On Posterior consistency of Bayesian Changepoint models

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Abstract

While there have been a lot of recent developments in the context of Bayesian model selection and variable selection for high dimensional linear models, there is not much work in the presence of change point in literature, unlike the frequentist counterpart. We consider a hierarchical Bayesian linear model where the active set of covariates that affects the observations through a mean model can vary between different time segments. Such structure may arise in social sciences/ economic sciences, such as sudden change of house price based on external economic factor, crime rate changes based on social and built-environment factors, and others. Using an appropriate adaptive prior, we outline the development of a hierarchical Bayesian methodology that can select the true change point as well as the true covariates, with high probability. We provide the first detailed theoretical analysis for posterior consistency with or without covariates, under suitable conditions. Gibbs sampling techniques provide an efficient computational strategy. We also consider small sample simulation study as well as application to crime forecasting applications.

1 Introduction

In many applications such as economics, social science, the observed variable depends on covariates through mean structure, where the mean structure changes with time, based on changes in some latent unobserved factor such as an economic phenomenon/public policy change (Datta et al., 2019). The dependence on the covariate may be local (only in some time segments) or global. Selecting the change point and covariates consistently remains an important problem which provides an insight to underlying sociological and economical factors.

There is a huge and influential literature, both Bayesian and frequentist, on changepoint detection with application in diverse areas that dates back several decades. Initial attempts for changepoint detection using cumulative sums date back to Page (1955, 1957). Shortly after, changepoint for the location parameter, primarily within the Gaussian observation model, was studied by several authors including Chernoff and Zacks (1964); Gardner (1969); Srivastava
The problem of multiple changepoints were addressed by several people, notably Talwar (1983); Stephens (1994); Chib (1998). Vostrikova (1981) introduced the popular binary segmentation method that recursively partitions the observation window to estimate the number of changepoints. The early survey by Zacks (1983) provides a detailed account of these early innovations. Early Bayesian methodological contributions include Carlin et al. (1992), who provide a hierarchical Bayes framework for changepoint model with applications to changing regressions and changing Markov structures and Raftery (1994), who propose a Markov Transition Distribution model and provide Bayes factors for testing whether a change-point has occurred in a given segment. Csorgo and Horváth (1997) provides a detailed review of changepoint methods and some new contributions based on likelihood based tests. There has been a resurgence of changepoint literature in the last decade with a renewed focus on both running time (e.g. Killick et al., 2012) and suitable handling of multi-resolution and multidimensional nature of modern experiments and data collection routines (e.g. Frick et al., 2014; Fryzlewicz et al., 2014).

It is worthwhile to note that the majority of Bayesian changepoint estimation literature considers an offline, retrospective approach while online changepoint detection methods are somewhat more prevalent in the frequentist regime, as pointed out by Adams and MacKay (2007), who propose an online Bayesian method based on recursive run length estimation.

With high-throughput data becoming routine in modern scientific studies, the problem of variable selection in a high-dimensional changing linear model becomes important where a key inferential goal is to identify the potentially different set of ‘active’ variables within each segment. Recent papers that address this problem include frequentist approach such as Lee et al. (2016), and Bayesian treatment in Datta et al. (2019). In Lee et al. (2016), a lasso penalization approach is used for changing high-dimensional linear regression while selecting relevant regressors under sparsity assumption. While in Datta et al. (2019), this is handled by using the shrinking and diffusing prior (Narisetty et al., 2014) in each segment for variable selection. As Datta et al. (2019) points out, the decomposability of the likelihood for changing linear regression model also makes it easy to incorporate other Bayesian variable selection priors: for example, one could use a spike-and-slab prior (Mitchell and Beauchamp, 1988) or a variety of shrinkage priors that have become quite popular for sparse variable selection and estimation (Polson and Scott, 2010b; Bhadra et al., 2019).

1.1 Recent theoretical results

As the main focus of this article is theoretical guarantees, it is worthwhile to briefly mention a few recent theoretical results that are relevant to our present discourse. Roughly speaking, the recent theoretical advances can be broadly classified into two major thrusts: (a) optimality properties for the estimator for the underlying piecewise mean and (b) optimality properties
for the estimator for changepoint locations.

For the first kind, Gao et al. (2017) established sharp nonasymptotic risk bounds for least squares estimators when the underlying mean has a piecewise constant structure, and observed a phase change phenomenon when the number of changepoints goes beyond 2. Let $\Theta_k$ denote the model with all piecewise constant $\theta$ with maximum $k - 1$ changepoints and $\hat{\theta}(\Theta_k)$ is the least squares estimator (LSE) under this model. Now consider a possibly misspecified LSE $\hat{\theta}(\Theta_k)$, when the true $\theta_0 \in \Theta_k$. Gao et al. (2017) provided sharp risk bounds that are minimax when $k = k_0$, i.e., $\inf_{\hat{\theta}} \sup_{\theta_0 \in \Theta_k} E \left\| \hat{\theta} - \theta_0 \right\|^2 \approx \sigma^2 \{1 + \log \log n 1(k = 2) + k \log(en/k)1(k > 2)\}$.

Martin and Shen (2017) developed an efficient empirical Bayes strategy for the piecewise constant sequence model and showed that the resulting posterior distribution attains a similar optimal rate as in Gao et al. (2017). Theorem 1 of Martin and Shen (2017) states that under a data-driven prior on the elements of $\theta$ and a truncated geometric prior distribution on the number of changepoints, the empirical Bayes posterior distribution $\Pi_n$ of $\theta \in \mathbb{R}^n$ will satisfy:

$$\sup_{\theta_0} E_{\theta_0} \Pi_n \left( \{ \theta \in \mathbb{R}^n : \| \theta - \theta_0 \|^2 > M_n \epsilon_n(\theta_0) \} \right) \to 0, \ n \to \infty,$$

where $M_n$ is any sequence with $M_n \to \infty$, and $\epsilon_n(\theta_0)$ is the target rate similar to the one in Gao et al. (2017). Martin and Shen (2017) point out that the concentration rate for the empirical Bayes posterior distribution will have one phase transition from $k = 1$ to $k \geq 2$, unlike two phase transitions noted by Gao et al. (2017), and conjecture that this phenomenon might be a characteristic of all Bayesian approaches for piecewise constant changepoint detection. Liu et al. (2019) extends Gao et al. (2017) to the case of multidimensional change-point detection, where the location $\theta \in \mathbb{R}^{p \times n}$ can change in at most $s$ out of $p$ coordinates at some time-point $t_0 \in \{1, \ldots, n\}$.

For estimating the location of change points involving exponential families, Frick et al. (2014) proposed the multiscale estimator SMUCE that attains the minimax rate $O(n^{-1})$ up to a logarithmic factor. Frick et al. (2014) also constructed asymptotically honest confidence sets for the number and location of change points, and provided sufficient conditions for the SMUCE method to detect change points with probability approaching 1 in the presence of ‘vanishing signals’ for $n \to \infty$.

### 1.2 Our contributions and outline

Despite these remarkable advances, there is essentially no theoretical guarantees for changepoints in high-dimensional linear models concerning consistency in model selection. Here we provide the following theoretical substantiations:

(i) We show that under the default Bayesian hierarchy, it is possible to recover both the true change point locations and the true non-zero covariates with high probability under mild conditions on the covariates and the maximum model size.

(ii) Specifically, we prove formal posterior consistency results for model selection and change
point recovery via Bayes factor for both piecewise constant model as well as changing high-dimensional linear regression. We also prove that the minimax rate of $O(n^{-1})$ (Frick et al., 2014) is attained by the Bayes estimators for change point recovery.

(iii) Finally, we show that the empirical Bayes estimator attains the same optimal rate of convergence as the full Bayes solution, under the assumption of same, but unknown, error variance $\sigma^2$ across different segments.

To our knowledge, this is the first theoretical substantiation of the superior performance of Bayesian methods in this specific methodological context.

2 Mathematical/Asymptotic Framework

Consider the canonical high-dimensional regression set-up with an $n$-dimensional response $y$ and an $n \times p$ design matrix $X$, with $p \gg n$, where $y_i \sim N(x_i' \beta_j, \sigma^2)$ for covariate vector $x_i$ at time point $t_i$, $i = 1, \ldots, n$. Let $\beta_1, \beta_2, \ldots, \beta_l ; \beta_i \neq \beta_{i+1}, i = 1, \ldots, l - 1$ are the values of the coefficient vector, where $\beta_{k+1}$ is the value of the coefficient vector between $nt_k < i \leq \lfloor nt_{k+1} \rfloor, k = 1, \ldots, l - 1$, and $t_1, \ldots, t_{l-1}$ are locations of change points and $t_0 = 0$, and let $\beta_{j,m}$ be the $m^{th}$ component of $\beta_j$.

2.1 Changing linear model with variable selection

**Spike-and-slab priors:** For the changing linear regression, we want to incorporate covariates in the model with selection of relevant predictors for each time segment between two change-points. Our aim is to simultaneously select the true non-zero covariates as well as infer the correct number and positions of the changepoints. Our framework allows for using different priors that enable variable selection. The natural Bayesian solution is to put a spike-and-slab prior on $\beta_j$ that will ensure selection of covariates (Mitchell and Beauchamp, 1988).

Let $I_i^y$ be the indicator function associated with $y_i$ where $I_i^y = 1$ denotes a change point at the $i^{th}$ epoch/location. Given $I_i^y$’s $i = 1, \ldots, n$, let $l = 1 + \sum_{i=1}^n I_i^y$ be the number of partition based on change points and $P_j$ denote the $j^{th}$ partition. The hierarchical model can be written as:

\[
\begin{align*}
    y_i \mid \beta_j, \sigma^2 &\sim N(x_i' \beta_j, \sigma^2); \; i \in P_j; \quad \pi(\sigma^2) \propto \frac{1}{\sigma^2}; \\
    \beta_{j,m} \mid I_m^\beta, \pi, \tau_j^2 &\sim (1 - I_m^\beta)\delta_{\{0\}} + I_m^\beta N(0, \tau_j^2), \; \text{for } i \in P_j, \; j = 1, \ldots, l; \\
    I_m^\beta &\sim \text{Bernoulli}(\tilde{p}_m), \; \text{and } I_i^y \sim \text{Bernoulli}(p_n)
\end{align*}
\]

(2.1)

where $I_i^y$’s are independent Bernoulli indicators of whether the $i^{th}$ observation $y_i$ is associated with a ‘change-point’, and $I_m^\beta$’s are independent indicator Bernoulli random variables, indicating whether the $m^{th}$ covariate is included in the true model, or analogously if the $m^{th}$ parameter
in the $j^{th}$ partition, $\beta_{j,m}$ is non-zero.

