GLOBAL EXISTENCE OF SOLUTIONS FOR SEMILINEAR WAVE EQUATIONS IN FRIEDMANN-LEMAÎTRE-ROBERTSON-WALKER SPACETIME

MARCELO REMPEL EBERT AND JORGE MARQUES

Abstract. We consider the nonlinear massless wave equation belonging to some family of the Friedmann–Lemaître–Robertson–Walker (FLRW) spacetime. We prove the global in time small data solutions for supercritical powers in the case of decelerating expansion universe.

1. Introduction

In this paper, we prove the global existence (in time) of small data solutions to the Cauchy problem for the semilinear wave equation with scale-invariant damping and decreasing in time propagation speed

\[
\begin{align*}
&u_{tt}(t,x) - (1+t)^{-2\ell}\Delta u(t,x) + \frac{\beta}{1+t}u_t(t,x) = f(u(t,x)), \quad t \geq 0, \quad x \in \mathbb{R}^n, \\
&u(0,x) = u_0(x), \\
&u_t(0,x) = u_1(x),
\end{align*}
\]

with \(\ell \in (0,1)\) and \(\beta > 0\). We assume that \(f(u) = |u|^p\) for some \(p > 1\) or, more in general, \(f\) verifies the following local Lipschitz-type condition

\[
|f(u) - f(v)| \leq C |u - v| \left(|u|^{p-1} + |v|^{p-1}\right).
\]

The case \(\beta = 2\) in (1) is well known as FLRW spacetime model for the decelerating expansion universe, whereas in the particular case \(\ell = \frac{2}{3}\) (1) is the nonsingular covariant massless field in the Einstein–de Sitter spacetime (see [15]).

Let us start with the state of the art in the case \(\ell = 0\). If \(\beta \geq \frac{3}{2}\) for \(n = 1\), \(\beta \geq 3\) for \(n = 2\), or \(\beta \geq n + 2\) for \(n \geq 3\), by assuming data in the energy spaces with additional regularity \(L^1(\mathbb{R}^n)\), the global (in time) existence result for (1) was proved in [2] for \(p > p_F(n) = 1 + \frac{2}{n}\), the well known Fujita index [14]. The exponent \(p_F(n)\) is critical for this model, that is, for \(p \leq p_F(n)\) and suitable, arbitrarily small data, there exists no global weak solution [7]. As conjectured in [6] and [8], if \(\beta\) becomes smaller with respect to the space dimension \(n\), the critical exponent increase to \(\max\{p_S(n + \beta), p_F(n)\}\), where \(p_S\) is the Strauss exponent for the semilinear undamped wave equation [18], [22]. In [20] the authors proved a blow-up result and gave the upper bound for the lifespan of solutions to (1) for \(1 < p \leq p_S(n + \beta)\) and \(\beta \in [0, \beta_*]\), with \(\beta_* = \frac{n^2 + n + 2}{n + 2}\). It is worth noticing that if \(\beta \in [0, \beta_*)\), then \(p_F(n) < p_S(n + \beta)\) and, \(p_F(n) = p_S(n + \beta)\).

Recently D’Abbicco (see [3] and [11]) proved his conjecture in which the critical exponent is equals to \(\max\{p_S(n + \beta), p_F(n)\}\) for \(n = 1\) and, also proved the global

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existence of small data solutions for \( p > p_F(n) \) and \( \beta \geq n \) in space dimension \( 2 \leq n \leq 5 \). As far as we know, it is still an open problem to prove global existence of small data solutions for \( p \geq p_F(n) \) in the cases \( \beta_s < \beta < n \) for \( n \geq 3 \) and for \( p > p_S(n + \beta) \) for \( 0 < \beta < \beta_s \) for \( n \geq 2 \).

For \( \ell \in [0, 1) \), \( \beta \geq 0 \) and \( n \geq 2 \), let \( p_S(n, \ell, \beta) \) be the positive root of the quadratic equation

\[
\left( n - 1 + \frac{\beta - \ell}{1 - \ell} \right) p^2 - \left( n + 1 + \frac{\beta + 3\ell}{1 - \ell} \right) p - 2 = 0.
\]

Recently, in [22] and [21] the authors have proved blow-up in a finite time and upper estimates of the lifespan for solutions to (1) for

\[
1 < p \leq \max \{p_F(n(1 - \ell)), p_S(n, \ell, \beta)\}.
\]

A blow-up result for \( \beta = 2 \) and \( \ell \in (0, 1) \) in (1) was also proved in [15]. It is worth noticing that if \( p_F(n(1 - \ell)) = p_S(n, \ell, \beta_c(n, \ell)) \), where

\[
\beta_c(n, \ell) \doteq \ell + (1 - \ell) \left( n + 1 - \frac{2}{p_c} \right) = \frac{n^2(1 - \ell)^2 + n(1 - \ell)(1 + 2\ell) + 2}{2 + n(1 - \ell)}.
\]

In particular, if \( \beta \geq \beta_c(n, \ell) \), then \( p_S(n, \ell, \beta) \leq p_F(n(1 - \ell)) \).

In [1], the authors proposed a classification of non-effective and effective dissipation, respectively, for the damped wave equation

\[
u_{tt}(t, x) - a^2(t)\Delta u(t, x) + b(t)u_t(t, x) = 0
\]

with increasing speed of propagation. The authors derived sharp estimates for solutions to the Cauchy problem and, in the case of effective dissipation, i.e.,

\[
b(t)\frac{A(t)}{a(t)} \to \infty, \text{ as } t \to \infty, \quad A(t) = 1 + \int_0^t a(\tau) \, d\tau,
\]

derived global existence (in time) results for the semilinear problem with power nonlinearities. A similar classification was introduced in [12] in the case \( a \in L^1 \).

A natural generalization for the model (1) is to consider a positive and decreasing speed of propagation \( a(t) \), with \( a \notin L^1 \). But in this paper we restrict ourselves to the case in which \( a \) is a rational function, since it includes interesting models by itself, for instance, if \( \ell = \frac{2}{n} \) in (1), the considered model coincides with the non-singular wave equation in the Einstein de Sitter space-time ([10], [17]).

The main goal in this paper is to prove, under the assumption of small initial data in \( L^1(\mathbb{R}^n) \times H^{k-1}(\mathbb{R}^n) \), \( k \geq 1 \), the global existence (in time) of solutions to (1) for supercritical powers \( p > p_F(n(1 - \ell)) \), by supposing that \( \beta \geq \beta_c(n, \ell) \).

Combine the obtained results in this paper with the blow-up results derived in [24] we conclude that \( p_F(n, \ell) = 1 + \frac{2}{n(1-\ell)} \) is the critical exponent for the global in time existence of solutions for \( \beta \geq \beta_c(n, \ell) \).

As far as we know, it is still an open problem to prove global existence of small data solutions to (1) for \( p > p_S(n, \ell, \beta) \) and \( 0 < \beta \leq \beta_c(n, \ell) \). It is expected that a similar approach to those used for the semilinear free wave equation may be appropriate to decrease values of \( \beta \) and to overcome some gaps that appear in this paper.
2. Main results

To simplify the writing, from now we consider

\[ p_c(n, \ell) = p_F(n(1 - \ell)) = 1 + \frac{2}{n(1 - \ell)}. \]

In the next two theorems, due to the fact that \( p_c(n, \ell) \to \infty \) as \( \ell \to 1 \), the choice of the spaces of solutions is related to fixed ranges for \( \ell \in [0, 1) \) and the space dimensions \( n \geq 2 \). To state our first result, let us define the following parameters

\[ q_1 = \frac{2(nc(n, \ell) - 1)}{n + 1}, \quad q_2 = \frac{2(n + 1)}{n - 1}. \]

**Theorem 2.1.** Let \( \ell \) be such that

\[
\begin{align*}
0 \leq \ell &< 1 - \frac{n-1}{2n}, & \text{if } 2 \leq n \leq 5 \\
1 - \frac{2(n+1)}{n(n-3)} &\leq \ell < 1 - \frac{n-1}{2n}, & \text{if } 6 \leq n \leq 8 
\end{align*}
\]

and

\[ \beta \geq \ell + (n+1)(1 - \ell) - \frac{2}{q}(1 - \ell), \]

with \( q \in [p_c(n, \ell), q_2] \), where \( p_c(n, \ell) \), \( q_2 \) and \( q \) are given by (3) and (4). If

\[ p_c(n, \ell) < p \leq \frac{4p_c(n, \ell)}{n + 3} + 1, \]

then there exists \( \delta > 0 \) such that for any initial data

\[ u_1 \in \mathcal{D} = L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n), \quad ||u_1||_{\mathcal{D}} \leq \delta, \]

there exists a unique weak solution \( u \in C([0, \infty), L^p(\mathbb{R}^n) \cap L^q(\mathbb{R}^n)) \) to (7). Moreover, the solution satisfies the following estimates for \( p_c \leq q \leq q_2 \):

If \( \beta > \ell + (n+1)(1 - \ell) - \frac{2}{q}(1 - \ell) \) then \[ \|u(t, \cdot)\|_{L^q} \lesssim (1 + t)^{-n(1 - \frac{1}{q})c(1 - \ell)} ||u_1||_{\mathcal{D}} \] whereas if \( \ell + (n+1)(1 - \ell) - \frac{2}{q}(1 - \ell) \leq \beta \leq \ell + (n+1)(1 - \ell) - \frac{2}{q}(1 - \ell) \), then for any \( \varepsilon > 0 \)

\[ ||u(t, \cdot)||_{L^q} \lesssim (1 + t)^{\varepsilon(n-1)}(\frac{1}{2} - \frac{1}{q})c(1 - \ell) - \frac{2}{q}(1 - \ell) ||u_1||_{\mathcal{D}}. \]

**Remark 2.1.** One of the crucial property in the proof of Theorem 2.1 is that \( r(q)p_c(n, \ell) < q_2 \), for all \( p_c(n, \ell) \leq q \leq q_2 \), with \( \frac{1}{r(q)} = \frac{1}{2n} + \frac{1}{2} + \frac{1}{nq} \). This condition is satisfied under some condition on \( \ell \), namely,

\[ r(q)p_c < q_2 \iff \ell < 1 - \frac{4}{q_2(n+1) - 2(n-1)} = 1 - \frac{n-1}{2n}. \]

Since \( r(q) \leq r(q_2) \) for all \( p_c \leq q \leq q_2 \), we also have

\[ r(q)p_c(n, \ell) < q_2 \iff \ell < 1 - \frac{n-1}{2n}. \]

\[ ^1 \text{Let } f, g : \Omega \subset \mathbb{R}^n \to \mathbb{R} \text{ be two functions. From now one we use the notation } f \lesssim g \text{ if there exists a constant } C > 0 \text{ such that } f(y) \leq Cy(y) \text{ for all } y \in \Omega. \]
there exists a unique energy solution $u$ satisfying $u < q_2$.

