Turbulence, cascade and singularity in a generalization of the Constantin-Lax-Majda equation

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We study numerically a Constantin-Lax-Majda-De Gregorio model generalized by Okamoto, Sakajo and Wunsch, which is a model of fluid turbulence in one dimension with an inviscid conservation law. In the presence of the viscosity and two types of the large-scale forcings, we show that turbulent cascade of the inviscid invariant, which is not limited to quadratic quantity, occurs and that properties of this model’s turbulent state are related to singularity of the inviscid case by adopting standard tools of analyzing fluid turbulence.

I. INTRODUCTION

Fostering a number of simpler nonlinear partial differential equations (PDE) is a defining feature of the Navier-Stokes (NS) or Euler equations. This is perhaps because a reduced equation is more insightful and direct in understanding one particular phenomenon than the whole NS equations which include countless facets of fluid phenomena. The Korteweg-de Vries equation derived via the water-wave equation from the Euler equations is a prominent example for understanding a peculiar behavior of the shallow water wave, which is now called solitary wave.

Reaching a good reduced model is not at all limited to systematic derivations from the NS or Euler equations. Phenomenological modeling of them with one-dimensional (1D) PDE or a set of ordinary differential equations has been proved to be fruitful. Famous examples include the Burgers’ equation\(^1\), the Constantin-Lax-Majda (CLM) equation\(^2\) and the shell models of turbulent cascade\(^3\).

The major interest behind these models is in statistical laws of incompressible high-Reynolds number turbulence, a putative singular solution of the incompressible NS or Euler equations and a possible relation between them (see e.g., \(^4\), \(^5\)). By statistical laws, we mean those of homogeneous isotropic turbulence such as the scaling laws of the energy spectrum and of the structure functions, with the turbulent cascade of the energy or other inviscid conserved quantity. Since these problems are known to be one of the toughest in physics and mathematics, approach from a simple model is indispensable. The influential CLM eq. yields the analytic solution of the vorticity analogue becoming infinite in a finite time\(^2\). However it does not have a turbulent solution with viscosity (see e.g., \(^6\)). There may not be commonly accepted reduced PDE models suitable for studying those points. Nevertheless, we here mention two recent studies to develop such models.

Zikanov, Thess and Grauer introduced a nonlocal generalization of the 1D Burgers’ equation. They showed that the solution has the energy spectrum \(E(k) \propto k^{-5/3}\), which is consistent with the Kolmogorov scaling, and that the scaling exponent of the \(p\)-th order velocity structure function, \(\langle |u(x+r,t)−u(x,t)|^p⟩ \propto r^{ζ_p}\) is without intermittency, namely \(ζ_p = p/3\) (here \(\langle \cdot \rangle\) denotes an ensemble average). In their model the degree of the nonlocality can be changed by one parameter. The above result is obtained for the maximally nonlocal case. For an intermediately nonlocal case, they found that the scaling exponent \(ζ_p\) deviates from \(p/3\) in the quantitatively same way as the three dimensional (3D) incompressible turbulence\(^7\).

Recently, Luo and Hou numerically found a potentially singular solution to the 3D axisymmetric Euler flow confined in a cylindrical surface, where the vorticity grew by \(10^8\) times larger\(^8\). To understand the nature of this, 1D PDE models have been developed by Luo and Hou\(^8\) and by Choi, Keselev and Yao\(^10\). It is proven that a solution to each model starting from a smooth initial condition does blow up in a finite time\(^11\).

In the same spirit of the two models with an emphasis on the statistical laws and the possible role of the singularity, we here study a generalization, proposed by Okamoto, Sakajo and Wunsch\(^12\), of the Constantin-Lax-Majda-De Gregorio (gCLMG) equation\(^13\) with the viscosity and a forcing term \(f(x,t)\)

\[
\partial_t \omega + au \partial_x u = \omega \partial_x u + ν \partial_{xx} \omega + f.
\]

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Here $\omega(x,t)$ is a scalar modeling of the vorticity in three dimensions and the velocity analogue is expressed with $u(x,t) = -(-\partial_{xx})^{-1}\omega$ and $\partial_x u = H(\omega)$ which is the Hilbert transform of $\omega$. The Hilbert transform first considered in these 1D modelings \cite{2} is one of the key ingredients, which was used also in the models we mentioned \cite{7,8}. Notice that the velocity is no longer incompressible in 1D. A historical background of the gCLMG eq. \cite{11} can be found in \cite{11}. The parameter $\alpha$ in front of the advection term introduced in \cite{12} enables the equation to have a conserved quantity in the inviscid ($\nu = 0$) and unforced ($f = 0$) setting. Specifically, for $\alpha \leq -1$, it is easily shown that

$$C_\alpha = \frac{1}{\alpha} \int \omega^{-\alpha}(x,t)dx$$

is a conserved quantity if there is no input or output on the boundary \cite{12} (If $\alpha$ is not integer, we take $|\omega|^{-\alpha}$ in the integrand. Furthermore if $\alpha$ is odd, $\int |\omega|^{-\alpha}dx/(-\alpha)$ is also a conserved quantity). We notice here that negative $\alpha$ has no physical origin and that the Galilean invariance is lost for $\alpha < 0$. However the inviscid conservation law leaves possibility of turbulent cascade. Indeed for the case of $a = -1$, where Eq. \cite{2} coincides dimensionally with the enstrophy, it has been numerically shown that the enstrophy cascade takes place \cite{14}. It may appear paradoxical that, for $a = -2$, we have two-dimensional (2D) turbulence analogue from the model \cite{11} with the vortex stretching term that is the essential ingredient of the 3D vorticity equation. This suggests that, regardless of the form of the equation, the conservation law matters most. In this paper, from a mathematical and theoretical viewpoint, we extend our previous study of the gCLMG eq. \cite{14} (which was limited to $a = -2$) to general negative $a$'s. The $a = -2$ case in \cite{14} is the baseline of our analysis.

We now summarize findings of the previous study \cite{14} and state the plan of the present paper. In \cite{14}, the gCLMG eq. was numerically studied in a periodic interval. It is observed that a turbulent state occurs as a statistically steady state if the large-scale forcing $f(x,t)$ is random and that, if the forcing is deterministic, a solution becomes stationary. The turbulent state exhibited the cascade of $C_{-2}$ (enstrophy cascade) and the energy spectrum close to that of the 2D enstrophy-cascade turbulence $k^{-3}$ but with a measurable deviation from it in the inertial range. Interestingly, the stationary solution had the energy spectrum which is indistinguishable from the turbulent spectrum in the inertial range. The vorticity structure functions of the turbulent state at high even orders were possibly characterized with negative scaling exponents, indicating infinite vorticity as $\nu \to 0$. Also the nonlinear stationary solution as we decreased the viscosity $\nu$ suggested infinite vorticity with the finite enstrophy dissipation rate. Lastly the phase-space orbit of the turbulence state normalized by the stationary solutions showed a peculiar self-similarity.

In this paper, we show numerically for general negative $a$’s that the same above holds. Furthermore we analyze in detail the cascade of the inviscid invariant \cite{2} in the turbulent state, the profile of the stationary solution with $\nu \to 0$ and compare the scaling of the viscous case with the inviscid case.

The organization of the paper is as follows. In Sec I, we study the turbulent state under the random forcing. Specifically, we characterize it with the energy spectrum and analyze the cascade with the filtering flux method. We also consider the Kármán-Howarth Monin relation and the vorticity structure functions. In Sec II, we study the nonlinear stationary state under the deterministic forcing. Specifically, we consider the energy spectrum and the vorticity profile and then compare the the energy spectrum to that of the inviscid solution. In Sec IV, we show the self-similarity of the phase-space orbit of the turbulent solution normalized by the stationary solution. A summary and concluding discussion are given in Sec V.

II. TURBULENCE UNDER THE RANDOM FORCING

Throughout the paper we consider the gCLMG eq. \cite{11} in a periodic interval of length $2\pi$. We hence use the Fourier spectral method for numerical simulation. We set the vorticity Fourier mode of the zero wavenumber to zero initially. The dealiasing is done with the two-third method. The time stepping scheme is the forth-order Runge-Kutta method. It is known that the round-off noise grows in the spectral simulation of the gCLMG eq. with the double precision. To suppress this, we use the same spectral filter as \cite{12}. Namely, if the absolute value of the vorticity Fourier modes $\tilde{f}$ is smaller than $10^{-12}$, we set it to zero at each time step.

