Strategyproof Preference Aggregation Mechanisms for Purchasing a Shared Resource

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Abstract

We design a class of mechanisms called monotone aggregation mechanisms that enable a group of buyers with private utilities to collectively purchase a divisible resource that they intend to share. We assume that the resource is being sold in a deterministic truthful auction which solicits bids for the entire resource. Our mechanism then truthfully elicits utility functions from the buyers, prescribes a joint bid, and prescribes a division of the payment and the resource in the event that they win the resource in the auction. This mechanism is group-rational, which means that if there is a buyer in the group who can afford to individually pay for the entire resource on his own, then the group buys the resource in the outcome of the mechanism. Further, this mechanism satisfies a popular notion of collusion-resistance known as group-strategyproofness. We give two explicit characterizations of this generic class for the naturally occurring case where the utility functions of the buyers are non-decreasing and concave. We finally consider the case where the resource is not divisible, but is of the type where each buyer has a utility value for having access to the resource, and once this resource is bought then no buyer in the group can be excluded from enjoying it. We propose a class of collective purchasing mechanisms that satisfy this non-excludability requirement. Assuming that the resource is being sold in a truthful auction (not necessarily deterministic), we show that this mechanism is group-strategyproof.

1 Introduction

Collective decision-making scenarios in which a group of entities would like to jointly purchase a particular resource with the intention of sharing it are frequently encountered in everyday life. For example, suppose two individuals who are sharing living space would like to purchase an object, say an air conditioner or a television, which is available in the market for a certain price. How do they agree upon a division of this price amongst themselves when their utilities for using that object are private? Any scheme that recommends some notion of fair division of this price has to rely on the ability of the scheme to elicit the true utilities of the individuals, which is not easy since each individual wants to minimize his share of the payment. More generally, the resource in question may be congestible, and the utilities may depend on the proportion in which it is shared between the two users. In that case they not only have to decide how to divide the price of the resource but also how it will be shared, and the two decisions would naturally have to go hand in hand. Moreover, it may not be a simple question of paying a given price, but the resource itself may be offered in an auction, in which case the two buyers need to decide how they will jointly bid in the auction, along with the terms of sharing the resource and the division of payment in the event that they win.

Our primary motivation to study this setting comes from the market for radio spectrum. Recently there has been a debate concerning the merits and demerits of allocating newly opened blocks of spectrum for

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free unlicensed use (e.g. WiFi) as opposed to selling them for exclusive licensed use (e.g., to cellular service providers) as has been done for the past few decades. In a proposal in [1] that was further analyzed in [2], the authors suggest that groups consisting of smaller content or service providers who benefit from additional access to spectrum could jointly submit bids for a shared license, which will compete with bids for exclusive licenses from the bigger firms. This calls for an effort towards designing mechanisms that would help the groups take these collective decisions.

We present a class of mechanisms called monotone aggregation mechanisms that truthfully elicit and aggregate the private preferences of a group of buyers, in order to enable them to collectively participate in a market for a single resource under a broad class of settings. Specifically, we assume that the resource is being sold in a dominant strategy truthful auction that accepts single bids for the entire resource. Note that this includes the special case where there is simply a fixed price to be paid for the resource. Further, we assume that the resource is divisible and the buyer’s preferences are captured by a utility function that assigns real non-negative values to different fractions of the resource. The same mechanism is also applicable to cases in which the resource may not be divisible but ‘shareable’ and in which the buyers assign utility values to different proportions in which they get access to the resource. The mechanism then truthfully elicits these utility functions from the buyers in the group, prescribes a joint bid, and prescribes a division of the payment and the resource in the event that they win in the auction. A desirable property of such a mechanism is that if there are buyers in the group who would have individually been able to purchase the entire resource on their own (for the price revealed ex-post in the auction), then the mechanism should lead to the group purchasing the resource. We call this property group rationality and show that it is satisfied by our mechanism. Further, it also satisfies a popular notion of collusion-resistance known as group-strategyproofness. A mechanism is group-strategyproof if no coalition of buyers can find a deviation from truthfulness such that no buyer is worse off and at least one buyer is strictly better off, irrespective of the reports of the buyers not in the coalition.

Finally, we also consider the case where the resource is neither divisible nor congestible, but rather each buyer has a particular utility for having an access to the resource. Typically, once that resource is purchased, then a buyer cannot be excluded from enjoying it. An example would be buying a block of spectrum for unlicensed use. Once the block has been assigned for unlicensed use, no user would be restricted from having an access to it. For this case we design a class of collective purchasing mechanisms that satisfy this additional requirement of non-excludability of buyers and we show that these mechanisms are also group-strategyproof under weaker conditions on the external auction mechanism.

1.1 Related work

The special case of our setting where there is a fixed cost for buying the resource is related to the setting considered by Moulin et. al. in [5] and [6] for sharing the cost of a service. In that model, a set of agents have certain valuations for a service and the cost for the service depends on the subset of agents that are provided the service. The problem is to design a mechanism that decides the set of agents who will be provided the service and their cost shares that will recover the corresponding cost. They propose a carving mechanism that sequentially offers the service to diminishing subsets of agents according to a fixed cost-sharing scheme, until it finds a subset such that all buyers are willing to pay their share of the cost. Under certain conditions on this sharing scheme this mechanism is group-strategyproof. Beginning with this work, mechanisms for cost sharing have been extensively studied and generalized, focusing on desired properties like budget-balance, efficiency and different notions of strategyproofness, see for example [9], [10], [11] and [8]. Our mechanism is a new addition to this repertoire of cost-sharing mechanisms, capturing the important case where the cost of a divisible resource needs to be shared between the set of buyers. The additional difficulty arises from the fact that now the mechanism also has to decide a division of the resource and moreover, the utilities of the buyers are not simply scalar values, but are functions that assign values to different shares of the resource. Although the structure of our mechanism is reminiscent of the carving scheme typical of the aforementioned mechanisms, the conditions that guarantee strategy-proofness are quite different. Further, since the elicited preferences are in the form of entire utility functions, the arguments for showing group-strategyproofness are considerably more involved. Finally, to the best of our knowledge, the general setting where the group of buyers jointly bid for the resource in an external auction has not been considered before.
2 Model: Buying a divisible resource

A set (or group) \( L \) of \( n \) agents would like to collectively buy a resource that they intend to share. The resource is being sold in a deterministic dominant strategy truthful auction, in which the payment for not winning the resource is 0. A deterministic dominant strategy truthful auction is one in which the mapping from the bids to an allocation is deterministic and bidding truthfully is a dominant strategy for all the buyers. An example is the second price auction with any reserve price. In such an auction, each buyer submits a bid \( b_j \), which conveys the maximum payment he is willing to make for the resource. Then the auction mechanism computes a minimum price \( p_j \) needed to be paid by a buyer \( j \) to win the resource, which depends on the bids of the other buyers. These prices are computed so that \( b_j > p_j \) for at most one buyer \( j \), in which case he wins the resource. For example, in a second price auction \( p_j = \max\{b_j - b^{\tau}\} \). See [2] for more details. We denote \( p^* \) to be this a-priori unknown minimum price that is needed to be paid by the group \( L \) in the external auction in order to win the resource.

