Berestycki-Lions type results for a class of generalized Kadomtsev-Petviashvili equation

Claudianor O. Alves∗
Universidade Federal de Campina Grande,
Unidade Acadêmica de Matemática
CEP: 58429-900 - Campina Grande-PB, Brazil
e-mail: coalves@mat.ufcg.edu

Olímpio H. Miyagaki†
Departamento de Matemática
Universidade Federal de Juiz de Fora
CEP: 36036-330 - Juiz de Fora - MG, Brazil
e-mail: ohmiyagaki@gmail.com

Alessio Pomponio ‡
Dipartimento di Meccanica, Matematica e Management
Politecnico di Bari
Via Orabona, 4, 70125 Bari, Italy
e-mail: alessio.pomponio@poliba.it

Abstract
In this paper we use variational methods to establish a Berestycki-Lions type result for a class of generalized Kadomtsev-Petviashvili \((GKP - I)\) equation in \(\mathbb{R}^2\). The positive and zero mass cases are considered. The main argument is to find a Palais Smale sequence satisfying a property related to Pohozaev identity, as in [21], which was used for the first time by [23].

2000 Mathematics Subject Classifications: 35A15, 35B65, 35Q51, 35Q53.

Key words. Variational methods, KdV like equations, soliton like solutions.

∗Research of C. O. Alves partially supported by CNPq/Brazil 304036/2013-7 and INCTMAT/CNPq/Brazil.
†O.H.M. was partially supported by INCTMAT/CNPq/Brazil, CNPq/Brazil 304015/2014-8.
‡A. P. was partially supported by a grant of the group GNAMPA of INdAM and FRA2016 of Politecnico di Bari.
1 Introduction

A generalized Kadomtsev-Petviashvili (GKP) equation with variable coefficients has been studied by many authors, see for instance, Gungör and Winternitz [20], Tian and Gao [33], Zhang, Xu and Ma [38] and references therein, where they focused in the study of solitary or soliton solutions, complete integrability, etc.

A (GKP) equation with constant coefficients of the type

\[
\begin{align*}
    u_t + h'(u)u_x + u_{xxx} + \beta v_y &= 0, \\
    v_x &= u_y,
\end{align*}
\]

with \((t, x, y) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{N-1}, N \geq 2,\) has been also considered by several papers. In some of the articles, the main goal is to prove the existence of a solitary wave for (1.1), that is, a solution \(u\) of the form \(u(t, x, y) = u(x - \tau t, y)\), with \(\tau > 0\). Hence, the function \(u\) must satisfy the problem

\[
\begin{align*}
    -\tau u_x + h'(u)u_x + u_{xxx} + \beta v_y &= 0, \\
    v_x &= u_y.
\end{align*}
\]

Classically, this problem is called \((GKP - I)\) when \(\beta = -1\), and \((GKP - II)\) when \(\beta = 1\).

We would like to point out that the equation (1.1) is a two-dimensional Korteweg-de Vries type equation when \(h(t) = t^2\), which is a model for long dispersive waves, essentially unidimensional, but having small transverse effects, see [26]. Still considering Kadomtsev-Petviashvili equation with constant coefficients, for Cauchy problem associated with equation (1.1), as for \((GKP - I)\) or \((GKP - II)\) we would like to cite, e.g. [11, 17, 22] and the survey [31]. Focusing on the case \((GKP - I)\), recently, another interesting question is studied, namely, the existence and multiplicity of solitary waves to equation (1.1). The pioneering work is due to De Bouard and Saut in [13, 14], where they treated a nonlinearity \(h(s) = |s|^p s\) assuming that \(p = \frac{m}{n}\), with \(m\) and \(n\) relatively prime, and \(n\) is odd and \(1 \leq p < 4\), if \(N = 2\), or \(1 \leq p < 4/3\), if \(N = 3\). In the mentioned paper, De Bouard and Saut obtained existence results for equation (1.1) by combining minimization with concentration compactness theorem [28]. In [15] the authors proved that the solutions obtained in former papers are cylindrically symmetric, In an interesting paper [27], Klein and Saut used a numerical simulation to analyse several quantitative properties of the De Bouard and Saut existence results, such as, blow-up, stability or instability of solitary waves. Also, in this paper, they study the zero-mass case and make a survey of various mathematical results on this subject. For the regularity of the solutions they assumed \(p = 2, 3, 4\) if \(N = 2\), and \(p = 2\) if \(N = 3\). In Willem [35] and Wang and Willem [34] the existence of a solitary waves for a class of \((GKP - I)\) problems of the type (1.1) were considered with an autonomous continuous nonlinearity \(h\) in \(N = 2\), and existence and multiplicity results have been proved, respectively. Their results were obtained by applying the mountain pass theorem [4] and Lusternik-Schnirelman theory, respectively. In [29], Liang and Su have proved the existence of solution for a class considered the \((GKP - I)\) problem (1.1), which involves a non autonomous continuous function with \(N \geq 2\), while Xuan [37] treated the autonomous case in higher dimension. Applying the linking theorem stated in [32], He and Zou in [16] obtained nontrivial solution for (1.1), in higher dimension without Ambrosetti-Rabinowitz growth condition given in [4] (see also [39] and [36] for multiplicity results).
We recall that in all the above papers, the regularity of the solitary waves have not been treated. Recently Alves and Miyagaki [3] have treated the case non autonomous, getting results similar to those obtained in, for instance, [13, 14], and also the regularity properties of the solutions.

