Modified Spin Wave Theory
of the Bilayer Square Lattice
Frustrated Quantum Heisenberg Antiferromagnet

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The ground state of the square lattice bilayer quantum antiferromagnet with nearest and
next-nearest neighbour intralayer interaction is studied by means of the modified spin wave
method. For weak interlayer coupling, the ground state is found to be always magnetically
ordered while the quantum disordered phase appear for large enough interlayer coupling.
The properties of the disordered phase vary according to the strength of the frustration.
In the regime of weak frustration, the disordered ground state is an almost uncorrelated
assembly of interlayer dimers, while in the strongly frustrated regime the quantum spin
liquid phase which has considerable Néel type short range order appears. The behavior of
the sublattice magnetization and spin-spin correlation length in each phase is discussed.
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I. INTRODUCTION

The spin-1/2 square lattice Heisenberg model is now widely believed to have an antiferromagnetic long range order in the ground state. [1–5] However, it is expected that the strong quantum fluctuation in this system may lead to the destruction of the long range order with the help of some additional mechanism. In this context, the square lattice antiferromagnetic Heisenberg model with nearest and next-nearest exchange interaction (hereafter called $J_1 - J_2$ model) [6–18] and the bilayer Heisenberg model [19–23] have been studied extensively. Both of these models are expected to have the quantum disordered ground state for appropriate parameter regime. However, because the mechanisms leading to the quantum disordered phase in these two models are of very different nature, it must be most interesting to study their interplay in the bilayer $J_1 - J_2$ model.

In the $J_1 - J_2$ model, the competition between the nearest neighbour interaction $J_1$ and the nearest neighbour interaction $J_2$ introduces the frustration in spin configuration which enhances the quantum fluctuation. However, the conclusion about the presence of the quantum disordered state in this model is still controvertial even in the most frustrated regime.

On the other hand, in the bilayer model, if the interlayer antiferromagnetic coupling is strong enough, the spins on both layers form interlayer singlet pairs and the quantum fluctuation is enhanced leading to the quantum disordered state. In other words, the antiferromagnetic interlayer coupling reduces the effective spin magnitude. Therefore this model may be regarded as the physical realization of the single layer Heisenberg model with spin less than 1/2. Actually, in the study of the single layer $J_1 - J_2$ model, there are considerable number of works which cast doubt on the presence of quantum disordered phase even for $S = 1/2$ and $J_2/J_1 = 0.5$. [7–10] But some of these works also predict the presence of quantum disordered phase for $S < 1/2$ which is unreachable within the single layer model. [7,8] The bilayer $J_1 - J_2$ model can effectively realize such situation.

This paper is organized as follows: The bilayer $J_1 - J_2$ model and its classical ground
state are explained in the next section. In section 3, the modified spin wave approximation [24] is applied to this model. The phase diagram and the behavior of physical quantities are presented in section 4. The last section is devoted to summary and discussion.

II. BILAYER $J_1 - J_2$ MODEL

The Hamiltonian of the bilayer $J_1 - J_2$ model is given as follows,

$$H = J_1 \sum_{\langle i,j \rangle_{nn}} (S^A_i S^A_j + S^B_i S^B_j) + J_2 \sum_{\langle i,j \rangle_{nnn}} (S^A_i S^A_j + S^B_i S^B_j) + J_3 \sum_i S^A_i S^B_i,$$

where $S^\mu_i$ is the spin operator with magnitude $S$ on the $i$-th site of the layer $\alpha(\mu = A \text{ or } B)$. The expression $\sum_{\langle i,j \rangle_{nn}}$ and $\sum_{\langle i,j \rangle_{nnn}}$ denote the summation over the intralayer nearest neighbour pairs and next nearest neighbour pairs, respectively. The last term represents the interlayer coupling. All exchange couplings are assumed to be antiferromagnetic. In the following, we denote the ratios $J_2/J_1 = \alpha$ and $J_3/J_1 = \beta$ and take the energy unit $J_1 = 1$.

In the classical limit, the ground state is the Néel state and the collinear state according as $\alpha < 0.5$ or $\alpha > 0.5$. Actually, if the quantum fluctuation is completely neglected, infinite number of ground state configurations are degenerate for $\alpha > 0.5$. [22][23] However, for the single layer $J_1 - J_2$ model, it is known that this degeneracy is lifted by the quantum fluctuation and the collinear ground state is chosen. [25] It is straightforward to extend this argument to the bilayer model. Therefore, in the following, we only consider the collinear order for $\alpha > 0.5$.

