Renormalization-group analysis of a spin-1 Kondo dot: Non-equilibrium transport and relaxation dynamics

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We study non-equilibrium transport through a spin-1 Kondo dot in a local magnetic field. To this end we perform a two-loop renormalization group analysis in the weak-coupling regime yielding analytic results for (i) the renormalized magnetic field and the g-factor, (ii) the time evolution of observables and the relevant decay rates, (iii) the magnetization and anisotropy as well as (iv) the current and differential conductance in the stationary state. In particular, we find that compared to a spin-1/2 Kondo dot there exist three additional decay rates resulting in an enhanced broadening of the logarithmic features observed in stationary quantities. Additionally, we study the effect of anisotropic couplings between reservoir and impurity spin.

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I. INTRODUCTION

The Kondo model consists of a localized spin-1/2 which is coupled to the electrons in the host metal via a spin exchange interaction. The original motivation to study the properties of this and related models was the observation of a resistance minimum in some metals in the 1930s. More than two decades ago it was realized that the Kondo model can also be applied to describe transport experiments through quantum dots, where the localized spin-1/2 models a minimal two-state quantum dot which is coupled via exchange interactions to two (or more) leads held at different chemical potentials. The tremendous developments in the ability to engineer nanoscale devices also led to the experimental observation of Kondo physics in transport experiments. This in turn triggered many theoretical studies of the non-equilibrium transport properties of Kondo quantum dots.

It is well-known that the Kondo model with larger spin on the dot shows strong deviations from standard Fermi liquid behavior at low temperatures. For example, at low temperatures the local spin is not fully screened by the electrons and a residual magnetization remains. Despite the recent observation of this underscreened Kondo effect in transport experiments in single molecules, the non-equilibrium transport properties of the spin-1 Kondo model have not been studied theoretically in the same detail as its spin-1/2 counterpart.

In this article we fill this gap by performing a two-loop renormalization group (RG) analysis for a spin-1 Kondo dot with finite bias voltage. To this end we apply the real-time renormalization group method in frequency space (RTRG-FS) to derive analytic results for the renormalized magnetic field and g-factor, decay rates and time evolution as well as magnetization, anisotropy and differential conductance in the stationary state. All results are obtained in a systematic weak-coupling expansion in the renormalized exchange coupling \( J(\Lambda) \ll 1 \). The latter condition is satisfied as long as the maximum of all occurring energy scales is much larger than the Kondo temperature \( T_K \),

\[
\Lambda_c = \max\{V, h_0\} \gg T_K, \tag{1}
\]

where \( V \) and \( h_0 \) are the applied bias voltage and magnetic field, respectively, and \( T_K \) denotes the Kondo temperature at which the system enters the strong-coupling regime. We find that the properties of the spin-1 Kondo model are similar to the spin-1/2 system with the main differences (i) that the larger local Hilbert space on the dot results in additional (quintet) decay rates beside the usual longitudinal and transverse spin relaxation rates, (ii) that these rates describe the decay of the spin anisotropy \( T \propto S_i^i S_j^j \) (which is trivial in the spin-1/2 model), where \( S_i \) are the components of the spin operator on the dot, and (iii) that the existence of these additional rates yields a broadening (relative to the spin-1/2 model) of logarithmic features in stationary quantities like the susceptibilities and the differential conductance.

The paper is structured as follows. In the next two sections we define the spin-1 Kondo dot and briefly review the applied RTRG-FS method. In Sec. IV, we present the results for the renormalized magnetic field and g-factor, decay rates and time evolution as well as magnetization, anisotropy and current in the stationary state. In particular, we highlight the differences to the spin-1/2 model studied in Ref. 22. In Sec. V, we investigate the effect of anisotropic exchange couplings. Our calculations mainly follow the analysis of the spin-1/2 Kondo model and discuss the required modification for the study of the spin-1 system in the appendix.

II. SPIN-1 KONDO DOT

In this article we consider a spin-1 Kondo dot, i.e. a quantum dot whose internal degree of freedom is given by a spin \( S = 1 \), which is attached to two electronic reservoirs via a spin exchange interaction \( J \). This
set-up is relevant for recent transport experiments on $C_{60}$ molecules and cobalt complexes. The spin-1 Kondo model can be regarded as a direct generalization of a spin-1/2 Kondo dot studied previously in Ref. 25, which itself is related to the single-orbital Anderson model by a Schrieffer–Wolff transformation.

Specifically, the Hamiltonian of the spin-1 Kondo dot is given by (see Fig. 1)

$$H = \sum_{\alpha k \sigma} (\epsilon_k - \mu_{\alpha}) c_{\alpha k \sigma}^\dagger c_{\alpha k \sigma} + h_0 S_z$$

$$+ \frac{J_0}{2\nu_0} \sum_{\alpha \alpha' \vec{k} \vec{k}' \sigma \sigma'} \vec{S} \cdot \vec{\sigma}_{\sigma' \sigma} c_{\alpha \vec{k} \sigma}^\dagger c_{\alpha' \vec{k}' \sigma'}.$$  

$$J_0 \geq 0.$$  

We assume that for $t < t_0$ the reservoirs and quantum dot are decoupled and hence that the initial density matrix at $t = t_0$ is of the form $\rho(t_0) = \rho_S(t_0)\rho_{\text{res}}^R\rho_{\text{res}}^L$. Here $\rho_{\text{res}}^R$ is the grand canonical density matrix of reservoir $\alpha$ including the chemical potential $\mu_{\alpha}$ and $\rho_S(t_0)$ denotes the initial density matrix of the dot system. The central object of our analysis is the reduced density matrix of the dot $\rho_S(t)$, which is obtained by tracing out the reservoir degrees of freedom. Instead of studying the time evolution of the density matrix directly we further perform a Laplace transform

$$\tilde{\rho}_S(z) = \int_{t_0}^{\infty} dt \ e^{iz(t-t_0)} \rho_S(t)$$

$$= \text{Tr}_{\text{res}}\left( \frac{i}{z-L(z)} \rho_S(t_0)\rho_{\text{res}} \right).$$

$$\tilde{\rho}_S(z) = \rho_S(t_0)\rho_{\text{res}}^{L,R} \frac{1}{z-L(z)}.$$ 

Here the effective dot Liouvillian is given by

$$L_S^{\text{eff}}(z) = L_S^{(0)} + \Sigma(z).$$

As it is well known for Kondo models like perturbation theory in the exchange coupling $J_0$ leads to logarithmic divergencies. In order to tackle this problem we use the RTRG-FS method, which is particularly suited to derive analytic results for the spin decay rates, the renormalized magnetic field, the dot magnetization as well as the differential conductance in non-equilibrium. Our calculations follow the analysis of the spin-1/2 Kondo model performed in Ref. 25; we present here a brief summary to set up the notations and discuss the essential modifications required for the spin-1 system in the appendix.

