A NEW WEAK SOLUTION TO AN OPTIMAL STOPPING PROBLEM

CONG QIN*
Center for Financial Engineering, Soochow University
Suzhou, Jiangsu 215006, China

XINFU CHEN
Department of Mathematics, University of Pittsburgh
Pittsburgh, PA 15260, USA

(Communicated by Bei Hu)

Abstract. In this paper, we propose a new weak solution to an optimal stopping problem in finance and economics. The main advantage of this new definition is that we do not need the Dynamic Programming Principle, which is critical for both classical verification argument and modern viscosity approach. Additionally, the classical methods in differential equations, e.g. penalty method, can be used to derive some useful results.

1. Introduction. The theory of optimal stopping is very important due to its numerous and various applications in economics and finance. It provides a suitable modeling framework for the evaluation of optimal investment decisions, see e.g. [6]. In this paper, we consider the following optimal stopping problem:

\[ v(t,x) = \sup_{\tau \in \mathcal{T}_{t,T}} \mathbb{E}\left[ \int_t^\tau f(\rho, S_\rho) d\rho + g(\tau, S_\tau) \right| S_t = s], \]

(1)

where \( \mathcal{T}_{t,T} \) is the set of stopping time valued in \([t,T]\), \( f : [0,T] \times \mathbb{R} \to \mathbb{R} \) and \( g : [0,T] \times \mathbb{R} \to \mathbb{R} \) are continuous functions representing the running benefit and the final gain respectively, and the stock price \( \{S_t\}_{t \geq 0} \) satisfies

\[ dS_\rho = S_\rho \alpha d\rho + S_\rho \sigma dW_\rho. \]

(2)

Here \( \{W_t\}_{t \geq 0} \) is a standard Brownian motion on a filtered probability space \((\Omega, \mathcal{F}, \{\mathcal{F}_\rho\}_{\rho \geq 0}, \mathbb{P})\) satisfying the usual conditions, constants \( \alpha \in \mathbb{R}, \sigma > 0 \) are expected return rate and volatility, respectively.

A classic example is an American put option, which bestows the owner a right but not an obligation to sell a stock with a predetermined exercise price before a maturity. Mathematically, this put option’s pricing model can be written as (1) and (2) with \( f \equiv 0 \) and \( g(t,s) = e^{-\alpha t} \max\{K-s,0\} \), where \( \alpha \geq 0 \) represents the interest rate, and \( K > 0 \) is the exercise price, \( T \) is the maturity, and the expectation \( \mathbb{E} \) is

2020 Mathematics Subject Classification. Primary: 35K86, 91G80; Secondary: 35Q93, 91G20.

Key words and phrases. Weak solution, viscosity solution, variational inequality, optimal stopping problem, free-boundary problem.

The first author received support from NSFC 11901416, NSF of Jiangsu BK20190812, and NSF for Universities in Jiangsu Province 19KJD100005.

* Corresponding author: Cong Qin.
interpreted as a risk neutral measure; see e.g., [5] for more information. Motivated by this classic example and for simplicity, we make the following two assumptions throughout the paper:

(A1) \( g \in C([0, \infty) \times (0, \infty)) \) satisfies the growth condition: for each \( T > 0 \), there exist \( n > 0 \) and \( K(T) > 0 \) such that

\[
|g(t, s)| \leq K(T)[s^{-n} + s^n] \quad \forall (t, s) \in \mathcal{D}_T := [0, T] \times (0, \infty).
\]

(A2) \( f \in C([0, \infty) \times (0, \infty)) \) and is uniformly bounded, i.e., there is a constant \( M_f > 0 \) such that \( |f(t, s)| \leq M_f \) for all \( (t, s) \in \mathcal{D}_T \).

We shall show that \( v \) is a solution of the following obstacle, or variational, problem:

\[
\begin{cases}
\min \{ -\frac{\partial}{\partial t} - \mathcal{L} v - f, v - g \} = 0 & \text{in } \mathcal{Q}_T := [0, T) \times (0, \infty), \\
v(T, \cdot) = g(T, \cdot) & \text{on } \{T\} \times (0, \infty), \\
\lim_{s \downarrow 0} s^m v(t, s) = 0, \lim_{s \to \infty} s^{-m} v(t, s) = 0 & \text{uniformly in } t \in [0, T].
\end{cases}
\]

where \( m \) is some positive constant, and

\[
\mathcal{L} v := \frac{1}{2} \sigma^2 s^2 \frac{\partial^2 v}{\partial s^2} + \alpha s \frac{\partial v}{\partial s}.
\]

Before we proceed to prove our assertion, it is necessary to clarify that what a “solution” exactly means. One candidate is so called classical solution that admits sufficient regularity, i.e. the derivatives in the equation (4) are in classical sense.

If the value function \( v \) defined as (1) is sufficiently smooth, one then can verify, typically with the help of Dynamic Programming Principle, that the value function \( v \) is the unique classical solution of equation (4). This is called verification theorem in the classical optimal stopping problem.

However, from the definition (1), it is very difficult (sometime impossible) to obtain the regularity of the function \( v \). Therefore, in general, \( v \) is not a classical solution of equation (4). To avoid this difficulty, in modern mathematics, viscosity solutions are introduced as follows: Denote

\[
F(t, s, v, q, p, M) = \min \left\{ -q - \frac{1}{2} \sigma^2 s^2 M - \alpha s - f, v - g \right\}
\]

for \((t, s) \in \mathcal{D}_T \subseteq \mathbb{R}^4\).

**Definition 1.1.** A function \( v : \mathcal{D}_T \to \mathbb{R} \) is called a viscosity subsolution (resp. viscosity supersolution) of (4) if the following holds:

(i) \( v \) is upper-semi-continuous (resp. lower-semi-continuous), \( v(T, \cdot) \leq g(T, \cdot) \) (resp. \( v(T, \cdot) \geq g(T, \cdot) \)), and \( v(t, s) \leq K[s^m + s^{-m}] \) (resp. \( v(t, s) \geq -K[s^m + s^{-m}] \)) for all \((t, s) \in \mathcal{D}_T \) and some positive constants \( K \) and \( m \);

(ii) for any triple \((\bar{t}, \bar{s}, \varphi) \in \mathcal{Q}_T \times C^{1,2}(\mathcal{Q}_T) \) satisfying \( v(\bar{t}, \bar{s}) = \phi(\bar{t}, \bar{s}) \) and \( v \leq \varphi \) (resp. \( v \geq \varphi \)) in \( \mathcal{Q}_T \), there holds

\[
F(\bar{t}, \bar{s}, \phi(\bar{t}, \bar{s}), \phi_t(\bar{t}, \bar{s}), \phi_s(\bar{t}, \bar{s}), \phi_{ss}(\bar{t}, \bar{s})) \leq 0 \quad \text{resp. } \geq 0.
\]

It is called a viscosity solution if it is both a viscosity supersolution and a subsolution.

The viscosity approach is very powerful, since the necessary condition of the function is locally bounded. Hence, the viscosity approach is widely used in applied mathematics, especially in mathematical finance. However, due to the relax
requirements of the value function, the classical methods in differential equations cannot be used anymore. On the other hand, it is not trivial to prove that the value function is a viscosity solution of the equation. Besides, it is well known that the uniqueness of viscosity solution (i.e., the comparison principle) is very hard to prove. Therefore and for comparison with our new weak solution introduced later, the first result of this paper is the following:

**Theorem 1.2.** Assume (A1) and (A2). Then $v$ defined in (1) is the unique viscosity solution of problem (4).

Because of the complexity of the proof of theorem 1.2, we shall find a simpler way to give a new definition of the solution. Following Perron’s idea of constructing harmonic function by the infimum of superharmonic functions, we introduce the following new weak solution of equation (4), which is the main contribution of this paper:

**Definition 1.3.** Assume (A1) and (A2). Fix $T > 0$.

1. $\zeta$ is called a classical subsolution (resp. supersolution) of (4) if (i) $\zeta \in C^{1,2}(D_T)$, (ii) $|s\zeta_s(t,s)| \leq K(s^n + s^{-m}) \forall (t,s) \in D_T$ where $K$ and $m$ are some positive constants, and (iii) \[
\begin{align*}
\min\{-\zeta_t - \mathcal{L}\zeta - f, \zeta - g\} &\leq 0 \quad \text{(resp.} \geq 0) \quad \text{in} \quad D_T, \\
\zeta(T,\cdot) - g(T,\cdot) &\leq 0 \quad \text{(resp.} \geq 0) \quad \text{on} \quad \{T\} \times (0,\infty).
\end{align*}
\]

2. $v : D_T \rightarrow \mathbb{R}$ is called a weak solution of (4) if $\underline{v} \leq v \leq \bar{v}$ for every classical subsolution $\underline{v}$ and every classical supersolution $\bar{v}$.

