VECTOR TRANSFORM OPERATORS FOR PIECE-WISE HARMONIC FUNCTIONS

O.YAREMKO, Y. PARFENOVA

Abstract. The vector transform operators are investigated; these operators are used at the solution of boundary value problems in piecewise homogeneous spherically symmetric areas. In particular, examples of transformation operators for vector boundary value problems are given for third vector boundary value problem in the unit circle and for the Dirichlet problem in the unit circle.

Keywords: harmonic functions, vector transform operator, vector boundary value problems.

Mathematics Subject Classification 2010: 65Nxx Boundary value problems, 35N30 Overdetermined initial-boundary value problems; 35Cxx Representations of solutions; 35A22 Transform methods.

1. Introduction.

The transform operator is the operator which translates the decision of one problem of mathematical physics in the decision of other problem. Transform operators of K. Weierstrass, S. Poisson, N. Ja. Sonin are known.

Our interest are the special cases of transform operators concerning different boundary value problems for the same equation. By the way of illustration it is considered the first and third boundary value problems of Dirichlet for the Laplace equation in unit circle:

\[ \Delta \tilde{u} = 0, \]
\[ u|_s = f(\varphi), \]
\[ h\tilde{u} + \tilde{u}|_{r=1} = f(\varphi) \quad h > 0. \]

It is possible to prove that the transform operator \( P : \tilde{u} \to u \), as shown I.I. Bavrin in work [1], has form

\[ u(x) = \int_0^1 \varepsilon^{h-1} \tilde{u}(\varepsilon x) \, d\varepsilon. \]

In the present work transform operators are constructed in a vector case which formally turns out replacement of function \( u \) by a vector function \( u \), and number \( h \) by a matrix \( H \).

The case of transform operators is studied, the transform operators connecting the decision of the first boundary value problem with internal conditions of interface on sphere and the decision of a problem of Dirichlet are constructed.

Statement of the first regional problem with interface conditions:

\[ \Delta u_k = 0, \quad x \in V_k; \quad k = 1, ..., n + 1. \]
The edge conditions are

\[ \Gamma_0 [u_1] = f_0 (\eta), \quad \eta \in S_0, \]

where

\[ u_k = \begin{pmatrix} u_{k1} \\ u_{k2} \\ \vdots \\ u_{km} \end{pmatrix}, \quad f_0 = \begin{pmatrix} f_{01} \\ f_{02} \\ \vdots \\ f_{0m} \end{pmatrix}. \]

There are the non-uniform contact on the hypersurfaces conjugation

\[ S_k, S_k = \{ \eta = (\eta_1, ..., \eta_N) : ||\eta|| = r_k \} : \]

\[ \Gamma_{j1}^k [u_k] - \Gamma_{j2}^k [u_{k+1}] = f_{jk} (\eta); \quad \eta \in S_k; \quad k = 1, ..., n; \quad j = 1, 2, \]

where

\[ f_{jk} = \begin{pmatrix} f_{jk1} \\ f_{jk2} \\ \vdots \\ f_{jkm} \end{pmatrix}, \]

Here

\[ \Delta u_k = \frac{1}{r^{N-1}} \frac{\partial}{\partial r} \left( r^{N-1} \frac{\partial u_k}{\partial r} \right) + \frac{1}{r^2} \Delta_\eta u_k, \]

\( \Delta_\eta \) is Laplace operator on the sphere \( S_0 \); \( \Gamma_0, \Gamma_{j1}^k, \Gamma_{j2}^k (j = 1, 2; k = 1, ..., n) \) are posed operators, permutable with operator \( L_0 = \sum_{i=1}^{N} x_i \frac{d}{dx_i} \).

The transform operator looks like:

\[ u = P_0 [\tilde{u}_0] + 2 \sum_{j=1}^{n} \sum_{s=1}^{n} P_{js} [\tilde{u}_{js}] \]

where \( P_0, P_{js} \) - vector transform operators.

