THE GEOMETRY OF AMBIGUITY IN ONE-DIMENSIONAL PHASE RETRIEVAL

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Abstract. We consider the geometry associated to the ambiguities of the one-dimensional Fourier phase retrieval problem for vectors in \( \mathbb{C}^{N+1} \).

Our first result states that the space of signals has a finite covering (which we call the root covering) where any two signals in the covering space with the same Fourier intensity function differ by a trivial covering ambiguity.

Next we use the root covering to study how the non-trivial ambiguities of a signal vary as the signal varies. This is done by describing of the incidence variety of pairs of signals with same fourier intensity function modulo global phase. As an application we give a criterion for a real subvariety of the space of signals to admit generic phase retrieval. The extension of this result to multi-vectors played an important role in the author’s work with Bendory and Eldar on blind phaseless short-time fourier transform recovery.

1. Introduction

The discrete fourier transform of a vector \( x \in \mathbb{C}^{N+1} \) is the polynomial \( S^1 \to \mathbb{C} \) defined by the formula \( \hat{x}(\omega) = \sum_{n=0}^{N} x[n] \omega^n \) where we take \( \omega = e^{-i\theta} \) to be a coordinate on the unit circle. It is well-known that any vector \( x \) is uniquely recoverable from its DFT.

The phase retrieval problem asks if it possible to recover a vector \( x \in \mathbb{C}^{N+1} \) from its Fourier intensity function \( |\hat{x}(\omega)|^2 \). Obviously, if \( x \) and \( e^{i\theta} x \) have the same fourier intensity function as does the vector \( \hat{x} \) obtained by reflection and conjugation since \( \hat{x} = \overline{x} \). However, even modulo these trivial ambiguities the phase retrieval problem has no solution [6, 12]. In fact it is known that for given \( x \) there are up to \( 2^{N-1} \) vectors modulo trivial ambiguities with the same fourier intensity function. These vectors are referred to as the non-trivial ambiguities of the phase retrieval problem [1, 2].

By contrast, the 2D and higher phase retrieval problem is known to have a solution [6, 11]. Precisely if \( f : \mathbb{Z}_N^2 \to \mathbb{C} \) is a discrete function then generic \( f \) is uniquely determined modulo trivial ambiguities by the fourier intensity function \( |f(\omega, \eta)|^2 \). The reason for the difference is that for generic \( f \) the Fourier polynomial \( f(\omega, \eta) \) is irreducible, while in one variable \( f(\omega) \) always factors into distinct linear factors.

In this paper we consider the geometry associated to the ambiguities of the one-dimensional Fourier phase retrieval problem for vectors in \( \mathbb{C}^{N+1} \). Our first results (Theorem 3.2, 3.3) state that, the space of signals has a finite covering (which we call the root covering) where any two signals in the covering with the same Fourier intensity differ by a trivial covering ambiguity. In other words, we prove that phase retrieval is possible on the root cover.

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Next we use these results to study how the non-trivial ambiguities of a signal vector vary as the signal varies. To do this we describe (Theorem 1.1) the incidence variety $I$ consisting of pairs $(x, x')$ with same Fourier intensity function modulo global phase. We show that $I$ consists of $N + 1$ connected irreducible components, $I_0, \ldots, I_N$ and that the component $I_k$ is a finite covering of degree $\binom{N}{k}$ of the space signals modulo global phase.

Theorem 4.8 gives a geometric refinement of an earlier result of Beinert and Plonka [1 Theorem 2.3]. Our result states that the connected irreducible component $I_k$ of the incidence variety $I$ corresponds to pairs $(x, x')$ where $x = x_1 \ast x_2$, $x' = x_1 \ast x_2'$ for some vectors $x_1 \in \mathbb{C}^{k+1}, x_2 \in \mathbb{C}^{N-k+1}$. As a consequence, if $k \neq 0, N$ then for a generic pair $(x, x') \in I_k$, $x'$ is not obtained from $x$ by a trivial ambiguity. We also prove that if $(x, x') \in I_k$ then $(x, x') \in I_{N-k}$ where $x'$ is obtained from $x'$ by reflection and conjugation.

As an application we give (Theorem 5.1) a criterion for a real subvariety $W$ of the space of signals to admit generic phase retrieval. Precisely we prove that if there exists a single signal $w_0 \in W$ with the property that any $w_0' \in W$ with the same fourier intensity function is obtained from $w_0$ by a trivial ambiguity then the generic $w \in W$ has the same property. In other words, the condition that a signal $w$ lies in the subvariety $W$ enforces uniqueness of phase retrieval provided their exists a single signal in $W$ with this property. Examples of interesting $W$ include subvarieties of signals with a fixed entry or sparse signals [1 2]. This result for tuples of signals the motivation motivation for writing this paper. It plays a crucial role in the author’s work with Bendory and Eldar [3] proving that a pair of signals can be recovered from their blind phaseless short-time Fourier transform measurements using $10N$ measurements where $N$ is the length of the signal.

1.1. Notation. To slightly simplify our notation we assume that our signals are vectors $x \in \mathbb{C}^{N+1}$ as opposed to vectors in $\mathbb{C}^N$ which is often used in the literature [1 2]. We use the notation $x[n]$ to refer to the $n$th coordinate of a vector $x \in \mathbb{C}^{N+1}$; i.e. if $x = (x_0, \ldots, x_N)$ then $x[n] = x_n$. All signals $x \in \mathbb{C}^{N+1}$ are assumed to have full support. This means that $x[0], x[N]$ are both assumed to be non-zero. The set of such signals is parametrized by the complex variety $\mathbb{C}^* \times \mathbb{C}^{N-1} \times \mathbb{C}^*$ which we view as a real variety of dimension $2N + 2$. Because we work in $\mathbb{C}^{N+1}$ the Fourier intensity function $| \sum_{n=0}^{N} x[n] \omega^n |^2$ is non-negative real trigonometric polynomial of degree $N$ which is uniquely determined by $2N + 1$ coefficients.

1.2. Acknowledgments. The results of this paper were inspired by the papers of Beinert and Plonka [1 2] on the topic of ambiguities in fourier phase retrieval. The author is also grateful to Tamir Bendory for useful discussions.

2. Background on algebraic geometry

To make the paper self-contained we include this background material on algebraic geometry.

2.1. Real algebraic sets. A real algebraic set is a subset $X = V(f_1, \ldots, f_r) \subset \mathbb{R}^M$ defined by the simultaneous vanishing of polynomial equations $f_1, \ldots, f_r \in \mathbb{R}[x_1, \ldots, x_M]$. Note that any real algebraic set is defined by the single polynomial $F = f_1^2 + \ldots + f_r^2$. Given an algebraic set $X = V(f_1, \ldots, f_m)$ we define the Zariski topology on $X$ by declaring closed sets to be the intersections of $X$ with other algebraic subsets of $\mathbb{R}^M$. An algebraic set is irreducible if it is not the union of proper algebraic subsets. An irreducible algebraic set is
called a real algebraic variety. Every algebraic set has a decomposition into a finite union of irreducible algebraic subsets.

An algebraic subset of \( X \subset \mathbb{R}^M \) is irreducible if and only if the ideal \( I(X) \subset \mathbb{R}[x_1, \ldots, x_M] \) of polynomials vanishing on \( X \) is prime. More generally we declare an arbitrary subset of \( X \subset \mathbb{R}^M \) to be irreducible if its closure in the Zariski topology is irreducible. This is equivalent to the statement that \( I(X) \) is a prime ideal.

