Improved Coulomb Correction Formulae 
for Bose-Einstein Correlations

M. Biyajima\textsuperscript{1}, T. Mizoguchi\textsuperscript{2}, T. Osada\textsuperscript{3} and G. Wilk\textsuperscript{4}\textsuperscript{†}

\textsuperscript{1}Department of Physics, Faculty of Liberal Arts, Shinshu University, Matsumoto 390, Japan 
\textsuperscript{2}Toba National College of Maritime Technology, Toba 517, Japan 
\textsuperscript{3}Department of Physics, Tohoku University, Sendai 980, Japan 
\textsuperscript{4}Soltan Institute for Nuclear Studies, Zd-PVIII, Hoża 69, PL-00-681 Warsaw, Poland

Abstract

We present improved Coulomb correction formulae for Bose-Einstein correlations including also exchange term and use them to calculate appropriate correction factors for several source functions. It is found that Coulomb correction to the exchange function in the Bose-Einstein correlations cannot be neglected.

\textit{SULDP-1995-3} \textit{TU477} \textit{SINS-1995-1} Introduction: Recently many experimental groups have investigated Bose-Einstein correlations (BEC) in high energy hadronic collisions \cite{1}, $e^+e^-$ annihilations \cite{2} and heavy-ion collisions \cite{3,4}. The high quality of data obtained already (and expected in near future, especially for heavy-ion collisions) make the analysis of experimental results sensitive to all possible corrections, especially to those due to final state interactions because of the Coulomb interactions and the strong interactions \cite{5}. In the present letter we shall concentrate on the Coulomb corrections only.

Several authors have calculated Coulomb corrections to the BEC by their own methods \cite{6,7,8,9,10} but this problem is still controversial \cite{10,11}. In this letter we shall concentrate on the formula provided by Bowler \cite{10}, because it includes all orders of the parameter $\eta$ (defined below). He demonstrated that widely used zero range Gamow factor substantially overestimates Coulomb corrections (and hence also the true magnitude of BEC). However, Bowler did not calculate the

\textsuperscript{*}e-mail: minoru44@jpnyitp.bitnet
\textsuperscript{†}e-mail: wilk@fuw.edu.pl
Coulomb correction to the exchange function present in BEC formulae. We investigate therefore this problem and in particular: (1) we calculate the Coulomb corrections using the exact formula (to all orders in parameter \( \eta \)) and (2) we use its approximate form (retaining only terms linear in \( \eta \)) to calculate explicit Coulomb wave function in this approximation \([7, 12, 13]\) and with its help we provide formulae for Coulomb corrections for the exchange function in the BEC for several source functions and compare some of them with those presented in ref.\([10]\). We demonstrate that the presence of exchange term diminishes correction factor even more than anticipated in \([10]\) and therefore it should be included in analysis of experimental data.

**Theoretical calculation of BEC with Coulomb wave function:** To write down an amplitude \( A_{12} \) satisfying Bose-Einstein statistics it is convenient to decompose the wave function of identical (charged in our case) bosons with momenta \( p_1 \) and \( p_2 \) into the wave function of the center-of-mass system (c.m.) with total momentum \( P = \frac{1}{2}(p_1 + p_2) \) and the inner wave function with relative momentum \( Q = (p_1 - p_2) = 2q \). It allows us to express \( A_{12} \) in terms of the confluent hypergeometric function \( \Phi \) \([14]\):

\[
A_{12} = \frac{1}{\sqrt{2}} \left[ \Psi(q, r) + \Psi_S(q, r) \right],
\]

\[
\Psi(q, r) = \Gamma(1 + i\eta)e^{-\pi\eta/2}e^{iqr \cdot \mathbf{r}}\Phi(-i\eta; 1; iqr(1 - \cos \theta)),
\]

\[
\Psi_S(q, r) = \Gamma(1 + i\eta)e^{-\pi\eta/2}e^{-iqr \cdot \mathbf{r}}\Phi(-i\eta; 1; iqr(1 + \cos \theta)),
\]

