VECTOR CONSTANTS OF MOTION FOR TIME-DEPENDENT KEPLER AND ISOTROPIC HARMONIC OSCILLATOR POTENTIALS

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Abstract
A method of obtaining vector constants of motion for time-independent as well as time-dependent central fields is discussed. Some well-established results are rederived in this alternative way and new ones obtained.

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1 Introduction

It is well known that in classical mechanics the knowledge of all first integrals of motion of a given problem is equivalent to finding its complete solution. Nowadays the search for first integrals has assumed an increasing importance in the determination of the integrability of a dynamical system. It is extremely important to know if a non-linear dynamical system will present chaotic behavior in some regions of the phase space. The notion of integrability is related to the existence of first integrals of motion. Several methods of finding first integrals are available in the literature for example, Lie’s method \[1\], Noether’s theorem \[2\], or the direct method \[3\]. Even if not all first integrals of motion associated with the problem at hand are found, it may happen that the ones which are obtained contribute to the discovery of the solution we are seeking for. Nevertheless, if we do find the solution we are after by solving the equations of motion in a straightforward way, it still may be profitable to look for additional constants of motion. Such is the case of the Kepler problem where the knowledge of the Laplace-Runge-Lenz vector \[4\], \[5\] allows us to obtain the orbit in a simple way.

Of the inexhaustible wealth of problems which we can find in classical mechanics one of the most aesthetically appealing and important is the central field problem. Energy and angular momentum associated with this type of field are well known conserved quantities. However, other vector and tensor conserved quantities have been associated with some particular central fields. The Laplace-Runge-Lenz vector is a vector first integral of motion for the Kepler problem; the Fradkin tensor \[7\] is conserved for the case of the harmonic oscillator and for any central field it is possible to find a vector first integral of motion as was shown in \[4\]. In the general case these additional integrals of motion turn out to be complicated functions of the position \(r\) and linear momentum \(p\) of the particle probing the central field. When orbits are closed and degenerated with respect to the mechanical energy, however, we should expect these additional constant of motion to be simple function of \(r\) and \(p\). In this article we wish to exploit further this line of reasoning by determining the existence of such additional vector first integrals of motion for the time-dependent Kepler and isotropic harmonic oscillator problems. In particular, we will show that for the time-dependent Kepler problem the
existence of a vector constant of motion coupled to a simple transformation of variables turns the problem easily integrable.

The structure of this paper goes as follows: in section 2 we establish the conditions which guarantee the existence of a vector first integral for a general central force field. In section 3 we put the method to test by rederiving some well known results such as the conservation of angular momentum in an arbitrary central field, the conservation of the Laplace-Runge-Lenz vector for the Kepler problem, and the conservation of the Fradkin tensor fixing en route these specific fields to which they correspond. In section 4 we consider the time-dependent case establishing generalizations of the examples considered before and presenting new ones. In section 5 we show that the existence of a vector first integral enable us to find the orbits of a test particle. This is accomplished for the case of harmonic oscillator, and the time-dependent Kepler problem. Also the period of the time-dependent Kepler problem is obtained. Finally, section 6 is reserved for final comments.

2 Constructing vector constants of motion

The force \( f(r,t) \) acting on a test particle moving in a central but otherwise arbitrary and possible time-dependent field of force \( g(r,t) \) can be written as

\[
f(r,t) = g(r,t) \mathbf{r},
\]

(1)

where \( r = r(t) \) is the position vector with respect to the center of force, \( r \) is its magnitude, and \( t \) is the time. To this test particle we assume that it is possible to associate a vector \( \mathbf{j} \) which in principle can be written in the form

\[
\mathbf{j}(p,r,t) = A(p,r,t) \mathbf{p} + B(p,r,t) \mathbf{r},
\]

(2)

where \( p = p(t) := m \dot{r}(t) \) is the linear momentum, \( m \) is the reduced mass and \( A, B \) are arbitrary scalar functions of \( p, r \) and \( t \). Taking the total time derivative of (2) and making use of Newton’s second law of motion we readily obtain

\[
\frac{d\mathbf{j}}{dt} = \left( Ag + \frac{dB}{dt} \right) \mathbf{r} + \left( \frac{dA}{dt} + \frac{B}{m} \right) \mathbf{p}.
\]

(3)

