Spin network setting of topological quantum computation

Annalisa Marzuoli
Dipartimento di Fisica Nucleare e Teorica, Università degli Studi di Pavia and
Istituto Nazionale di Fisica Nucleare, Sezione di Pavia,
via A. Bassi 6, 27100 Pavia (Italy)
annalisa.marzuoli@pv.infn.it

Mario Rasetti
Dipartimento di Fisica and Istituto Nazionale di Fisica della Materia,
Politecnico di Torino,
corso Duca degli Abruzzi 24, 10129 Torino (Italy)
mario.rasetti@polito.it

Abstract

The spin network simulator model represents a bridge between (generalised) circuit schemes for standard quantum computation and approaches based on notions from Topological Quantum Field Theories (TQFTs). The key tool is provided by the fiber space structure underlying the model which exhibits combinatorial properties closely related to SU(2) state sum models, widely employed in discretizing TQFTs and quantum gravity in low spacetime dimensions.

PACS numbers: 03.67.Lx, 03.65.Fd, 11.10.Kk
Keywords: Quantum Computation, Spin Networks, Topological Quantum Field Theory.

We discuss here some aspects of a novel setting for quantum computation [1] [2] based on the (re)coupling theory of SU(2) angular momenta [3] [4]. On the one hand, the "spin network simulator" can be thought of as a non–Boolean generalisation of the quantum circuit model, with unitary gates expressed in terms of:

i) recoupling coefficients (3nj symbols) between inequivalent binary coupling schemes of $N \equiv (n + 1)$ SU(2)–angular momenta (j–gates);

ii) Wigner rotations in the eigenspace of the total angular momentum (M–gates). The picture does contain the Boolean case as the particular case when...
all $N$ angular momenta are spin $\frac{1}{2}$. The combinatorial structure of the model, on the other hand, closely resembles $SU(2)$ “state sum models” employed in discretized approaches to 2 and 3–dimensional Topological Quantum Field Theories [3] (TQFT) and Quantum Gravity [6]. An explicit mapping relating the spin network to the topological approach to quantum computation based on modular functors [7] can be established as illustrated in Ref. 2, Section 6.

The basic ingredients of the kinematics of the simulator are computational Hilbert spaces (to be defined below) equipped with all unitary gates of types i) and ii) we may consistently apply to transform vectors belonging to such spaces. A specific computation will be implemented by picking up an “input” state to get an “output” generated by acting with a particular (finite) sequence of elementary gates. An elementary operation will be carried out in one unit of time since it is assumed that the intrinsic time variable of the simulator has a discrete character.

The architecture of the spin network is modelled as an $SU(2)$ fiber space structure over a discrete base space $V$

$$\{V, \mathbb{C}^{2^J+1}, SU(2)^J\}_n$$

which encodes all possible computational Hilbert spaces as well as gates for any fixed number $N = n + 1$ of incoming angular momenta.

The base space $V = \{v(b)\}$ represents the vertex set of a regular, $2(n-1)$–valent graph $\mathfrak{G}(V,E)$ with cardinality $|V| = (2n-1)!!$. There exists a 1:1 correspondence

$$\{v(b)\} \longleftrightarrow \{H^J_n(b)\}$$

between the vertices of $\mathfrak{G}(V,E)$ and the computational Hilbert spaces of the simulator. For a given value of $n$, $H^J_n(b)$ is the simultaneous eigenspace of the squares of $2n + 1$ Hermitean, mutually commuting angular momentum operators, namely

$$J_1, J_2, J_3, \ldots, J_{n+1} \equiv \{J_i\}; \quad J_1 + J_2 + J_3 + \ldots + J_{n+1} \doteq J;$$

