Q-ary non-overlapping codes: a generating function approach

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Abstract

Non-overlapping codes are a set of codewords in $\bigcup_{n \geq 2} Z_q^n$, where $Z_q = \{0, 1, \ldots, q - 1\}$, such that, the prefix of each codeword is not a suffix of any codeword in the set, including itself; and for variable-length codes, a codeword does not contain any other codeword as a subword. In this paper, we investigate a generic method to generalize binary codes to $q$-ary for $q > 2$, and analyze this generalization on the two constructions given by Levenshtein (also by Gilbert; Chee, Kiah, Purkayastha, and Wang) and Bilotta, respectively. The generalization on the former construction gives large non-expandable fixed-length non-overlapping codes whose size can be explicitly determined; the generalization on the later construction is the first attempt to generate $q$-ary variable-length non-overlapping codes. More importantly, this generic method allows us to utilize the generating function approach to analyze the cardinality of the underlying $q$-ary non-overlapping codes. The generating function approach not only enables us to derive new results, e.g., recurrence relations on their cardinalities, new combinatorial interpretations for the constructions, and the limit superior of their cardinalities for some special cases, but also greatly simplifies the arguments for these results. Furthermore, we give an exact formula for the number of fixed-length words that do not contain the codewords in a variable-length non-overlapping code as subwords. This thereby solves an open problem by Bilotta and induces a recursive upper bound on the maximum size of variable-length non-overlapping codes.

Index Terms: Non-overlapping code, variable-length code, generating function.

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I. INTRODUCTION

Motivated by applications for synchronization, non-overlapping codes were first defined by Levenshtein in 1964 under the name strongly regular codes, a.k.a. codes without overlaps [1], [2], [3]. A code \( S \subseteq \bigcup_{n \geq 2} \mathbb{Z}_q^n \) is called non-overlapping if \( S \) satisfies the following two conditions:

1. No non-empty prefix of each codeword is a suffix of any one, including itself;
2. For all distinct \( u, v \in S \), \( u \) does not contain \( v \) as a subword.

We say that \( S \) is a fixed-length non-overlapping code if \( S \subseteq \mathbb{Z}_q^n \), otherwise it is called a variable-length non-overlapping code. In this paper, we consider both fixed-length and variable-length cases. Fixed-length non-overlapping codes have been intensively studied in the literature.

Let \( C(n, q) \) be the maximum size of a \( q \)-ary non-overlapping codes of length \( n \). The main research problems are to construct non-overlapping codes as large as possible in size and to bound \( C(n, q) \). The first construction was proposed by Levenshtein in 1964 [1], [2] (Construction I, see also [4], [5]). Following the work by de Lind van Wijngaarden and Willink [6] in 2000, Bajic and Stojanovic [7] independently rediscovered binary fixed-length non-overlapping codes (under the name cross-bifix-free codes) in 2004. In 2012, Bilotta, Pergola, and Pinzani [8] provided a binary construction based on Dyck paths, by which the code size is smaller than Levenshtein’s. However, it reveals an interesting connection between non-overlapping codes and other combinatorial objects. In 2013, Chee, Kiah, Purkayastha, and Wang [5] rediscovered Levenshtein’s construction (Construction I), and verified that it is optimal for \( q = 2 \) and \( n \leq 16 \), except when \( n = 9 \) by computer search. In 2015, Blackburn [9] generalized Levenshtein’s construction and thereby provided a class of largest fixed-length non-overlapping codes (see Construction I.A) whenever \( n \) divides \( q \). In 2016, Barucci, Bilotta, Pergola, Pinzani, and Succi [10] proposed a construction for \( q \)-ary (\( q \geq 3 \)) fixed-length non-overlapping code based on colored Motzkin paths.

The best known lower bound of \( C(n, q) \) for fixed \( q \), proposed by Levenshtein [1] (see also [5], [4], [11]), states that,

\[
C(n, q) \gtrsim \frac{q - 1}{qe \cdot n} q^n, \\
\text{as } n \to \infty \text{ over the subsequence } n = \frac{q^i - 1}{q - 1} \text{ for } i = 1, 2, \ldots, \text{ where } e \text{ is the base of natural logarithm. This lower bound is derived from the cardinality of non-overlapping codes by Construction I. The best known upper bound, also by Levenshtein [2], states that}
\]

\[
C(n, q) \leq \left( \frac{n - 1}{n} \right)^{n-1} \frac{q^n}{n}. \tag{1}
\]
Blackburn [9] showed that this bound is tight if \( n \) divides \( q \). A weaker bound

\[
C(n, q) \leq \frac{q^n}{2n - 1}
\]  

was found by Chee et al. [5] independently, and Blackburn [9] further proved the equality cannot hold via a simple proof. Moreover, Blackburn [9] showed that for fixed \( n \geq 2 \),

\[
\liminf_{q \to \infty} \frac{C(n, q)}{q^n} = \frac{1}{n} \left( \frac{n - 1}{n} \right)^{n-1},
\]

and there exist absolute constants \( c_1, c_2 \) such that \( c_1(q^n/n) \leq C(n, q) \leq c_2(q^n/n) \) for all \( n, q \) larger than 1.

Recently, non-overlapping codes have also found applications in DNA storage systems [11], [12], pattern matching [13], and automata theory [14]. For other related work on fixed-length non-overlapping codes, see also [11], [15], [16], [17].

Variable-length non-overlapping codes, defined several decades ago, were not further studied until recently. In 2017, Bilotta [18] constructed a class of binary variable-length non-overlapping codes (see Construction II) and gave a recurrence relation on their cardinalities. It was proved that the cardinality is about \( [2(1 - \epsilon)]^n \) as \( n \to \infty \), where \( \epsilon \) is a small positive value. In addition, a recursive bound on the maximum size of binary variable-length non-overlapping codes was proposed. More precisely, let \( J_2 = \bigcup_{n \geq 2} J_2(n) \) be a binary variable-length code, where \( J_2(n) \subseteq \mathbb{Z}_2^n \). Denote by \( b_{J_2(n)}(i) \) the number of \( i \)-length binary words that do not contain codewords in \( J_2 \) as subwords. It was shown that

\[
|J_2(n)| < \frac{2^n}{n+1} - \sum_{i=1}^{n-2} 2^i |J_2(n-i)| - \frac{1}{2} \sum_{i=h}^{n+1-h} b_{J_2(n)}(i) \cdot |J_2(n+1-i)|.
\]

However, the exact expression for \( b_{J_2(n)}(i) \) was left open by Bilotta [18].

Variable-length non-overlapping codes could also be used in DNA storage systems as the address sequences [12]. More importantly, for two codewords with length \( m \) and \( m + d \) respectively, their Hamming distance is admitted at least \( d \). This is the major advantage of using variable-length non-overlapping codes for DNA storage systems.

The main contribution of this paper is summarized as follows. Firstly, a generic method to generalize binary non-overlapping codes to \( q \)-ary is provided; secondly, a generating function approach is utilized to analyze the generalized constructions of both Levenshtein’s and Bilotta’s. More precisely, for Construction I’ (the generalized Levenshtein’s construction), we find a 2-term recurrence relation of the cardinalities, and it reveals a new combinatorial interpretation for the
structure of this classic construction. In addition, the asymptotic behavior of the cardinality of codes by Construction I’ for a special case is also given. Numerical results suggest that Construction I’ may outperform other constructions of fixed-length non-overlapping codes, especially when $q$ is large. Meanwhile, for Construction II’ (the generalized Bilotta’s construction), we give the generating function and a 3-term recurrence relation of the cardinalities which cannot be trivially obtained by Bolotta’s method. Finally, an exact formula of $b_{J_2(n)}(i)$ is given and Eq. (3) is further generalized to the $q$-ary case.

The rest of this paper is organized as follows. In Section II, we provide some necessary notations and definitions. In Section III, we introduce the generic method to generalize binary codes to $q$-ary, and investigate how it works on the aforementioned two constructions. In Section IV, we give an exact formula of $b_{J_2(n)}(i)$ and generalize the recursive upper bound Eq. (3) for $q$-ary non-overlapping codes. Finally, we conclude this paper with some open problems in Section V.

II. NOTATIONS AND DEFINITIONS

In this paper, both $q$-ary codewords and words are vectors over $\mathbb{Z}_q = \{0, 1, \ldots, q - 1\}$. We use $[n]$ to denote the set $\{1, \ldots, n\}$ for an integer $n \geq 1$. It is convenient to write vectors as strings. For example, 02101 represents the vector $(0, 2, 1, 0, 1)$. A $q$-ary code is a set of codewords over $\mathbb{Z}_q$ and is called variable-length if its codewords have different lengths. The size of a $q$-ary code $S$ is the number of codewords in $S$, and is denoted by $|S|$. The empty set is denoted by $\emptyset$.