If we assume that \( \mathbf{j} \) is a constant of motion it follows that the functions \( A \) and \( B \) must satisfy
\[ Ag + \frac{dB}{dt} = 0, \quad (4) \]

\[ \frac{dA}{dt} + \frac{B}{m} = 0. \quad (5) \]

Eliminating \( B \) between (4) and (5) we obtain

\[ m\frac{d^2A}{dt^2} - gA = 0. \quad (6) \]

It follows from (5) that \( j \) can be written in the form

\[ j = A \mathbf{p} - m\frac{dA}{dt} \mathbf{r}. \quad (7) \]

Therefore, since (6) is equivalent to both (4) and (5) if the field \( g(r, t) \) is known any solution of (6) will yield a vector constant of motion of the form given by (4). Equation (6), however, is a differential equation whose solution may turn out to be a hard task to accomplish. Nevertheless, we can make progress if instead of trying to tackle it directly we make plausible guesses concerning \( A \) thereby linking \( j \) to specific forms of the field \( g(r, t) \). This procedure is tantamount to answering the following question: given \( j \) what type of central field will admit it as a constant of motion? The answer is given in the next section.

### 3 Simple examples

With \( \mathbf{r}, \mathbf{p} \) and a unit constant vector \( \mathbf{u} \) we can construct the following scalars: \( \mathbf{u} \cdot \mathbf{r}, \mathbf{u} \cdot \mathbf{p} \) and \( \mathbf{r} \cdot \mathbf{p} \). Other possibilities will be considered later on. For the moment let us consider some simple possibilities for the scalar function \( A(\mathbf{p}, \mathbf{r}, t) \).

Consider first \( A(\mathbf{p}, \mathbf{r}, t) = \mathbf{u} \cdot \mathbf{r} \). It is immediately seen that this choice for \( A \) satisfies (3) for any function \( g(r, t) \). The constant vector \( j \) reads

\[ j = (\mathbf{u} \cdot \mathbf{r}) \mathbf{p} - (\mathbf{u} \cdot \mathbf{p}) \mathbf{r}, \quad (8) \]

and it can be related to the angular momentum \( \mathbf{L} = \mathbf{r} \times \mathbf{p} \) as follows. Firstly we recast (8) into the form

\[ j = \mathbf{M} \cdot \mathbf{u}, \quad (9) \]
where $M = p \mathbf{r} - r \mathbf{p}$. Since $\mathbf{u}$ is a constant vector we conclude that the constancy of $\mathbf{j}$ is equivalent to the constancy of $M$ whose components are $M_{jk} = p_j x_k - x_j p_k$, with $i, j, k = 1, 2, 3$. The antisymmetrical tensor $M$ is closely related to angular momentum $L$ of the test particle. In fact, it can be easily shown that $-2L_i = \varepsilon_{ijk}M_{jk}$, where $L_i$ is the $i$-th angular momentum component and $\varepsilon_{ijk}$ is the usual permutation symbol or Levi-Civita density. Therefore, this simple choice for $A$ leads to conservation of angular momentum for motion under a central arbitrary field $g(r, t)$.

Consider now the choice $A(p, r, t) = \mathbf{u} \cdot p$. Then making use of Newton’s second law it follows that (7) is satisfied if we find a solution to

$$\frac{dg(r, t)}{dt} \mathbf{u} \cdot \mathbf{r} = 0.$$  

(10)

For arbitrary values of $\mathbf{u} \cdot \mathbf{r}$ we can find a solution to (10) if and only if $g(r, t) \equiv 0$, or $g = g_0$ where $g_0$ is a constant. In this case we can write

$$\mathbf{j} = (\mathbf{u} \cdot \mathbf{p}) \mathbf{p} - g_0 (\mathbf{u} \cdot \mathbf{r}) \mathbf{r}.$$  

(11)

If we choose the constant to be equal to $-k$ then the central force field will correspond to an isotropic harmonic oscillator, $\mathbf{f} = -k \mathbf{r}$. As before, (11) can be recasted into the form

$$\mathbf{j} = 2m \mathbf{F} \cdot \mathbf{u},$$  

(12)

where $\mathbf{F}$ is given by

$$\mathbf{F} = \frac{\mathbf{p} \mathbf{p}}{2m} + k \frac{\mathbf{r} \mathbf{r}}{2}.$$  

(13)