Thus we consider the following two types of spin configuration in the classical limit;

$$\begin{cases}
S^A_i = S(-1)^{m+n} \\
S^B_i = S(-1)^{m+n+1}
\end{cases} : \text{Néel (N-)configuration,} \tag{2.2}
$$

$$\begin{cases}
S^A_i = S(-1)^m \\
S^B_i = S(-1)^{m+1}
\end{cases} : \text{collinear (C-)configuration,} \tag{2.3}
$$
where the position of the $i$-th site $r_i$ is denoted by $(m,n)$. In the following, we treat the quantum fluctuations around these configurations by means of the modified spin wave method. [24]

### III. MODIFIED SPIN WAVE APPROXIMATION

Based on the classical configurations explained in the last section, let us introduce the Dyson-Maleev transformation [26,27] for each spin as follows:

\[
S_i^{A\uparrow} = (-1)^{m+n}(S - a_i^\dagger a_i), \\
S_i^{B\uparrow} = (-1)^{m+n+1}(S - b_i^\dagger b_i), \\
S_i^{A\downarrow} = \frac{1}{\sqrt{2}}(1 - a_i^\dagger a_i), \quad S_i^{B\downarrow} = -\sqrt{2}b_i \\
S_i^{A\uparrow} = -a_i^\dagger \frac{1}{\sqrt{2}}(1 - a_i^\dagger a_i), \quad S_i^{B\uparrow} = \sqrt{2}b_i \\
S_i^{A\downarrow} = -\sqrt{2}a_i, \quad S_i^{B\downarrow} = \sqrt{2}b_i^\dagger
\]

for the N-configuration (2.2), and

\[
S_i^{A\uparrow} = (-1)^m(S - a_i^\dagger a_i), \\
S_i^{B\uparrow} = (-1)^{m+1}(S - b_i^\dagger b_i), \\
S_i^{A\downarrow} = \frac{1}{\sqrt{2}}(1 - a_i^\dagger a_i), \quad S_i^{B\downarrow} = -\sqrt{2}b_i \\
S_i^{A\uparrow} = -a_i^\dagger \frac{1}{\sqrt{2}}(1 - a_i^\dagger a_i), \quad S_i^{B\uparrow} = \sqrt{2}b_i \\
S_i^{A\downarrow} = -\sqrt{2}a_i, \quad S_i^{B\downarrow} = \sqrt{2}b_i^\dagger
\]

for the C-configuration (2.3).

For the N-configuration (2.2), the Hamiltonian (2.1) is rewritten as,
For the C-configuration (2.3), we have
\[
\begin{align*}
H &= \sum \{-2S^2 + S(a_i^+a_i + a_j^+a_j - a_i^+a_j^+ - a_i a_j) \\
&+ b_i^+b_i + b_j^+b_j - b_i^+b_j - b_j^+b_i + a_i^+(a_j^+ - a_i^+)a_j - a_i(a_j^+ - a_i^+)a_j^+/2 + b_i^+(b_j^+ - b_i^+)b_j/2 \} \\
&+ \alpha \sum \{2S^2 - S(a_i^+ - a_j^+)(a_i - a_j) + a_i^+a_j^+(a_i - a_j)^2\delta_{\text{xy}}/2 - (a_i^+ - a_j^+)^2a_ia_j\delta_{\text{xy}}/2 - S(b_i^+ - b_j^+)(b_i - b_j) - b_i^+b_j^+(b_i - b_j)^2\delta_{\text{xy}}/2 - (b_i^+ - b_j^+)^2b_ib_j\delta_{\text{xy}}/2 \} \\
&+ \beta \sum \{-S^2 + 2S(a_i^+a_i + b_i^+b_i - a_i^+b_i^+ - a_i b_i^+) + a_i^+(b_i^+ - a_i)^2b_i/2 \},
\end{align*}
\]
where \(\delta_{\text{xy}}(\delta_{\text{xy}}) = 1\) or 0 according as \(m + n = \text{even (odd)}\) or odd(even) where \(r_i = (m, n)\).

