The Location of the First Ascent in a 123-Avoiding Permutation

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Abstract

It is natural to ask, given a permutation with no three-term ascending subsequence, at what index the first ascent occurs. We shall show, using both a recursion and a bijection, that the number of 123-avoiding permutations at which the first ascent occurs at positions $k, k+1$ is given by the $k$-fold Catalan convolution $C_{n,k}$ [1], [8], [9]. For $1 \leq k \leq n$, $C_{n,k}$ is also seen to enumerate the number of 123-avoiding permutations with $n$ being in the $k$th position. Two interesting discrete probability distributions, related obliquely to the Poisson and geometric random variables, are derived as a result.

1 Introduction

For $n \geq 0$, the Catalan numbers $C_n$ are given by

$$C_n = \frac{1}{n+1} \binom{2n}{n};$$

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generalizing this fact, Catalan \cite{3} proved the $k$-fold Catalan convolution formula

\[ C_{n,k} := \sum_{i_1 + \ldots + i_k = n} \prod_{r=1}^{k} C_{i_r - 1} = \frac{k}{2n-k} \binom{2n-k}{n}. \]

The theory of pattern avoidance in permutations is now well-established and thriving, and a survey of the many results in that area may be found in the text by Kitaev \cite{5}. One of the earliest and most fundamental results in the field is that the number of permutations of $[n] := \{1, 2, \ldots, n\}$ in which the longest increasing sequence is of length $\leq 2$, the so-called 123-avoiding permutations, is given by the Catalan numbers, and classical bijective techniques give that the each of the $ijk$-avoiding permutations with $\{i, j, k\} = \{1, 2, 3\}$ are equinumerous. In this paper, we ask a very natural question, namely in how many permutations in which the longest increasing subsequence is of length at most 2, does the first ascent occur in positions $k, k+1$. Actually, we were web-searching for the answer to this question to use in a different context, and were rather surprised to find that the solution appeared to be not known explicitly: Bousquet-Mélou \cite{2} addressed this question indirectly when she used the location of the first ascent as a “catalytical variable” in the “laziest proof, combinatorially speaking,” of the fact that there are $C_n$ 123-avoiding permutations. In Sections 2 and 3 of this paper, we will give recursive and bijective proofs respectively of the fact that there are $C_{n,k}$ 123-avoiding permutations on $[n]$ for which the first ascent occurs at positions $k, k+1$. Critical to the bijective proof are the various combinatorial interpretations of the Catalan convolutions due to Tedford \cite{9}, and the bijections between Dyck paths and avoiding permutations due to Krattenthaler \cite{7}. In Section 2, additionally, we show that $C_{n,k}$ also enumerates 123-avoiding permutations with the position of “$n$” being $k \in [1, n]$. Finally, in Section 4, we produce two interesting probability distributions on $\mathbb{Z}^+$ related to these issues. These are reminiscent of the geometric and Poisson random variables, and are studied systematically in \cite{4}.

## 2 Recursive Proof

Throughout, we refer to the first ascent as being in position $k$ if the ascent is at the $k$th and $k+1$st positions of the permutation. If a permutation of $[n]$
letters has no ascents at all (i.e., it is the permutation \{n, n-1, \ldots, 2, 1\}), we define it as having first ascent in position \(n\), as though it had a first ascent “after” the last letter in the permutation – perhaps using a number such as 1.5 in the \((n+1)\)st spot.

Any 123-avoiding permutation of \([n]\) becomes a 123-avoiding permutation of \([n-1]\) when the letter \(n\) is removed from it, so we can view each 123-avoiding permutation of \([n]\) as being grown uniquely by taking a 123-avoiding permutation of \([n-1]\) and inserting the letter \(n\) into certain positions.

Suppose we have a 123-avoiding permutation of \([n-1]\) with first ascent in position \(k\). How can this be grown into a longer 123-avoiding permutation by inserting \(n\)? If \(n\) is inserted at the very front of this permutation, the resulting permutation is still 123-avoiding with the original first ascent being “pushed forward” one position due to the presence of \(n\). If \(n\) is inserted anywhere between the very front of this permutation and the peak of the original first ascent, i.e., as anything from the 2nd letter to the \(k+1\)st letter of the permutation, then \(n\) becomes the peak of the new first ascent, which therefore has position 1 less than the position of \(n\). Furthermore, as there are no ascents before \(n\), and as \(n\) cannot be involved in any subsequent ascents, the new permutation is still 123-avoiding. Finally, if \(n\) is inserted after the original first ascent, then the resulting permutation is no longer 123-avoiding.

