LOG-MAJORIZATION OF THE MODULI OF THE EIGENVALUES OF A MATRIX POLYNOMIAL BY TROPICAL ROOTS

MARIANNE AKIAN ∗, STÉPHANE GAUBERT †, AND MEISAM SHARIFY ‡

Abstract. We show that the sequence of moduli of the eigenvalues of a matrix polynomial is log-majorized, up to universal constants, by a sequence of “tropical roots” depending only on the norms of the matrix coefficients. These tropical roots are the non-differentiability points of an auxiliary tropical polynomial, or equivalently, the opposites of the slopes of its Newton polygon. This extends to the case of matrix polynomials some bounds obtained by Hadamard, Ostrowski and Pólya for the roots of scalar polynomials. We also obtain new bounds in the scalar case, which are accurate for “fewnomials” or when the tropical roots are well separated.

Key words. Matrix polynomial, Tropical algebra, Majorization of eigenvalues, Tropical roots, Roots of polynomial, Bound of Pólya.

AMS subject classifications. 15A22,15A80,15A18,47J10

1. Introduction. Let \( p(x) = \sum_{j=0}^{n} a_j x^j \), \( a_j \in \mathbb{C} \) be a polynomial of degree \( n \) in a complex variable \( x \). Let \( \zeta_1, \ldots, \zeta_n \) denote the roots of \( p(x) \) arranged by non-decreasing modulus (i.e., \( |\zeta_1| \leq \ldots \leq |\zeta_n| \)). We associate with \( p \) the tropical polynomial \( \mathbf{t} p(x) \), defined for all nonnegative numbers \( x \) by

\[
\mathbf{t} p(x) := \max_{|a_j|} |a_j|x^j .
\]

The tropical roots of \( \mathbf{t} p, \alpha_1, \ldots, \alpha_n \), ordered by non-decreasing value (i.e., \( \alpha_1 \leq \ldots \leq \alpha_n \)), are defined as the non-differentiability points of the function \( \mathbf{t} p \), counted with certain multiplicities. They coincide with the exponential of the opposite of the slopes of the edges of a Newton polygon, defined by Hadamard [H93] and Ostrowski [Ost40a, Ost40b] as the upper boundary of the convex hull of the set of points \( \{(j, \log |a_j|) : 0 \leq j \leq n\} \). The logarithms of these roots were called the inclinaisons numériques by Ostrowski. One interest of these roots is that they can be easily computed (linear number of arithmetic operations and comparisons). See Section 3.1 below for details.

Hadamard was probably the first to prove a log-majorization type inequality for the modulus of the roots of a scalar polynomial by using what we call today the tropical roots. His result (page 201 of [H93], third inequality) can be restated as follows in tropical terms:

\[
\frac{|\zeta_1 \zeta_2 \ldots \zeta_k|}{\alpha_1 \cdots \alpha_k} \geq \frac{1}{k+1} .
\]

This bound, proved in passing in a memoir devoted to the Riemann zeta function, remained apparently not so well known. In particular, the special case \( |\zeta_1|/\alpha_1 \geq 1/2 \)

∗INRIA Saclay–Île-de-France & CMAP, Ecole Polytechnique, Route de Saclay, 91128 Palaiseau Cedex, France (Marianne.akian@inria.fr).
†INRIA Saclay–Île-de-France & CMAP, Ecole Polytechnique, Route de Saclay, 91128 Palaiseau cedex, France (Stephane.Gaubert@inria.fr).
‡School of Mathematics, Alan Turing Building, The University of Manchester, Manchester, M13 9PL, UK (Meisam.Sharify@manchester.ac.uk).

This work was performed when the third author was with INRIA Saclay–Île-de-France and CMAP, Ecole Polytechnique. The last two authors were partly supported by the Arpege programme of the French National Agency of Research (ANR), project “ASOPT”, number ANR-08-SEGI-005.
is equivalent to the homogeneous form of the classical bound of Cauchy, established later on by Fujiwara [Fuj16], and a weaker inequality, with $\alpha_k$ at the denominator instead of $\alpha_1 \cdots \alpha_k$, appeared later on in the work of Specht [Spe38].

Ostrowski proved several bounds on the roots of a polynomial in his work on the method of Graeffe [Ost40a, Ost40b], in which he used again the Newton polygon considered by Hadamard. In particular, he obtained the following upper bound (see [Ost40a §7]),

$$\frac{|\zeta_1 \zeta_2 \cdots \zeta_k|}{\alpha_1 \cdots \alpha_k} \leq \binom{n}{k},$$

(1.2)

which can be thought of as a generalization of a “reverse” of the Cauchy inequality due to Birkhoff [Bir15] (corresponding to the case $k = 1$ in (1.2)). He also gave a different proof of a variant of (1.1), with the constant $1/(2k)$ instead of $1/(k+1)$, and reported a private communication of Pólya, leading to a tighter constant

$$\frac{|\zeta_1 \zeta_2 \cdots \zeta_k|}{\alpha_1 \cdots \alpha_k} \geq \frac{1}{\sqrt{E(k)(k+1)}},$$

(1.3)

with

$$E(k) := \left(\frac{k+1}{k}\right)^k < e.$$  

(1.4)

In this paper, we generalize the bounds of Hadamard, Ostrowski, and Pólya, to the case of a matrix polynomial

$$P(\lambda) = A_0 + A_1 \lambda + \cdots + A_d \lambda^d, \quad A_j \in \mathbb{C}^{n \times n}, \quad 0 \leq j \leq d.$$  

(1.5)

We now associate with the matrix polynomial $P$ the tropical polynomial

$$tp(x) := \max_{0 \leq j \leq d} \|A_j\| x^j,$$  

(1.6)

where $\| \cdot \|$ is a norm on the space of matrices, and show that the moduli of the roots $\zeta_1, \ldots, \zeta_{nd}$ of $P$ can still be controlled in terms of the tropical roots $\alpha_1, \ldots, \alpha_n$ of $tp$. Our results give bounds on the ratio $|\zeta_1 \cdots \zeta_k|/(\alpha_1 \cdots \alpha_k)^n$, which extend and refine the above bounds. In particular, in Theorem 4.4 we extend the lower bound (1.3) of Pólya to the matrix polynomial case, and in Theorem 6.1 we extend the upper bound (1.2) of Ostrowski. Moreover, we obtain other lower bounds that are new even in the case of scalar polynomials. In particular, in Theorem 4.1 we obtain a lower bound which may be tighter for “fewnomials”. In Theorems 4.5 and 4.6 we obtain general lower bounds, which extend the bound of Pólya and its extension to the matrix case, and which may be much tighter when the tropical roots are sufficiently separated. Then, all together our results show that the tropical roots give tight estimates of the moduli of the eigenvalues if the tropical roots are sufficiently separated and if certain matrices are sufficiently well conditioned.

The results of the present paper combine ideas from max-plus algebra and tropical geometry, and numerical linear algebra. In [ABG98, ABG04, ABG05], Akian, Bapat, and Gaubert studied the eigenvalues and eigenvectors of matrices and matrix polynomials which entries (the $A_j$) are functions, for instance Puiseux series, of a (perturbation) parameter. It is shown there that the leading exponents of the
Puiseux series representing the different eigenvalues (resp. eigenvectors) coincide, under some genericity conditions, with the “tropical eigenvalues” (resp. eigenvectors) of the tropical matrix polynomial with entries equal to the leading exponents of the entries of the initial matrix polynomial. This can be interpreted in the light of tropical geometry (see [Vir01, IMS07, RGST05]), using the notion of non-archimedean amoeba [EKL06] with respect to the usual non-archimedean valuation on the field of Puiseux series (taking the leading exponent). Since amoebas with respect to the archimedean valuation $z \mapsto \log |z|$ on the field of complex numbers $\mathbb{C}$ can be approximated by non-archimedean amoebas [PR04, Pur08], tropical eigenvalues are expected to provide approximations of the log of moduli of the classical eigenvalues. Indeed, the above bounds of Hadamard, Ostrowski, and Pólya can be seen as the (archimedean) analogue of the results of [ABC04] in the particular one dimensional case ($n = 1$). Moreover, in the case of a matrix with non-negative coefficients, Friedland [Fri86] established a bound for its spectral radius (or its Perron eigenvalue), which can be interpreted as a bound of the maximal tropical eigenvalue of the valuation of $A$ [ABC06], and similar bounds for the other eigenvalues have been established by Akian, Gaubert and Marchesini in [AGM14].

Here, we replaced the valuation on $\mathbb{C}$ by the “valuation” $A \mapsto \log \|A\|$ on the ring of $n \times n$ matrices over $\mathbb{C}$, which leads to (1.6). The tropical roots obtained in this way are easy to compute and we shall see they provide a good approximation of the moduli of the eigenvalues under reasonable assumptions. The idea of using the norm instead of the usual valuation was inspired by several works in numerical linear algebra, suggesting that the information of the norms is relevant. For instance, Higham and Tisseur [HT03] extended to matrix polynomials the bound of Cauchy (related to the special case $k = 1$ in the Hadamard-Ostrowski-Pólya inequality), by using the norms of $A_0^{-1}A_j$ and $A_d^{-1}A_j$. Fan, Lin and Van Dooren [FLVD04] introduced a scaling based on the norms of the matrix coefficients of a matrix quadratic polynomial.

The interest of tropical roots is not limited to theoretical bounds, they can be used to perform scalings allowing one to improve the accuracy of the numerical computation of the eigenvalues of a matrix polynomial. In [GS09], the tropical polynomial of (1.6) was initially introduced to refine the results of [FLVD04]. It was shown there (see also [Sha11]) that the tropical roots of $tP$ can be used to perform scalings allowing one to improve the backward stability of eigenvalue computations for a matrix polynomial $P$. In the special case of quadratic pencils, this was confirmed by a work of Hamarling, Munro, and Tisseur [HMT13], who implemented the tropical scaling as an option of quadeig, and gave theoretical estimates showing that the tropical scaling does reduce the backward error under some conditions. We shall explain this scaling in more details in Section 7. We also demonstrate there by numerical examples that this scaling helps to computate the eigenvalues of a matrix polynomial, at least when the lower bounds of Section 4 are tight.

Some bounds on the modulus of the eigenvalues of $P$, involving the tropical roots of $tP$, also appeared in [GS09] in the case in which $d = 2$, and in [Sha11] for the smallest and largest tropical roots when $d \geq 2$, which may be seen as particular cases of the lower bounds of the present paper.

Let us finally point out some further related works. Bini used in [Bini96] what we call the tropical roots (from the Newton polygon technique) to initialize the Aberth method of computation of the roots of a scalar polynomial. Also, Malajovich and Zubelli applied Ostrowski’s analysis to effective root solving [MZ01]. Finally, Bini, Noferini, and Sharify recently proved some location results by using the tropical roots.
of (1.6) for the eigenvalues of a specific class of matrix polynomials such that the
coefficient matrices are unitary up to some constants. They also generalized the
classical Pellet’s theorem to the case of matrix polynomials \cite{BNS13}, which involve
the norms of the $A_k^{-1}A_j$, except in the specific above class, where they involve the
norms of the $A_j$. They also relate it to the Newton polygon method, thus to the
tropical roots. Moreover, Melman \cite{Mel12} proved a different generalization and some
variations of Pellet’s theorem, which are in general less costly since they involve
the norms of the $A_j$ and $A_k^{-1}$ only (we need to compute $O(d)$ norms instead of $O(d^2)$, each
of these computations requiring at least an inversion or a matrix product). Note that
the information obtained by Pellet type methods is complementary and incomparable
to the one obtained by the present log-majorization inequalities, see Remark 6.4.

The paper is organized as follows. We first recall a variation on Jensen formula
due to Landau (Section 2.1) and extend it to the matrix case (Section 2.3), exploiting
some technical results on matrix norms (Section 2.2). In Section 3.1, we recall the
construction of the Newton polygon and of tropical roots. Then, we derive in Sec-
ction 3.2 a general estimate for the modulus of the product of $nk$ smallest eigenvalues
of a matrix polynomial in terms in tropical roots. This leads to various explicit lower
bounds, stated in Section 4, which extend the Hadamard-Ostrowski-Pólya inequality
to the matrix case. These bounds are proved in Section 5.1 in the scalar case, and then
in Section 5.2 in general. An upper bound (reverse inequality) is stated in Section 6.1
and proved in Section 6.2. In Section 7 we provide examples showing the tightness of
the lower bounds. The application to matrix scaling is presented in Section 8.

2. Estimates of the eigenvalues of matrix polynomials using norms of
matrices.

2.1. An inequality of Landau. An ingredient of our results is a bound on
the modulus of the product of the $k$ smallest roots of a polynomial in terms of its
coefficients. It is a consequence of Jensen’s formula, derived by Landau in \cite{Lan05},
building on an earlier observation of Lindelöf \cite{Lin02}.

Lemma 2.1 \cite{Lan05}. Let $\zeta_1, \ldots, \zeta_d$ be the roots of a polynomial
$p(x) = \sum_{j=0}^{d} a_j x^j$, arranged by non-decreasing modulus, and assume that $a_0 \neq 0$.
For all $1 \leq k \leq d$, we have

$$\log |\zeta_1 \cdots \zeta_k| \geq - \inf_{r>0} \frac{1}{2} \log \left( \sum_{j=0}^{d} \frac{|a_j|^2}{|a_0|^2} r^{2(j-k)} \right) . \quad (2.1)$$

We include the short proof, as its idea will be used in the extension to the matrix
case.

Proof. The formula of Jensen \cite{Jen99} shows that if $\zeta_1, \ldots, \zeta_k$ are the roots of $p(z)$
in the closed disk of $C$ of radius $r$, counted with multiplicities, then

$$\log |\zeta_1 \cdots \zeta_k| = k \log r + \log |p(0)| - \frac{1}{2\pi} \int_0^{2\pi} \log |p(re^{i\theta})| d\theta .$$

It follows that, for all $r > 0$, and $k = 1, \ldots, d$:

$$\log |\zeta_1 \cdots \zeta_k| \geq k \log r + \log |p(0)| - \frac{1}{2\pi} \int_0^{2\pi} \log |p(re^{i\theta})| d\theta . \quad (2.2)$$
Using the comparison between the geometric and the $L^2$ mean, together with Parseval’s identity, we get

$$
\frac{1}{2\pi} \int_0^{2\pi} \log |p(re^{i\theta})|d\theta \leq \frac{1}{2} \log \left( \frac{1}{2\pi} \int_0^{2\pi} |p(re^{i\theta})|^2d\theta \right) = \frac{1}{2} \log \left( \sum_{j=0}^{\infty} |a_j|^2 r^{2j} \right).
$$

Gathering this inequality with (2.2) yields

$$
\log |\zeta_1 \cdots \zeta_n| \geq k \log r + \log |a_0| - \frac{1}{2} \log \left( \sum_{j=0}^{n} |a_j|^2 r^{2j} \right).
$$

Since this holds for all $r > 0$, this shows the inequality of the lemma. □

### 2.2. Preliminary results on matrix norms.

In order to generalize Lemma 2.1 to matrix polynomials, and to derive effective lower bounds for eigenvalues, we need to introduce technical assumptions on matrix norms, like the following one:

(A1) $|\det A| \leq ||A||^n$, for all $A \in \mathbb{C}^{n \times n}$.

The normalized Frobenius norm,

$$
||A||_{nF} := \left( \frac{1}{n} \sum_{i,j=1}^{n} |A_{ij}|^2 \right)^{\frac{1}{2}}, \quad \forall A \in \mathbb{C}^{n \times n},
$$

will be of special interest, as some of our estimates rely on $L_2$ methods. Therefore, the next assumption will also be considered:

(A2) There exists $Q, Q' \in \mathbb{C}^{n \times n}$, such that $\det Q = \det Q' = 1$ and $||AQQ'||_{nF} \leq ||A||$, for all $A \in \mathbb{C}^{n \times n}$.

