Spacetime scale-invariant super-$p$-brane actions on enlarged superspaces and the geometry of $\kappa$-symmetry

José A. de Azcárraga †a, José M. Izquierdo ‡b, C. Miquel-Espanya †c

† Departamento de Física Teórica and IFIC (Centro Mixto CSIC-UVEG), 46100-Burjassot (Valencia) Spain
‡ Departamento de Física Teórica, Facultad de Ciencias, E-47011-Valladolid, Spain

Abstract

We use the additional variables of suitably enlarged superspaces to write new actions for extended objects, with $\kappa$-symmetry, in such a way that the tension emerges from them as an integration constant. Our actions correspond to the spacetime scale-invariant ones previously considered by Bergshoeff et al. once the worldvolume forms introduced there are reinterpreted in terms of fields associated with the coordinates of the enlarged superspaces. It is shown that the $\kappa$-symmetry of the new actions is given by a certain type of right local transformations of the extended superspace groups. Further, we also show that the enlarged superspaces that allow for strictly invariant Wess-Zumino terms also lead to strict $\kappa$-invariance i.e., the Lagrangian itself (not only the action) is both supersymmetry- and $\kappa$-invariant.
1 Introduction

Super-$p$-brane actions in rigid superspace contain a kinetic term, that supersymmetrizes the Nambu-Goto-Dirac action, plus a Wess-Zumino (WZ) term. This last is the integral of a supersymmetry quasi-invariant Lagrangian (i.e., invariant but for a total derivative) \[1, 2\]. The WZ term has to be present, with the appropriate normalization factor, for the complete action to be invariant under $\kappa$-symmetry (see \[1\] for a general analysis). The quasi-invariance of the WZ terms leads to central terms in the algebra of charge densities and to topological 'central' charges \[3\] (in connection with super-$p$-brane charges, see also \[5, 6, 7, 8, 9\]). These charges bypass Witten's 'no-go' theorem \[10\] and allow for the partial breaking of rigid supersymmetry (PBRS) \[11\], as implied by the presence of $\kappa$-symmetry: WZ terms, $\kappa$-invariance of the action and PBRS are all related. In fact, it is possible to see all these phenomena already for the massive superparticle in \[12\] which, due to its WZ term, is the $p = 0$ analogue \[13, 14\] of the super-$p$-branes \[1\]; their tension $T$, of dimensions $[T]=ML^{-p}$ ($c=1$), corresponds to the superparticle mass $m$ for $p = 0$, and one might rather write $T(p)$ for $T$ with $T(0) = m$. Furthermore, the appearance of the central mass term in the algebra of Noether charges \[12\] constitutes the simplest example of the classical ‘anomalies’ that are present in the higher $p$ superbrane case, where $T$ enters \[15\]. Notice, nevertheless, that the properties of the $\kappa$-symmetry of the scalar branes \[1\] and of the massive superparticle \[12, 13\] are different in so far as this last allows for a covariant fixing of $\kappa$-symmetry (as for other $D$-branes see \[16, 17, 18\]).

The WZ term of the massive superparticle may be written in a strictly invariant form\(^2\) (see \[19\]) by using a new coordinate that corresponds to the central generator of the, say, $D = 4$, $N = 2$ supersymmetry, the mass being the central charge. This is seen to be equivalent to the reverse of the dimensional reduction procedure that may be used to obtain the massive superparticle action from that of the massless superparticle, the mass appearing as the integration constant that fixes the value of the extra momentum component. For the scalar super-$p$-branes one may also obtain invariant WZ terms \[20, 7\] by using suitable extensions\(^3\) $\tilde{\Sigma}$ of the standard superspaces $\Sigma$ \[7\]. Moreover, for all $p \geq 0$ cases one may find that the contribution to the Noether charges resulting from the quasi-invariance of the original WZ terms, responsible for the topological charges \[3\], is provided by that coming from the additional extended superspace group variables once the WZ terms become invariant. Both contributions are equal, but have a different origin: one results from the quasi-invariance, the other from the additional variables that make the WZ terms invariant under the extended symmetry group. This reflects the connection between the quasi-invariance of the Lagrangians under symmetry groups and their (central) extensions (see also \[19\] for the general theory).

However, in spite of the above similarities with the massive superparticle, the derivation of the string or of a higher $p$ superbrane action from a scale-invariant one, where $T$ does not enter, is more involved since $p \neq 0$. Indeed, for $p \geq 1$ it is not obvious how to derive by dimensional reduction the super-$p$-brane action from that of a null $p$-brane in such a way

\(^1\)The $p$-brane WZ terms are associated \[14\] with $p + 2$-cocycles on the standard graded translations algebra, and therefore produce extensions of the superalgebra of ($p$-dimensional) charge densities that have to be integrated (central extensions of the algebras of charges are directly produced by two-cocycles, the $p=0$ case).

\(^2\)In spite of being a somewhat self-contradicting terminology, we still refer to these invariant terms as WZ terms.

\(^3\)Throughout this paper the word ‘extension’ is used in its (super)group/algebra extension theory meaning, not in a ‘N-extended’ supersymmetry sense.
that the tension appears as an integration constant. Two possible solutions to this problem
were put forward in the past. The first \cite{21} introduced a new variable \( x^*_\tau(\tau) \) depending on
the first of the coordinates \( \xi = (\tau, \sigma^i), i = 1, \ldots, p \) of the worldvolume \( W \), playing the role
of the extra variable of the particle case. The second \cite{22, 23} (see \cite{21} for D-branes and also
\cite{25, 26}) avoided using variables not locally defined on the worldvolume, but was based on
the introduction of an independent \( p \)-form gauge potential \( A(\xi) \), directly defined on \( W \), such
that the variation of its field strength \( dA(\xi) \) cancels exactly the variation of the WZ part of
the Lagrangian. In both cases spacetime scale-invariant actions were written in such a way
that the tension arose as an integration constant.

In this paper we present a third possibility, although related to the second case above. We
construct new rigid superspace actions for \( p = 1, 2 \) (although there is no obstacle for higher
\( p \)) by adding worldvolume fields associated with the extra variables of suitably extended
superspace groups, generically denoted \( \tilde{\Sigma} \). In this way, we adhere to the enlarged superspace
variables/worldvolume fields correspondence principle for superbranes \cite{7}, by which all branes
worldvolume fields (and not only \( x^\mu(\xi), \theta^\alpha(\xi) \)) originate from the (enlarged) superspace \( \Sigma \) variables \( i.e., \) are obtained from pull-backs by the map \( \phi : W \to \tilde{\Sigma} \) that locates the \( p \)-brane
worldvolume in the enlarged superspace\(^4\). The rigid superspace \( \tilde{\Sigma} \) is the group manifold of an
extension of the standard supersymmetry group \( \Sigma \) that is in general non-central (except in the
case of the string Green algebra \cite{29}) and that contains new odd, spinorial generators. These
extended superspace group manifolds \( \tilde{\Sigma} \) (see \cite{7, 20}) trivialize the Chevalley-Eilenberg (CE)
cohomology \((p+2)\)–cocycles on the ordinary supersymmetry algebra that characterize \cite{4}
the quasi-invariant WZ terms of the scalar super-\( p \)-branes \cite{11}, making Lie (super)algebras \cite{20}
of the original free differential algebras. The new actions also reduce to the ordinary super-\( p \)-brane actions when the field equations of the additional superspace variables, that appear
only in the WZ term (see also \cite{7} for other cases), are used. They may be seen to correspond
to those in \cite{22, 23} once the worldvolume \( p \)-form \( A(\xi) \) introduced there is expressed as the
pull-back of a \( p \)-form on the appropriate enlarged superspace, much in the same way that the
world-volume Born-Infeld fields of the D-branes, directly introduced in their actions, may be
obtained from forms on suitably enlarged superspaces \cite{7, 30} (for an analysis of the
\( D \)-\( p \)-brane Noether charges see \cite{3, 6, 21, 7}). The above fact is perhaps not surprising because for \( p = 1 \)
the actions of \cite{23} can be written in a Born-Infeld form, and the relation of \( dA \) there with the
invariant form \( F, F = dA - b \), where \( db = dF \) is also invariant (and a CE cocycle), is that of
a Born-Infeld field \(^5\).

It was early noticed that the fermionic gauge symmetry found in \cite{12} for the massive and in
\cite{34} for the massless superparticle \((p = 0)\) cases, called \( \kappa \)-symmetry, had its origin in a kind of
‘right’ local supersymmetry transformation (with one-half spinor independent parameters).
This was exhibited \cite{12, 13, 15} by the fact that the fermionic constraints of the super-\( p \)-branes
(half of which are first class, generating the local fermionic invariance that removes the unphysical fermionic degrees of freedom) correspond to supercovariant derivatives (the

\(^4\)The idea of a ‘fundamental symmetry between coordinates and fields’ is explicitly stated in Berezin \cite{27}
and is implicit in earlier work of D. V. Volkov; see also \cite{28}.

\(^5\)There are other examples where fields introduced initially ‘by hand’ find an interpretation in terms of
variables of a suitably enlarged superspace, as that of the three-form \( A_3(x) \) field of \( D = 11 \) supergravity
\cite{31, 32, 33}, that support the idea of an extended superspace variables/fields correspondence. That the three
form \( A_3 \) can be ‘trivialized’ is not surprising: the \( D = 11 \) supermembrane WZ term is the pull-back to \( W \) of
the potential of the (CE four-cocycle on superspace) \( dA_3 \), and we know that the WZ terms may be trivialized
on extended superspaces. The above \cite{32, 35} trivializations are, however, different.
"D’s") that generate the right supersymmetry transformations (see [35]), a fact also seen in [36] for the \( p=0 \) case and considered in a general framework in [37] for super-\( p \)-branes. Another treatment of \( \kappa \)-symmetry was given in a twistor-like doubly supersymmetry formulation of the massless superparticle action [38], in which \( \kappa \)-symmetry was identified with proper time (worldline) supersymmetry. This was extended to super-\( p \)-branes (see, e.g. [26]) in the superembedding approach (see [29] and [30] for a review and further references.). Here we exhibit the geometrical character of \( \kappa \)-symmetry as right local transformations when enlarged superspaces \( \tilde{\Sigma} \) are considered. We shall devote a special attention to the role and definition of \( \kappa \)-transformations for the new actions on these enlarged superspaces (explicitly for \( p = 0, 1, 2 \)). It will be shown that these actions have both strict supersymmetry and \( \kappa \)-symmetry, the latter being given in terms of the standard one-half spinor local transformations \( \epsilon(\kappa) \) plus additional local bosonic \( b(\kappa) \)-translations.

