L(2,1)-LABELING OF THE CARTESIAN AND STRONG PRODUCT OF TWO DIRECTED CYCLES

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Abstract. The frequency assignment problem (FAP) is the assignment of frequencies to television and radio transmitters subject to restrictions imposed by the distance between transmitters. One of the graph theoretical models of FAP which is well elaborated is the concept of distance constrained labeling of graphs. Let \( G = (V,E) \) be a graph. For two vertices \( u \) and \( v \) of \( G \), we denote \( d(u,v) \) the distance between \( u \) and \( v \). An \( L(2,1) \)-labeling for \( G \) is a function \( f : V \to \{0,1,\ldots\} \) such that \( |f(u) − f(v)| \geq 1 \) if \( d(u,v) = 2 \) and \( |f(u) − f(v)| \geq 2 \) if \( d(u,v) = 1 \). The span of \( f \) is the difference between the largest and the smallest number of \( f(V) \). The \( \lambda \)-number for \( G \), denoted by \( \lambda(G) \), is the minimum span over all \( L(2,1) \)-labelings of \( G \). In this paper, we study the \( \lambda \)-number of the Cartesian and strong product of two directed cycles. We show that for \( m,n \geq 4 \) the \( \lambda \)-number of \( C_m \circ C_n \) is between 4 and 5. We also establish the \( \lambda \)-number of \( C_m \boxtimes C_n \) for \( m \leq 10 \) and prove that the \( \lambda \)-number of the strong product of cycles \( C_m \boxtimes C_n \) is between 6 and 8 for \( m,n \geq 48 \).

1. Introduction. The Frequency Assignment Problem, which is the assignment of frequencies to television and radio transmitters subject to restrictions imposed by the distance between transmitters, asks for assigning frequencies to transmitters in a broadcasting network with the aim of avoiding undesired interference. This problem was first formulated as a graph coloring problem by Hale [15] under the name \( T \)-coloring. Later, a variation of the channel assignment problem was proposed in which “close” transmitters must receive different channels and “very close” transmitters must receive channels at least two apart. The problem was modeled by one...
of the graph theoretical models which is very well elaborated and known under the notion of distance constrained labeling of graphs [21]. Motivated by this problem, Griggs and Yeh [14] studied the $L(2,1)$-labeling problem with a condition at distance two. For a comprehensive survey on the models on the frequency assignment $L$ has attracted a lot of interest [5,6,27]. Along the study of $L$ practical importance, the requirements the $L$-number of $G$, denoted by $\lambda(G)$, is the minimum value $\lambda$ such that $G$ admits an $L(2,1)$-labeling [17].

Griggs and Yeh [14] studied the $L(2,1)$-labeling problem with a condition at distance two. They put forward a conjecture that $\lambda(G) \leq \Delta^2$ for any graph with $\Delta \geq 2$, where $\Delta$ is the maximum degree of $G$, and they also proved that $\lambda(G) \leq \Delta^2+2\Delta$. Later, it was presented that $\lambda(G) \leq \Delta^2+\Delta$ by Chang and Kuo [7] in 1996, $\lambda(G) \leq \Delta^2+\Delta-1$ by Kráľ and Škrekovski [20] in 2003, and then $\lambda(G) \leq \Delta^2+\Delta-2$ by Goncalves [13] in 2008, however, the conjecture is still open.

There are also a number of studies on the algorithms for $L(2,1)$-labeling problem [2,11,21]. It is known to be NP-hard for general graphs [14], and even many special graphs such as planar graphs, bipartite graphs, chordal graphs [2] and graphs of treewidth two [12] are also NP-hard. Until now, only a few graph classes such as paths, cycles, and wheels are known to have polynomial time algorithms for this problem [2].

For undirected graphs $G = (V,E)$ and $H = (W,F)$, the strong product $G \boxtimes H$ and the Cartesian product $G \square H$ of $G$ and $H$ are defined as follows: $V(G \boxtimes H) = V(G) \times W$, $E(G \boxtimes H) = \{(a,x),(b,y)\}: \{a,b\} \in E$ and $x = y$, or $\{x,y\} \in F$ and $a = b$, and $E(G \square H) = E(G) \cup \{(a,x),(b,y)\}: \{a,b\} \in E$ and $\{x,y\} \in F$. The subgraph of $G \square H$ or $G \boxtimes H$ induced by $\{u\} \times V(H)$ is isomorphic to $H$. It is called an $H$-fiber and denoted $H^u$. 

In this paper, only directed and undirected graphs without multiple edges or loops are considered. For a graph $G = (V,E)$, $V(G)$ and $E(G)$ are the sets of vertices and edges of $G$, respectively. A directed graph $D$ consists of vertices $V(D)$ together with a set of arcs $A(D)$.

The path $P_n$ is the graph whose vertices are $0,1,\ldots,n-1$ and for which two vertices are adjacent precisely if their difference is $\pm 1$. For an integer $n \geq 3$, the cycle of length $n$ is the graph $C_n$ whose vertices are $0,1,\ldots,n-1$ and whose edges are the pairs $i, i+1$, where the arithmetic is done modulo $n$. A walk in a directed graph $D$ is a sequence of (not necessarily distinct) vertices $v_1,v_2,\ldots,v_n$ such that $v_tv_{t+1} \in A(D)$ for $1,2,\ldots,n-1$. If $v_1 = v_n$ we say it is a closed walk. If $P$ is a path (resp. walk), then its length is its number of edges (resp. arcs). Sometimes, the terms directed path and directed cycle are considered which are used in the directed case. We denote by $P_n^\geq$ and $C_n^\geq$ the directed path and directed cycle with $n$ vertices, respectively.

Let $G = (V,E)$ be a graph. The distance $d(u,v)$ between vertices $u$ and $v$ of $G$ is the length of a shortest path between $u$ and $v$ in $G$.

