Stochastic Orderings of the Location-Scale Mixture of Elliptical Distribution

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Abstract

We consider a class of location-scale mixture of elliptical distributions. This class of distributions takes skewness into consideration and presents a mathematically tractable extension of the multivariate elliptical distribution. We give some sufficient and/or necessary conditions for various integral stochastic orders such as usual stochastic order, convex order, increasing convex order and directionally convex order. Two important assumptions are given in this paper to compare random variables under usual stochastic order and increasing convex order.

Keywords: Elliptical distribution, Integral stochastic orderings, Location-scale mixture, Stochastic orderings.

1. Introduction and Motivation

Stochastic orders, which are partial orders on a set of random variables, are now used as a method of comparing random variables in many areas like statistics (Cal and Carcamo \[16\]), actuarial science (Denuit et al. \[19\]), operations research (Fábián et al. \[22\]), clinical trials (Bekele and Thall \[14\]) and other related fields. Different kinds of stochastic orders have different properties and applications, and interested readers may refer to Denuit et al. \[19\], Müller and Stoyan \[38\] and Shaked and Shanthikumar \[41\] for details.

Many stochastic orders are characterized by the integral stochastic orders, which seek the order between random vectors $X$ and $Y$ by comparing $\mathbb{E} f(X)$ and $\mathbb{E} f(Y)$, where $f \in F$ and $F$ is a certain class of functions. Integral stochastic orders include a wide range of stochastic orders like usual stochastic order and stop-loss order. Some important treatments for this class of orders can be found in Whitt \[45\], Müller \[34\] and Müller \[35\]. On the other hand, some integral stochastic orders are introduced to characterize the correlation, dependence and orthant probabilities of random vectors. Orthant orders (\[42\]), Hessian orders (Arlotto and Scarsini \[5\]) and concordance orders (Joe \[28\]) focus on the high order derivatives of $f$ and have proved to be useful in many fields.

Elliptical distributions, which can be seen as convenient extensions of multivariate normal distributions, were introduced by Kelker \[30\]. Some properties and characterizations of this family
of distributions was discussed in Fang et al. [24]. Elliptical distributions provide an attractive tool for statistics, economics, finance and actuarial science to describe fat or light tails of distributions because of the flexibility of density functions. Interested readers are referred to monographs of Fang et al. [24], Gupta et al. [25] and McNeil et al. [33].

However, the class of elliptical distributions fails to capture the skewness of data because of its symmetrical characteristic. Researchers try to extend the class of elliptical distributions in order to fill this gap. It is well-known that the scale mixture of elliptical distributions is still elliptical; therefore, adding a factor of skewness into the elliptical model seems to be a straightforward method. The mean-variance mixture of multinormal distributions, which is generated by stochastic representation

\[ Y \overset{d}{=} \mu + \sqrt{Z}X + Z\delta, \]  

(1)

where \( \mu, \delta \in \mathbb{R}^n \), \( X \sim N_n(0, \Sigma) \) and \( Z \) is a non-negative random quantity, was introduced Barndorff-Nielsen et al. [12]. This class of distributions plays an important role in statistical modeling and interested readers are referred to Jones [29], Kim and Kim [31] and Jamali et al. [26]. Recently, Zuo [47] extended this class of distribution by considering \( X \) elliptically distributed. An alternative way to take skewness into consideration is introduced by Arslan [7], which studied a random vector with stochastic representation

\[ Y \overset{d}{=} \mu + \frac{1}{\sqrt{Z}}X + \frac{1}{Z}\delta, \]  

(2)

where \( X \) follows elliptical distribution with location parameter 0 and scale parameter \( \Sigma \) and \( Z \) is a non-negative random quantity follows beta distribution. Arslan [7] call this class of distributions as the “Generalized Hyperbolic Skew-Slash Distributions”.

In the field of finance, Simaan [43] proposed that the vector of returns on financial assets should be represented as

\[ Y \overset{d}{=} \mu + X + Z\delta, \]  

(3)

where \( X \) follows elliptical distribution with location parameter 0 and scale parameter \( \Sigma \) and \( Z \) is a non-negative random quantity. In (1) and (2), both mean and variance are mixed with the same positive random variable \( Z \), while in (3) just the mean parameter is mixed with \( Z \). It is clear that the class in (3) cannot be obtained from the class in (1) or (2). Adcock [2] studied some special cases of the class in (3), including the normal–exponential and normal-gamma distributions. Adcock [2] showed some applications of this class of distributions in capital pricing, return on financial assets and portfolio selection.

The univariate and multivariate skew-normal distributions were introduced in Azzalini [8] and Azzalini and Dalla Valle [9]. An \( n \)-dimensional random vector \( Y \) follows multivariate skew-normal distribution if it has stochastic representation

\[ Y \overset{d}{=} \mu + \sigma (\delta Z + X), \]  

(4)

where \( X \sim N_n \left(0, \Sigma - \delta\delta'\right) \), random quantity \( Z \) has a standard normal distribution within the truncated interval \((0, +\infty)\), independently of \( X \), \( \delta = (\delta_1, \delta_2, \ldots, \delta_n)' \) and \(-1 < \delta_i < 1 \) for all \( 1 \leq i \leq n \). The square matrix \( \sigma \) is a diagonal matrix formed by \( \sigma = (\Sigma \odot I_n)^{1/2} \), where \( \odot \) stands for the
Hadamard product. Azzalini [10] illustrates various areas of application of skew-normal distribution, including selective sampling, models for compositional data, robust methods and non-linear time series. Recently, a general new family of mixture distributions of multivariate normal distribution was be introduced by Negarestani et al. [37] and Abdi et al. [1] based on arbitrary random variable \( Z \) in (4). Besides the three aforementioned approaches to model skewed data, Branco and Dey [15], Arnold et al. [6], Wang et al. [44] and Dey and Liu [20] extended elliptical distributions to skew-elliptical distributions under different perspectives. Their extensions are mainly based on the density functions, correlation of random vectors and conditional representations.

Stochastic orderings of multinormal random vectors have been studied by Scarsini [40], Müller [35], Denuit and Müller [18], Arlotto and Scarsini [5] among others. Müller [35] provided a general treatment on integral stochastic orders, with the main tool being an identity for \( E(f(Y)) - E(f(X)) \), where \( X \) and \( Y \) are multinormal random variables; necessary and sufficient conditions for some integral stochastic orders for multivariate normal distributions were obtained in Müller [35] by using this identity. Ding and Zhang [21] extended these results to Kotz-type distributions which form a special class of elliptical symmetric distributions. Some conditions under bivariate elliptical distributions are ordered through the convex, increasing convex and concordance orders were obtained in Landsman and Tsanakas [32]. Davidov and Peddada [17] showed an important result that the positive linear usual stochastic order coincides with the multivariate usual stochastic order for elliptically distributed random vectors. In recent years, Pan et al. [39] studied convex and increasing convex orderings of multivariate elliptical random vectors and derived some necessary and sufficient conditions. Later, some other integral stochastic orderings of multivariate elliptical distribution were studied in Yin [46]. Jamali et al. [26], Jamali et al. [27] and Amiri et al. [3] studied some conditions for stochastic orderings of skew-normal distributions ([26]), multivariate normal mean-variance mixtures ([27]), skew-normal scale-shape mixtures ([27]) and scale mixtures of the multivariate skew-normal distributions([3]).

The stochastic representations of (1), (2) and (3) are quite similar, therefore we seek a unified method to derive some sufficient and necessary conditions for various integral stochastic orderings of random vectors following the aforementioned distributions. We introduce the class of location-scale mixture of elliptical \((LSE)\) distributions which takes skewness into consideration and presents a mathematically tractable extension of the multivariate elliptical distribution. The distributions presented by (1), (2) and (3) are special cases of \(LSE\) distribution. Our method of ordering follows those of Müller [35] and Yin [46].

