Cartan’s structure of symmetry pseudo-group and coverings for the r-th modified dispersionless Kadomtsev–Petviashvili equation

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Abstract. We derive two non-equivalent coverings for the r-th mdKP equation from Maurer–Cartan forms of its symmetry pseudo-group. Also we find Bäcklund transformations between the covering equations.

AMS classification scheme numbers: 58H05, 58J70, 35A30
1. Introduction

The role of coverings in studying nonlinear differential equations (DEs) is well-known, [20, 21, 22, 23]. They lead to a number of useful techniques such as inverse scattering transformations, Bäcklund transformations, recursion operators, nonlocal symmetries and nonlocal conservation laws. For a given DE, a problem of constructing a covering is very difficult, see, e.g., [41, 8, 6, 14, 29, 30, 38, 15, 35, 36, 42, 40, 28, 12, 13]. One of the possible approaches to solution lies in the framework of Élie Cartan’s structure theory of Lie pseudo-groups, [27, 3, 33, 34].

In the present paper we apply the method of [33, 34] to the r-th modified dispersionless Kadomtsev–Petviashvili equation (r-mdKP), [1]. We use Élie Cartan’s method of equivalence, [4, 11, 17, 37], to compute Maurer–Cartan (MC) forms of the pseudo-group of contact symmetries of r-mdKP, and then find two linear combinations of these forms, whose horizontalizations provide covering equations of r-mdKP. Previously this approach was applied in [34] to a particular case of r-mdKP – modified dispersionless Kadomtsev–Petviashvili equation (mdKP), or modified Khokhlov–Zabolotskaya equation, [27, 25, 26]. Coverings for particular cases of r-mdKP were found in [10, 7, 18, 5] via other methods.

2. Preliminaries

2.1. Coverings of DEs

Let \( \pi_\infty : J^\infty(\pi) \to \mathbb{R}^n \) be the infinite jet bundle of local sections of the bundle \( \pi : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R} \). The coordinates on \( J^\infty(\pi) \) are \( (x^i, u_I) \), where \( I = (i_1, ..., i_k) \) are symmetric multi-indices, \( i_1, ..., i_k \in \{1, ..., n\} \), \( u_\emptyset = u \), and for any local section \( f \) of \( \pi \) there exists a section \( j_\infty(f) : \mathbb{R}^n \to J^\infty(\pi) \) such that \( u_I(j_\infty(f)) = \partial^{#I} f / \partial x^{i_1} ... \partial x^{i_k} \), \( #I = #(i_1, ..., i_k) = k \). The total derivatives on \( J^\infty(\pi) \) are defined in the local coordinates as

\[
D_i = \frac{\partial}{\partial x^i} + \sum_{#I \geq 0} u_I \frac{\partial}{\partial u_I}.
\]

We have \([D_i, D_j] = 0\) for \( i, j \in \{1, ..., n\} \). A DE \( F(x^i, u_K) = 0 \) defines a submanifold \( \mathcal{E}^\infty = \{ D_I(F) = 0 \mid #I \geq 0 \} \subset J^\infty(\pi) \), where \( D_I = D_{i_1} \circ ... \circ D_{i_k} \) for \( I = (i_1, ..., i_k) \). We denote restrictions of \( D_I \) on \( \mathcal{E}^\infty \) as \( \bar{D}_I \).

In local coordinates, a covering over \( \mathcal{E}^\infty \) is a bundle \( \bar{\mathcal{E}}^\infty = \mathcal{E}^\infty \times \Omega \to \mathcal{E}^\infty \) with fibre coordinates \( q^\alpha, \alpha \in \{1, ..., N\} \) or \( \alpha \in \mathbb{N} \), equipped with extended total derivatives

\[
\bar{D}_i = \bar{\mathcal{D}}_i + \sum_{\alpha} T_i^\alpha(x^j, u_I, q^\beta) \frac{\partial}{\partial q^\alpha}
\]

such that \([\bar{D}_i, \bar{D}_j] = 0\) whenever \( (x^i, u_I) \in \mathcal{E}^\infty \).

Dually, the covering is defined by the following differential 1-forms, [41],

\[
\omega^\alpha = dq^\alpha - T_i^\alpha(x^j, u_I, q^\beta) dx^i
\]
such that $d\omega^\alpha \equiv 0 \pmod{\omega^\beta, \vartheta_I}$ iff $(x^i, u_I) \in \mathcal{E}^\infty$, where $\vartheta_I$ are restrictions of the contact forms $\vartheta_I = du_I - u_{I,k} dx^k$ on $\mathcal{E}^\infty$.

2.2. Cartan’s structure theory of contact symmetry pseudo-groups of DEs

A pseudo-group on a manifold $M$ is a collection of local diffeomorphisms of $M$, which is closed under composition when defined, contains an identity and is closed under inverse. A Lie pseudo-group is a pseudo-group whose diffeomorphisms are local analytic solutions of an involutive system of partial differential equations. Élie Cartan’s approach to Lie pseudo-groups is based on a possibility to characterize transformations from a pseudo-group in terms of a set of invariant differential 1-forms called Maurer–Cartan forms.

The MC forms for a Lie pseudo-group can be computed by means of algebraic operations and differentiation. Expressions of differentials of the MC forms in terms of themselves give structure equations of the pseudo-group. The structure equations contain the full information about their pseudo-group.

EXAMPLE 1. Consider the bundle $J^2(\pi)$ of jets of the second order of the bundle $\pi$. A differential 1-form $\vartheta$ on $J^2(\pi)$ is called a contact form if it is annihilated by all 2-jets of local sections: $j_2(f)^*\vartheta = 0$. In the local coordinates every contact 1-form is a linear combination of the forms $\vartheta_0 = du - u_i dx^i$, $\vartheta_i = du_i - u_{ij} dx^j$, $i, j \in \{1, \ldots, n\}$, $u_{ij} = u_{ij}$.

