Theory of SU(N) Fermi liquid

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We generalized the Fermi liquid theory to \( N \) component systems with SU(N) symmetry. We emphasize the important role of fluctuations when \( N \) is large. These fluctuations dramatically modifies the properties for repulsive Fermi gases, in particular the spin susceptibility.

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I. INTRODUCTION

Recent advances in the field of cold atoms provide us with many new opportunities to study quantum multiparticle physics \([1,2]\), in systems or with methods often not available in traditional condensed matter settings. In this paper, we consider one such example, namely fermionic systems with SU(N) symmetry, with \( N > 2 \). This is naturally available in cold atomic gases of alkaline atoms \([3,4]\) without much fine tuning \([10–12]\). With two electrons in the outermost shell and in the total spin and orbital angular momentum zero state, the hyperfine splitting of the atom is due entirely to the nuclear spin, \( \hbar f \), with many new opportunities to study quantum multi-component systems. In particular both the Fermi gases \([5]\) and trapped \([6–8]\). With a lot of experimental progress has been made on these systems. In particular both the Fermi gases \([13,17]\) with \( f = 5/2 \) and \(^{87}\)Sr with \( f = 9/2 \) have been cooled to quantum degeneracy \([3,4]\). The Mott state of such a multi-component system has already been reached \([9]\). Many theorists have predicted that this Mott state should exhibit exotic ordering at low temperatures \([11–17]\). Also available are degenerate Bose-Fermi mixtures \([4,7]\) and Fermi-Fermi mixtures \([3]\). This latter system has SU(2) \( \times \) SU(6) symmetry and can in principle hosts a very interesting superfluid \([18]\), though this regime is yet to be achieved.

In this paper, we study one aspect of the SU(N) fermionic systems which should be readily accessible experimentally, namely their Fermi liquid properties. In its original form \([14,21]\), the Landau Fermi liquid theory was intended for a two-component system with SU(2) symmetry. We generalize the theory to SU(N). We evaluate the Fermi liquid parameters, in particular in the dilute limit. We obtain the compressibility and the generalized magnetic susceptibilities of this gas. These quantities can be readily measured by monitoring the density profile and relative number fluctuations. We illuminate in particular how these quantities depend on the number of components. We show that an increase in this number leads to a dramatic suppression of the spin susceptibility of repulsive gases. This is a consequence of an enhanced effective repulsion between identical particles, generated by induced density fluctuations of the other species.

II. FORMULATION

The Hamiltonian of our system is given by

\[
H = \sum_{\vec{k}\gamma} \left( \frac{k^2}{2M} - \mu \right) a_{\vec{k}\gamma}^\dagger a_{\vec{k}\gamma} + \frac{g}{2} \sum_{\vec{k},\vec{k}'\gamma_1,\gamma_2} a_{\vec{k}\gamma_1}^\dagger a_{\vec{k}\gamma_2}^\dagger a_{\vec{k}'\gamma_2} a_{\vec{k}'\gamma_1} \delta_{\vec{k}+\vec{k}',\gamma_1,\gamma_2} \delta_{\vec{k}-\vec{k}',\gamma_1,\gamma_2} \delta_{\gamma_1,\gamma_2}
\]

where the first term is the kinetic energy and the second, the interaction. \( a_{\vec{k}\gamma}^\dagger \) and \( a_{\vec{k}\gamma} \) are the creation and annihilation operators for a Fermion of the \( \gamma \) species with momentum \( \vec{k} \), \( M \) is the atomic mass, and \( \mu \) is the chemical potential, assumed identical for all the species at the moment. \( g \) characterizes the short range two-body interaction, which will be eliminated in favor of the s-wave scattering length between the particles. In the interaction term, it is sufficient to include \( \gamma_1 \neq \gamma_2 \) due to Fermionic antisymmetry. \( g \) is independent of the species involved, so we do not have spin conversion processes in the sense that the outgoing particles must have the same spin indices \( \gamma_1, \gamma_2 \) as the incoming particles. Eq (1) can be obtained from the more general Hamiltonian in \([22]\) if we set all the scattering lengths there to be equal. Hereafter, to avoid possible confusion with the particle number, the number of species will be denoted as \( N_c \). Eq (1) has SU(\( N_c \)) symmetry since it is invariant under any unitary transformation \( (a_\alpha \rightarrow \sum_\alpha' U_{\alpha\alpha'} a_\alpha' ) \) with \( U_{\alpha\alpha'} \) unitary) among the \( N_c \) species involved.