The rest of the paper is organized as follows. In Section 2 we review multivariate elliptical distribution and state some key properties and characterizations. We also present a brief review of integral stochastic orderings. In Section 3 we introduce the elliptical location-scale mixtures and some related properties. Section 4 provides the results of necessary and/or sufficient conditions for integral orderings and some actuarial applications. Section 5 concludes with a short discussion and some possible directions for future research.

2. Preliminaries

The following notations will be used throughout this paper. We will use lowercase letters, bold lowercase letters and bold capital letters to denote numbers, vectors and matrices, respectively;
\(\Phi(\cdot)\) and \(\phi(\cdot)\) to denote the cumulative distribution function and probability density function of the univariate standard normal distribution, respectively; and \(\Phi_n(\cdot; \mu, \Sigma)\) and \(\phi_n(\cdot; \mu, \Sigma)\) to denote the cumulative distribution function and probability density function of \(n\)-dimensional normal distribution with mean vector \(\mu\) and covariance matrix \(\Sigma\).

For twice continuously differentiable function \(f: \mathbb{R}^n \to \mathbb{R}\), we use
\[
\nabla f(x) = \left( \frac{\partial f(x)}{\partial x_i} \right)_{i=1}^n, \quad H_f(x) = \left( \frac{\partial^2 f(x)}{\partial x_i \partial x_j} \right)_{i,j=1}^n
\]
to denote the gradient vector and the Hessian matrix of \(f\), respectively. We use \(\text{tr}(C)\) to denote the trace of square matrix \(C\). For \(n\)-dimensional vectors \(a\) and \(b\), we define the inner product of two vectors \(\langle a, b \rangle = a' b\). For \(n \times p\)-dimensional matrices \(A\) and \(B\), we define the inner product of matrices \(\langle A, B \rangle = \text{tr}(A'B)\).

2.1. Introduction to Some Distribution

The class of multivariate elliptical distributions is a natural extension to the class of multivariate normal distributions. We follow the definition of Fang et al. [24].

**Definition 2.1.** An \(n\)-dimensional random vector \(X\) is said to have an elliptical distribution with location parameter \(\mu\) and scale parameter \(\Sigma\) (denoted by \(\text{ELL}_n(\mu, \Sigma, \psi)\)) if its characteristic function has the form
\[
\Psi_X(t) = \exp \left( it^T \mu \right) \psi \left( t^T \Sigma t \right), \tag{5}
\]
where \(\psi\) is called the characteristic generator satisfying \(\psi(0) = 1\). If \(X\) has a density function, then the density has the form
\[
f_X(x) = \frac{c_n}{\sqrt{\mid \Sigma \mid}} g_n \left( (x - \mu)^T \Sigma^{-1} (x - \mu) \right), \tag{6}
\]
\[c_n = \frac{\Gamma(n/2)}{\pi^{n/2}} \left( \int_0^{\infty} z^{n/2-1} g_n(z) dz \right)^{-1}, \tag{7}\]
for some nonnegative function \(g_n\) called the density generator and for some constant \(c_n\) called the normalizing constant. One sometimes writes \(\text{ELL}_n(\mu, \Sigma, g_n)\) for the \(n\)-dimensional elliptical distributions generated from the function \(g_n\). Some families of elliptical distributions with their density generators are presented in Table 2.

**Remark 1.** If \(X\) has a density function, if and only if its density generator \(g_n\) satisfies the condition
\[
0 < \int_0^{\infty} z^{n/2-1} g_n(z) dz < +\infty. \tag{8}\]

**Lemma 2.1.** Let \(X \sim \text{ELL}_n(\mu, \Sigma, \psi)\), then:

1. The mean vector \(E(X)\) (if it exists) coincides with the location vector and the covariance matrix \(\text{Cov}(X)\) (if it exists), being \(-2\psi'(0)\Sigma\).
Table 1: Some families of elliptical distributions with their density generator

| Family        | Density generator                                      |
|---------------|-------------------------------------------------------|
| Cauchy        | \( g_n(u) = (1 + u)^{-(n+1)/2} \)                   |
| Exponential power | \( g_n(u) = \exp\left(-\frac{1}{2}(u^{s/2})\right), \ s > 1 \) |
| Laplace       | \( g_n(u) = \exp\left(-\sqrt{u}\right) \)           |
| Normal        | \( g_n(u) = \exp\left(-u/2\right) \)                |
| Student       | \( g_n(u) = \left(1 + \frac{u}{m}\right)^{-(n+m)/2}, \ m \) is a positive integer |
| Logistic      | \( g_n(u) = \exp\left(-u\right)(1 + \exp\left(-u\right))^{-2} \) |

2. \( X \) admits the stochastic representation

\[
X \overset{d}{=} \mu + RA'U^{(n)},
\]

where \( A \) is a square matrix such that \( A'A = \Sigma \), \( U^{(n)} \) is uniformly distributed on the unit sphere \( S^{n-1} = \{ u \in \mathbb{R}^n | u' \mu = 1 \} \), \( R \geq 0 \) is the random variable with distribution function \( F \) called the generating variable and \( F \) is called the generating distribution function, \( R \) and \( U^{(n)} \) are independent.

3. Multivariate elliptical distribution is closed under affine transformations. Considering \( Y = BX + b \), where \( B \) is a \( m \times n \) matrix with \( m < n \) and \( \text{rank}(B) = m \) and \( b \in \mathbb{R}^m \), then \( Y \sim \text{Ell}_m(B\mu + b, B\Sigma B', \psi) \).

Yin [46] provided an important identity for multivariate elliptical distributions.

**Lemma 2.2.** (Yin [46]) Let \( X \sim E_n(\mu^x, \Sigma^x, \psi) \) and \( Y \sim E_n(\mu^y, \Sigma^y, \psi) \) with \( \Sigma^x \) and \( \Sigma^y \) positive definite. Let \( \phi_\lambda \) be the density function of

\[
E_n(\lambda \mu^y + (1 - \lambda) \mu^x, \lambda \Sigma^y + (1 - \lambda) \Sigma^x, \psi), 0 \leq \lambda \leq 1,
\]

and \( \phi_{1,1} \) be the density function of

\[
E_n(\lambda \mu^y + (1 - \lambda) \mu^x, \lambda \Sigma^y + (1 - \lambda) \Sigma^x, \psi_1), 0 \leq \lambda \leq 1,
\]

where

\[
\psi_1(u) = \frac{1}{E(r^2)} \int_0^{\infty} 0F_1\left(\frac{n}{2} + 1; -\frac{r^2u}{4}\right) R^2 P(R \in dr).
\]

Here

\[
0F_1(\gamma; z) = \sum_{k=0}^{\infty} \frac{\Gamma(\gamma)}{\Gamma(\gamma + k) k!} z^k,
\]

is the generalized hypergeometric series of order \((0, 1)\), \( R \) is defined by with \( E(R^2) < \infty \). Moreover, assume that \( f : \mathbb{R}^n \to \mathbb{R} \) is twice continuously differentiable and satisfies some polynomial growth conditions at infinity:

\[
f(x) = O(||x||), \nabla f(x) = O(||x||).
\]
Then,
\[
E[f(Y)] - E[f(X)] = \int_0^1 \int_{\mathbb{R}^n} (\mu^x - \mu^y)^T \nabla f(x) \phi_1(x) \, dx \, d\lambda
+ \frac{E(R^2)}{2p} \int_0^1 \int_{\mathbb{R}^n} \text{tr} \left( (\Sigma^y - \Sigma^x) H_f(x) \right) \phi_{1,1}(x) \, dx \, d\lambda.
\]  

(9)

2.2. Integral Stochastic Orders

Given two \(n\)-dimensional random vectors \(X\) and \(Y\). Integral stochastic orders seek orderings between \(X\) and \(Y\) by comparing \(Ef(Y)\) and \(Ef(X)\). Let \(F\) be a class of measurable functions \(f : \mathbb{R}^n \to \mathbb{R}\). Then, we say that \(X \leq_f Y\) if \(Ef(X) \leq Ef(Y)\) holds for all \(f \in F\), whenever the expectations are well defined. A general study of this type of orders has been given by Müller [3].

