q-SERIES AND L-FUNCTIONS RELATED TO HALF-DERIVATIVES OF THE
ANDREWS–GORDON IDENTITY

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ABSTRACT. Studied is a generalization of Zagier’s q-series identity. We introduce a generating function of L-functions at non-positive integers, which is regarded as a half-differential of the Andrews–Gordon q-series. When q is a root of unity, the generating function coincides with the quantum invariant for the torus knot.

1. Introduction

In Ref. [13], Zagier studied the q-series,

\[ X(q) = \sum_{n=0}^{\infty} (1 - q)(1 - q^2) \cdots (1 - q^n) \]

and proved that the asymptotic expansion is given by

\[ X(e^{-t}) = e^{t/24} \sum_{n=0}^{\infty} \frac{T(n)}{n!} \left( \frac{t}{24} \right)^n \]

Here \( T(n) \) is the Glaisher T-number,

\[ \frac{\text{sh}(2x)}{\text{ch}(3x)} = \sum_{n=0}^{\infty} \chi_{12}(n) e^{-nx} = 2 \sum_{n=0}^{\infty} (-1)^n \frac{T(n)}{(2n+1)!} x^{2n+1} \]

and is given in terms of the Dirichlet L-function as

\[ T(n) = \frac{1}{2} (-1)^{n+1} L(-2n - 1, \chi_{12}) \]

where \( \chi_{12}(n) \) is the Dirichlet character with modulus 12 defined by

| n mod 12 | 1 | 5 | 7 | 11 | others |
|-----------|---|---|---|----|-------|
| \( \chi_{12}(n) \) | 1 | -1 | -1 | 1 | 0 |

It was pointed out that the right hand side of eq. (2) is regarded as a half-differential of the Dedekind \( \eta \)-function with weight 1/2. Interesting is that the function \( X(q) \) is intimately connected with the knot theory; it is a generating function of an upper bound of the number of linearly independent Vassiliev invariants.

Purpose of this paper is to study a generalization of Zagier’s identity (see Refs. [4, 5, 10] for this attempt). Our motivation is based on an observation that the q-series \( X(q) \) with q being root of unity
appears as a colored Jones invariant of the trefoil \([8, 11]\). We shall show that the \(q\)-series, which reduces to the invariant of the torus knot in a case of \(q\) being root of unity, becomes the generating function of the \(L\)-function with negative integers. We note that a relationship between the modular form and the quantum invariant was discussed in Ref. [9], where the Witten–Reshetikhin–Turaev invariant of the Poincaré homology sphere was studied. Throughout this paper we use a standard notation,

\[
(x)_n = (x; q)_n = \prod_{i=1}^{n}(1 - x q^{i-1})
\]

\[(x_1, \ldots, x_j)_n = (x_1, \ldots, x_j; q)_n = (x_1)_n \cdots (x_j)_n
\]

\[
\left[\frac{n}{c}\right] = \begin{cases} 
\frac{(q)_n}{(q)_c (q)_{n-c}} & \text{for } 0 \leq c \leq n \\
0 & \text{otherwise}
\end{cases}
\]

We state the main result of this article. Let the \(q\)-series \(X_2^{(a)}(q)\) for \(a = 0, 1\) be

\[
X_2^{(0)}(q) = \sum_{n=0}^{\infty} \sum_{c=0}^{n} (q)_n q^{2c+c} \left[\frac{n}{c}\right] 
\]

\[
X_2^{(1)}(q) = \sum_{n=1}^{\infty} \sum_{c=0}^{n} (q)_{n-1} q^{2c} \left[\frac{n}{c}\right]
\]

**Theorem 1.**

\[
X_2^{(0)}(e^{-t}) = e^{9t/40} \sum_{n=0}^{\infty} \frac{T_2^{(0)}(n)}{n!} \left( \frac{t}{40} \right)^n
\]

\[
X_2^{(1)}(e^{-t}) = e^{t/40} \sum_{n=0}^{\infty} \frac{T_2^{(1)}(n)}{n!} \left( \frac{t}{40} \right)^n
\]

where

\[
T_2^{(a)}(n) = \frac{1}{2} (-1)^{n+1} L(-2n - 1, \chi_2^{(a)})
\]

We note that, using the Mellin transformation, we have the generating function of the \(T\)-series;

\[
\frac{\text{sh}(2(a+1)x)}{\text{ch}(5x)} = \sum_{n=0}^{\infty} X_2^{(a)}(n) e^{-nx} = 2 \sum_{n=0}^{\infty} (-1)^n \frac{T_2^{(a)}(n)}{(2n+1)!} x^{2n+1}
\]

where the periodic function \(X_2^{(a)}(n)\) are

| \(n \mod 20\) | 3 | 7 | 13 | 17 | other |
|---|---|---|---|---|---|
| \(X_2^{(0)}(n)\) | 1 | -1 | -1 | 1 | 0 |

| \(n \mod 20\) | 1 | 9 | 11 | 19 | other |
|---|---|---|---|---|---|
| \(X_2^{(1)}(n)\) | 1 | -1 | -1 | 1 | 0 |
To see a relationship with a modular form, we recall the well known Rogers–Ramanujan identity (see, e.g., Refs. [1, 2, 6]):

\[(q)_\infty \sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q)_n} = (q, q^4, q^5, q^5)_\infty \]

(8a)

\[= \sum_{n=0}^{\infty} \chi^{(0)}_{20}(n) q^{(n^2-9)/40} \]

\[(q)_\infty \sum_{n=0}^{\infty} \frac{n^2 q^2}{(q)_n} = (q^2, q^3, q^5, q^5)_\infty \]

(8b)

\[= \sum_{n=0}^{\infty} \chi^{(1)}_{20}(n) q^{(n^2-1)/40} \]

Those functions are the two-dimensional representation of the modular group with weight 1/2, and Theorem 1 indicates that the \( q \)-series \( X_2^{(a)}(q) \) is related with a “half-differential” of the Rogers–Ramanujan \( q \)-series;

\[ X_2^{(a)}(q) = -\frac{1}{2} \sum_{n=0}^{\infty} nX_{20}^{(a)}(n) q^{\frac{n^2-(3-2a)^2}{40}} \]

(9)

Here two sides cannot be defined simultaneously but the equality holds as the Taylor expansions of \( q \to e^{-t} \).

The Rogers–Ramanujan identity can be generalized to the Andrews–Gordon identity [2]; let \( m \in \mathbb{Z}_{>1} \) and \( 0 \leq a \leq m - 1 \), then

\[ \sum_{n_1 \geq \cdots \geq n_{m-1} \geq 0} \frac{q^{n_1^2+n_2^2+\cdots+n_{m-1}^2+n_{m-1}+\cdots+n_{m-1}}}{(q)_{n_1-n_2}(q)_{n_2-n_3} \cdots (q)_{n_{m-1}}} = \prod_{n=1}^{\infty} (1 - q^n)^{-1} \]

\[= \frac{1}{(q)_\infty} \sum_{n=0}^{\infty} \chi_{8m+4}^{(a)}(n) q^{\frac{n^2-(2m-2a-1)^2}{8(2m+1)}} \]

(10)

where we have introduced the periodic function \( \chi_{8m+4}^{(a)}(n) \) as

\[
\begin{array}{c|cccccc}
  n \mod (8m+4) & 2m - 2a - 1 & 2m + 2a + 3 & 6m - 2a + 1 & 6m + 2a + 5 & \text{others} \\
  \chi_{8m+4}^{(a)}(n) & 1 & -1 & 1 & 1 & 0 \\
\end{array}
\]

