Three Topics on Non-decomposability of Generalized Multiplicative Connectives

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Abstract

Danos and Regnier introduced generalized (non-binary) multiplicative connectives in (Danos and Regnier, 1989). They showed that there exist the generalized multiplicative connectives which cannot be defined by any combinations of the tensor and par rules in the multiplicative fragment of linear logic. These connectives are called non-decomposable generalized multiplicative connectives (ibid, p.192). In this short note, we investigate the notion of Danos and Regnier’s non-decomposability and give three results concerning (non-)decomposability of generalized multiplicative connectives.

1 Introduction

Danos and Regnier introduced generalized (non-binary) multiplicative connectives in [2]. They showed that there exist the generalized connectives which cannot be defined by any combinations of the tensor and par rules in the multiplicative fragment of linear logic that satisfy the main reduction step of the cut elimination theorem. These connectives are called non-decomposable generalized multiplicative connectives. Danos and Regnier defined generalized multiplicative connectives only for classical multiplicative linear logic. In this short note, we investigate the (non-)decomposability of generalized multiplicative connectives in the other fragments of linear logic and give three results concerning (non-)decomposability of generalized multiplicative connectives. In the first part of this note, we define intuitionistic generalized multiplicative connectives by employing the polarities which is inspired by [1]. We show that all of intuitionistic generalized multiplicative connectives in Intuitionistic Multiplicative Linear Logic IMLL are decomposable. Hence, there are no non-decomposable multiplicative connectives in IMLL. Secondly, we generalize the definition of Danos and Regnier’s (non-)decomposability and show that all of non-decomposable generalized multiplicative connectives are decomposable in Multiplicative Additive Linear Logic
MALL and Multiplicative Exponential Linear Logic MELL. Finally, we show that elementary light linear logic EMLL preserves non-decomposability of generalized multiplicative connectives.

2 Decomposability of intuitionistic generalized connectives

In this section, we define an intuitionistic generalized connective by modifying the definition of the (classical) generalized multiplicative connective. The main result of this section is as follows; all of intuitionistic generalized connectives in IMLL are decomposable. This result says that there are no non-decomposable generalized connectives in IMLL. We assume the knowledges of generalized multiplicative connectives (see [2, 8]).

In the following part, we define intuitionistic generalized connectives.

First, we define a polarized partition, then we define an intuitionistic polarized partition and an intuitionistic polarized meeting graph.

**Definition 1.** A polarity of a natural number s is positive if it is not checked and negative if it is checked. A partition p of a natural number n is a polarized partition if p takes a positive or negative polarity.

In IMLL, a partition set $P^C$ of a generalized connective $C$ corresponds to left-one-sided sequents. A formula $A_n$ which is in right hand side of a sequent corresponds to the checked number $\hat{n}$. Hence, a sequent of IMLL $A_1, \ldots, A_{n_1}, \sim A_n \vdash$ corresponds to the partition $\{1, \ldots, n - 1, \hat{n}\}$. When an atomic formula (a metavariable) $A$ has positive (resp. negative) polarity, we often denote $A$ as $A^+$ (resp. $A^-$).

**Definition 2.** A polarized partition p is an intuitionistic polarized partition if each class of p contains at most one checked element.

An intuitionistic polarized meeting graph $\mathcal{H}(p, q)$ is a polarized meeting graph which is obtained from two intuitionistic polarized partitions $p$ and $q$ in the following way; we draw an edge between two nodes $N \in p$ and $N' \in q$ if two numbers $n \in N$ and $n' \in N'$ are the same and one of them is checked and the other is not.

**Definition 3.** A n-ary intuitionistic generalized connectives $C$ is a pair of finite intuitionistic partition set of a natural number n $(P_L, P_R)$ such that $(P_R)^\perp = P_L$ and $(P_L)^\perp = P_R$ hold.
The definition of (non-)decomposability for the intuitionistic generalized multiplicative connectives is similar for the (classical) generalized multiplicative connectives.

