On a Finite Group Generated by Subnormal Supersoluble Subgroups

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Abstract. Supersolubility of a finite group $G = \langle A, B \rangle$ with the nilpotent derived subgroup $G'$ is established under the condition that the subgroups $A$ and $B$ are both subnormal and supersoluble.

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1 Main Result

All groups in this paper are finite. We use the standard notations and terminology of [3].

B. Huppert [4] and R. Baer [1] gave the first examples of nonsupersoluble groups that were a product of normal supersoluble subgroups. R. Baer [1] established supersolubility of a group $G = AB$ with the nilpotent derived subgroup $G'$ and normal supersoluble subgroups $A$ and $B$. A. F. Vasil’ev and T. I. Vasil’eva [7] showed that instead of nilpotency of the derived subgroup $G'$ it is enough to require nilpotency of the $A$-residual, where $A$ is the formation of all groups with abelian Sylow subgroups. In these results, normality of subgroups $A$ and $B$ can be weakened to subnormality of $A$ and $B$ [5]. This themes was developed in many papers, see for example [2].

If a group $G$ is generated by subnormal subgroups $X$ and $Y$, then it is not always true that $G = XY$. The simplest examples are the dihedral group of order 8 and nonabelian group of order $p^3$ and exponent $p$. The normal closure $X^G$ of a subnormal supersoluble subgroup $X$ in a group $G$ can be a nonsupersoluble subgroup. For example, $G = PSU_3(2) \times C_2$ [3, SmallGroup(144,182)] contains a nonsupersoluble maximal subgroup $H = S_3 \ltimes C_2$ [3, SmallGroup(72,40)] of index 2. A subgroup $X = S_3 \times S_3$ of $H$ is supersoluble and not normal in $G$. Since $|H : X| = 2$, $X^G = H$ and $X$ is subnormal in $G$.

Next, we write $G = \langle A, B \rangle$ when a group $G$ is generated by subgroups $A$ and $B$.

In this paper, we prove the following theorem.

Theorem 1.1. Let $A$ and $B$ be subnormal supersoluble subgroups of a group $G$ and let $G = \langle A, B \rangle$. Then $G$ is metanilpotent and has a Sylow tower of supersoluble type. Furthermore, $G$ is supersoluble when one of the following conditions holds.

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(1) \(G^A\) is nilpotent.

(2) \(|A : A'|, |B : B'|\) = 1.

Proof. Let \(p \in \pi(G)\) and let \(A_p\) and \(B_p\) be Sylow \(p\)-subgroups of \(A\) and \(B\), respectively. Suppose that \(p\) is the largest prime in \(\pi(G)\). Then \(A_p\) and \(B_p\) are normal in \(A\) and \(B\), respectively [3, VI.9.1]. Therefore \(A_p\) and \(B_p\) are subnormal in \(G\) and \(\langle A_p, B_p \rangle \leq O_p(G)\).

By induction, \(G/O_p(G)\) has a Sylow tower of supersoluble type. Since \(AO_p(G)/O_p(G)\) and \(BO_p(G)/O_p(G)\) are subnormal \(p'\)-subgroups of \(G/O_p(G)\),

\[
G/O_p(G) = \langle AO_p(G)/O_p(G), BO_p(G)/O_p(G) \rangle
\]

is a \(p'\)-group. Hence \(G\) has a Sylow tower of supersoluble type.

Since \(A'\) is nilpotent [3, VI.9.1] and subnormal in \(G\), we have \(A' \leq F(A) \leq F(A)^G \leq F(G)\). Consequently,

\[
AF(G)/F(G) \cong A/(A \cap F(G))
\]

is abelian and subnormal in \(G/F(G)\). Similarly,

\[
B' \leq F(B) \leq F(B)^G \leq F(G), BF(G)/F(G) \cong B/(B \cap F(G)),
\]

therefore \(BF(G)/F(G)\) is abelian and subnormal in \(G/F(G)\). It is clear that

\[
G/F(G) = \langle AF(G)/F(G), BF(G)/F(G) \rangle.
\]

So, \((AF(G)/F(G))^{G/F(G)}\) and \((BF(G)/F(G))^{G/F(G)}\) are normal in \(G/F(G)\) and nilpotent. Hence

\[
G/F(G) = (AF(G)/F(G))^{G/F(G)}(BF(G)/F(G))^{G/F(G)}
\]

is nilpotent, and \(G\) is metanilpotent.

(1) Let \(G^A\) be nilpotent. Then \(G^A \leq F(G)\). Since \(G\) is metanilpotent, we conclude that \(G/F(G)\) is nilpotent, hence \(G/F(G)\) is abelian and \(G' \leq F(G)\). Therefore \(AG'\) and \(BG'\) is normal in \(G\). According to [3, Lemma 10], \(AG'\) and \(BG'\) are supersoluble. Hence \(G = (AG')(BG')\) is supersoluble by Baer’s Theorem.

(2) Let \(|A : A'|, |B : B'|\) = 1. Since \(A'B' \subseteq F(G)\), we get

\[
(|AF(G)/F(G)|, |BF(G)/F(G)|) = 1.
\]

From (\(\star\)), it follows that

\[
G/F(G) = AF(G)/F(G) \times BF(G)/F(G)
\]

is abelian and \(G' \leq F(G)\). Therefore \(G = (AG')(BG')\) is supersoluble.

\(\square\)

**Corollary 1.1.1.** Let \(A\) and \(B\) be subnormal supersoluble subgroups of a group \(G\) and let \(G = \langle A, B \rangle\). If the derived subgroup of \(G\) is nilpotent, then \(G\) is supersoluble.
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