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1. Introduction

In this paper we study the $\sigma_2$ Yamabe problem on conic four-spheres. First we fix our notations. For a closed Riemannian manifold $(M^n, g)$, let Ric, $R$ and $A$ be the corresponding Ricci curvature, Scalar curvature, and Schouten tensor respectively. That means,

$$A = \frac{1}{n-2}(\text{Ric} - \frac{R}{2(n-1)}g),$$

$$R = g^{ij}\text{Ric}_{ij}.$$  

Let $\{\lambda(A)\}_{i=1}^n$ be the set of eigenvalues of $A$ with respect to $g$. Define $\sigma_k(g^{-1}A_g)$ to be the $k$-th symmetric function of $\{\lambda(A)\}$, that is,

$$\sigma_k(g^{-1}A_g) = \sum_{1\leq i_1 < \cdots < i_k \leq n} \lambda_{i_1} \lambda_{i_2} \cdots \lambda_{i_k}.$$  

We note that $\sigma_1(\lambda(A)) = \text{Tr} A = \frac{1}{2(n-1)}R$. When necessary, we use, for example, $R_g$ to denote $R$, when the metric needs to be specified. We define the smooth conformal metric class $[g] = \{g_u = e^{2u}g_0, u \in C^\infty(M)\}$.

The classical Yamabe problem is to look for constant scalar curvature metrics in a given conformal metric class. This was solved through works of Yamabe [Y], Trudinger [Tr], Aubin [A], and Schoen [S].

In [V1], Viaclovsky raised the $\sigma_k$ Yamabe problem, which is to look for constant $\sigma_k(g^{-1}A_g)$ curvature metric in a given conformal class. In particular, for $k \geq 2$, the corresponding fully nonlinear equation is the following

$$\sigma_k(\lambda(A_{e^{2u}g})) = \text{constant.} \hspace{1cm} (1.1)$$

To fully understand the relation between solutions of (1.1) and conformal geometry of $M$, a so-called positive cone condition is commonly required. We define

$$C_k^+ := \{[g], \exists g \in [g] \text{ s.t. } \sigma_1(A_g) > 0, \ldots, \sigma_k(A_g) > 0\}. $$

Note that when $[g] \in C_k^+$, equation (1.1) becomes a fully nonlinear elliptic partial differential equation when $k > 1$.

Of all cases, $\sigma_2$ Yamabe problem for four-manifold is of particular interest. From an analytical point of view, this problem is variational [V1] [BJ] [STW]. While from a geometric point of view, $\sigma_2(A)$ in four-manifold is connected with the Gauss-Bonnet-Chern integrand. For a closed four-manifold, we have the following

$$k_g := \int_M \sigma_2(A) dv_g = 2\pi^2\chi(M) - \frac{1}{16} \int_M |W|^2 dv_g,$$

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where $W$ is the Weyl tensor of $g$ and $\chi(M)$ is the Euler characteristic of $M$.

Chang-Gursky-Yang in [CGY1] proved that when four-manifold $M$ has a positive Yamabe constant and positive $k_g$, there exists a conformal metric $g_u = e^{2u}g$ such that $\sigma_2(A_g) > 0$. Furthermore, in [CGY2], they proved that under the same condition, for any prescribed function $f > 0$, there exists a smooth solution of (1.1) when $M$ is not conformally equivalent to the round sphere. When the manifold is conformally flat, Viaclovsky [V1] has shown that under natural growth conditions the solution is unique up to a translation. Gursky-Streets [GS] proved that solutions are unique unless the manifold is conformally equivalent to the standard round sphere. For the conformal class of standard round 4-sphere, see [CGY3, V1, LL1, LL2, CHY, CHY2].

Many related works further explore the relation between the geometry/topology of four-manifolds and solutions of $\sigma_2$ Yamabe problem or related PDEs. See, for example, [CGY1, CQY, GV3]. Also, many works are devoted to the general $\sigma_k$ Yamabe problem for $k \geq 2$. We refer readers to [GY0, GY1, GV2, GV3, GW1, GW2, GB, SC, L, LL1, LL2, LN, PVW, STW, TW, V1, V2, V3, V4, W1, SS, BG] and references therein.

At the same time, singular sets of locally conformally flat metrics with positive $\sigma_k$ curvature are widely studied. In [CHPY], Chang, Hang, and Yang proved that if $\Omega \subset S^n(n \geq 5)$ admits a complete, conformal metric $g$ with $\sigma_1(A_g) > c > 0$, $\sigma_2(A_g) \geq 0$, and $|R_g| + |\nabla gR|_g \leq c_0$, then $\dim(S^n \setminus \Omega) < (n - 4)/2$. This has been further generalized by González [G1] and Guan-Lin-Wang [GLW] to the case of $2 < k < n/2$. More related works can be seen in [G0, G2, L] and their references.

In this article, we concentrate on four-manifolds with isolated singularities. In particular, we concentrate on conic manifolds. For a closed manifold $M$ with a smooth background Riemannian metric $g_0$, a singular metric $g_1$ is said to have a conic singularity of order $\beta$ at a given point $p \in M$, if in a local coordinate centered at $p$, $g_1 = e^{2v(x, p)} g_0$, where $v$ is a locally bounded function and $d(x, p)$ is the distance function with respect to $g_0$.

To better describe our singularity, we define a conformal divisor

$$D = \sum_i^q \beta_i p_i,$$

where $M$ has $q$ distinct points $p_1, \ldots, p_q$, $q \in \mathbb{N}$, and $\beta_1, \ldots, \beta_q \in (-1, 0)$. A conic metric $g_1$ is said to represent the divisor $D$ with respect to the background metric $g_0$, if $g_1$ has conic singularity of order $\beta_i$ at $p_i$ and is smooth elsewhere. We also define the singular conformal metric class $[g_D]$ to be the collection of all such metrics. Note that $[g_D]$ is dependent on the triple $(M, g_0, D)$.

