Stability of Bianchi attractors in Gauged Supergravity

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Abstract

In this paper, we analyse the stability of extremal black brane horizons with homogeneous symmetry in the spatial directions in five dimensional gauged supergravity, under the fluctuations of the scalar fields about their attractor values. We examine the scalar fluctuation equations at the linearised level and demand that the fluctuations vanish as one approaches the horizon. Imposing certain restrictions on the Killing vectors used for gauging we find that the necessary conditions for stability are satisfied only by a subclass of the Bianchi metrics whose symmetry group factorises into a two dimensional Lifshitz symmetry and any homogeneous symmetry group given by the Bianchi classification. We apply these results to a simple example of a gauged supergravity model with one vector multiplet to find the stable attractors.
1 Introduction

The study of extremal black branes in anti-deSitter space has become a topic of recent interest because of their appearance as candidates for gravity duals of field theories describing condensed matter systems [1]. Several examples of such solutions in the context of gravity coupled to a dilaton as well as to various scalars and gauge fields have been studied [2,3].

In addition to their relevance as gravity duals to condensed matter field theories, such black brane geometries also appear in natural extensions of the attractor mechanism to gauged supergravities [4–10], there by providing a larger class of configurations for studying the thermodynamics of black holes. A classification of such configurations which are homogeneous but not isotropic has also been carried out in [11,12]. These attractor geometries, known as the Bianchi attractors, have a richer structure. They are characterised by constant anholonomy coefficients and are regular. In a number of cases belonging to Bianchi type I, they can be obtained either from a gauged supergravity theory or from string theory by a suitable choice of the internal manifold for compactification [13–15]. A prescription has been given to obtain such generalised attractors from gauged supergravity [16]. In this framework one sets all the fields and the curvature components to constants in the tangent space. Following this prescription, in a previous work [17], we embedded some of the Bianchi attractors [11] in five dimensional gauged supergravity. We considered a simple gauged supergravity model and constructed Lifshitz, Bianchi II and Bianchi VI solutions.

One of the important issues being investigated currently is the stability of such Lorentz violating geometries [18–23]. Instabilities due to scalar field fluctuations were found to exist in a class of charged black brane geometries [24,25]. Presence of such instabilities in these solutions plays a crucial role because they indicate that the geometry might get corrected in the deep infrared [20]. Though the stability analysis has been carried out in a number of examples, a common recipe (to figure out whether certain geometry has any instability) is still lacking.

Though the attractor mechanism has been studied quite extensively in the context of extremal black holes in Minkowski space with near horizon geometry $AdS_2 \times S^2$, the study of generalised attractors has not yet been explored thoroughly for the new class of Lorentz violating geometries arising as gravity duals of condensed matter systems. Especially, it is not at all obvious which among these entire class of new attractor geometries are stable and can survive in the deep infrared. Since a number of such geometries can be embedded in gauged supergravity, where the scalar couplings and potential term are determined by symmetry, it is natural to ask whether these gauged supergravity attractors are stable.

\footnote{See [26–29] for some early works and the reviews [30,31] for a detailed analysis of the attractor mechanism.}
In this paper, we analyse the stability of electrically charged Bianchi attractors in gauged supergravity. For attractors which asymptote to Minkowski space the conditions for stability is well understood [32]. In such cases the attractor values of the scalar fields must correspond to an absolute minimum of the black hole potential. In the present paper we derive the analogous condition for the generalised attractors. We consider scalar fluctuations about the attractor value and analyse the scalar field equations in the background of the Bianchi geometries. We take the fluctuations to depend on time and on the radial direction as we are mainly interested in determining the radial behaviour. We study the stress energy tensor in gauged supergravity and expand it in first order of scalar fluctuations. We find that the stress energy tensor in gauged supergravity depends on the scalar fluctuations even at first order perturbation due to non-trivial interaction terms in the theory. If there is a large backreaction due to scalar fluctuations, the geometry would significantly differ from the attractor geometry indicating an instability. Therefore, stable attractor geometries are those where the scalar fluctuations die out as one approaches the horizon.

We then study the scalar field equations with the fluctuations at first order, determine the general solution and the conditions under which these fluctuations can exist. These conditions are such that the generalised attractor geometries must exist at critical points which are maxima of the attractor potential. We then derive conditions for stability of the Bianchi attractors in gauged supergravity by studying the near horizon behaviour of the scalar fluctuations and demanding regularity. In particular, we find that this severely restricts the general form of these metrics.\footnote{We find that metrics which factorise as}

\begin{equation}
 ds^2 = L^2 \left[ -r^{2u_0} dt^2 + \frac{dr^2}{r^2} + \eta_{ij} \omega^i \otimes \omega^j \right], \tag{1}
\end{equation}

are stable under scalar fluctuations about the attractor value. The parameter $u_0$ must be positive in order to have a regular horizon, $i = 1, 2, 3$ corresponds to the $\hat{x}, \hat{y}, \hat{z}$ directions, $\eta_{ij}$ is a constant matrix independent of co-ordinates and $\omega^i$ are the one forms invariant under the homogeneous symmetries.\footnote{We call this special sub class of metrics as $Lif_{u_0}(2) \times M$.\footnote{Note that the symmetry group of (1) factorises into a direct product of a 1+1 dimensional non-relativistic conformal group times a homogeneous group of symmetries in three dimensions. The metric $Lif_{u_0}(d)$ has been used in the literature to denote the d dimensional Lifshitz metric with exponent $u_0$. The three dimensional submanifold $M$ is constructed from one forms invariant under the homogeneous groups given by the Bianchi classification, there are 9 such submanifolds each denoted as $M_I, M_{II}, \ldots M_{IX}$.}} We find that this severely restricts the general form of these metrics.\footnote{For more details on the homogeneity and invariant one forms, see [11,33,34].} In deriving this result, we make certain technical assumption on the killing vectors used in gauging, as well as on the nature of the critical points giving rise to the attractor geometry which will be discussed in due course.
has the scaling symmetry,

\[ \hat{r} \to \alpha \hat{r}, \quad \hat{t} \to \frac{\hat{t}}{\alpha u_0}, \]  

(2)

and homogeneous symmetries generated by the Bianchi groups along the \( \hat{x}, \hat{y}, \hat{z} \) directions. When \( u_0 = 1 \), we get an AdS\(_2\) factor and the symmetry is enhanced to \( SO(2,1) \times M \). This factorisation is reminiscent of extremal black holes in four dimensions where the near horizon geometries factorise as \( AdS_2 \times S^2 \).

The organisation of the paper is as follows. In §2 we give a brief background on gauged supergravity and generalised attractors. Subsequently, in §3.1 we describe the field equations satisfied by the gauge field fluctuations. Following which, we describe the scalar fluctuations about the attractor values followed by an expansion of the stress energy tensor with the fluctuations in §3.2. We then derive the general solutions for the scalar fluctuations and describe the conditions under which these fluctuations exist in §3.3. Following this we study the near horizon behaviour of the fluctuations and derive stability conditions for the Bianchi attractors in §3.4. In appendices §A and §B we give some background material on a simple gauged supergravity model. In §B.1 we have summarised some examples of Bianchi attractors in gauged supergravity from our earlier work. Finally, in §C we have provided some new examples of \( Lif_{u_0}(2) \times M \) class of metrics in a \( U(1)_R \) gauged supergravity.

