Abstract—We present a novel data-driven MPC approach to control unknown nonlinear systems using only measured input-output data with closed-loop stability guarantees. Our scheme relies on the data-driven system parametrization provided by the Fundamental Lemma of Willems et al. We use new input-output measurements online to update the data, exploiting local linear approximations of the underlying system. We prove that our MPC scheme, which only requires solving strictly convex quadratic programs online, ensures that the closed loop (practically) converges to the (unknown) optimal reachable equilibrium that tracks a desired output reference. As intermediate results of independent interest, we extend the Fundamental Lemma to affine systems and we propose a data-driven tracking MPC scheme with guaranteed robustness. The theoretical analysis of this MPC scheme relies on novel robustness bounds w.r.t. noisy data for the open-loop optimal control problem, which are directly transferable to other data-driven MPC schemes in the literature. The applicability of our approach is illustrated with a numerical application to a continuous stirred tank reactor.

Index Terms—Data-driven control, predictive control for linear systems, nonlinear systems, time varying systems.

I. INTRODUCTION

Data-driven control has received significant attention in recent years due to the abundance of available data, the potential difficulties in obtaining accurate models, and the simplicity of data-driven approaches, see [1] for an overview of different methods. Our paper relies on the Fundamental Lemma by Willems et al. [2] which shows that one persistently exciting input-output trajectory can be used to parametrize all trajectories of a linear time-invariant (LTI) system. This provides a promising foundation for data-driven control of LTI systems and it can, e.g., be used to design data-driven model predictive control (MPC) schemes [3], [4]. Different contributions have analyzed such schemes in the presence of noise with regard to open-loop robustness [5], [6], [7], [8] or closed-loop stability/robustness both with [9] and without [10] terminal ingredients. Although different successful applications to complex nonlinear systems have been reported in the literature, see, e.g., [11], [12], providing theoretical guarantees of data-driven MPC for nonlinear systems remains a widely open research problem. The literature contains various extensions and variations of [2] for specific classes of nonlinear systems such as Hammerstein and Wiener systems [13], Volterra systems [14], polynomial systems [15], [16], systems with rational dynamics [17], flat systems [18], and linear parameter-varying systems [19]. However, all of these works assume that the system is linearly parametrized in known basis functions, which restricts their practical applicability.

In summary, although data-driven MPC is well-explored for LTI systems, there exists no unifying framework for nonlinear data-driven control.

In this paper, we propose an MPC approach to control unknown nonlinear systems with closed-loop stability guarantees by updating the data used in the data-driven system parametrization of [2] online, thereby exploiting that nonlinear systems can be approximated locally via linearization. The basic idea is depicted in Figure 1.

As in our companion paper [20], the goal is to stabilize the optimal reachable output equilibrium $y^{sr}$ corresponding to a given setpoint $y^f$, which may not lie on the output equilibrium manifold $Z^*_y$. To this end, we employ a tracking MPC formulation with an artificial equilibrium $y^*(t)$ which is optimized online, similar to [21], [22], [23]. In our companion paper [20], we show that, if the current linearization is used for prediction, then, under suitable assumptions on the design parameters and for initial conditions close to $Z^*_y$, $y^*(t)$ and thus the closed-loop output $y_t$ slide along $Z^*_y$ towards $y^{sr}$, see [20] for details. In the present paper, the key difference to [20] is that no model of the nonlinear system or its linearization is available and we use past $N$ input-output measurements to predict future trajectories based on the Fundamental Lemma [2]. Since these
measurements originate from the nonlinear system, they do not provide an exact description of the linearized dynamics which poses additional challenges if compared to the model-based MPC in [20]. In this paper, we show that, if the initial distance between the data points is not too large, then it remains small in closed loop and the predictions are sufficiently accurate such that practical stability can be guaranteed. The implementation requires solving strictly convex quadratic programs (QPs) and the only prior knowledge about the nonlinear system is a (potentially rough) upper bound on its order. For our theoretical analysis, we assume that the closed-loop input generated by the MPC scheme is persistently exciting, but we discuss multiple practical approaches for ensuring this property and we plan to investigate this issue in more detail in future research.

The works [24], [25] are related to our approach since they estimate linear time-varying models of nonlinear systems from data online, but they do not provide closed-loop stability guarantees. Controlling nonlinear systems using linear models via Koopman operator arguments has received increasing attention in recent years, see, e.g., [26], but typically no closed-loop guarantees can be given. Further recent approaches at the intersection of Koopman operators and the Fundamental Lemma have been developed in [27], [28], but, again, without closed-loop guarantees. Moreover, data-driven control methods based on machine learning techniques, compare, e.g., [29], [30], have been successfully applied but, also, they often do not provide closed-loop guarantees.

An obvious alternative to our results is provided by sequential system identification and model-based MPC. A simple approach would be online LTI system identification or recursive least squares estimation [32] followed by model-based MPC. Data-driven MPC has the advantage of being more direct, only requiring to tune and solve one optimization problem, the parameters of which can even be interpreted as an implicit system identification step [33]. Regardless, to the best of our knowledge, there are no results on closed-loop stability based on linearization arguments under similar assumptions as we consider for either identification-based or data-driven MPC. Alternative system identification approaches for nonlinear systems rely on Lipschitz continuity-like properties [34], [35], possibly leading to increasingly complex models, or they require appropriately chosen basis functions, see, e.g., [36] or approaches from nonlinear adaptive MPC [37]. In contrast, our data-driven MPC is direct, simple, applicable to a broad class of nonlinear systems, and it provides closed-loop stability guarantees, thereby indicating potential advantages over the classical approaches based on system identification. This is possible since we implicitly encourage the closed-loop trajectory to remain in vicinity of the steady-state manifold, where our data-driven prediction model is a good approximation of the underlying nonlinear dynamics. Initial ideas in this direction have been discussed in [38], however, without any theoretical analysis. Finally, the presented results are also related to offset-free MPC [39], which deals with setpoint tracking based on an uncertain model, whereas our approach achieves asymptotic convergence to the setpoint using only input-output data.

The remainder of the paper is structured as follows. Since the linearization generally leads to an affine dynamical system, we first extend the Fundamental Lemma to affine systems in Section II. Next, as a technical intermediate step, we present a robust data-driven tracking MPC scheme to control unknown affine systems based on noisy input-output data in Section III. Based on a separation argument of data-driven MPC with noise-free and noisy data, we show closed-loop practical stability w.r.t. the noise level. We then propose our data-driven MPC scheme for nonlinear systems in Section IV and we prove desirable closed-loop properties. The data-driven MPC approach is applied to a nonlinear numerical example in Section V and the paper is concluded in Section VI.

**Notation:** We denote the set of nonnegative integers by \( \mathbb{I}_{\geq 0} \), the set of integers in the interval \([a, b]\) by \( \mathbb{I}_{[a,b]} \), and the nonnegative real numbers by \( \mathbb{R}_{\geq 0} \). Moreover, \( \|\cdot\|_p \) denotes the \( p \)-norm of a vector, or the induced \( p \)-norm if the argument is a matrix. For a matrix \( P = P^\top \), we denote by \( \lambda_{\min}(P) (\lambda_{\max}(P)) \) the minimum (maximum) eigenvalue of \( P \), we write \( P > 0 \) if \( P \) is positive definite, and we define \( \|x\|_p^2 := x^\top P x \) for some vector \( x \). For matrices \( P_1 = P_1^\top \), \( P_2 = P_2^\top \), we define \( \lambda_{\min}(P_1, P_2) := \min\{\lambda_{\min}(P_1), \lambda_{\min}(P_2)\} \) and similarly for \( \lambda_{\max}(P_1, P_2) \). Further, \( A^\dagger \) denotes the Moore-Penrose inverse of a matrix \( A \) and \( \otimes \) denotes the Kronecker product. The interior of a set \( X \) is denoted by \( \text{int}(X) \). We define \( K_{\alpha, \epsilon} \) as the class of functions \( \alpha : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0} \) which are continuous, strictly increasing, unbounded, and satisfy \( \alpha(0) = 0 \). For a sequence \( \{x_k\}_{k=0}^{N-1} \), we define the Hankel matrix

\[
H_L(x) := \begin{bmatrix}
    x_0 & x_1 & \cdots & x_{N-L} \\
    x_1 & x_2 & \cdots & x_{N-L+1} \\
    \vdots & \vdots & \ddots & \vdots \\
    x_{L-1} & x_L & \cdots & x_{N-1}
\end{bmatrix},
\]

we denote a stacked window by \( x_{[a,b]} := [x_a^\top \cdots x_b^\top]^\top \), and we write \( x := x_{[0,N-1]} \). Finally, we write \( \mathbb{I}_n \) for an \( n \)-dimensional column vector with all entries equal to 1.

Throughout this paper, we use the inequalities

\[
\begin{align*}
    \|a + b\|_p^2 & \leq \|a\|_p^2 + 2\|b\|_p^2, \\
    \|a\|_p^2 - \|b\|_p^2 & \leq \|a - b\|_p^2 + 2\|a - b\|_p \|a\|_p, \\
    \|a\|_p^2 - \|b\|_p^2 & \leq \|a - b\|_p^2 + 2\|a - b\|_p \|b\|_p,
\end{align*}
\]

which hold for any vectors \( a, b \) and matrix \( P = P^\top > 0 \).

**II. Fundamental Lemma for affine systems**

In this section, we provide a data-driven parametrization of unknown systems with affine dynamics, i.e.,

\[
x_{k+1} = Ax_k + Bu_k + e,
\]

\[
y_k = Cx_k + Du_k + r
\]

with state \( x_k \in \mathbb{R}^n \), input \( u_k \in \mathbb{R}^m \), and output \( y_k \in \mathbb{R}^p \), all at time \( k \in \mathbb{I}_{\geq 0} \). We assume that the matrices \( A, B, C, D \) and the offsets \( e, r \) are unknown, but one input-output trajectory \( \{u^d_k, y^d_k\}_{k=0}^{N-1} \) of (4) is available. The Fundamental Lemma [2] shows that, if \( e = 0, r = 0 \) (i.e., the system is linear) and certain persistence of excitation and controllability conditions
hold, then a sequence \( \{u_k, y_k\}_{k=0}^{L-1} \) is a trajectory of (4) if and only if there exists a vector \( \alpha \in \mathbb{R}^{N-L+1} \) such that

\[
\begin{bmatrix}
H_L(u^d) \\
H_L(y^d)
\end{bmatrix} \alpha = \begin{bmatrix} u \\ y \end{bmatrix}.
\] (5)

In the following, we provide an extension of this result to the class of affine systems (4). We note that such an extension is not trivial since, without knowledge of the vectors \( e \) and \( r \), they cannot be set to zero without loss of generality. We propose the following definition of persistence of excitation, where we write \( \{x^d_k\}_{k=0}^{N-1} \) for the state sequence corresponding to the available input-output data \( \{u_k, y_k\}_{k=0}^{N} \).

**Definition 1.** We say that the data \( \{u_k\}_{k=0}^{N-1}, \{x^d_k\}_{k=0}^{N-L} \) are persistently exciting of order \( L \) if

\[
\text{rank} \begin{bmatrix} H_L(u^d) \\ H_1(x^d_{[0,N-L]}) \end{bmatrix} = mL + n + 1.
\] (6)

The persistence of excitation condition used for the Fundamental Lemma in [2] requires that \( H_{L+1}(u^d) \) has full rank which in turn, assuming controllability, implies that

\[
\text{rank} \begin{bmatrix} H_L(u^d) \\ H_1(x^d_{[0,N-L]}) \end{bmatrix} = mL + n.
\] (7)

In the present paper, we require the stronger condition in Definition 1 to account for the affine system dynamics (4) with \( e \neq 0 \), \( r \neq 0 \). In general, it is non-trivial to verify (6) from given input-output data. However, (7) can be ensured for controllable systems by choosing a persistently exciting input, cf. [2], and any input-state trajectory satisfying (7) and violating (6) satisfies (6) after an arbitrarily small perturbation.

**Theorem 1.** Suppose the data \( \{u^d_k\}_{k=0}^{N-1}, \{x^d_k\}_{k=0}^{N-L} \) are persistently exciting of order \( L \). Then, \( \{u_k, y_k\}_{k=0}^{N-1} \) is a trajectory of (4) if and only if there exists \( \alpha \in \mathbb{R}^{N-L+1} \) such that

\[
\sum_{i=0}^{N-L} \alpha_i = 1, \quad \begin{bmatrix} H_L(u^d) \\ H_L(y^d) \end{bmatrix} \alpha = \begin{bmatrix} u \\ y \end{bmatrix}.
\] (8)

The proof of Theorem 1 is straightforward and it is provided for completeness in Appendix A. Theorem 1 extends the Fundamental Lemma [2] to systems with affine dynamics. The key difference to the linear case is the condition \( \sum_{i=0}^{N-L} \alpha_i = 1 \) in (8), i.e., the vector \( \alpha \) sums up to one. Intuitively, this implies that the offsets \( e \) and \( r \) in (4) are carried through from the data \( (u^d, y^d) \) to the new trajectory \((u, y)\) in (8). We note that a condition similar to \( \sum_{i=0}^{N-L} \alpha_i = 1 \) also appears in [40], albeit in a different problem setting with the objective of offset-free data-driven control. Further, instead of considering this additional condition, it is also possible to utilize the fact that the trajectory \( \{u_{k+1} - u_k, y_{k+1} - y_k\}_{k=0}^{N-2} \) corresponds to an LTI system, compare, e.g., [41].

**III. ROBUST DATA-DRIVEN TRACKING MPC FOR AFFINE SYSTEMS**

In this section, we use Theorem 1 to design a tracking MPC scheme for affine systems based only on input-output data affected by noise. The presented results combine and extend the linear data-driven MPC results in [9] and [42] regarding robustness to noise and nominal setpoint tracking, respectively.

After stating the problem setting in Section III-A, we present the MPC scheme in Section III-B. Next, we provide a theoretical analysis of the scheme by bounding the distance of the optimal input to the input resulting from a nominal (noise-free) MPC in Section III-C, which we then use to prove closed-loop practical stability in Section III-D.

**A. Problem setting**

Similar to other works on data-driven MPC with closed-loop guarantees (see, e.g., [9]) we assume that (4) is controllable and observable.

**Assumption 1.** (Controllability and observability) The pair \((A, B)\) is controllable and the pair \((A, C)\) is observable.

Since we only have access to input-output data of (4), we define an input-output equilibrium as follows (cf. [9], [42]).

**Definition 2.** We say that an input-output pair \((u^s, y^s)\) is an equilibrium of (4), if the sequence \( \{\tilde{u}_k, \tilde{y}_k\}_{k=0}^{n} \) with \((\tilde{u}_k, \tilde{y}_k) = (u^s, y^s)\) for \( k \in [0, n] \) is a trajectory of (4).

For a given equilibrium \((u^s, y^s)\), we write \( u^s_n := \mathbb{I}_n \otimes u^s \) and similarly for \( y^s_n \). In this section, the goal is to track an output target setpoint \( y^T \in \mathbb{R}^p \), which need not be an equilibrium of (4), while satisfying pointwise-in-time input constraints \( u_t \in U \subseteq \mathbb{R}^n \), \( t \in \mathbb{I}_{\geq 0} \), with a compact, convex polytope \( U \). We do not consider output constraints to avoid the additional challenges in the required robust constraint tightening, compare [43]. We follow the approach to tracking MPC originally proposed in [21], wherein an artificial setpoint is included in the online optimization, thereby guaranteeing closed-loop stability of the optimal reachable equilibrium. Due to a local controllability argument required for our theoretical results, we consider only equilibria whose input component lies in the interior of the constraints, i.e., in some convex polytope \( U^s \subseteq \text{int}(U) \). Given a matrix \( S \succ 0 \) and a data trajectory \( \{u^d_k, y^d_k\}_{k=0}^{N-1} \) of (4) with persistently exciting input and state according to Definition 1, we define the optimal reachable equilibrium \((u^r, y^r)\) as the minimizer of

\[
J^r_{eq} := \min_{u^r, y^r, \alpha^r} \|y^s - y^T\|^2_S \text{ s.t. } \begin{bmatrix} H_{L+n+1}(u^d) \\ H_{L+n+1}(y^d) \end{bmatrix} L_{N-L} \alpha^r = \begin{bmatrix} u^s_{L+n+1} \\ y^s_{L+n+1} \end{bmatrix}, \ u^r \in U^s
\] (9)

with some \( L \in \mathbb{I}_{\geq 0} \). We note that \( y^r \) is unique due to \( S \succ 0 \) and \( u^r \) is unique due to Assumption 2 below. On the other hand, the corresponding solution \( \alpha^r \) is in general not unique and we denote the solution with minimum 2-norm by \( \alpha^r \).

**Assumption 2.** (Unique steady-state) The matrix \( \begin{bmatrix} A - I & B \\ C & D \end{bmatrix} \) has full column rank.

