A HYPERBOLIC MODEL OF SPATIAL EVOLUTIONARY GAME THEORY

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Abstract. We present a one space dimensional model with finite speed of propagation for population dynamics, based both on the hyperbolic Cattaneo dynamics and the evolutionary game theory. We prove analytical properties of the model and global estimates for solutions, by using a hyperbolic nonlinear Trotter product formula.

1. Introduction. We study the evolution of different populations that are in the same environment. Following Maynard Smith, [14], we model the interactions between different populations by using game theory: each species is a player who takes a precise strategy in evolutionary game. The relationship between populations and their strategies are expressed by a payoff matrix whose entries represent the payoffs that each species have in the game. The most investigated model has been so far the so called prisoner’s dilemma, where the population is composed by the cooperators which are friendly and the defectors which are hostile. One of the main problems has been about the possibility of success for cooperation, which is impossible in the spatially homogeneous case. One possible issue is considering spatial effects. Up to now the main concern has been around reaction-diffusion models, see for instance [6, 7]. A different approach is given by cellular automata, which exhibit a complex and interesting behavior, [20, 8]. However, this approach is still oversimplified and not suitable for a realistic description of living natural populations.

In the present paper we introduce a continuous finite propagation speed model for population dynamics, which is based on a hyperbolic Cattaneo dynamics for the flux function [4, 10, 17, 16]. This model is intermediate between the discrete states approach of cellular automata and the continuous reaction-diffusion models.

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In [1], we presented some numerical simulations for this model, which shown a somewhat different behavior with respect to the diffusion models. In particular, cooperators show a consistent longer time-life with respect to defectors. The main goal of the present work is to start a first analytical study of these hyperbolic dynamics, which is not limited at the global well-posedness of the problem, but it is also aimed in showing some insights on the rigorous qualitative behavior of the solutions.

The paper is organized as follows. In Section 2, we review and discuss shortly some previous models for this problem. Then, in the following sections we introduce our model and its analytical properties, prove existence of global solutions and show the convergence of a hyperbolic nonlinear Trotter formula, mostly adapting the arguments in [21]. Finally, in the last section, we use the Trotter formula to analyze the asymptotic behavior of our solutions, according to the different values of the parameters. Some more technical arguments and estimates are deferred to the Appendix.

2. Evolutionary game theory and models. Evolutionary game dynamics is based on the use of ideas coming from game theory in the investigation of population dynamics, following the ideas introduced in 1973 by Maynard Smith and Price [14]. Traditional game theory analyzes the interactions between two or more players: an individual (player) is engaged in a given interaction (game) with other players and has to decide between different options (strategies) in order to maximize her performance (payoff), which depends on the strategies of co-players.

Evolutionary game theory considers each population like a set of players taking a clear fixed strategy in the game of life: the players are entire populations having different behaviors (strategies). The payoffs depend on the actions of co-players and, hence, on the frequencies of the strategies within the population. The payoff is interpreted as a fitness, and the success in the game is translated into reproductive success. The strategies with high payoff reproduce faster; the strategies that do poorly are outcompeted. This is straightforward natural selection. The relationships between populations and their strategies are expressed by a payoff matrix whose entries represent the payoffs that each species has in the game.

Maynard Smith formalized a central concept in ecology called evolutionarily stable strategy (ESS), [14]. In behavioral ecology, an ESS is a strategy which, if adopted by a population of players, cannot be invaded by any alternative strategy. It turns out that every ESS is a Nash equilibrium [19].

2.1. The replicator dynamics. Following [11, 12, 20], we consider a population with two phenotypes A and B. Evolutionary game theory assumes that the fitness of individuals is not constant, but depends linearly on the relative frequencies of the different phenotypes in the population through a payoff matrix. The space homogeneous general model of frequency-dependent selection between two strategies A and B reads

\[
\begin{align*}
\dot{x}_A &= x_A [f_A(x_A, x_B) - \Phi] \\
\dot{x}_B &= x_A [f_B(x_A, x_B) - \Phi],
\end{align*}
\] (1)

where \(x_A\) and \(x_B\) are respectively the frequency of A and B in population, \(f_A\) and \(f_B\) are respectively the fitness of A and B:

\[
\begin{align*}
f_A(x_A, x_B) &= ax_A + bx_B \\
f_B(x_A, x_B) &= cx_A + dx_B,
\end{align*}
\]
where \(a, b, c, d\) are the entries of the payoff matrix. The function \(\Phi\) is the average fitness and is given by \(\Phi = x_A f_A(x_A, x_B) + x_B f_B(x_A, x_B)\), and it is needed to keep the system at equilibrium. The equations (1) are called replicator equations and were introduced in 1978 by Taylor and Jonker [22]. Since \(x_A + x_B = 1\) at all times, we can introduce the variable \(x\) with \(x = x_A\) and \(1 - x = x_B\). We can write the fitness functions as \(f_A(x), f_B(x)\). We obtain the equation
\[
\dot{x} = x(1-x)(f_A(x) - f_B(x)).
\]
The equilibria of this equation are given by \(x = 0, x = 1\), and all values \(x^* \in (0,1)\) that satisfy \(f_A(x^*) = f_B(x^*)\). The equilibrium \(x = 0\) is stable if \(f_A(0) < f_B(0)\); conversely \(x = 1\) is stable if \(f_A(1) > f_B(1)\). The interior equilibria \(x^*\) are stable if \(f_A' < f_B'\). All these conditions on the fitness functions are conditions on the entries of the payoff matrix. We can have five situations: (i) \(A\) dominates \(B\) if \(a > c\) and \(b > d\). (ii) \(B\) dominates \(A\) if \(a < c\) and \(b < d\). (iii) \(A\) and \(B\) are bistable if \(a > c\) and \(b < d\). (iv) \(A\) and \(B\) coexist if \(a < c\) and \(b > d\). (v) \(A\) and \(B\) are neutral if \(a = c\) and \(b = d\).

One interesting example is given by the so-called Prisoner’s dilemma game in which there are two players and two possible strategies. The players have two options, cooperate or defect. The payoff matrix is the following
\[
\begin{pmatrix}
R & S \\
T & P
\end{pmatrix}
\]
If both players cooperate both obtain \(R\) fitness units (reward payoff); if both defect, each receives \(P\) (punishment payoff); if one player cooperates and the other defects, the cooperator gets \(S\) (sucker’s payoff) while the defector gets \(T\) (temptation payoff). The payoff values are ranked \(T > R > P > S\) and \(2R > T + S\). According to replicator dynamics of prisoner’s dilemma, we have that cooperators are always dominated by defectors, which actually have an ESS strategy.

2.2. Reaction-diffusion models. To include spatial effects, the classical approach is to introduce partial differential models. The first reaction-diffusion models for population dynamics had no relation with game theory and were simply obtained by adding a diffusion term to the famous Lotka-Volterra system, see for instance [18].

To introduce diffusive models including game theory, consider games in which the number of strategies, \(r\), is finite and the game is regulated by the payoff matrix \(A = (a_{ij})\), where \(a_{ij}\) is the payoff of a population which plays the strategy labelled \(i\) against a population which plays the strategy labelled \(j\). In [6], Vickers proposed the following reaction-diffusion equation
\[
\frac{\partial n_i}{\partial t} = n_i \left[ \frac{e_i \cdot An}{N} - \frac{n \cdot An}{N^2} \right] + D_i \Delta n_i \quad 1 \leq i \leq r,
\] (2)
where \(n = (n_1, \ldots, n_r)^T\), \(n_i(x, t)\) is the population density for each strategy and \(N(x, t) = \sum n_i\) is the total density. The \(i\)th strategy is denoted by the vector \(e_i\) that has 1 in the \(i\)th component and zeros elsewhere; \(D_i\) is the diffusion coefficient. The term
\[
\frac{e_i \cdot An}{N}
\]
is the fitness of a population related to the \(i\)th strategy; the term
\[
\frac{n \cdot An}{N^2}
\]
is introduced into model to regulate the growth of $N$: if we neglect the diffusion term $D_i \triangle n_i$ in the equation (2) and we consider the reaction equation

$$\dot{n}_i = n_i \left[ \frac{e_i \cdot A n}{N} - \frac{n \cdot A n}{N^2} \right],$$

we have that $\partial N/\partial t = 0$, thus the density regulation term maintains the total density at constant size $N^*$, that can be interpreted as the species’ carrying capacity. The term $D_i \triangle n_i$ is a diffusion term. The standard method to include spatial effects assumes that the spatial domain is homogeneous and so the nonnegative diffusion coefficients $D_i$ is constant. Vickers and her co-workers investigated two important aspects of the reaction-diffusion equation (2). The first concerns the existence of spatial patterns that can arise through Turing instabilities; the second aspect is to analyze the existence and properties of travelling waves. In particular they proved that no such spatial patterns exist when the matrix $A$ has an ESS $p^*$, according to the definition given by Maynard Smith: the spatially uniform population with frequency $p^*$ and density at the population carrying capacity $N^*$ is globally stable. For all details, see [23]. When there are only two possible strategies which are both ESSs, there are travelling wave solutions, [13]. However in the case of the prisoner’s dilemma the only ESS strategy is again played by the defectors.

2.3. **Discrete spatial games.** An alternative approach to spatial games stands in modelling the networks of interacting players. In the simplest case, the players are assumed to be located at each vertexes of a given lattice. At each of the (discrete) time steps, each player engages in pairwise interactions with all co-players from some neighborhood. Then, players update their strategies according to some rules. If rules are deterministic, the resulting process describing the evolution of the distribution of strategies over the lattice is a deterministic cellular automaton, [20]. For instance, in each round, every individual plays the game with its immediate neighbors and after this, each site is occupied by who scored the highest payoff in the round and player keeps his current strategy or adopts one of his neighbors’ strategies according to who has the highest payoff in the round.