where \( r = x_1 - x_2 \) and the parameter \( \eta = m\alpha/2q \). Using now the Kummer’s first formula for the confluent hypergeometric functions that appears in the cross term of \( |A_{12}| \),

\[
\Phi(\alpha; \gamma; z) = e^z\Phi(\gamma - \alpha; \gamma; -z),
\]

we calculate first the exact formula of Coulomb correction (i.e., the one that is exact to all orders of parameter \( \eta \)) including also the exchange function in BEC. Namely, assuming factorization in the source functions, \( \rho(r_1)\rho(r_2) = \rho(R)\rho(r) \) (here \( R = \frac{1}{2}(x_1 + x_2) \)), one obtains the following expression for theoretical BEC formula:

\[
N^{\pm\pm}/N^{BG} = \frac{1}{G(q)} \int \rho(R)d^3R \int \rho(r)d^3r |A_{12}|^2
\]

\[
= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{(-i)^n(i)^m}{n + m + 1} (2q)^{n+m} I_R(n, m) A_n A_m^* \times \left[ 1 + \frac{n!m!}{(n+m)!} \left( 1 + \frac{n}{i\eta} \right) \left( 1 - \frac{m}{i\eta} \right) \right],
\]

where \( G(q) = 2\pi\eta/(e^{2\pi\eta} - 1) \) is the Gamow factor and

\[
I_R(n, m) = 4\pi \int dr r^{2+n+m}\rho(r), \quad A_n = \frac{\Gamma(i\eta + n)}{\Gamma(i\eta)} \frac{1}{(n!)^2}.
\]
The second term in the squared parenthesis in eq.(2) is a new term which is due to the exchange function in the BEC. The above expression represents our main result. However, it has disadvantage that it is difficult to separate in it the exchange function and its Coulomb correction. Therefore, in order to know the separate contributions we must either subtract from it the exchange function or to use the approximate formula for the Coulomb correction. To calculate it we expand function $\Phi$ in powers of $\eta$ and retaining only linear terms we get

$$
\Phi(-i\eta; 1; ix) = 1 + \eta \text{Si}(x) - i\eta(Ci(x) - \gamma_E - \ln(x)) + O(\eta^2),
$$

$$
\Phi(-i\eta; 1; i\tilde{x}) = 1 + \eta \text{Si}($$

$$\tilde{x}) - i\eta(Ci($$

$$\tilde{x}) - \gamma_E - \ln(\tilde{x})) + O(\eta^2)
$$

where $x = iqr(1 - \cos \theta)$, $\tilde{x} = iqr(1 - \cos \theta)$, $\gamma_E$ is the Euler’s number and $\text{Si}(x)$ and $\text{Ci}(x) - \gamma_E - \ln(x)$ are the sine and cosine integral, respectively, defined as:

$$
\text{Si}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!(2n+1)},
$$

$$
\text{Ci}(x) - \gamma_E - \ln(x) = \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!(2n)}.
$$

With such approximated $\Phi$’s we obtain the following approximate formula for BEC with Coulomb corrections included:

$$
N^{(\pm)}_{(x)/N^{BG}} = \frac{1}{G(q)} \int \rho(R)d^3R \int \rho(r)d^3r|A_{12}|^2
$$

$$
= I_1(q) + I_2(q)
$$

$$
= 1 + \delta_{1C} + \delta_{EC} + E_{2B}
$$

$$
= (1 + \delta_{1C} + \delta_{EC})\left(1 + \frac{E_{2B}}{1 + \delta_{1C} + \delta_{EC}}\right)
$$

where $(A = 2qr)$

$$
I_1(q) = 4\pi \int \rho(r)r^2dr \left\{1 + \eta \sum_{n=0}^{\infty} \frac{(-1)^n (qr)^{2n+1}}{(2n+1)!(2n+1)} \int_{-1}^{1} (1 - \cos \theta)^{2n+1} d\cos \theta\right\}
$$

