Abstract

This paper presents an algorithmic method to study structural properties of nonlinear control systems in dependence of parameters. The result consists of a description of parameter configurations which cause different control-theoretic behaviour of the system (in terms of observability, flatness, etc.). The constructive symbolic method is based on the differential Thomas decomposition into disjoint simple systems, in particular its elimination properties.

1 Introduction

Symbolic computation allows to study many structural aspects of control systems, e.g., controllability, observability, input-output behaviour, etc. In contrast to a numerical treatment, the dependence of the results on parameters occurring in the system is accessible to symbolic methods.

An algebraic approach for treating nonlinear control systems has been developed during the last decades, cf., e.g., Fliess and Glad (1993), Diop (1991, 1992), Pommaret (2001), and the references therein. In particular, the notion of flatness has been studied extensively and has been applied to many interesting control problems (cf., e.g., Fliess et al. (1995)). The approach of Diop builds on the characteristic set method (cf. Kolchin (1973), Wu (2000)). The Rosenfeld-Gröbner algorithm (cf. Boulier et al. (2009)) can be used to perform the relevant computations effectively; cf. also Wang (2001) for alternative approaches.

Dependence of control systems on parameters has been examined, in particular, in Pommaret and Quadrat (1997), Pommaret (2001). In the case of linear systems, stratifications of the space of parameter values have been studied using Gröbner bases in Levandovskyy and Zerz (2007).

In the 1930s the American mathematician J. M. Thomas designed an algorithm which decomposes a polynomially nonlinear system of partial differential equations into so-called simple systems. The algorithm uses, in contrast to the characteristic set method, inequalities to provide a disjoint decomposition of the solution set (cf. Thomas (1937)). It precedes Kolchin (1973) and Seidenberg (1956) (building on Ritt (1950)). Recently a new algorithmic approach to the Thomas decomposition method has been developed (cf. Gerdt (2008); Bächler et al. (2012); Robertz (2012)), also using ideas of M. Janet (cf. Janet (1929)); cf. also Wang (1998) for an earlier implementation of the algebraic part.

So far the dependence of nonlinear control systems on parameters has not been studied by such a rigorous method as the Thomas decomposition. This paper demonstrates how the Thomas decomposition method can be applied in this context. In particular, the Thomas decomposition can detect certain structural properties of control systems by performing elimination and it can separate singular cases of behaviour in control systems from the generic case due to splitting into disjoint solution sets.

In Section 2 we sketch the Thomas decomposition method for algebraic and differential systems. The algorithm for the differential case builds on the al-




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gebraic part. Section 3 describes how the differential Thomas decomposition can be used to solve elimination problems that occur in our study of nonlinear control systems. In Section 4 we recall the notions of observability and flatness, which are addressed in the examples in Section 5 using our Maple implementation.

2 Thomas Decomposition

2.1 Simple Algebraic Systems

Let $K$ be a field of characteristic zero. We denote by $R$ the polynomial algebra $K[x_1, \ldots, x_n]$ and we fix a total ordering $<$ on $\{x_1, \ldots, x_n\}$. Then the greatest variable with respect to $<$ which occurs in a non-constant polynomial $p \in R$ is called the leader of $p$ and is denoted by $\text{ld}(p)$.

In what follows, we regard every $p \in R \setminus K$ as a polynomial in $\text{ld}(p)$ with coefficients in $K[x_i \mid 1 \leq i \leq n, x_i < \text{ld}(p)]$, and, recursively, each coefficient as a polynomial in its leader, etc. The coefficient of the highest power of $\text{ld}(p)$ in $p$ is called the initial of $p$, denoted by $\text{init}(p)$. Moreover, we denote the discriminant of $p$ (as a polynomial in $\text{ld}(p)$) by $\text{disc}(p)$. Both $\text{init}(p)$ and $\text{disc}(p)$ are elements of the polynomial ring $K[x_i \mid 1 \leq i \leq n, x_i < \text{ld}(p)]$.

Let $K$ be an algebraic closure of $K$. In this subsection we consider (finite) algebraic systems (of equations and inequations, defined over $K$)

$$S = \{p_1 = 0, \ldots, p_s = 0, q_1 \neq 0, \ldots, q_t \neq 0\},$$

where $p_i, q_j \in R$, $s, t \in \mathbb{Z}_{\geq 0}$. The set of solutions of $S$ (in $K^n$) is defined as

$$\text{Sol}_K(S) = \{a \in K^n \mid p_i(a) = 0, q_j(a) \neq 0 \text{ for all } i, j\}.$$

Finally, for $0 \leq k < n$, let $\pi_k : K^n \to K^{n-k}$ be the projection onto the coordinate subspace of dimension $n-k$ whose coordinates are ranked lowest with respect to $<$, i.e., $\pi_k(a_1, \ldots, a_n) = (a_{i_j} \mid 1 \leq j \leq n-k)$, where $1 \leq i_1 < \ldots < i_{n-k} \leq n$ and $x_{i_1}, \ldots, x_{i_{n-k}}$ are the smallest $n-k$ elements of $\{x_1, \ldots, x_n\}$ with respect to $<$. 

