Form Factors in $D_n^{(1)}$ Affine Toda Field Theories

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Abstract

We derive closed recursion equations for the symmetric polynomials occurring in the form factors of $D_n^{(1)}$ affine Toda field theories. These equations follow from kinematical- and bound state residue equations for the full form factor. We also discuss the equations arising from second and third order forward channel poles of the S-matrix. The highly symmetric case of $D_4^{(1)}$ form factors is treated in detail. We calculate explicitly cases with a few particles involved.

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1 Introduction

Form factors are matrix elements of local operators in a quantum field theory. Knowing them explicitly or at least perturbatively gives a deep knowledge of the quantum structure of a given classical Lagrangian field theory. For instance it might then in principle be possible to classify the local operator content of the theory or to calculate correlation functions. However, for a generic field theory it is a difficult problem to calculate these form factors.

In the class of two-dimensional field theories one is in a better situation. It is known that there exist models which enjoy the property that scattering is factorizable, i.e. any n-particle scattering process can be decomposed into a product of two-particle scattering processes.

For theories whose scattering is factorizable it can be shown that the form factors are subject to a set of axioms \[1\]. The axioms (equations) provide a machinery which in principle enables one to calculate form factors explicitly.

Within the last years based mainly on the studies \[2, 1\] form factors have been studied for some diagonal scattering theories \[3, 4, 5, 6, 7, 8, 9, 10\]. Among them a complete solution to the form factor equations was obtained for the scaling Lee-Yang-model (minimal $A_2^{(2)}$) \[5\] and for the sinh-Gordon model \[6\]. However, these models are simple in the sense that they do involve only one type of particle or that the S-matrices of the corresponding theories do only have poles of first order. We note that only these first order poles are covered directly by the axioms mentioned above \[1\].

Recently the two particle form factors for the magnetic perturbation of the Ising model have been investigated \[11\]. This work is the first which treats a model with several species of particles and an S-matrix with higher order poles.

Moreover some form factors in $A_n^{(1)}$ affine Toda field theories were studied in \[12\]. These theories do generically have more than one species of particles and the S-matrices have poles up to second order. Based on the analysis in \[12\] we would like to address the problem of form factors in $D_n^{(1)}$ affine Toda field theories.

In order to make this paper a bit more self contained let us define affine Toda field theories (ATFT). They are given at the classical level by the following Lagrangian.

\[
L = \frac{1}{2} \partial_\mu \Phi \cdot \partial^\mu \Phi - V(\Phi), \quad V(\Phi) = \frac{m^2}{\beta^2} \sum_{i=0}^{r} n_i \exp(\beta \alpha_i \cdot \Phi). \tag{1}
\]
\( \Phi = (\Phi_1, \Phi_2, \ldots, \Phi_r) \) is a vector of \( r \) scalar fields, \( \alpha_i \) and \( n_i \) denote for \( i = 1, 2, \ldots, r \) the simple root vectors and the Coxeter labels of the corresponding root of a simple Lie algebra \([13]\). We consider only Lie algebras of type A, D, and E.

The real constant \( m \) sets the mass scale of the theory while \( \beta \), which is assumed to be real throughout this paper, is the coupling of the theory. In fact it turns out that as long as we do not consider the perturbative structure we can work for our purposes with an “effective coupling” \([14]\)

\[
B(\beta) = \frac{1}{2\pi} \frac{\beta^2}{1 + \beta^2/4\pi}.
\]  

(2)

The exact S-matrices for ADE-ATFTs have been given in \([15]\) and serve as the main input for the form factor equations. It is known that S-matrices for the A-series possess poles of at most order two, while for the D-series they do have poles up to order four. In the latter case poles of first and third order correspond to forward channel physical particles.

We will come to that point below. A general reference on this might be \([16, 17]\). The form factor problem for these theories has first been addressed in \([18]\).

In this paper we are going to shed some light on the form factors for \( D_{(1)}^n \)-ATFT. As it was already mentioned they provide an interesting complication of the studies quoted above since they do contain several different particle species and have S-matrices which contain higher order poles.

The outline of the paper is the following. In section 2 we define form factors and review the axioms which they have to obey and some results of \([12]\) which will be needed for our study. Some well known facts about S-matrices and fusings in \( D_{(1)}^n \) are recalled in section 3. We then give a closed form of the kinematical and bound state residue equations and discuss the influence of the higher order poles to the form factor equations. In section 4 the case of \( D_{(1)}^4 \) will be discussed in detail. We point out how the symmetries of the Dynkin diagram enter the form factor equations and analyse in detail the poles of second and third order which are present in this particular model. We are going to calculate some examples of low particle form factors in \( D_{(1)}^4 \) in section 5. The last section is devoted to comments and discussions.
2 Some facts about the form factor bootstrap

As mentioned in the introduction a form factor is a matrix element corresponding to a local operator $O$. We take this matrix element in a special form which is commonly used in the literature:

$$F_{a_1 \ldots a_n}(\theta_1, \ldots, \theta_n) := \langle 0 | O(0) | \theta_1, \ldots, \theta_n \rangle. \quad (3)$$

In this expression $\theta_i$ labels the rapidity of the particle of species $a_i$, and $|0\rangle$ denotes the physical vacuum of the theory. The fact that we take the matrix element of $O$ at the origin is simply a matter of convenience.

The form factors are known to be subject to four axioms [1]. The first two of them are the so-called Watson’s equations.

$$F_{a_1 \ldots a_i a_{i+1} \ldots a_n}(\theta_1, \ldots, \theta_i, \theta_{i+1}, \ldots, \theta_n) = S_{a_1 a_{i+1}}(\theta_i - \theta_{i+1}) F_{a_1 \ldots a_{i+1} a_i \ldots a_n}(\theta_1, \ldots, \theta_{i+1}, \theta_i, \ldots, \theta_n),$$

$$F_{a_1 a_2 \ldots a_n}(\theta_1 + 2\pi i, \theta_2, \ldots, \theta_n) = F_{a_2 \ldots a_n a_1}(\theta_2, \ldots, \theta_n, \theta_1). \quad (4)$$

Needless to mention that $S_{ab}(\theta)$ denotes the S-matrix element.

Due to the structure of the theory the form factors do contain poles. There are two kinds of them. The first one is due to the kinematics. The following equation can be explained by a field theoretical reasoning [1] and links a particle of species $a$ with its antiparticle $\bar{a}$.

$$-i \left. \text{res}_{\theta=\theta+\pi} \right|_{\theta'=\theta+\pi} F_{\bar{a}d_1 \ldots d_n}(\theta', \theta, \theta_1, \ldots, \theta_n) = F_{d_1 \ldots d_n}(\theta_1, \ldots, \theta_n) \left( 1 - \prod_{j=1}^n S_{ad_j}(\theta - \theta_j) \right). \quad (5)$$

We will refer to this equation as kinematical residue equation.

