Bayesian based estimation of turbulent flow fields from lidar observations in a conventionally neutral atmospheric boundary layer

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Abstract. In this paper, we consider the reconstruction of 3D turbulent flow fields from a time series of lidar data in a conventionally neutral atmospheric boundary layer (CNBL). For the reconstruction we use the maximum a posteriori estimate of the flow field. This corresponds to an optimization problem, with a cost function that has two contributions; a first term originating from the prior belief on the probability of having a certain turbulent flow field without any observations. Flow field fluctuations are assumed normally distributed and thus statistically fully determined by the mean and two-point covariance of the velocity field. The second term, is related to the likelihood of the observations, influenced by model and measurement uncertainties. The two-point covariance is computed and found to be significantly altered by the Coriolis force, breaking up longer streamwise velocity streaks and veering spanwise structures by $\sim 45^\circ$ with respect to the mean flow direction. For the reconstruction, we consider two different scanning modes, a plan position indicator (PPI) mode and a trajectory which is based on a Lissajous curve. For the PPI scanning mode we find that the mean squared error of the reconstructed velocity field is around 10% of the background variance in the scanning plane, and quickly increases outside this region. The Lissajous curve on the other hand attains an average error of 40% over the scanning region, which spans almost the whole BL height.

1. Introduction

Three-dimensional turbulent flow field information can be used for a wide variety of applications such as analysis of experimental data, prediction of wind turbine power output and the cooperative control of wind farms, for tracking a predefined power signal [1] or optimizing the power output [2].

A wide variety of flow field reconstruction approaches exist, most of which can be related to a (simplified) Bayesian criterium. Flow field fluctuations are often assumed Gaussian, such that the prior is fully determined by the spatial covariance and the mean. Flow field reconstruction using only observations at a single time instance, combined with covariances based on the homogeneous isotropic turbulence spectrum was performed in [3]. In [4] flow field reconstruction was considered in a horizontal plane by using an unscented Kalman filter combined with a 2D Navier–Stokes model, where viscosity effects are ignored. [5] performed 3D flow field reconstruction, using a Taylor’s frozen turbulence flow model to relate measurements at different time instances, but ignoring prior flow field information. In [6] large-eddy simulations (LES) were used as a model, and the prior distribution was approximated by a Laplacian based
smoothing term. The spatial structures are however known to exhibit strong non-isotropic behaviour, see e.g. [7].

In the current work, we determine the flow field, by combining prior statistical knowledge of the flow field with a time series of lidar measurements. Instead of using real lidar measurement, we take virtual lidar measurements from a reference LES simulation. This methodology was already successfully applied to a pressure-driven boundary layer (PDBL) in [8]. In a real atmospheric boundary layer (ABL), stratification and Coriolis forces are known to significantly alter the BL. Therefore in this work, we demonstrate the methodology on a conventionally-neutral atmospheric boundary layer (CNBL).

The remainder of the paper is organized as follows. In §2 we discuss the general state estimation framework, the lidar observations and the case-setup. In §3 we start by discussing the one- and two-point statistics, which plays an import role in the flow field reconstruction. Next, we discuss the results of the state-estimation. Finally, §4 summarizes and concludes the paper.

2. Methodology

2.1. Bayesian inference

We are looking for an estimate of the velocity field at time \( t_0 \). To this end we will use a time series of lidar observations \( y_1, \ldots, y_{N_s} \) at \( t_1, \ldots, t_{N_s} \) with \( N_s \) the amount of samples in time. For the estimation of the velocity field we use the maximum a posterior probability criterium (MAP).

