A COMBINATORIAL FORMULA FOR HOMOGENEOUS MOMENTS

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Abstract. We establish a combinatorial formula for homogeneous moments and give some examples where it can be put to use. An application to the statistical mechanics of interacting gauged vortices is discussed.

1. Introduction

In this article we shall prove the following.

Theorem 1. Suppose we are given \( n \) integrable real-valued functions \( J_1, J_2, \ldots, J_n \) on a measure space \( M \), an integer \( m \geq 0 \), and a non-zero constant \( C \), so that

\[
\int_M (v_1 J_1 + v_2 J_2 + \cdots + v_n J_n)^{2m} = C,
\]

for all \( (v_1, v_2, \ldots, v_n) \in \mathbb{R}^n \) with \( v_1^2 + v_2^2 + \cdots + v_n^2 = 1 \). Then there is a rational number

\[
I_{m,n} = \frac{\Gamma(\frac{1}{2})\Gamma(m + \frac{n}{2})}{\Gamma(m + \frac{1}{2})\Gamma(\frac{n}{2})}
\]

(depending only on \( m \) and \( n \)) so that

\[
\int_M (J_1^2 + J_2^2 + \cdots + J_n^2)^m = I_{m,n} C.
\]

We refer to quantities such as the integrals in the left-hand side of (1) and (3) as homogeneous moments (in the given functions \( J_1, \ldots, J_n \)). For applications, we have in mind that \( M \) should be a compact smooth manifold equipped with a volume form and that \( J_1, J_2, \ldots, J_n \) should be smooth functions. A simple example may be given by taking \( M = S^{n-1} \) with its usual round metric and \( J_1, J_2, \ldots, J_n \) to be the coordinate functions for the standard embedding \( S^{n-1} \hookrightarrow \mathbb{R}^n \). In this case, \( v_1 J_1 + v_2 J_2 + \cdots + v_n J_n : S^{n-1} \to \mathbb{R} \) is simply another coordinate function on the sphere and (1) is evident by rotational invariance. (The constant \( C \) may always be realised by setting \( v_n = 1 \) and all other \( v_j \)'s equal to zero: \( C = \int_M J_n^{2m} \).) In this example, reflection symmetry directly implies that the integral on the left-hand side of (1) would vanish if \( 2m \) were replaced by an odd integer; an easy argument shows that this would also be true in the general case of our hypothesis. Thus we are not losing generality by assuming that the degree of homogeneity in (1) is even.

Our motivation, however, stems from a more substantial example in which \( M \) is a Kähler manifold whose structure is invariant under the action of \( \text{SO}(3) \). In this case \( n = 3 \) and the three functions \( J_1, J_2, J_3 : M \to \mathbb{R} \) are the components of the associated moment map \( M \to \mathfrak{so}(3)^* \). The action of \( \text{SO}(3) \) arises because \( M \) is the moduli space of gauged vortices on the round 2-sphere [1]. Further discussion of this

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case is provided at the end of this article. When \( n = 3 \), the rational numbers \( I_{m,3} \) are, in fact, integers: \( I_{m,3} = 2m + 1 \). That the \( I_{m,n} \) are well-defined rational numbers is most easily proved without the explicit formula (2). Having done this in §2, we shall establish (2) in §3. In §4, we give a geometric setting for our hypothesis (1) with some examples, and conclude in §5 with a brief discussion of vortices on the 2-sphere.

2. Combinatorics

We may evidently extend (1) by homogeneity to conclude that

\[
\int_M (v_1 J_1 + v_2 J_2 + \cdots + v_n J_n)^{2m} = C (v_1^2 + v_2^2 + \cdots + v_n^2)^m,
\]

for all \((v_1, v_2, \ldots, v_n) \in \mathbb{R}^n\). Also, by simultaneously rescaling all the functions \( J_1, J_2, \ldots, J_n \) we may suppose without loss of generality that \( C = 1 \) and our task is now to compute

\[
I_{m,n} \equiv \int_M (J_1^2 + J_2^2 + \cdots + J_n^2)^m
\]

from this information. To do this we may regard (4) as the equality of two polynomials in the \( u \)-variables and deduce the equality of their coefficients. Examining the left hand side of (4), this means that we may compute

\[
\int_M J_1^{r_1} J_2^{r_2} \cdots J_n^{r_n} \quad \text{for all } r_1 + r_2 + \cdots + r_n = 2m.
\]

Indeed, since only integers are involved in expanding the two sides of (4), it follows that these integrals are all rational. This is more than enough to compute \( I_{m,n} \) and to conclude that these quantities are also rational.