In discrete spatial games, the theory of cellular automata meets game theory. It may happen that strategies which are strictly dominated for replicator dynamics, can now resist elimination and survive, for instance by freezing into clusters. For instance, in the discrete spatial Prisoner’s dilemma, much more situations arise: defectors invading cooperators, cooperators invading defectors, coexistence. These different situations may arise depending on the values of the payoff matrix and on the initial conditions, see for instance the book [20] and the recent paper [8] for more references and examples.

3. **A hyperbolic approach for spatial evolutionary games.** Reaction-diffusion models are not the most appropriate to describe the population dynamics, because, as known, they have infinite speed of propagation: it is unrealistic to assume that any species is moving with infinite speed. Moreover the survival of cooperation is rapidly excluded by these diffusive models, in sharp contrast with typical ecological behaviors. The discrete approach of cellular automata is able to display a richer panel of different scenarios, but players have not an effective speed, because there is not an effective movement of each individual at each time step. Here, in a first attempt of considering more realistic models, we present a continuous finite speed model for population dynamics.
3.1. PDE modelling. Assume there are two populations adopting two different strategies, whose interplay is ruled by the payoff matrix

\[ A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}. \]

We denote by \( N \) the total density of the population, and by \( n_1 \) and \( n_2 \), respectively, the density of population adopting the first and the second strategy; thus

\[ N = n_1 + n_2. \]

We will also use the notation \( \mathbf{n} = (n_1, n_2)^T \) for the vector of partial densities.

We adopt an hyperbolic model to describe the dynamics of the partial densities:

\[ \partial_t n_i + \partial_x w_i = n_i \left( \frac{e_i \cdot A n}{N} - \frac{n \cdot A n}{N^2} \right), \quad i = 1, 2. \]

Here, \( w = (w_1, w_2)^T \) is the flux vector of the partial densities and is an unknown function. We need an equation for the evolution of the fluxes \( w_i, i = 1, 2 \). The Fick’s law for the flux and

\[ \lambda_i^2 \partial_x n_i = -w_i, \quad i = 1, 2, \]

yields precisely the Vickers’ parabolic model (2) with \( D_i = \lambda_i^2 \). To force a finite speed of propagation we consider a flux of Cattaneo type [4], namely

\[ \tau \partial_t w_i + \lambda_i^2 \partial_x n_i = -w_i, \quad i = 1, 2, \]

which is a Fick type law with a delay due to inertia and expressed by the relaxation time \( \tau \). See [10] for a review of the mathematical literature of this type of models and [16, 17] for an updated presentation of the physical derivation. Finally, the model reads as

\[
\begin{cases}
\partial_t n_i + \partial_x w_i = n_i \left( \frac{e_i \cdot A n}{N} - \frac{n \cdot A n}{N^2} \right), & i = 1, 2, \\
\tau \partial_t w_i + \lambda_i^2 \partial_x n_i = -w_i, & i = 1, 2.
\end{cases}
\] (3)

If the source term is put to zero, the model represents a correlated random walk of two families moving with two constant and opposite speeds. The term \( \frac{\tau}{\lambda} \) is the turning probability, which is the probability to change direction and to switch from one family to the other. Actually more realistic descriptions of the flux function could be considered, for instance by making hydrodynamic limits of suitable kinetic models, see [9, 10, 2], but here we are just interested in making a first analysis of the simplest example of a hyperbolic model in population dynamic.

3.2. The total density \( N \). We deduce the differential equation that drives the evolution of the total density \( N \). Summing the first pair of equations of the system (3) gives

\[ \partial_t N = -\partial_x (w_1 + w_2). \]

Next, summing the second pair of equations of the system (3) and differentiating w.r.t. \( x \) gives

\[ \tau \partial_t \partial_x (w_1 + w_2) + \lambda_1^2 \partial_{xx} n_1 + \lambda_2^2 \partial_{xx} n_2 = -\partial_x (w_1 + w_2), \]

and then, since \( \partial_t \partial_x (w_1 + w_2) = -\partial_{tt} N, \) we get

\[ \partial_t N + \tau \partial_{tt} N - \lambda_1^2 \partial_{xx} N = (\lambda_1^2 - \lambda_2^2) \partial_{xx} n_1. \] (4)
The total population living in the region of interest, to say the interval \([0, L]\), is described by the quantity

\[ \bar{N}(t) := \int_0^L N(x, t) \, dx \]

and its variation in time is given by

\[ \frac{d}{dt} \bar{N}(t) = \int_0^L \partial_t N(x, t) \, dx = - \int_0^L \partial_x (w_1 + w_2)(x, t) \, dx \]

\[ = w_1(0, t) + w_2(0, t) - w_1(L, t) - w_2(L, t). \]

(5)

3.3. The hyperbolic semilinear system. The system (3) can be rewritten in compact form as

\[ \partial_t U + A \partial_x U = G(U), \]

where \( U = (n_1, n_2, w_1, w_2)^T \) is a vector-valued function of \( (x, t) \in [0, L] \times [0, +\infty) \), the matrix

\[ A = \begin{pmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\lambda^2_1/\tau & 0 & 0 & 0 \\
0 & \lambda^2_2/\tau & 0 & 0
\end{pmatrix} \]

has constant entries, \( G(U) = (-g(n), g(n), -w_1/\tau, -w_2/\tau)^T \) is a vector field, and \( g(n) = \frac{n_1^2 + n_2^2}{(n_1 + n_2)^2} (\alpha_1 n_1 + \alpha_2 n_2) \) is a real valued function of \( n = (n_1, n_2) \), with \( \alpha_i = a_{2i} - a_{3i} \) as \( i = 1, 2 \).

Let us remark that the source term \( G \) is not defined at \( n_1 + n_2 = 0 \). In the set \( n_1, n_2 \geq 0 \), the function \( G \) can be extended in a global Lipschitz-continuous way by setting \( G(0) = 0 \). On the other hand, the quantities \( n_1 \) and \( n_2 \) are requested to be nonnegative, since they stay for densities of populations. Hence, when looking for solutions to system (3), we will keep in mind that initial data have to be chosen in such a way to guarantee that \( n_1 \) and \( n_2 \) do not become negative in finite time.

The differential system (3) is endowed with an initial condition

\[ U(x, 0) = U_0(x), \quad \text{for all } x \in [0, L], \]

(6)

and zero-flux at the space-boundary is imposed:

\[ w_i(\xi, t) = 0, \quad \text{for } \xi = 0, L, \text{ and } t \geq 0, \text{ and } i = 1, 2. \]

(7)

Inserting this boundary condition in (5) yields that the total mass \( \bar{N} \) is conserved. The boundary condition (7) brings informations about the normal derivative of \( n_i \). Indeed, since \( w_i(\xi, t) = 0 \) for all \( t \geq 0 \), we have that \( \partial_x n_i(\xi, t) = 0 \) and therefore the second equation in (3) yields

\[ \partial_x n_i(\xi, t) = 0, \quad \text{for } \xi = 0, L, \text{ and } t > 0, \quad \text{as } i = 1, 2. \]

We shall also assume that initial data satisfy a consistency condition:

\[ w_{0i}(0, t) = w_{0i}(L, t) = 0, \quad \text{as } i = 1, 2. \]

(8)

The system (3) is symmetrizable (and strictly hyperbolic): the eigenvalues of the matrix \( A \) are

\[ \theta_i^\pm = \pm \lambda_i/\sqrt{\tau}, \quad \text{as } i = 1, 2, \]

and the respective Riemann invariants are

\[ R_i^\pm = n_i \pm \sqrt{\tau} w_i, \quad \text{as } i = 1, 2. \]
The original variables can be recovered from the Riemann invariants as
\[ n_i = \frac{1}{\lambda_i} (R_i^+ + R_i^-), \quad w_i = \frac{\lambda_i}{2\sqrt{\tau}} (R_i^+ - R_i^-). \] (9)

W.r.t. the Riemann invariants, system (3) has the diagonal form:
\[ \partial_t \mathbf{R} + \Theta \partial_x \mathbf{R} = \mathbf{F}(\mathbf{R}), \] (10)
where \( \mathbf{R} = (R_1^+, R_1^-, R_2^+, R_2^-)^T \) is a vector-valued function of in \( \mathbb{R} \times [0, +\infty) \), \( \Theta = \text{diag}(\theta_1^+, \theta_1^-, \theta_2^+, \theta_2^-) \) is a real diagonal matrix, \( \mathbf{F} = (f_1^+, f_1^-, f_2^+, f_2^-)^T \) is a vector field in \( \mathbb{R}^4 \), and
\[ f_i^\pm(\mathbf{R}) = (-1)^i \frac{(R_1^+ - R_1^-)(R_2^+ - R_2^-)(\alpha_1(R_1^+ + R_1^-) + \alpha_2(R_2^+ + R_2^-))}{2(R_1^+ + R_1^- + R_2^+ + R_2^-)^2} \pm \frac{1}{2\tau}(R_i^+ - R_i^-) \] are real valued functions of \( \mathbf{R} \in \mathbb{R}^4 \).