The resource is assumed to be divisible and each agent \( i \) in the group has a utility \( U_i(x_i) \), expressed in monetary value, for a fraction \( x_i \) of the resource. We assume that \( U_i(x_i) \) is the maximum payment that a buyer is willing to make for the fraction \( x_i \) of the resource. This utility function \( U_i : [0,1] \rightarrow \mathbb{R}_+ \) is known only to buyer \( i \) and functions \( \{U_i\} \) of the buyers are assumed to belong to a class of utility functions \( \mathcal{C} \) that is publicly known. We further assume that \( U(0) = 0 \) for every \( U \in \mathcal{C} \). In order to participate in the auction, the group has to submit a single bid for the resource and make the required payment \( p^* \) computed by the external auction in the case that they win, i.e. if their bid is higher than \( p^* \). Our goal is to design a mechanism that accomplishes the following two tasks:

1. Elicit individual utility functions from the agents and then output a group bid to enter into the external auction.
2. Prescribe a division of the resource and that of the payment needed to be paid in the external auction amongst the buyers, in the event that they win the resource.

Such a mechanism will be called an aggregation mechanism since it aggregates the preferences of all the agents in a group to result in a single decision-making entity. We first define the following property of a mechanism.

**Definition 2.1. (Group Rationality)** Consider the set of buyers \( W \), that would have individually been willing to purchase the entire resource for the price \( p^* \) offered in the external auction, i.e. for all \( i \in W \), \( U_i(1) > p^* \). Then the aggregation mechanism is said to be group rational if \( W \neq \emptyset \) implies that the group ends up purchasing the resource.

Next, we define the following notion of collusion resistance.

**Definition 2.2. (Group-Strategyproofness)** An aggregation mechanism is group-strategyproof if for any coalition of buyers \( S \subseteq L \), fixing any feasible utility function reports of all buyers not in \( S \), for every feasible deviation of the buyers in \( S \) from truthful reporting, either all the buyers are indifferent between the original outcome and the new resulting outcome or at least one buyer is strictly worse off.

3 Monotone Aggregation Mechanism

We first give an informal description of our mechanism. For each subset \( A \subseteq L \), the mechanism fixes two vectors \( (x_1(A), \ldots, x_n(A)) \) and \( (y_1(A), \ldots, y_n(A)) \) corresponding to resource shares and payment shares respectively, such that they satisfy certain conditions. These shares are made public and the utility functions of the buyers are elicited. The mechanism then starts by first considering the entire set of buyers \( S = L \). Using the reported utility functions, it computes the maximum cumulative payment such that each buyer in the set under consideration can afford to buy his share of the resource in that set for the corresponding share of the payment (i.e. his payment is less than his utility for his share). Let \( \beta_1 \) denote this maximum cumulative payment. The subset of buyers whose utility for their share exactly equals their share of the payment \( \beta_1 \), i.e. the bottleneck buyers, are removed from the set. The mechanism continues with the remaining
set $S_2$, finds the corresponding payment $\beta_2$, and continues so on to find the rest of the vector $\bar{\beta}$ in a similar way until no buyer remains. The largest value in this vector is submitted to the auction. If a payment $p^*$ is to be made in the auction to win the resource, the mechanism looks for the largest subset $S_i$ (i.e. the one with the smallest index $i$) that can afford to pay the price, and both the price and the resource is divided according to the corresponding shares in that subset. Following is the formal definition of the mechanism.

Definition 3.1. Monotone aggregation mechanism with fixed shares for the class of utility functions $\mathcal{C}$:

For each subset $A \subseteq L$, fix two tuples of $n$ non-negative numbers $(x_1(A), \ldots, x_n(A))$ and $(y_1(A), \ldots, y_n(A))$, corresponding to the resource shares and the payment shares respectively, such that

1. $\sum_{i=1}^{n} x_i(A) = 1$ and $x_i(A) > 0$ only if $i \in A$.
2. $\sum_{i=1}^{n} y_i(A) = 1$ and $y_i(A) > 0$ only if $i \in A$.
3. (monotonicity) For any $C > 0$, for any two subsets $A$ and $B$ such that $A \subseteq B$, and for any $i \in A$, if
   \[ U_i(x_i(B)) < Cy_i(B) \]
   then
   \[ U_i(x_i(A)) < Cy_i(A) \]
   for every $U_i \in \mathcal{C}$.

The mechanism elicits utility functions $G_i \in \mathcal{C}$ from all the agents and computes a vector of values

\[ \bar{\beta} = (\beta_1, \beta_2, \ldots, \beta_m) \]

corresponding to diminishing subsets of agents $S_1 \supseteq S_2 \supseteq \cdots \supseteq S_m$ as follows.

- Let $S_1 = L$. For each subset $S_j$, define
  \[ \beta_j = \max\{k \geq 0 : (ky_1(S_j), ky_2(S_j), \ldots, ky_n(S_j)) \leq (G_1(x_1(S_j)), \ldots, G_n(x_n(S_j)))\} \]  
  where $\leq$ denotes a component-wise $\leq$ inequality.

- Let $P(S_j)$ be the set of agents with positive payment shares who force the inequality in the definition above, i.e. all agents $i$ such that $y_i(S_j) > 0$ and $\beta_j y_i(S_j) = G_i(x_i(S_j))$. Then $S_{j+1} = S_j \setminus P(S_j)$ and $m$ is the smallest integer such that $S_{m+1} = \phi$.

- Let $\beta^* = \max\{\beta_1, \ldots, \beta_m\}$. The mechanism submits the bid $\beta^*$ to the auction.

- Suppose the group has to make a payment $p^*$ in the auction. Let $r = \min\{i : \beta_i \geq p^*\}$. Then each agent $i$ pays $y_i(S_r)p^*$ and gets a fraction $x_i(S_r)$ if the group is allotted the resource.

$S_r$ will be called the winning set of buyers. The key requirement of the mechanism is the monotonicity condition. It says that if a buyer in a set cannot afford to pay his share of some price $C$, for his share of the resource in that set, then he should not be able to do so for any subset of that set. We then show the following main result.