Assumption 2 is a standard condition in tracking MPC (cf. [44, Lemma 1.8], [22, Remark 1]) and it implies the

\[1\] Input setpoints can be included via an augmented output \( y' := \begin{bmatrix} y \\ u \end{bmatrix} \).
existence of an affine (hence Lipschitz continuous) map \( \tilde{g} \) which uniquely maps output equilibria \( y^* \) to their corresponding input-state components \( (x^*, u^*) \), i.e., \( \tilde{g}(y^*) = (x^*, u^*) \) and

\[
\left\| x^*_1 \left[ u^*_1 \right. \right. \left. - x^*_2 \left[ u^*_2 \right. \right. \left. \right. \right\|_2^2 \leq c_g \left\| y^*_1 - y^*_2 \right\|_2^2 \quad (10)
\]

with some \( c_g > 0 \) for any two input-output equilibria \( (u^*_1, y^*_1), (u^*_2, y^*_2) \) with corresponding steady-states \( x^*_1, x^*_2 \). Since \( S \geq 0 \), the cost of (9) is strongly convex in \( y^* \) and for any \( y^* \) satisfying the constraints of (9) for some \( u^*, \alpha^* \), we have

\[
\left\| y^* - y^T \right\|_2^2 - J_{eq} > \left\| y^* - y^sr \right\|_2^2,
\]

(11)

compare \([23, \text{Inequality (11)}]\). Throughout this section and in contrast to Section II, we assume that a noisy input-output trajectory \( \{u^*_k, y^*_k\}_{k=0}^{N-1} \) of (4) is available, where the output

\[
y^*_k = y^d_k + \xi_k, k \in \mathbb{I}_{[0, N-1]},
\]

is perturbed by additive measurement noise \( \{\xi_k\}_{k=0}^{N-1} \). Additionally, the output measurements obtained online which are used to include initial conditions are also affected by noise \( \{\xi_{k}\}_{k=0}^{\infty} \) as \( y_k = y_k + \xi_k, k \in \mathbb{I}_{\geq 0} \).

**Assumption 3. (Noise bound)** It holds that \( \|\xi_{k}\|_{\infty} \leq \varepsilon, k \in \mathbb{I}_{[0, N-1]} \), and \( \|\xi_{k}\|_{\infty} \leq \varepsilon, k \in \mathbb{I}_{\geq 0} \), with \( \varepsilon > 0 \) known.

Further, we require that the data are persistently exciting.

**Assumption 4. (Persistence of excitation)** The data \( \{u^*_k, y^*_k\}_{k=0}^{N-1} \) are persistently exciting of order \( L + n + 1 \) in the sense of Definition 1.

### B. Proposed MPC scheme

Given data \( \{u^*_k, y^*_k\}_{k=0}^{N-1} \) as well as initial conditions \( \{y_{k,0} = y^*_1\}_{k=0}^{N-1} \), the following open-loop optimal control problem is the basis for our MPC scheme:

\[
\min_{\alpha(t), \sigma(t)} \sum_{t=0}^{L} \left\| u_k(t) - u^*(t) \right\|_Q^2 + \left\| \tilde{y}_k(t) - y^*(t) \right\|_Q^2 \quad (12a)
\]

\[
+ \left\| y^*(t) - y^T \right\|_2^2 + \lambda_{\alpha} \varepsilon \left\| \alpha(t) - \alpha^* \right\|_2^2 + \lambda_{\sigma} \varepsilon \left\| \sigma(t) \right\|_2^2
\]

s.t. \[
\begin{align*}
\hat{u}(t) = & \left[ \begin{array}{cc} H_{L+1} & (u(t)) \\ H_{L+n+1} & (y^d(t)) \end{array} \right] (t), \\
\tilde{y}_1(t) + \sigma(t) & = \left[ \begin{array}{c} u(t-n-1, t-n-1) \\ \bar{y}_{t-n-1, t} \end{array} \right], \\
\tilde{y}_2(t) & = \left[ \begin{array}{c} u(t, t-n-1) \\ \bar{y}_{t-n-1, t} \end{array} \right], \\
\bar{u}_k(t) & = \left[ \begin{array}{c} u^*(t) \\ y^*(t) \end{array} \right], k \in \mathbb{I}_{[0, L]},
\end{align*}
\]

(12b)

(12c)

(12d)

(12e)

The cost (12a) contains a tracking term with weights \( Q > 0 \), \( R > 0 \) w.r.t. an artificial setpoint \( (u^*(t), y^*(t)) \) which is optimized online and whose distance to the target setpoint \( y^T \) is penalized in the cost by the weight \( S > 0 \). The fact that the cost (12a) is summed over \( k \in \mathbb{I}_{[-n, L]} \) (instead of \( k \in \mathbb{I}_{[0, L]} \)) as, e.g., in \([9, 42]\) simplifies the theoretical analysis and is analogous to model-based MPC, where the initial condition usually enters the cost. We assume that the prediction horizon \( L \) satisfies \( L \geq 2n \). Similar to existing robust data-driven MPC schemes \([4, 9]\), Problem (12) uses a prediction model based on Hankel matrices (12b). In order to account for the model mismatch due to the noise, the slack variable \( \sigma(t) \) is introduced and the cost (12a) additionally contains regularization terms of \( \alpha(t) \) and \( \sigma(t) \) with parameters \( \lambda_{\alpha}, \lambda_{\sigma}, \beta_{\alpha}, \beta_{\sigma} > 0 \). The regularization depends on the noise bound \( \varepsilon \) such that, in the limit \( \varepsilon \to 0 \), a nominal MPC scheme with guaranteed exponential stability is recovered. In the literature, different choices for \( \beta_{\alpha}, \beta_{\sigma} \) have been considered such as \( \beta_{\alpha} = 1, \beta_{\sigma} = 0 \) (see \([5, 9]\)) or \( \beta_{\alpha} = 1, \beta_{\sigma} = 1 \) (see \([10]\)). In this paper, we require the additional flexibility provided by the parameters \( \beta_{\alpha} \) and \( \beta_{\sigma} \) for an argument in the nonlinear stability proof in Section IV. All theoretical results in the present section remain valid as long as \( \beta_{\alpha} + \beta_{\sigma} < 2 \).

In (12c), the first \( n \) components of the predictions are set to the past \( n \) input-output measurements in order to specify initial conditions, compare \([45]\). We note that \( n \) can be replaced by any upper bound on the system order (or, more specifically, on the system’s lag), i.e., the application of the proposed MPC does not require accurate knowledge of the system order. Moreover, (12d) represents the terminal equality constraint w.r.t. the artificial equilibrium \( (u^*(t), y^*(t)) \). It is defined over \( n + 1 \) steps since this ensures that \( (u^*(t), y^*(t)) \) is an (approximate) equilibrium, compare Definition 2. In order to parametrize trajectories of the unknown affine system (4), the last line of (12b) implies that \( \alpha(t) \) sums up to 1, compare Theorem 1. Further, the scheme contains constraints on the input equilibrium and on the input trajectory in (12e). Note that (12) is a strictly convex QP which can be solved efficiently.

**Remark 1.** Inspired by \([11]\), the regularization of \( \alpha(t) \) is not w.r.t. zero but depends on \( \alpha^sr \) since we want to track the generally non-zero equilibrium \( (u^sr, y^sr) \) and due to the affine system dynamics (cf. Theorem 1). Thus, an implementation of Problem (12) requires knowledge of \( \alpha^sr \). Note that \( \alpha^sr \) can be (approximately) computed as the least-squares solution of (9) based on the available noisy data. For practical purposes, it is beneficial to include a slack variable \( \sigma^* \) as well as regularization of \( \alpha^* \) and \( \sigma^* \) in the cost of (9), exactly as in Problem (12). If an approximation \( \alpha^sr \) of \( \alpha^sr \) with \( \|\alpha^sr - \alpha^sr\|_2 \leq c \) for some \( c > 0 \) is known, the following theoretical results remain true, albeit with more conservative bounds on the noise bound \( \varepsilon \) which deteriorate for increasing values of \( c \). In particular, the presented results hold qualitatively for \( \alpha^sr = 0 \).

The optimal solution of (12) at time \( t \) is denoted by \( \tilde{u}(t), \tilde{y}(t), \tilde{\alpha}(t), \tilde{\sigma}(t), u^*(t), y^*(t) \), and the closed-loop input, state, and output at time \( t \) are denoted by \( u_0, x_t, \) and \( y_t \) respectively. Algorithm 1 summarizes the proposed MPC scheme, which takes a multi-step form.

Considering a multi-step MPC scheme instead of a standard (one-step) MPC scheme simplifies the theoretical analysis with terminal equality constraints due to a local controllability argument in the proof. In \([42]\), guarantees for a one-step tracking MPC scheme for linear systems are provided for noise-free data. We conjecture that the results in this section hold locally close to the steady-state manifold if Algorithm 1 is executed in a one-step fashion, see also \([9, \text{Remark 4}] \) for a similar argument in data-driven MPC without online optimization of \( u^*(t), y^*(t) \).
Algorithm 1. Robust Data-Driven MPC Scheme

**Offline:** Choose upper bound on system order $n$, prediction horizon $L$, cost matrices $Q, R, S > 0$, regularization parameters $\lambda_\alpha, \lambda_\sigma > 0$, constraint sets $\mathbb{U}, \mathbb{U}^n$, noise bound $\bar{\varepsilon}$, setpoint $y^T$, and generate data $\{u^d_k, \tilde{y}^d_k\}_{k=0}^{N-1}$. Compute an approximation of $\alpha^{\sigma r}$ by solving (9), where $\{u^d_k, \tilde{y}^d_k\}_{k=0}^{N-1}$ is replaced by $\{u_k, \tilde{y}_k\}_{k=0}^{N-1}$.

**Online:**

1. At time $t$, take the past $n$ measurements $\{u_k, \tilde{y}_k\}_{k=t-n}^{t-1}$ and solve (12).
2. Apply the input sequence $u[t,t+n-1] = \tilde{u}^*_{[0,n-1]}(t)$ over the next $n$ time steps.
3. Set $t = t + n$ and go back to 1.

**Remark 2.** Although our main motivation for the above MPC scheme and its theoretical guarantees are the applicability to nonlinear systems, which we explore in Section IV, Algorithm 1 has significant advantages over the existing data-driven MPC schemes with closed-loop guarantees in [9], [10]. In particular, since the equilibrium $(u^a(t), y^a(t))$ is optimized online, it is not required to know a priori whether $(u^a, y^a)$ is a feasible equilibrium, the presented scheme is recursively feasible in case of online setpoint changes, and it possesses a significantly improved robustness and larger region of attraction.

A key difficulty in analyzing closed-loop properties of data-driven MPC based on Theorem 1 (and analogously for MPC based on [2]) is that no state measurements are available and the cost only penalizes the input and output. Therefore, our analysis uses the extended state $\xi_t$ and its noisy version $\tilde{\xi}_t$:

$$\xi_t := \begin{bmatrix} u_{[t-n,t-1]}^a \\ y_{[t-n,t-1]}^a \end{bmatrix}, \quad \tilde{\xi}_t := \begin{bmatrix} u_{[t-n,t-1]}^a \\ y_{[t-n,t-1]}^a \end{bmatrix},$$

and we write $J^*_L(\tilde{\xi}_t)$ for the optimal cost of (12). We denote the optimal reachable extended state by $\xi^{\sigma r} := \begin{bmatrix} u^r_n \\ y^r_n \end{bmatrix}$. Further, we note that, by observability, there exist $T_x, T_e, T_r$, such that

$$x_t = T_x \xi_t + T_e e + T_r r.$$  

Thus, for any two pairs of state and extended state vectors ($x^a_t, \xi^a_t$) and ($x^b_t, \xi^b_t$), it holds that

$$x^a_t - x^b_t = T_x (\xi^a_t - \xi^b_t).$$

**C. Continuity of the optimal input and cost w.r.t. noisy data**

In the following, we present our key technical result for the closed-loop robustness analysis. To be precise, we show that the distance of the optimal input/cost of Problem (12) to the corresponding noise-free optimal input/cost can be bounded in terms of $\bar{\varepsilon}$. Given a vector $\tilde{\sigma} = \begin{bmatrix} \bar{\sigma}_{\text{init}} \\ \bar{\sigma}_{\text{dyn}} \end{bmatrix} \in \mathbb{R}^{p(L+2n+1)}$, we define the optimization problem

$$\min \left\{ \frac{1}{2} \sum_{k=-n}^{L} \|\tilde{u}_k(t) - u^a(t)\|_Q^2 + \|\tilde{y}_k(t) - y^a(t)\|_Q^2 + \|\tilde{y}_k(t) - y^a(t)\|_Q^2 + \|y^a(t) - y^T\|^2 + \lambda_\varepsilon \bar{\varepsilon} \|\alpha(t) - \alpha^{\sigma r}\|^2 \right\}$$

$$s.t. \begin{bmatrix} \tilde{u}(t) \\ \tilde{y}(t) + \tilde{\sigma}_{\text{dyn}} \\ H_{L+n+1} \begin{bmatrix} u^d \end{bmatrix} + \tilde{\sigma}_{\text{init}} \end{bmatrix} = \begin{bmatrix} H_{L+n+1} \begin{bmatrix} y^d \end{bmatrix} \alpha, \end{bmatrix}$$

$$\tilde{u}[-n,-1] = \begin{bmatrix} u[0,n-1] \\ y[0,n-1] \end{bmatrix}, \quad \tilde{y}[-n,-1] = \begin{bmatrix} y[0,n-1] + \tilde{\sigma}_{\text{init}} \end{bmatrix}, \quad \tilde{u}[L+n,L] = \begin{bmatrix} u_{n+1}^* \\ y_{n+1}^r \end{bmatrix}, \quad \tilde{y}[L+n,L] = \begin{bmatrix} y_{n+1}^r \end{bmatrix}, \quad \tilde{u}_k(t) \in \mathbb{U}, \quad k \in [0,L], \quad u^a(t) \in \mathbb{U}^n.$$

We denote the optimal solution of (16) with $\alpha = \alpha^a(t)$, $\tilde{u}^a(t)$, $\tilde{y}^a(t)$, $\tilde{u}^s(t)$, $\tilde{y}^s(t)$, and the corresponding cost by $J^*_L(\tilde{\xi}_t)$. For $\alpha = 0$, the constraints of (16) correspond to those of Problem (12) with noise-free data and $\sigma(t) = 0$ such that $(\tilde{u}^*(t), \tilde{y}^*(t))$ is an input-output trajectory of (4). In the following result, we derive a bound on the difference between the nominal optimal cost $J^*_L(\xi(t))$ and input $\tilde{u}^a(t)$ corresponding to Problem (12).
where \(\|\tilde{y}_{t+k} - y_{t+k}\|_{Q}^2 \leq p\lambda_{\text{max}}(Q)\varepsilon^2\). Further, we have
\[
\|\sigma(t)\|_2^2 \leq 2\|H_{L+n+1}(\varepsilon^d)\|_2^2\|\hat{\alpha}^*(t)\|_2^2 + 2np\varepsilon^2
\]
(22)
for a suitably defined \(c_\sigma > 0\). Using \(\lambda_n\varepsilon^{\beta_n}\|\hat{\alpha}^*(t) - \alpha^{**}\|_2^2 \leq \tilde{J}_L(\xi)\) and (1), we also have
\[
\|\hat{\alpha}^*(t)\|_2^2 \leq \frac{2}{\lambda_n\varepsilon^{\beta_n}}\tilde{J}_L(\xi) + 2\|\alpha^{**}\|_2^2.
\]
(23)
Plugging (21)–(23) into (20) and using
\[
\varepsilon^{\beta_n} \sum_{k=-n}^{-1}\|\tilde{y}_{t+k} - u^{**}(t)\|_2^2 \leq \varepsilon^{\beta_n}\tilde{J}_L(\xi),
\]
we obtain (17) for appropriately defined \(c_{I,a}, c_{I,b} > 0\).

(ii) **Proof of (18)**

(ii.a) **Strong convexity of Problem (12)**

Exploiting the initial condition (12c) and the terminal constraint (12d), the terms in (12a) involving \(\tilde{u}(t)\) and \(u^*(t)\) are
\[
\sum_{k=-n}^{-1}\|\tilde{u}_{t+k} - u^*(t)\|_R^2 + \sum_{k=0}^{L-n-1}\|\tilde{u}_k(t) - u^*(t)\|_R^2.
\]
(24)

The Hessian w.r.t. \([\tilde{u}_{0:L-n-1}(t) u^*(t)]\) is equal to
\[
[I_{L-n} \otimes R - \hat{I}_{L-n} \otimes R]L R
\]
which is positive definite since its Schur complement w.r.t. the left upper block is
\[
LR - (I_{L-n} \otimes R)(I_{L-n} \otimes R)^{-1}(I_{L-n} \otimes R) = nR > 0.
\]
(25)

Thus, Problem (12) is strongly convex in \([\tilde{u}_{0:L-n-1}(t) u^*(t)]\) and hence, using \(\tilde{u}_k(t) = u^*(t)\) for \(k \in \{L-n, L\}\) due to (12d), it is strongly convex in \(u^*(t)\). We note that \(\tilde{u}_k(t) = u^*(t)\) for \(k \in \{L-n, L\}\) such that (18) trivially holds over the first \(n\) time steps and therefore, we focus on \(k \in \{0, L\}\) in the following.

