First–Forbidden Continuum- and Bound-State $\beta^–$–Decay Rates of Bare

$^{205}$Hg$^{80+}$ and $^{207}$Tl$^{81+}$ Ions

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We calculate the decay rates $\lambda_{\beta^–}$ and $\lambda_{\beta^–}$ of the continuum- and bound-state $\beta^–$–decays for bare $^{205}$Hg$^{80+}$ and $^{207}$Tl$^{81+}$ ions. For the ratio of the decay rates $R_{b/c} = \lambda_{\beta^–}^b / \lambda_{\beta^–}^c$ we obtain the values $R_{b/c} = 0.161$ and $R_{b/c} = 0.190$ for bare $^{205}$Hg$^{80+}$ and $^{207}$Tl$^{81+}$ ions, respectively. The theoretical value of the ratio $R_{b/c} = 0.190$ for the decays of $^{207}$Tl$^{81+}$ agrees within 1 % of accuracy with the experimental data $R_{b/c} = 0.188(18)$, obtained at GSI. The theoretical ratio $R_{b/c} = 0.161$ for $^{205}$Hg$^{80+}$ is about 20 % smaller than the experimental value $R_{b/c} = 0.20(2)$, measured recently at GSI.

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As has been shown in [1], the standard theory of weak interactions of heavy ions [2], allows to describe well the K–shell electron capture (EC) and $\beta^+$ decays of the H–like and He–like ions $^{140}$Pr$^{50+}$ and $^{140}$P$^{57+}$, agreeing with the experimental data, obtained at GSI [3], with an accuracy better than 20 %.

In this letter we apply the standard theory of weak interactions of heavy ions [2] and technique developed in [1] to the calculation of the decay rates $\lambda_{\beta^–}$ and $\lambda_{\beta^–}$ of the decays

$$
\begin{align*}
205\text{Hg}^{80+} & \rightarrow 205\text{Tl}^{81+} + e^- + \bar{\nu}_e, \\
205\text{Hg}^{80+} & \rightarrow 205\text{Tl}^{81+} + \bar{\nu}_e, \\
207\text{Tl}^{81+} & \rightarrow 207\text{Pb}^{82+} + e^+ + \nu_e, \\
207\text{Tl}^{81+} & \rightarrow 207\text{Pb}^{81+} + \nu_e,
\end{align*}
$$

(1)

where $^{205}$Hg$^{80+}$, $^{207}$Pb$^{82+}$ and $^{205}$Tl$^{81+}$, $^{207}$Tl$^{81+}$ are two pairs of bare ions with quantum numbers $I^P = \frac{1}{2}^−$ and $I^P = \frac{1}{2}^+$, respectively, and $^{205}$Pr$^{50+}$ and $^{207}$P$^{57+}$ are the H–like ions. The continuum- and bound-state $\beta^–$–decays in Eq. (1) satisfy the selection rule $\Delta I^P = 0^−$, which corresponds to the selection rule of the first–forbidden $\beta^–$–decays [4]. The bound-state $\beta^–$–decay to the K–shell is the time reversed orbital K–shell Electron Capture (EC) decay, which we analysed in [1]. The main distinction of the bound state $\beta^–$–decay from the EC–decay is that the bound electron can be not only on the K–shell in the 1s state but on any other shells in any excited ns state, the contribution of which is about 20 %.

A measurement of the continuum- and bound-state $\beta^–$–decays of bare $^{207}$Tl$^{81+}$ ion was reported by Ohtsubo et al. [6]. The experimental value of the ratio of the decay rates $R_{b/c} = 0.188(18)$ agrees within one standard deviation with the theoretical value $R_{b/c} = 0.171(1)$ [8], obtained from the theory employing spectra of allowed transitions [2]. Very recently an experimental value $R_{b/c} = 0.20(2)$ of the ratio of the continuum- and bound-state $\beta^–$–decays of bare $^{205}$Hg$^{80+}$ has become known [9]. This has motivated us to carry out a model–independent calculation of the first–forbidden continuum- and bound-state $\beta^–$–decays of bare $^{205}$Hg$^{80+}$ and $^{207}$Tl$^{81+}$ ions.