Theorem 3.1. Suppose that the resource is being sold in a deterministic dominant strategy truthful auction, in which the payment for not winning the resource is 0. Also, assume that a buyer strictly prefers the outcome where he obtains a non-zero fraction of the resource with a payment equal to his utility for that fraction of the resource, to the outcome where he does not obtain anything and makes no payment. Then any monotone aggregation mechanism with fixed shares is group-strategyproof.
The proof of this theorem is given in the appendix. We argue that the assumption that any buyer strictly prefers winning a fraction of the resource with zero net utility to not obtaining any of the resource is mild. Given the type of auction assumed and the definition of the aggregation mechanism, from the perspective of any coalition of buyers, under a plausible subjective prior over the outcome (derived from their common beliefs about the utilities of all the agents involved in the auction), some buyer in the coalition winning a positive share of the resource with zero net utility is a zero probability event under truthful reporting. One can indeed modify the definition of group-strategyproofness by assuming a probability space over utilities of the agents and having group deviations to not be profitable (in the sense defined) on all but zero probability events. The mechanism would then be group-strategyproof in this sense without requiring the additional assumption. We further prove the following.

**Theorem 3.2.** Any monotone aggregation mechanism with fixed shares is group-rational. Further if $W$ is the set of buyers for whom $U_i(1) > p^*$, then all the buyers in $W$ are in the winning set of buyers $S_r$.

**Proof.** Suppose there is a buyer $i$ for whom $U_i(1) \geq p^*$. From the definition of the resource and payment shares, $x_i\{\{i\}\} = y_i\{\{i\}\} = 1$ and thus this inequality is the same as $U_i(x_i\{\{i\}\}) \geq y_i\{\{i\}\}p^*$. But the monotonicity condition conversely implies that if

$$U_i(x_i\{\{i\}\}) \geq y_i\{\{i\}\}p^*,$$

then

$$U_i(x_i(B)) \geq y_i(B)p^*$$

(2)

for any $B$ such that $i \in B$. Now $i \in S_1 = L$ and further (2) implies that $i \notin P(S_j)$ for any $S_j$ such that the corresponding $\beta_j < p^*$. Further $U_i(x_i\{\{i\}\}) \geq y_i\{\{i\}\}p^*$ implies that there is some $j^*$ such that $\beta_j^* \geq p^*$ and $\beta_j < p^*$ for all $j < j^*$ and further $\{i\} \in S_{j^*}$. Thus the resource is bought in the outcome of the mechanism. This argument is valid for every buyer $i \in W$. Thus each buyer in $W$ is in the winning set of buyers.

Note that the mechanism guarantees that the entire set $W$ is in the winning set of buyers $S_r$. But the share of the resource for a buyer in $W$ in the winning set could be 0 (and hence the payment is also 0). This can be easily avoided by choosing resource shares that satisfy $x_i(A) > 0$ for all $i \in A$, for every $A \in L$.

### 4 Explicit Characterizations

In this section we give two explicit characterizations of the monotone aggregation mechanism with fixed shares for the class $\mathcal{C}$ of non-negative, concave and non-decreasing utility functions defined on $[0, 1]$ such that $U(0) = 0$ for each $U \in \mathcal{C}$. For the first mechanism, consider the following choice of shares in the monotone aggregation mechanism.

**Definition 4.1.** (Cross-monotonic shares aggregation mechanism (CSAM) for the class of utility functions $\mathcal{C}$)

For each subset $A \subseteq L$, fix $n$ non-negative numbers $(x_1(A), \ldots, x_n(A))$ which are the resource as well as the payment shares, such that they satisfy

1. $\sum_{i=1}^n x_i(A) = 1$ and $x_i(A) > 0$ only if $i \in A$.

2. (Cross-monotonicity) If $A \subseteq B$, then $x_i(A) \geq x_i(B)$ for all $i \in A$.

Thus the cross-monotonic shares aggregation mechanism is a monotone aggregation mechanism in which the payment shares and the resource shares are equal for each subset of buyers, and these shares satisfy the property of cross-monotonicity. We can then show that this choice of shares also satisfies the monotonicity requirement with respect to the class $\mathcal{C}$.

**Theorem 4.1.** The choice of resource and payment shares in the cross-monotonic shares aggregation mechanism satisfies the monotonicity condition with respect to the class $\mathcal{C}$.
Proof. By the property of any concave function $U \in \mathcal{C}$, if $U(x) < Cx$ for some $C > 0$ and some $x \in [0, 1]$, then $U(x') < Cx'$ for any $x \geq x'$ (see figure). Then the result follows from the cross-monotonicity of the shares.

Example 4.1. Let us consider an example to illustrate the mechanism in this case. Consider a resource $A$ which is being sold in a second price auction. Suppose the group $L$ that intends to buy the resource consists of three buyers $1$, $2$ and $3$ with utility functions $U_1(x) = x$, $U_2(x) = \sqrt{x}$ and $U_3(x) = \ln(1 + x)$. Let the bid-aggregation mechanism prescribe equal shares for every subset of the buyers, i.e. $x_i(A) = \frac{1}{|A|}$ if $i \in A$ and $0$ otherwise. These shares clearly satisfy cross-monotonicity. Assume that the buyers truthfully report their utility functions to the mechanism. Set $S_1 = L = \{1, 2, 3\}$. The mechanism thus computes

\[ \beta_1 = \max\{k \geq 0 : \left(\frac{k}{3}, \frac{k}{3}, \frac{k}{3}\right) \preceq \left(\frac{1}{3}, \frac{1}{\sqrt{3}}, \ln(1 + \frac{1}{3})\right)\} \]
\[ = \min\{1, \sqrt{3}, 3\ln\left(\frac{4}{3}\right)\} \]
\[ = 3\ln\left(\frac{4}{3}\right) \approx 0.86. \]

Thus buyer 3 is removed from the group to form $S_2 = \{1, 2\}$. Next, the mechanism computes

\[ \beta_2 = \max\{k \geq 0 : \left(\frac{k}{2}, \frac{k}{2}\right) \preceq \left(\frac{1}{2}, \frac{1}{\sqrt{2}}\right)\} \]
\[ = \min\{1, \sqrt{2}\} = 1. \]

Thus buyer 1 is removed from $S_2$ to result in $S_3 = \{2\}$. We thus finally have

\[ \beta_3 = \max\{k \geq 0 : k \leq 1\} = 1. \]

Hence the vector $\bar{\beta} = (0.86, 1, 1)$. Thus $\beta^* = 1$ is submitted as a bid in the second price auction. Now assume that there is a single other competing buyer in the second price auction and suppose that his bid is 0.6. Thus the minimum payment required to win the auction for the group is 0.6, which is feasible. Now $r = \min\{i : \beta_i \geq 0.6\} = 1$. Thus the entire group, i.e. $S_1 = L$ is allotted equal shares of the resource and each buyer pays 0.2 to the seller. Suppose instead that the other buyer in the auction submitted a bid of 0.9. Then in that case $r = \min\{i : \beta_i \geq 0.9\} = 2$. Thus the group $S_2 = \{1, 2\}$ is allotted equal shares of the resource while buyer 3 does not get any share of the resource. Both the winning buyers pay 0.45 to the seller. Thus in short, the resource is shared between the largest subset of buyers who can jointly afford to pay the price, divided according to the prescribed shares.
Thus we need to show that
\[ \sum_{i} \] and let \((\tilde{A}, \tilde{B})\) satisfies the monotonicity condition with respect to the class \(i\).

