IRREDUCIBILITY OF THE SET OF CUBIC POLYNOMIALS WITH ONE PERIODIC CRITICAL POINT

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Abstract. The space of monic centered cubic polynomials with marked critical points is isomorphic to \( \mathbb{C}^2 \). For each \( n \geq 1 \), the locus \( \mathcal{S}_n \) formed by all polynomials with a specified critical point periodic of exact period \( n \) forms an affine algebraic set. We prove that \( \mathcal{S}_n \) is irreducible, thus giving an affirmative answer to a question posed by Milnor.

1. Introduction

In their celebrated articles [BH88, BH92], Branner and Hubbard layed the foundations for the study of complex cubic polynomials as dynamical systems. These articles shed light on the structural importance of certain dynamically defined complex one dimensional slices of cubic parameter space [Bra93]. The systematic study of such slices was initiated by Milnor in a 1991 preprint later published as [Mil09].

Milnor worked in the parameter space of monic centered cubic polynomials with marked critical points. This parameter space is isomorphic to \( \mathbb{C}^2 \). For each \( n \geq 1 \) he considered the affine algebraic set \( \mathcal{S}_n \) formed by all such polynomials with a specified critical point periodic of exact period \( n \). Milnor established that \( \mathcal{S}_n \) is smooth. Therefore, connectedness and irreducibility of \( \mathcal{S}_n \) are equivalent properties. Computing explicit parametrizations he proved that \( \mathcal{S}_1, \mathcal{S}_2 \) and \( \mathcal{S}_3 \) are connected and asked in general: Is \( \mathcal{S}_n \) connected? ([Mil09, Question 5.3]).

Our aim here is to answer this question in the positive:

**Theorem 1.** For all \( n \geq 1 \), the affine algebraic set \( \mathcal{S}_n \) is connected and irreducible.

Our result is similar to one known for quadratic dynatomic curves:

\[ \{(z,c) \in \mathbb{C}^2 \mid z \text{ has exact period } n \text{ for } z \mapsto z^2 + c \}. \]

Dynatomic curves were shown to be smooth by Douady and Hubbard [DH85] and irreducible by Bousch [Bou92]. Nowadays several proofs for the irreducibility of these curves are available. The proofs range from those relying more on algebraic methods by Morton [Mor96] to those more strongly based

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on dynamical techniques by Schleicher-Lau [Sch94] and Buff-Tan [BL14].
However, these techniques have failed to adapt for the study of \( S_n \).

In analogy with complex cubic polynomials, the dynamics of quadratic rational maps also strongly depends on the behavior of two free critical points. Therefore, both cubic and quadratic rational parameter spaces are two complex dimensional. The one dimensional algebraic set analogous to \( S_n \) is often denoted by \( V_n \) and has been under intensive study during the last 25 years as well (e.g. see [Ree92, Ree95, Rec03, AY09, Tim10]). Nevertheless, the irreducibility of \( V_n \) is still to be determined. One of the key ingredients in the proof of our main result is the Polynomial Type Theorem [Ree03, §5.9] by Rees. The quadratic rational version of this result is the Topographer’s View Theorem [Ree03, §5.10] which implies that a study for \( V_n \) similar to the one presented here for \( S_n \) would be more complicated. (See also the recent works [FKS16, HK16]).

There are still plenty of open questions about the global topology of \( S_n \). As anticipated by Milnor [Mil09, Section 5D] once we know that \( S_n \) is connected it becomes meaningful to determine its genus. A formula for the Euler characteristic of \( S_n \) was obtained in [BKM10, Theorem 7.2]. However the formula for the number of punctures of \( S_n \) remains unknown. Nevertheless, De Marco and Schiff [DS10] gave an algorithm to compute this number based on the work by De Marco and Pilgrim [DP11].

There is a rich interplay between the global topology of \( S_n \) and how polynomials are organized within \( S_n \) according to dynamics. Since the curves \( S_n \) are complex one dimensional parameter spaces, they have been natural grounds to employ and further develop the wealth of ideas available for quadratic polynomials (cf. [Fau92, Roe07, Mil09]). However, a relevant novelty here is that \( S_n \) has non-trivial global topology for \( n \geq 2 \). The usual dichotomy between connected and disconnected Julia sets takes place on \( S_n \) as well. Based on this dichotomy, conjectural cell subdivisions for \( S_n \) were proposed in [Mil09] and in [BKM10] where a concrete model for \( S_4 \) is given together with computer generated evidence supporting its validity. Recently, in [Arf15] a proof of part of this model was announced and used to deduce that \( S_4 \) is connected.

The importance of \( S_n \) is also ratified in [DF08] where it is shown that these curves distribute nicely towards the bifurcation current introduced by De Marco [DeM01]. Moreover, recently, also using pluripotential theoretic techniques, Ghioca-Ye [GY16] and Favre-Gauthier [FG16] have shown that the curves \( S_n \) form part of the short list of proper Zariski closed subsets of parameter space containing infinitely many postcritically finite polynomials.

**Overview of the proof.**

Rather than working in the space of monic centered cubic polynomials as in [Mil09], for us it will be convenient to work in the moduli space of critically marked cubic polynomials Poly\(_3\). This moduli space is also an affine algebraic surface (cf. [Sil98]). Our focus will be on the affine algebraic
set $\mathcal{S}_n$ formed by all the elements in $\text{Poly}_3$ for which an specified critical point is periodic of exact period $n \geq 2$. As we will discuss in Section 2.3, $\mathcal{S}_n$ is a double cover of $\mathcal{G}_n$. Moreover, from [Mil09] it immediately follows that $\mathcal{S}_n$ is connected if and only if $\mathcal{G}_n$ is connected (see Proposition 2.1).