Kelvin reflection method for solutions of mathematical physics boundary value problem with symmetric boundary from served as the basis for method of operators progresses in mathematical physics, complex analysis, harmonic analysis[2]. In this paper operator method is developed for vector problems of heterogeneous pattern mathematical physics.

2. The common boundary value problem for the Laplace equation in unit sphere with non-uniform internal conditions of interface.

Let \( B_n \) be pieszewise homogeneous unit ball of \( R^N \):

\[ B_n = \bigcup_{i=1}^{n+1} V_i; V_i = \{ x \in R^N : r_i < ||x|| < r_{i-1} \} ; i = 1, ..., n + 1, \]

\[ B_n = S_0 \times I_n^+, S_0 = \{ \eta \in R^N : ||\eta||^2 = 1 \}, \]
Let us consider a problem about construction set separate Laplace combined equations solution, bounded on $B_n$.

(1) \( \Delta u_k = 0, x \in V_k; k = 1, \ldots, n + 1; \)

(2) \( \Gamma_0 [u_1] = f_0(\eta), \quad \eta \in S_0 \)

(3) \( \Gamma_{j1}^k [u_k] - \Gamma_{j2}^k [u_{k+1}] = f_{jk}(\eta); \quad \eta \in S_k; k = 1, \ldots, n, \quad j = 1, 2, \)

where \( f_{jk} = \begin{pmatrix} f_{jk1} \\ f_{jk2} \\ \vdots \\ f_{jkm} \end{pmatrix} \)

Here

\[
\Delta u_k = \frac{1}{r^{N-1}} \frac{\partial}{\partial r} \left( r^{N-1} \frac{\partial u_k}{\partial r} \right) + \frac{1}{r^2} \Delta_\eta u_k,
\]

\( \Delta_\eta \) is the Laplace operator on sphere $S_0$; \( \Gamma_0, \Gamma_{j1}^k, \Gamma_{j2}^k \) \((j = 1, 2; k = 1, \ldots, n)\) are permutable with \( L_0 = \sum_{i=1}^{N} x_i \frac{d}{dx_i} \) defined operators.

3. Method of influence function.

Fourier transform on sphere $S_0$ with nonseparated variables [3] reduces the problem (1)-(3) to form: find the separate combined differential equations solution

\[
\frac{1}{r^{N-1}} \frac{d}{dr} \left( r^{N-1} \frac{d\tilde{u}_k,l}{dr} \right) - l(l + N - 2) \frac{1}{r^2} \tilde{u}_k,l = 0; l \in \mathbb{Z}, r_i < r < r_{i-1}
\]

by boundary conditions

\( \Gamma_0 [\tilde{u}_{1,l}]_{r=r_0} = r_0^{N-1} \tilde{f}_{0,l} \)

There are the heterogeneous contact conditions in points joint $r = r_k$

\( \Gamma_{j1}^k [\tilde{u}_{k,l}] - \Gamma_{j2}^k [\tilde{u}_{k+1,l}] = r_0^{N-1} \tilde{f}_{jk,l}, k = 1, \ldots, n; j = 1, 2. \)

Assign formula by immediate checking:
\[ \tilde{u}_{j,l}(r) = H^*_{j,1,l}(r, r_0) \begin{pmatrix} 0 \\ 1 \end{pmatrix} r_0^{N-1} \hat{f}_{0,l} + \sum_{s=1}^{n} H^*_{j,s,l}(r, r_s) r_s^{N-1} \begin{pmatrix} \hat{f}_{1,s,l} \\ \hat{f}_{2,s,l} \end{pmatrix}, \]

Here \( H^*_{k,s,l} = H^*_{k,l,s}(r, \rho) \) are matrix-valued \( m \times m \) functions, that defined by formulas:

When \( k < s \)

\[ H^*_{k,s,l} = \begin{pmatrix} \varphi_{k,l}(r) - \psi_{k,l}(r) \psi^{-1}_{1,l} & \varphi^{-1}_{1,l} \\ \varphi_{1,l} & \psi_{1,l} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \Omega^{-1}_{s,l}(\rho) \end{pmatrix}, r_k < r < r_{k-1}, r_s < \rho < r_{s-1}, \]

