On Disjoint Holes in Point Sets

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Abstract

Given a set of points $S \subseteq \mathbb{R}^2$, a subset $X \subseteq S$, $|X| = k$, is called $k$-gon if all points of $X$ lie on the boundary of the convex hull $\text{conv}(X)$, and $k$-hole if, in addition, no point of $S \setminus X$ lies in $\text{conv}(X)$. We use computer assistance to show that every set of 17 points in general position admits two disjoint 5-holes, that is, holes with disjoint respective convex hulls. This answers a question of Hosono and Urabe (2001). We also provide new bounds for three and more pairwise disjoint holes.

In a recent article, Hosono and Urabe (2018) present new results on interior-disjoint holes – a variant, which also has been investigated in the last two decades. Using our program, we show that every set of 15 points contains two interior-disjoint 5-holes.

Moreover, our program can be used to verify that every set of 17 points contains a 6-gon within significantly smaller computation time than the original program by Szekeres and Peters (2006).

1 Introduction

A set of points in the Euclidean plane $S \subseteq \mathbb{R}^2$ is in general position if no three points lie on a common line. Throughout this paper all point sets are considered to be in general position. A subset $X \subseteq S$ of size $|X| = k$ is a $k$-gon if all points of $X$ lie on the boundary of the convex hull of $X$. A classical result from the 1930s by Erdős and Szekeres asserts that, for fixed $k \in \mathbb{N}$, every sufficiently large point set contains a $k$-gon [ES35, Mat02]. They also constructed point sets of size $2^{k-2}$ with no $k$-gon. Recently, Suk [Suk17] significantly improved the upper bound by showing that every set of $2^{k+o(k)}$ points contains a $k$-gon. However, the precise minimum number $g(k)$ of points needed to guarantee the existence of a $k$-gon is still unknown for $k \geq 7$ (cf. [SP06]).

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1 Erdős offered $500 for a proof of Szekeres’ conjecture that $g(k) = 2^{k-2} + 1$. 

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In the 1970s, Erdős [Erd78] asked whether every sufficiently large point set contains a $k$-hole, that is, a $k$-gon with no other points of $S$ lying inside its convex hull. Harborth [Har78] showed that every set of 10 points contains a 5-hole and Horton [Hor83] introduced a construction of large point sets without 7-holes. The question, whether 6-holes exist in sufficiently large point sets, remained open until 2007, when Nicolas [Nic07] and Gerken [Ger08] independently showed that point sets with large $k$-gons also contain a 6-hole. In particular, Gerken proved that every point set that contains a 9-gon also contains a 6-hole. The currently best bound is by Koshelev [Kos09], who showed that every set of 463 points contains a 6-hole. However, the largest set without 6-holes currently known has 29 points and was found using computer-assistance by Overmars [Ove02].

In 2001, Hosono and Urabe [HU01] started the investigation of disjoint holes, where two holes $X_1, X_2$ of a given point set $S$ are said to be disjoint if their respective convex hulls are disjoint (that is, $\text{conv}(X_1) \cap \text{conv}(X_2) = \emptyset$). This led to the following question: What is the smallest number $h(k_1, \ldots, k_l)$ such that every set of $h(k_1, \ldots, k_l)$ points determines a $k_i$-hole for every $i = 1, \ldots, l$, such that the holes are pairwise disjoint [HU08]? As there are arbitrarily large point sets without 7-holes, only parameters $k_i < 7$ are of interest. Moreover, since the gap between the upper bound and the lower bound for $h(6)$ is still huge, mostly values with parameters $k_1, \ldots, k_l \leq 5$ were investigated. Also note that, if all $k_i$ are at most 3, then the value $h(k_1, \ldots, k_l) = k_1 + \ldots + k_l$ is straightforward because every set of $k_1 + \ldots + k_l$ points can be cut into blocks of $k_1, \ldots, k_l$ points (from left to right), which clearly determine the desired holes.

In Sections 2 and 3 we summarize the current state of the art for two- and three-parametric values and we present some new results that were obtained using computer-assistance. Moreover, we describe some direct consequences for multi-parametric values in Section 4. The basic idea behind our computer-assisted proofs is to encode point sets and disjoint holes only using triple orientations (see Section 5), and then to use a SAT solver to disprove the existence of sets with certain properties (see Section 6).

In the Final Remarks (Section 7) we outline how our SAT model can be adapted to tackle related questions on point sets. For interior-disjoint holes, we show that every set of 15 points contains two interior-disjoint 5-holes. Also it is remarkable, that our SAT model can be used to prove $g(6) = 17$ with significantly smaller computation time than the original program from Szekeres and Peters [SP06].

