High orders perturbation theory and dual models for Yang-Mills theories

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Abstract

We start with the QCD sum rules which are originally based on the idea that it is the power-like corrections to the parton model which are related to the confinement. The naive use of the Operator Product Expansion ensures that there is a ‘gap’ in the powers of \(\Lambda_{\text{QCD}}\) which miss the quadratic terms and start with the quartic term, proportional to the gluon condensate, \(< (G^a_{\mu \nu})^2 >\). We review how this hypothesis stood against various checks through the last three decades and how it was modified through inclusion of the missing link, that is quadratic corrections. In field theoretic language the quadratic corrections are dual to long perturbative series. In the dual description, the quadratic corrections are conveniently parameterized in terms of the metric in extra dimensions. We emphasize that the dual models do not incorporate the so called infrared renormalon.

Keywords: Quantum Chromodynamics, Perturbation theory

1. Introduction

This talk was given at a special session of the QCD10 conference at Montpellier, devoted to (30+1) year of the QCD sum rules [1]. I am thankful to Stephan Narison for organizing this event and, also, for overtaking the most difficult part of the job, that is reviewing the present status of the sum rules [2]. As for myself, I chose for myself to talk more about not yet fully settled issues. The emphasize is mainly on the quadratic corrections, simply missing from the standard sum rules. To a large extent the talk is based on the published papers [3, 4] but we add a few remarks as well.

Originally the sum rules were applied to two-point functions induced by external currents. More specifically, one considers integrals of the kind

\[
f(M^2) \equiv \int ds \exp(-s/M^2) \text{Im}\Pi(s),
\]

(1)

where \(M^2\) is a large mass parameter, \(\text{Im}\Pi(s)\) is imaginary part of a polarization operator and, speaking generically, can be measured by studying transition induced by currents. The sum rules equate the observable (1) to power corrections:

\[
f(M^2) \approx [f(M^2)]_{\text{parton model}} \left(1 + c_{g\text{,cond}} \frac{\Lambda_{\text{QCD}}^4}{M^8}\right)
\]

(2)

where the product \(c_{g\text{,cond}}\Lambda_{\text{QCD}}^4\) is calculable in terms of the gluon condensate, or matrix element \(< (G^a_{\mu \nu})^2 >\).

The form (2) is oversimplified since it omits, in particular, the first perturbative correction of order \(\alpha_s(M^2) \sim \ln^{-1} M^2\), the quark-condensate terms and so on. But nevertheless, eq. (2) does summarize correctly the basic idea that in the crucial region of \(M^2\) it is the power corrections which are related to the confinement (manifested through resonance masses and widths entering (1)). The power corrections are given in terms of matrix elements of local gauge invariant operators. In particular, there is no correction of order \(\Lambda_{\text{QCD}}^2\) since there is no corresponding operator of dimension \(d=2\). Eq. (2) is in no way obvious and was introduced on phenomenological grounds [1]. In particular, dropping the perturbative corrections which are powers of \(\alpha_s(M^2)\) and keeping the power correction which is of order \(\exp(-\text{const}/\alpha_s)\) might look confusing.

2. Quadratic corrections:

unifying continuum- and lattice-languages

The expansion (2) works well in many cases [2]. However, there exist reasons to revisit it:

1. Trying to improve the quality of the sum rules one starts to calculate perturbative corrections. The question is, whether the numerical value of \(< G^2 >\)
is kept independent of these corrections or should it vary from order to order of perturbation theory?

2. The Cornell potential for heavy quark interaction:

\[ V_{Qar{Q}}(R) \approx -\frac{\text{const}}{R} + \sigma R, \quad (3) \]

where \( R \) is the distance between the quarks and \( \sigma \) is the string tension holds numerically at all the distances measured on the lattice including small distances. In fact, this is an ideal example of a kind of expansion \( (\text{4}) \), with a single power-like correction present. The problem is that the correction to the leading, Coulomb-like term at short distances is quadratic, \( \sim (\sigma R^2) \). And quadratic corrections are not originally included into \( (\text{4}) \) and things look mysterious.

Although the questions are straightforward, convincing answers are difficult to get phenomenologically. The reason is that, after all, we talk about relatively small power-like corrections. Each particular case of such phenomenology is tedious, and more important, very difficult to follow from outside. Nevertheless, after a few years of hard work the answer seems to be unique: quadratic corrections do exist \( (\text{5}) \).

