ON APPROXIMATION OF SOLUTIONS TO THE HEAT EQUATION FROM LEBESGUE CLASS $L^2$ BY MORE REGULAR SOLUTIONS

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Abstract. Let $s \in \mathbb{N}$, $T_1, T_2 \in \mathbb{R}$, $T_1 < T_2$, and $\Omega, \omega$ be bounded domains in $\mathbb{R}^n$, $n \geq 1$, such that $\omega \subset \Omega$ and the complement $\Omega \setminus \omega$ has no (non-empty) compact components in $\Omega$. We prove that this is the necessary and sufficient condition for the space $H^{2s,2s}_{(\mathcal{H})}(\Omega \times (T_1, T_2))$ of solutions to the heat operator $\mathcal{H}$ in a cylinder domain $\Omega \times (T_1, T_2)$ from the anisotropic Sobolev space $H^{2s,2s}_{(\mathcal{H})}(\Omega \times (T_1, T_2))$ to be dense in the space $L^2_{(\mathcal{H})}(\omega \times (T_1, T_2))$, consisting of solutions in the domain $\omega \times (T_1, T_2)$ from the Lebesgue class $L^2(\omega \times (T_1, T_2))$. As an important corollary we obtain the theorem on the existence of a basis with the double orthogonality property for the pair of the Hilbert spaces $H^{2s,2s}_{(\mathcal{H})}(\Omega \times (T_1, T_2))$ and $L^2_{(\mathcal{H})}(\omega \times (T_1, T_2))$.

Introduction

The problem of the uniform approximation on subcompacts of a domain in $\mathbb{R}^{n+1}$ of solutions to the heat equation was solved in the papers by [1], [2] (see also [3] for some refinement related to the so-called rational approximation). It appeared that for this purpose one may use an approach quite similar to the Runge type approximations of solutions to an elliptic system in a lesser domain by solutions in a bigger domain (these include the theorem by C. Runge [4] for the holomorphic functions, the theorem by S.N. Mergelyan [5] for the harmonic functions and their generalizations for spaces of solutions to various systems of partial differential equations (see, for instance, [6] for operators with constant coefficients or [7], ch. 4, ch. 5 for elliptic operators with sufficiently smooth coefficients). More precisely, the key assumption, providing that the space $S_{(\mathcal{H})}(D)$ of solutions to the heat operator $\mathcal{H}$ to be dense in the space $S_{(\mathcal{H})}(D')$ for a pair of domains $D' \subset D \subset \mathbb{R}^{n+1}$, is the absence of compact components of $D \setminus D'$ in $D'$ for any sections $D, D'$ of the domains $D$ and $D'$, respectively, by hyperplanes parallel to the coordinate hyperplane $\{\tau = 0\}$ in $\mathbb{R}^{n+1}$ and containing the point $t$.

However, more delicate approximation problems appeared in the middle of the last century in the theory of the analytic functions. They were related to the approximation in the function spaces where the behaviour of the elements are controlled up to the boundaries of the considered sets, see, for instance, pioneer papers by A.G. Vitushkin [8] and V.P. Havin [9]. Later on it was discovered that these problems of approximation by solutions of various differential equations are closely related to the theory of non-linear potential, see, for instance, [10].

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In the present paper we will discuss approximation of solutions to the heat equation from the normed Lebesgue space $L^2_N(D)$. This imposes certain restrictions on the domain $D \subset \mathbb{R}^{n+1}$. To simplify the exposition we will concentrate our efforts on cylinder domains with the forming side surfaces parallel to the time axis.

One of the reason to consider this problem is the need to construct bases with the double orthogonality property related the spaces of solutions to the heat equations in a pair cylinder domains. We recall that systems with the double orthogonality property are used to investigate operator equations in Hilbert spaces since the middle of the XX-th century, see, for instance, [11]. They were especially useful in the situations where the linear operator $A : H_1 \to H_2$, acting between Hilbert spaces $H_1, H_2$, is injective, compact and it has a dense image. In this case, the basis with the double orthogonality property with respect to two inner products $(\cdot, \cdot)_{H_1}$ and $(A \cdot, A \cdot)_{H_2}$ on the space $H_1$, is the complete system of eigen-vectors of the compact self-adjoint operator $A^* A : H_1 \to H_1$. This allows to construct regularising operators for the operator equation $Au = f$ with given $f \in H_2$ and looked for $u \in H_1$, see, for instance, [12], ch. 12.

At the first part of the XX-th century, long before the formation of the this standard scheme of the functional analysis, S. Bergman suggested to use such systems for spaces of holomorphic functions in the problem of the analytic continuation from a lesser plane domain to a bigger one (see later exposition by [13]). Later, in 1980', L. Aizenberg, using integral representation method, reduced the Cauchy problem for holomorphic functions (of one and several variables) to the problem of the analytic continuation. This opened the way for applying the systems with the double orthogonality property are used to investigate operator equations in Hilbert spaces since the middle of the XX-th century, see, for instance, [11]. They were especially useful in the situations where the linear operator $A : H_1 \to H_2$, acting between Hilbert spaces $H_1, H_2$, is injective, compact and it has a dense image. In this case, the basis with the double orthogonality property with respect to two inner products $(\cdot, \cdot)_{H_1}$ and $(A \cdot, A \cdot)_{H_2}$ on the space $H_1$, is the complete system of eigen-vectors of the compact self-adjoint operator $A^* A : H_1 \to H_1$. This allows to construct regularising operators for the operator equation $Au = f$ with given $f \in H_2$ and looked for $u \in H_1$, see, for instance, [12], ch. 12.

At the first part of the XX-th century, long before the formation of the this standard scheme of the functional analysis, S. Bergman suggested to use such systems for spaces of holomorphic functions in the problem of the analytic continuation. This opened the way for applying the systems with the double orthogonality for the construction of the Carleman formulas, see [15]. This approach was successfully use in order to investigate the Cauchy problem for a wide class of elliptic (including the overdetermined ones) systems with real analytic coefficients, see [13] ch. 10, ch. 12, [16], and even to elliptic differential complexes, see [17]. Taking in the account the requirements described above for a continuous operator $A : H_1 \to H_2$, the key issues for this theory were the Uniqueness Theorems for solutions to elliptic systems with real analytic coefficients, providing its injectivity, the Sobolev Embedding Theorems, Rellich-Kondrashov Theorem and/or Stiltjes-Vitaly Theorem, guaranteeing its compactness, and the Runge type Theorems on the approximation of solutions in a lesser domain by the solutions in a bigger one.

It appeared that the Cauchy problem for the heat operator $H = \frac{\partial}{\partial \tau} - \sum_{j=1}^n \frac{\partial^2}{\partial x_j^2}$ in a cylinder domain $\mathbb{R}^n \times \mathbb{R}$ with the Cauchy data on a part of its lateral surface (that naturally arises in the diffusion problems, for instance in the inverse problem of the electrocardiography using models of the charge diffusion in the heart tissues) can be also reduced to the continuation problem for solutions of the heat equations from a lesser cylinder domain to a bigger one, see [18], [19]. Since the solutions to the heat equation are real analytic with respect to the space variables (see, for instance, [20] ch. VI, §1, theorem 1]), then for domains $\omega \subset \Omega \subset \mathbb{R}^n$ and numbers $T_1, T_2 \in \mathbb{R}$, $T_1 < T_2$, the natural embedding operator

$$A : H^{2s,s}_H(\Omega \times (T_1, T_2)) \to H^{2s',s'}_H(\omega \times (T_1, T_2)), \; s \geq s', s, s' \in \mathbb{Z}_+,$$

between anisotropic spaces $H^{2s,s}_H(\Omega \times (T_1, T_2))$ and $H^{2s',s'}_H(\omega \times (T_1, T_2))$, consisting of solutions to the heat operator $H$ from the Sobolev class $H^{2s,s}_H(\Omega \times (T_1, T_2))$ in a cylinder domain $\Omega \times (T_1, T_2) \subset \mathbb{R}^{n+1}$ and the Sobolev $H^{2s',s'}_H(\omega \times (T_1, T_2))$ in the
domain $\omega \times (T_1, T_2)$, respectively, is injective. The compactness of the operator $A$ may be easily extracted from general embedding theorems for anisotropic Sobolev type spaces, see, for instance, [21, ch. III and ch. VI], or from the results by J.-L. Lions [22, ch. 1, §5], see [22] below. As for the density of the range of the operator $A$, it is precisely connected with the approximation theorems for the solutions to the heat equation.

