Dynamics of the Batch Minority Game with Inhomogeneous Decision Noise

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We study the dynamics of a version of the batch minority game, with random external information and with different types of inhomogeneous decision noise (additive and multiplicative), using generating functional techniques à la De Dominicis. The control parameters in this model are the ratio α = p/N of the number p of possible values for the external information over the number N of trading agents, and the statistical properties of the agents’ decision noise parameters. The presence of decision noise is found to have the general effect of damping macroscopic oscillations, which explains why in certain parameter regions it can effectively reduce the market volatility, as observed in earlier studies. In the limit N → ∞ we (i) solve the first few time steps of the dynamics (for any α), (ii) calculate the location αc of the phase transition (signaling the onset of anomalous response), and (iii) solve the statics for α > αc. We find that αc is not sensitive to additive decision noise, but we arrive at non-trivial phase diagrams in the case of multiplicative noise. Our theoretical results find excellent confirmation in numerical simulations.

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I. INTRODUCTION

One of the more recent application domains of equilibrium and non-equilibrium statistical mechanics is the analysis of simplified models describing large markets of competing traders (or agents). One such model, which in spite of its apparent simplicity was found to exhibit highly non-trivial behaviour and has therefore attracted much attention, is the so-called Minority Game (MG) [1,2], which is a variation on the so-called El-Farol bar problem [3] which mimics in a highly idealized fashion a market of speculators attempting to profit by buying when most others wish to sell or selling when others wish to buy, without individual knowledge of their fellows but only of their collective consequences and external information available to all. An extensive overview of the literature on the MG and its many variations and extensions can be found in [4]. The striking feature of the MG, clearly observed in numerical simulations, is the nontrivial dependence of the market volatility (measuring global fluctuations) on the dimensionality of the information supplied to the agents (which is defined as the relative number α of different values which the information can take). For large α the volatility approaches the value corresponding to random trading, and the system is ergodic. As α is reduced, also the volatility is found to decrease beneath random, which is indicative of a more efficient market, where agents have ‘learned’ to improve the effectiveness of their selection of trading strategies. A further decrease of α will at some critical point αc force the system to undergo a phase transition to a highly non-ergodic regime, where both a high-volatility stage and a low-volatility state can emerge, dependent on initial conditions (this was only appreciated later).

In the original minority game, the information supplied to the agents consisted of the actual history of the market. However, it was soon realized [1] that the dynamics of the MG remains largely unaltered if, instead of the true history of the market, random information is supplied to the agents; given α, the only relevant condition is that all agents must be given the same information (whether sensible or otherwise). This led to a considerable simplification of theoretical approaches to the MG, since it reduced the process to a Markovian one. A further generalization of the game was the introduction of agents’ decision noise [5], which was shown not only to improve worse than random behaviour but also, more surprisingly, to be able to make it better than random. The study [6] was followed by a number of papers aiming to develop a solvable statistical mechanical theory, either by using decision noise to ‘regularize’ the stochastic equations and replace these by deterministic ones (followed by an equilibrium analysis of the ergodic regime, built on the construction and exploitation of a Lyapunov function) [6,7], or by concentrating further on analysis of the stochastic equations themselves [8]. Since the MG process does not obey detailed balance, such studies (which also involved different implementations of the decision noise) proved to be hard, and their results partly controversial and (especially with regard to the questions of whether and when the stochastic MG equations can be replaced by suitable deterministic ones).

More recently, in [9] the analysis of the MG was approached from a different angle: all problems and debates regarding microscopic determinism were simply avoided by re-defining the MG dynamics directly in the form of discrete-time deterministic equations, without decision noise (the so-called ‘Batch Minority Game’). This allowed for an exact solution of the model using generating functional techniques à la De Dominicis [10], which was found to be in excellent agreement with numerical simulations, and which (due to it being dynamical in nature) even applied to the non-ergodic regime. The present
study, which can be regarded as a natural follow-up on \[12\], achieves the following objectives. We generalize
the ‘Thermal Minority Game’ such as to allow different
groups to have different levels of decision noise. This
introduces inhomogeneity into the agent population, which
leads to interesting new phenomena and phase diagrams.
We generalize and apply the (exact) formalism of \[12\]
(which was developed for the deterministic MG) to the
case of having inhomogeneous decision noise, within the
context of the discrete-time deterministic (‘batch’) equa-
tions. All our theoretical results are shown to find excel-
lent confirmation in extensive numerical simulations.

II. MODEL DEFINITIONS

The minority game involves \(N\) agents, labeled with
Roman indices \(i,j,k,\ldots\). At each round \(\ell\) of the game,
all agents act on the basis of the same piece of external
information \(I(\ell)\). In the original model \[1\] the his-
tory of the actual market was used as the information
given to the agents. In view of the observation in \[12\] that random information is equally efficacious we here con-
der that at each round \(\ell\) the agents are given the infor-
mation \(I(\ell) = I_\mu(\ell)\), where for each \(\ell\) the label \(\mu(\ell)\) is
chosen randomly and independently from \(p = \alpha N\)
possible values, i.e. \(\mu(\ell) \in \{1, \ldots, \alpha N\}\). To determine
how to convert the external information into a trading
decision, each agent \(i\) has at his/her disposal \(S\) strategies
\(\mathbf{R}_{ia} = (R_{ia1}, \ldots, R_{iaN}) \in \{-1,1\}^{\alpha N}; \ a \in \{1, \ldots, S\}\).
Each component \(R_{ia}\) is selected randomly and indepen-
dently from \(-1,1\) before the start of the game, with
uniform probabilities, and remains fixed throughout the
game. The strategies introduce quenched disorder into
the model. Each strategy \(\mathbf{R}_{ia}\) of every agent \(i\) is given
an initial valuation or point-score \(p_{ia}(0)\). In the deter-
mensional version of the game, given a choice \(\mu(\ell)\) made
for the information presented at round \(\ell\), every agent \(i\)
selects the strategy which for trader \(i\) has the highest
valuation at that point in time, i.e. the strategy with
label \(\tilde{a}_i(\ell) = \arg \max p_{ia}(\ell)\), and subsequently makes
a binary bid \(b_i(\ell) = R_{ia}\tilde{a}_i(\ell)\). The (re-scaled) total bid
at stage \(\ell\) is defined as \(A(\ell) = N^{-1/2} \sum b_i(\ell)\). Each
agent subsequently updates the pay-off values of each of
his/her strategies \(a\) on the basis of comparing the bid
which would have resulted from playing that strategy
with the actual outcome:

\[
p_{ia}(\ell + 1) = p_{ia}(\ell) - R_{ia}\mu(\ell) A(\ell).
\]

(1)

The minus sign in this expression ensures that strate-
gies that would have produced a minority decision are
rewarded. Since the qualitative behaviour of the market
fluctuations was found to be very much the same for all
non-extensive numbers of strategies per agent larger than
one \[12\], we restrict our discussion to the \(S = 2\) model,
where the equations can be simplified upon introducing
for each agent the instantaneous difference between the
two strategy valuations, \(q_i(\ell) = [p_{i1}(\ell) - p_{i2}(\ell)]/2\), as
well as the average strategy \(\omega_i = (R_{i1} + R_{i2})/2\) and
the difference between the strategies \(\xi_i = (R_{i1} - R_{i2})/2\).
The actually selected strategy in round \(\ell\) can now be
written explicitly as a function of a binary variable
\(s_i(\ell) = \pm 1\), which in the original model takes the value
\(s_i(\ell) = \text{sgn}(q_i(\ell))\), viz. \(R_{ia}(\ell) = \omega_i + s_i(\ell)\xi_i\), and the
evolution of the difference will now be given by:

\[
q_i(\ell + 1) = q_i(\ell) - \epsilon_i^{(\mu)} [\Omega(\mu) + \frac{1}{\sqrt{N}} \sum_j \epsilon_j^{(\mu)} s_j(\ell)],
\]

(2)

with \(\Omega = N^{-1/2} \sum_j \omega_j \in \mathbb{R}^{\alpha N}\).

In the so-called thermal minority game \[12\], the deter-
mensional decision rule \(s_j(\ell) = \text{sgn}(q_j(\ell))\) was replaced by
a stochastic recipe; another recipe was employed in \[13\].
Here we generalise this idea further, by allowing different
traders to have different levels of stochasticity in their
decision making. We will consider decision noise of the
general form

\[
s_j(\ell) = \sigma[q_j(\ell), z_j(\ell)|T_j],
\]

(3)

in which the \(z_j(\ell)\) are independent and zero-average ran-
dom numbers, described by some symmetric distribu-
tion \(P(z)\) which is normalised according to \(\int dz \ P(z) = 1\).
The function \(\sigma[q, z|T] \in \{-1,1\}\) is chosen to interpola-
te smoothly via a single control parameter \(T\) between \(\sigma[q, z|0] = \text{sgn}[q]\) for \(T = 0\) and
\(\sigma[q, z|\infty] = \pm 1\) (randomly, with equal probabilities) for
\(T = \infty\), so that \(T\) provides a measure of the degree of
stochasticity in the traders’ decision making (with ran-
dom choice in the case \(q = 0\)). Typical examples are
additive and multiplicative noise definitions such as

\[
\text{additive: } \quad \sigma[q, z|T] = \text{sgn}[q + Tz]
\]

(4)

\[
\text{multiplicative: } \quad \sigma[q, z|T] = \text{sgn}[q] \sgn[1 + Tz].
\]

(5)

In the first case \(\sigma[1]\) the noise has the potential to be
overruled by the so-called ‘frozen’ agents \[12\], who have
\(q_i(t) \sim \tilde{q}_i t\) for \(t \to \infty\) \[10\]. In the second case the
decision noise will even retain its effect for ‘frozen’ agents
(if they exist). The above definitions represent situations
where for \(T_i > 0\) a trader \(i\) need not always use his/her
‘best’ strategy; for \(T_i \to 0\) we revert back to the deter-
mensional model. The impact of the multiplicative noise
\[13\] can be characterised by the monotonic function

\[
\lambda(T) = \int dz \ P(z) \sgn[1 + Tz],
\]

(6)

with \(\lambda(0) = 1\) and \(\lambda(\infty) = 0\). For example, for a Gaus-
sian \(P(z)\) one has \(\lambda(T) = \text{erf}[1/\sqrt{2T}]\).

