DOMINANT CLASSES OF PROJECTIVE VARIETIES

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Abstract. We give evidence for a uniformization-type conjecture, that any algebraic
variety can be altered into a variety endowed with a tower of smooth fibrations of relative
dimension one.

The problem of constructing a resolution of singularities of projective varieties is one of
the most fundamental obstructions to our understanding of their analytic, arithmetic and
germonic properties. A tremendous amount of work has shed light on the this problem, yet
in its full generality it is still wide open over fields of positive characteristic. A cornerstone
result, albeit conjecturally not optimal, is de Jong’s [dJ96] 3.1, stating that any variety
can be altered into a smooth projective one. Allowing alterations, other than birational
modifications, comes along with a profusion of new natural questions, in the spirit of: which
further properties can we require, for a class of smooth projective varieties, to be dominant?
Recall from [BH00] that a class $\mathcal{C}(k, n)$ of $n$-dimensional projective varieties over a field
$k$ is dominant if for every projective $n$-dimensional variety $X$, there exists $Y \in \mathcal{C}(k, n)$
and a surjective $k$-morphism $Y \to X$. In this terminology, de Jong’s result says that
smooth projective varieties form a dominant class over any field and in any dimension.
Constructing minimal classes of dominant varieties is a problem that attracted attention,
and a satisfactory answer is still unknown even in the case of curves over fields that are
finitely generated over their prime subfield. Some results, questions and speculations in
this direction can be found in [BT02],[BT02'],[BT05]. In this paper we give evidence for
the following conjecture:

Conjecture 1 ([BH00]). For any field $k$ and positive integer $n$, the class of $n$-dimensional
smooth projective varieties $X$, endowed with a tower of smooth fibrations

$$X \to X_1 \to \ldots \to X_n$$

with $\dim X_i = n - i$, is dominant.

Section II will be devoted to the proof of:
Theorem 1. The following classes $\mathcal{C}(k,n)$ are dominant:

(i) For $n = 3$, smooth threefolds with a smooth connected morphism onto a smooth curve.

(ii) For any $n$ and $k$ a finite field, projective varieties admitting a connected morphism onto a smooth curve, with only one singular fiber whose singular locus consists of one ordinary double point.

Let us give a quick indication of the proof. The idea is to construct fibrations using Lefschetz pencils. In fact, the existence of Lefschetz pencils on smooth projective varieties, [SGA7.2] XVII, combined with de Jong’s alteration result, [dJ96] 3.1, immediately gives:

Fact. For any field $k$ and integer $n \geq 2$, the class of projective varieties admitting a connected morphism onto $\mathbb{P}^1_k$, with isolated singular fibers whose singular locus consists of one ordinary double point, is dominant.

Statement (i) of the Theorem is then an immediate consequence of the Brieskorn-Tyurina’s simultaneous resolution of surface ordinary double points, which in fact provides a simultaneous resolution of the fibers of the fibration induced by the Lefschetz pencil. Statement (ii) is more delicate. We are given $X$ a smooth and projective over a finite field, with $X^\vee$ the dual variety of singular hyperplane sections. We construct a curve $C$, in the space of hyperplanes of $X$, that intersects $X^\vee$ in a single point, which is a smooth point for both $X^\vee$ and $C$. In order to perform this construction, the crucial assumption, that $k$ is a finite field, manifests itself in that the Picard group - of degree zero divisor classes - is always finite for projective curves. The total space of the induced family of hyperplane sections has a natural fibration onto $C$ with the required conditions on the fibers, and by construction it has a surjective morphism onto $X$.

In section II we focus our attention on surfaces. In this case the dual variety $X^\vee$ stratifies according to geometric genus of the generic member, and therefore one might try to understand the geometry and the modularity of such strata. The general idea is outlined in Proposition II.10. The geometric structure of the stratification provides satisfactory answers for surfaces of negative Kodaira dimension, on which complete families of curves with constant geometric genus are easily constructed in Proposition II.11. Such families induce, upon normalization of the total space, an equigeneric fibration with smooth general member. Unfortunately the situation becomes complicated in non-negative Kodaira dimensions, where the method doesn’t provide any obvious answer on general K3’s and hypersurfaces in $\mathbb{P}^3$.
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of degree at least 4. The difficulty here is that the strata might be nested into each other as divisors, and it seems hard to insert a complete curve between two consecutive ones. Therefore we turn our attention to the modularity of the stratification. Less vaguely, each stratum $S^k$, the closure of the set of curves with geometric genus $g^k$, admits a rational map $S^k \to \overline{M}_{g^k}$, and one might try to lift complete curves from $\overline{M}_{g^k}$ to $S^k$. Question 14 summarizes the difficulty with this approach, due essentially to the non-genericity of the image of this rational map, whose behaviour near the Deligne-Mumford boundary is, a priori, arbitrary.

In section III we continue our discussion by pointing out that the category of surfaces that can be dominated by complete families of smooth curves, which conjecturally is everything, is in fact rather flexible. By this we mean that it is somewhat natural and easy to create new smooth fibrations out of old ones, using ideas inspired by Kodaira’s construction, [Ko67], of non-isotrivial smooth fibrations. First we show in Proposition 15, that any product of two curves can be dominated by a non-isotrivial smooth fibration. Finally we prove, in Proposition 17, that any two smooth fibrations can be dominated simultaneously by a third one.

