Geometric representation of graphs in low dimension using axis parallel boxes

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Abstract

An axis-parallel $k$-dimensional box is a Cartesian product $R_1 \times R_2 \times \cdots \times R_k$ where $R_i$ (for $1 \leq i \leq k$) is a closed interval of the form $[a_i, b_i]$ on the real line. For a graph $G$, its boxicity $\text{box}(G)$ is the minimum dimension $k$, such that $G$ is representable as the intersection graph of (axis-parallel) boxes in $k$-dimensional space. The concept of boxicity finds applications in various areas such as ecology, operations research etc.

A number of NP-hard problems are either polynomial time solvable or have much better approximation ratio on low boxicity graphs. For example, the max-clique problem is polynomial time solvable on bounded boxicity graphs and the maximum independent set problem for boxicity 2 graphs has a $1 + \frac{1}{e} \log n$ approximation ratio for any constant $c$. In most cases, the first step usually is computing a low dimensional box representation of the given graph. Deciding whether the boxicity of a graph is at most 2 itself is NP-hard.

We give an efficient randomized algorithm to construct a box representation of any graph $G$ on $n$ vertices in $[(\Delta + 2) \ln n]$ dimension, where $\Delta$ is the maximum degree of $G$. This algorithm implies that $\text{box}(G) \leq [(\Delta + 2) \ln n]$ for any graph $G$. Our bound is tight up to a factor of $\ln n$.

We also show that our randomized algorithm can be derandomized to get a polynomial time deterministic algorithm.

Though our general upper bound is in terms of maximum degree $\Delta$, we show that for almost all graphs on $n$ vertices, their boxicity is $O(d_{av} \ln n)$ where $d_{av}$ is the average degree.

Key words: Boxicity, randomized algorithm, derandomization, random graph, intersection graphs

Preprint submitted to Elsevier
1 Introduction

Let $\mathcal{F} = \{S_x \subseteq U : x \in V\}$ be a family of subsets of a universe $U$, where $V$ is an index set. The intersection graph $\Lambda(\mathcal{F})$ of $\mathcal{F}$ has $V$ as vertex set, and two distinct vertices $x$ and $y$ are adjacent if and only if $S_x \cap S_y \neq \emptyset$. Representations of graphs as the intersection graphs of various geometrical objects is a well studied topic in graph theory. Probably the most well studied class of intersection graphs are the interval graphs, where each $S_x$ is a closed interval on the real line.

A well known concept in this area of graph theory is the boxicity, which was introduced by F. S. Roberts in 1969 [16]. This concept generalizes the concept of interval graphs. A $k$–dimensional box is a Cartesian product $R_1 \times R_2 \times \cdots \times R_k$ where $R_i$ (for $1 \leq i \leq k$) is a closed interval of the form $[a_i, b_i]$ on the real line. For a graph $G$, its boxicity is the minimum dimension $k$, such that $G$ is representable as the intersection graph of (axis–parallel) boxes in $k$–dimensional space. We denote the boxicity of a graph $G$ by $\text{box}(G)$. The graphs of boxicity 1 are exactly the class of interval graphs. The boxicity of a complete graph is 0 by definition.

It was shown by Cozzens [10] that computing the boxicity of a graph is NP–hard. This was later improved by Yannakakis [21], and finally by Kratochvil [15] who showed that deciding whether the boxicity of a graph is at most 2 itself is NP–complete.

In many algorithmic problems related to graphs, the availability of certain convenient representations turns out to be extremely useful. Probably, the most well-known and important examples are the tree decompositions and path decompositions [5]. Many NP-hard problems are known to be polynomial time solvable given a tree(path) decomposition of bounded width for the input graph. Similarly, the representation of graphs as intersections of “disks” or “spheres” lies at the core of solving problems related to frequency assignments in radio networks, computing molecular conformations etc. For the maximum independent set problem which is hard to approximate within a factor of $n^{(1/2)-\epsilon}$ for general graphs [13], a PTAS is known for disk graphs given the disk representation [111]. In a similar way, the availability of box representation in low dimension make some well known NP hard problems polynomial time solvable. For example, the max-clique problem is polynomial time solvable in boxicity $k$ graphs since there are only $O((2n)^k)$ maximal cliques in such graphs. It was shown in [13] that the complexity of finding the maximum independent set is hard to approximate within a factor $n^{(1/2)-\epsilon}$ for general graphs. In fact, [13] gives the stronger inapproximability result of $n^{1-\epsilon}$, for any $\epsilon > 0$, under the assumption that $NP \neq ZPP$. Though this problem is NP-hard even for boxicity 2 graphs, it is approximable to a log $n$ factor for boxicity 2 graphs.
Thus, it is interesting to design efficient algorithms to represent small boxicity graphs in low dimensions. Roberts [16] had given a general upper bound of $n/2$ for the boxicity of any graph on $n$ vertices. In this paper, we show an upper bound of $\lceil(\Delta + 2)\ln n\rceil$ for the boxicity for any graph $G$ on $n$ vertices and having maximum degree $\Delta$ by giving a randomized algorithm that yields a box representation for $G$ in $\lceil(\Delta + 2)\ln n\rceil$ dimension in $O(\Delta n^2 \ln^2 n)$ time with high probability. We also derandomize our randomized algorithm and obtain a deterministic polynomial time algorithm to do the same. Very recently, the authors had shown that for any graph $G$ with maximum degree $\Delta$, $\text{box}(G) \leq 2\Delta^2$ [23]. However, it may be noted that the result in this paper yields better bounds on boxicity for graphs where $\Delta \geq \ln n$.