Note that in real algebraic geometry irreducible algebraic sets need not be connected in the classical topology. For example the real variety defined by the equation \( y^2 - x^3 + x \) consists of two connected pieces.

### 2.2. Semi-algebraic sets and their maps.

In real algebraic geometry it is natural to also consider subsets of \( \mathbb{R}^M \) defined by inequalities of polynomials. A semi-algebraic subset of \( \mathbb{R}^M \) is a finite union of subsets of the form

\[
\{ x \in \mathbb{R}^M ; P(x) = 0 \text{ and } Q_1(x) > 0 \text{ and } \ldots Q_\ell(x) > 0 \}
\]

(Note that if \( f \in \mathbb{R}[x_1, \ldots, x_M] \) the set \( f(x) \geq 0 \) is semi-algebraic since it is the union of the set \( f(x) = 0 \) with the set \( f(x) > 0 \).

The reason for considering semi-algebraic sets is that the image of an algebraic set under a real algebraic map need not be real algebraic. For a simple example consider the algebraic map \( \mathbb{R} \to \mathbb{R}, x \mapsto x^2 \). This map is algebraic but its image is the semi-algebraic set \( \{ x \geq 0 \} \subset \mathbb{R} \). A basic result in real algebraic geometry states that the image of a semi-algebraic set under a polynomial map is semi-algebraic \[4\] Proposition 2.2.7.

A map \( f : X \subset \mathbb{R}^N \to Y \subset \mathbb{R}^M \) of semi-algebraic sets is semi-algebraic if the graph \( \Gamma_f = \{(x, f(x))\} \) is a semi-algebraic subset of \( \mathbb{R}^N \times \mathbb{R}^M \). For example the map \( \mathbb{R}_{>0} \to \mathbb{R}_{>0}, x \mapsto \sqrt{x} \) is semi-algebraic since the graph \( \{(x, \sqrt{x})| x \geq 0 \} \) is the semi-algebraic subset of \( \mathbb{R}^2 \) defined by the equation \( x = y^2 \) and inequality \( x \geq 0 \). Again the image of a semi-algebraic set under a semi-algebraic map is semi-algebraic.

### 2.3. Dimension of a semi-algebraic sets.

A result in real algebraic geometry \[4\] Theorem 2.3.6] states that any real semi-algebraic subset of \( \mathbb{R}^n \) is homeomorphic as a semi-algebraic set to a finite disjoint union of sets diffeomorphic to hypercubes. Thus we can define the dimension of a semi-algebraic set \( X \) to be the maximal dimension of a hypercube in this decomposition. This can be shown to equal equals to the Krull dimension of the Zariski closure of \( X \) in \( \mathbb{R}^M \) \[4\], Corollary 2.8.9].

As a consequence we obtain the important fact that if \( Y \) is a semi-algebraic subset of an algebraic set \( X \) with \( \dim Y < \dim X \) then \( Y \) is a contained in a proper algebraic subset of \( X \).

### 2.4. Finite coverings of semi-algebraic sets.

Following \[8\] we say that a map of \( f : X \to Y \) of locally connected, connected Hausdorff topological spaces is a finite or ramified cover if it is open and closed and \( f^{-1}(y) \) is a finite non-empty set. Define the degree of \( f \) to be \( \sup\{|f^{-1}(y)|, y \in Y \} \) with the convention that that \( \deg f = \infty \) if the supremum does not exist. A result in point-set topology \[5\] Theorem I.10.2.1] states that these conditions are equivalent to the map \( f \) being proper with finite fibers.

1A map \( f : X \to Y \) of topological spaces is proper if for any topological space \( Z \) the induced map \( f : X \times Z \to Y \times Z \) is closed. This is analogous to the notion of universally closed in algebraic geometry. When \( X, Y \) are locally compact this is equivalent to the more familiar notion that the inverse image of any compact set is compact. \[5\] Proposition I.3.7]
In this paper all examples finite coverings come from group actions. If $X$ is a connected Hausdorff topological space and $G$ is a finite group acting discretely on $X$ then set of orbits $X/G$ is also a Hausdorff topological space and the orbit map $f : X \to X/G$ is a finite covering. This follows from a result in general topology [5] Proposition III.4.2.2] that states if $G$ is compact (for example finite) then $X/G$ is Hausdorff and the orbit map $X \to X/G$ is proper.

If $G$ acts with almost freely, meaning that the set of points with trivial stabilizer is dense then the degree of $f$ is $|G|$ because the fibers are orbits and the assumption implies a dense set of orbits has cardinality equal to $|G|$. A key fact about finite coverings is the following:

**Proposition 2.1.** Let $X \subset \mathbb{R}^N$, $Y \subset \mathbb{R}^M$ be semi-algebraic sets and let $f : X \to Y$ be a semi-algebraic map which is a finite covering. Then $\dim X = \dim Y$.

**Proof.** By the semi-algebraic triviality theorem [4] Theorem 9.3.2], $Y$ can be partitioned into a finite number of semi-algebraic sets $Y_1, \ldots, Y_r$ such that $f^{-1}(Y_\ell)$ is homeomorphic to $F_\ell \times Y_\ell$, where $F_\ell$ is the fiber over a point of $Y_\ell$. Since $f$ is a finite cover, $F_\ell$ is a finite set and we conclude that $\dim f^{-1}(Y_\ell) = \dim Y_\ell$ since two homeomorphic semi-algebraic sets have the same dimension [4] Theorem 2.8.8]. This also gives a partition of $X$ into a finite number of semi-algebraic sets of the same dimensions. Since the partition is finite we necessarily have that $\dim Y = \max_\ell \dim Y_\ell$ and likewise $\dim X = \max_\ell \dim f^{-1}(Y_\ell)$. Therefore $\dim X = \dim Y$. \hfill \Box

### 2.5. Finite coverings and quotients by finite groups in complex algebraic geometry.

We briefly consider finite covers of complex algebraic varieties. A complex algebraic subset $X \subset \mathbb{C}^M$ is the subset defined by the simultaneous vanishing of polynomial $f_1, \ldots, f_r \in \mathbb{C}[x_1, \ldots, x_M]$. As in the real case we define the Zariski topology by declaring algebraic sets to be closed. An algebraic subset which is irreducible is called a variety. Unlike the real case, any complex algebraic variety is connected.

If $X$ is a complex algebraic variety then the ring $\mathbb{C}[x_1, \ldots, x_M]/I(X)$ is called the coordinate ring of $X$ where $I(X)$ is the ideal of functions vanishing on $X$. We denote this ring by $\mathbb{C}[X]$. The ring $\mathbb{C}[X]$ is the ring of polynomial functions on $X$. Because $X$ is irreducible, $I(X)$ is a prime ideal so $\mathbb{C}[X]$ is an integral domain. Any polynomial map of complex varieties $f : X \subset \mathbb{C}^M \to Y \subset \mathbb{C}^M$ is induced by a ring homomorphism $f^* : \mathbb{C}[Y] \to \mathbb{C}[X]$. It is defined by the formula $f^*(h)(x) = h(f(x))$ where $h \in \mathbb{C}[Y]$.

