Inertial Forces - á la Newton in General Relativity

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1 Introduction

Einstein, generalising the special theory of relativity to include gravitational force gave the beautiful theory of general relativity wherein the spacetime geometry took the center stage with its curvature personifying gravitation. As a covariant theory General relativity is completely independent of observers and their state of motion. Thus it is always interesting to look for adapting systems of coordinates wherein one can find some new interpretation that could help in understanding the physical behaviour of a system better. In fact it is well known that in the full 4 dim. space-time theory General Ralativity dispensed away with the Newtonian notion of inertial forces. However in 1988, Abramowicz Carter and Lasota showed that if one introduces the so called optical geometry with a 3+1 splitting of space-time using a conformal slicing, then one can reintroduce the notion of Newtonian forces, which can be useful in clarifying the origin of certain GTR effects that were known in the 4-dim. theory. For a free particle only under the influence of gravity the equations of motion are given by the geodesics of the given spacetime manifold describing the gravitational field. Much as one admires the language of geometry the question is, does one get all the information inherent in the system within this framework or could there be something overlooked or missed in the interpretation. Further, the language of forces has indeed been very useful in describing the other interactions and thus it could be useful to bring it back into the realms of General Relativity, which might focus the non-Newtonian features inherent in the geometry more explicitly. This approach would particularly be of help in studying physical aspects very close to ultra compact objects and blackholes, a region which is never available for the weak field Newtonian physics.
2 Formalism

As the idea is to bring in Newtonian language into a geometric theory of spacetime, one needs to slice the 4-space into a (3 space + time) structure and look at the features on the absolute 3 space so obtained. In fact, such a 3+1 split of spacetime is nothing new in general relativity, as, long ago Arnowitt, Deser and Misner (1962) introduced such a scheme while looking for a method to give a Hamiltonian descriptin of general relativity. As Misner, Throne and Wheeler (1972) mention, “The slicing of spacetime into a one parameter family of space-like hypersurfaces is called for, not only by the analysis of the dynamics along the way, but also by the boundary conditions as they pose themselves in any action principle of the form - give the 3-geometries on the two faces of a sandwich of spacetime and adjust the 4-geometry in between to extremize the action”.

In fact, such a procedure of studying the dynamics effectively paved the way for setting up numerical methods to study evolution equations of the Cauchy data given on an initial hyper surface, and thus became a standard procedure for numerical relativity (Seidel (1996) and York (1979)).

Instead of a fully dynamical system, suppose one has a stationary system wherein a time-like Killing vector exists then one can get a lower dimensional quotient space through an isometry group action and one can study certain dynamical features within a given geometrical background. Abramowicz, Carter and Lasota (1988, hereafter referred to as ACL) used such a prescription with a conformal rescaling factor and showed that one can indeed obtain a 3+1 splitting wherein the 3-space is the quotient space obtained from the action of the time-like Killing vector and the metric conformal to the spatial geometry of the original four-space. As they
realised the most significant feature of a conformal reslicing was that the normally geometrical geodesic equation for a test particle would separate into language of Newtonian forces wherein one can directly interpret terms as gravitational, centrifugal and Coriolis accelerations.

Abramowicz, Nurowski and Wex (1993, hereafter referred to as ANW) later gave a covariant approach to this formulation which does not depend upon any particular symmetry and is as follows:

In the given spacetime manifold $M$ with the metric

$$ds^2 = g_{ij}dx^i dx^j$$ (1)

introduce a congruence of world lines which is globally orthogonal to $t = \text{const.}$, hypersurface which ensures that the vorticity of the congruence to be zero. In fact, Bardeen (1972) adopted such a congruence in axisymmetric, stationary spacetimes defining what are called locally non-rotating observers or zero angular momentum observers (ZAMO). The advantage of having such a congruence is that these local observers ‘rotate with the geometry’ and the connecting vectors between two such observers with adjacent trajectories do not precess with respect to Fermi-Walker transport.