We next show that Assumption (A2) implies Assumption (A1) and that a number of commonly used norms satisfy Assumption (A1) or Assumption (A2).

For all $p \in [1, \infty)$, and $n \geq 1$, we shall denote by $|| \cdot ||_p$ the $\ell^p$ norm of $\mathbb{C}^n$: $||v||_p := \left( \sum_{i=1}^{n} |v_i|^p \right)^{1/p}$ for $p < \infty$ and $||v||_\infty := \max_{i=1, \ldots, n} |v_i|$. In particular, $|| \cdot ||_2$ is the Euclidean norm. The norm on the space of matrices $\mathbb{C}^{n \times n}$ induced by the norm $|| \cdot ||_p$ on $\mathbb{C}^n$ will also be denoted by $|| \cdot ||_p$ (the same norm is used for the domain and the range of $A$). In particular, the norm $|| \cdot ||_2$ on $\mathbb{C}^{n \times n}$ is the spectral norm. Moreover, for all $p \in [1, \infty)$, and $n \geq 1$, we shall denote by $|| \cdot ||_{sp}$ the following normalized Schatten $p$-norm on the space of matrices $\mathbb{C}^{n \times n}$: $||A||_{sp} := \left( \frac{1}{n} \sum_{i=1}^{n} \sigma_i^p \right)^{1/p}$, where $\sigma_1, \ldots, \sigma_n$ are the singular values of $A$ (the eigenvalues of $\sqrt{A^*A}$, where $A^*$ denotes the conjugate transpose of $A$). Then, the normalized Schatten 2-norm coincides with the normalized Frobenius norm. Recall that the (unnormalized) Schatten 1-norm is also called the trace norm or the Ky Fan $n$-norm. We shall also denote by $||A||_{\infty} := \max_{i=1, \ldots, n} \sigma_i$ the Schatten $\infty$-norm, which coincides with the spectral norm.

**Property 2.2.** The normalized Frobenius norm satisfies Assumption (A1).

Therefore, Assumption (A2) implies Assumption (A1).

**Proof.** From Hadamard’s inequality, we get that $|\det A| \leq ||A^{(1)}||_2 \cdots ||A^{(n)}||_2$, for all $A \in \mathbb{C}^{n \times n}$, where $A^{(j)}$ denotes the $j$-th column of $A$. Then, since the geometric mean is less than or equal to the arithmetic mean (or simply by the concavity of the logarithm), we get that $|\det A|^2/n \leq \frac{1}{n} (||A^{(1)}||_2^2 + \cdots + ||A^{(n)}||_2^2) = ||A||_{nF}^2$, which implies that the normalized Frobenius norm satisfies Assumption (A1). Since for
all $Q, Q' \in \mathbb{C}^{n \times n}$ such that $\det Q = \det Q' = 1$, we get $|\det A| = |\det(QAQ')| \leq \|QAQ'\|_{nF}$ for all $A \in \mathbb{C}^{n \times n}$. Assumption \((A2)\) implies Assumption \((A1)\). \(\Box\)

\textbf{Property 2.3.} For all $p \in [1, \infty]$, the norm $\| \cdot \|_p$ on $\mathbb{C}^{n \times n}$ satisfies \((A2)\).

\textit{Proof.} Indeed, let us denote by $\|A\|_{p,q}$ the norm induced by the norms $\| \cdot \|_p$ and $\| \cdot \|_q$ on the range and domain $\mathbb{C}^n$ of $A$ respectively, which means that $\|A\|_{p,q} = \max\{\|Av\|_q/\|v\|_p, v \in \mathbb{C}^n, v \neq 0\}$. Since for all $v \in \mathbb{C}^n$, $p \in [1, \infty]$ we have $\|v\|_p, q \geq p'$ when $p \leq p'$ and $q \geq q'$.

In particular $\|A\|_p = \|A\|_{p,p} \geq \|A\|_{2,\infty}$ when $2 \leq p$, and $\|A\|_p = \|A\|_{p,p} \geq \|A\|_{1,2}$ when $p \leq 2$. In addition, it is easy to show that $\|A\|_{2,\infty} = \max(\|(A^*)^{(i)}\|_2, i = 1, \ldots, n)$. Since the last expression is greater or equal to $\|A\|_{nF}$, we deduce that $\|A\|_p \geq \|A\|_{2,\infty} \geq \|A\|_{nF}$ for all $A \in \mathbb{C}^{n \times n}$ and $2 \leq p \leq \infty$. Similarly, $\|A\|_{1,2} = \max\{\|A^{(i)}\|_2, i = 1, \ldots, n\} \geq \|A\|_{nF}$, hence $\|A\|_p \geq \|A\|_{1,2} \geq \|A\|_{nF}$ for all $A \in \mathbb{C}^{n \times n}$ and $1 \leq p \leq 2$. This shows that $\| \cdot \|_p$ satisfies \((A2)\) with $Q = Q' = I$ the identity matrix. \(\Box\)

This result implies that all norms induced by the norm $\|v\|_d = \|Qv\|_p$ in the domain of $A$ and the norm $\|v\|_p, = \|Q^*v\|_p$ in the range of $A$ with the same $p$, but possibly different matrices $Q, Q'$ such that $\det Q = \det Q' = 1$, satisfy \((A2)\).

\textbf{Property 2.4.} If $\| \cdot \|$ is the norm on $\mathbb{C}^{n \times n}$ induced by a norm on $\mathbb{C}^n$ (the same for the domain and the range of matrices), then it satisfies \((A1)\).

\textit{Proof.} Since $\det A$ is the product of the eigenvalues of $A$ counted with multiplicities, we get that $|\det A| \leq \rho(A)^n$, where $\rho(A)$ is the spectral radius of $A$. Since $\rho(A) \leq \|A\|$ for all $A \in \mathbb{C}^{n \times n}$ and all induced norms $\| \cdot \|$ on the space of matrices, we get the result. \(\Box\)

\textbf{Property 2.5.} The normalized Schatten p-norm $\| \cdot \|_{sp}$ on $\mathbb{C}^{n \times n}$ satisfies \((A1)\) for all $p \in [1, \infty]$. It satisfies \((A2)\) if and only if $p \geq 2$. Moreover, for $p \in [1, 2]$, the least constant $\eta$ such that $\eta \cdot \| \cdot \|_{sp}$ satisfies \((A2)\) is given by $\eta = n^{1/p-1/2} > 1$.

\textit{Proof.} Since $|\det A|$ is the product of the singular values of $A$ counted with multiplicities, we get that $|\det A|^p = \sigma_1^p \cdots \sigma_n^p$, and using that the geometric mean is less than or equal to the arithmetic mean, we obtain that $|\det A|^p/n \leq \frac{1}{n} (\sigma_1^p + \cdots + \sigma_n^p) = \|A\|_{sp}^p$, which implies that the normalized Schatten p-norm satisfies Assumption \((A1)\).

We have that $p \mapsto \|A\|_{sp}^p$ is a nondecreasing function. Hence, when $p \in [2, \infty]$, $\|A\|_{sp} \geq \|A\|_{2} = \|A\|_{nF}$, for all $A \in \mathbb{C}^{n \times n}$, which implies that $\| \cdot \|_{sp}$ satisfies \((A2)\) when $p \geq 2$. Let $\eta > 0$ be the least constant such that $\eta \cdot \| \cdot \|_{sp}$ satisfies \((A2)\) and let us show that $\eta = n^{1/p-1/2}$ when $p < 2$. This will imply in particular that $\eta > 1$ hence $\| \cdot \|_{sp}$ does not satisfy \((A2)\) for $p < 2$, which will finishes the proof of the equivalence $\| \cdot \|_{sp}$ satisfies \((A2)\) if and only if $p \geq 2$. Since $n^{1/p} \cdot \| \cdot \|_{sp}$ is the unnormalized Schatten norm, which is nonincreasing with respect to $p$, we get that, for all $p \in [1, 2]$, and $A \in \mathbb{C}^{n \times n}$, $n^{1/2}/\|A\|_{nF} \leq n^{1/p}/\|A\|_{sp}$, which implies that $\eta \leq n^{1/p-1/2}$. Let us fix $p, Q, Q' \in \mathbb{C}^{n \times n}$ such that $\det Q = \det Q' = 1$ and $\|QAQ'|_{nF} \leq \|A\|_{sp}$, for all $A \in \mathbb{C}^{n \times n}$. For all $i = 1, \ldots, n$, let us consider the matrix $A$ whose entries are all zero but the entry $ii$ equal to 1. Then, the singular values of $A$ are all zero except one which is equal to 1, so that $\|A\|_{sp} = (1/n)^{1/p}$. Moreover $QAQ' = Q(i)Q'(i)$, where $Q'(i)$ denotes the $i$-th raw of $Q'$. Hence $\|QAQ'\|_{nF} = (1/n)^{1/2}/\|Q(i)\|_2/\|Q(i')\|_2$. From $\|QAQ'\|_{nF} \leq \|A\|_{sp}$, we deduce that $\|Q(i)\|_2/\|Q(i')\|_2 \leq \eta(1/n)^{1/p-1/2}$. Since $\det Q = \det Q' = 1$, we get using Hadamard’s inequality, $1 \leq \|Q(i)\|_2/\|Q(i')\|_2 \cdots \|Q(n)\|_2/\|Q(n')\|_2 \leq (\eta(1/n)^{1/p-1/2})^n$, which shows that $\eta \geq n^{1/p-1/2}$. We deduce that $\eta = n^{1/p-1/2} > 1$ for $p > 2$, which completes the proof. \(\Box\)
2.3. Generalization of the inequality of Landau to matrix polynomials.  

The following generalization of Lemma 2.1 will be a key tool to establish lower bounds for the eigenvalues of matrix polynomials.  

**Lemma 2.6.** Consider the matrix polynomial $P$ with degree $d$ defined in (1.5), and let $\zeta_1, \ldots, \zeta_n$ denote its eigenvalues, arranged by non-decreasing modulus. Assume that $\| \cdot \|$ is any norm on the space of matrices satisfying (A1), we get that $\det Q$ satisfies (A1), and let $c = \frac{|\det A_0|}{\|A_0\|^{\|\cdot\|}}$. Then, for $1 \leq k \leq d$, we have

$$
\log |\zeta_1 \cdots \zeta_n| \geq \log c - n \inf_{r > 0} \log \left( \sum_{j=0}^{d} \frac{\|A_j\| r^{j-k}}{\|A_0\|} \right). \quad (2.3)
$$

When $\| \cdot \|$ satisfies (A2) the previous bound can be improved as follows

$$
\log |\zeta_1 \cdots \zeta_n| \geq \log c - n \inf_{r > 0} \frac{1}{2} \log \left( \sum_{j=0}^{d} \frac{\|A_j\|^2 r^{2(j-k)}}{\|A_0\|^2} \right). \quad (2.4)
$$

**Proof.** From Inequality (2.2) applied to $\hat{p}(\lambda) = \det P(\lambda)$, we get, for all $r > 0$,

$$
\log |\zeta_1 \cdots \zeta_n| \geq nk \log r + \log |\det A_0| - \frac{1}{2\pi} \int_0^{2\pi} \log |\det P(re^{i\theta})| d\theta. \quad (2.5)
$$

Since $\| \cdot \|$ satisfies (A1) we have

$$
|\det P(re^{i\theta})| \leq \|P(re^{i\theta})\|^n = \| \sum_{j=0}^{d} A_j (re^{i\theta})^j \|^n \leq \left( \sum_{j=0}^{d} \|A_j\|^r \right)^n, \quad (2.6)
$$

for all $\theta \in [0,2\pi]$. Gathering (2.6) with (2.5), we obtain (2.3).

Assume now that $\| \cdot \|$ satisfies (A2) with some matrices $Q, Q'$ such that $\det Q = \det Q' = 1$, and let us prove (2.4). Since from Property 2.2 the normalized Frobenius norm satisfies (A1), we get that $|\det P(re^{i\theta})| = |\det(QP(re^{i\theta})Q')| \leq \|QP(re^{i\theta})Q'|_{nF}^n$. Now using the comparison between geometric and $L^2$ means, we deduce

$$
\frac{1}{2\pi} \int_0^{2\pi} \log |\det P(re^{i\theta})| d\theta \leq \frac{n}{2} \log \left( \frac{1}{2\pi} \int_0^{2\pi} \log \|QP(re^{i\theta})Q'\|_{nF}^2 d\theta \right).
$$

From the formula of the normalized Frobenius norm, we get by applying Parseval’s identity to each coordinate $(QP(re^{i\theta})Q')_{\ell,m}$

$$
\frac{1}{2\pi} \int_0^{2\pi} \|QP(re^{i\theta})Q'\|_{nF}^2 d\theta = \frac{1}{n} \sum_{\ell,m=1}^{n} \left( \frac{1}{2\pi} \int_0^{2\pi} \|QP(re^{i\theta})Q'\|_{\ell,m}^2 d\theta \right)
$$

$$
= \frac{1}{n} \sum_{\ell,m=1}^{n} \left( \sum_{j=0}^{d} \|QA_jQ'\|_{\ell,m}^2 r^{2j} \right) = \sum_{j=0}^{d} (\|QA_jQ'\|_{nF}^2 r^{2j}) \leq \sum_{j=0}^{d} (\|A_j\|^2 r^{2j}).
$$

Gathering this inequality with Inequalities (2.7) and (2.5), we obtain (2.4). □
3. Tropical polynomials and numerical Newton polygons.

3.1. Preliminary results on tropical roots. We recall here basic results on tropical polynomials of one variable. See for instance [BCOQ92, Vir01, IMS07] for more background on tropical polynomials from different perspectives.

Let \( \mathbb{R}_{\text{max}} \) denote the set \( \mathbb{R} \cup \{-\infty\} \). A (max-plus) tropical polynomial \( f \) is a function of a variable \( x \in \mathbb{R}_{\text{max}} \) of the form

\[
f(x) = \max_{0 \leq j \leq d} (f_j + jx),
\]

where \( d \) is an integer, and \( f_0, \ldots, f_d \) are given elements of \( \mathbb{R}_{\text{max}} \). We say that \( f \) is of degree \( d \) if \( f_d \neq -\infty \). We shall assume that at least one of the coefficients \( f_0, \ldots, f_d \) is finite (i.e., that \( f \) is not the tropical “zero polynomial”). Then, \( f \) is a real valued convex function, piecewise affine, with integer slopes.

Cuninghame-Green and Meijer showed [CGM80] that the analogue of the fundamental theorem of algebra holds in the tropical setting, i.e., \( f(x) \) can be written uniquely as

\[
f(x) = f_d + \sum_{j=1}^d \max(x, \alpha_j),
\]

where \( \alpha_1 \leq \cdots \leq \alpha_d \in \mathbb{R}_{\text{max}} \). The numbers \( \alpha_1, \ldots, \alpha_d \) are called the tropical roots. The finite tropical roots can be checked to be the points at which the maximum in the expression (3.1) of \( f(x) \) is attained at least twice, whereas \(-\infty\) arises as a tropical root if \( f_0 = -\infty \). The multiplicity of a root \( \alpha \) is defined as the cardinality of the set \( \{ j \in \{1, \ldots, d\} \mid \alpha_j = \alpha \} \).

The multiplicity of finite root \( \alpha \) can be checked to coincide with the variation of the derivative of the map \( f \) at point \( \alpha \), whereas the multiplicity of the root \(-\infty\) is given by \( \inf\{ j \mid f_j \neq -\infty \} \) or by the slope of the map \( f \) at \(-\infty\). The notion of tropical roots is an elementary special case of the notion of tropical variety which has arisen recently in tropical geometry [IMS07].