1. A density theorem

Let $\mathbb{R}^n$ be $n$-dimensional Euclidean space with the coordinates $x = (x_1, \ldots, x_n)$, and $\Omega \subset \mathbb{R}^n$ be a bounded domain. As usual, denote by $\overline{\Omega}$ the closure of $\Omega$, and by $\partial \Omega$ its boundary. We assume that $\partial \Omega$ is piece-wise smooth hypersurface. We denote also by $\Omega_{T_1, T_2}$ a bounded open cylinder $\Omega \times (T_1, T_2)$ in $\mathbb{R}^{n+1}$ with $T_1 < T_2$.

We consider functions over $\mathbb{R}^n$ and $\mathbb{R}^{n+1}$. For $s \in \mathbb{Z}_+$, we denote by $C^s(\Omega)$ the space of all $s$-times continuously differentiable functions over $\Omega$, and by $C^s(\Omega)$ the subset of $C^s(\Omega)$ such that for each function $u \in C^s(\Omega)$ and each multi-index $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}_+^n$ there is a function $u_{\alpha}$, continuous on $\overline{\Omega}$ and satisfying $\partial^\alpha u = u_{\alpha}$ in $\Omega$. Besides, let $L^2(\Omega)$ be the Lebesgue space over $\Omega$ with the standard inner product $(u, v)_{L^2(\Omega)}$, and $H^s(\Omega)$ be the Sobolev space, $s \in \mathbb{N}$, with the standard inner product $(u, v)_{H^s(\Omega)}$. As it is well known, both $L^2(\Omega)$ and $H^s(\Omega)$ are Hilbert spaces.

Investigating spaces of solutions to the heat equation, we need the anisotropic Sobolev spaces $H^{2s, s}(\Omega_{T_1, T_2})$, $s \in \mathbb{Z}_+$, see, for instance, [23, ch. 2], i.e. the set of all measurable functions $u$ over $\Omega_{T_1, T_2}$ such that (generalised) partial derivatives $\partial_t^j \partial_x^\alpha u$ belong to the Lebesgue space $L^2(\Omega_{T_1, T_2})$ for all multi-indexes $(\alpha, j) \in \mathbb{Z}_+^n \times \mathbb{Z}_+$ with $|\alpha| + 2j \leq 2s$. This is a Hilbert space with the inner product

$$(u, v)_{H^{2s, s}(\Omega_{T_1, T_2})} = \sum_{|\alpha|+2j \leq 2s} \int_{\Omega_{T_1, T_2}} \partial_t^j \partial_x^\alpha v(x, t) \partial_t^j \partial_x^\alpha u(x, t) dx dt.$$ 

For $s = 0$ we obtain $H^{0, 0}(\Omega_{T_1, T_2}) = L^2(\Omega_{T_1, T_2})$. In particular, $C^\infty(\Omega_{T_1, T_2})$ can be considered as the intersection $\cap_{s=0}^{\infty} H^{2s, s}(\Omega_{T_1, T_2})$.

Finally, for $k \in \mathbb{Z}_+$, we denote by $H^{k, 2s, s}(\Omega_{T_1, T_2})$ the set of all functions $u$ in $H^{2s, s}(\Omega_{T_1, T_2})$ such that $\partial_x^\alpha u \in H^{2s, s}(\Omega_{T_1, T_2})$ for all $|\beta| \leq k$. This is a Hilbert space with the inner product

$$(u, v)_{H^{k, 2s, s}(\Omega_{T_1, T_2})} = \sum_{|\beta| \leq k} (\partial^\beta u, \partial^\beta v)_{H^{2s, s}(\Omega_{T_1, T_2})}.$$ 

We also will use the so-called Bocher spaces of functions depending on $(x, t)$ from the strip $\mathbb{R}^n \times [T_1, T_2]$. Namely, for a Banach space $B$ (for example, the space of functions on a subdomain of $\mathbb{R}^n$) and $p \geq 1$, we denote by $L^p(I, B)$ the Banach space of all the measurable mappings $u : [T_1, T_2] \to B$ with the finite norm

$$\|u\|_{L^p([T_1, T_2], B)} := \|\|u(\cdot, t)\|_B\|_{L^p([T_1, T_2])},$$ 

see, for instance, [22, ch. §1.2], [21, ch. III, §1].

The space $C([T_1, T_2], B)$ is introduced with the use of the same scheme; this is the Banach space of all the measurable mappings $u : [T_1, T_2] \to B$ with the finite norm

$$\|u\|_{C([T_1, T_2], B)} := \sup_{t \in [T_1, T_2]} \|u(\cdot, t)\|_B.$$
Now let $S_H(\Omega_{T_1,T_2})$ be the set of all generalised functions over $\Omega_{T_1,T_2}$, satisfying the (homogeneous) heat equation
\begin{equation}
Hu = 0 \text{ in } \Omega_{T_1,T_2}
\end{equation}
in the sense of distributions. First of all, we note that according to the hypoellipticity of the operator $H$, solutions to equation (1.1) are infinitely differentiable on their domains (see, for instance, [20, ch. VI, §1, theorem 1]), i.e.

$$S_H(\Omega_{T_1,T_2}) \subset C^\infty(\Omega_{T_1,T_2}).$$

As it is known, this is a closed subspace in the space $C^\infty(\omega_{T_1,T_2})$ with the standard Fréchet topology (inducing the uniform convergence together with all the partial derivatives on compact subsets of $\omega_{T_1,T_2}$).

Also, we need the space $S_H(\Omega_{T_1,T_2})$, defined as the union of the spaces

$$\bigcup_{\Omega_{T_1,T_2} \supset \tilde{\Omega}_{T_1,T_2}} S_H(\tilde{\Omega}_{T_1,T_2}),$$

where the union is with respect to all the domains $\tilde{\Omega}_{T_1,T_2}$, containing the closure of the domain $\Omega_{T_1,T_2}$. Usually, this space is endowed with the topology of the inductive limit associated with a decreasing sequences of neighbourhoods of the compact $\tilde{\Omega}_{T_1,T_2}$; however, we will not use any topology of this space, considering it as a set of functions.

Let $H^{k,2s,s}_{H}(\Omega_{T_1,T_2}) = H^{k,2s,s}(\Omega_{T_1,T_2}) \cap S_H(\Omega_{T_1,T_2})$, $s \in \mathbb{Z}_+, k \in \mathbb{Z}_+$. As it is known, this is a closed subspace of the Sobolev space $H^{k,2s,s}(\Omega_{T_1,T_2})$. Similarly, $C^\infty(\Omega_{T_1,T_2}) = C^\infty(\Omega_{T_1,T_2}) \cap S_H(\Omega_{T_1,T_2})$ is a closed subspace, consisting of solutions to equation (1.1), of the space $C^\infty(\Omega_{T_1,T_2})$ with the standard Fréchet topology. The hypoellipticity of the operator $H$ provides the following (continuous) embeddings

\begin{equation}
S_H(\Omega_{T_1,T_2}) \subset C^\infty(\Omega_{T_1,T_2}) \subset H^{k,2s,s}_{H}(\Omega_{T_1,T_2})
\end{equation}

for all $k, s \in \mathbb{Z}_+$.

Now we may formulate the main results of this paper.

**Theorem 1.1.** If $\omega \subset \Omega \subset \mathbb{R}^n$, $\partial \omega, \partial \Omega \in C^2$ then $S_H(\Omega_{T_1,T_2})$ is everywhere dense in $L^2_H(\omega_{T_1,T_2})$ if and only if the complement $\Omega \setminus \omega$ has no compact components in $\Omega$.

**Proof.** Sufficiency. Clearly, the set $S_H(\Omega_{T_1,T_2})$ is everywhere dense in $L^2_H(\omega_{T_1,T_2})$ if and only if the following relation
\begin{equation}
(u, w)_{L^2(\omega_{T_1,T_2})} = 0 \text{ for all } w \in S_H(\Omega_{T_1,T_2})
\end{equation}
means precisely for a function $u \in L^2_H(\omega_{T_1,T_2})$ that $u = 0$ in $\omega_{T_1,T_2}$. Of course, the zero function of the space $L^2_H(\omega_{T_1,T_2})$ satisfies (1.3).

