Density functional theory of the trapped Fermi gas in the unitary regime

Brandon P. van Zyl, D. A. W. Hutchinson, and Melodie Need

Department of Physics, St. Francis Xavier University, Antigonish, Nova Scotia
The Jack Dodd Centre for Photonics and Ultra-Cold Atoms, Department of Physics, University of Otago, Dunedin, New Zealand
Laboratoire Kastler Brossel, École Normale Supérieure, 24 rue Lhomond, 75231 Paris Cedex 05, France

Abstract

We investigate a density-functional theory (DFT) approach for an unpolarized trapped dilute Fermi gas in the unitary limit. A reformulation of the recent work of T. Papenbrock [Phys. Rev. A, 72, 041602(R) (2005)] in the language of fractional exclusion statistics allows us to obtain an estimate of the universal factor, $\xi_{3D}$, in three dimensions (3D), in addition to providing a systematic treatment of finite-$N$ corrections. We show that in 3D, finite-$N$ corrections lead to unphysical values for $\xi_{3D}$, thereby suggesting that a simple DFT applied to a small number of particles may not be suitable in 3D. We then perform an analogous calculation for the two-dimensional (2D) system in the infinite-scattering length regime, and obtain a value of $\xi_{2D} = 1$. Owing to the unique properties of the Thomas-Fermi energy density-functional in 2D our result, in contrast to 3D, is exact and therefore requires no finite-$N$ corrections.

Keywords: fermions, unitarity, two-dimensions
I. INTRODUCTION

Harmonically trapped ultracold atomic Fermi gases have received considerable attention within the last few years owing to the experimental realization of Fermi degeneracy by DeMarco and Jin \[1\], the pioneering work of the Thomas group in the unitary regime \[2, 3\] and other experimental efforts including, for example, Refs. \[4, 5, 6, 7\]. Using the phenomenon of a Feshbach resonance, the two-body inter-particle interactions between the atoms can be continuously tuned according to the magnitude of the magnetic field across the resonance region. Of particular interest is the so-called unitary regime which occurs at the midpoint of this crossover, and is characterized by the divergence of the scattering length due to the existence of a zero-energy bound state for the two-body system \[8\]. In this regime, the only relevant length scale is set by the Fermi momentum, \(k_f^{-1}\), and so the corresponding energy scale is the Fermi kinetic energy. Therefore, in three-dimensions (3D), the energy density should be proportional to that of a free Fermi gas

\[
\frac{E}{N} = \xi_{3D} \left( \frac{E}{N} \right)_{\text{free}} = \xi_{3D} \frac{3k_f^2}{10m}
\]

where \(\xi_{3D}\) is a dimensionless proportionality constant, and \(k_f = (3\pi^2 \rho)^{1/3}\). In the unitary limit, the system details do not contribute to the bulk thermodynamic properties \[9, 10, 11\], and so \(\xi_{3D}\) is usually called the universal factor. Owing to the lack of a small expansion parameter, i.e., \((ak_f)\), the exact determination of the universal factor \(\xi_{3D}\) is a challenging problem because the usual Green’s function techniques for the many body system are completely unreliable. Moreover, as the unitary Fermi gas may be relevant for the physics of neutron stars and high-\(T_c\) superconductivity, the extraction of the universal factor is also an important task.

To date, theoretical interest in the universal factor has been primarily limited to the 3D unitary Fermi gas \[9, 10, 11, 12, 13, 14, 15, 16, 17, 18\]. These theoretical approaches vary in their level of sophistication, ranging from simple applications of density-functional theory (DFT), to relativistic QED-like theories with hidden local symmetry, and quantum Monte Carlo (QMC) simulations. Unfortunately, the wide variety of theoretical approaches employed for the determination of \(\xi_{3D}\) have also led to a broad range of possible values for \(\xi_{3D}\); namely, theory predicts \(\xi_{3D} \sim 0.3 - 0.6\), although Engelbrecht \textit{et al.} have established an upper bound of \(\xi_{3D} < 0.59\) from a BCS treatment \[12\]. The disagreement between theoretical
approaches not withstanding, it is generally accepted that the recent QMC simulations of Carlson et al. 15 yield the most reliable estimate for the universal factor, which is $\xi_{3D} = 0.44 \pm 0.01$.