As a generalization of Theorem 1 to the \( q \)-series related to a half-derivative of the Andrews–Gordon \( q \)-series in a sense of eq. (9), we define the function \( X_m^{(a)}(q) \) with \( m \in \mathbb{Z}_{>0} \) and \( a = 0, 1, \ldots, m - 1 \) by

\[ X_m^{(a)}(q) = \sum_{k_1, k_2, \ldots, k_m = 0}^{\infty} (q)_{k_1}^m q^{k_1^2+\cdots+k_{m-1}^2+k_{m-1}+\cdots+k_{m-1}} \prod_{i=1}^{m-1} \left[ \frac{k_{i+1}}{k_i} \right] \left[ \frac{k_{a+1} + 1}{k_a} \right] \]

(11)
Theorem 2.

\[
X_m^{(a)}(e^{-t}) = e^{\frac{(2m+2m+1)^2}{8(2m+1)^2}} \sum_{n=0}^{\infty} \frac{T_m^{(a)}(n)}{n!} \left( \frac{t}{8(2m+1)} \right)^n
\]

where T-series is given by the L-function

\[
T_m^{(a)}(n) = \frac{1}{2} (-1)^{n+1} L(-2n-1, \chi_{8m+4}^{(a)}) = (-1)^n 2^{4n} \frac{(2m+1)^{2n+1}}{n+1} \sum_{r=1}^{8m+4} X_{8m+4}^{(a)}(r) B_{2n+2} \left( \frac{r}{8m+4} \right)
\]

with the Bernoulli polynomial \( B_n(x) \).

In this case the generating function of the T-series is written as

\[
\frac{\text{sh}(2(a+1)x)}{\text{ch}((2m+1)x)} = \sum_{n=0}^{\infty} \chi_{8m+4}^{(a)}(n) e^{-nx} = 2 \sum_{n=0}^{\infty} (-1)^n \frac{T_m^{(a)}(n)}{(2n+1)!} x^{2n+1}
\]

See that a case of \( m = 2 \) corresponds to Theorem 1 and that \( m = 1 \) is nothing but Zagier’s identity (2). Furthermore above Theorem shows that

\[
X_m^{(a)}(q) = -\frac{1}{2} \sum_{n=0}^{\infty} n \chi_{8m+4}^{(a)}(n) q^{n^2-(2m+2m+1)^2} \chi_{8m+4}^{(a)}(n)
\]

as a generalization of eq. (9).

For our later convention we collect q-series identities as follows (see, e.g., Refs. [1, 6]);

- q-binomial coefficient

\[
\begin{bmatrix} n+1 \\ c \end{bmatrix} = q^c \begin{bmatrix} n \\ c \end{bmatrix} + \begin{bmatrix} n \\ c-1 \end{bmatrix}
\]

- q-binomial formula

\[
\sum_{n=0}^{\infty} \frac{(a)_n}{(q)_n} z^n = \frac{(az)_\infty}{(z)_\infty} = \frac{(az)_\infty}{z_{\infty}} x = \frac{(z)^k}{x^k}
\]

- the Jacobi triple product identity

\[
\sum_{k=-\infty}^{\infty} (-1)^k q^{k^2} x^k = (q)_\infty (x^{-1} q^\frac{1}{2})_\infty (x q^\frac{1}{2})_\infty
\]

In Section 2 we prove Theorem 1. Section 3 is for the proof of Theorem 2. Strategy to prove these theorems is essentially same with a proof of eq. (2) in Ref. [13]; we define the function \( H_m^{(a)}(x) \) and derive the q-series \( X_m^{(a)}(q) \) as a differential of \( H_m^{(a)}(x) \). We comment on a relationship between the quantum knot invariant and our q-series in Section 4.
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2. PROOF OF THEOREM[1]

We define

(19) \[ H(x) = \sum_{n=0}^{\infty} \sum_{c=0}^{n} (x)_{n+1} x^n \cdot q^{c+2c} z^{2c} \binom{n}{c} \]

Proposition 3.

(20) \[ H(x) = \sum_{n=0}^{\infty} \chi_{20}^{(0)}(n) q^{(n^2-9)/40} x^{(n-3)/2} \]

Proof. We prove this identity by showing that both hand sides satisfy the same difference equation. For the right hand side, we have

\[ H(x) = \sum_{n=0}^{\infty} \chi_{20}^{(0)}(n) q^{(n^2-9)/40} x^{(n-3)/2} \]

\[ = 1 - q x^2 + \sum_{n=10}^{\infty} \chi_{20}^{(0)}(n) q^{(n^2-9)/40} x^{(n-3)/2} \]

(21)

where we have used \( \chi_{20}(n+10) = -\chi_{20}(n) \).

To study the difference equation for the left hand side, we further define

(22) \[ H(x, y, z) = \sum_{n=0}^{\infty} \sum_{c=0}^{n} (x)_{n} y^n \cdot q^{c+2c} z^{2c} \binom{n}{c} \]

and we investigate the difference equation of \( H(x, y, z) \).

We have

\[ H(x, y, q z) = \sum_{n=0}^{\infty} \sum_{a=0}^{n+1} (x)_{n} y^n q^{a^2+a-2} z^{2a-2} \binom{n}{a-1} \]

\[ = \sum_{m=1}^{\infty} \sum_{a=0}^{m} (x)_{m-1} y^{m-1} q^{a^2+a-2} z^{2a-2} \binom{m}{a} \]

\[ - q^{-2} z^{-2} \sum_{n=0}^{\infty} \sum_{a=0}^{n} (x)_{n} y^n q^{a^2+a} (q^{1/2} z)^{2a} \binom{n}{a} \]
= \sum_{m=1}^{\infty} \sum_{a=0}^{m} \frac{(q^{-1})_m}{1-q^{-1}x} y^{m-1} q^{a^2 + a - 2} z^{2a - 2} \left[ \begin{array}{c} m \\ a \end{array} \right] 
- q^{-2} z^{-2} H(x, y, q^{1/2} z) 
= \frac{y^{-1} q^{-2} z^{-2}}{1-q^{-1}x} \left( H(q^{-1} x, y, z) - 1 \right) - q^{-2} z^{-2} H(x, y, q^{1/2} z) 
(23)

In the same manner, we have following;

\[ H(x, qy, z) = \sum_{n=0}^{\infty} \sum_{c=0}^{n} (x)_n \left( 1 - (1 - q^n x) \right) x^{-1} y^n q^{n^2 + c} z^{2c} \left[ \begin{array}{c} n \\ c \end{array} \right] \]

= x^{-1} H(x, y, z) - \sum_{n=0}^{\infty} \sum_{c=1}^{n+1} (x)_{n+1} x^{-1} y^n q^{n^2 + n + 1} z^{2c} \left[ \begin{array}{c} n \\ c - 1 \end{array} \right] 
+ \sum_{n=0}^{\infty} \sum_{c=1}^{n+1} (x)_{n+1} x^{-1} y^n q^{n^2 + n + 1} z^{2c} \left[ \begin{array}{c} n \\ c - 1 \end{array} \right] 
= x^{-1} H(x, y, z) - \sum_{m=1}^{\infty} \sum_{c=0}^{m} (x)_m x^{-1} y^{m-1} q^{2^2 + c} z^{2c} \left[ \begin{array}{c} m \\ c \end{array} \right] 
+ \sum_{n=0}^{\infty} \sum_{a=0}^{n} (x)_{n+1} x^{-1} y^n q^{a^2 + a + 1} z^{2a + 2} \left[ \begin{array}{c} n \\ a \end{array} \right] 
= x^{-1} y^{-1} + x^{-1} (1 - y^{-1}) H(x, y, z) + q^2 (1 - x) x^{-1} z^{2} H(q x, q y, q^{1/2} z) 
(24)