We need to show that for any \(f\) and resource shares. We can again show that under the assumptions on the properties satisfied by \(f\), the resource shares is such that \(x_i(L) = (1 - \sum_{r=2}^{i} x_j(L)), x_j(L), \ldots, x_j(L)\).

The choice of resource and payment shares in the priority based aggregation mechanism
\[ \text{Proof.} \]

Fix a priority ordering of the buyers \(\{1, \ldots, n\}\). Fix an initial set of resource shares \(x_1(L), \ldots, x_n(L)\) corresponding to set \(L\) satisfying
\[ \sum_{i=1}^{n} x_i(L) = 1. \]

For each subset \(A \subseteq L\), the ordered vector of resource shares of buyers in \(A\) is defined to be
\[ (x_{j_1}(A), \ldots, x_{j_l}(A)) = ((1 - \sum_{r=2}^{l} x_j(L)), x_j(L), \ldots, x_j(L)). \]

The resource-sharing scheme of this mechanism has the following interpretation. When a set of buyers diminishes to a smaller subset in the monotone aggregation mechanism, the shares of the buyers that are removed are allocated to the buyer with the highest rank in the subset. Once the shares of the buyers in the largest set \(L\) is fixed, this rule determines the shares corresponding to all of its subsets. The payment shares in any subset are defined to be proportional to the values of the function \(f\) evaluated at the corresponding resource shares. We can again show that under the assumptions on the properties satisfied by \(f\), the choice of the shares in this mechanism satisfies the monotonicity condition.

**Theorem 4.2.** The choice of resource and payment shares in the priority based aggregation mechanism satisfies the monotonicity condition with respect to the class \(\tilde{C}\).

**Proof.** We need to show that for any \(C > 0\), for any two subsets \(A\) and \(B\) such that \(A \subseteq B\), and for any \(i \in A\), if \(U_i(x_i(B)) < C \frac{f(x_i(B))}{\sum_{j \in B} f(x_j(B))}\) then \(U_i(x_i(A)) < C \frac{f(x_i(A))}{\sum_{j \in A} f(x_j(A))}\) for every \(U_i \in \tilde{C}\). Now, the choice of the resource shares is such that \(x_i(A) \geq x_i(B)\) and so from the single crossing property of the function \(f\), we have that
\[ U_i(x_i(A)) < C \frac{f(x_i(A))}{\sum_{j \in B} f(x_j(B))} \text{ for every } U_i \in \tilde{C}. \]

Thus we need to show that \(\sum_{j \in B} f(x_j(B)) \geq \sum_{j \in A} f(x_j(A))\). Let \(i^*\) be the highest ranked buyer in the set \(A\) and let \((x_{j_1}(b), \ldots, x_{j_{|B\setminus A|}})\) be the ordered vector of shares of buyers in \(B \setminus A\). Then from the definition
of the resource shares, since the shares of the buyers in $A \setminus i^*$ are the same in the sets $A$ and $B$ and since in $A$, $i^*$ gets the all the shares of the buyers in $B \setminus A$, we have that

$$\sum_{j \in B} f(x_j(B)) - \sum_{j \in A} f(x_j(A)) = \sum_{r=1}^{\left|B\setminus A\right|} f(x_{j_r}) + f(x_{i^*}) - f\left(\sum_{r=1}^{\left|B\setminus A\right|} x_{j_r} + x_{i^*}\right)$$

$$= \sum_{r=1}^{\left|B\setminus A\right|} \left(f(x_{j_r}) - f(0)\right) - \left(f\left(\sum_{l=1}^{r-1} x_{j_l} + x_{i^*}\right) - f\left(\sum_{l=1}^{r-1} x_{j_l} + x_{i^*}\right)\right) + f(0)$$

$$\geq \sum_{r=1}^{\left|B\setminus A\right|} f(0) \geq 0.$$

The first inequality holds since $f$ is concave and the shares are non-negative, and the second holds since $f$ is non-negative. \hfill \Box

Note that the resource shares in the class B mechanism are cross-monotonic. Thus we can naturally compare this mechanism to the CSAM with the same resource shares.

**Example 4.2.** Consider the class of utility functions

$$\tilde{c} = \{g(x) = cx^k : c \geq 0, k \in \left[0, \frac{1}{2}\right]\}.$$  

One can easily show that the function $f(x) = \sqrt{x}$ satisfies the single crossing property with respect to this class. Consider three buyers 1, 2 and 3 with utility functions $U_1(x) = x^{\frac{1}{2}}$, $U_2(x) = x^{\frac{1}{3}}$ and $U_3(x) = \sqrt{x}$ respectively. Suppose that the priority order of the buyers is $\{1, 2, 3\}$. Let the resource shares corresponding to the entire set of the buyers be $\left(\frac{1}{2}, \frac{1}{4}, \frac{1}{4}\right)$. This determines the resource and payment shares for all the subsets, as described in the priority based aggregation mechanism. The following table shows these shares for the three buyers. The last two columns show the sequence of $\{\beta_j\}$ computed using the class B shares and the CSAM shares (in which the payment shares are the same as the resource shares) respectively. The $\{\beta_j\}$ are only computed for the sets $\{S_j\}$ that are encountered in the mechanism. Thus as shown in the table, for the class B shares, $S_1 = \{1, 2, 3\}$, $S_2 = \{1, 2\}$ and $S_3 = \{2\}$ while for the CSAM shares, $S_1 = \{1, 2, 3\}$, $S_2 = \{2, 3\}$ and $S_3 = \{3\}$. Note that the ordered vector of $\beta$ values computed in Class B mechanism dominates the vector of values computed in the CSAM. This suggests that, depending on the utility functions, for the same resource shares, a class B mechanism can lead to a strictly higher bid than the CSAM.

| Subsets | Resource shares | Class B Payment shares | Class B $\{\beta_j\}$ | CSAM $\{\beta_j\}$ |
|---------|-----------------|------------------------|------------------------|---------------------|
| $\{1, 2, 3\}$ | $\left(\frac{1}{2}, \frac{1}{4}, \frac{1}{4}\right)$ | $\left(\frac{1}{1+\sqrt{2}}, \frac{1}{1+\sqrt{2}}, \frac{1}{1+\sqrt{2}}\right)$ | 1.707 | 1.68 |
| $\{1, 2\}$ | $\left(\frac{1}{3}, \frac{1}{3}, 0\right)$ | $\left(\frac{1}{1+\sqrt{3}}, \frac{1}{1+\sqrt{3}}, 0\right)$ | 1.467 | - |
| $\{2, 3\}$ | $\left(0, \frac{1}{3}, \frac{1}{3}\right)$ | $\left(0, \frac{1}{1+\sqrt{3}}, \frac{1}{1+\sqrt{3}}\right)$ | - | 1.21 |
| $\{1, 3\}$ | $\left(\frac{1}{3}, 0, \frac{1}{3}\right)$ | $\left(\frac{1}{1+\sqrt{3}}, 0, \frac{1}{1+\sqrt{3}}\right)$ | - | - |
| $\{1\}$ | $\left(1, 0, 0\right)$ | $\left(1, 0, 0\right)$ | - | - |
| $\{2\}$ | $\left(0, 1, 0\right)$ | $\left(0, 1, 0\right)$ | 1 | - |
| $\{3\}$ | $\left(0, 0, 1\right)$ | $\left(0, 0, 1\right)$ | - | 1 |

