On the Drinfeld Centers of Fusion 2-Categories

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Abstract

We prove that the Drinfeld center of a fusion 2-category is invariant under Morita equivalence and under taking the 2-Deligne tensor product with an invertible fusion 2-category. We go on to show that the concept of Morita equivalence between connected fusion 2-categories recovers exactly the notion of Witt equivalence between braided fusion 1-categories. Then, we introduce the notion of separable fusion 2-category. Conjecturally, separability ensures that a fusion 2-category is 4-dualizable. We define the dimension of a fusion 2-category, which is a scalar whose non-vanishing is equivalent to separability. In addition, we prove that a fusion 2-category is separable if and only if its Drinfeld center is finite semisimple. We then establish the separability of every strongly fusion 2-category, that is fusion 2-category whose braided fusion 1-category of endomorphisms of the monoidal unit is Vect or SVect. We proceed to show that every fusion 2-category is Morita equivalent to the 2-Deligne tensor product of a strongly fusion 2-category and an invertible fusion 2-category. Finally, we prove that every fusion 2-category is separable.

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Introduction

Given a monoidal 1-category $\mathcal{C}$, one can form its Drinfeld center $\mathcal{Z}(\mathcal{C})$, that is the braided monoidal 1-category whose objects are pairs consisting of an object of $\mathcal{C}$ together with a coherent half-braiding. This construction categorifies the notion of the center of an algebra, and was first considered by Drinfeld in unpublished notes as an abstraction of the double construction for Hopf algebras [Dri87]. The construction of Drinfeld center as braided monoidal 1-category subsequently appeared in the category theory literature in [JS91], and its relation with the double of a Hopf algebra was made precise in [Maj91]. Namely, it was shown by Müger that the Drinfeld center of a fusion 1-category of non-zero global dimension is a braided fusion 1-category. Further, it was then proven in [ENO05] that every fusion 1-category over an algebraically closed field of characteristic zero has non-zero global dimension, so that the Drinfeld center of any fusion 1-category is a braided fusion 1-category. In addition, it was shown in [ENO09] that the Drinfeld center of a fusion 1-category completely characterizes the Morita equivalence class of the fusion 1-category.

The non-vanishing of the global dimension of fusion 1-categories, is a crucial property for the constructions of invariants of 3-manifolds. Namely, this hypothesis is needed in order to extend the Turaev-Viro construction to spherical fusion 1-categories [BW96], and to extend the Reshetikhin-Turaev construction to modular 1-categories [Tur92]. In fact, it was established in [DSPS21] that a fusion 1-category has non-zero global dimension if and only if its Drinfeld center is finite semisimple. More precisely, through the cobordism hypothesis of [BD95] and [Lur10], framed fully extended TQFTs can be constructed by specifying a fully dualizable object in a symmetric monoidal higher category. Conceptually, full dualizability should be thought of as a very strong finiteness condition. In [DSPS21], the authors considered the symmetric monoidal 3-category $\mathbf{TC}$ of finite tensor 1-categories, and exhibited fully dualizable objects, i.e. 3-dualizable objects, therein. More precisely, they showed that a fusion 1-category $\mathcal{C}$ is a fully dualizable object of $\mathbf{TC}$ if and only if it is separable, meaning that $\mathcal{C}$ as a $\mathcal{C}$-$\mathcal{C}$-bimodule 1-category is equivalent to the 1-category of modules over a separable algebra in $\mathcal{C} \boxtimes \mathcal{C}^\text{nop}$. In addition, it was shown in [DSPS21] that a
fusion 1-category is separable if and only if its Drinfeld center is finite semisimple. But, over an algebraically closed field of characteristic zero, it follows from [ENO05] that every fusion 1-category has non-vanishing global dimension. Thus, every fusion 1-category is separable and therefore yields a framed fully extended 3-dimensional TQFT.

Fusion 2-categories were introduced in [DR18] as a categorification of the notion of a fusion 1-category over an algebraically closed field of characteristic zero, and spherical fusion 2-categories were used to define a state-sum invariant of 4-manifolds. Further, Douglas and Reutter conjectured that fusion 2-categories are fully dualizable objects, that is 4-dualizable objects, in an appropriate symmetric monoidal 4-category. Using the theory of higher condensations introduced in [GJF19a], and under some assumptions on the behaviour of higher colimits, the construction of the appropriate symmetric monoidal 4-category was given in [JF22]. Further, Johnson-Freyd sketched a characterization of the fully dualizable objects of this 4-category. In a different context, it was proven in [BJS21] that every braided fusion 1-category is a fully dualizable object of the symmetric monoidal 4-category $\text{BrFus}$ of braided fusion 1-categories. This is related to the theory of fusion 2-categories. Namely, to any braided fusion 1-category $\mathcal{B}$, one can associate the fusion 2-category $\text{Mod}(\mathcal{B})$ of finite semisimple $\mathcal{B}$-module 1-categories. Such fusion 2-categories are called connected, and it was shown in [JFR21] that connected fusion 2-categories have finite semisimple Drinfeld centers.

In the present article, we study the properties of the Drinfeld centers of arbitrary, i.e. not necessarily connected, fusion 2-categories using the previous work of the author on fusion 2-categories in [Dec21a], [Dec21c], [Dec22c], and [Dec22a]. In particular, we show that a fusion 2-category is separable if and only if its Drinfeld center is finite semisimple. Moreover, we prove that, over an algebraically closed field of characteristic zero, every fusion 2-category is separable.

The Drinfeld Centers of Fusion 1-Categories

We now recall in more detail some of the relevant properties of the Drinfeld center of a fusion 1-category. Given any fusion 1-category $\mathcal{C}$, we can canonically view $\mathcal{C}$ as a left $\mathcal{C} \boxtimes \mathcal{C}^{\text{mop}}$-module 1-category. Thanks to a result of Ostrik [Ost03b], there exists an algebra $A$ in $\mathcal{C} \boxtimes \mathcal{C}^{\text{mop}}$ such that $\mathcal{C}$ as a left $\mathcal{C} \boxtimes \mathcal{C}^{\text{mop}}$-module 1-category is equivalent to the 1-category of right $A$-modules in $\mathcal{C} \boxtimes \mathcal{C}^{\text{mop}}$. We say that $\mathcal{C}$ is a separable fusion 1-category if $A$ is a separable algebra. In fact, $A$ admits a canonical Frobenius algebra structure, and $A$ is a special Frobenius algebra if and only if it is a separable algebra. It was established in [DSPS21] that $A$ is special if and only if the quantum dimension of $\mathcal{C}$ is non-zero. But, over an algebraically closed field of characteristic zero, it was proven in [ENO05] that every fusion 1-category has non-vanishing quantum dimension, so that every fusion 1-category is separable. On the one hand, it was shown in [Ost03a] that the Drinfeld center $\mathcal{Z}(\mathcal{C})$ is equivalent to the tensor 1-category of $\mathcal{C} \boxtimes \mathcal{C}^{\text{mop}}$-module endofunctors on $\mathcal{C}$. In particular, this implies that $\mathcal{Z}(\mathcal{C})$ is equivalent to the 1-category of
A-A-bimodules in $C \boxtimes C^{mop}$. On the other hand, the algebra $A$ is separable if and only if the associated 1-category of A-A-bimodules in $C \boxtimes C^{mop}$ is finite semisimple. This shows that, over an algebraically closed field of characteristic zero, the Drinfeld center of any fusion 1-category is a finite semisimple 1-category. Another key property of the Drinfeld center is that it is invariant under Morita equivalence of fusion 1-categories \cite{Ost03a}. This result was refined further in \cite{ENO11}, where it is shown that two fusion 1-categories are Morita equivalent if and only if their Drinfeld centers are equivalent as braided fusion 1-categories.

**The Drinfeld Centers of Fusion 2-Categories**

Let $C$ be a multifusion 2-category over an algebraically closed field of characteristic zero. The Drinfeld center $Z(C)$ of $C$ was defined in \cite{BN95}. The objects of $Z(C)$ are given by objects of $C$ equipped with a half braiding and coherence data. In particular, $Z(C)$ is canonically a braided monoidal 2-category. We establish our first theorem using the notion of Morita equivalence between multifusion 2-categories developed in \cite{Dec22a}.

**Theorem 2.3.2** Let $C$ and $D$ be Morita equivalent multifusion 2-categories, then there is an equivalence $Z(C) \cong Z(D)$ of braided monoidal 2-categories.

Given any two multifusion 2-categories $C$ and $D$, we can consider $C \boxtimes D$, the 2-Deligne tensor product of $C$ and $D$ as defined in \cite{Dec21a}, which is a multifusion 2-category. Using the above theorem we show that if $D$ is an invertible fusion 2-category, i.e. $Z(D) \cong 2\text{Vect}$, then $Z(C \boxtimes D) \cong Z(C)$.

We also investigate the notion of Morita equivalence between connected fusion 2-categories, that is fusion 2-categories of the form $\text{Mod}(B)$ with $B$ a braided fusion 1-category. Specifically, for any symmetric fusion 1-category $\mathcal{E}$, a notion of Witt equivalence between braided fusion 1-categories with symmetric center $\mathcal{E}$ was introduced in \cite{DNO13}. We obtain a 2-categorical characterization of this concept.

**Theorem 3.2.3** Let $B_1$ and $B_2$ be two braided fusion 1-categories. The fusion 2-categories $\text{Mod}(B_1)$ and $\text{Mod}(B_2)$ are Morita equivalent if and only if $B_1$ and $B_2$ have the same symmetric center $\mathcal{E}$ and they are Witt equivalent over $\mathcal{E}$.

This theorem generalizes example 5.4.6 of \cite{Dec22a}, where it is shown that if $B_1$ and $B_2$ are non-degenerate braided fusion 1-categories, then $\text{Mod}(B_1)$ and $\text{Mod}(B_2)$ are Morita equivalent fusion 2-categories if and only if $B_1$ and $B_2$ are Witt equivalent in the sense of \cite{DMNO13}. We also use the above result to classify Morita autoequivalences of connected symmetric fusion 2-categories.

Additionally, we study the properties of separable multifusion 2-categories. Any multifusion 2-category $C$ may be viewed as a left $C \boxtimes C^{mop}$-module 2-category. By the main theorem of \cite{Dec21c}, there exists a canonical rigid algebra $R_C$ in $C \boxtimes C^{mop}$ such that $C$ is equivalent to the 2-category of right $R_C$-modules in $C \boxtimes C^{mop}$. We say that $C$ is separable if $R_C$ is a separable algebra. Moreover, if $C$ is a fusion 2-category, we can use a construction from \cite{Dec22a} to obtain the
dimension of the algebra $R_C$. Accordingly, we define the dimension $\text{Dim}(\mathcal{C})$ of the fusion 2-category $\mathcal{C}$ to be the dimension of the canonical algebra $R_C$. Using these definitions, we establish the following categorification of theorem 2.6.7 of [DSPS21].

**Theorem 4.1.6** For a multifusion 2-category $\mathcal{C}$, the following are equivalent:

1. The multifusion 2-category $\mathcal{C}$ is separable.
2. The Drinfeld center $Z(\mathcal{C})$ of $\mathcal{C}$ is finite semisimple.

Further, if $\mathcal{C}$ is a fusion 2-category, this is also equivalent to:

3. The dimension $\text{Dim}(\mathcal{C})$ of $\mathcal{C}$ is non-zero.

Further, we outline an argument showing that separability is equivalent to 4-dualizability for fusion 2-categories. We then use the above theorem to give an alternative proof of the fact that the Drinfeld center of every connected fusion 2-category is finite semisimple, which was first established in [JFR21]. Further, recall from [JFY21] that a strongly fusion 2-category is a fusion 2-category whose associated braided fusion 1-category of endomorphisms of the monoidal unit is equivalent to the symmetric monoidal 1-category of (super) vector spaces. We prove that every strongly fusion 2-category is separable. Additionally, every invertible fusion 2-category is separable by definition, and we show that every invertible fusion 2-category is of the form $\text{Mod}(\mathcal{B})$ with $\mathcal{B}$ a non-degenerate braided fusion 1-category.

We prove our main theorems by assembling the above results. We note that the proof of the first theorem below is inspired by the classification of topological orders studied in [JF22], and that its proof relies on the minimal non-degenerate extension conjecture established in [JFR21].

**Theorem 4.2.3** Every fusion 2-category is Morita equivalent to the 2-Deligne tensor product of a strongly fusion 2-category and an invertible fusion 2-category.

But, we have shown that the Drinfeld center of a fusion 2-category is invariant under Morita equivalence and under taking the 2-Deligne tensor product with an invertible fusion 2-category. Thus, we find that the Drinfeld center of every fusion 2-category is equivalent to the Drinfeld center of a strongly fusion 2-category. In particular, this shows that, unlike in the case of fusion 1-categories, Morita equivalence classes of fusion 2-categories are not completely characterized by their Drinfeld centers. This failure comes from the existence of invertible fusion 2-categories whose Morita equivalence class is not trivial. Namely, there are non-degenerate braided fusion 1-categories whose Witt-equivalence class is non-trivial. In another direction, we also note that distinct strongly fusion 2-categories may have braided equivalent Drinfeld centers [JF]. Finally, as we have shown that every strongly fusion 2-category is separable, we derive the following theorem.

**Theorem 4.2.4** Every fusion 2-category is separable.
This last result has many immediate consequences. In particular, it implies that the Drinfeld center commutes with the 2-Deligne tensor product for fusion 2-categories.

Outline

In section 1, we review the definitions of a finite semisimple 2-category and of a fusion 2-category. We survey some examples of fusion 2-categories including connected fusion 2-categories and strongly fusion 2-categories. In addition, we recall the construction of the 2-Deligne 2-tensor product, as well as the definition of a separable algebra in a fusion 2-category. We also review the Morita theory of fusion 2-categories.

Next, in section 2, we begin by recalling the definition of the Drinfeld center of a monoidal 2-category, and go on to uncover its elementary properties. For instance, we show that the Drinfeld center of a rigid monoidal 2-category is a rigid monoidal 2-category. We go on to study more specifically the properties of the Drinfeld centers of multifusion 2-categories over an algebraically closed field of characteristic zero. In particular, we show that the Drinfeld center is invariant under Morita equivalence of fusion 2-categories. Furthermore, we also prove that the Drinfeld center is unaffected by taking the 2-Deligne tensor product with an invertible multifusion 2-category.

Then, in section 3 we examine in detail some examples of Morita equivalence between fusion 2-categories. More precisely, given a finite 2-group $G$, we show that the fusion 2-categories of $G$-graded 2-vector spaces and of 2-representations of $G$ are Morita equivalent. We proceed to establish that the notion of Morita equivalence between connected fusion 2-categories recovers exactly the concept of Witt equivalence between braided fusion 1-categories. We also classify Morita autoequivalences of connected symmetric fusion 2-categories.

In section 4, we define separable multifusion 2-categories, and we construct the dimension of a fusion 2-category. Then, we show that a multifusion 2-category is separable if and only if its Drinfeld center is a finite semisimple 2-category. Further, we prove that a fusion 2-category is separable if and only if its dimension is non-zero. In addition, we sketch an argument showing that separability is equivalent to 4-dualizability for fusion 2-categories. We also show that every strongly fusion 2-category is separable. We then establish that every fusion 2-category is Morita equivalent to the 2-Deligne tensor product of a strongly fusion 2-category with an invertible fusion 2-category. As a consequence, we show that every fusion 2-category is separable. We end by spelling out various corollaries of the fact that every fusion 2-category is separable. For instance, we deduce that every multifusion 2-category is separable, and we show that the Drinfeld center commutes with the 2-Deligne tensor product.

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1 Preliminaries

1.1 Finite Semisimple 2-Categories

We begin by reviewing the definition of a 2-condensation monad introduced in [GJF19a] as a categorification of the notions of an idempotent. More precisely, we recall the unpacked version of this definition given in section 1.1 of [Dèc22b].

**Definition 1.1.1.** A 2-condensation monad in a 2-category $C$ is an object $A$ of $C$ equipped with a 1-morphism $e: A \to A$ and two 2-morphisms $\mu: e \circ e \Rightarrow e$ and $\delta: e \circ e \Rightarrow e$ such that $\mu$ is associative, $\delta$ is coassociative, the Frobenius relations hold (i.e. $\delta$ is a 2-morphism of $e$-$e$-bimodules) and $\mu \cdot \delta = \text{Id}_e$.

