Current status of relativistic core collapse simulations

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Abstract. With the first generation of ground-based gravitational wave laser interferometers already taking data, the availability of reliable waveform templates from astrophysical sources, which may help extract the signal from the anticipated noisy data, is urgently required. Gravitational stellar core collapse supernova has traditionally been considered among the most important astrophysical sources of potentially detectable gravitational radiation. Only very recently the first multidimensional simulations of relativistic rotational core collapse have been possible (albeit for models with simplified input physics), thanks to the use of conservative formulations of the hydrodynamics equations and advanced numerical methodology, as well as stable formulations of Einstein’s equations. In this paper, the current status of relativistic core collapse simulations is discussed, with the emphasis given to the modelling of the collapse dynamics and to the computation of the gravitational radiation in the existing numerical approaches. Work employing the conformally-flat approximation (CFC) of the 3+1 Einstein’s equations is reported, as well as extensions of this approximation (CFC+) and investigations within the framework of the so-called BSSN formulation of the 3+1 gravitational field equations (with no approximation for the spacetime dynamics). On the other hand, the incorporation of magnetic fields and the MHD equations in numerical codes to improve the realism of core collapse simulations in general relativity, is currently an emerging field where significant progress is bound to be soon achieved. The paper also contains a brief discussion of magneto-rotational simulations of core collapse, aiming at addressing the effects of magnetic fields on the collapse dynamics and on the gravitational waveforms.

1. Introduction
Understanding the dynamics of the gravitational collapse of the core of massive stars leading to supernova explosions still remains one of the primary problems in relativistic astrophysics, despite the continuous theoretical efforts towards that goal for a period of time spanning about the last 40 years (zeroing the counter in the seminal work of May and White in 1966 [1]). This problem stands as a distinctive example of a research field where essential progress has been accomplished through numerical modeling with increasing levels of complexity in the input physics and mathematics: hydrodynamics, gravity, magnetic fields, nuclear physics, equation of state, neutrino transport, etc. While studies based upon Newtonian physics are highly developed nowadays, state-of-the-art simulations still fail, broadly speaking, to generate successful supernova explosions under generic conditions (see e.g. [2, 3] for details on the degree of sophistication achieved in present-day supernova modelling, and [4] and references therein for a review on the mechanism of core collapse supernovae).

Progress in core collapse modelling has been gradually achieved at the expense of accuracy in the treatment of the hydrodynamics and of the gravitational field, by using Newtonian
physics. On the other hand, contributions towards addressing this problem from a “numerical relativity” perspective have also been made, where the focus has been to highlight those aspects of gravitational core collapse having to do with relativistic gravity, in particular, the emission of gravitational radiation from nonspherical collapse. Most of the effort has been spent in developing stable numerical schemes to solve the gravitational field equations and less significance has been given to additional aspects of the physics and microphysics of the collapse.

These efforts have paid off to the extent that the tide may have started to turn in recent years, as multidimensional relativistic approaches have become possible thanks to conservative formulations of the equations of general relativistic hydrodynamics and long-term stable formulations of Einstein’s gravitational field equations. Sophisticated numerical technology is now available to render relativistic simulations on the same footing, regarding numerical stability of the time-dependent, partial differential equations to integrate, than Newtonian simulations. While such advances also hold true in the case of the magneto-hydrodynamics (MHD) equations, the development still awaits here for a thorough numerical exploration. This paper, hence, aims at presenting an overview of such recent advances on the numerical modelling of relativistic stellar core collapse.

The standard model of gravitational core collapse supernovae (Types Ib/Ic/II) stands on the shoulders of the wealth of information increasingly available from numerical simulations. The fate of stellar cores can be characterised by the initial mass of the progenitor and by its metallicity (see [5]). For stars with masses below $9M_\odot$ stellar evolution stops at the white dwarf stage and no gravitational collapse follows. On the other hand Giant stars with radii $R \sim 50R_\odot$ and masses in the range $9M_\odot \leq M \leq 25M_\odot$, proceed during their lifetime through the entire nuclear burning stages, developing an onion-like burning-shell structure. At the centre an iron-group nuclei core develops, with roughly $1M_\odot$ and 1000 km radius. In the core support against gravity is provided by the pressure of a relativistic degenerate fermion gas, which can be described by an ideal fluid equation of state (EOS hereafter) with adiabatic index $\Gamma = 4/3$. The iron core, however, is unstable due to photo-disintegration of nuclei and electron captures. The neutrinos generated in such processes become trapped inside matter when the average density exceeds $10^{12}$ g cm$^{-3}$. Deleptonization leads to remarkable reductions in the pressure ($\Gamma < 4/3$) and the gravitational collapse of the core to nuclear matter densities in a timescale of $\sim 100$ ms is inevitable. Shortly after this, repulsive nuclear forces start dominating. The EOS stiffens, which leads to a large pressure increase, a bounce of the infalling matter, and the subsequent formation of a prompt shock. The initial kinetic energy of the shock is $\sim 5 - 8 \times 10^{51}$ erg, and thus sufficient, in principle, to power a supernova explosion. However, the prompt shock suffers severe energy losses as it propagates out, namely dissociating Fe nuclei into free nucleons, typically $\sim 8$ MeV/nucleon or $\sim 1.6 \times 10^{51}$ erg/0.1$M_\odot$. As a result the shock consumes its entire kinetic energy while it is still inside the iron core, transforming into a standing accretion shock in barely $t \sim 3$ ms.

It is widely accepted nowadays that, in general, the prompt shock mechanism just outlined is insufficient to power a supernova explosion. The responsible for the explosion are neutrinos. At about $t \sim 10$ ms after core bounce neutrinos are no longer trapped and start streaming out as the newly-born, hot proto-neutron star (PNS hereafter) further deleptonizes. Moreover, convective (Ledoux-type) instabilities in the PNS enhance remarkably the neutrino luminosity. As a result, the stalled shock is revived by neutrinos which transfer their energy and momentum causing a delayed shock, strong enough to propagate out and eventually disrupt the envelope of the star.

The role played by rotation in core collapse progenitors is still primarily an open issue, only being currently incorporated in stellar evolution codes [6]. Observations of surface velocities support the fact that a large fraction of progenitor cores is rapidly rotating. However, magnetic torques can spin down the core by dynamo action which couples to the outer layers of the star [7, 8, 9, 6]. On the other hand PNS have periods of $\sim 10 - 15$ ms, consistent...
with observations. There are various possible ways to rapid rotation: a) Massive stars with $M \geq 25M_\odot$ may evolve so rapidly that there is not sufficient time to efficiently slow down the core. In such Wolf-Rayet stars strong winds could expel the envelope, supressing dynamo action; b) Deep rotational mixing in massive OB stars could prevent the RSG phase; c) Loss of envelope in binary evolution could be a way to also avoid the RSG phase; and d) accretion induced collapse can also naturally provide rapidly-rotating cores. A recent estimation by [10] indicates that $\sim 1\%$ of all stars with $M \geq 10M_\odot$ could have rapidly rotating cores.