For the calculation of the rates of the $\beta^–$–decays we use the Hamilton density operator [1, 2]

$$
\mathcal{H}_W(x) = \frac{G_F}{\sqrt{2}} V_{ud} \left[ \bar{\psi}_d(x) \gamma^\mu (1 - g_A \gamma^5) \psi_u(x) \right] \\
\times \left[ \bar{\psi}_e(x) \gamma_\mu (1 - \gamma^5) \psi_\nu_e(x) \right] + \text{h.c.}
$$

(2)

with standard notations [1, 2, 8]. In our calculations the anti–neutrino is assumed to be a massless Dirac anti–particle as in [1].

The continuum-state $\beta^–$–decay

In the rest frame of the mother ion the amplitudes of the continuum-state $\beta^–$–decay are defined by

$$
M_{II_\alpha - II_\beta} =
= -i \bar{\psi}_e(\vec{k}) e^- (\vec{q}_-) d(\vec{q}) \mathcal{H}_W(0) |m(\vec{0})\rangle_{II_\alpha},
$$

(3)
where $II_z = \frac{1}{2}, \pm \frac{1}{2}$ and $I'I_z' = \frac{1}{2}, \pm \frac{1}{2}$ determine the spinorial states of the mother $m$ and daughter $d$ ions. The decay rate $\lambda_{\beta^+}$ of the continuum-state $\beta^-$-decay is defined by

$$\lambda_{\beta^+} = \frac{1}{2M_m} \int \frac{d^3q}{(2\pi)^3\tilde{E}_d} \frac{d^3\tilde{p}_-}{(2\pi)^32\tilde{E}_-} \frac{d^3\tilde{E}}{(2\pi)^32\tilde{E}_{d\bar{E}}(2\pi)^32\tilde{E}} \times (2\pi)^3\delta(k + p_- + q - k_m) F(Z + 1, E_-) \times \frac{1}{2} \sum_{I_z, I_z', \sigma_-} |M_{II_z \rightarrow I'I_z'}|^2, \quad (4)$$

where $k = (E_{\bar{E}_d}, \vec{k})$, $p_- = (E_-, \vec{p}_-)$, $q = (E_d, \vec{q})$ and $k_m = (M_m, \vec{0})$ are 4-momenta of the interacting particles, $F(Z + 1, E_-)$ is the Fermi function

$$F(Z + 1, E_-) = \left(1 + \frac{1}{2} \gamma \right) \frac{4(2R_{p_0}^{-2\gamma})}{(3 + 2\gamma)} \times e^{\frac{\pi(Z+1)\alpha E}{\alpha \gamma}} \left[1 + \gamma + \frac{\alpha(Z + 1)\gamma E_-}{p_-} \right]^2, \quad (5)$$

with $\gamma = \sqrt{1 - ((Z + 1)\alpha)^2} - 1$, $p_- = \sqrt{E_-^2 - m_e^2}$, $Z = 80$ and $Z = 81$ for $^{205}\text{Hg}^{80+}$ and $^{207}\text{Tl}^{81+}$, respectively. The summation in Eq. (4) should be carried out over all polarisations of the interacting particles, where $\sigma = \pm \frac{1}{2}$ is a polarisation of the electron. The anti-neutrino is polarised parallel to the momentum $\vec{k}$. Following [11], for the amplitudes of the continuum-state $\beta^-$-decay we get the expressions

$$M_{\frac{1}{2} - \frac{1}{2} \rightarrow \frac{1}{2} - \frac{1}{2} +} = \sqrt{2M_m2E_d}M_{m \rightarrow d}$$

$$\times \left[\bar{u}_e(\vec{p}_-, \sigma_-)(g_0\gamma^0 - \gamma^3)(1 - \gamma^5)\bar{v}_{\bar{E}}(\vec{k}, +\frac{1}{2})\right],$$

