From particle systems to the Landau equation: a consistency result

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Consider a mechanical particle system. \( N \) (large) identical particles of unitary mass. Positions and velocities: \( q_1 \ldots q_N, q_i \in \mathbb{R}^3 \), \( v_1 \ldots v_N, v_i \in \mathbb{R}^3 \).
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Statistical description: $W_N(Z_N)$ symmetric probability measure $Z_N = (q_1 \ldots q_N; v_1 \ldots v_N)$.
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Statistical description: \( W_N(Z_N) \) symmetric probability measure \( Z_N = (q_1 \ldots q_N; v_1 \ldots v_N) \).

Time evolution: \( W_N(Z_N; t) = W_N(\Phi^{-\tau}(Z_N)), \Phi^\tau(Z_N) \) is the flow with initial datum \( Z_N \).
Instead of looking at $Z_N$, construct the random measure

$$\mu_N(dz; \tau) = \frac{1}{N} \sum_j \delta(z - z_j(\tau))dz$$

empirical distribution. $\{z_i(\tau)\}_{i=1}^N = \Phi^\tau(Z_N)$. 
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Look for an evolution equation for $\mu_N$ or $E(\mu_N) = f_1(z; \tau)$ to have a one-particle description.
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Look for an evolution equation for $\mu_N$ or $\mathbb{E}(\mu_N) = f_1(z; \tau)$ to have a one-particle description. Dynamics creates correlations. Closure problem. Suitable scaling limits could recover the statistical independence, provided it is assured at time zero: $W_N = f_0^\otimes N$
General strategy of the Kinetic Theory

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A full derivation proof is a challenging open problem, even for short times.
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\[(\partial_t + \nu \cdot \nabla_x)f = Q_L(f, f)\]
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\[ Q_L(f, f) = \int d\nu_1 \nabla_\nu a(\nu - \nu_1)(\nabla_\nu - \nabla_{\nu_1}) f(\nu)f(\nu_1), \]
The Landau equations

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\]

\[ a = a(v - v_1) \] is the matrix

\[
a_{i,j}(V) = \frac{B}{|V|} (\delta_{i,j} - \hat{V}_i \hat{V}_j), \quad a(V) = \frac{B}{|V|} P^\perp_V.
\]

\[
\hat{V} = \frac{V}{|V|}
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From the mathematical side very little is known about the Landau equation even for the homogeneous case.
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$$\alpha = 0, 1, 2$$ and the Entropy production is given by the following expression:

$$- (\log f, Q_L(f, f)) = \frac{1}{2} \int \frac{1}{f f_1} |v - v_1|^2 P_{\perp} (\nabla v - \nabla v_1) ff_1.$$
The Landau equations

From the mathematical side very little is known about the Landau equation even for the homogeneous case. The main difficulty is due to the presence of the diverging factor $\frac{1}{|V|}$. Same properties as for the Boltzmann equation.

\[(v^\alpha, Q_L(f, f)) = 0\]

for $\alpha = 0, 1, 2$ and the Entropy production is given by the following expression $(f = f(v), f_1 = f(v_1))$

\[-(\log f, Q_L(f, f)) = \frac{1}{2} \int dv \int dv_1 \frac{1}{ff_1} \frac{1}{|v - v_1|} |P_{v - v_1}(\nabla v - \nabla v_1)ff_1|^2.\]
The weak coupling limit

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Here $F = -\nabla \phi$, $\phi$ the smooth, two-body, spherically symmetric interaction potential and $\tau$ the time. In this regime $N$ is very large and the interaction strength quite moderate. $\varepsilon > 0$ a small parameter = the ratio between the macro and microscales. $N = O(\varepsilon^{-3})$, the density is $O(1)$.

Rescale $x = \varepsilon q$, $t = \varepsilon \tau$, $\phi \rightarrow \sqrt{\varepsilon} \phi$.

\[
\frac{d}{dt} x_i = v_i, \quad \frac{d}{dt} v_i = \frac{1}{\sqrt{\varepsilon}} \sum_{\substack{j=1\ldots N: \ j \neq i}} F\left(\frac{x_i - x_j}{\varepsilon}\right).
\]
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The total momentum variation for unit time is $O\left(\frac{1}{\sqrt{\varepsilon}}\right)$.
But zero in the average.
The variance $= \frac{1}{\varepsilon} O(\sqrt{\varepsilon})^2 = O(1)$. 
The weak coupling limit

\[ X_N = x_1 \ldots x_N \quad V_N = v_1 \ldots v_N. \]