Categorifying the notion of split surjection, [GJF19a] gave the definition of a 2-condensation, which we recall below. Further, we also review the definition of the splitting of a 2-condensation monad by a 2-condensation, which is spelled out in [Dèc22b].

**Definition 1.1.2.** A 2-condensation in a 2-category $C$ is a pair of objects $A, B$ in $C$ together with two 1-morphisms $f: A \to B$ and $g: B \to A$ and two 2-morphisms $\phi: f \circ g \Rightarrow \text{Id}_B$ and $\gamma: \text{Id}_B \Rightarrow f \circ g$ such that $\phi \cdot \gamma = \text{Id}_B$.

We say that a 2-condensation monad splits if it can be extended to a 2-condensation (see definition 1.1.4 of [Dèc22b] for details).

Let us now fix a field $k$. Following [GJF19a], we will call a $k$-linear 2-category locally Cauchy complete if its $\text{Hom}$-categories are Cauchy complete, that is they have direct sums and idempotents split. Furthermore, a Cauchy complete linear 2-category is a locally Cauchy complete 2-category that has direct sums for objects and 2-condensation monads split. Given an arbitrary locally Cauchy complete 2-category $C$, one can form its Cauchy completion $\text{Cau}(C)$, which is Cauchy complete (see [GJF19a], and also [DR18] for a slightly different approach). Further, it was shown in proposition 1.2.5 of [Dèc22b] that $\text{Cau}(C)$ is 3-universal with respect to linear 2-functors from $C$ to a Cauchy complete linear 2-category. Let us also mention that, given two Cauchy complete linear 2-category $C$ and $D$, one can form their Cauchy completed tensor product $C \hat{\otimes} D$, which is 3-universal with respect to bilinear 2-functor from $C \times D$ to a Cauchy complete linear 2-category (see proposition 3.4 of [Dèc22a]).

From now on, we will assume that $k$ is an algebraically closed field of characteristic zero. We are ready to review the definitions of a semisimple 2-category and of a finite semisimple 2-category introduced in [DR18] (more precisely, we review the variants given in [Dèc22b]).

**Definition 1.1.3.** A linear 2-category is semisimple if it is locally semisimple, has right and left adjoints for 1-morphisms, and is Cauchy complete.

An object $C$ of a semisimple 2-category $C$ is called simple if the identity 1-morphism $\text{Id}_C$ is a simple object of the semisimple 1-category $\text{End}_C(C)$.

**Definition 1.1.4.** A semisimple linear 2-category is finite if it is locally finite semisimple and it has finitely many equivalence classes of simple objects.
Given a multifusion 1-category $\mathcal{C}$, the 2-category $\text{Mod}(\mathcal{C})$ of finite semisimple right $\mathcal{C}$-module 1-categories is a finite semisimple 2-category (see theorem 1.4.8 of [DR18]). Moreover, by theorem 1.4.9 of [DR18], every finite semisimple 2-category is of this form. If $\mathcal{C}$ and $\mathcal{D}$ are finite semisimple, then it follows from theorem 3.7 of [Déc21a] that $\mathcal{C} \boxtimes \mathcal{D}$ is a finite semisimple 2-category, which we denote by $\mathcal{C} \Join \mathcal{D}$, and call the 2-Deligne tensor product of $\mathcal{C}$ and $\mathcal{D}$.

Given two simple objects $C$ and $D$ of a finite semisimple 2-category $\mathcal{C}$. The finite semisimple 1-category $\text{Hom}_\mathcal{C}(C, D)$ may be non-zero. If this is the case, we say that $C$ and $D$ are connected. We emphasize that two non-equivalent simple objects may be connected. This is in sharp contrast with the decategorified setting, in which the Schur lemma prevents such behaviour. Further, it was shown in section 1.2.3 of [DR18] that being connected defines an equivalence relation on the set of simple objects.

**Definition 1.1.5.** Let $\mathcal{C}$ be a finite semisimple 2-category. We write $\pi_0(\mathcal{C})$ for the quotient of the set of simple objects of $\mathcal{C}$ under the relation of being connected.

We say that a finite semisimple 2-category $\mathcal{C}$ is connected if $\pi_0(\mathcal{C})$ is a singleton. For instance, if $\mathcal{C}$ is a fusion 1-category, then $\text{Mod}(\mathcal{C})$ is connected. Further, we will abuse the terminology, and identify a connected component, i.e. an equivalence class in $\pi_0(\mathcal{C})$, with the finite semisimple sub-2-category of $\mathcal{C}$ that is spanned by the simple objects in this equivalence class and full on 1-morphisms.

### 1.2 Fusion 2-Categories

We recall the definition of a fusion 2-category first introduced in [DR18] (more precisely, we review the variant considered in [Déc22d]), and give a number of key examples. Throughout, we work over $k$, an algebraically closed field of characteristic zero.

**Definition 1.2.1.** A multifusion 2-category is a rigid monoidal finite semisimple 2-category. A fusion 2-category is multifusion 2-category whose monoidal unit is a simple object.

**Example 1.2.2.** We write $\mathbf{2Vect}$ for the 2-category of finite semisimple ($k$-linear) 1-categories, also called 2-vector spaces. The Deligne tensor product endows $\mathbf{2Vect}$ with the structure of a fusion 2-category.

**Example 1.2.3.** Let $\mathcal{B}$ be a braided fusion 1-category, then the relative Deligne tensor product over $\mathcal{B}$ endows the 2-category $\text{Mod}(\mathcal{B})$ with a rigid monoidal structure, so that $\text{Mod}(\mathcal{B})$ is a connected fusion 2-category. In fact, every connected fusion 2-category is of this form (see section 2.4 of [Déc22d]). Further, connected fusion 2-categories are ubiquitous. Namely, if $\mathcal{C}$ is an arbitrary fusion 2-category, then $\mathcal{C}^0$, the connected component of the identity of $\mathcal{C}$, is a connected fusion 2-category by proposition 2.4.5 of [Déc22d]. More precisely, if we let $\Omega \mathcal{C}$ denote the braided fusion 1-category of endomorphisms of the monoidal of $\mathcal{C}$, then $\mathcal{C}^0 \cong \text{Mod}(\Omega \mathcal{B})$ as fusion 2-categories.
Example 1.2.4. Let $\mathcal{G}$ be a finite 2-group. We write $B\mathcal{G}$ for the 2-category with one object, and endomorphisms given by $\mathcal{G}$. We can consider the finite semisimple 2-category $\text{Fun}(\mathcal{G}, 2\text{Vect})$ of (finite) 2-representations of $\mathcal{G}$, denoted by $2\text{Rep}(\mathcal{G})$. The symmetric monoidal structure of $2\text{Vect}$ endows $2\text{Rep}(\mathcal{G})$ with the structure of a symmetric fusion 2-category (see construction 2.1.12 of [DR18]).

The following classes of fusion 2-categories were first singled out in [JFY21], and will play a key role in the proof of our main theorem.

Definition 1.2.5. A bosonic strongly fusion 2-category is fusion 2-category $\mathcal{C}$ for which $\Omega\mathcal{C} \cong \text{Vect}$ as braided fusion 1-categories. A fermionic strongly fusion 2-category is fusion 2-category $\mathcal{C}$ for which $\Omega\mathcal{C} \cong \text{SVect}$, the symmetric fusion 1-category of super vector spaces.

Example 1.2.6. Let $G$ be a finite group. We use $2\text{Vect}_G$ to denote the finite semisimple 2-category of $G$-graded 2-vector spaces. The convolution product turns $2\text{Vect}_G$ into a fusion 2-category. Furthermore, given a 4-cocycle $\pi$ for $G$ with coefficients in $k$, we can form the fusion 2-category $2\text{Vect}_G^\pi$ by twisting the structure 2-isomorphisms of $2\text{Vect}_G$ using $\pi$ (see construction 2.1.16 of [DR18]). In fact, it follows from theorem A of [JFY21] that every bosonic strongly fusion 2-category is of this form.

Remark 1.2.7. Let us write $2\text{SVect} := \text{Mod}(\text{SVect})$ for the fusion 2-category of super 2-vector spaces. Give a finite group $G$, we can consider the fusion 2-category $2\text{SVect}_G$ of $G$-graded super 2-vector spaces. One can use a 4-cocycle for $G$ with coefficients in the (extended) supercohomology of $[GJF19b]$ to obtain another fusion 2-category by twisting the coherence 2-isomorphisms of $2\text{SVect}_G$. Such fusion 2-categories are examples of fermionic strongly fusion 2-categories. However, not every strongly fusion 2-category is of this form as can be seen from example 2.1.27 of [DR18]. Nonetheless, theorem B of [JFY21] shows that any connected component of a fermionic strongly fusion 2-category is equivalent to $2\text{SVect}$.

Example 1.2.8. Given a finite 2-group $\mathcal{G}$, we can consider the fusion 2-category $2\text{Vect}_\mathcal{G}$ of $\mathcal{G}$-graded 2-vector spaces (see construction 2.1.13 of [DR18]). Succinctly, $2\text{Vect}_\mathcal{G}$ is constructed as follows. We define a monoidal 2-category by $\mathcal{X} := \mathcal{G} \times B^2k$, the free $k$-linear monoidal 2-category on $\mathcal{G}$. We then write $\tilde{\mathcal{X}}$ for its local Cauchy completion, and finally set $2\text{Vect}_\mathcal{G} := \text{Cau}(\tilde{\mathcal{X}})$.

Example 1.2.9. Let $G$ be a finite group, and $\pi$ a 4-cocycle for $G$ with coefficients in $k^\times$. We may consider the braided monoidal 2-category $\mathcal{Z}(2\text{Vect}_G^\pi)$, the Drinfeld center of the fusion 2-category $2\text{Vect}_G^\pi$. We review this definition in section 2.1 below. It was shown directly in [KTZ20] that $\mathcal{Z}(2\text{Vect}_G^\pi)$ is a finite semisimple 2-category. In fact, $\mathcal{Z}(2\text{Vect}_G^\pi)$ is rigid as we will show in corollary 2.2.2 so that it really is a fusion 2-category.

Example 1.2.10. Let $\mathcal{C}$ and $\mathcal{D}$ be two fusion 2-categories. It follows from theorem 5.6 and lemma 5.7 of [Déc21a] that their 2-Deligne tensor product,
which we denote by $\mathcal{C} \boxtimes \mathcal{D}$, is a fusion 2-category. In addition, it follows from lemma 4.3 of [Dec21a] that $(\mathcal{C} \boxtimes \mathcal{D})^0 \simeq \mathcal{C}^0 \boxtimes \mathcal{D}^0$. Further, by proposition 4.1 of [Dec21a], we have that $\Omega(\mathcal{C} \boxtimes \mathcal{D}) \simeq \Omega \mathcal{C} \boxtimes \Omega \mathcal{D}$ as braided fusion 1-categories.

### 1.3 Rigid and Separable Algebras

Let us fix $\mathcal{C}$ a fusion 2-category (over an algebraically closed field of characteristic zero), with monoidal product $\Box$ and monoidal unit $I$. An algebra, also called pseudo-monoid in [DS97], in $\mathcal{C}$ is an object $A$ of $\mathcal{C}$ equipped with a multiplication 1-morphism $m : A \Box A \to A$, a unit 1-morphism $i : I \to A$, and coherence 2-isomorphisms ensuring that $m$ is associative and unital. We focus our attention on the following refinement of the notion of an algebra, which was introduced in [Gai12], and was first considered in the context of fusion 2-categories in [JFR21].

**Definition 1.3.1.** A rigid algebra in $\mathcal{C}$ is an algebra $A$ whose multiplication 1-morphism $m : A \Box A \to A$ has a right adjoint $m^*$ as an $A$-$A$-bimodule 1-morphism.

For our purposes, it is necessary to single out the following class of rigid algebras introduced in [JFR21], and whose origin can be traced back to [GJF19a].

**Definition 1.3.2.** A separable algebra in $\mathcal{C}$ is a rigid algebra $A$ in $\mathcal{C}$ equipped with a section of the counit $m \circ m^* \Rightarrow Id_A$ as an $A$-$A$-bimodule 2-morphism.

As we will briefly recall below, separable algebras can be used to define the notion of Morita equivalence between fusion 2-categories. This is thanks to the following result, which is a combination of theorem 3.1.6 of [Dec22c] and corollary 2.2.3 of [Dec21b].

**Theorem 1.3.3.** Let $A$ be a rigid algebra in a fusion 2-category $\mathcal{C}$. Then, $A$ is separable if and only if $\text{Bimod}_\mathcal{C}(A)$, the 2-category of $A$-$A$-bimodules in $\mathcal{C}$, is finite semisimple. Further, if either of these conditions is satisfied, $\text{Mod}_\mathcal{C}(A)$, the 2-category of right $A$-modules in $\mathcal{C}$, is finite semisimple.

It is in general difficult to prove that a rigid algebra is separable. Nonetheless, if $A$ is a connected rigid algebra, meaning that $i : I \to A$ is a simple 1-morphism, then one can associate to $A$ a scalar $\text{Dim}_\mathcal{C}(A)$ (see section 3.2 of [Dec22c]). It was established in theorem 3.2.4 of [Dec22c] that $A$ is separable if and only if $\text{Dim}_\mathcal{C}(A)$ is non-zero. This criterion can be effectively used to give examples of separable algebras.

**Example 1.3.4.** Rigid algebras in $2\text{Vect}$ are precisely multifusion 1-categories. It follows from corollary 2.6.8 of [DSPS21] that every such rigid algebra is separable. More generally, if $G$ is a finite group, rigid algebras in $2\text{Vect}_G$ are exactly $G$-graded multifusion 1-categories. It was argued in example 5.4.4 of [Dec21c] that every such rigid algebra is separable.

**Example 1.3.5.** Let $\mathcal{B}$ be a braided fusion 1-category. In the terminology of [BJS21], a $\mathcal{B}$-central monoidal 1-category is a monoidal linear 1-category $\mathcal{C}$.
equipped with a braided monoidal functor \( F : \mathcal{B} \to \mathcal{Z}(\mathcal{C}) \) to the Drinfeld center of \( \mathcal{C} \). By lemma 2.1.4 of [Dec22c], rigid algebras in \( \text{Mod}(\mathcal{B}) \) are exactly \( \mathcal{B} \)-central multifusion 1-categories. By combining proposition 3.3.3 of [Dec22c] together with theorem 2.3 of [ENO05], we find that every \( \mathcal{B} \)-central fusion 1-category is separable. More generally, every \( \mathcal{B} \)-central multifusion 1-category is separable as we explain below.

Namely, let \( \mathcal{C} \) be a \( \mathcal{B} \)-central multifusion 1-category. We use \( \mathcal{C}^{\text{mop}} \) to denote the multifusion 1-category obtained from \( \mathcal{C} \) by reversing the order of the monoidal product. Note that there is a canonical \( \mathcal{B} \)-central structure on \( \mathcal{C}^{\text{mop}} \). Then, inspection shows that there is an equivalence

\[
\text{Bimod}_{\text{Mod}(\mathcal{B})}(\mathcal{C}) \simeq \text{Mod}(\mathcal{C}^{\text{mop}} \boxtimes \mathcal{B} \mathcal{C})
\]

of 2-categories. Moreover, it follows from theorem 6.2 of [Gre10] that \( \mathcal{C}^{\text{mop}} \boxtimes \mathcal{B} \mathcal{C} \) is a rigid monoidal 1-category. Further, it is finite semisimple by proposition 3.5 of [ENO09]. This shows that \( \mathcal{C} \) is indeed separable.

**Remark 1.3.6.** The 2-category \( \text{Bimod}_{\text{Mod}(\mathcal{B})}(\mathcal{C}) \) has essentially already appeared in [Lau20]. In particular, the objects of \( \text{Bimod}_{\text{Mod}(\mathcal{B})}(\mathcal{C}) \) are exactly finite semisimple \( \mathcal{B} \)-balanced \( \mathcal{C} \)-\( \mathcal{C} \)-bimodule 1-categories in the sense of definition 3.24 of [Lau20].

