On equidistribution theorem for multi-sequences of holomorphic line bundles

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Abstract

Given several sequences of Hermitian holomorphic line bundles \( \{(L_{kp}, h_{kp})\}_{p=1}^{\infty} \), we establish the distribution of common zeros of random holomorphic sections of \( L_{kp} \) with respect to singular measures. We also study the dimension growth for a sequence of pseudo-effective line bundles.

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1 Introduction

The study of equidistribution of zeros of random holomorphic sections has become intensively active recently, which can be applied to quantum chaotic eigenfunctions [1] [13] and shed a light on the Quantum unique ergodicity conjecture [14] [12].

Shiffman-Zelditich [17] established an equidistribution theorem for high powers of a positive line bundle. More results on equidistribution for singular metrics of line bundles were obtained. For example, Dinh-Ma-Marinescu [10] explored the equidistribution of zeros of random holomorphic sections for singular Hermitian metrics with a convergence speed. Coman-Marinescu-Nguyêñ [7] studied the equidistribution of common zeros of sections of several big line bundles. Coman-Marinescu-Nguyêñ [9] studied equidistribution for spaces of \( L^2 \)-holomorphic sections vanishing along subvarieties. Dinh-Sibony [11] first extended equidistribution with respect to general measures with a good convergence speed. Shao

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provided large family of singular measures to satisfy equidistribution theorems. See [5, 3, 2, 6, 12] for more references.

Recently, Coman-Lu-Ma-Marinescu [4] studied equidistribution for a sequence $(L_p, h_p)$ instead of $(L⊗^p, h⊗^p)$ of a single line bundle $L$. They imposed a natural convergence, that is, the first Chern curvature currents $c_1(L_p, h_p)$ converge to a (non-integral) Kähler form $\omega$, which can be regarded as a ”prequantization” of $\omega$ in the setting of geometric quantization.

In this paper, we establish an equidistribution theorem for several sequences of line bundles. Now we formulate our setting and state our main result. Let $(X, \omega)$ be a Kähler manifold of dim$_{\mathbb{C}} X = n$ with a fixed Kähler form $\omega$. Let $\{ (L_{kp}, h_{kp}) \}_{p=1}^{\infty}$ be $m$ sequences of Hermitian holomorphic line bundles on $X$ with (possibly singular) Hermitian metrics $h_{kp}$, where $1 \leq k \leq m \leq n$. We endow the space $\mathcal{C}^\infty(X, L_{kp})$ of smooth sections $L_{kp}$ with the inner product

$$\langle s_1, s_2 \rangle := \int_X \langle s_1, s_2 \rangle h_{kp} \frac{\omega^n}{n!}, s_1, s_2 \in \mathcal{C}^\infty(X, L_{kp}),$$

and we set $||s||^2 = \langle s, s \rangle$. We denote by $L^2(X, L_{kp})$ the completion of $\mathcal{C}^\infty(X, L_{kp})$ with respect to this norm. Denote by $H^0_{(2)}(X, L_{kp})$ the Bergman space of $L^2$ holomorphic sections of $L_{kp}$ and let $B_{kp} := L^2(X, L_{kp}) \rightarrow H^0_{(2)}(X, L_{kp})$ be the orthogonal projection. The integral kernel $B_{kp}(x, x')$ of $B_{kp}$ is called the Bergman kernel. The restriction of the Bergman kernel to the diagonal of $X$ is the Bergman kernel function of $H^0_{(2)}(X, L_{kp})$, which we still denote by $B_{kp}$, i.e., $B_{kp}(x) = B_{kp}(x, x)$. The first assumption is the following:

**Assumption 1:** There exist a constant $M_0 > 1$ and $p_0 > 0$ such that

$$\frac{A_{kp}^n}{M_0} \leq B_{kp}(x) \leq M_0 A_{kp}^n,$$

for any $x \in X, p \geq p_0, 1 \leq k \leq m$, where $A_{kp}$ are positive numbers, $\lim_{p \to \infty} A_{kp} = \infty$ for $1 \leq k \leq m$ with the same order of infinite.

Denote by $\mathbb{CP}H^0_{(2)}(X, L_{kp})$ the associated projective space of $H^0_{(2)}(X, L_{kp})$. Set $d_{kp} := \dim \mathbb{CP}H^0_{(2)}(X, L_{kp})$. By Assumption 1, we have

$$d_{kp} = \int_X B_{kp}(x) \omega^n - 1 \approx A_{kp}^n.$$

There exist $M_1 > 1$ and $p_0 > 0$, such that

$$\frac{A_{kp}^n}{M_1} \leq d_{kp} \leq M_1 A_{kp}^n.$$

Consider the multi-projective spaces

$$X_p := \mathbb{CP}H^0_{(2)}(X, L_{1p}) \times \cdots \times \mathbb{CP}H^0_{(2)}(X, L_{mp}),$$

(1)
equipped with a probability (singular) measure $\sigma_p$. The standard measure $\sigma_p$ is just the product of the Fubini-study volume forms on all components of $X_p$. In our main theorem, $\sigma_p$ is the product of moderate measures, which is a generalization of the standard one (cf. [16]). The product space with product measure is $\mathbb{P}^X := \prod_{p=1}^{\infty} X_p, \sigma = \prod_{p=1}^{\infty} \sigma_p$. For $s_p = (s_{1p}, \ldots, s_{mp}) \in X_p$, we define $[s_p = 0] := [s_{1p} = 0] \land \cdots \land [s_{mp} = 0]$. With a minor change of the proof for Bertini type theorem with respect to a singular measure in [16] Section 2. We can deduce that $[s_p = 0]$ is well-defined for a.e. $\{s_p\}$ with respect to $\sigma_p$. In fact, if $\sigma_p$ has no mass on any proper analytic subset of $X_p$, the above statement still holds true, see the proof of [16] Lemma 2.2, Proposition 2.3.

Now, we give the second natural assumption on the convergence of Chern curvature currents $c_1(L_{kp}, h_{kp})$ associated to $h_{kp}$, which can be thought as a ”prequantization” process.

**Assumption 2:** There exist positive closed $(1, 1)$-currents $\omega_k$ ($\omega_k \geq 0$ in the sense of currents) with positive measure ($i.e. \int \omega_k \wedge \omega^{n-1} > 0$) such that the norms of currents satisfy

$$\left\| \frac{1}{A_{kp}} c_1(L_{kp}, h_{kp}) - \omega_k \right\| \leq C_0 A_{kp}^{-a_k},$$

where $C_0, a_k > 0$ are constants. Moreover, $\omega_{j_1} \land \cdots \land \omega_{j_l}$ are well-defined with positive measures, for any multi-index $(j_1, \ldots, j_l) \subset \{1, 2, \ldots, m\}$.

Note that one trivial example is $L_{kp} = L_{k}^{\otimes p}, h_{kp} = h_{k}^{\otimes p}, \omega_k = c_1(L_k, h_k)$ and the non-continuous set of $h_k$ are in general position (cf. [7]).