## 2 Gauged supergravity and generalised attractors

In this section, we briefly summarise some necessary background in gauged supergravity and generalised attractors. We switch off the tensor as well as the hyper multiplets and turn on only the vector multiplets. The bosonic part of the \( \mathcal{N} = 2, d = 5 \) gauged supergravity is then given by by [35]:

\[
\hat{e}^{-1} \mathcal{L}_{\text{Bosonic}}^{\mathcal{N}=2} = - \frac{1}{2} R - \frac{1}{4} a_{IJ} F_{I\mu}^I F_{J\mu}^J - \frac{1}{2} g_{xy} D_\mu \phi^x D_\mu \phi^y \\
- \mathcal{V}(\phi) + \frac{\hat{e}^{-1}}{6 \sqrt{6}} C_{IJK} \epsilon^{\mu\nu\rho\sigma\tau} F_{I\mu}^I F_{J\nu}^J A^K_\tau.
\]  

(3)

Here \( \hat{e} = \sqrt{-\text{det}g_{\mu\nu}} \) and \( a_{IJ} \) is the ambient metric used to raise and lower the vector indices and \( g_{xy} \) is the metric on the moduli space. The moduli space spanned by the real scalar fields \( \phi^x \) is a very special manifold. It is completely specified in terms of constant symmetric tensors \( C_{IJK} \) by the hypersurface:

\[ C_{IJK} h^I h^J h^K = 1, \quad h^I \equiv h^I(\phi). \]  

(4)
The group of isometries of this manifold is denoted by $G$. The gauging is specified in terms of a subgroup $K \subset G$ generated by the Killing vectors $K^I_x$. The covariant derivatives on the scalars are given in terms of the Killing vectors $K^I_x$ as:

$$D_\mu \phi^x \equiv \partial_\mu \phi^x + gA^I_\mu K^I_\phi(\phi).$$ (5)

The potential term is given by,

$$V(\phi) = -g^2 R[P^2 - \bar{P}\bar{P}],$$ (6)

where $g_R$ is the gauge coupling constant associated with gauging the R symmetry group and

$$P^I_{ij} \equiv h^I P^I_{ij}, \quad \bar{P}^I_{ij} \equiv h^{\bar{I}} P^\bar{I}_{ij}.$$ (7)

Here $i, j = 1, 2, \bar{a} = 0, 1, \ldots n_V$ and for the case of $U(1)_R \subset SU(2)_R$ gauging, $P^I_{ij} = V_I \delta_{ij}$.

In the following, we give various field equations of this theory that are necessary for the stability analysis. All the equations are written in position space for the purpose of this paper. It was shown in [17] that all the field equations become algebraic at the attractor points defined by,

$$\phi^x = \text{const} ; A^I_a = \text{const} ; c^a_{bc} = \text{const},$$ (8)

where $c^a_{bc}$ are called as the anholonomy coefficients (see §A for a detailed discussion).

The gauge field equation is:

$$\partial_\mu (\hat{e}a_{IJ}F^{I\mu\nu}) = \hat{e}(g^2K_{IJ}A^{\mu J} + gK_{Iy}\partial^{\nu}\phi^y),$$ (9)

where we have defined $K_{IJ} = K^I_x K^J_y g_{xy}$. Note that we have ignored the Chern-Simons term, since it vanishes for the Bianchi attractors [17]. At the attractor points, the equation simplifies to,

$$a_{IJ\phi} \nabla_\mu F^{I\mu\nu} = g^2 K_{IJ\phi} A^{\nu J}$$ (10)

The equation of motion for the scalar fields $\phi^x$ is given by:

$$\hat{e}^{-1}\partial_\mu [\hat{e} g_{xy} D^\mu \phi^y] - \frac{1}{2}\partial_\phi^z D^\mu \phi^x D^{\mu} \phi^y - gA^I_\mu g_{xy} \frac{\partial K^I_z}{\partial \phi^z} D^{\mu} \phi^y$$

$$- \frac{1}{4}\partial_\phi^x F^{I\mu\nu} F^{J\mu\nu} - \frac{\partial V(\phi)}{\partial \phi^x} = 0,$$ (11)

where as the Einstein’s equation is:

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = T_{\mu\nu}.$$ (12)
The stress energy tensor $T_{\mu\nu}$ has the expression:

$$T_{\mu\nu} = g_{\mu\nu} \left[ \frac{1}{4} a_{IJ} F^I_{\mu\sigma} F^J_{\nu\sigma} + \frac{1}{2} a_{xy} D_\mu \phi^x D_\nu \phi^y + \mathcal{V}(\phi) \right] - \left[ a_{IJ} F^I_{\mu\tau} F^J_{\nu\lambda} + a_{xy} D_\mu \phi^x D_\nu \phi^y \right]. \quad (13)$$

At the attractor points the scalar field equation reduces to minimisation of the attractor potential,

$$\frac{\partial \mathcal{V}_{\text{attr}}(\phi, A)}{\partial \phi} = 0, \quad (14)$$

which has the expression:

$$\mathcal{V}_{\text{attr}}(\phi, A) = \mathcal{V}(\phi) + \frac{1}{2} g^2 K_{IJ} A^I \mu A^{J \mu} + \frac{1}{4} a_{IJ} F^I_{\mu\nu} F^J_{\mu\nu}. \quad (15)$$

The critical points of the attractor potential are denoted as $\phi_c$, and solving (14) relates $\phi_c$ in terms of the charges. In [17] we considered a simple gauged supergravity model with one vector multiplet and constructed examples of Bianchi attractors. Some details of the gauged supergravity model and the Bianchi attractor solutions are provided in appendices §B and §B.1 respectively.

3 Scalar Perturbation about attractor values

In this section, we consider the fluctuations of the scalar fields about their attractor value. We take the fluctuation to be of the form,

$$\phi_c + \epsilon \delta \phi(r, t), \quad (16)$$

where $t$ denotes the time, $r$ is the radial direction, $\phi_c$ are the attractor values of the scalars and $\delta \phi$ is the perturbation with $\epsilon < 1$. We have taken the fluctuation to not depend on the $(x, y, z)$ directions to respect the Bianchi type symmetries along these directions. Besides, we are primarily interested in the radial behaviour of the fluctuation as one approaches the horizon. We also assume that the black brane metric can be expanded about the near horizon geometries as follows,

$$\tilde{g}_{\mu\nu} \sim g_{\mu\nu}(r - r_h) + \epsilon \gamma_{\mu\nu}(r - r_h) + O(\epsilon^2) + \ldots, \quad (17)$$

where $g_{\mu\nu}$ is the near horizon metric given by the Bianchi type geometries. The higher order terms like $\gamma_{\mu\nu}$ are due to the back reaction of the scalar field fluctuations on the attractor geometry.
3.1 Gauge field fluctuations

We also consider gauge field fluctuations together with scalar and metric fluctuations. In this subsection, we derive the gauge field fluctuation equation in terms of scalar fluctuations. Let us consider fluctuations of the form

$$A^I_{\mu} + \epsilon \delta A^I_{\mu},$$

where $A^I_{\mu}$ are the attractor values of the gauge field for any given Bianchi metric. The field strength expands as,

$$F^I_{\mu\nu} + \epsilon F^I_{\mu\nu},$$

We now expand the scalars and gauge fields about the attractor value in (9) to get,

$$a_{IJ}|_{\phi_c} \nabla_\mu F^I_{\mu\nu} - g^2 K_{IJ}|_{\phi_c} \delta A^\nu J = - \left( \frac{\partial a_{IJ}}{\partial \phi^z} \right|_{\phi_c} \nabla_\mu (F^I_{\mu\nu} \delta \phi^z) - g^2 \partial K_{IJ} \partial \phi^z \left|_{\phi_c} \delta \phi^z A^\nu J \right) + g K_{IJ}|_{\phi_c} \partial^\nu \delta \phi^y,$$

where we have used (10) for simplification. Note that we did not have to consider metric fluctuations in the above equation, since it would lead to second order terms. It can be seen that regular behaviour of the gauge field fluctuations depend on regularity of the scalars and their derivatives near the horizon. In other words, the gauge fluctuation is not independent of the scalar fluctuation but varies as $\delta A^I_{\mu} \sim \delta \phi^x$. Subsequently in §3.3 we will argue that, for stable attractors the scalar field fluctuation $\delta \phi^x$ must vanish at the attractor point. The above analysis shows that the attractor which is stable against scalar field fluctuation is also stable against gauge field fluctuation.