In section IV, we conclude our discussion by analyzing several aspects of surfaces of general type that carry an everywhere smooth foliation. It turns out, Proposition 19, that such a surface must have positive topological index, which leads one to think about Kodaira fibrations. Indeed Brunella, in [Br97], has set the the foundations of the uniformization theory of such foliated surfaces, by establishing that the universal cover has the structure of a disk bundle over a disk. This, together with Corlette-Simpson’s classification, [CS08], of Zariski dense Kahler representations in $PSL_2(\mathbb{R})$ leads us to Theorem 21: a smoothly foliated surface of general type is either a Kodaira fibration, or a foliated subvariety of a polydisk quotient. It is extremely reasonable that, in the second case, our initial surface is itself a bidisk quotient.

We remark that the existence of ball and bidisk quotients can be used to prove, as in Proposition 22, that surfaces with an étale cover which is a Stein submanifold in a 3-dimensional ball form a dominant class.

Switching our attention to smooth foliations on surfaces over fields of characteristic $p > 0$, the situation becomes as different from the complex case, as pleasant. Indeed we have:
**Theorem 2.** Let $k$ be a field with $p := \text{char } k > 2$, and let $X/k$ be an algebraic surface. Then there exists a birational modification of $X$, followed by an inseparable cyclic cover, such that the resulting surface $Y$ carries a smooth $p$-closed foliation.

It is worth remarking that the construction of $Y$ is extremely generic, in that we start with a general Lefschetz pencil in $X$, pick a general curve going through the nodes in the pencil, and finally take an inseparable cyclic cover branched along such curve. The foliation defining the Lefschetz pencil is shown to pull back, upon saturation, to a smooth and $p$-closed foliation, via a trivial local computation. Due to the lack of Brunella’s theorem in positive characteristic, the uniformization-type consequences that can be deduced from this statement, if any, are completely mysterious.

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**I. Proof of theorem 1**

Let $X$ be a smooth projective variety of dimension $n$ over a field $k$, and let $P_X$ denote the projective space of hyperplane sections of $X$, with universal family

\[
\mathcal{U}_X \longrightarrow X \\
u \downarrow \\
P_X
\]

Let us prove (i). Assume $n = 3$, and consider a Lefschetz pencil $f : P^1_k \to P_X$. There is a Zariski-closed subset $S \subset P^1_k$ such that $(f^*\mathcal{U}_X)_s$ has a single ordinary double point if and only if $s \in S$. By the Brieskorn-Tyurina’s simultaneous resolution of ordinary double points of surfaces, [AT74], there is a ramified cover $C \to P^1_k$, and a birational morphism $Z \to C \times_{P^1_k} f^*\mathcal{U}_X$ such that the composite $Z \to C$ is a smooth morphism. \qed
What follows is the proof of (ii). Assume $k$ is a finite field, and denote by $X^\vee \subset P_X$ the dual variety of singular hyperplanes. It is well known, [SGA 7.2] XVII, that upon replacing the projective embedding of $X$ with a multiple, $X^\vee$ is an irreducible divisor inside $P_X$, whose smooth locus corresponds to hyperplane sections with a unique singular point, which is an ordinary double point.

**Claim 3.** In order to conclude the proof of the theorem, it is enough to find an irreducible curve $C \subset P_X$ such that the intersection $C \cap X^\vee$ is supported in a single point, which is smooth for both $C$ and $X^\vee$.

**Proof.** Let $f : C \to P_X$ be such curve, and let $u_C : f^*\mathcal{U}_X \to C$. $u_C$ has a unique singular fiber, whose singular locus is a single ordinary double point, lying over a smooth point of $C$. Therefore the induced fibration $f^*\mathcal{U}_X \times_C C^{\text{norm}} \to C^{\text{norm}}$ is the required one. \qed

The rest of the proof will be devoted to the construction of such curve $C$. Let $S \sim_{P^2} P_X$ be a general linear plane inside $P_X$, intersecting the divisor $X^\vee$ along an irreducible, reduced curve $D$. Denote by $d = \deg_S(D)$. Recall the well known

**Fact 4.** Since $k$ is finite, the group $\text{Pic}^0_D/k$ of degree zero divisor classes on $D$ is finite.

Let $N > d$ be an integer that kills $\text{Pic}^0_D/k$. Denote by $H$ an hyperplane section of $S$ and by $x$ a smooth point of $D$. By Fact 4 there exists an isomorphism of sheaves

$$\mathcal{O}_D(NH) \sim \mathcal{O}_D(Mx)$$

where of course $M = Nd$, inducing a commutative diagram

$$
\begin{array}{cccccc}
0 & \longrightarrow & \mathcal{O}_S(NH - D) & \longrightarrow & \mathcal{O}_S(NH) & \longrightarrow & \mathcal{O}_D(NH) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \text{Ker} \gamma & \longrightarrow & \mathcal{O}_S(NH) & \gamma & \longrightarrow & \mathcal{O}_D(Mx) & \longrightarrow & 0
\end{array}
$$

whose vertical arrows are all isomorphisms. By our choice of $N$ we deduce that $\text{Ker} \gamma$ has non-trivial global sections, and the points of $P(H^0(S, \text{Ker} \gamma))$ correspond to curves of degree $N$ inside $S$, whose intersection with $D$ is supported on $x$. All is left to do is to check that the generic member of the linear system $P(H^0(S, \text{Ker} \gamma))$ is smooth at $x$. This is achieved by way of:

**Lemma 5.** Let $G, L \in k[X,Y,Z]$ be homogeneous polynomials of degrees $d$ and $N > d$ respectively. Assume that they define irreducible curves intersecting only at $x = [0 : 0 : 1]$
and that the curve defined by $G$ is smooth at $x$. If the curve defined by $L$ is singular at $x$, then $F = Z^{N-d}G + L$ satisfies

(i) the curves defined by $F$ and $G$ meet only in $x$, and

(ii) the curve defined by $F$ is smooth at $x$.