Here the variable selection can be done via the posterior inclusion probability (PIP) for each $\beta_j$ within each time segment. The inclusion probability $P(\beta_j \neq 0 \mid y)$ is used to select the relevant predictors. Spike-and-slab priors have proven optimality properties for high-dimensional linear models as shown by (Castillo et al., 2015), although they come with a higher computational burden due to the need for exploring a high-dimensional parameter space. Alternatively, the global-local shrinkage priors (Polson and Scott, 2010a; Bhadra et al., 2019) that have been proven optimal for variable selection (Datta and Ghosh, 2013; Ghosh et al., 2016) can be used.

The simpler case of a Gaussian sequence model with no covariates is presented first to build the intuition, which will later be generalized for covariates. This leads to the following hierarchical model:

$$
\begin{align*}
    y_i \mid \theta_i, \sigma^2 &\sim \mathcal{N}(\theta_i, \sigma^2); \quad \pi(\sigma^2) \propto \frac{1}{\sigma^2}; \\
    \theta_i \mid \mu_j, \tau^2 &\sim \mathcal{N}(\mu_j, \tau^2), \text{ for } i \in P_j, j = 1, \ldots, l; \\
    \mu_j &\sim \mathcal{N}(0, V), \quad V \gg 1 \text{ and } \pi(\tau^2) \propto \frac{1}{\tau^2}; \quad I_i^y \sim \text{Bernoulli}(p_n);
\end{align*}
$$

where $I_i^y$ is the indicator whether $y_i$ is a change point, $\mu_j$’s are independent given $I_i^y$’s, $\theta_i$’s are independent given $\mu_j$’s and $\tau^2$, and $y_i$’s are independent given $\theta_i$ and $\sigma^2$. It is assumed that $p_n \sim n^{-1}c_n$. For $n$ observations the expected number of change points $E[\sum_{i=1}^{n} I_i] = c_n$ is therefore of the order of $c_n$. Here, $c_n$ controls the expected numbers of change points apriori, and letting $c_n$ increase to infinity provides flexibility to model multiple and potentially large number of change points for large $n$.

3 Theoretical Properties

In this section, we establish the change-point detection consistency, in presence of covariate, and establish the rates of convergence. Let $l^*$ be the true number of change-point and $l^* + 1$ be the corresponding number of the partitions. Let $m^*_\beta$ denote the true model for the covariate combination. Let $M^* = M^*_{l^*, m^*_\beta} = M^*_{l^*, m^*_\beta}$ be denote the model with both true covariate combination and true change-point location, where for convenience we drop the suffix $t^*$ in $l^*_t$. Similarly, $M^*_{l^*, \tilde{m}^*_\beta}$ be a model with true change-point locations and $\tilde{m}^*_\beta \supset m^*_\beta$. For a generic model with change-point locations $t_l = (t_1, \cdots, t_l)$ and the covariate combination $m_\beta$, we write as $M^*_{l, m_\beta}$ or $M_{l, m_\beta}$ dropping the index $t_l$ for notational convenience.

For a model $M_{l, m_\beta}$ and for partition $P_j$ corresponding to change points $t_{j-1}$ and $t_j$, let $X_j$ be the corresponding design matrix. Let $q_n = o(n)$ be the maximum number of covariates permissible in a model. Then, we can state the following relatively straightforward result about the marginal distribution.
Proposition 1. Under model given in (2.1) we have for $\tau_j^2 = O(1)$, and known $\sigma^2$

$$\log L(Y_n \mid M_{l,m}) = -\frac{1}{2} \sum_j \log(|X_j'X_j|) - \frac{1}{2\sigma^2} \sum_j \sum_{i \in P_j} (y_i - x_i\hat{\beta}_j)^2 + c_{l,n} + O(1) \quad (3.1)$$

where $\hat{\beta}_j$ is the least squares estimator of the coefficient vector based only on the observations lying in $P_j$, and $c_{l,n} = -\frac{n}{2} \log \sigma^2 + c_l$, $c_l = O(kl')$ where $k = \#\{m_\beta\}$ denotes the size of the model $m_\beta$.

Remark 2. For $\tau_j^2 \gg 1$ in (2.1), we have

$$\log L(Y_n \mid M_{l,m}) = -\frac{1}{2} \sum_j \log(|X_j'X_j|) - \frac{1}{2\sigma^2} \sum_j \sum_{i \in P_j} (y_i - x_i\hat{\beta}_j)^2 + c'_{l,n},$$

where $c'_{l,n} = -\frac{n}{2} \log \sigma^2 + c'_l$, $c'_l = O(kl')$. Using the above mentioned result, it will be sufficient to prove our result for the flat normal prior on the coefficients, which will help us streamline the derivation of the following results.

Since many important applications as well as recent developments such as Frick et al. (2014) use the equal and known variance set-up: we establish our result first for this set-up. Then we extend our results to the case of unknown variance.

For a model $M_{l*,m_\beta}$ with the same number of change point as the true model let $P_j$ be the $j^{th}$ partition of $i = 1, \ldots, n$. Let $P_j*$ be the $j^{th}$ partition for $M^*$. Suppose, $X_{j\cap j*}$ be the design matrix corresponding to $P_j \cap P_j*$ and covariate combination $m_\beta \cup m_{\beta*}$, and $X_{j- j*}$ corresponds to $P_j - P_j*$, $X_{j\cap k*}$ corresponds to $P_j \cap P_k^*$, and $\beta^*_i$ be the true coefficient vector for $y_i$ and $\beta^*_i = \beta^{*j}$ for $P_j \cap P_j*$. Note that the design matrices are constructed when $P_j \cap P_j^*$, $P_j - P_j^*$, $P_j \cap P_k^*$ are non empty, respectively. Let $\hat{\beta}_j$ be the least square estimator based on $P_j$ based on model covariate choice given by $m_\beta$, and the entries corresponding to $m_\beta \cup m_{\beta*} - m_\beta$ is zero. Similarly, $\hat{\beta}_i^*$ and $\hat{\beta}^{*j}$ is defined by filling the entries not in $m_{\beta*}$ by zero.

Then, it follows after some calculations,

$$P(Y_n \mid M_{l*,m_\beta}) = -\frac{1}{2} \sum_j \log(|X_j'X_j|) - \frac{1}{2\sigma^2} \sum_i (y_i - \theta_i)^2 + \sum_{j: P_j \cap P_j^* \neq \emptyset} (\beta^{*j} - E[\hat{\beta}])'X_{j\cap j*}X_{j\cap j*}(\beta^{*j} - E[\hat{\beta}])$$

$$+ \sum_j \sum_{k: k \neq j: P_j \cap P_k^* \neq \emptyset} (\beta^{*k} - E[\hat{\beta}])'X_{j\cap k*}X_{j\cap k*}(\beta^{*k} - E[\hat{\beta}])$$

$$+ \sum_j (\hat{\beta}_j - E[\hat{\beta}])'X_j'X_j(\hat{\beta}_j - E[\hat{\beta}]) - \frac{1}{\sigma^2} [E_1 + E_2 + E_3] + c_{l,n} + O(1) \quad (3.2)$$
where, \( E_1, E_2, E_3 \) are the cross product terms, and \( \theta_t^* \) is the true value of \( E[y_t] \). Hence, accounting for the bias terms and bounding the cross product terms we can show the Bayes factor consistency if the proposed partition is not a refinement of the true partition for some covariate combination. If \( l > l^* \) then the expression similar to in equation 3.2 can be derived by comparing \( M_{l,m,\beta} \) with model with partition/change points corresponding to a refinement of the true partition, with \( l \) change points. A refinement of the true partition is a partition corresponding to change points \( \tilde{t}_1, \ldots, \tilde{t}_l \), with \( l > l^* \) and \( t^*_i \in \{\tilde{t}_1, \ldots, \tilde{t}_l\} \), for \( i = 1, \ldots, l^* \).

If the proposed partition is a refinement of the true partition, and the covariate combination contains the true set of covariates, then we will have terms asymptotically similar to Schwarz’s BIC (Schwarz et al., 1978) which will guarantee Bayes factor consistency.

Next, we make the following assumptions on covariates and model size:

(A1) The covariates \( X_j \)'s are uniformly bounded.

(A2) Coefficients \( \beta_j \)'s are uniformly bounded below and above.

(A3) \( (q_n \log n)^2 < n\epsilon_n \) and \( q_n^2 \leq n^c, c < 0.5 \); for a sequence \( \epsilon_n \to 0 \) such that \( n\epsilon_n \to \infty \). For two positive sequences \( \{a_n\}_{n\geq1}, \{b_n\}_{n\geq1}, a_n < b_n \) if \( a_n/b_n \to 0 \) as \( n \uparrow \infty \).

(A4) \( \tau_j^2 = V_n = O(1) \).

(A5) \( 0 < t^*_1 < \cdots < t^*_l < 1 \) are the true locations of change points (i.e. \( nt^*_j \leq \tau^*_j \leq nt^*_j + 1 \)).

(A6) Let \( \tilde{P}_1, \tilde{P}_2, \ldots, \tilde{P}_k \) be \( k \) disjoint subset of \( \{1, \ldots, i, \ldots, n\} \) of size \( n_1, n_2, \ldots, n_k \). Let \( X_1, X_2 \cdots, X_k \) be design matrices corresponding to observations indexed by \( \tilde{P}_1 \cdots \tilde{P}_k \) for some model based on \( q_n = o(n) \) many coefficients.

For, \( n_1, \ldots, n_k \to \infty, q_n/n_1, \ldots, q_n/n_k \to 0 \), assume \( a_n < \lambda_A < b_n \), \( a_{n_1+n_2+\cdots+n_k}/(n_1+n_2+\cdots+n_k) < \lambda_C \) \( \leq b_{n_1+n_2+\cdots+n_k}/(n_1+n_2+\cdots+n_k) \) where \( \lambda_X \) is singular value of \( X \) and, \( A = X'_iX_i \) and \( C = (X'_iX_i + X'_iX_j + \cdots + X'_kX_k)^{-1}X'_iX_i \) and \( a, b > 0 \).