**Definition 1.** Let $n, q$ be integers larger than 1. For $u \in \mathbb{Z}_q^n$, the set of prefixes of $u$ is $\text{Pre}(u) = \{(u_1, u_2, \ldots, u_i) \mid 1 \leq i \leq n - 1\}$, and the set of suffixes of $u$ is $\text{Suf}(u) = \{(u_i, u_{i+1}, \ldots, u_n) \mid 2 \leq i \leq n\}$.

For example, $\text{Pre}(0011) = \{0, 00, 001\}$ and $\text{Suf}(0011) = \{1, 11, 011\}$.

**Definition 2** (Non-overlapping codes). A $q$-ary non-overlapping code is a finite or countable code $S \subseteq \bigcup_{n \geq 2} \mathbb{Z}_q^n$ which satisfies the following two conditions:

1. For all $u, v \in S$, $\text{Pre}(u) \cap \text{Suf}(v) = \emptyset$;
2. For all distinct $u, v \in S$, $u$ does not contain $v$ as a subword.
For example, \{00101, 00111\}, \{11101000, 111011000\} are both binary non-overlapping codes, while \{0111, 0011\} is not since 011 \in \text{Pre}(0111) \cap \text{Suf}(0011). In addition, \{10, 1100\} is not a non-overlapping code since 10 is a subword of 1100.

### III. A GENERIC METHOD TO GENERALIZE BINARY CODES TO $q$-ARY

We now define a generic way to generalize binary codes to $q$-ary codes.

**Definition 3.** Let $I, J$ form a bipartition of $\mathbb{Z}_q$ with $q \geq 2$ and $\omega = (\omega_1, \ldots, \omega_n) \in \mathbb{Z}_2^n$. We define a map $\phi_{I,J}(\cdot)$ from a binary codeword to a set of $q$-ary codewords as follows.

$$\phi_{I,J}(\omega) = A_1 \times A_2 \times \cdots \times A_n,$$

where $\times$ is the Cartesian product and for $1 \leq i \leq n$,

$$A_i = \begin{cases} I & \text{if } \omega_i = 0, \\ J & \text{if } \omega_i = 1. \end{cases}$$

Define the map $\Phi_{I,J}(\cdot)$ from a binary code $S \subseteq \bigcup_{n \geq 2} \mathbb{Z}_2^n$ to a $q$-ary code as

$$\Phi_{I,J}(S) = \bigcup_{\omega \in S} \phi_{I,J}(\omega).$$

**Example 1.** Let $I = \{0, 2\}, J = \{1, 3\}$. Then

$$\phi_{I,J}(001) = \{0, 2\} \times \{0, 2\} \times \{1, 3\}$$

is a code over $\mathbb{Z}_4$ with length 3 and size 8.

The non-overlapping property is preserved by the map we defined above. We give the following lemma.

**Lemma 1.** Let notations be the same as above. If $S$ is a binary non-overlapping code, then $\Phi_{I,J}(S)$ is a $q$-ary non-overlapping code.

**Proof.** Suppose to the contrary that $\Phi_{I,J}(S)$ is overlapping. Then there are two possible cases:

**Case 1:** There exist two $q$-ary codewords $a' = (a_1, \ldots, a_{n_1})$ and $b' = (b_1, \ldots, b_{n_2})$ from $\Phi_{I,J}(S)$ such that $\text{Pre}(a') \cap \text{Suf}(b') \neq \emptyset$. W.l.o.g., assume that $(x_1, \ldots, x_k) \in \text{Pre}(a') \cap \text{Suf}(b')$ for a certain integer $k$. Thus we have $(a_{k'}, \ldots, a'_{k'}) = (b'_{n_2-k+1}, \ldots, b'_{n_2})$. By Definition 3, there exist two $q$-ary codewords $a = (a_1, \ldots, a_{n_1}), b = (b_1, \ldots, b_{n_2})$ from $S$ such that
\( a' \in \phi_{I,J}(a) \) and \( b' \in \phi_{I,J}(b) \). It then follows that \((a_1, \ldots, a_k) = (b_{n_2 - k + 1}, \ldots, b_{n_2})\). This is impossible since \( S \) is non-overlapping.

Case 2): There exist two \( q \)-ary codewords \( a', b' \) from \( \Phi_{I,J}(S) \) such that \( a' \) contains \( b' \) as a subword. By Definition 3, there exist two \( q \)-ary codewords \( a, b \) from \( S \) such that \( a' \in \phi_{I,J}(a) \), \( b' \in \phi_{I,J}(b) \), and \( a \) contains \( b \) as a subword. This is again impossible since \( S \) is non-overlapping.

The proof is then completed. \( \square \)

In the following, we investigate how the generalization in Definition 3 works on two known constructions. Lemma 2 will be useful to analyze the cardinalities of the codes by the two generalized constructions.

**Lemma 2.** [19, Lemma 3.6, Theorem 3.9] Let \( k \) be an integer larger than 1. The equation \( y^k - \sum_{i=0}^{k-1} y^i = 0 \) has one positive real root \( y_0 \) in the open interval \((1, 2)\). If \( k \) is even, it also has a negative real root \( y_1 \) belonging to \((-1, 0)\). Moreover, \( y_0 = 2(1 - \epsilon_k) > 2(1 - 2^{-k}) \) and approaches to 2 as \( k \to \infty \), where

\[
\epsilon_k = \sum_{i \geq 1} \frac{(k + 1)i - 2}{i} \frac{1}{i^{2(k+1)}}.
\]

The root \( y_1 \) and each complex root have modulus lying in the open interval \((3^{-k}, 1)\).

**A. The first construction**

We first review the classic construction given by Levenshtein [1], [2], which was also considered by Gilbert [4], and rediscovered by Chee et al. [5].

**Construction I** ([1], [2], [4], [5]). Let \( n, q \) be integers larger than 1 and \( 1 \leq k \leq n - 1 \). Denote by \( S_q^{(k)}(n) \) the set of all codewords \( s = (s_1, s_2, \ldots, s_n) \in \mathbb{Z}_q^n \) such that

- \( s_1 = s_2 = \cdots = s_k = 0 \), and \( s_{k+1} \neq 0, s_n \neq 0 \);
- \( (s_{k+1}, \ldots, s_n) \) does not contain \( k \) consecutive 0's.

Then \( S_q^{(k)}(n) \) is a non-overlapping code.

**Example 2.** Take \( k = 2, q = 2 \) and \( n = 6 \). Construction I gives the following non-overlapping code.

\[
S_2^{(2)}(6) = \{001011, 001101, 001111\}.
\]
Chee et al. [5] verified that this construction gives largest possible non-overlapping codes for \( q = 2 \) and \( n \leq 16 \), expect for \( n = 9 \). Let \( C \) be a subset of \( \mathbb{Z}_q^k \), we say that \((x_1, x_2, \ldots, x_r) \in \mathbb{Z}_q^r\) is \( C \)-free if \( r < k \), or if \( r \geq k \), \((x_i, x_{i+1}, \ldots, x_{i+k-1}) \notin C\) holds for all \( i = 1, 2, \ldots, r - k + 1 \). To obtain non-overlapping codes as large as possible in size, Construction I was generalized by Blackburn [9] as follows.

Construction I.A ([9]). Let \( n, q \) be integers larger than 1 and \( 1 \leq k \leq n - 1 \). Let \( I, J \) form a bipartition of \( \mathbb{Z}_q^k \), and \( C \subseteq I^k \). Denote by \( S^C_{I,J}(n) \) the set of all codewords \( s = (s_1, s_2, \ldots, s_n) \in \mathbb{Z}_q^n \) such that

- \((s_1, s_2, \ldots, s_k) \in C\), and \( s_{k+1} \in J \), \( s_n \in J \);
- \((s_{k+1}, \ldots, s_n)\) is \( C \)-free.

Then \( S^C_{I,J}(n) \) is a non-overlapping code.

Blackburn [9] showed that \( S^1_{I,J}(n) = I \times J^{n-1} \) achieves Levenshtein’s bound (see Eq. (1)) if \( n \) divides \( q \) and \(|I| = q/n\). However, the cardinality of \( S^C_{I,J}(n) \) for general \( C \) is hard to decide and left open.

