MULTI-POLE EXTENSION OF THE ELLIPTIC MODELS OF INTERACTING INTEGRABLE TOPS

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We review and give a detailed description of the $gl_{NM}$ Gaudin models related to holomorphic vector bundles of rank $NM$ and degree $N$ over an elliptic curve with $n$ punctures. We introduce their generalizations constructed by means of $R$-matrices satisfying the associative Yang–Baxter equation. A natural extension of the obtained models to the Schlesinger systems is also given.

Keywords: elliptic integrable system, elliptic Schlesinger system, Gaudin model

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1. Introduction

In this paper, we discuss elliptic integrable systems of classical mechanics that are described by $gl_{NM}$-valued Lax matrices with a spectral parameter $z$ (which is a coordinate on an elliptic curve $Σ_τ$ with moduli $τ$) and have simple poles at $n$ marked points. We restrict ourself to nonrelativistic models governed by linear classical $r$-matrix structures based on either canonical or Poisson–Lie brackets on the phase space.

Our aim here is to present the full classification of this type of integrable systems by summarizing the previously obtained results and by introducing the model of the most general type, which includes all the known ones as particular cases. The classification scheme is given in Fig. 1, with the most general model—the general $gl_{NM}$ Gaudin model—placed in the box number 1 at the top.

In what follows, we use matrix-valued spin variables $S$. The condition rank $S = 1$ means selecting the corresponding minimal coadjoint orbit, as we explain below. We briefly review the elliptic nonrelativistic models involved in the scheme in Fig. 1. All these models can be roughly subdivided into two families: those governed by dynamical classical $r$-matrices (with an explicit dependence on the dynamical variables) and those described by nondynamical $r$-matrices (they depend on spectral parameters, but do not depend on the dynamical variables). The first family is presented in the right column, and the second is in the left column. The middle column includes the intermediate $gl_{NM}$ cases, which turn into the respective first or second family when $N = 1$ or $M = 1$.

The first family includes the spinless $gl_M$ Calogero–Moser (CM) model [1] as its basis element (box 9). The Lax representation was introduced by Krichever [2], and his ansatz can naturally be extended to more complicated models including the most general one in the considered class of integrable systems. The next

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is the spin generalization of the CM model [3] (box 6). For the Lax representation, we use the approach suggested by Billey, Avan, and Babelon [4]. The spin part of the phase space is the symplectic quotient space $O//H$ of a coadjoint orbit $O$ by the action of the Cartan subgroup $H \subset GL(M, \mathbb{C})$. The Poisson bivector and the Lax pair depend on the chosen gauge-fixing conditions entering the Hamiltonian reduction with respect to $H$. To avoid these complications, we can describe the model on the nonreduced phase space $O$ endowed with the simple Poisson–Lie structure. Then the spinless CM system is described as the nonreduced spin CM model on the minimal-dimension orbit. This is shown by the down arrow between boxes 6 and 9 in Fig. 1. The price for this lift from $O//H$ to $O$ is the appearance of unwanted terms in the Lax equation, which vanish on the constraints of the Hamiltonian reduction. We provide more details in the next section. Finally, a Gaudin-type generalization of the spin CM model was suggested by Nekrasov [5]. It is in box 3 in Fig. 1, and is called the multispin CM model. The spin component of the phase space $O^1 \times \cdots \times O^n//H$ includes $n$ coadjoint orbits attached to $n$ marked points $z_1, \ldots, z_n$ on the punctured elliptic curve $\Sigma \setminus \{z_1, \ldots, z_n\}$.

The second family arises from quantum anisotropic $(XYZ)$ exactly solvable models [6] and their semiclassical description [7]. The Lax pairs are constructed using the elliptic Baxter-type $R$-matrix and the Sklyanin $L$-operator. The underlying $gl_N$ mechanical integrable systems (boxes 5 and 8 in Fig. 1) are special (elliptic) tops of the Euler–Arnold type [8]. The elliptic top can be viewed as a multidimensional generalization of the complexified Euler top in $\mathbb{C}^3$ with a certain tensor of inertia. Its phase space is a coadjoint orbit $O$ of the $GL(N, \mathbb{C})$ Lie group. The Gaudin-type generalization was introduced by Reiman and Semenov-Tian-Shansky [9]. The Lax matrix has simple poles at $n$ points on the elliptic curve, and the phase space is the direct product of $n$ coadjoint orbits $O^1 \times \cdots \times O^n$. In the rational limit, the Lax matrix takes the form

$$L(z) = \sum_{k=1}^{n} \frac{S^k}{z - z_k}, \quad S^k \in O^k.$$  \hspace{1cm} (1.1)

Models of this type are usually called the Gaudin models, and in physical literature they are often called Hitchin systems. We remark that both names are somewhat misleading from the historical standpoint.
Gaudin [10] studied quantum models in a special limit. The monodromy matrix of the generalized spin chain on \( n \) sites with inhomogeneous parameters \( z_k \) can be represented in form (1.1), where \( S^k \) are quantum operators (representing Lie algebra generators). The Hitchin approach deals with integrable systems on the moduli space of Higgs bundles over curves [11]. Originally, neither low-genus curves nor marked points were considered. This was done later in [5], and all the models in Fig. 1 can indeed be described in the framework of Hitchin’s construction [5], [8], [12].

The middle family consists of intermediate \( gl_{NM} \) (box 4) models turning into the first- or second-family models when \( N = 1 \) or \( M = 1 \). Originally, they were introduced by Polychronakos in his studies of matrix models [13]. Later, the Lax representation for these models was found [14] using the Hitchin approach and Atiyah’s classification of vector bundles on elliptic curves [15]. In the special case where the \( GL_{NM} \) coadjoint orbit is of minimal dimension, the model can be represented in the form of \( M \) interacting \( gl_N \) elliptic tops (box 7). Finally, the most general model in Fig. 1 is in box 1. It is the subject of this paper.

We also note that all the families can be unified into the so-called symplectic Hecke correspondence [8]. A set of models (from different columns) of the same rank and of the same structure of the underlying coadjoint orbits are gauge equivalent because each model comes from vector bundles of different degrees, and all the bundles can be related by a modification procedure. At the level of the Lax equations, this means that the corresponding Lax matrices are related by (singular) gauge transformation that degenerate at some point. An explicit construction of such gauge transformations in the general case is a complicated problem. But this can be done in some particular cases. For example, the spinless CM model (box 9) is gauge equivalent to the elliptic top with the minimal-dimension orbit (box 8) when \( M = N \). A similar phenomenon in statistical exactly solvable models is known as the IRF–vertex correspondence [16] (using this analogy, we could have called the first family the IRF-type models, and the second family, the vertex-type models).

In the rational and trigonometric cases, the models of the (spin) Calogero and Gaudin types are classified by the same scheme. But there are more different models due to the variety of possibilities appearing in the limit procedures applied to a given elliptic model. Instead of specifying all these possibilities (which is a nontrivial task), we use the \( R \)-matrix formulation, with the Lax pair written in terms of an \( R \)-matrix that satisfies the associative Yang–Baxter equation and has some set of properties. In the elliptic case, the only possible \( R \)-matrix is the Baxter–Belavin one. With this \( R \)-matrix, we just reproduce the elliptic models featuring in Fig. 1. In the rational and trigonometric limits, the models are therefore classified by the same scheme supplied also with the classification of possible (trigonometric or rational) \( R \)-matrices satisfying the associative Yang–Baxter equation and some additional properties. Such models were previously discussed in [17]–[19]. We note that in this way we describe not all possible trigonometric and rational limits but only those that can be represented in the form of spin Calogero and/or Gaudin systems. For example, the class of Toda-type models is absent in the \( R \)-matrix formulation, although it can be derived starting from the elliptic models by means of the Inozemtsev limit procedure [20].

This paper is organized as follows. In Sec. 2, we recall the constructions of the Lax representations for the CM model and its spin generalization. We then proceed to the most general model in Sec. 3 and describe some particular cases in Sec. 4, including the model of interacting tops and the multispin CM system. The generalized formulation of the obtained results by means of quantum \( R \)-matrices is given in Sec. 5. Finally, in Sec. 6, we discuss the Schlesinger systems, which are nonautonomous versions of the elliptic integrable models.

2. Calogero–Moser model and its spin extension

2.1. The spinless \( gl_M \) Calogero–Moser model. The phase space is \( \mathbb{C}^{2M} \) parameterized by the canonical variables (positions and momenta of particles) with the canonical Poisson brackets

\[
\{ p_i, q_j \} = \delta_{ij}, \quad \{ p_i, p_j \} = \{ q_i, q_j \} = 0.
\]
The Hamiltonian
\[ H^{CM} = \sum_{i=1}^{M} \frac{p_i^2}{2} - \nu^2 \sum_{i>j}^{M} \varphi(q_i - q_j) \] (2.2)
describes the pairwise interaction with the potential being the Weierstrass \( \varphi \)-function and the coupling constant \( \nu \in \mathbb{C} \). We then have the equations of motion
\[ \dot{q}_i = p_i, \quad \ddot{q}_i = \nu^2 \sum_{k \neq i} \varphi'(q_{ik}). \] (2.3)

The Lax pair with a spectral parameter was introduced in [2]. It is an explicitly specified pair of \( M \times M \) matrices\(^1\)
\[ L^{CM}_{ij}(z) = (p_i + \nu E_1(z))\delta_{ij} + \nu(1 - \delta_{ij})\phi(z, q_{ij}), \quad q_{ij} = q_i - q_j, \] (2.4)
\[ M^{CM}_{ij}(z) = \nu d_i \delta_{ij} + \nu(1 - \delta_{ij})f(z, q_{ij}), \quad d_i = \sum_{k \neq i} E_2(q_{ik}), \] (2.5)
which provides equations of motion (2.3) via the Lax equation
\[ \dot{L}(z) = [L(z), M(z)] \] (2.6)
(identically in the spectral parameter \( z \)). In fact, Krichever’s ansatz for Lax representation (2.4), (2.5) underlies the Lax pair for the most general model as well.

2.2. Spin generalization of the Calogero–Moser model [3] (box 6 in Fig. 1). For the Lax representation, we use the approach proposed in [4]. We now explain the main idea. Besides the many-body component \( \mathbb{C}^{2M} \), the phase space also includes the space (as a component in the direct product) parameterized by variables \( S_{ij}, i, j = 1, \ldots, M \) treated as the classical spin variables. They are naturally arranged into a \( gl(M, \mathbb{C}) \)-valued matrix \( S = \sum_{ij} E_{ij}S_{ij} \). The spin component of the phase space is \( \mathcal{O} // H \), where \( \mathcal{O} \) is a coadjoint orbit of the \( GL(N, \mathbb{C}) \) Lie group, \( H \subset GL(N, \mathbb{C}) \) is its Cartan subgroup, and the double quotient \( // \) means performing the Hamiltonian (or Poisson) reduction of \( \mathcal{O} \) with respect to the adjoint action of \( H \). This action (the conjugation \( S \to hSh^{-1} \), where \( h \in H \) is a diagonal \( M \times M \) matrix) provides the moment map constraints
\[ S_{ii} = \text{const}, \quad i = 1, \ldots, M. \] (2.7)
Supplied with some gauge-fixing conditions \( \varsigma_k, k = 1, \ldots, M \) (fixing the action \( S \to hSh^{-1} \)), they form \( 2M \) Dirac second-class constraints. The Poisson bivector on the reduced phase space \( \mathcal{O} // H \) depends on the choice of the gauge-fixing conditions \( \varsigma_i \) because they enter the Dirac bracket formula (for a pair of functions \( f_1, f_2 \) on the reduced space)
\[ \{f_1, f_2\}_{\text{red}} = (\{f_1, f_2\} - \{f_1, \chi\} C^{-1}(\chi^T, f_2))|_{\text{on shell}}, \] (2.8)
where \( \chi \) is a \( 2M \)-dimensional row \( (S_{11}, \ldots, S_{MM}, \varsigma_1, \ldots, \varsigma_M) \), and \( C \in \text{Mat}_{2M} \) is the matrix with elements \( C_{kl} = \{\chi_k, \chi_l\} \), \( k, l = 1, \ldots, 2M \). The “on shell” subscript means the restriction to the constraint surface.