Numerical simulations of stellar core collapse are highly motivated nowadays by the prospects of direct detection of the gravitational waves emitted. In core collapse events gravitational waves are dominated by a burst associated with the hydrodynamical bounce. If rotation is present, the post-bounce wave signal shows large amplitude oscillations associated with pulsations in the collapsed core [12, 13] and (possibly) low-$T/|W|$ rotational dynamical instabilities [14, 27] (here $T$ stands for the kinetic rotational energy and $W$ for the potential energy).

Müller [15] obtained the first numerical evidence of the low gravitational wave efficiency of the core collapse scenario. His Newtonian, axisymmetric numerical simulations revealed that an amount of energy smaller than $10^{-6}M_\odot c^2$ is radiated as gravitational waves. Subsequently, Bonazzola and Marck [16] attempted the first 3D simulations of the infall phase using pseudospectral methods, confirming the axisymmetric results of [15] on the low amount of energy radiated in gravitational waves, irrespective of the initial conditions. Later on, Zwerger and Müller [12] performed a comprehensive parameter study of the gravitational radiation produced in core collapse events using a simple EOS [17] and Newtonian physics. On the other hand, it has recently been shown that gravitational waves from convection have preeminence over the purely burst signal of the core bounce on longer timescales [18]. In this case the inherent long emission timescale can yield high energy in a continuous signal. We also note that as general relativity counteracts the stabilizing effect of rotation, a bounce caused by rotation has to occur at larger densities than in the Newtonian case. This motivated relativistic simulations, which have become possible since the early 2000s [19, 20, 21, 22, 23, 25, 26, 24, 27].

The distinctive burst gravitational waveform signals present in core collapse events reflect the underlying collapse dynamics. Two waveform types were first identified and cataloged as Type I and Type II by [11]. In the subsequent investigation of [12] the waveform types were extended to four (Types III and $\Pi_{\text{rot}}$ were added). These types of signals are depicted in Fig. 1. A core collapse supernova is a very energetic event, involving a binding energy of about $E_{\text{bind}} \sim 3 \times 10^{53}$ erg, i.e. a significant fraction of $M_{\text{core}} c^2 \sim M_\odot c^2 \sim 2 \times 10^{54}$ erg. However, the deformation of spacetime and the signal amplitude are very small. The typical (dimensionless) signal strength of these waves is $h \sim 10^{-20}$ for a (Galactic) core collapse event up to a distance of 10 kpc, while their frequencies lie roughly between 500 Hz and 1 kHz. Realistic models of core collapse supernovae yield $E_{\text{GW}} \sim 10^{-6} - 10^{-9} E_{\text{bind}}$. These values are some orders of magnitude smaller than those of the most promising astrophysical sources of gravitational radiation such as binary black hole mergers, where $E_{\text{GW}} \sim 10^{-2} E_{\text{bind}}$. As a result the energy released in gravitational waves is unimportant for core collapse dynamics. Successful detection of the gravitational radiation from core collapse faces, hence, two main problems: 1) The smallness of the signal strength, and 2) the complexity of the burst signal from bounce. Signal analysis may be painstakingly close to searching for a needle in a haystack, a situation which can be alleviated with the aid of numerical simulations. These are essential to pave the road for successful detection as they can provide waveform templates to match-filter the experimental data.

Neutron stars, furthermore, have intense magnetic fields ($\sim 10^{12} - 10^{13}$ G) or even larger at birth ($\sim 10^{14} - 10^{15}$ G), as inferred from studies of anomalous X-ray pulsars and soft gamma-ray repeaters [28]. In extreme cases such as magnetars the magnetic field can be so strong to affect the internal structure of the star [29]. The presence of such intense magnetic fields render magneto-rotational core collapse simulations mandatory. We note that as early as 1979
magneto-rotational core collapse was already proposed by [30] as a plausible supernova explosion mechanism.

The weakest point of all existing magneto-rotational core collapse simulations to date is the fact that the strength and distribution of the initial magnetic field in the core are basically unknown. If the magnetic field is initially weak, there exist several mechanisms which may amplify it to values which can have impact on the dynamics, among them differential rotation (Ω-dynamo) and the magneto-rotational instability (MRI hereafter). The former transforms rotational energy into magnetic energy, winding up any seed poloidal field into toroidal field, while the latter leads to an exponential growth of the field strength. MRI is present as long as the radial gradient of the angular velocity of the fluid is negative, a situation which holds in core collapse simulations (see below).

In recent years, an increasing number of authors have performed axisymmetric magneto-rotational core collapse simulations (within the so-called ideal MHD limit) employing Newtonian treatments of the magneto-hydrodynamics, the gravity, and of the microphysics [31, 32, 33, 34, 35, 36, 37, 38, 39]. We note, however, that no single relativistic magneto-rotational core collapse simulations is yet available. Specific magneto-rotational effects on the gravitational wave signature were first studied in detail by [33] and [37], who found differences with purely hydrodynamical models only for very strong initial fields ($\geq 10^{12}$ G). The most exhaustive parameter study of magneto-rotational core collapse to date has been carried out very recently by [40, 41]. The axisymmetric simulations of [40, 41] have employed rotating polytropes, Newtonian (and modified Newtonian) gravity and hydrodynamics, and ad-hoc initial poloidal magnetic field distributions (as no self-consistent solution is yet known). These authors have found that for weak initial fields ($\leq 10^{11}$ G, which is the most relevant case, astrophysics-wise) there are no differences in the collapse dynamics nor in the resulting gravitational wave signal, when comparing with purely hydrodynamical simulations. However, strong initial fields

Figure 1. Types of burst gravitational waveforms present in stellar core collapse, as cataloged by [11] and [12].
manage to slow down the core efficiently (leading even to retrograde rotation in the PNS) which causes qualitatively different dynamics and gravitational wave signals. For some models [40] even find highly bipolar, jet-like outflows.

As mentioned before the aim of this paper is to present an overview of the status of the numerical modelling of relativistic stellar core collapse to neutron stars. Within such scope, emphasis will be given to those aspects of core collapse related to the emission of gravitational radiation from such astrophysical source. The interested reader is addressed to [42, 43] and references therein for additional material. The paper is organized as follows: Section 2 presents the framework to perform relativistic simulations of core collapse, namely the equations of general relativistic hydrodynamics and magnetohydrodynamics, and the gravitational field equations in ways suitable for numerical work. Next, Section 3 presents an overview of the existing core collapse simulations and their most significant results. The paper closes with a short summary in Section 4.

2. Framework for relativistic core collapse simulations

We turn to discuss the basic ingredients that need to be considered in order to perform relativistic core collapse simulations, both for the magnetohydrodynamics and for the gravity.

2.1. General relativistic hydrodynamics

The general relativistic hydrodynamics (GRHD) equations are the local conservation laws of momentum and energy, encoded in the stress-energy tensor \( T^{\mu\nu} \), and of the matter density, \( J^\mu \) (the continuity equation)

\[
\nabla_\mu T^{\mu\nu} = 0, \quad \nabla_\mu J^\mu = 0, \quad (1)
\]

where \( \nabla_\mu \) stands for the 4-dimensional covariant derivative. (Throughout the paper Greek indices run from 0 to 3 while Latin indices run from 1 to 3; geometrized units \( G = c = 1 \) are used.) The density current is given by \( J^\mu = \rho u^\mu \), where \( u^\mu \) is the 4-velocity of the fluid and \( \rho \) its rest-mass density. We assume a perfect fluid stress-energy tensor

\[
T^{\mu\nu} = \rho h u^\mu u^\nu + pg^{\mu\nu}, \quad (2)
\]

where \( p \) is the pressure, \( h \) is the specific enthalpy, \( h = 1 + \varepsilon + p/\rho \), \( \varepsilon \) being the specific internal energy, and \( g^{\mu\nu} \) is the spacetime metric tensor.