The general form of a right rule of the (two-sided) intuitionistic generalized multiplicative connectives is as follows;

\[
\Gamma_1, A_{11}, \ldots, A_{1m_1-1}, \Gamma_k, A_{k1}, \ldots, A_{km_k-1} \vdash A_{km_k}
\]

where \( \Gamma = \bigcup_{i=1}^k \Gamma_i \) and \( jm_j \in \{1, \ldots, n\} \).

We show that all of intuitionistic generalized multiplicative connectives are decomposable.

**Theorem 1.** Let \( C \) be an arbitrary intuitionistic multiplicative generalized connective. Then, \( C \) is IMLL-decomposable.

**Proof.** Let \( L_1, \ldots, L_k \) be an enumeration of left introduction rules of \( C \) and \( R_1, \ldots, R_m \) be that of right introduction rules. The main reduction step of the cut-elimination for a \( C \)-connective holds if and only if the two intuitionistic partition sets of \( C \), \( (P_L, P_R) \) are orthogonal. This equivalence is proved as [2, Lemma 2]. By definition of \( C \), \( C \) satisfies the main reduction step of the cut-elimination. We prove this theorem by case analysis on the number and the form of right introduction rules.

Case \( #P_R = 1 \): \( C \) is decomposed as follows. For each class \( x_i \) \( (i = 1, \ldots, k) \), we apply the left \( \otimes \)-rules as much as possible. Then, we use \( \not\to \) and obtain a formula \( \alpha_i \). We connect these formulas by the tensor rules \( \alpha = \alpha_1 \otimes \ldots \otimes \alpha_k \). In the following, we call this form of formulas normal. By construction, \( P_\alpha = P_\beta \) holds. Hence, \( C \) is decomposable.

Case \( #P_R > 1 \): Consider two arbitrary right rules of \( C \) (say, \( R_1, R_2 \in \{R_1, \ldots, R_k\} \)).

Case 1: For some formula \( A \), a polarity of \( A \) is changed between \( R_1 \) and \( R_2 \) (say, \( A^+ \in R_1 \) and \( A^- \in R_2 \)). In this case, the cut-elimination does not hold. For example, let \( R_1 = \{(A, B \vdash E) \ (C, D \vdash F)\} \) and \( R_2 = \{(F, B \vdash E) \ (C, D \vdash A)\} \). The dual of \( R_1 \) should be \( L_1 = \{(F \vdash (\not\to A) \cdot \cdot \cdot )\} \), where we omitted irrelevant formulas. The meeting graph between \( R_2 \) and \( L_1 \) is cyclic and the cut-elimination does not hold.

Case 2: For all formulas \( A_i \), a polarity of \( A_i \) does not change in \( R_k \). In this case, if we apply the right \( \otimes \)-rules as much as possible, we obtain one sequent. After that, we apply the left \( \otimes \)-rules as much as possible and then apply one right \( \not\to \)-rule. By this construction, we can obtain the same formula \( \alpha \) from \( R_k \) for each \( k \). We show that \( \alpha \) is the decomposition of \( C \). We assume that \( #P_\alpha > #P_R \) and show a contradiction. Consider arbitrary two elements \( p \in (P_\alpha \setminus P_R) \) and \( q \in P_\sim \alpha \).
By the relation $P_\alpha \supset P_R, P_L \supset P_{\neg \alpha}$ holds. Hence $q \in P_L$ holds. It contradicts with the definition of an intuitionistic generalized connective $(P_L)^\bot = P_R$ because $p \perp q$ and $p \notin P_R$ hold. Therefore, $\#P_\alpha = \#P_R$ holds. □

**Corollary 1.** There are no non-decomposable multiplicative connectives in IMLL.

### 3 Decomposability of generalized multiplicative connectives in MALL and MELL

Danos and Regnier defined (non-)decomposability of a generalized connective \cite{2} only in the it multiplicative framework.

To investigate preservation of non-decomposability for various systems, we extend the definition of Danos and Regnier’s non-decomposability. After that, we show that Multiplicative Additive Linear Logic MALL and Multiplicative Exponential Linear Logic MELL does not preserve non-decomposability.

