Hardy’s inequality in the scope of Dirichlet forms

Nedra Belhadjrhouma∗ & Ali BenAmor†‡

Abstract

We revisit Hardy’s inequality in the scope of regular Dirichlet forms following an analytical method. We shall give an alternative necessary and sufficient condition for the occurrence of Hardy’s inequality. A special emphasis will be given for the case where the Dirichlet form under consideration is strongly local, extending therefore some known results in the Euclidean case.

Key words: Hardy’s inequality, Dirichlet form, energy measure.

1 Introduction

By Hardy’s inequality we mean an inequality of the type

\[ \int f^2 \, d\mu \leq C \mathcal{E}[f], \quad \forall f \in D(\mathcal{E}), \]  (1.1)

where \( \mathcal{E} \) is a Dirichlet form and \( \mu \) a positive measure charging no set having zero capacity.

Discussions of such type of inequalities and its consequences in the scope of Dirichlet forms was made by several authors and the subject has gained much more interest in the last years [Kai92, Von96, Fit00, FU03, BA04, Rv06], due to their relevance to many areas of mathematics (spectral theory, PDE’s, potential theory,...etc).

In the literature there are many type of necessary and sufficient conditions for the validity of inequality (1.1) (especially for the gradient energy form on Euclidean domains [Ada73, Anc86, Maz85, AH96, Tid05]): capacitary conditions, functional conditions...etc.

In [Anc86], Ancona proved that Hardy’s inequality holds true on Euclidean domains \( \Omega \) for the measure \( (\text{dist}(x, \partial \Omega))^{-2} \, dx \) and where the energy is the gradient energy form if and only if \( \Omega \) possesses a ‘strong barrier’. Years after Fitzsimmons [Fit00], proved that this deep result holds true in a very large generality, namely for quasi-regula Dirichlet form.

Being inspired by the papers of Ancona [Anc86] and Fitzsimmons [Fit00] we shall give new necessary and sufficient condition ensuring the validity of Hardy’s inequality. In fact, using Beurling-Deny formula, we shall write Ancona’s condition in a variational form, without assuming the barrier to be superharmonic.

∗Faculté des Sciences de Tunis, Department of Mathematics. Tunisia
†corresponding author
‡Department of Mathematics, Uni.Gafsa, Tunisia. E-mail: ali.benamor@ipeit.rnu.tn
We shall also show that our condition is equivalent to the one due Fitzsimmons. In the special (but relevant) case where the Dirichlet form is strongly local (of diffusion type), using the intrinsic metric induced by $\mathcal{E}$, we shall generalize and improve the known Hardy [Anc86] and improved Hardy inequality on bounded Euclidean domains [FLA07] in our general setting.

Our method is rather direct and analytic. It is based upon the use of the celebrated Beurling-Deny formula.

2 Preliminaries

We first shortly describe the framework in which we shall state our results.

Let $\mathcal{E}$ be a regular symmetric transient Dirichlet form, with domain $\mathcal{F} := D(\mathcal{E})$ w.r.t. the space $L^2 := L^2(X, m)$. We assume that $X$ is a separable metric space and that $m$ is a reference measure.

In this stage we would like to emphasize that our assumptions on the Dirichlet form are not very restrictive. Indeed, every quasi-regular Dirichlet form is quasi-homeomorphic to a regular Dirichlet form [CMR94]. So that our results are true for quasi-regular Dirichlet forms as well.

The local Dirichlet space related to $\mathcal{E}$ will be denoted by $\mathcal{F}_{\text{loc}}$. A function $f$ belongs to $\mathcal{F}_{\text{loc}}$ if for every relatively compact subset $\Omega \subset X$ there is $\tilde{f} \in D(\mathcal{E})$ such that $f = \tilde{f}$-a.e. on $\Omega$.

We recall the known fact that every element from $\mathcal{F}_{\text{loc}}$ has a quasi-continuous modification. We shall always implicitly assume that elements from $\mathcal{F}_{\text{loc}}$ has been modified so as to become quasi-continuous.

We also designate by $\mathcal{F}_b := \mathcal{F} \cap L^\infty(X, m)$ and $\mathcal{F}_{b,\text{loc}} := \mathcal{F}_{\text{loc}} \cap L^\infty_{\text{loc}}(X, m)$.

We shall denote respectively by $\kappa$, $J$ the killing and the jumping measures related to $\mathcal{E}$ and $\mathcal{E}^c$ its strong part both given by Beurling-Deny formula (See [FOT94, Theorem 4.5.2,p.164] (for quasi-regular Dirichlet forms, see [Kuw98]).

$$J(f, g) = \int_{X \setminus d \times X \setminus d} (f(x) - f(y))(g(x) - g(y)) J(dx, dy), \ \forall f, g \in \mathcal{F}.$$  

Given $f, g \in \mathcal{F}$ set $\mu_{\Delta f}^c$ the energy measure of $f$ and $\mu_{\Delta f, g}^c$ the mutual energy measure of $f, g$ (see [FOT94, pp.110-114]). Furthermore the strong local part of $\mathcal{E}$ possesses the representation

$$\mathcal{E}^c[f] = \frac{1}{2} \mu_{\Delta f}^c(X), \ \forall f \in \mathcal{F}. \quad (2.1)$$

The representation goes as follows: for $f \in \mathcal{F}_b$ its energy measure is defined by

$$\int \phi \, d\mu_{\Delta f}^c = 2\mathcal{E}(f, \phi) - \mathcal{E}(f^2, \phi), \ \forall 0 \leq \phi \in \mathcal{F} \cap C_c(X). \quad (2.2)$$
Truncation and monotone convergence allow then to define $\mu^{c}_{<f>}$ for every $f \in \mathcal{F}$. Furthermore with the help of strong locality

$$\int \{f = c\} \, d\mu^{c}_{<f>} = 0, \ \forall \ f \in \mathcal{F},$$

(2.3)

it is possible to define $\mu^{c}_{<f>}$ for every $f \in \mathcal{F}_{loc}$ as follows: for every relatively compact subset $\Omega \subset X$

$$1_{\Omega} \mu^{c}_{<f>} = 1_{\Omega} \mu^{c}_{<f>},$$

(2.4)

where $\tilde{f} \in \mathcal{F}$ and $f = \tilde{f}$-q.e. on $\Omega$.

