A SIMPLE CHARACTERIZATION OF CHAOS FOR WEIGHTED COMPOSITION $C_0$-SEMIGROUPS ON LEBESGUE AND SOBOLEV SPACES

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ABSTRACT. We give a simple characterization of chaos for weighted composition $C_0$-semigroups on $L^p_\rho(\Omega)$ for an open interval $\Omega \subseteq \mathbb{R}$. Moreover, we characterize chaos for these classes of $C_0$-semigroups on the closed subspace $W^{1,p}_\rho(\Omega)$ of the Sobolev space $W^{1,p}(\Omega)$ for a bounded interval $\Omega \subseteq \mathbb{R}$. These characterizations simplify the characterization of chaos obtained by Aroza, Kalmes, and Mangino (2014) for these classes of $C_0$-semigroups.

1. INTRODUCTION

The purpose of this article is to give a simple characterization of chaos for certain weighted composition $C_0$-semigroups on Lebesgue spaces and Sobolev spaces over open intervals. Recall that a $C_0$-semigroup $T$ on a separable Banach space $X$ is called chaotic if $T$ is hypercyclic, i.e. there is $x \in X$ such that $\{T(t)x; t \geq 0\}$ is dense in $X$, and if the set of periodic points, i.e. $\{x \in X; \exists t > 0 : T(t)x = x\}$, is dense in $X$.

The study of chaotic $C_0$-semigroups has attracted the attention of many researchers. We refer the reader to Chapter 7 of the monograph by Grosse-Erdmann and Peris [9] and the references therein. Some recent papers on the topic are [1, 4, 5, 8, 15].

For $\Omega \subseteq \mathbb{R}$ open and a Borel measure $\mu$ on $\Omega$ admitting a strictly positive Lebesgue density $\rho$ we consider $C_0$-semigroups $T$ on $L^p(\Omega, \mu)$, $1 \leq p < \infty$, of the form

$$T(t)f(x) = h_t(x)f(\varphi(t, x)),$$

where $\varphi$ is the solution semiflow of an ordinary differential equation

$$\dot{x} = F(x)$$

in $\Omega$ and

$$h_t(x) = \exp \left( \int_0^t h(\varphi(s, x))ds \right),$$

with $h \in C(\Omega)$. Such $C_0$-semigroups appear in a natural way when dealing with initial value problems for linear first order partial differential operators. While a characterization of chaos for such $C_0$-semigroups was obtained for open $\Omega \subseteq \mathbb{R}^d$ for arbitrary dimension $d$ in [10], evaluation of these conditions in concrete examples is sometimes rather involved. In contrast to general dimension, the case...
d = 1 allows for a significantly simplified characterization (see [3]). However, this characterization of chaos still depends on the knowledge of the solution semiflow ϕ which might be difficult to determine in concrete examples.

In section 2 we give, under mild additional assumptions on F and h, a characterization of chaos which only depends on the ingredients F, h, and ρ, without referring to the semiflow ϕ.

In section 3 we use this result to obtain a similarly simple characterization of chaos for the above kind of $C_0$-semigroups acting on the closed subspace

$$W_1^{1,p}[a,b] = \{ f \in W_1^{1,p}[a,b]; f(a) = 0 \}$$

of the Sobolev spaces $W_1^{1,p}[a,b]$, where $(a, b) \subseteq \mathbb{R}$ is a bounded interval. It was shown in [3] that such $C_0$-semigroups cannot be hypercyclic, a fortiori chaotic, on the whole Sobolev space $W_1^{1,p}(a,b)$.

In order to illustrate our results, several examples are considered.

2. Chaotic weighted composition $C_0$-semigroups on Lebesgue spaces

Let $\Omega \subseteq \mathbb{R}$ be open and let $F : \Omega \to \mathbb{R}$ be a $C^1$-function. Hence, for every $x_0 \in \Omega$ there is a unique solution $\varphi(\cdot, x_0)$ of the initial value problem

$$\dot{x} = F(x), \; x(0) = x_0.$$

Denoting its maximal domain of definition by $J(x_0)$ it is well known that $J(x_0)$ is an open interval containing 0. We make the general assumption that $\Omega$ is forward invariant under $F$, i.e. $[0, \infty) \subseteq J(x_0)$ for every $x_0 \in \Omega$, that is $\varphi : [0, \infty) \to \Omega$. This is true, for example, if $\Omega = (a, b)$ is a bounded interval and if $F$ can be extended to a $C^1$-function defined on a neighborhood of $[a, b]$ such that $F(a) \geq 0$ and $F(b) \leq 0$ (cf. [2 Corollary 16.10]).

From the uniqueness of the solution it follows that $\varphi(t, \cdot)$ is injective for every $t \geq 0$ and $\varphi(t+s, x) = \varphi(t, \varphi(s, x))$ for all $x \in \Omega$ and $s, t \in J(x)$ with $s+t \in J(x)$. Moreover, for every $t \geq 0$ the set $\varphi(t, \Omega)$ is open and for $x \in \varphi(t, \Omega)$ we have $[-t, \infty) \subseteq J(x)$ as well as $\varphi(-s, x) = \varphi(s, \cdot)^{-1}(x)$ for all $s \in [0, t]$. Since $F$ is a $C^1$-function it is well known that the same is true for $\varphi(t, \cdot)$ on $\Omega$ and $\varphi(-t, \cdot)$ on $\varphi(t, \Omega)$ for every $t \geq 0$.