The escape locus $\mathcal{E}(\mathcal{G}_n)$ is the subset of $\mathcal{G}_n$ formed by the polynomials which have disconnected Julia sets or equivalently which have a critical point escaping to infinity. Since $\mathcal{E}(\mathcal{G}_n)$ is a neighborhood of infinity in $\mathcal{G}_n$ and irreducible components of affine algebraic sets are not compact, it follows that every irreducible component of $\mathcal{G}_n$ contains a connected component of $\mathcal{E}(\mathcal{G}_n)$. The connected components of $\mathcal{E}(\mathcal{G}_n)$ are called escape regions. It follows that $\mathcal{G}_n$ is connected if and only if all the escape regions lie in the same connected component of $\mathcal{G}_n$. Thus, in order to show that $\mathcal{G}_n$ is connected we will show that within $\mathcal{G}_n$ there exist paths joining all of the escape regions.

Our proof stands on the results of Rees (cf. Corollary 2.3) which allows us to work in a moduli space $\mathcal{B}_n$ of topological polynomials having a period $n$ branched point (cf. Section 2.1). Although $\mathcal{B}_n$ is significantly larger than $\mathcal{G}_n$, her result asserts that escape regions joined through paths within $\mathcal{B}_n$ can also be joined within $\mathcal{G}_n$.

Our route in the collection of escape regions is guided by kneading sequences. Associated to a polynomial in the escape locus, the kneading sequence is a periodic sequence of symbols “0” and “1” that encodes the itinerary of the periodic critical value with respect to a suitable partition. This sequence is constant within each escape region. Although there are several escape regions sharing the same kneading sequence, the itinerary $(1^{n-1}0)^\infty$ is realized by a unique one which we call the distinguished escape region.

In $\mathcal{B}_n$ the notion of kneading sequence extends to a subspace $\mathcal{E}(\mathcal{F}_n)$ containing $\mathcal{E}(\mathcal{G}_n)$. This set $\mathcal{E}(\mathcal{F}_n)$ is formed by the conjugacy classes of topological polynomials which are conformal near infinity and have one escaping critical point.

We travel towards the distinguished region from another one via the concatenation of two paths in $\mathcal{B}_n$: the twisting path and the geometrization path. The twisting path ends at a map in $\mathcal{E}(\mathcal{F}_n)$ with kneading sequence “closer” to $(1^{n-1}0)^\infty$. The geometrization path ends at a polynomial in $\mathcal{E}(\mathcal{G}_n)$ which is provided by a Theorem of Cui and Tan. Along the geometrization path, which is fully contained in $\mathcal{E}(\mathcal{F}_n)$, the kneading sequence remains unchanged. We complete the proof by successively moving in the aforementioned manner to reach our distinguished final destination.

Organization.

We start by introducing the relevant spaces of dynamical systems in Section 2 as well as the homotopy equivalence given by Rees’ Polynomial Type Theorem. Then we discuss kneading sequences in Section 3. The twisting
path and its effect on kneading is the content of Section 4 while the geometrization path is the content of Section 5. We assemble the proof of Theorem 1 in Section 6.

2. Spaces

As discussed in the introduction, the interplay among several spaces of dynamical systems is essential for the proof of our main result. We are mainly concerned with spaces formed by conjugacy classes of topological polynomials $B_n$, of semi-rational maps $F_n$ and of cubic polynomials $\text{Poly}_3$. The aim of this section is to introduce these spaces as well as relevant subspaces of them. The section ends with the statement of Rees’ Polynomial Type Theorem.

For the rest of the paper we let $n$ be an integer such that $n \geq 2$.

2.1. Topological Polynomials. Let $B_n$ be the space formed by triples $(f,c,c')$ where

$$f : \mathbb{C} \to \mathbb{C}$$

is a degree 3 topological branched covering with branched points $\infty, c, c'$ such that all of the following statements hold:

1. $\infty$ is a fixed point of $f$ and $f$ is locally 3-to-1 around $\infty$.
2. $c$ has period exactly $n$ under $f$.
3. The branched value $f(c')$ is disjoint from the forward orbit of $c$.

Let $B_n$ be the quotient of $B_n$ under the equivalence relation that identifies $(f,c,c')$ and $(g,c,g,c')$ if there exists an affine conjugacy $A$ between $f$ and $g$ (i.e. $A^{-1} \circ f \circ A = g$) such that $A(c) = c_f$ and $A(c') = c'_f$. We endow $B_n$ with the topology of uniform convergence and $B_n$ with the corresponding quotient topology.

2.2. Semi-rational maps. Let us recall the notion of semi-rational map introduced in [CT11]. Consider a topological branched covering $f : \mathbb{C} \to \mathbb{C}$ and denote by $P_f$ the closure of

$$\{ f^m(c) : m > 0, c \text{ branched point of } f \}.$$

We say that $f : \mathbb{C} \to \mathbb{C}$ is a semi-rational map if the following statements hold:

1. The set of accumulation points $P_f'$ of $P_f$ is finite or empty.
2. If $P_f' \neq \emptyset$, then $f$ is holomorphic in a neighborhood of $P_f'$.
3. Every periodic orbit in $P_f'$ is either attracting or super-attracting.

We denote by $\mathcal{F}_n$ the subset of $B_n$ formed by conjugacy classes having semi-rational representatives. Endow $\mathcal{F}_n$ with the subspace topology. We are interested in the set $\mathcal{E}(\mathcal{F}_n)$ of elements $[(f,c,c')] \in \mathcal{F}_n$ such that $P_f' = \{ \infty \}$. That is, $[(f,c,c')] \in \mathcal{E}(\mathcal{F}_n)$ if and only if the orbit of the critical point $c'$ tends to $\infty$. It follows that elements in $[(f,c,c')] \in \mathcal{E}(\mathcal{F}_n)$ have a unique periodic branched point, so often we will simply write $[f] \in \mathcal{E}(\mathcal{F}_n)$. 

2.3. Critically marked cubic polynomials. A critically marked cubic polynomial is a triple \((f, c, c')\) where \(f\) is a cubic polynomial with critical points \(c, c' \in \mathbb{C}\), and \(c = c'\) only when \(f\) has a double critical point in \(\mathbb{C}\). Following Milnor [Mil09], the space of monic centered critically marked cubic polynomials \(\text{Poly}_3^{cm}\) is identified with \(\mathbb{C}^2\) via the family \((P_{a,v}, +a, -a)\) where

\[
P_{a,v}(z) = z^3 - 3a^2 z + 2a^3 + v, \quad (a, v) \in \mathbb{C}^2.
\]

He defines \(S_n\) to be the algebraic subset of \((a, v) \in \mathbb{C}^2\) for which \(+a\) has exact period \(n\) under the iterations of \(P_{a,v}\). According to [Mil09, Theorem 5.2], \(S_n\) is a smooth affine algebraic set.