By polarization and regularity we can thereby define a Radon-measures-valued bilinear form on $\mathcal{F}_{loc}$ so that

$$\mathcal{E}^{c}(f, g) = \frac{1}{2} \mu^{c}_{<f,g>}(X), \ \forall \ f \in \mathcal{F}_{loc}.$$ 

(2.5)

The truncation property for $\mathcal{E}^{c}$ reads as follows: For every $a \in \mathbb{R}$, every $f \in \mathcal{F}_{loc}$ and every $g \in \mathcal{F}_{b,loc}$ we have

$$\mathcal{E}^{c}((f-a)_{+}, g) = 1_{\{f>a\}} \mathcal{E}^{c}(f, g) \text{ and } \mathcal{E}^{c}[(f-a)_{+}] = 1_{\{f>a\}} \mathcal{E}^{c}[f].$$

(2.6)

Furthermore the following product formula holds true

$$d\mu^{c}_{<fh,g>} = fd\mu^{c}_{<h,g>} + hd\mu^{c}_{<f,g>}, \ \forall \ f, g, h \in \mathcal{F}_{b,loc}.$$ 

(2.7)

By the regularity assumption the latter formula extends to every $f, g, h \in \mathcal{F}_{loc}$.

Another rule that we shall occasionally use is the chain rule (See [FOOT94, pp.11-117]): For every function $\phi : \mathbb{R} \to \mathbb{R}$ of class $C^{1}$ with bounded derivative ($\phi \in C^{1}_{b}(\mathbb{R})$), every $f \in \mathcal{F}_{loc}$ and every $g \in \mathcal{F}_{b,loc}$, $\phi(f) \in \mathcal{F}_{loc}$ and

$$d\mu^{c}_{<\phi(f),g>} = \phi'(f)d\mu^{c}_{<f,g>}. $$

(2.8)

We improve a bit the chain rule.

**Lemma 2.1.** Let $\phi : (0, \infty) \to \mathbb{R}$ of class $C^{1}$ be such that for every $a > 0$, $f \in C^{1}_{b}([a, \infty))$.

Let $f \in \mathcal{F}_{loc}$ such that for every kompact subset $K \subset X$, there is $C_{K} > 0$ such that $f \geq C_{K}$-q.e. on $K$. Then $\phi(f) \in \mathcal{F}_{loc}$ and

$$d\mu^{c}_{<\phi(f),g>} = \phi'(f)d\mu^{c}_{<f,g>}, \ \forall \ g \in \mathcal{F}_{b,loc}.$$ 

(2.9)

**Proof.** Let $K \subset X$, kompact and $C_{K} > 0$ as in the lemma. Let $\tilde{f} \in \mathcal{F}$ s.t. $\tilde{f} = f$-q.e.on $K$. We extend the restriction of $\phi$ to $[C_{K}, \infty)$ by a function $\tilde{\phi} \in C^{1}_{b}(\mathbb{R})$. Then $\tilde{\phi}(f) \in \mathcal{F}$ and $\tilde{\phi}(\tilde{f}) = \phi(f)$-q.e.on $K$ and by formula(2.8)

$$1_{K}d\mu^{c}_{<\phi(f),g>} = 1_{K}\tilde{\phi}'(f)d\mu^{c}_{<f,g>},$$

$$= 1_{K}d\mu^{c}_{<\phi(f),g>} = 1_{K}\phi'(f)d\mu^{c}_{<f,g>} \forall \ g \in \mathcal{F}_{b,loc}.$$ 

(2.10)

which was to be proved.
We shall also make use of the following fact.

**Lemma 2.2.** Let $w$ be a q.c. function such that $w > 0$-q.e. Then $w^{-1}$ is locally quasi-bounded.

**Proof.** By [FOT94, Theorem 2.1.2], there is a nest $(F_k)$ s.t. $w \in C(F_k)$, for every $k$. Set $Y := \cup_k F_k$, then $X \setminus Y$ has zero capacity.

For every integer $k$, we set $G_k := F_k \cap \{w \geq \frac{1}{k}\}$. (2.11)

Then $G_k$ is closed as well as for the topology of $X$ and that of $Y$ inherited from $X$. Also $K' := K \cap Y$ is compact w.r.t. to the trace topology of $X$ on $Y$. Since $(G_k)$ is a covering for $Y$ of closed sets, there is a finite number of $G_k$'s s.t. $K = \cup_{finite} G_k$. Thus $\inf_{K'} w > 0$. On the other hand $\text{Cap}(K \cap Y^c) \leq \text{Cap}(X \setminus Y) = 0$, yielding

$$w(x) \geq \inf_{K'} w > 0 \text{ - q.e. on } K,$$

which was to be proved. \hfill \square

### 3 Hardy’s inequality

We are in position now to assert the first part of the main result.

**Theorem 3.1.** Let $\mathcal{E}$ be a transient Dirichlet form and $\mu$ be a positive Radon measure on Borel subsets of $X$, charging no sets having zero capacity. Assume that there is $C > 0$ and a function $w \in \mathcal{F}_{loc}$, $w > 0$-q.e., such that

$$\frac{1}{2}\mu_{\mathcal{E}}(X) + J(w, f) + \int f w \, d\kappa - C^{-1} \int w f \, d\mu \geq 0, \forall 0 \leq f \in \mathcal{F}_{loc}. \quad (3.1)$$

Then the following Hardy’s inequality holds true

$$\int f^2 \, d\mu \leq C\mathcal{E}[f], \forall f \in \mathcal{F}. \quad (3.2)$$

**Remark 3.1.** Condition (3.1) is fulfilled if there is a function $0 < w$-q.e. $w \in \mathcal{F}$ such that

$$\mathcal{E}(w, f) - C^{-1} \int w f \, d\mu \geq 0, \forall 0 \leq f \in \mathcal{F} \cap C_0(X). \quad (3.3)$$

In particular, if $w$ is the potential of a positive measure $\mu$ charging no sets having zero capacity and such that $\|w\|_{\infty} < \infty$, we get

$$\mathcal{E}(w, f) = \int f \, d\mu \geq \|w\|_{\infty}^{-1} \int w f \, d\mu, \forall 0 \leq f \in \mathcal{F} \cap C_0(X). \quad (3.4)$$

Obtaining therefore, the known inequality [Von96, SV96, Fit00, BA04]

$$\int f^2 \, d\mu \leq \|w\|_{\infty} \mathcal{E}[f], \forall f \in \mathcal{F}. \quad (3.5)$$
We shall say that a measure \( \mu \) satisfies Hardy’s inequality if inequality (3.2) holds true.

