THE SMALL-IS-VERY-SMALL PRINCIPLE

ALBERT VISser

Abstract. The central result of this paper is the small-is-very-small principle for restricted sequential theories. The principle says roughly that whenever the given theory shows that a property has a small witness, i.e., a witness in every definable cut, then it shows that the property has a very small witness: i.e., a witness below a given standard number.

We draw various consequences from the central result. For example (in rough formulations): (i) Every restricted, recursively enumerable sequential theory has a finitely axiomatized extension that is conservative w.r.t. formulas of complexity $\leq n$. (ii) Every sequential model has, for any $n$, an extension that is elementary for formulas of complexity $\leq n$, in which the intersection of all definable cuts is the natural numbers. (iii) We have reflection for $\Sigma^2_0$-sentences with sufficiently small witness in any consistent restricted theory $U$. (iv) Suppose $U$ is recursively enumerable and sequential. Suppose further that every recursively enumerable and sequential $V$ that locally interprets $U$ globally interprets $U$. Then, $U$ is mutually globally interpretable with a finitely axiomatized sequential theory.

The paper contains some careful groundwork developing partial satisfaction predicates in sequential theories for the complexity measure depth of quantifier alternations.

1. Introduction

Some proofs are like hollyhocks. If you are nice to them they give different flowers every year. This paper is about one such proof. I discovered it when searching for alternative, more syntactic, proofs of certain theorems by Harvey Friedman (discussed in [Sma85]) and by Jan Krajíček (see [Kra87]). The relevant theorem due to Harvey Friedman tells us that, if a finitely axiomatized, sequential, consistent theory $A$ interprets a recursively enumerable theory $U$, then $A$ interprets $U$ faithfully. Krajíček’s theorem tells us that a finitely axiomatized, sequential, consistent theory cannot prove its own inconsistency on arbitrarily small cuts. There is a close connection between these two theorems.

The quest for a syntactic proof succeeded and the results were reported in [Vis93]. One advantage of having such a syntactic proof is clearly that it can be ‘internalized’ in the theories we study. I returned to the argument in a later paper [Vis05], which contains improvements and, above all, a better theoretical framework. In my papers

Date: May 4, 2018.

2010 Mathematics Subject Classification. 03C62, 03F30, 03F40, 03H15.

Key words and phrases. interpretations, degrees of interpretability, sequential theories, Rosser argument.

We thank Lev Beklemishev for enlightening discussions. We thank Ali Enayat for suggesting some important references. We are grateful to Joost Joosten for sharing his insights on the Friedman-Goldfarb-Harington Theorem. The main result of Section 7 is due to the previous 2006 version of me. I thank my previous self for its gracious permission to publish the result here.
and \[Vis15\], the argument is employed to prove results about provability logic and about degrees of interpretability, respectively.

The syntactic argument in question is a Rosser-style argument or, more specifically, a Friedman-Goldfarb-Harrington-style argument. It has all the mystery of a Rosser argument: even if every step is completely clear, it still retains a feeling of magic trickery.

1.1. Contents of the Paper. In the present paper, we will obtain more information from the Friedman-Goldfarb-Harrington-style argument discussed above. In previous work, the basic conclusion of the argument is that, given a consistent, finitely axiomatized, sequential theory $A$, there is an interpretation $M$ of the basic arithmetic $S^1_2$ in $A$ that is $\Sigma^0_2$-sound. In the present paper, we extend our scope from \textit{finitely axiomatized sequential theories} to \textit{restricted sequential theories} — this means that we consider theories with axioms of complexity below a fixed finite bound. Secondly, we replace the $\Sigma^0_2$-soundness by the more general small-is-very-small principle (SIVS).

The improved results have a number of consequences. In Section 4, we show that, for any $n$, every consistent, restricted, recursively enumerable, sequential theory has a finitely axiomatized extension that is conservative w.r.t. formulas of complexity $\leq n$. In Section 5, we show that, for any $n$, every sequential model model has an elementary extension w.r.t. formulas of $\leq n$, such that the intersection of all definable cuts consists of the standard numbers. In Section 6, we indicate how results concerning $\Sigma^0_2$-soundness can be derived from our main theorem. Finally, in Section 7, we prove a result in the structure of the combined degrees of local and global interpretability of recursively enumerable, sequential theories. We show that if a local degree contains a minimal global degree, then this global degree contains a finitely axiomatized theory. Thus, finite axiomatizability has a natural characterization, modulo global interpretability, in terms of the double degree structure.

Section 2 provides the necessary elementary facts. Unlike similar sections in other papers of mine, this section also contains something new. In \[Vis93\], I provided groundwork for the development of partial satisfaction predicates for the complexity measure \textit{depth of quantifier alternations}. Our present Subsection 2.3 gives a much better treatment of the complexity measure than the one in \[Vis93\]. Subsection 2.5 develops the facts about sequential theories and partial satisfaction predicates in greater detail than previously available in the literature. Moreover, we provide careful estimates of the complexities yielded by the various constructions. On the one hand, these subsections contain ‘what we already knew’, on the other hand, as I found, even if you already know how things go, it can still be quite a puzzle to get all nuts and bolts at the precise places where they have to go. Of course, the present treatment is still not fully explicit, but we are further on the road.

Section 3 contains the central result of the paper. As the reader will see, after all is said and done, the central argument is amazingly simple. The work is in creating the setting in which the result can be comfortably stated.

\footnote{I presented this result in a lecture for the Moscow Symposium on Logic, Algebra and Computation in 2006. However, I was not able to write down the proof, since I lacked the necessary groundwork on partial satisfaction. This groundwork is provided in Section 2 of the present paper.}
In the present section, we provide the basics needed for the rest of the paper. As pointed out in the introduction the development of partial satisfaction predicates is done in more detail here than elsewhere. For this reason this section may also turn out to be useful for subsequent work. Of course, the reader who wants to get on quickly to more exciting stuff could briefly look over the relevant subsections and, if needed, return to them later.

2.1. Theories. In this paper we will study theories with finite signature. In most of our papers, theories are intensional objects equipped with a formula representing the axiom set. In the present paper, to the contrary, a theory is just a set of sentences of the given signature closed under deduction. This is because most of the results in the paper are extensional.

Also we do not have any constraints on the complexity of the axiom set of the theory. If a theory is finitely axiomatizable, par abus de langage, we use the variables like $A$ and $B$ for it, making the letters do double work: they both stand for the theory and for a single axiom.

When we diverge from our general format this will always be explicitly mentioned.

In the paper, we will meet many concrete theories, to wit $\text{AS}$, $\text{PA}^-$, $\text{EA}$, $\text{PRA}$, $\text{PA}$. We refer the reader to the textbooks [HP93] and [Kay91] for an introduction to these theories.

2.2. Translations and Interpretations. We present the notion of $m$-dimensional interpretation without parameters. There are two extensions of this notion: we can consider piecewise interpretations and we can add parameters. We will give some details on parameters in Appendix A. We will not describe piecewise interpretations here.

Consider two signatures $\Sigma$ and $\Theta$. An $m$-dimensional translation $\tau : \Sigma \rightarrow \Theta$ is a quadruple $\langle \Sigma, \delta, \mathcal{F}, \Theta \rangle$, where $\delta(v_0, \ldots, v_{m-1})$ is a $\Theta$-formula and where, for any $n$-ary predicate $P$ of $\Sigma$, $\mathcal{F}(P)$ is a formula $A(\vec{v}_0, \ldots, \vec{v}_{n-1})$ in the language of signature $\Theta$, where $\vec{v}_i = v_{i0}, \ldots, v_{i(m-1)}$. Both in the case of $\delta$ and $A$ all free variables are among the variables shown. Moreover, if $i \neq j$ or $k \neq \ell$, then $v_{ik}$ is syntactically different from $v_{j\ell}$.

We demand that we have $\vdash \mathcal{F}(P)(\vec{v}_0, \ldots, \vec{v}_{n-1}) \rightarrow \bigwedge_{i<n} \delta(\vec{v}_i)$. Here $\vdash$ is provability in predicate logic. This demand is inessential, but it is convenient to have.

We define $B^\tau$ as follows:

- $(P(x_0, \ldots, x_{n-1}))^\tau := \mathcal{F}(P)(\vec{x}_0, \ldots, \vec{x}_{n-1})$.
- $\cdot$ commutes with the propositional connectives.
- $(\forall x A)^\tau := \forall \vec{x}(\delta(\vec{x}) \rightarrow A^\tau)$.
- $(\exists x A)^\tau := \exists \vec{x}(\delta(\vec{x}) \land A^\tau)$.

There are two worries about this definition. First, what variables $\vec{x}_i$ on the side of the translation $A^\tau$ correspond with $x_i$ in the original formula $A$? The second worry is that substitution of variables in $\delta$ and $\mathcal{F}(P)$ may cause variable-clashes. These worries are never important in practice: we choose ‘suitable’ sequences $\vec{x}$ to correspond to variables $x$, and we avoid clashes by $\alpha$-conversion. However, if we want to give precise definitions of translations and, for example, of composition of
translations, these problems come into play. The problems are clearly solvable in a systematic way, but this endeavor is beyond the scope of this paper.

We allow the identity predicate to be translated to a formula that is not identity.

A translation \( \tau \) is direct, if it is one-dimensional and if \( \delta_\tau(x) := (x = x) \) and if it translates identity to identity.

There are several important operations on translations.

- \( \text{id}_\Sigma \) is the identity translation. We take \( \delta_{\text{id}_\Sigma}(v) := v = v \) and \( F(P) := P(\bar{v}) \).
- We can compose translations. Suppose \( \tau : \Sigma \to \Theta \) and \( \nu : \Theta \to \Lambda \). Then \( \nu \circ \tau \) or \( \tau \nu \) is a translation from \( \Sigma \) to \( \Lambda \). We define:
  \[
  \begin{align*}
  \delta_{\tau \nu}(\bar{v}_0, \ldots, \bar{v}_{m-1}) & := \bigwedge_{i=0}^{m-1} \delta_{\nu}(\bar{v}_i) \land (\delta_{\tau}(\bar{v}_0, \ldots, \bar{v}_{m-1}))^\nu, \\
  P_{\tau \nu}(\bar{v}_0, \ldots, \bar{v}_{m-1}) & := \bigwedge_{i=0}^{m-1} \delta_{\nu}(\bar{v}_i) \land (P(\bar{v}_0, \ldots, \bar{v}_{m-1}))^\nu.
  \end{align*}
  \]
- Let \( \tau, \nu : \Sigma \to \Theta \) and let \( A \) be a sentence of signature \( \Theta \). We define the disjunctive translation \( \sigma := \tau(A) \nu : \Sigma \to \Theta \) as follows. We take \( m_\tau := \max(m_\tau, m_\nu) \). We write \( \bar{v} \upharpoonright n \), for the restriction of \( \bar{v} \) to the first \( n \) variables, where \( n \leq \text{length}(\bar{v}) \).
  \[
  \begin{align*}
  \delta_\sigma(\bar{v}) & := (A \land \delta_\tau(\bar{v} \upharpoonright m_\tau) \lor (\neg A \land \delta_\nu(\bar{v} \upharpoonright m_\nu))), \\
  P_\sigma(\bar{v}_0, \ldots, \bar{v}_{n-1}) & := (A \land P_\tau(\bar{v}_0 \upharpoonright m_\tau, \ldots, \bar{v}_{n-1} \upharpoonright m_\tau)) \lor \\
  & (\neg A \land P_\nu(\bar{v}_0 \upharpoonright m_\nu, \ldots, \bar{v}_{n-1} \upharpoonright m_\nu)).
  \end{align*}
  \]

Note that in the definition of \( \tau(A) \nu \) we used a padding mechanism. In case, for example, \( m_\tau < m_\nu \), the variables \( v_{m_\tau}, \ldots, v_{m_\nu-1} \) are used ‘vacuously’ when we have \( A \). If we had piecewise interpretations, where domains are built up from pieces with possibly different dimensions, we could avoid padding by building the domain directly of disjoint pieces with different dimensions.

A translation relates signatures; an interpretation relates theories. An interpretation \( \mathcal{K} : U \to V \) is a triple \( \langle U, \tau, V \rangle \), where \( U \) and \( V \) are theories and \( \tau : \Sigma_U \to \Sigma_V \).

We demand: for all theorems \( A \) of \( U \), we have \( V \vdash A^\tau \). Here are some further definitions.

- \( \text{ID}_U : U \to U \) is the interpretation \( \langle U, \text{id}_\Sigma, U \rangle \).
- Suppose \( K : U \to V \) and \( \mathcal{M} : V \to \mathcal{W} \). Then, \( KM := M \circ K : U \to W \) is \( \langle U, \tau_M \circ \tau_K, W \rangle \).
- Suppose \( K : U \to (V + A) \) and \( \mathcal{M} : V \to (V + \neg A) \). Then \( K(\mathcal{A})M : U \to V \) is the interpretation \( \langle U, \tau_K(\mathcal{A}) \tau_M, V \rangle \). In an appropriate category \( K(\mathcal{A})M \) is a special case of a product.

A translation \( \tau \) maps a model \( \mathcal{M} \) to an internal model \( \mathcal{K}(\mathcal{M}) \) provided that \( \mathcal{M} \models \exists \bar{x} \delta_\tau(\bar{x}) \). Thus, an interpretation \( K : U \to V \) gives us a mapping \( \mathcal{K} \) from \( \text{MOD}(V) \), the class of models of \( V \), to \( \text{MOD}(U) \), the class of models of \( U \). If we build a category of theories and interpretations, usually \( \text{MOD} \) with \( \text{MOD}(K) := \mathcal{K} \) will be a contravariant functor.

We use \( U \xrightarrow{\mathcal{K}} V \) or \( K : U \preccurlyeq V \) or \( K : V \succcurlyeq U \) as alternative notations for \( K : U \to V \). The alternative notations \( \preccurlyeq \) and \( \succcurlyeq \) are used in a context where we are interested in interpretability as a preorder or as a provability analogue.

We write: \( U \preccurlyeq V \) and \( V \succcurlyeq U \), for: there is an interpretation \( K : U \preccurlyeq V \). We use \( U \equiv V \), for: \( U \preccurlyeq V \) and \( V \preccurlyeq U \).

The arrow notations are mostly used in a context where we are interested in a category of interpretations, but also simply when they improve readability.
We write \( U \prec_{\text{loc}} V \) or \( V \succ_{\text{loc}} U \) for: for all finite subtheories \( U_0 \) of \( U \), \( U_0 \prec V \). We pronounce this as: \( U \) is locally interpretable in \( V \) or \( V \) locally interprets \( U \). We use \( \equiv_{\text{loc}} \) for the induced equivalence relation of \( \prec_{\text{loc}} \).

2.3. Complexity and Restricted Provability. Restricted provability plays an important role in the study of interpretability between sequential theories. An \( n \)-proof is a proof from axioms with Gödel number smaller or equal than \( n \) only involving formulas of complexity smaller or equal than \( n \). To work conveniently with this notion, a good complexity measure is needed. Such a measure should satisfy three conditions.

i. Eliminating terms in favor of a relational formulation should raise the complexity only by a fixed standard number.

ii. Translation of a formula via the translation \( \tau \) should raise the complexity of the formula by a fixed standard number depending only on \( \tau \).

iii. The tower of exponents involved in cut-elimination should be of height linear in the complexity of the formulas involved in the proof.

Such a good measure of complexity together with a verification of desideratum (iii)—a form of nesting degree of quantifier alternations—is supplied in the work of Philipp Gerhardy. See [Ger03] and [Ger05]. A slightly different measure is provided by Samuel Buss in [Bus15]. Buss also proves that (iii) is fulfilled for his measure. In fact, Buss proves a sharper result. He shows that the bound is \( d + O(1) \) for \( d \) alternations. In the present paper, we will follow Buss’ treatment.

We work over a signature \( \Theta \). The formula-classes we define are officially called \( \Sigma^*_n(\Theta) \) and \( \Pi^*_n(\Theta) \). However, we will suppress the \( \Theta \) when it is clear from the context. Let \( \overline{\text{AT}} \) be the class of atomic formulas for \( \Theta \), extended with \( \top \) and \( \bot \). We define:

* \( \Sigma^*_0 := \Pi^*_0 := \emptyset \).