The second type of singularity is due to the possible bound states of the theory. As long as the singularities of the S-matrix which lead to bound states are of first order the following bound state residue equation for the fusion $a + b \rightarrow c$ at fusion angle $\theta_{ab}^c$ and with on shell three-point vertex

$$(\Gamma_{ab}^c)^2 = -i \left. \text{res}_{\theta=\theta_{ab}} \right|_{\theta'=\theta_{ab}^c} S_{ab}(\theta), \quad (6)$$

holds

$$-i \left. \text{res}_{\theta=\theta_{ab}+\theta_{bc}} \right|_{\theta'=\theta_{ab}^c} F_{abd_1 \ldots d_n}(\theta', \theta, \theta_1, \ldots, \theta_n) = \Gamma_{ab}^c F_{\bar{c}d_1 \ldots d_n}(\theta + i\theta_{bc}^c, \theta_1, \ldots, \theta_n). \quad (7)$$
We have set $\bar{\theta} = \pi - \theta$.
The S-matrix of $D_n^{(1)}$-affine Toda theories is known to possess poles of order 2, 3 and 4 as well. The residue equations corresponding to these higher order poles need to be clarified and will be discussed below.

The form factors are in general meromorphic functions in $n$ variables. It is possible to split the full factor into a minimal $F_{\text{min}}$ part which is analytic in the strip $0 \leq \text{Im}\theta < 2\pi$ and has no zeros in $0 < \text{Im}\theta < 2\pi$. It can be shown in generalization of a theorem in [2] that a solution of Watson’s equations (4) takes the form

$$F_{a_1\ldots a_n}^{\theta_1,\ldots,\theta_n} = K_{a_1\ldots a_n}(\theta_1, \ldots, \theta_n) \prod_{i<j} F_{a_i a_j}^{\theta_i - \theta_j}, \quad (8)$$

where it has to hold that $F_{a_i a_j}^{\theta}(\theta) = S_{a_i a_j}(\theta) F_{a_j a_i}^{\theta}(\theta)$ and $F_{a_i a_j}^{\theta + 2\pi i}(\theta) = F_{a_j a_i}^{\theta}(\theta)$. These minimal form factors are well known for ADE-ATFT. They can be presented in an integral form [13] or equivalently in an infinite product expansion of $\Gamma$-functions (see e.g. [12]).

Now that these are known the work to be done is to solve for the form factor equations for the object $K$ which contains poles and zeros in the above mentioned region. By the construction (8) $K$ automatically satisfies (4) by requiring some obvious monodromy properties.

It has been shown in [12] that from the kinematical (5) and bound state (7) residue equations we get the following equations for $K$.

$$-i \text{ res}_{\theta' = \theta + i\pi} K_{a_1\ldots a_n}^{\theta'\theta}(\theta', \theta, \theta_1, \ldots, \theta_n) = K_{d_1\ldots d_n}(\theta_1, \ldots, \theta_n) \left( \prod_{j=1}^{n} \xi_{a_j d_j}(\theta - \theta_j) - \prod_{j=1}^{n} \xi_{d_j a_j}(\theta - \theta_j) \right) / F_{\text{min}}^{\theta\theta}(i\pi). \quad (9)$$

Here we have set

$$\xi_{ab}(\theta)^{-1} = \prod_{x \in A_{ab}} \langle x \rangle_{+(\theta)}. \quad (10)$$

This notation is taken from [12] and will be explained in the beginning of the next section (16). Here we would only like to state that the S-matrix is given by

$$S_{ab}(\theta) = \frac{\xi_{ab}(\theta)}{\xi_{ab}(\theta)}. \quad (11)$$

This form should be compared with the expressions for the S-matrix at the beginning of the next section.

The bound state residue equation is
\[ -i \text{ res}_{\theta' = \theta + i \theta_{c}} K_{\theta, \theta_{1}, \ldots, \theta_{n}} = \Gamma_{ab}^{c} K_{\theta_{1}, \ldots, \theta_{n}} (\theta + i \theta_{c}) \prod_{j=1}^{n} \lambda_{abj}^{c} (\theta + i \theta_{c} - \theta_{j}) / F_{ab}^{\min} (i \theta_{c}). \]  

The object \( \lambda \) is via Watson’s equation very closely related to the S-matrix bootstrap equation. Since its explicit form will be of importance later we mention:

\[ \lambda_{abcd}^{c}(\theta) = \frac{F_{ad}^{\min}(\theta + i \theta_{ac}) F_{bd}^{\min}(\theta - i \theta_{bc})}{F_{cd}^{\min}(\theta)} \prod_{x \in \mathcal{A}_{ad}, x \leq u_{ac}^{b}} \langle \bar{u}_{ac}^{b} - x \rangle_{+(\theta)} \prod_{x \in \mathcal{A}_{bd}, x < u_{bc}^{a}} \langle x - u_{bc}^{a} \rangle_{+(\theta)}. \]  

The \( \theta_{bc} \)'s (\( \bar{\theta}_{bc} = \pi - \theta_{bc} \)) are the fusion angles of the theory and we set \( u_{bc}^{a} = \theta_{bc}^{a} h / \pi \), with \( h \) being the Coxeter number.

We will give details on this notation at the beginning of the next section.

We have now reduced the problem of calculating form factors in ATFT to that of making a proper ansatz for \( K \) and solving the two equations \((9)\) and \((12)\). In cases where the S-matrix has higher order poles we do have to supplement these equations with additional ones.

## 3 Form Factors in \( D_{n}^{(1)} \) Toda theory

Let us recall some facts about \( D_{n}^{(1)} \)-ATFT \([15]\). These theories contain \( n \) kinds of particles which will be labeled by the elements of the set \( \{1, 2, \ldots, n-2, s, s'\} \). While the particles labeled by ordinary numbers in this set correspond to the points on the straight line, \( s \) and \( s' \) correspond to the particle upside and downside of the fishtail part of the Dynkin diagram of \( D_{n} \).

In this class of ATFT we have to distinguish between \( n \) odd and \( n \) even. In the even case all particles are self-conjugate while in the odd case we have \( \bar{s} = s' \).

Let \( h = 2(n-1) \) denote the Coxeter number of the theory in question. The masses of the particles are then given by

\[ m_{s}^{2} = m_{s'}^{2} = 2m^{2}, \quad m_{k}^{2} = 8m^{2} \sin^{2} \left( \frac{k \pi}{h} \right), \quad k = 1, 2, \ldots, n-2. \]  

(14)
\[ m^2 \] sets the mass scale of the theory. We remark that the case of \( D_4^{(1)} \) is special in the sense that here we do have three particles \( \{1, s, s'\} \) of mass \( \sqrt{2}m \) and one heavy particle \( \{2\} \) of mass \( \sqrt{6}m \) which corresponds to the central point of the Dynkin diagram of \( D_4 \).

