Deformation of half-BPS solution in ABJM model and instability of supermembrane

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Abstract

It is well-known that a supermembrane in the light-cone gauge has a continuous spectrum and is unstable. Physical interpretation of this instability is that a supermembrane can have a long thin tube without cost of energy and consequently it becomes a spiky configuration in which multiple membranes are connected by thin tubes.

On the other hand, the ABJM model was proposed as a low-energy description of multiple M2-branes in the static gauge. It is natural that an M2-brane is also unstable in this gauge if we believe the physical picture in the light-cone gauge. In order to examine this, we construct a BPS solution explicitly both in the Nambu-Goto action of a supermembrane in the static gauge and in the $U(1) \times U(1)$ ABJM model, which represents intersecting M2-branes. Since this configuration is regarded as a single M2-brane emitting another one, we study the instability of an M2-brane by analyzing fluctuations around it. We show that a zero mode exists which can deform the configuration. For comparison, we also examine a similar configuration on the D2-brane and check that it does not have such zero modes under a fixed string charge. Furthermore we confirm that the novel Higgs mechanism translates our BPS solution in the ABJM model into that in the D2-brane world volume theory, where the winding number of the former around the fixed point of the orbifold becomes the number of strings ending on the D2-brane in the latter.

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1 Introduction

In nonperturbative formulation of string theory, identification of fundamental degrees of freedom is one of the most important problems. M-theory [1] defined in eleven dimensions is proposed as a unifying framework of string theory, where a membrane (an M2-brane) is believed to play an essential role like a fundamental string in string theory. In fact, if the eleventh dimension is compactified on a circle, an M2-brane wrapped in this direction is regarded as a fundamental string in ten dimensions. Therefore it is natural to expect that M-theory should be formulated in terms of a supermembrane as a fundamental object.

However, it is well-known that a quantum supermembrane theory has a serious problem that it has a continuous spectrum [2]. This means that it is unstable quantum mechanically. More precisely, if we formulate a supermembrane theory in eleven dimensions in the light-cone gauge, the action is reduced to a matrix quantum mechanics with the supersymmetric Yang-Mills type action, where the gauge group is the area preserving diffeomorphism on the world volume [3]. Since this action has eight scalar fields and the scalar potential is given in terms of their commutators, there are flat directions where the scalar fields have arbitrary large values as long as they are diagonal. It was shown
rigorously in [2] that the presence of these flat directions makes a spectrum of supermembrane continuous. This fact implies that a supermembrane is pathological quantum mechanically. As explained in [2], the physics behind this instability is as follows: a supermembrane has mass proportional to its 2-dimensional area. Hence it can have a long tube (spike) without cost of energy as long as it is sufficiently thin. Thus a supermembrane can emit a tube which becomes thinner and longer, and eventually by an entropic effect, a supermembrane becomes a spiky configuration in which multiple supermembranes are connected with each other by thin tubes. Namely, quantum mechanically a single supermembrane does not make sense and it is a multi-body problem in nature, which can be regarded as the origin of the continuous spectrum.

On the other hand, some years ago a low energy effective action of multiple M2-branes was proposed [4]. It is defined as three-dimensional $\mathcal{N} = 6$ superconformal $U(N) \times U(N)$ Chern-Simons theory with level $(k, -k)$. It is conjectured to describe $N$ M2-branes located at the fixed point of the $\mathbb{C}^4/\mathbb{Z}_k$ orbifold in the static gauge. If this conjecture is true, one of the most important issues would be whether the ABJM model resolves the instability of a supermembrane shown in [2]. Namely, if we believe the physical picture mentioned above behind the instability, the supermembrane should be also unstable in other gauges than the light-cone gauge. From this point of view, it is natural to expect that the $U(1) \times U(1)$ ABJM model would show the instability of a single M2-brane as well. Moreover the ABJM model is formulated in the background which is different from the flat space. Thus it is interesting to examine the instability of M2-brane in the context of the ABJM model.

Our idea is quite simple: we first construct a classical configuration both in the Nambu-Goto action of a supermembrane and in the $U(1) \times U(1)$ ABJM model, which represents a single M2-brane emitting another M2-brane. Fortunately it preserves a half of supersymmetries in both models. This BPS solution has a scale corresponding to the size of the configuration, thus it breaks the conformal symmetry in the ABJM model explicitly. Then we construct quadratic Lagrangian for fluctuations around it. If this Lagrangian has a flat direction which is associated with the deformation of the original configuration, it would be a sign of instability of the classical configuration itself against a small fluctuation. It is however worth noticing that such a perturbative result of the flat direction is not direct evidence of instability, but only its sign. A good example is provided by a bosonic membrane, where it has a flat direction and thus looks unstable at first sight, but it is known to be stable [2]. Therefore even if there exists a flat direction in the fluctuation Lagrangian, it would need careful analysis to confirm that a supermembrane is indeed unstable due to it by an entropic effect. Notice that in the present case we should not have tachyonic fluctuations because we are considering a BPS solution. Then instead of looking for a flat direction of the fluctuation Lagrangian explicitly, we construct a Hamiltonian from the Lagrangian and examine whether it has zero modes in both Nambu-Goto and ABJM. In both models the Hamiltonian is actually represented as a sum of squares, which makes identification of zero modes easier, where in the ABJM we do not have to take any gauge of $U(1) \times U(1)$ gauge symmetry. In fact, if we substitute the zero mode found in this way into the fluctuation Lagrangian, it vanishes up to boundary contributions. Thus the zero mode would give rise to a flat direction at least in a perturbative sense. Hence their existence can be considered as one of signs of instability of the membrane.
Moreover, in the ABJM model the zero mode we identify corresponds to a fluctuation which makes the configuration thin in accordance with the physical picture behind the instability of a supermembrane in the light-cone gauge mentioned above. In the ABJM model, the existence of such a zero mode is guaranteed by the fact that the BPS solution breaks the conformal symmetry. Here it should be noticed that as far as zero modes are concerned, the ABJM model should incorporate them correctly even if it is a low-energy effective theory of an M2-brane. We further confirm that the situation does not change in a nonlinear extension of the ABJM model proposed recently in [5], thus higher derivative corrections do not stabilize an M2-brane.

For comparison and in order to check the validity of our analysis, we also consider a spike solution which is again half-BPS in the D2-brane world volume theory [6,7]. Since it is regarded as a fundamental string ending on the D2-brane which has mass proportional to its length, the spectrum of fluctuations\(^1\) around it should not contain a zero mode corresponding to its deformation, which we check in the same way as in the ABJM case. In the D2-brane case, we find that a zero mode is not allowed if the string charge is fixed. On the other hand, in the supermembrane case, the zero mode exists even if we fix its winding number around the fixed point of the orbifold. Thus they show a clear difference in the existence of a zero mode under fixed corresponding charges. We regard such a sharp contrast as a manifestation of a difference in stability of a membrane and string.

We also clarify how both BPS solutions in the ABJM model and in the D2-brane world volume theory are connected by the novel Higgs mechanism [10–13].\(^2\) It is explicitly shown that the winding number of the BPS solution in the ABJM model around the fixed point of the orbifold are translated into the number of strings ending on the D2-brane in accordance with the M-theory interpretation. In fact, this observation plays a crucial role in the discussion of stability of a membrane and a string.

This paper is organized as follows: in the next section we consider the Nambu-Goto action of a supermembrane and construct a half-BPS solution. Then we show that fluctuations around it contain a zero mode which deforms the configuration in the case of the noncompact target space, but that this is not the case with the compact one. In section 3 we turn to the ABJM model and repeat the same argument to show that the half-BPS configuration also has a zero mode which indicates instability of an M2-brane perturbatively. In section 4 we construct a spike solution on a D2-brane which is identified as a fundamental string ending on the D2-brane, and show that it does not contain a zero mode if the string charge is fixed, which makes manifest a difference in stability of an M2-brane and a string. The final section is devoted to conclusions and discussions. In appendix A we give a half-BPS condition in the Dirac-Born-Infeld action for a D2-brane. In appendix B we discuss possible boundary contributions to the energy of the BPS solution.

\(^1\)Relations between fluctuations around the spike solution given in [6] and the boundary conditions the D-brane imposes on a string are discussed e.g. in [6,8,9].

\(^2\)Correspondence at the level of the half-BPS equations in both sides is demonstrated in [14].
2 Deformation of the Nambu-Goto supermembrane

Based on the physical interpretation for the instability of a supermembrane, we construct a BPS configuration in the Nambu-Goto action, and examine whether it allows deformations which make the configuration thinner. We start with a supermembrane in the flat noncompact 11-dimensional spacetime. Next we compactify the 11th direction on an $S^1$ and study a supermembrane winding around it.

2.1 Noncompact target space

The bosonic part of the action for a single supermembrane on the flat 11-dimensional spacetime is given by the Nambu-Goto action

$$S_{M2} = -T_{M2} \int d^3x \sqrt{- \det g_{\mu \nu}}$$

where each index takes values as $\mu, \nu = 0 \sim 2$, $M, N = 0 \sim 9, 11$ and $A = 1 \sim 4$. $x^\mu$ are the world volume coordinates and $X^M$ are the target ones. $\eta^{(11)}_{MN}$ and $\eta_{\mu \nu}$ are the flat Minkowski metric, diag($-1, 1, 1, \cdots$). In the third line we took the static gauge, $x^\mu = X^\mu$, and introduced the complex coordinates as

$$Y_A = \sqrt{\pi T_{M2}} (X^3 + iX^4), \quad Y^A = \sqrt{\pi T_{M2}} (X^5 + iX^6),$$

$$Y^3 = \sqrt{\pi T_{M2}} (X^7 + iX^8), \quad Y^4 = \sqrt{\pi T_{M2}} (X^9 + iX^{11}).$$

$Y_A$ are the complex conjugates of $Y^A$.

