A Model–Independent and Rephase–Invariant Parametrization of CP–Violation

D. Colicchio\(^{(1,2)}\) and M. Viggiano\(^{(1)}\)

1) Dipartimento di Matematica, Univ. Basilicata, Potenza, Italy
   Via N. Sauro 85, 85100 Potenza, Italy

2) Istituto Nazionale di Fisica Nucleare, Sezione di Milano, Italy
   Via G. Celoria 16, 20133 Milano, Italy

ABSTRACT

The phenomenological description of the neutral \(B\) meson system is proposed in terms of the fundamental \(CP\)–violating observables and within a rephasing invariant formalism. This generic formalism can select the time–dependent and time–integrated asymmetries which provide the basic tools to discriminate the different kinds of possible \(CP\)–violating effects in dedicated experimental \(B\)–meson facilities.

PACS numbers: 12.15.Ff, 13.20.Jf, 14.40.Jz.

Keywords: \(CP\)–Violation.
The time evolution of neutral meson decays can probably check whether $CP$–violation arises from $CP$–violating phases in the mixing (indirect $CP$–violation) or in the weak decay amplitudes (direct $CP$–violation). Recently, there has been much interest particularly on measuring the quark–mixing angles of the unitarity triangle by means of the study of $B$–meson decays [1]. Although, it has been observed that there are some limitations in extracting these angles by using the isospin $SU(3)$ relations, nevertheless it is still tantalizing to investigate carefully the time–dependence of correlative decay rates, in order to extract the penguin effects into the $CP$–violating asymmetries for neutral $B$ meson decays. In this paper, we propose a model–independent and rephase–invariant formalism which provides the fundamental parameters directly in terms of the measured quantities. The rephase–invariant formalism [2] will result more useful from a phenomenological viewpoint, since it explicitly intends to separate the different forms of $CP$–violation which a modern gauge theory can account for. In a neutral meson system, a meaningful classification of the three possible realizations of the $CP$–violating asymmetries (through mixing, decay or mixing–and–decay) is related to some phenomenological parameters which in principle might be detected by studying the time evolution of the neutral meson and some rate asymmetries. The $B^0$–$\bar{B}^0$ interfering effects into the $CP$–asymmetries can be described in terms of an effective matrix Hamiltonian

$$H = \begin{cases} M - \frac{i}{2} \Gamma \\ \mathcal{H}_\mu \sigma_\mu = (E_1 \sigma_1 + E_2 \sigma_2 E_3 \sigma_3) - i D \end{cases} \begin{pmatrix} id \\ b^2 \\ id' \end{pmatrix} \begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \end{pmatrix} \quad [3]$$

$$[4] \quad [5] \quad (1)$$

In Table I, we propose the transformation properties of the elements of the Hamiltonians, with respect to some important combinations like $CP$ and $CPT$ of the discrete space–time symmetries.
TABLE I. The restrictions imposed by combinations of charge conjugation $C$, parity $P$ and, time reversal $T$ on the elements of the Hamiltonian matrix.

| Form of $H$ | $CPT$ | $T$ | $CP$ |
|------------|-------|-----|------|
| $M_{11} = M_{22}$ | $M_{12} = M_{12}^* = M_{21} = M_{21}^*$ | $M_{12} = M_{12}^* = M_{21} = M_{21}^*$ | $M_{11} = M_{22}^*$, $\Gamma_{11} = \Gamma_{22}^*$ |
| $M - \frac{i}{2}\Gamma$ | $\Gamma_{11} = \Gamma_{22}$ | $\Gamma_{12} = \Gamma_{21}^* = \Gamma_{21}^*$ | $\Gamma_{12}^* = \Gamma_{21}^* = \Gamma_{21}^*$ |
| $\mathcal{H}_\mu \sigma_\mu$ | $\mathcal{H}_z = 0$ | $\mathcal{H}_y = 0$ | $\mathcal{H}_y = \mathcal{H}_z = 0$ |
| $d, d', b, b'$ | $d = d'$ | $b^2 = b'^2$ | $d = d'$, $b^2 = b'^2$ |

The effective Hamiltonian matrix is determined by eight real parameters, but only seven are physical meaningful because the absolute phase of $H_{12}$ or $H_{21}$ is meaningless, being the relative phase of $B^0$ and $\bar{B}^0$ arbitrary. They can be substituted by the two complex eigenvalues

$$\lambda_{H,L} = \frac{1}{2} (C \mp D)$$

where, in a general theory, $C = \lambda_L + \lambda_H = H_{11} + H_{22} = \text{tr} H$ and $D^2 = (\lambda_L - \lambda_H)^2 = (H_{11} - H_{22})^2 + 4H_{12}H_{21} = (\text{tr} H)^2 - 4(\det H)$, and by the two complex mixing parameters $\epsilon_{H,L}$ which are given by:

$$\epsilon_H = \left( \frac{2H_{12} - D}{2H_{12} + D} \right) - \left( \frac{4H_{12}}{2H_{12} + D} \right) \left( \frac{H_{11} - H_{22}}{H_{11} - H_{22} + D + 2H_{12}} \right) = \epsilon_B - \delta_H$$

$$\epsilon_L = \left( \frac{2H_{12} - D}{2H_{12} + D} \right) - \left( \frac{4H_{12}}{2H_{12} + D} \right) \left( \frac{H_{11} - H_{22}}{H_{11} - H_{22} - D - 2H_{12}} \right) = \epsilon_B - \delta_L,$$ (3)

where
\[\epsilon_B = \frac{2H_{12} - D}{2H_{12} + D} = \frac{\sqrt{H_{12} - H_{21}}}{\sqrt{H_{12} + H_{21}}},\]

\[\delta_H = \left( \frac{2H_{12}}{2H_{12} + D} \right) \left( \frac{H_{11} - H_{22}}{H_{11} + H_{12} - \lambda_H} \right),\]

\[\delta_L = \left( \frac{2H_{12}}{2H_{12} + D} \right) \left( \frac{H_{11} - H_{22}}{H_{11} + H_{12} - \lambda_L} \right).\]