We start with the time evolution of the density matrix of the full system $\rho(t)$, which is governed by the von Neumann equation

$$\dot{\rho}(t) = -i \left[ H, \rho(t) \right].$$

This equation is formally solved by

$$\rho(t) = e^{-iH(t-t_0)}\rho(t_0)e^{iH(t-t_0)} = e^{-iL(t-t_0)}\rho(t_0),$$

where the Liouvillian $L$ is defined via $L = [H, \cdot \cdot \cdot]$. We assume that for $t < t_0$ the reservoirs and quantum dot are decoupled and hence that the initial density matrix at $t = t_0$ is of the form $\rho(t_0) = \rho_S(t_0)\rho_{\text{res}}$.

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$$\tilde{\rho}_S(z) = \int_{t_0}^{\infty} dt \ e^{iz(t-t_0)} \rho_S(t)$$

$$= \text{Tr}_{\text{res}}\left( \frac{i}{z-L(z)} \rho_S(t_0)\rho_{\text{res}} \right).$$

where $z$ is the Laplace variable. In order to calculate $\tilde{\rho}_S(z)$ we expand in the exchange interaction, which yields a diagrammatic representation in Liouville space. Resumming the resulting series one obtains

$$\tilde{\rho}_S(z) = \frac{i}{z-L_S^{\text{eff}}(z)}\rho_S(t_0),$$

where the effective dot Liouvillian is given by

$$L_S^{\text{eff}}(z) = L_S^{(0)} + \Sigma(z).$$

Here the first term $L_S^{(0)} = h_0 [S^z, \cdot \cdot \cdot]$ describes the action of the applied magnetic field on the spin, whereas the second term $\Sigma(z)$ incorporates all interaction effects of the
coupling to the leads. From \( L_0^{\text{eff}}(z) \) we obtain the renormalized magnetic field, the decay rates, and the reduced density matrix of the dot (and thus the dot magnetization) which are presented in Secs. [IV A], [IV B], and [IV C] respectively. A similar analysis can be performed for the current kernel yielding the current and differential conductance through the system, where we in particular focus on the differences to the spin-1/2 model.

In the following we present the results for the renormalized magnetic field and g-factor, decay rates and time evolution, magnetization and anisotropy as well as the differential conductance through the system, where we in particular focus on the differences to the spin-1/2 model.

**IV. RESULTS**

The systematic weak-coupling expansions for the effective dot Liouvillian \( L_0^{\text{eff}}(z) \) and the current kernel \( \Sigma^I(z) \) involve terms of the order \( J_0^2 \ln \), where \( \ln \) denotes terms of the form \( \ln[i\Lambda_c/(z + \Delta + i\Gamma_1)] \) with some physical scale \( \Delta = V, h_0, V \pm h_0 \) and \( \Gamma_1 \sim J_2^2 \Delta \) the decay rates (see Sec. [IV B]). From the dot Liouvillian one can derive the renormalized magnetic field as well as the decay rates by determining the poles of the resolvent \( 1/[z - L_0^{\text{eff}}(z)] \). If we use the spectral decomposition \( L_0^{\text{eff}}(z) = \sum \lambda_i(z) P_i(z) \), where \( \lambda_i(z) \) and \( P_i(z) \) denote the eigenvalues and corresponding projectors respectively, they are determined by solving \( z_i = \lambda_i(z) \)

where the real parts of \( z_i \) yield oscillation frequencies in the time evolution of observables and thus the renormalized magnetic field, while the imaginary parts lead to exponential decay and thus determine the decay rates (see App. [A] for details). Furthermore, the stationary reduced density matrix \( \rho_0^z \) is obtained by solving

\[
L_0^{\text{eff}}(i0^+) \rho_0^z = 0,
\]

thus yielding the stationary dot magnetization and anisotropy (see Sec. [IV C]). Finally the current through the system is derived from \( \rho_0^z \) and the current kernel.

**A. Renormalized magnetic field and g-factor**

The renormalized magnetic field \( h \), which emerges from the externally applied magnetic field \( h_0 \) due to screening processes from the electrons in the leads, is determined by the real parts of the poles \( z_i \) derived from the self-consistency equations (13). We present explicit relations for the \( z_i \)'s in Eqs. (A13)–(A18). To first order in the exchange coupling we find

\[
h^{(1)} = [1 - (J_c - J_0)] h_0,
\]

which is identical to the renormalization of the magnetic field in the spin-1/2 Kondo model. Differences in the models manifest themselves in the logarithmic corrections to (13), which yield

\[
h = h^{(1)} - \frac{J_2^2}{2} h \mathcal{L}^-(h) + \frac{J_2^2}{4} (V - h) \mathcal{L}^-(V - h).
\]

In the derivation of (16) we have neglected all terms in order \( J_0^2 \) that do not contain logarithms at either \( h = \)
0 or $h = V,$ performed the scaling limit $J_0 \to 0,$ and further replaced $h_0 \to h$ (which only leads to higher-order corrections). In addition, we have introduced the short-hand notation

$$L_{-}(x) = \ln \frac{\Lambda_c}{\sqrt{x^2 + (\Gamma^1_q - \Gamma^2_q)^2}} + \ln \frac{\Lambda_c}{\sqrt{x^2 + (\Gamma^1_q - \Gamma^2_q)^2}},$$

where the appearing decay rates $\Gamma^1_q$, $\Gamma^2_q$, and $\Gamma^2_S$ will be discussed in detail in the next section (see Fig. 3). Here we already note that at resonance $V = h$ they satisfy $\Gamma^1_q \neq \Gamma^2_q$ and $\Gamma^2_q \neq \Gamma^2_S$, i.e. (17) is well-defined.

