Randomized and Fault-tolerant Method of Subspace Corrections

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Abstract In this paper, we consider the iterative method of subspace corrections with random ordering. We prove identities for the expected convergence rate, which can provide sharp estimates for the error reduction per iteration. We also study the fault-tolerant feature of the randomized successive subspace correction method by simply rejecting all the corrections when error occurs and show that the results iterative method converges with probability one. Moreover, we also provide sharp estimates on the expected convergence rate for the fault-tolerant, randomized, subspace correction method.

Keywords Method of Subspace Corrections · Randomized Method · Fault-Tolerant Method

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1 Introduction

In this paper, we consider iterative methods for solving the following model problem: Given $f \in V$, find $u \in V$ such that

$$Au = f,$$

where $V$ is a Hilbert space and $A : V \mapsto V$ is a symmetric positive definite (SPD) linear operator. The class of iterative methods we are interested fall into the category of the so-called the methods of subspace corrections (MSC) which have been widely studied in the past several decades, see [24, 1, 25]. MSC is a general framework for linear iterative methods for the solution of linear problems in Hilbert spaces. Many well-known iterative methods can be viewed in the MSC framework and, therefore, can be studied using the general theory of MSC framework, for example, multigrid (MG) method [6, 11, 24] and domain decomposition (DD) method [14, 22].

There are basically two kinds of MSC depending on how the subspaces are corrected. Parallel subspace corrections (PSC) method corrects all the subspaces simultaneously while the successive subspace corrections (SSC) method corrects one after another. The standard SSC method traverses the subspace problems in a fixed order, but one of the interesting features of the SSC method is that the ordering need not be fixed from the start and it can be chosen dynamically during the iterations. A classical example in this direction is the greedy ordering algorithm for Gauss-Seidel method by Southwell [19]. Recently, the effects of the greedy ordering on the convergence of the multiplicative Schwarz method have been studied in [5]. Other algorithms, such as randomized Kaczmarz iterative method have been recently studied in detail (see [9, 10, 3, 11, 21, 13]). A randomized Schwarz method has been discussed in [5], and a randomized coordinate decent methods for certain class of convex optimization problems was in the focus of several recent works [15, 16, 12].

One of our main results is the proof of an identity for the expected error reduction in energy norm per iteration step of the randomized SSC method. In addition, we propose and analyze the convergence of a novel SSC method with $J$ subspaces in which the ordering of the subspace corrections is chosen every $J$ iterations by randomly selecting a permutation of $J = \{1, \ldots, J\}$. We further provide a generalization of the XZ-identity [25] which applies to the error reduction rate in energy norm for such randomized SSC method. Next, we consider a special feature of this, namely, its convergence in case of hardware and/or software failures. On one hand, when such error occur, there is no guarantee that the iterative method can produce a reasonable approximation of the solution. On the other hand, for many PDE-based applications, solving the linear system of equations dominates the overall simulation time (more than 80% of the simulation time for large-scale simulations). Therefore, the development and analysis of fault-tolerant linear solvers with low overhead is an important and urgent issue for improving the overall reliability of the huge pool of PDE-based applications. The standard approaches for constructing fault-tolerant iterative methods usually belong to the so-called ABFT
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(Algorithm-Based Fault Tolerance) category and basic linear and fault tolerant versions of nonlinear iterative methods, such as successive over-relaxation (SOR) method, conjugate gradient (CG) method, and general minimal residual (GMRes) method have been studied in \[17,16,7,18\]. Another approach, proposed in \[20\], relies on rejecting large hardware error propagation and improves the resilience of iterative methods with respect to silent errors. More recently, in \[2\], an intrinsic fault/error tolerant feature of the MSC has been explored. The idea is based on introducing redundant subspaces and working with specially designed mappings between subspaces and processors.

Our results also show how the randomization can be used to improve the reliability of the SSC method in this paper. We built our fault-tolerant, randomized SSC method on a procedure which rejects the faulty subspace corrections when errors occur. Basically, we only update the solution when there is no error and, naturally, we are able to show that this simple procedure, together with randomization, converges almost surely (with probability 1). Our results demonstrate the potential of the SSC method as a natural fault-tolerant iterative method and provide theoretical justification of the usage of the SSC method in improving the reliability of long-running large-scale PDE applications.

The reminder of the paper is organized as follows. We recall the PSC method, the SSC method, the XZ-identity, and some basic notions from probability theory in Section 2.1. In Section 3, we describe the randomized SSC method and its fault-tolerant variant. Section 4 presents our main results, i.e. the sharp identity estimates of the convergence rate and almost sure convergence of the proposed SSC methods. At the end, we give some remarks in Section 6 to conclude the paper.

2 Preliminaries

In this section, we introduce the notation and review some basic results and definitions from the theory of the subspace correction methods and basic probability.