For instance, for $\ell \in [0, \frac{2}{3})$ if $n = 2$, and for $\ell \in [0, \frac{2}{3})$ if $n = 3$.

Moreover,

$$r(q_2)p_c(n, \ell) < q_2 \iff p_c(n, \ell) \left( 1 - \frac{r(q_2)}{q_2} \right) + 1 = \frac{4p_c(n, \ell)}{n + 3} + 1 < \frac{q_2}{r(q_2)} = \frac{n + 3}{n - 1}.$$ 

Remark 2.2. Taking into account that $L^1 - L^q$ linear estimates in Corollary 2 of [3] hold only for $\frac{2(n-1)}{n+1} \leq q \leq q_2$, in the proof of Theorem 2.1 we have to assume

$$p_c(n, \ell) \geq \frac{2(n-1)}{n+1}.$$ 

Hence a restriction from below in $\ell$ is also needed, namely, $\ell \geq 1 - \frac{2(n+1)}{n(n-3)}$, $n \geq 4$. This condition is true for $\ell = 0$ under the assumption $2 \leq n \leq 5$ in Theorem 2.1.

Remark 2.3. Let $\frac{n}{r(q)} = \frac{1}{2} + \frac{n}{q} + \frac{1}{q}$. Condition (1) means that $\bar{q}$ is defined by $p_c(n, \ell) r(\bar{q}) = q$. In particular, thanks to $(n-1)p_c(n, \ell) \geq 1$ for $n \geq 2$ it holds

$$\frac{1}{q} - \frac{n}{p_c r(q)} + \frac{n - 1}{q} = \frac{n}{p_c r(q)} - \frac{n - 1}{q} - \frac{n - 1}{q} = \frac{n}{p_c r(q)} - \frac{n - 1}{q} + \frac{1}{q} \frac{1}{q} \leq 0.$$ 

In the case $\bar{q} = p_c$, Theorem 2.1 yields the threshold value

$$\beta \geq \ell + (1 - \ell) \left( n + 1 - \frac{2}{p_c} \right) = \beta_c(n, \ell).$$

If $\ell = 0$ then $p_c(n, 0) = 1 + \frac{2}{n}$ and $\bar{q} = 2$, so we have to assume $\beta \geq n + 1 - \frac{2}{p_c} = n$. In particular for $\ell = 0$ and $n = 2$, this condition coincides with the threshold value $\beta \geq \beta_c(2, 0) = 2$.

In the next result, the novelty is to use higher regularity $H^k(\mathbb{R}^n), k > \frac{n}{2}$, in order to consider larger values on the parameter $\ell$ and to relax the condition in the upper bound for $p$ in Theorem 2.1. In this way one can also consider values of $\ell \in \left[ \frac{n+1}{2n}, 1 \right]$ for $n = 3, 4$, in particular include the decreasing speed of propagation $a(t) = (1 + t)^{-\frac{2}{n}}$, that appears in the well known Einstein de Sitter model for decelerating expanding universe [10].

Theorem 2.2. Let $\ell \in \left( 1, \frac{2}{n+1} \right)$ for $2 \leq n \leq 4$ and $\ell \in \left[ \frac{1}{2} \left( 1 + \sqrt{1 - 16/n^2} \right), 1 \right]$ for $n \geq 5$. If $\beta \geq \ell + (1 - \ell)(1 + \ell)$ and $p > p_c(n, \ell)$, with $p_c(n, \ell)$ given by (8), then there exists $\delta > 0$ such that for any initial data

$$u_1 \in D = H^{k-1}(\mathbb{R}^n) \cap L^1(\mathbb{R}^n), \quad ||u_1||_D \leq \delta,$$

there exists a unique energy solution $u \in C([0, \infty), H^k(\mathbb{R}^n))$ to (7) with $1 + \frac{2k}{n} = k \leq p_c$, which satisfies the following estimates

$$||u(t, \cdot)||_{L^2} \lesssim (1 + t)^{\frac{1}{2}(\ell-1)}||u_1||_D;$$
and

$$\|u(t, \cdot)\|_{H^k} \leq \|u_1\|_{\mathcal{F}} \begin{cases} (1 + t)\beta + (\frac{4}{\tau} + k), & \beta > \ell + n(1 - \ell) + 2k(1 - \ell) \\ (1 + t)\frac{\ell^2}{2}, & \beta = \ell + n(1 - \ell) + 2k(1 - \ell) \\ (1 + t)\frac{\ell^2}{2}, & \beta < \ell + n(1 - \ell) + 2k(1 - \ell). \end{cases}$$

(8)

**Remark 2.4.** For $n \geq 2$ we point out that

$$\ell > 1 - \frac{2}{n} \iff k > \frac{n}{2} \iff \rho_c(n, \ell) > 2.$$  

For $n \geq 5$, we have

$$1 + \frac{n\ell}{2} \leq \rho_c \iff \ell \geq \frac{1}{2} \left(1 + \sqrt{1 - 16/n^2}\right).$$

The last condition is equivalent to

$$k \geq \frac{n + \sqrt{n^2 - 16}}{4} + 1$$

and $\frac{\ell}{2} < \frac{n + \sqrt{n^2 - 16}}{4} + 1$, $n \geq 5$, so that

$$\ell \geq \frac{1}{2} \left(1 + \sqrt{1 - 16/n^2}\right) \iff k > \frac{n}{2}.$$  

**Example 2.1.** If $\ell = \frac{2}{3}$, the conclusion of Theorem 2.2 holds for $n = 2, 3, 4$ with $\beta \geq \frac{1}{3} \left(2 + \frac{5n}{3}\right)$.

### 3. Representation of the Solution to the Linear Cauchy Problem

Let $s \geq 0$ be a parameter. We need to solve a family of parameter dependent linear ($f(u) = 0$) Cauchy problems corresponding to (1):

$$\begin{cases} u_{tt}(t, x) - (1 + t)^{-2\ell} \Delta u(t, x) + \frac{\beta}{1 + t} u(t, x) = 0, & t \geq s \\ u(s, x) = g_1(s, x) \\ u_t(s, x) = g_2(s, x). \end{cases}$$

(9)

We begin by applying Fourier transform to the solution of the problem (9). We denote the partial Fourier transform of a tempered distribution or of a function $u : \mathbb{R}_0^+ \times \mathbb{R}^n \to \mathbb{C}$ with respect to $x$, by $\hat{u} = \mathcal{F}u$ or $\hat{u}(t, \cdot) = \mathcal{F}u(t, \cdot)$. The notation $\mathcal{F}^{-1}$ denotes the inverse Fourier transform, in the appropriate sense. Following as in [12], we make the change of variables $\tau = \frac{(1 + t)^{1-\ell}}{1-\ell} |\xi|$ and $v(\tau, s) = \hat{u}(t, s, \xi)$. If $u(t, s, x)$ is a solution of (9) then $v(\tau, s)$ satisfies

$$\begin{cases} v''(\tau) + \frac{\beta - \ell}{1 - \ell} v'(\tau) + v(\tau) = 0 \\ v(1 + s)^{1-\ell}\left(\frac{\xi}{1 - \ell}\right) = \hat{g}_1(s, \xi) \\ v'(1 + s)^{1-\ell}\left(\frac{\xi}{1 - \ell}\right) = \hat{g}_2(s, \xi). \end{cases}$$

Moreover, if we are looking for a solution in the product form $v(\tau, s) = \tau^\rho w(\tau, s)$, then $w(\tau, s)$ is a solution of the Bessel’s differential equation of order $\pm \rho$:

$$\tau^2 w''(\tau) + \tau w'(\tau) + (\tau^2 - \rho^2) w(\tau) = 0$$

(11)
We consider the cut-off function $\psi$ and the zones of low frequencies $\rho$ for $\tau$ small. We define the zone of high frequencies $\gamma, \delta, s$ where $\s, \s, \s$ and $\s, \s, \s, \s, \s$.

First, according to Lemma 3.1, we introduce the general solution of the ODE (11). First, according to [25], we introduce $u$ with Fourier multipliers and the first order partial derivatives with respect to $t$. We then determine the Fourier multipliers and the first order partial derivatives with respect to $t$, respectively. We then determine the Fourier multipliers and the first order partial derivatives with respect to $t$ to represent $\dot{u}$ and $\ddot{u}$ in the explicit form.

Lemma 3.1. (see [9]) Let $u(t, x)$ be the solution of (4). Then the partial Fourier transform of $u$ with respect to $x$, $\hat{u}$, is represented by

$$\hat{u}(t, s, \xi) = m_0(t, s, \xi)\hat{g}_1(s, \xi) + m_1(t, s, \xi)\hat{g}_2(s, \xi)$$

with Fourier multipliers and the first order partial derivatives with respect to $t$ given by

$$\partial_t^k m_k = \frac{(-1)^k \pi i}{4(1-t)^{\ell}}(1+s)^{(1+\beta-1)/2}(1+t)^{(1-\beta)/2-j}\psi_{1+j-k, \rho+k-1,1-j-k}$$

where $\rho = \frac{1-\beta}{2(1-t)^{\ell}}$, $k, j = 0, 1$.

4. $L^p - L^q$ estimates

In order to obtain an estimate of (15), we have to distinguish between large and small $\tau$ values. We divide the extended phase space $R^+_0 \times R^+_0 \times R^+_0$ into three zones. We define the zone of high frequencies

$Z_1 = \{(t, s, |\xi|) : |\xi| \geq (1+s)^{\ell-1}\}$;

and the zones of low frequencies

$Z_2 = \{(t, s, |\xi|) : (1+t)^{\ell-1} \leq |\xi| \leq (1+s)^{\ell-1}\}$;

$Z_3 = \{(t, s, |\xi|) : |\xi| \leq (1+t)^{\ell-1}\}$.