Initially it was difficult experimentally to obtain a value for $\xi_{3D}$, because the precise location of the required Feshbach resonance was not known and a range of estimates $\xi_{3D} \sim 0.3 - 0.8$ 19 were thus obtained. With the position of the resonance now accurately determined, more recent experimental measurements of $\xi_{3D}$ are in better agreement and have yielded $\xi_{3D} = 0.51 \pm 0.04$ 20 and $\xi_{3D} = 0.46 \pm 0.05$ 21. These results strongly suggest that the QMC estimate of $\xi_{3D} \approx 0.44$ is indeed very reliable.

In this paper we determine the universal factor, $\xi$ for a trapped two-dimensional (2D) and 3D Fermi gas. In 3D, our formulation of the problem in the language of fractional exclusion statistics (FES) allows us not only to extract a value for $\xi_{3D}$, but also provides a systematic treatment of finite-$N$ corrections. Our determination of $\xi_{2D}$ is likewise obtained within an exceedingly simple mathematical framework, which nevertheless leads to an exact result for the universal factor in two-dimensions. We are aware of no experimental results for the (quasi) 2D Fermi gas in, or near, the appropriate scattering regime and only two recent theoretical studies with the explicit aim of extracting $\xi_{2D}$, both agreeing on $\xi_{2D} = 1$ 22, 23. It has also been pointed out in passing by Tonini et al. 24 that for large scattering length in 2D, the effective interactions are weakly attractive, thereby yielding the non-interacting Fermi gas in the infinite scattering length limit 8; this observation plays a critical role in our subsequent determination of $\xi_{2D}$.

II. TRAPPED FERMI GAS IN THE UNITARY REGIME

In this section, we apply a simple DFT in order to determine the universal factor for an unpolarized trapped Fermi gas. We begin by reformulating the work of Papenbrock in 3D, with particular attention paid to finite-$N$ corrections, and what impact they have on the estimate of $\xi_{3D}$. We also present an analogous analysis in 2D, and show that it yields an exact result for $\xi_{2D}$ without requiring any finite-$N$ corrections.
A. Three Dimensions

As stated in the introduction, Carlson *et al.* have performed simulations for a uniform 3D Fermi gas in the unitary regime. Dimensional arguments suggest that in this regime, the energy must be proportional to that of a free Fermi gas,

\[ E(N) = \xi_{3D} E_{\text{free}}(N), \]

(2)

where \( E_{\text{free}}(N) \) is the Thomas-Fermi (TF) energy of \( N \) noninteracting spin-1/2 fermions. Surprisingly, the QMC simulations with \( N = 10 - 40 \) fermions agree very well with Eq. (2) in spite of the fact that the exact quantum mechanical energies are known to differ from the corresponding TF energies due to shell and finite size effects. Motivated by this result, and again on dimensional grounds, Papenbrock [13] has suggested that a density functional from TF theory,

\[ \varepsilon_{\text{3DHO}}^{\text{TF}}(\rho) = \xi_{3D} \frac{\hbar^2}{m} \rho^{5/3} + \frac{1}{2} m \omega^2 r^2 \rho, \]

(3)

should be a good approximation to the exact density functional. In general, gradient corrections should be included, but Eq. (3) represents the simplest possible form for the energy density-functional. In the above, \( \omega \) is the trapping frequency of an isotropic harmonic oscillator (HO) potential, and \( c = \frac{3}{10} (3\pi^2)^{2/3} \). It is straightforward to show that a minimization of this functional, subject to a fixed number of particles, leads to

\[ \rho_{\text{3DHO}}^{\text{TF}} = \frac{1}{3\pi^2} \left( \frac{2}{\xi_{3D} l^2} \right)^{3/2} \left( (3N)^{1/3} \xi_{3D}^{1/2} - \frac{r^2}{2l^2} \right)^{3/2}, \]

(4)

where \( l = (\hbar/m\omega)^{1/2} \) is the oscillator length. Integrating this energy functional up to the TF radius yields the total TF energy

\[ E_{\text{3DHO}}^{\text{TF}} = \frac{1}{4} (3N)^{4/3} \xi_{3D}^{1/2} \hbar \omega. \]

(5)

The energy can be calculated exactly for the case \( N = 2 \), where there is a binding energy of \( \hbar \omega \) in the unitary limit such that total energy for two fermions in a harmonic trap \( E_{\text{ex}} = 2\hbar \omega \) [25]. Comparison with the TF expression (which is expected to be reasonable for all even numbers of fermions) with \( N = 2 \) gives a value for the universal constant of \( \xi_{3D} \approx 0.54 \). Although it is not explicitly mentioned in Ref. [13], if one uses the exact two-particle density

\[ \rho_{\text{ex}}(r) = \frac{4}{\pi^{3/2} l^3} e^{-2(r/l)^2} \int_0^r dx e^{x^2}, \]