First, we set the large-scale forcing to be random. Specifically, we set the Fourier mode of the forcing to Gaussian random variable without temporal correlation having the following mean and variance

$$\langle \tilde{f}(k,t) \rangle = 0,$$

$$\langle \tilde{f}(k',t')\tilde{f}(k,t) \rangle = 2\sigma_f^2 \delta_{k',k} \delta(t' - t).$$

To make the forcing effective in a large scale, we set non-zero $\tilde{f}$ only for the wavenumbers $k = \pm 1$. We set $\sigma_f = 1.0 \times 10^{-2}$ leading to the average enstrophy-input rate $2\sigma_f^2 = 2.0 \times 10^{-4}$. The initial condition of the simulation is $\omega = 0$. 

Next, we present the vorticity profile and the energy spectrum for a wide range of $a$’s as an overview of gCLMG turbulence. After that, by limiting to a smaller range of $a$’s, we study its property in more detail.

### A. Appearance of the vorticity and the energy spectrum

In Fig. 1 we plot vorticity snapshots in statistically steady states for $a = -0.01, -0.5, -1.0, -2.0, -3.0, -4.0, -10.0$ and $-100.0$. They are normalized by the temporally averaged enstrophy

$\langle Q \rangle \propto \langle -a \rangle^{-1}$

(although $\nu$ is not the same for all the nine cases). This is consistent to the blow-up of vorticity of the viscous CLM eq. (see e.g., [3]).
In Fig. 2 we plot the corresponding energy spectra

\[ E(k) = \left\langle \sum_{k \leq |k'| < k + 1} \frac{1}{2} |\tilde{u}(k', t)|^2 \right\rangle, \]  

which are normalized with the temporally averaged energy

\[ \langle E \rangle = \left\langle \frac{1}{2\pi} \int_0^{2\pi} \frac{1}{2} u^2(x, t) dx \right\rangle = \left\langle \sum_k \frac{1}{2} |\tilde{u}(k, t)|^2 \right\rangle. \]  

There are two ranges which are analogous to the inertial range and the dissipation range in the NS turbulence. If we fit \( E(k) \) in the inertial range with a power-law scaling \( k^{-q} \), the scaling exponent \( q \) varies probably from 0 \((a \to 0)\) to around 4 \((a \to -\infty)\). The former is again consistent with a blowup solution to the CLM eq., seemingly having a flat \( (k^0) \) energy spectrum. The latter limit is consistent with the shock like, or step-function like, vorticity profile for large \(|a|\).

Now we present Kolmogorov-type dimensional analysis about the scaling exponent of the energy spectra. Notice that the inviscid conservation of \( C_\alpha \) is not guaranteed for \(-1 < a < 0\). We first assume that the inviscid invariant \([2]\) is cascading down to smaller scales and that its “dissipation rate”,

\[ \beta_a = (-a - 1) \nu \frac{1}{2\pi} \int_0^{2\pi} \omega^{-a-2} (\partial_x \omega)^2 dx, \]  

determines the inertial range quantity for \( a \leq -1 \) (\( \beta_-2 \) is dimensionally the same as the enstrophy dissipation rate of the NS turbulence). Since \( \beta_a \) has the dimension \([\text{(time)}^{1-a}]\), the inertial-range spectrum behaves as

\[ E(k) \propto \beta_a^{\frac{3}{a-1}} k^{-3}, \]  

which can be obtained by application of the Kraichnan-Leith-Bachelor argument on the 2D enstrophy-cascade NS turbulence \([16, 18]\). However Eq. 9 does not agree well with the numerical result plotted in Fig. 2 even if we omit the cases with \(-1 < a \leq 0\) since they do not have the inviscid conservative quantity. More precisely, the numerical result shows a-dependence of the spectrum such that \( E(k) \) seems to take the power law \( k^{-q} \) with exponent \( 2 \leq q \leq 4 \) by choosing some \( a < 0 \). Nevertheless \( E(k) \) seems concentrating around \( k^{-3} \) in the intermediate wavenumber range.

One way to understand the discrepancy from \( k^{-3} \) is the logarithmic correction that was first proposed by Kraichnan \([13]\) for the 2D enstrophy-cascade NS turbulence. If we apply his derivation to the gCLMG turbulence, the logarithmic correction takes the form

\[ E(k) \propto \beta_a^{\frac{3}{a-1}} k^{-3} \left[ \log \left( \frac{k}{k_f} \right) \right]^{-\frac{a}{a-1}}, \]  

where \( k_f \) is the wavenumber in which the forcing is added. Here we make a wildly heuristic assumption that the flux of \( C_{-a} \) in the Fourier space can be expressed with \( \Omega(k) \) \([11, 18]\) where \( \Omega(k) \) is the non-local frequency \( \Omega(k)^2 \sim \int_{k_f}^k \nu^2 E(p) dp \). Obviously Eq. (10) with \( a = -2 \) coincides with the log-corrected spectrum of the 2D enstrophy-cascade NS turbulence. As observed in \([14]\), for \( a = -2 \) the log-corrected form, \( k^{-3} \log^{-1/3}(k/k_f) \) does not agree with \( E(k) \). The same is true for other \( a's \) as we will see later.

What can be inferred from behavior of the \( E(k) \) then? Is the cascade of \( C_{-a} \) in the gCLMG turbulence just a coincidence for certain \( a's \)? For small \( a's \) \((a \ll -1)\) and large \( a's \) \((a \sim -0)\), the assumption of the cascade may be invalid since \( E(k) \) is rather distinct from \( k^{-3} \). Indeed it may appear strange that the high-order quantity, such as \( C_{-10} \) or \( C_{-100} \), determines \( E(k) \) which is the second order quantity of \( \tilde{u} \). Therefore we study in detail the cases of \( a's \) in which \( E(k) \) is around \( k^{-3} \). Specifically, we take five cases, \( a = -1.0, -1.5, -2.0, -3.0 \) and \(-4.0 \) \((a = -1.5 \) is taken as a representative of the fractional cases).

First, we check behavior of \( E(k) \) as decreasing \( \nu \). The energy spectra with a smaller viscosity are shown in Fig. 3. We observe that the “inertial-range behavior” of each \( a \) extends to the larger wavenumber region than Fig. 2 without changing the wavenumber dependence. In particular, the energy spectrum \( E(k) \) is close to \( k^{-3.0} \) for \( a = -1.0 \) and to \( k^{-4.0} \) for \( a = -4.0 \). Comparing to Eq. (10), this dependence of \( E(k) \) on the parameter \( a \) indicates that turbulent cascade of the inviscid conservative quantity \( C_{-a} \) is unlikely and that the enstrophy cascade for \( a = -2 \) analyzed previously in \([14]\) is just coincidental. In the next subsection we numerically analyze directly whether or not a scale-wise nonlinear transport of \( C_{-a} \) is considered to be turbulence cascade with a spatial-filter method \([14]\).
FIG. 3: The time-averaged energy spectra for various $a$’s normalized with the energy, $\langle E \rangle$. Here the viscosity is $\nu = 1.6 \times 10^{-6}$ and the number of grid points is $2^{15}$. The time average is taken over 3150 equi-spaced instances between $700 \leq t \leq 7000$ from three different realizations of the random forcing. The scaling laws, $k^{-3}$ and $k^{-4}$, are shown as a guide.

B. Analysis of the cascade

Working in the periodic domain, the most convenient method to analyze the cascade is the transfer function or the flux in the Fourier space, if the cascade quantity is quadratic. For the case of $a = -2$, the enstrophy flux in the Fourier space was used to show the cascade of the enstrophy [14]. Here, for general $a$’s where the quantity is no longer quadratic, we adopt a more versatile method introduced in [19] to investigate whether or not the turbulent cascade of the conservative quantity $C_{-a}$ occurs.