Finally in the formulation of the model, we replace the
above ‘on-line’ version of the microscopic dynamics
\[13\], following \[12\], by a so-called ‘batch’ version, where,
rather than modifying the \(\{q_i\}\) after every observation
of an individual piece of external information, they are
modified according to the average effect of the possible choices for the external information:

\[ q_i(\ell + 1) = q_i(\ell) - \frac{1}{p} \sum_{\mu=1}^P \xi_i^{\mu} [\Omega^\mu + \frac{1}{\sqrt{N}} \sum_j \xi_j^{\mu} s_j(\ell)], \quad (7) \]

with

\[ q_i(t + 1) = q_i(t) - h_i - \sum_j J_{ij} \sigma[q_j(t), z_j(t)|T_j], \quad (8) \]

where \( J_{ij} = 2N^{-1} \xi_i \cdot \xi_j \) and \( h_i = 2N^{-2} \xi_i \cdot \Omega \). The specific choice of time scaling in (8) has been made for later convenience. The batch dynamics (8) has the advantage of being sufficiently simple and transparent to allow for a straightforward exact dynamical solution of the model, using generating functional techniques \( [12] \). It is important to note that \( \xi_i \) is not exactly equivalent to \( \xi_i \) in \( [3] \), even for large \( N \rightarrow \infty \) (see \( [3] \) for the generating functional analysis of the online dynamics and its relation to the batch alternative), but it does present qualitatively similar features.

There are many ways to introduce a stochastic element into the traders’ decision making. For instance, the two versions of the minority game studied in \( [3,14] \) correspond to the forms \( [14] \) and \( [3] \) with \( P(z) = 1/2[K(1 - \tanh^2(Kz))] \) and \( T_i = T \) for all \( i \) (giving strategy selection probabilities of the form \( \text{Prob}(\sigma = \pm 1) \approx e^{\pm h_i} \text{and} \text{Prob}(\sigma = \pm 1) \approx e^{\pm \text{sgn}(q)} \), respectively).

The magnitude of the market fluctuations, or volatility, is given by

\[ \sigma^2 = \left( \frac{1}{p} \sum_{\mu} (A^{\mu})_z \right)^2 - \left( \frac{1}{p} \sum_{\mu} A^{\mu} \right)^2_z, \quad (9) \]

where \( A^\mu = N^{-\frac{1}{2}} \sum_i [w_i^\mu + s_i^\mu] \) and where \((\ldots) \) denotes an average over the random numbers \( \{z_i\} \). One easily derives

\[ \left( \frac{1}{p} \sum_{\mu} A^\mu \right)_z = \frac{1}{\alpha N^{\frac{1}{2}}} \sum_i (s_i)_z \sum_j \xi_j^\mu + O(N^{-\frac{1}{2}}), \quad (10) \]

\[ \left( \frac{1}{p} \sum_{\mu} (A^\mu)^2 \right)_z = \frac{1}{2} + \frac{1}{\alpha N^{\frac{1}{2}}} \sum_i (s_i)_z + \frac{1}{2} \sum_{ij} J_{ij} (s_i s_j)_z \] + \( O(N^{-\frac{1}{2}}) \), \quad (11)

Purely random trading corresponds to \( \left( \frac{1}{p} \sum_{\mu} A^\mu \right)_z = 0 \) and \( \sigma^2 = 1 \). Following \( [12] \) we also define the volatility matrix \( \Xi_{it} \):

\[ \Xi_{it} = \left( \frac{1}{p} \sum_{\mu} [A^\mu_i - \left( \frac{1}{p} \sum_{\nu} A^\nu_i \right)_z] [A^\mu_t - \left( \frac{1}{p} \sum_{\nu} A^\nu_t \right)_z] \right)_z, \quad (12) \]

which measures the temporal correlations of the market fluctuations. Note that \( \sigma^2 = \Xi_{it} \). In the case where the average bid \( \langle A \rangle \) is zero (as in the present model), the volatility measures the efficiency of the market.

## III. Generating Functional Analysis

The canonical tool to deal with the dynamics of the present problem is generating functional analysis à la De Dominicis \( [13] \), which allows one to carry out the disorder average (here: the average over all strategies) and take the \( N \rightarrow \infty \) limit exactly. The final result of the analysis is a set of closed equations, which can be interpreted as describing the dynamics of an effective ‘single agent’ \( [14,12] \). Due to the disorder in the process, this single agent will acquire an effective ‘memory’, i.e. he/she will evolve according to a non-trivial non-Markovian stochastic process. Here we will follow closely the steps taken in \( [12] \), and we refer to the latter paper for full details of the calculation. In contrast to the situation in \( [12] \), for the present noisy version of the game one finds a microscopic transition probability density operator \( W(q|q') \):

\[ W(q|q') = \int \frac{d\hat{q}}{(2\pi)^N} e^{\left( \sum_i \hat{q}_i (q_i - q'_i) + \sum_j J_{ij} (s_j') \right)} , \quad (13) \]

with the short-hand \( s'_j = \sigma[q'_j, z_j|T_j] \). The moment generating functional for a stochastic process of the present type is defined as

\[ Z[\psi] = \langle e^{\sum_i \hat{\psi}_i (q_i - q_i(t))} \rangle \quad = \int \prod \left[ dq(t) W(q(t + 1)|q(t)) \right] p_0(q(0)) \times e^{\sum_i \hat{\psi}_i (q_i - q_i(t))} . \quad (14) \]

Derivation of the generating functional with respect to the conjugate variables \( \psi \) generates all moments of \( q \) at arbitrary times. Upon introducing the two short-hands:

\[ w^\mu_i = \sqrt{\frac{7}{N}} \sum_j \hat{q}_j (q^\mu_i), \quad x^\mu_i = \sqrt{\frac{3}{N}} \sum_j s_j (q^\mu_i) , \quad (15) \]

as well as \( Dq = \prod \left[ dq(t)/\sqrt{2\pi} \right] \), \( Dw = \prod \left[ dw^\mu(t)/\sqrt{2\pi} \right] \) and \( Dx = \prod \left[ dx^\mu(t)/\sqrt{2\pi} \right] \) (with similar definitions for \( Dq, Dw \) and \( Dx \), respectively), the generating functional takes the following form:

\[ Z[\psi] = \int Dw Dw Dw Dx Dx x e^{\sum_i \hat{\psi}_i (w^\mu_i + x^\mu_i + \sqrt{2} w^\mu_i (t^{\mu} + t^{\mu}))} \times \int Dq Dq p_0(q(0)) \langle e^{\sum_i \hat{\psi}_i (q_i (q_i(t) - q_i(t)) + \hat{\psi}_i (s_i(t))))} \rangle \] \times \langle e^{\sum_i \hat{\psi}_i (q_i (q_i(t) - q_i(t)) + \hat{\psi}_i (s_i(t))))} \rangle , \quad (16) \]

where, as in \( [12] \), we introduced external ‘forces’ \( \theta_i(t) \) to generate response functions.

To describe typical behaviour, and in view of the self-averaging character of the large \( N \) limit, at this stage we average over over the explicit choices of the quenched random parameters \( \{ R \} \). These averages are not affected in any way by the introduction of the noise variables \( \{ z_i \} \) or the independent temperatures \( T_i \), and the further procedure of \( [12] \) still applies here, generating the
dynamical order parameters $C_{tt'} = N^{-1}\sum_i s_i(t)s_i(t')$, $K_{tt'} = N^{-1}\sum_i \tilde{s}_i(t)\tilde{s}_i(t')$, and $L_{tt'} = N^{-1}\sum_i \tilde{q}_i(t)\tilde{q}_i(t')$ and their conjugates. For times which do not scale with $N$ and for simple initial conditions of the form $p_0(q) = \prod_i p_0(q_i)$ one thus finds:

$$Z[\psi] = \int [D\hat{C}D\hat{C}][DKD\hat{K}][DLD\hat{L}] e^{N[\psi + \Phi + \Omega] + \mathcal{O}(N^0)}.$$  

(17)

The $\mathcal{O}(N^0)$ term in the exponent is independent of the fields $\{\psi_i(t)\}$ and $\{\theta_i(t)\}$. The three relevant exponents in (17) are given by the following expressions:

$$\Psi = \alpha \ln \left[ \int D\omega D\hat{\omega} D\phi D\hat{\phi} e^\phi \int d\omega_0 d\hat{\omega}_0 d\phi_0 d\hat{\phi}_0 \right]$$

$$\Phi = \alpha \ln \left[ \int D\omega D\hat{\omega} e^\phi \int d\omega_0 d\hat{\omega}_0 \right]$$

(18)

$$\Omega = \frac{1}{N} \sum_i \ln \left[ \int Dq D\hat{q} p_0(q(0)) \right. \times e^{i\sum_i [\bar{q}_i(t)(q(t+1)-q(t))-\bar{q}_i(t)+\psi_i(t)q(t)-i\sum_i \bar{q}_i(t)L_{tt'}\tilde{q}(t')]} \times \left. e^{-i\sum_i \bar{q}_i(t)C_{tt'}(s(t)+s(t)\theta_i(t)+\tilde{s}_i(t)\tilde{q}_i(t'))} \right] \right].$$

(19)

Here $s_i(t) = \sigma[q(t),z_i(T)]$ and the average $\langle \ldots \rangle_z$ has now been reduced to a single site (but many-time) one: $\{g[z_1,z_2,\ldots]\}_z = \prod_i \{d[z_t]p(z_t)\} g[z_1,z_2,\ldots]$. Following (12) we have also introduced the short-hands $Dq = \prod_i\{dq_i(t)/\sqrt{2\pi}\}$, $D\omega = \prod_i\{d\omega_i(\sqrt{2\pi})\}$. Note that all the quantities appearing in (17) are macroscopic; all the microscopic variables have been integrated out.