(6)
in Eq. (3) rather than Eq. (4), and then performs the spatial integration, a much better value of $\xi_{3D} \approx 0.48$ is obtained. Thus, a very simple DFT already yields a value for $\xi_{3D}$ that is within 8% of the generally accepted QMC result. It is then natural to ask what effect finite-$N$ corrections will have on this simple result; after all, TF DFT is generally only valid for $N \gg 1$, and/or slowly spatially varying potentials, which is certainly not the case in the present analysis. In order to investigate the finite-$N$ corrections, it turns out to be more convenient to reformulate the DFT of Ref. 13 in a slightly different manner.

Bhaduri, et al. [26] have argued that the uniform 3D interacting Fermi gas at unitarity can be described in terms of a non-interacting gas obeying fractional exclusion statistics (FES). This generalization of the standard Bose-Einstein and Fermi-Dirac particle statistics was developed by Haldane [27] in the context of the quasiparticle excitations of the fractional quantum Hall effect, and later generalized by Wu [28]. In these Haldane-Wu statistics the concept of the Pauli exclusion principle is extended such that the distribution function for a single particle state with energy $\varepsilon$ is given by

$$n(\varepsilon) = \frac{1}{w + \alpha}, \quad (7)$$

where the temperature dependent function $w$ is defined through

$$w(1 + w)^{1-\alpha} = e^{\beta(\varepsilon - \mu)}. \quad (8)$$

In the limit $\alpha = 0$ this then reduces to the usual Bose distribution and for $\alpha = 1$, the Fermi-Dirac distribution. At zero temperature the distribution reduces to

$$n(\varepsilon) = \frac{1}{\alpha}, \quad \varepsilon < \mu$$
$$n(\varepsilon) = 0, \quad \varepsilon > \mu. \quad (9)$$

The parameter $\alpha$ that defines the statistics gives a modified exclusion principle which states that the maximum occupancy of a single state is $1/\alpha$ — hence fractional exclusion statistics.

Let us now consider the density of states, $D(\varepsilon)$, for the 3D HO (including the spin degeneracy factor of 2) [29]

$$D(\varepsilon) = \frac{\varepsilon^2}{(\hbar\omega)^3} - \frac{1}{4(\hbar\omega)}. \quad (10)$$

Following Bhaduri et al. [26], we now utilize the Haldane-Wu statistics to write the total number of particles, $N$, in the system as

$$N = \frac{1}{\alpha} \int_{0}^{\mu} D(\varepsilon) \, d\varepsilon, \quad (11)$$
where, as above, $1/\alpha$ replaces the usual zero temperature occupancy factor of unity for fermions. Rescaling all energies with respect to the oscillator energy $\hbar \omega$ gives the following expression for the chemical potential upon integration

$$\frac{\mu^3}{3} - \frac{\mu}{4} = (\alpha N) = 0.$$  

(12)

This equation has one physically acceptable root, viz.,

$$\mu = \frac{1}{2} \left( 12\alpha N + \sqrt{144\alpha^2 N^2 - 1} \right)^{1/3} + \frac{1}{2} \left( 12\alpha N + \frac{1}{\sqrt{144\alpha^2 N^2 - 1}} \right)^{1/3}$$

$$= (3\alpha N)^{1/3} + \frac{1}{4} (3\alpha N)^{1/3} - \frac{1}{192 (3\alpha N)^{5/3}} + \cdots$$  

(13)

to $O(N^{-5/3})$ corrections. Similarly, the total energy of the system is given by

$$E = \frac{1}{\alpha} \int_0^\mu D(\varepsilon) \varepsilon \, d\varepsilon$$

$$= \frac{1}{\alpha} \left( \frac{\mu^4}{4} - \frac{\mu^2}{8} \right).$$  

(14)

Rewriting Eq. (14) explicitly in terms of $N$ results in the expression (to $O(N^{-2/3})$ corrections)

$$E = \alpha^{1/3} \frac{(3N)^{4/3}}{4} + \alpha^{-1/3} \frac{(3N)^{2/3}}{8} + \alpha^{-1} \frac{1}{32} + \alpha^{-5/3} \frac{1}{384 (3N)^{2/3}} + \cdots$$  