Instead of dealing with reduced brackets (2.8), which requires some choice of \( \varsigma_k \), we can describe the spin CM model on the nonreduced phase space \( \mathbb{C}^{2M} \times \mathcal{O} \). Then the spin component of the phase space is equipped with a natural and simple Poisson brackets, the Poisson–Lie structure on \( gl^\ast(M, \mathbb{C}) \):
\[ \{S_{ij}, S_{kl}\} = -S_{il}\delta_{kj} + S_{kj}\delta_{il}. \] (2.9)
\(^1\)See the Appendix for the definitions of elliptic functions.
We represent it in block-matrix form with $E$

\[ L_{ij}^{\text{spin}}(z) = \delta_{ij}(p_i + S_{ii}E_1(z)) + (1 - \delta_{ij})S_{ij}\phi(z, q_{ij}), \quad (2.10) \]

\[ M_{ij}^{\text{spin}}(z) = (1 - \delta_{ij})S_{ij}f(z, q_i - q_j). \quad (2.11) \]

The Hamiltonian

\[ H^{\text{spin CM}} = \sum_{i=1}^{N} \frac{p_i^2}{2} - \sum_{i>j}^{N} S_{ij}S_{ji}q_i - q_j, \quad (2.12) \]

obtained from $\text{tr}(L^{\text{spin}}(z))^2$ and Poisson brackets (2.1), (2.9) give rise to the equations of motion

\[ \dot{q}_i = p_i, \quad \dot{p}_i = \sum_{j\neq i}^{M} S_{ij}S_{ji}q_j - q_j, \quad (2.13) \]

\[ \dot{S}_{ii} = 0, \quad \dot{S}_{ij} = \sum_{k\neq i,j}^{M} S_{ik}S_{kj}(q_i - q_k) - q_j - q_k, \quad i \neq j. \quad (2.14) \]

They can be equivalently represented in the form of a Lax equation with an additional unwanted term:

\[ \dot{L}(z) = [L(z), M(z)] + \sum_{i,j=1}^{M} E_{ij}(S_{ii} - S_{jj})S_{ij}E_1(z)f(z, q_{ij}). \quad (2.15) \]

Thus, the spin CM model is not integrable on the nonreduced space, and it becomes integrable when constraints (2.7) are imposed, ensuring the vanishing of the unwanted term. However, equations of motion (2.13), (2.14) are no longer valid on the reduced phase space because the reduction includes not only the on-shell constraint but also some other (Dirac) terms coming from the second term in (2.8). This can easily be seen by considering the example of the minimal-dimension coadjoint orbit. In that case, $S$ is a rank-one matrix $S_{ij} = \xi_\eta_{ij}$, and hence $N - 1$ of the $N$ eigenvalues of $S$ coincide. The constraints $S_{ii} = \nu$ taken together with the gauge-fixing condition $\xi_i = 1$ $\forall i$ lead to a trivial spin space after the reduction: $S_{ij} = \nu$, and therefore the spin variables are absent in this case and we return to the spinless case in Eq. (4.2). The Dirac terms result in a nontrivial diagonal part of matrix (2.5), while they vanish for the nonreduced $M$-matrix (2.11). In this way, we describe the spinless CM system as the nonreduced spin CM model on the minimal-dimension orbit.

3. Lax pair in the general case

3.1. Lax matrix in the general case. In the general case, the Lax matrix has the size $NM \times NM$. We represent it in block-matrix form with $M \times M$ blocks of size $N \times N$ each:

\[ \mathcal{L}(z) = \sum_{i,j=1}^{M} E_{ij} \otimes L_{ij}(z) \in \text{Mat}(NM, \mathbb{C}), \quad \mathcal{L}^{ij}(z) \in \text{Mat}(N, \mathbb{C}), \quad (3.1) \]

where $E_{ij}$ is the standard basis in $\text{Mat}(M, \mathbb{C})$. Inside each $N \times N$ block, we use another matrix basis $T_\alpha$,  

\[ T_\alpha = e^{a_1 a_2 i/N}Q^{a_1}A^{a_2}, \quad \alpha = (a_1, a_2) \in \mathbb{Z}_N \times \mathbb{Z}_N, \quad T_0 = T(0,0) = 1_N, \quad (3.2) \]

defined in terms of the generators of the noncommutative torus (a finite-dimensional representation of the Heisenberg group)

\[ Q_{jk} = \delta_{jk}e^{2\pi i k/N}, \quad A_{jk} = \delta_{j-k+1=0 \mod N}, \quad Q^N = A^N = 1_N. \quad (3.3) \]

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The commutation relations take the form

\[ T_\alpha T_\beta = \kappa_{\alpha,\beta} T_{\alpha+\beta}, \quad \kappa_{\alpha,\beta} = e^{(\pi i/N)(\alpha_2 \beta_1 - \alpha_1 \beta_2)}, \quad \kappa_{\alpha,\beta} = \kappa_{\alpha,\beta}, \quad \kappa_{-\alpha,\beta} = \kappa_{\beta,\alpha}, \quad (3.4) \]

\[ \text{tr}(T_\alpha T_\beta) = N \delta_{\alpha+\beta}, \quad \delta_\alpha = \delta_{\alpha_1,0} \delta_{\alpha_2,0}, \quad (3.5) \]

\[ [T_\alpha, T_\beta] = (\kappa_{\alpha,\beta} - \kappa_{-\beta,\alpha}) T_{\alpha+\beta} = 2i \sin(\frac{\pi}{N}(\alpha_1 \beta_2 - \alpha_2 \beta_1)) T_{\alpha+\beta}. \quad (3.6) \]

The matrix blocks in (3.1) are of the form

\[ L^{ij}(z) = \delta_{ij} \left( p_i 1_N + \sum_{a=1}^{N} S_{ij}^{\alpha,a} z E_1(z - z_a) + \sum_{a=1, \alpha \neq 0}^{n} S_{ij}^{\alpha,a} T_{\alpha,\alpha}(z - z_a, \omega_\alpha) \right) + \]

\[ + (1 - \delta_{ij}) \sum_{a=1}^{n} \sum_{\alpha} S_{ij}^{\alpha,a} T_{\alpha,\alpha}(z - z_a, \omega_\alpha + \frac{q_{ij}}{N}). \quad (3.7) \]

The index \( \alpha = (\alpha_1, \alpha_2) \) takes values in \( \mathbb{Z}_N \times \mathbb{Z}_N \). The sum over \( \alpha \neq 0 \) means that the index value \( \alpha = (0,0) \) corresponding to \( T_{(0,0)} = 1_N \) (the identity matrix) is excluded, and that \( \omega_\alpha = 0 \) in (A.19).

Lax matrix (3.7) is a natural extension of the \( gl_{NM} \) model to the multi-pole case (with \( n \) marked points on the elliptic curve \( \Sigma_\tau \)):

\[ S^{ij,a} = \text{Res}_{z=z_a} L^{ij}(z) \in \text{Mat}(N, \mathbb{C}). \quad (3.8) \]

When \( n = 1 \) we return to the mixed-type \( gl_{NM} \) model (box 4 in Fig. 1).

The origin of the explicit expression for the Lax matrix (3.1), (3.7) is as follows. As was shown in [12], [14], Lax matrices are classified by the structure of the underlying (Higgs) bundles over an elliptic curve. The general classification of bundles is known from [15]. We here deal with a holomorphic vector bundle \( V \) of degree \( M \) and rank \( NM \). The Lax matrix is a section of the \( \text{End}(V) \) bundle with the transition functions

\[ L(z + 1) = g_1 L(z) g_1^{-1}, \quad L(z + \tau) = g_\tau L(z) g_\tau^{-1}, \quad (3.9) \]

where \( g_1 \) and \( g_\tau \) are \( (NM \times NM) \) matrices with the block diagonal structure

\[ g_1 = \bigoplus_{k=1}^{M} Q^{-1}, \quad g_\tau = \bigoplus_{k=1}^{M} e^{-2\pi i k \phi / N \Lambda}^{-1}. \quad (3.10) \]

The residues of \( L(z) \) are fixed as

\[ \text{Res}_{z=z_a} L(z) = S^a \in \text{Mat}(NM, \mathbb{C}). \quad (3.11) \]

The solution of (3.11) with the quasiperiodicity conditions defined by (3.10) is given by (3.1), (3.7).

The Poisson brackets for momenta and positions of particles are canonical, Eq. (2.1). The set of classical spin variables \( S^{ij,a}_\alpha, i, j = 1, \ldots, M, \alpha \in \mathbb{Z}_N \times \mathbb{Z}_N, a = 1, \ldots, n \) parameterizes \( n \) Lie coalgebras \( gl_{NM}^* \). Hence, the Poisson brackets are given by the Lie–Poisson structure that is dual to the basis \( E_{ij} \otimes T_\alpha \) in \( \text{Mat}(NM, \mathbb{C}) \):

\[ \{ S^{ij,a}_\alpha, S^{km,b}_\beta \} = \frac{\delta^{ab}}{N} (\delta^{im} \kappa_{\alpha,\beta} S^{kj,a}_{\alpha+\beta} - \delta^{kj} \kappa_{\beta,\alpha} S^{im,a}_{\alpha+\beta}). \quad (3.12) \]

### 3.2. Hamiltonian description

The generating function of the Hamiltonians has the standard form

\[ \frac{1}{2N} \text{tr}(L^2(z)) = H_0 + \sum_{a=1}^{n} H_{1,a} E_1(z - z_a) + \]

\[ + \sum_{a=1}^{n} H_{2,a} E_2(z - z_a) + \sum_{b=1}^{n} S^{ii,b}_{0,0} \sum_{a=1}^{n} S^{ii,a}_{0,0} \rho(z - z_a), \quad (3.13) \]

where the last term contains a non-double-periodic function of \( z \), Eq. (A.8). This term can be eliminated using additional constraints. We discuss this below.
To compute the left-hand side of (3.13), we use property (3.5). This yields
\[
\frac{1}{2N} \text{tr}(C^2(z)) = \sum_{i=1}^{M} \left( \frac{p_i^2}{2} + \frac{1}{2} \sum_{a,b=1}^{n} S_{0,0}^{i,a} S_{0,0}^{i,b} E_1(z-z_a) E_1(z-z_b) + \right.
\]
\[
\left. + p_i \sum_{a=1}^{n} S_{0,0}^{i,a} E_1(z-z_a) + \frac{1}{2} \sum_{a,b=1}^{n} \sum_{\alpha \neq 0} S_{\alpha}^{i,a} S_{-\alpha}^{i,b} \varphi_\alpha(z-z_a,\omega_\alpha) \varphi_{-\alpha}(z-z_b,\omega_{-\alpha}) \right) + \right.
\]
\[
\left. + \frac{1}{2} \sum_{i \neq j, a,b=1}^{n} \sum_{\alpha} S_{\alpha}^{i,a} S_{\alpha}^{j,a} \varphi_\alpha(z-z_a,\omega_\alpha + \frac{q_{ij}}{N}) \varphi_{-\alpha}(z-z_b,\omega_{-\alpha} + \frac{q_{ij}}{N}). \right) \tag{3.14}
\]