Note that all of the above hold exactly even if the original permutation is the descending permutation \{\(n-1, n-2, \ldots, 2, 1\}\} with first ascent in position \(k = n - 1\). To summarize, a 123-avoiding permutation of length \(n - 1\) with first ascent in position \(k\) gives rise to exactly one 123-avoiding permutation of length \(n\) with first ascent in position \(i\) for each \(1 \leq i \leq k + 1\), and each 123-avoiding permutation of length \(n\) must be grown in such a way.

Turning this around, the number \(A_{n,k}\) of 123-avoiding permutations of length \(n\) with first ascent in position \(k\) is equal to the number of 123-avoiding permutations of length \(n - 1\) with first ascent in position \(k - 1\) or later, as it is these permutations that give rise to them. Thus

\[
A_{n,k} = \sum_{i=k-1}^{n} A_{n-1,i}
\]

In particular, we note that
\[ A_{n,k} = \sum_{i=k-1}^{n} A_{n-1,i} = A_{n-1,k-1} + \sum_{i=k}^{n} A_{n-1,i} = A_{n-1,k-1} + A_{n,k+1}. \]

Since \( A_{n,0} = 0 \) for all \( n \), the above recursion indicates that \( A_{n,1} = A_{n,2} \) for all \( n \), with both being equal to the total number of 123-avoiding permutations of length \( n - 1 \). We can see that this is true: any such \( n - 1 \) permutation can be uniquely grown into a 123-avoiding permutation with first ascent in position 2 by inserting \( n \) either as the third letter (if the original permutation did not have first ascent in position 1) or as the first letter (otherwise); or it can be uniquely grown into a 123-avoiding permutation with first ascent in position 1 by inserting \( n \) as the second letter. Combined with the base cases \( A_{n,0} = 0 \) and \( A_{n,n} = 1 \) for all \( n \geq 1 \), this recurrence relation is sufficient to fully characterize \( A_{n,k} \) for any \( 1 \leq k \leq n \).

Note that \( C_{n,0} = 0 \) and \( C_{n,n} = 1 \) for all \( n \geq 1 \). Moreover, we find that the Catalan convolutions \( C_{n,k} \) obey the same recurrence relation as above:

\[
C_{n-1,k-1} + C_{n,k+1} = \frac{k-1}{2n-k-1} \binom{2n-k-1}{n-1} + \frac{k+1}{2n-k-1} \binom{2n-k-1}{n} \\
= \frac{k-1}{2n-k-1} \left( \frac{n}{2n-k} \right) \binom{2n-k}{n} + \frac{k+1}{2n-k-1} \left( \frac{n}{2n-k} \right) \binom{2n-k}{n} \\
= \frac{k}{2n-k} \binom{2n-k}{n} = C_{n,k}.
\]

Since \( C_{n,k} \) obey the same recurrence relation as the \( A_{n,k} \), and they have the same base cases (which generate their values for all \( 1 \leq k \leq n \)), we find that \( C_{n,k} = A_{n,k} \) everywhere.

**Corollary 2.1.** The number of 123-avoiding permutations where \( n \) is in the \( k \)th spot are also given by the Catalan convolutions \( C_{n,k} \).

**Proof.** The result is obvious for \( k = 1 \) where the answer equals \( C_{n-1} = C_{n,1} \). Let \( k = 2 \). We have that \( n \) is in position 2 in a 123-avoiding permutation
iff the first ascent is at positions (1, 2), necessarily to \( n \). Thus again there are \( C_{n,1} = C_{n-1} \) such possibilities. For \( k \geq 3 \) let \( \alpha_{n,k} \) be the number of 123-avoiding permutations on \([n]\) with \( n \) in the \( k \)th spot. Since, as will be emphasized in Section 3, for the first ascent to be at spots \((k-1, k)\), \( n \) must either be in position 1, or the first ascent must be to \( n \), we have that

\[
\alpha_{n,k} = C_{n,k-1} - C_{n-1,k-2} = C_{n,k},
\]

by the above recursion.

