GLOBAL MONGE-AMPÈRE EQUATION WITH ASYMPTOTICALLY PERIODIC DATA

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Abstract. Let \( u \) be a convex solution to \( \det(D^2 u) = f \) in \( \mathbb{R}^n \) where \( f \in C^\alpha(\mathbb{R}^n) \) is asymptotically close to a periodic function \( f_p \). We prove that the difference between \( u \) and a parabola is asymptotically close to a periodic function at infinity, for dimension \( n \geq 4 \). We also prove corresponding weaker estimates if the dimension is 2 or 3.

1. Introduction

In this article we study convex, entire viscosity solutions \( u : \mathbb{R}^n \to \mathbb{R} \), to the Monge-Ampère equation

\[
\det(D^2 u) = f(x), \quad \text{in} \quad \mathbb{R}^n.
\]

The forcing term \( f \) is assumed to be positive and asymptotically close to a periodic function at infinity. Our main goal is to establish a classification theorem for such solutions.

Monge-Ampère equation with periodic data can be found in various topics in applied mathematics such as homogenization theory, optimal transportation problems, vorticity arrays, etc. Equation (1.1) also occurs in differential geometry, when it is lifted from a Hessian manifold, [12]. In spite of the profusion in application, Monge-Ampère equation is well known for its analytical difficulty and it is no exception for equation (1.1) when the right hand side is close to a periodic function. In [9] Caffarelli and Li proved that if \( f \) is a positive periodic function then \( u \) has to be a parabola plus a periodic function with the same periodicity as \( f \). This classification theorem can be viewed as an extension to classical ones: Jörgens [28], Calabi [13], Pogorelov [35], Caffarelli-Li [8], etc.

The aim of this article is to establish an optimal perturbation result from the Caffarelli-Li’s classification theorem, as to cover forcing terms \( f \) that are asymptotically a periodic function at infinity.
In more precise terms, the assumption on $f$ is as follows: Let $f_p$ be a positive, $C^{1,\alpha}$ periodic function in $\mathbb{R}^n$, i.e.:

\begin{align}
\exists d_0 > 0, \alpha \in (0,1), a_1, \ldots, a_n > 0 \text{ such that } \\
d_0^{-1} \leq f_p \leq d_0, \quad \|f_p\|_{C^{1,\alpha}(\mathbb{R}^n)} \leq d_0, \\
f_p(x + a_ie_i) = f_p(x), \quad \forall x \in \mathbb{R}^n.
\end{align}

where $e_1 = (1,0,\ldots,0), \ldots, e_n = (0,\ldots,0,1)$. We will assume that $f \in C^{1,\alpha}(\mathbb{R}^n)$ is asymptotically close to $f_p$ in the following sense:

\begin{align}
\exists d_1 > 0 \text{ and } \beta > 2, \text{ such that } \\
d_1^{-1} \leq f(x) \leq d_1, \quad \forall x \in \mathbb{R}^n, \\
\|f\|_{C^{1,\alpha}(\mathbb{R}^n)} \leq d_1, \\
|\nabla^j (f-f_p)(x)| \leq d_1 (1 + |x|)^{-\beta-j}, \quad \forall x \in \mathbb{R}^n, \quad j = 0, 1, 2, 3.
\end{align}

Under the above framework, our main theorem is:

**Theorem 1.1.** Let $n \geq 4$ and $u \in C^{3,\alpha}(\mathbb{R}^n)$ be a convex solution to

\begin{equation}
\det(D^2u) = f, \quad \text{in } \mathbb{R}^n
\end{equation}

where $f$ satisfies (1.3). Then there exist $b \in \mathbb{R}^n$, a symmetric, positive definite matrix $A$ with $\det(A) = \Pi_{1 \leq i \leq n} f$, and $v \in C^{3,\alpha}(\mathbb{R}^n)$, which is $a_i$-periodic in the $i$-th variable, such that

\begin{equation}
|u(x) - \left(\frac{1}{2} x' Ax + b \cdot x + v(x)\right)| \leq C (1 + |x|)^{-\sigma}, \quad \forall x \in \mathbb{R}^n
\end{equation}

for some $C(d_0, d_1, n, \beta, a_1, \ldots, a_n) > 0$ and $\sigma := \min\{\beta, n-2\}$.

**Remark 1.1.** If $n = 3$, the estimate is weakened to

$$|u(x) - \frac{1}{2} x' Ax| \leq C (1 + |x|) \log(2 + |x|) \quad \forall x \in \mathbb{R}^3.$$ 

For $n = 2$ it is already established by Caffarelli-Li in [8] that

$$|u(x) - \frac{1}{2} x' Ax| \leq C (1 + |x|)^{2-\delta}, \quad \forall x \in \mathbb{R}^2$$

for $C, \delta > 0$ only depending on the usual parameters. So the optimal estimates for $n = 2, 3$ are still open.

**Remark 1.2.** Theorem [17] does not imply corresponding (better) estimates on higher order derivatives because it is not possible to obtain an oscillation control on $D^2u$.

Caffarelli and Li [9] proved that if $f = f_p$ in (1.1) then

$$u(x) = \frac{1}{2} x' Ax + b \cdot x + v(x), \quad \mathbb{R}^n, \quad n \geq 2.$$ 

Thus Theorem [17] is an extension of the classification theorem of Caffarelli-Li.
The assumption $\beta > 2$ is essentially optimal, as one can observe from the following example: let $f$ be a radial, smooth, positive function such that $f(r) \equiv 1$ for $r \in [0, 1]$ and $f(r) = 1 + r^{-2}$ for $r > 2$. Let

$$u(r) = n^{\frac{1}{2}} \int_0^r \left( \int_0^t s^{n-1} f(t) dt \right)^{\frac{1}{2}} ds, \quad r = |x|.$$ 

It is easy to check that $\det(D^2 u) = f$ in $\mathbb{R}^n$. Moreover for $n \geq 3$,

$$u(x) = \frac{1}{2} |x|^2 + O(\log |x|)$$

at infinity, which means by taking $f_p \equiv 1$ the estimate in Theorem 1.1 is violated for $n \geq 4$.