A local diffeomorphism $\Delta : J^2(\pi) \to J^2(\pi)$, $\Delta : (x^i, u, u_{ij}) \mapsto (\bar{x}^i, \bar{u}, \bar{u}_{ij})$, is called a contact transformation if for every contact 1-form $\vartheta$ the form $\Delta^*\vartheta$ is also contact. We denote by $\text{Cont}(J^2(\pi))$ the pseudo-group of contact transformations on $J^2(\pi)$. Consider the following 1-forms

$$\Theta_0 = a \vartheta_0, \quad \Theta_i = g_i \Theta_0 + a B^k_i \vartheta_k, \quad \Xi^i = c^i \Theta_0 + f^{ik} \Theta_k + b^i_k d x^k,$$

$$\Theta_{ij} = a B^j_k B^i_l (du_{kl} - u_{kln} dx^n) + s_{ij} \Theta_0 + w^{k}_{ij} \Theta_k + z_{ijk} \Xi^k,$$

deﬁned on $J^2(\pi) \times \mathcal{H}$, where $\mathcal{H}$ is an open subset of $\mathbb{R}^{(2n+1)(n+3)(n+1)/3}$ with local coordinates $(a, b^i_k, c^i, f^{ik}, g_i, s_{ij}, u^{k}_{ij}, u^{jk}_{ij}, i, j, k \in \{1, \ldots, n\}$, $i \leq j$, such that $a \neq 0$, det$(b^i_k) \neq 0$, $f^{ik} = f^{ki}$, $u^{ijk} = u^{ikj} = u^{jik}$, while $(B^i_k)$ is the inverse matrix for the matrix $(b^i_k)$. As it is shown in [32], the forms (1) are MC forms for $\text{Cont}(J^2(\pi))$, that is, a local diffeomorphism $\hat{\Delta} : J^2(\pi) \times \mathcal{H} \to J^2(\pi) \times \mathcal{H}$ satisfies the conditions $\hat{\Delta}^* \overline{\Theta}_0 = \Theta_0$, $\hat{\Delta}^* \overline{\Theta}_i = \Theta_i$, $\hat{\Delta}^* \overline{\Xi} = \Xi$, and $\hat{\Delta}^* \overline{\Theta}_{ij} = \Theta_{ij}$ if and only if $\hat{\Delta}$ is projectable on $J^2(\pi)$, and its projection $\Delta : J^2(\pi) \to J^2(\pi)$ is a contact transformation. The structure equations for $\text{Cont}(J^2(\pi))$ have the form

$$d\Theta_0 = \Phi^0_0 \wedge \Theta_0 + \Xi^i \wedge \Theta_i,$$

$$d\Theta_i = \Phi^0_i \wedge \Theta_0 + \Phi^k_i \wedge \Theta_k + \Xi^k \wedge \Theta_k,$$

$$d\Xi^i = \Phi^0_i \wedge \Xi^i - \Phi^k_i \wedge \Xi^k + \Psi^0_0 \wedge \Theta_0 + \Psi^{ik} \wedge \Theta_k,$$

$$d\Theta_{ij} = \Phi^k_{ij} \wedge \Theta_{kj} + \Phi^k_j \wedge \Theta_{ki} - \Phi^0_{ij} \wedge \Theta_0 + \Gamma^0_{ij} \wedge \Theta_0 + \Gamma^0_{ij} \wedge \Theta_k + \Lambda_{ijk} \wedge \Xi^k,$$

where the additional forms $\Phi^0_0, \Phi^0_i, \Phi^k_i, \Psi^0_0, \Psi^{ij}, \Gamma^0_{ij}, \Gamma^k_{ij}$, and $\Lambda_{ijk}$ depend on differentials of the coordinates of $\mathcal{H}$.
EXEMPLARY 2. Suppose $E$ is a second-order differential equation in one dependent and $n$ independent variables. We consider $E$ as a submanifold in $J^2(\pi)$. Let $\text{Cont}(E)$ be the group of contact symmetries for $E$. It consists of all the contact transformations on $J^2(\pi)$ mapping $E$ to itself. Let $\iota_0 : E \to J^2(\pi)$ be an embedding, and $\iota = \iota_0 \times \text{id} : E \times \mathcal{H} \to J^2(\pi) \times \mathcal{H}$. The mc forms of $\text{Cont}(E)$ can be derived from the forms $\theta_0 = \iota^* \Theta_0$, $\theta_i = \iota^* \Theta_i$, $\xi^i = \iota^* \Xi^i$, and $\theta_{ij} = \iota^* \Theta_{ij}$ by means of Cartan’s method of equivalence, see details and examples in [9, 31, 32].

3. Cartan’s structure of the contact symmetry pseudo-group for r-mdKP

The r-th mdKP

$$u_{tx} = \frac{- (3 - r)(1 - r)}{2} u^2_{xx} + \frac{r(3 - r)}{2 - r} u_x u_{xy} + \frac{3 - r}{2 - r} u_{yy} + \frac{(3 - r)(1 - r)}{2 - r} u_y u_{xx},$$

$r \in \mathbb{Z} \setminus \{2\}$, was derived in [1]. For a convenience of computations we use the following change of variables:

$$\tilde{t} = (3 - r)t, \quad \tilde{x} = x, \quad \tilde{y} = (2 - r)y, \quad \tilde{u} = -(1 - r)u,$$

where $r \not\in \{1, 2, 3\}$. Then we have

$$\tilde{u}_{\tilde{y}\tilde{y}} = \tilde{u}_{\tilde{t}\tilde{x}} + \left( \frac{1}{2(1 - r)} \tilde{u}_x^2 + \tilde{u}_{\tilde{y}} \right) \tilde{u}_{\tilde{x}\tilde{x}} + \frac{r}{1 - r} \tilde{u}_{\tilde{z}\tilde{z}} \tilde{u}_{\tilde{x}\tilde{y}}.$$

We drop tildes and denote $\kappa = \frac{r}{1 - r}$; this yields

$$u_{yy} = u_{tx} + \left( \frac{\kappa + 1}{2} u_x^2 + u_y \right) u_{xx} + \kappa u_x u_{xy}. \quad (2)$$

The exceptional cases $r = 2$ and $r = 3$ correspond to $\kappa = -2$ and $\kappa = -\frac{3}{2}$, respectively. We will not consider the case of $r = 1$. The case of $\kappa = -1$ is exceptional, too, since $r \to \infty$ when $\kappa \to -1$. In the cases of $\kappa = 0$, $\kappa = 1$, and $\kappa = -1$ Eq. (2) gets the forms of the mdKP equation, [27, 25, 26],

$$u_{yy} = u_{tx} + \frac{1}{2} u_x^2 + u_y \quad u_{xx}, \quad (3)$$

the dBKP equation, [39, 18],

$$u_{yy} = u_{tx} + \left( u_x^2 + u_y \right) u_{xx} + u_x u_{xy}. \quad (4)$$

and the equation describing Lorentzian hyper-CR Einstein–Weil structures, [7, 10],

$$u_{yy} = u_{tx} + u_y u_{xx} - u_x u_{xy}. \quad (5)$$

We use the method outlined in the previous section to compute mc forms and structure equations of the pseudo-group of contact symmetries for Eq. (2). The results depend on $\kappa$. 