Definition 2.2. For any function \(f : \mathbb{R}^p \to \mathbb{R}\), the difference operator \(\Delta_i^f\), \(1 \leq i \leq p, \epsilon > 0\) is defined as \(\Delta_i^f(x) = f(x + \epsilon e_i) - f(x)\), where \(e_i\) stands for the \(i\)-th unit basis vector of \(\mathbb{R}^n\). Then

1. \(f\) is supermodular if \(\Delta_i^f \Delta_j^f(x) \geq 0\) holds for all \(x \in \mathbb{R}^n, \epsilon_1, \epsilon_2 \geq 0\) and \(1 \leq i < j \leq n\);

2. \(f\) is directionally convex if \(\Delta_i^f \Delta_j^f(x) \geq 0\) holds for all \(x \in \mathbb{R}^n, \epsilon_1, \epsilon_2 \geq 0\) and \(1 \leq i, j \leq n\);

3. \(f\) is \(\Delta\)-monotone if \(\Delta_i^f \Delta_j^f \cdots \Delta_k^f(x) \geq 0\) holds for all \(x \in \mathbb{R}^n, \epsilon_i \geq 0\) for \(1 \leq i \leq k\) and for any subset \(\{i_1, i_2, \ldots, i_k\} \subseteq \{1, 2, \ldots, n\}\).

Remark 2. These three classes of functions can be characterized by their derivatives:

1. \(f\) is supermodular if and only if \(\frac{\partial^2}{\partial x_i \partial x_j} f(x) \geq 0\) holds for all \(x \in \mathbb{R}^n\) and \(1 \leq i < j \leq n\);

2. \(f\) is directionally convex if and only if \(\frac{\partial^2}{\partial x_i \partial x_j} f(x) \geq 0\) holds for all \(x \in \mathbb{R}^n\) and \(1 \leq i, j \leq n\);

3. \(f\) is \(\Delta\)-monotone if and only if \(\frac{\partial^k}{\partial x_{i_1} \cdots \partial x_{i_k}} f(x) \geq 0\) holds for all \(x \in \mathbb{R}^n, 1 \leq k \leq n\) and \(1 \leq i_1 < \cdots < i_k \leq n\).

Definition 2.3. (Arlotto and Scarsini [5]) An \(n \times n\) matrix \(A\) is called copositive if the quadratic form \(x^T A x \geq 0\) for all \(x \geq 0\), and \(A\) is called completely positive if there exists a nonnegative \(m \times n\) matrix \(B\) such that \(A = B^T B\).

We use \(C_{\text{cop}}\) to denote the cone of copositive matrices and \(C_{\text{cp}}\) to denote the cone of completely positive matrices. We use \(C^*\) to denote the dual of the closed convex cone \(C\), i.e. \(C^* = \{B : \langle A, B \rangle \geq 0, \forall A \in C\}\). Arlotto and Scarsini [5] proved that the cones of \(C_{\text{cop}}\) and \(C_{\text{cp}}\) are both closed and convex and

\[
C_{\text{cop}}^* = C_{\text{cp}}, C_{\text{cp}}^* = C_{\text{cop}}^*.
\]

(10)

Definition 2.4. We say \(X\) is smaller than \(Y\) in:

1. Usual stochastic order, i.e. \(X \leq_{st} Y\), if \(Ef(X) \leq Ef(Y)\) for all increasing functions;
2. Positive linear usual stochastic order, i.e. $X \leq_{plst} Y$, if $\langle a, X \rangle \leq_{st} \langle a, Y \rangle$ for all $a \in \mathbb{R}^n$;

3. Convex order, i.e. $X \leq_{cx} Y$, if $Ef(X) \leq Ef(Y)$ for all convex functions;

4. Linear convex order, i.e. $X \leq_{lcx} Y$, if $\langle a, X \rangle \leq_{cx} \langle a, Y \rangle$ for all $a \in \mathbb{R}^n$;

5. Positive linear convex order, i.e. $X \leq_{ilcx} Y$, if $\langle a, X \rangle \leq_{cx} \langle a, Y \rangle$ for all $a \in \mathbb{R}^n$;

6. Increasing convex order, i.e. $X \leq_{icx} Y$, if $Ef(X) \leq Ef(Y)$ for all increasing convex functions;

7. Directionally convex order, i.e. $X \leq_{dcx} Y$, if $Ef(X) \leq Ef(Y)$ for all directionally convex functions;

8. Componentwise convex order, i.e. $X \leq_{ccx} Y$, if $Ef(X) \leq Ef(Y)$ for all componentwise convex functions;

9. Upper orthant order, i.e. $X \leq_{uo} Y$, if $Ef(X) \leq Ef(Y)$ for all $\Delta$-monotone functions;

10. Supermodular order, i.e. $X \leq_{sm} Y$, if $Ef(X) \leq Ef(Y)$ for all supermodular functions;

11. Copositive order, i.e. $X \leq_{cp} Y$, if $Ef(X) \leq Ef(Y)$ for all functions such that $H_f(x) \in C_{cp}$;

12. Completely positive order, i.e. $X \leq_{cop} Y$, if $Ef(X) \leq Ef(Y)$ for all functions such that $H_f(x) \in C_{cop}$.

The following implications are well known (see Scarsini [40], Pan et al. [39] and Amiri et al. [3]):

$$X \leq_{cx} Y \Rightarrow X \leq_{icx} Y \Leftrightarrow X \leq_{ilcx} Y$$

$$\Downarrow$$

$$X \leq_{plst} Y \Leftrightarrow X \leq_{st} Y \Rightarrow X \leq_{icx} Y \Rightarrow X \leq_{iplcx} Y.$$

3. Location-scale mixture of elliptical distributions

Consider the $n$-dimensional random vector $Y$ that can be expressed as

$$Y \overset{d}{=} \mu + \alpha(Z)X + \beta(Z)\delta,$$

where $\mu, \delta \in \mathbb{R}^n$, $\alpha, \beta : \mathbb{R}^q \to \mathbb{R}^+$, $X \sim ELL_n(0, \Sigma, \psi)$ with a positive definite matrix $\Sigma$ and it has density generator $g_n$, and $Z$ is a $q$-dimensional random vector with CDF $H(z)$ and independent to $X$. Then, the random vector $Y$ is said to have a location-scale mixture of elliptical(LES) distributions, which will be denoted by $LS(\mu, \Sigma, \delta, \psi, \alpha, \beta, H)$ in this paper. The conditional representation of $Y$ can be expressed as

$$Y|Z \sim ELL_n \left( \mu + \beta(Z)\delta, \alpha^2(Z)\Sigma, \psi \right).$$
Therefore, the density and characteristic functions of $Y$ take the forms

$$f(y) = \int_{\mathbb{R}^n} \frac{c_n}{\alpha(z) \sqrt{\det(\Sigma)}} g_n(y, \mu + \beta(z)\delta, \alpha^2(z)\Sigma) \, dH(z),$$

where $c_n$ follows (7) and

$$\Psi(t) = \exp(it^t\mu)E_Z\left(\exp(i\beta(Z)t^t\delta) \psi(\alpha^2(Z)t^t\Sigma t)\right)$$

respectively. Provided that $E(\alpha^2(Z))$, $E(\beta(Z))$, $Var(\beta(Z))$, the mean vector and the covariance matrix of $Y$ exist, then the mean vector and the covariance matrix of $Y$ are given by

$$E(Y) = \mu + E(\beta(Z))\delta,$$

and

$$Cov(Y) = -2\psi'(0)E(\alpha^2(Z)\Sigma + Var(\beta(Z))\delta\delta^T).$$

The following lemma presents that $LSE$ distribution is closed under affine transformations.