Liouville equation

\[
(\partial_t + V_N \cdot \nabla_N) W^N(X_N, V_N) = \frac{1}{\sqrt{\epsilon}} (T^\epsilon_N W^N)(X_N, V_N)
\]

where \( V_N \cdot \nabla_N = \sum_{i=1}^{N} v_i \cdot \nabla x_i \)

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(T^\epsilon_N W^N)(X_N, V_N) = \sum_{0<k<\ell \leq N} (T^\epsilon_{k,\ell} W^N)(X_N, V_N),
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\[ (T_N^\varepsilon W^N)(X_N, V_N) = \sum_{0<k<\ell\leq N} (T_{k,\ell}^\varepsilon W^N)(X_N, V_N), \]

\[ T_{k,\ell}^\varepsilon W^N = \nabla \phi\left(\frac{x_k - x_\ell}{\varepsilon}\right) \cdot (\nabla v_k - \nabla v_\ell) W^N. \]
The weak coupling limit

BBKGY hierarchy of equations for the marginals $f_j^N$ (for $1 \leq j \leq N$):

$$(\partial_t + \sum_{k=1}^j v_k \cdot \nabla_k) f_j^N = \frac{1}{\sqrt{\varepsilon}} T_j^\varepsilon f_j^N + \frac{N - j}{\sqrt{\varepsilon}} C_{j+1}^\varepsilon f_{j+1}^N.$$
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The operator $C_{j+1}^{\varepsilon}$ is defined as:

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$$C_{k,j+1}^\varepsilon f_{j+1}(x_1 \ldots x_j; v_1 \ldots v_j) =$$

$$- \int dx_{j+1} \int dv_{j+1} F \left( \frac{x_k - x_{\ell}}{\varepsilon} \right) \cdot \nabla_{v_k} f_{j+1}(x_1, x_2, \ldots, x_{j+1}; v_1, \ldots, v_{j+1}).$$
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$$- \int dx_{j+1} \int dv_{j+1} F\left(\frac{x_k - x_\ell}{\varepsilon}\right) \cdot \nabla v_k f_{j+1}(x_1, x_2, \ldots, x_{j+1}; v_1, \ldots, v_{j+1}).$$

The initial value $\{f_j^0\}_{j=1}^N$ factorizes

$$f_j^0 = f_0 \otimes^j,$$ for some $f_0$. 

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The weak coupling limit

Duhamel formula:

\[(S(t)f_j)(X_j, V_j) = f_j(X_j - V_j t, V_j),\]
The weak coupling limit

Duhamel formula:

\[(S(t)f_j)(X_j, V_j) = f_j(X_j - V_j t, V_j),\]

\[f_j^N(t) = S(t)f_j^0 + \frac{N - j}{\sqrt{\varepsilon}} \int_0^t \frac{S(t - t_1)}{\sqrt{\varepsilon}} C_{j+1}^\varepsilon f_{j+1}^N(t_1) dt_1 + \]

\[\frac{1}{\sqrt{\varepsilon}} \int_0^t \frac{S(t - t_1)}{\sqrt{\varepsilon}} T_j^\varepsilon f_j^N(t_1) dt_1.\]

Assuming that the time evolved \(j\)-particle distributions \(f_j^N(t)\) are smooth

\[C_{j+1}^\varepsilon f_{j+1}^N(X_j; V_j; t_1) =\]

\[- \varepsilon^3 \sum_{k=1}^j \int dr \int d\nu_{j+1} F(r) \cdot \nabla_{\nu_k} f_{j+1}(X_j, x_k - \varepsilon r; V_j, \nu_{j+1}, t_1) = O(\varepsilon^4)\]

because \(\int dr F(r) = 0\). Also the third term is vanishing.
The weak coupling limit

Hence \( f_j^N(t) \) cannot be smooth!

We conjecture

\[
f_j^N = g_j^N + \gamma_j^N
\]

where \( g_j^N \) is the main part of \( f_j^N \) and is smooth, while \( \gamma_j^N \) is small, but strongly oscillating.

\[
(\partial_t + \sum_{k=1}^{j} v_k \cdot \nabla x_k)g_j^N = \frac{N - j}{\sqrt{\varepsilon}} C_{j+1}^\varepsilon g_{j+1}^N + \frac{N - j}{\sqrt{\varepsilon}} C_{j+1}^\varepsilon \gamma_{j+1}^N
\]

\[
(\partial_t + \sum_{k=1}^{j} v_k \cdot \nabla x_k)\gamma_j^N = \frac{1}{\sqrt{\varepsilon}} T_j^\varepsilon \gamma_j^N + \frac{1}{\sqrt{\varepsilon}} T_j^\varepsilon g_j^N,
\]