Symmetries and coverings for the r-th mdKP equation
When $\kappa \not\in \{-2, -1\}$, the structure equations have the form
\[
d\theta_0 = (\eta_1 + \xi^2 - \frac{\kappa^2 - 4}{8} \xi^3 - \frac{\kappa - 4}{\kappa + 2} \theta_{22}) \wedge \theta_0 + \xi^1 \wedge \theta_1 + \xi^2 \wedge \theta_2 + \xi^3 \wedge \theta_3,
\]
\[
d\theta_1 = \left(\frac{3}{2} \eta_1 - \frac{\kappa^2 - 4}{8} \xi^3 - \frac{3(\kappa + 1)}{\kappa + 2} \theta_{22}\right) \wedge \theta_1 + \left((\kappa + 1) \theta_2 + (\kappa + 2) \xi^2\right) \wedge \theta_3 + \xi^1 \wedge \theta_{11}
+ \left(\frac{2\kappa^2 + 15\kappa + 4}{8} \theta_2 - \frac{\kappa^2 - 10\kappa + 8}{8} \xi^2 + \kappa \theta_{23}\right) \wedge \theta_0 + \xi^2 \wedge \theta_{12} + \xi^3 \wedge \theta_{13},
\]
\[
d\theta_2 = \left(\frac{1}{2} \eta_1 - \frac{\kappa + 1}{\kappa + 2} \theta_{22} + \frac{3\kappa + 4}{8} \xi^3\right) \wedge \theta_2 + \xi^1 \wedge \theta_{12} + \xi^2 \wedge \theta_{22} + \xi^3 \wedge \theta_{23},
\]
\[
d\theta_3 = \left(\eta_1 - \frac{2(\kappa + 1)}{\kappa + 2} \theta_{22} + \frac{\kappa + 4}{8} \xi^3\right) \wedge \theta_3 + \left(\frac{\kappa - 4}{8} \theta_{22} - \frac{\kappa^2 - 16}{64} \xi^3\right) \wedge \theta_0 + \frac{\kappa + 2}{2} \xi^2 \wedge \theta_2
+ \xi^1 \wedge \theta_{13} + \xi^2 \wedge \theta_{23} + \xi^3 \wedge \theta_{12},
\]
\[
d\xi^1 = -\left(\frac{1}{2} \eta_1 - \frac{2\kappa + 3}{\kappa + 2} \theta_{22}\right) \wedge \xi^1,
\]
\[
d\xi^2 = \left(\frac{1}{2} \eta_1 + \frac{1}{\kappa + 2} \theta_{22} + \xi^3\right) \wedge \xi^2 + \left(\frac{\kappa - 4}{8} \theta_0 - \theta_3\right) \wedge \xi^1 - \theta_2 \wedge \xi^3,
\]
\[
d\xi^3 = -\left(\kappa + 2\right) \left(\theta_2 + \xi^2\right) \wedge \xi^1 + \theta_{22} \wedge \xi^3,
\]
\[
d\theta_{11} = 2\eta_1 \wedge \theta_{11} - \kappa \eta_2 \wedge \theta_0 + \eta_3 \wedge \xi^2 + \eta_4 \wedge \xi^3 + \eta_5 \wedge \xi_5 + \kappa \left(\theta_3 + \frac{\kappa + 8}{4} \theta_{12}\right) \wedge \theta_0
+ \left((4 \kappa + 3) \theta_{23} - \frac{\kappa^2 - 18 \kappa^2 + 20}{4} \xi^2\right) \wedge \theta_1 - \left((2 \kappa + 3) \theta_{13} + \frac{5\kappa^2 + 31 \kappa + 24}{4} \theta_1\right) \wedge \theta_2
- \kappa \theta_3 \wedge \theta_{12} - \left(\frac{5 \kappa + 6}{\kappa + 2} \theta_{22} + \frac{\kappa - 4}{8} \xi^3\right) \wedge \theta_{11} + (2 \kappa + 4) \xi^2 \wedge \theta_{13},
\]
\[
d\theta_{12} = \eta_1 \wedge \theta_{12} + \eta_2 \wedge \xi^3 + \eta_3 \wedge \xi^1 + \theta_{12} \wedge \left(\theta_{23} - \xi^2\right) + \left(\theta_3 - \frac{\kappa - 4}{8} \theta_0 + \frac{3 \kappa + 4}{\kappa + 2} \theta_{12}\right) \wedge \theta_{22}
+ \frac{3 \kappa + 4}{8} \xi^3 \wedge \theta_{12},
\]
\[
d\theta_{13} = \frac{3}{2} \eta_1 \wedge \theta_{13} + \eta_2 \wedge \xi^2 + \eta_3 \wedge \xi^3 + \eta_4 \wedge \xi^1 + \frac{3 + 10 \kappa^2 - 32 \kappa - 96}{64} \theta_0 \wedge \xi^2
- \left(\frac{\kappa^2 + 14 \kappa - 24}{16} \theta_2 + \frac{\kappa^2 - 3 \kappa - 4}{8} \theta_{23}\right) \wedge \theta_0 + \left(\frac{\kappa - 4}{8} \theta_{22} - \frac{\kappa^2 - 16}{64} \xi^3\right) \wedge \theta_1
- \frac{7 \kappa^2 + 32 \kappa + 24}{8} \theta_3 + (\kappa + 2) \theta_{12}\right) \wedge \theta_2 + \left((2 \kappa + 1) \theta_{23} + \frac{\kappa^2 + 22 \kappa + 24}{8} \xi^2\right) \wedge \theta_3
+ \frac{3(\kappa + 2)}{4} \xi^2 \wedge \theta_{12} - \left(\frac{4 \kappa + 5}{\kappa + 2} \theta_{22} - \frac{\kappa + 4}{8} \xi^3\right) \wedge \theta_{13},
\]
\[
d\theta_{22} = \left(\frac{3 \kappa^2 + 18 \kappa + 24}{8} \theta_2 + (\kappa + 2) \left(\theta_{23} + \xi^2\right)\right) \wedge \xi^1.
\]
\[
d\theta_{23} = \frac{1}{2} \eta_1 \wedge \theta_{23} + \eta_2 \wedge \xi^1 - \left(\frac{3 \kappa + 4}{8} \theta_{22} + \frac{9 \kappa^2 + 48 \kappa + 112}{64} \xi^3\right) \wedge \theta_2 + \left(\frac{3 \kappa + 4}{8} \theta_{23} - \xi^2\right) \wedge \xi^3
+ \left(\frac{2 \kappa + 3}{\kappa + 2} \theta_{23} - \frac{3 \kappa - 4}{8} \xi^2\right) \wedge \theta_{22},
\]
\[
d\eta_1 = 0,
\]
\[
d\eta_2 = \eta_6 \wedge \xi^1 + \left(\eta_1 + \frac{2(\kappa + 3)}{\kappa + 2} \theta_{22} - \frac{3(\kappa + 4)}{8} \xi^3\right) \wedge \eta_2 + \theta_3 + \left(\frac{3(\kappa - 4)}{8} \theta_{22} - \xi^3\right)
+ \left(\frac{3 \kappa^2 - 16 \kappa + 16}{64} \theta_{22} - \frac{\kappa^2 - 4}{8} \xi^3\right) \wedge \theta_0 + \left(\frac{3(\kappa + 4)}{8} \theta_2 + \xi^2\right) \wedge \theta_{23}
+ \left(\frac{3(\kappa + 4)}{8} \theta_{22} + \frac{9 \kappa^2 + 48 \kappa + 112}{64} \xi^3\right) \wedge \theta_{12},
\]
\[
d\eta_3 = \eta_6 \wedge \xi^3 + \eta_7 \wedge \xi^1 + (\kappa + 3) \theta_2 + (\kappa + 2) \xi^2\right) \wedge \eta_2 + \left(\frac{\kappa - 4}{8} \theta_1 - \theta_{13}\right) \wedge \theta_{22}
+ \left(\frac{3}{2} \eta_1 + \frac{5 \kappa + 7}{\kappa + 2} \theta_{22} + \frac{3 \kappa + 4}{8} \xi^3\right) \wedge \eta_3 + (\kappa + 2) \theta_3 \wedge (\theta_{23} + \xi^2)
+ \frac{\kappa - 4}{64} \left((3 \kappa^2 + 18 \kappa + 32) \theta_2 + 8 (\kappa + 2) (\theta_{23} + \xi^2)\right) \wedge \theta_0
+ \left(\frac{3 \kappa^2 + 18 \kappa + 32}{8} \theta_3 + \frac{6 \kappa^2 + 29 \kappa + 28}{8} \theta_{12}\right) \wedge \theta_2 - \left(3 (\kappa + 1) \theta_{23} - \frac{3 \kappa^2 + 31 \kappa + 32}{8} \xi^2\right) \wedge \theta_{12},
\]
\[
d\eta_4 = \eta_6 \wedge \xi^2 + \eta_7 \wedge \xi^3 + \eta_8 \wedge \xi^1 + \left(2 \eta_1 - \frac{2(3 \kappa + 4)}{\kappa + 2} \theta_{22} + \frac{\kappa + 4}{8} \xi^3\right) \wedge \eta_4
and coverings for the r-th mKdP equation