In a recent manuscript [7] the authors showed that for any graph $G$, $\text{box}(G) \leq \text{tw}(G) + 2$, where $\text{tw}(G)$ is the treewidth of $G$. This result implies that the class of ‘low boxicity’ graphs properly contains the class of ‘low treewidth graphs’. It is well known that almost all graphs on $n$ vertices and $m = cn$ edges (for a sufficiently large constant $c$) have $\Omega(n)$ treewidth [14]. In this paper we show that almost all graphs on $n$ vertices and $m$ edges have boxicity $O(d\text{av}\ln n)$ where $d\text{av} = 2m/n$. An implication of this result is that for almost all graphs on $m = cn$ edges, there is an exponential gap between their boxicity and treewidth. Hence it is interesting to reconsider those NP-hard problems that are polynomial time solvable in bounded treewidth graphs and see whether they are also polynomial time solvable for bounded boxicity graphs.

Researchers have also tried to bound the boxicity of graph classes with special structure. Scheinerman [17] showed that the boxicity of outer planar graphs is at most 2. Thomassen [19] proved that the boxicity of planar graphs is bounded above by 3. Upper bounds for the boxicity of many other graph classes such as chordal graphs, AT-free graphs, permutation graphs etc. were shown in [7] by relating the boxicity of a graph with its treewidth. Researchers have also tried to generalize or extend the concept of boxicity in various ways. The poset boxicity [20], the rectangle number [8], grid dimension [3], circular dimension [12,18] and the boxicity of digraphs [9] are some examples.

1.1 Our Results

We summarize below the results of this paper.

1. We show that for any graph $G$ on $n$ vertices with maximum degree $\Delta$, $\text{box}(G) \leq \lceil(\Delta + 2)\ln n\rceil$. This bound is tight up to a factor of $2\ln n$.
2. In fact, we show a randomized algorithm to construct a box representation of $G$ in $\lceil(\Delta + 2)\ln n\rceil$ dimension, that runs in $O(\Delta n^2 \ln^2 n)$ time with high
probability.

(3) Next we show a polynomial time deterministic algorithm to construct a box representation in \([(\Delta + 2)\ln n]\) dimension by derandomizing the above randomized algorithm.

(4) Though the general upper bound that we show is in terms of the maximum degree \(\Delta\), we also investigate the relation between boxicity and average degree. We show that for almost all graphs on \(n\) vertices and \(m\) edges, the boxicity is \(O(d_{av}\ln n)\), where \(d_{av} = 2m/n\) is the average degree.

(5) We also derive an upper bound for boxicity in terms of \(m\) and \(n\). We show that for any connected graph \(G\), \(\text{box}(G) \leq 5\sqrt{m\ln n}\), which is tight up to a factor of \(b\sqrt{\ln n}\) for a constant \(b\).

1.2 Definitions and Notations

Let \(G\) be an undirected simple graph on \(n\) vertices. The vertex set of \(G\) is denoted as \(V(G) = \{1, \ldots, n\}\) (or \(V\) in short). Let \(E(G)\) denote the edge set of \(G\). We denote by \(\overline{G}\), the complement of \(G\). We say that the edge \(e\) is missing in \(G\), if \(e \in E(\overline{G})\). A graph \(G'\) is said to be a supergraph of \(G\) where \(V(G) = V(G')\), if \(E(G) \subseteq E(G')\). For a vertex \(u \in V\), let \(N(u)\) denote the set of neighbors of \(u\) in \(G\) i.e., \(N(u) = \{v \in V \mid u \neq v \text{ and } (u, v) \in E(G)\}\). Let \(d(u)\) denote the degree of \(u\) in \(G\), i.e. \(d(u) = |N(u)|\). Let \(\Delta\) denote the maximum degree of \(G\).

**Definition 1 (Projection)** Let \(\pi\) be a permutation of the set \(\{1, \ldots, n\}\). Let \(X \subseteq \{1, \ldots, n\}\). The projection of \(\pi\) onto \(X\) denoted as \(\pi_X\) is defined as follows. Let \(X = \{u_1, \ldots, u_r\}\) such that \(\pi(u_1) < \pi(u_2) < \ldots < \pi(u_r)\). Then \(\pi_X(u_1) = 1, \pi_X(u_2) = 2, \ldots, \pi_X(u_r) = r\).

**Definition 2 (Interval Representation)** An interval graph can be represented as the intersection graph of closed intervals on the real line \(\mathbb{R}\). Such a representation is called an interval representation. Equivalently and in fact more precisely, an interval representation of an interval graph \(G\) can be defined by two functions \(l: V \to \mathbb{R}\) and \(r: V \to \mathbb{R}\). The interval corresponding to a vertex \(v\) denoted as \(I(v)\) is given by \([l(v), r(v)]\), where \(l(v)\) and \(r(v)\) are the left and right end points of the interval corresponding to \(v\).

**Definition 3** Given a graph \(G\) and a permutation \(\pi\) of its vertices \(\{1, \ldots, n\}\), we define a map \(M(G, \pi)\), which associates to the pair \((G, \pi)\), an interval supergraph \(G'\) of \(G\), as follows: consider any vertex \(u \in V(G)\). Let \(n_u \in N(u) \cup \{u\}\) be the vertex such that \(\pi(n_u) = \min_{w \in N(u) \cup \{u\}} \pi(w)\). Then associate the interval \([\pi(n_u), \pi(u)]\) to the vertex \(u\). Let \(G'\) be the resulting interval graph. It is easy to verify that \(G'\) is a super graph of \(G\). We define \(M(G, \pi) = G'\).
1.3 Box Representation and Interval Graph Representation

Let $G$ be a graph and let $I_1, \ldots, I_k$ be $k$ interval graphs such that each $I_j$ is defined on the same set of vertices $V$. That is, $V(I_j) = V(G)$ for $j = 1, \ldots, k$. If

$$E(G) = E(I_1) \cap \cdots \cap E(I_k),$$

then we say that $I_1, \ldots, I_k$ is an interval graph representation of $G$. The following equivalence is well-known.

