Exact Oscillation Probabilities of Neutrinos in Three generations
derived from Relativistic Equation

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In three generations or more, we derive the oscillation probabilities of both Dirac and Majorana neutrinos relativistically by using the Dirac equation. We present various oscillation probabilities for including wrong-helicity neutrinos, right-handed neutrinos, and anti-neutrinos. We summarize the relations between these probabilities. As neutrinos have finite mass, there are two components for each chirality corresponding to positive and negative helicities. We show that the probability is different for each component even if neutrinos have the same chirality. The probabilities derived by the relativistic equation depend on not only the mass squared differences but also the absolute masses of neutrinos. Besides, the new CP phases appear in the probabilities of oscillations with chirality-flip. These new CP phases are equivalent to the Majorana CP phases in the case of Majorana neutrinos. We investigate the CP dependence of oscillation probabilities in vacuum. There are no direct CP violation in $\nu_\alpha \leftrightarrow \nu_\beta^c$ oscillations even if the flavors, $\alpha$ and $\beta$, are different as in the same as two generations. In other words, the difference between the CP-conjugate probabilities vanishes. However, in three generations or more, the sine terms of new CP phases appear in the probabilities in addition to the cosine terms. This is different from the result obtained in two generations. Furthermore, the zero-distance effect does not appear in our formulation.

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I. INTRODUCTION

The idea of neutrino-antineutrino oscillations proposed by Pontecorvo in 1957 \cite{1}. After the discovery of muon neutrino, Maki, Nakagawa and Sakata \cite{2} proposed the oscillations between neutrinos with different flavors in 1962, and the oscillations have been confirmed in the Super-Kamiokande atmospheric neutrino experiment in 1998 \cite{3}. In the period of twenty years after the experiment, the evidence of neutrino oscillations has been accumulating in the solar neutrino experiments \cite{4-6}, the long-baseline experiments \cite{7,8}, and the reactor experiments \cite{9-12}. The understanding of the neutrino mass squared differences and mixing angles has proceeded through these experiments and we are getting the clue of the leptonic Dirac CP phase at present \cite{13,14}. To estimate the value of the Dirac CP phase as precisely as possible, the exact formulation of the oscillation probabilities including matter effect has been developed \cite{15-19}.

On the other hand, $0\nu\beta\beta$ decay experiments have been performed to determine whether the neutrino is the Dirac particle or the Majorana particle \cite{20}, and the absolute value of neutrino mass \cite{21-27}. The possibility for the transition from neutrinos to anti-neutrinos with different flavor were also discussed to investigate the Majorana CP phases \cite{28-33}.

In our previous papers, we have derived the exact neutrino oscillation probabilities relativistically by using the Dirac equation to analyze future neutrino experiments as precisely as possible. In the first paper, we gave the formulation for the Dirac neutrinos in two generations \cite{34}. As the result, a new CP phase different from the Dirac CP phase appears in the oscillations with chirality-flip even in the framework of two-generation Dirac neutrinos. We have also shown that the terms dependent on the absolute value of neutrino mass also appear in our formulation. In the second paper, we applied the relativistic formulation for the Majorana neutrinos \cite{35}. We have shown the new phase that appeared in the case of the Dirac neutrinos becomes the Majorana CP phase. This is because $\nu_\alpha^c$ in Majorana neutrinos plays a role of $\nu_R$ in Dirac neutrinos. We can interpret that the Majorana CP phase is not accompanied by the lepton number violation but with the chirality-flip.

In this paper, we extend our relativistic formulation to three generations or more. We derive various oscillation probabilities for including wrong-helicity neutrinos, right-handed neutrinos, and anti-neutrinos in a unified way. We summarize the relations in these probabilities. In the Dirac equation, there are two components for each chirality and each generation corresponding to positive and negative helicities. We show that the probabilities with different helicities are not the same even if the neutrinos have the same chirality and the same flavor. The probabilities derived by the relativistic equation depend on not only the mass squared differences but also the absolute masses of neutrinos. Besides, the new CP phases appear in the probabilities of oscillations with chirality-flip. These new CP phases are equivalent to the Majorana CP phases in the case of Majorana neutrinos. We investigate the CP dependence of oscillation probabilities.
in vacuum. In the case of Majorana neutrinos, there is no direct CP violation in $\nu_\alpha \leftrightarrow \nu_\beta^c$ oscillations even if the flavors, $\alpha$ and $\beta$, are different as in the same as two generations [32]. In other words, the difference between the CP-conjugate probabilities, $P(\nu_{\alpha L} \rightarrow \nu_{\beta L}^c) - P(\nu_{\beta L} \rightarrow \nu_{\alpha L}^c)$, vanishes. However, in three generations or more, the sine terms of new CP phases appear in the probabilities in addition to the cosine terms. This is different from the result obtained in two generations. Furthermore, the zero-distance effect [30], which was known as the phenomena for neutrinos instantly changing to anti-neutrinos, cannot be occurred from our calculation. These results are different from the previous ones.

We give the number of the independent CP phases in $n$-generations. For both Dirac and Majorana neutrinos, the number of the Dirac CP phases is given by

$$\frac{(n-1)(n-2)}{2},$$  \hspace{1cm} (1)

and the number of the CP phases accompanied to the oscillations with chirality-flip is

$$n-1.$$  \hspace{1cm} (2)

Therefore, the total number of the independent CP phases becomes

$$\frac{n(n-1)}{2},$$  \hspace{1cm} (3)

and in accordance with the result of the Majorana neutrinos [36]. If neutrinos are the Dirac particles and the flavors of $\nu_R$ cannot be distinguished beyond the Standard Model, $(n-1)$ CP phases originated from the oscillations with the chirality-flip are not observable and coincide with the previous result.

The paper is organized as follows. In section II, we define our notations used in this paper. In section III, we review the non-relativistic derivation of neutrino oscillation probabilities developed in the previous papers by using the Schrödinger equation. In section IV, we present the relativistic derivation of various neutrino oscillation probabilities for Dirac neutrinos including wrong-helicity neutrinos, right-handed neutrinos and anti-neutrinos by using the Dirac equation. We also investigate the CP dependence of the probabilities, in particular on the new CP phases, and count the number of independent CP phases. In section V, we also present the relativistic derivation of oscillation probabilities for Majorana neutrinos. In section VI, we summarize the relation of these oscillation probabilities. In section VII, we compare our result of the Majorana neutrinos with the previous one. In section VIII, we summarize our results obtained in this paper.

II. NOTATION

In this section, we write down the notation used in this paper. We mainly use the chiral representation because neutrinos are measured through weak interaction. In chiral representation, the gamma matrices with 4-dimensional representation, the gamma matrices with 4 components are given by

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \gamma^i = \begin{pmatrix} 0 & -\sigma_i \\ \sigma_i & 0 \end{pmatrix}, \gamma_5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$  \hspace{1cm} (4)

where $2 \times 2$ $\sigma_i$ matrices are defined by

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$  \hspace{1cm} (5)

We also define 4-component spinors $\psi, \psi_L$ and $\psi_R$ as

$$\psi = \begin{pmatrix} \xi \\ \eta \end{pmatrix},$$  \hspace{1cm} (6)

$$\psi_L = \frac{1 - \gamma_5}{2} \psi = \begin{pmatrix} 0 \\ \eta \end{pmatrix}, \psi_R = \frac{1 + \gamma_5}{2} \psi = \begin{pmatrix} \xi \\ 0 \end{pmatrix},$$  \hspace{1cm} (7)

and 2-component spinors $\xi$ and $\eta$ as

$$\xi = \begin{pmatrix} \nu_R^c \\ \nu_R \end{pmatrix}, \eta = \begin{pmatrix} \nu_L^c \\ \nu_L \end{pmatrix}.  \hspace{1cm} (8)$$

Furthermore, we use the subscript $\alpha$ and $\beta$ for flavor, $L$ and $R$ for chirality, the number $j$ and $k$ for generation and superscript $\pm$ for energy. Because of negligible neutrino mass, mass eigenstate has been often identified with energy eigenstate in the previous papers. But in the future, we should distinguish these two kinds of eigenstates for the finite neutrino mass. More concretely, we use the following eigenstates;

$$\text{chirality-flavor eigenstates} : \nu_{\alpha L}, \nu_{\alpha R}, \nu_{\beta L}^c, \nu_{\beta R}^c,$$  \hspace{1cm} (9)

$$\text{chirality-mass eigenstates} : \nu_{j L}, \nu_{j R}, \nu_{k L}, \nu_{k R},$$  \hspace{1cm} (10)

$$\text{energy-helicity eigenstates} : \nu_j^+, \nu_j^-, \nu_k^+, \nu_k^-.$$  \hspace{1cm} (11)

It is noted that chirality-mass eigenstates are not exactly the eigenstates of the Hamiltonian. We use the term, eigenstates, in the sense that the mass submatrix in the Hamiltonian is diagonalized. Judging from common sense, one may think it strange that the chirality and the mass live in the same eigenstate. Details will be explained in the subsequent section.

We also define the spinor for anti-neutrino as charge conjugation of neutrino $\psi^c = i\gamma^2 \psi^*$. The charge conjugations for left-handed and right-handed neutrinos are
defined by

\[ \psi_L^c = (\psi_L)^c = \begin{pmatrix} \nu_L^c \\ \nu_R^c \\ 0 \\ 0 \end{pmatrix} = i\gamma^2\psi_L^c = i\gamma^2\frac{1-\gamma_5}{2}\psi^c \]

\[ = \frac{1+\gamma_5}{2}(i\gamma^2\psi^c) = (\psi^c)_R = \begin{pmatrix} i\sigma_2\eta_i \\ 0 \end{pmatrix} = \begin{pmatrix} \nu_L^c \\ 0 \\ 0 \end{pmatrix} = \nu_R^c \]

\[ \psi_R^c = (\psi_R)^c = \begin{pmatrix} 0 \\ 0 \\ \nu_R^c \\ \nu_L^c \end{pmatrix} = i\gamma^2\psi_R^c = i\gamma^2\frac{1+\gamma_5}{2}\psi^c \]

It is noted that the chirality is flipped by taking the charge conjugation.

