Out-of-equilibrium relaxation of the Edwards–Wilkinson elastic line

Sebastian Bustingorry\textsuperscript{1}, Leticia F Cugliandolo\textsuperscript{2} and José Luis Iguain\textsuperscript{3}

\textsuperscript{1} DPMC-MaNEP, Université de Genève, 24 Quai Ernest Ansermet, 1211 Genève 4, Switzerland
\textsuperscript{2} Université Pierre et Marie Curie—Paris VI, LPTHE UMR 7589, 4 Place Jussieu, 75252 Paris Cedex 05, France
\textsuperscript{3} Departamento de Física, FCEyN, Universidad Nacional de Mar del Plata, Deán Funes 3350, 7600 Mar del Plata, Argentina
E-mail: Sebastian.Bustingorry@physics.unige.ch, leticia@lpt.ens.fr and iguin@mdp.edu.ar

Received 23 May 2007
Accepted 29 July 2007
Published 12 September 2007

Online at stacks.iop.org/JSTAT/2007/P09008
doi:10.1088/1742-5468/2007/09/P09008

Abstract. We study the non-equilibrium relaxation of an elastic line described by the Edwards–Wilkinson equation. Although this model is the simplest representation of interface dynamics, we highlight that many (though not all) important aspects of the non-equilibrium relaxation of elastic manifolds are already present in such quadratic and clean systems. We analyze in detail the ageing behavior of several two-times averaged and fluctuating observables taking into account finite size effects and the crossover to the stationary and equilibrium regimes. We start by investigating the structure factor and extracting from its decay a growing correlation length. We present the full two-times and size dependence of the interface roughness and we generalize the Family–Vicsek scaling form to non-equilibrium situations. We compute the incoherent scattering function and we compare it to the one measured for other glassy systems. We analyze the response functions, the violation of the fluctuation-dissipation theorem in the ageing regime, and its crossover to the equilibrium relation in the stationary regime. Finally, we study the out-of-equilibrium fluctuations of the previously studied two-times functions and we characterize the scaling properties of their probability distribution functions. Our results allow us to obtain new insights into other glassy problems such as the ageing behavior in colloidal glasses and vortex glasses.

Keywords: kinetic growth processes (theory), self-affine roughness (theory), slow dynamics and ageing (theory)
1. Introduction

1.1. Aim

Interfaces are important in many physical, chemical and biological phenomena. In the physical context they appear in stochastic surface growth [1], domain growth and

References
coarsening phenomena (as domain walls) [2], type-II superconductivity (as magnetic flux lines) [3], fracture cracks [4], and fluid invasion in porous media [5], among others realizations. Although specific problems involve different levels of complexity, as the inclusion of non-linear or disorder contributions, the main features of interface dynamics are already contained in the simplest theoretical formulation of the problem: the Edwards–Wilkinson (EW) equation [6].

The aim of this article is to exhibit, in a concrete and fully solvable example, several generic properties of the averaged and fluctuating ageing dynamics of finite and infinite elastic manifolds. The problem we analyze is the relaxation of an elastic line described by the Edwards–Wilkinson equation [6]. The embedding space dimension plays an important role. In the particular one-dimensional transverse space case on which we focus here the dynamics has aspects of diffusion, glassiness and saturation depending on the time and length scales observed.

Previous studies of the ageing dynamics of linear interfaces without disorder focused on the infinite size limit and explored some particular correlation and response functions [7,8]. In a recent work, Röthlein et al [7] analyzed the non-equilibrium space–time correlation and response functions by using a space–time symmetry approach. Here, we explore the dynamics of both finite and infinite lines, extending the results of these authors to the space–time height–height correlation function as well as other two-times correlation functions, thus giving a thorough description of the out-of-equilibrium dynamics of the EW equation.

The relaxation process we are interested in is the following [9]. After equilibration at a temperature $T_0$ the system is suddenly taken to a different working temperature $T$ that can be higher or lower than $T_0$. Time is then set to zero. The line subsequently tries to adapt to the working temperature. The relaxation is characterized by the time dependence of correlation and linear response functions. One lets the line relax until a waiting time, $t_w$, when the quantities of interest are first recorded and later compared to their values at a subsequent time $t$.

A series of glassy properties of elastic lines evolving in disordered environments have been recently reported in different contexts, related to directed polymers in random media [10,11], the vortex glass dynamics in high temperature superconductors [12]–[14], and domain wall motion in magnetic systems [15,16]. All these studies focused on models including disorder, which non-disordered counterparts belong to the EW universality class of interface dynamics [1], except for [16] in which the analysis is focused on the non-linear contribution in the Kardar–Parisi–Zhang (KPZ) [17] universality class. In order to disentangle the effects of elasticity and disorder it is then important to study in detail the dynamics of the non-disordered counterparts.

In general, the glassy phenomenon in finite dimensional models of elastic lines with and without quenched disorder appears as a dynamic crossover [10]–[12]. For all waiting times, $t_w$, that are longer than a size, $L$, and eventually also temperature, $T$, dependent equilibration time, $t_L$, the dynamics is stationary. Instead, for $t_w < t_L$ the system is in the preasymptotic regime and the relaxation occurs out of equilibrium as demonstrated by two-times correlations and linear responses that age. For each waiting time the dependence on the time delay, $\Delta t \equiv t - t_w$, then shows a growth regime for $\Delta t < t_L$ and saturation regime at longer time, $\Delta t > t_L$, where the correlation functions saturate to a size dependent value.

\[ \text{doi:10.1088/1742-5468/2007/09/P09008} \]
Figure 1. Two line configurations at different times, e.g. $x(z, t_w)$ and $x(z, t_w + \Delta t)$, showing the relevant displacements defining the mean squared displacement of the line segments, $B$, the roughness, $w^2$, and the mean squared displacement of the center of mass, $D$.

Here we compute, analytically, the averaged two-times roughness and we relate it to the displacement field, which was the main focus of previous studies of the dynamics of elastic manifolds in random environments [10,12,18], and the center of mass evolution of the interface (see the sketch in figure 1). The quadratic character of the model allows us to control the crossover to equilibration and saturation of finite lines. We next deduce several other correlators that are of interest in glassy dynamics. In particular, we analyze the incoherent scattering function and compare it to light scattering measurements in clay colloidal suspensions [19]. We interpret our results in terms of a two-times correlation length that we evaluate and confront to the one measured in other ageing glassy systems [20, 21]. We also study linear responses and their relation to the companion correlations. We discuss in detail the special features of these two-times observables linked to the multiplicative—diffusive—scaling form.

Finally, we focus on thermally induced fluctuations. Recently, the importance of studying fluctuations—and not only averaged quantities—in dynamic phenomena was stressed in several contexts. Rácz proposed to use scaling functions characterizing the fluctuations of global observables in *non-equilibrium steady states* to classify systems in ‘universality classes’ dictated by symmetries and dynamic mechanisms [22]. In *ageing glassy* systems the study of fluctuations seems to be fundamental to understand the mechanism for the dramatic slowing down and non-equilibrium relaxation. Chamon *et al* [23, 24] proposed a symmetry based sigma model like theory for fluctuations in conventional glassy systems. For a number of reasons this theory is not expected to apply, without modification, to interface dynamics. In more technical terms, the averaged interface dynamics is characterized by a multiplicative ageing scaling that should result in the need to modify the approach in [23] to take this feature into account. We thus wish to confront the fluctuations of conventional glassy systems to those of interface models searching for similarities and differences. With this purpose we derive the probability density functions (pdfs) of several two-times correlation functions. We compare our results to previous studies [25], which focused on the time delay dependence of the...
fluctuations\(^4\), and to predictions of the time reparametrization invariance scenario of glassy systems \[23, 24\].

The problem addressed here is related to a number of other models that have already been studied in the literature; among the pure cases one has the Langevin dynamics of the Gaussian and massless scalar field \[8\], the \(XY\) ferromagnet \[8, 26\], the \(p = 2\) spherical spin glass \[27\] and the \(O(N)\) ferromagnet in the large \(N\) limit \[28\]. Related models with quenched disorder are problems of elastic manifolds in quenched random environments \[10, 11, 15, 18\]. In all these studies the finite size dependence was not taken into account and the dynamic fluctuations were not studied. We explain in the conclusions how our results relate to the ones in these papers.

1.2. The model

The EW equation for a scalar field \(x\) representing the height of a surface over a one-dimensional substrate parametrized by the coordinate \(z\) (a one-dimensional directed interface) is

\[
\partial_t x(z, t) = \nu \partial_z^2 x(z, t) + f(z, t) + \xi(z, t),
\]

\[
\langle \xi(z, t) \rangle = 0, \quad \langle \xi(z, t) \xi(z', t') \rangle = \frac{2T}{\gamma} \delta(z - z') \delta(t - t'),
\]

with \(\nu = c/\gamma\), \(c\) the elastic constant, \(\gamma\) the friction coefficient, \(T\) the temperature of the thermal bath and \(<\cdot \cdot \cdot >\) the average over the white noise \(\xi\), i.e. the thermal average. The term \(f = h/\gamma\) represents the effect of a perturbing field that couples linearly and locally to the height, \(-h(z, t)x(z, t)\). One can also consider other types of perturbations that couple to more complicated functions of the height, as discussed in section 7.

We are interested in characterizing the dynamics of elastic lines with finite and infinite length \(L\). Thus for convenience we shall take periodic boundary conditions in the \(z\)-direction. Following Antal and Rácz we introduce a Fourier representation of the position dependent distance of the line from its average \[25\]

\[
\delta x(z, t) \equiv x(z, t) - \overline{x}(t) = \sum_{n=-\infty}^{\infty} c_n(t) e^{i k_n z},
\]

with \(k_n = 2\pi n/L\). The overline indicates an average over the full line’s configuration;

\[
\overline{x}(t) = \frac{1}{L} \int_0^L dz \ x(z, t)
\]

being then the mean height—or center of mass—of the interface. The Fourier coefficients are given by

\[
c_n(t) = \frac{1}{L} \int_0^L dz \ [x(z, t) - \overline{x}(t)] e^{-i k_n z}
\]

and their evolution is given by

\[
\partial_t c_n(t) = -\nu k_n^2 c_n(t) + f_n(t) + \xi_n(t),
\]

\[
\langle \xi_n(t) \rangle = 0, \quad \langle \xi_n(t) \xi_n(t') \rangle = \frac{2T}{\gamma L} \delta_{n, -n} \delta(t - t'),
\]

\(^4\) We note that the solution presented in \[25\] corresponds actually to the \(t_w = 0\) solution in this paper.
for all \( n \neq 0 \) while \( c_0(t) = 0 \) at all \( t \). Note that equation (6) effectively describes the position of a particle in a harmonic potential with spring constant \( \nu k_n^2 \) which implies an elastic constant softening with decreasing \( n \) or increasing system size \( L \). The solution to equation (6) is

\[
c_n(t) = c_n(0)e^{-\nu k_n^2 t} + e^{-\nu k_n^2 t} \int_0^t dt' e^{\nu k_n^2 t'} [\xi_n(t') + f_n(t')],
\]

where we set the initial time to \( t = 0 \). The coefficients \( c_n(0) \) encode the structure of the initial conditions. We are interested in the evolution after an instantaneous quench from equilibrium at a generic temperature \( T_0 \) to the working temperature \( T \). We are then considering that the initial conditions satisfy

\[
\langle c_n(0)c_m(0) \rangle_{ic} = |c_n(0)|^2 \delta_{n,-m},
\]

where \( |c_n(0)|^2 \) should reflect the equilibrium at \( T_0 \) (see below). When the initial temperature is higher than the working temperature, \( T_0 > T \), the initial state is more disordered than the equilibrium one at \( T \) and the line tends to ‘order’ as time elapses. On the other hand, when \( T_0 < T \), temperature fluctuations roughen the initial—more ordered—configuration of the line. A special case is \( T_0 = 0 \), corresponding to a perfectly flat initial configuration.