**IV. THE SADDLE-POINT EQUATIONS**

We can now evaluate (17) by saddle-point integration, in the limit $N \to \infty$. We define $G_{tt'} = -iK_{tt'}$. Taking derivatives with respect to the generating fields and using the normalisation $Z[0] = 1$ then gives (at the physical saddle-point) the usual identifications

$$C_{tt'} = \lim_{N \to \infty} \frac{1}{N} \sum_i \langle s_i(t)s_i(t') \rangle,$$

(21)

$$G_{tt'} = \lim_{N \to \infty} \frac{1}{N} \sum_i \frac{\partial}{\partial \theta_i(t)} \langle s_i(t) \rangle,$$

(22)

and also

$$L_{tt'} = - \lim_{N \to \infty} \frac{1}{N} \sum_i \frac{\partial^2}{\partial \theta_i(t)\partial \theta_i(t')} = 0.$$  

(23)

Putting $\psi_i(t) = 0$ (they are no longer needed) and $\theta_i(t) = \theta(t)$ then simplifies (21) to

$$\Omega = \int_0^\infty d\tau W(T) \ln \left[ \int Dq D\hat{q} p_0(q(0)) \right. \times e^{i\sum_i \bar{q}_i(t)[q(t+1)-q(t)]-i\sum_i \bar{q}_i(t)L_{tt'}\tilde{q}(t')} \times \left. \langle e^{-i\sum_i [s(t)C_{tt'}(s(t)+s(t)\theta(t)+\tilde{s}_i(t)\tilde{q}_i(t'))]_z} \right],$$

(24)

in which now $s(t) = \sigma[q(t),z_i(T)]$, and where $W(T)$ denotes the distribution of local noise strengths:

$$W(T) = \lim_{N \to \infty} \frac{1}{N} \sum_i \delta[T-T_i].$$

(25)

Extremisation of the extensive exponent $N[\Psi + \Phi + \Omega]$ of (17) with respect to $\{C, \tilde{C}, K, \tilde{K}, L, \tilde{L}\}$ gives the saddle-point equations

$$C_{tt'} = \langle s(t)s(t') \rangle_\star,$$

(26)

$$G_{tt'} = \frac{\partial}{\partial \theta(t')} \langle s(t) \rangle_\star,$$

(27)

$$\tilde{C}_{tt'} = \frac{i}{\partial \theta(t')} \tilde{K}_{tt'} = \frac{i}{\partial \theta(t')} \tilde{L}_{tt'} = \frac{i}{\partial \theta(t')} \tilde{L}_{tt'}. $$

The effective single-trader averages $\langle \ldots \rangle_\star$, generated by taking derivatives of (21), are defined as

$$\langle f[q,s] \rangle_\star = \int_0^\infty d\tau W(T) \left\{ \frac{\int Dq \langle M[q,s]f[q,s] \rangle_z}{\int Dq \langle M[q,s] \rangle_z} \right\},$$

(28)

$$M[q,s] = p_0(q(0)) e^{-i\sum_i s(t)C_{tt'}(s(t)+s(t)\theta(t)+\tilde{s}_i(t)\tilde{q}_i(t'))} \times \int D\hat{q} e^{i\sum_i \hat{q}_i(t)L_{tt'}\hat{q}(t')} \times e^{i\sum_i \hat{q}_i(t)[q(t+1)-q(t)]-i\sum_i \hat{q}_i(t)L_{tt'}\hat{q}(t')} \times \langle e^{-i\sum_i [s(t)C_{tt'}(s(t)+s(t)\theta(t)+\tilde{s}_i(t)\tilde{q}_i(t'))]_z} \right],$$

(29)

Upon elimination of the trio $\{\tilde{C}, \tilde{K}, \tilde{L}\}$ via (22) we obtain exact closed equations for the disorder-averaged correlation- and response functions in the $N \to \infty$ limit: equations (24), with the effective single trader measure (25). One recovers the theory of (12) upon putting $W(T) = \delta(T)$. Since the introduction of decision noise into the dynamics has only affected the term $\Omega$ (24), compared to the analysis in (12), the simplifications of the term $\Phi$ (reflecting the statistical properties of the trading strategies) derived in (12) apply unaltered, so that at the physical saddle-point we again find

$$\tilde{L} = -\frac{1}{2}i\alpha(1+G)^{-1}D(1+G^T)^{-1},$$

(30)

$$\tilde{K}^T = -\alpha(1+G)^{-1},$$

(31)

$$\tilde{C} = 0,$$

(32)

where $A^T$ denotes the transpose of the matrix $A$, and the entries of the matrix $D$ are given by $D_{tt'} = 1 + C_{tt'}$. We now find our effective single trader measure $M[q,s]$ of (29) reducing further to
\[ M\{q, s\} = p_0(q(0)) \times \int Dq \, e^{-\frac{1}{2} \sum_{t'} \hat{q}(t') \left( (\mathbf{1} + G)^{-1} D(\mathbf{1} + G T^{-1}) \right)_{t't'} \hat{q}(t')} \times e^{\sum_{t'} \hat{q}(t') \{ q(t+1) - q(t) - \theta(t) + \alpha \sum_{t'} (\mathbf{1} + G)^{-1}_{tt'} s(t') \}}. \] (33)

For a given value of \( T \), this describes a stochastic single-agent process of the form

\[ q(t+1) = q(t) - \alpha \sum_{t' \leq t} \{ (\mathbf{1} + G)^{-1}_{tt'} \sigma[q(t')], z_t | T \} + \theta(t) + \sqrt{\alpha} \eta(t). \] (34)

Causality ensures that \((\mathbf{1} + G)^{-1}_{tt'} = 0\) for \( t' > t \). The variable \( z_t \) represents the original single-trader decision noise, with \( \langle z_t \rangle = 0 \) and \( \langle z_t z_{t'} \rangle = \delta_{tt'} \), and \( \eta(t) \) is a disorder-generated Gaussian noise with zero mean and with temporal correlations given by \( \langle \eta(t) \eta(t') \rangle = \Sigma_{tt'} \).

\[ \Sigma = (\mathbf{1} + G)^{-1} D(\mathbf{1} + G T^{-1}). \] (35)

The correlation- and response functions (21, 22) are the dynamic order parameters of the problem, and must be solved self-consistently from the closed equations

\[ C_{tt'} = \langle \sigma[q(t)], z_t | T \} \sigma[q(t')], z_{t'} | T \rangle, \] (36)

\[ G_{tt'} = \frac{\partial}{\partial \theta(t')} \langle \sigma[q(t)], z_t | T \rangle, \] (37)

which, following (28), now also involve averaging over the distribution of the noise strengths \( T \). Note that \( M\{q, s\} \) as given by (33) is normalised, i.e. \( \int Dq \, M\{q, s\} = 1 \), so the associated averages reduce to

\[ \langle f\{q, s\} \rangle = \int_0^\infty dT \, W(T) \int Dq \, \langle M\{q, s\} f\{q, s\} \rangle z. \] (38)

The calculation in (12) of the disorder-averaged average bid and the volatility matrix (including the single-time volatility \( \sigma_t^2 = \Sigma_{tt'} \)) still hold, and hence

\[ \lim_{N \to \infty} \langle A \rangle_t = 0, \quad \lim_{N \to \infty} \Sigma_{tt'} = \frac{1}{2} \Sigma_{tt'}. \] (39)

**V. THE FIRST TIME STEPS**

For the first few time steps one can calculate quite easily the order parameters (correlation- and response functions) and the volatility, from (33), using the simplifications which follow from causality such as

\[ [G^\alpha]_{tt'} = 0 \quad \text{for} \quad t' > t - n. \] (40)

At \( t = 0 \) this immediately allows us to conclude that \( \Sigma_0 = D_0 = 2 \). We now obtain from (33) the joint statistics at times \( t = 1 \), given a value for \( T \):

\[ p(q(1)|q(0)) = \int dz_0 \, P(z_0) \frac{e^{-\frac{1}{2} [q(1) - q(0) - \theta + \alpha \sigma[q(0), z_0 | T]]^2/\sqrt{\alpha}}}{2\sqrt{\alpha \pi}}. \] (41)

Equation (41) allows us to calculate \( C_{10} \) and \( \Sigma_{10} \), although the presence of the decision noise induces expressions which are significantly more difficult to work out explicitly than those of the noise-free case in (12), and which will depend on the choice made for \( \sigma[q, z | T] \):

\[ C_{10} = \int_0^\infty dt \, W(T) \int dz_0 dz_1 P(z_0) P(z_1) \int dq(0) p_0(q(0)) \times \int dq(1) e^{-\frac{1}{2} [q(1) - q(0) - \theta + \alpha \sigma[q(0), z_0 | T]]^2/\sqrt{\alpha}} \times \sigma[q(0), z_0 | T] \sigma[q(1), z_1 | T], \] (42)

\[ G_{10} = \int_0^\infty dt \, W(T) \int dz_0 dz_1 P(z_0) P(z_1) \int dq(0) p_0(q(0)) \times \int dq(1) e^{-\frac{1}{2} [q(1) - q(0) - \theta + \alpha \sigma[q(0), z_0 | T]]^2/\sqrt{\alpha}} \times \frac{\partial}{\partial q(1)} \sigma[q(1), z_1 | T]. \] (43)

We can now move to the next time step, again using (40), where we need the noise covariances \( \Sigma_{11} \) and \( \Sigma_{10} \):

\[ \Sigma_{10} = 1 + C_{10} - 2G_{10}, \] (44)

\[ \Sigma_{11} = 2 - 2G_{10}[1 + C_{01}] + 2[G_{10}]^2. \] (45)

This procedure can in principle be repeated for an arbitrary number of time steps.