Under the above definition, we shall prove the following:

**Theorem 1.4.** Assume (A1) and (A2). Then $v$ defined in (1) is the unique weak solution of the variational inequality (4). In addition, the following holds:

1. $v$ is continuous and $g(t,s) \leq v(t,s) \leq K^T[s^n + s^{-m}] \forall (t,s) \in D_T$, where $K^T$ is a constant;

2. For each $(t,s) \in D_T$, $v(t,s) = \mathbb{E}^{t,s}[\int_t^{\tau^*} f(\rho,S_\rho) d\rho + g(\tau^*,S_{\tau^*})]$ where the optimal stopping time $\tau^*$ is given by
   \[
   \tau^* := \min\{\rho \in [t,T] \mid (\rho, S_\rho) \notin N\}, \quad N := \{(\rho, x) \in D_T \mid v(\rho, s) > g(\rho, s)\}.
   \]

One advantage of this definition is (as we will see in Section 3) that we do not need the Dynamic Programming Principle which is critical for both classical and viscosity approach. Secondly, since the sub-supersolution are defined in classical way, comparison principle can be applied directly. Additionally, the classical methods in differential equations, e.g. penalty method, can be used to derive some useful results, see e.g. [2, 3]. As an illustration, with the modification of classical penalty method, we have a key estimate (see Theorem 3.2 in Section 3) which is pivotal for the uniqueness and approximation of the solution. Also, we expect that many other classical techniques can be used for further applications.

Thus, the rest of this paper is organized as follows: In Section 2, we shall prove Theorem 1.2 by a standard viscosity approach; Then, Section 3 is devoted to the proof of Theorem 1.4. Finally, we give a short conclusion in Section 4.
2. The modern viscosity solution approach.

2.1. The idea of the proof.

We shall prove Theorem 1.2 by the following steps:

1. \(|v|\) is locally bounded and has at most polynomial growth (i.e. bounded by \(K[s^n + s^{-n}]\) for some positive constants \(K\) and \(n\));
2. \(v\) is continuous on \(\{T\} \times (0, \infty)\);
3. \(v_*(t,s) := \lim \inf_{D_T \ni (\tau,y) \to (t,s)} v(\tau, y)\) is a supersolution of (4a) and \(v_*(T, \cdot) = g(T, \cdot)\);
4. \(v^*(t,s) := \lim \sup_{D_T \ni (\tau,y) \to (t,s)} v(\tau, y)\) is a subsolution of (4a) and \(v^*(T, \cdot) = g(T, \cdot)\);
5. If \(U\) is a subsolution, \(V\) is a supersolution, \(U(T, \cdot) \leq V(T, \cdot)\), and \(U, V\) have at most polynomial growth (i.e. \(|U| + |V|\) is bounded by \(K[s^n + s^{-n}]\) for some positive constants \(K\) and \(n\)), then \(U \leq V\) in \(D_T\).

Note that in \(D_T = [0, T] \times (0, \infty)\), \(v_*\) is lower-semi-continuous, \(v^*\) is upper-semi-continuous, and \(v_* \leq v \leq v^*\). The above five properties, especially the last one, called comparison principle, imply that \(v^* \leq v_*\). Hence, \(v = v^* = v_*\) is a viscosity solution, which is unique in the class of continuous functions with polynomial growth.

In the sequel, \(T > 0\) is fixed.

2.2. Dynamic programming principle. The key to link optimal stopping problem and viscosity solution is the following:

**Lemma 2.1 (Dynamic Programming Principle).** For each \((t, s) \in D_T\) and \(\theta \in T_{t,T}\),

\[
v(t, s) = \sup_{\tau \in T_{t,T}} E^{t,s}[ \int_t^{\tau \wedge \theta} f(\rho, S_\rho) d\rho + g(\tau, S_\tau) 1_{\tau < \theta} + v(\theta, S_\theta) 1_{\theta \leq \tau} ]
\]  

\((6)\)

**Proof.** Let \((t, s) \in D_T\) and \(\theta \in T_{t,T}\) be fixed.

1. For each \(\varepsilon > 0\) there exists a \(\tau_\varepsilon \in T_{t,T}\) such that

\[
v(t, s) \leq \varepsilon + E^{t,s}[ \int_t^{\tau_\varepsilon} f(\rho, S_\rho) d\rho + g(\tau_\varepsilon, S_{\tau_\varepsilon}) ]
\]

\[
= \varepsilon + E^{t,s}[ \int_t^{\tau_\varepsilon} f(\rho, S_\rho) d\rho \left( 1_{\tau_\varepsilon < \theta} + 1_{\theta \leq \tau_\varepsilon} \right) g(\tau_\varepsilon, S_{\tau_\varepsilon}) 1_{\tau_\varepsilon < \theta} + g(\tau_\varepsilon, S_{\tau_\varepsilon}) 1_{\theta \leq \tau_\varepsilon} ]
\]

\[
= \varepsilon + E^{t,s}[ \int_t^{\tau_\varepsilon \wedge \theta} f(\rho, S_\rho) d\rho + g(\tau_\varepsilon, S_{\tau_\varepsilon}) 1_{\tau_\varepsilon < \theta} ]
\]

\[+ E^{t,s}[ E \left[ \int_{\theta}^{\tau_\varepsilon} f(\rho, S_\rho) d\rho + g(\tau_\varepsilon, S_{\tau_\varepsilon}) 1_{\theta \leq \tau_\varepsilon} \bigg| \mathcal{F}_{\theta} \right] ] \]

\[\leq \varepsilon + E^{t,s}[ \int_t^{\tau_\varepsilon \wedge \theta} f(\rho, S_\rho) d\rho + g(\tau_\varepsilon, S_{\tau_\varepsilon}) 1_{\tau_\varepsilon < \theta} ] + E^{t,s}[ v(\theta, S_\theta) 1_{\theta \leq \tau_\varepsilon} ]
\]

\[= \varepsilon + E^{t,s}[ \int_t^{\tau_\varepsilon \wedge \theta} f(\rho, S_\rho) d\rho + g(\tau_\varepsilon, S_{\tau_\varepsilon}) 1_{\tau_\varepsilon < \theta} + v(\theta, S_\theta) 1_{\theta \leq \tau_\varepsilon} ]
\]

\[\leq \varepsilon + \sup_{\tau \in T_{t,T}} E^{t,s}[ \int_t^{\tau \wedge \theta} f(\rho, S_\rho) d\rho + g(\tau, S_\tau) 1_{\tau < \theta} + v(\theta, S_\theta) 1_{\theta \leq \tau} ]
\]
Lemma 2.2. \( v \) is continuous on \( \{T\} \times (0, \infty) \), i.e.,
\[
\lim_{D_T \ni (t,y) \to (T,s)} v(t,y) = g(T,s) \quad \forall s > 0.
\]

Proof. Fix \( s_0 \in (0, \infty) \). Set \( \eta(s) = \frac{n+1}{2\varepsilon^{n+1}} + \frac{n+1}{2\varepsilon^{n+1}} \). Then \( \eta(s_0) = 1, \eta'(s_0) = 0 \), and \( \eta'' > 0 \) in \( (0, \infty) \). Fix \( \varepsilon > 0 \). There exists \( M_\varepsilon > 0 \) such that when \( m \geq M_\varepsilon \),
\[
\zeta^m(t,s) := g(T,s_0) + \left( \varepsilon + M_\varepsilon |\eta(s) - 1| \right) e^{m(T-t)} \geq g(t,s) \quad \forall (t,s) \in D_T.
\]

Set \( k = \max \{ \frac{1}{2}\sigma^2(n+1)n + \alpha(n+1), \frac{1}{2}\sigma^2(n+1)(n+2) - \alpha(n+1) \} \). Then
\[
-G^m \zeta^m \geq e^{m(T-t)} \left\{ m\varepsilon + mM_\varepsilon |\eta(s) - 1| - kM_\varepsilon \eta(s) \right\} \geq M_f > f \quad \text{on } D_T
\]
if
\[
m \geq M_{1\varepsilon} := \sup_{s > 0} \frac{kM_\varepsilon |\eta(s) + M_f |}{\varepsilon + M_\varepsilon |\eta(s) - 1|}.
\]

Hence, setting \( m = \max \{ M_\varepsilon, M_{1\varepsilon} \} \) we see that \( \zeta^m \) is a classical supersolution, so \( v \leq \zeta^m \). This implies that
\[
v^*(T,s_0) := \limsup_{D_T \ni (t,s) \to (T,s_0), t \leq T} v(t,s) \leq g(T,s_0) + \varepsilon.
\]
Sending \( \varepsilon \searrow 0 \) we obtain \( v^*(T, s_0) \leq g(T, s_0) \). Since \( v \geq g \), the assertion of the lemma thus follows.