(15)

We can now make a formal connection with Ref. 13 by identifying $\xi_{3D} = \alpha^{2/3}$. Retaining only the first term in Eq. (15), we then obtain

$$E = \frac{1}{4} (3N)^{4/3} (\xi_{3D})^{1/2},$$  

(16)

which is identical to the result for the total TF energy given in [13]. For the $N = 2$ case, we have $E_{ex} = 2$, and we obtain as a first approximation to $\alpha$

$$2 = \alpha^{1/3} \frac{6^{4/3}}{4} \approx 2.725 \alpha^{1/3},$$  

(17)

which immediately gives $\alpha = 32/81$, or $\xi_{3D} \approx 0.54$, in complete agreement with the DFT approach of Papenbrock. If we keep the first leading order correction in Eq. (15), we obtain (again, for the $N = 2$ case) a second approximation to $\alpha$

$$2 = \alpha^{1/3} \frac{6^{4/3}}{4} + \alpha^{-1/3} \frac{6^{2/3}}{8},$$  

(18)

which has no real roots. However, if we retain the first and third (i.e., $\alpha^{-1}/32$) terms in Eq. (15), we obtain a much improved value of $\xi_{3D} \approx 0.49$. Nevertheless, beyond the first and
this latter approximation, we find that there is no real value of $\alpha$ which yields $E = 2$. Thus, a systematic treatment of the finite-$N$ corrections (sometimes referred to as the extended Thomas-Fermi theory (ETF)), with the goal of yielding an improved value for $\xi_{3D}$, fails in 3D (for the $N = 2$ case at least). Of course this could also be an indication that the identification of the strongly interacting Fermi gas with a gas of non-interacting particles obeying FES is not valid (in 3D) for small particle numbers once finite-$N$ corrections are considered. Bhaduri et al. [26] have however used this approach, even for small numbers of particles, to obtain thermodynamic properties of the unitary gas at finite-temperature which compare well with the Monte Carlo results [16, 17]. It is also worth noting that for $\alpha = 1$ (i.e., Fermi-Dirac statistics), the exact energy and chemical potential are $E = 3$ and $\mu = 2$, respectively. In this case, the systematic corrections discussed above do yield improved values. For example, the first approximation gives $E = 2.73$ and $\mu = 1.82$, while the second approximation already gives $E = 3.14$ and $\mu = 1.96$. This said, the fact that the first approximation, and its augmented form (i.e., including the $N$-independent term) yield values for the universal factor so close to the QMC result is perhaps fortuitous. Interestingly, the requirement that $E$ be real leads to a lower bound of $\xi_{3D} > 0.12$, and the energy takes on a minimum value $E \approx 2.45$ for $\xi_{3D} \approx 0.3$.

B. Two Dimensions

Turning now to the two-dimensional isotropic harmonic trap, the TF density functional reads (for the noninteracting Fermi gas)

$$\varepsilon_{\text{TF}}^{2\text{DHO}}(\rho) = \frac{\hbar^2}{2m} \rho^2 + \frac{1}{2} m \omega^2 r^2 \rho. \tag{19}$$

At zero temperature, when filling $M + 1$ oscillator shells, Brack and van Zyl [30] have provided an exact, closed form solution for the zero-temperature particle density in 2D,

$$\rho(r) = \sum_{n=0}^{M} (M - n + 1)(-1)^n L_n \left( \frac{2m\omega}{\hbar} r^2 \right) e^{-m\omega r^2}, \tag{20}$$

where $L_n(x)$ is a Laguerre polynomial. Inserting Eq. (20) into Eq. (19), and integrating over all space yields the total TF energy

$$E_{\text{TF}}^{2\text{DHO}} = \frac{\hbar \omega}{3} [2M^3 + 9M^2 + 13M + 6]. \tag{21}$$
The number of particles in \( M + 1 \) filled shells is given by \( N(M) = M^2 + 3M + 2 \). What is truly remarkable about Eq. (21) is that it is also the exact quantum mechanical energy of the system. In other words, the TF functional (which is exact only in the homogeneous limit), without gradient corrections yields the exact total energy of the inhomogeneous non-interacting system.