* \( \Sigma^*_{n+1} := \overline{\text{AT}} \cap \neg \Pi^*_{n+1} \cap (\Sigma^*_{n+1} \land \Sigma^*_{n+1}) \cap (\Sigma^*_{n+1} \lor \Sigma^*_{n+1}) \cap (\Pi^*_{n+1} \to \Sigma^*_{n+1}) \cap \exists v \Sigma^*_{n+1} \cap \forall v \Pi^*_{n} \).

* \( \Pi^*_{n+1} := \overline{\text{AT}} \cap \neg \Sigma^*_{n+1} \cap (\Pi^*_{n+1} \land \Pi^*_{n+1}) \cap (\Pi^*_{n+1} \lor \Pi^*_{n+1}) \cap (\Sigma^*_{n+1} \to \Pi^*_{n+1}) \cap \forall v \Pi^*_{n+1} \cap \exists v \Sigma^*_{n} \).

Buss uses \( \Sigma^*_{n+1} \) and \( \Pi^*_{n+1} \) where we use \( \Sigma^*_n \) and \( \Pi^*_n \). We employ the asterix to avoid confusion with the usual complexity classes in the arithmetical hierarchy where bounded quantifiers also play a role. Secondly, we modified Buss’ inductive definition a bit in order to get unique generation histories. For example, Buss adds \( \Pi^*_n \) to \( \Sigma^*_n \) in stead of \( \forall v \Pi^*_n \). In addition our \( \Sigma^*_0 \) and \( \Pi^*_0 \) are empty, where Buss’ corresponding classes consist of the quantifier-free formulas.

Here is the parse-tree of \( \forall x (\forall y \exists z Pxyz \to \exists u \exists v Qxuv) \) as an element of \( \Sigma^*_4 \).
The extensional equivalence, for \( n > 0 \) of our definition to Buss’s is immediate from the following:

**Fact 2.1.** The quantifier-free formulas are in \( \Sigma^*_n \cap \Pi^*_n \) and \( \Sigma^*_n \cup \Pi^*_n \subseteq \Sigma^*_{n+1} \cap \Pi^*_{n+1} \).

The proof is by five simple inductions. We define:

\[
\begin{align*}
\Delta^*_n & := \\
\forall \sigma & \neg \Delta^*_n, (\Delta^*_n \land \Delta^*_n) \lor (\Delta^*_n \lor \Delta^*_n) \lor (\Delta^*_n \rightarrow \Delta^*_n) \lor \exists \sigma \Sigma^*_n \lor \forall \sigma \Pi^*_n.
\end{align*}
\]

We have:

**Theorem 2.2.** \( \Delta^*_{n+1} = \Sigma^*_n \cap \Pi^*_n \).

**Proof.** That \( \Delta^*_{n+1} \subseteq \Sigma^*_n \cap \Pi^*_n \) is an easy induction based on Fact 2.1. We prove the converse by ordinary induction on formulas. The atomic case and the propositional cases are immediate. Suppose \( \Delta^*_n \cap \Pi^*_n \) has the form \( \exists \sigma \sigma \). Then \( \sigma \) must be in \( \Sigma^*_n \). It follows that \( \Delta^*_n \) and, thus, that \( \Delta^*_n \) is in \( \Delta^*_n \).

We want a complexity measure \( \rho(A) \) such that \( \rho(A) \) is the smallest \( n \) such that \( A \) is in \( \Sigma^*_n \). This measure is very close to the measure that was employed in [Vis93]. We recursively define this measure by taking \( \rho := \rho_2 \), where \( \rho_2 \) is defined as follows:

\[
\begin{align*}
\rho_2(A) & := \rho_2(A) = 1, \text{ if } A \text{ is atomic} \\
\rho_2(\neg B) & := \rho_2(B), \quad \rho_2(B) := \rho_2(B). \\
\rho_2(B \land C) & := \max(\rho_2(B), \rho_2(C)), \quad \rho_2(B \land C) := \max(\rho_2(B), \rho_2(C)). \\
\rho_2(B \lor C) & := \max(\rho_2(B), \rho_2(C)), \quad \rho_2(B \lor C) := \max(\rho_2(B), \rho_2(C)). \\
\rho_2(B \rightarrow C) & := \max(\rho_2(B), \rho_2(C)), \quad \rho_2(B \rightarrow C) := \max(\rho_2(B), \rho_2(C)). \\
\rho_2(\exists \sigma B) & := \rho_2(B), \quad \rho_2(\exists \sigma B) := \rho_2(B) + 1. \\
\rho_2(\forall \sigma B) & := \rho_2(B) + 1, \quad \rho_2(\forall \sigma B) := \rho_2(B). \\
\rho(A) & := \rho_2(A), \quad \rho_0(A) := \max(\rho_2(A), \rho_2(A)).
\end{align*}
\]

We verify the basic facts about \( \rho. \)

**Theorem 2.3.** \( \rho_\theta(A) \leq \rho_2(A) + 1 \) and \( \rho_2(A) \leq \rho_\theta(A) + 1. \)

**Proof.** The proof is by induction on \( A. \) We treat the case that \( A = \exists \sigma \sigma. \) We have:

\[
\rho_\theta(\exists \sigma B) = \rho_2(B) + 1 = \rho_2(B) + 1.
\]

Note that this does not use the induction hypothesis.

**Theorem 2.4.** \( \Sigma^*_n = \{ A \mid \rho_2(A) \leq n \} \) and \( \Pi^*_n = \{ A \mid \rho_\theta(A) \leq n \}. \) It follows that, for \( n > 0 \), we have \( \Delta^*_n = \{ A \mid \rho_0(A) \leq n \} \)

**Proof.** We prove, by induction on \( n \), that: \( A \in \Sigma^*_n \) iff \( \rho_2(A) \leq n \) and \( A \in \Pi^*_n \) iff \( \rho_\theta(A) \leq n \).

The case of 0 is clear. We prove by induction on the definition of \( \Sigma^*_n \), that \( A \in \Sigma^*_n \) iff \( \rho_2(A) \leq n + 1 \). The atomic case, the propositional cases and the existential case are clear. Suppose \( A = \forall \sigma \sigma. \) If \( A \) is in \( \Sigma^*_n \), then \( B \) is in \( \Pi^*_n. \) By the Induction Hypothesis, \( \rho_\theta(B) \leq n, \) so \( \rho_\theta(A) \leq n + 1. \) If \( \rho_\theta(A) \leq n, \) then \( \rho_\theta(B) \leq n. \) Hence, by the Induction Hypothesis, \( B \in \Pi^*_n, \) so \( A \in \Sigma^*_n. \) The case of \( \Pi^*_n \) is similar.

Let \( \tau : \Sigma \rightarrow \Theta \) be a translation. We define \( \rho^*(\tau) \) to be the maximum of \( \rho_0(\delta_\tau) \) and the \( \rho_0(P_{\tau}), \) for \( P \) in \( \Sigma. \) If \( K \) is an interpretation, then \( \rho^*(K) := \rho^*(\tau_K). \)

**Theorem 2.5.** Let \( \tau : \Sigma \rightarrow \Theta. \) We have:

\[
\rho_2(A^\tau) \leq \rho_2(A) + \rho^*(\tau) \quad \text{and} \quad \rho_\theta(A^\tau) \leq \rho_\theta(A) + \rho^*(\tau).
\]
Proof. The proof is by induction on $A$. The case of the atoms is trivial.

We treat the case of implication and $\rho_\exists$. Suppose $A$ is $B \rightarrow C$. We have:

$$\rho_\exists(A^\tau) = \max(\rho_\forall(B^\tau), \rho_\exists(C^\tau))$$

$$\leq \max(\rho_\forall(B) + \rho^*(\tau), \rho(C) + \rho^*(\tau))$$

$$= \max(\rho_\forall(B), \rho_\exists(C)) + \rho^*(\tau)$$

$$= \rho_\exists(A) + \rho^*(\tau)$$

The other cases concerning the propositional connectives are similar.

We treat the case for universal quantification and $\rho_\exists$. Suppose $A$ is $\forall v B$. We have:

$$\rho_\exists(A^\tau) = \rho_\exists(\forall \vec{v} (\delta_\tau(\vec{v}) \rightarrow B^\tau))$$

$$= \rho_\forall(\delta_\tau(\vec{v}) \rightarrow B^\tau) + 1$$

$$= \max(\rho_\exists(\delta_\tau(\vec{v})), \rho_\forall(B^\tau)) + 1$$

$$\leq \rho_\forall(B) + \rho^*(\tau) + 1$$

$$= \rho_\exists(A) + \rho^*(\tau)$$

The remaining cases for the quantifiers are similar or easier.

\[
\]

2.4. Sequential Theories. The notion of sequentiality is due to Pavel Pudlák. See, e.g., [Pud83], [Pud85], [MPS90], [HP93].

To define sequentiality we use the auxiliary theory $\text{AS}^+$ (Adjunctive Set Theory with extras). The signature $\mathfrak{A}$ of $\text{AS}^+$ consists of unary predicate symbols $N$ and $Z$, binary predicate symbols $\in$, $E$, $\leq$, $<$, $S$, ternary predicate symbols $A$ and $M$.

$\text{AS}^+$ 1 We have a set of axioms that provide a relative interpretation $N^1$ of $S_2^1$ in $\text{AS}^+$, where $N$ represents the natural numbers, $E$ represents numerical identity, $Z$ stands for zero modulo $E$, $A$ stands for addition modulo $E$, and $M$ stands for multiplication modulo $E$.

$\text{AS}^+$ 2 $\vdash \exists x \forall y y \notin x$,

$\text{AS}^+$ 3 $\vdash \forall x \exists y \forall z (z \in y \leftrightarrow z = x)$,

$\text{AS}^+$ 4 $\vdash \forall x, y \exists z \forall u (u \in z \leftrightarrow (u \in x \lor u \in y))$,

$\text{AS}^+$ 5 $\vdash \forall x, y \exists z \forall u (u \in z \leftrightarrow (u \in x \land u \in y))$,

$\text{AS}^+$ 6 $\vdash \forall x, y \exists z \forall u (u \in z \leftrightarrow (u \in x \land \neg u \in y))$.

An important point is that we do not demand extensionality for our sets. A many-sorted version of $\text{AS}^+$ would be somewhat more natural. We refrain from developing it in this way here to avoid the additional burden of working with interpretations between many-sorted theories.

A theory is sequential iff it interprets the theory $\text{AS}^+$ via a direct interpretation $S$. We call such an $S$ a sequence scheme.

It is possible to work with an even simpler base theory. The theory $\text{AS}$ is given by the following axioms.

$\text{AS}1 \vdash \exists y \forall x x \notin y$,

$\text{AS}2 \vdash \forall x \forall y \exists z \forall u (u \in z \leftrightarrow (u \in y \lor u = x))$.

One can show that $\text{AS}$ is mutually directly interpretable with $\text{AS}^+$. For details concerning the bootstrap see e.g. the textbook [HP93] and also [MPS90], [Vis09], [Vis11b], [Vis13].
Remark 2.6. We could work in a somewhat richer class of theories, the polysequential theories. See [Vis13].

Let’s say that an interpretation is \( m \)-direct, if it is \( m \)-dimensional, if its domain consists of all \( m \)-tuples of the original domain, and if identity is interpreted as component-wise identity. A theory \( U \) is \( m \)-sequential, if there is an \( m \)-direct interpretation of \( \text{AS}^+ \) in \( U \). A theory is polysequential, if it is \( m \)-sequential for some \( m \geq 1 \). Note that if we want the \( \text{AS}^+ \) format, the interpretation of the natural numbers should also be chosen to be \( m \)-dimensional for the given \( m \). The development given in the present paper also works with minor adaptations in the polysequential case.

It is known that there are polysequential theories that are not sequential. However, I only have an artificial example. Every polysequential theory is polysequential without parameters, where a sequential theory may essentially need an interpretation with parameters to witness its sequentiality. (One raises the dimension to ‘eat up’ the parameters.) Polysequential theories are closed under bi-interpretability. Moreover, every polyquential theory is bi-interpretable with a sequential one.

2.5. Satisfaction & Reflection. In this subsection, we develop partial satisfaction predicates for sequential theories with some care. We prove the corresponding partial reflection principles. This subsection is rather long because it provides many details. The impatient reader could choose to proceed to Theorem 2.17 since that is the main result of the subsection that we will use in the rest of the paper.

Consider any signature \( \Theta \). We extend the signature \( \mathfrak{A} \) of \( \text{AS}^+ \) in a disjoint way with \( \Theta \) to, say, \( \mathfrak{A} + \Theta \). Call the resulting theory (without any new axioms) \( \text{AS}^+ / \Theta \).

We work towards the definition of partial satisfaction predicates. We provide a series of definitions illustrative of what we need to get off the ground.

- \( \text{pair}(u, v, w) : \iff \exists a \exists b (\forall c (c \in w \iff (c = a \lor c = b)) \land \forall d (d \in u \iff d = u) \land \forall e (e \in b \iff (e = u \lor e = v))) \).
  We can easily show that for all \( u \) and \( v \) there is a \( w \) such that \( \text{pair}(u, v, w) \) and, whenever \( \text{pair}(u, v, w) \) and \( \text{pair}(u', v', w) \), then \( u = u' \) and \( v = v' \). Note that there may be several \( w \) such that \( \text{pair}(u, v, w) \).

- \( \text{Pair}(w) : \iff \exists u \exists v \text{pair}(u, v, w) \).

- \( \pi_0(w, u) : \iff \exists v \text{pair}(u, v, w) \).

- \( \pi_1(w, v) : \iff \exists u \text{pair}(u, v, w) \).

- \( \text{fun}(f) : \iff \forall w \in f (\text{pair}(w) \land \forall u' \in f \forall u (\pi_0(w, u) \land \pi_0(w', u) \rightarrow w = w')) \).

Note that we adapt the notion of function to our non-extensional pairing. We demand that there is at most one witnessing pair for a given argument. This choice makes resetting a function on an argument where it is defined a simple operation: we subtract one pair and we add one.

- \( \text{dom}(f, x) : \iff \text{fun}(f) \land \exists w \in f \pi_0(w, x) \).

- \( f(u) \approx v : \iff \text{fun}(f) \land \exists w \in f \text{pair}(u, v, w) \).

- \( \text{nfun}(\alpha) : \iff \forall w \in \alpha (\text{pair}(w) \land \forall u (\pi_0(w, u) \rightarrow \text{N}(u)) \land \forall w' \in \alpha \forall u \forall u' ((\pi_0(w, u) \land \pi_0(w', u') \land u \in u' \rightarrow w = w')) \).

We note that \( \text{nfun}(f) \) implies \( \text{fun}(f) \).

- \( \text{ndom}(\alpha, a) : \iff \text{nfun}(\alpha) \land \exists w \in \alpha \exists b (a \in b \land \pi_0(w, b)) \).

- We fix a parameter \( x^* \).

\[ \alpha[a] \approx v : \iff \text{nfun}(\alpha) \land (\exists w \in \alpha \exists b (a \in b \land \text{pair}(b, v, w)) \lor (\neg \text{ndom}(\alpha, a) \land v = x^*)) \].
So outside of $\text{ndom}$ we set the value of $\alpha$ to a default value. In this way we made it a total function on the natural numbers\(^2\)

\[ \alpha [a] \beta : \iff \forall b (\text{ndom}(\beta, b) \iff (\text{ndom}(\alpha, b) \lor b \in a)) \land \\
\forall b \forall x ((N(b) \land \neg a \in b) \rightarrow (\alpha[b] \approx x \iff \beta[b] \approx x)). \]

\[ \alpha[a : y] \approx \beta : \iff \alpha [a] \beta \land \beta[a] \approx y. \]

We will use $\alpha, \beta$ to range over functions with numerical domains (elements of $\text{nfun}$).

We note that $\text{AS}^+(\Theta)$ proves that the ‘operation’ $\alpha \mapsto \alpha[a : y]$ is total and that any output sequences are extensionally the same.