Next, we use identities (A.17), (A.13), and (A.25), which lead to the following answer for the Hamiltonians in the right-hand side of (3.13):
\[
H_0 = \sum_{i=1}^{M} \frac{p_i^2}{2} + \frac{1}{2} \sum_{i=1}^{M} \sum_{a,b=1}^{n} S_{0,0}^{i,a} S_{0,0}^{i,b} \rho(z_{ab}) + \frac{1}{2} \sum_{i=1}^{M} \sum_{a,b=1}^{n} \sum_{\alpha \neq 0} S_{\alpha}^{i,a} S_{-\alpha}^{i,b} f_\alpha(z_{ba},\omega_\alpha) + \right.
\]
\[
\left. + \frac{1}{2} \sum_{i \neq j, a,b=1}^{n} \sum_{\alpha} S_{\alpha}^{i,a} S_{\alpha}^{j,a} f_\alpha \left(z_{ba},\omega_\alpha + \frac{q_{ij}}{N}\right), \quad z_{ab} = z_a - z_b. \right) \tag{3.15}
\]
\[
H_{1,a} = \sum_{i=1}^{M} p_i S_{0,0}^{i,a} + \sum_{i=1}^{M} \sum_{a,b=1}^{n} S_{0,0}^{i,a} S_{0,0}^{i,b} E_1(z_{ab}) - \sum_{i=1}^{M} \sum_{a,b=1}^{n} \sum_{\alpha \neq 0} S_{\alpha}^{i,a} S_{-\alpha}^{i,b} \varphi_\alpha(z_{ba},\omega_\alpha) - \right.
\]
\[
\left. - \sum_{i,j: i \neq j, b \neq a}^{n} \sum_{\alpha} S_{\alpha}^{i,a} S_{\alpha}^{j,a} \varphi_\alpha \left(z_{ba},\omega_\alpha + \frac{q_{ij}}{N}\right), \right) \tag{3.16}
\]
\[
H_{2,a} = \frac{1}{2} \sum_{i,j} S_{\alpha}^{i,a} S_{\alpha}^{j,a}, \tag{3.17}
\]

where the function \(\rho(z)\) in (A.8) and the function \(f_\alpha(z, u)\) is defined by (A.4), (A.13). We note that because \(f(z, u)\) is a derivative of \(\phi(z, u)\) with respect to the second argument, it does not have a pole at \(z = 0\), i.e., \(f(0, u)\) is well defined (see (A.7)).

It is easy to verify that the Hamiltonians \(H_{2,a}\) in (3.17) are in fact Casimir functions. They give rise to trivial dynamics. By fixing their levels, we restrict the spin part of the phase space to the product of \(n\) orbits \(O^1 \times \cdots \times O^n\) of the coadjoint action of \(GL(NM, \mathbb{C})\).

In the spin CM case, we had additional constraints (2.7) generated by the action of the Cartan subgroup of \(GL(M, \mathbb{C})\). Here, we deal with the \(M\)-dimensional Cartan subgroup \(H_M \subset H \subset GL(NM, \mathbb{C})\) in the Cartan subgroup of \(GL(NM, \mathbb{C})\) [12]. Its common action on all orbits gives rise to the moment map generalizing the additional constraints in (2.7):
\[
\sum_{a=1}^{n} S_{0,0}^{k,a} = \text{const} \quad \forall k = 1, \ldots, M. \tag{3.18}
\]

Imposing certain gauge-fixing conditions, we arrive at the final description of (the spin part of) the phase space: \((O^1 \times \cdots \times O^n)/H_M\). But similarly to the spin CM case, we do not perform this reduction. Instead, we write the Lax equations with an additional unwanted term, similarly to what we did in (2.15).

It follows from the behavior of the function \(E_1(z-z_a)\) on the lattice \(\mathbb{Z} \oplus \mathbb{Z}\) (see (A.9)) that Lax matrix (3.7) becomes quasiperiodic when the constant in the right-hand side of (3.18) is chosen to be zero. In this case, expression (3.13) is a double-periodic function of \(z\). Therefore, the sum of its residues is equal to zero:
\[
\sum_{a=1}^{n} H_{1,a}\big|_{(3.18), \text{const}=0} = 0. \tag{3.19}
\]
Alternatively, we could redefine the Lax matrix by making the shift

\[ \mathcal{L}(z) \rightarrow \mathcal{L}(z) - 1_{NM} \sum_{a=1}^{n} \frac{\text{const}}{n} E_1(z - z_a). \] (3.20)

On one hand, such a modified Lax matrix satisfies the same Lax equation because it is unaffected by the scalar nondynamical term. On the other hand, this is equivalent to a redefinition of residues \( S_{0,0}^{ii,a} \rightarrow S_{0,0}^{ii,a} - \text{const}/n \), and hence constraints (3.18) with the zero right-hand side hold for the new set of \( S_{0,0}^{ii,a} \).

### 3.2.1. Lax pair for the flow of the Hamiltonian \( H_0 \)

We consider the dynamics generated by Hamiltonian (3.15) on the nonreduced phase space \( O^1 \times \cdots \times O^n \) with Poisson brackets (2.1) and (3.12). The equations of motion take the form

\[
\begin{align*}
\dot{q}_i & = p_i, \\
\dot{p}_i & = \frac{1}{N} \sum_{k: k \neq i, a=1}^{M} \sum_{\alpha} S_{\alpha}^{ki,a} S_{-\alpha}^{ik,b} f_\alpha \left( z_{ba}, \omega_\alpha + \frac{q_{ki}}{N} \right),
\end{align*}
\] (3.21)

\[
\begin{align*}
\hat{S}_{\alpha}^{ij,a} & = \frac{1}{N} \sum_{b: b \neq a}^{n} \sum_{k: k \neq i}^{M} S_{\alpha}^{ij,a} \left( S_{0,0}^{ik,b} - S_{0,0}^{ii,b} \right) \rho(z_{ba}) + \\
+ \frac{1}{N} \sum_{b=1}^{n} \sum_{b \neq 0} \sum_{k: \beta \neq k} \left( \sum_{\alpha} \kappa_{\alpha,\beta} S_{\alpha}^{ik,a} S_{\beta}^{kj,b} \right) f_\beta \left( z_{ab}, \omega_\beta + \frac{q_{ki}}{N} \right) - \\
- \sum_{k: k \neq i}^{M} \kappa_{\beta,\alpha} S_{\alpha}^{ik,a} S_{\beta}^{kj,b} f_\beta \left( z_{ab}, \omega_\beta + \frac{q_{ki}}{N} \right).
\end{align*}
\] (3.22)

The function \( f'(z, u) \) entering (3.21) is a derivative of \( f_\alpha(z, u) \) with respect to the second argument, and therefore \( f'(z, u) = \partial_2 \phi_\alpha(z, u) \). It also follows from (A.7) that \( f'(0, u) = -E_2'(u) = -\varphi'(u) \). We also write Eqs. (3.22) in some particular cases:

\[
\begin{align*}
\hat{S}_{\alpha}^{ii,a} & = \frac{1}{N} \sum_{b=1}^{n} \sum_{\beta \neq 0} \left( \kappa_{\alpha,\beta} - \kappa_{\beta,\alpha} \right) S_{\alpha}^{ii,a} S_{\beta}^{ij,b} f_\beta \left( z_{ab}, \omega_\beta + \frac{q_{ki}}{N} \right) + \\
+ \frac{1}{N} \sum_{k: k \neq i}^{M} \sum_{\beta} \left( \kappa_{\alpha,\beta} S_{\alpha}^{ik,a} S_{\beta}^{kj,b} \right) f_\beta \left( z_{ab}, \omega_\beta + \frac{q_{ki}}{N} \right) - \\
- \kappa_{\beta,\alpha} S_{\alpha}^{ik,a} S_{\beta}^{kj,b} f_\beta \left( z_{ab}, \omega_\beta + \frac{q_{ki}}{N} \right),
\end{align*}
\] (3.23)

\[
\begin{align*}
\hat{S}_{0,0}^{ii,a} & = \frac{1}{N} \sum_{k: k \neq i}^{M} \sum_{\alpha} \left( S_{-\alpha}^{ik,a} S_{\alpha}^{ji,b} f_\alpha \left( z_{ab}, \omega_\alpha + \frac{q_{ki}}{N} \right) - \\
- S_{-\alpha}^{ik,a} S_{\alpha}^{ji,b} f_\alpha \left( z_{ab}, \omega_\alpha + \frac{q_{ki}}{N} \right) \right).
\end{align*}
\] (3.24)

For the spin CM model, we saw that the constraints \( S_{ii} = \text{const} \) are preserved by the dynamics, i.e., \( \hat{S}_{ii} = 0 \) in (2.14). The same happens in the general model. To see this, we sum Eqs. (3.24) over \( a = 1, \ldots, n \):

\[
\begin{align*}
\frac{d}{dt} \sum_{a=1}^{n} S_{0,0}^{ii,a} & = \frac{1}{N} \sum_{k: k \neq i}^{M} \sum_{a=1}^{n} \left( S_{-\alpha}^{ik,a} S_{\alpha}^{ji,b} f_\alpha \left( z_{ab}, \omega_\alpha + \frac{q_{ki}}{N} \right) - \\
- S_{-\alpha}^{ik,a} S_{\alpha}^{ji,b} f_\alpha \left( z_{ab}, \omega_\alpha + \frac{q_{ki}}{N} \right) \right),
\end{align*}
\] (3.25)
where \( t_0 \) is the time of the Hamiltonian \( H_0 \). The expression in the right-hand side of (3.25) vanishes. Indeed, by interchanging the summation indices \( a \leftrightarrow b \) and \( \alpha \leftrightarrow -\alpha \) and using property (A.4), we easily obtain that the right-hand side is equal to itself with the opposite sign.

We introduce the \( M \)-matrix

\[
M_0(z) = \sum_{i,j=1}^{M} E_{ij} \otimes M_{ij}^0(z) \in \text{Mat}(NM, \mathbb{C}), \quad M_{ij}^0(z) \in \text{Mat}(N, \mathbb{C}),
\]

\[
M_{ij}^0(z) = \frac{\delta_{ij}}{N} \sum_{a=1}^{n} S_{0,0}^{i,a} \rho(z - z_a) + \frac{\delta_{ij}}{N} \sum_{a=1, a \neq 0}^{n} S_{0}^{i,a} T_{a} f_{a}(z - z_a, \omega_{a}) +
\]

\[
\frac{1}{N}(1 - \delta_{ij}) \sum_{a=1, a \neq 0}^{n} S_{0}^{i,a} T_{a} f_{a} \left( z - z_a, \omega_{a} + \frac{q_{ij}}{N} \right).
\]

We now give the main statement in this subsection.