(ii.b) **Bound on \(\|u^*(t) - \tilde{u}(t)\|_2^2\)**

We denote the optimal solution of (16) with
\[
\tilde{\sigma} = \tilde{\sigma}_\varepsilon := \left[\sigma^*(t) - H_{L+n+1}(\varepsilon^d)\alpha^*(t)\right]
\]
(26)
by \(\tilde{\alpha}(t), \tilde{u}^*(t), \tilde{y}^*(t), \tilde{u}(t), \tilde{y}(t)\), and the corresponding cost by \(\tilde{J}_L\). Since \(\tilde{J}_L(\xi) \leq \tilde{J}\) by assumption and using (17) by Part (i), we infer \(\tilde{J}_L(\xi) < \infty\), i.e., Problem (12) is feasible. The optimal solution of Problem (12) is a feasible (but not necessarily optimal) solution of (16) with \(\tilde{\sigma} = \tilde{\sigma}_\varepsilon\), i.e.,
\[
\tilde{J}_L \leq J^*_L(\xi) - \frac{\lambda_\sigma}{\varepsilon^{\beta_n}}\|\sigma^*(t)\|_2^2
\]
(27)

We now construct a candidate solution for Problem (12) based on the optimal solution of (16) with \(\tilde{\sigma} = \tilde{\sigma}_\varepsilon\). To this end, we choose \(\tilde{u}(t) = \tilde{u}(t), \tilde{y}(t) = \tilde{y}(t), \alpha(t) = \tilde{\alpha}(t), u^*(t) = \tilde{u}^*(t), y^*(t) = \tilde{y}^*(t), \) and
\[
\sigma(t) = \sigma^*(t) + H_{L+n+1}(\varepsilon^d)(\tilde{\alpha}(t) - \alpha^*(t)).
\]
(28)

Note that \((\alpha(t), u^*(t), y^*(t), \tilde{u}(t), \tilde{y}(t))\) is a feasible solution of (12), whose cost we denote by \(J'_L\), which satisfies \(J'_L \geq J^*_L(\xi)\) by optimality. By definition,
\[
J'_L - J_L = \frac{\lambda_\sigma}{\varepsilon^{\beta_n}}\|\sigma(t)\|_2^2.
\]
(29)

Moreover, it follows from the definition of \(J^*_L(\xi)\) that
\[
\|\tilde{\alpha}^*(t) - \alpha^*(t)\|_2^2 \leq J'_L(\xi)\frac{J^*_L(\xi)}{\lambda_\sigma},
\]
(30)

Similarly, we have
\[
\|\tilde{\sigma}^*(t) - \sigma^*(t)\|_2^2 \leq \frac{J'_L(\xi)}{\lambda_\sigma},
\]
(31)

we have
\[
\|\tilde{u}(t) - u^*(t)\|_2^2 \leq \frac{J'_L(\xi)}{\lambda_\sigma},
\]
(32)

Using that the solution with cost \(J'_L\) is feasible for Problem (12), strong convexity as shown in Part (i) implies
\[
\|\tilde{u}(t) - u^*(t)\|_2^2 \leq c_{u,a}(J'_L - J^*_L(\xi))
\]
for some \(c_{u,a} > 0\), compare [23, Inequality (11)]. Together with \(\tilde{u}(t) = \tilde{u}(t)\), this implies
\[
\|\tilde{u}(t) - u^*(t)\|_2^2 \leq c_{u,a}(J'_L - J^*_L(\xi))
\]
(33)

(ii.c) **Bound on \(\|\tilde{u}(t) - \tilde{u}^*(t)\|_2^2\)**

Denoting \(U = \{u \in \mathbb{R}^m \mid H_u u \leq g_u\}\), we assume w.l.o.g. that all rows of \(H_u\) are linearly independent, and similarly for \(U^*\). Since Problem (16) is a multi-parametric QP, the optimal solution of (16) is continuous and piecewise affine in \((\xi, \tilde{\sigma})\), compare [46]. In particular, using that the partition of the optimal solution admits finitely many regions, it is uniformly Lipschitz continuous in \(\tilde{\sigma}\). Moreover, Problem (16) is feasible for \(\tilde{\sigma} = 0\) by assumption, i.e., by \(J^*_L(\xi) \leq \tilde{J}\), and for \(\tilde{\sigma} = \tilde{\sigma}_\varepsilon\) since Problem (12) is feasible, cf. Part (i). Thus, there exists \(c_{u,4} > 0\) such that
\[
\|\tilde{u}(t) - u^*(t)\|_2^2 \leq c_{u,4}\|\tilde{\sigma}_\varepsilon\|_2^2
\]
(34)

The definition of \(\tilde{\sigma}_\varepsilon\) in (24) together with (28), (29), and the triangle inequality implies
\[
\|\tilde{\sigma}_\varepsilon\|_2^2 \leq c_{u,4}\varepsilon + \varepsilon^{\beta_n} \frac{J^*_L(\tilde{\xi})}{\lambda_\sigma} + c_{u,5}\varepsilon^{1 - \frac{\beta_n}{4}}\frac{J^*_L(\tilde{\xi})}{\lambda_\sigma}
\]
(35)
for some $c_{g,4}, c_{g,5} > 0$. Combining (17), (35), (36), and $\bar{J}_L(\xi_{t}) \leq \bar{J}$, the term $\|\bar{u}(t) - \bar{u}^*(t)\|_2$ is bounded by $\beta_t(\xi)$ with some function $\beta_t$. It is simple to verify that $\beta_t \in K_c$ if $\beta_\alpha + \beta_\sigma < 2$. Together with (17), (34), and $\bar{J}_L(\xi_{t}) \leq \bar{J}$, this implies the existence of $\beta_t \in K_c$, such that (18) holds.

Proposition 1 bounds the difference between the optimal input/cost for the robust MPC problem (12) with noisy data and the optimal input/cost for the nominal MPC problem (16) with noise-free data, i.e., with $\tilde{\sigma} = 0$. The bounds depend on the noise level $\tilde{\sigma}$ and thus, the result can be seen as a continuity property of the optimal solution of Problem (12) w.r.t. noise. In Part (i) of the proof, we bound the optimal cost $\bar{J}_L(\xi_{t})$ in terms of $\bar{J}_L(\xi_{t})$ using a simple candidate solution for $\sigma(t)$, i.e., we show that the robust MPC problem (12) is feasible whenever the nominal problem is feasible. Next, strong convexity of Problem (12) w.r.t. the input is shown (Part (ii).a). In Part (ii).b, we employ the optimal input $\bar{u}(t)$ for Problem (16) with $\tilde{\sigma} = \bar{\sigma}_t$ (with cost $\bar{J}_L$) to construct a feasible solution for Problem (12) with cost $\bar{J}_L$ close to $\bar{J}_L + \frac{\bar{\sigma}_t}{\varrho}\|\sigma^*(t)\|_2^2$. Using the sandwich inequality (32), $\bar{J}_L$ is close to $\bar{J}_L(\xi_{t})$, which, together with strong convexity, implies that $\bar{u}(t)$ is close to $\bar{u}^*(t)$. In Part (iii).c, we exploit that the optimal solution of Problem (16) depends in a piecewise affine fashion on $\sigma$ and thus, the difference between the optimal input for $\tilde{\sigma} = 0$ (i.e., $\bar{u}^*(t)$) and $\tilde{\sigma} = \bar{\sigma}_t$ (i.e., $\bar{u}(t)$) is bounded in terms of $\tilde{\sigma}$, which then implies that $\bar{u}^*(t)$ and $\bar{u}^*(t)$ are close, cf. (18). In Proposition 2 (see Appendix C), we show that a result analogous to Proposition 1 holds in the presence of additional input disturbances affecting the initial conditions.

### D. Closed-loop guarantees

Proposition 1 shows that the noisy output measurements in Problem (12) can be translated into an additive input disturbance for data-driven MPC with noise-free data. In the following, we show that the latter scheme is robust w.r.t. input disturbances (Theorem 2) which we then combine with Proposition 1 to conclude practical exponential stability of the closed loop under the robust data-driven MPC scheme in Algorithm 1 (Corollary 1). For the stability analysis, we consider the Lyapunov function candidate $V(\xi_{t}) := \bar{J}_L(\xi_{t}) - J_{\min}$. 

**Theorem 2.** Let Assumptions 1, 2, 3, and 4 hold and suppose $\beta_\alpha + \beta_\sigma < 2$. Then, for any $V_{\text{ROA}} > 0$, there exist $\tilde{\varepsilon}_{\text{max}} > 0$ and $\beta_V \in K_c$ such that, for all initial conditions with $V(\xi_{0}) \leq V_{\text{ROA}}$ and all $\tilde{\varepsilon} \leq \tilde{\varepsilon}_{\text{max}}$, the closed-loop trajectory under Algorithm 1 satisfies

$$V(\xi_{t+n}) \leq \tilde{c}_V V(\xi_{t}) + \beta_V(\tilde{\varepsilon})$$

(40) for all $t = n, \ i \in \mathbb{I}_{\geq 0}$, with $\tilde{c}_V$ as in (39).

**Proof.** If $\tilde{\varepsilon} \leq \tilde{\varepsilon}_{\text{max}}$ is sufficiently small, then Proposition 1 and Theorem 2 imply (40) for $t = 0$ with $\beta_V := \beta_\alpha \circ \beta_\sigma$. Further, with $\tilde{\varepsilon}$ sufficiently small, we have $V(\xi_{t+n}) \leq V_{\text{ROA}}$ such that the argument can be applied recursively and (40) holds for all $t = n, \ i \in \mathbb{I}_{\geq 0}$.

Corollary 1 shows that the closed loop under Algorithm 1 exponentially converges to a neighborhood of the optimal reachable equilibrium whose size increases with $\tilde{\varepsilon}$. The result is a simple consequence of the facts that the noise in Problem (12) can be translated into an input disturbance for nominal data-driven MPC (Proposition 1) and the closed loop under the latter is practically stable w.r.t. the disturbance bound (Theorem 2). Roughly speaking, the analysis presented in this section reveals a separation principle of data-driven MPC with noise-free and noisy data. That is, any data-driven MPC scheme whose nominal version is robust w.r.t. input disturbances will also lead to a practically stable closed loop in the presence of noisy output measurements affecting both the offline data in the Hankel matrices and the online data used to specify initial conditions. More precisely, this analysis can
also be applied to simplify the robust stability proofs for data-driven MPC schemes with [9] or without [10] terminal equality constraints. Similarly, we conjecture that robustness of a one-step data-driven MPC scheme with the terminal ingredients from [47] can be proven, given the inherent robustness results due to terminal ingredients shown in [48].

The main contribution of this paper is a data-driven MPC scheme to control unknown nonlinear systems based only on measured input-output data with closed-loop guarantees. In this context, the analysis in this section provides a technical intermediate step since similar ideas and in particular Proposition 2 (see Appendix C), which extends Proposition 1 to input disturbances, will be used in the following section, where ε takes the role of the error induced by the local linear approximation of a nonlinear system. Nevertheless, the theoretical guarantees provided by Corollary 1 are also a significant improvement over existing works [9], [10] on closed-loop stability and robustness in data-driven MPC for linear systems (cf. Remark 2).

IV. DATA-DRIVEN MPC FOR NONLINEAR SYSTEMS

In this section, we present a data-driven MPC scheme to control unknown nonlinear systems with closed-loop guarantees. After describing the problem setting in Section IV-A, we introduce assumptions on the system in Section IV-B. The MPC scheme is presented in Section IV-C and its theoretical properties are analyzed in Sections IV-D and IV-E. The results in this section are built on the MPC approach based on linearized models in our companion paper [20] with the additional difficulty that the model of the linearization is unknown and only past input-output measurements are available.

A. Problem setting

We consider unknown nonlinear systems of the form
\[
\begin{align*}
x_{k+1} &= f(x_k, u_k) = f_0(x_k) + g(x_k)u_k, \\
y_k &= h(x_k, u_k) = h_0(x_k) + h_1(x_k)u_k
\end{align*}
\]  
(41)

with state \(x_k \in \mathbb{R}^n\), input \(u_k \in \mathbb{R}^m\), and output \(y_k \in \mathbb{R}^p\), all at time \(k \in \mathbb{N}_0\), and vector fields \(f_0 : \mathbb{R}^n \to \mathbb{R}^n\), \(g : \mathbb{R}^n \to \mathbb{R}^{n \times m}\), \(h_0 : \mathbb{R}^n \to \mathbb{R}^p\), \(h_1 : \mathbb{R}^n \to \mathbb{R}^{p \times m}\).

System (41) is subject to pointwise-in-time input constraints \(u_t \in \mathbb{U}\) for \(t \in \mathbb{N}_0\) with some convex, compact polytope \(\mathbb{U}\). Similar to [20], output constraints can be satisfied in closed loop if the steady-state manifold lies strictly inside the constraints. For some convex polytope \(\mathbb{U}^o \subseteq \text{int}(\mathbb{U})\), which is required for a local controllability argument, we define the steady-state manifold and its projection onto the output by
\[
\begin{align*}
Z^o_x &:= \{ (x^*, u^*) \in \mathbb{R}^n \times \mathbb{U}^o | x^* = f(x^*, u^*) \}, \\
Z^o_y &:= \{ y^* \in \mathbb{R}^p | y^* = h(x^*, u^*), (x^*, u^*) \in Z^o_x \}.
\end{align*}
\]

Our control goal is tracking of a target setpoint \(y^T \in \mathbb{R}^p\). In general, \(y^T \notin Z^o_y\) such that our scheme will guarantee stability of the optimal reachable equilibrium \(x^{sr}\), which is the minimizer of
\[
J^*_{eq, NL} := \min_{y^* \in Z^o_y} \|y^* - y^T\|^2_2
\]  
(42)

with some \(S > 0\). Assumptions made later will imply that this minimizer and the corresponding input-state pair \((x^{sr}, u^{sr})\) are unique, and we denote by \(\xi^s\) the corresponding extended state, cf. (13). We assume that all vector fields in (41) are continuously differentiable and we define
\[
\begin{align*}
A_x &:= \frac{\partial f_0}{\partial x}, \quad B_x := g(x), \quad e_x := f_0(x) - A_x x, \\
C_x &:= \frac{\partial h_0}{\partial x}, \quad D_x := h_1(x), \quad r_x := h_0(x) - C_x x
\end{align*}
\]  
(43)

for some linearization point \(x \in \mathbb{R}^n\). Moreover, we define the affine dynamics resulting from the linearization of (41) at \((x, u) = (\hat{x}, 0)\) by \(f_x(x, u) := A_x \hat{x} + B_x u + e_x\) and \(h_x(x, u) := C_x \hat{x} + D_x u + r_x\). Let now \(u'_k, y'_k\in \mathbb{R}^{m+1}\) be a trajectory of the linearized dynamics at some \(\hat{x} \in \mathbb{R}^n\) (i.e., a trajectory of (41), replacing \(f\) and \(h\) by \(f_x\) and \(h_x\), respectively), whose input-state component is persistently exciting in the sense of Definition 1. We then define the optimal steady-state problem for the linearized dynamics by
\[
J_{eq}^*(\hat{x}) := \min_{u^s, y^s, \alpha^s} \left\| y^s - y^T \right\|^2_2
\]
\[
\text{s.t.} \quad \begin{bmatrix} H_{L+n+1}(u') \\ H_{I_{L+n+1}}(y') \end{bmatrix} \alpha^s = \begin{bmatrix} u^s_{L+n+1} \\ 1 \end{bmatrix}, \quad u^s \in \mathbb{U}^s.
\]  
(44)

We write \(u^{sr}(\hat{x}), y^{sr}(\hat{x})\) for the optimal solution, \(x^{sr}(\hat{x})\) for the (extended) steady-state of the linearized dynamics, and \(\alpha^{sr}(\hat{x})\) for the corresponding optimal solution with minimum 2-norm.

B. Assumptions

Since our analysis relies on similar arguments as our companion paper [20], we require the following assumptions.

**Assumption 5.** System (41) satisfies [20, Assumptions 1-5].

Through [20, Assumption 1], we assume that all vector fields in (41) are twice continuously differentiable. Moreover, [20, Assumptions 2 and 3] require that the linearized dynamics are controllable (cf. Assumption 1) and satisfy the tracking condition in Assumption 2 at any linearization point. In [20, Assumption 4], we assume that \(I - A_x\) is non-singular for any \(\hat{x} \in \mathbb{R}^n\), and [20, Assumption 5] requires that the union of all steady-state manifolds of the linearized dynamics are compact. We refer to [20] for a more detailed discussion of these assumptions. We also work with the extended state vector \(\xi_t\) in (13) on which we make the following assumption.

**Assumption 6.** (Observability) There exists a locally Lipschitz continuous map \(T_L : \mathbb{R}^{n(m+p)} \to \mathbb{R}^n\) such that
\[
x_t = T_L(\xi_t).
\]  
(45)

Assumption 6 is connected to (final state) observability of the nonlinear system (41), compare, e.g., [44, Definition 4.29] and [49, Definition 2.4] for similar conditions. In this section, Assumption 6 replaces Assumption 1 on observability, i.e., (15), which was used for the analysis in Section III.
Assumption 7. (Convex steady-state manifold) For any compact set \( \Xi \), there exist constants \( c_{\text{eq,1}}, c_{\text{eq,2}} > 0 \) such that, for any extended state \( \hat{x} \in \Xi \) it holds that
\[
c_{\text{eq,1}} \| \hat{x} - \hat{x}^{*r}(\hat{x}) \|_2^2 \leq \| \hat{x} - \xi^{*r}(\hat{x}) \|_2^2 \leq c_{\text{eq,2}} \| \hat{x} - \xi^{*r}(\hat{x}) \|_2^2,
\]
(46)
where \( \hat{x} = T_L(\hat{x}) \), compare (45).