In the following, we give a generalization of Construction I by the method in Definition 3. We remark that this is a special case of Construction I.A. However, it is worth noting that this leads to more transparent analysis on the cardinality using a generating function approach. Recalling that Chee et al. [5] showed \( S^{(k)}(n) \) is non-overlapping for \( q \geq 2 \), then by Lemma 1, \( \Phi_{I,J}(S^{(k)}_2(n)) \) is also non-overlapping. It can be easily verified that \( \Phi_{I,J}(S^{(k)}_2(n)) \) is equivalent to the following construction.

Construction I’. Let \( n, q \) be integers larger than 1 and \( 1 \leq k \leq n - 1 \). Let \( I, J \) form a bipartition of \( \mathbb{Z}_q^k \). Denote by \( S^{(k)}_{I,J}(n) \) the set of all codeword \( s = (s_1, s_2, \ldots, s_n) \in \mathbb{Z}_q^n \) such that

- \((s_1, s_2, \ldots, s_k) \in I^k\), and \( s_{k+1} \in J \), \( s_n \in J \);
- \((s_{k+1}, \ldots, s_n)\) is \( I^k \)-free.

Then \( S^{(k)}_{I,J}(n) \) is a non-overlapping code.

Remark 1. Construction I can be viewed as a special case of Construction I’. More precisely, taking \( I = \{0\}, J = \{1, 2, \ldots, q - 1\} = [q - 1] \) in Construction I’, we have \( S^{(k)}_{\{0\},[q-1]}(n) = S^{(k)}_q(n) = \Phi_{\{0\},[q-1]}(S^{(k)}_2(n)) \); Construction I’ can be viewed as a special case of Construc-
tion I.A. More precisely, taking $C = I^k$ in Construction I.A, we have $S_{I,J}^{(k)}(n) = S_{I,J}^{(k)}(n)$. Hence, $S_{I,J}^{(1)}(n)$ also achieves Levenshtein’s bound (see Eq. (1)) when $n$ divides $q$ and $|I| = q/n$.

A fixed-length non-overlapping code $S \subseteq \mathbb{Z}_q^n$ is called non-expandable if $S \cup \{x\}$ is overlapping for each $x \in \mathbb{Z}_q^n \setminus S$. A non-expandable non-overlapping code must be maximal but may not be of largest size. In [5, Theorem 3.1], it is claimed that $S_q^{(k)}(n)$ is non-expandable for all $2 \leq k \leq n-2$. However, a key condition was ignored in the proof and the statement was thereby incorrect. To see this, by Construction I, $S_2^{(3)}(6) = \{000101, 000111\}$, but $S_2^{(3)}(6) \cup \{001101\}$ is still non-overlapping. Similarly, $S_2^{(4)}(7) = \{0000101, 000111\}$ but $S_2^{(4)}(7) \cup \{0001001\}$ is still non-overlapping. In the following, we give a generalized result on the non-expandability of the codes by Construction I'.

**Theorem 1.** The $q$-ary non-overlapping code $S_{I,J}^{(k)}(n)$ by Construction I’ is non-expandable if $k = n - 1$ or $1 \leq k < n/2$.

**Proof.** By Construction I’, the $q$-ary word $x = (x_1, \ldots, x_n)$ overlaps with every codeword from $S_{I,J}^{(k)}(n)$ if $x_1 \in J$ or $x_n \in I$. Therefore, we only need to consider $x \in (I \times \mathbb{Z}_q^{n-2} \times J) \setminus S_{I,J}^{(k)}(n)$. Note that $S_{I,J}^{(n-1)}(n) = I^{n-1} \times J$ is clearly non-expandable. For $1 \leq k < n/2$, there are two cases:

Case 1): The word $x$ is $I^k$-free. In this case, it belongs to $I^l \times J \times \mathbb{Z}_q^{n-l-1}$ for some $0 \leq l \leq k-1$. The $(l+1)$-length prefix of $x$ is also a suffix for some $y$ in $I^k \times J^{n-k-l-1} \times I^l \times J \subseteq S_{I,J}^{(k)}(n)$ since $n > 2k \geq k + l + 1$. Hence, $S_{I,J}^{(k)}(n) \cup \{x\}$ is overlapping.

Case 2): The word $x$ contains $\omega \in I^k$ as a subword. We consider the shortest suffix of $x$ which is not $I^k$-free. In this case, $x$ belongs to $\mathbb{Z}_q^{n-m-k-1} \times I^k \times J \times T^{(k)}(m)$, where $T^{(k)}(m)$ is the set of $q$-ary $I^k$-free words of length $m < n - k - 1$. It then follows that the $(k+1+m)$-length suffix of $x$ is also a prefix for some $y$ in $I^k \times J \times T^{(k)}(m) \times J^{n-m-k-1} \subseteq S_{I,J}^{(k)}(n)$. Hence, $S_{I,J}^{(k)}(n) \cup \{x\}$ is overlapping.

Therefore, no additional word $x$ can be appended to the set $S_{I,J}^{(k)}(n)$ such that $S_{I,J}^{(k)}(n) \cup \{x\}$ is still non-overlapping. The proof is then completed. \qed

**Remark 2.** In the proof of [5, Theorem 3.1], the condition $n - k - l - 1 > 0$ in Case 1) of the proof of Theorem 1 was ignored.

We now clarify the result of [5, Theorem 3.1] in the following Corollary.
Corollary 1. The code $S_{q}^{(k)}(n)$ by Construction I is non-expandable if $k = n - 1$ or $1 \leq k < n/2$.

In the rest of this subsection, we analyze the cardinality of codes by Construction I’ using a generating function approach. Let $S_{I,J}^{(k)}(n+k)$ be a code defined by Construction I’, and $U_{I,J}^{(k)}(n)$ be the set of $I^k$-free $q$-ary codewords with length $n$ that start and end both with an element in $J$. Note that $S_{I,J}^{(k)}(n+k) = I^k \times U_{I,J}^{(k)}(n)$, $U_{I,J}^{(1)}(n) = J^n$, $U_{I,J}^{(k)}(1) = J$ and $U_{I,J}^{(k)}(2) = J \times J$. Therefore, the cardinality of $S_{I,J}^{(k)}(n+k)$ is determined by that of $U_{I,J}^{(k)}(n)$. Define

$$u_{I,J}^{k}(n) = \begin{cases} 0 & \text{if } n \leq 0, \\ |U_{I,J}^{k}(n)| & \text{if } n \geq 1, \end{cases}$$

then we have the following lemma.

Lemma 3. Let $I, J$ form a bipartition of $\mathbb{Z}_q$ and $u_{I,J}^{(k)}(n)$ be defined in (5). For any fixed $k \geq 1$, we have

$$\sum_{n=0}^{\infty} u_{I,J}^{(k)}(n)x^n = \frac{|J|x(1 - |I|x)}{1 - qx + |I|^k|J|x^{k+1}},$$

Therefore, we have

$$u_{I,J}^{(k)}(n) = \begin{cases} 0 & \text{if } n \leq 0, \\ |J|^n & \text{if } n = 1, 2, \\ q \cdot u_{I,J}^{(k)}(n-1) - |I|^k|J|u_{I,J}^{(k)}(n - k - 1) & \text{if } n \geq 3. \end{cases}$$

Proof. We first show that Eq.s (6) and (7) hold for $k \geq 2$, then we consider the remaining case $k = 1$. Let $k \geq 2$. By definition, each codeword from $\bigcup_{n \geq 1} U_{I,J}^{(k)}(n)$ can be uniquely constructed by appending letters from the alphabet $\mathbb{Z}_q$ to the empty word as follows.

1. Repeat the following two steps for $i \geq 0$ times;
   (a) append arbitrarily many consecutive letters from $J$;
   (b) append less than $k$ consecutive letters from $I$;

2. Append arbitrarily many consecutive letters from $J$.

Therefore, we have
Then Eq. (6) follows from the relation that

\[ \sum_{n=0}^{\infty} u_{I,J}^{(k)}(n) x^n = \left\{ \sum_{i=0}^{\infty} [(|J| x + |J|^2 x^2 + \ldots)(|I| x + |I|^2 x^2 + \ldots + |I|^{k-1} x^{k-1})]^i \right\} \cdot (|J| x + |J|^2 x^2 + \ldots) \]

Next we show Eq. (7) holds when \( k \geq 2 \). For \( n \leq 2 \), Eq. (7) follows from Eq. (5). For \( n \geq 3 \), multiplying \( 1 - qx + |I|^k |J| x^{k+1} \) to both sides of Eq. (6) and comparing the coefficient of \( x^n \), we get Eq. (7).