We say that a map of complex varieties is a *finite algebraic cover* if the map $f^*$ is injective and $\mathbb{C}[X]$ is finitely generated as a $\mathbb{C}[Y]$ module. The degree of $f$ is the degree of the necessarily finite field extension $[k(X) : k(Y)]$. Any finite algebraic cover $f : X \to Y$ of degree $d$ is also a finite cover in the sense of topology where $X, Y$ are given the subspace topologies induced by $\mathbb{C}^M$ and $\mathbb{C}^N$ respectively. To see this we first note that a finite algebraic cover has finite fibers [10] Exercise 4.1] and is also projective. Hence, if $f : X \subset \mathbb{C}^M \to Y \subset \mathbb{C}^N$ is a finite algebraic map of varieties, then $f$ can be identified with a closed algebraic subset of $\mathbb{P}^s \times Y$ for some $s \geq 0$. Since complex projective space is compact in the analytic topology, the projection $\mathbb{P}^s \times Y \to Y$ is proper as a map of topological spaces. The map $f : X \to Y$ is the composition of a closed immersion with a proper map which implies that it is also proper.

A finite group $G$ acts algebraically on a variety $X$ if for each $g \in G$ the automorphism $X \to X$, $x \mapsto gx$ is a polynomial map. In particular, the group $G$ acts on the coordinate ring $\mathbb{C}[X]$. A fundamental result in invariant theory [9] states that the invariant subring
\( \mathbb{C}[X]^G := \{ h \in \mathbb{C}[X] | gh = h \ \forall g \in G \} \) is a finitely generated algebra, and that \( \mathbb{C}[X] \) is finitely generated \( \mathbb{C}[X]^G \) module. This means that there is a complex variety \( Y \) whose coordinate ring is \( \mathbb{C}[X]^G \) and the map \( X \to Y \) is a finite cover. In addition, if we view \( Y \) as a subset of \( \mathbb{C}^N \) then it can be identified with the set of orbits \( X/G \). As in topology, the degree of the finite cover \( X \to X/G \) equals to \( |G| \) when the set of points with trivial stabilizer is dense. (Indeed a deep result on actions of algebraic groups states that if there exists a single point with trivial stabilizer, then the set of points with trivial stabilizer is dense.)

3. Phase retrieval on the root covering

To understand the non-trivial ambiguities we pass to an auxiliary variety which we call the root covering. It parametrizes all orderings of the roots of the fourier polynomials of signals in \( \mathbb{C}^* \times \mathbb{C}^{N-1} \times \mathbb{C}^* \). The root covering has a bigger group of trivial ambiguities and we demonstrate that every vector in the root covering is determined modulo trivial ambiguities from the fourier intensity function of the corresponding signal.

3.1. The group of trivial ambiguities of the space of signals. We begin by identifying a group of trivial ambiguities acting on \( \mathbb{C}^* \times \mathbb{C}^N \times \mathbb{C}^* \) which preserves the fourier intensity function.

There is a natural free action of the circle group \( S^1 \) on \( \mathbb{C}^* \times \mathbb{C}^{N-1} \times \mathbb{C}^* \) where \( e^{i \theta} \) acts on a vector \( x \) by scalar multiplication. This action of \( S^1 \) clearly preserves the fourier intensity function \( |\hat{x}(\omega)|^2 \).

There is also an action of the group \( \mu_2 = \{ \pm 1 \} \) where the non-trivial element \( -1 \in \mu_2 \) takes \( x \) to \( \hat{x} \) where \( \hat{x} \) is obtained from \( x \) by reflection and conjugation. The action of \( \mu_2 \) is not free (since it fixes vectors \( x \in \mathbb{C}^{N+1} \) with the property that where \( x[k] = x[N-k] \) but it also preserves the fourier intensity function since \( \hat{x} = \bar{x} \).

The group generated by \( S^1 \) and the conjugation reflection involution is the continuous dihedral group \( S^1 \ltimes \mu_2 \). We refer to this group as the group of trivial ambiguities of the phase retrieval problem. In classical fourier phase retrieval (cf. [1] Proposition 2.1) shifts are also considered to be trivial ambiguities. However, we eliminate the shift ambiguity from the outset by assuming our signals have fixed support \([0, N]\).

The basic difficulty in phase retrieval is that the map
\[
(\mathbb{C}^* \times (\mathbb{C}^{N-1}) \times \mathbb{C}^*) / (S^1 \ltimes \mu_2) \to \mathbb{R}_{\geq 0}^{2N+1},
\]
x \mapsto |\hat{x}(\omega)|^2 is not injective. Indeed the generic fiber has \( 2^{N-1} \) points. The elements of the fiber are referred to as non-trivial ambiguities. In this paper we study how the non-trivial ambiguities vary with the signal.

3.2. The root covering. If \( x \in \mathbb{C}^{N+1} \) with \( x[N] \neq 0 \), then fourier transform can be encoded in the fourier polynomial \( \hat{x}(\omega) = \sum x[n] \omega^n \) where \( \omega = e^{i \theta} \) is a coordinate on the unit circle. By the fundamental theorem of algebra we can factor \( \hat{x}(\omega) = a_0(\omega-\beta_1)(\omega-\beta_2)\ldots(\omega-\beta_N) \).

If we assume that \( x \) has full support, the \( x[0] = (-1)^N a_0 \beta_1 \ldots \beta_N \neq 0 \), so none of the roots of \( \hat{x}(\omega) \) are 0.

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The reason we take the semi-direct product rather than the product is the actions of \( S^1 \) and \( \mathbb{Z}_2 \) do not commute. The semi-direct product consists of pairs \((\lambda, \pm 1)\) but the multiplication is non-commutative. Precisely, that \((\lambda, -1)(\mu, 1) = (\lambda \bar{\mu}, -1)\) while \((\mu, 1)(\lambda, -1) = (\lambda \mu, -1)\).
We denote $\mathbb{C}^* \times (\mathbb{C}^*)^N$ parametrizing tuples $(a_0, \beta_1, \ldots, \beta_N)$ as the root covering of the space of signals $\mathbb{C}^* \times \mathbb{C}^{N-1} \times \mathbb{C}^*$. The reason for this terminology is that we show (Proposition 3.1) that the map $\Phi: \mathbb{C}^* \times (\mathbb{C}^*)^N \to \mathbb{C}^* \times \mathbb{C}^{N-1} \times \mathbb{C}^*$ defined by the formula
\begin{equation}
(a_0, \beta_1, \ldots, \beta_N) \mapsto a_0(e_N(\beta_1, \ldots, -\beta_N), e_{N-1}(\beta_1, \ldots, -\beta_N), \ldots e_1(\beta_1, \ldots, -\beta_N), 1)
\end{equation}
where $e_n(\beta_1, \ldots, -\beta_N)$ indicates the $n$-th elementary symmetric polynomial in $(-\beta_1, \ldots, -\beta_N)$ is a finite algebraic covering.

By construction, the map $\Phi$ associates to $(N + 1)$-tuple $(a_0, \beta_1, \ldots, \beta_N)$ a vector $x = (a_0, x_1, \ldots, x_N)$ whose Fourier transform factors as
\[ \hat{x}(\omega) = \omega^{-N} a_0(\omega - \beta_1)(\omega - \beta_2)\ldots(\omega - \beta_N). \]

Note that the map $\Phi$ is multiple-to-one since any permutation of $(\beta_1, \beta_2, \ldots, \beta_N)$ produces the same vector.