The previous system of equations is closed once an EOS is chosen, i.e. a constitutive relation of the form \( p = p(\rho, \varepsilon) \). In the so-called test-fluid approximation the dynamics of the matter fields is completely described by the previous conservation laws and the EOS. If such approximation does not hold, these equations must be solved in conjunction with Einstein’s equations for the gravitational field which describe the evolution of a dynamical spacetime.

The approach most commonly employed to solve Einstein’s equations in Numerical Relativity is the so-called Cauchy or 3+1 formulation (IVP). Introducing a coordinate chart \((x^0, x^i)\) the 3+1 line element reads

\[
ds^2 = -(\alpha^2 - \beta^i\beta_i)dx^0dx^0 + 2\beta_idx^idx^0 + \gamma_{ij}dx^idx^j, \quad (3)
\]

where \( \alpha \) is the lapse function, \( \beta^i \) is the shift vector, and \( \gamma_{ij} \) is the spatial 3-metric induced on each spacelike slice. Hence, the above conservation equations read

\[
\frac{\partial}{\partial x^\mu} \sqrt{-g} J^\mu = 0, \quad \frac{\partial}{\partial x^\mu} \sqrt{-g} T^{\mu\nu} = -\sqrt{-g} \Gamma^\nu_{\mu\lambda} T^{\mu\lambda}, \quad (4)
\]

where \( g = \det(g_{\mu\nu}) = \alpha \sqrt{\gamma} \), \( \gamma = \det(\gamma_{ij}) \) and \( \Gamma^\nu_{\mu\lambda} \) are the so-called Christoffel symbols.
In [44] the GRHD equations were written as a first-order, flux-conservative, hyperbolic system, amenable to numerical work [42]:

\[
\frac{1}{\sqrt{-g}} \left( \frac{\partial \sqrt{\gamma} U(w)}{\partial x^0} + \frac{\partial \sqrt{-g} F^i(w)}{\partial x^i} \right) = S(w).
\]

(5)

With respect to an Eulerian observer and in terms of the primitive variables, \( w = (\rho, v^i, \varepsilon) \), where \( v^i \) is the 3-velocity of the fluid, the state vector \( U \) (conserved variables) and the vectors of fluxes \( F \) and source terms \( S \), are given by

\[
U(w) = (D, S_j, \tau),
\]

(6)

\[
F^i(w) = \left( D\tilde{v}^i, S_j \tilde{v}^i + \rho \delta_j^i, \tau \tilde{v}^i + p v^i \right),
\]

(7)

\[
S(w) = \left( 0, T^{\mu\nu} \left( \frac{\partial g_{\nu j}}{\partial x^\mu} - \Gamma^j_{\nu\delta} \right), \alpha \left( T^{\mu\nu} \frac{\partial \ln \alpha}{\partial x^\mu} - T^{\mu\nu} \Gamma^0_{\nu\mu} \right) \right),
\]

(8)

with \( \tilde{v}^i = v^i - \beta^i/\alpha \) and \( \delta_j^i \) is the Kronecker delta. The conserved quantities in Eq. (6) are the relativistic densities of mass, momenta, and energy, respectively, defined as \( D = \rho W \), \( S_j = \rho h W^2 v_j \), and \( \tau = \rho h W^2 - p - D \). In these definitions \( W \) stands for the Lorentz factor, defined as \( W = \alpha u^0 \) (not to confuse with the potential energy of a rotating star mentioned before for which the same letter is usually employed).

### 2.2. General relativistic magneto-hydrodynamics

General relativistic MHD is concerned with the dynamics of relativistic, electrically conducting fluids (plasma) in the presence of magnetic fields. Here, we concentrate on purely ideal GRMHD, neglecting the presence of viscosity and heat conduction in the limit of infinite conductivity (perfect conductor fluid). As the GRHD equations discussed before, the GRMHD equations can also be cast in first-order, flux-conservative, hyperbolic form. The discussion reported here follows the derivation of these equations as presented in [45] to which the reader is addressed for details.

In terms of the (Faraday) electromagnetic tensor \( F^{\mu\nu} \), Maxwell’s equations read

\[
\nabla_\nu \ast F^{\mu\nu} = 0, \quad \nabla_\nu F^{\mu\nu} = \mathcal{J}^\mu,
\]

(9)

where \( F^{\mu\nu} = U^\mu E^\nu - U^\nu E^\mu - \eta^{\mu\rho\lambda\delta} U_{\lambda} B_{\delta} \), its dual \( \ast F^{\mu\nu} = \frac{1}{2} \eta^{\mu\nu\lambda\delta} F_{\lambda\delta} \), and \( \eta^{\mu\nu\lambda\delta} = \frac{1}{\sqrt{-g}} [\mu\nu\lambda\delta] \), where [\( \mu\nu\lambda\delta \)] is the completely antisymmetric Levi-Civita symbol. \( E^\mu \) and \( B^\mu \) stand for the electric and magnetic fields measured by an observer with 4-velocity \( U^\mu \), respectively, and \( \mathcal{J}^\mu \) is the electric 4-current, defined as \( \mathcal{J}^\mu = \rho_q u^\mu + \sigma F^{\mu\nu} u_\nu \), where \( \rho_q \) is the proper charge density and \( \sigma \) is the electric conductivity.

Maxwell’s equations can be simplified if the fluid is a perfect conductor. In this case \( \sigma \) is infinite and, to keep the current finite, the term \( F^{\mu\nu} u_\nu \) must vanish, which results in \( E^\mu = 0 \) for a comoving observer. This case corresponds to the so-called ideal MHD condition. Under this assumption the electric field measured by the Eulerian observer has components

\[
E^0 = 0, \quad E^i = -\alpha \eta^{0ij} v_j B_k,
\]

(10)

and Maxwell’s equations \( \nabla_\nu \ast F^{\mu\nu} = 0 \) reduce to the divergence-free condition plus the induction equation for the evolution of the magnetic field

\[
\frac{\partial (\sqrt{\gamma} B^i)}{\partial x^i} = 0, \quad \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^i} (\sqrt{\gamma} B^i) = \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^j} \left\{ \sqrt{\gamma} (\alpha \tilde{v}^i B^j - \alpha \tilde{v}^j B^i) \right\}.
\]