$$
= 1 + 4\pi \cdot 2\eta \int \rho(r)r^2dr \sum_{n=0}^{\infty} \frac{(-1)^n A^{2n+1}}{(2n+1)!(2n+1)(2n+2)}
$$

$$
= 1 + \delta_{1C},
$$

$$
I_2(q) = 4\pi \int \rho(r)r^2dr \left\{\sin \frac{A}{A} + \eta[\text{Sp}(qr) + \text{Cp}(qr)]\right\}
$$

$$
= E_{2B} + \delta_{EC},
$$

and

$$
\text{Sp}(qr) = \sum_{n=0}^{\infty} \frac{(-1)^{2n+1}(qr)^{2n+1}\Theta(2n+1)}{(2n+1)!(2n+1)},
$$

$$
\text{Cp}(qr) = \sum_{n=1}^{\infty} \frac{(-1)^{2n}(qr)^{2n}\Theta(2n)}{(2n)!(2n)}.
$$

(7)
with

$$\Theta(2n+1) = \int (1 - \cos \theta)^{2n+1} \cos(A \cos \theta) \, d \cos \theta,$$

$$\Theta(2n) = \mp \int (1 + \cos \theta)^{2n} \sin(A \cos \theta) \, d \cos \theta.$$  \hspace{1cm} (8)

(Notice that $\Theta(1) = \frac{2 \sin A}{A}$, $\Theta(3) = \frac{2^2 \cos A}{A^2} + \frac{12 \sin A}{A^3} \ldots$ and $\Theta(2) = \frac{2^2 \cos A}{A^2} - \frac{2^2 \sin A}{A^3} \ldots$).

The normalization-like factor $(1 + \delta_{1C} + \delta_{EC})$ in eq.(4) differs from that in ref. [10] only by the additional exchange term $\delta_{EC}$. There is therefore the following (approximate) relation between our correction factor, $C_{Ours}$, and that of Bowler [10], $C_{Bowler}$:

$$C_{Ours} = C_{Bowler} + \delta_{EC} \cdot G(q) \hspace{1cm} (9)$$

where $G(q)$ is Gamow factor.

**Coulomb corrections and source functions:** In order to provide numerical estimations of our new correction factor we calculate analytical expressions for necessary ingredients of both exact result as given by eq.(2), $I_R$, and the approximate one provided by eqs.(4) and (9), $\delta_{1C}$, $E_{2B}$ and $\delta_{EC}$ for several different source functions (superscripts below denote the type of source considered):

**S1) Exponential source function,** $\rho(r) = \frac{\beta^3}{\sqrt{\pi}} \exp(-\beta r)$:

Taking into account only leading terms in $\Theta(2n+1)$ and $\Theta(2n)$ in eqs. (8) one has:

$$I_R^E(n, m) = \left( \frac{1}{\beta} \right)^{n+m} \frac{(n+m+2)!}{2}, \hspace{1cm} (10)$$

$$\delta_{EC}^E = \eta \sum_{n=0}^{\infty} \frac{(-1)^n(2n+3)}{2n+1} \frac{(2q/\beta)^{2n+1}}{(2q/\beta)^{2n+1}}, \hspace{1cm} (11)$$

$$E_{2B}^E = \frac{1}{(1 + (2q/\beta)^2)^2}, \hspace{1cm} (12)$$

$$\delta_{EC}^E = \frac{\eta}{2} \sum_{n=1}^{\infty} \frac{(-1)^n(2n+1)}{2n} \frac{(2q/\beta)^{2n-1}}{(1 + (2q/\beta)^2)^{n+1}} \cos((2n+2) \arctan(2q/\beta)) \hspace{1cm} (13)$$

Our results for this source function are given in Fig. 1. Notice that there is significant systematic difference between $C_{Bowler}$ and $C_{Ours}$ correction terms whereas $C_{Ours}$ is practically the same whenever calculated using exact formula (2) or approximate one (4) (in the range of $q$ considered the differences are of the order of 1%).