Moreover, the quadratic correction is directly related to confinement. Although the statement might look too strong and vague, actually, it has a well defined content. Namely, on the lattice one is able to clarify what kind of field configurations are responsible for the confinement. In the lattice nomenclature these configurations are called monopoles and vortices, for a review see, e.g., \( (\text{6}) \). Most amusing, they occupy a small fraction of the lattice which tends to zero with the vanishing lattice spacing \( a \rightarrow 0 \). Closer to our story, it was demonstrated that the non-perturbative, i.e. monopole- or vortex- related potential is indeed linear at all the distances beginning with a single lattice spacing:

\[ \left( V_{Qar{Q}}(R) \right)_{\text{non-perturbative}} = \sigma \cdot R. \quad (4) \]

Details and references can be found in \( (\text{7}) \). Thus, the short-distance quadratic correction is certainly there !

3. Power corrections vs perturbative series

What is the relation between the power corrections and perturbative series? Concentrate on the gluon condensate itself. Then perturbatively:

\[ a^4 \frac{\pi^2}{12N_c} \left( \frac{\bar{b}_0 g^3}{\beta(g)} \right) < \frac{\alpha_s}{\pi} g G G \gg \text{pert} = \sum_{n=1}^{\infty} \left( 1 + a_0 a^2 \right), \quad (5) \]

where \( \alpha_s \equiv g^2 / 4\pi \) is the strong interaction coupling, \( \beta(g) \) is the beta function, \( a_n \) are perturbative coefficients. Moreover, \( a \) is the lattice spacing and the factor \( a^4 \) in the l.h.s. of Eq \((\text{5})\) is introduced to cancel the UV divergence inherent to the quantity considered.

The perturbative expansion \( (\text{5}) \) is expected to be asymptotic. Namely, for \( n \) large enough the expansion coefficients are expected to grow factorially:

\[ \lim_{n \rightarrow \infty} a_{nIR} = n! \left( \frac{b_0}{2} \right)^n \quad (6) \]

The series \( (\text{6}) \) is called infrared renormalon. If we estimate the uncertainty of the asymptotic expansion due to the factorial growth \( (\text{6}) \) we find contribution of order \( (\Lambda_{\text{QCD}} \cdot a)^2 \) compared to the leading perturbative term. Such an uncertainty would introduce \( < G^2 > \sim \Lambda_{\text{QCD}}^4 \) which is the physical gluon condensate entering, in particular QCD sum rules \( (\text{7}) \). This is the standard wisdom on the relation between divergences of the perturbative series in large orders \( n \) and the Operator Product Expansion.

Where is then the hypothetical quadratic correction? Well, there is no pronounced role for such a correction in the set up considered. It should be buried within the still-convergent orders of the perturbative series. In other words, keeping the quadratic correction might be reasonable only as far as the perturbative series is not long enough. If we keep many terms, then the quadratic correction is to be eaten up by the perturbative terms. Note that within such a logic the quadratic correction (if any) is inferior to many orders of perturbation series and uninteresting. Moreover, the very language of power corrections seems rather irrelevant if we need to keep explicit many orders of perturbation theory.

Things become, however, much more interesting if we turn to the example of the longest perturbative series known. It is indeed for the gluon condensate \( (\text{5}) \) and contains 20 (no mistake: twenty) first terms in the expansion, see \( (\text{8}) \) and references therein. Moreover, the full value of the gluon condensate is known from the lattice measurements since it is simply the plaquette action and can be measured to a very high precision. As a result one can use the following fitting procedure:

\[ (\Delta P)_N = P_{\text{full}} - \sum p_\alpha a_\alpha^\rho \approx (\Lambda_{\text{QCD}} \cdot a)^{\rho(N)}, \quad (7) \]

where \( P_{\text{full}} \) is the exact (or full) plaquette action, \( p_\alpha \) are perturbative coefficients evaluated explicitly and the difference between the partial (up to order \( N \)) perturbative series and the full value is fitted by a power-like correction where the index \( \rho(N) \) depends on \( N \) itself.
The results [8] concerning the index $\rho(N)$ are remarkable. Namely, the fits produce:

$$\rho(N) \approx 2 \text{ if } N \leq 10$$

$$\rho(N) \approx 4 \text{ if } N \geq 10$$

Thus the perturbative series does know about the sacred quadratic and quartic corrections although this knowledge is very difficult to express analytically. Everything is numerical.