Assumption 7 is related to convexity of the steady-state manifold and it is analogous to [20, Assumption 6], where Inequality (46) is assumed for the state \( \hat{x} \) and the optimal reachable steady-states \( x^{*r}, x^{*r}(\hat{x}) \) (see [20] for details and for sufficient conditions). In the data-driven framework, we require Assumption 7 since the proposed data-driven MPC scheme only involves input-output values and hence, our theoretical analysis relies on the extended state \( \hat{x} \).

C. Proposed MPC scheme

At time \( t \geq N \), given past \( N \) input-output measurements \( \{u_k, y_k\}_{k=-N}^{-1} \) of the nonlinear system (41), we define the following open-loop optimal control problem
\[
\min_{\alpha(t), \sigma(t)} \sum_{n=-N}^{N} \| u_k(t) - u^*(t) \|_I^2 + \| y_k(t) - y^*(t) \|_Q^2 + \| y^*(t) - y^T \|_S^2 + \lambda_\alpha \| \alpha(t) - \alpha^{*r}(x_t) \|_2^2 + \lambda_\sigma \| \sigma(t) \|_2^2 \nonumber
\]
\[
\text{s.t.} \quad \begin{bmatrix}
\hat{u}(t) \\
\hat{y}(t) + \sigma(t)
\end{bmatrix} = \begin{bmatrix}
H_{L+1} & u_{[t-N,t-1]} \\
H_{L+1} & y_{[t-N,t-1]}
\end{bmatrix} \begin{bmatrix}
\alpha(t) \\
1
\end{bmatrix},
\]
\[
\begin{bmatrix}
\hat{u}_{[t-n,1]}(t) \\
\hat{y}_{[t-n,1]}(t)
\end{bmatrix} = \begin{bmatrix}
u_{[t-n,1]} \\
y_{[t-n,1]}
\end{bmatrix},
\]
\[
\begin{bmatrix}
\hat{u}_{L,n}(t) \\
\hat{y}_{L,n}(t)
\end{bmatrix} = \begin{bmatrix}
u_{n+1} \\
y_{n+1}
\end{bmatrix},
\]
\[
u_k(t) \in U, \quad k \in \mathbb{I}_{[0,L]}, \quad u^*(t) \in U^*. \quad (47a)
\]

Problem (47) is similar to the robust MPC scheme for affine systems (12) with weighting matrices \( Q, R, S > 0 \), prediction horizon \( L \geq 2n \), and regularization parameters \( \lambda_\alpha, \lambda_\sigma > 0 \). The key difference is that the data used for prediction in (47a) consist of past \( N \) measurements of the nonlinear system (41). The prediction model of Problem (47) can be seen as an approximation of the affine dynamics resulting from the linearization at \( x_t \), which in turn provides a local approximation of the nonlinear dynamics (41). As we show later in this section, under suitable assumptions on the design parameters and the initially collected data, and if the initial state is sufficiently close to the steady-state manifold, then this approximate model is sufficiently accurate such that closed-loop practical stability can be guaranteed.

Note that the variable \( \alpha \) is regularized w.r.t. \( \alpha^{*r}(x_t) \), i.e., the minimum norm solution of (44). While \( \alpha^{*r}(x_t) \) is generally unknown, it can be computed approximately based on (44) similar to the affine MPC in Section III by inserting the past \( N \) input-output measurements of the nonlinear system (41). As for the results in Section III, our qualitative theoretical results remain true if \( \alpha^{*r}(x_t) \) is replaced by an approximation, and satisfactory performance can often be achieved in practice by simply replacing \( \alpha^{*r}(x_t) \) in (47) with 0. Finally, Problem (47) requires noise-free input-output data of the nonlinear system (41). This is mainly assumed for simplicity and we conjecture that an extension of our results to practical stability w.r.t. noise affecting the measured output is straightforward.

We denote the optimal solution of (47) at time \( t \) by \( \hat{u}^*(t), \hat{y}^*(t), \alpha^*(t), \sigma^*(t), u^*(t), y^*(t), \) and the closed-loop input, state, and output at time \( t \) by \( u_t, x_t, \) and \( y_t \), respectively. Further, we write \( J^*_t(\xi_t) \) for the corresponding optimal cost.

Algorithm 2. Nonlinear Data-Driven MPC Scheme

Offline: Choose upper bound on system order \( n \), prediction horizon \( L \), cost matrices \( Q, R, S > 0 \), regularization parameters \( \lambda_\alpha, \lambda_\sigma > 0 \), constraint sets \( U, U^* \), setpoint \( y_T \), and generate data \( \{u_k, y_k\}_{k=-N}^{N-1} \).

Online:
1. At time \( t \geq N \), compute an approximation of \( \alpha^{*r}(x_t) \) by solving (44), where \( \{u_k, y_k\}_{k=-N}^{N-1} \) is replaced by \( \{u_k, y_k\}_{k=-N}^{N-1} \).
2. Solve (47).
3. Apply the input sequence \( u[t,t+n-1] = \hat{u}^*_{[0,n-1]}(t) \) over the next \( n \) time steps.
4. Set \( t = t + n \) and go back to 1.

D. Prediction error bound due to nonlinearity

A key issue, which complicates the analysis compared to the model-based MPC in [20], is that the prediction model (47a) is not an accurate description of the linearized dynamics of (41) at \( x_t \). In the following, we quantify the prediction error due to this mismatch. We write \( \{x^*_k(t)\}_{k=-N}^{N}, \{\xi^*_k(t)\}_{k=-N}^{N}, \) and \( \{y^*_k(t)\}_{k=-N}^{N-1} \) for the state, extended state, and output corresponding to the dynamics linearized at \( x_t \) resulting from an application of the closed-loop input \( \{u_k\}_{k=-N}^{t} \) with the initial state of the nonlinear system \( x_{t-N} \) at time \( t - N \). To be precise, \( x_{t-N} = x_{t-N} \) and
\[
x_{k+1} = A x_k + B x_k u_{t+k} + C x_k, \quad y_k = C x_k u_k + y_{t+k} + r_x,
\]
for \( k \in \mathbb{I}_{[-N,n-1]} \). The following result provides a bound on the difference between the (known) output of the nonlinear system \( y \) and the (unknown) output of the affine dynamics \( y^* \).
Lemma 1. Let Assumption 5 hold. For any compact set \( X \subset \mathbb{R}^n \), there exists \( c_\Delta > 0 \) such that for any \( t \geq N \), \( k \in \mathbb{I}_{[-N,N-1]} \), and \( x_k, x_{t} \in X \), \( \Delta_{t,k} := y_{t+k} - y_{t+k}^*(t) \) satisfies

\[
\| \Delta_{t,k} \|_2 \leq c_\Delta \sum_{j=t-N}^{t+k} \| x_{t+j} - x_{j} \|_2^2.
\]

Proof. Using \( f(x,u) = f_x(x,u), h(x,u) = h_x(x,u) \), we have for \( k \in \mathbb{I}_{[t-N,t+n-1]} \)

\[
x_{k+1} = A_{x,k} x_{k} + B_{x,k} u_{k} + e_{x,k},
\]

(49)

\[
= A_{x,k} x_{k} + B_{x,k} u_{k} + e_{x,k} + (A_{x,k} - A_{x,k}) x_{k} + (B_{x,k} - B_{x,k}) u_{k} + e_{x,k} - e_{x,k},
\]

(50)

\[
y_{k} = C_{x,k} x_{k} + D_{x,k} u_{k} + r_{x,k}
\]

Using smoothness of the dynamics in (41), we can apply [20, Inequality (7)] and an analogous inequality for \( h \) to derive

\[
\| \Delta_{x,k} \|_2 = \| f_{x,k}(x_{k}, u_{k}) - f_{x}(x_{k}, u_{k}) \|_2 \leq c_X \| x_{t} - x_{k} \|_2^2,
\]

\[
\| \Delta_{y,k} \|_2 = \| h_{x,k}(x_{k}, u_{k}) - h_{x}(x_{k}, u_{k}) \|_2 \leq c_{X,h} \| x_{t} - x_{k} \|_2^2
\]

for \( k \in \mathbb{I}_{[t-N,t+n-1]} \) and with suitable \( c_X, c_{X,h} > 0 \), which implies the existence of \( c_\Delta > 0 \) such that (48) holds. \( \square \)

Lemma 1 shows that the difference between \( y_{t+k} \) and \( y_{t+k}^*(t) \), i.e., the “output measurement noise” in Problem (47), is bounded by the squared distance of \( x_{t} \) to the past states. This means that, if the closed loop does not move too rapidly, then the prediction model of the proposed MPC scheme is an accurate approximation of the dynamics linearized at \( x_{t} \).

E. Closed-loop guarantees

Before presenting our main theoretical result, we make an additional assumption on closed-loop persistence of excitation.

Assumption 8. (Closed-loop persistence of excitation) There exists \( c_H > 0 \) such that, for all \( t \geq N \), the matrix

\[
H_{ux,t} := \begin{bmatrix}
H_{L+N+1}(u_{[t-N,N-1]})
\end{bmatrix},
\]

(51)

has full row rank and \( \| H_{ux,t}^T \|_2 \leq c_H \), where \( x'(t) \) is the state trajectory of the dynamics linearized at \( x_{t} \) with initial condition \( x_{t-N} \) (compare Section IV-D).

Assumption 8 ensures that the persistence of excitation condition in Assumption 4, which is required for the results in Section III, holds uniformly in closed loop. To be precise, in addition to \( H_{ux,t} \) having full row rank, we also require a uniform upper bound on \( \| H_{ux,t}^T \|_2 \) which holds if the singular values of \( H_{ux,t} \) are uniformly lower and upper bounded. Assumption 8 is crucial for our theoretical results since, if the data used for prediction are updated online, they may in general not be persistently exciting and can thus lead to inaccurate predictions. We note that Assumption 8 can be restrictive and may not be satisfied in practice, in particular upon convergence of the closed loop. Nevertheless, there are various pragmatic approaches to ensure the availability of persistently exciting data for our purposes such as (i) stopping the data updates as soon as a neighborhood of the setpoint is reached, (ii) adding suitable excitation signals to the closed-loop input applied to the system, or (iii) incentivizing persistently exciting inputs similar to adaptive MPC approaches [50]. Ensuring closed-loop persistence of excitation within our data-driven MPC framework for nonlinear systems is beyond the scope of this paper and we plan to investigate this issue in future research.

For our analysis, we define an extended state \( \xi_{t} \) corresponding to \( x_{t} \) in the dynamics linearized at \( x_{t} \), i.e.,

\[
x_{t} = T_{x}(x_{t}) \xi_{t} + T_{cr}(x_{t}),
\]

(52)

where \( T_{x}(x_{t}) \) and \( T_{cr}(x_{t}) \) correspond to \( T_{x} \) and \( T_{cr} \) in (14) for the affine dynamics of the linearization at \( x_{t} \), compare the proof of Theorem 3 for the precise definition. In the following result, we use \( V(\xi_{t}, D_{t}) := J_{t}^{*}(\xi_{t}, D_{t}) - J_{t}^{*}(x_{t}) \), where \( J_{t}^{*}(\xi_{t}, D_{t}) \) is the optimal cost of the nominal data-driven MPC problem (16) with \( \bar{s} = 0 \) and with initial condition \( \xi_{t} \), where the data set \( D_{t} := \{ u_{t+k}, y_{t+k}(t) \}_{k=-N}^{N-1} \)

(53)

for \( k \in \mathbb{I}_{[-N,t+n-1]} \) and with suitable \( c_{X}, c_{X,h} > 0 \), which implies the existence of \( c_\Delta > 0 \) such that (48) holds. \( \square \)

Theorem 3. Let Assumptions 5, 6, and 7 hold. There exists \( \theta > 0 \) such that for any \( \theta \in (0, \bar{\theta}) \), if Assumption 8 holds with \( c_{H} = \frac{c^{\sigma,0}}{\theta} \) for some \( c_{\sigma,0} > 0 \), then there exist \( V_{ROA}, S, C > 0 \), regularization parameters \( \lambda_{\alpha}, \lambda_{\sigma} > 0 \), and \( 0 < c_{V} < 1 \), \( \beta_{0} \in \mathbb{K}_{\infty} \), such that, if

\[
V(\xi_{N}, D_{N}) \leq V_{ROA}, \lambda_{\max}(S) \leq \bar{S}
\]

(54)

then, for any \( t = N + ni, i \in \mathbb{I}_{[0,N-1]} \), Problem (47) is feasible and the closed loop under Algorithm 2 satisfies

\[
\| \xi_{t} - \xi_{t+1} \|_2^2 \leq c_{V} C \| \xi_{N} - \xi_{t+1} \|_2^2 + \beta_{0}(\theta).
\]

(55)

Proof. After stating some preliminaries in Part (i), we show in Part (ii) that, if \( \| \xi_{t+k} - \xi_{t+k-j} \|_{2}, j \in \mathbb{I}_{[0,n-1]} \) is bounded by some constant for \( k = t - N, t - N + n, \ldots, t - n \), then it is bounded by the same constant for \( k = t \). In combination with Lemma 1, this serves as a bound on the “noise” affecting the data in Problem (47), which we combine with the robustness of data-driven MPC in Section III and the results in our companion paper [20] to prove practical stability in Part (iii).

For simplicity, we assume w.l.o.g. that \( N = N_{i} \) in \( \mathbb{I}_{[0,N-1]} \), i.e., \( N \) is divisible by \( n \). Note that \( \xi_{N} \) lies in the set \( \{ \xi' | V(\xi', D_{N}) \leq V_{ROA} \} \), which is compact due to the lower bound (38) and Assumption 5, i.e., compactness of the (linearized) steady-state manifold [20, Assumption 5]. Similar to the proof of [20, ...
Proposition 2]. Lipschitz continuity of the dynamics (41) and compactness of \( U \) imply that the union of the \( N \)-step reachable sets of the linearized and the nonlinear dynamics (compare [51]) starting in \( \{ \xi' \mid V(\xi', D_N) \leq V_{ROA} \} \), which we denote by \( X \), is compact. In the proof, we show that \( \{ \xi' \mid V(\xi', D_N) \leq V_{ROA} \} \) is positively invariant and thus, the bound in Lemma 1 as well as Lipschitz continuity of the map (45) hold uniformly throughout the proof. Similarly, whenever we apply Proposition 2, we use that the bounds hold uniformly for all linearized dynamics considered in the proof (after potential modification of some constants).

(i) Preliminaries
Let (47) be feasible at time \( t = ni \geq N \) and suppose
\[
\|\xi_{k+n} - \xi_{k+1}\|_2 \leq c_{\xi, 0} \theta, \quad j \in \mathbb{I}_{[0, n-1]},
\]
(56)
for \( k = t - N, t - N + n, \ldots, t - n \) with some \( c_{\xi, 0} > 0 \). Note that this holds at initial time \( t = N \) with \( c_{\xi, 0} = 2 \) due to \( \|\xi_N - \xi_k\|_2 \leq \theta \) for \( k \in \mathbb{I}_{[0, N]} \). Clearly, (56) implies
\[
\|\xi_t - \xi_{k+n}\|_2 \leq c_{\theta} \theta, \quad k \in \mathbb{I}_{[t-N, t-1]},
\]
(57)
for some \( c_{\theta} > 0 \). Using Lemma 1, the difference between \( y_{t+k} \) and \( y'_{t+k} \) is bounded for \( k \in \mathbb{I}_{[-N, 1]} \) by
\[
\|y_{t+k} - y'_{t+k}(t)\|_\infty \leq \|y_{t+k} - y'_{t+k}(t)\|_2 = \|\Delta t_{k, 1}\|_2
\]
(58)
\[
\leq c_{\Delta} \sum_{t=-N}^{t-1} \|x_t - x_t\|_2 \leq c_{\Delta} \sum_{t=-N}^{t-1} \|\xi_t - \xi_t\|_2
\]
\[
\leq c_{\Delta} \sum_{t=-N}^{t-1} \|\xi_t - \xi_t\|_2 \leq c_{\theta} \theta^2
\]
(57)
for some \( c_{\Delta} > 0 \), using that \( T_L \) in (45) is Lipschitz continuous.

Next, we provide an extended state \( \hat{\xi}_t = \left[ \begin{array}{c} \hat{u}_{[t-n, t-1]} \\ y_{[t-n, t-1]} \end{array} \right] \) satisfying (52) with a suitable bound on \( \|\hat{\xi}_t - \xi_t\|_2 \). We write \( \hat{x}_{[t-n, t-1]} \) for the state trajectory resulting from an application of the closed-loop input \( \hat{u}_{[t-n, t-1]} \) to the initial condition \( x_{t-n} \) for the dynamics linearized at \( x_t \). We define the input component \( \bar{u}_{[t-n, t-1]} \) of \( \hat{\xi}_t \) as
\[
\bar{u}_{[t-n, t-1]} = u_{[t-n, t-1]} + \Gamma_c(x_t)^\dagger (x_t - \hat{x}_t),
\]
(59)
where \( \Gamma_c(x_t) \) is the controllability matrix of the dynamics linearized at \( x_t \). According to Assumption 5, i.e., [20, Assumption 3], the Moore-Penrose inverse \( \Gamma_c(x_t)^\dagger \) is uniformly bounded. Further, we define \( \bar{y}_{[t-n, t-1]} \) as the output trajectory for the dynamics linearized at \( x_t \) with input \( \bar{u}_{[t-n, t-1]} \) and initial condition \( x_{t-n} \). Similar to Lemma 1, we can show that
\[
\|x_t - \hat{x}_t\|_2 \leq c_{\theta, 3} \sum_{j=t-n}^{t-1} \|x_t - x_j\|_2 \leq c_{\theta, 4} \theta^2
\]
(60)
for some \( c_{\theta, 3}, c_{\theta, 4} > 0 \). Combining this with (59), we obtain
\[
\|u_{[t-n, t-1]} - \bar{u}_{[t-n, t-1]}\|_2 \leq c_{\theta, 5} \theta^2
\]
(61)
with some \( c_{\theta, 5} > 0 \). We write \( \bar{x}_{[t-n, t-1]} \) and \( \bar{y}_{[t-n, t-1]} \) for the state and output resulting from an application of \( \bar{u}_{[t-n, t-1]} \) to the nonlinear system (41) with initial state \( x_{t-n} \).