Definition 1. An algebraic system $S$ as in (A) is simple (with respect to $<$) if the following conditions hold.

1. $p_i \notin K, q_j \notin K$ for all $i$ and $j$.
2. $|\{\text{ld}(p_1), \ldots, \text{ld}(p_s), \text{ld}(q_1), \ldots, \text{ld}(q_t)\}| = s + t$.
3. For every $r \in \{p_1, \ldots, p_s, q_1, \ldots, q_t\}$, if $\text{ld}(r)$ is the $k$-th greatest variable w.r.t. $<$, then neither $\text{init}(r) = 0$ nor $\text{disc}(r) = 0$ has a solution in $\pi_k(\text{Sol}_K(S))$.  


Remark 2. Condition (2) implies that the leaders of the left hand sides of the equations and inequations in a simple algebraic system $S$ are pairwise distinct, i.e., $S$ is a triangular set (cf., e.g., Aubry et al. (1999); Hubert (2003); Wang (2001); cf. Lemaire et al. (2005) for a related implementation).

The meaning of condition (3) is that for every $1 \leq k < n$ and each $\langle a_1, \ldots, a_{n-k} \rangle \in \pi_k(\text{Sol}_K(S))$ there exists $a \in K$ such that $\langle a_1, \ldots, a_{n-k}, a \rangle \in \pi_{k-1}(\text{Sol}_K(S))$, and the number of possible values for $a$ is either finite or co-finite, determined by the degree of the leader in the corresponding equation or inequation in $S$, if any (cf. also Plesken (2009)).

Definition 3. Let $S$ be an algebraic system as in (A). A Thomas decomposition of $S$ (with respect to $<$) consists of finitely many simple algebraic systems $S_1, \ldots, S_k$ such that $\text{Sol}_K(S)$ is the disjoint union of $\text{Sol}_K(S_1), \ldots, \text{Sol}_K(S_k)$.

Remark 4. Euclidean pseudo-division and splitting of systems allow to compute a Thomas decomposition for any given (finite) algebraic system (defined over a computable field $K$ of characteristic zero) in finitely many steps.

In each round the algorithm chooses a system $S$ from a set of systems to be processed. If $S$ contains an equation whose left hand side is a non-zero constant or an inequation with zero left hand side, then $S$ is discarded because its solution set is empty. Moreover, we assume that equations $0 = 0$ and inequations $k \neq 0$, where $k \in K \setminus \{0\}$ are removed from each system, so that now the leader of every equation and inequation is well-defined.

The algorithm applies Euclidean pseudo-division to each pair $p_1 = 0$, $p_2 = 0$ of distinct equations in $S$ with $\text{ld}(p_1) = \text{ld}(p_2) =: x$; i.e., if $\deg_x(p_1) \geq \deg_x(p_2)$, then polynomial division is performed on $c \cdot p_1$ and $p_2$, where $c$ is a suitable power of $\text{init}(p_2)$. In order not to change the solution set, when $p_1 = 0$ is replaced with the result of the polynomial division, non-vanishing of $\text{init}(p_2)$ on the set of solutions of $S$ is assumed. To this end, a preparatory step splits a system into two and adds $\text{init}(p_2) \neq 0$ and $\text{init}(p_2) = 0$, respectively, if necessary.

Each pair $q_1 \neq 0$, $q_2 \neq 0$ of distinct inequations in $S$ with $\text{ld}(q_1) = \text{ld}(q_2)$ is replaced with $\text{lcm}(q_1, q_2) \neq 0$. The computation of the least common multiple depends on case splittings according to vanishing of initials, similar to the previous remarks.

For each pair $p = 0$, $q \neq 0$ in $S$ with $\text{ld}(p) = \text{ld}(q)$, the algorithm computes $r := \text{gcd}(p, q)$ using Euclidean pseudo-division. This process again relies on non-vanishing initials, which is ensured by distinguishing cases appropriately in advance. If $p$ divides $q$, then $S$ has no solution and is discarded. If $r$ is a non-zero constant, then $q \neq 0$ is removed from $S$. Else, $p = 0$ and $q \neq 0$ are replaced with $p/r = 0$ and $q/r \neq 0$, respectively.

