Entanglement is one of the most interesting and prominent phenomena which distinguish the classical and quantum worlds. It has been recognized as a resource for quantum communication and teleportation, as well as computational tasks [1]. In practice, a significant obstacle to the realization of quantum technologies utilizing entanglement is the inevitable interaction between system and environment, which can lead to decoherence. Therefore, how entanglement between atoms in external environment evolves and how to avoid the influence of environment on entanglement become important issues in quantum information science.

In the past few years, many proposals have been suggested for fighting against the deterioration of entanglement under the impact of environment, such as decoherence-free subspaces [2–4], quantum error correction codes [5–7], dynamical decoupling [8–10] and quantum Zeno dynamics [11, 12, 15]. However, when the time scale characterizing the undesired interaction is too short, dynamical decoupling will not work due to the lack of memory [14, 15]. And the efficiency of Zeno dynamics is restricted by the requirement of high measurement frequency.

Recently it was found that, with the presence of a boundary, the quantum Fisher information of the parameters of the initial atomic state can be shielded from the influence of the vacuum fluctuations in certain circumstances as if it were a closed system [16]. And it was shown that quantum coherence of a two-level atom in the presence of a boundary could be effectively inhibited when the atom is transversely polarizable and near the boundary [17, 18].

It is natural for us to wonder whether entanglement between two atoms can be protected for a long time in the presence of a reflecting plate. By investigating the dynamical evolution of entanglement between two two-level atoms interacting with electromagnetic vacuum fluctuations, we demonstrate that, with the presence of a boundary, the entanglement can indeed be protected from decreasing as if it were isolated from the environment. Since two atoms are considered and the existence of one atom will have an influence on another atom, so we also wonder how this influence is reflected in the evolution of entanglement. Besides, we will

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**Protection of entanglement between two two-level atoms**

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**Abstract**

The dynamical evolution of entanglement between two polarizable two-level atoms in weak interaction with electromagnetic vacuum fluctuations is investigated. We find that, for initial Bell state \( \psi^\pm \), the decay rate of entanglement between atoms is just the superradiant spontaneous emission rate, which depends not only on the spontaneous emission rate of each atom but also on the modulation of the spontaneous emission rate due to the presence of another atom. It is shown that, with the presence of a boundary, the entanglement between transversely polarizable atoms can be protected for a very long time. It is pointed out that when the two atoms in initial Bell state \( \psi^- \) are put close enough, the entanglement between them can also be protected.

**Keywords:** protecting entanglement, modulation of spontaneous emission, superradiant

(Some figures may appear in colour only in the online journal)
also consider the role of atomic polarization in protecting entanglement.

Let us consider two identical two-level atoms interacting with fluctuating electromagnetic fields in vacuum. In such a case, the total Hamiltonian of the coupled system can be described by \( H = H_0 + H_1 + H_2 \). Here \( H_0 \) is the free Hamiltonian of the two atoms and its explicit expression is given by \( H_0 = \sum_{j=1}^{2} \hbar \omega_j \sigma_j^+ \sigma_j^- \), where \( \omega_j \) is the level spacing of atom, \( \sigma_j^+ = |e_j\rangle \langle g_j| \) and \( \sigma_j^- = |g_j\rangle \langle e_j| \) are the raising and lowering operators, respectively, of the atom \( j \). \( H_1 \) is the free Hamiltonian of the quantum field, which takes the form \( H_1 = \sum_{\lambda} \hbar \omega_{\lambda} \Delta_{\lambda}^+ \Delta_{\lambda}^- \). Here \( \Delta_{\lambda}^+ \), \( \Delta_{\lambda}^- \) are the creation and annihilation operators for a photon with momentum \( \lambda \), frequency \( \omega_{\lambda} \) and polarization \( \lambda \). Finally, the atom-field interaction Hamiltonian \( H_2 \) can be written in the electric dipole approximation