The EW energy is just the one of a massless scalar field and the model does not have a finite temperature static phase transition.

Through out the paper we shall present some figures which highlight the main analytical results. Without loss of generality we set \( \gamma = \nu = 1 \) in all the figures. We distinguish noise averaged from fluctuating quantities by enclosing the former with angular brackets. In our expressions we do not write explicitly the \( T, T_0 \) and \( L \) dependences but one has to keep in mind that they are, in principle, always present.

1.3. Background

Before proceeding further let us briefly summarize some previous results on the ageing dynamics of elastic lines in disordered media. It will be useful to compare systems with and without quenched disorder in order to identify the role of disorder on the ageing behavior.

The out-of-equilibrium dynamics of different elastic line models in random environments were analyzed in [10,11], for a directed polymer with solid-on-solid constraints, and in [12], for the vortex glass matter in high temperature superconductors. The main features described in these works that are of relevance here are the following. The ageing of the two-times correlation functions, such as the roughness, mean square displacements, and integrated response functions, can be described in terms of a multiplicative ageing scenario. Thus, for example, the roughness scales as \( w^2 \sim t^\alpha \tilde{w}^2(t/t_w) \), where \( \alpha \) is a small exponent which depends on the disorder intensity and the bath temperature. This scaling leads to special fluctuation-dissipation relations which allow us to obtain a well defined effective temperature \( T_{\text{eff}} \). Interestingly, it was also shown that \( T_{\text{eff}} \), that depends on the disorder intensity but not on the working temperature within numerical accuracy, is correlated with the dynamical glass transition temperature in a model for the vortex glass in high temperature superconductors. The roughness
fluctuations (i.e. the probability density function of the two-times roughness) can also be properly scaled with the inclusion of the multiplicative ageing scaling. In the following, all these features will be discussed in detail and compared to the results for the simple EW equation which is the object of this paper.

1.4. Organization of the paper

The organization of the paper is the following. In section 1.2 we had introduced the model and our conventions. Section 2 deals with the two-times structure factor and section 3 with the associated two-times correlation length. In section 4 we analyze the line’s roughness and in section 5 we study the displacement field as well as the motion of the center of mass. In section 6 we derive a wavevector dependent correlation inspired in the incoherent scattering function typical of particles in interaction. Section 7 analyzes different linear response functions and their relation to the associated correlations that is to say the modifications of the fluctuation-dissipation theorem. In section 8 we study thermally induced fluctuations by computing the probability distribution function of the roughness, displacement and linear responses and we relate to previous studies of elastic lines in equilibrium as well as with the proposal for fluctuations in ageing systems based on the development of time reparametrization invariance. Finally, in section 9 we summarize our results and we present our conclusions. The main features of the chosen observables are shown in figures that display the analytic solution.

2. Two-times structure factor

A key quantity in the two-times evolution of the elastic line is the local displacement of the surface height, defined as $u(z, t, t_w) \equiv x(z, t) - x(z, t_w)$, which allows for the two-times generalization of several quantities. The structure factor associated to this displacement is

$$\langle S_n \rangle(t, t_w) = L \langle |c_n(t) - c_n(t_w)|^2 \rangle$$

$$= \frac{T_0}{\gamma \nu k_n^2} \left(1 - e^{-\nu k_n^2|\Delta t|}\right)^2 e^{-2\nu k_n^2 t_w}$$

$$+ \frac{T}{\gamma \nu k_n^2} \left[2 \left(1 - e^{-\nu k_n^2|\Delta t|}\right) \left(1 - e^{-2\nu k_n^2 t_w}\right) + \left(1 - e^{-2\nu k_n^2|\Delta t|}\right) e^{-2\nu k_n^2 t_w}\right].$$

(10)

The structure factor defined in this way is the two-times generalization of the one commonly used in the solution of the EW equation [15, 29]; by definition, it is symmetric under $t \leftrightarrow t_w$. Hereafter we take $t \geq t_w$, dropping the absolute value in the exponentials.

One can easily obtain several limits that show the complexity of the ageing behavior. Let first consider the stationary limit, which is reached when $\nu k_n^2 t_w \gg 1$ for all $n$ in (10). This condition is fulfilled when $t_w \gg t_L$, where

$$t_L \equiv \frac{L^2}{4\pi^2 \nu}$$

(11)

is a characteristic time marking the onset of finite size equilibration, and which will play an important role in the expressions below. Then, in the stationary limit, the structure
factor becomes
\[ \lim_{\nu k_n^2 t_w \to 1} \langle S_n \rangle(t, t_w) = \frac{2T}{\gamma \nu k_n^2} (1 - e^{-\nu k_n^2 \Delta t}). \quad (12) \]

If one subsequently takes the small wavevector limit for a given \( \Delta t \), i.e. \( \nu k_n^2 \Delta t \ll 1 \), the structure factor reaches the generic asymptote \( \langle S_n \rangle = 2T \Delta t / \gamma \). On the other hand, if one takes the large wavevector limit for a given \( \Delta t \), i.e. \( \nu k_n^2 \Delta t \gg 1 \), one finds the power-law decay of the structure factor
\[ \lim_{\nu k_n^2 \Delta t \gg 1} \langle S_n \rangle(t, t_w) = \frac{2T}{\gamma \nu k_n^2}, \quad (13) \]
typically found in harmonic processes. Thus, this is the typical stationary solution of the structure factor, displaying the \( 1/k_n^2 \) behavior in the large wavevector limit corresponding to the modes already equilibrated at the temperature \( T \) [15]. When \( \Delta t \gg t_L \) all the modes are equilibrated at the working temperature and the structure factor shows the power-law behavior in (13) for all \( k_n \), without the saturation regime. This behavior indicates that the large wavevectors equilibrate first at the working temperature, and then, for increasing time delay, the number of equilibrated modes increases. This is obviously related to a growing correlation length, which will be analyzed in section 3, together with its ageing behavior.

Now, let see how the waiting time dependence modifies the behavior of the structure factor. First one notices that the small wavevector saturation asymptote does not depend on the waiting time, i.e. the saturation value
\[ \lim_{\nu k_n^2 \Delta t \ll 1} \langle S_n \rangle(t, t_w) = \frac{2T}{\gamma} \Delta t \quad (14) \]
is \( t_w \)-independent but time delay dependent. On the other hand, the waiting time is most prominent in the asymptotic power-law regime, \( \nu k_n^2 \Delta t \gg 1 \), where one finds
\[ \lim_{\nu k_n^2 \Delta t \gg 1} \langle S_n \rangle(t, t_w) = \frac{T_0}{\gamma \nu k_n^2} e^{-2\nu k_n^2 t_w} + \frac{T}{\gamma \nu k_n^2} (2 - e^{-2\nu k_n^2 t_w}). \quad (15) \]

From this expression one observes that the asymptotic power-law regime in the large wavevector limit changes from the \( (T_0 + T) / (\gamma \nu k_n^2) \) asymptote at small waiting time, to the \( 2T / (\gamma \nu k_n^2) \) asymptote in the long waiting time limit. For a fixed time delay value such that \( \Delta t > t_w \), and for modes such that \( \nu k_n^2 \Delta t \gg 1 \), the power-law behavior changes from \( (T_0 + T) / (\gamma \nu k_n^2) \) at lower values of \( k_n \) to \( 2T / (\gamma \nu k_n^2) \) at higher values, indicating that the large wavevectors \( \nu k_n^2 t_w \gg 1 \) are equilibrated at the working temperature, while the other modes still reflect the initial condition. This also implies the existence of a waiting time dependent mode, \( k_w \sim (\nu t_w)^{-1/2} \), which separates these two regimes, allowing us to write the structure factor, in the asymptotic time delay limit, in a scaling form:
\[ \lim_{\nu k_n^2 \Delta t \gg 1} \langle S_n \rangle(t, t_w) = k_w^{-2} S(k / k_w). \quad (16) \]

In figures 2–4 we display the analytic results explained above. In figure 2 we show the \( T = 1 \) evolution of the structure factor \( \langle S_n \rangle \) after heating a perfectly flat initial condition, i.e. from \( T_0 = 0 \). The first panel illustrates the time delay saturation value and the
out-of-equilibrium Edwards–Wilkinson equation

Figure 2. $T = 1$ evolution of the structure factor from a flat initial condition ($T_0 = 0 < T$). (a) $t_w = 10$ and the curves correspond to the time delay values $\Delta t = 1, 10, 10^2, 10^3, 10^4$, and $10^5$, from bottom to top. The upper and lower dashed lines are $2T/(\gamma \nu k^2)$ and $T/(\gamma \nu k^2)$, respectively. The crossover between these asymptotes at $\Delta t \sim t_w$ ($k_n \sim k_w$) is clear. (b) Asymptotic time delay limit, $\Delta t \gg t_L$, for $t_w = 1, 10, 10^2, 10^3, 10^4$, and $10^5$, from bottom to top. Dashed lines as in (a).

power-law decay at large $k_n$, where the change between the two asymptotes, which are proportional to $T_0 + T$ and $2T$, is clear. Figure 2(b) shows the crossover at $k_w \sim (\nu t_w)^{-1/2}$, between the two asymptotes using different waiting times and the large time delay limit $\Delta t \gg t_L$. In this case, i.e. $T > T_0$, the equilibrated asymptote, proportional to $2T$, is the upper dashed line. In figure 3(a) we show the dynamics after a quench from $T_0 > T$; the approach to the saturation value at small $k_n$ is non-monotonic. In the asymptotic time delay limit shown in panel (b), $\langle S_n \rangle$ crosses over from a higher value asymptote at small wavevector to a lower value at large wavevector, proportional to $2T$ and corresponding to equilibration, at the waiting time dependent value $k_w$ in equation (16). In figure 4 we study the dependence on the initial condition, $T_0$, finding that the height and width of the bump increase with $T_0$.

3. Two-times correlation length

A practical definition of a growing two-times correlation length, $l(\Delta t, t_w)$, is given by the fact that it marks the transition between the saturation regime at small $k_n$ and the $1/k_n^2$ behavior at large $k_n$:

$$\langle S_{n=1} \rangle(\Delta t, t_w) = \lim_{\nu k_n^2 \Delta t \gg 1} \langle S_{n=L/t} \rangle(\Delta t, t_w).$$

(17)

This is similar to the numerical study in [15] for disordered elastic lines that, however, focuses only on a flat initial condition and does not take into account the waiting time dependence. In that case, for $t_w = 0$ and $T_0 = 0$, a crossover from a power-law growth to a logarithmic growth was found, which is related to the disordered energy landscape. In our case, without disorder, we found a waiting time crossover between two regimes.
marking the separation between non-equilibrated and equilibrated modes, as detailed in the following subsections.

3.1. Stationary limit

In the stationary limit $t_w \gg t_L$ one finds

$$\lim_{t_w \gg t_L} I(\Delta t, t_w) = L \sqrt{1 - e^{-\Delta t/2t_L}}.$$  \hspace{1cm} (18)

doi:10.1088/1742-5468/2007/09/P09008
This expression varies from $\lim_{\Delta t \leq t_L} \lim_{t_w \geq t_L} l(\Delta t, t_w) = \sqrt{4\pi^2 \nu \Delta t}$ in the growing regime to $\lim_{\Delta t \geq t_L} \lim_{t_w \leq t_L} l(\Delta t, t_w) = L$ in the saturation regime. These limits correspond to the increases of the saturation value of the structure factor and the limit in which all the modes are equilibrated at the working temperature, respectively. The saturation time in the stationary regime corresponds to the time delay when these limits are equal, $\Delta t^*_w = 2t_L$, and it is independent of $T$ and $T_0$.