Finally, there is some flexibility when to take care of non-vanishing discriminants. Since $K$ is of characteristic zero, this property is established for any equation $r = 0$ or inequation $r \neq 0$ by examining $g := \text{gcd}(r, \frac{\partial r}{\partial q})$, again with appropriate case distinctions, and thus determining the square-free part $r/g$.  

3
We refer to Bächler et al. (2012); Plesken (2009); Robertz (2012) for more information and to Bächler and Lange-Hegermann (2012) for an implementation in Maple. In practice, we apply subresultants for the computation of gcds and the related case distinctions.

2.2 Simple Differential Systems

Let $K$ be a differential field of characteristic zero with pairwise commuting derivations $\delta_1, \ldots, \delta_n$, i.e., $K$ is a field of characteristic zero, and each $\delta_i$ is a map $K \to K$ satisfying $\delta_i(k_1 + k_2) = \delta_i(k_1) + \delta_i(k_2)$ and the Leibniz rule $\delta_i(k_1 k_2) = k_1 \delta_i(k_2) + \delta_i(k_1) k_2$ for all $k_1, k_2 \in K$.

In this subsection we denote by $R$ the differential polynomial ring $K\{u_1, \ldots, u_m\}$ in the differential indeterminates $u_1, \ldots, u_m$ with pairwise commuting derivations $\partial_1, \ldots, \partial_n$ whose restrictions to $K$ are $\delta_1, \ldots, \delta_n$, i.e., $R = K[(u_i)_j \mid 1 \leq i \leq m, J \in (Z_{\geq 0})^n]$ is the polynomial ring in the algebraically independent variables $(u_i)_j$, and $\partial_j : R \to R$ is the derivation defined by extending $\partial_j((u_i)_j) := (u_i)_{j+1}$ additively to $R$ such that it satisfies the Leibniz rule on $R$ and restricts to $\delta_j$ on $K$. Here $1_j$ is the multi-index of length $n$ whose $i$-th entry is $1$ if $i = j$ and $0$ otherwise.

Each variable $(u_i)_j$ is thought of as representing the partial derivative, corresponding to the multi-index $J$, of a smooth (or rather analytic) $K$-valued function of $n$ arguments. When dealing with differential polynomials $p \in R$ algorithmically, we compare terms with respect to a ranking, which is defined next. We agree that applying derivations should make terms larger with respect to the ranking. This is taken into account as follows.

**Definition 5.** A ranking on $R$ is a total ordering $<$ on $\{(u_i)_j \mid 1 \leq i \leq m, J \in (Z_{\geq 0})^n\}$ such that $u_j < \partial_j u_i$ for all $i$ and $j$, and such that $(u_i)_j < (u_i)_k$ implies $\partial_j(u_i)_j < \partial_j(u_i)_k$ for all $j, i_1, i_2, J_1, J_2$.

Every ranking is a well-ordering, i.e., there exist no infinitely decreasing sequences of variables $(u_i)_j$.

We define $\partial^J := \partial^{J_1} \ldots \partial^{J_n}$, $J \in (Z_{\geq 0})^n$, and write $|J| := J_1 + \ldots + J_n$ for the length of the multi-index $J$.

**Example 6.** The degree-reverse lexicographical ranking on the differential polynomial ring $K\{u\}$ (i.e., $m = 1$) with pairwise commuting derivations $\partial_1, \ldots, \partial_n$ is defined by

$$u_j < u_{j'} \iff \bigl\{ |J| < |J'| \text{ or } (|J| = |J'| \text{ and } J \neq J') \text{ and } J_i > J_i' \text{ for } i = \max\{1 \leq k \leq n \mid J_k \neq J_k'\}\bigr\}.$$  

In this example, we have $\partial_n u < \partial_{n-1} u < \ldots < \partial_1 u$. There are in fact $n!$ different degree-reverse lexicographical rankings according to the ordering of the $\partial_i u$.

In this subsection we consider (finite) differential systems (i.e. systems of differential equations and inequations)

$$S = \{p_1 = 0, \ldots, p_s = 0, q_1 \neq 0, \ldots, q_t \neq 0\}, \quad (D)$$
where \( p_i, q_j \in R, s, t \in \mathbb{Z}_{\geq 0} \).

In what follows, we assume that a ranking \(<\) on \( R \) is fixed. Carrying over the concepts of Subsection 2.1 the leader \( \text{ld}(p) \) of \( p \in R \setminus K \) is defined to be the greatest variable with respect to \(<\) which occurs in \( p \). The initial of \( p \), denoted by \( \text{init}(p) \), is the coefficient of the highest power of \( \text{ld}(p) \) in \( p \), and the separant of \( p \), denoted by \( \text{sep}(p) \), is the formal partial derivative of \( p \) with respect to \( \text{ld}(p) \). The initial of \( p \) is an element of the polynomial ring \( K[[u_i] \mid 1 \leq i \leq m, J \in (\mathbb{Z}_{\geq 0})^n, (u_i)_J < \text{ld}(p)] \).