\[ \begin{align*}
+ \kappa \left( \frac{5}{8} \theta_0 - 2 \theta_3 \right) \wedge \eta_2 + \left( (2 \kappa + 4) \theta_2 + \frac{5(\kappa+2)}{2} \xi^2 \right) \wedge \eta_3 \\
- \frac{\kappa}{16} \left( 2 (\kappa - 4) \theta_3 - (\kappa^2 - 2 \kappa - 4) \theta_{12} \right) \wedge \theta_0 + \frac{\kappa}{8} (7 \kappa + 5) \theta_3 \wedge \theta_{12} \\
+ \frac{1}{16} \left( (2 \kappa^3 + 3 \kappa^2 - 30 \kappa - 112) \theta_2 - 2 (2 \kappa^2 - 5 \kappa + 4) \theta_{23} \right) \wedge \theta_1 \\
- \frac{1}{32} (3 \kappa^3 + 10 \kappa^2 - 32 \kappa - 96) \theta_1 \wedge \xi^2 \\
- (6 (\kappa + 1) \theta_{23} + \frac{1}{4} (\kappa^2 + 30 \kappa + 36) \xi^2) \wedge \theta_{13} \\
- \frac{1}{9} (20 \kappa^2 + 101 \kappa + 92) \theta_2 \wedge \theta_{13} - \frac{1}{144} (8 (\kappa - 4) \theta_{22} - (\kappa^2 - 16) \xi^3) \wedge \theta_{11},
\end{align*} \]

where

\[ \begin{align*}
\xi^1 &= q^{-1} dt, \\
\xi^2 &= q u^2_{xx} \left( \left( \frac{\kappa+1}{2} u^2_x - u_y \right) dt + dx - u_x dy \right), \\
\xi^3 &= u_{xx} \left( dy - (\kappa + 2) u_x dx \right), \\
\eta_1 &= 2 \frac{dq}{q} + \frac{2 (2 \kappa + 3)}{\kappa + 2} \frac{du_{xx}}{u_{xx}}, \quad (6)
\end{align*} \]

and \( q = B^1 \neq 0 \). We need not explicit expressions for the other \( \mathcal{M}_\mathcal{C} \) forms in the sequel.

