Symmetries of finite Heisenberg groups for \( k \)-partite systems

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Abstract. Symmetries of finite Heisenberg groups represent an important tool for the study of deeper structure of finite-dimensional quantum mechanics. This short contribution presents extension of previous investigations to composite quantum systems comprised of \( k \) subsystems which are described with position and momentum variables in \( \mathbb{Z}_n^i \), \( i = 1, \ldots, k \). Their Hilbert spaces are given by \( k \)-fold tensor products of Hilbert spaces of dimensions \( n_1, \ldots, n_k \). Symmetry group of the corresponding finite Heisenberg group is given by the quotient group of a certain normalizer. We provide the description of the symmetry groups for arbitrary multipartite cases. The new class of symmetry groups represents very specific generalization of finite symplectic groups over modular rings.

1. Introduction
The Heisenberg Lie algebra and the Heisenberg-Weyl group lie at the heart of quantum mechanics [1]. Therefore their symmetries induced by unitary automorphisms play very important role in quantum kinematics as well as quantum dynamics. The growing interest in quantum communication science has pushed the study of quantum systems with finite-dimensional Hilbert spaces to the forefront, both single systems and composite systems. For them the finite Heisenberg groups provide the basic quantum observables. It is then clear that symmetries of finite Heisenberg groups uncover deeper structure of finite-dimensional quantum mechanics.

Our continuing interest in finite-dimensional quantum mechanics goes back to the paper [2] where finite-dimensional quantum mechanics was developed as quantum mechanics on configuration spaces given by finite sets equipped with the structure of a finite Abelian group. In our recent paper [3] detailed characterization was given of the symmetry groups of finite Heisenberg groups for composite quantum systems consisting of two subsystems with arbitrary dimensions \( n, m \). In this contribution these results for bipartite systems are extended to the general situation where the composite systems are multipartite, consisting of an arbitrary finite number \( k \) of subsystems with arbitrary dimensions \( n_1, \ldots, n_k \). Their Hilbert spaces are given by \( k \)-fold tensor products of Hilbert spaces of dimensions \( n_1, \ldots, n_k \). The exposition starts with relevant facts of quantum mechanics. After introductory material on finite-dimensional quantum mechanics in sections 2–3 the new symmetry groups are described in section 4. They deserve to be called generalized finite symplectic groups.

Symmetries in Hamiltonian mechanics belong to the class of canonical transformations of the phase space. The symmetries considered here have their simplest occurrence as the canonical transformations which leave the form of fundamental Poisson brackets unchanged. In the phase space \( \mathbb{R}^{2n} \) they are the linear canonical transformations forming the symplectic group \( \text{Sp}(2n, \mathbb{R}) \) [4]. For \( n = 1 \) degree of freedom it reduces to \( \text{Sp}(2, \mathbb{R}) \cong \text{SL}(2, \mathbb{R}) \) with the action on the
canonical variables \( q, p \)

\[
(q', p') = (q, p) \mathbb{A} = (q, p) \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \det \mathbb{A} = ad - bc = 1. \tag{1}
\]

In quantum mechanics the canonical observables satisfying the canonical commutation relations \([Q_j, \hat{P}_k] = i\hbar \delta_{jk} \mathbb{I}\) are represented by essentially self-adjoint operators unitarily equivalent to well-known operators in the Hilbert space \(L^2(\mathbb{R}^n, d^n q)\) of the Schrödinger representation. In the simplest case of the Hilbert space \(L^2(\mathbb{R}, dq)\) \((n = 1)\)

\[
\hat{Q}\psi(q) = q\psi(q), \quad \hat{P}\psi(q) = -i\hbar \frac{\partial \psi(q)}{\partial q}.
\]

Unitary representation of linear transformations of \(\hat{Q}\) and \(\hat{P}\) in the same Hilbert space \(L^2(\mathbb{R}^n, d^n q)\) is known as metaplectic representation \([5, 6, 4]\). It is a unitary irreducible representation \(X(\mathbb{A})\) of the double covering of \(\text{Sp}(2n, \mathbb{R})\) (called the metaplectic group) which for \(n = 1\) reduces to the double covering of \(SL(2, \mathbb{R})\),

\[
(\hat{Q}', \hat{P}') = (\hat{Q}, \hat{P}) \mathbb{A}, \quad \det \mathbb{A} = 1. \tag{2}
\]