We combine these two difference equations; from eqs. (24) we eliminate \( H(q x, q y, q^2 z) \) using eq. (23). We get

\[ (x y - q z^2) H(x, q y, z) + (1 - y) H(x, y, z) = -q^4 (1 - x) y z^4 H(q x, q y, q z) + 1 - q z^2 \]

When we substitute \((x, y, z) \rightarrow (q x, x, x)\) to this equation, the first term vanishes. Recalling that

\[ H(x) = (1 - x) H(q x, x, x) \]

we obtain eq. (21). \(\square\)

Setting \( x \rightarrow 1 \) in Prop. 3, we see that the right hand side reduces to the Rogers–Ramanujan \(q\)-series due to the Jacobi triple identity,

\[ H(x = 1) = (q, q^4, q^5; q^5)_\infty \]

For the left hand side (19), we see this fact from the following lemma, which can be proved by use of the binomial formula (17).
Lemma 4.

\[ H(x) = (q x)_{\infty} \sum_{c=0}^{\infty} \frac{q^{c^{2} + c}}{(q x)^{c}} x^{3c} + (1 - x) \sum_{n=0}^{\infty} \sum_{c=0}^{n} \left( (q x)_n - (q x)_{\infty} \right) x^n q^{c^{2} + c} x^{2c} \left[ \frac{n}{c} \right] \]

Before proceeding to the proof of Theorem 1, we recall the known result on the Mellin transformation;

**Proposition 5.** Let \( \chi_p \) be a periodic function with modulus \( p \) with mean value zero, and

\[ L(s, \chi_p) = \sum_{n=1}^{\infty} \frac{\chi_p(n)}{n^s} \]

As \( t \downarrow 0 \) we have

\[ \sum_{n=0}^{\infty} n \chi_p(n) e^{-n^2 t} \sim \sum_{n=0}^{\infty} L(-2 n - 1, \chi_p) \frac{(-t)^n}{n!} \]

**Proof.** Assumptions of \( \chi_p \) support that \( L(s, \chi_p) \) has an analytic continuation to \( \mathbb{C} \). We apply the Mellin transformation to

\[ \sum_{n=0}^{\infty} n \chi_p(n) e^{-n^2 t} \sim \sum_{n=0}^{\infty} \gamma_n t^n \]

From the left hand side, we have

\[ \sum_{n=0}^{\infty} n \chi_p(n) \int_{0}^{\infty} t^{-1} e^{-n^2 t} dt = \sum_{n=0}^{\infty} \chi_p(n) \frac{\Gamma(s)}{n^{2s-1}} = \Gamma(s) L(2 s - 1, \chi_p) \]

We also have from the right hand side that

\[ \int_{0}^{\infty} \left( \sum_{n=0}^{N-1} \gamma_n t^n + O(t^N) \right) t^{-1} dt = \sum_{n=0}^{N-1} \gamma_n \frac{n}{n + s} + R_N(s) \]

where \( R_N(s) \) is analytic in \( \Re(s) > -N \). Thus \( \gamma_n \) is the residue of \( \Gamma(s) L(2 s - 1, \chi_p) \) at \( s = -N \), and we get

\[ \gamma_n = (-1)^n \frac{L(-2 n - 1, \chi_p)}{n!} \]

**Proof of eq. (6) in Theorem 1.** We equate eq. (20) with eq. (27), and we set \( x \to 1 \) after differentiating with respect to \( x \). Using eq. (8a), we get

\[ \sum_{n=0}^{\infty} n x_2^{(0)}(n) q^{(n^2 - 9)/40} = - \sum_{n=0}^{\infty} \sum_{c=0}^{n} ((q)_n - (q)_{\infty}) q^{c^{2} + c} \left[ \frac{n}{c} \right] + (q)_{\infty} \sum_{c=0}^{\infty} \frac{q^{c^{2} + c}}{(q)_c} \left( 3 c + \sum_{i=1}^{c} \frac{q^i}{1 - q^i} \right) + (q, q^4, q^5; q^5)_{\infty} \left( \frac{3}{2} - \sum_{i=1}^{\infty} \frac{q^i}{1 - q^i} \right) \]
We substitute $q = e^{-t}$ to eq. (29), and study the Taylor expansion of $t$. Therein terms including infinite products such as $(q)_{\infty}$ and $(q, q^4, q^5; q^5)_{\infty}$ vanish as they induce an infinite order of $t$. Then we get eq. (9). Using Prop. 5 we recover eq. (6) in Theorem 1.

For the proof of the rest of Theorem 1 we define the function $G(x)$ by

\[(30)\quad G(x) = \sum_{n=1}^{\infty} \sum_{c=0}^{n} (x)_n x^{n-1} q^{c^2} x^{2c} \left[ \frac{n}{c} \right] \]

**Proposition 6.**

\[(31)\quad G(x) = \sum_{n=0}^{\infty} \chi_2 \left( \frac{1}{20} (n) \right) q^{(n^2-1)/40} x^{(n-1)/2} \]

**Proof.** We show that both hand sides satisfy the same $q$-difference equation. It is easy to see that the right hand side satisfies

\[(32)\quad G(x) = 1 - q^2 x^4 - q^3 x^5 G(q x) \]

For the left hand side, we substitute $(x, y, z) \rightarrow (x, x, q^{-1/2} x)$ in eq. (25). Recalling that the function $G(x)$ in eq. (30) is given by

\[G(x) = \frac{1}{x} \left( H(x, x, q^{-1/2} x) - 1 \right)\]

we obtain eq. (32). \(\square\)

One sees from eq. (31) using eq. (8b) that the function $G(x)$ gives the Rogers–Ramanujan $q$-series

\[G(x = 1) = (q^2, q^3, q^5; q^5)_{\infty}\]

For expression (30), we can rewrite as follows using the binomial formula (17). This lemma supports above equality.

**Lemma 7.**

\[(33)\quad G(x) = (q x)_{\infty} \sum_{c=0}^{\infty} \frac{q^{c^2}}{(q x)_c} x^{3c-1} - (1 - x) x^{-1} (q x)_{\infty} + (1 - x) \sum_{n=1}^{\infty} \sum_{c=0}^{n} ((q x)_{n-1} - (q x)_{\infty}) x^{n-1} q^{c^2} x^{2c} \left[ \frac{n}{c} \right] \]
Proof of eq. (7) in Theorem 1. We differentiate both eqs. (31) and (33) w.r.t. $x$ and substitute $x \to 1$. Using eq. (8b) we have

\[
\sum_{n=0}^{\infty} \frac{n}{2} x^{20(n^2-1)/40} = - \sum_{n=1}^{\infty} \sum_{c=0}^{n} ((q)_{n-1} - (q)_{\infty}) q^{c^2} \left[ \frac{n}{c} \right] 
\]

\[
- (q^2, q^3, q^5, q^6) \left( \frac{1}{2} + \sum_{i=1}^{\infty} \frac{q^i}{1-q^i} \right) + (q)_{\infty} \sum_{c=0}^{\infty} \frac{q^{c^2}}{(q)_{\infty}} \left( 3c + \sum_{i=1}^{c} \frac{q^i}{1-q^i} \right)
\]

We substitute $q = e^{-t}$, and we obtain eq. (7). Prop. 5 proves eq. (7) in Theorem 1. \(\square\)