**Proposition 3.1.** The map $\Phi$ is a finite algebraic covering of degree $N!$.

**Proof.** The map $\Phi$ is the composition $\sigma \circ \pi$ where $\pi: \mathbb{C}^* \times \mathbb{C}^N \to \mathbb{C}^*$ is the map

\[ (a_0, \beta_1, \ldots, \beta_N) \mapsto (a_0, e_N(\beta_1, \ldots, -\beta_N), \ldots, e_1(\beta_1, \ldots, -\beta_N)) \]

and $\sigma(a_0, a_1, \ldots, a_N) = (a_0, a_0 a_1, \ldots, a_0 a_N)$. The map $\sigma$ is an isomorphism of complex varieties with inverse given by $(a_0, b_1, \ldots, b_N) \mapsto (a_0, a_0^{-1} b_1, \ldots, b_N)$.

The map $\pi$ a finite algebraic cover of complex varieties of degree $N!$ since it is the $S_N$ quotient map $\mathbb{C}^* \times \mathbb{C}^N \to \mathbb{C}^* \times \mathbb{C}^N$ where $S_N$ acts by permuting the last $N$ coordinates. This fact follows from the classical fundamental theorem of symmetric polynomials which states that every symmetric polynomial can be unique expressed as a polynomial in the elementary symmetric functions and the polynomial ring is a free module over the ring of symmetric functions of degree $N!$. For a reference see [7, Chapter 7, Theorem 3].

Since $\sigma$ is an isomorphism this means we can identify the map $\mathbb{C}^* \times \mathbb{C}^{N-1} \times \mathbb{C}^*$ as the quotient of $(\mathbb{C}^*)^{N+1}$ by the action of the symmetric group $S_N$ where $S_N$ acts by permuting the last $N$ factors. Since $S_N$ acts with generically trivial stabilizer $\Phi$ is a finite algebraic covering of degree $N!$.

If we view $\Phi$ as a map of real algebraic varieties then Proposition (need to state or prove) implies that $\Phi$ is a finite covering of degree $N!$ in the sense of real algebraic geometry.

### 3.3. The group of trivial ambiguities of the root covering.

We now consider the group of trivial ambiguities of the root cover. Precisely we consider a group $G$ acting faithfully on $(\mathbb{C}^*)^{N+1}$ such that for all $\tilde{x} \in (\mathbb{C}^*)^{N+1}$ with $\Phi(\tilde{x}) = x$ and $g \in G$ with $\Phi(g \tilde{x}) = x'$ then $|\tilde{x}(\omega)|^2 = |\tilde{x}'(\omega)|^2$.

**Theorem 3.2** (The ambiguity group of the root cover). The group $G = S^1 \ltimes ((\mu_2)^N \ltimes S_N)$ is a group of trivial ambiguities for phase retrieval on $(\mathbb{C}^*)^{N+1}$.

We refer to $G$ as the root ambiguity group.

**Proof of Theorem 3.2** We first describe the fourier intensity preserving action of $G = S^1 \ltimes ((\mu_2)^N \ltimes S_N)$ on $(\mathbb{C}^*)^{N+1}$.

The action of $S^1$ is given as follows: If $\tilde{x} = (a_0, \beta_1, \ldots, \beta_N)$ then $\lambda \cdot \tilde{x} = (\lambda a_0, \beta_1, \ldots, \beta_N)$. The effect of the action of $S^1$ on $\Phi(\tilde{x})$ is to multiply each entry of $\Phi(\tilde{x})$ by the scalar $\lambda$. Since $\lambda \in S^1$ this does not change the fourier intensity function.
We now describe the action of \((\mu_2)^N \ltimes S_N\). The symmetric group \(S_N\) acts by permuting \(\beta_1, \ldots, \beta_N\). Since the elementary symmetric polynomials are invariant under permutations of \(\beta_1, \ldots, \beta_N\), if \(\tau \in S_N\) then \(\Phi(a_0, \beta_1, \ldots, \beta_N) = \Phi(a_0, \beta_{\tau(1)}, \ldots, \beta_{\tau(N)})\), so \(\Phi(\tau \cdot \tilde{x}) = \Phi(\tilde{x})\).

The group \((\mu_2)^N\) is generated by elements \(s_i = (1, \ldots, 1, -1, 1, \ldots, 1)\) where the \(-1\) is in the \(i\)th position. The element \(s_i\) acts on \(\tilde{x} = (a_0, \beta_1, \ldots, \beta_N)\) by

\[\tilde{x}_i = (a_0|\beta_i|, \beta_1, \ldots, \beta_{i-1}, \overline{\beta_i}, \beta_{i+1}, \ldots, \beta_N).\]

The actions of \(S_N\) and \(\mu_2^N\) do not commute since \(\tau s_i\tilde{x} = s_{\tau(i)}\tau \tilde{x}\). Thus we have an action of the semi-direct of \(\mu_2^N \ltimes S_N\) where \(S_N\) acts on \(\mu_2^N\) by permutation. (Note that the action of \(\mu_2\) is only semi-algebraic because we need to multiply by \(|a_i|\) in order to ensure that \(s_i^2\) acts as the identity.)

Let us verify that if \(x' = \Phi(s_i \cdot \tilde{x})\) and \(x = \Phi(x)\) then \(x'\) and \(x\) have the same fourier intensity function. The fourier transform of \(x\) is \(\hat{x}(\omega) = a_0\omega^{-N} \prod_{i=1}^{N} (\omega - \beta_i)\). Thus

\[|\hat{x}(\omega)|^2 = (|a_0|^2 \prod_{i=1}^{N} (\omega - \beta_i)(\omega^{-1} - \overline{\beta_i})\]

while

\[\hat{x}'(\omega) = a_0\beta_i\omega^{-N}(\omega - \beta_1)\cdots(\omega - \beta_{i-1})(\omega - \overline{\beta_i}^{-1})(\omega - \beta_{i+1})\cdots(\omega - \beta_N)\]

so

\[|\hat{x}'(\omega)|^2 = |a_0\beta_i|^2(\omega + \beta_1)(\omega^{-1} + \overline{\beta_1})\cdots(\omega + \beta_{i-1})(\omega^{-1} + \overline{\beta_{i-1}})(\omega + \beta_{i+1})(\omega^{-1} + \overline{\beta_{i+1}})\cdots(\omega + \beta_N)(\omega^{-1} + \overline{\beta_N})\]

Since

\[(\omega + \overline{\beta_i}^{-1})(\omega^{-1} + \beta_i^{-1}) = \frac{1}{\beta_i\overline{\beta_i}}(\omega^{-1} + \overline{\beta_i})(\omega + \beta_i)\]

we see that the two fourier intensity functions are the same.

Finally, note the actions of \(S_N\) and \(\mu_2^N\) do not commute since \((\tau s_i)\tilde{x} = (s_{\tau(i)}\tau)\tilde{x}\) which corresponds to an action of the semi-direct product \((\mu_2)^N \ltimes S_N\) where \(S_N\) acts on \((\mu_2)^N\) by permutations.

\[\square\]

3.4. Phase retrieval on the root cover. Our next result shows that phase retrieval is possible on the root coverings modulo its larger group of trivial ambiguities. In other words, every vector \(\tilde{x} \in (\mathbb{C}^*)^{N+1}\) can be recovered from the corresponding fourier intensity function up to the action of the group \(G = S^1 \ltimes (\mu_2^N \ltimes S_N)\).

**Theorem 3.3** (Phase retrieval on the root cover). Every \(\tilde{x}\) can be uniquely determined modulo the root ambiguity group \(G\) from the Fourier intensity function of \(\Phi(x)\).

In other words the map \((\mathbb{C}^*)^{N+1}/G \to \mathbb{R}_{\geq 0}^{2N+1}\) which sends the orbit of \(\tilde{x}\) to the coefficients of the Fourier intensity function of \(\Phi(x)\) is well defined and injective.