**Lemma 3.1.** Let $Y \sim LSE_n(\mu, \Sigma, \delta, \psi, \alpha, \beta, H)$, and $B$ be a $m \times n$ matrix with $m < n$ and rank($B$) = $m$ and $b \in \mathbb{R}^m$, then $BY + b \sim LSE_m(B\mu + b, B\Sigma B^T, B\delta, \psi, \alpha, \beta, H)$.

**Proof.** The characteristic function of $BY + b$ is obtained as

$$\Psi(t) = \exp(it^tB\mu)\exp(it^t b)E_Z\left(\exp(i\beta(Z)t^tB\delta) \psi(\alpha^2(Z)t^tB\Sigma B^t t)\right),$$

which shows the result. \hfill $\square$

The family of $LSE$ distributions is large enough to contain several subfamilies of symmetric and non-symmetric distributions. A considerable amount of well known distributions can be seen as special cases of $LSE$ distribution and we introduce some of them here.

1. Mean-variance mixture of multinormal distributions (Barndorff-Nielsen et al. [12]) when setting $\psi = \exp(-u/2)$, $q = 1$, $\alpha(z) = \sqrt{\lambda}$, $\beta(z) = z$.

2. Generalized hyperbolic skew-slash distribution (Arslan [7]) when $\psi = \exp(-u/2)$, $q = 1$, $Z \sim beta(\lambda, 1)$, $\alpha(z) = z^{-\frac{1}{2}}$, $\beta(z) = z^{-1}$. This distribution will be denoted by $GHS(\mu, \Sigma, \delta, \lambda)$ in this paper.

3. Generalized Hyperbolic distribution (Barndorff-Nielsen and Blaesild [11]) when setting $\psi = \exp(-u/2)$, $q = 1$, $\alpha(z) = \sqrt{\chi}$, $\beta(z) = z$ and $Z$ follows Generalized inverse Gussian distribution with density

$$h(\omega) = \frac{(\tau/\chi)^{\frac{1}{2}}}{2K_{\lambda}(\sqrt{\chi} \tau)} \omega^{\lambda-1} \exp\left(-\frac{1}{2}\left(\frac{\chi}{\omega} + \tau \omega\right)\right), \omega > 0,$$

where parameters follow

$$\begin{cases} 
\chi > 0 & \text{and} \quad \tau \geq 0, \quad \text{if} \quad \lambda < 0, \\
\chi > 0 & \text{and} \quad \tau > 0, \quad \text{if} \quad \lambda = 0, \\
\chi \geq 0 & \text{and} \quad \tau > 0, \quad \text{if} \quad \lambda > 0
\end{cases}$$

and $K_{\lambda}$ being the Bessel function of the third kind with index $\lambda$. 

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4. The model of nonsymmetric security returns (Simaan [43]) when setting $\alpha(z) = 1, \beta(z) = z$ and $Z$ be a univariate random variable with any non-elliptical distribution.

To illustrate the tail behavior and the skewness of the $LSE$ distribution, we draw density curves of univariate $GHSS$ distributions in Fig. 1. We can see that the skewness of distributions gets heavier when $\delta$ gets larger while the tails get longer when $\lambda$ gets smaller. Detailed characterizations about the relationship between $\lambda$ and $\delta$ and skewness and tail behavior could be found in Arslan [7].

The following lemma could be proved by using Lemma 2.1 and applying double expectation formula.

**Lemma 3.2.** Assume $Y_1 \sim LSE_n(\mu_1, \Sigma_1, \delta_1, \psi, \alpha, \beta, H)$ and $Y_2 \sim LSE_n(\mu_2, \Sigma_2, \delta_2, \psi, \alpha, \beta, H)$. If all the conditions in Lemma 2.1 are satisfied, then

$$
E[f(Y_1)] - E[f(Y_2)] = \int_{\mathbb{R}^n} \int_0^1 \int_{\mathbb{R}^n} (\mu_2 - \mu_1 + \beta(z)(\delta_2 - \delta_1))' \nabla f(x) \phi_{1,1}(x) \, dx \, d\lambda \, dH(z)
$$

$$
+ \frac{E(\tau^2)}{2p} \int_{\mathbb{R}^n} \int_0^1 \int_{\mathbb{R}^n} \alpha^2(z) \text{tr} \left( (\Sigma_2 - \Sigma_1) H_f(x) \right) \phi_{1,1}(x) \, dx \, d\lambda \, dH(z).
$$

(17)

If one set $\delta = 0$ in (11), then the part of location mixture part of $LSE$ distribution vanishes and $LSE$ distribution will degenerate to scale mixture of elliptical distributions. In other words, the family of scale mixture of elliptical distributions is set up by stochastic representation $Y \overset{d}{=} \mu + \alpha(Z)X$, where the parameters are set in parallel with (11). The random vector $Y$ will be denoted...
by $SME(\mu, \Sigma, \psi, \alpha, H)$. Obviously, the identities presented in this section are still valid in SME case. Setting $\delta = 0$, the PDF in (13) has the form

$$f(y) = \frac{c_n}{\alpha(z) \sqrt{\Sigma}} g_n \left( (y - \mu)'\Sigma^{-1}(y - \mu) \right),$$

where $g_n(y) = \int_{[0,1]} g_n \left( \frac{1}{\sigma(z)} (y - \mu)'\Sigma^{-1}(y - \mu) \right) dH(z)$. One can observe from the density that the characteristic functions of SME distributions is of the form (5).

4. Results of Stochastic Ordering

In some cases, the density generators are arbitrarily chosen and are too generalized to study the properties of $LS E$ distribution. We have to narrow the variety of density generator $g$ under some specific situations. To this end, the following technical assumptions are necessarily needed.

**Assumption 1.** Let $t_i = \frac{t_i - \mu_i - \beta(z)\delta_i}{\alpha(z)}$ for $i = 1, 2$. We assume density generator $g$ satisfies if $\sigma_1 \neq \sigma_2$, then

$$\lim_{t \to \pm \infty} \frac{\sigma_1 g(t_2^2)}{\sigma_2 g(t_1^2)} = \frac{\sigma_1 g(t_2^2)}{\sigma_2 g(t_1^2)} = C,$$

where $C \in \mathbb{R}^+ \setminus \{1\}$.

**Assumption 2.** Let $t_i = \frac{t_i - \mu_i - \beta(z)\delta_i}{\alpha(z)}$ for $i = 1, 2$. We assume density generator $g$ satisfies if $\sigma_1 > \sigma_2$, then

$$\lim_{t \to \pm \infty} \frac{\sigma_1 g(t_2^2)}{\sigma_2 g(t_1^2)} = \frac{\sigma_1 g(t_2^2)}{\sigma_2 g(t_1^2)} = C',$$

where $C' \in [0, 1)$.

Note that Assumptions 1 and 2 are not strict and all the density generators presented in Table 1 follow Assumptions 1 and 2. We will prove this proposition later in Appendix. They are needed for characterizing the limit behavior of the density generator. Based on the foregoing assumptions, the conditions for usual stochastic order of univariate $LS E$ distribution can first be established.

**Lemma 4.1.** Assume $Y_1 \sim LSE_1(\mu_1, \sigma_1, \delta_1, g, \alpha, \beta, H)$ and $Y_2 \sim LSE_1(\mu_2, \sigma_2, \delta_2, g, \alpha, \beta, H)$.

1. If $\mu_2 - \mu_1 + \beta(z)(\delta_2 - \delta_1) \geq 0$ for all $z$ and $\sigma_1 = \sigma_2$, then $Y_1 \leq_{st} Y_2$.
2. If $Y_1 \leq_{st} Y_2$ and $g$ satisfies Assumption 1 then $\mu_1 + E(\beta(z)) \delta_1 \leq \mu_2 + E(\beta(z)) \delta_2$ and $\sigma_1 = \sigma_2$.