In order to avoid technicalities associated with infinite dimensional probability density functions (PDFs), we discretize our velocity field using a finite set of proper orthogonal decomposition (POD) modes \( \Psi = [\psi_1, \ldots, \psi_{N_m}] \), where \( \psi_k(x) \) is the \( k \)-th mode and \( N_m \) is the total amount of modes. The POD modes are found as the eigenvectors of the two-point covariance \( R_{ij}(x, \hat{x}) \), defined as

\[
R_{ij}(x, \hat{x}) = \langle u_i(x, t_0) u_j(\hat{x}, t_0) \rangle,
\]

with \( \langle \cdot \rangle \) the ensemble average and \( u' = u - U \) the deviations from the mean \( U = \langle u \rangle \). The POD modes are orthogonal and normalized such that \( \langle \Omega \rangle^{-1} \int_{\Omega} \psi_i^T(x) \psi_j(x) \, dx = \delta_{ij} \), with \( |\Omega| \) the domain volume. The modes are ordered such that the corresponding eigenvalues \( \lambda_k \) are in decreasing order, i.e. \( \lambda_k \geq \lambda_{k+1} \), which are grouped in a diagonal matrix \( \Lambda = \text{diag}(\lambda_1, \ldots, \lambda_{N_m}) \).

The POD expansion of the velocity is then written as

\[
u(x, t_0) = \sum_{k=1}^{N_m} \psi_k(x) \lambda_k a_k + U(x) = \Psi(x) \Lambda^{1/2} a + U(x),
\]

with \( a = [a_1, \ldots, a_{N_m}]^T \) the vector of coefficients \( a_k \), which have a zero mean \( \langle a_k \rangle = 0 \) and are uncorrelated with unit variance, i.e. \( \langle a_i a_j \rangle = \delta_{ij} \).

The MAP estimate is then equal to

\[
u^* = \arg \max \ p(a|y),
\]

where for ease of notation all the measurements are concatenated in a single vector \( y = [y_1, \ldots, y_{N_s}]^T \) and \( p(a|y) \) is the conditional PDF of the coefficients \( a \) given the set of measurements \( y \). In order to find an expression for \( p(a|y) \) we apply Bayes’ theorem, which gives

\[
p(a|y) \propto p(y|a)p(a).
\]

Equation (4) has two contributions: the prior knowledge of the state \( p(a) \), and the added information of the observations \( p(y|a) \).
In order to elaborate \( p(a) \), we assume that the turbulent velocity fluctuations in the ABL are normally distributed. The PDF of the POD coefficients, \( p(a) \) is simply given by

\[
p(a) \propto \exp \left(-\frac{1}{2} \| a \|^2 \right). \tag{5}
\]

In order to obtain an expression for \( p(y|a) \), we introduce the observation operator \( h(u(x, t)) \), which maps the velocity field to the measurements. The velocity fields at \( t > t_0 \) are approximately linked to \( t_0 \) by a state-space model. In this way we can introduce the solution operator \( M_t \) integrating a solution to time \( t \), i.e. \( \tilde{u}(x, t) = M_t(u(x, t_0)) \). The mismatch between the modelled observations and the true observations is given by \( y - h(M_t(u(x, t_0))) = \eta \), where \( \eta \) contains contributions from model uncertainties and measurement errors. Since an exact PDF of \( \eta \) is very hard to come by, we simply assume a normal distribution with zero mean \( \langle \eta \rangle = 0 \) and covariance \( \langle \eta_i \eta_j \rangle = \delta_{ij} \gamma^2 \). This leads to

\[
p(y|a) \sim \exp \left(-\frac{\| y - h(M_t(\Psi(x)\Lambda^{1/2}a + U(x))) \|^2}{2\gamma^2} \right). \tag{6}
\]

Inserting (5) and (6) in (4) and taking the logarithm which does not change the optimum and multiplying by \(-1\), we end up with

\[
a^* = \arg \min_a J(a) = \frac{1}{2} \| a \|^2 + \frac{1}{2\gamma^2} \| y - h(\Psi(x)\Lambda^{1/2}a + U(x)) \|^2. \tag{7}
\]

The coefficient vector \( a^* \) is directly related to the velocity field by \( u^*(x, t_0) = \Psi(x)\Lambda^{1/2}a^* + U(x) \).