In order to ensure formal integrability for the kind of differential systems we are heading for, we apply the concept of Janet division (cf. [Janet (1929); Gerdt and Blinkov (1998)], which restricts the usual divisibility relation on the free commutative monoid \( \Theta \) generated by \( \partial_1, \ldots, \partial_n \).

**Definition 7.** Let \( M \subset (\mathbb{Z}_{\geq 0})^n \) be finite. Janet division associates with each \( I \in M \) a partition of \( \{\partial_1, \ldots, \partial_n\} \) into the subsets of admissible and non-admissible derivations as follows. Let \( I = (I_1, \ldots, I_n) \in M \). Then \( \partial_i \) is admissible for \( I \) if and only if
\[
I_i = \max\{I'_i \mid (I'_1, \ldots, I'_n) \in M, I'_j = I_j \text{ for all } 1 \leq j < i\};
\]
otherwise \( \partial_i \) is non-admissible for \( I \). We denote by \( \mu(I, M) \) the set of derivations that are admissible for \( I \) (with respect to \( M \)), \( \mathcal{P}(I, M) := \{\partial_1, \ldots, \partial_n\} \setminus \mu(I, M) \), and
\[
\Theta(I, M) := \{\partial^I \mid J \in (\mathbb{Z}_{\geq 0})^n, J_j = 0 \text{ if } \partial_j \in \mathcal{P}(I, M)\}.
\]

Janet division defines for each \( \partial^I \) with \( I \in M \) a cone of multiples \( \Theta(I, M)\partial^I \) such that each two cones are disjoint. By enlarging \( M \) appropriately, these cones cover the set of all multiples of \( \partial^I, I \in M \).

**Definition 8.** A finite subset \( M \) of \( (\mathbb{Z}_{\geq 0})^n \) is said to be Janet-complete if \( \bigcup_{I \in M} \Theta\partial^I = \bigcup_{I \in M} \Theta(I, M)\partial^I \).

**Definition 9.** Let \( p_1, \ldots, p_r \in R \setminus K \), \( \text{ld}(p_j) = \partial^{I_j} u_{i_j} \), be such that each \( M_k = \{I_j \mid 1 \leq j \leq r, i_j = k\} \) is Janet-complete. A differential polynomial \( p \in R \) is Janet-reduced modulo \( p_1, \ldots, p_r \) if, for all \( 1 \leq j \leq r \), \( \deg_x(p) < \deg_x(p_j) \), where \( x := \text{ld}(p_j) \), and no variable \( \partial^I \text{ld}(p_j) \), where \( \partial^I \in \Theta(I_j, M_{i_j}), |I| > 0 \), occurs in \( p \).

**Remark 10.** Any \( p \in R \) can be transformed into a differential polynomial which is Janet-reduced modulo \( p_1, \ldots, p_r \) by applying Euclidean pseudo-division modulo \( p_1, \ldots, p_r \) and their derivatives repeatedly. The obvious strategy is to first eliminate the greatest variable \( \partial^I \text{ld}(p_j) \) with respect to \(<\) that occurs in \( p \) and to proceed to lower variables. If \( \partial^I \text{ld}(p_j) \) is a proper derivative of \( \text{ld}(p_j) \), then, before reducing modulo \( p_j \), the pseudo-division multiplies \( p \) with \( \text{sep}(p_j) \) (which, in fact, is the coefficient of the leader of any proper derivative of \( p_j \)). If \( |J| = 0 \), then the pseudo-division multiplies \( p \) with \( \text{init}(p_j) \). As in Subsection 2.4.
cases of vanishing initials or separatons have to be examined separately, which
denote the result of the pseudo-reduction by NF(p, {p_1, \ldots, p_r}).

**Definition 11.** In the situation of Definition 9, \{p_1, \ldots, p_r\} is passive if, for all
1 \leq j \leq r, NF(d p_j, \{p_1, \ldots, p_r\}) = 0 for all d \in \pi(I_j, M_j).

**Definition 12.** We call a differential system S as in (D) simple (with respect to <) if the following conditions hold.

1. S is simple as an algebraic system (in the finitely many variables (u_i)_J, which occur in S, totally ordered by the ranking <).
2. \{p_1, \ldots, p_s\} is passive.
3. q_1, \ldots, q_t are Janet-reduced modulo p_1, \ldots, p_s.