**Example 1.3.7.** Let \( G \) be a finite group. Inspection shows that rigid algebras in \( \mathcal{Z}(\mathcal{2Vect}_G) \) precisely \( G \)-crossed multifusion 1-categories in the sense of [Gal17] (see [Tur00] for the original definition). By corollary 3.3.6 of [Dec22c], every \( G \)-crossed fusion 1-categories is a separable algebra in \( \mathcal{Z}(\mathcal{2Vect}) \). Moreover, braided rigid algebras in \( \mathcal{Z}(\mathcal{2Vect}_G) \) are exactly \( G \)-crossed braided multifusion 1-categories in the sense of [Gal17] (see also [Tur00]).

### 1.4 The Morita Theory of Fusion 2-Categories

We end this section by reviewing the notion of Morita equivalence between fusion 2-categories introduced in [Dec22a]. Given a fusion 2-category \( \mathcal{E} \), a left \( \mathcal{E} \)-module 2-category is a 2-category \( \mathcal{M} \) equipped with a coherent left action by \( \mathcal{E} \) (see definition 2.1.3 of [Dec21c] for details). If \( \mathcal{M} \) is a finite semisimple 2-category, then it follows from theorems 5.1.5 and 5.4.7 of [Dec21c] that there exists a rigid algebra \( A \) in \( \mathcal{E} \), together with an equivalence \( \mathcal{M} \simeq \text{Mod}_\mathcal{E}(A) \) of left \( \mathcal{E} \)-module 2-categories. We say that \( \mathcal{M} \) is separable if \( A \) is separable, and note that this property does not depend on the algebra \( A \) (see proposition 5.1.3 of [Dec22a]).

Given a separable left \( \mathcal{E} \)-module 2-category \( \mathcal{M} \), we use \( \mathcal{E}_\mathcal{M} \) to denote \( \text{End}_\mathcal{E}(\mathcal{M}) \), the monoidal 2-category of left \( \mathcal{E} \)-module 2-endofunctors on \( \mathcal{M} \) (see theorem 4.1.7 of [Dec22a]), and call it the dual tensor 2-category to \( \mathcal{E} \) with respect to \( \mathcal{M} \). On the other hand, given a separable algebra \( A \) in \( \mathcal{E} \), it follows from theorem 3.2.8 of [Dec22a] that the finite semisimple 2-category \( \text{Bimod}_\mathcal{E}(A) \) has a canonical monoidal structure given by the relative tensor product over \( A \). The following result is then given by theorem 5.3.2 of [Dec22a] combined with corollary 2.2.4 of [Dec21b].
Theorem 1.4.1. Let $A$ be a separable algebra in a fusion 2-category $\mathcal{C}$. Then,

$$\text{End}_\mathcal{C}(\text{Mod}_\mathcal{C}(A)) \simeq \text{Bimod}_\mathcal{C}(A)^{mop}$$

is a multifusion 2-category.

In order to ensure that the multifusion 2-category of theorem 1.4.1 is fusion, we need to impose a condition on the algebra. More precisely, we say that an algebra is indecomposable if it cannot be written as the direct sum of two non-trivial algebras. For instance, any connected algebra is indecomposable. Likewise, we use an analogous terminology for module 2-categories. The next result is obtained by combing theorem 5.4.3, which characterizes Morita equivalence between multifusion 2-categories, and section 5.2 of [Déc22a].

Theorem 1.4.2. For any two fusion 2-categories $\mathcal{C}$ and $\mathcal{D}$, the following are equivalent:

1. There exists an indecomposable separable left $\mathcal{C}$-module 2-category $\mathcal{M}$, and an equivalence of monoidal 2-categories $\mathcal{D}^{mop} \simeq \mathcal{C}_\mathcal{M}$.

2. There exists an indecomposable separable algebra $A$ in $\mathcal{C}$, and an equivalence of monoidal 2-categories $\mathcal{D} \simeq \text{Bimod}_\mathcal{C}(A)$.

If either of the above conditions is satisfied, we say that $\mathcal{C}$ and $\mathcal{D}$ are Morita equivalent. Furthermore, Morita equivalence defines an equivalence relation between fusion 2-categories.

For later use, we also record the following technical lemmas.

Lemma 1.4.3. Let $\mathcal{C}$ and $\mathcal{D}$ be two multifusion 1-categories. For any separable algebras $A$ in $\mathcal{C}$ and $B$ in $\mathcal{D}$, there is an equivalence

$$\text{Mod}_\mathcal{C}(A) \boxtimes \text{Mod}_\mathcal{D}(B) \simeq \text{Mod}_{\mathcal{C} \boxtimes \mathcal{D}}(A \boxtimes B)$$

of left $\mathcal{C} \boxtimes \mathcal{D}$-module 2-categories.

Proof. Thanks to the 3-universal property of the 2-Deligne tensor product obtained in theorem 3.7 of [Déc21a], the bilinear 2-functor

$$(M, N) \mapsto M \boxtimes N$$

induces a linear 2-functor $K : \text{Mod}_\mathcal{C}(A) \boxtimes \text{Mod}_\mathcal{D}(B) \to \text{Mod}_{\mathcal{C} \boxtimes \mathcal{D}}(A \boxtimes B)$. Furthermore, this 2-functor is compatible with the left $\mathcal{C} \boxtimes \mathcal{D}$-module structures. Now, it follows from the proof of proposition 3.1.2 of [Déc22c] that $\text{Mod}_{\mathcal{C} \boxtimes \mathcal{D}}(A \boxtimes B)$ is generated under Cauchy completion by the full sub-2-category on the objects of the form $(C \boxtimes D) \boxtimes (A \boxtimes B)$ with $C$ in $\mathcal{C}$ and $D$ in $\mathcal{D}$. Likewise, it follows from its construction that $\text{Mod}_\mathcal{C}(A) \boxtimes \text{Mod}_\mathcal{D}(B)$ is generated under Cauchy completion by the full sub-2-category on the objects of the form $(C \boxtimes A) \boxtimes (D \boxtimes B)$ for every $C$ in $\mathcal{C}$ and $D$ in $\mathcal{D}$. Furthermore, $K ((C \boxtimes A) \boxtimes (D \boxtimes B)) = (C \boxtimes D) \boxtimes (A \boxtimes B)$, and it follows from proposition 4.1 of [Déc21a] that $K$ induces an equivalence between these two full sub-2-categories. This establishes that $K$ is an equivalence as desired. \[\square\]
Lemma 1.4.4. Let $\mathcal{C}$ and $\mathcal{D}$ be two multifusion 1-categories. For any separable algebras $A$ in $\mathcal{C}$ and $B$ in $\mathcal{D}$, there is an equivalence

$$\text{Bimod}_\mathcal{C}(A) \boxtimes \text{Bimod}_\mathcal{D}(B) \simeq \text{Bimod}_{\mathcal{C} \boxtimes \mathcal{D}}(A \boxtimes B)$$

of multifusion 2-categories.

Proof. The proof is analogous to that of the previous lemma. \qed

Remark 1.4.5. In particular, the 2-Deligne tensor product is compatible with Morita equivalences of multifusion 2-categories.

2 The Drinfeld Center

In this section, we begin by reviewing the definition of the Drinfeld center of a monoidal 2-category. We go on to prove some of its elementary properties. Finally, we show that the Drinfeld center of a multifusion 2-category is invariant under Morita equivalence and taking the 2-Deligne tensor product with an invertible multifusion 2-category.

2.1 Definition

Let $\mathcal{C}$ be a monoidal 2-category in the sense of [SP11] with monoidal product $\Box$ and monoidal unit $I$. Thanks to [Gur13], we may assume without loss of generality that $\mathcal{C}$ is strict cubical in that it satisfies definition 2.26 of [SP11]. This ensures that almost all of the coherence data for $\mathcal{C}$ is given by identities, so that we will often omit the symbols $\Box$ and $I$. The only exception being the interchanger 2-isomorphism $\phi_{\Box}$ witnessing that the 2-functor $\Box$ respects the composition of 1-morphisms. We now recall from [Cra98] the definition of the Drinfeld center $\mathcal{Z}(\mathcal{C})$ of $\mathcal{C}$, which is a braided monoidal 2-category (see also [BN95]). We do so using the variant of the graphical calculus of [GS16] introduced in [Dec21c]. For a version the Definition of the Drinfeld center of a general monoidal 2-category, we refer the reader to [KTZ20].

An object of $\mathcal{Z}(\mathcal{C})$ consists of an object $Z$ of $\mathcal{C}$ equipped with a half braiding, that is an adjoint 2-natural equivalence $b_Z$ given on $C$ in $\mathcal{C}$ by

$$(b_Z)_C : Z \Box C \xrightarrow{\simeq} C \Box Z,$$

together with an invertible modification $R_Z$ given on $C, D$ in $\mathcal{C}$ by

such that, for every objects $C, D, E$ in $\mathcal{C}$, we have
A 1-morphism in $Z(C)$ from $(Y, b_Y, R_Y)$ to $(Z, b_Z, R_Z)$ consists of a 1-morphism $f : Y \to Z$ in $C$ together with an invertible modification $R_f$ given on $C$ in $C$ by

$$\begin{array}{ccc}
Y \square C & \overset{b_Y}{\longrightarrow} & C \square Y \\
\downarrow f1 & \Rightarrow & \downarrow 1f \\
Z \square C & \overset{b_Z}{\longrightarrow} & C \square Z
\end{array}$$

such that, for every objects $C, D$ in $C$, we have

$$\begin{array}{ccc}
f1 & \gamma & b_Y \\
b_g & \Rightarrow & 1f \\
R_f & \Rightarrow & b_Y \\
\end{array} = \begin{array}{ccc}
f1 & \gamma & b_Y \\
b_g & \Rightarrow & 1f \\
R_f & \Rightarrow & b_Y \\
\end{array}$$

in $Hom_C(YC, CDZ)$.

A 2-morphism in $Z(C)$ from $(f, R_f)$ to $(g, R_g)$, two 1-morphisms from $(Y, b_Y, R_Y)$ to $(Z, b_Z, R_Z)$, is a 2-morphism $\gamma : f \Rightarrow g$ in $C$ such that, for every object $C$ in $C$, we have

$$\begin{array}{ccc}
f1 & \gamma & b_Y \\
b_g & \Rightarrow & 1f \\
R_f & \Rightarrow & b_Y \\
\end{array} = \begin{array}{ccc}
f1 & \gamma & b_Y \\
b_g & \Rightarrow & 1f \\
R_f & \Rightarrow & b_Y \\
\end{array}$$

in $Hom_C(YC, CZ)$.

The identity 1-morphism on the object $(Z, b_Z, R_Z)$ of $Z(C)$ is given by $(Id_Z, Id_{b_Z})$. Given two 1-morphisms $(f, R_f)$ from $(X, b_X, R_X)$ to $(Y, b_Y, R_Y)$ and $(g, R_g)$ from $(Y, b_Y, R_Y)$ to $(Z, b_Z, R_Z)$ in $Z(C)$ their composite is the 1-morphism $(g \circ f, (R_g \circ f1) \cdot (1g \circ R_f))$. This endows $Z(C)$ with the structure of a strict 2-category.

The monoidal structure of $Z(C)$ is given as follows. The monoidal unit is $I$, the monoidal unit of $C$, equipped with the identity adjoint 2-natural equivalence, and identity modification. The monoidal product of two objects $(Y, b_Y, R_Y)$ and $(Z, b_Z, R_Z)$ of $Z(C)$ is given by

$$(Y, b_Y, R_Y) \square (Z, b_Z, R_Z) = (Y \square Z, (b_Y \square 1) \circ (1b_Z), R_{YZ}),$$

where $R_{YZ}$ is the invertible modification given on $C, D$ in $C$ by
in \( \text{Hom}_\mathcal{C}(YZCD, CDYZ) \). The monoidal product of two 1-morphisms \((f, R_f)\) and \((g, R_g)\) of \( \mathcal{Z}(\mathcal{C}) \) is given by \((f, R_f) \Box (g, R_g) = (f \Box g, (R_f \Box g) \cdot (f \Box R_g))\). Finally, the monoidal product of two 2-morphisms \(\gamma\) and \(\delta\) in \( \mathcal{Z}(\mathcal{C}) \) is given by \(\gamma \Box \delta\). Using the interchange \(\phi^\Box\) of the monoidal product of \(\mathcal{C}\), we can upgrade the above assignment to a 2-functor that defines a strict cubical monoidal structure on \( \mathcal{Z}(\mathcal{C}) \) as in \cite{Cra98}. Furthermore, it follows from the definitions that the forgetful 2-functor \( Z(\mathcal{C}) \to \mathcal{C} \) is monoidal.

Finally, in the notations of \cite{SP11}, the braiding on \( \mathcal{Z}(\mathcal{C}) \) is given as follows.

For any two objects \((Y, b_Y, R_Y)\), and \((Z, b_Z, R_Z)\) of \( \mathcal{Z}(\mathcal{C}) \), we define the braiding \(b\) by \(b_{Y,Z} : (b_Y)_Z\). Further, \(b\) is promoted to an adjoint 2-natural equivalence in an obvious using the definitions of the objects and 1-morphisms in \( \mathcal{Z}(\mathcal{C}) \). For any three objects \((X, b_X, R_X), (Y, b_Y, R_Y),\) and \((Z, b_Z, R_Z)\) of \( \mathcal{Z}(\mathcal{C}) \), we also define invertible modifications \(R_{X,Y,Z} := (R_X)_{Y,Z}\) and \(S_{X,Y,Z} := \text{Id}_{X,Y,Z}\). It follows from \cite{Cra98} that these assignments define a braiding on \( \mathcal{Z}(\mathcal{C}) \).

### 2.2 Elementary Properties

As \( \mathcal{C} \) is a monoidal 2-category, it has a canonical structure of \( \mathcal{C} \)-\( \mathcal{C} \)-bimodule 2-category (see definition 2.1.1 of \cite{Dec21c}). We begin by the following familiar observation.

**Lemma 2.2.1.** There is an equivalence \( \mathcal{Z}(\mathcal{C}) \simeq \text{End}_{\mathcal{C}-\mathcal{C}}(\mathcal{C}) \) of monoidal 2-categories.

*Proof.* This is a straightforward verification. We leave the details to the keen reader. \( \square \)

Using \( \mathcal{C}^{\text{op}} \) to denote \( \mathcal{C} \) with the opposite monoidal structure, we may equivalently think of \( \mathcal{C} \) as a \( \mathcal{C} \)-\( \mathcal{C} \)-bimodule 2-category or as a left \( \mathcal{C} \times \mathcal{C}^{\text{op}} \)-module 2-category. By combining the above lemma with propositions 4.2.3 and 4.2.4 of \cite{Dec22a}, we therefore obtain the following result.

**Corollary 2.2.2.** If \( \mathcal{C} \) is a rigid monoidal 2-category, then its Drinfeld center \( \mathcal{Z}(\mathcal{C}) \) is a rigid monoidal 2-category.

We also record the following lemma, where \( \mathcal{Z}(\mathcal{C})^{\text{rev}} \) denotes the braided monoidal 2-category obtained from \( \mathcal{Z}(\mathcal{C}) \) by using the opposite braiding. More precisely, given \((Y, b_Y, R_Y)\) and \((Z, b_Z, R_Z)\) in \( \mathcal{Z}(\mathcal{C})^{\text{rev}} \), the braiding on the objects is given by \( b_{Y,Z}^\ast : Y \Box Z \to Z \Box Y \), the prescribed adjoint equivalence of \( b_{Y,Z} \).
Lemma 2.2.3. There is an equivalence $\mathcal{Z}(\mathcal{C}^{\text{mop}}) \simeq \mathcal{Z}(\mathcal{C})^{\text{rev}}$ of braided monoidal 2-categories.

Proof. This follows by direct inspection.

From now on, we will work over a fixed field $k$. In particular, we will assume that $\mathcal{C}$ is a monoidal $k$-linear 2-category.

Lemma 2.2.4. Let $\mathcal{C}$ be a Cauchy complete monoidal 2-category. Then, $\mathcal{Z}(\mathcal{C})$ is Cauchy complete.