Since the supersymmetry plays a crucial role in the analysis of [2], we consider a configuration which preserves some supersymmetries. The residual supersymmetries preserved by the membrane on the flat background is given by [15]

$$\Gamma \epsilon = \epsilon, \quad \Gamma = \frac{1}{3!} \sqrt{- \det g_{\mu \nu}} \epsilon^{\mu \nu \rho} \partial_{\mu} X^M \partial_{\nu} X^N \partial_{\rho} X^P \Gamma_{MNP}.$$

Here $\Gamma_M$ are the 11-dimensional Dirac matrices which satisfy the relation $\{\Gamma_M, \Gamma_N\} = 2i \eta^{(11)}_{MN}$, and $\epsilon$ is a constant 32-component Majorana spinor. We consider the static configuration $\partial_0 Y^A = 0$ and further assume $Y^{1,2,3} = 0$. Then a holomorphic configuration $Y^4 = w(z)$, with $z = x^1 + ix^2$, or an anti-holomorphic one $Y^4 = \bar{w}(\bar{z})$ preserves the quarter of the 32 supersymmetries which are specified by $(1 - \Gamma_{012}) \epsilon = (1 - \Gamma_{09(11)}) \epsilon = 0$ or $(1 - \Gamma_{012}) \epsilon = (1 + \Gamma_{09(11)}) \epsilon = 0$, respectively. From the point of view of the world volume theory it is called the half-BPS. In the following we study the holomorphic case.

The holomorphic configuration $Y^4 = w(z)$ can be specified by the condition $f(w, z) = 0$ with some holomorphic function $f(w, z)$. We want a configuration which asymptotically
approaches a single flat membrane extended in the \((X^1, X^2)\)-plane in the limit \(|z| \to \infty\), while which is deformed around the origin. Based on this viewpoint, we assume that there is a unique zero of \(f(w, z)\) for any fixed value of \(z\). We also assume that the membrane does not go through the same point on the target space twice, namely we assume that for a fixed \(w\), \(f(w, z)\) has, if any, a unique zero. These requirements restrict the function \(f(w, z)\) to the form
\[ f(w, z) = awz + bw + cz + d. \]

The condition that the configuration is reduced to a flat membrane in the \((X^1, X^2)\)-plane in the limit \(|z| \to \infty\) also imposes \(a \neq 0\). Then by using the translation along the world volume and the target space, we take \(b = c = 0\). The rotation symmetry on the \((X^9, X^{11})\)-plane can be used to set the relative phase between \(a\) and \(d\) to be zero. After these considerations, the resulting configuration is given by
\[ Y^4 = w(z) = \frac{\alpha}{z}, \quad (2.6) \]
where \(\alpha\) is a real number.\(^3\)

Next we study fluctuations around the configuration. We consider that of the field \(Y^4\), assuming the rest to be zero, \(Y^{1,2,3} = 0\) for simplicity. For later convenience, we consider the fluctuation around a generic holomorphic configuration \(Y^4 = Y_{cl}(z)\) as \(Y^4 = Y_{cl}(z) + Y(x^0, z, \bar{z})\). By keeping only the quadratic terms, which is valid for small fluctuations, the action for the fluctuation becomes
\[ S_{M2}^{(2)} = -\frac{1}{2\pi} \int d^3x \left( -|\dot{Y}|^2 + 2|\bar{\partial}Y|^2 + \frac{2}{\pi T_{M2}} \frac{|\partial Y_{cl}|^2}{|\overline{\partial Y_{cl}}|^2} |\partial Y|^2 \right). \quad (2.7) \]

As explained in the introduction, we study a zero mode of the Hamiltonian since it would give rise to a flat direction, and can be a sign for the instability of the solution (2.6).

The conjugate momenta are given by
\[ P_Y = \frac{1}{2\pi} \dot{Y}^\dagger, \quad P_{Y^\dagger} = \frac{1}{2\pi} \dot{Y}, \quad (2.8) \]
and the Hamiltonian following from (2.7) is
\[ H = 2\pi P_Y P_{Y^\dagger} + \frac{2T_{M2}}{\pi T_{M2} + |\partial Y_{cl}|^2} |\bar{\partial}Y|^2. \quad (2.9) \]

In the final expression (2.9), we neglected total derivatives. However, this does not necessarily mean that they actually vanish, because we will allow a singular fluctuation \(Y\) as shown in a moment. In order to take account of total derivatives for a singular configuration, we would have ambiguities associated with regularizations and possible boundary terms. Hence we concentrate on the bulk contribution to the Hamiltonian. For some concrete consideration of boundary terms, see appendix B. (2.9) shows that the quadratic Hamiltonian vanishes for any static holomorphic fluctuation \(Y = Y(z)\). Hence it can be a candidate for the marginal deformation. We further argue that possible deformations should be no more singular than the original configuration \(Y_{cl}(z)\).\(^4\) In the case of the

\(^3\)In [16] this kind of analysis is applied to derive a different BPS configuration.

\(^4\)Another criterion for allowed fluctuations around a singular solution is found, e.g. in [17].
configuration \( Y_{cl}(z) = \alpha/z \), these requirements allow only the fluctuation of the overall factor, namely \( Y^4 = (\alpha + \delta\alpha)/z \). Then it is a candidate for the marginal deformation which destabilize the original configuration.\(^5\)

Let us clarify the relation between the above analysis and the stability analysis of [2]. First of all the solution (2.6) is not like a \textit{“spike”} in the sense that in a small \(|z|\) region it becomes a flat 2-dimensional plane extended in the transverse \( X^9, X^{11} \) directions. In the above analysis, the emphasis is placed not on the shape of the classical configuration but on the existence of its marginal deformation. In fact, the fluctuation of the overall constant do change the form of the membrane into the one which is thinner compared to the original one. (2.6) shows that under fixed \(|Y^4|\) if we reduce \( \alpha \), so does \(|z|\). This means that the configuration becomes thinner by the change of the overall constant. This is a reminiscent of the marginal deformation discussed in [2]. Notice that we can describe such a deformation through \( Y^4 \) as a result of the static gauge. In the analysis based on the ABJM model, in the following section, a solution similar to (2.6) will be discussed, which represents a spike-like configuration. Also one should notice that the existence of the marginal deformation in the above analysis does not immediately mean the instability. In order to judge whether the configuration is stable or not, we need to go beyond perturbative analysis and see whether or not the flat direction is lifted by quantum effect. Such an interesting and important analysis is beyond the scope of the present paper; we concentrate on the existence of the flat direction at the quadratic level.

\subsection*{2.2 Compact target space}

Next we consider the case where the 11th direction \( X^{11} \) is compactified on a circle and a supermembrane is wrapped around it. In this case, since the compact direction has a finite size, the membrane can not shrink to an arbitrary thin configuration. Then one may expect that there are no marginal deformations. We will see how this intuitive expectation is realized.

Let us compactify the 11th direction with the period \( 2\pi R \), \( X^{11} \sim X^{11} + 2\pi R \). Then a holomorphic configuration \( Y^4 = w(z) \) which is consistent with the compactification is specified by introducing the variable \( t = \exp(\frac{w}{\sqrt{\pi}\text{MSR}}) \) and require the condition \( f(t, z) = 0 \). We consider a membrane solution which is winding once around the \( S^1 \) direction. This requires the function \( f(t, z) \) to be linear in \( z \). Also we assume that there is only a unique zero of \( f(t, z) \) for a fixed value of \( z \), which seems to be natural for the spike-like configuration. Then again we have \( f(t, z) = atz + bt + cz + d \). We assume that \(|z| \to \infty \) corresponds to \( X^9 \to -\infty \). This results in the condition \( a \neq 0 \) and \( c = 0 \). Next by using the translation along the world volume direction and target space, we can set \( b = 0 \) and \( a + d = 0 \), respectively. Then, we obtain the following BPS configuration:

\[ t = \frac{1}{z}. \] (2.10)

\(^5\)Notice that the zero modes originating from the global symmetries like translations and rotations which are broken by the configuration \( Y = Y_{cl}(z) \) are also given by the holomorphic functions. Some of these are more singular than \( Y_{cl} \) at the origin and others are not. In any case we expect that these are not related to the instability of the supermembrane.
In terms of the original coordinates \((X^9, X^{11})\) and the polar coordinate on the world volume, \((x^1, x^2) = (r \cos \theta, r \sin \theta)\), it is expressed as

\[
X^9 = -R \log r, \quad (2.11)
\]

\[
X^{11} = -R \theta \quad \text{(mod } 2\pi R). \quad (2.12)
\]

The fluctuation analysis around this configuration is exactly the same as in the previous subsection. By requiring that the possible fluctuation should be no more singular than the original one, in particular both at \(|z| \to 0\) and at \(|z| \to \infty\), we find that the only possible candidate is the fluctuation of the overall constant, namely \(R \to R + \delta R\). However, such fluctuation conflicts with the periodicity of the target space and hence it is forbidden. This result agrees with the physical intuition explained at the beginning of this subsection.

From the viewpoint of the 10-dimensional string theory, the winding M2-brane solution considered in this section is interpreted as a fundamental string ending on a D2-brane. We will study the corresponding system in section 4.

3 Winding solution and fluctuation around it in the ABJM model

In the previous section we studied the fluctuation around the BPS configuration of a single membrane on the flat target spacetime. In this section we perform a similar analysis on the \(C^4/Z_k\) orbifold background based on the \(U(1) \times U(1)\) ABJM model.