2.1 Method of Subspace Corrections

In this subsection, we recall the standard method of subspace corrections. We consider a decomposition of the vector space \(V\) which consists subspaces \(V_i \subset V, \ i = 1, 2, \cdots, J\), such that

\[
V = \sum_{i=1}^{J} V_i.
\]

(2)

This means that, for each \(v \in V\), there exist \(v_i \in V_i, \ i = 1, 2, \cdots, J\), such that \(v = \sum_{i=1}^{J} v_i\). This representation of \(v\) may not be unique in general, namely (2) is not necessarily a direct sum.
For each $i$, we define $Q_i, P_i : V \mapsto V_i$ and $A_i : V_i \mapsto V_i$ by

$$(Q_i u, v_i) = (u, v_i), \quad (P_i u, v_i)_A = (u, v_i)_A, \quad \forall \ u \in V, \ v_i \in V_i,$$

and

$$(A_i u_i, v_i) = (A u_i, v_i), \quad \forall \ u_i, v_i \in V_i. \quad (4)$$

$Q_i$ and $P_i$ are both orthogonal projections and $A_i$ is the restriction of $A$ on $V_i$ and is SPD.

It follows from the definition that $A_i P_i = Q_i A$. Indeed, $\forall \ u, v \in V$, we have $(Q_i A u, v_i) = (A u, Q_i v_i) = (u, Q_i v_i)_A = (P_i u, Q_i v)_A = (P_i u, v_i)_A$, therefore $A_i P_i = Q_i A$.

Since $V_i \subset V$, we may consider the natural inclusion $I_i : V_i \mapsto V$ defined by

$$(I_i u_i, v) = (u_i, v), \quad \forall \ u_i \in V_i, \ v \in V. \quad (6)$$

We notice that $Q_i = I_i^T$ as $(Q_i u, v_i) = (u, v_i)_i = (u, I_i^T v_i) = (I_i^T u, v_i)$. Similarly, we have $P_i = I_i^*$, where $I_i^*$ is the transpose of $I_i$ with respect to the inner product $(\cdot, \cdot)_A$ induced by $A$.

If $u$ is the solution of (1), then

$$A_i u_i = f_i,$$

where $u_i = P_i u$ and $f_i = Q_i f$. (7) can be viewed as the restriction of (1) on the subspace $V_i$, $i = 1, 2, \ldots, J$. MSC solves these subspace equations (7) iteratively. In general, these subspace equations are solved approximated. More precisely, we introduce a non-singular operator $R_i : V_i \mapsto V_i$, $i = 1, 2, \ldots, J$, which assumed to be an approximation of $A_i^{-1}$ in certain sense. And then the subspaces equation (7) are solved approximated by $u_i \approx \hat{u}_i = R_i f_i$.

As we pointed out in the introduction, there are two major type MSC, depending on how the error is corrected by solving subspace problems. These are the parallel subspace corrections (PSC) method and the successive subspace corrections (SSC) method. The PSC method is similar in nature to the classical Jacobi method, where the subspace equations are solved in parallel as in Algorithm 1. From the definitions in Algorithm 1 it is easy to see that

**Algorithm 1 Parallel Subspace Correction Method**

1: Compute the residual by $r^m = f - Au^m$.
2: Approximately solve the subspace equations $A_i e_i = Q_i r^m$ by $\hat{e}_i = R_i Q_i r^m$ in parallel.
3: Update the iteration by $u^{m+1} = u^m + \sum_{i=1}^J I_i \hat{e}_i$.

$$u^{m+1} = u^m + B_u (f - Au^m),$$
with
\[ B_m = \sum_{i=1}^{J} I_i R_i Q_i = \sum_{i=1}^{J} I_i R_i I_i', \] (8)

which is the operator corresponds to the PSC method.

The SSC method is similar to the classical Gauss-Seidel iterative method
and the error is corrected successively in every subspace as outlined in Alg-

\begin{algorithm}
\caption{Successive Subspace Correction Method}
\begin{algorithmic}[1]
\State Compute the residual by \( r^m = f - A u^m \).
\State Set \( v^0 = u^m \).
\For {\( k = 0 \rightarrow J - 1 \)}
\State \( v^{k+1} = v^k + R_{k} Q_{k} (f - A v^k) \),
\EndFor
\State Update \( u^{m+1} = v^J \).
\end{algorithmic}
\end{algorithm}

Let us define \( T_i = R_i Q_i A \), and we note that \( T_i : V_i \mapsto V_i \) is symmetric with respect to \( (\cdot, \cdot)_A \), nonnegative definite, and satisfies \( T_i = R_i A_i P_i \). Moreover, \( T_i = P_i \) if \( R_i = A_i^{-1} \). Using this notation, we have
\[ I - B_m A = (I - T_J)(I - T_{J-1}) \cdots (I - T_1), \] (9)

where \( B_m \) is the operator approximating \( A^{-1} \) which corresponds to the SSC method.