\footnote{For a reasonably short proof for the existence of 6-holes we refer to [Val08].}

\footnote{Koshelev’s publication covers more than 30 pages (written in Russian).}
2 Two Disjoint Holes

For two parameters, the value $h(k_1, k_2)$ has been determined for all $k_1, k_2 \leq 5$ except for $h(5, 5)$ \cite{HU01, HU05, HU08, BD11}. Table 1 summarizes the currently best bounds for two-parametric values. Concerning the value $h(5, 5)$, the best bounds are $17 \leq h(5, 5) \leq 19$. The lower bound $h(5, 5) \geq 17$ is witnessed by the set of 16 points with no two disjoint 5-holes (taken from Hosono and Urabe \cite{HU08}), which is depicted Figure 1 and the upper bound $h(5, 5) \leq 19$ was shown by Bhattacharya and Das \cite{BD13} by an elaborate case distinction.

| 2 | 3 | 4 | 5 |
|---|---|---|---|
| 2 | 4 | 5 | 6 |
| 3 | 6 | 7 | 10 |
| 4 | 9 | 12 |
| 5 | 17* |

Table 1: Values of $h(k_1, k_2)$ \cite{HU01, HU05, HU08, BD11}. The entry marked with star (*) is new.

As our main result of this paper, we determine the precise value of $h(5, 5)$. The proof is based on a SAT model which we later describe in Section 6.

**Theorem 1** (Computer-assisted). Every set of 17 points contains two disjoint 5-holes, hence $h(5, 5) = 17$.
The computations for verifying Theorem 1 take about two hours on a single 3 GHz CPU using a modern SAT solver such as glucose (version 4.0)\(^4\) or picosat (version 965)\(^5\). Moreover, we have verified the output of glucose and picosat with the proof checking tool drat-trim\(^6\) (see Section 6.2).

### 3 Three Disjoint Holes

For three parameters, most values \(h(k_1, k_2, k_3)\) for \(k_1, k_2, k_3 \leq 4\) and also the values \(h(2, 3, 5) = 11\) and \(h(3, 3, 5) = 12\) are known \([HU08, YW15]\). Tables 2 and 3 summarize the currently best known bounds for three-parametric values.

| \(k_1\) | \(k_2\) | \(k_3\) | \(h(k_1, k_2, k_3)\) |
|-------|-------|-------|------------------|
| 2     | 3     | 4     | 11               |
| 3     | 3     | 5     | 12               |

**Table 2:** Values of \(h(k_1, k_2, 4)\) \([HU08, YW15]\).

| \(k_1\) | \(k_2\) | \(k_3\) | \(h(k_1, k_2, 5)\) |
|-------|-------|-------|------------------|
| 2     | 2     | 4     | 10               |
| 3     | 3     | 5     | 12               |

**Table 3:** Bounds for \(h(k_1, k_2, 5)\) \([HU08, YW15]\).

The values \(h(2, 2, 4)\), \(h(3, 3, 4)\), and \(h(2, 4, 4)\) have not been explicitly stated in literature. However, the former two can be derived directly from other values as follows:

\[
8 = 2 + 2 + 4 \leq h(2, 2, 4) \leq 2 + h(2, 4) = 8
\]
\[
10 = 3 + 3 + 4 \leq h(3, 3, 4) \leq 3 + h(3, 4) = 10
\]

To determine the value \(h(2, 4, 4) = 11\), observe that \(h(2, 4, 4) \leq 2 + h(4, 4) = 11\) clearly holds. Equality is witnessed by the double circle with 10 points (cf. Figure 2). This statement can be verified by computer or as follows: First, observe that no 4-hole contains two consecutive extremal points, thus every 4-hole contains at most two exterior points. Now consider two disjoint 4-holes. Since not both 4-holes can contain two extremal points, one of them contains two exterior points while the other one contains one exterior point. As illustrated in Figure 2, this configuration is unique up to symmetry and does not allow any further disjoint 2-hole. This completes the argument.

Also we could not find the value \(h(2, 2, 5)\) in literature, however, using a SAT instance similar to the one for Theorem 1 one can also easily verify that \(h(2, 2, 5) \leq 10\), and equality follows from \(h(5) = 10\) \([Har78]\). One can

\[^4\]http://www.labri.fr/perso/lsimon/glucose/\, see also [AS09]
\[^5\]http://fmv.jku.at/picosat/\, see also [Bie08]
\[^6\]http://cs.utexas.edu/~marijn/drat-trim\, see also [WHH14]
also use the order type database of 10 points to verify the existence of those particular disjoint holes for all possible configurations of 10 points.