(11)
In addition, for a fluid endowed with a magnetic field the stress-energy tensor is the sum of that of the fluid and that of the electromagnetic field, $T_{\mu \nu} = T_{\mu \nu}^{\text{Fluid}} + T_{\mu \nu}^{\text{EM}}$, where $T_{\mu \nu}^{\text{Fluid}}$ is given by Eq. (2) for a perfect fluid. On the other hand $T_{\mu \nu}^{\text{EM}}$ can be obtained from the Faraday tensor as follows:

$$T_{\mu \nu}^{\text{EM}} = F_{\mu \lambda} F_{\nu}^{\lambda} - \frac{1}{4} g_{\mu \nu} F_{\lambda \delta} F^{\lambda \delta},$$

which, in ideal MHD, can be rewritten as

$$T_{\mu \nu}^{\text{EM}} = \left( u^\mu u^\nu + \frac{1}{2} g^{\mu \nu} \right) b^2 - b^\mu b^\nu,$$

where $b^\mu$ is the magnetic field vector of fluxes of the GRMHD system of equations read:

$$U(\mathbf{w}) = (D, S_j, \tau, B^k),$$
$$F(\mathbf{w}) = (D \vec{v}^i, S_j \vec{v}^i + p^* \delta_j^i - b_j B^i/W, \tau \vec{v}^i + p^* v^i - \alpha b^0 B^i/W, \vec{v}^i B^k - \vec{v}^k B^i),$$

where the conserved quantities are now defined as $D = \rho W$, $S_j = \rho h^* W^2 v_j - \alpha b^0 b_j$, and $\tau = \rho h^* W^2 - p^* - \alpha^2 (b^0)^2 - D$. The corresponding vector of sources coincides with the one given by Eq. (8) save for the use of the complete (fluid plus electromagnetic field) stress-energy tensor (the magnetic field evolution equation is source-free).

### 2.3. CFC metric equations

Until recently the workhorse formulation of Einstein’s equations in Numerical Relativity has been the so-called ADM 3+1 formulation (see e.g. [46] and references therein). Given a choice of lapse $\alpha$ and shift vector $\beta^i$, Einstein’s equations in the 3+1 formalism split into evolution equations for the 3-metric $\gamma_{ij}$ and constraint equations that must be satisfied on any time slice. The evolution equations are

$$\partial_t \gamma_{ij} = -2 \alpha K_{ij} + \nabla_i \beta_j + \nabla_j \beta_i,$$

$$\partial_t K_{ij} = -\nabla_i \nabla_j \alpha + \alpha \left( R_{ij} + K K_{ij} - 2 K_{im} K^m_{\ j} \right) + \beta^m \nabla_m K_{ij} + K_{im} \nabla_j \beta^m + K_{mj} \nabla_i \beta^m - 8 \pi \alpha \left( T_{ij} - \frac{1}{2} \gamma_{ij} T^m_{\ m} + \frac{1}{2} \beta^m \gamma_{ij} \right),$$

where $K_{ij}$ is the extrinsic curvature of the 3-dimensional time slice, $R_{ij}$ is the Ricci tensor of the induced 3-metric, and $\nabla_i$ is the covariant three-space derivative.

In addition, the Hamiltonian constraint equation is

$$R + K^2 - K^{ij} K_{ij} = 16 \pi \rho_E,$$

with $R$ being the Ricci scalar, and $K$ the trace of the extrinsic curvature, and the three momentum constraint equations are

$$\nabla_i \left( K^{ij} - \gamma^{ij} K \right) = 8 \pi S^i.$$

...
In the above equations, the quantities $T^{ij}$, $S^i$, and $\rho_E$ are the spatial components of the stress-energy tensor, the momenta and the total energy, respectively, and are obtained by projecting the four stress-energy tensor using $n_\mu$, the normal to the slice.

If the spatial 3-metric is assumed to be conformally flat, i.e. $\gamma_{ij} = \phi^4 \delta_{ij}$, with $\phi$ being the conformal factor, and under the additional assumption of a maximally-sliced spacetime ($K = 0$), the ADM 3+1 equations reduce to a system of five coupled, nonlinear elliptic equations for the lapse function, the conformal factor, and the shift vector [47, 48]:

\[
\hat{\Delta} \phi = -2\pi \phi^5 \left( \rho h W^2 - P + \frac{K_{ij} K^{ij}}{16\pi} \right), \tag{21}
\]

\[
\hat{\Delta} (\alpha \phi) = 2\pi \alpha \phi^5 \left( \rho h (3W^2 - 2) + 5P + \frac{7K_{ij} K^{ij}}{16\pi} \right), \tag{22}
\]

\[
\hat{\Delta} \beta^i = 16\pi \alpha \phi^4 S^i + 2\phi^{10} K^{ij} \hat{\nabla}_j \left( \frac{\alpha}{\phi^5} \right) - \frac{1}{3} \hat{\nabla}^i \hat{\nabla}_k \beta^k, \tag{23}
\]

where $\hat{\nabla}_i$ and $\hat{\Delta}$ are the flat space Nabla and Laplace operators, respectively.

The simplified Einstein’s equations introduced by the conformally-flat condition (CFC) are adequate in situations for which the deviations from spherical symmetry are not extreme. They have been used to simulate rotational core collapse in [19, 21].

2.4. CFC+ metric equations

Recently, Cerdá-Durán et al. [24] have extended the above CFC system of equations (and the corresponding CFC simulations reported in [21]) by the incorporation of additional degrees of freedom in the approximation, which render the spacetime metric exact up to the second post-Newtonian order. Despite the extension of the five original elliptic CFC metric equations (21)-(23) by additional equations, the final system in the new formulation is still elliptic. This formulation is called CFC+.

The natural way of improving the CFC approximation is through a post-Newtonian expansion of the traceless part of the three-metric:

\[
\gamma_{ij} = \phi^4 \gamma_{ij} + h_{ij}^{TT} = \gamma_{ij}^{\text{CFC}} + \left[ h_{ij}^{2\text{PN}} + O \left( \frac{1}{c^4} \right) \right], \tag{24}
\]

where $h_{ij}^{2\text{PN}}$ is the leading order in the expansion of $h_{ij}^{TT}$. The CFC+ metric is obtained by truncating Eq. (24) at second post-Newtonian terms in the traceless and transverse part. We note that CFC+ becomes exact for spherically symmetric spacetimes, as CFC, since in this case the TT part vanishes, while in a general nonspherical spacetime, CFC+ behaves as a second post-Newtonian approximation of the metric.

The (lengthy) derivation of the CFC+ equations is described in detail in [24]. Here, we simply summarize in a nutshell the main steps in the procedure to calculate the CFC+ metric:

1. Calculate “Newtonian” potential (1 Poisson equation):

\[
\hat{\Delta} U = -4\pi GD^* \quad \text{with} \quad D^* = \sqrt{\bar{\gamma}} D, \tag{25}
\]

where $\gamma = \gamma / \gamma^\star$, $\gamma^\star$ being the determinant of the flat 3-metric.

2. Calculate intermediate (scalar, vector, and tensor) potentials (16 linear Poisson equations):

\[
\hat{\Delta} S = -4\pi \frac{S^i_i S^j_j}{D^*Ix^ix^j}, \tag{26}
\]
\[ \hat{\Delta} S_i = \left[ -4\pi \frac{S_i^* S_j^*}{D^*} - \hat{\nabla}_i U \hat{\nabla}_j U \right] x^j, \] (27)

\[ \hat{\Delta} T_i = \left[ -4\pi \frac{S_i^* S_j^*}{D^*} - \hat{\nabla}_i U \hat{\nabla}_k U \right] \hat{\gamma}^{ij} \hat{\gamma}^{kl} x^l, \] (28)

\[ \hat{\Delta} R_i = \hat{\nabla}_i (\hat{\nabla}_j U \hat{\nabla}_k U x^j x^k), \] (29)

\[ \Delta S_{ij} = -4\pi S_i^* S_j^* \hat{\nabla}_i U \hat{\nabla}_j U, \] (30)

where superscript * in the hydrodynamical fields follows the same notation of Eq. (25) (i.e. the fields are multiplied by \( \sqrt{\bar{\gamma}} \)).

