Original Paper

Dynamic behaviors of soliton solutions for a three-coupled Lakshmanan–Porsezian–Daniel model

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Abstract In this paper, we use the Riemann–Hilbert (RH) approach to examine the integrable three-coupled Lakshmanan–Porsezian–Daniel (LPD) model, which describes the dynamics of alpha helical protein with the interspine coupling at the fourth-order dispersion term. Through the spectral analysis of Lax pair, we construct the higher-order matrix RH problem for the three-coupled LPD model, when the jump matrix of this particular RH problem is a $4 \times 4$ unit matrix, the exact N-soliton solutions of the three-coupled LPD model can be exhibited. As special examples, we also investigate the nonlinear dynamical behaviors of the single-soliton, two-soliton, three-soliton and breather soliton solutions. Finally, an integrable generalized N-component LPD model with its linear spectral problem is discussed.

Keywords Lax pair · Riemann–Hilbert approach · Three-coupled Lakshmanan–Porsezian–Daniel model · Soliton solutions · Dynamic behaviors

Mathematics Subject Classification 35C08 · 35Q15 · 37K10 · 45D05

1 Introduction

Since the nonlinear evolution equations can be widely used to describe some of the physical phenomena, such as nonlinear optical, quantum mechanics, fluid physics, and plasma physics. The research on the methods of solving nonlinear evolution equations becomes a challenging and vital task, and has attracted more and more people’s attention. With the development of soliton theory, a series of methods for solving nonlinear development equations are proposed, such as the inverse scattering method [1], the Hirota’s bilinear method [2], the Bäcklund transformation method [3], the Darboux transformation (DT) method [4] and others [5–8]. Based on these available methods, we have obtained a series of solutions of nonlinear evolution equations, including komponon solutions, peakon solutions, periodic sharp wave solutions, lump solutions, breather solutions, bright soliton, dark soliton, rogue waves, etc. These solutions can further help to understand natural phenomena and laws. In recent years, more and more mathematical physicists have begun to pay attention to Riemann–Hilbert (RH) approach [9,10], which is a new powerful method to solve integrable linear and nonlinear evolution equations [11–27]. The main idea of this method is to establish a corresponding matrix RH problem on the Lax pair of integrable equations. Furthermore, the RH approach is also an effective way to examine the initial boundary value problems [28–
32] and the long-time asymptotic behaviors [33, 34] of the integrable nonlinear evolution equations.

The Lakshmanan–Porsezian–Daniel (LPD) model [35] is one of the most paramount integrable systems in mathematics and physics which reads

\[ i q_t + q_{xx} + 2q|q|^2 + \varepsilon (q_{xxxx} + 8|q|^2 q_{xx} + 2q^2 q_x^4 + 6q^2 q_t^2 + 4q q_x q_{xx}^* + 6q|q|^4) = 0. \] (1.1)

This model not only can simulate the nonlinear transmission and interaction of ultrashort pulses in high-speed optical fiber transmission systems, but also can describe the nonlinear spin excitation phenomenon of a one-dimensional Heisenberg ferromagnetic chain with octpole and dipole interactions [36]. When \( \varepsilon = 0 \), Eq. (1.1) can be reduced to the famous nonlinear Schrödinger (NLS) equation. However, a slice of phenomena have been observed by experiment which cannot be explained by Eq. (1.1). For example, the dynamics of the alpha helical protein with the intermittent collision at the fourth-order dispersion term. In order to explain these phenomena, the three-component case has been introduced as follows [37–39]:

\[
i q_{\alpha,t} + q_{\alpha,xx} + 2 \sum_{k=1}^{3} |q_k|^2 q_{\alpha} + \varepsilon \left[ q_{\alpha,xxxx} + 2 \sum_{k=1}^{3} |q_k|^2 q_{\alpha,k} + 2 \sum_{k=1}^{3} q_k q_{k,x}^* q_{\alpha,x} + 6 \sum_{k=1}^{3} q_k q_{k,x}^* q_{\alpha,xx} + 4 \sum_{k=1}^{3} |q_k|^2 q_{\alpha,xx} + 4 \sum_{k=1}^{3} q_k q_{k,x}^* q_{\alpha,xx} q_{\alpha} + 2 \sum_{k=1}^{3} q_k q_{k,x}^* q_{\alpha,xx} q_{\alpha} + 6 \left( \sum_{k=1}^{3} |q_k|^2 \right)^2 q_{\alpha} \right] = 0,
\] (1.2)

where \( q_{\alpha}(x,t), (\alpha = 1,2,3) \) represent amplitude of molecular excitation in the \( \alpha \)-th spine, the subscripts \( x \) and \( t \) represent the partial derivatives with respect to the scaled space variable and retarded time variable, respectively, the \( * \) denotes the complex conjugate, and \( \varepsilon \) is a real parameter standing for the strength of higher-order linear and nonlinear effects. When \( \varepsilon = 0 \), Eq. (1.2) can be reduced to the three-component NLS equation [40, 41]. Indeed, the three-coupled LPD model (1.2) is a completely integrable model, and quiet a few properties have been investigated, for instance, the Lax pair [37], the Hirota bilinear forms, the soliton solutions by the DT method and the binary Bell polynomial approach [38], and the semirational rogue waves by the generalized DT method [39]. In this paper, we aim to investigate the soliton solutions of three-coupled LPD model (1.2) via the RH approach, and discuss the dynamic behaviors of the soliton solutions.