In this paper, we focus on the randomized and fault-tolerant versions of the SSC method \( B_m \). For its convergence analysis, we have the following well-

\begin{theorem}[XZ-identity] \label{thm:XZ}
Assume that \( B_m \) is defined by the SSC method (Algorithm 2), then we have
\[ \| I - B_m A \|_A^2 = 1 - \frac{1}{1 + c_0} = 1 - \frac{1}{c_1}, \] (10)

where
\[ c_0 = \sup_{\| v \|_A = 1} \inf_{\sum v_i = v} \sum_{i=1}^{J} \| R_i^t w_i \|_{T_i^{-1}}^2, \] (11)

with \( w_i = A_i P_i \sum_{j \geq i} v_j - R_i^{-1} v_i \), and
\[ c_1 = \sup_{\| v \|_A = 1} \inf_{\sum v_i = v} \sum_{i=1}^{J} \left\| T_i^{-1/2} \left( v_i + T_i^* P_i \sum_{j \geq i} v_j \right) \right\|_A^2, \] (12)

with \( T_i = T_i + T_i^* - T_i^* T_i \) and \( T_i = R_i A_i P_i \), \( 1 \leq i \leq J \).
Corollary 1 In case that the subspace problems are solved exactly, i.e. \( R_i = A_i^{-1} \), the X-Z identity holds with

\[
c_0 = \sup_{\|v\|_A = 1} \inf_{\sum v_i = v} \sum_{i=1}^{J} \left\| P_i \sum_{j\geq i} v_j \right\|_{A_i}^2.
\]

and

\[
c_1 = \sup_{\|v\|_A = 1} \inf_{\sum v_i = v} \sum_{i=1}^{J} \left\| P_i \left( \sum_{j\geq i} v_j \right) \right\|_{A}^2.
\]

3 Randomized and Fault-tolerant SSC

Traditionally, the SSC method visits each subspace in a pre-determined ordering, i.e., it solves subspace problems one by one in a fixed, problem-independent order. Here we consider to choose the ordering randomly, which is a key component of the algorithms. Another component we introduce into the SSC method is the fault-tolerant ability enabled by randomization. In this section, we formulate those algorithms and their convergence analysis are discussed in the next section.

3.1 Randomized SSC

In the randomized SSC method, we randomly choose the next subspace in which the error needs to be corrected. We randomly choose the subspace, according to certain probability distribution as in Algorithm 3.

**Algorithm 3** SSC method with random ordering (Version 1)

1: Randomly choose an index \( i \in \{1, 2, \ldots, J\} \) with probability \( p_i = \frac{1}{J} \).
2: \( u^{k+1} = u^k + R_i Q_i (f - A u^k) \)

As discussed in [5], the cost of randomly picking \( i \) does not exceed \( O(\log J) \) and each update in the SSC method can be done in \( O(N) \) operations where \( N \) is the dimension of the vector space \( V \). Therefore, the overall computational cost of the randomized SSC method is comparable to the standard SSC method which is a very desirable feature.

Note that in Algorithm 3 there is no guarantee that all the \( J \) subspaces are all corrected in \( J \) iterations. Therefore, we propose the second version randomized SSC method, such that the \( J \) subspaces are guaranteed to be corrected within \( J \) iterations by randomly choosing the ordering in which the error is corrected. To do this, we first consider the set of all permutations of \( J = \{1, 2, \ldots, J\} \). Then a permutation of \( J \) is any bijective mapping \( \sigma : J \mapsto J \).
The idea is to randomly choose a permutation $\sigma$ from the set of permutations and apply the SSC following the correction order as specified by $\sigma$. We have the randomized SSC method presented in Algorithm 4.

**Algorithm 4** SSC method with random ordering (Version 2)

1: Compute the residual by $r^m = f - Au^m$,
2: $v^0 = u^m$,
3: Randomly choose a permutation $\sigma$ of the indexes $J = \{1, 2, \cdots, J\}$ with probability $\frac{1}{J!}$,
4: for $k = 0 \rightarrow J - 1$ do
5: $v^{k+1} = v^k + R_{\sigma(k)}Q_{\sigma(k)}(f - Av^k)$,
6: end for
7: Update the iteration by $u^{m+1} = v^J$.

We need to note that, in Algorithm 4, the cost of randomly picking the permutation $\sigma$ is $O(\log J!) = O(J \log J)$ which is expensive than Algorithm 3. But each update still can be done in $O(N)$ operations. It is reasonable to assume that $J = O(N)$ at the worst case, therefore, the overall computational cost of Algorithm 4 is $O(N \log N)$ which is slightly expensive than traditional SSC method and Algorithm 3.

3.2 Fault-tolerant Randomized SSC

Another feature pertaining to this randomized SSC method is its fault-tolerance. During the iterative process, the correction or update may fail due to hard and/or soft errors, which, if not handled correctly, may result in a stagnating iterative method. We propose a simple approach which can handle all such scenarios (see Algorithm 5). Basically, we do not update the approximation to the solution during the iterations when error occurs. The randomization of the ordering helps to guarantee that such simple treatment leads to theoretically convergent iterative method.

**Algorithm 5** Fault-tolerant SSC method with random ordering

1: if error occurs then
2: $u^{k+1} = u^k$,
3: else
4: Randomly choose an index $i \in \{1, 2, \cdots, J\}$ with probability $p_i = \frac{1}{J}$,
5: $u^{k+1} = u^k + R_i Q_i(f - Au^k)$,
6: end if

Similar to Algorithm 4, the cost of randomly picking $i$ is $O(\log J)$ and each correction costs $O(N)$. Therefore, the overall cost of Algorithm 5 is comparable to the cost of the traditional SSC method and Algorithm 3.
4 Convergence Analysis

In this section, we discuss the convergence analysis of the randomized and fault-tolerant SSC methods (Algorithm 3-5). We want to emphasize that, instead of usual upper bound estimation, we present identities to estimate the convergence rate of the randomized and fault-tolerant SSC methods. We note that in the analysis below we relate the expected convergence rate of the SSC method to the quality of the PSC preconditioner, which is independent of the ordering. This is not surprising and shows rigorously the fact that the expected (average) convergence rate of an SSC method is also independent of the ordering.