We employ a usual efficient coding of syntax in the interpretation $\mathcal{N}$ of $\mathcal{S}_1$. We have shown above how to formulate things in order to cope with the fact that each number, set, pair, function and sequence can have several representatives. The definition of satisfaction would be completely unreadable if we tried to adhere to this high standard. Hence, we will work more informally pretending, for example, that each number has just one representative. An assignment will simply be a numerical function where we restrict our attention to the codes of variables in the domain.

A sat-sequence is a triple of the form $\langle i, \alpha, A \rangle$, where: $i$ is $+$ or $-1$ (coded as, say, 1 and 0). $\alpha$ is an assignment and $A$ is a formula. An a-sat-sequence is a sat-sequence $\langle i, \alpha, A \rangle$, where $A$ is $\Sigma^*_\alpha$, if $i = +$ and $A$ is $\Pi^*_\alpha$, if $i = -$. We call the virtual class of all a-sat sequences $\mathcal{K}_\alpha$.

A set $X$ is good if, for all numbers $b$ the virtual class $X \cap \mathcal{K}_b$ exists as a set. We have:

**Lemma 2.7.** ($\text{AS}^+(\Theta)$). The good sets are closed under the empty sets, singletons, union, intersection and subtraction and are downwards closed w.r.t. the subset ordering.

**Proof.** We leave the easy proof to the reader. \(\square\)

We count two sat-sequences $\langle i, \alpha, A \rangle$ and $\langle j, \beta, B \rangle$ as extensionally equal if (i) $i$ and $j$ are $\mathcal{E}$-equal, (ii) $\alpha$ and $\beta$ have the same functional behaviour on the natural numbers $\mathbb{N}$, and (iii) $A$ and $B$ are $\mathcal{E}$-equal. We say that a sequence $\sigma$ is of the form $\langle i, \alpha, A \rangle$ if it is extensionally equal to a sequence $\tau$ with $\tau_0 = i$, $\tau_1 = \alpha$ and $\tau_2 = A$.

We define $n + 1$-adequacy and $\text{sat}_n$ by external recursion on $n$. We define $\text{sat}_0(+, \alpha, A) : \mapsto \bot$ and $\text{sat}_0(-, \alpha, A) : \mapsto \top$. We define $\text{sat}_{n+1}(i, \alpha, A)$ iff, for some $n + 1$-adequate set $X$, we have $\langle i, \alpha, A \rangle \in X$. A set $X$ is $n + 1$-adequate, if it is good, if its elements are sat-sequences, and if it satisfies the following clauses:

a. If a sequence of the form $\langle +, \beta, P(v_0, v_1) \rangle$ is in $X$, then $P(\beta(v_0), \beta(v_1))$.\(^3\)

Similarly, for other atomic formulas including $\top$ and $\bot$.

\(^2\)The need for a parameter is regrettable but in a sequential theory there need not be definable elements. Of course, we could set the value at $a$ outside the $\text{ndom}$ of $\alpha$ to $a$, but that would mean that when we reset our function we would change all the values outside the $\text{ndom}$ too. One way to eliminate the parameter would be to make the default value an extra part of the data for the function. Then, a reset would keep the default value in tact. The option of working with partial functions is certainly feasible. However, e.g., clause (e) of the definition of adequate set would be more complicated.

\(^3\)Our variables are coded as numbers. Conceivably not all numbers code variables. The values of $\beta$ on non-variables are simply don’t care.
b. If a sequence of the form $(-, \beta, P(v_0, v_1))$ is in $X$, then $\neg P(\beta(v_0), \beta(v_1))$.

Similarly, for other atomic formulas including $\top$ and $\bot$.

c. If a sequence of the form $(+, \beta, \neg B)$ is in $X$, then a sequence of the form $(-, \beta, B)$ is in $X$.

d. If a sequence of the form $(-, \beta, \neg B)$ is in $X$, then a sequence of the form $(+, \beta, B)$ is in $X$.

e. If a sequence of the form $(+, \beta, (B \land C))$ is in $X$, then sequences of the form $(+, \beta, B)$ and $(+, \beta, C)$ are in $X$.

f. If a sequence of the form $(-, \beta, (B \land C))$ is in $X$, then a sequence of the form $(-, \beta, B)$ is in $X$ or a sequence of the form $(-, \beta, C)$ is in $X$.

g. If a sequence of the form $(+, \beta, (B \lor C))$ is in $X$, then a sequence of the form $(+, \beta, B)$ is in $X$ or a sequence of the form $(+, \beta, C)$ is in $X$.

h. If a sequence of the form $(+, \beta, (B \lor C))$ is in $X$, then sequences of the form $(-, \beta, B)$ and $(-, \beta, C)$ are in $X$.

i. If a sequence of the form $(+, \beta, (B \rightarrow C))$ is in $X$, then a sequence of the form $(+, \beta, B)$ is in $X$ or a sequence of the form $(+, \beta, C)$ is in $X$.

j. If a sequence of the form $(-, \beta, (B \rightarrow C))$ is in $X$, then sequences of the form $(+, \beta, B)$ and $(+, \beta, C)$ are in $X$.

k. If a sequence of the form $(+, \beta, \exists v B)$ is in $X$, then, for some $\gamma$ with $\beta \models v \gamma$, a sequence of the form $(+, \gamma, B)$ is in $X$.

l. If a sequence of the form $(+, \beta, \exists v B)$ is in $X$, then $\neg \mathsf{sat}_n(+, \beta, \exists v B)$.

m. If a sequence of the form $(+, \beta, \forall v B)$ is in $X$, then $\neg \mathsf{sat}_n(-, \beta, \forall v B)$.

n. If a sequence of the form $(-, \beta, \forall v B)$ is in $X$, then, for some $\gamma$ with $\beta \models v \gamma$, a sequence of the form $(-, \gamma, B)$ is in $X$.

Note that if $\sigma$ and $\tau$ are extensionally equal and if $X$ is $n$-adequate and $\sigma$ is in $X$, then the result of replacing $\sigma$ in $X$ by $\tau$ is again $n$-adequate.

We will often write $\alpha \models^i_n A$ for $\mathsf{sat}_n(i, \alpha, A)$. The relation $\mathsf{sat}_n$, when restricted to $\mathcal{K}_n$, will have a number of desirable properties. We write $\mathsf{sat}_n^*$ for $\mathsf{sat}_n \cap \mathcal{K}_n$.

We note that we have implicitly given a formula $\Phi_0(\mathcal{X}, i, \alpha, A)$, where $\mathcal{X}$ is a second order variable with:

$$\mathsf{sat}_{n+1}(i, \alpha, A) = \Phi_0(\mathsf{sat}_n, i, \alpha, A).$$

Thus, for some fixed standard $c_0$, we have $\rho(\mathsf{sat}_{n+1}(u, v, w)) = \rho(\mathsf{sat}_n(u, v, w)) + c_0$.

It follows that $\rho(\mathsf{sat}_n(u, v, w)) = c_0n + c_1$, for some fixed standard number $c_1$.

**Remark 2.8.** We note that $\mathcal{X}$ occurs twice in the formula $\Phi_0(\mathcal{X}, k, \alpha, A)$ described above. This has no effect on the growth of the complexity of the formula $\mathsf{sat}_n$ but it makes the number of symbols of the formula $\mathsf{sat}_n$ grow exponentially in $n$. For the purposes of this paper, this is good enough. However, a slightly more careful rewrite of our definition reduces the number of occurrences of $\mathcal{X}$ to one. As a consequence, we can get the number of symbols of $\mathsf{sat}_n$ linear in $n$. So, the code of $\mathsf{sat}_n$ will be bounded by a polynomial in $n$, assuming we use an efficient Gödel numbering.

Our next step is to verify in $\mathsf{AS}^+(\Sigma)$ some good properties of $n + 1$-adequacy and $\mathsf{sat}_n$. We first show that $n$-adequacy is preserved under certain operations.

**Theorem 2.9.** The $n$-adequate sets are closed under unions and under intersection with the virtual class of $a$-sat-sequences, for any $a$. 


Proof. Closure under unions is immediate given that good sets are closed under unions. Closure under restriction to $\alpha$-sat-sequences is immediate by the definition of good and the fact that our formula classes are closed under subformulas.

We prove a theorem connecting $\text{sat}_k^*$ and $\text{sat}_n^*$, for $k < n$. We remind the reader that $K_k$ is the virtual class of all $k$-sat-sequences and $\text{sat}_n^* = \text{sat}_n \cap K_n$.

**Theorem 2.10 (AS$^+$($\Theta$)).** Suppose $k < n$. Then, $\text{sat}_k^* = \text{sat}_n^* \cap K_k$.

Proof. The proof is by external induction on $n$. The case that $n = 0$ is trivial.

Suppose $X$ is $n + 1$-adequate set. Let $Y := X \cap K_k$. We note that $Y$ is a set by Theorem 2.9. We claim that $(\neg_k, \neg_k)$ and $(\neg_k)$ is in $Y$. From this it follows that $k \neq 0$. Since $X$ is $n + 1$-adequate, it follows that: $\neg \text{sat}_n (+, \beta, \exists v B)$. By the induction hypothesis, we find that $\neg \text{sat}_{k-1} (+, \beta, \exists v B)$. Hence, the clause for $k$-adequacy is fulfilled. Clause (m) is similar.

Conversely, suppose $Z$ is $k$-adequate. Let $W := Z \cap K_k$. The argument that $W$ is also $n + 1$-adequate, is analogous to the argument above.

We note that in the proof of Theorem 2.10 we could as well do the induction on $k$. This observation is important in case we study models with a full satisfaction predicate. In this context, we can replace $n$ by a non-standard number and still get our result for standard $k$.

In the following theorem we prove the commutation conditions for $\text{sat}_{n+1}$.

**Theorem 2.11 (AS$^+$($\Theta$)).** We have:

a. $\beta \models_{n+1}^+ P(v_0, v_1)$ iff $P(\beta(v_0), \beta(v_1))$, and similarly for the other atomic formulas including $\top$ and $\bot$,

b. $\beta \models_{n+1}^- P(v_0, v_1)$ iff $\neg P(\beta(v_0), \beta(v_1))$, and similarly for the other atomic formulas including $\top$ and $\bot$,

c. $\beta \models_{n+1}^+ \neg B$ iff $\beta \models_{n+1}^- B$,

d. $\beta \models_{n+1}^- \neg B$ iff $\beta \models_{n+1}^+ B$,

e. $\beta \models_{n+1}^+ B \land C$ iff $\beta \models_{n+1}^+ B$ and $\beta \models_{n+1}^+ C$,

f. $\beta \models_{n+1}^- B \land C$ iff $\beta \models_{n+1}^- B$ or $\beta \models_{n+1}^+ C$,

g. $\beta \models_{n+1}^+ B \lor C$ iff $\beta \models_{n+1}^+ B$ or $\beta \models_{n+1}^+ C$,

h. $\beta \models_{n+1}^- B \lor C$ iff $\beta \models_{n+1}^- B$ and $\beta \models_{n+1}^+ C$,

i. $\beta \models_{n+1}^+ B \rightarrow C$ iff $\beta \models_{n+1}^- B$ or $\beta \models_{n+1}^+ C$,

j. $\beta \models_{n+1}^- B \rightarrow C$ iff $\beta \models_{n+1}^+ B$ and $\beta \models_{n+1}^- C$,

k. $\beta \models_{n+1}^+ \exists v B$ iff, for some $\gamma$ with $\beta [v] \gamma$, we have $\gamma \models_{n+1}^+ B$,

l. $\beta \models_{n+1}^- \exists v B$ iff $\beta \not\models_{n+1}^+ \exists v B$,

m. $\beta \models_{n+1}^+ \forall v B$ iff $\beta \not\models_{n+1}^- \forall v B$,

n. $\beta \models_{n+1}^- \forall v B$ iff, for some $\gamma$ with $\beta [v] \gamma$, we have $\gamma \models_{n+1}^- B$.

Proof. We will treat the illustrative clauses (e), (k) and (l). In the first two cases the right-to-left direction is trivial.

Ad (e). Suppose $X$ witnesses that $\beta \models_{n+1}^+ B$ and $Y$ witnesses that $\beta \models_{n+1}^- C$. Let $\sigma$ be a triple of the form $(\beta, +, (B \land C))$. Let $Z$ be a union of $X$ and $Y$ and a singleton with element $\sigma$. It is immediate that $Z$ witnesses $\beta \models_{n+1}^+ B \land C$. 


Ad (k). Suppose that $\beta [v] \gamma$ and that $X$ witnesses that $\gamma \models^+_{n+1} B$. Let $\sigma$ be of the form $\langle \beta, +, \exists v B \rangle$. Let $Y$ be a union of $X$ and a singleton with element $\sigma$. Then $Y$ witnesses $\beta \models^+_{n+1} \exists v B$.

Ad (l). Suppose $\beta \not\models^+_{n} B$. Let $\sigma$ be a triple of the form $\langle \beta, -, \exists v B \rangle$. Let $X$ be a singleton with element $\sigma$. Then $X$ witnesses $\beta \models^-_{n+1} \exists v B$. Conversely, if $Y$ witnesses $\beta \models^-_{n+1} \exists v B$, then we must have $\text{sat}_n (\beta, +, \exists v B)$. \hfill \Box

We note that the commutation conditions are inherited by $\text{sat}^*_n$, provided that the formulas in the conditions belong to the right classes.

The commutation conditions proven in Theorem 2.11 are not yet full commutation conditions. The defect is in the clauses (l) and (m). Let’s zoom in on (m):

* $\beta \models^+_{n+1} \forall v B$ iff $\beta \not\models^+_{n} \forall v B$.

The right-hand-side is equivalent to: for all $\gamma$ with $\beta [v] \gamma$, we have $\gamma \not\models^-_{n+1} B$. To get the desired commutation condition, we would like to move from $\gamma \not\models^-_{n+1} B$ to $\gamma \models^+_{n+1} B$. We have seen in Theorem 2.10 that to make our predicates behave in expected ways, it is better to consider the formulas in their ‘intended range’. So what if $\forall v B$ is in $\Sigma^*_n$? In this case $B$ must be in $\Pi^*_n$ and hence in $\Delta^*_n$. So is it true that if $C$ is in $\Delta^*_n$, then $\alpha \not\models^-_{n+1} C$ iff $\alpha \models^+_{n+1} C$? To prove this we need induction, which we do not have available in $\text{AS}^+(\Theta)$. The solution is to move to a cut. To realize this, we define a second measure of complexity $\nu$ (depth of connectives) as follows: $\nu (A) := 0$ if $A$ is atomic, $\nu (\neg A) := \nu (\exists v A) := \nu (\forall v A) := \nu (A) + 1$ and $\nu (A \circ B) := \max (\nu (A), \nu (B)) + 1$, where $\circ$ is a binary propositional connective. Let $\Gamma_x := \{ A \mid \nu (A) \leq x \}$. We define:

* $J^1_{n+1}$ is the virtual class of all numbers $x$ such that, for all $\alpha$ and for all $C \in \Delta^*_n \cap \Gamma_x$, we have $\alpha \not\models^-_{n+1} C$ iff $\alpha \models^+_{n+1} C$.

We have:

**Theorem 2.12 ($\text{AS}^+(\Theta)$).** $J^1_{n+1}$ contains 0, is closed under successor and is downwards closed w.r.t. $\leq$.

**Proof.** Downwards closure is immediate.

By definition, we have $\alpha \not\models^-_{n+1} C$ iff $\alpha \models^+_{n+1} C$ if $C$ is an atom. So 0 is in $J^1_{n+1}$.

Suppose $x$ is in $J^1_{n+1}$. We will show that $x+1$ is in $J^1_{n+1}$, i.e. for all $C \in \Delta^*_n \cap \Gamma_{x+1}$, we have $\alpha \not\models^-_{n+1} C$ iff $\alpha \models^+_{n+1} C$.