When \( k > s \)

\[ H^*_{k,s,l} = -\psi_{k,l}(r) \begin{pmatrix} \varphi^{-1}_{1,l} & \varphi_{1,l} \\ \varphi_{1,l} & \psi_{1,l} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \Omega^{-1}_{s,l}(\rho) \end{pmatrix}, r_k < r < r_{k-1}, r_s < \rho < r_{s-1}, \]

When \( k = s \)

\[ H^*_{k,s,l} = \begin{cases} \begin{pmatrix} \varphi_{k,l}(r) - \psi_{k,l}(r) \psi^{-1}_{1,l} & \varphi^{-1}_{1,l} \\ \varphi_{1,l} & \psi_{1,l} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \Omega^{-1}_{s,l}(\rho) \end{pmatrix}, & r_{k-1} < r < \rho < r_k, \\ -\psi_{k,l}(r) \begin{pmatrix} \varphi^{-1}_{1,l} & \varphi_{1,l} \\ \varphi_{1,l} & \psi_{1,l} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \Omega^{-1}_{s,l}(\rho) \end{pmatrix}, & r_{k-1} < \rho < r < r_k. \end{cases} \]

Let us define matrix-valued functions

\[ \varphi_{n+1,l}(r) = r^l \cdot E; \psi_{n+1,l}(r) = r^{-(l+N-2)} \cdot E; l \in \mathbb{Z}, \]

where \( E \) is identity matrix. Order \( n \) pairs of functions \( (\varphi_{k,l}, \psi_{k,l}) \), \( k = 1, \ldots, n \) are founded from the recurrence equations.

\[ \Gamma^k_{i,j} (\varphi_{k,l}, \psi_{k,l}) = \Gamma^k_{j,i} (\varphi_{k+1,l}, \psi_{k+1,l}) , k = 1, \ldots, n ; i, j = 1, 2. \]

Let us use the following notations

\[ \varphi_{1,l} = \Gamma_0 [\varphi_{1,l}(r)] \bigg|_{r=r_0} \psi_{1,l} = \Gamma_0 [\psi_{1,l}(r)] \bigg|_{r=r_0}; l \in \mathbb{Z}, \]

\[ \Gamma^k_{i,j} (\varphi_{k,l}(r), \psi_{k,l}(r)) = (\varphi^k_{i,j,l}(r), \psi^k_{i,j,l}(r)) ; i, j = 1, 2 ; k = 1, \ldots, n ; l \in \mathbb{Z}, \]

\[ \Omega_{k,l}(\rho) = \begin{pmatrix} \varphi^k_{1,l}(\rho) & \psi^k_{1,l}(\rho) \\ \varphi^k_{2,l}(\rho) & \psi^k_{2,l}(\rho) \end{pmatrix}. \]

Matrix-valued functions \( H^*_{k,s,l}(r, \rho) \) correctly defined, if following conditions satisfied:

i) when \( l \to \infty \), matrix sequences \( \alpha_{0,l}, \alpha^k_{j,1,l}, \alpha^k_{j,2,l} \), that defined by formulas

\[ \Gamma_0 [r^l] = \alpha_{0,l}, \Gamma^k_{i,j} [r^l] = \alpha^k_{i,j,l} r^l ; i, j = 1, 2 ; k = 1, \ldots, n ; l \in \mathbb{Z}, \]
have growth which no more than power-mode, and
\[
\det M_{k,l} = \det \left( \begin{bmatrix} \alpha_{1,j,l}^k & \alpha_{2,j,l}^k \\ \alpha_{2,j,-l}^k & \alpha_{1,j,-l}^k \end{bmatrix} \right) \neq 0; \ k = 1, \ldots, n; \ j = 1, 2; l \in \mathbb{Z},
\]
ii) for every \( l \in \mathbb{Z} \) following inequality are valid \( \det \Omega_{k,l} (\rho) \neq 0 \); \( k = 1, \ldots, n \);
\( \psi_{1,l} \neq 0 \).