Motivated by above articles dealing with Kadomtsev-Petviashvili equation as well as by the fact that the variational methods can be employed to find a solitary waves for (1.1), this paper concerns with the existence of solitary waves for the following class of variable coefficient GKP, of the type

\[
\begin{align*}
&u_t + h'(u)u_x + u_{xxx} + \beta v_y = 0 \\
v_x = u_y,
\end{align*}
\] (1.2)

where \( u = u(t, x, y) \) and \( v = v(t, x, y) \), with \( (t, x, y) \in \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R} \), and \( h : \mathbb{R} \to \mathbb{R} \) is a smooth function. A solitary wave of the system (1.2) is a solution of the form \( u(t, x, y) = u(x - \tau t, y) \), with \( \tau > 0 \). Hence, the function \( u \) must satisfy the problem

\[
\begin{align*}
&-\tau u_x + h'(u)u_x + u_{xxx} + \beta v_y = 0 \\
v_x = u_y.
\end{align*}
\]

In the sequel, we will treat the case \( \beta = -1 \) and \( \tau = 1 \).

By a simple calculus, it is easy to see that the above system corresponds to the following equation

\[-u_x + (h(u))_x + u_{xxx} - D^{-1}_x u_{yy} = 0, \quad \text{in } \mathbb{R}^2,
\]

or, equivalently,

\[-u_{xx} - h(u) + u + D^{-2}_x u_{yy})_x = 0, \quad \text{in } \mathbb{R}^2, \quad (1.3)
\]

where \( D^{-1}_x \) denotes the following operator

\[D^{-1}_x g(x, y) = \int_{-\infty}^x g(s, y)ds.\]

In whole this paper, the function \( h \) belongs to \( C^1(\mathbb{R}^2) \) and \( h(0) = 0 \). Here, we are going to work with two classes of problems:

**Problem 1: Positive Mass.** In this case, we assume that \( h \) satisfies the following conditions:

\((h_1)\) for some \( p \in (1, 5)\)

\[\lim_{|t| \to +\infty} \frac{h(t)}{|t|^p} = 0;\]

\((h_2)\) \( h'(0) = 0; \)

\((h_3)\) there exists \( \xi > 0 \) such that \( 2H(\xi) - \xi^2 > 0 \), where \( H(t) = \int_0^t h(r)dr. \)

Our main result for this class of problems is the following

**Theorem 1.1.** Suppose \((h_1)-(h_3)\), then the problem (1.3) possesses at least a nontrivial solution.
Our second class of problems is the following

**Problem 2: Zero Mass.**

We suppose \((h_3)\) and the conditions below

\[(h_4) \quad h(t) = f(t) + t;\]

\[(h_5) \quad \lim_{t \to 0} \frac{f(t)}{|t|^5} = \lim_{|t| \to +\infty} \frac{f(t)}{|t|^5} = 0.\]

Observe that, by \((h_3)\) and \((h_4)\), defining \(F(t) = \int_0^t f(r)dr\), we infer that there exists \(\xi > 0\) such that \(F(\xi) > 0\). \hfill (1.4)

For the Zero Mass case, the problem \((1.3)\) will be written of the form

\[(-u_{xx} - f(u) + D_{-2}^{-2}u_{yy})_x = 0, \quad \text{in} \quad \mathbb{R}^2.\] \hfill (1.5)

Related with this class of problems we have the following result

**Theorem 1.2.** Suppose \((h_3)-(h_5)\), then problem \((1.5)\) possesses at least a nontrivial solution.

One of the main motivations to study Problems 1 and 2 comes from the seminal papers due to Berestycki and Lions in [8], for \(N \geq 3\), and to Berestycki, Gallouët and Kavian in [9], for \(N = 2\), where the authors proved the existence of a ground state, namely a solution which minimizes the action among all the nontrivial solutions, for the problem

\[\begin{cases} -\Delta u = g(u), & \text{in} \ \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N), \end{cases}\]

under the following assumptions on the nonlinearity \(g\):

\[(g_1) \quad g \text{ is an odd and continuous function;}\]

\[(g_2) \quad -\infty < \lim \inf_{s\to 0^+} \frac{g(s)}{s} \leq \lim \sup_{s\to 0^+} \frac{g(s)}{s} = m \leq 0;\]

\[(g_3) \quad -\infty < \lim \sup_{s\to +\infty} \frac{g(s)}{s^{2/(2-1)}} \leq 0, \quad (N \geq 3), \lim \sup_{s\to +\infty} \frac{g(s)}{e^{\alpha s}} \leq 0, \forall \alpha > 0, \quad (N = 2);\]

\[(g_4) \quad \text{there exists } \tau > 0 \text{ such that } G(\tau) := \int_0^\tau g(s)ds > 0;\]

where \(2^* = 2N/(N-2)\). They considered two cases \(m < 0\) and \(m = 0\), so called “positive mass” and “zero mass” cases, respectively. The last case is related to the Yang-Mills equation, see, e.g. [18]. After these two pioneering papers, many researches worked in this subject, extending or improving in several ways, see, for instance, [1, 2, 5, 7, 19, 21, 25, 30], and references therein: clearly the list is not complete.
It is very important to point out that in the most part of the above mentioned papers, which involves the Laplacian operator, the radial functions space plays an important role because of its compactness embedding properties. At contrary, for problems involving the Kadomtsev-Petviashvili equation, we do not have a similar result for radial functions any longer and therefore we need to use different arguments to prove our main results. Here, we will adapt for our problem the variational approach explored in Jeanjean [24] and Hirata, Hikoma and Tanaka [21] (see also [6, 12]) by considering an auxiliary functional that allows to construct a suitable Palais-Smale sequence, which almost satisfies a Pohozaev type identity, see Sections 3 and 4 for more details.

The paper is organized as follows. In Section 2 we present our functional setting describing its embeddings properties. In the last two sections, instead, we treat the positive mass case and the zero mass case, proving our main existence results.