Moreover, let $h \in C(\Omega)$ and define for $t \geq 0$,

$$h_t : \Omega \to \mathbb{C}, \; h_t(x) = \exp(\int_0^t h(\varphi(s, x))ds).$$

For $1 \leq p < \infty$ and a measurable function $\rho : \Omega \to (0, \infty)$ let $L_p^p(\Omega)$ be as usual the Lebesgue space of $p$-integrable functions with respect to the Borel measure $\rho \lambda$, where $\lambda$ denotes Lebesgue measure. If $\Omega$ is forward invariant under $F$ the operators

$$T(t) : L_p^p(\Omega) \to L_p^p(\Omega), (T(t)f)(x) := h_t(x)f(\varphi(t, x)) \quad (t \geq 0)$$

are well-defined continuous linear operators defining a $C_0$-semigroup $T_{F,h}$ on $L_p^p(\Omega)$ if $\rho$ is $p$-admissible for $F$ and $h$, i.e. if there are constants $M \geq 1$, $\omega \in \mathbb{R}$ with

$$\forall t \geq 0, x \in \Omega : |h_t(x)|^p \rho(x) \leq M e^{\omega t} \rho(\varphi(t, x)) \exp\left(\int_0^t F'(\varphi(s, x))ds\right),$$

(see [3]). Because $|h_t(x)|^p = \exp(p \int_0^t \text{Re} h(\varphi(s, x))ds)$ it follows that $\rho = 1$ is $p$-admissible for any $p$ if $\text{Re} h$ is bounded above and $F'$ is bounded below, i.e. in this case the above operators define a $C_0$-semigroup $T_{F,h}$ on the standard Lebesgue

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spaces $L^p(\Omega)$. Under mild additional assumptions on $F$ and $h$ the generator of this $C_0$-semigroup is given by the first order differential operator $Af = Ff' + hf$ on a suitable subspace of $L^p(\Omega)$ (see [3, Theorem 15]).

In [3, Theorem 6 and Proposition 9] the authors characterize when the $C_0$-semigroup $T_{F,h}$ is chaotic on $L^p_\rho(\Omega)$. However, this characterization depends on a more or less explicit knowledge of the semiflow $\varphi$.

Our aim is to prove the following characterization of chaos for $T_{F,h}$ on $L^p_\rho(\Omega)$ valid under mild additional assumptions on $F$ and $h$ and which is given solely in terms of $F$, $h$, and $\rho$. Throughout this article, we use the following common abbreviation $\{F = 0\} := \{x \in \Omega; F(x) = 0\}$.

**Theorem 2.1.** For $1 \leq p < \infty$ let $\Omega \subset \mathbb{R}$ be an open interval which is forward invariant under $F \in C^1(\Omega)$, and let $h \in C(\Omega)$ be such that $F'$ and Re$h$ are bounded and

a) There is $\gamma \in \mathbb{R}$ such that $h(x) = \gamma$ for all $x \in \{F = 0\}$.

b) With $\alpha := \inf \Omega$ and $\omega := \sup \Omega$ the function

$$\Omega \to \mathbb{C}, y \mapsto \frac{\text{Im} h(y)}{F(y)}$$

belongs to $L^1((\alpha, \beta))$ for all $\beta \in \Omega$ or to $L^1((\beta, \omega))$ for all $\beta \in \Omega$.

Then for every $\rho$ which is $p$-admissible for $F$ and $h$ the following are equivalent.

i) $T_{F,h}$ is chaotic in $L^p_\rho(\Omega)$.

ii) $\lambda(\{F = 0\}) = 0$ and for every connected component $C$ of $\Omega\setminus\{F = 0\}$

$$\int_C \exp(-p \int_x^w \frac{\text{Re} h(y)}{F(y)} dy) \rho(w) d\lambda(w) < \infty$$

for some/all $x \in C$.

In order to prove Theorem 2.1 we define for $x \in \Omega$, $p \geq 1$, and $t \geq 0$,

$$\rho_{t,p}(x) = \chi_{\varphi(t,\Omega)}(x) |h_t(\varphi(-t,x))|^p \exp\left(\int_0^- t F'(\varphi(s,x)) ds\right) \rho(\varphi(-t,x))$$

$$= \chi_{\varphi(t,\Omega)}(x) \exp(p \int_0^t \text{Re} h(\varphi(s,\varphi(-t,x))) ds)$$

$$\cdot \exp\left(\int_0^- t F'(\varphi(s,x)) ds\right) \rho(\varphi(-t,x)),$$ 

as well as

$$\rho_{-t,p}(x) = |h_t(x)|^{-p} \exp\left(\int_0^t F'(\varphi(s,x)) ds\right) \rho(\varphi(t,x))$$

$$= \exp(-p \int_0^t \text{Re} h(\varphi(s,x)) ds) \exp\left(\int_0^t F'(\varphi(s,x)) ds\right) \rho(\varphi(t,x)).$$
Then $\rho_{0,p} = \rho$, $\rho_{t,p} \geq 0$ for every $t \in \mathbb{R}$, and for fixed $x \in \Omega$ the mapping $t \mapsto \rho_{t,p}(x)$ is Lebesgue measurable. Moreover, it follows that

$$
\rho_{-(t+s),p}(x) = \exp(-p \int_0^{t+s} \Re h(\varphi(r,x)) dr) \cdot \exp\left(\int_0^{t+s} F'(\varphi(r,x)) dr \rho(\varphi(t,\varphi(s,x)))\right)
$$

$$
= \exp\left(\int_0^s F'(\varphi(r,x)) - p \Re h(\varphi(r,x)) dr\right) \rho(-t,p)(\varphi(s,x))
$$

(2.1)

and analogously

$$
\rho_{(t+s),p}(x) = \chi_{\bar{\varphi}(t+s,\Omega)}(x) \exp(p \int_0^0 \Re h(\varphi(r,x)) - \frac{1}{p} F'(\varphi(r,x)) dr) \cdot \rho(-s,p)(\varphi(-t,x))
$$

(2.2)

The following lemma will be used in the proof of the first auxiliary result. We cite it for the reader’s convenience. For a proof see [11] Lemma 7.

**Lemma 2.2.** Let $\Omega \subseteq \mathbb{R}$ be open, let $F \in C^1(\Omega)$ be such that $\Omega$ is forward invariant under $F$, and let $h \in C(\Omega)$ be real valued. Moreover, for fixed $1 \leq p < \infty$ let $\rho$ be $p$-admissible for $F$ and $h$. For $[a,b] \subset \Omega \{F = 0\}$ set $\alpha := a$ and $\beta := b$ if $F|_{[a,b]} > 0$, respectively $\alpha := b$ and $\beta := a$ if $F|_{[a,b]} < 0$.