**Theorem 4 (Roberts [16])** Let $G$ be a simple, undirected, non-complete graph. Then, the minimum $k$ such that there exists an interval graph representation of $G$ using $k$ interval graphs $I_1, \ldots, I_k$ is the same as $\text{box}(G)$.

Recall that a $k$–dimensional box representation of $G$ is a mapping of each vertex $u \in V$ to $R_1(u) \times \cdots \times R_k(u)$, where each $R_i(u)$ is a closed interval of the form $[\ell_i(u), r_i(u)]$ on the real line. It is straightforward to see that an interval graph representation of $G$ using $k$ interval graphs $I_1, \ldots, I_k$, is equivalent to a $k$–dimensional box representation in the following sense. Let $R_i(u) = [\ell_i(u), r_i(u)]$ denote the closed interval corresponding to vertex $u$ in an interval representation of $I_i$. Then the $k$–dimensional box corresponding to $u$ is simply $R_1(u) \times \cdots \times R_k(u)$. Conversely, given a $k$–dimensional box representation of $G$, the set of intervals $\{R_i(u) : u \in V\}$ forms the $i$th interval graph $I_i$ in the corresponding interval graph representation.

When we say that a box representation in $t$ dimension is output by an algorithm, the algorithm actually outputs the interval graph representation: that is, the interval representation of the constituent interval graphs.

2 The randomized construction

Consider the following randomized procedure $\text{RAND}$ which outputs an interval super graph of $G$. Let $\Delta$ be the maximum degree of $G$.

\text{RAND}

Input: $G$.

Output: $G'$ which is an interval super graph of $G$.

begin
step 1. Generate a permutation \( \pi \) of \( \{1, \ldots, n\} \) uniformly at random.

step 2. Return \( G' = \mathcal{M}(G, \pi) \).

end.

**Lemma 5**  Let \( e = (u, v) \notin E(G) \). Let \( G' \) be the output of \( \text{RAND}(G) \). Then,

\[
\Pr[e \in E(G')] = \frac{1}{2} \left( \frac{d(u)}{d(u)+2} + \frac{d(v)}{d(v)+2} \right) \leq \frac{\Delta}{\Delta + 2}.
\]

**PROOF.**

We have to estimate the probability that \( u \) and \( v \) are adjacent in \( G' \). That is, \( I(u) \cap I(v) \neq \emptyset \).

Let \( n_u \in N(u) \cup \{u\} \) be a vertex such that \( \pi(n_u) = \min_{w \in N(u) \cup \{u\}} \pi(w) \). Similarly, let \( n_v \in N(v) \cup \{v\} \) be a vertex such that \( \pi(n_v) = \min_{w \in N(v) \cup \{v\}} \pi(w) \).

Clearly, \( I(u) = [\pi(n_u), \pi(u)] \) and \( I(v) = [\pi(n_v), \pi(v)] \). Thus, it is easy to see that \( I(u) \cap I(v) \neq \emptyset \) if (a) \( \pi(n_u) < \pi(v) < \pi(u) \) or (b) \( \pi(n_u) < \pi(u) < \pi(v) \).

On the other hand, \( I(u) \cap I(v) \neq \emptyset \) only if either (a) or (b) hold. To see this, first observe that \( \pi(u) \neq \pi(w) \) for any \( w \in N(v) \cup \{v\} \) and \( \pi(v) \neq \pi(w) \) for any \( w \in N(u) \cup \{u\} \). Without loss of generality, let \( \pi(u) < \pi(v) \). Now, it is obvious that \( I(u) \cap I(v) \neq \emptyset \), \( \pi(n_v) < \pi(u) < \pi(v) \) since \( \pi(n_v) \neq \pi(u) \). Since (a) and (b) are mutually exclusive,

\[
\Pr[e \in E(G')] = \Pr[\pi(n_u) < \pi(v) < \pi(u)] + \Pr[\pi(n_u) < \pi(u) < \pi(v)].
\]

We bound \( \Pr[\pi(n_u) < \pi(v) < \pi(u)] \) as follows. Let \( X = \{u\} \cup N(u) \cup \{v\} \). Let \( \pi_X \) be the projection of \( \pi \) onto \( X \). Clearly, the event \( \pi(n_u) < \pi(v) < \pi(u) \) translates to saying that \( \pi_X(v) < \pi_X(u) \) and \( \pi_X(v) \neq 1 \). Note that \( \pi_X \) can be any permutation of \( |X| \) elements with equal probability, which is \( \frac{1}{d(u)+2)!} \). The number of permutations where \( \pi_X(v) < \pi_X(u) \) equals \( (d(u)+2)!/2 \). Moreover, the number of permutations where \( \pi_X(v) = 1 \) equals \( (d(u)+1)! \). Note that the set of permutations with \( \pi_X(v) = 1 \) is a subset of the set of permutations with \( \pi_X(v) < \pi_X(u) \). It follows that

\[
\Pr[\pi_X(v) < \pi_X(u) \text{ and } \pi_X(v) \neq 1] = \frac{(d(u)+2)!/2 - (d(u)+1)!}{(d(u)+2)!}.
\]

which is \( \frac{d(u)}{2(d(u)+2)!} \). Using similar arguments, it follows that

\[
\Pr[\pi(n_v) < \pi(u) < \pi(v)] = \frac{d(v)}{2(d(v)+2)}.
\]

Summing the two bounds, the result follows.
Lemma 6 Let $I_1, I_2, \cdots, I_t$ be the output generated by $t$ invocations of $\text{RAND}(G)$. If $t \geq (\Delta + 2) \ln n$ then $\Pr[E(G) = E(I_1) \cap E(I_2) \cap \cdots \cap E(I_t)] \geq \frac{1}{2}$.