Finally, let \( k = 1 \), then we have

\[ u_{I,J}^{(1)}(n) = \begin{cases} 0 & \text{if } n \leq 0, \\ |J|^n & \text{if } n \geq 1. \end{cases} \tag{8} \]

It follows that

\[ \sum_{i=0}^{\infty} u_{I,J}^{(1)}(i) x^i = |J| x + |J|^2 x^2 + \ldots = \frac{|J| x}{1 - |J| x}. \]

Clearly, Eq. (8) coincides with Eq. (7) if \( k = 1 \) and \( n \leq 2 \). For \( n \geq 3 \), note that Eq. (7) yields \( u_{I,J}^{(1)}(n) = |I| \left( u_{I,J}^{(1)}(n-1) - |J| u_{I,J}^{(1)}(n-2) \right) + |J| u_{I,J}(n-1) = |J| u_{I,J}(n-1) \). Hence, Eq. (8) coincides with Eq. (7) for all \( n \geq 0 \) if \( k = 1 \).

Setting \( k = 1 \) in Eq. (6), we have

\[ \frac{|J| x (1 - |I| x)}{1 - qx + |I|^k |J| x^{k+1}} = \frac{|J| x}{1 - |J| x}. \]

The proof is then completed. \( \Box \)

**Theorem 2.** Let \( S_{I,J}^{(k)}(n) \) be the code given by Construction I’ with \( k \geq 1 \). Then we have

\[ \sum_{n=0}^{\infty} |S_{I,J}^{(k)}(n)| x^n = \sum_{n=0}^{\infty} |I|^k u_{I,J}^{(k)}(n-k) x^n = \frac{|I|^k |J| x^{k+1}(1 - |I| x)}{1 - qx + |I|^k |J| x^{k+1}}. \tag{9} \]
and

\[ |S_{I,J}^{(k)}(n)| = \begin{cases} 
0 & \text{if } n \leq k, \\
|I|^k |J|^{n-k} & \text{if } n = k + 1, k + 2, \\
q \cdot |S_{I,J}^{(k)}(n-1)| - |I|^k |J| \cdot |S_{I,J}^{(k)}(n-k-1)| & \text{if } n \geq k + 3.
\end{cases} \tag{10} \]

**Proof.** From the fact that $S_{I,J}^{(k)}(n) = I^k \times U_{I,J}^{(k)}(n-k)$ and Lemma 3, the conclusion follows. \(\square\)

We remark that Eq. (10) has a direct combinatorial interpretation. Let $n \geq k+3$. Define $P \subseteq \mathbb{Z}_q^n$ such that for any $p = (p_1, p_2, \ldots, p_n) \in P$, $(p_1, p_2, \ldots, p_{n-2}, p_n) \in S_{I,J}^{(k)}(n-1)$ and $p_{n-1} \in \mathbb{Z}_q$. In other words, $P$ is constructed by inserting an arbitrary $q$-ary letter in the second last position for each codeword in $S_{I,J}^{(k)}(n-1)$. Let $Q = S_{I,J}^{(k)}(n)$. Let $T = S_{I,J}^{(k)}(n-k-1) \times I^k \times J$. Note that $Q \subseteq P, P \setminus Q = T$. $|P| = q \cdot |S_{I,J}^{(k)}(n-1)|$, $|Q| = |S_{I,J}^{(k)}(n)|$, and $|T| = |I|^k |J| \cdot |S_{I,J}^{(k)}(n-k-1)|$. Eq. (10) then follows from $|P| - |Q| = |T|$.

Chee et al. [5] determined $|S_q^{(k)}(n)|$ by giving a recurrence relation of the set $S_q^{(k)}(n)$ in $k$ terms of $S_q^{(k)}(n-l)$ for $1 \leq l \leq k$. Hence we give a simpler recurrence of $S_q^{(k)}(n)$ involving only two terms: $S_q^{(k)}(n-1)$ and $S_q^{(k)}(n-k-1)$. The generating function approach is new and thereby provides us a more transparent understanding of this classic construction. It can be verified that Theorem 2 coincides with Chee et al’s result [5, Corollary 3.1] given that $I = \{0\}$ and $J = \{q-1\}$.

In the following, we estimate the asymptotic behavior of the cardinality of $S_{I,J}^{(k)}(n)$ when $|I| = |J|$. Clearly, $|S_{I,J}^{(1)}(n)| = (q/2)^n$. For $k > 1$, we give the following theorem.

**Theorem 3.** Let $S_{I,J}^{(k)}(n)$ be the code given by Construction I’ with $|I| = |J|$ and $k > 1$, we have

\[ \limsup_{n \to \infty} |S_{I,J}^{(k)}(n)|^{1/n} = q(1 - \epsilon_k), \]

where $\epsilon_k$ is given in Eq. (4).

**Proof.** It is well known in complex analysis (see [20, Theorem 2.5], for example) that for a power series $f = \sum_{n \geq 0} a_n x^n$, the limit superior of the sequence $\{|a_n|\}^{1/n}$ is $R^{-1}$ as $n \to \infty$, where $R$ is the radius of convergence of $f$.

Define $F(x) = \sum_{n=0}^{\infty} |S_{I,J}^{(k)}(n)| x^n$. By Eq. (9), we have
\[ F(x) = \frac{(r x)^{k+1}(1 - r x)}{1 - 2 r x + (r x)^{k+1}} = \frac{(r x)^{k+1}}{1 - \sum_{i=1}^{k} (r x)^i}, \]

where \( r = q/2 \).

The coefficient of \( F(x) \) are all non-negative and by Pringsheim’s Theorem (see [21, Theorem IV.6], for example), the radius of convergence of \( F(x) \) (denoted by \( R \)) is the smallest real root of the denominator. Let \( f(x) = 1 - \sum_{i=1}^{k} (r x)^i \), then \( f(R) = 0 \) and \( 0 < R < 1/r \) since \( f(0) > 0 \) and \( f(1/r) < 0 \). Let \( y = r x \), and \( f(x) = z(y) = 1 - \sum_{i=1}^{k} y^i \). The reciprocal polynomial of \( z(y) \) is

\[ z^*(y) = y^k - y^{k-1} - \cdots - y - 1. \]

By Lemma 2, \( z^*(2(1 - \epsilon_k)) = 0 \), and \( [2(1 - \epsilon_k)]^{-1} \) is the unique positive real root of \( z(y) \). Therefore \( R = [2r(1 - \epsilon_k)]^{-1} \) and we have

\[
\limsup_{n \to \infty} |S_{I,J}^{(k)}(n)|^{1/n} = R^{-1} = q(1 - \epsilon_k),
\]

where \( \epsilon_k \) is given in Eq. (4). The proof is then completed.

By Theorem 3, we know that, for \( |I| = |J|, k > 1 \) and all fixed \( \epsilon > 0 \), there exists an integer \( N \) such that \( |S_{I,J}^{(k)}(n)| < [q(1 - \epsilon_k) + \epsilon]^n \) holds for all \( n > N \), and \( |S_{I,J}^{(k)}(n)| > [q(1 - \epsilon_k) - \epsilon]^n \) holds for infinitely many values of \( n \).

Recall in Remark 1 that Construction I’ generates largest non-overlapping code when \( n \) divides \( q \). We further note that the code by Construction I’ is close to Levenshtein’s bound (see Eq. (1)) even if \( n \) does not divide \( q \). Let \( S_{I,J}^{(1)}(n) = I \times J^{n-1} \) with \( |I| = \lfloor q/n \rfloor \) be the code by Construction I’. Then by \( |J| = q - |I| \geq q\frac{n-1}{n} \), we have

\[
|S_{I,J}^{(1)}(n)| = \frac{1}{n} \left( \frac{n-1}{n} \right)^{n-1} q^n - O(q^{n-1}).
\]

B. The second construction

Before we give the generalization on variable-length non-overlapping codes, we first review the binary construction by Bilotta [18].

**Construction II** (Bilotta [18]). Let \( n, k \) be integer such that \( 3 \leq k \leq \lfloor n/2 \rfloor - 2 \). Denote by \( V_2^{(k)}(n) \) the set of all binary codewords \( v = (v_1, v_2, \ldots, v_n) \) with length \( n \) such that
• \(v_1 = v_2 = \cdots = v_k = 1, \text{ and } v_{k+1} = 0, v_{n-k} = 1;\)
• \(v_{n-k+1}, v_{n-k+2}, \ldots, v_n = 0;\)
• \((v_{k+1}, v_{k+2}, \ldots, v_{n-k})\) does not contain \(k\) consecutive 0’s or \(k\) consecutive 1’s.