The same commutators \([\hat{Q}', \hat{P}'] = i\hbar \mathbb{I}\) obviously lead, via the Stone–von Neumann theorem, to unitary equivalence

\[
\hat{Q}' = X(\mathbb{A})\hat{Q}X(\mathbb{A})^{-1}, \quad \hat{P}' = X(\mathbb{A})\hat{P}X(\mathbb{A})^{-1},
\]

also known in exponential form for the Weyl operators

\[
X(\mathbb{A})W(s, t)X(\mathbb{A})^{-1} = W((s, t)\mathbb{A}^T).
\]

2. Finite-dimensional quantum mechanics

In an \(N\)-dimensional Hilbert space with orthonormal basis \(\mathcal{B} = \{\{0\}, \{1\}, \ldots, \{N - 1\}\}\) unitary operators \(Q_N, P_N\) are defined \([1, 7, 8, 9]\) by the relations

\[
Q_N|j\rangle = \omega_N^j|j\rangle, \quad P_N|j\rangle = |j - 1 \pmod{N}\rangle,
\]

where \(\omega_N = \exp(2\pi i/N)\) and \(j = 0, 1, \ldots, N - 1\). If \(\mathcal{B}\) is the canonical basis of \(\mathbb{C}^N\), the operators \(P_N\) and \(Q_N\) are represented by matrices

\[
Q_N = \text{diag}(1, \omega_N, \omega_N^2, \ldots, \omega_N^{N-1}) \tag{3}
\]

and

\[
P_N = \begin{pmatrix}
0 & 1 & 0 & \cdots & 0 & 0 \\
0 & 0 & 1 & \cdots & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & 0 & \cdots & 0 & 1 \\
1 & 0 & 0 & \cdots & 0 & 0
\end{pmatrix} \quad \tag{4}
\]

They fulfil \(P_N^N = Q_N^N = I\) and commutation relation

\[
P_NQ_N = \omega_N Q_N P_N, \tag{5}
\]

which is analogous to the relation for Weyl’s exponential form of Heisenberg’s commutation relations. The finite Heisenberg group of order \(N^3\) is generated by \(\omega_N, Q_N\) and \(P_N\)

\[
\Pi_N = \left\{ \omega_N^i Q_N^j P_N^k | i, j = 0, 1, 2, \ldots, N - 1 \right\}. \tag{6}
\]
In a geometric interpretation [2] the cyclic group $\mathbb{Z}_N = \{0, 1, \ldots, N - 1\}$ is a configuration space for $N$-dimensional quantum mechanics. Elements of $\mathbb{Z}_N$ label the vectors of the basis $\mathcal{B} = \{|0\rangle, |1\rangle, \ldots, |N - 1\rangle\}$ with the physical interpretation that $|j\rangle$ is the normalized eigenvector of position at $j \in \mathbb{Z}_N$. The action of $\mathbb{Z}_N$ on $\mathbb{Z}_N$ via addition modulo $N$ is represented by unitary operators $U(k) = P_N^k$. Their action on vectors $|j\rangle$ from basis $\mathcal{B}$ is given by

$$U(k)|j\rangle = P_N^k|j\rangle = |j - k \pmod{N}\rangle. \quad (7)$$

3. Finite phase space and its automorphisms

For better understanding of symmetries for general composite systems to be described in section 4, we summarize briefly the basic ideas of our construction in the case of a single system with configuration space $\mathbb{Z}_N$ [10]. In this case the finite phase space [11, 12] (also called quantum configuration space) is the toroidal lattice $P_N = \mathbb{Z}_N \times \mathbb{Z}_N$. It is simply obtained from the finite Heisenberg group as the quotient group $\Pi_N/\mathbb{Z}(\Pi_N)$ where $\mathbb{Z}(\Pi_N)$ is the center of the finite Heisenberg group,

$$Z(\Pi_N) = \{(i, 0, 0)|i = 0, 1, \ldots, N - 1\}.$$

The correspondence

$$\Pi_N/\mathbb{Z}(\Pi_N) \to \mathbb{Z}_N \times \mathbb{Z}_N : Q^i P^j \mapsto (i, j), \quad (8)$$

is an isomorphism of Abelian groups, so the quotient group is identified with the finite phase space with elements $(i, j) \in \mathbb{Z}_N \times \mathbb{Z}_N$.