An $L(p,q)$-labeling of a graph $G$ is an assignment of labels $f$ from $\{0,1,\ldots,\lambda\}$ to the vertices of $G$ such that vertices at distance two get labels that are at least $q$ apart and adjacent vertices get labels that are at least $p$ apart. We call these requirements the $L(p,q)$-conditions and $f(v)$ the label of $v$ under $f$. Due to its practical importance, the $L(p,q)$-labeling problem has been widely studied, and it has attracted a lot of interest [5,6,27]. Along the study of $L(p,q)$-labeling problem, extensive attention has been paid in the case when $(p,q) = (2,1)$. The $\lambda$-number of $G$, denoted by $\lambda(G)$, is the minimum value $\lambda$ such that $G$ admits an $L(2,1)$-labeling [17].
For directed graphs $D_1 = (V_1, A_1)$ and $D_2 = (V_2, A_2)$, the strong product $D_1 \boxtimes D_2$ and the Cartesian product $D_1 \square D_2$ are defined analogous to that of undirected graphs: $V(D_1 \boxtimes D_2) = V_1 \times V_2$, $A(D_1 \boxtimes D_2) = \{(a, x), (b, y)\} : \{a, b\} \in A_1$ and $x = y$, or $\{x, y\} \in A_2$ and $a = b$, and $E(D_1 \boxtimes D_2) = E(D_1 \square D_2) \cup \{(a, x), (b, y)\} : \{a, b\} \in A_1$ and $\{x, y\} \in A_2$.

As an example, Figure 1 shows the Cartesian product of two directed $P_6$ and the Cartesian product of two directed $C_6$.

![Figure 1. (a) Cartesian product of $\overrightarrow{P}_6$ and $\overrightarrow{P}_6$ (b) Cartesian product of $\overrightarrow{C}_6$ and $\overrightarrow{C}_6$](image)

The Cartesian and strong product have a number of applications in engineering, computer science and related disciplines. They provide a setting in which to analyze many existing networks as well as to construct new and interesting networks [16, 17, 19, 22, 26]. Among various graphs products, the products which contains path and cycles have proved to be one of the most important [16, 17, 19].

The frequency assignments in which frequency inference has direction have also attracted attention in the literature [10]. They are modeled by digraphs, including ditrees [8], planar graphs [4, 23], graphs with distance two [9]. In this paper, we focus on the $L(2, 1)$-labeling problem of the strong and Cartesian product of path and cycle, which are two of the most common and important graph products. The main result of this paper is to provide bounds for the $\lambda$-number of $\overrightarrow{C}_m \boxtimes \overrightarrow{C}_n$ for $m, n \geq 4$ and bounds for $\lambda$-number in the infinite families of $\overrightarrow{C}_m \square \overrightarrow{C}_n$. We prove that the $\lambda$-number of the strong product of cycles $\overrightarrow{C}_m \boxtimes \overrightarrow{C}_n$ is between 6 and 8 for $m, n \geq 48$, and moreover, we obtain the $\lambda$-number of $\overrightarrow{C}_m \square \overrightarrow{C}_n$ for $m \leq 10$.

2. Preliminaries. We shall need the following well known lemma.

Lemma 1. If $H$ is a subgraph of $G$, then $\lambda(H) \leq \lambda(G)$.

Let $G$ be a graph. A function $f$ from $V(G)$ onto the set $\{0, 1, \ldots, k\}$ is called a $k$-labeling. If a $k$-labeling $f$ of $G$ is an $L(2, 1)$-labeling of $G$, then $f$ is a $k$-$L(2, 1)$-labeling of $G$. 


Let $f$ be a $k$-labeling of the strong or Cartesian product of two graphs. For the sake of brevity, we denote by $f(x, y)$ the value of $f(u)$ for $u = (x, y)$ such that $u \in V(C_m \square C_n)$ or $u \in V(C_m \square \overrightarrow{C}_n)$.

A $k$-labeling $f$ of $\overrightarrow{C}_m \square C_n$ or $C_m \square \overrightarrow{C}_n$ can be presented with a $n \times m$ labeling matrix $C$ of $f$, where $C_{i,j} = f(i-1, j-1)$, $i \in \{1, \ldots, n\}$, $j \in \{1, \ldots, m\}$.

Let $f$ denote a $k$-$L(2, 1)$-labeling of $\overrightarrow{C}_m \square C_n$. We denote by $f_{i,p}$ the restriction of $f$ to $V(\overrightarrow{C}_n^i), \ldots, V(\overrightarrow{C}_n^{i+p-1})$. We also write $f_i$ for $f_{i,1}$.

The following lemma plays an important role in the sequel.

**Lemma 2.** Let $m,n,p \geq 3$, $t \geq 1$ and $f$ be a $k$-$L(2, 1)$-labeling of $\overrightarrow{C}_m \square \overrightarrow{C}_n$. If $f_{0,p}$ is a $k$-$L(2, 1)$-labeling of $\overrightarrow{C}_p \square \overrightarrow{C}_n$, then $\lambda(\overrightarrow{C}_{m+(t-1)p} \square \overrightarrow{C}_n) \leq k$.

**Proof.** Let $f'$ be a $k$-labeling of $\overrightarrow{C}_{m+(t-1)p} \square \overrightarrow{C}_n$ and $f'_i$ the restriction of $f'$ to $V(\overrightarrow{C}_n^i)$. The function $f'$ is defined as follows:

$$f'_i = \begin{cases} f_i, & i < m \\ f_{(i-m) \bmod p}, & i \geq m \end{cases}$$

It can be seen that $f'$ is a $k$-$L(2, 1)$-labeling of $\overrightarrow{C}_{m+(t-1)p} \square \overrightarrow{C}_n$, and the result holds.

Given two integers $r$ and $s$, let $S(r,s)$ denote the set of all nonnegative integer combinations of $r$ and $s$:

$$S(r,s) = \{ \alpha r + \beta s : \alpha, \beta \in \mathbb{Z}^+ \}$$

Sylvester in [25] provide the following result which is useful to provide $L(2,1)$-labelings for infinite cases:

**Lemma 3.** If $r,s > 1$ are relatively prime integers, then $t \in S(r,s)$ for all $t \geq (s-1)(r-1)$.

If $f$ is a $k$-$L(2, 1)$-labeling of a graph or digraph $G$, then the mirror labeling of $f$, denoted by $\overline{f}$, is function from $\{0,1,\ldots,k\}$ to the vertices of $\overline{G}$ such that for every $v \in V(G)$ we have $\overline{f}(v) := k - f(v)$.