The tropical roots can be computed by the following variant of the classical Newton polygon construction. Define the Newton polygon \( \Delta(f) \) of \( f \) to be the upper boundary of the convex hull of the region

\[
\{(j, \lambda) \in \mathbb{N} \times \mathbb{R} \mid \lambda \leq f_j, \ 0 \leq j \leq d\} \subset \mathbb{R}^2.
\]

This boundary consists of (linear) segments.

The following result was established in [ABC05], it relies on standard Legendre-Fenchel duality argument.

**Proposition 3.1** ([ABC05, Proposition 2.10]). There is a bijection between the set of finite tropical roots of \( f \) and the set of segments of the Newton polygon \( \Delta(f) \): the tropical root corresponding to a segment is the opposite of its slope, and the multiplicity of this root is the length of this segment (measured by the difference of the abscissae of its endpoints). (Actually, min-plus polynomials are considered in [ABC05], but the max-plus case reduces to the min-plus case by an obvious change of variable.)

The Graham scan algorithm [Gra72] allows one to compute the convex hull of a finite set of points by making \( O(n) \) arithmetical operations and comparisons, provided that the given set of points is already arranged by increasing abscissae, which is the case in the present setting. It follows that the tropical roots, counted with multiplicities,
can be computed in linear time (see also [GS09, Proposition 1]). In particular, the maximal tropical root is given by

$$\alpha_d = \max_{0 \leq j \leq d} \frac{f_j - f_d}{d - j}.$$  

**Example 3.2.** Consider

$$f(x) = \max(0, 1 + x, 6 + 2x, 4 + 4x, 9 + 8x, 5 + 10x, 1 + 16x).$$

The graph of $f$ and the Newton polygon of $f$ are shown in Figure 3.1. The tropical roots are $-3$, $-1/2$, and $1$, with respective multiplicities $2$, $6$ and $8$.

![Graph of the tropical polynomial function](image)

**The notion of root also applies with trivial changes to the “max-times” model of the tropical structure, in which polynomial functions now have the form**

$$t_p(x) = \max_{0 \leq j \leq d} a_j x^j,$$

where $a_0, \ldots, a_d$ are nonnegative numbers, and the variable $x$ now takes nonnegative values. Then, the **tropical roots** of $t_p(x)$ are, by definition, the exponentials of the tropical roots of its log-exp transformation $f(x) := \log t_p(\exp(x)) = \max_{0 \leq j \leq d} (\log a_j + jx)$.

In the sequel we shall consider max-times polynomials associated with usual scalar or matrix polynomials. We shall need the following result which follows from the above definitions and properties.

**Proposition 3.3.** Let $t_p(x) = \max_{0 \leq j \leq d} a_j x^j$ be a max-times polynomial of degree $d$, with $a_j \geq 0$, $j = 0, \ldots, d$. Assume that $a_0 \neq 0$. Let $0 < \alpha_1 \leq \cdots \leq \alpha_d$ denote the tropical roots of $t_p$ arranged in non-decreasing order. Then,

$$a_j \leq a_0 \prod_{\ell=1}^{j} \alpha_{\ell}^{-1}, \quad j = 0, \ldots, d.$$  

Moreover, let $k$ be the abscissa of a vertex of the Newton polygon of $p$, then $a_k > 0$ and

$$a_j \leq a_k \prod_{\ell=k+1}^{d} \alpha_{\ell}^{-1}, \quad \text{for } j = k, \ldots, d,$$

$$a_j \leq a_k \prod_{\ell=j+1}^{d} \alpha_{\ell}, \quad \text{for } j = 0, \ldots, k.$$
Proof. Let \( f(x) := \max_{0 \leq j \leq d} (\log a_j + jx) \). By definition \( \log \alpha_j \), \( j = 1, \ldots, d \), are the tropical roots of \( f \). The upper boundary of the Newton polygon of \( f \) coincides with the graph of the concave hull \( \hat{a} \) of the map \( j \in \{0, \ldots, d\} \mapsto \log a_j \in \mathbb{R} \). By Proposition 3.1, the tropical roots are the opposites of the slopes of \( \hat{a} \) and their multiplicities are the lengths of the segments where \( \hat{a} \) has this slope. This means that \( \log \alpha_j = \hat{a}_j - 1 - \hat{a}_j \), hence \( \hat{a}_j = \hat{a}_0 - \sum_{\ell=1}^{j} \log \alpha_\ell \), and using that \( \hat{a} \) is above the map \( j \mapsto \log a_j \), and that both maps coincide at the boundary point \( j = 0 \), we get the first inequality of the proposition. If now \( k \) is the abscissa of a vertex of the Newton polygon of \( p \), then \((k, \hat{a}_k)\) is an exposed point of the Newton polygon, which implies that \( \hat{a}_k = \log a_k \). Since \( \hat{a}_j = \hat{a}_k - \sum_{\ell=k+1}^{j} \log \alpha_\ell \), for all \( j \geq k \), we get the two last inequalities of the proposition.

3.2. A general lower bound using tropical roots. Using the same method as in the proof of Pólya’s inequality reproduced in [Ost40a], we derive from Landau’s inequality the following result which involves now the tropical roots \( \alpha_k \) instead of the moduli \(|a_k|\) of the coefficients of \( p \).

**Proposition 3.4.** Let \( \zeta_1, \ldots, \zeta_d \) be the roots of a univariate scalar polynomial \( p(x) = \sum_{j=0}^{d} a_j x^j \), arranged by non-decreasing modulus, and assume that \( a_0 \neq 0 \). Let \( \alpha_1, \ldots, \alpha_d \) denote the tropical roots of the associated tropical polynomial \( t(p)(x) = \max_{0 \leq j \leq d} |a_j| x^j \), arranged in non-decreasing order. Then, for all \( 1 \leq k \leq d \), the inequality

\[
\frac{|\zeta_1 \cdots \zeta_k|}{\alpha_1 \cdots \alpha_k} \geq L_k , \tag{3.2}
\]

holds with

\[
(L_k)^{-2} = \inf_{\xi > 0} \left( \sum_{j=0, a_j \neq 0}^{d} \beta_{k,j}^2 \xi^{j-k} \right) , \tag{3.3}
\]

and

\[
\beta_{k,j} := \begin{cases} 
\prod_{\ell=j+1}^{k} \left( \frac{\alpha_\ell}{\alpha_k} \right) & \text{if } j < k , \\
\prod_{\ell=k+1}^{j} \left( \frac{\alpha_k}{\alpha_\ell} \right) & \text{if } j > k , \\
1 & \text{if } j = k ,
\end{cases} \tag{3.4}
\]

so that in particular \( \beta_{k,j} \leq 1 \) for all \( k, j = 0, \ldots, d \).

Proof. By applying the change of variable \( r = \alpha_k \sqrt{\xi} \) in the inequality of Lemma 2.1, we get

\[
\log |\zeta_1 \cdots \zeta_k| \geq \sup_{\xi > 0} \left( -\frac{1}{2} \log \left( \sum_{j=0}^{d} \frac{|a_j|^2}{|a_0|^2} (\alpha_k^2 \xi)^{j-k} \right) \right)
\geq \log(\alpha_1 \cdots \alpha_k) + \sup_{\xi > 0} \left( -\frac{1}{2} \log \left( \prod_{\ell=1}^{k} \alpha_\ell^2 \left( \sum_{j=0}^{d} \frac{|a_j|^2}{|a_0|^2} (\alpha_k^2 \xi)^{j-k} \right) \right) \right) .
\]
Applying Proposition 3.3 to the max-times polynomial $t^p$, we get
\[ |a_j| \leq |a_0| \prod_{\ell=1}^d \alpha^{-1}_\ell \]
for all $j = 0, \ldots, d$, which with the above inequality yields (3.2) with
\[
(L_k)^{-2} := \inf_{\xi > 0} \left( \sum_{j=0, a_j \neq 0}^d (\prod_{\ell=1}^d \alpha_\ell \prod_{j=1}^d \alpha^{-1}_\ell) \xi^{j-k} \right)
\]
which can be written in the form (3.3) with $\beta_{k,j}$ as in (3.4). Moreover, since $\alpha_j$ is nondecreasing with respect to $j$, we get that all the $\beta_{k,j}$ are less than or equal to 1.

The following is a matrix version of Proposition 3.4. It is proved along the same lines.

**Proposition 3.5.** Let $A_0, \ldots, A_d$, $P$, $\zeta_1, \ldots, \zeta_{nd}$, $\| \cdot \|$, $c$, be as in the first part of Lemma 2.6, and let $\alpha_1, \ldots, \alpha_d$ be the tropical roots of the tropical polynomial of (1.6), arranged in non-decreasing order. Then, for all $1 \leq k \leq d$, the inequality
\[
\frac{|\zeta_1 \cdots \zeta_{nk}|}{(\alpha_1 \cdots \alpha_k)^n} \geq c(L_k)^n,
\]
holds with $L_k$ such that:
\[
(L_k)^{-1} = \inf_{\xi > 0} \left( \sum_{j=0, A_j \neq 0}^d \beta_{k,j} \xi^{j-k} \right),
\]
with $\beta_{k,j}$ as in (3.4).

Moreover, when $\| \cdot \|$ satisfies (A2) the constant $L_k$ in (3.5) can be replaced by the greater one:
\[
(L_k)^{-2} = \inf_{\xi > 0} \left( \sum_{j=0, A_j \neq 0}^d \beta^2_{k,j} \xi^{j-k} \right).
\]

**Remark 3.6.** In the above proof, the only ingredient to deduce the result of Proposition 3.4 from that of Lemma 2.1 is the inequality $|a_j| \leq |a_0| \prod_{\ell=1}^d \alpha^{-1}_\ell$ for all $j = 0, \ldots, d$. By Proposition 3.3, this inequality is an equality for any abscissa $j$ of a vertex of the Newton polygon of $p$. Hence, if all the tropical roots of the tropical polynomial of (1.6) are simple (which means that all $k = 0, \ldots, d$ are abscissae of a vertex of the Newton polygon of $p$), the inequalities of Proposition 3.4 and Lemma 2.1 are equivalent. The same is true for the inequalities of Proposition 3.5 and Lemma 2.6.

**4. Explicit lower bounds in terms of tropical roots.** We now derive from Proposition 3.5 explicit lower bounds for the product of the $nk$ smallest eigenvalues of a matrix polynomial. Different bounds can be given, depending on the information available on the matrix polynomial. Our first bound is useful for “fewnomials” (polynomials with few non-zero coefficients).

**Theorem 4.1 (Bounds involving the number of nonzero coefficients).** Consider the matrix polynomial $P$ with degree $d$ defined in (1.5), and let $\zeta_1, \ldots, \zeta_{nd}$ denote its eigenvalues, arranged by non-decreasing modulus. Assume that $\| \cdot \|$ is any norm on the space of matrices satisfying (A1) that $\det A_0 \neq 0$ and let $c = \frac{|\det A_0|}{\|A_0\|^n}$. Let $\alpha_1, \ldots, \alpha_d$ be the tropical roots of the tropical polynomial of (1.6), arranged in non-decreasing order.
order. Also let \( \text{mon} P \) denote the number of nonzero monomials of \( P \). Then, for all \( 1 \leq k \leq d \), we have

\[
\frac{\left| \zeta_1 \cdots \zeta_{nk} \right|}{(\alpha_1 \cdots \alpha_k)^n} \geq c(L_k)^n, \tag{4.1}
\]

with

\[
L_k = \frac{1}{\text{mon} P}. \tag{4.2}
\]

Moreover, when \( \| \cdot \| \) satisfies [A2], the constant \( L_k \) can be replaced by the greater one:

\[
L_k = \frac{1}{\sqrt{\text{mon} P}}. \tag{4.3}
\]

The proof of all the results of this section will be given in Section 5.

**Remark 4.2.** When \( n = 1 \) (thus the matrices \( A_0, \ldots, A_d \) are scalars), any norm is proportional to the normalized Frobenius norm which is nothing but the modulus map \( | \cdot | \), satisfies [A2], and for which \( c = 1 \). Therefore the tropical roots of \( tp \) are the same for all norms, the best possible inequality (4.1) is

\[
\frac{\left| \zeta_1 \cdots \zeta_k \right|}{\alpha_1 \cdots \alpha_k} \geq L_k, \tag{4.4}
\]

and this inequality holds with \( L_k \) as in (4.3).

**Remark 4.3.** If a norm \( \| \cdot \| \) on the space of matrices satisfies [A1] but not [A2], we can still obtain a bound of the form (4.1,4.3) by changing the constant \( c \). Indeed, by the equivalence between norms on \( \mathbb{C}^{n \times n} \), for any norm \( \| \cdot \| \) on \( \mathbb{C}^{n \times n} \), there exists a constant \( \eta > 0 \) (which depends on \( n \)) such that \( \|A\|_{nF} \leq \eta \|A\| \) for all \( A \in \mathbb{C}^{n \times n} \). Then, the norm \( N \) obtained by multiplying \( \| \cdot \| \) by \( \eta \) satisfies [A2], hence (4.1,4.3). There, the constant \( L_k \) is given by (4.3), thus independent of the norm, whereas the constant \( c \) and the tropical roots \( \alpha_j \) are computed with respect to the norm \( N \). In particular \( c = \frac{\left| \det A_0 \right|}{\eta^{n}} = c_0/\eta^n \), where \( c_0 \) is the constant \( c \) associated to the norm \( \| \cdot \| \). Moreover, if \( tp \) denotes the tropical polynomial associated to the norm \( \| \cdot \| \), then the tropical polynomial associated to the norm \( N \) is equal to \( \eta tp \), and its tropical roots are equal to the ones of \( tp \). Hence, we deduce that (4.1,4.3) holds for \( \| \cdot \| \) with \( c/\eta^n \) instead of \( c \). However, if \( \| \cdot \| \) satisfies [A1] but not [A2], then \( \eta > 1 \), so that the inequality derived from (4.1,4.3) with \( c/\eta^n \) instead of \( c \) may be weaker than (4.1,4.2): this is indeed the case if and only if \( \eta > \sqrt{\text{mon} P} \). The same type of conclusions can be obtained for the lower bounds that are stated in the next theorems.

The following theorem provides a lower bound generalizing the lower bound of Pólya to the matrix case (since, as said in Remark 4.2 when \( n = 1 \), the modulus map is a norm satisfying [A2], and for which Inequality (4.1) reduces to (4.4)). Up to the constant \( c \), the following bounds are independent of the coefficients of the matrix polynomial \( P \).