**Proof.** Without loss of generality we may and shall neglect the killing term in \( E \).

Let \( f \) be s.t. \( wf \in \mathcal{F} \). Since, by Lemma 2.2, for every compact subset \( K \subset X \) there is \( C_K \) s.t. \( w^{-1} \leq C_K \)-q.e. on \( K \), we obtain by Lemma 2.1 that \( w^{-1} \in \mathcal{F}_{b,loc} \) and \( f = w^{-1}wf \in \mathcal{F}_{loc} \).

By formula (2.1) together with the product formula (2.7), we obtain

\[
E[wf] = \frac{1}{2} \mu_{<wf,wf>} + J[wf] \\
= \frac{1}{2} \int w^2 d\mu_{<f>} + \int wf d\mu_{<w,f>} + \frac{1}{2} \int f^2 d\mu_{<w>} + J[wf] \tag{3.6}
\]

Yielding

\[
E[wf] - C^{-1} \int (wf)^2 d\mu = \frac{1}{2} \int w^2 d\mu_{<f>} + \int wf d\mu_{<w,f>} + \frac{1}{2} \int f^2 d\mu_{<w>} - C^{-1} \int (wf)^2 d\mu + J[wf]
\]

Replacing \( f \) by \( wf^2 \in \mathcal{F}_{loc} \) in Eq. (3.1), we get

\[
0 \leq \frac{1}{2} \int d\mu_{<wf^2,w>} - C^{-1} \int (wf)^2 d\mu + J(w,wf^2) = \frac{1}{2} \int f^2 d\mu_{<w>}
+ \frac{1}{2} \int wd\mu_{<f^2,w>} - C^{-1} \int (wf)^2 d\mu + J(w,wf^2). \tag{3.7}
\]

Observing that

\[
\int wd\mu_{<f^2,w>} = 2 \int wf d\mu_{<f,w>}, \tag{3.8}
\]

and that

\[
J(w,wf^2) \leq J[wf] \tag{3.9}
\]

we achieve

\[
\int w^2 f^2 d\mu \leq C E[wf], \tag{3.10}
\]

for every \( f \) as given in the beginning of the proof.

Now let \( f \in \mathcal{F} \cap C_0(X) \). Then \( f = w^{-1}f \). We set \( g := w^{-1}f \). Then \( wg \in \mathcal{F} \). Applying the first part of the proof and using the regularity assumption, we get the result. \( \square \)

As an example of measures for which Hardy’s inequality holds true we give

**Corollary 3.1.** Let \( 0 < w \) be a superharmonic function and \( \mu \) its Riesz charge. Then

\[
\int w^{-1} f^2 d\mu \leq E[f], \quad \forall f \in \mathcal{F}. \tag{3.11}
\]
Proof. Assume first that \( w \in \mathcal{F} \). Then for all \( f \in \mathcal{F} \cap C_c(X) \) we have

\[
\mathcal{E}(w, w^{-1}f) = \int w^{-1}f \, d\mu, \tag{3.12}
\]

which yields inequality (3.11) by Theorem 3.1.

For general \( w \), let \( \mu_k \uparrow \mu \), be such that \( w_k := U\mu_k \in \mathcal{F} \). Then by the first step

\[
\int w^{-1}f^2 \, d\mu = \lim_{k \to \infty} \int w_k^{-1}f^2 \, d\mu_k \leq \lim_{k \to \infty} \int w_k^{-1}f^2 \, d\mu_k \leq \mathcal{E}[f], \forall f \in \mathcal{F}, \tag{3.13}
\]

which finishes the proof \( \square \)

Remark 3.2. On the light of Corollary 3.1, Theorem 3.1 has the following consequence: every measure which is dominated by a constant times the inverse of a nonnegative superharmonic function times its Riesz charge satisfies Hardy’s inequality.

This result is exactly Fitzsimmons’s result [Fit00].

Example 3.1. Improved Hardy inequality in the half-space: In this example we shall rediscover an improved Hardy inequality proved in [Tid05, Corollary 3.1]. Let \( d \geq 3 \). Set \( \mathbb{R}^d_+ \) the upper half-space. Set

\[
\psi(x) := x_1^d (x_{d-1}^2 + x_d^2)^{\frac{d}{4}}, \quad x \in \mathbb{R}^d_+.
\]

Let \( 0 < \epsilon < 1/4 \). Then with \( w := \psi \), we get

\[
-\Delta w - \frac{1}{8} w^{-2} - (\frac{1}{4} - \epsilon) w x_d^{-2} = \frac{1}{4} w \left( \frac{1}{(x_{d-1}^2 + x_d^2)^2} \right) + \frac{4\epsilon}{x_d^2} - \frac{1}{2x_d(x_{d-1}^2 + x_d^2)^{\frac{d}{2}}} \geq 0 \tag{3.14}
\]

Thus by Theorem 3.1, we obtain

\[
(\frac{1}{4} - \epsilon) \int_{\mathbb{R}^d_+} x_d^{-2} f^2 \, dx + \frac{1}{8} \int_{\mathbb{R}^d_+} \frac{f^2}{x_d(x_{d-1}^2 + x_d^2)^{\frac{d}{2}}} \, dx \leq \int_{\mathbb{R}^d_+} |\nabla f|^2 \, dx, \forall f \in C_0^\infty(\mathbb{R}^d_+). \tag{3.15}
\]

Letting \( \epsilon \to 0 \), we derive

\[
\frac{1}{4} \int_{\mathbb{R}^d_+} x_d^{-2} f^2 \, dx + \frac{1}{8} \int_{\mathbb{R}^d_+} \frac{f^2}{x_d(x_{d-1}^2 + x_d^2)^{\frac{d}{2}}} \, dx \leq \int_{\mathbb{R}^d_+} |\nabla f|^2 \, dx, \forall f \in C_0^\infty(\mathbb{R}^d_+). \tag{3.15}
\]

As in the context of Ancona and Fitzsimmons (See [Anc86, Proposition 1],[Fit00]) we proceed to show that a sort of converse to Theorem (3.1) holds true.