Rewriting the relation about the fields to one particle states by using the production operator, we obtain

\[ |\nu_{\alpha L}(t)\rangle = \sum_{j=1}^{3} U_{\alpha j}^* e^{-iE_j t} |\nu_j^+\rangle \]

and also their conjugate states,

\[ \langle \nu_{\beta L}| = \sum_{j=1}^{3} U_{\beta j} |\nu_j^+\rangle. \]

III. REVIEW OF OSCILLATION PROBABILITIES FROM NON-RELATIVISTIC EQUATION

In this section, we review how the neutrino oscillation probabilities in vacuum were derived in the previous papers. For example, in ref. [37], the flavor eigenstates are given as the linear combination of the energy (mass) eigenstates,

\[ \begin{pmatrix} \nu_{\alpha L} \\ \nu_{\mu L} \\ \nu_{\tau L} \end{pmatrix} = \begin{pmatrix} U_{e1} & U_{e2} & U_{e3} \\ U_{\mu 1} & U_{\mu 2} & U_{\mu 3} \\ U_{\tau 1} & U_{\tau 2} & U_{\tau 3} \end{pmatrix} \begin{pmatrix} \nu_1^+ \\ \nu_2^+ \\ \nu_3^+ \end{pmatrix}. \]

The energy eigenstates evolve following the equation

\[ \frac{d}{dt} \begin{pmatrix} \nu_1^+ \\ \nu_2^+ \\ \nu_3^+ \end{pmatrix} = \begin{pmatrix} E_1 & 0 & 0 \\ 0 & E_2 & 0 \\ 0 & 0 & E_3 \end{pmatrix} \begin{pmatrix} \nu_1^+ \\ \nu_2^+ \\ \nu_3^+ \end{pmatrix}, \]

and after the time \( t \), the flavor eigenstates become

If we take a certain flavor \( e, \mu \) or \( \tau \) as \( \alpha \) and \( \beta \), the amplitude for \( \nu_{\alpha L} \) to \( \nu_{\beta L} \) is given by

\[ A(\nu_{\alpha L} \rightarrow \nu_{\beta L}) = \langle \nu_{\beta L}|\nu_{\alpha L}(t)\rangle = \sum_{j=1}^{3} U_{\alpha j}^* U_{\beta j} e^{-iE_j t} \]

The oscillation probability for \( \nu_{\alpha L} \) to \( \nu_{\beta L} \) becomes

\[ P(\nu_{\alpha L} \rightarrow \nu_{\beta L}) = |A(\nu_{\alpha L} \rightarrow \nu_{\beta L})|^2 = \sum_j |U_{\alpha j}^* U_{\beta j}|^2 + 2\text{Re}[U_{\alpha j}^* U_{\beta j} U_{\alpha k} U_{\beta k} e^{-i(E_j - E_k)t}] \]

\[ = \sum_j |U_{\alpha j}^* U_{\beta j}|^2 + 2 \sum_{j<k} \text{Re}[U_{\alpha j}^* U_{\beta j} U_{\alpha k} U_{\beta k}^*] \cos \Delta E_{jk} t + \text{Im}[U_{\alpha j}^* U_{\beta j} U_{\alpha k} U_{\beta k}^*] \sin \Delta E_{jk} t \]

\[ = \sum_j |U_{\alpha j}^* U_{\beta j}|^2 - 2 \sum_{j<k} \text{Re}[U_{\alpha j}^* U_{\beta j} U_{\alpha k} U_{\beta k}^*](1 - \cos \Delta E_{jk} t) + 2 \sum_{j<k} \text{Im}[U_{\alpha j}^* U_{\beta j} U_{\alpha k} U_{\beta k}^*] \sin \Delta E_{jk} t \]

\[ = \delta_{\alpha \beta} - 4 \sum_{j<k} \text{Re}[U_{\alpha j}^* U_{\beta j} U_{\alpha k} U_{\beta k}^*] \sin^2 \left( \frac{\Delta E_{jk} t}{2} \right) - 2 \sum_{j<k} \text{Im}[U_{\alpha j}^* U_{\beta j} U_{\alpha k} U_{\beta k}^*] \sin \Delta E_{jk} t, \]

(20)
where $\Delta E_{jk} = E_j - E_k$. Writing the survival probability and the transition probability separately, we obtain

\[ P(\nu_{\alpha L} \rightarrow \nu_{\alpha L}) = 1 - \sum_{j<k} 4 |U_{\alpha j} U_{\alpha k}|^2 \sin^2 \left( \frac{\Delta E_{jk} t}{2} \right). \]

\[ P(\nu_{\alpha L} \rightarrow \nu_{\beta L}) = -4 \sum_{j<k} \text{Re}[U_{\alpha j} U_{\beta j}^* U_{\alpha k} U_{\beta k}] \sin^2 \left( \frac{\Delta E_{jk} t}{2} \right) - 2 \sum_{j<k} \text{Im}[U_{\alpha j} U_{\beta j}^* U_{\alpha k} U_{\beta k}] \sin \Delta E_{jk} t. \]  

These representations are valid regardless of whether neutrinos are the Dirac particles or the Majorana particles and can be extended to $n$ generations. In the usual oscillations without chirality-flip, the oscillation probability does not depend on the Majorana CP phase and depends only on the Dirac CP phase.

**IV. OSCILLATION PROBABILITIES OF DIRAC NEUTRINOS FROM RELATIVISTIC EQUATION**

In this section, we derive the neutrino oscillation probabilities from the relativistic equation in the case of three-generation neutrinos with only Dirac mass term. At first, we calculate the oscillation probabilities of left-handed neutrinos to other neutrinos. Next, we investigate the CP dependence of these probabilities and check the unitarity. Second, we calculate the probabilities of also left-handed but wrong-helicity neutrinos. After that, we derive the probabilities of right-handed neutrinos and antineutrinos.

**A. Oscillation Probabilities of Left-Handed Neutrinos**

In three generations, the lagrangian for the Dirac neutrinos is represented by the spinors with four components as

\[ L = \sum_{\alpha} (i \bar{\psi}_{\alpha L} \gamma^\mu \partial_\mu \psi_{\alpha L} + i \bar{\psi}_{\alpha R} \gamma^\mu \partial_\mu \psi_{\alpha R}) - \frac{1}{2} m^2 \sum_{(\alpha, \beta)} (\overline{\psi}_{\alpha L} m^*_{\beta \alpha} \psi_{\beta R} + \overline{\psi}_{\alpha R} m_{\alpha \beta} \psi_{\beta L}), \]

where $(\alpha, \beta)$ means the sum over all combinations of $e$, $\mu$, and $\tau$. The Eular-Lagrange equation for $\psi_{\alpha L}$,

\[ \frac{\partial L}{\partial \psi_{\alpha L}} - \partial_\mu \left( \frac{\partial L}{\partial (\partial_\mu \psi_{\alpha L})} \right) = 0, \]

leads to the equation,

\[ i \gamma^\mu \partial_\mu \psi_{\alpha L} - \overline{\psi}_{\beta R} m^*_{\alpha \beta} \psi_{\beta R} = 0. \]

Multiplying $\gamma^0$ from the left, the equation becomes

\[ i \partial_0 \psi_{\alpha L} + i \gamma^0 \gamma^i \partial_i \psi_{\alpha L} - \sum_{\beta} m^*_{\alpha \beta} \gamma^0 \psi_{\beta R} = 0. \]  

This equation is represented by two-component spinors $\xi$ and $\eta$ as

\[ i \partial_0 \left( \begin{array}{c} \xi \\ \eta \end{array} \right) + i \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right) \left( \begin{array}{c} 0 \\ \sigma_i \end{array} \right) \partial_i \left( \begin{array}{c} \xi \\ \eta \end{array} \right) - \sum_{\beta} m^*_{\alpha \beta} \left( \begin{array}{c} 0 \\ 0 \end{array} \right) = 0. \]

Taking the lower two components, we obtain the equation,

\[ i \partial_0 \eta - i \sigma_i \partial_i \eta - \sum_{\beta} m^*_{\alpha \beta} \xi_\beta = 0. \]

In the same way, the Eular-Lagrange equation

\[ \frac{\partial L}{\partial \psi_{\alpha R}} - \partial_\mu \left( \frac{\partial L}{\partial (\partial_\mu \psi_{\alpha R})} \right) = 0, \]

leads to

\[ i \gamma^\mu \partial_\mu \psi_{\alpha R} - \sum_{\beta} m_{\alpha \beta} \psi_{\beta L} = 0. \]

Multiplying $\gamma^0$ from the left, the equation becomes

\[ i \partial_0 \psi_{\alpha R} + i \gamma^0 \gamma^i \partial_i \psi_{\alpha R} - \sum_{\beta} m_{\alpha \beta} \gamma^0 \psi_{\beta L} = 0. \]  

This equation is also represented by two-component spinors, $\xi$ and $\eta$ as

\[ i \partial_0 \left( \begin{array}{c} \xi \\ \eta \end{array} \right) + i \left( \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right) \left( \begin{array}{c} \sigma_i \end{array} \right) \partial_i \left( \begin{array}{c} \xi \\ \eta \end{array} \right) - \sum_{\beta} m_{\alpha \beta} \left( \begin{array}{c} 0 \\ 0 \end{array} \right) = 0. \]

Taking the upper two components, we obtain the equation,

\[ i \partial_0 \xi + i \sigma_i \partial_i \xi - \sum_{\beta} m_{\alpha \beta} \eta_\beta = 0. \]
Here, we assume the equal momentum for different flavors and factor out the dependence of the distance as

\[ \eta_\alpha(x, t) = e^{i\vec{p} \cdot \vec{x}} \eta_\alpha(t) = e^{i\vec{p} \cdot \vec{x}} \begin{pmatrix} \nu_{\alpha L} \\ \nu_{\alpha L} \end{pmatrix}, \]

(36)

\[ \xi_\alpha(x, t) = e^{i\vec{p} \cdot \vec{x}} \xi_\alpha(t) = e^{i\vec{p} \cdot \vec{x}} \begin{pmatrix} \nu'_{\alpha R} \\ \nu_{\alpha R} \end{pmatrix}. \]

(37)

Furthermore, if we choose \( \vec{p} = (0, 0, p) \), the equations \( 36 \) and \( 37 \) are rewritten as

\[ i\partial_0 \begin{pmatrix} \nu'_{\alpha L} \\ \nu_{\alpha L} \end{pmatrix} + p \begin{pmatrix} \nu'_{\alpha L} \\ -\nu_{\alpha L} \end{pmatrix} - \sum_\beta m_{\beta\alpha}^* \begin{pmatrix} \nu'_{\beta R} \\ \nu_{\beta R} \end{pmatrix} = 0, \]

(38)

\[ i\partial_0 \begin{pmatrix} \nu'_{\alpha R} \\ \nu_{\alpha R} \end{pmatrix} - p \begin{pmatrix} \nu'_{\alpha R} \\ -\nu_{\alpha R} \end{pmatrix} - \sum_\beta m_{\alpha\beta} \begin{pmatrix} \nu'_{\beta L} \\ \nu_{\beta L} \end{pmatrix} = 0. \]

(39)

If we write the equations \( 38 \) and \( 39 \) for three flavors together in a matrix form, the time evolution of the chirality-flavor eigenstates is represented by

\[ \frac{d}{dt} \begin{pmatrix} \nu_{eR} \\ \nu_{\mu R} \\ \nu_{\tau R} \\ \nu_{eL} \\ \nu_{\mu L} \\ \nu_{\tau L} \end{pmatrix} = \begin{pmatrix} p \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} m_{ee} & m_{e\mu} & m_{e\tau} \\ m_{\mu e} & m_{\mu\mu} & m_{\mu\tau} \\ m_{\tau e} & m_{\tau\mu} & m_{\tau\tau} \end{pmatrix} \begin{pmatrix} p \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} \nu_{eR} \\ \nu_{\mu R} \\ \nu_{\tau R} \\ \nu_{eL} \\ \nu_{\mu L} \\ \nu_{\tau L} \end{pmatrix}, \]

(40)

as same as the two generations. As the lower-right part is separated from the upper-left part completely in the case of Dirac neutrinos, they cannot mix each other even if the time has passed. Below, we consider the lower-right part,

\[ \frac{d}{dt} \begin{pmatrix} \nu_{eR} \\ \nu_{\mu R} \\ \nu_{\tau R} \\ \nu_{eL} \\ \nu_{\mu L} \\ \nu_{\tau L} \end{pmatrix} = \begin{pmatrix} -p \\ 0 \\ 0 \\ p \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} m_{ee} & m_{e\mu} & m_{e\tau} \\ m_{\mu e} & m_{\mu\mu} & m_{\mu\tau} \\ m_{\tau e} & m_{\tau\mu} & m_{\tau\tau} \end{pmatrix} \begin{pmatrix} -p \\ 0 \\ 0 \\ p \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} \nu_{eR} \\ \nu_{\mu R} \\ \nu_{\tau R} \\ \nu_{eL} \\ \nu_{\mu L} \\ \nu_{\tau L} \end{pmatrix}. \]

(41)

The chirality-flavor eigenstates are represented as the linear combination of the chirality-mass eigenstates,

\[ \begin{pmatrix} \nu_{eR} \\ \nu_{\mu R} \\ \nu_{\tau R} \\ \nu_{eL} \\ \nu_{\mu L} \\ \nu_{\tau L} \end{pmatrix} \begin{pmatrix} V_{e1} & V_{e2} & V_{e3} \\ V_{\mu 1} & V_{\mu 2} & V_{\mu 3} \\ V_{\tau 1} & V_{\tau 2} & V_{\tau 3} \end{pmatrix} \begin{pmatrix} \nu_{1R} \\ \nu_{2R} \\ \nu_{3R} \end{pmatrix}, \]