Initial data

\[
g_j^N(X_j, V_j) = f_j^0(X_j, V_j), \quad \gamma_j^N(X_j, V_j) = 0.
\]

Note that \( \gamma_1^N = 0 \) since \( T_1^\varepsilon = 0 \).
The weak coupling limit

The remarkable fact of this decomposition is that $\gamma$ can be eliminated. Let $(X_j(t), V_j(t)) = (\{x_1(t) \ldots x_j(t), v_1(t) \ldots v_j(t)\})$ be the solution of the $j$-particle flow (in macro variables)

\[
\frac{d}{dt} x_i = v_i \\
\frac{d}{dt} v_i = -\frac{1}{\sqrt{\varepsilon}} \sum_{k=1 \ldots j: k \neq i} \nabla \phi \left( \frac{x_i - x_k}{\varepsilon} \right).
\]

Initial datum $(X_j, V_j) = (\{x_1 \ldots x_j, v_1 \ldots v_j\})$. $U_j(t)$ is the operator solving the Liouville equation

\[
(\partial_t + V_j \cdot \nabla_j) h(X_j, V_j; t) = \frac{1}{\sqrt{\varepsilon}} (T_j^{\varepsilon} h)(X_N, V_N; t)
\]

namely

\[
h(X_j, V_j, t) = U_j h(X_j, V_j) = h(X_j(-t), V_j(-t)).
\]
Then
\[ \gamma_j^N(t) = -\frac{1}{\sqrt{\varepsilon}} \int_0^t ds U(s) T_j^\varepsilon g_j^N(t - s). \]

\[ \gamma_j^N(X_j, V_j, t) = -\frac{1}{\sqrt{\varepsilon}} \int_0^t ds \sum_{1 \leq i < k \leq j} \nabla \phi\left( \frac{x_i(-s) - x_k(-s)}{\varepsilon} \right). \]

\[ (\nabla v_i - \nabla v_k) g_j^N(X_j(-s), V_j(-s); t - s). \]

Finally we arrive to a closed hierarchy for \( g^N \):

\[ (\partial_t + \sum_{k=1}^j v_k \cdot \nabla x_k) g_j^N(X_j, V_j; t) = \frac{N - j}{\sqrt{\varepsilon}} C_{j+1}^\varepsilon g_{j+1}^N(X_j, V_j; t) + \]

\[ \frac{N - j}{\varepsilon} \sum_{k=1}^j \sum_{i, r=1}^{j+1} \int_0^t ds \int dv_{j+1} \int dx_{j+1} \text{div}_v F\left( \frac{x_k - x_{j+1}}{\varepsilon} \right) F\left( \frac{x_i(-s) - x_r(-s)}{\varepsilon} \right) \]

\[ (\nabla v_i - \nabla v_r) g_{j+1}^N(X_{j+1}(-s), V_{j+1}(-s); t - s). \]
The weak coupling limit

We now present a formal derivation of the Landau eq.n (assuming $g_2^N$ smooth).

$$(\partial_t + v_1 \cdot \nabla_{x_1}) g_1^N(t) = \frac{N - 1}{\sqrt{\varepsilon}} C_2^\varepsilon g_2^N(t)$$

$$+ \frac{N - 1}{\varepsilon} C_2^\varepsilon \int_0^t ds U_2(s) T_2 g_2^N(t - s).$$

Let $u \in D$ be a test function.

$$\frac{N - 1}{\sqrt{\varepsilon}} (u, C_2^\varepsilon g_2^N(t)) = O(\sqrt{\varepsilon}).$$
The weak coupling limit

Last term:

\[- \frac{N - 1}{\varepsilon} \int dx_1 \int dx_2 \int dv_1 \int dv_2 \int_0^t ds \quad \nabla v_1 u(x_1, v_1)\]

\[F\left(\frac{x_1 - x_2}{\varepsilon}\right)F\left(\frac{x_1(-s) - x_2(-s)}{\varepsilon}\right) \cdot (\nabla v_1 - \nabla v_2) g_2^N(X_2(-s), V_2(-s); t-s) \approx\]

\[- \int dx_1 \int dr \int dv_1 \int dv_2 \int_0^\infty ds \quad \nabla v_1 u(x_1, v_1)\]

\[F(r)F\left(\frac{x_1(-\varepsilon s) - x_2(-\varepsilon s)}{\varepsilon}\right) \cdot (\nabla v_1 - \nabla v_2) g_2^N(x_1, x_2, v_1, v_2; t).\]

\[(r = \frac{x_1-x_2}{\varepsilon}) \text{ and } s \to \frac{s}{\varepsilon}.\]
The weak coupling limit

\[ w = v_1 - v_2 \] the relative velocity:

\[ \frac{x_1(-\varepsilon s) - x_2(-\varepsilon s)}{\varepsilon} = r + ws + \frac{1}{\varepsilon} \int_0^{-\varepsilon s} d\tau (v_1(\tau) - v_1) - (v_2(\tau) - v_2). \]