Using Lipschitz continuity of (41) by Assumption 5 (i.e., [20, Assumption 1]), there exists \( c_{\theta, 6} > 0 \) such that
\[
\|\bar{y}_{[t-n, t-1]} - y_{[t-n, t-1]}\|_2 \leq c_{\theta, 6} \|u_{[t-n, t-1]} - \bar{u}_{[t-n, t-1]}\|_2.
\]
(62)
Using again similar arguments as in Lemma 1, we can show
\[
\|\bar{y}_{[t-n, t-1]} - y_{[t-n, t-1]}\|_2 \leq c_{\theta, 7} \sum_{j=t-n}^{t-1} \|x_t - x_j\|_2 \leq 2c_{\theta, 7} \sum_{j=t-n}^{t-1} \|x_t - x_j\|_2 \|x_j - x_j\|_2
\]
\[
\leq 2c_{\theta, 7} \sum_{j=t-n}^{t-1} \|x_t - x_j\|_2 + \|x_j - x_j\|_2 \leq 2c_{\theta, 7} \sum_{j=t-n}^{t-1} \|x_t - x_j\|_2 + \|x_j - x_j\|_2
\]
(62), (63)
for suitably defined \( c_{\theta, 7} > 0 \). In this part of the proof, we show that, under suitable conditions, (56) holds for \( k = t \) if it holds for \( k = t - N, t - N + n, \ldots, t - n \). We denote the extended state (13) corresponding to \( (u^*(t), y^*(t)) \) by \( \xi^*(t) \). For \( j \in \mathbb{I}_{[0, n-1]} \), it holds that
\[
\|\xi_{t-n} - \xi_{t-j}\|_2 \leq \|\xi_{t-n} - \xi^*(t)\|_2 + \|\xi_{t-j} - \xi^*(t)\|_2
\]
\[
\leq c_{\xi, 1} (\|\xi_{t-n} - \xi^*(t)\|_2 + \|\xi_{t-j} - \xi^*(t)\|_2)
\]
(65)
for some \( c_{\xi, 1} > 0 \), where we use that \( \xi_{t-n} \) and \( \xi_{t-j} \) contain all elements of \( \xi_{t-j} \). Using optimality in Problem (47), we have
\[
\|\xi_{t-n} - \xi^*(t)\|_2 \leq \frac{1}{\lambda_{min}(Q, R)} F^*_\xi(\xi_t)
\]
(66)
Moreover, using \( u_{t+k} = \bar{u}_i(t) \) for \( k \in \mathbb{I}_{[0, n-1]} \) due to the \( n \)-step scheme, cf. Algorithm 2, we can derive
\[
\|\xi_{t-n} - \xi^*(t)\|_2 = \sum_{k=0}^{n-1} \|u_{t+k} - u^*(t)\|_2 + \|y_{t+k} - y^*(t)\|_2
\]
\[
\leq \sum_{k=0}^{n-1} \|\bar{u}_i(t) - u^*(t)\|_2 + \|y_{t+k} - y^*(t)\|_2
\]
\[
\leq \sum_{k=0}^{n-1} \|\bar{y}_i(t) - y^*(t)\|_2 + \|y_{t+k} - y^*(t)\|_2
\]
(66)
Inequalities (75) and (76) together with Proposition 2 imply
\[ \|y_k(t) - \overline{y}_k(t)\|_2 \leq c_{\xi,3}\theta^2 \|\alpha^* (t)\|_2 + c_{\xi,4}\theta^2 + c_{\xi,5}\|\sigma^* (t)\|_2 \]
for \( k \in I_{[0,n-1]} \) with some \( c_{\xi,i} > 0, i \in \{4,6\} \). Finally, using the definition of \( J^*_L(\xi_t) \), we have
\[ \|\alpha^* (t)\|_2 \leq \sqrt{\frac{1}{\lambda_\alpha}} J^*_L(\xi_t) + \|\alpha^{sr} (x_t)\|_2, \]
\[ \|\sigma^* (t)\|_2 \leq \sqrt{\frac{1}{\lambda_\sigma}} J^*_L(\xi_t), \]
where \( \|\alpha^{sr}(x_t)\|_2 \) is uniformly bounded due to compactness of the steady-state manifold of the linearization which holds by Assumption 5, i.e., by [20, Assumptions 4 and 5].

Letting \( \lambda_\alpha = \frac{\lambda_\alpha}{\theta^2g_\alpha} \) and \( \lambda_\sigma = \frac{\lambda_\sigma}{\theta^2g_\sigma} \) for some \( \lambda_\alpha, \lambda_\sigma, g_\alpha, g_\sigma > 0 \) with \( \beta_\alpha > 1, \beta_\alpha + \beta_\sigma < 2 \), there exist \( c_{\xi,6}, c_{\xi,7} > 0 \) such that, for \( k \in I_{[0,n-1]} \),
\[ \|y_k(t) - \overline{y}_k(t)\|_2 \leq c_{\xi,6}\theta^2 + c_{\xi,7}(\theta^2 - \beta_\alpha + \theta^2g_\sigma) \sqrt{J^*_L(\xi_t)}. \]
Combining (65)–(68) and (72), we infer for \( j \in I_{[0,n-1]} \)
\[ \|\xi_{t+n} - \xi_{t+j}\|_2 \leq c_{\xi,1} \left( \frac{2}{\sqrt{\lambda_{\min}(Q,R)}} \right) J^*_L(\xi_t) + c_{\xi,2} \sum_{k=0}^{n-1} \|\xi_{t+k} - \xi_t\|_2 \]
\[ + c_{\xi,2}Nc_{\alpha,1}\theta^2 + a \left( c_{\xi,6}\theta^2 + c_{\xi,7}(\theta^2 - \beta_\alpha + \theta^2g_\sigma) \right) \sqrt{J^*_L(\xi_t)}. \]
Let now
\[ \bar{S} = \theta^4, \quad V_{\text{ROA}} = \theta^2 + \eta \]
for some \( 0 < \eta < 2(\beta_\alpha - 1) \) and suppose \( \bar{\theta} < 1 \). This implies \( \theta^3 < \theta^2g_\alpha \) if \( \eta_1 > \eta_2 \), which will be used throughout the proof. Using Assumption 5, i.e., [20, Assumptions 4 and 5], \( J^*_e(x_t) \) is uniformly bounded by \( \bar{S} \), i.e.,
\[ J^*_e(x_t) \leq c_{J,S} \bar{S} = c_{J,S} \theta^4 \]
with \( c_{J,S} > 0 \). Hence, \( J^*_L(\xi_t, D_t) \leq V_{\text{ROA}} + J^*_e(x_t) \) implies
\[ J^*_L(\xi_t, D_t) \leq \xi_{t,b} \theta^2 + \eta \]
for some \( \xi_{t,b} > 0 \). We now apply Proposition 2 (see Appendix C), which generalizes the robustness results in Proposition 1 to also account for disturbances in the initial condition (not only in the output). In this context, the noise / disturbance is bounded by \( \varepsilon = c_\theta \theta^2 \) for some \( c_\theta > 0 \), compare (58) for the “noise” entering the output Hankel matrix and (64) for the “disturbance” in the initial conditions. Inequalities (75) and (76) together with Proposition 2 imply
\[ J^*_L(\xi_t) \leq c_{J,2}(c_\theta \theta^2)^\beta_s c_{J,3} \theta^4 + c_{J,3}((c_\theta \theta^2)^2 - \beta_s + (c_\theta \theta^2)^2) \]
\[ + (1 + c_{J,1}((c_\theta \theta^2)^\beta_s + (c_\theta \theta^2)^2)) \xi_{t,b} \theta^2 + \eta < \xi_{t,b} \theta^2 + \eta \]
for some \( c_{J,\theta} > 0 \), where we use \( \eta < 2(\beta_s - 1) < 2(1 - \beta_\sigma) < 2 \) and \( \bar{\theta} < 1 \). Plugging this into (73), we obtain
\[ \|\xi_{t+n} - \xi_{t+j}\|_2 \leq \beta_1(\theta) + c_{\xi,1}c_{\xi,2} \sum_{k=0}^{n-1} \|\xi_{t+k} - \xi_t\|_2^2 \]
for a suitably defined \( \beta_1 \in \mathcal{K}_\infty \) containing only terms with order strictly larger than 1. Each of the terms \( \|\xi_{t+k} - \xi_t\|_2, k \in I_{[0,n-1]} \) can be bounded analogously to \( \|\xi_{t+n} - \xi_{t+j}\|_2 \) in (78), using the same bounds as above leading to (78). To be precise, following the same steps as above, there exist \( \bar{\beta} \in \mathcal{K}_\infty, c_\xi > 0 \) such that, for all \( k \in I_{[0,n-1]} \),
\[ \|\xi_{t+k} - \xi_t\|_2^2 \leq \bar{\beta}(\theta) + c_\xi \sum_{n=0}^{k-1} \|\xi_{t+n} - \xi_t\|_2^2, \]
where \( \bar{\beta} \) contains only terms with order strictly larger than 1. Applying this bound recursively \( n - 1 \) times and plugging the result into (78), we obtain
\[ \|\xi_{t+n} - \xi_{t+j}\|_2 \leq \beta_2(\theta) \]
for \( j \in I_{[0,n-1]} \) with \( \beta_2 \in \mathcal{K}_\infty \) containing only terms with order strictly larger than 1. Hence, if \( \bar{\theta} \) is sufficiently small such that \( \beta_2(\theta) < c_\theta \theta^2 \), then (56) holds for \( k = t \). To conclude, we have shown that (56) holds recursively, assuming that \( V(\xi_t, D_t) \leq V_{\text{ROA}} \). Thus, using (56) for \( k = t + n - N, n = 2, \ldots, t \), we infer (cf. (57), (64))
\[ \|\xi_{t+n} - \xi_t\|_2 \leq c_{\xi,1} \theta, \quad k \in I_{[t+n-N,t+n-1]}, \]
\[ \|\xi_{t+n} - \xi_{t+n-1}\|_2 \leq c_{\xi,10} \theta^2. \]
With \( c_{\theta,2} \) as in (58), we have for \( k \in I_{[-N-1]} \)
\[ \|y_{t+n-k} - \bar{y}_k(t + n)\|_2 \leq c_{\Delta} \sum_{i=t+n-N}^{t+n-1} \|x_{i+n} - x_i\|_2^2 \leq c_{\theta,2} \theta^2, \]
(82)
For \( \eta_{t+n-1}(t) = u_{t+n-1}(t) \) \leq \beta_\eta(c_\theta \theta^2), \]
where \( \bar{u}^*(t) \) is the nominal optimal input corresponding to the optimal cost \( J^*_1(\xi_t, D_t) \). Since the nominal MPC problem (16) with \( \bar{\sigma} = 0 \) contains an exact model of the linearization, we can apply the main technical result in our companion paper [20, Proposition 2] to arrive at
\[ V(\xi_{t+n}, D_{t+n}) \leq c_{\xi,1} V(\xi_t, D_t) \]
\[ + \lambda_\alpha(c_\theta \theta^2)^\beta_s \|\ddot{u}(t + n) - c_{\alpha,1}(c_\theta \theta^2)^\beta_s \|_2^2 + c_{d,1} \beta_\eta(c_\theta \theta^2)^2 \]
(84)
2The result can be applied despite the fact that the cost of Problem (16) depends on the output, i.e., in general only positive semidefinite in the state, since we perform an \( n \)-step analysis and the cost is positive definite over \( n \) steps.
for some $0 < c_{V,1} < 1$, $c_{d,1} > 0$. Here, $\alpha(t+n)$ is a later specified candidate solution corresponding to the optimal control problem with optimal cost $J_E^*(\xi_{t+n}, D_{t+n})$. The additional term depending on $\alpha(t+n)$ is due to the regularization of $\alpha$ in the cost, which is not present in the model-based MPC scheme in [20]. Further, the term $c_{d,1}\beta_u(c_\theta \beta^2)^2$ is due to the fact that $x_{t+n}$ results from applying the optimal input $u_{[t,t+n-1]} = \hat{u}_0(t;\alpha,\beta,\gamma)$ of Problem (47) to the state $x_t$, whereas [20, Proposition 2] considers the closed loop under the nominal MPC input $\hat{u}_0(t;\alpha,\beta,\gamma)$. To be more precise, with minor adaptations of the proof of [20, Proposition 2], it can be shown that, if the closed-loop state at time $t + n$ results from applying an input which differs from $u^*(t)$ by no more than $\beta_u(c_\theta \beta^2)^2$ (compare (83)), then the Lyapunov function decay shown in [20] remains true if the squared disturbance bound $c_{d,1}\beta_u(c_\theta \beta^2)^2$ is added on the right-hand side (i.e., (84)) holds. This is possible since the main proof idea of [20, Proposition 2] does not rely on the exact state value at time $t + n$, but rather on the fact that it remains close to $x_t$.

(iii).b Bound on $\|\hat{a}(t+n) - \alpha \sigma^r(\hat{x}(t+n))\|_2^2$

We bound this term using the candidate solution of the model-based MPC in [20]. We write $\hat{u}(t+n)$ for a candidate input used in the proof of [20, Proposition 2], which corresponds to a feasible candidate solution to Problem (16) with $\delta = 0$ for $\alpha(t+n) = H_{\alpha,x,t+n}^\top [\hat{u}(t+n) \alpha \gamma^{t+1}]$. Thus, we infer

\[ \|\hat{a}(t+n) - \alpha \sigma^r(\hat{x}(t+n))\|_2^2 \leq \|H_{\alpha,x,t+n}^\top\|_2^2 = (85) \]

The analysis in [20, Proposition 2] implies that both candidate solutions used in [20] satisfy

\[ \|\hat{u}(t+n) - u^*(x(t+n))\|_2^2 \leq c_{V,2} V(\xi, D_t) \] (86)

for some $c_{V,2} > 0$. Further, there exist $c_{V,3}, c_{V,4} > 0$ such that

\[ \|x_t - x^*(x(t+n))\|_2^2 \leq \|x_t - x^*(x(t+n))\|_2^2 + 2\|x_{t+n} - x^*(x(t+n))\|_2^2 \] (38),(45),(80)

\[ \leq c_{V,3} \beta^2 + c_{V,4} V(\xi_{t+n}, D_{t+n}) \] (87)

Assumption 8 and $c_{H} = \frac{c_\theta \beta^2}{\gamma}$ imply $\|H_{\alpha,x,t+n}^\top\|_2^2 \leq c_{\theta \beta}\gamma$. Plugging (85)-(87) into (84), we thus infer

\[ V(\xi_{t+n}, D_{t+n}) \leq (c_{V,1} + c_{V,5} \beta^2 \beta^2 \beta^2 - 2)V(\xi, D_t) + c_{d,1}\beta_u(c_\theta \beta^2)^2 \] (88)

\[ + c_{V,7} \beta \beta^2 \beta^2 - 2V(\xi_{t+n}, D_{t+n}) \]

for suitably defined $c_{V,i} > 0$, $i \in [5,7]$. Given that $\beta_u > 1$, $c_{V,1} < 1$, we can choose $\theta$ sufficiently small such that

\[ c_{V,7} \beta \beta^2 \beta^2 - 2 < 1, \quad \frac{c_{V,1} + c_{V,5} \beta^2 \beta^2 \beta^2}{1 - c_{V,7} \beta \beta^2 \beta^2 \beta^2} < 1. \]

Multiplication of both sides of (88) by $\frac{1}{1 - c_{V,7} \beta \beta^2 \beta^2 \beta^2}$ yields

\[ V(\xi_{t+n}, D_{t+n}) \leq c_{V,8} V(\xi, D_t) \] (89)

\[ + c_{V,9}\beta_u(c_\theta \beta^2)^2 + \theta^2 \theta \beta^2 \]

for suitably defined $0 < c_{V,8} < 1$, $c_{V,9} > 0$. (ii).c Practical stability

Plugging (77) and $\bar{e} = c_\theta \theta^2$ into (34)-(36), straightforward algebraic calculations reveal that (18) also holds with

\[ \beta_u(c_\theta \beta^2)^2 \leq c_{V,10} (\theta^2 / 2) + \theta^2 / 2 + \theta^2 / 2 \]

for some $c_{V,10} > 0$. Using $\beta_u + \beta_u > 2$ and $\eta < 2(\beta_u - 1)$, it is straightforward to verify that the smallest exponent appearing in $(\beta_V(\theta))^2 + \theta^2 \beta_u$ is strictly larger than $2 + \eta$. This implies that, if $\theta$ is sufficiently small, then (89) yields $V(\xi_{t+n}, D_{t+n}) \leq \theta^2 \eta = V_{\text{ROA}}$. Hence, all bounds in this proof hold recursively for $t = N + ni$, $i \in [0, \infty)$. In particular, the nominal MPC problem with cost $J_E^*(\xi, D_t)$ and, using Proposition 2, Problem (47) are recursively feasible. Finally, a recursive application of (89) yields

\[ V(\xi, D_t) \leq c_{V,15} V(\xi, D_N) + \beta_{V,3}(\theta) \] (90)

for some $\beta_{V,3} \in K_{\infty}$ and any $t = N + ni$, $i \in [0, \infty)$. Exploiting the lower and upper bounds in (38), we obtain

\[ \|\tilde{\xi}_t - \xi^r\|_2^2 \leq c_{V,4} c_{V,15} \|\tilde{\xi}_N - \xi^r(x_N)\|_2^2 + 1 - \beta_{V,3}(\theta). \]

Using Assumption 7, i.e., (46), as well as (64), we obtain (55) with $c_V = c_{V,15}$ and some $C > 0$, $\beta_0 \in K_{\infty}$.

