ON THE ERROR TERM OF A LATTICE COUNTING PROBLEM, II

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ABSTRACT. Under the Riemann Hypothesis, we improve the error term in the asymptotic formula related to the counting lattice problem studied in a first part of this work. The improvement comes from the use of Weyl’s bound for exponential sums of polynomials and a device due to Popov allowing us to get an improved main term in the sums of certain fractional parts of polynomials.

1. INTRODUCTION AND RESULT

This work is the continuation of the paper [3] in which the following problem is studied. For integer $T \geq 1$, we let

$$F(T) := \{a/b : (a,b) \in \mathbb{Z}^2, \ 0 \leq a < b \leq T, \ (a,b) = 1\}$$

be the set of Farey fractions. We also define

$$I(T) = F(T) \cap [0, \frac{1}{2}]$$

and consider the quantity

$$C(T) = \sum_{a/b \in I(T)} \text{#}C_{a,b}(T),$$

where

$$C_{a,b}(T) := F(T) \cap \left[1 - \frac{a^2}{b^2}, 1\right].$$

As it is mentioned in [3], this quantity $C(T)$ appears naturally in some counting problems for two-dimensional lattices and the main term of the asymptotic formula for $C(T)$ can be expressed via the cardinality

$$F(T) := \#F(T)$$

of the set of Farey fractions and also second moment of the Farey fractions in $[0, \frac{1}{2}]$:

$$G(T) := \sum_{\xi \in F(T)} \xi^2.$$

More precisely, it is shown [3, Theorem 1.1] that, unconditionally

$$C(T) = F(T)G(T) + O\left(T^3 \delta(T^{1/2}) \log T\right)$$

where $\delta(t)$ is the usual number-theoretic remainder function defined as

$$\delta(t) := \exp(-c(\log t)^{3/5}(\log \log t)^{-1/5}) \quad (c > 0).$$

Under the Riemann Hypothesis, one can improve on the error term by using the well-known estimate [1]

$$M(t) := \sum_{n \leq t} \mu(n) \ll t^{1/2} \rho(t)$$

where

$$\rho(t) := \exp \left((\log t)^{1/2}(\log \log t)^{5/2+o(1)}\right)$$

and the authors derived in [3, Theorem 1.4] the estimate

$$C(T) = F(T)G(T) + O\left(T^{752/283} \rho(T) \log T\right)$$

under RH, with the help of Bourgain’s exponent pair $(k, \ell) = (\frac{13}{84}, \frac{55}{84})$. Furthermore, it is pointed out that, if the exponent pair conjecture is true, then the error-term may be sharpened to $O(T^{5/2+o(1)})$. Note that

$$\frac{752}{283} \approx 2.6572.$$
The aim of this work is to show that there is no need to assume this very difficult conjecture in order to get this estimate. More precisely, we prove

**Theorem 1.1.** Assume the Riemann Hypothesis. Then

\[ C(T) = F(T)G(T) + O(T^{5/2+o(1)}) \]

The idea is to estimate a sum of fractional parts using Weyl’s bound for exponential sums of polynomials [6] and a device of Popov [7], also used by Fomenko [4], which allows us to improve the main term in sums of the shape

\[ \sum_{N<n \leq 2N} \psi(P(n)) \]

where \( P \) is any polynomial of degree \( \geq 2 \) and

\[ \psi(x) := x - \lfloor x \rfloor - \frac{1}{2}. \]

2. Notation

We take all the notation of [3] into account, in particular

\[ E(T) := C(T) - F(T)G(T) \]

is the error term in the lattice counting problem considered here. Let \( \psi(x) := x - \lfloor x \rfloor - \frac{1}{2} \) be the 1st Bernoulli function and \( \|x\| := \min (\frac{1}{2} - \psi(x), \frac{1}{2} + \psi(x)) \) is the distance of \( x \) to its nearest integer.

For any \( \beta \geq 0 \), let \( F_\beta \) be the multiplicative function defined by

\[ F_\beta(n) := \sum_{d|n} \frac{\mu(d)^2}{d^\beta} = \prod_{p|n} \left(1 + \frac{1}{p^\beta}\right). \]

Note that \( F_0 = 2^x \) and

\[ \sum_{n \leq x} F_\beta(n) \ll \begin{cases} x \log x, & \text{if } \beta = 0 \\ x, & \text{if } \beta > 0. \end{cases} \]

Finally, if \( f : (M, M+N] \rightarrow \mathbb{R} \) is any map and \( \delta \in (0, \frac{1}{2}] \), define

\[ R(f, N, \delta) := \# \{ n \in (M, M+N] \cap \mathbb{Z} : \|f(n)\| < \delta \}. \]

3. Sums of fractional parts of polynomials

This section is devoted to the proof of the following proposition, generalizing [4, Theorems 1 and 2], and which may have its own interest (see Remark 3.3 below).