**Theorem 4.4 (Universal bound).** Let \( A_0, \ldots, A_d, P, \zeta_1, \ldots, \zeta_{nd}, \| \cdot \|, c, \) and \( \alpha_1, \ldots, \alpha_d \) be as in the first part of Theorem 4.1. Then, for all \( 1 \leq k \leq d \), Inequality (4.1) holds with \( L_k \) defined as follows:

\[
L_k = \frac{1}{E(k)(k+1)}, \tag{4.5}
\]
where $E$ is defined as in (1.4). Moreover, when $\| \cdot \|$ satisfies (A2), Inequality (4.1) holds with the greater constant:

$$ L_k = \frac{1}{\sqrt{E(k)(k+1)}}. \quad (4.6) $$

The lower bound of Pólya, and its matrix version given above, are tight only for $k$ small. By symmetry, one can obtain a tight lower bound when $k$ is close to $d$ (this was already noted by Ostrowski in the scalar case [Ost40a]). Theorem 4.5 below will allow us to obtain a tight lower bound when $k$ lies “in the middle” of the interval $[0,d]$, by using the comparison between the tropical roots. Unlike the lower bound of Pólya, this theorem gives a bound which is not anymore independent of the coefficients of the polynomial $P$, although it depends only on a small information, namely a ratio measuring the separation between some tropical roots. Thus, the bound will involve a coefficient $U(k,\delta)$ depending both on the index $k$ and on the parameter $\delta$ (the ratio). It will be specially useful in situations in which $\delta$ is small. This coefficient $U(k,\delta)$ is defined as follows:

$$ U(k,\delta) := E(k) \left( \frac{1 + \delta + \sqrt{k^2(1-\delta)^2 + 4\delta}}{1 - \delta} \right), \quad \text{for } k \geq 0 \text{ and } 0 \leq \delta \leq 1, \quad (4.7) $$

where $E(k)$ is defined by (1.4) for $k \geq 1$, and $E(0) := 1$, and with the convention that $1/0 = +\infty$, so that $U(k,1) = +\infty$. It is easy to check that

$$ k + \frac{1 + \delta}{1 - \delta} \leq U(k,\delta) \leq e \left( k + \frac{1 + \sqrt{\delta}}{1 - \sqrt{\delta}} \right). \quad (4.8) $$

The following asymptotic regime should be kept in mind:

$$ U(k,\delta) \sim E(k)(k+1) < e(k+1), \quad \delta \to 0. $$

**Theorem 4.5 (Master lower bound).** Let $A_0, A_1, \ldots, A_{d-1}, A_d$, $P$, $\zeta_1, \ldots, \zeta_n$, $\| \cdot \|$, $c$, and $\alpha_1, \ldots, \alpha_d$ be as in the first part of Theorem 4.4, and denote $\alpha_0 = 0$ and $\alpha_{d+1} = +\infty$. Let $0 \leq k^- < k < k^+ \leq d+1$, and denote $\delta_- := \frac{\alpha_-}{\alpha_k} \leq 1$ and $\delta_+ := \frac{\alpha_k}{\alpha_+} \leq 1$. Then, Inequality (4.1) holds with

$$ L_k := \max(L_k^\pm), \quad L_k^+ := \frac{1}{U(k^+ - k - 1, \delta_+^-)}, \quad L_k^- := \frac{1}{U(k - k^-, \delta_-)}, \quad (4.9) $$

with the convention that $1/\infty = 0$. Moreover, when $\| \cdot \|$ satisfies (A2), the constant $L_k$ of (4.1) can be replaced by the greater constant $L_k^*$:

$$ L_k^* := \max(L_k^{\pm*}), \quad L_k^+ := \frac{1}{\sqrt{U(k^+ - k - 1, \delta_+^*)}}, \quad L_k^- := \frac{1}{\sqrt{U(k - k^-, \delta_-^*)}}. \quad (4.10) $$

Theorem 4.5 generalizes Theorem 4.4 and thus the lower bound of Pólya. Indeed, taking $k^- = 0$, $k^+ = d+1$, and using that $U(k,0) = E(k)(k+1)$, we get that the constants $L_k^- \leq L_k$ and $L_k^- \leq L_k^*$ of Theorem 4.5 are exactly the constants $L_k$ of (4.5) and (4.6) of Theorem 4.4 respectively. Moreover, Theorem 4.5 is already new in the scalar case ($n = 1$).
Note that when $\delta_- = 1$ in Theorem 4.5, $U(k-k^-,\delta_-) = +\infty$, so that $L_k^- = 0$ and $L_k = L_k^+$. Similarly when $\delta_+ = 1$, we get $L_k^+ = 0$. Hence, if $\delta_- = 1 = \delta_+$, we get $L_k = 0$ so that (4.1) does not provide any information, although it is true. However, if for instance $\delta_- = 1$ and $\delta_+ < 1$, we get that $L_k^- = L_k^+ > 0$, which gives a positive lower bound in (4.1).

Applying Theorem 4.5 to the particular case when $k = k^- + 1 = k^+ - 1$ leads to the following formula for the constants of (4.9) and (4.10):

$$L_k := \max(L_k^\pm), \quad L_k^+ := \frac{1 - \sqrt{\delta_+}}{1 + \sqrt{\delta_+}}, \quad L_k^- := \frac{1 - \delta_-}{4(1 + \delta_-)}, \quad (4.11)$$

and

$$L^*_k := \max(L_k^{\pm*}), \quad L_k^{+*} := \sqrt{\frac{1 - \delta_+}{1 + \delta_+}}, \quad L_k^{-*} := \frac{1}{2} \sqrt{\frac{1 - \delta_-^2}{1 + \delta_-^2}}. \quad (4.12)$$

In this special case, we obtain the following stronger lower bounds.

**Theorem 4.6.** Let us use the notations of Theorem 4.5, and assume that $k = k^- + 1 = k^+ - 1$. Then, the statements of Theorem 4.5 hold with the constants $L_k$ and $L^*_k$ (given in (4.9) and (4.10), or (4.11) and (4.12)) replaced respectively by the greater constants $L_k^\#$ and $L^*_k^\#$ given by:

$$L_k^\# := \frac{1 - \delta_-}{(1 + \sqrt{\delta_+})^2}, \quad (4.13)$$

and

$$L^*_k^\# := \frac{\sqrt{1 - \delta_-^2 \delta_+^2}}{1 + \delta_+}. \quad (4.14)$$

We next indicate how the indices $k^-$ and $k^+$ should be chosen, for each $1 \leq k \leq d$, in order to get the best lower bound $L_k$.

Let $k_0 = 0, k_1, \ldots, k_q = d$ be the sequence of abscissae of the vertices of the Newton polygon of $t p(x)$, as shown in Figure 4.1. For $j = 1, \ldots, q$, we have, $\alpha_{k_{j-1}+1} = \cdots = \alpha_{k_j} < \alpha_{k_{j+1}}$, with the convention $\alpha_{d+1} = +\infty$. We also denote by

$$\delta_j = \frac{\alpha_{k_j}}{\alpha_{k_{j+1}}}, \quad (4.15)$$

for $j = 0, \ldots, q$, the parameters measuring the separation between the tropical roots, in particular $\delta_0 = \delta_q = 0$.

**Proposition 4.7.** Each of the constants $L_k$ and $L_k^*$ appearing in Theorem 4.5 is maximized by choosing $k^- = k_r$ for some $0 \leq r \leq q$ such that $k_r < k$ and $k^+ = k_s + 1$ for some $0 \leq s \leq q$ such that $k_s \geq k$.

This proposition shows that, to apply Theorem 4.5, we may always require $k^-$ and $k^+ - 1$ to be abscissae of vertices of the Newton polygon. Optimizing the choice of $k^\pm$, we readily arrive at the following corollary.

**Corollary 4.8.** Let us use the notations of Theorem 4.5 and let $k_0 = 0, k_1, \ldots, k_q = d$ be the sequence of abscissae of the vertices of the Newton polygon of $t p(x)$, as shown in Figure 4.1. Then, the statements of Theorem 4.5 hold with the
Log-majorization of the moduli of the eigenvalues of a matrix polynomial by tropical roots

Fig. 4.1: Newton polygon corresponding to $t p(x)$.

constants $L_k, L_k^\pm, L_k^\ast$ and $L_k^\pm\ast$ (given in (4.9) and (4.10)) replaced respectively by the following optimal ones:

\[ L_k^{\text{opt}} := \max(L_k^\pm, \text{prox}), \]  
\[ L_k^{+, \text{opt}} := \max \frac{1}{U(k_j - k, \frac{\alpha_k}{\alpha_{k+1}})}, \]  
\[ L_k^{-, \text{opt}} := \max \frac{1}{U(k - k_{j-1}, \frac{\alpha_{j-1}}{\alpha_k})}, \]

and

\[ L_k^{*, \text{opt}} := \max(L_k^{\pm*, \text{opt}}), \]  
\[ L_k^{+,*, \text{opt}} := \max \frac{1}{\sqrt{U(k_j - k, \delta_{2j})}}, \]  
\[ L_k^{-*, \text{opt}} := \max \frac{1}{\sqrt{U(k - k_{j-1}, \delta_{2j-1})}}. \]

However, a simpler choice of $k^\pm$ consists in taking the nearest vertices of the Newton polygon in (4.16) and (4.17), which lead to the following corollary.

**Corollary 4.9.** Let us use the notations of Theorem 4.5, let $k_0 = 0, k_1, \ldots, k_q = d$ be the sequence of abscissæ of the vertices of the Newton polygon of $t p(x)$, as shown in Figure 4.1, and let $\delta_0, \ldots, \delta_q$ be defined by (4.15). For $0 \leq k \leq d$, let us consider the unique $j \in \{1, \ldots, q\}$ such that $k_{j-1} < k \leq k_j$, so that $\alpha_k = \alpha_{k_{j-1}+1} = \alpha_{k_j}$. Then, the statements of Theorem 4.5 hold with the constants $L_k, L_k^\pm, L_k^\ast$ and $L_k^{\pm\ast}$ (given in (4.9) and (4.10)) replaced respectively by the following ones:

\[ L_k^{\text{prox}} := \max(L_k^\pm, \text{prox}), \]  
\[ L_k^{+,\text{prox}} := \frac{1}{U(k_j - k, \delta_j)}, \]  
\[ L_k^{-,\text{prox}} := \frac{1}{U(k - k_{j-1}, \delta_{j-1})}, \]

and

\[ L_k^{*,\text{prox}} := \max(L_k^{\pm*, \text{prox}}), \]  
\[ L_k^{+,*,\text{prox}} := \frac{1}{\sqrt{U(k_j - k, \delta_{2j})}}, \]  
\[ L_k^{-*,\text{prox}} := \frac{1}{\sqrt{U(k - k_{j-1}, \delta_{2j-1})}}. \]
Moreover, in the particular case where \( k = k_j \), we have

\[
L_k^\text{prox} \geq L_k^{\text{prox}*} = \frac{1 - \sqrt{\delta_j}}{1 + \sqrt{\delta_j}}, \quad L_k^{\text{prox}*} \geq L_k^{+\text{prox}*} = \frac{1 - \delta_j}{1 + \delta_j}.
\]

**Remark 4.10.** When all the ratios \( \delta_1, \ldots, \delta_q \) are small, the maxima in (4.16) and (4.17) are attained by taking \( j \) as in Corollary 4.9, that is \( L_k^{\text{opt}} = L_k^{\text{prox}} \) and \( L_k^{* \text{opt}} = L_k^{* \text{prox}} \) for all \( 0 \leq k \leq d \).

**Remark 4.11.** Since \( U(k, \delta) \) is increasing in \( k \) and \( \delta \), the maximizing \( j \) in the definition of \( L_k^{+\text{opt}} \) arises from a compromise between keeping \( k_j \geq k \) close to \( k \) and \( \delta_j \) small. In particular, when \( k \) belongs to an edge of the Newton polygon such that several consecutive edges have almost the same slope than this edge, the maximizing \( j \) may be the one corresponding to the first vertex at which the slope changes significantly, i.e., the first one such that \( \delta_j \) is small. Similar considerations apply to \( L_k^{-\text{opt}} \).

### 5. Proof of the lower bounds.

#### 5.1. Proof of the lower bounds for the roots of scalar polynomials.

In this section, we prove the main lower bounds of Section 4 for scalar polynomials, since the arguments are more transparent in this case. The generalization to the matrix case will be given in the next section.

From Remark 4.2, in the scalar case, all lower bounds reduce to (4.4) with the constant \( L_k \) obtained under Assumption (A2), that is (4.3) in Theorem 4.1, (4.6) in Theorem 4.5, (4.10) in Theorem 4.6 and (4.14) in Theorem 4.7.

**Proof of the scalar version of Theorem 4.4.** We note first that Pólya’s inequality is an immediate consequence of Proposition 3.4. Indeed, using the property that all the \( \beta_{k,j} \) are less than or equal to 1, we obtain:

\[
(L_k)^{-2} \leq \sum_{j=0}^{\infty} \xi^{j-k} \leq \sum_{j=0}^{\infty} \xi^{j-k} = \frac{1}{\xi^{k}(1-\xi)},
\]

for all \( \xi > 0 \). The minimum of the right hand side of the previous inequality for \( 0 < \xi < 1 \), is attained for \( \xi = k/(k+1) \), from which we deduce (4.3), which is also the scalar version of Theorem 4.1.

We can now deduce similarly, from Proposition 3.4, the scalar versions of the lower bounds of Theorems 4.1, 4.5 and 4.6. Since in the scalar case, we are reduced to show (4.4) for some constants \( L_k \), and that this inequality is precisely the statement of Proposition 4.4, we only need to show that these constants \( L_k \) are lower bounds of the constant \( L_k \) of (3.3).

**Proof of the scalar version of Theorem 4.1.** Using the property that \( \beta_{k,j} \leq 1 \) for all \( k,j \), we obtain that the constant \( L_k \) of (3.3) satisfies \( (L_k)^{-2} \leq \sum_{j=0,\xi_{j,0}\neq 0}^{\infty} \xi^{j-k} \) for all \( \xi > 0 \). When \( \xi = 1 \), the right hand side of this inequality is equal to the number of non-zero coefficients of \( p \), which shows that the constant \( L_k \) of (3.3) is lower bounded by the constant \( L_k \) of (4.3), which implies (4.4) with this lower bound \( L_k \).

**Proof of the scalar version of Theorem 4.5.** Assume that \( 0 \leq k^- < k < k^+ \leq d+1 \). Denote \( \delta_- = \alpha_{k-}/\alpha_k \) and \( \delta_+ = \alpha_k/\alpha_{k+} \). Then, for all \( 1 \leq \ell < k, \alpha_{\ell} \leq \alpha_k \), and for all \( 1 \leq \ell \leq k^- \), \( \alpha_{\ell} \leq \alpha_{k^-} = \alpha_k \delta_- \). This implies that \( \beta_{k,j} \leq \delta_-^{k-j} \), for all \( j \leq k^- \). Similarly, for all \( k \geq \ell / k^+ \), \( \alpha_{\ell} \geq \alpha_{k^+} = \alpha_k / \delta_+ \), hence, for all
The last formula gives \( \delta^j < \xi \). Since we also have \( \beta_{k,j} \leq \delta^{k^+} \), we obtain that the constant \( L_k \) of (5.3) satisfies, for all \( \delta^2 < \xi < \delta^2_+\), \( \xi \neq 1 \),

\[
(L_k)^{-2} \leq \left( \sum_{j=0}^{k^{-}} \delta_\vartriangle^{2(k^{-}-j)} \xi^{j-k} \right) + \left( \sum_{j=k^{-}+1}^{k^{-}+1} \xi^{j-k} \right) + \left( \sum_{j=k^{-}+2}^{d} \delta_\vartriangle^{2(j-k^+)+1} \xi^{j-k} \right)
\]

\[
\leq \xi^{k^{-}-k+1} \frac{1}{\xi - \delta^2_\vartriangle} + \frac{\xi^{k^{-}+1}}{1 - \xi} + \frac{\xi^{k+1}}{1 - \delta^2_\vartriangle} \frac{\delta^2_\vartriangle}{\delta^2_\vartriangle + \xi} .
\]

Since \( \frac{\delta^2_\vartriangle}{1 - \delta^2_\vartriangle} \leq \frac{\xi}{1 - \delta^2_\vartriangle} \), the later inequality can be written as

\[
(L_k)^{-2} \leq g(\xi), \quad \forall \delta^2_\vartriangle < \xi < \delta^2_+, \xi \neq 1,
\]

with

\[
g(\xi) := g_-(\xi) + g_+\left(\xi\right) ,
\]

\[
g_-(\xi) := \xi^{k^{-}-k+1} \left( \frac{1}{\xi - \delta^2_\vartriangle} + \frac{1}{1 - \xi} \right) ,
\]

\[
g_+\left(\xi\right) := \xi^{k+1} \frac{1}{1 - \delta^2_\vartriangle} \frac{1 - \delta}{\left(\xi - 1\right)(1 - \delta^2_\vartriangle)} .
\]

Note that (5.1) also holds when \( k^{-} = 0 \) or \( k^+ = d + 1 \), since then \( \delta_- = 0 \) or \( \delta_+ = 0 \) respectively. When \( \delta_- = 1 \) and \( \delta_+ < 1 \), the conditions on \( \xi \) in (5.1) are equivalent to \( 1 < \xi < \delta^2_+ \), whereas when \( \delta_- = \delta_+ = 1 \), these conditions are never satisfied, but in this case the constant \( L_k \) of (4.10) is equal to 0 so there is nothing to prove.