Then there is a constant $C > 0$ such that

$$
\forall x \in [a,b] : \frac{1}{C} \leq \rho(x) \leq C,
$$

as well as

$$
\forall t \in \mathbb{R}, x \in [a,b] : \frac{1}{C} \rho_{t,p}(\alpha) \leq \rho_{t,p}(x) \leq C \rho_{t,p}(\beta).
$$

The next lemma generalizes parts of [12] Lemma 3.2] where the special case of the translation semigroup is considered, i.e. $F = 1$ and $h = 0$.

**Lemma 2.3.** Let $\Omega \subseteq \mathbb{R}$ be open and forward invariant under $F \in C^1(\Omega)$, let $h \in C(\Omega)$ be such that $F'$ and $\Re h$ are bounded. Moreover, let $\rho$ be $p$-admissible for $F$ and $h$, $1 \leq p < \infty$. Then the following are equivalent.

i) For all $x \in \Omega \{F = 0\}$ there is $t_0 > 0$ such that $\sum_{k \in \mathbb{Z}} \rho_{k_0,p}(x) < \infty$.

ii) For all $x \in \Omega \{F = 0\}$ : $\int_{a}^{b} \rho_{x,p}(x) d\lambda(t) < \infty$.

iii) For all $x \in \Omega \{F = 0\}$ and $t_0 > 0$ : $\sum_{k \in \mathbb{Z}} \rho_{k_0,p}(x) < \infty$.

**Proof.** In order to show that i) implies ii) fix $x \in \Omega \{F = 0\}$ and choose $t_0 > 0$ according to i) for $x$. We distinguish two cases. If $x$ belongs to $\bigcap_{t \geq 0} \varphi(t,\Omega)$, it
follows by equation \([2.2]\) and the boundedness of \(\text{Re} \ h\) and \(F'\) that

\[
\int_{[0, \infty)} \rho_{t,p}(x) d\lambda(t) = \sum_{k=0}^{\infty} \int_{[0, t_0]} \rho_{kt_0+s,p}(x) d\lambda(s)
\]

\[
= \sum_{k=0}^{\infty} \int_{[0, t_0]} \chi_{\varphi(s,\Omega)}(x) \exp(p \int_{-s}^{0} \text{Re} \ h(\varphi(r, x)) - \frac{1}{p} F'(\varphi(r, x)) dr) \rho_{kt_0,p}(\varphi(-s, x)) d\lambda(s)
\]

\[
\leq C \sum_{k=0}^{\infty} \int_{[0, t_0]} \chi_{\varphi(s,\Omega)}(x) \rho_{kt_0,p}(\varphi(-s, x)) d\lambda(s)
\]

\[
= C \sum_{k=0}^{\infty} \int_{[0, t_0]} \rho_{kt_0,p}(\varphi(-s, x)) d\lambda(s)
\]

\[
\leq \begin{cases} 
\hat{C} \sum_{k=0}^{\infty} \rho_{kt_0,p}(x), & F(x) > 0, \\
\hat{C} \sum_{k=0}^{\infty} \rho_{kt_0,p}(\varphi(-t_0, x)), & F(x) < 0,
\end{cases}
\]

where \(C\) and \(\hat{C}\) depend on \(t_0\) and where in the last step we used Lemma \([2.2]\) for \(F\) and \(\text{Re} \ h\). Since by equation \([2.2]\) together with the boundedness of \(F'\) and \(\text{Re} \ h\) we also have with suitable \(D > 0\) that for all \(k \geq 0\)

\[
\rho_{kt_0,p}(\varphi(-t_0, x)) \leq D \rho_{(k+1)t_0,p}(x).
\]

The above shows the existence of \(\hat{C} > 0\) such that

\[
\int_{[0, \infty)} \rho_{t,p}(x) d\lambda(t) \leq \hat{C} \sum_{k=0}^{\infty} \rho_{kt_0,p}(x).
\]

If \(x\) does not belong to \(\bigcap_{t \geq 0} \varphi(t, \Omega)\), then \(\int_{[0, \infty)} \rho_{t,p}(x) d\lambda(t) = \int_{[0, r]} \rho_{t,p}(x) d\lambda(t)\) for some \(r > 0\). Combining Lemma \([2.2]\) for \(F\) and \(\text{Re} \ h\) with equation \([2.2]\), the boundedness of \(F'\) and \(\text{Re} \ h\) gives for suitable \(C > 0\)

\[
\int_{[0, \infty)} \rho_{t,p}(x) d\lambda(t)
\]

\[
= \int_{[0, r]} \chi_{\varphi(t,\Omega)}(x) \exp(p \int_{-s}^{0} \text{Re} \ h(\varphi(s, x)) - \frac{1}{p} F'(\varphi(s, x)) ds) \rho_{0,p}(\varphi(-t, x)) d\lambda(t)
\]

\[
\leq C \int_{[0, r]} \chi_{\varphi(t,\Omega)}(x) \rho_{0,p}(\varphi(-t, x)) d\lambda(t) < \infty.
\]

Thus, if i) holds, then \(\int_{[0, \infty)} \rho_{t,p}(x) d\lambda(t) < \infty\) for all \(x \in \Omega \setminus \{F = 0\}\).
Moreover, by equation (2.1) we obtain for every \(x \in \Omega \setminus \{F = 0\}\) together with the boundedness of \(F'\) and \(\text{Re } h\)

\[
\int_{(-\infty,0]} \rho_{t,p}(x) d\lambda(t) = \sum_{k=0}^{\infty} \int_{[-t_0,0]} \rho_{-kt_0,p}(x) d\lambda(s)
\]

\[
= \sum_{k=0}^{\infty} \int_{[-t_0,0]} \exp\left(\int_{0}^{-s} F'(\varphi(r,x)) - p \text{Re } h(\varphi(r,x)) \right) \rho_{-kt_0,p}(\varphi(-s,x)) d\lambda(s)
\]

\[
\leq C \sum_{k=0}^{\infty} \rho_{-kt_0,p}(\varphi(-s,x)) d\lambda(s)
\]

where \(C\) and \(\tilde{C}\) again depend on \(t_0\) and where in the last step we again used Lemma 2.2 for \(F\) and \(\text{Re } h\). Equation (2.1) and the fact that \(F'\) and \(\text{Re } h\) are bounded yield the existence of \(D > 0\) such that for all \(k \geq 0\)

\[
\rho_{-kt_0,p}(\varphi(t_0,x)) \leq D \rho_{-(k+1)t_0,p}(x).
\]

So the above gives

\[
\int_{(-\infty,0]} \rho_{t,p}(x) d\lambda(t) \leq \tilde{C} \sum_{k=0}^{\infty} \rho_{-kt_0,p}(x)
\]

for some \(\tilde{C} > 0\). Hence, i) implies ii).