**Proposition 1.** Equations of motion (3.21)–(3.24) are equivalent to the Lax equations with an additional term:

\[
\frac{d}{dt_0} \mathcal{L}(z) = [\mathcal{L}(z), M_0(z)] + \frac{1}{2N} \sum_{i,j=1}^{M} \sum_{a,b=1}^{n} \sum_{\alpha} S_{ij}^{i,b}(S_{0,0}^{i,a} - S_{0,0}^{j,a}) E_{ij} \otimes T_{a} f_{a} \left( z - z_b, \omega_{a} + \frac{q_{ij}}{N} \right).
\]

The additional term vanishes on constraints (3.18).

The proof is straightforward, although cumbersome. It is based on the use of Eqs. (A.11)–(A.18).

### 3.2.2. Lax pairs for the flows of Hamiltonians \( H_{1,a} \)

We consider the dynamics generated by the Hamiltonian \( H_{1,a} \) in (3.16). Similarly to Sec. 3.2.1, we assume Poisson structure (2.1), (3.12), which means that constraints (3.18) are not yet imposed. The equations of motion are of the form (the dot denotes the derivative with respect to the \( t_{a,1} \) time variable)

\[
\dot{q}_i = S_{0,0}^{i,a}, \quad \dot{p}_i = \frac{1}{N} \sum_{k: k \neq i, b; b \neq a}^{M} \sum_{\alpha} \left( S_{\alpha}^{k,a} S_{-\alpha}^{k,b} f_{a} \left( z_{ba}, \omega_{a} + \frac{q_{ab}}{N} \right) \right) -
\]

\[
- S_{\alpha}^{k,a} S_{-\alpha}^{k,b} f_{a} \left( z_{ba}, \omega_{a} + \frac{q_{ki}}{N} \right),
\]

\[
\dot{S}_{\alpha}^{i,b} = \frac{1}{N} S_{\alpha}^{i,b}(S_{0,0}^{i,a} - S_{0,0}^{j,a}) E_{1}(z_{ab}) +
\]

\[
\frac{1}{N} \sum_{\beta \neq 0} S_{\alpha-\beta}^{i,b} \left( \kappa_{\beta, \alpha} S_{\beta}^{i,a} - \kappa_{\alpha, \beta} S_{\beta}^{j,a} \right) \varphi_{\beta}(z_{ba}, \omega_{\beta}) +
\]

\[
\frac{1}{N} \sum_{\beta \neq 0} \left( \sum_{k \neq i}^{M} \kappa_{\beta, \alpha} S_{\alpha-\beta}^{k,b} S_{\beta}^{k,a} \varphi_{\beta}(z_{ba}, \omega_{\beta} + \frac{q_{ki}}{N}) -
\]

\[
- \sum_{k \neq j}^{M} \kappa_{\alpha, \beta} S_{\alpha-\beta}^{k,b} S_{\beta}^{j,a} \varphi_{\beta}(z_{ba}, \omega_{\beta} + \frac{q_{kj}}{N}) \right).
\]

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for $b \neq a$, and

$$S_{i,j,a} = -\frac{1}{N}(p_i - p_j)S_{i,j,a} + \frac{1}{N} \sum_{c : c \neq a} S_{\alpha}^{i,j,a} (S_{\alpha}^{i,j,c} - S_{\alpha}^{i,c})E_1(z_{ac}) +$$

$$+ \frac{1}{N} \sum_{c : c \neq a, \beta \neq 0} S_{\alpha - \beta}^{i,j,a} (\kappa_{\alpha, \beta} S_{\beta}^{i,j,c} - \kappa_{\beta, \alpha} S_{\beta}^{i,c}) \varphi_\beta(z_{ac}, \omega_\beta) +$$

$$+ \frac{1}{N} \sum_{c : c \neq a} \left( M_{k,j} \delta_{i,b}^{k,i} \varphi_\beta \left( z_{ac}, \omega_\beta + \frac{q_{ik}}{N} \right) -$$

$$- \sum_{k : k \neq 1} \kappa_{\beta, a} S_{\alpha - \beta}^{i,j,a} \left( z_{ca}, \omega_\beta + \frac{q_{ik}}{N} \right) \right) +$$

$$(3.29)$$

In some particular cases ($i = j$ and $\alpha = 0$), we have

$$\delta_{i,j} = \frac{1}{N} \sum_{k : k \neq 1} (1 - \delta_{ab}) S_{\alpha - \beta}^{i,j} \varphi_\beta(z_{ba}, \omega_\beta) -$$

$$- \sum_{c : c \neq a} \delta_{ab} S_{\alpha + \beta}^{i,j,a} \left( z_{ca}, \omega_\beta + \frac{q_{ik}}{N} \right) +$$

$$+ \frac{1}{N} \sum_{k : k \neq 1} \sum_{\beta} (1 - \delta_{ab}) \left( \kappa_{\beta, a} S_{\alpha - \beta}^{i,j,a} \varphi_\beta \left( z_{ba}, \omega_\beta + \frac{q_{ik}}{N} \right) -$$

$$- \kappa_{\beta, a} S_{\alpha + \beta}^{i,j,a} \left( z_{ca}, \omega_\beta + \frac{q_{ik}}{N} \right) \right) +$$

$$(3.30)$$

and

$$\delta_{0,0} = \frac{1}{N} \sum_{k : k \neq 1} (1 - \delta_{ab}) S_{\alpha - \beta}^{i,j,a} \varphi_\beta \left( z_{ca}, \omega_\beta + \frac{q_{ik}}{N} \right) -$$

$$- \delta_{ab} \left( \kappa_{\beta, a} S_{\alpha - \beta}^{i,j,a} \varphi_\beta \left( z_{ca}, \omega_\beta + \frac{q_{ik}}{N} \right) \right) +$$

$$+ \frac{1}{N} \sum_{k : k \neq 1} \sum_{\beta} \left( \delta_{ab} \left( \kappa_{\beta, a} S_{\alpha - \beta}^{i,j,a} \varphi_\beta \left( z_{ca}, \omega_\beta + \frac{q_{ik}}{N} \right) \right) -$$

$$- \delta_{ab} S_{\alpha - \beta}^{i,j,a} \left( z_{ca}, \omega_\beta + \frac{q_{ik}}{N} \right) \right), \quad (3.31)$$

where we unified the cases $a = b$ and $a \neq b$.

We consider the $M$-matrix

$$M_{1,a}(z) = \sum_{i,j=1}^M E_{ij} \otimes M_{1,a}(z),$$

$$M_{ij}^{1,a}(z) = -\delta ij S_{\alpha}^{i,j,a} 1_N E_1(z - z_a) - \delta ij \sum_{a \neq 0} S_{\alpha}^{i,j,a} T_{\alpha} \varphi_\alpha(z - z_a, \omega_\alpha) -$$

$$- \frac{1}{N}(1 - \delta ij) \sum_{a \neq 0} S_{\alpha}^{i,j,a} T_{\alpha} \varphi_\alpha \left( z - z_a, \omega_\alpha + \frac{q_{ij}}{N} \right). \quad (3.32)$$

The following statement holds.
Proposition 2. Equations of motion (3.28), (3.29) can be equivalently written in the form of the Lax equation with an additional term

\[ \dot{L}(z) = [L(z), M_{1,a}(z)] + \frac{1}{N} \sum_{i,j=1}^{M} \sum_{b=1}^{n} \sum_{\alpha} S_{\alpha}^{i,a} (S_{\alpha}^{i,b} - S_{\alpha}^{j,b}) E_{ij} \otimes T_{a} f_{\alpha} \left( z - z_{\alpha}, \omega_{\alpha} + \frac{q_{ij}}{N} \right), \]  

(3.33)

where the dot is the derivative with respect to the time variable \( t_{1,a} \). The additional term vanishes on constraints (3.18).

4. Particular cases

4.1. Tops and Gaudin models.

4.1.1. The \( gl_{N}^{n} \) Gaudin model (box 2 in Fig. 1). We start with the second integrable family in our scheme, the top-like models. In the case \( M = 1 \), our general \( gl_{N}^{n} \) model turns into the elliptic Gaudin model \([9]\). In this model, we have only the spin part of the phase space, which is now isomorphic to the direct product of \( n \) orbits: \( O_1 \times \cdots \times O_n \). Poisson structure (3.12) then becomes

\[ \{ S_{a}^{a}, S_{b}^{b} \} = \delta^{ab} (\kappa_{a,b} - \kappa_{b,a}) S_{a}^{a} \]  

(4.1)

where we drop the factor \( 1/N \) introduced in (3.12). We note that the Poisson brackets of diagonal scalar elements of the spin matrix \( S_{0,0} \) with any other spin variable are equal to zero. This, together with the fact that all terms in the matrices \( L \) and \( M \) with this scalar diagonal spin commute with any terms (\( S_{0,0}^{a} \) is a coefficient in front of the identity matrix), allows us to eliminate all such terms from the Hamiltonians and the Lax matrices. For the Lax matrix, we then have

\[ L(z) = \sum_{a} \sum_{\alpha \neq 0} S_{a}^{a} T_{\alpha} \varphi_{\alpha}(z - z_{\alpha}, \omega_{\alpha}). \]  

(4.2)

Using (3.13) we obtain the Hamiltonians (see [21] for the details of the calculation):

\[ H_{0} = \frac{1}{2} \sum_{a,b=1}^{n} \sum_{\alpha \neq 0} S_{a}^{a} S_{\alpha}^{b} f_{\alpha}(z_{ba}, \omega_{\alpha}), \] \[ H_{1,a} = \sum_{b: b \neq a}^{n} \sum_{\alpha \neq 0} S_{a}^{a} S_{\alpha}^{b} \varphi_{\alpha}(z_{ba}, \omega_{\alpha}). \]  

(4.3)

The equations of motion corresponding to these Hamiltonians are

\[ \dot{S}_{a}^{a} = \sum_{b=1}^{n} \sum_{\beta} [S_{a}^{a}, S_{\beta}^{b} T_{\beta} \varphi_{\beta}(z_{ab}, \omega_{\beta})], \] \[ \dot{S}_{a}^{a} = - \sum_{b: b \neq a}^{n} \sum_{\beta} [S_{a}^{a}, S_{\beta}^{b} T_{\beta} \varphi_{\beta}(z_{ab}, \omega_{\beta})], \] \[ \dot{S}_{b}^{b} = \sum_{\beta} [S_{b}^{b}, S_{\beta}^{a} T_{\beta} \varphi_{\beta}(z_{ba}, \omega_{\beta})], \] \[ b \neq a. \]  

(4.4)

(4.5)

The Lax equations for \( L \)-matrix (4.2) and \( M \)-matrices,

\[ M_{0}(z) = - \sum_{a=1}^{n} \sum_{\alpha \neq 0} S_{a}^{a} T_{\alpha} f_{\alpha}(z - z_{\alpha}, \omega_{\alpha}), \] \[ M_{1,a}(z) = \sum_{\alpha \neq 0} S_{a}^{a} T_{\alpha} \varphi_{\alpha}(z - z_{\alpha}, \omega_{\alpha}) \]  