The functions \( g_- \) and \( g_+ \) satisfy \( g_-(\xi) = g_\vartriangle^{k-k_\vartriangle}(\xi^{-1}) \) and \( g_+\left(\xi\right) = g_\vartriangle^{k-k_\vartriangle}(\delta^2_\vartriangle) \) where \( g_\vartriangle \) is defined for \( \xi \neq 1 \) and \( \delta^2_\vartriangle \) by:

\[
g_\vartriangle(\xi) := \xi^{k_\vartriangle} \left( \frac{1}{\xi - 1} + \frac{1}{1 - \delta^2_\vartriangle} \right) = \xi^{k_\vartriangle} \frac{1 - \delta}{\left(\xi - 1\right)(1 - \delta^2_\vartriangle)} .
\]

We have \( g_\vartriangle(\xi) \leq 0 \) for \( \xi < 1 \), hence \( g_-\left(\xi\right) \leq 0 \) for all \( \xi > 1 \), and \( g_+\left(\xi\right) \leq 0 \) for all \( \xi < 1 \). When \( \delta < 1 \), the minimum of \( g_\vartriangle \) on \( \left(1, \delta^{-1}\right) \) is attained at

\[
\xi_{k_\vartriangle,\delta} := \begin{cases} \frac{k(\delta^{-1}) - \sqrt{2}\sqrt{2}(\delta^{-1})^2 - 4\delta}{2(k-1)\delta} & \text{when } k \neq 1, \delta \neq 0 , \\ \frac{k+1^2}{k} & \text{when } k = 1 , \\ \frac{1+1}{1+1} & \text{when } \delta = 0 . 
\end{cases}
\]

The last formula gives \( \xi_{k_\vartriangle,\delta} = \infty \) when \( \delta = 0 \) and \( k = 0 \), which is the point of infimum of \( g_\vartriangle \) for \( 1 < \xi < \infty = \delta^{-1} \), since \( g_\vartriangle \) is decreasing. It also gives \( \xi_{k,1} = 1 \) for all \( k \geq 0 \). Hence extending \( g_\vartriangle \), \( g_- \) and \( g_+ \) by \( +\infty \) at point 1, we get that the infimum of \( g_\vartriangle \) on \( \left(1, \delta^{-1}\right) \) equals \( g_\vartriangle(\xi_{k,\delta}) \), and denoting \( \xi_- = (\xi_{k-k_\vartriangle,\vartriangle})^{-1} \) and \( \xi_+ = (\delta_{k-k_\vartriangle,\vartriangle})^{-1} \), we obtain

\[
(L_k)^{-2} \leq \min(g(\xi_-), g(\xi_+)) \leq \min(g_-(\xi_-), g_+(\xi_+)) \leq \min(g_-(\xi_-), g_+(\xi_+)) .
\]

In order to simplify this bound, we need to find good estimates of \( \xi_{k,\delta} \).

When \( k \neq 1 \) and \( \delta \neq 0 \), we have:

\[
\xi_{k,\delta} = \frac{k(\delta + 1) - \sqrt{2}+\delta^{2}(1+\delta)^{2}+4\delta}{2(k-1)\delta} = \frac{2(k + 1)}{k(\delta + 1) + \sqrt{2}+\delta^{2}(1+\delta)^{2}+4\delta} .
\]
Using the property that \( \delta \geq 0 \) in the last formula, we get that \( \xi_{k,\delta} \leq \frac{k+1}{k} = \xi_{k,0} \). Moreover, this inequality also holds for \( k = 1 \). In particular \( (\xi_{k,\delta})^k \leq \mathcal{E}(k) \), for all \( k \geq 0 \) and \( \delta \geq 0 \) (taking the convention that \( \xi^0 = 1 \) for all \( \xi \in (1,\infty) \)). We also have, for \( k \geq 0 \) and \( 0 \leq \delta < 1 \),

\[
\frac{1}{\xi_{k,\delta} - 1} + \frac{1}{1 - \delta \xi_{k,\delta}} = \frac{1 + \delta + \sqrt{k^2(1-\delta)^2 + 4\delta}}{1 - \delta} \leq k + \frac{1 + \sqrt{\delta}}{1 - \sqrt{\delta}}.
\]

This yields

\[
g_{k,\delta}(\xi_{k,\delta}) \leq \mathcal{U}(k, \delta) \quad \forall k \geq 0, \ 0 \leq \delta \leq 1,
\]

with \( \mathcal{U} \) as in (4.7). Indeed, this inequality holds for \( \delta < 1 \) by the above arguments. It also holds for \( \delta = 1 \) since \( \mathcal{U}(k,1) = +\infty \). This implies in particular that \( g_+(\xi_+) \leq \mathcal{U}(k^+ - k - 1, \delta^+ \xi^+ \) and \( g_-(\xi-) \leq \mathcal{U}(k - k^-, \delta^- \xi^- \). Combining these two inequalities with (5.2), we obtain that the constant \( L_k \) of (3.3) is lower bounded by the constant \( L_k^* \) of (4.10), which implies (4.4) with this lower bound \( L_k^* \) instead of \( L_k \).

**Remark 5.1.** In the previous proof, obtaining the minimum of \( g \) instead of \( g_+ \) and \( g_- \) would have led to a better lower bound for \( L_k \) than the one of Theorem 4.5. However, such a bound is more difficult to estimate for general values of \( k^+ - k^- \). We can use for instance the first inequality in (5.2), which will give better estimates in some particular cases. For instance, when \( k = 0 \), we get that \( g_{k,\delta}(\xi_{k,\delta}) = \mathcal{U}(k, \delta) = (1 + \sqrt{\delta})/(1 - \sqrt{\delta}) \). In particular, when \( k = k^+ - 1 \), the bound \( (L_k)^{-2} \leq g_+(\xi^+) \) gives the lower bound \( L_k^* \) of (4.12), with \( \delta = \delta^+ \). However (5.2) gives the slightly better bound:

\[
(L_k)^{-2} \leq g(\xi^+) = \frac{1 + \delta_+}{1 - \delta_+} - \frac{1 - \delta_-^2}{(1 - \delta_+ \delta_+^2)(1 - \delta_+^2 / \delta_+^2)} \delta_+^k - k^-.
\]

**Proof of the scalar version of Theorem 4.6.** Let us use the notation of the previous proof. When \( k = k^+ - 1 = k^- + 1 \), Inequality (5.1) is true for all \( \xi \in (\delta_-^2, \delta_+^2) \) since \( g \) can be well defined at 1 (by continuity for instance). When \( \delta_- \delta_+ < 1 \), the interval \( (\delta_-^2, \delta_+^2) \) is nonempty, the map \( g \) is convex there and its minimum is achieved at the point \( \xi = (1 + \delta_-^2 \delta_+)/(\delta_+ + 1 + \delta_+) \). Then (5.1) yields \( (L_k)^{-2} \leq (1 + \delta_+^2)^2/(1 - \delta_+^2 \delta_+^2) \), which implies that the constant \( L_k \) of (3.3) is lower bounded by the constant \( L_k^* \) of (4.14), which implies (4.4) with this lower bound \( L_k^* \) instead of \( L_k \). Moreover, since the minimum of \( g \) is less or equal to the right hand side of (5.2), the proof of the scalar version of Theorem 4.5 implies that \( L_k^* \leq L_k^* \).}

**5.2. Proofs of the lower bounds for the eigenvalues of matrix polynomials.** In this section, we prove the lower bounds which were stated in Section 4 for matrix polynomials, and proved in Section 5.1 in the special scalar case.

**Proof of Theorems 4.4, 4.5, and 4.6.** In Proposition 3.5, the formula (3.7) of \( L_k \) coincides with (3.3), except that the condition \( a_j \neq 0 \) is replaced by \( A_j \neq 0 \). Hence, the lower bounds of the constant \( L_k \) of (3.3) obtained in the proofs of the scalar versions of Theorems 4.1, 4.5, and 4.6 give similar lower bounds for the constant \( L_k \) of (3.7), which combined with the assertion that Inequality (4.1) holds with this constant, now provide the lower bounds of Theorems 4.1, 4.5, and 4.6 respectively in the case where \( \| \cdot \| \) satisfies (A2) that is Inequality (4.1) with \( L_k \) replaced by the constant \( L_k \) of (4.3), the constant \( L_k^* \) of (4.10), and the constant \( L_k^* \) of (4.14), respectively.
In the case of a norm satisfying only (A1), the formula (3.7) of $L_k$ is replaced by (4.6), which means that everything behave as if $L_k$ were replaced by its square root and the numbers $\alpha_k$ were replaced by their square roots in (3.7). Hence the assertions of Theorems 4.4, 4.5, and 4.6 in the case of Assumption (A2) are still true under Assumption (A1), up to this transformation of the constant $L_k$ of (4.3), the constant $L_k^* \leq L_k$ of (4.4) and the constant $L_k^{q+1}$ of (4.11), respectively.

The constant $L_k$ of (4.3) does not depend on the values of the constants $\alpha_k$, hence the above transformation leads to the square of $L_k$, which is exactly the constant $L_k$ of (4.2), which finishes the proof of Theorem 4.4. The constant $L_k^*$ of (4.4) is a function of $\delta_+ = \frac{\alpha_k}{\alpha_k}$ and $\delta_- = \frac{\alpha_k}{\alpha_k}$. Since replacing all numbers $\alpha_i$, $i = 0, \ldots, d + 1$, by their square roots, reduces to replace the numbers $\delta_-$ and $\delta_+$ by their square roots too, the above transformation of the constant $L_k$ of (4.3) consists in taking its square and replacing $\delta_-$ and $\delta_+$ by their square roots, which leads to the constant $L_k$ of (4.9), which finishes the proof of Theorem 4.5. Similarly, the above transformation of the constant $L_k^{q+1}$ of (4.11) leads to the constant $L_k^{q+1}$ of (4.13), which finishes the proof of Theorem 4.6.

Now Theorem 4.4 is an immediate corollary of Theorem 4.5. Indeed as said in Section 4 taking $k^- = 0$, $k^+ = d + 1$ in Theorem 4.5 and using $L_k^- \leq L_k$, $L_k^{q+} \leq L_k^*$ and $\mathcal{U}(k, 0) = \mathcal{E}(k)(k + 1)$, we get Theorem 4.4. Corollaries 4.8 and 4.9 are straightforward consequences of Theorem 4.5 and Proposition 4.7. We only need to prove the latter proposition.

**Proof of Proposition 4.7.** Let us use the notations of Theorem 4.5 and let $k_0 = 0, k_1, \ldots, k_d = d$ be the sequence of abscissae of the vertices of the Newton polygon of $tp(x)$, as shown in Figure 4.1. Then, the ratio $\delta_-$ does not change when $k^-$ moves inside an interval $[k_{r-1} + 1, k_r]$. Since $\mathcal{U}(\cdot, \delta)$ is a nondecreasing function, in order to maximize $L_k^-$ or $L_k^{q+}$ and thus $L_k$, with $\delta_-$ constant (and $\delta_+$ and $k^+$ constant), we need to minimize $k - k^-$, which implies that $k^- = k_ r$ for some $0 \leq r \leq q$ such that $k_r < k$. Similarly, $k^+ = k_s + 1$ for some $0 \leq s < q$ such that $k_s \geq k$.

**Remark 5.2.** All the lower bounds in Theorems 4.1, 4.4, 4.5, and 4.6 are equivalent to inequalities of the form

$$
(|\zeta_1 \cdots \zeta_{nk}|^{-1} \det A_0)^{1/n} L_k \leq \|A_0\|\alpha_1^{-1} \ldots \alpha_k^{-1}. 
$$

By definition of the tropical roots, $\|A_0\|\alpha_1^{-1} \ldots \alpha_k^{-1} = \exp \hat{p}_k$, where $\hat{p}_k$ is the value in $k$ of the concave hull of the map $j \in \{0, \ldots, d\} \mapsto \log \|A_j\|$. Hence, if $\|\cdot\|$ and $\|\cdot\|$ are two norms on the space of matrices such that $\|A\| \leq \|A\|$ for all $A \in \mathbb{C}^{n \times n}$, then the right hand side in (5.3) is smaller for $\|\cdot\|$ than for $\|\cdot\|$. Moreover, in (4.2), (4.3), (4.5), and (4.6), $L_k$ depends only on $P$ or $k$ and $n$, but not on the norm $\|\cdot\|$, so that (5.3) is necessarily a tighter inequality for $\|\cdot\|$ than for $\|\cdot\|$. In particular if (A2) holds with $Q$ and $Q'$, then the lower bounds (4.3) and (4.6) for $\|\cdot\|$ are weaker than the corresponding ones for $A \mapsto \|QAQ'\|_F$. For the lower bounds (4.9), (4.10), (4.13), and (4.14) of Theorems 4.3 and 4.6 and the ones of Corollaries 4.8 and 4.9, based on the separation between the tropical roots, the comparison is not so simple, because changing the norm changes the separation between the tropical roots.

6. Upper bound.

6.1. Statement of the upper bound and corollary. The previous results provide lower bounds for the product of the $nk$ smallest eigenvalues, in terms of tropical roots. We next state a reverse inequality.
Theorem 6.1 (Upper bound). Let $\| \cdot \|$ be any norm on the space of matrices $\mathbb{C}^{n \times n}$. For all $i = 1, \ldots, n$, we denote by $A^{(i)}$ the $i$-th column of $A$, by $\eta_i$ the least positive constant such that $\| A^{(i)} \|_2 \leq \eta_i \| A \|$ for all $A \in \mathbb{C}^{n \times n}$, where $\| \cdot \|_2$ is the Euclidean norm of $\mathbb{C}^n$, and we set $\eta := \eta_1 \cdots \eta_n$. Let $A_0, \ldots, A_d$, $P$, $\xi_1, \ldots, \xi_{nd}$, $c$, $t$, $\alpha_1, \ldots, \alpha_d$, be as in Theorem 4.4. Let $k_0 = 0, k_1, \ldots, k_q = d$ be the sequence of all abscissae of the vertices of the Newton polygon of $tp(x)$, and define $\delta_0, \ldots, \delta_q$ be as in (4.15). Denote by $C_{n,d,k}$ the number of maps $\phi : \{0, \ldots, n\} \to \{0, \ldots, d\}$ such that $\sum_{j=1}^n \phi(j) = k$, so that $C_{n,d,k} \leq (\min(d, k, nd - k) + 1)^{n-1}$.

Then, for every $j = 1, \ldots, q$, if

$$c_j := \frac{|\det A_{j_k}|}{\|A_{j_k}\|^n} - (C_{n,d, nk_j} - 1) \eta \delta_j > 0,$$

we have

$$\frac{|\xi_{j_k} \cdots \xi_{nk_j}|}{(\alpha_1 \cdots \alpha_{k_j})^n} \leq \frac{c}{c_j} \left( \frac{nd}{nk_j} \right)^n.$$

Using the results of Corollary 6.9 and Theorem 6.1, we are able to show that the modulus of a group of eigenvalues corresponding to an edge of the Newton polygon is bounded from above and below by the corresponding tropical roots. The following immediate corollary gives an example of such bounds.