This method uses a low-pass spatial filter with a filtering scale $\ell$ in the physical space. Let us write the filter function with $g_\ell(x)$. The filtered quantity of a function $A(x,t)$ is then expressed as

$$
\overline{A}_\ell(x,t) = \int_0^{2\pi} A(x',t)g_\ell(x' - x)dx'.
$$

(11)

Now the filtered gCLMG eq. can be written as

$$
\partial_t \overline{\omega} + a\overline{u}_\ell \partial_x \overline{\omega}_\ell = \overline{u}_\ell \partial_x \overline{u}_\ell + \zeta + \nu \partial_x \overline{\omega}_\ell + \overline{f}_\ell,
$$

(12)

where the vorticity input from the scales smaller than $\ell$ (the subgrid scales) is

$$
\sigma^{(a)}_\ell = -a \partial_x [\overline{u} \overline{\omega} - \overline{u}_\ell \overline{\omega}_\ell] + (1 + a) \left[ \overline{\omega}(\partial_x u)_\ell - \overline{\omega}_\ell \partial_x \overline{u}_\ell \right].
$$

(13)

This leads to the equation of the low-pass filtered $(-a)$-th power of the vorticity as

$$
\partial_t \left( \frac{1}{-a} \overline{\omega}^{-a}_\ell \right) = \partial_x \left( \overline{u}_\ell \overline{\omega}_\ell^{-a-1} + \nu \overline{\omega}_\ell^{-a-1} \partial_x \overline{\omega}_\ell \right)
= -Z^{(a)}_\ell - \nu(-a - 1)\overline{\omega}^{-a-2}_\ell (\partial_x \overline{\omega}_\ell)^2 + \overline{\omega}^{-a-1}_\ell f_{\ell}.
$$

(14)

Here $Z^{(a)}_\ell(x,t)$ is the flux of the grid-scale moment $\overline{\omega}^{-a}_\ell$ being transferred to smaller scales than $\ell$, which is expressed as

$$
Z^{(a)}_\ell(x,t) = \overline{\omega}^{-a-1}_\ell \sigma^{(a)}_\ell.
$$

(15)

With $Z^{(a)}_\ell(x,t)$, we can analyze the cascade in a precise way [19]. By cascade, it is understood that the space-time average of $Z^{(a)}_\ell(x,t)$, which we denote $\langle Z^{(a)}_\ell \rangle$, becomes independent of the filter scale $\ell$ in some range of $\ell$. If such a range of $\ell$ exists, we call it inertial range. Notice that we assume spatial homogeneity and statistical steadiness of the flux, $Z^{(a)}_\ell(x,t)$. It is known that the expressions of the flux is not unique. This non-uniqueness does not matter since we consider the spatial average of the flux.

In Fig.4 we show $\langle Z^{(a)}_\ell \rangle$ for even $a$ cases, where the inviscid conservative quantities $C_{-a}$ are positive definite. As the filter function $g_\ell(x)$, we use the Gaussian filter $g_\ell(x) = \frac{1}{\sqrt{2\pi} \ell} \exp[-(x/\ell)^2/2]$. For $a = -2$, there is a plateau that
indicates the average dissipation rate \langle \beta_a \rangle. The viscosity and the resolution \((\nu, N)\) are \((2.5 \times 10^{-5}, 2^{13})\), \((6.25 \times 10^{-6}, 2^{14})\) and \((1.5625 \times 10^{-6}, 2^{15})\). The method of time average is the same as in Figs. 2 and 3. For the \(N = 2^{14}\) case, the time average is taken over equi-spaced 3700 snapshots from \(4000 \leq t \leq 41000\).

![Figure 4](image1.png)

**FIG. 4:** Averaged flux \(\langle Z_t^{(a)} \rangle\) of the inviscid conservative quantity \(C_a\) for \(a = -2\) (left) and \(a = -4\) (right). The horizontal line indicates the average dissipation rate \(\langle \beta_a \rangle\). The viscosity and the resolution \((\nu, N)\) are \((2.5 \times 10^{-5}, 2^{13})\), \((6.25 \times 10^{-6}, 2^{14})\) and \((1.5625 \times 10^{-6}, 2^{15})\). The method of time average is the same as in Figs. 2 and 3. For the \(N = 2^{14}\) case, the time average is taken over equi-spaced 3700 snapshots from \(4000 \leq t \leq 41000\).

![Figure 5](image2.png)

**FIG. 5:** Same as Fig. 4 but for \(a = -3\). For the \(N = 2^{15}\) and \(2^{14}\) cases, \(C_{-3} > 0\) while \(C_{-3} < 0\) for the \(N = 2^{13}\) case. The horizontal line indicates the average dissipation rate \(\langle \beta_a \rangle\) for each case.

amounts to the \(\ell\)-independent flux. This implies that the cascade of the enstrophy, \(C_{-2}\), occurs, as indicated with the equivalent flux in the Fourier space in [14]. While for \(a = -4\) such a plateau is not well developed in comparison. Instead of plateau, the flux in the intermediate range is a mildly decreasing function of \(\ell\). The variation shown in Fig. 4 may suggest \(\langle Z_t^{(-4)} \rangle \propto -\log(\ell)\). This indicates that \(C_{-4}\) does not cascade at least for the ranges of \(\nu\) considered here. Nevertheless, if we decrease \(\nu\) furthermore, a plateau may appear in small scales. Hence the cascade is not completely ruled out for \(a = -4.0\).

For an odd \(a\) case, \(a = -3\), the flux is shown in Fig. 5. The wiggliness variation in the intermediate range of \(\ell\) indicates evidence against the cascade. This wiggliness may be caused because \(\beta_{-3}\) and \(\beta_{-3}\) are not sign definite and hence fluctuation effects are strong. There may be cascade for smaller \(\nu\) but the way of extension of the possible inertial range shown in Fig. 5 is not convincing. Thus we do not have a numerical evidence for the cascade of \(C_{-3}\). If we can control the input rate of \(C_{-3}\) with the forcing, a clearer result may be obtained. For the other odd case, \(a = -1\), the average flux \(\langle Z_t^{(a)} \rangle\) is by definition zero since the nonlinearity vanishes in the equation of \(C_{-1}\). Therefore the cascade of \(C_{-1}\) is not possible. However recall that the energy spectrum for \(a = -1\) is broad and close to a power law \(k^{-3}\) in Fig. 8. At least for \(a = -1\), the spectrum has nothing to do with the cascade of \(C_{-1}\).

Now we move to a different form of the inviscid conservation law. For negative odd integer \(a\), the absolute \((-a)\)-th moment

\[
\hat{C}_a = \frac{1}{-a} \int_0^{2\pi} |\omega(x, t)|^{-a} dx
\]  

(16)
Although this cascade of values among the three resolutions is large. However they are consistent with values of the dissipation rate $\tilde{\nu}$ since $E$ presented in Fig.3 has a slight but measurable deviation from the dimensional result.

The horizontal line indicates the averaged dissipation rate $\langle \tilde{\beta}_a \rangle$ measured in the simulation with $2^{15}$ grid points.

is also an inviscid conserved quantity. The dissipation rate of $\tilde{C}_a$ can be defined as

$$\tilde{\beta}_a = -\nu \frac{1}{2\pi} \int_0^{2\pi} \text{sgn}(\omega)|\omega|^{-a-1}\partial_x^2 \omega dx,$$

where $\text{sgn}(\omega)$ denotes the sign of $\omega$. The corresponding flux can be obtained via the equation of $|\nabla\ell|^a$. One expression is

$$\tilde{Z}_\ell^{(a)} = \text{sgn}(\omega)|\nabla\ell|^{-a-1} \left[ a\partial_x(\nabla\ell) - (a+1)\omega(\partial_xu)\right].$$

For $a = -1$ and $-3$ cases, the averaged flux $\langle \tilde{Z}_\ell^{(a)} \rangle$ is shown in Fig.6. Comparing with Fig.5 the flux of the absolute third order moment appears quite different for $a = -3$ and looks similar to the flux $\langle Z_\ell^{(4)} \rangle$ for $a = -4$. We begin to see the plateau for the smallest $\nu$ case. For $a = -1$ case, a well-developed plateau is seen. From this, the cascade of the absolute moment is plausible for $a = -1$ and $-3$. Is this consistent with the dimensional analysis of the energy spectrum, Eq.4, provided that the dissipation rate is now $\tilde{\beta}_a$? As seen in Fig.6, for $a = -1$, it may be consistent since $E(k)$ is close to $k^{-3}$. While $a = -3$, it is not since $E(k)$ is closer to $k^{-4}$. This point will be revisited with a stationary solution under a deterministic forcing in the next section.

As a non-integer value, we here take $a = -1.5$ as a representative case. We plot the flux of the absolute moment $\langle \tilde{Z}_\ell^{(-1.5)} \rangle$ in Fig.7. It shows a well-developed plateau as in the previous case $a = -1.0$. The difference in the plateau values among the three resolutions is large. However they are consistent with values of the dissipation rate $\tilde{\beta}_{-1.5}$. Although this cascade of $a = -1.5$ indicated by the plateau implies $E(k) \propto k^{-3}$ dimensionally, the measured $E(k)$ presented in Fig.6 has a slight but measurable deviation from the dimensional result.
of the enstrophy conjecture (for the Onsager’s conjecture on the 3D Euler equations, see, e.g., [21, 22]). By following the Duchon-ǫ-equation

Kármán-Howarth-Monin (KHM) relation for the gCLMG turbulence is

For the case of indicated by the plateau of the averaged flux \( \langle \beta_{-2} \rangle \) calculated in the simulation with \( N = 2^{15} \) grid points. Both terms on the right hand side of Eq.(19) have the same order of magnitude in the inertial range.