Let us remark that, similarly to the original vector field \( \mathbf{G} \), also the vector field \( \mathbf{F} \) is Lipschitz-continuous in the closed cone \([0, \infty)^4\), but it is not of class \( C^1 \) at the origin. As for the initial and boundary conditions, they become
\[ \mathbf{R}(x, 0) = \mathbf{R}_0(x), \quad \text{for all } x \in [0, L], \] (11)
\[ R_i^+(\xi, t) = R_i^-(\xi, t), \quad \text{for } t \geq 0 \text{ and } \xi = 0, L, \] (12)
as \( i = 1, 2 \). Here, \( \mathbf{U}_0 \) and \( \mathbf{R}_0 \) are related by means of the formula (9); in particular, the consistency condition
\[ R_{0i}^+(\xi) = -R_{0i}^-(\xi), \quad \text{for } \xi = 0, L \] (13)
is satisfied.

3.4. Broad solutions with boundary conditions. We begin by looking into a Cauchy problem for a semilinear system:
\[ \begin{cases} \partial_t \mathbf{U} + \mathbf{A} \partial_x \mathbf{U} = \mathbf{G}(\mathbf{U}), & x \in \mathbb{R}, \ t > 0, \\ \mathbf{U}(x, 0) = \mathbf{U}_0(x), & x \in \mathbb{R}, \end{cases} \] (14)
where \( \mathbf{A} \) is a (constant) matrix with distinct eigenvalues and \( \mathbf{G} \) a locally Lipschitz, vector valued function. This system is strictly hyperbolic and symmetrizable. Let \( \mathbf{R} \) be its Riemann invariants: we can write the diagonal form of the Cauchy problem (14)
\[ \begin{cases} \partial_t \mathbf{R} + \Theta \partial_x \mathbf{R} = \mathbf{F}(\mathbf{R}), & x \in \mathbb{R}, \ t > 0, \\ \mathbf{R}(x, 0) = \mathbf{R}_0(x). & x \in \mathbb{R}. \end{cases} \] (15)

Cauchy problems for semilinear hyperbolic systems have been object of extensive studies, and different notion of solutions have been proposed and discussed. Here, mostly following [3], we focus our attention on broad solutions, which are defined by integrating the differential equations along characteristic lines.

**Definition 3.1.** A vector valued function \( \mathbf{U} \) is a **broad solution** to the Cauchy problem (14) if its Riemann invariants \( \mathbf{R} \) are locally integrable, as well as the functions \( f_i^\pm(R_1^+, \cdots, R_2^-), \) and satisfy
\[ R_i^+(x, t) = R_{0i}^+(x - \theta_i^+ t) + \int_0^t f_i^+(R_1^+(x - \theta_1^+ (t - \sigma), \cdots, R_2^- (x - \theta_2^- (t - \sigma), \sigma)) d\sigma, \]
almost everywhere in \( \mathbb{R} \times [0, T] \).
If the differential system is studied on a bounded interval \(x \in [0, L]\), it should be endowed by a boundary condition that allows the characteristic lines to reflect in the boundary and cover all the strip \([0, L] \times [0, T]\). In particular, the boundary condition (12) (respectively, (7)) gives rise to an equivalent Cauchy problem with periodic datum. Given an initial datum \(R_0\) defined for \(x \in [0, L]\), we extend it in a periodic way as follows:

\[
\tilde{R}_{0i}(x) = \begin{cases} 
R_{0i}^+(x), & \text{for } x \in [0, L] \\
R_{0i}^-(x), & \text{for } x \in [-L, 0] \\
2L - \text{periodic}, & \text{otherwise}, 
\end{cases} 
\tag{16}
\]

as \(i = 1, 2\), and look into the respective Cauchy problem (15). If \(R \in C^1([0, L] \times [0, T])\) is a classical solution to the initial-boundary value problem (10)–(12), satisfying the consistency condition (13), we can build a solution to the Cauchy problem (15) as follows. Let \(\tilde{R}(t)\) be the function obtained by extending the function \(R(t)\) like in (16) (for each time \(t\)): it is clearly continuous by (12), (13), and joins the initial datum of (15). The differential equality in (15) holds true also for \(x \in (-L, 0)\) because

\[
\partial_t \tilde{R}_{i}^{\pm}(x, t) + \theta_i^{\pm} \partial_x \tilde{R}_{i}^{\pm}(x, t) - f_i^{\pm}(\tilde{R}(x, t)) \\
= \partial_t R_{i}^{\pm}(-x, t) - \theta_i^{\pm} \partial_x R_{i}^{\pm}(-x, t) - f_i^{\pm}(R(-x, t)) \\
= \partial_t R_{i}^{\pm}(-x, t) + \theta_i^{\pm} \partial_x R_{i}^{\pm}(-x, t) - f_i^{\pm}(R(-x, t)) = 0.
\]

At \(x = 0\) we have by construction that \(\partial_x \tilde{R}_{i}^{\pm}(0-, t) = -\partial_x R_{i}^{\pm}(0+, t)\). On the other hand, the differential system (10) gives

\[
-\partial_x R_{i}^{\pm}(0+, t) = \frac{1}{\theta_i^{\pm}} \left( \partial_t R_{i}^{\pm}(0, t) - f_i^{\pm}(R(0, t)) \right) \\
= -\frac{1}{\theta_i^{\pm}} \left( \partial_t R_{i}^{\pm}(0, t) - f_i^{\pm}(R(0, t)) \right) \\
= \partial_x R_{i}^{\pm}(0+, t) = \partial_x \tilde{R}_{i}^{\pm}(0+, t).
\]

Hence the function \(\tilde{R}\) is continuously differentiable also at \(x = 0\). The other points \(|x| \geq L\) can be handled in similar way.

Viceversa, if an initial datum is symmetric as in (16), the solution to the relative Cauchy problem (14) has the same symmetry (via uniqueness of solution), and therefore solves the initial-boundary value problem (10)–(12).

In view of this facts, we shall define a broad solution to the initial-boundary value problem (10)–(12) as a broad solution to the related Cauchy problem (15), whose initial datum has been extended as in (16). Obviously the transfer from an initial-boundary value problem into a Cauchy problem with periodic datum should rather be done for the biological differential system (3)–(6)–(7); in this case, the extension of the initial datum need for an even/odd reflection, as \(i = 1, 2\)

\[
(\tilde{u}_{0i}(x), \tilde{w}_{0i}(x)) = \begin{cases} 
(u_{0i}(x), w_{0i}(x)), & \text{for } x \in [0, L], \\
(w_{0i}(-x), -w_{0i}(-x)), & \text{for } x \in [-L, 0], \\
2L - \text{periodic}, & \text{otherwise}. 
\end{cases} 
\tag{17}
\]

We end up with the following definition.

**Definition 3.2.** A vector valued function \(U\) is a **broad solution** to the initial boundary value problem (3)–(6)–(7) if it can be extended to a broad solution of the
Cauchy problem (14) with initial datum $U_0$ given by (17). This, in turns, means that its Riemann invariants $R$ are locally integrable and satisfy Definition 3.1, with initial datum $R_0$ given by (16), at almost any point $(x, t) \in [0, L] \times [0, T]$.

Since the domain of the source term $G$ in (3) does not contain the hyperplane $n_1 + n_2 = 0$, well posedness can be addressed only after establishing a-priori that solutions do not cross such hyperplane. Afterward all results concerning globally Lipschitz sources can be easily adapted: in particular, broad solutions exist globally in time, they are continuous and depend continuously by the initial datum (see [3]). We mention in passing that the general theory for hyperbolic systems with global Lipschitz continuous sources allow solutions to grow up in time with exponential rate. This behavior is unexpected in our particular setting, also according to the numerical results in [1], and we provide better estimates in next sections.

By now we address well posedness, by producing an invariant set contained in the region where the flux is Lipschitz-continuous. We actually prove that the quantities $n_1$ and $n_2$ cannot become negative. This property is a very important consistency condition in view of the applications, since $n_1$ and $n_2$ stand for population densities.

**Proposition 3.3.** Assume that

$$\tau \leq \frac{1}{\max(|\alpha_1|, |\alpha_2|)},$$

(18)

and let $U = (n_1, n_2, w_1, w_2)$ be a solution to (3) with

$$n_0(x) \geq \frac{\sqrt{\tau}}{\lambda_i}|w_0(x)| \quad \text{for a.a. } x \in \mathbb{R}, \text{ as } i = 1, 2.$$  (19)

Then

$$n_i(x, t) \geq \frac{\sqrt{\tau}}{\lambda_i}|w_i(x, t)| \quad \text{for a.a. } x \in \mathbb{R}, \text{ as } i = 1, 2.$$  (20)

In particular, we have $n_i(x, t) \geq 0$, as $i = 1, 2$.

**Proof.** Remembering the relation (9) between $U$ and $R$, assumption (19) translates into

$$R_{0+}^i(x) \geq 0 \quad \text{for a.a. } x \in \mathbb{R}, \text{ as } i = 1, 2,$$  (21)

and we can equivalently prove that the positive cone $R_i^+ \geq 0$ is invariant for (10). Following [5], it suffices to check that the vector field $F$ points towards the interior of the positive cone $R_i^+ \geq 0$ (at any boundary points). For instance, at the boundary $R_1^+ = 0$ we have to prove that

$$f_i^+(0, R_1^-, R_2^+, R_2^-) = \frac{R_1^-}{2(R_1^- + R_2^+ + R_2^-)^2} \left( \left( \frac{2}{\tau} - \alpha_1 \right) R_1^- (R_2^+ + R_2^-) + \left( \frac{1}{\tau} - \alpha_2 \right) (R_2^+ + R_2^-)^2 + \frac{1}{\tau} (R_1^-)^2 \right) \geq 0$$

for all $R_1^-, R_2^+, R_2^- \geq 0$, which in turn follows by assumption (18). The other sides behave similarly.  