The organization of this paper is as follows. In Sect. 2, we will construct a specific RH problem based on the inverse scattering transformation. In Sect. 3, we will compute N-soliton solutions of the three-coupled LPD model (1.2) from the given RH problem in Sect. 2, which possesses the identity jump matrix on the real axis. In Sect. 4, the spatial structures and collision dynamics behaviors of soliton solutions will be examined. In Sect. 5, as a promotion, we will briefly explain an integrable generalized multicomponent LPD model.

2 The Riemann–Hilbert problem

The three-coupled LPD model (1.2) possesses the following Lax pair [39]

\[
\begin{align*}
\Phi_x &= U(x,t,\xi)\Phi = (i\xi \sigma_4 + M(x,t))\Phi, \\
\Phi_t &= V(x,t,\xi)\Phi = (-8i\varepsilon \xi^4 \sigma_4 + 2i\xi^2 \sigma_4 + Q(x,t,\xi))\Phi,
\end{align*}
\] (2.1a, 2.1b)

where \( \xi \) is a complex spectral parameter, \( \Phi = (\Phi_1, \Phi_2, \Phi_3, \Phi_4)^T \) is a vector function, the superscript \( T \) represents the transpose of a matrix or vector and the \( 4 \times 4 \) matrices \( \sigma_4, M(x,t) \) are defined by

\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & -1
\end{pmatrix},
\] (2.2)

\[
\begin{pmatrix}
0 & q_1^* & q_2^* & q_3^* \\
q_1 & 0 & 0 & 0 \\
q_2 & 0 & 0 & 0 \\
q_3 & 0 & 0 & 0
\end{pmatrix},
\]

and \( Q(x,t,\xi) = Q_{4\times4} \) is a matrix-value function defined by

\[
\begin{align*}
Q(x,t,\xi) &= -8i\varepsilon M\xi^3 + 4\varepsilon(iM^2\sigma_4 - \sigma_4 M_t)\xi^2 \\
&+ 2i\varepsilon M_{tx} + 2i\varepsilon M^3 \\
&- \varepsilon(MM_x - M_x M) + iM)\xi + \varepsilon\sigma_4 M_{xxx} \\
&- i\varepsilon\sigma_4(MM_{xx} + M_{xx} M)
\end{align*}
\]
\[-3i\varepsilon\sigma_4 M^4 - i\sigma_4 M^2 \\
i\varepsilon M_1^2\sigma_4 + 3\varepsilon\sigma_4(M_2 M_1 + M_3 M^2) \\
+\sigma_4 M_8. \quad (2.3)\]

Equation (2.1a)–(2.1b) is rewritten as
\[
\Phi_x - i\xi\sigma_4\Phi = M(x,t)\Phi, \quad (2.4a) \\
\Phi_t + (8i\varepsilon\xi^4\sigma_4 - 2i\xi^2\sigma_4) \\
\Phi = Q(x, t, \xi)\Phi. \quad (2.4b) 
\]

since \(\tilde{A}(x, t, \xi) = e^{i\xi(x-\xi^4 t+2i\xi^2)}\sigma_4\) is a solution for Eq. (2.4a)–(2.4b), by setting \(\Phi(x, t, \xi) = \Psi(x, t, \xi)\tilde{A}(x, t, \xi)\), we have the following Lax pair forms:
\[
\Psi_x - i\xi[\sigma_4, \Psi] = M(x, t)\Psi, \quad (2.5a) \\
\Psi_t + (8i\varepsilon\xi^4 - 2i\xi^2) [\sigma_4, \Psi] = Q(x, t, \xi)\Psi. \quad (2.5b)
\]

Now, we construct two Jost solutions \(\Psi_{\pm} = \Psi_{\pm}(x, \xi)\) of Eq. (2.5a)
\[
\Psi_+ = ([\Psi_+, 1], [\Psi_+, 2], [\Psi_+, 3], [\Psi_+, 4]), \quad (2.6a) \\
\Psi_- = ([\Psi_-, 1], [\Psi_-, 2], [\Psi_-, 3], [\Psi_-, 4]), \quad (2.6b)
\]

with the boundary conditions
\[
\Psi_+ \to I, \quad x \to +\infty, \quad (2.7a) \\
\Psi_- \to I, \quad x \to -\infty, \quad (2.7b)
\]

where \(I = \text{diag}(1, 1, 1, 1)\) is a \(4 \times 4\) identity matrix, \([\Psi_{\pm}]_n(n = 1, 2, 3, 4)\) denote the \(n\)-th column vector of \(\Psi_\pm\). Indeed, for \(\xi \in \mathbb{R}\), the two Jost solutions \(\Psi_{\pm} = \Psi_{\pm}(x, \xi)\) are determined by Volterra integral equations as follows:
\[
\Psi_+(x, \xi) = I - \int_x^{+\infty} e^{i\xi\hat{\sigma}_4(x-\zeta)} M(\zeta)\Psi_+(\zeta, \xi)d\zeta, \quad (2.8a) \\
\Psi_-(x, \xi) = I + \int_{-\infty}^{x} e^{i\xi\hat{\sigma}_4(x-\zeta)} M(\zeta)\Psi_-(\zeta, \xi)d\zeta, \quad (2.8b)
\]

where \(\hat{\sigma}_4\) represents a matrix operator (see [28]).