We say that two critically marked cubic polynomials \((f, c_f, c'_f)\) and \((g, c_g, c'_g)\) are affinely conjugated if there exists an affine conjugacy \(A\) between \(f\) and \(g\) (i.e. \(A^{-1} \circ f \circ A = g\)) such that \(A(c_g) = c_f\) and \(A(c'_g) = c'_f\). The moduli space of critically marked cubic polynomials \(\text{Poly}_3\) is the space of affine conjugacy classes of critically marked cubic polynomial.

The critically marked polynomials \((P_{a,v}, +a, -a)\) and \((P_{a',v'}, +a', -a')\) are affinely conjugated if and only if \((a', v') = (-a, -v)\). Thus, \(\text{Poly}_3\) is the algebraic surface identified with the quotient of \(\mathbb{C}^2\) by the action of the involution \(I : (a, v) \mapsto (-a, -v)\). That is,

\[
\text{Poly}_3 \equiv \mathbb{C}^2/I.
\]

We consider the algebraic set \(\mathcal{S}_n\) formed by all \([(f, c, c')] \in \text{Poly}_3\) such that \(c\) has exact period \(n\) under iterations of \(f\). It follows that

\[
\mathcal{S}_n \equiv \mathcal{S}_n/I.
\]

Since the unique fixed point of \(I\) corresponds to \(z \mapsto z^3\) which lies in \(S_1\) and we are working with an integer \(n \geq 2\), the map \(S_n \to \mathcal{S}_n\) is a regular double cover. Therefore, \(\mathcal{S}_n\) is also smooth.

Recall that the escape locus \(E(\mathcal{S}_n)\) consists of all \([(f, c, c')] \in \mathcal{S}_n\) such that \(c'\) lies in the basin of infinity. Connected components of \(E(\mathcal{S}_n)\) are called escape regions. According to [BKM10, Lemma 5.16], there exist escape regions \(\mathcal{U}\) such that its preimage \(\mathcal{U}'\) under the double cover \(S_n \to \mathcal{S}_n\) is connected (i.e. \(\mathcal{U}'\) is an escape region of \(S_n\) fixed under the involution \(I\)). Therefore, \(\mathcal{S}_n\) is connected if and only if \(\mathcal{S}_n\) is. Due to the smoothness of these algebraic sets we also have:

**Proposition 2.1.** The algebraic set \(\mathcal{S}_n\) is irreducible if and only if \(\mathcal{S}_n\) is irreducible.

The topology induced by the uniform convergence of maps \(f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}\) coincides with the one of \(\text{Poly}_3\) as a complex orbifold, thus we have the following inclusions of topological spaces:

\[
E(\mathcal{S}_n) \subset E(\mathcal{F}_n) \subset \mathcal{B}_n.
\]
2.4. Homotopy equivalence. With this notation the Polynomial Type Theorem of Rees [Ree03, § 5.9] says the following.

Theorem 2.2. Let $B'_n$ be a connected component of $B_n$ containing an element of $S_n$. Then

$$B'_n \cap S_n \hookrightarrow B'_n$$

is a homotopy equivalence.

Corollary 2.3. If there exists a path $((f_t, c_t, c'_t))_{t \in [0,1]}$ in $B_n$ with $[(f_0, c_0, c'_0)]$ and $[(f_1, c_1, c'_1)]$ in $S_n$, then $[(f_0, c_0, c'_0)]$ and $[(f_1, c_1, c'_1)]$ are the endpoints of a path in $S_n$.

3. Kneading sequence

Here we generalize the notion of kneading sequences originally introduced in [Mil09, Definition 5.4] for elements of $E(S_n)$ to elements of $E(F_n)$. Then we show that such a sequence is constant on connected components of $E(F_n)$.

Given $[(f, c, c') \in E(F_n)]$ we consider a small neighborhood $U$ of $\infty$ such that $f|_{U}$ is holomorphic. According to the general theory of holomorphic functions with a super-attractive fixed point we may choose $U$ so that $g_f(z) := \lim_{m \to \infty} \frac{1}{3^m} \log |f^m(z)|,$ is a well defined continuous function $g_f : U \to [0, +\infty]$ (e.g. see [Mil06, §9]). The level curves $g_f = r$ for $r$ sufficiently large are Jordan curves in $U$ surrounding $\infty$. Moreover, $g_f$ satisfies the functional relation

$$g_f \circ f = 3 \cdot g_f.$$

Via this relation, after declaring $g_f(z) = 0$ for all $z$ in the complement of the basin of infinity, we obtain a continuous extension $g_f : \overline{U} \to [0, +\infty]$ where $[0, +\infty]$ is endowed with the order topology.

The following lemma shows that some basic features of cubic polynomials with one escaping critical point are also present in the dynamical space of the semi-rational map $f$ (see Figure 1).

Lemma 3.1. Let $[(f, c, c') \in E(F_n)]$. Then

$$D := \{ z \in \mathbb{C} : g_f(z) < g_f(f(c')) \}$$

is a Jordan domain and

$$f^{-1}(D) = \{ z \in \mathbb{C} : g_f(z) < g_f(c') \}$$

is the disjoint union of two Jordan domains $D_0^f$ and $D_1^f$ such that $f : D_0^f \to D$ is a degree 2 covering branched at $c$ and $f : D_1^f \to D$ is a homeomorphism.