The exact S-matrices and the fusion angles for the \( D_n^{(1)} \)-theories have been given in \cite{[15]}. We recall them for convenience. In doing that we borrow the following notations from \cite{[12]}.

\[
(r)_{+}(\theta) := \frac{1}{i\pi} \sinh \frac{1}{2} \left( \theta + \frac{i\pi}{h} r \right), \quad (r)_{-(\theta)} := \frac{(r)_{+}(\theta)}{(-r)_{+}(\theta)},
\]

and

\[
\langle r \rangle_{+}(\theta) = \frac{(r + 1)_{+}(\theta)(r - 1)_{+}(\theta)}{(r + 1 - B)_{+}(\theta)(r - 1 + B)_{+}(\theta)}, \quad \langle r \rangle_{-(\theta)} = \frac{\langle r \rangle_{+}(\theta)}{\langle -r \rangle_{+}(\theta)}.
\]

Using these notations the S-matrices for \( n \) even are given by

\[
S_{ab}(\theta) = \prod_{\{a-b\}+1}^{a+b-1} \langle p \rangle_{(\theta)} \langle h - p \rangle_{(\theta)}, \quad S_{sa}(\theta) = S_{s'\, a}(\theta) = \prod_{0}^{2a-2} \langle n - a + p \rangle_{(\theta)},
\]

with \( a, b \in 1, 2, \ldots, n - 2 \), and

\[
S_{ss}(\theta) = S_{s'\, s'}(\theta) = \prod_{1}^{h-1} \langle p \rangle_{(\theta)}, \quad S_{ss'}(\theta) = \prod_{3}^{h-3} \langle p \rangle_{(\theta)}.
\]

For odd \( n \) similar expressions are obtained \cite{[15]}. We denote by \( A_{ab} \) the set of integers actually appearing in the brackets of the corresponding S-matrix above and by \( \hat{A}_{ab} = A_{ab} \setminus \{h - 1\} \). For later calculations it turns out to be useful to know the number of elements in these sets.

\[
\#A_{aa} = 2a, \quad h - 1 \in A_{aa}, \quad h - 1 \notin A_{ab}, \quad a < b,
\]

\[
\#A_{ab} = 2a, \quad h - 1 \notin A_{ab}, \quad \#A_{as} = a, \quad h - 1 \notin A_{as}, \quad \#A_{ss} = \#A_{s's'} = n/2, \quad h - 1 \in A_{ss}, A_{s's'},
\]

\[
\#A_{ss'} = (n - 2)/2, \quad h - 1 \notin A_{ss'}. \quad \#A_{s} \neq b \notin \{s, s'\}. \quad \text{For odd } n \text{ these results have to be slightly modified.}
\]

The fusion angles can be divided into three classes. The first one involves the particles \( s \) and \( s' \).
\[ u^a_{ss} = u^a_{ss'} = h - 2a, \quad u^a_{sa} = u^a_{s'a} = h/2 + a. \]  

(20)

The second and third class do not incorporate \( s \) and \( s' \) and are distinguished by a relation of the three particles in the the fusion process. We have

\[
\begin{align*}
  u^c_{ab} &= h - c, & u^a_{ac} &= h - b, & u^b_{ac} &= h - a, & \text{if } a + b + c = h, \\
  u^c_{ab} &= h - c, & u^a_{bc} &= h - a, & u^b_{ac} &= b, & \text{if } a - b + c = 0.
\end{align*}
\]

(21)

Up to now we had to review a lot of known things and to introduce a bulk of notation. We are now in a position to come to the main part of this paper.

There might be several ways to write down the part of the full form factor which contains poles and zeros in the physical region. We prefer a slightly modified version of what has been presented in [12]. This version is based on a factorization of the pole part of \( K \) while the part containing zeroes is the quantity to be determined by the form factor equations. We take \( x_i = e^{i \theta_i} \) and define the following vectors

\[
\mathbf{x}^{(k)} = (x_1^{(k)}, x_2^{(k)}, \ldots, x_{N_k}^{(k)}), \quad k \in \{1, 2, \ldots, n - 2, s, s'\}.
\]

(22)

The \( N_k \) denote the number of particles of type \( k \) in the form factor (cf. (3)). We can then write down the following parametrization:

\[
K_{[N_1, \ldots, n-2, N_s, N_{s'}]}(\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n-2)}, \mathbf{x}^{(s)}, \mathbf{x}^{(s')}) = Q_{[N_1, \ldots, N_{s'}]}(\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(s')})
\times \left( \prod_{k=1}^{s'} \prod_{i<j}^{N_k} \frac{1}{x_i^{(k)} - x_j^{(k)}} \frac{1}{W_{kl}(x_i^{(k)}, x_j^{(k)})} \right) \prod_{k=1}^{s'-1} \Pi_{l=k+1}^{N_k} \Pi_{j=1}^{N_{s'}} \frac{1}{W_{kl}(x_i^{(k)}, x_j^{(s')})}.
\]

(23)

The factors \( 1/(x_i^{(k)} + x_j^{(k)}) \) parametrize the kinematical singularities. The product in front of them is already chosen for \( D_n^{(1)} \) with \( n \) even, i.e. all particles are self conjugate. Obviously we can write down \( K \) without any difficulties for odd \( n \) as well. In what follows we give the explicit formulas for the cases with even \( n \) only.

The objects \( W_{kl}(x_i^{(k)}, x_j^{(l)}) \) do parametrize the fusion poles. In [12] it was shown that a particular ansatz for them does lead from (9) and (12) to a polynomial equation for the \( Q \)'s at least for \( A_n^{(1)} \)-ATFT. We take this ansatz in a slightly modified form.

\[
W_{kl}(x_i^{(k)}, x_j^{(l)}) = \prod_{r \in A_{kl}} (x_i^{(k)} - \Omega^{r+1} x_j^{(l)}) (x_i^{(k)} - \Omega^{-r-1} x_j^{(l)}), \quad \Omega = e^{i \pi/h}.
\]

(24)
In the ansatz (23) above $Q_{[N_1, N_2, \ldots, N_s]}$ is assumed to be a polynomial which is due to the structure of ATFT symmetric at least in each of the vector components of any $x^{(k)}$. Of course this assumption has to be justified case by case especially in the presence of higher order poles in ATFT. Since our form factors are required to be Lorentz invariant it is straightforward to calculate the degree of the polynomials $Q$. A Lorentz transformation consists simply of a linear shift of the rapidities. The denominator of (23) is not Lorentz invariant. However its degree determines the total degree of $Q$ to be:

$$\text{deg } Q_{[N_1, \ldots, N_s]} = \sum_{k=1}^{n-2} \left( N_k (N_k - 1) (2k - \frac{1}{2}) + 2N_k (N_s + N_s') k \right) + 4 \sum_{k=1}^{n-3} \sum_{l=k+1}^{n-2} k N_k N_l$$

$$+ \frac{n-1}{2} \sum_{k=s, s'} N_k (N_k - 1) + N_s N_s' (n - 2).$$

(25)

The partial degree, i.e. the maximal degree of a variable $x_i^{(k)}$ in $Q$ can be calculated only after having established the recursion equations for the polynomials below. In general this leads to a quite complicated formula. We treat the problem of calculating the partial degree for the special case of $D_4^{(1)}$ in section 5.

It is now clear that we have chosen an ansatz for the form factor in such a way that we have to solve only for the polynomials $Q$. However, as can be read off from (23) these objects do have a quite high degree already in cases when only a few particles are present. This makes the general solution technically quite difficult.