In the case of \( \kappa = -1 \) the contact symmetry pseudo-group of Eq. (5) has the following structure equations

\[ \begin{align*}
d\theta_0 &= (\eta_1 + \frac{3}{8} \xi^3 + \theta_{22}) \wedge \theta_0 + \xi^1 \wedge \theta_1 + \xi^2 \wedge \theta_2 + \xi^3 \wedge \theta_3, \\
d\theta_1 &= \frac{3}{2} \eta_1 \wedge \theta_1 + \eta_2 \wedge \left( \frac{1}{2} \theta_0 + \theta_3 \right) - \left( \frac{9}{8} \theta_2 + \theta_{23} - \frac{1}{2} \xi^2 \right) \wedge \theta_0 - \frac{5}{8} \theta_1 \wedge \xi^3 + \xi^1 \wedge \theta_{11} \\
&+ \xi^2 \wedge \theta_{12} + \xi^3 \wedge \theta_{13}, \\
d\theta_2 &= \frac{1}{8} \left( 4 \eta_1 + \frac{7}{8} \xi^3 \right) \wedge \theta_2 + \xi^1 \wedge \theta_{12} + \xi^2 \wedge \theta_{22} + \xi^3 \wedge \theta_{23}, \\
d\theta_3 &= \eta_1 \wedge \theta_3 + \frac{1}{12} \theta_2 \wedge \theta_2 - \frac{5}{8} \left( 8 \theta_{22} + 3 \xi^3 \right) \wedge \theta_0 + \frac{3}{8} \xi^2 \wedge \theta_3 + \xi^1 \wedge \theta_{13} + \xi^2 \wedge \theta_{23} \\
&+ \xi^3 \wedge \theta_{12}, \\
d\xi^1 &= - \frac{1}{2} \left( \eta_1 - 2 \theta_{22} \right) \wedge \xi^1, \\
d\xi^2 &= - \frac{1}{8} \left( 5 \theta_0 + 8 \theta_3 \right) \wedge \xi^1 + \frac{1}{2} \left( \eta_1 + 2 \theta_{22} + \xi^3 \right) \wedge \xi^2 - \frac{1}{2} \left( \eta_2 + \theta_2 \right) \wedge \xi^3, \\
d\xi^3 &= - \left( \eta_2 + \theta_2 \right) \wedge \xi^1 + \theta_{22} \wedge \xi^3, \\
d\theta_{11} &= 2 \eta_1 \wedge \theta_{11} + \eta_2 \wedge \left( \frac{3}{8} \theta_1 + 2 \theta_{13} \right) + \eta_4 \wedge \xi^2 + \eta_6 \wedge \xi^3 + \eta_6 \wedge \xi^1 - \left( \theta_{22} - \frac{5}{8} \xi^3 \right) \wedge \theta_{11} \\
&+ \left( \theta_3 - 3 \theta_3 - \frac{7}{4} \theta_{12} \right) \wedge \theta_0 - \left( \frac{1}{4} \theta_2 - \theta_{23} - \frac{1}{2} \xi^2 \right) \wedge \theta_1 + \theta_2 \wedge \theta_{13} + \theta_3 \wedge \theta_{12}, \\
d\theta_{12} &= \eta_1 \wedge \theta_{12} + \eta_2 \wedge \left( \frac{3}{8} \theta_2 + \theta_{23} + \xi^3 \right) + \eta_3 \wedge \xi^1 + \eta_4 \wedge \xi^1 + \frac{5}{8} \theta_0 \wedge \theta_{22} - \xi^2 \wedge \theta_{23} \\
&- \left( \theta_{23} - \frac{9}{8} \xi^2 \right) \wedge \theta_2 + \theta_{32} + \frac{1}{8} \xi^3 \wedge \theta_{12}, \\
d\theta_{13} &= \frac{3}{2} \eta_1 \wedge \theta_{13} + \frac{1}{12} \theta_2 \wedge (35 \theta_0 + 24 \theta_3 + 96 \theta_{12}) + \eta_3 \wedge \xi^2 + \eta_4 \wedge \xi^3 + \eta_5 \wedge \xi^1
\end{align*} \]
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\[ + \frac{5}{16} \left( 2 \theta_2 + \xi^2 \right) \wedge \theta_0 - \frac{5}{64} \left( 8 \theta_{22} - 3 \xi^3 \right) \wedge \theta_1 + \left( \frac{1}{8} \theta_3 + \theta_{12} \right) \wedge \theta_2 + \theta_3 \wedge \theta_{23} - \left( \theta_{22} - \frac{3}{8} \xi^3 \right) \wedge \theta_{13}, \]

\[ d \theta_{22} = \frac{1}{8} \left( 4 \eta_2 + 9 \theta_2 + 4 \xi^2 + 8 \theta_{23} \right) \wedge \xi^1, \]

\[ d \theta_{23} = \frac{1}{8} \eta_1 \wedge \theta_{23} + \frac{1}{2} \eta_2 \wedge \left( \theta_{22} + \xi^3 \right) + \eta_3 \wedge \xi^1 + \frac{1}{64} \theta_2 \wedge \left( 72 \theta_{22} - 73 \xi^3 \right) + \left( \theta_{23} + \frac{3}{8} \xi^3 \right) \wedge \theta_{22} + \frac{1}{8} \left( 9 \theta_{23} + 4 \xi^2 \right) \wedge \xi^3, \]

\[ d \eta_1 = 0, \]

\[ d \eta_2 = \frac{1}{2} \left( \eta_1 + \xi^3 \right) \wedge \eta_2 - \frac{5}{8} \theta_0 \wedge \xi^1 - \left( \theta_2 + \frac{1}{2} \xi^2 \right) \wedge \xi^3 - \theta_3 \wedge \xi^1 - \theta_{22} \wedge \xi^2, \]

\[ d \eta_3 = \eta_2 \wedge \xi^1 + \eta_1 \wedge \eta_3 + \frac{1}{64} \eta_2 \wedge \left( 41 \theta_2 + 72 \theta_{23} + 36 \xi^3 \right) + \frac{1}{8} \eta_3 \wedge \left( 16 \theta_{22} + 9 \xi^3 \right) + \frac{5}{64} \left( 7 \theta_{22} + 8 \xi^3 \right) \wedge \theta_0 - \frac{1}{64} \left( 72 \theta_{23} + 41 \xi^2 \right) \wedge \theta_2 - \left( \frac{1}{8} \theta_{22} + \xi^3 \right) \wedge \theta_3 + \frac{1}{64} \left( 72 \theta_{22} + 73 \xi^3 \right) \wedge \theta_{12} + \frac{1}{8} \theta_{23} \wedge \xi^2, \]