Before applying the above result to the trapped 2D Fermi gas at unitarity, we must first make clear what is meant by “unitarity” in two-dimensions. If one considers the 2D case (see our comments for 3D in [8]) the \( s \)-wave scattering cross-section takes the form \((\hbar = 2m = 1)\)

\[
\sigma_{2D} = \frac{4\pi^2}{k \left\{ \pi^2 + 4 \left[ \ln \left( \frac{1}{a_{2D}k} \right) \right]^2 \right\}},
\]

with \( a_{2D} \) now the 2D scattering length. The maximal value for the cross-section is then obtained when the logarithmic term vanishes. Following the equivalent argument to that in 3D, the unitarity limit is therefore obtained when \( \frac{1}{a_{2D}k} \to 1 \). In general, this is not what is taken as the unitary limit in the literature. For example, Refs. 22 and 23 refer to the limit \( a_{2D} \to \infty \). Clearly, if we hold that unitarity occurs when \( \sigma_{2D} \) reaches maximal value, the dimensional arguments leading to fact that the energy of the interacting gas must be proportional to the energy of the noninteracting gas are not applicable. In particular, the hierarchy: \( a \gg k_f^{-1} \gg r_0 \), implicitly used in obtaining Eq. (2) for 3D, is not sufficient in 2D as the range of the potential, \( r_0 \), cannot generally be taken to be zero. Indeed, unitarity in 2D implies that \( a_{2D} = k_f^{-1} \). Thus, if we wish to preserve an analogous notion of unitarity in 2D, we require that the 2D scattering length become very large such that \( a_{2D}k_f \gg 1 \). Obviously, this does not correspond to a maximum in \( \sigma_{2D} \), but the large (i.e., essentially infinite) scattering length limit is what is often regarded as the unitary regime in two-dimensions 22, 23.

With the above qualification regarding unitarity in 2D in mind, we now consider the binding energy of a dimer in free space, \( E_0 \),

\[
E_0 = \frac{4\hbar^2}{ma_{2D}^2 e^{2\gamma}}
\]

where \( \gamma = 0.57721..... \) is Euler’s constant, along with the coupling constant for \( s \)-wave scattering between two particles,

\[
\frac{1}{g_0} = \frac{m}{2\pi \hbar^2} \left[ \log \left( \frac{r_0}{\pi a_{2D}} \right) - \gamma + \frac{2G}{\pi} \right],
\]
where $G = 0.91596...$ is Catalan’s constant. In the limit $a_{2D} \to \infty$, for a fixed density and range $r_0$, $g_0 \to 0^-$ and $E_0 \to 0^+$. Thus, in the limit $a_{2D}k_f \gg 1$, the trapped 2D gas corresponds to a weakly attractive Fermi gas with a zero energy bound state. In other words, the threshold for the appearance of the first zero-energy bound state corresponds to zero coupling. In this sense, the infinite scattering length limit in 2D corresponds to a noninteracting Fermi gas, which is in stark contrast to 3D, where the $(a_{3D}k_f) \gg 1$ regime requires a non-trivial treatment of interactions.

We are now in a position to apply the simple DFT approach to the trapped 2D Fermi gas in this limit. Arguing as in Ref. 13, the appropriate 2D TF density functional is given by

$$
\varepsilon_{\text{TF}}^{2\text{DHO}}(\rho) = \xi_{2D} \frac{\hbar^2 \pi}{2m} \rho^2 + \frac{1}{2} m \omega^2 r^2 \rho.
$$

Again, gradient corrections generally have to be introduced, but as we will show, they are not required for the 2D case. Minimizing with respect to $\rho$, subject to the constraint that the total number of particles is held fixed, yields a chemical potential

$$
\mu_{\text{TF}}^{2\text{DHO}} = \xi_{2D}^{1/2} N^{1/2} \hbar \omega
$$

with TF density

$$
\rho_{\text{TF}}^{2\text{DHO}} = \frac{m}{\xi_{2D} \hbar^2 \pi} \left( \mu_{\text{TF}}^{2\text{DHO}} - \frac{1}{2} m \omega^2 r^2 \right).
$$

Inserting Eq. (27) into the energy functional and integrating to the TF radius yields a total TF energy

$$
E_{\text{TF}}^{2\text{DHO}} = \frac{2}{3} N^{3/2} \xi_{2D}^{1/2} \hbar \omega.
$$