From the renormalized magnetic field (16) we can directly obtain the g-factor

$$g = 2 \frac{dh}{dh_0} = 2(1 - J_c) - J_c^2 L_{-}(h) - \frac{J_c^2}{2} L_{-}(V - h),$$

which is plotted in Fig. 2. Compared to its bare value the g-factor is reduced due to the screening of the spin on the dot by the electron spins in the leads. We further observe two logarithmic features at $h = V$ and $h = V/2$. Due to the reduction of the renormalized magnetic field compared to $h_0$ the latter appears at $h_0 > V$, however, up to logarithmic corrections in order $J_c^2$ the position equals the spin-1/2 situation. Differences show up in the line shapes close to $h = 0$ and $h = V$, where the spin-1 result clearly shows a larger broadening. The reason is that in the spin-1 case both broadenings $\Gamma^1_q$ and $\Gamma^1_t$ (which satisfy $\Gamma^1_q > \Gamma^1_t$), appear in the logarithms, while in the spin-1/2 case one finds $\Gamma^2_q = \Gamma^2_t$. Away from these positions, however, the g-factors are identical.

**B. Decay rates and time evolution**

In this section we investigate the imaginary parts of the poles $z_i$ obtained by solving (13). As we will show they lead to an exponential decay in the time evolution of observables and thus constitute the decay rates of the system. As the local Hilbert space as well as the corresponding Liouville space of the spin-1 dot has a larger dimension than its spin-1/2 analog, we expect the appearance of additional decay rates in the spin-1 case. In the following we first analyze these rates in the case of vanishing magnetic field and then consider the case of a finite field.

1. **No magnetic field $h_0 = 0$**

For vanishing magnetic field $h_0 = 0$ the model is fully spin isotropic. Thus the local Liouville space of the dot can be decomposed into irreducible representations of SU(2). For a spin-1/2 system these are a singlet and a triplet (according to $\frac{1}{2} \otimes \frac{1}{2} = 0 \oplus 1$), while for the spin-1 system one finds a singlet, triplet, and quintet $(1 \oplus 1 = 0 \oplus 1 \oplus 2)$. Therefore the effective dot Liouvillian of the spin-1 model (2) decomposes as

$$L_{eff}^V(z) = -i f_s(z) L^s - i f_l(z) L^t - i f_q(z) L^q,$$

where the superoperators $L^s$, $L^t$, and $L^q$ project onto the respective subspaces (see App. A). In leading order the coefficients in (19) can be decomposed into irreducible representations (see (13)). The difference to the spin-1/2 model is the appearance of additional decay rates in the spin-1 case. Similarly the decay rates in a general spin-S Kondo dot in the absence of a magnetic field are given by

$$\Gamma_j = \frac{\pi}{2} j (j + 1) J_c^2 V \quad \text{for } j = 0, 1, \ldots, 2S.$$

In order to interpret the result (20) we investigate the time evolution of the reduced density matrix, which is given by the inverse Laplace transform of (7)

$$\rho_D(t) = \frac{i}{2\pi} \int_{-\infty + i0}^{\infty + i0} \frac{dz}{z - L_{eff}^V(z)} e^{-izt} \rho_D(0).$$

The initial density matrix is assumed to be of the form

$$\rho_S(0) = \begin{pmatrix} \rho_{11} & \rho_{10} & \rho_{1-1} \\ \rho_{01} & \rho_{00} & \rho_{0-1} \\ \rho_{-11} & \rho_{-10} & \rho_{-1-1} \end{pmatrix}.$$

We perform the integral in (22) by using the decomposition (19) for the Liouvillian. In leading order this yields

$$\rho_S(t) = \rho_S(0) e^{-\Gamma^j t} L^j + e^{-\Gamma^q t} L^q \rho_S(0).$$
We note that the time evolution is governed by purely exponential decays. This is due to the absence of a $z$-dependence of $L^{eff}_S$ in leading order; in general the branch cuts of $L^{eff}_S$ due to logarithmic terms will lead to additional power-law corrections. The time evolution of the spin operator is now readily obtained from

$$\langle \vec{S}(t) \rangle \propto e^{-\Gamma_\delta t}, \text{ e.g.}$$

$$\langle S^z(t) \rangle = (\rho_{11} - \rho_{01} - 1) e^{-\Gamma_\delta t}.$$  

(25)

Thus the magnetization decays only with the triplet rate $\Gamma_\delta$, which is identical to the spin-1/2 model. However, in the spin-1 model one can construct a second non-trivial operator on the dot, namely the anisotropy operator (or spherical quadrupole tensor) with components

$$T_{2 \pm 2} = \frac{1}{2} [S^x S^x - S^y S^y \pm i(S^x S^y + S^y S^x)],$$

(26)

$$T_{2 \pm 1} = \mp i \frac{1}{2} [S^x S^z + S^z S^x \pm i(S^y S^z + S^z S^y)],$$

(27)

$$T_{20} = \sqrt{\frac{3}{2}} S^x S^z - \sqrt{\frac{3}{2}} S^y.$$

(28)

($T$ is related to the traceless quadrupole tensor $Q_{ij}$ introduced in Refs. 46 and 47 to study the transport through spin valves.) Physically $T_{20}$ measures the anisotropy along the $z$-direction. ($T_{20} > 0$ shows a tendency to align the impurity-spin along the $z$-axis, while $T_{20} < 0$ indicates an alignment in the xy-plane. We further note that in the spin-1/2 case one finds $T_{20} \propto \bar{I}$, i.e. the anisotropy operator is trivial. From (24) we can easily infer the time evolution of $T$

$$\langle T(t) \rangle \propto e^{-\Gamma_\delta t}, \text{ e.g.}$$

$$\langle T_{20}(t) \rangle = \frac{1}{\sqrt{6}} (\rho_{11} - 2 \rho_{01} + \rho_{-11}) e^{-\Gamma_\delta t}.$$  

(29)

Thus the situation in the absence of a magnetic field is quite simple: The magnetization decays with the triplet rate $\Gamma_\delta$, whereas the quintet rate $\Gamma_q$ governs the time evolution of the anisotropy operator. In the next section we consider how this changes if the spin rotational invariance is broken by the application of a finite magnetic field.

2. Finite magnetic field $h_0 > 0$

The breaking of the spin rotational invariance results in a decomposition of the triplet and quintet subspaces in the local Liouville space of the dot. Hence the dot Liouvillian does no longer take the simple form (19) and the system possesses more than two decay rates. Specifically we find that the rates (20) split up as

$$\Gamma_t \rightarrow \Gamma^0_t, \Gamma^1_t, \quad \Gamma_q \rightarrow \Gamma^0_q, \Gamma^1_q, \Gamma^2_q.$$  

(30)

Compared to the spin-1/2 model, where at finite magnetic field two different rates (the longitudinal and transverse spin relaxation rates usually denoted by $\Gamma_1$ and $\Gamma_2$ respectively) exist, we find five decay rates in the spin-1 system. As each rate corresponds to a relaxation or decoherence time, the time evolution towards the stationary state will be more complex in the spin-1 model (see below).