Assume that the complement $\Omega \setminus \omega$ has no (non-empty connected) compact components in $\Omega$. In order to prove the sufficiency of the statement we will use the fact that the heat operator $H$ admits the bilateral fundamental solution of the convolution type, see, for instance, [20, 25]:

\[
\Phi(x,t) = \begin{cases} 
\frac{e^{-\frac{|x|^2}{4t}}}{(2\sqrt{\pi}t)^n} & \text{if } t > 0, \\
0 & \text{if } t \leq 0.
\end{cases}
\]
By the definition,
\begin{equation}
H_{x,t}\Phi(x-y, t-\tau) = \delta(x-y, t-\tau), \tag{1.4}
\end{equation}
\begin{equation}
H'_{y,\tau}\Phi(x-y, t-\tau) = \delta(x-y, t-\tau), \tag{1.5}
\end{equation}
where $H'_{y,\tau} = -\frac{\partial}{\partial \tau} - \sum_{j=1}^{n} \frac{\partial^2}{\partial y_j^2}$ is the formal adjoint operator for $H$ and $\delta(x,t)$ is the Dirac functional with the support at the point $(x,t)$.

Let for a function $u \in L^2_H(\omega_{T_1, T_2})$ relation \eqref{1.3} holds true.

Consider an auxiliary function
\begin{equation}
v(y, \tau) = \int_{\omega_{T_1, T_2}} u(x,t) \Phi(x-y, t-\tau) \, dx \, dt. \tag{1.6}
\end{equation}
According to \eqref{1.4},
\[ H_{x,t}\Phi(x-y, t-\tau) = 0 \text{ if } (x,t) \neq (y,\tau). \]
Hence
\[ H_{x,t}\Phi(x-y, t-\tau) = 0 \text{ in } \Omega_{T_1, T_2} \]
for any fixed pair $(y, \tau) \notin \Omega_{T_1, T_2}$. Then, using the hypoellipticity of the operator $H$, we conclude that $\Phi(x-y, t-\tau) \in S_H(\Omega_{T_1, T_2})$ with respect to $(x,t) \in \Omega_{T_1, T_2}$ for any fixed pair $(y, \tau) \notin \Omega_{T_1, T_2}$. In particular, relation \eqref{1.3} implies
\begin{equation}
v(y, \tau) = 0 \text{ in } \mathbb{R}^{n+1} \setminus \bar{\omega}_{T_1, T_2}. \tag{1.7}
\end{equation}
On the other hand, by property \eqref{1.3} of the fundamental solution,
\begin{equation}
H'v = \chi_{\omega_{T_1, T_2}} u \text{ in } \mathbb{R}^{n+1}, \tag{1.8}
\end{equation}
where $\chi_{\omega_{T_1, T_2}}$ is the characteristic function of the domain $\omega_{T_1, T_2}$. Obviously,
\[ H'_{y,\tau}v(y, \tau) = 0 \text{ in } D \times (t_1, t_2) \]
for some domain $D \subset \mathbb{R}^n$ and numbers $t_1 < t_2$ if and only if
\[ H_{y,\tau}v(y, -\tau) = 0 \text{ in } D \times (-t_2, -t_1). \]
Therefore, according to \cite{23} ch. VI, §1, theorem 1), the function $v$ is real analytic with respect to the space variables of $\mathbb{R}^{n+1} \setminus \bar{\omega}_{T_1, T_2}$.

Since the domain $\omega$ has a smooth boundary, each component of $\mathbb{R}^n \setminus \omega$ is itself a non-empty open domain with a smooth boundary and the similar fact is true for the domain $\Omega$. However the complements $\Omega \setminus \omega$ has no compact components in $\Omega$ and hence each component of $\mathbb{R}^n \setminus \omega$ intersects with $\mathbb{R}^n \setminus \Omega$ by a non-empty open set. Thus, \eqref{1.8} and the uniqueness theorem for real analytic functions yields
\begin{equation}
v(y, \tau) = 0 \text{ in } \mathbb{R}^{n+1} \setminus \bar{\omega}_{T_1, T_2}. \tag{1.9}
\end{equation}
Besides, \eqref{1.8}, \eqref{1.9} mean that the function $\tilde{v}(y, \tau) = v(y, -\tau)$ is a solution to the Cauchy problem
\[
\begin{align*}
&H\tilde{v} = \chi_{\omega_{T_1, T_2}} u \text{ in } \mathbb{R}^n \times (-T_2 - 1, 1 - T_1), \\
&\tilde{v}(y, -T_2 - 1) = 0 \text{ on } \mathbb{R}^n.
\end{align*}
\]
Then, according to \cite{23} ch. 2, §5, theorem 3], $\tilde{v} \in H^{2,1}(\mathbb{R}^n \times (-T_2 - 1, 1 - T_1))$, and the solution in this class is unique. Moreover, the regularity of this unique solution to the Cauchy problem may be expresses in terms of the Bochner spaces, too. Namely, $\tilde{v} \in C([-T_2 - 1, 1 - T_1], \mathcal{H}^1(\mathbb{R}^n)) \cap L^2([-T_2 - 1, 1 - T_1], \mathcal{H}^2(\mathbb{R}^n))$, see,
for instance, \[20\] ch. 3, §1], where similar linear problems for Stokes equations are considered. In particular, the function \(v\) belongs to the space
\[(1.10)\]
\(C([T_{1} - 1, T_{2} + 1], H^{1}(\mathbb{R}^{n})) \cap L^{2}([T_{1} - 1, T_{2} + 1], H^{2}(\mathbb{R}^{n})) \cap H^{2, 1}(\mathbb{R}^{n} \times (T_{1} - 1, T_{2} + 1)).\)

**Lemma 1.2.** Any function of the type \((1.6)\), satisfying \((1.9)\), may be approximated by elements of \(C_{0}^{\infty}(\omega_{T_{1}, T_{2}})\) in the topology of the Hilbert space \(H^{2, 1}(\omega_{T_{1}, T_{2}})\).

**Proof.** First of all, we note that such a function may be approximated by functions of the class \(C_{0}^{\infty}(\mathbb{R}^{n+1})\) in the topology of the space \(H^{2, 1}(\mathbb{R}^{n+1})\). This fact can be extracted from \[20\] ch. 3, §7, property 6], but it can be proved directly, too.

Indeed, denote by \(h_{\delta}(x)\) the standard compactly supported function with the support in the ball \(B(x, \delta) \subset \mathbb{R}^{n}\) with the centre at the point \(x\) and of the radius \(\delta > 0:\)

\[
h_{\delta}(x) = \begin{cases} 0, & \text{if } |x| \geq \delta, \\ c(\delta) \exp \left(1/(|x|^{2} - \delta^{2})\right), & \text{if } |x| < \delta, \end{cases}
\]

where \(c(\delta)\) is the constant providing equality

\[
\int_{\mathbb{R}^{n}} h(x) \, dx = 1.
\]

Then, as it is well known, the standard regularisation

\[
(R_{\delta}v)(x, t) = \int_{\mathbb{R}^{n+1}} h_{\delta}(x - y, t - \tau)v(y, \tau) \, dy \, d\tau
\]

belongs to the space \(C_{0}^{\infty}(\mathbb{R}^{n+1})\) for any positive number \(\delta\) and

\[
\lim_{\delta \to +0} \|v - R_{\delta}v\|_{L^{2}(\mathbb{R}^{n+1})} = 0,
\]

see, for instance, \[20\]. Since the standard regularization is defined with the use of the convolution, and the function \(v\) belongs to \(H^{2, 1}(\mathbb{R}^{n+1})\) and supported in \(\overline{\omega_{T_{1}, T_{2}}},\)

\[
\partial_{x}^{\alpha} \partial_{t}^{\beta}(R_{\delta}v)(x, t) = (R_{\delta} \partial_{x}^{\alpha} \partial_{t}^{\beta} v)(x, t),
\]

if \(|\alpha| + 2j \leq 2\). Therefore

\[
\lim_{\delta \to +0} \|\partial_{x}^{\alpha} \partial_{t}^{\beta} v - \partial_{x}^{\alpha} \partial_{t}^{\beta}(R_{\delta}v)\|_{L^{2}(\mathbb{R}^{n+1})} = 0 \text{ if } |\alpha| + 2j \leq 2,
\]

and then

\[
\lim_{\delta \to +0} \|v - R_{\delta}v\|_{H^{2, 1}(\mathbb{R}^{n+1})} = 0.
\]

Next, we may continue the proof with the use standard scheme, see, for example. Indeed, denote by \(\partial_{\nu} = \sum_{j=1}^{n} \nu_{j} \partial_{x_{j}}\) the normal derivative, where \(\nu(x) = (\nu_{1}(x), ..., \nu_{n}(x))\) is the unit external normal vector to the surface \(\partial \Omega\) at the point \(x\). If \(\partial \omega\) is a surface of class \(C^{2}\), then, as the function \(v\) belongs to space \((1.10)\), we see that there are the traces

\[
v|_{\partial(\omega_{T_{1}, T_{2}})} \in H^{1/2}(\partial(\omega_{T_{1}, T_{2}})),
\]

\[
v|_{(\partial \omega)_{T_{1}, T_{2}}} \in L^{2}([T_{1}, T_{2}], H^{1/2}(\partial \omega)), \quad \partial_{\nu} v|_{(\partial \omega)_{T_{1}, T_{2}}} \in L^{2}([T_{1}, T_{2}], H^{1/2}(\partial \omega)),
\]

cf. \[20\] ch. 3, §7, property 7].