We have thus proved that, under assumption (18), the solutions $R$ of the diagonal Cauchy problem (15) stay nonnegative for all times, if they are nonnegative at $t = 0$. As the flux $F$ is globally Lipschitz-continuous in the positive cone $R_i^+ \geq 0$, existence and uniqueness of solutions follows readily.
Theorem 3.4 (Well-posedness). Assume (18), take $T > 0$ and $U_0 \in (L^\infty(0,L;\mathbb{R}))^4$ satisfying (8), (19). Then the initial-boundary value problem (3)–(6)–(7) has an unique broad solution satisfying (20). Finally, if $L$ is the Lipschitz constant of $F$, we have the following estimate

$$\|U(t)\|_{\infty} \leq \|U(0)\|_{\infty} e^{Lt}, \quad \text{for all } t \in [0,T].$$  \hspace{1cm} (22)

Proof. In view of Definition 3.2, we argue in terms of the Cauchy problem (15), and assume that its initial datum satisfies (13), (21). We cut the vector field $F$ in order to have a new vector field, say it $\bar{F}$, which coincides with $F$ in the closed positive cone $[0,\infty)^4$ and is globally Lipschitz continuous. By standard theory [3, Theorem 3.3], the obtained Cauchy problem has an unique bounded broad solution, which turns out to have nonnegative components by the same arguments of Proposition 3.3. Hence actually it solves the starting problem (15). Uniqueness follows by applying the uniqueness part of [3, Theorem 3.3] to the modified problem. Finally, estimate (22) follows by the Definition 3.1 and Gronwall’s Lemma. \hfill \Box

4. The nonlinear Trotter formula. In this section, we show that the solutions of the initial-boundary value problem (3)–(6)–(7) can be approximated by a splitting technique, inspired by the classical Trotter product formula. We next use such approximation to show that for some values of the parameters it is possible to improve estimate (22).

Let us present the assumptions that shall be in force in the following part of the paper. We assume that $U_0 \in (L^\infty(0,L;\mathbb{R}))^4 \cap BV(0,L;\mathbb{R}^4)$ satisfies (19). As a consequence, the initial datum for the Riemann invariants is $R_0 \in (L^\infty(0,L;\mathbb{R}))^4 \cap BV(0,L;\mathbb{R}^4)$ and satisfies (21). In the following, we shall refer for simplicity to the Cauchy problem for the Riemann invariants (15), whose initial datum is obtained by $R_0$ via (16). Since the domain of determinacy is contained in a strip $IT \times [0,T]$, we may assume without loss of generality that $R_0$ has compact support, so that $R_0 \in L^1(\mathbb{R};\mathbb{R}^4) \cap (L^\infty(\mathbb{R};\mathbb{R}))^4 \cap BV(\mathbb{R};\mathbb{R}^4)$. We shall always assume that the relaxation coefficient $\tau$ is not too large, namely that (18) holds.

We denote by $S_t$ the semigroup associated to the Cauchy problem for the diagonal linear hyperbolic system

$$\left\{ \begin{array}{ll} \partial_t R + \Theta \partial_x R = 0, & x \in \mathbb{R}, \quad t > 0, \\ R(x,0) = R_0(x), & x \in \mathbb{R}, \end{array} \right.$$ \hspace{1cm} (23)

and by $F_t$ the flow associated to Cauchy problem for the O.D.E.:

$$\left\{ \begin{array}{l} \partial_t R = F(R), \quad t > 0, \\ R(0) = R_0. \end{array} \right.$$ \hspace{1cm} (24)

The classical Trotter product formula aims to approximate the solution of (15) by the limit of the fractional step procedure:

$$R(t) = \lim_{n \to \infty} \left( S_{\frac{t}{n}} F \right)^n (R_0).$$

Here we face two technical difficulties: on one hand the initial datum is not continuous, on the other hand the equation is nonlinear. As a consequence, the approximating sequence needs to be a bit more refined. Concerning the initial datum, we
shall take a smooth approximating sequence \( \{ R_{0n} \} \subseteq C_0^\infty(\mathbb{R}; \mathbb{R}^4) \) such that 
\[
R_{0n} \to R_0 \quad \text{in } L^1(\mathbb{R}; \mathbb{R}^4) \text{ and pointwise a.e.,}
\]
\[
DR_{0n} \to DR_0 \quad \text{weakly in the sense of measures,}
\]
\[
\| R_{0n}(x) \|_{L^1} \leq 2 \| R_0(x) \|_{L^1}, \quad \| DR_{0n}(x) \|_{L^1} \leq 2 \| DR_0(x) \|_{L^1}, \quad 0 \leq R_{0n}(x) \leq \text{esssup} R_{0n}^+, \quad \text{for a.a. } x \in \mathbb{R}, i = 1, 2.
\]

Next, for any integer \( n \), we partition the time interval into small steps of type \( [k/n, (k + 1)/n) \), as \( k \in \mathbb{N} \). At the beginning of each time step, we update the initial datum by means of 
\[
R_n^k(x) = \left( S_{\frac{k}{n}} F_{\frac{k}{n}} \right)^k R_{0n}(x).
\]
Inside the interval \( (k/n, (k + 1)/n) \), we follow the evolution operator by setting 
\[
R_n(x, t) = (S_{\sigma_n} F_{\sigma_n}) R_n^k(x) \quad \text{for } \sigma_n(t) = t - k/n \in [0, 1/n).
\]

The convergence result is stated as follows.

**Theorem 4.1** (A nonlinear hyperbolic Trotter formula). Assume that (18) holds, take \( T > 0 \) and \( R_0 \in L^1(\mathbb{R}; \mathbb{R}^4) \cap (L^\infty(\mathbb{R}; \mathbb{R}))^4 \cap BV(\mathbb{R}; \mathbb{R}^4) \) satisfying (21). Then the sequence \( R_n \) defined by (25) converges to the broad solution of (15) in the space \( C(0, T; L^1(\mathbb{R}; \mathbb{R}^4)) \).

In order to prove this result, we need to recollect some well-known facts about the semigroup \( S_t \) and to check some properties of the flow \( F_t \). The following Lemma follows by standard arguments.

**Lemma 4.2.** \( S_t \) is a contraction semigroup in \( W^{1,1} \) and in \( L^\infty \), that is, for every function \( R \) with enough regularity, it holds 
\[
\| S_t R \|_{L^p} \leq \| R \|_{L^p} \quad \text{as } p = 1, \infty, \quad (26)
\]
\[
\| D S_t R \|_{L^1} \leq \| DR \|_{L^1}. \quad (27)
\]
We have also that \( S_t R \in [0, M]^4 \) if \( R \in [0, M]^4 \) and moreover 
\[
\| (S_t - I) R \|_{L^1} \leq \max(\theta^+, \alpha^+) t \| DR \|_{L^1}. \quad (28)
\]

Next we describe the evolution of hyper-rectangles according to the flow \( F_t \).

**Lemma 4.3.** Let \( K_0 = [0, M]^4 \), \( K_t = [0, M e^{ct})^4 \), with \( c = \max\{|\alpha_1|, |\alpha_2|\} \). Then \( F_t(K_s) \subseteq K_{s+t} \) for all \( s, t \geq 0 \).

We defer the proof of Lemma 4.3 to the appendix, when the flow \( F_t \) will be analyzed in more details. For the time being, we investigate the function \( F_t(R)(x) \), in terms of a given function \( R(x) \).

**Lemma 4.4.** There is a positive constant \( c \) such that 
\[
F_t(R)(x) \in [0, \infty)^4 \quad \text{for a.a. } x \in \mathbb{R}, \quad (29)
\]
\[
\| F_t(R) \|_p \leq e^{ct} \| R \|_p \quad \text{as } p = 1, \infty, \quad (30)
\]
\[
\| DF_t(R) \|_1 \leq e^{ct} \| DR \|_1, \quad (31)
\]
for every function \( R(x) \in W^{1,1}(\mathbb{R}; \mathbb{R}^4) \cap (L^\infty(\mathbb{R}; \mathbb{R}))^4 \), with nonnegative components. Moreover, if \( R_1, R_2 \in L^1 \cap L^\infty \) have nonnegative components, there is another constant \( \delta \) such that 
\[
\| F_t(R_1) - F_t(R_2) \|_{L^p} \leq e^{ct} \| R_1 - R_2 \|_{L^p} \quad \text{as } p = 1, \infty. \quad (32)
\]
Proof. Statements (29) and (30) come trivially by Lemma 4.3.

Concerning the estimate (31), we set $P = D_x(F_t(R))$ and compute

$$\frac{d}{dt} P = D_x(\partial_t R) = D_x(F(F_t(R))) = DF(F_t(R)) P.$$ 

Because the function $DF(F_t(R))$ is bounded, the thesis follows by Gronwall’s inequality. With respect to (32), we set $V = F_t(R_1) - F_t(R_2)$ and compute

$$\frac{d}{dt} V = F(F_t(R_1)) - F(F_t(R_1)) = a V,$$

where $a = \int_0^1 DF(\rho F_t(R_1) + (1 - \rho) F_t(R_2)) d\rho$.

Again, the function $a$ is bounded and Gronwall’s inequality yields the thesis. \qed

We are now ready to give the proof of Theorem 4.1.