3.1 \(U(1) \times U(1)\) ABJM model

We follow the convention of [18], in which the action of the \(U(1) \times U(1)\) ABJM model is given by

\[
S_{\text{ABJM}} = \frac{k}{2\pi} \int d^3 x \left( -D^\mu Y^A D_\mu Y_A + i \bar{\Psi}_A \gamma^\mu D_\mu \Psi_A + \frac{1}{2} \epsilon^{\mu \nu \lambda} (A^{(1)}_\mu \partial_\nu A^{(1)}_\lambda - A^{(2)}_\mu \partial_\nu A^{(2)}_\lambda) \right). \quad (3.1)
\]

Here \(Y^A\) and \(Y_A\) \((A = 1 \sim 4)\) are the complex scalars and their complex conjugates. The Dirac fermions \(\Psi^A\) and \(\Psi_A\) are also the complex conjugates of each other. The lower (upper) index \(A\) are the 4 (4) representation of the SU(4) R-symmetry. The two \(U(1)\) gauge fields \(A^{(1)}_\mu\) and \(A^{(2)}_\mu\) are the Chern-Simons gauge fields with level \(k\) and \(-k\), respectively. The fields \(Y_A\) and \(\Psi^A\) are the bi-fundamental representation of the \(U(1) \times U(1)\) gauge group with charge \((+, -)\). Then the covariant derivatives for the scalars are defined as

\[
D_\mu Y^A = \partial_\mu Y^A - i (A^{(1)}_\mu - A^{(2)}_\mu) Y^A, \quad (3.2)
\]

\[
D_\mu Y_A = \partial_\mu Y_A + i (A^{(1)}_\mu - A^{(2)}_\mu) Y_A. \quad (3.3)
\]
Since the scalars and fermions are coupled only to the specific combination of the gauge fields, it is sometimes convenient to introduce the notation $A_{\mu}^{\pm} \equiv A_{\mu}^{(1)} \pm A_{\mu}^{(2)}$. With this, the Chern-Simons term is written as

$$\frac{1}{2} \epsilon^{\mu \nu \rho} (A_{\mu}^{(1)} \partial_{\nu} A_{\rho}^{(1)} - A_{\mu}^{(2)} \partial_{\nu} A_{\rho}^{(2)}) = \frac{1}{2} \epsilon^{\mu \nu \rho} A_{\mu}^{+} \partial_{\nu} A_{\rho}^{-} + \text{(total derivatives)}. \quad (3.4)$$

This model is proposed as the IR limit of the world volume theory on a single M2-brane on the fixed point of the $C^4/Z_k$ orbifold. The three directions of the target space are supposed to be identified with the world volume coordinates, $x^0$, $x^1$ and $x^2$ by taking the static gauge. Then the remaining eight target space coordinates are combined into four complex scalars as

$$Y^1 = \sqrt{\frac{\pi T_{M2}}{k}} (X^3 + iX^4), \quad Y^2 = \sqrt{\frac{\pi T_{M2}}{k}} (X^5 + iX^6),$$

$$Y^3 = \sqrt{\frac{\pi T_{M2}}{k}} (X^7 + iX^8), \quad Y^4 = \sqrt{\frac{\pi T_{M2}}{k}} (X^9 + iX^{11}), \quad (3.5)$$

where the normalization is fixed by comparing quadratic part of the Nambu-Goto action (2.1) and the ABJM model (3.1). The $Z_k$ orbifold identification is defined as $Y^A \sim e^{2\pi i k} Y^A$.

### 3.2 Winding BPS M2-brane

Let us consider an M2-brane solution which is winding around the orbifolding direction. As in the previous section, we set $Y^{1,2,3} = 0$. Then the BPS conditions coming from the requirement that the SUSY transformation of the gaugino fields $\Psi_A$ should vanish become

$$\delta \Psi_1 = \gamma^\mu (i \varepsilon^2 + \varepsilon^5) D_\mu Y^4 = 0, \quad (3.6)$$

$$\delta \Psi_2 = \gamma^\mu (\varepsilon^1 + i \varepsilon^3) D_\mu Y^4 = 0, \quad (3.7)$$

$$\delta \Psi_3 = \gamma^\mu (i \varepsilon^4 - \varepsilon^6) D_\mu Y^4 = 0, \quad (3.8)$$

where $\varepsilon^1, \ldots, \varepsilon^6$ are all two component Majorana spinors in three dimensions. The condition coming from $\delta \Psi_4 = 0$ is satisfied trivially. Since these conditions can be summarized as $\gamma^\mu D_\mu Y^4 \varepsilon = 0$ with some spinor $\varepsilon$, the necessary condition is given by\(^6\)

$$\det(\gamma^\mu D_\mu Y^4) = \det \left( \begin{array}{cc} D_2 Y^4 & D_0 Y^4 + D_1 Y^4 \\ -D_0 Y^4 + D_1 Y^4 & -D_2 Y^4 \end{array} \right)$$

$$\quad = (D_0 Y^4)^2 - (D_1 Y^4)^2 - (D_2 Y^4)^2$$

$$\quad = 0. \quad (3.9)$$

It is easy to check that this is also the sufficient condition for the half-BPS configuration. We further assume the static solution $\partial_0 Y^4 = 0$ and also $A_{\mu}^- = 0$. The latter assumption is consistent with the equation of motion. Then the BPS condition is simplified as \(^5\)

$$(\partial_1 + i \partial_2) Y^4 = 0, \quad \text{or} \quad (\partial_1 - i \partial_2) Y^4 = 0, \quad (3.10)$$

\(^6\)We have used $\gamma^0 = i\sigma^2$, $\gamma^1 = \sigma^1$, $\gamma^2 = \sigma^3$, where $\sigma^a$ are the Pauli matrices.
namely the configuration $Y^4 = Y^4(z, \bar{z})$ is a half-BPS solution if it is either holomorphic or anti-holomorphic. In the rest of the paper, we consider the holomorphic case.

We specify the holomorphic function in a similar manner as in the previous section. A general form of the holomorphic function $Y^4 = Y^4(z)$ which is consistent with the orbifolding identification $Y^4 \sim e^{2\pi i Y^4}$ can be studied by introducing the coordinate $t = (Y^4)^k$ and impose the condition $f(t, z) = 0$. First we consider a configuration which is winding once around the orbifold direction. By assuming also a unique zero of $f(t, z)$ for a fixed value of $z$, the form of $f(t, z)$ is restricted to $f(t, z) = atz + bt + cz + d$. The coefficient $a$ is again not equal to zero for a configuration which is reduced to a flat membrane in the $(X^1, X^2)$-plane in the limit $|z| \to \infty$. The translational symmetry along the world volume direction is used to set $b = 0$. Since the solution with winding number one satisfies $Y^4(e^{2\pi i z}) = e^{\pm \frac{2\pi i}{k} Y^4(z)}$, we impose $c = 0$. Then finally we end up with $f(t, z) = atz + d = 0$, namely $Y^4 \propto z^{-\frac{1}{k}}$. Now it is easy to generalize the configuration to an M2-brane with winding number $n$ as

$$Y^4 = \frac{\alpha}{z^\frac{1}{k}}.$$  \hspace{1cm} (3.11)

This configuration satisfies the winding property $Y^4(e^{2\pi i z}) = e^{-\frac{2\pi i n}{k}} Y^4(z)$. The overall constant $\alpha$ can be taken to be real by using the phase of the $Y^4$ coordinate, namely the rotation in the $(X^9, X^{11})$-plane. The equation of motion coming from the variation of (3.1) with respect to $A^-_{\mu}$ fixes the non-dynamical field $A^+_{\mu}$ in terms of the solution (3.11) as

$$A^+_0 = 2Y^4 + c, \quad A^+_1 = A^+_2 = 0.$$  \hspace{1cm} (3.12)

The integration constant $c$ is introduced for the later convenience.

For large $k$, the orbifolding identification effectively compactifies the $X^{11}$-direction and the above solution becomes much like a spike configuration extended in the $X^9$-direction. In fact, as we will see in the next section, the winding solution is reduced to the D2-brane with a spike corresponding to the fundamental string.

It is easy to check that the spike-like configuration (3.11) has the membrane tension $T_{M2}$, which trivially follows from the normalization in (3.5). Thus (3.11) can be regarded as a deformation of the membrane itself. This is quite different from the situation in the spike solution on a D-brane discussed in [6], where it is regarded not as the D-brane itself but as a fundamental string. Namely, our solution becomes massless when its thickness is zero, while the spike solution given in [6] still has a finite string tension even if it is sufficiently thin. We note here that in the latter the nontrivial gauge field configuration plays an essential role. We also notice that our normalization (3.5) coincides with that adopted in [19] because we fix it essentially in the same way. What is important here is that under this normalization the one-loop effective action in the ABJM model also exactly agrees with the result of the supergravity computation up to the first nontrivial order of $v^4$ terms as shown in [19].
3.3 Fluctuations around BPS configuration