3. Proof of Theorem 2

We define the function $H_m^{(a)}(x)$ for $m \in \mathbb{Z}_{>0}$ and $a = 0, 1, \ldots, m - 1$ by

\[
H_m^{(a)}(x) = \sum_{k_1, \ldots, k_m=0}^{\infty} (x)_{k_{m+1}} x^{k_m} \left( \prod_{i=1}^{m} q_i^{k_i} x^{2k_i} \left[ \frac{k_{i+1}}{k_i} \right] \right)
\]

\[
\times q^{k_2} x^{k_a} \left[ \frac{k_{a+1}+1}{k_a} \right] \cdot \left( \prod_{i=a+1}^{m-1} q_i^{k_i} \right) x^{2k_a} \left[ \frac{k_{i+1}}{k_i} \right]
\]

Proposition 8.

\[
H_m^{(a)}(x) = \sum_{n=0}^{\infty} \chi_{8m+4}^{(a)}(n) q^{\frac{x^{2-(2m-2a-1)^2}}{8(2m+1)}} x^{-\frac{(2m-2a-1)^2}{2}}
\]

Proof: Method is essentially same with a proof of Prop. 3. We prove that both sides satisfy the same difference equation as a function of $x$. Anti-periodicity, $\chi_{8m+4}^{(a)}(n+4m+2) = -\chi_{8m+4}^{(a)}(n)$, shows that the r.h.s. satisfies the difference equation

\[
H_m^{(a)}(x) = 1 - q^{x+1} x^{2a+2} - q^{2m-a} x^{2m+1} H_m^{(a)}(q x)
\]

For the l.h.s. we prepare several difference equations for the following functions;

\[
H_m^{(a)}(x, y, z_1, \ldots, z_{m-1}) = \sum_{k_1, \ldots, k_{m}=0}^{\infty} (x)_{k_{m+1}} y^{k_m} \times \left( \prod_{i=1}^{m-1} q_i^{k_i} z_i \left[ \frac{k_{i+1}}{k_i} \right] \right) q^{k_a} z_a \left[ \frac{k_{a+1}+1}{k_a} \right] \times \left( \prod_{i=a+1}^{m-1} q_i^{k_i} \right) z_a \left[ \frac{k_{i+1}}{k_i} \right]
\]

We note that

\[
H_m^{(a)}(x) = (1 - x) H_m^{(a)}(q x, x, x)
\]
where for brevity we have used a notation, \( x = (x, \ldots, x) \).

By applying eq. (16a) to \( \begin{bmatrix} k_{a+1} + 1 \\ k_a \end{bmatrix} \) in the definition (35) of \( H_m^{(a)}(x, y, z_1, \ldots, z_{m-1}) \), we obtain a following equation:

\[
H_m^{(a)}(x, y, z_1, \ldots, z_{m-1}) = H_m^{(0)}(x, y, q^{\frac{1}{2}}z_1, \ldots, q^{\frac{1}{2}}z_{a-1}, z_a, \ldots, z_{m-1}) \\
+ q z_a^2 H_m^{(a-1)}(x, y, z_1, \ldots, z_{a-1}, q^{\frac{1}{2}}z_a, z_{a+1}, \ldots, z_{m-1})
\]

When we apply eq. (16b) in place of eq. (16a), we get for \( a = 0, \ldots, m - 2 \)

\[
H_m^{(a)}(x, y, z_1, \ldots, z_{m-1}) = H_m^{(0)}(x, y, q^{\frac{1}{2}}z_1, \ldots, q^{\frac{1}{2}}z_{a+1}, z_{a+1}, \ldots, z_{m-1}) \\
+ q z_a^2 H_m^{(a-1)}(x, y, z_1, \ldots, z_a, q^{\frac{1}{2}}z_{a+1}, z_{a+2}, \ldots, z_{m-1})
\]

For \( a = m - 1 \) we have

\[
H_m^{(m-1)}(x, y, z_1, \ldots, z_{m-1}) = H_m^{(0)}(x, y, q^{\frac{1}{2}}z_1, \ldots, q^{\frac{1}{2}}z_{m-1}) + q z_{m-1}^2 H_m^{(m-2)}(x, q, y, z_1, \ldots, z_{m-1})
\]

Next we have

\[
H_m^{(m-1)}(q x, y, z_1, \ldots, z_{m-1})
= \sum_{k_m=1}^{\infty} \sum_{k_{m-1}=0}^{k_m} \cdots \sum_{k_1=0}^{k_2} (q x)_{k_m-1} y^{k_{m-1}} \left( \prod_{i=1}^{m-1} q^{k_i^2} z_i^{2k_i} \binom{k_{i+1}}{k_i} \right)
= \frac{1}{1 - x} \left( H_m^{(0)}(x, y, q^{\frac{1}{2}}z_1, \ldots, q^{\frac{1}{2}}z_{m-1}) - 1 \right)
\]

Further we have

\[
H_m^{(0)}(q x, y, z_1, \ldots, z_{m-1})
= \sum_{k_m=0}^{\infty} \sum_{k_{m-1}=0}^{k_m} \cdots \sum_{k_1=0}^{k_2} (q x)_{k_m} (1 - q^{k_m} x) y^{k_{m-1}} \left( \prod_{i=1}^{m-1} q^{k_i^2+k_i} z_i^{2k_i} \binom{k_{i+1}}{k_i} \right)
= \frac{1}{1 - x} \left( H_m^{(0)}(x, y, z_1, \ldots, z_{m-1}) - x H_m^{(0)}(x, q, y, z_1, \ldots, z_{m-1}) \right)
\]

We use these difference equations to prove Theorem 2.

A recursive use of eq. (40) gives

\[
H_m^{(m-1)}(q x, y, q^{\frac{1}{2}}z_1, \ldots, q^{\frac{1}{2}}z_{m-1}) = H_m^{(0)}(q x, y, z_1, \ldots, z_{m-2}, q^{\frac{1}{2}}z_{m-1}) \\
+ q^2 z_{m-1}^{2} H_m^{(0)}(q x, y, z_1, \ldots, z_{m-3}, q^{\frac{1}{2}}z_{m-2}, q z_{m-1}) \\
+ q^4 z_{m-2}^2 z_{m-1}^2 H_m^{(0)}(q x, y, z_1, \ldots, z_{m-4}, q^{\frac{1}{2}}z_{m-3}, q z_{m-2}, q z_{m-1}) \\
+ \cdots + q^{2m-2} z_{m-2}^2 \cdots z_{m-1}^2 H_m^{(0)}(q x, y, q z_1, \ldots, q z_{m-1})
\]
while a recursive use of eq. (41) gives

\begin{equation}
H^{(m-2)}(x, y, y^2 z_1, \ldots, y^{2m-3} z_{m-1}) = H^{(0)}(x, y, z_1, \ldots, z_{m-2}, y^2 z_{m-1}) \\
+ q^2 z_{m-2}^2 H^{(0)}(x, y, z_1, \ldots, z_{m-3}, q^2 z_{m-2}, q z_{m-1}) \\
+ q^4 z_{m-3}^2 z_{m-2}^2 H^{(0)}(x, y, z_1, \ldots, z_{m-4}, q^2 z_{m-3}, q z_{m-2}, q z_{m-1}) \\
+ \cdots + q^{2m-4} z_1^2 z_{m-2}^2 H^{(0)}(x, y, q^2 z_1, q z_2, \ldots, z_{m-1})
\end{equation}

Then we get

\begin{equation}
H^{(0)}_m(x, y, z_1, \ldots, z_{m-1}) = x H^{(0)}_m(x, q y, z_1, \ldots, z_{m-1})
\end{equation}

\begin{equation}
= (1 - x) H^{(0)}_m(x, y, z_1, \ldots, z_{m-1})
\end{equation}

\begin{equation}
= (1 - x) \left( H^{(m-1)}_m(x, y, q^2 z_1, \ldots, q^2 z_{m-1}) - q^2 z_{m-1}^2 H^{(m-2)}_m(x, q y, q^2 z_1, \ldots, q^2 z_{m-1}) \right)
\end{equation}

\begin{equation}
= y^{-1} \left( H^{(0)}_m(x, y, z_1, \ldots, z_{m-1}) - 1 \right) - q^2 (1 - x) z_{m-1}^2 H^{(m-2)}_m(x, q y, q^2 z_1, \ldots, q^2 z_{m-1})
\end{equation}