Proof of Theorem 4.1. The function $R_n$ introduced in (25) is continuous w.r.t. $t$ since

$$\lim_{t \to t+1} R_n(x, t) = (S_\sigma F_{\frac{1}{2}}) R_n^k(x) = R_n^k(x) = R_n(x, \frac{k + 1}{n}).$$

It also has nonnegative components and is bounded in $W^{1,1}$ and $L^\infty$ (uniformly w.r.t. $t \in [0, T]$) because of Lemmas 4.2 and 4.4. In order to show that $R_n$ tends to $R$, the broad solution to (15), we denote by $\tilde{R}_n$ the (classical) solution to (15) with initial datum $R_{0n}$ instead of $R_0$, and we split

$$R - R_n = P_n + Q_n, \quad P_n = R - \tilde{R}_n, \quad Q_n = \tilde{R}_n - R_n.$$ 

The sequence $P_n \to 0$ as $n \to \infty$ in $C(0, T; L^1(\mathbb{R}; \mathbb{R}^4))$, by using standard results about stability with respect to initial data, see [3, Theorem 3.6]. Therefore we have to show that also the sequence $Q_n$ vanishes in $C(0, T; L^1(\mathbb{R}; \mathbb{R}^4))$. With this aim, we notice that $Q_n$ solves a linear Cauchy problem with homogeneous initial condition

$$\begin{cases}
\partial_t Q_n + \Theta \partial_x Q_n = b_n Q_n + e_n, \\
Q_n(0) = 0,
\end{cases}$$

with

$$b_n(x, t) = \int_0^1 DF(\rho \tilde{R}_n(x, t) + (1 - \rho) R_n(x, t)) d\rho,$$

$$e_n(x, t) = S_{\sigma_n} F(\sigma_n R_n^k(x, t)) - F(R_n(x, t)).$$

As $F$ is Lipschitz continuous in the positive cone where both $\tilde{R}_n$ and $R_n$ live, it is clear that $\|b_n(t)\|_\infty$ is bounded, uniformly w.r.t. $t \in [0, T]$ and $n \in \mathbb{N}$. In view of estimating the error term $e_n$, we write

$$e_n(t) = (S_{\sigma_n} - I) F(\sigma_n R_n^k) + F(\sigma_n R_n^k) - F(R_n(t)),$$

and notice that

$$\| (S_{\sigma_n} - I) F(\sigma_n R_n^k) \|_{L^1} \leq \max(\theta_1^+) \sigma_n \| D(\sigma_n R_n^k) \|_{L^1},$$

$$\leq c \max(\theta_1^+ \sigma_n \| D(\sigma_n R_n^k) \|_{L^1},$$

$$\| F(\sigma_n R_n^k) - F(R_n) \|_{L^1} = \| F(\sigma_n R_n^k) - F(\sigma_n R_n^k) \|_{L^1},$$

$$\leq c \| (S_{\sigma_n} - I)(\sigma_n R_n^k) \|_{L^1} \leq c \max(\theta_1^+ \sigma_n \| D(\sigma_n R_n^k) \|_{L^1},$$

$$\leq c \| (S_{\sigma_n} - I)(\sigma_n R_n^k) \|_{L^1} \leq c \max(\theta_1^+ \sigma_n \| D(\sigma_n R_n^k) \|_{L^1},$$

$$\leq c \| (S_{\sigma_n} - I)(\sigma_n R_n^k) \|_{L^1} \leq c \max(\theta_1^+ \sigma_n \| D(\sigma_n R_n^k) \|_{L^1},$$

$$\leq c \| (S_{\sigma_n} - I)(\sigma_n R_n^k) \|_{L^1} \leq c \max(\theta_1^+ \sigma_n \| D(\sigma_n R_n^k) \|_{L^1},$$
bifurcations. Moreover, if the actual growth has to be slower, and most of the solutions stay actually exponential in time, (22). However, numerical simulations in [1] indicate

5. Long-time behavior of solutions. Since our source term is Lipschitz continuous in the positive cone, we can show a general boundary of our solutions, which grow exponentially in time, (22). However, numerical simulations in [1] indicate that the actual growth has to be slower, and most of the solutions stay actually bounded. Moreover, if \( \lambda_1 = \lambda_2 \), the equation (4) satisfied by the total density \( N = n_1 + n_2 \) is just the dissipative wave equation, and therefore \( N \) is conserved and consequently \( n_1, n_2 \) stay bounded. Eventually also \( w_1, w_2 \) stay bounded by (20).

On the other hand, the Trotter formula established by Theorem 4.1 allows to deduce estimate for the equation (15) from estimates for the O.D.E. (24), since \( S_t \) is a contraction semigroup. To this aim, a relevant role is played by the sign of the parameters \( \alpha_1, \alpha_2 \). Hence before entering the technical details we make some remarks about their evolutionary meaning. If \( \alpha_1 \) and \( \alpha_2 \) have the same sign, then one species dominates the other one: if they are both positive (as for the Prisoner’s dilemma) species 2 is dominating, and vice versa. Otherwise, if the two parameters have opposite sign, then there is not a perfect dominance and the relative replicator equation has an internal equilibrium point at

\[
\alpha_2/\alpha_1,
\]

which is stable if \( \alpha_2 < 0 < \alpha_1 \), or unstable if \( \alpha_1 < 0 < \alpha_2 \). On the other hand, switching the role of population 1 and 2 yields a new payoff matrix:

\[
\tilde{A} = \begin{pmatrix}
\tilde{a}_{11} & \tilde{a}_{12} \\
\tilde{a}_{21} & \tilde{a}_{22}
\end{pmatrix},
\]

so \( \tilde{\alpha}_1 = -\alpha_2, \tilde{\alpha}_2 = -\alpha_1 \) and \( \max\{\tilde{\alpha}_1, \tilde{\alpha}_2\} = -\min\{\alpha_1, \alpha_2\} \). In view of this fact, one can suppose without loss of generality that one of the following items take place:

i) \( \min\{\alpha_1, \alpha_2\} \geq 0 \) (dominance),

ii) \( \alpha_2 = \min\{\alpha_1, \alpha_2\} < 0 < \max\{\alpha_1, \alpha_2\} = \alpha_1 \) (stable internal equilibrium),

iii) \( \alpha_1 = \min\{\alpha_1, \alpha_2\} < 0 < \max\{\alpha_1, \alpha_2\} = \alpha_2 \) (bistability).

We next rewrite system (24) componentwise. Set

\[
N = \left( R_1^+ + R_1^- + R_2^+ + R_2^- \right) / 2, \quad x = \left( R_1^+ + R_1^- \right) / 2N
\]

and

\[
h(x) = -x(1-x)(\alpha_1 x + \alpha_2 (1-x)).
\]
Then we have
\[
\begin{align*}
    \frac{d}{dt} R_1^\pm &= -\frac{1}{\tau} R_1^\pm + \frac{N}{\tau} x + N h(x), \quad t > 0, \\
    \frac{d}{dt} R_2^\pm &= -\frac{1}{\tau} R_2^\pm + \frac{N}{\tau} (1 - x) - N h(x), \quad t > 0, \\
    R_i^\pm (0) &= R_{0i}^\pm, \quad t = 0, \quad i = 1, 2.
\end{align*}
\]

Summing all equations in (33) yields
\[
\begin{align*}
    \frac{d}{dt} N &= 0, \\
    \frac{d}{dt} x &= h(x). \quad \text{(34)}
\end{align*}
\]

It follows that \(N\) is constant and so \(0 \leq R_i^\pm (t) \leq 2N\), as the functions \(R_i^\pm\) are nonnegative for the previous arguments. Unfortunately, such simple, but rough estimate is not sufficient to conclude an uniform bound by using the Trotter formula, since, at any step, the norm doubles. A better estimate is obtained by looking into the replicator equation (34) and taking into account the respective values of the parameters \(\alpha_1, \alpha_2\).

**Proposition 5.1.** Assume
\[
\tau \leq 1 / \max \left\{ |\alpha_1|, |\alpha_2|, \left| \frac{\alpha_1 \alpha_2}{\alpha_1 - \alpha_2} \right| \right\}. \quad (35)
\]

Take \(R_0 \in [0, M_1]^2 \times [0, M_2]^2\), and let \(\mathcal{F}_i(R_0)\) the solution to (33) computed at time \(t \geq 0\).

i) If \(\alpha_1, \alpha_2 > 0\), then
\[
\mathcal{F}_i(R_0) \in [0, M_1]^2 \times [0, M_2 + M_1(1 - e^{-\tau t})]^2
\]
as \(\overline{\alpha} = \max\{\alpha_1, \alpha_2\}\).

ii) If \(\alpha_2 < 0 < \alpha_1\), then
\[
\mathcal{F}_i(R_0) \in \left[0, \max \left\{ M_1, \left| \frac{\alpha_2}{\alpha_1} M_2 \right| \right\} \right]^2 \times \left[0, \max \left\{ M_2, \left| \frac{\alpha_1}{\alpha_2} M_1 \right| \right\} \right]^2.
\]

iii) If \(\alpha_1 < 0 < \alpha_2\), then
\[
\mathcal{F}_i(R_0) \in \left[0, M_1 + M_1 \left| \frac{\alpha_1}{\alpha_2} \right| (1 - e^{-|\alpha_1| t}) \right]^2 \times \left[0, M_2 + M_2 \left| \frac{\alpha_2}{\alpha_1} \right| (1 - e^{-|\alpha_2| t}) \right]^2.
\]

The proof of Proposition 5.1 is complicated by various technical details and is deferred to the appendix. For the time being, we use this result to give some more accurate estimates of the long-time behavior of the solutions to the hyperbolic system (15).