4.1 Convergence Rate of the Randomized SSC

First, we consider Algorithm 3 and the main result is stated in the following theorem. Here, we use \( B_a \) to denote the operator corresponding to the PSC method with \( R_i = R_i^0 + R_i - R_i^0 A_i R_i \) as the inexact subspace solver.

**Theorem 2** The Algorithm 3 converges with the expected error decay rate,

\[
E(\|u - u^{k+1}\|_A^2) = (1 - \frac{\delta_k}{J}) E(\|u - u^k\|_A^2) = \prod_{\ell=0}^{k}(1 - \delta_\ell^J)\|u - u^0\|_A^2, \tag{15}
\]

where \( \delta_k = \frac{E((B_a A e_k, e_k) A)}{E((e_k, e_k) A)} > 0 \) and \( e_k = u - u^k \). Moreover, if \( \| I - T_i \|_A < 1 \), then \( \delta_k < J \).

**Proof** It is easy to see that, given \( i \), we have

\[
\|e^{k+1}\|_A^2 = \|(I - T_i)e^k\|_A^2 = ((I - \bar{T}_i)e^k, e^k)_A,
\]

where \( \bar{T}_i \) is the symmetrized version of \( T_i \). According to the way we pick the index \( i \), at iteration \( k \), consider the probability space \((\Omega_k, \mathcal{F}_k, P)\) where

\[
\Omega_k = \Omega \times \Omega \times \cdots \times \Omega, \quad \Omega := \{1, 2, \cdots, J\},
\]

\( \mathcal{F}_k \) contains all the subset of \( \Omega_k \), and \( P(a) = |a|/|\Omega_k|, a \in \mathcal{F}_k \). We define random variables \( S_k(x) : \Omega_k \mapsto \mathbb{R}^k \) such that,

\[
S_k(\omega_k) = \omega_k, \quad \omega_k = (i_1, i_2, \cdots, i_k) \in \Omega_k, \quad i_m \in \Omega, \quad m = 1, \cdots, k.
\]

Because we choose an index \( i \in \{1, 2, \cdots, J\} \) uniformly with probability \( 1/J \) and picking \( i \) is independent of the iteration number \( k \), we have,

\[
P(S_{k+1} = \omega_{k+1} | S_k = \omega_k) = \begin{cases} \frac{1}{J}, & \text{if } \omega_{k+1} = (\omega_k, i), \ i = 1, 2, \cdots, J \\ 0, & \text{otherwise} \end{cases}
\]
Then we define functions
\[ g_k(\omega) := \|(I - T_{i_k})(I - T_{i_{k-1}}) \cdots (I - T_{i_1}) e^0\|_A^2, \quad \omega_k = (i_1, i_2, \ldots, i_k) \]
and the compositions of \( g_k \) and \( S_k \) define other random variables which we denote by
\[ X_k = g_k \circ S_k, \]
and, for \( \omega_k \in \Omega_k \)
\[ X_k(\omega_k) = g_k(S_k(\omega_k)) = \|(I - T_{i_k})(I - T_{i_{k-1}}) \cdots (I - T_{i_1}) e^0\|_A^2 := \|e^k\|_A^2. \]

Then we compute the conditional expectation as following,
\[
E(X_{k+1} | S_k)(\omega_k) = E(X_{k+1} | S_k = \omega_k) \\
= \sum_{\omega_{k+1} \in \Omega_{k+1}} X_{k+1}(\omega_{k+1}) P(S_{k+1} = \omega_{k+1} | S_k = \omega_k) \\
= \sum_{i=1}^J \|(I - T_i)e^k\|_A^2 \frac{1}{J} \\
= \sum_{i=1}^J \frac{1}{J}((I - T_i)e^k, e^k)_A \\
= \|e^k\|_A^2 - \frac{1}{J} \left( \sum_{i=1}^J (T_i e^k, e^k)_A \right) \\
= \|e^k\|_A^2 - \frac{1}{J} (B_a A e^k, e^k)_A.
\]

Apply \( E(X) = E(E(X|Y)) \), use the linearity of the expectation, and let \( X = X_{k+1}, Y = S_k \), we then have
\[
E(\|e^{k+1}\|_A^2) = E(X_{k+1}) = E(E(X_{k+1}|S_k)) \\
= E(\|e^k\|_A^2 - \frac{1}{J} (B_a A e^k, e^k)_A) \\
= E(\|e^k\|_A^2) - \frac{1}{J} E((B_a A e^k, e^k)_A) \\
= \left(1 - \frac{1}{J} \frac{E((B_a A e^k, e^k)_A)}{E(\|e^k\|_A^2)}\right) E(\|e^k\|_A^2).
\]

Note that, if \( \|I - T_i\|_A < 1 \), we have \((B_a A e^k, e^k)_A = \left(\sum_{i=1}^J (T_i e^k, e^k)_A \right) < J(e^k, e^k)_A \) which implies that \( E((B_a A e^k, e^k)_A) < J E(e^k, e^k)_A \), i.e. \( \delta_k < J \). This completes the proof.