**Lemma 2.3.** \( v^* \) is a viscosity subsolution of (4).

**Proof.** 1. First of all, \( v^*(t, s) = \limsup_{\tau \to t} Q_{\tau} \) is upper-semi-continuous in \( D_T = [0, T] \times (0, \infty) \). Next, we have \( v^*(T, \cdot) = g(T, \cdot) \). Also, by the classical supersolution, \( v^* \leq K[\varepsilon^n + s^{-n}]e^{k(T-t)} \) for some constants \( n \), \( k \) and \( K \).

2. Let \( Q_T = [0, T] \times (0, \infty) \). Suppose \( (\bar{t}, \bar{s}, \varphi) \in Q_T \times C^{1,2}(Q_T) \) is a triple satisfying

\[
0 = \max_{Q_T} (v^* - \varphi)(t, s) = (v^* - \varphi)(\bar{t}, \bar{s}).
\]

We want to show that \( \min \{-\varphi_t - \mathcal{L}\varphi - f, \varphi - g\}(\bar{t}, \bar{s}) \leq 0 \). We argue by contradiction by assuming

\[
(\varphi_t - \mathcal{L}\varphi - f)(\bar{t}, \bar{s}) > 0, \quad (\varphi - g)(\bar{t}, \bar{s}) > 0.
\]

By the continuity, there exists \( \delta_0 > 0 \) such that \( \Omega := (\bar{t} - \delta_0, \bar{t} + \delta_0) \times (\bar{s} - \delta_0, \bar{s} + \delta_0) \subset Q_T \) and

\[
\min \{-\varphi_t - \mathcal{L}\varphi - f, \varphi - g\} > \delta_0 \text{ in } \Omega.
\]

By the definition of \( v^* \), there exist a sequence \( \{(t_m, s_m, v(t_m, s_m))\}_{m=1}^{\infty} \) which converges to \( (\bar{t}, \bar{s}, v^*(\bar{t}, \bar{s})) \), as \( m \to \infty \). Without loss of generality, we can assume \( \{(t_m, s_m)\}_{m=1}^{\infty} \subset (\bar{t} - \delta_0/2, \bar{t} + \delta_0/2) \times (\bar{s} - \delta_0/2, \bar{s} + \delta_0/2) \). By the continuity of \( \varphi \) we have

\[
\gamma_m := \varphi(t_m, s_m) - v(t_m, s_m) \to \varphi(\bar{t}, \bar{s}) - v^*(\bar{t}, \bar{s}) = 0 \text{ as } m \to \infty.
\]

Now fix \( (t_m, s_m) \) and let \( \{S_u^{t_m, s_m}\}_{u \geq t_m} \) be the Itô process with initial condition \( S_{t_m} = s_m \). We define

\[
\tau_m := \inf\{u \geq t_m \mid (u, S_u^{t_m, s_m}) \notin \Omega\}.
\]

For each \( \tau \in T_{t_m, T} \) we can apply Itô’s formula to \( \varphi \) between \( (t_m, s_m) \) and \( (\tau \wedge \tau_m, S_{\tau \wedge \tau_m}) \) to obtain

\[
v(t_m, s_m) + \gamma_m
= \varphi(t_m, s_m)
= E\left[ \varphi(\tau \wedge \tau_m, S_{\tau \wedge \tau_m}) + \int_{t_m}^{\tau \wedge \tau_m} (-\varphi_t - \mathcal{L}\varphi)du + \int_{t_m}^{\tau \wedge \tau_m} \sigma s \varphi_s dW_u \bigg| S_{t_m} = s_m \right]
\]

\[
= E\left[ \varphi(\tau, S_\tau)1_{\tau < \tau_m} + \varphi(\tau, S_{\tau_m})1_{\tau_m < \tau} + \int_{t_m}^{\tau \wedge \tau_m} (-\varphi_t - \mathcal{L}\varphi)du \bigg| S_{t_m} = s_m \right]
\]

\[
\geq E\left[ (g(\tau, S_\tau) + \delta_0)1_{\tau < \tau_m} + v(\tau, S_{\tau_m})1_{\tau_m < \tau} + \int_{t_m}^{\tau \wedge \tau_m} (f + \delta_0)du \bigg| S_{t_m} = s_m \right]
\]

\[
= E\left[ \int_{t_m}^{\tau \wedge \tau_m} fdu + g(\tau, S_\tau)1_{\tau < \tau_m} + v(\tau, S_{\tau_m})1_{\tau_m < \tau} \bigg| S_{t_m} = s_m \right]
\]

\[
+ \delta_0 E\left[ 1_{\tau < \tau_m} + \int_{t_m}^{\tau \wedge \tau_m} du \bigg| S_{t_m} = s_m \right]
\]

\[
\geq E\left[ \int_{t_m}^{\tau \wedge \tau_m} fdu + g(\tau, S_\tau)1_{\tau < \tau_m} + v(\tau, S_{\tau_m})1_{\tau_m < \tau} \bigg| S_{t_m} = s_m \right]
\]

\[
+ \delta_0 E\left[ \min\{1, \tau_m - t_m\} \bigg| S_{t_m} = s_m \right]
\]

where we used the fact that the stochastic integral on the right-hand side of second equation is a martingale, that \( \varphi \geq v \) in \( Q_T \) and \( \varphi \geq g + \delta_0, -\varphi_t - \mathcal{L}\varphi - f \geq \delta_0 \) in
\(\Omega\) in the first inequality. Take the supreme over all \(\tau \in \mathcal{T}_{m,T}\) and use the principle of dynamic programming, we have

\[
v(t_m, s_m) + \gamma_m \geq \sup_{\tau \in \mathcal{T}_{m,T}} \mathbb{E} \left[ \int_{t_m}^{\tau \wedge \tau_m} fdu + g(\tau, S_\tau)1_{\tau < \tau_m} + v(\tau, S_\tau)1_{\tau_m \leq \tau} \right] + \delta_0 \mathbb{E} \left[ \min\{1, \tau_m - t_m\} \big| S_{t_m} = s_m \right]
\]

\[
= v(t_m, s_m) + \delta_0 \mathbb{E} \left[ \min\{1, \tau_m - t_m\} \big| S_{t_m} = s_m \right]
\]

Hence,

\[
\gamma_m \geq \delta_0 \mathbb{E} \left[ \min\{1, \tau_m - t_m\} \big| S_{t_m} = s_m \right]
\]

Let \(m\) go to infinity and use \(S_t = s_m e^{(\alpha - \sigma^2/2) (t-t_m) + \sigma \left(W_t - W_{t_m}\right)}\) for \(t > t_m\). We derive that

\[
0 \geq \delta_0 \lim_{m \to \infty} \mathbb{E} \left[ \min\{1, \tau_m - t_m\} \big| S_{t_m} = s_m \right]
\]

where \(\tau = \inf\{t > \bar{t} \mid (t, S_t) \notin \Omega\}\). Subject to \(S_{\bar{t}} = \bar{s}\), we know that \(\tau > \bar{t}\) a.s., i.e., \(\mathbb{E} \left[ \min\{1, \tau - \bar{t}\} \big| S_{\bar{t}} = \bar{s} \right] > 0\). Hence we obtain a contradiction. This completes the proof. \(\square\)

2.4. Supersolution property.

**Lemma 2.4.** \(v_*\) is a viscosity supersolution of \((4)\).

**Proof.** 1. We know that \(v_*(t, s) := \lim\{u, y) \in \mathcal{D}_{T \to (t, s)} v(u, y)\) is lower-semi-continuous. Also, \(v_*(T, \cdot) = g(T, \cdot)\) and \(v_* \geq g \geq -K[s^n + s^{-n}]e^{k(T-t)}\) for some constants \(n, k\) and \(K\).