We now use Theorem 1 to derive new bounds on the value $h(k, 5, 5)$ for $k = 2, 3, 4, 5$.

**Corollary 1.** We have

$$h(2, 5, 5) = 17,$$
$$17 \leq h(3, 5, 5) \leq 19,$$
$$17 \leq h(4, 5, 5) \leq 23,$$
$$22 \leq h(5, 5, 5) \leq 27.$$

**Proof.** To show $h(2, 5, 5) \leq 17$, observe that, due to Theorem 1, every set of 17 points contains two disjoint 5 holes that are separated by a line $\ell$. By the pigeonhole principle there are at least 9 points on one of the two sides of such a separating line $\ell$. Again, using a SAT instance similar to the one for Theorem 1, one can easily verify that every set of 9 points with a 5-hole also contains a 2-hole which is disjoint from the 5-hole. This completes the argument. We remark that one can also use the order type database of 9 points to verify this statement.

To show $h(3, 5, 5) \leq 2 \cdot h(3, 5) - 1 = 19$, observe that, due to Theorem 1, every set of 19 points contains two disjoint 5 holes that are separated by a line $\ell$. Now there are at least 10 points on one side of such a separating line $\ell$, and since $h(3, 5) = 10$, there is a 3-hole and a 5-hole that are disjoint on that particular side.

An analogous argument shows $h(4, 5, 5) \leq 2 \cdot h(4, 5) - 1 = 23$.

The set of 21 points depicted in Figure 3 witnesses $h(5, 5, 5) > 21$ (can be easily verified by computer), while $h(5, 5, 5) \leq h(5) + h(5, 5) = 27$. We

\footnote{The database of all combinatorially different sets of $n \leq 10$ points is available online at \url{http://www.ist.tugraz.at/staff/aichholzer/research/rp/triangulations/ordertypes/} and requires roughly 550 MB of storage. For more information we refer to [Kra03] [AAK02] [AK09].}
remark that this point set was found using local search techniques, implemented in our framework *pyotlib*.

**Figure 3:** A set of 21 points with no three disjoint 5-holes.

### 4 Many Disjoint Holes

As introduced by Hosono and Urabe [HU01, HU08], we use the following notation: Given positive integer $k$ and $n$, let $F_k(n)$ denote the maximum number of pairwise disjoint $k$-holes that can be found in every set of $n$ points, that is,

$$F_k(n) := \max(\{0\} \cup \{t \in \mathbb{N} : h(k; t) \leq n\}) \quad \text{with} \quad h(k; t) := h(k, k, \ldots, k).$$

In the following, we revise and further improve results by Hosono and Urabe [HU01, HU08] and by Bárány and Károlyi [BK01]. The currently

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8 The “*python order type library*” was initiated during the Bachelor’s studies of the author [Sch14] and provides many features to work with (abstract) order types such as local search techniques, realization or proving non-realizability of abstract order types, coordinate minimization and “beautification” for nicer visualizations. For more information, please consult the author.
best bounds are the following:

\[ F_k(n) = \lfloor n/k \rfloor \text{ for } k = 1, 2, 3 \]
\[ 3n/13 + o(n) \leq F_4(n) < n/4 \]
\[ \lfloor 2n/17 \rfloor \leq F_5(n) < n/6 \]
\[ \lfloor n/h(6) \rfloor \leq F_6(n) < n/12 \]
\[ F_k(n) = 0 \text{ for } k \geq 7. \]

Hosono and Urabe [HU01] showed that \( F_4(n) \geq (3n - 1)/13 \) holds for an infinite sequence of integers \( n \). Moreover, since we have

\[ h(k; s + t) \leq h(k; s) + h(k; t), \]

Fekete’s subadditivity lemma (see for example [Sch03, Chapter 14.5]) asserts

\[ \lim_{t \to \infty} \frac{h(k; t)}{t} = \inf_{t \in \mathbb{N}} \frac{h(k; t)}{t}, \]

and consequently \( 3n/13 + o(n) \leq F_4(n) \) holds.

Concerning the lower bound on \( F_5(n) \), Theorem [1] clearly implies that \( F_5(n) \geq \lfloor 2n/17 \rfloor \) holds.