Theorem 3.2. Assume that inequality (3.2) holds true. Then for every \( 0 < \Lambda < C^{-1} \) there is \( w \in \mathcal{F} \), \( w > 0 \)-a.e., and fulfills condition (3.1).
Proof. Suppose that (3.2) holds true. Then by [BA04, Theorem 3.1], the operator

\[ K^\mu := L^2(\mu) \to L^2(\mu), \quad f \mapsto U f \mu, \]  

(3.16)

where \( U f \mu \) is the potential of \( f \mu \) is bounded and \( \|K^\mu\| \leq C \). Thus for every \( 0 < \Lambda < C^{-1} \) the operator \( 1 - \Lambda K^\mu \) is invertible on \( L^2(\mu) \).

Let \( \varphi \in \mathcal{F} \), s.t. \( 0 < \varphi \leq 1 \). Then there is \( \psi \in L^2(\mu) \) with \( \psi - \Lambda K^\mu \psi = \varphi - \mu \) a.e. Thus

\[ K^\mu \psi - \Lambda K^\mu (K^\mu \psi) = K^\mu \varphi - q.e. \]  

(3.17)

Since \( \varphi > 0 \), \( K^\mu \) is positivity preserving and

\[ \psi = \sum_{k=0}^{\infty} \Lambda^k (K^\mu)^k \varphi, \]  

(3.18)

we conclude that \( \psi > 0 \)-a.e. and \( w := K^\mu \psi > 0 \)-q.e., which by Lemma 2.2 yields that \( w^{-1} \) is quasi-bounded.

For the rest of the proof, observe that for every \( 0 \leq f \in \mathcal{F} \)

\[
E(\psi, f) - \Lambda \int w f \, d\mu = E(K^\mu \psi, f) - \Lambda \int w f \, d\mu \\
= E(K^\mu \varphi, w) + \Lambda E(K^\mu w, f) - \Lambda \int w f \, d\mu \\
= \int w \varphi \, d\mu \geq 0,
\]  

(3.19)

which finishes the proof.

The proof of Theorem 3.2 shows that if the operator \( 1 - K^\mu \) is invertible, then the conclusion holds true with \( \Lambda = C^{-1} \) as well.

We shall add an alternative assumption (which is fulfilled in many cases) on the form

\[ \mathcal{E}_\mu, D(\mathcal{E}_\mu) = \mathcal{F}, \mathcal{E}_\mu[f] = \mathcal{E}[f] - \int f^2 \, d\mu, \]  

(3.20)

that ensures that the case \( \Lambda = C^{-1} \) is included as well.

**Proposition 3.1.** Let \( \mu \) be a positive Radon measure on Borel subsets of \( X \) that satisfies the Hardy’s inequality with best constant 1. Assume that there is \( \Lambda > 0 \) s.t.

\[ \int f^2 \, d\mu \leq \Lambda \mathcal{E}_\mu[f], \quad \forall f \in D(\mathcal{E}). \]  

(3.21)

Then for every \( g \in \mathcal{F} \) there is \( f \in \mathcal{F} \) s.t.

\[ \mathcal{E}_\mu(\varphi, f) = \int \varphi g \, d\mu \forall \varphi \in \mathcal{F}. \]  

(3.22)

If in particular \( g > 0 \)-q.e. then there is \( 0 < w \)-q.e., \( w \in \mathcal{F} \) and satisfies condition (3.1) with \( C = 1 \).
The proof is easy, so we omit it.

Let $\mu$ be a positive Radon measure on Borel subsets of $X$ charging no sets having zero capacity. Assume that there is $w$ satisfying the assumptions of Theorem 3.1 with best constant $C = 1$. Then Theorem 3.1 yields that the quadratic form defined by

$$D(\mathcal{E}^w) = \{ f \in L^2(w^2 dm) : wf \in \mathcal{F}, \mathcal{E}^w[f] = \mathcal{E}_\mu[wf] \},$$

is a positive quadratic form. We shall prove that $\mathcal{E}^w$ is, in fact, a Dirichlet form. A proof of this result was shortly quoted by Fitzsimmon [Fit00] using a probabilistic method. We shall, however prove it using an analytical one.

**Proposition 3.2.** Under the above assumptions the form $\mathcal{E}^w$ is a Dirichlet form.

**Proof.** We develop the proof by steps.

Step 1: $\mathcal{E}_\mu$ is closable. Indeed, we associate to $\mathcal{E}_\mu$ a positive symmetric operator $H_\mu$ such that $D(H_\mu) = D(H)$ and $(H_\mu f, g) = \mathcal{E}_\mu(f, g), \forall f \in D(H_\mu), g \in \mathcal{F}$. Since $\mathcal{F}$ is dense in $L^2$ then so is $D(H_\mu)$. Thus by [Dav89, Theorem 1.2.8], $\mathcal{E}_\mu$ is closable. We still denote by $\mathcal{E}_\mu$ its closure and $H_\mu$ the operator associated to it via Kato’s representation theorem.

Step 2: $\mathcal{E}_\mu^w$ is closed. The operator $H_\mu^w := w^{-1}H_\mu w$ is closed and for every $f, g$ s.t. $wf, wg \in D(H_\mu^{1/2}) = \mathcal{F}$ we have

$$(H_\mu^w)^{1/2}f, g)_{L^2(w^2 dm)} = \mathcal{E}_\mu^w(f, g).$$

Thus $\mathcal{E}_\mu^w$ is closed.

Step 3: $\mathcal{E}_\mu^w$ is a Dirichlet form. Set

$$\hat{\mathcal{E}} : D(\hat{\mathcal{E}}) = D(\mathcal{E}_\mu^w), \hat{\mathcal{E}}[f] = \frac{1}{2} \int w^2 d\mu\mathcal{E}_\mu^{w^2}$$

Then $\hat{\mathcal{E}}$ is a densely defined closable positive quadratic form satisfying the truncation property (by property (2.6)). Hence its closure is a Dirichlet form, which we still denote by $\hat{\mathcal{E}}$. We denote by $\hat{H}$ its related operator.