Denoting such a vector field by $n^i \ (n_i n^i = -1 \ \text{time-like})$ it can be verified that the corresponding four-acceleration is proportional to the gradient of a scalar potential.

$$n^k \nabla_k n_i = \nabla_i \phi \ ; \ n^i \nabla_i \phi = 0$$ (2)

Though the vector field $n^i$ is not uniquely determined by (2), locally each particular choice of $n^i$ uniquely defines a foliation of the spacetime into slices each of which represents space at a particular instant of time, whose geometry is given by

$$h_{ik} = g_{ik} + n_in_k \ ; \ h^i_k = \delta^i_k + n^i n_k$$ (3)
(2) also ensures that the special observers $n^i$ see no change in the potential as their proper time passes by and thus have fixed positions that help them in distinguishing between different ‘inertial forces’.

Consider a particle of rest mass $m_o$ and four-velocity $U^i$, which can be expressed as

$$U^i = \gamma \left( n^i + v\tau^i \right)$$

wherein $\tau^i$ is the unit tangent vector (space-like) orthogonal to $n^i$ and parallel to the 3-velocity $v$ of the particle in the 3-space (Lorentz speed) and $\gamma$ the Lorentz factor $\left( = \frac{1}{\sqrt{1-v^2}} \right)$. The four-acceleration of the particle $a^i$ may now be obtained through direct computation (Abramowicz, 1993)

$$a_k = u^i \nabla_i u_k = -\gamma^2 \nabla_k \phi + \gamma^2 v (n^i \nabla_i \tau_k + \tau^i \nabla_i n_k)$$

$$+ \gamma^2 v^2 \tau^i \nabla_i \tau_k + (v\gamma) \tau_k + \dot{\gamma} n_k$$

Let us consider the motion of the particle in a circular orbit in a general stationary axisymmetric spacetime. From the given symmetries, there exist two Killing vectors $\eta^i$ the time-like having open trajectories, and $\zeta^i$ the space-like with closed trajectories. If the particle has a constant angular velocity $\Omega$ as measured by the stationary observer at infinity then its four-velocity $u^i$ may be expressed as

$$u^i = A \left( \eta^i + \Omega \zeta^i \right)$$

$A$ being the redshift factor

$$A^2 = \left[ \langle \eta \eta > + 2\Omega \langle \eta \zeta > + \Omega^2 \langle \zeta \zeta > \right]^{-1}$$

In terms of the Killing vectors one can express $n^i$ and $\phi$ consistently as

$$n^i = e^\phi \left( \eta^i + \omega \zeta^i \right)$$

$$\omega = - \langle \eta, \zeta > / \langle \zeta, \zeta \rangle$$

and

$$\phi = -\frac{1}{2} \ell n \left[ -\langle \eta, \eta > - 2\omega \langle \zeta, \eta > - \omega^2 \langle \zeta, \zeta \rangle \right] $$
From (4) and (6) one can evaluate the particle speed $V$ to be:

$$V \tau^i = e^{\phi}(\Omega - \omega)\zeta^i$$

(10)

Using now the ACL approach of conformal rescaling of the 3-metric, one can define the projected metric and the vectors

$$\tilde{h}_{ij} = e^{2\phi}h_{ij}, \quad \tilde{\tau}^i = e^\phi \tau^i$$

(11)

such that the acceleration may now be written as

$$a_k := -\nabla_k \phi + \gamma^2 V \left(n^i \nabla_i \tau_k + \tau^i \nabla_i n_k \right) + (\gamma V)^2 \tilde{\tau}^i \tilde{\nabla}_i \tilde{\tau}_k$$

(12)

as the last two terms in (5) become zero for a particle with constant speed, and conserved energy. $\tilde{\nabla}$ in (12) refers to the covariant derivative with respect to the metric $\tilde{h}_{ij}$. As may be seen, the acceleration is made up of three distinct terms, (i) gradient of a scalar potential, (ii) a term proportional to $V$, and (iii) one proportional to $V^2$. The first and the third terms may be immediately recognised as the gravitational and centrifugal accelerations. Further, as $n^i$ and $\tau^i$ are parallel to the Killing vector $\eta^i$ and $\zeta^i$ respectively from the equation for Lie derivative, one has

$$L_{n^i} \tau^j \equiv n^i \nabla_i \tau^j + \tau^i \nabla_k n^i = 0$$

(13)

Using this along with the fact that $n^i$ and $\tau^i$ are orthogonal, the second term of (12) may be written as

$$\gamma^2 \left[n^i (\nabla_i \tau_k - \nabla_k \tau_i) \right]$$

(14)

which represents the Lense-Thirring effect of the inertial drag and thus identified as the generalisation of the Coriolis acceleration.