\[ d \eta_4 = \eta_2 \wedge \xi^3 + \eta_8 \wedge \xi^1 + \frac{1}{8} \left( 5 \theta_0 + 8 \theta_3 + 6 \theta_{12} \right) \wedge \eta_2 + \left( 2 \eta_2 + 2 \theta_2 - \xi^2 \right) \wedge \eta_3 + \frac{1}{8} \left( 12 \eta_1 - 16 \theta_{22} + \xi^3 \right) \wedge \eta_4 + \frac{5}{64} \left( 17 \theta_2 + 8 \theta_{23} \right) - \frac{5}{8} \theta_1 \wedge \theta_{22} + \frac{1}{8} \left( 17 \theta_3 + 10 \theta_{12} \right) \wedge \theta_2 + \theta_3 \wedge \theta_{23} - \frac{5}{8} \theta_{12} \wedge \xi^2 - \theta_{13} \wedge \theta_{22}, \]

\[ d \eta_5 = \eta_2 \wedge \xi^2 + \eta_8 \wedge \xi^3 + \eta_9 \wedge \xi^1 - \frac{5}{32} \eta_2 \wedge \left( 7 \theta_1 + 8 \theta_{13} \right) + \left( \frac{5}{8} \theta_0 + 2 \theta_3 \right) \wedge \eta_3 + \left( \frac{5}{8} \eta_2 + 2 \theta_2 \right) \wedge \eta_4 + \left( 2 \eta_1 - 2 \theta_{22} + \frac{3}{8} \xi^3 \right) \wedge \eta_5 + \frac{13}{64} \left( \theta_{12} - 2 \theta_3 \right) \wedge \theta_0 - \frac{5}{16} \left( 5 \theta_2 + 2 \theta_{23} + 2 \xi^2 \right) \wedge \theta_1 + \frac{13}{8} \eta_{12} \wedge \theta_3 - \frac{5}{64} \left( 8 \theta_{22} + 3 \xi^3 \right) \wedge \theta_{11} - \frac{1}{8} \left( 11 \theta_2 + 4 \xi^2 \right) \wedge \theta_{13}, \]

\[ d \eta_6 = \eta_2 \wedge \theta_0 + \eta_8 \wedge \xi^2 + \eta_9 \wedge \xi^3 + \eta_{10} \wedge \xi^1 + \eta_2 \wedge \left( \eta_5 - \frac{15}{8} \theta_1 \right) - 2 \eta_3 \wedge \theta_1 - \eta_4 \wedge \left( \frac{10}{8} \theta_0 + 2 \theta_3 \right) - 2 \eta_5 \wedge \theta_2 - \frac{1}{8} \eta_6 \wedge \left( 20 \eta_1 + 16 \theta_{22} - 5 \xi^3 \right) + \frac{1}{32} \left( 5 \theta_1 + 72 \theta_{13} \right) \wedge \theta_0 + \frac{3}{4} \left( \theta_3 + 3 \theta_{12} \right) \wedge \theta_1 + \frac{5}{4} \theta_{11} \wedge \theta_2 + 2 \theta_{13} \wedge \theta_3 \]

with

\[ \xi^1 = q^{-1} dt, \]

\[ \xi^2 = q u_{xx}^2 \left( (s^2 u_{xx} - s u_x u_{xx} - u_y) \ dt + dx - s u_{xx} \ dy \right), \]

\[ \xi^3 = u_{xx} \left( (u_x - 2 s u_{xx}) \ dt + dy \right), \quad (7) \]

\[ \eta_1 = 2 \frac{dq}{q} + 2 \frac{du_{xx}}{u_{xx}}, \]

where \( s = B_3^2 \in \mathbb{R} \).

4. Coverings of \( r \)-mdKP

Following [33, 34], we find linear combinations of the MC forms (6) and (7), which provide coverings of Eq. (2) in the cases of \( \kappa \notin \{-2, -\frac{3}{2}, -1\} \), \( \kappa = -\frac{3}{2} \), and \( \kappa = -1 \), respectively.

4.1. General case

When \( \kappa \notin \{-2, -\frac{3}{2}, -1\} \), we take the following linear combination of the MC forms (6)

\[ \omega = \eta_1 - \lambda_1 \xi^1 - \lambda_2 \xi^2 - \lambda_3 \xi^3 \]
with \( \lambda_1, \lambda_2, \lambda_3 \in \mathbb{R} \), and put

\[
q = -\frac{(\kappa + 1) \lambda}{(\kappa + 2) v_1^{2+2}}, \quad u_{xx} = \frac{\nu_1^{\kappa+2}}{v},
\]

where \( v \) and \( v_1 \) are new independent variables. This gives

\[
\omega = -\frac{2(\kappa + 1)}{(\kappa + 2) v} (dv - A \, v_1 \, dt - v_1 \, dx - B \, v_1 \, dy)
\]

with

\[
A = -\frac{(\kappa + 2)^2}{4(\kappa + 1)} (\lambda_1 \lambda_2 v_1^{2+2(\kappa+1)} + 2 \lambda_3 (\kappa + 1) u_x v_1^{\kappa+1}) + \frac{\kappa + 1}{2} u_x^2 - 2 u_y,
\]

\[
B = -u_x - \frac{\lambda_3 (\kappa + 2)}{2(\kappa + 1)} v_1^{\kappa+1}.
\]

The form (8) is equal to zero whenever \( v_1 = v_x \) and

\[
v_t = \left( \frac{\lambda_1 \lambda_2 (\kappa + 2)^2}{4(\kappa + 1)^2} v_x^{2(\kappa+1)} + \frac{\lambda_3 (\kappa + 2)^2}{2(\kappa + 1)} v_x^{\kappa+1} + \frac{\kappa + 1}{2} u_x^2 - u_y \right) v_x,
\]

\[
v_y = -\left( \frac{\lambda_3 (\kappa + 2)}{\kappa + 1} v_x^{\kappa+1} + u_x \right) v_x.
\]