Besides, according to \((1.9)\),

\[
v = 0 \text{ on } \partial(\omega_{T_{1}, T_{2}}), \quad \partial_{\nu} v = 0 \text{ on } (\partial \omega)_{T_{1}, T_{2}}.
\]
Hence, the spectral synthesis theorem, see [20], implies that \( v \) belongs to both the space \( C^1 \) and the space
\[
C([T_1, T_2], H_0^1(\omega)) \cap L^2([T_1, T_2], H_0^1(\omega)) \cap H_0^1(\omega) \cap H^{2,1}(\omega),
\]
where \( H_0^1(\omega) \) is the closure in \( H^* \) of the space \( C_0^\infty(\omega) \) of infinitely differentiable functions with compact supports in \( \omega \).

On the other hand, if \( \partial \omega \in C^2 \), then there is a real valued function \( \rho \), two times continuously differentiable in a neighbourhood \( U \) of the surface \( \partial \omega \) and such that
\[
\omega = \{ x \in \mathbb{R}^n : \rho(x) < 0 \}, \quad \nabla \rho \neq 0 \text{ in } U.
\]

Hence, for all sufficiently small numbers \( \varepsilon > 0 \) the sets
\[
\omega^\varepsilon = \{ x \in \mathbb{R}^n : \rho(x) < -\varepsilon \}
\]
are domains with boundaries of class \( C^2 \) and
\[
\omega^\varepsilon \subset \omega^\varepsilon' \subset \omega,
\]
if \( 0 < \varepsilon' < \varepsilon \); moreover for the Lebesgue measure of the domain \( \omega \setminus \overline{\omega^\varepsilon} \) we have
\[
\lim_{\varepsilon \to 0^+} \text{mes}(\omega \setminus \overline{\omega^\varepsilon}) = 0.
\]

According to [20] ch. 3, §5, lemma 1], if \( \partial \omega \in C^1 \) then there is a constant \( C_0(\partial \omega) \), depending on the square of the surface \( \partial \omega \), only, and such that
\[
\| \tilde{v} \|_{L^2(\omega \setminus \overline{\omega^\varepsilon})} \leq C_0(\partial \omega) \varepsilon \| \tilde{v} \|_{H^1(\omega \setminus \overline{\omega^\varepsilon})}
\]
for any function \( \tilde{v} \in H^1(\omega) \) with zero trace \( \tilde{v}|_{\partial \omega} \) on \( \partial \omega \).

Similarly, if \( \partial \omega \in C^2 \), then there are constants \( C_1(\partial \omega) \), \( C_2(\partial \omega) \), depending on the square of the surface \( \partial \omega \), and such that
\[
\begin{align*}
\| \tilde{v} \|_{L^2(\omega \setminus \overline{\omega^\varepsilon})} & = C_1(\partial \omega) \varepsilon^2 \| \tilde{v} \|_{H^2(\omega \setminus \overline{\omega^\varepsilon})}, \\
\| \nabla \tilde{v} \|_{L^2(\omega \setminus \overline{\omega^\varepsilon})} & = C_2(\partial \omega) \varepsilon \| \tilde{v} \|_{H^2(\omega \setminus \overline{\omega^\varepsilon})}
\end{align*}
\]
for any function \( \tilde{v} \in H^2(\omega) \) with zero traces \( \tilde{v}|_{\partial \omega} \) and \( \partial_v \tilde{v}|_{\partial \omega} \) on \( \partial \omega \).

Set
\[
R^{(1)}_\varepsilon(x) = \int_{\omega \setminus \overline{\omega^\varepsilon}} h_{\varepsilon/3}(x - y) dy,
\]
\[
R^{(2)}_\varepsilon(t) = \int_{T_1 + \varepsilon/2}^{T_2 - \varepsilon/2} h_{\varepsilon/3}(t - \tau) d\tau.
\]

It is known, see, for instance, [20] ch. 3, §5], that
\[
0 \leq R^{(1)}_\varepsilon \leq 1, \quad 0 \leq R^{(2)}_\varepsilon \leq 1,
\]
\[
|\partial^\alpha R^{(1)}_\varepsilon| \leq c^\alpha \varepsilon^{-|\alpha|}, \quad |\partial^\alpha R^{(2)}_\varepsilon| \leq c^\alpha \varepsilon^{-|\alpha|}
\]
for all \( x \in \mathbb{R}^n \), \( t \in \mathbb{R} \), \( \alpha \in \mathbb{Z}^n_+ \), \( j \in \mathbb{Z}^n_+ \), with some positive constants \( c^\alpha, c^\alpha_j \), independent on \( x \) and \( t \).

Fix a sequence \( \{ v_k \} \subset C_0^\infty(\mathbb{R}^{n+1}) \), converging to \( v \) in the space \( H^{2,1}(\mathbb{R}^{n+1}) \).

Then the functional sequence
\[
\{ v_k, \varepsilon(x, t) = R^{(2)}_\varepsilon(t) R^{(2)}_\varepsilon(x) v_k(x, t) \}
\]
lies to \( C_0^\infty(\omega_{T_1, T_2}) \).

By the triangle inequality,
\[
\| v - v_k, \varepsilon \|_{H^{2,1}(\omega_{T_1, T_2})} \leq \| v - v_k \|_{H^{2,1}(\omega_{T_1, T_2})} + \| v - v_k, \varepsilon \|_{H^{2,1}(\omega_{T_1, T_2})},
\]
As
\[
\lim_{k \to +\infty} \|v - v_k\|_{H^{2,1}(\omega_{T_1+\varepsilon}, T_2)} = 0,
\]
then we need to estimate the second summand in the right hand side of formula (1.16), only. However,
\[
v_k(x, t) - v_{k, \varepsilon}(x, t) = (1 - R^{(1)}_\varepsilon(t)R^{(2)}_\varepsilon(x))v_k(x, t)
\]
and, in particular,
\[
v_k(x, t) - v_{k, \varepsilon}(x, t) = 0 \text{ for all } (x, t) \in \omega^\varepsilon \times (T_1 + \varepsilon, T_2 - \varepsilon).
\]
Hence,
\[
2^{-1}\|v_k - v_{k, \varepsilon}\|_{H^{2,1}(\omega_{T_1, T_2})} \leq \sum_{|\alpha| + 2\beta \leq 2} \|((1 - R^{(1)}_\varepsilon R^{(2)}_\varepsilon)\partial_x^\alpha \partial_t^\beta v_k\|_{L^2(\omega_{T_1, T_2})}^2 + \|
(\partial^\alpha R^{(1)}_\varepsilon)R^{(2)}_\varepsilon v_k\|_{L^2((\omega^\varepsilon)_{T_1, T_2})}^2 + \sum_{|\beta| = 1, |\gamma| = 1} \|\partial^\beta R^{(1)}_\varepsilon R^{(2)}_\varepsilon \partial^\gamma v_k\|_{L^2((\omega^\varepsilon)_{T_1, T_2})}^2.
\]
Since \( v \) belongs to the space (1.11), then (1.12), (1.14), (1.15) and the Fubini theorem imply that
\[
\sum_{1 \leq |\alpha| \leq 2} \|\partial^\alpha R^{(1)}_\varepsilon R^{(2)}_\varepsilon v\|_{L^2((\omega^\varepsilon)_{T_1, T_2})}^2 \leq \tilde{C} \sum_{1 \leq |\alpha| \leq 2} \varepsilon^{-|\alpha|} \int_{T_1}^{T_2} \|v\|_{H^2(\omega^\varepsilon)}^2 dt \leq C\|v\|_{H^{2,1}(\omega_{T_1, T_2})}^2
\]
and, similarly,
\[
\sum_{|\beta| = 1, |\gamma| = 1} \sum_{1 \leq |\alpha| \leq 2} \|\partial^\beta R^{(1)}_\varepsilon R^{(2)}_\varepsilon \partial^\gamma v\|_{L^2((\omega^\varepsilon)_{T_1, T_2})}^2 \leq \tilde{C} \sum_{|\beta| = 1} \varepsilon^{-|\beta|} \int_{T_1}^{T_2} \|v\|_{H^2(\omega^\varepsilon)}^2 dt \leq C\|v\|_{H^{2,1}(\omega_{T_1, T_2})}^2 \leq C\|v\|_{H^{2,1}(\omega_{T_1, T_2})}^2
\]
with constants \( C, \tilde{C} \), independent on \( v \) and \( \varepsilon \).