We now specialise to the case where the game is initialised in a tabula rasa manner, i.e. \( p(q(0)) = \delta[q_0] \), and where we have no perturbation fields, i.e. \( \theta(t) = 0 \). Now, upon also using the symmetry of \( P(z) \), we can reduce the above results to

\[ C_{10} = \int_0^\infty dt \, W(T) \int dz \, P(z) \int dq \frac{e^{-\frac{1}{2} [q + \alpha z]^2/\sqrt{\alpha}}}{4\sqrt{\alpha \pi}} \times \{ \sigma[q, z | T] - \sigma[-q, -z | T] \}. \] (46)

\[ G_{10} = \int_0^\infty dt \, W(T) \int dz \, P(z) \int dq \frac{e^{-\frac{1}{2} [q + \alpha z]^2/\sqrt{\alpha}}}{4\sqrt{\alpha \pi}} \times \frac{\partial}{\partial q} \{ \sigma[q, z | T] - \sigma[-q, -z | T] \}. \] (47)

Inspection of these expressions for large and small \( \alpha \), and for the specific choices (13) reveals the following. For \( \alpha \to \infty \) one finds

\[ \lim_{\alpha \to \infty} G_{10} = 0, \quad \lim_{\alpha \to \infty} \Sigma_{11} = 2, \] (48)

for both noise types. The order parameters \( C_{10} \) and \( \Sigma_{10} \), in contrast, are sensitive to the type of noise chosen. For additive noise of the form (11) one has

\[ \lim_{\alpha \to \infty} C_{10} = -1, \quad \lim_{\alpha \to \infty} \Sigma_{10} = 0. \] (49)
whereas for multiplicative noise (5) one has
\[ \lim_{\alpha \to \infty} C_{10} = -\frac{\alpha \sqrt{2}}{\sqrt{\pi}} \int_0^\infty dT \ W(T) T^{-1} + O(\alpha^{3/2}) , \quad (50) \]
\[ \lim_{\alpha \to \infty} \Sigma_{10} = 1 - \frac{\alpha \sqrt{2}}{\sqrt{\pi}} \int_0^\infty dT \ W(T) T^{-1} + O(\alpha^{3/2}) \quad (51) \]
In both cases the negativity of \( C_{10} \) shows that the tabula-rasa initialised system immediately enters an oscillation, with the \( q_i(1) \) on average having opposite sign to the corresponding \( q_i(0) \). Initially, additive noise is found not to play a role, and the effective disorder-generated noise components \( \eta(t) \) decorrelate, compared with the deterministic case of [12]. Multiplicative noise, on the other hand, is seen to retain an impact, even for short times and large \( \alpha \), and to cause a reduction of the oscillation amplitude.

Now we turn to small \( \alpha \), where we make the choice \( P(z) = (2\pi)^{-\frac{1}{2}} e^{-z^2/2} \) in order to work out integrals explicitly. For additive noise (5) we find
\[ C_{10} = -\frac{\alpha \sqrt{2}}{\sqrt{\pi}} \int_0^\infty dT \ W(T) T^{-1} + O(\alpha^{3/2}) , \quad (52) \]
\[ G_{10} = \frac{\sqrt{2}}{\sqrt{\pi}} \int_0^\infty dT \ W(T) T^{-1} + O(\alpha^{3/2}) \quad (53) \]
(provided the above integrals over \( T \) exist; if they do not, we revert back to the leading orders of the \( T = 0 \) case [12], i.e. \( C_{10} = O(\sqrt{\alpha}) \) and \( G_{10} = O(1/\sqrt{\alpha}) \)). Combination with the expressions (44,45) shows that in leading order
\[ \eta(1) = \left( \frac{1}{2} - G_{10} \right) \eta(0) + w + \ldots \quad (54) \]
in which \( w \) is a zero-average Gaussian variable, independent of \( \eta(0) \), with variance \( \langle w^2 \rangle = 3/2 \). Hence we find from the effective single spin equation (44):
\[ q(1) = \sqrt{\alpha} \ \eta(0) + O(\alpha) , \quad (55) \]
\[ q(2) = \sqrt{\alpha} \left( \frac{3}{2} - G_{10} \right) \eta(0) + w + O(\alpha) . \quad (56) \]
We observe, as in [12], that for small \( \alpha \) and additive decision noise the first two time steps are driven predominantly by the disorder-generated noise component in (54). However, whether this noise component starts oscillating in sign is, in the case of decision noise, crucially dependent on the distribution of temperatures; only when \( \int_{\Lambda} dT \ W(T) T^{-1} \) is sufficiently large should we expect the system to enter the high volatility state. For multiplicative noise, on the other hand, we arrive for small \( \alpha \) at the leading orders
\[ C_{10} = -\frac{\sqrt{\alpha}}{\sqrt{\pi}} \int_0^\infty dT \ W(T) \lambda(T) + O(\alpha^{3/2}) , \quad (57) \]
\[ G_{10} = \frac{1}{\sqrt{\alpha \pi}} \int_0^\infty dT \ W(T) \lambda(T) + O(\sqrt{\alpha}) + \ldots \quad (58) \]
Here the oscillation is much stronger (provided we do not scale the temperatures with \( \alpha \)). Combination with the expressions (44,45) shows that in leading order the disorder-generated noise not only drives the oscillation, but is also being amplified by a factor of order \( \alpha^{-1/2} \):
\[ \eta(1) = -G_{10} \eta(0) + O(\alpha^{3/2}) \quad (59) \]
The effective single trader equation subsequently gives:
\[ q(1) = \sqrt{\alpha} \ \eta(0) + O(\alpha) , \quad (60) \]
\[ q(2) = -\frac{\eta(0)}{\sqrt{\alpha}} \int_0^\infty dT \ W(T) \lambda(T) + O(\sqrt{\alpha}) . \quad (61) \]
Thus, for small \( \alpha \) and tabula rasa initialisation additive decision noise has the most drastic effect on the dynamics, changing the leading order of the relevant observables by a factor \( \sqrt{\alpha} \) (in contrast to multiplicative noise).

VI. STATIONARY STATE FOR \( \alpha > \alpha_C(W(T)) \)

If the game has reached a time-translation invariant stationary state without long-term memory, then \( C_{1t} = C(t - t') \), \( C_{t\tau} = C(t - t') \) and \( \Sigma_{t\tau} = \Sigma(t - t') \). In this section we assume that the stationary state is one without anomalous response, i.e. \( \lim_{t \to \infty} \Sigma_{t \leq t'}^\infty G(t) = k \) exists. The lower limit of such behaviour in \( \alpha \) defines \( \alpha_C(W(T)) \).

In a stationary state one generally finds agents who change strategy frequently, but also agents who are consistently in the minority group. For the latter ‘frozen’ agents, the values of \( q_i \) will grow linearly in time. We follow [12] and separate the two groups by introducing \( \tilde{q}_i(t) = q_i(t)/t \); frozen agents will be those for whom \( \lim_{t \to \infty} \tilde{q}_i(t) \neq 0 \), and the quantity \( \phi = \lim_{t \to 0} \lim_{t \to \infty} \{ \theta \mid \tilde{q}(t) - \epsilon \} \) gives the fraction of ‘frozen’ agents in the original \( N \)-agent system, for \( N \to \infty \). Transformation of the process (34) gives, for a given \( T \):
\[ \tilde{q}_T(t) = \frac{1}{t} \tilde{q}_T(1) + \sqrt{\alpha} \sum_{t' < t} q(t') \frac{\alpha}{t} \sum_{t' < t' < t''} (1 + G)_{t''}^{-1} \sigma[q(t''), z_{t''} | T] . \quad (62) \]
We now define \( \tilde{q}_T = \lim_{t \to \infty} \tilde{q}_T(t) \) (assuming this limit exists) and take the limit \( t \to \infty \) in (62), giving
\[ \tilde{q}_T = -\frac{\alpha}{1 + k} m_T + \sqrt{\alpha} \ \eta , \quad (63) \]
with the time averages \( m_T = \lim_{t \to \infty} \frac{1}{t} \sum_{t \leq t} \sigma[q, z_t | T] \) and \( \eta = \lim_{t \to \infty} \frac{1}{t} \sum_{t \leq t} \eta(t) \). The variance of \( \eta \) follows from (33):

\[ \text{Note that the small } \alpha \text{ expansions in this section are made for fixed } W(T); \text{ the observed behaviour is likely to be different when } W(T) \text{ is allowed to scale with } \alpha. \]
\[ \langle \eta^2 \rangle = (1 + k)^{-2} \left[ 1 + \lim_{\tau, \tau' \to \infty} \frac{1}{\tau \tau'} \sum_{\tau' \leq \tau} \sum_{\tau''} C_{\tau'} \right] \]
\[ = [1 + \langle m_T^2 \rangle_*] / (1 + k)^2. \tag{64} \]

Note that \( \langle m_T^2 \rangle_* = \lim_{\tau \to \infty} \tau^{-1} \sum_{t \leq \tau} C(t) = c. \)