**PROOF.** For $e = (u, v) \notin E(G)$, let $Z_e$ denote the event

$$e \in E(I_1) \land e \in E(I_2) \ldots \land e \in E(I_t)$$

That is, $Z_e$ denote the event that $e \in E(I_1) \cap E(I_2) \cdots \cap E(I_t)$. Note that for any $i, j \in \{1, \ldots, t\}$, $i \neq j$, the events $e \in E(I_i)$ and $e \in E(I_j)$ are independent. It follows from Lemma 5 that

$$\Pr[Z_e] = \prod_{j \in \{1, \ldots, t\}} \Pr[e \in E(I_j)] \leq \left(\frac{\Delta}{\Delta + 2}\right)^t.$$

Hence by union bound,

$$\Pr\left[\bigvee_{e \notin E(G)} Z_e\right] \leq \sum_{e \notin E(G)} \Pr[Z_e] \leq \frac{n^2}{2} \left(\frac{\Delta}{\Delta + 2}\right)^t \leq \left(n^2/2\right)e^{\frac{2t}{\Delta + 2}}.$$

If we choose $t = (\Delta + 2) \ln n$ then the above probability is upper bounded by $1/2$. Recall that $E(I_j) \subseteq E(G)$ for all $j$. It follows that $\Pr\left[E(G) = \bigcap_j E(I_j)\right] = 1 - \Pr\left[\bigvee_{e \notin E(G)} Z_e\right]$. Thus the result follows.

Since the set of interval graphs generated by $\lceil(\Delta + 2) \ln n \rceil$ invocations of $\text{RAND}(G)$ is a valid interval graph representation of $G$ with non-zero probability as shown above, we have the following corollary.

Corollary 7 Let $G$ be a graph on $n$ vertices and with maximum degree $\Delta$. Then $\text{box}(G) \leq \lceil(\Delta + 2) \ln n \rceil$.

Lemma 8 The $\text{RAND}$ procedure can be implemented in $O(m + n)$ time assuming that a permutation of $\{1, \ldots, n\}$ can be generated uniformly at random in $O(n)$ time.

**PROOF.** The vertex set of $G$ is $\{1, \ldots, n\}$. We make the standard assumption that $G$ is available as an adjacency list representation. First generate $\pi$ and store the mapping $\pi^{-1}()$ in an array indexed from 1 to $n$. Maintain another array $D$ indexed from 1 to $n$ such that $D[i] = 1$ if $l(i)$ for vertex $i$ is already defined and $D[i] = 0$ otherwise. We now construct $G' = \mathcal{M}(G, \pi)$ in $n$ steps as follows. In $i$th step, consider the vertex $u = \pi^{-1}(i)$. Define $r(u) = i$. If $D[u] = 0$ then $l(u) = i$. For each element $w \in N(u)$, if $D[w] = 0$ then define $l(w) = i$ and assign $D[w] = 1$. Thus total time taken in step $i$ is $O(|d(u)| + 1)$ steps. It follows that the interval representation of $G'$ can be constructed in $O(m + n)$ steps.
Theorem 9  Given a graph $G$ on $n$ vertices and $m$ edges, with high probability, a box representation of $G$ in $\lceil (\Delta + 2) \ln n \rceil$ dimension can be constructed in $O(\Delta n^2 \ln^2 n)$ time, where $\Delta$ is the maximum degree of $G$.

**Proof.** We construct an algorithm that takes the graph $G$ as input and tries to compute an interval graph representation for $G$. It repeatedly computes a set $S$ of $\lceil (\Delta + 2) \ln n \rceil$ interval supergraphs of $G$ until it generates a set that is a valid interval graph representation of $G$. If the algorithm fails to find such a set after $(\log_2 e) \ln n$ tries, it reports a failure. Each $S$ is computed through $\lceil (\Delta + 2) \ln n \rceil$ invocations of $\text{RAND}(G)$. From lemma 6, the probability of failure of this algorithm is at most $(1/2)^{(\log_2 e) \ln n} = 1/n$ and hence it computes a valid interval graph representation of $G$ with high probability.

Using lemma 8, it is easily verified that to compute a set of $(\Delta + 2) \ln n$ interval graphs, our algorithm takes $O(\Delta (m + n) \ln n)$ time. To verify whether a generated set of $(\Delta + 2) \ln n$ interval graphs is a valid interval representation or not, it takes $O(n^2 \Delta \ln n)$ time. Thus the overall complexity is $O(\Delta n^2 \ln^2 n)$. 

2.1 Almost tight example

We remark that for any given $\Delta$ and $n > \Delta + 1$, we can construct a graph $G$ on $n$ vertices and with maximum degree $\Delta$ such that $\text{box}(G) \geq \lfloor (\Delta + 2)/2 \rfloor$. We assume that $\Delta$ is even for the ease of explanation. Roberts [16] has shown that for any even number $k$, there exists a graph on $k$ vertices with degree $k - 2$ and boxicity $k/2$. We call such graphs as Roberts graphs. The Roberts graph on $n$ vertices is obtained by removing the edges of a perfect matching from a complete graph on $n$ vertices. We take such a graph by fixing $k = \Delta + 2$ and we let the remaining $n - (\Delta + 2)$ vertices be isolated vertices. Clearly, the boxicity of such a graph is also $k/2 = (\Delta + 2)/2$, whereas the maximum degree is $\Delta$. Thus our upper bound is tight up to a factor of $2 \ln n$.

3 Derandomization

In this section we derandomize the above randomized algorithm to obtain a deterministic polynomial time algorithm to output the box representation in $(\Delta + 2) \ln n$ dimensional space for a given graph $G$ on $n$ vertices with maximum degree $\Delta$.