Theorem 3 shows that the closed-loop state trajectory converges close to the optimal reachable equilibrium (cf. (55)) if the cost matrix $S$ is sufficiently small, the regularization parameters are chosen suitably (roughly speaking, $\lambda_\alpha$ is small and $\lambda_\gamma$ is large), and if the initial data satisfy $\|\xi_N - \xi|_2 \leq \theta$ for $i \in [0, N-1]$ with a sufficiently small $\theta$. Inequality (55) implies that the optimal reachable setpoint $\xi^r$ is practically exponentially stable, i.e., the closed-loop trajectory exponentially converges to a region around $\xi^r$ whose size increases with $\theta$. In particular, under suitable assumptions, the proposed MPC scheme steers the closed loop arbitrarily close to $\xi^r$ if the initial state evolution, i.e., the parameter $\theta$, is sufficiently small. Similar to the affine system case in Section III, the proof relies on combining the continuity of data-driven MPC w.r.t. noise with the corresponding model-based MPC analysis from [20], where the linearization error takes the place of the output measurement noise.

In the following, we provide an interpretation for this stability result. As shown in Section IV-D, the prediction accuracy of the implicit prediction model in (47) depends on the distance of past state values to the current state. The parameter $\theta$ provides a bound on this distance at initial time $t = N$, which is shown to hold recursively for all $t \geq N$ in the proof. Thus, if $\theta$ is sufficiently small at $t = N$, then the closed loop does not move too rapidly such that the prediction model remains accurate and stability can be shown. The conditions $\lambda_{\text{max}}(S) \leq S$ and $V(\xi_N(0), D_N) \leq V_{\text{ROA}}$ for sufficiently small $S$ and $V_{\text{ROA}}$ are similar to the results for model-based MPC in [20], and they guarantee that the closed-loop trajectory remains close to the steady-state manifold and that the bound $\|\tilde{\xi}_t - \xi^r\|_2 \leq c_{\xi,0}\theta$, $t \in [0, \infty)$ holds recursively. Further, the proof of Theorem 3 shows that the choice of the regularization parameters $\lambda_\alpha$ and $\lambda_\gamma$ leading to closed-loop practical stability depends on $\theta$. In particular, we have $\lambda_\alpha = \lambda_\alpha \theta^2 \beta_u$ and $\lambda_\gamma = \frac{\lambda_\gamma}{\theta^2 \beta^2}$ for some $\lambda_\alpha, \lambda_\gamma, \beta_u, \beta_\gamma > 0$. 
Note that Assumption 8 requires a minimum amount of persistence of excitation, i.e., a uniform bound $\|H_{u,x,t}\|_2 \leq c_H$. On the other hand, Theorem 3 also requires (an initial) upper bound on $\|\xi_t - \xi_i - N+i\| \leq \theta$, $i \in [0,N-1]$, which in turn limits the amount of persistence of excitation. By allowing $c_H$ to depend on $\theta$ as $c_H = \frac{s}{\theta}$, our analysis takes into account the fact that $\theta$ cannot be arbitrarily small for a fixed $c_H$, i.e., for a fixed lower bound on the amount of persistence of excitation. Assumption 8 is the main reason why Theorem 3 only provides a practical stability result, i.e., asymptotic stability can in general not be proven if we assume uniform closed-loop persistence of excitation. This is in contrast to the results in our companion paper [20] which showed closed-loop exponential stability assuming that an exact model of the linearization is available, i.e., without any requirements on persistence of excitation for the closed loop.

For the numerical example in Section V, we observe that the closed loop indeed converges very closely to the optimal reachable equilibrium $\xi^*$ and, moreover, closed-loop persistence of excitation can be ensured by stopping the data updates. Finally, the condition $\|\xi_t - \xi_i\|_2 \leq \theta$, $i \in [0,N-1]$, is generally easier to satisfy for smaller values of the data length $N$ and the prediction model in (47a) is less accurate for too large values of $N$. This is different to linear data-driven MPC, where larger values of $N$ usually improve the closed-loop performance in case of noisy data. For nonlinear systems, on the other hand, “older” data points correspond to an “older” approximate linear model of the system which typically yields a worse approximation at the current state.

To summarize, Theorem 3 provides closed-loop stability guarantees for the MPC scheme in Algorithm 2, which uses no model knowledge but only measured input-output data and which relies on solving strictly convex QPs online. The guarantees hold for suitable parameters and for initial conditions in a neighborhood of the steady-state manifold. We have thus demonstrated that the data-driven parametrization of affine systems (Section II) with online data updates allows us to control unknown nonlinear systems with closed-loop guarantees under reasonable assumptions on smooth system dynamics, controllability, observability, as well as convexity and compactness of the steady-state manifold.

V. NUMERICAL EXAMPLE

We apply the presented data-driven MPC approach (see Algorithm 2) to the nonlinear continuous stirred tank reactor from [52]. We consider the same discretized system as in our companion paper [20] (see [20] for a detailed description of the system dynamics, parameters, and satisfaction of Assumption 5 as well as the counterpart of Assumption 7). To implement the MPC, we choose the parameters

$$Q = 1, R = 5 \cdot 10^{-2}, S = 5, \lambda_\alpha = 5 \cdot 10^{-6}, \lambda_\sigma = 10^5,$$

and the prediction horizon $L = 20$. Further, the constraints are $U = [0,2]$, $U^* = [0.01,1.99]$. In comparison to [20], where we chose $S = 100$, we use the smaller choice $S = 5$ for the data-driven MPC scheme considered in this paper to ensure that the closed loop does not change too rapidly and the data-driven prediction model in Problem (47) remains accurate. Additionally, we (approximately) compute the optimal solution $x(t) \in (44)$ at each time $t$ based on the past $N$ input-output measurements, where we add a slack variable $\sigma$ in the constraints and regularizations $10^{-4}||\alpha||_2^2$ and $10^2||\sigma||_2^2$ in the cost. The data length is chosen as $N = 70$ and we generate initial data samples for $t \in [0,2000]$ by sampling the input uniformly from $u_t \in [0,1]$. Finally, to ensure that the data used for prediction are persistently exciting, we stop updating the data in (47a) at time $t = 2000$ when the closed loop is close to the setpoint, i.e., we use the input-output data \{uk,yk\}k=2000-1 for prediction at any $t > 2000$.

We now apply Algorithm 2 with these parameters and modifications in a one-step fashion. While the theoretical analysis in Section IV only applies for a multi-step MPC scheme, we conjecture that similar results can be derived locally close to the steady-state manifold for a one-step MPC scheme, provided that a stronger version of the controllability assumption [20, Assumption 3] holds, see [20] for details. First, we note that updating the data in Problem (47) online is a crucial ingredient of our approach. In particular, a data-driven MPC using the data \{uk,yk\}k=0-1 for prediction at all times (compare Algorithm 1) leads to a huge tracking error, i.e., the output converges to $1.08 \neq y^T = 0.6519$. The closed-loop trajectory under our MPC approach is depicted in Figure 2, along with the trajectory resulting from the one-step MPC based on a linearized model in our companion paper [20] (using the same parameters as above, except for $S = 100$ instead of $S = 5$). In the data-driven MPC, the input is more unsteady which can be explained by the combination of the less accurate prediction model and terminal equality constraints. On the other hand, the data-driven MPC yields faster convergence although the matrix $S$, which is the key ingredient related to convergence speed, is chosen smaller. This can be explained via the slack variable $\sigma(t)$, which implicitly relaxes the terminal equality constraint (47c) such that the artificial output equilibrium $y^*(t)$ (and thus, the closed loop) converges more quickly towards the setpoint $y^T$. A model-based MPC with a relaxed terminal set constraint and terminal cost also exhibits a significantly faster convergence towards $y^T$. Further, we observe that the norm of $H_{u,x,t}^*$ indeed increases in closed loop under the data-driven MPC when the trajectory approaches the setpoint, compare Assumption 8 and the condition $c_H = \frac{s}{\theta}$ in Theorem 3. Finally, we note that the proof of Theorem 3 relies on multiple conservative estimates and should therefore be interpreted as a qualitative stability result. Correspondingly, the magnitude of the (initial) excitation bound $\theta$ and the choices of the tuning variables $S$, $\lambda_\alpha$, $\lambda_\sigma$ considered in the example are not necessarily chosen small (for $\theta$, $S$, $\lambda_\alpha$) or large (for $\lambda_\sigma$) enough to satisfy the corresponding bounds in the proof. To summarize, the proposed data-driven MPC scheme can successfully control an unknown nonlinear system based only on input-output data. It should be noted that the parameters in Problem (47) have a crucial influence on the closed-loop performance and performing a detailed case study of our MPC approach is a
very interesting issue, which goes however beyond the scope of this paper.

VI. CONCLUSION

We presented a novel data-driven MPC approach to control unknown nonlinear systems based on input-output data with closed-loop stability guarantees. Our approach relied on the data-driven system parametrization of affine systems stated in Section II, which can (approximately) describe nonlinear systems by updating the data online. We proved that the proposed MPC scheme practically stabilizes the closed loop under the assumptions that the design parameters are chosen suitably, the initial condition is close to the steady-state manifold, and the data generated during initialization are not too far apart. As intermediate results of independent interest, we extended the Fundamental Lemma of [2] to affine systems and we presented a robust data-driven tracking MPC scheme with superior closed-loop robustness properties compared to existing data-driven MPC approaches. The theoretical analysis of the latter MPC scheme relied on a novel separation argument of nominal and robust data-driven MPC, which can be easily applied to other data-driven MPC formulations in the literature.

We did not address the issue of how to ensure closed-loop persistence of excitation (Assumption 8) and, in particular, we only showed practical stability of the closed loop. Tackling these issues by developing appropriate modifications of our MPC algorithm is a highly interesting and relevant future research direction, potentially leading to both superior theoretical guarantees and more reliable practical performance.

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Proof. Proof of “ii”:
It clearly holds that
\[ y_{d,i,i+L-1} = \Phi L x_i + \Gamma u, L u^{d,i,i+L-1} + \Gamma e, L e_L + r_L \] (91)
for suitably defined matrices \( \Phi L, \Gamma c, L, \) and \( \Gamma u, L \) depending on \( A, B, C, D, \) and for \( e_L := I_L \otimes e, r_L := I_L \otimes r. \) Hence,
\[ y_{0,L-1} = \sum_{i=0}^{N-L} y_{0,i,i+L-1}^{d,i} \alpha_i \] (92)
\[ = \sum_{i=0}^{N-L} \alpha_i (\Phi L x_i^{d,i} + \Gamma u, L u^{d,i,i+L-1} + \Gamma e, L e_L + r_L) \] (93)
\[ = \sum_{i=0}^{N-L} \alpha_i y_{d,i,i+L-1}^{d,i} + \Gamma u, L u^{0,0} + \Gamma e, L e_L + r_L. \]
This implies that \( \{u_k, y_k\}_{k=0}^{L-1} \) is a trajectory of (4) with initial condition \( x_0 = \sum_{i=0}^{N-L} \alpha_i x_i^{d,i}. \)

Proof of “only if”: Let \( \{u_k, y_k\}_{k=0}^{L-1} \) be a trajectory of (4) with initial condition \( x_0 \). Using (6), there exists \( \alpha \in \mathbb{R}^{N-L+1} \) such that
\[ \begin{bmatrix} H_L (u^d) \\ H_1 (x_{[0,N-L]}^d) \end{bmatrix} \alpha = \begin{bmatrix} u_{[0,L-1]} \\ x_0 \\ 1 \end{bmatrix} \] (94)
Note that the last row implies \( \sum_{i=0}^{N-L} \alpha_i = 1. \) Moreover,
\[ y_{0,L-1} = \Phi L x_0 + \Gamma u, L u_{[0,L-1]} + \Gamma e, L e_L + r_L \]
\[ = \sum_{i=0}^{N-L} \alpha_i (\Phi L x_i^{d,i} + \Gamma u, L u^{d,i,i+L-1} + \Gamma e, L e_L + r_L) \]
\[ = \sum_{i=0}^{N-L} \alpha_i y_{d,i,i+L-1}^{d,i} + \Gamma u, L u^{0,0} + \Gamma e, L e_L + r_L. \]
\[ + \Gamma u, L u^{0,0} + \Gamma e, L e_L + r_L. \]
\[ = \sum_{i=0}^{N-L} \alpha_i y_{d,i,i+L-1}^{d,i} + \Gamma u, L u^{0,0} + \Gamma e, L e_L + r_L. \]
\[ = \sum_{i=0}^{N-L} \alpha_i y_{d,i,i+L-1}^{d,i} + \Gamma u, L u^{0,0} + \Gamma e, L e_L + r_L. \]

B. Proof of Theorem 2
Proof. The proof is divided into four parts. We first show the lower and upper bounds (38) on the Lyapunov function candidate in Part (i). In Parts (ii) and (iii), we propose two different candidate solutions for (16) at time \( t + n \) for two complementary scenarios, assuming that (16) is feasible at time \( t. \) In Part (iv), we combine the bounds to prove (39).

(i) Lower and upper bound on \( V(\xi_t) \)
Using that \( (\tilde{u}^{st}(t), \tilde{y}^{st}(t)) \) is feasible for (9), we have
\[ \frac{(a)}{2} \lambda_{\min}(S) \] (95)
\[ \geq \frac{1}{c_g} ||\tilde{u}^{st}(t) - u^{st}||_2 \] (96)
In combination with (1), this implies
\[
V(\xi_t) \geq \sum_{k=-n}^{-1} \|u_{t+k} - \bar{u}^{*}(t)\|_2^2 + \|y_{t+k} - \bar{y}^{*}(t)\|_Q^2 \quad (95)
\]
\[
+ \|\bar{y}^{*}(t) - y^T\|_S^2 - \|y^{sr} - y^T\|_S^2 \geq c_1 \sum_{k=-n}^{-1} \left( \|u_{t+k} - \bar{u}^{*}(t)\|_2^2 + \|y_{t+k} - \bar{y}^{*}(t)\|_Q^2 \right)
\]
and therefore, the lower bound in (38) with
\[
c_t := \frac{1}{2} \min \left\{ \lambda_{\min}(Q, R), \frac{\lambda_{\min}(S)}{2} \right\} \min \left\{ 1, \frac{1}{c_1} \right\}.
\]

(i).b Upper bound on \(V(\xi_t)\)

Suppose \(\|\xi_t - \xi^{sr}\|_2 \leq \delta\) for a sufficiently small \(\delta > 0\). We define a candidate for the artificial equilibrium by \((u^*(t), y^*(t)) = (u^{sr}, y^{sr})\). Using controllability and \(u^{sr} \in \mathbb{U}^{sr} \in \text{int}(U)\), there exists a feasible input-output trajectory \(\{\bar{u}(t), \bar{y}(t)\}_{k=-n}^{k} \) for Problem (16) with \(\tilde{\sigma} = 0\) satisfying the terminal constraint (16d) as well as
\[
\sum_{k=-n}^{L} \|\bar{u}_k(t) - u^{sr}\|_2^2 + \|\bar{y}_k - y^{sr}\|_2^2 \leq \Gamma_{\xi} \|\xi_t - \xi^{sr}\|_2^2 \quad (96)
\]
for some \(\Gamma_{\xi} > 0\). The vector \(\alpha(t)\) is chosen as
\[
\alpha(t) = H_{ux}^{\dagger} \begin{bmatrix} \bar{u}(t) \\ x_{t-n} \\ 1 \end{bmatrix},
\]
where \(H_{ux}\) is defined as
\[
H_{ux} := \begin{bmatrix} H_{L+n+1}^{T}(u^d) \\ H_{1}(x_{0:n-L-n-1}^{2}) \end{bmatrix} \quad (97)
\]
This implies that all constraints of Problem (16) are satisfied (compare the proof of “only if” in Theorem 1) and thus, the Lyapunov function candidate \(V(\xi_t)\) is upper bounded as
\[
V(\xi_t) \leq \Gamma_{\xi} \lambda_{\max}(Q, R) \|\xi_t - \xi^{sr}\|_2^2 + \lambda_{\alpha} \varepsilon^{\beta_n} \|\alpha(t) - \alpha^{sr}\|_2^2 \quad (98)
\]
Further, using \(\alpha^{sr} = H_{ux}^{\dagger} \begin{bmatrix} u_{L+n+1}^{sr} \\ x^{sr} \end{bmatrix} \), we infer
\[
\|\alpha(t) - \alpha^{sr}\|_2^2 \leq \|H_{ux}^{\dagger}\|_2^2 (\Gamma_{\xi} \|\xi_t - \xi^{sr}\|_2^2 + \|x_{t-n} - x^{sr}\|_2^2) \quad (99)
\]
Finally, using observability, there exists a matrix \(M\) such that
\[
x_{t-n} - x^{sr} = M(\xi_t - \xi^{sr}) \quad (100)
\]
Combining (98)–(100), we deduce that the upper bound in (38) holds for all \(\xi_t\) satisfying \(\|\xi_t - \xi^{sr}\|_2 \leq \delta\) with
\[
c_u := \Gamma_{\xi} \lambda_{\max}(Q, R) + \lambda_{\alpha} \varepsilon^{\beta_n} \|H_{ux}^{\dagger}\|_2^2 (1 + \Gamma_{\xi} + \|M\|_2^2).
\]
Since (16) is a multi-parametric QR, the optimal cost is piecewise quadratic and thus, (38) holds for any feasible \(\xi_t\) (with a modified constant \(c_u\)).