Before proving this lemma let us define the kneading word $\kappa^f \in \{0, 1\}^n$ by

$$\kappa^f := \kappa_1 \ldots \kappa_n \quad \text{where} \quad \kappa_j = i \iff f^j(c) \in D_i^f.$$
The kneading word does not depend on the representative of \([\{f, c, c'\}]\) and on the choice of \(U\). Always \(\kappa_n = 0\) since the period of \(c\) is \(n\). The kneading sequence of \(f\) is the element of \([0, 1]^\infty\) obtained as the infinite repetition of the kneading word. When \(f\) is a polynomial this agrees with the definition introduced by Milnor.

The non-critical preimage of the critical value \(f(c)\) (resp. \(f(c')\)) is called the co-critical point of \(c\) (resp. \(c'\)). Note that the co-critical point of \(c\) always lies in \(D_1\).

**Proof of Lemma 3.1.** The Böttcher map furnishes a local conjugacy \(\phi\) between \(f\) and \(z \mapsto z^3\) near infinity (e.g. see [Mil06, §9]). That is, there exist a punctured neighborhood \(V\) of \(\infty\) and a conformal isomorphism \(\phi : V \to \{z \in \mathbb{C} : |z| > r_0\}\) for some \(r_0 > 1\) such that \(\phi \circ f(z) = \phi(z)^3\) for all \(z \in V\). It follows that

\[
g_f(z) = \log |\phi(z)|.
\]

Therefore \(\{z : g_f(z) = r\}\) is a Jordan curve around \(\infty\) for all \(r > r_0\).

Given \(r > 0\), let \(V_r = \{z : r < g_f(z) < +\infty\}\) and observe that \(f : V_r \to V_{3r}\) is a branched covering of degree 3. For all \(r > 0\), it follows that \(V_r\) is connected and \(\mathbb{C} \setminus V_r\) is a disjoint union of Jordan domains.

Now if \(r > g_f(c')\), then the map \(f : V_r \to V_{3r}\) is branched point free and thus it is a topological covering of degree 3. Hence, we deduce that \(V_r\) is a disk punctured at infinity and bounded by a Jordan curve. Therefore \(D\) is a Jordan domain. By the Riemann-Hurwitz formula, the Euler characteristic of \(f^{-1}(D)\) is 2, since \(f : f^{-1}(D) \to D\) has exactly one branched point \(c\). Hence \(f^{-1}(D)\) is the union of two disjoint disks \(D_0\) and \(D_1\), each one mapping onto \(D\).

\[ \square \]

**Proposition 3.2.** The kneading sequence is constant on each connected component of \(\mathcal{E}(F_n)\).

In order to prove the proposition we first establish the continuous dependence of \(g_f\) on \(f\):
Lemma 3.3. Let $F_n$ be the subspace of $B_n$ formed by all $(f, c, c') \in B_n$ such that $f$ is holomorphic near infinity. Given $[(f_0, c_0, c'_0)] \in \mathcal{E}(F_n)$, there exists a neighborhood $V$ of $(f_0, c_0, c'_0)$ in $F_n$ such that if $(f, c, c') \in V$ then $[(f, c, c')] \in \mathcal{E}(F_n)$ and $g_f : \mathbb{C} \to [0, +\infty]$ depends continuously on $(f, c, c') \in F_n$.

Proof. It is easy to check that the conjugacy class $[(f, c, c')]$ lies in $\mathcal{E}(F_n)$ for all $(f, c, c')$ in an open neighborhood $V \subset F_n$ of $(f_0, c_0, c'_0)$. Thus, $g_f : \mathbb{C} \to [0, +\infty]$ is well defined and depends continuously on $(f, c, c') \in V$. Then, the functional relation

$$g_f(z) = \frac{g_f(f(z))}{3}$$

spreads the continuous dependence of $g_f : U \to [0, +\infty]$ to the basin of infinity under iterations of $f_0$.

Now the continuous dependence extends to $\mathbb{C}$ as follows. Given $\varepsilon > 0$ small and a compact set $X$ such that $g_{f_0}(X) \subset (0, \varepsilon]$ it is sufficient to show that for $f$ close to $f_0$, also $g_f(X) \subset (0, \varepsilon]$. In fact, since $X$ is compact, we may choose $0 < \lambda < 1$ such that $g_{f_0}(X) \subset (0, \varepsilon \lambda]$. Now let $N$ be large enough and $f$ be sufficiently close to $f_0$ such that $g_{f_0}(f^N_0(z)) < 3^N \varepsilon \lambda$ implies that $g_f(f^N(z)) < 3^N \varepsilon \lambda'$ for some $\lambda < \lambda' < 1$ and such that this latter inequality implies that $g_f(f^N(z)) < 3^N \varepsilon$, equivalently $g_f(z) < \varepsilon$. Hence, $g_f(X) \subset (0, \varepsilon]$ as desired.

Proof of Proposition 3.2. As the kneading word lies in a discrete set, it is sufficient to prove that it is locally constant in $\mathcal{E}(F_n)$. Consider $[(f_0, c_0, c'_0)] \in \mathcal{E}(F_n)$ and $1 \leq m \leq n - 1$. Assume that $f^m_0(c_0) \in D^0_i$ for $i = 0$ (resp. $i = 1$). Denote by $c_{0i}$ the co-critical point of $c_0$ and recall that $c_0 \in D^0_i$. Let $X \subset D^0_i$ be a compact connected set containing $f^m_0(c_0)$ and $c_0$ (resp. $c_{0i}$) in its interior. For some $\varepsilon > 0$ small, $g_{f_0}(z) < g_{f_0}(c'_0) - \varepsilon$, for all $z \in X$. Hence, there exists a neighborhood $W$ of $(f_0, c_0, c'_0)$ in $F_n$ such that if $(f_1, c_1, c'_1) \in W$, then $f^m_1(c_1) \in X$, $c_1 \in X$ (resp. $c_{0i} \in X$, where $c_{0i}$ is the co-critical point of $c_1$), $g_{f_1}(c'_1) > g_{f_0}(c'_0) - \varepsilon/2$ and $g_{f_0}(c'_0) - \varepsilon/2 > g_{f_1}(z)$ for all $z \in X$. Therefore, $X$ is contained in $D^1_i$ and $\kappa^m_{f_0} = \kappa^m_{f_1}$.