We can now state the following result for the number of change points:

**Theorem 3.** Let \( 0 < t^*_1 < \cdots < t^*_l < 1 \) be the locations of true change points, \( M^* \) be the true model corresponding to true covariates and true change points, and \( 0 < t_1 < \cdots < t_l < 1 \) be the locations of change points for the alternative with \( l \neq l^* \). Let \( M_l \) be the corresponding model with change points at \( t_1 < \cdots < t_l \) and some covariate combinations \( m_\beta \) with at most \( q_n \) many covariates. Then under \( A1 - A6, \) for (2.1)

\[
BF(M_{l,m_\beta}, M^*) \to 0 \text{ in probability as } n \to \infty.
\]
Let $0 < t_1^* < \cdots < t_l^* < 1$ be the locations of true change points and $M^*$ be the true model corresponding to true covariates and true change points. Let $0 < t_1 < \cdots < t_l < 1$ be the location of change points for an alternative model with some covariate combination and assume $\max_i |t_i^* - t_i| > \epsilon_n$. Let $M_{l,m}^{*n}$ be the corresponding model with change points at $t_1, \cdots, t_l$ and some covariate combinations $m_\beta$ with at most $q_n$ many covariates. Then, we show that even for $\epsilon_n \sim n^{-1}$ up to some log factors, if $q_n$ increases in logarithmic rate, we have consistency., where for two positive sequences $\{a_n\}_{n \geq 1}, \{b_n\}_{n \geq 1}, a_n \sim b_n$ if $a_n/b_n$ is bounded away from zero and infinity. In particular:

**Theorem 4.** Under $A1 - A6, for (2.1)$

$$BF(M_{l,m_\beta}^{*}, M^*) \to 0 \text{ in probability as } n \to \infty.$$ 

Note that $q_n \log n \prec \sqrt{n \epsilon_n}$. If $q_n = O(\log n)$, $\epsilon_n \sim n^{-1}$ (up to logarithmic factors), satisfies this condition. Under stronger conditions, the Bayes factor consistency holds uniformly over covariate choice. The results hold for misspecification of variance parameter.

**Remark 5.** The condition of bounded covariates given in (A1) can be relaxed. For example, if we use sub-exponential type tail bound conditions on the distributions of the covariates, that is, if $F_m(\cdot)$ is the CDF for $m^{th}$ covariate and $-\log(1 - F_m(t)) \geq t^\alpha$ for some $\alpha > 0$, then Theorem 4 holds under slightly modified (A3) (up to log factors). For two positive sequences $\{a_n\}_{n \geq 1}, \{b_n\}_{n \geq 1}, a_n \leq b_n$ if $a_n \leq K b_n$ for some constant $K > 0$.

Instead of using known variance $\sigma^2$, if we use $\hat{\sigma}^2 = n^{-1} \sum_j \| Y_j - \hat{Y}_j \|^2$, it can be shown that the conclusion of Theorems 3 and 4 hold. Here $Y_j$ is the observation vector for $P_j$ and the $\hat{Y}_j$ is the least square fit for the proposed model $m_\beta$ based on the observations and covariates in $P_j$. This result is addressed in the following Theorems.

**Theorem 6.** Let $0 < t_1^* < \cdots < t_l^* < 1$ be the locations of true change points, $M^*$ be the true model corresponding to true covariates and true change points, and $0 < t_1 < \cdots < t_l < 1$ be the locations of change points for the alternative with $l \neq l^*$. Let $M_{l}^{*}$ be the corresponding model with change points at $t_1 < \cdots < t_l$ and some covariate combinations with at most $q_n$ many covariates. Then under $A1 - A6, for (2.1)$, for the empirical estimator of $\sigma^2$,

$$BF(M_{l,m_\beta}^{*}, M^*) \to 0 \text{ in probability as } n \to \infty.$$ 

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For the rate calculation, we have the following result.

**Theorem 7.** Under $A_1 - A_7$, for (2.1), for the empirical estimator of $\sigma^2$,

$$BF(M_{t*}^{(n)}, M^*) \to 0 \text{ in probability as } n \to \infty.$$ 

**Remark 8.** Theorem 6 and Theorem 7 hold under misspecification, that is if the true variance parameter is not same over different partitions $P_j$'s.

Next, we address the variable selection issue. We have already shown that under any covariate combination, the model with incorrectly selected change points has Bayes factor converging to zero with respect to the model with true change points and covariate combination. Showing variable selection consistency with the change points set to true change points then boils down to showing the consistency of variable selection. Let $\Pi(M_{t*}^{(m_{\beta})} | \cdot)$ be the posterior probability of a model with covariate combinations given $m_{\beta}$ and change points at true change point locations $t_1^*, \cdots, t_l^*$, and let $P_n(M_{t*}^{(m_{\beta})} : M^*)$ be the ratio of the posterior probabilities $\Pi(M_{t*}^{(m_{\beta})} | \cdot)$ and $\Pi(M^* | \cdot)$, based on $n$ observations. Similarly, we can define $P_n(M_{t,m_{\beta}} : M^*)$. We have the following result regarding variable selection.

**Theorem 9.** Under $A_1 - A_7$, for (2.1), $BF(M_{t*}^{(m_{\beta})}, M^*) \to 0$ in probability for $m_{\beta} \neq m_{\beta^*}$.

For showing consistency over all possible variable choices we assume the following.

- **(B1)** $- \log P(I_{m}^\beta = 1) \sim (\log n)^{1+\alpha_1}$, for any $\alpha_1 > 0$.
- **(B2)** $\sqrt{n} > q_2^{2.5} \log n$ and $\sqrt{n}e_n > q_1^{1.5} \log n$.

**Remark 10.** Condition B1 imposes a stronger penalty on the larger model, which induces selection consistency uniformly over covariate choice and model size. Similarly B2 is needed to bound the Bayes factors uniformly over covariate choices.

We consider the models $M_{t*,m_{\beta}}^{(\epsilon_n)}$ defined as earlier. Then we have the following.

**Theorem 11.** Under the assumptions $A_1 - A_7$ and B1, B2, for (2.1), $\sup_{m_{\beta} \neq m_{\beta^*}} P_n(M_{t*,m_{\beta}}^{(\epsilon_n)} : M^*) \to 0$ in probability, where the supremum is over the possible change point selections and $\epsilon_n$ converges to zero.

For any model with change points $t_1 < \cdots < t_l$ we can state the following result for variable
Theorem 12. Let $M_{l,m_\beta}$ be the corresponding model with change points at $t_1 < \cdots < t_l$ and some covariate combinations $m_\beta$. Under $A1 - A7, B1, B2,$ for (2.1), $\sum_{m_\beta : m_\beta \neq m_\beta^*} P_n(M_I, m_\beta : M^*) \to 0$ in probability.

Remark 13. Theorems 11 and Theorem 12 hold under unknown $\sigma^2$, for the empirical estimator $\sigma^2$ as in Theorem 6 and 7, and under misspecification of equal variance as in Remark 8.

Remark 14. Covariate free cases. One special case of the model given in (2.2) is the covariate free cases, which is the simple mean model. For such model the result given in Theorem 4 regarding Bayes factor consistency holds for $\epsilon_n$ going to zero for $\epsilon_n \geq n^{-1}(\log n)^2$, which gives us rate equivalent to frequentist minimax rate up to logarithmic factors.

4 Simulation

In this section, we demonstrate the performance of the Bayesian hierarchical model described in (2.2) and (2.1), for changing linear model and the simpler special case of piecewise constant mean model, respectively. First, we show the recovery of true mean as well as the true change-point locations the simple changing mean model, and then we consider a case with number of covariates $d_n \sim n$ case where we show that we can achieve accuracy in both variable selection and change point estimation under the Bayesian model. Our goal here is not to establish superiority of the Bayesian method used here over extant methods, but rather to show that the methods are not just theoretically optimal, they also have satisfactory small sample performance.

4.1 Example 1

We consider an example originally reported in (Frick et al., 2014) and compared against an empirical Bayes procedure in (Martin and Shen, 2017), with 6 change points for piecewise constant Gaussian sequence model. Here the data-generating model is given as follows:

$$y_i = \theta_i + \epsilon_i, \quad \epsilon_i \overset{iid}{\sim} \mathcal{N}(0, 0.04), \quad i = 1, \ldots, n \ (= 497),$$

(4.1)
with the true mean being:

\[
\theta_i = \begin{cases} 
-0.18; & 1 \leq i \leq 138 \\
0.08; & 139 \leq i \leq 225 \\
1.07; & 226 \leq i \leq 242 \\
-0.53; & 243 \leq i \leq 299 \\
0.16; & 300 \leq i \leq 308 \\
-0.69; & 309 \leq i \leq 333 \\
-0.16; & 334 \leq i \leq 497 
\end{cases}
\]

The sequence of true means \(\theta_i\)'s are depicted in Fig. 1.

![Figure 1: True mean parameter values used in (Frick et al., 2014)](image)

We compare the recovery and estimation performance of the method proposed here with two candidates: the first is a frequentist method (the pruned exact linear time method or PELT, (Killick et al., 2012)) and the second is an empirical Bayes approach from (Martin and Shen, 2017) (EB). We describe these two comparative candidates briefly.

For PELT, consider ordered data-points: \(y_1, y_2, \ldots, y_n\) and \(m\) change-points \(\tau_1, \ldots, \tau_m\) that divide the data into \(m + 1\) partitions. The change-point detection methods then seek to minimize a function:

\[
\sum_{i=1}^{m} C(y_{\tau_{i-1}+1: \tau_i}) + \text{pen}(n) f(m),
\]

where \(C\) is a cost-function and \(\text{pen}(n) \times f(m)\) is the penalty applied to prevent over-fitting. For observations \(y_1, \ldots, y_n \sim f(y | \theta)\) for some unknown underlying parameter \(\theta\). The PELT method uses the negative log-likelihood as the cost function:

\[
C(y_{(t+1):s}) = -\max_{\theta} \sum_{i=t+1}^{s} f(y_i | \theta).
\]

The penalty is chosen based on the inferential goal, e.g. \(\text{pen}(n) = n \log(n)\) is the popular BIC penalty and \(f(m) = m\) assumes that penalization is linear with the number of change-
points. When $m$ is not too large, BIC favors a parsimonious model and can be shown to be model selection consistent.

The empirical Bayes (EB) approach in (Martin and Shen, 2017) works via specification of priors on block-specific parameter vectors ($\theta_B$) and block configurations, where the prior centers on mean parameters are data-dependent. In particular, the mean parameter in each block is assumed to be Gaussian centered on maximum likelihood estimates based on observations in that block, and the block-configuration follows a discrete uniformly distributed partition points, with the configuration size or number of blocks following a truncated geometric distribution.

For the results shown below, we do not assume known $\sigma^2$ and assume that they can be different over partitions. We use $V = 1$, $I_i \sim \text{Bernoulli}(1/n)$, and calculate the posterior summaries based on 8,000 Markov chain monte carlo samples with first 4,000 burn-ins. The fitted mean and the posterior probabilities for partitions are given in Fig. 2.