**Theorem 5.2** (Global estimates). Assume (35), take \(R_0 \in L^1(\mathbb{R}; \mathbb{R}^4) \cap (L^\infty(\mathbb{R}; \mathbb{R}))^4 \cap BV(\mathbb{R}; \mathbb{R}^4)\) satisfying
\[
R_0(x) \in [0, M_1]^2 \times [0, M_2]^2, \quad \text{for a.a. } x \in \mathbb{R},
\]
and let \(R(t)\) be the global broad solution to (15).

i) If the second population dominates the first one, i.e. if \(\alpha_1, \alpha_2 > 0\), then
\[
0 \leq R_i^\pm (x, t) \leq M_1,
0 \leq R_2^+ (x, t) \leq M_2 + M_1 \overline{\alpha} t
\]
for a.a. $x \in \mathbb{R}$ and all $t \geq 0$.

ii) If the replicator equation (33) has a stable internal equilibrium point, i.e. if $\alpha_2 < 0 < \alpha_1$, then

$$0 \leq R_1^\pm (x,t) \leq \max \left\{ M_1, \frac{\alpha_2}{\alpha_1} M_2 \right\},$$

$$0 \leq R_2^\pm (x,t) \leq \max \left\{ M_2, \frac{\alpha_1}{\alpha_2} M_1 \right\},$$

for a.a. $x \in \mathbb{R}$ and all $t \geq 0$.

iii) If the replicator equation (33) is bistable, i.e. if $\alpha_1 < 0 < \alpha_2$, then we can just refine our exponential bounds

$$0 \leq R_1^\pm (x,t) \leq M_1 e^{\frac{\alpha_2^2}{2\pi^2} t},$$

$$0 \leq R_2^\pm (x,t) \leq M_2 e^{\frac{\alpha_1^2}{2\pi^2} t},$$

for a.a. $x \in \mathbb{R}$ and all $t \geq 0$.

**Proof.** By Theorem 4.1, the solution to (15) $R$ is the limit of the sequence

$$R_n(x,t) = (S_{\sigma_n} F_{\sigma_n}) \left( S_1 F_1 \right)^k R_0(x)$$

with $k = n(t - \sigma_n)$, $0 \leq \sigma_n < 1/n$.

We first suppose that $\alpha_1$ and $\alpha_2$ are both nonnegative, so Proposition 5.1.i) holds. Since $S_t$ is a contractive semigroup, we have that

$$0 \leq R_{1n}^\pm (x,t) \leq M_1,$$

$$0 \leq R_{2n}^\pm (x,t) \leq M_2 + M_1 (t - \sigma_n) \left( 1 - e^{-\frac{\pi}{n}} \right) n + M_1 (1 - e^{-\frac{\pi}{n} \sigma_n}).$$

The thesis follows by letting $n$ to infinity, because $(1 - e^{-\frac{\pi}{n}}) n \to \pi$ and $\sigma_n \to 0$.

Otherwise, if $\alpha_2 < 0 < \alpha_1$, we enlarge the starting hyper-rectangle by taking $R_0 \in [0, M_1]^2 \times [0, M_2]^2 \subset [0, \overline{M_1}]^2 \times [0, \overline{M_2}]^2$ with

$$\overline{M_1} = \max \left\{ M_1, \frac{\alpha_2}{\alpha_1} M_2 \right\}, \quad \overline{M_2} = \max \left\{ M_2, \frac{\alpha_1}{\alpha_2} M_1 \right\}.$$ 

Now Proposition 5.1.ii) implies that at any step the hyper-rectangle $[0, \overline{M_1}]^2 \times [0, \overline{M_2}]^2$ is invariant, namely

$$0 \leq R_{1n}^\pm (x,t) \leq \overline{M_1}, \quad 0 \leq R_{2n}^\pm (x,t) \leq \overline{M_2},$$

and the same inequality holds after sending $n$ to infinity.

If, conversely, $\alpha_1 < 0 < \alpha_2$, after using iteratively Proposition 5.1.iii) one gets

$$0 \leq R_{1n}^\pm (x,t) \leq M_1 \left( 1 + \frac{\alpha_1}{\alpha_2} \left( 1 - e^{-\frac{\alpha_1}{n}} \right) \right)^{n(t-\sigma_n)+\sigma_n},$$

Sending $n$ to infinity gives the thesis since

$$\left( 1 + \frac{\alpha_1}{\alpha_2} \left( 1 - e^{-\frac{\alpha_1}{n}} \right) \right)^n = \exp \left( n \log \left( 1 + \frac{\alpha_1}{\alpha_2} \left( 1 - e^{-\frac{\alpha_1}{n}} \right) \right) \right) \to \exp \left( \frac{\alpha_1^2}{\alpha_2 \alpha_1} \right)$$

and $\sigma_n \to 0$, as $n \to \infty$. The other variables $R_2^\pm$ can be dealt with in the same way. \qed
6. Conclusions. In this paper we have presented and analyzed a hyperbolic model of population dynamics based on the ideas of evolutionary game theory. We have established a result of global existence of solutions, and, for some special choices of the parameters, we have established some uniform bounds, which improve the natural exponential growth. Clearly, these are only some preliminary results, since, in view of the numerical simulations in [1], we can imagine that a more complex behavior of the solutions is expected. In particular, it should be interesting to show the exact rate of decay for the cooperators population in the Prisoner’s Dilemma problem, also in comparison with the behavior in the diffusive models.

Appendix: proof of Lemma 4.3 and Proposition 5.1. The proof of Lemma 4.3 and Proposition 5.1 are split in several lemmas. We begin by establishing upper and lower bounds for the dynamics (33). Let us introduce the notations
\[ \alpha = \min\{\alpha_1, \alpha_2\}, \quad \bar{\alpha} = \max\{\alpha_1, \alpha_2\}, \]
and, for every \( \alpha \in \mathbb{R} \),
\[ \phi_\alpha(x, y, z, t) = \frac{x(x + y)}{2(\alpha e^{\alpha t} + y)} e^{\alpha t} + \frac{x - 2z}{2} e^{-\frac{t}{\tau}}. \]

Lemma 6.1. Under the assumption (35), the solution of (33) satisfies
\[
\phi_{-\bar{\alpha}} \left( R^+_{01} + R^+_{02} + R^-_{02} + R^-_{01}, t \right) \leq R^+_1(t)
\]
\[
\leq \phi_{-\alpha} \left( R^+_{01} + R^-_{01}, R^+_{02} + R^-_{02}, t \right) \,
\]
\[
\phi_{\bar{\alpha}} \left( R^+_{02} + R^-_{01} + R^-_{01} + R^+_{02}, t \right) \leq R^-_2(t)
\]
\[
\leq \phi_{\alpha} \left( R^+_{02} + R^-_{02} + R^+_{01} + R^-_{01}, t \right),
\]
for all \( t \geq 0 \).

Proof. For all \( 0 \leq x \leq 1 \), the function \( h(x) \) is bounded by
\[ -\bar{\alpha}x(1 - x) \leq h(x) \leq -\alpha x(1 - x). \]
So (34) gives
\[ \frac{x \alpha e^{-\alpha t}}{1 - x \alpha(1 - e^{-\alpha t})} \leq x(t) \leq \frac{x \alpha e^{-\alpha t}}{1 - x \alpha(1 - e^{-\alpha t})}. \]

(36)

We next address to \( R^\pm_1 \) and remark that the function \( x \mapsto \frac{Nx}{\tau} + Nh(x) \) is increasing under assumption (35). So inserting (39) and (38) in (33) allows to control the dynamics for \( R^\pm_1(t) \) as follows
\[ a_{-\bar{\alpha}}(x_o, t) \leq \frac{d}{dt}R^\pm_1 + \frac{R^\pm_1}{\tau} \leq a_{-\alpha}(x_o, t), \]
with
\[ a_{\alpha}(x_o, t) = \alpha e^{\alpha t} \left( 1 + \frac{1}{\bar{\alpha}} \right) (1 - x_o + x_o e^{\alpha t}) - x_o e^{\alpha t} \frac{N x_o}{(1 - x_o + x_o e^{\alpha t})^2}. \]

Integrating these differential inequalities gives (36).

Concerning \( R^\pm_2 \), it suffices to switch the role of \( R^\pm_1 \) and \( R^\pm_2 \) to get an equivalent system (33) with new parameters \( \alpha_1 = -\alpha_2 \) and \( \bar{\alpha}_2 = -\alpha_1 \). In particular,
\[ \max\{\bar{\alpha}_1, \bar{\alpha}_2\} = -\alpha \quad \text{and} \quad \min\{\bar{\alpha}_1, \bar{\alpha}_2\} = -\bar{\alpha}. \]
Therefore the estimate (37) for \( R^\pm_2 \) follows by applying (36) to the new system. \( \square \)
We now use estimates (36), (37) to analyze the evolution of hyper-rectangles under the flux of $F_i$.

**Lemma 6.2.** Assume (35) and let $R_0 \in [0, M_1]^2 \times [0, M_2]^2$. Then the solution of (33) fulfills

$$R_{i}^\pm(t) \geq 0, \quad \text{for all } t \geq 0, \quad \text{as } i = 1, 2.$$

If, in addition, $\alpha_1, \alpha_2 \geq 0$, then

$$R_1^+(t) \leq M_1 \quad \text{for all } t \geq 0,$$

$$R_2^+(t) \leq M_2 + M_1(1 - e^{-\eta t}) \quad \text{for all } t \geq 0.$$

Otherwise, if $\alpha_1$ and $\alpha_2$ have opposite sign,

$$R_1^+(t) \leq M_1 + M_2(1 - e^{\eta t}) \quad \text{for all } t \geq 0,$$

$$R_2^+(t) \leq M_2 + M_1(1 - e^{-\eta t}) \quad \text{for all } t \geq 0,$$

where $\alpha, -\eta < 0$.