4. Twisting path

This section is devoted to the construction of a “twisting” path with the desired effect on the kneading word:

**Proposition 4.1.** Let $m$ be such that $1 \leq m \leq n - 1$. Given $[f_0] \in \mathcal{E}(\mathcal{G}_n)$ with kneading word $\kappa([f_0]) = \kappa_1 \ldots \kappa_{n-1} 0$, there exists $[f_1] \in \mathcal{E}(\mathcal{F}_n)$ with kneading word $\kappa([f_1]) = \kappa'_1 \ldots \kappa'_{n-1} 0$ such that the following hold:
does not affect the results in Proposition 4.1. Following statements hold: Construction of a suitable loop

Step 1: Construction of a twisting loop

\[ \kappa_j = \begin{cases} 
\kappa_j & \text{if } j \neq m, \\
1 - \kappa_m & \text{if } j = m.
\end{cases} \]

The rest of this section is devoted to the proof of the proposition which consists first on constructing a path in \( B_n \) (the twisting path) and then proving that the endpoint of this path has the desired kneading word.

4.1. Construction of the twisting path. Consider \( f_0 \) and \( m \) as in the statement of the proposition. First we will construct an appropriate loop in the dynamical space of \( f_0 \) and then we will thicken the loop to an annulus where we will postcompose \( f_0 \) by twists to obtain a twisting path.

Consider \( ([f_0, c_0, c'_0]) \in B_n \) such that \([f_0] \in \mathcal{E}(\mathfrak{S}_n)\). Let \( \mathcal{O}(c_0) \) denote the critical periodic orbit. Recall that \( D = \{ z \in \mathbb{C} : g_{f_0}(z) < g_{f_0}(c'_0) \} \) and let \( D_0, D_1 \) be the connected components of \( f_0^{-1}(D) \). We follow the notation of Lemma 3.1. That is, \( D_0 \) contains the critical point \( c_0 \) and \( D_1 \) contains the co-critical point \( c_0 \).

Step 1: Construction of a twisting loop. This step consists of the construction of a suitable loop \( \tau \) surrounding \( f_0^{m+1}(c_0) \) in \( D \). More precisely a twisting loop \( \tau : [0,1] \to \overline{D} \) is a continuous function such that all of the following statements hold:

(L1) \( \tau \) is a loop with endpoints at the critical value \( f_0(c'_0) \) (i.e. \( \tau(0) = \tau(1) = f_0(c'_0) \)).

(L2) \( \tau([0,1]) \subset D \setminus \mathcal{O}(c_0) \).

(L3) \( \tau \) bounds a Jordan domain \( V \subset D \) such that \( V \cap \mathcal{O}(c_0) = \{ f_0^{m+1}(c_0) \} \).

(L4) The connected component of \( c'_0 \) in \( f_0^{-1}(\tau) \) bounds two Jordan domains \( V'_0 \subset D_0 \) and \( V'_1 \subset D_1 \) with \( f_0^m(c_0) \in V'_0 \cup V'_1 \).

Such a path \( \tau \) exists. Indeed, let \( Y : [0,1] \to \overline{D} \) be an arc such that \( Y([0,1]) \subset D \setminus \mathcal{O}(c_0) \) with initial point \( f_0(c'_0) \) and endpoint \( f_0(c_0) \). Abusing of notation, let \( Y = Y([0,1]) \). The set \( f_0^{-1}(Y) \) separates \( f_0^{-1}(D) \) into 3 connected components (cf. Figure 2), each mapping bijectively onto \( D \setminus Y \). Let \( U \) denote the one containing \( f_0^m(c_0) \). Let \( \tau' \) be a loop with endpoints at \( c'_0 \) such that \( \tau'([0,1]) \subset U \setminus \mathcal{O}(c_0) \) and \( \tau' \) bounds a Jordan domain \( V' \) with \( V' \cap \mathcal{O}(c_0) = \{ f_0^m(c_0) \} \) (cf. Figure 2). The path \( \tau := f_0 \circ \tau' \) and the domain \( V := f_0(V') \) satisfy the desired conditions.\(^1\)

Step 2: Twisting. Thicken \( \tau([0,1]) \) in order to obtain an open annulus \( A \) bounded by Jordan curves such that:

\begin{itemize}
  \item (A1) \( A \subset f_0(D) \).
  \item (A2) \( A \) is disjoint from \( \mathcal{O}(c_0) \).
  \item (A3) \( D \setminus (A \cup V) \) is connected.
\end{itemize}

\(^1\)The choice of different orientations for \( \tau([0,1]) \) may give different twisting paths but does not affect the results in Proposition 4.1.
Figure 2. Illustration of the construction of the twisting loop corresponding to \( m = 3 \) and kneading word 1000. The exterior curve in black is the level curve \( g_{f_0} = g_{f_0}(c'_0) \). The set \( f_0^{-1}(Y) \) is drawn in gray.

Figure 3. Illustration of the annulus \( A \) around the twisting loop \( \tau \) (left) and its preimage (right).

Observe that each component of \( \partial A \) is homotopically equivalent to \( \tau([0,1]) \) rel \( \mathcal{O}(c_0) \). Moreover, \( f_0^{-1}(A) \) has exactly two connected components, one containing \( c'_0 \) and the other containing the corresponding co-critical point \( c_{0}' \) (see Figure 3).

The loop \( \tau \) cuts \( A \) into two sub-annuli \( A_{ext} := A \setminus \overline{V} \) and \( A_{int} := A \cap V \). For \( t \in [0,1] \), let \( T_t : \mathbb{C} \rightarrow \mathbb{C} \) be a continuous family of quasiconformal maps such that all of the following hold:

(T1) \( T_t(\tau(s)) = \tau(s - t \mod 1) \) for all \( s \in [0,1] \).
(T2) \( T_0 = id_A \) and \( T_t \) is the identity on \( \overline{\mathbb{C}} \setminus A \).
(T3) \( T_1 \) is the inverse of a Dehn twist in \( A_{ext} \) and a Dehn twist in \( A_{int} \).