4.2 Example 2: a case with covariates

Consider a changing linear regression problem where the underlying linear model changes between different observation windows or epochs. Here the parameters of interest are both the
parameter vector $\beta$ as well as the number and location of change points. Let us fix the dimensions of observations and covariates to be $n = 250, p = 250$. Suppose the true locations of change-points as a fraction of the total number of observations are given by: $t^*_1 = 0.3, t^*_2 = 0.7$. Finally, let the covariates for the $i^{th}$ observation $x_{i,1}, \ldots, x_{i,p}$ ($i = 1, \ldots, n$), are generated from independent standard normal distribution. The data-generating model used for this experiment is given as follows:

$$y_i = \begin{cases} 
3 + x_{i,2} + 2x_{i,12} + 1.2e_i & \text{if } i \leq 75 \\
1 + 2x_{i,2} + .8e_i & \text{if } 75 < i \leq 175, i = 1, \ldots, 250. \\
-2.5 + 2x_{i,2} - x_{i,3} + e_i & \text{if } 175 < i \leq 250.
\end{cases}$$

Here, $e_i \overset{iid}{\sim} \mathcal{N}(0, 1)$ and the true change points occur in positions $i = 75$ and $i = 175$ as mentioned before, and the proportion of change-points in observations is $p_n = \frac{1}{n}$.

Figure 3: Changing linear regression example showing performance of the hierarchical Bayesian model

We use a spike-slab prior on the regression coefficients to detect the global set of covariates and the change points. We let the Markov chain Monte Carlo chain run for 8,000 iterations and calculate the posterior modes and means for the two indicator variables for the change-points.
and the non-null $\beta_j$s. The posterior mean estimates are plotted in Fig. 3. Figure 3a shows that the both the two change-points at $i/n = 0.3$ and 0.7 can be recovered with high posterior probability. Figures 3b and 3c show that the global set of active $\beta_j$’s can also be recovered with high probability. In particular, from Fig. 4, almost 93% of the posterior samples give the correct number of partitions (3), and almost 90% of the posterior samples select the right model. Thus, the numerical results are in concurrence with our theoretical proofs of consistency in model selection and change-point detection for a changing linear regression in §3.

![Posterior distribution for number of change-points](image1)

(a) Posterior distribution for number of change-points

![Posterior distribution for number of active covariates](image2)

(b) Posterior distribution for number of active covariates

**Figure 4: Regression example: posterior distribution of model size and partition size.**

### 4.3 Example 3: a case with covariates and time dependent component

This example considers a change in linear structure when an autoregressive time dependent component is present. Such scenario may arise in economic application, when for example housing price index may change with the covariates stock market return, but an autoregressive structure may be present for the response variable, i.e. price. We use a similar model as before with $n = 300$, $p = 250$ but add an autoregressive component of first order (AR(1)) with autocorrelation equal to $\rho$. As before, the true change-points occur at positions $i = 90$ and $i = 210$, at relative positions $t^* = 0.3$ and $t^* = 0.7$ as before. We generate data from the following model:

\[
y_i = \begin{cases} 
3 + \rho y_{i-1} + x_{i,2} + 3x_{i,12} + 1.2e_i & \text{if } i \leq 90 \\
1 + \rho y_{i-1} + 2x_{i,2} + .8e_i & \text{if } 90 < i \leq 210 \\
-2 + \rho y_{i-1} + x_{i,2} - x_{i,3} + e_i & \text{if } 210 < i \leq 300,
\end{cases}
\]

with $\rho = 0.5$, $x_{i,j}$, $e_i$’s are i.i.d. $\mathcal{N}(0, 1)$. It should be noted that our theoretical results from §3 will continue to hold for this situation, as long as condition (A6) holds for the new design matrix, accounting for the autoregressive structure.

We use the spike and slab prior on the coefficients and the parameter $\rho$, and use a computational scheme similar to the example in §4.2. The posterior probabilities for the variable selection and
change-point detection are given in Fig. 5, where it can be seen that the true change point locations (vide Fig. 5a) and the active variables (vide Fig. 5b and Fig. 5c) are selected with high probability. We also note that the estimated posterior inclusion probability for the AR(1) component is 1.

Figure 5: Changing linear regression example showing performance of the hierarchical Bayesian model, in the presence of the AR(1) component.
5 Real Data Applications

Next the proposed method is applied to detecting change points in crime data from Little Rock, AR. Among all cities in the United States with at least 100,000 residents, Little Rock is ranked in the top 10 for the highest violent crime (7th) and property crime (4th) rates in 2015 (Chillar and Drawve, 2020).

For a piecewise constant mean model, there are competing methods against which the proposed method will be compared, as mentioned earlier there is essentially no comprehensive framework for model selection under change point. The data is preprocessed by a square root transformation and standardization.

For comparing and contrasting the proposed method on the weekly burglary and breaking and entering data from Little Rock from 2017, the empirical Bayes method proposed by Martin and Shen (2017), as well as the PELT (Pruned Exact Linear Timing) method by Killick et al., 2012, are used. As it is impossible to know if there should be any “true” change-points in 2017 data in the absence of additional information, this can be regarded as a preliminary exploratory analysis. As Fig. 6 suggests, the change-points recovered by the proposed method mostly agree with those by the PELT method, while the Empirical Bayes method seems to be conservative and does not detect any change points in the data.

![Figure 6: Changepoints detected by the PELT-BIC method (top panel) and the proposed approach and Empirical Bayes approach (Martin and Shen, 2017) (bottom panel) for the burglary and breaking and entering activities in Little Rock in 2017. Data is standardized for the bottom panel.](image)
6 Appendix

Proof of Theorem 3

We show our result first for a model \( M_{l,m_{\beta}} = M_{l,m_{\beta}}^* \) with \( l \) change points, where, the \( P_j^* \)'s are a refinement of \( P_j \)'s, the true partition, and \( m_{\beta} \) contains the true covariate combination, i.e. \( m_{\beta} \supset m_{\beta^*} \). That is for \( t_1, \ldots, t_l \) be the proposed change point, and \( \tilde{t}_1, \ldots, \tilde{t}_l \) be the change points corresponding to some refinement of the true partition \( t_{1}^*, \ldots, t_{l}^* \) such that \( t_i = \tilde{t}_i, \forall i \). Then we show that \( BF(M_{l,m_{\beta}}^*, M_{l,m_{\beta}}^*) \to 0 \) and \( BF(M_{l,m_{\beta}}^*, M^*) \to 0 \). (Case i)

Then we show the case where \( t_1, \ldots, t_l \) be the proposed change point, and \( \tilde{t}_1, \ldots, \tilde{t}_l \) be change point corresponding to any refinement of the true partition \( t_{1}^*, \ldots, t_{l}^* \) and \( \max_i |t_i - \tilde{t}_i| > \epsilon > 0 \) (Case ii). Without loss of generality, we assume \( \epsilon < \min_i |t_i - t_{i-1}|; i = 1, \ldots, l \).

Then we show the case where the \( m_{\beta} \) does not contain the true covariate combination for some partition for \( l \geq l^* \). (Case iii)

Finally we show for the case where \( l < l^* \). (Case iv)

Case (i)

We show \( BF(M_{l,m_{\beta}}^*, M_{l,m_{\beta}}^*) \to 0 \) and \( BF(M_{l,m_{\beta}}^*, M^*) \to 0 \) in probability.

\[
BF(M_{l,m_{\beta}}^*, M_{l,m_{\beta}}^*) \to 0
\]

Writing the \( P_j^* \) for some \( j \) for the true partition as the union of \( P_j^* \)'s, \( P_j^* \) \( k > 1 \), where \( P_j^* = P_j^{'k} \) for some \( j^{'k} \in \{1, \ldots, l + 1\} \). Let \( Y_n^{j_*} \) be the vector of \( y_i \)'s in \( P_j^* \), and \( X_j^* \) be the corresponding covariate matrix. Similarly, \( X_j^* \) be the covariate matrix for \( P_j^* \). Let \( \hat{Y}_n^{j_*} \) and \( \hat{Y}_n^{j_*} \) be their least square fit based on \( P_j^* \) and \( P_j^*'s \). Let \( \hat{Y}_{n,i}^{j_*} \) be the sub-vector of \( \hat{Y}_n^{j_*} \) corresponding to the observations in \( P_j^{i_*} \).

Now

\[
- \log L(Y_n^{j_*} | M_{l,m_{\beta}}^*) + \log L(Y_n^{j_*} | M_{l,m_{\beta}}^*) = -\frac{1}{2} \det(X_j^* X_j^*) + \frac{1}{2} \sum_{i=1}^{k} \det(X_j^* X_j^*) - \frac{1}{2\sigma^2} \|Y_n^{j_*} - \hat{Y}_n^{j_*}\|^2
\]

\[
+ \sum_{i=1}^{k} \frac{1}{2\sigma^2} \|Y_n^{j_*} - \hat{Y}_n^{j_*}\|^2 + c_{l,l^*}
\]

where \( c_{l,l^*} = O(k(l - l^*)) \), where \( k \) is te number of covariates in \( m_{\beta} \). Now,

\[
\|Y_n^{j_*} - \hat{Y}_n^{j_*}\|^2 = \sum_{i=1}^{k} \|Y_n^{j_*} - \hat{Y}_n^{j,*}\|^2 = \sum_{i=1}^{k} \|Y_n^{j_*} - \hat{Y}_n^{j,*} + \hat{Y}_n^{j,*} - \hat{Y}_n^{j,*}\|^2 = \sum_{i=1}^{k} \|Y_n^{j,*} - \hat{Y}_n^{j,*}\|^2 + \|\hat{Y}_n^{j,*} - \hat{Y}_n^{j,*}\|^2.
\]
Next, we consider the determinant term. Let
\[
\delta_l \equiv \sum_{\iota \in P^*_{j_l}} (y_\iota - x_\iota \hat{\beta}_l) x_\iota (\hat{\beta}_l - \check{\beta}_l) = \sum_m \delta_{l,m} \sum_{\iota \in P^*_{j_l}} (y_\iota - x_\iota \hat{\beta}_l) x_\iota = 0.
\]

Next, we consider the determinant term. Let \(n^*_j\) be the number of observations in \(P^*_j\). Similarly we define \(n^*_j\) for \(P^*_j\)'s, and \(\sum_i n^*_j = n^*_j\). Let \(\lim_{n \to \infty} \frac{n^*_j}{n} = \alpha_j \). From the fact that \(\log \det X_{j^*_j} = q \log n^*_j + \log \alpha_j + e_q\), for large \(n\) where \(|e_q| \leq q(|\log a| + |\log b|)\), we have
\[
-\log \left( \frac{Y_{n}^{j^*_j}}{M_{l,m}} \right) + \log \left( \frac{Y_{n}^{j^*_j}}{M_{l,m}} \right) \geq \frac{1}{2}(k - 1)q \log n - qO_p(1) + C_0
\]
for a generic constant \(C_0\) and hence, \(BF(M_{l,m}, M_{l^*,m}) \to 0\) in probability.

\[
BF(M_{l^*,m}, M^*) \to 0
\]

For each partition \(P^*_j\),
\[
2 \log L(Y_{n}^{j^*_j} | M^*) - 2 \log L(Y_{n}^{j^*_j} | M_{l,m}) \sim (q - \#\{m_{\beta^*}\}) \log n + \frac{1}{\sigma^2} \chi^2_{\{q - \#\{m_{\beta^*}\}\}} + d_{m} \chi^2_{\{q - \#\{m_{\beta^*}\}\}} C_0
\]
for a generic constant \(C_0\), where \(q = \#\{m_{\beta^*}\}\). Hence, the result follows.