In order to show that ii) implies iii) we fix \(t_0 > 0\) and \(x \in \Omega \setminus \{F = 0\}\) and distinguish again two cases. If \(x\) does not belong to \(\bigcap_{t \geq 0} \varphi(t, \Omega)\), there is \(t_1 > 0\) such that \(\rho_{t,p}(x) = 0\) for all \(t > t_1\). Therefore, \(\sum_{k=0}^{\infty} \rho_{kt_0,p}(x) < \infty\).

In the case of \(x \in \bigcap_{t \geq 0} \varphi(t, \Omega)\) it follows from equation (2.2) together with the boundedness of \(F'\) and \(\text{Re } h\) that for some \(C > 0\)

\[
\int_{[0,\infty)} \rho_{t,p}(x) d\lambda(t) = \sum_{k=0}^{\infty} \int_{[0,t_0]} \rho_{kt_0+t,p}(x) d\lambda(t)
\]

\[
= \sum_{k=0}^{\infty} \int_{[0,t_0]} \exp(p \int_{0}^{t} \text{Re } h(\varphi(r,x)) - \frac{1}{p} F'(\varphi(r,x)) dr) \rho_{kt_0,p}(\varphi(-t,x)) d\lambda(t)
\]

\[
\geq \sum_{k=0}^{\infty} C \int_{[0,t_0]} \rho_{kt_0,p}(\varphi(-t,x)) d\lambda(t)
\]

\[
\geq \begin{cases} 
\tilde{C} \sum_{k=0}^{\infty} \rho_{kt_0,p}(x), & F(x) < 0, \\
\rho_{kt_0,p}(\varphi(-t_0,x)), & F(x) < 0,
\end{cases}
\]

where we used Lemma 2.2 in the last step. By equation (2.2) and the boundedness of \(F'\) and \(\text{Re } h\) we have \(\rho_{kt_0,p}(\varphi(-t_0,x)) \geq D \rho_{(k+1)t_0,p}(x)\) for suitable \(D > 0\) such that the above gives

\[
(2.3) \quad \int_{[0,\infty)} \rho_{t,p}(x) d\lambda(t) \geq \tilde{C}_1 \sum_{k=0}^{\infty} \rho_{kt_0,p}(x)
\]

for some \(\tilde{C}_1\).
Additionally, applying Lemma 2.2 for \( F \) and \( \text{Re} \ h \) we also obtain from the boundedness of \( F' \) and \( \text{Re} \ h \), together with equation (2.1)
\[
\int_{(0,\infty)} \rho_{t,p}(x) d\lambda(t) = \sum_{k=0}^{\infty} \int_{(-t_0,0]} \rho_{-kt_0,t,p}(x) d\lambda(t)
\]
\[
= \sum_{k=0}^{\infty} \int_{(-t_0,0]} \exp \left( \int_{0}^{t} F'(\varphi(r,x)) - p\text{Re} \ h(\varphi(r,x)) dr \right) \rho_{-kt_0,p}(\varphi(-t,x)) d\lambda(t)
\]
\[
\geq C \sum_{k=0}^{\infty} \int_{(-t_0,0]} \rho_{-kt_0,p}(\varphi(-t,x)) d\lambda(t)
\]
\[
\geq \begin{cases} 
\hat{C} \sum_{k=0}^{\infty} \rho_{-kt_0,p}(x), & F(x) > 0, \\
\hat{C} \sum_{k=0}^{\infty} \rho_{-kt_0,p}(\varphi(-t_0,x)), & F(x) < 0
\end{cases}
\]
\[
\geq \hat{C} \sum_{k=0}^{\infty} \rho_{-kt_0,p}(x).
\]

Hence, together with (2.3), iii) follows from ii), and as iii) obviously implies i), the lemma is proved.

The applicability of the previous lemma depends on an explicit knowledge of \( \varphi \). The next lemma shows that the integrals appearing in the previous result can be expressed in terms of \( F \), \( h \), and \( \rho \).

**Lemma 2.4.** Let \( \Omega \subseteq \mathbb{R} \) be open and forward invariant under \( F \in C^{1}(\Omega), h \in C(\Omega) \) and let \( \rho \) be \( p \)-admissible for \( F \) and \( h \), \( 1 \leq p < \infty \). Then for every \( x \in \Omega \setminus \{F = 0\} \) we have
\[
\int_{\mathbb{R}} \rho_{t,p}(x) d\lambda(t) = \frac{1}{|F(x)|} \int_{C(x)} \exp(p \int_{w}^{x} \frac{\text{Re} \ h(y)}{F(y)} dy) \rho(w) d\lambda(w),
\]
where \( C(x) \) denotes the connected component of \( \Omega \setminus \{F = 0\} \) containing \( x \).

**Proof.** Fix \( x \in \Omega \setminus \{F = 0\} \) and let \( C(x) \) be as in the lemma. Observe that \( \varphi(t,x) \in C(x) \) for all \( t \in J(x) \) and that \( \varphi(J(x), x) = C(x) \), where \( J(x) \) is the domain of the maximal solution \( \varphi(\cdot, x) \) of the initial value problem \( \dot{y} = F(y), y(0) = x \). Obviously,
\[
\int_{\mathbb{R}} \rho_{t,p}(x) d\lambda(t) = \int_{[0,\infty)} \rho_{t,p}(x) d\lambda(t) + \int_{[0,\infty)} \rho_{-t,p}(x) d\lambda(t).
\]

We set \( C^{+}(x) = \{\varphi(t,x); t \geq 0\} \). Applying the Transformation Formula for Lebesgue integrals we obtain with equation (2.1)
\[
\int_{[0,\infty)} \rho_{-t,p}(x) d\lambda(t)
\]
\[
= \int_{[0,\infty)} \exp \left( \int_{0}^{t} F'(\varphi(r,x)) - p\text{Re} \ h(\varphi(r,x)) dr \right) \rho(\varphi(t,x)) d\lambda(t)
\]
\[
= \int_{[0,\infty)} \exp \left( \int_{0}^{t} F'(\varphi(r,x)) - p\text{Re} \ h(\varphi(r,x)) \right) \frac{\partial_{1} \varphi(r,x) dr}{F(\varphi(r,x))} \rho(\varphi(t,x)) d\lambda(t)
\]
\[
= \int_{[0,\infty)} \exp \left( \int_{x}^{\varphi(t,x)} F'(y) - p\text{Re} \ h(y) \right) \frac{dy}{F(y)} \rho(\varphi(t,x)) d\lambda(t).
\]
\[
1568 \quad T. \text{ KALMES}
\]

which proves the lemma.