**Corollary 6.2.** Under the assumptions and notations of Theorem 6.1, together with (A2), we have, for all $1 \leq j \leq q$, such that $c_j > 0$ and $c_{j-1} > 0$,

$$\frac{c_{j-1}}{c_j} \left( \frac{1 - \delta_j}{1 + \delta_j} \right)^{n/2} \leq \frac{|\xi_{nk_j+1} \cdots \xi_{nk_j}|}{(\alpha_{k_j})^n(k_j - k_{j-1})} \leq \frac{(nd)}{nk_j} \left( \frac{1 + \delta_{j-1}}{1 - \delta_{j-1}} \right)^{n/2}.$$

Similar results can be obtained when $\| \cdot \|$ only satisfies (A1) and also for general $1 \leq k \leq d$. Note also that the bounds in the above corollary do not depend on the constant $c$, except if $j = 1$ (in which case $c_{j-1} = c_0 = c$). Hence, by an argument of continuity, the assumption that $\det A_0 \neq 0$, which is present there (since the notations and assumptions of Theorem 6.1 include the ones of the first part of Theorem 4.1), can be dispensed with, except when $j = 1$.

**Remark 6.3.** Since $1 \geq \frac{|\det A_{k_j}|}{\|A_{k_j}\|} \geq (\kappa(A_{k_j}))^{-n}$, where $\kappa$ denotes the condition number of a matrix with respect to the norm $\| \cdot \|$, we get that Condition (6.1) of Theorem 6.1 holds if the matrix $A_{k_j}$ is nonsingular and has a sufficiently small condition number, and if the tropical roots $\alpha_{k_j}$ and $\alpha_{k_{j+1}}$ are sufficiently separated, so that $\delta_j = \alpha_{k_j}/\alpha_{k_{j+1}} \ll 1$. Similarly, the tightness of the bounds in Corollary 6.2 depends on the parameters $\delta_{j-1}, \delta_j$ and on the condition number of the matrices $A_{k_{j-1}}, A_{k_j}$.

**Remark 6.4.** The bounds of Corollary 6.2 require the $\delta_j$ to be very small. In this case, a different approach would be to use the extensions of Pellet’s theorem to the case of matrix polynomials [BNS13, Mel12].
interval \((\log r(\alpha), \log R(\alpha))\). If Pellet’s theorem can be applied to every abscissa \(k\) of a vertex of the Newton polygon of \(p\), then each moduli of a root of \(p\) can be estimated using the corresponding tropical root. This is in particular possible if \(\delta_j < 1/9\) for all \(j = 1, \ldots, q\), by a result of Gaubert and Sharify (see \cite[Theorem 3.3.3]{Sha11}).

Using an extension of Pellet’s theorem to the case of matrix polynomials, one may obtain a similar localisation of the eigenvalues when all the \(j\) and condition numbers of the matrices \(A_{k_j}\) are small enough, see for instance \cite{BNS13} for the case where the coefficient matrices are unitary up to some constants.

However, it may happen that Pellet’s theorem cannot be applied for any vertex of the Newton polygon except the first and last one. In this case, the log-majorization type inequalities of Section \ref{section:4} do provide a better information. Note that there is some duality between Pellet type estimates and log-majorization estimates: the former hold conditionally, and give exclusion annuli, whereas the latter hold inconditionally, and allow one to infer inclusion regions for the eigenvalues. For instance, it does not seem that tight constants like the one in (1.3) are achievable by Pellet type approaches.

6.2. Proof of the upper bound. To prove Theorem \ref{6.1}, we first prove the following lemma which provides a lower bound for the modulus of the coefficients of the polynomial \(\det(P(\lambda))\).

**Lemma 6.5.** Let \(A_0, \ldots, A_d, P, \|\cdot\|, \eta, \alpha_1, \ldots, \alpha_d, q, \delta_0, \ldots, \delta_q\), and \(C_{n,d,k}\) be as in Theorem \ref{6.1} and denote by \(\tilde{p}\) the polynomial:

\[
\tilde{p}(\lambda) = \det P(\lambda) = \sum_{l=0}^{nd} \hat{p}_l \lambda^l. \tag{6.3}
\]

Then, for \(j = 0, \ldots, q\), we have \(\|A_{k_j}\| > 0\) and

\[
|\hat{p}_{nk_j}| \geq |\det A_{k_j}| - (C_{n,d,nk_j} - 1)\eta \|A_{k_j}\|^{\delta_j}.
\]

**Proof.** Let \(k = 0, \ldots, d\). Using the multilinearity of the determinant we get

\[
\tilde{p}(\lambda) = \det \sum_{j=0}^{d} A_j \lambda^j = \sum_{\phi} \det(A_{\phi(1)}^{(1)}, A_{\phi(2)}^{(2)}, \ldots, A_{\phi(n)}^{(n)}) \lambda^{\sum_{m=1}^{n} \phi(m)},
\]

where the sum is taken over all maps \(\phi : \{1, \ldots, n\} \to \{0, \ldots, d\}\). Denoting by \(\Phi_k\) the set of all such maps that satisfy \(\sum_{m=1}^{n} \phi(m) = nk\), we obtain that the \(nk\)-th coefficient of the polynomial \(\tilde{p}\) is equal to:

\[
\hat{p}_{nk} = \sum_{\phi \in \Phi_k} \det(A_{\phi(1)}^{(1)}, A_{\phi(2)}^{(2)}, \ldots, A_{\phi(n)}^{(n)})
\]

\[
= \det A_k + \sum_{\phi \in \Phi_k, \phi \neq k} \det(A_{\phi(1)}^{(1)}, A_{\phi(2)}^{(2)}, \ldots, A_{\phi(n)}^{(n)}).
\]

Using Hadamard’s inequality together with the definition of \(\eta\) yields

\[
|\det(A_{\phi(1)}^{(1)}, A_{\phi(2)}^{(2)}, \ldots, A_{\phi(n)}^{(n)})| \leq \|A_{\phi(1)}^{(1)}\|_2 \cdots \|A_{\phi(n)}^{(n)}\|_2 \leq \eta \|A_{\phi(1)}\|_2 \cdots \|A_{\phi(n)}\|.
\]

Assume now that \(k = k_j\) for some \(j = 0, \ldots, q\). Then, \(k\) is the abscissa of a vertex of the Newton polygon of the tropical polynomial \(t p\) defined in Theorem \ref{4.1}. By Proposition \ref{3.3} applied to \(t p\), we get that \(\|A_k\| > 0\), \(\|A_m\| \leq \|A_k\| \prod_{\ell=m+1}^{k} \alpha_\ell \leq \|A_k\|^{k-m}\).
for all \( m \leq k \) and \( \|A_m\| \leq \|A_k\| \prod_{\ell=k+1}^{n} \alpha_{\ell}^{-1} \leq \|A_k\| \alpha_{k+1}^{k-m} \leq \|A_k\| \alpha_{k}^{k-m} \delta_{j}^{-k} \) for all \( m > k \). Hence

\[
\|A_{\phi(1)}\| \ldots \|A_{\phi(n)}\| \leq \|A_k\|^{n} \sum_{m=1}^{n} (k-\phi(m)) \sum_{m=1,\phi(m)>k}^{n} (\phi(m)-k) \delta_{j}^{-m} 
\]

when \( \sum_{m=1}^{n} \phi(m) = nk \). When in addition \( \phi \neq k \), there exists \( m = 1, \ldots, n \) such that \( \phi(m) > k \), thus \( \sum_{m=1,\phi(m)>k}^{n} (\phi(m)-k) \geq 1 \), which yields

\[
\|A_{\phi(1)}\| \ldots \|A_{\phi(n)}\| \leq \|A_k\|^{n} \delta_{j}^{-m} 
\]

From all the above inequalities, we deduce that

\[
|\tilde{p}_{nk}| \geq |\det A_k| - \sum_{\phi \in \Phi_k, \phi \neq k} \eta_{j} \|A_k\|^{n} \delta_{j}^{-m} 
\]

Since by definition \( C_{n,d,nk} \) is the cardinality of \( \Phi_k \), and there exists exactly one element of \( \Phi_k \) such that \( \phi \equiv k \), we obtain the inequality of the lemma for \( k = k_j \).

**Proof of Theorem 6.1** Consider the polynomial \( \hat{p} \) of \( (6.3) \) and let \( \gamma_1, \ldots, \gamma_{nd} \) be the tropical roots of the tropical polynomial \( t\hat{p}(x) = \max_{0 \leq l \leq nd} |\hat{p}_l| x^l \) arranged in non-decreasing order. Let \( j = 0, \ldots, q \) and denote \( k = k_j \). From \( (1.2) \), we have

\[
|\gamma_1 \ldots \gamma_{nk}| \leq \left( \begin{array}{c} nd \\ nk \end{array} \right) \gamma_1 \ldots \gamma_{nk}, \tag{6.4}
\]

By the first part of Proposition 6.3 applied to the tropical polynomial \( t\hat{p} \), we get \( |\tilde{p}_{nk}| \gamma_1 \ldots \gamma_{nk} \leq |\tilde{p}_0| \). Applying the result of Lemma 6.5 and using the assumption of Theorem 6.1 on \( c_j \), we get that \( \tilde{p}_{nk} \geq c_j \|A_k\|^{n} > 0 \). Since \( \tilde{p}_0 = |\det A_0| \), we deduce that \( \gamma_1 \ldots \gamma_{nk} \leq |\det A_0| / (c_j \|A_k\|^{n}) \). Gathering this with \( (6.4) \), we get

\[
|\gamma_1 \ldots \gamma_{nk}| \leq \frac{\left( \begin{array}{c} nd \\ nk \end{array} \right) |\det A_0| \|A_0\|^{n}}{c_j \|A_k\|^{n}} \|A_k\|^{n}. 
\]

Since \( k = k_j \) is the abscissa of a vertex of the Newton polygon of the tropical polynomial \( t\hat{p} \), we get from Proposition 3.3 applied to \( t\hat{p} \) that \( \frac{\|A_0\|}{\|A_k\|^{n}} = \alpha_1 \ldots \alpha_k \), hence the previous inequality shows Theorem 6.1.

**7. Numerical examples illustrating the lower bounds.** In this section, we illustrate the bounds by numerical examples. We shall see in particular that the lower bounds stated in Section 4 can be tight when the tropical roots are well separated and when the input matrices are well conditioned, according to Remark 6.3. All the numerical results of the present paper were obtained using Scilab 4.1.2. Note that similar results are obtained using Matlab, specially the polyeig function.

**7.1. Illustration of the lower bounds for the roots of scalar polynomials.** In the following examples, for each scalar polynomial, using the notations of Section 4, we shall show in tables or plots the following quantities with respect to \( k = 1, \ldots, d \) for comparison: - the ratio \( \frac{|\gamma_1 \ldots \gamma_{nk}|}{\alpha_1 \ldots \alpha_k} \), denoted RATIO; - the lower bound of Pólya given in \( (1.3) \), which coincides with the constant \( L_k \) of \( (4.6) \), denoted PÓLYA; - the lower bound \( L_k \) of \( (4.3) \) in Theorem 4.1, denoted FEW NOM; - the lower bound
Consider the following scalar polynomial
\[
p(x) = 1 - \exp(1)x - \exp(6)x^2 + \exp(4)x^4 + \exp(9)x^8 + \exp(5)x^{10} + \exp(1)x^{16}.
\] (7.1)

The log-exp transformation of its tropical polynomial \(tp\) is the tropical polynomial \(f\) of Example 3.2. The graph of \(f\) and the associated Newton polygon were shown in Figure 3.1. Hence, the tropical roots of \(tp\) are the exponentials of \(-3\), \(-1/2\), and 1, with multiplicities 2, 6 and 8, respectively. Table 7.1 shows the ratios and lower bounds, as explained at the beginning of the section.

| \(k\) | \(\text{RATIO}\) | \(\text{FEWWOM}\) | \(\text{POLYA}\) | \(\text{SEPAR}\) |
|-------|----------------|----------------|--------------|-------------|
| 1     | 0.93475        | 0.37796        | 0.5          | 0.5         |
| 2*    | 1.00034        | "              | 0.3849       | 0.92102     |
| 3     | 0.98789        | "              | 0.32476      | 0.49664     |
| 4     | 0.97926        | "              | 0.28622      | 0.38360     |
| 5     | 0.98348        | "              | 0.25880      | 0.32403     |
| 6     | 0.98771        | "              | 0.23802      | 0.37508     |
| 7     | 0.99383        | "              | 0.22156      | 0.47570     |
| 8*    | 1.00000        | "              | 0.20810      | 0.79696     |
| 9     | 0.98811        | "              | 0.19663      | 0.47570     |
| 10    | 0.97636        | "              | 0.18721      | 0.37508     |
| 11    | 0.96476        | "              | 0.17888      | 0.31917     |
| 12    | 0.95329        | "              | 0.17158      | 0.28622     |
| 13    | 0.96476        | "              | 0.16510      | 0.32476     |
| 14    | 0.97636        | "              | 0.15930      | 0.3849      |
| 15    | 0.98811        | "              | 0.15407      | 0.5         |
| 16*   | 1              | "              | 0.14933      | 1           |

Table 7.1: Comparison of the lower bounds for the scalar polynomial \(p\) of (7.1).

**Example 7.2.** Consider now the following scalar polynomial
\[
p(x) = 1 - \exp(3)x - \exp(6)x^2 - \exp(7)x^4 - \exp(8)x^6 + \exp(9)x^8 + \exp(7)x^{10} - \exp(4)x^{13} - \exp(3)x^{14} + \exp(1)x^{16}.
\] (7.2)

The log-exp transformation of its tropical polynomial \(tp\) is again the tropical polynomial \(f\) of Example 3.2, but now there are some points of the graph of the map \(k \mapsto \log|p_k|\), where \(p_k\) is the \(k\)th coefficient of \(p\), that are on the edges of its Newton polygon, that is its concave hull. Table 7.2 shows the ratios and lower bounds for \(p\). In Figure 7.1, we plot in the same graph the values of the ratios and lower bounds for the scalar polynomials \(p\) of (7.1) and (7.2). This indicates that the lower bound \(L_k\) of Corollary 4.9 based on the separation between tropical roots, may become tighter when the polynomial has non zero coefficients between the vertices of the Newton polygon.

**7.2. Illustration of lower bounds for the eigenvalues of matrix polynomials.** Note that to compute the eigenvalues of the following examples we had to use the tropical scaling algorithm that we recall in the next section.

In the following examples, we are comparing the value of the ratio \(|\zeta_1 \cdots \zeta_k|\) with the lower bounds (4.1) of Section 4 for some matrix polynomials \(P\). In view of Remark 5.2, we computed only the lower bounds obtained under (A2) for the normalized Frobenius norm \(\|\cdot\|_{nF}\), and the lower bounds obtained under (A1) for the
| $k$ | RATIO | FEWNUM | POLYA | SEPAR |
|----|--------|--------|-------|-------|
| 1  | 0.61759 | 0.31623 | 0.5   | 0.5   |
| 2  | 0.98684 | "      | 0.3849| 0.92102|
| 3  | 0.84948 | "      | 0.32476| 0.49664|
| 4  | 0.73124 | "      | 0.28622| 0.38360|
| 5  | 0.63280 | "      | 0.23802| 0.37508|
| 7  | 0.72118 | "      | 0.22156| 0.47570|
| 8  | 0.95684 | "      | 0.20810| 0.79696|
| 9  | 0.70959 | "      | 0.19683| 0.47570|
| 10 | 0.61899 | "      | 0.18721| 0.37508|
| 11 | 0.57724 | "      | 0.17888| 0.31917|
| 12 | 0.53832 | "      | 0.17158| 0.28622|
| 13 | 0.92117 | "      | 0.16510| 0.32476|
| 14 | 0.71677 | "      | 0.15900| 0.3849 |
| 15 | 0.84662 | "      | 0.15407| 0.5   |
| 3  | 1      | "      | 0.14933| 1     |

Table 7.2: Comparison of the lower bounds for the scalar polynomial $p$ of (7.2).