The boundaries of the cylinder domains \( \omega_{T_1, T_1+\varepsilon} \) and \( \omega_{T_2-\varepsilon, T_2} \) are not smooth, but combining results [20] ch. 3, §5] related to a function \( v \), having the trace vanishing on surfaces \( \omega \times T_1 \) and \( \omega \times T_2 \), with bounds (1.14), (1.15), we see that
\[
\sum_{|\alpha| + 2\beta \leq 2} \|((1 - R^{(1)}_\varepsilon R^{(2)}_\varepsilon)\partial_x^\alpha \partial_t^\beta v_k\|_{L^2(\omega_{T_1, T_2})}^2 \leq C\|v\|_{H^{2,1}(\omega_{T_1, T_2})}^2
\]
with a constant \( C \), independent on \( v \) and \( \varepsilon \).
Using the continuity of the Lebesgue integral with respect to the measure of the integration set, we conclude that
\begin{equation}
\lim_{\varepsilon \to +0} \|v\|_{H^{2,1}(\omega_{T_1,T_2})} = 0.
\end{equation}

Fix a number \( E > 0 \). Relation (1.17) means that there is a number \( N(E, \varepsilon) \in \mathbb{N} \) such that for all \( k \geq N(E, \varepsilon) \) we have
\begin{equation}
\|v - v_k\|_{H^{2,1}(\omega_{T_1,T_2})} < E\varepsilon^2.
\end{equation}

In this case, (1.23) implies that for such \( k \) we have
\begin{equation}
2^{-1} \sum_{|\alpha|+|\beta| \leq 2} \|(1 - R^{(1)}_{\varepsilon} R^{(2)}_{\varepsilon}) \partial_{x}^{\alpha} \partial_{t}^{\beta} v_k\|_{L^2(\omega_{T_1,T_2})}^2 \leq \\
\sum_{|\alpha|+|\beta| \leq 2} \|(1 - R^{(1)}_{\varepsilon} R^{(2)}_{\varepsilon}) \partial_{x}^{\alpha} \partial_{t}^{\beta} (v_k - v)\|_{L^2(\omega_{T_1,T_2})}^2 + \|(1 - R^{(1)}_{\varepsilon} R^{(2)}_{\varepsilon}) \partial_{x}^{\alpha} \partial_{t}^{\beta} v\|_{L^2(\omega_{T_1,T_2})}^2 \leq \\
C(E\varepsilon^2 + \|v\|_{H^{2,1}(\omega_{T_1,T_2})})
\end{equation}
with a constant \( C \), independent on \( v \) and \( \varepsilon \).

Besides, (1.20) and (1.21) yield
\begin{equation}
2^{-1} \sum_{1 \leq |\alpha| \leq 2} \|(\partial_{x}^{\alpha} R^{(1)}_{\varepsilon} R^{(2)}_{\varepsilon}) v_k\|_{L^2(\omega_{T_1,T_2})}^2 \leq \\
\sum_{1 \leq |\alpha| \leq 2} \|(\partial_{x}^{\alpha} R^{(1)}_{\varepsilon} R^{(2)}_{\varepsilon})(v_k - v)\|_{L^2(\omega_{T_1,T_2})}^2 + \|(\partial_{x}^{\alpha} R^{(1)}_{\varepsilon} R^{(2)}_{\varepsilon}) v\|_{L^2(\omega_{T_1,T_2})}^2 \leq \\
C(E + \|v\|_{H^{2,1}(\omega_{T_1,T_2})})
\end{equation}
and
\begin{equation}
2^{-1} \sum_{|\beta| = 1} \sum_{|\gamma| = 1} \|(\partial_{x}^{\beta} R^{(1)}_{\varepsilon} R^{(2)}_{\varepsilon}) (\partial_{x}^{\gamma} v_k)\|_{L^2(\omega_{T_1,T_2})}^2 \leq \\
\sum_{|\beta| = 1} \sum_{|\gamma| = 1} \|(\partial_{x}^{\beta} R^{(1)}_{\varepsilon} R^{(2)}_{\varepsilon})(\partial_{x}^{\gamma} (v_k - v))\|_{L^2(\omega_{T_1,T_2})}^2 + \|(\partial_{x}^{\beta} R^{(1)}_{\varepsilon} R^{(2)}_{\varepsilon})(\partial_{x}^{\gamma} v)\|_{L^2(\omega_{T_1,T_2})}^2 \leq \\
C(\varepsilon + \|v\|_{H^{2,1}(\omega_{T_1,T_2})})
\end{equation}
with a constant \( C \), independent on \( v \) and \( \varepsilon \).

Similarly, using (1.22), we obtain
\begin{equation}
\|R^{(1)}_{\varepsilon}(\frac{dR^{(2)}_{\varepsilon}}{dt}) v_k\|_{L^2(\omega_{T_1,T_2})}^2 \leq \\
\|R^{(1)}_{\varepsilon}(\frac{dR^{(2)}_{\varepsilon}}{dt})(v_k - v)\|_{L^2(\omega_{T_1,T_2})}^2 + \|R^{(1)}_{\varepsilon}(\frac{dR^{(2)}_{\varepsilon}}{dt}) v\|_{L^2(\omega_{T_1,T_2})}^2 \leq \\
C(\varepsilon + \|v\|_{H^{2,1}(\omega_{T_1,T_2})})
\end{equation}
with a constant \( C \), independent on \( v \) and \( \varepsilon \).

Finally, combining estimates (1.19), (1.23) - (1.28) and taking in account (1.24), we conclude that the statement of the lemma is fulfilled. \(\square\)
Now, using lemma [122] and fixing a sequence \( \{v_k\} \subset C_0^\infty(\omega_{T_1, T_2}) \), converging to the function \( v \) in \( H^{2, 1}(\omega_{T_1, T_2}) \) we see that
\[
\|u\|_{L^2(\omega_{T_1, T_2})}^2 = (u, \mathcal{H}v)_{L^2(\omega_{T_1, T_2})} = \lim_{k \to +\infty} (u, \mathcal{H}v_k)_{L^2(\omega_{T_1, T_2})} = 0,
\]
because \( \mathcal{H}u = 0 \) in \( \omega_{T_1, T_2} \) in the sense of distributions.

Thus, \( u = 0 \) in \( \omega_{T_1, T_2} \), that was to be proved.

**Necessity.** This part of the proof inspired by the arguments from the classical approximation theorem for spaces of solutions to the heat equation in domains from \( \mathbb{R}^{n+1} \) with the topology of the uniform convergence on subcompacts, see \[1\]. More precisely, let the complement \( \Omega \setminus \omega \) has at least one compact component. As we noted before, since the domains \( \Omega \), \( \omega \) has smooth boundaries, this component is the closure of a non-empty domain \( \omega^{(0)} \). Moreover, the set \( \omega \cup \overline{\omega^{(0)}} \) is a domain with smooth boundary in \( \mathbb{R}^n \).

Fix a point \( (x_0, t_0) \in \omega^{(0)} \times (T_1, T_2) \). According to \[1\] lemma from \( \S 1 \), for any \( \delta > 0 \) there is a function \( v_0 \in \mathcal{S}_{\mathcal{H}}(\mathbb{R}^{n+1}) \) such that \( v(x_0, t_0) \neq 0 \) and \( v_0(x, t) = 0 \) for all \( t, |t - t_0| \geq \delta \).

Next, there is an infinitely times differentiable function \( \phi \) supported in \( \omega \) such that \( \phi(x_0) \equiv 1 \) in a neighbourhood \( U \) of the compact \( \overline{\omega^{(0)}} \). Then the function \( v_1(x, t) = \phi(x)v_0(x, t) \) is infinitely smooth in \( \mathbb{R}^{n+1} \), supported in \( \omega \times [t_0 - \delta, t_0 + \delta] \) and, moreover,
\[
\mathcal{H}v_1 = 2\nabla \phi \cdot \nabla v_0 + v_0 \Delta \phi \text{ in } \mathbb{R}^{n+1}.
\]
In particular, since \( \nabla \phi = 0 \) in \( U \), then
\[
\mathcal{H}v_1 = 0 \text{ in } U \times (T_1, T_2).
\]

Denote by \( \Pi_0 \) the orthogonal projection from \( L^2(\omega_{T_1, T_2}) \) onto \( L^2_{\mathcal{H}}(\omega_{T_1, T_2}) \).