3.2 Backreaction at first order

We now expand the stress energy tensor (13) up to first order in $\epsilon$ under the scalar perturbations (16), and simplify, to get:

$$T_{\mu\nu} (\phi_c + \delta \phi) = T_{\mu\nu}^{attr}|_{\phi_c} + g K_{IJ}|_{\phi_c} \left( A^I_\mu A^J_\lambda - A^I_\mu A^J_\lambda \right) + g^2 K_{IJ} F^{I}_{\mu\nu} F^{J}_{\mu\nu} - F^{I}_{\mu\nu} F^{J}_{\mu\nu} \lambda,$$

where we have introduced

$$T_{\mu\nu}^{attr}|_{\phi_c} = V^{attr}(\phi_c) g_{\mu\nu} - \left[ a_{IJ}|_{\phi_c} F^I_{\mu\lambda} F^J_{\nu\lambda} + g^2 K_{IJ}|_{\phi_c} A^I_{\mu} A^J_{\nu} \right].$$
The attractor equations (14) can be used for further simplification to get,
\[
T_{\mu\nu}(\phi_c + \delta \phi) = T_{\mu\nu}^{\text{attr}}|_{\phi_c} + gK_{gI}|_{\phi_c} \left( A^M \partial_\lambda (\delta \phi^y) g_{\mu\nu} - A^I_{\mu} \partial_\nu (\delta \phi^y) - A^I_{\nu} \partial_\mu (\delta \phi^y) \right) \\
- \left[ \partial_{aIJ} \right] \left| \frac{\partial K_{IJ}}{\partial \delta \phi^z} \right|_{\phi_c} \left[ A^I_{\mu} A^J_{\nu} \right] \delta \phi^z.
\]

It is already clear that for general perturbations of the scalar field, there is backreaction at first order even after using the attractor equations. In particular this requires the fluctuations and their derivatives to be well behaved as one approaches the horizon. Any divergent fluctuation would cause infinite backreaction and deviation from the attractor geometry indicating an instability. Taking the trace of (23) we get,
\[
T^{\mu}_{\mu}(\phi_c + \delta \phi) = T^{\text{attr}}_{\mu\mu}|_{\phi_c} + (d - 2)gK_{gI}|_{\phi_c} A^M \partial_\lambda (\delta \phi^y) \\
- \left[ \partial_{aIJ} \right] \left| \frac{\partial K_{IJ}}{\partial \delta \phi^z} \right|_{\phi_c} \left[ A^I_{\mu} A^J_{\nu} \right] \delta \phi^z.
\]

where \( d \) is the space time dimension. Once again we can use the attractor equations (14) to simplify, and the Einstein equations take the form,
\[
R \frac{(2 - d)}{2} = T^{\text{attr}}_{\mu\mu}|_{\phi_c} + (d - 2)gK_{gI}|_{\phi_c} A^M \partial_\lambda (\delta \phi^y) \\
+ \left[ g^2 \frac{\partial K_{IJ}}{\partial \delta \phi^z} \right] A^I_{\mu} A^J_{\nu} + 4 \left[ \frac{\partial \mathcal{V}}{\partial \delta \phi^z} \right] \delta \phi^z.
\]

Suppose the critical points of the attractor potential are also simultaneous critical points of the gauged supergravity scalar potential (as was the case with all the examples discussed in [17]), we see that the terms relevant for the backreaction are proportional to \( g \):
\[
R \frac{(2 - d)}{2} = T^{\text{attr}}_{\mu\mu}|_{\phi_c} + (d - 2)gK_{gI}|_{\phi_c} A^M \partial_\lambda (\delta \phi^y) + g^2 \frac{\partial K_{IJ}}{\partial \delta \phi^z} \left| \frac{\partial \mathcal{V}}{\partial \delta \phi^z} \right| A^I_{\mu} A^J_{\nu} \delta \phi^z.
\]

Thus, for gauging of \( R \) symmetry, \( g = 0 \) and hence the backreaction is absent:
\[
R \frac{(2 - d)}{2} = T^{\text{attr}}_{\mu\mu}|_{\phi_c}.
\]

(See §C for some examples of generalised attractor in gauged supergravity with just \( R \) symmetry gauging). However, in gauged supergravity with a generic gauging of symmetries of the scalar manifold, the equation depends on the first order fluctuations in the scalar fields. Thus, the generalised attractor geometries in gauged supergravity with a generic gauging can get backreacted by fluctuations
of scalar fields. It then follows that the relevant boundary conditions to have stable attractors should be such that the fluctuations and derivatives of fluctuations vanish as one approaches the horizon.

The main point of the above calculation was to indicate that there are first order scalar fluctuation terms present in the expansion of the stress energy tensor. We now include the metric fluctuation as well and write down the linearised Einstein equation:

\[
\nabla^\alpha \nabla_\alpha \bar{\gamma}_{\mu\nu} + 2R_{\langle \mu \nu \rangle}^\alpha \bar{\gamma}_{\beta\alpha} - 2R_{\langle \mu \beta \rangle}^\alpha \bar{\gamma}_{\nu\beta} + g_{\mu\nu} \left( R_{\alpha\beta} \bar{\gamma}_{\alpha\beta} + \frac{2}{2 - D} R \bar{\gamma} \right) + R \bar{\gamma}_{\mu\nu} + 2\dot{T}^\text{attr}_{\mu\nu} \left( g_{\alpha\beta} + \epsilon \gamma_{\alpha\beta} \right) |_{\epsilon=0} = 0. \tag{28}
\]

Our conventions are as in [39]. We denote \( \gamma_{\mu\nu} \) to be the first order perturbation about the metric, the dot indicates derivative w.r.t \( \epsilon \), \( \bar{\gamma}_{\mu\nu} = \gamma_{\mu\nu} - \frac{1}{2} \gamma g_{\mu\nu} \), \( \bar{\gamma} = \frac{2 - D}{2} \gamma \), and \( \gamma = g^{\mu\nu} \gamma_{\mu\nu} \). All the covariant derivatives, Riemann tensor, Ricci tensor and curvature terms are with respect to the unperturbed metric \( g_{\mu\nu} \). The terms in the stress energy tensor dependent on the metric fluctuations are given by,

\[
\dot{T}^\text{attr}_{\mu\nu} \left( g_{\alpha\beta} + \epsilon \gamma_{\alpha\beta} \right) |_{\epsilon=0} = V^\text{attr}_{\phi_c}(\bar{\gamma}_{\mu\nu} + \frac{2\bar{\gamma}}{2 - D} g_{\mu\nu}) - (\bar{\gamma}_{\lambda\sigma} + \frac{\bar{\gamma}}{2 - D} g_{\lambda\sigma}) (\frac{1}{2} T^\lambda_{\text{attr}} g_{\mu\nu} + a_{IJ} \partial_{\phi_c} F^\lambda_{\mu} F^J_{\nu} \sigma). \tag{29}
\]

To include the scalar fluctuations one just need to add (23) to (29). The condition of stability boils down to the existence of a one parameter family of solutions to (28) (see chapter 8 of [39]). The equation (28) is a very complicated set of differential equations and exact analysis is possible only in the case of a flat background or for a very specialised set of background metric [39]. Proof of existence of a one parameter family of solutions in our attractor background is beyond the scope of the current analysis and we leave a detailed analysis of this for future work.

### 3.3 Scalar fluctuations

In this section, we will analyse the scalar fluctuations in detail using the equation of motion for the scalar fields. The field equation (11) can be rewritten as:

\[
\hat{\epsilon}^{-1} \partial_{\mu} \left[ \hat{\epsilon} g_{\rho\sigma} D^\mu \phi^\rho \right] - \frac{1}{2} \frac{\partial g_{\rho\sigma}}{\partial \phi^\rho} \nabla_\mu \phi^\rho \nabla_\mu \phi^\sigma - g \frac{\partial K_{\rho\sigma}}{\partial \phi^\rho} A^I_{\mu} \nabla_\mu \phi^\rho - \frac{\partial V^\text{attr}}{\partial \phi^\rho} = 0. \tag{30}
\]
We will now expand the scalar fields about their attractor values and keep terms of \( O(\epsilon) \) to get:

\[
g_{xy}\left|_{\phi_c} \nabla_\mu \delta \phi^y - \frac{\partial^2 V_{\text{attr}}}{\partial \phi^x \partial \phi^y} \right|_{\phi_c} \delta \phi^y + g \left[ \frac{\partial K_{Iz}}{\partial \phi^y} - \frac{\partial K_{Iy}}{\partial \phi^x} \right] \left|_{\phi_c} \right. A^\mu_\nu \nabla_\mu \delta \phi^y \\
+ g \left[ K_{Iz} \left|_{\phi_c} + \frac{\partial K_{Iz}}{\partial \phi^y} \delta \phi^y \right|_{\phi_c} \right] \nabla_\mu A^\mu_\nu = 0. \quad (31)
\]

Here the covariant derivative \( \nabla_\mu \) is taken with respect to the zeroth order metrics which represent the near horizon Bianchi geometries. Note that the higher order metric terms which are undetermined are not required at \( O(\epsilon) \). We choose the gauge condition \( \nabla_\mu A^\mu_\nu = 0 \) to eliminate the last term. Finally we get,

\[
\nabla_\mu \nabla^\mu - g_{xz} \left|_{\phi_c} \partial^2 V_{\text{attr}} \right|_{\phi_c} \delta \phi^y + 2g \left( g_{xz} \nabla_y K_{Iz} \right) \left|_{\phi_c} \right. A^\mu_\nu \nabla_\mu \delta \phi^y = 0, \quad (32)
\]

where \( \nabla \) is the covariant derivative with respect to the metric on the scalar manifold \( g_{xy} \). The Laplacian operator can be written as,

\[
\nabla_\mu \nabla^\mu = g^{\tilde{r}\tilde{t}} \partial_{\tilde{r}}^2 + g^{\tilde{t}\tilde{t}} \partial_{\tilde{t}}^2 + (g^{\tilde{r}\tilde{r}} \partial_{\tilde{r}} \tilde{e} + \partial_{\tilde{r}} g^{\tilde{r}\tilde{r}}) \partial_{\tilde{t}}, \quad (33)
\]

since the scalar fluctuations depend only on the radial and time co-ordinates.