The paper is organized as follows. In Sec. 2 we first review the massive particle case and consider in Sec. 2.2 an action in the first order formalism (which is actually that of a massless superparticle in one more dimension) defined on the superspace of a centrally extended supersymmetry algebra, to recall how the mass appears. In Secs. 3 and 4 we consider the \( p = 1, 2 \) super-\( p \)-brane actions in enlarged superspaces \( \tilde{\Sigma} \) on which the Wess-Zumino terms are strictly invariant under supersymmetry. We rewrite them as counterparts of the first order action of the superparticle, suitable for our purposes, and then present in Secs. 3.2 and 4.2 new actions for \( p = 1 \) and \( p = 2 \) respectively in a way analogous to the \( p = 0 \) case. Next we show in Secs. 3.3 and 4.3 that these actions are \( \kappa \)-invariant, and study the geometry of the \( \kappa \)-symmetry transformations, both fermionic and bosonic, in the context of the extended supersymmetry groups.

We note that all considerations in this paper are made for actions on rigid superspaces which, in that case, are supergroup manifolds. For supergravity backgrounds (curved superspaces), \( \kappa \)-transformations may be related to one-half of the fermionic general coordinate transformations of standard superspace (pulled-back to the worldvolume) preserved by the brane action. The group-theoretical structure appears there when the supergravity constraints describing the standard rigid superspace are considered. We shall not discuss this here and refer instead to e.g. [41] and references therein, where the local symmetries of the dynamical supergravity-superbrane interacting system are also considered.

2 The massive superparticle and its \( \kappa \)-symmetry

2.1 The massive superparticle action with invariant WZ term

The action for the massive superparticle [12] is

\[
I = \int d\tau \left\{ -m \sqrt{-\omega_\mu \omega^\mu} \right. + \lambda \epsilon_{IJ} C_{\alpha\beta} \theta^\alpha_I \dot{\theta}^\beta_J \left. \right\}
\]

where \( m \) is the mass of the particle, \( I = 1, 2 \ (N=2) \), \( \epsilon_{IJ} = -\epsilon_{JI} \), \( \epsilon_{12} = 1 \), \( \theta_I \) are Majorana spinors, \( \omega_\mu \) is given by

\[
\omega_\mu = \dot{x}_\mu + \theta^I (C\Gamma_\mu)_{\alpha\beta} \dot{\theta}^\beta_I
\]

and \( \lambda \) is fixed so that the action has the required \( \kappa \)-invariance [12] [13] [36]; we use mostly plus signature. The key feature of the WZ term in [11] is that it is the worldline expression of the superspace quasi-invariant one-form \( \epsilon_{IJ} C_{\alpha\beta} \theta^\alpha_I d\theta^\beta_J \), the differential of which, \( \epsilon_{IJ} C_{\alpha\beta} d\theta^\alpha_I \wedge d\theta^\beta_J \),
is a non-trivial two-cocycle\(^6\) (here, for the \(N=2\) supersymmetry algebra cohomology; we shall omit wedge products henceforth). By construction, the potential WZ one-form \(\epsilon_{IJ}C_{\alpha\beta}\theta^I_\alpha\theta^J_\beta\) cannot be an exact differential. This non-trivial two-cocycle property \([12]\) is the property that determines all massive superparticle (0-brane) actions in any dimension \(D\), much in the same way as the non-trivial higher order CE cocycles characterize \([4]\) (see also [7]) other \(p > 0\) branes. For instance, for \(D = 4\) the charge conjugation matrix \(C\) is antisymmetric, and the bilinears \(\theta_1\theta_2, \theta_1\gamma^5\theta_2\), where \(\theta_{1,2}\) are Majorana spinors, are symmetric under the exchange \(1 \leftrightarrow 2\). This is already sufficient to discard \(m\theta^\dagger\theta, m\theta\gamma^5\theta\) as \(N=1, D=4, p=0\) WZ terms (they are total derivatives). Hence a \(N=2\) superspace was used for the original massive superparticle action with WZ term \([12]\) since it was specifically written for \(D = 4\). In contrast, since \(C\) is symmetric for \(D=9\), \(m\theta^\dagger\theta\) is a non-trivial two-cocycle and \(m\theta\gamma^5\theta\) is an \(N=9\) \(D0\)-brane) WZ term. All other cases follow analogously.

To look for the value of \(\lambda\) in the action \([11]\) we shall first introduce a new coordinate \(\phi\) in such a way that the second (WZ) term in \([11]\) becomes invariant. The action \([11]\) is built using the pull-backs of the left-invariant Maurer-Cartan (MC) one-forms, \(\Pi^\mu \equiv dx^\mu + \theta^\alpha_\mu (C\theta^\alpha_\beta)\theta^\beta_\mu\) and \(\Pi^\beta_I \equiv d\theta^\beta_I\), defined on \(N = 2\) superspace \(\Sigma^2\), to the particle worldline. These forms satisfy the MC equations of the standard \(N = 2\) supersymmetry algebra

\[
d\Pi^\beta_I = 0 , \quad d\Pi^\mu = \Pi^\beta_I (C\theta^\mu_\alpha)\theta^\beta_\mu (I = 1, 2) .
\]

This superalgebra can be centrally extended, a fact that is equivalent to the existence of a closed invariant two-form on \(\Sigma^2\), \(\Omega_2 = \epsilon_{IJ}C\theta^I_\alpha\theta^J_\beta\), which is not the exterior differential of a left-invariant one-form. However, its quasi-invariant potential one-form \(\epsilon_{IJ}C\theta^I_\alpha\theta^J_\beta\) can be supplemented with a piece \(d\phi\) in such a way that the transformation properties of \(\phi\) make the resulting one-form left-invariant. In this way, a larger superspace group (a central extension of the \(N = 2\) superspace) is defined, on which the one-form \(\Pi^\phi \equiv d\phi + \epsilon_{IJ}C\theta^I_\alpha\theta^J_\beta\), \(d\Pi^\phi = \Omega_2\), is the MC form associated with the central charge coordinate \(\phi\) of the \(N = 2\) enlarged superspace \(\Sigma^2\). The one-form \(\Pi^\phi\) can be pulled back to the worldline to define an invariant WZ term for the massive superparticle:

\[
I = \int d\tau \{-m\sqrt{-\omega^\mu\omega^\mu} + \lambda \omega^\phi\} \quad (4)
\]

where

\[
\phi^*(\Pi^\phi) \equiv \phi^*(d\phi + \epsilon_{IJ}C\theta^I_\alpha\theta^J_\beta) = (\dot{\phi} + \epsilon_{IJ}C\theta^I_\alpha\theta^J_\beta\dot{\theta}^\beta) d\tau \equiv \omega^\phi d\tau .
\]

It is easy to see here that \(\kappa\)-symmetry is a right local supersymmetry translation depending of one-half of the parameters. The MC forms \(\Pi^\alpha, \Pi^\beta\) of \([3]\) and \(\Pi^\phi\) of \([5]\) are all left-invariant under rigid extended supersymmetry transformations. Under right local transformations, the MC forms transform as gauge fields. Indeed, the canonical left-invariant form on a Lie group \(G, \ g^{-1}dg = \omega^I \circ T_i\) (where \(T_i\) is a basis of \(G\)), transforms under local right translations \(h\) of coordinates \(h^i\) as

\[
g^{-1}dg \rightarrow (gh)^{-1}d(gh) = h^{-1}(g^{-1}dg)h + h^{-1}dh ,
\]

\(^6\)A non-trivial \((p+2)\)-cocycle is given by a closed and left-invariant \((p+2)\)-form that it is not the exterior derivative of a left-invariant \((p+1)\)-form.
and the component MC forms $\omega^j$ as
$$\delta_h \omega^j \simeq dh^j + C^j_{ik} \omega^i \omega^k$$  \hspace{1cm} (7)$$

So we do not need to write the explicit form of $\Pi^\alpha$, $\Pi^\mu$ and $\Pi^\varphi$ in terms of the $\tilde{\Sigma}^2$ coordinates $\theta^\alpha$, $x^\mu$ and $\varphi$ to know the gauge transformation of the Lagrangian; it suffices to know the extended superalgebra structure constants.

Here we need the transformation of the MC one-forms under the local fermionic super-translation $\epsilon^\alpha_I(\kappa)$,
$$\delta_\kappa \Pi^\alpha_I = d\epsilon^\alpha_I(\kappa), \quad \delta_\kappa \Pi^\mu = 2\epsilon^\alpha_I(\kappa)(C\Gamma^\mu)_{\alpha\beta}\Pi^\beta_I, \quad \delta_\kappa \Pi^\varphi = 2\epsilon^{IJ}\epsilon^\alpha_I(\kappa)\Pi^\alpha_J.$$  \hspace{1cm} (8)

Using (8) in (4), one obtains
$$\delta I = \int d\tau \{ m \omega^\mu \sqrt{-\omega^\nu \omega_\nu} (C\Gamma^\mu)_{\alpha\beta} \delta_{I\beta} + \lambda \epsilon_{IJ} \epsilon^\alpha_I(\kappa) \} \dot{\theta}^\beta_J.$$  \hspace{1cm} (9)

It is then straightforward to check that if $\lambda = \pm m$ and
$$\epsilon^\alpha_I = \frac{\omega^\mu}{\sqrt{-\omega^\nu \omega_\nu}} (\Gamma^\mu)^\alpha_{\beta \gamma} \delta^\gamma_I = \pm \epsilon_{IK} \frac{\omega^\mu}{\sqrt{-\omega^\nu \omega_\nu}} (\Gamma^\mu)^\alpha_{\beta} \delta^\beta_J,$$  \hspace{1cm} (10)

where the expression in brackets is twice a projector with one-half eigenvalues equal to zero (its second term squares to unity and is traceless), the variation of $I$ vanishes identically. Therefore the $\kappa$-invariant action is
$$I = \int d\tau \{-m \sqrt{-\omega^\nu \omega_\nu} \pm m\omega^\varphi\}.$$  \hspace{1cm} (11)

It is interesting to note that the bosonic and fermionic first class constraints for the Lagrangian in eq. (11), that produce the Klein-Gordon and Dirac equation for the fields of the first quantized model [12, 13, 36], give fields of equal mass (as required by supersymmetry) due to the presence of $\kappa$-symmetry, which requires $\lambda = \pm m$.