Proof. Let $(Y,e,\mu,\delta)$ be a 2-condensation monad in $\mathcal{Z}(\mathcal{C})$. By hypothesis, the underlying 2-condensation monad in $\mathcal{C}$ can be split by a 2-condensation $(Y,Z,f,g,\phi,\gamma)$ and 2-isomorphism $\theta : g \circ f \cong e$. We begin by upgrading $Z$ to an object of $\mathcal{Z}(\mathcal{C})$. To this end, recall that splittings of 2-condensation monads are preserved by all 2-functors. In particular, for any $C$ in $\mathcal{C}$, $b_{Y,C} : Y \Box C \simeq C \Box Y$ induces an equivalence between two 2-condensation monads. In particular, by the 2-universal property of the splitting of 2-condensation monads (see theorem 2.3.2 of [GJF19a]), this induces an equivalence $b_{Z,C} : Z \Box C \simeq C \Box Z$. Further, thanks to the 2-universal property these 1-morphisms assemble to given an adjoint 2-natural equivalence $b_{Z,C}$. Likewise, the invertible modification $R_{Y,C}$ induces an invertible modification $R_{Z,C}$ on $Z$, which satisfies equation (1). Similarly, one can endow both $f$ and $g$ with the structures of 1-morphisms in $\mathcal{Z}(\mathcal{C})$, and check that $\phi$, $\gamma$, and $\theta$ are 2-morphisms in $\mathcal{Z}(\mathcal{C})$. This concludes the proof.

Let $\mathcal{C}$ and $\mathcal{D}$ be two Cauchy complete monoidal linear 2-categories. It follows from theorem 5.2 of [Dec21a] that the completed tensor product $\mathcal{C} \hat{\otimes} \mathcal{D}$ inherits a monoidal structure. It is therefore natural to attempt to compare its Drinfeld center with that of $\mathcal{C}$ and $\mathcal{D}$. In this direction, we prove the following technical result.

Lemma 2.2.5. Let $\mathcal{C}$ and $\mathcal{D}$ be two Cauchy complete monoidal 2-categories. Then, there is a canonical braided monoidal 2-functor $\mathcal{Z}(\mathcal{C}) \hat{\otimes} \mathcal{Z}(\mathcal{C}) \to \mathcal{Z}(\mathcal{C} \hat{\otimes} \mathcal{D})$ that commutes with the forgetful monoidal 2-functors to $\mathcal{C} \hat{\otimes} \mathcal{D}$ up to equivalence.

Proof. We claim that there is a bilinear 2-functor

$$
\mathcal{Z}(\mathcal{C}) \times \mathcal{Z}(\mathcal{D}) \to \mathcal{Z}(\mathcal{C} \hat{\otimes} \mathcal{D}).
$$

Namely, given $Y$ in $\mathcal{Z}(\mathcal{C})$ and $Z$ in $\mathcal{Z}(\mathcal{D})$, we can equip the object $Y \hat{\otimes} Z$ of $\mathcal{C} \hat{\otimes} \mathcal{D}$ with the adjoint 2-natural equivalence $b_Y \otimes b_Z$, which is defined for all objects of $\mathcal{C} \hat{\otimes} \mathcal{D}$ of the form $C \otimes D$. But, by the construction given in the proof of proposition 3.4 of [Dec21a], $\mathcal{C} \hat{\otimes} \mathcal{D}$ is the Cauchy completion of its full sub-2-category on the objects of the form $C \otimes D$ with $C$ in $\mathcal{C}$ and $D$ in $\mathcal{D}$. Thus, thanks to the 3-universal property of the Cauchy completion (see [Dec22a]), $b_Y \otimes b_Z$ can be canonically extended to an adjoint 2-natural equivalence defined on all the objects of $\mathcal{C} \hat{\otimes} \mathcal{D}$. An analogous argument shows that the invertible modification $R_Y \otimes R_Z$ can be canonically extended to all of $\mathcal{C} \hat{\otimes} \mathcal{D}$. Further, it follows from its
construction that this extension satisfies equation \([1]\), so that \(Y \otimes Z\) is indeed an object of \(\mathcal{F}(\mathcal{C} \otimes \mathcal{D})\). This assignment is extended to morphisms using the same techniques.

Thanks to lemma \(2.2.4\) above and the 3-universal property of the completed tensor product obtained in proposition 3.4 of [Déc21a], the bilinear 2-functor above induces a linear 2-functor \(\mathcal{F}(\mathcal{C}) \otimes \mathcal{F}(\mathcal{D}) \rightarrow \mathcal{F}(\mathcal{C} \otimes \mathcal{D})\). By construction, this is a braided monoidal 2-functor, and it is compatible with the forgetful monoidal 2-functors. □

2.3 Morita Invariance

Let us now fix \(\mathcal{C}\) a multifusion 2-category over an algebraically closed field \(k\) of characteristic zero, and \(\mathcal{M}\) a separable left \(\mathcal{C}\)-module 2-category. Thanks to proposition 2.2.8 of [Déc21c], we may assume without loss of generality that \(\mathcal{C}\) and \(\mathcal{M}\) are strict cubical in the sense of definition 2.2.7 of [Déc21c]. Now, it is shown in proposition 4.1.10 of [Déc22a] that \(\mathcal{M}\) can be canonically viewed as a left \(\mathcal{C} \times \mathcal{C}_\mathcal{M}^\ast\)-module 2-category. Further, this left action is multilinear, so that we may view \(\mathcal{M}\) as a left \(\mathcal{C} \boxtimes \mathcal{C}_\mathcal{M}^\ast\)-module 2-category.

**Proposition 2.3.1.** Let \(\mathcal{M}\) be a faithful separable left \(\mathcal{C}\)-module 2-category. Then, there is an equivalence

\[(\mathcal{C} \boxtimes \mathcal{C}_\mathcal{M}^\ast)^\ast_\mathcal{M} \simeq \mathcal{F}(\mathcal{C})\]

of monoidal 2-categories.

**Proof.** By definition, we have that \((\mathcal{C} \boxtimes \mathcal{C}_\mathcal{M}^\ast)^\ast_\mathcal{M} = \text{End}_{\mathcal{C} \boxtimes \mathcal{C}_\mathcal{M}^\ast}(\mathcal{M})\). In fact, as the left action of \(\mathcal{C} \times \mathcal{C}_\mathcal{M}^\ast\) on \(\mathcal{M}\) is multilinear, it follows from the 3-universal property of the 2-Deligne tensor product obtained in [Déc21c] that \((\mathcal{C} \boxtimes \mathcal{C}_\mathcal{M}^\ast)^\ast_\mathcal{M} \simeq \text{End}_{\mathcal{C} \times \mathcal{C}_\mathcal{M}^\ast}(\mathcal{M})\). Further, note that left \(\mathcal{C} \times \mathcal{C}_\mathcal{M}^\ast\)-module 2-endofunctors on \(\mathcal{M}\) are exactly left \(\mathcal{C}_\mathcal{M}^\ast\)-module 2-endofunctors on \(\mathcal{M}\) equipped with a left \(\mathcal{C}\)-module structure, whose coherence data consists of left \(\mathcal{C}_\mathcal{M}^\ast\)-module 2-natural transformations and left \(\mathcal{C}_\mathcal{M}^\ast\)-module modifications. More succinctly, \(\mathcal{C} \times \mathcal{C}_\mathcal{M}^\ast\)-module 2-endofunctors are exactly \(\mathcal{C}_\mathcal{M}^\ast\)-module left \(\mathcal{C} \times \mathcal{C}_\mathcal{M}^\ast\)-module left actions by \(\mathcal{C}\) and \(\mathcal{C}_\mathcal{M}^\ast\). Similar descriptions hold for left \(\mathcal{C} \times \mathcal{C}_\mathcal{M}^\ast\)-module 2-natural transformations, as well as left \(\mathcal{C} \times \mathcal{C}_\mathcal{M}^\ast\)-module modifications.

Given \((Z, b_Z, R_Z)\) in \(\mathcal{F}(\mathcal{C})\), we can consider the 2-endofunctor \(J(M)\) on \(\mathcal{M}\) given by \(M \mapsto Z \square M\). For any \(F\) in \(\mathcal{C}_\mathcal{M}^\ast\), the left \(\mathcal{C}\)-module structure on \(F\) yields an adjoint 2-natural equivalence \((\chi^{(Z)}_{F,M})^\ast : F(Z \square M) \simeq Z \square F(M)\) and invertible modification, which endow \(J(Z)\) with a canonical left \(\mathcal{C}_\mathcal{M}^\ast\)-module structure. Furthermore, \(J(Z)\) carries a compatible left \(\mathcal{C}\)-module structure given by \([\psi_{Z,C,C} : C \square Z \square M \simeq Z \square \square M]\). This assignment can be extended to morphisms in the obvious way, so that we have a 2-functor \(J : \mathcal{F}(\mathcal{C}) \rightarrow \text{End}_{\mathcal{C} \times \mathcal{C}_\mathcal{M}^\ast}(\mathcal{M})\). Further, it follows by inspection that \(J\) is monoidal.

Now, corollary 5.4.4 of [Déc22a] implies that the canonical monoidal 2-functor \(\mathcal{C} \rightarrow \text{End}_{\mathcal{C}_\mathcal{M}^\ast}(\mathcal{M})\) given by \(C \mapsto \{M \mapsto C \square M\}\) is an equivalence. In particular, objects of \(\text{End}_{\mathcal{C} \times \mathcal{C}_\mathcal{M}^\ast}(\mathcal{M})\) are identified with 2-endofunctors \(M \mapsto Z \square M\) for
some $Z$ in $\mathcal{C}$ equipped with a left action by $\mathcal{C}$. More precisely, for every $C$ in $\mathcal{C}$, we have a left $\mathcal{C}^*_M$-module adjoint 2-natural equivalence $C \square Z \square M \simeq Z \square C \square M$. By the above equivalence of monoidal 2-categories, this corresponds exactly to an adjoint 2-natural equivalence $b^*_Z, C : C \square Z \simeq Z \square C$. An analogous argument shows that the invertible modification witnessing the coherence of the left $\mathcal{C}$-action corresponds precisely to the data of an invertible modification $R_Z$ satisfying (1). This shows that $J$ is essentially surjective on objects. One proceeds similarly to show that $J$ is essentially surjective on 1-morphisms and fully faithful on 2-morphisms. This concludes the proof.

We are now ready to prove the main theorem of this section.

**Theorem 2.3.2.** Let $\mathcal{C}$ and $\mathcal{D}$ be Morita equivalent multifusion 2-categories, then there is an equivalence $\mathcal{Z}(\mathcal{C}) \simeq \mathcal{Z}(\mathcal{D})$ of braided monoidal 2-categories.

**Proof.** Thanks to theorem 5.4.3 of [Déc22a], there exists a faithful separable left $\mathcal{C}$-module 2-category $\mathcal{M}$ and an equivalence $\mathcal{D} \simeq (\mathcal{C}^*_M)^{\text{mop}}$ of monoidal 2-categories. Thus, by lemma [2.2.3] it is enough to prove that $\mathcal{Z}(\mathcal{C}^*_M)^{\text{rev}} \simeq \mathcal{Z}(\mathcal{C})$. Thanks to the proof of proposition [2.3.1] we know that the canonical monoidal 2-functor $J : \mathcal{Z}(\mathcal{C}) \to \text{End}_{\mathcal{C} \times \mathcal{C}^*_M}(\mathcal{M})$ that sends $Z$ in $\mathcal{Z}(\mathcal{C})$ to $M \mapsto Z \square M$ is an equivalence. But, thanks to corollary corollary 5.4.4 of [Déc22a], we have that $\mathcal{D}^*_M \simeq \mathcal{C}^{\text{mop}}$ as monoidal 2-categories. This implies that the canonical monoidal 2-functor $K : \mathcal{Z}(\mathcal{C}^*_M) \to \text{End}_{\mathcal{C} \times \mathcal{C}^*_M}(\mathcal{M})$ that sends $F$ in $\mathcal{Z}(\mathcal{C}^*_M)$ to $M \mapsto F(M)$ is an equivalence.

Now, given any object $Z$ in $\mathcal{Z}(\mathcal{C})$, the left $\mathcal{C}$-module 2-endofunctor $M \mapsto Z \square M$ can be canonically upgraded to an object of $\mathcal{Z}(\mathcal{C}^*_M)$. Namely, for every object $F$ of $\mathcal{C}^*_M$, the left $\mathcal{C}$-module structure on $F$ provides us with an adjoint 2-natural equivalence $\chi^*_Z, M : Z \square F(M) \simeq F(Z \square M)$, and an invertible modification satisfying (1). This define a half braiding on $M \mapsto Z \square M$, and yields a monoidal 2-functor $L : \mathcal{Z}(\mathcal{C}) \to \mathcal{Z}(\mathcal{C}^*_M)$. Further, observe that the following diagram of monoidal 2-functors is weakly commutative:

\[
\begin{array}{ccc}
\mathcal{Z}(\mathcal{C}) & \xrightarrow{L} & \mathcal{Z}(\mathcal{C}^*_M) \\
J & \text{L} & \text{K} \\
\text{End}_{\mathcal{C} \times \mathcal{C}^*_M}(\mathcal{M}).
\end{array}
\]

It follows from the first part of the proof that both $J$ and $K$ are equivalences, so that $L$ is an equivalence of monoidal 2-categories.

Finally, let $(Y, b_Y, R_Y)$ and $(Z, b_Z, R_Z)$ be objects of $\mathcal{Z}$. By the definition recalled in section [2.1] above, the braiding on $L(Y)$, and $L(Z)$ in $\mathcal{Z}(\mathcal{C}^*_M)$ is given by

\[
b_{L(Y), L(Z)}(M) = \chi^*_{Y, M} = b^*_Z, Y \square M : Y \square Z \square M \to Z \square Y \square M.
\]

Thus, $L$ yields an equivalence $\mathcal{Z}(\mathcal{C}) \to \mathcal{Z}(\mathcal{C}^*_M)^{\text{rev}}$ of braided monoidal 2-categories. This concludes the proof. □
Remark 2.3.3. The proof of theorem 2.3.2 constructs for every Morita equivalence between \(\mathcal{C}\) and \(\mathcal{D}\) an equivalence \(\mathcal{Z}(\mathcal{C}) \simeq \mathcal{Z}(\mathcal{D})\) of braided monoidal 2-categories. With \(\mathcal{C} = \mathcal{D}\), we expect that this assignment defines a group homomorphism \(\Phi : \text{BrPic}(\mathcal{C}) \to \text{Aut}^{br}(\mathcal{Z}(\mathcal{C}))\), from the group of Morita autoequivalences of \(\mathcal{C}\) to the group of braided monoidal autoequivalences of \(\mathcal{Z}(\mathcal{C})\). This is a partial categorification of theorem 1.1 of [ENO09]. On the other hand, it is well-known that two fusion 1-categories are Morita equivalent if and only if their Drinfeld centers are braided equivalent (see theorem 3.1 of [ENO11]). This is not the case for fusion 2-categories! Namely, it follows from lemma 2.16 and theorem 2.5.2 of [JFR21] that, for any non-degenerate braided fusion 1-category \(\mathcal{B}\), there is an equivalence \(\mathcal{Z}(\text{Mod}(\mathcal{B})) \simeq \mathcal{2Vect}\) of braided fusion 2-category (see also proposition 4.17 of [DN21]). Further, by example 5.4.6 of [Dec22a], \(\text{Mod}(\mathcal{B})\) is Morita equivalent to \(\mathcal{2Vect}\) if and only if \(\mathcal{B}\) is Witt equivalent to \(\mathcal{Vect}\). But, the Witt group of non-degenerate braided fusion 1-categories is known to be non-trivial (see [DMNO13]).

Corollary 2.3.4. There is an equivalence of braided fusion 2-categories
\[ \mathcal{Z}(\mathcal{2Vect}_G) \simeq \mathcal{Z}(\mathcal{2Rep}(G)) .\]

Remark 2.3.5. Corollary 2.3.4 has one particularly noteworthy consequence, which we now explain. Namely, as \(\mathcal{Z}(\mathcal{2Vect}_G)\) and \(\mathcal{Z}(\mathcal{2Rep}(G))\) are equivalent as braided monoidal 2-categories, the associated (monoidal) 2-categories of algebras, and their homomorphisms are equivalent. In particular, this induces an equivalence between the full sub-2-categories on the rigid algebras. On one hand, rigid algebras in \(\mathcal{Z}(\mathcal{2Vect}_G)\) are precisely \(G\)-crossed multifusion 1-categories. On the other hand, rigid algebras in \(\mathcal{Z}(\mathcal{2Rep}(G))\) are exactly multifusion 1-categories \(\mathcal{C}\) equipped with two braided monoidal functor \(\text{Rep}(G) \Rightarrow \mathcal{Z}(\mathcal{C})\) lifting \(\text{Rep}(G) \to \mathcal{C}\). This correspondence appears to be new.