**Proof.** 1. The implication follows Lemma 3.2.

2. If $Y_1 \leq_{st} Y_2$, then $EY_1 \leq EY_2$, obviously we have $\mu_1 + E(\beta(z)) \delta_1 \leq \mu_2 + E(\beta(z)) \delta_2$.

We claim $\sigma_1 = \sigma_2$. If $\sigma_1 \neq \sigma_2$, according to Assumption 1 we have

$$\lim_{y \to \pm \infty} \frac{r(y, z)}{p_1(y, z)} = C,$$

where $p_2(y, z) = \lim_{y \to \pm \infty} p_1(y, z) = C$. 

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Fig. 2: Survival functions and stoploss transformations of univariate GHSS distributions

where $C \in \mathbb{R}^+ \setminus \{1\}$. If $C \in [0, 1)$, then for sufficiently large positive $t$, $p_2(y, z) < p_1(y, z)$. Consider the CDF of $Y_1$ and $Y_2$, we have

$$F_2(t) = \int_{\mathbb{R}^q} \int_{t}^{+\infty} p_2(x, z) dx dH(z) < \int_{\mathbb{R}^q} \int_{t}^{+\infty} p_1(x, z) dx dH(z) = F_1(t),$$

which contradicts $Y_1 \leq_{sl} Y_2$. In parallel, if $C \in (1, +\infty]$, then for sufficiently large negative $t$, $p_2(y, z) > p_1(y, z)$. So

$$F_2(t) = \int_{\mathbb{R}^q} \int_{-\infty}^{t} p_2(x, z) dx dH(z) > \int_{\mathbb{R}^q} \int_{-\infty}^{t} p_1(x, z) dx dH(z) = F_1(t),$$

leads a contradiction to $Y_1 \leq_{sl} Y_2$. Hence, we conclude $\sigma_1 = \sigma_2$.

We consider the special case of univariate LSE distribution to illustrate the result of Lemma 4.1. Plots of the survival functions of two cases of GHSS distribution are shown in Figure 2. The GHSS distributed variables are ordered under usual stochastic order if the conditions in Lemma 4.1 (1) are satisfied. Also, Figure 2 (b) provides the counter example.

**Theorem 4.1.** Assume that

$$Y_1 \sim LSE_n(\mu_1, \Sigma_1, \delta_1, g, \alpha, \beta, H),$$

$$Y_2 \sim LSE_n(\mu_2, \Sigma_2, \delta_2, g, \alpha, \beta, H).$$

1. If $\mu_2 + \beta(z)\delta_2 \geq \mu_1 + \beta(z)\delta_1$ for all $z$ and $\Sigma_1 = \Sigma_2$, then $Y_1 \leq_{sl} Y_2$. 

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2. If \( Y_1 \leq_{st} Y_2 \) and the density generator of \( a'Y_1 \) satisfies Assumption [I] for all \( a \in \mathbb{R}^n \), then \( \mu_1 + E(\beta(z)) \delta_1 \leq \mu_2 + E(\beta(z)) \delta_2 \) and \( \Sigma_1 = \Sigma_2 \).

**Proof.**

1. The proof is routine and will be omitted.

2. It follows from \( Y_1 \leq_{st} Y_2 \) that \( Y_1 \leq_{plst} Y_2 \), which means \( Y_{1,i} \leq_{st} Y_{2,j} \) and \( Y_{1,j} + Y_{1,i} \leq_{st} Y_{2,j} + Y_{2,i} \) for all \( 1 \leq i, j \leq p \), where \( Y_{1,i}(Y_{2,i}) \) stands for the \( i \)-th component of \( Y_1(Y_2) \). Note that \( Y_1, Y_2 \) following (18) leads to

\[
\begin{align*}
Y_{1,i} & \sim LS E_1(\mu_{1,i}, \sigma_{1,i}, \delta_{1,j}, g, \alpha, \beta, H), \\
Y_{2,j} & \sim LS E_1(\mu_{2,j}, \sigma_{2,j}, \delta_{2,j}, g, \alpha, \beta, H),
\end{align*}
\]

(19)

(20)

Applying Lemma \[4.1\] then the desired result is obtained.

We know \( Y_1 \leq_{st} Y_2 \Rightarrow Y_1 \leq_{plst} Y_2 \), so if one change “\( \leq_{st} \)” to “\( \leq_{plst} \)” in Theorem \[4.1\] the result is still valid. For any weight vector \( \alpha = (\alpha_i) \in \mathbb{R}_n^+ \), let random variables \( S_1 = \sum_{i=1}^n \alpha_i Y_{1,i} \), and \( S_2 = \sum_{i=1}^n \alpha_i Y_{2,i} \). In actuarial science, we consider the components of \( Y_1 \) and \( Y_2 \) to be risks, then random variables \( S_1 \) and \( S_2 \) are seen as collective risks. An interesting result is that \( S_1 \) and \( S_2 \) can be compared under usual stochastic order as well.

**Corollary 1.** If \( \mu_2 + \beta(z) \delta_2 \geq \mu_1 + \beta(z) \delta_1 \) for all \( z \) and \( \Sigma_1 = \Sigma_2 \), then \( S_1 \leq_{st} S_2 \).

The following results generalizes Theorem 3.2. in Yin \[46\] and Proposition 5 in Jamali et al. \[27\] for the \( LSE \) distribution case.

**Theorem 4.2.** Let \( Y_1, Y_2 \) follows (18).

1. \( \mu_1 = \mu_2, \delta_1 = \delta_2 \) and \( \Sigma_2 - \Sigma_1 \) is positive semi-definite, then \( Y_1 \leq_{cx} Y_2 \).

2. If \( \mu_1 = \mu_2 \), then \( Y_1 \leq_{cx} Y_2 \) if and only if \( \delta_1 = \delta_2 \) and \( \Sigma_2 - \Sigma_1 \) is positive semi-definite.

3. If \( \delta_1 = \delta_2 \), then \( Y_1 \leq_{cx} Y_2 \) if and only if \( \mu_1 = \mu_2 \) and \( \Sigma_2 - \Sigma_1 \) is positive semi-definite.

**Proof.**

1. The proof is routine and will be omitted.

2. & 3. It can be derived from \( Y_1 \leq_{cx} Y_2 \) that \( EY_1 = EY_2 \); therefore, if we know \( \mu_1 = \mu_2 \), then \( \delta_1 = \delta_2 \) can be obtained and vice versa. We claim \( \Sigma_2 - \Sigma_1 \) is positive semi-definite. Otherwise, there exist \( a \in \mathbb{R}^n \) such that \( a' (\Sigma_2 - \Sigma_1) a < 0 \). Let \( f(x) = (a'x)^2 \), which is convex. According to Definition \[2.4\] we have \( E(a' Y_1' Y_1 a) \leq E(a' Y_1' Y_1 a) \). It can be derived by considering (16) that \( a' (\Sigma_2 - \Sigma_1) a \geq 0 \), which leads a contradiction.

We know \( Y_1 \leq_{cx} Y_2 \Rightarrow Y_1 \leq_{lcx} Y_2 \Leftrightarrow Y_1 \leq_{plst} Y_2 \), so if one change “\( \leq_{cx} \)” to “\( \leq_{lcx} \)” or “\( \leq_{plst} \)” in Theorem \[4.2\] the result is still valid.

Increasing convex order is also known as stop-loss order because of the fact that random variable \( X \leq_{lcx} Y \) if and only if their stop-loss transform \( E ((X - t)_+) \leq E ((Y - t)_+) \) for all \( t \in \mathbb{R} \). In the perspective of insurance, stop-loss order can be interpreted as a comparison of stop-loss premiums. The following theorem provides the necessary and sufficient conditions for increasing convex order of univariate \( LSE \) distribution and some related conditions of elliptical distribution can be found in Pan et al. \[39\].