**S2) Gaussian source distribution,** $\rho(r) = \frac{\beta^3}{\sqrt{\pi}} \exp(-\beta^2 r^2)$:

For this type of source function we have:

$$I_R^G(n, m) = \frac{2}{\sqrt{\pi}} \left( \frac{1}{\beta} \right)^{n+m} \frac{(n+m+1)}{2} \Gamma \left( \frac{n+m+1}{2} \right), \hspace{1cm} (14)$$
\[ \delta_{IC}^G = \frac{4\eta}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n(n+1)!}{(2n+2)!(2n+1)} \left( \frac{2q}{\beta} \right)^{2n+1}, \]

\[ E_{2B}^{G} = \exp \left( -\frac{q^2}{\beta^2} \right), \]

\[ \delta_{EC}^G = \frac{2\eta}{\sqrt{\pi}} \sum_{n=1}^{\infty} \frac{(-1)^n n!}{(2n)!2n} \left( \frac{2q}{\beta} \right)^{2n-1} \Phi \left( n+1, \frac{1}{2}, -\frac{q^2}{\beta^2} \right) \]

\[ + \frac{2\eta}{\sqrt{\pi}} \exp \left( \frac{q^2}{\beta^2} \right) \sum_{n=0}^{\infty} \frac{(-1)^n(n+1)!}{(2n+1)!(2n+1)} \left( \frac{2q}{\beta} \right)^{2n+1} \Phi \left( n+\frac{3}{2}, \frac{3}{2}, -\frac{q^2}{\beta^2} \right). \]

In Fig. 2 we show typical examples of our correction term for Gaussian source distribution. Also here exact and approximate formulas lead practically to the same results at lower values of \( q \) (compatible with those in Fig. 1) but differ substantially for larger \( q \)'s. The difference between our results and the result obtained using Bowler’s formula (cf. eq. (27) below) exists also here.

(S3) Modified Bessel source functions, \( \rho(r) = \frac{e^3}{2\pi} K_0(\beta r) \) and \( \rho(r) = \frac{e^4}{2\pi} r K_1(\beta r) \):

In the case of source functions described by the Modified Bessel functions \( K_0 \) or \( K_1 \) [13, 14] we have the following expressions:

\[ I_R^{K_0}(n,m) = \frac{2^{n+m+2}}{\pi} \left( \frac{1}{\beta} \right)^{n+m} \Gamma \left( \frac{n+m+3}{2} \right) \Gamma \left( \frac{n+m+3}{2} \right), \]

\[ \delta_{1IC}^{K_0} = \frac{8n}{\pi} \sum_{r=0}^{\infty} \frac{(-1)^n((n+1)!)^2}{(2n+2)!(2n+1)} \left( \frac{4q}{\beta} \right)^{2n+1}, \]

\[ E_{2B}^{K_0} = \frac{1}{(1+(2q/\beta)^2)^{3/2}}, \]

\[ \delta_{EC}^{K_0} = \frac{4\eta}{\sqrt{\pi}} \sum_{n=1}^{\infty} \frac{(-1)^n((n+1)!)^2}{(2n)!2n} \left( \frac{4q}{\beta} \right)^{2n-1} \sum_{m=0}^{\infty} \frac{(-1)^m((n+m)!)^2}{\Gamma(m+\frac{3}{2})m!} \left( \frac{2q}{\beta} \right)^{2m} \]

\[ + \frac{2\eta}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n((n+1)!)^2}{(2n+1)!(2n+1)} \left( \frac{4q}{\beta} \right)^{2n+1} \sum_{m=0}^{\infty} \frac{(-1)^m((n+m+1)!)^2}{\Gamma(m+\frac{5}{2})m!} \left( \frac{2q}{\beta} \right)^{2m}. \]