But

\[ v_1(\tau) - v_1 = \frac{1}{\sqrt{\varepsilon}} \int_0^\tau ds F\left(\frac{x_1(s) - x_2(s)}{\varepsilon}\right) = O(\sqrt{\varepsilon}). \]

The time spent when the two particles are at distance less that \( \varepsilon \) is \( O(\varepsilon) \), (if the relative velocity \( w \) not too small). Thus:

\[ \approx - \int dx_1 \int dr \int dv_1 \int dv_2 \int_0^\infty ds \ \nabla v_1 u(x_1, v_1) F(r) F(r + ws) \]

\[ (\nabla v_1 - \nabla v_2) g_2^N(x_1, x_1, v_1, v_2; t) \]

\[ \approx (u, Q_L(g_1^N, g_1^N)). \]

Invoking propagation of chaos.
The weak coupling limit

Actually it can be proven that

$$\int dr \int_0^\infty ds F(r) F(r - ws) = \frac{1}{2} \int dr \int_{-\infty}^\infty ds F(r) F(r - ws) = a(w)$$
The weak coupling limit

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$$a(w)_{\alpha,\beta} = \frac{B}{|w|} (\delta_{\alpha,\beta} - \frac{w_\alpha w_\beta}{|w|^2})$$

and

$$B = C \int_0^\infty d\rho \rho^3 \phi^2(\rho).$$
The weak coupling limit

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$$B = C \int_0^\infty d\rho \rho^3 \hat{\phi}^2(\rho).$$
Consider the first order (in time) approximation $\tilde{g}^N$ of $g^N$:

$$(\partial_t + \sum_{k=1}^{j} v_k \cdot \nabla x_k)\tilde{g}_j^N(X_j, V_j; t) = \frac{N - j}{\sqrt{\varepsilon}} C_{j+1}^\varepsilon S(t) f_{j+1}^0(X_j, V_j) +$$

$$\sum_{k=1}^{j} \sum_{i,r=1}^{j+1} \int_0^t ds \int dv_{j+1} \int dx_{j+1} \text{div}_v F\left(\frac{x_k - x_{j+1}}{\varepsilon}\right) F\left(\frac{x_i(-s) - x_r(-s)}{\varepsilon}\right)$$

$$\left(\nabla_{v_i} - \nabla_{v_r}\right) S(t - s) f_{j+1}^0(X_{j+1}(-s), V_{j+1}(-s)).$$
Suppose $f_0 \in C^3_0(\mathbb{R}^3 \times \mathbb{R}^3)$ be the initial probability density satisfying:

$$|D^r f_0(x, v)| \leq Ce^{-b|v|^2} \quad \text{for} \quad r = 0, 1, 2 \quad (1)$$

where $D^r$ is any derivative of order $r$ and $b > 0$. $\phi \in C^2(\mathbb{R}^3)$, $\phi \geq 0$ and $\phi(x) = 0$ if $|x| > 1$. Assume factorization at time zero, then

$$\lim_{\varepsilon \to 0} \tilde{g}_1^N(t) = S(t)f_0 + \int_0^t d\tau S(t - \tau)Q_L(S(\tau)f_0, S(\tau)f_0)$$

where $N\varepsilon^3 = 1$ and the above limit is considered in $\mathcal{D}'$. 

Assume factorization at time zero,
The weak coupling limit

Propagation of chaos

**Theorem**

*Under the same hypotheses*

\[
\lim_{\varepsilon \to 0} \tilde{g}_j^N(t, x_1, v_1, \ldots, x_j, v_j) = \prod_{i=1}^{j} S(t)f_0(x_i, v_i)
\]

\[
+ \sum_{i=1}^{j} \prod_{k=1, k \neq i}^{j} S(t)f_0(x_k, v_k) \int_{0}^{t} d\tau S(t - \tau)Q_L(S(\tau)f_0, S(\tau)f_0)(x_i, v_i)
\]

*in \( \mathcal{D}' \).*