**Proposition 3.1.** Let \( q \in \mathbb{Z}_{\geq 1}, k \in \mathbb{Z}_{\geq 2}, N \in \mathbb{Z}_{\geq 1} \) large and \( \alpha > 0 \). Then, for any \( \varepsilon > 0 \)

\[ \sum_{N<n \leq 2N} \psi(n^k\alpha) \ll_{k, \varepsilon} (N\kappa)^\varepsilon \left( N\alpha^{2^{-k}-k} F_{1-k2^{-1-k}}(q) + N^{1-2^{1-k}} F_{1-2^{-1-k}}(q) + N^{1-k2^{-1-k}} F_1(q) \right) \]

where \( \kappa := \max(\alpha, \alpha^{-1}) \).

For the proof, the following lemma is needed. This result is similar to [6, (9)] but the statement does not need any rational approximation of \( \alpha \) and the method of proof is quite different.

**Lemma 3.2.** Let \( M \in \mathbb{Z}_{\geq 0}, N \in \mathbb{Z}_{\geq 1}, L \in \mathbb{Z}_{\geq 4} \) and \( \alpha > 0 \). Then

\[ \sum_{M<n \leq M+N} \min \left( L, \frac{1}{\|na\|} \right) \ll LN\alpha + (N + \alpha^{-1}) \log L + L. \]

**Proof.** From [2, Lemma 6.45], we first have

\[ \sum_{M<n \leq M+N} \min \left( L, \frac{1}{\|na\|} \right) \ll N + L \sum_{k=0}^{K-2} 2^{-k} R(n\alpha, N, 2^kL^{-1}) \]
where \( K := \left\lfloor \frac{\log L}{\log 2} \right\rfloor \), and using [2, Theorem 5.6] we get

\[
\sum_{M < n \leq M + N} \min \left( L, \frac{1}{\| n \alpha \|} \right) \ll N + L \sum_{k=0}^{K-2} 2^{-k} \left( N^\alpha + 2^k N L^{-1} + 2^k (L \alpha)^{-1} + 1 \right)
\ll N + L N \alpha + (N + \alpha^{-1}) K + L
\]

implying the asserted result. \( \square \)

We now are in a position to prove Proposition 3.1.

**Proof of Proposition 3.1.** One may assume \( \alpha \in (0,1) \), otherwise \( N^\alpha \gg N \). For any \( H \in \mathbb{Z}_{\geq 1} \), the left-hand side does not exceed

\[
\ll \frac{\varphi(q)}{q} N + \sum_{h \leq H} \frac{1}{h} \sum_{N < n \leq 2N} e(h \alpha n)
\ll \frac{N}{H} + \sum_{h \leq H} \frac{1}{h} \sum_{d \mid q} \mu(d)^2 \sum_{\frac{N}{d} < n \leq \frac{2N}{d}} e(h d \alpha)
\ll \frac{N}{H} \log H + \sum_{d \mid q} \mu(d)^2 \sum_{h \leq H} \frac{1}{h} \sum_{\frac{N}{d} < n \leq \frac{2N}{d}} e(h d \alpha)
\ll \frac{N}{H} + 2^{\omega(q)} \log H + \sum_{d \mid q} \mu(d)^2 \sum_{h \leq H} \frac{1}{h} \sum_{\frac{N}{d} < n \leq \frac{2N}{d}} e(h d \alpha)
\]

and using Weyl's bound [6, (4) p. 40] we get

\[
\ll \frac{N}{H} + 2^{\omega(q)} \log H + \sum_{d \mid q} \mu(d)^2 \sum_{h \leq H} \frac{1}{h} \left( \frac{N}{d} \right)^{2^{k-1}-1}
\]