Though the preceding argument is straightforward in principle, in practise it is almost useless in establishing formulae for \( I_{m,n} \). We shall rectify this deficiency in the following section. In the meantime, let us compute \( I_{3,3} \) by bare hands. We are supposing that

\[
\int_M (v_1 J_1 + v_2 J_2 + v_3 J_3)^6 = (v_1^2 + v_2^2 + v_3^2)^3
\]

and, by expanding as polynomials in \( v_1, v_2, v_3 \), we conclude that

\[
\int_M J_1^6 = \int_M J_2^6 = \int_M J_3^6 = 1
\]

\[
\int_M J_1^4 J_2^2 = \int_M J_1^2 J_2^4 = \cdots = \int_M J_2^2 J_4^4 = 1/5
\]

\[\int_M J_2^2 J_3^2 = 1/15\]

(and more besides). Therefore,

\[
\int_M (J_1^2 + J_2^2 + J_3^2)^3 = \int_M (J_1^6 + \cdots + 3J_1^4 J_2^2 + \cdots + 6J_1^2 J_2^2 J_3^2) = 3 \times 1 + 6 \times 3 \times 1/5 + 6 \times 1/15 = 7.
\]

Such naïve computations of \( I_{m,n} \) rapidly get out of hand for large \( m \) and \( n \).
3. Calculation of $I_{m,n}$

Given that the argument we used above for the existence of $I_{m,n}$ is purely combinatorial, it is clear that this quantity does not depend on the measure space $M$. Thus we can calculate $I_{m,n}$ by evaluating both the integral and the constant $C$ in (3) for a specific model where the hypothesis (1) is satisfied. We shall do this for the first example mentioned in the Introduction, where $M$ is $S^{n-1}$ with metric induced from the embedding $\iota : S^{n-1} \rightarrow \mathbb{R}^n$ and $J_j : S^{n-1} \rightarrow [-1, 1]$ are the standard cartesian coördinates.

In this example, the integral on the left-hand side of (3) is just the well-known volume of $S^{n-1}$,

\begin{equation}
\int_{S^{n-1}} (J_1^2 + \cdots + J_n^2)^m = \int_{S^{n-1}} 1 = \frac{2\pi^{n/2}}{\Gamma\left(\frac{n}{2}\right)}.
\end{equation}

To evaluate the constant $C = \int_{S^{n-1}} J_n^{2m}$ on the right-hand side, we start by using cylindrical coördinates to write the euclidean metric $g$ on in $\mathbb{R}^n$ as

\begin{equation}
g = dx^2 + dr^2 + r^2 g_{S^{n-2}}.
\end{equation}

Here, $x = J_n$, $r = \sqrt{J_1^2 + \cdots + J_{n-1}^2}$ and $g_{S^{n-2}}$ denotes the metric on $S^{n-2}$. On $S^{n-1} \subset \mathbb{R}^n$, we have the relation

\[ x^2 + r^2 = 1 \Rightarrow x \, dx = -r \, dr, \]

therefore (6) pulls back to $S^{n-1}$ as

\[ g_{S^{n-1}} = \iota^* g = \frac{dx^2}{1-x^2} + (1-x^2) g_{S^{n-2}}. \]

In these coördinates, the volume form on $S^{n-1}$ can then be written as

\begin{equation}
dvol_{S^{n-1}} = \sqrt{\det(g_{S^{n-1}})} \, dx \wedge dvol_{S^{n-2}} = (1-x^2)^{(n-3)/2} \, dx \wedge dvol_{S^{n-2}}.
\end{equation}