**Remark 1** Now we discuss about the constant \( \delta_k \), we have
\[
\lambda_{\min}(B_a A) \|e^k\|_A^2 \leq (B_a A e^k, e^k)_A \leq \lambda_{\max}(B_a A) \|e^k\|_A^2.
\]
Use the linearity and monotonicity of expectation, we have
\[ \lambda_{\text{min}}(B_a A) E(\|e^k\|_A^2) \leq E((B_a A e^k, e^k)_A) \leq \lambda_{\text{max}}(B_a A) E(\|e^k\|_A^2). \]

Therefore, we have
\[ \lambda_{\text{min}}(B_a A) \leq \delta_k \leq \lambda_{\text{max}}(B_a A), \]
and
\[ 1 - \frac{\delta_k}{J} \leq 1 - \frac{\lambda_{\text{min}}(B_a A)}{J}. \]

**Remark 2** After \( J \) steps, we have \( E(\|e_J\|_A^2) \leq \left(1 - \frac{\lambda_{\text{min}}(B_a A)}{J}\right)^J \|e^0\|_A^2 \) and the energy error reduction is bounded by \( \left(1 - \frac{\lambda_{\text{min}}(B_a A)}{J}\right)^J \leq \exp(-\lambda_{\text{min}}(B_a A)) \).

**Remark 3** Taking the multigrid method as an example, if we use deterministic approach, for any given \( i \), we can only get a reduction that is given by a smoother (such as Gauss-Seidel), namely
\[ \|e^{k+1}\|_A \leq (1 - c h^2)\|e^k\|_A. \]
But for randomized method, we have
\[ E(\|e^{k+1}\|_A) \leq (1 - c/|\log h|)E(\|e^k\|_A) \text{ since } J = O(|\log h|). \]
This is a significantly improvement.

**Remark 4** Here we choose next subspace used for correction based on a uniformly distributed random variable with probability constant probability \( p_i = 1/J \). However, one can use other choices as long as the probability does not depend on the iteration number \( k \) and similar results can be derived. Same statement also applied to the theoretical results that follow.

\[ E(\|u - u^{k+1}\|_A^2) \leq \delta_k E(\|u - u^0\|_A^2), \quad \delta = e^{-\lambda_{\text{min}}(B_a A)} \]

A special case of Algorithm 3 is that the subspace corrections are exact, i.e., \( R_i = A_i^{-1} \). And the following corollary is a direct consequence of Theorem 2.

**Corollary 2** Assume that the probabilities \( p_i, i \in J \) are independent of the iteration number \( k \). Then Algorithm 3 with \( R_i = A_i^{-1} \) converges with the following expected error reduction:
\[ E(\|u - u^{k+1}\|_A^2) = (1 - \frac{\delta_k}{J})E(\|u - u^k\|_A^2) = \prod_{\ell=0}^k (1 - \frac{\delta_k}{J})\|u - u^0\|_A^2, \quad (16) \]
where \( 0 < \delta_k = \frac{E(\sum_{i=1}^J |P_i e^k|_A^2)}{E(\|e^k\|_A^2)} < J. \)
Next theorem shows that the randomized SSC method converges almost surely, i.e., converges with probability 1, if all the subspace corrections are convergent.

**Theorem 3** If \( \|I - T_i\|_A < 1 \) for \( i = 1, 2, \ldots, J \), then Algorithm 3 converges almost surely or with probability 1, i.e., \( \|e_k\|_A^{2} \overset{a.s.}{\longrightarrow} 0 \).

**Proof** In order to show the almost sure convergence, we need to show that
\[
\sum_{k=1}^{\infty} P(\|e_k\|_A^2 \geq \varepsilon) < +\infty
\]
for any \( \varepsilon > 0 \). Note that, by Markov’s inequality, we have
\[
P(\|e_k\|_A^2 \geq \varepsilon) \leq \frac{1}{\varepsilon} E(\|e_k\|_A^2) = \frac{1}{\varepsilon} \prod_{\ell=0}^{k-1} \left( 1 - \frac{\delta_{\ell}}{J} \right) \|e_0\|_A^2,
\]
Since \( \|I - T_i\|_A < 1 \) and according to Remark 1, we have \( \lambda_{\min}(B_a A) < \delta_{\ell} < \lambda_{\max}(B_a A) < J \) and then,
\[
\sum_{k=1}^{\infty} \frac{1}{\varepsilon} \prod_{\ell=0}^{k-1} \left( 1 - \frac{\delta_{\ell}}{J} \right) \|e_0\|_A^2 \leq \frac{1}{\varepsilon} \|e_0\|_A^2 \sum_{k=1}^{\infty} \left( 1 - \frac{\lambda_{\min}(B_a A)}{J} \right)^{k-1}
\]
\[
= \frac{1}{\varepsilon} \|e_0\|_A^2 \frac{J}{\lambda_{\min}(B_a A)} < +\infty.
\]
Therefore, we have that the series \(\sum_{k=1}^{\infty} P(\|e_k\|_A^2 \geq \varepsilon)\) converges.