$$K_1, K_2, K_3, \ldots, K_{n-1} \equiv \{K_h\}$$

and of the operator $J_z$ (the projection of the total angular momentum $J$ along the quantization axis). The associated quantum numbers are $j_1, j_2, \ldots, j_{n+1}$
\(J; k_1, k_2, \ldots, k_{n-1}\) and \(M\), where \(-J \leq M \leq J\) in integer steps. If \(\mathcal{H}^{j_1} \otimes \mathcal{H}^{j_2} \otimes \cdots \otimes \mathcal{H}^{j_n} \otimes \mathcal{H}^{j_{n+1}}\) denotes the factorized Hilbert space, namely the \((n + 1)\)-fold tensor product of the individual eigenspaces of the \((J_i)^2\)’s, the operators \(K_h\)’s represent intermediate angular momenta generated, through Clebsch–Gordan series, whenever a pair of \(J_i\)’s are (binarily) coupled. As an example, by coupling sequentially the \(J_i\)’s according to the scheme \((\cdots ((J_1 + J_2) + J_3) + \cdots + J_{n+1}) = J\) which generates \((J_1 + J_2) = K_1, (K_1 + J_3) = K_2, \ldots\) we should get a binary bracketing structure of the type \((\cdots (((\mathcal{H}^{j_1} \otimes \mathcal{H}^{j_2})_{k_1} \otimes \mathcal{H}^{j_3})_{k_2} \otimes \cdots \otimes \mathcal{H}^{j_{n+1}})_{k_{n-1}})_J\), where we add an overall bracket labeled by the quantum number of the total angular momentum \(J\).

Note that, as far as \(j_i\)’s quantum numbers are involved, any value belonging to \(\{0, 1/2, 1, 3/2, \ldots\}\) is allowed, while the ranges of the \(k_h\)’s are suitably constrained by Clebsch–Gordan decompositions (e.g. if \((J_1 + J_2) = K_1 \Rightarrow |j_1 - j_2| \leq k_1 \leq j_1 + j_2\)). We denote a binary coupled basis of \((n + 1)\) angular momenta in the \(JM\)–representation and the corresponding Hilbert space appearing in (2) according to

\[
\{ |[j_1, j_2, j_3, \ldots, j_{n+1}]^b; k_1^b, k_2^b, \ldots, k_{n-1}^b; JM \rangle, -J \leq M \leq J\} = \mathcal{H}^J_n(b) = \text{span} \{ |b; JM \rangle_n \},
\]

where the string inside \([j_1, j_2, j_3, \ldots, j_{n+1}]^b\) is not necessarily an ordered one, \(b\) indicates the current binary bracketing structure and the \(k_h\)’s are uniquely associated with the chain of pairwise couplings selected by \(b\). For a given value of \(J\) each \(\mathcal{H}^J_n(b)\) has dimension \((2J + 1)\) over \(\mathbb{C}\) and thus there exists one isomorphism

\[
\mathcal{H}^J_n(b) \cong_v \mathbb{C}^{2J+1}
\]

for each admissible binary coupling scheme \(b\) of \((n + 1)\) incoming spins. The vector space \(\mathbb{C}^{2J+1}\) is interpreted as the typical fiber attached to each vertex \(v(b) \in V\) of the fiber space structure (1) through the isomorphism (5).

For what concerns unitary operations acting on the computational Hilbert spaces (4), we examine first \(j\)–gates associated with recoupling coefficients (3nj symbols) of \(SU(2)\) as anticipated in i). It can be shown [3] that any such coefficient can be splitted into "elementary" \(j\)–gates, the Racah transforms, possibly apart from phases and weight factors. A Racah transform applied to a basis vector is defined formally as

\[
\mathcal{R} : |\ldots ((a b)_{d c})_f \ldots; JM \rangle \mapsto |\ldots (a (b c)_e)_f \ldots; JM \rangle,
\]

(6)
where we are using here Latin letters $a, b, c, \ldots$ to denote both incoming ($j_i$'s in the previous notation) and intermediate ($k_i$'s) spin quantum numbers. The explicit expression of (6) reads

$$
\langle (a (b c)e) ; M \rangle
= \sum_d (-1)^{a+b+c+f} [(2d + 1)(2e + 1)]^{1/2} \left\{ \begin{array}{ccc} a & b & d \\ c & f & e \end{array} \right\} \langle (a b d c) ; M \rangle, \quad (7)
$$

where there appears the Racah–Wigner $6j$ symbol of $SU(2)$ and $f$ plays the role of the total angular momentum quantum number. Recall that the square of the $6j$ symbol in (7) represents the probability that a system prepared in the state $\langle (a b d c) ; M \rangle$ will be measured in the state $\langle (a (b c)e) ; M \rangle$. Moreover, Racah transforms are the key ingredients to complete the construction of the Rotation graph $\mathcal{G}_n(V, E)$ introduced above (note that in this case "Rotation" refers to a topological operation on binary tree structures and not to rotation operators referred to in (ii), see Ref. 2, Appendix A). The edge set $E = \{e\}$ of $\mathcal{G}_n(V, E)$ is a subset of the Cartesian product $(V \times V)$ selected by the action of elementary $j$–gates. More precisely, an (undirected) arc between two vertices $v(b)$ and $v(b')$