(42)
The mass submatrix in the Hamiltonian is diagonalized by the mixing matrix $\mathcal{U}$ defined above as

$$
\mathcal{U} = \begin{pmatrix}
-p & 0 & 0 & m_{ee} & m_{e\mu} & m_{e\tau} \\
0 & -p & 0 & m_{\mu e} & m_{\mu\mu} & m_{\mu\tau} \\
0 & 0 & -p & m_{\tau e} & m_{\tau\mu} & m_{\tau\tau}
\end{pmatrix}
\begin{pmatrix}
-p & 0 & 0 & m_1 & 0 & 0 \\
0 & -p & 0 & 0 & m_2 & 0 \\
0 & 0 & -p & 0 & 0 & m_3
\end{pmatrix}.
$$

(43)

The time evolution of the chirality-mass eigenstates is given by

$$
\frac{d}{dt} \begin{pmatrix}
\nu_{1R} \\
\nu_{1L} \\
\nu_{2R} \\
\nu_{2L} \\
\nu_{3R} \\
\nu_{3L}
\end{pmatrix} = \begin{pmatrix}
-p & 0 & 0 & m_1 & 0 & 0 \\
0 & -p & 0 & 0 & m_2 & 0 \\
0 & 0 & -p & 0 & 0 & m_3
\end{pmatrix}
\begin{pmatrix}
\nu_{1R} \\
\nu_{2R} \\
\nu_{3R} \\
\nu_{1L} \\
\nu_{2L} \\
\nu_{3L}
\end{pmatrix}.
$$

(44)

In order to diagonalize the Hamiltonian in eq. (44) completely, let us rewrite this into the equation for the energy-helicity eigenstates. Exchanging some rows and some columns of eq. (44) and grouping by each generation, it can be rewritten as

$$
\frac{d}{dt} \begin{pmatrix}
\nu_{1R} \\
\nu_{1L} \\
\nu_{2R} \\
\nu_{2L} \\
\nu_{3R} \\
\nu_{3L}
\end{pmatrix} = \begin{pmatrix}
-p & m_1 & 0 & 0 & 0 & 0 \\
0 & -p & m_2 & 0 & 0 & 0 \\
0 & 0 & -p & m_3 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
\nu_{1R} \\
\nu_{2R} \\
\nu_{3R} \\
\nu_{1L} \\
\nu_{2L} \\
\nu_{3L}
\end{pmatrix}.
$$

(45)

The chirality-mass eigenstates are represented as the linear combination of the energy-helicity eigenstates,

$$
\begin{pmatrix}
\nu_{1R} \\
\nu_{1L} \\
\nu_{2R} \\
\nu_{2L} \\
\nu_{3R} \\
\nu_{3L}
\end{pmatrix} = 
\begin{pmatrix}
\sqrt{\frac{E_1 + p}{2E_1}} & \sqrt{\frac{E_1 - p}{2E_1}} & 0 & 0 & 0 & 0 \\
0 & 0 & \sqrt{\frac{E_2 + p}{2E_2}} & \sqrt{\frac{E_2 - p}{2E_2}} & 0 & 0 \\
0 & 0 & \sqrt{\frac{E_3 + p}{2E_3}} & \sqrt{\frac{E_3 - p}{2E_3}} & 0 & 0 \\
0 & 0 & 0 & 0 & \sqrt{\frac{E_1 + p}{2E_1}} & \sqrt{\frac{E_1 - p}{2E_1}} \\
0 & 0 & 0 & 0 & \sqrt{\frac{E_2 + p}{2E_2}} & \sqrt{\frac{E_2 - p}{2E_2}} \\
0 & 0 & 0 & 0 & \sqrt{\frac{E_3 + p}{2E_3}} & \sqrt{\frac{E_3 - p}{2E_3}}
\end{pmatrix}
\begin{pmatrix}
\nu_1^- \\
\nu_1^+ \\
\nu_2^- \\
\nu_2^+ \\
\nu_3^- \\
\nu_3^+
\end{pmatrix}.
$$

(46)

Then, the time evolution of the energy-helicity eigenstates is given by

$$
\frac{d}{dt} \begin{pmatrix}
\nu_1^- \\
\nu_1^+ \\
\nu_2^- \\
\nu_2^+ \\
\nu_3^- \\
\nu_3^+
\end{pmatrix} = 
\begin{pmatrix}
-E_1 & 0 & 0 & 0 & 0 & 0 \\
0 & E_1 & 0 & 0 & 0 & 0 \\
0 & 0 & -E_2 & 0 & 0 & 0 \\
0 & 0 & 0 & E_2 & 0 & 0 \\
0 & 0 & 0 & 0 & -E_3 & 0 \\
0 & 0 & 0 & 0 & 0 & E_3
\end{pmatrix}
\begin{pmatrix}
\nu_1^- \\
\nu_1^+ \\
\nu_2^- \\
\nu_2^+ \\
\nu_3^- \\
\nu_3^+
\end{pmatrix}.
$$

(47)

where

$$
E_j = \sqrt{p^2 + m_j^2} \quad (j = 1, 2, 3).
$$

(48)
Connecting eqs. (12) and (46), the chirality-flavor eigenstates are represented by the energy-helicity eigenstates,

\[
\begin{pmatrix}

\nu_{\alpha R} \\
\nu_{\mu R} \\
\nu_{\tau R} \\
\nu_{e L} \\
\nu_{\mu L} \\
\nu_{\tau L}
\end{pmatrix} = U \begin{pmatrix}

1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0
\end{pmatrix} \begin{pmatrix}

\nu_1^+ \\
\nu_2^+ \\
\nu_3^+ \\
\nu_1^- \\
\nu_2^- \\
\nu_3^-
\end{pmatrix}
\]

\[
\left(\begin{array}{c}
\sqrt{E_1 - p_{\nu_1}} V_{\nu_1} \\
\sqrt{E_1 + p_{\nu_1}} V_{\nu_1} \\
\sqrt{E_2 - p_{\nu_2}} V_{\nu_2} \\
\sqrt{E_2 + p_{\nu_2}} V_{\nu_2} \\
\sqrt{E_3 - p_{\nu_3}} V_{\nu_3} \\
\sqrt{E_3 + p_{\nu_3}} V_{\nu_3}
\end{array}\right) = U \begin{pmatrix}

E_1 & -E_1 & E_2 & -E_2 & E_3 & -E_3 \\
E_1 & E_1 & E_2 & E_2 & E_3 & E_3 \\
E_2 & -E_2 & E_3 & -E_3 & E_2 & -E_2 \\
E_2 & E_2 & E_3 & E_3 & E_2 & E_2 \\
E_3 & -E_3 & E_3 & E_3 & E_3 & E_3 \\
E_3 & E_3 & E_3 & E_3 & E_3 & E_3
\end{pmatrix} \begin{pmatrix}

\nu_1^+ \\
\nu_2^+ \\
\nu_3^+ \\
\nu_1^- \\
\nu_2^- \\
\nu_3^-
\end{pmatrix} .
\]

(49)

As same as the non-relativistic case, rewriting the relation about the fields to one particle states, we obtain

\[
|\nu_{\alpha L}(t)| = \sum_{j=1}^{3} \left( -\sqrt{\frac{E_j - p_j}{2E_j}} U_{\alpha j}^* e^{iE_j t} |\nu_j^-\rangle + \sqrt{\frac{E_j + p_j}{2E_j}} U_{\alpha j} e^{-iE_j t} |\nu_j^+\rangle \right),
\]

(50)

and their conjugate states,

\[
\langle \nu_{\beta L} | = \sum_{j=1}^{3} \left( -\sqrt{\frac{E_j - p_j}{2E_j}} U_{\beta j} |\nu_j^-\rangle + \sqrt{\frac{E_j + p_j}{2E_j}} U_{\beta j} |\nu_j^+\rangle \right)
\]

(51)

\[
\langle \nu_{\beta R} | = \sum_{j=1}^{3} \left( \sqrt{\frac{E_j + p_j}{2E_j}} V_{\beta j} |\nu_j^-\rangle + \sqrt{\frac{E_j - p_j}{2E_j}} V_{\beta j} |\nu_j^+\rangle \right)
\]

(52)

Note that a one particle state of $\nu_{\alpha L}$ includes not only the positive energy parts but also the negative energy parts. This is the difference between the relativistic and non-relativistic case. In the same way as the non-relativistic case, we calculate the amplitudes,

\[
A(\nu_{\alpha L} \rightarrow \nu_{\beta L}) = \langle \nu_{\beta L} | \nu_{\alpha L}(t) \rangle = \sum_{j} U_{\alpha j}^* U_{\beta j} \left( \sqrt{\frac{E_j - p_j}{2E_j}} e^{iE_j t} + \sqrt{\frac{E_j + p_j}{2E_j}} e^{-iE_j t} \right)
\]

\[
= \sum_{j} U_{\alpha j}^* U_{\beta j} \left( \frac{e^{iE_j t} + e^{-iE_j t}}{2} - \frac{p_j}{2E_j} \right) = \sum_{j} U_{\alpha j}^* U_{\beta j} \left\{ \cos(E_j t) - i \frac{p_j}{E_j} \sin(E_j t) \right\},
\]

(53)

\[
A(\nu_{\alpha L} \rightarrow \nu_{\beta R}) = \langle \nu_{\beta R} | \nu_{\alpha L}(t) \rangle = \sum_{j} -U_{\alpha j}^* V_{\beta j} \frac{m_j}{2E_j} (e^{iE_j t} - e^{-iE_j t}) = \sum_{j} -iU_{\alpha j}^* V_{\beta j} \frac{m_j}{E_j} \sin(E_j t).
\]

(54)
Furthermore, we derive the oscillation probabilities by squaring the absolute value of the amplitudes,

\[ P(\nu_{\alpha L} \rightarrow \nu_{\beta L}) = \sum_j |U_{\alpha j}U_{\beta j}|^2 \left\{ 1 - \frac{E_j^2 - p^2}{E_j^2} \sin^2(E_j t) \right\} \]

\[ + 2 \sum_{j<k} \text{Re} \left[ U_{\alpha j}^*U_{\beta j}U_{\alpha k}U_{\beta k}^* \right] \left\{ \cos(\Delta E_{jkt}) - \frac{E_jE_k - p^2}{E_jE_k} \sin(E_jt) \sin(E_kt) \right\} \]

\[ + 2 \sum_{j<k} \text{Im} \left[ U_{\alpha j}^*U_{\beta j}U_{\alpha k}U_{\beta k}^* \right] \left\{ \sin(\Delta E_{jkt}) + \frac{E_k - p}{E_k} \cos(E_jt) \sin(E_kt) - \frac{E_j - p}{E_j} \cos(E_kt) \sin(E_jt) \right\} \]

\[ = \delta_{\alpha\beta} - \sum_j |U_{\alpha j}U_{\beta j}|^2 \left\{ \frac{m_j^2}{E_j^2} \sin^2(E_j t) \right\} \]

\[ - 2 \sum_{j<k} \text{Re} \left[ U_{\alpha j}U_{\beta j}U_{\alpha k}^*U_{\beta k}^* \right] \left\{ 2 \sin^2 \left( \frac{\Delta E_{jkt}}{2} \right) + \frac{E_jE_k - p^2}{E_jE_k} \sin(E_jt) \sin(E_kt) \right\} \]

\[ - 2 \sum_{j<k} \text{Im} \left[ U_{\alpha j}^*U_{\beta j}U_{\alpha k}^*U_{\beta k}^* \right] \left\{ \sin(\Delta E_{jkt}) + \frac{E_k - p}{E_k} \cos(E_jt) \sin(E_kt) - \frac{E_j - p}{E_j} \cos(E_kt) \sin(E_jt) \right\} \] \hspace{1cm} (55)

\[ P(\nu_{\alpha L} \rightarrow \nu_{\beta R}) = \sum_j |U_{\alpha j}V_{\beta j}|^2 \frac{m_j^2}{E_j^2} \sin^2(E_j t) + 2 \sum_{j<k} \text{Re}[U_{\alpha j}^*V_{\alpha k}^*V_{\beta j}V_{\beta k}^*] \frac{m_jm_k}{E_jE_k} \sin(E_jt) \sin(E_kt). \] \hspace{1cm} (56)