The complex scaling matrix \( \mathcal{R} \), which diagonalizes the effective Hamiltonian, is then given by

\[\mathcal{R} = \begin{pmatrix} N_H (1 + \epsilon_H) & -N_H (1 - \epsilon_H) \\ N_L (1 + \epsilon_L) & -N_L (1 - \epsilon_L) \end{pmatrix} = \begin{pmatrix} p & -q \\ p' & q' \end{pmatrix},\]

where \( N_{H,L}^{-2} = 2(1 + |\epsilon_{H,L}|^2) \) are fixed only once the eigenvectors normalization is realized and

\[\eta_H = -\frac{q}{p} = -\frac{(1 - \epsilon_H)}{(1 + \epsilon_H)} = -\frac{(H_{11} - H_{22} + D)}{2H_{12}},\]

\[\eta_L = \frac{q'}{p'} = \frac{(1 - \epsilon_L)}{(1 + \epsilon_L)} = -\frac{(H_{11} - H_{22} - D)}{2H_{12}}.\]

In any CPT-invariant theory, we consider from now onward, \( H_{11} = H_{22} \) and there is only one mixing parameter \( \epsilon_B = \epsilon_H = \epsilon_L \) and therefore \( \eta_H = -\eta_L, N_{H,L}^{-2} = N^{-2} = 2(1 + |\epsilon_B|^2), p = p', q = q' \). In this case, we have that

\[\lambda_H = H_{11} - \sqrt{H_{12}H_{21}} = M_{11} - \frac{i}{2} \Gamma_{11} - \frac{D}{2} = m_H - \frac{i}{2} \gamma_H\]

\[\lambda_L = H_{11} + \sqrt{H_{12}H_{21}} = M_{11} - \frac{i}{2} \Gamma_{11} + \frac{D}{2} = m_L - \frac{i}{2} \gamma_L\]

with

\[D = 2\sqrt{H_{12}H_{21}} = 2\sqrt{(M_{12} - \frac{i}{2} \Gamma_{12})(M_{12}^* - \frac{i}{2} \Gamma_{12}^*)} = -\left( \Delta m - \frac{i}{2} \Delta \gamma \right),\]

being \( \Delta m = m_H - m_L \) and \( \Delta \gamma = \gamma_H - \gamma_L \). These real \((m_{H,L})\) and imaginary \((\gamma_{H,L})\) components will define the masses and the decay widths of the eigenstates \( B_H \) and \( B_L \) in the narrow width approximation. These heavy and light particles are then a linear combination of the flavour \( B^0 \) and \( \bar{B}^0 \) states:

\[\begin{pmatrix} |B_H\rangle \\ |B_L\rangle \end{pmatrix} = \mathcal{R} \begin{pmatrix} |B^0\rangle \\ |\bar{B}^0\rangle \end{pmatrix},\]

\[4\]
where usually $\mathcal{R}$ is preferably parameterized according to the following relations

$$
\mathcal{R} = \begin{cases}
\frac{1}{\sqrt{2(1+|\epsilon_B|^2)}} \begin{pmatrix} (1 + \epsilon_B) & -(1 - \epsilon_B) \\ (1 + \epsilon_B) & (1 - \epsilon_B) \end{pmatrix} \\
\frac{|1 - \eta|}{1 - \eta} \frac{1}{\sqrt{1 + |\eta|^2}} \begin{pmatrix} 1 & \eta \\ 1 & -\eta \end{pmatrix} \\
\begin{pmatrix} p & -q \\ p & q \end{pmatrix}.
\end{cases}
$$

(11)

After the corresponding normalization of the eigenvectors

$$
\langle B_L| B_L \rangle = \langle B_H| B_H \rangle = |p|^2 + |q|^2 = 1,
$$

(12)

due to the state vectors $|B^0\rangle$ and $|\bar{B}^0\rangle$. Therefore both $p$ and $q$ are not measurable quantities. Thus, it results evident that the $CP$-violation parameter $\epsilon_B$ arises from a relative imaginary part between the off-diagonal elements $M_{12}$ and $\Gamma_{12}$ i.e. if $\delta = arg(M_{12}\Gamma_{12}^*) \neq 0$. To this end, we introduce the ratio between these relevant variables

$$
\frac{M_{12}}{\Gamma_{12}} = \frac{|M_{12}|}{|\Gamma_{12}|} e^{i\delta} = -r e^{i\delta},
$$

(14)

where the relative phase is $\delta = (\delta_M - \delta_G)$. In terms of $r$ and $\delta$, the $CP$–violating parameter $\epsilon_B$ is obtained as follows

$$
\epsilon_B \simeq i \sin \frac{\delta_M}{1 + \cos \delta_M} + \left( \frac{1}{1 + \cos \delta_M} \right) \left( \frac{2r - i}{4r^2 + 1} \right) \delta.
$$

(15)

To the extent that

$$
M_{12} = |M_{12}| e^{i\delta_M} \quad \text{and} \quad \Gamma_{12} = |\Gamma_{12}| e^{i\delta_G},
$$

(16)

share the same phase $\delta_M = \phi = \delta_G$, we have no $CP$–violation and we obtain that

$$
\epsilon_B = i \frac{|M_{12}| \sin \delta_M - \frac{i}{2} |\Gamma_{12}| \sin \delta_G}{|M_{12}| \cos \delta_M - \frac{i}{2} |\Gamma_{12}| \cos \delta_G + \frac{D}{2}} = i \frac{\sin \phi}{1 + \cos \phi}
$$