From now on we assume that K is a differential field of complex meromorphic functions on a connected open subset \Omega of \mathbb{C}^n with coordinates z_1, \ldots, z_n, and that the derivations \delta_1, \ldots, \delta_n are defined by partial differentiation with respect to z_1, \ldots, z_n, respectively. Differential equations p = 0 (and their derivatives) translate into algebraic equations for the Taylor coefficients of a power series expansion of u_1, \ldots, u_m around an arbitrary point in \Omega. A differential inequation q \neq 0 can be interpreted as the disjunction of algebraic inequations for all Taylor coefficients of the analytic expression that is obtained from q by substitution of power series expansions for u_1, \ldots, u_m.

In considering complex analytic functions on \Omega as solutions to differential systems, we assume that \Omega is chosen appropriately with regard to given differential systems, more precisely, that for every problem instance treated by the Thomas decomposition method, for each of the resulting simple differential systems there exists an analytic solution on \Omega. Which subsets \Omega of \mathbb{C}^n are appropriate can often be decided only after the formal treatment of the given differential systems by the Thomas algorithm. Questions concerning the radius of convergence of analytic solutions are ignored here.

For any differential system S as in (D) we denote by Sol_{\Omega}(S) the set of complex analytic functions f : \Omega \to \mathbb{C} satisfying p_i(f) = 0 and q_j(f) \neq 0 for all i and j.

**Definition 13.** Let S be a differential system as in (D). We call a family of finitely many simple differential systems S_1, \ldots, S_k such that Sol_{\Omega}(S) is the disjoint union of the solution sets Sol_{\Omega}(S_1), \ldots, Sol_{\Omega}(S_k) a Thomas decomposition of S (with respect to <).
Remark 14. Given any (finite) differential system (with coefficients in a computable differential subfield of \(K\)), a Thomas decomposition into simple differential systems can be computed in finitely many steps by a process which interweaves the algebraic Thomas algorithm (cf. Remark 4) and Janet reduction (cf. Remark 10).

Theorem 15. (cf. Robertz 2012) Let \(S\) as in (D) be a simple differential system, \(E\) the differential ideal of \(R\) generated by \(p_1, \ldots, p_s\), and let \(q\) be the product of all \(\text{init}(p_i)\), \(\text{sep}(p_i)\). Then

\[
E : q^\infty := \{ p \in R \mid q^r \cdot p \in E \text{ for some } r \in \mathbb{Z}_{\geq 0} \}
\]

is a radical differential ideal, which consists of all differential polynomials in \(R\) vanishing on \(\text{Sol}_\Omega(S)\). Given \(p \in R\), we have \(p \in E : q^\infty\) if and only if \(\text{NF}(p, \{p_1, \ldots, p_s\}) = 0\).

We refer to Bächler et al. 2012; Gerdt 2008; Robertz 2012 for more information and to Bächler and Lange-Hegermann 2012 for an implementation in Maple.

3 Elimination

We continue to use the same notation as in the previous section. Our objective is to perform a projection of the solution set of a differential system onto the space which is addressed by only certain of the components of the solution tuples. In other words, we would like to determine all differential consequences of the given system involving selected differential indeterminates only.

Definition 16. Let \(B_1, \ldots, B_k\) form a partition of the set of differential indeterminates \(\{u_1, \ldots, u_m\}\). The block ranking on \(R\) with blocks \(B_1, \ldots, B_k\) is defined as follows, where \(u_{i_1} \in B_{j_1}, u_{i_2} \in B_{j_2}, J_1, J_2 \in (\mathbb{Z}_{\geq 0})^n\):

\[
\partial^{J_1} u_{i_1} < \partial^{J_2} u_{i_2} \iff j_1 > j_2 \text{ or } (j_1 = j_2 \text{ and } \partial^{J_1} < \partial^{J_2}).
\]

The comparison of \(\partial^{J_1}\) and \(\partial^{J_2}\) is defined by a choice of a degree-reverse lexicographical ordering as in Example 6. Such a ranking is said to satisfy \(B_1 \gg B_2 \gg \ldots \gg B_k\).

Using the above notation, for any \(1 \leq i \leq k\), we define \(K\{B_i, \ldots, B_k\} := K\{u \mid u \in B_i \cup \ldots \cup B_k\} \subseteq R\).

Proposition 17. (cf. Robertz 2012) In the situation of Theorem 15, suppose that \(<\) is a block ranking with blocks \(B_1, \ldots, B_k\). For \(1 \leq i \leq k\), let \(E_i\) be the differential ideal of \(K\{B_i, \ldots, B_k\}\) generated by \(\{p_1, \ldots, p_s\} \cap K\{B_i, \ldots, B_k\}\), and let \(q_i\) be the product of all initials and separatants of the elements in this intersection. Then, for every \(1 \leq i \leq k\),

\[
(E : q^\infty) \cap K\{B_i, \ldots, B_k\} = E_i : q_i^\infty.
\]
Hence, computing a Thomas decomposition with respect to a block ranking enables us to extract generating sets for the differential consequences satisfied by the projection of the solution set of the given differential system.