We observe in passing that the case $n = 3$ is special, as (7) then gives an identification of the volume forms on $S^2$ and on the cylinder in $\mathbb{R}^3$ defined by $r = 1$ and $|x| \leq 1$; this is the celebrated hat-box theorem of Archimedes, of which our equation (7) may be regarded as a generalisation to arbitrary dimensions. The constant on the right-hand side of (3) can now be evaluated as

\begin{equation}
\int_{S^{n-1}} J_n^{2m} = \int_{[-1,1] \times S^{n-2}} x^{2m} (1-x^2)^{(n-3)/2} \, dx \wedge dvol_{S^{n-2}}
\end{equation}

\[ = \int_{S^{n-2}} 1 \times \int_0^1 t^{m-1/2} (1-t)^{(n-3)/2} \, dt \]

\[ = \frac{2\pi^{(n-1)/2}}{\Gamma\left(\frac{m+1}{2}\right)} B\left(m + \frac{1}{2}, \frac{n-1}{2}\right) \]

\[ = \frac{2\pi^{(n-1)/2} \Gamma(m + \frac{1}{2})}{\Gamma(m + \frac{n+1}{2})}, \]

(8) where $B$ denotes Euler’s Beta-function [2]. Finally, we obtain $I_{m,n}$ as a quotient of (5) and (8), which yields (2) using $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$.

We would like to make two remarks on the formula (2):
A) Notice that (2) gives \( I_{0,n} = 1 \), and so it follows from the recursion relation \( \Gamma(z + 1) = z \Gamma(z) \) that \( I_{m,n} \) is obviously a rational number for all \( m \in \mathbb{N} \). More explicitly, we can write

\[
I_{m,n} = \begin{cases} 
\frac{2^{2m-1} (m + \frac{n}{2} - 1)! (m - 1)!}{(2m - 1)! (\frac{n}{2} - 1)!} & \text{if } n \text{ is even,} \\
\frac{(2m + n - 2)!(m - 1)! (\frac{n-2}{2})!}{2(2m - 1)!(n - 2)! (m + \frac{n-2}{2})!} & \text{if } n \text{ is odd.}
\end{cases}
\]

B) Our result for \( I_{m,n} \) as expressed in equation (2) turns out to be a value of Gauß’s hypergeometric function,

\[
I_{m,n} = \binom{2m, n-1; m + \frac{n}{2}; \frac{1}{2}}{m,n},
\]

which comes as a consequence of Gauß’s second summation theorem (cf. [3], p. 32). The property \( I_{m,n} \in \mathbb{Q} \) provides examples of the curious fact that the hypergeometric function sometimes assumes rational values when its argument and parameters are rational. This behaviour is general for the geometric series, but it is not understood under which circumstances it generalises to hypergeometric series.

4. Geometry and examples

We shall now describe a geometric setup where the hypothesis (1) naturally arises, and which inspires generalisations of Theorem 1. Let us suppose that \( V \) is a real inner product space and that a compact Lie group \( G \) has an orthogonal representation on \( V \), \( \rho : G \to O(V) \). Typically, we take as \( M \) a smooth manifold where \( G \) acts, equipped with an invariant volume form, which may in turn be induced by a Riemannian or symplectic structure on \( M \). We also assume that \( J : M \to V \) is a \( G \)-equivariant mapping. At this point, symmetry considerations will severely restrict the integrals

\[
\Phi_m(v) := \int_M \langle v, J \rangle^{2m}
\]

with \( v \in V \). In fact, for all \( g, \rho(g)v = \int_M \langle \rho(g)v, J \rangle^{2m} = \int_M \langle v, \rho(g^{-1})J \rangle^{2m} = \Phi_m(v),
\]

and this implies that \( \Phi(v) \) is an invariant polynomial restricted by Weyl’s classical invariant theory [4]. Because \( G \) is represented by orthogonal transformations, \( ||v||^{2m} \) is one of the invariant polynomials of degree \( 2m \); however, these ingredients are still not enough to enforce (1). In the following, we discuss three realisations of this general setup that lead to condition (1) being satisfied, or to more general conditions that still allow us to use the arguments in sections 2 and 3 to determine \( \int_M \langle J, J \rangle^{2m} \) by combinatorial means.