In summary of this cascade analysis, we observe that, for \( a = -1.0, -1.5, -2.0 \) and \(-3.0\), the cascade of \( \hat{C}_a \) is indicated by the plateau of the averaged flux \( \langle \hat{Z}_r^{(\beta)} \rangle \) and that, for \( a = -4.0 \), the indication of the cascade becomes weaker. Therefore the inertial range, in the sense of the range of scales where the flux becomes constant, is likely to exist at least for \( a \geq -3.0 \). At the same time we see the systematic change of the energy spectrum \( E(k) \) in the inertial range as a function of \( a \) in Figs.2–3. This is not consistent with the dimensional result, Eq.(9) although the change is around \( k^{-3} \) except for extreme values of \( a \). Our view on the discrepancy between the cascade analyzed here and the dimensional form of \( E(k) \) is that the variation around \( k^{-3} \) can be understood as a non-dimensional correction such as the logarithmic correction proposed by Kraichnan for 2D enstrophy-cascade turbulence. This point will be studied in the next section with the stationary solution under the deterministic forcing. For extreme values of \( a \)’s (close to 0 and smaller than \(-3\)), the behavior of \( E(k) \) may be inferred from the corresponding limits, such as the CLM eq. for \( a \to -0 \) and the advection equation for \( a \to -\infty \), not from the cascade.

C. Kármán-Howarth-Monin relation and dissipative weak solution

Having obtained an evidence of the cascade of \( \hat{C}_{-a} \) for certain \( a \)’s, we now consider a gCLMG analogue of the Kármán-Howarth-Monin (KHM) relation of the NS turbulence [4]. For the case of \( a = -2.0 \), an expression of the KHM relation for the gCLMG turbulence is

\[
B_{-2}(r) = -\partial_r \langle \omega(x)\omega(x')[u(x') - u(x)] \rangle + \frac{1}{2} \langle \omega(x)\omega(x')[\partial_x u(x) + \partial_x u(x')] \rangle,
\]

where \( x' = x + r \). This \( B_{-2}(r) \) is interpreted as the enstrophy flux across scale \( r \), which is similar to \( \langle Z_r^{(-2)} \rangle \). In other words, the KHM relation is yet another device to look at the cascade. Although it is closely related to the filtering flux used previously, the KHM relation does not involve spacial filtering. Here we assume that the correlation functions on the right hand side of Eq.(19) are homogeneous (depending only on \( r \)) and in a statistically steady state. Due to the compressibility, expression of the KHM relation is not unique. Furthermore, it cannot be expressed in terms of divergence of certain products of the velocity and vorticity increments. This means that the analogue is not like the 4/5-law of the energy cascade of the 3D NS turbulence (see, e.g., [4]) or the 2-law of the enstrophy cascade of the 2D NS turbulence (e.g., [20]). A numerical confirmation of the KHM relation (19) is shown in Fig.8. The flux \( B_{-2}(r) \) in the inertial range is \( r \) independent and close to the enstrophy dissipation rate \( \langle \beta_{-2} \rangle \), again demonstrating the cascade of the enstrophy \( C_{-2} \).

This fact has an interesting consequence on dissipation without viscosity, namely gCLMG analogue of the Onsager’s conjecture (for the Onsager’s conjecture on the 3D Euler equations, see, e.g., [21, 22]). By following the Duchon-Robert formalism [23], let us now consider a weak solution of the inviscid and unforced gCLMG eq. with \( a = -2.0 \). With a \( \epsilon \)-scale mollifier \( \varphi_\epsilon(x' - x) \), the regularized vorticity \( \omega_\epsilon = \int \omega(x')\varphi_\epsilon(x' - x)dx' \) of the weak solution obeys the equation

\[
\partial_t \omega_\epsilon = 2\partial_x(\omega u)_\epsilon - (\omega \partial_x u)_\epsilon.
\]
The local enstrophy budget equation becomes
\[
\partial_t \frac{1}{2}(\omega_\epsilon) - \partial_x(u\omega_\epsilon) = \frac{1}{2} \omega \left[ -2 \partial_x(u\omega)_\epsilon + \partial_x(u\omega_\epsilon) + u\partial_x\omega_\epsilon + (\omega \partial_x u)_\epsilon \right].
\] (21)

This motivates us to put the right hand side as “dissipation”
\[
D^{(-2)}_\epsilon(\omega) = \frac{1}{2} \omega \left[ -2 \partial_x(u\omega)_\epsilon + \partial_x(u\omega_\epsilon) + u\partial_x\omega_\epsilon + (\omega \partial_x u)_\epsilon \right].
\] (22)

If \(D^{(-2)}_\epsilon(\omega) > 0\) as \(\epsilon \to 0\), the weak solution to the inviscid gCLMG eq. can be called dissipative. Under what conditions it becomes dissipative is an interesting question. Unlike the 3D Euler case, condition for \(D^{(-2)}_\epsilon(\omega) = 0\) (or > 0) may not be characterized by the Hölder exponent of \(\omega\). Another formal analysis leads to an expression of \(D^{(-2)}_\epsilon(\omega)\) as
\[
\int dr \ \omega(x)\omega(x') \left[ (u(x') - u(x))(\partial_r \varphi_\epsilon(r) \right] + \frac{1}{2} \varphi_\epsilon(r)(\partial_x u(x') + \partial_x u(x)) \right) = D^{(-2)}_\epsilon(\omega),
\] (23)
which is an equivalent of the KHM relation of the weak solution.

For the general \(a\), the formal expression of the KHM flux of the gCLMG turbulence is
\[
B_a(r) = -\partial_t \langle \omega^{-a-1}(x)\omega(x')[u(x') - u(x)] \right] \left[ -\frac{1}{a} \langle \omega^{-a-1}(x)\omega(x')[\partial_x u(x) - (a + 1)\partial_x u(x')] \right],
\] (24)
and the “dissipation” of the weak solution to the inviscid, unforced gCLMG eq. is
\[
D^{(a)}_\epsilon(r) = \frac{1}{-a} \omega^{-a-1} \left[ a \partial_x(u\omega)_\epsilon + \partial_x(u\omega_\epsilon) \right.
\left. - (a + 1)u\partial_x\omega_\epsilon - (a + 1)(\omega \partial_x u)_\epsilon \right].
\] (25)

Consequently, the KHM relation of the weak solution can be
\[
\int dr \ \omega^{-a-1}(x)\omega(x') \left[ (u(x') - u(x))(\partial_r \varphi_\epsilon(r) \right] \left[ \varphi_\epsilon(r)(\partial_x u(x) - (1 + a)\partial_x u(x')) \right] = D^{(a)}_\epsilon(\omega).
\] (26)

This can also be interpreted as a law which holds local in time and space without assuming the homogeneity, isotropy and statistical steadiness.

**D. Vorticity structure function**

Now, coming back to the gCLMG turbulent solution under the random forcing, we look into the \(p\)-th order structure function of the vorticity
\[
S^{(a)}_p(r) = \langle [\omega(x + r, t) - \omega(x, t)]^p \rangle
\] (27)
and its logarithmic local slope
\[
\frac{\log S^{(a)}_p(r + \Delta r) - \log S^{(a)}_p(r)}{\log(r + \Delta r) - \log r}.
\] (28)

The purpose here is to analyze the possible presence of the (non-power law) correction to the simple scaling law. Recall that, if the energy spectrum is a power law with an exponent \(\gamma\), namely, \(E(k) \propto k^{-\gamma}\), the structure function
FIG. 9: Second order structure function of the vorticity (left) and its logarithmic local slope (right) for various a’s, which are calculated with the same data sets used in Figs. 3-7. Shorter curves for a = −2.0, −3.0, and −4.0 are data with the larger ν and the smaller 2^14 grid points.

can be predicted as S_2^{(a)}(r) \propto r^{(3−γ)/p/2} with the dimensional analysis. As we discussed at the end of the previous section, the correction to E(k) \propto k^{-3} (γ = 3) can be observed with the second order vorticity structure function. Here we focus on even orders, p = 2, 4 and 6.

The second order structure function is shown in Fig.9. For a = −1.0 and −1.5, the local slope does not become flat for the range 0.01 ≤ r ≤ 0.1 where the absolute flux Z_0^{(a)} is reasonably flat (the inertial range). This indicates that S_2^{(a)}(r) is not characterized with a power-law scaling, which supports a non-dimensional correction to the energy spectrum E(k) \propto k^{-3}. For a = −2.0, −3.0 and −4.0, the local slope has a local minimum around r ≈ 0.1, which may become a flat region with smaller ν. However comparing with a different ν case, we see that the minimum cannot be interpreted as the beginning of such a flat region. If it is the beginning, the minimum value is unchanged as we change ν. This indicates that S_2^{(a)} is not a simple power law also for a = −2.0, −3.0 and −4.0.