Concerning the upper bounds, it was remarked in [BK01] that \( F_5(n) < n/6 \) is not too difficult to prove but no explicit construction was given. We now outline how the upper bounds \( F_5(n) < n/6 \) and \( F_6(n) < n/12 \) can obtained from the double circle on \( 2n \) points with an additional “center point”: Every 5-hole (6-hole) in this “dotted double circle” is incident to at most 2 extremal points, and therefore, at most 2/3 (2/4) of the exterior points – that is less than 5/6 (3/4) of all points – can be covered by disjoint 5-holes (6-holes). An analogous statement shows that the dotted double circle on \( 4k + 1 \) points has no \( k \) disjoint 4-holes, hence \( F_4(n) < n/4 \). In particular, we obtain that \( h(4, 4, 4, 4) = 18 \) since \( 17 < h(4, 4, 4, 4) \leq 2h(4, 4) = 18 \).

It is also worth to note that the double circle (sometimes with the additional center point, sometimes without) is a maximal configuration also for other settings; see for example [HU01, Figure 5] and [HU08, Figures 1(a), 4, 10(a), and 10(b)].

5 Encoding with Triple Orientations

In this section we describe how point sets and disjoint holes can be encoded only using triple orientations. This combinatorial description allows us to get rid of the actual point coordinates and to only consider a discrete parameter-space. This is essential for our SAT model of the problem.
5.1 Triple Orientations

Given a set of points \( S = \{s_1, \ldots, s_n\} \) with \( s_i = (x_i, y_i) \), we say that the triple \((a, b, c)\) is \( \text{positively (negatively) oriented} \) if

\[
\chi_{abc} := \text{sgn det} \begin{pmatrix} 1 & 1 & 1 \\ x_a & x_b & x_c \\ y_a & y_b & y_c \end{pmatrix} \in \{-1, 0, +1\}
\]

is positive (negative)\(^9\). Note that \( \chi_{abc} = 0 \) indicates collinear points, in particular, \( \chi_{aaa} = \chi_{aab} = \chi_{aba} = \chi_{baa} = 0 \). It is easy to see, that convexity is a combinatorial rather than a geometric property since \( k \)-gons can be described only by the relative position of the points: If the points \( s_1, \ldots, s_k \) are the vertices of a convex polygon (ordered along the boundary), then, for every \( i = 1, \ldots, k \), the cyclic order of the other points around \( s_i \) is \( s_{i+1}, s_{i+2}, \ldots, s_{i-1} \) (indices modulo \( k \)). Similarly, one can also describe containment (and thus \( k \)-holes) only using relative positions: A point \( s_0 \) lies inside a convex polygon if the cyclic order around \( s_0 \) is precisely the order of the vertices along the boundary of the polygon.

To observe that the disjointness of two point sets can be described solely using triple orientations, suppose that a line \( \ell \) separates point sets \( A \) and \( B \). Then, for example by rotating \( \ell \), we can find another line \( \ell' \) that contains a point \( a \in A \) and a point \( b \in B \) and separates \( A \setminus \{a\} \) and \( B \setminus \{b\} \). In particular, we have \( \chi_{aba'} \leq 0 \) for all \( a' \in A \) and \( \chi_{abb'} \geq 0 \) for all \( b' \in B \), or the other way round. Altogether, the existence of disjoint holes can be described solely using triple orientations.

Even though, for fixed \( n \in \mathbb{N} \), there are uncountable possibilities to choose \( n \) points from the Euclidean plane, there are only finitely many equivalence classes of point sets when point sets inducing the same orientation triples are considered equal. As introduced by Goodman and Pollack [GP83], these equivalence classes (sometimes also with unlabeled points) are called \textit{order types}.

5.2 An Abstraction of Point Sets

We consider a point set \( S = \{s_1, \ldots, s_n\} \) where \( s_1, \ldots, s_n \) have increasing \( x \)-coordinates. Using the \textit{unit paraboloid duality transformation}, which maps a point \( s = (a, b) \) to the line \( s^* : y = 2ax - b \), we obtain the arrangement of dual lines \( S^* = \{s^*_1, \ldots, s^*_n\} \), where the dual lines \( s^*_1, \ldots, s^*_n \) have increasing slopes. By the increasing \( x \)-coordinates and the properties of the unit paraboloid duality (see e.g. [Kra03 Chapter 1.3]), the following three statements are equivalent:

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\(^9\) The letter \( \chi \) is commonly used in literature to denote triple orientations as the word “chirality” is derived from the Greek word for “hand”.

(i) The points $s_i, s_j, s_k$ are positively oriented.