It is straightforward to see

**Lemma 4.** If $f$ is a $k$-$L(2, 1)$-labeling of a graph or digraph $G$, then $\overline{f}$ is also a $k$-$L(2, 1)$-labeling of $G$.

Let $D_1$ be a digraph with the vertex set $V(D_1) = \{x, z, w, u, v\}$ and the set of arcs $A(D_1) = \{(x, w), (z, w), (w, u), (w, v)\}$.

**Fact 1.** If $f$ is a $L(2,1)$-labeling of $D_1$ such that $f(u) \neq f(v)$, then the span of $f$ is at least four.

**Proof.** Since $u$ and $v$ are at distance two from $z$, we have $f(z) \neq f(v)$ and $f(z) \neq f(u)$. Moreover, since $w$ is adjacent to all other four vertices, it can be verified that the span of $f$ is at least four.

The following lemma is the basis for our results:

**Lemma 5.** Let $f$ be a $4$-$L(2,1)$-labeling of $\overrightarrow{C}_m \square \overrightarrow{C}_n$, where $m,n \geq 4$. Then $f(i,j) = f(i+1, j-1)$, where $1 \leq i \leq m-1, 1 \leq j \leq n-1, i+1$ is taken modulo $m$ and $j-1$ is taken modulo $n$.\n
Thanks to Lemma 2, we have
\[ \lambda \text{ (leftmost columns) of } P \]
(1 − 5-
Finally, Lemma 3 shows that \( \{f \} \). Due to Lemma 4 it suffices to consider the case with \( f(0,2) \) and \( f(1,1) \) from \{3,4\}. Moreover, we may set w.l.o.g. \( f(0,2) = 3 \) and \( f(1,1) = 4 \). We then distinguish the following two cases.

If \( f(0,1) = 0 \), then \( f(1,2) = 1 \). Let \( H \) denote the subgraph of \( C_m \Box C_n \) induced by the vertices \((0,2),(1,1),(0,1),(1,2),(0,3),(1,3)\). Note that \( H \) is isomorphic to \( P_2 \Box P_3 \). By Lemma 1, the restriction of \( f \) to \( H \) has to be a 4-L(2,1)-labeling of \( H \). Since \( f(1,2) = 1 \), the label of \((1,3)\) can be either 3 or 4. Note that \((1,3)\) is at distance two from \((0,2)\) and \((1,1)\). Since \( f(0,2) = 3 \) and \( f(1,1) = 4 \), the label of \((1,3)\) cannot be from \{3,4\} and we obtain a contradiction.

If \( f(0,1) = 1 \), then \( f(1,2) = 0 \). Let \( H' \) denote the subgraph of \( C_m \Box C_n \) induced by the vertices \((0,2),(1,1),(0,1),(1,2),(0,0),(1,0)\). Note that \( H' \) is isomorphic to \( P_2 \Box P_3 \). By Lemma 1, the restriction of \( f \) to \( H' \) has to be a 4-L(2,1)-labeling of \( H' \). Since \( f(0,1) = 1 \), the label of \((0,0)\) can be either 3 or 4. Note that \((0,0)\) is at distance two from \((0,2)\) and \((1,1)\). Since \( f(0,2) \) and \( f(1,1) \) are from \{3,4\}, the label of \((0,0)\) cannot be from \{3,4\} and we again obtain a contradiction.

3. \( \lambda \)-number of \( C_m \Box C_n \).

Proposition 1. If \( m, n \geq 3 \), then \( \lambda(C_m \Box C_n) \geq 4 \).

Proof. Suppose to the contrary that there exists a 3-L(2,1)-labeling \( f \) of \( P_m \Box P_n \). Since the vertex \((1,1)\) is adjacent to both of \((0,1)\) and \((1,0)\), we have either \( f(1,1) \in \{0,1\} \) or \( f(1,1) \in \{2,3\} \). Due to Lemma 4, it suffices to consider only \( f(1,1) \in \{0,1\} \). Note that the subgraph of \( C_m \Box C_n \) induced by the vertices \{\((0,1),(1,1),(1,0),(1,2),(2,1)\}\) is isomorphic to the graph \( D_1 \).

Case 1. If \( f(1,1) = 0 \), then we have \( f(u) \in \{2,3\} \) for any \( u \in \{(0,1),(1,0),(1,2),(2,1)\} \). From Fact 1 it follows that \( f(1,2) = f(2,1), f(0,1) = f(1,0) \) and \( f(0,1) = f(1,2) \). If \( f(0,1) = 2 \), then the vertex \((0,1)\) has no label to assign and we obtain a contradiction. Analogously, if \( f(1,2) = 2 \), we cannot label the vertex \((2,2)\) and we again obtain a contradiction.

Case 2. If \( f(1,1) = 1 \), then we have \( f(u) = 3 \) for any \( u \in \{(0,1),(1,0),(1,2),(2,1)\} \). Since \((0,1)\) and \((1,0)\) are at distance two, \( f(0,1) = f(1,0) \) and we obtain a contradiction.

Since we settled both cases, the assertion follows.

Theorem 1. If \( m, n \geq 40 \), then \( \lambda(C_m \Box C_n) \leq 5 \).

Proof. In order to construct a 5-L(2,1)-labeling of \( C_{11} \Box C_{11} \), we use the 11 × 11 pattern depicted in Figure 2 and denoted by \( P \). It is easy to check that \( P \) induces a 5-L(2,1)-labeling of \( C_{11} \Box C_{11} \). Moreover, the five uppermost rows (resp. five leftmost columns) of \( P \) induce a 5-L(2,1)-labeling of \( C_5 \Box C_5 \) (resp. \( C_5 \Box C_{11} \)). Thanks to Lemma 2, we have \( \lambda(C_{5\alpha+11\beta} \Box C_{5\gamma+11\delta}) \leq 5 \) for integers \( \alpha, \beta, \gamma, \delta \). Finally, Lemma 3 shows that \( \lambda(C_m \Box C_n) \leq 5 \) for every \( m, n \geq (11-1)(5-1) = 40 \). This assertion completes the proof of the theorem.
Figure 2. A 5-L(2, 1)-labeling of $\overrightarrow{C_{11}} \boxtimes \overrightarrow{C_{11}}$

4. $\lambda$-number of $\overrightarrow{C_m} \boxtimes \overrightarrow{C_n}$.