(4.6)

are equivalent to Eqs. (4.4), (4.5) on the constraints: \( \sum_{a} \text{tr} S_{a}^{a} = \text{const.} \)
4.1.2. The $gl_N$ integrable top (boxes 5 and 8 on Fig. 1). In the case of a single marked point, the Gaudin model turns into the integrable $gl_N$ top [8]. The phase space of this model is a single coadjoint orbit of the $Gl_N$ Lie group: $\mathcal{O}_N$. The Lax pair takes the form

$$L(z, S) = \sum_{\alpha \neq 0} S_\alpha T_\alpha \varphi_\alpha(z, \omega_\alpha), \quad M(z, S) = \sum_{\alpha \neq 0} S_\alpha T_\alpha f_\alpha(z, \omega_\alpha),$$

and the Lax equation is equivalent to the equation of motion,

$$\dot{S} = [S, J], \quad J(S) = \sum_{\alpha \neq 0} S_\alpha T_\alpha J_\alpha, \quad S = \sum_{\alpha \neq 0} S_\alpha T_\alpha,$$

where the inverse tensor of inertia $J$ has the components

$$J_\alpha = -E_2(\omega_\alpha).$$

The Hamiltonian corresponding to Eqs. (4.8) is

$$H_{\text{top}} = \frac{1}{2} \text{tr}(S \cdot J(S)).$$

The special case of the $gl_N$ top is the case of the minimal coadjoint orbit $\mathcal{O}_N^{\text{min}}$ (box 8 in Fig. 1). The dimension of the orbit (and therefore, of the phase space) depends on the eigenvalues of $S$, which are fixed by the Casimir functions $\text{tr} S^k$. The case of the minimal orbit corresponds to $N - 1$ coincident eigenvalues, i.e., rank $S = 1$, and hence the dimension of the phase space is

$$\dim \mathcal{O}_N^{\text{min}} = 2(N - 1).$$

4.2. Multispin Calogero models. We consider the special case $N = 1$. In this case, Lax matrix (3.7) loses its block structure and becomes an $M \times M$ matrix:

$$L^G_{ij} = \delta_{ij} \left( p_i + \sum_{a=1}^n S^a_{ii} E_1(z - z_a) \right) + (1 - \delta_{ij}) \sum_{a=1}^n S^a_{ij} \phi(z - z_a, q_{ij}) \in \text{Mat}(N, \mathbb{C}), \quad i, j = 1, M.$$
This Poisson structure gives rise to the equations of motion

\[ \dot{q}_i = p_i, \quad \dot{p}_i = \sum_{k: \ k \neq i \ a, b} S^a_{ki} S^b_{ik} f'(z_{ba}, q_{ki}), \]

\[ \dot{S}^a_{ij} = \sum_{b: \ b \neq a} S^a_{ij}(S^b_{jj} - S^b_{ii}) p(z_{ba}) + \sum_{b=1}^n \left( \sum_{k: \ k \neq j} S^a_{ik} S^b_{kj} f(z_{ab}, q_{kj}) - \sum_{k: \ k \neq i} S^b_{kj} S^b_{ik} f(z_{ab}, q_{ik}) \right) \]

for the Hamiltonian \( H_0 \) and

\[ \dot{q}_i = S^{ii,a}_{0,0}, \quad \dot{p}_i = \sum_{k: \ k \neq i \ b: \ b \neq a} (S^a_{ik} S^b_{ki} f(z_{ba}, q_{ki}) - S^b_{ik} S^a_{ki} f(z_{ba}, q_{ki})), \]

\[ \dot{S}^a_{ij} = -p_{ij} S^a_{ij} + \sum_{c: \ c \neq a} \left( S^a_{ij}(S^c_{jj} - S^c_{ii}) E_i(z_{ac}) + \sum_{k \neq j} S^a_{ik} S^c_{kj} \phi(z_{ac}, q_{kj}) - \sum_{k \neq i} S^c_{kj} S^a_{ik} \phi(z_{ac}, q_{ik}) \right), \]

\[ \dot{S}^b_{ij} = S^a_{ij} (S^{ij,a}_{0,0} - S^{ii,a}_{0,0}) E_1(z_{ab}) + \sum_{k \neq i} S^b_{kj} S^a_{ik} \phi(z_{ba}, q_{ik}) - \sum_{k \neq j} S^b_{kj} S^a_{ik} \phi(z_{ba}, q_{ik}), \quad b \neq a \]

for \( H_{1,a} \). These equations of motion are equivalent to the Lax equations with an additional term,

\[ \frac{d}{dt_0} L(z) = [L(z), M_0(z)] + \frac{1}{2} \sum_{i,j=1}^M \sum_{a,b=1}^n S^b_{ij} (S^a_{ii} - S^a_{jj}) E_{ij} f'(z - z_b, q_{ij}), \]

\[ \frac{d}{dt_a} L(z) = [L(z), M_{1,a}(z)] + \sum_{i,j=1}^M \sum_{b=1}^n S^a_{ij} (S^b_{ii} - S^b_{jj}) E_{ij} f(z - z_a, q_{ij}), \]

where the corresponding \( M \)-matrices are

\[ M_0 = \delta_{ij} \sum_{a=1}^n S^a_{ii} \rho(z - z_a) + (1 - \delta_{ij}) \sum_{a=1}^n S^a_{ij} f(z - z_a, q_{ij}), \]

\[ M_{1,a} = -\delta_{ij} S^a_{ii} E_1(z - z_a) - (1 - \delta_{ij}) S^a_{ij} \phi(z - z_a, q_{ij}). \]

The additional terms disappear under the spin constrains, which for the multispin CM model take form

\[ \sum_{a=1}^n S^a_{ii} = \text{const}, \quad i = 1, M. \]

**4.3. Interacting tops.** For \( n = 1 \), the general \( gl_{NM} \) model degenerates into the elliptic \( gl_{NM} \) mixed-type model [13], [14] (see box 4 in Fig. 1). In this case, we have only one pole on the elliptic curve and therefore only one type of spin variables. However, in contrast to the CM spin model, the spin variables are matrix valued, which makes this model a top-like model.

The Lax matrix of the mixed-type model preserves the block structure,

\[ L(z) = \sum_{i,j=1}^M E_{ij} \otimes L^{ij}(z) \in \text{Mat}(NM, \mathbb{C}), \quad L^{ij}(z) \in \text{Mat}(N, \mathbb{C}), \]

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where each block is defined as

\[
\mathcal{L}^{ij}(z) = \delta_{ij} \left( p_i 1_N + S_{0,0}^{ii} 1_N E_1(z) + \sum_{\alpha \neq 0} S_{\alpha}^{ii} T_\alpha \varphi_\alpha (z, \omega_\alpha) \right) + \\
+ (1 - \delta_{ij}) \sum_{\alpha} S_{\alpha}^{ij} T_\alpha \varphi_\alpha \left( z, \omega_\alpha + \frac{q_{ij}}{N} \right).
\]

(4.23)

In the case of a single pole, the second term in expression (3.13) vanishes and we are left with the Hamiltonian

\[
H = \sum_{i=1}^{M} \frac{p_i^2}{2} - \frac{1}{2} \sum_{i=1}^{M} S_{\alpha}^{ii} S_{-\alpha}^{ii} E_2(\omega_\alpha) - \frac{1}{2} \sum_{i \neq j}^{M} \sum_{\alpha} S_{\alpha}^{ij} S_{-\alpha}^{ji} E_2 \left( \omega_\alpha + \frac{q_{ij}}{N} \right).
\]

(4.24)

The equations of motion for this Hamiltonian are of the form

\[
\dot{q}_i = p_i, \quad \dot{p}_i = -\frac{1}{N} \sum_{k: k \neq i}^{M} S_{k}^{i} S_{-k}^{i} E_2 \left( \omega_\alpha + \frac{q_{ik}}{N} \right),
\]

\[
\dot{S}_{\alpha}^{ij} = -\frac{1}{N} \sum_{\beta \neq 0} S_{\alpha - \beta}^{ij} (\kappa_{\alpha, \beta} S_{\beta}^{ij} + \kappa_{\beta, \alpha} S_{\beta}^{ii}) E_2(\omega_\beta) + \\
+ \frac{1}{N} \sum_{\beta} \left( \frac{M}{\sum_{k: k \neq i}^{M} \kappa_{\beta, \alpha} S_{\alpha - \beta}^{k} S_{\beta}^{kj} E_2 \left( \omega_\beta + \frac{q_{ik}}{N} \right) - \\
- \frac{M}{\sum_{k: k \neq i}^{M} \kappa_{\beta, \alpha} S_{\alpha - \beta}^{k} S_{\beta}^{kj} E_2 \left( \omega_\beta + \frac{q_{ik}}{N} \right)} \right).
\]

(4.25)

These equations are equivalent to the Lax equation for (4.23) with the \(M\)-matrix

\[
\mathcal{M}^{ij}(z) = \frac{\delta_{ij}}{N} S_{0,0}^{ii} 1_N z + \frac{\delta_{ij}}{N} \sum_{\alpha \neq 0} S_{\alpha}^{ij} T_\alpha f_\alpha (z, \omega_\alpha) + \\
+ \frac{1}{N} (1 - \delta_{ij}) \sum_{\alpha} S_{\alpha}^{ij} T_\alpha f_\alpha \left( z, \omega_\alpha + \frac{q_{ij}}{N} \right).
\]

(4.26)

on constraints (3.18), which now take the form \(\text{tr} S^{ii} = \text{const} \forall i\).

In the special case of rank \(S = 1\), the mixed-type model turns into the model of interacting tops [14] (box 7 in Fig. 1). For this model, after the reduction, the spin part of the phase space becomes isomorphic to a product of \(M\) minimal coadjoint orbits:

\[
\mathcal{O}_{NM}^{\min} // H_{NM} \cong \mathcal{O}_{N}^{\min} \times \cdots \times \mathcal{O}_{N}^{\min}.
\]

(4.27)

Comparing this with the integrable top model, we see that the interacting top model has the same phase space as \(M\) tops of the minimal orbit.

We now discuss how Hamiltonian (4.24) changes in the rank-1 case. As we have mentioned, for rank \(S = 1\) the spin variables can be parameterized as \(S^{ij} = \xi^i \eta^j\). Taking into account that \(S_{\alpha}^{ij} = \text{tr}(S^{ij} T_{-\alpha})/N\), we obtain

\[
S_{\alpha}^{ij} S_{-\alpha}^{ij} = \frac{\text{tr}(\eta^j T_{-\alpha} \xi^i)}{N^2} \frac{\text{tr}(\eta^j T_{-\alpha} \xi^i)}{N^2} = \frac{\text{tr}(\xi^j T_{-\alpha} \xi^i T_{\alpha})}{N^2} = \\
= \frac{\text{tr}(S^{ij} T_{-\alpha} S_{\alpha}^{ij} T_{\alpha})}{N^2} = \sum_{\beta} \frac{\kappa_{\alpha,\beta}^{ij} S_{\beta}^{ij} S_{-\beta}^{ii}}{N}.
\]

(4.28)
Substituting this expression in (4.24), we obtain the Hamiltonian of the form

\[
H^{\text{top}} = \sum_{i=1}^{M} \frac{p_i^2}{2} - \frac{1}{2} \sum_{i=1}^{M} \sum_{\alpha \neq 0} S_{ii}^\alpha S_{\alpha i}^\alpha E_2(\omega_\alpha) - \\
- \frac{1}{2N} \sum_{i \neq j}^{M} \sum_{\alpha,\beta} \kappa_{\alpha,\beta}^2 \delta_{ij}^\alpha \delta_{ij}^\beta E_2\left(\omega_\alpha + \frac{q_{ij}}{N}\right).
\]

(4.29)

This Hamiltonian has a clear physical interpretation: the first two terms describe the kinetic (and internal) energy of \(M\) tops and the last term can be interpreted as the interaction between the tops.

