KNOTTED HOLOMORPHIC DISCS IN $\mathbb{C}^2$

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Abstract. We construct knotted proper holomorphic embeddings of the unit disc in $\mathbb{C}^2$.

1. Introduction

Every classical knot type can be represented by a polynomial embedding of $\mathbb{R}$ in $\mathbb{R}^3$ [9]. In particular, there exist topologically distinct polynomial embeddings of $\mathbb{R}$ in $\mathbb{R}^3$. Crossing these with another coordinate, we obtain topologically distinct polynomial embeddings of $\mathbb{R}^2$ in $\mathbb{R}^4$. In contrast, all polynomial embeddings of $\mathbb{C}$ in $\mathbb{C}^2$ are topologically equivalent, in fact even connected by a polynomial automorphism of $\mathbb{C}^2$. The first complete algebraic proof of this fact is due to Abhyankar and Moh [1]; later a purely knot theoretical proof was found by Rudolph [6]. It is an open question whether proper holomorphic embeddings of the complex plane $\mathbb{C}$ or of the unit disk $\Delta \subset \mathbb{C}$ in $\mathbb{C}^2$ can be topologically knotted (see Problem 1.102 (A/B) of Kirby’s list [4]). In this note we construct knotted proper holomorphic embeddings of the unit disc in $\mathbb{C}^2$.

Theorem 1. There exist topologically knotted proper holomorphic embeddings of the unit disc in $\mathbb{C}^2$.

Since the image of a holomorphically embedded disc in $\mathbb{C}^2$ is a minimal surface, we obtain the following corollary, which solves Problem 1.102 (C) of Kirby’s list.

Corollary 1. There exists a proper embedding $f : \mathbb{R}^2 \to \mathbb{R}^4$ whose image is a topologically knotted complete minimal surface.

Our construction is based on the existence of locally well-behaved Fatou-Bieberbach domains and on the existence of knotted holomorphic discs in the 4-ball. A Fatou-Bieberbach domain in $\mathbb{C}^2$ is an open subset of $\mathbb{C}^2$ which is biholomorphically equivalent to $\mathbb{C}^2$. Fatou-Bieberbach domains tend to have wild shapes; the first Fatou-Bieberbach domain with smooth boundary was constructed by Stensønes [10]. In [3], Globevnik constructed Fatou-Bieberbach domains $\Omega \subset \mathbb{C}^2$ whose intersections with $\mathbb{C} \times \Delta$ are arbitrarily small $C^1$-perturbations of $\Delta \times \Delta$. 


Here $\Delta$ stands for the open unit disc in $\mathbb{C}$. These domains are well-adapted for constructing knotted holomorphic discs in $\mathbb{C}^2$. Indeed, all we need is a knotted proper holomorphic embedding of the closed unit disc $\varphi : \overline{\Delta} \to \overline{\Delta} \times \overline{\Delta}$ that maps $\partial \Delta$ to $\partial \overline{\Delta} \times \Delta$. Composing the restriction of this embedding $\varphi|_{\Delta} : \Delta \to \Delta \times \Delta \subset \Omega$ with a biholomorphic map $h : \Omega \to \mathbb{C}^2$ yields a knotted proper holomorphic embedding of $\Delta$ in $\mathbb{C}^2$. The existence of knotted proper holomorphic embeddings $\varphi : \overline{\Delta} \to \overline{\Delta} \times \overline{\Delta}$ is easily established by using the theory of complex algebraic curves in $\mathbb{C}^2$.

We describe the relevant features of the Fatou-Bieberbach domains needed for our purpose in Section 2. The proof of Theorem 1 is completed in Section 3, where we give an explicit example of a knotted holomorphic disc in $\overline{\Delta} \times \overline{\Delta}$.

2. Fatou-Bieberbach domains

Fatou-Bieberbach domains in $\mathbb{C}^2$ are usually described by certain infinite processes, for example as domains of convergence of maps defined by sequences of automorphisms. This technique was used by Globevnik to construct Fatou-Bieberbach domains with controlled shape inside $\mathbb{C} \times \Delta$.

**Theorem 2** (Globevnik [3]). Let $Q \subset \mathbb{C}$ be a bounded open set with boundary of class $\mathcal{C}^1$ whose complement is connected. Let $0 < R < \infty$ be such that $\overline{Q} \subset R\Delta$. There are a domain $\Omega \subset \mathbb{C}^2$ and a volume-preserving biholomorphic map from $\Omega$ onto $\mathbb{C}^2$ such that

(i) $\Omega \subset \{(z, w) \in \mathbb{C}^2 : |z| < \max\{R, |w|\}\}$,

(ii) $\Omega \cap R(\Delta \times \Delta)$ is an arbitrarily small $\mathcal{C}^1$-perturbation of $Q \times R\Delta$.

The assumptions of Theorem 2 are verified when $Q$ is an open disc with smooth boundary (and $R > 0$ large enough). This is the version we will need. As mentioned in the introduction, we will insert knotted discs in $\Delta \times \Delta$ with boundary in $\partial \overline{\Delta} \times \Delta$. It is a priori not clear whether such discs stay knotted in the larger domain $\Omega$. The following lemma allows us to control knottedness on the level of the fundamental group.