2. Let \((\bar{t}, \bar{s}, \varphi) \in \Omega_T \times C^{1,2}(\Omega_T)\) satisfy \(0 = (v_* - \varphi)(\bar{t}, \bar{s}) = \min_{\Omega_T} (v_* - \varphi)(t, s)\). There exists a sequence \(\{(t_m, s_m\})_{m=1}^\infty\) such that

\[
\lim_{m \to \infty} (t_m, s_m, v(t_m, s_m)) = (\bar{t}, \bar{s}, v_*(\bar{t}, \bar{s})).
\]

By the continuity of \(\varphi\),

\[
\gamma_m := v(t_m, s_m) - \varphi(t_m, s_m) \to 0 \text{ as } m \to \infty.
\]

Since \(v \geq v_* \geq \varphi\), \(\gamma_m \geq 0\). We can find a positive sequence \(\{h_m\}\) such that

\[
h_m \to 0 \text{ and } \frac{\gamma_m}{h_m} \to 0 \text{ as } m \to \infty.
\]

Let

\[
\tau_m := \inf\{u \geq t \mid S_{t_m}^{u, s_m} - s_m \geq \eta\}\text{ where } \eta \text{ is a positive constant.}
\]

From the Dynamical Programming equation by taking \(\tau = \theta = \theta_m := \tau_m \wedge (t_m + h_m)\), we have

\[
v(t_m, s_m) \geq \mathbb{E} \left[ \int_{t_m}^{\theta_m} fdu + v(\theta_m, S_{\theta_m}) \big| S_{t_m} = s_m \right]
\]

Then

\[
\varphi(t_m, s_m) + \gamma_m
\]

\[
= v(t_m, s_m) \geq \mathbb{E} \left[ \int_{t_m}^{\theta_m} fdu + v(\theta_m, S_{\theta_m}) \big| S_{t_m} = s_m \right]
\]
\[ \gamma_m \geq \mathbb{E} \left[ \int_{t_m}^{\theta_m} f \, du + \varphi(t_m, S_{\theta_m}) \bigg| S_{t_m} = s_m \right] - \mathbb{E} \left[ \varphi(t_m, s_m) + \int_{t_m}^{\theta_m} (\varphi_t + \mathcal{L} \varphi + f) \, du + \int_{t_m}^{\theta_m} \sigma s \varphi \, dW_u \bigg| S_{t_m} = s_m \right] \]

Note that the above stochastic integral is a martingale since the integrand is bounded. Hence, dividing by \( h_m \) on both sides, we have

\[ \frac{\gamma_m}{h_m} \geq \mathbb{E} \left[ \frac{1}{h_m} \int_{t_m}^{\theta_m} (\varphi_t + \mathcal{L} \varphi + f) \, du \bigg| S_{t_m} = s_m \right] - \mathbb{E} \left[ \frac{\theta_m - t_m}{h_m} \bigg| S_{t_m} = s_m \right] = \mathbb{E} \left[ \frac{\theta_m - t_m}{h_m} \bigg| S_{t_m} = s_m \right] \]

where \( p = \varphi_t + \mathcal{L} \varphi + f \) and \( o(1) \to 0 \) as \( \eta \searrow 0 \). Sending \( m \to \infty \) and using the definition of \( \theta_m \) we find that \( \lim_{m \to \infty} \mathbb{P}[(\theta_m = t_m + h_m)] = 1 \), so by Lebesgue’s dominated convergence theorem,

\[ 0 \geq p + o(1). \]

Send \( \eta \searrow 0 \) we find that \( p \leq 0 \), i.e.,

\[ (-\varphi_t - \mathcal{L} \varphi - f)(\bar{t}, \bar{s}) \geq 0. \]

In conclusion, we have

\[ \min \{-\varphi_t - \mathcal{L} \varphi - f, \varphi - g\}(\bar{t}, \bar{s}) \geq 0. \]

Hence \( v_* \) is a supersolution of (4).

2.5. Comparison principle. In this subsection, we shall discuss the comparison principle which is critical to the uniqueness. First of all, let us give an alternative definition of viscosity solution based on jets whose proof can be found in [7].

Lemma 2.5 (Alternative definition of viscosity solution). An upper semi-continuous (resp. lower semi-continuous) function \( w \) in \([0, T] \times \mathbb{R}^n\) is a viscosity subsolution (resp. supersolution) of the following equation

\[ F(t, x, w, D_t w, D_x w, D^2_x w) = 0, \text{ in } [0, T] \times \mathbb{R}^n, \]

if and only if for all \((t, x) \in [0, T] \times \mathbb{R}^n\), and all \((q, p, M) \in \mathcal{P}^{2,+} w(t, x)\) (resp. \( \mathcal{P}^{2,-} w(t, x)\)),

\[ F(t, x, w(t, x), q, p, M) \leq (\text{resp.} \geq) 0. \]

Here \( \mathcal{P}^{2,+} w \) and \( \mathcal{P}^{2,-} w \) are the superjets and subjets of \( w \) respectively.

The following Ishii’s Lemma plays a pivotal role in the proof of comparison principle. One can find the proof in [4].

Lemma 2.6 (Ishii’s Lemma). Let \( U \) and \( -V \) be a upper-semi-continuous functions on \([0, T] \times \mathbb{R}^n, \phi \in C^{1,1,2,2}(\mathbb{R}^2 \times \mathbb{R}^n \times \mathbb{R}^n) \), and \((\bar{t}, \bar{r}, \bar{s}, \bar{x}) \in [0, T] \times \mathbb{R}^n \times \mathbb{R}^n\) a local maximum of \( U(t, s) - V(\tau, x) - \phi(t, r, s, x) \). Then for each \( \eta > 0 \), there exist \( P, Q \in \mathcal{S}_n \) such that

\[ \left( \frac{\partial \phi}{\partial t}(\bar{t}, \bar{r}, \bar{s}, \bar{x}), D_t \phi(\bar{t}, \bar{r}, \bar{s}, \bar{x}), P \right) \in \mathcal{P}^{2,+} U(\bar{t}, \bar{s}), \]

\[ \left( -\frac{\partial \phi}{\partial \tau}(\bar{t}, \bar{r}, \bar{s}, \bar{x}), -D_x \phi(\bar{t}, \bar{r}, \bar{s}, \bar{x}), Q \right) \in \mathcal{P}^{2,-} V(\bar{r}, \bar{x}), \]
and
\[
\begin{pmatrix}
  P & 0 \\
  0 & -Q
\end{pmatrix} \leq D_{x,x}^2 \phi(\tilde{t}, \tilde{r}, \tilde{s}, \tilde{x}) + \eta(D_{x,x}^2 \phi(\tilde{t}, \tilde{r}, \tilde{s}, \tilde{x}))^2.
\]

Now we can present the following comparison principle:

**Lemma 2.7.** Suppose \( u \) is a viscosity subsolution and \( v \) is a viscosity supersolution of (4). Then \( u \leq v \) in \([0, T] \times (0, \infty)\).

**Proof.** **Step 1.** Fix \( \beta > n \), set \( \zeta = K[s^\beta + s^{-\beta}]e^{k(T-t)} \) with \( k = \min\{\frac{\beta}{2}\sigma(\beta - 1) + \alpha\beta, \frac{1}{2}\sigma^2\beta(\beta + 1) - \alpha\beta\} \) and \( K \) is sufficiently large. Then it is easy to show that \( -\zeta_t - \mathcal{L} \zeta - f > 0 \). Hence, for each \( \epsilon > 0 \), \( u_\epsilon = u - 2\epsilon\zeta \) is a subsolution and \( v_\epsilon := v + 2\epsilon\zeta \) is a supersolution. In addition, there exists \( 0 < r_\epsilon < R_\epsilon < \infty \) such that \( u_\epsilon(t, s) < -\epsilon[s^{-\beta} + s^\beta] \) and \( v_\epsilon(t, s) > \epsilon[s^{-\beta} + s^\beta] \) for all \( s \in (0, r_\epsilon) \cup [R_\epsilon, \infty) \) and \( t \in (0, T) \).