Now we are in a position to state our main theorem:

**Theorem 1.1.** Let $\{(L_{kp}, h_{kp})\}_{p=1}^{\infty}$ be $m$ sequences of Hermitian holomorphic line bundles on a compact $\mathbb{C}$-Kähler manifold $X$ of $\dim_{\mathbb{C}} X = n$. If Assumption 1 and Assumption 2 hold, then for $\sigma$-a.e. $\{s_p\} \in \mathbb{P}^X$, we have $\frac{1}{m} \prod_{k=1}^{m} [s_p = 0] \to \omega_1 \land \cdots \land \omega_m$ in the sense of currents.

**Theorem 1.2.** With the same notations and assumptions in Theorem 1.1, for any $\alpha > 0$, there exist $C_1 > 0, C_2 > 0, p_1 > 0$ and $E_{p}^{\alpha}$ such that

(i) $\sigma_p(E_{p}^{\alpha}) \leq C_1 (\sum_{k=1}^{m} A_{kp})^{-\alpha}$ for any $p > p_1$;

(ii) for $s_p \in X_p \setminus E_{p}^{\alpha}$ and any $(n - m, n - m)$ form $\phi$ of class $\mathcal{C}^2$,

$$\left\| \left( \frac{1}{m} \prod_{k=1}^{m} [s_p = 0] - \omega_1 \land \cdots \land \omega_m, \phi \right) \right\| \leq C_2 \left( \sum_{k=1}^{m} \frac{\log A_{kp}}{A_{kp}} + \frac{\log \left( \sum_{k=1}^{m} A_{kp} \right)}{\sum_{k=1}^{m} A_{kp}} + \sum_{k=1}^{m} A_{kp}^{-a_k} \right) \left\| \phi \right\|_{\mathcal{C}^2}. $$
In [4], \( L_{kp} = L_p, \omega_k = \omega_1 \) and \( h_{kp} = h_p \) are smooth metrics. In this case, Assumption 1 is automatically satisfied under Assumption 2. In fact, they gave the asymptotic expansion of the Bergman kernel \( B_{kp}(x) = B_p(x) \). Our theorem is a generalization of [4, Theorem 0.4]. If \( L_{kp} = L^\otimes_k \), \( \{ L_k \}_{k=1}^m \) are all big line bundles and the non-continuous set of \( h_{kp} = h^\otimes_k \) are in general position and of Hölder continuous with singularities, our theorems recover [16, Theorem 1.2, Theorem 1.3] partially.

In the classical setting of equidistribution theorems, there is a single sequence \( \{ L_p \} \) of a positive line bundle \( L \). Assumption 1 is satisfied due to the uniform estimate of \( B_p(x) \) for \( H^0(X,L_p) \), then we derive the classical result by Shiffman-Zelditch [17].

**Corollary 1.3.** Let \( m = 1, \{ L_p = L^\otimes_p \} \) be a sequence of high powers of a positive line bundle. Take \( \omega = c_1(L,h) \), where \( h \) is a smooth Hermitian metric, \( \sigma_p \) is the Fubini-Study volume on \( \mathbb{CP}H^0(X,L_p) \). Then, for \( \sigma \)-a.e. \( \{ s_p \} \in \mathbb{P}^X, \frac{1}{p}[s_p = 0] \to \omega \) in the sense of currents.

Assumption 1 is indeed strong. We next provide a dimension growth result for sequences of pseudo-effective line bundles, which can shed a light on this assumption. To simplify, let \( (L_{1p}, h_{1p}) = (L_p, h_p) \), where all \( L_p \) are pseudo-effective line bundles, i.e. \( c_1(L_p, h_p) \geq 0 \) in the sense of currents. \( A_p > 0, \lim_{p \to \infty} A_p = +\infty \) and \( \frac{1}{A_p} c_1(L_p, h_p) \to \omega_1 \). Suppose that \( h_p \) is continuous on \( X \setminus \Sigma \), where \( \Sigma \) is a proper analytic subset, \( c_1(L_p, h_p) \geq A_p \eta_p \omega_p \), where \( \eta_p : X \to [0, +\infty) \). For any \( x \in X \setminus \Sigma \), there exists a neighborhood \( U_x \) of \( x \) and a constant \( c_x > 0 \) such that \( \eta_p(x) \geq c_x \) on \( U_x \) for any \( p \) large.

**Theorem 1.4.** With the above setting, there exists a constant \( C_3 > 0 \), we have \( \dim H^0_{(2)}(X, L_p) \geq C_3 A_p^n \).

In the rest of the paper, we abusively use \( C \) to denote positive constants, where \( C \) is not necessary to be the same in different places. The paper is organized as follows. In Section 2 we recall Dinh-Sibony’s technique and the notion of moderate measure with estimates for capacities in multi-projective spaces. In Section 3 we prove a variant of convergence result of Fubini-Study currents for multi-sequences of holomorphic line bundles. Section 4 is devoted to proving the main theorem. We conclude Section 5 with a dimension growth estimate for pseudo-effective line bundles.

## 2 Preliminaries

### 2.1 Dinh-Sibony’s technique

Let \((X, \omega)\) (resp. \((Y, \omega_Y)\)) be a compact Kähler manifold of dimension \( n \) (resp. \( n_Y \)). Recall that a meromorphic transform \( F : X \to Y \) is the graph of
an analytic subset $\Gamma \subset X \times Y$ of pure dimension $n_Y + k$ such that the natural projections $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$ restricted to each irreducible component of the analytic subset $\Gamma$ are surjective. We write $F = \pi_2 \circ (\pi_1|_\Gamma)^{-1}$.

The dimension of the fiber $F^{-1}(y) := \pi_1(\pi_2^{-1}|_\Gamma(y))$ is equal to $k$ for the point $y \in Y$ generic. This is the codimension of the meromorphic transform $F$. If $T$ is a current of bidegree $(l, l)$ on $Y$, $n_Y + k - n \leq l \leq n_Y$, we define

$$F^*(T) := (\pi_1)_*(\pi_2^*(T) \wedge [\Gamma]),$$

where $[\Gamma]$ is the current of integration over $\Gamma$. We introduce the notations of intermediate degrees of $F$,

$$\delta^1(F) := \int_X F^*(\omega^{n_Y}) \wedge \omega^k,$$

$$\delta^2(F) := \int_X F^*(\omega^{n_Y-1}) \wedge \omega^{k+1}.$$

To introduce more notions and notations, we first recall the following lemma in [11, Proposition 2.2].