These are separated by the boundary $\{(t, |\xi|) : (1+t)^{1-\ell}|\xi| = N(1-\ell)\}$.

We consider the cut-off function $\chi \in C^\infty(R^n)$ with $\chi(r) = 1$ for $r \leq \frac{1}{2}$ and $\chi(r) = 0$ for $r \geq 1$ and define

$$\chi_1(s, \xi) = 1 - \chi((1+s)^{1-\ell}|\xi|),$$

$$\chi_2(t, s, \xi) = \chi((1+s)^{1-\ell}|\xi|) \chi((1+t)^{1-\ell}|\xi|),$$

$$\chi_3(t, s, \xi) = \chi((1+s)^{1-\ell}|\xi|) \chi((1+t)^{1-\ell}|\xi|),$$
such that $\chi_1 + \chi_2 + \chi_3 = 1$.

**Lemma 4.1.** Let $\ell \in (0, 1)$, $\gamma \neq 0$, and $k \geq 0$. It holds

\begin{equation}
|\xi|^k |\psi_{0,\gamma,0}(t,s,\xi)| \lesssim \begin{cases} 
|\xi|^{k-1}(1+t)^{(\ell-1)/2}(1+s)^{(\ell-1)/2} & \text{if } (t,s,\xi) \in Z_1 \\
|\xi|^{k-|\gamma|-1/2}(1+s)^{(\ell-1)/2}(1+t)^{|\gamma|} & \text{if } (t,s,\xi) \in Z_2 \\
|\xi|^{k}(1+s)^{(\ell-1)/2}(1+t)^{(1-\ell)/|\gamma|} & \text{if } (t,s,\xi) \in Z_3.
\end{cases}
\end{equation}

for all $s \geq 0$ and $t \geq s$.

**Proof.** For any $N \in (0,1)$, the following properties hold:

1. \begin{equation}
|H_\gamma^\pm(\tau)| \lesssim \tau^{-1/2}, \quad \tau \in [N, \infty);
\end{equation}
2. \begin{equation}
|H_\gamma^\pm(\tau)| \lesssim \tau^{-|\gamma|}, \quad \tau \in (0,N), \quad \gamma \neq 0;
\end{equation}
3. \begin{equation}
|J_\gamma(\tau)| \lesssim \tau^\gamma, \quad \tau \in (0,N);
\end{equation}
4. \begin{equation}
|Y_\gamma(\tau)| \lesssim \tau^{-\gamma}, \quad \tau \in (0,N), \quad \gamma \neq 0.
\end{equation}

To conclude the estimates in zones $Z_1$ and $Z_2$ we may use the representation (12), estimates (18) and (19), whereas in the zone $Z_3$ we use (13)-(14) and (20)-(21). □

**Proposition 4.1.** Let $0 \leq \ell < 1$, $\beta > 1$, $k \geq 0$, $n \in \mathbb{N}$ and $q \geq 2$. Assume that $g_1 = 0$ and $g_2 \in L^1(R^n) \cap H^{[k-1],1}(R^n) \cap L^m(R^n)$, with $m \in [1,2]$ such that

\begin{equation}
m = m(k,n,q) > \frac{nq}{n+q(1-k)}, \quad k \in [0,1).
\end{equation}

Then the solution $u$ of the problem (9) satisfies the following a priori estimates:

(i): If $1 < \beta < \ell + 2n(1-\ell) \left( 1 - \frac{1}{q} \right) + 2k(1-\ell)$ then

\begin{equation}
||D^k u(t,s,\cdot)||_{L^q} \lesssim (1+t)^{-\frac{\alpha}{2} + (\ell-1)(n-\beta)} \left( ||g_2||_{L^1} + (1+s)^{n(1-\ell)(1-\beta)} \right) ||g_2||_{L^m}
\end{equation}
for $k \in [0,1)$ and $2 \leq q < \frac{nm}{n+nm+k}$, whereas for $k \geq 1$

\begin{equation}
||u(t,s,\cdot)||_{\dot{H}^k} \lesssim (1+t)^{-\frac{\alpha}{2} + (\ell-1)(\frac{1}{2}+\frac{k}{2})} \left( ||g_2||_{L^1} + (1+s)^{(1-\ell)(\frac{1}{2}+k-1)} \right) ||g_2||_{\dot{H}^{k-1}}.
\end{equation}

(ii): If $\beta = \ell + 2n(1-\ell) \left( 1 - \frac{1}{q} \right) + 2k(1-\ell)$ then

\begin{equation}
||D^k u(t,s,\cdot)||_{L^q} \lesssim (1+s)(1+t)^{(\ell-1)(n-\beta)} \left( \ln \left( \frac{e+t}{e+s} \right) \right)^{1-\frac{\beta}{2n}} \left( ||g_2||_{L^1} + (1+s)^{n(1-\ell)(1-\beta)} \right) ||g_2||_{L^m}
\end{equation}
for $k \in [0,1)$ and $2 \leq q < \frac{nm}{n+nm+k}$, whereas for $k \geq 1$

\begin{equation}
||u(t,s,\cdot)||_{\dot{H}^k} \lesssim (1+s)(1+t)^{(\ell-1)(\frac{1}{2}+k)} \left( \ln \left( \frac{e+t}{e+s} \right) \right)^{\frac{1}{2}} \left( ||g_2||_{L^1} + (1+s)^{(1-\ell)(\frac{1}{2}+k-1)} \right) ||g_2||_{\dot{H}^{k-1}}.
\end{equation}

(iii): If $\beta > \ell + 2n(1-\ell) \left( 1 - \frac{1}{q} \right) + 2k(1-\ell)$ then

\begin{equation}
||D^k u(t,s,\cdot)||_{L^q} \lesssim (1+s)(1+t)^{(\ell-1)(n-\beta)} \left( ||g_2||_{L^1} + (1+s)^{n(1-\ell)(1-\beta)} \right) ||g_2||_{L^m}
\end{equation}
for $k \in [0,1)$ and $2 \leq q < \frac{nm}{n+nm+k}$, whereas for $k \geq 1$. 

\begin{equation}
||u(t,s,\cdot)||_{\dot{H}^k} \lesssim (1+s)(1+t)^{(\ell-1)(\frac{1}{2}+k)} \left( \ln \left( \frac{e+t}{e+s} \right) \right)^{\frac{1}{2}} \left( ||g_2||_{L^1} + (1+s)^{(1-\ell)(\frac{1}{2}+k-1)} \right) ||g_2||_{\dot{H}^{k-1}}.
\end{equation}
If $\beta$ By using Haussdorff-Young inequality and Hölder inequality, setting

$$\|u(t, s, \cdot)\|_{L^2} \lesssim (1 + s)(1 + t)^{(1 - \beta)}(1 + s)^{\frac{(\beta - 1)}{2} - 1} \|g_2\|_{L^1} \beta \left(1 + s\right)^{(1 - \beta)}(1 + t)^{(1 - \beta)} + \|g_2\|_{L^1};$$

Proof. Let $k \geq 0$, $q \geq 2$, $\ell \in (0, 1)$ and $\beta > 1$.

Considerations in $Z_3$: In the zone $Z_3$, from Lemma 4.1 we may estimate

$$|\xi|^k m_1(t, s, \xi) \lesssim |\xi|^k (1 + s).$$

By using Haussdorff-Young inequality and Hölder inequality, setting

$$\frac{1}{r} = 1 - \frac{1}{q},$$

for $q \geq 2$, one may estimate

$$\|F^{-1}(\chi_3(s, \xi)|\xi|^k m_1(t, s, \xi)) * g_2\|_{L^r} \lesssim \|\chi_3(s, \xi)|\xi|^k m_1(t, s, \xi)\|_{L^r} \lesssim \|\chi_3(s, \xi)|\xi|^k m_1(t, s, \xi)\|_{L^r} \lesssim \|\chi_3(s, \xi)|\xi|^k m_1(t, s, \xi)\|_{L^r} \lesssim (1 + s)(1 + t)^{(1 - \beta)(n(1 - \frac{\beta}{2}) + k)} \|g_2\|_{L^1}.$$
we may gluing the estimates in zones $Z_1$ and $Z_3$, namely, for $2 \leq q < \frac{nm}{(n-m+m_{k})}$ and $k \in [0, 1)$, we get

$$
\|\mathcal{F}^{-1}(\chi_1(s, \xi)\xi^k m_1(t, s, \xi)) \ast g_2\|_{L^q} \lesssim (1 + t)^{(\ell-1)(n(1-\frac{1}{2})+k)}(1 + s)^{1+n(1-\ell)(1-\frac{1}{2})}\|g_2\|_{L^m},
$$

whereas for $k \geq 1$ we get

$$
\|\mathcal{F}^{-1}(\chi_1(s, \xi)\xi^km_1(t, s, \xi)) \ast g_2\|_{L^2} \lesssim (1 + t)^{(\ell-1)(n(1-\frac{1}{2})+k)}(1 + s)^{\ell+n(1-\ell)(1-\frac{1}{2})}\|g_2\|_{H^{k-1}}.
$$

Considerations in $Z_2$: In the zone $Z_2$, from Lemma [11] we may estimate

$$
|\xi|^k |m_1(t, s, \xi)| \lesssim |\xi|^{-(\alpha)}(1 + s)(1 + t)^{\ell(\beta-\ell)/2},
$$

where $\alpha = \frac{\beta-\ell}{2(1-\ell)}$. Setting

$$
\frac{1}{r} = 1 - \frac{1}{q},
$$

for $q \geq 2$ then thanks to

$$
\|\mathcal{F}^{-1}(\chi_2(s, \xi)\xi^k m_1(t, s, \xi)) \ast g_2\|_{L^q} \lesssim \|\mathcal{F}^{-1}(\chi_2(s, \xi)\xi^k m_1(t, s, \xi)) \ast g_2\|_{L^r}
$$

one may estimate

$$
\|\mathcal{F}^{-1}(\chi_2(s, \xi)\xi^k m_1(t, s, \xi)) \ast g_2\|_{L^q} \lesssim \|\mathcal{F}^{-1}(\chi_2(s, \xi)\xi^k m_1(t, s, \xi)) \ast g_2\|_{L^r}
$$

for $q \geq 2$ then thanks to

$$
\|\mathcal{F}^{-1}(\chi_2(s, \xi)\xi^k m_1(t, s, \xi)) \ast g_2\|_{L^q} \lesssim \|\mathcal{F}^{-1}(\chi_2(s, \xi)\xi^k m_1(t, s, \xi)) \ast g_2\|_{L^r}
$$