For the C-configuration (2.3), we have
\[
\begin{align*}
H &= \sum \{-2S^2 + S(a_i^+a_i + a_j^+a_j - a_i^+a_j^+ - a_i a_j) \\
&+ b_i^+b_i + b_j^+b_j - b_i^+b_j - b_j^+b_i + a_i^+(a_j^+ - a_i^+)a_j - a_i(a_j^+ - a_i^+)a_j^+/2 + b_i^+(b_j^+ - b_i^+)b_j/2 \} \\
&+ \alpha \sum \{2S^2 - S(a_i^+ - a_j^+)(a_i - a_j) - a_i^+a_j^+(a_i - a_j)^2\delta_{\text{xy}}/2 - (a_i^+ - a_j^+)^2a_ia_j\delta_{\text{xy}}/2 - S(b_i^+ - b_j^+)(b_i - b_j) - b_i^+b_j^+(b_i - b_j)^2\delta_{\text{xy}}/2 - (b_i^+ - b_j^+)^2b_ib_j\delta_{\text{xy}}/2 \} \\
&+ \alpha \sum \{-2S^2 + S(a_i^+a_i + a_j^+a_j - a_i^+a_j^+ - a_i a_j) \\
&+ b_i^+b_i + b_j^+b_j - b_i^+b_j - b_j^+b_i + a_i^+(a_j^+ - a_i^+)a_j - a_i(a_j^+ - a_i^+)a_j^+/2 + b_i^+(b_j^+ - b_i^+)b_j/2 \} \\
&+ \beta \sum \{-S^2 + 2S(a_i^+a_i + b_i^+b_i - a_i^+b_i^+ - a_i b_i^+) + a_i^+(b_i^+ - a_i)^2b_i/2 \},
\end{align*}
\]
where \(\delta_{\text{xy}}(\delta_{\text{xy}}) = 1\) or 0 according as \(m + n = \text{even (odd)}\) or odd(even). The notations \(\sum \) and \(\sum \) denote the summation over the intralayer nearest neighbour pairs in \(x\) and \(y\) direction, respectively.

Following Takahashi, 24 we assume the constraint that the sublattice magnetization vanish as expected for the two dimensional spin system with continuous symmetry at finite temperatures. 28

\[
S = \sum_i < a_i^+a_i >= \sum_i < b_i^+b_i > .
\]
We impose this condition even in the ground state where the long range sublattice magnetization may be present. This means that the average is taken over the direction of the
sublattice magnetization even in the ordered phase. Nevertheless, we can calculate the sub-
lattice magnetization from the long range part of the correlation function which originate
from the Bose condensate of the bose fields $a_i$ and $b_i$. Although the validity of this
procedure is not well founded, these are the common features of the modified spin wave
method and we do not discuss this point further.

We treat the nonlinear terms in (3.14) and (3.15) by the mean field approximation as.

$$H^{MF} = \sum_{i,j>nn} \{ \Delta(a^+_ia_i + a^+_ja_j - a^+_ia_j^+ - a^+_ja_i) + b^+_ib_i + b^+_jb_j - b^+_ib_j + 2\Delta^2 - 4S\Delta \}$$

$$+ \sum_{i,j>nn} \{ -q_{xy}(a^+_i - a^+_j)(a_i - a_j) + 4Sq_{xy} - 2q_{xy}^2 \}$$

$$+ \beta \sum_i \{ \Delta_{AB}(a^+_ia_i + b^+_ib_i - a^+_ib_i - a_i) + \Delta^2_{AB} - 2S\Delta_{AB} \}$$

$$- \sum_i \{ \mu(S - a^+_ia_i) + \mu(S - b^+_ib_i) \}, \quad (3.17)$$

for the N-configuration phase. Here $\mu$ is the Lagrangian multiplier corresponding to the
constraint (3.16). The order parameters are defined by

$$< a^+_ia_j > = < b^+_ib_j > = q_{xy} \text{ for } \mathbf{r}_j = \mathbf{r}_i + \delta, \quad (3.18)$$

$$< a^+_ia_j^+ > = < b^+_ib_j^+ > = \Delta \text{ for } \mathbf{r}_j = \mathbf{r}_i + \rho, \quad (3.19)$$

$$< a^+_ib_i^+ > = < a_ib_i > = \Delta_{AB}, \quad (3.20)$$

where $\rho$ is the vector to the nearest neighbour sites and $\delta$ to the next nearest sites.