In Fermi liquid theory, the low energy Fermionic excitations are described by quasiparticle excitations \( \delta n_{\delta\gamma}(\vec{k}) \) at \( \vec{k} \). \( \delta n_{\delta\gamma}(\vec{k}) \) is an \( N_c \times N_c \) matrix in the species indices. The change in quasiparticle energy at \( \vec{k} \) is \( \delta \epsilon_{\alpha\beta}(\vec{k}) \), also an \( N_c \times N_c \) matrix, related to \( \delta n_{\delta\gamma} \) by
\[ \delta \epsilon_{\alpha \beta}(\vec{k}) = \sum_{\vec{k}', \gamma \delta} f_{\alpha \gamma, \beta \delta}(\vec{k}, \vec{k}') \delta n_{\gamma}(\vec{k}') \]  

(2)

where it is understood that all \( \vec{k}, \vec{k}' \) are near the Fermi surface. Our notations are generalization of those in [23] from two to \( N_c \) species. Slightly different notations were used in [10]. Ref [10] parameterized \( f_{\alpha \gamma, \beta \delta} \) as

\[ f_{\alpha \gamma, \beta \delta}(\vec{k}, \vec{k}') = f_s(\vec{k}, \vec{k}') \delta_{\alpha \beta} \delta_{\gamma \delta} + 4 f_m(\vec{k}, \vec{k}') \sum_a T_{a \alpha}^a T_{a \beta}^a \]  

(3)

where the matrices \( T^a \), \( a = 1, \ldots, N^2_c - 1 \) are the (traceless) generators of \( SU(N_c) \) with the normalization \( \text{Tr}[T^a T^b] = \frac{1}{2} \delta^{ab} \), and \( \text{Tr} \) denotes the trace. Here, we have defined \( f_m \) which differs from [10] by a factor of 2 so that eq (3) will reduce to the form of [23] if \( N_c = 2 \) where \( T^a \) will become \( \sigma^a/2 \) with \( \sigma^a \) being the Pauli matrices. Eq (3) implies that, if we write

\[ \delta n_{\gamma}(\vec{k}) = \frac{\delta n(\vec{k})}{N_c} \delta_{\gamma \gamma} + \frac{1}{2} \delta m_{\gamma \gamma} \]  

(4)

with

\[ \delta m_{\gamma \gamma} = \sum_a m^a T_{a \gamma}^a \]  

(5)

and

\[ \delta \epsilon_{\alpha \beta}(\vec{k}) = \delta \epsilon_s(\vec{k}) \delta_{\alpha \beta} - \delta h_{\alpha \beta} \]  

(6)

with

\[ \delta h_{\alpha \beta} = \sum_a \delta h^a T_{a \alpha}^a \]  

(7)

that is, we separate out, for both \( \delta m_{\alpha \beta} \) and \( \delta \epsilon_{\alpha \beta} \) the parts that are proportional to the identity matrix and those which are linear combinations involving \( T^a \)'s , we get

\[ \delta \epsilon_s(\vec{k}) = \sum_{\vec{k}'} f_s(\vec{k}, \vec{k}') \delta n(\vec{k}') \]  

(8)

and

\[ \delta h^a(\vec{k}) = - \sum_{\vec{k}'} f_m(\vec{k}, \vec{k}') m^a(\vec{k}') \]  

(9)

The above form is suggested by the fact that \( \delta n(\vec{k}) = \sum_\gamma \delta n_{\gamma}(\vec{k}) \) is the total density change at \( \vec{k} \), and \( \delta m^a(\vec{k}) \) is a generalized magnetization. Under a unitary transformation among the components, \( \delta n(\vec{k}) \) is invariant whereas \( \delta m^a(\vec{k}) \) would transform among each other. Similarly, \( \delta \epsilon_s(\vec{k}) \) is the part of \( \delta \epsilon_{\alpha \beta} \) that is invariant under \( SU(N_c) \), whereas \( \delta h^a(\vec{k}) \) is a kind of generalized magnetic field.