Let us now write the action in first order form by introducing $p^\mu(\tau)$ and the einbein $E(\tau) = e(\tau)d\tau$:
$$I = \int d\tau \{ p^\mu \omega^\nu - \frac{1}{2} e(p^\mu p^\mu + m^2) \pm m\omega^\varphi \}.$$  \hspace{1cm} (12)

This action is classically equivalent to (11). Indeed, the $p^\mu$ equation gives $p^\mu = \omega^\mu/e$. This can be substituted into the action, which now depends on $\omega^\mu, e, \omega^\varphi$. The $e$-dependence can be removed by using its Lagrange equation. Choosing the positive sign, $e = \frac{1}{m} \sqrt{-\omega^\nu \omega_\nu}$, and introducing this again into the action gives (11).

### 2.2 The superparticle mass as an integration constant

The new variable $\varphi$ appears in the action (12) as a total derivative, but if we write instead the action
$$I_* = \int d\tau \{ p^\mu \omega^\nu - \frac{1}{2} e(p^\mu p^\mu + p^2_\varphi) + p_\varphi \omega^\varphi \}$$  \hspace{1cm} (13)

$\varphi$ is no longer dynamically trivial, and its Lagrange equation gives $\dot{p}_\varphi = 0$. Then, setting $p_\varphi = \pm m$ and substituting in (13), one recovers (12): the mass of the particle can be viewed
as an integration constant. By doing this, we restrict the variable \( p_* \) to a particular classical solution, so that in moving to \( I_* \) we have lost a dynamical degree of freedom present in \( I_{13} \); obviously, the massive superparticle action \( I_* \) is not equivalent to the action \( I_{13} \). This is clear at the quantum level, where one has to integrate over all degrees of freedom. The same observation will apply to the higher order branes to be considered later.

The action \( I_* \) of \( I_{13} \) can be re-interpreted as an action for a massless superparticle in one dimension higher, if \( \varphi \) is treated as the \((D + 1)\)-th new spatial coordinate and \( p_* \) as its associated momentum. Inserting the \( p^\mu \) and \( p_* \) equations

\[
p^\mu = \frac{\omega^\mu}{e}, \quad p_* = \frac{\omega^\varphi}{e},
\]

into \( I_{13} \), we obtain

\[
I_* = \int d\tau \frac{1}{2e} \left\{ \omega^\mu \omega_\mu + (\omega^\varphi)^2 \right\} \equiv \int d\tau \frac{1}{2e} \omega^\mu_\mu \omega^\varphi_\varphi .
\]

### 3 The \( N = 1, p = 1 \) superstring case

#### 3.1 Superstring actions in enlarged superspace

As in the massive superparticle case, the Lagrangian of the \( D = 10, N = 1 \) superstring has a quasi-invariant Wess-Zumino term, which also can be made strictly invariant by enlarging the standard superspace to one \( \Sigma \) with coordinates \( x^\mu, \theta^\alpha, \varphi_\alpha \), the variables of the extended superspace group corresponding to the Green algebra \[29\]. Its MC equations are\(^7\)

\[
d\Pi^\alpha = 0 ,
\]

\[
d\Pi^\mu = \frac{1}{2} (\zeta \Gamma^\mu)_{\alpha\beta} \Pi^\alpha \Pi^\beta ,
\]

\[
d\Pi_\alpha = (\zeta \Gamma_\mu)_{\alpha\beta} \Pi^\beta + (\zeta \Gamma_\varphi)_{\alpha\beta} \Pi^\beta .
\]

Notice that \( \Pi^\alpha \) and \( \Pi_\alpha \) are associated with the different (Majorana-Weyl) fermionic variables \( \theta^\alpha \) and \( \varphi_\alpha \) and that are hence unrelated (no charge conjugation matrix is used to move the index \( \alpha \)). Since we shall use them to construct the new action, we give below the MC equations \([27, \text{eq. (48)})\) of a generalization of the above Green algebra

\[
d\Pi^\alpha = 0 , \quad d\Pi^\mu = \frac{1}{2} (\zeta \Gamma^\mu)_{\alpha\beta} \Pi^\alpha \Pi^\beta ,
\]

\[
d\Pi_\varphi = \frac{1}{2} (\zeta \Gamma^\mu)_{\alpha\beta} \Pi^\beta + (\zeta \Gamma_\mu)_{\alpha\beta} \Pi^\beta ,
\]

\[
d\Pi_\alpha = (\zeta \Gamma_\mu)_{\alpha\beta} \Pi^\beta + (\zeta \Gamma_\varphi)_{\alpha\beta} \Pi^\beta .
\]

Adding the Lorentz automorphisms part, eqs. \[17\] determine an enlarged \( N = 1 \) super-Poincaré group \( \tilde{\mathbb{SP}} \) parametrized by \( (x^\mu, \theta^\alpha, \varphi_\alpha, \varphi_\mu, \alpha^\mu) \). Thus, \( (x^\mu, \theta^\alpha, \varphi_\alpha, \varphi_\mu) \) parametrize an extended \( D = 10 \) superspace group manifold \( \Sigma = \tilde{\mathbb{SP}}/L \).

Using the MC forms in \[17\], we may construct a strictly invariant WZ term for the superstring following \[16\]. Writing \( \phi^\ast(\Pi^\mu) = \Pi^\mu_i d\xi^i \), \( \phi^\ast(\Pi^\varphi_\alpha) = \Pi^\varphi_\alpha d\xi^i \), \( \phi^\ast(\Pi^\varphi_\mu) = \Pi^\varphi_\mu d\xi^i \), \( \phi^\ast(\Pi^\alpha) = \Pi^\alpha_i d\xi^i \),

\(^7\)In Sec. 3 all appropriate expressions have the chiral projector understood, so that all spinors are Majorana-Weyl \( (\theta^\alpha \equiv P^+ \theta^\alpha, \zeta \Gamma^\mu \text{ corresponds to } \sigma^\mu, \text{ etc}) \).
\( \phi^* (\Pi_\alpha) = \Pi_\alpha d\xi^i \) for the pull-backs of the MC forms on the worldsheet, and \( M_{ij} = \Pi^\mu_i \Pi^\nu_j \) for the induced metric on \( \mathcal{W} \), the invariant superstring action is (see [7])

\[
I = - \int d^2 \xi \left\{ T \sqrt{-\det M} + \lambda \epsilon^{ij} (\Pi^\mu_i \Pi^\nu_j - \frac{1}{2} \Pi_\alpha \Pi_\alpha^0) \right\},
\]

where again \( \lambda \) is to be fixed by \( \kappa \)-symmetry. Note that in this action the new variables \( \varphi_\alpha \), \( \varphib^\mu \) (i.e., those beyond the standard superspace \( \Sigma \) coordinates \( (x^\mu, \theta^a) \)) appear only in the invariant WZ term and through a total derivative. This is evident once one computes its exterior differential with the help of (16), since the result involves only \( \Pi^\alpha \) and \( \Pi^\mu \):

\[
d(\Pi^\mu \Pi^\nu - \frac{1}{2} \Pi_\alpha \Pi^\alpha) \equiv dB = -(C\Gamma^\mu)_{\alpha\beta} \Pi^\alpha \Pi^\beta \Pi^\alpha. \tag{19}\]

To show that \( \kappa \)-symmetry is a right local supersymmetry depending on one-half of the fermionic parameters, we need to know the form of these transformations. Either taking

\[
\delta_\kappa \theta^\alpha = \epsilon^\alpha (\kappa), \quad \delta_\kappa x^\mu = -\frac{1}{2} (C\Gamma^\mu)_{\alpha\beta} \theta^\alpha \epsilon^\beta (\kappa), \quad \delta_\kappa \varphib^\mu = -\frac{1}{2} (C\Gamma^\mu)_{\alpha\beta} \theta^\alpha \epsilon^\beta (\kappa), \quad \delta_\kappa \varphi_\alpha = -\frac{1}{2} (C\Gamma^\mu)_{\alpha\beta} x^\mu \epsilon^\beta (\kappa) - \frac{1}{2} (C\Gamma^\mu)_{\alpha\beta} \varphib^\mu \epsilon^\beta (\kappa)
\]

\[+ \frac{1}{6} (C\Gamma^\mu)_{\alpha\beta} (C\Gamma^\nu)_{\gamma\delta} \theta^\alpha \theta^\beta \epsilon^\gamma (\kappa) \tag{20}\]

(see [7] for the extended superspace group law corresponding to the superalgebra [17]) or directly from the transformation properties of the left invariant MC one-forms under local right supertranslations, eq. [7], we obtain

\[
\delta_\kappa \Pi^\alpha = d(\epsilon^\alpha (\kappa)), \quad \delta_\kappa \Pi^\mu = -(C\Gamma^\mu)_{\alpha\beta} \Pi^\alpha \epsilon^\beta (\kappa), \quad \delta_\kappa \Pi^\nu = -(C\Gamma^\nu)_{\alpha\beta} \Pi^\alpha \epsilon^\beta (\kappa), \quad \delta_\kappa \Pi_\alpha = -(C\Gamma^\mu)_{\alpha\beta} \Pi^\mu \epsilon^\beta (\kappa) - (C\Gamma^\mu)_{\alpha\beta} \Pi_\beta \epsilon^\gamma (\kappa). \tag{21}\]