For any differential system \( S \) let \( S^\square \) (resp. \( S^\neq \)) denote the set of left hand sides of equations (resp. inequations) in \( S \).

**Corollary 18.** (cf. Robertz (2012)) Let \( S \) be a (finite, not necessarily simple) differential system, and let \( S_1, \ldots, S_r \) be a Thomas decomposition of \( S \) with respect to a block ranking with blocks \( B_1, \ldots, B_k \). Let \( E \) be the differential ideal of \( R \) generated by \( S^\square \) and define the product \( q \) of all elements of \( S^\neq \). Let \( i \in \{1, \ldots, k\} \). For \( 1 \leq j \leq r \), let \( E_j \) be the differential ideal of \( K\{B_1, \ldots, B_k\} \) generated by \( S^\square_j \cap K\{B_1, \ldots, B_k\} \), and let \( q_j \) be the product of all initials and separants of the elements in this intersection. Then,

\[
\sqrt{E : q^\infty} \cap K\{B_1, \ldots, B_k\} = (E_1 : q_1^\infty) \cap \ldots \cap (E_r : q_r^\infty).
\]

### 4 Systems Theory

In this section we adapt well-known concepts of nonlinear control theory to our framework (cf., e.g., Glad (1989); Diop (1992); Fliess and Glad (1993); Fliess et al. (1995); Pommaret (2001) and the references therein).

Let \( R \) be the differential polynomial ring \( K\{u_1, \ldots, u_m\} \) in the differential indeterminates \( u_1, \ldots, u_m \) with pairwise commuting derivations \( \partial_1, \ldots, \partial_n \) as in Subsection 2.2.

We assume that a (nonlinear) control system is given by a simple differential system \( S \) as in (1). Let \( E \) be the differential ideal of \( R \) generated by \( p_1, \ldots, p_s \), and let \( q \) be the product of all \( \text{init}(p_i), \text{sep}(p_i) \). Let \( U := \{u_1, \ldots, u_m\} \). (No distinction is made a priori between input, output, state variables, etc.)

**Definition 19.** Let \( x \in U \) and \( Y \subseteq U \setminus \{x\} \). Then \( x \) is **observable with respect to** \( Y \) if there exists \( p \in (E : q^\infty) \setminus \{0\} \) such that \( p \) is a polynomial in \( x \) (not involving any proper derivative of \( x \)) with coefficients in \( K\{Y\} \) and such that neither its leading coefficient nor \( \frac{\partial p}{\partial x} \) is in \( E : q^\infty \).

**Remark 20.** Given the components of a solution to \( S \) corresponding to the variables in \( Y \), the implicit function theorem yields the component corresponding to \( x \), if there exists a polynomial \( p \) as in Definition 19. If \( S \) is simple with respect to a block ranking \( U \setminus (Y \cup \{x\}) \gg \{x\} \gg Y \) by Corollary 18 there exists such a \( p \in (E : q^\infty) \setminus \{0\} \) if and only if such a \( p \) exists in \( S^\square \cap K\{Y, x\} \). Otherwise, one has to compute a Thomas decomposition of \( S \) with respect to such a ranking and inspect each simple system.

**Definition 21.** Let \( Y \subseteq U \). Then \( Y \) is called a **flat output** if \( (E : q^\infty) \cap K\{Y\} = \{0\} \) and every \( x \in U \setminus Y \) is observable with respect to \( Y \). The control system given by \( S \) is said to be **flat** if a flat output exists.
Remark 22. Whereas, again, computation of a differential Thomas decomposition of \( S \) with respect to a block ranking satisfying \( U \setminus Y \gg Y \) allows to decide whether \( Y \) is a flat output, deciding whether a given nonlinear control system is flat is a difficult problem in general.

Many further applications of the Thomas decomposition method to systems theory (e.g., computation of the input-output behaviour from a state space representation, parameter identification, realization, inversion) can be realized and will be studied in the future (cf. also Diop (1992)).

5 Applications

In this section we demonstrate the Thomas decomposition method on two examples.

Example 23. As an application we consider the model of a continuous stirred-tank reactor taken from Kwakernaak and Sivan (1972).