Proof. $\{x\} = \text{Supp}((G = 0) \cap (L = 0)) = \text{Supp}((G = 0) \cap (F = 0))$, which is (i). In order to prove (ii), observe that $x$ is the origin of the affine space $Z = 1$, so then denoting by $u = X/Z$, $v = Y/Z$ and $g(u,v) = Z^{-d}G(X,Y,Z)$, $l(u,v) = Z^{-N}L(X,Y,Z)$ we see that $(F = 0)$ is defined, around $x$, by the vanishing of $f = g + l$. Since $\nabla f(0,0) = \nabla g(0,0) \neq 0$ the proof is complete. $\square$

Consequently the generic member of $\mathbf{P}(H^0(S, \text{Ker} \gamma))$ is smooth at $x$, and can be taken to be the curve $C$ we are looking for. $\square$

II. An approach to the conjecture for surfaces

Let us describe some simple examples of surfaces for which Conjecture 1 holds:

**Example 6.** Minimal models of surfaces of negative Kodaira dimension: apart from $\mathbf{P}^2$, these are $\mathbf{P}^1$-bundles over smooth curves.

**Example 7.** Some surfaces of Kodaira dimension 0:

- **Abelian varieties:** Let $A$ be a $d$-dimensional abelian variety, and $f : C \to A$ any non-constant algebraic curve. The sum morphism $C^d \to A$ given by $(c_1, \ldots, c_d) \mapsto f(c_1) + \ldots + f(c_d)$ is surjective.

- **Kummer K3 surfaces:** Let $C$ be a genus 2 curve with hyperelliptic involution $\iota$. Consider the sequence of morphisms

  $C \times C \to \text{Sym}^2 C \to \text{Jac}^2(C)$, $(c_1, c_2) \mapsto c_1 + c_2 \mapsto [c_1 + c_2]$

The graph $\Gamma_\iota$ of the hyperelliptic involution is projected onto a rational curve by the first morphism, and it is contracted by the second. Pulling back the above diagram under a degree 16 cover $m_2 : \text{Jac}^2(C) \to \text{Jac}^2(C)$ - obtained by identifying $\text{Jac}^2(C) \xrightarrow{\sim} \text{Pic}(C)$ via $K_C$ - we obtain morphisms

$m_2^*(C \times C) \to m_2^*\text{Sym}^2 C \to m_2^*\text{Sym}^2 C/(-1) = K3(C)$

Since $m_2$ is étale, $m_2^*(C \times C)$ is still a product of curves.
The previous example generalizes as follows:

**Proposition 8.** Let $k$ be a finite field, and $S_0 \subset \mathbf{P}^2(k)$ any finite set. Then there exist curves $C_1, C_2$ and a morphism $C_1 \times C_2 \to \text{Bl}_{\mathbf{P}^2}(S_0)$.

**Proof.** Observe that, for any such $S_0$, there exists a finite set $S \subset \mathbf{P}^1(k)$ and a morphism $\text{Bl}_{\mathbf{P}^1 \times \mathbf{P}^1}(S \times S) \to \text{Bl}_{\mathbf{P}^2}(S_0)$. Let $E_1$ be an elliptic curve, and $C$ a curve of genus 2 with a non-constant morphism $C \to E_1$. We can assume wlog that the Jacobian of $C$ is isomorphic to $E_1 \times E_2$, for some elliptic curve $E_2$. Consider non-constant morphisms $g_i : E_i \to \mathbf{P}^1$ and the resulting $(g_1, g_2) : \text{Bl}_{E_1 \times E_2}(g_1^* S \times g_2^* S) \to \text{Bl}_{\mathbf{P}^1 \times \mathbf{P}^1}(S \times S)$

By our assumption on $k$, any finite set of points in $E_1 \times E_2$ is torsion, hence there exists an integer $N$ and a morphism $m_N : \text{Bl}_{E_1 \times E_2}(0 \times 0) \to \text{Bl}_{E_1 \times E_2}(g_1^* S \times g_2^* S)$

where $m_N$ is the multiplication by $N$ map on $E_1 \times E_2$. Finally, composing $C \times C \to \text{Sym}^2 C \to \text{Bl}_{E_1 \times E_2}(0 \times 0)$

with the morphisms constructed above, yields $m_N^*(C \times C) \to m_N^* \text{Bl}_{E_1 \times E_2}(0 \times 0) \to \text{Bl}_{E_1 \times E_2}(g_1^* S \times g_2^* S) \to \text{Bl}_{\mathbf{P}^1 \times \mathbf{P}^1}(S \times S)$

In order to conclude, we employ [SGA1] X.1.7 to find curves $C_1, C_2$ and an isomorphism $C_1 \times C_2 \cong m_N^*(C \times C)$. □

Since every Hirzebruch surface is dominated by a blow-up, in a finite set of points, of $\mathbf{P}^1 \times \mathbf{P}^1$, we obtain:

**Corollary 9.** Let $X$ be a Hirzebruch surface over a finite field, and $S \subset X(k)$ any finite set of points, then there exist curves $C_1, C_2$ and a morphism $C_1 \times C_2 \to \text{Bl}_X(S)$.