**Proof.** Let us introduce the notation

$$D_{i,j} = \{(x, y, z, t) : z \leq x \leq z + M_i, 0 \leq y \leq 2M_j, 0 \leq z \leq M_i, t \geq 0\}.$$

In view of Lemma 6.1, bounds for $R_{i}^\pm(t)$ are obtained by computing the bounds of the functions $\phi_{\alpha}$ in the sets $D_{i,j}$. To this aim, we begin by remarking that the function $\phi_{\alpha}$ is always increasing with respect to $x$, and increasing or decreasing w.r.t. $y$ according if $\alpha$ is positive or negative.

As we may assume without loss of generality that $\eta \geq 0$, we get

$$R_1^+(t) \geq \phi_{-\eta}(z, 2M_2, z, t) = \frac{z}{2} \left(\frac{z + 2M_2}{ze^{-\eta t} + 2M_2}e^{-\eta t} - e^{-\frac{z}{2}}\right) \geq \frac{z}{2} \left(e^{-\eta t} - e^{-\frac{z}{2}}\right) \geq 0$$

by (35). Concerning $R_2^+$, if $\alpha \leq 0$ we get similarly

$$R_2^+(t) \geq \phi_{\alpha}(z, 2M_1, z, t) \geq 0.$$

Otherwise, if $\alpha \geq 0$, we have

$$R_2^+(t) \geq \phi_{\alpha}(z, 0, z, t) = \frac{z}{2} \left(1 - e^{-\frac{z}{2}}\right) \geq 0.$$

We next pass to upper bounds; if $\alpha \geq 0$ we have

$$R_2^+(t) \leq \phi_{-\alpha}(z + M_1, 0, z, t) = \frac{M_1}{2} \left(1 + e^{-\frac{z}{2}}\right) + \frac{z}{2} \left(1 - e^{-\frac{z}{2}}\right) \leq M_1.$$

Conversely, $\alpha < 0$ gives

$$R_1^+(t) \leq \phi_{-\alpha}(z + M_1, 2M_2, z, t) = \frac{(z + M_1)(z + M_1 + 2M_2)}{2((z + M_1)e^{-\alpha t} + 2M_2)^2} e^{-\alpha t} + \frac{M_1 - z}{2} e^{-\frac{z}{2}}.$$

The last expression increases with $z$ because

$$\frac{d}{dz} \phi_{-\alpha}(z + M_1, 2M_2, z, t) = \frac{2M_2^2(e^{-\alpha t} - 1)}{(z + M_1)e^{-\alpha t} + 2M_2)^2} + \frac{1}{2} \left(1 - e^{-\frac{z}{2}}\right) \geq 0.$$

Hence

$$R_1^+(t) \leq \phi_{-\alpha}(2M_1, 2M_2, M_1, t) = \frac{M_1 M_2(e^{-\alpha t} - 1)}{M_1 e^{-\alpha t} + M_2} \leq M_1 + M_2(1 - e^{\alpha t}),$$

and similarly

$$R_2^+(t) \leq M_2 + M_1(1 - e^{-\eta t}),$$

because $\eta \geq 0$ in any case. \qed
Now the estimates stated by Lemma 4.3 follow readily.

**Proof of Lemma 4.3.** If $M_1 = M_2 = M$, Lemma 6.2 implies
\[
0 \leq R_1^\pm(t) \leq \max\{M, M(2 - e^{ct})\} \leq M(2 - e^{-ct}),
\]
\[
0 \leq R_2^\pm(t) \leq M(2 - e^{-\pi t}) \leq M(2 - e^{-ct}),
\]
as $c = \max\{|\alpha_1|, |\alpha_2|\}$. The thesis of Lemma 4.3 follows readily since $2 - e^{-ct} \leq e^{ct}$.

Lemma 6.2 also provides Proposition 5.1.i), and the lower estimates of 5.1.ii,iii). Obtaining estimates from above requests some more work. In the following we shall assume that $\alpha_1$ and $\alpha_2$ have opposite sign and make use of the following notations
\[
x^* = \frac{\alpha_2}{\alpha_2 - \alpha_1} \in (0, 1),
\]
\[
\beta = \max_{0 \leq x \leq 1} \{-\alpha_2(1 - x^*x)\} = \begin{cases} 
-\alpha_2 > 0 & \text{if } \alpha_2 < 0 < \alpha_1 \\
\frac{\alpha_1\alpha_2}{\alpha_1 - \alpha_2} < 0 & \text{if } \alpha_1 < 0 < \alpha_2
\end{cases}
\]
\[
\gamma = \max_{0 \leq x \leq 1} \{-\alpha_1(1 - (1 - x^*)x)\} = \begin{cases} 
\frac{\alpha_1\alpha_2}{\alpha_2 - \alpha_1} < 0 & \text{if } \alpha_2 < 0 < \alpha_1 \\
-\alpha_1 > 0 & \text{if } \alpha_1 < 0 < \alpha_2
\end{cases}
\]
\[
\Phi(x, y, z, s) = \frac{x^*x(x + y)s}{2((s - 1 + x^*)x + x^*y)} + \left(\frac{x}{2} - z\right) s^{-\frac{1}{\beta}},
\]
\[
\Psi(x, y, z, s) = \frac{x^*(x + y) + \frac{1}{2}((1 - x^*)(1 - x^*)x - x^*y)(x + y)s}{2(y + ((1 - x^*)x - x^*y)s)}
\]
\[
+ \left(\frac{x}{2} - z\right) s^{-\frac{1}{\beta}}.
\]

**Lemma 6.3.** Let $\alpha_1$, $\alpha_2$ have opposite sign. If (35) holds, then the solution of (33) satisfies
\[
R_1^+(t) \leq \Phi \left( R_0^+ + R_0^- + R_0^+, R_0^+ + R_0^-, e^{\beta t} \right), \tag{40}
\]
provided that $(1 - x^*)(R_0^+ + R_0^-) \leq x^*(R_0^+ + R_0^-)$, or
\[
R_1^-(t) \leq \Psi \left( R_0^+ + R_0^- + R_0^+, R_0^+- R_0^-, e^{\beta t} \right), \tag{41}
\]
if $(1 - x^*)(R_0^+ + R_0^-) \geq x^*(R_0^+ + R_0^-)$.

**Proof.** We recall that the dynamics (34) has an equilibrium point at $x^*$, which is stable if $\alpha_1 < 0 < \alpha_2$ and unstable if $\alpha_2 < 0 < \alpha_1$.

Let $x_0 = (R_0^+ + R_0^-)/(R_0^+ + R_0^- + R_0^+ + R_0^-)$ the starting point of the dynamics (34). It is clear that $0 \leq x_0 \leq 1$; we next split the interval $0 \leq x_0 \leq 1$ into two sub-intervals $0 \leq x_0 \leq x^*$ and $x^* \leq x_0 \leq 1$.

If $x_0 \in [0, x^*]$, i.e. if $(1 - x^*)(R_0^+ + R_0^-) \leq x^*(R_0^+ + R_0^-)$, we perform the change of variables $X = x/x^*$ and notice that
\[
\frac{d}{dt} X = -\alpha_2(1 - x^*X)X(1 - X).
\]

Hence
\[
\frac{d}{dt} X \leq \beta X(1 - X), \tag{42}
\]
and so
\[
X(t) \leq \frac{x_0 e^{\beta t}}{1 - x_0(1 - e^{\beta t})}, \tag{43}
\]
Besides
\[ \frac{d}{dt} R_1^\pm = -\frac{1}{\tau} R_1^\pm + N x^* \left( \frac{X}{\tau} - \alpha_2 (1 - x^* X) X (1 - X) \right). \]

By assumption (35) the function \( X \mapsto \frac{X}{\tau} - \alpha_2 (1 - x^* X) X (1 - X) \) is monotone increasing; so inserting (43), (42) and integrating gives (40).

Similarly, if \( x_0 \in [x^*, 1] \) we perform the change of variables \( X = (x - x^*)/(1 - x^*) \) and notice that
\[ \frac{d}{dt} X = \alpha_1 (1 - (1 - x^*)(1 - X)) X (1 - X). \]

Hence \( \frac{d}{dt} X \leq \gamma X (1 - X) \), and arguing as in the first part of the proof yields (41). \( \square \)

We next use (40), (41) to control the evolution of hyper-rectangles under the flux \( \mathcal{F}_t \).

**Lemma 6.4.** Let \( \alpha_1, \alpha_2 \) have opposite sign, and assume (35). Take \( R_0 \in [0, M_1]^2 \times [0, M_2]^2 \), with
\[ (1 - x^*)(R_0^+ + R_0^-) \leq x^*(R_0^+_1 + R_0^-_1). \]

Then the solution of (33) fulfills
\[ R_1^\pm(t) \leq \max \{ M_1, |\alpha_2/\alpha_1| M_2 \}, \text{ if } \alpha_2 < 0 < \alpha_1, \quad (44) \]
\[ R_1^\pm(t) \leq M_1, \text{ if } \alpha_1 < 0 < \alpha_2. \quad (45) \]

**Proof.** We look for an upper bound for the function \( \Phi(x, y, z, s) \) when \( (x, y, z) \) belongs to the set
\[ D = \{(x, y, z) : z \leq x \leq z + M_1, (1 - x^*) x/x^* \leq y \leq 2M_2, 0 \leq z \leq M_1\}, \]
and \( s \geq 1 \) (if \( \alpha_2 < 0 < \alpha_1 \)) or \( 0 < s \leq 1 \) (if \( \alpha_1 < 0 < \alpha_2 \)).