Next, we discuss the convergence rate for Algorithm 4. We introduce \( B_{\sigma} \) to denote the corresponding operator for \( J \) iterations using permutation \( \sigma \), i.e.,
\[
I - B_{\sigma} A = (I - T_{\sigma(J)}) (I - T_{\sigma(J-1)}) \cdots (I - T_{\sigma(1)}).
\]
We have the following theorem

**Theorem 4** Consider \( B_{\sigma} \) defined by Algorithm 4, we have
\[
E(\|I - B_{\sigma} A\|_A^2) = 1 - \frac{1}{J!} \sum_{i=1}^{J!} \frac{1}{c_{\sigma_i}}.
\]

**Proof** According to XZ-identity (10), for a permutation \( \sigma_i \), we have
\[
\|I - B_{\sigma_i} A\|_A^2 = 1 - \frac{1}{c_{\sigma_i}}.
\]
Here, we need to introduce a different probability space \( (\Omega, F, P) \) where \( \Omega = \{\sigma_1, \sigma_2, \ldots, \sigma_{J!}\} \), i.e., the set of all possible permutations, \( F \) is again the \( \sigma \)-algebra of \( \Omega \), and the probability \( P \) is defined by \( P(a) = |a|/J! \), \( a \in F \). Based on this probability space, we define a random variable \( X : \Omega \rightarrow \mathbb{R} \) as following,
\[
X(\sigma_i) = \|I - B_{\sigma_i} A\|_A^2 = 1 - \frac{1}{c_{\sigma_i}} =: x_{\sigma_i}, \quad i = 1, 2, \ldots, J!
\]
$X$ is a discrete random variable and its expectation can be computed as following

$$E(||I - B\sigma A||_A^2) = E(X) = \sum_{i=1}^{J!} x_{\sigma_i} P(X = x_{\sigma_i})$$

$$= \sum_{i=1}^{J!} \left(1 - \frac{1}{c_{\sigma_i}} \right) \frac{1}{J!}$$

$$= 1 - \frac{1}{J!} \sum_{i=1}^{J!} \frac{1}{c_{\sigma_i}},$$

which completes the proof.

**Remark 5** Note that

$$1 - \frac{1}{J!} \sum_{i=1}^{J!} \frac{1}{c_{\sigma_i}} \leq 1 - \frac{1}{c_{\text{max}}},$$

where $c_{\text{max}} = \max_{1 \leq i \leq J} c_{\sigma_i}$. The equality holds if and only if $c_{\sigma_i} = c_{\text{max}}$ for all $\sigma_i$. Therefore randomized SSC method Algorithm 4 “improves” the convergence rate of the traditional SSC method which only considers the worst case scenario.

### 4.2 Fault-tolerant Randomized SSC

In this section, we discuss the convergence rate of the fault-tolerant randomized SSC method (Algorithm 5). The main assumption is that the errors occur with probability $\theta \in [0, 1)$. Next theorem says that the fault-tolerant randomized SSC method converges in expectation. Again, we use $B_a$ to denote the operator corresponding to the PSC method with $R_i = R_i^t + R_i - R_i^t A R_i$ as the inexact subspace solver.

**Theorem 5** Assume that error occurs with probability $\theta \in [0, 1)$ which is independent of $k$ and how $i$ is picked, then the Algorithm 5 converges with the expected convergence rate,

$$E(||u - u^{k+1}||_A^2) = \left(1 - \frac{(1-\theta)\delta_k}{J} \right) E(||u - u^k||_A^2)$$

$$= \prod_{\ell=0}^{k} \left(1 - \frac{(1-\theta)\delta_\ell}{J} \right) ||u - u^0||_A^2, \quad (17)$$

where $\delta_k = \frac{E((B_a A e_k, e_k)_{A})}{E((e^*, e^*)_{A})} > 0$ and $e^k = u - u^k$. Moreover, if $||I - T_i||_A < 1$, then $\delta_k < J$. 
Proof Note that, if there is no error (with probability $1 - \theta$), for a given $i$, we have
\[ \|e^{k+1}\|_A^2 = \|(I - T_i)e^k\|_A^2 = ((I - \bar{T}) e^k, e^k)_A, \]
otherwise, we have
\[ \|e^{k+1}\|_A^2 = \|e^k\|_A^2. \]
Let us consider the probability space $(\Omega_k, \mathcal{F}_k, P)$ where
\[ \Omega_k = \Omega \times \Omega \times \cdots \times \Omega, \Omega := \{0, 1, 2, \cdots, J\}, \]
where $\{0\}$ means error occurs, which corresponds to the case that the if statement (line 1 of Algorithm 5) branching with “true” condition and following the line after then. The choice $\{i\}$, $i = 1, 2, \cdots, J$, means there is no error and that index $i$ is picked at random. This, latter case corresponds to the else statement (line 1 of Algorithm 5) and index $i$ is picked (line 4 of Algorithm 5).

$\mathcal{F}_k$ contains all the subset of $\Omega_k$. The probability $P$ is given in the following way: We set $P(\emptyset) = 0$. Next, $P(\{0\}) = \theta \in [0, 1)$ is the probability that error occurs. Further, $P(\{i\})$ is the probability that there is no error and index $i$ is picked. Assuming that all hard/soft errors occur independently from how $i$ is picked, we have $P(\{i\}) = P(\text{no error} \cap \text{index } i \text{ is picked}) = P(\text{no error}) P(i \text{ is picked}) = (1 - \theta)(1/J) = (1 - \theta)/J$. For other events $a \in \mathcal{F}$, $P(a)$ is defined using the countable additivity property.