$$M_{\frac{1}{2} - \frac{1}{2} \rightarrow \frac{1}{2} - \frac{1}{2} -} = \sqrt{2M_m2E_d}M_{m \rightarrow d}$$

$$\times \left[\bar{u}_e(\vec{p}_-, \sigma_-)(\gamma^1 + i\gamma^2)(1 - \gamma^5)\bar{v}_{\bar{E}}(\vec{k}, +\frac{1}{2})\right],$$

$$M_{\frac{1}{2} - \frac{1}{2} \rightarrow \frac{1}{2} - \frac{1}{2} +} = \sqrt{2M_m2E_d}M_{m \rightarrow d}$$

$$\times \left[\bar{u}_e(\vec{p}_-, \sigma_-)(\gamma^1 - i\gamma^2)(1 - \gamma^5)\bar{v}_{\bar{E}}(\vec{k}, +\frac{1}{2})\right],$$

$$M_{\frac{1}{2} - \frac{1}{2} \rightarrow \frac{1}{2} - \frac{1}{2} -} = \sqrt{2M_m2E_d}M_{m \rightarrow d}$$

$$\times \left[\bar{u}_e(\vec{p}_-, \sigma_-)(g_0\gamma^0 + \gamma^3)(1 - \gamma^5)\bar{v}_{\bar{E}}(\vec{k}, +\frac{1}{2})\right]. \quad (6)$$

where $\bar{u}_e$ and $\bar{v}_{\bar{E}}$ are Dirac bispinors of the electron and the anti-neutrino, $M_{m \rightarrow d}$ is the nuclear matrix element defined by

$$M_{m \rightarrow d} = -\frac{G_F}{\sqrt{2}} V_{ud} \int d^3x \Psi_d^*(\vec{r}) \Psi_m(\vec{r}), \quad (7)$$

where $\Psi_d(\vec{r})$ and $\Psi_m(\vec{r})$ are the wave functions of the daughter and mother nuclei. For the numerical calculations we assume that the product $\Psi_d^*(\vec{r}) \Psi_m(\vec{r})$ has the Wood–Saxon shape  [1]. Substituting the amplitudes Eq.(6) into Eq.(4) and carrying out the summation over polarisations we get

$$\lambda_{\beta^+} = (3 + g^2_d) |M_{m \rightarrow d}|^2 f(Q_{\beta^+}, Z + 1), \quad (8)$$

where the Fermi integral $f(Q_{\beta^+}, Z + 1)$ is

$$f(Q_{\beta^+}, Z + 1) = \int_{m_e}^{Q_{\beta^+} + m_e} (Q_{\beta^+} + m_e - E_-)^2$$

$$\times F(Z + 1, E_-) \sqrt{E_-^2 - m_e^2} E_- dE_- \quad (9)$$

The $Q$–values of the continuum-state $\beta^-$–decays are equal to $Q_{\beta^+} = 1515.734\text{keV}$ and $Q_{\beta^-} = 1407.471\text{keV}$ for $^{205}\text{Hg}^{80+}$ and $^{207}\text{Tl}^{81+}$, respectively [10, 11].

The bound-state $\beta^-$–decay

In the bound-state $\beta^-$–decay the electron in the final state can be in any bound $ns$–state. Due to hyperfine interactions the $ns$–state is splitted into two hyperfine states $(ns)_{F=0}$ and $(ns)_{F=1}$ with a total spin $F = 0$ and $F = 1$, respectively [12].

The experimental value of the hyperfine energy level splitting for the $1s$–state $\Delta E_{1s}^{\text{Exp}} = E_{(1s)F=0} - E_{(1s)F=1} = -3.24409 \pm 0.00029\text{eV}$ [13] agrees well with the theoretical one $\Delta E_{1s}^{\text{Th}} = -3.275\text{eV}$ [12]. The hyperfine splitting $\Delta E_{ns}^{\text{H}}$ of the energy level of the excited $ns$ state is related to $\Delta E_{1s}^{\text{H}}$ as

$$\Delta E_{ns}^{\text{H}} = \frac{\Delta E_{1s}^{\text{Exp}}}{(n+1)(n+2)(n+3)} \times \left[\frac{3(n+1)}{2(n+2)} - \frac{\sqrt{(n+2)(n+3)}}{2(n+1)}\right]. \quad (10)$$