Example 1. Let \( G = SO(V) \), \( \rho \) be the defining representation, and \( J : M \to V \) be the inclusion of any invariant measurable set. Then necessarily \( \Phi_m(v) = C||v||^{2m} \). A particular case is when \( M \) is a single \( SO(V) \)-orbit embedded in \( V \), which yields the example that we used in the calculations of section 3.
Example 2. Let $G = \text{SO}(4)$, $V = \mathfrak{so}(4) \cong \bigwedge^2 \mathbb{R}^4$ and $\rho$ be the adjoint representation. The ring of invariants is freely generated by two polynomials of degree two \cite{5}, the squared norm $||v||^2$ and the pfaffian $\text{Pf}(v)$. Thus we can write
\[ \Phi_m(v) = \sum_{j=0}^{m} C_j ||v||^{2(m-j)} \text{Pf}(v)^j \]
for suitable real constants $C_j$. Our hypothesis in the form (4) will hold at least for $v$ in the hypersurface defined by $\text{Pf}(v) = 0$, and the argument in section 2 will still lead to a combinatorial formula for the homogeneous moments in (3). Whether (4) holds more generally depends on the choice of $M$ and $J$. For instance, one can show that, for $m = 1$, (4) holds for all $v \in V$ if and only if, say,
\[ \int_M J_{12} J_{34} = 0, \]
where the indices refer to the standard basis of $\bigwedge^2 \mathbb{R}^4$. An interesting example is when $M$ is the 4-manifold of simple 2-vectors in $\bigwedge^2 \mathbb{R}^4$ of unit norm, the adjoint orbit of $\text{SO}(4)$ given by the algebraic equations $\text{Pf}(v) = 0$ and $||v||^2 = 1$, and $J : M \hookrightarrow \bigwedge^2 \mathbb{R}^4$ is the inclusion; an easy check shows that (10) is satisfied, hence our hypothesis (1) holds true in this case.

Example 3. We take $M$ to be a symplectic manifold with moment map $J : M \to \mathfrak{g}^*; \rho$ will be the coadjoint representation, and we can use the Killing form to identify $\mathfrak{g}$ with $\mathfrak{g}^*$. Again, $\Phi_m$ is restricted to be a linear combination of the $G$-invariants in $\text{Sym}^2 m(\mathfrak{g})$ and $||v||^{2m}$ is one of them. In particular, if $G = \text{SO}(3)$, then any invariant must be a scalar multiple of $||v||^{2m}$ and our hypothesis (1) must hold. A particular case of this situation that illustrates the usefulness of our formula (3) will be discussed in the next section.

5. Interacting vortices on a 2-sphere

In this section, we describe an application of Theorem 1 to a natural setting where our hypothesis holds true. Let $N$ be a positive integer. We consider the measure space $M$ to be the moduli space $\mathcal{M}_N$ of $N$-vortices on a 2-sphere of radius $R > \sqrt{N}$ \cite{1}. This is just $\mathbb{C}P^N$ as a complex manifold, but equipped with a Kähler structure $\omega_{L^2}$ induced from a gauge-theoretic version of the $L^2$ norm on the space of fields. The associated Kähler metric $g_{rs}$ encodes information about the physics of vortices at low energies; for example, its geodesic flow gives a good approximation to the slow dynamics of the abelian Higgs model close to critical coupling \cite{6}. For $N > 1$, this Kähler structure is distinct from the Fubini–Study structure $\omega_{FS}$ on $\mathbb{C}P^N$, although it has been argued that \cite{7}
\[ \omega_{L^2} = 2\pi(R^2 - N)\omega_{FS} + o(R^2 - N) \]
as $R^2 \downarrow N$.

There is a local description of $\omega_{L^2}$, which we briefly recall here \cite{8, 9}. This uses the fact that $\mathbb{C}P^N \cong \text{Sym}^N(S^2) := (S^2)^N/\mathfrak{S}_N$. We denote by $\Delta \subset (S^2)^N$ the set of fixed points of elements of $\mathfrak{S}_N$, and let $z$ be a complex stereographic coordinate.
local quantities like $SO(3)$ on $M$ which in turn can be used to show that (11) is preserved by the induced action of $J$. Thus the Liouville measure associated to $\omega_{L^2}$ is SO(3)-invariant. In addition, there exists a moment map $J = (J_1, J_2, J_3) : \mathcal{M}_N \rightarrow \mathfrak{so}(3)^*$, for which our hypothesis (1) is obviously satisfied. Its components can be calculated as [10]

$$J_1 = 2\pi \sum_{r=1}^{N} \left( R^2 \frac{z_r + \bar{z}_r}{1 + |z_r|^2} + \frac{i}{2}(b_r + \bar{b}_r) \right),$$

$$J_2 = -2\pi i \sum_{r=1}^{N} \left( R^2 \frac{z_r - \bar{z}_r}{1 + |z_r|^2} - \frac{i}{2}(b_r - \bar{b}_r) \right),$$

$$J_3 = 2\pi \sum_{r=1}^{N} \left( R^2 \frac{1 - |z_r|^2}{1 + |z_r|^2} - (z_r b_r + 1) \right).$$