(17)
independently of any phase convention. However, it is clear that only the magnitude of
\[ \eta = -\frac{q}{p} = -\frac{1 - \epsilon_B}{1 + \epsilon_B} = -\sqrt{\frac{M_{12}^* - \frac{i}{2} \Gamma_{12}^*}{M_{12} - \frac{i}{2} \Gamma_{12}}} \] (18)
results a measurable quantity and it results connected to the following overlap parameter
\[ s = \langle B_H|B_L \rangle = \frac{2 \text{Re} \epsilon_B}{1 + |\epsilon_B|^2} = \frac{1 - |\eta|^2}{1 + |\eta|^2} \simeq \frac{2r}{4r^2 + 1} \delta . \] (19)
This means that \( CP \)-nonconservation is determined by the relative phase between \( M_{12} \) and \( \Gamma_{12} \). The value of the parameter \( |\eta| \) is significant, in the sense that \( \eta = \frac{1 - \epsilon_B}{1 + \epsilon_B} \neq 1 \) does not necessarily imply \( CP \)-violation. \( CP \) is violated in the mixing matrix if \( |\eta| \neq 1 \).
Remember that, since flavour is conserved in strong interactions, there is some freedom in defining the phases of flavour eigenstates. This means that \( \eta \) is a phase dependent quantity manifesting its presence in the phase of \( \epsilon_B \) which must only satisfy Eq. (19).
In turn, Eq. (19) reduces to the equation of a circumference
\[ \left( \text{Re} \epsilon_B - \frac{1}{s} \right)^2 + (\text{Im} \epsilon_B)^2 = \left( \frac{1}{s} \right)^2 - 1 \] (20)
of radius \( \sqrt{\left(\frac{1}{s}\right)^2 - 1} \simeq \frac{1}{s} - \frac{s}{2} \) centered in \( \left( \frac{1}{s}, 0 \right) \) of the Gauss complex \( \epsilon_B \)-plane. This geometric interpretation of the dependence of \( \epsilon_B \) on the choice of the phase convention means that \( \epsilon_B \) picks a point on the circumference of this circle according to each possible convention. The relative pure phase \( \delta \) can be derived from the fact that
\[ \eta \simeq -e^{i\delta} \left[ 1 - \frac{2r}{4r^2 + 1} (1 + 2ir) \delta \right] . \] (21)
Assuming the \( \Delta B = \Delta Q \) rule conserved [6], the amount of its magnitude can then be extracted from the decay rate asymmetry between the semileptonic channels \( B \to \ell^+ \nu_\ell X^- \) and \( \overline{B} \to \ell^- \overline{\nu}_\ell X^+ \). The related \( CP \)-violating asymmetry
\[ A_{SL}^{\ell \ell} = \frac{1 - |\eta|^4}{1 + |\eta|^4} = \frac{4r \sin \delta}{4r^2 + 1} , \] (22)
has been predicted theoretically [7] and somewhat detected experimentally [8]. However, it is worth noting that the experimental results [8] of the decay rate asymmetry \( A_{SL}^{\ell \ell} \) do not constrain \( \epsilon_B \) in a sensible way, due to the lack of available data of other direct \( CP \)-violating effects. Analogously to the case of the neutral kaon mixing, the
magnitude and the phase of the complex parameter $\epsilon_B$ depend on the specific phase-convention adopted to describe the $B^0 - \bar{B}^0$ system. Indeed, this phase-convention dependence could induce errors to provide the experimental information on $\epsilon_B$ [9]. For a particular phase-convention, $M_{12}$ may turn out to have a large phase, but without also knowing the phase of $\Gamma_{12}$, no conclusion can be reached as to the size of indirect $CP$–nonconservation. In the Wu-Yang convention $\text{Im} \Gamma_{12} = 0$, we obtain that

$$\arg(\epsilon_B) \simeq \begin{cases} \pi + \Phi_{SW} & \text{for } \text{Im} \epsilon_B > 0, \\ \Phi_{SW} & \text{for } \text{Im} \epsilon_B < 0, \end{cases}$$

being the superweak phase $\Phi_{SW} = \tan^{-1}(2r)$. The question we refer here arises whatever we introduce an a priori assumption for a phase convention dependent observable, like in the case of $\epsilon_B$. Assuming the box diagram dominance of the Standard Model, the situation is still unclear, being $|M_{12}^B|$ very large and $|\eta| \simeq 1$. From the available experimental inputs of $x_d \simeq \Delta m_{B\tau} \Delta m_B \simeq 2|M_{12}^B|$, we find that $\delta \simeq \left(\frac{\text{Im} M_{12}^B}{|M_{12}^B|}\right)$ is in a wide range. Consequently, we may say that the rephasing dependence of the impurity parameter $\epsilon_B$ is of no use for the study of the $CP$–violating effects in the $B^0 - \bar{B}^0$ system. This example suggests to develop a formalism more transparent from a phenomenological point of view and expressible directly in terms of observables. Therefore, a suitable choice of the relative phase between $CP|B^0\rangle = e^{i\delta_{CP}}|\bar{B}^0\rangle$ has to be adopted for the specification of the parameter $\epsilon_B$. Usually, we fix this relative phase between the two states and then we determine the $CP$–nonconserving effects from the relative phase between $M_{12}$ and $\Gamma_{12}$. But, as we stressed before, this approach can induce some ambiguities if not errors and, therefore, a more general formalism is needed. In order to develop a generalized rephasing invariant method we cannot forget that, although the properties of the particle mixing are connected to the solution of a Schrödinger equation of an effective Hamiltonian $H = M - \frac{i}{2}\Gamma$, nevertheless the essential tool of the description is represented by the transition amplitude that is given in matrix notation by

$$\tau_{FI} = \sum_{ij} A^*_{F,i}[sI - H]^{-1}_{ij} A_{j,I} = A^*_D[sI - H]^{-1}_P A_P$$

in terms of the production $A_P = (A_{i,I})$ and decay $A_D = (A_{j,F})$ amplitudes of the production vertex $P$ of the initial $I$ channel mode and the decay vertex $D$ of the final $F$–decay mode. An initial $B$–meson in a pure state decays into an $F$–channel mode.
with an amplitude

$$A_{iF}(t) = \langle F|B_i(t)\rangle = \langle F|U_{ij}(t)|B_j\rangle$$  \hspace{1cm} (25)$$

being

$$A_D = (A_{iF}) = \left(\begin{array}{c} \langle F|B^0\rangle \\ \langle F|\bar{B}^0\rangle \end{array}\right) = \left(\begin{array}{c} A(B^0 \rightarrow F) \\ A(\bar{B}^0 \rightarrow F) \end{array}\right)$$  \hspace{1cm} (26)$$

and

$$U(t) = \left(\begin{array}{cc} g_1(t) & \eta g_2(t) \\ \frac{1}{\eta} g_2(t) & g_1(t) \end{array}\right)$$  \hspace{1cm} (27)$$

written in terms of $g_{1,2}(t) = \frac{1}{2}(e^{-i\lambda_H t} \pm e^{-i\lambda_L t})$ [10]. If $\bar{F}$ denotes the charge–conjugate final decay state, we have the conjugate amplitude

$$A_{i\bar{F}} = \langle \bar{F}|B_i(t)\rangle \quad \text{and} \quad \bar{A}_D = (A_{i\bar{F}}).$$  \hspace{1cm} (28)$$