The case for atomic $C$ follows by previous reasoning. Suppose, for example, that $C := (D \rightarrow E)$ is in $\Delta^*_n \cap \Gamma_{x+1}$. Then $D$ and $E$ are both in $\Delta^*_n \cap \Gamma_x$. By the fact that $x$ is in $J^1_{n+1}$ we find:

\[
\begin{align*}
\alpha \models^+_{n+1} (D \rightarrow E) & \iff (\alpha \models^-_{n+1} D) \text{ or } (\alpha \models^+_{n+1} E) \\
& \iff (\alpha \not\models^+_{n+1} D) \text{ or } (\alpha \not\models^-_{n+1} E) \\
& \iff \neg ((\alpha \models^+_{n+1} D) \text{ and } (\alpha \models^-_{n+1} E)) \\
& \iff \neg \alpha \models^-_{n+1} (D \rightarrow E).
\end{align*}
\]

The other unary and binary propositional connectives are similar. Now suppose $C$ is of the form $\exists v D$ and $C \in \Delta^*_n \cap \Gamma_{x+1}$. Since $C$ is in $\Pi^*_n$, we must have
Let $A$ be of the form $(C \land D)$. Suppose $w$ is free for $v$ in $A$. Let $B$ be (of the form) $A[v := w]$. Suppose further that $\alpha$ and $\beta$ assign the same values to the free variables of $A$ except $v$ and $\beta[w] = \alpha[v]$. Clearly $B$ is of the form $E \land F$, where $E$
is of the form \( C[v := w] \) and \( F \) is of the form \( D[v := w] \). We have:

\[
\alpha \models^+_{n+1} C \land D \iff \alpha \models^+_{n+1} C \land \alpha \models^+_{n+1} D
\]

\[
\iff \beta \models^+_{n+1} E \land \beta \models^+_{n+1} F
\]

Similarly for the \( \models^- \)-case. The other propositional cases are similar.

We treat the case of the existential quantifier, the case of the universal quantifier being similar. Let \( A \) be of the form \( \exists z C \). Suppose \( w \) is free for \( v \) in \( A \). Let \( B \) be (of the form) \( A[v := w] \). Suppose further that \( \alpha \) and \( \beta \) assign the same values to the free variables of \( A \) except \( v \) and \( \beta[w] = \alpha[v] \).

We first address the \( \models^+ \)-case. The argument splits into two subcases. First we have the case that \( z \) is (of the form) \( v \), we find that \( B \) is of the form \( A \). Hence, replacing \( z \) by \( v \), we have:

\[
\alpha \models^+_{n+1} \exists C \iff \exists \gamma (\alpha[v] \gamma \models^+_{n+1} C)
\]

\[
\iff \exists \delta (\beta[v] \delta \models^+_{n+1} C)
\]

\[
\iff \beta \models^+_{n+1} \exists C
\]

For example, in the left-to-right direction of the second step we can take \( \delta \) of the form \( \beta[v : \gamma(v)] \). We can use the fact that \( C \) has \( \nu \)-complexity \( x \) and \( x \in J^\nu_{n+1} \). The property \( \mathcal{Q}_{n+1} \) is applied with \( v \) in the role of both \( v \) and \( w \).

Next we have the case that \( z \) and \( w \) are different variables. Let \( D \) be of the form \( C[v : w] \). So \( B \) is of the form \( \exists z D \). We have:

\[
\alpha \models^+_{n+1} \exists C \iff \exists \gamma (\alpha[z] \gamma \models^+_{n+1} C)
\]

\[
\iff \exists \delta (\beta[z] \delta \models^+_{n+1} D)
\]

\[
\iff \beta \models^+_{n+1} \exists D
\]

E.g., in the left-to-right direction of the second step we can again take \( \delta \) of the form \( \beta[z : \gamma(z)] \).

Finally we address the \( \models^- \)-case. We have:

\[
\alpha \models^-_{n+1} \exists C \iff \neg \alpha \models^+_{n} \exists C
\]

\[
\iff \neg \beta \models^+_{n} \exists D
\]

\[
\iff \beta \models^-_{n+1} \exists D
\]

Here we use the fact that \( x \in J^\nu_{n} \), so that also \( x + 1 \in J^\nu_{n} \).

We turn to the second approach. We define:

\[
* \ J^\nu_{n+1} := \{ x \in \mathbb{N} \mid (\Gamma_x \cap \Delta^\nu_{n+1}) \subseteq \mathcal{Q}_{n+1}\}
\]

We have:

**Theorem 2.14 (\( \text{AS}^+(\Theta) \)).** The virtual class \( J^\nu_{n+1} \) is closed under 0, successor and is downwards closed w.r.t. \( \leq \).

**Proof.** The cases of closure under 0 and downwards closure are trivial. Suppose \( x \) is in \( J^\nu_{n+1} \). The cases of the propositional connectives use the same argument as we saw in the proof of theorem 2.13. We consider
Theorem 2.15. We have full commutation of $J$ for $\exists$ and all elements of $\Xi$. Moreover, the $Y_c$ is estimated by a linear term of the form $n$. The first and the fifth step use the commutation conditions for $\exists$. The second and the fourth step use Theorem 2.10. The third step uses the previous case for $\models^+$. We note that the definition of $J^{n+1}$ is of the form $\Phi_3(a, \text{Sat})$. So, its $p_0$-complexity is estimated by a linear term of the form $n + c_6$. Here the use of $J^{n+1}$ has an advantage over $J^n$, since construction of the $J^{n+1}$ gives us a linear complexity but conceivably with a higher constant as coefficient of $n$.

Let us take stock of what we accomplished. We have defined virtual classes $J^{n+1}$ that are closed under $0$ and $\mathbb{S}$ and that are downwards closed such that for all formulas $A$ in $\Gamma^{J^{n+1}}$, we have, for all $\alpha$, that $\alpha \models^+_n A$ if $\alpha \not\models^+_{n+1} A$. Here $\Gamma^{J^{n+1}} := \bigcup_{x \in J^{n+1}} \Gamma_x$.

Also, we have developed virtual classes $J^+_n$ and $J^+_{n+1}$ such that all $A$ in $\Gamma^{J^+_n}$, and, similarly, all $A$ in $\Gamma^{J^+_{n+1}}$ have the property $Q_{n+1}$ defined above.

So, if we take $J^{n+1}_+$ either $J^{n+1}_+ \cap J^+_n$ or $J^+_n \cap J^{n+1}_+$ then $J^{n+1}_+$ is progressive and all elements of $\Xi^{n+1} := \Gamma^{J^{n+1}_+} \cap \Delta^{n+1}$ have both good properties. Let us choose for $J^*$ in the definition of $\Xi^{n+1}$, so that its $p$-complexity is estimated by $n + c_7$.

We summarize the result in a theorem.

Theorem 2.15. We have full commutation of $\text{sat}(+, \cdot)$ for the $\Xi^{n+1}$-formulas. Moreover, the $\Xi^{n+1}$-formulas have property $Q_{n+1}$.

We write $\mathcal{A} \succ_n A$ for the formalization of $\mathcal{A} \vdash_n A$, where $\mathcal{A}$ codes a finite set of formulas and $\vdash_n$ is provability in predicate logic where we restrict ourselves in the proof to $\Xi^{n+1}$-formulas. We choose to code the set of formulas in the natural numbers. This is a bit unnatural since $AS(\Theta)$ contains sets as first-class citizens. However, if we code sets of formulas in the sets provided by $AS(\Theta)$ directly we do not know, for example, that $\text{ass}(p)$ the set of assumptions of a proof $p$ is a set. Of course, this problem can be evaded by shortening $N$ in such a way that any set coded in the natural numbers maps to first-class set. If the reader prefers this other road, we think it is sufficiently clear how to adapt the results below to this alternative approach.

We write $p : \mathcal{A} \succ_n A$ for $p$ is the code of a $n$-proof witnessing $\mathcal{A} \succ_n A$. We write $\Lambda_{n,y}$ for the class of $n$-proofs $p$ where the number of steps of $p$ is $\leq y$. On the semantical side, we define, for $\mathcal{A} \cup \{A\} \subseteq \Xi^{n+1}$:

$$\mathcal{A} \models_{n+1} A : \iff \forall \alpha (\forall A' \in \mathcal{A} \text{sat}_{n+1}(+, \alpha, A') \rightarrow \text{sat}_{n+1}(+, \alpha, A)).$$

We work in $\text{AS}^+(\Theta)$. We write $\text{ass}(p)$ for the assumption set of (proof code) $p$. Let $Y_n$ be the class of $y$ such that, for all $p \in \Lambda_{n,y}$, if $p : \text{ass}(p) \succ_n A$, then $\text{ass}(p) \models_{n} A$. The case of $\models^-$. Suppose $\exists v C$ is in $\Delta^{n+1}_+$. In this case $\exists v C$ must be in $\Sigma^n_+$. We have:

$$\alpha \models^+_{n+1} \exists v C \iff \neg \alpha \models^+_n \exists v C$$

$$\iff \neg \alpha \models^+_n \forall v C$$

$$\iff \neg \beta \models^+_n \exists v D$$

$$\iff \beta \models^+_n \exists v D$$

The first and the fifth step use the commutation conditions for $\exists$. The second and the fourth step use Theorem 2.10. The third step uses the previous case for $\models^+$. We note that the definition of $J^{n+1}_+$ is of the form $\Phi_3(a, \text{Sat})$. So, its $p_0$-complexity is estimated by a linear term of the form $n + c_6$. Here the use of $J^{n+1}_+$ has an advantage over $J^n$, since construction of the $J^{n+1}_+$ gives us a linear complexity but conceivably with a higher constant as coefficient of $n$.
Theorem 2.16. The virtual class \( Y_n \) is downwards closed under \( \leq \), contains 0, and is closed under successor.

Proof. Downwards closure under \( \leq \) is trivial. We show that \( Y_n \) is progressive. We follow the system for Natural Deduction in sequent style as given in [TS00, Subsection 2.1.4]. By Theorem 2.15, the propositional cases are immediate. We will treat the introduction and the elimination rule of the universal quantifier. This follows mainly the usual text book proof. For the convenience of the reader, we repeat the property \( Q_{n+1} \):

- The formula \( C \) has the property \( Q_{n+1} \) if the following holds. Consider any sat-sequence \( (i, \alpha, C) \). Suppose \( u \) is free for \( z \) in \( C \). Suppose further that \( \alpha[a] = \beta[a] \) for all free variables \( a \) of \( A \) except \( z \) and that \( \beta[u] = \alpha[z] \). Then,

\[
\alpha \models^{+}_{n+1} C \iff \beta \models^{+}_{n+1} C[z := u].
\]

We treat the case of universal instantiation. Suppose \((\star)\): Suppose, for all \( A \) in \( \mathcal{A} \), we may conclude that \( \alpha \models^{+}_{n+1} \forall A \).

Consider any \( \alpha \) and suppose \( \alpha \models^{+}_{n+1} A' \), for all \( A' \) in \( \mathcal{A} \). We want to show that \( \alpha \models^{+}_{n+1} \forall A \). Let \( d \) be any element. It is clearly sufficient to show that \( \alpha[v : d] \models^{+}_{n+1} A \).

We first note that \( \alpha[w : d] \) and \( \alpha \) are the same on the free variables of \( A' \) in \( \mathcal{A} \). So, by \( Q_{n+1} \) in a degenerate case, we conclude that \( \alpha[w : d] \models^{+}_{n+1} A' \), for all \( A' \) in \( \mathcal{A} \). By \((\dagger)\), we find \( \alpha[w : d] \models^{+}_{n+1} A[v := w] \). We note that \( \alpha[w : d] \) and \( \alpha[v : d] \) assign the same values to all free variables \( u \) of \( A \), except possibly \( v \). This uses that \( w \) does not occur in \( A \). Moreover, \( \alpha[w : d][v] = \alpha[v : d][v] \). So, by \( Q_{n+1}(A) \), we find \( \alpha[v : d] \models^{+}_{n+1} A \) iff \( \alpha[w : d] \models A[v := w] \). Thus, we may conclude \( \alpha[v : d] \models^{+}_{n+1} A \) as desired.

We treat the case of universal instantiation. Suppose \((\ddagger)\): Suppose, for all \( A' \in \mathcal{A} \), we have \( \alpha \models^{+}_{n+1} A' \). It follows that \((\S)\): \( \alpha \models^{+}_{n+1} \forall A \). We want to conclude that \( \alpha \models A[v := w] \). From \((\S)\), we have \( \alpha[v := \alpha[w]] \models^{+} A \). We note that \( \alpha \) and \( \alpha[v := w] \) assign the same values to all free variables \( A \) except possibly \( v \). Moreover \( \alpha[v := \alpha[w]][v] = \alpha[w] \). By \( Q_{n+1}(A) \), we may conclude that \( \alpha \models^{+}_{n+1} A[v := w] \).

Inspecting the construction of \( Y_n \) we see that it is of the form \( \Phi_{\eta}(\mathbf{sat}_n, J^t_{n+1}) \), where \( \Phi_{\eta}(x, \mathcal{A}, \mathcal{Y}) \) is a fixed formula. Thus \( \rho(Y_n) \) is estimated by \( v + c_8 \).

We now have a refined result involving separate restrictions on \( \rho \) on \( \nu \) and on the lengths of the proofs. For other applications this refinement may be useful, however, in the present paper, we will simply demand that our proofs are in a cut \( \exists_n(\Theta) \) that is obtained by taking the intersection of \( J^t_{n+1} \) and \( Y_n \), and shortening to obtain downwards closure and closure under \( 0, \mathcal{S}, +, \times, \text{and} \omega_1 \). Since the shortening procedure only adds a standardly finite depth to the input formula \( J^t_{n+1} \cap Y_n \), \( \rho(\exists_n(\Theta)) \) will have complexity \( v + c_9 \). Moreover, when \( p \in \exists_n(\Theta) \), then ipso facto its length is in \( Y_n \) and its \( \nu \)-complexity is in \( J^t_{n+1} \).

We write \([\mathcal{A}, \models_{\eta}^{J^t_{n}} A] \) for provability in predicate logic involving only \( \Delta_n^1 \)-formulas where the proof is constrained to be in the cut \( J \). We write \([\mathcal{A}, \models_{\eta}^{J^{\nu}_{n}} A] \) when the witness for \([\mathcal{A}, \models_{\eta}^{J^t_{n}} A] \) is constrained to the cut \( J \). We write \( \Box_{\Theta, \eta, n} A \) for \([\emptyset, \models_{\eta}^{J^t_{n}} A] \), and \( \Box_{\Theta, \eta, n}^{J} A \) for \([\emptyset, \models_{\eta}^{J^{\nu}_{n}} A] \). For sentences \( A \), we will write \( \text{true}_{\Theta, \eta, n}(A) \) for \( \forall \alpha \text{ sat}_n(+, \alpha, A) \).
Theorem 2.17. We can find an $\omega_1$-cut $\exists_n(\Theta)$ such that $\rho(\exists_n(\Theta))$ is of order $\aleph_0 + \aleph_0$ and such that:

$$\text{AS}^+(\Theta) \vdash \forall \mathcal{A}, A ([\mathcal{A} \vdash \exists_n^-(\Theta) A] \rightarrow \mathcal{A} \models_n A).$$

As a special case, we have:

$$\text{AS}^+(\Theta) \vdash \forall A \in \text{sent}^\exists_n(\Theta) (\square^\exists_n(\Theta) A \rightarrow \text{true}_{\Theta,n}(A)).$$

3. SMALL-IS-VERY-SMALL PRINCIPLES

In this section we present the central argument of this paper. It is a simple Rosser argument. The bulk of the work has already been done in creating the setting for the result. I choose to give the pure argument in Theorem 3.1 rather than proceed immediately to the somewhat more complicated Theorem 3.2. The more complicated version is needed for application in model theory.

First some preliminaries and notations, in order to avoid too heavy notational machinery.

We will work in sequential theories $U$ of signature $\Theta$ with sequence scheme $\mathcal{S}$. So, $\mathcal{S} : \text{AS}^+ \rightarrow U$. We can lift $\mathcal{S}$ to a direct interpretation $\mathcal{S}_0 : \text{AS}^+(\Theta) \rightarrow U$ by translating $\Theta$ identically.

We remind the reader that $\mathcal{N} : \exists_1 \rightarrow \text{AS}^+$. We will write $N := \mathcal{S} \circ \mathcal{N} : \exists_1 \rightarrow U$. So, e.g., $\delta_N = N^S$. We write $\exists_n$ for $(\exists_n(\Theta))^S_{\Theta}$. When we write numerals these are always numerals w.r.t. $N$. We note that the numerals really are eliminated using the term elimination algorithm. However, this elimination just gives an overhead on 1 in $\rho_0$-complexity.