If we describe the survival probabilities (the case for \( \alpha = \beta \)) and the transition probabilities (the case for \( \alpha \neq \beta \)) separately, we obtain

\[ P(\nu_{\alpha L} \rightarrow \nu_{\alpha L}) = 1 - \sum_j |U_{\alpha j}|^4 \left\{ \frac{m_j^2}{E_j^2} \sin^2(E_j t) \right\} \] \hspace{1cm} (57)

\[ - 2 \sum_{j<k} |U_{\alpha j}U_{\alpha k}|^2 \left\{ 2 \sin^2 \left( \frac{\Delta E_{jkt}}{2} \right) + \frac{E_jE_k - p^2}{E_jE_k} \sin(E_jt) \sin(E_kt) \right\}, \] \hspace{1cm} (58)

\[ P(\nu_{\alpha L} \rightarrow \nu_{\beta L}) = - \sum_j |U_{\alpha j}U_{\beta j}|^2 \left\{ \frac{m_j^2}{E_j^2} \sin^2(E_j t) \right\} \] \hspace{1cm} (59)

\[ - 2 \sum_{j<k} \text{Re} \left[ U_{\alpha j}U_{\beta j}^*U_{\alpha k}U_{\beta k}^* \right] \left\{ 2 \sin^2 \left( \frac{\Delta E_{jkt}}{2} \right) + \frac{E_jE_k - p^2}{E_jE_k} \sin(E_jt) \sin(E_kt) \right\} \] \hspace{1cm} (60)

\[ - 2 \sum_{j<k} \text{Im} \left[ U_{\alpha j}^*U_{\beta j}U_{\alpha k}^*U_{\beta k}^* \right] \left\{ \sin(\Delta E_{jkt}) + \frac{E_k - p}{E_k} \cos(E_jt) \sin(E_kt) - \frac{E_j - p}{E_j} \cos(E_kt) \sin(E_jt) \right\} \] \hspace{1cm} (61)

\[ P(\nu_{\alpha L} \rightarrow \nu_{\alpha R}) = \sum_j |U_{\alpha j}V_{\alpha j}|^2 \frac{m_j^2}{E_j^2} \sin^2(E_j t) + 2 \sum_{j<k} \text{Re}[U_{\alpha j}^*V_{\alpha k}^*V_{\alpha j}V_{\alpha k}] \frac{m_jm_k}{E_jE_k} \sin(E_jt) \sin(E_kt), \] \hspace{1cm} (62)

\[ P(\nu_{\alpha L} \rightarrow \nu_{\beta R}) = \sum_j |U_{\alpha j}V_{\beta j}|^2 \frac{m_j^2}{E_j^2} \sin^2(E_j t) + 2 \sum_{j<k} \text{Re}[U_{\alpha j}^*V_{\alpha k}^*V_{\beta j}V_{\beta k}] \frac{m_jm_k}{E_jE_k} \sin(E_jt) \sin(E_kt). \] \hspace{1cm} (63)

In three or more generations, the terms proportional to the imaginary part of the product for four matrix elements represented in (61) is added. The results, (57)-(61), derived by the relativistic method should be compared by the results, (21) and (22), derived by the non-relativistic method. Comparing these equations, one can see that the first terms of (57), (58), (60) and (61) are equal to the results by the non-relativistic method and the remaining terms is new terms appeared as the correction. It is also noted that the representation of the above probabilities is parameter independent of the unitary matrix.

**B. CP dependence of Oscillation probability**

Next, let us consider the CP dependence of the oscillation probabilities. We would like to show how the
new CP phases appear with chirality-flip in the oscillation probabilities. The $3 \times 3$ unitary matrix is in general represented by nine parameters as

$$ U = \begin{pmatrix} e^{i\rho_L} & 0 & 0 \\ 0 & e^{i\rho_R} & 0 \\ 0 & 0 & e^{i\rho_L} \end{pmatrix} \tilde{U} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{i\phi_2 L} & 0 \\ 0 & 0 & e^{i\phi_3 L} \end{pmatrix}, \quad (64) $$

$$ V = \begin{pmatrix} e^{i\phi_R} & 0 & 0 \\ 0 & e^{i\phi_R} & 0 \\ 0 & 0 & e^{i\phi_3 R} \end{pmatrix} \tilde{V} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{i\phi_2 R} & 0 \\ 0 & 0 & e^{i\phi_3 R} \end{pmatrix}. \quad (65) $$

The dependence of overall phases remains in the oscillation probabilities without chirality-flip. On the other hand, the product of four matrix elements in the probabilities (62) and (63) becomes

$$ U_{\alpha j} U_{\beta k} = e^{i\rho_{\alpha L}} U_{\alpha j} e^{i\phi_{j L}} e^{-i\phi_{j L}} U_{\beta j} e^{-i\phi_{j L}} U_{\beta k} e^{i\phi_{k L}} = U_{\alpha j} \tilde{U}_{\beta j} U_{\beta k} $$

$$ V_{\alpha j} V_{\beta k} = e^{i\phi_{R}} V_{\alpha j} e^{i\phi_{R}} e^{-i\phi_{R}} V_{\beta j} e^{-i\phi_{R}} V_{\beta k} e^{i\phi_{R}} = V_{\alpha j} \tilde{V}_{\beta j} V_{\beta k}. \quad (66) \quad (67) $$

One can see that the overall phases cancel out and only the Dirac CP phases included in $\tilde{U}$ and $\tilde{V}$ remain in the oscillation probabilities without chirality-flip. On the other hand, the product of four matrix elements in the probabilities (62) and (63) becomes

$$ U_{\alpha j} U_{\beta k} = e^{i\rho_{\alpha L}} U_{\alpha j} e^{i\phi_{j L}} e^{-i\phi_{j L}} U_{\beta j} e^{-i\phi_{j L}} U_{\beta k} e^{i\phi_{k L}} = U_{\alpha j} \tilde{U}_{\beta j} U_{\beta k} $$

$$ V_{\alpha j} V_{\beta k} = e^{i\phi_{R}} V_{\alpha j} e^{i\phi_{R}} e^{-i\phi_{R}} V_{\beta j} e^{-i\phi_{R}} V_{\beta k} e^{i\phi_{R}} = V_{\alpha j} \tilde{V}_{\beta j} V_{\beta k}. \quad (68) $$

The dependence of overall phases remains in the oscillation probabilities with chirality-flip. In principle, these overall phases can be observed if we can distinguish the flavor of right-handed neutrinos beyond the Standard Model and then these phases become new CP phases. The probabilities, (62) and (63) depend on the new CP phases only through $\text{Re} \left[ U_{\alpha j} U_{\beta k} \right]$. Therefore, the effect of the new CP phases can be measured indirectly from the oscillation probabilities and there is no direct CP violation related to the new CP phases, namely the difference, $P(\nu_{\alpha L} \rightarrow \nu_{\beta R}) - P(\nu_{\beta R} \rightarrow \nu_{\alpha L})$ vanishes. However, the real part of the product of four mixing matrix elements is decomposed as

$$ \text{Re} \left[ U_{\alpha j} U_{\beta k} \right] = \text{Re} \left[ \tilde{U}_{\alpha j} \tilde{U}_{\beta k} \right] e^{i(\phi_{j L} - \phi_{k L} - \phi_{j R} + \phi_{k R})} $$

$$ = \text{Re} \left[ \tilde{U}_{\alpha j} \tilde{U}_{\beta k} \right] \cos(\phi_{j L} - \phi_{k L} - \phi_{j R} + \phi_{k R}) $$

$$ - \text{Im} \left[ \tilde{U}_{\alpha j} \tilde{U}_{\beta k} \right] \sin(\phi_{j L} - \phi_{k L} - \phi_{j R} + \phi_{k R}). \quad (69) $$

If the matrix elements of $\tilde{U}$ and $\tilde{V}$ are investigated by the oscillations of left-handed and right-handed neutrinos, we can obtain the information of the new CP phases through both cosine and sine terms. The cosine term alone cannot determine the value of a new CP phase as one of 360 degrees, but it can be determined by measuring both cosine and sine terms.

Let us count the number of independent parameters related to the new CP phases. If we define $\Delta \phi_{j k L} = \phi_{j L} - \phi_{k L}$ and $\Delta \phi_{j k R} = \phi_{j R} - \phi_{k R}$, the probabilities depend through the form of $\Delta \phi_{j k L} - \Delta \phi_{j k R}$, where both $j$ and $k$ run from 1 to 3 in three generations. As the relation $\Delta \phi_{1 3 L} = \Delta \phi_{1 2 L} - \Delta \phi_{2 3 L}$ etc. holds for example, the number of independent parameters including the new CP phases is two in three generations. Extending the above discussion to the case of $n$ generations, the number of new phases becomes

$$ n - 1, $$

and the number of the CP phases included in the MNS matrix is

$$ \frac{(n - 1)(n - 2)}{2}. \quad (70) $$

as in the previous case. Summing up these two kinds of phases, the total number of the CP phases is

$$ \frac{n(n - 1)}{2}. \quad (71) \quad (72) $$

C. Unitary Check of Oscillation Probabilities

Next, let us confirm the unitarity in the framework of three generations. In the Standard Model, a right-handed neutrino can be chosen as the mass eigenstate because $\nu_{R}$ does not interact through weak interactions.
In this case, the matrix $V$, which mixes right-handed neutrinos, becomes the unit matrix and the mixing angles and the CP phase corresponding to $\nu_{R}$ do not appear. Then, only the sum of the oscillation probabilities for $\nu_{\alpha L} \rightarrow \nu_{R}$,

$$\begin{align*}
P(\nu_{\alpha L} \rightarrow \nu_{R}) &= \sum_{\beta} P(\nu_{\alpha L} \rightarrow \nu_{\beta R}) \\
&= \sum_{\beta} \sum_{j} |U_{\alpha j}V_{j\beta}|^{2} \left\{ \frac{m_{j}^{2}}{E_{j}} \sin^{2}(E_{j}t) + 2 \sum_{k} \text{Re}[U_{\alpha k}^{*}U_{\beta k}] \frac{m_{j}m_{k}}{E_{j}E_{k}} \sin(E_{j}t) \sin(E_{k}t) \right\} \\
&= \sum_{j} |U_{\alpha j}|^{2} \frac{m_{j}^{2}}{E_{j}} \sin^{2}(E_{j}t),
\end{align*}$$

is observable, where we use the unitarity of $V$. We can also calculate the sum of the probabilities for $\nu_{\alpha L} \rightarrow \nu_{L}$,

$$\begin{align*}
P(\nu_{\alpha L} \rightarrow \nu_{L}) &= \sum_{\beta} P(\nu_{\alpha L} \rightarrow \nu_{\beta L}) \\
&= \sum_{\beta} \delta_{\alpha\beta} - \sum_{\beta} \sum_{j} |U_{\alpha j}U_{\beta j}|^{2} \left\{ \frac{m_{j}^{2}}{E_{j}} \sin^{2}(E_{j}t) \right\} \\
&- 2 \sum_{\beta} \sum_{j<k} \text{Re} \left[ U_{\alpha j}U_{\beta j}^{*}U_{\alpha k}^{*}U_{\beta k} \right] \left\{ 2 \sin^{2} \left( \frac{\Delta E_{jk}t}{2} \right) + \frac{E_{j}E_{k} - p^{2}}{E_{j}E_{k}} \sin(E_{j}t) \sin(E_{k}t) \right\} \\
&+ 2 \sum_{\beta} \sum_{j<k} \text{Im} \left[ U_{\alpha j}U_{\beta j}^{*}U_{\alpha k}^{*}U_{\beta k} \right] \left\{ \sin(\Delta E_{jk}t) + \frac{E_{k} - p}{E_{k}} \cos(E_{j}t) \sin(E_{k}t) - \frac{E_{j} - p}{E_{j}} \cos(E_{j}t) \sin(E_{k}t) \right\} \\
&= 1 - \sum_{j} |U_{\alpha j}|^{2} \left\{ \frac{m_{j}^{2}}{E_{j}} \sin^{2}(E_{j}t) \right\},
\end{align*}$$

and adding (73) and (74), we obtain

$$P(\nu_{\alpha L} \rightarrow \nu_{R}) + P(\nu_{\alpha L} \rightarrow \nu_{L}) = 1,$$

and therefore we have confirmed the unitarity by adding all probabilities with and without chirality-flip.