**Lemma 2.1.** There exists a constant $r > 0$ such that for any positive closed current $T$ of bidegree $(1, 1)$ with mass 1 on $(X, \omega)$, there is a smooth $(1, 1)$-form $\alpha$ which depends only on the cohomology class of $T$ and a q.p.s.h. function $\varphi$ satisfying that

$$-r\omega \leq \alpha \leq r\omega, \quad d^c \varphi - T = \alpha.$$

Denote by $r(X, \omega)$ the smallest $r$ in Lemma 2.1. For example, $r(\mathbb{CP}^N, \omega_{FS}) = 1$. Consider a positive measure $\mu$ on $X$. $\mu$ is said to be a PLB measure if all q.p.s.h. functions are integrable with respect to $\mu$. It is easy to see that all moderate measures are PLB. Now given a PLB probability measure $\mu$ on $X$ and $t \in \mathbb{R}$, we define,

$$Q(X, \omega) := \{\varphi \text{ q.p.s.h. on } X, dd^c \varphi \geq -r(X, \omega) \omega\},$$

$$R(X, \omega, \mu) := \sup \{\max \varphi : \varphi \in Q(X, \omega), \int_X \varphi d\mu = 0\}$$

$$= \sup \{-\int_X \varphi d\mu : \varphi \in Q(X, \omega), \max \varphi = 0\},$$

$$S(X, \omega, \mu) := \sup \{|\int \varphi d\mu| : \varphi \in Q(X, \omega), \int_X \varphi \omega^n = 0\},$$

$$\Delta(X, \omega, \mu, t) := \sup \{\mu(\varphi < -t) : \varphi \in Q(X, \omega), \int_X \varphi d\mu = 0\}.$$

These constants are related to Alexander-Dinh-Sibony capacity, see [11, A.2].
Let $\Phi_p$ be a sequence of meromorphic transforms from a projective manifold $(X, \omega)$ into the compact Kähler manifolds $(X_p, \omega_p)$ of the same codimension $k$, where $X_p$ is defined in (11). Let
\[ d_p = d_{1,p} + \ldots + d_{m,p} \]
be the dimension of $X_p$. Consider a PLB probability measure $\mu_p$ on $X_p$, for every $p > 0$, $\epsilon > 0$, we define
\[ E_p(\epsilon) := \bigcup_{\|\phi\|_{C^2} \leq 1} \{ s_p \in X_p : \langle (\Phi_p^*(\delta_{s_p}) - \Phi_p^*(\mu_p)), \phi \rangle \geq \delta^1(\Phi_p)\epsilon \}, \tag{4} \]
where $\delta_{s_p}$ is the Dirac measure at the point $s_p$. By the definition of the pullback of $\Phi_p$ on currents, we see that $\Phi_p^*(\delta_{s_p})$ and $\Phi_p^*(\mu_p)$ are positive closed currents of bidimension $(k, k)$ on $X$. Moreover, $\Phi_p^*(\delta_{s_p})$ is well-defined for $s_p \in X_p$ generic.

Recall that $\omega_{|\mu_p}$ and $c_p$ were defined in [16 (18),(19)]. The following estimate from Dinh-Sibony equidistribution theorem [11] is crucial in our paper.

**Theorem 2.2.** Let $\eta_{k,p} := \epsilon \delta^2(\Phi_p)^{-1} \delta^1(\Phi_p) - 3R(X_p, \omega_{M_p}, \mu_p)$, then
\[ \mu_p(E_p(\epsilon)) \leq \Delta(X_p, \omega_{M_p}, \mu_p, \eta_{k,p}). \]

We also need the following important estimate, which was deduced from [11] Lemma 4.2(c),Proposition 4.3).

**Theorem 2.3.** In the above setting, we have
\[ \langle \delta^1(\Phi_p)^{-1}(\Phi_p^*(\mu_p) - \Phi_p^*(\omega_{M_p}^{d_p})), \phi \rangle \leq 2S(X_p, \omega_{M_p}, \mu_p)\delta^2(\Phi_p)\delta^1(\Phi_p)^{-1}\|\phi\|_{C^2} \]
for any $(k, k)$-form $\phi$ of class $C^2$ on $X$.

**Theorem 2.4.** Suppose that the sequence \{ $R(X_p, \omega_{M_p}, \mu_p)\delta^2(\Phi_p)\delta^1(\Phi_p)^{-1}$ \} tends to 0 and
\[ \Sigma_{p \geq 1} \Delta(X_p, \omega_{M_p}, \mu_p, \delta^2(\Phi_p)^{-1}\delta^1(\Phi_p)t) < \infty \]
for all $t > 0$. Then for almost everywhere $s = (s_p) \in \mathbb{P}X$ with respect to $\mu = \prod_{p=1}^{\infty} \mu_p$, the sequence $\langle \delta^1(\Phi_p)^{-1}(\Phi_p^*(\delta_{s_p}) - \Phi_p^*(\mu_p)), \phi \rangle$ converges to 0 uniformly on the bounded set of $(k - 1, k - 1)$-forms on $X$ of class $C^2$.

Consider the Kodaira map
\[ \Phi_{k,p} : X \to \mathbb{CP}(H^0_{(2)}(X, L_{kp})^*). \]
Here $H^0_{(2)}(X, L_{kp})^*$ is the dual space of $H^0_{(2)}(X, L_{kp})$. Choose \{ $\xi_{k,p}^{j}$ \} as an orthonormal basis of $H^0_{(2)}(X, L_{kp})$. By an identification via the basis, it boils down to a meromorphic map
\[ \Phi_{k,p} : X \to \mathbb{CP}^d_{kp}. \]
Now we give a local analytic description of the above map. Let $U \subset X$ be a contractible Stein open subset, $e_{kp}$ be a local holomorphic frame of $L_{kp}$ on $U$. Then there exists a holomorphic function $s_j^{k,p}$ on $U$ such that $S_j^{k,p} = s_j^{k,p} e_{kp}$. Then the map is expressed locally as

$$
\Phi_{k,p}(x) = [s_0^{k,p}(x) : \ldots : s_{d_{kp}}^{k,p}(x)], \quad \forall x \in U
$$

(5)

It is called the Kodaira map defined by the basis $\{S_j^{k,p}\}_{j=0}^{d_{kp}}$. Denote by $B_{kp}$ the Bergman kernel function defined by

$$
B_{kp}(x) = \sum_{j=0}^{d_{kp}} |S_j^{k,p}(x)|^2_{h_{kp}}, \quad |S_j^{k,p}(x)|^2_{h_{kp}} = h_{kp}(S_j^{k,p}(x), S_j^{k,p}(x)).
$$

(6)

It is easy to see that this definition is independent of the choice of basis.