3.2. Decay of the initial condition

Setting $t_w = 0$ (or, more generally, $t_w \ll t_L$) one finds

$$\lim_{t_w \ll t_L} l(\Delta t, t_w) = L \sqrt{\frac{T_0 (1 - e^{-\Delta t/2t_L})^2 + T (1 - e^{-\Delta t/t_L})}{T_0 + T}},$$

(19)

which varies between $\lim_{\Delta t \ll t_L} \lim_{t_w \ll t_L} l(\Delta t, t_w) = \sqrt{8\pi^2 \nu \Delta t T/(T_0 + T)}$ in the growing regime and $\lim_{\Delta t > t_L} \lim_{t_w \leq t_L} l(\Delta t, t_w) = L$ in the saturation regime. Equating these limits one finds the saturation time associated to the growing correlation length $\Delta t^*_w = 0 = (T_0/T + 1) t_L$, which depends on the initial condition.

3.3. Ageing scaling

In between the initial and the late stationary growths the two-times correlation length ages; i.e. it depends also on $t_w$. Equation (17) can be recast in a way that makes the scaling solution apparent

$$\frac{l^2}{t_w} \left[ \left( \frac{T_0}{T} - 1 \right) e^{-4\pi^2 \nu t_w/T^2} + 2 \right] = g \left( \frac{T_0}{T}, \frac{\Delta t}{t_L}, \frac{t_w}{t_L} \right).$$

(20)

Indeed, this equation has a unique solution for $l^2/t_w$ for each set of parameters in the right-hand side. In the growing regime, $\Delta t \ll t_L$, for $T_0 > T$ the two-times length $l$ moves from the upper asymptote, that corresponds to the $t_w = 0$ form in equation (19), to the lower one, that corresponds to the $t_w \gg t_L$ form in equation (18), at a $\Delta t$ that increases with $t_w$. For $T_0 < T$ the trend reverses: $l$ moves from the lower asymptote to the upper one, representing the $t_w \ll t_L$ and $t_w \gg t_L$ limits, respectively. The crossover between the two asymptotes occurs at a $t_w$ dependent $\Delta t$. More precisely, in the growing regime one has

$$\frac{l^2}{t_w} (\Delta t, t_w, T, T_0) = \begin{cases} \frac{c_0}{2} \Delta t/\left( T_0/T + 1 \right) & \Delta t \ll t_w, \\
\frac{c_\infty}{2} \Delta t/\left( T_0/T + 1 \right) & \Delta t \gg t_w, \\
\end{cases}$$

(21)

with $c_\infty = 4\pi^2 \nu$. It is clear that the relative value of the prefactors depends on $T_0 > T$ or $T_0 < T$. The prefactor $c_0 = 2c_\infty/(T_0/T + 1)$ vanishes at $T/T_0 \ll 1$, equals $c_\infty$ at $T_0 = T$ and approaches $2c_\infty$ at $T/T_0 \gg 1$. The behavior of the two-times dependent correlation length is summarized in figure 5 which displays the waiting time dependent growth and subsequent saturation regime for two initial conditions (a) $T_0 < T$ and (b) $T_0 > T$.

One should notice that $l$ increases monotonically with $\Delta t$ and $t_w$ before reaching saturation for both $T_0 > T$ and $T_0 < T$. This increase displays a multiplicative ageing scaling behavior between two asymptotes with opposite trend depending on $T_0 > T$ or...
Figure 5. Ageing of the two-times correlation length $l(\Delta t, t_w)$ for $L = 1000$. (a) $T_0 = 1 < T = 5$ and (b) $T_0 = 5 > T = 1$. The continuous curves are the full solution with $t_w = 1, 10, 10^2, 10^3$ from left to right in (a) and from bottom to top in (b). The thick curves correspond to $\lim_{t_w \gg t_L} l$ and $\lim_{t_w \ll t_L} l$; the former is located below for $T > T_0$ and above for $T < T_0$. The dashed straight lines correspond to the infinite size limit $L \to \infty$ taken at the outset and describe the $\Delta t^{1/2}$ growth law.

$T_0 < T$. These kind of results where obtained, with additive ageing scaling, in the out-of-equilibrium relaxation of mixtures of soft spheres and Lennard-Jones particles [20] and the 3d EA spin glass [23, 21] after a quench from a high temperature (although the finite size saturation was not reached in these numerical studies). The heating case was not considered in these models.

4. Two-times roughness

The statics and equilibrium dynamics of elastic manifolds is usually understood in terms of the time, temperature and system size dependence of their averaged roughness or width [1]. In most studies of interface dynamics one compares the time dependent configuration to the initial state, typically taken to be perfectly flat (equilibrium at zero temperature). The thermal averaged (one-time dependent) roughness is then

$$\langle w^2 \rangle(t) = L^{-d} \int d^dz \langle [x(z, t) - x(z, 0)]^2 \rangle,$$

(22)

where $x$ is the height of the surface and $z$ is the position in the $d$-dimensional substrate typically with cubic geometry and linear size $L$.

The initial grow of the roughness is linear with time,

$$\lim_{\Delta t \to 0} \langle w^2 \rangle(t) = 2Tt,$$

(23)

which corresponds to a normal diffusion regime in which the beads on the line are still uncorrelated. Note that there is no ballistic regime since the EW equation represents overdamped Langevin dynamics. However, these inertial effects could be relevant in
polymer molecular dynamics studies. The normal diffusion regime can be considered as a transient before the correlated dynamics of the interface is reached. In the single particle regime the roughness does not age. In the following we concentrate on the correlated ageing dynamics of the line.

The thermal (and disorder averaged if random interactions are present) roughness follows the Family–Vicsek scaling [30], which means that it crosses over from growth to saturation at \( t_x \sim L^z \) [1]:

\[
\langle w^2 \rangle(t) \sim L^z f(t/t_x),
\]

where the scaling function obeys \( f(y) \sim y^\beta \) for \( y \ll 1 \) and \( f(y) \sim 1 \) for \( y \gg 1 \), with \( \zeta, \beta \) and \( z = \zeta/\beta \) the roughness, growth and dynamic exponents, respectively. For the Edwards–Wilkinson (EW) line in \( 1+d \) dimensions \( \zeta = 2-d, \beta = 1-d/2 \) and \( z = 2 \). In the presence of disorder, \( \zeta \) is expected to take a ‘thermal’ value, \( \zeta_{\text{th}} \) for \( L < L_c(T) \) and a larger ‘disorder’ dominated value, \( \zeta_{\text{dis}} \) for \( L > L_c(T) \), both exponents being \( T \)-independent [1]. The other exponents, \( \beta \) and \( z \) may depend on \( T \) [13] or even logarithmic time dependences may exist [15].

We wish to take into account the waiting time \( t_w \) and consider also more general initial conditions. To this end we generalize the definition in equation (22) to

\[
\langle w^2 \rangle(t, t_w) \equiv \frac{1}{L} \int_0^L dz \langle [\delta x(z, t) - \delta x(z, t_w)]^2 \rangle = \frac{2}{L} \sum_{n=1}^{\infty} \langle S_n \rangle(t, t_w),
\]

with \( \delta x(z, t) \) defined in equation (3) and specialized to \( d = 1 \). Note that the zero mode is not included in the sum.

In general, one expects the dynamics to become stationary after an equilibration time \( t_L \); the generalized thermal averaged roughness should then scale as in equation (24) with \( t \) replaced by \( \Delta t \). For not too short \( L \), \( t_L \) may become very long and the dynamics might remain non-stationary with \( \langle w^2 \rangle \) depending on \( t_w \) explicitly for \( t_w < t_L \). In [11] we conjectured that the scaling of the roughness in the non-equilibrium relaxation of infinitely long elastic lines with or without quenched disordered potentials follows the law

\[
\langle w^2 \rangle(\Delta t, t_w) \sim \ell^\zeta(\Delta t) \mathcal{F} \left[ \frac{\ell(t)}{\ell(t_w)} \right],
\]

with \( \ell(t) \) a growing length (dimensions are restored by prefactors that we omit) and \( \mathcal{F} \) a scaling function. For each waiting time this form approaches a stationary growth regime \( \langle w^2 \rangle \sim \ell^\zeta(\Delta t) \) when \( t_w \ll \Delta t \ll t_L \) if \( \mathcal{F}(y) \sim y^\zeta \) for \( y \gg 1 \). It is also reasonable to assume \( \mathcal{F}(\ell(t)/\ell(t_w)) \sim \ell^\zeta(\Delta t) \) for \( \Delta t \ll t_w \), that leads to a stationary growth of the averaged roughness at very short time delays. This result is explicitly realized in the power-law case. One may extend this conjecture to apply to manifolds with internal dimension \( D \) in a transverse space with dimension \( d \). The functional form of the growing length, \( \ell \), the scaling function, \( \mathcal{F} \), and the values of the exponent are expected to vary from case to case.

In a series of numerical studies one established that, in the numerically accessible times, the quenched dynamics of a disordered \( 1+1 \) lattice model [10,11] satisfies the scaling in equation (26) with \( \ell(t) \sim t^{\nu/\zeta} \) and \( \alpha/\zeta \) a rather small exponent. A crossover to a logarithmic time dependence [15] is not excluded although it was not seen in the
out-of-equilibrium relaxation. It was shown that the roughness ages, by crossing over between two asymptotes, for $\Delta t \gg t_w$ and $\Delta t \ll t_w$, thus having

$$\langle w^2 \rangle \sim c_{1,2}(T) \Delta t^{\alpha(T)}, \quad \text{with } \alpha(T) < \beta_{\text{EW}} = 0.5,$$

(27)

and different proportionality constants. The waiting time dependence appears in the way these asymptotes are approached. $\alpha(T)$ is a generalization of the growth exponent, $\beta$, in surface growth literature, and $\alpha(T) < \beta_{\text{EW}}$, with $\beta_{\text{EW}}$ the Edwards–Wilkinson value, reflects that quenched disorder slows down the dynamics.

In order to better understand this ageing behavior, trying to separate the effects due to disorder from those related to the intrinsic elastic character of the line, we present here results for the EW case in 1 + 1 dimensions without disorder. In this case the two-times averaged roughness is given by

$$\langle w^2 \rangle(\Delta t, t_w) = \frac{6w_0^2}{\pi^2} \sum_{n=1}^{\infty} b_n(\Delta t, t_w) + \frac{6w_0^2}{\pi^2} \sum_{n=1}^{\infty} a_n(\Delta t, t_w),$$

(28)

$$n^2a_n(\Delta t, t_w) = 2(1 - e^{-n^2\Delta t/2L})(1 - e^{-n^2t_w/t_L}) + (1 - e^{-n^2\Delta t/2L})e^{-n^2t_w/t_L},$$

(29)

$$n^2b_n(\Delta t, t_w) = (1 - e^{-n^2\Delta t/2L})^2e^{-n^2t_w/t_L},$$

(30)

where we used the definitions

$$w_0^2 \equiv T_0 L/(12 \gamma \nu), \quad w_\infty^2 \equiv TL/(12 \gamma \nu).$$

(31)

The averaged two-times roughness can be written in terms of the scaled times

$$\langle w^2 \rangle(\Delta t, t_w) = \langle w^2 \rangle \left( \frac{\Delta t}{t_L}, \frac{t_w}{t_L} \right) = \langle w^2 \rangle \left( \frac{\Delta t}{\Delta t}, \frac{\Delta t}{t_w} \right),$$

(32)

while temperatures appear separately, through $w_0^2$ and $w_\infty^2$. We first focus on the time dependence of the roughness of infinite lines, $L \to \infty$, before testing the dynamics at different asymptotic limits, and we later reverse the order of limits by considering finite lines, $L < \infty$. We display the analytic results in figures 6–8 in which we approximate the series in the analytic expressions by using finite sums, $\sum_{n=0}^{\infty} \to \sum_{n=0}^{M}$, with $M = 1000$ in all cases, except for lines with $L = 3000$ for which we used $M = 500$.