One major difficulty in the study of (1.1) is that the right hand side oscillates wildly when it is scaled. The regularity theory for Monge-Ampère equations with oscillating right hand side is very challenging (see [18, 19]). In this respect, we found that the strategy implemented by Caffarelli and Li in [9] is essentially optimal, as we cannot simplify any step of the program followed in their work. In turn, our arguments are essentially based on the corresponding steps from [9] as well as previous works of Caffarelli and Li [5, 6, 8]. The main difference, though, is that in the proof of Theorem 1.1 one needs to take care of perturbational terms in a sharp manner. In order to handle all the perturbations, we need to make use of intrinsic structures implied by Monge-Ampère equations (such as (2.8) below) and also estimates on Green’s functions, such as the Littman-Stampacchia-Weinberger Theorem, [33], etc.

Theorem 1.1 is closely related to the exterior Dirichlet problem: Given a strictly convex set $D$ and the value of $u$ on $\partial D$, can one solve the Monge-Ampère equation in $\mathbb{R}^n \setminus D$ if the asymptotic behavior of $u$ at infinity is prescribed? Clearly Theorem 1.1 must be established before such a question can be attacked. We plan to address the exterior Dirichlet problem in a future work. The traditional (interior) Dirichlet problem has been fairly well understood through the contribution of many people (see [1, 2, 34] [13, 35, 14, 11, 6, 24, 25, 26, 30, 27, 37, 39, 19, 18] and the references therein). If $f$ is equal to a positive constant outside a compact set, Delanoë [17], Ferrer-Martínez-Milán [20, 21] and Bao-Li [3] studied the exterior Dirichlet problem for $n = 2$, Caffarelli-Li [8] studied the case of $n \geq 3$. If $f$ is a perturbation of a positive constant at infinity, Bao-Li-Zhang studied the exterior Dirichlet problem in [4].

The organization of this article is as follows: The proof of Theorem 1.1 consists of five steps. First in step one we employ the argument in [6, 8] to show that the growth of the solution of (1.1) is roughly similar to that of a parabola. Then in step two we prove that $D^2 u$ is positive definite, which makes (1.1) uniformly elliptic. The key point in this step is to consider a second order incremental of $u$ as a subsolution to an elliptic equation. In step three we prove a pointwise estimate of the second incremental of
The proof of Theorem 1.1 for \( n \geq 5 \) is placed in step four since all the perturbations in this case are bounded. Finally in step five we prove the case \( n = 4 \), which relies on an application of Krylov-Safonov Harnack inequality, as to handle perturbations with logarithmic growth.

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2. Proof of Theorem 1.1

Since the Monge-Ampère equation is invariant under affine transformation, we assume \( a_1 = ... = a_n = 1 \) and \( f_{[0,1]^n} = 1 \).

In step one we prove that \( u \) grows like a quadratic polynomial at infinity.

First we normalize \( u \) to make \( u(0) = 0 \) and \( u \geq 0 \) in \( \mathbb{R}^n \). Since \( f \) is bounded above and below by two positive constants, we use the argument in Caffarelli-Li \([8]\), see also \([4]\). Let \( \Omega_M = \{ x \in \mathbb{R}^n, \ u(x) < M \} \).

Then the following properties hold:

1. \( C^{-1}M^{\frac{n}{2}} \leq |\Omega_M| \leq CM^{\frac{n}{2}} \),
2. There exists \( A_M(x) = a_M \cdot x + b \) such that \( det(a_M) = 1 \) and \( B_R \subset A_M(\Omega_M) \subset B_R \) and
3. \( \frac{1}{C} \sqrt{M} \leq R \leq C \sqrt{M} \),
4. \( 2nR \geq \text{dist}(a_M(\Omega_M), \partial a_M(\Omega_M)) \geq \frac{\delta}{2} \)

where all the constants \( C \) only depend on \( d_1 \) and \( n \).

The properties listed above are proved in \([8]\) only based on the assumption that \( f \) is bounded above and below. Here for the convenience of the reader we mention the idea of the proof: First Caffarelli-Li used the following lemma (Lemma 2.9 in \([8]\))

Lemma A (Caffarelli-Li): Let \( e_1 = (1, 0, ..., 0) \) and

\[
B'_\delta = \{ (0, x_2, ..., x_n); \quad |(0, x_2, ..., x_n)| < \delta \}.
\]

Let \( K \) be the convex hull of \( B'_\delta \cup \{re_1\} \), and let \( u \) be a nonnegative convex viscosity solutions of \( det(D^2u) \geq \lambda > 0 \) in the interior of \( K \). Assume \( u \leq \beta \) on \( B'_\delta \). Then there exists \( C(n) > 1 \) such that

\[
\max\{ \beta, u(re_1) \} \geq \frac{\lambda^{1/n} \delta^{2(a-1)/n} \gamma^{2/n}}{C}.
\]

In other words, \( u \) cannot be small in one direction for too long. Lemma A is important since it says Pogorelov’s famous example of non-strictly convex solution does not exist if the domain is large. A simple application of Lemma A leads to \( \Omega_M \subset B_{CM^{\frac{n}{2}}} \). A volume preserving affine transformation can be used to make the image of \( \partial \Omega_M \) between two balls with comparable radii. A comparison with a parabola gives \( |\Omega_M| \sim M^{\frac{n}{2}} \) using only the
upper bound and lower bound of \( f \). In [5] Caffarelli proved that \( u \) must depart from its level set in a non-tangential manner, using this we have \( \text{dist}(A_M(\partial \Omega_M), A_M(\Omega_M/2)) \sim M^{1/2} \) where \( A_M \) is a volume-preserving affine transformation:

\[
A_M(x) = a_Mx + b_M, \quad \det(a_M) = 1.
\]

