Research Article

Hermite–Hadamard Inequalities for Harmonic \((s, m)\)-Convex Functions

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The objective of this article is to establish some Hermite–Hadamard-type inequalities via harmonic \((s, m)\)-convex functions in the framework of conformal fractional integral.

1. Introduction and Preliminaries

Let \(C \subset \mathbb{R}\) be an interval. Then, \(C\) is said to be convex, if
\[
(1 - t)u + tv \in C,
\]
holds \(\forall u, v \in C\) and \(t \in [0, 1]\).

Let \(C \subset \mathbb{R}\) be an interval. Then, a function \(f: C \to \mathbb{R}\) is said to be convex (concave), if
\[
f((1 - t)u + tv) \leq (\geq) (1 - t)f(u) + tf(v),
\]
holds for all \(u, v \in C\) and \(t \in [0, 1]\).

It can be easily seen in [1–7] that the convex (concave) functions have extensive applications in pure and applied mathematics, and in the literature [8–15], many eminent inequalities and other properties can be found in the framework of convexity. One of the renowned inequalities in the literature of Hermite–Hadamard Integral Inequality is given below:
\[
f\left(\frac{u + v}{2}\right) \leq \frac{1}{v - u} \int_{u}^{v} \frac{f(x)}{x} \, dx \leq \frac{f(u) + f(v)}{2}.
\]

These both inequalities hold in reverse if the function is concave. Now, the harmonic convex set is defined as follows.

**Definition 1.** Let \(C \subset \mathbb{R}\) be an interval. Then, \(C\) is said to be harmonic convex, if
\[
\frac{uv}{(1 - t)u + tv} \in C,
\]
holds \(\forall u, v \in C\) with \((u, v) \neq (0, 0)\) and \(t \in [0, 1]\).

Iscan [8] introduced the concept of harmonic convex function.

**Definition 2** (see [8]). Let \(C \subset \mathbb{R}\) be an interval. Then, a function \(f: C \to \mathbb{R}\) is called harmonic convex (concave), if
\[
f\left(\frac{uv}{(1 - t)u + tv}\right) \leq (\geq) tf(u) + (1 - t)f(v),
\]
holds for all \(u, v \in C\) with \((u, v) \neq (0, 0)\) and \(t \in [0, 1]\).

Iscan by using the concept of harmonic convex function gave a new refinement of Hermite–Hadamard inequality as
\begin{align}
f\left(\frac{2uv}{u+v}\right) \leq \frac{uv}{v-u} \int_u^v f(x) \, dx \leq \frac{f(u) + f(v)}{2}, \quad (6)
\end{align}

**Definition 3.** Let \( \mathbb{C} \subseteq R \) be an interval and \( s, m \in (0, 1) \). Then, a function \( f: \mathbb{C} \rightarrow R \) is called harmonic \((s, m)\)-convex (concave), if
\[
f\left(\frac{muv}{(1-t)u + mtv}\right) \leq (\geq) t^s f(u) + m(1-t)^s f(v), \quad (7)
\]
holds \( \forall u, v \in \mathbb{C} \) with \( (u, v) \neq (0, 0) \) and \( t \in [0, 1] \).

If \( s = m = 1 \), then harmonic \((s, m)\)-convex function becomes the classical harmonic convex function. So harmonic convex function is a special case of harmonic \((s, m)\)-convex function.

The main purpose of this article is to establish some conformable fractional estimates of Hermite–Hadamard–convex function. Before going further towards our main results, let us have a brief review of the previously well known concepts and results. These preliminaries will be highly helpful in acquiring the main results.

The eminent gamma and beta functions are defined as
\[
\Gamma(u) = \int_0^\infty e^{-t} t^{u-1} \, dt, \quad \text{for } u > 0,
\]
\[
\beta(u, v) = \int_0^1 t^{u-1} (1-t)^{v-1} \, dt = \frac{\Gamma(u) \Gamma(v)}{\Gamma(u+v)}, \quad \text{for } u, v > 0.
\]

(8)

The integral form of hypergeometric function is defined as
\[
_2F_1(u, v; w; z) = \frac{1}{\beta(v, w-v)} \int_0^1 t^{w-1} (1-t)^{v-1} (1-zt)^{-u} \, dt,
\]
for \( |z| \geq 1 \).