\[
H_2 = -e \sum_{j=1}^{2} \vec{r}_j \cdot \vec{E} (\vec{x}_j) = -e \sum_{j=1}^{2} (\vec{d}_j \sigma_j^+ + \vec{d}_j^* \sigma_j^-) \cdot \vec{E} (\vec{x}_j),
\]

where \( e \) is the electron electric charge, \( e \vec{r}_j \) is the electric dipole moment for atom \( j \), \( \vec{d}_j = (\vec{e}_j | \vec{g}_j) \), and \( \vec{E} (\vec{x}_j) \) is the electric field strength evaluated at the position \( \vec{x}_j \) of atom \( j \).

The time evolution of the system is governed by the Schrödinger equation, which in the interaction picture has the form

\[
\frac{i \hbar}{\tilde{\lambda}} \partial_t |\varphi(t)\rangle = H_1(t) |\varphi(t)\rangle,
\]

where \( H_1(t) = -e \sum_{j=1}^{2} (\vec{d}_j \sigma_j^+ + \vec{d}_j^* \sigma_j^-) \cdot \vec{E}(\vec{x}_j, t) \). \( \vec{E}(\vec{x}_j, t) = e^{i \omega_{\lambda} t / \hbar} \vec{E}(\vec{x}_j) e^{-i \omega_{\lambda} t / \hbar} \) and we have let \( \tilde{\lambda}_1 = \tilde{\lambda}_2 = \tilde{\lambda} \) for simplicity. Now decomposing \( \vec{E} (\vec{x}_j, t) \) in \( H_1(t) \) into positive- and negative-frequency parts: \( \vec{E} (\vec{x}_j, t) = \vec{E}^+ (\vec{x}_j, t) + \vec{E}^- (\vec{x}_j, t) \) with \( \vec{E}^+ (\vec{x}_j, t) | 0 \rangle = 0 \) and \( \langle 0 | \vec{E}^- (\vec{x}_j, t) = 0 \), we have \( H_1(t) \) in rotating-wave approximation

\[
H_1(t) = -e \sum_{j=1}^{2} (\vec{d} \cdot \vec{E}^+ (\vec{x}_j, t) \sigma_j^+ e^{i \omega_{\lambda} t} + \vec{d}^* \cdot \vec{E}^- (\vec{x}_j, t) \sigma_j^- e^{-i \omega_{\lambda} t}).
\]

(3)

Taking the initial states of atoms as the Bell state \( \psi^+ \) and of the environment as vacuum state,

\[
|\varphi(0)\rangle = \frac{1}{\sqrt{2}} (|e_1 g_2\rangle + |g_1 e_2\rangle) | 0 \rangle,
\]

(4)

The state vector at time \( t \) can be written as

\[
|\varphi(t)\rangle = b_1(t) |e_1 g_2\rangle | 0 \rangle + b_2(t) |g_1 e_2\rangle | 0 \rangle + \sum_{\lambda} b_{\lambda}^0 (t) |g_1 e_2\rangle | 1_{\lambda}\rangle,
\]

where \( |1_{\lambda}\rangle \) denoting one photon in the mode \( (\lambda, \lambda) \). Now inserting (5) and (3) into (2), we can obtain

\[
\begin{align*}
b_1(t) | 0 \rangle & = \frac{ie}{\hbar} \vec{d} \cdot \vec{E}^+ (\vec{x}_1, t) e^{i \omega_{\lambda} t} \sum_{\lambda} b_{\lambda}^0 (t) | 1_{\lambda}\rangle, \\
b_2(t) | 0 \rangle & = \frac{ie}{\hbar} \vec{d}^* \cdot \vec{E}^- (\vec{x}_2, t) e^{-i \omega_{\lambda} t} \sum_{\lambda} b_{\lambda}^0 (t) | 1_{\lambda}\rangle.
\end{align*}
\]

(6)