The \( \kappa \)-variation of the action may now be calculated from eq. [21] as follows. For the kinetic (‘Dirac’) part one may use \( \delta \det M = \det MM^{ij} \delta M_{ij} \). That of the WZ term is calculated by noticing that the second term in (18) is the pull-back of \( \phi^* (\lambda B) \) to the worldsheet, where \( B \) is defined in (19), and that the variations (21) are given by Lie derivatives \( L_X \) where the vector field has only the \( \epsilon^\alpha (\kappa) \) components, \( i_X (\Pi^\alpha) = \epsilon^\alpha (\kappa) \) (the other contractions are zero). Then, the relevant term of \( \delta_\kappa B = L_X B = (i_X d + d i_X)B \) in the action is \( i_X dB \) which, by (19), is given by \( 2(C\Gamma^\mu)_{\alpha\beta} \Pi^\mu \epsilon^\beta (\kappa) \Pi_\alpha \). It is useful to introduce the worldvolume Dirac matrices and their antisymmetrized products by the definitions

\[
\Gamma_i \equiv \Pi^\mu_i \Gamma^\mu \quad (\{\Gamma_i, \Gamma_j\} = 2M_{ij}), \quad \Gamma_{ij} = \Gamma_{\mu\nu} \Pi^\mu_i \Pi^\nu_j. \tag{22}\]

Then,

\[
\delta_\kappa I = - \int d^2 \xi \left\{ -T \sqrt{-\det M} M^{ij} (C\Gamma_i)_{\alpha\beta} \Pi^\alpha_j \epsilon^\beta (\kappa) + 2\lambda \epsilon^{ij} (C\Gamma_i)_{\alpha\beta} \epsilon^\beta (\kappa) \Pi^\alpha_j \right\}. \tag{23}\]
If we now set
\[ e^\alpha (\kappa) = \frac{1}{2} \left( \delta^\alpha_\beta \pm \frac{1}{2} \epsilon^{ij} (\Gamma_{ij})^\alpha_\beta \right) \kappa^\beta , \] (24)
the matrix acting on \( \kappa \) is again a projection operator since \( \frac{1}{2} \epsilon^{ij} (\Gamma_{ij})^\alpha_\beta \) squares to unity, and, being traceless, \( \epsilon(\kappa) \) depends on half of the \( \kappa \) parameters. Then, choosing \( \lambda = \pm \frac{1}{2} T \) we see that \( \delta_\kappa I = 0 \), and the \( \kappa \)-invariant action reads
\[ I = - \int d^2 \xi \left\{ T \sqrt{-\text{det}M} \pm \frac{T}{2} \epsilon^{ij} \left( \Pi_{\mu_i} \Pi^\mu_{\nu_j} \mp \frac{1}{2} \Pi_{\alpha_i} \Pi^\alpha_{\beta_j} \right) \right\} . \] (25)

We can now write the action (25) in a ‘first order-like’ form analogous to (12) by introducing \( p^{\mu\nu}(\xi) = p^{[\mu\nu]}(\xi) \) and \( e(\xi) \), which nevertheless lacks the interpretation in terms of the canonical formalism that is applicable to the particle case:
\[ I = - \int d^2 \xi \left\{ \frac{1}{2} \epsilon^{\mu\nu} e^{ij} \Pi_{\mu_i} \Pi_{\nu_j} + \frac{1}{2} e(p^{\mu\nu} p_{\mu\nu} + 2T^2) \pm T \epsilon^{ij} \left( \Pi_{\mu_i} \Pi^\mu_{\nu_j} \mp \frac{1}{2} \Pi_{\alpha_i} \Pi^\alpha_{\beta_j} \right) \right\} . \] (26)

In differential form language, and using \( B \) (eq. (19)), \( I \) reads
\[ I = - \int \frac{1}{2} \left\{ \epsilon^{\mu\nu} \Pi_{\mu} \Pi_{\nu} + \frac{1}{2} E p^{\mu\nu} p_{\mu\nu} + T^2 E \pm TB \right\} , \] (27)
where
\[ e(\xi) = \frac{1}{2} \epsilon^{ij} E_{ij}(\xi) , \quad E(\xi) = \frac{1}{2} E_{ij}(\xi) d\xi^i d\xi^j . \] (28)

It is easy to prove that this action is classically equivalent to (25). First, the Euler-Lagrange (E-L) equation for \( p^{\mu\nu} \) tells us that \( p_{\mu\nu} = -\frac{1}{2} \epsilon^{ij} \Pi_{\mu_i} \Pi_{\nu_j} \). Substituting this into the action (26), solving for \( e, e = \frac{1}{2} \sqrt{-\text{det}M} \), and substituting again yields (25). In the process, one uses the identity
\[ \epsilon^{ij} e^{\mu\nu} \Pi_{\mu_i} \Pi_{\nu_j} = 2\text{det}M . \] (29)

We conclude this subsection with a remark on one more form of the action (27). The evident (‘rest-like system’ solution \( p^{(a)(b)} = T \delta^{(a)(b)} \) \( (\delta^{D-1}_{(a)})^\pm \) of the constraint \( p_{\mu\nu} p^{\mu\nu} = -2T^2 \) (provided by the E-L equation for \( e \)) may be made covariant by means of the Lorentz group matrix
\[ \Lambda^\mu_{(a)} = \left[ \frac{1}{2} (u^{+\mu} + u^{-\mu}) , u^{\mu}_{+} , 1/2 (u^{\mu}_{+} - u^{\mu}_{-}) \right] \in SO(1, D - 1) , \] (30)
where \( u^{+\mu} (u^{-\mu}) \) corresponds to \( u^{0} + u^{(D-1)} \) \( (u^{0} - u^{(D-1)}) \) and \( i = 1, \ldots, D - 2 \). Inserting the solution
\[ p_{\mu\nu} = p^{(a)(b)} \Lambda^\mu_{(a)} \Lambda^\nu_{(b)} = T u^{+\mu}_{[\mu} u^{\nu}_{] -} , \] (31)
into the action (27) we arrive at
\[ I = -T \int \left( \frac{1}{2} \Pi^\mu u^{+\mu}_{[\mu} \Pi^\nu u^{\nu}_{] -} \pm B \right) . \] (32)

This is the so-called Lorentz-harmonics formulation of the superstring action 42 (see also 33, 39), but now with a strictly invariant WZ term.
3.2 A new action in the superstring case

In Sec. 2.2 we substituted a new momentum $p_*$ for $m$ to obtain the generalized action \[ (13) \]. Here we shall do something similar, by adding to the action a new worldsheet field (also denoted $p_*$ but obviously different, $p_* = p_*(\xi)$) that will replace the tension. This new, spacetime scale-invariant form $I_*$ of the action \[ (27) \] has the expression

$$I_* = - \int d^2 \xi \left\{ \frac{1}{2} \partial^\mu \epsilon^{ij} \Pi_{\mu i} \Pi_{\nu j} + \frac{1}{4} \epsilon(p^{\mu \nu} p_{\mu \nu} + 8 p_*^2) + p_* \Phi \right\} ,$$

(33)

where, for convenience, we have written $\phi^*(B) = \frac{1}{2} B_{ij} d\xi^i d\xi^j = \Phi d^2 \xi$

$$\Phi \equiv \frac{1}{2} \epsilon^{ij} B_{ij} = \epsilon^{ij}(\Pi_{\mu i} \Pi_{\nu j} - \frac{1}{2} \Pi_{\alpha i} \Pi_{\beta j})$$

(34)

(recall that $\Pi_\alpha$ and $\Pi^\alpha$ are unrelated).

Let us show now that $T$ in \[ (26) \] appears as an integration constant (see the comment after \[ (13) \]). To this aim, let us first compute the E-L equations for the enlarged superspace worldsheet fields. These result from setting equal to zero the coefficients of $i_X \Pi^\alpha$, $i_X \Pi_\mu$, $i_X \Pi_\alpha$, $i_X \Pi_\mu$ in a general variation $\delta = \Delta \Pi = \Pi(\delta) = \Phi d^2 \xi$

$$X = \Pi(\delta)^\alpha D_\alpha + \Pi(\delta)^\mu D_\mu + \Pi(\delta)^\alpha + \Pi(\delta)_\beta D^\beta$$

(35)

where the $D$’s are the left-invariant vector fields on $\tilde{\Sigma}$ (so that $\Pi^A(D_B) = \delta^A_B$) and the $\Pi(\delta)$’s are the corresponding MC forms in which $d\theta, dx^\mu, d\varphi_\alpha, d\varphi^\mu$ have been replaced by the variations $\delta \theta, \delta x^\mu, \delta \varphi_\alpha, \delta \varphi^\mu$. Adding the equations for $p^{\mu \nu}$, $p_*$ and $E$, the complete set of E-L equations in differential form is given by

$$-\frac{1}{2} d(p_* \Pi_\alpha) + p^{\mu \nu} \Pi_\nu (\Sigma \mu)_{\alpha \beta} \Pi^\beta - \frac{1}{2} p_*(\Sigma \mu)_{\alpha \beta} \Pi^\beta \Pi_\varphi$$

$$- \frac{3}{2} p_*(\Sigma \mu)_{\alpha \beta} p^{\mu \nu} \Pi^\beta = 0$$

(36)

$$-d(p^{\mu \nu} \Pi_\nu) - d(p_* \Pi_\varphi) - \frac{1}{2} p_*(\Sigma \mu)_{\alpha \beta} \Pi^\alpha \Pi^\beta = 0$$

(37)

$$dp_* \Pi^\alpha = 0$$

(38)

$$dp_* \Pi_\mu = 0$$

(39)

$$\Pi^\mu \Pi^\nu + E p^{\mu \nu} = 0$$

(40)

$$\Pi^\mu \Pi_\varphi - \frac{1}{2} \Pi_\alpha \Pi^\alpha + 4 E p_* = 0$$

(41)

$$p^{\mu \nu} p_{\mu \nu} + 8 p_*^2 = 0 .$$

(42)

Equation \[ (39) \] in worldsheet coordinates is $\epsilon^{ij} \partial_i p M^\mu_{jk} = 0$, which contracted with $\Pi_{\mu k}$ gives

$$\epsilon^{ij} \partial_i p M^\mu_{jk} = 0 .$$

(43)

If the induced metric $M_{ij}$ is non-degenerate, one immediately concludes that $p_*$ is a constant; if not, eqs. \[ (12) \] and \[ (40) \] (or \[ (45) \] below) tell us that $p_* = 0$ necessarily, so in any case $p_*$ is a constant that we can set equal to $\pm \frac{T}{2}$ (with $T = 0$ for $p_* = 0$). If we use this solution in $I_*$,
the result is \[26\]. When \(p^{\mu\nu}\) and \(p_*\) are eliminated using their E-L equations, the resulting action is

\[
I_* = \int d^2\xi \frac{1}{e} \left\{ \det M + \frac{\Phi^2}{8} \right\},
\]

(44)

where \(\Phi\) is given in [23]. This recovers the action of [22] although there \(\Phi\) involves the field strength of an independent worldsheet field \(A_i(\xi)\). In the present framework, \(A_i(\xi)\) is expressed as a composite of fields associated with the enlarged superspace coordinates in the spirit of the mentioned extended superspace group variables/fields correspondence principle for branes [7].