4.1. Infinite system size

By taking $L \to \infty$ at the outset $t_L$ diverges and

$$\lim_{L \to \infty} \langle w^2 \rangle(\Delta t, t_w) = \sqrt{\frac{2t_w}{\pi \gamma^2 \nu}} \left[ (T - T_0) \left( 1 + \sqrt{\frac{\Delta t}{t_w}} + 1 - \sqrt{\frac{2\Delta t}{t_w}} + 4 \right) + 2T \sqrt{\frac{\Delta t}{2t_w}} \right],$$

(33)

which can be written in the scaling form

$$\lim_{L \to \infty} \langle w^2 \rangle(\Delta t, t_w) = \tilde{t}_w^{1/2} \bar{w}^2 \left( \frac{\Delta t}{t_w} \right),$$

(34)

and admits the scaling form in equation (26) with $\ell(t) \sim t^{1/2}$. Comparing now $\Delta t$ to a long waiting time, that is to say taking $\Delta t \ll t_w$, one finds that the waiting time
Figure 6. (a) Stationary roughness \( (t_w \gg t_L) \) as a function of \( \Delta t \). Thin lines correspond to different system sizes \( L = 100, 300, 1000, 3000 \), from bottom to top all at \( T = 1 \). Thick lines correspond to \( L = 1000 \) and \( T = 2 \) (bottom) and \( T = 5 \) (top). Note that the saturation time \( t_s \) does not depend on \( T \). The dashed line is the limit \( L \to \infty \) for \( T = 1 \). (b) Roughness with \( t_w = 0 \), \( T = 1 \), and different initial conditions corresponding, from bottom to top, to \( T_0 = 0 \), \( 1 \), and \( 5 \). In the \( t_w \gg t_L \) limit, the roughness goes to its stationary solution, also given by the curve in the middle. Dashed lines correspond to \( \lim_{L \to \infty} \langle w^2 \rangle \) given by equation (35).

dependence determines the crossover between two square-root dependences in \( \Delta t \) with different, temperature dependent, prefactors:

\[
\langle w^2 \rangle(\Delta t, t_w) = \begin{cases} 
  c_w^2(T) \Delta t^{1/2} & \Delta t \ll t_w, \\
  c_w^2(T, T_0) \Delta t^{1/2} & \Delta t \gg t_w,
\end{cases}
\]

with \( c_w^2 = [T + T_0(\sqrt{2} - 1)]\sqrt{2/(\pi \gamma^2 \nu)} \) and \( c_w^2 = 2T/\sqrt{\pi \gamma^2 \nu} \). Note that \( c_0^2 < c_w^2 \) for \( T > T_0 \), the two constants are identical at \( T = T_0 \), and \( c_0^2 > c_0^2 \) for \( T < T_0 \). The two trends are shown in figure 7(b). These results are of the generic form proposed in [11], see equation (26), with \( \alpha = \beta_{EW} = 1/2 \) independently of temperature, \( \ell(t) \sim t^{1/2} \), and temperature dependences of the constants made explicit.

4.2. Finite lines

For finite \( L \), by considering the long waiting time limit \( t_w \gg t_L \) the stationary Family–Vicsek scaling form (24) is recovered. Indeed, one has that

\[
\lim_{t_w \gg t_L} \langle w^2 \rangle(\Delta t, t_w) = w_\infty^2 \sum_{n=1}^{\infty} \frac{12}{\pi^2 n^2} \left( 1 - e^{-n^2 \Delta t/2 t_L} \right). 
\]

This form for the stationary roughness contains the growing regime where \( \lim_{\Delta t \ll t_L} \lim_{t_w \gg t_L} \langle w^2 \rangle = c_w^2(T) \Delta t^{1/2} \) as in equation (35), and later crosses over to the saturation value

\[
\lim_{\Delta t \gg t_L} \lim_{t_w \gg t_L} \langle w^2 \rangle(\Delta t, t_w) = 2w_\infty^2 = \frac{TL}{6\gamma \nu}. 
\]
Figure 7. Ageing of the two-times roughness, $t_w = 0, 10^0, 10^1, 10^2, 10^3, 10^4, 10^5$. (a) $L = 1000$, $T_0 = 5$ and $T = 1$, $t_w$ increases from top to bottom. The upper and lower dashed lines correspond to $L \to \infty$ with $t_w = 0$ and $t_w \gg t_L$, respectively. (b) $L \to \infty$; $t_w$ increases from top to bottom in the lower set of curves ($T_0 = 5$ and $T = 1$) and from bottom to top in the upper set of curves ($T_0 = 1$ and $T = 5$).

On the other hand, the averaged roughness at $t_w = 0$ for finite $L$, or more precisely for $t_w \ll t_L$, also follows the familiar Family–Vicsek scaling form (24). In this case, the roughness crosses over from a growing regime $\lim_{\Delta t \ll t_L} \lim_{t_w \ll t_L} w^2 (T, T_0) \Delta t^{1/2}$ to a saturation value which depends on the initial condition,\[ \lim_{\Delta t \gg t_L} \lim_{t_w \ll t_L} \langle w^2 \rangle (\Delta t, t_w) = w_0^2 + w_\infty^2 = \frac{(T_0 + T)L}{12 \gamma \nu}. \]Note that the saturation value reached in the stationary regime, after having taken $t_w \gg t_L$, is twice the saturation value obtained from the flat initial condition $T_0 = 0$. This result demonstrates how important it is to be careful with the choice of $t_w$ to ensure that one has reached the stationary regime when using numerical simulations.

When considering all the waiting time dependence the roughness interpolates between the different limiting values given above. For sufficiently large but finite systems it is possible to find a well defined ageing of the growing regime when both $\Delta t$ and $t_w$ are smaller than the saturation time. In this case one recovers the scaling behavior found in the infinite size system, equations (33) and (35). For finite $L$ though this scaling form terminates at a characteristic time delay, $t_x(t_w) \propto t_L$—see below—and $\langle w^2 \rangle$ later saturates at a waiting time dependent value, which interpolates between the two limiting saturation values, equations (37) and (38). These results are the same as the ones obtained using the reverse order of time limits, see (37) and (38). The $t_w$ dependence in the saturation regime is shown in figure 8.

Although in the previous analysis we used $t_L$ as a reference time to determine the long time delay limit. One can be more precise and define a waiting time dependent saturation time, $t_x(t_w)$. Equating the growing and saturation behavior of the roughness, one can...
extract the saturation times at two extreme values of $t_w$:

$$t_x(t_w \ll t_L) = \frac{\pi}{288\nu} \left[ \frac{T + T_0}{T + T_0(\sqrt{2} - 1)} \right]^2 L^2,$$

$$t_x(t_w \gg t_L) = \frac{\pi}{144\nu} L^2. \quad (40)$$

Both results are proportional to $t_L$, and $t_x(t_w)$ interpolates between these two values. This justifies the use of $t_L$ as the reference time delay before saturation; the prefactors are finite for all $T$ and $T_0$, the former is larger (smaller) for $T < T_0$ ($T > T_0$) and become identical at $T = T_0$. The saturation time and its dependence on $L$ are visible in figures 6 and 7(a).

**4.3. Scaling relations**

Here we present the main results concerning the ageing properties of the roughness. We have shown for the ageing regime that

$$t_x \sim a(T) L^z, \quad z = 2,$$

$$\langle w^2 \rangle_{\Delta t > t_x} \sim L^\zeta, \quad \zeta = 1,$$

$$\langle w^2 \rangle_{\Delta t \ll t_x} \sim c(T, T_0, \Delta t/t_w) t_w^3, \quad \beta = 1/2,$$

see equations (39) and (40), equation (31) and equation (35), respectively. The prefactor $c(T, T_0, \Delta t/t_w)$ approaches $c_{w^2}(T, T_0) (\Delta t/t_w)^{1/2}$ and $c_0^{w^2}(T, T_0) (\Delta t/t_w)^{1/2}$, for $\Delta t \ll t_w$ and $\Delta t \gg t_w$, respectively. These relations imply

$$t_x \sim a(T)(w_\infty^2)^{z/\zeta} \sim \left[ a(T)^{\xi/z} w_\infty^2 \right]^{z/\zeta}, \quad (41)$$

where we have used that $\langle w^2 \rangle_{\Delta t > t_x} \sim w_\infty^2$ for simplicity. From matching the end of the growth regime with saturation at $t_w \ll \Delta t = t_x$ one has

$$c(T, T_0, \Delta t/t_w) L^3 \sim c_{0}^{w^2}(T, T_0) l_3^3 \sim w_\infty^2 \quad \text{and} \quad \beta = \zeta/z, \quad (42)$$

$$\text{doi:10.1088/1742-5468/2007/09/P09008} \quad 17$$
Out-of-equilibrium Edwards–Wilkinson equation

(43)

Note that the exponents take simple values in the EW case but, in general, they can be $T$ dependent (not $\zeta$) [13].

5. Mean squared and center of mass displacements

The two-times roughness, equation (25), may also be written as

$$
\langle w^2 \rangle(t, t_w) = \langle B \rangle(t, t_w) - \langle D \rangle(t, t_w)
$$

$$
= \left\langle \left[ x(z, t) - x(z, t_w) \right]^2 \right\rangle - \left\langle \left[ x(t) - x(t_w) \right]^2 \right\rangle,
$$

(44)

where $\langle B \rangle$ and $\langle D \rangle$ represent the averaged mean squared displacement of the differential line segments and the center of mass of the line, respectively, see figure 1. This relation simply states that the roughness is a measure of the fluctuations around the center of mass of the line.

It is simple to show that the center of mass diffuses normally. Indeed, integrating equation (1) over the line length one has

$$
\partial_t x(t) = \xi'(t),
$$

with $\xi'(t) = (1/L) \int_0^L dz \xi(z, t)$, $\langle \xi'(t) \rangle = 0$, $\langle \xi'(t) \xi'(t') \rangle = (2T/\gamma L) \delta(t-t')$, and

$$
\langle D \rangle(t, t_w) = \langle D \rangle(\Delta t) = \frac{2T}{\gamma L} \Delta t.
$$

(45)

The diffusion constant is an inverse function of the line length.

In previous studies of the elastic line out-of-equilibrium dynamics [10, 18] one focused on the mean squared displacement that, in the case of the EW line, is just given by $\langle B \rangle(t, t_w) = \langle w^2 \rangle(t, t_w) + 2T/\gamma L \Delta t$. At short time delay $w^2$ and $B$ are practically identical while in the saturation regime the displacement is just given by the normal diffusion law.

6. The incoherent scattering function

The dynamics of glassy systems is usually analyzed in terms of the wavevector dependent incoherent scattering function:

$$
\langle C_q \rangle(t, t_w) = N^{-1} \sum_{i=1}^N \langle e^{i\bar{q} \cdot \vec{r}_i(t) - \bar{q} \cdot \vec{r}_i(t_w)} \rangle,
$$

(46)

with $N$ the total number of particles, $\vec{r}_i(t)$ the time dependent position of particle $i$ and $\bar{q}$ a wavevector. $C_q$ is measured numerically and experimentally. In the context of elastic lines, one defines

$$
\langle C_q \rangle(\Delta t, t_w) = L^{-1} \int d\vec{x} \left\langle e^{i\bar{q} \cdot [\delta x(z, t) - \delta x(z, t_w)]} \right\rangle.
$$

(47)

In the EW case the displacement in the exponential is a Gaussian random variable and

$$
\langle C_q \rangle(\Delta t, t_w) = \exp \left( -\frac{q^2}{2} \langle w^2 \rangle(\Delta t, t_w) \right).
$$

(48)
The incoherent scattering function $\langle C_q \rangle$ is simply related to the roughness, $\langle w^2 \rangle$, analyzed in section 4. Figure 9 displays the time delay decay of $\langle C_q \rangle$ at $q = 0.1$ using several waiting times. For finite lines the saturation in $\langle w^2 \rangle$ is attained at sufficiently long $\Delta t$ and the correlation reaches a waiting time dependent plateau with its height increasing with $t_w$, figure 9(a). For sufficiently long lines the saturation regime can be pushed beyond the observed time delay window as exemplified in the limit $L \to \infty$, figure 9(b). In figure 10 we display the dependence of $\langle C_q \rangle$ on $\Delta t$ for fixed $t_w$ and different values of $q$. One notices that the small $q$ correlations saturate in the $\Delta t$ window while large $q$ correlations relax to zero. This is to be expected since $q^2 w^2_\infty$ scales as $q^2 L$ and the effect of decreasing $q$ is like decreasing $L$. Note the similarity between these plots and light scattering measurements in clay suspensions (laponite) [19].