If \(\mathbb{C}_+\) and \(\mathbb{C}_-\) denote the upper half \(\xi\)-plane and the lower half \(\xi\)-plane, respectively, then \([\Psi_-]_1, [\Psi_-]_2, [\Psi_-]_3\) and \([\Psi_+]_4\) admit analytic extensions to \(\mathbb{C}_-\), \([\Psi_+]_1, [\Psi_-]_2, [\Psi_-]_3\) and \([\Psi_-]_4\) admit analytic extensions to \(\mathbb{C}_+\). Furthermore, the determinants of \(\Psi_{\pm}\) are constants for all \(x\) because of the Abel’s identity and \(\text{Tr}(M) = 0\). It follows from the boundary conditions Eq.(2.8) that
\[
\det\Psi_{\pm} = 1, \quad \xi \in \mathbb{R}. \quad (2.9)
\]

Next, we introduce a new function \(A(x, \xi) = e^{i\xi\hat{\sigma}_4}\); then, Eq. (2.5a) exists two fundamental matrix solutions \(\Psi_+ A\) and \(\Psi_- A\) which satisfy the following relationship:
\[
\Psi_- A = \Psi_+ A \cdot S(\xi), \quad \xi \in \mathbb{R}, \quad (2.10)
\]

where \(S(\xi)\) is a \(4\times4\) scattering matrix. From Eqs. (2.9)–(2.10), we can get
\[
\det S(\xi) = 1. \quad (2.11)
\]

Let \(x \to +\infty\), we have the \(4 \times 4\) scattering matrix \(S(\xi)\)
\[
S(\xi) = (s_{ij})_{4 \times 4} = \lim_{x \to +\infty} A^{-1}\Psi_+^{-1} \Psi_- A = I + \int_{-\infty}^{+\infty} e^{-i\xi\hat{\sigma}_4} M\Psi_- d\xi, \quad \xi \in \mathbb{R}. \quad (2.12)
\]

It follows from the analytic property of \(\Psi_-\) that \(s_{11}\) allows analytic extensions to \(\mathbb{C}_-\), and the others \(s_{ij}, i, j \neq 1\) allow analytic extensions to \(\mathbb{C}_+\).

In order to discuss behavior of Jost solutions for large \(\xi\), we assume that
\[
\Psi = \Psi_0 + \frac{\Psi_1}{\xi} + \frac{\Psi_2}{\xi^2} + \frac{\Psi_3}{\xi^3} \\
+ \frac{\Psi_4}{\xi^4} + \cdots, \quad \xi \to \infty, \quad (2.13)
\]

Substituting the above expansion Eq. (2.13) into Eq. (2.5a) and comparing the coefficients of the same order of \(\xi\) yields
\[
O(\xi^1) : -i[\sigma_4, \Psi_0] = 0, \quad (2.14a) \\
O(\xi^0) : \Psi_{0,x} - i[\sigma_4, \Psi_1] - M\Psi_0 = 0, \quad (2.14b) \\
O(\xi^{-1}) : \Psi_{1,x} - i[\sigma_4, \Psi_2] - M\Psi_1 = 0. \quad (2.14c)
\]

From \(O(\xi^1)\) and \(O(\xi^0)\), we have
\[
i[\sigma_4, \Psi_1] = -M\Psi_0, \quad \Psi_{0,x} = 0. \quad (2.15)
\]

On the one hand, we define another new Jost solution for Eq. (2.5a) by
\[
H_+ = ([\Psi_-]_1, [\Psi_+]_2, [\Psi_+]_3, [\Psi_+]_4). \quad (2.16)
\]

Taking Eq. (2.6a)–(2.6b) into Eq. (2.10) gives rise to
\[
([\Psi_-]_1, [\Psi_-]_2, [\Psi_-]_3, [\Psi_-]_4) = \Psi_+ AS(\xi)A^{-1}
\]
Obviously, the inverse matrices $\Psi_1^{-1}$ admit analytic extensions to the counter part of $H_-$ and admits asymptotic behavior for very large $\xi$ as

$$H_+ \to I, \xi \to +\infty, \xi \in \mathbb{C}_-.$$  \hfill (2.19)

On the other hand, in order to obtain the analytic counterpart of $H_+$ denoted by $H_-$ in $\mathbb{C}_+$, we consider the adjoint scattering equation of Eq. (2.25a) as follows:

$$G_x - i\xi [\sigma_4, G] = -GM.$$ \hfill (2.20)

Obviously, the inverse matrices $\Psi_\pm^{-1}$ is defined as:

$$[\Psi_+]^{-1} = \left(\begin{array}{c} [\Psi_1^{-1}]^1 \\
[\Psi_1^{-1}]^2 \\
[\Psi_1^{-1}]^3 \\
[\Psi_1^{-1}]^4 \end{array}\right), \quad [\Psi_-]^{-1} = \left(\begin{array}{c} [\Psi_1^{-1}]^1 \\
[\Psi_1^{-1}]^2 \\
[\Psi_1^{-1}]^3 \\
[\Psi_1^{-1}]^4 \end{array}\right).$$ \hfill (2.21)

which satisfy the adjoint Eq. (2.20), where $[\Psi_\pm^{-1}]^n (n = 1, 2, 3, 4)$ denote the $n$-th row vector of $\Psi_\pm^{-1}$, then, $[\Psi_+^{-1}]^1, [\Psi_+^{-1}]^2, [\Psi_+^{-1}]^3$ and $[\Psi_+^{-1}]^4$ admit analytic extensions to $\mathbb{C}_-, [\Psi_-^{-1}]^1, [\Psi_-^{-1}]^2, [\Psi_-^{-1}]^3$ and $[\Psi_-^{-1}]^4$ admit analytic extensions to the $\mathbb{C}_+$. In addition, it is not difficult to find that the inverse matrices $\Psi_+^{-1}$ and $\Psi_-^{-1}$ satisfy the following boundary conditions:

$$\Psi_+^{-1} \to I, \ x \to +\infty,$$ \hfill (2.22a)

$$\Psi_-^{-1} \to I, \ x \to -\infty.$$ \hfill (2.22b)
• $H_+(x, \xi)$ and $H_-(x, \xi)$ are analytic for $\xi$ in $\mathbb{C}_-$ and $\mathbb{C}_+$, respectively.