In the small-field limit $h_0 \ll V$ the splitting of the rates is explicitly given by

$$\Gamma^0_t = \pi J^2_e V + \pi J^2_e h, \quad \Gamma^1_t = \pi J^2_e V + \frac{\pi}{2} J^2_e h,$$

(31)

and

$$\Gamma^0_q = 3 \pi J^2_e V + 3 \pi J^2_e h, \quad \Gamma^1_q = 3 \pi J^2_e V + \frac{5 \pi}{2} J^2_e h,$$

(32)

$$\Gamma^2_q = 3 \pi J^2_e V + \pi J^2_e h.$$  

We note that the magnetic-field dependence of the triplet rates (31) is identical to the spin-1/2 model with the identifications $\Gamma^0_t \rightarrow \Gamma_1$ and $\Gamma^1_t \rightarrow \Gamma_2$. For arbitrary magnetic fields the rates have been obtained by numerically solving the self-consistency equations (C1) and (C3), the result is shown in Fig. 3. We observe in particular that two of the rates ($\Gamma^0_t$ and $\Gamma^0_q$) merge for $h = V$. We note that (17) is well-defined at resonance as $\Gamma^1_t \neq \Gamma^2_t$ and $\Gamma^1_q \neq \Gamma^2_q$. The time evolution of the reduced density matrix follows from (22) using the spectral decomposition $L^{eff}_S(z) = \sum_i \lambda_i(z) P_i(z)$ of the dot Liouvillian. If we further approximate this by expanding around the poles of the resolvent, $L^{eff}_S(z) \approx \sum_i \lambda_i(z) P_i(z)$ and use Eq. (13), we obtain

$$\rho_S(t) = \sum_i e^{-iz_i t} P_i(z_i) \rho_S(t = 0).$$

(33)

Here the real parts of the $z_i$ yield oscillations with the oscillation frequency determined by the renormalized magnetic field, while the imaginary parts correspond to the
decay rates \[ \text{[30]} \] [see Eqs. \{A13\}–\{A18\}]. The first pole \( z_1 = 0 \) yields a stationary contribution, i.e. \( P_1(0) \) is the projector onto the stationary density matrix \( \rho^n_{st} \). From \[ \text{[33]} \] we obtain for example
\[
(T_{2+\pm1}(t)) = e^{\pm 2iht} e^{-\Gamma^t_\pm t} \rho^n_{\mp1,\pm1}.
\] \[ \text{(34)} \]
Corrections to \[ \text{[33]} \] can be calculated along the lines of Refs. \{44\} \{48\} and \{49\}. We expect additional terms oscillating with the freqencies \( V, h \pm V, \) and \( 2h \pm V \) as well as power-law corrections to the exponential decays.

A better understanding of the time evolution and the relevant decay rates can be obtained by studying the large-field limit, \( h_0 \gg V \). In this regime we find only two effective rates which are related to the five original ones by (see inset in Fig. \[ \text{3} \])
\[
\Gamma_\pm, \Gamma_q \rightarrow 2\pi J^2_q h,
\]
\[ \text{(35)} \]
[We note that although \( \Gamma_\pm \rightarrow \Gamma_q^2 \) the logarithmic terms in \[ \text{[16]} \] are well defined as we are off resonance.] The time evolution of the magnetization is now given by
\[
\langle S^z(t) \rangle = -(1 - e^{-\Gamma_\pm t})^2 + e^{-\Gamma_\pm t} (\rho_{11} - \rho_{-1-1}),
\]
\[ \text{(37)} \]
\[
\langle S^\pm(t) \rangle = \frac{e^{\pm iht}}{\sqrt{2}} \left( e^{-\Gamma_\pm t} + e^{-\Gamma_\pm t} \right) (\rho_{10} + \rho_{0 \pm 1}),
\]
\[ \text{(38)} \]
with \( S^\pm = S^x \pm iS^y \) and the initial density matrix of the dot given by \[ \text{[23]} \] while the anisotropy operator relaxes as
\[
\langle T_{20}(t) \rangle = \frac{1}{\sqrt{6}} (1 - e^{-\Gamma_\pm t})
\]
\[ \text{(39)} \]
\[
+ \frac{e^{-\Gamma_\pm t}}{\sqrt{6}} (\rho_{11} - 2\rho_{00} + \rho_{-1-1}),
\]
\[ \text{(39)} \]
\[
\langle T_{2\pm\pm1}(t) \rangle = \frac{e^{\pm iht}}{2\sqrt{2}} \left( e^{-\Gamma_\pm t} + e^{-\Gamma_\pm t} \right) (\rho_{10} - \rho_{0 \pm 1}),
\]
\[ \text{(40)} \]
\[
\langle T_{2\pm\pm2}(t) \rangle = \frac{e^{\pm iht}}{2\sqrt{2}} e^{-\Gamma_\pm t} \rho_{\mp1,\pm1}.
\]
\[ \text{(41)} \]
We observe that the decay of the expectation values of the diagonal operators \( S^z \) and \( T_{20} \) is purely governed by the rate \( \Gamma_z \), while \( (T_{2\pm\pm1}(t)) \) decays purely with \( \Gamma_\pm \). Thus we conclude that \( \Gamma_z \) corresponds to the longitudinal spin relaxation rate of the spin-1/2 system (usually denoted by \( \Gamma_1 \)), while \( \Gamma_\pm \) corresponds to the transverse one (usually denoted by \( \Gamma_2 \)). Comparing the explicit values \[ \text{[35]} \] and \[ \text{[30]} \] with the spin-1/2 result \[ \text{[23]} \] shows that both rates \( \Gamma_\pm \) and \( \Gamma_z \) are twice as large as their spin-1/2 counterparts (i.e. their ratio is identical in the spin-1/2 and spin-1 model). We stress, however, that the time evolution in the spin-1 model is more complex, e.g. in the decay of \( \langle S^\pm(t) \rangle \) and \( \langle T_{2\pm\pm1}(t) \rangle \) both rates \( \Gamma_\pm \) and \( \Gamma_z \) appear.