Now, if \( f \in L_1[u, v] \) with \( u \geq 0 \), then Riemann–Liouville integrals \( I_u^a f \) and \( I_v^a f \) of any positive order \( a \) are defined as
\[
I_u^a f(t) = \frac{1}{\Gamma(a)} \int_0^a (a-t)^{a-1} f(t) \, dt, \quad a > u,
\]
\[
I_v^a f(t) = \frac{1}{\Gamma(a)} \int_0^v (t-a)^{a-1} f(t) \, dt, \quad a < v.
\]

(10)

For more details, see [11].

Recently, Abdeljawad [16] introduced the notation of right and left conformable fractional integrals for any positive order \( \alpha \) as follows.

**Definition 4.** Let \( \alpha \in (n, n+1] \). Then, the left and right conformable fractional integrals starting from \( u \) of any positive order \( \alpha \) is given as
\[
I_u^\alpha f(t) = \frac{1}{n!} \int_u^t (t-a)^{\alpha-n-1} f(a) \, da,
\]
\[
I_v^\alpha f(t) = \frac{1}{n!} \int_t^v (a-t)^{\alpha-n-1} f(a) \, da.
\]

(11)

**Lemma 1.** Let \( f : \mathbb{C} = [a, b] \subseteq R \setminus \{0\} \rightarrow R \) be a harmonic \((s, m)\)-convex function such that \( f \in L_1[u, v] \) and \( s, m \in (0, 1] \). Then,
\[
\left(\frac{2uv}{u+v}\right) \leq \frac{1}{n!} \int_u^v f(x) \, dx \leq \frac{f(u) + f(v)}{2},
\]
\[
\leq \frac{1}{n!} \left\{ f(x) + m f\left(\frac{x}{m}\right) \right\}.
\]

(12)

**2. Main Results**

In this section, we will present our main results.

**Theorem 1.** Let \( f : \mathbb{C} = [a, b] \subseteq R \setminus \{0\} \rightarrow R \) be a harmonic \((s, m)\)-convex function for \( t = 1/2 \), we have
\[
\left(\frac{xy}{x+y}\right) \leq \frac{1}{n!} \left\{ f(x) + m f\left(\frac{y}{m}\right) \right\}.
\]

(14)

Put \( x = uv/\mathbb{t} + (1-t)v \) and \( y = uv/\mathbb{t} + (1-t)v \). Then,
\[
\left(\frac{2uv}{u+v}\right) \leq \frac{1}{n!} \left\{ f\left(\frac{uv}{tu + (1-t)v}\right) + m f\left(\frac{uv}{mtu + (1-t)v}\right) \right\},
\]
\[
\leq \frac{1}{n!} \left\{ \int_0^1 f\left(\frac{uv}{tu + (1-t)v}\right) \, dt + m \int_0^1 f\left(\frac{uv}{mtu + (1-t)v}\right) \, dt \right\}.
\]

(15)

We know that
\[
\int_0^1 \left(\frac{1}{2}\right)^s \left(\frac{uv}{tu + (1-t)v}\right) \, dt = \frac{uv}{v-u} \int_u^v \frac{f(x)}{x^2} \, dx.
\]
\[
\Rightarrow \quad f\left(\frac{2uv}{u+v}\right) \leq \frac{1}{n!} \left\{ \int_0^1 f\left(\frac{x}{u}\right) \, dx + m \int_0^1 f\left(\frac{x}{m}\right) \, dx \right\}.
\]