7. Response functions and FDT

We now compute the linear response function of several observables related to the two-times correlations studied above.

7.1. Center of mass response

In order to evaluate the linear response we have to switch on a perturbing field coupled to the observable of interest. Let us start with the linear response function associated with the center of mass diffusion, $\langle \chi^D \rangle$. The effect of a perturbing field, $h$, applied on the center of mass after time $t_w$ is described by the term

$$\mathcal{H}^{t_D} = -h L \bar{\pi}(t) \theta(t - t_w) = -h \int_0^L dz \, x(z,t) \, \theta(\Delta t),$$

(49)
that is added to the energy. Calling $\overline{\mathbf{r}}^h(t)$ and $\overline{\mathbf{r}}(t)$ the center of mass position with and without field, respectively, the linear response function is

$$\langle \chi^D \rangle (t, t_w) = \frac{1}{hL} \langle \overline{\mathbf{r}}^h(t) - \overline{\mathbf{r}}(t) \rangle,$$

which depends only on the time difference and satisfies the FDT for any $t$ and $t_w$.

$$\langle D \rangle (\Delta t) = 2T \langle \chi^D \rangle (\Delta t).$$

### 7.2. Roughness response

The energy contribution of a field conjugated to the roughness is

$$\mathcal{H}'^{w^2} = -h \int_0^L dz \left[ x(z, t) - \overline{\mathbf{r}}(t) \right] s(z) \theta(\Delta t).$$

As usual, $s(z)$ are i.i.d. quenched random variables taking values $s(z) = \pm 1$ with equal probability: $\langle s(z) \rangle = 0$ and $\langle s(z) s(z') \rangle = \delta(z - z')$. The associated response function is

$$\langle \chi^{w^2} \rangle (t, t_w) = \frac{1}{hL} \left\langle \int_0^L dz \left[ \delta x^h(z, t) - \delta x(z, t) \right] s(z) \right\rangle$$

$$= \frac{2}{h} \sum_{n=1}^{\infty} \left\langle \left[ c_n^h(t) - c_n(t) \right] s_n \right\rangle,$$

with $s_n$ defined through $\delta s(z) = s(z) - \overline{s} = \sum_{n=-\infty}^{\infty} s_n e^{ik_nz}$. Here $\langle \cdots \rangle$ also indicates the average over the $s_n$ distribution. One finds

$$\langle \chi^{w^2} \rangle (\Delta t) = (1 - e^{-\nu k^2 \Delta t}),$$

where we used $\langle s_n^2 \rangle = 1/L$. Interestingly enough, the linear response is stationary for all $t_w$ while the roughness is not. Therefore, the FDT is not respected for waiting times $t_w \ll t_L$, and its modification is discussed in section 7.4 and figure 11. In the stationary limit $t_w \gg t_L$, the roughness becomes stationary and the FDT holds, i.e.

$$\lim_{t_w \gg t_L} \langle w^2 \rangle (\Delta t, t_w) = 2T \langle \chi^{w^2} \rangle (\Delta t).$$

Figure 10. $\langle C_q \rangle$ for $L = 1000$, $t_w = 10^2$ and different wavevectors $q$ as indicated. $T_0 = 5$ and $T = 1$. 

doi:10.1088/1742-5468/2007/09/P09008
Figure 11. Violation of the FDT in the ageing regime of the EW equation. The parametric plots \( \tilde{\chi}^{w^2} = 1/(2T_{\text{eff}}) \tilde{w}^2 \) are constructed from the scaled variables defined by \( \tilde{w}^2 = t_w^{-1/2}\langle w^2 \rangle \) and \( \tilde{\chi}^{w^2} = t_w^{-1/2}\langle \chi^{w^2} \rangle \). (a) The parametric plot in linear scale showing the departure from the FDT (dotted line). Upper and lower sets of curves correspond to \( T_0 = 1, T = 5 \) and \( T_0 = 5, T = 1 \), respectively. Thick dashed lines represent the \( L \to \infty \) limit. Thin lines are for \( L = 1000 \) and different waiting times, and show the finite size signature in the FDT parametric plot. The inset shows the large scale violation of FDT with two straight lines, which corresponds to effective temperatures larger (for \( T_0 > T \)) and smaller (for \( T_0 < T \)) than the working temperature \( T \). The initial FDT regime is not clearly observed in this scale. (b) The parametric plot in log–log scale showing that the violation of the FDT is given by equation (63) at \( \Delta t \gg t_w \).

This statement also implies that the FDT does not hold for \( t_w = 0 \) and \( T = 0 \), pointing again that one should be careful with the choice of \( t_w \) and the stationary limit.

7.3. Mean squared displacement response

The effect of a perturbing field conjugated to the mean squared displacement is represented by

\[
\mathcal{H}' = -h \int_0^L dz \, x(z,t) s(z) \theta(\Delta t),
\]

with the random \( s(z) \) distributed as above. The linear response function is defined as

\[
\langle \chi^B(t, t_w) \rangle = \frac{1}{hL} \left\langle \int_0^L dz \left[ x^h(z,t) - x(z,t) \right] s(z) \right\rangle
= \langle \chi^{w^2}(t, t_w) \rangle + \frac{\langle s \rangle}{h} \left[ x^h(t) - x(t) \right].
\]

The last term of this expression represents the center of mass response to a field of intensity \( h' = h/\langle s \rangle \). Thus, in the long waiting time limit, \( \langle \chi^B \rangle \) is also stationary and simply related to the roughness and center of mass responses,

\[
\langle \chi^B(\Delta t) \rangle = \langle \chi^{w^2}(\Delta t) \rangle + \langle \chi^D(\Delta t) \rangle.
\]
In this case, the FDT is not satisfied in general but in the stationary regime it is:

$$\lim_{t_w \gg t_L} \langle B \rangle(t, t_w) = \langle B \rangle(\Delta t) = 2T \langle \chi^2 \rangle(\Delta t).$$  \hspace{1cm} (59)

7.4. FDT and effective temperature

Since FDT holds for $t_w \gg t_L$, it is interesting to study the violation of the FDT at finite $t_w$. To this end we use the $L \rightarrow \infty$ limit, where the roughness takes the scaling form in equation (33), and

$$\lim_{L \rightarrow \infty} \langle \chi_w^2 \rangle(\Delta t) = \frac{1}{2T_{\text{eff}}} \tilde{w}^2,$$  \hspace{1cm} (60)

which, in this case, depends on $T$, $T_0$, $\Delta t$ and $t_w$. From equation (33) we obtain

$$T_{\text{eff}} = T \left[ 1 + \frac{T - T_0}{T} \left( \sqrt{\frac{t_w}{2\Delta t}} + \frac{1}{\sqrt{2}} \sqrt{1 + \frac{t_w}{\Delta t}} - \sqrt{1 + \frac{2t_w}{\Delta t}} \right) \right].$$  \hspace{1cm} (62)

In the case $T = T_0$ one recovers $T_{\text{eff}} = T$ as expected. One can check that $T_{\text{eff}} > T$ or $T_{\text{eff}} < T$ whenever $T_0 > T$ or $T_0 < T$. In the two extreme cases $\Delta t \ll t_w$ and $\Delta t \gg t_w$ one finds two waiting time independent values of $T_{\text{eff}}$:

$$T_{\text{eff}} = \begin{cases} T, & \Delta t \ll t_w, \\ T + (T_0 - T) \left( 1 - \frac{1}{\sqrt{2}} \right), & \Delta t \gg t_w, \end{cases}$$  \hspace{1cm} (63)

see figures 11 and 12, indicating that fast modes are equilibrated while the slow ones are not. These results are very similar to what has been found numerically for the dynamics of elastic lines in a quenched random potential [10] and in models of interacting elastic lines in quenched random environments that describe the vortex glass in high $T_c$ superconductors [12], see figure 12. Note that the waiting time dependence in $T_{\text{eff}}$ only marks the crossover between the two asymptotic regimes. At still longer $\Delta t$ such that $\Delta t \gg t_x \sim t_L$ the FDT result, $T_{\text{eff}} = T$, is recovered. This is shown in figure 13, a result that, once again, resembles what was found in laponite [31].

8. Fluctuations

Up to this point we studied the scaling properties of several two-times quantities averaged over the length of the line and thermal noise. A more refined investigation of interface dynamics, currently done theoretically and experimentally, deals with the fluctuating quantities such as the distribution functions of the width of heights [22], the density of local maxima or minima of heights [32,33], the statistics of first passage times, etc. It is also clear by now that to gain a complete understanding of glassiness one should also
understand the dynamic fluctuations \cite{23,24}. In the glassy context one expects that the competition between different time or length scales in the system reflects in the way the probability distribution functions (pdfs) behave.

Rácz proposed that, for elastic systems, the scaling form of the distribution function characterizing the roughness fluctuations may serve to classify the systems into different universality classes \cite{22}. For a given system size the pdf of the roughness, $P_L(w^2)$, scales as \cite{34}

$$w^2_\infty P_L(w^2) = \Phi \left( \frac{w^2}{w^2_\infty} \right),$$  \tag{64}

in the saturation regime, and the form of the scaling function was shown to be useful to differentiate between the EW and KPZ universality classes \cite{22}. The full stationary pdf of the EW roughness was also computed, showing the same scaling behavior as in equation (64). These works considered only the $t_w = 0$ case with a flat initial condition ($T = 0$). Recently, a simulation study of a disordered elastic line model defined on the lattice demonstrated that the scaling form of the distribution function must be modified to include the ageing effects \cite{11}.

In the body of this section we analyze the thermal noise-induced dynamic fluctuations of the two-times quantities defined previously during the ageing relaxation.
where the independent Fourier modes defined in equation (3)

\[ G = \text{Using the independent Fourier modes defined in equation (3)} \]

8.1. Roughness distribution

Rewriting the roughness \( w^2 \) in terms of \( u(z,t,t_w) = x(z,t) - x(z,t_w) \), the pdf of the two-times roughness, \( P_L(u^2) \), is given by

\[
P_L(u^2) = \int D[x]D[x'] \delta \left[ u^2 - \left( \frac{u(t,t_w) - u(t,t_w)}{2} \right) \right] p(x,t; x', t_w),
\]

where \( D[x] \) is the measure over \( x \) configurations; \( x \) and \( x' \) represent the configurations at time \( t \) and \( t_w \), respectively; and \( p(x,t; x', t_w) \) is their joint probability density. The Laplace transform \( G_L(\lambda) = \int_0^\infty \alpha P_L(\alpha) e^{-\lambda \alpha} \), can be written as the path integral

\[
G_L(\lambda) = \frac{1}{\mathcal{N}} \prod_{n=1}^\infty \int D[c_n]D[c_n'] \exp \left[ -\lambda \left( \frac{\int_{t}^{t_w} \left( \frac{c_n(t) - c_n(t_w)}{2} \right)^2 dt}{2} \right) \right] P[\{c_n(t), c_n(t_w)|c_n(0)\}] e^{-\lambda |c_n(t)-c_n(t_w)|^2},
\]

Figure 13. The effective temperature \( T_{\text{eff}} \) as a function of time delay for \( t_w = 1, 10, 10^2, 10^3, 10^4 \) and \( 10^5 \) from top to bottom; \( L = 1000, T_0 = 5 \) and \( T = 1 \). The dashed line is \( \lim_{t_w \to \Delta t} \lim_{L \to \infty} T_{\text{eff}} \) (the same dashed curve as in figure 12(a)) and justifies the shoulder for small \( t_w \). The inset shows \( T_{\text{eff}}(\Delta t = 10^6 \gg t_L, t_w) \) as a function of \( t_w \); it is clear that at sufficiently long \( t_w \) one recovers \( T_{\text{eff}} = T \).