3. Calculate \( h_{ij}^{TT} \):
\[
\begin{align*}
  h_{ij}^{TT} &= \frac{1}{2} S_{ij} - 3x^k \hat{\nabla}((S_{lj})_k) + \frac{5}{4} \hat{\gamma}_{jm} x^m \hat{\nabla}_i \left( \hat{\gamma}^{kl} S_{kl} \right) + \frac{1}{4} x^k x^l \hat{\nabla}_i \hat{\nabla}_j S_{kl} \\
  &\quad + 3 \hat{\nabla}_j (S_{ij}) - \frac{1}{2} x^k \hat{\nabla}_i \hat{\nabla}_j S_k + \frac{1}{4} \hat{\nabla}_i \hat{\nabla} S - \frac{5}{4} \hat{\nabla}_i T_j - \frac{1}{4} \hat{\nabla}_i R_j \\
  &\quad + \hat{\gamma}_{ij} \left[ \frac{1}{4} \hat{\gamma}^{kl} S_{kl} + x^k \hat{\gamma}^{lm} \hat{\nabla}_l S_{kl} - \hat{\gamma}^{kl} \hat{\nabla}_k S_l \right].
\end{align*}
\] (31)

4. Calculate the modified CFC equations (5 nonlinear Poisson-like equations):
\[
\begin{align*}
  \hat{\Delta}(\alpha \phi) &= 2\pi \alpha \phi^5 \left( E + 2S + \frac{7K_{ij} K^{ij}}{16\pi} \right) - \frac{1}{c^2} \hat{\gamma}^{ik} \hat{\gamma}^{jl} h_{ij}^{TT} \hat{\nabla}_k \hat{\nabla}_l U, \\
  \hat{\Delta} \phi &= -2\pi \phi^5 \left( E + \frac{K_{ij} K^{ij}}{16\pi} \right), \\
  \hat{\Delta} \beta^i &= 16\pi \alpha \phi^4 S^i + 2\tilde{K}^{ij} \hat{\nabla}_j \left( \frac{\alpha}{\phi^6} \right) - \frac{1}{3} \hat{\nabla}^i \hat{\nabla} \beta^k,
\end{align*}
\] with
\[
\begin{align*}
  \tilde{K}_{ij} &= \frac{1}{\alpha} \left( \hat{\nabla}_i \beta_j + \hat{\nabla}_j \beta_i - \frac{2}{3} \gamma_{ij} \hat{\nabla} \beta^k \right), \\
  K_{ij} K^{ij} &= \frac{1}{\alpha^2} \tilde{K}_{ij} \tilde{K}^{ij}.
\end{align*}
\] (34) (35)

5. Finally, calculate the metric:
\[
\begin{align*}
  g_{ij} &= \gamma_{ij} = \phi^4 \hat{\gamma}^{ij} + h_{ij}^{TT}, \\
  g_{00} &= \gamma_{ij} \beta^i \beta^j - \alpha^2, \\
  g_{0i} &= \beta_i.
\end{align*}
\] (36) (37) (38)

We note that the main limitation of both CFC and CFC+ is the fact that the gravitational wave degrees of freedom are lost in the approximations made. As a result, gravitational waves have to be extracted in an approximate way directly from the sources (e.g. resorting to the standard quadrupole formula).
2.5. BSSN metric equations

The ADM 3+1 metric equations have been shown over the years to be intrinsically unstable for long-term numerical simulations, especially for those dealing with black hole spacetimes. Recently, there have been diverse attempts to reformulate those equations into forms better suited for numerical investigations (see [49, 50, 46] and references therein). Among the various approaches proposed, the so-called BSSN reformulation of the ADM system [49, 50] stands as very appropriate for long-term stable numerical work.

The main idea underlying this approach is to remove the mixed second order derivatives appearing in the Ricci tensor (see Eq. (18)) by introducing auxiliary variables so that the evolution equations look like wave equations for the 3-metric and the extrinsic curvature. To achieve this the BSSN formulation makes use of a conformal decomposition of the 3-metric, $\tilde{\gamma}_{ij} = e^{-4\phi}\gamma_{ij}$ and the trace-free part of the extrinsic curvature, $A_{ij} = K_{ij} - \gamma_{ij}K/3$, with the conformal factor $\phi$ chosen to satisfy $e^{4\phi} = \gamma^{1/3} \equiv \det(\gamma_{ij})^{1/3}$. In this formulation, as shown in Ref. [50], in addition to the evolution equations for the conformal 3-metric $\tilde{\gamma}_{ij}$ and the conformal-traceless extrinsic curvature variables $\tilde{A}_{ij} = e^{-4\phi}(K_{ij} - \gamma_{ij}K/3)$, there are evolution equations for the conformal factor $\phi$, the trace of the extrinsic curvature $K$, and the “conformal connection functions” $\tilde{\Gamma}^i$ [50], defined as

$$\tilde{\Gamma}^a = \tilde{\gamma}^{ij}\tilde{\gamma}^a_{ij} = -\partial_j\tilde{\gamma}^{ai}.$$  (39)

The final set of BSSN evolution equations reads:

\[ (\partial_t - L_{\beta})\tilde{\gamma}_{ij} = -2\alpha\tilde{A}_{ij}, \]

\[ (\partial_t - L_{\beta})\phi = -\frac{1}{6}\alpha K, \]

\[ (\partial_t - L_{\beta})K = -\tilde{\gamma}^{ij}\nabla_i \nabla_j \alpha + \alpha \left[ \tilde{A}_{ij}\tilde{A}^{ij} + \frac{1}{3}K^2 + \frac{1}{2}(\rho + S) \right], \]

\[ (\partial_t - L_{\beta})\tilde{A}_{ij} = e^{-4\phi} \left[ -\nabla_i \nabla_j \alpha + \alpha \left( R_{ij} - S_{ij} \right) \right]_{TF} + \alpha \left( K\tilde{A}_{ij} - 2\tilde{A}_{ik}\tilde{A}^k_j \right), \]

\[ (\partial_t - L_{\beta})\tilde{\Gamma}^i = -2\tilde{A}^{ij}\partial_j \alpha + 2\alpha \left( \tilde{\Gamma}^j_k \tilde{A}^{kj} - \frac{2}{3}\tilde{\gamma}^{ij}\partial_j K - \tilde{\gamma}^{ij}S_j + 6\tilde{A}^{ij}\partial_j \phi \right) \]

\[ + \partial_j \left( \beta^k\tilde{\gamma}^{ij} - 2\tilde{\gamma}^{m(j}\partial_m \beta^i) + \frac{2}{3}\tilde{\gamma}^{ij}\partial_i \beta^j \right), \]  (44)

where $L_{\beta}$ stands for the usual Lie derivative.