Before substituting the details we would like to make some comments on the co-ordinate system used for writing the Bianchi attractor geometries. In [11] the horizon for the Bianchi metrics was located at \( r = -\infty \), where as in [17] we have chosen the co-ordinate \( \tilde{r} = e^r \) such that the horizon lies at \( \tilde{r} = 0 \) instead (also see \( \tilde{y}(B.1) \)). As can be seen from the general form of the Bianchi metrics,

\[
ds^2 = L^2 \left[ -\tilde{r}^{2u_0} d\tilde{t}^2 + \frac{d\tilde{r}^2}{\tilde{r}^2} + \tilde{e}^{2(u_1 + u_2)} \eta_{ij} \omega^i \otimes \omega^j \right], \quad (34)
\]

the constants \( u_0, u_i \) must be positive in order to have a regular horizon. Here the index \( i = 1, 2, 3 \) corresponds to the \( \hat{x}, \hat{y}, \hat{z} \) directions, \( \eta_{ij} \) is a constant matrix independent of the co-ordinates and \( \omega^i \) are the one forms invariant under the homogeneous symmetries [11].

Thus one can see that the general form of the determinant is

\[
\tilde{e} = \sqrt{-\det g_{\mu\nu}} \sim L^5 \tilde{r}^m f(x, y, z), \quad (35)
\]

where \( m = -1 + \sum c_i u_i \), \( u_i \) are the various exponents and \( c_i \) is a positive number with \( c_0 = 1 \) for all Bianchi attractors. For example, in the Bianchi II case (see (84)) \( m = -1 + u_0 + 2(u_1 + u_2) \). We can also see that,

\[
g^{\tilde{r}\tilde{r}} = \frac{\tilde{r}^2}{L^2}, \quad g^{\tilde{t}\tilde{t}} = -\frac{1}{L^2 \tilde{r}^{2u_0}}, \quad (36)
\]
for all Bianchi attractors. Using the above data, the Laplacian (33) can be expressed as,
\[
\nabla_\mu \nabla^\mu = \frac{1}{L^2} \left[ \hat{r}^2 \partial_{\hat{r}}^2 + (m + 2) \hat{r} \partial_{\hat{r}} - \frac{1}{\hat{r}^2 u_0} \partial_{\hat{t}}^2 \right].
\]
(37)
Substituting (37) in (32) and using the ansatz (82) for \( A^I_\mu \) we get,
\[
\left[ \hat{r}^2 \partial_{\hat{r}}^2 + (m + 2) \hat{r} \partial_{\hat{r}} - \frac{1}{\hat{r}^2 u_0} \partial_{\hat{t}}^2 \right] \delta \phi^x - M^x_y |_{\phi_c} \delta \phi^y + N^x_y |_{\phi_c} \frac{1}{\hat{r}^2 u_0} \partial_{\hat{t}} \delta \phi^y = 0,
\]
(38)
where,
\[
M^x_y |_{\phi_c} = L^2 g^{xx} \frac{\partial^2 V_{\text{attr}}}{\partial \phi^x \partial \phi^y} |_{\phi_c}, \quad N^x_y |_{\phi_c} = 2g L A^I 0 (g^{xy} \tilde{\nabla}_x K_{Iz}) |_{\phi_c}.
\]
(39)
The metric on the moduli space \( g_{xy} \) is chosen to be positive definite and the nature of the critical point is given by the sign of the double derivative of the attractor potential. We further assume that \( M^x_y |_{\phi_c} \) is diagonal so that,
\[
M^x_y |_{\phi_c} \delta \phi^y = \lambda \delta \phi^x.
\]
(40)
The term \( N^x_y \) can be non zero in general, but vanishes trivially for the gauged supergravity model where we found some examples Bianchi attractors (see §(B)). There is only one Killing vector (78) that generates the SO(2) isometry on the scalar manifold, and the critical point is such that \( \phi^2_c = \phi^3_c = 0 \). Therefore one is left with just the \( \tilde{\nabla}_x K_{Ix} \) component which vanishes due to the Killing vector equation on the manifold.\(^5\)

Thus, the scalar fluctuation equation (32) has the final form,
\[
\left[ \hat{r}^2 \partial_{\hat{r}}^2 + (m + 2) \hat{r} \partial_{\hat{r}} - \frac{1}{\hat{r}^2 u_0} \partial_{\hat{t}}^2 - \lambda \right] \delta \phi^x = 0.
\]
(41)

The above equation admits a simple solution when the fluctuations \( \delta \phi^x \) are time independent. In this case, we have
\[
\delta \phi^x = C_1 r \left( \sqrt{\lambda + (m + 1)^2} - (1 + m) \right) / 2 + C_2 r \left( -\sqrt{\lambda + (m + 1)^2} - (1 + m) \right) / 2.
\]
(42)
Thus, one of the modes vanishes as \( r \to 0 \) provided \( \lambda \) is positive and hence we can get stable attractors upon setting \( C_2 = 0 \). Unfortunately, the examples we consider in this paper do not admit a critical point with \( \lambda > 0 \). Thus such fluctuations are not stable.

\(^5\)Here, the single surviving component of the Killing vector is along the direction of \( \phi^1 \) on the scalar manifold.
Now we turn to the case of time dependent fluctuations. Since the equation for $\delta\phi^x$ is separable, we try the ansatz $\delta\phi(\hat{r}, \hat{t}) = f(\hat{r}) e^{ik\hat{t}}$ (with $k$ real) to get the Bessel equation:

$$\left[ \hat{r}^2 \partial_{\hat{r}}^2 + (m + 2)\hat{r} \partial_{\hat{r}} + \left( \frac{k^2}{\hat{r} u_0} - \lambda \right) \right] f(\hat{r}) = 0 . \quad (43)$$

The general solutions for this equation are given by the standard Bessel functions (see, for example, [36], page 932):

$$f(X) = \left( \frac{X}{2} \right)^{\nu_0} \left[ C_1 \Gamma(1 - \nu_\lambda) J_{-\nu_\lambda}(X) + C_2 \Gamma(1 + \nu_\lambda) J_{\nu_\lambda}(X) \right] , \quad (44)$$

where,

$$X = \frac{k}{u_0 \hat{r} u_0} , \quad \nu_\lambda = \frac{\sqrt{(1 + m)^2 + 4\lambda}}{2u_0} , \quad \nu_0 = \frac{(1 + m)}{2u_0} \quad , \quad (45)$$

$C_1$ and $C_2$ are arbitrary constants, and

$$J_{\nu_\lambda}(X) = \left( \frac{X}{2} \right)^{\nu_\lambda} \sum_{j=0}^{\infty} \frac{(-1)^j}{j!\Gamma(j + \nu_\lambda + 1)} \left( \frac{X}{2} \right)^{2j} ,$$

$$J_{-\nu_\lambda}(X) = \left( \frac{X}{2} \right)^{-\nu_\lambda} \sum_{j=0}^{\infty} \frac{(-1)^j}{j!\Gamma(j - \nu_\lambda + 1)} \left( \frac{X}{2} \right)^{2j} . \quad (46)$$