Next, for \( m > 0 \) let \( v_{\epsilon,m} = e^{m(T-t)}v_\epsilon \), \( u_{\epsilon,m} = e^{m(T-t)}u_\epsilon \), \( f_m = e^{m(T-t)}f \) and \( g_m = e^{m(T-t)}g \). Then one can check \( \tilde{v}_\epsilon, \tilde{u}_\epsilon \) are super/subsolutions of the following problem
\[
\min\{mw - w_t - \mathcal{L}w - f_m, w - g_m\} = 0 \quad \text{in} \quad [0, T] \times (0, \infty).
\]

Suppose \( u - v \leq 0 \) in \( \mathcal{D}_T \) is not true. Then there exists \( \epsilon > 0 \) such that \( u_\epsilon - v_\epsilon \leq 0 \) in \( \mathcal{D}_T \) is not true. Since \( u_\epsilon - v_\epsilon \) is upper-semi-continuous, bounded from above and \( u(T, \cdot) - v(T, \cdot) \leq 0 \), by taking \( m \) large enough, there exists \( (\tilde{t}, \tilde{s}) \in (0, T) \times (0, \infty) \) such that
\[
M := \max_{\mathcal{D}_T}(u_{\epsilon,m} - v_{\epsilon,m}) = u_{\epsilon,m}(\tilde{t}, \tilde{s}) - v_{\epsilon,m}(\tilde{t}, \tilde{s}) > 0.
\]

We now fix \( \epsilon \) and \( m \) and denote \( U = u_{\epsilon,m} \) and \( V = v_{\epsilon,m} \).

**Step 2.** Now for each fixed \( \epsilon > 0 \), consider the functions
\[
\Phi_\epsilon(t, \tau, s, x) = U(t, s) - V(\tau, x) - \phi_\epsilon(t, \tau, s, x),
\]
\[
\phi_\epsilon(t, \tau, s, x) = \frac{1}{2\epsilon} \left[(t - \tau)^2 + (s - x)^2\right].
\]

The u.s.c. function \( \Phi_\epsilon \) attains its maximum, denoted by \( M_\epsilon \), on \([0, T]^2 \times (0, \infty)^2\) at \((t_\epsilon, \tau_\epsilon, s_\epsilon, x_\epsilon)\). We claim that
\[
M_\epsilon \to M \quad \text{and} \quad \phi_\epsilon(t_\epsilon, \tau_\epsilon, s_\epsilon, x_\epsilon) \to 0, \quad \text{as} \ \epsilon \to 0.
\]

For this, we note that
\[
M \leq M_\epsilon = U(t_\epsilon, s_\epsilon) - V(\tau_\epsilon, x_\epsilon) - \phi_\epsilon(t_\epsilon, \tau_\epsilon, s_\epsilon, x_\epsilon) \leq U(t_\epsilon, s_\epsilon) - V(\tau_\epsilon, x_\epsilon).
\]

Now the sequence \((t_\epsilon, \tau_\epsilon, s_\epsilon, x_\epsilon)\) converges, up to a subsequence, to some \((\tilde{t}, \tilde{r}, \tilde{s}, \tilde{x}) \in [0, T]^2 \times [1/B, B]^2\) for some large constant \( B \). And since the sequence \( \{U(t_\epsilon, s_\epsilon) - V(\tau_\epsilon, x_\epsilon)\}_\epsilon \) is bounded, we see from the above inequality that \( \{\phi_\epsilon(t_\epsilon, \tau_\epsilon, s_\epsilon, x_\epsilon)\}_\epsilon \) is also bounded, which implies \( \tilde{t} = \tilde{r}, \tilde{s} = \tilde{x} \). By sending \( \epsilon \) to zero, we get \( M \leq \lim \sup_{\epsilon \to 0} M_\epsilon \leq (U - V)(\tilde{t}, \tilde{s}) \leq M \). Hence \( M = (U - V)(\tilde{t}, \tilde{s}) \) and \( \lim_{\epsilon \to 0} M_\epsilon = M \).

In addition, by Step 1 we have \( (\tilde{t}, \tilde{s}) \in (0, T) \times (0, \infty) \).

**Step 3.** Note that \((t_\epsilon, s_\epsilon)\) is a global maximum of \((t, s) \to U(t, s) - \phi_\epsilon(t, \tau_\epsilon, s_\epsilon, x_\epsilon)\) in \([0, T] \times (0, \infty)\), \((\tau_\epsilon, x_\epsilon)\) is a global minimum of \((\tau, x) \to V(\tau, x) + \phi_\epsilon(t_\epsilon, \tau_\epsilon, s_\epsilon, x)\) in \([0, T] \times (0, \infty)\).
From Ishii’s lemma, by taking $\eta = \epsilon$ there exist $P_\epsilon$ and $Q_\epsilon \in S_1$ such that
\[
\left(\frac{t_\epsilon - \tau_\epsilon}{\epsilon}, \frac{s_\epsilon - x_\epsilon}{\epsilon}, P_\epsilon\right) \in \mathcal{P}^{2,+}U(t_\epsilon, s_\epsilon),
\]
and
\[
\left(\frac{t_\epsilon - \tau_\epsilon}{\epsilon}, \frac{s_\epsilon - x_\epsilon}{\epsilon}, Q_\epsilon\right) \in \mathcal{P}^{2,-}V(\tau_\epsilon, x_\epsilon).
\]
and
\[
\begin{pmatrix} P_\epsilon & 0 \\ 0 & -Q_\epsilon \end{pmatrix} \leq \frac{1}{\epsilon} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} + \eta \epsilon \frac{1}{\epsilon^2} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \leq \frac{3}{\epsilon} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}
\]
then for any $a, b \in \mathbb{R}$,
\[
a^2 P_\epsilon - b^2 Q_\epsilon = \text{Tr} \left[ \begin{pmatrix} a^2 & ab \\ ab & b^2 \end{pmatrix} \begin{pmatrix} P_\epsilon & 0 \\ 0 & -Q_\epsilon \end{pmatrix} \right] \leq \frac{3}{\epsilon} \text{Tr} \left[ \begin{pmatrix} a^2 & ab \\ ab & b^2 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \right] = \frac{3}{\epsilon} (a - b)^2.
\]

**Step 4.** Note the viscosity sub-supersolution properties of $U$, $V$, we have
\[
\min \left\{ mU(t_\epsilon, s_\epsilon) - \frac{t_\epsilon - \tau_\epsilon}{\epsilon} - \alpha s_\epsilon \frac{s_\epsilon - x_\epsilon}{\epsilon} - \frac{\sigma^2}{2} s_\epsilon^2 P_\epsilon - f_m(t_\epsilon, s_\epsilon), \right.
\]
\[
U(t_\epsilon, s_\epsilon) - g_m(t_\epsilon, s_\epsilon) \bigg\} \leq 0, \quad (7)
\]
\[
\min \left\{ mV(\tau_\epsilon, x_\epsilon) - \frac{t_\epsilon - \tau_\epsilon}{\epsilon} - \alpha x_\epsilon \frac{s_\epsilon - x_\epsilon}{\epsilon} - \frac{\sigma^2}{2} x_\epsilon^2 Q_\epsilon - f_m(\tau_\epsilon, x_\epsilon), \right.
\]
\[
V(\tau_\epsilon, x_\epsilon) - g_m(\tau_\epsilon, x_\epsilon) \bigg\} \geq 0. \quad (8)
\]

(i) If $U(t_\epsilon, s_\epsilon) - g_m(t_\epsilon, s_\epsilon) > 0$ for $m$ large enough, then the first term of (7) is nonpositive, i.e.
\[
mU(t_\epsilon, s_\epsilon) - \frac{t_\epsilon - \tau_\epsilon}{\epsilon} - \alpha s_\epsilon \frac{s_\epsilon - x_\epsilon}{\epsilon} - \frac{\sigma^2}{2} s_\epsilon^2 P_\epsilon - f_m(t_\epsilon, s_\epsilon) \leq 0.
\]
On the other hand, (8) implies
\[
mV(\tau_\epsilon, x_\epsilon) - \frac{t_\epsilon - \tau_\epsilon}{\epsilon} - \alpha x_\epsilon \frac{s_\epsilon - x_\epsilon}{\epsilon} - \frac{\sigma^2}{2} x_\epsilon^2 Q_\epsilon - f_m(\tau_\epsilon, x_\epsilon) \geq 0.
\]

