarieties $Z_S$ and $Z_{S'}$ with distinct quadrics $Q$ and $Q'$, then these quadrics are Grassmannians $\text{Gr}(2,W)$ and $\text{Gr}(2,W')$ for 4-dimensional subspaces $W$ and $W'$ of $H^0(X,E)\ast$. So $Q$ and $Q'$ have a plane or a point in common. But $S$ and $S'$ are linear sections of $X$ and the corresponding quadric surfaces may be chosen to be smooth with exactly a conic section in common. This is a contradiction. Therefore the subvarieties $Z$ for distinct surface sections of $X$ have the same 4-dimensional quadric $Q$.

The linear span of $X$ intersects the quadric $Q$ in a linear section of dimension $\dim X - 1$ of the quadric. Therefore the linear span of $X$ and $Q$ has dimension 11 and cuts $\text{Gr}(2,6)$ in a subvariety projectively equivalent to $Z$.

By [29] any smooth Fano 3-fold $X = X_{16}$ of degree 16, of rank $\text{Pic} = 1$ and of index 1 (or – simply – a prime Fano 3-fold of degree 16) is a linear 3-fold section of $\Sigma$, and the hyperplane class $H$ of $X_{16} = \Sigma \cap \mathbb{P}^{10}$ is the ample generator of $\text{Pic} X$ over $\mathbb{Z}$.

If $X = X_{16}$ be general, then the $\text{Sp}(3)$-dual linear section $\tilde{F}(X)$ is a smooth plane quartic curve which does not intersect $\tilde{\Omega}$.

**Proposition 3.5.3** Let $X = X_{16} \subset \mathbb{P}^{10}$ be a general prime Fano threefold of degree 16. Then the $\text{Sp}(3)$-dual to $X$ plane quartic curve $\tilde{F}(X)$ is isomorphic to an irreducible component of the moduli space $M_X(2;H,\sigma_X)$ of stable rank 2 vector bundles on $X$ with $c_1 = [h]$ and $c_2 = \sigma_X$, where $[h]$ is the class of the hyperplane section and $\sigma_X$ is the class of a sextic elliptic curve on $X$.

**Proof.** Now the condition in 3.5.2 is certainly an open one, so the complement of the image of the map $e_X$ is closed. Since $e_X$ is injective by 1.4.1 and its image is closed the theorem follows. Q.E.D.

We end with an easy corollary in the fourfold case:

**Corollary 3.5.4** For a general linear fourfold section $X \subset \Sigma$ the $\text{Sp}(3)$-dual $\tilde{F}(X)$, consists of four points. Whenever $X$ has no automorphisms, these four points define precisely the four isomorphism classes of stable rank 2 vector bundles $E$ with $c_1(E) = H$ and $c_2(E) = \sigma_X$ where $\sigma_X$ is the class of a Del Pezzo surface of degree 6 on $X$ such that the natural map $\wedge^2 H^0(X,E) \to H^0(\mathcal{O}_X(H))$ is surjective.
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