(ii) The point $s_k$ lies above the line $s_is_j$.

(iii) The intersection-point of the two lines $s_i^*$ and $s_j^*$ lies above the line $s_k^*$.

Due to Felsner and Weil [FW01] (see also [BFK15]), for every 4-tuple $s_i, s_j, s_k, s_l$ with $i < j < k < l$ the sequence

$$
\chi_{ijk}, \chi_{ijl}, \chi_{ikl}, \chi_{jkl}
$$

(index-triples are in lexicographic order) changes its sign at most once. These conditions are the signotope axioms.

It is worth to note that the signotope axioms are necessary conditions but not sufficient for point sets. There exist $\chi$-configurations which fulfill the conditions above – so-called abstract point sets, abstract order types, abstract oriented matroids (of rank 3), or signotopes – that are not induced by any point set, and in fact, deciding whether an abstract point set has a realizing point set is known to be $\exists\mathbb{R}$-complete. For more information we refer to [FG18].

5.3 Increasing Coordinates and Cyclic Order

In the following, we see why we can assume, without loss of generality, that in every point set $S = \{s_1, \ldots, s_n\}$ the following three conditions hold:

- the points $s_1, \ldots, s_n$ have increasing $x$-coordinates,
- in particular, $s_1$ is an extremal point, and
- the points $s_2, \ldots, s_n$ are sorted around $s_1$.

When modeling a computer program, one can use these constraints (which do not affect the output of the program) to restrict the search space and to possibly get a speedup. This idea, however, is not new and was already used for the generation of the order type database, which provides a complete list of all order types of up to 11 points [Kra03, AAK02, AK06].

Lemma 1. Let $S = \{s_1, \ldots, s_n\}$ be a point set where $s_1$ is extremal and $s_2, \ldots, s_n$ are sorted around $s_1$. Then there is a point set $\tilde{S} = \{\tilde{s}_1, \ldots, \tilde{s}_n\}$ of the same order type as $S$ (in particular, $\tilde{s}_2, \ldots, \tilde{s}_n$ are sorted around $\tilde{s}_1$) such that the points $\tilde{s}_1, \ldots, \tilde{s}_n$ have increasing $x$-coordinates.

Proof. We can apply an appropriate affine-linear transformation onto $S$ so that $s_1 = (0,0)$ and $x_i, y_i > 0$ holds for $i \geq 2$. Moreover, we have that $x_i/y_i$ is increasing for $i \geq 2$ since $s_2, \ldots, s_n$ are sorted around $s_1$. Since $S$ is in general position, there is an $\varepsilon > 0$ such that $S$ and $S' := \{(0,\varepsilon)\} \cup \{s_2, \ldots, s_n\}$ are of the same order type. We apply the projective
transformation \((x, y) \mapsto (x/y, -1/y)\) to \(S'\) to obtain \(\tilde{S}\). By the multilinearity of the determinant, we obtain

\[
\det \begin{pmatrix}
1 & 1 & 1 \\
x_i & x_j & x_k \\
y_i & y_j & y_k
\end{pmatrix} = y_i \cdot y_j \cdot y_k \cdot \det \begin{pmatrix}
1 & 1 & 1 \\
x_i/y_i & x_j/y_j & x_k/y_k \\
-1/y_i & -1/y_j & -1/y_k
\end{pmatrix}.
\]

Since the points in \(S'\) have positive \(y\)-coordinates, \(S'\) and \(\tilde{S}\) have the same triple orientations. Moreover, as \(\tilde{x}_i = x'_i/y'_i\) is increasing for \(i \geq 1\), the set \(\tilde{S}\) fulfills all desired properties.

It is worth to mention that the transformation \((x, y) \mapsto (x/y, -1/y)\) is the concatenation of the (inverse of the) unit paraboloid duality transformation and unit circle duality transformation which – under the given conditions – both preserve the triple orientations (see e.g. [Kra03, Chapters 1.3 and 2.2]).

6 SAT Model

In this section we describe the SAT model that we use to prove Theorem 1. The basic idea of the proof is to assume towards a contradiction that a point set \(S = \{s_1, \ldots, s_{17}\}\) with no two disjoint 5-holes exists. We formulate a SAT instance, where boolean variables indicate whether triples are positively or negatively oriented and clauses encode the necessary conditions introduced in Section 5. Using a SAT solver we verify that the SAT instance has no solution and conclude that the point set \(S\) does not exist. This contradiction then completes the proof of Theorem 1.