We study fluctuations around the winding M2-brane configuration (3.11). As in the case of the Nambu-Goto action, we concentrate on the fluctuation of the field $Y^4$ with taking $Y^1, Y^2, Y^3 = 0$, while in the present case there exist also the fluctuations of the gauge fields

$$Y^4 = Y_{cl} + Y, \quad Y_{cl} = Y_{cl}^+ + Y^+,$$

$$A_\mu^- = 0 + a_\mu^-, \quad A_\mu^+ = A_{cl, \mu}^+ + a_\mu^+. \quad (3.13)$$

By inserting (3.13)–(3.15) into (3.1), we obtain the quadratic part of the action for the fluctuations as

$$\mathcal{L} = \frac{k}{2\pi} \left\{ - \partial^\mu Y \partial_\mu Y^+ + i \left( Y \partial^\mu Y_{cl}^+ - Y^+ \partial^\mu Y_{cl} + Y_{cl} \partial_\mu Y^+ - Y_{cl}^+ \partial_\mu Y \right) a^-_\mu \\
- Y_{cl}^+ Y_{cl} a^-_\mu + \frac{1}{2} \epsilon^{\mu\nu\rho} a_\mu^+ \partial_\nu a_\rho \right\}, \quad (3.16)$$

where this action is valid when $Y$ is small compared to $Y_{cl}$ or $k$ is large. The latter condition accords with the novel Higgs mechanism discussed in subsection 4.2. The conjugate momenta are

$$P_Y = \frac{\partial \mathcal{L}}{\partial \dot{Y}} = \frac{k}{2\pi} (\dot{Y}^+ - i Y_{cl}^- a^-_0), \quad (3.17)$$

$$P_{Y^+} = \frac{\partial \mathcal{L}}{\partial \dot{Y}^+} = \frac{k}{2\pi} (\dot{Y} + i Y_{cl} a^-_0), \quad (3.18)$$

$$P_{a^-} = \frac{\partial \mathcal{L}}{\partial \dot{a}^-_\mu} = \frac{k}{2\pi} \frac{1}{2} \epsilon^0_{\mu\nu} a^+_{\nu}, \quad (3.19)$$

and the canonical Hamiltonian for the fluctuations is given by

$$\mathcal{H} = P_Y \dot{Y} + P_{Y^+} \dot{Y}^+ + P_{a^-} \dot{a}^-_\mu - \mathcal{L}$$

$$= \frac{k}{2\pi} \left\{ \left( \frac{2\pi}{k} \right)^2 P_{Y^+} P_Y - i \left( Y \partial^\mu Y_{cl}^+ - Y^+ \partial^\mu Y_{cl} + Y_{cl} \partial_\mu Y^+ - Y_{cl}^+ \partial_\mu Y \right) a^-_\mu \\
+ Y_{cl}^+ Y_{cl} \left( -2(a^-_0)^2 + (a^-_i)^2 \right) + \partial_i Y^+ \partial_i Y - \frac{1}{2} \epsilon^{\mu\nu\rho} a^+_{\nu} \partial_\rho a^-_\mu \right\}. \quad (3.20)$$

Now we want to find a zero mode by rewriting the Hamiltonian in terms of sum of squares. For this purpose we first note that if the gauge fields $a^\pm_\mu$ are integrated out in (3.16) separately, they amount to imposing conditions

$$0 = i \left( Y \partial^\mu Y_{cl}^+ - Y^+ \partial^\mu Y_{cl} + Y_{cl} \partial_\mu Y^+ - Y_{cl}^+ \partial_\mu Y \right) - 2 Y_{cl}^+ Y_{cl} a^-_\mu + \frac{1}{2} \epsilon^{\mu\nu\lambda} \partial_\nu a^+_{\lambda}, \quad (3.21)$$

$$0 = -\frac{1}{2} \epsilon^{\mu\nu\lambda} \partial_\nu a^-_{\lambda}. \quad (3.22)$$
This suggests that it would be sufficient to look for a zero mode under these conditions. Then omitting total derivative terms, we have

$$
\mathcal{H} = \frac{k}{2\pi} \left\{ \left( \frac{2\pi}{k} \right)^2 |P_Y|^2 - i(Y \partial_i Y^\dagger_{cl} - Y^\dagger \partial_i Y_{cl} + Y_{cl} \partial_i Y^\dagger - Y^\dagger_{cl} \partial_i Y) a_i^- + |Y_{cl}|^2 (a_i^-)^2 + |\partial_i Y|^2 \right\}.
$$

(3.23)

We can bring it to a sum of square terms by using that $Y_{cl}$ is the holomorphic function of $z = x^1 + ix^2$. Indeed the problematic second term can be rewritten as

$$
-i(Y \partial_i Y^\dagger_{cl} - Y^\dagger \partial_i Y_{cl} + Y_{cl} \partial_i Y^\dagger - Y^\dagger_{cl} \partial_i Y) a_i^- = 4i\{(\partial_z Y) Y^\dagger_{cl} a_z^- - (\partial_z Y^\dagger) Y_{cl} a_z^-\} + \text{(total derivative)}.
$$

(3.24)

Here, in addition to the holomorphy $\partial_z Y_{cl} = 0$, we have also utilized the constraint $\partial_1 a_2^- = \partial_2 a_1^- = 0$. Omitting the total derivative terms, the Hamiltonian becomes the following form

$$
\mathcal{H} = \frac{2\pi}{k} P_{Y^\dagger} P_Y + \frac{2k}{\pi} (\partial_z Y^\dagger + i a_z^+ Y^\dagger_{cl})(\partial_z Y - i a_z^+ Y_{cl})).
$$

(3.25)

This clearly shows that the Hamiltonian is semi-positive definite for the BPS configuration $\partial_z Y_{cl} = 0$. The flat direction can be specified by

$$
\partial_z Y - i a_z^+ Y_{cl} = 0, \quad P_{Y^\dagger} = \frac{k}{2\pi} (\dot{Y} - i Y_{cl} a_0^-) = 0,
$$

(3.26)

with the conditions (3.21) and (3.22). Now it is clear that the static holomorphic function $Y = Y(z)$ with vanishing gauge fields $a_\mu^- = 0$ satisfies these equations. The conditions (3.22) are trivially satisfied and by solving (3.21), $a_\mu^+$ is determined. The solution of $Y$ in (3.26) under (3.22) is shown to satisfy the equation of motion. For large $k$ and fixed $n$, the fluctuation $Y(z)$ should be no more singular than $Y_{cl}(z)$, in particular both at the origin and at the infinity. A generic form of such holomorphic functions which is consistent with the orbifolding condition $Y^\dagger (e^{2\pi i} z) = e^{-2\pi^2 i/\kappa} Y^\dagger(z)$ is given by $Y(z) = \Sigma_m c_m z^{-\frac{2}{n}} z^m$. As a result we can single out a possible fluctuation as $Y(z) = z^{-\frac{2}{n}}$. This means the fluctuation of the overall factor is again the possible marginal deformation, $Y^\dagger = (\alpha + \delta \alpha) z^{-\frac{2}{n}}$.

Remembering that the ABJM model is the conformal field theory, the origin of the flat direction is clear. It originates from the scale invariance which is now broken by the classical solution $Y_{cl}(z)$. We may expect the existence of such a flat direction is rather robust. It is also natural that this fluctuation makes the configuration thinner as it does. This shows a contrast with the case of the zero modes originating from the translation and rotation where they do not deform the classical configuration.

### 3.4 Nonlinear ABJM

Since our solution is singular at $|z| = 0$, the argument based on the ABJM model might be modified if we take account of the nonlinear contributions. In fact, a nonlinear extension of the ABJM model is proposed in [5]. We address effect of the nonlinearity on the analysis of the previous section.
The bosonic part of the nonlinear action proposed in \cite{5} is given by
\[ S_{\text{NL-ABJM}} = S_{\text{NG}} + S_{\text{CS}}. \] (3.27)

Here \( S_{\text{NG}} \) is given by
\[ S_{\text{NG}} = -T_{M2} \int d^3x \sqrt{- \det \left( \eta_{\mu\nu} + \frac{k}{2\pi T_{M2}} (D_\mu Y^A D_\nu Y_A + D_\nu Y^A D_\mu Y_A) \right)}, \] (3.28)

where the covariant derivatives in (3.28) are the same as those in the linear ABJM model. The second term \( S_{\text{CS}} \) is the same Chern-Simons term as in (3.1). If we set \( Y^{1,2,3} = 0 \), \( S_{\text{NG}} \) is simplified as
\[ S_{\text{NG}} = -T_{M2} \int d^3x \sqrt{1 + \frac{k}{\pi T_{M2}} |D_\mu Y^4|^2 - \left( \frac{k}{2\pi T_{M2}} \right)^2 (|D_\mu Y^4|^2 - |D_\mu Y^4|^4)}. \] (3.29)

By considering the fluctuation as (3.14) and (3.15) with holomorphic \( Y_{cl} \), the quadratic action for the fluctuation is given by
\[ \mathcal{L} = -\frac{k}{2\pi} \left\{ |\partial_\mu Y|^2 + |Y_{cl}|^2 (a^-_\mu)^2 - i(Y \partial^\mu Y^\dagger_{cl} - Y^\dagger_{cl} \partial^\mu Y - (c.c.)) a^-_\mu - \frac{1}{2} \epsilon^{\mu\nu\rho} a^+_\mu \partial_\nu a^-_\rho \right. \]
\[ - \left. \frac{2k}{2\pi T_{M2} + k|\partial_j Y_{cl}|^2} |\partial_j Y_{cl} (\partial_j Y - ia^-_j Y_{cl})|^2 \right\}. \] (3.30)

Repeating a similar analysis from (3.16) to (3.25), we obtain the following Hamiltonian:
\[ \mathcal{H} = \frac{2\pi}{k} P_Y P_{Y^\dagger} + \frac{k}{2\pi \pi T_{M2}} \frac{4\pi T_{M2}}{k|\partial_j Y_{cl}|^2} (\partial_j Y^\dagger + iY^\dagger_{cl} a^-_j) (\partial_j Y - iY_{cl} a^-_j). \] (3.31)

It is clear that the zero mode analysis is not affected by the nonlinear extension of the ABJM model.