$$
e (b, b') \doteq (v(b), v(b')) \in (V \times V) \quad (8)
$$

exists if, and only if, the underlying Hilbert spaces are related to each other by an elementary unitary operation of the type (6). Note also that elements in $E$ can be considered as mappings

$$(V \times \mathbb{C}^{2J+1})_n \longrightarrow (V \times \mathbb{C}^{2J+1})_n$$

connecting each given decorated vertex to one of its nearest $2(n - 1)$ vertices and thus define a "transport prescription in the horizontal sections" belonging to the total space $(V \times \mathbb{C}^{2J+1})_n$ of the fiber bundle $\mathbb{P}$. The crucial feature that characterises the graph $\mathcal{G}_n(V, E)$ arises from compatibility conditions satisfied by $6j$ symbols appearing in (7). The Racah (triangular) identity, the Biedenharn–Elliott (pentagon) identity and the orthogonality conditions for
$6j$ symbols \cite{1} ensure that any simple path in $\mathcal{G}_n(V, E)$ with fixed endpoints can be freely deformed into any other, providing identical quantum transition amplitudes (and of course probabilities) at the kinematical level (cfr. Ref. 2, Section 3.1 for more details).

To complete the description of the structure $(V, \mathbb{C}^{2J+1}, SU(2)^J)_n$, we call into play $M$–gates which act on the angular dependence of vectors in $\mathcal{H}_n^J(b)$. By expliciting such dependence according to

$$\mathcal{H}_n^J(b) \equiv \text{span} \{ |b; \theta, \phi; JM \rangle_n \},$$

we write the action of a rotation on a basis vector as

$$|b; \theta', \phi'; M'J \rangle_n = \sum_{M=-J}^{J} D^J_{MM'}(\alpha\beta\gamma) |b; \theta, \phi; JM \rangle_n,$$

where $(\theta, \phi)$ and $(\theta', \phi')$ are polar angles in the original and rotated coordinate systems, respectively. $D^J_{MM'}(\alpha\beta\gamma)$ are Wigner rotation matrices in the $JM$ representation expressed in terms of Euler angles $(\alpha\beta\gamma)$ which form a group under composition \cite{1} (see also Ref. 2, Section 3.2 and Appendix B1, for a discussion on the possibility of selecting "elementary" $M$–gates in some specific cases). In what follows we shall assume for simplicity that, for given $n$ and $J$, it takes just one unit of time to perform one rotation. The shorthand notation $SU(2)^J$ employed in \cite{1} actually refers to the group of $W$–rotations, which in turn can be interpreted as the automorphism group of the "fiber" $\mathbb{C}^{2J+1}$. Since rotations in the $JM$ representation do not alter the binary bracketing structure of vectors in computational Hilbert spaces we can interpret actions of $W$–matrices as "transport prescriptions along the fiber."

The framework outlined above makes it manifest the fact that we can switch independently $j$ and $M$–gates without mixing spin and magnetic quantum numbers. This feature, which relies on the discreteness of the base space $V$ and on the "triviality" of the total space $(V \times \mathbb{C}^{2J+1})_n$, makes it easy with kinematics, although the simulator will exhibit complex behaviors at the dynamical level.

A pictorial representation of the computational fiber space $(V, \mathbb{C}^{2J+1}, SU(2)^J)_n$ for $(n+1) = 4$ incoming spins is sketched in Fig. 1. Each vertex is associated, through the mapping \cite{2}, with a binary coupled Hilbert space $\mathcal{H}_n^J(b)$ depicted as a plane. Edges of the graph represent Racah transforms.
which actually move each space viewed as a whole into one of its nearest
$2(n - 1) = 4$ ones. Inside each $\mathcal{H}_n^J(b) \cong \mathbb{C}^{2^J+1}$ (see (5)) we can pick up a
particular vector (as shown in the drawing) and rotate its components by
means of the Hermitean conjugate of the matrix $D^I_{MM'}(\alpha\beta\gamma)$ introduced in
(11) for some choice of $(\alpha\beta\gamma)$. 
Figure 1: The fiber space structure of the spin network simulator for \((n+1) = 4\) incoming spins. Vertices and edges on the perimeter of the Rotation graph \(\mathcal{S}_3(V, E)\) have to be identified through the antipodal map. A couple of vertices are "blown up" into planes representing the associated computational Hilbert spaces.