2.2. Lidar observations

We assume the lidar observations are made by a pulsed lidar sensor. The modelled observation of the \( i \)-th range gate at \( t_n \) is given by

\[
h_{n,i} = \frac{1}{T_s} \int_{t_n-1}^{t_n} \int_{\Omega} u(x, t) \cdot e_i(t) \ G_t(Q(t)(x - x_i(t))) \, dx \, dt,
\]

with \( x_i \) the location of the \( i \)-th range gate, \( e_i(t) \) the unit vector in the direction of the lidar, \( Q(t) \) a rotation matrix which rotates the main axis frame \( [e_1, e_2, e_3] \), such that \( e_1 \) is aligned with the lidar, i.e. \( e_1(t) = Q(t)e_1 \). Finally, \( G_t \) is a filter which takes into account the finite pulse- and range gate width of the lidar (see [9, 8] for further details).

We use a laser which is based on the Lockheed Martin WindTracer. The most important specifications are: an initial blind zone of 436 m, a range gate and pulse width of 105 m, 100 active range gates and a sampling frequency of 5 Hz. Further, we consider two different scanning modes. First, a plan-position indicator mode with zero elevation and an azimuthal deflection of 10.6° (as can be seen on figure 1). Secondly, a scanning mode based on a Lissajous curve with the same spanwise extent and a vertical extent up to 900 m (see figure 8). The period of the scanning is taken as \( T_p = 100 \) s. The lidar mount is located at \( x_m = [0, 0, z_m] \), where we take the mount height \( z_m = 100 \) m.

2.3. Simulation and optimization setup

The simulation setup is schematically represented in figure 1. All simulations are performed using our in-house LES code SP-Wind. We shortly summarize the case-setup, and refer to [10]
for further implementational details. The LES equations are solved on a cuboidal domain, where we use periodic boundary conditions in the horizontal directions, a high-Reynolds wall stress model at the wall combined with impermeability, and symmetry boundary conditions at the top of the domain.

The reference measurements are taken from a fine-grid reference LES simulation. The geostrophic velocity is assumed horizontal and decomposed as $U_g = U_{g,1}e_1 + U_{g,2}e_2$. We choose a geostrophic wind speed $G = (U_{g,1}^2 + U_{g,2}^2)^{1/2} = 12\,\text{ms}^{-1}$, angled at $-5^\circ$ with the x-axis, i.e. $\arctan(U_{g,2}/U_{g,1}) = -5^\circ$. Since we consider a neutral BL, we have a zero heat flux at the bottom boundary. Further, we set the strength of the capping inversion at 1 K, which approximately corresponds with an equilibrium BL-height $H = 1\,\text{km}$ [11]. The total domain height is chosen as $L_3 = 1.75\,\text{km}$ and the top 500 m is reserved for a Rayleigh damping layer with damping coefficient of 0.016s$^{-1}$ to prevent reflection of upward propagating gravity waves [12]. The Coriolis parameter is set to $f = 1 \times 10^{-4}\,\text{s}^{-1}$, corresponding to latitude of 43°N. A surface roughness $z_0 = 2 \times 10^{-3}\,\text{m}$ is selected, which corresponds to a typical value found over desert plains or sea [13]. The dimensions of the reference domain, based on which the correlations and the reference observations are computed, is chosen relatively big ($42\,\text{km} \times 12\,\text{km}$) to avoid spurious periodic effects of the correlations. Further a grid resolution of $15\,\text{m} \times 15\,\text{m} \times 5\,\text{m}$ is used.

We use a spin-up period of 20 h, until a statistical steady state is reached and successively average the two-point covariance matrix over 5000 s. As an illustration, a snapshot of an instantaneous velocity field is shown on figure 2.