Substituting eq. (46) into above equation, we get a difference equation for \( H^{(0)}_m(x, y, z_1, \ldots, z_{m-1}) \);

\begin{equation}
(1 - y^{-1}) H^{(0)}_m(x, y, z_1, \ldots, z_{m-1}) = -q^2 (1 - x) z_{m-1}^2 \left( H^{(0)}_m(x, q y, z_1, \ldots, z_{m-2}, q^2 z_{m-1}) + y^{-1} \right)
\end{equation}

\begin{equation}
+ q^2 z_{m-2}^2 H^{(0)}_m(q x, q y, z_1, \ldots, z_{m-3}, q^2 z_{m-2}, q z_{m-1}) \\
+ \cdots + q^{2m-4} z_1^2 z_{m-2}^2 H^{(0)}_m(q x, q y, q^2 z_1, q z_2, \ldots, q z_{m-1})
\end{equation}

Therewith by substituting eq. (45) into eq. (43), we obtain another difference equation for \( H^{(0)}_m(x, y, z_1, \ldots, z_{m-1}) \);

\begin{equation}
\frac{1}{(1 - x) y} \left( H^{(0)}_m(x, y, z_1, \ldots, z_{m-1}) - 1 \right) - q^{2m-2} z_1^2 \cdots z_{m-1}^2 H^{(0)}_m(x, y, q z_1, \ldots, q z_{m-1})
\end{equation}

\begin{equation}
= H^{(0)}_m(x, y, z_1, \ldots, z_{m-2}, q^2 z_{m-1}) + q^2 z_{m-1}^2 H^{(0)}_m(q x, y, z_1, \ldots, z_{m-3}, q^2 z_{m-2}, q z_{m-1}) \\
+ \cdots + q^{2m-4} z_1^2 z_{m-2}^2 H^{(0)}_m(q x, y, q^2 z_1, q z_2, \ldots, q z_{m-1})
\end{equation}

We set \( z_1 = z_2 = \cdots = z_{m-1} = z \) in eqs. (47) and (48). We can eliminate the right hand side of eq. (48) by use of eq. (47), and we get

\begin{equation}
(1 - y) H^{(0)}_m(x, y, z) + 1 - q z^2
\end{equation}

\begin{equation}
= (x y - q z^2) H^{(0)}_m(x, q y, z) + q^{2m} (1 - x) y z^{2m} H^{(0)}_m(q x, q y, q z)
\end{equation}

Setting \((x, y, z) \rightarrow (q x, x, x)\), and recalling eq. (39), we find \( H^{(0)}_m(x) \) satisfies \( q \)-difference equation (37) with \( a = 0 \).
In the case of $a \neq 0$, we first recall that

\begin{equation}
H_m(x, y, z_1, \ldots, z_{m-1}) - H_0(x, y, q^{-1}z_1, \ldots, q^{-1}z_{a-1}, z_a, \ldots, z_{m-1})
+ \cdots + q^a z_a^2 \cdots z_2^2 H_m(x, y, q^{2}z_1, \ldots, q^{2}z_a, z_{a+1}, \ldots, z_{m-1})
\end{equation}

which is given by an iterated use of eq. (40). We rewrite this identity as

\begin{equation}
H_m(x, y, z_1, \ldots, z_{m-1}) = \left( \hat{D}_m \hat{H}_m \right)(x, y, z_1, \ldots, z_{m-1})
\end{equation}

Here the difference operator is defined by

\begin{equation}
\hat{D}_m = \hat{T}_1^{-1} \cdots \hat{T}_{a-1}^{-1} + q z_a^2 \hat{T}_1^{-1} \cdots \hat{T}_{a-2}^{-1} \hat{T}_a + \cdots + q^a z_a^2 \cdots z_2^2 \hat{T}_1 \cdots \hat{T}_a
\end{equation}

where we have used the $q$-shift operator

\begin{equation}
\left( \hat{T}_k \right)^{\pm 1} f(z_1, \ldots, z_{m-1}) = f(z_1, \ldots, q^{\pm 1}z_k, \ldots, z_{m-1})
\end{equation}

It can be seen by a direct computation that for $1 \leq b \leq a$ we have

\begin{equation}
\hat{D}_m z_a^2 \cdots z_b^2 H_m(x, y, z_1, \ldots, z_{b-1}, q^2 z_b, q z_{b+1}, \ldots, q z_a, z_{a+1}, \ldots, z_{m-1})
= q^{b-a} z_b^2 \cdots z_a^2 H_m(x, y, z_1, \ldots, z_{b-1}, q^{2}z_b, q z_{b+1}, \ldots, q z_a, q_{a+1}, \ldots, q_{m-1})
\end{equation}

We apply $\hat{D}_m$ to eq. (47). Using eq. (53), we get

\begin{equation}
(1 - y^{-1}) H_m(x, y, z_1, \ldots, z_{m-1}) - x H_m(x, y, q y, z_1, \ldots, z_{m-1})
+ y^{-1} \left( 1 + q z_a^2 + \cdots + q^a z_a^2 \cdots z_2^2 \right)
= -q^2 \left( 1 - x \right) z_{m-1}^2 \left( H_m(x, y, q, y, z_1, \ldots, z_{m-2}, q^2 z_{m-1}) \right) + \cdots +
+ q^{2(m-a-1)} z_a^2 \cdots z_{m-2}^2 H_m(x, y, q, y, z_1, \ldots, z_{a-1}, q^{2}z_a, q z_{a+1}, \ldots, q z_{m-1})
+ q^{2(m-a-1)} z_{a-1}^2 \cdots z_{m-2}^2 H_m(x, y, q, y, z_1, \ldots, z_{a-2}, q^{2}z_{a-1}, q z_{a}, \ldots, q z_{m-1})
+ \cdots + q^{2(m-a-1)} z_{m-2}^2 H_m(x, y, q, y, q^{2}z_1, q z_2, \ldots, q z_{m-1})
\end{equation}

Using eq. (50), eq. (48) can be rewritten as

\begin{equation}
\frac{1}{1 - x y} \left( H_m^0(x, y, z_1, \ldots, z_{m-1}) - 1 \right)
= H_m^0(x, y, z_1, \ldots, z_{m-2}, q^{2}z_{m-1}) + q^2 z_{m-1}^2 H_m^0(x, y, z_1, \ldots, z_{m-3}, q^{2}z_{m-2}, q z_{m-1})
+ \cdots + q^{2(m-a-2)} z_{a-2}^2 \cdots z_{m-2}^2 H_m^0(x, y, z_1, \ldots, z_{a-1}, q^{2}z_{a}, q z_{a+1}, \ldots, q z_{m-1})
+ q^{2(m-a-1)} z_{a-1}^2 \cdots z_{m-2}^2 H_m^0(x, y, q^{2}z_1, q z_2, \ldots, q z_{m-1})
\end{equation}
Applying $\hat{D}_m^{(a)}$ to above equation, we get

\begin{equation}
(56) \quad \frac{1}{(1-x)y} \left( H_m^{(a)}(x, y, z_1, \ldots, z_{m-1}) - (1 + qz_1^2 + \cdots + q^a z_1^2 \cdots z_{m-1}^2) \right)
\end{equation}

\begin{align*}
&= H_m^{(a)}(q x, y, z_1, \ldots, z_{m-2}, q^{\frac{1}{m-1}} z_{m-1}) + q^2 z_{m-1}^2 \left( H_m^{(a)}(q x, y, z_1, \ldots, z_{m-3}, q^{\frac{1}{m-2}} z_{m-2}, q z_{m-1}) \right) \\
&\quad + \cdots + q^{2(m-a-2)} z_{a+2}^2 \cdots z_{m-1}^2 \left( H_m^{(a)}(q x, y, z_1, \ldots, z_{m-a-1}, q^{\frac{2}{m-a}} z_{m-a}, q z_{m-a+2}, \ldots, q z_{m-1}) \right) \\
&\quad + q z_a^2 \left( H_m^{(a)}(q x, y, z_1, \ldots, z_{a-2}, q^{\frac{2}{a-1}} z_{a-1}, q z_a, \ldots, q z_{m-1}) \right)
\end{align*}

\begin{equation}
\quad + \cdots + q^a z_1^2 \cdots z_a^2 \left( H_m^{(a)}(q x, y, q z_1, \ldots, q z_{m-1}) \right)
\end{equation}