**Proof of Theorem 3.3** Suppose that \(x = \Phi(\tilde{x})\) and \(x' = \Phi(\tilde{x}')\) have the same fourier intensity function where \(\tilde{x} = (a_0, \beta_1, \ldots, \beta_N)\) and \(\tilde{x}' = (a_0', \beta_1', \ldots, \beta_N')\). We wish to show that \(\tilde{x}\) can be obtained from \(\tilde{x}'\) by the action of the root ambiguity group \(G\).
Expanding out the fourier intensity functions we have
\[ |\hat{x}(\omega)|^2 = \omega^{-N}(|a_0|^2 \prod_{i=1}^|\beta_i| \prod (\omega - \beta_i)(\omega - \bar{\beta}_i^{-1}) \]
and
\[ |\hat{x}'(\omega)|^2 = \omega^{-N}(|a_0|^2 \prod_{i=1}^|\beta_i'| \prod (\omega - \beta'_i)(\omega - \bar{\beta}'_i^{-1}) \]
Since the polynomial ring in one-variable is a unique factorization domain we must have that
\[ |a_0|^2 \prod \beta_i = |a_0|^2 \prod \bar{\beta}_i \] and an equality of sets \{\beta_1, \bar{\beta}_1^{-1}, \ldots, \beta_N, \bar{\beta}_N^{-1}\} = \{\beta_1', \bar{\beta}_N', \ldots, \beta_N', \bar{\beta}_N'\}
Hence after reordering the \(\beta_1, \ldots, \beta_N\) which corresponds to applying a permutation to \(\hat{x}\) we may assume that \(\beta_i' = \beta_i^{-1}\) if \(i \in S\) and \(\beta_i' = \beta_j\) if \(j \in S^c\). (Note that \(S\) is uniquely determined if and only none of the \(\beta_i\) lies on the unit circle.) Let \(S = \prod_{i \in S} s_i\). Then \(s \hat{x}' = (a_0' \prod_{i \in S} \bar{\beta}_i^{-1}, \bar{\beta}_1^{-1}, \ldots, \bar{\beta}_N^{-1})\). Since \(\Phi(s \hat{x}')\) and \(\hat{x}\) have the same fourier intensity function we conclude that \(|a_0' \prod_{i \in S} \bar{\beta}_i^{-1}| = |a_0|\). Hence there is a scalar \(\lambda \in S^1\) such that \(\lambda s \hat{x}' = x\).
This concludes the proof. □

3.5. The fourier intensity map for signals modulo trivial ambiguities. The space of signals modulo trivial ambiguities is the quotient of the variety \(\mathbb{C}^* \times \mathbb{C}^{N-1} \times \mathbb{C}^*\) by the group \(S^1 \times \mu_2\). Since \(\mathbb{C}^* \times \mathbb{C}^{N-1} \times \mathbb{C}^*\) is the quotient of \(\mathbb{C}^* \times (\mathbb{C}^*)^N\) by \(S_N\) we can realize the quotient of \(\mathbb{C}^* \times \mathbb{C}^{N-1} \times \mathbb{C}^*\) by its group of ambiguities as a quotient of the root cover \(\mathbb{C}^* \times (\mathbb{C}^*)^N\).

**Proposition 3.4.** The space of signals modulo trivial ambiguities \((\mathbb{C}^* \times \mathbb{C}^{N-1} \times \mathbb{C}^*)/S^1 \times \mu_2\) is homeomorphic to the quotient of \((\mathbb{C}^* \times (\mathbb{C}^*)^N)\) by a subgroup \(H\) of the root ambiguity group \(G\) of index \(2^{N-1}\).

**Proof.** Let \(H\) be the subgroup of \(G = S^1 \times (\mu_2^N \times S_N)\) consisting triples \((\lambda, s, \tau)\) where \(\lambda \in S^1, \tau \in S_N\) and \(s = (1, \ldots, 1)\) or \(s = (-1, \ldots, -1)\). Since \((-1, \ldots, -1)\) and \((1, \ldots, 1)\) are invariant under the action of permutations, this subgroup is isomorphic to the semi-direct product \(S^1 \times (\mu_2 \times S_N)\). Moreover, the group \(S_N\) acts trivially on \(S^1\) so this semi-direct product is the same as \(S_N \times (S^1 \times \mu_2)\). In particular \(S_N\) is a normal subgroup. Taking the quotient by the action of \(S_N\) produces \(\mathbb{C}^* \times \mathbb{C}^{N-1} \times \mathbb{C}^*\) with a residual action of the quotient group \(S^1 \times \mu_2\).

To complete the proof of Proposition 3.4 we need show that the involution of \(\mathbb{C}^* \times \mathbb{C}^{N-1} \times \mathbb{C}^*\) coming from \((-1) \in \mu_2\) is the involution \(x \mapsto \hat{x}\). This follows from the following lemma. We omit the proof because it follows from the arguments used in Beinert and Plonka’s paper [1] cf. Proof of Corollary 3.3.

**Lemma 3.5.** If \(\beta_1, \ldots, \beta_N\) the roots of \(\hat{x}(\omega)\) then the roots of \(\hat{x}(\omega)\) are \(\bar{\beta}_1^{-1}, \ldots, \bar{\beta}_N^{-1}\).

**Remark 3.6.** One might hope that there is a larger group \(G\) of ambiguities acting on \(\mathbb{C}^* \times \mathbb{C}^{N-1} \times \mathbb{C}^*\) such that the fourier intensity function is injective modulo this group. Such a group would necessarily be a quotient of the root ambiguity group \(G\) by the symmetric group \(S_N\). However, \(H\) is not a normal subgroup of the full root ambiguity group \(G\) so the quotient \(G/H\) is not a group. There is a map of quotients
\[(\mathbb{C}^* \times \mathbb{C}^{N-1} \rightarrow \mathbb{C}^*)/(S^1 \times \mu_2) = (\mathbb{C}^* \times (\mathbb{C}^*)^N)/H \rightarrow (\mathbb{C}^* \times (\mathbb{C}^*)^N)/G\]
which is $G/H$ fibration. This is a finite covering of connected, irreducible semi-algebraic degree $|G/H| = 2^{N-1}$ corresponding to the $2^{N-1}$ vectors modulo trivial ambiguities with the same fourier intensity function.

Precisely the fourier intensity map $(\mathbb{C}^* \times (\mathbb{C}^*)^N)/H = (\mathbb{C}^* \times \mathbb{C}^{N-1} \times \mathbb{C}^*)/(S^1 \times \mu_2) \to \mathbb{R}^{2N+1}$ factors as

\[
\begin{array}{c}
(\mathbb{C}^* \times \mathbb{C}^{N-1} \times \mathbb{C}^*)/(S^1 \times \mu_2) \\
\downarrow \\
(\mathbb{C}^* \times (\mathbb{C}^*)^N)/G \\
\to \mathbb{R}^N
\end{array}
\]

where the bottom arrow is injective and the diagonal arrow is a finite covering of degree $2^{N-1}$.