As an example, let us describe conic metrics on standard spheres. We may use the Euclidean model by the standard stereographic projection. Let $g_E$ be the standard Euclidean metric on $\mathbb{R}^n$. A conic metric $g_1 = e^{2u} g_E$ representing a conformal divisor $D = \sum_i^q \beta_i p_i$, where $p_i \in \mathbb{R}^n$, $i = 1, \ldots, q - 1$, $p_q = \infty$, and all $\beta_i \in (-1, 0)$, if and only if $u$ satisfies the following

- $u(x) = \beta_i \ln |x - p_i| + v_i(x) \text{ as } x \to p_i \text{ for } i = 1, \ldots, q - 1$;
- $u(x) = (-2 - \beta_q) \ln |x| + v_\infty(x) \text{ as } |x| \to \infty$,

where $v_i(x)$ and $v_\infty(x)$ are bounded in their respective neighborhoods.

Chang-Han-Yang [CHY] classified global radial solutions to $\sigma_k$ Yamabe problem. Especially, in the case $k = \frac{n}{2}$, solutions of (1.1) has two conic points with identical
cone angles, which we call American footballs. Li studied the radial symmetry of solutions in the punctured Euclidean space in [L], which, by our definition, is equivalent to the study on the conic sphere with two isolated singular points. These two results imply that a conical sphere with two singularities and a constant $\sigma_{n/2}$ curvature can only be a football.

Locally, a deep theorem of Han-Li-Teixeira [HLT] describes the behavior of the conformal factor $u$ near the singularity when the $\sigma_k$ curvature is constant.

**Theorem 1.** [HLT] Let $u(x)$ be a smooth solution of $\sigma_n^2(g^{-1}A_g) = c$ on $B_R \setminus \{0\}$ in the $\Gamma^+_{q\sigma}$ class, where $c$ is a positive constant and $n$ is even. Then there exists some constant $\beta$ with $-1 < \beta \leq 0$ and a $C^\alpha$ function $v(x)$ such that $u(x) = v(x) + \beta \log |x|$ and $v(0) = 0$.

When $k = 1$, the above theorem was first proved in [CGS], where Caffarelli-Gidas-Spruck used the radial average to approximate the solution. Many related works followed. In particular, Korevaar-Mazzeo-Pacard-Schoen [KMPS] have successfully applied the methods of Caffarelli-Gidas-Spruck [CGS] and Korevaar-Mazzeo-Pacard-Schoen [KMPS] to the $\sigma_k$ Yamabe problem ($k \geq 2$) and described the regularity of the singular solution near isolated singularity. Theorem [HLT] is a special case of their main results. One of our motivations for this work is to obtain a global description of solutions of $\sigma_k^2(g^{-1}A_g) = c$. As we can see later, Theorem [HLT] is also the starting point of our analysis.

Another motivation for our study is from the conclusion of conic surfaces by Troyanov. In his now classical work [T], Troyanov presented the following

**Definition 2.** Let $S$ be a conic 2-sphere with divisor $D = \sum_{i=1}^q \beta_i p_i$. Define $\chi(S, D) := 2 + \sum_{i=1}^q \beta_i$, where 2 is the Euler characteristic of the 2-sphere. Then

- $(S, D)$ is called subcritical if $0 < \chi(S, D) < \min\{2, 2 + 2 \min_{1 \leq i \leq q} \beta_i\}$;
- $(S, D)$ is called critical if $0 < \chi(S, D) = \min\{2, 2 + 2 \min_{1 \leq i \leq q} \beta_i\}$;
- $(S, D)$ is called supercritical if $\chi(S, D) > \min\{2, 2 + 2 \min_{1 \leq i \leq q} \beta_i\} > 0$.

Accordingly, Troyanov proved the existence of a unique solution to the conic Yamabe problem when $(S, D)$ is subcritical. Later Luo-Tian [LT] showed that the subcritical condition is both necessary and sufficient when $q \geq 3$. When $q = 2$, Chen-Li [CLT] proved that only the case $\beta_1 = \beta_2$ has the solution and the corresponding manifold is the football.

Note that Troyanov’s theory introduces the supercritical case, which does not exist in the smooth category. In some sense, a sphere is the only critical case when $M$ is smooth. Troyanov also studied surfaces of higher genus. See [T] for more details.

Our first result is a Gauss-Bonnet-Chern formula for conic spheres of general even dimension $n$ and constant positive $\sigma_m$ curvature, where $m = \frac{n}{2}$.

**Theorem 3.** Let $(S^n, D)$ be a conic sphere with standard round background metric $g_0$ and positive constant $\sigma_m$ curvature. We assume that $[g_D] \in C^+_m$. Then we have

$$
\frac{1}{|S_{n-1}|} \int_{S^n} \frac{m}{2m-1} \sigma_m(g^{-1}A_g) dv_g = 2 - \sum_{i=1}^q f(\beta_i),
$$

(1.2)
\[ f(\beta) = \frac{1}{2m-2} \sum_{k=0}^{m-1} C_{n-k} \beta^k |\beta|^{n-k-1}. \]

While \( \sigma_m \) is known as the Phaffian curvature for smooth locally conformally flat manifolds, equation (1.2) gives precise Gauss-Bonnet-Chern defect for each conic singular point. Note that when \( n = 2 \), equation (1.2) is the classical Gauss-Bonnet-Chern formula obtained by Troyanov [T].

For \( m = 2 \) and \( n = 4 \), equation (1.2) becomes

\[
\frac{1}{|S|} \int_{S^4} \sigma_2(g^{-1}A_g)dv_g = 2 - \sum_{j=1}^{q} \frac{\beta_j^3 + 3\beta_j^2}{2}.
\]

While it is tempting to use \( \chi(S^4, D) := \left( 2 - \sum_{i=1}^{q} \frac{\beta_i^3 + 3\beta_i^2}{2} \right) \) to classify conic 4-spheres as Troyanov has done in 2-dimension, the 4-dimension case is far more complicated. In order to obtain a proper definition for conic manifolds of dimension 4 or higher, we consider the case of conic 4-spheres with the standard round sphere background metric. While it is geometrically and topologically simple, it is often the most difficult case in terms of analysis, which is indicated by previous studies of the smooth case. Our next result is the following definition.