For the C-configuration, we have

$$H^{MF} = \sum_{i,j>nn} \{ \Delta(a^+_ia_i + a^+_ja_j - a^+_ia_j^+ - a^+_ja_i) + b^+_ib_i + b^+_jb_j - b^+_ib_j + 2\Delta^2 - 4S\Delta \}$$

$$+ \sum_{i,j>nn} \{ -q_y(a^+_i - a^+_j)(a_i - a_j) + 4Sq_y - 2q_y^2 \}$$

$$+ \alpha \sum_{i,j>nn} \{ \Delta_{xy}(a^+_ia_i + a^+_ja_j - a^+_ia_j^+ - a^+_ja_i) + b^+_ib_i + b^+_jb_j - b^+_ib_j + 2\Delta^2_{xy} - 4S\Delta_{xy} \}$$

$$+ \beta \sum_i \{ \Delta_{AB}(a^+_ia_i + b^+_ib_i - a^+_ib_i - a_i) + \Delta^2_{AB} - 2S\Delta_{AB} \}$$

$$- \sum_i \{ \mu(S - a^+_ia_i) + \mu(S - b^+_ib_i) \}. \quad (3.21)$$

Here, the order parameters are defined by
\[ < a_i^\dagger a_j > = < b_i^\dagger b_j > = q_y \text{ for } r_j = r_i + \rho_y, \]  
\[ (3.22) \]

\[ < a_i^\dagger a_j > = < b_i^\dagger b_j > = \Delta_x \text{ for } r_j = r_i + \rho_x, \]  
\[ (3.23) \]

\[ < a_i^\dagger a_j > = < b_i^\dagger b_j > = \Delta_{xy} \text{ for } r_j = r_i + \delta, \]  
\[ (3.24) \]

\[ < a_i^\dagger b_j > = < a_i b_i > = \Delta_{AB}, \]  
\[ (3.25) \]

where \( \rho_x \) and \( \rho_y \) is the vector to the nearest neighbour sites in \( x \) and \( y \) direction, respectively.

These mean field Hamiltonians are transformed into the fourier space as

\[ H^{MF} = \sum_k > \{ \Lambda(k)(a_k^\dagger a_k + a_{-k}^\dagger a_{-k} + b_k^\dagger b_k + b_{-k}^\dagger b_{-k}) \]  
\[ \quad - \Gamma(k)(a_k^\dagger a_{-k} + a_{-k}^\dagger a_k + b_k^\dagger b_{-k} + b_{-k}^\dagger b_k) \]  
\[ \quad + \beta \Delta_{AB}(a_k^\dagger a_k + a_{-k}^\dagger a_{-k} + b_k^\dagger b_k + b_{-k}^\dagger b_{-k} - a_k^\dagger b_{-k} - a_k b_{-k}^\dagger - b_k a_{-k}^\dagger - b_k a_{-k}) \} \]  
\[ + E_0, \]  
\[ (3.26) \]

where

\[ \Lambda(k) = 4(\Delta - \alpha q_{xy}(1 - \cos(k_x) \cos(k_y))) + \mu, \]  
\[ (3.28) \]

\[ \Gamma(k) = 4\Delta \gamma(k), \]  
\[ (3.29) \]

\[ \gamma(k) = \frac{1}{2}(\cos(k_x) + \cos(k_y)), \]  
\[ (3.30) \]

\[ E_0 = N\{4\Delta^2 - 8S\Delta + 4\alpha(4S q_{xy} - 2q_{xy}^2) \]  
\[ + \beta(\Delta_{AB}^2 - 2S\Delta_{AB}) - 2\mu S\}, \]  
\[ (3.31) \]

for the N-configuration and

\[ \Lambda(k) = -2q_y(1 - \cos(k_y)) + \mu, \]  
\[ (3.33) \]

\[ \Gamma(k) = 2\Delta_x \cos(k_x) + 4\alpha \Delta_{xy} \cos(k_x) \cos(k_y), \]  
\[ (3.34) \]