**Notations and terminologies:** The letter $L$ represents some logical system containing MALL as a subsystem. We call the formulas of the form $(A_{11} \otimes \ldots \otimes A_{1i_1}) \otimes \ldots \otimes (A_{m1} \otimes \ldots \otimes A_{mi_m})$ normal (where $1 \leq m, 0 \leq i_j$ and $n$ is a natural number).

We define the order relation $\preceq_L$ as follows; for any formulas $P, Q \in L, P \preceq_L Q$ if there is a proof of $Q$ from that of $P$ in the one-sided sequent calculus $L$ using the rules of $L$. We omit the subscript $L$ for readability.

**Definition 4.** Let $I_1, \ldots, I_k$ ($k$ is a natural number) be an enumeration of the introduction rules of a generalized multiplicative connective $C$ and $S_1, \ldots, S_m_i_i$ ($i = 1, \ldots, k$) be the premises of each introduction rule $I_i$. A generalized connective $C$ is $L$-decomposable if either one of the following holds:

1. there exists some formula $\alpha \in MLL$ such that $P_C = P_\alpha$ holds by using only the inference rules of MLL,
2. there exists some formula $\alpha \in L$ such that
   (1) for all $i$ ($i = 1, \ldots, k$), $\vdash \alpha$ is derivable from $S_{i1}, \ldots, S_{im_i}$ by using at least one inference rules of $L$ other than those of MLL,
   (2) $S_{i1}, \ldots, S_{im_i}$ is obtainable from $\vdash \alpha$ by bottom-up construction and using at least one inference rules of $L$ other than those of MLL,

We show that all of generalized multiplicative connectives are decomposable in MALL and MELL.

**Proposition 1.** Let $C$ be an arbitrary MLL-non-decomposable generalized connective. $C$ is MALL-decomposable.
Proof. Let $P_C = \{p_1, \ldots, p_s\}$ ($s \in \mathbb{N}$) be the partition set of $C(A_1, \ldots, A_m)$. For each partition $p_j$, the unique (up to commutativity and associativity) normal formula $\alpha_j$ ($j \in \{1, \ldots, s\}$) which have the same assumptions as $p_j$ is determined. Put $\alpha = \alpha_1 \oplus \cdots \oplus \alpha_s$. $\vdash \alpha_1 \oplus \cdots \oplus \alpha_s$ is derivable from $p_j$ ($j = 1, \ldots, s$). For each $j$, $p_j$ is obtainable by bottom-up construction.

The Figure 1 is an example of decomposition for the Danos and Regnier’s non-decomposable connective in MALL.

\[
\begin{array}{ccc}
\vdash A, B, \Gamma & \vdash C, D, \Delta \\
\vdash A \otimes B, \Gamma & \vdash C \otimes D, \Delta \\
\vdash (A \otimes B) \otimes (C \otimes D), \Gamma, \Delta \\
\vdash ((A \otimes B) \otimes (C \otimes D)) \oplus ((A \otimes C) \otimes (B \otimes D)), \Gamma, \Delta \\
\vdash A, C, \Gamma & \vdash B, D, \Delta \\
\vdash A \otimes C, \Gamma & \vdash B \otimes D, \Delta \\
\vdash (A \otimes C) \otimes (B \otimes D), \Gamma, \Delta \\
\vdash ((A \otimes B) \otimes (C \otimes D)) \oplus ((A \otimes C) \otimes (B \otimes D)), \Gamma, \Delta
\end{array}
\]

Figure 1: Decomposition of a generalized connective in MALL

Remark 1. When we consider only cut-free proofs, we can regard a generalized connective $C(A_1, \ldots, A_m)$ as the formula $\alpha_1 \oplus \cdots \oplus \alpha_s$ as the above proof. However, when we consider a proof containing the cut-rules, we cannot identify $C(A_1, \ldots, A_m)$ with $\alpha_1 \oplus \cdots \oplus \alpha_s$. The reason is as follows: the dual connective $C^*(\sim A_1, \ldots, \sim A_m)$ of $C(A_1, \ldots, A_m)$ can be decomposed as $\beta_1 \oplus \cdots \oplus \beta_s$ by the same method. However, $\alpha_1 \oplus \cdots \oplus \alpha_s$ and $\beta_1 \oplus \cdots \oplus \beta_s$ are not de Morgan dual and the cut is not defined between these formulas (Figure 2).