5 Non-excludable buyers

The monotone aggregation mechanism is allowed to exclude buyers from enjoying the resource by assigning them a zero fraction of the resource. But in many cases, the resource is neither divisible nor congestible, but rather each buyer has a particular utility for having an access to the resource. Typically, once that resource
is purchased, then a buyer cannot be excluded from enjoying it. An example would be buying a block of spectrum for unlicensed use. Once the block has been assigned for unlicensed use, no user would be restricted from having an access to it. Another example would be buying a centralized air-conditioning system for the house. Once it is bought, a roommate cannot be excluded from enjoying it. Such setting is thus highly vulnerable to free-riding attempts by the agents. Nevertheless we propose a simple aggregation mechanism for this setting that is also group-strategyproof. As before, consider a group $L$ of $n$ buyers. Suppose each buyer $i$ has a utility value $V_i$ for enjoying access to the resource. We define the following class of mechanisms.

**Definition 5.1.** *(Non-excludable aggregation mechanism)* Fix $n$ non-negative numbers $(x_1, \cdots, x_n)$ such that $\sum_{i=1}^{n} x_i = 1$. The mechanism elicits bids $(b_1, \cdots, b_n)$ from the buyers. It then submits a bid $\beta^*$ defined as:

$$\beta^* = \max\{k \geq 0 : (kx_1, kx_2, \cdots, kx_n) \preceq (b_1, \cdots, b_n)\}.$$  

If the group wins the auction and it is supposed to pay a price $p^* \leq \beta^*$, then each buyer $j$ pays $x_j p^*$.

The non-excludable aggregation mechanism has a natural increasing prices implementation. Starting from $k = 0$, each buyer $i$ is offered an increasing price $kx_i$. Each buyer can continue to accept or reject at any point. If the first buyer to reject his price does so at $k = \beta^*$, then $\beta^*$ is the submitted bid. We have the following result:

**Theorem 5.1.** Suppose that the resource is being sold in a dominant strategy truthful auction. Then any non-excludable aggregation mechanism is group-strategyproof.

Note that in contrast to Theorem 3.2, the auction mechanism for selling the resource is only required to be dominant strategy truthful and not necessarily deterministic.

### 6 Conclusion and future work

We designed a class of preference aggregation mechanisms that we call monotone aggregation mechanisms that enable a group of agents with private utilities to make purchasing decisions for a shared resource. The key properties of the mechanism are group rationality, group-strategyproofness and the ability to exactly recover the price of the resource in the market. We gave two explicit characterizations of this mechanism for the case where the utility functions of the buyers are concave and non-decreasing. Apart from these characterizations, one may find other explicit characterizations of the monotone aggregation mechanism for specific classes of utility functions. Generalized techniques of designing resource and payment sharing schemes that satisfy the monotonicity condition for broad classes of utility functions would be very useful. The monotone aggregation mechanism is not efficient. It may be the case that the group of agents may jointly be able to pay for the resource with some payment and resource division, but the mechanism leads to an outcome in which they do not buy the resource. Although, this efficiency loss is unavoidable in mechanisms that satisfy participation constraints and require an exact recovery of the fixed cost, an insight into the worst-case welfare properties of our mechanism would be very helpful and it forms an important part of our on-going work.

### 7 Appendix

#### 7.1 Proof of Theorem 3.1

As discussed earlier, in a dominant strategy truthful auction such that the payment for not winning the resource is 0, there is a minimum price needed to be paid by an agent to win the resource which depends on the bids of the other agents, see [7]. For our group of buyers, let this price be $p^*$. For two different sets of reports of the utility functions by the buyers in the group, we say that outcome of the auction remains the same if the same set of buyers $W \subseteq L$ is the winning set. Since the resource and
payment shares depend only on the set of winning buyers, these shares in the two outcomes are the same for all the buyers. Consider a coalition of buyers \( C \subseteq L \). Assume that the reports of the utility functions of all the other buyers are fixed. Suppose that if all the agents in \( C \) report their utilities truthfully (keeping all other reports fixed), then the vector of values generated is \( \beta^0 = (\beta_1^0, \beta_2^0, \ldots, \beta_m^0) \) and let the winning set of buyers be \( W^0 \) (which may be empty). Also let \( \mathcal{C} \) be the set of agents in \( C \) who are in \( W^0 \) and let \( C' = C \setminus \mathcal{C} \).

We first prove the following pair of lemmas

**Lemma 7.1.**

1. Suppose that starting from a set of buyers \( S \), under a fixed report from all the buyers, the set of winning buyers is some set \( W \subseteq S \) (which may be empty). Then starting from a set of buyers \( S \setminus M \) where \( M \subseteq S \setminus W \), under the same reports, the set of winning buyers is also \( W \).

2. Suppose that starting from a set of buyers \( S \), under a fixed report from all the buyers, the set of winning buyers is some set \( W \subseteq S \). Then starting from a set of buyers \( S \setminus \{i\} \) where \( i \in W \), the set of winning buyers \( W' \) satisfies \( W' \subseteq G \setminus \{i\} \).

**Proof.** We first prove the first claim. First, we can easily show that \( W \) is a subset of the new set of winning buyers \( W' \). This is because, if any buyer \( i \) in \( W \) claims to be able to afford to pay \( y_i(W)p^* \) for \( x_i(W) \) fraction of the resource, he can also pay \( y_i(W \cup Q)p^* \) for \( x_i(W \cup Q) \) fraction of the resource. This is because, by the converse implication of the monotonicity assumption, we have that if \( G_i(x_i(W)) \geq C y_i(W) \) then \( G_i(x_i(W \cup Q)) \geq C y_i(W \cup Q) \) for every \( i \) and every \( G_i \in \mathcal{C} \).

Thus we just need to prove that \( Q = W' \setminus W \) is empty. Suppose not. Then there is some buyer \( i \in Q \) who in the original case was removed from the set of buyers \( S' \subseteq S \) where \( (W \cup Q) \subseteq S' \). This in particular implies that \( G_i(x_i(S')) < p^* y_i(S') \). But this again implies, by the monotonicity assumption that \( G_i(x_i(W \cup Q)) < p^* y_i(W \cup Q) \). This contradicts the assumption that \( i \) is in the new winning set. Thus \( Q \) has to be empty. For the second claim, we just need to prove that \( W' \setminus (W \setminus \{i\}) \) is empty, which again follows from a similar argument.