The decay rate $\lambda_{\beta^-}$ of the bound-state $\beta^-$–decay into any hyperfine states $(ns)_{F=0}$ and $(ns)_{F=1}$ is defined by

$$\lambda_{\beta^-} = \sum_{n=1}^{\infty} \sum_{F=0,1} \lambda_{\beta^-}(n) \times \frac{1}{2M_m} \left|\Psi_d^*(\vec{r})\right|^2 \Psi_m(\vec{r}) \times \int \frac{d^3q}{(2\pi)^32E_d} \frac{d^3\tilde{E}}{(2\pi)^32E_{\bar{E}}(2\pi)^32E_{\bar{E}}(2\pi)^4\delta(4)(q + k - k_m)} \times \frac{1}{2} \sum_{n=1}^{\infty} \sum_{F=0,1} |M_{II_z \rightarrow MF}^{(n)}|^2, \quad (11)$$

where $k = (E_{\bar{E}}, \vec{k})$, $q = (E_d, \vec{q})$ and $k_m = (M_m, \vec{0})$ are 4-momenta of the interacting particles. In the rest frame of the mother ion the amplitudes of the bound-state $\beta^-$–decay are determined
by

\[ M_{I_\mu \rightarrow FM_F}^{(n)} = -F_{MF'} \langle \bar{\nu}_c(\tilde{k})d^{(n)}(\tilde{q})|H_W(0)|\tilde{b}\rangle_{I_\mu}, \tag{12} \]

Following [1] for the amplitudes of the transitions \( I_{\mu} \rightarrow FM_F \) we get the expressions

\[
M_{I_{\mu} \rightarrow FM_F}^{(n)} = \sqrt{2}M_{m\rightarrow d}(\psi_{ns}(Z+1)),
\]

\[
M_{I_{\mu} \rightarrow FM_F}^{(n)} = 2\mathcal{M}_{m\rightarrow d}(\psi_{ns}(Z+1)),
\]

\[
M_{I_{\mu} \rightarrow FM_F}^{(n)} = 2\mathcal{M}_{m\rightarrow d}(\psi_{ns}(Z+1)),
\]

where \( \psi_{ns}(Z+1) \) is the wave function of the bound electron in the \( ns \)-state, averaged over the nuclear density \( \Psi \). The wave function \( \psi_{ns}(Z+1) \) is the solution of the Dirac equation [13, 16]. Substituting the amplitudes Eq. (12) into Eq. (11) we get the decay rate

\[
\lambda_{\beta^-} = (3 - g_A)^2\mathcal{M}_{m\rightarrow d}(2\sum_{n=1}^{\infty} |\psi_{ns}(Z+1)|^2 Q_{ns}^2)
\]

\[
+3(1 + g_A)^2\mathcal{M}_{m\rightarrow d}(2\sum_{n=1}^{\infty} |\psi_{ns}(Z+1)|^2 Q_{ns}^2)
\]

\[
= 3(1 + g_A)^2\mathcal{M}_{m\rightarrow d}(2\sum_{n=1}^{\infty} |\psi_{ns}(Z+1)|^2 Q_{ns}^2).
\]

The numerical values of the ratio of the \( \beta^- \)-decays of \( ^{205} \text{Hg}^{80+} \) and \( ^{207} \text{Tl}^{81+} \) are

\[
^{205} \text{Hg}^{80+} : R_{\beta^-/\beta^-}^{\text{th}} = 0.161,
\]

\[
^{207} \text{Tl}^{81+} : R_{\beta^-/\beta^-}^{\text{th}} = 0.190.
\]

These results, obtained for \( R = 1.1 \times A^{1/3} \) fm [1], are stable under reasonable variations of \( R \) as it has been observed for the \( EC \) and \( \beta^+ \) decays of the \( \text{H} \)-like \( ^{140} \text{Pd}^{58+} \) and \( \text{He} \)-like \( ^{140} \text{Pd}^{57+} \) ions in [1].