Squaring, we obtain the time–dependent rates

$$\Gamma(B_i(t) \rightarrow \bar{F}) = \sum_{kj} U_{ki}^* U_{kj} |A_j(\bar{F})|^2$$  \hspace{1cm} (29)$$

where the summation over repeated indices is tacitly assumed. The time–integrated rates of interest

$$\hat{\Gamma}(B_i \rightarrow \bar{F}) = \int_0^\infty dt \Gamma(B_i(t) \rightarrow \bar{F}) = \int_0^\infty dt |A_{i\bar{F}}(t)|^2,$$  \hspace{1cm} (30)$$

can be expressed in terms of the rephase–invariant complex parameters

$$\xi_F = \frac{q A(B^0 \rightarrow \bar{F})}{p A(\bar{B}^0 \rightarrow \bar{F})} = \frac{q}{p} \frac{\bar{A}_{i\bar{F}}}{A_{i\bar{F}}},$$  \hspace{1cm} (31)$$

with the following resulting expressions

$$\hat{\Gamma}(B^0 \rightarrow \bar{F}) = |A_{i\bar{F}}|^2 \left[\hat{M}_{11} + |\xi_F|^2 \hat{M}_{22} - 2\Re\left(\xi_F \hat{M}_{21}\right)\right]$$

$$\hat{\Gamma}(\bar{B}^0 \rightarrow \bar{F}) = |A_{i\bar{F}}|^2 \left[\hat{M}_{22} + |\xi_F|^2 \hat{M}_{11} - 2\Re\left(\xi_F \hat{M}_{12}\right)\right] |\eta|^{-2},$$  \hspace{1cm} (32)$$

being $\hat{M}_{ij} = \int_0^\infty dt M_{ij}(t)$ with $M_{ij} = g_i g_j^*$ [10] and, supplemented by the unitarity sum rule of Bell and Steinberger [11],

$$\langle B_H|\hat{\Gamma}|B_L\rangle = \left[\left(\frac{\gamma_H + \gamma_L}{2}\right) - i(m_H - m_L)\right] \langle B_H|B_L\rangle =$$

$$= \sum_F \langle F|H|B_H\rangle^* \langle F|H|B_L\rangle = \sum_F \langle \Gamma_F \rangle (A + iB)$$  \hspace{1cm} (33)$$

8
where the last expression has been obtained by choosing the final decay modes $F$ to be $CP$–eigenstates and the integration with respect to the phase space must be understood. We have defined

$$\langle \Gamma_F \rangle = \frac{1}{1 + |\eta|^2} \left[ \Gamma(B^0 \to F) + |\eta|^2 \Gamma(\overline{B}^0 \to \overline{F}) \right] \simeq \frac{1}{2} \left[ \Gamma(B^0 \to F) + \Gamma(\overline{B}^0 \to \overline{F}) \right].$$ (34)

The two independent $CP$–violating parameters are given by

$$A = \frac{1 - |\xi_F|^2}{1 + |\xi_F|^2},$$

$$B = \frac{2 \text{Im} \xi_F}{1 + |\xi_F|^2},$$

(35)

and $(B_L|B_H) = \frac{1 - |\eta|^2}{1 + |\eta|^2} \simeq 10^{-3}$ imposes large cancellations in the sum.

In the most general case of a state $\overline{F}$ which is not $CP$ self–conjugate of the final decay mode $F$ [12], we can introduce the following relevant parameters

$$\epsilon'_{ij} = \frac{1 - f_{ij}}{1 + f_{ij}},$$

$$\epsilon''_{ij} = \frac{1 - \overline{f}_{ij}}{1 + \overline{f}_{ij}},$$

$$\epsilon^{CP}_{ij} = \frac{1 - f^{CP}_{ij}}{1 + f^{CP}_{ij}},$$

(36)

being

$$f_{ij} = \frac{A_iF}{\overline{A}_jF}, \quad \overline{f}_{ij} = \frac{A_i\overline{F}}{\overline{A}_j\overline{F}} \quad \text{and} \quad f^{CP}_{ij} = \frac{A_iF}{\overline{A}_j\overline{F}},$$

(37)

or alternatively we can introduce the parameters

$$\eta_{\alpha\beta} = \frac{A_{\alpha F}}{A_{\beta F}} = \frac{R_{\alpha i}f_{ij} + R_{\alpha j}}{R_{\beta i}\overline{f}_{ij} + R_{\beta j}} \quad \text{and} \quad \overline{\eta}_{\alpha\beta} = \frac{A_{\alpha \overline{F}}}{A_{\beta \overline{F}}} = \frac{R_{\alpha i}\overline{f}_{ij} + R_{\alpha j}}{R_{\beta i}f_{ij} + R_{\beta j}}.$$ (38)

The above parameters $f_{ij}$ and $\overline{f}_{ij}$ can describe the direct $CP$–violating effects, but indeed, they are not physical observables since they are not rephase–invariant. Indeed, the relevant phase–independent observables are represented only by

$$\eta_F = \frac{\langle F|B_H \rangle}{\langle F|B_L \rangle} = \frac{1 - \xi_F}{1 + \xi_F},$$

$$\overline{\eta}_F = \frac{\langle \overline{F}|B_H \rangle}{\langle \overline{F}|B_L \rangle} = \frac{1 - \overline{\xi}_F}{1 + \overline{\xi}_F}.$$ (39)
Physically, we can introduce the following possible rate asymmetries

\[ a_{ij} = \frac{|A_{iF}(t)|^2 - |A_{jF}(t)|^2}{|A_{iF}(t)|^2 + |A_{jF}(t)|^2} = -\frac{2\text{Re} \, \epsilon_{ij}'}{1 + |\epsilon_{ij}'|^2} \]

\[ \bar{a}_{ij} = \frac{|A_{iF}(t)|^2 - |A_{jF}(t)|^2}{|A_{iF}(t)|^2 + |A_{jF}(t)|^2} = -\frac{2\text{Re} \, \epsilon_{ij}^{CP}}{1 + |\epsilon_{ij}^{CP}|^2} \]