5. Global existence results

By Duhamel’s principle, a function $u \in X$, where $X$ is a suitable space, is a solution to (11) if, and only if, it satisfies the equality

$$
(30) \quad u(t, x) = u^0(t, x) + \int_0^t \int K(t, s, x) \ast_{(x)} f(u(s, x)) \, ds, \quad \text{in } X,
$$

where $u^0(t, x)$ is the solution to the linear Cauchy problem

$$
(31) \quad \begin{cases}
\partial_t u(t, x) - (1 + t)^{-2\ell} \Delta u(t, x) + \frac{\pi}{4(1-\ell)} u(t, x) = 0, & t \geq 0 \\
u(0, x) = 0 \\
u_t(0, x) = u_1(x)
\end{cases}
$$

and $K(t, s, x) \ast_{(x)} f(u(s, x))$ is the solution to the linear Cauchy problem [9] with $g_1 \equiv 0$ and $g_2 \equiv f(u)$, being $K(t, s, x) = \mathcal{F}^{-1}(m_1)(t, s, x)$, i.e.

$$
K(t, s, x) = \frac{-\pi i}{4(1-\ell)(1+s)^{1+\beta-1/2}(1 + t)^{1-\beta/2}} \mathcal{F}^{-1}(\psi_{0,\beta,0})(t, s, x).
$$

The proof of our global existence results is based on the following scheme: We define an appropriate data function space $D$ and an evolution space for solutions $X(T)$.
equipped with a norm relate to the estimates of solutions to the linear problem \(31\) such that
\[
\|u^0\|_X \leq C \|u_1\|_D.
\]
For any \(u \in X\), we define the operator \(P\) by
\[
P : u \in X(T) \rightarrow Pu(t, x) := u^0(t, x) + Fu(t, x),
\]
with
\[
Fu(t, x) := \int_0^t K(t, s, x) * f(u(s, x)) \, ds,
\]
then we prove the estimates
\[
\|Pu\|_X \leq C \|u_1\|_D + C_1(t) \|u\|_X^p,
\]
\[
\|Pu - Pv\|_X \leq C_2(t) \|u - v\|_X (\|u\|_{X}^{p-1} + \|v\|_{X}^{p-1}).
\]
The estimates for the image \(Pu\) allow us to apply Banach’s fixed point theorem. In this way we get simultaneously a unique solution to \(Pu = u\) locally in time for large data and globally in time for small data. To prove the local (in time) existence we use that \(C_1(t), C_2(t)\) tend to zero as \(t\) goes to zero, whereas to prove the global (in time) existence we use \(C_1(t) \leq C\) and \(C_2(t) \leq C\) for all \(t \geq 0\).

### 5.1. Proof of Theorem 2.2

**Proof.** (Theorem 2.2) We define the space
\[
X(T) \doteq C([0, \infty), H^k(\mathbb{R}^n)), \quad \frac{n\ell}{2} + 1 \leq k \leq p_\ell(n, \ell)
\]
equipped with the norm
\[
\|u\|_{X(T)} \doteq \begin{cases}
\sup_{t \in [0, T]} (1 + t)^{(1-\ell)\frac{\ell}{2}} \left( \|u(t, \cdot)\|_{L^2} + (1 + t)^{(1-\ell)k} \|u(t, \cdot)\|_{H^k} \right), & \bar{k} > k, \\
\sup_{t \in [0, T]} (1 + t)^{(1-\ell)\frac{\ell}{2}} \left( \|u(t, \cdot)\|_{L^2} + (1 + t)^{(1-\ell)\bar{k}} (\ln(e + t))^{-\frac{k}{2}} \|u(t, \cdot)\|_{\dot{H}^k} \right), & \bar{k} = k, \\
\sup_{t \in [0, T]} \left(1 + t\right)^{(1-\ell)\frac{\ell}{2}} \|u(t, \cdot)\|_{L^2} + (1 + t)^{\frac{\alpha - \ell}{2}} \|u(t, \cdot)\|_{\dot{H}^k} \right), & \frac{n\ell}{2} \leq \bar{k} < k,
\end{cases}
\]
where \(\bar{k} \doteq \frac{\beta - \ell}{2} + \frac{\ell}{2}\).

We have to prove the global existence in time of the solution \(u\) assuming that there exists \(\delta > 0\) such that
\[
u_1 \in \mathcal{D} \doteq H^{k-1}(\mathbb{R}^n) \cap L^1(\mathbb{R}^n), \quad \|u_1\|_D \leq \delta.
\]
Thanks to Proposition 4.1, \(u^0 \in X(T)\) and it satisfies
\[
\|u^0\|_X \leq C \|u_1\|_D.
\]
It remains to show the estimates
\[
(32) \quad \|Fu\|_X \leq C \|u\|_X^p,
\]
\[
(33) \quad \|Fu - Fv\|_X \leq C \|u - v\|_X (\|u\|_{X}^{p-1} + \|v\|_{X}^{p-1}).
\]
Let us begin by prove \((32)\). Taking into account the definition of the norm in the function space \(X(T)\), we split the proof accordingly to size of \(\beta\):
The case $\tilde{k} > k$, i.e., $\beta > \ell + n(1-\ell) + 2k(1-\ell)$:

Applying Proposition 4.1, we have

$$
\|F(t, \cdot)\|_{L^2} \lesssim \int_0^t (1 + s)(1 + t)^{(\ell - 1)\frac{p}{2}} \left( \|u(s, \cdot)^p\|_{L^1} + (1 + s)^{(1-\ell)n(1 - \frac{1}{k})}\|u(s, \cdot)^p\|_{L^n} \right) ds
$$

and

$$
\|F(t, \cdot)\|_{H^k} \lesssim \int_0^t (1 + s)(1 + t)^{(\ell - 1)\frac{p}{2} + k} \left( \|u(s, \cdot)^p\|_{L^1} + (1 + s)^{(1-\ell)n(\frac{p}{2} + k - 1)}\|u(s, \cdot)^p\|_{H^{k-1}} \right) ds.
$$

First, we use Gagliardo-Nirenberg inequality

$$
\|u(s, \cdot)^p\|_{L^2} \lesssim \|u(s, \cdot)^p\|_{L^{\frac{p}{2}}} \|u(s, \cdot)^p\|_{H^k}, \quad \theta = \frac{n}{k} \left( \frac{1}{2} - \frac{1}{q} \right),
$$

by taking $q = p$ and $q = mp$, where $m \in (1, 2]$ such that $m > \frac{2n}{n+2}$. Since $u \in X(T)$ we may estimate

$$
\|u(s, \cdot)^p\|_{L^2} \lesssim \|u(s, \cdot)^p\|_{L^{\frac{p}{2}}} \|u(s, \cdot)^p\|_{H^k}^{(1-\theta)p}
$$

and

$$
(1 + s)^{(\ell - 1)\frac{p}{2} + (\ell - 1)\frac{1}{k} - \frac{p}{2}q}\|u\|_{X(T)}^{(1 + s)^n(\ell - 1)(p - \frac{p}{2})}\|u\|_{X(T)},
$$

for all $p \geq 2$, $q = p$ and $q = mp$. Therefore, we obtain

$$
\|F(t, \cdot)\|_{L^2} \lesssim (1 + t)^{(\ell - 1)\frac{p}{2}} \int_0^t (1 + s)^{1 + n(\ell - 1)(p - 1)} ds \|u\|_{X(T)}^p
$$

and

$$
\|F(t, \cdot)\|_{H^k} \lesssim (1 + t)^{(\ell - 1)\frac{p}{2} + k}\|u\|_{X(T)}^p;
$$

for $p > 1 + \frac{2}{n(1-\ell)}$.

Then, in order to estimate $\|F(t, \cdot)\|_{H^{k}}$, we may use that $H^k(\mathbb{R}^n)$, with $k > \frac{n}{2}$, is imbedded into $L^\infty(\mathbb{R}^n)$. Indeed, thanks to Corollary 5.2 for $p > \max\{1, k - 1\}$ we may estimate

$$
\|u(s, \cdot)^p\|_{H^{k-1}} \leq C\|u(s, \cdot)^p\|_{H^{k-1}}\|u(s, \cdot)^p\|_{L^\infty}^{k-1}.
$$

Since $u \in X(T)$ we have

$$
\|u(s, \cdot)^p\|_{H^{k-1}} \lesssim (1 + s)^{(\ell - 1)(\frac{n}{2} + k - 1)}\|u\|_{X(T)},
$$

and thanks to Lemma 5.1 for $k < \frac{n}{2} < k$ it follows

$$
\|u(s, \cdot)^p\|_{L^\infty} \lesssim \|u(s, \cdot)^p\|_{H^k} + \|u(s, \cdot)^p\|_{H^{k-1}} \lesssim (1 + s)^{(\ell - 1)(\frac{n}{2} + k)}\|u\|_{X(T)}.
$$

If we choose $k = \frac{n}{2} - \epsilon_0$, with $\epsilon_0$ sufficiently small, then

$$
\|u(s, \cdot)^p\|_{H^{k-1}} \lesssim (1 + s)^{(\ell - 1)(\frac{n}{2} + k - 1) + (\ell - 1)(n - \epsilon_0)(p - 1)}\|u\|_{X(T)},
$$

and

$$
\int_0^t (1 + s)^{1 + n(\ell - 1)(p - 1)} ds \|u\|_{X(T)}^p
$$

for $p > 1 + \frac{2}{n(1-\ell)}$. 