Receive formula for problem solutions(1)-(3) by returning fo Fourier original:
\[
\begin{align*}
\hat{u}_j (r \xi) &= \frac{1}{\omega_N} \int_{S_0} \left( H_{j,1} (r, r_0, (\eta, \xi)) \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) r_0^{N-1} f_0 (\eta) + \\
&+ \sum_{s=1}^{n} H_{j,s} (r, r_s, (\eta, \xi)) r_s^{N-1} \begin{bmatrix} f_{1s} (\eta) \\ f_{2s} (\eta) \end{bmatrix} dS_0,
\end{align*}
\]
(4)
where
\[
H_{j,s} (r, r_s, (\eta, \xi)) = \sum_{l=0}^{\infty} \frac{2l + N - 1}{N - 1} C_l^{(N-1)/2} (\langle \eta, \xi \rangle) H_{j+s,l}^* (r, r_s)
\]
- \( \omega_N \) - (N-1)- \( l \rightarrow \infty \) -dimensional volume of unit sphere \( S_0 \) from \( R^N \); \( C_l^{(N-1)/2} \) are Gegenbauer polynomials \[1\], \( \langle \eta, \xi \rangle \) - is scalar product of vectors \( \eta, \xi \)

4. TRANSFORM OPERATORS

Let \( \hat{u}_0, \hat{u}_{jq} \) are harmonic vector-functions in the unit ball \( B_0, B_0 = \{ x \in R^N : \| x \|^2 < 1 \} \)
and continuous on \( \bar{B}_0 \) Boundary values of that functions are vectors \( f_0 (\eta), f_{jq} (\eta) \) respectively. Let us define vector transform operators \( P_0, P_{jq} \) by using rules:
if
\[
u_0 (r \xi) = \sum_{k=1}^{n+1} \chi (V_k) \ u_{0k} (r \xi),
\]
\[
u_{0k} (r \xi) = \int_{S_0} H_{k,1} (r, r_0, (\eta, \xi)) \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) r_0^{N-1} \hat{u}_0 (r \eta) dS_0,
\]
then \( P_0 : \hat{u}_0 \rightarrow u_0 \); similarly, if
\[
u_j (r \xi) = \sum_{k=1}^{n+1} \chi (V_k) \ u_{j,k} (r \xi),
\]
\[
u_{jq,k} (r \xi) = \int_{S_0} H_{k,q,l} (r, r_q, (\eta, \xi)) \begin{bmatrix} \delta_{1j} \\ \delta_{2j} \end{bmatrix} r_q^{N-1} \hat{u}_{jq} (r \eta) dS_0,
\]
then \( P_{jq} : \hat{u}_{jq} \rightarrow u_{jq} \); \( j = 1, 2; q = 1, \ldots, n \), where \( \chi (V_k) \) is a characteristic function in \( V_k \) \( \delta_{ij} \) is Kronecker symbol.
\[
\chi (V_k) = \begin{cases} 1, & r \xi \in V_k, \\ \chi (V_k) - 1, & r \xi \notin V_k \end{cases}, \quad \delta_{ij} = \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases}
\]

Basic formula (4) can be brought to the following form by using \( P_0, P_{jq} \) vector transform operators:
\[
u = P_0 [\hat{u}_0] + \sum_{j=1}^{n} \sum_{s=1}^{n} P_{js} [\hat{u}_{js}],
\]
where \( \mathcal{E} \) is the Sturm–Liouville problem for the Fourier operator in piecewise homogeneous axis. Eigenfunction \( \varphi \) here continues in \( V_k \). Here of existence i)-ii) vector-valued function \( H_{k,x,l} \) \((r, \rho)\), satisfied, then transformation operator \( P_0 \) \((P_{jq})\) compare \( \hat{u}_0 \) harmonic vector function \( u_0 \) \((u_{jq})\) in piecewise homogeneous ball \( B_n \). The components \( u_{0,k} \) \((u_{jq,k})\) of that function continued in \( V_k \), ball layer and satisfy the boundary condition (2) and conjugation condition (3).