8.1. Roughness distribution

Using the independent Fourier modes defined in equation (3)

\[
G_L(\lambda, t, t_w) = \mathcal{N} \prod_{n=1}^\infty \int dc_n dc_n' \exp \left[ -\lambda \left( \frac{\int_{t}^{t_w} \left( \frac{c_n(t) - c_n(t_w)}{2} \right)^2 dt}{2} \right) \right] P[\{c_n(t), c_n(t_w)|c_n(0)\}] e^{-\lambda |c_n(t)-c_n(t_w)|^2},
\]

where \( \mathcal{N} \) is a normalization factor ensuring \( G_L(0) = 1 \) at all times. The quantity \( p[c_n(t), c_n(t_w)|c_n(0)] \) is the joint probability density of having \( c_n \) at time \( t \) and \( c_n' = c_n(t_w) \) at time \( t_w \), given the initial condition \( c_n(0) \). This quantity can be expressed as \( p[c_n(t), c_n(t_w)|c_n(0)] = p[c_n(t)|c_n(t_w)]p[c_n(t_w)|c_n(0)] \), where \( p[c_n(t)|c_n(t')] \) is the conditional probability of evolving from \( c_n(t') \) to \( c_n(t) \) in the time interval \( t - t' \). This states that the joint probability is simply the product of the probabilities of evolving from the initial condition to the intermediate configuration \( c_n(t_w) \), and from there to the configuration.
c_n(t), and satisfies
\[ p[c_n(t)|c_n(0)] = \int dc'_n dc''_n p[c_n(t), c_n(t')|c_n(0)]. \] (67)

The conditional probability \(p[c_n(t)|c_n(t')]\) is given by the complex Gaussian
\[ p[c_n(t)|c_n(t')] = \frac{1}{2\pi \sigma^2_n(t-t')} \exp\left(-\frac{|c_n(t) - c_n(t') e^{-\nu k^2_n(t-t')}|^2}{2\sigma^2_n(t-t')}\right), \] (68)
where
\[ \sigma^2_n(t-t') = \frac{T}{L^2 \nu k^2_n} \left[1 - e^{-2\nu k^2_n(t-t')}\right]. \] (69)

Therefore, the joint probability function for the Fourier modes at times \(t\) and \(t_w\) becomes
\[ p[c_n(t), c_n(t_w)|c_n(0)] = \frac{e^{-|c_n(t) - c_n(t_w) e^{-\nu k^2_n t_w}|2/2\sigma^2_n(t_w)}}{(2\pi)^2 \sigma^2_n(\Delta t) \sigma^2_n(t_w)}. \] (70)

After some algebra one finds the normalization factor \(\mathcal{N} = \prod_{n=1}^{\infty} 16\pi^2 \sigma^2_n(\Delta t) \sigma^2_n(t_w)\) and
\[ G_L(\lambda) = \prod_{n=1}^{\infty} \frac{e^{-2\lambda|c_n(0)|2(1-e^{-\nu k^2_n \Delta t})^2 e^{-2\nu k^2_n t_w} / (1+\lambda w^2_n a_n(\Delta t, t_w))}}{1+\lambda w^2_n a_n(\Delta t, t_w)}, \] (71)

with the coefficients \(a_n\) defined in equation (29). This is the two-times generalization of the result in [25], including arbitrary initial conditions. The averaged two-times roughness follows from \(G_L\) as \(\langle w^2(\Delta t, t_w) = -\partial_\lambda G_L(\lambda, t, t_w)|_{\lambda=0}\rangle\), which allows to recover the result in equation (28) for the initial condition \(|c_n(0)|^2 = T_0/(L^2 \nu k^2_n)\).

Since the two-times roughness pdf is given by
\[ P_L(w^2) = \int_{-\infty}^{\infty} \frac{d\lambda}{2\pi} e^{\lambda w^2} G_L(\lambda, t, t_w), \] (72)
we can extract its scaling properties from the ones of \(G_L(\lambda)\) in equation (71). Using \(y = \lambda w^2_\infty\) and \(2|c_n(0)|^2/w^2_\infty = 6w^2_\infty/(\pi^2 n^2 w^2_\infty) = 6/(\pi^2 n^2) s^2_0\) we find
\[ w^2_\infty P_L(w^2) = \Phi \left(\frac{w^2}{w^2_\infty}, \frac{\Delta t}{t_w}, \frac{t_w}{t_L}, \frac{s^2_0}{s^2_0}\right), \] (73)
\[ \Phi \left(x; \frac{\Delta t}{t_L}, \frac{t_w}{t_L}, s^2_0\right) = \int_{-\infty}^{\infty} \frac{dy}{2\pi i} e^{iy x} \prod_{n=1}^{\infty} \frac{e^{-y s^2_0 b_n(\Delta t/t_L, t_w/t_L)/(1+y a_n(\Delta t/t_L, t_w/t_L))}}{1 + y a_n(\Delta t/t_L, t_w/t_L)}, \] (74)
with the coefficients \(a_n\) and \(b_n\) defined in equations (29) and (30). For a flat initial condition, \(T_0 = 0\) and \(s^2_0 = w^2_0/w^2_\infty = T_0/T = 0\). With a very similar calculation to the one explained in [25] for the stationary case, we rewrite the function \(\Phi\) as
\[ \Phi \left(x; \frac{\Delta t}{t_L}, \frac{t_w}{t_L}, 0\right) = \sum_{n=1}^{\infty} \frac{e^{-x/a_n}}{a_n} \prod_{m=1, m \neq n}^{\infty} \frac{a_n}{a_n - a_m}, \] (75)
which is essentially the same result in [25] but with two-times dependent coefficients \(a_n(\Delta t/t_L, t_w/t_L)\).

doi:10.1088/1742-5468/2007/09/P09008
By using now a different independent variable, \( x' = w^2 / \langle w^2 \rangle \), one has
\[
\langle w^2 \rangle P_L(w^2) = \Phi' \left( \frac{w^2}{\langle w^2 \rangle}; \frac{\Delta t}{t_L}, \frac{t_w}{t_L}, s_0^2 \right),
\]
(76)
\[
\Phi' \left( x'; \frac{\Delta t}{t_L}, \frac{t_w}{t_L}, s_0^2 \right) = \int_{-\infty}^{\infty} \frac{dy'}{2\pi} e^{iy'x'} \prod_{n=1}^{\infty} \frac{e^{-y' \sigma_n^2 (\Delta t/t_L, t_w/t_L)/(1+y' \rho_n (\Delta t/t_L, t_w/t_L))}}{1+y' \rho_n (\Delta t/t_L, t_w/t_L)}
\]
(77)
and
\[
da'_n = \frac{a_n}{\sum_{n=1}^{\infty} a_n + s_0^2 \sum_{n=1}^{\infty} b_n}, \quad db'_n = \frac{b_n}{\sum_{n=1}^{\infty} a_n + s_0^2 \sum_{n=1}^{\infty} b_n}.
\]
(78)
Equations (73) and (74) are the generalization of equation (15) in [25] that takes into account the ageing regime. Equation (76) is a rewriting of the latter using the more convenient normalized variable \( w^2 / \langle w^2 \rangle \). The parameters are, in both cases, \( \Delta t/t_L \), \( t_w/t_L \) and \( s_0^2 \). In the growth and ageing regime in which \( \Delta t/t_w \) is finite and the two parameters are very small, i.e. \( \Delta t/t_L, t_w/t_L \ll 1 \), one formally has
\[
\Delta t/t_w = \epsilon, \quad \tilde{w}^2 = \mathcal{G}(\epsilon), \quad \langle w^2 \rangle = t_w^{1/2} \mathcal{G}(\epsilon).
\]
(79)
We can now easily exchange \( \Delta t/t_L \) and \( t_w/t_L \) by the more convenient set \( \tilde{w}^2 \) and \( \langle w^2 \rangle / \langle w_\infty^2 \rangle \). First, we exchange \( \Delta t/t_L \) and \( t_w/t_L \) by \( \Delta t/t_w \) and \( t_w/t_L \). Second, on the one hand \( \Delta t/t_w \) is an exclusive function of \( \tilde{w}^2 \). On the other hand, using the results in section 4.3 one can show that
\[
t_w/t_L = \left( \frac{\langle w^2 \rangle c_0(T_0, T_0)}{\tilde{w}^2 / w_\infty^2} \right)^2.
\]
(80)
The factor \( \tilde{w}^2 \) enters the last equation, but we can ignore it by redefining the scaling function. We used the EW exponents but this relation can be easily rewritten for generic \( \beta, z \) and \( \zeta \). We thus have
\[
\langle w^2 \rangle P_L(w^2) = \Phi'' \left( \frac{w^2}{\langle w^2 \rangle}; \frac{\langle w^2 \rangle}{w_\infty^2}, \tilde{w}^2, T_0 \right),
\]
(81)
as proposed in [11] for the generic disordered case.

Let us list and illustrate in some plots different trends in the scaling function \( \Phi \) evaluated for the flat initial condition. In the numerical evaluations we approximate the infinite sums and products, as in (75), by finite sums and products with different cut-off values, \( M_1 \) and \( M_2 \) respectively. The main panel in figure 14(a) shows the time delay evolution of the scaling function \( \Phi \) for \( t_w = 0 \) and fixed system size. This corresponds essentially to the results obtained in [25], and shows how the pdf is broader for increasing \( \Delta t \) until the saturation regime is reached. The inset shows the system size dependence, indicating that at fixed \( t_w \) the function \( \Phi \), tends to a delta function in the infinite size limit. In figure 14(b) one observes how the pdf is spread at fixed \( L \) and \( \Delta t \) while increasing the waiting time. These results correspond to a flat initial condition \( T_0 = 0 \).

Since it is hard to compute the scaling function \( \Phi \) for a \( T_0 > 0 \) initial condition we just present the skewness and kurtosis,
\[
\sigma = \frac{\mu_3}{\mu_2^{3/2}}, \quad \kappa = \frac{\mu_4}{\mu_2^2} - 3,
\]
(82)
respectively, with the centered moments defined as 

\[ \mu_2 = \langle w^4 \rangle - \langle w^2 \rangle^2, \quad \mu_3 = \langle w^6 \rangle - 3\langle w^4 \rangle \langle w^2 \rangle + 2\langle w^2 \rangle^3, \quad \mu_4 = \langle w^8 \rangle - 4\langle w^6 \rangle \langle w^2 \rangle + 6\langle w^4 \rangle \langle w^2 \rangle^2 - 3\langle w^2 \rangle^4. \]

The moments of the roughness pdf are given by

\[ \langle (w^2)^m \rangle = (-1)^m \partial^m \lambda G_L(\lambda)|_{\lambda=0}; \]

then after some algebra one finds

\[ \sigma = \frac{\sum_{n=1}^{\infty} [2a_n^3 + 6s_0^2a_n^3b_n]}{[\sum_{n=1}^{\infty} (a_n^2 + 2s_0^2a_n^3b_n)]^{3/2}}, \quad \kappa = \frac{\sum_{n=1}^{\infty} [6a_n^4 + 24s_0^2a_n^3b_n]}{[\sum_{n=1}^{\infty} (a_n^2 + 2s_0^2a_n^3b_n)]^2}. \]

Figure 15 displays the time delay evolution of the skewness and kurtosis for the cases \( T > T_0 = 0 \) (figure 15(a)) and \( T < T_0 \) (figure 15(b)). One can observe that the pdfs are broader and more asymmetric until saturation. Generally, both \( \sigma \) and \( \kappa \) age with a similar time dependence as the correlation length \( l \) or the averaged roughness \( \langle w^2 \rangle \). The peculiarities are that the asymptotes corresponding to the infinite size limits grow as \( \Delta t^{1/4} \) for the skewness and \( \Delta t^{1/2} \) for the kurtosis. The approach to saturation is non-monotonic, showing a bump around \( \Delta t \approx t_w \). Finally, figure 16 shows the \( T_0 \) dependence of the skewness for \( t_w = 10^3 \ll t_L \) (the kurtosis behaves in a similar way).