The integrated response (or static susceptibility) \( k = \lim_{\tau \to \infty} \sum_{t \leq \tau} G(t) \) is also calculated along the lines of \( \text{[13]} \). One writes the response function as
\[ G_{\nu'} = \alpha^{-1} \langle \partial \sigma[q(t), z_t[T]/\partial \eta(t')] \rangle_* \]
Integration by parts in this expression generates
\[ \langle \partial \sigma[q(t), z_t[T]/\partial \eta(t')] \rangle_* = \sum_{\nu'} \eta_{\nu'} \langle \sigma[q(t), z_t[T]|\eta(t'') \rangle_* \]
and hence
\[ \sqrt{\alpha} \sum_{\nu'} \langle \eta(t') \eta(t'') \rangle G^{T}_{\nu'} = \langle \sigma[q(t), z_t[T]|\eta(t') \rangle_* \]. \tag{66} \]

Averaging over the two times \( t \) and \( t' \) now gives in a stationary state without anomalous response:
\[ \langle m_T \eta \rangle_* = k \sqrt{\alpha} \langle \eta^2 \rangle. \tag{67} \]

Inserting the variance \( \langle \eta^2 \rangle \), as given in \( \text{[24]} \), then gives the general relation
\[ \langle \eta m_T \rangle_* = k \sqrt{\alpha} (1 + c) / (1 + k)^2. \tag{68} \]

### A. Additive Decision Noise

In the case of additive decision noise \( \text{[1]} \), we have \( \sigma[q(z[T] = \text{sgn}[q(z[T] + \eta] \). The effective agent is frozen if \( q \neq 0 \), in which case \( m_T = \text{sgn}[q_t] \). This solves equation \( \text{[13]} \) if and only if \( |\eta| > \sqrt{\alpha}/(1 + k) \). If \( |\eta| < \sqrt{\alpha}/(1 + k) \), on the other hand, the agent is not frozen; now \( \eta_T = 0 \) and \( m_T = (1 + k) |\eta|/\sqrt{\alpha} \). As a result, we can calculate \( c = \langle m_T^2 \rangle_* \) and the fraction \( \phi = [\theta(\eta] - \sqrt{\alpha}/(1 + k)] = 1 - \text{erf}(\sqrt{\alpha}/(2(1 + c))) \) of frozen agents exactly as in the case \( \text{[2]} \) without decision noise, giving the deterministic (i.e. \( W(T) = \delta(T) \)) result
\[ c = 1 - \left( 1 - \frac{1 + c}{\alpha} \right) \text{erf} \left[ \frac{\sqrt{\alpha}}{2(1 + c)} \right] - 2 \frac{1 + c}{2\pi \alpha} e^{-\alpha/2}. \tag{69} \]

We use \( \text{[68]} \) and calculate the covariance \( \langle \eta m_T \rangle_* \) exactly as in \( \text{[12]} \). The final result is
\[ \frac{1}{k} = \text{erf} \left[ \frac{\sqrt{\alpha}}{2(1 + c)} \right] - 1, \tag{70} \]
with the value of \( c \) to be determined by solving Eqn. \( \text{[69]} \). We find exactly the same transition point \( \alpha_c \approx 0.33740 \),

signaling the divergence of the integrated response \( k \), as was found in the noise-free case.

Numerical simulations of the (batch) dynamics of the present model (which we will not present here, for brevity) confirm quite convincingly that, upon measuring objects such as \( c \) or \( \phi \), in the case of additive decision noise one indeed exactly recovers the graphs of \( \text{[12]} \), without any dependence on the noise parameters. This, however, will turn out to be quite different in the case of multiplicative noise.

### B. Homogeneous Multiplicative Decision Noise

Next we turn to the case of multiplicative noise \( \text{[6]} \), at first with the simplest distribution \( W(T) = \delta[T - \overline{T}] \), where \( \sigma[q(z[T] = \text{sgn}[q] \text{sgn}[1 + \lambda \overline{T} z_t] \), and where \( m_T = \lim_{\tau \to \infty} \tau^{-1} \sum_{t \leq \tau} \text{sgn}[q_t(t)] \text{sgn}[1 + \lambda \overline{T} z_t] \). Since there is now only one noise strength in the system, \( \overline{T} \), we may drop the subscripts \( T \) for variables such as \( q(t) \) or \( m_t \), without danger of confusion. For a frozen agent one now finds
\[ m = \lambda \overline{T} \text{sgn}[q]. \tag{71} \]

This solves equation \( \text{[16]} \) when \( |\eta| > \sqrt{\alpha} \lambda \overline{T}/(1 + k) \). If \( |\eta| < \sqrt{\alpha} \lambda \overline{T}/(1 + k) \), on the other hand, the agent is...
FIG. 2. The asymptotic fraction of frozen agents $\phi$ as a function of $\alpha = p/N$, for multiplicative noise with $W(T) = \delta[T - T]$. Dotted line represents $\phi = 0$ and $m = (1 + k)/\sqrt{\alpha}$. We can again calculate $c = \langle m^2 \rangle$, self-consistently, upon distinguishing between the two possibilities:

$$c = \lambda^2(T)\langle \theta \left[ \left| \eta - \frac{\sqrt{\alpha} \lambda(T)}{1 + k} \right| \right] \rangle$$

$$+ \frac{(1 + k)^2}{\alpha}\langle \theta \left[ \sqrt{\alpha} \lambda(T) - |\eta| \right] \eta^2 \rangle.$$  

(72)

(73)

Working out the Gaussian integrals describing the statics of $\eta$, with variance (64), subsequently gives

$$c = \lambda^2(T) - \left[ \lambda^2(T) - \frac{1 + c}{\alpha} \right] \text{erf}\left[ \sqrt{\frac{\alpha \lambda^2(T)}{2(1 + c)}} \right]$$

$$- 2\lambda(T) \sqrt{\frac{1 + c}{2\pi \alpha}} e^{-\frac{\alpha^2 \lambda^2(T)}{2(1 + c)}}.$$  

(74)

From this equation the value of $c$ is solved numerically. The fraction $\phi$ of frozen agents is given by

$$\phi = \langle \theta \left[ \left| \eta - \frac{\sqrt{\alpha} \lambda(T)}{1 + k} \right| \right] \rangle = 1 - \text{erf}\left[ \sqrt{\frac{\alpha \lambda^2(T)}{2(1 + c)}} \right].$$  

(75)

We calculate the remaining object $\langle \eta m \rangle$, in (62), by again distinguishing between frozen and non-frozen agents and

FIG. 3. Phase diagram in the $(\alpha, 1 - \lambda(T))$ plane for homogenous multiplicative noise, i.e. $W(T) = \delta[T - T]$. The solid line separates a non-ergodic phase with anomalous response (left) from an ergodic one without anomalous response (right). For additive noise our theory predicts the $T$-independent transition given by the dashed line.

by using the two identities $m = \lambda(T)\text{sgn}[\eta]$ (for frozen agents) and $m = \eta(1 + k)/\sqrt{\alpha}$ (for fickle ones), both of which follow from (63), giving

$$\langle \eta m \rangle = \lambda(T)\langle \theta \left[ |\eta| - \frac{\sqrt{\alpha} \lambda(T)}{1 + k} \right] \rangle$$

$$+ \frac{1 + k}{\sqrt{\alpha}}\langle \theta \left[ \sqrt{\alpha} \lambda(T) - |\eta| \right] \eta^2 \rangle.$$  

$$= \frac{1 + c}{(1 + k)\sqrt{\alpha}} \text{erf}\left[ \sqrt{\frac{\alpha \lambda^2(T)}{2(1 + c)}} \right].$$  

(76)

Insertion into (68), together with (64), then gives the desired expression for the integrated response:

$$\frac{1}{k} = \frac{\alpha}{\text{erf}\left[ \sqrt{\frac{\alpha \lambda^2(T)}{2(1 + c)}} \right]} - 1,$$

with the value of $c$ to be determined by solving Eqn. (74). Equivalently, using (74) we find, as in the $T = 0$ case [14]

$$k = \frac{1 - \phi}{\alpha - 1 + \phi}.$$  

(77)

The integrated response $k$ is positive and finite, and our solution exact, for $\alpha > \alpha_c(W(T))$. At $\alpha_c(W(T))$ one finds that $k$ diverges; this transition is, as for $T = 0$, found to happen when the fraction of fickle agents equals $\alpha_c$. Finally, according to (74), we can write $\alpha_c(W(T))$ as $\alpha_c(W(T)) = \text{erf}[x]$, where $x$ is the solution of the transcendental equation

$$\lambda^2(T) \left\{ \text{erf}[x] - 1 + \frac{1}{x\sqrt{\pi}} e^{-x^2} \right\} = 1.$$  

(78)
Equivalently, we can write our transition line explicitly in terms of the inverse error function as

$$
\lambda(T_c) = \left\{ \alpha_c + e^{-\frac{\text{erf}[\alpha_c]}{\text{erf}[\alpha_c] \sqrt{\pi}}} - 1 \right\}^{-\frac{1}{2}}, \quad (79)
$$

where \( \lambda(T) \in [0, 1] \), see (3).