C. Magnetization and Anisotropy

In this section we analyze the stationary values of magnetization and anisotropy before turning to the differential conductance in the next section. The stationary density matrix of the dot is obtained by solving \[ \text{[14]} \]. Introducing the observables magnetization and anisotropy by
\[
M = \langle S^z \rangle_{st}, \quad A = \langle T_{20} \rangle_{st},
\]
\[ \text{(42)} \]
the stationary density matrix can be written as
\[
\rho^n_{st} = \frac{1}{3} \mathbb{1} + \frac{M}{2} S^2 + A T_{20}.
\]
\[ \text{(43)} \]

The knowledge of the effective dot Liouvillian enables us to derive analytic results for the magnetization and anisotropy including the leading logarithmic corrections, which are given by
\[
M = \frac{4f_1f_2}{(f_1)^2 + 3(f_2)^2}, \quad A = \frac{1}{\sqrt{6}} \frac{4(f_1)^2}{(f_1)^2 + 3(f_2)^2}.
\]
\[ \text{(44)} \]
with
\[
f_1 = 2\pi J^2_q h + 2\pi J^3_q h \mathcal{L}_1(h) - \pi J^3_q (V - h) \mathcal{L}_1(V - h),
\]
\[ \text{(45)} \]
\[
f_2 = -\frac{\pi}{4} J^2_q (2V + 6h + [V - h])
\]
\[ \text{(46)} \]
\[
\mathcal{L}_1(x) = \frac{\Lambda_e}{\sqrt{x^2 + (\Gamma^l_1)^2}} + \frac{\Lambda_e}{\sqrt{x^2 + (\Gamma^q_1)^2}},
\]
\[ \text{(47)} \]
\[
|x|_1 = \frac{2}{\pi} \left( \arctan \frac{x}{\Gamma_t^l} + \arctan \frac{x}{\Gamma_t^q} \right).
\]
\[ \text{(48)} \]
In the derivation of \[ \text{[44]} \] we have neglected all terms in order \( J^3_q \) that do not contain logarithms at either \( h = 0, V = 0 \), or \( h = V \). We note that the decay rates \( \Gamma_1 \) and \( \Gamma_2 \) cut-off the logarithmic divergencies present in bare perturbation theory. As we will see below these two rates also appear in the logarithmic corrections to the differential conductance.

Specifically we find that for large magnetic fields \( h > V \) the spin is completely aligned along the field, i.e. \( M = -1 \) and \( A = 1/\sqrt{6} \). Close to the resonance, i.e. at \( h < V, V - h \ll h \) we find by expanding \[ \text{[14]} \] up to \( O(\Delta \ln \Delta) \)
\[
M = -1 \mp \frac{1}{4} \frac{V - h}{h} \mathcal{L}_1(V - h),
\]
\[ \text{(49)} \]
\[
A = \frac{1}{\sqrt{6}} \left[ 1 - \frac{3V - h}{4h} \frac{J_c V - h}{h} \mathcal{L}_1(V - h) \right].
\]
\[ \text{(50)} \]
Similarly in the limit of small magnetic fields, \( h \ll V \), we obtain
\[
M = -\frac{8h}{3V} \left[ 1 + J_c \mathcal{L}_1(h) \right],
\]
\[ \text{(49)} \]
\[
A = \frac{16}{3\sqrt{6} V^2} \left[ 1 + 2J_c \mathcal{L}_1(h) \right].
\]
\[ \text{(50)} \]
The magnetization (together with the spin-1/2 result) and anisotropy are shown in Fig. \[ \text{4} \].
The corresponding magnetic susceptibility is readily obtained from $M$. At resonance and for small fields we find

$$
\chi_{h=V} = -\frac{\partial M}{\partial h_0} \bigg|_{h=V} = \frac{1}{4V} [1 + J_c \mathcal{L}_1(V-h)],
$$

(51)

$$
\chi_{h=0} = -\frac{\partial M}{\partial h_0} \bigg|_{h=0} = \frac{8}{3V} [1 + J_c \mathcal{L}_1(h)].
$$

(52)

Except for the logarithmic broadening the susceptibility at $h = V$ is identical to the spin-1/2 case, whereas in the limit $h \to 0$ they differ by a relative factor $8/3$. We note that the same factor also appears in the susceptibility of an isolated spin at finite temperature. In Fig. 4 we plot the susceptibility for the spin-1 model as well as the spin-1/2 system. In order to enhance the visibility of the logarithmic terms we further plot $\chi / \chi^{(0)}$, where $\chi^{(0)}$ is the result in $O(J_c^0)$. In comparison to the spin-1/2 model the logarithmic corrections at $h \to 0$ and $h \approx V$ are broadened as they contain two decay rates [see (45)] of which one is larger than the rate appearing in the spin-1/2 susceptibility. We expect this to be a generic feature of the spin-S Kondo dot, i.e. the logarithmic corrections will be most pronounced in the $S = 1/2$ model and become more and more broadened in the large-$S$ limit.

**D. Differential Conductance**

Finally, we discuss the current and conductance through the system. We define the current as the change of the electron number in the left reservoir, $I^L = -\frac{d}{dt} N^L_{\text{res}}$. Because the number of electrons on the dot is fixed, the current in the right reservoir is accordingly given by $I^R = -I^L$. Thus we will omit the superscript in the following, $I \equiv I^L$. Within the RTRG-FS formalism the stationary current is given by

$$
I \equiv (I)_{\text{st}} = -i \text{Tr}_S \Sigma^I (i0^+) \rho^S_{\text{st}},
$$

(53)

where the current kernel $\Sigma^I (z)$ can be derived in complete analogy to the effective dot Liouvillian. We calculate $\Sigma^I$ up to order $J_c^0 \ln$, thus together with the knowledge of $\rho^S_{\text{st}}$ up to this order we are able to derive analytic results for current and conductance including the leading logarithmic corrections. The current is explicitly given by

$$
I = f_I^1 + M f_M^1 + \frac{1}{\sqrt{6}} A f_A^1,
$$

(54)

where $M$ and $A$ are given by (44) and

$$
f_I^1 = 2\pi J_c^2 V + \frac{4}{3} \pi J_c^3 \left[(V-h)\mathcal{L}_1(V-h) + V\mathcal{L}_0(V)\right],
$$

(55)

$$
f_M^1 = \frac{\pi}{4} J_c^2 \left[2V + 2h - |V-h|\right],
$$

(56)

$$
f_A^1 = \frac{\pi}{4} J_c^3 \left[4V\mathcal{L}_0(V) + 4h\mathcal{L}_1(h) - |V-h|\mathcal{L}_1(V-h)\right],
$$

(57)

with the logarithmic terms given by Eq. (45) and

$$
\mathcal{L}_0(x) = \ln \frac{\Lambda_c}{x^2 + (\Gamma_0)^2} + \ln \frac{\Lambda_c}{x^2 + (\Gamma_q^0)^2},
$$

(58)

Fig. 6 shows the differential conductance as a function of the applied bias voltage $V$ derived from Eq. (54).
We find for small bias voltage $V \ll h$ that there are no logarithmic corrections and the current is governed solely by elastic cotunneling processes,

$$I = \pi J_c^2 V, \quad G = \frac{dI}{dV} = \pi J_c^2. \quad (59)$$

This is four times the result in the spin-1/2 model due to the additionally available transport channels. (In the spin-S dot the elastic cotunneling current will be given by $I = \pi/12(2S+1)S(S+1)J_c^2V$.)