It is folklore that satisfiability is \(NP\)-hard in general, thus it is challenging for SAT solvers to terminate in reasonable time for certain inputs of SAT instances. We now highlight the two crucial parts of our SAT model, which are indeed necessary for reasonable computation times: First, due to Lemma 1, we can assume without loss of generality that the points are sorted from left to right and also around the first point \(s_1\). Second, we teach the solver that every set of 10 points gives a 5-hole, that is, \(h(5) = 10\) [Har78]. By dropping either of these two constraints (which only give additional information to the solver and do not affect the solution space), none of the tested SAT solvers terminated within days.

In the following, we give a detailed description of our SAT model. For the sake of readability, we refer to points also by their indices. Moreover, we use the relation “\(a < b\)” simultaneously to indicate a larger index, a larger \(x\)-coordinate, and the later occurrence in the cyclic order around \(s_1\).

6.1 A Detailed Description

1) Alternating axioms For every triple \((a, b, c)\), we introduce the variable \(O_{a,b,c}\) to indicate whether the triple \((a, b, c)\) is positively oriented. Since
we have that
\[ \chi_{abc} = \chi_{bca} = \chi_{cba} = -\chi_{b,a,c} = -\chi_{a,c,b} = -\chi_{c,b,a}, \]

we formulate clauses to assert
\[ O_{a,b,c} = O_{b,c,a} = O_{c,a,b} \neq O_{b,a,c} = O_{a,c,b} = O_{c,b,a} \]
by using the fact \( A = B \iff (\neg A \lor B) \land (A \lor \neg B) \), and \( A \neq B \iff (A \lor B) \land (\neg A \lor \neg B) \).

(2) **Signotope Axioms** As described in Section 5.2 for every 4-tuple \( a < b < c < d \), the sequence
\[ \chi_{abc}, \chi_{abd}, \chi_{acd}, \chi_{bcd} \]
changes its sign at most once. Formally, to forbid such sign-changes (that is, “− + −” and “+ − +”), we add the constraints
\[ O_I \lor \neg O_J \lor O_K \quad \text{and} \quad \neg O_I \lor O_J \lor \neg O_K \]
for every lexicographically ordered triple of index triples, that is, \( \{I, J, K\} \subset \binom{\{a,b,c,d\}}{3} \) with \( I \prec J \prec K \).

(3) **Sorted around first point** Since, for every triple \( a < b < c \), the points are sorted from left to right and also around \( s_1 \), we have that all triples \((1, a, b)\) are positively oriented for \( 1 < a < b \).

(4) **Bounding segments** For a 4-tuple \( a, b, c, d \), we introduce the auxiliary variable \( E_{a,b,c,d} \) to indicate whether the segment \( ab \) spanned by \( a \) and \( b \) bounds the convex hull \( \text{conv} \{a, b, c, d\} \). Since the segment \( ab \) bounds the convex hull \( \text{conv} \{a, b, c, d\} \) if and only if \( c \) and \( d \) lie on the same side of the line \( \overline{ab} \), we add the constraints
\[ \neg E_{a,b,c,d} \lor O_{abc} \lor \neg O_{abd}, \]
\[ \neg E_{a,b,c,d} \lor \neg O_{abc} \lor O_{abd}, \]
\[ E_{a,b,c,d} \lor O_{abc} \lor O_{abd}, \]
\[ E_{a,b,c,d} \lor \neg O_{abc} \lor \neg O_{abd}. \]

(5) **4-Gons and containments** For every 4-tuple \( a < b < c < d \), we introduce the auxiliary variable \( G^4_{a,b,c,d} \) to indicate whether the points \( \{a, b, c, d\} \) form a 4-gon. Moreover we introduce the auxiliary variable \( I_{i, a, b, c} \) for every 4-tuple \( a, b, c, i \) with \( a < b < c \) and \( a < i < c \) to indicate whether the point \( i \) lies inside the triangular convex hull \( \text{conv} \{a, b, c\} \).
Four points $a < b < c < d$, sorted from left to right, form a 4-gon if and only if both segments $ab$ and $cd$ bound the convex hull $\text{conv}\{(a, b, c, d)\}$. Moreover, if $\{a, b, c, d\}$ does not form a 4-gon, then either $b$ lie inside the triangular convex hull $\text{conv}\{(a, c, d)\}$ or $c$ lies inside $\text{conv}\{(a, b, d)\}$, and consequently not both edges $ab$ and $cd$ bound the convex hull $\text{conv}\{(a, b, c, d)\}$. Pause to note that $a$ and $d$ are the left- and rightmost points, respectively, and that not both points $b$ and $c$ can lie in the interior of $\text{conv}\{(a, b, c, d)\}$. Formally, we assert

\[
G_{a,b,c,d}^4 = E_{a,b,c,d} \land E_{c,d,a,b},
I_{b,a,c,d} = \neg E_{a,b,c,d} \land E_{c,d,a,b},
I_{c,a,b,d} = E_{a,b,c,d} \land \neg E_{c,d,a,b}.
\]