**Definition 4.** Let \((S^4, D, g_0)\) be a conic 4-sphere with the standard round background metric \(g_0\). For all \( j = 1, \ldots, q \), we denote \( \tilde{\beta}_j := \left( \sum_{1 \leq i \neq j \leq q} \beta_i^3 \right)^{1/3} \).

- We call \((S^4, D)\) subcritical for \( \sigma_2 \) Yamabe equation if for any \( j = 1, \ldots, q \)
  \( \frac{3}{8} \beta_j^2 (\beta_j + 2)^2 < \frac{3}{8} \beta_j^2 (\tilde{\beta}_j + 2)^2 + (\tilde{\beta}_j + 3 \beta_j^2 \left( \sum_{1 \leq i \neq j \leq q} \beta_i^2 \right) ) \).
- We call \((S^4, D)\) critical for \( \sigma_2 \) Yamabe equation if there exists a \( j \in \{1, \ldots, q\} \) such that
  \( \frac{3}{8} \beta_j^2 (\beta_j + 2)^2 = \frac{3}{8} \beta_j^2 (\tilde{\beta}_j + 2)^2 + (\tilde{\beta}_j + 3 \beta_j^2 \left( \sum_{1 \leq i \neq j \leq q} \beta_i^2 \right) ) \).
- Otherwise, we call \((S^4, D)\) supercritical for \( \sigma_2 \) Yamabe equation, which means that there exists a \( j \in \{1, \ldots, q\} \),
  \( \frac{3}{8} \beta_j^2 (\beta_j + 2)^2 > \frac{3}{8} \beta_j^2 (\tilde{\beta}_j + 2)^2 + (\tilde{\beta}_j + 3 \beta_j^2 \left( \sum_{1 \leq i \neq j \leq q} \beta_i^2 \right) ) \).

We remark that Definition 3 is purely numerical and independent of the geometric configuration of points \( \{p_i\} \).

Finally, we present our main theorem.

**Theorem 5.** Let \((S^4, D, g_0)\) be defined as above. Assume that \([g_D] \in C^+ \). If \((S^4, D)\) is supercritical, there does not exist a conformal metric \(g \in [g_D] \) with constant \( \sigma_2 \) curvature. If \((S^4, D)\) is critical with constant \( \sigma_2 \) curvature, then it is a football as defined in [CHY].

Theorem 5 justifies Definition 3 while both should be considered as a 4-dimensional generalization of Troyanov’s Definition 2 and corresponding results in [T, L, LT, CL1]. As noted earlier, when the metric is smooth, i.e., \( D \) is empty, Theorem 5 was first proved by Viaclovsky [V1]; when \( D \) contains 1 or 2 points, Theorem 5 was first proved by Li [L]. Our approach is different from those of earlier works and may be considered as an alternative proof. For general \( D \), Theorem 5 is new.

Our method to prove this theorem is to do a careful analysis of geometric quantities on level sets of the conformal factor \( u \). Such an approach was first used in
the 2-dimensional case by [B, ChLin, CL, CL1, BDM, BM, BD] and later used by
the first named author and Lai [FL1, FL2, FL3] to obtain various sharp geometric
bounds. However, $\sigma_2$ Yamabe equation is now a fully nonlinear equation of
Monge-Ampère type. New quantities and more importantly, new ideas to treat the
non-linearity are needed. For dimension 4, we find a surprising integrable quantity
which leads to a new monotonicity formula that is key to our proof.

We would like to remark that Troyanov’s theory on conic surfaces may be inter-
preted both in conformal geometry and Kähler geometry. In particular, Definition
2 is identical to stability conditions on singular algebraic curves. See [CHY] for de-
tails. The recently settled Yau-Tian-Donaldson conjecture connects the existence of
the Kähler-Einstein metric to the stability conditions [CDS1, CDS2, CDS3, T1, T2].
The study of conic metrics on Kähler manifolds with singularities over algebraic
divisors is crucial in the solution of the Yau-Tian-Donaldson conjecture. It is our
intention to develop a parallel theory in terms of conformal geometry. See also
Fang-Ma [FM], where Branson’s Q curvature is considered in dimension 4.

In a subsequent work, we would like to address the $\sigma_2$ Yamabe problem for sub-
critical conic 4-spheres. As we have only treated locally conformally flat manifolds
in this paper, we hope that in future works, we can generalize the correspond-
ing definitions and results to the general 4-manifolds. Higher dimensional case is
another direction that we would like to explore.

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We organize the paper as follows. In Section 2, we prove the Gauss-Bonnet-
Chern formula for conic conformally flat manifold. In Section 3, we introduce
the quantities related to the level set and obtain some fundamental equalities. In
Section 4, we establish a key inequality and then prove our main theorem.

2. GAUSS-BONNET-CHERN FORMULA

In this Section, we prove a Gauss-Bonnet-Chern formula for conic spheres. Here we
consider any even natural number $n$. Let $g_E$ be the standard Euclidean met-
ric on $\mathbb{R}^n$. It is known that the standard metric on $S^n$ can be represented as
$(S^n, 4/(1+|x|^4)g_E)$, where $x$ is the coordinate function of $\mathbb{R}^n$ with $|x|$ being its Eu-
clidean norm. For future use, we note that for $n = 2m$,

$$|S_n| = |S_{n-1}| \frac{2^{n-1}(m-1)!^2}{(n-1)!}.$$

Let $(S^n, g_u, D)$ be a conical sphere as defined in the Introduction. We assume
that $[g_D] \in C_m^+$, where $m = \frac{n}{2}$. Note that for $g = g_u = e^{2u}g_E$, we have the following:

$$\text{Ric}_{ij} = -(n-2)u_{ij} + (n-2)u_iu_j + (-\Delta u - (n-2)|\nabla u|^2)g_{ij},$$

$$R = e^{-2u}(-2(n-1)\Delta u - (n-1)(n-2)|\nabla u|^2),$$

and

$$A_{ij} = -u_{ij} + u_iu_j - \frac{\nabla u|^2}{2}\delta_{ij}.$$

Here all derivatives are with respect to the Euclidean metric $g_E$ and $\delta_{ij}$ is the
Kronecker delta function.
Similarly, we have the following:

$$\sigma_k(g^{-1}A_g) = \frac{1}{k!} \sum_{i_1, \ldots, i_k} \delta(i_1 \ldots i_k) A_{i_1}^i \ldots A_{i_k}^i,$$

and define

$$T_l(g^{-1}A)^i_j = \frac{1}{n} \sum_{i_1, \ldots, i_l, j_1, \ldots, j_l} \delta(i_1 \ldots i_l) \delta(j_1 \ldots j_l) A_{i_1}^{i_1} \ldots A_{i_l}^{i_l},$$

where $A_{i_1}^{i_1} = g^{1k_1}A_{k_1i_1}$ and $\delta$ is the Kronecker delta function.