**Example 1.** Transform operator \( P_0 \) for the third vector boundary value problem \( Hu + E \frac{\partial u}{\partial n} \bigg|_{S_0} = \hat{u} \bigg|_{S_0} \) in unit circle have form

\[
P_0 : \hat{u} \to u \ (x) = \int_0^1 \varepsilon^{H-E} \hat{u} (\varepsilon x) \, d\varepsilon.
\]

Here
\[
\varepsilon^{H-E} = e^{(H-E) \ln \varepsilon},
\]
\( E \) is identity matrix, \( H \) is symmetric and positive-definite matrix [4].

**Example 2.** Transform operator \( P_0 \) for Dirichlet problem in the unit circle with the internal conjunction conditions

\[
u^- (\eta) = u^+ (\eta), \quad K \frac{\partial}{\partial n} u^- (\eta) = \frac{\partial}{\partial n} u^+ (\eta), \ |\eta| = r
\]

have form \( P_0 : \hat{u} \to u \),
\[
u (x) = \begin{cases} \sum_{j=0}^{\infty} \left[ (E - K)(E + K)^{-1} \right]^j \left( \hat{u} (x r^{2j}) - (E - K)(E + K)^{-1} \hat{u} \left( \frac{x}{r^{j+2}} \right) \right), & r < |x| < 1, \\ 2K (E + K)^{-1} \sum_{j=0}^{\infty} \left[ (E - K)(E + K)^{-1} \right]^j \hat{u} (x r^{2j}), & |x| < r. \end{cases}
\]

Where \( (E + K)^{-1} \) is inverse matrix, \( u^- (\eta), u^+ (\eta) \) are limit of function \( u = u (x) \) when \( x \to h \) from without and from within respectively. Similarly,
\[
\frac{\partial}{\partial n} u^- (\eta), \frac{\partial}{\partial n} u^+ (\eta) \text{ are limits } u' (x),
\]

when \( x \to h \) from without and from within respectively.

4.1. **Transform operators in half-plane.** Method of transformation operators is used to solve the problem [1]. Necessary definitions from [3], [4], [5]. The direct \( J : \hat{f} \to f \) and inverse \( J^{-1} : f \to \hat{f} \) transformation operators are set equalities:
\[
f(x) = \int_{-\infty}^{\infty} \varphi(x, \lambda) \left( \int_{-\infty}^{\infty} e^{-i\lambda \xi} \hat{f} (\xi) d\xi \right) d\lambda,
\]
\[
\hat{f}(x) = \int_{-\infty}^{\infty} e^{-i\lambda \xi} \left( \int_{-\infty}^{\infty} \varphi^*(\xi, \lambda) f(\xi) d\xi \right) d\lambda.
\]

Here \( \varphi(x, \lambda), \varphi^*(x, \lambda) \) - are the eigenfunctions [13], [14] of the direct and coupling Sturm–Liouville problems for the Fourier operator in piecewise-homogeneous axis in. Eigenfunction
\[
\varphi(x, \lambda) = \sum_{k=2}^{n} \theta(x - l_{k-1}) \theta(l_k - x) \varphi_k(x, \lambda) +
\]
VECTOR TRANSFORM OPERATORS