4. The incidence variety of ambiguities

Let $X$ be the quotient of the space $\mathbb{C}^* \times \mathbb{C}^{N-1} \times \mathbb{C}^*$ by the the free action of $S^1$. The semi-algebraic map $(a_0, a_1, \ldots, a_{N-1}, a_N) \to (|a_0|, \frac{a_0}{|a_0|} a_1, \ldots, \frac{a_0}{|a_0|} a_N)$ identifies $X$ with the semi-algebraic variety $\mathbb{R}_{>0} \times \mathbb{C}^{N-1} \times \mathbb{C}^*$. The space $X$ is the space of equivalence classes of signals modulo global phase. Since $S^1$ is a normal subgroup of $S^1 \times \mu_2$ there is an action of $\mu_2$ on $X$.

If $x \in X$ is represented by $(a_0, a_1, \ldots, a_{N-1}, a_N)$ with $a_0 \in \mathbb{R}_{>0}$. Then $(-1) \cdot x$ is represented by $(|a_N|, \frac{a_N}{|a_N|} a_{N-1}, \ldots, \frac{a_N}{|a_N|} a_0)$.

Let $I \subset X \times X$ be the subset of pairs $(x, x')$ of equivalence classes of signals such that $|\hat{x}(\omega)| = |\hat{x}'(\omega)|$. We call $I$ the fourier intensity incidence correspondence. Since $I$ is defined by real algebraic equations we say that $I$ is a real algebraic subset of the semi-algebraic set $X \times X$.

The goal of this section is to describe the decomposition of $I$ into irreducible components.

**Theorem 4.1.** (i) The real algebraic subset $I \subset X \times X$ decomposes into $N + 1$ irreducible components $I_0, \ldots, I_N$ each of which is connected.

(ii) The projection $I_k \to X$ is a finite cover of degree $\binom{N}{k}$.

(iii) The total degree of the map $I \to X$ is $\sum_{n=0}^{N} \binom{N}{k} = 2^N$ and $I_0$ and $I_N$ are both isomorphic to $X$ as semi-algebraic sets.

(iv) There is an additional action of $\mathbb{Z}_2$ on $I$ given by $(x, x') \mapsto (x, \hat{x}')$. Under this action $I_k \mapsto I_{N-k}$.

(v) If $(x, x') \in I_k \setminus (I_k \cap (I_0 \cup I_n))$ then $x'$ is not obtained from $x$ by a trivial ambiguity.

**Remark 4.2.** We denote the union $\bigcup_{k \neq 0,N} I_k$ by $I^0$. The generic point of $I^0$ is a pair of $S^1$-equivalence classes $(x, x')$ such that $|\hat{x}(\omega)| = |\hat{x}'(\omega)|$ but $x'$ is not obtained from $x$ by a trivial ambiguity.

**Remark 4.3.** For a generic vector $x \in \mathbb{C}^{N+1}$ there are, modulo trivial ambiguities, $2^{N-1}$ vectors $x'$ such that $|\hat{x}(\omega)|^2 = |\hat{x}'(\omega)|^2$. This follows from our result since the finite covering $(I/\mathbb{Z}_2) \to X$ has degree $2^N/2 = 2^{N-1}$ so the generic fiber has $2^{N-1}$ points.

Our result explains how the $2^{N-1}$ points are partitioned into $\lceil (N+1)/2 \rceil$ components.
4.1. The incidence correspondence in the root covering. To prove Theorem 4.1 we again pass to the root covering.

Let $\tilde{X}$ be the quotient of $\mathbb{C}^* \times (\mathbb{C}^*)^N$ by the free action of $S^1$ on $(\mathbb{C}^*)^{N+1}$ given by $e^\theta (a_0, \beta_1, \ldots, \beta_N) = (e^\theta a_0, \beta_1, \ldots, \beta_N)$. The semi-algebraic map $(a_0, \beta_1, \ldots, \beta_N) \mapsto (|a_0|, \beta_1, \ldots, \beta_N)$ identifies $\tilde{X}$ with $\mathbb{R}_{>0} \times (\mathbb{C}^*)^N$.

The map $\Phi$ is $S^1$-equivariant where $S^1$ acts on $(\mathbb{C}^*)^{N+1}$ and $\mathbb{C}^* \times \mathbb{C}^{N-1} \times \mathbb{C}^*$ as above. Since the action of $S_N$ also commutes with the $S^1$ action. Hence there is an induced map $\tilde{\Phi}: \tilde{X} \to X$ which identifies $X$ as the quotient $\tilde{X}$ by $S_N$.

As a consequence of Proposition 3.3 we have

**Proposition 4.4.** The map $\tilde{\Phi}$ is a finite algebraic covering of degree $N!$.

Let $\tilde{I}$ be the inverse image of the incidence $I$ under the product map $(\tilde{X} \times \tilde{X}) \xrightarrow{\tilde{\Phi} \times \tilde{\Phi}} (X \times X)$. As a set

$$\tilde{I} = \{(\tilde{x} = (a_0, \beta_1, \ldots, \beta_N), \tilde{x}' = (a'_0, \beta'_1, \ldots, \beta'_N)) | |\tilde{x}(\omega)|^2 = |\tilde{x}'(\omega)|^2 \text{ and } \forall n \beta'_n \in \{\beta_n, \overline{\beta_n}^{-1}\}\} \subset \tilde{I} \times \tilde{I}$$

where $x = \tilde{\Phi}(\tilde{x})$ and $x' = \tilde{\Phi}(\tilde{x}')$. We refer to $\tilde{I}$ as the root incidence variety.

Since $I \subset X \times X$ is a real algebraic subset of $X \times X$ and $\tilde{\Phi}$ is a polynomial map, $\tilde{I}$ is a real algebraic subset of $\tilde{X} \times \tilde{X}$ because the inverse image of an algebraic set under a polynomial map is always algebraic.

**Proposition 4.5.** The incidence $\tilde{I}$ decomposes into $2^N$ irreducible components each isomorphic (via a semi-algebraic isomorphism) to $\tilde{X}$ embedded as the diagonal in $\tilde{X} \times \tilde{X}$. (In particular each irreducible component is connected.)

Proof. Let $\tilde{x} = (a_0, \beta_1, \ldots, \beta_N)$ and $\tilde{x}' = (a'_0, \beta'_1, \ldots, \beta'_N)$ be vectors in $\tilde{X}$ and let $x = (a_0, a_1, \ldots, a_N)$ and $x' = (a'_0, a'_1, \ldots, a'_N)$ be their images in $X$. By the proof of Theorem 3.3 we know that $|\tilde{x}(\omega)|^2 = |\tilde{x}'(\omega)|^2$ if and only if after possibly reordering the $\beta_i$ there exists a subset $S \subset \{1, \ldots, N\}$ such that $\beta'_i = \overline{\beta_i}^{-1}$ for $i \in S$ and $\beta'_i = \beta_i$ if $i \in S^c$ and $\prod_{i=1}^N \frac{\beta_i}{\beta'_i} = (a_0/a'_0)^2$.