In this section, we assume that $(S^n, g_u, D)$ satisfies the following curvature equation

$$\sigma_m(g^{-1}A_g) = \left( \frac{n}{m} \right) \left( \frac{1}{2} \right)^m. \quad (2.1)$$

Notice that the right hand side is the corresponding value of the standard sphere $(S^n, \frac{2}{n+2} |dx|^2)$.

It is well known that in the conformally flat case, the Gauss-Bonnet-Chern integral is just $\sigma_m$ up to a constant. In addition, it has a special divergence structure as follows (see also [H]):

**Lemma 6.** On a conformally flat manifold $(M^n, g)$,

$$\sigma_m(g^{-1}A_g) = - \frac{1}{m} \text{div}_g \left\{ \sum_{j=1}^m T_{m-j}(g^{-1}A_g)^a_b |\nabla_g u|_g^{2(j-1)} \nabla^b u \right\}.$$

Especially, for $n = 4$, $m = 2$, we have

$$\sigma_2(g^{-1}A_g) = - \frac{1}{2} \partial_t (\nabla u \delta_{ij} + u_{ij} - u_i u_j)u_j),$$

where all derivatives are with respect to the Euclidean metric of $\mathbb{R}^4$.

Before we establish the Gauss-Bonnet-Chern formula, we first need to know the asymptotic behavior of conformal factor $u$ near singularities.

**Lemma 7.** Assume $(S^n, g_u, D)$ has positive constant $\sigma_m$ curvature; then we have, for $1 \leq l \leq p - 1$, as $|x - p_l| \to 0$,

$$u_l(x) = \frac{\beta_l}{|x - p_l|^2} (x_l - p_{l,i}) + o\left( \frac{1}{|x - p_l|^2} \right), \quad (2.2)$$

$$u_{ij}(x) = \beta_l \frac{\delta_{ij}}{|x - p_l|^2} - 2\beta_l \frac{(x_l - p_{l,i})(x_j - p_{l,j})}{|x - p_l|^4} + o\left( \frac{1}{|x - p_l|^2} \right), \quad (2.3)$$

$$H(x) = \frac{3}{|x - p_l|^2} + o\left( \frac{1}{|x - p_l|^2} \right), \quad (2.4)$$

where $H(x)$ is the mean curvature of level set $\{ x, u(x) = t \}$ near $p_l$ and $t$ is sufficiently large.

**Proof.** These are direct consequences of the main theorem of Han-Li-Teixeira [HLT]. By Theorem 1 (Theorem 1’ or Corollary 1 in [HLT]), we have for some small $R_0 > 0$,

$$|x - p_l|^k (u(x) - \beta_l \log |x - p_l|) \in C^{k,\alpha}(B_{R_0}(p_l)).$$
Near \( p_i \), we then have
\[
\partial_i(|x - p_i|(u(x) - \beta_i \log |x - p_i|)) = \frac{x_i - p_{i,i}}{|x - p_i|}(u(x) - \beta_i \log |x - p_i|) + |x - p_i|\partial_{x_i}(u(x) - \beta_i \log |x - p_i|) = o(1), \quad |x - p_i| \to 0.
\]
Thus, we get
\[
\begin{aligned}
  u_i(x) &= \frac{\beta_i}{|x - p_i|^2}(x_i - p_{i,i}) + o\left(\frac{1}{|x - p_i|}\right), \quad |x - p_i| \to 0. \\
\end{aligned}
\tag{2.5}
\]
A similar computation shows that as \( |x - p_i| \to 0 \),
\[
  u_{ij}(x) = \beta_{ij} \frac{\delta_{ij}}{|x - p_i|^2} - 2\beta_i \frac{(x_i - p_{i,i})(x_j - p_{j,j})}{|x - p_i|^4} + o\left(\frac{1}{|x - p_i|^2}\right). \tag{2.6}
\]
Furthermore, the mean curvature of level set \( \{x : u(x) = t\} \) for a sufficiently large \( t \) can be computed as below
\[
H(x) = -\text{div}\left(\frac{\nabla u}{|\nabla u|}\right) = \frac{3}{|x - p_i|^2} + o\left(\frac{1}{|x - p_i|}\right) \quad \text{as} \quad |x - p_i| \to 0, \tag{2.7}
\]
for the singular point \( p_i, \ l = 1, \cdots, q - 1 \). We have finished the proof. \( \Box \)

The main result of this section is the following Gauss-Bonnet-Chern formula with defects.

**Theorem 8.** Conditions as in Lemma 4. Then we have
\[
\frac{1}{|S^n_{\lambda-1}|} \int_{S^n} \frac{m}{2^{m-1}} \sigma_m(g^{-1}A_g)dv_g = 2 - \sum_{l=1}^{q} f(\beta_l), \tag{2.8}
\]
where
\[
f(\beta) = \frac{1}{2^{n-2}} \sum_{k=0}^{m-1} \binom{n-1}{k} (2 + \beta)^k |\beta|^{n-k-1}.
\]