**Lemma 1.** Let $\Omega \subset \mathbb{C}^2$ be an open domain homeomorphic to $\mathbb{C}^2$ with $\Omega \cap \mathbb{C} \times \Delta = \Delta \times \Delta$, and let $X \subset \Delta \times \Delta$ be a subset with $\overline{X} \subset \overline{\Delta} \times \Delta$. The inclusion $i : (\Delta \times \Delta) \setminus X \to \Omega \setminus X$ induces an injective map $i_* : \pi_1((\Delta \times \Delta) \setminus X) \to \pi_1(\Omega \setminus X)$ (suppressing base points).
Remark. At this point it does not matter whether $\Omega \cap \Delta \times \Delta$ is precisely $\Delta \times \Delta$ or a small $C^1$-perturbation of it. Further, the corresponding statement stays true if the intersection $\Omega \cap (\mathbb{C} \times \Delta)$ is a small $C^1$-perturbation of $D \times \Delta$, where $D \subset \mathbb{C}$ is any embedded disc with smooth boundary.

Proof. Choose $\epsilon > 0$ such that $X \subset \Delta \times (1-\epsilon)\Delta$. Setting

$$U := (\Delta \times \Delta) \setminus X$$

and

$$V := \Omega \setminus (\Delta \times (1-\epsilon)\Delta),$$

we have $\Omega \setminus X = U \cup V$. The theorem of Seifert-van Kampen tells us that $\pi_1(\Omega \setminus X)$ is the free product of $\pi_1(U)$ and $\pi_1(V)$ amalgamated over $\pi_1(U \cap V)$. The latter is isomorphic to $\mathbb{Z}$, since

$$U \cap V = \Delta \times (\Delta \setminus (1-\epsilon)\Delta)$$

is homotopy equivalent to a circle. Let $\gamma : S^1 \to U \cap V$ be a loop that generates $\pi_1(U \cap V)$. If the induced map $i_* : \pi_1((\Delta \times \Delta) \setminus X) \to \pi_1(\Omega \setminus X)$ were not injective, then a certain non-zero multiple of $[\gamma] \in \pi_1(V)$ would have to vanish. This is impossible since the inclusion $j : V \to \mathbb{C} \times \mathbb{C}^*$ maps $\gamma$ onto a generator of $\pi_1(\mathbb{C} \times \mathbb{C}^*) \cong \mathbb{Z}$. $\square$

3. Complex plane curves

A complex plane curve is the zero level $V_f = \{(z, w) \in \mathbb{C}^2 : f(z, w) = 0\} \subset \mathbb{C}^2$ of a non-constant polynomial $f(z, w) \in \mathbb{C}[z, w]$. Complex plane curves form a rich source of examples in algebraic geometry and topology. For example, the intersection of a complex plane curve $V_f$ with the boundary of a small ball centred at an isolated singularity of $f$ forms a link which is often called algebraic. The class of algebraic links has been generalized by Rudolph to the larger class of quasipositive links [7]. Using the theory of quasipositive links, it is easy to construct knotted proper holomorphic embedded discs in $\Delta \times \Delta$. In the following, we will give a short description of Rudolph’s theory; more details are contained in [7] and [8].

Let $f(z, w) = f_0(z)w^n + f_1(z)w^{n-1} + \ldots + f_n(z) \in \mathbb{C}[z, w]$ be a non-constant polynomial, $f_0(z) \neq 0$. Under some generic conditions on $f$, the set $B$ of complex numbers $z$ such that the equation $f(z, w) = 0$ has strictly less than $n$ solutions $w$, is finite. Let $\gamma \subset \mathbb{C} \setminus B$ be a smooth simple closed curve. The intersection $L = V_f \cap \gamma \times \mathbb{C}$ is a smooth closed 1-dimensional manifold, i.e. a link, in the solid torus $\gamma \times \mathbb{C}$. More precisely, the link $L$ is an $n$-stranded braid in $\gamma \times \mathbb{C}$,
which becomes a link in $S^3$ via a standard embedding of $\gamma \times \mathbb{C}$ in $S^3$. Links that arise in this way are called quasipositive.

If $D \subset \mathbb{C}$ is the closed disc bounded by $\gamma$, then the intersection $X = V_f \cap D \times \mathbb{C}$ is a piece of complex plane curve bounded by the link $L$. Choosing the polynomial $f$ and the curve $\gamma$ appropriately, it is possible to arrange $X$ to be an embedded disc with a non-trivial quasipositive knot as boundary (see [7], Example 3.2). Assuming that $X$ is compact, there exists $R > 0$, such that $X \subset D \times R\Delta$. In order to establish Theorem 1, it remains to find an example where $X$ is knotted in $D \times R\Delta$, more precisely

$$\pi_1(D \times R\Delta \setminus X) \neq \mathbb{Z}.$$ 

In view of Lemma [1] this will then give rise to a knotted proper holomorphic embedding of the unit disc in $\Omega$, hence in $\mathbb{C}^2$.