The power series representation is valid in the small $X$ or equivalently, in the large $r$ regime. We can rewrite the solution in terms of the Hankel functions

$$J_{\nu_\lambda}(X) = \frac{1}{2} (H^1_{\nu_\lambda}(X) + H^2_{\nu_\lambda}(X)),$$

$$J_{-\nu_\lambda}(X) = \frac{1}{2} H^1_{\nu_\lambda}(X) e^{i\nu_\lambda \pi} + H^2_{\nu_\lambda}(X) e^{-i\nu_\lambda \pi}) , \quad (47)$$

to get,

$$f(X) = \left( \frac{X}{2} \right)^{\nu_0} \left[ C_1 H^1_{\nu_\lambda}(X) [\Gamma(1 - \nu_\lambda)e^{i\nu_\lambda \pi} + \Gamma(1 + \nu_\lambda)] \right. \left. + C_2 H^2_{\nu_\lambda}(X) [\Gamma(1 - \nu_\lambda)e^{-i\nu_\lambda \pi} + \Gamma(1 + \nu_\lambda)] \right] . \quad (48)$$

As one can see from above equation, there is already a restriction on $\nu_\lambda$ from the Gamma function that appears in the general solution. First let us consider the case $\nu_\lambda$ real, then we have the condition,

$$\nu_\lambda = \sqrt{\frac{(1 + m)^2 + 4\lambda}{2u_0}} = \sqrt{\sum_l c_i u_l^2 + 4\lambda} \leq 1 , \quad (49)$$
for
\[- \left( \sum_l c_l u_l \right)^2 \leq \lambda < 0. \tag{50}\]

Note that only negative \( \lambda \) can satisfy (49).\(^6\) Since \( c_l > 0 \) and all the \( u_l \) have to be positive for the existence of a regular horizon, we conclude that \( \lambda \) has to be negative. Remember that the sign of \( \lambda \) is provided by the double derivative of the attractor potential eqs. (39,40). This implies that the critical points correspond to maxima of the attractor potential. For the case of imaginary \( \nu_\lambda \) we have,
\[\lambda < - \frac{\left( \sum_l c_l u_l \right)^2}{4}, \tag{51}\]

and hence, even in this case the critical points correspond to a maxima of the attractor potential. Thus we have determined the general solution for the scalar fluctuation (48) and and we find that they are well behaved at large distance provided they satisfy the conditions (50,51). This may be useful for the study of attractor flow equations for black holes in \( AdS \).

### 3.4 Stable Bianchi attractors

In this section we will analyse the stability of the Bianchi attractors by studying the behaviour of the solution in the \( r \to 0 \) limit. We are interested in the question which class of the Bianchi attractors can be stable attractor geometries in gauged supergravity. This can be answered by looking at the near horizon behaviour of the scalar fluctuations (48). From our analysis of the stress energy tensor in gauged supergravity (26) we found that there is dependence on the fluctuations and their derivatives at first order perturbation. Hence, we only require that the fluctuations do not blow up near the horizon as that would backreact strongly and deviate from the geometry. This requirement places some constraints on the form of the metric itself as we explain in the rest of the section.

Both the solutions in (48) are given in terms of the Hankel functions, the behaviour near the horizon can be determined by considering the asymptotic expansions of the Hankel functions. Remember that the horizon for the Bianchi metrics (34) is located at \( \hat{r} = 0 \) The form of the solution (48) makes it convenient to use the asymptotic expansions of the Hankel functions, since from (45) \( X \to \infty \) as \( \hat{r} \to 0 \). The asymptotic expansions are given by,
\[
H^1_{\nu_\lambda}(X) \sim \sqrt{\frac{2}{\pi X}} e^{i\left(X-\frac{\pi}{4}(\nu_\lambda+\frac{1}{2})\right)}
\]
\[
H^2_{\nu_\lambda}(X) \sim \sqrt{\frac{2}{\pi X}} e^{-i\left(X-\frac{\pi}{4}(\nu_\lambda+\frac{1}{2})\right)}.
\tag{52}\]

\(^6\)We do not consider \( \lambda = 0 \) as that would leave the nature of the critical point undetermined.
Substituting (52) in (48) we determine the behaviour of the fluctuation near the horizon as,

$$f(X) \sim \left(\frac{X}{2}\right)^{\nu_0 - \frac{1}{2}} \sqrt{\pi} \left[ C_1 e^{i(X - \frac{\pi}{2}(\nu_0 + \frac{1}{2}))} \left[ \Gamma(1 - \nu_\lambda) e^{i\nu_\lambda \pi} + \Gamma(1 + \nu_\lambda) \right] + C_2 e^{-i(X - \frac{\pi}{2}(\nu_0 + \frac{1}{2}))} \left[ \Gamma(1 - \nu_\lambda) e^{-i\nu_\lambda \pi} + \Gamma(1 + \nu_\lambda) \right] \right].$$  

(53)

Since $X \sim \frac{1}{r u_0}$ and $u_0 > 0$, there is a leading divergent term as $\hat{r} \to 0$ unless

$$\frac{1 - 2\nu_0}{2} \geq 0,$$

(54)

which can be rewritten as,

$$\nu_0 = \frac{(1 + m)}{2u_0} = \frac{\sum_l c_l u_l}{2u_0} \leq \frac{1}{2}.$$  

(55)

Since $c_0 = 1$, this implies,

$$\sum_{l, l \neq 0} c_l u_l \leq 0,$$

(56)

which can never be satisfied without some of the exponents $u_l$ being negative. Since we require a regular horizon, all the exponents have to be positive. Thus the only possibility for which eq. (56) can be satisfied is

$$u_0 \neq 0, \quad u_l = 0 \quad \forall \ l \neq 0.$$  

(57)

The conditions on $\lambda$ (50),(51) for the general solution (48) to exist can now be written as,

$$-\frac{u_0^2}{4} \leq \lambda < 0,$$

(58)

for real $\nu_\lambda$, and

$$\lambda < -\frac{u_0^2}{4},$$

(59)

for imaginary $\nu_\lambda$. To summarise, Bianchi attractors are stable against scalar fluctuations about the attractor value for the class of metrics which satisfy the condition (57).

The condition (57) is highly restrictive on the form of the Bianchi metrics. In particular it follows from (57) that $\nu_0 = \frac{1}{2}$ for any $u_0 > 0$ and the scalar fluctuations (53) do not diverge near the horizon.\(^7\) In particular this restricts the metrics (34) to be of the form,

$$ds^2 = L^2 \left[ -\dot{r}^2 u_0 dt^2 + \frac{d\hat{r}^2}{\hat{r}^2} + \eta_{ij} \omega^i \otimes \omega^j \right].$$  

(60)

\(^7\)Note that there are still oscillatory terms in the fluctuation.
Geometry | $\lambda$ | $u_0$ | $u_l, l \neq 0$ | Stability
--- | --- | --- | --- | ---
Lifshitz | $-34$ | $3$ | $1$ | no
Bianchi II | $-\frac{22}{3}$ | $\sqrt{2}$ | $u_1 = u_2 = \frac{1}{2\sqrt{2}}$ | no
Bianchi VI $h < 0$ | $-1 + \frac{14}{3} - h^2$ | $\frac{1}{\sqrt{2}}(1 - h)$ | $u_1 = -\frac{1}{\sqrt{2}}h, u_2 = \frac{1}{\sqrt{2}}$ | no
$Lif_{u_0}(2) \times M_I$ | $-\frac{5u_0^2}{3}$ | any $u_0 > 0$ | $0$ | yes
$AdS_2 \times M_I$ | $-\frac{5}{3}$ | $1$ | $0$ | yes
$Lif_{u_0}(2) \times M_{II}$ | $-\frac{61}{6}$ | $\sqrt{\frac{11}{2}}$ | $0$ | yes
$Lif_{u_0}(2) \times M^*$ | $\lambda < 0$ | any $u_0 > 0$ | $0$ | yes

Table 1: Bianchi attractor geometries in gauged supergravity, nature of critical points and stability. The first three entries are for the solutions found in [17]. The next three entries are generalised attractors in $U(1)_R$ gauged supergravity (C). The last entry with the * is the most general possible Bianchi attractor geometry (60) that satisfies our stability criteria.

It is very interesting to note that the symmetry group of this metric form factorises into a direct product of the $(1 + 1)$ dimensional Lifshitz group and a group in the Bianchi classification. This is similar to what happens for example in four dimensional extremal black holes where the near horizon geometry factorises as $AdS_2 \times S^2$.