**Theorem 2.** If $m, n \geq 48$, then $6 \leq \lambda(\overrightarrow{C_m} \boxtimes \overrightarrow{C_n}) \leq 8$.

**Proof.** A backtracking procedure is first applied in order to find a 5-L(2, 1)-labeling of $\overrightarrow{P_6} \boxtimes \overrightarrow{P_6}$. Since the procedure fails, we have $\lambda(\overrightarrow{C_m} \boxtimes \overrightarrow{C_n}) \geq 6$ for $m, n \geq 6$. The upper bound is obtained by using the $13 \times 13$ pattern depicted in Figure 3 and denoted by $P$. It is easy to check that $P$ induces an 8-L(2, 1)-labeling of $\overrightarrow{C_{13}} \boxtimes \overrightarrow{C_{13}}$. Moreover, the five uppermost rows (resp. five leftmost columns) of $P$ induce a 8-L(2, 1)-labeling of $\overrightarrow{C_{13}} \boxtimes \overrightarrow{C_5}$ (resp. $\overrightarrow{C_5} \boxtimes \overrightarrow{C_{13}}$). Thanks to Lemma 2, we have $\lambda(\overrightarrow{C_m} \boxtimes \overrightarrow{C_n}) \leq 7$ for every $m, n \geq (13 - 1)(5 - 1) = 48$ and the proof is complete.

Figure 3. An 8-L(2, 1)-labeling of $\overrightarrow{C_{13}} \boxtimes \overrightarrow{C_{13}}$

**Theorem 3.** (i) If $n \geq 48$, then $\lambda(\overrightarrow{C_5} \boxtimes \overrightarrow{C_n}) \leq 8$.

(ii) If $n \geq 60$, then $\lambda(\overrightarrow{C_6} \boxtimes \overrightarrow{C_n}) \leq 8$.

(iii) If $n \geq 220$, then $\lambda(\overrightarrow{C_7} \boxtimes \overrightarrow{C_n}) \leq 8$.

(iv) If $n \geq 112$, then $\lambda(\overrightarrow{C_8} \boxtimes \overrightarrow{C_n}) \leq 8$.

(v) If $n \geq 112$, then $\lambda(\overrightarrow{C_9} \boxtimes \overrightarrow{C_n}) \leq 8$.

(vi) If $n \geq 180$, then $\lambda(\overrightarrow{C_{10}} \boxtimes \overrightarrow{C_n}) \leq 8$. 
Proof. (i) The $5 \times 13$ pattern depicted in Figure 4 is an $8-L(2,1)$-labeling of $C_5 \boxtimes C_{13}$. Moreover, its five leftmost columns induce an $8-L(2,1)$-labeling of $C_5 \boxtimes C_5$. Thanks to Lemma 2, we have $\lambda (C_5 \boxtimes C_{5\gamma+13\delta}) \leq 8$ for integers $\gamma, \delta$. Finally, Lemma 3 shows that $\lambda(C_5 \boxtimes C_n) \leq 8$ for every $m,n \geq (13 - 1)(5 - 1) = 48$. This assertion completes the proof of this case.

(ii) The $6 \times 13$ pattern depicted in Figure 5 is an $8-L(2,1)$-labeling of $C_6 \boxtimes C_{13}$. Moreover, its five leftmost columns induce an $8-L(2,1)$-labeling of $C_6 \boxtimes C_6$. Thanks to Lemma 2, we have $\lambda (C_6 \boxtimes C_{6\gamma+13\delta}) \leq 8$ for integers $\gamma, \delta$. Finally, Lemma 3 shows that $\lambda(C_6 \boxtimes C_n) \leq 8$ for every $m,n \geq (13 - 1)(6 - 1) = 60$. This assertion completes the proof of this case.

(iii) The $7 \times 23$ pattern depicted in Figure 6 is an $8-L(2,1)$-labeling of $C_7 \boxtimes C_{23}$. Moreover, its 11 leftmost columns induce an $8-L(2,1)$-labeling of $C_7 \boxtimes C_7$. Thanks to Lemma 2, we have $\lambda (C_7 \boxtimes C_{11\gamma+23\delta}) \leq 8$ for integers $\gamma, \delta$. Finally, Lemma 3 shows that $\lambda(C_7 \boxtimes C_n) \leq 8$ for every $m,n \geq (11 - 1)(23 - 1) = 220$. This assertion completes the proof of this case.

(iv) The $8 \times 17$ pattern depicted in Figure 7 is an $8-L(2,1)$-labeling of $C_8 \boxtimes C_{17}$. Moreover, its eight leftmost columns induce an $8-L(2,1)$-labeling of $C_8 \boxtimes C_8$. Thanks to Lemma 2, we have $\lambda (C_8 \boxtimes C_{8\gamma+17\delta}) \leq 8$ for integers $\gamma, \delta$. Finally, Lemma 3
shows that $\lambda(\overrightarrow{C_8} \boxtimes \overrightarrow{C_n}) \leq 8$ for every $m, n \geq (8 - 1)(17 - 1) = 112$. This assertion completes the proof of this case.

\[
\begin{array}{cccccccccccccccc}
7 & 0 & 2 & 4 & 6 & 8 & 1 & 3 & 5 & 7 & 2 & 0 & 6 & 8 & 1 & 3 & 5
\end{array}
\]
\[
\begin{array}{cccccccccccccccc}
0 & 2 & 4 & 6 & 8 & 0 & 3 & 5 & 7 & 2 & 0 & 4 & 8 & 1 & 3 & 5
\end{array}
\]
\[
\begin{array}{cccccccccccccccc}
2 & 4 & 6 & 8 & 0 & 3 & 5 & 7 & 2 & 4 & 6 & 8 & 1 & 3 & 5 & 7
\end{array}
\]
\[
\begin{array}{cccccccccccccccc}
4 & 6 & 8 & 1 & 3 & 5 & 7 & 2 & 0 & 6 & 8 & 1 & 3 & 5 & 7 & 0
\end{array}
\]
\[
\begin{array}{cccccccccccccccc}
6 & 8 & 1 & 3 & 5 & 7 & 2 & 0 & 6 & 8 & 1 & 3 & 5 & 7 & 0 & 2
\end{array}
\]
\[
\begin{array}{cccccccccccccccc}
8 & 3 & 5 & 7 & 0 & 2 & 4 & 6 & 8 & 1 & 3 & 5 & 7 & 0 & 2 & 4 & 6
\end{array}
\]
\[
\begin{array}{cccccccccccccccc}
3 & 5 & 7 & 0 & 2 & 4 & 6 & 8 & 1 & 3 & 5 & 7 & 0 & 2 & 4 & 6
\end{array}
\]
\[
\begin{array}{cccccccccccccccc}
5 & 7 & 0 & 2 & 4 & 6 & 8 & 1 & 3 & 5 & 7 & 2 & 4 & 6 & 8 & 1
\end{array}
\]