Since the finite Heisenberg group was introduced as a subgroup of $\text{GL}(N, \mathbb{C})$, finite phase space $P_N$ can equivalently be seen as an Abelian subgroup of $\text{Int}(\text{GL}(N, \mathbb{C}))$. For $M \in \text{GL}(N, \mathbb{C})$ let $\text{Ad}_M \in \text{Int}(\text{GL}(N, \mathbb{C}))$ be the inner automorphism of the group $\text{GL}(N, \mathbb{C})$,

$$\text{Ad}_M(X) = MXM^{-1} \quad \text{for} \quad X \in \text{GL}(N, \mathbb{C}).$$

Then the finite phase space $P_N$ is isomorphic to the Abelian group

$$P_N \cong \{\text{Ad}_{Q_N^i P_N^j} | (i, j) \in \mathbb{Z}_N \times \mathbb{Z}_N\}. \quad (9)$$

In this mathematical model $P_N$ is the Abelian subgroup of $\text{Int}(\text{GL}(n, \mathbb{C}))$ generated by two commuting automorphisms $\text{Ad}_{Q_N}$, $\text{Ad}_{P_N}$ [10].

Consider those inner automorphisms acting on elements of $\Pi_N$ which induce permutations of phase space points, i.e. of cosets in $\Pi_N/\mathbb{Z}(\Pi_N)$:

$$\text{Ad}_X(\omega_N^i Q_N^j P_N^j) = X \omega_N^i Q_N^j P_N^j X^{-1}, \quad (10)$$

where $X$ are unitary matrices from $\text{GL}(N, \mathbb{C})$. These inner automorphisms are equivalent, if they define the same transformation of cosets,

$$\text{Ad}_Y \sim \text{Ad}_X \iff YQ^j P^j Y^{-1} = XQ^j P^j X^{-1} \quad (11)$$

for all $(i, j) \in \mathbb{Z}_N \times \mathbb{Z}_N$. Modulo this equivalence, they are elements of the normalizer of $P_N$ as Abelian subgroup of $\text{Int}(\text{GL}(n, \mathbb{C}))$.

Now the group $P_N = \Pi_N/\mathbb{Z}(\Pi_N)$ has two generators, the cosets $Q$ and $P$ (or $\text{Ad}_{Q_N}$ and $\text{Ad}_{P_N}$, respectively). Hence, if $\text{Ad}_Y$ induces a permutation of cosets in $\Pi_N/\mathbb{Z}(\Pi_N)$, then there must exist elements $a, b, c, d \in \mathbb{Z}_N$ such that

$$YQY^{-1} = Q^a P^b \quad \text{and} \quad YPY^{-1} = Q^c P^d. \quad (12)$$

It follows that to each equivalence class of inner automorphisms $\text{Ad}_Y$ a quadruple $(a, b, c, d)$ of elements in $\mathbb{Z}_N$ is assigned.
**Proposition** [10] For integer $N \geq 2$ there is an isomorphism $\Phi$ between the set of equivalence classes of inner automorphisms $\text{Ad}_Y$ which induce permutations of cosets and the group $\text{SL}(2, \mathbb{Z}_N)$ of $2 \times 2$ matrices with determinant equal to 1 (mod $N$),

$$
\Phi(\text{Ad}_Y) = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right), \quad a, b, c, d \in \mathbb{Z}_N.
$$

(13)

The action of these automorphisms on $\Pi_N / \mathbb{Z}(\Pi_N)$ is given by the right action of $\text{SL}(2, \mathbb{Z}_N)$ on elements $(i, j)$ of the phase space $\mathcal{P}_N = \mathbb{Z}_N \times \mathbb{Z}_N$,

$$(i', j') = (i, j) \left( \begin{array}{cc} a & b \\ c & d \end{array} \right).$$

(14)

**4. Symmetries for multipartite systems**

In our paper [3] we presented detailed description of the symmetry group of the finite Heisenberg group in the case of a bipartite quantum system consisting of two subsystems with arbitrary dimensions $n, m$. The corresponding finite Heisenberg group is embedded in $\text{GL}(N, \mathbb{C})$, $N = nm$. Via inner automorphisms it induces an Abelian subgroup in $\text{Int}(\text{GL}(N, \mathbb{C}))$. The normalizer of this Abelian subgroup in the group of inner automorphisms of $\text{GL}(N, \mathbb{C})$ contains all inner automorphisms transforming the phase space $\mathcal{P}_N$ onto itself, hence necessarily contains $\mathcal{P}_N$ as an Abelian semidirect factor. The true symmetry group is then given by the quotient group of the normalizer with respect to this Abelian subgroup.