Using \( u(0) = 0 \) and \( u \geq 0 \) one can further conclude \( B_{R/C} \subset a_M(\Omega_M) \subset B_{2nR} \)

Let

\[
u_M(x) = \frac{1}{R^2} u(a_M^{-1}(Rx)), \quad x \in O_M := \frac{1}{R} a_M(\Omega_M),
\]

Using \( u(0) = 0 \) and \( u \geq 0 \) we have

\[
B_{1/C} \subset O_M \subset B_{2n}.
\]

The equation for \( u_M \) is

\[
(2.1) \quad \det(D^2u_M(x)) = f(a_M^{-1}(Rx)), \quad \text{in} \quad O_M.
\]

We compare \( u_M \) with

\[
(2.2) \quad \left\{ \begin{array}{l}
\det(D^2w_p) = f_p(a_M^{-1}(Rx)), \quad \text{in} \quad O_M, \\
w_p = M/R^2, \quad \text{on} \quad \partial O_M.
\end{array} \right.
\]

Let \( h = u_M - w_p \), using Alexandrov estimate (see [4])

\[
\max_{O_M} (h^-) \leq C \left( \int_{S^+} \det(D^2(u_M - w_p)) \right)^{\frac{1}{n}}
\]

where

\[
S^+ = \{ x \in O_M; \quad D^2(u_M - w_p) > 0 \}.
\]

On \( S^+ \) by the concavity of \( \frac{1}{n} \) on positive definite matrices we have

\[
\det(\frac{1}{2} D^2u_M) \geq \frac{1}{2} \det(\frac{1}{2} D^2(u_M - w_p)) + \frac{1}{2} \det(\frac{1}{2} D^2w_p).
\]

Thus

\[
\max_{O_M} h^- \leq C \left( \int_{O_M} |f(\frac{1}{2} a_M^{-1}(Rx)) - f_p(\frac{1}{2} a_M^{-1}(Rx))|^n dx \right)^{\frac{1}{n}}.
\]

By the assumption on \( f \) ([1,3]), the right hand side is \( O(1/R) \). Next we cite the homogenization theorem of Caffarelli-Li (Theorem 3 of [9]): Let \( w \) satisfy

\[
\det(D^2w) = 1, \quad \text{in} \quad O_M
\]

with \( w = M/R^2 \) on \( \partial O_M \), then

\[
|w - w_p| \leq CR^{-\delta}
\]

for some \( \delta > 0 \). Thus

\[
(2.3) \quad |u_M(x) - w(x)| \leq CR^{-\delta}
\]
for some $\delta > 0$. Set
\[ E_M := \{x; \ (x - \bar{x})' D^2 w(\bar{x})(x - \bar{x}) \leq 1\} \]
where $\bar{x}$ is the unique minimum of $w$ (note that Caffarelli [5] proved that the minimum point of $w$ is unique) that satisfies $dist(\bar{x}, \partial O_M) > C(d_1, n)$. By the same argument in [6, 8] there exist $\bar{k}$ and $C$ that only depend on $n$ and $d_1$ such that for $\epsilon << \delta$, $M = 2(1+\epsilon)k^k, 2^{k-1} \leq M' \leq 2^k, R \sim M^2$,
\[ \left(\frac{2M'}{R^2} - C^2\frac{a_k}{2^{k/2}}\right)^2 E_M \subset \frac{1}{R} a_M(\Omega_{M'}) \subset \left(\frac{2M'}{R^2} + C^2\frac{a_k}{2^{k/2}}\right)^2 E_M, \ \forall k \geq \bar{k}, \]
which can be translated as
\[ \sqrt{2M'(1 - \frac{C}{2^{k/2}})} E_M \subset a_M(\Omega_{M'}) \subset \sqrt{2M'(1 + \frac{C}{2^{k/2}})} E_M. \]
Let $Q$ be a positive definite matrix satisfying $Q^2 = D^2 w(\bar{x}), O$ be an orthogonal matrix that makes $T_k = OQa_M$ upper triangular, then $det(T_k) = 1$ and by Proposition 3.4 of [8]
\[ \|T_k - T\| \leq C^2 \frac{a_k}{2^{k/2}} \]
for some matrix $T$. By setting
\[ v = u \cdot T \]
we have
\[ det(D^2 v(x)) = f(Tx) \]
and
\[ \sqrt{2M'(1 - \frac{C}{2^{k/2}})} B \subset \{x; \ v(x) < M'\} \subset \sqrt{2M'(1 + \frac{C}{2^{k/2}})} B \]
for all $M' \geq 2^k$. Here $B$ stands for the unit ball. Consequently
\[ |v(x) - \frac{1}{2}|x|^2| \leq C|x|^2 - \epsilon. \]
The equation for $v$ is
\[ det(D^2 v) = f_v(x) \]
where $f_v(x) = f(T(x))$.
\[ \textbf{Step two: Uniform Ellipticity} \]
The purpose of this step is to show: There exist $c_1$ and $c_2$ depending only on $d_0, d_1, \beta, a_1, ..., a_n$ and $n$ that
\[ c_1 I \leq D^2 v \leq c_2 I. \]
First we choose $M > 10$ so that for $R = M^2$ and
\[ v_R(y) = \frac{1}{R^2} v(Ry), \]
\[ \Omega_{1,v_R} := \{y; \ v_R(y) \leq 1\} \]
Consider the following second order incremental for $v$ where $\vec{v}$

On the other hand

(2.6)

where $e \in \mathbb{R}^n$ and $\|e\|$ is its Euclidean norm. Later we shall always choose $e \in E$ which is defined as

$$E \triangleq \{a_1 \vec{v}_1 + ... + a_n \vec{v}_n; \quad a_1, ..., a_n \in \mathbb{Z}\}$$

where $\vec{v}_i$ satisfies $f_v(x + \vec{v}_i) = f_v(x)$ for all $x \in \mathbb{R}^n$.