4 Fundamental string attached to D2-brane

In the previous section, we studied the zero mode around a winding M2-brane in the ABJM model. In this section we examine that around a fundamental string for comparison. We consider a bunch of fundamental strings ending on a D2-brane which is represented by a spike solution of the Dirac-Born-Infeld (DBI) action \cite{6}. In subsection 4.2 we will show that this spike D2-brane is actually related to the winding M2-brane (3.11) through the novel Higgs mechanism. Here we see that the winding number of the M2-brane around the orbifold corresponds to the number of the fundamental strings. This observation plays a crucial role in the later discussion on the quantum fluctuation around the spike solution. In fact, we find a sharp difference between the zero mode analysis for an M2-brane and a fundamental string. Notice that the fluctuation analysis in the present section should have implications on the stability of not a D2-brane but a fundamental string.
4.1 Spike solution

We begin with a review of the spike solution derived in [6] in the D2-brane case. This spike represents a bunch of fundamental strings ending on the D2-brane and it should satisfy the charge quantization condition corresponding to the string charge, namely the number of the strings.

The D2-brane world volume theory is given by the U(1) DBI action

\[ S_{D2} = -T_{D2} \int d^3 x \sqrt{-\det (g_{\mu\nu} + 2\pi\alpha' F_{\mu\nu})} \]

where the tension of the D2-brane is defined by

\[ T_{D2} = \frac{2\pi}{g_s - \frac{3}{2}} \]

and \( g_s \) is the string coupling constant. \( g_{\mu\nu} \) and \( F_{\mu\nu} \) are the induced metric and the U(1) gauge field strength, respectively. \( \mu, \nu = 0, 1, 2 \) are world volume indices. In the second line, we took the static gauge and \( X^i (i = 3, \ldots, 9) \) are the transverse target coordinates.

The spike solution in [6] has a nontrivial configuration only for one of the scalars, say \( X^9 \), and also the gauge potential \( A_0 \). Since the solution corresponds to D2-F1 system which preserves the eight supersymmetry, its derivation is based on the study of the BPS condition. As we explain in the appendix A, if we assume the \( S^1 \) symmetric static ansatz, \( X^9 = X^9(r) \) and \( F_{0r} = F_{0r}(r) \), then the configuration preserves half of the supersymmetry provided it satisfies the condition \((\partial_r X^9)^2 = (2\pi\alpha' F_{0r})^2\). The specific solution which indeed represents a fundamental string ending on a D2-brane is given by

\[ X^9 = 2\pi\alpha' A_0 = c \log \frac{r}{\ell}, \]

where \( c \) and \( \ell \) are dimensionful constants.

The overall constant \( c \) needs to be tuned so that (4.3) represents fundamental strings. Moreover, \( c \) is quantized corresponding to the string charge. In order to find the correct quantization, we study how the spike solution is coupled to the NS-NS \( B \) field. In the case of the \( n \) fundamental strings extended into \( X^9 \) direction, the string world sheet is coupled to the \( B \) field as

\[ n T_F \int dX^0 dX^9 B_{09}, \]

where \( T_F = \frac{1}{(2\pi\alpha')} \) and we identified \( X^0 \) and \( X^9 \) with the world volume coordinates. Next we consider the case of the above spike D2-brane. Let us introduce the notation

\[ S_{D2} = -T_{D2} \int dt dr d\theta r \sqrt{-\det (g_{\mu\nu} + B_{\mu\nu} + 2\pi\alpha' F_{\mu\nu})} = \int dX^0 dX^9 \mathcal{L}, \]

where \( B_{\mu\nu} \) is the \( B \) field induced on the world volume. We assumed \( S^1 \) symmetric ansatz \( dX^9 = dr \partial_r X^9 \). Then in order for the spike solution to represent the \( n \) fundamental strings, the differential of \( \mathcal{L} \) with respect to \( B_{09} \) should be equal to the string charge\(^7\)

\[ n = T_F^{-1} \frac{\partial \mathcal{L}}{\partial B_{09}} \bigg|_{B=0} = \partial_r X^9(r) \frac{\partial \mathcal{L}}{\partial F_{0r}} \bigg|_{B=0} = (2\pi\alpha')^2 T_{D2} \int_{S^1} F_{0r}, \]

\(^7\)The resulting flux quantization condition is the same as the one given in [6].
where in the last equality we used the BPS condition \((\partial_r X^9(r))^2 = (2\pi\alpha' F_{0r}(r))^2\). For the spike solution (4.3), this means the following quantization of the factor \(c^8\)

\[
n = -(2\pi)^2 \alpha' T_{D2} c \quad \rightarrow \quad c = -ng_s \ell_s.
\] (4.7)

Consequently, the half BPS solution of the DBI action representing \(n\) fundamental strings is

\[
X^9 = 2\pi \alpha' A_0 = -ng_s \ell_s \log \frac{r}{\ell}.
\] (4.8)

Since the combination \(R = g_s \ell_s\) is the radius of the M-theory circle the solution corresponds to the M2-brane solution winding around the M-theory circle which is discussed in 2.2.

Finally, we calculate the energy of the spike to confirm that it can be identified with a fundamental string. From the DBI action (4.2) its energy\(^9\) is

\[
E = T_{D2} \int_{\delta}^{\Lambda} dr d\theta r \left( \frac{ng_s \ell_s}{r} \right)^2 = n T_F \left[ X^9(r = \delta) - X^9(r = \Lambda) \right],
\] (4.9)

where we have introduced cutoffs to regularize divergences both at the origin and infinity of \(r\). The energy of the spike (4.9) tells us that \(n\) is in fact interpreted as the number of fundamental strings, because the energy of the spike with \(n = 1\) is exactly the tension of a fundamental string times its length.

### 4.2 Novel Higgs mechanism

In this subsection we clarify how the spike solution with string (electric) charge \(n\) discussed in the previous subsection is related to the half-BPS solution in the ABJM model (3.11), (3.12) via the novel Higgs mechanism.

#### 4.2.1 A brief review

Let us begin with a brief review of the novel Higgs mechanism in the case of \(U(1) \times U(1)\) ABJM model. For simplicity, we consider only the bosonic part of the ABJM model

\[
S = \frac{k}{2\pi} \int d^3 x \left( -D^\mu Y^A D_\mu Y_A + \frac{1}{2} \epsilon^{\mu\nu\rho} A^+_\mu \partial_\nu A^-_\rho \right).
\] (4.10)

We assume a vacuum expectation value \(v\) for the real component of \(\sqrt{k/(2\pi)} Y^4\) and rename the fields as

\[
\sqrt{\frac{k}{2\pi}} Y^a = \sqrt{\frac{T_{M2}}{2}} (\widetilde{X}^{2a+1} + i \widetilde{X}^{2a+2}) \quad (a = 1, 2, 3),
\] (4.11)

\(^8\)Here the sign is not important because it depends on the orientation of the string and the D2-brane.

\(^9\)Note that in order to obtain the energy of the spike, we have to subtract the energy of the D2-brane itself from the Hamiltonian evaluated with the spike solution.
\[ \sqrt{\frac{k}{2\pi}} Y^4 = v + \sqrt{\frac{T_{M2}}{2}} (\tilde{X}^9 + i\tilde{X}^{11}), \]  
(4.12)

\[ A^-_\mu = \frac{B_\mu}{v}, \quad A^+_\mu = 2A_\mu. \]  
(4.13)

Here \( \tilde{X}^i \) is related to the target space coordinate \( X^i \) by the shift as \( X^i = \sqrt{2/T_{M2}v}\delta^9 + \tilde{X}^i \).

Then by taking large \( k \) and \( v \) limit with keeping \( \frac{v}{k} \) finite,\(^{10} \) the action becomes

\[ S = \int d^3x \left\{ -\frac{T_{M2}}{2} (\partial^\mu \tilde{X}^i \partial_\mu \tilde{X}^i + \partial^\mu \tilde{X}^{11} \partial_\mu \tilde{X}^{11}) - B^\mu B_\mu \\
+ \sqrt{2T_{M2}} \partial^\mu \tilde{X}^{11} B_\mu + \frac{k}{2\pi v} \epsilon^{\mu\nu\rho} A_\mu \partial_\nu B_\rho \right\}, \]  
(4.14)

where the summation over \( i \) is taken for \( i = 3 \sim 9 \).

Next, by using the equation of motion for \( B_\mu \)

\[ B_\mu = \sqrt{T_{M2}/2} \partial_\mu \tilde{X}^{11} + \frac{k}{4\pi v} \epsilon^{\mu\nu\rho} \partial_\nu A_\rho, \]  
(4.15)

and neglecting the boundary terms, the action becomes

\[ S = -\int d^3x \left( \frac{T_{M2}}{2} \partial^\mu \tilde{X}^i \partial_\mu \tilde{X}^i + \frac{1}{2} \left( \frac{k}{4\pi v} \right)^2 F_{\mu\nu} F^{\mu\nu} \right). \]  
(4.16)

Now recall that around \( X^9 \sim \sqrt{2/T_{M2}v} \), the radius of the M-theory circle is given by \( R = \sqrt{\frac{2}{T_{M2}k}} \times [10–13] \). Then by using the relation

\[ g_s = (R/\ell_P)^{3/2}, \quad \alpha' = R^{-1} \ell_P^3, \]  
(4.17)

and the definition of the membrane tension \( T_{M2} = (2\pi)^{-2} \ell_P^{-3} \), (4.16) becomes

\[ S = -\frac{\ell_s}{g_s} \int d^3x \left( \frac{1}{2} \partial^\mu \Phi^i \partial_\mu \Phi^i + \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right), \]  
(4.18)

where the scalar field \( \Phi^i \) is defined in the standard manner by

\[ \Phi^i = \frac{1}{2\pi \alpha'} \tilde{X}^i, \]  
(4.19)

and the overall factor \( \ell_s/g_s \) should be identified with the three-dimensional Yang-Mills coupling constant as \( \ell_s/g_s = 1/g_{YM}^2 \).

---

\(^{10}\) As mentioned just before the equation (4.17), the scale of the ratio \( \frac{v}{k} \) is given by \( \frac{v}{k} = R \sqrt{\frac{T_{M2}}{2}} \sim (\frac{g}{\ell_P})^{\frac{3}{2}} \).