For our later purposes, we shall instead use

\[ f_{\alpha \gamma, \beta \delta}(\vec{k}, \vec{k}') = \left( f_s(\vec{k}, \vec{k}') - \frac{2}{N_c} f_m(\vec{k}, \vec{k}') \right) \delta_{\alpha \beta} \delta_{\gamma \delta} + 2 f_m(\vec{k}, \vec{k}') \delta_{\alpha \delta} \delta_{\beta \gamma} \]  

(10)

That this parametrization is possible is suggested by the fact that \( \delta_{\alpha \beta} \delta_{\gamma \delta} \) and \( \delta m_{\alpha \beta} \) are two linearly independent invariants under the \( SU(N_c) \) transformations, and eq (3) also just contains two such invariants. Indeed, it is simple to verify that, when eq (10) are substituted in eq (10), we obtain the same \( \delta \epsilon_s(\vec{k}) \) and \( \delta h^a(\vec{k}) \) as in eq (8) and (9). Note that we simply have

\[ \delta h_{\alpha \beta}(\vec{k}) = - \sum_{\vec{k}'} f_m(\vec{k}, \vec{k}') \delta m_{\alpha \beta}(\vec{k}') \]  

Eq (10) is particularly easy to understand if we consider the special case where all distribution functions are diagonal, that is, \( \delta n_{\gamma}(\vec{k}) = \delta n_{\gamma}(\vec{k}) \delta_{\gamma \gamma} (\gamma \text{ not summed}) \). Then \( \delta \epsilon_{\alpha \beta}(\vec{k}) \) is also diagonal and can be written as \( \delta \epsilon_{\alpha \alpha} \delta_{\beta \beta} \). By \( SU(N_c) \) symmetry, we expect

\[ \delta \epsilon_s(\vec{k}) = \sum_{\vec{k}'} f_{\alpha \alpha}(\vec{k}, \vec{k}') \delta n_{\alpha}(\vec{k}') + \sum_{\vec{k} \neq \vec{k}_0} f_{\alpha \beta \neq \alpha}(\vec{k}, \vec{k}') \delta n_{\beta}(\vec{k}') + \sum_{\vec{k} \neq \vec{k}_0} f_{\alpha \gamma}(\vec{k}, \vec{k}') \delta n_{\gamma}(\vec{k}') \]  

(11)

with \( f_{\alpha \alpha} \) independent of \( \alpha \) and \( f_{\alpha \beta \neq \alpha} \) independent of the species involved (so long as they are distinct). \( f_{\alpha \alpha} \) and \( f_{\alpha \beta \neq \alpha} \) play the role of effective interaction between identical and distinguishable species, respectively. Indeed, eq (10) and eq (11) are equivalent if

\[ f_{\alpha \alpha}(\vec{k}, \vec{k}') = f_s(\vec{k}, \vec{k}') + 2 \left( 1 - \frac{1}{N_c} \right) f_m(\vec{k}, \vec{k}') \]  

(12)

\[ f_{\alpha \beta \neq \alpha}(\vec{k}, \vec{k}') = f_s(\vec{k}, \vec{k}') - \frac{2}{N_c} f_m(\vec{k}, \vec{k}') \]  

(13)

We shall make use of these expressions below.

With the Landau liquid formulation, it is straightforward to evaluate the response of the system to external perturbations. Consider uniform, \( \vec{k} \) independent external potentials parameterized in the form \( \delta e_{ext}^s = \delta e_{ext}^s \delta \alpha \beta - \delta h_{\alpha \beta}^s \) with \( \delta h_{\alpha \beta}^s = \sum_a h^s_{ext} T_{a \alpha}^a \) in analogy with eq (6) and (7). One can easily show that (see Supplemental Material [24]) the density change \( \delta n = \sum_{\vec{k}} \delta n(\vec{k}) \equiv \left( \frac{dn}{d\epsilon_{ext}^s} \right) \delta e_{ext}^s \) and is thus linear in \( \delta e_{ext}^s \) and independent of \( \delta h_{ext}^s \), and similarly \( m^a = \chi h_{ext}^s \) independent of \( \delta e_{ext}^s \). Since a uniform increase in energy for all the species is equivalent to a lowering of the chemical potential \( \mu \), we shall write \( \frac{dn}{d\mu} = -\frac{dn}{d\epsilon_{ext}^s} \). We obtain

\[ \frac{dn}{d\mu} = \frac{N_c N(0)}{1 + N_c N(0) f_{0,s}} \]  

(14)

and

\[ \chi = \frac{2 N(0)}{1 + 2 N(0) f_{0,m}} \]  

(15)