**Figure 7.** An 8-L(2,1)-labeling of $\overrightarrow{C_8} \boxtimes \overrightarrow{C_{17}}$

(v) The $9 \times 17$ pattern depicted in Figure 8 is an 8-L(2,1)-labeling of $\overrightarrow{C_9} \boxtimes \overrightarrow{C_{17}}$. Moreover, its eight leftmost columns induce an 8-L(2,1)-labeling of $\overrightarrow{C_9} \boxtimes \overrightarrow{C_8}$. Thanks to Lemma 2, we have $\lambda(\overrightarrow{C_9} \boxtimes \overrightarrow{C_{9+177}}) \leq 8$ for integers $\gamma, \delta$. Finally, Lemma 3 shows that $\lambda(\overrightarrow{C_9} \boxtimes \overrightarrow{C_n}) \leq 8$ for every $m, n \geq (8 - 1)(17 - 1) = 112$. This assertion completes the proof of this case.

\[
\begin{array}{cccccccccccccccc}
5 & 7 & 0 & 2 & 4 & 6 & 8 & 1 & 3 & 5 & 7 & 0 & 2 & 4 & 8 & 1
\end{array}
\]
\[
\begin{array}{cccccccccccccccc}
7 & 0 & 2 & 4 & 6 & 8 & 1 & 3 & 5 & 7 & 0 & 2 & 4 & 6 & 1 & 3
\end{array}
\]
\[
\begin{array}{cccccccccccccccc}
0 & 2 & 4 & 6 & 8 & 1 & 3 & 5 & 7 & 0 & 2 & 4 & 6 & 8 & 3 & 5
\end{array}
\]
\[
\begin{array}{cccccccccccccccc}
2 & 4 & 6 & 8 & 1 & 3 & 5 & 7 & 0 & 2 & 4 & 6 & 8 & 1 & 5 & 7
\end{array}
\]
\[
\begin{array}{cccccccccccccccc}
4 & 6 & 8 & 1 & 3 & 5 & 7 & 0 & 2 & 4 & 6 & 8 & 1 & 3 & 7 & 0
\end{array}
\]
\[
\begin{array}{cccccccccccccccc}
6 & 8 & 1 & 3 & 5 & 7 & 0 & 2 & 4 & 6 & 8 & 1 & 3 & 5 & 7 & 2
\end{array}
\]
\[
\begin{array}{cccccccccccccccc}
8 & 1 & 3 & 5 & 7 & 0 & 2 & 4 & 6 & 8 & 1 & 3 & 5 & 7 & 0 & 4
\end{array}
\]
\[
\begin{array}{cccccccccccccccc}
1 & 3 & 5 & 7 & 0 & 2 & 4 & 6 & 8 & 1 & 3 & 5 & 7 & 0 & 2 & 6
\end{array}
\]
\[
\begin{array}{cccccccccccccccc}
3 & 5 & 7 & 0 & 2 & 4 & 6 & 8 & 1 & 3 & 5 & 7 & 0 & 2 & 6 & 8
\end{array}
\]

**Figure 8.** An 8-L(2,1)-labeling of $\overrightarrow{C_9} \boxtimes \overrightarrow{C_{17}}$

(vi) The $10 \times 21$ pattern depicted in Figure 9 is an 8-L(2,1)-labeling of $\overrightarrow{C_{10}} \boxtimes \overrightarrow{C_{21}}$. Moreover, its ten leftmost columns induce an 8-L(2,1)-labeling of $\overrightarrow{C_{10}} \boxtimes \overrightarrow{C_{10}}$. Thanks to Lemma 2, we have $\lambda(\overrightarrow{C_{10}} \boxtimes \overrightarrow{C_{10\gamma+171}}) \leq 8$ for integers $\gamma, \delta$. Finally, Lemma 3 shows that $\lambda(\overrightarrow{C_{10}} \boxtimes \overrightarrow{C_n}) \leq 8$ for every $m, n \geq (10 - 1)(21 - 1) = 180$. This assertion completes the proof of this case.

\[
\begin{array}{cccccccccccccccc}
1 & 8 & 6 & 2 & 7 & 5 & 3 & 0 & 6 & 3 & 1 & 8 & 6 & 3 & 0 & 7 & 5 & 0 & 2 & 6 & 4
\end{array}
\]
\[
\begin{array}{cccccccccccccccc}
7 & 5 & 2 & 0 & 5 & 3 & 8 & 6 & 4 & 1 & 7 & 5 & 2 & 0 & 7 & 5 & 2 & 8 & 6 & 4 & 1
\end{array}
\]
\[
\begin{array}{cccccccccccccccc}
5 & 3 & 0 & 7 & 3 & 1 & 6 & 4 & 1 & 7 & 5 & 2 & 0 & 7 & 5 & 1 & 8 & 6 & 3 & 1 & 7
\end{array}
\]
\[
\begin{array}{cccccccccccccccc}
3 & 0 & 6 & 4 & 1 & 8 & 4 & 0 & 7 & 3 & 0 & 8 & 6 & 4 & 1 & 8 & 6 & 3 & 0 & 7 & 5
\end{array}
\]
\[
\begin{array}{cccccccccccccccc}
8 & 6 & 4 & 1 & 8 & 5 & 0 & 7 & 3 & 5 & 8 & 6 & 4 & 1 & 8 & 6 & 2 & 0 & 7 & 5 & 2
\end{array}
\]
\[
\begin{array}{cccccccccccccccc}
6 & 4 & 1 & 7 & 3 & 0 & 7 & 3 & 1 & 8 & 2 & 4 & 1 & 7 & 5 & 3 & 0 & 7 & 4 & 1 & 8
\end{array}
\]
\[
\begin{array}{cccccccccccccccc}
4 & 0 & 7 & 5 & 0 & 6 & 4 & 1 & 8 & 4 & 6 & 0 & 7 & 5 & 2 & 0 & 7 & 4 & 1 & 8 & 6
\end{array}
\]
\[
\begin{array}{cccccccccccccccc}
2 & 7 & 5 & 2 & 8 & 4 & 1 & 8 & 5 & 2 & 0 & 3 & 5 & 2 & 0 & 6 & 4 & 1 & 8 & 5 & 0
\end{array}
\]
\[
\begin{array}{cccccccccccccccc}
7 & 5 & 1 & 8 & 4 & 1 & 7 & 5 & 2 & 0 & 7 & 5 & 1 & 8 & 6 & 4 & 1 & 8 & 5 & 0 & 2
\end{array}
\]
\[
\begin{array}{cccccccccccccccc}
4 & 1 & 8 & 4 & 0 & 7 & 5 & 2 & 0 & 6 & 4 & 1 & 8 & 6 & 3 & 1 & 7 & 5 & 0 & 2 & 6
\end{array}
\]