Hence the condition that \( \frac{v}{k} \) is kept finite means that we take the scale \( \frac{g}{\ell_P} \) to be finite and that in this unit all dimensionful quantities are described.
4.2.2 Novel Higgs mechanism for the winding solution

Next we consider the novel Higgs mechanism for the M2-brane solution with winding number $n$, (3.11) and (3.12),

$$
Y^A = 0 \quad (A = 1 \sim 3), \quad A^- = 0,
Y^4 = Y_{cl} = \alpha z^{-\frac{2}{k}},
A^+_0 = 2Y_{cl}Y_{cl}^\dagger + c, \quad A^+_1 = A^+_2 = 0.
$$

(4.20)

Namely, we interpret it in terms of the D2-brane world volume fields $\Phi^i$ and $A_\mu$ in the limit $k \to \infty$. In the limit, $\sqrt{k/(2\pi)}Y_{cl}$ is expanded as

$$
\sqrt{\frac{k}{2\pi}}Y_{cl}(z) = \sqrt{\frac{k}{2\pi}}\alpha \left( 1 - \frac{n}{k} \log r - i\frac{n}{k} \theta + O(k^{-2}) \right).
$$

(4.21)

The novel Higgs mechanism tells us that if the real part of $\sqrt{k/(2\pi)}Y^4$ develops the large constant expectation value $v (\sim k)$, the remaining $O(1)$ part of it can be identified as the scalar field $\Phi^9$ on the D2-brane world volume. Therefore we consider the large $\alpha$ and identify it as

$$
\sqrt{\frac{k}{2\pi}}\alpha = v \to \infty.
$$

(4.22)

Then the remaining part of (4.21) can be regarded as the classical configuration for the scalar field $\Phi^9$ and the compact direction as

$$
\tilde{X}^9 \to -ng_s \ell_s \log r, \quad \tilde{X}^{11} \to -ng_s \ell_s \theta,
$$

(4.23)

where we used $\frac{2}{k} = \sqrt{\frac{T_{M2}}{2R}}$. Similarly the D2-brane gauge field $A_\mu$ can be deduced from (4.13) and (4.20) as

$$
A_0 = Y_{cl}Y_{cl}^\dagger + \frac{1}{2}c = \alpha^2 z^{-\frac{n}{k}} \bar{z}^{-\frac{n}{k}} + \frac{1}{2}c
$$

(4.24)

$$
= \frac{2\pi v^2}{k} \left( 1 - \frac{2n}{k} \log r + O(k^{-2}) \right) + \frac{1}{2}c.
$$

(4.25)

This means

$$
A_0 \to -\frac{g_s}{2\pi \ell_s} n \log r,
$$

(4.26)

where we have chosen $c = -4\pi v^2/k$ for simplicity, because a constant of the gauge field does not play any role. Since the configuration for the fields $Y^1$, $Y^2$ and $Y^3$ are trivial, the remaining scalar fields are zeros $\Phi^i = 0 \quad (i = 3 \sim 8)$. Recalling the computation (4.9), the resulting spike D2-brane solution obviously reproduces the correct string tension.

From (4.23) and (4.26), we see that our winding solution (4.20) is correctly reduced to the spike solution on a D2-brane given in (4.8) as long as configurations around $X^9 = \sqrt{\frac{2}{T_{M2}}}v$ are concerned. In fact, they coincide even including the coefficients. It should be noticed that the winding number $n$ appears as the string charge after the novel Higgs
mechanism. This is consistent with the M-theory interpretation of a fundamental string. Here we remember that the string charge appears as the overall factor of the solution, while in the case of the winding M2-brane solution (4.20), the winding number is encoded into the power. In the next subsection, we see that a possible zero mode of fluctuations around the spike on the D2-brane can shift slightly the overall factor of the original spike solution. However, such a zero mode is forbidden in contrast to the M2-brane case because we have fixed the string charge as the winding number of the M2-brane. Therefore, although the spike solutions on the M2-brane and D2-brane are directly related by the novel Higgs mechanism as we have seen above, the difference in their dependence of the charge yields a clear difference in existence of a zero mode.

4.3 The stability of a fundamental string

Let us consider fluctuations around the spike solution (4.8) in (4.2) and examine stability of a fundamental string attached to a D2-brane. Then we point out a difference between an M2-brane and a fundamental string.

Fluctuations we are interested in are

\[ \frac{X^9}{2\pi\alpha'} = \frac{R}{2\pi\alpha'} \log \frac{r}{\ell} + y(t, r, \theta), \quad A_0 = \frac{R}{2\pi\alpha'} \log \frac{r}{\ell} + \phi(t, r, \theta), \quad A_r(t, r, \theta), \quad A_\theta(t, r, \theta), \]  

(4.27)

where for later convenience, we take the polar coordinate system. Substituting these fluctuations for the action (4.2) and keeping terms of second order with respect to them, the resulting quadratic action is

\[ S_2 = \int dtdr d\theta \left[ \frac{1}{2} y'^2 - \frac{1}{2} y''^2 - \frac{1}{2r^2} (\partial_\theta y)^2 - \frac{1}{2r^2} (A'_\theta - \partial_\theta A_r)^2 + \frac{1}{2r^2} (A_\theta - \partial_\theta \phi)^2 \right. 

\[ + \frac{1}{2} (A_r - \phi')^2 + \frac{R^2}{2r^2} (A_\theta + \partial_\theta y - \partial_\theta \phi)^2 + \frac{R^2}{2r^2} (A_r + y' - \phi')^2 \left. \right] \]  

(4.28)

When \( R = 0 \), \( S_2 \) becomes the Yang-Mills action. It is easy to see that up to this order the other fluctuations than those in (4.27) decouples. We consider the Hamiltonian obtained from \( S_2 \)

\[ H_2 = \int d\theta \left\{ P_y y + P_{A_r} A_r + P_{A_\theta} A_\theta - \mathcal{L} \right\} \]

\[ = \int d\theta \left\{ \frac{g_s}{2\ell_s r} P_y^2 + \frac{\ell_s}{2g_s r^2} F_{r\theta}^2 + \frac{\ell_s}{2g_s r} \left( 1 + \frac{R^2}{r^2} \right)^{-1} \left( \frac{g_s}{\ell_s} P_{A_r} + ry' \right)^2 \right. 

\[ + \frac{\ell_s r}{2g_s} \left( 1 + \frac{R^2}{r^2} \right)^{-1} \left( \frac{g_s}{\ell_s} P_{A_\theta} + \frac{1}{r} \partial_\theta y \right)^2 + (P_{A_r} \phi' + P_{A_\theta} \partial_\theta \phi - P_{A_\theta} \partial_\theta y - P_{A_r} y') \right\}, \]  

(4.29)

where the conjugate momenta are

\[ P_y = \frac{\partial \mathcal{L}}{\partial y'} = \frac{\ell_s r}{g_s} y'. \]  

(4.30)
\[ P_{A_r} = \frac{\partial L}{\partial \dot{A}_r} = \ell_s r \left\{ \left( 1 + \frac{R^2}{r^2} \right)(\dot{A}_r - \phi') + \frac{R^2}{r^2} y' \right\}, \quad (4.31) \]
\[ P_{A_\theta} = \frac{\partial L}{\partial \dot{A}_\theta} = \ell_s r \frac{1}{r^2} \left\{ \left( 1 + \frac{R^2}{r^2} \right)(\dot{A}_\theta - \partial_\theta \phi) + \frac{R^2}{r^2} \partial_\theta y \right\}. \quad (4.32) \]

From variations of (4.29) with respect to \( \phi \), the Gauss' law constraint reads
\[ 0 = \partial_r P_{A_r} + \partial_\theta P_{A_\theta} = \frac{\ell_s}{g_s} r \left\{ r \left( 1 + \frac{R^2}{r^2} \right)(\dot{A}_r - \phi') + \frac{R^2}{r^2} y' \right\} + \partial_\theta \left\{ \frac{1}{r} \left( 1 + \frac{R^2}{r^2} \right)(\dot{A}_\theta - \partial_\theta \phi) + \frac{R^2}{r^2} \partial_\theta y \right\}. \quad (4.33) \]

Using the partial integration and the Gauss' law constraint, the Hamiltonian becomes
\[ H_2 = \int dr d\theta \left\{ g_s \frac{P^2}{2\ell_s r} + \frac{\ell_s}{2g_s r} F^2 + \frac{\ell_s}{2g_s r} \left( 1 + \frac{R^2}{r^2} \right)^{-1} \left( \frac{g_s}{\ell_s} P_{A_r} + ry' \right)^2 \right\} + \frac{\ell_s}{2g_s r} \left( 1 + \frac{R^2}{r^2} \right)^{-1} \left( \frac{g_s}{\ell_s} P_{A_\theta} + \frac{1}{r} \partial_\theta y \right)^2, \quad (4.34) \]
where contributions from boundary terms are ignored. Thus the Hamiltonian is positive semi-definite up to boundary contributions. In appendix B, we will consider a contribution from a boundary to the Hamiltonian.