Now suppose for some fixed reports of all buyers, the vector of values computed by the mechanism is \( \bar{\beta} \). Now the effect of any deviation from these fixed reports by a coalition \( C \), manifests itself for the first time by a change in some \( \beta_k \) corresponding to some subset \( S_k \), to a new value \( \beta_k' \). If \( \beta_k' < \beta_k \) then this change has been effectively implemented by one buyer \( i \) in coalition \( C \) by under-reporting. If \( \beta_k' > \beta_k \), then there must be at least one buyer \( i \) in the coalition \( C \) who was forcing the constraint in the set \( S_k \) under truthful reporting, i.e. \( G_i(x_i(S_k)) = \beta_k y_i(S_k) \) and who over-reports. In either case, we say that such a buyer in \( C \) is responsible for causing this first change. Knowing this, we decompose the deviation by coalition \( C \) into a sequence of deviations by individual agents that sequentially bring out the transformation:

\[
\beta^0 \rightarrow \beta^1 \rightarrow \beta^2 \rightarrow \cdots \beta^n.
\]

This sequence is constructed in the following way. First, \( \beta^0 \) is the vector of values computed under truthful reports from all agents in \( C \) and certain assumed fixed reports of the other buyers. The individual agent in \( C \) whose untruthful report brings out the first change in \( \beta^0 \) is denoted by \( i_1 \). Next \( \beta^1 \) is the vector of values computed under truthful reports by all the buyers in \( C \setminus i_1 \), the untruthful report of agent \( i_1 \) and under the assumed fixed reports of the other buyers. Then recursively, we denote \( i_j \) to be the individual agent in \( C \) whose untruthful report brings out the first change in \( \beta^{j-1} \). Then define \( \beta^j \) to be the vector of values computed under truthful reports by all the buyers in \( C \setminus \{i_0, i_1, \ldots, i_{j-1}\} \), the untruthful report of agents \( \{i_0, i_1, \ldots, i_{j-1}\} \) and under the assumed fixed reports of the other buyers. Finally \( \beta^n \) is the vector of values computed using the deviated reports of all the agents in \( C \) (note that not all agents in \( C \) are necessarily untruthful) and the assumed fixed reports of the other buyers. With each \( \beta^j = (\beta^j_1, \ldots, \beta^j_{m_j}) \), are the associated subsets \( \{S^j_1, \ldots, S^j_{m_j}\} \) encountered by the mechanism. Further denote \( W^j = S^j_{m_j} \) to be the winning set of buyers corresponding to \( \beta^j \), where \( r_j = \min\{i : \beta^j_i \geq p^* \} \). We first prove the following.

**Lemma 7.2.** For \( j \in \{1, \ldots, n\} \), if \( i_j \notin W^{j-1} \) then either \( W^j = W^{j-1} \) or \( W^j = W^{j-1} \cup M \cup \{i_j\} \), where \( M \cap W^{j-1} = \emptyset \) and \( i_j \) makes a strict loss being in \( W^j \).
Proof. Let \( k \) be the smallest index at which \( \beta_j \) differs from \( \beta^{j-1} \). Note that \( S_k^{j-1} = S_k^j \). Now if agent \( i_j \) changes the value of \( \beta_k^{j-1} \) to \( \beta_k^j < \beta_k^{j-1} \), then he forces the constraint to have \( G_i(x_i(S_k^j)) = \beta_k^j y_i(S_k^j) \). Since \( i_j \) is not in the winning set of buyers under \( \beta^{j-1} \), \( \beta_k^j < p^* \) and thus \( \beta_k^j < p^* \). Thus agent \( i_j \) is removed from the group and the remaining set of buyers is \( S_k^j \setminus i_j \). By the first claim in lemma 7.1, the set of winning buyers is again \( W^{j-1} \). Next, assume that the agent \( i_j \)'s report changes the value of \( \beta_k^j \) to \( \beta_k^j > \beta_k^{j-1} \). This implies that \( i_j \) was forcing the constraint under truthful reporting in step \( j-1 \), i.e. \( U_i(x_i(S_k^{j-1})) = \beta_k^{j-1} y_i(S_k^{j-1}) < p^* y_i(S_k^{j-1}) \). Now if \( \beta_k^j > p^* \), then the resource is allotted to the group \( S_k^{j-1} = S_k^j \) and thus agent \( i_j \) is a part of the new group of winning buyers \( W^j \), resulting in a strict loss \( p^* y_i(S_k^j) < U_i(x_i(S_k^j)) \). Thus \( W^j = W^{j-1} \cup \{ C \} \) where \( W^j \cap W^{j-1} = \emptyset \) and \( i_j \) makes a strict loss again in \( W^j \).

Next suppose that \( \beta_k^j < p^* \), then in the case that \( i_j \) is forcing the constraint by his report, i.e. \( G_i(x_i(S_k^j)) = \beta_k^j y_i(S_k^j) \), he is still removed from the set of buyers and by the first claim in lemma 7.1, the set of winning buyers is again \( W^{j-1} \). In the case that \( i_j \) does not force the constraint by his report, i.e. \( G_i(x_i(S_k^j)) > \beta_k^j y_i(S_k^j) \), it follows that some other buyer \( m \) was removed from the set of buyers. It cannot be one of the buyers in \( W^{j-1} \), since if \( y_m(W^{j-1}) p^* \leq G_m(x_m(W^{j-1})) \), by the converse implication of monotonicity, \( y_m(S_k^j) p^* \leq G_m(x_m(S_k^j)) \) also and thus, since \( \beta_k^j < p^* \), we have that \( y_m(S_k^j) \beta_k^j \leq G_m(x_m(S_k^j)) \). Thus the agent \( m \) is in \( S_k^j \setminus W^{j-1} \cup i_j \). Now with the new \( \beta^j \) computed, either \( i_j \) is in the winning set of buyers, in which case he makes a strict loss since \( p^* y_i(S_k^j) > U_i(x_i(S_k^j)) \), then he cannot afford to pay \( p^* y_i(S_k^j) > U_i(x_i(S_k^j)) \), for any \( S_k^j \subseteq S_k^j \). Also \( W^{j-1} \) has to be in the winning set again by the converse implication of the monotonicity assumption. Thus the claim holds true. Or he is not in the winning set of buyers. In that case suppose he was removed from some set \( S_k^j \) at some stage \( q \) by the mechanism. Then note that since \( S_k^j \) is not the winning set, the entire set \( W^{j-1} \) is has to be in the set of active buyers \( S_k^{j+1} \), again because of the converse implication of the monotonicity assumption. Thus after stage \( q \), the set of remaining buyers is \( S_k^{j+1} \setminus W^{j-1} \) (which may be empty). Thus by the first claim in lemma 7.1, the set of winning buyers is again \( W^{j-1} \). Thus again the claim holds true.