On the other hand we have (by Theorem 1.1)

$$0 \leq \hat{\mathcal{E}} \leq \mathcal{E}_\mu^w,$$

yielding, for every $\alpha > 0$

$$0 \leq (H_\mu^w + \alpha)^{-1} \leq (\hat{H} + \alpha)^{-1}.$$

Now since $(H_\mu^w + \alpha)^{-1}$ is positivity preserving (because $(H_\mu + \alpha)^{-1}$ is) and $(\hat{H} + \alpha)^{-1}$ is Markovian, we derive that $(H_\mu^w + \alpha)^{-1}$ is Markovian as well and $\mathcal{E}_\mu^w$ is a Dirichlet form, which finishes the proof.

$\square$
4  Examples for strongly local Dirichlet forms

In this section we shall concentrate on giving general and concrete examples of measures satisfying the Hardy inequality provided the Dirichlet form is strongly local. Furthermore in some positions we shall even improve Hardy’s inequality. These examples are mainly inspired from classical Hardy’s on Euclidean domains having strong barriers [Anc86].

\[ \int_{\Omega} \frac{f^2(x)}{\text{dist}(x, \partial \Omega)} \, dx \leq C_{\Omega} \int_{\Omega} |\nabla f(x)|^2 \, dx, \quad \forall f \in W^1_0(\Omega). \]  

(4.1)

and from an example given by Fitzsimmons [Fit00, Example4.2].

For the sake of completeness, we recall some basic concepts related to strongly local Dirichlet forms.

Every strongly local Dirichlet form, $E$ induces a pseudo-metric on $X$ known as the intrinsic metric and defined by

\[ \rho(x, y) := \sup \{ f(x) - f(y), \, \, f \in F_{loc}, \frac{1}{2} \mu_c^{<f>} \leq m \text{ on } X \}, \]  

(4.2)

where the inequality $1/2 \mu_c^{<f>} \leq m$ in the above definition means that the energy measure $\mu_c^{<f>}$ is absolutely continuous w.r.t. the reference measure $m$ with Radon-Nikodym derivative smaller that 1.

Throughout this section we shall assume that $\rho$ is a true metric whose topology coincides with the original one and that $(X, \rho)$ is complete.

For a given closed subset $F \subset X$, we set

\[ \rho_F(x) := \rho(x, F), \quad \forall x \in X. \]  

(4.3)

Then under the above assumption (See [Stu95, Remark after Lemma1.9]),

\[ \rho_F \in F_{loc} \cap C(X) \text{ and } \frac{1}{2} d\mu_c^{<\rho_F>} \leq dm. \]

Now let $\Omega \subset X$ be an open fixed subset, $\mathcal{E}_\Omega$ the form defined by

\[ \mathcal{F}_\Omega := D(\mathcal{E}_\Omega) = \{ f \in D(\mathcal{E}) : f = 0 - \text{q.e. on } X \setminus \Omega \}, \mathcal{E}_\Omega[f] = \mathcal{E}[f]. \]

Then $\mathcal{E}_\Omega$ is a regular strongly local Dirichlet form on $L^2(\Omega, m)$ ([FOT94, Theorem4.4.3]). Set $F = X \setminus \Omega$ or any closed subset of $\Omega$ having zero capacity and $\mathcal{F}_{\Omega, loc}$ the local domain of $\mathcal{E}_\Omega$.

We are in position now to extend inequality (4.1) in our framework.

**Theorem 4.1.** Assume that

\[ \int d\mu_c^{<\rho_F,f>} \geq 0, \quad \forall 0 \leq f \in \mathcal{F}_{\Omega, loc}. \]  

(4.4)

Then

\[ \frac{1}{2} \int_{\Omega} f^2 \frac{d\mu_c^{<\rho_F>}}{\rho_F} \leq 4 \mathcal{E}[f], \quad \forall f \in \mathcal{F}_\Omega. \]  

(4.5)
For the gradient energy form on Euclidean domains, condition (4.4) expresses the fact that $\rho_F$ is superharmonic, under which the constant $C_\Omega$ appearing in inequality (4.1) may be chosen to be equal 4. On the light of this observation, our extension seems to be quite natural.

Proof. Set $w = \frac{1}{2}$. By Theorem 3.1, it suffices to prove

$$\frac{1}{2} \int_\Omega d\mu^c_{<w,f>} - \frac{1}{8} \int_\Omega w f \frac{d\mu^c_{<\rho_F>}}{\rho^2_F} \geq 0, \quad \forall 0 \leq f \in \mathcal{F}_{\Omega,\text{loc}},$$

(4.6)

or equivalently

$$\frac{1}{2} \int_\Omega d\mu^c_{<w,wf>} - \frac{1}{8} \int_\Omega w^{-2} f d\mu^c_{<\rho_F>} \geq 0, \quad \forall 0 \leq f \in \mathcal{F}_{\Omega,\text{loc}}.$$  

(4.7)

Let $0 \leq f \in \mathcal{F}_{\Omega,\text{loc}}$. Owing to the product formula together with the chain rule given by Lemma 2.1, a straightforward computation yields

$$\int_\Omega d\mu^c_{<w,wf>} = \int_\Omega f d\mu^c_{<w>} + \int_\Omega w d\mu^c_{<w,f>} = \frac{1}{4} \int_\Omega f w^{-2} d\mu^c_{<\rho_F>} + \frac{1}{2} \int_\Omega d\mu^c_{<\rho_F,f>},$$

(4.8)

obtaining thereby

$$\frac{1}{2} \int_\Omega d\mu^c_{<w,wf>} - \frac{1}{8} \int_\Omega f w^{-2} d\mu^c_{<\rho_F>} \geq 0,$$

(4.9)

which completes the proof.