We can also confirm the unitarity for a right-handed neutrino by replacing $U \leftrightarrow V$, for an anti-neutrino by replacing $U \rightarrow U^{*}$, $V \rightarrow V^{*}$ and for a Majorana neutrino by replacing $V \rightarrow U^{*}$ in equations, (73)-(74).

### D. Oscillation Probabilities of Wrong-Helicity Neutrinos

Next, let us consider the upper-left part of (10), namely the oscillation probabilities for $\nu'$. The time evolution of the chirality-flavor eigenstates is given by

$$\begin{align*}
\frac{d}{dt} \begin{pmatrix}
\nu'_{eR} \\
\nu'_{\mu R} \\
\nu'_{\tau R} \\
\nu'_{\tau L} \\
\nu'_{\mu L} \\
\nu'_{eL}
\end{pmatrix}
&= \begin{pmatrix}
p & 0 & 0 & m_{ee} & m_{e\mu} & m_{e\tau} \\
0 & p & 0 & m_{\mu e} & m_{\mu\mu} & m_{\mu\tau} \\
0 & 0 & p & m_{\tau e} & m_{\tau\mu} & m_{\tau\tau} \\
m_{e\bar{e}} & m_{e\bar{\mu}} & m_{e\bar{\tau}} & -p & 0 & 0 \\
m_{\bar{\mu}e} & m_{\bar{\mu}\mu} & m_{\bar{\mu}\tau} & 0 & -p & 0 \\
& & & & & & 0 & 0 & -p
\end{pmatrix}
\begin{pmatrix}
\nu'_{eR} \\
\nu'_{\mu R} \\
\nu'_{\tau R} \\
\nu'_{\tau L} \\
\nu'_{\mu L} \\
\nu'_{eL}
\end{pmatrix}.
\end{align*}$$

Comparing (76) to (11), we can see that the sign of $p$ in the Hamiltonian is reversed from the case for $\nu$. Therefore, the probabilities of $\nu'$ are obtained by changing the sign of $p$ in eqs.(57)-(63). As changing the sign of $p$ does not change the energy $E_{j} = \sqrt{p^{2} + m_{j}^{2}}$, the probabilities except for (63) do not change. On the other hand, the eq.(63)
can be expressed as

\[
-2 \sum_{j<k} \text{Im} \left[ U_{\alpha j} U_{\beta j}^{\ast} U_{\alpha k} U_{\beta k} \right] \left\{ \sin(\Delta E_{jk} t) + \frac{E_k - p}{E_k} \cos(E_j t) \sin(E_k t) - \frac{E_j - p}{E_j} \cos(E_j t) \sin(E_k t) \right\},
\]

\[
= 2p \sum_{j<k} \text{Im} \left[ U_{\alpha j} U_{\beta j}^{\ast} U_{\alpha k} U_{\beta k} \right] \left\{ \frac{1}{E_k} \cos(E_j t) \sin(E_k t) - \frac{1}{E_j} \cos(E_j t) \sin(E_k t) \right\},
\]

and is proportional to the momentum \( p \). Namely, the sign of (61) changes according to reversing the sign of \( p \). As a result, the probabilities of \( \nu' \) are given by

\[
P(\nu'_{\alpha L} \to \nu'_{\alpha L}) = 1 - \sum_{j} |U_{\alpha j}|^2 \left\{ \frac{m_j^2}{E_j} \sin^2(E_j t) \right\},
\]

\[
-2 \sum_{j<k} |U_{\alpha j} U_{\beta j}|^2 \left\{ 2 \sin^2 \left( \frac{\Delta E_{jk} t}{2} \right) + \frac{E_j E_k - p^2}{E_j E_k} \sin(E_j t) \sin(E_k t) \right\},
\]

\[
P(\nu'_{\alpha L} \to \nu'_{\beta L}) = -\sum_{j} |U_{\alpha j} U_{\beta j}|^2 \left\{ \frac{m_j^2}{E_j} \sin^2(E_j t) \right\},
\]

\[
-2 \sum_{j<k} \text{Re} \left[ U_{\alpha j} U_{\beta j}^{\ast} U_{\alpha k} U_{\beta k} \right] \left\{ 2 \sin^2 \left( \frac{\Delta E_{jk} t}{2} \right) + \frac{E_j E_k - p^2}{E_j E_k} \sin(E_j t) \sin(E_k t) \right\}
\]

\[
+2 \sum_{j<k} \text{Im} \left[ U_{\alpha j} U_{\beta j}^{\ast} U_{\alpha k} U_{\beta k} \right] \left\{ \sin(\Delta E_{jk} t) + \frac{E_k - p}{E_k} \cos(E_j t) \sin(E_k t) - \frac{E_j - p}{E_j} \cos(E_j t) \sin(E_k t) \right\}
\]

\[
P(\nu'_{\alpha L} \to \nu'_{\alpha R}) = \sum_{j} |U_{\alpha j} V_{\beta j}|^2 \frac{m_j^2}{E_j} \sin^2(E_j t) + 2 \sum_{j<k} \text{Re}[U_{\alpha j} U_{\beta j}^{\ast} V_{\alpha k} V_{\beta k}] \frac{m_j m_k}{E_j E_k} \sin(E_j t) \sin(E_k t),
\]

\[
P(\nu'_{\alpha L} \to \nu'_{\beta R}) = \sum_{j} |U_{\alpha j} V_{\beta j}|^2 \frac{m_j^2}{E_j} \sin^2(E_j t) + 2 \sum_{j<k} \text{Re}[U_{\alpha j} U_{\beta j}^{\ast} V_{\beta k}] \frac{m_j m_k}{E_j E_k} \sin(E_j t) \sin(E_k t).
\]

It is noted that in the oscillation probabilities for \( \nu'_{\alpha L} \) the sign of the term proportional to \( \text{Im} \left[ U_{\alpha j} U_{\beta j}^{\ast} U_{\alpha k} U_{\beta k} \right] \) is reversed to those for \( \nu_{\alpha L} \) although \( \nu'_{\alpha L} \) and \( \nu_{\alpha L} \) have the same chirality. This is because the Hamiltonians in (41) and (76) have opposite sign of momentum.

It is noted that the oscillation probabilities for \( \nu \) and \( \nu' \) are different in three or more generations. To the best of our knowledge, this is the first paper to point this out.

### E. Oscillation Probabilities of Right-Handed Neutrinos

Next, we consider the oscillation of right-handed neutrinos. If the flavor of the right-handed neutrinos can be distinguished beyond the Standard Model, the oscillations of the right-handed neutrinos can be occured and we can calculate the oscillation probabilities. For completeness, we describe these probabilities. In the same way as the case for left-handed neutrinos, from (69), the amplitudes are given by

\[
A(\nu_{\alpha R} \to \nu_{\beta R}) = \sum_{j} \bar{V}_{\alpha j} V_{\beta j} \left( \frac{E_j + p}{2E_j} e^{iE_j t} + \frac{E_j - p}{2E_j} e^{-iE_j t} \right)
\]

\[
= \sum_{j} \bar{V}_{\alpha j} V_{\beta j} \left( \frac{e^{iE_j t} + e^{-iE_j t}}{2} + \frac{p}{E_j} \frac{e^{iE_j t} - e^{-iE_j t}}{2} \right)
\]

\[
= \sum_{j} \bar{V}_{\alpha j} V_{\beta j} \left\{ \cos(E_j t) + i \frac{p}{E_j} \sin(E_j t) \right\},
\]

\[
A(\nu_{\alpha R} \to \nu_{\beta L}) = \sum_{j} -V_{\alpha j} U_{\beta j}^{\ast} \frac{m_j}{2E_j} \left( e^{iE_j t} - e^{-iE_j t} \right) = \sum_{j} -i V_{\alpha j} U_{\beta j}^{\ast} \frac{m_j}{E_j} \sin(E_j t).
\]
Then, the oscillation probabilities of right-handed neutrinos are calculated by squaring the absolute value of corresponding amplitude,

\[
P(\nu_{\alpha R} \rightarrow \nu_{\alpha R}) = 1 - \sum_j |V_{\alpha j}|^4 \left\{ \frac{m_j^2}{E_j} \sin^2(E_j t) \right\} - 2 \sum_{j<k} |V_{\alpha j} V_{\alpha k}|^2 \left\{ 2 \sin^2 \left( \frac{\Delta E_{jk} t}{2} \right) + \frac{E_j E_k - p^2}{E_j E_k} \sin(E_j t) \sin(E_k t) \right\},
\]

(84)

\[
P(\nu_{\alpha R} \rightarrow \nu_{\beta R}) = - \sum_j |V_{\alpha j} V_{\beta j}|^2 \left\{ \frac{m_j^2}{E_j} \sin^2(E_j t) \right\} - 2 \sum_{j<k} \text{Re} [V_{\alpha j} V_{\beta j}^* U_{\alpha j} U_{\beta k}] \left\{ 2 \sin^2 \left( \frac{\Delta E_{jk} t}{2} \right) + \frac{E_j E_k - p^2}{E_j E_k} \sin(E_j t) \sin(E_k t) \right\} + 2 \sum_{j<k} \text{Im} [V_{\alpha j} V_{\beta j}^* V_{\alpha k} V_{\beta k}] \left\{ \sin(\Delta E_{jk} t) + \frac{E_k - p}{E_j} \cos(E_j t) \sin(E_k t) - \frac{E_j - p}{E_k} \cos(E_k t) \sin(E_j t) \right\},
\]

(85)

\[
P(\nu_{\alpha R} \rightarrow \nu_{\alpha L}) = \sum_j |V_{\alpha j} U_{\alpha j}|^2 \sin^2(E_j t) + 2 \sum_{j<k} \text{Re} [V_{\alpha j} V_{\alpha k}^* U_{\alpha j} U_{\alpha k}] \frac{m_j m_k}{E_j E_k} \sin(E_j t) \sin(E_k t),
\]

(86)

\[
P(\nu_{\beta R} \rightarrow \nu_{\beta L}) = \sum_j |V_{\beta j} U_{\beta j}|^2 \sin^2(E_j t) + 2 \sum_{j<k} \text{Re} [V_{\beta j} V_{\beta k}^* U_{\beta j} U_{\beta k}] \frac{m_j m_k}{E_j E_k} \sin(E_j t) \sin(E_k t).
\]

(87)

These probabilities are obtained by exchanging the left-handed mixing matrix \( U \) and the right-handed mixing matrix \( V \) in eqs. (57)-(59). There is a possibility that the new mixing angles for right-handed neutrinos could be measured through \( \nu_R \rightarrow \nu_R \) oscillations if the flavors of right-handed neutrino are distinguished. In this case, high energy pion can decay through \( W_R \) and produce \( \nu_R \) beam. If this \( \nu_R \) reacts with some matter in a detector and puts it back to the right-handed charged lepton through \( W_R \), we can observe the charged lepton. Thus, we measure the new mixing angles for right-handed neutrinos without the suppression of order \( (m/E)^2 \).