This model describes a tank with a dissolved material of concentration \( c \), which is assumed to be the same everywhere in the tank due to stirring. Two input feeds with flow rates \( F_1 \) and \( F_2 \) feed this material into the tank with constant concentrations \( c_1 \) and \( c_2 \), respectively. There exists an outward flow with a flow rate proportional to the square root of the volume \( V \) of liquid in the tank. The system is modelled by the two differential equations

\[
\dot{V}(t) = F_1(t) + F_2(t) - k \sqrt{V(t)} \\
\dot{c}(t)V(t) = c_1 F_1(t) + c_2 F_2(t) - c(t) k \sqrt{V(t)} ,
\]

for an experimental constant \( k \).

In the following we want to describe the properties of the system in dependence of the constants \( c_1 \) and \( c_2 \). The Thomas algorithm performs this analysis if we model these constants as functions \( c_1(t) \) and \( c_2(t) \) satisfying the differential equations \( \dot{c_1}(t) = 0 \) and \( \dot{c_2}(t) = 0 \). Additionally, the square root \( \sqrt{V(t)} \) of \( V(t) \) appears in the equations. Our formalism cannot handle roots of functions directly; instead, we introduce \( \sqrt{V(t)} \) as new differential indeterminate and substitute \( V(t) \) by \( (\sqrt{V(t)})^2 \). To exclude trivial cases, we assume \( c_1(t) \neq 0 \), \( c_2(t) \neq 0 \), and \( V(t) \neq 0 \).

We apply our implementation (cf. Bächler and Lange-Hegermann (2012)). The command ComputeRanking sets the ranking. Its first parameter is the list of independent variables and the second parameter is the list of differential indeterminates; a list of lists of differential indeterminates indicates a block ranking. We input a derivative of a differential indeterminate as the name of the differential indeterminate indexed by its order; for example \( sV[1] \) stands for \( \frac{d}{dt} \sqrt{V(t)} \). The main command DifferentialThomasDecomposition computes a differential Thomas decomposition for a given list of equations and a list of inequations with respect to the fixed ranking.
In the following we compute a Thomas decomposition of the system using a ranking with \( F_1, F_2 \gg \sqrt{V}, c \gg c_1, c_2 \).

```plaintext
ivar:=\{t\};
dvar:=\{[F1,F2],[sV,c],[c1,c2]\};

ComputeRanking(ivar,dvar);

L:=\{2*sV[1]*sV[0]-F1[0]-F2[0]+k*sV[0],
       c[1]*sV[0]^2-c2[0]*F2[0]+c[0]*k*sV[0],
       -c1[0]*F1[0]+2*c[0]*sV[1]*sV[0],
       c1[1], c2[1]\}:
res:=DifferentialThomasDecomposition(L,[sV[0],c1[0],c2[0]]);
```

This yields a decomposition consisting of three simple systems. We print the first system; for better legibility we have underlined the leaders of equations.

```plaintext
subs(sV(t)=sqrt(V(t)),
     PrettyPrintDifferentialSystem(res[1]));
```

\[
\begin{align*}
[c_2(t) - c_1(t)]F_1(t) + \left(\frac{d}{dt}c(t)\right)\left(\sqrt{V(t)}\right)^2 \\
+ (c(t) - c_2(t)) \left(2 \frac{d}{dt}\sqrt{V(t)} + k\right)\sqrt{V(t)} = 0, & \quad (1) \\
(c_1(t) - c_2(t))F_2(t) + \left(\frac{d}{dt}c(t)\right)\left(\sqrt{V(t)}\right)^2 \\
+ (c(t) - c_1(t)) \left(2 \frac{d}{dt}\sqrt{V(t)} + k\right)\sqrt{V(t)} = 0, & \quad (2) \\
\frac{d}{dt}\sqrt{V(t)} = 0, \quad \frac{d}{dt}c_2(t) = 0, & \quad (3) \\
c_1(t) - c_2(t) \neq 0, \quad c_2(t) \neq 0, \quad c_1(t) \neq 0
\end{align*}
\]

The equations (1) and (2) allow us to solve for \( F_1(t) \) and \( F_2(t) \) given any \( c(t) \) and \( V(t) \). Thus, we consider \( c(t) \) and \( V(t) \) a flat output of the system under the additional condition \( c_1 - c_2 \neq 0 \) on the constants. Note that the two equations \( \dot{c}_1(t) = 0 \) and \( \dot{c}_2(t) = 0 \) in (3) just model the parameters \( c_1 \) and \( c_2 \) as constants.

The other two systems of this decomposition have the additional condition \( c_1 = c_2 \) on the constants. This condition prohibits to control the concentration in the tank as both input feeds are equivalent. Thus, these systems do not admit \( c(t) \) and \( V(t) \) as a flat output.