On the other hand, close to the resonance $V \approx h$ we observe a jump in the differential conductance which is due to the onset of inelastic cotunneling processes (see also Refs. [35] and [37] and [39] and [40]). Explicitly we find

$$\frac{G}{G_0} = \begin{cases} 2\pi J_c^2 + 2\pi^2 J_c^3 L_1(V - h) & \text{for } V < h, \\ \frac{9}{2}\pi J_c^2 + \frac{9}{2}\pi^2 J_c^3 L_1(V - h) & \text{for } V > h, \end{cases} \quad (60)$$

where we assumed $|V - h| \ll h$ in both cases, and $G_0 = 1/2\pi$ is the conductance quantum. Furthermore, from (60) we deduce the height of the jump at $V = h$

$$\Delta G = \frac{5}{2}\pi J_c^2 [1 + J_c L_1(V - h)]. \quad (61)$$

We note that the analog prefactor in the spin-1/2 model equals $\pi^2/3/2$. The physical origin of this jump is the onset of inelastic cotunneling processes: For finite magnetic field the spin states on the dot are separated by the energy $h$. Thus for $V < h$ only elastic cotunneling processes are possible in which electrons are transferred from the left to the right lead via virtual states on the dot without flipping its spin state. Hence the initial and final state of the dot have the same energy. For $V > h$, however, also inelastic cotunneling processes will contribute in which the spin on the dot is flipped (thus requiring the energy $h$). In analogy to the susceptibility we observe that the logarithmic corrections are less pronounced in the spin-1 model.

Finally we would like to add that the current noise in the spin-1/2 and spin-1 models are qualitatively similar, i.e., differences appear only in the decay rates broadening the logarithmic corrections.[38]

V. ANISOTROPIC MODEL

So far we have restricted ourselves to the case of isotropic exchange couplings. Dropping this restriction we have to study the spin-1 Kondo model [2] with the generalized exchange interaction

$$\frac{1}{2} \sum_{\alpha \kappa \kappa' \sigma \sigma'} [J^{\uparrow}_0 (S^x \sigma_{\sigma'} + S^y \sigma_{\sigma'} y') + J^{\downarrow}_0 S^z \sigma_{\sigma'} \sigma_0] \epsilon \epsilon_{\alpha \kappa \kappa' \sigma \sigma'}. \quad (62)$$

As explained in detail in Refs. [25] and [42] the RTRG-FS method integrates out the reservoir degrees of freedom in a two-step procedure. In the first step one removes the symmetric part of the Fermi function in the reservoir contractions in a single (discrete) RG step, which yields a well-defined perturbative expansion in $J_0$ and $1/D$, where $D$ denotes the bandwidth. For the anisotropic spin-1/2 Kondo model this step only yields negligible perturbative corrections to the Liouvillian and current kernel. In the second (continuous) step one then resums the logarithmic divergencies by integrating out infinitesimal energy shells in the remaining asymmetric part of the reservoir contractions. This yields the PMS equations (10) as well as similar RG equations for the Liouvillian and the current kernel.

However, the situation is completely different for the anisotropic spin-1 Kondo model [62]. The presence of a finite anisotropy

$$c^2 = (J^z)^2 - (J^z)^2 \quad (63)$$

produces the term [see Eq. (74) in Ref. [25]]

$$-\frac{4}{\pi} D G^2 \frac{\Delta}{G_2} = -\pi c^2 D L \Delta \quad (64)$$

in the Liouvillian. Here $L_{\Delta}$ denotes the superoperator corresponding to the square of the local spin, $L_{\Delta} = [S^z S^z, .]$. We note that for spin-1/2 this operator vanishes since $S^z S^z = 1/4$, thus the term (64) does not appear.

Since Eq. (64) contains the initial band width $D \gg h_0$, $V$ it will completely dominate the physics of the system. We distinguish two cases: (i) For $c^2 > 0$ the term (64) is negative, thus locking the dot in one of the fully polarized states with $M = \pm 1$ and $A = 1/\sqrt{6}$. As the
energy spacing to the other states on the dot is proportional to $D$ only elastic cotunneling processes are possible and the current is given by $I = \pi (J_0^2) V$. (ii) For $e^2 < 0$ the term $\{44\}$ is positive and the dot magnetization takes the value $M = 0$. Perturbation theory in the exchange couplings then yields $I = 0$.

This behavior was previously observed in the anisotropic spin-1 Kondo model at equilibrium via a PMS analysis as well as an NRG calculation. In fact, the term $S^z S^z$ was shown to be a relevant perturbation on the grounds of general symmetry considerations.

VI. CONCLUSIONS

In conclusion, we have studied non-equilibrium transport through a quantum dot modeled by a spin-1 Kondo model in a magnetic field. We obtained analytic results for the renormalized magnetic field and g-factor, the magnetization and anisotropy as well as the differential conductance. The latter shows non-monotonic behavior as a function of the bias voltage with a pronounced jump at $V = h$ due to the onset of inelastic cotunneling processes. In particular, this jump is strongly enhanced by logarithmic corrections in the logarithmic terms of the perturbative expansions. Furthermore, the relaxation dynamics of the magnetization and anisotropy as well as the differential conductance (see Fig. 6), which are of stationary quantities like the susceptibility (see Fig. 5) have been studied previously using the same formalism. As a consequence, the presence of additional rates leads to an enhanced broadening of the logarithmic corrections.