**Notations:** Throughout the paper, unless explicitly stated, the symbol $C$ will always denote a generic positive constant, which may vary from line to line. The symbols “→” and “$\rightharpoonup$” denote, respectively, strong and weak convergence, and all the convergences involving sequences in $n \in \mathbb{N}$ are as $n \to \infty$. Moreover we denote by $| \cdot |_q$ the usual norm of Lebesgue space $L^q(\mathbb{R}^2)$, for $q \in [1, +\infty]$.

# 2 Functional setting

We intend to study our problem using variational methods and, as first step, we introduce our functional setting.

**Definition 2.1.** On $Y = \{g_x : g \in C_0^\infty(\mathbb{R}^2)\}$ define the inner product

$$(u, v) = \int_{\mathbb{R}^2} (u_x v_x + D_x^{-1} u_y D_x^{-1} v_y + u v) \, dx \, dy$$

with the corresponding norm

$$\|u\| = \left( \int_{\mathbb{R}^2} (|u_x|^2 + |D_x^{-1} u_y|^2 + |u|^2) \, dx \, dy \right)^{\frac{1}{2}}.$$

We say that $u : \mathbb{R}^2 \to \mathbb{R}$ belongs to $X$ if there exists a sequence $\{u_n\} \subset Y$ such that

a) $u_n \to u$ a.e. on $\mathbb{R}^2$,

b) $\|u_j - u_k\| \to 0$, as $j, k \to \infty$.

**Definition 2.2.** On $Y = \{g_x : g \in C_0^\infty(\mathbb{R}^2)\}$ define the inner product

$$(u, v)_0 = \int_{\mathbb{R}^2} (u_x v_x + D_x^{-1} u_y D_x^{-1} v_y) \, dx \, dy$$

with the corresponding norm

$$\|u\|_0 = \left( \int_{\mathbb{R}^2} (|u_x|^2 + |D_x^{-1} u_y|^2) \, dx \, dy \right)^{\frac{1}{2}}.$$

We say that $u : \mathbb{R}^2 \to \mathbb{R}$ belongs to $X_0$ if there exists a sequence $\{u_n\} \subset Y$ such that
a) $u_n \to u$ a.e. on $\mathbb{R}^2$, \\
b) $\|u_j - u_k\|_0 \to 0$, as $j, k \to \infty$.

From definition of $X$ and $X_0$, the embedding $(X, \| \cdot \|) \hookrightarrow (X_0, \| \cdot \|_0)$ is continuous.

The spaces $X$ and $X_0$ endowed with inner products and norms given above are Hilbert spaces. Moreover, we have the following continuous embeddings, whose proof can be found in [10, Theorem 15.7 p. 323] and [29, Lemma 2.1],

$$X \hookrightarrow L^q(\mathbb{R}^2), \quad \text{for } 1 \leq q \leq 6.$$  \hfill (2.1)

Regarding to compact embeddings, De Bouard and Saut in [14, Remark 1.1] have proved that the embeddings below

$$X \hookrightarrow L^q_{\text{loc}}(\mathbb{R}^2), \quad \text{for } 1 \leq q < 6,$$  \hfill (2.2)

are compacts. For higher dimensions, see [37, Lemma 2.4].

The following lemma is a Lions’ type result, see [28], for the space $X$.

**Lemma 2.1.** ([35, Lemma 7.4]) If $\{u_n\}$ is a sequence bounded in $X$ and if

$$\sup_{(x,y)\in\mathbb{R}^2} \int_{B_r((x,y))} |u_n|^2 \, dx \, dy \to 0,$$

then $u_n \to 0$ in $L^q(\mathbb{R}^2)$ for all $q \in (2, 6)$.

With relation to the continuous embedding $X_0 \hookrightarrow L^6(\mathbb{R}^2)$, we have the following result

**Lemma 2.2.** ([37, Lemma 2.3]) There exists a constant $S > 0$ such that

$$|u|_6 \leq S \left( \int_{\mathbb{R}^2} (|u_x|^2 + |D_x^{-1}u_y|^2) \, dx \, dy \right)^{\frac{1}{2}}, \quad \forall u \in X_0.$$  \hfill (2.3)

Finally, before concluding this section, we would like to point out that the same approach explored in [37, Lemma 2.4], or [35, Theorem 7.3], gives

$$X_0 \hookrightarrow L^q_{\text{loc}}(\mathbb{R}^2), \quad \text{for } 1 \leq q < 6,$$  \hfill (2.3)

are compact. Moreover, it is very important to say that if $\Omega \subset \mathbb{R}^2$ is a smooth bounded domain and $q \in [1, 6]$, there exists $C > 0$ such that

$$|u|_{L^q(\Omega)} \leq C \left( \int_{\Omega} (|u_x|^2 + |D_x^{-1}u_y|^2 + |u|^2) \, dx \, dy \right)^{\frac{1}{2}}, \quad \forall u \in X_0.$$  \hfill (2.4)

The above information follows from properties involving anisotropic Sobolev spaces, for more details see Besov, Il’in, and Nikolski [10, Chapter 3].
3 The existence of solution for positive mass case

Through this section we will assume \( (h_1)-(h_3) \). We will find solutions of equation (1.3) as critical points of the energy functional \( I : X \to \mathbb{R} \) given by

\[
I(u) = \frac{1}{2} \|u\|^2 - \int_{\mathbb{R}^2} H(u) \, dxdy.
\]

**Lemma 3.1.** The functional \( I : X \to \mathbb{R} \) verifies the mountain pass geometry, that is,

(i) there are \( \alpha, \rho > 0 \) such that \( I(u) \geq \alpha \), for \( \|u\| = \rho \);

(ii) there is \( e \in X \setminus \{0\} \) such that \( I(e) < 0 \), with \( \|e\| > \rho \).