The simplest non-trivial example of this class is the $Lif_{u_0}(2) \times M_I$ solution,

$$ds^2 = L^2 \left[ -\hat{r}^{2u_0}dt^2 + \frac{d\hat{r}^2}{\hat{r}^2} + (d\hat{x}^2 + d\hat{y}^2 + d\hat{z}^2) \right],$$

one obtains the $AdS_2 \times \mathbb{R}^3$ solution when $u_0 = 1$. Another less trivial example is the $Lif_{u_0}(2) \times M_{II}$ solution,

$$ds^2 = L^2 \left[ -\hat{r}^{2u_0}dt^2 + \frac{d\hat{r}^2}{\hat{r}^2} + (d\hat{x}^2 + d\hat{y}^2 - 2\hat{x}\hat{y}d\hat{z} + (\hat{x}^2 + 1)d\hat{z}^2) \right].$$

We have constructed the $Lif_{u_0}(2) \times M_I$ for any $u_0 > 0$ and a $Lif_{u_0}(2) \times M_{II}$ in a simple $U(1)_R$ gauged supergravity theory with one vector multiplet. These solutions and the details of the theory are summarised in Appendix (C). It can be seen from Table (1) that these solutions satisfy the stability criteria (57) and hence are examples of stable Bianchi attractors in gauged supergravity.

The examples we constructed earlier in [17] have $\lambda < 0$ and exist at maxima of the attractor potential. Therefore the condition (49) allows scalar fluctuations about the attractor values. However as one can see from Appendix B.1 all the metrics have some $u_l \neq 0$ for $l \neq 0$ and do not satisfy (57). Hence the radial
fluctuation of the scalar field diverges near the horizon for all these metrics. To complicate matters further, as one can see from (26) the fluctuations and their derivatives backreact on the geometry strongly. Thus there would be significant deviation of the geometry even at the first order and we conclude that these geometries are unstable attractors in the theory. These results are summarised in Table (1).

4 Summary

We have studied the stability of Bianchi attractors in gauged supergravity by considering scalar fluctuations about the attractor value. In general, the stress energy tensor in a generic gauged supergravity depends on the scalar fluctuations and their derivatives even at first order perturbation. Therefore, it is important that the scalar fluctuations are well behaved near the horizon. In particular, if there is a large backreaction then the geometry would deviate from the attractor geometry. Hence the fluctuations must vanish as one approaches the horizon for the attractor geometry to be stable.

We analysed the scalar fluctuation equations and found that the fluctuations can exist in general when the attractor geometries in consideration exist at critical points which, in the present case, correspond to maxima of the attractor potential. By demanding that the fluctuations vanish as one approaches the horizon we determined the conditions of stability for the metric. We found that the Bianchi attractors are stable if the metric factorises as,

\[ ds^2 = L^2 \left( -\hat{r}^{2u_0} d\hat{t}^2 + \frac{d\hat{r}^2}{\hat{r}^2} \right) + L^2 \left( \eta_{ij} \omega^i \otimes \omega^j \right), \]  

which is a subclass of the Bianchi attractors constructed by [11]. We refer to this class of metrics as \( Li_f u_0(2) \times M \), where \( M \) refers to three dimensional manifolds invariant under the nine groups given by the Bianchi classification. We also constructed explicit examples of \( Li_f u_0(2) \times M_I \) and \( Li_f u_0(2) \times M_{II} \) in a \( U(1)_R \) gauged supergravity using the generalised attractor procedure. As stated before, these solutions exist for critical points which are maxima of the attractor potential and they satisfy all the conditions of stability. It would be interesting to explore whether this is a generic feature of attractors in gauged supergravity or an artifact of the models we are considering in this paper.

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A Tangent space and constant anholonomy

In this section of the appendices we describe notations and conventions used in tangent space. We use Greek indices $\mu, \nu, \ldots = 0, 1, \ldots, 4$ to denote the space time coordinates where as Latin indices $a, b, \ldots = 0, 1, \ldots, 4$ to denote the tangent space coordinates. The tangent space and space time metrics are denoted by $\eta_{ab}$ and $g_{\mu\nu}$ respectively with signature $\lbrace -, +, +, + \rbrace$. The vielbeins $e^a_\mu(x)$ are related to the space time metric $g_{\mu\nu}$ by

$$g_{\mu\nu} = e^a_\mu e^b_\nu \eta^{ab}.$$  \hfill (64)

The covariant derivative with respect to the spin connection $\omega_{a,bc}$ is denoted by $D_a(\omega)$. The covariant derivative is defined in terms of its action on spinors $\chi_\alpha$

$$D_a(\omega)\chi_\alpha = \partial_a \chi_\alpha - \frac{1}{4} \omega^{bc}_{a} \eta_{bc} \chi_\alpha,$$  \hfill (65)

and on vectors $V^a$

$$D_a(\omega)V^b = \partial_a V^b + \omega^b_{a, c} V^c.$$  \hfill (66)

We introduce the one forms $e^a \equiv e^a_\mu dx^\mu$ associated with the vielbeins $e^a_\mu$ and the corresponding dual vector fields $\tilde{e}_a \equiv e^c_\mu \partial_\mu$. The anholonomy coefficients are defined to be the structure constants of Lie algebra associated with the duals vector fields $\tilde{e}_a$:

$$[\tilde{e}_a, \tilde{e}_b] \equiv c_{a b}^c \tilde{e}_c.$$  \hfill (67)

They can also be expressed in terms of the vielbeins $e^a_\mu$ as

$$c_{a b}^c = e^\mu_a e^c_\nu (\partial_\mu e^\nu_b - \partial_\nu e^\nu_a).$$  \hfill (68)

In the absence of torsion the spin connection can uniquely be determined in terms of the anholonomy coefficients using the relation:

$$\omega_{a,bc} = \frac{1}{2} [c_{a b, c} - c_{a c, b} - c_{b c, a}],$$  \hfill (69)

where $\omega_{a,bc} = -\omega_{b,c a}$ and $c_{ab,c} = -c_{ba,c}$. It is straightforward to compute the tangent space components of the Riemann tensor. It can be written in terms of the anholonomy coefficients and the spin connection as:

$$R_{abcd} = \partial_a \omega_{bc}^d - \partial_b \omega_{ac}^d - \omega_{ac} \omega_{bc}^d + \omega_{bc} \omega_{ac}^d - c_{ab} \omega_{ec}^d.$$  \hfill (70)

For constant anholonomy coefficients the partial derivatives acting on the spin connections vanish and as result the curvature tensor is expressed entirely as a function of the anholonomy coefficients.
B Gauged supergravity with one vector multiplet:

In this section we will discuss one of the simplest gauged supergravity model in five dimensions coupled to one vector multiplet constructed by Gunaydin and Zagermann [37,38]. We will outline some of the important results derived by them which are useful to study the generalised attractors. This gauged supergravity model consists of one gravity multiplet, one vector multiplet and two tensor multiplets with field contents:

\[
\{e^a_{\mu}, \psi^i_{\mu}, A^I_{\mu}, B^M_{\mu\nu}, \lambda^{\tilde{a}}, \phi^{\tilde{x}}\},
\]

with the indices taking values such that \(i = 1, 2; \mu = 0, \ldots, 4; a = 0, \ldots, 4; I = 0, 1; M = 2, 3; \tilde{x} = 1, 2, 3\) and \(\tilde{a} = 0, 1, 2, 3\). Here \(\psi^i_{\mu}\) are the gravitinos, \(A^I_{\mu}\) are the vectors in the gravity and vector multiplets, \(\lambda^{\tilde{a}}\) are the gaugini and \(B^M_{\mu\nu}\) are antisymmetric tensors in the tensor multiplets. The scalars in the vector and tensor multiplet are collectively written as \(\phi^{\tilde{x}}\). We also use the index \(\tilde{I}\) to label the vector and tensor multiplet indices collectively.

In five dimensional supergravity the moduli space of scalars \(h^{\tilde{I}} \equiv h^{\tilde{I}}(\phi)\) in the vector and tensor multiplets parametrise a very special manifold \(\mathcal{S}\) which is described by a cubic surface:

\[
N \equiv C_{IJK} h^{\tilde{I}} h^{\tilde{J}} h^{\tilde{K}} = 1.
\]

In the present case the scalar manifold is given by the coset space:

\[
\mathcal{S} = SO(1, 1) \times \frac{SO(2, 1)}{SO(2)}
\]

The symmetry group of the scalar manifold \(\mathcal{S}\) is \(G = SO(1, 1) \times SO(2, 1)\). We can gauge a subgroup of this symmetry group \(G\) as well as a subgroup of the \(R\)-symmetry group \(SU(2)_R\). One possibility for gauging is the \(SO(2) \subset SO(2, 1)\) symmetry of the scalar manifold using the graviphoton \(A^{0\mu}\) and the subgroup \(U(1)_R \subset SU(2)_R\) using \(A_{\mu}(U(1)_R) = V_I A^I_{\mu}\).