Turning to a more general discussion, let $k$ be a field, $X/k$ a smooth projective surface, and $H$ an ample divisor such that $H^1(X, \mathcal{O}_X(nH)) = H^2(X, \mathcal{O}_X(nH)) = 0$ for all $n \geq 1$. We have the linear system $\mathbf{P}_{X,n} = \mathbf{P}(H^0(X, \mathcal{O}_X(nH)))$ of dimension $d_n = \dim(\mathbf{P}_{X,n}) = \dim H^0(X, \mathcal{O}_X(nH)) - 1 = nH \cdot (nH - K_X)/2 + \chi(\mathcal{O}_X) - 1$
and its generic member is smooth of genus \( g_n = 1 + nH \cdot (nH + K_X) / 2 \). Observe that \( P_{X,n} \)
admits a natural stratification
\[
\emptyset =: S_n^{N(n)+1} \subsetneq S_n^{N(n)} \subsetneq \ldots \subsetneq S_n^0 := P_{X,n}
\]
by, not necessarily irreducible, closed subvarieties, such that the generic member of each irreducible component of \( S_n^k \), \( 0 \leq k \leq N(n) \), corresponds to a curve with geometric genus at most \( g_n^k \), and of course \( 0 \leq g_n^{N(n)} < \ldots < g_n^0 = g_n \). We will call it the Severi stratification. For \( g_n^k \geq 2 \) there is rational map
\[
p : S_n^k \rightarrow \overline{M}_{g_n^k}
\]
into the Deligne-Mumford compactification of the moduli stack of curves. The reason to introduce the Severi stratification is:

**Proposition 10.**

(i) Assume that, for some \( 0 \leq k \leq N(n) \) such that \( g_k \geq 2 \), there exists a smooth proper curve \( C \) and a non-constant morphism \( f : C \rightarrow S_n^k \) such that the rational map \( p \circ f : C \rightarrow \overline{M}_{g_n^k} \) extends, after a finite cover \( C' / C \), to a morphism \( C' \rightarrow \overline{M}_{g_n^k} \). Then, upon replacing \( C \) by a finite cover, there exists a smooth surface \( Y \), a smooth fibration \( Y \rightarrow C \), and a surjective morphism \( Y \rightarrow X \).

(ii) For some \( 0 \leq k \leq N(n) \), there exists a smooth proper curve \( C \) and a non-constant morphism \( f : C \rightarrow S_n^k \setminus S_n^{k+1} \). Then, upon replacing \( C \) by a finite cover, there exists a smooth surface \( Y \), a fibration \( Y \rightarrow C \) whose fibers have constant geometric genus, and a surjective morphism \( Y \rightarrow X \).

**Proof.**

(i) As in the previous section, \( P_{X,n} \) is equipped with a universal space \( u : U_{X,n} \rightarrow P_{X,n} \), and we can consider the fibration \( u_C : f^*U_{X,n} \rightarrow C \). The normalization \( u_C^{\text{norm}} : (f^*U_{X,n})^{\text{norm}} \rightarrow C \) induces the moduli morphism \( p \circ f : C \rightarrow \overline{M}_{g_n^k} \) and therefore after replacing \( C \) by \( C' \), the resulting fibration is smooth.

(ii) Since the curves in our family \( u_C \) are equigeneric - meaning that the geometric genus is constant along the fibers - we have that \( u_C^{\text{norm}} \) is again an equigeneric family, with smooth general member.

\( \square \)
The natural problem becomes to investigate to what extent Proposition 10 can be applied. The next proposition shows how the geometry of the moduli \( \mathcal{M}_3 \) can be used to construct complete families of smooth curves mapping to the plane, using 10.(i):

**Fact.** There exists a complete family \( p: Y \to C \) of smooth genus 3 curves, whose fibrewise canonical map \( |p_*\omega_{Y/C}|: Y \to C \times \mathbb{P}^2_k \) realizes the generic curve as a smooth quartic, and the special ones as double conics with smooth support.

**Proof.** The Torelli map \( \mathcal{M}_3 \to \mathcal{A}_3 \) is bijective on closed points, so there is a canonical Baily-Borel compactification \( \mathcal{M}_3^{BB} \), which is projective and with boundary in codimension 2. Therefore, a generic complete intersection curve \( C \subset \mathcal{M}_3^{BB} \) is contained in \( \mathcal{M}_3 \) and intersects the hyperelliptic locus \( \mathcal{H}_3 \subset \mathcal{M}_3 \) transversely. Moreover the canonical map \( |\omega_C|: C \to \mathbb{P}^2 \) realizes points of \( \mathcal{H}_3 \) as plane double conics, and points of \( \mathcal{M}_3 \setminus \mathcal{H}_3 \) as smooth plane quartics. \( \square \)

We refer to [HM98], pp.133 for a discussion around the modular behavior of families plane quartics degenerating to double conics.