The case $\frac{n\ell}{n} < \tilde{k} < k$, i.e., $\ell + n(1 - \ell)(1 + \ell) \leq \beta < \ell + n(1 - \ell) + 2k(1 - \ell)$: Applying again Proposition 4.1, we have

$$\|F(u(t, \cdot))\|_{L^2} \lesssim \int_0^t (1 + s)(1 + t)^{(1 - \ell)} \left( \|u(s, \cdot)\|_{L^1} + (1 + s)^{(1 - \ell)n(1 - \frac{1}{p})}\|u(s, \cdot)\|_{L^m} \right) ds.$$ 

Now we use the fractional Sobolev embedding

$$\|u(s, \cdot)\|_{L^p} \lesssim \|u(s, \cdot)\|_{\dot{H}^k}, \quad k(q) = n \left( \frac{1}{2} - \frac{1}{q} \right), \quad 2 \leq q < \infty,$$

by taking $q = p$ and $q = mp$, where $\frac{2n}{n + 2} < m \leq 2$. On the one hand, if $\beta > \ell + n(1 - \ell) + 2k_2(1 - \ell)$, with $k_2 = k(mp) = n \left( \frac{1}{2} - \frac{1}{mp} \right)$ we may estimate

$$\|u(s, \cdot)\|_{L^p}^\frac{2}{p} = \|u(s, \cdot)\|_{\dot{H}^k}^\frac{2}{p} \lesssim \|u(s, \cdot)\|_{\dot{H}^k}^\frac{2}{p}, \quad k(q) = n \left( \frac{1}{2} - \frac{1}{q} \right), \quad 2 \leq q < \infty,$$

for $q = p$ and $q = mp$, with $\tilde{k} = k(q)$ and $p \geq 2$. Hence, as before we conclude

$$\|F(u(t, \cdot))\|_{L^2} \lesssim (1 + t)^{\frac{2}{p}(1 - \ell)}\|u\|_{X(T)}^p,$$

for $p > 1 + \frac{2}{n(1 - \ell)}$. On the other hand, if $1 < \beta \leq \ell + n(1 - \ell) + 2k_2(1 - \ell)$ we may estimate

$$\|u(s, \cdot)\|_{L^1} = \|u(s, \cdot)\|_{L^p} \lesssim \|u(s, \cdot)\|_{\dot{H}^k} \lesssim (1 + s)^{p\max(\ell - 1, 0)} (\ln(e + s))^{\frac{2}{2}} \|u\|_{X(T)}^p,$$

with $k_1 = k(p) = n \left( \frac{1}{2} - \frac{1}{p} \right)$, whereas

$$\|u(s, \cdot)\|_{L^m} = \|u(s, \cdot)\|_{L^m} \lesssim \|u(s, \cdot)\|_{\dot{H}^k} \lesssim (1 + s)^{\frac{m}{2} + p(\ln(e + s))^{\frac{2}{2}}} \|u\|_{X(T)}^p,$$

with $k_2 = n \left( \frac{1}{2} - \frac{1}{mp} \right)$. Therefore

$$\|F(u(t, \cdot))\|_{L^2} \lesssim (1 + t)^{\frac{2}{p}(1 - \ell)} \int_0^t (1 + s)^{1 + \max(\ell - 1, 0)} (\ln(e + s))^{\frac{2}{2}} ds \|u\|_{X(T)}^p$$

$$+ (1 + t)^{\frac{2}{p}(1 - \ell)} \int_0^t (1 + s)^{(1 + \ell)(1 - \ell)n(1 - \frac{1}{p}) + \frac{m}{2}} (\ln(e + s))^{\frac{2}{2}} ds \|u\|_{X(T)}^p$$

$$\lesssim (1 + t)^{\frac{2}{p}(1 - \ell)}\|u\|_{X(T)}^p,$$

for $p > 1 + \frac{2}{n(1 - \ell)}$, $m > \frac{2n}{n + 2}$ and

$$\beta > \ell + n(1 - \ell) + \frac{2n(1 - \ell)}{2 + n(1 - \ell)}.$$

Moreover, applying again Proposition 4.1 for $1 < \beta < \ell + n(1 - \ell) + 2k(1 - \ell)$, we have

$$\|F(u(t, \cdot))\|_{\dot{H}^k} \lesssim \int_0^t (1 + t)^{\frac{2}{p}(1 + s)^{1 + \frac{m}{2}} + (\ell - 1)(\frac{2}{2} + k)} \left( \|u(s, \cdot)\|_{L^1} + (1 + s)^{(1 - \ell)(\frac{2}{2} + k)}\|u(s, \cdot)\|_{L^m} \right) ds.$$
As before we may estimate
\[ \|u(s, \cdot)^p\|_{L^1} = \|u(s, \cdot)^p\|_{L^p} \lesssim \|u(s, \cdot)^p\|_{\dot{H}^{k_1}} \]
\[ \lesssim \|u\|_{X(T)}^p \begin{cases} (1 + s)^{(e-1)(\frac{p}{2})} & \text{if } k_1 \leq \bar{k} \\ (1 + s)^{(e-\beta)} & \text{if } k_1 > \bar{k} \end{cases} \]
with \( k_1 = n \left( \frac{1}{2} - \frac{1}{p} \right) \), then
\[ \int_0^t (1 + s)^{1 + \frac{\beta}{2} + (\ell-1)\left(\frac{p}{2} + k\right)} \|u(s, \cdot)^p\|_{L^1} ds \lesssim \|u\|_{X(T)}^p \]
for \( p > 1 + \frac{2}{n(1-\ell)} \) and \( \ell + \frac{4(n-1)}{2 + n(1-\ell)} < \beta < \ell + n(1-\ell) + 2k(1-\ell) \).

If \( p > \max\{1, k-1\} \) we may estimate
\[ \|u(s, \cdot)^p\|_{\dot{H}^{k-1}} \lesssim \|u\|_{X(T)} \|u(s, \cdot)^p\|_{L^\infty}^{-1} \]
\[ \lesssim \|u\|_{X(T)} \begin{cases} (1 + s)^{(e-1)(\frac{p}{2} - 1)} & \text{if } k - 1 \leq \bar{k} \\ (1 + s)^{\frac{\beta - e}{2}} & \text{if } k - 1 > \bar{k} \end{cases} \]

The condition \( k - 1 \leq \bar{k} \) is equivalent to \( \beta \geq \ell + n(1-\ell) + 2(k-1)(1-\ell) \).

For \( \ell > 1 - \frac{2}{n}, k \geq 1 + \frac{n\ell}{2} \) and \( \beta \geq \ell + n(1-\ell) + 2(k-1)(1-\ell) \) we obtain
\[ \int_0^t (1 + s)^{1 + \frac{\beta}{2} + (\ell-1)\left(\frac{p}{2} + k\right)} (1 + s)^{(1-\ell)(\frac{p}{2} + k-1)} \|u(s, \cdot)^p\|_{\dot{H}^{k-1}} ds \lesssim \|u\|_{X(T)}^p \]
for \( p > 1 + \frac{2}{n(1-\ell)} \), hence
\[ \|Fu(t, \cdot)^p\|_{\dot{H}^{k}} \lesssim (1 + t)^{\frac{\beta - e}{2}} \|u\|_{X(T)}^p. \]

Here we remark that
\[ \ell + n(1-\ell) + 2(k-1)(1-\ell) \geq \ell + n(1-\ell) + \frac{2n(k(\ell - 1) + 1)(1-\ell)}{2 - n(1-\ell)} \]
for all \( k \geq 1 + \frac{4\ell}{2} \).

• The case \( \bar{k} = k \), i.e., \( \beta = \ell + n(1-\ell) + 2k(1-\ell) \):

In this case one may conclude that
\[ \|Fu(t, \cdot)^p\|_{L^2} \lesssim (1 + t)^{\frac{\beta}{2} + (\ell-1)} \|u\|_{X(T)}^p, \]
and
\[ \|Fu(t, \cdot)^p\|_{\dot{H}^{k}} \lesssim (1 + t)^{\frac{\beta - e}{2} + (1-\ell)n(1-\frac{1}{2})} \|u\|_{X(T)}^p \]
for \( p > 1 + \frac{2}{n(1-\ell)} \).

Finally, let us discuss the proof of (33) only in the case \( \beta > \ell + n(1-\ell) + 2k(1-\ell) \).

Applying Proposition 4.1 we have
\[ \|Fu(t, \cdot) - Fv(t, \cdot)^p\|_{L^2} \lesssim (1 + t)^{(\ell-1)\frac{\beta}{2}} \int_0^t (1 + s)^{(f(u) - f(v))(s, \cdot)} ds 
+ (1 + t)^{(\ell-1)\frac{\beta}{2}} \int_0^t (1 + s)^{1 + (1-\ell)n(1-\frac{1}{2})} \|f(u) - f(v)(s, \cdot)\|_{L^\infty} ds. \]
Applying again Proposition 4.1 we have
\[ p > k \]
and
\[ \text{Here, we may take } \]

By using (2) and Hölder inequality, we find that

\[ H^{(\ell-1)}(\frac{p}{2}) \|u - v\|_{X(T)}\left(\|u\|_{X(T)}^{-1} + \|v\|_{X(T)}^{-1}\right), \]

for any \(1 \leq \alpha \leq m\). Therefore

\[ \|Fu(u, \cdot) - Fv(t, \cdot)\|_{L^p} \lesssim (1 + t)^{(\ell-1)(\frac{p}{2} + \frac{1}{2k} + \frac{1}{2} + \frac{1}{2})) \int \|f(u) - f(v)\|(s, \cdot)\|_{L^1} ds + (1 + t)^{(\ell-1)(\frac{p}{2} + \frac{1}{2k} + \frac{1}{2} + \frac{1}{2})) \int \|f(u) - f(v)\|(s, \cdot)\|_{\dot{H}^{k-1}} ds. \]

From now we assume that \(f(u) = |u|^p\), without lose of generality. In order to estimate \(\|(f(u) - f(v))(s, \cdot)\|_{\dot{H}^{k-1}}\), we use

\[ |u(s, x)|^p - |v(s, x)|^p = p \int_0^1 |v + \tau(u - v)|^{p-2}(v + \tau(u - v))(s, x) d\tau(u - v)(s, x). \]

Hence, applying Proposition 5.3 gives

\[ \|u(s, x)|^p - |v(s, x)|^p\|_{\dot{H}^{k-1}} \lesssim \|u - v\|(s, \cdot)\|_{\dot{H}^{k-1}} \int_0^1 \|v + \tau(u - v)|^{p-2}(v + \tau(u - v))(s, \cdot)\|_{\dot{H}^{k-1}} d\tau + \|u - v\|(s, \cdot)\|_{\dot{H}^{k-1}} \int_0^1 \|v + \tau(u - v)|^{p-2}(v + \tau(u - v))(s, \cdot)\|_{\dot{H}^{k-1}} d\tau. \]

Now, since \(u, v \in X(T)\) we have

\[ \|(u - v)(s, \cdot)\|_{\dot{H}^{k-1}} \lesssim (1 + s)^{(\ell-1)(\frac{p}{2} k + k-1)} \|u - v\|_{X(T)}, \]

Applying Lemma 5.1 for \(k < \frac{2}{p} < k\) it follows

\[ \|(u - v)(s, \cdot)\|_{\dot{H}^{k}} \lesssim (1 + s)^{(\ell-1)(\frac{p}{2} + k-1)} \|u - v\|_{X(T)}, \]

and

\[ \|v + \tau(u - v)|^{p-2}(v + \tau(u - v))(s, \cdot)\|_{\dot{H}^{k-1}} \lesssim (1 + s)^{(\ell-1)(\frac{p}{2} + k-1)(p-1)} \|u\|_{X(T)}^{-1} + \|v\|_{X(T)}^{-1}, \]

with \(k = \frac{2}{p} - \varepsilon_0\) and \(\varepsilon_0\) sufficiently small.