(40)

\[ \Gamma(B^0(t) \rightarrow \ell^-) = \frac{1}{4} |A_{\ell}^F(t)|^2 \left[ f_1(t)|\xi_F|^2 + f_2(t) + 2f_3(t)\text{Re} \, \xi_F + 2f_4(t)\text{Im} \, \xi_F \right] \]

\[ + \frac{1}{4} |A_{\ell}^F(t)|^2 \left[ h_1(t)|1 - \xi_F|^2 + h_2(t)|1 + \xi_F|^2 + h_3(t)\text{Re}[(1 + \xi_F)(1 - \xi_F)] + h_4(t)\text{Im}[(1 + \xi_F)(1 - \xi_F)] \right] \]

(41)

and

\[ \Gamma(B^0(t) \rightarrow \ell^-) = \frac{1}{4} |A_{\ell}^{CP}(t)|^2 \left[ f_1'(t)|\xi_F|^2 + f_2'(t) + 2f_3'(t)\text{Re} \, \xi_F + 2f_4'(t)\text{Im} \, \xi_F \right] \]

\[ + \frac{1}{4} |A_{\ell}^{CP}(t)|^2 \left[ h_1'(t)|1 - \xi_F|^2 + h_2'(t)|1 + \xi_F|^2 + h_3'(t)\text{Re}[(1 + \xi_F)(1 - \xi_F)] + h_4'(t)\text{Im}[(1 + \xi_F)(1 - \xi_F)] \right] \]

(42)
where the time-dependent functions are defined by

\[
\begin{align*}
    f_1^{(t)}(t) &= e^{-\gamma_L t} + e^{-\gamma_H t} e^{-\gamma^{+} t} 2 e^{-\Gamma t} \cos(\Delta mt), \\
    f_2^{(t)}(t) &= e^{-\gamma_L t} + e^{-\gamma_H t} e^{-\gamma^{+} t} 2 e^{-\Gamma t} \cos(\Delta mt), \\
    f_3^{(t)}(t) &= e^{-\gamma_L t} - e^{-\gamma_H t}, \\
    f_4^{(t)}(t) &= e^{-\gamma^{+} t} 2 e^{-\Gamma t} \sin(\Delta mt);
\end{align*}
\]

\[
\begin{align*}
    h_1^{(t)}(t) &= e^{-\gamma_H t}, \\
    h_2^{(t)}(t) &= e^{-\gamma_L t}, \\
    h_3^{(t)}(t) &= e^{-\gamma^{+} t} 2 e^{-\Gamma t} \cos(\Delta mt), \\
    h_4^{(t)}(t) &= e^{-\gamma^{+} t} 2 e^{-\Gamma t} \sin(\Delta mt).
\end{align*}
\]

(43)

Here, we have preferred to use the following two dimensionless parameters

\[
x = \frac{\Delta m}{\Gamma}, \quad y = \frac{\Delta \gamma}{2\Gamma},
\]

(44)

being \(\Delta m = m_H - m_L\), \(\Delta \gamma = \gamma_H - \gamma_L\) and \(\Gamma = (\gamma_H + \gamma_L)/2\). It is worth noting that the eight parameters

\[
\begin{align*}
    \left( \begin{array}{c} C_y \\ \end{array} \right) &= 1 + |\xi_F|^2, \\
    \left( \begin{array}{c} C_x \\ \end{array} \right) &= 1 - |\xi_F|^2, \\
    \left( \begin{array}{c} S_y \\ \end{array} \right) &= 2\text{Re} \left( \begin{array}{c} \xi_F \\ \end{array} \right), \\
    \left( \begin{array}{c} S_x \\ \end{array} \right) &= -2\text{Im} \left( \begin{array}{c} \xi_F \\ \end{array} \right); \\
    \left( \begin{array}{c} C_y' \\ \end{array} \right) &= \frac{1 + |\xi_F|^2}{|\eta|^2}, \\
    \left( \begin{array}{c} C_x' \\ \end{array} \right) &= \frac{-1 - |\xi_F|^2}{|\eta|^2}, \\
    \left( \begin{array}{c} S_y' \\ \end{array} \right) &= \frac{2\text{Re} \left( \begin{array}{c} \xi_F \\ \end{array} \right)}{|\eta|^2}, \\
    \left( \begin{array}{c} S_x' \\ \end{array} \right) &= \frac{2\text{Im} \left( \begin{array}{c} \xi_F \\ \end{array} \right)}{|\eta|^2};
\end{align*}
\]

(45)

and the coefficients of \(f_i^{(t)}(t)\) and \(h_i^{(t)}(t)\) in formulas (41) and (42) are all rephase-invariant quantities. We apply the above general analyses to some specific cases. \(CP\)-violating asymmetries can be realized in semileptonic and nonleptonic decays. Hadronic \(CP\)-asymmetries may be classified according to the final decay states. Final states may be pure \(CP\)-eigenstates \(|F\rangle = CP|F\rangle = |\overline{F}\rangle\), such as \(F = \pi^+\pi^-, \pi^0\pi^0, \ldots\) or \(CP\)-mixed states, such as \(K^0\pi^0, D^0\pi^0\) which can be recombined into \(CP\)-eigenstates.

In the most general case, both \(F\) and \(\overline{F}\) are common final states of \(B^0\) and \(\overline{B}^0\), but they are not \(CP\)-eigenstates, such as \(D^-\rho^+\). A particular interest deserves the \(CP\)-violating asymmetry between \(B^0 \to F\) and \(\overline{B}^0 \to \overline{F}\) (in the case \(F\) results a hadronic \(CP\)-eigenstate). In this case the relevant asymmetry is

\[
a_{CP} = \frac{\Gamma(B^0(t) \to F) - \Gamma(\overline{B}^0(t) \to \overline{F})}{\Gamma(B^0(t) \to F) + \Gamma(\overline{B}^0(t) \to \overline{F})} = \frac{\left| f_{00} \frac{A_{\pi p}}{A_{\pi F}} \right|^2 - 1}{\left| f_{00} \frac{A_{\pi p}}{A_{\pi F}} \right|^2 + 1},
\]