Recall that $\omega_{FS}$ is the normalized Fubini-Study form on $\mathbb{CP}^{d_{kp}}$. We define the Fubini-Study currents $\gamma_{k,p}$ of $H^0_{(2)}(X, L_{kp})$ as pullbacks of $\omega_{FS}$ by Kodaira map,

$$
\gamma_{k,p} = \Phi_{k,p}^* (\omega_{FS}).
$$

(7)

We have in the local Stein open subset $U$,

$$
\gamma_{k,p} \big|_U = \frac{1}{2} dd^c \log \sum_{j=0}^{d_{kp}} |s_j^{k,p}|^2.
$$

This yields

$$
\frac{1}{p} \gamma_{k,p} = c_1(L_k, h_k) + \frac{1}{2p} dd^c \log B_{kp}.
$$

Since $\log B_{kp}$ is a global function which belongs to $L^1(X, \omega^n)$, $\frac{1}{p} \gamma_{k,p}$ has the same cohomology class as $c_1(L_k, h_k)$. We focus on the special meromorphic transforms $\Phi_p : X \to \mathbb{X}_p$ induced by the product map of Kodaira maps $\Phi_{kp} : X \to \mathbb{CP} H^0_{(2)}(X, L_{kp})$. $\Phi_p$ is indeed a meromorphic transform with a graph

$$
\Gamma_{kp} = \{(x, s_1p, \ldots, s_mp) \in X \times \mathbb{X}_p : s_1p(x) = \ldots = s_mp(x) = 0\},
$$

see [16 Section 3]. We also need the following

**Lemma 2.5.** $\Phi_p^*(\delta_{s_p}) = [s_p = 0]$.

**Proposition 2.6.** [7 Lemma 4.5] $\Phi_p^*(\omega_{M_p}^{d_p}) = \gamma_{1,p} \wedge \ldots \wedge \gamma_{m,p}$ for all $p$ sufficiently large.
2.2 Moderate measures on multi-projective spaces

We say that a function \( \phi \) on \( X \) is quasi-plurisubharmonic (q.p.s.h) if it is \( c\omega \)-p.s.h. for some constant \( c > 0 \). Consider a measure \( \mu \) on \( X \), \( \mu \) is said to be PLB if all the q.p.s.h. functions are \( \mu \)-integrable.

Let
\[
F = \{ \phi \text{ q.p.s.h. on } X : dd^c \phi \geq -\omega, \max_X \phi = 0 \}.
\] (8)

\( F \) is compact in \( L^p(X) \) and bounded in \( L^1(\mu) \) when \( \mu \) is a PLB measure, see [11].

**Definition 2.7.** Let \( \mu \) be a PLB measure on \( X \). We say that \( \mu \) is \((c, \alpha)\)-moderate for some constants \( c, \alpha > 0 \) if
\[
\int_X \exp(-\alpha \phi) d\mu \leq c
\]
for all \( \phi \in F \). The measure \( \mu \) is called moderate if there exist constants \( c, \alpha > 0 \) such that it is \((c, \alpha)\)-moderate.

For example, \( \omega^n \) is moderate in \( X \). In particular, the Fubini-Study volume form is moderate in a projective space. We introduce product of moderate measures used in the main theorem. We define singular moderate measures \( \sigma_p \) as perturbations of standard measures on \( X_p \). For each \( p \geq 1, 1 \leq k \leq m, 1 \leq j \leq d_{k,p} \), let \( u_{j,p}^{k,p} : \mathbb{CP}^{H^0_2}(X, L_{kp}) \to \mathbb{R} \) be an upper-semi continuous function. Fix \( 0 < \rho < 1 \) and a sequence of positive constants \( \{c_p\}_{p \geq 1} \). We call \( \{u_{j,p}^{k,p}\} \) a family of \((c_p, \rho)\)-functions if all \( u_{j,p}^{k,p} \) satisfy the following two conditions:

- \( u_{j,p}^{k,p} \) is of class \( \mathcal{C}^\rho \) with modulus \( c_p \),
- \( u_{j,p}^{k,p} \) is a \( c_p \omega_{FS}\)-p.s.h.

Then for each \( p \geq 1 \), there is a probability measure
\[
\sigma_p = \prod_{k=1}^{m} \prod_{j=1}^{d_{k,p}} \pi_{k,p}^*(dd^c u_{j,p}^{k,p} + \omega_{FS})
\] (9)
on \( X_p \). By [15] Theorem 1.1, Remark 2.12, \( \bigwedge_{j=1}^{d_{k,p}}(dd^c u_{j,p}^{k,p} + \omega_{FS}) \) is a moderate measure on \( \mathbb{CP}^{H^0_2}(X, L_{kp}) \) when \( c_p \leq 1/c(\sum_{k=1}^{m} A_{k,p})^n \) for a suitable constant \( c > 1 \), \( \forall 1 \leq k \leq m, p \geq 1 \). We call
\[
\sigma = \prod_{p=1}^{\infty} \sigma_p = \prod_{p=1}^{\infty} \prod_{k=1}^{m} \prod_{j=1}^{d_{k,p}} \pi_{k,p}^*(dd^c u_{j,p}^{k,p} + \omega_{FS})
\] (10)

a probability measure on \( \mathbb{P}^X \) generated by a family of \((c_p, \rho)\)-functions \( \{u_{j,p}^{k,p}\} \) on \( \{\mathbb{CP}^{H^0_2}(X, L_{kp})\} \).
2.3 Capacity estimate on multi-projective space

Now we study the estimates on multi-projective spaces. Let $\mathbb{CP}^{\ell_1}, \ldots, \mathbb{CP}^{\ell_m}$ be $m$ projective spaces. Let $\pi_k : \mathbb{CP}^{\ell_1} \times \cdots \times \mathbb{CP}^{\ell_m} \rightarrow \mathbb{CP}^{\ell_k}$ be the natural projection map. Let $\sigma_k$ be a probability moderate measure with respect to a family of $(c_{\ell_k}, \rho)$-functions $\{u_{k,j}\}_{j=1}^{\ell_k}$ on $\mathbb{CP}^{\ell_k}$ defined in (9). Let $\ell = \ell_1 + \cdots + \ell_m$.

Recall that the notation $r(\mathbb{CP}^{\ell_1} \times \cdots \times \mathbb{P}^{\ell_m}, \omega_{M_p})$ is defined after Lemma 2.1.

We have the following lemma [7, Lemma 4.6].

Lemma 2.8. Under the above hypotheses,

$$r(\mathbb{CP}^{\ell_1} \times \cdots \times \mathbb{CP}^{\ell_m}, \omega_{M_p}) \leq r(\ell_1, \ldots, \ell_m) := \max_{1 \leq k \leq m} \frac{\ell_k}{\ell}.$$

The following proposition is taken from [16, Proposition 3.14].