For \(p > k\) Corollary 5.2 implies

\[ \|v + \tau(u - v)|^{p-2}(v + \tau(u - v))(s, \cdot)\|_{\dot{H}^{k-1}} \lesssim (1 + s)^{(\ell-1)(\frac{p}{2} + k-1)(1 + s)^{(\ell-1)(\frac{p}{2} + k-1)(p-2)}(\|u\|_{X(T)}^{-1} + \|v\|_{X(T)}^{-1})}. \]
Therefore
\[ \| Fu(t, \cdot) - Fv(t, \cdot) \|_{H^k} \lesssim (1 + t)^{(\ell - 1)}(\frac{2}{\ell} + k) \int_0^t (1 + s)^{1 + n(\ell - 1)(p - 1)} ds \| u - v \|_{X(T)} (\| u \|_{X(T)}^{p-1} + \| v \|_{X(T)}^{p-1}) \]
\[ + (1 + t)^{(\ell - 1)}(\frac{2}{\ell} + k) \int_0^t (1 + s)^{(\ell - 1)(n - \varepsilon_0)(p - 1)} ds \| u - v \|_{X(T)} (\| u \|_{X(T)}^{p-1} + \| v \|_{X(T)}^{p-1}) \]
\[ \leq (1 + t)^{(\ell - 1)}(\frac{2}{\ell} + k) \| u - v \|_{X(T)} (\| u \|_{X(T)}^{p-1} + \| v \|_{X(T)}^{p-1}), \]
for \( p > 1 + \frac{2}{n - \varepsilon_0}. \)

5.2. The proof of Theorem 2.1 By applying the change of variable
\[ v(\tau, x) = u(t, x), \quad 1 + \tau = \frac{(1 + t)^{1 - \ell}}{1 - \ell}, \]
the Cauchy problem \((1)\) takes the form
\[ v_{\tau\tau} - \Delta v + \frac{\mu}{1 + \tau} v_\tau = g(v), \quad \tau \geq s, \ x \in \mathbb{R}^n, \]
\[ v(s, x) = 0, \quad x \in \mathbb{R}^n, \]
\[ v_\tau(s, x) = u_1(x), \quad x \in \mathbb{R}^n, \tag{35} \]
with \( s = \frac{\ell}{1 - \ell}, \ g(v) = [(1 - \ell)(1 + \tau)]^{\frac{2\ell}{1 - \ell}} |v|^p \)
and
\[ \mu = \frac{\beta - \ell}{1 - \ell}. \]

We now enunciate Corollary 2 from D’Abbicco’s paper \([4]\), which will be useful in the proof of the Theorem 2.1. There was introduced the following notation: For any \( 1 \leq r \leq q \leq \infty, \) let be
\[ d(r, q) = \begin{cases} \frac{n}{r} - \frac{n + 1}{q} - \frac{1}{q} & \text{if } r < q' \\ \frac{1}{r} + \frac{n - 1}{2} - \frac{1}{q} & \text{if } r \geq q' \end{cases} \]

Corollary 5.1. (see \([4]\)) Let \( \mu \geq 2. \) Let \( n = 2 \) and \( 2 < q \leq q_1, \) or \( n = 3 \) and \( q \in (1, 4) \) or \( n \geq 4 \) and \( \frac{2(n - 1)}{n + 1} \leq q \leq q_1. \) Then there exists \( r_2 \in (1, \min\{q, q_1\}) \) such that \( d(r_2, q) = 1 \) and the solution to \((35)\) verifies the following \((L^1 \cap L^2) - L^q\) decay estimate
\[ \| v(t, \cdot) \|_{L^q} \lesssim (1 + s)(1 + \tau)^{-n(\frac{\mu}{2} - \frac{\mu}{q})} (\| u_1 \|_{L^1} + (1 + s)^{\frac{n - 1}{2}} \| u_1 \|_{L^2}) \]
if \( \mu > n + 1 - \frac{2}{q}, \) and for any \( \varepsilon > 0 \) verifies the \((L^1 \cap L^2) - L^q\) estimate
\[ \| v(t, \cdot) \|_{L^q} \lesssim (1 + s)^{\frac{\mu}{2} - \varepsilon} (1 + \tau)^{\varepsilon - (n - 1)(\frac{\mu}{2} - \frac{\mu}{q})} (1 + s)^{\frac{n - 1}{2} - \frac{n - 1}{q}} \| u_1 \|_{L^1} + \| u_1 \|_{L^2} \]
if \( \mu \leq n + 1 - \frac{2}{q}. \)

Proof. (Theorem 2.1) We define the space
\[ X(T) = C([0, \infty), L^p(\mathbb{R}^n) \cap L^q(\mathbb{R}^n)), \]
equipped with the norm
\[ \|v\|_{X(T)} \doteq \sup_{\tau \in [0,T]} \left\{ (1 + \tau)^{n(1 - \frac{1}{q})} \|v(\tau, \cdot)\|_{L^p} + (1 + \tau)^n \|v(\tau, \cdot)\|_{L^q} \right\} \]

for \( \mu > n + 1 - \frac{2}{q_2} \) and,
\[ \|v\|_{X(T)} \doteq \sup_{\tau \in [0,T]} \left\{ (1 + \tau)^{n(1 - \frac{1}{q_2})} \|v(\tau, \cdot)\|_{L^p} + (1 + \tau)^n \|v(\tau, \cdot)\|_{L^q} \right\} \]

for \( n + 1 - \frac{2}{q} < \mu \leq n + 1 - \frac{2}{q_2} \) and,
\[ \|v\|_{X(T)} \doteq \sup_{\tau \in [0,T]} \left\{ (1 + \tau)^{n(1 - \frac{1}{q_2})} \|v(\tau, \cdot)\|_{L^p} + (1 + \tau)^n \|v(\tau, \cdot)\|_{L^q} \right\} \]

for \( \mu = n + 1 - \frac{2}{q} \).

In the following we only verify how to prove the global existence in time assuming that there exists \( \delta > 0 \) such that
\[ u_1 \in \mathcal{D} \doteq L^2(\mathbb{R}^n) \cap L^1(\mathbb{R}^n), \quad \|u_1\|_{\mathcal{D}} \leq \delta. \]

Thanks to Corollary 5.1 if \( \beta \geq 2 \), then \( v^0 \in X(T) \) and it satisfies
\[ \|v^0\|_X \leq C \|u_1\|_p. \]

Let us prove (3.2). Applying Corollary 5.1 we have for \( \mu > n + 1 - \frac{2}{q_2} \) and for all \( p_c \leq q \leq q_2 \)
\[ \|Fv(\tau, \cdot)\|_{L^q} \lesssim \int_0^\tau (1 + s)^{1 + \frac{2\beta}{r(q)}(1 + \tau)^{-n(1 - \frac{1}{q})}} \left[ \left( \|v(s, \cdot)\|_{L^p}^{p-1} \right)^{\frac{1}{p-1}} \|v(s, \cdot)\|_{L^{r(q)}}^p \right] ds \]

with \( r(q) \in [1,2] \) given by \( \frac{n}{r(q)} = \frac{1}{2} + \frac{2}{\beta} + \frac{1}{q} \). Taking into account that \( u \in X(T) \), thanks to \( r(q_2)p_c < q_2 \) for all \( p_c \leq q \leq q_2 \), we may estimate for \( p_c < p \leq \frac{q}{r(q_2)} \)
\[ \|v(s, \cdot)\|_{L^r}^p \lesssim (1 + s)^{-n(1 - \frac{1}{r(q_2)})p} \|v\|_{X(T)}^p \]
\[ \lesssim (1 + s)^{-n(1 - \frac{1}{r(q_2)})p} \|v\|_{X(T)}^p. \]