Hence the relaxation dynamics of the magnetization and anisotropy of three additional decay rates which govern the relaxation of the spin anisotropy and the exchange coupling it is proportional to an enhanced broadening of the logarithmic corrections. As a consequence, the presence of additional rates leads to a decomposition of the dot magnetization and anisotropy as well as the differential conductance. The latter shows non-monotonic behavior as a function of the bias voltage with a pronounced jump at $V = h$ due to the onset of inelastic cotunneling processes. In particular, this jump is strongly enhanced by logarithmic corrections in the logarithmic terms of the perturbative expansions.

Finally we note that as our approach is based on an expansion in the renormalized exchange coupling it is not possible to derive reliable results in the low-energy regime $\Lambda_x = \max\{V, h\} < T_K$ where the differences between the fully screened spin-1/2 model and the under-screened spin-1 model are most pronounced. As the under-screened Kondo model can be mapped to an effective ferromagnetic Kondo model, a perturbative study of the transport properties in this regime might be possible using a Majorana diagramatic theory as recently performed for the investigation of the magnetotransport.

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Appendix A: Liouvillian

Here we present the main modifications required to generalize the calculations of Ref. [25] to the spin-1 Kondo model. Further details can be found in Ref. [56].

Superoperators.—The basis for the local Liouville space of the dot can be built up by the superoperators $\vec{L}^+$ and $\vec{L}^-$ defined by

$$\vec{L}^+ A = \vec{S} A, \quad \vec{L}^- A = -A \vec{S}. \quad (A1)$$

An explicit representation is given by

$$\vec{L}^+ = \vec{S} \otimes 1, \quad \vec{L}^- = -\vec{c} \otimes \vec{S} \quad (A2)$$

where $\vec{c} = (1, -1, 1)$. In general the action of a superoperator $O$ on an arbitrary operator $B$ (acting itself in the Hilbert space) can be written in a matrix notation as

$$(OB)_{ss'} = O_{ss'; \bar{s}\bar{s}'} B_{\bar{s}\bar{s}'} \quad (A3)$$

Symmetries.—The effective dot Liouvillian obeys the same symmetries as the Hamiltonian: hermiticity relation, spin-conservation, and probability conservation

$$H = H^\dagger \Leftrightarrow (L^b_\vec{S}(z))^c = -L^b_\vec{S}(z^*) \quad (A4)$$

$$s - s' \neq \bar{s} - \bar{s}' \Rightarrow (L^b_\vec{S})_{ss'; \bar{s}\bar{s}'} = 0 \quad (A5)$$

$$\text{Tr}_S \left( \rho_S(t) - \rho_S(t_0) \right) = 0 \Leftrightarrow \text{Tr}_S L^b_S(z) = 0 \quad (A6)$$

Here the c-transform is defined by

$$(A^c)_{ss', \bar{s}\bar{s}'} = A^{s's', \bar{s}\bar{s}} \quad (A7)$$

and the trace over the system states is given by

$$\text{Tr}_S L_{\bar{s}\bar{s}} = \sum_s L_{ss, \bar{s}\bar{s}} \quad (A8)$$

If we choose an ordered basis in the Liouville space of the dot by

$$\{|11\}, \{|00\}, \{|-1-1\}, \{|10\}, \{|0-1\}, \{|01\}, \{|-10\}, \{|1-1\}, \{|-1\} \} \quad (A9)$$

the relations (A5) leads to a decomposition of $L^b_\vec{S}$ into one $3 \times 3$ block with $|s - s'| = 0$, two $2 \times 2$ blocks with $|s - s'| = 1$, and two $1 \times 1$ blocks with $|s - s'| = 2$. The restrictions (A4) and (A6) then yield further relations between the matrix elements in these blocks.

For vanishing magnetic field $h_0 = 0$ the model is fully rotational invariant and the dot Liouvillian decomposes...
L
el for the spin-1 Kondo model starts with the results of the logarithmic corrections to the renormalized magnetic field \( \lambda_i \)

\[
L^s = -\frac{1}{3} t + \frac{1}{3} (\mathbf{L}^+ \cdot \mathbf{L}^-)^2, \\
L^l = 1 - \frac{1}{2} \mathbf{L}^+ \cdot \mathbf{L}^- - \frac{1}{2} (\mathbf{L}^+ \cdot \mathbf{L}^-)^2, \\
L^n = \frac{1}{3} t + \frac{1}{2} \mathbf{L}^+ \cdot \mathbf{L}^- + \frac{1}{6} (\mathbf{L}^+ \cdot \mathbf{L}^-)^2.
\]

Poles of the resolvent.—The poles of the resolvent \( 1/[z - L_{\text{eff}}^S(z)] \) can be parametrized by

\[
z_1 = 0, \\
z_2 = -z_2^* \equiv -i t_0^q, \\
z_3 = -z_3^* \equiv -i t_0^q, \\
z_4/6 = -z_4^*/4 \equiv \pm h - i \Gamma_1^q, \\
z_5/7 = -z_5^*/5 \equiv \pm h - i \Gamma_1^q, \\
z_6/9 = -z_6^*/6 \equiv \pm 2h - i \Gamma_2^q.
\]

where the real parts are related to the renormalized magnetic field \( h \) discussed in Sec. [V A] while the imaginary parts yield the decay rates discussed in Sec. [IV B].

Appendix B: Evaluation of RG equations

The derivation of the Liouvillian and current kernel for the spin-1 Kondo model starts with the results (184)–(187) of Ref. 25. For the explicit evaluation we decompose the effective dot Liouvillian as \( L_{\text{eff}}^S(z) = \sum_i \lambda_i(z) P_i(z) \), where \( \lambda_i(z) \) denote the eigenvalues and the corresponding projectors \( P_i(z) \)’s are determined by

\[
G \left\{ \mathcal{H}_2(z) P_2(z) + \mathcal{H}_3(z) P_3(z) \right\} = \frac{1}{2} G \left\{ \mathcal{H}_2(z) [P_2(z) + P_3(z)] + \mathcal{H}_3(z) [P_2(z) + P_3(z)] \right\} G \\
+ \frac{1}{2} \mathcal{H}_2(z) [P_2(z) - P_3(z)] - \mathcal{H}_3(z) [P_2(z) - P_3(z)] G = \frac{1}{2} [\mathcal{H}_2(z) + \mathcal{H}_3(z)] G \frac{1}{2} \mathcal{H}_2(z) - \mathcal{H}_3(z)] G [P_2(z) - P_3(z)] G + O(J_c^4),
\]

where we have used \( G[P_2(z) + P_3(z)] G = G \sum_{i=1}^2 P_i(z) G = G \mathbf{1}_{3 \times 3} G \) with \( \mathbf{1}_{3 \times 3} \) denoting the projector onto the \( 3 \times 3 \) subspace with \( |s - s'| = 0 \), and \( P_i(z) = P_i(z) + O(J_c) \). Now the first term in (B4) yields the logarithmic corrections in \( O(J_c^2) \), e.g. the real parts of the analog terms with \( i = 4, 5 \) and \( i = 6, 7 \) yield the logarithmic corrections to the renormalized magnetic field \( t \).