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Lemma 4.2. Assume $Y_1 \sim LSE_1(\mu_1, \sigma_1, \delta_1, g, \alpha, \beta, H)$ and $Y_2 \sim LSE_1(\mu_2, \sigma_2, \delta_2, g, \alpha, \beta, H)$.

1. If $\mu_2 - \mu_1 + \beta(z) (\delta_2 - \delta_1) \geq 0$ for all $z$ and $\sigma_1 \leq \sigma_2$, then $Y_1 \leq_{icx} Y_2$;

2. If $Y_1 \leq_{icx} Y_2$ and $g$ satisfies Assumption 2, then $\mu_2 - \mu_1 + E\beta(z) (\delta_2 - \delta_1) \geq 0$ and $\sigma_1 \leq \sigma_2$.

Proof. 1. The implication follows Lemma 3.2.

2. $EY_1 \leq EY_2$ can be derived from $Y_1 \leq_{icx} Y_2$; therefore, $\mu_2 - \mu_1 + E\beta(z) (\delta_2 - \delta_1) \geq 0$. We claim $\sigma_1 \leq \sigma_2$. If $\sigma_1 > \sigma_2$, then $F_2(t) < F_1(t)$ for sufficiently large positive $t$ can be proved as shown in the proof of Lemma 4.1. Then, for sufficiently large positive $t$, we have

$$E(Y_1 - t)_+ = \int_t^{+\infty} F_1(x)dx > \int_t^{+\infty} F_2(x)dx = E(Y_2 - t)_+,$$

which leads a contradiction to $Y_1 \leq_{icx} Y_2$.

To illustrate Lemma 4.2 visually, the stop-loss premiums for the GHSS distributions are plotted in Figure 3 to graphically show that the variables are ordered under increasing convex order if the conditions in Lemma 4.2 (1) are satisfied. Also, Figure 3 (b) provides the counter example.

The following theorem generalizes Theorem 7 in Müller and Theorem 3.3 in Yin [46] for LSE distribution case. Some necessary and sufficient conditions for orderings of multinormal mean–variance mixture for was studied in Jamali et al. [27] but the sufficient condition for multivariate increasing convex order was not given. The following theorem filled this gap.

Theorem 4.3. Let $Y_1, Y_2$ follows (18).
1. If \( \mu_2 + \beta(z)\delta_2 \geq \mu_1 + \beta(z)\delta_1 \) for all \( z \) and \( \Sigma_2 - \Sigma_1 \) is positive semi-definite, then \( Y_1 \leq_{icx} Y_2 \):

2. If \( Y_1 \leq_{icx} Y_2 \) and the density generator of \( a'Y_1 \) satisfies Assumption 2 for all \( a \in \mathbb{R}^n \), then 
\[ \mu_2 + E\beta(z)\delta_2 \geq \mu_1 + E\beta(z)\delta_1 \] and \( \Sigma_2 - \Sigma_1 \) is copositive.

Proof. 1. The implication follows Lemma 3.2.

2. \( EY_1 \leq EY_2 \) can be derived from \( Y_1 \leq_{icx} Y_2 \); therefore, \( \mu_2 + E\beta(z)\delta_2 \geq \mu_1 + E\beta(z)\delta_1 \). \( Y_1 \leq_{icx} Y_2 \) means that \( Y_1 \leq_{iplex} Y_2 \), i.e. \( a'Y_1 \leq ax \) \( a'Y_2 \) is valid for all \( a \in \mathbb{R}^n \). From Lemma 3.1 we have \( a'Y_1 \sim \text{LS}E_1 (a'\mu_1, a'\Sigma_1 a, a'\delta_1, \psi, \alpha, \beta, H), i = 1, 2 \). Both \( a'Y_1 \) and \( a'Y_2 \) are quantities and the density generator they share satisfies Assumption 2, thus, \( a'\Sigma_2 a - a'\Sigma_1 a \geq 0 \) can be derived from Lemma 4.2.

It is worth noting that \( Y_1 \leq_{icx} Y_2 \) do not necessarily imply \( Y_1 \leq_{iplex} Y_2 \) for all distributions \( Y_1 \) and \( Y_1 \) follows. A counter example can be found in Pan et al. [39]. But \( Y_1 \leq_{icx} Y_2 \) do imply \( Y_1 \leq_{iplex} Y_2 \) because \( g_\theta(x) = g(a'x) \) is increasing convex for any increasing convex function \( g \) and \( a \in \mathbb{R}^n \). If one change every \( \leq_{icx} \) to \( \leq_{iplex} \) in Theorem 4.3, the result is still valid. As a result, the collective risks \( S_1 \) and \( S_2 \) can be compared under increasing convex order as well.

Corollary 2. If \( \mu_2 + \beta(z)\delta_2 \geq \mu_1 + \beta(z)\delta_1 \) for all \( z \) and \( \Sigma_2 - \Sigma_1 \) is PSD, then \( S_1 \leq_{st} S_2 \).

The following result generalizes Theorem 12 in Müller [35], Theorem 3.6 in Yin [46] and Proposition 6 in Amiri et al. [3] in which the multivariate \( \text{LS}E \) case was considered.

Theorem 4.4. Let \( Y_1, Y_2 \) follows (18).

1. \( \mu_1 = \mu_2, \delta_1 = \delta_2 \) and \( \Sigma_2 \geq \Sigma_1 \), then \( Y_1 \leq_{dcrx} Y_2 \);

2. If \( \mu_1 = \mu_2 \), then \( Y_1 \leq_{dcrx} Y_2 \) if and only if \( \delta_1 = \delta_2 \) and \( \Sigma_2 \geq \Sigma_1 \);

3. If \( \delta_1 = \delta_2 \), then \( Y_1 \leq_{dcrx} Y_2 \) if and only if \( \mu_1 = \mu_2 \) and \( \Sigma_2 \geq \Sigma_1 \).

Proof. 1. The proof is routine and will be omitted.

2. & 3. Note that the functions \( f_i(x) = x_i \) and \( f_2(x) = -x_i \) are directionally convex for all \( 1 \leq i \leq n \); therefore, \( EY_1 = EY_2 \). Then the equivalence between \( \delta_1 \) and \( \delta_2 \) (alternatively, \( \mu_1 \) and \( \mu_2 \)) can be established by using the same method in the proof of Theorem 4.2.

Let \( f_j(x) = x_jx_i \), which is directionally convex for all \( 1 \leq i, j \leq n \). It can be derived that 
\[ \text{Cov}(Y_1) \leq \text{Cov}(Y_2) \], then we claim \( \Sigma_2 \geq \Sigma_1 \) on the ground that \( \delta_1 = \delta_2 \).

The following theorem considers the componentwise convex order. The multinormal case can be found in Arlotto and Scarsini [5] while the multivariate elliptical case can be found in Yin [46].

Theorem 4.5. Let \( Y_1, Y_2 \) follows (18).

1. If \( \mu_1 = \mu_2, \delta_1 = \delta_2, \sigma_{1,ii} \leq \sigma_{2,ii} \) for \( 1 \leq i \leq n \) and \( \sigma_{1,ij} = \sigma_{2,ij} \) for \( 1 \leq i < j \leq n \) then 
\( Y_1 \leq_{ccx} Y_2 \);

2. If \( \mu_1 = \mu_2 \), then \( Y_1 \leq_{ccx} Y_2 \) if and only if \( \delta_1 = \delta_2, \sigma_{1,ii} \leq \sigma_{2,ii} \) for \( 1 \leq i \leq n \) and \( \sigma_{1,ij} = \sigma_{2,ij} \) for \( 1 \leq i < j \leq n \).
3. If \( \delta_1 = \delta_2 \), then \( Y_1 \preceq_{cx} Y_2 \) if and only if \( \mu_1 = \mu_2 \), \( \sigma_{1,ii} \leq \sigma_{2,ii} \) for \( 1 \leq i \leq n \) and \( \sigma_{1,ij} = \sigma_{2,ij} \) for \( 1 \leq i < j \leq n \).