Fig. 7.1: Plot of RATIO as a function of $k$ for the scalar polynomials of (7.1) (blue circles) and (7.2) (red squares), together with the lower bound FEWNUM (blue and red lines resp.), POLYA (plus signs) and SEPAR (stars signs).

normalized Schatten 1-norm (or normalized trace norm) $\| \cdot \|_1$ (which satisfies (A1) but not (A2) see Property 2.5).

Then, using the notations of Section 4 we shall show in tables or plots the following quantities with respect to $k = 1, \ldots, d$ for comparison: - the ratio $\frac{|\zeta_1 \ldots \zeta_{nk}|}{\alpha_1 \ldots \alpha_k}$, denoted RATIO; - the universal lower bound generalizing the lower bound of Pólya to the matrix case, given by $cL_k^n$ (see (4.1)) with $L_k$ of (4.5) or (4.6), denoted G-PÓLYA; - the lower bound involving the number of monomials, given by $cL_k^n$ with $L_k$ of (4.2) or (4.3), denoted FEWNUM; - the lower bound based on the separation between the tropical roots, given by $cL_k^{\text{prox}} n$ or $c(L_k^{\text{prox}})^n$ with $L_k^{\text{prox}}$ and $L_k^{\text{prox}}$ of (4.18) and (4.19) respectively, denoted SEPAR. Also, in tables, the abscissae of the vertices of the Newton polygon of $t p$ are indicated by the symbol $^\ast$.

Example 7.3. Consider the following matrix polynomial

$$P_1(\lambda) = 10^{-7} U_0 + 10^2 \lambda^2 U_2 + 10^7 \lambda^4 U_4 + 10^7 \lambda^7 U_7 + 10^{-8} \lambda^9 U_9,$$
where all the matrices $U_j$, $j \in \{0, 2, 4, 7, 9\}$, are unitary of dimension 3. Here, we shall only consider the normalized Frobenius norm since for a unitary matrix $U$, we have $\|U\|_F = \|U\|_1 = 1$, so the tropical polynomial associated with $P_1$ are the same for both norms, and the lower bounds under (A1) are the same, and are thus weaker than the lower bounds under (A2) for the normalized Frobenius norm. The Newton polygon of the tropical polynomial corresponding to $P_1$ (for the normalized Frobenius norm) is shown in Figure 7.2, and its tropical roots are $10^{-9/2}$, $10^{-5/2}$, $10^2$, and $10^{9/2}$ with multiplicities 2, 2, 3, and 2, respectively. Figure 7.3 shows the lower bounds for all values of $1 \leq k \leq 9$, and the maximum of the ratios for a sample of 1000 random choices of the unitary matrices $U_j$. The ratios slightly change for different random choices of the unitary matrices $U_j$, but with a difference smaller than $6 \times 10^{-4}$. Note that when $1 \leq k \leq 9$ increases, the generalized lower bound of Pólya decreases, and should be replaced by its symmetrized version with $k$ replaced by $d - k$. However the bounds using the separation between tropical roots are the best ones, and they are tight at the vertices of the Newton polygon of the tropical polynomial associated with $P_1$.

Example 7.4. In the following example, we perturb the previous polynomial by adding coefficients to $P_1$, in such a way that the Newton polygon of the tropical polynomial associated with $P_1$ does not change, either for the normalized Frobenius norm or at least for the normalized Schatten 1-norm. Here we fix the unitary matrices
to be either the identity matrix or its opposite, and we consider the following matrices:

\[
B = b \begin{pmatrix} 10^{-15} & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 2 & 3 \end{pmatrix}, \quad C = \frac{\sqrt{3}}{14} \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{pmatrix}
\]

where \( b \) is chosen so that \( \|B\|_{nF} = 1 \), and \( C \) is also such that \( \|C\|_{nF} = 1 \). The matrix \( C \) is singular, since it has rank 1, and it satisfies \( \|C\|_{*,1} = 1/\sqrt{3} \). The matrix \( B \) is ill-conditioned, so that \( \|B\|_{*,1} < \|B\|_{nF} = 1 \) too.

We shall consider the two polynomials:

\[
P_2(\lambda) = 10^{-7} U_0 + 10^{-5/2} \lambda C_1 + 10^2 \lambda^2 U_2 + 10^7 \lambda C_1 + 10^7 \lambda^3 U_4 + 10^5 \lambda^5 C_1 \\
+ 10^8 \lambda^6 B_1 + 10^7 \lambda U_7 + 10^{-7/2} 8 B_1 + 10^{-8} \lambda^9 U_9
\]

\[
P_3(\lambda) = 10^{-7} U_0 + 10^{-5/2} \lambda C_2 + 10^2 \lambda^2 U_2 + 10^7 \lambda C_2 + 10^7 \lambda^3 U_4 + 10^5 \lambda^5 C_2 \\
+ 10^8 \lambda^6 B_2 + 10^7 \lambda U_7 + 10^{-7/2} 8 B_2 + 10^{-8} \lambda^9 U_9,
\]

with \( U_0 = U_4 = U_7 = -I, U_2 = U_9 = I, B_1 = B, C_1 = C, B_2 = \|B\|_{*,1} B, C_2 = \sqrt{3} C \). Since for any unitary matrix \( U \) with dimension 3, \( \|U\|_{*,1} = \|U\|_{nF} = 1 \), and since for any complex matrix \( A \) of dimension 3, we have \( \|A\|_{*,1} \leq \|A\|_{nF} \), the tropical polynomial associated with \( P_2 \) for either the normalized Frobenius norm or the Schatten 1-norm coincides with the one associated with \( P_1 \). Hence, as for \( P_1 \), the bounds for \( P_2 \) with the normalized Schatten 1-norm are necessarily weaker than the ones with the normalized Frobenius norm. Moreover, all the bounds presented in Table 7.3 for \( P_1 \) remain the same for \( P_2 \) except the bounds based on the number of nonzero coefficients. We present all these bounds in Figure 7.4 together with the new ratios. Since these ratios are different from the ones of Table 7.3, one can see that the lower bounds based on the separation between tropical roots may be tight.

![Fig. 7.4: Comparison of the bounds for \( P_2 \) using the normalized Frobenius norm.](image)

Let us consider now the polynomial \( P_3 \). The normalized Schatten 1-norm of the coefficients of \( P_3 \) coincide with the normalized Frobenius norm of the coefficients of \( P_2 \), hence the tropical polynomial associated with \( P_3 \) for the normalized Schatten 1-norm coincides with the one associated with \( P_1 \) for the normalized Frobenius norm, so that the tropical roots are the same. However, since the Schatten 1-norm satisfies (A1) but not (A2), we can only get the bounds based on (A1) which are necessarily lower than the one presented above for \( P_1 \) with the normalized Frobenius norm. Finally, the tropical polynomial associated with \( P_3 \) for the normalized Frobenius norm differs from
the one associated with $P_1$, so that the tropical roots and the bounds differ too. All tropical roots of this new tropical polynomial have multiplicity 1, which means that all indices are abscissae of vertices of its Newton polygon. We present in Figure 7.5 the ratios and the lower bounds for both normalized Frobenius and Schatten 1-norms. Here the ratios and lower bounds for the normalized Schatten 1-norm are lower than the ones for the normalized Frobenius norm. However, one can see that the results for the normalized Frobenius norm are still better than the ones for the normalized Schatten 1-norm.

| $k$ | RATIO | FEWNO | G-PÓLYA | SEPAR | RATIO | FEWNO | G-PÓLYA | SEPAR |
|-----|-------|-------|---------|-------|-------|-------|---------|-------|
| 1   | 1.57223 | 0.03162 | 0.125 | 0.35355 | 0.30258 | 0.001 | 0.01563 | 0.01563 |
| 2   | 1.11466 | *      | 0.05702 | 0.91391 | 1.11466 | *      | 0.00325 | 0.54771 |
| 3   | 1.57223 | *      | 0.03425 | 0.35355 | 0.30258 | *      | 0.00117 | 0.01562 |
| 4   | 1.00028 | *      | 0.02345 | 0.99972 | 1.00028 | *      | 0.00055 | 0.96682 |
| 5   | 1.45041 | *      | 0.01733 | 0.18345 | 0.27913 | *      | 0.00030 | 0.01562 |
| 6   | 1.01082 | *      | 0.01348 | 0.05989 | 0.27913 | *      | 0.00018 | 0.01533 |
| 7   | 1.02104 | *      | 0.01088 | 0.97788 | 1.02104 | *      | 0.00012 | 0.71337 |
| 8   | 1.26686 | *      | 0.00990 | 0.25721 | 0.34983 | *      | 0.00008 | 0.01563 |
| 9   | 1      | *      | 0.00763 | 1      | 1      | *      | 0.00006 | 1      |

Fig. 7.5: Comparison of the bounds for $P_3$ using the normalized Frobenius (on left) and Schatten 1-norms (on right). The symbol * corresponds to the abscissae of the vertices of the Newton polygon of the tropical polynomial associated to the normalized Frobenius norm.

**Example 7.5.** In order to obtain a polynomial for which the lower bounds for the normalized Schatten 1-norm are better than the ones for the normalized Frobenius norm, one need to take matrices $A_i$ such that $\|A_i\|_{nF}/\|A_i\|_1$ is as large as possible. This means that the dimension $n$ is large and that some of the singular values of $A_i$ are zero, or at least that the ratio between the maximal and minimal singular values is large, which implies that $A_i$ is singular or at least has a large condition number. Moreover some parameters of the lower bounds need to be smaller than $\|A_i\|_{nF}/\|A_i\|_1$, which is possible for instance for the lower bounds based on the separation between tropical roots, when these tropical roots have a small multiplicity and are well separated. We shall show here such examples.

We consider a matrix polynomial with degree 3 and dimension $n = 10$:

$$P_3(\lambda) = A_0 + A_1 \lambda + A_2 \lambda^2 + A_3 \lambda^3,$$

where all the matrices $A_i, i = 0, \ldots, 3$ are diagonal matrices: $A_i = \text{diag}(d_i)$ with $d_i \in \mathbb{C}^n$. We choose $A_0 = A_3 = I$, so that $c = 1$, $\alpha_1 \alpha_2 \alpha_3 = 1$ (for norms such that $\|I\| = 1$) and for $k = 3$ the ratio $\left|\frac{\zeta_1 \cdots \zeta_{3k}}{\alpha_{1 \cdots \alpha_k}}\right|$ is equal to 1, which is equal to the lower
bound based on separation of tropical roots. We choose the diagonal elements of the matrices $A_1, A_2$ as follows:

$$d_1 := \frac{a_1}{1 + b/n} (1, \ldots, 1, 1 + b), \quad d_2 := \frac{a_2}{1 + b/n} (1 + b, 1, \ldots, 1),$$

with $a_1, a_2, b > 0$. Then

$$\|A_1\|_{*1} = \frac{\|d_1\|_1}{n} = a_1, \quad \|A_2\|_{*1} = \frac{\|d_2\|_1}{n} = a_2,$$

$$\|A_1\|_{nF} = \frac{\|d_1\|_2}{\sqrt{n}} = a_1 f_b, \quad \|A_2\|_{nF} = \frac{\|d_2\|_2}{\sqrt{n}} = a_2 f_b,$$

with

$$f_b := \sqrt{\frac{1 + 2b + b^2}{n}} > 1.$$

The constant $f_b$ is large when $b$ is large, that is when the condition numbers of $A_1$ and $A_2$ are large.

When $a_1 = a_2 > 1$, the tropical polynomial associated with $P_4$ for the normalized Schatten 1-norm (resp. the normalized Frobenius norm) has 3 different tropical roots equal to $1/a_1$, 1 and $a_1$ (resp. $1/(a_1 f_b)$, 1 and $a_1 f_b$). We present in Tables 7.3 and 7.4 the ratios and the lower bounds for both normalized Frobenius and Schatten 1-norms, when $a_1 = 10^4$, and $b = 2$ and $b = 10$ respectively. When $b = 10$, the matrices have a large condition number, so that the ratios $|\zeta_{i,\ldots,\zeta_{nk}}|/|\alpha_{1,\ldots,\alpha_k}|$ for $k = 1$ or 2 are very large.

When either $b = 2$ or $b = 10$, we see that the lower bounds based on the separation between the tropical roots are the best ones, and that they are nearer the ratios $|\zeta_{i,\ldots,\zeta_{nk}}|/|\alpha_{1,\ldots,\alpha_k}|$ in the case of the Schatten 1-norm than in the case of the Frobenius norm, although they are far in the case where $b = 10$.

| norm. Frobenius norm | norm. Schatten 1-norm |
|---------------------|----------------------|
| k | RATIO | FEW Nom | G-Polya | SEPAR | RATIO | FEW Nom | G-Polya | SEPAR |
| 1 | 6.30697 | 0.00098 | 0.00007 | 0.99911 | 2.06667 | 9.5D-07 | 9.5D-07 | 0.81873 |
| 2* | 6.30697 | 0.00098 | 0.00007 | 0.99911 | 2.06667 | 9.5D-07 | 5.1D-09 | 0.81873 |
| 3* | 1 | 0.00098 | 0.00001 | 1 | 1 | 9.5D-07 | 1.7D-10 | 1 |

Table 7.3: Comparison of the bounds for $P_4$ with $a_1 = a_2 = 10^4$ and $b = 2$.

| norm. Frobenius norm | norm. Schatten 1-norm |
|---------------------|----------------------|
| k | RATIO | FEW Nom | G-Polya | SEPAR | RATIO | FEW Nom | G-Polya | SEPAR |
| 1 | 3.38827 | 0.00098 | 0.00007 | 0.99945 | 9.34462 | 9.5D-07 | 9.5D-07 | 0.81873 |
| 2* | 3.38827 | 0.00098 | 0.00007 | 0.99945 | 9.34462 | 9.5D-07 | 5.1D-09 | 0.81873 |
| 3* | 1 | 0.00098 | 0.00001 | 1 | 1 | 9.5D-07 | 1.7D-10 | 1 |

Table 7.4: Comparison of the bounds for $P_4$ with $a_1 = a_2 = 10^4$ and $b = 10$.

When $a_1 > 1$ and $a_2 = \sqrt{a_1/f_b}$, the tropical polynomial associated with $P_4$ for the normalized Schatten 1-norm (resp. the normalized Frobenius norm) has only 2 different tropical roots equal to $1/a_1$ and $\sqrt{a_1}$ (resp. $1/(a_1 f_b)$, 1 and $\sqrt{a_1 f_b}$) with respective multiplicities 1 and 2. We present in Tables 7.6 and 7.7 the ratios and the
lower bounds for both normalized Frobenius and Schatten \(1\)-norms, when \(a_1 = 10^4\), and \(b = 2\) and \(b = 10\) respectively. When \(b = 10\), the matrices have a large condition number, so that for \(k = 1\) or \(2\) the distances (ratios) between the ratio \(|\zeta_1\cdots\zeta_n|/\alpha_1\cdots\alpha_k\) and its lower bounds are very large, although for \(k = 2\) this ratio is smaller than in the case where \(a_1 = a_2 = 10^4\) above. When either \(b = 2\) or \(b = 10\), we see that the lower bounds based on the separation between the tropical roots are the best ones. For \(k = 2\) which is the abscissa of a vertex of the Newton polygon, the lower bound obtained in the case of the Schatten \(1\)-norm is nearer the ratio \(|\zeta_1\cdots\zeta_n|/\alpha_1\cdots\alpha_k\) than the one obtained in the case of the Frobenius norm, whereas for \(k = 3\) the contrary holds.