Define
\[
V_2^{(k)}(n) = \bigcup_{i \geq 2k+2} V_2^{(k)}(i).
\]

Then \(V_2^{(k)}(n)\) is a binary variable-length non-overlapping code with maximum length \(n\).

**Example 3.** Take \(k = 3\) and \(n = 10\), then Construction II gives the code as
\[
V_2^{(3)}(10) = \{11101000, 11101100, 11100100, 11101010, 11100110\}.
\]

By the generic method, we now generalize Construction II by \(\Phi_{I,J}(V_2^{(k)}(n))\). It is readily seen that \(\Phi_{I,J}(V_2^{(k)}(n))\) is equivalent to the code by following construction.

**Construction II’**. Let \(I, J\) form a bipartition of \(\mathbb{Z}_q\) with \(q \geq 2\). Let \(n, k\) be integers such that \(3 \leq k \leq \lfloor n/2 \rfloor - 2\). Denote by \(V_{I,J}^{(k)}(n)\) the set of all \(q\)-ary codewords \(v = (v_1, v_2, \ldots, v_n)\) with length \(n\) such that
• \((v_1, v_2, \ldots, v_k) \in J^k, \text{ and } v_{k+1} \in I, v_{n-k} \in J;\)
• \((v_{n-k+1}, v_{n-k+2}, \ldots, v_n) \in I^k;\)
• \((v_{k+1}, v_{k+2}, \ldots, v_{n-k})\) is \((I^k \cup J^k)\)-free.

In other words, \(V_{I,J}^{(k)}(n) = J^k \times R_{I,J}^{(k)}(n-2k) \times I^k\), where \(R_{I,J}^{(k)}(l)\) is the set of \(q\)-ary \((I^k \cup J^k)\)-free codewords of length \(l\), that start with an element from \(I\), and end with an element from \(J\). Note that \(R_{I,J}^{(k)}(0) = \emptyset, R_{I,J}^{(k)}(1) = \emptyset,\) and \(R_{I,J}^{(k)}(2) = I \times J\). Define
\[
V_{I,J}^{(k)}(n) = \bigcup_{i \geq 2k+2} V_{I,J}^{(k)}(i).
\]

Then by Lemma 1, \(V_{I,J}^{(k)}(n)\) is a \(q\)-ary variable-length non-overlapping code with maximum length \(n\).

**Remark 3.** Construction II can be viewed as a special case of Construction II’. More precisely, taking \(I = \{0\}\) and \(J = \{1\}\) in Construction II’, we have \(V_{\{0\},\{1\}}^{(k)}(n) = V_2^{(k)}(n)\).

In the rest of this subsection, we again analyze the cardinality of codes by Construction II’ using a generating function approach. The cardinality is related to the intermediate variable \(r_{I,J}^{(k)}(l)\) defined as follows. Let \(k \geq 2\), define
\[
\begin{array}{ll}
I, J \in \mathbb{L} & = \\
\begin{cases}
0 & \text{if } l < 0, \\
1 & \text{if } l = 0, \\
|R_{l}^{(k)}| & \text{if } l > 0, \text{ where } R_{l}^{(k)} \text{ is defined in Construction II}'.
\end{cases}
\end{array}
\] (11)

By Construction II’, for \( k \geq 3 \) and \( n \geq 2k + 2 \),
\[
|V_{l, J}^{(k)}(n)| = \sum_{i \geq 2k+2} |V_{l, J}^{(k)}(i)|, (12)
\]

and for \( i \geq 2k + 2 \),
\[
|V_{l, J}^{(k)}(i)| = |I|^{|J|^k} r_{l, J}^{(k)}(i - 2k), (13)
\]

It remains to determine the values of \( r_{l, J}^{(k)}(\cdot) \). For \( I = \{0\} \) and \( J = \{1\} \), it was solved by Barcucci, Bernini, Bilotta, and Pinzani [22] as follows. Firstly, they showed
\[
r_{l}^{(k)}(n) = \sum_{j=1}^{k} r_{0, j}^{(k)}(n-j) - \sum_{j=k+1}^{2k-1} r_{0, j}^{(k)}(n-j),
\]

by giving a recurrence relation of the set \( R_{0, 1}^{(k)}(n) \) in \( 2k - 1 \) terms of \( R_{0, 1}^{(k)}(n-j) \) for \( 1 \leq j \leq 2k - 1 \). Then, they proved by induction that \( r_{0, 1}^{(k)}(n) \) is equal to or differ by 1 the sum of the \( k - 1 \) preceding terms. More precisely, for \( k \geq 3, n \geq 2 \), it was proved that
\[
r_{0, 1}^{(k)}(n) = \sum_{j=1}^{k-1} r_{0, j}^{(k)}(n-j) + d^{(k)}(n), (14)
\]

where \( d^{(k)}(n) \) is defined as
\[
\begin{array}{ll}
d^{(k)}(n) & = \\
1 & \text{if } n \equiv 0 \pmod{k}, \\
-1 & \text{if } n \equiv 1 \pmod{k}, \\
0 & \text{otherwise}.
\end{array}
\]

Finally, by the generating functions of \( d^{(k)}(n) \) and \( k \)-generalized Fibonacci numbers (see [22, Proposition 3.2]), the following was given.
\[
\sum_{n=0}^{\infty} r_{0, 1}^{(k)}(n)x^n = \frac{1 - 2x + x^2}{1 - 2x + 2x^{k+1} - x^{2k}}, (15)
\]
Remark 4. In [22], it is claimed that
\[
\sum_{n=0}^{\infty} r_{\{0\},\{1\}}^{(k)}(n)x^n = \frac{x^2 - 2x^{k+1} + x^{2k}}{1 - 2x + 2x^{k+1} - x^{2k}} = \frac{1 - 2x + x^2}{1 - 2x + 2x^{k+1} - x^{2k}} - 1.
\]
Taking \(x = 0\), it yields that \(r_{\{0\},\{1\}}^{(k)}(0) = 0\), which should be 1. We correct this equation in Eq. (15).

However, Eq. (14) cannot be easily generalized for other \(I, J\). Instead of trying to follow Barcucci et al.’s argument, we find a simple way to determine the generating function of \(r_{I,J}^{(k)}(n)\) directly.

Lemma 4. Let \(I, J\) form a bipartition of \(\mathbb{Z}_q\). For any integer \(n, k, q\) larger than 1, we have
\[
\sum_{n=0}^{\infty} r_{I,J}^{(k)}(n)x^n = \frac{|I||J|x^2 - qx + 1}{1 - qx + (|I||J| + |I||J|^k)x^{k+1} - |I||J|^kx^{2k}}, \tag{16}
\]
Therefore, \(r_{I,J}^{(k)}(n)\) is given as follows.

\[
r_{I,J}^{(k)}(n) = \begin{cases} 
0 & \text{if } n < 0 \text{ or } n = 1, \\
1 & \text{if } n = 0, \\
|I||J| & \text{if } n = 2, \\
qr_{I,J}^{(k)}(n - 1) - (|I||J| + |I||J|^k)r_{I,J}^{(k)}(n - k - 1) + |I||J|^k r_{I,J}^{(k)}(n - 2k) & \text{if } n \geq 3.
\end{cases} \tag{17}
\]

Proof. As in the proof of Lemma 3, each codeword from \(\bigcup_{n \geq 1} R_{I,J}^{(k)}(n)\) can be uniquely constructed by appending \(q\)-ary letters to the empty word as follows.

Repeat the following two steps for \(i \geq 0\) times.

(a) append less than \(k\) consecutive letters from \(I\);
(b) append less than \(k\) consecutive letters from \(J\);
Hence, we have
\[
\sum_{n=0}^{\infty} r_{I,J}^{(k)}(n)x^n = \sum_{i=0}^{\infty} \left( (|I|x + |I|^2x^2 + \cdots + |I|^{k-1}x^{k-1})(|J|x + |J|^2x^2 + \cdots + |J|^{k-1}x^{k-1}) \right)^i \\
= \sum_{i=0}^{\infty} \left( \frac{|I|x - |I|^kx^k}{1 - |I|x} \cdot \frac{|J|x - |J|^kx^k}{1 - |J|x} \right)^i \\
= \frac{1}{1 - \left( \frac{|I|x - |I|^kx^k}{1 - |I|x} \cdot \frac{|J|x - |J|^kx^k}{1 - |J|x} \right)} \\
= \frac{|I||J|x^2 - (|I| + |J|)x + 1}{1 - (|I| + |J|)x + (|I|^k|J| + |I||J|^k)x^{k+1} - |I|^k|J|^kx^{2k}}.
\]
Then Eq. (16) follows from the relation that \(|I| + |J| = q\).