Example 4.1. Let $\Omega$ be a convex subset of the Euclidean space $\mathbb{R}^d$ ($d \geq 3$ if $\Omega$ is unbounded) and $\varphi$ a function s.t. $\varphi > 0$-q.e. on $\Omega$ and $\varphi, \varphi^{-1} \in L^2_{\text{loc}}(\Omega, dx)$. We define the Dirichlet form on $L^2(\Omega, dx)$ by

$$\mathcal{E}[f] = \int_\Omega |\nabla f|^2 \varphi^2 \, dx, \quad \forall f \in C^\infty_0(\Omega),$$

(4.10)

and $\mathcal{F}$ being the closure of $C^\infty_0(\Omega)$ w.r.t. $\mathcal{E}$. Then it is known that

$$\rho(x, y) = |x - y|, \quad \forall x, y \in \Omega.$$  

(4.11)

Set $F = \mathbb{R}^d \setminus \Omega$. Assume that $\varphi$ satisfies condition (4.3) which reads

$$-\Delta \rho_F - 2 \varphi^{-1} \nabla \varphi \nabla \rho_F \geq 0.$$  

(4.12)

(Its the case if for example $\varphi = \rho_F^{-\alpha}$, $\alpha \geq 0$). Then conditions of Theorem 4.1 are fulfilled and we get

$$\int_\Omega \frac{f^2}{\rho^2_F} \, dx \leq 4 \int_\Omega |\nabla f|^2 \varphi^2 \, dx, \quad \forall f \in \mathcal{F}.$$  

(4.13)
Another general example is the following

**Theorem 4.2.** Let $\psi \in \mathcal{F}_{\text{loc}}$ be s.t. $\psi > 0$ q.e., $\frac{d\mu^c_{\psi,>}}{2dm} \leq 1$ and for some constant $C > 1/2$

$$\mathcal{E}(\psi, f) \leq -2C \int \psi^{-1} f \, dm, \forall 0 \leq f \in \mathcal{F}_{\Omega,\text{loc}}. \quad (4.14)$$

Set $\beta := C - \frac{1}{2}$. Then

$$\int f^2 \psi^{-2} \, dm \leq \beta^{-2} \mathcal{E}[f], \forall f \in \mathcal{F}. \quad (4.15)$$

**Proof.** Let $0 \leq f \in \mathcal{F}_{\Omega,\text{loc}}$. Changing $f$ by $\psi^{-2\beta - 1} f$ in inequality (4.14) and applying the chain rule we achieve

$$\frac{1}{2} \int \psi^{-2\beta - 1} \, d\mu^c_{\psi,>} - \frac{1}{2} (2\beta + 1) \int \psi^{-2\beta - 2} f \, d\mu^c_{\psi} \leq -2C \int \psi^{-2\beta - 2} f \, dm. \quad (4.16)$$

Using the latter inequality together with the assumption $\frac{d\mu^c_{\psi,>}}{2dm} \leq 1$, we obtain

$$\frac{1}{2} \int d\mu^c_{\psi^{-\beta},\psi^{-\beta}f} = -\frac{\beta}{2} \int \psi^{-2\beta - 1} \, d\mu^c_{\psi,>} + \frac{\beta^2}{2} \int \psi^{-2\beta - 2} f \, d\mu^c_{\psi} \geq 2C\beta \int \psi^{-2\beta - 2} f \, dm - \frac{\beta}{2} (2\beta + 1) \int \psi^{-2\beta - 2} \, d\mu^c_{\psi}$$

$$+ \frac{\beta^2}{2} \int \psi^{-2\beta - 2} f \, d\mu^c_{\psi} \geq 2(\beta + \frac{1}{2}) \beta \int \psi^{-2\beta - 2} f \, dm - \beta (\beta + 1) \int \psi^{-2\beta - 2} f \, dm. \quad (4.17)$$

Thus

$$\frac{1}{2} \int d\mu^c_{\psi^{-\beta},\psi^{-\beta}f} - \beta^2 \int \psi^{-2\beta - 2} \, dm \geq 0, \quad (4.18)$$

and $w = \psi^{-\beta}$ satisfies condition (3.1), with $d\mu = \psi^{-2} \, dm$, which completes the proof.

**Example 4.2.** We take an other time the Dirichlet form of Example 4.1, with $d \geq 3$. We suppose that $\Omega$ is star-shaped around one of its points $x_0 \in \Omega$. We choose $F = \{x_0\}$ and assume that points have zero capacity. Then

$$\rho(x) := \rho_F(x) = |x - x_0|.$$ 

We choose $\psi(x) = \rho(x)$ and $\varphi(x) = e^{\rho(x)}$. Then $\frac{d\mu^c_{\psi,>}}{2dm} \leq 1$.

On the other hand condition (4.25) reads

$$d - 1 + 2\rho(x) \geq 2Ce^{-2\rho(x)}, \quad (4.19)$$

which is fulfilled with $C = \frac{d-1}{2}$. Thus we get

$$\int \Omega \frac{f(x)^2}{|x - x_0|^2} e^{2|x-x_0|} \, dx \leq \left( \frac{d-2}{2} \right)^{-2} \int \Omega |\nabla f(x)|^2 e^{2|x-x_0|} \, dx, \forall f \in \mathcal{F}. \quad (4.20)$$
Example 4.3. We investigate in this example the Dirichlet form given by: Set $\sigma(x) = (1 + |x|^2)^{\frac{d}{2}}$ and

$$ \mathcal{E}[f] = \int_{\mathbb{R}^d} |\nabla f|^2 \, dx + \int_{\mathbb{R}^d} f^2 \sigma^\lambda(x) \, dx, \quad (4.21) $$

considered on the space $L^2(\mathbb{R}^d, \sigma^\lambda \, dx)$. In this situation the intrinsic metric is given by \cite{CG98}

$$ \rho(0, x) = \ln(|x| + \sqrt{1 + |x|^2}). \quad (4.22) $$

We set $\psi(x) := \rho(0, x), \ \forall x \in \mathbb{R}^d$ and suppose that $d \geq 3$.