**Remark.** Here we choose $\Omega \subset \mathbb{C}^2$ to be a Fatou-Bieberbach domain whose intersection $\Omega \cap \mathbb{C} \times R\Delta$ is a small $C^1$-perturbation of $D \times R\Delta$. If this perturbation is small enough, then the intersection $\tilde{X} = V_f \cap \mathbb{C} \times R\Delta \cap \Omega$ is still a proper holomorphic embedded disc, as knotted as $X$ in $D \times R\Delta$.

We conclude this section with a concrete example, in fact Rudolph’s Example 3.2 of [7]. Let

$$f(z, w) = w^3 - 3w + 2z^4.$$ 

The complex plane curve $V_f$ is easily seen to be non-singular. The equation $f(z_0, w) = 0$ fails to have three distinct solutions $w$, if and only if the two equations $f(z_0, w) = 0$ and $\frac{\partial}{\partial w} f(z_0, w) = 3w^2 - 3 = 0$ have a simultaneous solution. This happens precisely when $z_0^8 = 1$. Thus the set $B$ consists of the 8th roots of unity. For $z \in \mathbb{C} \setminus B$, the equation $f(z, w) = 0$ has three distinct solutions $w_1, w_2, w_3 \in \mathbb{C}$, which we index by increasing real parts. Further, it will be convenient to determine the set $B_+$ of complex numbers $z$ such that the equation $f(z, w) = 0$ has two distinct solutions $w$ with coinciding real parts. In our example, $B_+$ consists of 8 rays emanating from the points of $B$, as shown in Figure 1. These rays carry labels 1 or 2, depending on whether the real parts of $w_1$ and $w_2$ or those of $w_2$ and $w_3$ coincide. There is a way of orienting the rays that corresponds to choosing positive standard generators of the braid group. Figure 1 also indicates a curve $\gamma$ that bounds a disc $D$. We claim that the piece of complex plane curve $X = V_f \cap D \times \mathbb{C}$ is a disc with $\pi_1((D \times \mathbb{C}) \setminus X) \neq \mathbb{Z}$.

First, we observe that if we ‘cut off’ the two ends of the disc $D$ of Figure 1, we obtain another disc $D'$ disjoint from $B$. The intersection
Figure 1.

$V_f \cap D' \times \mathbb{C}$ is therefore a disjoint union of three discs. Adding the ends to $D'$ again gives rise to two identifications along the boundaries of these discs. It is easy to see that these identifications result in a single disc whose boundary $L = V_f \cap (\gamma \times \mathbb{C})$ is knotted (actually $L$ is the quasipositive ribbon knot $8_{20}$). However, the mere fact that $L$ is knotted does not imply that the fundamental group of $(D \times \mathbb{C}) \setminus X$ is not isomorphic to $\mathbb{Z}$ (see the last paragraph of [2]). We will describe $\pi_1((D \times \mathbb{C}) \setminus X)$ explicitly, in terms of generators and relations. Hereby we will use Orevkov’s method for presenting the fundamental group of the complement of complex plane curves in $\mathbb{C}^2$ [3]. To every connected component of $\mathbb{C} \setminus B_+$ (in our case $D \setminus B_+$), we assign $n$ (in our case 3) generators corresponding to meridians of the $n$ discs lying over that component. Every oriented edge of the graph $B_+$ gives rise to $n$ relations between the generators of the two (not necessarily distinct) components adjacent to that edge. All these relations are of Wirtinger-type (see [5], Lemma 3.1).

Let us denote the four connected component of $D \setminus B_+$ by $U_1, U_2, U_3, U_4$, starting at the bottom left and going clockwise around $D$. To every region $U_j$ correspond three generators $\alpha_{ij} = \alpha_i(U_j), 1 \leq i \leq 3$. The relations among these generators read as follows:

1. edge: $\alpha_{11} = \alpha_{21}$
2. edge: $\alpha_{11} = \alpha_{12}, \alpha_{31} = \alpha_{22}, \alpha_{21} = \alpha_{31}\alpha_{32}\alpha_{31}^{-1}$
3. edge: $\alpha_{12} = \alpha_{13}, \alpha_{32} = \alpha_{23}, \alpha_{22} = \alpha_{32}\alpha_{33}\alpha_{32}^{-1}$
4. edge: $\alpha_{13} = \alpha_{14}, \alpha_{33} = \alpha_{24}, \alpha_{23} = \alpha_{33}\alpha_{34}\alpha_{33}^{-1}$
5. edge: $\alpha_{14} = \alpha_{24}$

From this it is easy to verify that the assignment $\alpha_{11} \mapsto (23), \alpha_{31} \mapsto (12)$ defines a surjective homomorphism $\varphi : \pi_1((D \times \mathbb{C}) \setminus X) \to S_3$, whence $\pi_1((D \times \mathbb{C}) \setminus X)$ is not a cyclic group.
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