The tensor $\mathbf{F}$ is symmetrical and is known as the Fradkin tensor [7]. Finally, consider $A = \mathbf{r} \cdot \mathbf{p}$. For this choice of $A$ (7) yields

$$\frac{1}{g} \frac{dg}{dt} + 3 \frac{dr}{r} \frac{dr}{dt} = 0,$$  

(14)

where we have made use of (11) and also of the fact that $d\mathbf{r}/dt$ and $\mathbf{r}$ are perpendicular vectors. Equation (14) can be easily integrated if $g$ is considered to be a function of the radial distance $r$ only. If this is the case we obtain the Kepler field $g(r) = -k/r^3$. The constant vector $\mathbf{j}$ is then given by

$$\mathbf{j} = (\mathbf{r} \cdot \mathbf{p}) \mathbf{p} - (\mathbf{p} \cdot \mathbf{p}) \mathbf{r} + mk \frac{r}{r},$$  

(15)
Making use of a well known vector identity we can recast (15) into the form,

\[ j = L \times p - mk \frac{r}{r}. \]  

(16)

Therefore \( j \) can be equaled to minus the Laplace-Runge-Lenz vector \( A \). From (16) and the condition \( j \cdot r = -A \cdot r = 0 \) the allowed orbits for the Kepler problem can be obtained in a simple way, see for example [5].

4 Time-dependent fields

Let us now consider time-dependent central force fields for which we can build more general vector first integrals of motion. As with the time-independent case there are of course several possibilities when it comes to the choice of a function \( A \) for a time-dependent central field. Here is one

\[ A = \phi(t) r \cdot p + \psi(t) r \cdot r. \]  

(17)

Evaluating the second derivative of (17) we obtain

\[ \frac{d^2A}{dt^2} = \left( \frac{d^2\phi}{dt^2} + \frac{4g\phi}{m} + \frac{4}{m} \frac{d\psi}{dt} \right) r \cdot p + \left( 2 \frac{d\phi}{dt} g + \phi \frac{dg}{dt} + \frac{d^2\psi}{dt^2} + 2 \frac{\psi g}{m} \right) r \cdot r \]

\[ + \frac{2}{m^2} \left( m \frac{d\phi}{dt} + \psi \right) p \cdot p. \]  

(18)

Where we have made use of (1). If we impose the additional condition

\[ \psi + m \frac{d\phi}{dt} = 0, \]  

(19)

we eliminate the quadratic term in \( p \). With the condition given by (19) we can substitute for \( \psi \) in (17) and (18) and take the results into (3) thus obtaining

\[ 3 \left( -m \frac{d^2\phi}{dt^2} + g\phi \right) r \cdot \frac{dr}{dt} + \left( \phi \frac{dg}{dt} - m \frac{d^3\phi}{dt^3} + g \frac{d\phi}{dt} \right) r \cdot r = 0. \]  

(20)

Equation (20) can be rewritten as

\[ \frac{3}{2} \left( -m \frac{d^2\phi}{dt^2} + g\phi \right) \frac{d(r^2)}{dt} + \left[ \frac{d}{dt} \left( -m \frac{d^2\phi}{dt^2} + g\phi \right) \right] r^2 = 0, \]  

(21)
and easily integrated so as to yield

\[ g(r, t) = \frac{m}{\phi} \frac{d^2 \phi}{dt^2} + \frac{C}{\phi r^3}. \]  \hfill (22)

The vector first integral of motion associated with (22) is

\[ \mathbf{j} = \left( \phi \mathbf{r} \cdot \mathbf{p} - m \frac{d\phi}{dt} \mathbf{r} \cdot \mathbf{r} \right) \mathbf{p} + \left( m \frac{d\phi}{dt} \mathbf{r} \cdot \mathbf{p} - \phi \mathbf{p} \cdot \mathbf{p} - \frac{mC}{r} \right) \mathbf{r}, \]  \hfill (23)

which can be simplified and written in the form

\[ \mathbf{j} = m\phi^2 \mathbf{L} \times \frac{d}{dt} \left( \frac{\mathbf{r}}{\phi} \right) - \frac{mC}{r} \mathbf{r}. \]  \hfill (24)