### 5. Generalized models: description in terms of \(R\)-matrices

In this section, we construct a generalization of the Lax pairs \(L(z), M_0(z)\) in Eqs. (3.7), (3.26) and \(L(z), M_{1,a}(z)\) in Eqs. (3.7), (3.32). This generalization is based on the \(R\)-matrix formulation. The Lax pairs can be written in terms of \(R\)-matrix data, and the Lax equations hold due to a set of identities.

The main identity for the \(R\)-matrix that we use is the associative Yang–Baxter equation

\[
R_{12}^z R_{23}^w = R_{13}^w R_{12}^{w-z} + R_{23}^{w-z} R_{13}^z, \quad R_{ab} = R_{ab}(q_a - q_b),
\]

(5.1)

where we use the standard tensor notation for \(R\)-matrices, which are assumed here to be in the fundamental representation of the \(GL_N\) Lie group. Formally, a solution of (5.1) is not a quantum \(R\)-matrix because the latter (by definition) satisfies the quantum Yang–Baxter equation

\[
R_{12}^z R_{23}^w = R_{23}^w R_{12}^{w-z}.
\]

(5.2)

The sets of solutions of (5.1) and (5.2) are different, although they have a nonzero overlap, which includes the elliptic quantum Baxter–Belavin \(GL_N\) \(R\)-matrix [6] (in the fundamental representation). We briefly describe it in the Appendix. It is easy to see that in the scalar case, Eq. (5.2) is an empty condition while (5.1) is a nontrivial functional equation, the genus-one Fay identity (A.10). In the general case, it can be shown that the solution of (5.1) that also satisfies the unitarity and skew-symmetry conditions is a true \(R\)-matrix, i.e., satisfies (5.2).

The similarity between the addition theorem for the \(\phi\)-function and Eq. (5.1) allows regarding the \(R\)-matrix as a noncommutative analogue of the elliptic Kronecker function. This leads to a set of \(R\)-matrix identities similar to those known for the ordinary scalar elliptic functions [17], [22]. We describe some of them below.

The \(R\)-matrix formulation of integrable tops was proposed in [18], and it was then shown in [23] that relation (5.1) underlies the Lax equations. In the elliptic case, the \(R\)-matrix formulation does not give rise to new models but reproduces those described in the previous sections. At the same time, this formulation allows including the trigonometric and rational degenerations of the described models into consideration.

Another application of the \(R\)-matrix identities comes from the above-mentioned treatment of the \(R\)-matrix as a matrix analogue of the \(\phi\)-function. This leads to \(R\)-matrix-valued Lax pairs [17]. Models of this type turn out to be closely related to the models of interacting tops [19]. More precisely, they are the models of interacting tops with the quantized spin part of the phase space, while the many-body degrees of freedom remain classical. This is also related to applications to the long-range spin chains [24]. We thus see that Eq. (5.1) unifies the quantum and classical integrable structures.
5.1. **R-matrix properties and identities.** The $R$-matrix has the local expansion near $z = 0$

\[ R^{z}_{12}(x) = \frac{1}{z} 1_N \otimes 1_N + r_{12}(x) + zm_{12}(x) + O(z^2), \] (5.3)

where $r_{12}$ is the classical $r$-matrix satisfying the **classical Yang–Baxter equation**

\[ [r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = 0, \quad r_{12} = r_{12}(q_1 - q_2), \] (5.4)

\[ r_{12}(z) = \frac{1}{z} P_{12} + r^{(0)}_{12} + zr^{(1)}_{12}(x) + O(z^2). \] (5.5)

In calculations, we also use degenerations of Eq. (5.1). In particular, we use the following identity, which can be regarded as “half” the classical Yang–Baxter equation

\[ r_{12}(x)r_{13}(x + y) - r_{23}(y)r_{12}(x) + r_{13}(x + y)r_{23}(y) = m_{12}(x) + m_{23}(y) + m_{13}(x + y). \] (5.6)

The last identity has the following degeneration:

\[ r_{12}(x)r_{13}(x) - r^{(0)}_{23} r_{12}(x) + r_{13}(x)r^{(0)}_{23} + \partial_x r_{13}(x) P_{23} = m_{12}(x) + m_{23}(0) + m_{13}(x). \] (5.7)

We also need the following expression obtained by taking three consecutive limits ($z \to 0$, $q_3 \to 0$, $q_2 \to 0$) of (5.1):

\[ [r_{12}(x), m_{13}(x)] = [r^{(0)}_{23}, m_{12}(x)] + [r^{(0)}_{23}, m_{13}(x)] + [m_{23}(0), r_{12}(x)] + [\partial_x m_{12}(x), P_{23}]. \] (5.8)

We use the $R$-matrix that satisfies the following set of properties.

1) **Expansion near** $x = 0$:

\[ R^{z}_{12}(x) = \frac{1}{x} P_{12} + R^{z,(0)}_{12} + xR^{z,(1)}_{12} + O(x^2), \] (5.9)

where $P_{12}$ is the permutation operator,

\[ P_{12} = \frac{1}{N} \sum_\alpha T_\alpha \otimes T_{-\alpha}. \] (5.10)

2) The Fourier symmetry

\[ R^{z}_{12}(x) P_{12} = R^{z}_{12}(z). \] (5.11)

3) Unitarity

\[ R^{z}_{12}(x) R^{z}_{21}(-x) = (\varphi(z) - \varphi(x))1_N \times 1_N. \] (5.12)

4) Skew-symmetry

\[ R^{z}_{12}(x) = - R^{z}_{21}(-x), \quad r_{12}(z) = - r_{21}(-z), \]

\[ r^{(0)}_{12} = - r^{(0)}_{21}, \quad m_{12}(z) = m_{21}(-z). \] (5.13)

\[ ^2\text{“Half” refers to the fact that taking the difference of two such equations does yield the classical Yang–Baxter equation.} \]
From these properties, we deduce the following identities for the coefficients of expansions (5.3), (5.4), and (5.8):
\[
R_{12}^{z(0)} = r_{12}^{(0)} P_{12}, \quad r_{12}^{(0)} = r_{12}^{(0)} P_{12}, \\
R_{12}^{z(1)} = m_{12}^{(0)} P_{12}, \quad r_{12}^{(1)} = m_{12}^{(0)} P_{12}.
\] (5.13)

The special notation is used for the derivative of the \( R \)-matrix
\[
F_{12}^{z}(q) = \partial_q R_{12}^{z}(q). 
\] (5.14)

We list other degenerations of associative Yang–Baxter equation (5.1):
\[
R_{12}^{z-za}(x) R_{23}^{z-zyb}(0) = R_{13}^{z-za}(x) R_{12}^{za}(x) + R_{23}^{zyb}(0) R_{13}^{z-za}(x) + P_{23} R_{13}^{z-za}(x), \\
R_{12}^{z-za}(0) R_{23}^{z-zyb}(x) = R_{13}^{z-za}(x) R_{12}^{za}(0) + R_{23}^{zyb}(x) R_{13}^{z-za}(x) + F_{13}^{z-za}(x) P_{12}, \\
R_{12}^{z-za}(x) R_{23}^{z-zyb}(-x) = R_{13}^{z-za}(0) R_{12}^{za}(x) + R_{23}^{zyb}(-x) R_{13}^{z-za}(0) + F_{32}^{za}(x) P_{13}. 
\] (5.15)

Differentiating the last expression with respect to \( x \) yields
\[
F_{12}^{z-za}(x) R_{23}^{z-zyb}(-x) - R_{12}^{z-za}(x) F_{23}^{z-zyb}(-x) = \\
= R_{13}^{z-za}(0) F_{12}^{za}(x) - F_{23}^{zyb}(-x) R_{13}^{z-za}(0) + \partial_x F_{32}^{za}(x) P_{13}. 
\] (5.16)

Again, by differentiating the associative Yang–Baxter equation (5.1) with respect to \( q_2 \) and taking the limit \( q_3 \to q_2 \), we obtain
\[
R_{12}^{z-za}(x) R_{23}^{z-zyb}(1) - F_{12}^{z-za}(x) R_{23}^{z-zyb}(0) = \\
= F_{23}^{zyb}(1) R_{13}^{z-za}(x) - R_{13}^{z-za}(x) F_{12}^{za}(x) - \frac{1}{2} P_{23} \partial_x F_{13}^{z-za}(x), 
\] (5.17)

while in the limit \( q_1 \to q_2 \),
\[
R_{12}^{z-za}(0) F_{23}^{z-zyb}(x) - R_{12}^{z-za}(1) R_{23}^{z-zyb}(x) = \\
= F_{23}^{zyb}(x) R_{13}^{z-za}(x) - R_{13}^{z-za}(x) R_{12}^{za}(1) + \frac{1}{2} \partial_x F_{13}^{z-za}(x) P_{12}. 
\] (5.18)

Finally, we assume the following \( R \)-matrix traces:
\[
tr_1 R_{12}^{z}(x) = tr_2 R_{12}^{z}(x) = \phi(z, x) 1_N, \\
tr_1 r_{12}(x) = E_1(x) 1_N, \quad tr_1 m_{12}(x) = \rho(x) 1_N. 
\] (5.19)

**5.2. Lax matrix and Hamiltonians.** Following [17], [23], we recall the \( R \)-matrix formulation of the integrable top. The inverse tensor of inertia is of the form
\[
J(S) = tr_2 (m_{12}(0) S_2), \quad S_2 = 1_N \otimes S, 
\] (5.20)

and the corresponding Hamiltonian is given by
\[
H^{\text{top}} = \frac{1}{2} tr_{12}(m_{12}(0) S_1 S_2). 
\] (5.21)

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Equations of motion (4.8) for the $J$ tensor in (5.20) are equivalent to the Lax equation with the Lax pair

$$L(z, S) = \text{tr}_2(r_{12}(z)S_2), \quad M(z, S) = \text{tr}_2(m_{12}(z)S_2).$$

(5.22)

The case of the previously described elliptic $gl_N$ top model corresponds to the Baxter–Belavin $R$-matrix. Substituting the corresponding $r$ and $m$ matrices in (5.20)–(5.22), we exactly obtain expressions (4.7)–(4.10) (up to constants not included in the equations of motion; see the Appendix for details).

In the general case of the $gl_{NM}^{\times n}$ model, we deal with the Lax pair for the mixed-type model introduced in [19]. We extend it to the case of multiple poles. The Lax matrix still has a block structure and the size $NM \times NM$. It takes the form

$$\mathcal{L}(z) = \sum_{i,j=1}^{M} E_{ij} \otimes \mathcal{L}^{ij}(z) \in \text{Mat}(NM, \mathbb{C}), \quad \mathcal{L}^{ij}(z) \in \text{Mat}(N, \mathbb{C}),$$

(5.23)

Here, $P_{12}$ is the permutation operator in $\text{Mat}(N, \mathbb{C})^{\otimes 2}$. In terms of the matrix basis $T_\alpha$, it is $P_{12} = 1/N \sum_\alpha T_\alpha \otimes T_{-\alpha}$.