Next, we consider the minimum of the Hamiltonian and find a zero mode. From (4.34) \( H_2 \) has the minimum when
\[ P_y = 0, \quad F_r = 0, \quad P_{A_r} = -\frac{\ell_s}{g_s} y', \quad P_{A_\theta} = -\frac{\ell_s}{g_s r} \partial_\theta y. \quad (4.35) \]
Fluctuations that satisfy these conditions and the Gauss' law constraint are shown to satisfy the equation of motion and also are candidates for zero modes. Let us first find a zero mode of the fluctuation \( y \). Using \( P_y = 0 \) and (4.30), \( y \) is time-independent:
\[ y = y(r, \theta). \quad (4.36) \]
From the last two equations in (4.35) and (4.33), we find that \( y(r, \theta) \) should satisfy the Laplace equation:
\[ r \partial_r (r \partial_r y) + \partial^2_\theta y = 0, \quad (4.37) \]
whose solution is given by
\[ y(r, \theta) = \sum_{m=-\infty}^{\infty} \left( y_m r^m e^{im\theta} + y^*_m r^m e^{-im\theta} \right), \quad (4.38) \]
where we have imposed the reality condition. In general fluctuations must be smaller than the classical solution (4.8) both at the origin and infinity within our approximation.
of taking only the second order of fluctuations. Then all fluctuations in (4.38) are not allowed because they are more singular than (4.8) either at the origin or infinity. However, if we relax (4.37) at the origin where the original classical solution itself is ill-defined, we may allow a solution of (4.37) corresponding to $m = 0$ case in (4.38)

$$y(r, \theta) = \alpha \log \frac{r}{\ell},$$  \hspace{1cm} (4.39)

where $\alpha \ll R$ is a constant. From now on, we consider (4.39) as a candidate for the zero mode of $y$.

Let us find zero modes of the fluctuations $A_0, A_r, A_\theta$. From $F_r \theta = 0$, $A_r$ and $A_\theta$ can be rewritten as

$$A_r = \partial_r a(t, r, \theta), \quad A_\theta = \partial_\theta a(t, r, \theta),$$  \hspace{1cm} (4.40)

using a scalar field $a(t, r, \theta)$. Then $P_{A_r} = -\frac{\ell}{gs} \partial_r y$, (4.31) and $A_r = \partial_r a$ yield

$$\phi = y + \partial_t a - f(t, \theta),$$  \hspace{1cm} (4.41)

where a function $f$ is independent of $r$. Furthermore, $P_{A_\theta} = -\frac{\ell}{gs} \partial_\theta y$ and (4.32) tell us that $f$ depends only on time: $f = f(t)$. In summary we find that a candidate for the zero mode is given in general as

$$y = \alpha \log \frac{r}{\ell}, \quad \phi = y + \partial_t a - f(t), \quad A_r = \partial_r a, \quad A_\theta = \partial_\theta a.$$  \hspace{1cm} (4.42)

By a gauge transformation, these equations can be simplified into

$$y = \phi = \alpha \log \frac{r}{\ell}, \quad A_r = 0, \quad A_\theta = 0.$$  \hspace{1cm} (4.43)

These imply that fluctuations giving the minimum of the Hamiltonian preserve the half-BPS condition, which is similar to the winding M2-brane case in the ABJM model. In fact if we plug (4.43) into $S_2$, we find that it vanishes up to boundary contributions\textsuperscript{11}. Thus it may be possible that this mode would be a flat direction and indicate instability of the spike solution. However there is really an important difference between an M2-brane and a fundamental string. Namely, the spike on the D2-brane corresponds to a bunch of fundamental strings, where the number of them are related to the overall factor of the spike solution. Since (4.39) takes exactly the same form as the classical solution (4.8), it makes a slight change of the overall factor of the solution, or equivalently the number of strings. Hence such a mode cannot be allowed under the condition of fixed string charge. On the other hand, in the ABJM model the winding number is not related to the overall factor of the solution (3.11) and consequently the zero mode which takes the same form as the classical solution is allowed unlike the case of a fundamental string. We can, therefore, reasonably conclude that a fundamental string ending on a D2-brane is stable at least perturbatively, because a zero mode of fluctuations does not exist.

\textsuperscript{11}However, in the case of (4.43) boundary terms have nontrivial values in general according to boundary conditions. See appendix B.
Finally, we comment on zero modes related to symmetries. The original DBI action (4.2) has the translational symmetry on the world volume which the spike solution (4.8) breaks. As a consequence, a zero mode associated with this breaking should exist. It is actually included in the solution (4.38), but it is forbidden from behaviour of divergence at the origin. Likewise, a zero mode associated with a broken symmetry would be in general more singular at the origin than the classical solution itself, because it would correspond to a deformation of the classical solution by the broken symmetry and therefore take a form of a derivative of the classical solution. So we cannot find an allowed zero mode related to a symmetry which the spike breaks. This situation is quite different from the winding M2-brane in that it has a zero mode related to the broken scale symmetry, which does not make the classical solution more singular.

4.4 Novel Higgs mechanism for zero mode

In the analysis of the ABJM model, we found the zero mode which deforms the original spike-like configuration, while in subsection 4.3 we found no such fluctuation around the spike D2-brane. Since these two solutions are related through the novel Higgs mechanism, one may ask how the zero mode is mapped. In order to clarify it, we consider the deformation $Y^4(z) = (\alpha + \delta\alpha)z^{-\frac{n}{k}}$ and reexamine the novel Higgs mechanism. We concentrate on the real $\delta\alpha$ since the imaginary part corresponds to the rotation of the configuration and does not deform it.

The expansion of $\sqrt{k/(2\pi)}Y^4$ is now given by

$$\sqrt{\frac{k}{2\pi}} Y^4(z) = v\left(1 + \frac{\delta\alpha}{\alpha}\right)\left(1 - \frac{n}{k}\log r - i\frac{n}{k}\theta + O(k^{-2})\right). \quad (4.44)$$

Here $v$ is defined by $v = \sqrt{k/(2\pi)}\alpha$ as before. By subtracting the large expectation value $v$, the coordinates $\tilde{X}^9$ and $\tilde{X}^{11}$ are identified as

$$\sqrt{\frac{T_{M2}}{2}} \tilde{X}^9 = -\frac{v}{k}\log r + \frac{\delta\alpha}{\alpha}\frac{v}{\alpha} - \frac{n}{k}\log r + \cdots, \quad (4.45)$$
$$\sqrt{\frac{T_{M2}}{2}} \tilde{X}^{11} = -\frac{v}{k}\theta - \frac{\delta\alpha}{\alpha}\frac{v}{\alpha}n\frac{\theta}{k} + \cdots. \quad (4.46)$$

The first terms in these equations are the classical configurations given in (4.23). When we derived (4.14), we assumed that the order of the fields are as $\sqrt{T_{M2}}\tilde{X}^i \sim O(1)$.\textsuperscript{12} Hence as long as $k$ and $v$ dependences are concerned, the leading fluctuation, namely the constant term $v\frac{\delta\alpha}{\alpha}$ in (4.45) should be of the same order, $O(1)$. Otherwise the fluctuation $\delta\alpha$ cannot be regarded as the fluctuation in the D2-brane theory. This requirement is equivalent to the condition $\frac{\delta\alpha}{\alpha} = O(k^{-1})$ and then the remaining terms in (4.45) and (4.46) vanish in the large $k$ limit. This clearly shows that the zero mode found in the ABJM model corresponds to the constant shift along the $\tilde{X}^9$ direction in the D2-brane theory. In terms of $y$ given in (4.27), the fluctuation is expressed as $y = 2\alpha\delta\alpha$.

\textsuperscript{12}Recall this order counting is under the fixed scale $\frac{v}{k}$ as mentioned in footnote 10.
In the similar manner, the corresponding fluctuation for the gauge field $A_0$ can be deduced from (4.24). By taking account of the fluctuation for $Y^4$ as $Y^4 = Y_{cl} + Y = (\alpha + \delta \alpha)z^{-\frac{\pi}{k}}$ we have

$$A_0 = (Y_{cl} + Y)(Y_{cl}^\dagger + Y^\dagger) - \frac{2\pi v^2}{k}$$

$$= \left(Y_{cl}Y_{cl}^\dagger - \frac{2\pi v^2}{k}\right) + Y_{cl}Y^\dagger + YY_{cl}^\dagger + YY^\dagger$$

$$\rightarrow -\frac{g_s}{2\ell_s} n \log r + 2\alpha \delta \alpha. \quad (k \to \infty)$$

The first term in (4.48) is the classical configuration (4.26) and the remaining constant shift $2\alpha \delta \alpha$ is the fluctuation corresponding to $\phi$ defined in (4.27).

One should notice that the above result does not mean that the constant shift along the $\tilde{X}^9$ direction in the D2-brane theory implies any instability contained in the D2-brane theory itself. The novel Higgs mechanism connects the ABJM model and the D2-brane theory only locally around $\tilde{X}^9 = \sqrt{\frac{2}{l_M^2}} v$. Therefore the precise meaning of the above observation is that, around this region, the deformation of the M2-brane is described by the constant shift of the D2-brane. On the other hand, the constant shift in the D2-brane theory itself does not deform the shape of the spike D2-brane, and hence there is no reason to expect that it implies the instability of the fundamental string.

As a final check, if we allow a small change of the winding number in the ABJM model as $Y^4(z) = \alpha z^{-\frac{\pi}{k} + \delta n}$, then under the novel Higgs mechanism, it is reduced to the change of the overall constant of the fields as $X^9 = 2\pi \alpha' A_0 = -(n + \delta n)g_s\ell_s \log r$ in the D2-brane theory. This means that such fluctuations are forbidden by essentially the same reason both in the ABJM model and the D2-brane theory.