We can now plug the above ansatz into the residue equations to obtain recursion relations for the polynomials $Q$. In doing this we first make the following observation

$$K_{ab \ [N_1, \ldots, N_s]} (x^{(a)}, x^{(b)}, x^{(1)}, \ldots, x^{(s')}) = (x^{(a)} + x^{(b)})^{-\delta_{a b}} \prod_{i=1}^{N_a} (x^{(a)} + x_i^{(a)})^{-1} \prod_{i=1}^{N_b} (x^{(b)} + x_i^{(b)})^{-1}$$

$$\times \prod_{k=1}^{N_k} W_{ak} (x^{(a)}, x_i^{(k)})^{-1} W_{bk} (x^{(b)}, x_i^{(k)})^{-1} \cdot W_{ab} (x^{(a)}, x^{(b)})^{-1}$$

$$\times Q_{ab\ [N_1, \ldots, N_s]} (x^{(a)}, x^{(b)}, x^{(1)}, \ldots, x^{(s')}) \frac{K_{[N_1, \ldots, N_s]} (x^{(1)}, \ldots, x^{(s')})}{4^{[N_1, \ldots, N_s]}}.$$  

(26)

This equation indicates how the two particles $a$ and $b$ undergoing either a kinematical or a bound state process are separated from all other particles in the form factor. The last factor in the product does not contain any dependence on the particles $a$ and $b$, i.e. the coordinates $x^{(a)}$ and $x^{(b)}$ do not appear in that expression. A similar expression can of course be written down if we want to separate only one particle.
By using the structure of the index sets $A_{ab}$ of the exact S-matrices for $D_n^{(1)}$-ATFT and introducing the following piece of notation to make expressions a bit more transparent

$$[r]^{(k)}_i = x - \Omega^r x^{(k)}_i, \quad k \in \{1, 2, \ldots, n-2, s, s'\}, \quad i = 1, 2, \ldots N_k,$$

we arrive from the kinematical residue equation (21) at the following recursion equation which links an $N+2$-particle polynomial to an $N$-particle polynomial.

$$Q_{aa[N_1 \ldots N_{a'}](-x^{(a)}, x^{(a)}, x^{(1)}, x^{(2)}, \ldots, x^{(s')})} = -i(1)^{N_0} (x^{(a)})^{d_a-1} \prod_{r \in A_{aa}} \frac{2\cos((r+1)x)}{F_{\text{min}}(x)}$$

$$\times \left(\prod_{k=1}^{N_k} \prod_{i=1}^{N_i} \prod_{r \in A_{ak}} [-r + h_i^{(k)} [r + 1]_i^{(k)} \prod_{r \in A_{ak}} [1 - r - B_i^{(k)} [B - r - 1]_i^{(k)}

- \prod_{k=1}^{N_k} \prod_{i=1}^{N_i} \prod_{r \in A_{ak}} [-h + r + 1]_i^{(k)} [-r - 1]_i^{(k)} \prod_{r \in A_{ak}} [r - 1 + B]_i^{(k)} [r + 1 - B]_i^{(k)}

$$

$$\times Q_{[N_1 \ldots N_{a'}]}(x^{(1)}, x^{(2)}, \ldots, x^{(s')}).$$

Obtaining a closed recursion relations for the polynomials $Q$ coming from the first order bound state equation (12) is not so straightforward. It was mentioned above that the fusings in $D_n^{(1)}$-ATFT can be divided into three classes (20), (21). In the second class of (21) the three particles entering appear to be unsymmetric in the sense $a - b + c = 0$. In order to calculate the desired recursion relation these three classes have to be treated seperately and especially the third class carefully. We also have to take into account the number of elements in the sets $A_{ab}$ (19). The first order equation for the process $a + b \to c$ is then

$$Q_{ab[N_1 \ldots N_{a'}]}(x, x, x^{(1)}, x^{(2)}, \ldots, x^{(s')}),$$

$$\times \prod_{k=1}^{N_k} \prod_{i=1}^{N_i} \prod_{r \in A_{ak}} [1 + r + u_{ac_i}^{b} [r + 1 + r + u_{ac_i}^{b} [r + u_{bc_i}^{b} + r - B + 1]_i^{(k)} [-u_{bc_i}^{b} + r + B - 1]_i^{(k)}

$$

$$\times \prod_{r \in A_{ab}} [-u_{bc} + r - 1]_i^{(k)} [-u_{bc} + r - 1]_i^{(k)} \prod_{r \in A_{ab}} [1]_i^{(k)} [-u_{bc} + r + B - 1]_i^{(k)}

$$

$$\times Q_{c[N_1 \ldots N_{a'}]}(x, x^{(1)}, x^{(2)}, \ldots, x^{(s')}).$$

(29)
The last two lines should be compared with the explicit expression for \( \lambda \) given in (13). 

\( u_{ab}^c \) etc. are the fusion angles for the possible processes which involve the particles \( a, b, \) and \( c \). We used the abbreviation \( \Theta_{ab}^c = u_{ab}^c + u_{bc}^a - h \). For self-conjugate theories it holds of course that \( \Theta_{ab}^c = \bar{u}_{bc}^a \).

Even though we do have negative powers in the above expression the equation is a polynomial recursion equation. It can be checked that these negative powers are canceled for any ATFT. We have, however, chosen to write them in order to make the remaining products a bit more handy.

We note already at this stage that the recursion coefficients in (28) and (29) can be expressed as a sum of elementary symmetric polynomials. We will treat this point more explicitly in section 5 when we calculate some solutions to these equations for the \( D^{(1)}_4 \) case.

In order to be able to obtain solutions to these equations without going into tedious case by case calculations it would be desirable to have a Lie algebraic interpretation of the two recursion equations for instance in the sense of \([17]\). We were informed \([20]\) that an equation similar to (29) can also be obtained for the \( Q \)-polynomials arising in \( A^{(1)}_n \)-form factors.

It is important to note that (29) is written if the particles are in order. This means that we impose an ordering in the set of particles \( \{1, 2, \ldots n-2, s, s'\} \) by \( 1 < 2, \ldots 1 < s', \ldots, s < s' \). Then in (29) we have \( a \leq b \).

If we change this order of the particles in the process, i.e. we consider \( b + a \to c \) rather than \( a + b \to c \) the coefficient in (29) will change. This change is simply obtained by exchanging the first two arguments of \( Q \) and replacing \( \Omega \to \Omega^{-1} \) in the coefficient. Since both processes are physically equivalent this artifact leads to an identity for one particular polynomial \( Q \) at specific values of the arguments. This will become more transparent in section 5.

Let us now come to a discussion of the form factor equations in the presence of higher order poles of the S-matrix. The content of equation (29) can be stated graphically by the following diagram which should not strictly be interpreted in the perturbative sense.

![Fig. 1: First order pole diagram in the form factor](image)

The bubble in this figure depicts the local operator in the conventions of [3].
Let us consider the third order forward channel poles of the S-matrix first. These poles correspond to fusings which are already covered by the fusion angle analysis at the beginning of this section. For our purposes this means the following. The perturbative analysis shows that in general several Feynman diagrams are needed in order to explain the third order poles of the S-matrix. But the S-matrix in our case is known to satisfy the bootstrap equations. Hence, for the forward channel fusings at the position of a third order pole of the S-matrix we can in analogy to fig.1 draw the following effective picture.

![FIG. 2: Effective picture for the third order pole](image)

The circle at the position of the fusing indicates that we do not need to know about the perturbative processes which lead to the production of the physical particle running into the local operator.

This means that we should be able to derive equations for the $Q$-polynomials corresponding to fusings of the type in fig.2 in analogy to what has been done for the first order pole fusings.