This system is compatible, i.e., \((v_t)_y = (v_y)_t\), whenever

\[
u_{yy} - u_{tx} = \left( \frac{\kappa + 1}{2} u_x^2 + u_y \right) u_{xx} - \kappa u_x u_{xy} + \frac{(\kappa + 2)^2}{4(\kappa + 1)^2} v_x^{2(\kappa+1)} (\lambda_1^2 (\kappa + 2)^2 - \lambda_1 \lambda_2 (2 \kappa + 3)) = 0.
\]

This equation coincides with Eq. (2) iff \( \lambda_3^2 (\kappa + 2)^2 - \lambda_1 \lambda_2 (2 \kappa + 3) = 0 \). So we put

\[
\lambda_1 = \frac{\lambda_3^2 (\kappa + 2)^2}{\lambda_2 (2 \kappa + 3)}.
\]

This yields

\[
v_t = \left( \frac{\lambda_3^2 (\kappa + 2)^4}{(2 \kappa + 3)(\kappa + 1)^2} v_x^{2(\kappa+1)} + \frac{\lambda_3 (\kappa + 2)^2}{2(\kappa + 1)} u_x v_x^{\kappa+1} + \frac{\kappa + 1}{2} u_x^2 - u_y \right) v_x,
\]

\[
v_y = -\left( \frac{\lambda_3 (\kappa + 2)}{\kappa + 1} v_x^{\kappa+1} + u_x \right) v_x.
\]

When \( \lambda_3 = 0 \), we have

\[
v_t = \left( \frac{\kappa + 1}{2} u_x^2 - u_y \right) v_x, \quad v_y = -u_x v_x.
\]

In the case of \( \kappa = 0 \) this covering for Eq. (3) was obtained in [34]. When \( \lambda_3 \neq 0 \), we put

\[
v = \left( \frac{\lambda_3 (\kappa + 2)}{2(\kappa + 1)} \right)^{1/(\kappa+1)} w.
\]

Then

\[
w_t = \left( \frac{(\kappa + 2)^2}{2 \kappa + 3} w_x^{2(\kappa+1)} + (\kappa + 2) u_x w_x^{\kappa+1} + \frac{\kappa + 1}{2} u_x^2 - u_y \right) w_x,
\]

\[
w_y = -\left( w_x^{\kappa+1} + u_x \right) w_x.
\]
Symmetries and coverings for the r-th mdKP equation

For $\kappa = 0$ this covering of (3) was found in [5] by means of another technique and in [34] via the method described above. For $\kappa = 1$ the covering (10) of Eq. (4) was obtained in [18].

From (9) we have

$$u_x = -\frac{v_y}{v_x}, \quad u_y = \frac{\kappa + 1}{2} \left( \frac{v_y}{v_x} \right)^2 - \frac{v_t}{v_x}. \quad (11)$$

The integrability condition $(u_x)_y = (u_y)_x$ of this system gives

$$v_{yy} = v_{tx} + \left( \frac{(\kappa + 1) v_y^2}{v_x^2} - \frac{v_t}{v_x} \right) v_{xx} - \frac{\kappa v_y}{v_x} v_{xy}. \quad (12)$$

For $\kappa = 0$ this equation was obtained in [2]. Also, from (10) we get

$$u_x = -\frac{w_y}{w_x} - w_x^{\kappa + 1}, \quad u_y = -\frac{w_t}{w_x} + \frac{(\kappa + 1) w_y^2}{2w_x^2} - w_x^\kappa w_y - \frac{(\kappa + 1) w_x^{2(\kappa + 1)}}{2(2\kappa + 3)} \quad (13)$$

This system yields

$$w_{yy} = w_{tx} + \left( \frac{(\kappa + 1) w_y^2}{w_x^2} - \frac{w_t}{w_x} + \kappa w_x^\kappa w_y + \frac{(\kappa + 1)^2}{2\kappa + 3} w_x^{2(\kappa + 1)} \right) w_{xx}$$

$$- \kappa \left( \frac{w_y}{w_x} + w_x^\kappa \right) w_{xy}. \quad (14)$$

Substitution for (11) in (10) gives a Bäcklund transformation

$$w_t = \frac{(\kappa + 2)^2}{2\kappa + 3} w_x^{\kappa + 3} - \frac{(\kappa + 2) v_y}{v_x} w_x^{\kappa + 2} + \frac{v_t}{v_x} w_x, \quad w_y = -w_x^{\kappa + 2} + \frac{v_y}{v_x} w_x$$

from Eq. (12) to Eq. (14). The inverse Bäcklund transformation appears from substitution for (13) in (9).

4.2. Case of $\kappa = -\frac{3}{2}$

In the case of $\kappa = -\frac{3}{2}$ we take the following combination of the MC forms (6)

$$\omega = \eta_1 - \lambda_1 \xi^1 - 4 \xi^2 = 2 \frac{dq}{q} + \left( q u_x^2 (u_x^2 + 4 u_y) - \lambda_1 q^{-1} \right) dt - 4 q u_{xx} (dx - u_x dy)$$

and the following change of variables: $q = -v^{-2}, \quad u_{xx} = (v v_1)^{1/2}$. Then we have

$$\omega = -4 \frac{dv}{v} - \left( \frac{u_x^2 + 4 u_y}{v} \right) v_1 - \lambda_1 v^2 \right) dt + \frac{4 v_1}{v} dx - \frac{4 u_x v_1}{v} dy.$$}

This form is equal to zero whenever $v_1 = v_x$ and

$$v_t = \frac{1}{4} \lambda_1 v^3 - \left( \frac{1}{4} u_x^2 + u_y \right) v_x, \quad v_y = -u_x v_x.$$}

This system is compatible for every value of $\lambda_1$ whenever Eq. (2) with $\kappa = -\frac{3}{2}$ is satisfied. When $\lambda_1 = 0$, we have Eqs. (9) with $\kappa = -\frac{3}{2}$:

$$v_t = -\left( \frac{1}{4} u_x^2 + u_y \right) v_x, \quad v_y = -u_x v_x. \quad (15)$$

When $\lambda_1 \neq 0$, we put $v = 2 \lambda_1^{-1/2} w$. Then we get

$$w_t = w^3 - \left( \frac{1}{4} u_x^2 + u_y \right) w_x, \quad w_y = -u_x w_x. \quad (16)$$
Symmetries and coverings for the r-th mdKP equation