The symmetries of the scalar manifold can be made manifest by going to a suitable basis such that \(h^{\tilde{I}} = \sqrt{2} \xi^I|_{N=1}\) and \(h_{\tilde{I}} = \frac{\partial}{\partial \xi^I} N|_{N=1}\). In such a parametrisation, the constraint (72) takes the form

\[
N(\xi) = \sqrt{2} \xi^0 [(\xi^1)^2 - (\xi^2)^2 - (\xi^3)^2] = 1,
\]

where,

\[
\xi^0 = \frac{1}{\sqrt{2}||\phi||}; \quad \xi^1 = \phi^1; \quad \xi^2 = \phi^2; \quad \xi^3 = \phi^3,
\]

and,

\[
||\phi||^2 = (\phi^1)^2 - (\phi^2)^2 - (\phi^3)^2
\]
The scalar potential of this model is given by, for convenience, we also give the matrix representation for $a_{IJ}$.

The non-vanishing components of $C_{IJK}$ are given by $C_{011} = \frac{\sqrt{3}}{2}, C_{022} = C_{033} = -\frac{\sqrt{3}}{2}$. The $h^I$ are related to the fields $\phi$ in the Lagrangian through the following relations,

$$h^0 = \frac{1}{\sqrt{3}||\phi||^2}, \quad h^1 = \sqrt{\frac{2}{3}}\phi^1, \quad h^2 = \sqrt{\frac{2}{3}}\phi^2, \quad h^3 = \sqrt{\frac{2}{3}}\phi^3.$$

The moduli space metric has the expression

$$g_{\tilde{J}\tilde{K}} = \begin{pmatrix} 4(\phi^1)^2||\phi||^{-4} - ||\phi||^{-2} & -4\phi^1\phi^2||\phi||^{-4} & -4\phi^1\phi^3||\phi||^{-4} \\ -4\phi^1\phi^2||\phi||^{-4} & 4(\phi^2)^2||\phi||^{-4} + ||\phi||^{-2} & 4\phi^2\phi^3||\phi||^{-4} \\ -4\phi^1\phi^3||\phi||^{-4} & 4\phi^2\phi^3||\phi||^{-4} & 4(\phi^3)^2||\phi||^{-4} + ||\phi||^{-2} \end{pmatrix}.$$

The Killing vector that generates the $SO(2)$ symmetry is given by

$$K_0^2 = \left\{ -\frac{\phi^1}{||\phi||^2}, \frac{\phi^2}{||\phi||^2}, \frac{\phi^3}{||\phi||^2} \right\}.$$

For convenience, we also give the matrix representation for $a_{IJ}$:

$$a_{IJ} = \begin{pmatrix} ||\phi||^4 & 0 & 0 \\ 0 & 2(\phi^1)^2||\phi||^{-4} - ||\phi||^{-2} & -2\phi^1\phi^2||\phi||^{-4} \\ 0 & -2\phi^1\phi^2||\phi||^{-4} & 2(\phi^2)^2||\phi||^{-4} + ||\phi||^{-2} \end{pmatrix}.$$

The scalar potential of this model is given by,

$$\mathcal{V}(\phi) = \frac{g^2}{8} \left[ \frac{[(\phi^2)^2 + (\phi^3)^2]}{||\phi||^6} \right] - 2g_R^2 \left[ 2\sqrt{2} \frac{\phi^1}{||\phi||^2} V_0 V_1 + ||\phi||^2 V_1^2 \right].$$

**B.1 Generalised attractor solutions in Gauged supergravity with one vector multiplet.**

We summarise the generalised attractor solutions found earlier in [17] for reference. Using the generalised attractor procedure, some explicit examples of Bianchi attractors were constructed within the gauged supergravity model described in the previous section. Firstly, the vacuum $AdS_5$ solution is given by,
\[ ds^2 = L^2 \left[ -\hat{r}^2 d\hat{r}^2 + \frac{d\hat{r}^2}{\hat{r}^2} + \hat{r}^2 (d\hat{x}^2 + d\hat{y}^2 + d\hat{z}^2) \right] \]

\[ \phi^2_c = 0, \quad \phi^3_c = 0, \quad \phi^1_c = \frac{\sqrt{2}V_0}{V_1} \]  
\[ \Lambda = -6g_R^2 V_1^2 (\phi^1_c)^2, \]

\[ V_0 V_1 > 0, \quad 32g_R^2 V_0^2 \leq 1, \quad L^2 = \frac{6}{\Lambda}, \]  

(81)

where \( \Lambda \) is the cosmological constant. All the Bianchi examples exist at the same critical values of the scalars for which there is also an \( AdS \) solution. All of them are electrical and are sourced by a single time like gauge field (the graviphoton \( A^{0i} \))

\[ A^{0i} = \epsilon^i_0 A^{00} = \frac{\hat{r} - u_0}{L} A^{00}. \]  

(82)

The Lifshitz solution is given by

\[ ds^2 = L^2 \left[ -\hat{r}^{2u_0} d\hat{r}^2 + \frac{d\hat{r}^2}{\hat{r}^2} + \hat{r}^2 (d\hat{x}^2 + d\hat{y}^2 + d\hat{z}^2) \right], \]

\[ u_0 = 3, \quad L = \sqrt{3} \left( \frac{\phi^1_c}{g} \right)^4, \quad A^{00} = \frac{\sqrt{2}}{3} \frac{1}{(\phi^1_c)^2}, \]

\[ \phi^1_c = \left( \frac{\sqrt{2}V_0}{V_1} \right)^{\frac{1}{2}}, \quad V_0 V_1 > 0, \quad \frac{32g_R^2 V_0^2}{3(\phi^1_c)^4} \leq 1. \]  

(83)

The Bianchi II solution is given by,

\[ ds^2 = L^2 \left[ -\hat{r}^{2u_0} d\hat{r}^2 + \frac{d\hat{r}^2}{\hat{r}^2} + \hat{r}^{2u_2} d\hat{x}^2 + \hat{r}^{2(u_1 + u_2)} d\hat{y}^2 \right. \]

\[ - 2\hat{r}\hat{r}^{2(u_1 + u_2)} d\hat{y} d\hat{z} + \left[ \hat{r}^{2(u_1 + u_2)} \hat{x}^2 + \hat{r}^{2u_1} d\hat{z}^2 \right], \]

\[ u_0 = \sqrt{2}, \quad u_1 = u_2 = \frac{1}{2\sqrt{2}}, \quad L = \sqrt{2} \left( \frac{\phi^1_c}{g} \right)^4, \quad A^{00} = \sqrt{5} \frac{1}{8(\phi^1_c)^2}, \]

\[ \phi^1_c = \left( \frac{\sqrt{2}V_0}{V_1} \right)^{\frac{1}{2}}, \quad V_0 V_1 > 0, \quad \frac{23}{2(\phi^1_c)^4} \leq 1. \]  

(84)

whereas the Bianchi VI solution is given by,
\begin{align*}
\text{ds}^2 &= L^2 \left[ -\hat{r}^2 dt^2 + \frac{dr^2}{\hat{r}^2} + \hat{d}x^2 + e^{-2\hat{r}\hat{u}_1}dy^2 + e^{-2h\hat{r}\hat{u}_2}dz^2 \right], \\
u_0 &= \frac{1}{\sqrt{2}}(1 - h), \quad u_1 = -\frac{1}{\sqrt{2}}h, \quad u_2 = \frac{1}{\sqrt{2}}, \quad L = \frac{(\phi_1^c)^4}{\sqrt{6}g}(1 - h), \\
A_{00} &= \sqrt{\frac{-2h}{(-1 + h)^2 (\phi_1^c)^2}}, \quad h < 0, \quad h \neq 0, 1, \\
\phi_1^c &= \left( \frac{\sqrt{2} V_0}{V_1} \right)^{\frac{1}{2}}, \quad V_0 V_1 > 0, \quad \frac{8(3 - h + 3h^2)}{(\phi_1^c)^4(-1 + h)^2} \leq 1
\end{align*}