We set $z_1 = z_2 = \cdots = z_{m-1} = z$ in eqs. (55) and (56). Combining these two equations, we obtain

\begin{equation}
(57) \quad - (1 - y) H_m^{(a)}(x, y, z) - (xy - qz)^2 H_m^{(a)}(x, q y, z) + 1 - q^{a+1} z^{2a+2}
\end{equation}

\begin{align*}
&= q^{2m-a} (1 - x) y z^{2m} H_m^{(a)}(q x, q y, q z)
\end{align*}

When we set $(x, y, z) \rightarrow (q x, x, x)$, and by definition (39) we can conclude that $H_m^{(a)}(x)$ satisfies eq. (37).

\begin{proof}
\end{proof}

\textbf{Corollary 9.} \textit{Let the function $\tilde{H}_m^{(a)}(x)$ be}

\begin{equation}
\tilde{H}_m^{(a)}(x) = \frac{1}{x} \left( H_m^{(a)}(x, x, q^{-\frac{1}{m}} x) - 1 \right)
\end{equation}

\textit{It satisfies the difference equation;}

\begin{equation}
(58) \quad \tilde{H}_m^{(a)}(x) = 1 + x + \cdots + x^{2a} - q^{m-a} x^{2m} - q^{m-a+1} x^{2m+1} \tilde{H}_m^{(a)}(q x)
\end{equation}

\begin{proof}
\end{proof}

We see from the right hand side of eq. (36) that

\begin{equation}
(59) \quad H_m^{(a)}(x = 1) = (q^{a+1}, q^{2m-a}, q^{2m+1}, q^{2m+1})_{\infty}
\end{equation}

For the left hand side we have the following identity which follows from eq. (17);
Lemma 10.

\[ H_m^a(x) = (q \times) \sum_{k_1, k_2, \ldots, k_{m-1}=0}^{\infty} \frac{q^{k_1^2 + \cdots + k_{m-1}^2 + k_{a+1} + \cdots + k_{m-1}}}{(xq)_{k_{m-1}}} x^{2\sum_{i=1}^{m-1} k_i + k_{a+1}} \]

\[ \times \left( \prod_{i=1}^{m-2} [k_i]_{k_{i+1}} \right) \left[ k_{a+1} + 1 \right] \]

\[ + (1 - x) \sum_{k_1, k_2, \ldots, k_{m}=0}^{\infty} ((q \times)_{k_{m}} - (q \times)_{\infty}) x^{k_{m}} \left( \prod_{i=1}^{a-1} q^{k_{i}^2} x^{2k_{i}} [k_{i+1}]_{k_{i}} \right) \]

\[ \times q^{k_{m}^2} x^{2k_{m}} \left[ k_{a+1} + 1 \right] \left( \prod_{i=a+1}^{m-1} q^{k_{i}^2+k_{i}} x^{2k_{i}} [k_{i+1}]_{k_{i}} \right) \]

To check eq. (59) from eq. (60), we need a variant of the Andrews–Gordon identity;

Proposition 11.

\[ \frac{(q^{a+1}, q^{2m-a}, q^{2m+1}; q^{2m+1})_{\infty}}{(q)_{\infty}} \]

\[ = \sum_{k_1, \ldots, k_{m-1}=0}^{\infty} \frac{q^{k_1^2 + \cdots + k_{m-1}^2 + k_{a+1} + \cdots + k_{m-1}}}{(q)_{k_{m-1}}} \left( \prod_{i=1}^{m-2} [k_i]_{k_{i+1}} \right) \left[ k_{a+1} + 1 \right] \]

To prove this proposition we need a certain limit of the Bailey lemma (see, e.g., Ref. [2, 12, 3]);

Proposition 12 (Bailey lemma). If for \( n \geq 0 \)

\[ \beta_n = \sum_{r=0}^{n} \frac{\alpha_r}{(q)_{n-r} (xq)_{n+r}} \]

then

\[ \beta_n' = \sum_{r=0}^{n} \frac{\alpha'_r}{(q)_{n-r} (xq)_{n+r}} \]

where

\[ \alpha'_r = \frac{(\rho_1)_r (\rho_2)_r}{(xq/\rho_1)_r (xq/\rho_2)_r} \left( \frac{xq}{\rho_1 \rho_2} \right)^r \alpha_r \]

\[ \beta_n' = \sum_{j=0}^{\infty} \frac{(\rho_1)_j (\rho_2)_j}{(q)_{n-j} (xq/\rho_1)_n (xq/\rho_2)_n} \left( \frac{xq}{\rho_1 \rho_2} \right)^j \beta_j \]

Corollary 13.

\[ \sum_{k=0}^{n} \frac{a_k x^k}{(q)_{n-k} (xq)_{n+k}} = \sum_{j=0}^{n} q^j x^j \sum_{k=0}^{j} \frac{a_k q^{-k^2}}{(q)_{j-k} (xq)_{j+k}} \]
Proof. We take a limit \( n \to \infty \) in Prop. 13. Then, we set \( \rho_1 = q^{-m} \), \( \rho_2 = q^{-n} \), and \( \alpha_k = q^{-x^2} a_k \) and take \( m \to \infty \).

\[ \square \]

**Corollary 14.**

\[
(66) \quad \sum_{k=0}^{\infty} \frac{c_k}{(q)_{n-k} (q)_{n+k}} = \sum_{j=0}^{\infty} q^{j^2} \sum_{k=0}^{\infty} \frac{c_k}{(q)_{j-k} (q)_{j+k}}
\]

\[
(67) \quad \sum_{k=0}^{\infty} \frac{c_k}{(q)_{n-k} (q)_{n+k-1}} = \sum_{j=0}^{\infty} q^{j^2-j} \sum_{k=0}^{\infty} \frac{c_k}{(q)_{j-k} (q)_{j+k-1}}
\]

**Proof.** We set \( x = 1, q^{-1} \) in Corollary 13 and take a symmetrization for \( a_k \).

\[ \square \]

**Proof of Proposition 11.** We set the left hand side of Prop. 11 as \( A_m^{(a)} \). We apply the triple Jacobi identity to \( A_m^{(a)} \), and then use the Bailey chain recursively;

\[
A_m^{(a)} = \frac{1}{(q)_\infty} \sum_{k=0}^{\infty} (-1)^k q^{(m+\frac{1}{2})^2-(m-a-\frac{1}{2}) k}
\]

\[
(68) \quad \sum_{k=0}^{\infty} \left[ \frac{2n-1}{n-k} \right] (-1)^k q^{ck^2-\frac{1}{2}k}
\]

which gives

\[
(69) \quad \sum_{k=0}^{\infty} (-1)^k \frac{q^{ck^2-\frac{1}{2}k}}{(q)_{n-k} (q)_{n+k-1}} = (1 - q^n) \sum_{k=0}^{\infty} (-1)^k \frac{q^{ck^2-\frac{1}{2}k}}{(q)_{n-k} (q)_{n+k}}
\]