The oscillation probabilities for \( \nu'_R \) are also obtained by changing the sign of the momentum \( p \) in eqs. (84)-(87) as

\[
P(\nu'_{\alpha R} \rightarrow \nu'_{\alpha R}) = 1 - \sum_j |V'_{\alpha j}|^4 \left\{ \frac{m_j^2}{E_j} \sin^2(E_j t) \right\} - 2 \sum_{j<k} |V'_{\alpha j} V_{\alpha k}|^2 \left\{ 2 \sin^2 \left( \frac{\Delta E_{jk} t}{2} \right) + \frac{E_j E_k - p^2}{E_j E_k} \sin(E_j t) \sin(E_k t) \right\},
\]

(88)

\[
P(\nu'_{\alpha R} \rightarrow \nu'_{\beta R}) = - \sum_j |V'_{\alpha j} V_{\beta j}|^2 \left\{ \frac{m_j^2}{E_j} \sin^2(E_j t) \right\} - 2 \sum_{j<k} \text{Re} [V'_{\alpha j} V_{\beta j}^* U_{\alpha j} U_{\beta k}] \left\{ 2 \sin^2 \left( \frac{\Delta E_{jk} t}{2} \right) + \frac{E_j E_k - p^2}{E_j E_k} \sin(E_j t) \sin(E_k t) \right\} - 2 \sum_{j<k} \text{Im} [V'_{\alpha j} V_{\beta j}^* V_{\alpha k} V_{\beta k}] \left\{ \sin(\Delta E_{jk} t) + \frac{E_k - p}{E_j} \cos(E_j t) \sin(E_k t) - \frac{E_j - p}{E_k} \cos(E_k t) \sin(E_j t) \right\},
\]

(89)

\[
P(\nu'_{\alpha R} \rightarrow \nu'_{\alpha L}) = \sum_j |V'_{\alpha j} U_{\alpha j}|^2 \sin^2(E_j t) + 2 \sum_{j<k} \text{Re} [V'_{\alpha j} V_{\alpha k}^* U_{\alpha j} U_{\alpha k}] \frac{m_j m_k}{E_j E_k} \sin(E_j t) \sin(E_k t),
\]

(90)

\[
P(\nu'_{\beta R} \rightarrow \nu'_{\beta L}) = \sum_j |V'_{\beta j} U_{\beta j}|^2 \sin^2(E_j t) + 2 \sum_{j<k} \text{Re} [V'_{\beta j} V_{\beta k}^* U_{\beta j} U_{\beta k}] \frac{m_j m_k}{E_j E_k} \sin(E_j t) \sin(E_k t).
\]

(91)

In the case of also right-handed neutrinos, the sign of the term proportional to \( \text{Im} [V_{\alpha j} V_{\beta j}^* V_{\alpha k} V_{\beta k}] \) becomes opposite for \( \nu \) and \( \nu' \).
F. Oscillation Probabilities of Anti-Neutrinos

Next, we consider the oscillation probabilities for anti-neutrinos, which are defined as the charge conjugation of neutrinos. The charge conjugation of \( \psi_\alpha \),

\[
\psi^c_\alpha = (\psi_{\alpha L})^c + (\psi_{\alpha R})^c = \begin{pmatrix} \nu^c_{\alpha L} \\ \nu^c_{\alpha L} \\ \nu^c_{\alpha R} \\ \nu^c_{\alpha R} \end{pmatrix}
\]

also satisfies the Dirac equation,

\[
i\gamma^\mu \partial_\mu \psi^c_{\alpha L} - \sum_\beta m_{\beta \alpha} \psi^c_{\beta R} = 0. \tag{93}
\]

In general, the Dirac equation for \( \psi^c \) is slightly different from that for \( \psi \) because the mass term is complex in general. Namely, the mass term in the Dirac equation for \( \psi^c \) becomes complex conjugate of that for \( \psi \). Multiplying \( \gamma^0 \) from the left, we obtain

\[
i\partial_0 i\gamma^\mu \partial_\mu \psi^c_{\alpha L} + i\gamma^0 \gamma^\mu \partial_\mu \psi^c_{\alpha L} - \sum_\beta m_{\beta \alpha} \gamma^0 \psi^c_{\beta R} = 0. \tag{94}
\]

If we use two components spinors \( \xi \) and \( \eta \), the above equation can be rewritten as

\[
i\partial_0 \begin{pmatrix} i\sigma_2 \eta^*_\alpha & 0 \\ 0 & i\sigma_2 \xi^*_\alpha \end{pmatrix} + i \begin{pmatrix} \sigma_1 \partial_1 (i\sigma_2 \eta^*_\alpha) \\ \sigma_1 \partial_1 (i\sigma_2 \xi^*_\alpha) \end{pmatrix} - \sum_\beta m_{\beta \alpha} \begin{pmatrix} -i\sigma_2 \xi^*_\beta \\ 0 \end{pmatrix} = 0. \tag{95}
\]

Taking out the upper two components, we obtain

\[
i\partial_0 (i\sigma_2 \eta^*_\alpha) + i\sigma_1 \partial_1 (i\sigma_2 \eta^*_\alpha) - \sum_\beta m_{\beta \alpha} (-i\sigma_2 \xi^*_\beta) = 0. \tag{96}
\]

In the same way, we also obtain

\[
i\partial_0 (-i\sigma_2 \xi^*_\alpha) - i\sigma_1 \partial_1 (-i\sigma_2 \xi^*_\alpha) - \sum_\beta m_{\alpha \beta} (i\sigma_2 \eta^*_\beta) = 0. \tag{97}
\]

Here, we take the complex conjugate of (37).

\[
\eta^*_\alpha (x, t) = e^{-i\vec{p} \cdot \vec{x}} \eta^*_\alpha (t) = e^{-i\vec{p} \cdot \vec{x}} \begin{pmatrix} \nu^*_L \\ \nu^*_L \end{pmatrix}, \tag{98}
\]

\[
\xi^*_\alpha (x, t) = e^{-i\vec{p} \cdot \vec{x}} \xi^*_\alpha (t) = e^{-i\vec{p} \cdot \vec{x}} \begin{pmatrix} \nu^*_R \\ \nu^*_R \end{pmatrix}, \tag{99}
\]

and we choose \( \vec{p} = (0, 0, p) \). Then, the Dirac equations are rewritten as

\[
i\partial_0 \begin{pmatrix} \nu^c_{\alpha L} \\ \nu^c_{\alpha L} \end{pmatrix} + p \begin{pmatrix} \nu^c_{\alpha L} \\ \nu^c_{\alpha L} \end{pmatrix} - \sum_\beta m_{\beta \alpha} \begin{pmatrix} \nu^c_{\beta R} \\ \nu^c_{\beta R} \end{pmatrix} = 0, \tag{100}
\]

\[
i\partial_0 \begin{pmatrix} \nu^c_{\alpha R} \\ \nu^c_{\alpha R} \end{pmatrix} - p \begin{pmatrix} \nu^c_{\alpha R} \\ \nu^c_{\alpha R} \end{pmatrix} - \sum_\beta m^*_{\alpha \beta} \begin{pmatrix} \nu^c_{\beta L} \\ \nu^c_{\beta L} \end{pmatrix} = 0. \tag{101}
\]

Combining the above equations for three flavors to one matrix form, the time evolution of the chirality-flavor eigenstates is given by

\[
\begin{pmatrix} \nu^c_{eR} \\ \nu^c_{\mu R} \\ \nu^c_{\tau R} \\ \nu^c_{eL} \\ \nu^c_{\mu L} \\ \nu^c_{\tau L} \end{pmatrix} = \begin{pmatrix} p & 0 & 0 & m_{ee} & m_{e\mu} & m_{e\tau} & 0 & 0 & 0 & 0 \\ 0 & p & 0 & m_{ee} & m_{e\mu} & m_{e\tau} & 0 & 0 & 0 & 0 \\ 0 & 0 & -p & m_{ee} & m_{e\mu} & m_{e\tau} & 0 & 0 & 0 & 0 \\ m_{ee} & m_{e\mu} & m_{e\tau} & p & 0 & 0 & 0 & 0 & 0 & 0 \\ m_{e\mu} & m_{e\mu} & m_{e\tau} & p & 0 & 0 & 0 & 0 & 0 & 0 \\ m_{e\tau} & m_{e\tau} & m_{e\tau} & p & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \nu^c_{eR} \\ \nu^c_{\mu R} \\ \nu^c_{\tau R} \\ \nu^c_{eL} \\ \nu^c_{\mu L} \\ \nu^c_{\tau L} \end{pmatrix}. \tag{102}
\]
Comparing this with (40), we can see that the replacements of $\nu \rightarrow \nu'$, $\nu' \rightarrow \nu''$, $m^* \rightarrow m^*$ in (104) lead to (103). According to this correspondence, we derive the oscillation probabilities for anti-neutrinos by replacing $U \rightarrow U^*$ and $V \rightarrow V^*$ in (57)-(63). Namely, we obtain the probabilities for anti-neutrinos with positive momentum $p$ as:

\[
P(\nu'^L \rightarrow \nu^L) = 1 - \sum_j |U_{\alpha j}|^4 \left\{ \frac{m_j^2}{E_j} \sin^2(E_j t) \right\}
- 2 \sum_{j<k} |U_{\alpha j} U_{\alpha k}|^2 \left\{ 2 \sin^2 \left( \frac{\Delta E_{jk} t}{2} \right) + \frac{E_j E_k - p^2}{E_j E_k} \sin(E_j t) \sin(E_k t) \right\},
\]

\[
P(\nu'^L \rightarrow \nu'^R) = - \sum_j |U_{\alpha j} U_{\beta j}|^2 \left\{ \frac{m_j^2}{E_j} \sin^2(E_j t) \right\}
- 2 \sum_{j<k} \text{Re} \left[ U_{\alpha j} U_{\alpha k}^* U_{\beta j} U_{\beta k} \right] \left\{ 2 \sin^2 \left( \frac{\Delta E_{jk} t}{2} \right) + \frac{E_j E_k - p^2}{E_j E_k} \sin(E_j t) \sin(E_k t) \right\}
+ 2 \sum_{j<k} \text{Im} \left[ U_{\alpha j} U_{\alpha k}^* U_{\beta j} U_{\beta k} \right] \left\{ \sin(\Delta E_{jk} t) + \frac{E_j - p}{E_k} \cos(E_j t) \sin(E_k t) - \frac{E_j - p}{E_j} \cos(E_j t) \sin(E_k t) \right\}.
\]

\[
P(\nu'^L \rightarrow \nu'^R) = \sum_j |U_{\alpha j} V_{\alpha j}|^2 \left\{ \frac{m_j^2}{E_j} \sin^2(E_j t) \right\}
+ 2 \sum_{j<k} \text{Re} \left[ U_{\alpha j} U_{\alpha k}^* V_{\alpha j} V_{\alpha k} \right] \frac{m_j m_k}{E_j E_k} \sin(E_j t) \sin(E_k t),
\]

\[
P(\nu'^L \rightarrow \nu'^R) = \sum_j |U_{\alpha j} V_{\beta j}|^2 \left\{ \frac{m_j^2}{E_j} \sin^2(E_j t) \right\}
+ 2 \sum_{j<k} \text{Re} \left[ U_{\alpha j} U_{\alpha k}^* V_{\beta j} V_{\beta k} \right] \frac{m_j m_k}{E_j E_k} \sin(E_j t) \sin(E_k t),
\]