To see that \[33\] leads to \[44\] explicitly, let us solve the algebraic equations for \(p_{\mu\nu}\) and \(p_*\),

\[
p_{\mu\nu} = -\frac{1}{e} \varepsilon^{ij} \Pi_{\mu i} \Pi_{\nu j}
\]

(45)

\[
p_* = -\frac{1}{4e} \varepsilon^{ij} \left( \Pi_{\mu i} \Pi_{\nu j} - \frac{1}{2} \Pi_{\alpha i} \Pi_{\alpha j} \right).
\]

(46)

Inserting eqs. \([45], [46]\) into \(I_*\) (eq. \([33]\)) one obtains

\[
I_* = \int d^{p+1}\xi \left\{ -\frac{1}{2e} \varepsilon^{ij} \Pi_{\mu i} \Pi_{\nu j} \varepsilon^{kl} \Pi_{\mu k} \Pi_{\nu l} + \frac{1}{4e} \varepsilon^{ij} \Pi_{\mu i} \Pi_{\nu j} \varepsilon^{kl} \Pi_{\mu k} \Pi_{\nu l} 
\]

\[
-\frac{1}{4e} \varepsilon^{ij} \left( \Pi_{\mu i} \Pi_{\nu j} - \frac{1}{2} \Pi_{\alpha i} \Pi_{\alpha j} \right) \varepsilon^{kl} \left( \Pi_{\mu k} \Pi_{\nu l} - \frac{1}{2} \Pi_{\alpha k} \Pi_{\alpha l} \right) 
\]

\[
+\frac{1}{8e} \varepsilon^{ij} \left( \Pi_{\mu i} \Pi_{\nu j} - \frac{1}{2} \Pi_{\alpha i} \Pi_{\alpha j} \right) \varepsilon^{kl} \left( \Pi_{\mu k} \Pi_{\nu l} - \frac{1}{2} \Pi_{\alpha k} \Pi_{\alpha l} \right) \right\} 
\]

\[
= \int d^{p+1}\xi \left\{ \frac{1}{4e} \varepsilon^{ij} \varepsilon^{kl} M_{ik} M_{jl} + \frac{1}{8e} \Phi^2 \right\},
\]

(47)

which is eq. \([44]\). We note, for later use, the E-L for \(e\),

\[
\det M + \Phi^2/4 = 0.
\]

(48)

### 3.3 \(\kappa\)-invariance of the new actions

We now exhibit the \(\kappa\)-symmetry of the new superstring action on the enlarged superspace both in the forms of eq. \([33]\) and \([44]\).

**(a) \(\kappa\)-invariance of the action \([33]\)**

To see that this action is invariant under the \(\kappa\)-symmetry transformations, one may prove that the equations for \(\theta^\alpha\), eqs. \([35]\), are not independent from the other E-L equations. This, by virtue of the second Noether’s theorem, reflects the presence of a fermionic gauge symmetry of the action.

The first step is to realize that

\[
p^{\mu\nu} \Gamma_{\mu\nu} + 4p_*
\]

(49)

is \((8p_*)\) times a projector. One has to compute the square of the matrix \(p^{\mu\nu} \Gamma_{\mu\nu}\), which has a piece proportional to \(p_{\mu\nu} p^{\mu\nu}\) which by eq. \([12]\) is equal to \(-8p_*^2\), but also has contributions of
the form $\Gamma^{\mu\nu\rho\sigma}p_{\mu\nu}p_{\rho\sigma}$ and $\Gamma^{\mu\nu}p_{\mu}p_{\nu}$. These two types of contributions vanish, due to index symmetry arguments, if one takes into account eq. (40) in the form

$$p^{\mu\nu} = -\frac{1}{e^{ij}} \Pi^{\mu} \Pi^{\nu}.$$  

(50)

Next, after using $dp_* = 0$ and the last equation of (17), the fermionic equation (56) reads

$$[2p_*(CT_\mu)_{\alpha\beta} \Pi^\mu + p^{\mu\nu}(CT_\nu)_{\alpha\beta} \Pi_{\nu}] \Pi^\beta = 0.$$  

(51)

To show that some of these equations (actually half of them) are trivially satisfied we multiply them by (19). Again, there are terms that vanish due to index symmetry considerations once (50) is used. The only two surviving terms are proportional to $p^2 \Gamma^\mu_\mu$ and $p^\rho \beta p^{\mu\nu} \Gamma_\rho \Pi_\mu$ respectively. The latter may, for instance, be re-expressed as a term of the form $p_{\mu\nu} p^{\mu\nu} \Gamma_\mu \Pi_\mu$ by making the substitution (50), then using that the antisymmetrization of three worldsheet indices vanishes, and finally going back to an expression involving $p_{\mu\nu}$. Then both terms cancel each other due to eq. (12). This shows the presence of the standard number of fermionic gauge $(\kappa)$-symmetries under which the action is invariant.

(b) $\kappa$-invariance of the action (44)

We now find explicitly the form of the gauge $\kappa$-transformations under which the action (44) is invariant, in order to exhibit its geometrical nature as right local supersymmetries in the context of our enlarged superspace. We could do this in the case of action (53), but the transformation of the auxiliary field $p^{\mu\nu}(\xi)$ is non-trivial, and this obscures the geometrical interpretation.

Under a local right translation of parameters $\epsilon^\alpha(\kappa)$, $b^\mu(\kappa)$ associated with the right transformations of $\theta^\alpha$, $\varphi^\mu$, the variation of the MC forms is read from the structure constants of the extended algebra,

$$\delta_\kappa \Pi^\alpha = d\epsilon^\alpha(\kappa)$$  

(52)

$$\delta_\kappa \Pi^\mu = -(CT_\mu)_{\alpha\beta} \Pi^{\alpha} \epsilon^{\beta}(\kappa)$$  

(53)

$$\delta_\kappa \Pi^\mu_\varphi = -(CT_\mu)_{\alpha\beta} \Pi^{\alpha} \epsilon^{\beta}(\kappa) + db^\mu(\kappa)$$  

(54)

$$\delta_\kappa \Pi_\alpha = -(CT_\mu)_{\alpha\beta} \Pi^\beta(\kappa) - (CT_\mu)_{\alpha\beta} \Pi^{\alpha} \epsilon^{\beta}(\kappa) + (CT_\mu)_{\alpha\beta} \Pi^\beta b^\mu(\kappa).$$  

(55)

Using these variations we may now compute $\delta M_{jl} = -(CT_\mu)_{\alpha\beta} \Pi^{\alpha} \epsilon^{\beta}(\kappa) + (l \leftrightarrow j)$, from which we obtain

$$\delta_\kappa \det M = \epsilon^{ij} e^{kl} M_{ik} \delta_\kappa M_{jl} = 2\epsilon^{ij} e^{kl} M_{ik} \Pi_{\mu} \delta_\kappa \Pi^\mu_\mu$$

$$= -2\epsilon^{ij} e^{kl} M_{ik} (CT_\mu)_{\alpha\beta} \Pi^{\alpha} \epsilon^{\beta}(\kappa).$$  

(56)

There are no terms in $\delta_\kappa \det M$ containing $b^\mu(\kappa)$, exactly as in the standard superstring action. But since $\Phi$ does depend on the the new variables, there are contributions both from $\epsilon^\alpha(\kappa)$ and $b^{\mu}(\kappa)$ to $\delta_\kappa \Phi$. The $\epsilon^\alpha(\kappa)$ contribution gives

$$\delta_\kappa(\kappa) \Phi = \epsilon^{ij} \left( -(CT_\mu)_{\alpha\beta} \Pi_\mu^\alpha \epsilon^\beta(\kappa) + \Pi_{\mu\mu}(CT_\mu)_{\alpha\beta} \Pi^{\alpha} \epsilon^{\beta}(\kappa) + \frac{1}{2}(CT_\mu)_{\alpha\beta} \Pi^{\alpha} \epsilon^\beta(\kappa) \Pi_\mu^\beta + \frac{1}{2}(CT_\mu)_{\alpha\beta} \Pi^\beta(\kappa) + \frac{1}{2} \partial_\alpha \Pi_\alpha \epsilon^\alpha(\kappa) - \frac{1}{2} \partial_j(\Pi_\alpha \epsilon^\alpha(\kappa)) \right).$$  

(57)
In the standard superstring action the total derivative term in eq. (57) is dealt with integrating by parts, which is not possible here. To cancel it we shall use instead the $b^\mu(\kappa)$ part of the transformation in (52) - (55) associated with $\varphi^\mu$, which is

$$\delta b(\kappa) = \epsilon^{ij}(\Pi_{\mu i}b^\mu(\kappa))$$

Let us assume that there is a $b^\mu(\kappa)$ such that $\Pi_{\mu i}b^\mu(\kappa) = \frac{1}{2}\Pi_{\alpha i}\epsilon^\alpha(\kappa)$. Then,

$$\delta_\kappa \Phi = 2\epsilon^{ij}(\Gamma_i)_{\alpha \beta}\epsilon^\beta(\kappa)\Pi_j^\alpha$$