### Table 7.5: Comparison of the bounds for \(P_4\) with \(a_1 = a_2 = 10^4\) and \(b = 10^3\).

| \(k\) | \(\text{RATIO}\) | \(\text{FEW NOM}\) | \(\text{G-POLYA}\) | \(\text{SEPAR}\) | \(\text{RATIO}\) | \(\text{FEW NOM}\) | \(\text{G-POLYA}\) | \(\text{SEPAR}\) |
|-------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| 1*    | 1.1D+21         | 0.00098         | 0.000007        | 0.99968         | 1.2D+16         | 9.5D-07         | 9.5D-07         | 0.981873        |
| 2*    | 1.1D+21         | 0.00098         | 0.000007        | 0.99968         | 1.2D+16         | 9.5D-07         | 5.1D-09         | 0.81873         |
| 3*    | 1               | 0.00098         | 0.000001        | 1               | 1               | 9.5D-07         | 1.7D-10         | 1               |

### Table 7.6: Comparison of the bounds for \(P_4\) with \(a_1 = 10^4\), \(a_2 = \sqrt{a_1/f_b}\), and \(b = 2\).

| \(k\) | \(\text{RATIO}\) | \(\text{FEW NOM}\) | \(\text{G-POLYA}\) | \(\text{SEPAR}\) | \(\text{RATIO}\) | \(\text{FEW NOM}\) | \(\text{G-POLYA}\) | \(\text{SEPAR}\) |
|-------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| 1     | 6.29864         | 0.00098         | 0.000089        | 0.99999         | 2.06394         | 9.5D-07         | 9.5D-07         | 0.99820         |
| 2     | 6.29864         | 0.00098         | 0.000007        | 0.99999         | 2.06394         | 9.5D-07         | 9.5D-07         | 0.99820         |
| 3     | 1               | 0.00098         | 0.000001        | 1               | 1               | 9.5D-07         | 1.7D-10         | 1               |

### Table 7.7: Comparison of the bounds for \(P_4\) with \(a_1 = 10^4\), \(a_2 = \sqrt{a_1/f_b}\), and \(b = 10\).

| \(k\) | \(\text{RATIO}\) | \(\text{FEW NOM}\) | \(\text{G-POLYA}\) | \(\text{SEPAR}\) | \(\text{RATIO}\) | \(\text{FEW NOM}\) | \(\text{G-POLYA}\) | \(\text{SEPAR}\) |
|-------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| 1     | 3.3754.9        | 0.00098         | 0.00009         | 1.00009         | 93.0935         | 9.5D-07         | 9.5D-07         | 0.98020         |
| 2     | 9.86557         | 0.00098         | 0.000007        | 0.99999         | 93.0935         | 9.5D-07         | 5.1D-09         | 0.98020         |
| 3     | 1               | 0.00098         | 0.000001        | 1               | 1               | 9.5D-07         | 1.7D-10         | 1               |

8. The tropical scaling and its relation with the lower bounds. Since we considered matrix polynomials the coefficients of which have different order of magnitude, their eigenvalues need to be computed by using some scaling, otherwise their relative error may become very large. We used here the “tropical scaling” introduced in [GS09], that we next recall. We shall see that it improves the relative error, by comparison with a usual method (QZ on a companion linearization without scaling), and that the tightness of our lower bounds is related with the efficiency of the tropical scaling.

8.1. Tropical scaling. Let \(\alpha_k\) be a tropical root of \(tp(x) = \max_{0\leq j\leq d} \|A_j\| x^j\) corresponding to an edge of Newton polygon (see Figure 4.1) such that \(k_{i-1}, k_i\) are the abscissae of the start and the end points of this edge. So \(tp(\alpha_k) = \|A_{k_{i-1}}\| \|\alpha_k\|^{k_{i-1}}\). The tropical scaling is defined by setting \(\lambda = \alpha_k/\mu\) and defining \(\tilde{P}(\mu) := (tp(\alpha_k))^{-1} P(\alpha_k/\mu)\).
In this way,

\[ \tilde{P}(\mu) = \sum_{j=0}^{d} \tilde{A}_j \mu^j, \tag{8.1} \]

where \( \tilde{A}_j := (\|A_{k_{i-1}}\alpha_{k_i}^{k_i-1})^{-1} A_j \alpha_{k_i}^{k_i-1} \). Note that \( \mu \) is an eigenvalue of \( \tilde{P}(\mu) \) if and only if \( \alpha_{k_i} \mu \) is an eigenvalue of \( P(\lambda) \) and this scaling will not affect the condition number of any coefficient matrix. Since \( \text{tp}(\alpha_{k_i}) = \max_{0 \leq j \leq d} \|A_j\| \alpha_{k_i}^{k_i} \), we have \( \|\tilde{A}_{k_i}\| = 1 \) and \( \|\tilde{A}_j\| \leq 1 \) for any \( j = 0, \ldots, d \).

It has been observed numerically in [GS09], and proved in the special case of quadratic pencils, that when the coefficient matrices are well conditioned, the tropical roots are good approximations of the moduli of the eigenvalues. This observation can be further justified, to some extent, by our bounds of Section 4 and 6. So we expect to get \((k_i - k_{i-1})n\) eigenvalues of \( P \) with a modulus of order \( \alpha_{k_i} \), or equivalently, \((k_i - k_{i-1})n\) eigenvalues of \( \tilde{P}(\mu) \) with a modulus of order one.

The tropical scaling algorithm consists in doing a different scaling for every expected order of magnitude of the eigenvalues. It works as follows: for every tropical root \( \alpha_{k_i} \), we compute the scaled matrix polynomial \( \tilde{P} \) in (8.1), compute a companion linearization of it and apply the QZ algorithm to this linearization. We get \( nd \) eigenvalues, that we multiply by \( \alpha_{k_i} \), which yields candidate eigenvalues of \( P \). Then, we keep among these candidate eigenvalues only the \( n(k_i - k_{i-1}) \) ones having the closest modulus to \( \alpha_{k_i} \), and we say that these candidate eigenvalues are assigned to \( \alpha_{k_i} \). The other eigenvalues returned by the QZ algorithm are discarded. Note that we measure the distance between two moduli by the absolute value of the difference of the log of these moduli. Finally, the whole spectrum is obtained as the union of the different groups of \( n(k_i - k_{i-1}) \) candidate eigenvalues assigned to the different tropical roots, for \( 1 \leq i \leq q \). If the moduli of the values of some groups interleave, meaning that for \( i < j \), a candidate eigenvalue \( \lambda \) assigned to a tropical root \( \alpha_{k_i} \) is such that \( |\lambda| > |\lambda'| \) for some candidate eigenvalue \( \lambda' \) assigned to a tropical root \( \alpha_{k_j} \), the algorithm returns “fail” (the assignment is inconsistent).

In the following we compare the numerical accuracy of this method with the one of a standard method consisting in applying the QZ algorithm to the same companion linearization but without scaling. Recall that the accuracy of the computation of an eigenvalue can be evaluated by the product of its condition number and of its backward error. We can compute the condition number and the backward error directly from the matrix polynomial or from its linearization, leading to different measures of the accuracy. Here, condition numbers and backward errors are defined with respect to the relative errors and left and right eigenvectors as in [Tis00, Theorems 4 and 5]. Note that the condition numbers are computed using the result of the computation of the eigenvalues and eigenvectors, so only approximate condition numbers will be obtained. Actually, without scaling some of the eigenvalues will be aberrant.

8.2. Condition numbers and backward errors with or without tropical scalings. Below we consider the matrix polynomials of Section 7.2.

In Figure 8.1 we plot the condition numbers and backward errors for the polynomial \( P_1 \) of Example 7.4. The number of eigenvalues is 27, but they moduli are close to the 4 tropical eigenvalues as shown in Figure 7.3. So, we see only 4 points in each graph corresponding to the computations using the tropical scaling. However, in the graphs corresponding to the computations without scaling, we see an additional
point with approximate modulus $10^{-3}$ corresponding to aberrant values (there are no eigenvalues with this order of magnitude). We see also that the condition number of the linearization and the backward error of the matrix polynomial are much improved by using the tropical scaling, in particular for the smallest eigenvalues.

Similar conclusions are obtained for the polynomials $P_2$ and $P_3$ of Example 7.4.

We finally consider the polynomial $P_4$ of Example 7.5 with $a_1 = a_2 = 10^{4}$. In this case, we can compute the eigenvalues using the diagonal structure of the matrices, for verification. We saw that the tightness of the lower bound decreases when $b$ increases. The efficiency of the tropical scaling also decreases with $b$. Compare the results for $b = 10$ and $b = 100$ in Figures 8.4 and 8.5, respectively. When $b = 10^3$, the tropical scaling fails, leading to an error for the logarithm of the modulus greater than 1.99.

REFERENCES

[ABG98] M. Akian, R. B. Bapat, and S. Gaubert. Asymptotics of the Perron eigenvalue and eigenvector using max algebra. C. R. Acad. Sci. Paris, Sér. I 327:927–932, 1998.

[ABG04] M. Akian, R. Bapat, and S. Gaubert. Perturbation of eigenvalues of matrix pencils and
Fig. 8.3: Eigenvalues condition numbers and backward errors for $P_3$ of Example 7.4 without scaling or with scaling with respect to Schatten 1-norm.

Fig. 8.4: Eigenvalues condition numbers and backward errors for $P_4$ with $b = 10$, without scaling or with scaling with respect to Frobenius norm.

Fig. 8.5: Eigenvalues condition numbers and backward errors for $P_4$ with $b = 100$, without scaling or with scaling with respect to Frobenius norm.
optimal assignment problem. *C. R. Acad. Sci. Paris, Série I*, 339:103–108, 2004. Also arXiv:math.SP/0402438.

[ABG05] M. Akian, R. Bapat, and S. Gaubert. Min-plus methods in eigenvalue perturbation theory and generalised Lidskii-Vishik-Ljusternik theorem. arXiv:math.SP/0402090, 2005.

[ABG06] M. Akian, R. Bapat, and S. Gaubert. Max-plus algebras. In L. Hogben, editor, *Handbook of Linear Algebra (Discrete Mathematics and Its Applications)*, volume 39. Chapman & Hall/CRC, 2006. Chapter 25.

[AGM14] M. Akian, S. Gaubert, and A. Marchesini. Tropical bounds for eigenvalues of matrices. *Linear Algebra and its Applications*, January 2014. See also arXiv:arXiv:1309.7319.

[BCOQ92] F. Baccelli, G. Cohen, G.J. Olsder, and J.P. Quadrat. *Synchronization and Linearity*. Wiley, 1992.

[Bin96] D. A. Bini. Numerical computation of polynomial zeros by means of Aberth’s method. *Numerical Algorithms*, 13:179–200, 1996.

[Bir15] G. D. Birkhoff. An elementary double inequality for the roots of an algebraic equation having greatest absolute value. *Bull. Amer. Math. Soc.*, 21(10):494–495, 1915.

[BNS13] D. A. Bini, Y. Noferini, and M. Sharify. Locating the Eigenvalues of Matrix Polynomials. *SIAM J. Matrix Anal. Appl.*, 34(4):1708–1727, 2013.

[CGM80] R. A. Cuninghame-Green and P. F. J. Meijer. An algebra for piecewise-linear minimax problems. *Discrete Appl. Math.*, 2(4):267–294, 1980.

[EKL06] M. Einsiedler, M. Kapranov, and D. Lind. Non-Archimedean amoebas and tropical varieties. *J. Reine Angew. Math.*, 601:139–157, 2006.

[FLVD04] H.-Y. Fan, W-W. Lin, and P. Van Dooren. Normwise scaling of second order polynomial matrices. *SIAM J. Matrix Anal. Appl.*, 26(1):252–256, 2004.

[Fri86] S. Friedland. Limit eigenvalues of nonnegative matrices. *Linear Algebra Appl.*, 74:173–178, 1986.

[Fuj16] M. Fujiwara. Über die obere schranke des absoluten betrages der wurzeln einer algebraischen gleichung. *Tohoku Math. J.*, 1(10):167–171, 1916.

[Had93] J. Hadamard. Étude sur les propriétés des fonctions entières et en particulier d’une fonction considérée par Riemann. *Journal de Mathématiques Pures et Appliquées, 4ième série*, 9:171–216, 1893.

[HMT13] S. Hammarling, C. J. Munro, and F. Tisseur. An algorithm for the complete solution of quadratic eigenvalue problems. *ACM Transactions on Mathematical Software (TOMS)*, 39(3), 2013.

[HT03] N. J. Higham and F. Tisseur. Bounds for eigenvalues of matrix polynomials. *Linear Algebra Appl.*, 358:5–22, 2003. Special issue on accurate solution of eigenvalue problems (Hagen, 2000).

[IMS07] I. Itenberg, G. Mikhalkin, and E. Shustin. *Tropical algebraic geometry*. Oberwolfach seminars. Birkhäuser, 2007.

[Jun99] J. Jensen. Sur un nouvel et important théorème de la théorie des fonctions. *Acta Mathematica*, 22:359–364, 1899. 10.1007/BF02417878.

[Lan05] E. Landau. Sur quelques théorèmes de m. petrovitch relatifs aux zéros des fonctions analytiques. *Bulletin de la S. M. F.*, 33:251–261, 1905.

[Lin02] E. L. Lindelöf. Mémoire sur la théorie des fonctions entières de genre fini. *Acta Societatis Scientiarum Fennicae, XXXI*, 1902.

[Mel12] A. Melman. Generalization and variations of Pellets theorem for matrix polynomials, 2012. arXiv:1210.0172.

[MZ01] G. Malajovich and J. P. Zubelli. Tangent Graeffe iteration. *Numer. Math.*, 89(4):749–782, 2001.

[Ost40a] A. Ostrowski. Recherches sur la méthode de Graeffe et les zéros des polynomes et des séries de Laurent. *Acta Math.*, 72:99–155, 1940.

[Ost40b] A. Ostrowski. Recherches sur la méthode de Graeffe et les zéros des polynomes et des séries de Laurent. *Acta Math.*, 72:157–257, 1940.

[PR04] M. Passare and H. Rullgård. Amoebas, Monge-Ampère measures, and triangulations of the Newton polytope. *Duke Math. J.*, 121(3):481–507, 2004.
[Pur08] K. Purbhoo. A Nullstellensatz for amoebas. *Duke Math. J.*, 141(3):407–445, 2008.

[RGST05] J. Richter-Gebert, B. Sturmfels, and T. Theobald. First steps in tropical geometry. In *Idempotent mathematics and mathematical physics*, volume 377 of *Contemp. Math.*, pages 289–317. Amer. Math. Soc., Providence, RI, 2005.

[Sha11] M. Sharify. *Scaling Algorithms and Tropical Methods in Numerical Matrix Analysis: Application to the Optimal Assignment Problem and to the Accurate Computation of Eigenvalues*. PhD thesis, Ecole Polytechnique, France, 2011.

[Spe38] W. Specht. Zur theorie der algebraischen gleichungen. *Jahr. DMV*, 48:142–145, 1938.

[Tis00] Françoise Tisseur. Backward error and condition of polynomial eigenvalue problems. *Linear Algebra Appl.*, 309:339–361, 2000.

[Vir01] O. Viro. Dequantization of real algebraic geometry on logarithmic paper. In *European Congress of Mathematics, Vol. I (Barcelona, 2000)*, volume 201 of *Progr. Math.*, pages 135–146. Birkhäuser, Basel, 2001.