Another remarkable result is the simplicity of the coefficients $p_n$. The expansion is close to a geometric series with the ratio

$$r_n \equiv \frac{p_{n+1}}{p_n} = \text{Const}.$$  \hspace{1cm} (9)

and we do not quote the numerical values of the fit parameters here. Observation (9) implies that there is no sign of the infrared renormalon at least in the first 20 orders of expansion. This is the reality which we have to confront and appreciate theoretically.

In conclusion of this section it is worth mentioning that similar interplay between perturbative and power-like corrections was observed earlier by analyzing much shorter perturbative expansions. For example, it was observed in Ref. [9] that the quadratic correction at short distances to the Coulomb-like potential (see Eq. (4)) is reproduced in fact by higher orders of perturbation theory. Another example is provided by Ref. [10] where sum rules for the structure function $xF_2(x)$ measurable in deep inelastic neutrino scattering were analyzed. It turned out that including higher orders of the perturbative expansion reduces considerably the numerical value of the quadratic correction. Note that in case of the $xF_2(x)$ function the quadratic correction is commonly associated with the infrared renormalon while in case of the quark potential the quadratic correction corresponds to the ultraviolet renormalon. Strong correlation between the quadratic correction and perturbative series is found in both cases. The limitation of the analyses just mentioned is that the perturbative series known explicitly are relatively short, say 3-4 terms, and to reach conclusions one relies on theoretical estimates of higher-order terms. Signals for importance of quadratic corrections were obtained in quite a few other analyses, see in particular [11].

To summarize, there is strong evidence in favor of crucial role of the unconventional quadratic corrections in correlators of two currents. Moreover, the quadratic correction is dual to a long perturbative series. Thus, one is to use either of them but not both. "Unconventional" means that there is no operator of the dimension $d = 2$ which would enter the OPE. Thus, it is difficult to even parameterize the quadratic correction within the field-theoretic approach: the quadratic correction is a part of the coefficient function in front of the unit operator but we are lacking means to distinguish it from the rest of the coefficient function.

4. Dual models and power-like corrections

4.1. Finding a dimension-two parameter

The situation with the quadratic correction is quite paradoxical. On one hand, there is accumulating and strong phenomenological evidence in favor of such a correction. On the other hand, the field-theoretic framework does not even provide us with a suitable parameterization of the quadratic correction. The most successful phenomenological model [5] introduces a "short-distance gluon mass" $m_g$, so that one replaces the gluon propagator by

$$\frac{1}{q^2} \to \frac{1}{q^2 + m_g^2}.$$  \hspace{1cm} (10)

where the gluon mass turns to be tachyonic. Clearly, the replacement (10) is consistent with the gauge invariance only in the Born approximation. In higher orders, there is no way to introduce a dimension-two quantity.

The resolution of the paradox seems to be that the quadratic correction belongs rather to the realm of the dual models for the Yang-Mills theories. As is well known, the dual or stringy formulations of non-Abelian gauge theories is derived only in some supersymmetric cases. In QCD case, there are educated guesses on the universality class [12, 13] and more explicit but rather arbitrary models, see, in particular, [14, 16]. The main point now is that the models allow to introduce a dimension-two parameter to the theory in a gauge invariant and natural way.

To begin with, the stringy models readily incorporate confinement. Indeed, linear potential at large distances, see (3), is commonly related to existence of a string connecting the quarks. To be more ambitious and describe the potential at all the distances one can introduce a running string tension $\sigma(l)$ where $l$ is the length of the string and $\sigma(l)$ is its tension as function of the length. Moreover, one can trade the running tension for a string living in an extra dimension $z$ with non-trivial geometry,
The action of the string is the Goto-Nambu action,
\[ S = \frac{1}{2\pi a} \int d^2 \xi \sqrt{\det G_{mn} \partial_\alpha X^m \partial_\beta X^n}. \] (11)