\[ E_0 = N\{(2\Delta_x^2 - 4S\Delta_x + 4S q_{xy} - 2q_{xy}^2 + 2\alpha(2\Delta_{xy}^2) \]  
\[ - 4S\Delta_{xy}) + \beta(\Delta_{AB}^2 - 2S\Delta_{AB}) - 2\mu S\}, \]  
\[ (3.35) \]

for the C-configuration. The summation \( \sum_k > \) is taken over the left half of the Brillouin zone \( (k_x > 0) \) because the momenta \( k \) and \( -k \) are explicitly written in (3.26). The number of the lattice sites in each layer is denoted by \( N \).
These Hamiltonians are diagonalized as

\[ H_{\text{MF}} = \sum_k E_+(k) \alpha_k^\dagger \alpha_k + E_-(k) \beta_k^\dagger \beta_k + E_G, \]  

(3.37)

where

\[ E_\pm(k) = \sqrt{\eta(k)^2 - (\Gamma(k) \pm \delta)^2}, \quad \delta \equiv \beta \Delta_{\text{AB}}, \quad \eta(k) \equiv \Lambda(k) + \delta, \]  

(3.38)

by the Bogoliubov transform

\[ a_k = \{ \text{ch} \theta_k^+ \alpha_k + \text{ch} \theta_k^- \beta_k + \text{sh} \theta_k^+ \alpha_k^\dagger + \text{sh} \theta_k^- \beta_k^\dagger \} / \sqrt{2}, \]  

(3.39)

\[ b_k = \{ \text{ch} \theta_k^- \alpha_k - \text{ch} \theta_k^- \beta_k + \text{sh} \theta_k^+ \alpha_k^\dagger - \text{sh} \theta_k^- \beta_k^\dagger \} / \sqrt{2}, \]  

(3.40)

\[ a_k^\dagger = \{ \text{sh} \theta_k^+ \alpha_k + \text{sh} \theta_k^- \beta_k + \text{ch} \theta_k^+ \alpha_k^\dagger + \text{ch} \theta_k^- \beta_k^\dagger \} / \sqrt{2}, \]  

(3.41)

\[ b_k^\dagger = \{ \text{sh} \theta_k^+ \alpha_k - \text{sh} \theta_k^- \beta_k + \text{ch} \theta_k^+ \alpha_k^\dagger - \text{ch} \theta_k^- \beta_k^\dagger \} / \sqrt{2}, \]  

(3.42)

where

\[ \text{ch} \theta_k^\pm = \sqrt{\frac{1}{2} \left( \frac{\eta(k)}{E_\pm(k)} + 1 \right)}, \quad \text{sh} \theta_k^\pm = \sqrt{\frac{1}{2} \left( \frac{\eta(k)}{E_\pm(k)} - 1 \right) \text{sgn}(\Gamma_k \pm \delta)}. \]  

(3.43)

The self consistent equations for the order parameters are

\[ \Delta = \frac{1}{N} \sum_{k,\pm} \frac{\Gamma(k) \pm \delta}{4E_\pm(k)} \gamma(k) + N_0^N, \]  

(3.44)

\[ \Delta_{\text{AB}} = \frac{1}{N} \sum_{k,\pm} \frac{\delta \pm \Gamma(k)}{4E_\pm(k)} + N_0^N, \]  

(3.45)

\[ q_{xy} = \frac{1}{N} \sum_{k,\pm} \frac{\eta(k)}{4E_\pm(k)} \cos(k_x) \cos(k_y) + N_0^N, \]  

(3.46)

for the N-configuration and

\[ \Delta_{xy} = \frac{1}{N} \sum_{k,\pm} \frac{\Gamma(k) \pm \delta}{4E_\pm(k)} \cos(k_x) \cos(k_y) + N_0^C, \]  

(3.47)

\[ \Delta_x = \frac{1}{N} \sum_{k,\pm} \frac{\Gamma(k) \pm \delta}{4E_\pm(k)} \cos(k_x) + N_0^C, \]  

(3.48)

\[ \Delta_{\text{AB}} = \frac{1}{N} \sum_{k,\pm} \frac{\delta \pm \Gamma(k)}{4E_\pm(k)} + N_0^C, \]  

(3.49)

\[ q_y = \frac{1}{N} \sum_{k,\pm} \frac{\eta(k)}{4E_\pm(k)} \cos(k_y) + N_0^C, \]  