Note that Eq. (17) holds for \(n \leq 2\) by Eq. (11). Multiplying the denominator to both sides of Eq. (16) and comparing the coefficient of \(x^n\), we get the rest part of Eq. (17).

Setting \(I = \{0\}, J = \{1\}\) in Eq. (16), we reobtain Eq. (15). Moreover, Eq. (17) gives a simpler recurrence relation on \(r_{I,J}^{(k)}(n)\).

**Theorem 4.** Let \(V_{I,J}^{(k)}(n)\) be the code given by Construction II’ with \(3 \leq k \leq \lfloor n/2 \rfloor - 2\). We have
\[
\sum_{n=2k+2}^{\infty} |V_{I,J}^{(k)}(n)|x^n = \frac{|I|^k|J|^k x^{2k}}{1 - x} \left( \frac{|I||J|x^2 - qx + 1}{1 - qx + (|I|^k|J| + |I||J|^k)x^{k+1} - |I|^k|J|^kx^{2k}} - 1 \right). \quad (18)
\]

**Proof.** By Eqs (13) and (16), we have
\[
\sum_{i=2k+2}^{\infty} |V_{I,J}^{(k)}(i)|x^i = \sum_{i=2k+2}^{\infty} |I|^k|J|^k r_{I,J}^{(k)}(i - 2k)x^i \\
= |I|^k|J|^k x^{2k} \sum_{n=2}^{\infty} r_{I,J}^{(k)}(n)x^n \\
= |I|^k|J|^k x^{2k} \left( \frac{|I||J|x^2 - qx + 1}{1 - qx + (|I|^k|J| + |I||J|^k)x^{k+1} - |I|^k|J|^kx^{2k}} - 1 \right).
\]
The desired result then follows from Eq. (12).}

In particular, by Theorem 4, we have
\[
\sum_{n=2k+2}^{\infty} |V_{\{0\},\{1\}}^{(k)}(n)|x^n = \frac{x^{2k}(x - x^k)^2}{(1 - x)(1 - x^k)(1 - 2x + x^k)},
\]
which coincides with the generating function given in [18, Eq. (9)]. Moreover, Bilotta [18] showed that
\[
\limsup_{n \to \infty} |V_{\{0\},\{1\}}^{(k)}(n)|^{1/n} = 2(1 - \epsilon_{k-1}),
\]
where \(\epsilon_{k-1}\) is given in Eq. (4). We generalize this result as follows.

**Theorem 5.** Suppose that \(n, k, q\) are integers larger than 1 such that \(8 \leq 2k + 2 \leq n\). Let \(I, J\) form a bipartition of \(\mathbb{Z}_q\) such that \(|I| = |J|\), then we have
\[
\limsup_{n \to \infty} |V_{I,J}^{(k)}(n)|^{1/n} = q(1 - \epsilon_{k-1}),
\]
where \(\epsilon_{k-1}\) is given in Eq. (4).

**Proof.** The proof idea is similar to that of Theorem 3. Let \(q = 2r\). By Eq. (18), we have
\[
\sum_{n=2k+2}^{\infty} |V_{I,J}^{(k)}(n)|x^n = \frac{(rx)^{2k}(rx - (rx)^k)^2}{(1-x)(1-(rx)^k)(1-2rx+(rx)^k)}
\]
\[
= \frac{(rx)^{2k}(rx - (rx)^k)^2}{(1-x)(1-(rx)^k)(1-rx)(1-\sum_{i=0}^{k-1}(rx)^i)}.
\]

By Pringsheim’s Theorem, the covering radius \(R\) of the series \(\sum_{n=2k+2}^{\infty} |V_{I,J}^{(k)}(n)|x^n\) is the smallest positive real root of the denominator. By considering the reciprocal polynomial of \(1 - \sum_{i=0}^{k-1}(rx)^i\) and Lemma 2, the unique real root of \(1 - \sum_{i=0}^{k-1}(rx)^i = 0\) is \([2p(1-\epsilon_{k-1})]^{-1} = [q(1-\epsilon_{k-1})]^{-1}\), where \(\epsilon_{k-1}\) is given in Eq. (4). Therefore \(R = [q(1-\epsilon_{k-1})]^{-1}\) and
\[
\limsup_{n \to \infty} |V_{I,J}^{(k)}(n)|^{1/n} = R^{-1} = q(1 - \epsilon_{k-1}).
\]
The proof is then completed. \(\square\)

For \(|I| = |J|\) and all fixed \(\epsilon > 0\), there exists an integer \(N\) such that \(|V_{I,J}^{(k)}(n)| < [q(1 - \epsilon_{k-1}) + \epsilon]^n\) holds for all \(n > N\), and \(|V_{I,J}^{(k)}(n)| > [q(1 - \epsilon_{k-1}) - \epsilon]^n\) holds for infinitely many values of \(n\).

Note that our generalization also works for other constructions for binary non-overlapping codes. Numerical results show that our generalization indeed produces large \(q\)-ary non-overlapping codes, especially when \(q\) is large. Details are presented in Appendix.
IV. Recursive Upper Bounds on Variable-Length Non-Overlapping Codes

Throughout this section, denote by $J_q(n) = \bigcup_{i=h}^{n} J_q(i)$ a $q$-ary non-overlapping code, and suppose that each subset $J_q(i)$ contains all the codewords in $J_q(n)$ with length $i$. In particular, $J_q(n)$ is a fixed-length non-overlapping code if $h = n$. Let $B_{J_q(n)}(m)$ be the set of $q$-ary words with length $m$ that do not contain codewords from $J_q(n)$ as subwords. Define

$$b_{J_q(n)}(m) = \begin{cases} 0, & \text{if } m < 0 \\ 1, & \text{if } m = 0 \\ |B_{J_q(n)}(m)|, & \text{if } m > 0 \end{cases}$$

Recall that Bilotta [18] showed a recursive upper bound on $J_2(n)$ that

$$|J_2(n)| < \frac{2^n}{n+1} - \sum_{i=1}^{h-2} 2^i |J_2(n - i)| - \frac{1}{2} \sum_{i=h}^{n+1-h} b_{J_2(n)}(i) \cdot |J_2(n + 1 - i)|,$$

where the exact expression of $b_{J_2(n)}(i)$ is left open. In the following, we first give an exact expression of $b_{J_q(n)}(m)$ with $q \geq 2$, and then generalize this recursive bound.

**Theorem 6.** Let notations be the same as above. We have

$$b_{J_q(n)}(m) = \sum (-1)^r \frac{(t_1 + r)!}{t_1! t_h! \cdots t_n!} q^{t_1} |J_q(h)|^{t_h} |J_q(h+1)|^{t_{h+1}} \cdots |J_q(n)|^{t_n}, \quad (20)$$

where $r = t_h + t_{h+1} + \cdots + t_n$ and the summation is over all nonnegative integers $t_1, t_h, \ldots, t_n$ such that

$$t_1 + ht_h + (h+1)t_{h+1} + \cdots + nt_n = m \geq 0.$$

**Proof.** Let $m \geq h$. Let $P = B_{J_q(n)}(m-1) \times \mathbb{Z}_q$ be the set of $q$-ary words of length $m$ that begin with a codeword from $B_{J_q(n)}(m-1)$ and are followed by an arbitrary $q$-ary letter. Clearly, $B_{J_q(n)}(m) \subseteq P$. Let $T_i = B_{J_q(n)}(m - i) \times J_q(i)$ for $h \leq i \leq n$. Thus $T_i$’s are pairwise disjoint and $P \setminus B_{J_q(n)}(m) = \bigcup_{i=h}^{n} T_i$ since $J_q(n)$ is non-overlapping. Note that $|P| = q \cdot b_{J_q(n)}(m-1)$, and $|T_i| = b_{J_q(n)}(m - i) \cdot |J_q(i)|$ for $h \leq i \leq n$. Hence, for any $m \geq h$, we have

$$q \cdot b_{J_q(n)}(m-1) - b_{J_q(n)}(m) = \sum_{i=h}^{n} b_{J_q(n)}(m - i) \cdot |J_q(i)|,$$
and \( b_{\mathcal{J}_q(n)}(m) = q^m \) for \( 0 \leq m < h \). Therefore
\[
\sum_{m=0}^{\infty} b_{\mathcal{J}_q(n)}(m)x^m = \frac{1}{1 - qx + \sum_{i=h}^{n} |J_q(i)|x^i}
= 1 + \left( qx - \sum_{i=h}^{n} |J_q(i)|x^i \right) + \left( qx - \sum_{i=h}^{n} |J_q(i)|x^i \right)^2 + \ldots .
\] (21)

Let \( b_k \) be the coefficient of \( x^m \) in \((qx - \sum_{i=h}^{n} |J_q(i)|x^i)^k\) for \( k \geq 0 \). Then we have
\[
b_k = \sum (-1)^{k-t_i} \frac{k!}{t_1! t_h! \ldots t_n!} q^{t_1} |J_q(h)|^{t_h} |J_q(h+1)|^{t_{h+1}} \ldots |J_q(n)|^{t_n},
\]
where \( t_1 + t_h + \ldots + t_n = k \) and the summation is over all nonnegative integers \( t_1, t_h, \ldots, t_n \) such that
\[
t_1 + h t_h + (h+1) t_{h+1} + \ldots + n t_n = m \geq 0.
\]
It implies \( k \leq m \), and Eq. (20) then follows from the relation \( b_{\mathcal{J}_q(n)}(m) = \sum_{k=0}^{m} b_k \).