By substracting the above two inequalities, we have
\[
mU(t_\epsilon, s_\epsilon) - mV(\tau_\epsilon, x_\epsilon)
\]
\[
\leq \alpha s_\epsilon \frac{s_\epsilon - x_\epsilon}{\epsilon} + \frac{\sigma^2}{2} s_\epsilon^2 P_\epsilon - \alpha x_\epsilon \frac{s_\epsilon - x_\epsilon}{\epsilon} - \frac{\sigma^2}{2} x_\epsilon^2 Q_\epsilon + f_m(t_\epsilon, s_\epsilon) - f_m(\tau_\epsilon, x_\epsilon)
\]
\[
\leq \frac{\alpha}{\epsilon} (s_\epsilon - x_\epsilon)^2 + 3 \frac{\sigma^2}{2\epsilon} (s_\epsilon - x_\epsilon)^2 + f_m(t_\epsilon, s_\epsilon) - f_m(\tau_\epsilon, x_\epsilon)
\]
\[
\leq \frac{1}{2\epsilon} \left[ (t_\epsilon - \tau_\epsilon)^2 + (s_\epsilon - x_\epsilon)^2 \right] + f_m(t_\epsilon, s_\epsilon) - f_m(\tau_\epsilon, x_\epsilon)
\]
\[
= C \phi_\epsilon(t_\epsilon, \tau_\epsilon, s_\epsilon, x_\epsilon) + f_m(t_\epsilon, s_\epsilon) - f_m(\tau_\epsilon, x_\epsilon),
\]
where $C = \max \{ 2\alpha, 3\sigma^2 \}$. By the property of $\phi_\epsilon$ and continuity of $f_m$, let $\epsilon$ go to zero, we obtain $mM \leq 0$, a contradiction with $M > 0$.

(ii) Otherwise, up to a subsequence, $U(t_\epsilon, s_\epsilon) - g_m(t_\epsilon, s_\epsilon) \leq 0$ for all $m$, and since $V(\tau_\epsilon, x_\epsilon) - g_m(\tau_\epsilon, x_\epsilon) \geq 0$ by (8), we obtain that
\[
U(t_\epsilon, s_\epsilon) - V(\tau_\epsilon, x_\epsilon) \leq g_m(t_\epsilon, s_\epsilon) - g_m(\tau_\epsilon, x_\epsilon).
\]
By sending $m$ to infinity, and from the continuity of $g_m$, we also have required contradiction $M \leq 0$. This completes the proof.

**Proof of Theorem 1.2.**

*Proof.* We know that $v_*$ is a viscosity supersolution and $v^*$ is a viscosity subsolution. By comparison, obtain $v^* \leq v_$. On the other hand, $v_* \leq v \leq v^*$. Hence $v = v_ = v^*$ is a viscosity solution. By comparison, viscosity solutions, if it exists, is unique. The assertion of the theorem thus follows.

3. A classical approach with new ingredients.

3.1. *v is a weak solution.* First we show that $v$ is a weak solution and is locally bounded.

**Lemma 3.1.** The function $v$ defined in (1) is a weak solution of (4) and satisfies

$$g(t,s) \leq v(t,s) \leq K|s^n + s^{-m}|e^{M(T-t)} \quad \forall (t,s) \in D_T$$

where $K$ and $M$ are large constants.

The lemma implies that any classical supersolution is always bigger than or equal to any classical subsolution, a fact that can be shown directly by a standard PDE comparison technique.

*Proof.* Suppose $\zeta \in C^{1,2}(D_T)$ and $|s^z| \leq K(s^m + s^{-m})$ for some positive constants $K$ and $m$. Then we can apply Itô's formula to obtain, for any $\tau \in T_t,T$,

$$\zeta(t,s) = \mathbb{E}^{t,s}\left[\zeta(\tau,S_\tau) + \int_t^\tau (-\zeta_t - \mathcal{L}\zeta_t)du - \int_t^\tau \sigma S_u \zeta_u (u,S_u) dW_u \right]$$

$$= \mathbb{E}^{t,s}\left[\zeta(\tau,S_\tau) + \int_t^\tau (-\zeta_t - \mathcal{L}\zeta_t)du \right],$$

since the stochastic integral is a martingale ($\int_0^T \mathbb{E}(\sigma S_u \zeta_u (u,S_u))^2)du < \infty$ derived from the assumption that $|x\zeta_u(u,x)| \leq K|x^m + x^{-m}|$ and that $\{S_u\}_{u \in [t,T]}$ is a geometric Brownian motion).

Now suppose $\zeta$ is a classical supersolution. Then $\min\{-\zeta_t - \mathcal{L}\zeta - f, \zeta - g \} \geq 0$, so we obtain $\zeta(t,s) \geq \mathbb{E}^{t,s}[\int_t^\tau f(\rho,\sigma)\rho d\rho + g(\tau,S_\tau)]$. Taking the supreme over $\tau \in T_t,T$ we obtain $\zeta(t,s) \geq v(t,s)$.

Next suppose $\zeta$ is a classical subsolution. Fix $(t,s) \in D_T$ and condition on $S_t = s$. If $\zeta(t,s) \leq g(t,s)$, we have $\zeta(t,s) \leq v(t,s)$ since $\zeta \geq g$ (by taking $\tau = t$ in (1)).

If $\zeta(t,s) > g(t,s)$, let $\tilde{\Omega} = \{(u,x) \in D_T \mid \zeta(u,x) > g(u,x)\}$ and define a stopping time

$$\tau = \min\{u \in [t,T] \mid (u,S_u) \notin \tilde{\Omega}\}.$$  

Since $\zeta(T,\cdot) \leq g(T,\cdot)$, we see that $t \leq \tau \leq T$ and $\zeta(\tau,S_\tau) = g(\tau,S_\tau)$. In addition, as $\min\{-\zeta_t - \mathcal{L}\zeta - f, \zeta - g \} \leq 0$ in $D_T$, for any $u \in [t,\tau)$, $(-\zeta_t - \mathcal{L}\zeta - f)(u,S_u) \leq 0$. Hence,

$$\zeta(t,s) = \mathbb{E}^{t,s}\left[\zeta(\tau,S_\tau) + \int_t^\tau (-\zeta_t - \mathcal{L}\zeta_t)du \right] \leq \mathbb{E}^{t,s}\left[\int_t^\tau f(u,S_u)du + \zeta(\tau,S_\tau) \right]$$

$$= \mathbb{E}^{t,s}\left[\int_t^\tau f(u,S_u)du + g(\tau,S_\tau) \right] \leq v(t,s).$$

In summary, by Definition 1.3, $v$ is a weak solution of (4).
Finally, since \( g \) satisfies (A1), then we can pick large constants \( K \) and \( M \) such that \( \zeta^*(t, s) := K[s^n + s^{-n}] e^{M(T-t)} \geq g(t, s) \) for all \( t \in [0, T] \) and \( s \in (0, \infty) \). On the other hand, since \( f \) is uniformly bounded,

\[
-\zeta^t - \mathcal{L} \zeta^* - f = \left\{ \left( K M - \frac{1}{2} \sigma^2 n (n-1) - \alpha n \right) s^n + \left( K M - \frac{1}{2} \sigma^2 n (n+1) + \alpha n \right) s^{-n} \right\} e^{M(T-t)} - f \geq 0,
\]

provided \( K \) and \( M \) large enough.

Therefore, for sufficient large constants \( K \) and \( M \), \( \zeta^* \) is a classical supersolution. The assertion of the lemma thus follows.

\[ \square \]

3.2. A new usage of the old penalty method.

In this subsection, we modify the penalty method to get a priori estimates of the weak solution. Then using the estimates we can derive the uniqueness of the solution.