5 Conclusions and discussions

In order to study the instability of a membrane and a string, we first constructed half-BPS solutions in the Nambu-Goto action of a supermembrane, the $U(1) \times U(1)$ ABJM model, and the DBI action of the D2-brane. Then we looked for zero modes which deform them under fixed charges, namely the winding number for a wrapped membrane, or the number of strings. In the case of a supermembrane, a zero mode is indeed found even if we fix its winding number. In particular, in the case of the $U(1) \times U(1)$ ABJM model with large $k$, the BPS solution becomes a configuration like the spike and the zero mode we found can make it thinner. This is in accordance with the physical picture proposed in [2]. On the other hand, the spike solution in the DBI action of the D2-brane does not allow a zero mode under a fixed string charge. The situation is similar to the Nambu-Goto action of a compactified supermembrane with the winding number fixed. They are consistent with the fact that a string or a wrapped membrane has mass proportional to its length. The difference of the existence of an allowed zero mode between a membrane and a string originates from the form of the BPS solutions in that in the former the winding number is encoded in the power, while in the latter its coefficient corresponds to the number of
strings. We regard such a sharp contrast as a manifestation of a difference in the stability of a membrane and a string. Moreover we clarify how the BPS solutions we considered are connected with each other via the novel Higgs mechanism, in particular how the winding number of a membrane is translated into the string charge.

It is evident that our approach provides only a sign of instability of a supermembrane in the static gauge and is far from a proof. If we try to prove it rigorously as in [2], we have to regularize an action of the M2-brane in the static gauge which should be away from the IR fixed point. Therefore a kind of nonperturbative formulation of a Yang-Mills-Chern-Simons system may be necessary for it.

As an application of our approach, it would be interesting to apply it to the BLG model [25, 26]. In fact, we have stressed the difference between the half-BPS solutions in the ABJM model and in the DBI action in whether the gauge field plays a nontrivial role. It would be intriguing to study what happens to a half-BPS solution in the BLG model.

Another interesting question is a relation to stability of other classical solutions representing a fundamental string. For example, in [27] a fundamental string attached to a D-brane is realized as flux tube solutions attached to domain walls in various models. It is quite interesting to examine them from the point of view of world volume theories on domain walls.

Recalling that our BPS solutions and zero modes are singular at the origin and hence they have ambiguities coming from regularizations or boundary contributions. In order to avoid them, it would be better to consider a configuration of a membrane which corresponds to a fundamental string connecting two D2-branes in the IIA picture as constructed in [16]. Applying our considerations to it would clarify issues in our approach like the tension of the classical solutions and so on.

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A Supersymmetry condition for spike D2-brane

The supersymmetries preserved by a D2-brane are specified by the projection condition \( \Gamma \epsilon = \epsilon \) where \( \epsilon \) is a 32 component Majorana spinor and \( \Gamma \) is given by [20, 21]

\[
\Gamma = \frac{\sqrt{- \det g_{\mu \nu}}}{\sqrt{- \det (g_{\mu \nu} + 2 \pi \alpha' F_{\mu \nu})}} \sum_{n=0}^{\infty} \frac{(2 \pi \alpha')^n}{2^n n!} \gamma_{\mu_1 \nu_1} \cdots \gamma_{\mu_n \nu_n} F_{\mu_1 \nu_1} \cdots F_{\mu_n \nu_n} (\Gamma_{11})^n \Gamma_{(0)},
\]  

(A.1)
\[ \Gamma(0) = \frac{1}{3! \sqrt{-\det g_{\mu\nu}} \epsilon^{\mu\nu\rho\gamma} \gamma_{\mu\nu\rho}}, \quad (A.2) \]

where \( \gamma_{\mu} \) are defined by \( \gamma_{\mu} = \partial_{\mu} X^{m} \Gamma_{m} \) (\( m = 0, \ldots, 9 \)). We take the static gauge \( x^{\mu} = X^{\mu} \) and assume the following ansatz

\[ X^{i} = 0 \quad (i = 3, \ldots, 8), \quad X^{9} = X^{9}(r), \quad F_{0r} = F_{0r}(r), \quad F_{0\theta} = F_{r\theta} = 0. \quad (A.3) \]

Then only terms with \( n = 0, 1 \) are nontrivial in (A.1) and \( \Gamma \) becomes

\[ \Gamma = \frac{1}{\sqrt{1 + (\partial_r X^{9})^2 - (2\pi \alpha' F_{0r})^2}} \left\{ \Gamma_{012} + \left( \cos \theta \Gamma_{2} - \sin \theta \Gamma_{1} \right)(\partial_r X^{9} \Gamma_{09} + 2\pi \alpha' F_{0r} \Gamma_{11}) \right\}. \quad (A.4) \]

The condition \( \Gamma \epsilon = \epsilon \) is now rewritten as

\[ (\partial_r X^{9})^2 = (2\pi \alpha' F_{0r})^2, \quad \Gamma_{012} \epsilon = \epsilon, \quad (\partial_r X^{9} \Gamma_{09} + 2\pi \alpha' F_{0r} \Gamma_{11}) \epsilon = 0. \quad (A.5) \]

Then, depending on the branch \( \partial_r X^{9} = \beta \times 2\pi \alpha' F_{0r}, \beta = \pm 1 \), the D2-brane preserves the 8 supersymmetries specified by the following conditions

\[ \beta = +1 : \quad \Gamma_{012} \epsilon = \epsilon, \quad \Gamma_{09(11)} \epsilon = -\epsilon, \quad (A.6) \]

\[ \beta = -1 : \quad \Gamma_{012} \epsilon = \epsilon, \quad \Gamma_{09(11)} \epsilon = \epsilon. \quad (A.7) \]

**B Boundary contributions to energy**

In this appendix we show that if we try to reproduce the energy, it is necessary to take account of boundary contributions in the world volume theory.

For illustration we consider the D2-brane world volume theory (4.2), where the spike solution (4.8) is supposed to represent a fundamental string ending on the D2-brane. In section 4 we show that fluctuations around (4.8) contain a zero mode which takes the same form as in (4.8)

\[ y = \phi = \alpha \log \frac{r}{\ell'}, \quad (B.1) \]

where \( \alpha \ll R = g_s \ell_s \). Adding this zero mode to the original classical solution amounts to replacing \( R \rightarrow R + 2\pi \alpha' \alpha \) as seen in (4.27) and hence it is expected that the total tension should become that multiplied by \( (R + 2\pi \alpha' \alpha)/R \). Thus the total energy of this configuration should be

\[ F_{\text{tot}} = \frac{1}{2\pi \alpha'} \left( 1 + \frac{2\pi \alpha' \alpha}{R} \right) \left( \Delta X_{d}^{9} + 2\pi \alpha' \Delta y \right), \quad (B.2) \]

where \( \Delta X_{d}^{9} = X^{9}(r = \delta) - X^{9}(r = \Lambda), \Delta y = y(r = \delta) - y(r = \Lambda) \). However, since (B.1) is the zero mode of the Hamiltonian (4.34), it does not seem to increase the energy of the classical solution.
In order to resolve this puzzle, we take care of contributions from boundary terms neglected in deriving (4.34) from (4.29):

\[ H_b = \int drd\theta \left\{ P_{A_r} \phi' + P_{A_\theta} \partial_\theta \phi - P_{A_r} y' - P_{A_\theta} \partial_\theta y \right\}. \] (B.3)

The Gauss' law constraint (4.33) tells us that the integrand in \( H_b \) is indeed total derivative. If we integrate it naively, it would give rise to divergences both in \( r \to 0 \) and \( r \to \infty \). We therefore regularize it by introducing, for example, a cutoff at \( r = \delta \ll 1 \) and \( r = \Lambda \gg 1 \) as above, which in turn introduces a boundary. Thus we have to take care of contributions from these boundaries in order to calculate the energy of the configuration. Note that (B.1) of course satisfies the constraint (4.33) and hence we can add (4.33) to the Hamiltonian. In fact, the Hamiltonian in itself has ambiguity of adding a term proportional to (4.33). Since we are dealing with the quadratic Hamiltonian for fluctuations (4.34), in the following we show that by taking account of boundary contributions, we can reproduce the quadratic part of the total energy in (B.2), namely \( \frac{2\pi \alpha'}{R} \Delta y \).

At first sight, since our BPS configuration satisfies \( y = \phi \), the boundary term does not seem to contribute which in fact vanishes for \( y = \phi \). However, it is important to notice that they should obey different boundary conditions. Namely, \( y \) has the usual Dirichlet boundary condition, while \( \phi \) satisfies the Neumann boundary condition, because we have fixed the string charge\(^{13}\)

\[ \int_{S^1} F_{0r} = \text{constant}, \] (B.4)

which is imposed on both boundaries at \( r = \Lambda \) and \( r = \delta \). In order to switch the boundary condition, we should make the Legendre transformation for \( \phi \) which amounts to adding a boundary term to the Lagrangian (4.28)

\[ S_b = -\int dt \left[ \dot{\phi}(r = \Lambda) \int_{r = \Lambda}^{r = \delta} d\theta \frac{\delta L}{\delta (\partial_\theta \phi)} - \dot{\phi}(r = \delta) \int_{r = \delta}^{r = \Lambda} d\theta \frac{\delta L}{\delta (\partial_\theta \phi)} \right], \] (B.5)

where we have assumed that \( \phi \) is independent of \( \theta \). If we substitute (B.1) into (B.3) with (B.5) taken into account, it is easy to see that (B.5) exactly cancels the contribution from \( \phi \) to \( H_b \) given by the first two terms in (B.3). Thus we are left with the boundary terms only for \( y \) and by using (4.31), it is straightforward to check that it reproduces \( \frac{2\pi \alpha'}{R} \Delta y \). Thus if we regularize the fluctuation Hamiltonian and take account of boundary contributions arising as a result of the regularization, we can reproduce the correct value of the quadratic part of the total energy\(^{14}\) even if the BPS solution satisfies \( y = \phi \).

\(^{13}\)In section 4, we discuss this Neumann condition prohibits the zero mode (4.43), but in this appendix we argue that if it exists, what happens to the energy in order to clarify a role of boundary contributions.

\(^{14}\)Boundary terms which are necessary for the Legendre transformation to flip boundary conditions play important roles, in particular in the calculation of the expectation value of the Wilson loop in the AdS/CFT correspondence, see e.g. [22–24].
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