Then we have

\[
A_m^{(a)} = \sum_{k_m \geq k_{m-1} \geq \cdots \geq k_1 \geq 0} q^{\sum_{i=1}^{m-1} (k_i^2-k_i)} (1 - q^{k_{m+1}}) 
\]

\[
\times \sum_{k=0}^{\infty} (-1)^k \frac{q^{(a+\frac{1}{2})k^2-\frac{1}{2}k}}{(q)_{k+1-k} (q)_{k+1+k}}
\]

\[
\sum_{k_m \geq k_{m-1} \geq \cdots \geq k_1 \geq 0} q^{\sum_{i=1}^{m-1} (k_i^2-k_i) + \sum_{k=1}^{k_{m+1}} k_i^2} \frac{1}{(q)_{k_m-1} (q)_{k_m+1} (q)_{k_1}} (1 - q^{k_{m+1}})
\]
We note that, in the last equality, we have also used
\[ \sum_{k=-\infty}^{\infty} \frac{(-1)^k q^{\frac{1}{2} k^2 - \frac{1}{2} k}}{(q)_{n-k} (q)_{n+k}} = \delta_{n,0} \]
After we shift parameters; \((k_{m+1}, \ldots, k_m) \to (k_{m+1} + 1, \ldots, k_m + 1)\), we get a statement of Prop. 11.

**Proof of Theorem 2** We differentiate eqs. (60) and (36) with respect to \(x\), and substitute \(x \to 1\). We obtain

\[
(q_{a+1}, q_{2m-a}, q_{2m+1}; q_{2m+1})_{\infty} \left( \frac{2m - 2a - 1}{2} - \sum_{i=1}^{\infty} \frac{q^i}{1 - q^i} \right) + (q)_{\infty} \sum_{k_1, \ldots, k_{m-1}=0}^{\infty} q^{k_1^2 + \cdots + k_{m-1}^2 + k_{m+1}^2 + \cdots + k_{m-1}} \left( 2 \sum_{i=1}^{m-1} k_i + k_{m-1} + \sum_{i=1}^{k_{m-1}} \frac{q^i}{1 - q^i} \right) - \sum_{k_1, \ldots, k_{m-1}=0}^{\infty} ((q)_{k_{m+1}} - (q)_{\infty}) q^{k_1^2 + \cdots + k_{m-1}^2 + k_{m+1} + \cdots + k_{m-1}} \left( \prod_{i \neq d}^{m-1} \left[ \begin{array}{c} k_{i+1} \\ k_i \end{array} \right] \right) \left[ \begin{array}{c} k_{m+1} + 1 \\ k_{m} \end{array} \right] = \sum_{n=0}^{\infty} \frac{n}{2} X_{m+2}^{(a)}(n) q^{\frac{2 - (2m - 2a + 1)^2}{8(m+1)}}
\]
We substitute \(q \to e^{-t}\), and find eq. (15). As a result we obtain Theorem 2 applying Prop. 5.

4. **Knot Invariant and Nearly Modular Form**

We comment on a relationship between our \(q\)-series and the knot invariant. Generally the \(q\)-series \(X_m^{(a)}(q)\) does not converge in any open set of \(q\), but it reduces to the finite number in a case of \(q\) being root of unity. Furthermore this finite value coincides with Kashaev’s invariant (or, the colored Jones polynomial with a specific value) for the \((2m + 1, 2)\)-torus knot.

We prepare the following \(q\)-series identity;

**Lemma 15.** We set \(a \geq c \geq 0\). Then we have

\[
\sum_{b=c}^{a} (-1)^{b+c} q^{\frac{1}{2} (b(b+1) + \frac{1}{2}(c+1))} (q)_{a-b} (q)_{b-c} = q^{c^2 + c} \left[ \begin{array}{c} a \\ c \end{array} \right]
\]
Proof. Using \( b = c + n \), we have

\[
\text{l.h.s.} = q^{c^2+c} \sum_{n=0}^{a-c} (-1)^n \frac{q^{\frac{1}{2}(n^2+n)+cn}}{(q)_{a-c-n} (q)_n}
\]

\[
= \frac{q^{c^2+c}}{(q)_{a-c}} \sum_{n=0}^{a-c} \frac{q^{-(a-c)}_n}{(q)_n} q^{m(1+a)}
\]

\[
= \frac{q^{c^2+c}}{(q)_{a-c}} \cdot \frac{(q^c)_\infty}{(q^a\omega^1)_\infty} = q^{c^2+c} \frac{(q^c)_\infty}{(q)_{a-c}}
\]

which proves eq. (71).

We set \( N \in \mathbb{Z}_{>0} \) and define

\[
\omega = \exp(2 \pi i / N)
\]

Hereafter we mean that

\[
(\omega)_n = \prod_{a=1}^{n} (1 - \omega^a)
\]

\[
\begin{bmatrix} n \\ c \end{bmatrix} = \frac{(\omega)_n}{(\omega)_c (\omega)_{n-c}}
\]

and \( \ast \) denotes a complex conjugate.

**Proposition 16.** When \( q \) being the \( N \)-th root of unity, \( X^{(0)}_m(q = \omega) \) defined in eq. (11) coincides with Kashaev’s invariant for the \((2, 2m + 1)\)-torus knot.

Proof. We take an example for \( m = 2 \) case;

\[
X^{(0)}_2(\omega) = \sum_{0\leq k_1, k_2 \leq N-1} (\omega)_{k_2} \omega^{k_1^2+k_1} \begin{bmatrix} k_2 \\ k_1 \end{bmatrix}
\]

\[
= \sum_{0\leq k_1, k_2 \leq N-1} (-1)^{n+k_1} (\omega)_{k_2} \omega^{\frac{1}{2}n(n+1)+\frac{1}{2}k_1(k_1+1)} (\omega)_{k_2-n} (\omega)_{n-k_1}
\]

\[
= \sum_{0\leq k_1, k_2 \leq N-1} (\omega)_{k_2} \omega^{k_1(1+n)}
\]

\[
= \sum_{0\leq k_1, n-k_1 \leq N-1} \frac{N}{(\omega)_{n-k_1}} \omega^{k_1(1+n)}
\]

\[
= \sum_{0\leq k_1 \leq N-n-k_1 \leq N-1} (\omega)_{N-n+k_1} \omega^{k_1(1+n)}
\]

\[
= \sum_{\omega} (\omega)_{a+b} \omega^{-ab}
\]

\[
\text{We use } i = \sqrt{-1}.
\]
Theorem 17 (Conjecture in Ref. [7]).

We can check for other $m$’s that $X_m^{(0)}(\omega)$ reduces to the invariant for the torus knot given in Ref. [7].