On the other hand, the second term in (B4) does not lead to logarithmic corrections in \( O(J_c^2) \) as

\[
\mathcal{H}_2(z) - \mathcal{H}_3(z) \simeq \ln \frac{t_0^q}{\Gamma_1^q} = O(1).
\]

The approximation (B4) is the main technical modification required for the treatment of the spin-1 Kondo model. The possibility to neglect the contributions from (B5) is essential to derive the analytic results presented in the main text. The approximation (B4) obviously yields the combination \( \mathcal{H}_2(z) + \mathcal{H}_3(z) \) (similar for the other pairs \( i = 4, 5 \) and \( i = 6, 7 \)). We note, however, the direct relations between the functions \( \mathcal{H}_i(z) \) (and thus the rates \( \Gamma_i \)) and the projectors \( P_i(z) \) are lost.

We stress again that our calculations rely on (B5). Thus if (B4) were to contain \( \mathcal{H}_1(z) \) (as may be the case in the calculation of two-point correlation functions) the terms of the form (B5) become divergent and the approx-
We obtain the following self-consistency equations for the stationary quantities, only appearing in the logarithmic corrections to the decay rates and stationary quantities, only appear in $O(G^3)$. They can be evaluated by using a relation similar to (B4).

**Appendix C: Self-consistency equations for decay rates**

From the poles $z_i$ of the resolvent [see (A14)–(A18)] we obtain the following self-consistency equations for the decay rates (see Fig. 3)

\[
\Gamma^0_{i/q} = 2C_1 \mp \sqrt{(C_1 + C_2)(C_1 - C_2)}, \quad (C1)
\]

\[
\Gamma^1_{i/q} = C_3 + \frac{1}{2}C_4 + C_5 \mp \frac{1}{2}\sqrt{(C_5 + 2C_3)^2 - 3(C_3)^2}, \quad (C2)
\]

\[
\Gamma^2_q = 4C_3 + C_4, \quad (C3)
\]

where the right-hand side (RHS) of (C1) has to be evaluated at $z = z_2/3$, the RHS of (C2) at $z = z_4/5$, and the RHS of (C3) at $z = z_8$, respectively, and the appearing expressions read

\[
C_1(z) = \frac{\pi}{8} J_z^2 \left(2|z - h|_1 + 2|z + h|_1 + |z - h + V|_1 + |z + h + V|_1 + |z - h - V|_1 + |z + h - V|_1\right),
\]

\[
C_2 = 2\pi J^2 h,
\]

\[
C_3(z) = \frac{\pi}{4} J_z^2 \left(2|z - 2h|_2 + |z + 2h|_2 + |z - 2h - V|_2 + |z + 2h - V|_2\right),
\]

\[
C_4(z) = \frac{\pi}{4} J_z^2 \left(2|z - h|_1 + |z + h|_1 + |z - h - V|_1 + |z + h - V|_1\right),
\]

\[
C_5(z) = \frac{\pi}{4} J_z^2 \left(2|z|_0 + |z + V|_0 + |z - V|_0\right).
\]

The absolute values are broadened by the decay rates and explicitly given by (46) as well as

\[
|x|_0 = \frac{2}{\pi} x \left(\arctan \frac{2}{\Gamma^0_q} + \arctan \frac{x}{\Gamma^0_q}\right), \quad (C4)
\]

\[
|x|_2 = \frac{2}{\pi} x \arctan \frac{x}{\Gamma^2_q R^0_q}, \quad (C5)
\]
A. Posazhennikova and P. Coleman, Phys. Rev. Lett. 94, 036802 (2005).

J. Panske, A. Rosch, P. Wölfle, N. Mason, C. M. Marcus, and J. Nygård, Nature Phys. 2, 460 (2006).

A. Posazhennikova, B. Bayani, and P. Coleman, Phys. Rev. B 75, 245329 (2007).

P. Roura Bas and A. A. Aligia, Phys. Rev. B 80, 035308 (2009).

P. Roura-Bas and A. A. Aligia, J. Phys. Condens. Matter 22, 025602 (2010).

C. J. Wright, M. R. Galpin, and D. E. Logan, Phys. Rev. B 84, 115308 (2011).

H. Schoeller, Eur. Phys. J. Special Topics 168, 179 (2009).

M. Garst, P. Wölle, L. Borda, J. von Delft, and L. Glazman, Phys. Rev. B 72, 205125 (2005).

M. Pletyukhov, D. Schuricht, and H. Schoeller, Phys. Rev. Lett. 104, 106801 (2010).

D. A. Varshalovich, A. N. Moskalev, and V. K. Khersonskii, Quantum Theory of Angular Momentum (World Scientific, Singapore, 1988).

B. Sothmann and J. König, Phys. Rev. B 82, 245319 (2010).

M. M. E. Baumgärtel, M. Hell, S. Das, and M. R. Wegewijs, Phys. Rev. Lett. 107, 087202 (2011).

C. Karrasch, S. Andergassen, M. Pletyukhov, D. Schuricht, L. Borda, V. Meden, and H. Schoeller, Europhys. Lett. 90, 30003 (2010).

S. Andergassen, M. Pletyukhov, D. Schuricht, H. Schoeller, and L. Borda, Phys. Rev. B 83, 205103 (2011); ibid. 84, 039905(E) (2011).

S. Y. Müller, private communication.

R. M. Konik, H. Saleur, and A. W. W. Ludwig, Phys. Rev. B 66, 075105 (2002).

A. Schiller and L. De Leo, Phys. Rev. B 77, 075114 (2008).

R. Zitko, R. Peters, and T. Pruschke, Phys. Rev. B 78, 224404 (2008).

I. Affleck, A. W. W. Ludwig, H.-B. Pang, and D. L. Cox, Phys. Rev. B 45, 7918 (1992).

M. C. Cano and S. Florens, arXiv:1112.0925.

C. B. M. Hörig, Diploma thesis, RWTH Aachen University (2011).