### 8.2. Center of mass displacement distribution

In section 5 we showed that the line’s center of mass undergoes Brownian motion with mean \( \bar{x}(0) \) and variance \( \langle D \rangle (\Delta t) = 2T/(\gamma L)\Delta t \). Thus, the center of mass position is Gaussian distributed and, after a change of variables, the distribution of the mean squared displacement of the center of mass is

\[ P_L(D) = \frac{e^{-D/(2\langle D \rangle)}}{\sqrt{2\pi \langle D \rangle} D} = \sqrt{\frac{\gamma L}{4\pi T \Delta t D}} \exp \left( -\frac{\gamma L}{4T \Delta t} D \right), \]

doi:10.1088/1742-5468/2007/09/P09008
Out-of-equilibrium Edwards–Wilkinson equation

Figure 15. Two-times skewness \( \sigma(\Delta t, t_w) \) and kurtosis \( \kappa(\Delta t, t_w) \) for the distribution function \( P_L(w^2) \) (the kurtosis is rescaled by a factor 10 for clarity). The sums are truncated with \( M_3 = 100 \). \( T = 1, L = 1000 \), and \( t_w = 1, 10, 10^2, 10^3, 10^4, \) and \( 10^5 \) as indicated. Different initial conditions correspond to (a) \( T_0 = 0 \) and (b) \( T_0 = 5 \). The thick lines correspond to the \( t_w \to \infty \) limit.

which can also be written in the scaled form

\[
\langle D \rangle P_L(D) = \Omega \left( \frac{D}{\langle D \rangle} \right), \quad \text{with } \Omega(x) = \frac{e^{-x^2/2}}{\sqrt{2\pi x}}.
\] (85)

8.3. Mean squared displacement distribution

The mean squared displacement satisfies \( B(\Delta t, t_w) = w^2(\Delta t, t_w) + D(\Delta t) \). \( w^2 \) and \( D \) are independent variables (note that the roughness is independent of the zero mode), then the probability function for \( B \) can be formally obtained from the inverse Laplace transform of the product of Laplace transforms,

\[
P_L(B) = \int_{-\infty}^{\infty} \frac{d\lambda}{2\pi i} e^{\lambda B} K_L(\lambda, \Delta t, t_w),
\] (86)

where \( K_L = G_L J_L \) is the Laplace transform of \( P_L(B) \), \( G_L(\lambda, \Delta t, t_w) \) is given by equation (71), and \( J_L(\lambda, \Delta t) \) is the Laplace transform of \( P_L(D) \),

\[
J_L(\lambda, \Delta t) = \frac{1}{\sqrt{2\langle D \rangle \lambda + 1}} = \sqrt{\frac{\gamma L}{4T\Delta t\lambda + \gamma L}}.
\] (87)

Then one finds that

\[
K_L(\lambda, \Delta t, t_w) = \frac{1}{\sqrt{2(\langle B \rangle - \langle w^2 \rangle) \lambda + 1}} \prod_{n=1}^{\infty} \frac{e^{-2\lambda c_n(0)^2(1 - e^{-\kappa_2(\Delta t)} - 2e^{-2\kappa_2(\Delta t)} + 1)} + \lambda w_n^2 a_n(\Delta t, t_w)}}{1 + \lambda w_n^2 a_n(\Delta t, t_w)}.
\] (88)

doi:10.1088/1742-5468/2007/09/P09008
We now define $y'' = \lambda \langle B \rangle / \lambda w_\infty^2 a_n(\Delta t, t_w) = y'' a''_n(\Delta t, t_w)$, and

$$a''_n(\frac{\Delta t}{t_L}, \frac{t_w}{t_L}) = \frac{a_n}{\langle B \rangle / w_\infty^2}, \quad b''_n(\frac{\Delta t}{t_L}, \frac{t_w}{t_L}) = \frac{b_n}{\langle B \rangle / w_\infty^2},$$

and we obtain

$$\langle B \rangle P_L(B) = \Psi \left( \frac{B}{\langle B \rangle}, \frac{\Delta t}{t_L}, \frac{t_w}{t_L}, \sigma^2_0 \right).$$

The scaling function is given by

$$\Psi \left( x''; \frac{\Delta t}{t_L}, \frac{t_w}{t_L}, \sigma^2_0 \right) = \int_{-i\infty}^{+i\infty} \frac{dy''}{2\pi i} e^{iy''x''} \prod_{n=1}^{\infty} \frac{1}{1 + 2c''_n y''} \frac{e^{-y''s_0^2(\Delta t,t_w) / (1 + y''a''_n(\Delta t,t_w))}}{1 + y''a''_n(\Delta t,t_w)},$$

with

$$c''_n(\frac{\Delta t}{t_L}, \frac{t_w}{t_L}) = \frac{\langle D \rangle}{\langle B \rangle}.$$  \hfill (92)

The pdf of the mean squared displacement $B$ can also be written in a scaling form similar to the one found for the roughness.

8.4. The incoherent scattering function

In order to obtain the pdf of the incoherent scattering function we need to use the moment expansion:

$$p(z) = \int \frac{du}{2\pi} e^{iuz} p(u),$$

\hfill (93)

\hfill doi:10.1088/1742-5468/2007/09/P09008
with
\[ p(u) = \sum_{p=0}^{\infty} \frac{u^p}{p!} \frac{\partial^p p(u)}{\partial u^p} \bigg|_{u=0} = \sum_{p=0}^{\infty} \frac{(-iu)^p}{p!} \langle z^p \rangle. \] (94)

This expression assumes that the series converges and the moments exist. The moments of \( C_q \) are
\[ \langle C_q^p \rangle = L^{-p} \int dz_1 \ldots dz_p \exp \left( -\frac{q^2}{2} \sum_{r,r'=1}^{p} \langle H \rangle(z_r - z_{r'}; t, t_w) \right), \] (95)
where the function
\[ \langle H \rangle(z; t, t_w) = \frac{2}{L} \sum_{n=1}^{\infty} \langle S_n \rangle(\Delta t, t_w) e^{ijkz} \]
\[ = L^{-1} \int_0^L dz' \left[ \delta x(z', t) - \delta x(z', t_w) \right] \left[ \delta x(z' - z, t) - \delta x(z' - z, t_w) \right] \] (96)
is the two-times generalization of the height–height correlation function [29].

Using the fact that \( \langle H \rangle(0; t, t_w) = \langle w^2 \rangle(\Delta t, t_w) \) and equation (48), one can show that \( \langle C_q^p \rangle = \langle C_q \rangle^p I_p \), where the \( q \) dependent function \( I_p(t, t_w) \) is given by
\[ I_p(\Delta t, t_w, q^2 L) = L^{-p} \int dz_1 \ldots dz_p \exp \left( -\frac{q^2}{2} \sum_{r,r'=1}^{p} \langle H \rangle(z_r - z_{r'}; \Delta t, t_w) \right). \] (97)

From the moment expansion, using \( u' = u \langle C_q \rangle \), the fact that \( Tq^2L \) is a function of \( \langle C_q^\infty \rangle \) and calling \( T_0q^2L = \langle C_q^0 \rangle \), one can write the pdf of the incoherent scattering function in the scaled form
\[ \langle C_q \rangle P(C_q) = \Theta \left( \frac{C_q}{\langle C_q \rangle}; \frac{\Delta t}{t_L}, \frac{t_w}{t_L}, \langle C_q^\infty \rangle, \langle C_q^0 \rangle \right), \] (98)
with
\[ \Theta \left( x; \frac{\Delta t}{t_L}, \frac{t_w}{t_L}, \langle C_q^\infty \rangle, \langle C_q^0 \rangle \right) = \sum_{p=0}^{\infty} \frac{(-i)^p}{p!} \int \frac{du}{2\pi} e^{iu'x} \langle C_q \rangle^p I_p \left( \frac{\Delta t}{t_L}, \frac{t_w}{t_L}, q^2 L \right). \] (99)

Although in the last expression the functional form of \( \Theta(x) \) is not evident, the scaling properties are clear.

### 8.5. Fluctuations of the response functions

In quadratic models as the EW elastic line the response functions do not fluctuate. This can be easily shown as follows. Take the roughness integrated linear response (53) without the thermal average. Replacing \( \zeta_n^h(t) \) and \( \zeta_n(t) \) by their functional form, and using the fact that the two copies evolve with the same thermal noise, one has
\[ \chi_{\zeta_n}^w(t, t_w) = 2L \sum_{n=1}^{\infty} S_n^2 e^{-\nu k_n^2(t-t_w)}. \] (100)
This result depends on the random fields \( s_n \) but it is independent of the thermal noise. For \( \textit{fixed} \) random fields this quantity does not fluctuate and the pdf of \( \chi^{w^2} \) is a delta function. Similarly, one can prove that the displacement and center of mass responses are delta distributed. The same ‘trivial’ result was found for the ferromagnetic coarsening in the O(\( N \)) model with \( N \rightarrow \infty \) [28].

9. Summary and conclusions

We studied the averaged and fluctuating dynamics of the Edwards–Wilkinson elastic line in one transverse dimension.

Firstly, we analyzed the evolution of correlation functions in terms of the different time scales involved in the problem: the total time, \( t \), the waiting time, \( t_w \), and the saturation time \( t_L \). In particular, for the two-times roughness we found \( \langle w^2 \rangle \sim F_{\omega^2}[\Delta t/t_w, t_w/t_L, TL, T_0 L] \sim F_{\omega^2}[t/t_w, t_w/t_L, TL, T_0 L] \) (the scaling function in the third member is not identical to the one in the second member but we use the same name to simplify the notation). As mentioned earlier, the problem can also be analyzed in terms of associated length scales obtained from the growing correlation length \( \ell(t) = 4\pi^2 \nu t^{1/2} \).

Then, the relevant length scales are \( \ell(t), \ell(t_w) \), and \( \ell(t_L) \sim L \). This is reflected for instance in the scaling form of the structure factor in the asymptotic time delay limit (16), using \( k_w \sim \ell(t_w)^{-1} \). The scaling form of the two-times averaged roughness that describes the ageing, saturation and equilibrium regimes can be written as

\[
\langle w^2 \rangle(\Delta t, t_w) \sim F_{\omega^2} \left[ \frac{\ell(t)}{\ell(t_w)}, \frac{\ell(t_w)}{\ell(t_L)}, TL, T_0 L \right],
\]

with \( \ell(t) = 4\pi^2 \nu t^{1/2} \) and \( t_L = L^2/(4\pi^2 \nu) \) in the \( 1 + 1 \) EW case. This form extends the proposal in (26) to include another scaling variable and thus describe the dynamics of finite lines. It then generalizes the celebrated Family–Vicsek scaling [30] to include the preasymptotic non-equilibrium regime. The ageing regime corresponds to \( \ell(t_w) \ll \ell(t_L) \) and the function \( F_{\omega^2}[\ell(t)/\ell(t_w), 0, T_0 L, TL] \sim \ell^c F[\ell(t)/\ell(t_w)] \), leading to (26) with all the temperature dependent asymptotic properties already detailed in the central part of the manuscript. The saturation regime is reached by taking \( t \gg t_L \) at fixed \( t_w \); this means \( \ell(t_w)/\ell(t) \ll 1 \) and \( \ell(t)/\ell(t_L) \gg 1 \). Finally, the usual stationary equilibrium regime corresponds to \( t_w \gg t_L \), and for a power-law growth one recovers the Family–Vicsek scaling \( \langle w^2 \rangle \sim L^c F_{\omega^2} \ell(\Delta t)/L \).