If we consider only cut-free proofs, the introduction of non-decomposable connectives amount to the introduction of additive disjunction in the restricted form.

4 Non-decomposability preservation result

In this section, we show that Elementary Multiplicative Linear Logic EMLL satisfies non-decomposability preservation. Namely, if a generalized multiplicative
connective $C$ is non-decomposable in MLL, then $C$ is also non-decomposable in EMLL.

The multiplicative fragment of Elementary Linear Logic EMLL is defined as follows.

**Definition 5.** Formulas of EMLL are defined as follows;

$A := P | \neg P | A \otimes A | A \boxtimes A | ! A | ? A$

Negation is inductively defined as follows;

$\neg \neg P := P, \neg (A \otimes B) := \neg A \boxtimes \neg B, \neg (A \boxtimes B) := \neg A \otimes \neg B, \neg (! A) := \neg \neg A, \neg (? A) = ! \neg A$

(where $P$ is an atomic formula)

The inference rules of EMLL is as Figure 3.

The following is the general form of an introduction rule for a generalized multiplicative connective. Note that each premise of a generalized connective contains at least one principal formula.

$$
\vdash \Gamma, A_1, \ldots, A_{i_1}, \ldots, A_{i_l}, \Gamma_1, \ldots, \Gamma_m, C(A_1, \ldots, A_n)
$$

where $ji_j \in \{1, \ldots, n\}$ and $i_k \neq 0$ ($1 \leq k \leq m$).

**Theorem 2.** Let $C$ be an arbitrary MLL-non-decomposable connective. Then $C$ is EMLL-non-decomposable.
Proof. We assume that \( \mathcal{C} \) is EMLL-decomposable and show that it implies a contradiction. By definition, there is an EMLL-formula \( \alpha \) such that \( \vdash \Gamma, \alpha \) (where \( \Gamma = \bigcup_{i=1}^{n} \Gamma_i \) and “\( \bigcup \)” is a multiset union) is derivable from the same assumptions for \( \mathcal{C} \) in EMLL. The formula \( \alpha \) must contain at least one modal operator. Otherwise, it contradicts with the assumption that \( \mathcal{C} \) is MLL-non-decomposable. The modal operator contained in \( \alpha \) cannot be a bang connective. If that is a case, \( \Gamma \) must have the form \( ?\Delta \). It contradicts the assumption that \( \Gamma \) is arbitrary. If \( \alpha \) contains a why-not connective, an introduction of a why-not connective is by either \( K \)-rule or \( ? \)-weakening rule. It cannot be the \( K \)-rule as the above explanation. Hence, some formula \( ?A \) is introduced by the weakening rule. If an operand of a tensor in the formula \( \alpha \) has the form \( ?A \), then there is some decomposition of the formula \( \alpha \) such that some sequent has only one formula \( (\cdots \vdash ?A \cdots) \). We apply \( ? \)-weakening rule to this sequent and obtain the empty sequent \( \vdash \). This contradicts with the definition of \( \mathcal{C} \)-introduction rule. If any operands of all tensor connectives in the formula \( \alpha \) have the form \( ?A \), then we can construct the formula \( \alpha' \) from \( \alpha \) by deleting the formulas having the form \( ?A \) and the superfluous par connectives. This formula \( \alpha' \) is a decomposition of the connective \( \mathcal{C} \) in MLL, contradiction.

\[\square\]

Preservation of non-decomposability in Multiplicative Light Linear Logic MLLL can be also proved by the similar way as above proof.
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