**Figure 9.** An 8-L(2,1)-labeling of $\overrightarrow{C_{10}} \boxtimes \overrightarrow{C_{21}}$
In what follows, we use the dynamic algorithm introduced in [18] to search the \(\lambda\)-number in some infinite families of \(\overrightarrow{C}_m \otimes \overrightarrow{C}_n\). The idea has been introduced in a more general framework and later used several times, e.g. [19,24]. In order to make the paper self contained, we describe basic definitions and results.

We define a digraph \(D_{n,k}\) as follows. The vertices of \(D_{n,k}\) are the \(k\)-\(L(2,1)\)-labelings of \(\overrightarrow{P}_3 \otimes \overrightarrow{C}_n\). Let \(u = u_1u_2\) be a vertex of \(D_{n,k}\). Then \(u_1\) and \(u_2\) represent the \(k\)-\(L(2,1)\)-labeling of \(\overrightarrow{P}_3 \otimes \overrightarrow{C}_n\) restricted to the first and second copy of \(\overrightarrow{C}_n\), respectively.

Let \(u\) and \(v\) be two vertices of \(D_{n,k}\). Denote by \(\overrightarrow{uv}\) the labeling of \(\overrightarrow{P}_3 \otimes \overrightarrow{C}_n\) obtained by applying \(u_2\) and \(v_1\) to the consecutive copies of \(\overrightarrow{C}_n\). (Note that \(\overrightarrow{uv}\) is not always a \(k\)-\(L(2,1)\)-labeling of \(\overrightarrow{P}_3 \otimes \overrightarrow{C}_n\)). We make an arc from \(u\) to \(v\) in \(D_{n,k}\) if and only if, the following two conditions are fulfilled:

(i) \(u_2\) equals \(v_1\) and

(ii) \(\overrightarrow{uv}\) is a \(k\)-\(L(2,1)\)-labeling of \(\overrightarrow{P}_3 \otimes \overrightarrow{C}_n\).

We now have the following result that follows from the results presented in [18].

**Theorem 4.** \(\overrightarrow{C}_m \otimes \overrightarrow{C}_n\) admits a \(k\)-\(L(2,1)\)-labeling if and only if \(D_{n,k}\) contains a closed directed walk of length \(m\).

In order to find closed walks in \(D_{n,k}\), we first try to enumerate all cycles in the graph. Directed cycles of \(D_{n,k}\) can be found by the breadth first search or depth first search procedure if the graph of interest is not too large. As an alternative, recall that the number of distinct closed walks of length \(p\) in a digraph \(D\) can be computed via the \(p\)-th power of the adjacency matrix of \(D\). In particular, if \(A\) is the adjacency matrix of \(D\), then the entry \((u, u)\) of \(A^p\) equals the number of distinct closed walks of length \(p\) through \(u\) in \(D\).

We denote by \(\mathbb{N}\) the set of natural numbers and by \(\mathbb{M}_n(\{0, 1\})\) the set of \(n \times n\) binary matrices for \(n \in \mathbb{N}\). Given two elements \(A, B\) in \(\mathbb{M}_n(\{0, 1\})\), we define the product operation on \(A\) and \(B\) whose result is a matrix of \(\mathbb{M}_n(\{0, 1\})\) denoted by \(A \cdot B\) such that: \((A \cdot B)_{ij} = \bigvee_{k=1}^n (A_{ik} \land B_{kj})\) for \(i, j \in \{1, \ldots, n\}\), where \(\lor\) and \(\land\) are the logical operators “or” and “and”. We can use the product operation on \(A\) to determine whether the corresponding graph contains a closed walk of a given length. The following lemma by Bouznif et al. [3] is useful for us:

**Lemma 6.** Let \(n \in \mathbb{N}\) and \(A \in \mathbb{M}_n(\{0, 1\})\). There exists two integers \(u\) and \(P\) such that, starting from \(A^u\), the sequence of powers of \(A\) is periodic of period \(P\): \(\forall k \geq u, A^k = A^{u+(k-u) \mod (P)}\).

The following result is based on Lemma 6.

**Theorem 5.** (i) If \(n \geq 3\) and \(S = \{3, 4, 5, 7, 8, 11, 14, 17\}\), then \(\lambda(\overrightarrow{C}_3 \otimes \overrightarrow{C}_n) \leq 9\) for any \(n \notin S\).

(ii) If \(n \geq 4\) and \(S = \{4, 5, 6, 7, 8, 9, 10, 11, 12, 14, 15, 17, 18, 20, 21, 23, 24, 25, 27, 28, 30, 31, 33, 34, 37, 40, 43, 46, 50, 53, 56, 59\}\), then \(\lambda(\overrightarrow{C}_4 \otimes \overrightarrow{C}_n) \leq 8\) for any \(n \notin S\).