**Proof.** By Lemma
\[
\int_{S^n \setminus \cup_i(p_i)} m\sigma_m(g^{-1}A_g)dv_g
\]
\[
= \lim_{\varepsilon \to 0} \sum_{i=1}^{q-1} \int_{\{|x - p_i| < \varepsilon\}} m \sum_{k=1}^{m} T_{m-k}(A_g^{q} |\nabla u|^{2(k-1)}) (2k-1) u_{k} d\sigma ds
\]
\[
- \lim_{R \to \infty} \int_{\{|x| = R\}} m \sum_{k=1}^{m} T_{m-k}(A_g^{q} |\nabla u|^{2(k-1)}) (2k-1) u_{k} d\sigma ds
\]
\[
:= \lim_{\varepsilon \to 0} \sum_{l=1}^{q-1} I_{l} - \lim_{R \to \infty} I_{R},
\]
where
\[
I_{\varepsilon} = \int_{\{|x - p_i| < \varepsilon\}} m \sum_{k=1}^{m} T_{m-k}(A_g^{q} |\nabla u|^{2(k-1)}) (2k-1) u_{k} d\sigma ds.
\]
Thus we get

\[ I_R = \int_{|x|=R} \sum_{k=1}^{m} T_{m-k}(A)^a_b |\nabla u|^2 u_b^a ds. \]

For each \( l \), on the hypersurface \( \{ x, |x-p_l|=\varepsilon \} \), \( \nu = (\frac{x_1-p_{l,1}}{|x-p_l|}, \ldots, \frac{x_n-p_{l,n}}{|x-p_l|}) \). By Theorem 1 there exists a \( C^0 \) function \( v_l \) satisfying \( v_l(p_l) = 0 \) and \( u(x) - \beta l \log |x-p_l| = v_l(x) \). By Lemma 7 we have

\[ -u_{ij} + u_iu_j - \frac{|\nabla u|^2}{2} \delta_{ij} = (-\beta l - \frac{\beta^2}{2}) \frac{\delta_{ij}}{|x-p_l|^2} + (2\beta l + \beta^2) \frac{(x_i-p_{l,i})(x_j-p_{l,j})}{|x-p_l|^4} + o(1) \frac{1}{|x-p_l|^2}. \]

Thus we get

\[
\sum_{a,b=1}^{n} T_{m-k}(A)^a_b |\nabla u|^2 u_b^a \\
= \frac{1}{2k-1(m-k)!} \sum_{i_1 \cdots i_{m-k}, a} \delta(i_j \cdots i_{m-k} a) \\
\quad \cdot \left( (-\beta l - \frac{\beta^2}{2}) \delta_{i_1j_1} \frac{1}{|x-p_l|^2} + (2\beta l + \beta^2) \frac{(x_{i_1}-p_{l,i_1})(x_{j_1}-p_{l,j_1})}{|x-p_l|^4} + o(1) \right) \\
\quad \cdot \left( (-\beta l - \frac{\beta^2}{2}) \delta_{i_{m-k}j_{m-k}} \frac{1}{|x-p_l|^2} + (2\beta l + \beta^2) \frac{(x_{i_{m-k}}-p_{l,i_{m-k}})(x_{j_{m-k}}-p_{l,j_{m-k}})}{|x-p_l|^4} + o(1) \right) \\
\quad \cdot \left( \delta_{i_{m-k}j_{m-k}} \frac{1}{|x-p_l|^3} + \frac{1}{|x-p_l|^2} + o(1) \right) \\
= \frac{1}{2k-1(m-k)!} \sum_{i_1 \cdots i_{m-k}, a, b} \delta(i_1 \cdots i_{m-k} a) (-\beta l - \frac{\beta^2}{2}) \delta_{i_1j_1} \right) \]
Therefore,

\[
\lim_{\epsilon \to 0} \sum_{l=1}^{q-1} I_l^l = \lim_{\epsilon \to 0} \sum_{l=1}^{q-1} \int_{|x-p_l| = \epsilon} \sum_{k=1}^{m} \frac{T_{m-k}(A)_{k}[\nabla u]^2(k-1)}{2^{k-1}} u_b v_{\alpha} ds \\
= \lim_{\epsilon \to 0} \sum_{l=1}^{q-1} \sum_{k=1}^{m} \left(-\frac{\beta^2}{2}\right) m-k \beta^2(k-1)+1 (n-1)(n-2) \cdots (n-m+k) \frac{1}{2^{k-1}(m-k)!} o(\frac{1}{|x|^n-1})|S_{n-1}| \\
= \sum_{l=1}^{q-1} \sum_{k=1}^{m} \left(-\frac{\beta^2}{2}\right) m-k \beta^2(k-1)+1 (n-1)(n-2) \cdots (n-m+k) \frac{1}{2^{k-1}(m-k)!} |S_{n-1}| \\
= \sum_{l=1}^{q-1} \sum_{k=1}^{m} \left(-\frac{\beta^2}{2}\right) m-k \beta^2(k-1)+1 |S_{n-1}| \\
= |S_{n-1}| \sum_{l=1}^{q-1} (-2^{m-1} f(\beta_l)).
\]

As \(u(x) = (-2 - \beta_l) \log |x| + v_{\infty}(x)\), as \(|x| \to \infty\), where \(v_{\infty}\) is locally bounded, by a similar computation which we will omit here, we get that

\[
\lim_{R \to \infty} I_R = |S_{n-1}| (-2^{m-1} f(-2 - \beta_l)).
\]

Finally, to finish the proof of Theorem 8 we note the following simple fact

\[2 = f(-2 - \beta) + f(\beta).\]

Equality (2.8) is thus proved.

Remark 9. It is obvious that the Gauss-Bonnet-Chern formula for singular manifolds is highly dependent on the asymptotic behavior of the metric near the singularity. In our case, the partial differential equation is required to obtain the needed local regularity to compute the Gauss-Bonnet-Chern defect near each singularity.

3. Level set and Related functions

In this section, we study global solutions of (2.1) via the level set method. In [HIT], the authors proved the radial average of the solution is a good approximation to the solution and satisfies an ODE, which is an approximation to the ODE satisfied by a radial solution to (2.1). In this paper, we will instead use some quantities related to the level set and obtain an ordinary differential inequality, which is inspired by [FL1] [FL2] [FL3]. As the \(\sigma_2\) equation is fully nonlinear and quantities used in dimension 2 are not sufficient, we introduce some new functions constructed by curvatures of the level set and gradient of the solution.