If the general axisymmetric and stationary spacetime is represented by the metric

$$ds^2 = g_{tt} dt^2 + 2g_{t\phi} dt d\phi + g_{\phi\phi} d\phi^2 + g_{r r} dr^2 + g_{\theta \theta} d\theta^2$$

(15)
with $g_{ij}$s being functions of $r$ and $\theta$ only then the total force acting on a particle in circular orbit with constant speed $\Omega$ may be expressed as (Prasanna (1997))

$$F_i := (Gr)_i + (Co)_i + (Cf)_i$$  \hspace{1cm} (16)

wherein

$$(Gr)_i := -\nabla_i \phi = \frac{1}{2} \partial_i \left\{ \ln \left[ \left( g^{t\phi} - g^{tt} g^{\phi\phi} \right) / g^{\phi\phi} \right] \right\}$$

$$(Co)_i := \gamma^2 V \eta^k \left( \nabla_k \tau_i - \nabla_i \tau_k \right)$$

$$= -a^2(\Omega - \omega)\sqrt{g^{\phi\phi}} \left\{ \partial_i \left( \frac{g^{t\phi}}{\sqrt{g^{\phi\phi}}} \right) + \omega \partial_i \sqrt{g^{\phi\phi}} \right\}$$ \hspace{1cm} (17)

and

$$(Cf)_i := (\gamma V)^2 \tilde{\tau}^k \nabla_k \tilde{\tau}_i$$

$$= -A^2(\Omega - \omega) \frac{g^{\phi\phi}}{2} \pi_i \left\{ \ln \left[ g^{2\phi\phi} \left( g^{t\phi} - g^{tt} g^{\phi\phi} \right) \right] \right\}$$ \hspace{1cm} (18)

with

$$\Phi = -\frac{1}{2} \ell \left[ -g_{tt} - 2\omega g_{t\phi} - \omega^2 g^{\phi\phi} \right] ; \quad \omega = -g_{tq} / g^{\phi\phi}$$

and

$$A^2 = - \left[ g_{tt} + 2\Omega g_{t\phi} + \Omega^2 g^{\phi\phi} \right]^{-1}$$ \hspace{1cm} (19)

In order to understand this splitting of the total force, let us consider the simple case of static spacetimes (Abramowicz and Prasanna (1990)). Then, by definition the Coriolis term would be absent, and the force acting on a test particle of mass $m_o$ and 3-momentum $p^\alpha$ is given by (ACL)

$$m_o \tilde{f}_\alpha = p^\mu \nabla_\mu \tilde{p}_\alpha + \frac{m_o^2}{2} \nabla_\alpha \Phi$$  \hspace{1cm} (20)

wherein $\tilde{p}^\mu = m_o V \gamma \tilde{\tau}^\mu$ and thus consistent with what one gets from (15) in the projected 3-space. It is now clear that in the static space time, the trajectories of photons (rest mass zero particle) are given by the curves

$$\tilde{\tau}^\mu \nabla_\mu \tilde{\tau}_\alpha = 0$$ \hspace{1cm} (21)

which by definition are geodesics of this 3-space. Thus, one finds that the ‘null trajectories’ of the 4-space project onto the geodesics of the quotient space obtained through conformal slicing and indicates that particles on
these trajectories do not experience any ‘force’ in the Newtonian sense.

It is for this reason that ACL called this slicing as optical reference geometry meaning that the null lines project onto straight lines in the Euclidean sense. However, as was seen later this is true only for static spacetimes, as rotation would influence the photons differently for prograde and retrograde motion. In static spacetimes, one finds that particles acted on by forces other than gravity would deviate from the geodesics of the quotient 3-space obtained by conformal slicing, as these geodesics behave like straight lines of Newtonian geometry and thus follow Newtonian laws of motion.

The most important aspect of the formalism is that while the particle kinematics is expressed in Newtonian language, the general relativistic effects are all present and thus help in understanding results which were earlier known in general relativistic analysis but were hidden in the geometry and not accessible for visualisation.