As we have already discussed, in the infinite scattering length regime the exact energy for two fermions in a harmonic trap is zero energy bound state. Hence the exact energy for a pair of fermions in a 2D harmonic trap is also $E_{\text{ex}} = 2\hbar \omega$. Comparison of the TF energy, viz., (28) with Eq. (21), which is exact in the infinite scattering length limit, gives a value for $\xi_{2D}$ such that

$$
\xi_{2D} = 1 + \frac{1}{4N}
$$

For example, with $N = 2$, we obtain $\xi_{2D} = 9/8 = 1.125$. Thus, while an analysis of finite-$N$ corrections in 2D is not applicable here, because $D(\varepsilon) = \varepsilon/(\hbar \omega)^2$, Eq. (29) illustrates that employing the 2D TF density in Eq. (25) leads to (what turns out to be) an overestimate of $\xi_{2D}$, with $\xi_{2D} \to 1^+$ as $N \to \infty$.  

9
We have, however already established that Eq. (19) yields the exact quantum mechanical energy provided the exact density is used as input. Since the 2D gas in the infinite scattering length limit corresponds to the noninteracting system, if we use the exact N-particle density, instead of the TF density, viz., Eq. (27), and then perform the spatial integration of (19), we immediately obtain $\xi_{2D} = 1$ identically. This is true for any $N = M^2 + 3M + 2$, and thus establishes that the simple 2D TF functional does indeed lead to a universal $\xi_{2D} = 1$, independent of the particle number, when the exact density is used. Recall that in the 3D case, the use of the exact two-particle density, Eq. (6), does lead to an improved value of $\xi_{3D} \approx 0.48$ as compared to that obtained using the TF density. However, as the 3D TF functional is not exact in the unitary regime, neither is the extracted universal factor.

### III. CONNECTION WITH FES IN THE 2D FERMI GAS.

In a relatively recent paper, Srivastava et al. have considered a gas of $N$ fermions interacting via a repulsive two-body zero-range (psuedo) potential within a mean-field approach. Specifically, they showed that within the TF formalism, the zero temperature density functional (with a spin-degeneracy factor of 2 included) is given by

$$\varepsilon^{2D}_{FES} = \alpha \frac{\hbar^2}{2m} \pi \rho^2 + \frac{1}{2} m \omega^2 r^2 \rho,$$

(30)

where $\alpha = (1 + \frac{mM_0}{2\pi\hbar^2})$ is a dimensionless statistical parameter, and $M_0$ is the zeroth moment of the two-body potential. The above equation implies that a gas of fermions in 2D, interacting by a repulsive two-body zero-range potential, appear as particles obeying ideal FES (at least within the TF framework). Notice that the form of Eq. (30) is identical to Eq. (25), but here, $\alpha$ is a statistical parameter unrelated upon being in the unitary or infinite scattering length regimes. In fact, the scaling of the kinetic energy functional here relies entirely on the details of the two-body potential, whereas in the unitary limit, the details of the potential are unimportant. The similarities between Eqs. (25) and (30), and the analogous expressions for $\rho$, $\mu$ and $E$, naturally lead to the question of what connection the Fermi gas in the infinite scattering length limit has with FES. Unfortunately, the answer is quite anticlimactic in 2D because it is only in the limit $\alpha = 1$, that there exists any correspondence. This, unsurprisingly, leads immediately to $\xi_{2D} = 1$ since $\alpha = \xi_{2D} + \frac{mM_0}{2\pi\hbar^2}$. 

10
IV. CONCLUSIONS

In this paper we have reformulated the simple DFT of Papenbrock in the language of fractional exclusion statistics to obtain an estimate of the universal factor $\xi_{3D}$ which is in good agreement with that obtained via QMC simulations, and experiments. We have shown, however, that trying to go beyond this estimate by using a systematic treatment of finite $N$ corrections fails in 3D.

In contrast, when we apply the DFT technique to the 2D trapped Fermi gas in the infinite scattering length limit (and use the exact density in the 2D TF energy density functional), we have shown that the exact quantum mechanical energy is obtained for any number of closed shells. The TF result thus obtained is therefore not limited to $N \gg 1$ as in the 3D case and we are able to establish that the universal factor for the harmonically trapped 2D Fermi gas is exactly $\xi_{2D} = 1$. We therefore conclude that a simple DFT approach, as originally proposed by Papenbrock, is only strictly valid in 2D, where the TF energy functional, Eq. (19), has the remarkable property of being exact, without gradient corrections.