In figures 3 and 2 we show the solution of equation (24) and the corresponding fraction \( \phi \) of frozen agents as functions of \( \alpha \), together with the values for \( c \) and \( \phi \) as obtained by carrying out numerical solutions of the batch minority game (8) with homogeneous multiplicative decision noise. The two figures for \( c \) and \( \phi \) both show excellent agreement between theory and experiment above \( \alpha_c(W(T)) \). One observes that, in addition to a reduction in the persistent correlation, another effect of the introduction of multiplicative decision noise is an overall increase in the fraction of frozen agents. This is consistent with our solution of the first few iteration steps, where introducing decision noise had the effect of dampening the oscillations. In figure 3 we show the system’s phase diagrams for \( W(T) = \delta[T - T] \), defined by the transition line, where \( k = \infty \). This line is given by the solution of equation (79) in the case of multiplicative noise, and by \( \alpha_c(W(T)) \approx 0.33740 \) (i.e. the value corresponding to \( \lambda(0) = 1 \) for additive noise. Below \( \alpha_c(W(T)) \) our simulations show, as has been observed and reported earlier for the deterministic case, that in the anomalous response region the stationary state reached by the system depends critically on the initial conditions. For small values of the \( |q_i(0)| \) (i.e. weak initial strategy preferences) the system enters a high-volatility state with low \( c \) and \( \phi \), whereas for large values of the \( |q_i(0)| \) (i.e. strong initial strategy preferences) the system enters a low-volatility state with large \( c \) and \( \phi \).

C. Inhomogeneous Multiplicative Decision Noise

Finally we turn to the more complicated situation of multiplicative noise (3) with arbitrary distributions. For a frozen agent and for a given value of \( T \) one has

$$
m_T = \lambda(T) \text{ sgn}[\dot{q}]. \quad (80)
$$

As before, this solves equation (24) if \( |\eta| > \sqrt{\alpha \lambda(T)/(1 + k)} \), whereas for \( |\eta| < \sqrt{\alpha \lambda(T)/(1 + k)} \) the agent is fickle, i.e. \( \dot{q}_T = 0 \) and \( m_T = (1 + k)\eta/\sqrt{\alpha} \). According to (36) the calculation of persistent order parameters will now also involve averaging over the noise distribution. Since the macroscopic dynamics turns out to depend on \( T \) only via \( \lambda(T) \), it will be advantageous to define

$$
w(\lambda) = \int_0^\infty dT W(T) \delta[\lambda - \lambda(T)], \quad (81)
$$

From this equation the value of \( c \) is solved numerically. The fraction \( \phi \) of frozen agents is given by

$$
\phi = 1 - \int_0^1 d\lambda w(\lambda) \text{ erf} \left[ \frac{\alpha \lambda^2}{2(1 + c)} \right]. \quad (83)
$$

We calculate the remaining object \( \langle \eta m_T \rangle_* \) in (38) by again distinguishing between frozen and non-frozen agents and by using the two identities \( m_T = \lambda(T) \text{ sgn}[\eta] \) (for frozen agents) and \( m_T = \eta(1 + k)/\sqrt{\alpha} \) (for the non-frozen ones), both of which follow from (36), giving

$$
\langle \eta m_T \rangle_* = \frac{1 + c}{(1 + k)\sqrt{\alpha}} \int_0^1 d\lambda w(\lambda) \text{ erf} \left[ \frac{\alpha \lambda^2}{2(1 + c)} \right].
$$
with the value of $c$ to be determined by solving Eqn. (82). Using (83) this can again be written in the familiar form (7), which suggests that the $k = \infty$ transition is of a geometrical nature.

Unless we revert back to uniform noise levels, a transformation like $\alpha_\epsilon(W(T)) = \text{erf}[x]$ will now longer be helpful; to find the location of the phase transition one has to solve (82), together with the condition $k = \infty$.

Upon putting $y^* = \alpha/2(1 + c)$ one can, however, compactify these two coupled equations to

$$1 = \int_0^1 d\lambda \, w(\lambda) \lambda^2 \left\{ \text{erf}[y\lambda] - 1 + \frac{e^{-y^2\lambda^2}}{y\lambda\sqrt{\pi}} \right\},$$

$$\alpha = \int_0^1 d\lambda \, w(\lambda) \text{erf}[y\lambda].$$

We will finally work out our equations describing the system with inhomogeneous multiplicative decision noise explicitly for the following simple bi-modal distribution

$$W(T') = \epsilon \, \delta[T' - T] + (1 - \epsilon) \delta[T'],$$

with $\epsilon \in [0, 1]$. For $\epsilon = 1$ we revert back to the homogeneous case studied earlier in this section; for $\epsilon = 0$ we return to the model of [12]. Here we have

$$w(\lambda) = \epsilon \delta[\lambda - \lambda(T)] + (1 - \epsilon) \delta[\lambda - 1],$$

with the function $\lambda(T)$ as defined in (8). The general equations (83,84) from which to solve $c$ and $\phi$ reduce to

$$c = \epsilon \left\{ \lambda^2(T) - 2\lambda(T) \sqrt{\frac{1 + c}{2\pi \alpha}} e^{-\frac{\lambda^2(T)}{2(1 + c)}} \right\}
- \left[ \lambda^2(T) - \frac{1 + c}{\alpha} \right] \text{erf} \left\{ \frac{\alpha \lambda^2(T)}{2(1 + c)} \right\}
+ (1 - \epsilon) \left\{ 1 - 2 \sqrt{\frac{1 + c}{2\pi \alpha}} e^{-\frac{\lambda^2(T)}{2(1 + c)}} \right\}
- \left[ 1 - \frac{1 + c}{\alpha} \right] \text{erf} \left\{ \frac{\alpha}{2(1 + c)} \right\},$$

$$\phi = 1 - \epsilon \text{erf} \left[ \frac{\alpha \lambda^2(T)}{2(1 + c)} \right] - (1 - \epsilon) \text{erf} \left[ \frac{\alpha}{2(1 + c)} \right].$$

Similarly, the two coupled equations (83,84) which define the phase transition reduce to

$$1 = \int_0^1 d\lambda \, w(\lambda) \lambda^2 \left\{ \text{erf}[y\lambda] - 1 + \frac{e^{-y^2\lambda^2}}{y\lambda\sqrt{\pi}} \right\},$$

$$\alpha = \int_0^1 d\lambda \, w(\lambda) \text{erf}[y\lambda].$$

We will finally work out our equations describing the system with inhomogeneous multiplicative decision noise explicitly for the following simple bi-modal distribution

$$W(T') = \epsilon \, \delta[T' - T] + (1 - \epsilon) \delta[T'],$$

with $\epsilon \in [0, 1]$. For $\epsilon = 1$ we revert back to the homogeneous case studied earlier in this section; for $\epsilon = 0$ we return to the model of [12]. Here we have

$$w(\lambda) = \epsilon \delta[\lambda - \lambda(T)] + (1 - \epsilon) \delta[\lambda - 1],$$

with the function $\lambda(T)$ as defined in (8). The general equations (83,84) from which to solve $c$ and $\phi$ reduce to

$$c = \epsilon \left\{ \lambda^2(T) - 2\lambda(T) \sqrt{\frac{1 + c}{2\pi \alpha}} e^{-\frac{\lambda^2(T)}{2(1 + c)}} \right\}
- \left[ \lambda^2(T) - \frac{1 + c}{\alpha} \right] \text{erf} \left\{ \frac{\alpha \lambda^2(T)}{2(1 + c)} \right\}
+ (1 - \epsilon) \left\{ 1 - 2 \sqrt{\frac{1 + c}{2\pi \alpha}} e^{-\frac{\lambda^2(T)}{2(1 + c)}} \right\}
- \left[ 1 - \frac{1 + c}{\alpha} \right] \text{erf} \left\{ \frac{\alpha}{2(1 + c)} \right\},$$

$$\phi = 1 - \epsilon \text{erf} \left[ \frac{\alpha \lambda^2(T)}{2(1 + c)} \right] - (1 - \epsilon) \text{erf} \left[ \frac{\alpha}{2(1 + c)} \right].$$

FIG. 5. The asymptotic fraction of frozen agents $\phi$ as a function of $\alpha = p/N$, for multiplicative noise with $W(T') = \epsilon \delta[T' - T] + (1 - \epsilon) \delta[T']$, for $T = 1$ and different choices of the width ($\epsilon = 0$, 0.5, 1 from bottom to top). Markers: individual simulation runs, with $pN = \alpha N^2 = 10^6$ and homogeneous initial conditions where $q_0(0) = q(0)$ (circles: $q(0) = 0$, squares: $q(0) = 10$) and in excess of 1000 iteration steps. Solid curves for $\alpha > \alpha_c(W(T))$: analytical predictions. For $\alpha < \alpha_c(W(T))$, where they should no longer be correct, they have been continued as dashed lines.

$$1 = \epsilon \lambda^2(T) \left\{ \text{erf}[y\lambda(T)] - 1 + \frac{e^{-y^2\lambda^2(T)}}{y\lambda(T)\sqrt{\pi}} \right\}
+ (1 - \epsilon) \left\{ \text{erf}[y] - 1 + \frac{e^{-y^2}}{y\sqrt{\pi}} \right\},$$

$$\alpha = \epsilon \text{erf}[y\lambda(T)] + (1 - \epsilon) \text{erf}[y].$$

Note that for $T \rightarrow 0$ our transition line equations reduce once more to those of the noise-free case, as derived in [12], giving $\alpha_c \approx 0.33740$. For $T \rightarrow \infty$, in contrast, we find a strong dependence on $\epsilon$ (the fraction of traders who experience decision noise). In particular, there is a qualitative difference between $\epsilon < 1$ and $\epsilon = 1$ (where one of the two noise levels in the system becomes zero).