In addition, we also get an equivalence between the 2-categories of braided rigid algebras. More precisely, we get an equivalence between the 2-category of \(G\)-crossed braided multifusion 1-categories (see example 1.3.7) and that of braided multifusion 1-categories equipped with a braided monoidal functor from \(\text{Rep}(G)\) (see table 1 of [DN21]). This is a well-known result in the theory of fusion 1-categories (see [Kir01] and [Müg04]).

We end this section by establishing the following technical result, which is one of the key ingredients of the proof of our main theorem. In order to state this corollary, we will need the following definition.

Definition 2.3.6. A multifusion 2-category \(\mathcal{C}\) is invertible if \(\mathcal{Z}(\mathcal{C}) \simeq \mathcal{2Vect}\).

Remark 2.3.7. Let \(\mathcal{C}\) be an invertible multifusion 2-category. It follows from lemma 2.2.1 that the Morita equivalence class of \(\mathcal{C} \boxtimes \mathcal{C}^{mop}\) is trivial. This justifies the name. In fact, thanks to theorem 2.3.2, the latter gives an equivalent characterization of invertibility.

Corollary 2.3.8. Let \(\mathcal{C}\) be an arbitrary multifusion 2-category, and \(\mathcal{D}\) be an invertible multifusion 2-category. Then, there is an equivalence of braided monoidal 2-categories \(\mathcal{Z}(\mathcal{C} \boxtimes \mathcal{D}) \simeq \mathcal{Z}(\mathcal{C})\).
Proof. We show that the canonical braided monoidal 2-functor $F : \mathcal{Z}(C) \hat{\otimes} \mathcal{Z}(D) \to \mathcal{Z}(C \boxtimes D)$ of lemma 2.2.5 is an equivalence. To see this, note that the canonical braided monoidal 2-functor $H : \mathcal{Z}(C) \hat{\otimes} \mathcal{Z}(D) \hat{\otimes} \mathcal{Z}(D^{mop}) \to \mathcal{Z}(C \boxtimes D \boxtimes D^{mop})$ is an equivalence. Namely, by assumption, we have $\mathcal{Z}(C) \hat{\otimes} \mathcal{Z}(D) \hat{\otimes} \mathcal{Z}(D^{mop}) \simeq \mathcal{Z}(C)$. Further, $D \boxtimes D^{mop}$ is Morita equivalent to $2Vect$ by hypothesis, so that, thanks to theorem 2.3.2, we have $\mathcal{Z}(C \boxtimes D \boxtimes D^{mop}) \simeq \mathcal{Z}(C)$. Inspection shows that, up to these equivalences, the 2-functor $H$ is identified with the identity braided monoidal 2-functor on $\mathcal{Z}(C)$.

Furthermore, $H$ can be factored as the composite of both

$$F \simeq F \hat{\otimes} Id : \mathcal{Z}(C) \hat{\otimes} \mathcal{Z}(D) \hat{\otimes} \mathcal{Z}(D^{mop}) \to \mathcal{Z}(C \boxtimes D) \hat{\otimes} \mathcal{Z}(D^{mop}),$$

$$G : \mathcal{Z}(C \boxtimes D) \hat{\otimes} \mathcal{Z}(D^{mop}) \to \mathcal{Z}(C \boxtimes D \boxtimes D^{mop}).$$

This shows that $G$ is a left pseudo-inverse to $F$. On the other hand, we can run the above argument starting with $C \boxtimes D$ and $D^{mop}$, which shows that $G$ has a left pseudo-inverse. This implies that $G$, and therefore also $F$, is an equivalence. □

3 Examples of Morita Equivalences

In this section, we examine two classes of examples of Morita equivalences between fusion 2-categories. Throughout, we work over an algebraically closed field $k$ of characteristic zero.

3.1 Fusion 2-Categories Associated to Finite 2-Groups

Let $G$ be a finite 2-group, and observe that there is a monoidal 2-functor $U : 2Vect_G \to 2Vect$. Namely, by definition, $2Vect_G$ is the Cauchy completion of the monoidal 2-category $B^G \times B^2 k$. The monoidal 2-functor $U$ is induced by taking the Cauchy completion of the canonical monoidal 2-functor $B^G \times B^2 k \to B^2 k$. In particular, $2Vect$ is canonically a left $2Vect_G$-module 2-category.

Proposition 3.1.1. The dual of $2Vect_G$ with respect to $2Vect$ is $2Rep(G)$.

Proof. As $2Vect_G$ is the Cauchy completion of the monoidal 2-category $B^G \times B^2 k$, it follows from the 3-universal property of the Cauchy completion (see [Dec21a]) that the data of a left $2Vect_G$-module structure on a 2-functor, 2-natural transformation, or modification is equivalent to the data of a left $B^G \times B^2 k$-module structure. Moreover, all the 2-functors that we consider are linear, so that the latter structures are completely determined by their underlying $B^G$-module structures. This means that we have an equivalence of monoidal 2-categories

$$\text{End}_{2Vect_G}(2Vect) \simeq \text{End}_{B^G}(2Vect).$$
Furthermore, as $\mathbf{2Vect} \simeq \text{Mod}(\mathbf{Vect})$, any left $B\mathcal{G}$-module 2-endofunctor on $\mathbf{2Vect}$ is completely determined by its value on $\mathbf{Vect}$, which is a 2-representation of $\mathcal{G}$. Thus, there is an equivalence of 2-categories

$$\text{End}_{B\mathcal{G}}(\mathbf{2Vect}) \to \mathbf{2Rep}(\mathcal{G}).$$

Its pseudo-inverse is given by the 2-functor sending the 2-representation $R$ of $\mathcal{G}$ to the 2-endofunctor $F_R$ on $\mathbf{2Vect}$ given by $F_R(V) := R \boxtimes V$ with left $B\mathcal{G}$-module structure induced by the action of $\mathcal{G}$ on $R$. Direct inspection shows that this pseudo-inverse is a monoidal 2-functor, so that there is an equivalence of monoidal 2-categories

$$\text{End}_{B\mathcal{G}}(\mathbf{2Vect}) \simeq \mathbf{2Rep}(\mathcal{G}).$$

This concludes the proof as $\mathbf{2Rep}(\mathcal{G})$ is symmetric monoidal.

Remark 3.1.2. The above proposition can be slightly generalized. Namely, monoidal (linear) 2-functors $B\mathcal{G} \times B^2 k \to B^2 k$ are parametrised up to equivalence by $H^3(B\mathcal{G}, k^\times)$. Thus, given a 3-cocycle $\gamma$ for $\mathcal{G}$ with coefficients in $k^\times$, we can define a monoidal 2-functor $U^\gamma : \mathbf{2Vect}_G \to \mathbf{2Vect}$. This endows $\mathbf{2Vect}$ with a left $\mathbf{2Vect}_G$-module structure, and we write $\mathbf{2Vect}^\gamma$ for this module 2-category. We again find that the dual of $\mathbf{2Vect}_G$ with respect to $\mathbf{2Vect}^\gamma$ is $\mathbf{2Rep}(G)$. Namely, a left $B\mathcal{G}$-module 2-endofunctor on $\mathbf{2Vect}^\gamma$ is completely determined by its evaluation on $\mathbf{Vect}$ together with its left action by $B\mathcal{G}$, which is exactly the data of a 2-representation of $\mathcal{G}$.

3.2 Connected Fusion 2-Categories

Before stating the main theorem of this section, we need to recall some definitions and notations from [DNO13], to which we refer the reader for a detailed discussion.

Given a braided fusion 1-category $\mathcal{B}$ with braiding $\beta$. We use $\mathcal{Z}(\mathcal{B})$ to denote the symmetric center of $\mathcal{B}$, that is the full fusion sub-1-category of $\mathcal{B}$ on those objects $B$ for which

$$\beta_{C,B} \circ \beta_{B,C} = \text{Id}_{B \otimes C}$$

for every object $C$ of $\mathcal{B}$. It follows from the definition that $\mathcal{Z}(\mathcal{B})$ is a symmetric fusion 1-category.

Let $\mathcal{C}$ be a $\mathcal{B}$-central fusion 1-category $\mathcal{C}$. We write $\beta$ for the braiding on $\mathcal{Z}(\mathcal{C})$, and $F : \mathcal{B} \to \mathcal{Z}(\mathcal{C})$ for the braided monoidal functor supplying $\mathcal{C}$ with its central structure. Then, we use $\mathcal{Z}(\mathcal{C}, \mathcal{B})$ to denote the centralizer of the image of $\mathcal{B}$ in $\mathcal{Z}(\mathcal{C})$, that is the full fusion sub-1-category of $\mathcal{Z}(\mathcal{C})$ on those objects $Z$ for which

$$\beta_{F(B),Z} \circ \beta_{Z,F(B)} = \text{Id}_{Z \otimes F(B)}$$

for every object $B$ of $\mathcal{B}$. 

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Definition 3.2.1. Let \( \mathcal{B}_1 \) and \( \mathcal{B}_2 \) be two braided fusion 1-categories whose symmetric centers are given by the symmetric fusion 1-category \( \mathcal{E} \). We say \( \mathcal{B}_1 \) and \( \mathcal{B}_2 \) are Witt equivalent over \( \mathcal{E} \) provided that there exists a fusion 1-category \( \mathcal{C} \), together with a fully faithful braided embedding \( \mathcal{B}_1 \hookrightarrow \mathcal{Z}(\mathcal{C}) \) and a braided monoidal equivalence \( \mathcal{B}_2 \cong \mathcal{Z}(\mathcal{C}, \mathcal{B}_1) \). Because our definition is slightly different from that given in [DNO13], we begin by proving that they agree.

Lemma 3.2.2. The notion of Witt equivalence introduced in definition 3.2.1 above agrees with that given in definition 5.1 of [DNO13].

Proof. If there exists a fusion 1-category \( \mathcal{C} \), together with a fully faithful braided embedding \( \mathcal{B}_1 \hookrightarrow \mathcal{Z}(\mathcal{C}) \) and a braided monoidal equivalence \( \mathcal{B}_2 \cong \mathcal{Z}(\mathcal{C}, \mathcal{B}_1) \), then

\[
\mathcal{B}_1 \boxtimes \mathcal{E} \mathcal{B}_2^{\text{rev}} \cong \mathcal{B}_1 \boxtimes \mathcal{E} \mathcal{Z}(\mathcal{C}, \mathcal{B}_1) \cong \mathcal{Z}(\mathcal{C}, \mathcal{E})
\]

by proposition 4.3 of [DNO13]. This shows that \( \mathcal{B}_1 \) and \( \mathcal{B}_2^{\text{rev}} \) define opposite elements in the Witt group \( \mathcal{W}(\mathcal{E}) \) in the sense of [DNO13], so that \( \mathcal{B}_1 \) and \( \mathcal{B}_2 \) are Witt equivalent over \( \mathcal{E} \) in the sense of definition 5.1 of [DNO13]. The converse follows by running the argument in reverse. \( \square \)

We can now state the main theorem of this section, which completely characterizes which connected fusion 2-categories are Morita equivalent. In particular, it generalizes example 5.4.6 of [Dec22a]. The proof relies on the technical results derived in section 3.3 below.

Theorem 3.2.3. Let \( \mathcal{B}_1 \) and \( \mathcal{B}_2 \) be two braided fusion 1-categories. Then, \( \text{Mod} (\mathcal{B}_1) \) and \( \text{Mod} (\mathcal{B}_2) \) are Morita equivalent fusion 2-categories if and only if \( \mathcal{B}_1 \) and \( \mathcal{B}_2 \) have the same symmetric center \( \mathcal{E} \) and they are Witt equivalent over \( \mathcal{E} \).

Proof. Assume that \( \text{Mod} (\mathcal{B}_1) \) and \( \text{Mod} (\mathcal{B}_2) \) are Morita equivalent fusion 2-categories. Then, it follow from theorem 2.3.2 that

\[
\mathcal{Z}(\text{Mod}(\mathcal{B}_1)) \cong \mathcal{Z}(\text{Mod}(\mathcal{B}_2))
\]

as braided fusion 2-categories. But, it follows from lemma 2.16 of [JFR21] or corollary 3.3 above that

\[
\mathcal{Z}(\mathcal{B}_1) \cong \Omega \mathcal{Z}(\text{Mod}(\mathcal{B}_1)) \cong \Omega \mathcal{Z}(\text{Mod}(\mathcal{B}_2)) \cong \mathcal{Z}(\mathcal{B}_2)
\]

as symmetric fusion 1-categories. We denote by \( \mathcal{E} \) this symmetric fusion 1-category. Now, as \( \text{Mod} (\mathcal{B}_1) \) and \( \text{Mod} (\mathcal{B}_2) \) are Morita equivalent fusion 2-categories, there exists a fusion 1-category \( \mathcal{C} \) equipped with a braided monoidal functor \( F: \mathcal{B}_1 \to \mathcal{Z}(\mathcal{C}) \), and a monoidal equivalence \( \text{Bimod} (\text{Mod}(\mathcal{B}_1), \mathcal{C}) \cong \text{Mod}(\mathcal{B}_2) \) of fusion 2-categories. Let us assume that the functor \( F \) is not fully faithful, and write \( \mathcal{A} \) for its image in \( \mathcal{Z}(\mathcal{C}) \), which is a braided fusion 1-category.
by definition. Then, it follows from proposition 4.3 of [DNO13] and from the definitions that

$$\mathcal{Z}_{(2)}(\mathcal{A}) \simeq \mathcal{Z}_{(2)}(\mathcal{Z}(\mathcal{C}, \mathcal{A})) \simeq \mathcal{Z}_{(2)}(\mathcal{Z}(\mathcal{C}, \mathcal{B}_1))$$

as symmetric fusion 1-categories. On the other hand, by lemma [3.3.1] below, we find that $\mathcal{Z}(\mathcal{C}, \mathcal{B}_1)^{rev} \simeq \mathcal{B}_2$ as braided fusion 1-categories. This implies that $\mathcal{Z}_{(2)}(\mathcal{A}) \simeq \mathcal{E}$. Thence, it follows from corollary 3.24 of [DMNO13] that $F$ is fully faithful, so that $\mathcal{B}_1$ and $\mathcal{B}_2$ are Witt equivalent over $\mathcal{E}$.

On the other hand, by lemma 3.3.1 below, we find that $\mathcal{Z}(\mathcal{C}, \mathcal{B}_1)^{rev} \simeq \mathcal{B}_2$ as braided fusion 1-categories. This implies that $\mathcal{Z}_{(2)}(\mathcal{A}) \simeq \mathcal{E}$. Thence, it follows from corollary 3.24 of [DMNO13] that $F$ is fully faithful, so that $\mathcal{B}_1$ and $\mathcal{B}_2$ are Witt equivalent over $\mathcal{E}$.

Corollary 3.2.4. Let $\mathcal{B}$ be a braided fusion 1-category, and $\mathcal{C}$ a fusion 1-category with $\mathcal{B}$-central structure given by $F : \mathcal{B} \to \mathcal{Z}(\mathcal{C})$. Then, $\text{Bimod}_{\text{Mod}(\mathcal{B})}(\mathcal{C})$ is connected if and only if $F$ is fully faithful.

Remark 3.2.5. More generally, it would be interesting to understand the connected components of the fusion 2-category $\text{Bimod}_{\text{Mod}(\mathcal{B})}(\mathcal{C})$. For instance, with $\mathcal{B} = \text{Rep}(G)$ for some finite group $G$, and $\mathcal{C} = \text{Vect}$, it was shown in example 5.1.9 of [Dec22a] that $\text{Bimod}_{\text{Mod}(\mathcal{B})}(\mathcal{C}) \simeq 2\text{Vect}_G$, which has $|G|$ connected components. On the other hand, with $\mathcal{B} = \text{Rep}(\mathcal{G} \times \mathcal{G}^{op})$ and $\mathcal{C} = \text{Rep}(G)$, it follows from lemma 2.2.1 and corollary 2.3.4 that $\text{Bimod}_{\text{Mod}(\mathcal{B})}(\mathcal{C}) \simeq \mathcal{Z}(2\text{Vect}_G)$. It was shown in [KTZ20] that the connected components of this fusion 2-category are parametrised by the conjugacy classes of $G$.