(46)

\[
= \frac{C_y \cosh(y\tau) + C_x \cos(x\tau) + S_y \sinh(y\tau) + S_x \sin(x\tau)}{C_y \cosh(y\tau) + C_x \cos(x\tau) + S_y \sinh(y\tau) + S_x \sin(x\tau)}.
\]
The convenient notation $\tau = \Gamma t$ is used and the superscripts $\Sigma$ and $\Delta$ denote, respectively, the sum or the difference between $C$ and $C'$, $S$ and $S'$. The Standard Model predicts $|\eta| \simeq 1$ and, it holds to a good degree of accuracy. As a consequence, the following much simpler expression for Eq. (46) can be written as

$$a_{CP} \simeq \frac{[1 - |\xi_F|^2] \cos(x\tau) - 2\text{Im}\xi_F \sin(x\tau)}{[1 + |\xi_F|^2] \cosh(y\tau) + 2\text{Re}\xi_F \sinh(y\tau)}. \quad (47)$$

Due to the smallness of $y$ ($y \simeq 0$), we derive the further standard result

$$a_{CP} \simeq A \cos(x\tau) - B \sin(x\tau) \quad (48)$$

in terms of the two independent $CP$–violating parameters $A$ and $B$ of Eq. (35). In the case of a nonvanishing $A$, we speak of direct $CP$–violation in the decay amplitude. On the other side, the second parameter $B$ vanishes in the absence of $CP$–violating effects into $B^0 - \bar{B}^0$ mixing. The importance of this expression is related to the possibility of performing a classification of the different forms of $CP$–violation in terms of $A$ and $B$. In the Standard Model, $CP$–violating effects can arise through the interference between at least two independent amplitudes with different $CP$–phases. Besides the charged $W$ currents, some non zero $CP$–violating effects are provided by loop induced transitions involving strong (gluon) and electroweak ($\gamma, Z^0$ and $H^0$) interactions. Due to the presence of these penguin contributions, we cannot extract straightforwardly the $CP$–angle, which characterizes the $CP$–asymmetry between the two $CP$ conjugate decay modes. In order to clarify this point, we decompose the conjugated decay amplitudes $A_F$ and $\overline{A}_F$ as

$$A_F = A_1 e^{i\phi_1} e^{i\delta_1} + A_2 e^{i\phi_2} e^{i\delta_2}$$

$$\overline{A}_F = A_1 e^{-i\phi_1} e^{i\delta_1} + A_2 e^{-i\phi_2} e^{i\delta_2}, \quad (49)$$

where $\phi_1$, $\phi_2$ are weak phases associated with the different quark mixing elements, whereas $\delta_1$, $\delta_2$ are unitarity $CP$–conserving strong phases, usually, associated with the absorptive part of the penguin diagrams or also related to inelastic final state interactions. For convenience, we introduce the following quantities

$$R = \frac{A_2}{A_1}, \quad \phi_{12} = \phi_1 - \phi_2, \quad \delta_{12} = \delta_1 - \delta_2 \quad (50)$$

being $A_1$ and $A_2$ the magnitudes of the hadronic matrix elements. Without any loss of generality, one may suppose that a single weak amplitude (or rather a single weak
phase) dominates the decay process \((A_1 > A_2)\). Therefore, the above parameters are found to be

\[
A \simeq -2R \sin \phi_{12} \sin \delta_{12} \\
B \simeq -\left[ \sin 2(\phi_M + \phi_1) - 2R \cos 2(\phi_M + \phi_1) \sin \phi_{12} \cos \delta_{12} \right],
\]

where we have used the more convenient notation \(\eta = -e^{-2i\phi_M}\) and

\[
\xi_F = e^{-2i\phi_M} \frac{\overline{A}_F}{A_F} = e^{-2i(\phi_M + \phi_1)} \left[ 1 + \frac{2iR \sin \phi_{12} e^{-i\delta_{12}}}{1 + R e^{-i\phi_{12}} e^{-i\delta_{12}}} \right],
\]

\[
\text{Im} \xi_F = -\sin 2(\phi_M + \phi_1) + \Delta_F,
\]

with \(\Delta_F = -2R \sin \phi_{12} \cos \left[ \delta_{12} + 2(\phi_M + \phi_1) \right]\). We first consider the particular case \(R = 0\) or \(\phi_1 = \phi_2\). It is clear that it yields the box diagram dominant result

\[
A = 0 \\
B \simeq -\sin 2(\phi_M + \phi_1)
\]

which is peculiar of indirect \(CP\)–violation in the mixing, and it is due to the equality \(\overline{A}_F = A_F^*\). The mixing-and-decay \(CP\)–violating effects \((R \neq 0, \phi_1 \neq \phi_2)\) induce a nonvanishing \(A\) (i.e. \(\delta_{12} \neq 0\)) with a direct \(CP\)–violation in the decay amplitude \((A_F\) or \(\overline{A}_F\)). As we have pointed out, the existence of the direct \(CP\)–violation requires that both a strong and a weak phase difference exist. Usually, we calculate the decay amplitudes by using the effective weak Hamiltonian and the factorization approximation [13]. The evaluation of the relative strong phase \(\delta_{12}\) is more problematic as we lack a quantitative understanding of nonperturbative QCD and of the effects due to the final state interactions or to the production of coupled resonant decay modes. Neglecting the final state interactions which produce the strong phase difference \(\delta_{12}\), we can consider the typical \(B_d^0 \rightarrow \pi^+\pi^-\) decay process which is characterized by the \(CP\)–angle \(\alpha\) of the unitarity triangle [1], being

\[
\sin 2\alpha = -\sin 2(\phi_M + \phi_1)
\]

with

\[
\Delta_F = -R \left\{ \frac{2 \cos \delta_{12} \sin \alpha + \sin 2\alpha [R - 2 \cos (\alpha - \delta_{12})]}{1 + R^2 - 2R \cos (\alpha - \delta_{12})} \right\}.
\]

Adopting the usual valence–quark convention, the \(SU(3)\) invariance can be used to isolate the gluon penguin contamination and determine \(\alpha\) up to a two–fold ambiguity.
The penguin effects on $CP$–violation are supposed to be extracted by means of the knowledge of relevant branching ratios, without observing the time–dependence of the decay rates. Therefore, the $CP$–violation in the specimen case of $B_d^0 \to \pi^+ \pi^-$ is characterized by the following observables

\[
\begin{align*}
A_{\pi^+ \pi^-} &= \frac{1 - |\xi_{\pi^+ \pi^-}|^2}{1 + |\xi_{\pi^+ \pi^-}|^2} \\
B_{\pi^+ \pi^-} &= \frac{2\text{Im}\xi_{\pi^+ \pi^-}}{1 + |\xi_{\pi^+ \pi^-}|^2},
\end{align*}
\]