**Lemma 10** Let $G = (V, E)$ be the graph. Let $E(\overline{G})$ be the edge set of the
complement of \( G \). Let \( H \subseteq E(\overline{G}) \). Then we can construct an interval super graph \( G' \) of \( G \) in polynomial time such that \( |E(\overline{G'}) \cap H| \geq \frac{2}{\Delta+2} |H| \).

**PROOF.** We derandomize the RAND algorithm to devise a deterministic algorithm to construct \( G' \).

Our deterministic strategy defines a permutation \( \pi \) on the vertices \( \{1, \ldots, n\} \) of \( G \). The desired \( G' \) is then obtained as \( \mathcal{M}(G, \pi) \). Let the ordered set \( V_n = < v_1, \ldots, v_n > \) denote the final permutation given by \( \pi \). We construct \( V_n \) in a step by step fashion. At the end of step \( i \), we have already defined the first \( i \) elements of the permutation, namely the ordered set \( V_i = < v_1, \ldots, v_i > \), where each \( v_j \) is distinct. Let \( V_0 \) denote the empty set. Having obtained \( V_i \) for \( i \geq 0 \), we compute \( V_{i+1} \) in the next step as follows.

Given an ordered set \( V_i \) of \( i \) vertices \( < v_1, v_2, \ldots, v_i > \), let \( V_i \diamond u \) denote the ordered set of the \( i+1 \) vertices \( < v_1, v_2, \ldots, v_i, u > \). (We will abuse notation and use \( V_i \) to denote the underlying unordered set also, when there is no chance of confusion.) Let \( V_0 \diamond u \) denote \( < u > \).

Consider an invocation of the randomized algorithm RAND whose output is denoted as \( G'' \). For each \( e \in H \), let \( x_e \) denote the indicator random variable which is 1 if \( e \notin E(G'') \), and 0 otherwise. Let \( X_H = \sum_{e \in H} x_e \).

Given an ordered set \( S = < u_1, \ldots, u_r > \), let \( Z(S) \) denote the event that the first \( r \) elements of the random permutation generated by RAND is given by the ordered set \( S = < u_1, \ldots, u_r > \). Note that \( \Pr[Z(V_0)] = 1 \) since the first 0 elements of any permutation is the empty set \( V_0 \).

Let \( x_e | Z(V_i) \) denote the indicator random variable corresponding to \( x_e \) conditioned on the event \( Z(V_i) \).

Similarly, let the random variable \( X_H | Z(V_i) \) denote \( |E(\overline{G''}) \cap H| \) conditioned on the event \( Z(V_i) \).

For \( i \geq 0 \), let \( f_e(V_i) \) denote \( \Pr[x_e = 1 | Z(V_i)] \) and let \( F(V_i) \) denote \( E[X_H | Z(V_i)] \).

Note that \( f_e(V_0) \) denote \( \Pr[x_e = 1] \) and \( F(V_0) \) denote \( E[X_H] \).

Clearly

\[
F(V_i) = \sum_{e \in H} f_e(V_i).
\]

By Lemma \[5\] we know that for any \( e \in H \), \( f_e(V_0) \geq \frac{2}{\Delta+2} \). Thus \( F(V_0) \geq \frac{2|H|}{\Delta+2} \).

Clearly,

\[
E[X_H|Z(V_i)] = \frac{1}{|V-V_i|} \sum_{u \in V-V_i} E[X_H|Z(V_i \diamond u)].
\]
Let \( u \in V - V_i \) be such that
\[
\mathbb{E}[X_H|\mathcal{Z}(V_i \diamond u)] = \max_{w \in V - V_i} \mathbb{E}[X_H|\mathcal{Z}(V_i \diamond w)].
\]

Define \( V_{i+1} = V_i \diamond u \). It follows that
\[
F(V_{i+1}) = \mathbb{E}[X_H|\mathcal{Z}(V_{i+1})] \geq \mathbb{E}[X_H|\mathcal{Z}(V_i)] = F(V_i).
\]

In particular, it is also true that \( F(V_1) \geq F(V_0) \).

After \( n \) steps, we obtain the final permutation \( V_n \). Applying the above inequality \( n \) times, it follows that
\[
\mathbb{E}[X_H|\mathcal{Z}(V_n)] = F(V_n) \geq F(V_0).
\]

Recalling that \( F(V_0) \geq 2^{|H|}/\Delta + 2 \), we have \( F(V_n) \geq 2^{|H|}/\Delta + 2 \).

Let \( \pi \) be the permutation that corresponds to the ordered set \( V_n \). The final interval super graph \( G' \) output by our deterministic strategy is \( \mathcal{M}(G, \pi) \). Note that, \( F(V_n) = \mathbb{E}[X_H|\mathcal{Z}(V_n)] \) is the same as \( |E(G') \cap H| \). Thus we have shown that \( |E(G') \cap H| \geq 2^{\Delta+2} \) as claimed.

It remains to show that the above deterministic strategy takes only polynomial time. This can be shown as follows using Lemma 11 (see below). Recall that, given a vertex \( w \in V - V_i \), \( F(V_i \diamond w) \) is simply \( \sum_{e \in H} f_e(V_i \diamond w) \). It follows from Lemma 11 that \( F(V_i \diamond w) \) can be computed in polynomial time. Recall that given \( V_i, V_{i+1} = V_i \diamond u \) where \( u \) maximizes \( F(V_i \diamond w) \) where \( w \in V - V_i \). Clearly such a \( u \) can also be found in polynomial time. Since there are only \( n \) steps before computing \( V_n \), the overall running time is still polynomial.

Lemma 11 For any ordered set \( V_i = \langle v_1, \ldots , v_i \rangle \) and any \( e \in H \), \( f_e(V_i) \) can be computed in polynomial time.