Moreover, denoting \( \alpha = \sup \{ t \geq 0 ; x \in \varphi(t, \Omega) \} \) we have \(- \alpha = \inf J(x) \). With \( C^-(x) = \varphi((-\alpha, 0], x) \) it follows that \( C(x) = C^+(x) \cup C^-(x), C^+(x) \cap C^-(x) = \{ x \}, \) and

\[
\int_{[0, \alpha]} \rho_{t, p}(x) d\lambda(t) = \int_{[0, \alpha]} \rho_{t, p}(x) d\lambda(t)
\]

\[
= \int_{[0, \alpha]} \exp(\int_{-t}^{0} p \text{Re} h(\varphi(r, x)) - F'(\varphi(r, x)) dr) \rho(\varphi(-t, x)) d\lambda(t)
\]

\[
= \int_{(-\alpha, 0]} \exp(\int_{-t}^{0} p \text{Re} h(\varphi(r, x)) - F'(\varphi(r, x)) dr) \rho(\varphi(t, x)) d\lambda(t)
\]

\[
= \int_{(-\alpha, 0]} \exp(\int_{t}^{x} \frac{p \text{Re} h(y) - F'(y)}{F(y)} dy) \rho(\varphi(t, x)) d\lambda(t)
\]

\[
= \int_{(-\alpha, 0]} \exp(\int_{\varphi(t, x)}^{x} \frac{p \text{Re} h(y) - F'(y)}{F(y)} dy) \rho(\varphi(t, x)) d\lambda(t)
\]

Combining these equations yields

\[
\int_{\mathbb{R}} \rho_{t, p}(x) d\lambda(t)
\]

\[
= \int_{C(x)} \exp(\int_{w}^{x} \frac{p \text{Re} h(y) - F'(y)}{F(y)} dy) \rho(w) d\lambda(w)
\]

\[
= \int_{C(x)} \exp(\int_{w}^{x} \frac{\text{Re} h(y)}{F(y)} dy) \exp[\log |F(w)| - \log |F(x)|] \rho(w) d\lambda(w)
\]

\[
= \frac{1}{|F(x)|} \int_{C(x)} \exp(p \int_{w}^{x} \frac{\text{Re} h(y)}{F(y)} dy) \rho(w) d\lambda(w),
\]

which proves the lemma. \( \square \)
Remark 2.5. The last step in the above proof shows that for $x \in \Omega \setminus \{F = 0\}$ and all $v \in C(x)$ we have for every $1 \leq p < \infty$

$$\int_{\mathbb{R}} \rho_{t,p}(x) d\lambda(t)$$

$$= \frac{1}{|F(x)|} \int_{C(x)} \exp(p \int_{w}^{x} \frac{\text{Re} \, h(y)}{F(y)} \, dy) \rho(w) d\lambda(w)$$

$$= \frac{|F(v)|}{|F(x)|} \exp(p \int_{v}^{x} \frac{\text{Re} \, h(y)}{F(y)} \, dy) \frac{1}{|F(v)|} \int_{C(x)} \exp(p \int_{x}^{v} \frac{\text{Re} \, h(y)}{F(y)} \, dy) \rho(w) d\lambda(w)$$

$$= \frac{|F(v)|}{|F(x)|} \exp(p \int_{v}^{x} \frac{\text{Re} \, h(y)}{F(y)} \, dy) \int_{\mathbb{R}} \rho_{t,p}(v) d\lambda(t).$$

Thus, applying Remark 2.5, Theorem 2.1 follows.

Thus, under the hypotheses of Lemma 2.4 the following are equivalent for every connected component $C$ of $\Omega \setminus \{F = 0\}$ and all $1 \leq p < \infty$.

i) $\exists x \in C : \int_{\mathbb{R}} \rho_{t,p}(x) d\lambda(t) < \infty,$

ii) $\forall x \in C : \int_{\mathbb{R}} \rho_{t,p}(x) d\lambda(t) < \infty,$

iii) $\exists x \in C : \int_{C} \exp(-p \int_{x}^{w} \frac{\text{Re} \, h(y)}{F(y)} \, dy) \rho(w) d\lambda(w) < \infty,$

iv) $\forall x \in C : \int_{C} \exp(-p \int_{x}^{w} \frac{\text{Re} \, h(y)}{F(y)} \, dy) \rho(w) d\lambda(w) < \infty.$

We have now everything at hand to prove Theorem 2.1.

Proof of Theorem 2.1 By [3] Theorem 6 and Proposition 9] $T_{F,h}$ is chaotic on $L_{p}^{\rho}(\Omega)$ if and only if $\lambda(\{F = 0\}) = 0$ as well as for every $m \in \mathbb{N}$ for which there are $m$ different connected components $C_1, \ldots, C_m$ of $\Omega \setminus \{F = 0\}$, for $\lambda^m$-almost all choices of $(x_1, \ldots, x_m) \in \prod_{j=1}^{m} C_j$ there is $t > 0$ such that

$$\sum_{j=1}^{m} \sum_{\pi \in \mathbb{Z}} \rho_{t,p}^{1}(x_{\pi}) < \infty.$$ 

By Lemma 2.3 this holds precisely when $\lambda(\{F = 0\}) = 0$ and when for $\lambda$-almost every $x \in \Omega \setminus \{F = 0\}$

$$\int_{\mathbb{R}} \rho_{t,p}(x) d\lambda(t) < \infty.$$

Thus, applying Remark 2.5 Theorem 2.1 follows. □

Remark 2.6. a) Inspection of the proof of Theorem 2.1 yields the following. Under the hypothesis of Theorem 2.1 the following are equivalent for $\rho$ $p$-admissible for $F$ and $h$.