The reconstruction domain on the other hand only needs to encompass the measurement area, including a margin due to information being transported by the flow during the reconstruction time horizon. To this end domain dimensions of $21\,\text{km} \times 6\,\text{km}$ are chosen. Since the BL is neutral, thermal effects are very small inside the measurement region. Therefore as a simplification we omit all thermal effects, and choose a domain height of $L_3 = 1\,\text{km}$ as a first order approximation for the turbulent damping effect of the capping inversion. Scales finer than the range gate width are not captured by the lidar sensor and therefore a coarser grid $50\,\text{m} \times 50\,\text{m} \times 66\,\text{m}$ is chosen.

For the optimization, we consider a horizon of two full scanning cycles $T = 2T_p = 200\,\text{s}$. A good value of the model-measurement uncertainty is found by a Pareto front analysis as $\gamma^2 = 1\,\text{m}^2\text{s}^{-2}$. For the optimization we use the L-BFGS optimization algorithm. The gradients are computed using a continuous adjoint approach (see [8] for further details).

3. Results

3.1. One and two point statistics

Here we demonstrate one and two point statistics. Since the influence of the Coriolis force on the two-point statistics has to the author’s knowledge not been reported, we compare the results with a canonical PDBL. The PDBL is simulated using the same grid resolution, domain size in the horizontal directions, and BL-height as the CNBL. The domain height equals the BL-height such that $L_3 = H = 1\,\text{km}$. Results are scaled by the friction velocity $u_*$ such that the wall stress is the same for both cases.

Figure 3 represents the horizontally averaged one-point statistics. These were also reported in [10], but with more distance between the inversion and the damping layer and a thicker damping layer and were found to be quasi-identical. Panel (a) shows the average velocity magnitude. The CNBL velocity is higher than the PDBL and develops a supergeostrophic jet. The higher velocity can be attributed to the lower shear stresses (e) in the CNBL. The capping inversion is easily identified in figure (e). This is a highly stable region, such that the turbulent kinetic energy (TKE), and as a consequence also the horizontal shear stresses (e) quickly reduce to zero. The lower shear leads to a lower production of TKE and levels of TKE (d).

The two-point correlation between velocities $u_i(x)$ and $u_j(x)$ is the normalized version of the
Figure 1. Schematic overview of a horizontal section and side view of the domain. (■): lidar mount location; (-----): denotes the lidar scanning area; (-----): extent of the optimization domain. In the $x_1$–$x_3$ cross-section the inversion layer (slightly exaggerated for clarity) and Rayleigh damping layer.

Figure 2. Instantaneous horizontal flow field. The bottom and top respectively represent a $x_1$–$x_2$ cross-section at $x_3 = z_m$ and a $x_1$–$x_3$ cross-section.
Figure 3. Vertical profiles of velocity magnitude (a), veering angle w.r.t. the geostrophic wind direction, i.e. $\Delta \phi = \arctan(U_2/U_1) - \arctan(U_{g,2}/U_{g,1})$ (b), potential temperature (c), turbulent kinetic energy (d), shear stress magnitude (e) for (---): PDBL and (-----): CNBL.

two-point covariance (1) and is defined as

$$C_{ij}(x, \bar{x}) = \frac{R_{ij}(x, \bar{x})}{(R_{ii}(x, x)R_{jj}(\bar{x}, \bar{x}))^{1/2}}.$$  

(9)

The two-point correlation contains a lot of information about the structure of the flow field and has therefore been extensively studied in the past. In figure 4 a comparison is made for $C_{ij}$ between CNBL and a PDBL. In panels (a-c) the streamwise component $C_{11}$ is shown and it is observed that the PDBL correlation lengths are very long extending up to 20H similar to what was found in [14]. For the CNBL case it is found that streamwise correlation lengths are approximately five times shorter, such that velocity structures in the inner layer are well influenced by the presence of the Coriolis force, in contrast with the mean velocity profile. This is in accordance with experimental data from [15], where measurements are taken in an ABL in neutral conditions. The extend of the correlation was of order 4H at $x_3/H = 0.05$, which is comparable with our results, the BL-height was however significantly lower (60 m vs. 1 km), such that attribution of the effect to the Coriolis force is premature.

