An alternative recursive formula for the sums of powers of integers

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1. Introduction

The sums of powers of the first $n$ positive integers $S_p(n) = 1^p + 2^p + \cdots + n^p, (p = 0, 1, 2, \ldots)$ satisfy the fundamental identity

$$1 + \sum_{r=0}^{p} \binom{p+1}{r} S_r(n) = (n+1)^{p+1}, \quad p \geq 0,$$

(1)

from which we can successively compute $S_0(n), S_1(n), S_2(n), \ldots$. Identity (1) can easily be proved by using the binomial theorem; see e.g. [1, 2]. Several variations of (1) are also well known [3, 4, 5].

In this note, we derive the following lesser-known recursive formula for $S_p(n)$:

$$p! + \sum_{t=0}^{p} \binom{p+1}{t+1} S_t(n) = p! \left( \frac{n + p + 1}{p + 1} \right), \quad p \geq 0,$$

(2)

where $\left[ \begin{array}{c} p \\ t \end{array} \right]$ denote the (unsigned) Stirling numbers of the first kind, also known as the Stirling cycle numbers (see e.g. [6, Chapter 6]). Table 1 shows the first few rows of the Stirling number triangle. Although the recursive formula (2) is by no means new, our purpose in dealing with recurrence (2) in this note is two-fold. On one hand, we aim to provide a new algebraic proof of (2) by making use of two related identities involving the harmonic numbers. On the other hand, as we will show, using (2) in conjunction with the principle of strong mathematical induction yields the identity

$$\sum_{r=0}^{p} \binom{p+1}{t+1} = \frac{(p + r)!}{r!},$$

(3)

| $p \times t$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|--------------|---|---|---|---|---|---|---|---|---|
| 0            | 1 |   |   |   |   |   |   |   |   |
| 1            | 0 | 1 |   |   |   |   |   |   |   |
| 2            | 0 | 1 | 1 |   |   |   |   |   |   |
| 3            | 0 | 2 | 3 | 1 |   |   |   |   |   |
| 4            | 0 | 6 | 11 | 6 | 1 |   |   |   |   |
| 5            | 0 | 24 | 50 | 35 | 10 | 1 |   |   |   |
| 6            | 0 | 120 | 274 | 225 | 85 | 15 | 1 |   |   |
| 7            | 0 | 720 | 1764 | 1624 | 735 | 175 | 21 | 1 |   |
| 8            | 0 | 5040 | 13068 | 13132 | 6769 | 1960 | 322 | 28 | 1 |

TABLE 1
which holds for every \( r = 1, 2, 3, \ldots \). Interestingly enough, for the special case in which \( r = p \), identity (3) leads right away to an explicit formula for the Catalan numbers that seems not to have been noticed hitherto (see (16) below).

2. Derivation of the recursive formula

To prove formula (2), we proceed in two steps. In the first step, we state the identity:

- **Identity 1**

\[
\sum_{k=0}^{n} \binom{k+p}{p} H_k = \binom{n+p+1}{p+1} H_n - \frac{1}{(p+1)!} \sum_{t=0}^{p} \left[ \binom{n+p+1}{p+1} - 1 \right],
\]

where \( H_k = 1 + \frac{1}{2} + \ldots + \frac{1}{k} \) is the \( k \) th harmonic number. In the second step, we state the identity:

- **Identity 2**

\[
\sum_{k=0}^{n} \binom{k+p}{p} H_k = \binom{n+p+1}{p+1} H_n - \frac{1}{p+1} \binom{n+p+1}{p+1}.
\]

Formula (2) then follows by equating the right-hand sides of Identities 1 and 2.

2.1 Proof of the Identity 1

To prove Identity 1, we shall use the following lemma.

**Lemma:** For any non-negative integer \( p \) and for \( t = 0, 1, \ldots, p + 1 \), we have

\[
\sum_{k=0}^{p+1} (-1)^{p+1-k} \binom{p+1-k}{k} \binom{p+1}{t} = \binom{p+1}{t}.
\]

**Proof:** Let \([x]_p \) denote the falling factorial \( x(x-1)(x-2)\ldots(x-p+1) \).

Recall that the numbers \( \binom{p}{t} \) can be defined algebraically by the relation [6, equation (6.14)]

\[
[x]_p = \sum_{k=0}^{p} (-1)^{p-k} \binom{p}{k} x^{k}.
\]

Thus, we can evaluate \([x+p]_{p+1} \) as

\[
[x+p]_{p+1} = \sum_{k=0}^{p+1} (-1)^{p+1-k} \binom{p+1}{k} (x+p)^k.
\]
RECURSIVE FORMULA FOR THE SUMS OF POWERS OF INTEGERS