We evaluate the Hamiltonians:

$$\frac{1}{2N} \text{tr} \mathcal{L}^2(z) = H_0 + \sum_{a=1}^{n} (H_{1,a}E_1(z - z_a) + H_{2,a}E_2(z - z_a)) + \frac{1}{N^2} \sum_{i,a,b} \text{tr}(S^{i,a}_1)\text{tr}(S^{i,b}_1)\rho(z - z_a).$$

(5.24)

The last term here is treated in the same way as for the elliptic model. We then obtain the Hamiltonians

$$H_0 = \sum_{i=1}^{M} \frac{p_i^2}{2} + \frac{1}{2} \sum_{i=1}^{M} \sum_{a,b} \text{tr}_{12}(S_{1}^{i,a} S_{2}^{i,b} m_{12}(z_{ab})) + \frac{1}{2} \sum_{i,j: \ i \neq j} \sum_{a,b} \text{tr}_{12}(S_{1}^{i,a} S_{2}^{i,b} F_{21}(z_{ab}) (q_{ij})P_{12}),$$

(5.25)

$$H_{1,a} = \sum_{i=1}^{M} \frac{p_i}{N} \text{tr}(S^{i,a}_1) + \frac{1}{N} \sum_{i=1}^{M} \sum_{b: \ b \neq a} \text{tr}_{12}(S_{1}^{i,a} S_{2}^{i,b} r_{12}(z_{ab})) + \frac{1}{N} \sum_{i,j: \ i \neq j} \sum_{b: \ b \neq a} \text{tr}_{12}(S_{1}^{i,a} S_{2}^{i,b} R_{12}^{ab} (q_{ij})P_{12}),$$

(5.26)

$$H_{2,a} = \frac{1}{2N} \sum_{i,j} \text{tr}_{12}(S_{1}^{i,a} S_{2}^{i,a} P_{12}).$$

(5.27)

These are generalized formulas for the Hamiltonians of the $gl_{NM}^{\times n}$ model. Indeed, substituting the Belavin–Baxter $R$-matrix in expressions (5.25)–(5.27), we obtain Hamiltonians (3.15)–(3.17).
The nonreduced Poisson structure for the spin variables, Eqs. (3.12), and the moment map in Eq. (3.18) remain the same, but for our purposes it is more convenient to write them in terms of Mat($N, \mathbb{C}$)-valued blocks $S^{ij}$. Then the Poisson brackets become

$$\{S^{ij,a}, S^{km,b}\} = \delta^{ab}(\delta^{im}P_{12}S^{kj,a} - \delta^{kj}S^{im,a}P_{12}).$$  \hspace{1cm} (5.28)

As in the scalar case (3.17), Hamiltonian (5.27) turns out to be a Casimir function. The moment map can be represented as follows:

$$\text{tr} \left( \sum_{a=1}^{n} S^{kk,a} \right) = \text{const} \quad \forall k = 1, \ldots, M. \hspace{1cm} (5.29)$$

5.2.1. Lax pair for the flow of the Hamiltonian $H_0$. We apply the same procedure as we did in the scalar case in Sec. 3. Using Poisson brackets (2.1) and (5.28) for the nonreduced spin part of the phase space, we obtain the following equations of motion for the $H_0$ flow:

$$\dot{q}_i = p_i, \quad \dot{p}_i = -\sum_{k: k \neq i} \sum_{a,b} \text{tr}_{12}(S^{ik,a}S^{kj,b}m_{12}(z_{ab})) + \sum_{k: k \neq j} \sum_{a,b} \text{tr}_{12}(S^{kj,b}F^{z_{ab}}_{12}(q_{kj})P_{12}) - \sum_{k: k \neq i} \sum_{a,b} \text{tr}_{12}(S^{ij,a}F^{z_{ab}}_{12}(q_{ij})P_{12})S^{kj,a}. \hspace{1cm} (5.30)$$

As the $M$-matrix, we take

$$M_0(z) = \sum_{i,j=1}^{M} E_{ij} \otimes M_{0}^{ij}(z) \in \text{Mat}(NM, \mathbb{C}), \quad M_{0}^{ij}(z) \in \text{Mat}(N, \mathbb{C}), \hspace{1cm} (5.31)$$

Then the following statement holds.

**Proposition 3.** Equations of motion (5.30) are equivalent to the Lax equations with an additional term

$$\frac{d}{dt}L(z) = [L(z), M_0(z)] + \frac{1}{2} \sum_{i,j=1}^{M} \sum_{a,b=1}^{n} E_{ij} \otimes \text{tr}_{23}(S^{ij,b}(S^{ji,a} - S^{ji,a}) \partial_{q_{ij}}F^{z_{ij}}_{12}(q_{ij})P_{12}). \hspace{1cm} (5.32)$$

The additional term vanishes on constraints (5.29).

We note all equations of motion (5.30) and the $M_0$ matrix in (5.31) reproduce the corresponding equations of motion in the scalar case, Eqs. (3.21), (3.22), and the $M_0$-matrix in (3.26) for the Baxter–Belavin $R$-matrix. The proof of the above statement relies on the set of the $R$-matrix identities described above.
5.2.2. Lax pairs for the flows of Hamiltonians $\mathcal{H}_{1,a}$. We consider the $\mathcal{H}_{1,a}$ Hamiltonian flow. Again, we start from the nonreduced spin part of the phase space and use Poisson structure (2.1), (5.28) to obtain the equations of motion

$$\dot{q}_i = \frac{1}{N} \text{tr}(S^{ij,a}), \quad \dot{p}_i = \frac{1}{N} \sum_{k : k \neq i} \sum_{b : b \neq a} \text{tr}_{12}(S^{ik,a}_1 S^{ki,b}_2) \partial_{q^i_r} F^{z_{12} a}_{21}(q_{ik}) P_{12} -$$

$$- S^{ik,b}_2 \partial_{q^i_r} F^{z_{12} b}_{21}(q_{ik}) P_{12},$$

$$\dot{S}^{ij,a} = -\frac{p_{ij}}{N} S^{ij,a} + \frac{1}{N} \sum_{b : b \neq a} (S^{ij,a} \text{tr}_{2}(S^{ij,b}_2 r_{12}(z_{ab})) - \text{tr}_{2}(S^{ij,b}_2 r_{12}(z_{ab}))) S^{ij,a} +$$

$$+ \frac{1}{N} \sum_{b : b \neq a} \left( \sum_{k : k \neq j} S^{ik,a} \text{tr}_{2}(S^{kj,b}_2 R^{z_{ab}}_{12}(q_{ik}) P_{12}) -$$

$$- \sum_{k : k \neq j} \text{tr}_{2}(S^{ik,b}_2 R^{z_{ab}}_{12}(q_{ik}) P_{12}) S^{kj,a} \right), \quad (5.33)$$

$$\dot{S}^{ij,b} = \frac{1}{N} (\text{tr}_{2}(S^{ij,a}_2 r_{12}(z_{ba}))) S^{ij,b} - S^{ij,b} \text{tr}_{2}(S^{ij,a}_2 r_{12}(z_{ba})) +$$

$$+ \frac{1}{N} \left( \sum_{k : k \neq i} \text{tr}_{2}(S^{ik,a}_2 R^{z_{ab}}_{12}(q_{ik}) P_{12}) S^{kj,b} -$$

$$- \sum_{k : k \neq j} S^{ik,b}_2 \text{tr}_{2}(S^{kj,a}_2 R^{z_{ab}}_{12}(q_{kj}) P_{12}) \right), \quad b \neq a.$$  

As in the preceding case, these equations transform into the equations of motion for the scalar case (3.28) when we choose the $R$-matrix to be the Baxter–Belavin one. The same is true for the $M$-matrix

$$\mathcal{M}_{1,a}(z) = \sum_{i,j=1}^{M} E_{ij} \otimes M^{ij}_{1,a}(z) \in \text{Mat}(NM, \mathbb{C}), \quad M^{ij}_{1,a}(z) \in \text{Mat}(N, \mathbb{C}),$$

$$M^{ij}_{1,a}(z) = -\frac{\delta_{ij}}{N} \text{tr}_{2}(S^{ij,a}_2 r_{12}(z - z_a)) - \frac{1}{N} (1 - \delta_{ij}) \text{tr}_{2}(S^{ij,a}_2 R^{z_{12} - z_a}_{12}(q_{ij}) P_{12}). \quad (5.34)$$

Then the following statement holds.

**Proposition 4.** Equations of motion (5.33) can be equivalently written in the form of the Lax equation with an additional term

$$\dot{L}(z) = [L(z), \mathcal{M}_{1,a}(z)] + \frac{1}{N} \sum_{i,j=1}^{M} \sum_{b=1}^{n} E_{ij} \otimes \text{tr}_{23}(S^{ij,a}_2 (S^{ij,b}_3 - S^{ji,b}_3) F^{z_{12} - z_a}_{12}(q_{ij}) P_{12}), \quad (5.35)$$

where the dot is the derivative with respect to the time variable $t_{1,a}$. The additional term vanishes on constraints (5.29).

6. Schlesinger systems

Schlesinger systems on elliptic curves [25] can be treated as a nonautonomous generalization of the Gaudin models. The positions of marked points $z_a$ and the elliptic modular parameter $\tau$ become the time variables related to the respective Hamiltonians $H_a$ and $H_0$.  

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Lax equations (2.6) are replaced with monodromy-preserving equations having the form of zero-curvature equations. It turns out that these last equations can be formulated in terms of the Lax pair of the corresponding Gaudin model. The phenomenon is known as the classical Painlevé–Calogero correspondence [26]. For example, we consider the Lax pair of the CM model, Eqs. (2.4) and (2.5). Then the monodromy-preserving equation

\[ 2\pi i \frac{d}{d\tau} L_{\text{CM}}(z) - \frac{d}{dz} M_{\text{CM}}(z) = [L_{\text{CM}}(z), M_{\text{CM}}(z)] \] (6.1)

is equivalent to the nonautonomous equations

\[ \frac{d^2}{d\tau^2} q_i = \nu^2 \sum_{k \neq i} \varphi'(q_{ik}). \] (6.2)

The derivation of the last statement is almost the same as for Lax equation (2.6). The only additional tool is the heat equation

\[ 2\pi i \partial_\tau \phi(z,u) = \partial_z \partial_u \phi(z,u). \] (6.3)

This means that the Painlevé–Calogero correspondence is a gauge-dependent phenomenon, once it is based on a special choice of the gauge and normalization of the Lax pair, such that the matrix elements of the Lax matrix are given by the \( \phi \) functions. This is the case we are dealing with in this paper.

Similarly, for the Gaudin model, we have

\[ \frac{d}{dz} a L(z) - \frac{d}{dz} M_a(z) = [L(z), M_a(z)]. \] (6.4)

The Lax pairs described in the preceding sections can be straightforwardly used for the construction of Schlesinger systems. In fact, this statement is known for generic elliptic models related to bundles with arbitrary characteristic class [27]. For example, for the most general elliptic model, we have the following statement.

**Proposition 5.** Equations of motion (3.21)–(3.24), where the dot denotes the derivative with respect to \( \tau \), are equivalent to the monodromy-preserving equations with an additional term

\[ 2\pi i \frac{d}{d\tau} L(z) - \partial_z M_0(z) = [L(z), M_0(z)] + \frac{1}{2N} \sum_{i,j=1}^{M} \sum_{a,b=1}^{n} \sum_{\alpha} \mathcal{S}^i_j{}_{a}^{b} (\mathcal{S}^{ii,a}_{0,0} - \mathcal{S}^{ij,b}_{0,0}) \mathcal{E}_{ij} \otimes T_{\alpha} f'_{\alpha} \left( z - z_b, \omega_\alpha + \frac{q_{ij}}{N} \right). \] (6.5)

The additional term vanishes on constraints (3.18).