(ii) Candidate solution 1

Assume
\[
\sum_{k=-n}^{-1} \|u_{t+k} - \bar{u}^{*}(t)\|_R^2 + \|y_{t+k} - \bar{y}^{*}(t)\|_Q^2 \geq \gamma \|\bar{y}^{*}(t) - y^{sr}\|_S^2 \quad (101)
\]
for a constant \(\gamma > 0\) which will be fixed later in the proof.

(ii).a Definition of candidate solution

We choose both the input and output equilibrium candidate as the previously optimal solution \(u^{sr}(t + n) = \bar{u}^{*}(t), y^{sr}(t + n) = \bar{y}^{*}(t)\). The first \(L - 2n\) elements of the predicted input trajectory are a shifted version of the previously optimal trajectory, i.e., \(u_k^*(t + n) = u_k(t + n + k)\) and \(y_k^*(t + n) = y_k(t + n + k)\). Denote by \(\{y_k^*(t + n)\}_{k=0}^{L-1}\) the output resulting from an application of \(\tilde{u}_{[n,L]}(t)\) to the system (4) initialized at \((u_{[t,n+1]}, y_{[t,n+1]})\). For \(k \in \{0, L-2n\}\), we let \(y_k^*(t + n) = \bar{y}_k(t + n)\). We write \(x_l^{\bar{u}^*}(t + n)\) for the state at time \(L - 2n\) corresponding to \((\bar{u}_k(t + n), \bar{y}_k(t + n))\) and \(x_l^{\bar{y}^*}(t + n)\) for the state at time \(t\). Using controllability, there exists an input-output trajectory \(\{\bar{u}_k(t + n), \bar{y}_k(t + n)\}_{k=0}^{L-1}\) steering the system to the steady-state \(\hat{x}^{sr}(t)\) corresponding to \((\bar{u}^{*,}(t), \bar{y}^{*,}(t))\) while satisfying
\[
\sum_{k=L-2n}^{L-n-1} \|\bar{u}_k(t + n) - \bar{a}^{sr}(t)\|_2^2 + \|\bar{y}_k(t + n) - \bar{y}^{sr}(t)\|_2^2 \leq \Gamma \|x_{L-2n}(t + n) - \hat{x}^{sr}(t)\|_2^2 \quad (102)
\]
for some \(\Gamma > 0\). In the following, we show that \(\bar{u}_k(t + n) \in \mathbb{U}\), \(k \in \{L-2n, L-n-1\}\), if \(\varepsilon\) is sufficiently small. Recall that \(\hat{x}^{sr}(t)\) is the steady-state corresponding to \((\bar{u}^{*,}(t), \bar{y}^{*,}(t))\), whereas the output \(y_l^{\bar{u}^*}(t + n)\) results from applying \(\bar{u}_k^*(t + n)\) to the system at initial state \(\hat{x}_l^{\bar{u}^*}(t + n)\). Hence, using observability, there exists \(c_{x,1} > 0\) such that
\[
\|\hat{x}_{L-2n}(t + n) - \bar{x}^{sr}(t)\|_2^2 \leq c_{x,1} \|y_{L-2n,L-n-1}(t + n) - \bar{y}^{sr}(t)\|_2^2.
\]
The output trajectories \(y_{L-2n,L-n-1}(t + n)\) result from applying the input \(\bar{u}_{[n,L]}(t)\) to the system (4) with initial conditions \((u_{[t,n+1]}, y_{[t,n+1]})\) and \((\bar{u}_{[0,n-1]}(t), \bar{y}_{[0,n-1]}(t))\), respectively. Since the difference between these initial conditions is linear in the disturbance \(d_{[t,n+1]}\) (compare (37)), there exists \(c_{x,2} > 0\) such that
\[
\|\hat{x}_{L-2n}(t + n) - \bar{x}^{sr}(t)\|_2^2 \leq c_{x,2} \varepsilon^2.
\]
Together with (102) and \(\bar{u}^{*,}(t) \in \text{int}(U)\), this shows that \(\bar{u}_k(t + n) \in \mathbb{U}\) for \(k \in \{L-2n, L-n-1\}\) if \(\varepsilon\) is sufficiently small. Finally, we let \((\tilde{u}_k(t + n), \tilde{y}_k(t + n)) = (\bar{u}^{*,}(t), \bar{y}^{*,}(t))\) for \(k \in \{L-n-1\}\). Using Assumption 4, we choose
\[
\tilde{\alpha}'(t + n) = H_{ux}^{\dagger} \begin{bmatrix} \bar{u}(t + n) \\ x_t \\ 1 \end{bmatrix} \quad (104)
\]
with \(H_{ux}\) as in (97). This implies that (16b) and thus, all constraints of (16) hold.
Lyapunov function decay

Using the above candidate solution, we have

\[
V(\xi_{t+n}) - V(\xi_t) \leq \sum_{k=-n}^L \|u_k'(t+n) - \tilde{u}^s(t)\|_R^2 + \|\tilde{y}_k(t+n) - \tilde{y}^s(t)\|_Q^2 + \lambda_{\max}(Q)\|\tilde{y}_k(t+n) - \tilde{y}^s(t)\|_Q^2
\]

(105)

The terms involving the input are

\[
\sum_{k=-n}^L \|u_k'(t+n) - \tilde{u}^s(t)\|_R^2 - \|u_k(t) - \tilde{u}^s(t)\|_R^2 = \sum_{k=-n}^L (\|u_k'(t+n) - \tilde{u}^s(t)\|_R^2 - \|u_k(t) - \tilde{u}^s(t)\|_R^2)
\]

(106)

For \(k \in [-n,L]\), it holds that

\[
\|u_k'(t+n) - \tilde{u}^s(t)\|_R \leq V(\xi_t).
\]

(107)

Together with the fact that \(\|u_k(t) - \tilde{u}^s(t)\|_R \leq \varepsilon\) for \(k \in [0,n-1]\) (compare (37)) and using (3), this leads to

\[
\sum_{k=0}^{n-1} \|\tilde{u}_k(t+n) - u_k^s(t)\|_R^2 - \|\tilde{u}_k(t) - u_k^s(t)\|_R^2 \leq c_{u,1}\varepsilon^2 + c_{u,2}\varepsilon V(\xi_t)
\]

(108)

for some \(c_{u,1}, c_{u,2} > 0\). Further, the second term on the right-hand side of (106) is bounded as

\[
\sum_{k=L-2n}^{L-n-1} \|\tilde{u}_k'(t+n) - u_k^s(t)\|_R^2 \leq \lambda_{\max}(R)\|\tilde{u}_k(t+n) - u_k^s(t)\|_R^2
\]

(109)

for some \(c_{u,1}, c_{u,2} > 0\). Further, the second term on the right-hand side of (106) is bounded as

\[
\sum_{k=L-2n}^{L-n-1} \|\tilde{u}_k'(t+n) - u_k^s(t)\|_R^2 \leq \lambda_{\max}(R)\|\tilde{u}_k(t+n) - u_k^s(t)\|_R^2
\]

(102),(103)

Next, we analyze the terms in (105) depending on the output trajectory. Inequalities (102) and (103) imply

\[
\sum_{k=L-2n}^{L-n-1} \|\tilde{y}_k(t+n) - \tilde{y}_k(t)\|_Q^2 \leq \lambda_{\max}(Q)\|\tilde{y}_k(t+n) - \tilde{y}_k(t)\|_Q^2 \leq \lambda_{\max}(Q)\|\tilde{y}_k(t+n) - \tilde{y}_k(t)\|_Q^2
\]

(110)

The trajectories \(\{\tilde{y}_k(t+n)\}_{k=0}^{L-2n-1}\) and \(\{\tilde{y}_k(t)\}_{k=0}^{L-2n-1}\) differ only in terms of their initial conditions which in turn differ linearly in terms of \(d_{[t,t+n-1]}\). Hence, following the arguments above leading to (108), there exist \(c_{y,1}, c_{y,2} > 0\), such that for \(k \in [1,n,L-2n-1]\)

\[
\|\tilde{y}_k'(t+n) - \tilde{y}_k(t)\|_Q^2 \leq c_{y,1}\varepsilon^2 + c_{y,2}\varepsilon V(\xi_t)
\]

(109)

Finally, by definition of \(\tilde{a}'(t+n)\) in (104), we have

\[
\|\tilde{a}'(t+n) - \alpha^{sr}\|_Q^2 \leq c_{a,1}\|\tilde{y}_k(t) - y^{sr}\|_Q^2 \leq c_{a,2}\varepsilon^2 + c_{a,3}\varepsilon V(\xi_t)
\]

(112)

and

\[
\|\tilde{a}'(t+n) - \alpha^{sr}\|_Q^2 \leq c_{a,1}\|\tilde{y}_k(t) - y^{sr}\|_Q^2 \leq c_{a,2}\varepsilon^2 + c_{a,3}\varepsilon V(\xi_t)
\]

(113)

Note that

\[
\|\tilde{y}_k(t) - y^{sr}\|_Q^2 \leq c_{a,1}\|\tilde{y}_k(t) - y^{sr}\|_Q^2 \leq c_{a,2}\varepsilon^2 + c_{a,3}\varepsilon^2 + c_{a,4}\varepsilon V(\xi_t)
\]

(114)

for some \(c_{a,1}, c_{a,2}, c_{a,3}, c_{a,4} > 0\).

Plugging the bounds (108), (109) for the output, and (114) for \(\tilde{a}'(t+n)\) into (105), and using (15), we obtain

\[
V(\xi_{t+n}) - V(\xi_t) \leq -\sum_{k=-n}^{n-1} \|u_k(t+n) - u_k^s(t)\|_R^2 + c_{J,1}\varepsilon^2 + c_{J,2}\varepsilon^2 + c_{J,3}\varepsilon^2 + c_{J,4}\varepsilon^2 + \lambda_{\max}(R)\|\tilde{u}_k(t+n) - \tilde{u}_k(t)\|_R^2
\]

(115)

for some \(c_{J,1}, c_{J,2}, c_{J,3}, c_{J,4} > 0\). Note that

\[
\sum_{k=-n}^{L-n-1} \|u_k(t+n) - u_k^s(t)\|_R^2 \leq \frac{1}{2} \sum_{k=-n}^{n-1} \|u_k(t+n) - u_k^s(t)\|_R^2 \leq \frac{1}{2}\left(\lambda_{\min}(S)\|\tilde{y}_k(t) - y^{sr}\|_Q^2 + \frac{1}{2}\|\tilde{y}_k(t) - y^{sr}\|_Q^2\right)
\]

(109),(110)

for some \(c_{J,6} > 0\). This implies

\[
V(\xi_{t+n}) - V(\xi_t) \leq -\sum_{k=-n}^{L-n-1} \|u_k(t+n) - u_k^s(t)\|_R^2 + c_{J,1}\varepsilon^2 + c_{J,2}\varepsilon^2 + c_{J,3}\varepsilon^2 + c_{J,4}\varepsilon^2 + \lambda_{\max}(R)\|\tilde{u}_k(t+n) - \tilde{u}_k(t)\|_R^2
\]

(116)

with some \(c_{J,6} > 0\). This implies
(iii) Candidate solution 2
Assume
\[
\sum_{k=-n}^{-1} \left\| u_{t+k} - \tilde{u}_s^*(t) \right\|_R^2 + \left\| y_{t+k} - \tilde{y}_s^*(t) \right\|_Q^2
\leq \gamma \left\| \tilde{y}_s^*(t) - y^{sr} \right\|_S^2.
\]  
(117)

(iii).a Definition of candidate solution
We choose the equilibrium candidate as a convex combination of \((\tilde{u}_s^*(t), \tilde{y}_s^*(t))\) and the optimal reachable equilibrium, i.e.,
\[
\tilde{u}_s^*(t + n) = \lambda \tilde{u}_s^*(t) + (1 - \lambda) u^{sr},
\]
\[
\tilde{y}_s^*(t + n) = \lambda \tilde{y}_s^*(t) + (1 - \lambda) y^{sr}
\]
for some \(\lambda \in (0, 1)\) which will be fixed later in the proof, and we denote the corresponding state by \(x^s(t + n)\). By controllability, there exists an input steering the system from \(x_{t+n}\) to \(x^s(t + n)\) in \(L - n \geq n\) steps while satisfying
\[
\sum_{k=0}^{L} \left\| \tilde{u}_k(t + n) - \tilde{u}_s^*(t + n) \right\|_2^2 + \left\| \tilde{y}_k(t + n) - \tilde{y}_s^*(t + n) \right\|_2^2 
\leq \Gamma \left\| x_{t+n} - \tilde{x}_s^*(t + n) \right\|_2^2
\]  
(119)

for some \(\Gamma > 0\). In the following, we show that \(\tilde{u}_k(t + n) \in U, k \in \mathbb{I}_{[0,L]}\) if \(\gamma, (1 - \lambda), \text{ and } \bar{\varepsilon}\) are sufficiently small. Denoting the extended state (13) corresponding to \((\tilde{u}_s^*(t), \tilde{y}_s^*(t))\) by \(\xi(t)\), we have
\[
\sum_{k=0}^{n-1} \left\| \tilde{u}_k(t) - \tilde{u}_s^*(t) \right\|_2^2 + \left\| \tilde{y}_k(t) - \tilde{y}_s^*(t) \right\|_2^2
\leq \bar{c}_u \left( \frac{\lambda_{\max}(S)}{\lambda_{\min}(Q, R)} \bar{c}_u \gamma + c_g \right) \left\| \tilde{y}_s^*(t) - y^{sr} \right\|_2^2
\]  
(120)

for a suitable constant \(\bar{c}_u > 0\). The second inequality in (120) can be shown analogously to the upper bound in Part (i).b of the proof, using a controllability argument based on (117) with a sufficiently small \(\gamma\) and bounding \(\|\alpha_s^*(t) - \alpha^{sr}\|_2^2\) with
\[
\alpha_s^*(t) := H_{ux} \begin{pmatrix} \bar{u}_s^*_{L_t+n+1}(t) \\ \tilde{x}_s(t) \end{pmatrix}
\]  
in terms of \(\|\tilde{x}_s^*(t) - x^{sr}\|_2^2\) and \(\|\tilde{u}_s^*(t) - u^{sr}\|_2^2\). Moreover, for \(k \in \mathbb{I}_{[0,n-1]}\), the difference
\[
\left\| u_{t+k} - \tilde{u}_s^*(t) \right\|_2^2 - \left\| \tilde{u}_k(t) - \tilde{u}_s^*(t) \right\|_2^2
\]
is bounded as in (108), and similarly for the output, cf. (111). Thus, adding and subtracting \(\left\| \tilde{u}_k(t) - \tilde{u}_s^*(t) \right\|_2^2\) and \(\left\| \tilde{y}_k(t) - \tilde{y}_s^*(t) \right\|_2^2\), we obtain
\[
\|\tilde{y}_s^*(t) - y^{sr}\|_2^2 \leq 2\|\tilde{y}_k(t) - \tilde{y}_s^*(t)\|_2^2 + \|\tilde{y}_k(t) - \tilde{y}_s^*(t)\|_2^2
\]  
(121)

and
\[
\sum_{k=0}^{n-1} \left\| \tilde{u}_k(t) - \tilde{u}_s^*(t) \right\|_2^2 + \left\| \tilde{y}_k(t) - \tilde{y}_s^*(t) \right\|_2^2
\leq \bar{c}_x,1,\gamma \|\tilde{y}_s^*(t) - y^{sr}\|_2^2 + c_r,2\bar{c}_x + c_r,3\bar{c}_x \sqrt{V(\xi)}
\]  
(122)

for some \(\bar{c}_r > 0, i \in \mathbb{I}_{[1,3]}\). Writing \(\tilde{\xi}_s(t + n)\) for the extended state (13) corresponding to \((\tilde{u}_s^*(t + n), \tilde{y}_s^*(t + n))\), it holds that
\[
\|\tilde{\xi}_s(t + n) - \tilde{\xi}_s(t + n)\|_2^2 \leq (1 - \lambda)^2 \|\tilde{\xi}_s(t + n) - \tilde{\xi}_s(t + n)\|_2^2
\]  
(123)

Combining these bounds, we obtain
\[
\|\tilde{y}_s^*(t) - y^{sr}\|_2^2 \leq \frac{2}{\lambda_{\min}(S)} (V(\xi(t)) + 2J_{ca}).
\]  
(124)

Thus, using \(V(\xi(t)) \leq V_{\text{ROA}}\), if \(\gamma, (1 - \lambda), \text{ and } \bar{\varepsilon}\) are sufficiently small, then \(x_{t+n}\) is sufficiently close to \(x^s(t + n)\) such that (119) and \(\tilde{u}_s^*(t + n) \in \text{int}(U)\) ensure \(\tilde{u}_k(t + n) \in U\) for \(k \in \mathbb{I}_{[0,L]}\).