235

\[ \sum_{k=0}^{p+1} (-1)^{p+1-k} \binom{p+1}{k} \sum_{j=0}^{k} (k + 1) \binom{k}{j} x^j \]

\[ = \sum_{j=0}^{p+1} \sum_{k=0}^{p+1} (-1)^{p+1-k} \binom{k + 1}{k} \binom{p}{j} x^j. \] (5)

Now, denoting by \([x]^p\) the rising factorial \(x(x + 1)(x + 2) \ldots (x + p - 1)\), it is clear that \([x + 1]^p = [x]^p + 1\). Furthermore, \([x]^p\) can be expressed as [6, equation (6.16)]

\[ [x]^p = \sum_{t=0}^{p+1} \binom{p+1}{t} x^t. \] (6)

Therefore, equating coefficients of \(x^j\) on the right-hand sides of (5) and (6), we end up with relation (4).

Next, we proceed with the proof of the Identity 1:

\[ \sum_{k=1}^{n} \binom{k + p}{p} H_k = \sum_{k=1}^{n} \sum_{j=1}^{k} \binom{k + p}{p} j^{-1} = \sum_{j=1}^{n} \sum_{k=j}^{n} \binom{k + p}{p} j^{-1} \]

\[ = \sum_{j=1}^{n} j^{-1} \sum_{k=1}^{n} \binom{k + p}{p} - \sum_{j=1}^{n} j^{-1} \sum_{k=1}^{j-1} \binom{k + p}{p} \]

\[ = \left( \frac{n + p + 1}{p + 1} \right) H_n - \sum_{j=1}^{n} \binom{j + p}{p + 1} j^{-1}. \]

where in the last step the following ‘hockey stick’ identity [6, problem 1.7.9.] is used twice:

\[ \sum_{k=1}^{n} \binom{k + p}{p} = \binom{n + p + 1}{p + 1} - 1. \] (8)

Furthermore, we have

\[ \sum_{j=1}^{n} \binom{j + p}{p + 1} j^{-1} = \sum_{j=1}^{n} \frac{1}{(p + 1)!} \sum_{k=0}^{p+1} (-1)^{p+1-k} \binom{p + 1}{k} (j + p)^k \]

\[ = \frac{1}{(p + 1)!} \sum_{j=0}^{p+1} \sum_{k=0}^{j} (-1)^{p+1-k} \binom{k + 1}{k} \binom{p + 1}{j} \binom{j}{t} x^t. \]

By relation (4), this reduces to

\[ \sum_{j=1}^{n} \binom{j + p}{p + 1} j^{-1} = \frac{1}{(p + 1)!} \sum_{t=0}^{p+1} \binom{p + 1}{t} S_{t-1}(n). \]
Since \( \begin{bmatrix} p + 1 \\ 0 \end{bmatrix} = 0 \), this is in turn equivalent to
\[
\sum_{j=1}^{k} \binom{j + p}{p + 1} j^{-1} = \frac{1}{(p + 1)!} \sum_{t=0}^{p} \binom{p + 1}{t + 1} S_t(n). \tag{9}
\]
Finally, combining equations (7) and (9) gives Identity 1.

### 2.2 Proof of the Identity 2

To prove Identity 2, we employ the following version of Abel’s lemma on summation by parts (see, e.g., [7]) which states:

Let \( \{u_k\}_{k \geq 1} \) and \( \{v_k\}_{k \geq 1} \) be two sequences of real numbers with partial sums \( U_n = \sum_{k=1}^{n} u_k \) and \( V_n = \sum_{k=1}^{n} v_k \). Further define \( U_0 = V_0 = 0 \). Then
\[
\sum_{k=1}^{n} u_k v_k + \sum_{k=1}^{n} v_k U_{k-1} = U_n V_n. \tag{10}
\]

Hence, letting
\[
v_k = \frac{1}{k}, \quad V_k = \sum_{j=1}^{k} \frac{1}{j} = H_k, \]
\[
u_k = \binom{k + p}{p}, \quad U_k = \sum_{j=1}^{k} \binom{j + p}{p} = \binom{k + p + 1}{p + 1} - 1,
\]
(where we use (8) to get the closed form of \( U_k \)), and plugging into (10), we obtain
\[
\sum_{k=1}^{n} \binom{k + p}{p} H_k = \binom{n + p + 1}{p + 1} H_n - \sum_{k=1}^{n} \binom{k + p}{p + 1}. \tag{11}
\]
Furthermore, noting that \( \binom{n}{k+1} = \frac{n-k}{k+1} \binom{n}{k} \) and using (8) again, it follows that
\[
\sum_{k=1}^{n} \frac{1}{k} \binom{k + p}{p + 1} = \frac{1}{p + 1} \sum_{k=1}^{n} \binom{k + p}{p} = \frac{1}{p + 1} \left[ \binom{n + p + 1}{p + 1} - 1 \right]. \tag{12}
\]
and thus, from equations (11) and (12), we obtain Identity 2.