From the property of the intrinsic metric we derive \( \frac{d\mu_{c, \psi}}{2dm} \leq 1 \). The second condition imposed on $\psi$ reads

$$ - \Delta \psi + \psi \sigma^\lambda \leq -C \psi^{-1} \sigma^\lambda, \quad (4.23) $$

or equivalently

$$ \frac{d - 1}{|x|} + \frac{d}{(1 + |x|^2)^{3/2}} \geq \ln(|x| + \sqrt{1 + |x|^2}) \sigma^\lambda(x) \geq C \frac{\sigma^\lambda(x)}{\ln(|x| + \sqrt{1 + |x|^2})}, \ \forall x \in \mathbb{R}^d \setminus \{0\}, \quad (4.24) $$

with $C > 1/2$. Obviously this condition can not be fulfilled if $\lambda \geq 0$. However if $\lambda < 0$ and $-\lambda$ is big enough then the latter condition is satisfied and we obtain for such $\lambda$

$$ \int_{\mathbb{R}^d} f^2(x) \ln^{-2}(|x| + \sqrt{1 + |x|^2}) \, dx \leq \beta^{-2} \left( \int_{\mathbb{R}^d} |\nabla f|^2 \, dx + \int_{\mathbb{R}^d} f^2 \sigma^\lambda(x) \, dx \right), \ \forall f \in D(\mathcal{E}). $$

The latter theorem may be improved in the following way

**Theorem 4.3.** Let $\psi \in \mathcal{F}_{\text{loc}}$ be s.t. $\psi > 0$-q.e. and for some constant $C > 1/2$

$$ \mathcal{E}(\psi, f) \leq -C \int \psi^{-1} f \, d\mu_{c, \psi}^\circ, \ \forall 0 \leq f \in \mathcal{F}_{\Omega, \text{loc}}. \quad (4.25) $$

Set $\beta := C - \frac{1}{2}$. Then

$$ \frac{1}{2} \int f^2 \psi^{-2} \, d\mu_{c, \psi}^\circ \leq \beta^{-2} \mathcal{E}[f], \ \forall f \in \mathcal{F}_{\Omega}. \quad (4.26) $$

The proof runs as the previous one so we omit it.

**Remark 4.1.** Inequality (4.25) is fulfilled with $C = 1$ if

$$ \mathcal{E}(\log \psi, f) \leq 0, \ \forall 0 \leq f \in \mathcal{F}_{\Omega, \text{loc}}. \quad (4.27) $$
On the light of Theorems 4.2-4.3 and being inspired by a result due Filippas-Moschini-Tertikas [FLA07, Theorem 3.2], we shall improve, in some respect, the Hardy inequality.

**Theorem 4.4.** Assume that conditions imposed on $\rho_F$ in Theorem 4.1 and on $\psi$ in Theorem 4.3 are fulfilled. Then the following improved Hardy’s inequality

$$\frac{1}{2} \int_{\Omega} f^2 \frac{d\mu_{\rho_F}}{\rho_F} \leq 4(\mathcal{E}[f] - \frac{\beta^2}{2} \int f^2 \psi^{-2} d\mu_{\psi}^c), \ \forall \ f \in \mathcal{F}_\Omega, \quad (4.28)$$

holds true, provided

$$\int_{\Omega} \psi^{-2\beta} d\mu_{\rho_F,\psi}^c \geq 0 \ \forall \ 0 \leq f \in \mathcal{F}_{\Omega,\text{loc}}. \quad (4.29)$$

**Proof.** Set $w_1 = \psi^{-\beta}$, $w_2 = \rho_F^\frac{1}{2}$ and $f = w_1 w_2 g \in \mathcal{F}_\Omega$. Then

$$\mathcal{E}[w_1 w_2 g] = \frac{1}{2} \int d\mu_{w_1 w_2 g}^c = \frac{1}{2} \int (w_1 w_2)^2 d\mu_{g}^c + \frac{1}{2} \int \psi^{-2\beta} g d\mu_{\rho_F,\psi}^c
- \beta \int \psi^{-(2\beta+1)} \rho_F g d\mu_{\rho_F,\psi}^c
+ \frac{1}{8} \int \psi^{-2\beta} g^2 \rho_F^{-1} d\mu_{\rho_F}^c + \frac{\beta^2}{2} \int \psi^{-(2\beta+1)} g^2 \rho_F d\mu_{\psi}^c. \quad (4.30)$$

Yielding

$$\mathcal{E}[w_1 w_2 g] = -\frac{\beta^2}{2} \int (w_1 w_2)^2 \psi^{-2} g^2 d\mu_{\psi}^c
- \frac{1}{8} \int (w_1 w_2)^2 \rho_F^{-2} g^2 d\mu_{\rho_F}^c
+ \frac{1}{2} \int (w_1 w_2)^2 d\mu_{\psi}^c
- \beta \int \psi^{-2\beta} g d\mu_{\rho_F,\psi}^c
+ \frac{1}{2} \int \psi^{-(2\beta+1)} \rho_F g d\mu_{\rho_F,\psi}^c \quad (4.31)$$

Observe that by assumptions the first two integrals in the latter equality are positive. We shall prove that the remainder which we denote by $R$ is positive as well. We rewrite $R$ with the help of the product formula

$$R = -\int \psi^{-(2\beta+1)} d\mu_{g^2 \rho_F,\psi}^c. \quad (4.32)$$

Owing to inequality (4.25), we achieve

$$R \geq \frac{C}{2} \int \psi^{-2\beta-2} \rho_F d\mu_{\psi}^c - \beta(\beta+1/2) \int \psi^{-2\beta-2} \rho_F d\mu_{\psi}^c = 0, \quad (4.33)$$

which was to be proved.
We illustrate the improved Hardy’s inequality by an example.

**Example 4.4.** We reconsider the Dirichlet form of Example 4.1. We suppose that \( \Omega = B_R \), the open Euclidean ball centered at 0 with radius \( R > 0 \). We set

\[
\rho(x) := R - |x|, \quad x \in B_R.
\]

We fix \( \alpha \in [0, 1/2] \) and choose

\[
\varphi(x) = \rho(x)^{-\alpha} \quad \text{and} \quad \psi(x) = |x|, \quad \forall \ x \in B_R.
\] (4.34)

Then condition (4.25) imposed on \( \psi \) reads

\[
\frac{1 - d}{|x|} + \frac{2}{|x|} \varphi^{-1} x \cdot \nabla \varphi \leq -C \frac{1}{|x|} \varphi^{2},
\] (4.35)

which is always satisfied. However the condition \( C > 1/2 \) is fulfilled if and only if

\[
(d - 1)R^{2\alpha} > 1.
\]

Whence from now on we assume in this example that \( d > 1 \) and \( R \) satisfies the latter condition (big \( R \)).