\[ I_R^{K_1}(n,m) = \frac{2^{n+m+3}}{3\pi} \left( \frac{1}{\beta} \right)^{n+m} \Gamma \left( \frac{n+m+5}{2} \right) \Gamma \left( \frac{n+m+5}{2} \right), \]

\[ \delta_{1IC}^{K_1} = \frac{16n}{3\pi} \sum_{r=0}^{\infty} \frac{(-1)^n((n+1)!)^2}{(2n+2)!(2n+1)} \left( \frac{4q}{\beta} \right)^{2n+1}, \]

\[ E_{2B}^{K_1} = \frac{1}{(1+(2q/\beta)^2)^{3/2}}, \]

\[ \delta_{EC}^{K_1} = \frac{8\eta}{3\sqrt{\pi}} \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n)!2n} \left( \frac{4q}{\beta} \right)^{2n-1} \sum_{m=0}^{\infty} \frac{(-1)^m((n+m)!)^2}{\Gamma(m+\frac{3}{2})m!} \left( \frac{2q}{\beta} \right)^{2m} \]

\[ + \frac{4\eta}{3\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!(2n+1)} \left( \frac{4q}{\beta} \right)^{2n+1} \sum_{m=0}^{\infty} \frac{(-1)^m((n+m+1)!)^2}{\Gamma(m+\frac{5}{2})m!} \left( \frac{2q}{\beta} \right)^{2m}. \]

The correction factors for these source functions are presented in Fig. 3 which closely resembles Fig. 1a (again, approximate formula practically does not differ from the exact one whereas both
differ substantially from that calculated according to [10].

**Concluding remarks:** We have calculated the exact and approximate formulae for Coulomb corrections used in the BEC (including exchange term) and specified them to the Exponential, Gaussian and Modified Bessel source functions. The difference between exact and approximate formulas (eq. (3) and eq.(4), respectively) turns out to be smaller than 1% (at least for \( q \leq 0.2 \) GeV/c). However, due to the presence of the new correction for the exchange term (\( \delta_{EC} \) above) our results are systematically lower than those derived in [10]. To visualize it better we check if relation

\[
\frac{E_{2B}\delta_{1C}}{\delta_{EC}} = 1
\]  

holds; \( E_{2B} \) and \( \delta_{1C} \) above are quantities used in ref. [10]:

\[
N^{\pm \pm}/N^{BG} = (1 + \delta_{1C})(1 + E_{2B}). \tag{27}
\]

It is clear from Fig. 4 that (26) does not hold (here for the standard value of \( \beta = 0.2 \) GeV used also in [10] but we have checked that the same is true for the whole possible range of this parameter). We conclude that correction term \( \delta_{EC} \) in eqs.(2) and (3) cannot be neglected and that in calculations of Coulomb corrections eq.(27) should be replaced by eq.(1) (or eq.(3)).

In ref.[10] the relative changes of the normalised Fourier transforms of a given source function, \( \frac{\Delta \tilde{\rho}}{\tilde{\rho}} \), were also introduced and investigated as yet another estimate of the importance of the Coulomb corrections:

\[
\frac{\Delta \tilde{\rho}}{\tilde{\rho}} = \frac{(1 + \delta_{1C})(1 + E_{2B}) - 1}{(1 + E_{2B}) - 1} - 1, \tag{28}
\]

for the ideal BEC and

\[
\frac{\Delta \tilde{\rho}^R}{\tilde{\rho}} = \frac{x(1 + \delta_{1C})(1 + E_{2B}) + G^{-1}(1 - x) - 1}{x[1 + E_{2B}] - 1} - 1 \tag{29}
\]