The metric on $\mathcal{M}_N$ has been used to study the statistical mechanics of a gas of vortices in the abelian Higgs model at critical coupling, both in the noninteracting case where the net forces experienced by the vortices are zero [11] and in presence of a background potential [9]. A more physically interesting situation would be the case where inter-vortex interactions are introduced. At the level of the dynamics on the moduli space, these would be described by a potential $V : \mathcal{M}_N \rightarrow \mathbb{R}$ invariant under the action of SO(3). The simplest nontrivial potential with this property is just (a multiple of) the square of the moment map,

$$V = \mu^2 ||J||^2 = \mu^2 (J_1^2 + J_2^2 + J_3^2),$$

where $\mu^2$ is just a coupling constant. In the corresponding model, the partition function for the gas of vortices is given by (see [9] for details)

$$Z = \frac{1}{(2\pi)^{2N}} \int_{T^*\mathcal{M}_N} \exp \left( -\left( \frac{1}{2\pi} \sum_{r,s} g_{rs} w_r w_s + \mu^2 ||J||^2 / T \right) \right) \frac{\omega^N_{L^2}}{(2N)!},$$

$$= \left( \frac{T}{2\pi} \right)^N \int_{\mathcal{M}_N} \exp \left( -\frac{\mu^2}{T} ||J||^2 \right) \frac{\omega^N_{L^2}}{N!}.$$
Here, \( w_r = \pi \sum_{s=1}^{N} g_{rs} \hat{z}_s \) denote canonical momenta to the moduli \( z_r \),

\[
\omega_{\text{can}} = \frac{1}{2} \sum_{r=1}^{N} (dz_r \wedge d\bar{w}_r + d\bar{z}_r \wedge dw_r)
\]
is the canonical symplectic form on the phase space \( T^*\mathcal{M}_N \), \( 2\pi h \) is Planck’s constant and \( T \) denotes the absolute temperature. It turns out that Theorem 1 can be used to find a closed expression for this partition function. In fact, after expanding the exponential, we can organise the remaining integral as a sum of integrals of powers of the angular momentum along a fixed axis by making use of (2) with \( n = 3 \):

\[
Z = \left( \frac{T}{2h^2} \right)^N \sum_{m=0}^{\infty} \frac{(-1)^m \mu^{2m}}{m! T^{2m}} (2m + 1) \int_{\mathcal{M}_N} J_3^{2m}.
\]

Now we apply the following result of [9], which is obtained as an application of the Duistermaat–Heckman formula to the circle action generated by \( J_3 \) on the symplectic manifold \( (\mathcal{M}_N, \omega_{L^2}) \):

\[
\int_{\mathcal{M}_N} J_3^{2m} = \sum_{j=0}^{N} \frac{(-1)^N-j(2m)!}{j!(N-j)!(N+2m)!} (2\pi(R^2 - N)(2j - N))^{N+2m}.
\]

The partition function can then be expressed as

\[
Z = \left( \frac{T}{2h^2} \right)^N \sum_{j=0}^{N} \sum_{m=0}^{\infty} \frac{(-1)^j m^{2m+1}}{j!(N-j)!(N+2m)!} \left( \frac{\mu^2}{T} \right)^m (4\pi(R^2 - N) \left( \frac{N}{2} - j \right))^{N+2m}
\]

\[
= \frac{1}{N!} \left( \frac{\tilde{A} T}{2h^2} \right)^N \sum_{j=0}^{N} \frac{(-1)^j}{j!(N-j)!} \left( \frac{N}{2} - j \right)^2 \mathcal{F}_2(1, \frac{3}{2}; \frac{N+1}{2}, \frac{N+2}{2}; -\frac{(\mu \tilde{A})^2}{T} \left( \frac{N}{2} - j \right)^2)
\]
in terms of the generalised hypergeometric function \( \mathcal{F}_2 \) [3], and where we introduced the area available for \( N \) vortices on the sphere

\[
\tilde{A} := 4\pi(R^2 - N).
\]

It is still a challenging problem to understand the physics determined by \( Z \) — in particular, it would be interesting to obtain an equation of state for this system (in some approximation) and analyse whether it allows for phase transitions.

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