In figure 5 (a-c), we zoom in on the CNBL section of the $C_{11}$ correlation. In panel (c) the $x_1-x_2$ section is shown, where it is observed that the correlations show significant veer compared with the mean wind direction. This effect was also observed qualitatively in [16] where a DNS study of an Ekman layer was considered, and in [17] studying the weakly convective ABL by looking at the instantaneous velocity field and is attributed to the symmetry braking effect of the Coriolis force. The effect is more pronounced in figure (b), where the classical picture of high and low velocity streaks lying next to each other is less pronounced.

When looking at $C_{22}$ (d-f), the scale difference between CNBL and PDBL is much less pronounced. The $x_1-x_3$ section (d) shows that the main lobe is steeper inclined with respect to
Figure 4. Visualization of streamwise component of the two-point correlation $C_{11}(\mathbf{x}, \hat{\mathbf{x}})$ with reference point $\hat{\mathbf{x}} = [0, 0, H/10]$ and contour lines are drawn for $C_{ij} = [-0.1, -0.05, 0.05, 0.1, 0.3]$. (b): $x_1-x_2$ cross-section at $x_3 = H/10$, (a): $x_1-x_3$ cross-section at $x_2 = 0$, (c): $x_2-x_3$ cross-section at $x_1 = 0$. The line colours are (red): CNBL; (black): PDBL. (----): $C_{11} < 0$; (----): $C_{11} > 0$.

the surface. In the $x_1-x_2$ section, it is seen that the principal direction of the correlation is veered by approximately $45^\circ$ with the $x_1$ direction. Finally, the $C_{33}$ (g-i) does not show significant anti-symmetries, which can be explained since in our implementation the Coriolis force has no direct influence on $\tilde{u}_3$, neglecting the wind direction. Recently in [18], it has been show, that the influence is not completely negligible, such that wind direction dependent inclination can be expected. The main difference observed is the generation of two laterally positioned side lobs.

3.2. Reconstruction of the velocity field

A sample of the reconstructed velocity field is shown on figure 7 at the middle of the reconstruction window for the PPI scanning mode. For now we concentrate on the horizontal plane of the lidar mount. It is observed that the large scale structures of the $u_1$ and $u_2$ velocity field are relatively well reconstructed inside the measurement region, and even in a slight up and downstream region. This can be attributed to the convective transport of measurement information, during the time horizon. Further it is observed that weak streamwise streaks along the principal axis of the correlation are added. In this region no measurement information is available, and this demonstrates the impact of the prior $p(\mathbf{a})$. The wall-normal component of the velocity lacks long correlation lengths (as was shown on figure 5 (i)), and is therefore less smooth, and the reconstruction quality is harder to judge.

To further assess the quality of the reconstruction, figure 7 shows the velocity profiles along two horizontal lines. The $u_1$ velocity is best reconstructed, since this is the main lidar measurement direction, and measured directly. The $u_2$ and $u_3$ velocities are mainly reconstructed via cross-correlations between the different velocity components, in combination with state-space model constraint. The large scale trends however remain accurately captured. Further, it is interesting to see what happens in the vertical direction. This is further analysed in figure 8. As a comparison, we add a second lidar scanning trajectory, which is based on a Lissajous curve, which also measures in the vertical direction. The quality quickly degrades in the vertical direction for the PPI mode, the Lissajous mode on the other hand captures the general large scale flow pattern for the three velocity components.