Proposition 2.9. In the above setting, let $\mathbb{CP}^{\ell_k}$ be a projective space endowed with a probability moderate measure $\sigma_k$ defined in (9), $\forall 1 \leq k \leq m$. Set $\sigma := \sigma_1 \times \cdots \times \sigma_m$. Suppose that $\ell_1, \ldots, \ell_m$ are chosen sufficiently large such that

$$\frac{r(\ell_1, \ldots, \ell_m) \log \ell}{\min(\ell_1, \ldots, \ell_m)} \ll 1,$$

$$\frac{\rho}{4 \min(\ell_1, \ldots, \ell_m)} \ll 1.$$ (11)

Then there exist positive constants $\beta_1, \beta_2, \xi$ depending only on $m$ such that for $0 \leq t \leq \min(\ell_1, \ldots, \ell_m)$, we have

$$R(\mathbb{CP}^{\ell_1} \times \cdots \times \mathbb{CP}^{\ell_m}, \omega_{M_p}, \sigma) \leq \beta_1 r(\ell_1, \ldots, \ell_m)(1 + \log \ell),$$

$$S(\mathbb{CP}^{\ell_1} \times \cdots \times \mathbb{P}^{\ell_m}, \omega_{M_p}, \sigma) \leq \beta_1 r(\ell_1, \ldots, \ell_m)(1 + \log \ell),$$

$$\Delta(\mathbb{CP}^{\ell_1} \times \cdots \times \mathbb{CP}^{\ell_m}, \omega_{M_p}, \sigma, t) \leq \beta_1 \ell^\xi \exp\left(-\frac{\beta_2 t}{r(\ell_1, \ldots, \ell_m)}\right).$$

3 Convergence of Fubini-study currents

Recall that the Fubini-Study current of $H^0_2(X, L_{kp})$ is

$$\gamma_{kp} = \frac{1}{2} d\bar{d} \log \sum_{j=0}^{d_{kp}} \|f_{kp,j}\|^2 = c_1(L_{kp}, h_{kp}) + \frac{1}{2} d\bar{d} \log B_{kp}.$$

In this section, we will prove the following.

Proposition 3.1. With the same notations and assumptions in Theorem 1, there exists $C > 0$ such that

$$\left| \langle \prod_{k=1}^{m} A_{kp}^{-\alpha_k} \gamma_{lp} \wedge \cdots \wedge \gamma_{mp} - \omega_1 \wedge \cdots \wedge \omega_n, \phi \rangle \right| \leq C \sum_{k=1}^{m} \left( \frac{\log A_{kp}}{A_{kp}} + A_{kp}^{-\alpha_k} \right) \|\phi\|_{C^2}.$$
for any \((n - m, n - m)\)-form \(\phi\) of class \(C^2\).

**Proof.** Let \(W_p := \frac{1}{\prod_{k=1}^m \alpha_{kp}} \gamma_1 \wedge \cdots \wedge \gamma_m - \omega_1 \wedge \cdots \wedge \omega_m, \alpha_{kp} = \frac{c_1(L_{kp}, h_{kp})}{A_{kp}} - \omega_k\).

By Assumption 2, \(\|\alpha_{kp}\| \leq \frac{C_0}{A_{kp}}\) in the norm of currents

\[
\frac{\gamma_{kp}}{A_{kp}} - \omega_k = \alpha_{kp} + \frac{1}{2A_{kp}} dd^c \log B_{kp}.
\]

We have

\[
| < W_p, \phi > | = \sum_{k=1}^m | < \omega_1 \wedge \cdots \wedge \omega_{k-1} \wedge (\frac{\gamma_{kp}}{A_{kp}} - \omega_k) \wedge \frac{\gamma_{k+1,p}}{A_{k+1,p}} \wedge \cdots \wedge \frac{\gamma_{m,p}}{A_{m,p}}, \phi > |
\]

\[
\leq \sum_{k=1}^m | < \omega_1 \wedge \cdots \wedge \omega_{k-1} \wedge (\alpha_{kp} + \frac{1}{2A_{kp}} dd^c \log B_{kp}) \wedge \frac{\gamma_{k+1,p}}{A_{k+1,p}} \wedge \cdots \wedge \frac{\gamma_{m,p}}{A_{m,p}}, \phi > |.
\]

Note that there exists a constant \(c > 0\) such that

\[
-\frac{cC_0}{A_{kp}} \omega^{n-m+1} \leq \alpha_{kp} \wedge \phi \leq \frac{cC_0}{A_{kp}} \omega^{n-m+1}
\]

in the sense of currents. Then,

\[
| < W_p, \phi > | = \sum_{k=1}^m \frac{cC_0}{A_{kp}} |\phi|C^0 \int_X | \omega_1 \wedge \cdots \wedge \omega_{k-1} \wedge \omega^{n-m+1} \wedge \frac{\gamma_{k+1,p}}{A_{k+1,p}} \wedge \cdots \wedge \frac{\gamma_{m,p}}{A_{m,p}} |
\]

\[
+ \sum_{k=1}^m \int_X \frac{\log B_{kp}}{2A_{kp}} dd^c \phi \wedge (\omega_1 \wedge \cdots \wedge \omega_{k-1} \wedge \frac{\gamma_{k+1,p}}{A_{k+1,p}} \wedge \cdots \wedge \frac{\gamma_{m,p}}{A_{m,p}} ) |
\]

\[= I + II.\]

We choose \(A_p \geq M_0\) for large \(p\),

\[
A_{kp}^{n-1} \leq B_{kp}(x) \leq A_{kp}^{n+1},
\]

so \(| \log B_{kp} | \leq (n + 1) \log A_{kp}\). Hence

\[
II = \sum_{k=1}^m \frac{nc \log A_{kp}}{A_{kp}} |\phi|C^2 \int_X | \omega_1 \wedge \cdots \wedge \omega_{k-1} \wedge \omega^{n-m+1} \wedge \frac{\gamma_{k+1,p}}{A_{k+1,p}} \wedge \cdots \wedge \frac{\gamma_{m,p}}{A_{m,p}} |.
\]
Then we have
\[
|< W_p, \phi > | = \left| \sum_{k=1}^{m} \left( \frac{nc \log A_{k \Phi}}{A_{k \Phi}} + \frac{cC_0}{A_{k \Phi}} \right) \| \phi \| c^2 \int_X \omega_1 \wedge \cdots \wedge \omega_{k-1} \wedge \omega^{n-m+1} \wedge \frac{\gamma_{k+1,p}}{A_{k+1,p}} \wedge \cdots \wedge \frac{\gamma_{m,p}}{A_{m,p}} \right|
\]
\[
= \left| \sum_{k=1}^{m} \left( \frac{nc \log A_{k \Phi}}{A_{k \Phi}} + \frac{cC_0}{A_{k \Phi}} \right) \| \phi \| c^2 \int_X |\omega_1 \wedge \cdots \wedge \omega_{k-1} \wedge \omega^{n-m+1} \wedge \frac{c_1(L_{k+1,p} h_{k+1,p})}{A_{k+1,p}} \wedge \cdots \wedge \frac{c_1(L_{m,p} h_{m,p})}{A_{m,p}} | \right|
\]
\[
\leq \sum_{k=1}^{m} 2^{m-k} \left( \frac{nc \log A_{k \Phi}}{A_{k \Phi}} + \frac{cC_0}{A_{k \Phi}} \right) \| \phi \| c^2 \int_X |\omega_1 \wedge \cdots \wedge \omega_{k-1} \wedge \omega^{n-m+1} \wedge \omega_{k+1} \wedge \cdots \wedge \omega_{m} |.
\]

The proof is completed. \(\square\)

4 Proof of main theorems

Our proof is based on Dinh-Sibony's technique on equidistribution theorems. The key estimate is the following.