A detailed analysis shows that this is can consistently be achieved. The result for a fusing $a + b \to c$ at a third order pole of the S-matrix is exactly equation (29) where we only have to replace the residue appearing in the denominator of the right hand side of (29) by the coefficient of the leading order singularity of $W_{ab}(x',x)^{-1}$.

In other words, due to the bootstrap properties of the S-matrix the fusings corresponding to third order poles of the S-matrix can be treated on exactly the same footing like the first order ones and the corresponding recursion equations for the $Q$-polynomials are structurally identical!

We conjecture that the same phenomenon is true for the higher order forward channel fusings in the ATFTs coming from the E-series as well.

The problem of deriving equations for the form factor in presence of higher odd order poles of the S-matrix has already been addressed in the context of Ising models in [1].

The question arises what happens in the case of the second and fourth order poles of the S-matrix. For a discussion of this question see also [1]. These poles are known
not to participate in the S-matrix bootstrap. It is therefore not possible to draw effective diagrams like the one in fig. 2 for these cases. One might therefore be tempted to think that these poles do not lead to any significant contribution on the form factor level. However, this is not true. We will see in the next section that it is possible to derive a consistent equation for the $Q$-polynomials corresponding to the second order pole processes of the S-matrix in $D_{4}^{(1)}$-ATFT. Moreover, it will turn out in section 5 that these equations are needed in order to constrain the solution spaces of the $Q$-polynomials. Hence it seems that for the even order poles the processes which are needed in order to explain the corresponding S-matrix element have an influence on the solution spaces of the form factors. It is, however, not clear how to derive in general equations for the $Q$-polynomials in these cases even though it might in principle be clear how the equations for the full form factors can be formulated (see e.g. [11, 12, 21] and (39)). The main reason for this mainly the fact that in $D_{n}^{(1)}$-ATFT (in contrast to the $A_{n}^{(1)}$ cases) we do not know a priori how many Feynman diagrams do contribute to an even order pole of the S-matrix and what kinds of particles are involved in a particular diagram. Therefore, up to our present knowledge the contributions of these diagrams have to be studied case by case. As mentioned above we are going to treat the case of the second order pole in $D_{4}^{(1)}$ in detail in the following sections.

4 The $D_{4}^{(1)}$ case

We are now going to discuss the form factors in the simplest $D_{4}^{(1)}$-ATFT. As mentioned above the $D_{4}^{(1)}$-ATFT describes four particles which we label 1, 2, s, and $s'$. The mass of particle 2 is $m_{h} = \sqrt{6}m$ while the masses of the other particles are the same with $m_{l} = \sqrt{2}m$. This means that we can divide the particle content into light particles $l \in \{1, s, s'\}$ and one heavy particle $h = 2$. This kind of symmetry follows from the symmetry of the Dynkin diagram of $D_{4}$ (see e.g. [3]). In this section we are going to present the form factor equations of this particular model. We will see that the symmetries of the Dynkin diagram are maintained at the level of the form factors. Even though this model is quite simple compared to the general $D_{n}^{(1)}$-models it already shows the generic features which are present in $D_{n}^{(1)}$-theories. Referring to the S-matrix we do have a third order fusion pole for the process $h + h \rightarrow h$ and a second order pole for the process $l + h \rightarrow l' + l''$. However, the S-matrix does not have fourth order poles which makes life a bit easier.
We are going to apply the machinery presented in the previous section. First we use (23) and obtain for the total degree of our $Q$-polynomials:

$$\deg Q_{[N_1,N_2,N_s,N_{s'}]} = \frac{3}{2} (N_1(N_1 - 1) + N_s(N_s - 1) + N_{s'}(N_{s'} - 1)) + \frac{7}{2}N_2(N_2 - 1)$$

$$+ 2 (N_1 N_s + N_1 N_{s'} + N_s N_{s'}) + 4N_2 (N_1 + N_s + N_{s'}).$$  (30)

Using the structure of the $S$-matrices we obtain the following equations for the kinematical poles.

$$Q_{ll'} [N_1,N_2,N_s,N_{s'}] (-x, x, x^{(1)}, \ldots, x^{(s')}) = x^{3-3i \frac{(-1)^{N_l}}{F_{ll'}^{\text{inv}}(\pi)}}$$

$$\left(\prod_{i=1}^{N_l} [4]^l_i [2]^l_i [-B]^l_i [B - 2]^l_i [-B - 4]^l_i [B - 6]^l_i \right)$$

$$\prod_{j=1}^{N_{l'}} ([3]^h_j)^2 [1]^h_j [5]^h_j [-B - 1]^h_j [B - 3]^h_j [-B - 3]^h_j [B - 5]^h_j$$

$$\prod_{k=k',h'} \prod_{m=1}^{N_k} [2]^k_m [4]^k_m [-B - 2]^k_m [B - 4]^k_m$$

$$- \ [r] \leftrightarrow [-r] \ Q_{[N_1,N_2,N_s,N_{s'}]} (x^{(1)}, \ldots, x^{(s')}).$$  (31)

The notation in the last line indicates that on the right side of the minus sign appears a polynomial structurally identical to the one on the left side with all the entries in the brackets replaced by their negative values.

In this expression $l$ can be either 1, $s$, or $s'$ and therefore shows the symmetry of the theory on the form factor equation level.

For the kinematical pole of particle $h$ we find

$$Q_{hh} [N_1,N_2,N_s,N_{s'}] (-x, x, x^{(1)}, \ldots, x^{(s')}) = x^{7-3i \frac{(-1)^{N_h}}{F_{hh}^{\text{inv}}(\pi)}}$$

$$\left(\prod_{k=k',h'} \prod_{m=1}^{N_k} [1]^k_i ([3]^k_i)^2 [5]^k_i [-B - 1]^k_i [B - 3]^k_i [-B - 3]^k_i [B - 5]^k_i \right)$$

$$\prod_{j=1}^{N_{h'}} ([2]^h_j)^3 [4]^h_j [3]_{-B - 2}^h_j [B - 4]^h_j [B - 2]^h_j ([B - 4]^h_j)^2 [B - 6]^h_j$$

$$- \ [r] \leftrightarrow [-r] \ Q_{[N_1,N_2,N_s,N_{s'}]} (x^{(1)}, \ldots, x^{(s')}).$$  (32)
To get the recursion equations for the first order bound-state residue equations we introduce the following notation to make the formulae as transparent as possible.

\[
\{r\}^{(k)}_i = [r + 1]^{(k)}_i [r - 1]^{(k)}_i [-h + r + 1 - B]^{(k)}_i [-h + r - 1 + B]^{(k)}_i.
\] (33)

Here \( h \) stands for the Coxeter number which is equal to six in the present case.

We can derive the following recursion relations in \( D_4^{(1)} \) where we put some constant factors appearing in (29) into a constant \( H \) for the particular process.