The properties of the projection \( \Pi_0 \), the function \( v_1 \) and the fundamental solution \( \Phi \) imply that for all \( (y, \tau) \notin \omega \times (T_1, T_2) \) we have
\[
\begin{align*}
\int_{T_1}^{T_2} \int_{\omega} (\Pi_0 \mathcal{H}v_1)(x, y)\Phi(x - y, t - \tau)dx \, dt = \\
\int_{T_1}^{T_2} \int_{\omega} (\mathcal{H}v_1)(x, y)\Phi(x - y, t - \tau)dx \, dt = \\
\end{align*}
\]
(1.29)
\[
\int_{T_1}^{T_2} \int_{\omega \cup \omega^{(0)}} (\mathcal{H}v_1)(x, y)\Phi(x - y, t - \tau)dx \, dt = v_1(y, \tau).
\]

As a corollary, the function \( \Pi_0 \mathcal{H}v_1 \in L^2_{\mathcal{H}}(\omega_{T_1, T_2}) \) is not \( L^2_{\mathcal{H}}(\omega_{T_1, T_2}) \)-orthogonal to the function \( \Phi(x - x_0, t - t_0) \in L^2_{\mathcal{H}}(\omega_{T_1, T_2}) \), but it is \( L^2(\omega_{T_1, T_2}) \)-orthogonal to the functions \( \Phi(x - y, t - \tau) \in L^2_{\mathcal{H}}(\omega_{T_1, T_2}) \) with any vectors \( (y, \tau) \notin (\omega \cup \omega^{(0)}) \times (T_1, T_2) \).

To finish the proof we need the integral Green formula for the heat equation. With this purpose, for functions \( f \in L^2(\Omega_{T_1, T_2}), v \in L^2([T_1, T_2], H^{1/2}(\partial \Omega)), w \in L^2([T_1, T_2], H^{3/2}(\partial \Omega)), h \in H^{1/2}(\Omega) \) we consider the following parabolic potentials:
\[
\begin{align*}
I_{\Omega, T_1}(h)(x, t) &= \int_{\Omega} \Phi(x - y, t)h(y, T_1)dy, \\
G_{\Omega, T_1}(f)(x, t) &= \int_{T_1}^{t} \int_{\Omega} \Phi(x - y, t - \tau)f(y, \tau)dy \, d\tau,
\end{align*}
\]
\[ V_{\partial \Omega, T_1}(v)(x,t) = \int_{T_1} \int_{\partial \Omega} \Phi(x - y, t - \tau)v(y, \tau)ds(y)d\tau, \]

\[ W_{\partial \Omega, T_1}(w)(x,t) = - \int_{T_1} \int_{\partial \Omega} \Phi(x - y, t - \tau)w(y, \tau)ds(y)d\tau \]

(see, for instance, [25, ch. 1, §3 and ch. 5, §2]. By the definition, these are (improper) integrals, depending on the parameter \((x,t)\).

**Lemma 1.3.** For any \(T_1 < T_2\) and any \(u \in H^{2,1}(\Omega_{T_1}, T_2)\), the following formula holds true:

\[ u(x,t) \text{ in } \Omega_{T_1, T_2} \]

\[ 0 \text{ outside } \Omega_{T_1, T_2} \]

\[ = I_{\Omega, T_1}(u) + G_{\Omega, T_1}(Hu) + V_{\partial \Omega, T_1}(\partial_y u) + W_{\partial \Omega, T_1}(u). \]

**Proof.** See, [27, ch. 6, §12] (and [7, theorem 2.4.8] for more general differential operators, having fundamental solutions or parametrices).

If a function \(u\) belongs to \(S_H(\Omega_{T_1, T_2})\), then it belongs to \(H^{2,1}_H(\Omega_{T_1', T_2'})\) for some numbers \(T_1' < T_1 < T_2 < T_2'\) and a bounded domain \(\Omega' \supseteq \Omega\). Then Green formula yields

\[ u(x,t) \text{ in } \Omega_{T_1', T_2'} \]

\[ 0 \text{ outside } \Omega_{T_1', T_2'} \]

\[ = I_{\Omega', T_1'}(u) + V_{\partial \Omega', T_1'}(\partial_y u) + W_{\partial \Omega', T_1'}(u). \]

In particular, Fubini theorem and formulas (1.29) for \((y, \tau) \in (\partial \Omega' \times [T_1', T_2']) \cap (\Omega' \times \{0\})\) give us possibility to conclude that the non-zero function \(\Pi_0 H' v_1 \in L^2(\omega_{T_1, T_2})\) is \(L^2(\omega_{T_1, T_2})\)-orthogonal to all the functions from \(S_H(\Omega_{T_1, T_2})\). This proves that \(S_H(\Omega_{T_1, T_2})\) is not everywhere dense set in the space \(L^2(\omega_{T_1, T_2})\) if there is a compact components of the set \(\Omega \setminus \omega\) in \(\Omega\).

**Corollary 1.4.** Let \(s, k \in \mathbb{Z}_+\) be arbitrary numbers, \(\omega \subset \Omega \subseteq \mathbb{R}^n\), \(\partial \omega, \partial \Omega \in C^2\) and let the complement \(\Omega \setminus \omega\) has no compact components in \(\Omega\). Then the spaces \(C^\infty(\Omega_{T_1, T_2})\) and \(H^{k,2s,s}_H(\Omega_{T_1, T_2})\) everywhere dense in \(L^2(\omega_{T_1, T_2})\).

**Proof.** Follows immediately from theorem 1.1 because of embeddings 1.2.

As we noted in the introduction, assumptions of theorem 1.1 are quite similar to that of the Runge type theorems related to the uniform approximation on compact subsets for solutions to the heat equation in a lesser domain by the solutions in a bigger one, see [1, 2] (and a refinement [3] related with a constructive way of approximation sequences with the use of the fundamental solution to the heat equation). It is appropriate to note, that instead of the cylinder domains of the type \(\omega_{T_1, T_2}\) one may consider more general domains with additional assumptions on the boundaries’ smoothness.

2. **Theorem on the basis with the double orthogonality property**

We continue with the theorem on the basis with the double orthogonality property in spaces of solutions to the heat equation.
Theorem 2.1. Let $s \in \mathbb{N}$, $k \in \mathbb{Z}_+$, and let $\omega$ be a subdomain in $\Omega \Subset \mathbb{R}^n$ with $C^2$-boundary and such that the complement $\Omega \setminus \omega$ has no compact components in $\Omega$. Then there is an orthonormal basis $\{b_v\}$ in the space $H^{k,2s,\omega}_H(\Omega_{T_1,T_2})$ such that its restriction $\{b_v|_{\omega_{T_1,T_2}}\}$ to $\omega_{T_1,T_2}$ is an orthogonal basis in $L^2_{H}(\omega_{T_1,T_2})$.

Proof. By the definition, for numbers $s \in \mathbb{N}$ and $k \in \mathbb{Z}_+$, the space $H^{k,2s,\omega}_H(\Omega_{T_1,T_2})$ is embedded continuously to the space $L^2_{H}(\omega_{T_1,T_2})$. Denote by $R_{\Omega,\omega}$ the natural embedding operator

$$R_{\Omega,\omega} : H^{k,2s,\omega}_H(\Omega_{T_1,T_2}) \to L^2_{H}(\omega_{T_1,T_2}).$$

The analyticity of the solutions to the heat equation with respect to the space variables implies that the operator $R_{\Omega,\omega}$ is injective. Moreover, according to theorem 2.1, the range of the operator $R_{\Omega,\omega}$ is everywhere dense in the space $L^2_{H}(\omega_{T_1,T_2})$.

By Fubini theorem, anisotropic Sobolev space $H^{2,1}_{H}(\Omega_{T_1,T_2})$ is embedded continuously into the Bochner space $\mathcal{B}((T_1, T_2), H^2(\Omega), L^2(\Omega))$, consisting of mappings $v : [T_1, T_2] \to H^2(\Omega)$ such that $\partial_t v \in L^2(\Omega)$, see [22] ch. 1, §5. Rellich-Kondrashov theorem implies that the embedding $H^2(\Omega) \to L^2(\Omega)$ is compact. Using the well known theorem on the compact embedding for the Bochner type spaces (see, for instance, [22] ch. 1, §5, theorem 5.1), we conclude that the space $\mathcal{B}((T_1, T_2), H^2(\Omega), L^2(\Omega))$ is embedded compactly into $L^2((T_1, T_2), \Omega) = L^2(\Omega_{T_1,T_2})$. Thus, the space $H^{2,1}_{H}(\Omega_{T_1,T_2})$ is embedded compactly into $L^2_{H}(\omega_{T_1,T_2})$ and, of course, into $L^2_{H}(\omega_{T_1,T_2})$. Therefore, the space $H^{k,2s,\omega}_H(\Omega_{T_1,T_2})$ is embedded compactly into $L^2_{H}(\omega_{T_1,T_2})$, too, i.e. the operator $R_{\Omega,\omega}$ is compact.