Now we may introduce the twisting path as

\[ (f_t, c_t, c'_t) := (T_t \circ f_0, c_0, c'_0) \]
where $t \in [0, 1]$.

The action of $f_t$ on the periodic critical orbit $O(c_0)$ remains unchanged for all $t$. Thus $(f_t, c_t, c_t') \in B_n$. Also, $f_0(c_0') = f_1(c_1')$ and $f_0(z) = f_t(z)$ for all $z \notin f_0(D)$ and all $t$. Since $f_0^k(c_0') \notin f_0(D)$ for all $k \geq 2$, the postcritical sets of $f_0$ and $f_1$ coincide. Moreover, $f_1^k(c'_1) \to +\infty$ and $f_1$ is a semi-rational map. That is, $[f_1] \in E(F_n)$.

4.2. Kneading after twisting. In order to finish the proof of Proposition 4.1, our aim now is to determine the kneading word of the endpoint $f_1$ of the twisting path constructed above.

By (A1) and (T2), $f_t(z) = f_0(z)$ for all $z \notin D$. In particular, $g_{f_t}(z) = g_{f_0}(z)$ for all $z \notin D$. In particular, $\partial D$ is the $g_{f_t}$-level curve containing the critical value $f_1(c_1') = T_1 \circ f_0(c_0') = f_0(c_0')$. Therefore, in order to determine the kneading word of $f_1$ we have to analyze the location of the periodic critical orbit in $f_1^{-1}(D) = D_0^f \sqcup D_1^f$. (See Figure 4.)

By (A3) the set $E := D \setminus (A \cup V)$ is connected. Moreover, by (T2), the maps $f_0$ and $f_1$ agree on the set $f_1^{-1}(E) = f_0^{-1}(T_1^{-1}(E)) = f_0^{-1}(E)$. Since $f_0(c_0) \in E$, we have that $f_0^{-1}(E)$ consists of two connected components $E_0 \subset D_0^f$ and $E_1 \subset D_1^f$ each mapping onto $E$. Since $c_1 = c_0 \in E_0$ we have that $E_0 \subset D_0^{f_0}$ and $E_1 \subset D_1^{f_0}$. In view of (L3) and (A2), with the exception of $f_0^m(c_0)$ all the periodic critical point orbit elements are in $E_0 \cup E_1$, therefore

$$\kappa'_i = \kappa'_i \quad \text{for all} \quad i \neq m.$$
According to (L4) a connected component of $f_0^{-1}(V)$ with $c'_0$ in its boundary is denoted by $V'_0$ or $V'_1$ according to whether it is contained in $D_i^{0}$ or in $D_i^{1}$. Observe that $V'_0$ and $V'_1$ are still contained in $f_1^{-1}(D)$ since

$$f_1(V'_i) = T_1(f_0(V'_i)) = T_1(V) = V \subset D.$$ 

However, we will show that $V'_{1-i} \subset D_i^{1}$ for $i = 0, 1$. That is, we will show that the elements of $V'_{1}$ “switch” label in $\{0, 1\}$.

For $t \in [0, 1]$, let $\tilde{T}_t : \overline{C} \rightarrow \overline{C}$ be the lift of $T_t$ by $f_0$ which is the identity in $C \setminus D$. That is:

$$T_t \circ f_0 = f_0 \circ \tilde{T}_t,$$

and

$$\tilde{T}_t(z) = z \text{ for all } z \in \overline{C} \setminus D.$$ 

Denote by $A'_{ext}$ the component of $f_0^{-1}(A_{ext})$ surrounding $V'_0$ and $V'_1$. Let $\gamma : [0, 1] \rightarrow A'_{ext}$ be an arc in $D$ starting at the inner boundary of $A_{ext}$ (i.e., $\partial V$) and ending at the outer boundary of $A_{ext}$. Since $f_0 : A'_{ext} \rightarrow A_{ext}$ is a regular covering of degree 2, the preimage of $\gamma$ under $f_0$ consists of two arcs (see Figure 4). We denote by $\gamma_0$ the one starting at $\partial V_0$ and by $\gamma_1$ the one starting at $\partial V_1'$. The lift $\tilde{T}_1 : A'_{ext} \rightarrow A'_{ext}$ is a half twist and therefore

$$\tilde{T}_1^{-1}(\gamma_i(0)) = \gamma_{1-i}(0) \in \partial V'_{1-i}.$$ 

As

$$f_0(\gamma_i) = f_0 \circ \tilde{T}_1(\tilde{T}_1^{-1}(\gamma_i)) = f_1(\tilde{T}_1^{-1}(\gamma_i)),$$

we deduce that

$$\tilde{T}_1^{-1}(\gamma_i) \subset f_1^{-1}(D).$$

That is, $\tilde{T}_1^{-1}(\gamma_i)$ connects $\partial V'_{1-i}$ to $\tilde{T}_1^{-1}(\gamma_i(1)) = \gamma_i(1) \in E_i \subset D_i^{1}$ within $f_1^{-1}(D)$ which implies that $V'_{1-i} \subset D_i^{1}$. From (L4), $f_0^{m}(c_0) \in V'_0 \cup V'_1$ and it follows that

$$\kappa'_m = 1 - \kappa_m.$$ 

Thus we have completed the proof of Proposition 4.1.

5. THE GEOMETRIZATION PATH

The aim of this section is to show that the endpoint of the twisting path can be joined through an appropriate path to a polynomial in $\mathcal{E}(\mathcal{S}_n)$. With the same effort we establish a slightly more general result:

**Theorem 5.1.** Every path connected component of $\mathcal{E}(\mathcal{F}_n)$ contains an element of $\mathcal{E}(\mathcal{S}_n)$.

Given an element of $\mathcal{E}(\mathcal{F}_n)$, a path within $\mathcal{E}(\mathcal{F}_n)$ joining it to an element of $\mathcal{E}(\mathcal{S}_n)$ will be called a geometrization path. The endpoint of such a geometrization path will be obtained from a Theorem by Cui and Tan [CT11]. Cui-Tan’s Theorem and its prerequisites are discussed in Section 5.1. To apply this theorem we rule out the presence of Thurston obstructions in Section 5.2. To obtain the geometrization path itself we show that $c$-equivalence
classes, defined in Section 5.1, are path-connected. In Section 5.3 we combine the above ingredients to prove Theorem 5.1.