According to the well-known rules of quantum mechanics, finite-dimensional quantum mechanics on $\mathbb{Z}_n$ can be extended in a straightforward way to arbitrary finite direct products $\mathbb{Z}_{n_1} \times \ldots \times \mathbb{Z}_{n_k}$ as configuration spaces. The cyclic groups involved describe independent quantum degrees of freedom. The Hilbert space of such a composite system is the tensor product

$$
\mathcal{H}_{n_1} \otimes \ldots \otimes \mathcal{H}_{n_k}
$$

of dimension $N = n_1 \ldots n_k$, where $n_1, \ldots, n_k \in \mathbb{N}$.

For such a $k$-partite system, quantum phase space is an Abelian subgroup of $\text{Int}(\text{GL}(N, \mathbb{C}))$ defined by

$$
\mathcal{P}_{(n_1, \ldots, n_k)} = \{ \text{Ad}_{M_1} \otimes \cdots \otimes M_k | M_i \in \Pi_{n_i} \}.
$$

(15)

Its generating elements are the inner automorphisms

$$
e_j := \text{Ad}_{A_j} \quad \text{for} \quad j = 1, \ldots, 2k,
$$

(16)

where (for $i = 1, \ldots, k$)

$$
A_{2i-1} := I_{n_1 \ldots n_{i-1}} \otimes P_{n_i} \otimes I_{n_{i+1} \ldots n_k}, \quad A_{2i} := I_{n_1 \ldots n_{i-1}} \otimes Q_{n_i} \otimes I_{n_{i+1} \ldots n_k}.
$$

(17)

The normalizer of $\mathcal{P}_{(n_1, \ldots, n_k)}$ in $\text{Int}(\text{GL}(n_1 \ldots n_k, \mathbb{C}))$ will be denoted

$$
\mathcal{N}(\mathcal{P}_{(n_1, \ldots, n_k)}) := \mathcal{N}_{\text{Int}(\text{GL}(n_1 \ldots n_k, \mathbb{C}))}(\mathcal{P}_{(n_1, \ldots, n_k)}),
$$

We need also the normalizer of $\mathcal{P}_n$ in $\text{Int}(\text{GL}(n, \mathbb{C}))$,

$$
\mathcal{N}(\mathcal{P}_n) := \mathcal{N}_{\text{Int}(\text{GL}(n, \mathbb{C}))}(\mathcal{P}_n),
$$

and

$$
\mathcal{N}(\mathcal{P}_{n_1}) \times \cdots \times \mathcal{N}(\mathcal{P}_{n_k}) := \{ \text{Ad}_{M_1} \otimes \cdots \otimes M_k | M_i \in \mathcal{N}(\mathcal{P}_{n_i}) \} \subseteq \text{Int}(\text{GL}(N, \mathbb{C})),
$$

satisfying

$$
\mathcal{N}(\mathcal{P}_{n_1}) \times \cdots \times \mathcal{N}(\mathcal{P}_{n_k}) \subseteq \mathcal{N}(\mathcal{P}_{(n_1, \ldots, n_k)}).
$$
Now the symmetry group $\mathcal{H}_{[n_1, \ldots, n_k]}$ is constructed in several steps. First let $S_{[n_1, \ldots, n_k]}$ be a set consisting of $k \times k$ matrices $H$ of $2 \times 2$ blocks

$$H_{ij} = \frac{n_i}{\gcd(n_i, n_j)} A_{ij}$$

(18)

where $A_{ij} \in M_2(\mathbb{Z}_{n_i})$ for $i, j = 1, \ldots, k$. Then $S_{[n_1, \ldots, n_k]}$ is — with usual matrix multiplication — a monoid. Next, for a matrix $H \in S_{[n_1, \ldots, n_k]}$, we define its adjoint $H^* \in S_{[n_1, \ldots, n_k]}$ by

$$(H^*)_{ij} = \frac{n_i}{\gcd(n_i, n_j)} A_{ji}^T.$$  

(19)