To give an alternate bijective proof for \( k \geq 2 \), we proceed as follows. Consider a 123-avoiding permutation \( \pi \) with first ascent at spots \((k, k+1)\), and move \( n \), originally in position 1 or \( k+1 \), into position \( k \), while keeping the relative order of the other numbers unchanged. Regardless of whether \( n \) was in position 1 or position \((k+1)\), the new permutation has first ascent at position \( k-1 \) and is still 123-free. Since just one of these original configurations is valid for a given relative ordering of \([n-1]\), we see that this map \( \varphi \) from the set of 123-avoiders with first ascent at \((k, k+1)\) to the set of 123-avoiding permutations with \( n \) in position \( k \) is one-to-one. Moreover the map has an inverse: If \( \varphi(\pi)(k-1) < \varphi(\pi)(k+1) \), \( n \) must have been at the beginning of \( \pi \), and \( n \) must have been in position \( k+1 \) if we find that \( \varphi(\pi)(k-1) > \varphi(\pi)(k+1) \).

\[\square\]

3 Bijective Proof

We have, from Tedford [9] that \( C_{n,k} \) is given (adjusting for his different indexing) by the number of lattice paths from \((k-1, 0)\) to \((n-1, n-1)\) consisting of steps of \((0, 1)\) and \((1,0)\) and never crossing above the line \( x = y \). Note that these paths, and hence, \( C_{n,k} \) are in bijection with these same types of paths between \((k, 1)\) and \((n, n)\). We will show that these paths are in bijection with the paths corresponding, by the Krattenthaler bijection, to 123-avoiding \( n \)-permutations with first ascent at \( k \).

Krattenthaler’s bijection between 123-avoiding permutations and Dyck paths can be described as follows [7]: Given a permutation \( \pi \) of \( n \) integers, denote the right to left maxima (RLM) of \( \pi \), reading from left to right, by \( \{m_s, m_{s-1}, \ldots, m_2, m_1\} \) and denote the (necessarily descending) word between \( m_{i+1} \) and \( m_i \) by \( w_i \). \( \pi \) will now read left to right as \( w_sm_s \ldots w_2m_2w_1m_1 \).
Read $\pi$ from left to right, and draw as follows, beginning at $(0,0)$: upon encountering $w_i$ add $|w_i| + 1$ steps in the $x$ direction, where $|w_i|$ is the length of $w_i$. Upon encountering $m_i$, add $m_i - m_{i-1}$ steps in the $y$ direction ($m_0$ is taken to be 0 by convention). This will give a lattice path between $(0,0)$ and $(n,n)$ consisting of steps of $(1,0)$ and $(0,1)$ and never crossing above the line $x = y$.

**Example 3.1.** The permutation 76584213 corresponds with the path encoded by XXXXYYYYXYXYYYYY where X, Y represent steps to the east (E) and north (N) respectively.

**Lemma 3.2.** If $\pi$ is a 123-avoiding $n$-permutation with first ascent at $k$, then $\mu$, the leftmost right to left maximum preceded by a non-empty word, is at position $k + 1$. Also, $\mu$ is either $n$, or $\mu$ is one less than the nearest right to left maximum to its left.

**Proof.** The first ascent of $\pi$ must be to a RLM, as follows: If the first ascent is either at positions $n-1$ or $n$, we easily or vacuously have the next symbol being a RLM. If the first ascent is at position $k \in [1,n-2]$ then the $k+1$st symbol must be a RLM, since otherwise the next RLM to the right would enable the formation of a 123. If the permutation does not begin with $n$, then $n$ must be at the top of the first ascent. This is because if $n$ is not the first number, it must be the top of an ascent, but if an ascent precedes that with the $n$, that ascent, along with $n$, would form a 123. This is the case in which $\mu = n$. If $n$ is the first term in $\pi$, then $\pi$ begins with the integers $n(n-1)...(n-a)$ for some $0 \leq a \leq k-2$ (we choose the maximum such $a$, and from now on we will say that “$\pi$ begins with a regular descent from $n$ of length $a + 1$”). The upper bound on $a$ comes from the fact that if $\pi$ began with regular descent from $n$ of length $k$ (i.e., if we had $a = k - 1$) then all integers greater than the one appearing at index $k$ have already appeared and so the first ascent could not be at $k$. After the end of the regular descent from $n$, we can think of the rest of the permutation as a 123-avoiding $(n-a-1)$-permutation, and so, by the same reasoning as above, along with the fact that $n-a-1$ cannot be the first term, or else it would lengthen the regular descent from $n$, $n-a-1$ must be at the top of the first ascent, and must be preceded by a non-empty word. 

**Theorem 3.3.** The Dyck paths that correspond, by the Krattenthaler bijection, with 123-avoiding $n$-perms with first ascent at $k$ are in bijection with the lattice paths given by Tedford [9], as counted by $C_{n,k}$. 