Therefore, for \( \mu > n + 1 - \frac{2}{q_2} \) we have for all \( p_c \leq q \leq q_2 \)
\[ \|Fv(t, \cdot)\|_{L^q} \lesssim (1 + \tau)^{-n(1 - \frac{1}{q_2})} \int_0^\tau (1 + s)^{1 + \frac{2\beta}{r(q_2)}(1 + s)^{-n(1 - \frac{1}{q_2})}} ds \|v\|_{X(T)}^p \]
\[ + (1 + \tau)^{-n(1 - \frac{1}{q_2})} \int_0^\tau (1 + s)^{1 + \frac{2\beta}{r(q_2)}(1 + s)^{-n(1 - \frac{1}{q_2})}} ds \|v\|_{X(T)}^p \]
\[ \lesssim (1 + \tau)^{-n(1 - \frac{1}{q_2})} \|v\|_{X(T)}^p. \]

for
\[ p > 1 + \frac{2}{n(1 - \ell)}. \]
For \( n + 1 - \frac{q}{p} < \mu \leq n + 1 - \frac{2}{q_1} \), applying again Corollary 5.1 for \( p_c \leq q \leq \bar{q} \) we may estimate

\[
\|Fv(t, \cdot)\|_{L_s} \lesssim (1 + \tau)^{-n(1 - \frac{\mu}{2})} \int_0^\tau (1 + s)^{1 + \frac{\mu}{2}} \left( \|v(s, \cdot)\|_{L^1} + (1 + s)^{\frac{n}{q_1} - \frac{\mu}{4}} \|v(s, \cdot)\|_{L_{\frac{q_1}{4}}} \right) ds
\]

with \( \frac{\mu}{q_1} = \frac{1}{p} + \frac{q}{p} + \frac{q}{q_1} \). For \( n + 1 - \frac{2}{q} < \mu \leq n + 1 - \frac{2}{q_1} \), we may estimate

\[
\|v(s, \cdot)\|_{L^p}^p = \|v(s, \cdot)\|_{L_{rc}^p}^p \lesssim \|v(s, \cdot)\|_{L_{rc}^{qpc}}^{(1 - \theta)p} \|v(s, \cdot)\|_{L_{rc}^{qpc}}^{\theta p} \lesssim (1 + s)^{-n(1 - \frac{\mu}{p}) + (\varepsilon - (n - 1)(\frac{1}{q_1} - \frac{1}{q}) + 1)p\theta} \|v\|_{X(T)}^p \lesssim (1 + s)^{-n(1 - \frac{\mu}{p})} \|v\|_{X(T)}^p,
\]

thanks to

\[
\varepsilon - (n - 1) \left( \frac{1}{2} - \frac{1}{q_1} \right) - \frac{\mu}{2} + n \left( 1 - \frac{1}{q_1} \right) \leq \varepsilon + (n - 1) \left( \frac{1}{q_1} - \frac{1}{p_c} \right) \leq 0,
\]

for \( \varepsilon > 0 \) and \( \theta = \left( \frac{1}{p_c} - \frac{1}{q_1} \right) / \left( \frac{1}{r(q)p_c} - \frac{1}{r(q)p} \right) \).

Thanks to \( r(q)p_c \leq \bar{q} \) for all \( p_c \leq q \leq \bar{q} \) and, for \( n + 1 - \frac{2}{q} < \mu \leq n + 1 - \frac{2}{q_1} \), we may estimate

\[
\|v(s, \cdot)\|_{L^{r(s)}_{\gamma}} = \|v(s, \cdot)\|_{L_{\gamma}^{r(s)p}}^p \lesssim \|v(s, \cdot)\|_{L_{\gamma}^{r(s)p}}^{(1 - \theta)p} \|v(s, \cdot)\|_{L_{\gamma}^{r(s)p}}^\theta \lesssim (1 + s)^{-n(1 - \frac{\mu}{r(s)p_c}) + (\varepsilon - (n - 1)(\frac{1}{r(s)p} - \frac{1}{r(q)p})) + 1)p\theta} \|v\|_{X(T)}^p
\]

with \( \theta = \left( \frac{r(s)p_c}{r(s)p} - \frac{1}{r(q)p} \right) / \left( \frac{1}{r(s)p_c} - \frac{1}{r(q)p} \right) \). Now,

\[
\gamma = -n \left( 1 - \frac{1}{r(q)p_c} \right) + \left( \varepsilon - (n - 1) \left( \frac{1}{2} - \frac{1}{r(q)p} \right) - \frac{\mu}{2} + n \left( 1 - \frac{1}{r(q)p_c} \right) \right) p\theta
\]

\[
\leq -np + \frac{n}{r(q)} + \left( \varepsilon - (n - 1) \left( \frac{1}{2} - \frac{1}{r(q)p} \right) - \frac{\mu}{2} + n \left( 1 - \frac{1}{r(q)p_c} \right) \right) p\theta
\]

\[
\leq -np + \frac{n}{r(q)} + \left( \frac{1}{r(q)p_c} - \frac{1}{r(q)p} \right) p\theta + \varepsilon p\theta
\]

\[
\leq -np + \frac{n}{r(q)} + \left( \frac{1}{r(q)} \left( \frac{1}{p_c} - \frac{1}{p} \right) \right) p + \varepsilon p\theta
\]

\[
\leq -np \left( 1 - \frac{1}{p_c} \right) + \frac{np}{p_c} + \frac{n}{r(q)} + \frac{p}{p_c} - 1 + \varepsilon p\theta.
\]

Therefore, for \( n + 1 - \frac{2}{q} < \mu \leq n + 1 - \frac{2}{q_1} \), and for \( p_c \leq q \leq \bar{q} \) conclude that

\[
\|Fv(t, \cdot)\|_{L_s} \lesssim (1 + \tau)^{-n(1 - \frac{\mu}{p})} \int_0^\tau (1 + s)^{1 + \frac{\mu}{2}} \left( \|v(s, \cdot)\|_{X(T)}^p + (1 + s)^{\frac{n}{q_1} - \frac{\mu}{4}} \|v(s, \cdot)\|_{X(T)} \right) ds
\]

\[
+ (1 + \tau)^{-n(1 - \frac{\mu}{p})} \int_0^\tau (1 + s)^{1 + \frac{\mu}{2}} \left( \|v(s, \cdot)\|_{X(T)}^p + (1 + s)^{\frac{n}{q_1} - \frac{\mu}{4}} \|v(s, \cdot)\|_{X(T)} \right) \|v\|_{X(T)}^p
\]

\[
\lesssim (1 + \tau)^{-n(1 - \frac{\mu}{p})} \|v\|_{X(T)}^p.
\]
for
\[ p > \frac{2}{n(1 - \ell)} \frac{p_c}{p_c - 1} = 1 + \frac{2}{n(1 - \ell)}. \]
Now, for \( \mu = n + 1 - \frac{2}{q} \), applying again Corollary 5.1, we may estimate \( \|Fv(\tau, \cdot)\|_{L^p_c} \)
as before, whereas for \( q = \bar{q} \) or \( q = q_2 \)
\[ \|Fv(\tau, \cdot)\|_{L^q} \lesssim (1 + \tau)^{\epsilon - n \left( \frac{1}{2} - \frac{\ell}{q} \right) - \frac{\mu}{2}} \int_0^\tau (1 + s)^{\frac{2\ell}{r(q) + 2 - \epsilon}} \left( 1 + s \right)^{\frac{\ell}{r(q) + 2 - \epsilon}} \|v(s, \cdot)\|_{L^1} + \|v(s, \cdot)\|_{L^{r(\tau)}(s)} \) \, ds \]
for any \( \epsilon > 0 \), with \( \frac{n}{r(q)} = \frac{1}{2} + \frac{\mu}{2} + \frac{1}{\bar{q}} \). Taking into account that \( u \in X(T) \), as before, we may estimate
\[ \|v(s, \cdot)\|_{L^1} \lesssim (1 + s)^{-n \left( \frac{1}{2} - \frac{\ell}{r(q)} \right)p} \|v\|_{X(T)}^p \]
and thanks to \( r(\bar{q})p_c \leq r(q_2)p_c < q_2 \) (see Remark 2.1), we may estimate for \( p_c < p \leq \frac{q}{r(q_2)} \)
\[ \|v(s, \cdot)\|_{L^{r(\tau)}(s)} \lesssim (1 + s)^{\epsilon - (n - 1) \left( \frac{1}{2} - \frac{1}{r(q)} \right)} \|v\|_{X(T)}^p \]
for any \( \epsilon > 0 \).
Now we may write
\[ \frac{2\ell}{1 - \ell} + \frac{\mu}{2} - \epsilon + \left( \epsilon - (n - 1) \left( \frac{1}{2} - \frac{1}{r(q)p} \right) - \frac{\mu}{2} \right) p = 1 + \frac{2\ell}{1 - \ell} + \epsilon(p - 1) + n - \frac{1}{r(q)} - (n - 1 + \mu) \frac{p}{2} + \gamma, \]
with
\[ \gamma = \frac{\mu}{2} - 1 + n \left( \frac{1}{r(q) - 1} \right) + \frac{1}{r(q)} - \frac{1}{r(q)}. \]
For \( \mu = n + 1 - \frac{2}{q} \) and \( q = \bar{q} \) we have that \( \gamma = 0 \), whereas for \( q = q_2 \) we have
\[ \gamma = \frac{\mu}{2} - 1 + n \left( \frac{1}{r(q_2) - 1} \right) + \frac{1}{r(q_2)} - \frac{1}{r(q_2)} = \frac{(n + 1)}{2} - \frac{1}{q} - 1 + \frac{1}{r(q_2)} + \frac{n}{q_2} + \frac{1}{n} \left( \frac{1}{q} - \frac{1}{q_2} \right) = \left( \frac{1}{n} - 1 \right) \left( \frac{1}{q} - \frac{1}{q_2} \right) < 0 \]
We conclude that for \( \mu = n + 1 - \frac{2}{q} \) and \( q = \bar{q} \) or \( q = q_2 \)
\[ \|Fv(\tau, \cdot)\|_{L^q} \lesssim (1 + \tau)^{\epsilon - (n - 1) \left( \frac{1}{2} - \frac{1}{r(q)} \right)} \int_0^\tau (1 + s)^{\frac{2\ell}{r(q) + 2 - \epsilon}} \left( 1 + s \right)^{\frac{\ell}{r(q) + 2 - \epsilon}} \|v(s, \cdot)\|_{L^1} + \|v(s, \cdot)\|_{X(T)}^p \] \, ds \]
for any \( \epsilon > 0 \), \( p > p_c = 1 + \frac{2}{n(1 - \ell)} \) and
\[ 1 + \frac{2\ell}{1 - \ell} + \epsilon(p - 1) + n - \frac{1}{r(q)} - (n - 1 + \mu) \frac{p}{2} < -1 \]
i.e.
\[ (n - 1 + \mu) \frac{p}{2} \geq \frac{2}{1 - \ell} + n - \frac{1}{r(q)}. \]
is equivalent to
\[ \mu \geq n + 1 - \frac{2}{\rho_c r(q)} = n + 1 - \frac{2}{q_f}. \]

Moreover, for \( n + 1 - \frac{2}{q_f} < \mu \leq n + 1 - \frac{2}{q_s} \), we have
\[
\| Fv(\tau, \cdot) \|_{L^{q_s}} \lesssim (1 + \tau)^{\varepsilon - (n-1)\left(\frac{1}{2} - \frac{1}{q_s}\right)} \int_0^\tau (1 + s)^{\frac{2\ell}{1-\ell} + \frac{p}{2} - \varepsilon} + (1 + s)^{\frac{2\ell}{1-\ell} + \frac{p}{2}} \| v(s, \cdot) \|_{L^1} + \| v(s, \cdot) \|_{L^r(q_s)} \, ds
\]
for any \( \varepsilon > 0 \), with \( \frac{n}{\rho_c r(q_f)} = \frac{1}{2} + \frac{n}{q_f} + \frac{1}{q_s} \).