(6) 3-Holes For every triple of points $a < b < c$, we introduce the auxiliary variable $H_{a,b,c}^3$ to indicate whether the points $\{a, b, c\}$ form a 3-hole. Since three points $a < b < c$ form a 3-hole if and only if every other point $i$ lies outside the triangular convex hull $\text{conv}\{(a, b, c)\}$, we add the constraint

\[
H_{a,b,c}^3 = \bigwedge_{i \in S \setminus \{a, b, c\}} \neg I_{i,a,b,c}.
\]

(7) 5-Holes For every 5-tuple $X = \{a, b, c, d, e\}$ with $a < b < c < d < e$, we introduce the auxiliary variable $H_{X}^5$ to indicate that the points from $X$ form a 5-hole. It is easy to see that the points from $X$ form a 5-hole if and only if every 4-tuple $Y \in \binom{X}{4}$ forms a 4-gon and if every triple $Y \in \binom{X}{3}$ forms a 3-hole. Therefore, we add the constraint

\[
H_{X}^5 = \left( \bigwedge_{Y \in \binom{X}{4}} G_{Y}^4 \right) \land \left( \bigwedge_{Y \in \binom{X}{3}} H_{Y}^3 \right).
\]

(8) Forbid disjoint 5-holes If that there were two disjoint 5-holes $X_1$ and $X_2$ in our point set $S$, then – as discussed in Section 5 – we could find two points $a \in X_1$ and $b \in X_2$ such that the line $\overline{ab}$ separates $X_1 \setminus \{a\}$ and $X_2 \setminus \{b\}$ – and this is what we have to forbid in our SAT model. Hence, for every pair of two points $a, b$ we introduce the variables

- $L_{a,b}$ to indicate that there exists a 5-hole $X$ containing the point $a$ that lies to the left of the directed line $\overrightarrow{ab}$, that is, the triple $(a, b, x)$ is positively oriented for every $x \in X \setminus \{a\}$, and
- $R_{a,b}$ to indicate that there exists a 5-hole $X$ containing the point $b$ that lies to the right of the directed line $\overrightarrow{ab}$, that is, the triple $(a, b, x)$ is negatively oriented for every $x \in X \setminus \{b\}$. 

\[
\]
For every 5-tuple $X$ with $a \in X$ and $b \notin X$ we assert

$$L_{a,b} \lor \neg H_X \lor \left( \bigvee_{c \in X \setminus \{a\}} \neg O_{a,b,c} \right),$$

and for every 5-tuple $X$ with $a \notin X$ and $b \in X$ we assert

$$R_{a,b} \lor \neg H_X \lor \left( \bigvee_{c \in X \setminus \{b\}} O_{a,b,c} \right).$$

Now we forbid that there are 5-holes on both sides of the line $\overline{ab}$ by asserting

$$\neg L_{a,b} \lor \neg R_{a,b}.$$ 

(9) Harborth’s result

Harborth [Har78] has shown that every set of 10 points gives a 5-hole, that is, $h(5) = 10$. Consequently, there is a 5-hole $X_1$ in the set $\{1, \ldots, 10\}$, and if $X_1 \subset \{1, \ldots, 7\}$, then there is another 5-hole $X_2$ in the set $\{8, \ldots, 17\}$. Analogously, if there is a 5-hole $X_3 \subset \{11, \ldots, 17\}$, then there is another 5-hole $X_4$ in the set $\{1, \ldots, 10\}$. Therefore, we can teach the SAT solver that

- there is a 5-hole $X$ with $X \subset \{1, \ldots, 10\}$,
- there is no 5-hole $X$ with $X \subset \{1, \ldots, 7\}$,
- there is a 5-hole $X$ with $X \subset \{8, \ldots, 17\}$, and
- there is no 5-hole $X$ with $X \subset \{11, \ldots, 17\}$.

We remark that the so-obtained SAT instance has $\Theta(n^5)$ variables and $\Theta(n^6)$ clauses. The source code of our python program which creates the instance is available online on our supplemental website [Sch].