Using eqs. (56) and (59) we obtain

$$\delta_\kappa \left( \det M + \frac{1}{4}\Phi^2 \right) = -2\epsilon^{ij}e^{kl}M_{ik}(\Gamma_j)_{\alpha \beta}\Pi_i^\alpha\epsilon^\beta(\kappa) + \Phi\epsilon^{ij}(\Gamma_i)_{\alpha \beta}\epsilon^\beta(\kappa)\Pi_j^\alpha$$

and we see that if $^8$

$$\epsilon^\beta(\kappa) = \frac{1}{e}\left( \Phi\delta^\beta_\gamma - \epsilon^{ij}(\Gamma_{ij})_{\gamma}^\beta \right)\kappa^\gamma,$$

then

$$\delta_\kappa \left( \det M + \frac{1}{4}\Phi^2 \right) = -\frac{2}{e}\epsilon^{ij}e^{kl}M_{ik}(\Gamma_j)_{\alpha \beta}\Pi_i^\alpha\Phi^\beta$$

$$+ \frac{2}{e}\epsilon^{ij}e^{kl}M_{ik}(\Gamma_j)_{\alpha \beta}\Pi_i^\alpha\epsilon^{rs}(\Gamma_{rs})_\gamma^\beta\kappa^\gamma$$

$$+ \frac{\Phi^2}{e}\epsilon^{ij}(\Gamma_i)_{\alpha \beta}\epsilon^\beta(\kappa)\Pi_j^\alpha - \frac{1}{e}\Phi\epsilon^{ij}(\Gamma_j)_{\alpha \beta}\epsilon^{kl}(\Gamma_{kl})_\beta^\gamma\kappa^\gamma\Pi_j^\alpha$$

$$= -\frac{4}{e}\epsilon^{ij}(\Gamma_i)_{\alpha \beta}\epsilon^\beta(\kappa)\Pi_j^\alpha$$

where the terms linear in $\Phi$ cancel each other. Therefore, the variation of the Lagrangian density is

$$\delta_\kappa \left\{ -\frac{1}{2e} \left( \det M + \frac{1}{4}\Phi^2 \right) \right\} =$$

$$\frac{1}{2e^2}(\delta_\kappa e) \left( \det M + \frac{1}{4}\Phi^2 \right) + \frac{2}{e^2}\epsilon^{ij}(\Gamma_i)_{\alpha \beta}\Pi_j^\alpha\kappa^\beta \left( \det M + \frac{1}{4}\Phi^2 \right),$$

which is equal to zero if

$$\delta_\kappa e = -4\epsilon^{ij}(\Gamma_i)_{\alpha \beta}\Pi_j^\alpha\kappa^\beta.$$

Let us return to the question of solving

$$\Pi_{\mu i}b^\mu(\kappa) = \frac{1}{2}\Pi_{\alpha i}\epsilon^\alpha(\kappa) = \frac{\Pi_{\alpha i}}{2e}\left( \Phi\kappa^\alpha - \epsilon^{j k}(\Gamma_{jk})_{\alpha \beta}\kappa^\beta \right).$$

$^8$For $\Phi \neq 0$, eq. (61) gives $\epsilon^\beta(\kappa) = \frac{1}{e}(\delta^\beta_\gamma - \epsilon^{ij}(\Gamma_{ij})_{\gamma}^\beta)\kappa^\gamma$ which exhibits its projector structure: the second term squares to unity on account of $(\epsilon^{ij}(\Gamma_{ij}))^2 = -(2!)^2\det M$ and eq. (48). Since $(\Gamma_{ij})_{\gamma}^\beta$ is traceless, half of the parameters are removed.
One might take
\[ b^\mu(\kappa) = \frac{1}{2} \Pi^\mu_i M^{ij} \Pi_{\alpha j} \epsilon^\alpha(\kappa) \]  
(66)
because then
\[ \Pi_{\mu s} b^\mu(\kappa) = \frac{1}{2} M_{si} M^{ij} \Pi_{\alpha j} \epsilon^\alpha(\kappa) = \frac{1}{2} \delta_j^s \Pi_{\alpha j} \epsilon^\alpha(\kappa) = \frac{1}{2} \Pi_{\alpha s} \epsilon^\alpha(\kappa) , \] 
(67)
which holds provided \( M^{ij} \) exists. This is not always the case because some solutions of the E-L equations (those corresponding to the tensionless string case) have degenerate induced metric. But these solutions also imply \( \Phi = 0 \) by virtue of the E-L equation for \( e \) (eq. (48)) so in that case we can write (65) with \( \Phi = 0 \). The solution of the resulting equation is
\[ b^\mu(\kappa) = \frac{1}{e} (\Gamma^\mu)^{\alpha}_\beta \Pi_{\alpha \kappa} \epsilon^{ij} \] .  
(68)
This completes the proof of the \( \kappa \)-symmetry for the form (44) of the action. We notice that the projector nature of the bracket in (61) (as it was the case for (10) and will be for (95) below) requires using the E-L eqs. for \( e \), eq. (28), although the \( \kappa \)-symmetry of the action itself does not. The full set of \( \kappa \)-symmetry transformations are expressed by eqs. (52)-(55) and (64) with the fermionic \( \epsilon(\kappa) \) and the bosonic \( b^\mu(\kappa) \) given by (61) and (66) or (68) respectively, and again we see that they correspond to right local transformations depending on a fermionic parameter \( \kappa^9 \). Furthermore, we see that the \( \kappa \)-symmetry of (44) is strict or manifest: the Lagrangian, and not only the action, is \( \kappa \)-invariant.

4 The \( N = 1 p = 2 D = 11 \) supermembrane case

4.1 The supermembrane with invariant WZ term

As we did in the last section, we consider here the rigid supermembrane action with an invariant WZ term written as a product of MC one-forms on an enlarged superspace. The particular superspace needed to do this is the group manifold (parametrized by \( \theta^\alpha, x^\mu, \varphi^{\mu \nu}, \varphi^{\mu \alpha}, \varphi_{\alpha \beta} \)) associated with the extended supersymmetry algebra defined by the MC equations
\[
\begin{align*}
d\Pi^\alpha &= 0, & d\Pi^\mu &= \frac{1}{2} (CT^\mu)_{\alpha \beta} \Pi^\alpha \Pi^\beta, \\
d\Pi^{\mu \nu} &= \frac{1}{2} (CT^{\mu \nu})_{\alpha \beta} \Pi^\alpha \Pi^\beta, \\
d\Pi_{\mu \alpha} &= (CT_{\nu \mu})_{\alpha \beta} \Pi^{\nu \beta} + (CT^{\nu}_{\beta \mu})_{\alpha \beta} \Pi^{\nu \beta}, \\
d\Pi_{\alpha \beta} &= -\frac{1}{2} (CT_{\mu \nu})_{\alpha \beta} \Pi^{\mu \nu} - \frac{1}{2} (CT^{\mu}_{\alpha \beta})_{\mu \nu} \Pi^{\mu \nu} + \frac{1}{4} (CT^{\mu \nu})_{\alpha \beta} \Pi^{\mu \nu} + (CT^{\mu \nu})_{\alpha \beta} \Pi^{\nu \delta} + (CT^{\mu \nu})_{\beta \delta} \Pi^{\mu \delta}. 
\end{align*}
\] 
(69)

9The presence of additional bosonic gauge transformations beyond reparametrization invariance is a feature of superspaces with additional bosonic coordinates; cf. [2] and [14] for the case of the \( \Sigma^{(n-1)/2n} \) superspaces, and references therein. The additional bosonic gauge 'b'-symmetries there should not be confused, however, with the \( \kappa \)-symmetry bosonic transformations discussed in the present paper, since the former depend on bosonic parameters that are \( \kappa \)-independent; they are, rather, bosonic superpartners of \( \kappa \)-symmetry.
Using these equations, it is straightforward to see that the CE three-cocycle on the standard supersymmetry algebra that characterizes the membrane, \((CT_{\mu\nu})_{\alpha\beta}\Pi^\mu\Pi^\nu\Pi^\alpha\Pi^\beta\), can be written as \(dB\), where \(B\) is now the three-form [20]

\[
B = \frac{2}{3} \Pi^\mu\Pi^\nu\Pi_{\mu\nu} + \frac{3}{5} \Pi^\mu\Pi^\alpha\Pi_{\mu\alpha} - \frac{2}{15} \Pi_{\alpha\beta}\Pi^\alpha\Pi^\beta.
\]

Since the three-form \(B\) is constructed out of MC forms of a (enlarged) supersymmetry algebra [20] it provides an invariant WZ term.

Let us now show that the \(\kappa\)-symmetry gauge transformations are also here a special type of right local transformations with parameters \(\epsilon^\alpha(\kappa)\). From the change (see [7]) of the group variables \(\theta^\alpha, x^\mu, \varphi^\mu\nu, \varphi_{\mu\alpha}\) and \(\varphi_{\alpha\beta}\) under local, right transformations \(\delta_\kappa\), we obtain

\[
\delta_\kappa \theta^\alpha = \epsilon^\alpha(\kappa),
\]

\[
\delta_\kappa x^\mu = -\frac{1}{2}(CT^\mu)_{\alpha\beta}\theta^\alpha e^\beta(\kappa),
\]

\[
\delta_\kappa \varphi_{\mu\nu} = -\frac{1}{2}(CT_{\mu\nu})_{\alpha\beta}\theta^\alpha e^\beta(\kappa),
\]

\[
\delta_\kappa \varphi_{\mu\alpha} = -\frac{1}{2}(CT^\nu)_{\alpha\beta}\varphi_{\mu\nu} e^\beta(\kappa) - \frac{1}{2}(CT_{\mu\alpha})_{\alpha\beta} x^\nu e^\beta(\kappa) + \frac{1}{12} (CT^\nu)_{\alpha\beta} (CT_{\mu\nu})_{\gamma\delta} \theta^\gamma \theta^\delta e^\epsilon(\kappa) + \frac{1}{12} (CT^\nu)_{\gamma\delta} (CT_{\mu\nu})_{\alpha\beta} \theta^\gamma \theta^\delta e^\epsilon(\kappa),
\]

\[
\delta_\kappa \varphi_{\alpha\beta} = -\frac{1}{2}(CT^\mu)_{\beta\gamma}\varphi_{\mu\alpha} e^\gamma(\kappa) - \frac{1}{2}(CT^\mu)_{\alpha\gamma}\varphi_{\mu\beta} e^\gamma(\kappa) - \frac{1}{48} (CT^\mu)_{\alpha\beta}(CT^\nu)_{\gamma\delta}\varphi_{\mu\nu} \theta^\gamma \theta^\delta e^\epsilon(\kappa) + \frac{1}{48} (CT^\mu)_{\alpha\beta}(CT_{\mu\nu})_{\gamma\delta} x^\nu \theta^\gamma \theta^\delta e^\epsilon(\kappa)
\]

\[
- \frac{1}{6} (CT^\mu)_{\alpha\gamma}(CT^\nu)_{\beta\delta}\varphi_{\mu\nu} \theta^\gamma \theta^\delta e^\epsilon(\kappa) + \frac{1}{6} (CT^\mu)_{\alpha\gamma}(CT_{\mu\nu})_{\beta\delta} x^\nu \theta^\gamma \theta^\delta e^\epsilon(\kappa)
\]

\[
+ \frac{1}{12} (CT^\mu)_{\gamma\delta}(CT_{\mu\nu})_{\alpha\beta} x^\nu \theta^\gamma \theta^\delta e^\epsilon(\kappa).
\]