Something can be said on surfaces with negative Kodaira dimension, indeed 10.(ii) quickly proves:

**Proposition 11.** Let \( X \) be a smooth surface with negative Kodaira dimension. Then for \( n \) sufficiently big, there exists a smooth proper curve \( C \) and a non-constant morphism \( f: C \to S^n \setminus S^n_{k+1} \). Therefore, \( X \) can be dominated by an equigeneric family of curves.

**Proof.** Consider the following inductive construction: set \( S_0 := S_n^0 \), and assuming defined \( S_k \), let \( S_{k+1} \subset S_k \) be an irreducible component of \( S_{n+1}^k \) of maximal dimension. We claim that, for some \( k \), \( \dim S_k - \dim S_{k+1} \geq 2 \): by assumption the canonical bundle of \( X \) is not pseudoeffective, therefore \( K_X \cdot H < 0 \), so then \( d_n - g_n = -nH \cdot K_X - 2 + \chi(\mathcal{O}_X) \) is positive for \( n \) sufficiently big. Hence it is impossible to have \( \dim S_k = \dim S_0 - k = d_n - k \) for every \( k \). \( \square \)

**Remark 12.** Unfortunately, this is still not enough to prove the conjecture in negative Kodaira dimension: obviously, there exist equigeneric families of curves with smooth generic member, yet carrying singular, necessarily not irreducible, members.

The situation becomes more interesting in non-negative Kodaira dimension, where the genus \( g_n \) tends to be bigger than the dimension \( d_n \), and the stratification might consist of strata of consecutive codimension one. In fact this happens on general hypersurfaces:
Theorem 13. Let $X$ be a general hypersurface of $\mathbb{P}^3$ of degree $d \geq 4$. For $n \geq d$ and any $0 \leq k \leq d$, the variety $S^k_n$ contains an irreducible component of dimension $d_n - k$ whose generic point parametrizes curves with $k$ nodes.

In this situation, the best we can hope in order to construct a smooth family is a positive answer to:

Question 14. Can we find $n, k$ such that the boundary divisor $\Delta_{g^k_n} \subset \overline{\mathcal{M}}_{g^k_n}$, restricted to the closure of $p(S^k_n)$, is not ample? Even better, admitting a contraction?

A positive answer to the above would provide us with a curve to which apply Proposition 10 (i), and hence prove the conjecture in dimension 2. The problem is that the image of $p$ in the moduli $\mathcal{M}_{g^k_n}$ is going to be of high codimension and extremely non-generic. For example, consider the natural rational map

$$p : \overline{\mathcal{M}}_{g_n} \rightarrow \mathcal{M}_{g_n}^{BB}$$

Then, unless $X$ is dominated by an isotrivial surface, $p(S^k_n) \cap \Delta_{g^k_n}$ is not contracted by $p$, albeit $p$ does restrict to a contraction along $\Delta_{g^k_n}$.

III. flexibility of Kodaira fibrations

In this section we emphasize that the class of Kodaira fibrations, i.e. those surfaces admitting a non-isotrivial smooth morphism onto a smooth curve, is remarkably flexible, and there plenty of smooth fibrations that can be constructed out of given ones. First, let us review Kodaira’s original construction, [Ko67]: given any curve $C_0$ of genus at least 2, let $C \rightarrow C_0$ be any non-trivial, finite étale cover with Galois group $\Gamma$. Consider, for any $m||\Gamma|$, the natural quotient $\pi_1(C) \rightarrow H_1(C, \mathbb{Z}/m\mathbb{Z})$, and the corresponding étale cover $f : C' \rightarrow C$ with Galois group $H_1(C, \mathbb{Z}/m\mathbb{Z})$. The crucial observation is that, by the Kunneth formula, the class of the graph $\Gamma_f$ inside $H^2(C' \times C, \mathbb{Z}/m\mathbb{Z})$ depends uniquely on the morphism $f^* : H^q(C, \mathbb{Z}/m\mathbb{Z}) \rightarrow H^q(C', \mathbb{Z}/m\mathbb{Z})$. By construction, this morphism is trivial when $q = 1, 2$, while it is an isomorphism when $q = 0$. In particular, the cohomology class of $\Gamma_{\gamma f}, \gamma \in \Gamma$ is independent of $\gamma$. Since $m||\Gamma|$, we deduce that $D := \cup_{\gamma} \Gamma_{\gamma f}$ is $m$-divisible in $H^2(C' \times C, \mathbb{Z})$. Let $X = X(C, m) \rightarrow C' \times C$ be the cyclic covering of order $m$, branched along $D$. Since the $\Gamma_{\gamma f}$ are pairwise disjoint and each of them is an étale multisection of the second projection $p_2 : C' \times C \rightarrow C$, we deduce that the composition
We now employ Kodaira’s construction as follows:

**Proposition 15.** Given two curves $C_1, C_2$, there exists a curve $C$, a smooth non-isotrivial fibration in smooth curves $Y \to C$ and a finite morphism $Y \to C_1 \times C_2$.