Exclusion of \( u_x \) and \( u_y \) from Eqs. (15) and (16) gives equations

\[
\begin{align*}
    v_{yy} &= v_{tx} - \left( \frac{v_y^2}{2v_x^2} + \frac{v_t}{v_x} \right) v_{xx} + 3 \frac{v_y}{v_x} v_{xy}, \\
    w_{yy} &= w_{tx} - \left( \frac{w_y^2}{2w_x^2} + \frac{w_t - w^3}{w_x} \right) w_{xx} + 3 \frac{w_y}{w_x} w_{xy} - 3 w^2 w_x,
\end{align*}
\]

(17) (18)

and a Bäcklund transformation from (17) to (18):

\[
\begin{align*}
    w_t &= w^3 + \frac{v_t}{v_x} w_x, \quad w_y = \frac{v_y}{v_x} w_x.
\end{align*}
\]

4.3. Case of \( \kappa = -1 \)

When \( \kappa = -1 \), we take the following combination of the MC forms (7):

\[
\omega = \eta_1 + \frac{1}{2} \lambda^2 \xi^4 + 2 \xi^2 - \lambda^3 \xi^3 = 2 q u_x^2 dx - u_{xx} \left( 2 q s u_{xx} + \lambda^3 \right) dy + 2 \frac{dq}{q} + 2 \frac{du_{xx}}{u_{xx}} + \frac{1}{2q} \left( (\lambda^3 + q u_{xx}(2s u_{xx} - u_x)^2 - q^2 u_{xx}^2 (u_x^2 + 4u_y)) \right) dt.
\]

Then we substitute for \( q = (v v_1)^{-1} \), \( s = u_x v_1^{-1} \), \( u_{xx} = v_1 \) and obtain

\[
\omega = -\frac{1}{2v} \left( 4 dv - \left( \lambda^3 v^2 + 2 \lambda^3 u_x v - 4 u_y \right) v_1 dt - 4 v_1 dx + 2 \left( \lambda^3 v + 2 u_x \right) v_1 dy \right).
\]

This form is equal to zero whenever \( v_1 = v_x \) and

\[
\begin{align*}
    v_t &= \left( \frac{\lambda^2}{4} v^2 + \frac{1}{2} u_x v - u_y \right) v_x, \quad v_y = -\left( \frac{1}{2} \lambda^3 v + u_x \right) v_x.
\end{align*}
\]

This system is compatible for every value of \( \lambda^3 \) whenever Eq. (5) is satisfied. For \( \lambda^3 = 0 \) we have

\[
\begin{align*}
    v_t &= -u_y v_x, \quad v_y = -u_x v_x.
\end{align*}
\]

(19)

When \( \lambda^3 \neq 0 \), we put \( v = 2 \lambda^{-1} \). Then we have

\[
\begin{align*}
    w_t &= (w^2 + u_x w - u_y) w_x, \quad w_y = -(w + u_x) w_x.
\end{align*}
\]

(20)

Excluding \( u_x \) and \( u_y \) from systems (19) and (20), we get equations

\[
\begin{align*}
    v_{yy} &= v_{tx} - \frac{v_t}{v_x} v_{xx} + \frac{v_y}{v_x} v_{xy}, \\
    w_{yy} &= w_{tx} - \frac{w_t + w w_x}{w_x} w_{xx} + \frac{w_y + w w_x}{w_x} w_{xy},
\end{align*}
\]

(21) (22)

correspondingly. The Bäcklund transformation between Eqs. (21) and (22) has the form

\[
\begin{align*}
    w_t &= \left( w^2 + \frac{v_t - w v_y}{v_x} \right) w_x, \quad w_y = \left( w - \frac{v_y}{v_x} \right) w_x.
\end{align*}
\]

REMARK 1. A one-parametric family of coverings with a nonremovable parameter

\[
\begin{align*}
    v_t &= -(u_y - \lambda u_x - \lambda^2) v_x, \quad v_y = -(u_x + \lambda) v_x,
\end{align*}
\]

(23)
for Eq. (5) is presented in [7]. This family can be obtained from (19) by means the following technique, [22, § 3.6] [19, 17, 30, 16]. Eq. (5) has the infinitesimal symmetry

$$X = y \frac{\partial}{\partial x} + 2x \frac{\partial}{\partial u},$$

which can’t be lifted into a symmtery of the covering (19). Then the deformation $e^{\lambda X}$ transforms the covering (19) into (23). Indeed, we have

$$\tilde{t} = e^{\lambda X}(t) = t, \quad \tilde{x} = e^{\lambda X}(x) = x + \lambda y, \quad \tilde{y} = e^{\lambda X}(y) = y,$$

and therefore

$$\tilde{u}_t = e^{\lambda X}(u_t) = u_t, \quad \tilde{u}_x = e^{\lambda X}(u_x) = u_x + 2\lambda, \quad \tilde{u}_y = e^{\lambda X}(u_y) = u_y - \lambda u_x - \lambda^2.$$

Since $\tilde{v} = e^{\lambda X}(v) = v$, for the form

$$\tilde{\omega}_1 = dv + (u_y - \lambda u_x - \lambda^2) v_x dt - v_x dx + (u_x + \lambda) v_x dy,$$

this form defines the family of coverings (23).

Similarly, we derive a new one-parametric family of coverings with a nonremovable parameter from the covering (20). We have $\tilde{w} = e^{\lambda X}(w) = w$, so the form

$$\tilde{\omega}_2 = dw - (w^2 + u_x w - u_y) w_x d\tilde{t} - w_x d\tilde{x} + (w + u_x) w_x d\tilde{y},$$

which defines the covering (20) in the tilded variables, provides

$$\left(e^{\lambda X}\right)^* \tilde{\omega}_2 = dw - (w^2 + (u_x + 2\lambda) w - u_y + \lambda u_x + \lambda^2) w_x dt - w_x dx + (w + u_x + \lambda) w_x dy.$$

This form defines a family of coverings

$$w_t = (w^2 + (u_x + 2\lambda) w - u_y + \lambda u_x + \lambda^2) w_x, \quad w_y = -(w + u_x + \lambda) w_x.$$

REMARK 2. In the case of $\kappa = -2$ Eqs. (9) define a covering for Eq. (2), too, while we can’t obtain this result by the method described above.

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