From eq. (71) or from the usual transformation properties of the left-invariant MC one-forms under right transformations (eqs. (69), (71)), it follows that

\[
\delta_\kappa \Pi^\alpha = de^\alpha(\kappa), \quad \delta_\kappa \Pi^\mu = -(CT^\mu)_{\alpha\beta}\Pi^\alpha e^\beta(\kappa),
\]

\[
\delta_\kappa \Pi^{\mu\nu} = -(CT^{\mu\nu})_{\alpha\beta}\Pi^\alpha e^\beta(\kappa),
\]

\[
\delta_\kappa \Pi_{\mu\alpha} = -(CT_{\nu\mu})_{\alpha\beta}\Pi^{\nu\beta}(\kappa) - (CT^\nu)_{\alpha\beta}\Pi_{\nu\mu} e^\beta(\kappa),
\]

\[
\delta_\kappa \Pi_{\mu\beta} = -\frac{1}{4} (CT^\mu)_{\alpha\beta}\Pi_{\mu\delta} e^\delta(\kappa) - (CT^\mu)_{\delta\alpha}\Pi_{\mu\beta} e^\delta(\kappa)
\]

\[
- (CT^\mu)_{\delta\beta}\Pi_{\mu\alpha} e^\delta(\kappa),
\]

for an appropriate \(e^\alpha(\kappa)\) to be determined from the invariance of the action.

Let us start by computing \(\delta_\kappa B\). To do that, one may use (70) and (72) or notice that the right transformations are generated by the left-invariant vector fields (dual to the MC forms in (69)). Thus, \(\delta_\kappa\) is nothing but the Lie derivative \(L_X\) with respect to a vector field \(X = X(\kappa)\) such that \(i_X \Pi = 0\) for all \(\Pi\)’s in eq. (69) but for \(i_X \Pi^\alpha = \Pi^\alpha(X) = e^\alpha(\kappa)\), so that
only \( \epsilon^\alpha(\kappa) \) enters in the components of \( X \). Then,

\[
\begin{align*}
\delta_\kappa B &= L_X B = i_X dB + di_X B \\
&= 2(C\Gamma_{\mu\nu})_{\alpha\beta} \Pi^\mu \Pi^\nu \epsilon^\alpha(\kappa) \Pi^\beta \\
&\quad + d \left( \frac{2}{5} \Pi^\rho \epsilon^\alpha(\kappa) \Pi_{\mu\rho} + \frac{4}{15} \Pi_{\alpha\beta} \epsilon^\alpha(\kappa) \Pi^\beta \right).
\end{align*}
\]

Let the strictly invariant supermembrane action be given by (cf. eq. (18) for \( p=1 \))

\[
I = -\int d^3\xi \left\{ T\sqrt{-\det M} + \lambda \frac{1}{3!} \epsilon^{ijk} B_{ijk} \right\},
\]

where \( M_{ij} = \Pi^\mu_\alpha \Pi^\mu_{\beta j}, i, j = 0, 1, 2, B_{ijk} \) are the coordinates of \( \phi^*(B) = B(\xi) = \frac{1}{3!} B_{ijk} d\xi^i d\xi^j d\xi^k \), and again \( \lambda \) is a constant to be fixed. Using the second equation in (72) and eq. (73) (ignoring the total derivative term) we obtain

\[
\delta_\kappa I = -\int d^3\xi \left\{ -T\sqrt{-\det M} (C\Gamma^l)^{\alpha\beta} \Pi^\alpha_l \left[ \epsilon^\beta(\kappa) \\
+ \frac{2\lambda}{3T\sqrt{-\det M}} \epsilon^{ijk} (\Gamma_{ijk})^\beta \epsilon^\gamma(\kappa) \right] \right\}.
\]

The expression between square brackets in (75) can be written as \( P^\beta_\gamma \epsilon^\gamma(\kappa) \), where \( \frac{1}{2} P^\beta_\gamma \) is a projector that projects into half the spinor space if the square of the (traceless) matrix \( 2\lambda \frac{1}{3T\sqrt{-\det M}} \epsilon^{ijk} (\Gamma_{ijk})^\beta \) is the unit matrix. This is the case when \( \lambda = \pm \frac{T}{4} \). Then one constructs \( \epsilon(\kappa) \) as for \( p = 1 \), and gets

\[
\epsilon^\alpha(\kappa) = \frac{1}{2} \left( \delta^\alpha_\beta + \frac{1}{6\sqrt{-\det M}} \epsilon^{ijk} (\Gamma_{ijk})^\alpha \right) \kappa^\beta.
\]

We remark that we differ from [20] in that the new coordinates of the appropriate enlarged superspace group ([7], eqs. (63)-(67)) are not inert under \( \kappa \)-symmetry, as seen in eqs. (74) or (72). We also note that we could have cancelled the second term in (73) to obtain strict \( \kappa \)-invariance by adding to the transformations in (72) a term in \( b^{\mu\nu}(\kappa) \). We will do this in Sec. 4.3, where this bosonic contribution to \( \kappa \)-symmetry will turn out to be necessary.

In order to obtain below the new action from that in eq. (74), we give its ‘first order-like’ formulation (cf. eq. (26) for \( p = 1 \)) by introducing \( p_{\mu\nu\rho} \) and \( e \):

\[
I = -\int d^3\xi \left\{ \frac{1}{3!} \epsilon^{ijk} p_{\mu\nu\rho} \Pi^\mu_i \Pi^\nu_j \Pi^\rho_k + e(p_{\mu\nu\rho} p_{\mu\nu\rho} + 6T^2) \pm \frac{T}{4} \Phi \right\},
\]

where (cf. eq. 321)

\[
\Phi = (1/3!) \epsilon^{ijk} B_{ijk}.
\]

This action is classically equivalent to (74) with \( \lambda = \pm \frac{T}{4} \), as can be checked by eliminating \( p_{\mu\nu\rho} \) and \( e \) via their algebraic E-L equations.
4.2 A new $p = 2$ action in enlarged superspace

Equation (78) suggests introducing a new action by the following expression

$$I_\ast = -\int d^3 \xi \left\{ \frac{1}{3!} \epsilon^{ijk} p_{\mu\nu\rho} \Pi^\mu_i \Pi^\nu_j \Pi^\rho_k + e(p^{\mu\nu\rho} p_{\mu\nu\rho} + 96p_\ast^2) + p_\ast \Phi \right\}. \quad (80)$$

The E-L equations are computed as in Sec 3.2, eq. (35) and below. To show that $dp_\ast = 0$ so that $p_\ast$ is constant it is sufficient to use the E-L equations corresponding to $\varphi^{\mu\nu}$. Specifically, we only need in the variation $\delta I_\ast$ the coefficient accompanying $i_X \Pi^{\mu\nu}$. Since $\Pi^{\mu\nu}$ appears only in $B$ (or $\Phi$, eqs. (70), (79)), we only require

$$L_X (p_\ast B) = i_X d(p_\ast B) + d(p_\ast i_X B) \simeq i_X d(p_\ast B), \quad (81)$$

where the total derivative is ignored under the integral sign. From the explicit form of $B$, we find that the $i_X \Pi^{\mu\nu}$ relevant term is

$$-\frac{2}{3} dp_\ast \Pi^{\mu}\Pi^{\nu}(i_X \Pi^{\mu\nu}) \quad (82)$$

so that, moving from differential forms to worldvolume fields, the $\varphi^{\mu\nu}$-associated E-L equation reads

$$\epsilon^{ijk} \partial_i p_\ast \Pi^{\mu}_j \Pi^{\nu}_k = 0. \quad (83)$$

If we now contract this expression with $\Pi_{\mu r} \Pi_{\nu s}$, we obtain

$$\epsilon^{ijk} \partial_i p_\ast M_{jr} M_{ks} = 0. \quad (84)$$

Then, as in the case of the string, if $det M = 0$ the $\delta e$ and $\delta p^{\mu\nu\rho}$ equations

$$p_{\mu\nu\rho} p^{\mu\nu\rho} + 96p_\ast^2 = 0, \quad p_{\mu\nu\rho} = -\frac{1}{12e} \epsilon^{ijk} \Pi_{\mu i} \Pi_{\nu j} \Pi_{\rho k}, \quad (85)$$

together with the fact that

$$det M = \frac{1}{3!} \epsilon^{ijk} \epsilon^{rst} M_{ir} M_{js} M_{kt}, \quad (86)$$

tell us that $p_\ast = 0$; if $det M \neq 0$ we may contract (81) with $M^{rr'} M^{ss'}$, to arrive at $dp_\ast = 0$. Thus, $p_\ast$ is a constant in any case. Setting $p_\ast = \pm \frac{T}{4}$ and introducing it into the action (80), one recovers the standard one, (78).