Integrating both sides of (7) over time, and substituting the result into (6), we get

\[
\begin{align*}
\tilde{b}_1(t) & = \frac{-e^2}{\hbar^2} \sum_{j=1}^{2} \frac{d_j d_j^*}{\hbar} \int_0^t dt' e^{i \omega_{\lambda} t' - \omega_{\lambda} t} \langle 0 | \vec{E}^+ (\vec{x}_1, t') \vec{E}^- (\vec{x}_1, t') | 0 \rangle b_1(t') + \langle 0 | \vec{E}^+ (\vec{x}_1, t) \vec{E}^- (\vec{x}_1, t') | 0 \rangle b_1(t'), \\
\tilde{b}_2(t) & = \frac{-e^2}{\hbar^2} \sum_{j=1}^{2} \frac{d_j d_j^*}{\hbar} \int_0^t dt' e^{i \omega_{\lambda} t' - \omega_{\lambda} t} \langle 0 | \vec{E}^+ (\vec{x}_1, t) \vec{E}^- (\vec{x}_1, t') | 0 \rangle b_1(t') + \langle 0 | \vec{E}^+ (\vec{x}_2, t) \vec{E}^- (\vec{x}_2, t') | 0 \rangle b_1(t').
\end{align*}
\]

(8)

Then we can take Laplace transformation of (8) to have

\[
\begin{align*}
\tilde{s}_1(s) & = \frac{-e^2}{\hbar^2} \sum_{j=1}^{2} \frac{d_j d_j^*}{\hbar} \int_0^\infty dt' e^{-st - \omega_{\lambda} t'} \langle 0 | \vec{E}^+ (\vec{x}_1, t) \vec{E}^- (\vec{x}_1, t') | 0 \rangle b_1(t'), \\
\tilde{s}_2(s) & = \frac{-e^2}{\hbar^2} \sum_{j=1}^{2} \frac{d_j d_j^*}{\hbar} \int_0^\infty dt' e^{-st - \omega_{\lambda} t'} \langle 0 | \vec{E}^+ (\vec{x}_2, t) \vec{E}^- (\vec{x}_2, t') | 0 \rangle b_1(t'),
\end{align*}
\]

(9)

where \( \tilde{b}_{1/2}(s) = \int_0^\infty dt \tilde{b}_{1/2}(t) e^{-st} \), \( L_{ab}(s) = \frac{e^2}{\hbar^2} \sum_{j=1}^{2} \frac{d_j d_j^*}{\hbar} \int_0^\infty dt e^{i \omega_{\lambda} t' - \omega_{\lambda} t'} \langle 0 | \vec{E}^+ (\vec{x}_a, t) \vec{E}^- (\vec{x}_b, t') | 0 \rangle b_1(t'). \)

Consider the two atoms separated by a distance \( r \) and positioned at the same side of the boundary, which is located at \( z = 0 \). The trajectories of atoms can be described by \( \vec{x}_1 = (x_0, y_0, z_0) \), \( \vec{x}_2 = (x_0 + r, y_0, z_0) \). Then the electric field correlation function for the trajectories can be calculated to get [19]
We assume the interaction between the atoms and the field to be weak. So the Wigner–Weisskopf approximation can be adopted by neglecting the z dependence of $L_{ab}(ic)$. Thus one has

$$L_{ab}(0) = \frac{\alpha^2}{\hbar} d_1^2 \left[ \frac{1}{2} G_{ab}(\omega_0) + i K_{ab}(\omega_0) \right]$$

with

$$G_{ab}(\omega_0) = \int_{-\infty}^{\infty} d\omega \left|\langle 0|E_r(\tilde{x}_a, t)E_r(\tilde{x}_b, 0)|0\rangle\right|^2,$$

$$K_{ab}(\omega_0) = \frac{P}{2\pi} \int_{-\infty}^{\infty} G_{ab}(\omega) d\omega.$$