**Proof.** The proof of (i) follows by using standard arguments involving the growth condition on \( h \), then it will be omitted. In order to prove (ii), from \( (h_3) \) there is \( \phi \in C_0^\infty(\mathbb{R}^2) \) such that

\[
\int_{\mathbb{R}^2} \left( H(\phi) - \frac{\phi^2}{2} \right) \, dxdy > 0.
\]

For \( t > 0 \), setting \( w_t(x, y) = \phi(x/t, y/t^2) \), we derive

\[
I_\lambda(w_t) = \frac{t}{2} \|\phi\|^2_0 - t^3 \int_{\mathbb{R}^2} \left( H(\phi) - \frac{\phi^2}{2} \right) \, dxdy \to -\infty \quad \text{as} \quad t \to +\infty.
\]

Therefore, (ii) follows by choosing \( e = w_t \) with \( t \) large enough. \( \square \)

We set \( \Gamma = \{ \gamma \in C([0, 1], X) : \gamma(0) = 0, \gamma(1) = e \} \) and

\[
\sigma = \inf_{\gamma \in \Gamma} \max_{t \in [0, 1]} I(\gamma(t)).
\]

Clearly, by Lemma 3.1, \( \sigma \geq \alpha > 0 \).

Following [21, 23], we introduce an auxiliary functional \( \bar{I} \in C^1(\mathbb{R} \times X, \mathbb{R}) \) given by

\[
\bar{I}(\theta, u) = \frac{e^\theta}{2} \|u\|^2_0 + \frac{e^{3\theta}}{2} |u|^2_2 - e^{3\theta} \int_{\mathbb{R}^2} H(u) \, dxdy.
\]

The following properties hold, for all \( (\theta, u) \in \mathbb{R} \times X \),

\[
\begin{align*}
\bar{I}(0, u) &= I(u), \\
\bar{I}(\theta, u) &= I(u(e^{-\theta}x, e^{-2\theta}y)).
\end{align*}
\]

We equip a standard product norm \( \|(\theta, u)\|_{\mathbb{R} \times X} = (|\theta|^2 + \|u\|^2)^{1/2} \) to \( \mathbb{R} \times X \).

By Lemma 3.1, it is easy to see that also the functional \( \bar{I} \) satisfies the mountain pass geometry. More precisely, the following holds
Lemma 3.2. The functional $\tilde{I} : \mathbb{R} \times X \to \mathbb{R}$ verifies the mountain pass geometry, that is,

(i) there are $\alpha, \rho > 0$ such that $\tilde{I}(\theta, u) \geq \alpha$, for $\|(\theta, u)\|_{\mathbb{R} \times X} = \rho$;

(ii) there is $\tilde{e} \in \mathbb{R} \times X \setminus \{0\}$ such that $\tilde{I}(\tilde{e}) < 0$, with $\|\tilde{e}\|_{\mathbb{R} \times X} > \rho$.

Proof. For (ii) it is sufficient to take $\tilde{e} = (0, e)$, while for (i) just follows by Lemma 3.1.

In what follows, we define the mountain pass level $\tilde{\sigma}$ for $\tilde{I}$ by

$$\tilde{\sigma} = \inf_{\tilde{\gamma} \in \tilde{\Gamma}} \max_{t \in [0,1]} \tilde{I}(\tilde{\gamma}(t)),$$

where

$$\tilde{\Gamma} = \{ \tilde{\gamma} \in C([0,1], \mathbb{R} \times X) : \tilde{\gamma}(0) = (0,0), \tilde{\gamma}(1) = (0,e) \}.$$

Hence, $\tilde{\sigma} \geq \alpha > 0$. Arguing as in [21, Lemma 4.1], we derive

Lemma 3.3. The mountain pass levels of $I$ and $\tilde{I}$ coincide, namely $\sigma = \tilde{\sigma}$.

Now, as a immediate consequence of Ekeland’s variational principle, we have the result below whose proof follows as in [24, Lemma 2.3].

Lemma 3.4. Let $\varepsilon > 0$. Suppose that $\tilde{\gamma} \in \tilde{\Gamma}$ satisfies

$$\max_{t \in [0,1]} \tilde{I}(\tilde{\gamma}(t)) \leq \sigma + \varepsilon,$$

then there exists $(\theta, u) \in \mathbb{R} \times X$ such that

1. $\text{dist}_{\mathbb{R} \times X}((\theta, u), \tilde{\gamma}([0,1])) \leq 2\sqrt{\varepsilon}$;

2. $\tilde{I}(\theta, u) \in [\sigma - \varepsilon, \sigma + \varepsilon]$;

3. $\|D\tilde{I}(\theta, u)\|_{\mathbb{R} \times X^*} \leq 2\sqrt{\varepsilon}$.

Arguing as in [21], by Lemma 3.4, the following proposition holds

Proposition 3.1. There exists a sequence $\{(\theta_n, u_n)\} \subset \mathbb{R} \times X$ such that, as $n \to +\infty$, we get

1. $\theta_n \to 0$;

2. $\tilde{I}(\theta_n, u_n) \to \sigma$;

3. $\partial_\theta \tilde{I}(\theta_n, u_n) \to 0$;

4. $\partial_u \tilde{I}(\theta_n, u_n) \to 0$ strongly in $X^*$.