Finally, the error is analysed more quantitatively. $e_1(\mathbf{x}, t) = \mathcal{I}_c^F \circ \tilde{u}_1(\mathbf{x}, t) - u_1(\mathbf{x}, t)$, where $\tilde{u}_1$ is the reconstructed velocity, $\mathcal{I}_c^F$ is a coarse-to-fine interpolation operator. We introduce
Figure 5. Visualization of the streamwise $C_{11}(x, \hat{x})$ (a-c), spanwise $C_{22}(x, \hat{x})$ (d-f) and wallnormal $C_{33}(x, \hat{x})$ (g-i) component of the two-point correlation with reference point $\hat{x} = [0, 0, H/10]$. Contour lines are drawn for $C_{ij} = [-0.1, -0.05, 0.05, 0.1, 0.3]$ with (red): CNBL; (black): PDBL; (-----): $C_{11} < 0$; (_______): $C_{11} > 0$. (c,f,i): $x_1-x_2$ cross-section at $x_3 = H/10$, (a,d,g): $x_1-x_3$ cross-section at $x_2 = 0$, (b,e,h): $x_2-x_3$ cross-section at $x_1 = 0$. 
Figure 6. The streamwise (a,b), spanwise (c,d) and wall-normal (e,f) velocity fluctuations at $t = t_i + T/2$. (a,c,e): reconstructed velocity field; (b,d,f): reference velocity field. (■): lidar mount location; (-----): scanning region; (------): lidar measurement points.

Figure 7. The streamwise (a,b), spanwise (c,d) and wall-normal (e,f) velocity component at $t = t_i + T/2$. (a,c,e): along a line in the streamwise direction through the lidar mount, i.e. $(x_1, 0, 0.1H)$; (b,d,f): along a line in the spanwise direction at mount height, and 8 km upstream of the mount, i.e. $(-8\text{ km}, x_2, 100\text{ m})$. (---): reconstructed velocity $\tilde{u}$, (---): reference velocity $u$. 
Figure 8. The streamwise (a-c), spanwise (d-f) and wall-normal (g-i) velocity fluctuations at $t = t_i + T/2$ in a $x_2-x_3$ plane 8 km upstream of the lidar mount. (a,d,g): PPI scanning mode reconstruction; (b,e,h): reference velocity field; (c,f,i): Lissajous scanning mode reconstruction. (- - - -): Lissajous scanning region; (-----): PPI scanning region.

the mean squared error over a horizontal region of interest $\Gamma$ as $\langle (\epsilon_1)^2 \rangle_\Gamma$. For $\Gamma$ we take the sweeping lidar region, where all points who are closer than $U_G T$ to the edge of the scanning region removed to avoid boundary effects. The results are summarized in figure 9. For the PPI scanning mode, the velocity in the scanning plane is reconstructed very well, with an error for the streamwise component of 10% compared with the background variance at $x_3 = 100$ m averaged over the scanning horizon. The error increases rapidly when looking at different heights, and is practically equal to the background variance at 400 m. The Lissajous curve outperforms the PPI at higher altitude, with an average error variance of 40%, which is remarkably higher than the 25% found in [8] considering a PDBL. A possible explanation could be the decreased presence of large scale structures, as discussed in 3.1.

4. Conclusion

We investigated reconstructing turbulent flow field from lidar measurements by using a Bayesian approach in combination with a LES model in a CNBL. The two-point velocity covariance was found to be significantly altered for the horizontal velocity components in comparison with a PDBL. First, for the streamwise velocity component correlation lengths were significantly reduced, and the anti-correlated lateral lobes due to the alternating high- and low speed streaks in a PDBL were not found. For the spanwise direction, the main structures seemed to be angled $45^\circ$ compared with the streamwise direction, illustrating the symmetry breaking effect of the Coriolis force. For the reconstruction, two different scanning modes were considered. In general large scale flow trends were well reconstructed. The reconstruction quality outside the lidar plane seemed to be reduced, compared with the PDBL. This may be explained by the above mentioned decrease in correlation length scales.

The current reconstruction was performed in an idealized testing environment. In practice, additional errors would appear due to e.g. imperfect knowledge of the atmospheric conditions, lidar measurement errors, etc. These elements are subject for future research.
Figure 9. Normalised error variances. (a,c,e): as function of height at $T = t_i + T/2$; (b,d,f): over the time window evaluated at height $x_3 = 100$ m. (a,b): Streamwise velocity component; (c,d): spanwise velocity component; (e,f): vertical velocity component. (△): PPI scanning mode; (○): Lissajous scanning mode; (□): background variance.

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