5.1. Cui-Tan Theorem. In order to state the aforementioned result by Cui and Tan we need to introduce the notions of $c$-equivalence and Thurston obstructions for semi-rational maps.

Following [CT11], we say that two semi-rational maps $f$ and $g$ are $c$-equivalent if there exist homeomorphisms $\varphi, \psi : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ and a neighborhood $U$ of $P_f$ such that all of the following statements hold:

- $\varphi \circ f = g \circ \psi$.
- $\varphi$ is holomorphic in $U$.
- $\varphi$ is isotopic to $\psi$ relative to $\overline{U} \cup P_f$.

Möbius conjugate semi-rational maps are $c$-equivalent. Hence, the notion of $c$-equivalence is well defined in $E(F_n)$. That is, two elements of $E(F_n)$ are said $c$-equivalent if they can be represented by $c$-equivalent semi-rational maps.

Consider a semi-rational map $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ with postcritical set $P_f$. We say that a Jordan curve $\gamma \subset \overline{\mathbb{C}} \setminus P_f$ is non-peripheral if each one of the two disks in $\overline{\mathbb{C}}$ bounded by $\gamma$ contains at least two elements of $P_f$. A pairwise disjoint and pairwise non-homotopic finite collection $\Gamma = \{\gamma_1, \ldots, \gamma_k\}$ of non-peripheral Jordan curves $\gamma_j \subset \overline{\mathbb{C}} \setminus P_f$ is called a multicurve in $\overline{\mathbb{C}} \setminus P_f$.

Given a multicurve $\Gamma = \{\gamma_1, \ldots, \gamma_k\}$ we say that $\Gamma$ is a $f$-stable multicurve if, for all $j = 1, \ldots, k$, each connected component of $f^{-1}(\gamma_j)$ either fails to be non-peripheral or is homotopic in $\overline{\mathbb{C}} \setminus P_f$ to some $\gamma_i \in \Gamma$.

The transition matrix $W_\Gamma$ associated to a $f$-stable multicurve $\Gamma = \{\gamma_1, \ldots, \gamma_k\}$ is the $k \times k$ non-negative matrix $(a_{ij})$ with rational entries:

$$a_{ij} = \sum_{\alpha} \frac{1}{\deg(\alpha \to \gamma_j)}$$

where the sum is taken over all connected components $\alpha$ of $f^{-1}(\gamma_j)$ such that $\alpha$ is homotopic in $\overline{\mathbb{C}} \setminus P_f$ to $\gamma_i$.

A $f$-stable multicurve is called a Thurston obstruction for $f$ if the leading eigenvalue $\lambda$ of $W_\Gamma$ is such that $\lambda \geq 1$.

**Theorem 5.2** (Cui and Tan). Let $f_0 : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ be a semi-rational map. Then, $f_0$ is $c$-equivalent to a rational map $f_1$ if and only if $f_0$ has no Thurston obstruction. In this case, $f_1$ is unique up to Möbius conjugacy.

5.2. No Thurston obstructions. A straightforward application of the ideas introduced by Levy [Lev86] to show that Thurston obstructions in polynomial dynamics contain “Levy cycles” allows us to establish the following:
Proposition 5.3. If $[(f,c,c')] \in \mathcal{E}(\mathcal{F}_n)$ is a class of semi-rational maps then $f$ has no Thurston obstructions.

Proof. We proceed by contradiction. Suppose that $\Gamma = \{\gamma_1, \ldots, \gamma_k\}$ is a Thurston obstruction for $f$. Let $W_\Gamma = (a_{ij})$ be the corresponding transition matrix.

We may assume that $\Gamma$ is minimal in the sense that if $\Gamma' \subseteq \Gamma$, then $\Gamma'$ is not a Thurston obstruction. Thus, for all $i$ there exists $j$ such that $a_{ij} \neq 0$, for otherwise we could erase $\gamma_i$ from $\Gamma$.

For $1 \leq i \leq k$, let $U_i$ be the bounded component of $\mathbb{C} \setminus \gamma_i$. We prove that

$$f^m(c') \notin U_i$$

for all $m \geq 1$ and for all $i$. For otherwise, there would exist $U_i$ and $m \geq 1$ such that $f^m(c') \in U_i$ and $f^{m+1}(c') \notin U_j$ for all $j = 1, \ldots, k$. Then, for all $j$ and for every connected component $\alpha$ of $f^{-1}(\gamma_j)$, we would have that $\alpha$ is not homotopic in $\mathbb{C} \setminus P_f$ to $\gamma_i$. Thus, we would have that $a_{ij} = 0$ for all $j$, which contradicts the minimality of $\Gamma$.

It follows that $P_f \cap U_i$ is a subset of the periodic orbit of $c$. Let $n_i \geq 2$ be the cardinality of $P_f \cap U_i$. Notice that if $a_{ij} \neq 0$, then $f(P_f \cap U_i) \subset P_f \cap U_j$ and therefore $n_i \leq n_j$.

Let $\lambda > 0$ be the leading eigenvalue of $W_\Gamma$ with eigenvector $(v_1, \ldots, v_k)$. Let $N$ be the maximal cardinality of $U_i \cap P_f$ with $i$ in the support of the eigenvector. That is,

$$N = \max \{n_i : v_i \neq 0, 1 \leq i \leq k\}.$$

Also let $I$ be the set of indices $i$ such that $v_i \neq 0$ and $n_i = N$.