As in the proof of Theorem 2, we define random variables $S_k(x) : \Omega_k \mapsto \mathbb{R}$ such that,
\[ S_k(\omega_k) = \omega_k, \quad \omega_k = (i_1, i_2, \cdots, i_k) \in \Omega_k, \quad i_m \in \Omega, \quad m = 1, \cdots, k. \]
Because we pick $i$ is independent of the iteration number $k$, we have,
\[ P(S_{k+1} = \omega_{k+1} | S_k = \omega_k) = \begin{cases} \theta, & \text{if } \omega_{k+1} = (\omega_k, 0), \\ \frac{1}{J}, & \text{if } \omega_{k+1} = (\omega_k, i), \quad i = 1, 2, \cdots, J \\ 0, & \text{otherwise} \end{cases} \]
Then we define functions
\[ g_k(\omega_k) := \|(I - T_{i_k})(I - T_{i_{k-1}}) \cdots (I - T_i)e^0\|_A^2, \quad \omega_k = (i_1, i_2, \cdots, i_k) \]
with $T_0 = 0$ and the compositions of $g_k$ and $S_k$ define other random variables which we denote by
\[ X_k = g_k \circ S_k, \]
and, for $\omega_k \in \Omega_k$
\[ X_k(\omega_k) = g_k(S_k(\omega_k)) = \|(I - T_{i_k})(I - T_{i_{k-1}}) \cdots (I - T_i)e^0\|_A^2 := \|e^k\|_A^2, \]
Therefore, then we compute the conditional expectation as following

\[ E(X_{k+1} | S_k) = E(X_{k+1} | S_k = \omega_k) = \sum_{\omega_{k+1} \in \Omega_{k+1}} X_{k+1}(\omega_{k+1})P(S_{k+1} = \omega_{k+1} | S_k = \omega_k) \]

\[ = \theta \|e^k\|^2_A + \sum_{i=1}^{J} \frac{(1 - \theta)}{J} \|(I - T_i)e^k\|^2_A \]

\[ = \theta \|e^k\|^2_A + \frac{1 - \theta}{J} J(e^k, e^k)_A - \sum_{i=1}^{J} \frac{1 - \theta}{J} (T_i e^k, e^k)_A \]

\[ = \|e^k\|^2_A - \frac{1 - \theta}{J} (B_A e^k, e^k)_A. \]

Following the proof of Theorem 2, we apply the identity

\[ E(X) = E(E(X|Y)) \]

and use the linearity of the expectation to derive (17). This completes the proof.

Remark 6 we can estimate the constant \( \delta_k \) as in Remark 1, i.e.

\[ \lambda_{\min}(B_A) \leq \delta_k \leq \lambda_{\max}(B_A). \]

Therefore, we have

\[ 1 - \frac{(1 - \theta) \delta}{J} \leq 1 - \frac{(1 - \theta) \lambda_{\min}(B_A)}{J}. \]

Remark 7 After \( J \) steps, we have

\[ E(\|e^J\|^2_A) \leq \left(1 - \frac{(1 - \theta) \lambda_{\min}(B_A)}{J}\right)^J \|e^0\|^2_A, \]

and the energy error reduction is bounded by

\[ \left(1 - \frac{(1 - \theta) \lambda_{\min}(B_A)}{J}\right)^J \approx \exp((\theta - 1) \lambda_{\min}(B_A)), \quad 0 \leq \theta < 1. \]

The following corollary consider a special case of Theorem 5 that all the subspace corrections are exact, i.e., \( R_i = A_i^{-1} \).

Corollary 3 Assume that errors occurs with probability \( \theta \in [0, 1] \) which is independent of \( k \), then Algorithm 5 with \( R_i = A_i^{-1} \) converges with the expected error decay rate,

\[ E(\|u - u^{k+1}\|^2_A) = \left(1 - \frac{(1 - \theta) \delta_k}{J}\right) E(\|u - u^k\|^2_A) \]

\[ = \prod_{\ell=0}^{\ell=k} \left(1 - \frac{(1 - \theta) \delta_\ell}{J}\right) \|u - u^0\|^2_A, \]  

(18)

where \( 0 < \delta_k = E(\sum_{i=1}^{J} ||P_i e^k||^2_A)/E(||e^k||^2_A) < J. \)
Next theorem shows that the fault-tolerant randomized SSC method converges almost surely or with probability 1 if all the subspace corrections are convergent.

**Theorem 6** Assume that $\|I - T_i\|_A < 1$ for $i = 1, 2, \cdots, J$. Moreover, assume that errors occur with probability $\theta \in [0, 1)$ and independent of $k$, Algorithm 5 converges almost surely or with probability 1, i.e., $\|e^k\|_A^2 \xrightarrow{a.s.} 0$.