• $H_+(x, \xi)$ and $H_-(x, \xi)$ satisfy the following jumping relationship on the real $x$-axis

$$H_-(x, \xi)H_+(x, \xi) = T(x, \xi), \quad \xi \in \mathbb{R},$$

(2.28)

where

$$T(x, \xi) = \begin{pmatrix}
1 & r_{12}e^{2i\xi} & r_{13}e^{2i\xi} & r_{14}e^{2i\xi} \\
s_{21}e^{-2i\xi} & 0 & 0 & 0 \\
s_{31}e^{-2i\xi} & 0 & 0 & 0 \\
s_{41}e^{-2i\xi} & 0 & 0 & 1
\end{pmatrix},$$

(2.29)

and $r_{11}s_{11} + r_{12}s_{21} + r_{13}s_{31} + r_{14}s_{41} = 1$.

• $H_+ \to I$, $\xi \to \infty$.

Furthermore, since $\Psi_-$ satisfies the temporal part of spectral equation

$$\Psi_{-,t} + (8i\varepsilon^4 - 2i\varepsilon^2)[\Psi_4, \Psi_-] = Q(x, t, \xi)\Psi_-, \quad (2.30)$$

from Eq. (2.10), we have

$$(\Psi_+AS)_t + (8i\varepsilon^4 - 2i\varepsilon^2)[\Psi_4, \Psi_+AS] = Q(x, t, \xi)\Psi_+AS, \quad (2.31)$$

for $x \to \infty$, assume that $q_\alpha$, $(\alpha = 1, 2, 3)$ are sufficient smoothness and fast decay, the matrix $Q(x, t, \xi) \to 0$ as $x \to \pm \infty$. Let $x \to +\infty$ in Eq. (2.31), we obtain

$$S_t = -(8i\varepsilon^4 - 2i\varepsilon^2)[\Psi_4, S], \quad (2.32)$$

which means that the scattering data $s_{11}$, $s_{22}$, $s_{33}$, $s_{23}$, $s_{32}$ are time independent

$$s_{11,t} = s_{22,t} = s_{33,t} = s_{44,t} = s_{23,t} = s_{24,t} = s_{32,t} = s_{34,t} = s_{42,t} = s_{43,t} = 0, \quad (2.33)$$

and the other scattering data satisfy

$$r_{12}(t, \xi) = r_{12}(0, \xi)e^{-4i(4\varepsilon^4 - \varepsilon^2)t},$$

$$r_{13}(t, \xi) = r_{13}(0, \xi)e^{-4i(4\varepsilon^4 - \varepsilon^2)t},$$

$$r_{14}(t, \xi) = r_{14}(0, \xi)e^{-4i(4\varepsilon^4 - \varepsilon^2)t},$$

$$s_{21}(t, \xi) = s_{21}(0, \xi)e^{4i(4\varepsilon^4 - \varepsilon^2)t},$$

$$s_{31}(t, \xi) = s_{31}(0, \xi)e^{4i(4\varepsilon^4 - \varepsilon^2)t},$$

$$s_{41}(t, \xi) = s_{41}(0, \xi)e^{4i(4\varepsilon^4 - \varepsilon^2)t}. \quad (2.34)$$

3 The soliton solutions for three-coupled LPD model

In this section, based on the RH problem constructed in Sect. 2, we would like to formulate the N-soliton solutions of three-coupled LPD model (1.2). In fact, the solution to this RH problem will not be unique unless the zeros of $\det H_+$ and $\det H_-$ in the upper and lower half of the $\xi$-plane are also specified, and the kernel structures of $H_\pm$ at these zeros are provided. It follows from the definitions of $H_+$ and $H_-$ that

$$\det H_+(x, \xi) = s_{11}(\xi), \quad \det H_-(x, \xi) = r_{11}(\xi), \quad (3.1)$$

which means that the zeros of $\det H_+$ and $\det H_-$ are the same as $s_{11}(\xi)$ and $r_{11}(\xi)$, respectively. Furthermore, the scattering data $s_{11}$ and $r_{11}$ are time independent (2.33), and we can get that the roots of $s_{11} = 0$ and $r_{11} = 0$ are also time independent, since

$$s_{11}M(x, t)\sigma_4 = -M(x, t), \quad M^\dagger(x, t) = M(x, t),$$

where $\dagger$ represents the Hermitian of a matrix. It is easy to see that

$$\Psi_{\pm}(x, t, \xi) = \psi_4\Psi_{\pm}(x, t, \xi)\sigma_4,$$

$$\Psi_+^{-1}(x, t, \xi) = \Psi_-^\dagger(x, t, \xi^*). \quad (3.2)$$

according to reduction conditions given by Eqs. (2.6a)–(2.6b), we obtain

$$S(-\xi) = \sigma_4S(\xi)\sigma_4, \quad R(\xi) = S^{-1}(\xi) = S^\dagger(\xi^*),$$