**Proof.** 1. The proof is routine and will be omitted.

2. & 3. Note that the functions \( f_1(x) = x_i \) and \( f_2(x) = -x_i \) are componentwise convex for all \( 1 \leq i \leq n \); therefore, \( EY_1 = EY_2 \). Then the equivalence between \( \delta_1 \) and \( \delta_2 \) (alternatively, \( \mu_1 \) and \( \mu_2 \)) can be established by using the same method in the proof of Theorem 4.2.

Let \( f_3(x) = x_i x_j \), \( f_4(x) = -x_i x_j \) and \( f_5(x) = x_i^2 \), they are all componentwise convex for all \( 1 \leq i < j \leq n \). Thus, we get \( \sigma_{1,ii} \leq \sigma_{2,ii} \) for \( 1 \leq i \leq n \) and \( \sigma_{1,ij} = \sigma_{2,ij} \) for \( 1 \leq i < j \leq n \) by considering (16). \( \square \)

Supermodular orders are important for a wide range of scientific and industrial processes. Several practical applications for supermodular orders, like applications in genetic selection, are presented in Bäuerle [13]. The following result generalizes Theorem 11 in Müller [15] for the multivariate normal case to the setting considered here.

**Theorem 4.6.** Let \( Y_1, Y_2 \) follow (18). \( Y_1 \preceq_{sm} Y_2 \) if and only if \( Y_1 \) and \( Y_2 \) have the same marginals and \( \sigma_{1,ij} \leq \sigma_{2,ij} \) for all \( 1 \leq i < j \leq n \).

**Proof.** Suppose \( Y_1 \preceq_{sm} Y_2 \). It can hold only if the random vectors have the same marginals, which means \( \mu_1 = \mu_2 \), \( \delta_1 = \delta_2 \) and \( \sigma_{1,ii} = \sigma_{2,ii} \) for any \( 1 \leq i \leq n \). Since the function \( f(x) = x_i x_j \) is supermodular for all \( 1 \leq i \neq j \leq n \), we see \( Y_1 \preceq_{sm} Y_2 \) implies \( \sigma_{1,ij} \leq \sigma_{2,ij} \) for all \( 1 \leq i \neq j \leq n \). Then Lemma 3.2 yields the converse, and hence the result. \( \square \)

From the perspective of correlation, Theorem 4.6 can be presented as

**Corollary 3.** Let \( Y_1 \sim LSE_n (\mu_1, \rho_1, \delta_1, g, \alpha, \beta, H) \) and \( Y_2 \sim LSE_n (\mu_2, \rho_2, \delta_2, g, \alpha, \beta, H) \) where \( \rho_1 \) and \( \rho_2 \) are correlation matrices. Then \( Y_1 \preceq_{sm} Y_2 \) if and only if \( \mu_1 = \mu_2 \), \( \delta_1 = \delta_2 \) and \( \rho_{1,ij} \leq \rho_{2,ij} \) for all \( 1 \leq i < j \leq n \).

The upper orthant order, given in Definition 2.4, can also be defined through a comparison of upper orthants, which means \( Y_1 \preceq_{uo} Y_2 \) if and only if \( P(Y_1 > t) \leq P(Y_2 > t) \) holds for all \( t \). These two definitions can be shown to be equivalent. The following lemma, which is presented in Müller and Scarsini [36], provides the fact that there is no difference between the upper orthant order and the supermodular order in the bivariate case.

**Lemma 4.3.** (Müller and Scarsini [36]) Let \( X, Y \) be two bivariate random vectors and have the same marginals, then \( X \preceq_{sm} Y \) is equivalent to \( X \preceq_{uo} Y \).

The following theorem provides conditions for comparing \( LSE \) distributed vectors under the upper orthant order.

**Theorem 4.7.** Let \( Y_1, Y_2 \) follow (18).

1. If \( \mu_1 + \beta(z) \delta_2 \geq \mu_1 + \beta(z) \delta_1 \) for all \( z \), \( \sigma_{1,ii} = \sigma_{2,ii} \) for all \( 1 \leq i \leq n \) and \( \sigma_{1,ij} \leq \sigma_{2,ij} \) for all \( 1 \leq i \neq j \leq n \), then \( Y_1 \preceq_{uo} Y_2 \);
2. If $Y_1 \leq_{wo} Y_2$ and $g$ satisfies Assumption [1], then $\mu_1 + E(\beta(z)) \delta_1 \leq \mu_2 + E(\beta(z)) \delta_2$ and $\sigma_{1,ii} = \sigma_{2,ii}$ for all $1 \leq i \leq n$.

3. If $Y_1$ and $Y_2$ have the same marginals and $Y_1 \leq_{wo} Y_2$, then $\sigma_{1,ij} \leq \sigma_{2,ij}$ for all $1 \leq i < j \leq n$.

Proof. 1. For any $\Delta$-monotone function $f$, $\nabla f(x) \geq 0$ and the off-diagonal elements in $H_f(x)$ are greater than 0. Then the results can be derived by using Lemma 3.2.

2. $Y_1 \leq_{wo} Y_2$ implies their components $Y_{1,i} \leq_{wo} Y_{2,i}$ for all $1 \leq i \leq n$. Note that $Y_1, Y_2$ following (18) leads to (19) and (20). Then the desired results can be proved by applying Lemma 4.1.

3. $Y_1 \leq_{wo} Y_2$ implies that $(Y_{1,i}, Y_{1,j})' \leq_{wo} (Y_{2,i}, Y_{2,j})'$, where $1 \leq i < j \leq n$ and it is quite obvious that $(Y_{1,i}, Y_{1,j})'$ and $(Y_{2,i}, Y_{2,j})'$ have the same marginals. It can be derived from Lemma 4.3 that $(Y_{1,i}, Y_{1,j})' \leq_{wo} (Y_{1,i}, Y_{1,j})'$. Then the required result follows from Theorem 4.6.

At the end of this section, we will consider the copositive and completely positive orders for random vectors follow multivariate LSE distribution. The multivariate normal and elliptical case can be found in Arlotto and Scarsini [5] and Yin [46].

**Theorem 4.8.** Let $Y_1, Y_2$ follows (18).

1. If $\mu_1 = \mu_2$, $\delta_1 = \delta_2$ and $\Sigma_2 - \Sigma_1$ is copositive, then $Y_1 \leq_{cp} Y_2$.

2. If $\mu_1 = \mu_2$, then $Y_1 \leq_{cop} Y_2$ if and only if $\delta_1 = \delta_2$ and $\Sigma_2 - \Sigma_1$ is copositive.

3. If $\delta_1 = \delta_2$, then $Y_1 \leq_{cop} Y_2$ if and only if $\mu_1 = \mu_2$ and $\Sigma_2 - \Sigma_1$ is copositive.

Proof. 1. For any function $f$ such that $H_f(x) \in C_{cp}$, using Lemma 3.2, together with (10), yields $E(f(Y_1)) \leq f(Y_2)$, which means that $Y_1 \leq_{cp} Y_2$.

2.-3. Note that the Hessian matrices of functions $f_1(x) = x_i$ and $f_2(x) = -x_i$ are completely positive for all $1 \leq i \leq n$. Thus, $Y_1 \leq_{cp} Y_2$ implies $\mu_1 + E\beta(z)\delta_2 = \mu_1 + E\beta(z)\delta_1$. If we know $\mu_1 = \mu_2$, then $\delta_1 = \delta_2$ can be obtained as well and vice versa. For any symmetric $n \times n$ matrix $A \in C_{cp}$, let

$$f_0(x) = \frac{1}{2}(x - EY_1)'A(x - EY_1).$$

Notice the fact that $H_f(x) = A$ for all $x \in \mathbb{R}^n$, and thus $Y_1 \leq_{cp} Y_2$ implies

$$E((Y_1 - EY_1)'A(Y_1 - EY_1)) \leq E((Y_2 - EY_2)'A(Y_2 - EY_2)).$$

This, together with (16), allows us to get $tr((\Sigma_2 - \Sigma_1)A) \geq 0$. Since $A \in C_{cp}$ is arbitrarily chosen, we conclude that $\Sigma_2 - \Sigma_1 \in C_{cp}$, i.e. $\Sigma_2 - \Sigma_1$ is copositive.