3 Specific Applications

We start from the simplest application of the methodology outlined above to study the particle kinematics in the static spacetimes, taking the Schwarzschild geometry as the first example (Abramowicz and Prasanna, 1990; hereafter referred to as AP).

The metric as expressed in the usual coordinates,

$$ds^2 = -\left(1 - \frac{2m}{r}\right) dt^2 + \left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2)$$  \hspace{1cm} (22)
would yield for the gravitational and centrifugal accelerations, acting on a test particle in circular orbit, the expressions:

\[(Gr)_r = \frac{m}{r^2} \left(1 - \frac{2m}{r}\right)^{-1}\] (23)

and

\[(Cf)_r = -\Omega^2 r \left(1 - \frac{2m}{r}\right)^{-1} \left(1 - \frac{2m}{r} - \Omega^2 r^2\right)^{-1} \left(1 - \frac{3m}{r}\right)\] (24)

From (24) it is clear that while \((Gr)\) and \((Cf)\) are in opposite directions up to \(r = 3m\) from infinity for \(r < 3m\) they both have the same direction. As gravitational acceleration is always undirectional, it is clear that the centrifugal acceleration reverses its sign at \(r = 3m\). In fact, if one looks at the photon effective potential in the Schwarzschild geometry, one finds that \(r = 3m\) is the location of the maximum and thus corresponds to an unstable circular orbit. What has been noticed now is that this null line is the straight line path for the photon in the quotient space and thus corresponds to a location at which the centrifugal acceleration is zero, and on either sides the centrifugal force acts in opposite directions.

As shown in AP, this feature has important kinematical implications in the study of accretion flows near ultra compact objects (black holes) like the Rayleigh criterion for stability of flow turns out to be

\[
\frac{2m(r^3 - 2mr^2)}{(r^3 - \ell^2r + 2m\ell^2)^2} \left(1 - \frac{3m}{r}\right) \frac{d\ell^2}{dr} > 0
\] (25)

This means \(\frac{d\ell^2}{dr} > 0\) for \(r > 3m\) and \(< 0\) for \(r < 3m\), indicating that for \(r < 3m\) the angular momentum has to be advected inwards for stability. In fact, this result clearly explained the findings of Anderson and Lemos (1988), who had obtained inward advection of angular momentum very close to black holes.

Another important implication of centrifugal reversal is borne out in the evaluation of ellipticity of slowly rotating fluid configuration represented
by a sequence of quasi-stationary solutions, with decreasing radii, keeping the mass and angular momentum conserved. Chandrasekhar and Miller (1974) had considered this problem and found that the ellipticity instead of increasing continuously, reached a maximum, a result which they had attributed to frame dragging of rotating systems. However, after the discovery of centrifugal reversal, Abramowicz and Miller (1990) analysed the equilibrium configuration in general relativity using the ‘inertial forces’ approach and found that the equilibrium demands

\[
\frac{GM}{R^2} = R\Omega_k^2 \left(1 - \frac{3m}{R}\right) \left(1 - \frac{2m}{R} - R^2\Omega_k^2\right)^{-1}
\]

(26)

which to the lowest order in \(\Omega\) gave the ellipticity function to be

\[
\epsilon(R) = \frac{125}{32} \left[1 - \frac{3}{2R}\right] \left(\frac{1}{R}\right)
\]

(27)

exhibiting a maximum at \(R = 3\).

Though they had obtained the ellipticity maximum, the lacuna in their approach was that they had used only the Schwarzschild exterior geometry for evaluating the inertial forces. Prasanna and coworkers (A. Gupta, S. Iyer and A.R. Prasanna (1996)) reconsidered this problem starting from the general conservation laws expressed in the 3+1 formalism and using the Hartle-Thorne approximation solution for the interior of a slowly rotating body. Defining the ellipticity through force balance equations at the pole and equator

\[
e^2 = 1 \left(\frac{F_{cf} - F_{ge}}{F_{gp}}\right), \quad \epsilon \approx \frac{e^2}{e}
\]

(28)

\(F_{ge}\) is the gravitational force at the equator and \(F_{gp}\) at the poles) they found the maximum to occur at \(r = 5.4m\), which is much closer to Chandrasekhar-Miller result of maximum occurring at \(r = 5m\).
Prasanna (1991) had considered another static spacetime viz. Ernst spacetime which represents the external field of a blackhole immersed in a uniform magnetic field. The photon effective potential in this metric has two extrema, the maxima corresponding to the unstable orbit very close to \( r = 3m \), while a minima located far away, governed by the strength of the magnetic field and yielding a stable photon orbit. As centrifugal force would go to zero at both these locations, it appears that the Newtonian orbits are possible only within the region bounded by these two points of extrema.