and

\[
P(\nu'^L \rightarrow \nu'^L) = 1 - \sum_j |U_{\alpha j}|^4 \left\{ \frac{m_j^2}{E_j} \sin^2(E_j t) \right\}
- 2 \sum_{j<k} |U_{\alpha j} U_{\alpha k}|^2 \left\{ 2 \sin^2 \left( \frac{\Delta E_{jk} t}{2} \right) + \frac{E_j E_k - p^2}{E_j E_k} \sin(E_j t) \sin(E_k t) \right\},
\]

\[
P(\nu'^L \rightarrow \nu'^L) = - \sum_j |U_{\alpha j} U_{\beta j}|^2 \left\{ \frac{m_j^2}{E_j} \sin^2(E_j t) \right\}
- 2 \sum_{j<k} \text{Re} \left[ U_{\alpha j} U_{\beta j}^* U_{\alpha k} U_{\beta k} \right] \left\{ 2 \sin^2 \left( \frac{\Delta E_{jk} t}{2} \right) + \frac{E_j E_k - p^2}{E_j E_k} \sin(E_j t) \sin(E_k t) \right\}
- 2 \sum_{j<k} \text{Im} \left[ U_{\alpha j} U_{\beta j}^* U_{\alpha k} U_{\beta k} \right] \left\{ \sin(\Delta E_{jk} t) + \frac{E_j - p}{E_k} \cos(E_j t) \sin(E_k t) - \frac{E_j - p}{E_j} \cos(E_j t) \sin(E_k t) \right\}.
\]
Comparing (104)-(111) with (57)-(63) and (78)-(81), we can see that only the sign of the term (105) and (109) proportional to \( \text{Im} U_{\alpha j} U^*_{\alpha k} U_{\beta j} U^*_{\beta k} \) is different for neutrinos and anti-neutrinos. The difference comes from the complex conjugate of \( U \) and \( V \) for anti-neutrinos. It is noted that the probabilities with chirality-flip are the same for neutrinos and anti-neutrinos.

Furthermore, the oscillation probabilities for right-handed anti-neutrinos are given by the replacement, \( U \rightarrow U^* \) and \( V \rightarrow V^* \) in (54)—(57) and (85)—(88),

\[
\begin{align*}
P(\nu^c_{\alpha R} \rightarrow \nu^c_{\alpha R}) &= 1 - \sum_j |V_{\alpha j}|^4 \left\{ \frac{m_j^2}{E_j^2} \sin^2(E_j t) \right\} \\
&\quad - 2 \sum_{j<k} |V_{\alpha j} V_{\alpha k}|^2 \left\{ 2 \sin^2 \left( \frac{\Delta E_{jk} t}{2} \right) + \frac{E_j E_k - p^2}{E_j E_k} \sin(E_j t) \sin(E_k t) \right\}, \\
&\quad - 2 \sum_{j<k} \text{Re} \left[ V_{\alpha j} V^*_{\alpha j} V^*_{\alpha k} V_{\beta k} \right] \left\{ 2 \sin^2 \left( \frac{\Delta E_{jk} t}{2} \right) + \frac{E_j E_k - p^2}{E_j E_k} \sin(E_j t) \sin(E_k t) \right\} \\
&\quad - 2 \sum_{j<k} \text{Im} \left[ V_{\alpha j} V^*_{\alpha j} V^*_{\alpha k} V_{\beta k} \right] \left\{ \sin(\Delta E_{jk} t) + \frac{E_j - p}{E_j} \cos(E_j t) \sin(E_k t) - \frac{E_j - p}{E_j} \cos(E_k t) \sin(E_j t) \right\}, \\
\end{align*}
\]

for \( j \neq k \).

\[
\begin{align*}
P(\nu^c_{\alpha R} \rightarrow \nu^c_{\beta R}) &= - \sum_j |V_{\alpha j}|^2 \left\{ \frac{m_j^2}{E_j^2} \sin^2(E_j t) \right\} \\
&\quad - 2 \sum_{j<k} |V_{\alpha j} V_{\beta j}|^2 \left\{ 2 \sin^2 \left( \frac{\Delta E_{jk} t}{2} \right) + \frac{E_j E_k - p^2}{E_j E_k} \sin(E_j t) \sin(E_k t) \right\} \\
&\quad - 2 \sum_{j<k} \text{Re} \left[ V_{\alpha j} V^*_{\beta j} V^*_{\alpha k} V_{\beta k} \right] \left\{ 2 \sin^2 \left( \frac{\Delta E_{jk} t}{2} \right) + \frac{E_j E_k - p^2}{E_j E_k} \sin(E_j t) \sin(E_k t) \right\} \\
&\quad + 2 \sum_{j<k} \text{Im} \left[ V_{\alpha j} V^*_{\beta j} V^*_{\alpha k} V_{\beta k} \right] \left\{ \sin(\Delta E_{jk} t) + \frac{E_j - p}{E_j} \cos(E_j t) \sin(E_k t) - \frac{E_j - p}{E_j} \cos(E_k t) \sin(E_j t) \right\}, \\
\end{align*}
\]

V. OSCILLATION PROBABILITIES OF MAJORANA NEUTRINO FROM RELATIVISTIC EQUATION

In this section, we derive the oscillation probabilities of the Majorana neutrinos in three generations or more. We count the number of measurable mixing
angles and CP phases in the case of \( n \)-generations. In the non-relativistic method, there was no difference between the probabilities of the Dirac neutrinos and the Majorana neutrinos. However, there appears a difference in the oscillations with chirality-flip when we use the relativistic equation.

### A. Oscillation Probabilities of Neutrinos

The lagrangian for Majorana neutrinos in three generations is given by

\[
L = \sum_{\alpha} \frac{1}{2} \left[ i \bar{\psi}_{\alpha L} \gamma^\mu \partial_\mu \psi_{\alpha L} + i \bar{\psi}_{\alpha R} \gamma^\mu \partial_\mu \psi_{\alpha R} \right] - \sum_{(\alpha, \beta)} \frac{1}{2} \left[ \bar{\psi}_{\beta L} \gamma^\mu M_{\beta \alpha} \psi_{\alpha L} - \bar{\psi}_{\alpha L} M^*_\beta \psi_{\beta L} \right].
\]  

(120)

About the kinetic term in the lagrangian, the relation,

\[
L_{\text{kin}} = i \bar{\psi}_{\alpha L} \gamma^\mu \partial_\mu \psi_{\alpha L} = i \bar{\psi}_{\alpha L} \gamma^\mu \partial_\mu \psi_{\alpha L},
\]

(121)

holds and the Eular-Lagrange equation for \( \bar{\psi}_{\alpha L} \),

\[
\frac{\partial L}{\partial \bar{\psi}_{\alpha L}} - \partial_\mu \left( \frac{\partial L}{\partial (\partial_\mu \psi_{\alpha L})} \right) = 0,
\]

(122)

leads to

\[
i \gamma^\mu \partial_\mu \bar{\psi}_{\alpha L} - \sum_{\beta} M^*_\beta \psi_{\beta L} = 0.
\]

(123)

Multiplying \( \gamma_0 \) from the left, this equation can be rewritten as

\[
i \partial_0 \bar{\psi}_{\alpha L} + i \gamma^0 \gamma^i \partial_i \bar{\psi}_{\alpha L} - \sum_{\beta} M^*_\beta \gamma^0 \psi_{\beta L} = 0.
\]

(124)

Substituting (4) and (7) into this equation, we obtain the equation for two-component spinor \( \eta \),

\[
i \partial_0 \left( \begin{array}{c} \eta_\alpha \\ \eta_\beta \end{array} \right) + i \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -\sigma_i \\ \sigma_i & 0 \end{pmatrix} \partial_i \left( \begin{array}{c} \eta_\alpha \\ \eta_\beta \end{array} \right) - \sum_{\beta} M^*_\beta \left( \begin{array}{c} i \sigma_2 \eta^*_\alpha \\ 0 \end{array} \right) = 0,
\]

(125)

\[
i \partial_0 \left( \begin{array}{c} \eta_\alpha \\ \eta_\beta \end{array} \right) - i \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \sigma_i \partial_i \eta_\alpha - \sum_{\beta} M^*_\beta \left( \begin{array}{c} 0 \\ i \sigma_2 \eta^*_\beta \end{array} \right) = 0.
\]

(126)

Taking out the lower two components, we obtain

\[
i \partial_0 \eta_\alpha - i \sigma_i \partial_i \eta_\alpha - \sum_{\beta} M^*_\beta (i \sigma_2 \eta^*_\beta) = 0.
\]

(127)

In the same way, the Eular-Lagrange equation for \( \bar{\psi}_{\alpha L} \),

\[
\frac{\partial L}{\partial \bar{\psi}_{\alpha L}} - \partial_\mu \left( \frac{\partial L}{\partial (\partial_\mu \psi_{\alpha L})} \right) = 0,
\]

(128)

leads to the equation,

\[
i \gamma^\mu \partial_\mu \psi_{\alpha L} - \sum_{\beta} M_{\alpha \beta} \psi_{\beta L} = 0.
\]

(129)

Multiplying \( \gamma_0 \) from the left, the above equation becomes

\[
i \partial_0 \psi_{\alpha L} + i \gamma^0 \gamma^i \partial_i \psi_{\alpha L} - \sum_{\beta} M_{\alpha \beta} \gamma^0 \psi_{\beta L} = 0.
\]

(130)

Substituting (4) and (7) into this equation, we obtain the equation for two-component spinor \( \eta \),

\[
i \partial_0 \left( \begin{array}{c} i \sigma_2 \eta^*_\alpha \\ 0 \end{array} \right) + i \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -\sigma_i \\ \sigma_i & 0 \end{pmatrix} \partial_i \left( \begin{array}{c} i \sigma_2 \eta^*_\alpha \\ 0 \end{array} \right) - \sum_{\beta} M_{\alpha \beta} \left( \begin{array}{c} \eta_\beta \\ 0 \end{array} \right) = 0,
\]

(131)

\[
i \partial_0 \left( \begin{array}{c} i \sigma_2 \eta^*_\alpha \\ 0 \end{array} \right) + i \sigma_i \partial_i (i \sigma_2 \eta^*_\alpha) - \sum_{\beta} M_{\alpha \beta} \left( \eta_\beta \right) = 0.
\]

(132)

Taking out the upper two components, we obtain

\[
i \partial_0 (i \sigma_2 \eta^*_\alpha) + i \sigma_i \partial_i (i \sigma_2 \eta^*_\alpha) - \sum_{\beta} M_{\alpha \beta} \eta_\beta = 0.
\]

(133)

Here, we take the equal momentum assumption for all flavors,

\[
\eta_\alpha (x, t) = e^{i \vec{p} \cdot \vec{x}} \eta_\alpha (t) = e^{i \vec{p} \cdot \vec{x}} \left( \nu^\alpha_{\nu L} \right),
\]

(134)

and we choose \( \vec{p} = (0, 0, p) \). The complex conjugate of these two-component spinors is given by

\[
\eta^*_\alpha (x, t) = e^{-i \vec{p} \cdot \vec{x}} \eta^*_\alpha (t) = e^{-i \vec{p} \cdot \vec{x}} \left( \nu_{\nu L} \right).
\]

(135)