The technical results used in the proof of the above theorem will be established below. But first, we record the following corollary. Let us fix $\mathcal{E}$ a symmetric fusion 1-category. A non-degenerate braided fusion 1-category $\mathcal{B}$ containing $\mathcal{E}$ fully faithfully is called minimal if the centralizer of $\mathcal{E}$ in $\mathcal{B}$ is exactly $\mathcal{E}$. The set of minimal non-degenerate extensions of $\mathcal{E}$ form a group (see [LKW17]), which is denoted by $\text{Mext}(\mathcal{E})$. Further, we say that a minimal non-degenerate extension $\mathcal{B}$ of $\mathcal{E}$ is Witt-trivial if the class of the non-degenerate braided fusion 1-category $\mathcal{B}$ is trivial in the Witt group $\mathcal{W}$. This is equivalent to requiring that $\mathcal{B}$ be braided equivalent to the Drinfeld center of a fusion 1-category. We expect that Witt-trivial minimal non-degenerate extensions form a subgroup of $\text{Mext}(\mathcal{E})$.

Corollary 3.2.6. Let $\mathcal{E}$ be a symmetric fusion 1-category. Then, there is a bijective correspondence between Morita autoequivalences of $\text{Mod}(\mathcal{E})$ and Witt-trivial minimal non-degenerate extensions of $\mathcal{E}$.

Proof. It follows readily from corollary 3.2.4 that every Morita autoequivalence of $\text{Mod}(\mathcal{E})$ is given by an $\mathcal{E}$-central fusion 1-category $\mathcal{C}$ with $\mathcal{Z}(\mathcal{C}, \mathcal{E}) = \mathcal{E}$. On the other hand, it follows from theorem 1.4.1 above that any two such $\mathcal{E}$-central fusion 1-category $\mathcal{C}_1$ and $\mathcal{C}_2$ give the same Morita equivalence if and only if $\text{Mod}(\mathcal{C}_1)$ and $\text{Mod}(\mathcal{C}_2)$ are equivalent as left $\text{Mod}(\mathcal{E})$-module 2-categories.
This implies that \( \mathcal{C}_1 \) and \( \mathcal{C}_2 \) are Morita equivalent fusion 1-categories so that we have an equivalence \( \mathcal{Z}(\mathcal{C}_1) \simeq \mathcal{Z}(\mathcal{C}_2) \) of braided fusion 1-categories (see theorem 1.1 of \cite{ENO09}). Finally, \( \text{Mod}(\mathcal{C}_1) \) and \( \text{Mod}(\mathcal{C}_2) \) are equivalent as left \( \text{Mod}(\mathcal{E}) \)-module 2-categories if and only if the diagram of braided monoidal functors

\[
\begin{array}{ccc}
\mathcal{E} & \longrightarrow & \mathcal{Z}(\mathcal{C}_1) \\
\downarrow & & \downarrow \\
\mathcal{Z}(\mathcal{C}_2) & \longrightarrow & \mathcal{Z}(\mathcal{C}_2)
\end{array}
\]

commutes. This concludes the proof.

\[\square\]

Remark 3.2.7. Let \( G \) be a finite group. It follows from the above corollary and example 2.1 of \cite{Nik22} that there is a bijection of sets \( \text{BrPic}(\text{Mod}(\text{Rep}(G))) \simeq H^3(G; k^*) \). Namely, in the Tannakian case, every minimal non-degenerate extension of \( \text{Rep}(G) \) is Witt-trivial. In this case, we expect that the map \( \Phi : \text{BrPic}(\text{Mod}(\text{Rep}(G))) \to \text{Aut}^{br}(\mathcal{Z}(\text{Mod}(\text{Rep}(G)))) \) of remark 2.3.3 is an isomorphism of groups.

On the other hand, the above corollary shows that \( \text{BrPic}(\text{2SVect}) \simeq \mathbb{Z}/2\mathbb{Z} \). Namely, the group of minimal non-degenerate extensions of \( \text{SVect} \) is identified with \( \mathbb{Z}/16\mathbb{Z} \) by example 2.2 of \cite{Nik22}, and only two out of the sixteen minimal non-degenerate extensions of \( \text{SVect} \) are Witt-trivial (see appendix A.3.2 of \cite{DGNO10}). But, it follows from corollary 2.3.8 that every minimal non-degenerate extension of \( \text{SVect} \) induces a braided autoequivalence of \( \mathcal{Z}(\text{2SVect}) \).

In fact, it was sketched in section 2.3 of \cite{JF21} that this construction induces a bijection of sets \( \text{Aut}^{br}(\mathcal{Z}(\text{2SVect})) \cong \mathbb{Z}/16\mathbb{Z} \). We therefore expect that the map \( \Phi : \text{BrPic}(\text{2SVect}) \to \text{Aut}^{br}(\mathcal{Z}(\text{2SVect})) \) is not an isomorphism. In particular, the full version of theorem 1.1 of \cite{ENO09} can not be na"ively categorified.

### 3.3 Technical Results

We establish the technical lemmas that were used in the proof of theorem 3.2.3.

**Lemma 3.3.1.** Let \( \mathcal{B} \) be a braided fusion 1-category, and \( \mathcal{C} \) a \( \mathcal{B} \)-central fusion 1-category. There is an equivalence

\[
\text{Bimod}_{\text{Mod}(\mathcal{B})}(\mathcal{C})^0 \simeq \text{Mod}(\mathcal{Z}(\mathcal{C}, \mathcal{B})^{rev})
\]

of monoidal 2-categories.

**Proof.** It follows from theorem 3.27 of \cite{Lau20} that there is an equivalence of 2-categories

\[
\text{Bimod}_{\text{Mod}(\mathcal{B})}(\mathcal{C}) \simeq \text{Mod}(\mathcal{C}^{mop} \boxtimes_\mathcal{B} \mathcal{C}).
\]

Namely, the 2-category on the left hand-side is the 2-category of finite semisimple \( \mathcal{B} \)-balanced \( \mathcal{C} \)-\( \mathcal{C} \)-bimodule 1-categories as considered in section 3.4 of \cite{Lau20}. Now, \( \mathcal{C} \) with its canonical \( \mathcal{B} \)-balanced \( \mathcal{C} \)-\( \mathcal{C} \)-bimodule structure is the monoidal unit of \( \text{Bimod}_{\text{Mod}(\mathcal{B})}(\mathcal{C}) \). Let us write \( \text{End}^{\mathcal{B}}_{\mathcal{C}, \mathcal{C}}(\mathcal{C}) \) for the braided fusion 1-category
of endomorphisms of $C$ in $\text{Bimod}_{\text{Mod}(B)}(C)$. Now, observe that the forgetful 2-functor

$$\text{Bimod}_{\text{Mod}(B)}(C) \to \text{Bimod}(C)$$

is monoidal. On the monoidal unit, this 2-functor induces a braided monoidal functor $\text{End}_{B C}^B(C) \to \text{End}_{C C}^C(C)$. But, by example 5.3.7 of [Déc22a], we know that there is an equivalence $\text{End}_{C C}^B(C) \simeq Z(C)^{rev}$ of braided fusion 1-categories. Moreover, it follows from the definitions in section 3.4 of [Lau20] that $\text{End}_{C C}^B(C)$ is the full fusion sub-1-category of $\text{End}_{C C}^C(C) \simeq Z(C)^{rev}$ on those objects centralizing the image of $B^{rev}$ in $Z(C)^{rev}$. Thus, we find that $\text{End}_{C C}^B(C) \simeq Z(C, B)^{rev}$ as braided fusion 1-categories. Finally, the proof is concluded by appealing to proposition 2.4.7 of [Déc22d].

Remark 3.3.2. Note that [Lau20] works with $B$-augmented tensor 1-categories. Nonetheless, the results of section 3.4 of [Lau20] hold for any $B$-central tensor 1-category. On the other hand, this is not the case for all the results of section 3 of [Lau20]. More precisely, corollary 3.46 of [Lau20] does not hold for an arbitrary $B$-central tensor 1-category. Namely, if $B = \text{Rep}(G)$, and $C = \text{Vect}$, then $C^{mop} \boxtimes_B C \simeq \boxtimes_{g \in G} \text{Vect}$ and $C$ is not a faithful $C^{mop} \boxtimes_B C$-module 1-category.

We recover lemma 2.16 of [JFR21].

Corollary 3.3.3. Let $B$ be a braided fusion 1-category, then we have

$$\Omega \mathcal{Z}(\text{Mod}(B)) \simeq Z_{(2)}(B).$$

Proof. Thanks to lemma 2.2.1, we can identify $\mathcal{Z}(\text{Mod}(B))$ with the dual to $\text{Mod}(B^{rev} \boxtimes B)$ with respect to $\text{Mod}(B)$. It follows from corollary 4.4 of [DNO13] and theorem 3.10 (ii) of [DGNO10] that $Z(B, B^{rev} \boxtimes B) \simeq Z_{(2)}(B)$, which establishes the result.

We now give a sufficient criterion for the fusion 2-category $\text{Bimod}_{\text{Mod}(B)}(C)$ to be connected.

Lemma 3.3.4. Let $B$ be a braided fusion 1-category, and $C$ a $B$-central fusion 1-category such that the composite $B \to C$ is fully faithful. Then, $\text{Bimod}_{\text{Mod}(B)}(C)$ is a connected fusion 2-category.

Proof. Let us identify $\text{Bimod}_{\text{Mod}(B)}(C) \simeq \text{Mod}(C^{mop} \boxtimes_B C)$ as 2-categories. Thanks to proposition 2.3.5 of [Déc22b], in order to prove that this finite semisimple 2-category is connected, it is enough to show that $C^{mop} \boxtimes_B C$ is a fusion 1-category. But, the canonical functor $C^{mop} \boxtimes C \to C^{mop} \boxtimes_B C$ is monoidal. Further, the image of $B$ in $C$ may be identified with $B$ by hypothesis. Thus, the monoidal unit of $C^{mop} \boxtimes_B C$ is given by the image of the monoidal unit $I \boxtimes I$ of $B^{mop} \boxtimes_B C$ under the canonical monoidal functor $B^{mop} \boxtimes B \to B^{mop} \boxtimes_B B \simeq B$. As $B$ is a fusion 1-category by hypothesis, the proof is complete.

We need the following slightly more general version of the previous lemma.
Lemma 3.3.5. Let $\mathcal{B}$ be a braided fusion 1-category, and $\mathcal{C}$ a fusion 1-category with $\mathcal{B}$-central structure $\mathcal{B} \to \mathcal{Z}(\mathcal{C})$ given by a fully faithful braided monoidal functor. Then, $\text{Bimod}_{\text{Mod}(\mathcal{B})}(\mathcal{C})$ is a connected fusion 2-category.

Proof. Let us consider the fusion 2-category $\text{Bimod}_{\text{Mod}(\mathcal{B})}(\mathcal{Z}(\mathcal{C}))$, where $\mathcal{Z}(\mathcal{C})$ is endowed with its canonical $\mathcal{B}$-central structure. Thanks to lemma 3.3.4 above, it is connected. On the other hand, by theorem 1.4.1 above, there is an equivalence of fusion 2-categories

$$\text{Bimod}_{\text{Mod}(\mathcal{B})}(\mathcal{Z}(\mathcal{C})) \simeq \text{Bimod}_{\text{Mod}(\mathcal{B})}(\mathcal{C}^{mop} \boxtimes \mathcal{C}).$$

More precisely, $\mathcal{C}^{mop} \boxtimes \mathcal{C}$ is equipped with the braided monoidal functor

$$\mathcal{B} \xrightarrow{\mathcal{F}} \mathcal{Z}(\mathcal{C}) \hookrightarrow \mathcal{Z}(\mathcal{C}^{mop} \boxtimes \mathcal{Z}(\mathcal{C})) \simeq \mathcal{Z}(\mathcal{C}^{mop} \boxtimes \mathcal{C}).$$

Then, as $\mathcal{Z}(\mathcal{C})$ and $\mathcal{C}^{mop} \boxtimes \mathcal{C}$ are Morita equivalent fusion 1-categories, $\text{Mod}(\mathcal{Z}(\mathcal{C}))$ and $\text{Mod}(\mathcal{C}^{mop} \boxtimes \mathcal{C})$ are equivalent finite semisimple 2-categories. Furthermore, this Morita equivalence of fusion 1-categories is compatible with the $\mathcal{B}$-central structures as the following diagram of braided monoidal functors commute

$$\begin{array}{ccc}
\mathcal{B} & \xrightarrow{\mathcal{F}} & \mathcal{Z}(\mathcal{C}^{mop} \boxtimes \mathcal{C}) \\
& & \downarrow \\
\mathcal{Z}(\mathcal{C}) & \hookrightarrow & \mathcal{Z}(\mathcal{C}).
\end{array}$$

But, as the $\mathcal{B}$-central structure on $\mathcal{C}^{mop} \boxtimes \mathcal{C}$ factors through $\mathcal{C}$, we have that

$$\text{Bimod}_{\text{Mod}(\mathcal{B})}(\mathcal{C}^{mop} \boxtimes \mathcal{C}) \simeq \text{Bimod}(\mathcal{C}^{mop}) \boxtimes \text{Bimod}_{\text{Mod}(\mathcal{B})}(\mathcal{C})$$

as fusion 2-categories. Finally, the fusion 2-categories $\text{Bimod}(\mathcal{C}^{mop})$ and $\text{Mod}(\mathcal{Z}(\mathcal{C}^{mop}))$ are equivalent (see example 5.3.7 of [Dec22a]). But, thanks to lemma 4.3 of [Dec21a], the set of connected components of a 2-Deligne tensor product is the product of the set of connected components, so that $\text{Bimod}_{\text{Mod}(\mathcal{B})}(\mathcal{C})$ is indeed connected. □

Corollary 3.3.6. Let $\mathcal{B}$ be a braided fusion 1-category, and $\mathcal{C}$ be a $\mathcal{B}$-central fusion 1-category such that the braided monoidal functor $\mathcal{B} \to \mathcal{Z}(\mathcal{C})$ is fully faithful. Then, the dual to $\text{Mod}(\mathcal{B})$ with respect to $\text{Mod}(\mathcal{C})$ is given by $\text{Mod}(\mathcal{Z}(\mathcal{C}, \mathcal{B}))$.

4 Separable Fusion 2-Categories

We begin by introducing a particularly important property of multifusion 2-categories called separability. Namely, it was shown in [JP22] that separable multifusion 2-categories are the fully dualizable objects of an appropriate symmetric monoidal 4-category (assuming some properties of enriched colimits in higher categories as in [GJF19a]). We begin by giving three characterizations of separability for a fusion 2-category. Then, through a careful analysis of the
Morita equivalence classes of fusion 2-categories, we will show that every fusion 2-category over an algebraically closed field of characteristic zero is separable. More precisely, we show that, up to taking the 2-Deligne tensor product with an invertible fusion 2-category, every fusion 2-category is Morita equivalent to a strongly fusion 2-category. Throughout, we work over a fixed algebraically closed field \( k \) of characteristic zero.

### 4.1 Definition & Characterizations

We work with a fixed multifusion 2-category \( \mathcal{C} \). The following definition categorifies definition 2.5.8 of [DSPS21].

**Definition 4.1.1.** The multifusion 2-category \( \mathcal{C} \) is called separable if it is separable as a finite semisimple left \( \mathcal{C} \boxtimes \mathcal{C}^{\text{op}} \)-module 2-category.

Our next objective is to categorify the notion of the global (also called categorical) dimension of a fusion 1-category. In order to do so, we begin by explaining how to associate a connected rigid algebra to any fusion 2-category.

**Construction 4.1.2.** As \( \mathcal{C} \boxtimes \mathcal{C}^{\text{op}} \) is a multifusion 2-category acting on the finite semisimple 2-category \( \mathcal{C} \), it follows from theorem 4.2.2 of [Dec21c] that \( \mathcal{C} \) can be given the structure of a \( \mathcal{C} \boxtimes \mathcal{C}^{\text{op}} \)-enriched 2-category. More precisely, for any \( C, D \) in \( \mathcal{C} \), the Hom-object \( \text{Hom}(C, D) \) in \( \mathcal{C} \boxtimes \mathcal{C}^{\text{op}} \) is characterized by the adjoint 2-natural equivalence

\[
\text{Hom}_{\mathcal{C}}(H \Box C, D) \simeq \text{Hom}_{\mathcal{C} \boxtimes \mathcal{C}^{\text{op}}}(E, \text{Hom}(C, D)),
\]

for every \( H \) in \( \mathcal{C} \boxtimes \mathcal{C}^{\text{op}} \). It was established in proposition 4.1.1 of [Dec21c] that such an object \( \text{Hom}(C, D) \) of \( \mathcal{C} \boxtimes \mathcal{C}^{\text{op}} \) always exists. Further, thanks to the 3-universal property of the Cauchy completion (see [Dec22a]), the above adjoint 2-natural equivalence is completely determined by its value on \( H = E \boxtimes F \), with \( E, F \) in \( \mathcal{C} \). Now, we write \( R_\mathcal{C} \) for the algebra \( \text{End}(I) \) in \( \mathcal{C} \boxtimes \mathcal{C}^{\text{op}} \), and recall that \( R_\mathcal{C} \) is rigid by theorem 5.4.7 of [Dec21c]. Now, if we assume that \( \mathcal{C} \) is a fusion 2-category, then \( R_\mathcal{C} \) is connected. Namely, its unit 1-morphism corresponds to the identity 1-morphism \( I \to I \) under the adjunction above.