**Case (ii)**

From (3.2), we have
\[
E_1 = \sum_j E_{1,j}, E_2 = \sum_j E_{2,j}, \text{ and } E_3 = \sum_j E_{3,j}, \text{ where for } e_i = y_i - \theta_i^*:
\]
\[
E_{1,j} = \sum_{i \in P^*_j} e_i (x_i E[\hat{\beta}_j] - x_i \hat{\theta}_j^*), \quad E_{2,j} = \sum_{i \in P^*_j} e_i (x_i \hat{\beta}_j - x_i E[\hat{\beta}_j]);
\]
\[
E_{3,j} = \sum_{i \in P^*_j} (x_i E[\hat{\beta}_j] - x_i \hat{\theta}_j^*) (x_i \hat{\beta}_j - x_i E[\hat{\beta}_j])
\]  
(6.1)

Let \(\delta_m^{n,i}\) be the \(m\) th component of \(E[\hat{\beta}_l] - \hat{\theta}_l^*\), and \(\delta_m^{n,E}\) be the \(m\) th component of \(\hat{\beta}_l - E[\hat{\beta}_l]\). Note that \(\delta_m^{n,i}\) can take \(l^* + 1\) many different values, as \(i \in P^*_j\) for some \(j \in \{1, \ldots , l^* + 1\}\). We denote it by \(\delta_m^{n,i} \text{ for } i \in P_j \cap P^*_i\).
Note that $E[\hat{\beta}]$ is linear combination of $\hat{\beta}^{*j}$’s of the form $(\sum_{i \in I_j} B'_iB_i)^{-1}(\sum_{i \in I_j} B'_i\hat{\beta}^{*i})$, where $I_j$ is a subset of $\{1, \ldots, l^* + 1\}$, and $i \in I_j$ if $P_i \cap P_j \neq \emptyset$ for some $i$, and $B_i$ be the corresponding covariate matrix for observations in $P_i \cap P_j$. Hence, $E[\hat{\beta}]$, bounded at each component by condition A6. For $E_{2,j}, E_{3,j}$ we use the fact that $n$ dimensional multivariate normal with bounded variance, the absolute value maximum is bounded by $\log n$ for large $n$. Then,

\[ |E_{1,j}| = \left| \sum_{m=1}^{q} \sum_{l'=1}^{l^{*}+1} \delta_{m,l'} \sum_{i \in P_j \cap P_j^*} e_i x_{i,m} \right| = o_p(q \sqrt{n} \log n), \]

\[ |E_{2,j}| = \left| \sum_{m=1}^{q} (\sqrt{n} \delta^n_{m,E}) \frac{1}{\sqrt{n}} \sum_{i \in P_j} e_i x_{i,m} \right| = o_p(q \log n), \]

\[ |E_{3,j}| = \left| \frac{1}{\sqrt{n}} \sum_{l'=1}^{l^{*}+1} \sum_{m,m'} \delta_{m,l'} (\sqrt{n} \delta_{m',E}) x_{i,m} x_{i,m'} \right| = o_p(\sqrt{n}q^2 \log n). \]

Without loss of generality we assume that $l = l^*$ (otherwise, we can show for a refinement of true partition for change points $\tilde{t}_1, \ldots, \tilde{t}_l$, such that $|t_j - \tilde{t}_j| > 0$ and $\tilde{t}_j \in \{t^*_1, \ldots, t^*_l\}$, then use the result proved in Case (i)).

Let $|t_j - t^*_j| > \epsilon$ and the model be denoted by $M_{t^*,m_B}^l$. Then,

\[ - \log L(Y_n|M_{t^*,m_B}^l) + \log L(Y_n|M_{t^*,m_B}^l) = -\frac{1}{2} \sum_j \log(\det(X'_j, X_j)) - \sum_j \log(\det(X'_j X_j)) + \frac{1}{2 \sigma^2} \sum_{j:P_j \cap P_j^* \neq \emptyset} (\hat{\beta}^{*j} - E[\hat{\beta}^{*j}])'(X'_j X_j)^{-1} (\hat{\beta}^{*j} - E[\hat{\beta}^{*j}]) + \frac{1}{2 \sigma^2} \sum_{j} \sum_{k:k \neq j, P_k \cap P_k^* \neq \emptyset} (\hat{\beta}^{*k} - E[\hat{\beta}^{*k}])'(X'_j X_j) (\hat{\beta}^{*k} - E[\hat{\beta}^{*k}]) + \frac{1}{2 \sigma^2} \sum_j (\hat{\beta}^{*j} - E[\hat{\beta}^{*j}])'(X'_j X_j) (\hat{\beta}^{*j} - E[\hat{\beta}^{*j}]) - R_n \]

(6.2)

where $R_n = o_p(\log n \sqrt{nq^2}),$ and $\sum_j (\hat{\beta}^{*j} - E[\hat{\beta}^{*j}])'(X'_j X_j) (\hat{\beta}^{*j} - E[\hat{\beta}^{*j}]) \sim \chi^2_{q^2j^*}$.

As, $\beta^{*j} \neq \beta^{*(j+1)}$ for $j = 1, \ldots, l^*$. Therefore, one of the first two quadratic form sums is computed for a nonzero vector for some $j$. The first quadratic is based on $\sim n(1 - \epsilon)$ observations.
and the second one is based on \( \sim n\epsilon \) observations, and therefore by A6

\[
- \log L(\mathbf{Y}_n | M_{l,m}^*) + \log L(\mathbf{Y}_n | M_{l',m}^*) \geq n\epsilon - R_n \to \infty
\]

which proves our claim.

**Case (iii)**

Case (iii) follows similar to last step in Case (ii), as the bounds on \( E_1, E_2, E_3 \) are of same order as the earlier part, and we have \( \mathbb{E}[\beta^j] \neq \beta^*_j \) for some \( j \).

**Case (iv)**

We have

\[
P(\mathbf{Y}_n \mid M_{l,m}) = \frac{1}{2} \sum_j \log(|X'_j X_j|) - \frac{1}{2\sigma^2} \sum_i (y_i - \theta^*_i)^2 + \sum_j \sum_{j_1 : P_j \cap P_j^* \neq \phi} (\beta^*_j - E[\beta^j])' X'_j \cap_{j_1} \cdot X_j \cap_{j_1} \cdot (\beta^*_j - E[\beta^j]) + \sum_j (\beta^j - E[\beta^j])' X'_j X_j (\beta^j - E[\beta^j]) - \frac{1}{\sigma^2} [E_1 + E_2 + E_3] + c_l + n + O(1).
\]

where \( X_j \cap_{j_1} \cdot \) is the design matrix corresponding to \( P_j \cap P_j^* \) if \( P_j \cap P_j^* \neq \phi \), \( j_1 \in \{1, \ldots, l^* + 1\} \). For \( l < l^* \), we have \( \min \{ \mathcal{L}(\{t_{i-1}, t_i \} \cap [t^*_j, t^*_j + 1]), \mathcal{L}(\{t_{i-1}, t_i \} \cap [t^*_j, t^*_j + 1]) \} > 0 \) for some \( i, j \), if \( t_1, \ldots, t_l \) correspond to the change points in \( M_{l,m} \), and \( \mathcal{L}(\cdot) \) is the Lebesgue measure. Then, \( \sum_j \sum_{j_1 : P_j \cap P_j^* \neq \phi} (\beta^*_j - E[\beta^j])' X'_j \cap_{j_1} \cdot X_j \cap_{j_1} \cdot (\beta^*_j - E[\beta^j]) \sim O(n) \). The bounds on \( E_1, E_2, E_3 \) from Case (ii) hold and hence, \(-P(\mathbf{Y}_n | M_{l,m}) + P(\mathbf{Y}_n | M_{l',m}) = O_p(n)\), which proves our claim.

**Proof of Proposition 1**

Marginalizing over the coefficient vector on \( P_j \) we have,

\[
\log L(\mathbf{Y}_n | M_{l,m}) = \frac{n}{2} \log \sigma^2 - \log \det(X'_j X_j + S_\beta) + \log \det(S_\beta) - \frac{1}{2\sigma^2} [Y_n' Y_n - Y_n' X'_j (X'_j X_j + S_\beta)^{-1} X_j Y_n] \quad (6.3)
\]

where \( \sigma^2 S_\beta^{-1} \) is prior variance covariance matrix for the coefficients in \( m_\beta \).

Next, we consider \((X'_j X_j + S_\beta)^{-1}\). Let \( AA = X'_j X_j \), where \( A \) is a positive definite matrix with eigenvalues of the order of \( \sqrt{m_j} \).
Then, using the Neumann series expansion (Horn and Johnson, 2012, p.348), we arrive at:

\[
(X_j'X_j + S_\beta)^{-1} = A^{-1}(I_q + A^{-1}S_\beta A^{-1})^{-1} A^{-1} = A^{-1}(I_q - B - B^2 - B^3 - \cdots )A^{-1}
\]

(6.4)

where \( B = A^{-1}S_\beta A^{-1} \) has Eigen values of the order \( n^{-1} \) and the above expression is valid for sufficiently large \( n \).

Hence,

\[
Y_n'X_j'(X_j'X_j + S_\beta)^{-1}X_jY_n = Y_n'X_j'A^{-1}A^{-1}X_jY_n - \sum_{k=1}^{\infty} Y_n'X_j'A^{-1}B^k A^{-1}X_jY_n.
\]

(6.5)

Note that \( \|Y_n\|^2 \leq 2\|\theta^n\|^2 + \|\epsilon^n\|^2 \leq n \) with probability one, \( \theta^n \) is the vector of \( \theta_i \)’s and \( \epsilon^n \) the vectors of \( \epsilon_i \)’s. The Eigen values of \( A^{-1}B A^{-1} \) is of the order of \( n^{-k-1} \).

Hence, \( \|Y_n'X_j'A^{-1}B^k A^{-1}X_jY_n\| \sim n^2n^{-k-1} \), and therefore, \( \sum_{k=1}^{\infty} Y_n'X_j'A^{-1}B^k A^{-1}X_jY_n \) is \( O(1) \) with probability one.