**Theorem 4.9.** Let $Y_1, Y_2$ follows (18).

1. If $\mu_1 = \mu_2$, $\delta_1 = \delta_2$ and $\Sigma_2 - \Sigma_1$ is completely positive, then $Y_1 \leq_{cop} Y_2$.

2. If $\mu_1 = \mu_2$, then $Y_1 \leq_{cop} Y_2$ if and only if $\delta_1 = \delta_2$ and $\Sigma_2 - \Sigma_1$ is completely positive.
3. If $\delta_1 = \delta_2$, then $Y_1 \leq_{cop} Y_2$, if and only if $\mu_1 = \mu_2$ and $\Sigma_2 - \Sigma_1$ is completely positive.

**Proof.** The proof of Theorem 4.9 is in parallel with the proof of Theorem 4.8, so we omit it. \hfill \square

In this section, we mainly study the conditions of stochastic orderings for location-scale mixture of elliptical distributions. If one set $\delta = 0$, then LSE distribution will degenerate to SME distribution and all the results in this section could be simplified. Let $Y_i \sim SME_2(\mu_i, \Sigma_i, \psi, \alpha, H)$, $i = 1, 2$, the conditions of stochastic orderings for scale mixture of elliptical distributions are presented in Table 2. As we mentioned in Section 1 that SME distributions are still elliptical, so the results in Table 2 can be considered as corollaries of the results in Yin [46] as well.

| Conditions                        | Relationships | Orders          |
|-----------------------------------|---------------|-----------------|
| $\mu_1 \leq \mu_2, \Sigma_1 = \Sigma_2$ | $\Leftrightarrow$ | $Y_1 \leq_{st} (\leq_{plst}) Y_2$ |
| $\mu_1 = \mu_2, \Sigma_2 - \Sigma_1$ is PSD | $\Leftrightarrow$ | $Y_1 \leq_{cx} (\leq_{icx}) Y_2$ |
| $\mu_1 \leq \mu_2, \Sigma_2 - \Sigma_1$ is copositive | $\Leftrightarrow$ | $Y_1 \leq_{icx} (\leq_{iplcx}) Y_2$ **|
| $\mu_1 \leq \mu_2, \Sigma_2 - \Sigma_1$ is PSD | $\Rightarrow$ | $Y_1 \leq_{icx} (\leq_{iplcx}) Y_2$ **|
| $\mu_1 = \mu_2, \sigma_{1,ij} \leq \sigma_{2,ij}$ | $\Leftrightarrow$ | $Y_1 \leq_{cx} Y_2$ |
| $\mu_1 = \mu_2, \sigma_{1,ii} \leq \sigma_{2,ii}, \sigma_{1,ij} = \sigma_{2,ij}$ | $\Leftrightarrow$ | $Y_1 \leq_{icx} Y_2$ |
| $\mu_1 = \mu_2, \sigma_{1,ii} = \sigma_{2,ii}, \sigma_{1,ij} \leq \sigma_{2,ij}$ | $\Leftrightarrow$ | $Y_1 \leq_{im} Y_2$ |
| $\mu_1 \leq \mu_2, \sigma_{1,ii} = \sigma_{2,ii}, \sigma_{1,ij} \leq \sigma_{2,ij}$ | $\Rightarrow$ | $Y_1 \leq_{ao} Y_2$ |
| $\mu_1 \leq \mu_2, \Sigma_2 - \Sigma_1$ is copositive | $\Leftrightarrow$ | $Y_1 \leq_{cx} Y_2$ |
| $\mu_1 = \mu_2, \Sigma_2 - \Sigma_1$ is completely positive | $\Leftrightarrow$ | $Y_1 \leq_{cop} Y_2$ |

* Here we assume the density generator of $aY_1$ satisfies Assumption 1 for all $a \in \mathbb{R}^n$.

** Here we assume the density generator of $aY_1$ satisfies Assumption 2 for all $a \in \mathbb{R}_+^n$.

5. Concluding Remarks

To conclude the article, we discuss here several interesting topics for future study. First, the assumptions for density generator $g$ proposed here is not strict but can still be simplified, and it is of low probability that the results in this paper can be derived with no prior assumptions of $g$. Then finding sufficient assumptions for $g$ should be a challenging and interesting topic for future study. Second, it will naturally be of interest to further generalize results established in this paper to other families of distributions such as skew-elliptical distributions.

Appendix

In this section, we prove that all the density generators presented in Table 1 follow Assumptions 1 and 2. Let $g^1 = (1 + \frac{u}{m})^{-(a+m)/2}$, where $m$ is a positive integer; $g^2(u) = \exp(-\frac{1}{2}(u)^{1/2})$, where $s > 1; g^3(u) = \exp(-u) \left(1 + \exp(-u)\right)^{-2}$. It is obvious that Cauchy distribution is a special case of Student distribution as normal distribution and Laplace distribution are special cases of exponential power distribution, so we just need to prove the aforementioned three density generators follow Assumptions 1 and 2.
Proof. For \( g^1 \), we have
\[
\lim_{t \to \pm \infty} \frac{\sigma_1}{\sigma_2} \frac{g^1(t_2)}{g^1(t_1^2)} = \lim_{t \to \pm \infty} \frac{\sigma_1}{\sigma_2} \left( \frac{m + t_2^1}{m + t_1^1} \right)^{-\frac{n+1}{2}} = \frac{\sigma_1}{\sigma_2} \left( \lim_{t \to \pm \infty} \frac{m + t_2^1}{m + t_1^1} \right)^{-\frac{n+1}{2}} = \left( \frac{\sigma_2}{\sigma_1} \right)^m \neq 1.
\]
If \( \sigma_1 > \sigma_2 \), then \( \left( \frac{\sigma_2}{\sigma_1} \right)^m < 1 \). For \( g^2 \), we have
\[
\lim_{t \to \pm \infty} \frac{\sigma_1}{\sigma_2} \frac{g^2(t_2)}{g^2(t_1)} = \lim_{t \to \pm \infty} \sigma_1 \exp \left( \frac{1}{s} (t_1^1 - t_2^1) \right) = \lim_{t \to \pm \infty} \sigma_1 \exp \left( \frac{1}{s} \left( \frac{1}{\sigma_1} - \frac{1}{\sigma_2} \right) t_1^1 \right).
\]
If \( \sigma_1 > \sigma_2 \), then \( \lim_{t \to \pm \infty} \frac{\sigma_1}{\sigma_2} \frac{g^2(t_2)}{g^2(t_1)} \) goes to zero otherwise goes to infinity.

For \( g^3 \), we have
\[
\lim_{t \to \pm \infty} \frac{\sigma_1}{\sigma_2} \frac{g^3(t_2)}{g^3(t_1)} = \lim_{t \to \pm \infty} \frac{\sigma_1}{\sigma_2} \exp \left( - \frac{t_2^3}{2} \right) \left( 1 + \exp \left( - \frac{t_1^3}{2} \right) \right)^2.
\]
We have
\[
\lim_{t \to \pm \infty} \left( 1 + \exp \left( - \frac{t_2^3}{2} \right) \right)^2 = 1
\]
If \( \sigma_1 > \sigma_2 \), then \( \lim_{t \to \pm \infty} \exp \left( t_1^2 - t_2^2 \right) \) goes to zero otherwise goes to infinity. So \( \lim_{t \to \pm \infty} \frac{\sigma_1}{\sigma_2} \frac{g^3(t_2)}{g^3(t_1)} \) behaves the same way. \( \square \)

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