**PROOF.** Let \( e = (u, v) \). We can compute \( f_e(V_i) \) as follows:

The easiest case is when both \( u, v \) are in \( V_i \). In this case, the intervals corresponding to \( u \) and \( v \), namely \( I(u) \) and \( I(v) \) are already defined. (Recall how \( \mathcal{M}(G, \pi) \) defines an interval supergraph of \( G \). See definition 3). Therefore either \( e \in E(G') \) or \( e \notin E(G') \). Therefore the conditional probability is either 0 or 1. This is summarized as:

Case 1.1: \( u, v \in V_i \) and \( I(u) \cap I(v) = \emptyset \).

Then \( f_e(V_i) = 1 \) since \((u, v) \notin E(G') \).

Case 1.2: \( u, v \in V_i \) and \( I(u) \cap I(v) \neq \emptyset \):
Then \( f_e(V_i) = 0 \) since \((u, v) \in E(G')\).

Next let us consider the case \( u \in V_i \) and \( v \notin V_i \). In this case \( I(u) = [l(u), r(u)] \) is already defined but \( I(v) \) is not yet determined since \( r(v) \) is still random. But it is clear that \( r(v) > r(u) \). Moreover, we can determine whether \( l(v) < r(u) \) or not, irrespective of \( r(v) \): If there is a vertex \( w \in N(v) \cap V_i \) such that \( r(w) < r(u) \), then \( l(v) < r(u) \), otherwise \( l(v) > r(u) \). Thus in this case also the conditional probability is either 0 or 1, as computed by the following procedure:

Case 2.1: \( u \in V_i, v \notin V_i \) and \( N(v) \cap V_i = \emptyset \):

Then \( f_e(V_i) = 1 \).

Case 2.2: \( u \in V_i, v \notin V_i \) and \( N(v) \cap V_i \neq \emptyset \):

Let \( t = \min_{w \in N(v) \cap V_i} r(w) \).

If \( r(u) > t \) then \( f_e(V_i) = 0 \) else \( f_e(V_i) = 1 \).

Now we examine the case when both \( u \) and \( v \) do not belong to \( V_i \). First consider the sub case when both \( N(u) \cap V_i \) and \( N(v) \cap V_i \) are non-empty. Then clearly \( I(u) \cap I(v) \neq \emptyset \). This is because of the following. Let us denote by \( v_i \), the \( i \)th vertex (the last vertex) in \( V_i \). Clearly, \( l(u) \leq r(v_i) < r(u) \) and \( l(v) \leq r(v_i) < r(v) \). Thus \( r(v_i) \in I(u) \cap I(v) \). It follows that in this case the conditional probability is 0 as given below.

Case 3.1: \( u, v \in V - V_i \) and \( N(v) \cap V_i \neq \emptyset \) and \( N(u) \cap V_i \neq \emptyset \):

Then \( f_e(V_i) = 0 \).

Next subcase is when \( N(u) \cap V_i \) and \( N(v) \cap V_i \) are both empty. That is, the set \( X = \{u\} \cup N(u) \cup \{v\} \cup N(v) \) has empty intersection with \( V_i \). Let \( \pi_X \) be the projection of \( \pi \) onto \( X \). Since \( X \cap V_i = \emptyset \), \( \pi_X \) can be any possible permutation of \( |X| \) elements with equal probability, namely \( \frac{1}{|X|!} \). We estimate the conditional probability \( f_e(V_i) \) as follows. Clearly, \((u, v) \notin E(G')\) if and only if \( \pi_X(u) < \min_{w \in \{u\} \cup N(v)} \pi_X(w) \) or \( \pi_X(v) < \min_{w \in \{u\} \cup N(u)} \pi_X(w) \). It is easy to see that the probability for the above condition to hold is \( \frac{1}{d(u)+2} + \frac{1}{d(v)+2} \). Thus, we have the following case:

Case 3.2: \( u, v \in V - V_i \) and \( N(v) \cap V_i = \emptyset \) and \( N(u) \cap V_i = \emptyset \):

Then \( f_e(V_i) = \frac{1}{d(u)+2} + \frac{1}{d(v)+2} \).

Now we are left with the last sub case: \( N(v) \cap V_i \neq \emptyset \) and \( N(u) \cap V_i = \emptyset \). Let \( X = \{u\} \cup N(u) \cup \{v\} \). Note that \( X \cap V_i = \emptyset \). Clearly \( \pi_X \) can be any possible permutation of \( |X| \) elements with equal probability, namely \( \frac{1}{|X|!} = \frac{1}{(d(u)+2)!} \).
We estimate the conditional probability \( f_e(V_i) \) as follows.

We claim that \((u, v) \notin E(G')\) if and only if \(\pi_X(v) = 1\). To see this, first observe that if \((u, v) \notin E(G')\) then \(\pi_X(v) < \pi_X(u)\) because otherwise \(r(u) < r(v)\) and since \(N(v) \cap V_i \neq \emptyset\), we get further \(l(v) < r(u) < r(v)\) leading to a contradiction. Therefore \(1 \leq \pi_X(v) < \pi_X(u)\). Now, if \(\pi_X(w) = 1\) for some \(w \in N(u)\) then \(l(u) < r(v) < r(u)\) and thus \(I(u) \cap I(v) \neq \emptyset\) leading to a contradiction. On the other hand, if \(\pi_X(v) = 1\) then clearly, \(r(v) < l(u)\) for any \(w \in N(u) \cup \{u\}\). It follows that \(r(v) < l(u)\) and therefore \(I(v) \cap I(u) = \emptyset\).

Now, having shown that \((u, v) \notin E(G')\) if and only if \(\pi_X(v) = 1\), we can compute \(f_e(V_i)\) by computing the conditional probability that \(\pi_X(v) = 1\), which is simply \(\frac{1}{d(u)+2}\). Thus we have the following final case:

**Case 3.3:** \(u, v \in V - V_i\) and \(N(v) \cap V_i \neq \emptyset\) and \(N(u) \cap V_i = \emptyset\):

Then \(f_e(V_i) = \frac{1}{d(u)+2}\).