As we increase the order to p = 4 and 6, a visible feature is a local maximum in the dissipative range. This is a reflection of the vorticity pulses shown in Fig.11 since the location of the maximum corresponds roughly to the width of the pulse. In Fig.11, as increasing a, we observe that the height of the pulse becomes larger and that the width becomes smaller. The observation is consistent with the fact that the maximum of the fourth order structure function appears only for large a cases, a = −1.0 and −1.5 and also with the fact that the location of the maximum in S_6^{(a)}(r) shifts to smaller r as we increase a.

Inevitably, for p = 4 and 6, the local slope becomes noisier since the high order structure functions are affected by rare events. Nevertheless in Figs.11 and 11 a power-law scaling is absent for a = −1.0 and −1.5 within the inertial range 0.01 ≤ r ≤ 0.1. In contrast, for a = −2.0, −3.0 and −4.0, the local slope appears to be flatter than that of the second order case, which may suggest a power-law scaling. But S_6^{(-2.0)}(r) is an exception. However the change from the larger ν data casts doubt on the scaling behavior.

Now let us assume that S_4^{(a)}(r) and S_6^{(a)}(r) are power-law functions for a = −2.0, −3.0 and −4.0. Then the scaling exponent of S_6^{(a)}(r) is smaller than that of S_4^{(a)}(r). This decrease of the exponent of the even-order vorticity structure function implies that the vorticity is not bounded as ν → 0 [1]. Moreover the possible scaling exponent of S_6^{(a)}(r) for a = −2.0 and −3.0 is negative.

With the structure function, which is a standard tool to probe scaling property of turbulence, we observed that, from the second order structure function, the possible correction to the dimensional-analysis scaling of the gCLMG turbulence is not a power-law type. From the higher order structure functions, we obtained an indication that the vorticity becomes infinite as ν → 0. Both points are next studied with a different large-scale forcing.

III. NONLINEAR STATIONARY SOLUTION UNDER THE DETERMINISTIC FORCING

In this section, we change the large-scale forcing to a deterministic and static forcing

$$f(x, t) = C_0 \sin x,$$

where we set C_0 = −0.1. As observed previously for a = −2.0 [14], with the deterministic forcing we obtain a nonlinear stationary solution for other a’s using the same time stepping method as in the previous section. Although
the numerical solution is not not at all turbulent, it has interesting properties from which we can get insights on the gCLMG turbulence realized under the random forcing as we will see.

The numerical method for the stationary solutions is as follows. Starting from the zero initial condition with \(2^{13}\) grid points, we run the simulation up to \(t = 50\) with the time step \(\Delta t = 1.25 \times 10^{-4}\). In this simulation, the 12 digits of the palinstrophy, \(\sum_k k^4 |\tilde{u}(k,t)|^2\), stays the same in \(30 \leq t \leq 50\) for the \(a = -1.0\) case (whose energy spectrum is the shallowest). Then we take the resultant Fourier modes as the initial condition of the next simulation with the doubled grid points and the one-half of the time step. The duration of each simulation of a given resolution is up to \(t = 50\). For \(a = -1.0\), the viscosity is set to \(\nu = 1.0 \times 10^{-4}\) with \(2^{13}\) grid points and \(\nu = 2.5 \times 10^{-5} \times 2^{14-m}\) with \(2^m\) grid points \((m = 14, \ldots, 18)\). For \(a = -1.5, -2.0, -3.0\) and \(-4.0\), it is set to \(\nu = 1.0 \times 10^{-4} \times 4^{13-m}\) with \(2^m\) grid points \((m = 13, \ldots, 18)\). With the largest number of grid points, \(2^{18}\), the palinstrophy for the case \(a = -1.0\) stays the same value in the 11 digits for \(15 < t \leq 50\).

### A. The energy spectrum

As shown in Fig.12 the stable nonlinear stationary solution for each \(a\) has the energy spectrum that is indistinguishable in the inertial range from the turbulent one realized with the random forcing. This implies that the possible correction \(E(k)\) to the \(k^{-3}\) scaling can be obtained through analysis of the stationary solution with the deterministic forcing. However, we do not succeed in such a theoretical analysis so far.

Instead, as we have done in [14] for the \(a = -2.0\) case, we do a curve fitting to empirically measure the correction. Now we fit the logarithmic local slope of \(E(k)\) with a functional form

\[
\log \frac{E(k+\Delta k)}{E(k)} \approx k \frac{d}{dk} \log E(k) = -c_0 - c_1 \left[ \log \left( \frac{k}{k_f} \right) \right]^{\delta},
\]

(30)
with the constant flux, which is again a similarity to the turbulent solution. Cases the differences are less than 1%. Apart from this discrepancy, we observe that the steady state is maintained the stationary solution (notice that such discrepancy is not seen in the randomly forced case, see Fig.7). In other view of the pulse for each expression of the energy spectrum, $k_0$ seems to be well parametrized with the form Eq.(31). Although Eqs.(30) and (31) are purely empirical, the functional energy spectrum. For the parameter values shown in Table I. The fitting with $\delta = 3$ and $\delta = -1$ correspond to the form of the Kraichnan’s logarithmic correction, $E(k) \propto k^{-3} [\log(k/k_f)]^{-c_1}$. For $\delta \neq -1$, integration of Eq.(30) leads to the expression of the energy spectrum,

$$E(k) \propto k^{-c_0} \exp \left\{-c_2 \left[ \log \left( \frac{k}{k_f} \right) \right]^\theta \right\},$$

where $c_2 = c_1/(1 + \delta)$ and $\theta = 1 + \delta$.

In the fitting, we fix the first parameter as $c_0 = 3$ by assuming that the power-law part of $E(k)$ is given by Eq.(9) since we have obtained an evidence for the cascade of $C_a$ with $a \geq -3$. For the fitting range, we take a range of $k$ in which the curves of different resolutions overlap in Fig.13 namely, $20 \leq k \leq 200$. A least square fitting yields parameter values shown in Table I. The fitting with $\delta = -1$, which corresponds to the logarithmic corrected form, Eq.(10), does not yield a better fit for every $a$. For $a = -4.0$, $\theta \approx 1$ suggests that $E(k)$ is simply proportional to $k_{-(c_0+c_2)} = k^{-3.763}$, which is consistent with the non-cascade of the inviscid conservative quantity. The spectrum seems to be well parametrized with the form Eq.(31). Although Eqs. (30) and (31) are purely empirical, the functional form with $c_0 = 3$ and $\theta = -1$ can be obtained theoretically with the incomplete self-similarity analysis, which is given in Appendix A.

We now formally calculate the spatially averaged flux, $\langle \tilde{Z}_\ell^{(a)} \rangle$, of the stationary solution. In spite of the inhomogeneity, we do this in order to look at nature of the nonlinear equilibrium. The averaged flux as a function of $\ell$ exhibits the $\ell$-independent range as shown in Fig.13. This supports the assumption of $c_0 = -3.0$ made in the fitting of the energy spectrum. For $a = -1.5$ case, the value of the plateau is about 10% larger than the dissipation rate $\beta_{-1.5}$ of the stationary solution (notice that such discrepancy is not seen in the randomly forced case, see Fig.7). In other cases the differences are less than 1%. Apart from this discrepancy, we observe that the steady state is maintained with the constant flux, which is again a similarity to the turbulent solution.

**B. Vorticity pulse and blowup of the inviscid limit**

The vorticity profile of the stationary solution for each $a$ consists of a single pulse around the origin. A magnified view of the pulse for each $a$ is shown in Fig.15. The energy spectrum of the stationary solution studied in the previous
subsection is a result of this single pulse. The question is now whether we can relate the form Eq.\((31)\) in the inertial range with the pulse profile in the physical space. Probably we cannot do so because the width of the pulse belongs to the dissipation range. Rather it is likely to be related with the profile far from the pulse, namely how the vorticity decreases from the pulse peaks.

Nevertheless, we study now the pulse profile closely because it is related to the constant \((\nu\text{-independent})\) dissipation rate \(\beta_\alpha\). As observed in Fig.\(\text{I}3\) the height and width of the pulse is affected by the viscosity \(\nu\). These profiles with various values of the viscosity are found to collapse to a single curve when the vorticity \(\omega(x)\) is scaled as \(\nu^{-\alpha}\omega(\nu^{\mu}x)\), where \(\alpha = -1/(1-2a)\) and \(\mu = (1-a)/(1-2a)\). The scaled profiles are shown in Fig.\(\text{I}4\). This viscous scaling implies that, for \(a < 0\), the vorticity becomes infinite, \(\omega(x) \to \nu^{\alpha} (\alpha < 0)\), as \(\nu \to 0\).