Now we turn our attention to the observability of \( \sqrt{V(t)} \) using Remark 20. Therefore, we choose a ranking with \( \sqrt{V} \gg \{c, F_1, F_2\} \gg \{c_1, c_2\} \). A Thomas decomposition with this ranking consists of seven systems:
ComputeRanking(ivar, dvar);

res := DifferentialThomasDecomposition(L, [sV[0], c1[0], c2[0]]);

res := [DifferentialSystem, DifferentialSystem, DifferentialSystem, DifferentialSystem, DifferentialSystem, DifferentialSystem]

In the first two systems an equation with \(\sqrt{V}(t)\) as leader appears and thus \(\sqrt{V}(t)\) is observable. For the first system the condition on the parameters for observability is

\[(c(t) - c_1)F_1(t) + (c(t) - c_2)F_2(t) \neq 0.\]

The second system is not physically feasible as it involves negative input feeds due to \(F_2(t) \neq 0\) and \(F_1(t) = -F_2(t)\).

The other five systems include an equation with \(\frac{d}{dt}\sqrt{V}(t)\) as leader and thus \(\sqrt{V}(t)\) is not observable. This follows for the first of these systems as it contains the Wronskian \(F_1(t)\dot{F}_2(t) - F_2(t)\dot{F}_1(t) = 0\) which makes the inputs linearly dependent. In the second system all three concentrations \(c(t), c_1, c_2\) are equal and constant. In the third system one input feed is zero and the concentration in the tank is equal to the concentration in the other input feed. The last two of these systems are not physically feasible because of negative values, as above.

**Example 24.** Now we consider a system of partial differential equations from Pommaret and Quadrat [1997]. The system consists of three linear pdes in \(\xi_i(z) = \xi_i(x_1, x_2, x_3)\) for \(i = 1, 2, 3\) with a parametric function \(a(x_2)\):

\[
0 = -a(x_2) \frac{\partial}{\partial x_1} \xi_1(z) + \frac{\partial}{\partial x_1} \xi_3(z) - \left( \frac{\partial}{\partial x_2} a(x_2) \right) \xi_2(z) + \frac{1}{2} a(x_2) \left( \nabla \cdot \xi(z) \right), \tag{4}
\]

\[
0 = -a(x_2) \frac{\partial}{\partial x_2} \xi_1(z) + \frac{\partial}{\partial x_2} \xi_3(z), \tag{5}
\]

\[
0 = -a(x_2) \frac{\partial}{\partial x_3} \xi_1(z) + \frac{\partial}{\partial x_3} \xi_3(z) - \frac{1}{2} \left( \nabla \cdot \xi(z) \right). \tag{6}
\]

We model the parameter \(a(x_2)\) as function \(a(x_1, x_2, x_3)\) in three independent variables satisfying the differential equations

\[
\frac{\partial}{\partial x_1} a(x_1, x_2, x_3) = 0 \text{ and } \frac{\partial}{\partial x_3} a(x_1, x_2, x_3) = 0. \tag{7}
\]

Note that names for derivatives of differential indeterminates now involve multi-indices; for example \(\text{xii1}[1, 0, 0]\) stands for \(\frac{\partial}{\partial x_1} \xi_1(z)\). We apply our implementation with a block ranking \(\{\xi_1, \xi_2, \xi_3\} \succ \{a\}\).
L := [-a[0,0,0]*xi1[1,0,0]+xi3[1,0,0]
> -a[0,1,0]*xi2[0,0,0]+(1/2)*a[0,0,0]*
> (xi1[1,0,0]+xi2[0,1,0]+xi3[0,0,1]),
> -a[0,0,0]*xi1[0,1,0]+xi3[0,1,0],
> -a[0,0,0]*xi1[0,0,1]+(1/2)*xi3[0,0,1]
> -(1/2)*xi1[1,0,0]-(1/2)*xi2[0,1,0],
> a[1,0,0],a[0,0,1]]:
> ivar := [x1,x2,x3]:dvar := [[xi1,xi2,xi3],[a]]:
> ComputeRanking(ivar,dvar);
> res := DifferentialThomasDecomposition(L,[]):

res := [DifferentialSystem, DifferentialSystem,
      DifferentialSystem]

The Thomas decomposition yields three systems. The first system contains no additional condition for \(a(x_2)\) except (7). In this generic case (4), (5), and (6) are not formally integrable leading to the compatibility condition \(\xi_2(x_1,x_2,x_3) = 0\). The second system includes the additional condition \(\frac{\partial^2}{\partial x_2^2}a(x_2) = 0\), which was calculated in Pommaret and Quadrat (1997) as the condition to ensure formally integrability of (4), (5), and (6). The third system is a special case of the second system with the new condition \(a(x_2) = 0\).

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