(56)

where, in the isospin decomposition, we have that

\[
\xi_{\pi^+ \pi^-} = e^{-2i\phi_M} \eta_{22} \left(\frac{1 + \sqrt{2} \eta_{02}}{1 + \sqrt{2} \eta_{00}}\right) = \left(1 - \frac{\eta_{22}}{1 + \eta_{22}}\right) \left(\frac{1 + \sqrt{2} \overline{\eta}_{02}}{\overline{\eta}_{00}} \frac{1 - \overline{\eta}_{00}}{1 - \overline{\eta}_{22}}\right) \simeq e^{2i\alpha} \left(1 + \frac{z}{1 + \overline{z}}\right)
\]

(57)

with

\[
\eta_{I_1, I_2} = \frac{A(B^0 \to \pi \pi, I = I_1)}{A(B^0 \to \pi \pi, I = I_2)}, \quad \overline{\eta}_{I_1, I_2} = \frac{A(B_H \to \pi \pi, I = I_1)}{A(B_L \to \pi \pi, I = I_2)},
\]

(58)

being $I_1, I_2 \in \{0, 2\}$ the isospin states. In this formula, we have introduced the quantities

\[
\overline{z} = \sqrt{2} \frac{\eta_{02}}{\eta_{00}}, \quad z = \sqrt{2} \frac{\eta_{02}}{\eta_{22}}.
\]

(59)

For such a decay mode, the relevant $CP$–violating observable becomes $\text{Im}\xi_{\pi^+ \pi^-} = \sin 2\alpha + \Delta_{+-}$, where $\alpha$ in terms of the Wolfenstein $\rho - \eta$ parameters [1] is given by

\[
\alpha = \arg \left(-\frac{V_{td} V_{*tb}}{V_{ud} V_{*ub}}\right) = \arctan \left(\frac{\eta}{\rho(\rho - 1) + \eta^2}\right),
\]

(61)

and

\[
\Delta_{+-} \simeq -2R \left[\cos \delta_{12} \sin \alpha - \sin 2\alpha \cos(\alpha - \delta_{12})\right].
\]

(62)
However, experimentally, it may be difficult to measure the deviation $\Delta_{+-}$ accurately, because it requires the measurement of the difficult decay $B_d^0 \to \pi^0\pi^0$, whose branching ratio is expected to be very small ($\sim 10^{-6}$), due to color suppression [15]. Within the flavor and isospin invariance, it has been pointed out that the isospin relations for amplitude differences in $B \to \pi K$ [16] may improve the situation. Furthermore, the inclusion of the $B \to \pi\rho$ and other decay modes, can also remove the gluon penguin contamination and solve the two–fold ambiguity [17], by studying the full Dalitz plot and the time–dependence for $3\pi$ decay channels [18]. The reliability of all these approaches is limited by possible $SU(3)$ breaking effects originating, for instance, from the presence of electroweak penguins which become important due to the fact that the $Z^0$ exchange depends on the square of the top quark mass [19]. Indeed, without any deep insight into the world of the penguin operators and into other isospin breaking effects [20], we can simply state that the influence of $SU(3)$–breaking effects (especially electroweak penguins) on the extraction of $\alpha$ is rather controversial [20]. The deviation $\Delta_{+-}$ in Eq. (62) seems sizeable as large as 0.12 using only factorization approximation. However, it is expected to be much smaller, if we include the further constraint $|A(B^- \to \pi^-\pi^0)| = |A(B^+ \to \pi^+\pi^0)|$ coming from the study of the decay rates of the charged $B$ mesons [20].

A better understanding of the $CP$–violating effects in $B$ meson decays originate from a full–fledged knowledge of the interference of the tree level amplitude with higher order corrections to vertex and masses. Concluding the paper, however, it is worth noting that this picture is obscured by other intriguing effects which support our need to consider a model–independent analysis. We mean the problems of the relative strong phase $\delta_{12}$ and the peak-dip structure which emerge when we consider the exchange of two (or more) resonances. The latter resembles a $s$-channel interaction between the initial and final states, and can be described by means of a $q^2$–dependent width in the Breit–Wigner intermediate propagators [21]. The particle width effects of the intermediate propagators should be taken into account since they introduce a $CP$ odd contribution. In this context, it is also possible to consider the interference effects in the breaking of some relevant isospin relations induced by the intermediate $\rho$–$\omega$ mixing [22]. On the other side, the question of how to calculate the relative strong phase $\delta_{12}$ has been
discussed both at the quark and at hadronic level. At the quark level, the necessary strong phases depend on the absorptive parts of $t$-channel vertex corrections (HARD FSI). These contributions are provided by different loop effects based both on the loop quark–rescattering (time like penguins) [23], and on final state hadronization (space-like penguins) [24], in dependence of the gluonic momentum transfer. This quark level approach is related to the influence of virtual gluons in the form of the interactions, and therefore, it is intimately connected to CPT–invariance. In fact, the stringent CPT constraints can be evaded by a partial sum of final states. Also the factorization consistency [25] is complicated by the presence of the strong rescattering of the intermediate virtual states (with the same quark gluon content of the final decay modes) just because they violate the cancellations imposed by unitarity and CPT–invariances [26]. Therefore, the danger of a nonvanishing FSI cannot be avoided if the factorization ansatz cannot be done. At hadronic level, the plot thickens when we consider the inelastic $t$-channel interactions (SOFT FSI) which induce the mixing of final decay products. Usually, final coupled decay modes are considered by means of a parametrization of the strong $S$–rescattering matrix with the pomeron dominance and Regge model. The S-matrix of two coupled resonant channels in B-meson decays is probably negligible [27], but in general the phase shift effects depend strongly on the kinematical configuration of the particles [28]. Finally, we can say that although, $CP$–violation has been observed only in the $K^0 - \overline{K}^0$ complex system, large $CP$–violating effects are expected in the $B^0_d - \overline{B}^0_d$ system. As we mentioned, the charge asymmetry in semileptonic decays Eq. (22) is predicted to be very small, without the inclusion of new physical effects in $B^0 - \overline{B}^0$ mixing [7]. But, in the nonleptonic decays, the relative asymmetries may be large due to the interplay of mixing and decay amplitude. This problem was widely discussed in the case the hadronic final states $F$ being $CP$–eigenstates. The predictions for the partial decay rate asymmetries seem to put in evidence some $CP$ breaking effects which, presumably, are sensitive to the strong interactions. In this letter, we propose a scattering theory to describe the $CP$–violation and we provide a complete set of the rephasing invariant observables. In particular, our proposal becomes a valid tool to provide the characterization of the $CP$–violating effects in neutral $B$ decays in two pseudoscalar mesons, and, in the same time, it can parameterize the difficult problem of the penguin unreliabilities.
References

[1] D. Cocolicchio, “CP–asymmetries in B decays”, Proc. Advanced Study Conference on Heavy Flavours, ed. G. Bellini et al. (Ed. Frontieres, 1993), p. 367.