Here \( f_{0,s} \) and \( f_{0,m} \) are the angular averages of \( f_s(\vec{k}, \vec{k}') \) and \( f_m(\vec{k}, \vec{k}') \) over the Fermi surface, and \( N(0) \) is the density of states at the Fermi level for a single species, given
by $\frac{M^*}{M}$ where $M^*$ is the effective mass of the quasiparticles, and $k_F$ is related to the equilibrium density of a single species by $n_a = k_F^3/6\pi^2$. Note that $\chi$ is independent of $a$, as expected from SU($N_c$) symmetry. Hence the matrix $\delta m_{a\beta}$ and $\delta h_{a\beta}$ are simply proportional to each other. $M^*$ can be obtained by considering Galilean invariance, as in the two-component case [22]. We obtain

$$M^*/M = 1 + N_c N(0)f_{1,s}/3$$

with $f_{1,s} \equiv 3((\vec{k} \cdot \vec{k}')f_{s}(\vec{k}, \vec{k}'))$ where $\vec{k}$ is the unit vector along $\vec{k}$ and the angular bracket denotes angular average. The speed of sound $u$ of the gas is related to $\frac{dn}{dp}$ by $u^2 = \frac{n}{M} \left(\frac{dn}{dp}\right)$, with $M$ the atomic (bare) mass.

$$f_{\alpha\alpha}(\vec{k}, \vec{k}') = -\left(\frac{4\pi a}{M}\right)^2 \left[ \frac{1}{V} \sum_{\vec{k}_1, \vec{k}_2, \gamma \neq \alpha} n_{\gamma}(\vec{k}_2) - n_{\gamma}(\vec{k}_1) \frac{K_2^2 - K_1^2}{2M} \delta_{K_2 + K', K_1 + K}\right]$$

and

$$f_{\alpha\beta \neq \alpha}(\vec{k}, \vec{k}') = \left(\frac{4\pi a}{M}\right)^2 \left[ \frac{1}{V} \sum_{\vec{k}_1, \vec{k}_2} n_{\alpha}(\vec{k}_2) - n_{\beta}(\vec{k}_1) \frac{K_2^2 - K_1^2}{2M} \delta_{K_2 + K', K_1 + K} + \frac{1}{V} \sum_{\vec{k}_1, \vec{k}_2} n_{\alpha}(\vec{k}_1) + n_{\beta}(\vec{k}_2) \frac{2K_2^2 - K_1^2}{2M} \delta_{K_1 + K_2, K'}\right]$$

Here $V$ is the volume, $a$ the s-wave scattering length, and $n_{\gamma}(\vec{k})$ is the equilibrium occupation number at $\vec{k}$ for $\gamma$ species (and thus equals unity if $k < k_F$ and zero otherwise). Compared with the spin 1/2 system, where we shall denote the interaction between identical species as $f_{+}^{(1/2)}(\vec{k}, \vec{k}')$ and distinguishable species as $f_{+}^{(1/2)}(\vec{k}, \vec{k}')$ as in [23], we see that

$$f_{\alpha\alpha}(\vec{k}, \vec{k}') = (N_c - 1) f_{+}^{(1/2)}(\vec{k}, \vec{k}')$$

and

$$f_{\alpha\beta \neq \alpha}(\vec{k}, \vec{k}') = f_{-}^{(1/2)}(\vec{k}, \vec{k}')$$

and thus they can be directly obtained from the results for the spin 1/2 system.

It is however more instructive to understand eq (18) and (20) in terms of Feynman diagrams. $f_{\alpha\gamma, \beta\delta}(\vec{k}, \vec{k}')$ can be expressed in terms of a special limit of the four point vertex function $\Gamma$ via [23, 24]

$$f_{\alpha\gamma, \beta\delta}(\vec{k}, \vec{k}') = Z^2 \lim_{\omega \to 0} \lim_{q \to 0} \Gamma_{\alpha\gamma, \beta\delta}(K + Q, K' - Q, K, K')$$

where $Q \equiv (q, \omega), K \equiv (\vec{k}, \epsilon)$ etc with implicitly $\epsilon \to 0$, $k \to k_F$ etc. Up to second order, $f_{\alpha\alpha}(\vec{k}, \vec{k}') = f_{\alpha\alpha, \alpha\alpha}(\vec{k}, \vec{k}')$ is given solely by the diagram in Fig [1a], which gives the expression