Now we argue that the exponents of the viscous scaling can be determined with a boundary-layer type analysis as \(\nu \to 0\). Let us first assume that the width of the pulse is given by \(\delta = \nu^{\mu}\). In the stretched coordinate \(X = x/\delta\), we assume that the solution is

\[
\omega(x) = \nu^{\alpha} \Omega(X), \quad u(x) = \nu^{\gamma} U(X).
\]

Then the stationary gCLMG eq. with the stationary forcing,

\[
a u \partial_x \omega = \omega \partial_x u + \nu \partial_x^2 \omega + C_0 \sin x,
\]

becomes

\[
a \frac{\nu^{\alpha+\gamma}}{\delta} U \partial_X \Omega = \frac{\nu^{\alpha+\gamma}}{\delta} \Omega \partial_X U + \frac{\nu^{1+\alpha}}{\delta^2} \partial_X^2 \Omega + C_0 \sin(\delta X).
\]

Here we assume that the dominant balance in Eq.\((34)\) holds between the nonlinear term and the viscous term, resulting
FIG. 15: The vorticity pulse of the stationary solution around the origin. Each panel has profiles of six different values of the viscosity $\nu$.

FIG. 16: Scaled vorticity pulse of the stationary solution around the origin, $\nu^{-\alpha}\omega(\nu^\mu x)$, with $\alpha = -1/(1 - 2a)$ and $\mu = (1 - a)/(1 - 2a)$. Each panel has profiles of six different values of the viscosity $\nu$. 
\[ \gamma + \mu = 1. \]  

(35)

We further assume the scaling relation \( \omega(x) \sim u(x)/\delta \), which gives

\[ \alpha - \gamma + \mu = 0. \]  

(36)

(the same scaling relation can be obtained through analysis of the Hilbert transform \( \partial_x u(x) = H(\omega)(x) \)). Lastly, we assume that the dissipation rate of the inviscid conserved quantity is \( \nu \) independent as \( \nu \to 0 \). The dissipation rate inside the pulse may be written as

\[ \beta_a \sim \nu^{1-\alpha-\mu}(-a-1) \int_{-\infty}^{\infty} \Omega^{-2} (\partial_X \Omega)^2 dX. \]  

(37)

Hence we have a relation

\[ 1 - a\alpha - \mu = 0. \]  

(38)

The three relations, Eqs. (35), (36) and (38), yield the exponents as a function of \( a \):

\[ \alpha = \frac{1}{1-2a}, \quad \gamma = \frac{a}{1-2a}, \quad \mu = \frac{1-a}{1-2a}. \]  

(39)

These exponents indeed scale well the numerical solutions as seen in Fig. 16 (not only \( \omega(x) \) but also \( u(x) \) and \( H(\omega)(x) \), figures not shown). The last assumption of the independence of the dissipation rate on \( \nu \) is not trivial [4, 24] but the result suggests that it is plausible including for the \( a = -4.0 \) case, where the turbulent cascade does not take place with the random forcing.

The width of the stationary pulse scales with the viscosity as \( \delta \propto \nu^{-\mu} = \nu^{(1-a)/(1-2a)} \). In contrast, the Kolmogorov-Kraichnan dissipation length scale of the gCLMG turbulence is \( \eta_a \propto \beta_a^{-1/2(a-1)} \nu^{1/2} \), which is the unique combination of the dissipation rate \( \beta_a \) and the viscosity \( \nu \) having the dimension of length. The two viscous length scales have different scaling exponents of \( \nu \) except for \( a \to -\infty \). This difference can be due to the fact that we determine the viscous scaling by considering only the “boundary layer”. It does not involve matching with the “outer layer” which corresponds to the inertial range. Recall that matching between the inertial and the dissipation ranges is the way to obtain the Kolmogorov-Kraichnan dissipation scale.

To see which viscous length scales is more relevant with respect to the spectrum in the dissipation range, we scale the enstrophy spectra of the randomly forced cases with \( 1/\delta \) and \( 1/\eta_a \) for different \( \nu \)’s (for the energy spectra, good collapse is not obtained for both viscous scales). A better collapse in the dissipation range for the enstrophy spectra is observed with \( 1/\delta \). This indicates that \( \delta \) is more relevant in the dissipation range than \( \eta_a \) for the turbulent cases under the random forcing. Additionally we observe numerically that \( E(k) \) of the stationary solution in the dissipation range decreases exponentially with the form \( \exp(-ck) \), with some constant \( c \).

The “inner solution” shown in Fig. 16 is a solution to the nonlinear and nonlocal equations

\[ aU \partial_X \Omega = \Omega \partial_X U + \partial_X^2 \Omega, \]  

(40)

\[ U(X) = \frac{1}{\pi} \int_{-\infty}^{\infty} \Omega(Y) \log |X - Y| dY, \]  

(41)

which we are not able to solve so far. Recall that the solution to the equations above may not be sufficient to determine the inertial-range properties which require the outer solution.

C. Energy spectra of inviscid blowup solution and stationary solution

Given the indication of the blowup of the vorticity of the stationary solution as \( \nu \to 0 \), comparison with the inviscid solution is of next interest. It is proven in [25] that an inviscid solution of the gCLMG eq. without a forcing term blows up in a finite time for \( a < 0 \) in an unbounded domain.

Now we compare the energy spectrum of the stationary solution to that of the inviscid gCLMG eq. without any forcing term starting from the initial condition \( \omega(x) = 0.1 \sin x \). Notice that the inviscid-limit case \( \nu \to 0 \) is different from the inviscid case \( \nu = 0 \).
FIG. 17: Time evolution of the energy spectrum $E(k, t)$ of the inviscid and unforced gCLMG eq. with $a = -2.0$ at $t = 1.0, 1.5, 2.0, \ldots, 7.5$ and $8.0, 8.1, 8.2, \ldots, 9.2$. The circles are the energy spectrum of the viscous stationary solution under the deterministic forcing for $a = -2.0$ calculated with $2^{18}$ grid points (the same one shown in Fig.12). The inviscid numerical solution is calculated with $2^{18}$ grid points and the time step $\Delta t = 7.81 \times 10^{-6}$.

FIG. 18: Instantaneous energy spectrum $E(k, t)$ of the inviscid and unforced gCLMG eq. for various $a$'s. Here the inviscid numerical solution is calculated with $2^{18}$ grid points and the time step $\Delta t = 7.81 \times 10^{-6}$. The solid line corresponds to the parametrization of $E(k)$ of the stationary solution, Eq.(31), with the parameter values in Table I. The inviscid and forceless gCLMG eq. is numerically solved with the same spectral method as in Sec.II. The inviscid solution does not become stationary and its Fourier modes in ever higher wavenumbers are generated in the course of time. With the finite resolution we hence should stop the numerical simulation at some time before the Fourier modes at the largest truncation wavenumber becomes larger than the filtering threshold (recall that we set the vorticity Fourier modes to zero if their magnitudes are smaller than the threshold value $10^{-12}$).

The time evolution of the energy spectrum obtained numerically is shown in Fig.17. The functional form of $E(k, t)$ does not change in the intermediate wavenumber range which corresponds to the inertial range of the turbulent solution. It is remarkable that the functional form of $E(k, t)$ in this range is quite close to that of the stationary solution for each $a$ case as seen in Fig.18. This implies that the inviscid $E(k, t)$ in the intermediate range is the same form as that of the randomly forced case in the inertial range as well. Such an agreement is not found in numerical solutions to the Euler and the Navier-Stokes equations in the 3D space (see, e.g.,[26]). As long as we run the simulation, $E(k, t)$ in the dissipation range decreases exponentially.

In the physical space, the vorticity profile of the inviscid solution looks quite similar to that of the stationary solution with the deterministic forcing except that the inviscid pulse becomes sharper and sharper as the time elapses. Its scaling analysis is done in Appendix B.
IV. SELF-SIMILARITY OF THE PHASE-SPACE ORBIT

We showed that, depending on the large-scale forcing, the gCLMG eq. has two classes of solutions: the turbulent one under the random forcing and the stationary one under the deterministic static forcing. The resemblance of the energy spectra of the two described in the previous section indicates that the turbulent solution is somehow fluctuating around the stationary solution. This point is now examined through a visualization of the phase-space orbit. As in [14], we consider the following 3D projection of the phase space:

\[
X(k, t) = (X_1(k_1, t), X_2(k_2, t), X_3(k_3, t)),
\]

\[
X_j(k_j, t) = \text{Im} \tilde{\omega}(k_j, t) \quad (j = 1, 2, 3),
\]

where we take two triplets of the wavenumbers \( \kappa = (k_1, k_2, k_3) \), in the inertial range as powers of two, \( \kappa_1 = (4, 8, 16) \) and \( \kappa_2 = (32, 64, 128) \). Here \( \text{Im} \tilde{\omega}(k_j, t) \) denotes the imaginary part of the vorticity Fourier coefficient of the turbulent solution normalized by the square root of the mean energy, \( \tilde{\omega}(k, t) = \sqrt{\langle E \rangle^{1/2}} \). Notice that the counterpart of the stationary solution, \( \tilde{\omega}_S(k, t) \), is purely imaginary.