BSSN (or slight modifications thereof) is currently the standard 3+1 formulation in Numerical Relativity [46]. Long-term stable applications include strongly-gravitating systems such as neutron stars (both, isolated [51, 52] and binaries [53]) and single and binary black holes. Such binary black hole evolutions, possibly the grandest challenge of Numerical Relativity since the very beginning of the field, have only been possible in the last few months (see [54] and references therein).

3. Relativistic core collapse simulations

Once the barebones framework to perform relativistic simulations of core collapse has been laid out, we turn to review the present status of such simulations. Before doing this, however, we notice that we do not consider investigations dealing with pulsations of relativistic, rotating stars (naturally excited in core collapse events) and the associated normal-mode-frequency characterization for asteroseismology, addressing the interested reader to [55, 56] and references therein for details. In addition, the nonrotating core collapse simulations of [23] performed using characteristic numerical relativity are not covered either in this review (see [23] for details).
3.1. CFC/CFC+ simulations with polytropes
The first axisymmetric simulations of rotational core collapse to neutron stars which incorporated a relativistic treatment of gravity (albeit approximate) were performed by [19, 20, 21]. These simulations employed simplified models to account for the thermodynamics of the process, in the form of a polytropic EOS conveniently modified to account for the stiffening of the matter once nuclear matter density is reached [17]. The inclusion of relativistic effects results primarily in the possibility to achieve deeper effective potentials. Moreover, higher densities than in Newtonian models are reached during bounce, and the resulting PNS is more compact.

A representative example of these simulations (model A3B2G4 in [21]) is depicted in Fig. 2 which shows the evolution of the maximum density (left panel), which for this model coincides with the central density, and the corresponding gravitational wave signal. The latter, as mentioned before, is characterised by a distinctive burst associated with the hydrodynamical bounce followed by a strongly-damped ringdown phase of the newly-born PNS. While indeed the maximum density reaches higher values in relativistic gravity (solid line in Fig. 2) than in Newtonian gravity (dashed line), the gravitational wave signals plotted in the right panel of Fig. 2 are of comparable amplitudes. In fact, the simulations performed by [21] revealed that this is a general trend, the gravitational radiation emitted in the relativistic models being strikingly similar to those previously obtained from Newtonian simulations [12]. On average, thus, the CFC simulations of [21] show that the gravitational wave signals of relativistic models have similar amplitude but somewhat higher frequency than the Newtonian counterparts [12].

Among the most interesting results of these investigations was the fact that in rotational core collapse, the collapse type can change from multiple centrifugal bounce (Type II) to standard single bounce collapse (Type I) (c.f. Fig. 1). Centrifugal bounce was highly suppressed in relativistic gravity, yet it was still possible for simple EOS (see next section for the situation regarding improved microphysics). The main conclusion of [21] was that only the gravitational signal of a Galactic supernova (i.e. within a distance of about 10 kpc) might be unambiguously detectable by first generation detectors. Signal recycling techniques in next generation detectors may be needed, however, for successful detection of more distant core collapse events (with increased event rate), up to distances of the Virgo cluster.

Relativistic simulations with improved dynamics and gravitational waveforms were reported
Figure 3. Time evolution of hydrodynamic and metric quantities for the rapid collapse model A1B3G5 of [24]. The top panel shows the central density $\rho_c$ (solid line) and the central lapse function $\alpha_c$ (dashed line). Both the CFC and the CFC+ results overlap. Nuclear matter density $\rho_{\text{nuc}}$ is indicated by the horizontal dotted line. The bottom panel displays the relative difference $\sigma$ for $\rho_c$ (solid line) and $\alpha_c$ (dashed line) between the CFC and CFC+ simulations. The vertical dotted line in both panels marks the time of bounce $t_b$.

by [24], who used the CFC+ framework outlined in Section 2.4. As in the CFC simulations of [21] a simple (modified polytrope) EOS was used, and the initial models were unmagnetized. The CFC+ simulations of [24] covered the basic morphology and dynamics of core collapse types studied by [21], including the extreme case of a core with strong differential rotation and torus-like structure. In either case no significant differences were found in all models investigated by [24]. Therefore, second post-Newtonian corrections to CFC do not significantly improve the results when simulating the dynamics of core collapse to a neutron star. (It remains to be checked whether this conclusion changes in strongest gravitational fields as, e.g., in collapse to black holes.) A representative model is shown in Fig. 3 which plots the time evolution of the central density and the lapse function for both CFC and CFC+ (the curves in the top panel actually overlap). This figure shows that the relative differences between CFC and CFC+ for the central density and the lapse (bottom panel) are of the order of $10^{-4}$ or smaller throughout the collapse and bounce (marked by a vertical dotted line in Fig. 3.

Regarding the gravitational wave extraction, no substantial differences were observed between CFC and CFC+ either, a fact which reflects the underlying similar dynamics. The comparison in [24] was carried out using the standard quadrupole formula, commonly employed in the literature to extract gravitational waveforms. In addition, the gravitational waves were also computed directly from the $h_{ij}^{\text{TT}}$ metric terms, which permits a straightforward use of the spacetime metric to study the gravitational wave generation mechanism from the near zone to the wave zone.

Comparisons of the CFC approach with fully general relativistic simulations (employing the BSSN formulation) have also been reported by [25] in the context of axisymmetric core collapse
simulations. As in the case of CFC+, the differences found in both, the collapse dynamics and the gravitational waveforms are minute, which highlights the suitability of CFC (and CFC+) for performing accurate simulations of these scenarios without the need for solving the full system of Einstein’s equations. We also point out the detailed comparison of a comprehensive set of models (some including collapse to black holes) reported recently by [57], to which the interested reader is addressed for further information. Owing to the excellent approximation offered, thus, by CFC for studying core collapse, extensions to improve the microphysics of the numerical modelling, through the incorporation of microphysical EOS and simplified neutrino treatments, as well as extensions to account for three spatial dimensions, have been reported in the last few months. These are discussed next.

3.2. CFC/BSSN simulations with physical EOS and deleptonization
The most realistic simulations of rotational core collapse in general relativity so far (both in axisymmetry and in three dimensions but without incorporating magnetic fields) are those performed by [27, 58]. These simulations employ both the CFC and BSSN formulations of Einstein’s equations, and have been carried out using state-of-the-art numerical codes, both in spherical polar coordinates (the CoCoNut code of [22]) and in Cartesian coordinates (the CACTUS/CARPET/WHISKY codes; see [59, 60, 51, 27] and references therein). As in the earlier works of [21, 25, 24] the gravitational wave information from these collapse simulations is extracted using (variations of) the Newtonian standard quadrupole formula.

Figure 4. CFC collapse simulation with microphysical EOS and deleptonization [58]. The time evolution of the central density shows the dramatic effects of deleptonization on the collapse type, which changes from multiple bounce to regular collapse.

The initial models of [27, 58] use nonrotating 20$M_\odot$ presupernova models built by [5, 61] to which an artificially parameterized rotation is added. In addition, the relevant microphysics of core collapse is accounted for by employing a finite-temperature nuclear EOS [62] together with an approximate (parameterized) neutrino treatment with deleptonization and neutrino pressure effects included, recently suggested by [63]. In the simulations of [27, 58] the electron fraction profiles are obtained from general relativistic spherical models with Boltzmann neutrino transport, and are used as input data for the time-dependent electron fraction prescription suggested by [63] (with no analytic fit). The physics included in the modelling provide a
reasonably good approximation up until shortly after core bounce. Therefore, the setup is valid for obtaining reliable gravitational wave signals from core bounce.