First we get for \( l + l \rightarrow h \):

\[
Q_{ll[N_1...N_s]}(x\Omega, x\Omega^{-1}, x^{(1)}, x^{(2)}, x^{(s)}, x^{(s')}) = H_{ll} x^3 \prod_{i=1}^{N_l} \{3\}^{(l)}_i \prod_{i=1}^{N_{h}} \{6\}^{(h)}_i \prod_{i=1}^{N_{l'}} \{5\}_i \prod_{i=1}^{N_{l''}} \{5\}_i Q_{ll[N_1...N_s]}(x, x^{(1)}, x^{(2)}, x^{(s)}, x^{(s')})
\] (34)

The next processes to be considered are \( l + h \rightarrow l \) and \( l + l' \rightarrow l'' \). Using (29) we arrive at the following recursion relations

\[
Q_{lh[N_1...N_s]}(x\Omega^4, x\Omega^{-1}, x^{(1)}, \ldots, x^{(s')}) = H_{lh} x^4 \Omega^{6N_l + 4N_{h} + 2(N_{l'} + N_{l''})} \times \prod_{i=1}^{N_l} \{3\}^{(l)}_i \prod_{i=1}^{N_{h}} \{4\}^{(h)}_i \prod_{i=1}^{N_{l'}} \{6\}_i \prod_{i=1}^{N_{l''}} \{5\}_i Q_{lh[N_1...N_s]}(x, x^{(1)}, \ldots, x^{(s')})
\] (35)

and

\[
Q_{ll'[N_1...N_s]}(x\Omega^2, x\Omega^{-2}, x^{(1)}, \ldots, x^{(s')}) = H_{ll'} x^2 \Omega^{2(N_l - N_{l'})} \times \prod_{i=1}^{N_l} \{5\}^{(l)}_i \prod_{i=1}^{N_{l'}} \{7\}_i \prod_{i=1}^{N_{l''}} \{6\}_i \prod_{i=1}^{N_{l''}} \{5\}_i Q_{ll'[N_1...N_s]}(x, x^{(1)}, \ldots, x^{(s')})
\] (36)

respectively. The next process which occurs is \( h + h \rightarrow h \) and is of third order with respect to the S-matrix which means that in the ansatz for \( K \) (23) a pole of second order appears. We evaluate according to the rules for third order forward channel fusion poles and arrive at the following recursion relation

\[
Q_{hh[N_1...N_s]}(x\Omega^2, x\Omega^{-2}, x^{(1)}, \ldots, x^{(s')}) = H_{hh} x^7 \times \prod_{i=1}^{N_{h}} \{6\}_i \prod_{i=1}^{N_{l'}} \{6\}_i \prod_{i=1}^{N_{l''}} \{6\}_i Q_{hh[N_1...N_s]}(x, x^{(1)}, \ldots, x^{(s')})
\] (37)

14
One recognizes that structurally there is no formal difference between the recursion relations which arise from a first order pole and the one coming from a third order pole respectively.

Let us now address the problem of the second order pole which occurs at $\theta = i\pi/2$. The diagram corresponding to this pole and the corresponding cut-diagram which leads to a form factor equation is as follows (see also [11, 12]).

![Diagram for the second order pole](image)

FIG. 3: Diagram for the second order pole

Our goal is to derive a relation for the $Q$-polynomials arising from the second order process. We take the full form factor (3) together with our parametrization of the pole part (23). Of course this ansatz can be taken for granted only after having established its consistency with the second order equation. This means that this ansatz is sufficient in the presence of a second order pole if and only if we derive a polynomial recursion relation for the $Q$’s. This means we do have to verify that the $Q$’s are actually polynomials.

First we have to calculate the momenta of the particles in fig.3. These are at the value of the pole $\theta_1 = \theta + i\pi/2$, $\theta_h = \theta$, and $\theta_v = \theta + i\pi/6$.

Using the explicit form of the minimal form factors [12, 19] we can deduce the following identity in $D_4^{(1)}$.

\[
\prod_{k=1}^{s'} \prod_{i=1}^{N_k} \frac{F_{l'_k}^\text{min}(\theta - \theta_i^{(k)} + i\pi/6)}{F_{l_k}^\text{min}(\theta - \theta_i^{(k)} + i\pi/2)} \frac{F_{l''_h}^\text{min}(\theta - \theta_i^{(k)} + i\pi/6)}{F_{h_k}^\text{min}(\theta - \theta_i^{(k)})} = \prod_{i=1}^{N_t} \frac{1}{\langle 2 \rangle^{(\theta_i)}} \prod_{i=1}^{N_h} \frac{1}{\langle 1 \rangle^{(\theta_i)}}. \tag{38}
\]

If we then write the equation for the full form factors in the following way

\[
-i \text{res}_{\theta = \theta + i\pi/2} F_{ld_{d_1}...d_n}(\theta', \theta, \theta_1, \ldots, \theta_n) = \Gamma^h_{\nu \nu} \Gamma^l_{\nu \nu} F_{\nu \nu \nu d_1...d_n}(\theta + i\pi/6, \theta + i\pi/6, \theta_1, \ldots, \theta_n), \tag{39}
\]
which is in accordance with the two particle form factor equations in an Ising model [11] and with [21], we arrive at another polynomial equation for the $Q$’s

$$Q_{lh[N_1,...,N_s]}(x\Omega^2, x\Omega^{-1}, x^{(1)}, \ldots, x^{s'}) = x^2 \frac{\Gamma_h^{(l)} \Gamma_l^{(l')}}{3\sqrt{3}} \frac{F_{lh}^{\text{min}(0)}}{F_{lh}^{\text{min}(i\pi/2)}} \Omega^{2N_l+N_h}$$

$$\times \prod_{i=1}^{N_l} [\frac{5}{4}]_i \prod_{i=1}^{N_h} [\frac{6}{7}]_i \prod_{i=1}^{N_u} [6]_i \prod_{i=1}^{N_v} [6]_i \prod_{i=1}^{N_w} \prod_{i=1}^{N_x} \Omega^{2N_l+N_h}$$

We can verify the following identity, which should be compared with an expression which occurred in [12] in connection with the second order poles in $A_n^{(1)}$

$$\frac{F_{lh}^{\text{min}(3i\pi/6)}}{F_{lh}^{\text{min}(5i\pi/6)}} = \frac{F_{lh}^{\text{min}(0)}}{F_{lh}^{\text{min}(4i\pi/6)}}.$$  \hfill (41)

This identity is necessary in order to guarantee the consistency of the solutions of (40) with the other equations in $D_4^{(1)}$ due to the presence of minimal form factors at particular values in the equations (28) and (29).

It is remarkable to note that even (39) can more or less be treated on the same footing like the first order equations for the form factor by realizing that if the fictitious vertex (cf. (6)) $\tilde{\Gamma}_{lh} = -i^{\text{res}}_{\theta=i\pi/2} S_{lh}(\theta)$ is equal to the expression $\Gamma_{l'l''}^{h} \Gamma_{l''}^{h}$ which occurs in (33). The quotation marks at the residue mean that we have to evaluate at the leading order of the singularity.

Moreover (40) looks very similar to the ones coming from first order processes. In particular one might have noticed above that the particles running into the local operator acquire a factor $[6] = [h]$ in the recursion coefficient. This also happens here where the two light particles running into the local operator just have this factor in the recursion coefficient.

To all the bound state equations the remark on the order of the particles which has been made after stating the general equation (29) applies. This means that we get additional relations for one and the same $Q$-polynomial at specific values of the arguments!

In the next section we will give some details on the calculation on solutions to the recursion relations derived above. It turns out that we do neccessarily need equation (10) which originates from the second order pole of the S-matrix. It is needed to reduce the degrees of freedom of the solutions.