where we have made used of the fact the angular momentum is constant for any arbitrary central field whether it is time-independent or not. If in (24) we set \( \phi = 1 \) and \( C \neq 0 \), then from (22) we see that \( g(r, t) \) is the Kepler field and \( \mathbf{j} \) is minus the Laplace-Runge-Lenz vector as before; the scalar function \( A(r, \mathbf{p}, t) \) reduces to \( \mathbf{r} \cdot \mathbf{p} \) which we have already employed in section 3. If we set \( (m/\phi) \frac{d^2 \phi}{dt^2} = -k(t) \), that is, if \( \phi \) is an arbitrary function of the time, and also \( C = 0 \), we have the time-dependent isotropic harmonic oscillator field, \( \mathbf{F}(r) = -k(t) \mathbf{r} \). In this case \( \mathbf{j} \) is equal to the first term on the R.H.S. of (24). If \( (m/\phi) \frac{d^2 \phi}{dt^2} = -k \) and \( C = 0 \), we have the time-independent isotropic harmonic oscillator field but this time \( \mathbf{j} \) is not the same vector as the one we have obtained before. The reason for this is our choice (17) for the scalar function \( A(r, \mathbf{p}, t) \) which is not reducible to the form \( \hat{\mathbf{u}} \cdot \mathbf{p} \) employed previously.

As a last example let us consider again the time-dependent isotropic harmonic oscillator and show how it is possible to generalize the Fradkin tensor for this case. Let the function \( A(r, \mathbf{p}, t) \) be written as

\[ A = \phi(t) \hat{\mathbf{u}} \cdot \mathbf{r} + \psi(t) \hat{\mathbf{u}} \cdot \mathbf{p}. \]  \hfill (25)

The first and the second derivative of \( A \) read

\[ \frac{dA}{dt} = \frac{d\phi}{dt} \hat{\mathbf{u}} \cdot \mathbf{r} + \frac{\phi}{m} \hat{\mathbf{u}} \cdot \mathbf{p} + \frac{d\psi}{dt} \hat{\mathbf{u}} \cdot \mathbf{p} + g \psi \hat{\mathbf{u}} \cdot \mathbf{r}, \]  \hfill (26)
and

\[ \frac{d^2 A}{dt^2} = \left( \frac{d^2 \phi}{dt^2} + 2g \frac{dq}{dt} + \frac{dg}{dt} \psi + \frac{g \phi}{m} \right) \hat{u} \cdot r + \left( \frac{d^2 \psi}{dt^2} + 2g \frac{d \phi}{dt} + \frac{g \psi}{m} \right) \hat{u} \cdot p \]

(27)

Taking (27) into (6) we obtain the condition

\[ m \left( \frac{d^2 \phi}{dt^2} + 2g \frac{d \psi}{dt} + \frac{dg}{dt} \psi \right) \hat{u} \cdot r + \left( 2 \frac{d \phi}{dt} + m \frac{d^2 \psi}{dt^2} \right) \hat{u} \cdot p = 0. \]  

(28)

Imposing the additional condition

\[ 2 \frac{d \phi}{dt} + m \frac{d^2 \psi}{dt^2} = 0, \]  

(29)

(28) becomes

\[ m \frac{d^3 \psi}{dt^3} - 4g \frac{d \psi}{dt} - 2 \frac{dg}{dt} \psi = 0. \]  

(30)

We can solve (30) thoroughly if \( g(r,t) \) is a function of the time \( t \) only. In this case, as before, we end up with the time-dependent isotropic harmonic oscillator. The vector \( \mathbf{j} \) associated with (25) can be obtained as follows: first we integrate (29) thus obtaining

\[ \phi = \frac{m \frac{d \psi}{2 dt}}{dt} + C, \]  

(31)

where \( C \) is an integration constant. Then making use of (24), (31) and (26) we arrive at

\[ \mathbf{j} = \left[ \left( -\frac{m \frac{d \psi}{2 dt}}{dt} + C \right) \hat{u} \cdot r + \psi \hat{u} \cdot p \right] \mathbf{p} \]

(32)

or in terms of components

\[ j_i = F_{ij} u_j, \]  

(33)

where \( F_{ij} \) is defined by
\[ F_{ij} = \left( -\frac{m}{2} \frac{d\psi}{dt} + C \right) p_i x_j + \psi p_i p_j \]  
\[ + \left( \frac{m^2}{2} \frac{d^2 \psi}{dt^2} - mg \psi \right) x_i x_j - \left( \frac{m}{2} \frac{d\psi}{dt} + C \right) x_i p_j. \]  