To this aim let us compute
\[ \partial_y \Phi = \frac{x^* s (s - 1) x^2}{2 ((s - 1 + x^*) x + x^* y)^2}. \]

As \( \partial_y \Phi \leq 0 \) in the set \( D \times (0, 1] \), then
\[ \Phi \leq \Phi(x, \frac{1 - x^*}{x^*} x, z, s) = \frac{x}{2} \left( 1 + s^{-\frac{1}{\alpha_1}} \right) - z s^{-\frac{1}{\alpha_2}} \]
\[ \leq \frac{z}{2} \left( 1 - s^{-\frac{1}{\alpha_2}} \right) + \frac{M_1}{2} \left( 1 + s^{-\frac{1}{\alpha_2}} \right) \leq M_1 \]
and (45) follows. Here, we have taken advantage from the inequality \( 0 < s^{-\frac{1}{\alpha_2}} < 1 \).

Conversely, \( \partial_y \Phi \geq 0 \) in the set \( D \times [1, \infty) \), and the maximum is attained at \( y = 2M_2 \). We thus go on and compute
\[ \partial_x \Phi = \frac{x^* s \left( (s - 1) x^2 + x^* (x + y)^2 \right)}{2 ((s - 1 + x^*) x + x^* y)^2} + \frac{1}{2} s^{-\frac{1}{\alpha_2}} \geq 0. \]

So the maximum w.r.t. \( x \) is attained at \( x = \min \{ z + M_1, 2 \frac{x^*}{1 - x^*} M_2 \} \). We may always suppose that \( M_1 = x^*/(1 - x^*)M_2 = |\alpha_2/\alpha_1| M_2 \), up to enlarge the starting hyper-rectangle into
\[ [0, \max \{ M_1, |\alpha_2/\alpha_1| M_2 \}]^2 \times [0, \max \{ M_2, |\alpha_1/\alpha_2| M_1 \}]^2. \]
As a consequence, \( z + M_1 \leq 2M_1 = 2 \frac{z}{1-x} M_2 \) and the maximum point is \( x = z + M_1 \). We thus address to
\[
\frac{d}{dz} \Phi(z + M_1, 2M_2, z, s) = \partial_z \Phi(z + M_1, 2M_2, z, s) - s^{-\frac{1}{\tau}}
\]
\[
= x^* s \left( (s-1)(z + M_1)^2 + x^*(z + M_1 + 2M_2)^2 \right) - \frac{1}{2} s^{-\frac{1}{\tau}}.
\]
At \( z = M_1 \), since \( M_1 = \frac{z}{1-x}, M_2 \) we have
\[
\frac{d}{dz} \Phi(2M_1, 2M_2, M_1, s) = \frac{1}{2s} \left( x^*(s-1) + 1 - s^{-\frac{1}{\tau}} \right).
\]
Recalling that \( s \geq 1 \) and \( 1/\beta \tau = 1/|\alpha_2|\tau > 1 \) by (35), one gets that \( \frac{d}{dz} \Phi \geq 0 \) at \( z = M_1 \). Besides \( \frac{d^2}{dz^2} \Phi \) is nondecreasing w.r.t. \( z \) because
\[
\frac{d^2}{dz^2} \Phi(z + M_1, 2M_2, z, s) = - \left( \frac{(2x^* M_2)^2 s(s-1)}{(s-1)(z + M_1) + x^*(z + M_1 + 2M_2))^3} \right) \leq 0,
\]
hence \( \frac{d}{dz} \Phi \geq 0 \) at all \( 0 \leq z \leq M_1 \). Eventually \( \Phi \leq \Phi(2M_1, 2M_2, M_1, s) \) on \( D \times [1, +\infty) \), and (44) follows because \( \Phi(2M_1, 2M_2, M_1, s) = M_1 \).

**Lemma 6.5.** Let \( \alpha_1, \alpha_2 \) have opposite sign, and assume (35). Take \( R_0 \in [0, M_1]^2 \times [0, M_2]^2 \), with
\[
(1-x^*)(R_0^+ + R_0^-) \geq x^*(R_0^+ + R_0^-).
\]
Then the solution of (33) fulfills
\[
R_{1+}^x(t) \leq M_1, \quad \text{if } \alpha_2 < 0 < \alpha_1, \quad (46)
\]
\[
R_{1+}^x(t) \leq M_1 + M_1 |\alpha_1/\alpha_2| (1 - e^{\alpha_1 t}) \quad \text{if } \alpha_1 < 0 < \alpha_2. \quad (47)
\]

**Proof.** Let \( O \) be the set
\[
O = \{(x, y, z) : z \leq x \leq z + M_1, \ (1-x^*)x \geq x^* y, \ 0 \leq y \leq M_2, \ 0 \leq z \leq M_1 \}.
\]
We look now into upper bounds for the function \( \Psi(x, y, z, s) \) as \( (x, y, z) \in O \) and \( 0 < s \leq 1 \) (if \( \alpha_2 < 0 < \alpha_1, \gamma < 0 \)) or, respectively, \( s \geq 1 \) (if \( \alpha_1 < 0 < \alpha_2, \gamma > 0 \)). Let us compute
\[
\partial_y \Psi = \frac{x^*(s-1) (x^* s ((1-x^*)x-x^* y)^2 - (x^* y)^2)}{2 (y + ((1-x^*)x-x^* y s)^2)}.
\]
For any given \( x, z, s \), the map \( y \mapsto \Psi(x, y, z, s) \) has a critical point at
\[
\eta(x, s) = \frac{\sqrt{x^* s}}{1+\sqrt{x^* s}} \frac{1-x^*}{x^*} x \leq \frac{1-x^*}{x^*} x.
\]
If \( \alpha_2 < 0 < \alpha_1, \ s - 1 \leq 0 \), therefore \( \eta \) is a minimum point and
\[
\Psi(x, y, z, s) \leq \max \left\{ \Psi(x, 0, z, s), \Psi \left( \frac{1-x^*}{x^*} x, z, s \right) \right\} \text{ on } O \times (0, 1].
\]
Eventually (46) follows since
\[
\Psi(x, 0, z, s) = \Psi(x, \frac{1-x^*}{x^*} x, z, s) = \frac{x}{2} \left( 1 + s^{-\frac{1}{\tau}} \right) - z s^{-\frac{1}{\tau}} \leq \frac{M_1}{2} (1 + s^{-\frac{1}{\tau}}) + \frac{z}{2} (1 - s^{-\frac{1}{\tau}}) \leq M_1,
\]
because \( 0 < s^{-\frac{1}{\tau}} \leq 1 \).
On the contrary, if \( s - 1 \geq 0 \), \( \eta \) is a maximum point and we get
\[
\Psi(x, y, z, s) \leq \Psi(x, \eta(x, s), z, s) \quad \text{on } O \times [1, +\infty).
\]
So we look into
\[
\frac{d}{dx} \Psi(x, \eta(x, s), z, s) = \frac{1}{2} + \frac{(1 - x^*)(s - 1)\eta^2}{2(\eta + ((1 - x^*)x - x^*\eta)s)^2} + \frac{1}{2}s^{-\frac{s}{\gamma}} \geq 0
\]
as \( s \geq 1 \). Hence
\[
\Psi \leq \Psi(z + M_1, \eta(z + M_1, s), z, s)
\]
\[
= \frac{z}{2} \left( \frac{\sqrt{x^*} + \sqrt{s}}{1 + \sqrt{x^*}} - s^{-\frac{1}{\gamma}} \right) + \frac{M_1}{2} \left( \frac{\sqrt{x^*} + \sqrt{s}}{1 + \sqrt{x^*}} + s^{-\frac{1}{\gamma}} \right).
\]
Since \( \frac{\sqrt{x^*} + \sqrt{s}}{1 + \sqrt{x^*}} \geq 1 \geq s^{-\frac{1}{\gamma}} \), it follows that
\[
\Psi \leq M_1 \left( \frac{\sqrt{x^*} + \sqrt{s}}{1 + \sqrt{x^*}} \right)^2 = M_1 \left( 1 + \frac{(1 - x^*)(s - 1)}{(1 + \sqrt{x^*})^2} \right) \leq M_1 \left( 1 + \frac{1 - x^*}{x^*}(1 - s^{-1}) \right),
\]
that is (47).

We eventually recollect all the partial estimates to conclude the proof of Proposition 5.1.

Proof of Proposition 5.1. Part i) is contained in Lemma 6.2.

Concerning part ii) and iii), the bounds from below have been proved in Lemma 6.2, and the ones from above concerning \( R_1^s(t) \) follows readily by Lemmas 6.4 and 6.5.

To get an upper bound for \( R_1^s(t) \), we switch the role of \( R_1 \) and \( R_2 \): it gives a system like (33) with new parameters \( \hat{\alpha}_1 = -\alpha_1 \) and \( \hat{\alpha}_2 = -\alpha_2 \), which has an equilibrium point at \( \hat{x}^* = \alpha_1/(\alpha_1 - \alpha_2) = 1 - x^* \), that is stable (or unstable) under the same condition \( \alpha_2 < 0 < \alpha_1 \) (respectively, \( \alpha_1 < 0 < \alpha_2 \)). So the previous arguments yield
\[
R_1^s(t) \leq \max \{ M_2, |\alpha_1/\alpha_2| M_1 \} \quad \text{if } \alpha_2 < 0 < \alpha_1,
\]
\[
R_2^s(t) \leq M_2 \left( 1 + |\alpha_2/\alpha_1| (1 - e^{-\alpha_2 t}) \right) \quad \text{if } \alpha_1 < 0 < \alpha_2.
\]

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