After the above study, we are ready to prove Theorem 1.1.
Proof of Theorem 1.1. By Proposition 3.1, there exists a sequence \( \{ (\theta_n, u_n) \} \subset \mathbb{R} \times X \) such that

\[
\begin{align*}
\frac{e^{\theta_n}}{2} \| u_n \|_0^2 + \frac{e^{3\theta_n}}{2} |u_n|_2^2 - e^{3\theta_n} \int_{\mathbb{R}^2} H(u_n) \, dx \, dy &= \sigma + o_n(1), \\
\frac{e^{\theta_n}}{2} \| u_n \|_0^2 + \frac{3e^{3\theta_n}}{2} |u_n|_2^2 - 3e^{3\theta_n} \int_{\mathbb{R}^2} H(u_n) \, dx \, dy &= o_n(1), \\
e^{\theta_n} \| u_n \|_0^2 + 3e^{3\theta_n} |u_n|_2^2 - e^{3\theta_n} \int_{\mathbb{R}^2} h(u_n) u_n \, dx \, dy &= o_n(1) \| u_n \|. 
\end{align*}
\]

(3.1)

From the first and the second equation of the previous system we get

\[ e^{\theta_n} \| u_n \|_0^2 = 3\sigma + o_n(1), \]

and so, since \( \theta_n \to 0 \), we infer that \( \{ u_n \} \) is bounded in \( X_0 \) and so also in \( L^6(\mathbb{R}^2) \), by Lemma 2.2. Observe that, by \((h_1)\) and \((h_2)\), we deduce that for any \( \delta > 0 \), there exists \( C_\delta > 0 \) such that

\[ |h(t)| \leq \delta |t| + C_\delta |t|^5, \quad \text{for all } t \in \mathbb{R}. \]

Hence, by the third equation of (3.1), using again the fact that \( \theta_n \to 0 \), we find

\[ \| u_n \|^2 \leq C e^{3\theta_n} \int_{\mathbb{R}^2} h(u_n) u_n \, dx \, dy + o_n(1) \| u_n \| \leq C \left( \delta |u_n|_2^2 + C_\delta |u_n|_6^6 \right) + o_n(1) \| u_n \|. \]

Then for \( \delta \) small enough, and using the fact that \( \{ |u_n|_6 \} \) is bounded, it follows that

\[ \| u_n \|^2 \leq C, \quad \forall n \in \mathbb{N}, \]

for some \( C > 0 \), showing that \( \{ u_n \} \) is actually bounded in \( X \). Moreover, by the continuous embedding \((2.1)\), we also have

\[ |u_n|_{p+1}^2 \leq C \| u_n \|^2 \leq C |u_n|_{p+1}^p, \quad \forall n \in \mathbb{N}, \]

and so there exists \( c > 0 \) such that \( |u_n|_{p+1}^p \geq c > 0 \) for all \( n \in \mathbb{N} \). Hence, by Lemma 2.1, there exist a sequence of points \( \{ (x_n, y_n) \} \subset \mathbb{R}^2 \) and \( r, \beta > 0 \) such that

\[ \int_{B_r((x_n, y_n))} |u_n|^2 \, dx \, dy \geq \beta > 0. \]

Hence, calling \( v_n = u(-x_n, -y_n) \), being \( \{ v_n \} \) a bounded sequence in \( X \), up to a subsequence, we must have

\[ v_n \rightharpoonup v \neq 0 \quad \text{weakly in } X. \]

By the invariance by translations of \( \tilde{I} \), we have that \( \partial_{\theta} \tilde{I}(\theta_n, v_n) \to 0 \) strongly in \( X^* \), and so, since \( \theta_n \to 0 \) and by the local compact embedding \((2.2)\), we conclude that \( I'(v) = 0 \), thus \( v \) is a non-trivial solution of \((1.3)\).  

\( \square \)
4 The existence of solution for the zero mass case

Through this section we will assume (h₃)-(h₅). We start with a technical lemma, which will be used later on.

Lemma 4.1. Let \( \{w_n\} \subset X_0 \) be a bounded sequence in \( X_0 \) with

\[
\lim_{n \to +\infty} \sup_{(x,y) \in \mathbb{R}^2} \int_{K(x,y)} |f(w_n)w_n| \, dx \, dy = 0,
\]

where \( K(x,y) = (x-1, x+1) \times (y-1, y+1) \). Then \( \lim_{n \to +\infty} \int_{\mathbb{R}^2} f(w_n)w_n \, dx \, dy = 0 \).

Proof. By (h₅), there is \( C > 0 \) such that

\[
|f(t)t| \leq C|t|^6, \quad \forall t \in \mathbb{R}.
\]

The above inequality combines with (2.4) to give

\[
\int_{K(x,y)} |f(w_n)w_n| \, dx \, dy \leq C \int_{K(x,y)} |w_n|^6 \, dx \, dy \leq C_1 \left[ \int_{K(x,y)} \left( |(w_n)_x|^2 + |D_x^{-1}(w_n)_y|^2 + |w_n|^2 \right) \, dx \, dy \right]^{\frac{3}{2}}.
\]

Thus, for all \( \lambda \in (0, 1) \),

\[
\int_{K(x,y)} |f(w_n)w_n| \, dx \, dy \leq C^{\lambda}_1 \left[ \int_{K(x,y)} \left( |(w_n)_x|^2 + |D_x^{-1}(w_n)_y|^2 + |w_n|^2 \right) \, dx \, dy \right]^{\frac{3\lambda}{2}} \left( \int_{K(x,y)} |f(w_n)w_n| \, dx \, dy \right)^{1-\lambda}.
\]

Setting \( \lambda = \frac{1}{3} \), we get

\[
\int_{K(x,y)} |f(w_n)w_n| \, dx \, dy \leq C^{1/3}_1 \left[ \int_{K(x,y)} \left( |(w_n)_x|^2 + |D_x^{-1}(w_n)_y|^2 + |w_n|^2 \right) \, dx \, dy \right]^{2/3} \left( \int_{K(x,y)} |f(w_n)w_n| \, dx \, dy \right)^{1/3}.
\]