This is an important fact because it is known [13] and it was mentioned above that the second order diagrams do not really contribute to the S-matrix bootstrap but they are needed for the corresponding form factor bootstrap!
Let us briefly comment on second order poles in general. In principle we can expect an equation for the entire form factor of the kind \((39)\). Even though the position of the second order poles of the S-matrices in the D-series are known there is no closed expression in the literature for the particles actually participating in the second order process. We have checked for the \(D_6\) case that an equation of the type \((39)\) gives a result comparable to \((40)\) for second order processes. However, since \((40)\) has a quite nice structure one might expect to be able to write down a closed recursion formula for all admissible second order processes in the D-series.

5 Some notes on solutions in the \(D_4^{(1)}\) case

Now we are coming to the construction of solutions to the recursion relations derived in the previous section.

It was already pointed out that due to the ansatz for the singular part \(K\) of the form factor \((23)\) the polynomials \(Q_{[N_1,N_2,N_s,N_{s'}]}(x^{(1)},x^{(2)},x^{(s)},x^{(s')})\) are symmetric at least in each of the components of one particular coordinate vector \(x^{(k)}\), with \(k = 1,2,s,s'\). It might therefore be useful to expand \(Q\) in symmetric polynomials. A basis in the space of these polynomials which is commonly used in form factor calculations (see e.g. [5, 6, 7]) are the so called elementary symmetric polynomials \(e_r^{(n)}\). However, even though we are going to use them in the calculations in this sections we have the impression that they might not be the appropriate basis for the problem because the actual calculations become quite tedious when using this basis. We will comment on that problem in the final section.

Let us introduce some simple facts about elementary symmetric polynomials \(p\). As functions of \(n\) variables \(x_1,x_2,\ldots,x_n\) they are defined by the following expression

\[
\prod_{i=1}^{n}(1 + t x_i) = \sum_{r=0}^{n} e_r^{(n)} t^r.
\]

Here the superscript on the \(e\)'s refers to the number of arguments. Explicitly the \(e\)'s are given by \(e_0^{(n)} = 1\), \(e_1^{(n)} = x_1 + x_2 + \ldots x_n\), \(e_2^{(n)} = x_1x_2\ldots x_n\). Comparing \((27)\) with \((42)\) one recognizes at once that all the recursion coefficients in the equations for the \(Q\)-polynomials can in principle be expressed in terms of the elementary symmetric polynomials.
The next entity we need is the concept of a partition \( \lambda \) of a positive integer \( m \). For our purposes this is a finite sequence of non-negative integers arranged in decreasing order \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r) \) with \( \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_r \). Characteristic quantities for \( \lambda \) are its length \( (\ell(\lambda) = r \text{ in our example}) \) and its weight \( |\lambda| = \sum_{i=1}^{r} \lambda_i \) (which is equal to \( m \) in our example).

Having these concepts we define an elementary symmetric function associated to a partition \( \lambda \) by

\[
E^{(n)}_{\lambda} = e_{\lambda_1}^{(n)} \cdots e_{\lambda_r}^{(n)}.
\] (43)

It is then clear that the weight of the partition gives the total degree of \( E^{(n)}_{\lambda} \) while its length gives the maximal power of a particular variable \( x_i \) in \( E^{(n)}_{\lambda} \). The latter is called partial degree.

Since we have recognized that the coefficients in the recursion equations are in fact elementary symmetric polynomials we can read off these equations in how much the degree of a variable \( x_i^{(k)} \) is changed when performing one step in the iteration of the polynomials \( Q \). Since the recursion equations are linked it is not quite a straightforward task to calculate the partial degree of the coordinates of \( (x^{(1)}, x^{(2)}, x^{(s)}, x^{(s')}) \). Given the polynomial \( Q[N_{l},N_{l'},N_{l''},N_{h}](x^{(l)}, x^{(l')}, x^{(l'')}, x^{(h)}) \) where \( l, l', l'' \in \{1, s, s'\} \) we find for the maximal length of the partitions \( \lambda^{(l)} \) corresponding to particle \( l \)

\[
l(\lambda^{(l)}) \leq 3(N_{l} - 1) + 2(N_{l'} + N_{l''}) + 4N_{h}.
\] (44)

The partial degree of \( l' \) and \( l'' \) resp. can be obtained by just replacing the letters in (44).

For the heavy particle we get

\[
l(\lambda^{(h)}) \leq 7(N_{h} - 1) + 4(N_{l} + N_{l'} + N_{l''}).
\] (45)

This result reflects again the symmetry of the Dynkin diagram at the level of form factors.

Due to the presence of four different particles in the theory we have to associate an elementary symmetric polynomial to each of the particles separately. We therefore introduce the following notation

\[
\Lambda = (\lambda^{(1)}|\lambda^{(2)}|\lambda^{(s)}|\lambda^{(s')}).
\] (46)

There are many different ways to choose initial conditions for the symmetric polynomials \( Q \), see e.g. [6]. From the degree-formulas (23), (30) one can see that the \( Q \)-polynomials with only one argument are of zero degree which means that they have to be constant. We therefore choose the following initial values for the \( Q \)’s

18
\[ Q_{[1,0,0,0]} = n_{(1)}, \quad Q_{[0,1,0,0]} = n_{(2)}, \quad Q_{[0,0,1,0]} = n_{(s)}, \quad Q_{[0,0,0,1]} = n_{(s')} \]  

(47)

With this data we can compute the two-particle polynomials. With \( c^{[a,b,c,d]} \) we denote constants which can not be determined by the recursion equations.

If two light particles are present we find

\[ Q_{[2,0,0,0]}(x_{1}^{(1)}, x_{2}^{(1)}) = (\frac{1}{\sqrt{3}} H_{ll} n_{(2)} - 3 c^{[2,0,0,0]} E_{(21|0|0|0)} + c^{[2,0,0,0]} E_{(1|1|0|0)}). \]  

(48)

We have written down the polynomial for two particles of type 1. The polynomials for particles \( s \) and \( s' \) are structurally identical. One has only to change the numbering at \( Q \) and the \( E' \) etc. to the appropriate place.

The next polynomial corresponds to two different light particles. We again write only one member of this family, the other ones can be obtained by symmetry.

\[ Q_{[1,0,1,0]}(x_{1}^{(1)}, x_{1}^{(s)}) = c^{[1,0,1,0]} (E_{(12|0|0|0)} + E_{(0|0|12|0)}) + (H_{ll'} n_{s'} + c^{[1,0,1,0]}) E_{(1|0|1|0)}. \]  

(49)

The polynomial for two heavy particles is already of degree 7 and it is on the two particle level not possible to eliminate most of the constants.

\[ Q_{[0,2,0,0]}(x_{1}^{(2)}, x_{2}^{(2)}) = c_{1}^{[2,0,0,0]} E_{(0|17|0|0)} + c_{2}^{[2,0,0,0]} E_{(0|21|2|0)} + c_{3}^{[2,0,0,0]} E_{(0|2|13|0|0)} + c_{4}^{[2,0,0,0]} E_{(0|2|31|0|0)}, \]  

\[ c_{1}^{[2,0,0,0]} + c_{2}^{[2,0,0,0]} + c_{3}^{[2,0,0,0]} + c_{4}^{[2,0,0,0]} = H_{hh} n_{(2)}. \]  