We next consider the situation in the Kerr geometry which is actually supposed to be the spacetime exterior to a rotating blackhole represented by the metric

\[
d s^2 = \left(1 - \frac{2mr}{\Sigma}\right) dt^2 - \frac{4amr}{\Sigma} \sin^2 \theta dtd\phi + \frac{\Sigma}{\Delta} dr^2 + \Sigma d\theta^2 \tag{29}
\]

with

\[
\Delta = r^2 - 2mr + a^2 \\
\Sigma = r^2 + a^2 \cos^2 \theta \\
B = (r^2 - a^2) - \Delta a^2 \sin^2 \theta
\]

Using the formulae presented in (16)-(18), one can get (Prasanna (1997))

\[
(\text{GR})_r = \frac{m (a^2 \Delta + r^4 + a^2 r^2 - 2ma^2 r)}{r \Delta (r^3 + a^2 r + 2ma^2)} \tag{30}
\]

\[
(\text{CF})_r = \frac{-(\Omega - \omega)^2 \left[ r^5 - 3mr^4 + a^2 (r^3 - 3mr^2 + 6m^2 r - 2ma^2) \right]}{r^2 \Delta \left[ 1 - \Omega^2 (r^2 + a^2) - \left( \frac{2m}{r} \right) (1 - \Omega a)^2 \right]} \tag{31}
\]

\[
(\text{Co})_r = \frac{2ma(\Omega - \omega) (3r^2 + a^2)}{r (r^3 + a^2 r + 2ma^2) \left[ 1 - \Omega^2 (r^2 + a^2) - \left( \frac{2m}{r} \right) (1 - \Omega a)^2 \right]} \tag{32}
\]

One can immediately see the difference in the nature of centrifugal and Coriolis forces, whereas the Coriolis depends on the coupling of the angular momentum of the central source with that of the particle \( a(\Omega - \omega) \), the
centrifugal can go to zero at different locations solely depending upon ‘a’ due to the zeros of the quintic equation

\[ r^5 - 3mr^4 + a^2 \left( r^3 - 3mr^2 + 6m^2r - 2ma^2 \right) = 0 \]  

(33)

Fig. 1. Location of the point where \( Cfg = 0 \), (——–), retrograde photon orbit (----) and of the prograde photon orbit (····) for different values of \( a \).

It may be easily seen that for \( 0 < a < 1 \) the equation (32) at best can have only three real roots of which, one is always definitely outside the event horizon (Iyer and Prasanna (1993)) as depicted in Fig. 1. However, as also shown in this figure this location does not coincide with the location of the unstable photon orbit, prograde or retrograde. The centrifugal force vanishes at a location between the two photon orbits and for the case \( a = 0 \), they all coincide at \( r = 3m \). Unfortunately, the direct link between the unstable photon orbit and the centrifugal force reversal, depicted in static spacetimes do not find a parallel in stationary spacetime. Rotation does indeed bring in some new features of which the frame dragging is the most important one.
Figs. 2 and 3 show the nature of the centrifugal and Coriolis forces at the location of retrograde and direct photon orbits as a function of $\Omega$ for $a = 0.5$. The first impression that one gets is that these forces change sign for different values of $\Omega$ across the asymptotes. However, one has to check that the asymptotes are caused by the infinity of the redshift factor $A^2$ at the roots of the equation

$$\Omega^2 g_{\phi\phi} + 2\Omega g_{t\phi} + g_{tt} = 0 \quad (34)$$

$$\Omega_{\pm} = \omega \pm \sqrt{\omega^2 - g_{tt}/g_{\phi\phi}} \quad (35)$$

Hence the only portion of the plots which is meaningful is the region corresponding to the values of $\Omega$: $\Omega_- < \Omega < \Omega_+$. As may be seen in this region the centrifugal is positive along the retrograde photon orbit ($rph_+$) and negative along the direct photon orbit ($rph_-$) as it should be according to Fig. 1. On the other hand, the coriolis changes sign in both cases at the point $\Omega = \omega$, wherein centrifugal also vanishes because of the fact that the angular velocity is just equal to the dragging of inertial frames, by the spacetime due to the rotation of the central object.