The dynamical behavior of the simulator in working as a quantum circuit can be discussed in terms of directed paths in the fiber space structure \((V, \mathbb{C}^{2^J+1}, SU(2)^J)_n\). By a directed path \(\mathcal{P}\) we mean a (time) ordered sequence

\[
|\psi_{\text{in}}\rangle_n \equiv |\psi_0\rangle_n \rightarrow |\psi_1\rangle_n \rightarrow \cdots \rightarrow |\psi_s\rangle_n \rightarrow \cdots \rightarrow |\psi_L\rangle_n \equiv |\psi_{\text{out}}\rangle_n , \tag{12}
\]

where we use the shorthand notation \(|\psi_s\rangle_n\) for computational states and \(s = 0, 1, 2, \ldots, L\) is the lexicographical labelling of the states along the given path with fixed endpoints. \(L\) is the length of the path \(\mathcal{P}\), which turns out to be proportional to the time duration of the computation process \(L \cdot \tau = T\) in terms of the discrete time unit \(\tau\). The integer \(L\) characterising the particular directed path in (12) equals the number of time–ordered elementary operations (computational steps) needed to get \(|\psi_{\text{out}}\rangle_n\) from \(|\psi_{\text{in}}\rangle_n\) following the path \(\mathcal{P}\). An elementary computational step, represented by an arrow in (12), is either a Racah transform or a W–rotation. A circuit–type computation consists in evaluating the expectation value of the unitary operator.
\( \mathcal{U}_\mathcal{P} \) associated with the path \( \mathcal{P} \), namely

\[
\langle v_{\text{out}} | \mathcal{U}_\mathcal{P} | v_{\text{in}} \rangle_n.
\]

By taking advantage of the possibility of decomposing \( \mathcal{U}_\mathcal{P} \) uniquely into an ordered sequence of elementary gates, (13) becomes

\[
\langle v_{\text{out}} | \mathcal{U}_\mathcal{P} | v_{\text{in}} \rangle_n = \lfloor \prod_{s=0}^{L-1} \langle v_{s+1} | \mathcal{U}_{s,s+1} | v_s \rangle_n \rfloor_{\mathcal{P}}
\]

with \( L \equiv L(\mathcal{P}) \) for short. The symbol \( \lfloor \rfloor_{\mathcal{P}} \) denotes the ordered product along the path \( \mathcal{P} \) and each elementary operation is now denoted by \( \mathcal{U}_{s,s+1} \) \( (s = 0, 1, 2, \ldots L(\mathcal{P})) \) to stress its "one-step" character with respect to computation. Consequently, each elementary transfer matrix in (14) turns out to be associated with a local Hamiltonian operator arising from

\[
\langle v_{s+1} | \mathcal{U}_{s,s+1} | v_s \rangle_n = \exp \{ i H_n (s, s + 1) \cdot \tau \}
\]

and representing the unitary evolution of the simulator in one unit of its intrinsic time variable \( \tau \). We indicate with the shorthand notation \( (s, s + 1) \) the dependence of \( H_n \) on its variables to make clear the local nature of this operator with respect to the computational space \( (V, \mathbb{C}^{2^J+1}, SU(2)^J)_n \).

When (15) is inserted in (14), such virtual Hamiltonians generally do not commute with each other but nonetheless the whole computational process may be identified with a well defined unitary evolution of the simulator, driven by a poly–local Hamiltonian, in the internal time interval \( T = L(\mathcal{P}) \cdot \tau \).

Different types of evolutions can be grouped into "computing classes" based on the choice of gates that a particular program \( \text{(i.e. a collection of directed paths)} \) has to employ. A computing class that alternates \( j \) and \( M \)–gates would be the most general one (as explained in Ref. 2, Sect. 4.2). However, two particular classes to be illustrated below come out to be quite interesting in view of the possibility of relating them to other models for quantum computation.

An \( M \)–computing class contains programs which employ only (finite sequences of) \( M \)–gates in their associated directed paths and it is not difficult to realize that such kind of computation, when applied to \( N \frac{1}{2} \) spins, reproduces the usual Boolean quantum circuit.