Here $P$ denotes the Cauchy principal value. Note that $\frac{\alpha^2}{\hbar} d_1^2 G_{ab}(\omega_0) = \gamma_{11}$ is the spontaneous emission rate of one two-level atom [20], $\frac{\alpha^2}{\hbar} d_1^2 K_{ab}(\omega_0) = \gamma_{12}$ is the modulation of the spontaneous emission rate of one atom due to the presence of another atom [21], $\frac{\alpha^2}{\hbar} d_1^2 K_{ab}(\omega_0) \propto \gamma_{11}$ corresponds to the level shift of the two-level atom which can be neglected since it is irrelevant to our purposes and $\frac{\alpha^2}{\hbar} d_1^2 K_{ab}(\omega_0) = V$ is the dipole–dipole interaction potential. Then from (12) and (13), the state probability amplitudes can be obtained

$$b_1(t) = b_2(t) = \frac{\sqrt{2}}{2} e^{-i(\gamma_{11} + \gamma_{12} + 2V)t}.$$

Now let us investigate the dynamics of entanglement between the two atoms. We take concurrence [22] as a measure of entanglement, which is defined by

$$C = \max \{0, \sqrt{\lambda_1 - \lambda_2 - \lambda_3 - \lambda_4} \},$$

where $\lambda_i$ are the eigenvalues (in descending order) of the Hermitian matrix $\hat{p} \rho \hat{p}$, in which $\rho = \sigma_\uparrow \otimes \sigma_\uparrow \rho' \sigma_\downarrow \otimes \sigma_\downarrow$ and $\sigma_\uparrow$ is a Pauli matrix. The concurrence is 1 for the maximally entangled states and 0 for separable states. The reduced density matrix $\rho'$, which is obtained by tracing the density matrix of the total system over the field degrees of freedom, can be written in the basis of the product states, $|1\rangle = |e_1e_2\rangle$, $|2\rangle = |g_1g_2\rangle$, $|3\rangle = |g_1e_2\rangle$, $|4\rangle = |g_1e_2\rangle$. In this basis, the concurrence is $C(t) = 2 \max \{0, |\gamma_{21}| \} = 2 |b_1(t)b_2^*(t)|$ [23]. Applying (15), we can find

$$C(t) = e^{-i(\gamma_{11} + \gamma_{12} + 2V)t},$$

where $\gamma_{ab}$ can be found by inserting (10) into (14)

$$\gamma_{11} = \gamma_{11}^{(0)} - 3 \gamma_{11}^{(0)} \frac{Z \cos Z + (Z^2 - 1) \sin Z}{2Z^2},$$

$$\gamma_{12} = \gamma_{12}^{(0)} \frac{\sin R - R \cos R}{R^3} - \frac{3 \gamma_{11}^{(0)}}{2(1 + \cos Z)^2} \left[ \frac{Z^2 - 2R^2}{\sqrt{R^2 + Z^2}} \cos \sqrt{R^2 + Z^2} + \left( \frac{2R^2 - Z^2}{\sqrt{R^2 + Z^2}} + Z^2 \right) \sin \sqrt{R^2 + Z^2} \right].$$

Here $\gamma_{11}^{(0)} = \frac{\alpha^2}{\hbar} d_1^2 G_{ab}(\omega_0)$ is the spontaneous decay rate of the atom in unbounded space, and the dimensionless parameters $Z \equiv 2(1 + \cos Z) / \hbar$, $R \equiv R \cos R / \hbar$ are introduced for simplicity. The second part of $\gamma_{11}$ is the correction induced by the boundary, and it can be seen that it is an oscillating function of the time required for a photon emitted by an atom to make a round trip between the atom and the boundary. The first term of $\gamma_{12}$, as the vacuum term of modulation of the spontaneous emission rate, is an oscillating function of the time needed by a photon to travel between the two atoms; the second term, as the correction induced by the boundary, is an oscillating function of the time required for a photon emitted by an atom to be reflected by the boundary and then be reabsorbed by the other atom. In the evolution of entanglement the influence of one atom on another is reflected in $\gamma_{12}$.