We also showed that the two-times dependent correlation length defined through the dynamics of the structure factor, satisfies a similar scaling law

\[
l(t, t_w) \sim F_l \left[ \frac{\ell(t)}{\ell(t_w)}, \frac{\ell(t_w)}{\ell(t_L)}, \frac{T}{T_0} \right].
\]

Note that although this expression gives the full ageing behavior of \( l(t, t_w) \), it can be completely rationalized using the simple length scale \( \ell(t) \). For instance, in the ageing regime, one finds that regions with \( \ell(t) < \ell(t_w) \) are equilibrated at the working temperature, while regions with \( \ell(t) > \ell(t_w) \) are still not at equilibrium.

Interestingly enough, we demonstrated that ordering or disordering non-equilibrium dynamics following a quench from higher temperature or a heating process from a lower
temperature are characterized by a higher or lower effective temperature than the one of the bath. This result is consistent with the intuitive interpretation of the effective temperature with higher (lower) values associated to more (less) disordered configurations than the equilibrium ones at the working temperature. A similar dependence on the initial condition was derived by Berthier et al in the 2d XY model [26].

The two-times length scale \( l \) also shows the latter property. In the ageing regime, for fixed \( t_w \) the length grows with \( \Delta t \) for all \( T \neq T_0 \) while for fixed \( \Delta t \) it grows with \( t_w \) when \( T_0 > T \) and it decreases with \( t_w \) when \( T_0 < T \). The former behavior is similar to what is observed in conventional glassy systems such as the 3d Edwards–Anderson spin glass [21] and models of particles in interaction [20]. The heating procedure was not studied in such cases.

The wavevector dependent correlation \( \langle C_q \rangle \) that plays the role of the incoherent scattering function in studies of glassy systems is particularly interesting. We showed that, although \( \langle C_q \rangle \) is simply related to the roughness, the characteristic multiplicative scaling is not easily detected in \( \langle C_q \rangle \). This might be the case in other systems, such as colloidal glasses where it was recently shown that diffusive correlations clearly display multiplicative ageing scaling [35], while this was not previously reckoned in the study of the incoherent scattering function [19].

One can also observe that the ageing behavior of \( \langle C_q \rangle \) for finite systems resembles strongly the experimental results in laponite [19]. In particular it was found in this system that the incoherent scattering function displays a waiting time dependent plateau at long time delay. This behavior is also observed in the EW line, and we showed that it is indeed a finite size equilibration effect. This suggests that a similar finite size equilibration mechanism might be at work in the relaxation dynamics of the rather complex laponite samples. The presence of the waiting time dependent plateau might be indicating that some competing length scale is confining the particle fluctuations, thus leading to equilibration within a finite small volume in the sample, which should be related to the saturation of the incoherent scattering function. The waiting time dependent saturation is in line with the fact that effective temperatures, measured through the FDT, should become the bath temperature at fixed time delay and sufficiently long waiting times where saturation is found. The measurements in [31] are such that the effective temperature does indeed tend to the bath temperature at fixed working frequency—equivalently time delay—when \( t_w \) is large enough, although this result remains controversial [36]. Although the laponite system is very different from the linear elastic model, we believe that some general trends, as the finite size equilibration effects on the scattering function discussed here, should be of similar nature.

We presented the first analytic calculation of finite size fluctuations during an out-of-equilibrium relaxation. This study complements the analysis in [34] and [25] for the width fluctuations at saturation and growth and in [32,33] for other quantities such as the maximum height displacement—indeed, it is also simple to include the \( t_w \) dependence in this calculation. Our results make explicit the dependence on the waiting time and display the crossover to equilibrium. They constitute a benchmark for Rácz proposal to classify interface dynamics into universality classes [22], now extending it to the non-equilibrium relaxation.

With this analytic study we showed that the qualitative ageing dynamics of the vortex glass [12,13] as well as the directed polymer model in a quenched disordered...
Out-of-equilibrium Edwards–Wilkinson equation

environment [10, 11] is mainly due to the non-equilibrium relaxation of the pure elastic lines. This means that the general non-equilibrium features of elastic lines in random media, as described in section 1.3 and discussed along the text, are already displayed by the EW line. The effect of quenched disorder and line–line interactions is to change the details of the scaling, more precisely the temperature and time dependence in \( \ell(t) \), but not the qualitative features. More precisely, this is observed in the exponent \( \alpha \) characterizing the slow growth of correlation length, i.e. \( \ell(t) \sim t^\alpha \), which is lower than the EW value, \( \alpha < \beta_{\text{EW}} \), and depends on the disorder intensity. Besides, the effective temperature on the disordered system does not depend on the working temperature within numerical accuracy, while it does for the EW line.

Regarding the relation of the present results with other models of interfaces out of equilibrium, the study of the ageing dynamics of the pure KPZ equation [37] will allow one to better rationalize the results in [16], where the ageing dynamics of this equation with a disordered potential—and driving force—were analyzed. It is also important to relate these studies with others where the relaxation of the KPZ equation was analyzed, showing a non-trivial stretched exponential relaxation of the dynamical structure factor [38]. Furthermore, the results obtained here could be strongly related to the non-equilibrium relaxation dynamics of confined polymers [39].

As regards the time reparametrization invariance scenario for glassy dynamics we do not expect it to hold, without modification, in models with multiplicative ageing scaling. Following the steps sketched in [24] to study the asymptotic averaged dynamics of the EW line (or massless scalar field) and its corresponding dynamic action, one soon realizes that the multiplicative \( t_w^{1/2} \) factor has to be extracted from the asymptotic analysis to search for time reparametrization invariance. This is similar to what was shown in [28] for the \( O(N) \) model in the large \( N \) limit. One should also notice that the dynamics of the EW line depends on the dimensionality of the transverse space. For instance, one can show that for infinite systems diffusion disappears and the ageing regime persists at infinite waiting times (i.e. the scaling becomes additive) in two transverse dimensions [8]. One has then the interesting possibility of testing the time reparametrization invariance scenario in the ‘critical’ EW equation with two transverse dimensions. The detailed analysis of the dynamic symmetries of the generic EW line goes beyond the scope of this paper.

We conclude with a note on the relevance of our results for coarsening phenomena. The domain walls between equilibrated regions during domain growth are usually described as elastic objects. Recently, the distribution of domain sizes and perimeter lengths in two-dimensional Ising ferromagnetic growth was shown to be unexpectedly non-trivial [40]. The wide distribution of domain sizes and domain wall lengths combined with the highly non-trivial fluctuating dynamics of finite length elastic lines derived here suggest that characterizing the fluctuations of standard two-times observables in domain growth phenomena can be a quite complicated problem.

Acknowledgments

We thank the Universidad Nacional de Mar del Plata, Argentina, for hospitality during the preparation of this work and C Chamon, D Domínguez, T Giamarchi, G Schehr and H Yoshino for very useful discussions. LFC acknowledges financial support from Secyt-ECOS P. A01E01 and PICS 3172, SB from the Swiss National Science Foundation under
MaNEP and Division II, and JLI from CONCIET PIP05-5648 and ANPCYT PICT04-20075. LFC is a member of Institut Universitaire de France.

References

[1] Barabási A-L and Stanley H E, 1995 Fractal Concepts in Surface Growth (Cambridge: Cambridge University Press)
Halpin-Healey T and Zhang Y-C, 1995 Phys. Rep. 254 215
[2] Bray A J, 1994 Adv. Phys. 43 357
[3] Blatter G, Feigel’man M V, Geshkenbein V B, Larkin A I and Vinokur V M, 1994 Rev. Mod. Phys. 66 1125
Nattermann T and Scheidt S, 2000 Adv. Phys. 49 607
[4] Hansen A, Himrichsen E L and Roux S, 1991 Phys. Rev. Lett. 66 2476
Bouchaud E, 1997 J. Phys.: Condens. Matter 9 4319
Alava M, Nukalaz P K V V and Zapperi S, 2006 Adv. Phys. 55 349
[5] Sahimi M, 1995 Flow and Transport in Porous Media and Fractured Rock (New York: Wiley)
Alava M, Dubé M and Rost M, 2004 Adv. Phys. 53 83
[6] Edwards S F and Wilkinson D R, 1982 Proc. R. Soc. A 381 17
Bouchaud E, 1997 J. Phys.: Condens. Matter 9 4319
Alava M, Nukalaz P K V V and Zapperi S, 2006 Adv. Phys. 55 349
[7] R¨othlein A, Baumann F and Pleimling M, 2006 Phys. Rev. E 74 061604
[8] Cugliandolo L F, 2004 Slow Relaxations and Nonequilibrium Dynamics in Condensed Matter
(Les Houches—Ecole d’Ete de Physique Theorique vol 77) ed J-L Barrat et al (Berlin: Springer)
[9] Cugliandolo L F, 1998 Europhys. Lett. 45 52
Tanaka H, Jabbari-Farouji S, Meunier J and Bonn D, 2005 Phys. Rev. E 71 021402
Castillo H E, Chamon C, Cugliandolo L F and Kennett M P, 2003 Phys. Rev. E 68 021402
Castillo H E, Chamon C, Cugliandolo L F, Iguain J L and Kennett M P, 2003 Phys. Rev. E 68 021402
Chamon C, Charbonneau P, Cugliandolo L F, Reichman D and Sellitto M, 2004 Chem. Phys. 311 1020
Cugliandolo L F and Dean D S, 1995 J. Stat. Mech.: Theory Exp. P09008 doi:10.1088/1742-5468/2007/09/P09008
Out-of-equilibrium Edwards–Wilkinson equation

[28] Chamon C, Cugliandolo L F and Yoshino H, 2006 J. Stat. Mech. P01006
[29] Yang H-N, Lu T-M and Wang G-C, 1992 Phys. Rev. Lett. 68 2612
Yang H-N, Lu T-M and Wang G-C, 1993 Phys. Rev. B 47 3911
[30] Family F and Vicsek T, 1985 J. Phys. A: Math. Gen. 18 L75
[31] Abou B and Gallet F, 2004 Phys. Rev. Lett. 93 160603
[32] Majumdar S N and Comtet A, 2004 Phys. Rev. Lett. 92 225501
Majumdar S N and Comtet A, 2005 J. Stat. Phys. 119 777
[33] Schehr G and Majumdar S N, 2006 Phys. Rev. E 73 056103
[34] Foltin G, Oerding K, Rácz Z, Workman R L and Zia R K P, 1994 Phys. Rev. E 50 R639
Plischke M, Rácz Z and Zia R K P, 1994 Phys. Rev. E 50 3589
Rácz Z and Plischke M, 1994 Phys. Rev. E 50 3530
Bramwell S T, Holdsworth P C W and Pinton J F, 1998 Nature 396 552
[35] Wang P, Song C and Makse H A, 2006 Nat. Phys. 2 526
[36] Greinert N, Wood T and Bartlett P, 2006 Phys. Rev. Lett. 97 265702
Jabbari-Farouji S, Mizuno D, Atakhorrami M, MacKintosh F C, Schmidt C F, Eiser E, Wegdam G H and Bonn D, 2007 Phys. Rev. Lett. 98 108302
[37] Bustingorry S, Aging dynamics of non-linear elastic interfaces: the Kardar-Parisi-Zhang equation, 2007 Preprint 0708.2615
[38] Katzav E and Schwartz M, 2004 Phys. Rev. E 69 052603
[39] Rahmani A, Castelnovo C, Schmit J and Chamon C, 2007 Preprint 0704.1663
[40] Arenzon J J, Bray A J, Cugliandolo L F and Sicilia A, 2007 Phys. Rev. Lett. 98 145701
Sicilia A, Arenzon J J, Bray A J and Cugliandolo L F, Domain growth morphology in curvature driven two dimensional coarsening, 2007 Preprint 0706.4314

doi:10.1088/1742-5468/2007/09/P09008

35