**Proof** Following the proof of Theorem 3, we have

$$P(\|e^k\|_A^2 \geq \varepsilon) \leq \frac{1}{\varepsilon} E(\|e^k\|_A^2) = \frac{1}{\varepsilon} \prod_{\ell=0}^{k-1} \left( 1 - \frac{(1 - \theta)\delta_{\ell}}{J} \right) \|e^0\|_A^2.$$  

Since $\|I - T_i\|_A < 1$, therefore, we have $\lambda_{\text{min}} < \delta_{\ell} < \lambda_{\text{max}} < J$ and,

$$\sum_{k=1}^{\infty} \frac{1}{\varepsilon} \prod_{\ell=0}^{k-1} \left( 1 - \frac{(1 - \theta)\delta_{\ell}}{J} \right) \|e^0\|_A^2 \leq \frac{1}{\varepsilon} \|e^0\|_A^2 \sum_{k=1}^{\infty} \left( 1 - \frac{(1 - \theta)\lambda_{\text{min}}(B_aA)}{J} \right)^{k-1} \frac{1}{\varepsilon} \|e^0\|_A^2 \frac{J}{(1 - \theta)\lambda_{\text{min}}(B_aA)} < +\infty.$$  

Therefore, we have $\sum_{k=1}^{\infty} P(\|e^k\|_A^2 \geq \varepsilon) < +\infty$ which completes the proof.

Theorem 5 and 6 suggest that, when error occurs, simply do not update the solution or reject the update. The randomized SSC method are guaranteed to converge to the correct solution. Such property is useful for developing error resilience algorithms.

### 5 Application of Markov Chains

In this section, we try to explain the theory in terms of the discrete time Markov chains which may allow more general consideration of the randomized method of subspace correction.

Let us first consider Algorithm 3. In this case, $\Omega = \{1, 2, \cdots, J\}$ is considered as the state space and choosing an index $i \in \Omega$ at each iteration $k$ gives a sequence $\{S_k\}_{k \geq 0}$ of random variables with value in the set $\Omega$. Because we choose index $i$ independent of $k$ at each step, for every sequence $i_0, i_1, \cdots, i_{k-1}, i_k, i_{k+1}$ of elements of $\Omega$, $k > 0$, we have the following Markov property holds for $\{S_k\}_{k \geq 0}$.

$$P(S_{k+1} = i_{k+1}|S_k = i_k, S_{k-1} = i_{k-1}, \cdots, S_0 = i_0) = P(S_{k+1} = i_{k+1}|S_k = i_k).$$

Moreover, since we choose index uniformly at each iteration, we have,

$$P(S_{k+1} = i_{k+1}|S_k = i_k) = \frac{1}{J}.$$
This gives the transition matrix $P \in \mathbb{R}^{J \times J}$ such that
\[ p_{ij} = P(S_{k+1} = j|S_k = i) = \frac{1}{J}. \]

Note that
\[ P = \frac{1}{J} \mathbf{1} \mathbf{1}^{T}, \quad \text{and} \quad P^n = P, \quad n = 2, 3, \ldots \]
where $\mathbf{1} = (1, 1, \ldots, 1)^T$. Based on those definitions, we can see that the sequence $\{S_k\}_{k \geq 0}$ is indeed a Markov chain.

Now we try to prove Theorem 2 again using the language of Markov chain.

**Proof (Proof of Theorem 2)** As before, we define functions
\[ g_k(i_k) := \|(I - T_i) e^{k-1}\|_A^2, \quad i_k \in \Omega, \]
and the composition of $g_k$ and $S_k$ define a random variable $X_k = g_k \circ S_k$. Denote $g_k^{i_k} := g_k(i_k)$, for $i_k \in \Omega$, we can compute the conditional expectation as following
\[
E(X_{k+1}|X_k)(g_k^{i_k}) = E(X_{k+1}|X_k = g_k^{i_k}) \\
= \sum_{i_{k+1} \in \Omega} g_{k+1}^{i_{k+1}} P(X_{k+1} = g_{k+1}^{i_{k+1}}|X_k = g_k^{i_k}) \\
= \sum_{i_{k+1} \in \Omega} g_{k+1}^{i_{k+1}} P(S_{k+1} = i_{k+1}|S_k = i_k) \\
= \sum_{i_{k+1} \in \Omega} \|(I - T_{i_{k+1}}) e^k\|_A^2 \frac{1}{J} \\
= \sum_{i \in \Omega} \|(I - T_i) e^k\|_A^2 \frac{1}{J} \\
= \|e^k\|_A^2 - \frac{1}{J}(B_a A e^k, e^k)_A.
\]

Then apply $E(X) = E(E(X|Y))$ as before, we finish the proof.

**6 Conclusion**

We study the convergence behavior of the randomized subspace correction methods. In stead of the usual upper bound for the convergence rate, we derived an identity for the estimation of the expect error decay rate in energy norm and also show the the randomized algorithm converges almost surely if all the subspace correction converges.

We also propose another version randomized subspace correction method in which each subspace is corrected once within $J$ iterations. We theoretically prove that it is convergent by using the XZ-identity and show how it improves the standard SSC method at the worst case in terms of the convergence rate.
In order to improve the error resilience of the subspace correction methods, we develop a fault-tolerant variant of the randomized method by rejecting any correction when error occurs. We show that the fault-tolerant iterative method based on such approach converges with probability 1 if all the subspace corrections are convergent and, moreover, we also derive a sharp identity estimate for the convergence rate. These results show the intrinsic fault-tolerant features of the subspace correction method and its potential in extreme-scale computing by introducing randomization.

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