Let $R_1 = |x|$ and

$$v_{R_1}(y) = \frac{1}{R_1} v(R_1 y).$$

Then by the closeness result of step one, the sections: $S_{v_{R_1}}(e_x, \frac{1}{2})$ and $S_{v_{R_1}}(e_x, \frac{1}{4})$ around $e_x := x/R_1$ are very close to those of the parabola $\frac{1}{2} |y|^2$. Here we recall that for a convex, $C^1$ function $v$,

$$S_v(x, h) := \{y; \quad v(y) \leq v(x) + \nabla v(x) \cdot (y - x) + h, \}.$$ 

Moreover the equation for $v_{R_1}$ is

(2.5)

$$\det(D^2 v_{R_1}(y)) = f_v(R_1 y)$$

where $f_v(R_1 y) = f(T(R_1 y))$. Let $e_{R_1} = e/R_1$, then direct computation shows

$$\Delta_{e_{R_1}}^2 v = \Delta_{e_{R_1}}^2 v_{R_1}.$$

Let

$$w(y) = \frac{v_{R_1}(y + e_{R_1}) + v_{R_1}(y - e_{R_1})}{2}.$$ 

Let $F = \det^{\frac{1}{2}}$. It is well known that $F$ is concave on positive definite matrices. On one hand, this concavity gives

$$\det^F(D^2 w)(y) \geq \frac{1}{2} \det^F(D^2 v_{R_1}(y + e_{R_1})) + \frac{1}{2} \det^F(D^2 v_{R_1}(y - e_{R_1}))$$

(2.6)

$$= \frac{1}{2} f_v^F(R_1 y + e) + \frac{1}{2} f_v^F(R_1 y - e).$$

On the other hand

(2.7)

$$F(D^2 w) \leq F(D^2 v_{R_1}) + F_{ij}(D^2 v_{R_1}) \partial_{ij}(w - v_{R_1})$$

$$= f_v^F(x) + F_{ij}(D^2 v_{R_1}) \partial_{ij}(w - v_{R_1})$$
where

\[ F_{ij}(x) = \frac{\partial F(D^2 v_{R_1})}{\partial_{ij} v_{R_1}} = \frac{1}{n} (\det(D^2 v_{R_1}))^{\frac{1}{n}}^{-1} \cof_{ij}(D^2 v_{R_1}). \]

Thus the combination of (2.6) and (2.7) gives

\[ a_{ij}\partial_{ij}(\Delta_{v_{R_1}}^2 v_{R_1}) \geq E_1 \]

where

\[ a_{ij} = \cof_{ij}(D^2 v_{R_1}), \]

\[ E_1 : = n\det(D^2 v_{R_1})^{(n-1)/n} \]

\[ \cdot \left( \frac{R_1^2}{2\|e\|^2} \left( f_{v}(R_1 y + e) + f_{v}(R_1 y - e) - 2f_{v}(R_1 y) \right) \right) \]

For applications later we state the following fact: For any smooth \( u \)

\[ (2.8) \quad \partial_{i}\left(\cof(D^2 u)_{ij}\right) = 0. \]

Let \( f_{v,p}(y) = f_{p}(Ty) \), then by choosing \( Te \in E \)

\[ (2.9) \quad f_{v,p}(R_1 y + e) + f_{v,p}(R_1 y - e) - 2f_{v,p}(R_1 y) = 0, \quad \text{for all } y \in \mathbb{R}^n. \]

By (1.3) we see that

\[ E_1(y) = n(\det(D^2 v_{R_1})^{(n-1)/n}) \frac{R_1^2}{2\|e\|^2} \left\{ (f_{v}(R_1 y + e) - f_{v,p}(R_1 y)) \right. \]

\[ \left. + (f_{v}(R_1 y - e) - f_{v,p}(R_1 y - e)) - 2(f_{v}(R_1 y) - f_{v,p}(R_1 y)) \right\} \]

\[ = O(R_1^{-\beta}). \]

Now we construct a function \( h \) that solves

\[ \begin{align*}
  a_{ij}\partial_{ij}h &= -E_1, \quad \text{in } B(e_x, \frac{1}{2}) \\
  h &= 0 \quad \text{on } \partial B(e_x, \frac{1}{2})
\end{align*} \]

Then we use the following classical estimate of Aleksandrov (see Page 220-222 of [22] for a proof):

**Theorem A:** Let \( \Omega \) be a domain in \( \mathbb{R}^n \) and let \( v \) be a solution in \( \Omega \) of the equation

\[ a_{ij}^*\partial_{ij}v = g \]

such that \( v = 0 \) on \( \partial \Omega \) and the coefficient matrix \( (a_{ij}^*)_{n \times n} \) satisfies

\[ c_1 \leq \det(a_{ij}^*) \leq c_2, \quad \text{and } \quad (a_{ij}^*) > 0, \]

then

\[ |v(x)| \leq C(n, c_1, c_2)\text{diam}(|\Omega|)\|g\|_{L^n(\Omega)}, \quad \forall x \in \Omega \]

Applying Theorem A to \( h \) we have

\[ (2.10) \quad |h(y)| \leq C(n, d_0)R_1^{-\beta}, \quad \text{for } \quad y \in B(e_x, \frac{1}{2}). \]
Remark 2.1. The estimate of Theorem A does not depend on constants of uniform ellipticity.