\[
\left| \langle \prod_{k=1}^{m} A_{k \Phi} \omega_1 \cdots \omega_m, \phi \rangle \right|
\]
\[
\leq \left| \langle \prod_{k=1}^{m} A_{k \Phi} (\Phi_{k \Phi}^* (\delta_{k \Phi}) - \Phi_{k \Phi}^* (\sigma_{k \Phi})), \phi \rangle \right|
\]
\[
+ \left| \langle \prod_{k=1}^{m} A_{k \Phi} (\Phi_{k \Phi}^* (\sigma_{k \Phi})), \phi \rangle \right|
\]
\[
+ \left| \langle \prod_{k=1}^{m} A_{k \Phi} (\Phi_{k \Phi}^* (\omega_{k \Phi}), \phi \rangle \right|
\]
\[
= I + II + III.
\]

The estimate of III is already done in Section 3. Now we deal with II. First, we compute \(\delta_1^p\) and \(\delta_2^p\), where

\[
\delta_1^p = \int_X (\Phi_{k \Phi}^* (\omega_{M \Phi}^{d_{k \Phi}}) \wedge \omega^{n-m}),
\]
\[
\delta_2^p = \int_X (\Phi_{k \Phi}^* (\omega_{M \Phi}^{d_{k \Phi}-1}) \wedge \omega^{n-m+1}.
\]

Lemma 4.1. There exists a constant \(C > 1\), such that

\[
\prod_{k=1}^{m} A_{k \Phi} \leq \delta_1^p \leq C \prod_{k=1}^{m} A_{k \Phi},
\]

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\[
\frac{1}{C} \prod_{k=1}^{m} A_{kp} \leq \delta_p^2 \leq \frac{1}{C} \sum_{k=1}^{m} A_{kp}.
\]

**Proof.** By Proposition 2.6, we have

\[
\delta_1^p = \int_X c_1(L_1 h_1) \wedge \cdots \wedge c_1(L_m h_m) \wedge \omega^{n-m} \approx \prod_{k=1}^{m} A_{kp} \int_X \omega_1 \wedge \cdots \wedge_m \wedge \omega^{n-m}.
\]

By Assumption 1, \(\int_X \omega_1 \wedge \cdots \wedge_m \wedge \omega^{n-m} > 0\), then we have \(\delta_1^p \approx \prod_{k=1}^{m} A_{kp}\).

To compute \(\delta_2^p\), we recall that

\[
\dim H^{\ell_1,\ell_2} (\mathbb{CP}^N) = 1,
\]

\[
\dim H^{\ell_1,\ell_2} (\mathbb{CP}^N) = 0, \ell_1 \neq \ell_2
\]

for cohomology groups associated to sheaf of currents. Then \(\omega_{FS}^{d_{kp}}\) and \(\delta_{s_{kp}}, \omega_{FS}^{d_{kp} - 1}\) and \([D_{kp}]\) have the same cohomology classes, where \(\delta_{s_{kp}}\) and \(D_{kp}\) are generic point and complex line respectively in \(\mathbb{CP}^{d_{kp}}\). By the definition of meromorphic transform, we have

\[
\langle \Phi^*_{kp}([D_{kp}]), \phi \rangle = \langle (\pi_1)_*(\pi_2)^* [D_{kp}] \wedge [\Gamma_{kp}], \phi \rangle
\]

\[
= (\pi_2)^* [D_{kp}] \wedge [\Gamma_{kp}], (\pi_1)^* \phi
\]

\[
= \langle [\pi_2^{-1} ([D_{kp}] \cap \Gamma_{kp})], \pi_1^* \phi \rangle
\]

\[
= \int_{\pi_2^{-1} (D_{kp}) \cap \Gamma_{kp}} \pi_1^* \phi.
\]

Note that \(\pi_2^{-1} (D_{kp}) \cap \Gamma_{kp} = \{(x, s_{kp}) \in X \times D_{kp} : s_{kp}(x) = 0\}\).

Since \(D_{kp}\) is generic, \(\forall x \in X\), there exists a unique \(s_{kp} \in D_{kp}\) such that \(s_{kp}(x) = 0\), where \(\Gamma_{kp} = \{(x, s_{kp}) \in X \times \mathbb{CP}^{d_{kp}} : s_{kp}(x) = 0\}\). So \(\pi_1 : \pi_2^{-1} (D_{kp}) \cap \Gamma_{kp} \rightarrow X\) is bijective. Hence, \(\langle \Phi^*_{kp}([D_{kp}]), \phi \rangle = \int_X \phi\), i.e. \(\Phi^*_{kp}([D_{kp}]) = [X] = 1\).

Note that

\[
\Phi^*_{kp} (\omega_{M_{kp}}^{d_{kp} - 1}) = \sum_{k=1}^{m} \frac{d_{kp}}{c_{kp} d_{kp}} \Phi^*_{kp}(s_{1p}) \times \cdots \times (s_{kp_1} \times \{D_{kp}\} \times \cdots \times \{s_{mp}\})
\]

\[
= \sum_{k=1}^{m} \frac{d_{kp}}{c_{kp} d_{kp}} [s_{1p} = 0] \wedge \cdots \wedge \Phi^*_{kp}([D_{kp}]) \wedge \cdots \wedge [s_{mp} = 0],
\]

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where the bounded sequence \( \{c_p\} \) is mentioned before Theorem 2.2. Then,

\[
\delta^2_p = \sum_{k=1}^{m} \frac{d_{kp}}{c_pd_p} \int_X c_1(L_{1p}, h_{1p}) \wedge \cdots \wedge c_1(L_{kp}, h_{kp}) \wedge \cdots \wedge c_1(L_{mp}, h_{mp}) \wedge \omega^{n-m+1}
\]

\[
\approx \sum_{k=1}^{m} \frac{d_{kp}}{c_pd_p} \prod_{j=1}^{m} A_{jp} \int_X \omega_1 \wedge \cdots \wedge \omega_k \wedge \omega_m \wedge \omega^{n-m+1}
\]

\[
\approx \prod_{k=1}^{m} A_{kp} \sum_{k=1}^{m} A_{kp}
\]

The last approximation follows from the fact that \( \{A_{kp}\}_{p=1}^{\infty} \) have the same infinity order. The proof is completed.

Now we are in a position to estimate the term \( II \).