Lemma 7.3. For each \( j \) one of the following holds.

1. \( W^j \subseteq W^0 \).
2. \( W^j \subseteq W^0 \cup M \) for some \( M \) such that \( M \cap W^0 = \phi \) and \( M \cap C \neq \phi \) and at least one \( i \in M \cap C \) makes a strict loss being in \( W^j \).

Proof. We will prove this lemma using induction. The claim clearly holds for \( j = 0 \). Now we assume that either of the following cases hold true for step \( j-1 \) for some \( j \geq 2 \):

1. \( W^{j-1} \subseteq W^0 \).
2. \( W^{j-1} \subseteq W^0 \cup M \) for some \( M \) such that \( M \cap W^0 = \phi \) and \( M \cap C \neq \phi \) and at least one \( i \in M \cap C \) makes a strict loss being in \( W^{j-1} \).

If the agent \( i_j \notin W^{j-1} \), then the previous lemma says that either \( W^j = W^{j-1} \) or \( W^j = W^{j-1} \cup \{ i_j \} \) where \( M' \cap W^{j-1} = \phi \) and \( i_j \) makes a strict loss. Since \( i_j \in C \), in either case the hypothesis holds for step \( j \).

Next, suppose that the agent \( i_j \) is in \( W^{j-1} \). Suppose that the first change in the vector happens at index \( k \) and the changed value \( \beta_k^j \) is such that \( \beta_k^j > \beta_k^{j-1} \). This means that \( i_j \) was forcing the constraint under truthful reporting in step \( j-1 \), and since he is amongst the winning set of buyers \( W^{j-1} \), this means that \( S_k^{j-1} \) is that winning set and still remains so under \( \beta^j \). Hence \( W^j = W^{j-1} \) and the hypothesis is true for step \( j \).

Now suppose that \( \beta_k^j < \beta_k^{j-1} \). Then either \( \beta_k^j \geq p^* \), in which case \( \beta_k^{j-1} \geq p^* \) also and \( S_k^{j-1} \) was and still remains the winning set of buyers, which again implies that the hypothesis is true for step \( j \). Or \( \beta_k^j < p^* \), in
which case agent \(i_j\) is removed from the subset. Now from the second claim of lemma 7.1, the set of winning
buyers satisfies \(W^j \subseteq W^{j-1} \setminus \{i_j\}\). In the case that \(W^{j-1} \subseteq W^0\), this means that \(W^j \subseteq W^0\) as well and
thus the hypothesis holds true for step \(j\).

In the case that \(W^{j-1} \subseteq W^0 \cup M\) for some \(M\) such that \(M \cap W^0 = \emptyset\) and where at least one \(i \in M \cap C\)
does a strict loss being in \(W^{j-1}\), we have that \(W^j = A \cup M'\) where \(A \subseteq W^0 \setminus \{i_j\}\) and \(M' \subseteq M \setminus \{i_j\}\) and
\(M' \cap W^0 = \emptyset\). Now each buyer \(i\) in \(M'\) is one of two types:

1. Type \(A\): \(i \in L \setminus C\).
2. Type \(B\): \(i \in \{i_1, \ldots, i_n\} \setminus \{i_j\}\).

Now since \(M'\) is not included in \(W^0\), there must be a buyer \(m \in M'\) who at step 0 was removed from the
set of buyers \(S'\) such that \(W^0 \cup M' \subseteq S'\). This in particular implies that \(G_m(x_m(S')) < p^* y_m(S')\). But this
again implies, by the monotonicity assumption that \(G_m(x_m(A \cup M')) < p^* y_m(A \cup M')\). Now if this buyer
\(m\) is of type \(A\), then his report at step \(j\) is the same as his report at step 0, which is \(G_m\). And thus this
contradicts the assumption that \(m\) is in the new winning set \(W^j\). Thus the buyer \(m \in M\) who at step 0
was removed from the set of buyers \(S'\), such that \(W^0 \cup M \subseteq S'\), has to be of type \(B\). But such a buyer
\(m\)‘s report at step 0 is truthful. Thus this implies that \(U_m(x_m(S')) < p^* y_m(S')\). Thus by the monotonicity
assumption, \(U_m(x_m(A \cup M')) < p^* y_m(A \cup M')\) and thus he makes a strict loss being in \(W^j\). Thus either
there is at least one buyer in \(M'\) who faces a strict loss being in \(W^j\) and he is in \(C\) (since a buyer of type \(B\)
is in \(C\)) or \(M' = \emptyset\). Thus the hypothesis holds for step \(j\).

Lemma 7.3 shows that that either \(W^n \subseteq W^0 = \overline{C}\) or \(W^n \subseteq W^0 \cup M\) for some \(M\) such that \(M \cap \overline{C} = \emptyset\) and
\(M \cap C \neq \emptyset\) and where at least one \(i \in M \cap C\) makes a strict loss being in \(W^n\). The former case implies that
either the outcome of the mechanism remains the same or there is a set of buyers who were in the winning
set \(\overline{C}\) under truthful reporting, thus making a non-negative profit, and who are not in the winning set \(W^n\)
after deviation. Note that by our assumption, a buyer strictly prefers being in the winning set while getting
a zero profit over not being in the winning set. The latter case implies that at least one of the players in the
coalition is strictly worse off by the deviation. Thus overall, any deviation by coalition \(C\) either does not change
the outcome of the auction or there is at least one buyer in \(C\) who is strictly worse off. Thus the
mechanism is group-strategyproof.

\[\square\]

7.2 Proof of theorem 5.1

In the external auction, let \(B_i\) be the set of possible bids of group/agent \(i\). For each \(i = 1, \ldots, n\), let
\(\pi_i : \times_{i=1}^n B_i \to [0, 1]\) be the allocation rule of the auction which maps a set of bids from the groups/agents
to a probability of allocating the resource to group/agent \(i\). Also let \(p_i : \times_{i=1}^n B_i \to \mathbb{R}\) be the payment made
by group/agent \(i\).