The condition imposed on \( \rho \) reads

\[
-\frac{1 + d}{|x|} + 2\alpha \rho^{-1}(x) \geq 0, \quad \text{on} \ B_R,
\] (4.36)

which is always true.

Lastly the condition (4.29) imposed jointly on \( \psi \) and \( \rho \) reads

\[
- \text{div} (\psi^{-2\beta} \rho^{-2\alpha} \nabla \rho) \geq 0,
\] (4.37)

or equivalently

\[
\frac{2\beta}{|x|} + \frac{2\alpha}{R - |x|} - \Delta \rho \geq 0,
\] (4.38)

which is always fulfilled.

Thus we get, with \( \beta := (d - 1)R^{2\alpha} - 1/2 \), for every \( f \in W^1_0(B_R) \)

\[
\int_{B_R} \frac{f^2}{(R - |x|)^2} \, dx \leq 4 \left( \int_{B_R} |\nabla f|^2 (R - |x|)^{-2\alpha} \, dx \right) - \beta^2 \int_{B_R} \frac{f^2}{|x|^2} (R - |x|)^{-2\alpha} \, dx.
\]

Other conditions may also lead to an improved Hardy’s inequality. Indeed, following the lines of the latter proof one get

**Proposition 4.1.** Assume that \( \rho_F \) satisfies conditions of Theorem 4.1, that

\[
\frac{1}{2} \frac{d\mu_{\rho_F}}{dm} = 1,
\]
and that $\psi$ satisfies conditions of Theorem 4.2. Then

$$\int_{\Omega} \frac{f^2}{\rho_F^2} \, dm \leq 4(\mathcal{E}[f] - \frac{\beta^2}{2} \int f^2 \psi^{-2} \, d\mu_{<\psi>}^c), \ \forall \ f \in \mathcal{F}_{\Omega}, \tag{4.39}$$

holds true, provided

$$\int_{\Omega} \psi^{-2\beta} \, d\mu_{<\rho_F,f>}^c \geq 0 \ \forall \ 0 \leq f \in \mathcal{F}_{\Omega,\text{loc}}. \tag{4.40}$$

Set $\text{Cap}_{\Omega}$, the capacity induced by $\mathcal{E}_{\Omega}$. In conjunction with the equivalence between isocapacitary inequality and Hardy’s inequality [Fit00, BA05], the latter proposition leads to the following lower estimate for the capacity of compact sets

$$\int_K \frac{1}{\rho_F^2} \, dm + \frac{\beta^2}{2} \int_K \psi^{-2} \, d\mu_{<\psi>}^c \leq 4\text{Cap}_{\Omega}(K), \ \forall K \subset \Omega, \text{ compact}. \tag{4.41}$$

References

[Ada73] David R. Adams. A trace inequality for generalized potentials. *Studia Math.*, 48:99–105, 1973.

[AH96] David R. Adams and Lars Inge Hedberg. *Function spaces and potential theory*. Springer-Verlag, Berlin, 1996.

[Anc86] Alano Ancona. On strong barriers and an inequality of Hardy for domains in $\mathbb{R}^n$. *J.London Math.Soc.*, 2(34):274–290, 1986.

[BA04] Ali Ben Amor. Trace inequalities for operators associated to regular Dirichlet forms. *Forum Math.*, 16(3):417–429, 2004.

[BA05] Ali Ben Amor. On the equivalence between trace and capacitary inequalities for the abstract space of Bessel potentials. *Osaka J.Math.*, 42:11–26, 2005.

[CG98] Fabio Cipriani and Gabriele Grillo. $L^p$-exponentiel decay for solutions to functional equations in local Dirichlet spaces. *J.reine angew.Math.*, 496:163–179, 1998.

[CMR94] Z.Q. Chen, Zhi-Ming Ma, and Michael Röckner. Quasi-homeomorphism of Dirichlet forms. *Nagoya Math.J.*, 136:1–15, 1994.

[Dav89] Eduard B. Davies. *Heat kernels and spectral theory*. Cambridge University Press, Cambridge, 1989.

[Fit00] P.J. Fitzsimmons. Hardy’s inequality for Dirichlet forms. *J. Math. Anal. Appl.*, 250(2):548–560, 2000.
[FLA07] Statis Filippas, Moschini Luisa, and Tertikas Achilles. Sharp two-sided heat kernel estimates for critical Schrödinger operators on bounded domains. *Comm.Math.Phys.*, 273:237–281, 2007.

[FÖT94] Masatoshi Fukushima, Yōichi Ōshima, and Masayoshi Takeda. *Dirichlet forms and symmetric Markov processes*. Walter de Gruyter & Co., Berlin, 1994.

[FU03] Masatoshi Fukushima and Toshihiro Uemura. Capacitary bounds of measures and ultracontractivity of time changed processes. *J.Math.Pures Appl.*, 82:553–572, 2003.

[Kai92] Vadim A. Kaimanovich. Dirichlet norms, capacities and generalized isoperimetric inequalities for Markov operators. *Potential Anal.*, 1(1):61–82, 1992.

[Kuw98] Kazuhiro Kuwae. Functional calculus for Dirichlet forms. *Osaka J.Math.*, 35:683–715, 1998.

[Maz85] Vladimir G. Maz’ja. *Sobolev spaces*. Springer-Verlag, Berlin, 1985.

[Rv06] Murali Rao and Hrvoje Šikić. Potential-theoretic nature of Hardy’s inequality for Dirichlet forms. *J.Math.Anal.Appl.*, 318:781–786, 2006.

[Stu95] Karl T. Sturm. Analysis on local Dirichlet spaces-II . Upper Gaussian estimates for the fundamental solutions of parabolic equations . *Osaka J.Math.*, 32:257–312, 1995.

[SV96] Peter Stollmann and Jürgen Voigt. Perturbation of dirichlet forms by measures. *Potential Anal.*, 5:109–138, 1996.

[Tid05] Jesper Tidblom. A Hardy inequality in the half-space. *J.Funct.Anal.*, 221:482–495, 2005.

[Von96] Zoran Vondracek. An estimate for the $L^2$-norm of a quasi continuous function with respect to a smooth measure. *Arch. Math.*, 67(5):408–414, 1996.