First of all, we make the change of variables

\[
\begin{align*}
x &= \ln s, & \tau &= \frac{1}{2} \sigma^2 (T-t), & \hat{T} &= \frac{1}{2} \sigma^2 T, \\
\hat{g} &\mathcal{T} \tau, x \mathcal{T} = g(T - \frac{2\tau}{\sigma^2}, e^\tau), & \hat{f} &\mathcal{T} \tau, x \mathcal{T} = 2\sigma^{-2} f(T - \frac{2\tau}{\sigma^2}, e^\tau), & \hat{v} &\mathcal{T} \tau, x \mathcal{T} = v(T - \frac{2\tau}{\sigma^2}, e^\tau).
\end{align*}
\]

Then problem (4) becomes

\[
\min \{ \hat{v} - \hat{v}_{xx} + [1 - 2\alpha \sigma^{-2}] \hat{v}_x - \hat{f}, \hat{v} - \hat{g} \} = 0 \text{ in } (0, \hat{T}] \times \mathbb{R}, \quad \hat{v}(0, \cdot) = \hat{g}(0, \cdot).
\]

To take care of the exponential growth, we take \( m = n + 1 \) and define

\[
\begin{align*}
V(x, \tau) &= \frac{v(T - 2\sigma^{-2} \tau, e^\tau)}{e^{m x + k(m) \tau} + e^{-m x + k(-m) \tau}}, \\
F(\tau, x) &= \frac{2\sigma^{-2} f(T - 2\alpha \sigma^{-2} \tau, e^\tau)}{e^{m x + k(m) \tau} + e^{-m x + k(-m) \tau}}, \\
G(\tau, x) &= \frac{g(T - 2\alpha \sigma^{-2} \tau, e^\tau)}{e^{m x + k(m) \tau} + e^{-m x + k(-m) \tau}},
\end{align*}
\]

where \( k(\lambda) = \lambda^2 + (2\alpha \sigma^{-2} - 1) \lambda \). Then (4) is equivalent to

\[ \begin{cases} 
\min \{ V - \mathcal{L} V - F, & V - G \} = 0 \text{ in } (0, \hat{T}] \times \mathbb{R}, \\
V(0, \cdot) = G(0, \cdot) \text{ on } \{ 0 \} \times \mathbb{R}, & \lim_{|x| \to \infty} \sup_{\tau \in [0, \hat{T}]} |V(\tau, x)| = 0
\end{cases} \quad (9) \]

where \( \mathcal{L} U = U_{xx} + b U_x \) with

\[
b = b(\tau, x) = 2\alpha \sigma^{-2} - 1 + 2m \frac{e^{m x + k(m) \tau} - e^{-m x + k(-m) \tau}}{e^{m x + k(m) \tau} + e^{-m x + k(-m) \tau}}.
\]

We now solve the variational problem (9) by the classical method of penalty.

1. Since \( \lim_{|x| \to \infty} \sup_{\tau \in [0, \hat{T}]} | G(\tau, x) | = 0 \), \( G \) is uniformly continuous on \( [0, \hat{T}] \times \mathbb{R} \). Hence, there exists a sequence \( \{ G^\varepsilon \}_{0 < \varepsilon < 1} \) of smooth functions such that, for some positive constant \( C_\varepsilon \),

\[
G \geq G^\varepsilon \geq G - \varepsilon \text{ on } [0, \hat{T}] \times \mathbb{R}, \quad \| G^\varepsilon - \mathcal{L} G^\varepsilon - F \|_{C([0, \hat{T}] \times \mathbb{R})} \leq C_\varepsilon.
\]

Let \( \{ \beta_\varepsilon \}_{0 < \varepsilon < 1} \) be a family of smooth functions having the following properties:

\[
\beta_\varepsilon = 0 \text{ in } [0, \infty), \quad \beta_\varepsilon' \leq 0 \text{ in } (-\infty, 0), \quad \beta_\varepsilon'' = 0 \text{ in } (-\infty, -\varepsilon), \quad \beta_\varepsilon(-\varepsilon) = C_\varepsilon.
\]
For each $\varepsilon \in (0,1]$, let $U^\varepsilon$ and $V^\varepsilon$ be respectively the unique bounded solution of the initial value problems
\begin{align}
U^\varepsilon_t - \mathcal{L}U^\varepsilon - F &= \beta_\varepsilon(U^\varepsilon - G^\varepsilon) \quad \text{in } \hat{\Omega}, \\
V^\varepsilon_t - \mathcal{L}V^\varepsilon - F &= \beta_\varepsilon(V^\varepsilon - G^\varepsilon - 2\varepsilon) \quad \text{in } \hat{\Omega},
\end{align}
(10)
where $\hat{\Omega} = (0,\hat{T}) \times (0,\infty)$. Since $\beta_\varepsilon$ has linear growth and $F$ is continuous, there are unique bounded solutions. The solutions have bounded first and second order derivatives.

2. Consider the function $U^\varepsilon$. At any point at which $U^\varepsilon > G$, we have $U^\varepsilon > G^\varepsilon$ (since $G \geq G^\varepsilon$). As $\beta_\varepsilon = 0$ on $[0,\infty)$, we have $U^\varepsilon_t - \mathcal{L}U^\varepsilon - F = \beta_\varepsilon(U^\varepsilon - G^\varepsilon) = 0$. This implies that
\[ \min\{U^\varepsilon_t - \mathcal{L}U^\varepsilon - F, U^\varepsilon - G\} \leq 0. \]
Hence, $U^\varepsilon$ is a classical subsolution of (9).

Next we consider the function $V^\varepsilon$. Note that $G^\varepsilon + \varepsilon$ is a subsolution, so $V^\varepsilon \geq G^\varepsilon + \varepsilon \geq G$. As $\beta_\varepsilon \geq 0$, we obtain
\[ \min\{V^\varepsilon_t - \mathcal{L}V^\varepsilon - F, V^\varepsilon - G\} \geq 0. \]
Thus, $V^\varepsilon$ is a classical supersolution of (9).

Finally we estimate $W^\varepsilon = V^\varepsilon - U^\varepsilon$. It is a bounded solution of
\[ W^\varepsilon_t - \mathcal{L}W^\varepsilon - \beta_\varepsilon(W^\varepsilon - 2\varepsilon + U^\varepsilon - G^\varepsilon) + \beta_\varepsilon(U^\varepsilon - G^\varepsilon) = 0 \quad \text{in } [0,\hat{T}] \times \mathbb{R}, \]
with $W^\varepsilon(0,\cdot) = \varepsilon$. It is easy to check that $2\varepsilon$ is a supersolution and $0$ is a subsolution. Hence, $0 < W^\varepsilon < 2\varepsilon$.

3. Since a classical subsolution is no bigger than a classical super-solution, we have
\[ V^\delta - 2\delta < U^\delta \leq V^\varepsilon < U^\varepsilon + 2\varepsilon \quad \forall \varepsilon \in (0,1], \delta \in (0,1]. \]
Hence, there exists a unique $V$ such that $U^\varepsilon \leq V \leq V^\varepsilon$ for all $\varepsilon \in (0,1]$. Then $|V - V^\varepsilon| \leq 2\varepsilon$, so as $\varepsilon \searrow 0$, $V^\varepsilon \to V$ uniformly. Consequently, $V \in C([0,\hat{T}] \times \mathbb{R})$.

Now if $\tilde{V}$ is a classical subsolution of (9), then $\tilde{V} \leq V^\varepsilon$ for every $\varepsilon$, so $\tilde{V} \leq \lim_{\varepsilon \searrow 0} V^\varepsilon = V$. Similarly, if $\check{V}$ is a classical supersolution, then $\check{V} \geq U^\varepsilon$ for every $\varepsilon \in (0,1]$ so $\check{V} \geq \lim_{\varepsilon \searrow 0} U^\varepsilon = V$. Thus, $\check{V}$ is a weak solution of (9). Now if $\tilde{V}$ is another weak solution, then $U^\varepsilon \leq \tilde{V} \leq V^\varepsilon$ for all $\varepsilon \in (0,1]$, so $\check{V} = V$. Thus $V$ is the unique weak solution of (9).