The orbit in the \( X \)-space for each \( a \) is shown in Fig. [19]. Qualitative observations are now in order. The orbit of the turbulent solution meanders a certain surface with a thickness, which is called here the attracting set. Its overall shape is the same for different \( a \)'s. An interesting question would be whether the thickness of the attracting set goes to zero as the amplitude of the random forcing tends to zero. The stationary solution, which is visualized as a big sphere in Fig. [19] is located at one edge of the attracting set, not in the middle. This implies that the precise form of the time-averaged \( E(k) \) of the turbulent solution in the inertial range can be slightly different from the energy spectrum of the stationary solution. Comparing the orbits between the two scale ranges, \( \kappa_1 \) and \( \kappa_2 \), we observe that the orbits \( X(\kappa_1) \) and \( X(\kappa_2) \) appears almost the same, which may be a manifestation of the near self-similarity of the energy spectrum within the inertial range. From this, it is tempting to seek a three-variable modeling of the inertial-range dynamics of the gCLMG turbulence.

V. SUMMARY AND CONCLUDING DISCUSSION

A. Summary

We have numerically studied solutions of the viscous gCLMG eq. under two kinds of large-scale monoscale forcing for certain range of negative \( a \)’s. Solutions strongly depend on the nature of the forcing. However the common characteristic structure independent on the forcing is vorticity pulses developed around stagnation points (velocity null point) with negative velocity gradient.

When the forcing is random, the solution become turbulent, which were analyzed with standard tools of studying NS turbulence. We observed that the energy spectra of the gCLMG turbulence appear to have power-law behaviors in the intermediate wavenumber range. However their scaling exponents are different from the dimensional prediction of the cascade of the inviscid invariant, Eq. (16).

We then looked for direct evidence for or against the cascade with the filtering flux method for the five cases \( a = -1.0, -1.5, -2.0, -3.0 \) and \( -4.0 \). We found that the invariant, Eq. (16), cascades down to smaller scales except for the \( a = -4.0 \) case. It showed that turbulent cascade occurs for non-quadratic conservative quantities. We then considered the Kármán-Howarth-Monin relation of the gCLMG turbulence and discussed possible dissipative weak solution of the inviscid gCLMG eq.

Through the structure functions of the vorticity in the inertial range, we observed that they were not simple power-law functions, supporting non power-law type correction seen in the energy spectra. Although, if we assume that the leading behavior of the structure function was power-law, the data indicated the negative Hölder exponent of the vorticity increment and hence blowup of the vorticity (of the vorticity pulses) in the inviscid limit.

When the forcing is deterministic and stationary, the solution becomes stationary. This nonlinear stationary solutions have almost identical energy spectra with those of the turbulent solutions. By increasing the resolution of the numerical simulation, we parametrized possible form of the asymptotic (as \( \nu \to 0 \)) energy spectra of the stationary solutions as Eq. (41) in the inertial range. This parametrization supported the presence of the correction to the dimensionally predicted \( k^{-3} \) due to the cascade of the inviscid invariant. However the functional form of the correction is different from the Kraichman’s log-correction proposed for the 2D enstrophy-cascade turbulence.

The stationary solution has single vorticity pulse. We found that its height and width scales with certain powers of the viscosity. We argued this viscous scaling with a boundary-layer analysis and obtained the scaling exponents as a function of \( a \) in Eq. (39). This viscous scaling also indicated blowup of the vorticity in the inviscid limit. Next we
FIG. 19: Phase-space orbit, $X(\kappa, t)$, of the turbulent solution under the random forcing for various $a$’s with $\kappa = (k_1, k_2, k_3) = (4, 8, 16)$ (left panels) and $(32, 64, 128)$ (right panels). The numerical solutions are the same as those in Sec[III] with $2^{14}$ grid points. The point (sphere) corresponds to the stationary solution under the deterministic forcing. The outline box in each panel is $−3 \leq X_j \leq 1$ $(j = 1, 2, 3)$. The duration of the orbit shown here is $T = 1600$ for the former $\kappa$ and $T = 400$ for the latter. The figure of the latter looks denser since the typical time scale of the variation of the orbit is smaller.
showed that the energy spectra of the stationary solution in the intermediate wavenumbers (which corresponds to the inertial range in the turbulent solution) is also close to that of the inviscid and unforced solution of the gCLMG eq.

Finally, by normalizing the turbulent solution with the nonlinear stationary solution, we found that the phase-space orbit of the turbulent solution is self-similar in the inertial range. This self-similarity is observed for all the five cases of \( a \) studied here. An important message here is that, not only this self-similarity, but all the other properties of the gCLMG solutions are qualitatively the same for all the cases of \( a \).

### B. Concluding discussion

Our motivation of studying the gCLMG equation is to obtain insights on the statistical laws of the NS turbulence and a possible role of singular behavior of the solutions to the Navier-Stokes or Euler equations.

With the suitable range of the parameter \( a \), the randomly forced cases exhibited certain similarities to the NS turbulence case, which are the cascade of the inviscid conservative quantities and the broad energy spectra. Due to the dimension of the dissipation rate of the conservative quantity, \((-a+1)\)-th power of time, the statistical laws were compared to those of the 2D enstrophy-cascade NS turbulence, in particular, the energy spectrum with the logarithmic correction and the vorticity structure functions.

One insight obtained here empirically, which may be useful to the 2D NS turbulence, is the expression with a high-order logarithmic correction of the energy spectrum, Eq. (31). This spectrum around \( k^{-3} \) implies that the cascade of the gCLMG turbulence is not local as argued for the 2D enstrophy-cascade turbulence, see, e.g., [27, 28]. By non-locality it is meant that effect of the large-scale motions on the flux is not diminished even if we have a sufficient scale separation. It can be shown that, for the \( a = -2 \) case, a large-scale effect on the cascade of the gCLMG turbulence is not negligible (infrared non-local in the language of [27]). However for other \( a \)'s, there is a possibility of local cascade. For example, for \( a = -3 \), let us now assume that the flux shown in Fig. 5 becomes flat and that the second order structure function is a power law, \( S_2(r) \propto r^{0.6} \) (which is contrary to our conclusion though). Then it can be inferred that the cascade of \( \tilde{C}_{-3} \) becomes infrared local from although its energy spectrum is around \( k^{-3} \).

Concerning the relation between the statistical laws and the singularity, the vorticity structure functions of the gCLMG turbulence indicated blowup of the vorticity as \( \nu \to 0 \). This blowup of the vorticity is also supported by the behavior of the nonlinear stationary solution. This implies that the qualitative aspect of the even-order structure functions allows us to detect signature of the inviscid-limit blowup as discussed already in, for example, [4]. We also speculate that the negative exponent of the sixth order structure function, provided that the non-power-law correction is small, corresponds to the spatial decay of the vorticity in the neighborhood of the pulse. However we are not able to identify quantitatively relation between the statistics and the singularity, such as the power-law exponent of the structure function and the scaling exponent of the blowup.

We have shown the strong dependence of numerical solutions of the gCLMG eq. on the forcing. It appears that this has nothing to do with the NS turbulence. However recently it became known that a numerical solution to the 3D NS equations with a large-scale deterministic forcing in the periodic cube reaches a near-stationary laminar state after an extremely long duration of the turbulent state [24]. If the large-scale forcing is random, our simulation shows that a solution to the NS equations does not become laminar at least in the same duration of the simulation of the deterministic forcing case. For the (2D) Kolmogorov flows, where the single Fourier-mode stationary forcing is added, it is found that stable stationary solutions exist at large Reynolds numbers [30] in certain cases. Therefore this sort of the forcing dependence is not limited to a small class of nonlinear PDE’s with periodic boundary condition. Furthermore, it may imply that a random dynamical system approach is fruitful when studying large-time asymptotic behavior of turbulent state.

We have seen that a number of the subtle properties of turbulent flows were realized in the gCLMG solutions. This suggests its role as a unique testing ground worth further rigorous and theoretical studies. Specifically, turbulent statistical laws can be understood via singularities characterized with vorticity pulse, which are present both in the nonlinear stationary solution and also the inviscid solution.

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Appendix A: Incomplete self-similarity analysis of the energy spectrum

Here we argue that the functional form of the energy spectrum, Eq. (31) with \( c_0 = 3 \) and \( \theta = -1 \), can be obtained with the incomplete self-similarity (see, e.g., Sec. 8.3 of [31]).