Further, we need a skew-symmetric matrix

$$J = \text{diag}(J_2, \ldots, J_2) \in S_{[n_1, \ldots, n_k]}, \text{ where } \quad J_2 = \left( \begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right).$$

(20)

Then the symmetry group is defined as a finite subgroup of the monoid $S_{[n_1, \ldots, n_k]}$,

$$\mathcal{H}_{[n_1, \ldots, n_k]} := \{ H \in S_{[n_1, \ldots, n_k]} | \ H^* J H = J \}$$

(21)

Our first theorem states the group isomorphism:

**Theorem 1**

$$\mathcal{N}(\mathcal{P}_{[n_1, \ldots, n_k]})/\mathcal{P}_{[n_1, \ldots, n_k]} \cong \mathcal{H}_{[n_1, \ldots, n_k]}.$$ 

(22)

Our second theorem describes the generating elements of the normalizer:

**Theorem 2** The normalizer $\mathcal{N}(\mathcal{P}_{[n_1, \ldots, n_k]})$ is generated by $\mathcal{N}(\mathcal{P}_{n_1}) \times \cdots \times \mathcal{N}(\mathcal{P}_{n_k})$ and $\{\text{Ad}_{R_{ij}}, \text{Ad}_{T_{ij}}\}$, where (for $1 \leq i < j \leq k$)

$$R_{ij} = I_{n_1 \cdots n_{i-1}} \text{diag}(I_{n_{i+1} \cdots n_j} T_{ij}, \ldots, T_{ij}^{n_i-1}) \otimes I_{n_{j+1} \cdots n_k}$$

and

$$T_{ij} = I_{n_1 \cdots n_{j-1}} \otimes (\mathbb{Z}_{n_j}^{n_i}).$$

Detailed proofs of these theorems will be published elsewhere. Due to (21) the new groups $\mathcal{H}_{[n_1, \ldots, n_k]}$ represent a very specific generalization of symplectic groups over modular rings, hence may be denoted $\text{Sp}_{[n_1, \ldots, n_k]}$. Important examples correspond to composite systems consisting of subsystems of equal dimensions $n_1 = \ldots = n_k$:

**Corollary** If $n_1 = \ldots = n_k = n$, i.e. $N = n^k$, the symmetry group is $\mathcal{H}_{[n, \ldots, n]} \cong \text{Sp}(2k, \mathbb{Z}_n)$.

These cases are of particular interest, since they uncover symplectic symmetry of $k$-partite systems composed of subsystems with the same dimensions. This circumstance was found, to our knowledge, first in [15] for $k = 2$ under additional assumption that $n = p$ is prime, leading to $\text{Sp}(4, \mathbb{F}_p)$ over the field $\mathbb{F}_p$. We have generalized this result in [3] to bipartite systems with arbitrary (non-prime) $n = m$ leading to the symmetry group $\text{Sp}(4, \mathbb{Z}_m)$ over the modular ring $\mathbb{Z}_m$. The above corollary extends this fact also to multipartite systems. Similar result has independently been obtained in [16], where symmetries of the tensored Pauli grading of $\text{sl}(n^k, \mathbb{C})$ were investigated.

5. Comments

Our motivation to study symmetries of finite Heisenberg groups not in prime or prime power dimensions as in [13, 17, 12], but for arbitrary dimensions stems from our previous research where we obtained results not restricted to finite fields. Especially recall our paper [19] on Feynman’s path integral and mutually unbiased bases. Also the recent paper [14] belongs to this direction, by dealing with quantum tomography over modular rings. The papers [20, 21] support our motivation, too, since they show that finite quantum mechanics with growing odd
dimensions yields surprisingly good approximations of ordinary quantum mechanics on the real line. This suggests a promising subject of research to extend the results of [13, 18] on the metaplectic representation of $\text{SL}(2, F_p)$ from finite fields to modular rings.

The symmetry groups $\text{Sp}_{[n_1,...,n_k]}$ described here can also serve as a starting point for an alternative proof of existence of the maximal set of mutually unbiased bases in Hilbert spaces of prime power dimensions [22, 23]. The group theoretical construction presented in [24] was based on the symmetry group $\text{SL}(2, F_p)$ of the finite Heisenberg group for Hilbert spaces of prime dimensions. In our forthcoming paper we shall present a new proof using the symmetry groups $\text{Sp}(2k, F_p)$ applied to Heisenberg groups for the Hilbert spaces of prime power dimensions.

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