**Proposition 6.** Equations of motion (3.28), (3.29) can be equivalently written in the form of the monodromy-preserving equation with an additional term

\[ \frac{d}{dz_a} L(z) - \partial_{z_a} M_{1,a}(z) = [L(z), M_{1,a}(z)] + \frac{1}{N} \sum_{i,j=1}^{M} \sum_{\alpha} \mathcal{S}^i_j{}_{a}^{b} (\mathcal{S}^{ii,a}_{0,0} - \mathcal{S}^{ij,b}_{0,0}) \mathcal{E}_{ij} \otimes T_{\alpha} f_{\alpha} \left( z - z_a, \omega_\alpha + \frac{q_{ij}}{N} \right), \] (6.6)

where the dot in the equations of motion is the derivative with respect to \( z_a \). The additional term vanishes on constraints (3.18).

For models written in the R-matrix formulation, all statements are the same if the heat equation

\[ 2\pi i \partial_{\tau} R_{12}^a(u) = \partial_{z_a} \partial_{u} R_{12}^a(u) \] (6.7)

holds. This is the case for the Baxter–Belavin R-matrix.
Appendix: Elliptic functions

The basic element for the construction of Lax pairs is the Kronecker elliptic function [28]

\[ \phi(z, u) = \frac{\vartheta'(0)\vartheta(z + u)}{\vartheta(z)\vartheta(u)}, \]  

(A.1)

defined in terms of the odd Riemann theta function

\[ \vartheta(z) = \vartheta(z|\tau) = -\sum_{k \in \mathbb{Z}} \exp\left(\pi i \tau \left(k + \frac{1}{2}\right)^2 + 2\pi i\left(z + \frac{1}{2}\right)\left(k + \frac{1}{2}\right)\right). \]  

(A.2)

Function (A.1) has the obvious properties

\[ \phi(z, u) = \phi(u, z), \quad \phi(-z, -u) = -\phi(z, u). \]  

(A.3)

We also need the derivative \( f(z, u) = \partial_u \varphi(z, u) \) given by

\[ f(z, u) = \phi(z, u)(E_1(z + u) - E_1(u)), \quad f(-z, -u) = f(z, u), \]  

(A.4)

where (the first and the second) Eisenstein functions are

\[ E_1(z) = \partial_z \ln \vartheta(z), \quad E_2(z) = -\partial_z E_1(z) = \vartheta(z) - \frac{\vartheta''(0)}{3\vartheta'(0)}, \]

\[ E_1(-z) = E_1(z), \quad E_2(-z) = E_2(z). \]

(A.5)

For these functions, the following local expansions hold near \( z = 0 \):

\[ \phi(z, u) = \frac{1}{z} + E_1(u) + z \rho(u) + O(z^2), \]  

(A.6)

\[ f(0, u) = -E_2(u), \]  

(A.7)

where we use the notation

\[ \rho(z) = \frac{E_1^2(z) - \vartheta(z)}{2}. \]  

(A.8)

We have the quasiperiodic behavior the lattice of periods 1 and \( \tau \)

\[ E_1(z + 1) = E_1(z), \quad E_1(z + \tau) = E_1(z) - 2\pi i, \]

\[ E_2(z + 1) = E_2(z), \quad E_2(z + \tau) = E_2(z), \]

\[ \phi(z + 1, u) = \phi(z, u), \quad \phi(z + \tau, u) = e^{-2\pi i u} \phi(z, u), \]  

(A.9)

\[ f(z + 1, u) = f(z, u), \quad f(z + \tau, u) = e^{-2\pi i u}(f(z, u) - 2\pi i \phi(z, u)). \]
The addition formula and its degenerations are

\[
\begin{align*}
\phi(z_1, u_1) \phi(z_2, u_2) &= \phi(z_1, u_1 + u_2) \phi(z_2 - z_1, u_2) + \phi(z_2, u_1 + u_2) \phi(z_1 - z_2, u_1), \\
f(z_1, u_1) \phi(z_2, u_2) - \phi(z_1, u_1) f(z_2, u_2) &= \phi(z_1, u_1 + u_2) f(z_2, u_2) - \phi(z_1, u_1 + u_2) f(z_2, u_2), \\
f(z, u_1) \phi(z, u_2) - \phi(z, u_1) f(z, u_2) &= \phi(z, u_1 + u_2) (E_2(u_2) - E_2(u_1)), \\
\phi(z, u) \phi(z, -u) &= E_2(z) - E_2(u) = \phi(z) - \phi(u), \\
\phi(z, u_1) \phi(z, u_2) &= \phi(z, u_1 + u_2) (E_1(z) + E_1(u_1) + E_1(u_2) - E_1(z + u_1 + u_2)), \\
\phi(z_1, u) \phi(z_2, u) &= \phi(z_1 + z_2, u) (E_1(z_1) + E_1(z_2)) - f(z_1 + z_2, u), \\
\phi(z_1, u) \rho(z_2) - E_1(z_2) f(z_1, u) + \phi(z_2, u) f(z_1, u) - \phi(z_1, u) \rho(z_2) = \frac{1}{2} \partial_u f(z_1, u), \\
(E_1(u + v) - E_1(u)) (E_1(v) - E_1(u)) = \varphi(u + v) + \varphi(u) + \varphi(v), \\
\phi(z, u) \rho(z) - E_1(z) f(z, u) - \phi(z, u) \rho(u) = \frac{1}{2} \partial_u f(z, u).
\end{align*}
\]

Using the Kronecker elliptic function and its derivative, we define the functions

\[
\begin{align*}
\varphi_\alpha(z, \omega_\alpha + u) &= e^{2 \pi i a z / N} \phi(z, \omega_\alpha + u), \\
\omega_\alpha &= \frac{a_1 + a_2 \tau}{N}, \\
f_\alpha(z, \omega_\alpha + u) &= e^{2 \pi i a z / N} f(z, \omega_\alpha + u), \\
f_\alpha(z, \omega_\alpha + u) &= \partial_u \varphi_\alpha(z, \omega_\alpha + u) = \varphi_\alpha(z, \omega_\alpha + u) (E_1(z + \omega_\alpha + u) - E_1(\omega_\alpha + u)).
\end{align*}
\]

Functions (A.19) are elements of a basis in the space of sections of the End(V) for a holomorphic vector bundle V (over the elliptic curve) of degree 1.

The addition formulas for the basis functions take the form

\[
\varphi_\alpha(z_1, \omega_\alpha + u_1) \varphi_\beta(z_2, \omega_\beta + u_2) = \varphi_\alpha(z_1 - z_2, \omega_\alpha + u_1) \varphi_\alpha + \beta(z_2, \omega_\alpha + \omega_\beta + u_1 + u_2) + \varphi_\beta(z_2 - z_1, \omega_\beta + u_1) \varphi_\alpha + \beta(z_1, \omega_\alpha + \omega_\beta + u_1 + u_2),
\]

In particular,

\[
\begin{align*}
\varphi_\alpha(z - z_\alpha, \omega_\alpha) \varphi_\beta(z - z_\beta, \omega_\beta) &= \varphi_\alpha(z_\alpha, \omega_\alpha) \varphi_\alpha + \beta(z - z_\alpha, \omega_\alpha + \omega_\beta) + \varphi_\beta(z_\beta, \omega_\beta) \varphi_\alpha + \beta(z - z_\alpha, \omega_\alpha + \omega_\beta), \\
\varphi_\alpha(z, \omega_\alpha + u_1) \varphi_\beta(z, \omega_\beta + u_2) &= \varphi_\alpha + \beta(z, \omega_\alpha + \omega_\beta + u_1 + u_2) \times \\
&\times (E_1(z) + E_1(\omega_\alpha + u_1) + E_1(\omega_\beta + u_2) - E_1(z + \omega_\alpha + \beta + u_1 + u_2)), \\
\varphi_\alpha(z_1, \omega_\alpha + u) \varphi_\alpha(z_2, \omega_\alpha + u) &= \varphi_\alpha(z_1 + z_2, \omega_\alpha + u) (E_1(z_1) + E_1(z_2)) - f_\alpha(z_1 + z_2, \omega_\alpha + u).
\end{align*}
\]

The Baxter–Belavin elliptic R-matrix [6]

\[
R_{12}^{BB}(z, x) = \sum_\alpha \varphi_\alpha(x, z + \omega_\alpha) T_\alpha \otimes T_{-\alpha} \in \text{Mat}(N, \mathbb{C})^{\otimes 2}
\]

satisfies all the required properties (5.3)–(5.19), but with a different normalization. We use an R-matrix slightly different from (A.26) to ensure that all properties hold with the correct normalization coefficients:

\[
R_{12}(x) = R_{12}^{BB} \left( \frac{z}{N}, x \right) = \frac{1}{N} \sum_\alpha \varphi_\alpha \left( x, \frac{z}{N} + \omega_\alpha \right) T_\alpha \otimes T_{-\alpha}.
\]
Using (A.5) and (5.3), we obtain the corresponding classical $r$- and $m$-matrices

$$r_{12}(z) = \frac{1}{N} E_1(z) 1_N \otimes 1_N + \frac{1}{N} \sum_{\alpha \neq 0} \varphi_\alpha(z, \omega_\alpha) T_\alpha \otimes T_{-\alpha},$$

$$m_{12}(z) = \frac{1}{N^2} \rho(z) 1_N \otimes 1_N + \frac{1}{N^2} \sum_{\alpha \neq 0} f_\alpha(z, \omega_\alpha) T_\alpha \otimes T_{-\alpha}.$$  \hfill (A.28)

Taking the derivative of the $r$-matrix gives

$$F_{12}^0(z) = \partial_z r_{12}(z) = -\frac{1}{N} E_2(z) 1_N \otimes 1_N +$$

$$+ \frac{1}{N^2} \sum_{\alpha \neq 0} \varphi_\alpha(z, \omega_\alpha)(E_1(z + \omega_\alpha) - E_1(z) + 2\pi i \partial_z \omega_\alpha) T_\alpha \otimes T_{-\alpha}.$$  \hfill (A.29)

The following identity for elliptic functions (finite Fourier transformation) is useful in proving the Fourier symmetry in (5.10) and other identities:

$$\frac{1}{N} \sum_{\alpha} \kappa^2_{\alpha,\beta} \varphi_\alpha(z, \omega_\alpha + \frac{z}{N}) = \varphi_\beta(z, \omega_\beta + x) \quad \forall \beta \in \mathbb{Z}_N \times \mathbb{Z}_N.$$  \hfill (A.30)

Its special cases are

$$\sum_{\alpha} E_2(\omega_\alpha + x) = N^2 E_2(N x)$$  \hfill (A.31)

and

$$\sum_{\alpha} \kappa^2_{\alpha,\beta} \varphi_\alpha(x, \omega_\alpha)(E_1(z + \omega_\alpha) - E_1(z) + 2\pi i \partial_z \omega_\alpha) - E_2(x) = -E_2\left(\omega_\beta + \frac{x}{N}\right).$$  \hfill (A.32)

Conflicts of interest. The authors declare no conflicts of interest.

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