**Proof.** (i) The graph \(D_{3,9}\) with 1800 vertices and the largest out degree 5 is created. Matrix multiplication is applied to confirm that there exists no closed walk of length in \(S\). For the adjacency matrix \(A\) of \(D_{3,9}\) we obtain that \(u = 41\) and \(P = 3\) such that \(\forall k \geq u, A^k = A^{u+(k-u) \mod (P)}\). Moreover, we find that there exist a closed walk for any length \(n\) for \(n \leq 40\) and \(n \notin S\) in \(D_{3,9}\). This assertion completes the
proof of this case.

(ii) The graph $D_{4,8}$ with 2664 vertices and the largest out degree 4 is created. Matrix multiplication is applied to confirm that there exists no closed walk of length from $S$. For the adjacency matrix $A$ of $D_{4,8}$ we obtain that $u = 89$ and $P = 1$ such that $\forall k \geq u, A^k = A^{u + (k-u) \mod (P)}$. Moreover, we find that there exists a closed walk of any length $n$ for $n \leq 89$ and $n \notin S$ in $D_{4,8}$. This assertion completes the proof of this case.

Table 1: Summary of results on $\lambda(C_m^r \boxtimes C_n^r)$

| $m$ | $k$ | $|D_{m,k}|$ | $\max\{d^+\}$ | cycle lengths | result |
|-----|-----|------------|-----------------|---------------|--------|
| 3   | 7   | 0          | 0               | $\emptyset$   | $\lambda(C_3^r \boxtimes C_n^r) \geq 8$. |
| 3   | 8   | 120        | 1               | $\{9\}$       | if $n \equiv 0 \mod 9$, then $\lambda(C_3^r \boxtimes C_n^r) \leq 8$; otherwise $\lambda(C_3^r \boxtimes C_n^r) \geq 9$. |
| 3   | 9   | 1800       | 4               | $\emptyset$   | $D_{3,9}$ contains no closed walk of length from $\{3, 4, 5, 7, 8, 11, 14, 17\}$, thus $\lambda(C_3^r \boxtimes C_n^r) \geq 10$ for $n \in \{3, 4, 5, 7, 8, 11, 14, 17\}$. |
| 4   | 6   | 0          | 0               | $\emptyset$   | $\lambda(C_4^r \boxtimes C_n^r) \geq 7$. |
| 4   | 7   | 72         | 1               | $\{16\}$      | if $n \equiv 0 \mod 16$, then $\lambda(C_4^r \boxtimes C_n^r) \leq 7$; otherwise $\lambda(C_4^r \boxtimes C_n^r) \geq 8$. |
| 4   | 8   | 2664       | 5               | $\emptyset$   | $D_{4,8}$ contains no closed walk of length from $S_4$, thus $\lambda(C_4^r \boxtimes C_n^r) \geq 9$ for $n \in S_4$. |
| 5   | 7   | 40         | 1               | $\emptyset$   | $\lambda(C_5^r \boxtimes C_n^r) \geq 8$. |
| 5   | 8   | 10200      | 10              | $\emptyset$   | $D_{5,8}$ contains no closed walk of length from $\{6, 7, 12\}$, thus $\lambda(C_5^r \boxtimes C_n^r) \geq 9$ for $n \in \{6, 7, 12\}$. |
| 6   | 6   | 0          | 0               | $\emptyset$   | $\lambda(C_6^r \boxtimes C_n^r) \geq 7$. |
| 6   | 7   | 540        | 4               | $\{6\}$       | if $n \equiv 0 \mod 6$, then $\lambda(C_6^r \boxtimes C_n^r) \leq 7$; otherwise $\lambda(C_6^r \boxtimes C_n^r) \geq 8$. |
| 6   | 8   | 72534      | 27              | $\emptyset$   | $D_{6,8}$ contains no closed walk of length 11, thus $\lambda(C_6^r \boxtimes C_{11}) \geq 9$. |
| 7   | 6   | 0          | 0               | $\emptyset$   | $\lambda(C_7^r \boxtimes C_n^r) \geq 7$. |
| 7   | 7   | 2296       | 8               | $\emptyset$   | $D_{7,7}$ contains no closed walk of length from $S_7$, thus $\lambda(C_7^r \boxtimes C_n^r) \geq 8$ for $n \in S_7$. |
| 8   | 6   | 0          | 0               | $\emptyset$   | $\lambda(C_8^r \boxtimes C_n^r) \geq 7$. |
Table 1 – continued from previous page

| $m$ | $k$ | $|D_{m,k}|$ | $\max\{d^+\}$ | cycle lengths | result |
|-----|-----|-------------|----------------|---------------|--------|
| 8   | 7   | 720         | 1              | $\{8,16\}$   | $n \equiv 0 \mod 8$, then $\lambda(\overrightarrow{C}_8 \boxtimes \overrightarrow{C}_n) \leq 7$; otherwise $\lambda(\overrightarrow{C}_8 \boxtimes \overrightarrow{C}_n) \geq 8$. |
| 9   | 7   | 1530        | 2              | $\emptyset$   | $\lambda(\overrightarrow{C}_9 \boxtimes \overrightarrow{C}_n) \geq 8$. |
| 10  | 6   | 0           | 0              | $\emptyset$   | $\lambda(\overrightarrow{C}_{10} \boxtimes \overrightarrow{C}_n) \geq 7$. |
| 10  | 7   | 16100       | 6              | $\emptyset$   | $D_{10,7}$ contains no closed walk of length from $S_{10}$, thus $\lambda(\overrightarrow{C}_{10} \boxtimes \overrightarrow{C}_n) \geq 8$ for $n \in S_{10}$. |

Table 1 reports the computational results of the dynamic algorithm on the graphs $\overrightarrow{C}_m \boxtimes \overrightarrow{C}_n$ for $m \leq 10$.