If \( n + 1 - \frac{2}{\rho_c r(q_f)} < \mu \leq n + 1 - \frac{2}{q_s} \) we may estimate
\[
\| v(s, \cdot) \|_{L^r(q_s)} \lesssim (1 + s)^{\varepsilon - (n-1)\left(\frac{1}{2} - \frac{1}{q_s}\right) - \frac{p}{2}} \| v \|_{X(T)}^p,
\]
hence
\[
(1 + s)^{\frac{2\ell}{1-\ell} + \frac{p}{2} - \varepsilon} \| v(s, \cdot) \|_{L^r(q_s)} \leq (1 + s)^{\frac{2\ell}{1-\ell} + \varepsilon (p - 1) - n\left(\frac{1}{2} - \frac{1}{q_s}\right) + 1 + (n - 1 + \mu) \frac{p}{2}}
\]
for \( \varepsilon > 0 \) sufficiently small and
\[
p_c < p \leq \left( \frac{1}{r(q_f)} - \frac{1}{q_f} \right) \rho_c r(q_f) = \frac{n}{\rho_c r(q_f)} - 1 \leq 0.
\]

Finally, if \( n + 1 - \frac{2}{q_f} < \mu \leq n + 1 - \frac{2}{\rho_c r(q_f)} \) we may estimate for \( p_c < p \leq \frac{q_f}{\rho_c r(q_f)} \)
\[
\| v(s, \cdot) \|_{L^r(q_s)} \| v(s, \cdot) \|_{L^r(q_s)} \lesssim (1 + s)^{\varepsilon - (n-1)\left(\frac{1}{2} - \frac{1}{q_s}\right) - \frac{p}{2}} \| v \|_{X(T)}^p
\]
for any \( \varepsilon > 0 \).

Now we may write
\[
\frac{2\ell}{1-\ell} + \frac{p}{2} - \varepsilon + (\varepsilon - (n-1)\left(\frac{1}{2} - \frac{1}{r(q_f)}\right)) = \frac{1}{r(q_f)} - 1 + \varepsilon (p - 1) + n\left(\frac{1}{2} - \frac{1}{r(q)}\right) - (n - 1 + \mu) \frac{p}{2} + \gamma,
\]
with
\[
\gamma = \frac{1}{r(q_f)} - 1 + n\left(\frac{1}{r(q_f)} - 1\right) + \frac{1}{r(q)} - \frac{1}{r(q_f)}.
\]
It remains to prove that \( \gamma \leq 0 \). Indeed, for \( \mu \leq n + 1 - \frac{2}{\rho_c r(q_f)} \) and for \( \frac{1}{q_f} \leq \frac{1}{n-1} \left( \frac{n}{\rho_c r(q_f)} - \frac{1}{q_f} \right) \) (see Remark 2.3) we have
\[
\gamma = \frac{1}{r(q_f)} - 1 + n\left(\frac{1}{r(q_f)} - 1\right) + \frac{1}{r(q)} - \frac{1}{r(q_f)}
\]
\[
= \frac{1}{r(q_f)} - 1 + \frac{1}{2} - \frac{1}{q_f} + \frac{1}{n} \left( \frac{1}{q} - \frac{1}{q_f} \right)
\]
\[
\leq \frac{1}{r(q_f)} - 1 + \frac{1}{2} - \frac{1}{q_f} + \frac{1}{n} \left( \frac{1}{q} - \frac{1}{q_f} \right) \leq 0.
\]
Therefore, if \( n + 1 - \frac{2}{q} < \mu \leq n + 1 - \frac{2}{q} \) we have proved that
\[
\| Fv(s, \cdot) \|_{L^\infty} \lesssim (1 + \tau)^{-\varepsilon(n-1)}(\varepsilon^{-\frac{1}{2}})^{-\varepsilon} \| v \|_{L^p(X(T))},
\]
for any \( \varepsilon > 0 \) sufficiently small, \( p > p_c = 1 + \frac{2}{m(1-\ell)} \).

\[\square\]

**APPENDIX**

In the Appendix we list some notations used through the paper and results of Harmonic Analysis which are important tools for proving results on the global existence of small data solutions for semi-linear models with power non-linearities. Through this paper, we use the following.

For any \( q \in [1, \infty] \), we denote by \( L^q(\mathbb{R}^n) \) the usual Lebesgue space over \( \mathbb{R}^n \). Let \( s \in \mathbb{R} \) and \( 1 < p < \infty \). Then
\[
H^{s,p}(\mathbb{R}^n) = \{ u \in S'(\mathbb{R}^n) : \| (D)^s u \|_{L^p(\mathbb{R}^n)} = \| u \|_{H^s_p(\mathbb{R}^n)} < \infty \},
\]
\[
\hat{H}^{s,p}(\mathbb{R}^n) = \{ u \in S'(\mathbb{R}^n) : \| (D)^s u \|_{L^p(\mathbb{R}^n)} = \| u \|_{\hat{H}^s_p(\mathbb{R}^n)} < \infty \}
\]
are called Bessel and Riesz potential spaces, respectively. If \( p = 2 \), then we use the notations \( H^s(\mathbb{R}^n) \) and \( \hat{H}^s(\mathbb{R}^n) \), respectively. In the definition of the Riesz potential spaces we use the space of distributions \( \mathcal{C}'(\mathbb{R}^n) \). This space of distributions can be identified with the factor space \( \mathcal{S}'/\mathcal{P} \), where \( \mathcal{S}' \) denotes the dual of Schwartz space and \( \mathcal{P} \) denotes the set of all polynomials.

We recall that \( H^{s,q}(\mathbb{R}^n) = W^{s,q}(\mathbb{R}^n) \), the usual Sobolev space, for any \( q \in (1, \infty) \) and \( s \in \mathbb{N} \).

The following inequality can be found in [13], Part 1, Theorem 9.3.

**Proposition 5.1 (Fractional Gagliardo-Nirenberg inequality).** Let \( 1 < p, p_0, p_1 < \infty \), \( \sigma > 0 \) and \( s \in (0, \sigma) \). Then it holds the following fractional Gagliardo-Nirenberg inequality for all \( u \in L^{p_0}(\mathbb{R}^n) \cap \hat{H}^{s,p_1}(\mathbb{R}^n) \):
\[
\| u \|_{H^{s,p}} \lesssim \| u \|_{L^{p_0}}^{1-\theta} \| u \|_{\hat{H}^{s,p_1}}^{\theta},
\]
where \( \theta = \theta_{s,\sigma}(p, p_0, p_1) = \frac{\frac{1}{p_0} - \frac{1}{p} + \frac{\sigma}{p_1}}{\frac{\sigma}{p_1} - \frac{1}{p_1}} \) and \( \frac{\sigma}{p_1} \leq \theta \leq 1 \).

We present here a result for fractional powers [21].

**Proposition 5.2.** Let \( p > 1 \), \( f(u) = |u|^p \) or \( f(u) = |u|^{p-1}u \) and \( u \in H^{s,m} \), where \( s \in (\frac{n}{m}, p) \), \( 1 < m < \infty \). Then the following estimate holds:
\[
\| f(u) \|_{H^{s,m}} \leq C \| u \|_{H^{s,m}} \| u \|_{L^\infty}^{p-1}.
\]

In [11] the following corollary was derived:

**Corollary 5.2.** Let \( f(u) = |u|^p \) or \( f(u) = |u|^{p-1}u \), with \( p > \max\{1, s\} \) and \( u \in H^{s,m} \cap L^\infty \), \( 1 < m < \infty \). Then the following estimate holds:
\[
\| f(u) \|_{H^{s,m}} \leq C \| u \|_{H^{s,m}} \| u \|_{L^\infty}^{p-1}.
\]

We refer to [5] for the next result:
Lemma 5.1. Let $0 < 2s_1 < n < 2s_2$. Then for any function $f \in \dot{H}^{s_1} \cap \dot{H}^{s_2}$ one has
\[
\|f\|_\infty \lesssim \|f\|_{\dot{H}^{s_1}} + \|f\|_{\dot{H}^{s_2}}.
\]

The next result combine in some sense some familiar results as Leibniz rule for the product of two function and Hölder’s inequality for derivatives of fractional order (Theorem 7.6.1 in [19]):

Proposition 5.3. Let us assume $s > 0$ and $1 \leq r \leq \infty$, $1 < p_1, p_2, q_1, q_2 \leq \infty$ satisfying the relation
\[
\frac{1}{r} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{q_1} + \frac{1}{q_2}.
\]

Then the following fractional Leibniz rules hold:
\[
\|D|^s(uv)\|_{L^r} \lesssim \|D|^s u\|_{L^{p_1}} \|v\|_{L^{p_2}} + \|u\|_{L^{q_1}} \|D|^s v\|_{L^{q_2}}
\]
for any $u \in \dot{H}^{s,p_1}(\mathbb{R}^n) \cap L^{q_1}(\mathbb{R}^n)$ and $v \in \dot{H}^{s,q_2}(\mathbb{R}^n) \cap L^{p_2}(\mathbb{R}^n)$,
\[
\|D|^s(uv)\|_{L^r} \lesssim \|D|^s u\|_{L^{p_1}} \|v\|_{L^{p_2}} + \|u\|_{L^{q_1}} \|D|^s v\|_{L^{q_2}}
\]
for any $u \in H^{s,p_1}(\mathbb{R}^n) \cap L^{q_1}(\mathbb{R}^n)$ and $v \in H^{s,q_2}(\mathbb{R}^n) \cap L^{p_2}(\mathbb{R}^n)$.

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Marcelo Rempel Ebert, Departamento de Computação e Matemática, Universidade de São Paulo, Ribeirão Preto, SP, 14040-901, Brazil, email: erbert@ffclrp.usp.br.br

Jorge Marques, CeBER and FEUC, University of Coimbra, Av. Dias da Silva 165, 3004-512 Coimbra, Portugal, email:jmarques@fe.uc.pt