\[
+ \left( \frac{N}{d} \right)^{2^{k-1}-k+\varepsilon} \sum_{\ell \leq k (N/d)^{k-1}} \min \left( \frac{N}{d}, \frac{1}{\| \ell hd \alpha \|} \right) \right\}^{2^{1-k}}
\ll \frac{N}{H} \left( N^{1-2^{1-k}} F_{1-2^{1-k}}(q) + 2^{\omega(q)} \right) \log H
+N^{1-k2^{1-k}+\varepsilon} \sum_{d \mid q} \mu(d)^2 \frac{1}{d^{1-k2^{1-k}}} S_{H,N}(d)
\]

where

\[
S_{H,N}(d) := \sum_{h \leq H} \frac{1}{h} \left( \sum_{\ell \leq k (N/d)^{k-1}} \min \left( \frac{N}{d}, \frac{1}{\| \ell hd \alpha \|} \right) \right)^{2^{1-k}}.
\]
Assume \( d \leq \frac{1}{4} N \). Hölder’s inequality with \( \lambda = \frac{2^k}{2^{k-2}} \) yields

\[
S_{H,N}(d) \leq \left( \sum_{h \leq H} \frac{1}{h} \right)^{1-2^{-1-k}} \left( \sum_{h \leq H} \frac{1}{h} \sum_{\ell \leq k/(N/d)^{k-1}} \min \left( \frac{N}{d}, \frac{1}{\|\ell h d^k\alpha\|} \right) \right)^{2^{1-k}}
\]

\[
\leq (\log eH)^{1-2^{-1-k}} \left( \sum_{n \leq k! H/(N/d)^{k-1}} \min \left( \frac{N}{d}, \frac{1}{\|nd^k\alpha\|} \right) \sum_{h \mid n \mid h \leq H} \frac{1}{h} \right)^{2^{1-k}}
\]

\[
= (\log eH)^{1-2^{-1-k}} \left( \sum_{j=0}^{k! H-1} \sum_{n/(j+1)(N/d)^{k-1} < n < j(N/d)^{k-1}} \min \left( \frac{N}{d}, \frac{1}{\|nd^k\alpha\|} \right) \sum_{h \mid n \mid h \leq H} \frac{1}{h} \right)^{2^{1-k}}
\]

Following [7, (13),(14)] (see also [4]), note that, in the inner sum

\[
\frac{1}{h} \leq \frac{k! N^{k-1}}{nd^{k-1}} < \frac{k!}{j} \quad (j \geq 1)
\]

so that

\[
S_{H,N}(d) \leq (\log eH)^{1-2^{-1-k}} \left( \sum_{n \leq (N/d)^{k-1}} \frac{\sigma(n)}{n} \min \left( \frac{N}{d}, \frac{1}{\|nd^k\alpha\|} \right) \right)
\]

\[
+ \sum_{j=1}^{k! H-1} \frac{k!}{j} \sum_{n/(j+1)(N/d)^{k-1} < n < j(N/d)^{k-1}} \tau(n) \min \left( \frac{N}{d}, \frac{1}{\|nd^k\alpha\|} \right)
\]

and the crude bounds \( \tau(n) \ll \epsilon n^\epsilon \) and \( \sigma(n) \ll \epsilon n^{1+\epsilon} \), along with Lemma (3.2) used with \( M = j(N/d)^{k-1} \), \( N \) replaced by \((N/d)^{k-1}\), \( \alpha \) replaced by \( d^k\alpha \) and \( L = \frac{N}{d^k} \geq 4 \), yield

\[
(NH)^{-\epsilon} S_{H,N}(d) \ll \left( N^k \alpha + \left( \frac{N}{d} \right)^{k-1} + d^{-k^2-k} \right) \left( \sum_{j=1}^{k! H-1} \frac{1}{j} + 1 \right)^{2^{1-k}}
\]

\[
\ll N^{k^2-k} \alpha^{2^{1-k}} + \left( \frac{N}{d} \right)^{(k-1)2^{1-k}} + d^{-k^2-k} \alpha^{2^{1-k}}.
\]