\[ + \theta(l_1 - x)\varphi_1(x, \lambda) + \theta(x - l_n)\varphi_{n+1}(x, \lambda) \]
is a solution of the system of separate differential equations
\[ \left( a_m^2 \frac{d^2}{dx^2} + \lambda^2 \right) \varphi_m(x, \lambda) = 0, \ x \in (l_m, l_{m+1}); \ m = 1, \ldots, n + 1, \]
on the coupling conditions
\[ \left[ \alpha_m^k \frac{d}{dx} + \beta_m^k \right] \varphi_k = \left[ \alpha_{m2}^k \frac{d}{dx} + \beta_{m2}^k \right] \varphi_{k+1}, \]
on the boundary conditions
\[ \varphi_1|_{x=-\infty} = 0, \ \varphi_{n+1}|_{x=\infty} = 0 \]
Similarly eigenfunction
\[ \varphi^*(\xi, \lambda) = \sum_{k=2}^{n} \theta(\xi - l_{k-1})\theta(\xi - \theta)\varphi^*(\xi, \lambda) + \theta(l_1 - \xi)\varphi_1^*(\xi, \lambda) + \theta(\xi - l_n)\varphi_{n+1}^*(\xi, \lambda) \]
is a solution of the system of separate differential equations
\[ \left( a_m^2 \frac{d^2}{dx^2} + \lambda^2 \right) \varphi^*_m(x, \lambda) = 0, \ x \in (l_m, l_{m+1}); \ m = 1, \ldots, n + 1, \]
With the coupling conditions
\[ \frac{1}{\Delta_{1,k}} \left[ \alpha_{m1}^k \frac{d}{dx} + \beta_{m1}^k \right] \varphi_k = \frac{1}{\Delta_{2,k}} \left[ \alpha_{m2}^k \frac{d}{dx} + \beta_{m2}^k \right] \varphi_{k+1}, \ x = l_k, \]
where
\[ \Delta_{i,k} = \det \left( \begin{array}{cc} \alpha_{1i}^k & \beta_{1i}^k \\ \alpha_{2i}^k & \beta_{2i}^k \end{array} \right) \ k = 1, \ldots, n; \ i, m = 1, 2, \]
on the boundary conditions
\[ \varphi_1|_{x=-\infty} = 0, \ \varphi_{n+1}|_{x=\infty} = 0 \]
Let for some \( \lambda \) of the considered boundary value problems have nontrivial solutions \( \varphi(x, \lambda), \varphi^*(x, \lambda) \), in this case the number \( \lambda \) is called the eigenvalue [13], [14], corresponding solutions \( \varphi(x, \lambda), \varphi^*(x, \lambda) \) - is called the eigenfunctions of the direct and coupling Sturm–Liouville problems, respectively. In the further we shall adhere to the following normalization of eigenfunctions:
\[ \varphi_{n+1}(x, \lambda) = e^{ia_{n+1}^x x \lambda} \varphi_{n+1}^*(x, \lambda) = e^{-ia_{n+1}^x x \lambda}. \]

5. Conclusion.

The vector transform operators are investigated; these operators are used at the solution of boundary value problems in piecewise homogeneous spherically symmetric areas in the article. Further it is supposed to extend results of work to a case of two and more internal conditions of interface.
References

[1] Bavrin I.I. Operatornyj metod v kompleksnom analize. -Moscow: Prometej, 1991. - 200 p.
[2] Bavrin I.I., Yaremko O.E. Integral Fourier transforms on compact of $\mathbb{R}^n$ and their applications to the moment problem. - Moscow: Doklady Mathematics, 2000. - 62, No.2, pp. 177-179. 374, No.2, p. 154-156.
[3] Yaremko O.E. The method of transformation operators as applied to boundary value problems in spherically symmetric domains. - Moscow: Doklady Mathematics, 2006. - 74, No. 1, pp. 507-511.
[4] Yaremko O.E. Transformation operator and boundary value problems Differential Equation. Vol.40, No. 8, 2004, pp.1149-1160
[5] Bavrin, I.I., Yaremko, O.E. Transformation Operators and Boundary Value Problems in the Theory of Harmonic and Biharmonic Functions (2003) Doklady Mathematics, 68 (??), pp. 371-375.
[6] Gantmacher F. R. The Theory of Matrices. 2 vols. Chelsea, New York, 1959.

Oleg Yaremko, Yulia Parfenova
Penza State University,
str. Lermontov, 37,
440038, Penza, Russia
E-mail address: yaremki@mail.ru