By using the fact that

\[
|w_n|_{L^2(K(x,y))} \leq C_* |w_n|_{L^6(K(x,y))}, \quad \forall n \in \mathbb{N},
\]

for some constant \( C_* > 0 \) independent of \( (x,y) \in \mathbb{R}^2 \), we get

\[
\int_{\mathbb{R}^2} |f(w_n)w_n| \, dx \, dy \leq C^{1/3}_1 \sup_{(x,y) \in \mathbb{R}^2} \int_{K(x,y)} |f(w_n)w_n| \, dx \, dy \]

\[
\leq C_2 \left( \sup_{(x,y) \in \mathbb{R}^2} \int_{K(x,y)} |f(w_n)w_n| \, dx \, dy \right)^{2/3},
\]

and so,

\[
\int_{\mathbb{R}^2} |f(w_n)w_n| \, dx \, dy \leq C_2 \left( \sup_{(x,y) \in \mathbb{R}^2} \int_{K(x,y)} |f(w_n)w_n| \, dx \, dy \right)^{2/3},
\]

for all \( n \in \mathbb{N} \) and for some \( C_2 > 0 \). From where it follows that

\[
\lim_{n \to +\infty} \int_{\mathbb{R}^2} |f(w_n)w_n| \, dx \, dy = 0
\]

and so the claim. \( \square \)
Associated with equation (1.5), by \((h_4)\), we have the energy functional \(I_0 : X_0 \to \mathbb{R}\) given by
\[
I_0(u) = \frac{1}{2}\|u\|_0^2 - \int_{\mathbb{R}^2} F(u) \, dx \, dy.
\]

**Lemma 4.2.** The functional \(I_0 : X_0 \to \mathbb{R}\) verifies the mountain pass geometry, that is,

(i) there are \(\alpha, \rho > 0\) such that \(I_0(u) \geq \alpha\), for \(\|u\|_0 = \rho\);

(ii) there is \(e \in X_0 \setminus \{0\}\) such that \(I_0(e) < 0\), with \(\|e\|_0 > \rho\).

**Proof.** The proof is similar as in Lemma 3.1, using \((h_5)\) and (1.4).

We set \(\Gamma_0 = \{\gamma \in C([0, 1], X_0) : \gamma(0) = 0, \gamma(1) = e\}\) and
\[
\sigma_0 = \inf_{\gamma \in \Gamma_0} \max_{t \in [0, 1]} I_0(\gamma(t)).
\]
Clearly, by Lemma 4.2, \(\sigma_0 \geq \alpha > 0\).

As in the previous section, we introduce the auxiliary functional \(\tilde{I}_0 \in C^1(\mathbb{R} \times X_0, \mathbb{R})\) given by
\[
\tilde{I}_0(\theta, u) = \frac{e^\theta}{2}\|u\|_0^2 - e^{3\theta} \int_{\mathbb{R}^2} F(u) \, dx \, dy.
\]

The following properties hold, for all \((\theta, u) \in \mathbb{R} \times X_0\),
\[
\tilde{I}_0(0, u) = I_0(u),
\]
\[
\tilde{I}_0(\theta, u) = I_0(u(e^{-\theta}x, e^{-2\theta}y)).
\]

We equip \(\mathbb{R} \times X_0\) with the standard product norm \(\|(\theta, u)\|_{\mathbb{R} \times X_0} = (|\theta|^2 + \|u\|_0^2)^{1/2}\). Arguing as in Lemma 3.2, we can see that \(\tilde{I}_0\) satisfies the mountain pass geometry. More precisely, we have

**Lemma 4.3.** The functional \(\tilde{I}_0 : \mathbb{R} \times X_0 \to \mathbb{R}\) verifies the mountain pass geometry, that is,

(i) there are \(\alpha, \rho > 0\) such that \(\tilde{I}_0(\theta, u) \geq \alpha\), for \(\|(\theta, u)\|_{\mathbb{R} \times X_0} = \rho\);

(ii) there is \(\tilde{e} \in \mathbb{R} \times X_0 \setminus \{0\}\) such that \(\tilde{I}_0(\tilde{e}) < 0\) with \(\|\tilde{e}\|_{\mathbb{R} \times X_0} > \rho\).

Hence we define the mountain pass level \(\tilde{\sigma}_0\) for \(\tilde{I}_0\) by
\[
\tilde{\sigma}_0 = \inf_{\tilde{\gamma} \in \tilde{\Gamma}_0} \max_{t \in [0, 1]} \tilde{I}_0(\tilde{\gamma}(t)),
\]
where \(\tilde{\Gamma} = \{\tilde{\gamma} \in C([0, 1], \mathbb{R} \times X_0) : \tilde{\gamma}(0) = (0, 0), \tilde{\gamma}(1) = (0, e)\}\).

From definition \(\sigma_0\), we have \(\tilde{\sigma}_0 \geq \alpha > 0\), \(\sigma_0 = \tilde{\sigma}_0\) and the following proposition

**Proposition 4.1.** There exists a sequence \(\{(\theta_n, u_n)\} \subset \mathbb{R} \times X_0\) such that, as \(n \to +\infty\), we get
1. $\theta_n \to 0$;
2. $\hat{I}_0(\theta_n, u_n) \to \sigma$;
3. $\partial_{\theta} \hat{I}_0(\theta_n, u_n) \to 0$;
4. $\partial_u \hat{I}_0(\theta_n, u_n) \to 0$ strongly in $X_0^*$.

Now, we are going to prove Theorem 1.2.