Thus \( \Delta^2_{\varepsilon_R} v_{R_1} + h \) is a super solution:

\[
a_{ij} \partial_{ij}(\Delta^2_{\varepsilon_R} v_{R_1} + h) \geq 0.
\]

In order to obtain a pointwise estimate for \( \Delta^2_{\varepsilon_R} v_{R_1} \) we apply Theorem 0.4 in [8] (a weak Harnack inequality of Caffarelli-Gutierrez [10]) to obtain

\[
\max_{y \in S(v_{R_1}, e, \frac{3}{4})} (\Delta^2_{\varepsilon_R} v_{R_1} + h) \leq C(n, d_0) \left( \Delta^2_{\bar{v}}(z) + h \right)
\]

Note that the distance between the two sections above is comparable to 1. It is also important to point out that the estimates in Theorem 0.4 of [9] does not depend on the regularity of \( v_{R_1} \). Let \( L \) be a line parallel to \( e \), then Lemma A1 of [9]

\[
\int_{S(v_{R_1}, e, \frac{3}{4}) \cap L} \Delta^2_{\varepsilon_R} v_{R_1} \leq C.
\]

Since \( h = O(|x|^{2-\beta}) \) we have proved

(2.11) \( 0 \leq \Delta^2_{\varepsilon_R} v(x) = \Delta^2_{\varepsilon_R} v_{R_1}(x) \leq C. \)

Note that \( \Delta^2_{\varepsilon_R} v \geq 0 \) because \( v \) is convex. Then the same argument as in [9] can be employed to prove that the level surfaces of \( v \) are like balls. For the convenience of the readers we describe the outline of this argument. Given \( x \in \mathbb{R}^n \) we set \( \gamma := \sup_{e \in E} \sum_{x \in \mathbb{R}^n} \Delta^2_{\varepsilon_R} v \) and

\[
\bar{v}(z) = v(z + x) - v(x) - \nabla v(x) z.
\]

Clearly \( \bar{v}(0) = 0 = \min_{\mathbb{R}^n} \bar{v} \). By (2.11) it is easy to see

\[
\sup_{B_r} \bar{v} \leq C(n) \gamma r^2, \quad \forall r > 1.
\]

On the other hand for any \( \bar{z} \in \partial B_r \) we show that for \( r \) large (but still only depending on \( n \) and \( d_0 \), \( \bar{v}(<\bar{z}) \geq \delta_0(n, d_0) \). Indeed, by (2.11),

(2.12) \( \bar{v}(\frac{\bar{z}}{2} + e) + \bar{v}(\frac{\bar{z}}{2} - e) - 2\bar{v}(\frac{\bar{z}}{2}) \leq \gamma ||e||^2 \)

for all \( e \in E \). Hence for \( z \in \frac{\bar{z}}{2} + (-2, 2)^n \), (2.12) implies ( using \( \bar{v} \geq 0 \))

\[
\bar{v}(z) \leq 2\bar{v}(\frac{\bar{z}}{2}) + C(n) \gamma.
\]

Further more, by \( \bar{v}(0) = 0 \) and the convexity of \( \bar{v} \) we have

\[
2\bar{v}(\frac{\bar{z}}{2}) \leq \bar{v}(\bar{z}).
\]
Therefore the following holds:

\[ \bar{v}(z) \leq \bar{v}(\bar{z}) + C(n)\gamma. \]

Consider

\[ w(z) = \frac{\bar{v}(\frac{\bar{z}}{2} + z)}{\bar{v}(\bar{z}) + C(n)\gamma}. \]

Clearly \( w \) satisfies \( \det(D^2w) \geq d_0^{-1}/(\bar{v}(\bar{z}) + C(n)\gamma)^n \) and

\[ w(z) \leq 1, \quad \text{for} \quad z \in \bar{z}/2 + (-2, 2)^n. \]

Applying Lemma A we have

\[ \max\{1, \bar{v}(\bar{z})\} \geq \left( \frac{d_0^{-\frac{1-n}{n}} C(n)}{\bar{v}(\bar{z}) + C(n)\gamma} \right) r^{\frac{2}{n}}. \]

Here we used the fact that \( \bar{v}(0) = 0 \) and \( \bar{v}(\bar{z}) \) is the maximum value of \( \bar{v} \) on the line segment connecting 0 and \( \bar{z} \). If \( \bar{v}(\bar{z}) \geq \gamma > 0 \), no need to do anything (here we assume \( \gamma > 1 \) without loss of generality), otherwise we obviously have

\[ \max\{1, \bar{v}(\bar{z})\} \geq \frac{d_0^{-\frac{1-n}{n}}}{C(n)\gamma} r^{\frac{2}{n}}. \]

Choose \( r \) large but only depending on \( n, \gamma \) and \( d_0 \) we have \( \bar{v}(\bar{z}) > 1 \). Thus for such \( r \), \( \min_{\partial B_r} \bar{v} \geq 1 \), then by standard argument the sections of \( \bar{v} \): \( S(\bar{v}, 0, h) \) are similar to balls if \( h \sim 1 \) and \( h < 1 \). Then interior estimate of Caffarelli, Jian-Wang [27] can be employed on \( S(\bar{v}, 0, h) \) to obtain the \( C^{2,\alpha} \) norm of \( \bar{v} \).

In particular \( |D^2v(x)| = |D^2\bar{v}(x)| \leq C \). Once the upper bound is obtained we also have the lower bounded from the Monge-Ampère equation. (2.4) is established.

**Remark 2.2.** In order to prove (2.4) the assumptions on the derivatives of \( f - f_p \) are not essential. In fact, as long as

\[ |(f - f_p)(x)| \leq C(1 + |x|)^{-\beta}, \quad \text{for} \quad x \in \mathbb{R}^n, \]

(2.4) still holds.