Let \( \lambda_{ij} \) be the Eigen values of \( B \) for \( i = 1, \cdots , q \), where \( \lambda_i > 0 \) and bounded. Again, \( \log \det((X_j'X_j + S_\beta)) = \log(det(A))^2 + \log(det(I + B)^{-1}) = \log det(X_j'X_j) - \sum_{k=1}^{q} \log(1 + \frac{\lambda_i}{n}) = \log det(X_j'X_j) - O(\frac{2}{n}) \).

Hence, combining the calculation of the determinant and the residual calculation from equation 6.5, the result follows.

**Proof of Theorem 4**

We assume that \( m_\beta \supseteq m_{\beta^*} \). For the case, where \( m_\beta \) does not contain the true covariates, the proof will follow similar to the proof of Case iii of Theorem 3. The proof for \( m_\beta \supseteq m_{\beta^*} \) is given as the following.

From equation (6.1), we decompose \( E_{1,j} \) and \( E_{2,j} \) in two parts \( E_{1,j \cap j^*}, E_{1,j - j^*}, \) and \( E_{3,j \cap j^*}, E_{3,j - j^*} \), where \( E_{1,j \cap j^*} = \sum_{i \in P_j \cap P_j^*} \epsilon_i (x_i E[\hat{\beta}] - x_i \hat{\beta}^j) \), and \( E_{1,j - j^*} = \sum_{i \in P_j - P_j^*} \epsilon_i (x_i E[\hat{\beta}] - x_i \hat{\beta}^j) \). Similarly, \( E_{3,j \cap j^*}, E_{3,j - j^*} \) are defined.

As in the proof of Theorem 3, \( \delta_{m,j}^i \) be the \( m \) th component of \( E[\hat{\beta}] - \hat{\beta}_i^j \), and \( \delta_{m,E}^i \) be the \( m \) th component of \( \hat{\beta}^j - E[\hat{\beta}] \), and \( \delta_{m,j}^i \) can take \( l^* + 1 \) many different values, as \( i \in P_j^* \) for some \( j \in \{1, \cdots , l^* + 1 \} \). It is denoted by \( \delta_{m,l^*}^i \) for \( i \in P_j \cap P_j^* \).

Note that, number of observations in \( P_j - P_j^* \) is \( \leq n\epsilon_n \), and \( \delta_{m,j}^i \) in \( P_j \cap P_j^* \) is of the order of \( \epsilon_n \) (follows from Lemma 15). Hence, \( |E_{1,j \cap j^*}| \leq \sum_{m} \left| \delta_{m,j}^n \right| n^{1/2}n^{-1/2} \sum_{i \in P_j \cap P_j^*} \epsilon_i x_i m \leq n^{1/2}q_n \log n \).
for large $n$ almost surely. Next,

$$|E_{1,j-j^*}| \leq \sum_{l'} \sum_m \left| \delta_{m,l'} \right| \frac{1}{\sqrt{n} \epsilon_n} \sum_{i \in P_j \cap P_{l'}^*} e_i x_{i,m} \leq q_n \sqrt{n} \epsilon_n \log n$$

almost surely.

Using, similar calculation, with probability one,

$$|E_{3,j-j^*}| \leq \sum_{m,m'} \left| \delta_{m,j} \right| n^{-1/2} \sum_{i \in P_j \cap P_{j'}^*} x_{i,m'} (n^{1/2} \delta_{m',E}) x_{i,m} \leq \epsilon_n n^{1/2} q_n^2 (\log n),$$

and,

$$|E_{3,j-j^*}| \leq \sum_{l'} \sum_{m,m'} \left| \delta_{m,l'} \right| n^{-1/2} \sum_{i \in (P_j - P_j') \cap P_{j'}^*} x_{i,m'} (n^{1/2} \delta_{m',E}) x_{i,m} \leq \epsilon_n n^{1/2} q_n^2 (\log n).$$

As in the proof of Theorem 3, here we use the fact that the absolute value of the maximum of an $n$-dimensional multivariate normal is less than $\log n$ for large $n$.

Hence, from equation (6.2), using the stricter bound for $E_1, E_2, E_3$ from the above derivation,

$$-\log L(Y_n|M_{t*,m\beta}) + \log L(Y_n|M_{t*,m\beta}) \geq n \epsilon_n - \sqrt{n} \epsilon_n q_n^2 \log n - q_n \sqrt{n} \epsilon_n \log n + q_n \log n \to \infty$$

(6.6)

as $n \to \infty$, which proves our result, as $BF(M_{t*,m\beta}, M^*) \to 0$ in probability, as in Theorem 3 (Case i, second part), by BIC type quantities for each partition $P_j^*.$

**Lemma 15.** Under the setting of Theorem 4, we have $\|E[\hat{\beta}_i] - \tilde{\beta}_i^*\|_{\infty} \leq \epsilon_n$, for $i \in P_j \cap P_{j}^*.$

**Proof.** Let $Z_j$ be the design matrix corresponding to observations in $P_j \cap P_{j}^*$, and $Z_{l'}$ be the matrix corresponding to observations in $P_j \cap P_{l'}^*$; $l' = 1, \cdots, l^* + 1, j \neq l'$. Note that number of observations corresponding to $P_j \cap P_{l'}^*$ is less than $n \epsilon_n + 1$.

Then,

$$E[\hat{\beta}] = (Z_j'Z_j + \sum_{l',l' \neq j} Z_{l'}Z_{l'})^{-1} (X_{j}E[Y_{n}]_j) = (CC + D)^{-1} (Z_j'Z_j \tilde{\beta}_j^* + \sum_{l',l' \neq j} Z_{l'}Z_{l}\tilde{\beta}_{l'}^*),$$

where $C$ is a positive definite matrix with Eigen values of the order of $\sqrt{n}$, when $\epsilon_n \to 0$, and $CC = Z_j'Z_j.$
Now, writing $(CC + D)^{-1} = C^{-1}(I + C^{-1}DC^{-1})^{-1}C^{-1}$ and from the fact the Eigen values of $B = C^{-1}DC^{-1}$ is of the order $\epsilon_n$, for large $n$ and therefore, similar to equation 6.4

$$(CC + D)^{-1} = C^{-1}(I - B - B^2 - \cdots)C^{-1}.$$ 

Hence,

$$E[\hat{\beta}] = C^{-1}CC\beta^* + C^{-1}C^{-1} \sum_{l' \neq j} Z_l^t Z_l^i \hat{\beta}^{*l'} - \sum_{k=1}^{\infty} C^{-1}B^k C^{-1}(Z_j^t Z_j^i \hat{\beta}^{*j} + \sum_{l' \neq j} Z_l^t Z_l^i \hat{\beta}^{*l'}).$$

We have $C^{-1}B^k C^{-1}$ with Eigen values at most of the order of $b^k \epsilon_n^{-1}$. Also, $C^{-1}Z_l^t Z_l^i$ has Eigen value at most of the order of $\epsilon_n$. We have $\beta^* j$ and $\beta^{*j}$ bounded. Hence,

$$\|C^{-1}C^{-1} \sum_{l' \neq j} Z_l^t Z_l^i \hat{\beta}^{*l'}\|_{\infty} < c^* \epsilon_n; \text{ and } \|C^{-1}B^k C^{-1}(Z_j^t Z_j^i \hat{\beta}^{*j} + \sum_{l' \neq j} Z_l^t Z_l^i \hat{\beta}^{*l'})\|_{\infty} \leq c^* b^k (\epsilon_n + b^k \epsilon_n^{-1}),$$

where $c^* > 0$ is an universal constant (using A6).

Hence, for $i \in P_j \cap P_j^*$, $\|E[\hat{\beta}] - \tilde{\beta}^*\|_{\infty} = \|E[\hat{\beta}] - \tilde{\beta}^*\|_{\infty} \leq \epsilon_n$.

\[\Box\]

**Proof of Theorem 6 and 7**

We use, $\hat{\sigma}_{M^*}^2 = \|Y_n - \hat{Y}_n\|^2 / n$ for the true model with change points at $t_i^*$, and $\hat{\sigma}_{M_i}^2 = \hat{\sigma}_{M_i,m^*_j}^2$ be the estimate corresponding to a model with change point $l$ change point, with covariate combination given by $m_j$.

From earlier calculation in Proposition 1, replacing the $\sigma^2$ in each partition $P_j$ by $\hat{\sigma}^2$,

$$\log L(Y_n|M_{l,m^*_j}) = -\frac{1}{2} \sum_j \log(\det(X_j^t X_j)) - \frac{1}{2\hat{\sigma}_{M_i}^2} n \hat{\sigma}_{M_i}^2 - \sum_j \frac{n_j}{2} \log \hat{\sigma}_{M_i}^2 + O(lq_n).$$

Hence,

$$\log L(Y_n|M_{l,m^*_j}) - \log L(Y_n|M^*)$$

$$= -\frac{1}{2} \sum_j \log(\det(X_j^t X_j)) - \sum_j \log(\det(X_j^t X_j^*)) - \sum_j \frac{n_j}{2} \log \hat{\sigma}_{M^*}^2(1 + \frac{\hat{\sigma}_{M_i}^2 - \hat{\sigma}_{M^*}^2}{\hat{\sigma}_{M^*}^2}) + \frac{n}{2} \log \hat{\sigma}_{M^*}^2 + O(lq_n)$$

$$= -\frac{1}{2} \sum_j \log(\det(X_j^t X_j)) + \frac{1}{2} \sum_j \log(\det(X_j^t X_j^*)) - \frac{n}{2} \log(1 + \frac{\hat{\sigma}_{M_i}^2 - \hat{\sigma}_{M^*}^2}{\hat{\sigma}_{M^*}^2}) + O(lq_n).$$
Note that $\tilde{\sigma}_{\beta}^2 \to \sigma^2$ in probability, and $\frac{\sigma^2_{M^*} - \tilde{\sigma}_{\beta}^2}{\sigma^2_{M^*}} > 0$, $\frac{\sigma^2_{M^*} - \tilde{\sigma}_{\beta}^2}{\sigma^2_{M^*}} > 0$ for large $n$ under the set up of Theorem 6 and Theorem 7, for $M^*_{l,m_\beta}$ and $M^*_{l^*,m_\beta}$, respectively. (from Theorems 3, 4 proofs).

For Theorem 6, for large $n$, we have,

$$\log L(Y_n|M^*_{l,m_\beta}) - \log L(Y_n|M^*) \leq -\frac{1}{2} \sum_j \log(\det(X_j^tX_j)) + \frac{1}{2} \sum_j \log(\det(X_j^tX_j)) - \frac{n}{2} c_0 \frac{\sigma^2_{M^*} - \tilde{\sigma}_{\beta}^2}{\sigma^2_{M^*}} \leq -n\epsilon,$$

as $\sigma^2_{M^*}$ is bounded with probability one, and $\log(1 + \frac{\sigma^2_{M^*} - \tilde{\sigma}_{\beta}^2}{\sigma^2_{M^*}}) > c_0 \frac{\sigma^2_{M^*} - \tilde{\sigma}_{\beta}^2}{\sigma^2_{M^*}}$ for some small positive constant $c_0$. Similarly, the proofs for the case $m_\beta$ not containing true covariate combination, and the case $l < l^*$ follow.