We remark that Eq. (20) has a combinatorial interpretation. Denote by \( E_{j,i} \) be the set of \( m \)-length \( q \)-ary words \((s_1, s_2, \ldots, s_m)\) that contain a subword \((s_j, \ldots, s_{j+i-1}) \in J_q(i)\). By the principle of inclusion and exclusion, Eq. (20) counts the number of elements in \( \mathbb{Z}_q^m \setminus \bigcup E_{j,i} = B_{\mathcal{J}_q(n)}(m) \) for all \( j + i - 1 \leq n \) and \( h \leq i \leq n \).

Given \( |J_q(i)| \) for \( h \leq i < n \), we are able to derive an upper bound on \( |J_q(n)| \) as follows.

**Theorem 7.** Let notations be the same as above and \( 1 \leq m < h \). We have
\[
|J_q(n)| < \frac{q^n}{m+n} - \frac{1}{q^m} \sum_{i=h}^{n-1} b_{\mathcal{J}_q(n)}(m + n - i) \cdot |J_q(i)|,
\] (22)
where \( b_{\mathcal{J}_q(n)}(m + n - i) \) is given in Eq. (20).

**Proof.** We follow the argument by Bilotta [18, Proposition 7]. Let \( X \subseteq \mathbb{Z}_q^{m+n} \) be the set of all \( q \)-ary words with length \((m+n)\) that each contains exactly one codeword from \( \mathcal{J}_q(n) \) as a cyclic subword. For every \( \omega \in X \), there are \((m+n)\) possible positions for the cyclic subword from \( \mathcal{J}_q(n) \). Since codewords in \( \mathcal{J}_q(n) \) do not overlap with each other, we have
\[
|X| = \sum_{i=h}^{n} (m+n) \cdot b_{\mathcal{J}_q(n)}(m + n - i) \cdot |J_q(i)|
= (m+n)q^m|J_q(n)| + \sum_{i=h}^{n-1} (m+n) \cdot b_{\mathcal{J}_q(n)}(m + n - i) \cdot |J_q(i)|.
\]

\[\]
On the other hand, \( |X| \leq q^{m+n} - q < q^{m+n} \) since the \( q \) constant words of length \( m + n \) (e.g., \( 00\ldots0 \)) cannot have a codeword from \( \mathcal{J}_q(n) \) as a cyclic subword. We have

\[
|X| = (m + n)q^m |J_q(n)| + \sum_{i=h}^{n-1} (m + n) \cdot b_{\mathcal{J}_q(n)}(m + n - i) \cdot |J_q(i)| < q^{m+n},
\]

and Eq. (22) then follows. Note that \( b_{\mathcal{J}_q(n)}(m + n - i) \) does not involve \( |J_q(n)| \).

We remark that Eq. (22) is a generalization of Eqs. (3) and (2). Take \( m = 1 \) in Eq. (22), we have

\[
|J_q(n)| < \frac{q^n}{n+1} - \frac{1}{q} \sum_{i=h}^{n-1} b_{\mathcal{J}_q(n)}(n + 1 - i) \cdot |J_q(i)|
\]

\[
= \frac{q^n}{n+1} - \frac{1}{q} \sum_{i=h}^{n-1} b_{\mathcal{J}_q(n)}(n + 1 - i) \cdot |J_q(i)| - \frac{1}{q} \sum_{i=n-h+2}^{n-1} q^{n+1-i} |J_q(i)|
\]

\[
= \frac{q^n}{n+1} - \sum_{i=1}^{h-2} q^{|J_q(n-i)|} - \frac{1}{q} \sum_{i=h}^{n-1} b_{\mathcal{J}_q(n)}(n + 1 - i) \cdot |J_q(i)|.
\]

On the other hand, take \( h = n \). Eq. (22) becomes \( q^n/(m + n) \) since the summation of the RHS vanishes.

Recall that \( C(n, q) \) is the maximum size of \( q \)-ary non-overlapping codes of length \( n \). Clearly, \( |J_q(n)| \leq C(n, q) \) for \( J_q(n) \) is also non-overlapping. Also, by Theorem 7, we have

\[
|J_q(n)| < \min_{1 \leq m < n} \left\{ \frac{q^n}{m+n} - \frac{1}{q} \sum_{i=h}^{n-1} b_{\mathcal{J}_q(n)}(m + n - i) \cdot |J_q(i)| \right\}. \tag{23}
\]

In particular, for \( h = n \),

\[
|J_q(n)| < \min_{1 \leq m < n} \left\{ \frac{q^n}{m+n} \right\} = \frac{q^n}{2n-1}.
\]

Ideally, we hope Eq. (23) provides a tighter upper bound on \( |J_q(n)| \) than Eqs. (1) and (2). However, Eq. (23) depends on the given non-overlapping code \( \mathcal{J}_q(n) \) and cannot be easily analysed. In fact, it seems hard to find a tight bound of \( |J_q(n)| \) only by \( |J_q(i)| \) for \( h \leq i < n \), and \( |J_q(n)| \) is highly related to the structure of \( J_q(i) \). For example, assume that \( 0001 \in \mathcal{J}_2(5) \). It can be readily seen that \( \mathcal{J}_2(5) = \{0001\} \) and \( |J_2(5)| = 0 \).

A more challenging problem is to find a direct upper bound for the maximum size of \( \mathcal{J}_q(n) \). Trivially, \( |\mathcal{J}_q(n)| = \sum_{i=h}^{n} J_q(i) \leq \sum_{i=h}^{n} C(i, q) \) and we are interested in finding a tighter bound. To see the difficulty of this problem, we remark that Eq. (1) cannot be easily generalized to bound \( |\mathcal{J}_q(n)| \). Let \( f(x) = 1 - qx + \sum_{i=h}^{n} |J_q(i)|x^i \) be the denominator of Eq. (21). By Pringsheim’s
Theorem and Descartes’ rule of signs, \( f(x) \) must have 2 positive real roots. Note that \( f(0) > 0 \) and \( f(x) \) has a unique minima at \( x = x_0 \) if \( h = n \), where

\[
x_0 = \left( \frac{q}{n|J_q(n)|} \right)^{1/(n-1)}.
\]

Hence Eq. (1) follows from \( f(x_0) \leq 0 \). However, when \( h < n \), the minima \( x_0 \) of \( f(x) \) cannot be explicitly given, and \( f(x_0) \leq 0 \) may not yield an upper bound of \( \sum_{i=h}^{n} |J_q(i)| \).

V. Conclusions

We give a generic method to extend binary non-overlapping codes to \( q \)-ary, and investigate this generalization on Construction I (to Construction I’) and Construction II (to Construction II’). Construction I’ provides a large non-expandable fixed-length non-overlapping codes whose sizes can be explicitly counted, and Construction II’ is the first construction for \( q \)-ary variable-length non-overlapping codes. By the generating function approach, we establish new results on their cardinalities and greatly simplify some previous arguments. In this process, we also find a new combinatorial interpretation and non-expandability for Construction I’. In addition, we answer an open problem by Bilotta [18] and give a recursive upper bound on the maximum size of \( q \)-ary variable-length non-overlapping codes.

Further studies on non-overlapping codes may be devoted to more constructions and tighter bounds on the cardinality of codes. In particular, it would be interesting but difficult to find a nontrivial direct upper bound on the maximum size of variable-length non-overlapping codes. Moreover, our generalization on other constructions for non-overlapping codes could be further studied.