Step three. Pointwise estimate of \( \Delta^2_v \)

Let \( e \in E \), recall the equation for \( \Delta^2_v \) is

(2.13)

\[ \tilde{a}_{ij}(x) \partial_{ij} \Delta^2_v \geq E_2 := n(det(D^2v)^{\frac{n-1}{n}}) \int_{\mathbb{R}^n} \frac{x_i}{\|x\|^2} \left[ f^{\frac{1}{n}}(x + e) + f^{\frac{1}{n}}(x - e) - 2f^{\frac{1}{n}}(x) \right]. \]

where

\[ \tilde{a}_{ij}(x) = cof_{ij}(D^2v(x)) \]
Without loss of generality we assume that $e$

Using because (2.13) can be written as

where $C$

we have

(2.15) $0 \leq G(x, y) \leq C|y - x|^{2-n}$, $x, y \in \mathbb{R}^n$.

because (2.13) can be written as

$$\partial_t (\tilde{a}_{ij} \partial_j \Delta_v^2 v) \geq E_2,$$

and $G$ satisfies

$$-\partial_{x_i} (\tilde{a}_{ij} \partial_{x_j} G(x, y)) = \delta_y.$$
Thus the result of Littman-Stampacchia-Weinberger [33] shows that (2.15) holds.

Let
\[ h(x) = -\int_{\mathbb{R}^n} G(x, y) E_2^-(y) dy, \]
we have, by standard estimates
\[ 0 \leq -h(x) \leq C(1 + |x|)^{-\beta_1}, \quad \beta_1 = \min\{n - 2, \beta\} \]
and
\[ \tilde{a}_{ij} \partial_{ij} h = E_2^-, \quad \text{in} \quad \mathbb{R}^n. \]

The estimate of (2.16) is rather standard, for \( x \in \mathbb{R}^n \), we divide \( \mathbb{R}^n \) into three regions:
\[ E_1 = \{ y; |y - x| < |x|/2 \}, \quad E_2 = \{ y; |y| < |x|/2 \}, \quad E_3 = \mathbb{R}^n \setminus (E_1 \cup E_2). \]
The estimate on each of these three regions is standard.

The main result of this step is:
\[ \text{(2.18)} \quad \text{Given} \quad e \in E; \quad \Delta^2 v(x) \leq 1 - h(x), \quad \text{for} \quad x \in \mathbb{R}^n. \]

Let \( v_\lambda(x) = v(\lambda x)/\lambda^2 \) and \( P(x) = \frac{1}{n+1} |x|^2 \). First we claim that
\[ \text{(2.19)} \quad D^j(v_\lambda - P(x)) \to 0, \quad j = 0, 1, \quad \forall x \in K \subset \subset \mathbb{R}^n, \quad \text{as} \quad \lambda \to \infty, \]
where \( K \) is any fixed compact subset of \( \mathbb{R}^n \).

To see (2.19), by step one
\[ \text{(2.20)} \quad |v_\lambda(y) - P(y)| \leq C \lambda^{-\epsilon}, \quad \forall y \in K \subset \subset \mathbb{R}^n. \]
and \( |D^2 v_\lambda(x)| \leq C \) for \( x \in K \) we obtain by Ascoli’s theorem that \( \partial_t v_\lambda(x) \) tends to a continuous function. By (2.20) this function has to be \( x_l \). Thus (2.19) holds.

**Remark 2.3.** We don’t have the estimate of \( \|Dv_\lambda - x\|_{L^\infty(K)} \) but we don’t need it.

The proof of (2.18) is as follows.

Let \( \alpha = \sup_{\mathbb{R}^n}(\Delta^2 v - h) \) for \( e \in E \). By Step two \( \alpha < \infty \). Let \( \hat{e} = e/\lambda \), then by (2.19) and Lebesgue’s dominated convergence theorem
\[ \text{(2.21)} \quad \lim_{\lambda \to \infty} \int_{B_1} \Delta^2 v_\lambda = \int_{B_1} 1 dx = |B_1|. \]
Indeed, the integral over \( B_1 \) can be considered as the collection of integration on segments all in the direction of \( \hat{e} \). Since \( Dv_\lambda \to DP \) in \( C^0 \) and \( DP \) is smooth, Lemma A.2 or [9] implies (2.21).

Let
\[ h_\lambda(x) = h(\lambda x)/\lambda^2. \]
It is easy to see that
\[ \lim_{\lambda \to \infty} \int_{B_1} (\Delta^2 v + h_\lambda) = |B_1|. \]
Obviously \( \alpha \geq 1 \) our goal is to show that \( \alpha = 1 \). If this is not the case we have \( \alpha > 1 \). Then

\[
\limsup_{\lambda} \left( \frac{\alpha + 1}{2} \right) \cap B_1 \leq \lim_{\lambda \to \infty} \int_{B_1} \Delta^2 v_\lambda + h_\lambda = |B_1|.
\]

Thus

\[
\left\| \frac{\{ \Delta^2 v_\lambda + h_\lambda \geq \frac{\alpha + 1}{2} \} \cap B_1}{|B_1|} \right\| \leq 1 - \mu
\]

for \( \mu = \frac{1}{2}(1 - \frac{2}{\alpha + 1}) > 0 \). Here we emphasize that it is important to have \( \mu > 0 \). Equivalently for large \( \lambda \)

\[
\left\| \frac{\{ \Delta^2 v_\lambda + h_\lambda \leq \frac{\alpha + 1}{2} \} \cap B_1}{|B_1|} \right\| \geq \mu.
\]