Under the setup of 6 for a refinement of true partition, and covariate combination containing the true covariates, $\frac{\sigma^2_{M^*} - \tilde{\sigma}_{\beta}^2}{\sigma^2_{M^*}} \to 0$ and $\log(1 + \frac{\sigma^2_{M^*} - \tilde{\sigma}_{\beta}^2}{\sigma^2_{M^*}}) \sim \frac{\sigma^2_{M^*} - \tilde{\sigma}_{\beta}^2}{\sigma^2_{M^*}}$. Hence, Theorem 3 proof (Case i) can be repeated, and we have $\log L(Y_n|M^*_{l,m_\beta}) - \log L(Y_n|M^*) \leq -n\epsilon$.

For Theorem 7, we note that $\log(1 + \frac{\sigma^2_{M^*} - \tilde{\sigma}_{\beta}^2}{\sigma^2_{M^*}}) \geq \frac{\sigma^2_{M^*} - \tilde{\sigma}_{\beta}^2}{\sigma^2_{M^*}}$ for large $n$, as $\sigma^2_{M^*} - \tilde{\sigma}_{\beta}^2 \geq 0$ and is of the order of $\epsilon_n$ for $m_\beta \supset m_{\beta^*}$, from the fact $\log(1+x) \sim x$ as $x \to 0$. Hence, Theorem 4 proof can be repeated.

**Addressing Remark 8**

If $\sigma^2 = \sigma_j^2$ for $P_{j^*}$, then $\tilde{\sigma}_{\beta}^2$ converges to $\sum_{j=1}^{n^*} \sigma_j^2 > 0$ in probability. Hence, the proofs of Theorem 6 and Theorem 7 hold.

**Proof of Theorem 12**

Let $M^*_{l,m_\beta}$ (denoted by $M^*_{l,m_\beta}$ for convenience) be the model with covariate combination given by $m_\beta$ and true change points $t^*_1, \ldots, t^*_l$. First we show that

$$\sum_{m_{\beta}:m_{\beta} \neq m_{\beta^*}} \mathbb{P}(M^*_{l,m_{\beta^*}}) \to 0$$

in probability (Part 1). Then, we show that

$$\sum_{m_{\beta}:m_{\beta} \neq m_{\beta^*}} \mathbb{P}(M^*_{l,m_{\beta}}) \to 0$$

where $M^*_{l,m_{\beta}}$ is a model with change points $t_1, t_2, \ldots, t_l$ such that for the refinements of true partition corresponding to change points, $\hat{t}_1, \ldots, \hat{t}_l$, such that $\inf \hat{t}_1, \ldots, \hat{t}_l \max_{i} |t_i - \hat{t}_i| = \xi$ (in Part 2). Next, we address the case where $t_1, \ldots, t_l$ are change points and $l < l^*$ (Part 3). For the case, $t_1, \ldots, t_l$ is a refinement of $t^*_1, \ldots, t^*_l$, the proof follows from Part 1, by considering $M^*_{t^*,m_{\beta}}$ instead of $M^*$ (true covariate and the change points $t_1, \ldots, t_l$) and concluding

$$\sum_{m_{\beta}:m_{\beta} \neq m_{\beta^*}} \mathbb{P}(M^*_{l,m_{\beta}}) \to 0$$

in probability and from the fact

$$\sum_{m_{\beta}:m_{\beta} \neq m_{\beta^*}} \mathbb{P}(M^*_{t^*,m_{\beta}}) \to 0$$
in probability.
Let \( R_t^{(j)} \) be the residual sum of square for \( P_j^* \) under \( M^* \). Let \( m_\beta \supset m_{\beta^*} \). For \(#\{m_\beta\} = q \) and \(#\{m_{\beta^*}\} = t \). Then we show that for residual sum of square for \( m_{\beta^*}, R_m^{(j)} \)

\[
P(R_t^{(j)} - R_m^{(j)}) \geq q - t + (q - t) \alpha (\log n)^{1+\alpha_1}; \text{ for some } m_\beta \supset m_{\beta^*}, \{m_\beta\} = q \text{ and for some } q \]

for some universal constant \( c > 0 \), for any \( \alpha > 0 \). Hence,

\[
P(R_t^{(j)} - R_m^{(j)}) \geq (q - t) + (q - t) \alpha (\log n)^{1+\alpha_1}; \text{ for some } m_\beta \supset m_{\beta^*}, \{m_\beta\} = q, \text{ for some } q \leq \sum_{q = t+1}^{q_n} d_n^{-t} e^{-c(q-t)(\log n)^{1+\alpha_1}}.
\]

Here, \( d_n = p \), the number of available covariates (depending on \( n \)) and \( \log d_n = O(\log n) \) (by assumption) and \( \delta_n = \sum_{q = t+1}^{q_n} d_n^{-t} e^{-c(q-t)(\log n)^{1+\alpha_1}} \to 0 \).

Similarly, for any \( m_\beta \not\supset m_{\beta^*}, \) let \( R_m^{(j)} \) be the residual sum of square for \( m_{\beta^*} \). A conservative bound is given by,

\[
P(R_t^{(j)} - R_m^{(j)}) \geq q + \alpha q (\log n)^{1+\alpha_1}; \text{ for some } m_\beta \not\supset m_{\beta^*}, \#\{m_\beta\} = q, \text{ for some } q \leq \sum_{q = 1}^{q_n} d_n^q e^{-c\alpha q(\log n)^{1+\alpha_1}}.
\]

For, any \( \tilde{m}_\beta \) missing at least one of the true covariates. Let, \( R_m^{(j)} \) be the residual sum of square for \( \tilde{m}_\beta \cup m_{\beta^*} \), and \( R_m^{(j)} \) for \( \tilde{m}_\beta \). Then,

\[
R_m^{(j)} - R_m^{(j)} = (\hat{\beta}_m^{(j)} - \hat{\beta}_m^{(j)}) X'_j X_j (\hat{\beta}_m^{(j)} - \hat{\beta}_m^{(j)}),
\]

where \( \hat{\beta}_m^{(j)}, \beta_m^{(j)} \) are corresponding least square estimates for \( \tilde{m}_\beta \cup m_{\beta^*} \) and \( \tilde{m}_\beta \), respectively, with entries corresponding to coefficients not in \( m_\beta \) but in \( \tilde{m}_\beta \) are filled with zero in \( \beta_m^{(j)} \), and \( X_j \) is the design matrix for \( P_j^* \) with covariates corresponding to \( \tilde{m}_\beta \cup m_{\beta^*} \). Let \( \delta > 0 \) be the minimum value of the true absolute coefficient vector over all \( P_j^* \). Then corresponding least square estimates for \( \tilde{m}_\beta \), for coefficients present in true model has absolute less than \( 3\delta/4 \) with probability less than \( e^{-c_1 n} \) for some universal constant \( c_1 > 0 \) and hence, the probability that some covariate belonging to true model has coefficient estimate less than \( 3\delta/4 \) for some \( \tilde{m}_\beta \cup m_{\beta^*} \) is less than \( d_n^\theta e^{-c_1 n} = o(\delta_n) \). Similarly, for a covariate not present in the true model the corresponding coefficient has absolute value less than \( \delta/4 \) with probability \( 1 - o(\delta_n) \) over all possible covariate combination. Hence,

\[
P(R_m^{(j)} - R_m^{(j)} \geq n, \text{ for all covariate combination } ) \geq 1 - \delta_n,
\]

\[25\]
Similarly, summing over universal constant. Hence, we have
\[ \sum_{\beta \neq \beta^*} \Pi(M_{\hat{t}, \beta}^* | \beta^*) \leq (q-t) \log \tilde{p}_n - \frac{1}{2} ((q-t)(1-2\alpha(\log n)^{\alpha_1}) \log n + d_0(q-t) \]
uniformly over \( \beta \), where \( d_0 \) does not depend on \( \beta \), and \( -\log \tilde{p}_n \sim (\log n)^{1+\alpha_1} \).

Hence, choosing \( \alpha > 0 \) small enough, summing over \( \beta \),
\[ \sum_{m_\beta \supseteq m_\beta^*} \Pi(M_{l_\beta, m_\beta^*}^* | \beta^*) \]
\[ \leq \sum_{q > t} e^{\log(\tilde{p}_n(q-t))} \left( \frac{d_n}{q-t} \right) e^{-\frac{1}{2} \alpha(\log n)^{\alpha_1}}(q-t) \log n \]
\[ d_1 \rightarrow 0 \]
as \( n \rightarrow \infty \) infinity, for some \( 0 < \alpha' < 1 \), choosing sufficiently small \( \alpha > 0 \) and here \( d_1 \) is a universal constant.

Similarly, summing over \( m_\beta \supseteq m_\beta^* \) gives, for outside of a set of probability \( 2\delta_n \), for some constant \( d' > 0 \),
\[ \sum_{m_\beta \supseteq m_\beta^*} \Pi(M_{l_\beta, m_\beta^*}^* | \beta^*) \]
\[ \leq e^{-d'n} \rightarrow 0. \]

As the results are shown outside a set of probability \( O(\delta_n) \) where \( \delta_n \rightarrow 0 \) set, this proves our claim.

**Part 2:**
\[ \frac{\sum_{m_\beta \supseteq m_\beta^*} \Pi(M_{l_\beta, m_\beta^*}^* | \beta^*)}{\Pi(M_{l, \beta}^* | \beta^*)} \rightarrow 0 \]

Using calculation from Theorem 11, using \( \xi \) in place of \( \epsilon_n \), or repeating the argument of Theorem 11 proof for Case (ii) of Theorem 3 proof, we can bound the cross product terms, and bound the Chi-square term \( \| X_j \hat{\beta}^j - X_j E[\hat{\beta}] \|^2 \) as in Part 1 in Theorem 12 or in Theorem 11, outside a set of probability approaching zero, and therefore, outside the small probability set, we have
\[ \log L(Y_n | M_{l_\beta, m_\beta^*}^*) - \log L(Y_n | M_{l_\beta}^*) \leq -n \]
uniformly over covariate choices. Here, \( M_{l_\beta}^* \) denote the model with true covariate combination and change points at \( \hat{t}_1, \ldots, \hat{t}_l \) for any refinement of true partition corresponding to \( t_1', \ldots, t_{l'} \).