As was proved in Ref. [7], the asymptotic expansion of knot invariant in a limit $N \to \infty$ can be written explicitly:

\[
X_m^{(0)}(\omega) \simeq \frac{2}{\sqrt{2m+1}} N^\frac{3}{2} e^{\pi i / \sqrt{2m+1}} \sum_{k=0}^{m-1} (-1)^k (m-k) \sin \left( \frac{2k+1}{2m+1} \pi \right) e^{-N\pi^2 m^2 / 4(2m+1)}
\]

\[
+ e^{\pi i / \sqrt{2m+1}} \sum_{n=0}^{\infty} T_m^{(0)}(n) \left( \frac{\pi}{4(2m+1)N} \right)^n.
\]

A case of $m = 1$ is given in Ref. [13] as “Kontsevich’s conjectural asymptotic formula”. In proving above asymptotic expansion, we used a previously known another expression for the colored Jones invariant for the torus knot. Correspondingly for a case of $a \neq 0$ we have a theorem;

**Theorem 17** (Conjecture in Ref. [7]).

(76) $X_m^{(a)}(\omega)$

\[
X_m^{(a)}(\omega) \simeq \frac{2}{\sqrt{2m+1}} N^\frac{3}{2} e^{\pi i / \sqrt{2m+1}} \sum_{k=0}^{m-1} (-1)^k (m-k) \sin \left( \frac{2k+1}{2m+1} \pi \right) e^{-N\pi^2 m^2 / 4(2m+1)}
\]

\[
+ e^{\pi i / \sqrt{2m+1}} \sum_{n=0}^{\infty} T_m^{(a)}(n) \left( \frac{\pi}{4(2m+1)N} \right)^n
\]

This theorem indicates a nearly modular property [13] of the function $X_m^{(a)}(q)$ with weight 1/2; we define functions $\Phi_m^{(a)}(\alpha)$ by

(77) $\Phi_m^{(a)}(\alpha) = e^{(2m-2a-1)^2 \pi i a / 4(2m+1)} X_m^{(a)}(e^{2\pi i a})$

and introduce a vector $\Phi_m(\alpha)$ by

(78) $\Phi_m(\alpha) = \begin{pmatrix} \Phi_m^{(m-1)}(\alpha) \\ \vdots \\ \Phi_m^{(1)}(\alpha) \\ \Phi_m^{(0)}(\alpha) \end{pmatrix}$
Eq. (76) is reformulated into

\[
\tilde{\Phi}_m \left( \frac{1}{N} \right) + (-i \, N)^{\frac{1}{2}} \mathbf{M}^{(2m+1)} \tilde{\Phi}_m (-N) = \sum_{n=0}^{\infty} \frac{T_m(n)}{n!} \left( \frac{\pi}{4 (2m + 1) i \, N} \right)^n
\]

where \( \mathbf{M}^{(2m+1)} \) is an \( m \times m \) matrix with an entry

\[
\left( \mathbf{M}^{(2m+1)} \right)_{a,b} = \frac{2}{\sqrt{2 \, m + 1}} \cos \left( \frac{(2 \, a - 1)(2 \, b - 1)}{2 (2m + 1)} \pi \right)
\]

and

\[
T_m(n) = \begin{pmatrix}
T_{m-1}^{(m-1)}(n) \\
\vdots \\
T_m^{(1)}(n) \\
T_m^{(0)}(n)
\end{pmatrix}
\]

**Proof of Theorem 17.** Eq. (70) indicates that the function \( \tilde{\Phi}_m^{(a)}(\alpha) \) coincides with a limit \( \tau \to \alpha \in \mathbb{Q} \) of the \( q \)-series

\[
\tilde{\Phi}_m^{(a)}(\tau) = \sum_{n=0}^{\infty} n \chi_{8m+4}(n) q^{\frac{1}{2(2m+1)}} n^2
\]

for \( q = e^{2\pi i \tau} \). This is regarded as the Eichler integral of

\[
\Phi_m^{(a)}(\tau) = \sum_{n=0}^{\infty} \chi_{8m+4}(n) q^{\frac{1}{2(2m+1)}} n^2
\]

which is modular with weight 1/2; it is straightforward to see that

\[
\Phi_m^{(a)}(\tau + 1) = e^{(2m-2a-1)^2 \pi i \tau} \Phi_m^{(a)}(\tau)
\]

and using the Poisson summation formula we obtain

\[
\Phi_m(\tau) = \sqrt{\frac{i}{\tau}} \mathbf{M}^{(2m+1)} \Phi_m(-1/\tau)
\]

where \( \mathbf{M}^{(2m+1)} \) is an \( m \times m \) matrix defined in eq. (80), and

\[
\Phi_m(\tau) = \begin{pmatrix}
\Phi_{m-1}^{(m-1)}(\tau) \\
\vdots \\
\Phi_{m}^{(1)}(\tau) \\
\Phi_{m}^{(0)}(\tau)
\end{pmatrix}
\]

To prove eq. (79) following Refs. [9, 13], we study an analogue of the Eichler integral defined by

\[
\tilde{\Phi}_m^{(a)}(z) = \sqrt{(2m + 1) i \, \pi} \int_{\mathbb{C}} \Phi_m^{(a)}(\tau) \frac{d\tau}{(\tau - z)^{\frac{1}{2}}}
\]
which is defined for \( z \) in the lower half plane \( z \in \mathbb{H}^- \). By performing an integration term by term, we have

\[
\tilde{\Phi}_m^{(a)}(z) = \frac{\sqrt{(2m+1)i}}{2\pi} \sum_{n=0}^{\infty} \chi_{8m+4}^{(a)}(n) \int_{z}^{\infty} e^{\pi \sqrt{\frac{2}{z}}(\tau - z)^{3/2}} d\tau
\]

\[
z \to \frac{1}{2} \sum_{n=0}^{\infty} n \chi_{8m+4}^{(a)}(n) e^{\frac{-z^2}{4(z^2-\alpha^2)}}
\]

which shows

\[(86)\]

\[
\tilde{\Phi}_m^{(a)}(z) = \tilde{\Phi}_m^{(a)}(\alpha)
\]

Note that l.h.s. is a limiting value from a lower half plane \( \mathbb{H}^- \) while r.h.s. is given from an upper half plane \( \mathbb{H} \).

To see a modular property of \( \tilde{\Phi}_m(\tau) \), we define the period function by

\[(87)\]

\[
r_m^{(a)}(z; \alpha) = \frac{\sqrt{(2m+1)i}}{2\pi} \int_{\alpha}^{\infty} \Phi_m^{(a)}(\tau) \frac{d\tau}{(\tau - z)^{3/2}}
\]

where \( \alpha \in \mathbb{Q} \). It is defined for \( z \in \mathbb{H}^- \), but it is analytically continued to \( \mathbb{R} \). We then have

\[
\sum_{b=1}^{m} (M^{2m+1})_{a,b} \tilde{\Phi}_m^{(m-b)}(-1/z) = \sum_{b=1}^{m} (M^{2m+1})_{a,b} \frac{\sqrt{(2m+1)i}}{2\pi} \int_{z}^{\infty} \Phi_m^{(m-b)}(-1/s) \frac{ds}{s^{3/2}}
\]

\[
= -(iz)^{3/2} \frac{\sqrt{(2m+1)i}}{2\pi} \int_{z}^{\infty} \Phi_m^{(m-a)}(s) \frac{ds}{(s-z)^{3/2}}
\]

\[(88)\]

We consider a limit \( z \to 1/N \) in eq. \( (88) \). We see that an asymptotic expansion of \( r_m^{(a)}(1/N; 0) \) in \( N \to \infty \) is given by

\[
r_m^{(a)}(1/N; 0) = -\frac{1}{2} \sum_{k=0}^{\infty} \frac{L(-2k - 1, \chi_{8m+4}^{(a)})}{k!} \left( \frac{\pi i}{4(2m+1)N} \right)^k
\]

and eq. \( (86) \) indicates that \( \tilde{\Phi}_m^{(a)}(z) \) coincides with \( \tilde{\Phi}_m^{(a)}(\alpha) \) for \( z = -N \) and \( z = 1/N \). Recalling eq. \( (13) \), we can conclude eq. \( (79) \). \( \square \)

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