$$H_-(x, \xi) = H_+^\dagger(x, \xi^*). \quad (3.3)$$

Assume that $s_{11}$ enjoys $N \geq 0$ possible zeros denoted as $\xi_m, 1 \leq m \leq N$ in $\mathbb{C}_-$, $r_{11}$ and enjoys $N \geq 0$ possible zeros denoted as $\tilde{\xi}_m, 1 \leq m \leq N$ in $\mathbb{C}_+$. For the convenience of later discussion, without loss of generality, we suppose that all zeros $\{\xi_m, \tilde{\xi}_m, m = 1, 2, \ldots, N\}$ of $s_{11}$ and $r_{11}$ are simple zeros. In this case, each of $\ker H_+(\xi_m)$ and $\ker H_-(\tilde{\xi}_m)$ includes only a single column vector $v_m$ and row vector $\tilde{v}_m$, respectively, i.e.,

$$H_+(\xi_m)v_m = 0,$$

$$\tilde{v}_m H_-(\tilde{\xi}_m) = 0, \quad (3.4)$$

because $H_+(\xi)$ is the solution of Eq. (2.5a). For large $\xi$, we suppose that $H_+(\xi)$ possesses the following asymptotic expansion:

$$H_+ = I + \frac{H_+^{(1)}}{\xi} + O(\xi^{-2}), \quad \xi \to \infty. \quad (3.5)$$
Taking Eq. (3.5) into Eq. (2.5a) and comparing $O(1)$ terms gets

$$M = -i[\sigma_4, H_+^{(1)}] = \begin{pmatrix}
0 & -2i(H_+^{(1)})_{12} & -2i(H_+^{(1)})_{13} & -2i(H_+^{(1)})_{14} \\
2i(H_+^{(1)})_{21} & 0 & 0 & 0 \\
2i(H_+^{(1)})_{31} & 0 & 0 & 0 \\
2i(H_+^{(1)})_{41} & 0 & 0 & 0 
\end{pmatrix}. \quad (3.6)$$

Then, the potential functions solutions $q_1(x, t)$, $q_2(x, t)$ and $q_3(x, t)$ of the three-coupled LPD model (1.2) can be reconstructed by

$$q_1(x, t) = 2i(H_+^{(1)})_{21},$$
$$q_2(x, t) = 2i(H_+^{(1)})_{31},$$
$$q_3(x, t) = 2i(H_+^{(1)})_{41}, \quad (3.7)$$

where $H_+^{(1)} = (H_+^{(1)})_{4 \times 4}$ and $(H_+^{(1)})_{ij}$ are the $(i, j)$-entry of $H_+^{(1)}$, $i, j = 1, 2, 3, 4$.

In order to get the spatial evolutions for vectors $v_m(x, t)$, on the one hand, taking the derivative of equation $H_+ v_m = 0$ with respect to $x$ and with Eq. (2.5a), we obtain

$$H_+ v_{m,x} + i \xi_m H_+ \sigma_4 v_m = 0. \quad (3.8)$$

Thus,

$$v_{m,x} = -i \xi_m \sigma_4 v_m; \quad (3.9)$$

on the other hand, taking the derivative of equation $H_+ v_m = 0$ with respect to $t$ and with Eq. (2.5b), we get

$$H_+ v_{m,t} - (8i \xi_m^4 - 2i \xi_m^2) H_+ \sigma_4 v_m = 0. \quad (3.10)$$

Thus,

$$v_{m,t} = (8i \xi_m^4 - 2i \xi_m^2) \sigma_4 v_m. \quad (3.11)$$

According to Eqs. (3.9) and (3.11), we have

$$v_m(x, t) = e^{-i \xi_m \sigma_4 t + (8i \xi_m^4 - 2i \xi_m^2) \sigma_4 t} v_{m0} e^{\int_{t_0}^t \rho_m(y)dy + \int_{t_0}^t \omega_m(\tau)d\tau}, \quad (3.12)$$

where $v_{m0}$ is constant, $\rho_m(y)$ and $\omega_m(\tau)$ are two scalar functions; at the same time, we have

$$\tilde{v}_m(x, t) = \tilde{v}_m^+(x, t). \quad (3.13)$$

In order to obtain multi-soliton solutions for the three-coupled LPD model (1.2), we assume that the jump matrix $T(x, \xi) = I$ is a $4 \times 4$ unit matrix in Eq. (2.28). In other words, the discrete scattering data $\{r_i\} = \{s_i\}$ would be reconstructed by

$$r_{12} = r_{13} = r_{14} = s_{21} = s_{31} = s_{41} = 0; \quad (3.14)$$

and consequently, the unique solution to this special RH problem can be described as:

$$H_+(\xi) = I + \sum_{m=1}^{N} \sum_{n=1}^{N} v_m \tilde{v}_n (P^{-1})_{mn}, \quad (3.14a)$$

$$H_- (\xi) = I - \sum_{m=1}^{N} \sum_{n=1}^{N} v_m \tilde{v}_n (P^{-1})_{mn}, \quad (3.14b)$$

where $P = (P_{mn})_{N \times N}$ is a $N \times N$ matrix defined by

$$P_{mn} = \frac{v_m v_n}{\xi_m - \xi_n}, \quad 1 \leq m, n \leq N; \quad (3.15)$$

therefore, from (3.14a) to (3.14b), we obtain

$$H_+^{(1)} = \sum_{m=1}^{N} \sum_{n=1}^{N} v_m (P^{-1})_{mn} \tilde{v}_n. \quad (3.16)$$