Denote by $R_{\Omega,\omega}^*$ the adjoint mapping for the operator $R_{\Omega,\omega}$ in the theory of the Hilbert spaces, i.e. $R_{\Omega,\omega}^* : L^2_{H}(\omega_{T_1,T_2}) \to H^{k,2s,\omega}_H(\Omega_{T_1,T_2})$. By the Hilbert-Schmidt Theorem, there is an orthonormal basis $\{b_v\}$ in the space $H^{k,2s,\omega}_H(\Omega_{T_1,T_2})$, consisting of the eigen-vectors of the compact self-adjoint operator $R_{\Omega,\omega}^* R_{\Omega,\omega} : H^{k,2s,\omega}_H(\Omega_{T_1,T_2}) \to H^{k,2s,\omega}_H(\Omega_{T_1,T_2})$. Finally, using results of [16] Example 1.9, we conclude that the system of vectors $\{b_v\}$ is the basis with the double orthogonality property looked for. □

Remark 2.1. It was shown in [16] theorem 6.5 that the operator $R_{\Omega,\omega}^* R_{\Omega,\omega}$ may be identified as an integral one. Indeed, by the Sobolev embedding theorem follow that for sufficiently large $s$ and $k$ the space $H^{k,2s,\omega}_H(\Omega_{T_1,T_2})$ is embedded continuously into the normed space of continuous functions $C(\Omega_{T_1,T_2})$ on the compact $\Omega_{T_1,T_2}$ from $\mathbb{R}^{n+1}$. Thus, it is a Hilbert space with the reproducing kernel (see [28]). Besides, as the heat operator is hypoelliptic, the elements of the space $H^{k,2s,\omega}_H(\Omega_{T_1,T_2})$ are smooth on $\Omega_{T_1,T_2}$. That is why there is a kernel $K(x, t, y, \tau) \in C^{\infty}_{loc}(\Omega_{T_1,T_2} \times \Omega_{T_1,T_2})$ such that

$$u(x) = (u, K(x, t, \cdot, \cdot))_{H^{k,2s,\omega}_H(\Omega_{T_1,T_2})}$$

for all $u \in H^{k,2s,\omega}_H(\Omega_{T_1,T_2})$, $(x, t) \in \Omega_{T_1,T_2}$.

If $\{e_v\}_{v=1}^{\infty}$ is an orthonormal basis in the Hilbert space $H^{k,2s,\omega}_H(\Omega_{T_1,T_2})$ then for all $(x, t) \in \Omega_{T_1,T_2}$ we have

$$K(x, t, y, \tau) = \sum_{j=1}^{\infty} e_j(x, t) e_j(y, \tau),$$
where the series converges in $H^{k,2}_{2π}(Ω_{T_1,T_2})$ with respect to variables $(x,t)$ for each pair $(y,τ) ∈ Ω_{T_1,T_2}$. As a series of variables $(x,t,y,τ) ∈ Ω_{T_1,T_2} × Ω_{T_1,T_2}$, it converges uniformly on compacts from $Ω_{T_1,T_2} ∈ Ω_{T_1,T_2}$.

Now, simple calculations show that

$$(R^*Ru)(x,t) = (u,K(x,t,·,·))H^{k,2}_{2π}(Ω_{T_1,T_2}), \quad (x,t) ∈ Ω_{T_1,T_2}.$$ 

However, it is not easy to construct an example of a basis with the double orthogonality property, provided by theorem 2.1. A non-complete double orthogonal countable (trigonometric) system was constructed in [19] for cubes $ω$ and $Ω$ in $\mathbb{R}^n$ if their centres coincide and the ratio of their edges equals to two. Let us indicate one more example; it is related to the case where the cylinder bases of $ω_{T_1,T_2}$, $Ω_{T_1,T_2}$ are concentric balls in $\mathbb{R}^n$.

**Example 2.1.** Let $0 < R_1 < R_2 < +∞$, and $ω = B(0,R_1)$, $Ω = B(0,R_2)$, where $B(0,R)$ is the ball of the radius $R$ in $\mathbb{R}^n$. In order to construct a system with the double orthogonality property for the cylinders $ω_{T_1,T_2}$ and $Ω_{T_1,T_2}$ we use eigenfunctions of the Laplace operator related to the Dirichlet and the Neumann problems in $B(0,R_2)$. More precisely, after passing to the spherical coordinates $x = r S(φ)$ with $φ$ being coordinates on the unit sphere $S$ in $\mathbb{R}^n$ one obtains

$$\Delta = \frac{1}{r^2} \left( \left( \frac{∂}{∂r} \right)^2 + (n−2) \left( \frac{∂}{∂r} \right)− Δ_S \right),$$

where $Δ_S$ is the Laplace–Beltrami operator on the unit sphere in $\mathbb{R}^n$. In order to solve the homogeneous equation

$$(-Δ + λ)u = 0 \quad \text{in} \quad B(0,R),$$

one usually uses Fourier method: look for $u$ in the form $u(r,φ) = g(r)h(φ)$, where $g$ and $h$ satisfy

$$\begin{cases} \left( \left( \frac{∂}{∂r} \right)^2 + (n−2) \left( \frac{∂}{∂r} \right)− λr^2 \right)g = a \ g \\ Δ_S h = a \ h, \end{cases}$$

with $a$ being an eigenvalue of the Laplace-Beltrami operator $Δ_S$. It is well known that these eigenvalues equal to $a = k(n+k−2)$, $k ∈ \mathbb{Z}+$, and the corresponding eigenfunctions are the spherical harmonics (see, for instance, [24] ch. 4, §3). Then the first equation takes the form

$$(2.1) \quad \left( \left( \frac{∂}{∂r} \right)^2 + (n−2) \left( \frac{∂}{∂r} \right)− (k(n+k−2) + λr^2) \right)g = 0,$$

and its solutions $g = gk(r,λ)$ can be expresses via the Bessel functions $J_p$, $Y_p$: 

$$g_k(r,λ) = r^{(2−n)/2} \left( C_1 J_{p_k}(√kr) + C_2 Y_{p_k}(√kr) \right),$$

$$p_k = \sqrt{(n−2)^2/4 + k^2(n+k−2)^2},$$

with arbitrary constants $C_1$, $C_2$, see [30], [24] appendix 2). For instance, for $λ = 0$ we obtain $g_k(r,0) = C_1 r^k + C_2 r^{2−k−n}$; in this case $r^k h_k(φ)$ are spherical harmonics (restrictions to the unit sphere of harmonic homogeneous polynomials $h_k$ of the degree $k$), forming a linear space of dimension $J(k) = \frac{(n+2k−2)(n+k−3)!}{k!(n−2)!}$. Choosing an $L^2(∂B(0,1))$-orthonormal basis $\{h_k^{(j)}(φ)\}$, $1 ≤ j ≤ J(k)$, in the space of spherical functions we obtain a typical basis $\{r^k h_k^{(j)}(φ)\}$ with the double orthogonality.
property in the spaces of Sobolev harmonic functions in any ball with the centre at
the origin, see [31]. Moreover, the functions
\[ B^{1,j}_{k,m}(r, \varphi) = r^{(2-n)/2} J_{p_k}(\sqrt{\lambda_{k,m}^{(1)}} r) h_k^{(j)}(\varphi) \]
form a system of eigenfunctions related to the Dirichlet problem for the Laplace
operator in the ball \( B(0, R) \) with the corresponding eigenvalues \( \lambda_{k,m}^{(1)} \), where \( \sqrt{\lambda_{k,m}^{(1)}} / R \)
is the \( m \)-th zero of the Bessel function \( J_{p_k} \). Similarly, the functions
\[ B^{2,j}_{k,m}(r, \varphi) = r^{(2-n)/2} J_{p_k}(\sqrt{\lambda_{k,m}^{(2)}} r) h_k^{(j)}(\varphi) \]
form a system of eigenfunctions related to the Neumann problem for the Laplace
operator in the ball \( B(0, R) \), with the corresponding eigenvalues, \( \lambda_{k,m}^{(2)} \), where \( \sqrt{\lambda_{k,m}^{(2)}} / R \)
is the \( m \)-th zero of the derivative \( J'_{p_k} \) of the Bessel function \( J_{p_k} \), see [29]
ch. 4, §3 and appendix 2). In the Cartesian coordinates we obviously have
\[ B^{2,j}_{k,m}(x) = |x|^{(2-n)/2-k} J_{p_k}(\sqrt{\lambda_{k,m}^{(2)} |x|}) h_k^{(j)}(x). \]