First we assume that the energy spectrum in the inertial range has the following form of correction to the dimensional result Eq. (31):

\[
E(k) = \beta_a^{3/2(1-a)} k^{-3} \Phi(k, \nu, k_f)
\]

where \( k_d \) is the dissipation wavenumber, which can be either the Kolmogorov-Kraichnan dissipation wavenumber \( k_d = \beta^{-2/[2(\alpha-1)]} \nu^{-1/2} \) or the inverse of the boundary-layer width \( 1/\delta \propto \nu^{-1-\alpha/(1-2\alpha)} \) that is the viscous length scale discussed in Sec III.B.

Second, based on this form, we assume that the small parameter is \( \varepsilon = \log^{-1}(k_f/k) \). This yields

\[
E(k) \approx \beta_a^{3/2(1-a)} k^{-3} \left[ A_0 + \frac{A_1}{\log \frac{k_f}{k}} \exp \left( \frac{\alpha_1 \log \frac{k_f}{k} \log \frac{k_f}{k}}{-\log \frac{k_f}{k_f}} \right) \right],
\]

where \( \alpha_0, \alpha_1 \) and \( A_0 \) are constants; \( A_1 \) can be a function of \( k/k_f \). In fact, the standard assumption on \( \Phi \) would be \( \Phi = (k/k_f)^{\alpha_0 + \varepsilon \alpha_1} [A_0 + \varepsilon A_1] \). We do not follow this since our intention is to obtain \( \Phi \) as a non-power-law function of \( k/k_f \).

Third, we assume \( \alpha_0 = 0 \) in order to have a nonvanishing limit of \( \Phi \) as \( k_d \to \infty \). Then the \( \alpha_1 \) part can be written as

\[
\left( \frac{k_f}{k_d} \right)^{\varepsilon \alpha_1} = \exp \left( \varepsilon \alpha_1 \left( \log \frac{k_f}{k} + \log \frac{k}{k_d} \right) \right).
\]

Fourth, based on this form, we assume that the small parameter is \( \varepsilon = \log^{-1}(k_f/k) \). This yields

\[
E(k) \approx \beta_a^{3/2(1-a)} k^{-3} \left[ A_0 + \frac{A_1}{\log \frac{k_f}{k}} \exp \left( \frac{\alpha_1 \log \frac{k_f}{k} \log \frac{k_f}{k}}{-\log \frac{k_f}{k_f}} \right) \right]
\]

which corresponds to Eq. (31) with \( \theta = -1 \). Notice that the constant \( c_2 \) in Eq. (31) is now proportional to \( \log(k_f/k_d) \).

Appendix B: Locally self-similar analysis of the inviscid blowup solution

With the initial data \( \omega(x, t = 0) = 0.1 \sin x \), the solution of the inviscid unforced gCLMG eq. has a single vorticity pulse at \( x = \pi \). Let us shift the coordinate so that the center of the pulse is at the origin. The time variation of the pulse profile is shown in Fig 20 for the case of \( a = -2.0 \) as an example (for other \( a \)'s the results are similar). Now we write the maximum of the vorticity as \( \omega^\ast_\alpha(t) \) and the location of the maximum of the vorticity as \( -x^\ast_\alpha(t) \) \( \langle t^\ast_\alpha(t) \rangle \). As shown in Fig 20, the pulse profiles at different times can be collapsed to a single curve by scaling the solution with \( x^\ast_\alpha(t) \) and \( \omega^\ast_\alpha(t) \). This suggests that the inviscid pulse can be described with a locally self-similar solution.

To analyze the rate of the blowup, let us here assume that the local self-similar form is

\[
\omega(x, t) = \omega^\ast_\alpha(t) F^\ast_\alpha \left( x/x^\ast_\alpha(t) \right)
\]

B1

(the scaled profile depends on the parameter \( a \)) and that the height and the width of the pulse have the following power-law dependence

\[
\omega^\ast_\alpha(t) \approx (t^\ast_\alpha - t)^{c_0}, \quad x^\ast_\alpha(t) \approx (t^\ast_\alpha - t)^{c_\alpha}
\]

B2
FIG. 20: Left: Time evolution of the vorticity pulse of the inviscid unforced gCLMG eq. with $a = -2.0$ (the coordinate is shifted so that the center of the pulse is at the origin). Right: scaled pulses together with the scaled nonlinear stationary solution of the viscous and forced gCLMG eq., which were shown in Fig. [16] for comparison. The functional form of the inviscid self-similar solution is different from that of the nonlinear stationary solution discussed in Sec. [III.B]

FIG. 21: Power-law relation between the pulse height $\omega_*(a)(t)$ and the pulse width $x_*(a)(t)$. The solid lines represent empirically fitted power-laws whose exponents are $\xi_a/\zeta_a = -1/1.6, -1/2.3, -1/2.9, -1/3.5, -1/5.0, -1/6.2$ from top to bottom. Different data points correspond to data at different times. The case $a = -0.5$ is included although the conservation of $\tilde{C}_{-0.5}$ is not guaranteed.

with the finite blowup time $t_*(a)$. The assumption [B2] is supported by the numerical data which shows the power-law behavior $\omega_*(a)(t) \propto [x_*(a)(t)]^{\xi_a/\zeta_a}$, plotted in Fig [21].

Now we argue that the exponent $\xi_a$ can be determined as $\xi_a = -1$ based on the conservation law. The local conservation law $\partial_t [\omega^{-a}/(-a)] = \partial_x [u|\omega|^{-a}]$ can be integrated in $0 \leq x \leq x_*(a)(t)$. The result is

$$\partial_t \left[ \int_0^{x_*(a)(t)} \frac{|\omega|^{-a}}{-a} \, dx \right] = (u|\omega|^{-a}) \bigg|_{x=x_*(a)(t)},$$

where we need some assumption on the behavior of the velocity at $x = x_*(a)(t)$. The numerical data indicates that the velocity around $x_*(a)(t)$ is not locally self similar with $\omega_*(a)(t)$ and $x_*(a)(t)$. This is as expected from the shape of the energy spectrum close to $k^{-3}$ since the velocity is dominated by the large-scale modes. Nevertheless the data shows that the time variation of the velocity is very close to the dimensional analysis result: $u(x_*(a)(t), t) \simeq \omega_*(a)(t)x_*(a)(t) \simeq (t_*(a) - t)^{\zeta_a}$. (a small discrepancy in the exponent is seen for $a = -3.0$ and $-4.0$, though). With this scaling of the velocity, Eq. [B3] yields the relation among the exponents as $-a\xi_a + \zeta_a - 1 = \xi_a + \zeta_a - a\zeta_a$. Hence $\xi_a = -1$.

This scaling $\omega_*(a)(t) \propto (t_*(a) - t)^{-1}$ can be observed as shown in Fig [22] provided that somewhat subjective choice of the unknown blowup time, $t_*(a)$, is made (recall that the power-law exponent emerged in such a figure is quite sensitive to choice of the origin $t_*(a)$). About the maximum of $H(\omega)$, the same temporal scaling $(t_*(a) - t)^{-1}$ is observed with the same choice of $t_*(a)$'s (figure not shown), indicating that the blowup criterion obtained in [12], $\int_0^{t_*(a)} \max \omega H(\omega)(t) dt = \infty$, is satisfied.

The numerical data about the scaling of the width of the pulse, $x_*(a)(t) \simeq (t_*(a) - t)^{\zeta_a}$, is shown in Fig [22]. One heuristic assumption leading to determination of the exponent $\zeta_a$ is that the left hand side of Eq. [B3], which is the
FIG. 22: Power-law behavior of $\omega^{(a)}(t) \simeq (t^{(a)} - t)^{-1}$ (Top) and of $x^{(a)}(t) \simeq (t^{(a)} - t)^{\zeta_a}$ (Bottom) with an empirical choice of $t^{(a)}_*$. The curves are shifted vertically for clarity. The solid lines in the bottom panel represent power laws with the exponents $\zeta_a = 1.6, 2.3, 2.9, 3.5, 5.0, 6.2$, which are determined in Fig. 21 independently on the choice of the blowup time $t^{(a)}_*$. The particular solution is different from the locally self-similar form.

rate of change of the part of the conservative quantity contained in the pulse, is time-independent. This gives us $-a\xi_a + \zeta_a - 1 = 0$. Together with $\xi_a = -1$, the assumptions yields $\zeta_a = 1 - a$. However this does not agree well with the numerical results.

In [23], a particular self-similar solution of the inviscid, unforced gCLM eq. for any $a$ was found, which corresponds formally to the exponents $\xi_a = -1$ and $\zeta_a = -a$. The particular solution is different from the locally self-similar form analyzed here.

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