Choosing $v_{m0} = (a_m, b_m, c_m, d_m)^T$ and $\rho_m = \omega_m = 0$, from (3.16), we get that the general $N$-soliton solution for the three-coupled LPD model (1.2):

$$q_1 = 2i \sum_{m=1}^{N} \sum_{n=1}^{N} b_m^* a_n \epsilon_{\theta_n + \theta_m}^{\alpha_n} (P^{-1})_{mn}, \quad (3.17a)$$

$$q_2 = 2i \sum_{m=1}^{N} \sum_{n=1}^{N} c_m^* a_n \epsilon_{\theta_n + \theta_m}^{\alpha_n} (P^{-1})_{mn}, \quad (3.17b)$$

$$q_3 = 2i \sum_{m=1}^{N} \sum_{n=1}^{N} d_m^* a_n \epsilon_{\theta_n + \theta_m}^{\alpha_n} (P^{-1})_{mn}, \quad (3.17c)$$

where $\theta_n = i \xi_m x - 8i \xi_m^4 t + 2i \xi_m^2 t$, $P = (P_{mn})_{N \times N}$ is a $N \times N$ matrix with elements given by

$$P_{mn} = \frac{a_m^* a_n \epsilon_{\theta_n + \theta_m}^{\alpha_n} + (b_m^* b_n + c_m^* c_n + d_m^* d_n) e^{-i(\theta_n + \theta_m)}}{\xi_m - \xi_n}, \quad 1 \leq m, n \leq N. \quad (3.18)$$
Furthermore, the N-soliton solutions Eq. (3.17a)–(3.17c) can be rewritten as the following determinant form:

\[
q_1(x, t) = -2i \frac{\det D_1}{\det P},
\]

\[
q_2(x, t) = -2i \frac{\det D_2}{\det P},
\]

\[
q_3(x, t) = -2i \frac{\det D_3}{\det P},
\]

(3.19)

where the \(N \times N\) matrix \(P\) is defined by Eq. (3.18), and \(D_1, D_2, D_3\) are \((N + 1) \times (N + 1)\) matrices given by

\[
D_1 = \begin{pmatrix}
    P_{11} & \cdots & P_{1N} & a_1 e^{-\theta_1} \\
    \vdots & \ddots & \vdots & \vdots \\
    P_{N1} & \cdots & P_{NN} & a_N e^{-\theta_N} \\
    b_1^1 e^{\theta_1} & \cdots & b_N^N e^{\theta_N} & 0
\end{pmatrix},
\]

\[
q_1(x, t) = 2b_1^1 \gamma_1 e^{-2i(\beta_1 x - 4(\beta_1^2 - 6\beta_1^2)\gamma_1 + \gamma_1^2)t + 2(\beta_1^2 - \gamma_1^2)\gamma_1 + \eta_1} \text{sech} \Delta(x, t),
\]

\[
q_2(x, t) = 2c_1^1 \gamma_1 e^{-2i(\beta_1 x - 4(\beta_1^2 - 6\beta_1^2)\gamma_1 + \gamma_1^2)t + 2(\beta_1^2 - \gamma_1^2)\gamma_1 + \eta_1} \text{sech} \Delta(x, t),
\]

\[
q_3(x, t) = 2d_1^1 \gamma_1 e^{-2i(\beta_1 x - 4(\beta_1^2 - 6\beta_1^2)\gamma_1 + \gamma_1^2)t + 2(\beta_1^2 - \gamma_1^2)\gamma_1 + \eta_1} \text{sech} \Delta(x, t),
\]

(4.1a)–(4.1c)

where \(\theta_1 = i\xi_1 x - 8i\varepsilon \xi_1^4 t + 2i\xi_1^2 t, \varepsilon = \frac{1}{2}, a_1 = 1, |b_1|^2 + |c_1|^2 + |d_1|^2 = e^{-2\eta_1}\), the single-soliton solution Eqs. (4.1a)–(4.1c) becomes

\[
q_1(x, t) = \frac{2ib_1^1 a_1 e^{\theta_1^* - \theta_1(i \xi_1^* - \xi_1)}}{|a_1|^2 e^{\theta_1^* + \theta_1} + (|b_1|^2 + |c_1|^2 + |d_1|^2) e^{-(\theta_1^* + \theta_1)}}.
\]

(4.2a)

\[
q_2(x, t) = \frac{2ic_1^1 a_1 e^{\theta_1^* - \theta_1(i \xi_1^* - \xi_1)}}{|a_1|^2 e^{\theta_1^* + \theta_1} + (|b_1|^2 + |c_1|^2 + |d_1|^2) e^{-(\theta_1^* + \theta_1)}}.
\]

(4.2b)

\[
q_3(x, t) = \frac{2id_1^1 a_1 e^{\theta_1^* - \theta_1(i \xi_1^* - \xi_1)}}{|a_1|^2 e^{\theta_1^* + \theta_1} + (|b_1|^2 + |c_1|^2 + |d_1|^2) e^{-(\theta_1^* + \theta_1)}}.
\]

(4.2c)

\[
D_2 = \begin{pmatrix}
    P_{11} & \cdots & P_{1N} & a_1 e^{-\theta_1} \\
    \vdots & \ddots & \vdots & \vdots \\
    P_{N1} & \cdots & P_{NN} & a_N e^{-\theta_N} \\
    c_1^1 e^{\theta_1} & \cdots & c_N^N e^{\theta_N} & 0
\end{pmatrix},
\]

D_3 = \begin{pmatrix}
    P_{11} & \cdots & P_{1N} & a_1 e^{-\theta_1} \\
    \vdots & \ddots & \vdots & \vdots \\
    P_{N1} & \cdots & P_{NN} & a_N e^{-\theta_N} \\
    d_1^1 e^{\theta_1} & \cdots & d_N^N e^{\theta_N} & 0
\end{pmatrix}.

(3.20)

4 Dynamic behaviors of soliton solutions

In what follows, we can examine the nonlinear dynamical behaviors of the breather and soliton solutions for the three-coupled LPD model (1.2).