(16)
Now, consider a function \( f: \mathbb{R} \to \mathbb{R} \) such that \( f(x) = 0 \). Then,
\[
 f \left( \frac{mxy}{mty + (1-t)x} \right) = 0. \tag{17}
\]
Also,
\[
 t'f(x) + m(1-t)'f(y) = 0. \tag{18}
\]
So \( f \) is harmonic \((s, m)\)-convex, and also, we have
\[
 f \left( \frac{2uv}{u+v} \right) = 0, \tag{19}
\]
which implies that the inequality holds. \( \square \)

**Theorem 2.** Let \( \mathcal{C} = [u, v] \subset \mathbb{R} \setminus \{0\} \to \mathbb{R} \) be a harmonic \((s, m)\)-convex function such that \( f \in L_1[u, v] \), where \( s, m \in (0, 1] \). Then,
\[
 \frac{\Gamma x - n}{\Gamma x + 1} f \left( \frac{2uv}{u+v} \right) \leq \left( \frac{1}{2} \right)^s \left( \frac{uv}{v-u} \right)^a I_{\alpha}^1 (fog) \left( \frac{1}{u} \right) \tag{20}
\]
Here, fog is the composition function.

**Proof.** From inequality 1, we have
\[
 f \left( \frac{2xy}{x+y} \right) \leq \left( \frac{1}{2} \right)^s \left\{ f(x) + m f \left( \frac{y}{m} \right) \right\}. \tag{21}
\]
Put \( x = uv/2 + (1-t)v \) and \( y = uv/2 + (1-t)u \). Then,
\[
 f \left( \frac{2uv}{u+v} \right) \leq \left( \frac{1}{2} \right)^s \left\{ f(x) + m f \left( \frac{y}{m} \right) \right\} + m I_{\alpha}^1 (fog) \left( \frac{1}{v} \right). \tag{22}
\]
By using change of variable technique of integration, we have
\[
 J_1 = \frac{1}{n!} \int_0^1 t^n (1-t)^{a-n-1} f \left( \frac{uv}{tu + (1-t)v} \right) dt \tag{23}
\]
where \( f(x) = 1/x \), and
\[
 J_2 = \frac{m}{n!} \int_0^1 t^n (1-t)^{a-n-1} f \left( \frac{uv}{mtv + (1-t)u} \right) dt \tag{24}
\]
where \( f(x) = 1/x \), and
\[
 | f(u+v) + f(x) | \leq \frac{uv}{v-u} \int_x^u f(x) dx \leq \frac{uv(v-u)}{2^{1/q} q!} (\Omega_1 | f'(u) |^q + m \Omega_2 | f'(v/m) |^q)^{1/q}, \tag{25}
\]
where
\[ \Omega_1 = \int_0^1 \frac{t^s}{(tv + (1 - t)u)^2} \, dt = \frac{u^{-2}}{s + 1} \, F_1(2,s + 1; s + 2; 1 - \frac{v}{u}), \]
\[ \Omega_2 = \int_0^1 \frac{(1 - t)^s}{(tv + (1 - t)u)^2} \, dt = \frac{-u^{-2}}{s + 1} \, F_1(2, 1; s + 2; 1 - \frac{v}{u}). \]  

Proof. Hölder’s inequality and Lemma 1 implies that
\[ |f'(u)|^q \leq \frac{uv(v - u)}{2} \left( \int_0^1 \left( \frac{1}{(tv + (1 - t)u)^2} \right)^{1/q} \, dt \right)^{1 - 1/q} \left( \int_0^1 \left( \frac{1}{(tv + (1 - t)u)^2} \right) \, dt \right)^{1/q}. \]  

(29)

Since \(|f'|^q\) is harmonically \((s,m)\)-convex on \([u,v/m]\), we have
\[ \frac{uv(v - u)}{2} \left( \int_0^1 \frac{(1 - t)^s}{(tv + (1 - t)u)^2} \, dt \right)^{1/q} \left( \int_0^1 \frac{1}{(tv + (1 - t)u)^2} \, dt \right)^{1 - 1/q} + m v \Omega_1 f' \left( \frac{v}{m} \right)^q \frac{1}{1/q}. \]  

(30)