**Proof.** Let $C_0$ be a curve of genus at least 2 with two surjective morphisms $f_i : C \to C_i$. Running the above construction, we obtain a Kodaira fibration $X$, with a natural sequence of finite morphism $X \to C' \times C \to C \times C \to C_1 \times C_2$. □

The class of surfaces of general type that are finite quotients of products of curves is vast. [BCF15], and references therein, point to a detailed study of the class of so-called product-quotient surfaces, which are, by the previous proposition, also dominated by non-isotrivial smooth fibrations.

Similarly to what has been done in Proposition 15, let $C$ of genus $g \geq 2$, and consider the divisor $\Delta_m \subset \text{Sym}^m C$ inside the $m$-fold symmetric product of $C$, of points $(c_1, ..., c_m)$ with $c_i = c_j$ for some $i \neq j$.

**Proposition 16.** Let $f : D \to \text{Sym}^m C \setminus \Delta_m$ be a non-constant morphism from a complete curve $D$. Then there exists a smooth non-isotrivial fibration $Y \to D$ and a finite morphism $Y \to C \times D$.

**Proof.** The morphism $f$ defines a divisor $D_f \subset C \times D$, whose fiber over $d \in D$ is the $m$-tuple of points $f(d) \subset C$, and by construction the projection $p : D_f \to D$ is étale. As before, upon replacing $D$ by a non-trivial Galois cover, one constructs Kodaira fibrations as cyclic covers of $D_f \times D$, branched along the Galois orbit of the graph of $p$. □

Observe that this construction provides many examples when $m = 2$, since $\Delta \subset \text{Sym}^2 C$ can be contracted via $\text{Sym}^2 C \to \text{Jac}^0(C)/(-1)$, $(c_1, c_2) \to [c_1 - c_2]$. The case $m \geq 3$ is more subtle, since the diagonal $\Delta_m \subset \text{Sym}^m C$ lies on the boundary of the effective cone of $\text{Sym}^m C$, and even its numerical properties seem rather mysterious.

In a deeper vein, the next proposition shows that pairs of Kodaira fibrations admit common refinements, meaning that they can dominated simultaneously by a third Kodaira fibration.

**Proposition 17.** Let $q_1 : X_1 \to C_1$ and $q_2 : X_2 \to C_2$ be smooth fibrations in curves. Then there exists a smooth curve $C_0$ with surjective morphisms $t_i : C_0 \to C_i$, a fibration in curves $X \to C_0$ and finite $C_0$-morphisms $X \to t_i^* X_i$, $i = 1, 2$. 
Proof. We first fix some notation: for a given curve $D$, denote by $\mathcal{M}_g(D)$ the substack of $\mathcal{M}_g$ parametrizing smooth genus $g$ curves $C$ that admit a surjective morphism $C \to D$. Similarly, for a given pair of curves $D_1, D_2$, denote by $\mathcal{M}_g(D_1, D_2)$ the substack of $\mathcal{M}_g$ parametrizing smooth genus $g$ curves $C$ that admit surjective morphisms $C \to D_1$ and $C \to D_2$. The above admits a relative analogue, in that if $D_1 \to B$ and $D_2 \to B$ are two families of curves, we have an induced fibration $\mathcal{M}_g(D_1, D_2)/B \to B$, whose fiber over $b \in B$ is $\mathcal{M}_g(D_1, b, D_2, b)$. A trick by Kodaira, [HM98] 2.33, shows that for every $n$, there exists $g$ such that $\mathcal{M}_g(D)$ contains complete subvarieties of dimension $n$, and therefore the same holds for $\mathcal{M}_g(D_1, D_2)$.

We are now ready to offer a proof of the proposition: let $C$ be a curve with surjective morphisms $t_i : C \to C_i$ and replace $q_i$ by $p_i : Y_i := t_i^* X_i \to C$. Consider the fibration $\pi : \mathcal{M}_g(Y_1, Y_2/C) \to C$. By the above remarks, for $g$ big enough there exists a surface $Z \subset \mathcal{M}_g(Y_1, Y_2/C)$ such that $\pi|_Z : Z \to C$ is surjective and the fiber $Z \cap \pi^{-1}(c)$ is a complete curve for generic $c \in C$. Such morphism $\pi|_Z$ clearly admits a multisection $C_0$ - for example by taking an ample divisor in $Z$, the closure of $Z$ in $\mathcal{M}_g(Y_1, Y_2/C)$, missing the isolated boundary points - and such multisection $C_0$ induces, by definition, a smooth fibration in curves $X \to C_0$ dominating $Y_1/C$ and $Y_2/C$.

IV. Smooth foliations on surfaces of general type

Motivated by the study of surfaces that carry smooth fibrations, we dedicate this final section to surfaces of general type carrying smooth foliations. In particular we look for restrictions the existence of a smooth foliation imposes on the ambient surface, and then try to understand what they might possibly look like. Recall that a smooth foliation $\mathcal{F}$ on an algebraic surface $X$ defines an exact sequence of vector bundles

$$0 \to T\mathcal{F} \to TX \to N\mathcal{F} \to 0$$

The next fact recollects some well known numerical properties of $K\mathcal{F} := T\mathcal{F}^\vee$.

Fact 18.

(i) $c_2(X) = K\mathcal{F} \cdot (K_X - K\mathcal{F})$.

(ii) $(K_X - K\mathcal{F})^2 = 0$.