i) $T_{F,h}$ is chaotic in $L_{p}^{\rho}(\Omega)$.

ii) $\lambda(\{F = 0\}) = 0$ and for all $x \in \Omega \setminus \{F = 0\}$ there is $t_0 > 0$ such that $\sum_{k \in \mathbb{Z}} \rho_{t_0,p}(x) < \infty$.

iii) $\lambda(\{F = 0\}) = 0$ and $\sum_{k \in \mathbb{Z}} \rho_{t_0,p}(x) < \infty$ for all $x \in \Omega \setminus \{F = 0\}$ and all $t_0 > 0$.

iv) $\lambda(\{F = 0\}) = 0$ and $\int_{\mathbb{R}} \rho_{t,p}(x) d\lambda(t) < \infty$ for all $x \in \Omega \setminus \{F = 0\}$.

v) $\lambda(\{F = 0\}) = 0$ and for every connected component $C$ of $\Omega \setminus \{F = 0\}$

$$\int_{C} \exp(-p \int_{x}^{w} \frac{\text{Re} \, h(y)}{F(y)} \, dy) \rho(w) d\lambda(w) < \infty$$

for some/all $x \in C$. 

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b) If \( h = 0 \) and if \( F \in C^1(\Omega) \) is as usual then the \( p \)-admissibility of \( \rho \) does not depend on \( p \). If moreover \( F' \) is bounded the following are then equivalent.

i) \( T_F = T_{F,0} \) is chaotic in \( L^p(\Omega) \) for some/all \( p \in [1, \infty) \).

ii) \( \lambda\{F = 0\} = 0 \) and for every connected component \( C \) of \( \Omega\{F = 0\} \) we have

\[
\int_C \rho(w)d\lambda(w) < \infty.
\]

**Example 2.7.** a) Let \( \Omega \in \{(0, \infty), \mathbb{R}\} \) and let \( F(x) = 1 \). Then \( \Omega \) is forward invariant under \( F \). Moreover, let \( h \in C(\Omega) \) be such that \( \text{Re} \ h \) is bounded. It follows from the definition, that \( \rho = 1 \) is \( p \)-admissible for \( F \) and \( h \) for every \( 1 \leq p < \infty \) so that \( T_{1,h} \) is a well-defined \( C_0 \)-semigroup on \( L^p(\Omega) \), the so-called perturbed translation semigroup. If \( h \) is bounded the generator of \( T_{1,h} \) in \( L^p(\Omega) \) is given by

\[
A_p : W^{1,p}(\Omega) \rightarrow L^p(\Omega), \ A_p f(x) = f' + hf,
\]

where \( f' \) denotes the distributional derivative of \( f \) (see e.g. [3, Theorem 15]).

If \( \text{Im} \ h \in L^1(0, \beta) \), resp. \( \text{Im} \ h \in L^1(-\infty, \beta) \), for all \( \beta \in \Omega \) or if \( \text{Im} \ h \in L^1(\beta, \infty) \) for all \( \beta \in \Omega \), by Theorem 2.11 this \( C_0 \)-semigroup is chaotic on \( L^p(\Omega) \) if and only if

\[
\int \exp(-p \int_1^w \text{Re} \ h(y)dy)d\lambda(w) < \infty.
\]

b) Consider again \( \Omega \in \{(0, \infty), \mathbb{R}\} \) and let \( F(x) = 1 \). Moreover, let \( \rho \) be \( p \)-admissible for \( F \) and \( h = 0 \) (which does not depend on \( p \) by Remark 2.7(b)). We then obtain the classical translation semigroup and Remark 2.6(a) gives the well-known characterizations of chaos for this semigroup due to Matsui, Yamada, and Takeo [13,14] and deLaubenfels and Emamirad [7], respectively.

c) Consider \( \Omega = (0, 1) \) and let \( F(x) = -x \). Then \( \Omega \) is forward invariant for \( F \). Additionally, let \( h \in C(0,1) \) be such that \( \text{Re} \ h \) is bounded. It follows again from the definition that \( \rho = 1 \) is \( p \)-admissible for \( F \) and \( h \) for every \( 1 \leq p < \infty \). Thus, we obtain a well-defined \( C_0 \)-semigroup \( T_{-id,h} \) on \( L^p(0,1) \). If \( h \) is bounded the generator of this semigroup in \( L^p(\Omega) \) is given by

\[
A_p : \{f \in L^p(0,1); xf'(x) \in L^p(0,1)\} \rightarrow L^p(\Omega), \ A_p f(x) = -xf'(x) + hf(x)f(x),
\]

where \( f' \) denotes again the distributional derivative of \( f \) (see e.g. [3, Theorem 15]).

If \( x \rightarrow \frac{\text{Im} h(x)}{x} \in L^1(0, \beta) \) for all \( \beta \in (0,1) \) or if \( x \rightarrow \frac{\text{Im} h(x)}{x} \in L^1(\beta, 1) \) for all \( \beta \in (0,1) \), by Theorem 2.11 this \( C_0 \)-semigroup is chaotic on \( L^p(\Omega) \) precisely when for some \( x \in (0,1) \),

\[
\int_{(0,1)} \exp(-p \int_x^w \frac{\text{Re} \ h(y)}{y}dy)d\lambda(w) < \infty.
\]

Because of

\[
\exp(p \int_x^w \frac{\text{Re} \ h(y)}{y}dy) = \left(\frac{w}{x}\right)^{p \text{Re} \ h(0)} \exp(p \int_x^w \frac{\text{Re} \ h(y) - \text{Re} \ h(0)}{y}dy),
\]

this generalizes a result of Dawidowicz and Poskrobko [6] who showed that in the case of a real valued \( h \in C[0,1] \) for which \( x \rightarrow \frac{h(x) - h(0)}{x} \in L^1(0,1) \) the above semigroup is chaotic on \( L^p(0,1) \) if and only if \( h(0) > -1/p \).