The relation to the potential is provided by the boundary condition that the string ends on the Wilson line \[ W(C) \approx \exp(-S(C)_{\text{min}}), \] (12)
and the Wilson line belongs to \( z = 0 \). One can then trade the potential \[ G_{mn}(z) \] for an explicit form of the metric \( G_{mn}(z) \). In particular, the following metric reproduces the Cornell potential \[ 4 \] quite satisfactorily \[ 16 \]:
\[ ds^2 = G_{mn} dx^m dx^n = R^2 e^{c z^2/2} (dx^i dx^j + dz^2), \] (13)
where the ordinary 4 dimensional space corresponds to \( z = 0 \), \( R^2 \) is a constant, the parameter \( c \) is of phenomenological nature and its numerical value, \( c = 0.9 \text{ GeV}^2 \), is obtained from the fitting procedure.

Most remarkably, the parameter \( c \) specifies the quadratic correction at small \( z \) in a perfectly gauge invariant way. Thus, the metric \[ 13 \] allows to introduce the quadratic correction in the universal geometric language. Upon fixation of the coefficient \( c \) in terms of the potential one can, in principle, evaluate quadratic corrections to other observables. For technical reasons, there are not many examples of this kind. However, one can say that the emerging phenomenology turns to be successful, see, in particular \[ 15 \] \[ 19 \].

To summarize, the use of the dual models promotes the numerical observation on the dominance of the quadratic correction at small \( R \) in the heavy-quark potential \[ 4 \] to the status of a universal feature of the short-distance physics.

4.2. On the sign of the quadratic correction

There is a puzzling question about the sign of the coefficient \( c \) in the metric \[ 14 \]. We fixed it as positive, by mapping the potential \[ 4 \] into the metric. Actually, the sign is fixed merely by the condition that we should have confinement, as first observed in \[ 16 \], see also \[ 17 \]. On the other hand, papers which introduced the quadratic correction originally \[ 14 \] \[ 15 \] are using rather negative value of the coefficient \( c \), \( h \sim \exp(-c z^2/2) \), instead of \( h \sim \exp(+c z^2/2) \) as in Eq. \[ 13 \].

This discrepancy might be finally settled in favor of, say, positive sign of \( c \) through further phenomenological analysis. One cannot rule out, however, that there is a matter of principle behind this apparent discrepancy. Namely, the papers \[ 14 \] \[ 15 \] refer rather to Minkowski space while numerically impressive successful applications \[ 16 \] \[ 19 \] of the metric refer to the Euclidean space. Then we would come to an intriguing possibility that in theories with confinement there is no analytical continuation of the quadratic correction from the Minkowski to Euclidean space.

The question is then, whether the insight we got on the nature of the quadratic correction as being dual to long perturbative series supports the idea of the non-analyticity or rather rejects it. Analytic properties of the running coupling are discussed in many papers, see, in particular, \[ 20 \] and it seems obvious that the continuation of the coupling from the Minkowski to Euclidean space and vice verse is not straightforward at all.

To put it even simpler, by continuing the log factor in the running coupling one gets an imaginary part in the Minkowski region, \( \ln Q^2 = \ln(-Q^2) + i\pi \). The imaginary part is big numerically and summing up higher orders in perturbation series at \( Q^2 > 0 \) and at \( q^2 = -Q^2 \) might well result in different quadratic corrections.

Basing on the example of the perturbative evaluation of the gluon condensate discussed above it is tempting to assume that geometric-series-type of expansion does incorporate the quadratic correction in other cases as well. And, indeed, one observes \[ 3 \] that existing examples of relatively long explicit perturbative expansions do not rule out that geometric series approximately apply in other cases as well. For example, using the results of Ref. \[ 21 \] one finds for the polarization operator associated with the scalar currents in QCD:
\[ -Q^2 \frac{d}{dQ^2} \Pi_s(Q^2) = \frac{N_c}{8\pi^2} \left( 1 + 5.67\alpha_s \right) + 45.85\alpha_s^2 + 465.8\alpha_s^3 + 5589\alpha_s^4. \] (14)

We see that this expansion numerically is quite close to a geometric series. In the context of our discussion, it is crucial that the expansion \[ 14 \] refers to the Euclidean coupling constant \( \alpha_s(Q^2) \). If one would use \( \alpha_s(-Q^2) \) as an expansion parameter, no similarity to a geometric series would persist.