---

**Theorem 12** Let \(G\) be a graph on \(n\) vertices with maximum degree \(\Delta\). The box representation of \(G\) in \(\lceil(\Delta + 2) \ln n\rceil\) dimension can be constructed in polynomial time.

**Proof.** Let \(h = |E(G)|\). It follows from Lemma 10 that we can construct \(t\) interval graphs such that the number of edges of \(E(G)\) which is present in all of these \(t\) interval graphs is at most \(\left(\frac{\Delta}{\Delta+2}\right)^t h\). If \(\left(\frac{\Delta}{\Delta+2}\right)^t h < 1\), then we are done. That is, we are done if \(t \ln \left(\frac{\Delta}{\Delta+2}\right) + \ln h < 0\) is true. Clearly this is true, if \(t > \ln h / \ln \left(\frac{\Delta}{\Delta+2}\right)\). We can assume that \(\Delta > 1\). If \(\Delta \leq 1\) then \(G\) is an interval graph and the theorem trivially holds. Using the fact that \(\ln \left(\frac{\Delta^2}{\Delta^2 - 1}\right) \leq \frac{\Delta^2}{2(\Delta - 1)} \ln h \leq (\Delta + 2) \ln n\), we obtain \(\text{box}(G) \leq \frac{\Delta^2}{2(\Delta - 1)} \ln h \leq (\Delta + 2) \ln n\). By Lemma 10, each interval graph is constructed in polynomial time. Hence the total running time is still polynomial. Thus the theorem follows.

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**4 In terms of average degree**

It is natural to ask whether our upper bound of \((\Delta + 2) \ln n\) still holds, if we replace \(\Delta\) by the average degree \(d_{av}\). Unfortunately this is not true in general: we show below an infinite family of graphs where the boxicity is exponentially higher than \((d_{av} + 2) \ln n\). Construct a graph \(G\) on \(n\) vertices as follows. First take a Roberts graph on \(n_1\) vertices where \(n_1 \leq n\). (Refer to section 2.1 for the
definition of Roberts graph.) Let the remaining \( n - n_1 \) vertices form a path which is connected by an edge to one of the vertices of the Roberts graph. The average degree of \( G \) is \( d_{av} = (n_1(n_1 - 2) + 2(n - n_1))/n \), whereas its boxicity is at least \( n_1/2 \geq \frac{1}{2} \sqrt{n(d_{av} - 2)} \). It is easy to see that the boxicity of \( G \) is exponentially larger than \((d_{av} + 2) \ln n \) for example when \( n_1 = \theta(\sqrt{n}) \).

Nevertheless we show the following general upper bound for boxicity in terms of \( d_{av} \). In fact, we will express the upper bound in terms of \( n \) and the number of edges \( m \).

**Theorem 13** For a connected graph \( G \) on \( n \) vertices and \( m \) edges, \( \text{box}(G) \leq 5\sqrt{m \ln n} \). Moreover, there exists a connected graph \( G \) with \( n \) vertices and \( m \) edges such that \( \text{box}(G) \geq \sqrt{mn}/2 \).

**Proof.** We show the upper bound as follows. Let \( x = \sqrt{m/\ln n} \). Let \( V' \) denote the set of vertices in \( G \) whose degree is at least \( x \). It is straightforward to verify that \( |V'| \leq 2m/x \). Let \( G'' \) be the induced subgraph on \( V - V' \). Each vertex in \( G'' \) has degree at most \( x \). By Corollary 7, we obtain that \( \text{box}(G'') \leq (x + 2) \ln n \). Since \( \text{box}(G'') + |V'| \) is a trivial upper bound for \( \text{box}(G) \), it follows that \( \text{box}(G) \leq (x + 2) \ln n + 2m/x \leq 5\sqrt{m \ln n} \) since \( m \geq n - 1 \). The example graph discussed in the beginning of this section serves as the example that illustrates the lower bound.

**4.1 Boxicity of Random Graphs**

Though in general boxicity of a graph is not upper bound by \((d_{av} + 2) \ln n \), where \( d_{av} \) is its average degree, we now show that for almost all graphs, the boxicity is \( O(d_{av} \ln n) \).

We show that for almost all graphs in the the \( G(n, m) \) model, \( \text{box}(G) \) is \( O(c \ln n) \) where \( c = 2m/n \). We shall only consider connected graphs here and so assume that \( m \geq n - 1 \). Thus, we have, \( c > 1 \). But we first show the result for the \( G(n, p) \) model setting \( p = c/(n - 1) \). As shown in [6], we can then carry over the result to the \( G(n, m) \) model since \( p = m/\binom{n}{2} \).

Consider the \( G(n, p) \) model with \( p = c/(n - 1) \). Let \( G \) denote a random graph drawn according to this model. For a vertex \( u \), define a random variable \( d_u \) that denotes the degree of \( u \), i.e., \( d_u = |N(u)| = \sum_{v \in V(G), v \neq u} e_{u,v} \) where \( e_{u,v} \) is an indicator random variable whose value is 1 if \((u, v) \in E(G)\) and 0 otherwise. Therefore, \( \mathbb{E}[d_u] = p(n - 1) = c \).

**Case 1:** \( c \geq \ln n \).
Since $d_u$ is the sum of independent Bernoulli random variables, we can use Chernoff bound to bound the probability of $d_u$ becoming large. In particular, we use the following form of the Chernoff bound given in [22] for the rest of the proof.

$$
\Pr[X \geq (1 + \delta)E[X]] \leq e^{-\frac{\delta^2E[X]}{2+\delta}}
$$

(1)

for all $\delta > 1$. Taking $\delta = 5$, we get, $\Pr[d_u \geq 6c] \leq 1/n^3$. Now, by union bound, it follows that $\Pr[\Delta(G) \geq 6c] = \Pr[\exists u \in V(G), d_u \geq 6c] \leq 1/n^2$. Using the result $\text{box}(G) \leq (\Delta + 2) \ln n$, we now have, $\text{box}(G) \leq (6c + 2) \ln n$ with probability at least $1 - 1/n^2$.