Although the concurrence between the two atoms decreases exponentially with time—as we show in (16)—when $Z$ is small, it can be found that the decay rate is proportional to $Z^2$. So in the case that the boundary is placed very close to atoms, $Z \to 0$, the decay rate will tend to zero, as is illustrated in figure 1. Thus the entanglement can be totally protected and remain constant for a long time. To show the efficiency of the presence of the boundary, we find that when $Z$ takes 0.5, 0.1, 0.05, respectively, the decay rate can reach to $5 \times 10^{-2}\gamma_{11}^{(0)}, 2 \times 10^{-3}\gamma_{11}^{(0)}$ and $5 \times 10^{-4}\gamma_{11}^{(0)}$.

Previously, we only consider the polarizations of the two atoms in $x$-direction. Next we will apply the above developed formalism to other cases. Here, let us note that the polarization directions of the atoms play a crucial role in the entanglement dynamics [24] and interatomic resonance interaction [25]. When the polarizations are along the $y$-axis, the spontaneous emission rate and the corresponding modulation can be written as

$$\gamma_{11} = \gamma_{11}^{(0)} - 3 \gamma_{11}^{(0)} \frac{Z \cos Z + (Z^2 - 1) \sin Z}{2Z^2},$$

$$\gamma_{12} = 3 \gamma_{11}^{(0)} \frac{\sin R - R \cos R}{R^3} - \frac{3 \gamma_{11}^{(0)}}{2(1 + \cos Z)^2} \left[ \cos \sqrt{R^2 + Z^2} + (\sqrt{R^2 + Z^2} - Z^2) \sin \sqrt{R^2 + Z^2} \right].$$

In the case that $Z$ is small, the decay rate is also proportional to $Z^2$. So the entanglement can also be shielded in the limit $Z \to 0$. For the polarizations in $z$-axis, similarly we have
The coherence for two atoms. In [17], the entanglement in our case is just coherence. So our results look, we can find that these two qubits have no interaction, so

If we put our two atoms very far from

The entanglement can also be protected, since in such a case \( \gamma_{12} = \gamma_{11} \) as can be verified from their definitions.

According to Dicke’s theory [27, 28], the two two-level atoms can be treated as a single four-level system, and the transition rate from collective states \( \psi^\pm \) to ground state \( |g_1g_2\rangle \) have superradiant decay rate \( \Gamma_{\pm} \). Meanwhile in this transition, the entanglement between atoms decreases from 1 to 0 with decay rate \( \gamma_{11} \pm \gamma_{12} \). These two decay rates should be in proportion. Besides, if we do not consider the influence of the boundary and then compare \( \gamma_{11} \pm \gamma_{12} \) in (18) or (19) with the formula (1) and figure 2 in [28], we can find that \( \gamma_{11} \pm \gamma_{12} \) are actually \( \Gamma_{\pm} \), respectively. Thus we can identify \( \gamma_{11} \pm \gamma_{12} \) with superradiant spontaneous emission rate. The appearance of superradiance in our model stems from the fact that light emitted by one atom can be absorbed by another atom, thus leading to cooperative processes in the emission [27–29].

In summary, we have investigated the dynamical evolution of entanglement between two polarizable two-level atoms in weak interaction with a bath of fluctuating vacuum electromagnetic fields. Under the condition that the initial state is Bell state \( \psi^\pm \), we find that the entanglement between atoms decreases exponentially with decay rate equal to their collective superradiant decay rate, which depends on the spontaneous emission rate of each atom and the modulation of the spontaneous emission rate due to the presence of another identical atom. It is shown that the entanglement is atomic polarization and position dependent. When the polarizations of atoms are in the xy plane and the distance between atoms and boundary is small, the entanglement will be protected for a very long time. It is all shown that the entanglement between atoms can also be protected if the atoms are in state \( \psi^- \) and positioned close enough.

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