We begin with the following definition about the level sets of the conformal factor \(u\):

\[
L(t) = \{ x : u = t \} \subset M, \\
S(t) = \{ x : u \geq t \} \subset M.
\]
We choose a local coordinate system related to our setup. For any point \( x \in M \), we choose an orthogonal coordinate \( x_i \) for \( i = 1, \ldots, n \) such that \( \frac{\partial}{\partial x_i} \) is the outer normal vector of \( L(t) \). We write \( u_i = \frac{\partial u}{\partial x_i} \), and \( \nabla_{ij} u \) or \( u_{ij} \) the Hessian of function \( u \) with respect to the Euclidean metric. Let \( \nabla^L_{ab} u \) be the Hessian of \( u \) with respect to the induced metric on \( L(t) \) and \( u_{n\alpha} = \nabla^L_{\alpha} u_n \). In the following, \( \alpha, \beta \) range from 1 to \( n - 1 \). Notice that \( \nabla^L_{\alpha} u = 0 \) on \( L(t) \).

In the following, \( n = 4 \). Without confusion, we sometimes write \( n = 4 \).

Let \( h_{ab} \) be the second fundamental form of the level set \( L(t) \) with respect to the outward normal vector. We have the following Gauss-Weingarten formula

\[
\nabla_{ab} u = \nabla^L_{ab} u + h_{ab} u_n,
\]

\[
\nabla_{an} u = \nabla^L_{a} u_n - h_{ab} u_b.
\]

We may now describe the Schouten tensor using our choice of coordinates. In particular, \( A_g \) can be written as

\[
A_g = \begin{pmatrix}
    h_{\alpha\beta} |\nabla u|^2 - \frac{|\nabla u|^2}{2} \delta_{\alpha\beta} & -\nabla_{41} u & -\nabla_{42} u & -\nabla_{43} u & -\nabla_{44} u + \frac{|\nabla u|^2}{2} \\
    -\nabla_{41} u & -\nabla_{42} u & -\nabla_{43} u & -\nabla_{44} u + \frac{|\nabla u|^2}{2} \\
    -\nabla_{42} u & -\nabla_{43} u & -\nabla_{44} u + \frac{|\nabla u|^2}{2} \\
    -\nabla_{43} u & -\nabla_{44} u + \frac{|\nabla u|^2}{2} \\
    -\nabla_{44} u + \frac{|\nabla u|^2}{2}
\end{pmatrix}.
\]

For future use, we also define

\[
\tilde{A} := (h_{\alpha\beta} |\nabla u|^2 - \frac{|\nabla u|^2}{2} \delta_{\alpha\beta}).
\]

For simplicity, we use \( \int_{S(t)} \), \( \int_{S(t)} \) to represent \( \frac{1}{|S(t)|} \int_{L(t)} \), \( \frac{1}{|S(t)|} \int_{S(t)} \), respectively. We define

\[
A(t) = \frac{1}{|S(t)|} \int_{S(t)} e^{4u} dx,
\]

\[
B(t) = \frac{1}{|S(t)|} \int_{S(t)} dx,
\]

\[
C(t) = e^{4t} B(t),
\]

and

\[
z(t) = -\left( \frac{1}{|S(t)|} \int_{S(t)} |\nabla u|^3 \right)^{1/3}.
\]

\[
\Sigma_0(t) = \frac{1}{|S(t)|} \int_{L(t)} |\nabla u|^3 \ dl = -z^3,
\]

\[
\Sigma_1(t) = \frac{1}{|S(t)|} \int_{L(t)} \{ 2H |\nabla u|^2 - 3|\nabla u|^3 \} \ dl,
\]

and finally,

\[
D(t) = \frac{1}{4} (\Sigma_0(t) + \Sigma_1(t)).
\]

Here \( H \) is the mean curvature of the level set \( L(t) \), and \( dl \) is the induced 3-dimensional measure on \( L(t) \). When no confusion arises, we may omit \( dl \).
Lemma 11. We have the following:

\[ D(+\infty) := \lim_{t \to +\infty} D(t) = \frac{1}{4} \sum_{i=1}^{q-1} (|\beta_i|^3 + 3(-2\beta_i - \beta_i^2)|\beta_i|) \]
\[ = \frac{3}{2} \sum_{i=1}^{q-1} |\beta_i|^2 - \frac{1}{2} \sum_{i=1}^{q-1} |\beta_i|^3, \quad \text{(3.4)} \]
\[ D(-\infty) := \lim_{t \to -\infty} D(t) = \frac{3}{2} (2 + \beta_\infty)^2 - \frac{1}{2} (2 + \beta_\infty)^3, \quad \text{(3.5)} \]
\[ \lim_{t \to +\infty} z(t) = \left( \sum_{i=1}^{q-1} \beta_i^3 \right)^{1/3}, \quad \lim_{t \to -\infty} z(t) = -2 - \beta_\infty, \quad \text{(3.6)} \]
\[ \lim_{t \to +\infty} C(t) = \lim_{t \to -\infty} \sum_{i=1}^{q-1} e^{4t(1+\beta_i)} = 0, \quad \lim_{t \to -\infty} C(t) = 0. \quad \text{(3.7)} \]

For future use, we prove the following

Lemma 11. Notations as above, we have

\[ C' = A' + 4C, \quad \text{(3.8)} \]
\[ \frac{1}{3} \frac{d}{dt} (z^4) = \int_{L(t)} \frac{H}{3} |\nabla u| - \nabla_{nn} u |\nabla u|, \quad \text{(3.9)} \]