Consider a group \(i\) with \(n\) buyers and consider a non-excludable aggregation mechanism with the share
values \((x_1, \ldots, x_n)\). We will first prove that the non-excludable aggregation mechanism is strategyproof, i.e.
it is not profitable for any single buyer in this group to deviate from truthful reporting. Consider a buyer \(j\).
Suppose that when all the buyers have submitted their bids to the mediator, \(j\) is the buyer who forces the
constraint in (3) so that \(v_j = kx_j\). Let \(\alpha^{-i}\) be the vector of bids of the other groups/agents in the external
auction. Now the strategy-proofness of this auction implies that

\[\pi_i(k, \alpha^{-i})k - p_i(k, \alpha^{-i}) \geq \pi_i(a, \alpha^{-i})k - p_i(a, \alpha^{-i}),\]  \hspace{1cm} (4)

for all \(k, a\) in \(B_i\). This in turn implies that

\[
\pi_i \left( \frac{v_j}{x_j}, \alpha^{-i} \right) \frac{v_j}{x_j} - p_i \left( \frac{v_j}{x_j}, \alpha^{-i} \right) \\
\geq \pi_i \left( \frac{u_j}{x_j}, \alpha^{-i} \right) \frac{v_j}{x_j} - p_i \left( \frac{u_j}{x_j}, \alpha^{-i} \right)
\]

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for all \( v_j, u_j \) in \( S_j^i \). Multiplying throughout by \( x_j \) we obtain

\[
\pi_i \left( \frac{v_i}{x_j}, \alpha^{-i} \right) v_j - x_j p_i \left( \frac{v_j}{x_j}, \alpha^{-i} \right) \\
\geq \pi_i \left( \frac{u_j}{x_j}, \alpha^{-i} \right) v_j - x_j p_i \left( \frac{u_j}{x_j}, \alpha^{-i} \right).
\]

But the LHS is exactly the utility of the buyer \( i \) if he truthfully reports his value to the mediator and the RHS is his utility if he decides to report some other value \( u_j \). This means that it is a dominant strategy for the buyer who forces the constraint in \( H \) to report his valuation truthfully. Now consider any other buyer \( j' \) whose valuation \( v_{j'} \) does not force the constraint. We have \( v_{j'} > k x_j' \). For this buyer, bidding higher than the true valuation has no effect on the submitted bid \( k \) and the payment made. Bidding lower than the true valuation also does not have any effect on the bid and the payment so long as the bid does not force the constraint. Let \( v_{j'} = k x_j' \) be the highest bid which forces the constraint for buyer \( j' \). For any bid \( u_{j'} \leq v_{j'} \), we would like to show that

\[
\pi_i \left( \frac{v_{j'}}{x_j'}, \alpha^{-i} \right) v_{j'} - x_j p_i \left( \frac{v_{j'}}{x_j'}, \alpha^{-i} \right) \\
\geq \pi_i \left( \frac{u_{j'}}{x_j'}, \alpha^{-i} \right) v_{j'} - x_j p_i \left( \frac{u_{j'}}{x_j'}, \alpha^{-i} \right).
\]

Here the first expression is the utility of the buyer if he bids truthfully, and the second expression is his utility if he bids \( u_{j'} \leq v_{j'} \), thus forcing the constraint and lowering the group bid to \( \frac{u_{j'}}{x_j'} \) from \( k \). To show this, we just need to show that the expression

\[
\pi_i \left( \frac{u_{j'}}{x_j'}, \alpha^{-i} \right) v_{j'} - x_j p_i \left( \frac{u_{j'}}{x_j'}, \alpha^{-i} \right)
\]

is non-decreasing in \( u \) for \( u \leq v_{j'} \). Now from the conditions for dominant strategy incentive compatibility in scalar domains, we know that truthful bidding is a dominant strategy for a single buyer \( i \), if and only if \( \pi_i(r, \alpha^{-i}) \) is non-decreasing in \( r \) and the payment satisfies

\[
p_i(r, \alpha^{-i}) = p_i(0, \alpha^{-i}) + r \pi_i(r, \alpha^{-i}) - \int_0^r \pi_i(s, \alpha^{-i}) ds.
\]

Substituting this expression for the payment in \( 3 \), we finally need to show that

\[
-p_i(0, \alpha^{-i}) + \int_0^{v_{j'}} \pi_i(s, \alpha^{-i}) ds + \left( \frac{v_{j'}}{x_j'} - \frac{u_{j'}}{x_j'} \right) \pi_i \left( \frac{u_{j'}}{x_j'}, \alpha^{-i} \right)
\]

is non-decreasing in \( u \) for \( u \leq v_{j'} \). Its derivative with respect to \( u \) is given by

\[
\left( \frac{v_{j'}}{x_j'} - \frac{u}{x_j'} \right) \frac{d\pi_i \left( \frac{u}{x_j'}, \alpha^{-i} \right)}{du},
\]

which is non-negative since \( u \leq v_{j'} \) and since \( \pi_i \left( \frac{u}{x_j'}, \alpha^{-i} \right) \) is non-decreasing in \( u \).

We now prove the group-strategyproofness. Suppose that under truthful bidding, the submitted bid is \( \beta \) and let \( H \) be the set of buyers who force the constraint, i.e. all buyers \( j \) who have \( v_j = \beta x_j \). Now any deviation of a coalition from truthfulness, that lowers the value of the submitted bid to \( \beta' < \beta \) can be effectively implemented by any single buyer \( j \) by lowering his bid to \( v_j' = \beta' x_j \). But as we saw, such a deviation cannot be profitable to buyer \( j \). Thus no coalition of buyers can all gain by effectively lowering the submitted bid. Now consider a deviation by a coalition that increases the value of the submitted bid to \( \beta' > \beta \). This implies that the coalition contains at least one buyer \( j \) in the set \( H \), and he has to submit a higher bid. But since such a deviation cannot be profitable for this buyer, we must have

\[
\pi_i \left( \frac{v_j}{x_j}, \alpha^{-i} \right) v_j - x_j p_i \left( \frac{v_j}{x_j}, \alpha^{-i} \right) = \pi_i \left( \frac{v_j}{x_j}, \alpha^{-i} \right) v_j - x_j p_i(\beta', \alpha^{-i}).
\]
Dividing throughout by \( \frac{x_j}{j} \) and using the fact that \( \frac{v}{x_j} = \beta < \beta' \), we obtain
\[
\pi_i(\beta, \alpha^{-i}) \beta - p_i(\beta, \alpha^{-i}) = \pi_i(\beta', \alpha^{-i}) \beta - p_i(\beta', \alpha^{-i}).
\]
Substituting the expression for payments from (7), we get
\[
\beta \left( \pi_i(\beta, \alpha^{-i}) - \pi_i(\beta', \alpha^{-i}) \right) = \beta \pi_i(\beta, \alpha^{-i}) - \beta' \pi_i(\beta', \alpha^{-i}) + \int_{\beta}^{\beta'} \pi_i(s, \alpha^{-i}) ds.
\]
Thus we have
\[
(\beta' - \beta) \pi_i(\beta', \alpha^{-i}) = \int_{\beta}^{\beta'} \pi_i(s, \alpha^{-i}) ds.
\]
But since \( \pi_i(s, \alpha^{-i}) \) is non-decreasing and since \( \beta' > \beta \), this implies that \( \pi_i(\beta, \alpha^{-i}) = \pi_i(\beta', \alpha^{-i}) \) and hence \( p_i(\beta, \alpha^{-i}) = p_i(\beta', \alpha^{-i}) \). Thus the deviation either does not change the outcome of the auction or the buyers in the coalition who are also in \( H \) bear a strict loss by the deviation.

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