Hence $\tilde{I}$ is the union of $2^N$ closed real algebraic subsets indexed by subsets of $\{1, \ldots, N\}$. Specifically if $S$ is a subset then we let

$$\tilde{I}_S = \{(a_0, \alpha_1, \ldots, \alpha_N), (a'_0, \alpha'_1, \ldots, \alpha'_N) | \alpha'_i = \overline{\alpha_i}^{-1} \text{ for } i \in S, \prod_{i=1}^N \frac{\beta_i}{\beta'_i} = (a_0/a'_0)^2\}$$

Each of the $I_S$ is connected and irreducible because there is a semi-algebraic isomorphism $\tilde{X} \to I_S$ given by $(a_0, \alpha_1, \ldots, \alpha_N) \mapsto \{(a'_0, \alpha'_1, \ldots, \alpha'_N)\}$ where $\alpha'_i = \overline{\alpha_i}^{-1}$ if $i \in S$ and $\alpha'_i = \alpha_i$ if $i \notin S$ and

$$a'_0 = a_0 \left(\prod_{i=1}^N \frac{\beta'_i}{\beta_i}\right)$$

and $\tilde{X}$ is connected and irreducible. \hfill \Box

**Remark 4.6.** Note that the $I_S \cap I'_S$ can be identified with the real subvariety of $\tilde{X}$ consisting of tuples $\tilde{x} = (a_0, \alpha_1, \ldots, \alpha_N)$ where $|\alpha_i| = 1$ for $i \in (S \cup S') \setminus (S \cap S')$. Hence $\cap S I_S$ can be identified with $\mathbb{R}_{>0} \times (S^1)^N$, corresponding to vectors all of whose fourier roots lie on the unit circle.
4.2. **Proof of Theorem 4.1.** To prove the theorem we need to understand the images in $I$ of the irreducible components $I_S$ of $\tilde{I}$.

**Lemma 4.7.** The image $\tilde{I}_S$ equals the image of $\tilde{I}_{S'}$ if and only $|S| = |S'|$.

**Proof.** If $|S| = |S'|$ then there is a permutation $\tau \in S_N$ such that $\tau(S) = S'$. Under the diagonal action of $S_N$ on $\tilde{I}$ given by

$$\tau((a_0, \alpha_1, \ldots, \alpha_N), (a'_0, \alpha'_1, \ldots, \alpha'_N)) = ((a_0, \alpha_{\tau(1)}, \ldots, \alpha_{\tau(N)}), (a'_0, \alpha'_{\tau(1)}, \ldots, \alpha'_{\tau(N)}))$$

$\tilde{I}_S$ is mapped to $\tilde{I}_{S'}$. Since the map $\tilde{I} \to I$ is $S_N$ invariant, it follows that $I_S$ and $I_{S'}$ have the same image.

Conversely suppose that $|S| \neq |S'|$. Without loss of generality we may assume that $|S| < |S'|$. Also we can find a permutation $\tau$ such that $\tau(S)$ is a proper subset of $\tau(S')$. Applying another permutation allows us to assume that $S = \{1, \ldots, k\}$ and $S' = \{1, \ldots, l\}$ with $l > k$.

If $\alpha_0, \ldots, \alpha_N$ are chosen to be distinct and none of them lie on the unit circle (for example we can take the $\alpha_i$ to be positive real numbers more than 1) then the image of the pair

$$(\tilde{x}, \tilde{x}') = \left((a_0, \alpha_1, \ldots, \alpha_N), (a'_0, \alpha^{-1}_1, \ldots, \alpha^{-1}_l, \alpha_{l+1}, \ldots, a_N)\right) \in I_{S'}$$

is not in the image of $I_S$.

Likewise,

$$(\tilde{x}, \tilde{x}') = \left((a_0, \alpha_1, \ldots, \alpha_N), (a'_0, \alpha^{-1}_1, \ldots, \alpha^{-1}_k, \alpha_{k+1}, \ldots, a_N)\right) \in I_S$$

is not in the image of $I_{S'}$.

\[\square\]

**Proof Theorem 4.1.**

(i) Since each $I_S$ is irreducible and connected, their images are irreducible so $I$ consists of $N + 1$ irreducible and connected components $I_0, \ldots, I_N$ where $I_k$ is the image of $I_S$ for any subset $S \subset \{1, \ldots, N\}$ such that $|S| = k$. (This includes the empty set.)

(ii,iii) We now compute the degree of the projection $I_k \to X$. We know that if $S$ is any subset with $|S| = k$ then the map $\tilde{I}_S \to I_k \to X$ has degree $N!$ since $I_S$ is homeomorphic to $X$. Two general elements of $\tilde{I}_S$ have the same image in $I_k$ if and only if there is a permutation $\tau \in S_N$ such that $\tau(S) = S$ and $\tau(S') = S'$. Hence $I_k$ may be identified with the quotient of $\tilde{I}_S$ by a subgroup of $S_N$ isomorphic to $S_k \times S_{N-k}$. Hence the degree of of the map $\tilde{I}_S \to I_k$ is $k!(N-k)!$. Since the degree of a finite map is multiplicative it follows that the degree of the map $\frac{N!}{k!(N-k)!} = \binom{N}{k}$.

(iv) The involution (order two automorphism) of $\tilde{I}$ given by

$$((a_0, \beta_1, \ldots, \beta_N), (a'_0, \beta'_1, \ldots, \beta'_N)) \mapsto ((a_0, \beta_1, \ldots, \beta_N), (a'_0, \overline{\beta'_1}, \overline{\beta'_N}, (\overline{\beta'_1})^{-1}, \ldots, (\overline{\beta'_N})^{-1}))$$

takes $\tilde{I}_S \to \tilde{I}_{S'}$. Given $(\tilde{x}, \tilde{x}') \in \tilde{I}$, let $(\tilde{x}, \tilde{x}')$ be it’s image under the involution. If $(x, x')$ is the image in $I$ of $(\tilde{x}, \tilde{x}')$ then the image in $I$ of $(\tilde{x}, \tilde{x}')$ is $(x, x')$ where $x'$ is obtained by conjugation and reflection. If $|S| = k$ then $|S'| = |N - k|$, so we see that $I_k \leftrightarrow I_{N-k}$ under the involution $(x, x') \mapsto (x, x')$.

(v) Given $x \in X$ let $\beta_1, \ldots, \beta_N$ be the roots of the Fourier polynomial $\hat{x}(\omega)$. For generic $x$ non of the roots $\beta_1, \ldots, \beta_N$ lie on the unit circle. If $(x, x') \in I_k$ and $\beta'_1, \ldots, \beta'_N$ are the
roots $\hat{x}'(\omega)$ then there is a subset $S \subset \{1, \ldots, N\}$ such that $\beta_i' = \beta_i^{-1}$ for $i \in S$ and $\beta_i' = \beta_i$ for $i \in S^c$. If none of $\beta_1, \ldots, \beta_N$ lie on the unit circle in the complex plane, then by Lemma 3.5, $x' \neq \hat{x}$ unless $S = \{1, \ldots, N\}$ meaning $|S| = N$. Hence if $0 < k < N$ then for generic pair $(x, x') \in I_k$ $x' \neq x$ and $x' \neq \hat{x}$.

4.3. Characterization of the components of $I$ in terms of convolution. In [1, Theorem 2.3], Beinert and Plonka prove that two signals $x$ and $y$ have the same fourier intensity function if and only there exists finite signals $x_1, x_2$ such that $x = x_1 \ast x_2$ and $y = \lambda x_1 \ast \hat{x}_2$ for some $\lambda \in S^1$.

Their result can be made more precise by using our analysis of the irreducible components of the incidence variety.

**Theorem 4.8.** The component $I_k \subset I$ parametrizes all pairs of equivalence classes $(x, x')$ such that there exist vectors $x_1 \in \mathbb{C}^{k+1}, x_2 \in \mathbb{C}^{N-k+1}$ such that $x = x_1 \ast x_2$ and $x' = x_1 \ast \hat{x}_2$.