(34)

The constant \( C \) in (34) can be made zero without loss of generality. A generalized Fradkin tensor can now be defined by

\[ F_{ij} = \left( -\frac{m}{2} \frac{d\psi}{dt} \right) p_i x_j + \psi p_i p_j + \left( \frac{m^2}{2} \frac{d^2 \psi}{dt^2} - mg \psi \right) x_i x_j - \frac{m}{2} \frac{d\psi}{dt} x_i p_j. \]  

(35)

From (35) we can read out the diagonal components of the generalized Fradkin tensor

\[ F_{ii} = -\frac{m}{2} \frac{d\psi}{dt} x_i \frac{dx_i}{dt} + \psi p_i^2 + \left( \frac{m^2}{2} \frac{d^2 \psi}{dt^2} - mg \right) x_i^2. \]  

(36)

It is not hard to see that the trace of this generalized Fradkin tensor \( F(\mathbf{r}, \mathbf{p}, t) \) becomes the energy of the particle when \( g(\mathbf{r}) \) is a constant field.

5 Obtaining explicit solutions: An alternative way

Now we wish to show how to take advantage of the vector constant \( \mathbf{j} \) to obtain the solution for the Kepler and the isotropic harmonic oscillator potentials. But firstly we must establish some very general relationships between the sought for solution \( \mathbf{r}(t) \) and \( A(\mathbf{r}, \mathbf{p}, t) \) and \( \mathbf{j} \). First notice that (10) can be recasted into the form

\[ \mathbf{j} = m A(\mathbf{r}, \mathbf{p}, t)^2 \frac{d}{dt} \left[ \frac{\mathbf{r}}{A(\mathbf{r}, \mathbf{p}, t)} \right], \]  

(37)

where it must be kept in mind that \( A(\mathbf{r}, \mathbf{p}, t) \) satisfies (3). As we have seen before in specific examples the form of the vector constant \( \mathbf{j} \) depends on the force acting on the particle. Integrating (37) we readily obtain
\[
\frac{\mathbf{r}(t)}{A(r,p,t)} - \frac{\mathbf{r}(0)}{A(r,p,0)} = \frac{\mathbf{j}}{m} \int_0^t \frac{d\tau}{A(r,p,\tau)^2}. \tag{38}
\]

Equation (38) can be given a simple but interesting geometrical interpretation. Assume that the initial conditions \(\mathbf{r}(0)\) and \(\mathbf{p}(0)\) are known and therefore the function \(A(r,p,0)\) can be determined. The vector \(\mathbf{r}(0)/A(r,p,0)\) is therefore a constant and completely determined vector. As time increases, the R.H.S. of (38) increases. The vector on the right side of (38) though varying in time has a fixed direction which is determined by \(\mathbf{j}\). Therefore, \(\mathbf{r}(t)/A(r,p,t)\) must increase in order to close the triangle whose sides are the three vectors involved in (38). If the orbit is unlimited then it is easy to see that the following property ensues: there is an asymptote if in the limit \(t \to \infty\) the definite integral \(\int_0^t \frac{d\tau}{A^2}\) is constant. On the other hand, if the orbit is limited, but not necessarily closed, there will be a position vector \(\mathbf{r}\) whose direction is parallel to that of the vector \(\mathbf{j}\) at the instant \(t^*\). If the length of the position vector \(\mathbf{r}\) is finite, we can conclude that at the same instant \(t^*\) the function \(A(r,p,0)\) must be zero. Thus, we can see that the vector \(\mathbf{r}(t)/A(r,p,t)\) must be reversed at this instant and its evolution is determined by the fact that its end is on the straight line that contains \(\mathbf{j}\). For \(t = t^* + \epsilon\), where \(\epsilon\) is a positive infinitesimal number, the vector \(\mathbf{r}(t^* + \epsilon)/A(r,p,t^* + \epsilon)\) changes its direction abruptly, so to speak, as shown in the figure, hence in the transition \(A(r,p,t^*) \rightarrow A(r,p,t^* + \epsilon)\) the scalar function must change its sign.