III. DILUTE GAS AND 1/$N_c$ EXPANSIONS

So far we only discussed the general formalism. Now we specialize to the dilute limit and evaluate the quasiparticle interaction $f_{\alpha\gamma, \beta\delta}(\vec{k}, \vec{k}')$ to second order in the interaction $g$. One way to proceed is to write down the total energy of the system up to $g^2$ and then take the derivative with respect to the occupation numbers (in the special case where all occupation numbers are diagonal), as done in [23] for the two-component system. We immediately obtain

written in eq (17). $f_{\alpha\beta \neq \alpha}(\vec{k}, \vec{k}') = f_{\alpha\beta, \alpha\beta}(\vec{k}, \vec{k}')$ has a Hartree contribution in first order, and second order contributions. They are depicted by the diagrams in Fig [1a]. These diagrams correspond respectively to the three terms written down in eq (15). (The quasiparticle weight $Z$ can be taken simply as unity since the correction is at least second order in $a$). The $N_c$ factor enhancement of $f_{\alpha\alpha}$ as compared with the spin 1/2 system is due to the increase in number of choices for the intermediate line labeled $\gamma$ in Fig [1]. It is also clear from Fig [1] that the effective interaction between distinguishable species is independent of $N_c$ to this order in $g$. We also note that Fig [1a] has the form of an interaction arising from an induced density fluctuation in the $\gamma$ component, as is also clear from the appearance of a density response function (the term in the square bracket in eq (17)), though however it involves an exchange in the sense that the incoming $\vec{k}$ line has become an outgoing $\vec{k}'$ line, and vice versa. (The corresponding diagram without the exchange vanishes under the limits in eq (21)). This exchange is responsible for the result that $f_{\alpha\alpha}(\vec{k}, \vec{k}') > 0$ (the term in the square bracket in Fig [1a]), which gives the expression in other contexts in cold atomic gases (e.g. [24]). Note that in mean-field theory, $f_{\alpha\alpha} = 0$ since identical particles do not interact, whereas $f_{\alpha\beta}$ is given entirely by the Hartree contribution $4\pi a/M$. We shall see that correc-
tions to mean-field results can be important in particular for large $N_c$.

The expressions for $f^{(1/2)}_{c+}$ and $f^{(1/2)}_{c-}$ can be found in textbooks. To save space we would not reproduce these equations, as well as the results for $f_{\alpha\alpha}$ and $f_{\alpha\beta \neq \alpha}$ here, but relegate them to Supplemental Material [24]. We directly give the results here for the effective mass

$$M^*/M = 1 + (N_c - 1) \frac{8}{15\pi^2} [7\ln 2 - 1](k_Fa)^2 ,$$

(22)

the inverse compressibility

$$\frac{(dn/d\mu)^{-1}}{(dn/d\mu)_{\text{free}}} = 1 + (N_c - 1) \frac{2k_Fa}{\pi}[1 + \frac{2k_Fa}{15\pi}(22 - 4\ln 2)]$$

(23)

and the inverse generalized spin-susceptibility

$$\chi^{-1}_{\text{free}} = 1 - \frac{2k_Fa}{\pi} - \frac{8(k_Fa)^2}{15\pi^2}[(11 - \frac{7N_c}{2}) + 2(N_c - 1)\ln 2]$$

(24)

where the subscript free denotes the non-interacting gas. The compressibility can also be obtained from the energy per particle given in [27, 28].

The results for $M^*/M$ and $\chi^{-1}/\chi^{-1}_{\text{free}}$ are plotted as dashed lines in Fig 2. The deviation of $M^*/M$ from unity is entirely due to non-mean-field effects, and is larger for larger $N_c$. The modification of $dn/d\mu$ from its free gas value however is dominated by the mean-field correction (not plotted). The spin susceptibility would be discussed in more detail below. The compressibility and the spin susceptibility can be readily measured in experiments. Both the compressibility and the magnetic susceptibility can be directly obtained from the density profile as a function of trap position if the local density approximation applies, and, if the trap is harmonic, even the equation of state can be directly deduced (e.g. [29, 32]).

They can also be measured via number fluctuations of the gas. [30, 37] For example, for any two components $\alpha$ and $\beta$, the total relative number fluctuation within a subvolume $V$ is related to $\chi$ via

$$\langle(N_\alpha - N_\beta)^2\rangle = k_BT V \chi$$

(25)

where $k_B$ is the Boltzmann constant and $T$ the temperature. Here the angular brackets denote averages over measurements. More general expressions can be found in Supplemental Material [24]. (In [30, 37], seemingly only the fluctuation integrated along the line of sight were reported. In principle, one can deduce also the local susceptibilities, in direct analogy to what has been done for the particle densities. Details can also be found in Supplemental Material [24].) $M^*/M$ can in principle also be obtained from entropy via the equation of state [38].