Acknowledgments

We would like to thank the NSERC of Canada, the Marsden Fund of New Zealand, and C.N.R.S. of France for fiscal support. For their warm hospitality, DAWH would like to thank the cold atoms group at ENS where this work was concluded and Yvan Castin in particular for useful discussions. MN would like to acknowledge an NSERC USRA award for additional financial support. We are also indebted to R. K. Bhaduri for suggesting the investigation of finite-$N$ corrections presented in Sec. II(A) of this paper.

[1] B. DeMarco and D. S. Jin, Science 285, 1703 (1999); M. J. Holland, B. DeMarco, and D. S. Jin, Phys. Rev. A 61, 053610 (2000).
[2] K. M. O’Hara et al., Science 298, 2179 (2002).
[3] J. E. Thomas, J. Kinast and A. Turlapov, Phys. Rev. Lett. 95, 120402 (2005).
[4] C. Chin et al., Science 305, 1128 (2004).
[5] Q. J. Chen et al., Phys. Rev. A 74, 011601(R) (2006).
[6] C. A. Regal and D. S. Jin, Phys. Rev. Lett. 90, 230404 (2003).

[7] C. A. Regal et al., Phys. Rev. Lett. 95, 250404 (2005).

[8] In 3D the unitary limit is defined as the point of maximal scattering cross-section. Setting $\hbar = 2m = 1$ the 3D scattering cross-section takes the form (for $s$-wave scattering)

$$\sigma_{3D} = \frac{4\pi}{(1/a_{3D})^2 + k^2},$$

where $a_{3D}$ is the 3D scattering length. This clearly takes a maximal value for all wavevectors $k$ as $a_{3D} \to \pm \infty$.

[9] T.-L. Ho. Phys. Rev. Lett. 92, 090402 (2004).

[10] H. Heiselberg, Phys. Rev. A 63, 043606 (2001).

[11] G. A. Baker, Phys. Rev C 60, 054311 (1999).

[12] J. R. Engelbrecht, M. Randeria and C. A. R. Sá de Melo, Phys. Rev. B 55, 15153 (1997).

[13] T. Papenbrock, Phys. Rev. A 72, 041603(R) (2005).

[14] J.-S Chen, preprint nucl-th/0602065 (2006).

[15] J. Carlson, S.-Y. Chang, V. R. Pandharipande, and K. E. Schmidt, Phys. Rev. Lett. 91, 050401 (2003).

[16] E. Burovski, N. Prokof’ev, B. Svistunov, and M. Troyer, Phys. Rev. Lett. 96, 160402 (2006).

[17] A. Bulgac, J. E Drut, and P. Magierski, Phys. Rev. Lett.96, 090404 (2006).

[18] In actual experiments, the gas is also trapped by an optical potential $V(r)$, and is thus inhomogeneous. Contact with experiments in this situation is usually facilitated by the use of the local-density approximation.

[19] M. W. Zwierlein et al., Phys. Rev. Lett. 92, 120403 (1994); ibid 94, 180401 (2005).

[20] J. Kinast, A. Turlapov, J. E. Thomas, Q. Chen. J. Stajic, and K. Levin, Science 307, 1296 (2005).

[21] G. B. Partridge, W. Li, R. I. Kamar, Y. Liao, and R. G. Hulet, Science 311, 503 (2006).

[22] Z. Nussinov and S. Nussinov, preprint cond-mat/0410597 (2004).

[23] Y. Nishida, and D. T. Son, preprint cond-mat/0607835 (2006).

[24] G. Tonini, F. Werner and Y. Castin, Eur. Phys. J. D 39, 283 (2006).

[25] Th. Busch, B. G. Englert, K. Rzazewski and M. Wilkens, Found. Phys. 28, 549 (1998).

[26] R. K. Bhaduri, M. V. N. Murthy and M. K. Srivastava, preprint cond-mat/0606117 (2006).

[27] F. D. M. Haldane, Phys. Rev. Lett. 67, 937 (1991).
A similar calculation can be performed for the one dimensional trapped Fermi gas in the infinite scattering length limit. In this case the Thomas-Fermi energy is given by

$$E_{\text{TF}}^{1\text{DHO}} = \frac{1}{4} N^2 \xi_{1\text{D}}^{1/2} \hbar \omega$$

and comparing to the exact result for $N = 2$ gives a value for the analogous “unitary” factor, $\xi_{1\text{D}}$, of unity. This happens to be also the exact result in 1D, but this is fortuitous because it is known that the TF functional in 1D is not exact, similar to the situation found in 3D. See, e.g., Ref. 30.