Let $\eta$ be a $\Sigma^b_1$-formula defining a set of axioms. We write $\square_\eta$ for provability from the axioms in $\eta$. We write $\square \eta$ for the result of restricting the witnesses for $\square_\eta$ to $J$.

We write $\square_{\eta,n}$ for the result of restricting the formulas in a witnessing proof to $\Delta^b_{\eta,n}$. Formulas like $\square_{\eta,n}$ have the obvious meanings. We suppress the information about the signature $\Theta$, which should be clear from the context. In case $\eta = (x = \UL{T}^A)$, we write $\square A$ for $\square_{\eta}$.4

We will employ witness comparison notation:

- $\exists x \in \delta_n A_0(x) \leq \exists y \in \delta_N B_0(y)$ iff $\exists x \in \delta_n (A_0(x) \land \forall y < N x \neq B_0(y)).$
- $\exists x \in \delta_n A_0(x) < \exists y \in \delta_N B_0(y)$ iff $\exists x \in \delta_n (A_0(x) \land \forall y < N x \neq B_0(y)).$
- $\exists x A_0(x) \leq \exists y B_0(y) \vdash : \exists y B_0(y) < \exists x A_0(x).$
- $\exists x A_0(x) < \exists y B_0(y) \vdash : \exists y B_0(y) < \exists x A_0(x).$

Theorem 3.1. Let $A$ be a finitely axiomatized sequential theory in a language with signature $\Theta$ with sequence scheme $\mathcal{S}$. Consider any sentence $B$ in the language of $A$ of the form $B := \exists x \in \delta_N B_0(x)$. Let $n := \max(\rho_0(A), \rho_0(B) + \epsilon_1, \rho_0(\mathcal{S}) + 11)$. Here 10 is a fixed finite constant that does not depend on $A$, $B$, and $\mathcal{S}$. We will determine 11 below.

Suppose $A \vdash \exists x \in \exists_n B_0(x)$. Then, for some $k$, we have $A \vdash \exists x \leq N k B_0(x)$, or, equivalently, $A \vdash \forall y < k B_0(y).$

4In case $\mathcal{S}$ would be a sequence scheme for a polysequential we would need a slight adaptation.

5In previous papers, I also used this notation to signal that the codes of the axioms were constrained to be $\leq n$. In this paper this extra demand is not made.

6Clearly, this introduces an ambiguity. E.g., does $\square_\top$ mean provability form all sentences or from the axiom $\top$? However, what we intend will be always clear from the context,
Proof. We work under the conditions of the theorem. Using the Gödel Fixed Point Lemma, we find a sentence \( R \) such that \( A \vdash R \iff B \leq \Box^N_{A,\Box} R \). We need that \( \rho_0(R) \leq n \).

Under the usual Gödel construction, \( R \) is of the following form:

\[
\exists x (\delta_N(x) \land B_0(x) \land \exists z (\delta_N(z) \land \text{sub}^N(\vec{z}, z) \land \forall y (y <^N x \rightarrow \neg \text{prov}^N_{A,\Box}(y, z)))).
\]

Thus, \( \rho_0(R) \) is estimated by

\[
\max(\rho_0(\text{sub}), \rho_0(\text{prov})) + \max(\rho_0(S), \rho_0(B_0)) + 3.
\]

Here the +3 is due to the additional quantifiers. We note that, if we unravel the numerals \( n \) wide scope, we even just need +2. So, we can take \( 10 = \max(\rho_0(\text{sub}), \rho_0(\text{prov})) + 3 \).

Reason in \( A \). We have \( \exists x \in \exists_n B_0(x) \). In case \( \neg \Box^3_{A,n} R \), we have \( R \). Suppose \( \Box^3_{A,n} R \). By reflection, as guaranteed by Theorem 2.17, we find \( R \). So, in both cases, we may conclude that \( R \). We leave \( A \) again.

Thus, we have shown (i) \( A \vdash R \). By cut-elimination, we find: \( A \vdash_n R \). Hence, (ii) for some \( k \), we find \( A \vdash \text{proof}^N_{A,n}(\vec{k}, R) \). Combining (i) and (ii), we may conclude that \( A \vdash \exists x \leq^N \vec{k} B_0(x) \), or, equivalently, \( A \vdash \exists x \leq^N \vec{k} B_0(q) \).

We note that, due to the use of cut-elimination, we need the totality of superexponentiation in the metatheory. Such theorems usually leave watered-down traces in weaker metatheories. We do not explore such possibilities in the present paper.

The above argument has some analogies with Harvey Friedman’s beautiful proof that, in a constructive setting, the disjunction property implies the existence property. See [Fri75]. I analyzed this argument in [Vis14b], having the benefit of many perceptive remarks by Emil Jeřábek. One surprising aspect of the above proof is that the minimization principle is not used. Joost Joosten pointed out to me in conversation that the closely related Friedman-Goldfarb-Harrington Theorem also can be proven without using minimization.

For our model theoretic applications we need a variant of Theorem 3.1 that adds domain constants. We allow for the domain constants the exceptional position that they are real constants rather than unary predicates posing as constants.

**Theorem 3.2.** Consider a finite set of domain constants \( C \). Let \( A_0 \) be any finitely axiomatized sequential theory with signature \( \Theta \) and sequence scheme \( S \). Let \( A_1 := A_1(\vec{c}) \) be any sentence in the language with signature \( \Theta + C \). Let \( A := A_0 \land A_1 \).

Consider any sentence \( B(\vec{c}) \) in the language of signature \( \Theta + C \) of the form

\[
B(\vec{c}) := \exists x \in \delta_N B_0(x, \vec{c}).
\]

Let \( n := \max(\rho_0(A), \rho_0(B)) + (10) \rho_0(S) + (10) \).

Suppose \( A(\vec{c}) \vdash \exists x \in \exists_n B_0(x, \vec{c}) \)\footnote{The fact that the constants in \( C \) do not occur in \( \exists_n = \exists^N_n(\Theta) \) is the whole point of the refined result.}. Then, for some \( k \), we have that \( A(\vec{c}) \vdash \exists x \leq^N k B_0(x, \vec{c}) \), or, equivalently, \( A(\vec{c}) \vdash \exists x \leq^N k B_0(q, \vec{c}) \).

**Proof.** We work under the conditions of the theorem. We find a sentence \( R(\vec{c}) \) such that \( A(\vec{c}) \vdash R(\vec{c}) \iff B(\vec{c}) \leq \Box^N_{A(\vec{c})} R(\vec{c}) \).

Reason in \( A(\vec{c}) \). In case \( \neg \Box^3_{A(\vec{c}),n} R(\vec{c}) \), we have \( R(\vec{c}) \). Suppose \( \Box^3_{A(\vec{c}),n} R(\vec{c}) \). Replacing the extra constants \( \vec{c} \) in \( A(\vec{c}) \) and \( R(\vec{c}) \) by fresh variables \( \vec{v} \), we get \( [A(\vec{c}) \vdash \Box^3_{A(\vec{c}),n} R(\vec{v})] \). Hence, \( A(\vec{v}) \models R(\vec{v}) \). Since \( A \) and \( R \) are standard and since we have \( A(\vec{c}) \), we find \( R(\vec{c}) \).
Thus, we have shown (i) $A(\bar{c}) \vdash R(\bar{c})$. By cut-elimination, we find: $A(\bar{c}) \vdash_n R(\bar{c})$. Hence, (ii) for some $k$, we have $A(\bar{c}) \vdash \text{proof}_{A(\bar{c}),n}^N(R(\bar{c}))$. Combining (i) and (ii), we get $A(\bar{c}) \vdash \exists x \leq N k B_0(x, \bar{c})$, or, equivalently, $A(\bar{c}) \vdash \bigvee_{q \leq k} B_0(q, \bar{c})$.

We call a theory $U$ restricted if, for some $m$ all its axioms are in $\Delta_{m}^*_b$.

**Theorem 3.3.** Suppose $A_0$ is a finitely axiomatized sequential theory in signature $\Theta$ with sequence scheme $S$. Let $m$ be any number such that $m \geq \rho_0(A_0)$. Let $C$ be a set of domain constants: $C$ is allowed to have any cardinality. Let $U$ be a restricted theory bounded by $m$ in the language of signature $\Theta + C$ extending $A_0$. The theory $U$ may have any complexity.

Consider any sentence $B$ in the language of signature $\Theta + C$ of the form $B := \exists x \in \delta N B_0(x)$. Let $n := \max(m, \rho_0(B) + \Box \rho_0(S) + \Box \rho_0(C))$.

Suppose $U \vdash \exists x \in \exists_n B_0(x)$. Then, for some $k$, we have $U \vdash \exists x \leq N k B_0(x)$, or, equivalently, $U \vdash \bigvee_{q \leq k} B_0(q)$.

**Proof.** The theorem is immediate from Theorem 3.2 using compactness.

It is of course trivial to take the contraposition of Theorem 3.3. However this contraposition has some heuristic value. So we state it here as a separate theorem.

**Theorem 3.4.** Suppose $A_0$ is a finitely axiomatized sequential theory in signature $\Theta$ with sequence scheme $S$. Let $m$ be any number such that $m \geq \rho_0(A_0)$. Let $C$ be a set of domain constants: $C$ is allowed to have any cardinality. Let $U$ be a restricted theory bounded by $m$ in the language of signature $\Theta + C$ extending $A_0$. The theory $U$ may have any complexity.

Consider any formula $C(x)$. Let $n := \max(m, \rho_0(C) + \Box \rho_0(S) + \Box \rho_0(C))$.

If the theory $U + \{C(q) \mid q \in \omega\}$ is consistent, then the theory $U + \forall x \in \exists_n C(x)$ is consistent.

**Proof.** We apply Theorem 3.3 to $\exists x \in \delta N \neg C(x)$ and take the contraposition.

4. A CONSERVATIVITY RESULT

We can use the machinery we built up to prove a Lindström-style result on conservative extensions.

Suppose $U$ is a restricted, sequential, recursively enumerable theory with sequence scheme $S$. Let $p$ be a bound for the complexity of the axioms of $U$. By Craig’s trick, we can give a $\Sigma^*_b$-axiomatization of $U$. Say the $\Sigma^*_b$-formula representing the axioms is $\eta$. Suppose $A_0$ is a finite subtheory of $U$ such that $S$ makes $A_0$ sequential. Clearly $U$ can be axiomatized by

$$A_0 + \{\eta^N(q) \rightarrow \text{true}_n(q) \mid q \in \omega\}.$$ 

This representation of the axiom set leads immediately to the following theorem.

**Theorem 4.1.** Suppose $U$ is a restricted, sequential, recursively enumerable theory. Consider any number $m$. Then there is a finitely axiomatized sequential theory $A$ in the same language that extends $U$ and is $\Delta^*_m$-conservative over $U$.

**Proof.** By our above observations there is a finitely axiomatized sequential theory $A_0$ and a formula $B(x)$ such that $U$ can be axiomatized as $A_0 + \{B(q) \mid q \in \omega\}$. Let $n := \max(\rho_0(A_0), \rho_0(B) + \Box \rho_0(S) + \Box \rho_0(C)).$
We take \( A := A_0 + \forall x \in \exists_n B(x) \). We note that \( A \) is a finitely axiomatized extension of \( U \). Consider any \( C \in \Delta^*_m \). Suppose \( U \not\models C \). Then, the theory \( A_0 + \{ (\exists q) (B(q)) \land \neg C \mid q \in \omega \} \) is consistent. It follows that \( A_0 + \forall x \in \exists_n (B(x)) \land \neg C \) is consistent. In other words, we find \( A \not\models C \).

We have the following corollary.

**Corollary 4.2.** Suppose \( U \) is a restricted, sequential, recursively enumerable theory. Suppose further that \( D \) is a finite extension of \( U \) such that \( U \not\models D \). Then, there is a finite extension \( D' \) of \( U \), such that \( D \vdash D' \) but \( D' \not\models D \).

**Proof.** Let \( m \) be a \( \rho_0 \)-bound on \( U \) and on \( D \). Let \( A \) be the sentence promised in Theorem 4.1 for \( \Delta^*_m \). Let \( D' := D \lor A \). Clearly \( D \vdash D' \) and \( D' \vdash U \). Suppose \( D' \vdash D \). Then, it follows that \( A \vdash D \), contradicting the \( \Delta^*_m \)-conservativity of \( A \) over \( U \).

Since, as is well-known, the finitely axiomatized sequential theories in the signature of \( U \) are dense w.r.t. \( \vdash \), it follows that we can add to the statement of the Corollary that \( U \not\models D' \): in case the \( D' \) provided the theorem would happen to axiomatize \( U \), we simply replace it by a \( D'' \) strictly between the original \( D' \) and \( D \).

Here is one more corollary.

**Corollary 4.3.** Consider any finitely axiomatized, sequential theory \( A \) in signature \( \Theta \). Suppose that for some class of \( \Theta \)-sentences \( \Omega \) we have a definable predicate \( \text{TRUE} \) such that, for any \( \Omega \)-sentence \( B \), we have \( A \vdash B \leftrightarrow \text{TRUE}(\langle B \rangle) \). Let \( X \) be any recursively enumerable set of \( \Omega \)-sentences. Then there is a finite extension \( A + X \) such that \( A + X \) is \( \Omega \)-conservative over \( A + X \).

**Proof.** Clearly \( A + \{ \text{TRUE}(B) \mid B \in X \} \) is restricted. We apply Theorem 4.1 taking \( m := \rho_0(\text{TRUE}(x)) + 1 \).

We give two examples of applications of the result.

**Example 4.4.** Let \( U \) be any recursively enumerable extension of PA. Then, there is a finite extension \( A \) of \( ACA_0 \) such that the arithmetical consequences of \( A \) are precisely the consequences of \( U \). Similarly for the pair \( ZF \) and \( GB \).

This result was previously proven by Robert van Wesep in his paper [Wes13].

**Example 4.5.** By Parsons’ result \( \Sigma_1 \) is \( \Pi^0_2 \)-conservative over \( PRA \)\(^8\). Since, over \( EA \), we have \( \Sigma_m \)-truth predicates. It follows that, for every \( m \), we have a finite extension \( A_m \) of \( PRA \) that is \( \Sigma_m \)-conservative. We can easily arrange that these extensions become strictly weaker when \( m \) grows.

We refer the reader to [PV18] for a number of results in the same niche using a different methodology.

## 5. Standardness Regained

Finiteness is Predicate Logic’s nemesis. However hard Predicate Logic tries, there is no way it can pin down the set of standard numbers. What happens when we invert the question? *Are there theories that interpret some basic arithmetic that do not have models in which the standard numbers are interpretable?* The answer is

\(^8\)We consider a version of \( PRA \) in the original arithmetical language here.
a resounding yes. For example, $\text{PA} + \text{incon(PA)}$ has no models in which the standard numbers are interpretable. More generally, consider any recursively enumerable consistent theory $U$ with signature $\Theta$. Suppose the signature of arithmetic is $\Xi$. Then, the theory $U + \{ (\forall \langle S^2 \rangle \tau \rightarrow \text{incon}^\tau(U)) \mid \tau : \Xi \rightarrow \Theta \}$ is consistent and does not have any models that have an internal model isomorphic to the standard numbers.

The situation changes when we put some restriction on the complexity of the axioms of the theory. The classical work concerning this idea the beautiful paper by Kenneth McAloon \[McA78\]. McAloon shows that arithmetical theories with axioms of restricted complexity that are consistent with $\text{PA}$ always have a model in which the standard integers are definable. McAloon’s work was further extended by Zofia Adamowicz, Andrés Cordón-Franco and Felix Lara-Martín. See [ACL16].

Our aim in this paper is to find an analogue of McAloon’s Theorem that works for all sequential theories. We prove a result that is more general in scope but, at the same time, substantially weaker in its statement. We show that any consistent restricted sequential theory $U$ has a model in which the intersection of all definable cuts is isomorphic to the standard natural numbers.

5.1. The Intersection of all Definable Cuts. In this subsection, we establish that the intersection of all definable cuts is a good notion.

Consider a sequential model $\mathcal{M}$. Let $\mathcal{N}$ be a $\mathcal{M}$-internal model satisfying $S^2_1$. Let $\mathcal{J}_{\mathcal{M},\mathcal{N}}$ be the intersection of all $\mathcal{M}$-definable $\mathcal{N}$-cuts in $\mathcal{M}$.