(85)

C Bianchi attractors in $U(1)_R$ gauged supergravity

We consider a truncated version of the gauged supergravity model discussed earlier with just $U(1)_R$ gauging. There is no gauging of the symmetries of the scalar manifold and hence there are no tensors as well. The field content of the reduced model is given by,

$$\{ e^a_\mu, \psi^i_\mu, A_I^\mu, \lambda^{i\bar{a}}, \phi^1 \},$$

with $I = 0, 1$ and $I = 0$ corresponds to the graviphoton as before. The field $\phi^1$ is the scalar in the single vector multiplet. The gauge field combination used for the $U(1)_R$ gauging is same as before. The potential of the $U(1)_R$ gauged supergravity contains only terms proportional to $g^2_R$ as there is no gauging of the symmetries of scalar manifold and can be obtained by setting $\phi^2 = \phi^3 = 0$ in (80),

$$\mathcal{V}(\phi^1) = -2g^2_R \left[ \frac{2\sqrt{2} V_0 V_1}{\phi_1^c} + (\phi_1^c)^2 V_1^2 \right].$$

(87)

We show the embedding of the $Li f_{w_0}(2) \times M_I$ solution, which has the form,

$$\text{ds}^2 = L^2 \left[ -\hat{r}^2 dt^2 + \frac{dr^2}{\hat{r}^2} + \hat{d}x^2 + dy^2 + dz^2 \right].$$

(88)

When $u_0 = 1$ we have the familiar $AdS_2 \times \mathbb{R}^3$ solution. We consider the gauge field ansatz as before,

$$A^{I\hat{I}} = e^I_0 A^{\hat{I}0} = \frac{1}{L^R} A^{\hat{I}0}. $$

(89)

For the $U(1)_R$ gauged supergravity the field equations at the attractor point for the gauge field, scalar field and Einstein equation are read off by setting $g = 0$ in
The corresponding field equations found for the general gauging considered in [17]. The gauge field equation has the form,

$$a_{IJ}[\omega_{a,c} F^{cbJ} + \omega_{a,c} F^{acJ}] = 0,$$

and is identically satisfied for the gauge field ansatz considered above. The scalar field equation is given by,

$$\frac{\partial}{\partial \phi^1} \left[ V_{\text{attr}}(\phi^1, A^{10}, A^{00}) \right] = 0, \quad V_{\text{attr}}(\phi^1, A^{10}, A^{00}) = V(\phi^1) + \frac{1}{4} a_{IJ} F^{I}_{ab} F^{J}_{ab}.$$  

At the critical point $\phi_c^1 = (\sqrt{\frac{V_0 V_1}{2}})^{\frac{1}{3}}$, it relates the parameters $V_0$ and $V_1$ to the charges,

$$(A^{10})^2 - 2(A^{00})^2(\phi_c^1)^6 = 0.$$  

The Einstein's equations are,

$$R_{ab} - \frac{1}{2} R \eta_{ab} = T_{ab}^{\text{attr}},$$

where

$$T_{ab}^{\text{attr}} = V_{\text{attr}}(\phi^1, A^{10}, A^{00}) \eta_{ab} - a_{IJ} F^{I}_{ab} F^{J}_{ab}.$$  

At the attractor point, there are only two independent equations in the above set,

$$3(A^{00})^2 u_0^2 (\phi_c^1)^5 - 12\sqrt{2} L^2 g_R^2 V_0 V_1 = 0,$$

$$u_0^2 (\phi_c^1)^{-2} + 3(A^{00})^2(\phi_c^1)^4) + 12\sqrt{2} L^2 g_R^2 V_0 V_1 = 0.$$  

Where we have used (92) for simplification. This can be solved to get,

$$L^2 = -\frac{u_0^2}{2\Lambda}, \quad \Lambda = -6g_R^2 V_1^2 (\phi_c^1)^2,$$

$$A^{00} = \frac{1}{\sqrt{3}(\phi_c^1)^2}, \quad A^{10} = \sqrt{\frac{2}{3}} \phi_c^1,$$

where $\Lambda$ is the AdS cosmological constant. Note that the Einstein equation does not place additional constraints on any of the gauged supergravity parameters $V_0, V_1, g_R$, unlike in the other Bianchi cases considered here. We summarise the solution,

$$ds^2 = L^2 \left[ -\dot{r}^2 + \frac{d\hat{r}^2}{\hat{r}^2} + d\hat{t}^2 + d\hat{x}^2 + d\hat{z}^2 \right],$$

$$A^{0i} = \frac{1}{L\hat{r}} A^{00}, \quad A^{i0} = \frac{1}{L\hat{r}} A^{10}, \quad \frac{A^{00}}{A^{10}} = \frac{V_1}{2 V_0}, \quad L^2 = -\frac{u_0^2}{2\Lambda},$$

$$\Lambda = -6g_R^2 V_1^2 (\phi_c^1)^2, \quad \phi_c^1 = \left(\frac{\sqrt{2} V_0}{V_1}\right)^{\frac{3}{2}}, \quad V_0 V_1 > 0.$$  

(97)
Note that the solution exists for any \( u_0 > 0 \) and in particular, when \( u_0 = 1 \) we get the familiar \( AdS_2 \times \mathbb{R}^3 \) solution.

The \( \text{Lif}_{u_0}(2) \times M_{II} \) metric has the form,

\[
ds^2 = L^2 \left[ - \dot{r}^2 u_0 \, dt^2 + \frac{dr^2}{r^2} + d\hat{x}^2 + d\hat{y}^2 + 2 \hat{x} \, d\hat{y} d\hat{z} + (\hat{x}^2 + 1) \, d\hat{z}^2 \right].
\]

(98)

We consider the same gauge field ansatz as for the previous case (89). As earlier, the gauge field equations are identically satisfied. The attractor equations are again same as (92) at the critical point. There are three independent Einstein equations given by,

\[
\begin{align*}
\phi_c^1 + 6(A^{0\bar{0}})^2 (\phi_c^1)^5 - 24\sqrt{2}L^2 g_R^2 V_0 V_1 &= 0, \\
\phi_c^1 - 4u_0^2 \phi_c^1 + 4(A^{0\bar{0}})^2 u_0^2 (\phi_c^1)^5 + 24\sqrt{2}L^2 g_R^2 V_0 V_1 &= 0, \\
-3\phi_c^1 - 4u_0^2 \phi_c^1 + 6(A^{0\bar{0}})^2 u_0^2 (\phi_c^1)^5 + 24\sqrt{2}L^2 g_R^2 V_0 V_1 &= 0,
\end{align*}
\]

(99)

where we have again used (92) for simplification. This set of algebraic equations can be solved to get,

\[
\begin{align*}
A^{0\bar{0}} &= \frac{\sqrt{2}}{u_0 (\phi_c^1)^2}, \\
A^{1\bar{0}} &= \frac{2\phi_c^1}{u_0}, \\
u_0 &= \sqrt{\frac{11}{2}}, \\
L^2 &= -\frac{13}{4\Lambda},
\end{align*}
\]

(100)

where \( \Lambda \) is the AdS cosmological constant. We summarise the \( \text{Lif}_{u_0}(2) \times M_{II} \) as,

\[
\begin{align*}
ds^2 &= L^2 \left[ - \dot{r}^2 u_0 \, dt^2 + \frac{dr^2}{r^2} + d\hat{x}^2 + d\hat{y}^2 + 2 \hat{x} \, d\hat{y} d\hat{z} + (\hat{x}^2 + 1) \, d\hat{z}^2 \right], \\
A^{0\bar{i}} &= \frac{1}{Lr} A^{0\bar{0}}, \quad A^{1\bar{i}} = \frac{1}{Lr} A^{1\bar{0}}, \quad \frac{A^{0\bar{0}}}{A^{1\bar{0}}} = \frac{V_1}{2 V_0}, \quad u_0 = \sqrt{\frac{11}{2}}, \\
L^2 &= -\frac{13}{4\Lambda}, \quad \Lambda = -6g_R^2 V_1 V_0 (\phi_c^1)^2, \quad \phi_c^1 = \left( \sqrt{\frac{2}{V_1}} \right)^{\frac{1}{5}}, \quad V_0 V_1 > 0.
\end{align*}
\]

(101)

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