(3.50)
for the C-configuration. The summation $\sum_{k,\pm}^{' }$ excludes $k$'s for which $E_{\pm}(k) = 0$. The quantities $N_0^N$ and $N_0^C$ are the amplitudes of the Bose condensate of the bose particles represented by the operators $\alpha_k$ and $\beta_k$ in the Néel phase and collinear phase, respectively. They correspond to the magnitude of the Néel or collinear long range order. These quantities vanish in the disordered phase. The constraint (3.16) is rewritten in the form

$$S = -\frac{1}{2} + \frac{1}{N} \sum_{k,\pm}^{'} \frac{\eta(k)}{4E_{\pm}(k)} + N_0, \quad (3.51)$$

in both cases where $N_0$ stands for $N_0^C$ or $N_0^C$. In order that $N_0$ remains finite, the excitation spectrum must have zero modes. This requirement fixes the value of $\mu$ in the ordered phase. On the other hand, for the solution corresponding to the disordered phase, the energy spectrum has no zero mode and the value of $\mu$ must be fixed so that the self-consistent equations are satisfied with $N_0 = 0$. The ground state energy $E_G$ is given as follows:

$$E_G = E_G^N \equiv N(-4\Delta^2 + 4\alpha_q^2 - \beta\Delta_{AB}^2) : \text{N-configuration}, \quad (3.52)$$

$$E_G = E_G^C \equiv N(-2\Delta_x^2 + 2q_y^2 - 4\alpha\Delta_{xy}^2 - \beta\Delta_{AB}^2) : \text{C-configuration}. \quad (3.53)$$

If the self consistent equations have more than two solutions, we choose the thermodynamically stabe solution comparing the ground state energy.

### IV. PHASE DIAGRAM

We have solved numerically the set of equations for $S = 1/2$ and found 4 kinds of phases:

i) Néel ordered phase : $\Delta, \Delta_{AB}, q_{xy}, N_0^N > 0$

ii) Collinear ordered phase : $\Delta_{xy}, \Delta_x, \Delta_{AB}, q_y, N_0^C > 0$

iii) Spin liquid phase with Néel-type correlation (NSL phase) : $\Delta, \Delta_{AB}, q_{xy} > 0, N_0^N = 0$

iv) Interlayer dimer phase (ILD phase) : $\Delta = q_{xy} = \Delta_{xy} = \Delta_x = q_y = N_0^C = N_0^N = 0, \Delta_{AB} = \frac{\sqrt{3}}{2}$

The spin fluid phase with collinear-type correlation ( $\Delta_{xy}, \Delta_x, \Delta_{AB}, q_y > 0, N_0^C = 0$ ) is only found as a metastable state. The ground state phase diagram is shown in Fig.
For small $\beta$, there appears no disordered phase. The ground state changes from the Néel ordered phase to the collinear ordered phase by the first order transition around $\alpha \gtrsim 0.5$. For larger values of $\beta$, however, there appears a NSL phase around $\alpha \sim 0.5$. In this phase, the short range intralayer singlet order parameter $\Delta$ remains finite, although the long range order is absent. The transition between the Néel phase and NSL phase is of the second order, while between the collinear ordered phase and NSL phase the transition is of the first order. As the value of $\beta$ is further increased, the intralayer correlation becomes weaker and the second order transition to the ILD phase takes place at $\beta = 4$. This is verified analytically from the stability analysis of Eqs. (3.44,3.45,3.46) around the ILD solution. Of course, this phase transition between the NSL phase and ILD phase is an artifact of the mean field approximation and it should be interpreted as a crossover. For very small or large value of $\alpha$, the direct first order transition from the Néel or collinear ordered phase to the ILD phase takes place as in the unfrustrated case. [20]

It should be remarked that the spin-fluid phase with considerable intralayer correlation is stabilized around $\alpha \sim 0.5$ where the effect of the frustration is most pronounced. In this sense, our NSL state is the first promising example of the non-trivial frustration induced quantum spin fluid phase in two dimensions.

Figures 2(a) and 2(b) show the amplitude of the sublattice magnetization $N^N_0$ and $N^C_0$ along the line with fixed $\alpha$. In general, the sublattice magnetization is enhanced for small $\beta$ and turns to decrease for larger $\beta$. Namely, the interlayer coupling strengthens the ordering as far as it is small. The same is true also in the unfrustrated case. [20,21] This feature can be already observed within the linear spin wave analysis. [21] Dotsenko [30] has also obtained the similar result using the mapping onto the nonlinear $\sigma$-model.