Let us obtain the solution \(\mathbf{r}(t)\) for the case of the isotropic harmonic oscillator. A particular solution of (17) for \(g = -\kappa\), where \(\kappa\) is the elastic constant is given by

\[
A(t) = \cos(\omega t), \tag{39}
\]

where \(\omega = \sqrt{\frac{\kappa}{m}}\) is the angular frequency. The solution given by (39) allows us to write

\[
\frac{\mathbf{r}(t)}{\cos(\omega t)} - \mathbf{r}(0) = \frac{\mathbf{p}(0)}{m} \int_0^t \frac{d\tau}{\cos^2(\omega \tau)} \tag{40}
\]

The integral can be readily performed and after some simplifications we finally obtain

\[
\mathbf{r}(t) = \cos \omega t \mathbf{r}(0) + \frac{1}{m\omega} \sin \omega t \mathbf{p}(0) \tag{41}
\]
Therefore we have obtained the solution of the time-independent harmonic oscillator in an alternative way from the knowledge of the initial conditions as it should be. Notice that the general solution \( A(t) = A(0) \cos(\omega t + \theta) \) would lead to the same general result. Another possible solution in the case of the time-independent isotropic harmonic oscillator is given by

\[
A(r, p, t) = \hat{u} \cdot p(t),
\]

as can be shown by substituting this solution into (33). This solution shows that the trajectories have no asymptotes.

Let us now show how we can obtain the orbits in the case of the time-dependent Kepler problem. Let us begin by rewriting (37) in polar coordinates on the plane. In these coordinates the angular momentum conservation law is written in the form

\[
l = mr^2 \frac{d\theta}{dt}
\]

and this allows to rewrite (37) as

\[
\mathbf{j} = mA^2 \frac{d}{d\theta} \left( \frac{r}{A} \right) \frac{d\theta}{dt} = l \left( \frac{A}{r} \right)^2 \frac{d}{d\theta} \left( \frac{r}{A} \right).
\]

(44)

Introducing the unitary vectors \( \hat{r} \) and \( \hat{\theta} \) we can write the above equation as

\[
\mathbf{j} = l \left[ -\frac{d}{d\theta} \left( \frac{A}{r} \right) \hat{r} + \frac{A}{r} \hat{\theta} \right].
\]

(45)

The components of the vector \( \mathbf{j} \) in the direction of \( \hat{r} \) and \( \hat{\theta} \) are given by

\[
\mathbf{j} \cdot \hat{r} = l \frac{A}{r},
\]

and

\[
\mathbf{j} \cdot \hat{\theta} = -l \frac{d}{d\theta} \left( \frac{A}{r} \right).
\]

(47)

Equation (45) can be obtained from (46) and therefore it is redundant. In section 4 we determined a generalized Laplace-Runge-Lenz vector for the time-dependent Kepler problem. The scalar function \( A(r, p, t) \) associated with this vector was found to be
\[ A = \phi(t) \ r \cdot p - m \frac{d\phi}{dt} r^2. \]  

Making use of (43) we can rewrite the linear momentum as a function of \( \theta \) as follows

\[ p = m \frac{dr}{dt} \frac{d\theta}{dt} = \frac{l}{r^2} \frac{dr}{d\theta}. \]  

Taking (49) into (48) and considering \( A \) as a function of \( \theta \) we obtain

\[ \frac{A}{r} = -l \frac{d}{d\theta} \left( \frac{\phi}{r} \right). \]  

Equations. (46) and (50) lead to

\[ \frac{d}{d\theta} \left( \frac{\phi}{r} \right) = \frac{\hat{\mathbf{j}}}{l^2} \sin(\theta - \alpha), \]  

where \( \alpha \) is the angle between \( \mathbf{j} \) and the \( OX \) axis (see figure 2). In order to integrate (51) we assume that the initial conditions at \( t = 0 \) are known vector functions, i.e.

\[ \mathbf{r}(0) = \mathbf{r}_0; \]  

and

\[ \mathbf{p}(0) = \mathbf{p}_0. \]  

In terms of polar coordinates these initial conditions are written as

\[ r(\theta_0) = r_0, \]  

and making use of (49)

\[ \left( \frac{dr}{d\theta} \right)_{\theta=\theta_0} = \frac{r_0^2}{l} \mathbf{p}_0. \]  