We now write the action that results from using the algebraic equations for the $\delta p_\ast$ and $\delta p^{\mu\nu\rho}$ variations in eq. (80). It is given by

$$I_\ast = \int d^3 \xi \frac{1}{24e} \left\{ det M + \frac{\Phi^2}{16} \right\}. \quad (87)$$

to be compared with that for the superstring, eq. (44). The E-L eq. for $e$ is

$$det M + \Phi^2/16 = 0. \quad (88)$$
4.3 \(\kappa\)-symmetry of the new action

The \(\kappa\)-symmetry for the new form of the action goes along the lines described in Sec. 3.3(a). We are more interested here, however, in the proof for the action (57) (the case of Sec. 3.3(b) above), since this allows us to give more easily the explicit variations of the enlarged superspace variables and to check that these variations are again right local transformations depending on \(\epsilon(\kappa)\). The novelty here is that, for \(p = 2\), they also depend on \(b^{\mu\nu}(\kappa)\), \(b^{\mu\nu}\) being the infinitesimal parameter associated with the new variable \(\varphi^{\mu\nu}\). So we now add to (72) the transformations that include the new \(b^{\mu\nu}\) parameter; the \(\kappa\)-symmetry vector field now has non-zero components \(\delta_\kappa \theta^\alpha = e^\alpha(\kappa)\), \(\delta_\kappa \varphi^{\mu\nu} = b^{\mu\nu}(\kappa)\). This gives

\[
\begin{align*}
\delta_\kappa \Pi^\alpha & = de^\alpha(\kappa) , \\
\delta_\kappa \Pi^\mu & = -(\text{CT}^\mu)_{\alpha\beta} \Pi^\alpha e^\beta(\kappa) , \\
\delta_\kappa \Pi^{\mu\nu} & = db^{\mu\nu}(\kappa) - (\text{CT}^{\mu\nu})_{\alpha\beta} \Pi^\alpha e^\beta(\kappa) , \\
\delta_\kappa \Pi_{\mu\alpha} & = (\text{CT}^{\alpha})_{\alpha\beta} b_{\mu\beta}(\kappa) \Pi^\beta - (\text{CT}_{\mu\alpha})_{\alpha\beta} \Pi^\beta + (\text{CT}^{\alpha})_{\alpha\beta} \Pi^\beta(\kappa) , \\
\delta_\kappa \Pi_{\mu\beta} & = -\frac{1}{2}(\text{CT}^{\mu})_{\alpha\beta} b_{\mu\beta}(\kappa) \Pi^\beta - \frac{1}{4}(\text{CT}^{\mu})_{\alpha\beta} \Pi^\beta e^\beta(\kappa) , \\
& \quad - (\text{CT}^{\mu})_{\delta\alpha} \Pi_{\mu\beta} e^\beta(\kappa) - (\text{CT}^{\mu})_{\delta\beta} \Pi_{\mu\alpha} e^\beta(\kappa) .
\end{align*}
\]

(89)

Note that the variations of \(\Pi^\alpha\) and \(\Pi^\mu\) are exactly the same as in the standard, unextended algebra case. This implies, in particular, that the variation of \(\text{det} M\) in eq. (56) is given by (cf. (56))

\[
\delta_\kappa \text{det} M = -\epsilon^{ijk} \epsilon^{rst} M_{ir} M_{js}(\text{CT}_k)_{\alpha\beta} \Pi^\alpha_i \Pi^\beta_j \epsilon^\beta(\kappa) ,
\]

(90)
since \(\delta \text{det} M = \frac{1}{2} \epsilon^{ijk} \epsilon^{rst} M_{ir} M_{js} \delta M_{kt}\). The variation of \(\Phi\) is now given by (cf. (63))

\[
\delta_\kappa \Phi = \epsilon^{ijk} \partial_i \left( \frac{3}{2} \Pi^\mu_j b_{\mu\nu}(\kappa) - \frac{3}{5} \Pi^\mu_j \epsilon^\alpha(\kappa) \Pi_{\mu\alpha\kappa} + \frac{4}{15} \Pi_{\alpha\beta} \epsilon^\alpha(\kappa) \Pi^\beta_k \right) + 2 \epsilon^{ijk} (\text{CT}_{ij})_{\alpha\beta} \epsilon^\alpha(\kappa) \Pi^\beta_k .
\]

(91)

However, in contrast with the action in Sec. 4.1, the total derivative in (91) cannot be ignored because it does not produce a total derivative in the action (77), since it depends on \(\Phi^2\) and \(\epsilon\). This is precisely why a non-vanishing \(b^{\mu\nu}(\kappa)\) is now needed: its role is to cancel the first term in (91). At the same time, we see that the use of enlarged superspaces not only produces strictly invariant WZ terms; it also leads to strict \(\kappa\)-invariance.

If we assume that we have found a \(b^{\mu\nu}(\kappa)\) such that the variation of \(\Phi\) in eq. (91) reduces to its second term, one checks that writing

\[
\epsilon^\alpha(\kappa) = \frac{1}{e} \left( \Phi \delta_\kappa^\alpha + \frac{2}{3} \epsilon^{ijk} (\Gamma_{ijk})_{\alpha}^\alpha \beta \right) \kappa^\beta
\]

(92)

the variation of the terms between brackets in (77) is given by

\[
\delta_\kappa \left( \text{det} M + \frac{\Phi^2}{16} \right) = \frac{4}{e} \epsilon^{ijk} (\text{CT}_{ij})_{\alpha\beta} \kappa^\alpha \Pi^\beta_k \left( \text{det} M + \frac{\Phi^2}{16} \right) ,
\]

(93)

so that choosing suitably the variation of the auxiliary field \(\epsilon\),

\[
\delta_\kappa \epsilon = 4 \epsilon^{ijk} (\text{CT}_{ij})_{\alpha\beta} \kappa^\alpha \Pi^\beta_k ,
\]

(94)
the action remains invariant. Again, for $\Phi \neq 0$ eq. (92) has a projector structure,

$$
\epsilon^\alpha(\kappa) = \frac{\Phi}{e} \left( \delta^\alpha_\beta + \frac{2}{3H} \epsilon^{ijk}(\Gamma_{ijk})^\alpha_\beta \right) \kappa^\beta
$$

which follows from $(\epsilon^{ijk}\Gamma_{ijk})^2 = -(3!)^2$ det $M$ and eq. (88).

To complete the proof of $\kappa$-invariance, we now have to show that the equation

$$
\epsilon^{ijk} \left( \frac{3}{2} \Pi^\mu_j \Pi^\nu_k b_{\mu\nu}(\kappa) - \frac{3}{5} \Pi^\mu_j \epsilon^\alpha(\kappa) \Pi_{\alpha\rho\kappa} + \frac{4}{15} \Pi_{\alpha \beta j} \epsilon^\alpha(\kappa) \Pi_{\kappa}^\beta \right) = 0
$$

(96)

does indeed have a solution for $b_{\mu\nu}$ when $\epsilon^\alpha(\kappa)$ is given by eq. (92). As in Sec. 3.3, we shall consider separately the det $M \neq 0$ and det $M = 0$ cases. When $M_{ij}$ is non-degenerate eq. (96) has the solution

$$
b_{\mu\nu}(\kappa) = \frac{1}{3\det M} \Pi^\rho_i \Pi^\tau_k b_{\mu\nu}(\kappa) - \frac{3}{5} \Pi^\mu_j \epsilon^\alpha(\kappa) \Pi_{\alpha\rho\kappa} + \frac{4}{15} \Pi_{\alpha \beta j} \epsilon^\alpha(\kappa) \Pi_{\kappa}^\beta
$$

(97)

We also have to worry about the $\kappa$-variation of configurations with $\det M = 0$, because these appear in the space of solutions of the E-L equations of the action (87), when eq. (88) gives $\Phi = 0$. In this case we have to solve eq. (96) for an $\epsilon^\alpha(\kappa)$ given by eq. (92) for $\Phi = 0$. It is then seen that

$$
b_{\mu\nu}(\kappa) = \frac{2}{3e} \epsilon^{ijk} \left( \frac{6}{5} \Pi^\rho_j \Pi_{\rho\kappa} + \frac{8}{15} \Pi_{\alpha \gamma j} \Pi^\gamma_k \right) (\Gamma_{\mu\nu})^\alpha_\beta \kappa^\beta
$$

(98)

where $\Gamma_{\mu\nu\kappa} = \Gamma_{\mu\nu\rho} \Pi^\rho_k$, cancels the first term in eq. (91), and the rest of the proof follows as for $\Phi \neq 0$. So the action (87) is $\kappa$-invariant, the variation of the Lagrangian components being given by (89), (92), (94) and (97) or (98).

5 Conclusions

We have obtained new rigid superspace actions explicitly for $p = 1, 2$ super-$p$-branes, starting from the usual superstring and supermembrane ones in which their WZ terms have been rewritten in a strictly invariant form by using MC forms defined on suitably extended superspace groups [20, 7]. The procedure generalizes the case of the massive superparticle as obtained from the massless superparticle in one higher dimension, by viewing the mass as an integration constant of this last one E-L equations. Specifically,

1) We present a ‘tension generating mechanism’ in a kind of first order formulation that extends the true first order formulation of the $p = 0$ case (cf. [22, 23]).

2) We do not include in the super-$p$-brane action any higher form fields directly defined on the worldvolume, only the auxiliary scalar ones. We achieve this by suitably enlarging ordinary superspace i.e., by adhering to the enlarged superspace variables/worldvolume fields correspondence for branes [7] as it is also done for the D-branes in [7, 30].

3) The use of the above enlarged superspaces allows us to characterize $\kappa$-symmetry transformations as local, $\kappa$-dependent right translations associated with the corresponding extended superspace group coordinates, as it is the case for the standard superspaces. $\kappa$-symmetry gauge transformations also include in our case $\kappa$-dependent bosonic transformations.
4) The new super-$p$-brane actions on the enlarged superspaces possess strict $\kappa$-invariance: their Lagrangian is itself $\kappa$-invariant. This requires that the additional field variables in $\tilde{\Sigma}$ be non-inert under $\kappa$-transformations. This strict $\kappa$-invariance is analogous to the strict supersymmetry invariance \[20\] of the standard Lagrangians (WZ terms) constructed on the enlarged superspaces. Strict $\kappa$-invariance is, however, optional for these super-$p$-brane actions (which explains why the new variables in $\tilde{\Sigma}$ may be inert under $\kappa$-symmetry, as in \[20\]) and is a necessary result for the new actions and Lagrangians discussed here.

Finally, we mention that our results may be generalized to $p > 2$. The actions \[33\] and \[80\] can be extended to higher $p$ (a) using the appropriate higher $p$ enlarged superspaces for the contribution of $\Phi$ from a manifestly invariant WZ form and (b) keeping the same structure of \[33\] and \[80\] but now with auxiliary variables $p^{\mu_1...\mu_{p+1}}, E = \frac{1}{(p+1)!} \epsilon_{i_1...i_{p+1}} d\xi^{i_1} ... d\xi^{i_{p+1}}$.

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