For $\epsilon = 1$ we return to the case of uniform decision noise, and equations (81,82) dictate that the transition line obeys $\alpha \rightarrow 0$ as $T \rightarrow \infty$. For $\epsilon < 1$ (i.e. a nonzero fraction of the traders take decisions deterministically), on the other hand, we find for $T \rightarrow \infty$ the equations (11,12) (which will now have a solution with finite $y$) reducing to

$$1 = (1 - \epsilon) \left\{ \text{erf}[y] - 1 + \frac{e^{-y^2}}{y\sqrt{\pi}} \right\},$$

$$\alpha = (1 - \epsilon) \text{erf}[y].$$

Equivalently:

$$\sqrt{\pi} \left[ \frac{2 - \epsilon - \alpha}{1 - \epsilon} \right] \text{erf}^\text{inv} \left[ \frac{\alpha}{1 - \epsilon} \right] = e^{-[\text{erf}^\text{inv} \left[ \frac{\alpha}{1 - \epsilon} \right]]^2}.$$
of the parameters \{z\} simulations. Here we have chosen Gaussian distributed given by (90), as functions of \( \alpha \), \( \phi \), \( c \), \( \alpha \). Finally, in figure 6 we show, in the \((\alpha, T)\) plane for different values of \( n \), \( \sigma \), \( T \), \( c \), \( T \), \( c \). Here we will translate the case of decision noise the (at least for the batch MG) slightly more accurate approximation proposed in [12]. We will abbreviate the double averages \( \langle \langle \odot \rangle \rangle \) as \( \langle \langle \odot \rangle \rangle \). In order to find the volatility we separate the correlations at stationarity in a ‘frozen’ and a ‘fickle’ contribution:

\[
C(t - t') = \phi \langle \langle \sigma(q(t), z_i(t)|\rangle \sigma(q(t'), z_i(t)|\rangle \rangle
+ (1 - \phi) \langle \langle \sigma(q(t), z_i(t)|\rangle \sigma(q(t'), z_i(t)|\rangle \rangle,
\]

which gives, using \( C(t - t') = C(t - t') - c \), and upon rewriting the ‘fickle’ contribution to the volatility:

\[
\sigma^2 = \frac{1}{2(1 + k)^2} + \frac{1}{2} (1 - \phi) \lim_{\tau \to \infty} \frac{1}{2\tau} \sum_{u \leq \tau} \sum_{t \leq t'} (1 + \tilde{G})^{-1}_{u} \tilde{G}_{u}^{-1} (1 + \tilde{G}^T) \langle \langle \sigma(q(t), z_i(t)|\rangle \sigma(q(t'), z_i(t)|\rangle \rangle.
\]

The approximation of [12] consists of retaining in the contribution from ‘fickle’ agents only the instantaneous \( u = t \) terms, the rationale being that the \( u \neq t \) ones represent in the original single-trader equation a retarded self-interaction, which is assumed to be significant only for ‘frozen’ agents. Hence we obtain

\[
\sigma^2 = \frac{1}{2(1 + k)^2} + \frac{1}{2} (1 - \phi) \lim_{\tau \to \infty} \frac{1}{2\tau} \sum_{u \leq \tau} \sum_{t \leq t'} (1 + \tilde{G})^{-1}_{u} \tilde{G}_{u}^{-1} (1 + \tilde{G}^T) \langle \langle \sigma(q(t), z_i(t)|\rangle \sigma(q(t'), z_i(t)|\rangle \rangle.
\]

Note that, according to (83,84), the integrated response \( k \) can be expressed in terms of the order parameter \( \phi \) as \( k = (1 - \phi)/(\alpha - 1 + \phi) \).

At this stage we again have to distinguish between additive noise and multiplicative noise, in order to work...
out the remaining averages. For additive noise one simply finds
\[ \langle \sigma(q(t), z_i T) \sigma(q(t'), z_i T') \rangle_{\text{i.r}} = \langle \sigma(q(t')) \sigma(q(t')) \rangle_{\text{i.r}} = 1, \]
and hence we recover the expression describing the noise-free case in [12]:
\[ \sigma^2 = \frac{1 + \phi}{2(1 + k)^2} + \frac{1}{2}(1 - \phi). \quad (101) \]
Since the order parameters \( \phi \) and \( k \) are, for additive noise, independent of the noise distribution, the same is true for the volatility. This independence of the noise parameters, at least for \( \alpha > \alpha_c \) (in line with [11], [14]), again finds confirmation in numerical simulations (that is, within the limits imposed by our approximation; one does observe some weak effect, which could either be due to excessive relation times or due to the retarded self-interaction of ‘fickle’ traders, which we neglected in deriving (101)).

The more interesting case, as before, is that of multiplicative noise. Here we have
\[ \langle \sigma(q(t), z_i T) \sigma(q(t'), z_i T') \rangle_{\text{i.r}} = \langle \lambda^2(T) \rangle_{\text{i.r}} + \delta_{\text{i.r}} [1 - \langle \lambda^2(T) \rangle_{\text{i.r}}]. \quad (102) \]
Hence the approximation [101] reduces to
\[ \sigma^2 = \left( \frac{2}{(1 + k)^2} \right) \left( \frac{1 + \phi}{2} + \frac{1}{2}(1 - \phi) \right) \quad (103) \]
and homogeneous initial conditions where \( q_i(0) = 0 \) (circles: \( q_i(0) = 0 \), squares: \( q_i(0) = 10 \)) and in excess of 1000 iteration steps. Thick solid curves for \( \alpha > \alpha_c(W(T)) \): analytical predictions for homogeneous multiplicative decision noise. For \( \alpha < \alpha_c(W(T)) \), where they should no longer be correct, they have been continued as thick dashed lines. For additive decision noise our theory predicts independence of \( T \), i.e. \( \sigma \) as given by the \( T = 0 \) curve of multiplicative noise.

FIG. 7. The asymptotic volatility \( \sigma \) as a function of \( \alpha \), for multiplicative noise with \( W(T) = \delta[T - T'] \) and different choices of the noise strength (\( T = 0, 1, 2 \) from bottom to top in \( \alpha > \alpha_c \) regime). Markers: individual simulation runs, with \( pN = \alpha N^2 = 10^6 \) and homogeneous initial conditions where \( q_i(0) = q(0) \) (circles: \( q(0) = 0 \), squares: \( q(0) = 10 \)) and in excess of 1000 iteration steps. Thick solid curves for \( \alpha > \alpha_c(W(T)) \): analytical predictions for homogeneous multiplicative decision noise. For \( \alpha < \alpha_c(W(T)) \), where they should no longer be correct, they have been continued as dashed lines.

FIG. 8. The asymptotic volatility \( \sigma \) as a function of \( \alpha \), for multiplicative noise with \( W(T') = \delta[T' - T] + (1 - \epsilon)\delta[T'] \), for \( T = 1 \) and different choices of the width (\( \epsilon = 0, 0.5, 1 \) from bottom to top in the \( \alpha > \alpha_c \) regime). Markers: individual simulation runs, with \( pN = \alpha N^2 = 10^6 \) and homogeneous initial conditions where \( q_i(0) = q(0) \) (circles: \( q(0) = 0 \), squares: \( q(0) = 10 \)) and in excess of 1000 iteration steps. Solid curves for \( \alpha > \alpha_c(W(T)) \): analytical predictions. For \( \alpha < \alpha_c(W(T)) \), where they should no longer be correct, they have been continued as dashed lines.

\[ \sigma^2 = \frac{1 + \phi \chi}{2(1 + k)^2} + \frac{1}{2}(1 - \phi) \]
\[ + \frac{1}{2} \phi(1 - \chi)[(1 + \lambda)^{-1}(\lambda + \lambda T)^{-1}](0). \quad (103) \]
Here we have used time-translation invariance of the stationary state, giving \( \ldots, t \rightarrow \ldots, (t - t) = \ldots, (0) \) for the relevant matrix elements in (103). The conditional average \( \chi = \langle \lambda^2(T) \rangle_{\text{i.r}} \), constrained by \( |\eta| > \sqrt{\lambda}(T)/(1 + k) \) (which, in the case of multiplicative noise, is the condition for an agent to be frozen) and calculated using the variance \( \langle \eta^2 \rangle = (1 + c)/\lambda^2(1 + k)^2 \) of the zero-average persistent noise term, is given by

\[ \chi = \langle \lambda^2(T) \rangle_{\text{i.r}} \]
\[ = \int_0^{\infty} dT \ W(T) \lambda^2(T) \int Dz \, \theta \left[ |z| - \frac{\sqrt{\lambda}(T)}{(1 + c)^{1/2}} \right] \]
\[ = \int_0^{\infty} dT \ W(T) \int Dz \, \theta \left[ |z| - \frac{\lambda T}{\sqrt{1 + c}} \right] \]
\[ = \int_0^{1} d\lambda \, w(\lambda) \lambda^2 \left[ 1 - \text{erf} \left( \frac{\lambda \sqrt{c}}{\sqrt{2(1 + c)}} \right) \right] \]
\[ + \int_0^{1} d\lambda \, w(\lambda) \left[ 1 - \text{erf} \left( \frac{\lambda \sqrt{c}}{\sqrt{2(1 + c)}} \right) \right]. \quad (104) \]
We note that only for \( W(T) = \delta(T) \) [12], i.e. \( w(\lambda) = \delta(\lambda - 1) \), where \( \chi = 1 \), will (103) involve only persistent observables. In the presence of decision noise, as in this study, one always has \( \chi < 1 \), and additional approximations are required to reduce also the last term in (103).
further to an expression in terms of persistent order parameters only. This is done in detail in the appendix, where we show that a reasonable approximation is obtained by simply putting \((\mathbb{1} + \hat{G})^{-1}(\mathbb{1} + \hat{G}^T)^{-1}(0)\rightarrow 1\). The end result is the following final approximation for the stationary state volatility:

\[
\sigma^2 = \frac{1 + \phi \chi}{2(1+k)^2} + \frac{1}{2}(1 - \phi) + \frac{1}{2}\phi(1-\chi), \quad (105)
\]

with \(\chi\) as given by \((104)\).