(iii) $c_2(X) = K_X^2 - K\mathcal{F} \cdot K_X$.

(iv) If $X$ is of general type then $K\mathcal{F}$ is pseudoeffective, and it is big unless $\mathcal{F}$ is either an isotrivial fibration, or a Hilbert modular foliation.
Proof.

(i) This follows by taking Chern numbers in the defining exact sequence of $T\mathcal{F}$, plus $c_2(X) > 0$.

(ii) This is the Baum-Bott index formula, $N\mathcal{F}^2 = 0$.

(iii) This is a formal consequence of (i) and (ii).

(iv) The pseudoeffectivity of $K\mathcal{F}$ follows from the Main Theorem of [BM16], since $X$ has general type. By the birational classification of foliations on surfaces, [McQ08], if a foliation on a surface of general type is such that $K\mathcal{F}$ is not big, then the foliation is an isotrivial or Hilbert modular.

Therefore, granted a decent amount of understanding of isotrivial and Hilbert modular surfaces, we concentrate on the strong topological and algebraic restrictions, for a surface of general type to carry a smooth foliation.

**Proposition 19.** Let $X$ a smooth surface of general type, and $\mathcal{F}$ an everywhere smooth foliation on $X$ with $K\mathcal{F}$ big. Then we have

- rank Pic$(X) \geq 2$.
- $X$ has positive topological index, i.e. $c_1(X)^2 > 2c_2(X)$.

**Proof.** Let $a := \frac{K_X \cdot K\mathcal{F}}{K_X}$, and $R := aK_X - K\mathcal{F}$. Since $R \cdot K_X = 0$ we deduce, by Hodge index theorem, that $R^2 < 0$ unless $R$ is numerically trivial.

**Claim 20.** $R \neq 0$

**Proof.** We have

$$0 = R \cdot K_X = -K\mathcal{F} \cdot K_X + K_X^2 + (a - 1)K_X^2 = c_2(X) + (a - 1)K_X^2$$

so then

$$c_2(X) = (1 - a)K_X^2$$

If $R = 0$ then $aK_X = K\mathcal{F}$, and $(K_X - K\mathcal{F})^2 = 0$ implies $a = 1$, from which $c_2(X) = 0$, impossible since $X$ has general type.

From which we deduce that rank Pic$(X) \geq 2$.

Let $P \subseteq NS(X)$ be the plane spanned by $K_X$ and $R$. We know that

$$Amp(X) = \{D : D^2 > 0, D \cdot K_X > 0\}$$
so then Fact 18 (i) and (ii) imply that $K_X - K_F = (1 - a)K_X + R$ lies on the boundary of $\text{Amp}(X) \cap P$. It follows that $\text{Amp}(X) \cap P$ is bounded by the rays $$(1 - a)K_X + R, \quad (1 - a)K_X - R$$ Since $$K_F = aK_X - R \in \text{Amp}(X) \cap P$$ by assumption, we have $a > 1 - a$, or $a > 1/2$. The identity $$c_2(X) = (1 - a)K_X^2$$ derived in the proof of Claim 20, concludes the proof. \[\square\]

More interestingly, Brunella [Br97] has initiated the uniformization theory on smoothly foliated surfaces $(X, F)$ of general type: the universal cover $\tilde{X}$ of such a surface, admits a smooth holomorphic fibration $p_\tilde{X} : \tilde{X} \to \Delta$ onto the unit disk, with disks as fibers, such that $\tilde{F}$ becomes tangent to $p_\tilde{X}$ on $\tilde{X}$. In particular, $X$ is a $K(\Gamma,1)$, for $\Gamma := \pi_1(X)$. With the aim of better understanding the geometry of $X$, observe that the fibration $p_\tilde{X}$ is naturally $\Gamma$-equivariant, and hence there is a natural representation $\rho : \Gamma \to PSL_2(\mathbb{R})$. Corlette and Simpson, in [CS08], have given a complete classification of what such a $\rho$ can be, and deduced a beautiful dichotomy for its geometric origin. We can summarize all of this in:

**Theorem 21** (Brunella, Corlette & Simpson). Let $(X, F)$ be a smoothly foliated surface of general type. Then at least one of the following happens:

- $X$ admits a smooth fibration $p : X \to C$ onto a smooth curve $C$ and $F$ is tangent to $p$;
- $\rho$ is rigid and integral, there exists a quasiprojective polydisk quotient, $Y$, and a natural morphism $X \to Y$ such that $F$ is induced by one of tautological codimension one foliations on $Y$.

The only issue to be settled here, is the relation between the dimensions of $X$ and $Y$. It seems plausible, thinking about the wild behaviour of Hilbert modular foliations, that indeed the dimensions must be the same, and $X$ itself is a bidisk quotient.

Let us remark that bounded symmetric domains can be used in our problem of finding dominant classes as follows:
**Proposition 22.** Let $X$ be a smooth algebraic surface. Then there exists a finite morphism $Y \to X$, such that an étale cover $Y'$ of $Y$ is a Stein submanifold of a 3-dimensional ball.