A \( j \)–computing class includes programs which employ only \( j \)-gates at each computational step. This class is particularly interesting since it shares many
features with suitable types of discretized TQFTs, the so–called state sum
models\[5\[6\[8\], as we already noticed in Ref. 1 and developed in details in
Ref. 2, Sect. 5. Now the combinatorial structure of Rotation graphs becomes
prominent owing to the existence of a 1:1 correspondence between allowed
elementary operations and the edge set $E$ of the graph $\mathcal{G}_n(V, E)$, for any $n$.
We denote the unitary operator associated with a program $P$ in this class by

$$ U_{3n} : |v_{\text{in}}\rangle_n \longrightarrow |v_{\text{out}}\rangle_n , \quad (16) $$

where, in a circuit–type setting, $|v_{\text{in}}\rangle_n$ is fixed and $|v_{\text{out}}\rangle_n$ is an accepted
state. However, in the $j$–computing class one may address other types of
problems. For instance, once selected two states, say $|v_{\text{in}}\rangle_n$ and $|v_{\text{out}}\rangle_n$, we
may consider all possible paths $P$ that compute $|v_{\text{out}}\rangle_n$ as the result of the
application of some $U_{3n}$ to $|v_{\text{in}}\rangle_n$. The functional which takes care of such
multiple choices is a "path sum" (a discretized Feynman’s path integral) which may be written formally as

$$ Z_{v_{\text{in}}, v_{\text{out}}} = \sum_P W_P \langle v_{\text{out}} | U_{3n} | v_{\text{in}} \rangle_n , \quad (17) $$

where the summation is over all paths with fixed endpoints and $W_P$ is a
weight to be assigned to each particular path. The explicit form of such
functional would contain sums over intermediate spin variables $k_h$’s of prod-
ucts of $6j$ symbols with suitable weights (dimensions of $SU(2)$ irreps labeled
by $k_h$’s) and phase factors. Note that if we give the same weight, say $W_P = 1$, to
each path, then the results on equi–probable amplitudes derived from the
algebraic identities for $6j$ symbols ensure that the functional (17) is a com-
binatorial invariant, namely it is actually independent of the particular path
connecting $v_{\text{in}}$ and $v_{\text{out}}$. The situation in this case resembles what happens
in studying classes of combinatorial invariants ($SU(2)$–state sum function-
als) for triangulated manifold originally defined in dimension 3 (see Refs. 5,6
for reviews) and extended (in all dimensions) to manifolds with boundary
\[8\]. Functionals like (17) can be actually associated with particular classes
of extended geometrical objects, namely triangulated surfaces and possibly
triangulated 3–manifolds. On the other hand, the number of incoming spins
must be kept fixed in a quantum circuit context, and thus we cannot con-
clude that such a computing machine is able to simulate TQFTs or gravit
where the path integrals contain sums over all configurations of fields (up
to regularization). Other intriguing possibilities open when we insert non
trivial weights $W_P$ in (17), since we may naturally address questions about most efficient algorithms and time complexity. For instance we could weigh paths with the inverse of their lengths $L(P)$; then the minimum–length path (the optimal algorithm) will be dynamically singled out in the path sum. As a matter of fact, even such simple example turns out to be highly non trivial owing to the combinatorial complexity of $\mathcal{G}_n(V, E)$, as pointed out in Ref. 2 (Sect. 4.3 and Appendix A).

As a final remark we mention the possibility of exploiting the fiber space structure $(V, \mathbb{C}^{2^{J+1}}, SU(2)^J)_n$ to provide a natural discrete–time implementation of holonomic quantum computation, at least when the Hamiltonians defined in (15) exhibit the required degeneracy.

References

[1] A. Marzuoli and M. Rasetti, Phys. Lett. A306, 79 (2002).

[2] A. Marzuoli and M. Rasetti, “Computing Spin Networks”, INFN Pavia Preprint FNT/T 2004/5, June 2004.

[3] L.C. Biedenharn and J.D. Louck, The Racah–Wigner Algebra in Quantum Theory, Topic 12. Coupling of $N$ Angular Momenta: Recoupling Theory, Encyclopedia of Mathematics and its Applications Vol 9, G–C. Rota Ed. (Addison–Wesley Publ. Co., Reading MA, 1981).

[4] D.A. Varshalovich, A.N. Moskalev and V.K. Khersonskii, Quantum Theory of Angular Momentum (World Scientific, Singapore, 1988).

[5] L.H. Kauffman, Knots and Physics (World Scientific, Singapore, 1991).

[6] T. Regge and R.M. Williams, J. Math. Phys. 41, 3964 (2000).

[7] M.H. Freedman, A. Kitaev, M. Larsen and Z. Wang, Bull. Amer. Math. Soc. 40, 31 (2002).

[8] G. Carbone, M. Carfora and A. Marzuoli, Nucl. Phys. B595, 654 (2001).

[9] V. Fack, S. Lievens and J. Van der Jeugt, Discr. Math. 245, 1 (2002).

[10] P. Zanardi and M. Rasetti, Phys. Lett. A 264, 94 (1999).