Proof. As \( A(t) \) and \( B(t) \) are non-increasing with respect to \( t \), \( A'(t) \) and \( B'(t) \) exist almost everywhere. Furthermore, we can deduce that \( A(t) \) and \( B(t) \) are absolutely continuous in any finite interval \([t_1, t_2] \) as in [FL2]. From the co-area formula (see Lemma 2.3 in [BZ]), we have, for given \( t_1 < t_2 \),

\[ B(t_1) - B(t_2) = |C \cap u^{-1}((t_1, t_2))| + \int_{t_1}^{t_2} \int_{L(s)} |\nabla u|^{-1} d\mathcal{H}^1 ds, \]

where \( C = \{ z | \nabla u(z) = 0 \} \). For any \( x = (x_1, x_2, x_3, x_4) \in C \cap u^{-1}((t_1, t_2)) \), we may assume, without loss of generality, that \( u_{x_1} \neq 0 \) because \( g_D \in \mathcal{C}_2^+ \) and \(-\Delta u > |\nabla u|^2 \geq 0 \). By the implicit function theorem, there exists some \( r_0 > 0 \) and \( g : (x_2, x_3, x_4) \to \mathbb{R} \) such that \( \{ u_{x_1}(x) = 0 \} \cap B_{r_0}(x) \) is the graph of the function \( x_3 = g(x_2, x_3, x_4) \). So \( |C \cap B_{r_0}(x)| \leq \{ y| u_{x_1}(y) = 0 \} \cap B_{r_0}(x) \leq 0 \). Therefore \( B(t) \) is absolutely continuous on any finite interval \([t_1, t_2] \) and so are \( A(t) \) and \( C(t) \). The computations below are done when \( A'(t), B'(t), C'(t) \) exist.

\[ B'(t) = \frac{1}{|S_3|} \frac{\partial}{\partial t} \int_{t}^{+\infty} \int_{L(s)} \frac{1}{|\nabla u|} dlds, \]
\[ = -\frac{1}{|S_3|} \int_{L(t)} \frac{1}{|\nabla u|} dl, \]
and

\[ A'(t) = \frac{1}{|S^3|} \frac{\partial}{\partial t} \int_{S^3} \int_{t}^{+\infty} e^{4u} \frac{dd}{dds} = e^{4t}B'(t). \]

Therefore, we have

\[ C'(t) = 4e^{4t}B(t) + e^{4t}B'(t) = 4C(t) + A'(t). \]

By the divergence theorem, co-area formula, and (3.1), we obtain

\[ \frac{d}{dt} \int_{L(t)} |\nabla u|^3 = \frac{d}{dt} \int_{S(t) \setminus S(t_0)} -\text{div}(\nabla u |\nabla u|^2) = \int_{L(t)} \text{div}(\nabla u |\nabla u|^2) \]

\[ = \int_{L(t)} \sum_{\alpha=1}^{n-1} \frac{\nabla_{\alpha\alpha} u |\nabla u|^2}{|\nabla u|} - \sum_{\alpha=1}^{n-1} \frac{\nabla_{\alpha} u \nabla_{\alpha} |\nabla u|^2}{|\nabla u|} + \frac{\nabla_{nn} u |\nabla u|^2}{|\nabla u|} + \frac{\nabla_{n} u \nabla_{n} |\nabla u|^2}{|\nabla u|} \]

\[ = \int_{L(t)} -H|\nabla u|^2 + 3\nabla_{nn} u |\nabla u|, \]

where \( t_0 \) is a fixed real number and \( u_0 = -|\nabla u| \) on \( L(t) \). We thus get (3.9). \( \square \)

Finally, we will use the divergence structure of \( \sigma_m \) and equation (2.1) to establish the relationship between \( A \) and \( D \).

**Theorem 12.** Let \( u(x) \) be a smooth solution to (2.1) on \( \mathbb{R}^4 \setminus \{p_1, \cdots, p_{q-1}, p_{\infty}\} \) in the \( C^+_2 \) class,

\[ \frac{1}{|S^3|} \int_{\{u \geq t\}} \sigma_2(A) \text{dvol} = D(t) - D(+\infty), \]

\[ A(t) = \frac{2}{3}(D(t) - D(+\infty)), \]

\[ A(t) = \frac{1}{6}(\Sigma_0 + \Sigma_1) - \frac{2}{3}D(+\infty), \]

\[ A'(t) = \frac{2}{3}D'(t) = \frac{1}{6}(\Sigma_0' + \Sigma_1'). \]

**Proof.** By Lemma 6 and (3.4),

\[ \int_{\{u \geq t\}} 2\sigma_2(g^{-1}A_g) \text{dvol} = \int_{\{u = t\}} H|\nabla u|^2 - \int_{\{u = t\}} |\nabla u|^3 - 2D(+\infty). \]

So we can get (3.10)(3.13) by equation (2.1). \( \square \)
4. Proof of main theorems

In this section, we prove our main theorems.

First, we state our key estimate.

Lemma 13. Let $u(x)$ be a smooth solution to equation (2.1) on $\mathbb{R}^4 \{p_1, \ldots, p_q\}$. Assume that in the $[g_D] \in C^+_\infty$. We have

$$z' \int_{L(t)} \sigma_1(A)|\nabla u|dl(zA')^2 \geq \frac{3}{2} (4C(t))^3. $$

Proof. By (3.3), we obtain

$$\sigma_2(A) = \sigma_2(\tilde{A}) + (-\nabla_{nn} u + \frac{[u]^2}{2})\sigma_1(\tilde{A}) - \sum_{a=1}^{3} (\nabla_{an} u)^2$$

$$\leq \sigma_2(\tilde{A}) + (-\nabla_{nn} u + \frac{[u]^2}{2})\sigma_1(\tilde{A}). \quad (4.1)$$

Here $n = 4$.

Especially, because $(\sum_{i=1}^{n-1} \lambda_i)^2 = \sum_{i=1}^{n-1} \lambda_i^2 + 2 \sum_{i<j} \lambda_i \lambda_j \geq \frac{2n-2}{n-1} \sigma_2$ holds even if $\tilde{A} \notin \Gamma_1^+$, we have

$$\sigma_2(\tilde{A}) \leq \frac{\sigma_2^2(\tilde{A})}{3}, \quad (4.2)$$

for any $\tilde{A}$.