Firstly, as a simple example, choosing \(N = 1\) in Eqs. (3.17a)–(3.17c) and with Eq. (3.15) gives rise to the following single-soliton solution of the three-coupled LPD model: (1.2)

\[
q_1(x, t) = 2ib_1^1 a_1 e^{\theta_1^* - \theta_1(i \xi_1^* - \xi_1)} (P^{-1})_{11} + 2ib_2^1 a_2 e^{\theta_1^* - \theta_2(i \xi_1^* - \xi_2)} (P^{-1})_{12} + 2ib_2^1 a_1 e^{\theta_2^* - \theta_1(i \xi_2^* - \xi_1)} (P^{-1})_{21} + 2ib_2^2 a_2 e^{\theta_2^* - \theta_2(i \xi_2^* - \xi_2)} (P^{-1})_{22}.
\]

(4.3a)
where $\theta_l = i\xi_l x - 8ie^{i\xi_l^2 t} + 2i\xi_l^2 t, (l = 1, 2)$, and $P = (P_{mn})_{2 \times 2}$ is a $2 \times 2$ matrix with the following elements:

$$P_{11} = \frac{|a_1|^2 e^{\theta_1 t} + (|b_1|^2 + |c_1|^2 + |d_1|^2) e^{i(\theta_1 + \theta_2)}}{\xi_1 - \xi_2},$$

$$P_{12} = \frac{a_2^* b_1 e^{\theta_1 + \theta_2} + (b_2^* a_1 + c_2^* c_1 + d_2^* d_1) e^{i(\theta_1 + \theta_2)}}{\xi_2 - \xi_1},$$

$$P_{21} = \frac{|a_2|^2 e^{\theta_2 t} + (|b_2|^2 + |c_2|^2 + |d_2|^2) e^{i(\theta_2 + \theta_1)}}{\xi_2 - \xi_1},$$

$$P_{22} = \frac{a_1^* b_2 e^{\theta_2 + \theta_1} + (a_2^* b_1 + c_1^* c_2 + d_1^* d_2) e^{i(\theta_2 + \theta_1)}}{\xi_1 - \xi_2}.$$

Let $\xi_l = \beta_l + i\gamma_l, (l = 1, 2), \epsilon = 1, a_1 = a_2 = 1, b_1 = b_2, c_1 = c_2, d_1 = d_2, |b|^2 + |c|^2 + |d|^2 = e^{-2\beta_2}$, the two-soliton solution Eqs. (4.3a)–(4.3c) becomes

$$q_1(x, t) = \frac{2b_1^*[e^{\theta_1 t} P_{22} - e^{\theta_2 t} P_{21} - e^{i(\theta_1 + \theta_2)} P_{12} + e^{i(\theta_1 + \theta_2)} P_{11}]}{P_{11} P_{22} - P_{12} P_{21}}.$$

By choosing different parameters, we can obtain the interaction solutions Eqs. (4.4a)–(4.4c) which are shown in Fig. 2 and Fig. 3. Indeed, Fig. 2 displays that soliton transmission, and the two solitons moving over each other, and their polarizations remaining unchanged. Figure 3 displays that soliton reflection produces the breather soliton solution.

Finally, as another special example, choosing $N = 3$ in Eqs. (3.17a)–(3.17c) and with Eq. (3.15) arrives at the following three-soliton solution of the three-coupled LPD model: (1.2)
Dynamic behaviors of soliton solutions

Two-soliton solution $q_1(x, t)$ (4.4a) for three-coupled LPD model (1.2) by choosing suitable parameters: a) $\eta_2 = 0$, $b_1 = 1, \beta_1 = -0.3$, $\gamma_1 = -0.3$, $\gamma_2 = 0.3$. b) $\eta_2 = 0, b_1 = 1, \beta_2 = -0.5, \gamma_1 = 0.2, \gamma_2 = -0.3$

Breather soliton solution $q_1(x, t)$ (4.4a) for three-coupled LPD model (1.2) by choosing suitable parameters: $\eta_2 = 0$, $b_1 = 0.5 - 0.5i, \beta_1 = 0.4$, $\beta_2 = -0.7, \gamma_1 = 0.5$, $\gamma_2 = -0.2$. a) Perspective view of modulus of breather soliton $q_1(x, t)$. b) Density plot of modulus of breather soliton $q_1(x, t)$

\[ q_1(x, t) = 2ib_1^*a_1e^{\theta_1 - \theta_1} (P^{-1})_{11} + 2ib_2^*a_2e^{\theta_1 - \theta_2} (P^{-1})_{12} + 2ib_2^*a_3e^{\theta_1 - \theta_3} (P^{-1})_{13} + 2ib_3^*a_1e^{\theta_2 - \theta_1} (P^{-1})_{21} + 2ib_2^*a_2e^{\theta_2 - \theta_2} (P^{-1})_{22} + 2ib_2^*a_3e^{\theta_2 - \theta_3} (P^{-1})_{23} + 2ib_3^*a_1e^{\theta_3 - \theta_1} (P^{-1})_{31} + 2ib_2^*a_2e^{\theta_3 - \theta_2} (P^{-1})_{32} + 2ib_3^*a_3e^{\theta_3 - \theta_3} (P^{-1})_{33}, \] (4.5a)