Here,
\[ \Omega_1 = \int_0^1 \frac{t^s}{(tv + (1 - t)u)^2} \, dt = \frac{u^{-2}}{s + 1} \, F_1(2,s + 1; s + 2; 1 - v/u), \]
\[ \Omega_2 = \int_0^1 \frac{(1 - t)^s}{(tv + (1 - t)u)^2} \, dt = \frac{-u^{-2}}{s + 1} \, F_1(2, 1; s + 2; 1 - v/u). \]  

(31)

\[ \square \]

Theorem 4. Let \( f : C = [u,v] \subset R \setminus \{0\} \longrightarrow R \) be differentiable on \( C \), \( u, v/m \in C, m \in [0, 1], \) and \( f' \in L[u,v]. \) If \(|f'|^q\) for \( q \geq 1 \) is harmonic \((s,m)\)-convex on \([u,v/m]\), then
\[ \frac{uv}{2} \left( \int_u^v \frac{f'(x)}{x^2} \, dx \right) \leq \frac{uv(v - u)}{2} \left( \frac{1}{u^2} \, F_1(2, 1; 2; 1 - \frac{v}{u}) \right)^{1 - 1/q} \left( \frac{uv}{m^2} \, F_1(2, 1; 2; 1 - \frac{v}{u}) \right)^{1/q}. \]  

where \( \Psi = s/(s + 1)(s + 2). \)

Proof. Hölder’s inequality and Lemma 1 implies that
\[ \frac{f(u) + f(v)}{2} - \frac{uv}{v - u} \int_u^v f(x) \, dx \leq \frac{uv(v - u)}{2} \int_0^1 \frac{1 - 2t}{(tv + (1 - t)u)^2} f' \left( \frac{uv}{tv + (1 - t)u} \right) dt \]

(33)

\[ \leq \frac{uv(v - u)}{2} \left( \int_0^1 \frac{dt}{(tv + (1 - t)u)^2} \right)^{1-1/q} \times \left( \int_0^1 (1 - 2t) \left| f' \left( \frac{uv}{tv + (1 - t)u} \right) \right|^q dt \right)^{1/q}. \]

Since \( |f'|^q \) is harmonically \((s,m)\)-convex on \([u,v/m]\), we have

\[ \leq \frac{uv(v - u)}{2} u^{-2} F_1 \left( 2, 1; 2; 1 - \frac{v}{u} \right)^{1-1/q} \times \left( \int_0^1 (1 - 2t) \left( t^{1/q} |f'(u)|^q + m(1 - t)^q \right)^{1/q} dt \right)^{1/q}, \]

(34)

where \( \Psi = s/(s + 1)(s + 2). \)

Theorem 5. Let \( f : I = [u, v] \subset \mathbb{R} \rightarrow R \) be differentiable on \( \mathbb{C}^n \), \( u, v/m \in \mathbb{C}, m \in [0,1], \) and \( f' \in L[u, v] \). If \( |f'|^q \) for \( q \geq 1 \) is harmonic \((s,m)\)-convex on \([u,v/m]\), then

\[ \left| \frac{f(u) + f(v)}{2} - \frac{uv}{v - u} \int_u^v f(x) \, dx \right| \leq \frac{uv(v - u)}{2} \left( \frac{1}{p + 1} \right)^{1/p} \times \left( \Phi_1 |f'(u)|^q + m \Phi_2 |f'(v/m)|^q \right)^{1/q}, \]

(35)

where

\[ \Phi_1 = \int_0^1 \frac{t^s}{(tv + (1 - t)u)^2} \, dt = \frac{u^{-2q}}{s + 1} F_1 \left( 2q, s + 1; s + 2; 1 - \frac{v}{u} \right), \]

\[ \Phi_2 = \int_0^1 \frac{(1 - t)^s}{(tv + (1 - t)u)^2} \, dt = \frac{u^{-2q}}{s + 1} F_1 \left( 2q, 1; s + 2; 1 - \frac{v}{u} \right). \]

Proof. Hölder's inequality and Lemma 1 implies that
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