6.2 Unsatifiability and Verification

Having the satisfiability instance generated, we used the following command to create an unsatisfiability certificate:

```
glucose instance.cnf -certified -certified-output=proof.out
```

The certificate created by glucose was then verified using the proof checking tool drat-trim by the following command:

```
drat-trim instance.cnf proof.out
```

The execution of each of the two commands (glucose and drat-trim), took about 1-2 hours and the certificate used about 3.1 GB of disk space.
We have also used pycosat to prove unsatisfiability:

```
pycosat instance.cnf -R proof.out
```

This command ran for about 6 hours and created a certificate of size about 2.1 GB. The verification of the certificate\(^\text{10}\) using drat-trim took about 9 hours.

7 Final Remarks

In (8), we have introduced the variable \(L_{a,b}\) to indicate that there exists a 5-hole \(X\) containing the point \(a\) that lies to the left of the directed line \(\overrightarrow{ab}\). By relaxing this to “... there exists a 5-hole \(X\), possibly containing the point \(a\), ...” and analogously for \(R_{a,b}\), the computation time reduces by factor of roughly 2 while the number of clauses raises by a factor of \(n\). The solution space, however, remains unaffected.

**Multi-parametric Values:** To determine multi-parametric values such as \(h(5,5,5)\), one can formulate a SAT instance as follows: Three 5-holes \(X_1, X_2, X_3\) are pairwise disjoint if there is a line \(\ell_{ij}\) for every pair \(X_i, X_j\) that separates \(X_i\) and \(X_j\). By introducing auxiliary variables \(Y_{i,j}\) for every pair of 5-tuples \(X_i, Y_i\) to indicate whether \(X_i\) and \(X_j\) are disjoint 5-holes, one can formulate an instance in \(\Theta(n^{10})\) variables with \(\Theta(n^{15})\) constraints. However, since this formulation is quite space consuming, a more compact formulation might be of interest.

**Interior-disjoint Holes:** Besides disjoint holes, also the variant of interior-disjoint holes has been investigated intensively by various groups of researchers (see e.g. [DHKS03, SU07, CGH+15, BMS17, HU18]). Two holes \(X_1, X_2\) are called **interior-disjoint** if their respective convex hulls are interior-disjoint. Interior-disjoint holes are also called **compatible** in literature. Note that a pair of interior-disjoint holes can share up to two vertices.

In a recent article, Hosono and Urabe [HU18] summarized the current status and presented some new results. By slightly adapting the SAT model from Section 6, we managed to show that every set of 15 points contains two interior-disjoint 5-holes. Moreover, this bound is best possible because, for example, the set of 14 points depicted in Figure 4 contains no two interior-disjoint 5-holes. This further improves Theorem 3 from [HU18].

Table 4 summarizes the best possible bounds for two interior-disjoint holes. We remark that, analogously to Section 3, one could further improve the bounds for three interior-disjoint holes.

\(^{10}\) In our experiments, pycosat wrote a comment “%RUPD32 ...” as first line in the RUP file. This line had to be removed manually to make the file parsable for drat-trim.
Figure 4: A set of 14 points with no two interior-disjoint 5-holes.

|   | 3 | 4 | 5 |
|---|---|---|---|
| 3 | 4 | 5 | 10 |
| 4 | 7 | 10 |
| 5 | 15* |

Table 4: Best possible bounds on the minimum number of points such that every set of that many points contains two interior-disjoint holes of sizes $k_1$ and $k_2$. The entry marked with star (*) is new.

To be more specific on the changes of the SAT model for this variant: we slightly relaxed the constraints “(8) Forbid disjoint 5-holes” so that each of the two points $a$ and $b$, which span a separating line $\ell$, can be contained in holes from both sides. The program creating the SAT instance is also available on our website [Sch].

Classical Erdős–Szekeres: The computation time for the computer assisted proof by Szekeres and Peters [SP06] for $g(6) = 17$ was about 1500 hours. By slightly adapting the model from Section 6 we have been able to confirm $g(6) = 17$ using glucose and drat-trim with about one hour of computation time. To be more specific with the adaption of the model from Section 6:

- The constraints “(6) 3-Holes” are removed.
- The constraints “(7) 5-Holes” are adapted to “(7*) 6-Gons” simply by testing 6-tuples instead of 5-tuples and by dropping the requirement that “triples form 3-holes”.

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• The constraints “(8) Forbid disjoint 5-holes” are removed.

Also this program is available on our website [Sch].

For determining the exact value of $g(7)$, however, further ideas or more advanced SAT solvers seem to be required.

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