\[ q_2(x, t) = 2ic_1^*a_1e^{\theta_1 - \theta_1} (P^{-1})_{11} + 2ic_1^*a_2e^{\theta_1 - \theta_2} (P^{-1})_{12} + 2ic_1^*a_3e^{\theta_1 - \theta_3} (P^{-1})_{13} + 2ic_2^*a_1e^{\theta_2 - \theta_1} (P^{-1})_{21} + 2ic_2^*a_2e^{\theta_2 - \theta_2} (P^{-1})_{22} + 2ic_2^*a_3e^{\theta_2 - \theta_3} (P^{-1})_{23} + 2ic_3^*a_1e^{\theta_3 - \theta_1} (P^{-1})_{31} + 2ic_3^*a_2e^{\theta_3 - \theta_2} (P^{-1})_{32} + 2ic_3^*a_3e^{\theta_3 - \theta_3} (P^{-1})_{33}, \] (4.5b)

\[ q_3(x, t) = 2id_1^*a_1e^{\theta_1 - \theta_1} (P^{-1})_{11} + 2id_2^*a_2e^{\theta_1 - \theta_2} (P^{-1})_{12} + 2id_1^*a_3e^{\theta_1 - \theta_3} (P^{-1})_{13} + 2id_2^*a_1e^{\theta_2 - \theta_1} (P^{-1})_{21} + 2id_2^*a_2e^{\theta_2 - \theta_2} (P^{-1})_{22} + 2id_2^*a_3e^{\theta_2 - \theta_3} (P^{-1})_{23} + 2id_3^*a_1e^{\theta_3 - \theta_1} (P^{-1})_{31} + 2id_3^*a_2e^{\theta_3 - \theta_2} (P^{-1})_{32} + 2id_3^*a_3e^{\theta_3 - \theta_3} (P^{-1})_{33}, \] (4.5c)

where $\theta_j = i\xi_j x - 8i\xi_j^2 t + 2i\xi_j^2 t, (j = 1, 2, 3)$, and $M = (P_{mn})_{3 \times 3}$ is a $3 \times 3$ matrix with following elements:
Indeed, as a promotion, the integrable three-coupled LPD model (1.2) can be extended to the integrable generalized N-component LPD model as follows:
where \( q_\alpha(x, t), (\alpha = 1, 2, \ldots, N) \) represent amplitude of molecular excitation in the \( \alpha \)-th spine, \( \varepsilon \) is a real parameter which stands for the strength of higher-order linear and nonlinear effects. Let \( q = (q_1, q_2, \ldots, q_N)^T \), Eq. (5.1) has the following vector form:

\[
iq_{\alpha,t} + q_{\alpha,xx} + 2 \sum_{k=1}^{N} |q_k|^2 q_{\alpha} + \varepsilon [q_{\alpha,xxxx} + 2 \sum_{k=1}^{N} |q_{k,x}|^2 q_{\alpha}] + 2 \sum_{k=1}^{N} q_k q_{k,x} q_{\alpha,x} + 6 \sum_{k=1}^{N} q_k^\alpha q_{k,x} q_{\alpha,x} + 4 \sum_{k=1}^{N} |q_k|^2 q_{\alpha,xx} + 4 \sum_{k=1}^{N} q_k^2 q_{k,xx} q_{\alpha} + 2 \sum_{k=1}^{N} q_k q_{k,x}^\alpha q_{\alpha} + 6 (\sum_{k=1}^{N} |q_k|^2)^2 q_{\alpha} = 0, \tag{5.1}
\]

Equation (5.1) possesses the following \((N+1) \times (N+1)\) matrix spectral problem:

\[
\Phi_x = [i \xi \Lambda + \tilde{M}(x, t)] \Phi, \tag{5.3a}
\]

\[
\Phi_t = [-8i \varepsilon \xi \Lambda^4 + 2i \varepsilon^2 \Lambda + \tilde{Q}(x, t, \xi)] \Phi, \tag{5.3b}
\]

where \( \xi \) is a complex spectral parameter, \( \Lambda, \tilde{M}(x, t) \) are \((N+1) \times (N+1)\) matrices defined by

\[
\Lambda = \begin{pmatrix} 1 & 0_{1 \times N} \\ 0_{1 \times N} & -I_{1 \times N} \end{pmatrix}, \quad \tilde{M}(x, t) = \begin{pmatrix} 0 & q^\dagger \\ q & 0_{N \times N} \end{pmatrix},
\tag{5.4}
\]

and \( \tilde{Q}(x, t, \xi) \) is a \((N+1) \times (N+1)\) matrix as follows:

\[
\tilde{Q}(x, t, \xi) = -8i \varepsilon \tilde{M}^3 + 4 \varepsilon (i \tilde{M}^2 \Lambda + \Lambda \tilde{M}) \xi^2
\]

Accordingly, we can also examine the N-soliton solutions to the integrable generalized N-component LPD model (5.1) by the same way in Sect. 3. However, we do not report them here since the procedure is mechanical.

In this work, we utilized the RH approach to study the three-coupled LPD model (1.2). By constructing a special matrix RH problem, we have obtained the multisoliton solution of the LPD model (1.2). In addition, some graphical analysis gave the dynamic characteristics of the soliton solution, including the interaction of single soliton, two solitons, three soliton and breather soliton solutions. It is hoped that our results can help enrich and explain other nonlinear integrable models.

Acknowledgements The authors would like to express our sincere thanks to every member in our discussion group for their valuable comments.

Funding information This study was funded by National Natural Science Foundation of China (grant numbers 12147115, 11835011 and 11975145), by Natural Science Foundation of Anhui Province (grant number 20180805QA09), by Postdoctoral Fund of Zhejiang Normal University (grant number ZC304021909), by Key Projects of Natural Science Research in Colleges and Universities in Anhui Province (grant number KJ2021B03).

Data availability The authors confirm that the data supporting the findings of this study are available within the article.

Declarations

Conflict of interest The authors declare that they have no conflict of interest.

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