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Proof. Let $\pi$ be a 123-avoiding $n$-permutation with first ascent at $k$. By Lemma 3.2, the following two cases are exhaustive: either (i) the first ascent is to $n$, and so $|w_s| = k$ meaning that the path begins with $k + 1$ $x$ steps for a word of length $k$, followed by a $y$ step for an RLM, or (ii) $\pi$ begins with a regular descent from $n$ of length $j$ where $1 \leq j \leq k - 1$ and subsequently contains a word of length $k - j$, and then a RLM with value less than the last term in the regular descent from $n$. In the latter case, the path will begin with $j$ iterations of the pattern ($x$ step, $y$ step) each representing an empty word followed by a RLM one greater than the following RLM, and then will have $k - j + 1$ $x$ steps, corresponding to the word of length $k - j$, and then a $y$ step corresponding to a RLM. In the first case, we have a specific path from $(0, 0)$ to $(k + 1, 1)$. In the second case, we have one specific path for each $1 \leq j \leq k - 1$ from $(0, 0)$ to $(k + 1, j + 1)$. This is to say that every Dyck path that gets to one of these points via the path associated with it represents a 123-avoiding permutation with first ascent at $k$, and vice versa. Therefore the number of 123-avoiding $n$-permutations with first ascent at $k$ is given by the number of unique ways to finish a Dyck path from each of these endpoints, i.e., denoting by “good” paths the ones that do not cross the line $x = y$,

$$C_{n,k} = \sum_{i=1}^{k} |\text{good lattice paths with } E/N \text{ steps from } (k + 1, i) \text{ to } (n, n)|.$$

A bijection between these paths and the paths from $(k, 1)$ to $(n, n)$ is given by taking a path from $(k, 1)$ to $(n, n)$, and disregarding every step up through the first $x$ step, so that a path from $(k + 1, i); 1 \leq i \leq k$ to $(n, n)$ remains. □

4 Limit Distributions

The probability that a random permutation on $[n]$ has its first ascent at position $k$ is given, for $1 \leq k \leq n - 1$, by $\frac{k}{(k+1)!}$. To see this, choose any one of the $k + 1$ elements in positions 1 through $k + 1$, except for the smallest, to occupy the $k + 1$st position, and then arrange the other elements in a monotone decreasing fashion. The chance that the first ascent is at position $n$ is, of course, $\frac{1}{n!}$. We will find it more convenient in this section to consider infinite analogs of the finite distributions we derive. An infinite permutation
may be realized, e.g., by considering the order statistics $X_1 < X_2 < \ldots$ of a sequence $X_1, X_2, \ldots$ of independent and identically distributed (i.i.d.) random variables with say a uniform distribution on $[0,1]$. Under this scheme we get the first ascent distribution as being

$$f(x) = \frac{x}{(x+1)!}, \ x = 1, 2, \ldots,$$

which is similar in form to the unit Poisson distribution with parameter $\lambda = 1$ – and mass function $g(y) = e^{-1/y!}; y = 0, 1, \ldots$, mean and variance equal to 1, and generating function $\mathbb{E}(s^X) = \exp\{s - 1\}$. By contrast, it is shown in [4] that the first ascent distribution above satisfies

$$\mathbb{E}(X) = e - 1; \mathbb{V}(X) = e(3 - e); \mathbb{E}(s^X) = \frac{(1 - e^s + se^s)}{s}.$$

What, on the other hand, can be said about the location distribution of the first ascent in a random 123-avoiding permutation? We see from our earlier results that for a randomly chosen 123-avoiding permutation on $[n]$ the distribution of the location of first ascent is given by

$$f(k) = \frac{C_{n,k}}{C_n} = k \frac{(2n - k - 1)!(n + 1)!}{(2n)!(n - k)!}, \ k = 1, 2, \ldots, n,$$

which, for small $k$ and large $n$, may be approximated by $f(k) = \frac{k}{2k+1}$. Accordingly, in [4] the authors studied the geometric-like distribution on $\mathbb{Z}^+ = 1, 2, \ldots$ defined by

$$f(w) = \frac{w}{2w+1}, w = 1, 2, \ldots,$$

showing that

$$\mathbb{E}(W) = 3; \mathbb{E}(s^W) = \frac{s}{s^2 - 4s + 4}.$$

Roughly speaking, the above facts indicate that for a random permutation on a large $[n]$, we expect the first ascent to be at position $e - 1 \approx 1.718$, whereas this value increases to 3 for a random 123-avoiding permutation.

5 Open Questions

A whole series of questions would relate to enumeration of permutations, free of a certain pattern, in which the first occurrence of another pattern
occurs at a certain spot. Another direction to pursue might be to consider a specific partial order on \([n]\) and answer the same question as that studied in this paper. Finally, can we use the notion of first ascents in the context of 123-avoiding permutations to give another combinatorial proof of Shapiro’s Catalan Convolution identity, as in [1], [6] (both papers were written in response to a query of R. M. Stanley)?

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