Now consider two $\mathcal{M}$-internal models $\mathcal{N}$ and $\mathcal{N}'$ satisfying $S^2_1$. By a result of Pavel Pudlák [Pud85], there is an $\mathcal{M}$-definable isomorphism $\mathcal{F}$ between an $\mathcal{M}$-definable cut $\mathcal{I}$ of $\mathcal{N}$ and an $\mathcal{M}$-definable cut $\mathcal{I}'$ of $\mathcal{N}'$. It is easily seen that $\mathcal{F}$ restricted to $\mathcal{J}_{\mathcal{M},\mathcal{N}}$ is an isomorphism between $\mathcal{J}_{\mathcal{M},\mathcal{N}}$ and $\mathcal{J}_{\mathcal{M},\mathcal{N}'}$.

Suppose $\mathcal{G}$ and $\mathcal{H}$ are two $\mathcal{M}$-definable partial functions between $\mathcal{N}$ and $\mathcal{N}'$ such that the restrictions of $\mathcal{G}$ and $\mathcal{H}$ to $\mathcal{J}_{\mathcal{M},\mathcal{N}}$ commute with zero and successor. Then it is easy to see that $\mathcal{G}$ and $\mathcal{H}$ are extensionally equal isomorphisms between $\mathcal{J}_{\mathcal{M},\mathcal{N}}$ and $\mathcal{J}_{\mathcal{M},\mathcal{N}'}$. Thus, in a sense, there is a unique definable isomorphism between $\mathcal{J}_{\mathcal{M},\mathcal{N}}$ and $\mathcal{J}_{\mathcal{M},\mathcal{N}'}$.

The above observations justify the notation $\mathcal{J}_{\mathcal{M}}$ for $\mathcal{J}_{\mathcal{M},\mathcal{N}}$ modulo isomorphism.

We note that, in the definition of $\mathcal{J}_{\mathcal{M}}$ it does not matter whether we allow parameters in the definition of the cuts. Every cut with parameters has a parameter-free shortening. Suppose $\mathcal{I}$ is an $\mathcal{N}$-cut that is given by $I(x, \vec{b})$. Then, $I^*(x) := \forall \vec{y} \left( \text{cut}(\{ z \mid I(x, \vec{y}) \}) \rightarrow I(x, \vec{y}) \right)$ defines a cut $I^*$ that is a shortening of $\mathcal{I}$.

Remark 5.1. What happens if the sequence scheme $S$ itself involves parameters? In [Vis13], it is shown that these parameters can be eliminated by raising the dimension of the interpretation. Since the standard development of an interpretation of $S^2_1$ in a sequential theory does not involve parameters, it follows that even in a sequential theory with a sequence scheme involving parameters, there is an interpretation of

\[9\] We assume that the axioms of identity are part of the axiomatization of $S^2_1$. 
S\textsubscript{1} that is parameter-free! However, the cost of this fact is that there may be no such interpretation that is one-dimensional.

Before going on with the main line of our story, I want to give some basic facts about \( \mathcal{J}_M \) in order to place it in perspective.

**Theorem 5.2.** Suppose \( \mathcal{M} \) is a sequential model. We have:

\[
\mathcal{J}_M \models \text{EA} + \text{B}\Sigma_1 + \{ \text{con}_n(A) \mid M \models A \}.
\]

**Proof.** Suppose \( M \) is, as before, the interpretation given by the sequence scheme \( S \) that defines an internal \( S_1 \)-model of \( \mathcal{M} \). We will consider \( \mathcal{J}_M \) as the intersection of all \( N \)-cuts. We again write \( \mathcal{I}_n \) for \( \mathcal{I}_{S_n}(\Theta) \).

Since there is an \( N \)-cut \( J \) such that on \( J \) we have \( W := I\Delta_0 + \Omega_1 + \text{B}\Sigma_1 \) and since this property is downwards preserved, we find that \( \mathcal{J}_M \models W \).

Suppose \( M \models A \). Without loss of generality, we can assume that \( n \geq \rho_0(A) \).

Clearly, we have \( M \models \text{con}_n(A) \). Hence, by downwards persistence, \( \mathcal{J}_M \models \text{con}_n(A) \).

Finally, consider any \( a \) in \( \mathcal{J}_M \). Consider any \( N \)-cut \( I \). There is an \( N \)-cut \( I' \) such that for every \( b \) in \( I' \), we have \( 2^b \) is in \( I \). Since \( a \) is in \( I' \), it follows that \( 2^a \) is in \( \mathcal{J}_M \).

\( \blacksquare \)

Let \( \mathfrak{J}_U := \text{Th}(\{ \mathcal{J}_M \mid M \models U \}) \). Then, we have:

**Corollary 5.3.** \( \mathfrak{J}_U \vdash \text{EA} + \text{B}\Sigma_1 + \overline{U} \).

The next theorem is a kind of overspill principle.

**Theorem 5.4.** Suppose \( \mathcal{M} \) is a sequential model and \( M \) defines an internal \( S_1 \)-model of \( \mathcal{M} \). We treat \( \mathcal{J}_M \) as the intersection of all \( M \)-cuts. Let \( B(x) \) be any formula. We have:

\[(\text{for all } b \text{ in } \mathcal{J}_M, \mathcal{M} \models B(b)) \iff \text{for some } M \text{-cut } J, \mathcal{M} \models \forall x \in J B(x).\]

**Proof.** The right-to-left direction is trivial. Suppose for all \( b \) in \( \mathcal{J}_M \), we have \( \mathcal{M} \models B(b) \). Let \( X := \{ x \in \delta_M \mid \forall y < M x B(y) \} \). If \( X \) is closed under successor, then we can shorten \( X \) to an \( M \)-cut \( J \) for which we have \( \forall x \in J B(x) \), and we are done. Otherwise, there is a \( c \) such that \( \neg B(c) \land \forall y < c B(y) \). Since \( c \) cannot be in \( \mathcal{J}_M \), it follows that there is a cut \( J \) below \( c \).

\( \blacksquare \)

Our overspill principle immediately gives:

**Theorem 5.5.** Suppose \( \mathcal{M} \) is a sequential model and \( M \) defines an internal \( S_1 \)-model of \( \mathcal{M} \). Let \( P \) be a \( \Pi_1 \)-formula. We have:

\[
\mathcal{J}_M \models P \iff \text{for some } M \text{-cut } J, \mathcal{M} \models P^J.
\]

**Corollary 5.6.** Let \( P \) be a \( \Pi_1 \)-formula. Then, \( \mathfrak{J}_U \vdash P \iff \text{for some } M : S_1 < U, \text{we have } U \vdash P^M \).

**Proof.** The proof of the corollary is by a simple compactness argument.

\( \blacksquare \)

**Open Question 5.7.** Any further information on \( \mathfrak{J}_U \) would be interesting. For example, what are the possible complexities of \( \mathfrak{J}_U \) for recursively enumerable sequential theories \( U \)?

\( \blacksquare \)
5.2. \( \omega \)-models. Before proceeding, we briefly reflect on the notion of \( \omega \)-model. The common practice is to say, e.g., that \( \mathcal{M} \) is an \( \omega \)-model of \( \mathsf{ZF} \) if the von Neumann numbers of \( \mathcal{M} \) are (order-)isomorphic to \( \omega \). Of course, there are other interpretations \( \mathcal{M} \) of arithmetic in \( \mathsf{ZF} \). However, we have the feature that if the \( M \)-numbers are isomorphic to \( \omega \), then so are the von Neumann numbers—but not vice versa. If we consider \( \mathsf{GB} \) in stead of \( \mathsf{ZF} \) we do not know whether this feature is preserved. It is conceivable that, in some model, a definable cut of the von Neumann numbers is isomorphic to \( \omega \) and the von Neumann numbers are not.

It seems to me that the proper codification of the common practice would be to say that an \( \omega \)-model is not strictly a model but a pair \( \langle \mathcal{M}, M \rangle \) of a model and an interpretation \( M \) of a suitable arithmetic in \( \mathcal{M} \) such that \( \tilde{M}(M) \) is isomorphic to \( \omega \).

Of course there is the option of existentially quantifying out the choice of the interpretation of arithmetic. Let’s say that \( \mathcal{M} \) is an e-\( \omega \)-model, if for some \( M \), \( \langle \mathcal{M}, M \rangle \) is an \( \omega \)-model.

Finally, in the sequential case, there is a third option. We define: a sequential model \( \mathcal{M} \) is an i-\( \omega \)-model if \( \mathcal{J}_M \) is isomorphic to the standard numbers. (i stands for: intersection.) In other words, \( \mathcal{M} \) is an i-\( \omega \)-model if, for some interpretation \( M \) of \( \mathbb{S}_1^2 \), for every non-standard element \( a \), there is an \( \mathcal{M} \)-definable \( M \)-cut \( I \) such that \( I < a \).

We have the following property of i-\( \omega \)-models.

**Theorem 5.8.** Suppose \( \mathcal{M} \) is a sequential i-\( \omega \)-model and \( M : \mathbb{S}_1^2 \prec \mathcal{M} \). Let \( X \) be a parametrically definable class of \( M \)-numbers. Suppose \( \omega \subseteq X \). (We confuse the standard part of \( M \) with \( \omega \).) Then there is a \( \mathcal{M} \)-definable \( M \)-cut \( J \) such that \( J \subseteq X \).

**Proof.** Suppose \( \omega \subseteq X \). Consider the class \( Y := \{ a \in \mathbb{N} \mid \forall \exists \leq a \ b \in X \} \). In case \( Y \) is closed under successor we can shorten it to a definable cut, and we are done. In case \( Y \) is not closed under successor, there is an \( a_0 \) such that \( \forall \exists \leq a_0 \ b \in X \) but \( \mathbb{S}a_0 \not\in X \). By our assumption \( \omega < a_0 \). Hence there must be a definable cut \( I \) with \( \omega \leq I < a_0 \). So, \( I \subseteq X \). \( \square \)

5.3. The Main Result. If we are content with the countable case, our main result is a simple application of the Omitting Types Theorem. We first give this easier proof.

**Theorem 5.9.** Let \( U \) be a consistent restricted sequential theory. Here \( U \) may be of any complexity. We allow countably many constants in \( U \). Then, \( U \) has a model \( \mathcal{M} \) in which \( \mathcal{J}_M \) is isomorphic to the standard natural numbers.

**Proof.** We fix a sequence scheme \( S \) for \( U \). We work with the interpretation \( N \) of \( \mathbb{S}_1^2 \) provided by this scheme. Suppose that in all countable \( U \)-models the type

\( \mathcal{T}(x) := \{ x \neq n \mid n \in \omega \} \cup \{ x \in \exists_n \mid n \in \omega \} \)

is realized. Then, by the Omitting Types Theorem, there is a formula \( A(x) \), such that, for a fresh constant \( c \), we have (i) \( U + A(c) \) is consistent and (ii) \( U + A(c) \vdash c \neq n \), for each \( n \in \omega \), and (iii) \( U + A(c) \vdash c \in \exists_n \), for each \( n \in \omega \).

We apply Theorem 5.4 to (i) and (ii) obtaining that, for some \( n^* \), the theory \( U + A(c) + \forall x \in \exists_{n^*} \ c \neq x \) is consistent. However, this directly contradicts (i) and (iii).
We may conclude that there is a countable model $\mathcal{M}$ in which $T(x)$ is omitted. Clearly, this tells us that $\mathcal{J}_M$ is isomorphic to the standard numbers.

We proceed to prove the stronger version of our theorem where the restriction to countability is lifted. We first prove a Lemma.

**Lemma 5.10.** Let $\mathcal{M}$ be any sequential model of signature $\Theta$ with domain $M$. Let $\mathcal{S}$ be a sequence scheme for $\mathcal{M}$. As usual, $N$ is the interpretation of $\mathcal{S}$ given by the sequence scheme.

Let $k$ be any number. We note that the $\rho_0$-complexity of the axioms of $\mathcal{S}^+$ is a fixed number, say $s$. So the sequentiality of $\mathcal{M}$ is witnessed by the satisfaction of a sentence $D$ of complexity below $s + \rho_0(\mathcal{S})$. Let $n := \max(k, s + \rho_0(\mathcal{S}), 1 + 10\rho_0(\mathcal{S}) + 10)$. Then, $\mathcal{M}$ has a sequential $\Delta^*_k$-elementary extension $\mathcal{K}$ with sequence scheme $\mathcal{S}$ such that $\mathfrak{S}_n \cap M = \omega$.

**Proof.** Without loss of generality we may assume that $k \geq s + \rho_0(\mathcal{S})$, so that $\mathcal{T}_{\Delta^*_k}(\mathcal{M})$ is sequential. Let $\Gamma := \mathcal{T}_{\Delta^*_k}(\mathcal{M})$. We claim that

$$\Gamma^* := \Gamma + \{2n < m \mid M \models m \in \delta_N \land \omega < m\}$$

is consistent (for $n$ as given in the statement of the Lemma). If not, then for some nonstandard $m_0, \ldots, m_{t-1}$ in $M$, we have

$$\Gamma \vdash m_0 \in \mathfrak{S}_n \lor \ldots \lor m_{t-1} \in \mathfrak{S}_n.$$ Let $m$ be the minimum of the $m_i$. We find $\Gamma \vdash m \in \mathfrak{S}_n$. On the other hand, the theory $\Gamma + 0 < m, 1 < m, \ldots$ is consistent. Hence, by Theorem 3.4, we have that $\Gamma + \mathfrak{S}_n < m$ is consistent. A contradiction.

Let $\mathcal{K}$ be a model of $\Gamma^*$. Clearly, in $\mathcal{K}$, we have that $\mathfrak{S}_n$ is below all non-standard elements inherited from $\mathcal{M}$ (but, of course, not necessarily below new non-standard elements). Also $\mathcal{K}$ is, by construction, a $\Delta^*_k$-elementary extension. Finally, since we have chosen $n \geq s + \rho_0(\mathcal{S})$, the model $\mathcal{K}$ is again sequential with the same sequence scheme.

With the Lemma in hand, we can now prove the promised theorem using a limit construction.

**Theorem 5.11.** Let $\mathcal{M}$ be any sequential model. Then, for any $k$, $\mathcal{M}$ has a $\Delta^*_k$-elementary extension $\mathcal{K}$ in which $\mathcal{J}_K$ is (isomorphic to) $\omega$.

**Proof.** Let $\mathcal{S}$ be a sequence scheme for $\mathcal{M}$. We work with the numbers $N$ provided by this scheme. Let $s$ be as in Lemma 5.10. We take:

* $n_0 := \max(k, s + \rho_0(\mathcal{S})).$
* $n_{j+1} := \max(\rho_0(\mathfrak{S}_j) + 1, s + \rho_0(\mathcal{S}), 1 + 10\rho_0(\mathcal{S}) + 10).$

(We note that $\rho_0(\mathfrak{S}_j) \approx 10^j$ and that for $j > 0$, we have $n_{j+1} := \rho_0(\mathfrak{S}_j) + 1$.) We construct a chain of models $\mathcal{M}_j$. Let $\mathcal{M}_0 = \mathcal{M}$. Suppose we have constructed $\mathcal{M}_j$. We now take as $\mathcal{M}_{j+1}$ a model that is a $\Delta^*_{n_{j+1}}$-elementary extension of $\mathcal{M}_j$ such that $\mathfrak{S}_{n_{j+1}} \cap M_j = \omega$.

Let $\mathcal{K}$ be the limit of $(\mathcal{M}_j)_{j \in \omega}$. Consider any non-standard element $a$ in $\tilde{\mathcal{N}}(\mathcal{K})$. We have to show that there is a $\mathcal{K}$-definable cut below it. Suppose $a$ occurs in $\mathcal{M}_j$. We have, by the construction of our sequence, that $(\downarrow) \mathfrak{S}^{\mathcal{M}_j}_{n_{j+1}} < a$. By the fact
that all $M_s$, with $s > j + 1$ are $\Delta^s_{n,j+1}$-elementary extensions of $M_{j+1}$, it follows that $(\dagger)$ is preserved to the limit: $\exists^J_n < a$.