### 3. Catalan numbers enter the scene

First we note that formula (2) can be written in the equivalent form
\[
\sum_{r=0}^{p} \binom{p + 1}{t + 1} = p! \left[ \binom{r + p + 1}{p + 1} - 1 \right] - \sum_{j=1}^{p} \sum_{r=0}^{j-1} \binom{p + 1}{t + 1}. \tag{13}
\]
where \( r \) stands for any arbitrary fixed positive integer. In particular, for \( r = 1 \) we retrieve the well-known relation for the Stirling cycle numbers \([6,\text{ equation (6.18)}]\)

\[
\sum_{t=0}^{p} \left[ \begin{array}{c} p + 1 \\ t + 1 \end{array} \right] = (p + 1)!, \tag{14}
\]

which constitutes the base case of the above identity (3). (Note, incidentally, that relation (14) also follows directly by letting \( x = 1 \) in equation (6), since \( [x]^{p+1} = (p + 1)! \) for \( x = 1 \).) Let us assume as a strong inductive hypothesis that

\[
\sum_{t=0}^{p} r^t \left[ \begin{array}{c} p + 1 \\ t + 1 \end{array} \right] = (p + 1)! \frac{(p+j)!}{j!}, \tag{15}
\]

for \( j = 1, 2, \ldots, r - 1 \). Thus, substituting (15) into (13), we have

\[
\sum_{t=0}^{p} r^t \left[ \begin{array}{c} p + 1 \\ t + 1 \end{array} \right] = p![\left( \frac{r + p + 1}{p + 1} \right) - 1] - \sum_{j=1}^{r-1} \frac{(p+j)!}{j!}.
\]

Since \( \binom{p+j}{j} = \binom{j+p}{p} \), we invoke (8) once more to get

\[
\sum_{t=0}^{p} r^t \left[ \begin{array}{c} p + 1 \\ t + 1 \end{array} \right] = p![\left( \frac{r + p + 1}{p + 1} \right) - \left( \frac{r + p}{p + 1} \right)] = \frac{(p+r)!}{r!}.
\]

This completes the inductive step and the proof of the identity (3).

Observe that, for \( r = p \), the said identity becomes

\[
\sum_{t=0}^{p} p^t \left[ \begin{array}{c} p + 1 \\ t + 1 \end{array} \right] = (p + 1)! C_p,
\]

where \( C_p = \frac{1}{p + 1} \binom{2p}{p} \) is the \( p \)th Catalan number [8]. Expressing \( C_p \) as

\[
C_p = \frac{\sum_{t=1}^{p+1} p^{-1} \left[ \begin{array}{c} p + 1 \\ t \end{array} \right]}{\sum_{t=1}^{p} p^{-1} \left[ \begin{array}{c} p + 1 \\ t \end{array} \right]}, \quad p \geq 1, \tag{16}
\]

we can therefore interpret \( C_p \) as the average of the function \( p^{-1} \) over all \((p + 1)!\) permutations of \( p + 1 \) elements, with \( t \) being the number of cycles...
of a permutation, and \( \binom{p + 1}{t} \) the number of permutations of \( p + 1 \) elements with exactly \( t \) cycles. As an illustrative example, let us apply (16) to calculate \( C_6 \). By using the entries in the seventh row of Table 1, from (16) we readily obtain

\[
C_6 = \frac{1}{7!} \left( 6^1 \binom{7}{1} + 6^1 \binom{7}{2} + 6^2 \binom{7}{3} + 6^3 \binom{7}{4} + 6^4 \binom{7}{5} + 6^5 \binom{7}{6} + 6^6 \binom{7}{7} \right)
\]

\[
= \frac{1}{5040} \left( 6^1 \cdot 720 + 6^1 \cdot 1764 + 6^2 \cdot 1624 + 6^3 \cdot 735 + 6^4 \cdot 175 + 6^5 \cdot 21 + 6^6 \cdot 1 \right)
\]

\[
= \frac{665280}{5040} = 132.
\]

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