**Remark 4.1.3.** For any multifusion 2-category \( \mathcal{C} \), note that the monoidal unit \( I \) of \( \mathcal{C} \) generates \( \mathcal{C} \) under the action of \( \mathcal{C} \boxtimes \mathcal{C}^{\text{op}} \) in the sense of definition 5.1.2 of [Dec21c]. Thus, by theorem 5.1.5 of [Dec22a], there is an equivalence of left \( \mathcal{C} \boxtimes \mathcal{C}^{\text{op}} \)-module 2-categories between \( \mathcal{C} \), and \( \text{Mod}_{\mathcal{C} \boxtimes \mathcal{C}^{\text{op}}}(R_\mathcal{C}) \). Furthermore, if \( \mathcal{C} \) is separable, then \( R_\mathcal{C} \) is separable thanks to corollary 5.1.7 of [Dec22a]. Thus, theorems 5.1.2 of [Dec22a] and lemma 2.2.1 above show that

\[
\text{Bimod}_{\mathcal{C} \boxtimes \mathcal{C}^{\text{op}}}(R_\mathcal{C}) \simeq \mathcal{Z}(\mathcal{C})
\]

as multifusion 2-categories.

If \( \mathcal{C} \) is a fusion 2-category, then we have argued above that \( R_\mathcal{C} \) is a connected rigid algebra in \( \mathcal{C} \boxtimes \mathcal{C}^{\text{op}} \). Then, via the construction given in section 3.2 of [Dec22c], one can associate to \( R_\mathcal{C} \) a well-defined scalar \( \text{Dim}_{\mathcal{C} \boxtimes \mathcal{C}^{\text{op}}}(R_\mathcal{C}) \), called its dimension.
Definition 4.1.4. The dimension of a fusion 2-category \( \mathcal{C} \) is
\[
\text{Dim}(\mathcal{C}) := \text{Dim}_{\mathcal{C} \boxtimes \mathcal{C}}(\mathcal{R}_\mathcal{C}).
\]

Remark 4.1.5. A notion of dimension for finite semisimple 2-categories was introduced in definition 1.2.28 of [DR18]. We do not know how this notion compares with ours. However, we point out that there is a priori no reason to expect that they agree, as the definition given in [DR18] does not take into account the rigid monoidal structure.

Theorem 4.1.6. For a multifusion 2-category \( \mathcal{C} \), the following are equivalent:

1. The multifusion 2-category \( \mathcal{C} \) is separable.
2. The Drinfeld center \( Z(\mathcal{C}) \) of \( \mathcal{C} \) is finite semisimple.

Further, if \( \mathcal{C} \) is a fusion 2-category, this is also equivalent to:

3. The dimension \( \text{Dim}(\mathcal{C}) \) of \( \mathcal{C} \) is non-zero.

Proof. The equivalence between 1 and 2 follows by combining proposition 2.3.1 and remark 4.1.3 above with proposition 5.1.6 of [Déc22a]. The equivalence between 2 and 3 is given by theorem 3.2.4 of [Déc22c].

Remark 4.1.7. We now comment on the relation between 4-dualizability and separability for multifusion 2-categories. We begin by recalling the setup considered in section 2 of [JF22]. Let us write \( \text{Cau2Cat}_k \) for the linear 3-category of Cauchy complete \( k \)-linear 2-categories, and note that the completed tensor product \( \hat{\otimes} \) endows this 3-category with a symmetric monoidal structure. We will assume that \( \text{Cau2Cat}_k \) is closed under colimits and that \( \hat{\otimes} \) preserves them. Then, thanks to the results of section 8 of [JFS17], we can consider the symmetric monoidal 4-category \( \text{Mor}_1(\text{Cau2Cat}_k) \) of Cauchy complete monoidal 2-categories, Cauchy complete bimodule 2-categories, and their morphisms. Theorem 1 of [JF22] then sketches a proof of the fact that a multifusion 2-category \( \mathcal{C} \) is a fully dualizable object, that is a 4-dualizable object, of \( \text{Mor}_1(\text{Cau2Cat}_k) \) if and only if it is 2-dualizable. More precisely, let \( T : \mathcal{C} \boxtimes \mathcal{C} \text{mop} \to \mathcal{C} \) denote the canonical left \( \mathcal{C} \boxtimes \mathcal{C} \text{mop} \)-module 2-functor. The proof of theorem 1 of [JF22] outlines that the multifusion 2-category \( \mathcal{C} \) yields a 4-dualizable object of \( \text{Mor}_1(\text{Cau2Cat}_k) \) if and only if there exists a left \( \mathcal{C} \boxtimes \mathcal{C} \text{mop} \)-module 2-functor \( \Delta : \mathcal{C} \to \mathcal{C} \boxtimes \mathcal{C} \text{mop} \) such that the composite left \( \mathcal{C} \boxtimes \mathcal{C} \text{mop} \)-module 2-endofunctor \( T \circ \Delta \) on \( \mathcal{C} \) can be extended to a 3-condensation monad in the sense of [GJF19a]. We now argue that this is possible if the multifusion 2-category \( \mathcal{C} \) is separable. Through the equivalence of left \( \mathcal{C} \boxtimes \mathcal{C} \text{mop} \)-module 2-categories of remark 4.1.3 the 2-functor \( T \) is identified with the left \( \mathcal{C} \boxtimes \mathcal{C} \text{mop} \)-module 2-functor \( \mathcal{C} \boxtimes \mathcal{C} \text{mop} \to \text{Mod}_{\mathcal{C} \boxtimes \mathcal{C} \text{mop}}(\mathcal{R}_\mathcal{C}) \) given by \( H \mapsto H \boxtimes \mathcal{R}_\mathcal{C} \). We let \( \Delta \) be the forgetful 2-functor \( \text{Mod}_{\mathcal{C} \boxtimes \mathcal{C} \text{mop}}(\mathcal{R}_\mathcal{C}) \to \mathcal{C} \boxtimes \mathcal{C} \text{mop} \). Then, the composite \( T \circ \Delta : \mathcal{C} \boxtimes \mathcal{C} \text{mop} \to \mathcal{C} \boxtimes \mathcal{C} \text{mop} \) is identified with the 2-functor \( H \mapsto H \boxtimes \mathcal{R}_\mathcal{C} \). But, theorem 5.1.2 of [Déc22a] exhibits an equivalence between the Morita 3-category of separable algebras.
in \( \mathcal{C} \boxtimes \mathcal{C} \) and the 3-category of separable left \( \mathcal{C} \boxtimes \mathcal{C} \)-module 2-categories. Under this equivalence, the composite left \( \mathcal{C} \boxtimes \mathcal{C} \)-module 2-functor \( T \circ \Delta \) corresponds to the \( I-I \)-bimodule \( \mathcal{R}_\mathcal{C} \). Finally, if \( \mathcal{C} \) is separable, then \( \mathcal{R}_\mathcal{C} \) is a separable algebra, so that it is in particular a 3-condensation monad. In this case, we find that \( T \circ \Delta \) can be extended to a 3-condensation monad, which proves the desired result. In fact, the converse also holds. Namely, if the multifusion 2-category \( \mathcal{C} \) is a 4-dualizable object of \( \text{Mor}_1(\text{Cau2Cat}_k) \). Then, it follows from remark II.1.4 of [JF22] that \( \mathcal{Z}(\mathcal{C}) \) is a finite semisimple 2-category. Thus, theorem 4.2.4 shows that \( \mathcal{C} \) is separable.

**Example 4.1.8.** Every connected fusion 2-category is separable. Finite semisimplicity of the Drinfeld center was established directly in lemma 2.18 of [JFR21]. Alternatively, every connected fusion 2-category is equivalent to \( \text{Mod}(B) \) for some braided fusion 1-category \( B \) (see [Dec22b]). Further, it follows from [Dec21a], that \( \text{Mod}(B) \boxtimes \text{Mod}(B)^{\text{mop}} \simeq \text{Mod}(B \boxtimes B^\text{rev}) \) is a connected fusion 2-category. Finally, by example 1.3.5 we know that every connected rigid algebra in a connected fusion 2-category is separable, so that \( \dim(\text{Mod}(B)) \) is non-zero as desired.

**Example 4.1.9.** Let \( \mathcal{C} \) be a bosonic strongly fusion 2-category. This implies that \( \mathcal{C} \) is equivalent to \( \text{2Vect}_G^\pi \) for some finite group \( G \) and 4-cocycle \( \pi \) for \( G \) with coefficients in \( \mathbb{C} \times \mathbb{C} \). The Drinfeld center \( \mathcal{Z}(\text{2Vect}_G^\pi) \) was studied in depth in [KTZ20]. In particular, they show that the underlying 2-category is finite semisimple, so that every bosonic strongly fusion 2-category is separable. We now give an alternative proof. Fix \( \mathcal{C} = \mathcal{Z}(\text{2Vect}_G^\pi) \), then, direct inspection shows that we have

\[
\mathcal{C} \boxtimes \mathcal{C}^{\text{mop}} = \text{2Vect}_G^\pi \boxtimes (\text{2Vect}_G^\pi)^{\text{mop}} \simeq \text{2Vect}_{G \times G^\text{op}}^\pi \times \pi^{-1}
\]

as fusion 2-categories. Following remark 4.1.3 we have that \( \mathcal{C} \simeq \text{Mod}_{\mathcal{C} \boxtimes \mathcal{C}^{\text{mop}}}(\mathcal{R}_\mathcal{C}) \). Now, given two elements \( g, h \in G \), we write \( \text{Vect}_{(g,h)} \) for the simple object of \( \text{2Vect}_{G \times G^\text{op}}^\pi \times \pi^{-1} \) given by \( \text{Vect} \) with grading \( (g, h) \). It follows from the definition above that

\[
\mathcal{R}_\mathcal{C} = \bigoplus_{g \in G} \text{Vect}_{(g, g^{-1})}
\]

as an object of \( \mathcal{C} \boxtimes \mathcal{C}^{\text{mop}} \). On the other hand, let \( \mathcal{D} \) be the fusion sub-2-category of \( \text{2Vect}_{G \times G^\text{op}}^\pi \times \pi^{-1} \) generated by the simple objects of the form \( \text{Vect}_{(g, g^{-1})} \) with \( g \in G \). It is clear that \( \mathcal{D} \simeq \text{2Vect}_G \), and that \( \mathcal{R}_\mathcal{C} \) is a connected rigid algebra in \( \mathcal{D} \). It thus follows from corollary 3.3.6 of [Dec22c] that \( \mathcal{R}_\mathcal{C} \) has dimension \( |G| \), so that \( \dim(\text{2Vect}_G^\pi) = |G| \). Thanks to theorem 4.1.6 this shows that \( \text{2Vect}_G^\pi \) is separable as desired.

More generally, the following result holds.

**Proposition 4.1.10.** Every strongly fusion 2-category is separable.

**Proof.** Thanks to example 4.1.9 it is enough to treat the case of fermionic strongly fusion 2-categories. Let us fix \( \mathcal{C} \) a fermionic strongly fusion 2-category.
Without loss of generality, we can assume that $C$ is skeletal. Now, thanks to theorem B of [JFY21], the set of connected components of $C$ inherits a group structure, and we write $G := \pi_0(C)$. Furthermore, they show that every simple object of $C$ is invertible, and we write $G$ for the group of simple objects of $C$. In fact, the fusion 2-category $2SVect$ has two simple objects both of which are invertible, so that $G$ is a double cover of $G$. Given an element $g \in G$, we use $V_g$ to denote the corresponding simple object of $C$.

It follows from proposition 4.1 of [Déc21c] that the underlying object of the connected rigid algebra $R_C$ in $C \boxtimes C^{\text{mop}}$ associated to $C$ is given by

$$R_C = \bigoplus_{g \in G} V_g \boxtimes V_{g^{-1}}.$$ 
Furthermore, it follows by inspection that the unit and multiplication 1-morphisms of $R_C$ are given by the direct sums of identity 1-morphisms. Now, write $\mathcal{D}$ for the diagonal fusion sub-2-category of $C \boxtimes C^{\text{mop}}$, that is the full sub-2-category generated under direct sums and splittings of 2-condensation monads by the simple objects of the form $V_g \boxtimes V_{g^{-1}}$ for $g \in G$.

By construction, the canonical left action of $C \boxtimes C^{\text{mop}}$ on $\mathcal{C}$ restricts to an action of $\mathcal{D}$ on $2SVect$. In particular, we get a monoidal 2-functor

$$F : \mathcal{D} \rightarrow \text{End}(2SVect)$$

between fusion 2-categories. But, we have argued above that $R_C$ is a connected rigid algebra in $\mathcal{D}$. Hence, by theorem 3.2.4 of [Déc22a] it is enough to prove that $R_C$ has non-zero dimension. Now, as the unit 1-morphism of $R_C$ is given by the inclusion of the identity 1-morphism on $\mathcal{D}$, $F(R_C)$ is necessarily a connected rigid algebra. Thus, we can apply lemma 3.3.5 of [Déc22a] to show that $R_C$ has non-zero dimension in $\mathcal{D}$ if and only if $F(R_C)$ has non-zero dimension in $\text{End}(2SVect)$. But, by theorem 1.4.1 there is an equivalence $\text{End}(2SVect) \simeq \text{Bimod}(SVec^{\text{mop}})$ of fusion 2-categories, and the right hand-side is connected by example 5.3.7 of [Déc22a]. Finally, it follows from example 1.3.5 that $F(R_C)$ has non-zero dimension, which establishes the result.

Thanks to the theorem above, we see that every invertible fusion 2-categories is separable. We now describe this class of fusion 2-categories explicitly.

**Lemma 4.1.11.** A fusion 2-category is invertible if and only if it is equivalent to $\text{Mod}(B)$ with $B$ a non-degenerate braided fusion 1-category.

**Proof.** Let $\mathcal{C}$ be an invertible fusion 2-category. This means that the dual to $\mathcal{C} \boxtimes C^{\text{mop}}$ with respect to $\mathcal{C}$ is $2Vec$. Thanks to corollary 5.4.4 of [Déc22a], this implies that the dual to $2Vec$ with respect to $\mathcal{C}$ is $\mathcal{C} \boxtimes C^{\text{mop}}$. Thus, by combining proposition 5.1.6 and corollary 5.2.5 of [Déc22a], we get that $\mathcal{C}$ is indecomposable as a finite semisimple 2-category, i.e. it is connected. Let $B$ be a braided fusion 1-category such that $\text{Mod}(B) \simeq \mathcal{C}$ as fusion 2-category. As $\mathcal{Z}(\text{Mod}(B)) \simeq 2Vec$, it follows from lemma 2.16 of [JFR21], or corollary 3.3.3 above, that $\mathcal{Z}_{(2)}(B) \simeq \text{Vect}$, so that $B$ is non-degenerate. 

$\square$
4.2 Main Results

Before stating our next theorem, we will establish a technical lemma. In order to do so, it is convenient to generalize the terminology of definition 1.2.5.

Definition 4.2.1. We say that a fusion 2-category \( \mathcal{C} \) is bosonic if the symmetric fusion 1-category \( \mathcal{Z}_{(2)}(\Omega \mathcal{C}) \) is Tannakian. Otherwise, we say that \( \mathcal{C} \) is fermionic.