**Proposition 4.2.** We have

\[
|\left\langle \prod_{k=1}^{m} \frac{1}{A_{kp}} (\Phi_p^*(\sigma_p) - \Phi_p^*(\omega_{M_p}^{d_p})), \phi \right\rangle| \leq C \log \left( \sum_{k=1}^{m} A_{kp} \right) \frac{C \log \left( \sum_{k=1}^{m} A_{kp} \right)}{m} \|\phi\|_c^2.
\]

**Proof.** Recall that \( d_{kp} \approx A_{kp}^n \) and \( \{d_{1p}, \ldots, d_{mp}\} \) satisfy the conditions in Proposition 2.9 due to Assumption 1. By Lemma 4.1 and Proposition 2.9 we deduce that \( S(X_p, \omega_{M_p}, \sigma_p) \leq C \log \left( \sum_{k=1}^{m} A_{kp} \right) \). By Lemma 4.1

\[
\delta_p^2/\delta_p^1 \approx \frac{1}{m} \sum_{k=1}^{m} A_{kp}
\]

Then it follows from Theorem 2.3 that

\[
|\left\langle \prod_{k=1}^{m} \frac{1}{A_{kp}} (\Phi_p^*(\sigma_p) - \Phi_p^*(\omega_{M_p}^{d_p})), \phi \right\rangle| \leq C S(X_p, \omega_{M_p}, \sigma_p) \delta^2_p/\delta^1_p \|\phi\|_c^2 \leq C \log \left( \sum_{k=1}^{m} A_{kp} \right) \frac{C \log \left( \sum_{k=1}^{m} A_{kp} \right)}{m} \|\phi\|_c^2.
\]

The proof is completed. \( \square \)
Next we study the estimate of the term $I$.

**Proposition 4.3.** For $\sigma$-a.e. $\{s_p\} \in \mathbb{P}^X$, we have

$$\frac{1}{\prod_{k=1}^{m} A_{kp}} (\Phi^*(\delta_{s_p}) - \Phi^*(\sigma_p))$$

tends to 0.

**Proof.** By Lemma 2.8 and Proposition 2.9 we have

$$R(X_p, \omega_{M_p}, \sigma_p) \leq C(1 + \log(\sum_{k=1}^{m} A_{kp})),$$

$$\Delta(X_p, \omega_{M_p}, \sigma_p, t) \leq C(\sum_{k=1}^{m} A_{kp})^\epsilon \exp(-\widetilde{\beta}_2 t),$$

where $\widetilde{\beta}_2$ is a positive constant. Then

$$R(X_p, \omega_{M_p}, \sigma_p)\delta_{s_p} / \delta_{p} \to 0,$$

$$\sum_{p=1}^{\infty} \Delta(X_p, \omega_{M_p}, \sigma_p, (\delta_{s_p} / \delta_{p}) t) \leq C \sum_{p=1}^{\infty} \left( \sum_{k=1}^{m} A_{kp} \right)^\epsilon \exp(-\widetilde{\beta}_2 (\sum_{k=1}^{m} A_{kp}) t) < \infty.$$

Then the proof is completed by applying Theorem 2.3.

**End of the proof of Theorem 1.1.** The theorem follows from Proposition 3.1, Proposition 4.2 and Proposition 4.3.

Now we prove Theorem 1.2 by applying Theorem 2.2, which gives also an alternative proof of Theorem 1.1.

**Proof.** We take $C_4 > 0$ to be determined later and set

$$\varepsilon_p := \frac{C_4 \log(\sum_{k=1}^{m} A_{kp})}{\sum_{k=1}^{m} A_{kp}},$$

and

$$\eta_{\varepsilon_p} = \varepsilon_p \delta_{s_p} / \delta_{p} - 3R_p$$

$$\geq C_5 \varepsilon_p \log(\sum_{k=1}^{m} A_{kp}) - C \log(\sum_{k=1}^{m} A_{kp})$$

$$\geq C_6 \log(\sum_{k=1}^{m} A_{kp}),$$

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where $C_6 > 0$ determined by $C_4$. Note that $\log(\sum_{k=1}^{m} A_{kp}) \leq \min\{\sum_{k=1}^{m} A_{kp}\}$ by Assumption 1 for large $p$. We can apply Theorem 2.2 and derive that

$$\sigma_p(E_p(\varepsilon_p)) \leq \Delta(X_p, \omega_{M_p}, \sigma_p, \eta_{\varepsilon_p})$$

$$\leq C_1 \left(\sum_{k=1}^{m} A_{kp}\right)^{\xi} \left(\sum_{k=1}^{m} A_{kp}\right)^{-\tilde{\beta}_2 C_6}$$

$$= C_1 \left(\sum_{k=1}^{m} A_{kp}\right)^{-\alpha},$$

where $C_4$ is chosen such that $\alpha = \tilde{\beta}_2 C_6 - \xi > 0$. Since $\tilde{\beta}_2$ and $\xi$ are fixed constants, $\alpha > 0$ can be arbitrarily chosen. Set $E^\alpha_p := E_p(\varepsilon_p)$. Hence for any $s_p \in X_p \setminus E^\alpha_p$, we have

$$\left| \frac{1}{m} \prod_{k=1}^{m} A_{kp} \right| \left| [s_p = 0] - \Phi^*_p(\sigma_p), \phi \right|$$

$$\leq \frac{C_7 \log(\sum_{k=1}^{m} A_{kp})}{\sum_{k=1}^{m} A_{kp}} \| \phi \|_{\psi^2}. \quad (14)$$

Combining Proposition 3.1, Proposition 4.2 and (14), we obtain

$$\left| \left( \frac{1}{m} \prod_{k=1}^{m} A_{kp} \right) [s_p = 0] - \omega_1 \wedge \omega_2 \cdots \wedge w_m, \phi \right|$$

$$\leq C_2 \left( \sum_{k=1}^{m} \log \frac{A_{kp}}{A_{kp}} + \log \left( \sum_{k=1}^{m} A_{kp} \right) + \sum_{k=1}^{m} A_{kp}^{-a_k} \right) \| \phi \|_{\psi^2}.$$

Then the proof of Theorem 1.2 is completed. When $\sum_{p=1}^{\infty} \left( \sum_{k=1}^{m} A_{kp}\right)^{-a} < \infty$, we can prove Theorem 1 by standard arguments using Borel-Cantelli lemma (cf. [16, Proposition 4.5]).

5 Dimension growth of a sequence of pseudo-effective line bundles
In this section, we provide a dimension growth result which sheds a light on Assumption 1. It is enough to consider one sequence of holomorphic line bundles \((L_p, h_p)\). We are devoted to proving Theorem 1.4.

We first recall the \(L^2\)-estimate for line bundles with singular metrics (cf. [8, Theorem 3.2]).

**Theorem 5.1.** Let \((X, \omega)\) be a Kähler manifold of dimension \(n\) which admits a complete Kähler metric. Let \((L, h)\) be a singular Hermitian holomorphic line bundle and let \(\lambda : X \to [0, \infty)\) be a continuous function such that \(c_1(L, h) \geq \lambda \omega\). Then for any form \(g \in L^2_{0,1}(X, L, \text{loc})\) satisfying

\[
\overline{\partial} g = 0, \quad \int_X \lambda^{-1} |g|^2_h \omega^n < \infty,
\]

there exists \(u \in L^2(M, L, \text{loc})\) with \(\overline{\partial} u = g\) and

\[
\int_X |u|^2_h \omega^n \leq \int_X \lambda^{-1} |g|^2_h \omega^n.
\]