**Proof.** Let $Z$ be a ball quotient, consider projections $X \to \mathbb{P}^2$, $Z \to \mathbb{P}^2$, and let $Y := X \times_{\mathbb{P}^2} Z$. For sufficiently generic projections, the branch loci in $\mathbb{P}^2$ are transverse, hence $Y$ is smooth. Let $p : B^2 \to Z$ be the universal cover, then $Y' := p^* Y$ is a complex manifold which is naturally a finite ramified cover of the ball $B^2$, and as such embeds into a product $B^2 \times \Delta$. □

We wish to conclude by turning our attention to the corresponding problem in characteristic $p > 0$, of understanding smooth foliations on algebraic surfaces. Perhaps not too surprisingly, the situation is drastically different from the complex case, and it turns out that it is extremely simple to construct a covering of a given surface that carries a smooth foliation. Before proceeding to our main result, let us recall a key difference between foliations in characteristic 0 and $p$, that lies in the notion of integrability: the celebrated Frobenius integrability theorem, as well known for integrable distributions over $\mathbb{C}$, fails over fields of positive characteristic: integrable distributions need not have a formal first integral. Observe, indeed, that the notion of leaf is rather badly behaved, and the closest we can get to formal integrability is $p$-closedness - that is, the algebra generated by the vector fields defining our foliation is closed under $p$-powers. This implies that the kernel of such algebra of differential operators defines a factorization of the Frobenius morphism on the ambient variety. And a modicum of thought shows that this is the best integrability condition one can hope for. In practice, probably the quickest way of appreciating the role of $p$-closedness in remediying the failure of formal integrability is:

**Fact 23.** [McQ08, II.1.6] Let $A$ be a complete regular local ring over an algebraically closed field of characteristic $p$, and let $\partial$ be a non-singular derivation of $A$. Then there exists a regular system $x, y_1, \ldots, y_n$ of parameters such that

$$\partial = \frac{\partial}{\partial x} + x^{p-1} \sum_{i=1}^n f_i(x^p, y) \frac{\partial}{\partial y_i}$$

Moreover the ideal defining the vanishing of $\partial \wedge \partial^p$ is generated by $f_1, \ldots, f_n$. In particular $\partial$ is $p$-closed iff $\partial = \frac{\partial}{\partial x}$, i.e. a smooth foliation by curves is $p$-closed iff it is formally integrable.
IV.1. Proof of Theorem \[2\] Let \( p : Y_0 \to P^1_k \) be a general Lefschetz pencil in \( X \), whose general member is smooth, and whose singular members have exactly one node. Observe that \( Y_0 \) is obtained by blowing up \( X \) along the base points of the pencil. Let \( S \subset P^1_k \) parametrize the singular fibers, and for each \( s \in S \) let \( y_s \in Y_s \) be the node in the fiber. Let \( H \) be a very ample divisor, and consider the sheaf \( \mathcal{O}(pH) \otimes I_{\cup y_s} \) whose local sections are those of \( \mathcal{O}(pH) \) vanishing along \( \cup y_s \). Its generic global section defines a curve, \( D \), which is smooth, transverse to the branches of \( Y_s \) at \( y_s \), and has simple tangencies with the fibers of \( p \) outside their nodes. The following claim will conclude:

**Claim 24.** Let \( r_D : Y := Y_0(\sqrt{D}) \to Y_0 \) denote the inseparable cyclic \( p \)-cover, branched along \( D \). If \( \mathcal{F} \) denotes the foliation defined by \( p \), then the saturation of \( r_D^* \mathcal{F} \) is smooth and \( p \)-closed.

**Proof.** First we deal with smoothness. We need to worry about what happens at:

(i) The nodes in the fibers of \( p \), and

(ii) The simple tangencies between \( D \) and the smooth locus of the fibers of \( p \).

Let us consider (i). In the local ring of \( Y_0 \) completed in a node in a singular fiber, the branches of the singular fiber give us local coordinates \( x, y \), while \( D \) is defined by the vanishing of a third local function, \( z \). \( \mathcal{F} \) is defined by the vanishing of the form \( d(xy) \), and our assumptions on \( D \) imply that there exists a formal function \( G \) such that \( y = G(x, z) \) holds. The cyclic cover \( r_D \) is defined by \( r_D^* z = z^p \), hence

\[
  r_D^* d(xy) = d(x \cdot G(x, z^p)) = (G + x \frac{\partial G}{\partial x}) dx
\]

and therefore, upon saturation, \( r_D^* \mathcal{F} \) is smooth around the pre-image, under \( r_D \), of nodes in the fibers of \( p \).

Let us deal with (ii). In the local ring of \( Y_0 \) completed in a point of simple tangency, we can pick local coordinates \( x, y \) such that our fibration is defined by the vanishing of the form \( dy \). If the vanishing of \( z \) defines \( D \) in such coordinates, simple tangency means that there exists a formal function \( g \) such that \( z = y - g(x) \) and \( \text{ord}_x g(x) = 2 \).

**In particular,** \( \frac{dg(x)}{dx} \) is not identically zero, since \( p > 2 \).

The cyclic cover \( r_D \) is defined by

\[
  r_D^* z = z^p, \quad r_D^* x = x, \quad r_D^* y = z^p + g(x)
\]
and the pullback foliation is then

$$r_D^* dy = d(z^p + g(x)) = \frac{dg(x)}{dx} dx$$

which is again smooth upon saturation.

Observe, as an output of our local computations, that the saturation of $r_D^* F$ is not only smooth, but also formally integrable. By Fact 23, it is $p$-closed. □

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