Then we cite the following well known result: For \( v \) satisfying

\[
a_{ij}^* \partial_{ij} v \geq 0, \quad \lambda I \leq (a_{ij}^*(x)) \leq \Lambda I
\]

and

\[
v \leq 1 \quad \text{in } B_1 \quad \text{and} \quad |\{ v \leq 1 - \epsilon \} \cap B_1 | \geq \mu |B_1|,
\]

then \( v \leq 1 - c(n, \lambda, \Lambda, \epsilon, \mu) \) over \( B_{1/2} \) (see Lemma 2.8 of [9] or Lemma 6.5 of [7]). It is easy to see that

\[
\tilde{a}_{ij} \partial_{ij} (\Delta^2 v_\lambda + h_\lambda) \geq 0, \quad \text{in } B_1.
\]

Applying this result to \( \Delta^2 v_\lambda + h_\lambda \) we have

\[
(2.22) \quad \Delta^2 v_\lambda + h_\lambda \leq \alpha - C, \quad \text{in } B_{1/2}.
\]

Since \( \Delta^2 v_\lambda \) is sub-harmonic,

\[
\alpha = \sup_{\mathbb{R}^n} \Delta^2 v + h = \lim_{\lambda \to \infty} \sup_{B_{1/2}} (\Delta^2 v_\lambda + h_\lambda) < \alpha.
\]

Then we get a contradiction, (2.18) is established.

Step four: The proof of the Liouville theorem for \( n \geq 5 \).

By a result of Li [32] there exists \( \xi \in C^{2,\alpha}(\mathbb{R}^n) \) such that

\[
\{ \det(I + D^2 \xi) = f_{v,p}, \quad \text{in } \mathbb{R}^n,
\}

\[
I + D^2 \xi > 0, \quad \xi \text{ is a periodic function with the same period as that of } f_{v,p}.
\]

Let \( P(x) = \frac{1}{2} |x|^2 \) and

\[
w(x) = v(x) - P(x) - \xi(x).
\]

By the periodicity of \( \xi \) we have

\[
\Delta^2 w \leq C(1 + |x|)^{-\beta_1}, \quad x \in \mathbb{R}^n.
\]

The equation for \( w \) is

\[
\tilde{a}_{ij} \partial_{ij} w = E_3, \quad \text{in } \mathbb{R}^n
\]

where \( \tilde{a}_{ij} \) is obtained from mean value theorem, uniformly elliptic and is divergence free.

\[
|E_3(x)| \leq C(1 + |x|)^{-\beta - 2} \quad \text{in } \mathbb{R}^n.
\]
Using Green’s function again we can eliminate the right hand side of the above by a small function $h_2$:

\[(2.23) \quad a_{ij}\partial_{ij}h_2 = E_3, \quad |h_2(x)| \leq C(1 + |x|)^{-\beta_1}, \quad \text{in } \mathbb{R}^n.\]

Thus

\[a_{ij}\partial_{ij}(w - h_2) = 0.\]

Our goal is to prove that

\[(2.24) \quad w - h_2 \leq C + \text{ a linear function}, \quad \text{in } \mathbb{R}^n.\]

Here we assume that $e_1, e_2, \ldots, e_n \in E$, otherwise we employ a linear transformation. Let $w_1 = w - (ax + b)$ where $a \in \mathbb{R}^n, b \in \mathbb{R}$ are chosen so that

\[w_1(Ne) = 0, \quad \text{for } e = e_1, \ldots, e_n, \text{ and } N = \pm 1.\]

For each $e_i$ we find $h_{e_i}$ such that $\Delta^2_{e_i}h_{e_i} = h$ on the axis $e_i$ ($x = (0, \ldots, x_i, 0, \ldots, 0)$). Since $|h(x)| \leq C(1 + |x|)^{-\beta_1}$ and $\beta_1 > 2$ (because $n \geq 5$) we can find $h_{e_i}$ to be bounded. The construction of $h_{e_i}$ is as follows: Let $a_N = h(Ne_i)$. Let

\[h_{e_i}(Ne_i) = \sum_{l=N+1}^{\infty} \left( \sum_{k=l}^{\infty} a_k \right), \quad N = 1, \ldots .\]

Then it is easy to see that $\lim_{N \to \infty} h_{e_i}(N) = 0,$

\[0 \leq h_{e_i}(Ne_i) \leq C, \quad \forall N = 1, 2, \ldots \]

where $C$ is independent of $N$; and

\[\Delta^2_{e_i}h_{e_i}(Ne_i) = h(Ne_i), \quad N = 1, 2, \ldots .\]

Note that we need $n \geq 5$ here because otherwise there is a logarithmic term. Thus the function $h_{e_i}$ is constructed at points $Ne_i$ for $N = 1, \ldots$. The construction of $h_{e_i}$ on $-Ne_i$ for $N = 1, \ldots$ is similar. Finally $h_{e_i}(0)$ is determined by $h_{e_i}(-e_i), h_{e_i}(e_i)$ and $a_0$.

Thus on the axis $e_i$

\[\Delta^2_{e_i}(w_1 + h_{e_i}) \leq 0,\]

which leads to

\[w_1(Ne_i) + h_{e_i}(Ne_i) \leq 0, \quad \text{for all } N = \pm 1, \pm 2, \pm 3, \ldots,\]

Then by the bound on $D^2v$ we further obtain

\[w_1(x) \leq C, \quad \text{for } x = Ne_1, Ne_2, \ldots, Ne_n, \quad \forall N \in \mathbb{R}.\]

Next we let

\[M_i = \sup_{B(0, i)} w_1.\]