From Theorem 5.11 we have immediately the desired strengthening of Theorem 5.9.

**Theorem 5.12.** Let $U$ be a consistent restricted sequential theory. Here $U$ may be of any complexity. We allow a number of constants in $U$ of any cardinality. Then $U$ has a model $M$ in which $J$ is isomorphic to the standard natural numbers.

**Remark 5.13.** From Theorem 5.12 we retrace our steps and derive a less explicit form of Theorem 5.11. Let $U$ be a restricted sequential theory and consider any $C(x)$. Suppose $V := U + \{C(n) | n \in \omega\}$ is consistent. Let $M$ be a model of $V$ in which $\mathcal{J}_M$ is isomorphic to the standard natural numbers. By Theorem 5.11 there is a definable $M$-cut $J$ so that, in $M$, we have $\forall x \in JC(x)$.

Now $J$ is a definable cut in $M$, but it need not automatically be a cut in $U$. There is a standard trick to remedy that. We define $J^o := J(\text{cut}(J))N$. Clearly, $J^o$ is a definable cut in $U$. Moreover, in the context of $M$, the cuts $J$ and $J^o$ coincide. By the above considerations, it follows that $U + \forall x \in J^oCx$ is consistent, where $J^o$ is a $U$-definable cut.

**Remark 5.14.** Consider any model $M$. We define $\text{DEF}(M)$ as the class of (parametrically) definable classes of $M$. We define $\text{DEF}^e(M)$ as the class of classes over $M$ that are definable without parameters. Also, $\text{DEF}_n(M)$ is the class (parametrically) definable $n$-ary relations and similarly for the parameter-free case.

It would seem that Theorem 5.11 gives us information about possible sequential models of the form $\langle M, \text{DEF}(M) \rangle$, since $\mathcal{J}_M$ is definable in $\langle M, \text{DEF}(M) \rangle$. However, this is not so, since we have a much stronger result for the models $\langle M, \text{DEF}(M) \rangle$, where $M$ is sequential.

We assume that $M$ has finite signature, where we may allow an infinity of constants. In each model $\langle M, \text{DEF}(M) \rangle$, where $M$ satisfies these demands, the natural numbers are definable. The argument is simple. Let $\text{comm}_X(Y)$ mean that $X$ satisfies the commutation conditions for satisfaction of $\Delta^s_X$-formulas in $\mathcal{J}_M$. Consider, in $\langle M, \text{DEF}(M) \rangle$ the class $Y := \{x \in N | \exists X \text{comm}_X(Y)\}$. Clearly, each standard $x$ is in $Y$. If a non-standard number $b$ would be in $Y$, the defining formula for the witnessing $X$ would violate Tarski’s Theorem of the undefinability of truth for $M$.

If the sequence scheme for $M$ is parameter-free, then the same argument works for $\langle M, \text{DEF}^e(M) \rangle$. If the sequence scheme contains parameters, we can make the argument work for $\langle M, \text{DEF}_n(M) \rangle$, for sufficiently large $n$.

Finally, note that if $M$ is a non-standard model of Peano Arithmetic, then $\mathcal{J}_M$ is simply isomorphic to $M$ itself. Thus, adding $\mathcal{J}_M$ (viewed as intersection of all cuts on the identical interpretation of $S^1_2$) to $M$ does not increase the expressiveness of the language. This consideration shows that adding the definable sets can be more expressive than adding $\mathcal{J}_M$ (as intersection of the cuts for a given interpretation of $S^1_2$).

6. Reflection

If we apply Theorem 5.3 to a formula of a special form, we get a reflection principle.

---

\(^{10}\) Ali Enayat tells me that the basic idea of this argument is originally due to Mostowski.
Theorem 6.1. Consider any consistent, restricted, sequential theory $U$ with sequence scheme $S$. Let $N$ be the interpretation of the numbers provided by $S$. Let $m_0$ be the bound for $U$ and let $m_1$ be any number. Let $n := \max(m_0, m_1 + \rho_0(S) + 11)$. Then, for every $\Sigma_2$-sentence $C$ of the form $C = \exists x C_0(x)$, where $C_0$ is $\Pi_1$ and $\rho_0(C) \leq m_1$, we have: if $U \vdash \exists x \in \Im C_0^N(x)$, then $C$ is true.

Proof. Under the assumptions of the theorem, we suppose $U \vdash \exists x \in \Im C_0^N(x)$. By Theorem 3.3 there is a $k$ such that $U \vdash \bigvee_{q \leq k} C_0^N(q)$. Suppose $C$ is false. Then, for each $q \leq k$, we have $\neg C_0(q)$. Hence, by $\Sigma_1$-completeness, for each $q \leq k$, we have $U \vdash \neg C_0^N(q)$. It follows that $U \vdash \bot$. Quod non.

We note that the above proof uses $\Sigma_1^0$-collection in the metalanguage.

If we take the formulas still simpler we can improve the above result. We fix a logarithmic cut $S_2$-cut $\mathcal{J}$. There is a $\Sigma_1$-truth predicate, say True, for $\Sigma_1$-sentences, such that, for any $\Sigma_1$-sentence $S$, we have $S_2 \vdash \text{True}(S) \rightarrow S$ and $S_2 \vdash S^J \rightarrow \text{True}(S)$. (See e.g. [HP93], Part V, Chapter 5b for details.)

Theorem 6.2. Consider any consistent, restricted, sequential theory $U$ with sequence scheme $S$ and bound $m$. Let $n := \max(m, \rho_0(\text{True}(x)) + \rho_0(S) + 11 + 1)$. If we start with the pair $\text{Glob}_{\text{seq}}$ and $\text{Loc}_{\text{seq}}$ and with the projection $\pi$, then we can define $\text{Fin}$ (using a first-order formula).

Theorem 6.2. Consider any consistent, restricted, sequential theory $U$ with sequence scheme $S$ and bound $m$. Let $n := \max(m, \rho_0(\text{True}(x)) + \rho_0(S) + 11 + 1)$. For all $\Sigma_1$-sentences $S$, we have: if $U \vdash S^{\mathcal{J}^m}$, then $S$ is true.

Proof. We have:

\[
U \vdash S^{\mathcal{J}^m} \quad \Rightarrow \quad U \vdash \text{True}^m(S) \\
\quad \Rightarrow \quad \text{True}(S) \text{ is true} \\
\quad \Rightarrow \quad S \text{ is true.}
\]

The second step is by theorem 6.1.

\[
\text{7. Degrees of Interpretability}
\]

In this section, we apply our results to study the joint degree structure of local and global interpretability for recursively enumerable sequential theories. The main result of this section is a characterization of finite axiomatizability in terms of the double degree structure. We will study the degree structures as partial pre-orderings.

7.1. The Basic Idea. The degree structures we are interested in are the degrees of global interpretability of recursively enumerable sequential theories $\text{Glob}_{\text{seq}}$ and the degrees of local interpretability of recursively enumerable sequential theories $\text{Loc}_{\text{seq}}$. It is well known that both structures are distributive lattices. We have the obvious projection functor $\pi$ from $\text{Glob}_{\text{seq}}$ onto $\text{Loc}_{\text{seq}}$.

Let $\text{Fin}$ be the property of global degrees of containing a finitely axiomatized theory. What we want to show is that, if we start with the pair $\text{Glob}_{\text{seq}}$ and $\text{Loc}_{\text{seq}}$ and with the projection $\pi$, then we can define $\text{Fin}$ (using a first-order formula).

The basic idea is simple. Zoom in on a local degree of $U$. This degree contains a distributive lattice, say, $L$ of global degrees. The lattice $L$ has a maximum, to wit $U_U$. Does it have a minimum? Well, if there is a finitely axiomatized theory $U_0$ in $L$, then its global degree will automatically be the minimum. We will see that (i) not in all cases a minimum degree of $L$ exists and (ii) if such a minimum exists it contains a finitely axiomatizable theory. In other words, the mapping $\phi$ with:
\[ \phi(U) \triangleleft_{\text{glob}} V \iff U \triangleleft_{\text{loc}} \pi(V) \]
is partial. However, if it has a value, this value contains a finitely axiomatized theory. So, we can define: \( \text{fin}(U) \) iff, for all \( V \), we have \( U \triangleleft_{\text{glob}} V \) iff \( \pi(U) \triangleleft_{\text{loc}} \pi(V) \).

**Open Question 7.1.** Can we define \( \text{fin} \) in \( \text{Glob}_{\text{seq}} \) alone? 

**Remark 7.2.** In the context of local degrees of arbitrary theories with arbitrarily large signatures, Mycielski, Pudlák and Stern characterize loc-finite as the same as compact in terms of the \( \triangleleft_{\text{loc}} \)-ordering. This will not work in our context of global interpretability and recursively enumerable sequential theories. We briefly give the argument that, in our context, every non-minimal globally finite degree is non-compact.

Let \( A \) be any \( \triangleleft \)-non-minimal, finitely axiomatized, sequential theory. Let \( B_i \) be an enumeration of all finitely axiomatized sequential theories. Let \( C_0 := S^1_2 \). We note that, by our assumption, \( C_0 \not\subseteq A \). Let \( C_{n+1} := B_n \) if \( C_n \not\subseteq B_n \not\subseteq A \) and \( C_{n+1} := C_n \) otherwise. Clearly, \( A \triangleright C_i \), for all \( i \). Consider any sequential recursively enumerable theory \( U \) such that \( U \triangleright C_i \), for all \( i \). Without loss of generality, we may assume that the signatures of \( U \) and \( A \) are disjoint. We easily see that the theory \( U^* \) axiomatized by \( \{D \triangleright A \mid D \text{ is an axiom of } U\} \) is the infimum in the degrees of global interpretability of \( U \) and \( A \). So \( C_i \not\triangleleft U^* \triangleleft A \). Suppose \( U^* \not\triangleleft A \). In this case, by [Vis17, Theorem 5.3], we can find a finitely axiomatized sequential \( B \) such that \( U^* \not\subseteq B \not\subseteq A \). Let \( B = B_j \). Since \( C_j \not\subseteq U^* \not\subseteq B \not\subseteq A \), we will have \( C_{j+1} := B_j \). A contradiction. It follows that \( U^* \equiv A \). We may conclude that \( A \not\triangleleft U \). So, \( A \) is the supremum of the \( C_i \). Clearly, \( A \) cannot be the supremum of a finite number of the \( C_i \). Thus, \( A \) is not compact.

**7.2. Preliminaries.** We consider the recursively enumerable theory \( U \). By Craig’s Theorem, we can give \( U \) a \( \Sigma^0_1 \)-definable axiomatization \( X \). Let this axiomatization is given by a \( \Sigma^0_1 \)-formula \( \eta \). We define \( U \upharpoonright n \) as the theory axiomatized by the axioms of \( U \), as given by \( \eta \) that are \( \leq n \).

An important funtor is \( \mathcal{U} \). We define: \( \mathcal{U} \upharpoonright U := S^2_2 + \{ \text{con}_n(U \upharpoonright n) \mid n \in \omega \} \). We note that \( \mathcal{U} \upharpoonright U \) is extensionally independent of the choice of \( \eta \).

One can show that \( \mathcal{U} \upharpoonright \pi(U) \). See e.g. [Vis11a] or [Vis17].

A theory is \( \text{glob-finite} \) iff it is mutually globally interpretable with a finitely axiomatized theory. A theory is \( \text{loc-finite} \) iff it is mutually locally interpretable with a finitely axiomatized theory. If we enrich \( \text{Glob}_{\text{seq}} \) with a predicate \( \text{Fin} \) for the globally finite degrees, we have a first-order definition of \( \text{Loc}_{\text{seq}} \) over this structure as follows:

\[ U \triangleleft_{\text{loc}} V \iff \forall A \in \text{Fin} \left( A \triangleleft_{\text{glob}} U \Rightarrow A \triangleleft_{\text{glob}} V \right). \]

Here is a first basic insight.

**Theorem 7.3.** The theory \( U \) is a-finite, for \( a \in \{\text{glob}, \text{loc}\} \), iff \( U \equiv_a U \upharpoonright n \), for some \( n \).

**Proof.** Suppose \( U \equiv_a V \), where \( V \) is finitely axiomatized. Clearly, \( U \upharpoonright n \triangleright V \), for some \( n \). We have: \( U \supseteq U \upharpoonright n \triangleright V \triangleright a U \), and we are done.
Some Examples. Before formulating and proving our main result, we briefly pause to provide a few examples.

Example 7.4. The theories $\text{I} \Delta_0$ and $\text{S}_2 = \text{I} \Delta_0 + \Omega_1$ are examples of theories of which the finite axiomatizability is an open problem, but which are, by an argument of Alex Wilkie, glob-finite. They are, for example, mutually interpretable with the finitely axiomatized sequential theory $\text{AS}$ and with the finitely axiomatized sequential theory $\text{PA}^{-}$.

We show that any global, recursively enumerable, sequential degree contains an element that is not finitely axiomatizable.

Theorem 7.5. Consider any consistent, sequential, recursively enumerable theory $U$. Then there is a $U^0 \equiv U$ such that $U^0$ is sequential and recursively enumerable and not finitely axiomatizable.

Proof. Consider a consistent, sequential and recursively enumerable theory $U$. In case $U$ is not finitely axiomatizable, we are done, taking $U^0 := U$. Suppose $U$ is finitely axiomatizable, say by a single sentence $A$. Par abus de langage, we write $A$ also for the theory axiomatized by $x = \code{A}$.

By Theorem 6.2 we can find $M : S_1 \triangleleft A$ for which $A$ is $\Sigma^0_1$-sound. Consider:

$$U^0 := A + \{(\text{con}(A) \rightarrow \text{con}^{n+1}(A))^M \mid n \in \omega\}. $$

We have, by Feferman’s version of the Second Incompleteness Theorem:

$$A \subseteq U^0 \subseteq (A + \text{incon}^M(A)) \triangleleft A.$$ 

So $U^0 \equiv A$. Suppose $U^0$ were finitely axiomatizable. Then, we would have, for some $n > 0$,

$$A + (\text{con}(A) \rightarrow \text{con}^n(A))^M \vdash (\text{con}(A) \rightarrow \text{con}^{n+1}(A))^M.$$ 

Hence $A \vdash (\Box^n \perp \rightarrow \Box^n \perp)^M$. So, by Löb’s Theorem, $A \vdash (\Box^n \perp)^M$, contradicting the $\Sigma^0_1$-soundness of $A$ w.r.t. $M$. 

Here is a sufficient condition for failure to be loc-finite. We define:

* $\mathcal{U}^+_U := S_1 + \{\text{con}(U \upharpoonright n) \mid n \in \omega\}.$

Here $U \upharpoonright n$ is defined with respect to a chosen $\Sigma^0_1$-formula $\eta$ that represents the axiom set of $U$.

We call $U$ strongly loc-reflexive if $U \triangleright \mathcal{U}^+_U$. We note that e.g. PRA is an example of a strongly loc-reflexive theory.

Theorem 7.6. Suppose that $U$ is strongly loc-reflexive. Then, $U$ is not loc-finite.
Proof. Suppose $U \triangleright_{\text{loc}} U_{i}^{\bot}$. Suppose, to obtain a contradiction, that $U$ is loc-finite. Then, for some $i$, we have $U \vdash_{\text{loc}} U$. Moreover, $U \triangleright (S^{1}_{2} + \text{con}(U | i))$. So, $U \vdash_{\text{loc}} (S^{1}_{2} + \text{con}(U | i))$, contradicting the second incompleteness theorem. 

7.4. Characterizations. The following characterization of loc-finiteness may look like an obscurum per obscurius. However, it is a central tool in what follows.

**Theorem 7.7.** Suppose $U$ is sequential. The variable $I$ will range over $S^{1}_{2}$-definable cuts. The following are equivalent:

1. $U$ is loc-finite.
2. $\exists i \forall j \geq i \exists I \ S^{1}_{2} + \text{con}_{i}(U | i) \vdash \text{con}_{j}(U | j)$.
3. $\exists i \forall j \geq i \exists I \ S^{1}_{2} + \text{con}_{i}(U | j) \vdash \text{con}_{j+1}(U | j)$.