Upon integrating (51) we find

\[ \frac{\phi}{r} = \frac{\phi_0}{r_0} + \frac{\hat{\mathbf{j}}}{l^2} \left[ \cos(\theta - \alpha) - \cos(\theta - \alpha) \right]. \]
For the usual time-independent Kepler problem, \( \phi = 1 \) and in this case \( (56) \) takes the form

\[
\frac{1}{r} = \frac{1}{r_0} + \frac{j}{l^2} \left[ \cos(\theta_0 - \alpha) - \cos(\theta - \alpha) \right]
\]  
\( (57) \)

The scalar product between \( j \) as given by \( (14) \) and \( \mathbf{r}_0 \) permit us to eliminate \( \cos(\theta_0 - \alpha) \) and leads to the usual orbit equation

\[
\frac{1}{r} = -\frac{mC}{l^2} \left[ 1 + \frac{j}{mC} \cos(\theta - \alpha) \right].
\]  
\( (58) \)

If we define a new position vector \( \mathbf{r}' \) according to

\[
\mathbf{r}' := \frac{\mathbf{r}}{\phi},
\]  
\( (59) \)

and redefine our time parameter according to

\[
dt' := \frac{dt}{\phi^2},
\]  
\( (60) \)

we can recast the equation of motion for the time-dependent Kepler problem, namely

\[
m \frac{d^2 \mathbf{r}}{dt^2} = \left( m \frac{d^2 \phi}{\phi \ dt^2} + \frac{C}{\phi r'^3} \right) \mathbf{r}
\]  
\( (61) \)

into a simpler form. According to \( (59) \) and \( (60) \) the velocity and the acceleration transform in the following way

\[
\frac{dr}{dt} = \mathbf{r}' \frac{d\phi}{dt} + \frac{1}{\phi} \frac{dr'}{dt'}
\]  
\( (62) \)

and

\[
\frac{d^2 \mathbf{r}}{dt^2} = \mathbf{r}' \frac{d^2 \phi}{dt^2} + \frac{1}{\phi^3} \frac{d^2 \mathbf{r}'}{dt'^2}.
\]  
\( (63) \)

where we have taken advantage of the fact that \( \mathbf{r}, \mathbf{r}' \) and \( \phi \) can be considered as functions of \( t \) or \( t' \). Making use of \( (63) \) the equation of motion \( (61) \) can be written as

\[
m \frac{d^2 \mathbf{r}'}{dt'^2} = \frac{C}{(r')^3} \mathbf{r}'.
\]  
\( (64) \)
Equation (64) corresponds to the usual time-independent Kepler problem whose solution is given by (58). Equations (59) and (60) show that the open solutions of (64) are transformed into the open solutions of (61) with the same angular size and that closed solutions of (64) are associated with spiraling solutions of (61). The period of the orbit of (64) is related to the time interval that the spiraling particle takes to cross a fixed straight line. Representing this time interval by $T_0$ we have

$$T = \int_{0}^{T_0} \frac{dt}{\phi^2}.$$  

(65)

As an application of the above remarks suppose we are looking for the form of the function $\phi$ which yields a circular orbit with radius $R$ as a solution to (61)? To find this function we see from (61) that we have to solve the following equation differential equation

$$\frac{d^2\phi}{dt^2} + \frac{|C|}{mR^3} \phi = \frac{|C|}{mR^3}.$$  

(66)

The solution is

$$\phi(t) = 1 + \phi_0 \cos(\omega t + \beta),$$  

(67)

where $\phi_0$ is a constant and $\omega = \sqrt{\frac{|C|}{mR^3}}$ and $\beta$ an arbitrary phase angle. The constant $\phi_0$ may be chosen so that the transformed solution will be a given ellipse as we show below. Equation (58) leads to

$$\frac{1}{r'} = R^{-1} [1 + \phi_0 \cos(\omega t + \beta)].$$

Comparing with (58) we obtain

$$R = \frac{l^2}{m|C|},$$  

(68)

and

$$\phi_0 = -\frac{j}{m|C|}.$$  

(69)

The period of this circular orbit is given
and using (65) we get

\[ T = \int_0^{2\pi} \frac{dt}{(1 + e \cos \omega t)^2} = \frac{2\pi}{\omega} \frac{1}{(1 - e^2)^{\frac{3}{2}}} \] (71)

where \( e = |\phi_0| \) is the eccentricity. Making use of \( 1 - e^2 = R/a \), where \( a \) is the major semi-axis we finally obtain the orbital period

\[ T = 2\pi a^{\frac{3}{2}} \sqrt{\frac{m}{C}}. \] (72)