Lemma 4.2.2. Every bosonic fusion 2-category \( \mathcal{C} \) is Morita equivalent to a fusion 2-category \( \mathcal{D} \) with \( \Omega \mathcal{D} \) a non-degenerate braided fusion 1-category. Every fermionic fusion 2-category \( \mathcal{C} \) is Morita equivalent to a fusion 2-category \( \mathcal{D} \) with \( \Omega \mathcal{D} \) a slightly-degenerate braided fusion 1-category, i.e. \( \mathcal{Z}_{(2)}(\Omega \mathcal{D}) = \mathbb{SVect} \).

Proof. Let \( \mathcal{C} \) be a bosonic fusion 1-category. We write \( \mathcal{E} := \mathcal{Z}_{(2)}(\Omega \mathcal{C}) \). Thanks to Deligne’s theorem (see [Del02]), there exists a symmetric monoidal functor \( \mathcal{E} \to \mathbb{Vect} \). We define a braided fusion 1-category \( \mathcal{D} := \Omega \mathcal{C} \boxtimes \mathcal{E} \mathbb{Vect} \). By proposition 4.30 of [DGNO10], \( \mathcal{D} \) is non-degenerate. Further, we view \( \mathcal{D} \) as a connected rigid algebra in \( \mathbb{Mod}(\mathcal{C}) \cong \mathcal{C}^0 \) using the canonical braided monoidal functor \( \Omega \mathcal{C} \to \mathcal{D} \to \mathcal{Z}(\mathcal{D}) \). It follows from example 1.3.5 that \( \mathcal{D} \) is a separable algebra. Thus, by theorem 5.4.3 of [Dec22a], the multifusion 2-category \( \mathbb{Bimod}_{\mathcal{C}}(\mathcal{D}) \) is Morita equivalent to \( \mathcal{C} \). Further, as \( \mathcal{D} \) is connected, it is necessarily indecomposable, so that it follows from section 5.2 of [Dec22a] that \( \mathbb{Bimod}_{\mathcal{C}}(\mathcal{D}) \) is in fact a fusion 2-category, and we set \( \mathcal{D} := \mathbb{Bimod}_{\mathcal{C}}(\mathcal{D}) \). It remains to analyze \( \mathcal{D}^0 \). As \( \mathcal{D} \) is an algebra in \( \mathcal{C}^0 \), corollary 2.3.6 of [Dec22a] shows that the underlying object of any simple \( \mathcal{D}\mathcal{D} \)-bimodule in \( \mathcal{C} \) is contained in a single connected component of \( \mathcal{C} \). In particular, we have an equivalence

\[
\mathcal{D}^0 \cong (\mathbb{Bimod}_{\mathcal{C}^0}(\mathcal{D}))^0 \cong (\mathbb{Bimod}_{\mathbb{Mod}(\mathcal{C})}(\mathcal{D}))^0
\]

of fusion 2-categories. Moreover, it was shown in lemma 3.3.1 above that

\[
(\mathbb{Bimod}_{\mathbb{Mod}(\mathcal{C})}(\mathcal{D}))^0 \cong \mathbb{Mod}(\mathcal{Z}(\mathcal{D}, \Omega \mathcal{C}))
\]

as fusion 2-categories. Finally, as \( \mathcal{D} \) is non-degenerate, we have that \( \mathcal{Z}(\mathcal{D}) \cong \mathcal{D} \boxtimes \mathcal{D}^{rev} \) by proposition 3.7 of [DGNO10]. Further, the canonical braided monoidal functor \( \Omega \mathcal{C} \to \mathcal{D} \) is surjective (in the sense of definition 2.1 of [DGNO10]), so that \( \mathcal{Z}(\mathcal{D}, \Omega \mathcal{C}) \cong \mathcal{D}^{rev} \) as braided fusion 1-categories. This concludes the proof in this case.

If \( \mathcal{C} \) is fermionic, then it follows from [Del02] that there exists a surjective symmetric monoidal functor \( \mathcal{E} \to \mathbb{SVect} \). We then define a braided fusion 1-category \( \mathcal{D} := \Omega \mathcal{C} \boxtimes \mathbb{E Vect} \). By proposition 4.30 of [DGNO10], \( \mathcal{D} \) is slightly-degenerate. Then, the argument used above shows that \( \mathcal{D} := \mathbb{Bimod}_{\mathbb{Mod}(\mathcal{C})}(\mathcal{D}) \) is a fusion 2-category with \( \mathcal{D}^0 \cong \mathbb{Mod}(\mathcal{Z}(\mathcal{D}, \Omega \mathcal{C})) \) as fusion 2-categories. It follows from proposition 4.3 of [DNO13] that \( \mathcal{Z}(\mathcal{D}, \Omega \mathcal{C}) \) is slightly-degenerate as desired, thereby finishing the proof.

We can now state and prove our main theorems.
Theorem 4.2.3. Every bosonic, respectively fermionic, fusion 2-category is Morita equivalent to the 2-Deligne tensor product of a bosonic, respectively fermionic, strongly fusion 2-category and an invertible fusion 2-category.

Proof. Let \( \mathcal{C} \) be a fusion 2-category, and write \( \mathcal{E} \) for the symmetric fusion 1-category \( Z(\mathcal{C}) \). We treat the cases of bosonic and fermionic fusion 2-categories separately.

Case 1: We assume that the fusion 2-category \( \mathcal{C} \) is bosonic. In that case, it follows from lemma 4.2.2 that there exists a fusion 2-category \( \mathcal{D} \) that is Morita equivalent to \( \mathcal{C} \), and such that \( \mathcal{B} := \Omega \mathcal{D} \) is a non-degenerate braided fusion 1-category. Let us now define \( \mathcal{E} := \mathcal{D} \boxtimes \text{Mod}(\mathcal{B}^{\text{rev}}) \), so that \( \Omega \mathcal{E} \simeq \mathcal{B} \boxtimes \mathcal{B}^{\text{rev}} \) by proposition 5.1 of [Dec21c]. In particular, equipping \( \mathcal{B} \) with the canonical braided monoidal functor \( \mathcal{B} \boxtimes \mathcal{B}^{\text{rev}} \to Z(\mathcal{B}) \), we can view \( \mathcal{B} \) as a separable algebra in the fusion 2-category \( \text{Mod}(\mathcal{B} \boxtimes \mathcal{B}^{\text{rev}}) \). Let us write \( \mathcal{G} \) for the fusion 2-category given by \( \text{Bimod}_\mathcal{D}(\mathcal{B}) \). By example 5.3.7 of [Dec22a], see also lemma 3.3.5 above, we find that

\[
\mathcal{E}^0 = \text{Bimod}_{\mathcal{G}^0}(\mathcal{B}) \simeq \mathbf{2Vect}.
\]

This shows that \( \mathcal{E} \) is a bosonic strongly fusion 2-category.

Finally, it follows from lemma 4.4.4 that Morita equivalence is compatible with the 2-Deligne tensor product. In particular, as \( \mathbf{2Vect} \) is Morita equivalent to \( \text{Mod}(\mathcal{B}^{\text{rev}}) \boxtimes \text{Mod}(\mathcal{B}) \), we have that \( \mathcal{C} \) is Morita equivalent to the fusion 2-category \( \mathcal{C} \boxtimes \text{Mod}(\mathcal{B}^{\text{rev}}) \boxtimes \text{Mod}(\mathcal{B}) \). Furthermore, the above discussion shows that \( \mathcal{C} \boxtimes \text{Mod}(\mathcal{B}^{\text{rev}}) \boxtimes \text{Mod}(\mathcal{B}) \) is Morita equivalent to \( \mathcal{E} \boxtimes \text{Mod}(\mathcal{B}) \). But, Morita equivalence is an equivalence relation by theorem 1.4.2, so that we get the desired result in this case.

Case 2: We assume that the fusion 2-category \( \mathcal{C} \) is fermionic. In that case, it follows from lemma 4.2.2 that there exists a fusion 2-category \( \mathcal{D} \) that is Morita equivalent to \( \mathcal{C} \), and such that \( \mathcal{B} := \Omega \mathcal{D} \) is a slightly degenerate braided fusion 1-category. Now, thank to the main theorem of [JFR21], there exists a non-degenerate braided fusion 1-category \( \mathcal{C} \) together with a braided embedding \( \mathcal{B} \subseteq \mathcal{C} \), such that the centralizer of \( \mathcal{B} \) in \( \mathcal{C} \) is exactly \( \text{SVect} \). Let us write \( \mathcal{G} \) for the fusion 2-category given by \( \text{Bimod}_\mathcal{D}(\mathcal{B}) \). By example 5.3.7 of [Dec22a], see also lemma 3.3.5 above, we find that

\[
\mathcal{G}^0 = \text{Bimod}_{\mathcal{G}^0}(\mathcal{B}) \simeq \text{Mod}(\text{SVect} \boxtimes \mathcal{C}^{\text{rev}}).
\]

Namely, as \( \mathcal{C} \) is a non-degenerate braided fusion 1-category, we have that \( Z(\mathcal{C}) \simeq \mathcal{C} \boxtimes \mathcal{C}^{\text{rev}} \) by proposition 3.7 of [DGNO10], so that \( Z(\mathcal{C}, \mathcal{B}) \simeq \text{SVect} \boxtimes \mathcal{C}^{\text{rev}} \) as braided fusion 2-categories. Then, we consider the fusion 2-category \( \mathcal{G} := \mathcal{C} \boxtimes \text{Mod}(\mathcal{C}) \). Using the same argument as the one used in the proof of the first case, we find that \( \mathcal{G} := \text{Bimod}_\mathcal{G}(\mathcal{C}) \) is a fermionic strongly fusion 2-category.

Finally, as \( \mathbf{2Vect} \) is Morita equivalent to \( \text{Mod}(\mathcal{C}) \boxtimes \text{Mod}(\mathcal{C}^{\text{rev}}) \), we have that \( \mathcal{C} \) is Morita equivalent to the fusion 2-category \( \mathcal{C} \boxtimes \text{Mod}(\mathcal{C}) \boxtimes \text{Mod}(\mathcal{C}^{\text{rev}}) \). Furthermore, the above discussion implies that \( \mathcal{C} \boxtimes \text{Mod}(\mathcal{C}) \boxtimes \text{Mod}(\mathcal{C}^{\text{rev}}) \) is Morita equivalent to \( \mathcal{G} \boxtimes \text{Mod}(\mathcal{C}^{\text{rev}}) \). This concludes the proof.

\[
\Box
\]

Theorem 4.2.4. Every fusion 2-category is separable.
Proof. By theorem 4.2.3 up to taking the Deligne tensor product with an invertible fusion 2-category, every fusion 2-category is Morita equivalent to a strongly fusion 2-category. But, thanks to theorem 2.3.2 and corollary 2.3.8, both of these operations induce braided monoidal equivalences at the level of the Drinfeld center. The result then follows by appealing to proposition 4.1.10. 

**Corollary 4.2.5.** The Drinfeld center of any fusion 2-category is equivalent as a braided fusion 2-category to the Drinfeld center of a strongly fusion 2-category.

**Remark 4.2.6.** We note that there are non-trivial braided monoidal equivalences between the Drinfeld centers of strongly fusion 2-categories. For instance, given a finite group $G$ and a 4-cocycle $\pi$ for $G$ with coefficients in $k^\times$, any outer automorphism $\alpha$ of $G$ induces an equivalence $\mathcal{Z}(\mathcal{2Vect}_G) \simeq \mathcal{Z}(\mathcal{2Vect}_{\alpha}(\pi))$ of fusion 2-categories, so that

$$\mathcal{Z}(\mathcal{2Vect}_G) \simeq \mathcal{Z}(\mathcal{2Vect}_{\alpha}(\pi))$$

as braided fusion 2-categories. We expect that any braided monoidal equivalence between the Drinfeld centers of bosonic strongly fusion 2-categories arises in this way. On the other hand, the classification of the Drinfeld centers of fermionic strongly fusion 2-categories up to braided monoidal equivalence is significantly more subtle (see [JFR]).

**Remark 4.2.7.** Let us write $\mathcal{M}$ for the set of Morita equivalence classes of fusion 2-categories. This is a commutative monoidal under the 2-Deligne tensor product. We use $\mathcal{M}^\times$ to denote the maximal subgroup of $\mathcal{M}$. It follows from lemma 4.1.11 and example 5.4.6 of [Déc22a] that $\mathcal{M}^\times$ is isomorphic to $W$, the Witt group of non-degenerate braided fusion 1-categories. Further, let us use $\mathcal{Z}\mathcal{F}\mathcal{C}$ to denote the set of braided monoidal equivalence classes of Drinfeld centers of fusion 2-categories. It follows from theorem 2.3.2 and 2.3.8 that the map $\mathcal{Z} : \mathcal{M} / W \rightarrow \mathcal{Z}\mathcal{F}\mathcal{C}$ sending a Morita equivalence class of fusion 2-categories to its Drinfeld center is well-defined. As a generalization of proposition 4.1 of [JFR21], we conjecture that the map $\mathcal{Z} : \mathcal{M} / W \rightarrow \mathcal{Z}\mathcal{F}\mathcal{C}$ is a bijection. We now give some additional corollaries of our main theorems.

**Corollary 4.2.8.** Every multifusion 2-category is separable.

**Proof.** Every multifusion 2-categories can be split into a direct sum of finitely many indecomposable multifusion 2-categories by lemma 5.2.10 of [Déc22a]. Moreover, by remark 5.4.2 therein, an indecomposable multifusion 2-category is Morita equivalent to a 2-category. The corollary then follows from the observation that the Drinfeld center commutes with finite direct sums, together with theorems 2.3.2 and 4.1.10 above.

**Corollary 4.2.9.** For every multifusion 2-category $\mathcal{C}$, the forgetful 2-functor $F : \mathcal{Z}(\mathcal{C}) \rightarrow \mathcal{C}$ is dominant, i.e., every object of $\mathcal{C}$ is the splitting of a 2-condensation monad in $\mathcal{C}$ supported on an object in the image of $F$. 

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Proof. The monoidal 2-functor $F$ is dominant if and only if $C$ is an indecomposable left $Z(C)$-module 2-category. But, the dual to $Z(C)$ with respect to $C$ is $C \boxtimes C^{mop}$, which is a fusion 2-category. The result thus follows from corollary 5.2.5 of [Dec22a].

Corollary 4.2.10. For every two multifusion 2-categories $C$ and $D$, we have $Z(C \boxtimes D) \simeq Z(C) \boxtimes Z(D)$ as braided monoidal 2-categories.

Proof. Let us write $E := C \boxtimes C^{mop} \boxtimes D \boxtimes D^{mop} \simeq C \boxtimes D \boxtimes (C \boxtimes D)^{mop}$. It follows from proposition 5.1 of [Dec21c] that the separable algebra $R_{C \boxtimes D}$ in $E$ is equivalent to $R_C \boxtimes R_D$. Now, thanks to remark 4.1.3, there is an equivalence of monoidal 2-categories $Z(C \boxtimes D) \simeq \text{Bimod}_E(R_C \boxtimes R_D)$. Furthermore, it follows from lemma 1.4.4 that there is an equivalence $\text{Bimod}_E(R_C \boxtimes R_D) \simeq Z(C \boxtimes D)$ of monoidal 2-categories. Finally, one checks that the composite monoidal 2-functor $Z(C \boxtimes D) \simeq \text{Bimod}_E(R_C \boxtimes R_D) \simeq Z(C \boxtimes D)$ agrees with the canonical braided monoidal 2-functor constructed in lemma 2.2.5. This finishes the proof of the result.

Corollary 4.2.11. Let $C$ be a multifusion 2-category, then $Z(C)$ is a factorizable braided multifusion 2-category, i.e. the canonical braided monoidal 2-functor $Z(C) \boxtimes Z(C)^{rev} \rightarrow Z(Z(C))$ is an equivalence.

Proof. The multifusion 2-categories $Z(C)$ and $C \boxtimes C^{mop}$ are Morita equivalent, so that $Z(C) \boxtimes Z(C)^{rev} \simeq Z(Z(C))$ as braided monoidal 2-category thanks to theorem 2.3.2. Inspection shows that the braided monoidal 2-functor constructed in this proof agrees with the canonical one.

Remark 4.2.12. Let $B$ be a braided multifusion 2-category. We use $Z(2)(B)$ to denote its sylleptic center, as in section 5.1 of [Cra98]. We expect that a braided multifusion 2-category $B$ is factorizable if and only if it is non-degenerate in the sense that there is an equivalence $Z(2)(B) \simeq 2\text{Vect}$ of sylleptic monoidal 2-categories.

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