(50)

The case of one light and one heavy particle is interesting because we can see there that the equation arising from the second order pole of the S-matrix is needed in order to reduce the number of degrees of freedom of the solutions. The basis for this particular case which is a polynomial of degree 4 is spanned by \( E_{(14|0|0|0)}, E_{(13|1|0|0)}, E_{(12|2|0|0)}, E_{(1|3|2|0|0)}, \) and \( E_{(0|14|0|0)} \). If we label the corresponding constants consecutively for these polynomials we see that the first order equations reduce to three degrees of freedom (one of them is actually reduced by the symmetry properties of the \( Q \)-polynomials at particular arguments mentioned above), while the "second order equations" remove another two degrees of freedom. We then find
\[ c_1^{[1,1,0,0]} = \frac{1}{2}(-c_3^{[1,1,0,0]} + P(1 + 2\Omega^{-3})), \quad c_2^{[1,1,0,0]} = (-\frac{1}{2} - \frac{\Omega^{-3}}{\sqrt{3}})c_3^{[1,1,0,0]} - \frac{\Omega^{-3}}{\sqrt{3}}(H - P), \]
\[ c_4^{[1,1,0,0]} = (-\frac{1}{2} - \frac{\Omega^{-3}}{\sqrt{3}})c_3^{[1,1,0,0]} - \frac{\Omega^{-3}}{\sqrt{3}}H - \frac{1}{2}\Omega^3 P, \quad c_5^{[1,1,0,0]} = -\frac{1}{2}c_3^{[1,1,0,0]}, \]

where we have set \( H = H_{th}n_{(1)} \) and \( P = 3c^{[0,0,1,1]} + H_{ll}n_{(s')}. \)

It is clear already at this point that the structure of the solutions becomes quite complicated already at the two particle level due to the high polynomial degrees.

Of course it is straightforward, however tedious, to compute the polynomials for higher particle content as well. For example the polynomial \( Q_{[2,1,0,0]} \) contains almost thirty elementary symmetric polynomials. We do prefer not to list the higher cases in this paper.

We would like to add the following remark. In (44) and (45) we already obtained a result on the partial degree of the \( Q \)-polynomials. It can be taken for granted that all the partitions for a specific \( Q \) which are constrained by the partial degree do contribute to the \( Q \)-polynomial. This means that we will not find a case where the coefficients \( c_{[a,b,c,d]}^{[a,b,c,d]} \) for the corresponding basis function do vanish. In other words all the elementary symmetric basis functions obtained by this general rule do and must nonvanishingly contribute to a consistent solution \( Q_{[a,b,c,d]} \).

We do believe that a basis using elementary symmetric polynomials is not the appropriate one for explicitly calculating the \( Q \)-part of the form factors and therefore would like to postpone a detailed discussion of the higher order polynomials. We are going to comment a bit more on that point in the next section.

### 6 Discussion

By using the form factor bootstrap and a specific (but generic) ansatz (23) for the singular part of the form factors we have been able to derive closed recursion relations for \( Q \)-polynomials within this ansatz. One of these equations arises from the kinematical residue equation of the form factor bootstrap. The other one which is at first a consequence of the first order bound state residue equation of the form factor bootstrap turned out with some
slight modifications to be applicable for the fusings in $D_n^{(1)}$-ATFT which come from third order poles of the S-matrices as well which is a consequence of the S-matrix bootstrap. We did comment on the consequences of the second and fourth order poles of the S-matrix on the $Q$-polynomials. Since at present there is, in contrast to the $A_n^{(1)}$ cases, no rule saying which particles do participate in the Feynman diagrams contributing to the even order poles of the S-matrix we cannot give a general closed equation for the $Q$-polynomials in these cases. However, it is clear that at least some of them are needed in a particular theory in order to reduce the number of degrees of freedom of the $Q$’s.

We treated $D_4^{(1)}$-ATFT as a special case of our general analysis. We have shown how the symmetries of the Dynkin diagram are in a sense maintained in the equations for the $Q$-polynomials in this case. Moreover we treated the influence of the second order poles of the S-matrix on the form factors in detail and derived another equation for the $Q$-polynomials.

In attempting to calculate solutions in the $D_4^{(1)}$ case we realized that due to the high polynomial degree of $Q$ the solution spaces are very complicated when a basis for $Q$ consisting of elementary symmetric polynomials is used.

In order to find solutions without going through tedious computations it would first be interesting to find a Lie algebraic interpretation of the recursion relations. Second one has to think of another polynomial basis in the solution spaces. A direct guess would be to use Macdonald symmetric polynomials \cite{22}. This is because these polynomials are symmetric polynomials which depend on two parameters. This fits well into the affine Toda case since we do have exactly two parameters in the recursion equations. One of them is $\Omega = e^{i\pi/\hbar}$ and for the other one we can take $\Omega^B$ with $B$ being the effective coupling \cite{1}. Due to the weak-strong duality $B \rightarrow 2 - B$ \cite{15} and the symmetry of the $Q$-polynomials which was pointed out in the discussion after equation \cite{23} we have some hint that the Macdonald polynomials might be a proper candidate (see \cite{22}). Also their connection to root systems of Lie algebra (see e.g. \cite{23}) might be of use for the present problem.

Let us make one last comment on the deformation aspect of the solution spaces. In \cite{3} form factors of the scaling Lee-Yang model were calculated. If one considers the closed solution of \cite{3} for the $Q$-polynomials one finds that a solution to the $n$ particle case is exactly one skew Schur function \cite{22}

$$Q_n \sim s_{\lambda/\mu}(x_1, \ldots, x_n), \quad \lambda' = (2n - 2, 2n - 3, \ldots, n - 2), \quad \mu' = (2n - 4, 2n - 6, \ldots, 0),$$

where $\lambda', \mu'$ are the conjugate partitions.
The scaling Lee-Yang model can in some sense be considered as a special case of the sinh-Gordon model (more or less by setting $B \to 0$). Or the latter one is a deformation of the first. In [6] it was shown that for the sinh-Gordon form factor problem it is possible to find a closed solution for the $Q$-polynomials as well. The result is

$$Q_n(k) = \det M_{ij}(k), \quad M_{ij}(k) = [i - j + k]e^{(n)}_{2i-j},$$

where $M$ is a $(n-1) \times (n-1)$ matrix and $[n] = \sin(nB/2)/\sin(B/2)$. This solution is a then a certain deformation of the skew Schur function above.

It would be interesting in the case of $D_n^{(1)}$ theories if in certain limits models exist for which a closed solution for the $Q$’s can be constructed easily which then would allow for hints for the generic cases.

**Acknowledgements**

The author is grateful to Professor R. Sasaki for many helpful and stimulating discussions and for advise concerning the higher order pole properties of the form factors, and to T. Oota for sharing his knowledge on form factors in the $A_n^{(1)}$ cases. Conversations with M. Niedermaier, P. Dorey, R. Tateo, S. Pratik Khastgir, and H.-C. Fu have been of much help as well.

This work was supported by a fellowship of the Japan Society for the Promotion of Science (JSPS) and also partially by the Alexander von Humboldt-Gesellschaft.

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