Further, choosing the input-output candidate \((\tilde{u}_s^*(t + n), \tilde{y}_s^*(t + n))\) such that \(\tilde{u}_k(t + n) = u_{t+n+k}, \tilde{y}_k(t + n) = y_{t+n+k}\) for \(k \in \mathbb{I}_{[0,n-1]}\) and \((\tilde{u}_k(t + n), \tilde{y}_k(t + n)) = (\tilde{u}_s^*(t + n), \tilde{y}_s^*(t + n))\) for \(k \in \mathbb{I}_{[L-n,L]}\) satisfies (16c) and (16d), respectively. Finally, with \(H_{ux}\) as in (97), we choose
\[
\hat{\alpha}(t + n) = H_{ux} \begin{pmatrix} \hat{u}(t + n) \\ t \end{pmatrix},
\]  
(125)

which implies that all constraints of (16) are fulfilled.

(iii).b Lyapunov function decay
Using the above candidate solution, we obtain
\[
V(\xi_{t+n}) - V(\xi_t)
\]  
(126)

\[
\sum_{k=n-n}^{L} \left\| u_{t+k} - \hat{u}_s^*(t + n) \right\|_R^2 + \left\| y_{t+k} - \hat{y}_s^*(t + n) \right\|_Q^2
\]
\[
+ \lambda_{\max}(S) \bar{c}_u \left( \frac{\lambda_{\max}(S)}{\lambda_{\min}(Q, R)} \bar{c}_u \gamma + c_g \right) \left\| \hat{y}_s^*(t) - y^{sr} \right\|_2^2
\]
\[
- \sum_{k=0}^{L} \left( \left\| \tilde{u}_k(t) - \hat{u}_s^*(t) \right\|_R^2 + \left\| \tilde{y}_k(t) - \hat{y}_s^*(t) \right\|_Q^2 \right)
\]
\[
+ \left\| y^s(t + n) - y^{sr}\right\|_S^2 - \left\| \hat{y}_s^*(t) - y^{sr}\right\|_S^2.
\]
Similar to [23, Inequality (19)], strong convexity of the cost in (9) implies
\[
\|\bar{y}^r(t + n) - y^T\|_2^2 - \|\bar{y}^{sr}(t) - y^T\|_2^2 
\leq - (1 - \lambda^2)\|\bar{y}^{sr}(t) - y^{sr}\|_2^2. \tag{127}
\]

The definition of \(\bar{\alpha}(t + n)\) implies
\[
\|\bar{\alpha}(t + n) - \alpha^{sr}\|_2^2 
\leq \|H^u_{ax}\|_2^2\|\bar{u}(t + n) - u_{2n+1}\|_2^2 + \|x_t - x^{sr}\|_2^2. \tag{128}
\]

Using \(\bar{u}(t + n) - u^{sr} = \lambda^{sr}(t + n) - u^{sr}\), as well as (1), (38), (113), (119), (123), and (124) we obtain
\[
\|\bar{\alpha}(t + n) - \alpha^{sr}\|_2^2 
\leq \hat{c}_{\alpha,1,J}^* + \hat{c}_{\alpha,2} + \|H^v_{ax}\|_2^2\|x_t - x^{sr}\|_2^2 + \|H^t_{ux}\|_2^2\|x_t - x^{sr}\|_2^2 \tag{129}
\]

for some \(\hat{c}_{\alpha,i} > 0, i \in \{1,4\}\). Plugging (119), (127), and (129) into (126) and using
\[
\sum_{k=-n}^{-1}\|\bar{u}(t + n) - \bar{u}^{sr}(t + n)\|_2^2 + \|\bar{y}(t + n) - \bar{y}^{sr}(t + n)\|_2^2 
\leq \lambda_{\text{max}}(Q, R)\|\xi(t + n) - \xi^*(t + n)\|_2^2 \tag{130}
\]
as well as (15), there exists \(\bar{c}_{J,t,i} > 0, i \in \{1,7\}\) such that
\[
V(\xi(t + n)) - V(\xi_t) \leq - \sum_{k=-n}^{-1}\left(\|u_{t+k} - \bar{u}^{sr}(t)\|_2^2 + \|y_{t+k} - \bar{y}^{sr}(t)\|_2^2\right) 
+ \bar{c}_{J,1}\|\xi(t + n) - \xi^*(t + n)\|_2^2 - (1 - \lambda^2)\|\bar{y}^{sr}(t) - y^{sr}\|_2^2
\]
\[+ \lambda_{\text{min}}(S)(1 - \lambda^2)\|\bar{y}(t) - y^{sr}\|_2^2 + \hat{c}_{\alpha,4}\|\xi(t) - \xi^{sr}\|_2^2 + \hat{c}_{\alpha,5}\|\bar{y}(t) - y^{sr}\|_2^2
\]
\[
+ \bar{c}_{J,7}\|x_{t+1} - x^{sr}\|_2^2 + \|x_t - x^{sr}\|_2^2 + \|H^t_{ux}\|_2^2\|x_t - x^{sr}\|_2^2 \tag{131}
\]

for some \(\hat{c}_{\alpha,4} > 0\).

**C. Robustness w.r.t. General Disturbances**

In the following, we extend Proposition 1 to disturbances in the input specifying the initial conditions, i.e., at time \(t \in \mathbb{N}_0\), we consider the disturbed input \(\bar{u}_k = u_k + \varepsilon_k\), \(k \in \{t-n,t-1\}\), where \(\|\varepsilon_k\|_\infty \leq \varepsilon\) with \(\varepsilon > 0\) as in Assumption 3. Throughout Appendix C, whenever we refer to Problem (12) or its ingredients, we assume that \(u_{[t-n,t-1]}\) in (12c) is replaced by \(\bar{u}_{[t-n,t-1]}\). Further, the noisy extended state is defined as \(\bar{\xi}_t = \frac{\bar{u}_{[t-n,t-1]}}{\bar{y}_{[t-n,t-1]}}\).

**Proposition 2.** Let Assumptions 1, 2, 3, and 4 hold and suppose \(\beta_\alpha + \beta_\sigma < 2\).

(i) There exist \(\bar{\varepsilon}_{\text{max}}, c_{\alpha,1}, c_{\alpha,2} > 0\) such that, for all \(\varepsilon \leq \bar{\varepsilon}_{\text{max}}\) and all \(\xi_t\),
\[
J_L(\bar{\xi}_t) \leq \left(1 + c_{\alpha,1}(2^\beta_\alpha - \beta_\alpha + 2\beta_\sigma)\right) J_L^*(\xi_t) + c_{\alpha,2}(2^\beta_\sigma + 2\beta_\sigma). \tag{133}
\]

(ii) For any \(\bar{J} > 0\), there exist \(\varepsilon_{\text{max}} > 0\), \(\beta_\alpha \in \mathbb{K}_{\text{eq}}\) such that, if \(J_L^*(\xi_t) \leq \bar{J}\), then it holds for all \(\varepsilon \leq \bar{\varepsilon}_{\text{max}}\) that
\[
\|\bar{u}^*(t) - \bar{u}^{sr}(t)\|_2 \leq \beta_\alpha(\varepsilon). \tag{134}
\]

**Proof.** (i) Proof of (133)

We construct a candidate solution for Problem (12) based on the optimal solution of Problem (16) for \(\sigma = 0\). To this end, we let
\[
\bar{u}(t) = \begin{cases} \tilde{u}_{[t-n,t-1]}(t) \quad t \in [t-n,t-1] \\ \tilde{u}_{[0,L]}(t) \quad t \in [0,L] \end{cases}, \quad \bar{y}(t) = \begin{cases} \tilde{y}_{[t-n,t-1]}(t) \quad t \in [t-n,t-1] \\ \tilde{y}_{[0,L]}(t) \quad t \in [0,L] \end{cases},
\]
and \(u^*(t) = \tilde{u}^{sr}(t), y^*(t) = \tilde{y}^{sr}(t)\). Further, we define
\[
\alpha(t) = H^t_{ux} \begin{bmatrix} \bar{u}(t) \\ x_{t-n} \end{bmatrix},
\]
with \(H_{ux}\) as in (97) as well as \(\sigma(t) = H_{L+n+1}(\tilde{y}^d)\alpha(t) - \tilde{y}(t)\).

This candidate fulfills all constraints of Problem (12).

Note that \(\hat{\alpha}^*(t) - \alpha^{sr}\) satisfies
\[
\begin{bmatrix} H_{L+n+1}(u^d) \\ H_{L+n+1}(y^d) \end{bmatrix}(\hat{\alpha}^*(t) - \alpha^{sr}) = \begin{bmatrix} \tilde{u}^*(t) - u_{2n+1}^T \\ \tilde{y}^*(t) - y_{L+n+1}^T \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \tag{135}
\]

Due to the minimization in (16a), \(\hat{\alpha}^*(t) - \alpha^{sr}\) is the vector satisfying (135) with minimum norm, which implies
\[
\hat{\alpha}^*(t) = H^t_{ux} \begin{bmatrix} \tilde{u}^*(t) \\ x_{t-n} \end{bmatrix}.
\]
Thus, it holds that
\[ \| \alpha(t) - \tilde{\alpha}^*(t) \|_2^2 \leq \| H_{ax}^T \|_2^2 \| u_{[t-n,t-1]} - \tilde{u}_{[t-n,t-1]} \|_2^2 \] (136)
\[ \leq \| H_{ax}^T \|_{mn} \| \bar{\varepsilon} \| =: c_n \bar{\varepsilon}^2. \]
Using \( \|a + b\|_2 \leq (1 + \bar{\varepsilon}^2)\|a\|_2 + (1 + \frac{1}{\bar{\varepsilon}^2})\|b\|_2 \) for arbitrary \( a, b \), this implies
\[ \| \alpha(t) - \alpha^*(t) \|_2^2 \leq \| \tilde{\alpha}^*(t) - \alpha^*(t) \|_2^2 + c_n^2 (\bar{\varepsilon}^2 + \bar{\varepsilon}^2 + \bar{\varepsilon}^2 - \beta \sigma). \]

Note that the output trajectories \( H_{L+n+1}(y^d)\alpha(t) \) and \( y^\ast(t) \) differ only due to the difference in the first \( n \) components of the corresponding input trajectory, which is in turn bounded by \( \bar{\varepsilon} \). Thus, there exists \( c_n^0 > 0 \) such that
\[ \| H_{L+n+1}(y^d)\alpha(t) - y^\ast(t) \|_2^2 \leq c_n^0 \bar{\varepsilon}^2. \]
Using (1), there exist \( c_n^1, c_n^2 > 0 \) such that
\[ \| \sigma(t) \|_2^2 \leq c_n^0 \bar{\varepsilon}^2 + 4\| H_{L+n+1}(\varepsilon^d)\alpha(t) \|_2^2 + 4\| \varepsilon_{[t-n,t-1]} \|_2^2 \leq c_n^0 \bar{\varepsilon}^2 + c_n^2 \bar{\varepsilon}^2 \| \alpha(t) \|_2^2. \]

Finally, analogously to Part (i) of the proof of Proposition 1 (compare (21) and the subsequent arguments), there exists \( c^3_n > 0 \) such that
\[ \sum_{k=-n}^{-1} \| \tilde{u}_{t+k} - \tilde{u}^\ast(t) \|_R^2 + \| u_{t+k} - \tilde{u}^\ast(t) \|_R^2 \] (139)
\[ + \sum_{k=-n}^{1} \| \tilde{y}_{t+k} - \tilde{y}^\ast(t) \|_Q^2 + \| y_{t+k} - \tilde{y}^\ast(t) \|_Q^2 \]
\[ \leq c_n^3 (\bar{\varepsilon}^2 + \bar{\varepsilon}^2 + \beta \sigma) \bar{J}_L^1(\xi_1), \]
where we also used
\[ \sum_{k=-n}^{-1} \| u_{t+k} - \tilde{u}^\ast(t) \|_R^2 + \| y_{t+k} - \tilde{y}^\ast(t) \|_Q^2 \leq \bar{J}_L^1(\xi_1). \]

Combining (137)–(139), we obtain
\[ J_L^1(\xi_1) - \bar{J}_L^1(\xi_1) \leq c_n^3 (\bar{\varepsilon}^2 + \bar{\varepsilon}^2 + \beta \sigma) \bar{J}_L^1(\xi_1) + \frac{\lambda_n \bar{\varepsilon}^2}{\beta \sigma} \| \sigma(t) \|_2^2 \] (140)
\[ + \lambda_n \bar{\varepsilon}^2 (\| \alpha(t) - \alpha^* \|_2^2 - \| \tilde{\alpha}^*(t) - \alpha^* \|_2^2) \]
\[ \leq c_n^3 (\bar{\varepsilon}^2 + \bar{\varepsilon}^2 + \beta \sigma) \bar{J}_L^1(\xi_1) + \lambda_n \bar{\varepsilon}^2 \bar{J}_L^1(\xi_1) + \frac{\lambda_n \bar{\varepsilon}^2}{\beta \sigma} \| \alpha(t) \|_2^2 + \lambda_n \bar{\varepsilon}^2 \bar{J}_L^1(\xi_1) + \frac{\lambda_n \bar{\varepsilon}^2}{\beta \sigma} \| \alpha(t) \|_2^2, \]
where we also use \( \lambda_n \bar{\varepsilon}^2 \| \tilde{\alpha}^*(t) - \alpha^* \|_2^2 \leq \bar{J}_L^1(\xi_1) \) for the second inequality. The term \( \| \alpha(t) \|_2^2 \) can be bounded as
\[ \| \alpha(t) \|_2^2 \leq 2\| \alpha(t) - \tilde{\alpha}^*(t) \|_2^2 + 4\| \tilde{\alpha}^*(t) - \alpha^* \|_2^2 + 4\| \alpha^* \|_2^2 \]
\[ \leq 2c_n^1 \bar{\varepsilon}^2 + 4\frac{1}{\lambda_n \bar{\varepsilon}^2} \bar{J}_L^1(\xi_1) + 4\| \alpha^* \|_2^2. \]

Plugging this into (140) and letting \( \bar{\varepsilon}_{\text{max}} < 1 \), we obtain (133) for appropriately defined constants \( c_{n,1}, c_{n,2} > 0 \).

**Proof of (134)**

Given some \( \tilde{\sigma} = \begin{bmatrix} \tilde{\sigma}_{\text{init}} \\ \tilde{\sigma}_{\text{dyn}} \end{bmatrix} \in \mathbb{R}^{p(L+2n+1)+mn} \), we define the following optimal control problem:

\[
\begin{aligned}
\min_{\alpha(t)} & \quad \sum_{k=-n}^{L} \left( \| \tilde{u}_k(t) - u^* (t) \|_R^2 + \| \tilde{y}_k(t) - y^* (t) \|_Q^2 \right) \\
\text{s.t.} & \quad \begin{bmatrix} H_{L+n+1} \left( u^d \right) \\ H_{L+n+1} \left( y^d \right) \end{bmatrix} \alpha(t), \\
& \quad \begin{bmatrix} \tilde{u}_{[t-n,t-1]}(t) \\ \tilde{y}_{[t-n,t-1]}(t) \end{bmatrix} = u_{[t-n,t-1]} + \tilde{\sigma}_{\text{init}}, \\
& \quad \begin{bmatrix} \tilde{u}_{[L-n,L]}(t) \\ \tilde{y}_{[L-n,L]}(t) \end{bmatrix} = u_{[L-n,L]}(t), \\
& \quad \tilde{u}_k(t) \in \mathbb{U}, \ t \in [0, L], \ u^* (t) \in \mathbb{U}^n.
\end{aligned}
\]

The only difference between Problem (141) and Problem (16) lies in (141c), where \( \tilde{\sigma}_{\text{init}} \) now acts on both the input and the output initial condition. Analogous to Part (ii) of the proof of Proposition 1, we consider Problem (141) with

\[ \tilde{\sigma} = \tilde{\sigma}_{\varepsilon} = \begin{bmatrix} \varepsilon_{[t-n,t-1]}^u \\ \varepsilon_{[t-n,t-1]}^\sigma \end{bmatrix}. \]

The optimal solution of this problem provides a feasible candidate solution to Problem (12) and vice versa. Using this fact in combination with a bound on the difference in the cost, we can proceed as in the proof of Proposition 1 to obtain (134) with a slightly different function \( \beta_u \in \mathcal{K}_\infty. \)

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