**Proof of Theorem 1.2.** By Proposition 4.1, there exists a sequence $\{(\theta_n, u_n)\} \subset \mathbb{R} \times X_0$ such that, as $n \to +\infty$, we have

$$\begin{align*}
\frac{e^{\theta_n}}{2} \|u_n\|_0^2 - e^{3\theta_n} \int_{\mathbb{R}^2} F(u_n) \, dx \, dy & = \sigma_0 + o_n(1), \\
\frac{e^{\theta_n}}{2} \|u_n\|_0^2 - 3e^{3\theta_n} \int_{\mathbb{R}^2} F(u_n) \, dx \, dy & = o_n(1), \\
\frac{e^{\theta_n}}{2} \|u_n\|_0^2 - e^{3\theta_n} \int_{\mathbb{R}^2} f(u_n) u_n \, dx \, dy & = o_n(1) \|u_n\|_0.
\end{align*}$$

(4.1)

From the first and the second equation of the previous system we get

$$e^{\theta_n} \|u_n\|_0^2 = 3\sigma_0 + o_n(1), \quad (4.2)$$

and so, since $\theta_n \to 0$, we infer that $\{u_n\}$ is bounded in $X_0$ and so also in $L^6(\mathbb{R}^2)$, by Lemma 2.2. Hence, by the third equation of (4.1) and (4.2), using again the fact that $\theta_n \to 0$, there exists $c > 0$ such that

$$\int_{\mathbb{R}^2} f(u_n) u_n \, dx \, dy \geq c > 0, \quad \forall n \in \mathbb{N}.$$ 

Hence, by Lemma 4.1, there exist a sequence of points $\{(x_n, y_n)\} \subset \mathbb{R}^2$ and $\bar{c} > 0$ such that

$$\int_{K(x_n, y_n)} |f(u_n) u_n| \, dx \, dy \geq \bar{c} > 0, \quad \forall n \in \mathbb{N}.$$ 

Hence, calling $v_n = u(\cdot - x_n, \cdot - y_n)$, and being $\{v_n\}$ a bounded sequence in $X_0$, up to a subsequence, we have

$$v_n \rightharpoonup v \quad \text{weakly in } X_0.$$ 

By $(h_5)$, there is $C > 0$ such that

$$|f(t)t| \leq \frac{\bar{c}}{2M} |t|^6 + C|t|^2, \quad \forall t \in \mathbb{R},$$

where $M = \sup_{n \in \mathbb{N}} \int_{\mathbb{R}^2} |v_n|^6 \, dx \, dy$. From this,

$$C \int_{K(0,0)} |v_n|^2 \, dx \, dy \geq \frac{\bar{c}}{2}, \quad \forall n \in \mathbb{N}$$

12
and so, by (2.3),
\[
\int_{K(0,0)} |v|^2 \, dx \, dy \geq \frac{c}{2C}, \quad \forall n \in \mathbb{N},
\]
implying that \( v \neq 0 \).

Now, we will prove that \( v \) is a solution for (1.5). To this end, without loss of generality, we can assume that
\[
v_n(x) \to v(x) \quad \text{a.e. in } \mathbb{R}^2,
\]
and so, by continuity,
\[
f(v_n(x)) \to f(v(x)) \quad \text{a.e. in } \mathbb{R}^2.
\]

By \((h_5)\), \( \{ f(v_n) \} \) is a bounded sequence in \( L^6_\infty(\mathbb{R}^2) \), and so, there exists \( g \in L^6_\infty(\mathbb{R}^2) \) such that \( f(v_n) \rightharpoonup g \) in \( L^6_\infty(\mathbb{R}^2) \). It is standard to prove that \( g = f(v) \). These informations yield, in particular, that
\[
\int_{\mathbb{R}^2} f(v_n) \phi \, dx \, dy \to \int_{\mathbb{R}^2} f(v) \phi \, dx \, dy, \quad \forall \phi \in X_0.
\]

This limit combines with \( \partial_u I_0(\theta_n, v_n)[\phi] \to 0 \) to give \( I'_0(v) \phi = 0 \), for any \( \phi \in X_0 \), showing that \( v \) is a nontrivial for (1.5).

\[\square\]

References

[1] C.O. Alves, M. Montenegro, and M.A.S. Souto, Existence of solution for three classes of elliptic problems in \( \mathbb{R}^N \) with zero mass, J. Differential Equations 252 (2012), 5735–5750.

[2] C.O. Alves, M. Montenegro, and M.A.S. Souto, Existence of a ground state solution for a nonlinear scalar field equation with critical growth, Calc. Var. Partial Differential Equations 43 (2012), 537–554.

[3] C.O. Alves and O.H. Miyagaki, Existence, regularity and concentration phenomenon of nontrivial solitary waves for a class of generalized variable coefficient Kadomtsev-Petviashvili equation, J. Math. Phys. 58 (2017), no. 8, 081503, 18 pp.

[4] A. Ambrosetti and P.H. Rabinowitz, Dual variational methods in critical point theory and applications, J. Funct. Anal. 14, (1973), 349–381.

[5] A. Azzollini and A. Pomponio, On the Schrödinger equation in \( \mathbb{R}^N \) under the effect of a general nonlinear term, Indiana Univ. Math. J. 58 (2009), no. 3, 1361–1378.

[6] A. Azzollini, P. d’Avenia, and A. Pomponio, Multiple critical points for a class of nonlinear functionals, Ann. Mat. Pura Appl. 190, (2011), 507–523.

[7] V. Benci, C.R. Grisanti, and A.M. Micheletti, Existence and non existence of the ground state solution for the nonlinear Schrodinger equations with \( V(\infty) = 0 \), Topol. Methods Nonlinear Anal., 26, (2005), 203–219.
[8] H. Berestycki and P.L. Lions, *Nonlinear scalar field equations, I-existence of a ground state*, Arch. Rat. Mech. Anal. **82**, (1983), 313–346.

[9] H. Berestycki, T. Gallouët, and O. Kavian, *Equations de Champs scalaires euclidiens non linéaires dans le plan*. C. R. Acad. Sci. Paris Ser. I Math. **297**, (1984), 307–310.

[10] O.V. Besov, V.P. Il’in, and S.M. Nikolski, *Integral representations of functions and imbedding theorems*, vol. I, Wiley, New York, 1978.

[11] J. Bourgain, *On the Cauchy problem for the Kadomtsev-Petviashvili equation*, Geom. Funct. Anal. **3**, (1993), 315-341.