**Proof.** (1) $\implies$ (2). Suppose $U$ is loc-finite. Then, $U \vdash_{\text{loc}} U$, for some $i$. Consider any $j \geq i$. For some $M$, we have $M : U | i \vdash U | j$. Using $M$, we can transform a $U | j$, $j$-inconsistency proof into an $U | i$, $k$-inconsistency proof, for a sufficiently large $k$. Thus, we have $S^{1}_{2} + \text{con}_{i}(U | i) \implies \text{con}_{j}(U | j)$. On the other hand, for some $I$, $S^{1}_{2} + \text{con}_{i}(U | i) \vdash \text{con}_{j+1}(U | i)$. This last step can be seen, e.g., from the fact that a cut-elimination that transforms a $k$-proof to an $i$-proof is multi-exponential (see [Bus15]). It suffices to take as $I$ an appropriate multi-logarithmic cut. We may conclude that $S^{1}_{2} + \text{con}_{i}(U | i) \vdash \text{con}_{j+1}(U | j)$.

(2) $\implies$ (3) and (3) $\implies$ (2) are trivial.

For (2) $\implies$ (1), one shows that, for $i$ as promised, $S^{1}_{2} + \text{con}_{i}(U | i)$ is mutually locally interpretable with $U$.

We note that in the above proof, the precise point where where sequentiality is used, is the insight that $U$ interprets $S^{1}_{2} + \text{con}_{i}(U | i)$. Thus, only (2) $\implies$ (1) depends on sequentiality.

By a result of Wilkie and Paris ([WPS7], see also [Vis11a]), we have, for $\Sigma_{1}$-sentences $P$ and $Q$, that $\text{EA} + P \vdash Q$ iff, for some $S^{1}_{2}$-cut $I$, we have $S^{1}_{2} + P \vdash Q$. Hence, it follows that:

**Corollary 7.8.** Suppose $U$ is sequential. The following are equivalent:

1. $U$ is loc-finite.
2. $\exists i \forall j \geq i \text{ EA} + \text{con}_{i}(U) \vdash \text{con}_{j}(U)$.
3. $\exists i \forall j \geq i \text{ EA} + \text{con}_{i}(U) \vdash \text{con}_{j+1}(U)$.

7.5. The Main Theorem. Consider the partial preorder, say $\mathcal{L}$, of sequential degrees of global interpretability contained in a given degree of local sequential interpretability. Note that $\mathcal{L}$ is closed under suprema and infima. Suppose our given local degree is not loc-finite. Then, $\mathcal{L}$ does not have a minimal. This insight is formulated in the following theorem.

**Theorem 7.9.** Suppose that $U$ is a recursively enumerable, sequential theory that is not loc-finite. Then, there is a theory $\bar{U}$, such that $\bar{U} \equiv_{\text{loc}} U$, but $\bar{U} \not\equiv_{\text{glob}} U$.

---

11Alternatively, we can prove the step by a combination of the Interpretation Existence Lemma ([Vis17]) and the local reflexiveness of $U | i$ that is a direct consequence of Theorem 2.17 of the present paper.

12This is not true anymore when we employ the argument of Footnote 11.
Proof. Suppose that $U$ is recursively enumerable, sequential and not loc-finite. Let the signature of $U$ be $\Theta$ and let a sequence scheme for $U$ be $S$. As usual $N$ is the standard interpretation of $S_1^2$ in $U$ given by $S$. We write $\mathcal{S}_i$ for $\mathbb{S}^i(\Theta)$. The complexities of the $\mathcal{S}_i$ are estimated by $\underline{1}^9 + \underline{1}^9 + \rho_0(\mathcal{S})$. Taking $c_{11} := \underline{1}^9 + \rho_0(\mathcal{S})$, our estimate becomes: $\underline{1}^9 + \underline{1}^9$ By the choice of the $\mathcal{S}_i$ we have: $U \vdash \con^\mathcal{S}_i (U \langle i \rangle)$.

Consider $S_2^1$. Say the signature of $S_2^1$ is $\Xi$ and a sequence scheme that interprets the numbers identically is $T$. We write $I_i$ for $\mathbb{S}_i^T(\Xi)$. The complexities of the $I_i$ are estimated by $\underline{1}^9 + \underline{1}^9 + \rho_0(T)$. Taking $c_{12} := \underline{1}^9 + \rho_0(T)$, our estimate becomes: $\underline{1}^9 + \underline{1}^9$ We define the theory $\tilde{U}$ as follows: Let $F(i) := (\underline{1}^9 + 1)i + 1^{19}$

$$\tilde{U} := S_1^2 + \{ \con^T_i (U \langle i \rangle) | i \in \omega \}.$$  

Clearly, $\tilde{U} \equiv_{\loc} U$.

Suppose, to obtain a contradiction, that, for some $K$, we have $K : \tilde{U} \nvdash U$. By Pudlák’s theorem ([Pud85]), there is a $\tilde{U}$-cut $J$ of $N$ and a $\tilde{U}$-cut $J'$ of $K \circ N$ and a $\tilde{U}$-definable isomorphism $G$ between $J$ and $J'$. We define $J_i := G^{-1}[\mathcal{S}_i^K \cap J']$. We clearly have: $J_i$ is a $\tilde{U}$-cut and $U \vdash \con^i_1(U)$. We note that:

$$x \in J_i :\iff \exists y (Gxy \land (y \in \mathcal{S}_i^K \land y \in J'),$$

so that $\rho_0(J_i)$ is estimated by a linear term of the form $\underline{1}^9 + c_{13}$. (Here $\underline{1}^{19}$ is dependent on $\rho_0(K)$.)

Consider any $s$. We will make $s$ more specific in the run of the argument. We have: $\tilde{U} \vdash \con^s_{s+1} (U \langle (s + 1) \rangle)$. Hence, by compactness, for some $p \geq s$:

1. $S_1^2 + \{ \con^T_i (U \langle i \rangle) | i \leq s \} + \{ \con^T_{j'} (U \langle j \rangle) | s < j \leq p \} \vdash \con^s_{s+1} (U \langle (s + 1) \rangle)$

Thus, it follows that:

2. $S_1^2 + \con^s_p (U \langle s \rangle) + \con^s_{s+1} (U \langle (s + 1) \rangle) \vdash \incon^T_p (U \langle p \rangle)$

Since, $U$ is not loc-finite, we have, by Theorem 7.7, arbitrarily large $s$’s such that $A_s := S_1^2 + \con^s_p (U \langle s \rangle) + \con^s_{s+1} (U \langle (s + 1) \rangle)$ is consistent. We note that $\rho_0(A_s)$ is estimated by $\underline{1}^9 + c_{14}$ for a suitable $\underline{1}^{19}$.

Consider any $s$ for which $A_s$ is consistent. We remind the reader of Theorem 6.1. Applied to the case at hand this tells us the following. Let $n := \max(\rho_0(A_s), c_{15})$ (where $c_{14}$ is a fixed constant). Then, if $A_s \vdash \incon^T_p (U \langle p \rangle)$, then $\incon^T_p (U \langle p \rangle)$ is true. Since we assumed that $U$ is consistent, it follows that $A_s \nvdash \incon^T_p (U \langle p \rangle)$.

We note that $n$ is estimated by $\underline{1}^9 + c_{16}$ for a suitable $\underline{1}^{19}$. We may choose $s$ large enough so that $F(s + 1) > \underline{1}^9 + c_{16}$ and such that $A_s$ is consistent. It follows that $I_{F(s+1)}$ is a subcut of $I_n$. Since, by Equation 2, $A_s \vdash \incon^T_p (U \langle p \rangle)$, it follows that $A_s \vdash \incon^T_p (U \langle p \rangle)$. A contradiction.

Note that Theorem 7.9 implies that any local sequential degree that is not loc-finite contains an infinity of global degrees.

\footnote{It is a sport to take the choice of $F$ as sharp as possible. The argument below becomes a bit more relax if we take $F(i) := i^2 + 1$ and just keep track of linear dependencies.}
Open Question 7.10. Every element of $\text{Loc}_{\text{seq}}$ contains an extension of $S_1^1$ (to wit an element of the form $\tilde{U}_U$). Does every element of $\text{Glob}_{\text{seq}}$ contain an extension of $S_2^1$?

REFERENCES

[ACL16] Z. Adamowicz, A. Cordón-Franco, and F.F. Lara-Martín. Existentially closed models in the framework of arithmetic. *The Journal of Symbolic Logic*, 81(2):774–788, 2016.

[Bus15] S. Buss. Cut elimination in situ. In Reinhard Kahle and Michael Rathjen, editors, *Gentzen’s Centenary*, pages 245–277. Springer International Publishing, 2015.

[Fri75] Harvey Friedman. The disjunction property implies the numerical existence property. *Proceedings of the National Academy of Sciences*, 72(8):2877–2878, 1975.

[Ger03] P. Gerhardy. Refined Complexity Analysis of Cut Elimination. In Matthias Baaz and Johann Makowsky, editors, *Proceedings of the 17th International Workshop CSL 2003*, volume 2803 of LNCS, pages 212–225. Springer-Verlag, Berlin, 2003.

[Ger05] P. Gerhardy. The Role of Quantifier Alternations in Cut Elimination. *Notre Dame Journal of Formal Logic*, 46(2):165–171, 2005.

[HP93] P. Hájek and P. Pudlák. Metamathematics of First-Order Arithmetic. Perspectives in Mathematical Logic. Springer, Berlin, 1993.

[Kay91] R. Kaye. *Models of Peano Arithmetic*. Oxford Logic Guides. Oxford University Press, 1991.

[Kra87] J. Krajiček. A note on proofs of falsehood. *Archiv für Mathematische Logik und Grundlagenforschung*, 26(1):169–176, 1987.

[McA78] K. McAloon. Completeness theorems, incompleteness theorems and models of arithmetic. *Transactions of the American Mathematical Society*, 239:253–277, 1978.

[MPS90] J. Mycielski, P. Pudlák, and A.S. Stern. A lattice of chapters of mathematics (interpretations between theorems), volume 84 of *Memoirs of the American Mathematical Society*. AMS, Providence, Rhode Island, 1990.

[Pud83] P. Pudlák. Some prime elements in the lattice of interpretability types. *Transactions of the American Mathematical Society*, 280:255–275, 1983.

[Pud85] P. Pudlák. Cuts, consistency statements and interpretations. *The Journal of Symbolic Logic*, 50(2):423–441, 1985.

[PV18] V. Pakhomov and A. Visser. On a question of Krajewski’s. *arXiv preprint arXiv:1712.01713*, 2018.

[Smo85] C. Smoryński. Nonstandard models and related developments. In L.A. Harrington, M.D. Morley, A. Scedrov, and S.G. Simpson, editors, *Harvey Friedman’s Research on the Foundations of Mathematics*, pages 179–229. North Holland, Amsterdam, 1985.

[Ts00] A.S. Troelstra and H. Schwichtenberg. *Basic proof theory*. Number 43 in Cambridge Tracts in Theoretical Computer Science. Cambridge University Press, Cambridge, 2000.

[Vis93] A. Visser. The unprovability of small inconsistency. *Archive for Mathematical Logic*, 32(4):275–298, 1993.

[Vis03] A. Visser. Faith & Falsity: a study of faithful interpretations and false $\Sigma^0_1$-sentences. *Annals of Pure and Applied Logic*, 131(1–3):103–131, 2005.

[Vis09] A. Visser. Cardinal arithmetic in the style of Baron von Münchhausen. *Review of Symbolic Logic*, 2(3):570–589, 2009, doi: 10.1017/S17550203090090261.

[Vis10] A. Visser. Can we make the Second Incompleteness Theorem coordinate free. *Journal of Logic and Computation*, 21(4):543–560, 2011. First published online August 12, 2009, doi: 10.1093/logcom/exp048.

[Vis11a] A. Visser. Hume’s principle, beginnings. *Review of Symbolic Logic*, 4(1):114–129, 2011.

[Vis11b] A. Visser. What is sequentiability? In P. Cégielski, Ch. Cornaros, and C. Dimitracopoulos, editors, *New Studies in Weak Arithmetics*, volume 211 of *CSLI Lecture Notes*, pages 229–269. CSLI Publications and Presses Universitaires du Pôle de Recherche et d’Enseignement Supérieur Paris-est, Stanford, 2013.

[Vis09a] A. Visser. Interpretability degrees of finitely axiomatized sequential theories. *Archive for Mathematical Logic*, 53(1-2):23–42, 2014.

[Vis14b] A. Visser. The interpretability of inconsistency, Feferman’s theorem and related results. Logic Group Preprint Series 318, Faculty of Humanities, Philosophy, Utrecht University, Janskerkhof 13, 3512 BL Utrecht, [http://www.phil.uu.nl/preprints/lgps/](http://www.phil.uu.nl/preprints/lgps/), 2014.
The parameters of the translation are given by a fixed sequence of variables \( \vec{w} \) that we keep apart from all other variables. A translation is defined as before, but for the fact that now the variables \( \vec{w} \) are allowed to occur in the domain-formula and in the translations of the predicate symbols in addition to the variables that correspond to the argument places. Officially, we represent a translation \( \tau_{\vec{w}} \) with parameters \( \vec{w} \) as a quintuple \( (\Sigma, \delta, \vec{w}, F, \Theta) \). The parameter sequence may be empty: in this case our interpretation is parameter-free.

An interpretation with parameters \( K : U \to V \) is a quadruple \( (U, \pi, E, \tau_{\vec{w}}, V) \), where \( \tau_{\vec{w}} : \Sigma_U \to \Sigma_V \) is a translation and \( \pi \) is a \( V \)-formula containing at most \( \vec{w} \) free. The formula \( \pi \) represents the parameter domain. For example, if we interpret the Hyperbolic Plane in the Euclidean Plane via the Poincaré interpretation, we need two distinct points to define a circular disk. These points are parameters of the construction, the parameter domain is \( \pi(w_0, w_1) = (w_0 \neq w_1) \). (For this specific example, we can also find a parameter-free interpretation.) The formula \( E \) represents an equivalence relation on the parameter domain. In practice this is always pointwise identity for parameter sequences, but for reasons of theory one must admit other equivalence relations too. We demand:

- \( \vdash \delta_{\tau, \vec{w}}(\vec{v}) \to \pi(\vec{w}) \),
- \( \vdash P_{\tau, \vec{w}}(\vec{v}_0, \ldots, \vec{v}_{n-1}) \to \pi(\vec{w}) \).
- \( V \vdash \exists \vec{w} \pi(\vec{w}) \);
- \( V \vdash E(\vec{w}, \vec{z}) \to (\pi(\vec{w}) \land \pi(\vec{z})) \);
- \( V \) proves that \( E \) represents an equivalence relation on the sequences forming the parameter domain;
- \( \vdash E(\vec{w}, \vec{z}) \to \forall \vec{x} (\delta_{\tau, \vec{w}}(\vec{x}) \leftrightarrow \delta_{\tau, \vec{z}}(\vec{x})) \);
- \( \vdash E(\vec{w}, \vec{z}) \to \forall \vec{x}_0, \ldots, \vec{x}_{n-1} (P_{\tau, \vec{w}}(\vec{x}_0, \ldots, \vec{x}_{n-1}) \leftrightarrow P_{\tau, \vec{z}}(\vec{x}_0, \ldots, \vec{x}_{n-1})) \);
- for all \( U \)-axioms \( A, V \vdash \forall \vec{w} (\pi(\vec{w}) \to A^\tau{\vec{w}}) \).

We can lift the various operations in the obvious way. Note that the parameter domain of \( N := M \circ K \) and the corresponding equivalence relation should be:

- \( \pi_N(\vec{w}, \vec{u}_0, \ldots, \vec{u}_{k-1}) := \pi_M(\vec{w}) \land \bigwedge_{i<k} \delta_{\tau M}(\vec{w}, \vec{u}_i) \land (\pi_K(\vec{u}_i))^\tau_M, \vec{w}. \)
- \( E_N(\vec{w}, \vec{u}_0, \ldots, \vec{u}_{k-1}, \vec{z}, \vec{v}_0, \ldots, \vec{v}_{k-1}) := E_M(\vec{w}, \vec{z}) \land \bigwedge_{i<k} \delta_{\tau M}(\vec{w}, \vec{u}_i) \land (E_K(\vec{u}_i))^\tau_M, \vec{w}. \)