The simulations reported by [27, 58] allow for a straight comparison with models with simple (polytropic) EOS and same rotation parameters as those of [21, 24, 25]. This sheds light on the influence of rotation and of the EOS on the gravitational waveforms. This comparison shows that for microphysical EOS the influence of rotation on the collapse dynamics and waveforms is somewhat less pronounced than in the case of simple EOS. In particular, the most important result of these investigations is the suppression of core collapse with multiple centrifugal bounces and its associated Type II gravitational waveforms (see Fig. 1).

The suppression of the multiple centrifugal bounce scenario is explained by [58, 27] as mainly due to the influence of deleptonization on the collapse dynamics, which removes energy from the system. This effect, not included in the previous relativistic simulations of [21, 23, 25, 24], needs to be incorporated, however, in numerical codes for a proper modelling of the collapse process, as its dynamics is mostly dominated by electron pressure, which is controlled by the electron captures on bound or free protons during the infall phase. Figure 4 shows the dramatic differences on the time evolution of the central density of a model with (black line) and without (red line) deleptonization included. While in the latter case the evolution shows the distinctive multiple centrifugal bounces of subnuclear matter density collapse models, once deleptonization is accounted for the outcome is a regular collapse instead.

As a result of the suppression of multiple bounces, and of its associated Type II gravitational waveforms, the improved simulations of [58, 27] reveal an enhanced uniformity on the

Figure 5. Spread of the characteristic gravitational wave strain for Newtonian and relativistic core collapse models with and without microphysical treatment of the EOS and deleptonization [58]. The lines are the first and advanced LIGO rms noise curves for a collapse source located at 10 kpc.
gravitational wave signal from core bounce. Figure 5 shows the distribution of the relativistic CFC models of [58] with microphysical EOS and deleptonization in a sensitivity diagram (characteristic strain versus frequency) for the first and advanced LIGO gravitational wave interferometers. Newtonian and relativistic models with simple EOS [21] are also included in the plot for comparison. It becomes apparent from this figure that the parameter space dependence of the gravitational wave signal is significantly smaller for the models with microphysical EOS and deleptonization as, in particular, the low frequency signals are suppressed (as they were associated with the multiple centrifugal bounce scenario). This important result has implications on signal analysis and signal inversion. In addition, signals for slow and moderate rotation have almost identical frequencies, being the signal peaked at a frequency of \( \sim 670 \) Hz.

On the other hand, the 3D simulations of general relativistic core collapse performed by [27] within the BSSN framework provide yet another confirmation of the satisfactory accuracy of CFC for the core collapse supernova problem. All models investigated in 3D are identical to those evolved in axisymmetry with the CoCoNut code in [58]. Typical simulation grids with 9 fixed mesh refinement levels and extending up to 3000 km in length were used. All models were followed up to \( \sim 20 \) ms after core bounce which showed that they all stay essentially axisymmetric through bounce, in good agreement with the models of [58]. One of the models, labelled E20A in [27], was further evolved until 70 ms after core bounce as it showed the largest gravitational wave amplitude of the sample and a value of the \( T/|W| \) ratio in the newly-formed PNS high enough to be prone to develop low \( T/|W| \) nonaxisymmetric (bar-mode-like) deformations. This had already been found in a similar model using Newtonian physics in [14].

The left panel of Fig. 6 displays the time evolution of the normalized mode amplitudes of the four lowest azimuthal density modes (\( \propto e^{im\phi} \)) in the equatorial plane, and shows that the one-armed \( m = 1 \) mode (red curve) becomes dominant from \( t \sim 20 \) ms after core bounce. Correspondingly, the right panel of Fig. 6 plots the gravitational wave strains \( h_+ \) and \( h_\times \) along the polar axis, where it becomes visible the late-time increase in amplitude during the growth of the instability. [27] find that the characteristic wave strain of this model peaks at a frequency of \( \sim 930 \) Hz, which is the pattern frequency associated with its nonaxisymmetric unstable one-armed structure. Such value is slightly larger than the typical frequencies of burst core collapse bounce signals (\( \sim 300 - 800 \) Hz). If confirmed when incorporating additional physics in the
modelling, namely magnetic fields, and such dynamic, low $T/|W|$ nonaxisymmetric instabilities turn out to be a generic feature of the early phase of post-bounce dynamics, the associated gravitational waves might be detectable out to much larger distances than the Milky Way.

3.3. Relativistic magneto-rotational core collapse

We end this section with a brief description of the recent simulations of relativistic magneto-rotational core collapse performed by [64]. The magnetic field structure and strength of the core collapse progenitors, needed as initial conditions for time-dependent numerical simulations, is still an open question. Models from stellar evolution predict that magnetic field strengths in iron cores probably do not exceed $10^9$ G and that the toroidal field component is expected to be much stronger than the poloidal one [6]. Therefore, for such weak fields ($p_{\text{mag}} \equiv b^2/\rho \ll p$), the assumption of a passive (test) magnetic field approximation seems sufficient to perform core collapse simulations and get some qualitative answers, as the magnetic fields are not going to affect the fluid evolution for the time scales involved.

This is the approximation adopted in the simulations of [64], which also employ the CFC approach for the formulation of Einstein’s equations. The inherent simplifications imply that the magnetic field evolution, given by the induction equation, does not affect the dynamics of the fluid, which is governed by the hydrodynamics equations only. However, the magnetic field evolution does depend on the fluid evolution due to the presence of the velocity components in the induction equation. In addition, such test magnetic field approximation simplifies the numerical procedure to solve the equations. The interested reader is addressed to [64] for technical details on the numerical approach, as well as for details on the different types of initial models considered and on tests of the GRMHD code employed in the core collapse simulations. Here, we focus our discussion on a single model, labelled A1B3G5 in [64], which is an almost rigidly rotating model with a magnetic field configuration given by a poloidal field generated by a circular current loop centered at $r = 400$ km.

The left panel of Fig. 7 shows the evolution of the energy parameters for the magnetic field $\beta_{\text{mag}}$ defined as $\beta_{\text{mag}} = E_{\text{mag}}/|W|$ where $E_{\text{mag}}$ and $W$ are the magnetic energy and the potential energy, respectively. The evolution of $\beta_{\text{mag}}$ allows to quantify the amount by which magnetic fields would affect the collapse dynamics. As the magnetic fields considered are weak, the resulting $\beta_{\text{mag}}$ is much less than unity throughout the simulation and much smaller than $\beta_{\text{rot}} = T/|W|$, where $T$ is the rotational kinetic energy. In order to analyze the growth of the magnetic field, we separate in Fig. 7 the effects of the different components of the magnetic field into $\beta_\phi$, for the toroidal component, and $\beta_{\text{polo}} = \beta_{\text{mag}} - \beta_\phi$, for the poloidal component. As the collapse proceeds, and leaving aside the effects of the magneto-rotational instability (MRI hereafter; see below), the magnetic field grows by at least two reasons: First, the radial flow compresses the magnetic field lines, amplifying the existing poloidal and toroidal magnetic field components. Second, during the collapse of a rotating star differential rotation is produced, even for rigidly rotating initial models. Hence, if a seed poloidal field exists, the $\Omega$-dynamo mechanism acts winding up poloidal field lines into the toroidal component. This (linear) amplification process generates a toroidal magnetic field component, even from purely poloidal initial configurations. The toroidal component of the magnetic field is affected by the two effects while the poloidal field is only amplified by the first effect (radial compression). Thus, even if the initial magnetic field configuration is purely poloidal, the toroidal component dominates after some dynamical time, as can be seen in Fig. 7 as $\beta_\phi$ (red line) grows much faster than $\beta_{\text{polo}}$, particularly after bounce ($t \sim 30$ ms) when the radial compression mechanism stops.