[12] P.L. Cunha, P. d’Avenia, A. Pomponio, and G. Siciliano, *A multiplicity result for Chern-Simons-Schrödinger equation with a general nonlinearity*, Nonlinear Differential Equations and Applications NoDEA **22**, (2015), 1831–1850.

[13] A. De Bouard and J.C. Saut, *Sur les ondes solitaires des équations de Kadomtsev-Petviashvili*, C. R. Acad. Sciences Paris, **320**, (1995), 315–318.

[14] A. De Bouard and J.C. Saut, *Solitary waves of generalized Kadomtsev-Petviashvili equations*, Ann. Inst. H. Poincaré Anal. Non Linéaire **14**, (1997), 211-236.

[15] A. De Bouard and J.C. Saut, *Symmetries and decay of the generalized Kadomtsev-Petviashvili equations solitary waves*, SIAM J. Math. Anal. **28**(1997), no. 5, 1064-1085.

[16] X.M. He and W.M. Zou, *Nontrivial solitary waves to the generalized Kadomtsev-Petviashvili equations*, Applied Mathematics and Computation **197**, (2008), 858–863.

[17] A. Faminskii, *The Cauchy problem for Kadomtsev-Petviashvili equation*, Russian Math. Surveys, **5** (1990), 203–204 and Siberian J. Math. **33** (1992), 133–143.

[18] B. Gidas, W.M. Ni, and L. Nirenberg, *Symmetry and related properties via the maximum principle*, Comm. Math. Phys. **68**, (1979), no. 3, 209–243.

[19] M. Ghimenti and A.M. Micheletti, *Existence of minimal nodal solutions for the Nonlinear Schrödinger equations with $V(\infty) = 0$*, Adv. Differential Equations **11**, (2006), no. 12, 1375–1396.

[20] F. Güngör and P. Winternitz, *Generalized Kadomtsev-Petviashvili equation with an infinite-dimensional symmetry algebra*, J. Math. Anal. Appl. **276**, (2002), 314–328.

[21] J. Hirata, N. Hikoma, and K. Tanaka, *Nonlinear scalar field equations in $\mathbb{R}^N$: mountain pass and symmetric mountain pass approaches*, Topol. Methods Nonlinear Anal. **35**, (2010), 253–276.

[22] P. Isaza and J. Mejia, *Local and global Cauchy problem for the Kadomtsev-Petviashvili (KP-II) equation in Sobolev spaces of negative indices*, Comm. Partial Differential Equations **26**, (2001), 1027–1054.
[23] L. Jeanjean, *Existence of solutions with prescribed norm for semilinear elliptic equations*, Nonlinear Anal. **28**, (1997), 1633–1659.

[24] L. Jeanjean, *On the existence of bounded Palais-Smale sequences and application to a Landesman-Lazer-type problem set on \( \mathbb{R}^N \)*, Proc. Royal Soc. Edin. 129A (1999), 787–809.

[25] L. Jeanjean and K. Tanaka, *A Remark on least energy solutions in \( \mathbb{R}^N \)*, Proc. Amer. Math. Soc. **131** (2002), 2399–2408.

[26] B.B. Kadomtsev and V.I. Petviashvili, *On the stability of solitary waves in weakly dispersing media*, Soviet Physics Doklady **15**(6), (1970), 539–541.

[27] C. Klein and J.C. Saut, *Numerical Study of Blow up and Stability of Solutions of Generalized Kadomtsev-Petviashvili Equations*, J Nonlinear Sci. **22** (2012), 763–811.

[28] P.L. Lions, *The concentration compactness principle in the calculus of variations. The locally compact case.*, Ann. Inst. H. Poincaré-AN **1** (2,4) (1984), 109–145; 223–283.

[29] Z.P. Liang and J.B. Su, *Existence of solitary waves to a generalized Kadomtsev-Petviashvili equation*, Acta Math. Scientia **32B**, (2012), 1149–1156.

[30] A. Pomponio and T. Watanabe, *Some quasilinear elliptic equations involving multiple \( p \)-Laplacians*, to appear on Indiana Univ. Math. J.

[31] J.C. Saut, *Recent results on the generalized Kadomtsev-Petviashvili equations*, Acta Appl. Math. **39**, (1995), 477–1487.

[32] A. Szulkin and W.M. Zou, *Homoclinic orbit for asymptotically linear Hamiltonian systems*, J. Funct. Anal. **187**, (2001), 25–41.

[33] B. Tian and Y.T. Gao, *Solutions of a variable-coefficient Kadomtsev-Petviashvili equation via computer algebra*, Applied Mathematics and Computation **84**, (1997), 125–130.

[34] Z.Q. Wang and M. Willem, *A multiplicity result for the generalized Kadomtsev-Petviashvili equation*, Topol. Meth. Nonlinear Anal. **7**(2), (1996), 261–270.

[35] M. Willem, *Minimax theorems*, Birkhäuser, Boston Basel Berlin, 1996.

[36] J. Xu, Z. Wei, and Y. Ding, *Stationary solutions for a generalized Kadomtsev-Petviashvili equation in bounded domain*, Electronic J. Qualitative Theory of Differential Equations **68**, (2012), 1–18.

[37] B. Xuan, *Nontrivial solitary waves of GKP equation in multi-dimensional spaces*, Revista Colombiana de Matemáticas **37**, (2003), 11–23.

[38] Y. Zhang, Y. Xu, and K. Ma, *New type of a generalized variable-coefficient Kadomtsev-Petviashvili equation with self-consistent sources and its Grammian-type solutions*, Communications in Nonlinear Science and Numerical Simulation. **37**, (2016), 77–89.
[39] W.M. Zou, *Solitary waves of the generalized Kadomtsev-Petviashvili equations*, Appl. Math. Lett. **15**, (2002), 35–39.