**Case 2:** $c < \ln n$.

Let $S_u = V(G) - N(u) - \{u\}$. Let $N'(u) = \{v \in S_u \mid \exists u' \in N(u) \text{ such that } (u', v) \in E(G)\}$.

In this case, we will use a different technique to upper bound boxicity. Let the graph $G^2$ denote the square of $G$. That is, $V(G^2) = V(G)$ and $(u, v) \in E(G^2)$ if there is a path of length 1 or 2 between $u$ and $v$. The authors showed in [23] that the boxicity of a graph $G$ is at most $2\Delta(G^2) + 2$. We will show below that if $c < \ln n$, then $\Delta(G^2) \leq c + 6 \ln n + 7c^2 + 42c \ln n$, with high probability. The reader may note that the degree of a vertex $u$ in $G^2$ equals $|N(u)| + |N'(u)|$. We will now show that for any vertex $u$, $\Pr[|N(u)| + |N'(u)| \notin O(c \log n)] \leq 3/n^3$.

Let $k = c + 6 \ln n$. We apply Chernoff bound (1) with $\delta = 6 \ln n / c$ to obtain

$$
\Pr[d_u \geq k] \leq e^{-\delta(6 \ln n)/(2+\delta)} \leq 1/n^3
$$

Let $A \subseteq V(G)$ such that $|A| < k$. Let $Z(A)$ denote the event that $N(u) = A$. Now, for each vertex $v \in S_u$, let $X_{v,A}$ denote an indicator random variable indicating whether $v \in N'(u)$ conditioned on the event $Z(A)$. Note that for any vertex $v \in S_u$, $\Pr[X_{v,A} = 1] \leq kp$. Let $X_A = \sum_{v \in S_u} X_{v,A}$. It follows that $E[X_A] \leq kp(n-1) = kc$. Since $X_A$ is the sum of independent Bernoulli random variables, we apply the Chernoff bound (1) by fixing $\delta = 6kc/E[X_A]$ to obtain

$$
\Pr[X_A \geq 7kc] \leq e^{-\delta(6kc)/(2+\delta)} \leq 1/n^3.
$$

Let the random variable $X_u = |N'(u)|$. We now have,

$$
\Pr[X_u \geq 7kc \mid d_u < k] = \sum_{A \subseteq V(G), |A| < k} \Pr[(X_u \geq 7kc) \land Z(A)]
$$

$$
= \sum_{A \subseteq V(G), |A| < k} \Pr[X_A \geq 7kc] \Pr[Z(A)] \leq 1/n^3
$$

It follows that
\[ \Pr[X_u \geq 7kc] = \Pr[X_u \geq 7kc \mid d_u < k] \Pr[d_u < k] \\
+ \Pr[X_u \geq 7kc \mid d_u \geq k] \Pr[d_u \geq k] \\
\leq (1/n^3) \Pr[d_u < k] + (1/n^3) \Pr[X_u \geq 7kc \mid d_u \geq k] \leq 2/n^3 \]

Let \( t_u = |N(u)| + |N'(u)| = d_u + X_u \). Combining the bounds on the values of \( d_u \) and \( X_u \), we get,

\[ \Pr[t_u \geq k + 7kc] \leq \Pr[d_u \geq k] + \Pr[X_u \geq 7kc] \leq 3/n^3 \]

Observe that \( \Delta(G^2) = \max_{u \in G} t_u \). Thus, by applying union bound, we obtain

\[ \Pr[\Delta(G^2) \geq k + 7kc] = \Pr[ \bigvee_{u \in V(G)} t_u \geq k + 7kc] \leq 3/n^2 \]

Thus, with high probability, \( \Delta(G^2) < k + 7kc = c + 6 \ln n + 7c^2 + 42c \ln n \). Recalling that \( \text{box}(G) \leq 2\Delta(G^2) + 2 \), we obtain \( \text{box}(G) \in O(c \ln n) \) with high probability, since \( c < \ln n \).

Having shown that in the \( G(n, p) \) model, \( \Pr[\text{box}(G) \notin O(c \ln n)] \leq 3/n^2 \), the following relation from page 35 of [6] helps us to extend our result to the \( G(n, m) \) model.

\[ P_m(Q) \leq 3m^{1/2}P_p(Q) \]

where \( Q \) is a property of graphs of order \( n \), and \( P_m(Q) \) and \( P_p(Q) \) are the probabilities of a graph chosen at random from the \( G(n, m) \) or the \( G(n, p) \) models respectively to have property \( Q \) given that \( p = m/\binom{n}{2} \). Using this result, we now have, for a graph \( G \) drawn randomly from the \( G(n, m) \) model,

\[ \Pr[\text{box}(G) \notin O(c \ln n)] \leq 9n^{-2} \sqrt{m} \leq 9/n \]

As \( c = 2m/n = d_{av} \), which is the average degree, we have shown that for almost all graphs with a given average degree \( d_{av} \), the boxicity is \( O(d_{av} \ln n) \).

Thus we have the following theorem:

**Theorem 14** For a random graph \( G \) on \( n \) vertices and \( m \) edges drawn according to \( G(n, m) \) model,

\[ \Pr[\text{box}(G) = O\left(\frac{2m}{n} \ln n\right)] \geq 1 - \frac{9}{n} \]

5 Acknowledgement

We thank the anonymous referee for carefully reading the paper and pointing out a serious mistake in the last section of the first version of the paper.
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