**Proof of Theorem 1.4.** We add additional local weights to the sequence. Let \(x \in U_\alpha \subseteq X \setminus \Sigma\), \(e_p\) is the local frame of \(L_p\) on \(U_\alpha\). Fix \(r_0 > 0\) so that the ball \(V := B(x, 2r_0) \subseteq U_\alpha\) and let \(U := B(x, r_0)\). Let \(\theta \in \mathcal{C}^\infty(\mathbb{R})\) be a cut-off function such that \(0 \leq \theta \leq 1, \theta(t) = 1\) for \(|t| \leq \frac{1}{2}, \theta(t) = 0\) for \(|t| \geq 1\). For \(z \in U\), define the quasi-psh function \(\varphi_z\) on \(X\) by

\[
\varphi_z(y) = \begin{cases} \theta\left(\frac{|y - z|}{r_0}\right) \log\left(\frac{|y - z|}{r_0}\right), & \text{for } y \in U_\alpha, \\ 0, & \text{for } y \in X \setminus B(z, r_0). \end{cases}
\]

Note that \(dd^c \varphi_z \geq 0\), on \(\{y : |y - z| \leq \frac{r_0}{2}\}\). Since \(V \subseteq U_\alpha\), it follows that there exists a constant \(c' > 0\) such that for all \(z \in U\) we have \(dd^c \varphi_z \geq -c' \omega\) on \(X\) and \(dd^c \varphi_z = 0\) outside \(\overline{V}\). Since

\[
c_1(L_p, h_p) \geq A_p \eta_p w \geq A_p c_z \omega = A_p \omega.
\]

We can find constants \(a, b\) with \(a = c - bc' > 0\), such that

\[
c_1(L_p, h_p e^{-bA_p \varphi_z}) \geq 0 \text{ on } X.
\]

\[
c_1(L_p, h_p e^{-bA_p \varphi_z}) = c_1(L_p, h_p) + bA_p dd^c \varphi_z
\]

\[
\geq A_p(c - bc')w = aA_p w \text{ near } \overline{V}.
\]

Consider a continuous function \(\lambda_p : X \to [0, +\infty)\) such that \(\lambda = aA_p\) on \(\overline{V}\),

\[
c_1(L_p, h_p e^{-bA_p \varphi_z}) \geq \lambda_p w. \text{ Set } \beta = (\beta_1, ..., \beta_n) \text{ with } \sum_{j=1}^n \beta_j \leq \lfloor bA_p \rfloor - n, \text{ and }
\]

\[
v_{z,p,\beta}(y) = (y_1 - z_1)^{\beta_1} ... (y_n - z_n)^{\beta_n}.
\]
Let 
\[ g_{z,p,\beta} = \overline{\partial}(v_{z,p,\beta}\theta(\frac{|y-z|}{r_0})e_p). \]

Then
\[
\int_X \frac{1}{\lambda} |g_{z,p,\beta}|^2 h_p e^{-2b_A p \varphi_z \omega^n} \\
= \int_V \frac{1}{\lambda} |g_{z,p,\beta}|^2 h_p e^{-2b_A p \varphi_z \omega^n} \\
= \frac{1}{a_A p} \int_{V \setminus B(z, \frac{r_0}{2})} |v_{z,p,\beta}|^2 |\partial \theta(\frac{|y-z|}{r_0})|^2 e^{-2\psi_p e^{-2b_A p \varphi_z \omega^n}},
\]

where \( \psi_p \) is the local weight of \( h_p \).

Note that \( \varphi_z \) is bounded on \( V \setminus B(z, \frac{r_0}{2}) \). Then
\[
\int_X \frac{1}{\lambda} |g_{z,p,\beta}|^2 h_p e^{-2b_A p \varphi_z \omega^n} < \infty, \forall p.
\]

By applying Theorem 5.1, there exists \( u_{z,p,\beta} \in L^2(X, L_p), \) such that
\[ \overline{\partial}u_{z,p,\beta} = g_{z,p,\beta}, \]

and
\[
\int_X |u_{z,p,\beta}|^2 h_p e^{-2b_A p \varphi_z \omega^n} \\
\leq \int_X \frac{1}{\lambda} |g_{z,p,\beta}|^2 h_p e^{-2b_A p \varphi_z \omega^n} \]

So we construct an element
\[ S_{z,p,\beta} = v_{z,p,\beta}\theta(\frac{|y-z|}{r_0})e_p - u_{z,p,\beta} \]
in \( H_{(2)}^0(X, L_p) \). For \( y \in B(z, \frac{r_0}{2}) \),
\[ S_{z,p,\beta}(y) = v_{z,p,\beta}(y)e_p - u_{z,p,\beta}(y), \]

we see that \( u_{z,p,\beta} \) is a holomorphic near \( z \).

Let \( \mathcal{L} \) be the sheaf of holomorphic functions on \( X \) vanishing at \( z \) and let \( m \subset \mathcal{O}_{X,z} \) the maximal ideal of the ring of germs of holomorphic function at \( z \). Consider the natural map
\[ L_p \rightarrow L_p \otimes \mathcal{O}_X / \mathcal{L}^{a+1}. \]

This map induces a map in the level of cohomology
\[ J_p^a : H_{(2)}^0(X, L_p) \rightarrow H_{(2)}^0(X, L_p \otimes \mathcal{O}_X / \mathcal{L}^{a+1}) = (L_p)_z \otimes \mathcal{O}_{X,z} / \mathcal{L}^{a+1}. \]
The right hand side of the above map is called the space of $a$-jets of $L^2$-holomorphic sections of $L_p$ at $z$. We recall the following fact:

$$\int |y_k - z_k|^{2r_k} |y - z|^{-2bA_p} i^n dy_1 \wedge d\bar{y}_1 \wedge \cdots \wedge dy_n \wedge d\bar{y}_n < \infty,$$

if and only if $\sum_{j=1}^n r_j \geq [bA_p] - n + 1$. Then for $u_{z,p,\beta} \in L^2(X, L_p)$, we have

$$\int_X |u_{z,p,\beta}|^2 e^{-2bA_p} \varphi_z \omega^n < \infty$$

if and only if $u_{z,p,\beta}$ has vanishing order of at least $[bA_p] - n + 1$ at $z$. So the $([bA_p] - n)$-jet of $S_{z,p,\beta}$ coincides with $v_{z,p,\beta}$. For any such $v_{z,p,\beta}$, $\sum_{j=1}^n \beta_j \leq [bA_p] - n$, we can construct $S_{z,p,\beta}$ as before such that

$$J_p^{[bA_p]-n}(S_{z,p,\beta}) = v_{z,p,\beta}. $$

So $J_p^{[bA_p]-n}$ is surjective. Hence

$$d_p = \dim H^0_{(2)}(X, L_p) - 1$$

$$\geq \dim(\mathcal{O}_{X,z}/L^{[bA_p]-n+1}) - 1$$

$$= \left(\frac{[bA_p]}{[bA_p] - n}\right) - 1 \geq C_3 A^n_p,$$

for some constant $C_3 > 0$. The proof is completed.

\begin{remark}
To get the upper estimate of $d_p$ by the spirit of Siegel’s lemma, we need impose more conditions on the transition functions of each $L_p$.
\end{remark}

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