Consequently

\[
\left| \sum_{N < n \leq 2N} \psi \left( n^k \alpha \right) \right| \ll \frac{N}{H} + \left( N^{1-2^{1-k}} F_{1-2^{1-k}}(q) + 2^{\omega(q)} \right) \log H
\]

\[
+ H^\epsilon N^{1-k^{2}-k+\epsilon} \sum_{d \mid q, d \leq \frac{1}{2} N} \frac{\mu(d)^2}{d^{k-2}} \left( N^{k^{2}-k} \alpha^{2^{1-k}} + \left( \frac{N}{d} \right)^{(k-1)2^{1-k}} + d^{-k^2-k} \alpha^{2^{1-k}} \right)
\]

\[
\ll \frac{N}{H} + \left( N^{1-2^{1-k}} F_{1-2^{1-k}}(q) + 2^{\omega(q)} \right) \log H
\]

\[
+ (NH)^\epsilon \left( N^{2^{1-k}} F_{1-2^{1-k}}(q) + N^{1-2^{1-k}} F_{1-2^{1-k}}(q) + \frac{N^{1-k^{2}-k} F_{1}(q)}{\alpha^{2^{1-k}}} \right)
\]

and the choice of \( H = \left\lfloor \alpha^{-2^{1-k}} \right\rfloor \) allows us to achieve the proof. \( \square \)
Remark 3.3. With \( q = 1 \), Proposition 3.1 yields
\[
(N\kappa)^{-\varepsilon} \sum_{N < n \leq 2N} \psi(n^{\alpha}) \ll_{k,\varepsilon} N\alpha^{2^{1-k}} + N^{1-2^{1-k}} + N^{1-k2^{1-k}}\alpha^{-2^{1-k}}
\]
whereas Van der Corput’s method [5, Theorem 2.8] provides
\[
\sum_{N < n \leq 2N} \psi(n^{\alpha}) \ll_k N\alpha^{1/(2^k - 1)} + N^{1-2^{1-k}} + N^{1-2^{1-k} - 2^{4-2k}}\alpha^{-2^{1-k}}
\]
so that, for any \( k \in \mathbb{Z}_{\geq 2} \) and \( \alpha \in (0, 1) \), Proposition 3.1 improves significantly the first term, sometimes called the main term, and the secondary terms are of the same strength.

4. Proof of Theorem 1.1

From [3, (2.2)] we get
\[
E(T) = - \sum_{a/b \in I(T)} M\left(\frac{T}{d}\right) \sum_{d \leq T} \psi\left(\frac{da^2}{b^2}\right) - \frac{1}{2} \sum_{a/b \in I(T)} M\left(\frac{T}{d}\right) \sum_{d \leq T} \psi\left(\frac{da^2}{b^2}\right) + O(T^2) := \Sigma(T) + O(T^2)
\]
say. Now from (1.1)
\[
|\Sigma(T)| = \left| \sum_{d \leq T} M\left(\frac{T}{d}\right) \sum_{b \leq T} \sum_{a \leq \frac{T}{b}} \psi\left(\frac{da^2}{b^2}\right) \right|
\]
\[
\ll T^{1/2} \rho(T) \sum_{d \leq T} \frac{1}{d^{1/2}} \sum_{b \leq T} \sum_{a \leq \frac{T}{b}, (a,b)=1} \psi\left(\frac{da^2}{b^2}\right)
\]
\[
\ll T^{1/2} \rho(T) \sum_{d \leq T} \frac{1}{d^{1/2}} \sum_{b \leq T} \max_{A \leq \frac{T}{b}} \sum_{A \leq n \leq 2A, (a,b)=1} \psi\left(\frac{da^2}{b^2}\right) \log b.
\]

We use Proposition 3.1 with \( k = 2 \), i.e.
\[
\sum_{N < n \leq 2N} \psi(n^{2\alpha}) \ll_{k,\varepsilon} (N\kappa)^{\varepsilon} \left( N\alpha^{1/2}2\omega(q) + N^{1/2}F_{1/2}(q) + \frac{F_1(q)}{a^{1/2}} \right)
\]
with \( N = A, q = b \) and \( \alpha = db^{-2} \), yielding
\[
|\Sigma(T)| \ll T^{1/2+o(1)} \sum_{d \leq T} \frac{1}{d^{1/2}} \sum_{b \leq T} \left( d^{1/2}2\omega(b) + b^{1/2}F_{1/2}(b) + \frac{bF_1(b)}{d^{1/2}} \right)
\]
\[
\ll T^{1/2+o(1)} \sum_{d \leq T} \frac{1}{d^{1/2}} \left( Td^{1/2} + T^{3/2} + T^2d^{-1/2} \right)
\]
\[
\ll T^{5/2+o(1)}
\]
where we used the bound (2.1) in the penultimate line. This completes the proof. \( \square \)

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