On the other hand, we have seen that there are locations in the given spacetime, wherein the centrifugal force is zero but $\Omega \neq \omega$ and thus the Coriolis is non-zero. In view of this, Prasanna (1997) defined an index of reference called the ‘Cumulative Drag Index’ as defined by the ratio

$$C' = \frac{(Co - Gr)}{(Co + Gr)}$$

which could characterise purely the rotational feature of a spacetime through its influence on a particle in circular orbit at the location where the centrifugal force is zero.
Fig. 2. Centrifugal (——) and Coriolis (- - -) force plots for $a = 0.5$, along the retrograde photon orbit for $-1 < \Omega < 1$.

Fig. 3. Same as (2) at the prograde photon orbit.
Fig. 4 shows the plot of $C$ as a function of $\Omega$, for a fixed $a$ and $R$. As may be seen, there are two zeros and two infinities for the function. As $a > 0$, $\Omega > 0$ represents the co-rotating particles and $\Omega < 0$ represents the counter-rotating particles.

![Plot of C vs Omega](image)

Fig. 4. Cumulative drag index $C$ for $a = 0.5$, $R = 2.9445$ as a function of $\Omega$ ($-1 < \Omega < 1$).

As the orbit chosen is the one where the centrifugal force is identically zero the infinities of $C$ refer to the trajectories along which the total force acting on the particle is zero as given by the definition of $C$. Fig. 5 shows the index $C$ for three distinct cases of $a = 0.1$, 0.5 and 1 at the respective locations where $Cf = 0$.

It is interesting to note that as $a$ increases, the co-rotating particles have to decrease their angular velocity $\Omega$ very little to stay in equilibrium, whereas the counter-rotating ones have to increase their $\Omega$ much more, as depicted explicitly in Table 1.
Fig. 5. $C$ for three different values of $a$ showing difference in $\Omega_-$ values for equilibrium orbit ($C_0 = -Gr$).

![Diagram](image)

**Table 1**

| $a$  | $R$      | $\Omega_+$ | $\Omega_-$ | $|\Omega_- - \Omega_+||$ | $\omega$ |
|------|----------|-------------|-------------|------------------------|---------|
| 0.1  | 2.9978   | 0.189022    | -0.1964     | 0.0074                 | 0.0074  |
| 0.5  | 2.9445   | 0.180094    | -0.21965    | 0.0395                 | 0.0374  |
| 1.0  | 2.7830   | 0.17722     | -0.27452    | 0.097                  | 0.076   |

However, it may be seen that, the excess adjustment on the part of counter-rotating particles is essentially due to the dragging of inertial frames $\omega$ which is always in the direction of rotation of the central source. Table 1 gives the values of $\omega$ for given $a$ and $R$, which matches quite closely with the difference in the corresponding $\Omega$ for given $a$ and $R$, between co- and counter-rotating particles. Thus one finds that using the language of forces within the framework of general relativity, one can exactly quantify the spacetime curvature effects which otherwise is hidden in the geometry.

There have been other approaches to describe particle motion in general relativity a general survey of which was provided by Bini et al. (1997).
However, the conformal slicing as described above helps directly in understanding certain physical features of fluid flow configurations near extremally compact objects and also visualise the inertial drag more clearly. It would indeed be very nice if one can use this in the context of ADM formalism for Hamiltonian dynamics. This could be of direct consequence to understand the dynamical collapse of fluid configurations, through numerical approach. It is indeed very necessary to follow the collapse, particularly to see whether the centrifugal reversal that is inherent would produce the strange behaviour of the ellipticity as evidenced in the work of Chandrasekhar and Miller. A sudden change in ellipticity could indeed produce noticeable signature on the emission of gravitational radiation from a collapsing star in the final stages before it gets into a blackhole. This is an open problem which could be of great interest for classical gravitation studies.
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