d) Consider \( \Omega = (0, 1) \) and \( F(x) = -x^3 \sin \left(\frac{1}{x}\right) \). Because we have \( \lim_{x \rightarrow 0} F(x) = 0 \) and \( \lim_{x \rightarrow 1} F(x) \leq 0 \) it follows that \( \Omega \) is forward invariant under \( F \) and since \( F' \)
is bounded $\rho = 1$ is $p$-admissible for $F$ and $h = 0$ for every $1 \leq p < \infty$. Thus, $T_F$ is a well-defined $C_0$-semigroup on $L^p(0, 1)$. By [3, Theorem 15] its generator is

$$A_p : \{ f \in L^p(0, 1); -x^3 \sin(\frac{1}{x})f'(x) \in L^p(0, 1) \} \to L^p(0, 1),$$

$$A_pf(x) = -x^2 \sin(\frac{1}{x})f'(x),$$

where $f'$ denotes the distributional derivative of $f$. By Remark 2.6 it follows that this $C_0$-semigroup is chaotic on $L^p(0, 1)$ for every $1 \leq p < \infty$.

3. Weighted composition $C_0$-semigroups on Sobolev spaces

For a bounded interval $(a, b)$, let $F \in C^1[a, b]$ with $F(a) = 0$ be such that $(a, b)$ is forward invariant under $F$, and let $h \in W^{1, \infty}[a, b]$ be such that

1) $\forall x \in \{ F = 0 \} : h(x) = h(a) \in \mathbb{R},$

2) the function $[a, b] \to \mathbb{R}, y \mapsto \frac{h(y) - h(a)}{F(y)}$ belongs to $L^\infty[a, b]$.

In [3] it is shown that under the above hypothesis the operator

$$A_p : \{ f \in W^{1, p}[a, b]; Ff'' \in L^p[a, b] \} \to W^{1, p}[a, b], A_pf = Ff'' + hf,$$

where the derivatives are taken in the distributional sense, is the generator of a $C_0$-semigroup $S_{F,h}$ on $W^{1, p}[a, b] (1 \leq p < \infty)$ which is given by

$$\forall t \geq 0, f \in W^{1, p}[a, b] : S(t)f(x) = h_t(x)f(\varphi(t, x)).$$

Moreover, it is shown in [3] that this $C_0$-semigroup $S_{F,h}$ is never hypercyclic on $W^{1, p}[a, b]$. In particular, $S_{F,h}$ cannot be chaotic on $W^{1, p}[a, b]$.

Because of $F(a) = 0$, the closed subspace

$$W^{1, p}_s[a, b] := \{ f \in W^{1, p}[a, b]; f(a) = 0 \}$$

of $W^{1, p}[a, b]$ is invariant under $S_{F,h}$ such that the restriction of $S_{F,h}$ to $W^{1, p}_s[a, b]$ defines a $C_0$-semigroup on $W^{1, p}_s[a, b]$ which we denote again by $S_{F,h}$. Its generator is given by

$$A_{p,*} : \{ f \in W^{1, p}_s[a, b]; Ff'' \in L^p[a, b] \} \to W^{1, p}[a, b], A_{p,*}f = Ff'' + hf$$

(see [3]). Using Theorem 2.1 we derive the following characterization of chaos for $S_{F,h}$ on $W^{1, p}_s[a, b]$.

**Theorem 3.1.** Let $(a, b)$ be a bounded interval, $F \in C^1[a, b]$ with $F(a) = 0$ such that $(a, b)$ is forward invariant under $F$. Moreover, let $h \in W^{1, \infty}[a, b]$ be such that

1) $\forall x \in \{ F = 0 \} : h(x) = h(a) \in \mathbb{R},$

2) the function $[a, b] \to \mathbb{C}, y \mapsto \frac{h(y) - h(a)}{F(y)}$ belongs to $L^\infty[a, b]$.

Then, for the $C_0$-semigroup $S_{F,h}$ on $W^{1, p}_s[a, b]$ the following are equivalent.

i) $S_{F,h}$ is chaotic.

ii) $\lambda(\{ F = 0 \}) = 0$ and for every connected component $C$ of $(a, b) \setminus \{ F = 0 \}$

$$\int_C \exp(-p \int_x^w F'(y) + h(a) \frac{dy}{F(y)})d\lambda(w) < \infty$$

for some/all $x \in C$. 

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Proof. Observe that by the boundedness of $F'$ on $[a,b]$, $p = 1$ is $p$-admissible for $F$ and $F' + h(a)$ for any $1 \leq p < \infty$. Under the above hypothesis 1) and 2) it is shown in [3] Theorem 20 and Proposition 24) that the $C_0$-semigroups $S_{F,h}$ on $W_*^{1,p}[a,b]$ and $T_{F,F' + h(a)}$ on $L^p[a,b]$ are conjugate, i.e. there is a homeomorphism $\Phi : L^p[a,b] \to W_*^{1,p}[a,b]$ such that $S_{F,h} (t) \circ \Phi = \Phi \circ T_{F,F' + h(a)} (t)$ for every $t \geq 0$. By the so-called Comparison Principle (see e.g. [9, Proposition 7.7]) it follows that $S_{F,h}$ is chaotic on $W_*^{1,p}[a,b]$ if and only if $T_{F,F' + h(a)}$ is chaotic on $L^p[a,b]$. Thus, an application of Theorem [2.1] proves the theorem. \hfill $\Box$

Example 3.2. a) We consider $(a,b) = (0,1)$ and $F(x) = -x$. Then, $(0,1)$ is forward invariant under $F$. For every $h \in W_*^{1,\infty}[0,1]$ with $h(0) \in \mathbb{R}$ and

$$[0,1) \to \mathbb{C}, y \mapsto \frac{h(y) - h(0)}{y} \in L^\infty[0,1],$$

the operator

$$A : \{f \in W_*^{1,p}[a,b]; \ x f''(x) \in L^p[a,b]\} \to W_*^{1,p}[a,b], \ A f(x) = -x f'(x) + h(x) f(x),$$

generates a $C_0$-semigroup on $W_*^{1,p}[0,1], 1 \leq p < \infty$. By Theorem 3.1 this semigroup is chaotic on $W_*^{1,p}[0,1]$ if and only if for some $x \in (0,1]

$$\int_{[0,1]} \left( \frac{w}{x} \right)^{p(h(0)-1)} d\lambda(w) = \int_{[0,1]} \exp(-p \int_{x}^{w} -1 + h(0) dy) d\lambda(w) < \infty,$$

which holds precisely when $p(h(0) - 1) > -1$, i.e. when $h(0) > 1 - \frac{1}{p}$ (see also [3 Theorem 27]).