Expression \((105)\), which reverts back to that of \([12]\) for \(T \rightarrow 0\) and which also reduces correctly to the random trading limit \(\sigma = 1\) for \(T \rightarrow \infty\) (where \(\phi = 1\), \(c = k = \chi = 0\), turns out to be a surprisingly accurate approximation of the volatility for \(\alpha > \alpha_c\) (i.e. in its regime of validity). This can be observed in Figures 7 and 8, where we compare the approximate prediction \((105)\) to the volatility as observed in numerical simulations, for both homogeneous multiplicative noise defined by \(W(T) = \delta[T - \tau]\) and for inhomogeneous multiplicative noise defined by \((77)(83)\), respectively. In all cases \(\lambda(T) = \text{erf}[1/T \sqrt{2}]\) (note: the persistent order parameters have already been calculated in the previous section).

The above results emphasize once more the qualitative difference between additive and multiplicative noise: in contrast to additive noise, the system remains sensitive to multiplicative noise even for \(\alpha > \alpha_c\). The resulting dependence of the volatility on the multiplicative noise strength is very similar to that reported in \([3]\) for additive noise (which was later understood to be caused by insufficient equilibration \([10]\)).

VIII. DISCUSSION

In this paper we have generalised the Thermal Minority Game \([3]\) to the case of inhomogeneous agent populations (where the decision noise, which can be either additive or multiplicative, is of non-uniform strength). We have solved the dynamics of the batch version of this model by generalizing the recent application \([2]\) to the Minority Game of the generating functional techniques of \([3]\) (note that in \([2]\) only the fully deterministic case was studied). This formalism reduces the \(N\)-agent dynamics, in the limit \(N \rightarrow \infty\), to a stochastic process for a single ‘effective agent’, with dynamic equations involving coloured noise and a retarded self-interaction. It leads to exact closed (but implicit and non-trivial) equations for correlation- and response-functions.

Our theory enables us to (i) obtain an analytical and quantitative understanding of previously observed but unexplained phenomena (e.g. suppression of the volatility by decision noise, even below random, due to damping of the ‘crowd anti-crowd’ oscillations), (ii) derive exact phase diagrams, and (iii) calculate macroscopic observables (e.g. the fraction of frozen agents and the persistent correlations) in ergodic stationary states exactly\(^2\). In the case of additive decision noise we find a phase diagram identical to that of deterministic decision making in the onset of and equilibrium properties of the higher \(\alpha\) ergodic phase, with non-ergodic behaviour at lower \(\alpha\). In the case of multiplicative decision noise, in contrast, we arrive at phase diagrams with non-trivial decision noise dependencies of the phase separation line as well as the behaviour of both phases. Here the control parameters are the relative number of possible value for the external information, \(\alpha = p/N\), and the parameters characterizing the noise statistics. In the non-ergodic regime of the model (i.e. for sufficiently small \(\alpha\)), our closed equations in terms of correlation- and response functions are still exact, and can be solved in principle iteratively for arbitrary times; however, finding the stationary states is hard (see e.g. the calculations for the simpler case \([3]\)). Here we have restricted our calculations in the non-ergodic regime to the the first few time-steps, finding noise dependence for both additive and multiplicative decision noise.

In the present paper we have only worked out explicitly two types of choices for the decision noise strengths statistics: a delta distribution (i.e. decision noise of uniform strength), and a parametrized class of bi-model distributions. Due to the general nature of our solution, however, there is no limit to the different types of noise statistics we could have studied. This emphasizes once more the remarkable potential and appropriateness to the Minority Games of the generating functional analysis methods of \([3]\). Two natural next steps would be to develop the generating functional formalism for the original ‘on-line’ formulation of the game, where the external information is fed to the agents sequentially (this is the subject of \([4]\)), or to analyze our present (exact) order parameter equations further in the non-ergodic region \(\alpha < \alpha_c(W(T))\).

\(^2\)Although the stationary state equations derived upon assuming ergodicity and absence of long-term memory are no longer valid in the non-ergodic regime, Figures 1, 2, 4 and 5 show that for \(\alpha < \alpha_c(W(T))\) their predictions regarding the persistent observables \(c\) and \(\phi\) nevertheless give good qualitative agreement with the results of simulations from a highly biased start (for the volatility \(\sigma\), which also involves non-persistent order parameters, this is no longer the case).

\(^3\)Note that a recently proposed procedure \([4]\) for calculating at least the high-volatility stationary state in the non-ergodic regime, based on assuming the integrated response function (which diverges exactly at the critical point) to remain infinite throughout the \(\alpha < \alpha_c\) region, is not likely to work for the case of decision noise. It would, for instance, predict the simple relation \(\phi = 1 - \alpha\) (i.e. \(\phi\) being independent of the noise parameters), which is clearly in conflict with the simulation experiments presented in this paper.
APPENDIX A: APPROXIMATION OF NON-PERSISTENT TERMS IN THE
STATIONARY VOLATILITY

The term \( Q = (\mathbb{1} + \hat{G})^{-1}(\mathbb{1} + \hat{G}^T)^{-1}(0) \) in \([103]\), which contains contributions of non-persistent order parameters, can be written as

\[
Q = \int \frac{d\omega}{2\pi} \frac{1}{|1 + G(\omega)|^2},
\]

(A1)

with the definition \( \hat{G}(\omega) = \sum_t \hat{G}(t)e^{-i\omega t} \). The simplest approximation for \( \hat{G}(t) \), which respects causality and also meets the requirement \( \sum_t \hat{G}(t) = k \), is an exponential expression of the form \( \hat{G}(t > 0) \approx k(1 - \gamma)^{t-1} \) (with \(-1 < \gamma < 1 \) and \( \hat{G}(t \leq 0) = 0 \)). This gives \( \hat{G}(\omega) = k(1 - \gamma)/(e^{i\omega} - \gamma) \), and thus

\[
Q = \int \frac{d\omega}{2\pi} \frac{|e^{i\omega} - \gamma|^2}{|e^{i\omega} - \gamma + k(1 - \gamma)|^2}.
\]

(A2)

We will obtain an estimate for \( \gamma \) by carrying out an approximate calculation of the one-step response function

\[
\hat{G}(1) = \frac{\partial}{\partial \theta(t)} \langle \sigma[q(t + 1), z_{t+1} | T] \rangle_*
\]

(A3)

We insert \([34]\), we use the fact that the response of frozen agents will be zero, we repeat our previous ansatz that fickle agents do not experience a retarded self-interaction, and we carry out the average over the decision noise variable \( z_t \). This is followed by carrying out the average over \( \eta(t) \) (which is Gaussian, with variance \( \langle \eta^2(t) \rangle = 2\sigma^2 \)), we assume, within the context of the present approximation, the correlations between \( \eta(t) \) and the persistent noise \( \eta \) not to be important for fickle agents). This gives

\[
\hat{G}(1) = \frac{1 - \phi}{\sigma\sqrt{\pi\alpha}} \langle \frac{\lambda(T)e^{-|q(t)|/\alpha + \alpha}}{4\sigma^2} \rangle_{\tilde{h}} \times \left[ \cosh \left( \frac{|q(t)|}{2\sigma^2} \right) + \lambda(T) \sinh \left( \frac{|q(t)|}{2\sigma^2} \right) \right]_{\tilde{h}}.
\]

(A4)

In this expression we simply replace \( |q(t)| \rightarrow 0 \) (fickle agents being described by values of \( q(t) \) which oscillate around zero), and we calculate the residual average \( \langle \lambda(T) \rangle_{\tilde{h}} \) similar to our calculation of \([104]\). Hence we arrive at the approximation

\[
\hat{G}(1) \approx \frac{1 - \phi}{\sigma\sqrt{\pi\alpha}} e^{-\alpha/4\sigma^2} \left\{ \int_0^1 d\lambda \, w(\lambda) \lambda \text{erf} \left( \frac{\lambda\sqrt{\pi}}{\sqrt{2}(1 + c)} \right) \right\}_{\tilde{h}}.
\]

(A5)

On the other hand, according to our ansatz \( \hat{G}(t > 0) = k(1 - \gamma)(1 - \gamma)^{t-1} \) we must demand \( \hat{G}(1) = k(1 - \gamma) \), so that \([13]\) leads to the following estimate of \( \gamma \):

\[
\gamma \approx 1 - \frac{1}{\sigma k\sqrt{\pi\alpha}} e^{-\alpha/4\sigma^2} \left\{ \int_0^1 d\lambda \, w(\lambda) \lambda \text{erf} \left( \frac{\lambda\sqrt{\pi}}{\sqrt{2}(1 + c)} \right) \right\}_{\tilde{h}}.
\]

(A6)

Since for \( \alpha \rightarrow \infty \) we must find \( \sigma \rightarrow 1 \) (random trading), and since \( k \approx \alpha^{-1} \), we conclude from \([10]\) that \( \gamma \rightarrow 1 \) for \( \alpha \rightarrow \infty \). Conversely, as \( \alpha \) is lowered, we find a divergence of \( k \) at finite \( \alpha_c \) (where also \( \phi \) is finite). Hence \([A6]\) also predicts that \( \gamma \rightarrow 1 \) for \( \alpha \rightarrow \alpha_c \). We now assume that \( \gamma \rightarrow 1 \) will give a sensible approximation in the whole range \( \alpha > \alpha_c \), and use \([A2]\) to arrive at the approximate result

\[
\langle (\mathbb{1} + \hat{G})^{-1}(\mathbb{1} + \hat{G}^T)^{-1}(0) \rangle \approx 1.
\]

(A7)

The above derivation is clearly far from rigorous, and not quite satisfactory; it simply appears the best one can do without actually solving the order parameter equations for finite time differences in the stationary state. Yet \([A3]\) turns out to lead to a surprisingly accurate approximation for the volatility (see the main text).

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