**Proof.** If $x = x_1 \ast x_2$ then $\hat{x}(\omega) = \hat{x}_1(\omega)\hat{x}_2(\omega)$. Thus if $\hat{x}_1(\omega) = \omega^{-k}a_0(\omega - \beta_1) \cdots (\omega - \beta_k)$ and $\hat{x}_2(\omega) = \omega^{N-k}a'_0(\omega - \beta_{k+1}) \cdots (\omega - \beta_{N-k})$ then $\hat{x}(\omega) = \omega^{-N}(a_0a'_0\omega^{-1}) \cdots (\omega - \beta_N)$. Similarly if $x' = x_1 \ast \hat{x}_2$ then $\hat{x}' = \omega^{-N}(\beta_{k+1} \cdots \beta_{N-k}) (\omega + \beta_1) \cdots (\omega + \beta_k)(\omega + \hat{\beta}_{k+1}^{-1}) \cdots (\omega + \hat{\beta}_N^{-1})$. Hence $(x, x') \in I_k$. The converse is similar. \qed

**Remark 4.9.** Theorem 4.8 above says that we can identify $I_k$ with the image of $\mathbb{C}^{k+1} \times \mathbb{C}^{N-k}$ under the map $(x_1, x_2) \mapsto (x_1 \ast x_2, (x_1 \ast \hat{x}_2))$.

5. Phase retrieval for vectors satisfying an algebraic condition

We can use our description of the incidence to prove that the generic vector satisfying any algebraic constraint can be uniquely recovered from its fourier intensity function, provided there exists one such vector. Examples include vectors with a fixed entry or sparse vectors. This technique for multi-vectors played in a crucial role in the paper [3] on STFT.

**Theorem 5.1** (Phase retrieval for vectors satisfying an algebraic condition). Let $W \subset X$ be an real subvariety of $X$ and suppose that there exists a point $w_0 \in W$ such that for all $(w_0, w'_0) \in \pi^{-1}(w_0) \setminus (I_0 \cup I_N)$, $w'_0 \notin W$ then a generic $w \in W$ can be recovered up to global phase from its fourier intensity function $|\hat{w}(\omega)|^2$.

If the condition holds for all $w'_0 \in \pi^{-1}(w_0) \setminus I_0$ then generic $w \in W$ can be recovered up to trivial ambiguities.

**Proof of Theorem.** Let $I^0 = \overline{I \setminus (I_0 \cup I_N)}$. Since $I^0$ is closed the map $I^0 \to X$ is still finite. Let $I_W = I^0 \cap (W \times W)$ be the real algebraic subset of $I^0$ consisting of pairs $(w, w')$ with $w, w'$ both in $W$. The image of $I_W$ under the projection $\pi: I \to X$ is the set of $w \in W$ which cannot be recovered up to trivial ambiguity from $|\hat{w}(\omega)|^2$. We will show that $W \setminus I_W$ is Zariski dense.

By assumption that there exists $w_0 \in W$ such that for all pairs $(w_0, w'_0) \in I^0$, $w'_0 \notin W$. This implies that $W \times W$ intersects each irreducible component of $\pi^{-1}(W)$ in a proper algebraic subset. Hence, $\dim I_W < \dim \pi^{-1}(W) = \dim W$. Thus, $\dim \pi(I_W) < W$ so $\pi(I_W)$ is contained in a proper algebraic subset of $W$. Hence the complement of $\pi(I_W)$ is dense in the real Zariski topology on $W$. \qed
5.1. **Imposing uniqueness with additional conditions.** Using Theorem 5.1, we can show that a signal can be recovered modulo trivial ambiguities from the Fourier intensity function and the absolute value of a single entry. We illustrate with the following Corollary which is also proved in [1 Corollary 4.4]. See the paper [2] for more conditions for which uniqueness can be imposed.

**Corollary 5.2.** [1 Corollary 4.4] For generic $x \in \mathbb{C}^n \times \mathbb{C}\times \mathbb{C}$ the system of equations

$$|\hat{x'}(\omega)|^2 = |\hat{x}(\omega)|^2$$

$$|x'[N]| = |x[N]|$$

has a unique solution modulo global phase. If $x'[N] = x[N]$ then the solution is unique.

**Proof.** Let $|x[N]| = a$ with $a > 0$. By Theorem 5 it suffices to find a single vector $x$ with $x[N] = a$ such that for all $(x, x') \in \pi^{-1}(x) \cap (I \setminus I_0 \cup I_N)$, $|x'[N]| = \neq a$.

We do this as follows: Let $x = (a', 0, \ldots, 0, a)$ with $a' > 0$ and not equal to $a$. The Fourier polynomial $\hat{x}(\omega) = a' + aw$ so $|\hat{x}(\omega)|^2 = (a^2 + (a')^2 + (aa')\omega^N + (a'a)\omega^{-N}$. If $x' = (a_0, \ldots, a_N)$ has the same Fourier intensity function then $a_0 a_N = a a'$. If $|a_0| = a$ then $|a_0| = a$. But the constant coefficient is $|a_0|^2 + \ldots + |a_N|^2 = a^2 + a^2$ so we conclude that all other entries are in $x'$ are 0. Hence, up to global phase $x' = (a_0, 0, \ldots, 0, a)$. But $a_0 a = a a'$ so $a_0 = a'$; ie $x' = x$. □

5.2. **Imposing uniqueness for multivectors.** Theorem 5.1 can easily be generalized to multi-vectors. It is this form of the theorem that was used in [3]. Given positive integers $N_1, \ldots, N_m$ let $X[n] = \mathbb{C}^{N_i+1}/S^1$. Let $I[n] \subset X[n] \times X[n]$ be the incidence variety and let $\pi[n] : I[n] \rightarrow X[n]$ the projection to the first factor. Let $X = X[1] \times \ldots \times X[m]$ and $I = I[1] \times \ldots I[m]$ be product of the incidences. Finally let $\pi : I \rightarrow X$ be the product of the projections $\pi[n]$.

**Theorem 5.3** (Imposing uniqueness for multivectors). Let $W$ be an irreducible algebraic subset of $X$. Suppose that there exists an $m$-tuple of vectors $w_0 \in W$ such that for all $(w_0, w'_0) \in \pi^{-1}(w), w'_0$ is not obtained from $w_0$ by a trivial ambiguity. Then the generic $m$-tuple $w \in W$ can be recovered (upto phase) from the Fourier intensity functions of its component vectors.

**Proof.** We use the same argument as in the proof of Theorem 5.1 to show that the set of $w \in W$ that can’t be recovered from their Fourier intensity function has strictly smaller dimension than $W$. □

**Example 5.4.** [3 Proposition B.1] In [3] we consider the problem of giving lower bounds on the number of measurements required for blind phaseless STFT for signals of length $N$, window of length $W$ and step size equal to $L$. The main result of that paper is that $\sim 10N$ measurements are sufficient for generic signal recovery modulo ambiguities and this is independent of the step size or window length.

As part of the proof we need to show that a generic triple $(y_1, y_2, y_3)$ in the subvariety $Z \subset \mathbb{C}^{L+1} \times \mathbb{C}^{2L+1} \times \mathbb{C}^{3L+1}$ defined by the system of quadratic equations

$$\{y_1[n]y_3[L+n] = y_2[n]y_2[L+n]\}_{n=0,\ldots,L}$$

is uniquely determined up to a global phase by the Fourier intensity functions of the vectors $y_1, y_2, y_3$. By Theorem 5.3 it suffices explicitly demonstrate one triple $(y_1, y_2, y_3) \in Z$ which is uniquely determined by the Fourier intensity functions of the vectors $y_1, y_2, y_3$.
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