for BEC containing contributions of the long lived resonances \( \{L\} = \{\eta, \omega, \eta'; c, b\} \) with \( 1 - x \) denoting the fraction of pairs involving a daughter of \( \{L\} \). In Figs. 5 and 6, using the exact formulae containing also the correction term \( \delta_{EC} \), we compare our results for \( \frac{\Delta \tilde{\rho}}{\tilde{\rho}} \) and \( \frac{\Delta \tilde{\rho}^R}{\tilde{\rho}} \) calculated for Exponential and Gaussian sources for different \( \beta \)'s (in Fig. 5) and different \( x \)'s (in Fig. 6). Contrary to the previous cases, for these quantities the introduction of Coulomb correction to the exchange term in BEC practically does not change the results obtained by using eq.(27) from [10], due to the large denominator.

**Acknowledgements:** The authors would like to thank S. Esumi, S. Nishimura, and S. D.
Pandey for useful correspondences. This work is partially supported by Japanese Grant-in-Aid for Scientific Research from the Ministry of Education, Science and Culture (#. 06640383).

References

[1] M. Agababyan et al. (EHS/NA22 Collaboration), Z. Phys. C59 (1993) 195.
[2] P. Abreu et al. (DELPHI Collaboration), Phys. Lett. 323 (1994) 242.
[3] H. Bøggild et al. (NA44 Collaboration), Phys. Lett. B302 (1993) 510.
[4] H. Beker et al. (NA44 Collaboration), Z. Phys. C64 (1994) 209.
[5] M. Biyajima, T. Mizoguchi, and G. Wilk, Z. Phys. C65 (1995) 511.
[6] M. Gyulassy, S. K. Kaufmann and L. W. Wilson, Phys. Rev. C20 (1979) 2267.
[7] S. Pratt, Phys. Rev. D33 (1986) 72.
[8] H.-U. Gersch, Z. Phys. 327 (1987) 115.
[9] D. H. Boal, C. -K. Gelbke and B. K. Jennings, Rev. Mod. Phys. 62 (1990) 553.
[10] M. G. Bowler, Phys. Lett. B270 (1991) 69.
[11] D. Anchishkin and G. Zinojev, Two-Pion Correlation Behaviour in Small Relative Momentum Region, BI-TP 94/19 (1994).
[12] M. Biyajima and T. Mizoguchi, talk given at the workshop on "Evolution from quarks to hadrons" at Yukawa Institute for Theoretical Physics. (Oct. 1994).
[13] M. Biyajima and T. Mizoguchi, preprint SULDP-94-9 (Dec.,1994). Several improper expressions in ref. [6] are pointed out and corrected therein.
[14] L. I. Schiff, Quantum Mechanics, 2nd Ed. (McGraw-Hill, New York, 1955) p.117.
[15] R. Shimoda, M. Biyajima, and N. Suzuki, Prog. Theor. Phys. 89 (1993) 697.
[16] T. Mizoguchi, M. Biyajima and T. Kageya, Prog. Theor. Phys. 91 (1994) 905.
Figure Captions

Fig. 1. Comparison of $C^{Bowler}$ with $C^{Ours}$ for exponential source function and for different choices of parameter $\beta$: (a) $\beta = 0.2$ GeV. (b) $\beta = 0.1$ GeV.

Fig. 2. The same as in Fig. 1 but for Gaussian source distribution.

Fig. 3. The same as in Fig. 1 but for Modified Bessel source functions ($K_0$ for (a) and $K_1$ for (b)) - only for $\beta = 0.2$ GeV in both cases.

Fig. 4. Examinations of eq. (26) for different choices of source functions.

Fig. 5. $\Delta \tilde{\rho}/\tilde{\rho}$ (cf. eq.(28)) for different $\beta$'s and for Exponential and Gaussian source distributions. In our calculations the exact formulae are used.

Fig. 6. $\Delta \tilde{\rho}^R/\tilde{\rho}$ (cf. eq.(29)) for Exponential and Gaussian source functions and for different values of the fraction of long lived resonances parameter $x$. In our calculations the exact formulae are used.