The three-dimensional structure of the magnetic field lines at the end of the simulation ($t \sim 60$ ms) is displayed in the right panel of Fig. 7. This figure shows that outside the formed PNS the magnetic field is approximately poloidal while a shell of entwined toroidal field lines has formed around the inner core of the PNS. In the innermost region of the PNS, however, the magnetic...
Magneto-rotational core collapse simulation [64]. Left panel: time evolution of the magnetic field energy parameters $\beta_{\text{mag}}$ (black line), $\beta_\phi$ (red line), and $\beta_{\text{polo}}$ (blue line). Right panel: 3D structure of the magnetic field lines at the end of the simulation, $t = 60$ ms. The box represents the region above the equatorial plane, and the axes are in km. Colors are proportional to the magnetic flux of each line.

field has a poloidal configuration, not visible in the plot as it is hidden by the shell of toroidal field lines.

Comparisons made by [64] with the Newtonian MHD simulations of [40, 41] show that both the dynamics and the morphology of the magnetic field is similar to those found in Newtonian gravity and Tolman-Oppenheimer-Volkoff (TOV) models, as long as the hydrodynamics is similar. The “magnetisation” ($\beta_{\text{mag}}$) of the final PNS is somewhat smaller in CFC than in Newtonian gravity and TOV models, irrespective of the numerical scheme and the resolution employed.

In the CFC magneto-rotational core collapse simulations of [64] gravitational waveforms have been computed using a modified standard quadrupole formula proposed by [40] that accounts for the magnetic terms. As the magnetic field remains low through the simulations, $b^2 \ll \rho$, the magnetic component of the gravitational wave signal is several orders of magnitude smaller than the corresponding hydrodynamical component. However, as the magnetic field amplification reaches saturation, the effects of the field on the waveforms are expected to be significant. In particular MRI may significantly alter the waveforms reported in [64], as it is the most efficient amplification mechanism.

MRI is a shear instability that generates turbulence and amplifies the magnetic field in rotating magnetized plasma, transporting angular momentum in the star (see [65] and references therein for generic details on the MRI). Results in Newtonian gravity have shown that magnetized collapse models are indeed susceptible of developing MRI [31, 40]. The (Newtonian) condition for the MRI to occur (neglecting buoyancy effects and assuming that the magnetic field strength is very low) is:

$$\frac{d\Omega^2}{d \ln \varpi} < 0,$$

(45)
where $\Omega$ is the angular velocity of the fluid and $\varpi \equiv r \sin \theta$ is the distance to the rotation axis. If this condition is fulfilled and the magnetic field has a poloidal component, the instability grows exponentially in time \cite{65}. The timescale of the fastest growing unstable mode is

$$\tau_{\text{MRI}} = 4\pi \left| \frac{d\Omega}{d \ln \varpi} \right|^{-1}, \quad (46)$$

which is independent of the magnetic field configuration and strength.

Since the simulations of \cite{64} are done in the passive magnetic field approximation there is no back-reaction of the magnetic field onto the dynamics, and only the potential effects of MRI during relativistic core collapse could be estimated. Figure 8 shows two snapshots of the collapse (around the time of bounce on the left and at the end of the simulation on the right). Depicted in white in this figure are the regions where condition (45) is not fulfilled or the timescale of the fastest growing mode exceeds 1 s. Clearly, a significant fraction of the newly-formed PNS, as well as the region behind the shock at the moment of its formation, are affected by the MRI. As a result, the magnetic field is going to grow exponentially on dynamical timescales and will reach saturation in those regions, becoming important for the dynamics. In these regions the passive field approximation is not valid, becoming necessary a full magnetic field treatment.

4. Summary
Gravitational stellar core collapse supernova has traditionally been considered among the most important astrophysical sources of potentially detectable gravitational radiation. In this paper we have discussed the current status of multidimensional relativistic core collapse simulations, giving emphasis to the numerical modelling of the collapse dynamics and to the computation of the gravitational radiation in the existing approaches. Only very recently the first such multidimensional simulations of relativistic rotational core collapse have become possible (albeit still for models with simplified input physics), thanks to the use of conservative formulations of the hydrodynamics and magnetohydrodynamics equations and advanced numerical methodology, as well as long-term stable formulations of Einstein’s equations. Work employing the CFC approximation of the 3+1 Einstein’s equations has been reported, as well as extensions of this approximation (CFC+) and investigations within the
framework of the so-called BSSN formulation of the 3+1 gravitational field equations, with no approximation for the spacetime dynamics.

With the first generation of ground-based, gravitational wave laser interferometers just taking the first data ever, the availability of reliable waveform templates from astrophysical sources, which may help extract the signal from the anticipated noisy data, is urgently required. Such gravitational waveform templates from relativistic core collapse have been built in recent years and they are being improved as additional ingredients are incorporated in the models. While early simulations used polytropic EOS, first steps towards significant improvement have been taken in recent modelling with the incorporation of microphysical (finite-temperature) EOS, deleptonization, and (simplified) neutrino transport. It has been shown that those effects have relevant implications in constraining the gravitational wave signature from core collapse. As of today, reliable waveform templates for most phases of core collapse - bounce, neutron star ringing, convection, nonaxisymmetric instabilities - are available.

On the other hand, the incorporation of magnetic fields and the MHD equations in numerical codes to further improve the realism of core collapse simulations in general relativity, is currently an emerging field where significant progress is bound to be soon achieved. The paper has also discussed magneto-rotational simulations of relativistic core collapse, aiming at addressing the effects of magnetic fields on the collapse dynamics and on the gravitational waveforms. The (yet incomplete) simulations yield no major differences with previous Newtonian studies, the strength of the final magnetic field being somewhat smaller in CFC gravity. Several magnetic field amplification mechanisms have been discussed, concluding that the magneto-rotational instability needs to be included in core collapse modelling as it dominates the post-bounce dynamics within a few ms and may have a major effect on the gravitational wave signal.

We hope to have shown in this review that relativistic treatments of the dynamics (hydrodynamics and MHD) and of the gravity are all ready and waiting for being incorporated into supernova codes with detailed microphysics and transport (somehow a bigger challenge on its own) to gain further progress in the understanding of the supernova explosion mechanism.

Acknowledgments

This research has been supported by the Spanish Ministerio de Educación y Ciencia (grant AYA2004-08067-C03-01). It is a pleasure to thank Pablo Cerdá-Durán, Harry Dimmelmeier, Christian Ott, and Nikolaos Stergioulas for stimulating insight and discussions, and for providing some of their (yet unpublished) figures for this article. Harry Dimmelmeier is further acknowledged for his meticulous reading of the manuscript.

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