To conclude consider the total mechanical energy associated with (64)

\[ E = \frac{p^2}{2m} + \frac{C}{r} = \text{const.} \] (73)

Since \( p' \) and \( p \) are related by

\[ p' = \phi(t) \ p - m \dot{\phi}(t) \ r \] (74)

and \( r' \) and \( r \) by (59) we easily obtain

\[ E = \phi^2 \frac{p^2}{2m} - 2 \phi \frac{d\phi}{dt} \cdot r + \left( \frac{d\phi}{dt} \right)^2 \frac{r^2}{2} + C \phi = \text{const.} \] (75)

which is a conserved quantity and can be interpreted as a generalization of the energy of the particle under the action of a time-dependent Kepler field.

6 Laplace-Runge-Lenz type of vector constants for arbitrary central fields

Equation (22) determines a time-dependent Kepler field \( g(r, t) \), where the variables \( r \) and \( t \) are independent. If, however, we consider the orbit equation \( r(t) \) we can eliminate the time variable and define the function \( g(r, t(r)) \) which can be understood as an arbitrary function of \( r \). For the sake of simplicity we denote this function by \( g(r) \). The function \( \phi(t) \) which transforms
the Kepler problem when understood as a function of \( r \) transforms the Kepler field in an arbitrary central field. Let us write Eq. (72) in the form

\[
m d^2 \phi \over dt^2 - g(r, t) \phi + C r^3 = 0
\]  

(76)

and consider the transformation

\[
d^2 \phi \over dt^2 = d^2 \phi \over dr^2 \left( dr \over dt \right)^2 + d\phi \over dr \frac{d^2 r}{dt^2}
\]  

(77)

Energy conservation and the equation of motion allow us to write

\[
\left( dr \over dt \right)^2 = 2 \frac{E - V(r)}{m} - \frac{l^2}{2mr^2}
\]  

(78)

and

\[
d^2 r \over dt^2 = \frac{r g(r)}{m} + \frac{l^2}{m^2 r^3}
\]  

(79)

where \( E \) is the energy of the particle and \( l \) its angular momentum. Taking these three last equations into account Eq. (76) reads now

\[
2 \left[ E - V(r) - \frac{l^2}{2mr^2} \right] d^2 \phi \over dr^2 + \left[ r g(r) + \frac{l^2}{m^2 r^3} \right] d\phi \over dr - g(r) \phi + C r^3 = 0
\]  

(80)

Equation (80) permit us to determine the function \( \phi(r) \) for any potential \( V(r) \). Thus, we conclude that a central field problem can be transformed into a time-dependent Kepler problem. When \( g(r) \) describes the Kepler field the solution of (80) is \( \phi(r) = 1 \). Sometimes it is convenient to perform a second change of variables by defining the transformation

\[
\phi = \psi - \frac{mC}{l^2}
\]  

(81)

Then Eq. (80) becomes

\[
2 \left[ E - V(r) - \frac{l^2}{2mr^2} \right] d^2 \psi \over dr^2 + \left[ r g(r) + \frac{l^2}{m^2 r^3} \right] d\psi \over dr - g(r) \psi = 0
\]  

(82)

As an example consider \( V(r) = k/r \). Then the solution of (82) is simply

\[
\psi(r) = c_1 \left( kmr + l^2 \right) + c_2 \sqrt{l^2 + 2kmr - 2mEr^2}
\]  

(83)
7 Conclusions

In this paper we have outlined a simple and effective method for treating problems related with time-dependent and time independent central force fields. In particular we have dealt with the Kepler problem and the isotropic harmonic oscillator fields. We have been able to rederive some known results from an original point of view and generalize others. The central force field has been discussed in the literature from many points of view. The difficulty in finding vector constants of motion for central fields stem from the fact that in general orbits for these type of problem are not closed, therefore any new ways to attack those problems are welcome. In our method this difficulty is transferred, so to speak, to the obtention for each possible central field, which can be time-dependent or not, of a certain scalar function of the position, linear momentum and time. For a given central field this scalar function is a solution of (6). In the general case, the obtention of the scalar function is a difficult task. Judicious guesses, however, facilitate the search for solution of (6) and this is what we have done here.

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