b) Let again $(a,b) = (0,1)$. We consider $F(x) = -x(1-x)$ so that $(0,1)$ is forward invariant under $F$. For each $h \in W_*^{1,\infty}[0,1]$ with $h(0) = h(1) \in \mathbb{R}$ and

$$[0,1) \to \mathbb{C}, y \mapsto \frac{h(y) - h(0)}{y(1-y)} \in L^\infty[0,1],$$

the operator

$$A : \{f \in W_*^{1,p}[a,b]; \ x(1-x) f''(x) \in L^p[a,b]\} \to W_*^{1,p}[a,b],$$

$$A f(x) = -x(1-x) f'(x) + h(x) f(x),$$

generates a $C_0$-semigroup on $W_*^{1,p}[0,1], 1 \leq p < \infty$. Since for any $x \in (0,1)$ the function

$$w \mapsto \exp \left( -p \int_{x}^{w} \frac{F'(y) - h(0)}{F(y)} dy \right)$$

$$= w^{-p(1+h(0))} (1 - w)^{-p(1-h(0))} (1 - x)^{p(1-h(0))} x^{p(1+h(0))}$$

does not belong to $L^1(0,1)$ for any value of $h(0)$ it follows from Theorem [3.1] that this semigroup is not chaotic.

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References

[1] Angela A. Albanese, Xavier Barrachina, Elisabetta M. Mangino, and Alfredo Peris, Distributional chaos for strongly continuous semigroups of operators, Commun. Pure Appl. Anal. 12 (2013), no. 5, 2069–2082, DOI 10.3934/cpaa.2013.12.2069. MR3015670

[2] Herbert Amann, Ordinary differential equations, An introduction to nonlinear analysis. de Gruyter Studies in Mathematics, vol. 13, Walter de Gruyter & Co., Berlin, 1990. Translated from the German by Gerhard Metzen. MR1071170 (91e:34001)

[3] Javier Aroz, Thomas Kalmes, and Elisabetta Mangino, Chaotic $C_0$-semigroups induced by semiflows in Lebesgue and Sobolev spaces, J. Math. Anal. Appl. 412 (2014), no. 1, 77–98, DOI 10.1016/j.jmaa.2013.10.002. MR3145782

[4] Javier Aroz and Alfredo Peris, Chaotic behaviour of birth-and-death models with proliferation, J. Difference Equ. Appl. 18 (2012), no. 4, 647–655, DOI 10.1080/10236198.2011.631535. MR2905288

[5] Jacek Banasiak and Marcin Moszyński, Dynamics of birth-and-death processes with proliferation—stability and chaos, Discrete Contin. Dyn. Syst. 29 (2011), no. 1, 67–79, DOI 10.3934/dcds.2011.29.67. MR2725281 (2012b:37137)

[6] Antoni Leon Dawidowicz and Anna Poskrobko, On chaotic and stable behaviour of the von Foerster-Lasota equation in some Orlicz spaces (English, with English and Estonian summaries), Proc. Est. Acad. Sci. 57 (2008), no. 2, 61–69, DOI 10.3176/proc.2008.2.01. MR2554563 (2010i:35034)

[7] Hassan Emamirad, Gisèle Ruiz Goldstein, and Jerome A. Goldstein, Chaotic solution for the Black-Scholes equation, Proc. Amer. Math. Soc. 140 (2012), no. 6, 2043–2052, DOI 10.1090/S0002-9939-2011-11069-4. MR2888192

[8] Karl-G. Grosse-Erdmann and Alfredo Peris Manguillot, Linear chaos, Universitext, Springer, London, 2011. MR2919812

[9] T. Kalmes, Hypercyclic, mixing, and chaotic $C_0$-semigroups induced by semiflows, Ergodic Theory Dynam. Systems 21 (2001), no. 5, 1411–1427, DOI 10.1017/S0143385701001675. MR1855839 (2002j:47030)

[10] R. deLaubenfels and H. Emamirad, Chaos for functions of discrete and continuous weighted shift operators, Ergodic Theory Dynam. Systems 27 (2007), no. 5, 1599–1631, DOI 10.1017/S014338570700144. MR2358980 (2008k:47020)

[11] T. Kalmes, Hypercyclic $C_0$-semigroups and evolution families generated by first order differential operators, Proc. Amer. Math. Soc. 137 (2009), no. 11, 3833–3848, DOI 10.1090/S0002-9939-09-09955-9. MR2529893 (2010k:47022)

[12] Elsabetta M. Mangino and Alfredo Peris, Frequently hypercyclic semigroups, Studia Math. 202 (2011), no. 3, 227–242, DOI 10.4064/sm202-3-2. MR2771652 (2012g:47030)

[13] Mai Matsui, Mino Yamada, and Fukiko Takeo, Supercyclic and chaotic translation semigroups, Proc. Amer. Math. Soc. 131 (2003), no. 11, 3535–3546 (electronic), DOI 10.1090/S0002-9939-03-06960-0. MR1991766 (2004c:47018)

[14] Mai Matsui, Mino Yamada, and Fukiko Takeo, Erratum to: “Supercyclic and chaotic translation semigroups” [Proc. Amer. Math. Soc. 131 (2003), no. 11, 3535–3546; MR 1991766], Proc. Amer. Math. Soc. 132 (2004), no. 12, 3751–3752 (electronic), DOI 10.1090/S0002-9939-04-07608-7. MR2084100 (2005d:47016)

[15] Ryszard Rudnicki, Chaoticity and invariant measures for a cell population model, J. Math. Anal. Appl. 393 (2012), no. 1, 151–165, DOI 10.1016/j.jmaa.2012.03.055. MR2921657