The table uses sets $S_4 = \{3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 14, 15, 17, 18, 20, 21, 23, 24, 25, 27, 28, 30, 31, 33, 34, 37, 40, 43, 46, 50, 53, 56, 59\}$, $S_7 = \{3, 4, 5, 6, 8, 9, 10, 11, 12, 13, 15, 16, 19, 20, 22, 23, 26, 27, 29, 30, 33, 37, 40, 44, 47\}$, and $S_{10} = \{3, 4, 5, 6, 7, 8, 9, 10, 12, 13, 14, 16, 17, 18, 20, 21, 23, 24, 25, 27, 28, 29, 31, 32, 35, 36, 39, 42, 43, 46, 47, 50, 54, 58, 61, 65, 69\}$. Column $\max\{d^+\}$ denotes the largest out degree in $D_{i,k}$, while column cycle lengths describes the set of cycles’ lengths in $D_{i,k}$ (obtained by the depth first search procedure). If the value of the entry in cycle lengths column is denoted by $\emptyset$, it means that we have failed to obtain the set by the depth first search procedure and that we have used matrix multiplication instead.

As an example, note that the graph $D_{7,7}$ has 2296 vertices with the largest out degree 8. Matrix multiplication is performed to confirm that $D_{7,7}$ contains no closed walk of length in $S_7$. It follows that $\lambda(\overrightarrow{C}_7 \boxtimes \overrightarrow{C}_n) \geq 8$ if $n \in S_7$. The other example is the graph $D_{3,8}$ with 120 vertices and with the largest out degree 1, which contains only cycles of length 9. Therefore, we have $\lambda(\overrightarrow{C}_3 \boxtimes \overrightarrow{C}_n) \leq 8$ if $n \equiv 0 \mod 9$, otherwise $\lambda(\overrightarrow{C}_3 \boxtimes \overrightarrow{C}_n) \geq 9$.

For $m \in \{5, 6, \ldots, 10\}$, we have established by using the dynamic algorithm that $\lambda(\overrightarrow{C}_m \boxtimes \overrightarrow{C}_n) \leq 8$ for every $n$ with the exception of a finite number of graphs. For example, we have $\lambda(\overrightarrow{C}_5 \boxtimes \overrightarrow{C}_n) \leq 8$ for $n \geq 48$ by Theorem 2. For $n \leq 47$ we can determine whether $D_{5,8}$ possesses a closed walk of length in $\{3, 4, \ldots, 47\}$ by matrix multiplication and thus determine the lower bounds of $\lambda(\overrightarrow{C}_5 \boxtimes \overrightarrow{C}_n)$ for $n \leq 47$.

By applying this approach, together with other results of Table 1, we obtain the following theorem:

**Theorem 6.** (i) If $n \geq 3$, then $\lambda(\overrightarrow{C}_3 \boxtimes \overrightarrow{C}_n) = \begin{cases} 8, & n \equiv 0 \mod 9, \\ 10, & n \in \{3, 5, 7, 8, 11, 14, 17\}, \\ 11, & n = 4, \\ 9, & \text{otherwise.} \end{cases}$

(ii) If $n \geq 4$ and $S = \{4, 5, 7, 8, 9, 10, 11, 12, 14, 15, 17, 18, 20, 21, 23, 24, 25, 27, 28, 30, 31, 33, 34, 37, 40, 43, 46, 50, 53, 56, 59\}$, then $\lambda(\overrightarrow{C}_4 \boxtimes \overrightarrow{C}_n) = \begin{cases} 7, & n \equiv 0 \mod 16, \\ 9, & n \in S, \\ 10, & n = 6, \\ 8, & \text{otherwise.} \end{cases}$
(iii) If \( n \geq 5 \), then
\[
\lambda(\overrightarrow{C}_5 \boxtimes \overrightarrow{C}_n) = \begin{cases} 
9, & n \in \{6, 7, 12\}, \\
8, & \text{otherwise}.
\end{cases}
\]

(iv) If \( n \geq 6 \), then \( \lambda(\overrightarrow{C}_6 \boxtimes \overrightarrow{C}_n) = \begin{cases} 
7, & n \equiv 0 \mod 6, \\
9, & n = 11, \\
8, & \text{otherwise}.
\end{cases} \)

(v) If \( n \geq 7 \) and \( S = \{8, 9, 10, 11, 12, 13, 15, 16, 19, 20, 22, 23, 26, 27, 29, 30, 33, 37, 40, 44, 47\} \), then
\[
\lambda(\overrightarrow{C}_7 \boxtimes \overrightarrow{C}_n) = \begin{cases} 
7, & n \notin S, \\
8, & n \in S.
\end{cases}
\]

(vi) If \( n \geq 8 \), then \( \lambda(\overrightarrow{C}_8 \boxtimes \overrightarrow{C}_n) = \begin{cases} 
7, & n \equiv 0 \mod 8, \\
8, & \text{otherwise}.
\end{cases} \)

(vii) If \( n \geq 9 \), then \( \lambda(\overrightarrow{C}_9 \boxtimes \overrightarrow{C}_n) = 8 \).

(viii) If \( n \geq 10 \) and \( S = \{10, 12, 13, 14, 16, 17, 18, 20, 21, 23, 24, 25, 27, 28, 29, 31, 32, 35, 36, 39, 42, 43, 46, 47, 50, 54, 58, 61, 65, 69\} \), then
\[
\lambda(\overrightarrow{C}_{10} \boxtimes \overrightarrow{C}_n) = \begin{cases} 
7, & n \notin S, \\
8, & n \in S.
\end{cases}
\]

Remark 1. An alternative approach for constructing explicit \( L(2,1) \)-labeling is the SAT reduction described in [24]. Most of the results in Theorem 6 are verified by solving satisfiability test reduction instances.

5. Conclusion. The frequency assignment problem for wireless networks is to assign a channel to each radio transmitter so that close transmitters are received channels, so as to avoid interference. This situation can be modeled by a graph whose vertices are the radio transmitters, and the adjacency indicate possible interference. Motivated by this problem, we studied the \( \lambda \)-number of the Cartesian and strong product of two directed cycles. We show that for \( m, n \geq 4 \) the \( \lambda \)-number of \( \overrightarrow{C}_m \boxtimes \overrightarrow{C}_n \) is between 4 and 5. The second part of the paper is devoted to the \( \lambda \)-number of the strong product of two directed cycles. We prove that the \( \lambda(\overrightarrow{C}_m \boxtimes \overrightarrow{C}_n) \) is between 6 and 8 for \( m, n \geq 48 \). Moreover, we obtain the \( \lambda \)-numbers of \( \overrightarrow{C}_m \boxtimes \overrightarrow{C}_n \) for every \( m \leq 10 \).

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