Hence, we have
\[ \frac{\sum_{m_\beta \supseteq m_\beta^*} \Pi(M_{l_\beta, m_\beta^*}^* | \beta^*)}{\Pi(M_{l_\beta}^* | \beta^*)} \leq \sum_{q = t+1}^{q_n} e^{\log(\tilde{p}_n(q-t))} \left( \frac{d_n}{q-t} \right) e^{-c'n} \rightarrow 0 \]
in probability, where \( c' > 0 \) is a constant. We already have shown that
\[ \frac{\Pi(M_{l_\beta}^* | \beta^*)}{\Pi(M_{l_\beta}^* | \beta^*)} \rightarrow 0 \]
in probability, which proves our claim.
Part 3: \[
\sum_{m_{\beta} \neq m_{\beta^*}} \frac{\Pi(M_{t,m_{\beta}})}{\Pi(M^*)} \to 0; \ l < l^*
\]

Using the calculation from the proof of Theorem 11 we bound the cross product terms and Chi-square term over all covariate choice, as in last part, and using Case (iv) in Theorem 3 proof, outside a set with probability approaching zero, \( \log L(Y_n | M_{t,m_{\beta}}) - \log L(Y_n | M^*) \leq -n \) uniformly over covariate choices. Hence, \( \frac{\sum_{m_{\beta} \neq m_{\beta^*}} \Pi(M_{t,m_{\beta}})}{\Pi(M^*)} \leq e^{-c'n} \to 0 \) in probability, where \( c'' > 0 \).

**Proof of Theorem 11**

Note that for a subset \( \mathcal{S} \) of \( i = 1, \ldots, n \), of cardinality \( n_s \) and for \( P(\sum_{i \in \mathcal{S}} e_i x_{ij} \geq \sqrt{q} \log n) \leq e^{-cq(\log n)^2} \) for a universal constant \( c > 0 \), for \( q \) many covariates. Similar bound can be derived for each coordinate for \( \sqrt{n}(\hat{\beta}^j - E[\hat{\beta}^j]) \). Hence, over all possible covariate and change point choices \( |E_1|, |E_2|, |E_3| \leq \max\{\sqrt{n}e_n q_n \sqrt{q_n} \log n, \sqrt{n}e_n q_n \sqrt{q_n} \log n\} \) outside a set of probability at most \( \tilde{n} \sim n^{s} \sum_{q=1}^{q_{\infty}} d_n e^{-c'(\log n)^2} \to 0 \).

Again, \( ||X_j \hat{\beta}^j - X_j E[\hat{\beta}^j]||^2 \leq q + q(\log n)^{1+\alpha} \) outside a set of probability approaching zero from Part 1 of Theorem 12. Therefore, outside a set of probability approaching zero,

\[
-\sigma^2 \log L(Y_n | M_{t,m_{\beta}}^{\epsilon_n}) + \sigma^2 \log L(Y_n | M^*) \geq n(e_n - \sqrt{n}e_n q_n^{1.5} \log n - \sqrt{n}e_n q_n^{1.5} \log n + q_n \log n \to \infty.
\]

Note that in the above equation, for \( \epsilon_n \), we have an universal lower bound of the order of \( n(e_n \) for the quadratic term corresponding to \( \frac{1}{2} \sum_{j: P_j \cap P_j^* \neq \emptyset} (\hat{\beta}^j - \hat{E}[\hat{\beta}^j])'X_j(X_j)^* \hat{\beta}^j - E[\hat{\beta}^j]) + \frac{1}{2} \sum_{j: k \neq j: P_j \cap P_j^* \neq \emptyset} (\hat{\beta}^j - \hat{E}[\hat{\beta}^j])'X_j(X_k)^* \hat{\beta}^j - E[\hat{\beta}^j]) \) from equations (6.2) and (6.6) over all covariate and change point choices, as a result of the Eigen value condition given in A6.

Hence, summing over possible covariate choices

\[
\sum_{m_{\beta} \neq m_{\beta^*}} P_n(M_{t,m_{\beta}}^{\epsilon_n} : M^*) \leq q_n e^{-\alpha' n + q_n \log d_n \to 0}
\]
in probability uniformly over change point choices, where \( \alpha' > 0 \) is a constant.

Similarly, for \( m_{\beta} \neq m_{\beta^*} \), outside a set with probability approaching zero, we have

\[
-\sigma^2 \log L(Y_n | M_{t,m_{\beta}}^{\epsilon_n}) + \sigma^2 \log L(Y_n | M^*) \geq n,
\]

from calculation similar to that of leading to equation (6.7). Hence, \( \sum_{m_{\beta} \neq m_{\beta^*}} P_n(M_{t,m_{\beta}}^{\epsilon_n} : M^*) \to 0 \) which concludes our claim.
Addressing Remark 13

Remark 13 follows from the proofs of Theorems 6 and 7.

Proof of Theorem 9

Proof of Theorem 9 follows directly from the proof of Theorem 12.

References

Adams, R. P. and MacKay, D. J. (2007). Bayesian online changepoint detection. *arXiv preprint arXiv:0710.3742*.

Bhadra, A., Datta, J., Polson, N. G., and Willard, B. T. (2019). Lasso meets horseshoe: A survey. *Statistical Science* forthcoming.

Carlin, B. P., Gelfand, A. E., and Smith, A. F. (1992). Hierarchical bayesian analysis of change-point problems. *Journal of the Royal Statistical Society: Series C (Applied Statistics)* 41, 389–405.

Castillo, I., Schmidt-Hieber, J., and van der Vaart, A. (2015). Bayesian linear regression with sparse priors. *The Annals of Statistics* 43, 1986–2018.

Chernoff, H. and Zacks, S. (1964). Estimating the current mean of a normal distribution which is subjected to changes in time. *The Annals of Mathematical Statistics* 35, 999–1018.

Chib, S. (1998). Estimation and comparison of multiple change-point models. *Journal of econometrics* 86, 221–241.

Chillar, V. F. and Drawve, G. (2020). Unpacking spatio-temporal differences of risk for crime: An analysis in little rock, ar. *Policing: A Journal of Policy and Practice* 14, 258–277.

Csorgo, M. and Horváth, L. (1997). *Limit theorems in change-point analysis*. John Wiley & Sons Chichester.

Datta, A., Zou, H., and Banerjee, S. (2019). Bayesian high-dimensional regression for change point analysis. *Statistics and its interface* 12, 253.

Datta, J. and Ghosh, J. K. (2013). Asymptotic properties of Bayes risk for the horseshoe prior. *Bayesian Analysis* 8, 111–132.

Frick, K., Munk, A., and Sieling, H. (2014). Multiscale change point inference. *Journal of the Royal Statistical Society: Series B (Statistical Methodology)* 76, 495–580.

Fryzlewicz, P. et al. (2014). Wild binary segmentation for multiple change-point detection. *The Annals of Statistics* 42, 2243–2281.
Gao, C., Han, F., and Zhang, C.-H. (2017). Minimax risk bounds for piecewise constant models. *arXiv preprint arXiv:1705.06386*.

Gardner, L. (1969). On detecting changes in the mean of normal variates. *The Annals of Mathematical Statistics* **40**, 116–126.

Ghosh, P., Tang, X., Ghosh, M., and Chakrabarti, A. (2016). Asymptotic properties of Bayes risk of a general class of shrinkage priors in multiple hypothesis testing under sparsity. *Bayesian Anal.* **11**, 753–796.

Horn, R. A. and Johnson, C. R. (2012). *Matrix analysis*. Cambridge university press.

Killick, R., Fearnhead, P., and Eckley, I. A. (2012). Optimal detection of changepoints with a linear computational cost. *Journal of the American Statistical Association* **107**, 1590–1598.

Lee, S., Seo, M. H., and Shin, Y. (2016). The lasso for high dimensional regression with a possible change point. *Journal of the Royal Statistical Society: Series B (Statistical Methodology)* **78**, 193–210.

Liu, H., Gao, C., and Samworth, R. J. (2019). Minimax rates in sparse, high-dimensional changepoint detection. *arXiv preprint arXiv:1907.10012*.

Martin, R. and Shen, W. (2017). Asymptotically optimal empirical bayes inference in a piecewise constant sequence model. *arXiv preprint arXiv:1712.03848*.

Mitchell, T. J. and Beauchamp, J. J. (1988). Bayesian Variable Selection in Linear Regression. *Journal of the American Statistical Association* **83**, 1023–1032.

Narisetty, N. N., He, X., et al. (2014). Bayesian variable selection with shrinking and diffusing priors. *The Annals of Statistics* **42**, 789–817.

Page, E. (1955). A test for a change in a parameter occurring at an unknown point. *Biometrika* **42**, 523–527.

Page, E. (1957). On problems in which a change in a parameter occurs at an unknown point. *Biometrika* **44**, 248–252.

Polson, N. G. and Scott, J. G. (2010a). Large-scale simultaneous testing with hypergeometric inverted-beta priors. *arXiv preprint arXiv:1010.5223*.

Polson, N. G. and Scott, J. G. (2010b). Shrink globally, act locally: Sparse Bayesian regularization and prediction. *Bayesian Statistics* **9**, 501–538.

Raftery, A. E. (1994). Change point and change curve modeling in stochastic processes and spatial statistics. *Journal of Applied Statistical Science* **1**, 403–423.

Schwarz, G. et al. (1978). Estimating the dimension of a model. *Annals of statistics* **6**, 461–464.
Sen, A. K. and Srivastava, M. S. (1973). On multivariate tests for detecting change in mean. *Sankhyā: The Indian Journal of Statistics, Series A* pages 173–186.

Sen, P. K. (1980). Asymptotic theory of some tests for a possible change in the regression slope occurring at an unknown time point. *Zeitschrift für Wahrscheinlichkeitstheorie und verwandte Gebiete* **52**, 203–218.

Smith, A. (1975). A bayesian approach to inference about a change-point in a sequence of random variables. *Biometrika* **62**, 407–416.

Srivastava, M. (1981). On tests for detecting change in the multivariate mean. In *Statistical Distributions in Scientific Work*, pages 181–191. Springer.

Stephens, D. (1994). Bayesian retrospective multiple-changepoint identification. *Journal of the Royal Statistical Society: Series C (Applied Statistics)* **43**, 159–178.

Talwar, P. P. (1983). Detecting a shift in location: Some robust tests. *Journal of Econometrics* **23**, 353–367.

Vostrikova, L. Y. (1981). Detecting “disorder” in multidimensional random processes. In *Doklady Akademii Nauk*, volume 259, pages 270–274. Russian Academy of Sciences.

Zacks, S. (1983). Survey of classical and bayesian approaches to the change-point problem: fixed sample and sequential procedures of testing and estimation. In *Recent advances in statistics*, pages 245–269. Elsevier.