Recently, there is a lot of interest on the ferromagnetic properties of the two component repulsive gas [38, 37, 39–42]. Hence, we here would like to give here a more detailed discussion on the spin-susceptibility and contrast the $N_c = 2$ case with general $N_c$. Fig 2 plots, in dashed lines, the term $2N(0)f_{0,m}$ which appears in the denominator of $\chi$. A negative (positive) sign of this term indicates an enhancement (suppression) of ferromagnetic tendency of the system. Within the mean-field approximation, it is simply $-2k_Fa/\pi$, and hence the susceptibility diverges at $k_Fa = \pi/2$. This divergence is independent of $N_c$, as also noted in [10]. However, the higher order terms in $k_Fa$ modify this result. The second order contribution can be easily evaluated to be

$$-\frac{2k_Fa}{\pi} 4k_Fa \left[\frac{2 - N_c}{2} - (N_c - 1)\ln 2\right]$$

For $N_c = 2$, this term is negative, thus the ferromagnetic tendency is enhanced. [42] (This qualitative trend is supported by numerical evaluations [43, 44]). However, for $N_c \geq 3$, the sign of this term is already reversed. This is a consequence of the fact that the effective repulsive interaction between identical species is proportional to $N_c - 1$ and hence enhanced. Using the above formulas, we find that, for $N_c \geq 3$, $2N(0)f_{0,m} > -1$ for arbitrary large $k_Fa$, and hence there is no ferromagnetic instability under our approximations (though a first order phase transition [10, 42, 50] to a ferromagnetic state is still possible). We note that, as can be seen from Fig 2, the correction to mean-field can already be substantial even for modest $k_Fa$ for large $N_c$.

Our expansion in $k_Fa$ gives a large and positive $2N(0)f_{0,m}$ for large $k_Fa$ and $N_c \geq 3$. However, this result is likely to be an artifact of the $k_Fa$ expansion. In fact, one expects that the range of $k_Fa$ values for our expressions above to be reliable should be restricted to $k_Fa < 1/N_c$, and hence become very small for $N_c$ such as 6 or 10. Hence we consider also the alternative where one regards $(N_c,k_Fa)$ as a new (fixed) parameter, and to perform an expansion in $1/N_c$. The details are given in Supplemental Material [24]. Both $f_{\alpha\alpha}$ and $f_{\alpha\beta}$ are formally of order $1/N_c$. They are both finite for large $N_c k_Fa > 0$ as a result of screening. The corresponding results for $2N(0)f_{0,m}$ are also plotted as full lines in Fig 2. We see that in general this factor is negative, saturates at large $k_Fa$, and to a smaller value for larger $N_c$. The results for $\chi^{-1}$, normalized to their free gas values, are shown as full-lines in Fig 2. It decreases fast with $k_Fa$, mainly due to the increase in $M^*/M$.

The behaviors of $\chi$ and $f_{0,m}$ are often considered as indicators of whether there is a ferromagnetic tendency of the system. If we examine these quantities alone, our results above would indicate that, for larger $N_c$, the system is further away from ferromagnetism. Interestingly however, an examination of the energy of the system indicates that the situation is more complex. Including the interaction energy up to first or second order in $a$, it can be shown [50, 51] that the unpolarized state actually becomes unstable at a smaller $k_Fa$ for larger $N_c$. This may indicate that, for larger $N_c$, the ferromagnetic transition actually becomes more strongly first order. It would be of great interest to examine whether this remains true when higher order interaction terms are included.
**IV. CONCLUSIONS**

In summary, we have considered the Fermi liquid theory of an SU(N) dilute gas. Fermi liquid parameters are evaluated beyond the mean field approximation. Though the corrections to mean-field are formally higher order in the gaseous parameter $k_F a$, they are nonetheless enhanced by the number of components $N_c$. Currently, $k_F a$ in the $^{173}$Yb experiments is $\approx 0.13$ with $N_c$ as large as 6. While $k_F a$ for the $^{87}$Sr experiments are currently somewhat smaller, $N_c$ can be as large as 10. In both cases, these beyond mean-field effects should be easily measurable, and they would be even more important if experiments can be carried out at higher densities.

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(a) 

K_\alpha \quad \gamma_1 \quad \gamma_2 \quad K' \alpha

K'_\alpha \quad \gamma_1 \quad \gamma_2 \quad K_+ \alpha

(b) 

K_\alpha \quad \gamma_1 \quad \gamma_2 \quad \gamma_3 \quad K' \alpha

K'_\alpha \quad \gamma_1 \quad \gamma_2 \quad \gamma_3 \quad K_+ \alpha