Then we claim that $M_i \leq 2M_{i/2} + C$. This can be observed by taking $e$ appropriately. Recall that there exists $h_3$ such that

\[a_{ij}\partial_{ij}(\Delta^2_{e_i}w_1 + h_3) \leq 0\]

and

\[|h_3(x)| \leq C(1 + |x|)^{-\beta_1}, \quad C \text{ is independent of } \|e\|.\]
Let $x_0$ be where $\Delta^2 w_1 + h_3$ attains its maximum on $B_i$, clearly $x_0 \in \partial B_i$. Let $e = \frac{2}{3} i + a$ where $a \in \mathbb{R}^n$ is chosen so that

$$|a| \sim 1, \quad f_{v,p}(x + e) = f_{v,p}(x), \quad \forall x \in \mathbb{R}^n.$$ 

Then for $i$ large

$$\Delta^2 e w_1(x - e) \leq C_i^{-\beta_1}$$

which is

$$w_1(x_0) \leq 2w_1(x_0 - e) - w(x_0 - 2e) + O(i^{2-\beta_1}).$$

Since $|x_0 - 2e|$ is in the neighborhood of 0, we have $|w_1(x_0 - 2e)| \leq C$. Thus

$$w_1(x_0) \leq 2 \max_{B(0,i/2)} w_1 + C.$$ 

The proof of Theorem 1.1 goes as follows. Let

$$g_i(y) = w_1(iy)/M_i.$$ 

We have

$$\max_{B_1} g_i \to 1, \quad \max_{B_{1/2}} g_i \geq \frac{1}{4}, \quad g_i(0) \to 0, \quad \text{as } i \to \infty.$$ 

On the other hand $1 - g_i$ is nonnegative in $B_1$ and the Harnack inequality guarantees that $g_i$ converges in $C^\alpha$ norm to $g$ in $B_1$. Also the Harnack inequality holds for $1 - g$. Moreover (2.25) and $\beta_1 \geq 2$ certainly lead to

$$g \text{ is concave in } B_1$$

because the perturbation term disappears as $i \to \infty$.

Let $l$ be a linear function that touches $g$ from above around 0 in $B_{1/2}$. This leads to a contradiction since on one hand $g(xe_i) \leq 0$ for all $i = 1, \ldots, n$ and $|x| < 1$. On the other hand $\max_{B_{1/8}} g \geq \frac{1}{4}$, there is no way to have $l - g \equiv C$ in $B_1$. Theorem 1.1 is established for $n \geq 5$.

**Step five: Proof of Theorem 1.1 for $n = 4$**

The main difference in the proof for $n = 4$ is on the construction of $h_{e_i}$. For $n = 4$ we first use a linear function to make $h_{e_i}(\pm e_i) = 0$ and set

$$h_{e_i}(Ne_i) = -\sum_{l=2}^{N} \sum_{k=l}^\infty a_k, \quad N = 1, 2, \ldots$$

The definition for $N = -1, -2, \ldots$ is similar. Then

$$0 \leq h_{e_i}(Ne_i) \leq C \log(2 + N), \quad n = 4.$$ 

Other than this there is no essential difference with the case $n \geq 5$. The same argument can still be employed to prove $M_i \leq C \log(2 + i)$ by way of contradiction. Suppose this is not the case, then $M_i / \log i \to \infty$ as $i$ is large. Then the same Thus we have obtained

$$w - h_2 = O(\log(2 + |x|)), \quad \text{in } \mathbb{R}^4.$$
Recall that \( w - h_2 \) satisfies
\[
\tilde{a}_{ij}\partial_{ij}(w - h_2) = 0, \quad \text{in} \quad \mathbb{R}^4.
\]
The following lemma says \( w - h_2 \) is a constant, which finishes the proof of Theorem 1.1 for \( n = 4 \).

**Lemma 2.1.** Let \( u \) solve
\[
a^*_ij \partial_{ij} u = 0, \quad \text{in} \quad \mathbb{R}^n
\]
where \( \lambda I \leq (a^*_ij(x)) \leq \Lambda I \) for all \( x \in \mathbb{R}^n \) and
\[
|u(x)| \leq C \log(2 + |x|), \quad x \in \mathbb{R}^n.
\]
Then \( u \equiv \text{constant} \).

**Proof of Lemma 2.1** For \( R > 1 \) let
\[
u_R(y) = u(Ry)/R^2, \quad |y| \leq 1.
\]
Then \( u_R \) satisfies
\[
a^*_ij(Ry)\partial_{ij} u_R = 0, \quad \text{in} \quad B_1
\]
and
\[
|u_R(y)| \leq C \log R/R^2, \quad \text{in} \quad B_1.
\]
By Krylov-Safonov’s [31] estimate
\[
(2.26) \quad \frac{|u_R(y) - u_R(0)|}{|y|^{\alpha}} \leq C \log R/R^2, \quad \forall y \in B_{1/2}
\]
where \( \alpha > 0 \) only depends on \( \lambda \) and \( \Lambda \). Clearly (2.26) can be written as
\[
|u(x) - u(0)| \leq C|x|^{\alpha}/R^\alpha, \quad \forall x \in B_{R/2}.
\]
Fix any \( x \in \mathbb{R}^n \), we let \( R \to \infty \), then \( u(x) = u(0) \). Lemma 2.1 is established. \( \square \)

The conclusion of Theorem 1.1 for \( n = 4 \) immediately follows from Lemma 2.1. \( \square \)

**Remark 2.4.** For \( n = 3 \) the construction of \( h_{e_i} \) should be
\[
h_{e_i}(Ne_i) = (N - 1)h_{e_i}(e_i) + (N - 2)h_{e_i}(0) + \sum_{l=2}^N \sum_{k=1}^{l-1} a_k, \quad N = 2, 3, ..
\]
then \( h_{e_i}(Ne_i) \leq CN \log N \). The construction for \( h(-Ne_i) \) is similar. Then one can prove by the same argument that
\[
w - h_2 = O\left((1 + |x|) \log(2 + |x|)\right), \quad \text{in} \quad \mathbb{R}^3.
\]
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