4. Suppose $V(\tau_0,x_0) > G(\tau_0,x_0)$, then by continuity and uniform convergence, there exists $\varepsilon_0 > 0$ such that $U^\varepsilon(\tau,x) > G^\varepsilon(\tau,x)$ and $V^\varepsilon(\tau,x) > G^\varepsilon(\tau,x) + 2\varepsilon$ for every $\varepsilon \in (0,\varepsilon_0]$ and every $(\tau,x) \in B := (\tau_0 - \varepsilon_0, \min\{\tau_0 + \varepsilon_0, \hat{T}\}) \times (x_0 - \varepsilon_0, x_0 + \varepsilon_0)$. This implies that $V^\varepsilon_t - \mathcal{L}V^\varepsilon - F = 0$ and $U^\varepsilon_t - \mathcal{L}U^\varepsilon - F = 0$ in $B$ for all $\varepsilon \in (0,\varepsilon_0]$. Consequently, we have $V \in C^{1,2}(B)$ and $\check{V}_t - \mathcal{L}V - F = 0$ in $B$.

5. Since classical sub-super solutions of (4) and (9) can be converted from one to the other, we also know that (4) admits a unique weak solution. We can now summarize our result as follows:

**Theorem 3.2.** Assume (A1) and (A2). There is a unique weak solution $V$ of (9) and the solution is continuous. In addition, for every $\varepsilon \in (0,1]$, the solution $U^\varepsilon$ of (10) is a classical subsolution of (9) and the solution $V^\varepsilon$ of (11) is a classical supersolution of (9) and they satisfy
\[ U_\varepsilon \leq V \leq V^\varepsilon, \quad V^\varepsilon - U^\varepsilon < 2\varepsilon. \]
Consequently, in terms of (4), it admits a unique weak solution, the weak solution is continuous and is given by \( v \) defined in (1). Furthermore, setting \( \mathcal{N} := \{(t, s) \in D_T \mid v(t, s) > g(t, s)\} \), we have \( v \in C^{1,2}(\mathcal{N}) \) and \( v_t + \mathcal{L}v = 0 \) in \( \mathcal{N} \).

As far as we know, our observation that solutions of (10) and (11) are sub/super solutions of (4) and the use of the property \( 0 < V^u - U^u < 2\varepsilon \) is new in the literature.

### 3.3. The optimal stopping time

To complete the proof of Theorem 1.4, it remains to prove the following:

**Lemma 3.3.** \( \tau^* \) defined in (5) is the optimal stopping time, i.e.,

\[
v(s, t) = \mathbb{E}^{t,s}\left[ \int_t^{\tau^*} f(\rho, S_\rho)d\rho + g(\tau^*, S_{\tau^*}) \right].
\]

**Proof.** Since \( v(T, \cdot) = g(T, \cdot) \), \( \Omega \subset [0, T) \times (0, \infty) \). For every \( \varepsilon \in (0, 1) \), we define

\[
\Omega_\varepsilon = \{(t, s) \in D_T \mid v(t, s) > g(t, s) + \varepsilon, \varepsilon < s < \varepsilon^{-1}\}.
\]

Then \( \overline{\Omega_\varepsilon} \subset \Omega \) and \( \Omega = \bigcup_{\varepsilon \in (0,1)} \Omega_\varepsilon \).

Now fix \( (t, s) \in D_T \) and condition on \( S_t = s \). Consider two cases:

(i) Suppose \( v(t, s) = g(t, s) \). Then \( \tau^* = t \) and \( \mathbb{E}^{t,s}[g(\tau^*, S_{\tau^*})] = g(t, S_t) = g(t, s) = v(t, s) \).

(ii) Suppose \( v(t, s) > g(t, s) \). Then \( t < T \) and \( (s, t) \in \mathcal{N} \). For each \( \varepsilon \in (0, 1] \) we define

\[
\tau_\varepsilon = \min\{u \in [t, T] \mid (u, S_u) \notin \Omega_\varepsilon\}.
\]

Then \( \tau_\varepsilon \in T_{t,T} \). Since \( v \in C^{1,2}(\mathcal{N}) \) and \( \overline{\Omega_\varepsilon} \) is a compact set contained in the open set \( \Omega \), we can apply Itô formula to obtain

\[
v(t, s) = \mathbb{E}^{t,s}\left[ v(\tau_\varepsilon, S_{\tau_\varepsilon}) + \int_t^{\tau_\varepsilon} (-v_t - \mathcal{L}v)du + \int_t^{\tau_\varepsilon} \sigma v_S dW_u \right]
\]

since the stochastic integral is a martingale and \( v_t + \mathcal{L}v + f = 0 \) in \( \mathcal{N} \). Note that \( \tau_\varepsilon \nearrow \tau^* \) as \( \varepsilon \searrow 0 \). Hence sending \( \varepsilon \searrow 0 \) and using Lebesgue’s dominated convergence Theorem (to be explained later) we obtain

\[
v(t, s) = \lim_{\varepsilon \searrow 0} \mathbb{E}^{t,s}\left[ \int_t^{\tau_*} f(u, S_u)du + v(\tau_\varepsilon, S_{\tau_\varepsilon}) \right]
\]

Thus, \( \tau^* \) is the optimal stopping time.

We now show the applicability of Lebesgue’s dominated convergence Theorem. Let us denote

\[
A(t, T) = \min_{u \in [t, T]} (W_u - W_t), \quad B(t, T) = \max_{u \in [t, T]} (W_u - W_t).
\]
Then conditional on $S_t = s$,

$$\min_{t \leq u \leq T} S_u \geq se^{(\alpha - \sigma^2/2)(T-t)} e^{-\sigma A(t,T)}, \quad \max_{t \leq u \leq T} S_u \leq se^{(\alpha - \sigma^2/2)(T-t)} e^{\sigma B(t,T)}.$$ 

Thus, for any stopping time $\tau \in \mathcal{T}_{t,T}$, $|v(\tau, S_\tau)| \leq K[S^n_\tau + S^{-n}_\tau] \leq K \max_{u \in [t,T]} [S^n_u + S^{-n}_u] \leq Ke^{nT} \left( e^{n\sigma A(t,T)} + e^{n\sigma B(t,T)} \right) =: F.$

For Brownian motion $\{W_u\}_{u \geq 0}$ and real number $x > 0$, defining $\tau_x = \min\{u \geq t \mid W_u - W_t \geq x\}$, we have

$$P(A(t, T) \geq x) = P(B(t, T) \geq x) = P(\tau_x \leq T) = 2P(W_T - W_t > x) = 2 \int_x^\infty \frac{e^{-z^2/2(T-t)}}{\sqrt{2\pi(T-t)}} dz$$

by the reflection principle. Thus, $E[F] < \infty$.

On the other hand, note that $\tau_x$ and $f$ are uniformly bounded, thus

$$\left| \int_t^{\tau_x} f(u, S_u) du \right| \leq MfT.$$ 

Therefore, we can use the dominated convergence theorem in the above proof. This completes the proof of the lemma and also Theorem 1.4.

4. **Conclusion.** In this paper, we have proposed a new weak solution to an optimal stopping problem in finance and economics. The main advantage of this new definition is that we do not need the Dynamic Programming Principle, which is critical for both classical verification argument and modern viscosity approach. Since the sub-supersolution are defined in classical way, comparison principle can be applied directly. Additionally, the classical methods in differential equations, e.g. penalty method, can be used to derive some useful results, and we expect that many other classical methods can be used for further applications.

**REFERENCES**

[1] D. Bertsekas and S. Shreve, *Stochastic Optimal Control: The Discrete-Time Case*, Math. in Sci. and Eng., Academic Press, 1978.
[2] A. Friedman, *Variational Principles and Free-Boundary Problems*, Dover Publications, 1982.
[3] D. Gilbarg and N. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Springer, 1998.
[4] H. Ishii, On uniqueness and existence of viscosity solutions of fully nonlinear second order elliptic PDE’s, *Comm. Pure. Appl. Math*, 42 (2009), 15–45.
[5] L. Jiang, *Mathematical Modeling and Methods of Option Pricing*, World Scientific Publication, 2005.
[6] B. Oksendal, *Stochastic Differential Equations*, 6th edition, Springer, 2003.
[7] H. Pham, *Continuous-Time Stochastic Control and Optimization with Financial Applications*, Springer, 2009.

Received for publication October 2019.

E-mail address: congqin@suda.edu.cn
E-mail address: xinfu@pitt.edu