[2] W. F. Palmer and Y. L. Wu, Phys. Lett. B350 (1995) 245 and refs. therein.

[3] L. Lavoura, Ann. Phys. (NY) 207 (1991) 428; C. D. Buchanan, R. Cousin, C. Dib, R. Peccei and J. Quackenbush, Phys. Rev. D45 (1992) 4088.

[4] M. Hayakawa and A. I. Sanda, Phys. Rev. D48 (1993) 1150; M. Kobayashi and A. I. Sanda, Phys. Rev. Lett. 69 (1992) 3139; see also D. Cocolicchio, L. Telesca and M. Viggiano, e-preprint archive hep-ph/9709486.

[5] S. H. Aronson, G. J. Bock, H.-Y. Cheng and E. Fishbach, Phys. Rev. D28 (1983) 495.

[6] G. V. Dass and K. V. L. Sarma, Phys. Rev. D54 (1996) 5880.

[7] D. Cocolicchio and L. Maiani, Phys. Lett. B291 (1992) 155.

[8] CLEO Collaboration, J. Bartelt et al., Phys. Rev. Lett. 71 (1993) 1680; CDF Collaboration, F. Abe et al., Phys. Rev. D55 (1997) 2546; OPAL Collaboration, K. Ackerstaff et al., Z. Phys. C76 (1997) 401.

[9] L. Lavoura, Mod. Phys. Lett. A7 (1992) 1367; A. I. Sanda and Zhi-zhong Xing, Phys. Rev. D56 (1997) 6866; Dan-Di Wu, Prairie View A & M Univ. preprint PVAM-HEP-9-97; M. C. Banuls and J. Bernabeu, preprint CERN-TH/97-216.

[10] A. Acuto and D. Cocolicchio, Phys. Rev. D47 (1993) 3945.

[11] J. S. Bell and J. Steinberger, Proceedings Oxford Int. Conf. on Elementary Particles 1965, ed. R. G. Moorehouse et al. (Rutherford HEP Lab., Chilton, Didcot, Berkshire, England, 1966) p. 195.

[12] H. J. Lipkin, Y. Nir, H. R. Quinn and A. E. Snyder, Phys. Rev. D44 (1991) 1454.
[13] M. Ciuchini, E. Franco, G. Martinelli and L. Reina, Nucl. Phys. B415 (1994) 403; G. Buchalla, A. J. Buras and M. E. Lautenbacher, Rev. Mod. Phys. 68 (1996) 1125.

[14] M. Gronau and D. London, Phys. Rev. Lett. 65 (1990) 3381.

[15] G. Kramer and W. F. Palmer, Phys. Rev. D52 (1995) 6411.

[16] M. Gronau, J. L. Rosner and D. London, Phys. Rev. Lett. 73 (1994) 21; J. P. Silva and L. Wolfenstein, Phys. Rev. D49 (1994) 1151; N. G. Deshpande and X.-G. He, Phys. Rev. Lett. 75 (1995) 1703.

[17] R. Aleksan et al., Phys. Lett. B356 (1995) 95; Y. Grossman and H. R. Quinn, Phys. Rev. D 56 (1997) 7259 and refs. therein.

[18] R. Aleksan, I. Dunietz, B. Kayser and F. Le Diberdier, Nucl. Phys. B361 (1991) 141.

[19] R. Fleischer, Int. J. Mod. Phys. A12 (1997) 2459.

[20] N. G. Deshpande and X. G. He, Phys. Rev. Lett. 74 (1995) 26; M. Gronau, O. F. Hernández, D. London and J. L. Rosner, Phys. Rev. D52 (1995) 6374; R. Fleisher, Phys. Lett. B365 (1996) 399; N. G. Deshpande, X. G. He and S. Oh, Phys. Lett. B384 (1996) 283; R. Fleischer and T. Mannel, Phys. Lett. B397 (1997) 269; N. G. Deshpande, X. G. He and S. Oh, Z. Phys. C74 (1997) 359.

[21] A. Pilaftsis and M. Nowakowski, Int. J. Mod. Phys. A9 (1994) 1097, (E) 5849; G. Eilam, M. Gronau and R. Mendel, Phys. Rev. Lett. 74 (1995) 4984.

[22] R. Enomoto and M. Tanabashi, Phys. Lett. B386 (1996) 413; S. Gardner, H. B. O’Connell, A. W. Thomas, Phys. Rev. Lett. 80 (1998) 1834.

[23] J. M. Gerard and W. S. Hou, Phys. Rev. Lett. 62 (1989) 855; Phys. Rev. D43 (1991) 2909; H. Simma, G. Eilam and D. Wyler, Nucl. Phys. B352 (1991) 367.

[24] M. Tanimoto et al. Phys. Rev. D42 (1990) 252; D.-S. Du, M.-Z. Yang, D.-Z. Zhang, Phys. Rev. D53 (1996) 249.
[25] N. Cabibbo and L. Maiani, Phys. Lett. B73 (1978) 418; J. D. Bjorken, Nucl. Phys. (Proc. Suppl.) 11 (1989) 287.

[26] J. M. Soares, Phys. Rev. Lett. 79 (1997) 1166.

[27] M. Wanninger and L. M. Sehgal, Z. Phys. C50 (1991) 47; B. Blok, M. Gronau and J. Rosner, Phys. Rev. Lett. 78 (1997) 3999.

[28] M. Simonius and D. Wyler, Z. Phys. C42 (1989) 471; C. Greub, H. Simma and D. Wyler, Nucl. Phys. B434 (1995) 39.