Then by (4.1) and (4.2),

$$\sigma_2(A) \leq \frac{\sigma_1(\tilde{A}) \sigma_1(\tilde{A})}{3} + (-\nabla_{nn} u + \frac{[u]^2}{2})\sigma_1(\tilde{A})$$

$$\leq \sigma_1(\tilde{A})(\frac{H}{3} |\nabla u| - \nabla_{nn} u).$$

As we have

$$\sigma_2(A) = \frac{3}{2} e^{4u},$$

we can then derive the following using Cauchy inequality

$$\int_{L(t)} \sigma_1(\tilde{A})|\nabla u|dl \int_{L(t)} \left(\frac{H}{3} |\nabla u| - \nabla_{nn} u\right) |\nabla u|$$

$$\geq \left(\int_{L(t)} \sigma_1(\tilde{A})|\nabla u|\left(\frac{H}{3} |\nabla u| - \nabla_{nn} u\right) |\nabla u|\right)^2$$

$$\geq \left(\int_{L(t)} \left(\frac{3}{2} e^{4u} |\nabla u|^2\right)^2\right)^2$$

$$\geq \frac{3}{2} e^{4t} \left(\int_{L(t)} |\nabla u|^2\right)^2. \quad (4.3)$$
Combine equality (1.3) with Lemma 11, we get that
\[
(A')^2 \int_{L(t)} \sigma_1(\tilde{A}) |\nabla u| \int_{L(t)} \left( \frac{H}{3} |\nabla u| - \nabla_{nn} u |\nabla u| \right) dl 
\geq \frac{3}{2} \left( \int_{L(t)} |\nabla u| \right)^2 e^4 t \cdot e^8 t \left( \int_{L(t)} \frac{1}{|\nabla u|} \right)^2 
= \frac{3}{2} e^{12t} \int_{L(t)} (|\nabla u|)^2 \left( \int_{L(t)} \frac{1}{|\nabla u|} \right)^2 
\geq \frac{3}{2} e^{12t} |L(t)|^4 \frac{1}{|S_3|^4} 
\geq \frac{3}{2} e^{12t} B(t)^3 4^4 |B_1| \frac{1}{|S_3|^4} 
= \frac{3}{2} (4C(t))^3, 
\]
where the third inequality is due to Cauchy inequality and the fourth inequality holds because of the isoperimetric inequality. We note that $|B_1| = \frac{1}{|S_3|}$. \qed

We remark that if $u$ is the radial solution to equation (2.1), all the above inequalities are equality.

We then prove the following

**Theorem 14.** Let $u(x)$ be a smooth solution to (2.1) on $\mathbb{R}^4 \setminus \{p_1, \cdots, p_{q-1}, p_\infty\}$ in the $C^2_+ \cap C^3$ class,
\[
C' \leq \frac{2}{3} D' + \frac{4}{9} (zD)' + \frac{1}{36} (z^4)'.
\]
In particular, the quantity
\[
M(t) = \frac{2}{3} D(t) + \frac{4}{9} D(t) z(t) + \frac{1}{36} z^4(t) - C(t)
\]
is monotonously increasing with respect to $t$.

**Proof.** By Lemma 13 and the inequality of arithmetic and geometric means, we have
\[
4C \leq \frac{1}{3} (2zA' + \frac{2}{3} z' \int_{L(t)} \sigma_1(\tilde{A}) |\nabla u|), \tag{4.4}
\]
where the identity holds if and only if $zA' = \frac{2}{3} z' \int_{L(t)} \sigma_1(\tilde{A}) |\nabla u|$.

Using fact that $C' = 4C + A'$, (4.3) then becomes
\[
C' \leq A' + \frac{1}{3} (2zA' + \frac{2}{3} z' \int_{L(t)} \sigma_1(\tilde{A}) |\nabla u|) 
\leq \frac{1}{3} (2D' + \frac{4}{3} z(D'(t)) + \frac{1}{3} z' \Sigma_1) 
= \frac{2}{3} D' + \frac{1}{3} (zD' - \frac{4}{3} z D + \frac{1}{3} z' \Sigma_1) 
= \frac{2}{3} D' + \frac{1}{3} (zD') - \frac{1}{3} z' (\Sigma_0 + \Sigma_1) + \frac{1}{3} z' \Sigma_1 
= \frac{2}{3} D' + \frac{1}{3} (4zD') + \frac{1}{3} z^3 \Sigma_1).
The key estimate is thus established. □

Finally, we are ready to prove our main result, Theorem 5.

Proof. By Theorem 14, we have

\[ C(\infty) - C(-\infty) \leq \frac{2}{3}(D(\infty) - D(-\infty)) + \frac{4}{9}(z(\infty)D(\infty) - z(-\infty)D(-\infty)) + \frac{1}{36}(z^4(\infty) - z^4(-\infty)). \]

Considering (3.7), (3.4), (3.5) and (3.6), we compute to get

\[ \frac{3}{8}\beta^2_\infty(\beta_\infty + 2)^2 \leq \frac{3}{8}\tilde{\beta}^2(\tilde{\beta} + 2)^2 + (\tilde{\beta} + \frac{3}{2})(\sum_{i=1}^{q-1}\beta^2_i - \tilde{\beta}^2). \]

Therefore, if \( \frac{3}{8}\beta^2_\infty(\beta_\infty + 2)^2 > \frac{3}{8}\tilde{\beta}^2(\tilde{\beta} + 2)^2 + (\tilde{\beta} + \frac{3}{2})(\sum_{i=1}^{q-1}\beta^2_i - \tilde{\beta}^2), \) such solutions do not exist.

Furthermore, when \( \frac{3}{8}\beta^2_\infty(\beta_\infty + 2)^2 = \frac{3}{8}\tilde{\beta}^2(\tilde{\beta} + 2)^2 + (\tilde{\beta} + \frac{3}{2})(\sum_{i=1}^{q-1}\beta^2_i - \tilde{\beta}^2), \) all the inequalities in Lemma 13 and 14 become equalities. In particular, the isoperimetric inequality being sharp implies that \( u(x) \) is radial, which means that \( u \) has at most two singular points \( p_1, p_2 = \infty. \) It is easy to check that \( \beta_1 = \beta_2. \) We have proved that \( M \) is the football described first by [CHY]. □

Remark 15. Li in [L] used the move sphere method to prove that a smooth solution \( u \) to equation (2.1) on \( \mathbb{R}^4 \{0\} \) in \( C^+ \) is radial. Combine with Chang-Han-Yang’s [CHY], the radial solution to equation (2.1) is a football. This is a generalization of the uniqueness result for the smooth case, which is first proved in [V1]. Our approach is different.

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