To summarize, the duality between the power corrections and long perturbative series rather supports the suggestion that the quadratic correction cannot be trivially continued from the Minkowskian to Euclidean domain and vice verse. Much more is to be done, however, to strengthen the argumentation.

4.3. Absence of infrared renormalon

Probably one of the most important changes brought by the dual models to the dogma of high-order corrections is the absence of the infrared renormalon \[ 4 \].
In more detail, to imitate the gluon condensate in the continuum theory one evaluates vacuum expectation value of a circular Wilson line of small radius \(r\), \(r^2 \Lambda_{QCD}^2 \ll 1\). In the dual model one uses eqs. (12), (11), (13) to perform the explicit calculation. The result can be represented as:

\[
<W(r) \approx \exp \left[ \text{const} \left( 1 - 0.84c r^2 - 0.035 c^2 r^2 \right) \right] \]

The first two terms in the r.h.s. of (15) correspond to the quartic and quadratic ultraviolet divergences in (5). The evaluation of these terms in the continuum theory is not unique and depends in fact on the regularization procedure. These terms cannot be compared directly with the results of the perturbative calculation in the lattice regularization.

The most interesting result is the term proportional to \(c^2\) in the r.h.s. of Eq. (15). It corresponds, in the continuum language, to \(<G^2> \sim C^2_{QCD}\) and is most relevant to the sum rules and phenomenological fits. Moreover, it is this term which is commonly related to the expected IR-renormalon divergence of the perturbative series in (5). The crucial observation is that in the dual-model approach the \(<G^2> \sim C^2_{QCD}\) term is related in fact to short distances and is calculable. It corresponds to the \(z^3\) term in the expansion of the factor \(h(z) = \exp(c z^2 / 2)\) in the metric (13) at small \(z, z \to 0\).

This is quite a remarkable possibility revealed by the dual models: the \(<G^2> \sim C^2_{QCD}\) piece in the gluon condensate is associated with the small, not large distances. If the piece \(<G^2> \sim C^2_{QCD}\) is indeed associated with short distances then it should be calculable in perturbation theory. This prediction of the dual models is in perfect agreement with the absence of the infrared renormalon from the explicit calculation in (5), see for discussion above. The difference between the standard field theoretic estimates which indicate the presence of the IR renormalon and dual approach which, does not incorporate the infrared-sensitive piece of \(<G^2>\) is of pure geometric nature. In the both approaches, the total value of the small Wilson line is dominated by small distances, of order \(r\). The infrared sensitive piece is anyhow relatively small and is due to either exceptional or suppressed configurations which involve distances much larger than \(r\). One can readily check that it is much easier to reach such distances for one virtual particle (gluon) than for a string, see (12), (13).

4.4. Choice of the dual model

We can summarize our discussion by saying that the mystery of the quadratic correction is resolved only by the dual models. Moreover, one can reverse the logic and try to clarify which dual models are fitting the knowledge on the power corrections in the best way. The conclusions then are:

1. the quadratic correction is to be built explicitly into the dual model
2. the model is to be formulated in terms of strings, not just fields living in the extra dimensions

It is interesting to note that an independent and much more thorough analysis of phenomenological implications of the existing dual models results in a similar conclusion [23]. Unfortunately, stringy models are more difficult to apply to phenomenology of the power corrections and mostly one considers models with fields (not strings) living in extra dimensions, see, e.g., [23].

5. Conclusions

To summarize, there is substantial progress in understanding the power corrections brought to light by the sum rules. Namely, the emphasis shifted to the quadratic correction absent from the original, simplified form of the sum rules. It turned out that it is just this correction which is most closely related to the confinement. Moreover, introduction of this correction allows for a straightforward interpretation of some lattice data and unifies the continuum-theory and lattice languages. Thus, phenomenologically the quadratic correction resolves quite a few puzzles. However, interpretation in the field-theoretic language is rather awkward: it is dual to a long perturbative series, as is confirmed by a number of perturbative calculations, see in particular [8–10]. With the advent of the dual models of QCD the quadratic correction found its interpretation in terms of the metric in an extra coordinate \(z\). Dual models which incorporate this quadratic correction at small \(z\) turn to be successful phenomenologically, probably even most successful.

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