Concluding discussion

Using the standard theory of weak interactions of heavy ions we have calculated the decay rates of the continuum- and bound-state \( \beta^- \) decays of bare \( ^{205} \text{Hg}^{80+} \) and \( ^{207} \text{Tl}^{81+} \) ions. These are first –forbidden \( \beta^- \) decays [4]. Our result for the ratio of the \( \beta^- \) decays of bare \( ^{207} \text{Tl}^{81+} \) ion \( R_{\beta^-/\beta^-}^{\text{th}} = 0.190 \) agrees with the experimental value \( R_{\beta^-/\beta^-}^{\text{exp}} = 0.188(18) \) within an accuracy of about 1%. This is much better than the theoretical value \( R_{\beta^-/\beta^-}^{\text{th}} = 0.171(1) \), obtained by Taka-hashi and Yokoi [4].

Our result for the ratio of the \( \beta^- \) decay rates of bare \( ^{205} \text{Hg}^{80+} \) ions deviates from the experimental value \( R_{\beta^-/\beta^-}^{\text{exp}} = 0.20(2) \) by about 20%. Since our agreement with the experimental value of \( R_{\beta^-/\beta^-}^{\text{exp}} = 0.20(2) \) for bare \( ^{207} \text{Tl}^{81+} \) ions is about of 1 %, we can argue that the experimental value for the ratio of the \( \beta^- \) decay rates of bare \( ^{205} \text{Hg}^{80+} \) ions maybe actually too high. Our assertion is based on the following. The dependence of the ratio \( R_{\beta^-/\beta^-} \) on the electric charge is rather weak. Indeed, for \( Z = 81 \) instead of \( Z = 80 \) the ratio of the \( \beta^- \) decays \( R_{\beta^-/\beta^-} \) of bare \( ^{205} \text{Hg}^{80+} \) ions changes to the value \( R_{\beta^-/\beta^-} = 0.167 \). Thus, the main distinction is due to different \( Q \) values of the continuum- and bound-state \( \beta^- \) decays. The \( Q \) values \( Q_{\beta^-} = 1515.734 \text{keV} \) and \( Q_{\beta^-} = 1407.471 \text{keV} \) of continuum-state \( \beta^- \) decays result in the Fermi integrals \( f(Q_{\beta^-}, Z + 1) = 22.119 \text{MeV} \) and \( f(Q_{\beta^-}, Z + 1) = 17.747 \text{MeV} \) for \( ^{205} \text{Hg}^{80+} \) and \( ^{207} \text{Tl}^{81+} \), respectively. On the other hand the bound-state \( \beta^- \) decay rates scale with \( Q_{\beta^-}^2 \), where \( Q_{1s} = 1614.557 \text{keV} \) and \( Q_{1s} = 1509.053 \text{keV} \) for \( ^{205} \text{Hg}^{80+} \) and \( ^{207} \text{Tl}^{81+} \), respectively. This implies that the ratio of the continuum- and bound-state \( \beta^- \) decay rates for bare \( ^{205} \text{Hg}^{80+} \) ions should be as minimum 1.1 times smaller relative to the ratio of the continuum- and bound-state \( \beta^- \) decay rates for bare \( ^{205} \text{Tl}^{81+} \) ions. This gives \( R_{\beta^-/\beta^-} \sim 0.171 \) for bare \( ^{205} \text{Hg}^{80+} \) ions at the nuclear radius \( R = 1.1 \times A^{1/3} \). A reduction of the value \( R_{\beta^-/\beta^-} \sim 0.171 \) to \( R_{\beta^-/\beta^-} \sim 0.161 \) is caused by the effective densities of electrons in the \( ns \)–states for different \( (Z + 1) \) values of electric charges of the ions \( ^{205} \text{Tl}^{80+} \) and \( ^{207} \text{Tl}^{81+} \) in the final state of the bound-state \( \beta^- \) decays.
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