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**APPENDIX**

A. *Fixed-length non-overlapping codes*

Tables I - IV list the largest cardinalities of \(q\)-ary (\(3 \leq q \leq 6\)) fixed-length non-overlapping codes by

1) Construction I;

2) Construction I’;

3) Bilotta *et al.*’s construction [8]; this is a binary construction based on Dyck paths. We generalize it to \(q\)-ary by the method in Definition 3.
4) Barcucci et al’s construction [10]; this is a $q$-ary ($q \geq 3$) construction based on colored Motzkin paths.

For each construction, the largest cardinality is obtained by properly choosing parameters. For the cardinalities of binary fixed-length non-overlapping codes, we refer to [15, Table 2].

B. Variable-length non-overlapping codes

Tables V and VI list the largest cardinalities of 3-ary and 4-ary $\gamma_{I,J}^{(k)}(n)$ for $8 \leq n \leq 23$ by properly choosing $I, J$. For the binary case, we refer to [18, Table II].

| $n$ | Construction I | Construction I' | Bilotta et al. [8] | Barcucci et al. [10] |
|-----|----------------|-----------------|--------------------|----------------------|
| 3   | 4              | 4               | 4                  | 4                    |
| 4   | 8              | 8               | 4                  | 7                    |
| 5   | 16             | 16              | 16                 | 16                   |
| 6   | 32             | 32              | 24                 | 36                   |
| 7   | 88             | 88              | 80                 | 87                   |
| 8   | 240            | 240             | 128                | 210                  |
| 9   | 656            | 656             | 448                | 535                  |
| 10  | 1792           | 1792            | 736                | 1350                 |
| 11  | 4896           | 4896            | 2688               | 3545                 |
| 12  | 13376          | 13376           | 4608               | 9205                 |
| 13  | 36544          | 36544           | 16896              | 24698                |
| 14  | 99840          | 99840           | 29056              | 65467                |
| 15  | 272768         | 272768          | 109824             | 178375               |
| 16  | 745216         | 745216          | 194560             | 480197               |
### TABLE II

**Cardinalities of 4-ary fixed-length non-overlapping codes**

| n  | Construction I | Construction I’ | Bilotta et al. [8] | Barucci et al. [10] |
|----|----------------|----------------|--------------------|--------------------|
| 3  | 9              | 9              | 9                  | 9                  |
| 4  | 27             | 27             | 16                 | 25                 |
| 5  | 81             | 81             | 64                 | 72                 |
| 6  | 243            | 243            | 192                | 223                |
| 7  | 729            | 729            | 640                | 712                |
| 8  | 2187           | 2187           | 2048               | 2334               |
| 9  | 7371           | 7371           | 7168               | 7868               |
| 10 | 27945          | 27945          | 23552              | 26731              |
| 11 | 105948         | 105948         | 86016              | 93175              |
| 12 | 401679         | 401679         | 294912             | 324520             |
| 13 | 1522881        | 1522881        | 1081344            | 1157031            |
| 14 | 5773680        | 5773680        | 3719168            | 4104449            |
| 15 | 21889683       | 21889683       | 14057472           | 14874100           |
| 16 | 82990089       | 82990089       | 49807360           | 53514974           |

### TABLE III

**Cardinalities of 5-ary fixed-length non-overlapping codes**

| n  | Construction I | Construction I’ | Bilotta et al. [8] | Barucci et al. [10] |
|----|----------------|----------------|--------------------|--------------------|
| 3  | 16             | 18             | 18                 | 16                 |
| 4  | 64             | 64             | 36                 | 61                 |
| 5  | 256            | 256            | 216                | 224                |
| 6  | 1024           | 1024           | 648                | 900                |
| 7  | 4096           | 4096           | 3240               | 3595               |
| 8  | 16384          | 16384          | 10368              | 15014              |
| 9  | 65536          | 65536          | 54432              | 63135              |
| 10 | 262144         | 278964         | 178848             | 271136             |
| 11 | 1048576        | 1219860        | 979776             | 1178677            |
| 12 | 4870144        | 5333364        | 3359232            | 5167953            |
| 13 | 23515136       | 23515136       | 18475776           | 22986100           |
| 14 | 113541120      | 113541120      | 63545472           | 102403229          |
| 15 | 548225024      | 548225024      | 360277632          | 463098075          |
| 16 | 2647064576     | 2647064576     | 1276508160         | 2089302415         |
### TABLE IV

Cardinalities of 6-ary fixed-length non-overlapping codes

| $n$ | Construction I | Construction I' | Bilotta *et al.* [8] | Barucci *et al.* [10] |
|-----|----------------|-----------------|---------------------|----------------------|
| 3   | 25             | 32              | 32                  | 25                   |
| 4   | 125            | 128             | 81                  | 121                  |
| 5   | 625            | 625             | 512                 | 550                  |
| 6   | 3125           | 3125            | 2187                | 2739                 |
| 7   | 15625          | 15625           | 10935               | 13260                |
| 8   | 78125          | 78125           | 52488               | 67740                |
| 9   | 390625         | 390625          | 275562              | 342676               |
| 10  | 1953125        | 1953125         | 1358127             | 1787415              |
| 11  | 9765625        | 10027008        | 7440174             | 9324647              |
| 12  | 48828125       | 54788096        | 38263752            | 49456240             |
| 13  | 244140625      | 299368448       | 210450636           | 263776127            |
| 14  | 1220703125     | 1635778560      | 1085733963          | 1417981855           |
| 15  | 7068828125     | 8938061824      | 6155681103          | 7688015908           |
| 16  | 41381640625    | 48838475776     | 32715507960         | 41785951916          |
| $n$ | $k$ | 3   | 4   | 5   | 6   | 7   | 8   | 9   | 10  |
|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| 8   |     | 16  |     |     |     |     |     |     |     |
| 9   |     | 64  |     |     |     |     |     |     |     |
| 10  |     | 128 | 32  |     |     |     |     |     |     |
| 11  |     | 320 | 128 |     |     |     |     |     |     |
| 12  |     | 800 | 416 | 64  |     |     |     |     |     |
| 13  |     | 1760| 992 | 256 |     |     |     |     |     |
| 14  |     | 4128| 2720| 832 | 128 |     |     |     |     |
| 15  |     | 9696| 7328| 2560| 512 |     |     |     |     |
| 16  |     | 22112| 19680| 6656| 1664| 256 |     |     |     |
| 17  |     | 51296| 51552| 18944| 5120| 1024|     |     |     |
| 18  |     | 119008| 137312| 53632| 15488| 3328| 512 |     |     |
| 19  |     | 274144| 365024| 151168| 42368| 10240| 2048|     |     |
| 20  |     | 634336| 969824| 425216| 123008| 30976| 6656| 1024|     |
| 21  |     | 1467616| 2571104| 1188608| 356480| 93184| 20480| 4096|     |
| 22  |     | 3389664| 6828896| 3341568| 1031552| 263168| 61952| 13312| 2048|
| 23  |     | 7837920| 18132320| 9388800| 2980736| 773120| 186368| 40960| 8192|
### TABLE VI

**Cardinalities of 4-ary $\mathcal{V}^{(k)}_{i,j}(n)$**

| $n$ | $k$ | 3   | 4   | 5   | 6   | 7   | 8   | 9   | 10  |
|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| 8   |     | 256 |     |     |     |     |     |     |     |
| 9   | 3328|     |     |     |     |     |     |     |     |
| 10  |     | 11520| 5120 |     |     |     |     |     |     |
| 11  |     | 40192| 21504| 4096 |     |     |     |     |     |
| 12  |     | 122112| 70656| 20480 |     |     |     |     |     |
| 13  |   400640 | 267264| 86016 | 16384 |     |     |     |     |     |
| 14  | 1318144 | 988160 | 348160 | 81920 |     |     |     |     |     |
| 15  | 4201728 | 3675136| 1265664| 344064 | 65536 |     |     |     |     |
| 16  | 13638912| 13374464| 4935680 | 1392640 | 327680 |     |     |     |     |
| 17  | 44309760| 49288192| 19091456| 5586944 | 1376256 | 262144 |     |     |     |
| 18  | 142875904| 181408768| 73617408 | 21315584 | 5570560 | 1310720 |     |     |     |
| 19  | 462691584| 667948032| 284381184 | 84230144 | 22347776 | 5505024 | 1048576 |     |     |
| 20  | 1498684672| 2454721536 | 1093881856 | 331694080 | 89456640 | 22282240 | 5242880 |     |     |
| 21  | 4845739264| 9031390208 | 4218638336 | 1304772608 | 349503488 | 89391104 | 22020096 | 4194304 |     |
| 22  | 15683820800| 33224135680| 16264679424 | 5129977856 | 1389690880 | 357826560 | 89128960 | 20971520 |     |
| 23  |     |     |     |     |     |     |     |     |     |