Given $i_0 \in I$, let $i_1$ be such that $a_{i_0,i_1} \neq 0$. Since $n_{i_0} \leq n_{i_1}$, we deduce that $i_1 \in I$ and $f(U_{i_0} \cap P_f) = U_{i_1} \cap P_f$. So such a $i_1$ is uniquely defined, and for $i \neq i_1$ we have $a_{i_0,i} = 0$. Thus $v_{i_0} = \lambda a_{i_0,i_1} v_{i_1}$, and therefore $v_{i_1} \neq 0$. Recursively we obtain distinct indices $i_0, \ldots, i_{p-1}$ in $I$, subscripts mod $p$ for some $p \geq 1$, such that

$$v_{i_m} = \lambda a_{i_m,i_{m+1}} v_{i_{m+1}} \neq 0.$$

It follows that $I = \{i_0, \ldots, i_{p-1}\}$ and $\{U_i \cap P_f\}_{i \in I}$ is a partition of the periodic critical orbit. Therefore, $a_{i_j,i_{j+1}} = 1/2$ for the unique $j$ such that $c \in U_{i_j}$ and, $a_{i_m,i_{m+1}} = 1$ for all $m \neq j$. Thus,

$$\lambda^p v_{i_0} = \frac{1}{2} v_{i_0} \neq 0.$$

Hence, $\lambda = 2^{-1/p} < 1$ and $\Gamma$ is not a Thurston obstruction.

5.3. Path connectedness of $c$-equivalence classes and Proof of Theorem 5.1. The absence of Thurston obstructions provided by Proposition 5.3 permits the application of Theorem 5.2:

Corollary 5.4. Every $c$-equivalence class in $\mathcal{E}(\mathcal{F}_n)$ contains a polynomial.
Therefore, to show that every path connected component of $\mathcal{E}(\mathcal{F}_n)$ contains a polynomial, that is to prove Theorem 5.1, it suffices to establish the following:

**Proposition 5.5.** The $c$-equivalence classes of elements in $\mathcal{E}(\mathcal{F}_n)$ are path connected.

**Proof.** Assume that $f_0$ and $f_1$ are $c$-equivalent semi-rational maps such that $[f_0]$ and $[f_1]$ lie in $\mathcal{E}(\mathcal{F}_n)$. Let $\phi$ and $\psi$ be homeomorphisms, holomorphic in a neighborhood $U$ of $\infty$, isotopic rel $\overline{U} \cup P_{f_0}$ such that

$$\psi \circ f_0 = f_1 \circ \phi.$$ 

An isotopy $\varphi_t$ rel $P_f \cup \overline{U}$ between $id_{\mathbb{C}}$ and $\phi^{-1} \circ \psi$ provides us a path $\varphi_t \circ f_0$ of $c$-equivalent semi-rational maps starting at $f_0$ and ending at $\phi^{-1} \circ f_1 \circ \phi$.

According to Lemma 5.6 below we may choose an isotopy $\phi_t$ between $\phi$ and $id_{\mathbb{C}}$ such that $\phi_t(\infty) = \infty$ and $\phi_t$ is holomorphic in a neighborhood of $\infty$, for all $t$. Therefore $\phi_t^{-1} \circ f_1 \circ \phi_t$ is a path of $c$-equivalent maps from $\phi^{-1} \circ f_1 \circ \phi$ to $f_1$.

Passing to affine conjugacy classes, the above paths determine a path joining $[f_0]$ to $[f_1]$ within a $c$-equivalence class. □

**Lemma 5.6.** The set formed by the homeomorphisms $\varphi : (\mathbb{C},0) \to (\mathbb{C},0)$ holomorphic in a neighborhood of 0, endowed with the uniform convergence topology, is path connected.

**Proof.** Consider such a homeomorphism $\varphi$. Without loss of generality, we may assume that $\varphi'(0) = 1$.

For $t \in [0,1]$ and $r \geq 0$ let

$$\rho_t(r) = \begin{cases} tr & \text{if } 0 \leq r < 1, \\ (2-t)r - 2 + 2t & \text{if } 1 \leq r \leq 2, \\ r & \text{if } 2 < r, \end{cases}$$

be a linear interpolation between $r \mapsto tr$ and the identity. Let $h_t : (\mathbb{C},0) \to (\mathbb{C},0)$ be given by

$$h_t(re^{i\theta}) = \rho_t(r)e^{i\theta}.$$ 

For $t \in [0,1]$, it follows that $\varphi_t = h_t^{-1} \circ \varphi \circ h_t : (\mathbb{C},0) \to (\mathbb{C},0)$ is a continuous family of homeomorphisms holomorphic in a neighborhood of 0. Moreover, $\varphi_t$ converges uniformly, as $t \searrow 0$, to a homeomorphism $\varphi_0 : (\mathbb{C},0) \to (\mathbb{C},0)$ which is the identity on the unit disk. Now $\varphi_0$ is isotopic to the identity rel the closed unit disk by Alexander’s trick (cf. [Ale23]). □

6. **Proof of Theorem 1**

Consider a class of polynomials $[f]$ in an escape component $\mathcal{U} \subset \mathcal{S}_n$ whose kneading word $\kappa_f = \kappa_1 \ldots \kappa_{n-1}0$ is such that $\kappa_m = 0$ for some $m \in \{1, \ldots, n - 1\}$. In view of Proposition 4.1, via twisting, there exists a path in $\mathcal{B}_n$ joining $[f]$ and some $[f_1] \in \mathcal{E}(\mathcal{F}_n)$ with kneading word obtained from
by replacing $\kappa_m = 0$ with 1. Now according to Theorem 5.1, there exists a geometrization path in $\mathcal{E}(\mathcal{F}_n) \subset \mathcal{B}_n$ from $[f_1]$ to some $[f_2] \in \mathcal{E}(\mathcal{S}_n)$. By Proposition 3.2, the kneading word of $f_2$ agrees with the one of $[f_1]$.

As we have a path in $\mathcal{B}_n$ between $[f]$ and $[f_2]$, we deduce from Corollary 2.3 that the escape components containing $[f]$ and $[f_2]$ are in the same connected component of $\mathcal{S}_n$.

By iterating this process a finite number of times we conclude that $\mathcal{U}$ is in the same connected component of $\mathcal{S}_n$ as the unique distinguished escape component with kneading word $1^{n-1}0$ (cf. [Mil09, Lemma 5.17] for the uniqueness). Hence all the escape components are in the same connected component of $\mathcal{S}_n$ and the theorem follows from Proposition 2.1.

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