The energy gap in the NLS phase $\Delta E$ is given by

$$\Delta E = E_+(\mathbf{k} = 0) = \sqrt{(8\Delta + \mu + \delta)(\mu - \delta)}. \quad (4.54)$$

Figure 3 shows the variation of $\Delta E$ for various values of $\alpha$. It grows from 0 starting from the Néel-NSL boundary and saturates at the NSL-ILD boundary. At the collinear-NSL boundary
the energy gap in the NSL phase remains finite reflecting the fact that this transition is of the first order. However, we may expect that this is also an artifact of the present approximation. Taking into account that $\Delta E$ is proportional to the inverse of the spin-spin correlation length $\xi$ for $\xi >> 1$, the antiferromagnetic short range correlation is highly enhanced in the NSL phase near the phase boundary.

V. SUMMARY AND DISCUSSION

The spin-1/2 bilayer $J_1 - J_2$ model is studied by means of the modified spin wave approximation and the ground state phase diagram is obtained. For small interlayer coupling, Néel or collinear type long range order exists for any value of $\alpha$. However, with the increase of the interlayer antiferromagnetic coupling $\beta$, the correlated quantum spin fluid phase appears around $\alpha \sim 0.5$. The width of the spin fluid phase becomes wider as $\beta$ increases. On the other hand, the correlation length decreases with the increase of $\beta$ and we only find the interlayer dimer phase for $\beta > 4$.

It has been widely expected that the frustration effect enhance the quantum fluctuation and leads to the quantum spin liquid phase in two dimensional system. Although the single layer $J_1 - J_2$ model is one of such candidates, the conclusion is rather sensitive to the approximations used, the method of numerical calculations and data analysis. On the other hand, we may expect the presence of the frustration induced highly correlated quantum spin liquid over a wide range of parameters for the bilayer $J_1 - J_2$ model.

We have also found that the Néel state remains stable for the value of $\alpha$ slightly larger than 0.5. This may be explained as follows: In the collinear phase, the classical ground state is continuously degenerate and therefore the quantum fluctuation is more pronounced than the Néel phase. Therefore the Néel state is stabilized rather than the collinear phase even for $\alpha > 0.5$.

Using the modified spin wave approximation, Nishimori and Saika obtained the result that the ground state energy jumps at the transition from the Néel phase to the collinear
phase in the spin-1/2 single layer $J_1 - J_2$ model, while in our calculation this transition is a usual first order transition even for $\beta = 0$. This is due to the fact that Nishimori and Saika expanded the ground state energy with respect to $1/S$ and truncated at the second order. Actually, such estimation is known to give the results better than the mean field type estimation in some cases. However, here we employ the naïve mean field ground state energy without expansion for the consistency within the present approximation. In any case, such details of the transition are beyond the scope of our approximation.

Although we expect that our approach captures the essential features of the ground state of the present model, our approximation is far from quantitative. Even in the unfrustrated case ($\alpha = 0$), modified spin wave theory predicts rather large Néel-ILS critical value of $\beta = \beta_c \sim 4.25$ \cite{20} compared to the more reliable estimation $\beta_c \sim 2.5$ by the dimer expansion \cite{21} and the quantum Monte Carlo simulation. \cite{23} Unfortunately, the quantum Monte Carlo simulation is not expected to be powerful enough for the frustrated model due to the negative sign problem. The dimer expansion method may be promising but higher order calculation requires excessive computational time and memory. Further investigation is thus required for the quantitative understanding of the present model.

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FIGURES

FIG. 1. The ground state phase diagram of the bilayer $J_1 - J_2$ model. The solid and broken lines are the lines of the first and second order transition, respectively.

FIG. 2. The $\beta$-dependence of the sublattice magnetization in the (a) Néel and (b) collinear ordered phases. The values of $\alpha$ are indicated in the figure. The values on the phase boundary are represented by the open circles.

FIG. 3. The $\beta$-dependence of the excitation gap in the disordered phase. The values of $\alpha$ are indicated in the figure. The values on the phase boundary are represented by the open circles.