Clearly, the functions
\[ E^{i,j}_{k,m}(x,t) = e^{i(k,m)\cdot x} B^{i,j}_{k,m}(x) \text{ and } h_k^{(j)}(x) \]
satisfy the heat equation in \( \mathbb{R}^{n+1} \). Then the double orthogonality property means
precisely that for \((k, j) \neq (k', j')\) and any numbers \( m', m \in \mathbb{N} \) the functions
\( E^{i,j}_{k,m} \) and \( E^{i',j'}_{k',m'} \), are orthogonal in the space \( H^{l,2s,s}(B(0, R) \times (T_1, T_2)) \) with any
\( 0 < R < R_2 \) and \( l, s \in \mathbb{Z}_+ \). This gives the possibility to choose from the set
\( \{ E^{i,j}_{k,m}(x,t), h_k^{(j)}(x) \} \) some finite or countable subsystems with non-repetitive pairs
\( (k, j) \), \( k \in \mathbb{Z}_+ \), \( 1 \leq j \leq J(k) \); obviously, these subsystems have the double orthogo-
nality property in the spaces \( H^{l,2s,s}(B(0, R_2) \times (T_1, T_2)) \) and \( H^{l',2s',s'}(B(0, R_1) \times
(T_1, T_2)) \). Unfortunately, such subsystems are not complete. For example, the
polynomial of the form
\[ t \Delta G_{k+2} + G_{k+2}, \]
where \( G_k \) is a homogeneous biharmonic polynomial of the degree \( k \in \mathbb{Z}_+ \), satisfies
the heat equation in \( \mathbb{R}^{n+1} \). However, if any of the discussed above subsystems are
complete \( H^{2,1}_R(B(0, R_2) \times (T_1, T_2)) \) then it is possible to approximate the homoge-
neous harmonic polynomial
\[ \Delta G_{k_0+2} = \Delta(t \Delta G_{k_0+2} + G_{k_0+2}) \]
of the degree \( k_0 \) in the space \( L^2(B(0, R_2) \times (T_1, T_2)) \) by linear combinations of
functions of the form
\[ \Delta E^{i,j}_{k,m}(x,t) = \lambda_{k,m}^{(i)} E^{i,j}_{k,m}(x,t), \Delta h_k^{(j)} = 0. \]

This is impossible because the pairs \((k, j)\) are non-repetitive, and hence the linear
combinations should be finite and containing the functions \( E^{i,j}_{k_0,m_j}(x,t), 1 \leq j \leq
J(k_0) \), \( m_j \in \mathbb{N} \), only, that corresponds to the homogeneous harmonic polynomials
\( h_k^{(j)} \) of the degree \( k_0 \). On the other hand, if we allow the pairs \((k, j)\) to enter these
subsystems for different \( i \) and \( m \), then the double orthogonality property will fail
and we will need an additional orthogonalisation.
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References

[1] Jones, B.F., Jr., An approximation theorem of Runge type for the heat equation, Proc. Amer. Math. Soc. 52 (1975), no. 1, 289–292.
[2] Diaz, R., A Runge theorem for solutions of the heat equation, Proc. Amer. Math. Soc. 80 (1980), no. 4, 643–646.
[3] Gauthier, P.M., Tarkhanov, N, Rational approximation and universality for a quasi-linear parabolic equation, Journal of Contemporary Mathematical Analysis, V. 43 (2008), 353–364.
[4] Runge, C., Zur Theorie der eindeutigen analytischen Funktionen, Acta Math. 6 (1885), 229–244.
[5] Mergelyan, S.N., Harmonic approximation and approximate solution of the Cauchy problem for the Laplace equation, Uspekhi Mat. Nauk, 11:5(71) (1956), 3–26.
[6] Malgrange, B., Existence et approximation des solutions des équations aux dérivées partielles et des équations de convolutions, Annales de l’Institut Fourier, V. 6 (1956), 271–355.
[7] Tarkhanov, N., The Analysis of Solutions of Elliptic Equations, Kluwer Academic Publishers, Dordrecht, NL, 1997.
[8] Vitushkin, A.G, The analytic capacity of sets in problems of approximation theory, Uspekhi Mat. Nauk, 22:6(138) (1967), 141–199.
[9] Havin, V. P., Approximation by analytic functions in the mean, Dokl. Akad. Nauk SSSR, 178:5 (1968), 1025–1028
[10] Hedberg, L.I., Nonlinear potential theory and Sobolev spaces. Nonlinear Analysis, Function Spaces and Applications. Proceedings of the Spring School held in Litomysl, 1986. Vol. 3. BSB B. G. Teubner Verlagsgesellschaft, Leipzig, 1986. pp. 5–30.
[11] Krasichkov, I.F., Systems of functions with the dual orthogonality property, Math. Notes, 4:5 (1968), 821–824.
[12] Tikhonov, A.N., Arsenin, V.Ya., Methods of solving ill-posed problems, Nauka, Moscow, 1986.
[13] Tarkhanov, N., The Cauchy Problem for Solutions of Elliptic Equations, Akademie-Verlag, Berlin, 1995.
[14] Bergman, S., The kernel function and conformal mapping: Second (revised) edition. (Mathematical Surveys, V), AMS, Providence, Rhode Island, 1970.
[15] Aizenberg L.A., Kytmanov, A.M., On the possibility of holomorphic continuation to a domain of functions given on a part of its boundary, Matem. sb., 182(1991), N. 5, 490–507.
[16] Shlapunov, A.A., Tarkhanov, N., Beases with double orthogonality in the Cauchy problem for systems with injective symbols. Proc. London. Math. Soc., 71 (1995), N. 1, p. 1–54.
[17] Fedchenko, D.P., Shlapunov, A.A., On the Cauchy problem for the elliptic complexes in spaces of distributions, Complex Variables and Elliptic Equations, V. 59, N. 5, 2014, 651–679.
[18] Makhmudov, K.O., Makhmudov, O.I., Tarkhanov, N.N. Non-standard Cauchy problem for the heat equation, Math. Notes, 102:2 (2017), 270–283.
[19] Kurilenko, I.A., Shlapunov, A.A., On Carleman-type Formulas for Solutions to the Heat Equation, Journal of Siberian Federal University, Math. and Phys., 12:4 (2019), 421–433.
[20] Mikhailov, V.P., Partial differential equations, Nauka, Moscow, 1976.
[21] Besov, O.V., Il’in, V.P., Nikol’skii, S.M., Integral representations and embedding theorems, Nauka, Moscow, 1975.
[22] Lions, J.-L., Quelques méthodes de résolution des problèmes aux limites non linéaire, Dunod/Gauthier-Villars, Paris, 1969.
[23] Krylov, N.V., Lectures on elliptic and parabolic equations in Sobolev spaces. Graduate Studies in Math. V. 96, AMS, Providence, Rhode Island, 2008.
[24] Temam, R., *Navier-Stokes Equations. Theory and Numerical Analysis*, North Holland Publ. Comp., Amsterdam, 1979.

[25] Friedman, A., *Partial differential equations of parabolic type*, Englewood Cliffs, N.J., Prentice-Hall, Inc., 1964.

[26] Hedberg, L.I., Wolff, T.H., *Thin sets in nonlinear potential theory*, Ann. Inst. Fourier (Grenoble) 33 (1983), no. 4, 161–187.

[27] Sveshnikov, A.G., Bogolyubov, A.N., Kravtsov, V.V., *Lectures on mathematical physics*, Nauka, Moscow, 2004.

[28] Aronszajn, N., *Theory of reproducing kernels*, Trans. Amer. Math. Soc. V. 68, 1950, 337–404.

[29] Tikhonov, A.N., Samarskii, A.A., *Equations of mathematical physics*, Nauka, Moscow, 1972.

[30] Bowman, F., *Introduction to Bessel Functions*. New York: Dover, 1958.

[31] Shlapunov, A.A., *Spectral Decomposition of Green’s Integrals and Existence of $W^{s,2}$-Solutions of Matrix Factorizations of the Laplace Operator in a Ball*, Rend. Sem. Mat. Univ. Padova, V. 96 (1996), 237–256.

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