It is noted that \( \nu^* \) included in \( \eta^* \) has the negative momentum. Substituting (134) and (135) into (127) and (133), the time evolution of the chirality-flavor eigenstates is given by
\[
\begin{pmatrix}
\nu_{eL}^c \\
\nu_{\mu L}^c \\
\nu_{\tau L}^c \\
\nu_{eL} \\
\nu_{\mu L} \\
\nu_{\tau L}
\end{pmatrix} =
\begin{pmatrix}
p & 0 & 0 & M_{ee} & M_{e\mu} & M_{e\tau} \\
0 & p & 0 & M_{\mu e} & M_{\mu\mu} & M_{\mu\tau} \\
0 & 0 & p & M_{\tau e} & M_{\tau\mu} & M_{\tau\tau} \\
M_{ee}^* & M_{e\mu}^* & M_{e\tau}^* & 0 & 0 & 0 \\
M_{\mu e}^* & M_{\mu\mu}^* & M_{\mu\tau}^* & 0 & 0 & 0 \\
M_{\tau e}^* & M_{\tau\mu}^* & M_{\tau\tau}^* & 0 & 0 & 0
\end{pmatrix}\begin{pmatrix}
p & 0 & 0 & M_{ee} & M_{e\mu} & M_{e\tau} \\
0 & p & 0 & M_{\mu e} & M_{\mu\mu} & M_{\mu\tau} \\
0 & 0 & p & M_{\tau e} & M_{\tau\mu} & M_{\tau\tau} \\
M_{ee} & M_{e\mu} & M_{e\tau} & 0 & 0 & 0 \\
M_{\mu e} & M_{\mu\mu} & M_{\mu\tau} & 0 & 0 & 0 \\
M_{\tau e} & M_{\tau\mu} & M_{\tau\tau} & 0 & 0 & 0
\end{pmatrix},
\]

(136)

This has the same construct as the equation (11) in the previous section. The only different point is that the mass matrix in the Hamiltonian is complex symmetric. Therefore, the mass matrix can be diagonalized by one unitary matrix \(U\). Then, the chirality-flavor eigenstates are represented as the linear combination of the chirality-mass eigenstates as

\[
\begin{pmatrix}
\nu_{eL}^c \\
\nu_{\mu L}^c \\
\nu_{\tau L}^c \\
\nu_{eL} \\
\nu_{\mu L} \\
\nu_{\tau L}
\end{pmatrix} =
\begin{pmatrix}
U_{e1}^* & U_{e2}^* & U_{e3}^* \\
U_{\mu1}^* & U_{\mu2}^* & U_{\mu3}^* \\
U_{\tau1}^* & U_{\tau2}^* & U_{\tau3}^* \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}\begin{pmatrix}
\nu_{eL}^c \\
\nu_{\mu L}^c \\
\nu_{\tau L}^c \\
\nu_{eL} \\
\nu_{\mu L} \\
\nu_{\tau L}
\end{pmatrix}.
\]

(137)

Namely, we obtain the oscillation probabilities for Majorana neutrino by the replacement \(V \to U^*\), \(\nu_{\alpha R} \to \nu_{\alpha L}^c\) in eqs. (17)-(63),

\[
P(\nu_{\alpha L} \to \nu_{\alpha L}) = 1 - \sum_j |U_{\alpha j}|^4 \left\{ \frac{m_j^2}{E_j^2} \sin^2(E_j t) \right\}
- 2 \sum_{j<k} |U_{\alpha j} U_{\alpha k}|^2 \left\{ 2 \sin^2 \left( \frac{\Delta E_{jk} t}{2} \right) + \frac{E_j E_k - p^2}{E_j E_k} \sin(E_j t) \sin(E_k t) \right\},
\]

(139)
\[ P(\nu_{\alpha L} \to \nu_{\beta L}) = -\sum_j |U_{\alpha j}U_{\beta j}|^2 \left\{ \frac{m_j^2}{E_j} \sin^2(E_j t) \right\} \]
\[ -2 \sum_{j<k} \text{Re} \left[ U_{\alpha j}U_{\beta j}^* U_{\alpha k}^* U_{\beta k} \right] \left\{ 2 \sin^2 \left( \frac{E_{jk} t}{2} \right) \right. + \left. \frac{E_j E_k - p^2}{E_j E_k} \sin(E_j t) \sin(E_k t) \right\} \]
\[ -2 \sum_{j<k} \text{Im} \left[ U_{\alpha j}U_{\beta j}^* U_{\alpha k}^* U_{\beta k} \right] \left\{ \sin(\Delta E_{jk} t) + \frac{E_k - p}{E_k} \cos(E_j t) \sin(E_k t) - \frac{E_j - p}{E_j} \cos(E_k t) \sin(E_j t) \right\} \].
\[ P(\nu_{\alpha L} \to \nu'_{\alpha L}) = \sum_j |U_{\alpha j}U_{\alpha j}|^2 \left\{ \frac{m_j^2}{E_j} \sin^2(E_j t) \right\} + 2 \sum_{j<k} \text{Re} \left[ U_{\alpha j}U_{\alpha k}^* U_{\alpha j}U_{\alpha k}^* \right] \frac{m_j m_k}{E_j E_k} \sin(E_j t) \sin(E_k t), \quad (141) \]
\[ P(\nu'_{\alpha L} \to \nu'_{\beta L}) = \sum_j |U_{\alpha j}U_{\beta j}|^2 \left\{ \frac{m_j^2}{E_j} \sin^2(E_j t) \right\} + 2 \sum_{j<k} \text{Re} \left[ U_{\alpha j}U_{\beta j}^* U_{\alpha k}^* U_{\beta k} \right] \frac{m_j m_k}{E_j E_k} \sin(E_j t) \sin(E_k t). \quad (142) \]

In this way, we can derive the oscillation probabilities for neutrinos and anti-neutrinos by entering \( \nu'_L \) in the same multiplet as \( \nu_L \) instead of \( \nu_R \). The probabilities obtained above are described by the parameter independent manner.

Next, let us derive the probabilities for \( \nu' \) and \( \nu'' \). The top-left part of the Hamiltonian in (130) is the same structure as the Hamiltonian in (79). We only have to replace \( V \to U^* \) in (78)-(81) according to the replacement \( m \to M \) in the Hamiltonian. Therefore, the oscillation probabilities for \( \nu' \) are given by
\[ P(\nu'_{\alpha L} \to \nu'_{\alpha L}) = 1 - \sum_j |U_{\alpha j}|^4 \left\{ \frac{m_j^2}{E_j} \sin^2(E_j t) \right\} \]
\[ -2 \sum_{j<k} |U_{\alpha j}U_{\alpha k}|^2 \left\{ 2 \sin^2 \left( \frac{E_{jk} t}{2} \right) \right. + \left. \frac{E_j E_k - p^2}{E_j E_k} \sin(E_j t) \sin(E_k t) \right\}, \quad (143) \]
\[ P(\nu'_{\alpha L} \to \nu''_{\alpha L}) = -\sum_j |U_{\alpha j}U_{\beta j}|^2 \left\{ \frac{m_j^2}{E_j} \sin^2(E_j t) \right\} \]
\[ -2 \sum_{j<k} \text{Re} \left[ U_{\alpha j}U_{\beta j}^* U_{\alpha k}^* U_{\beta k} \right] \left\{ 2 \sin^2 \left( \frac{E_{jk} t}{2} \right) \right. + \left. \frac{E_j E_k - p^2}{E_j E_k} \sin(E_j t) \sin(E_k t) \right\} \]
\[ + 2 \sum_{j<k} \text{Im} \left[ U_{\alpha j}U_{\beta j}^* U_{\alpha k}^* U_{\beta k} \right] \left\{ \sin(\Delta E_{jk} t) + \frac{E_k - p}{E_k} \cos(E_j t) \sin(E_k t) - \frac{E_j - p}{E_j} \cos(E_k t) \sin(E_j t) \right\}, \quad (144) \]
\[ P(\nu''_{\alpha L} \to \nu''_{\alpha L}) = \sum_j |U_{\alpha j}U_{\alpha j}|^2 \left\{ \frac{m_j^2}{E_j} \sin^2(E_j t) \right\} + 2 \sum_{j<k} \text{Re} \left[ U_{\alpha j}U_{\alpha k}^* U_{\alpha j}U_{\alpha k}^* \right] \frac{m_j m_k}{E_j E_k} \sin(E_j t) \sin(E_k t), \quad (145) \]
\[ P(\nu''_{\alpha L} \to \nu''_{\beta L}) = \sum_j |U_{\alpha j}U_{\beta j}|^2 \left\{ \frac{m_j^2}{E_j} \sin^2(E_j t) \right\} + 2 \sum_{j<k} \text{Re} \left[ U_{\alpha j}U_{\beta j}^* U_{\alpha k}^* U_{\beta k} \right] \frac{m_j m_k}{E_j E_k} \sin(E_j t) \sin(E_k t). \quad (146) \]

The difference between the probabilities for \( \nu \) and \( \nu' \) appears in (144). Namely, the sign of the term proportional to \( \text{Im} \left[ U_{\alpha j}U_{\alpha k}^* U_{\alpha j}U_{\alpha k}^* \right] \) is reversed. It is noted that eqs. (141), (142), (145) and (146) are the probabilities for oscillations from neutrinos with positive momentum to anti-neutrinos with negative momentum.

The probabilities of the Majorana neutrinos are obtained by the replacement \( \phi_{\alpha L} \to \phi_{j}, \phi_{\beta j} \to -\phi_{j} \) and \( V \to U^* \) in those of the Dirac neutrinos. The probabilities without chirality-flip are the same as those of the Dirac neutrinos. The probabilities with chirality-flip (141) and (142) depend on the Majorana CP phases through the four part of the product of four matrix elements,
\[ \text{Re} \left[ U_{\alpha j}U_{\alpha k}^* U_{\beta j}U_{\beta k}^* \right] \]
\[ = \text{Re} \left[ e^{i\phi_{\alpha j}}e^{-i\phi_{\beta j}}e^{-i\phi_{\alpha k}}e^{-i\phi_{\beta k}} \right] \]
\[ = \text{Re} \left[ e^{i\phi_{\alpha j}}e^{-i\phi_{j}}e^{-i\phi_{\beta j}}e^{-i\phi_{\beta k}} \right] \]
\[ = \text{Re} \left[ \tilde{\nu}_{\alpha j}U_{\alpha k}^* \tilde{\nu}_{\beta j}U_{\beta k}^* \cos(2(\phi_j - \phi_k)) \right] \]
\[ - \text{Im} \left( \tilde{\nu}_{\alpha j}U_{\alpha k}^* \tilde{\nu}_{\beta j}U_{\beta k}^* \right) \sin(2(\phi_j - \phi_k)). \quad (147) \]

As in the case of the Dirac neutrinos, we can obtain the information from both the sine and the cosine term through there is no direct CP violation. If we define \( \Delta \phi_{jk} = \phi_j - \phi_k \), the probabilities of the Majorana neutrinos depend on the new CP phase through the form \( \Delta \phi_{jk} \).
As there are the relations like \( \Delta \phi_{13} = \Delta \phi_{12} - \Delta \phi_{23} \), independent parameters related to the new CP phases is two in three generations. Namely, the number of CP phase appeared in the Majorana case is the same as that in the Dirac case. The result obtained here is the same as previously known in the case of the Majorana neutrinos.

B. Oscillation Probabilities of Anti-Neutrinos

Next, we consider the oscillation probabilities of anti-neutrinos with positive momentum. As the anti-neutrinos have negative momentum in eq. (137), let us change the sign of momentum \( p \) in order to derive the probabilities of anti-neutrinos with positive momentum. Namely, we start from the time evolution equation

\[
\frac{i}{\hbar} \frac{d}{dt} \left( \begin{array}{c}
\nu^c_{eL} \\
\nu^c_{\mu L} \\
\nu^c_{\tau L}
\end{array} \right) = \left( \begin{array}{ccc}
p & 0 & 0 \\
0 & p & 0 \\
0 & 0 & -p
\end{array} \right) \frac{1}{2} \left( \begin{array}{ccc}
M_{ee} & M_{e\mu} & M_{e\tau} \\
M_{\mu e} & M_{\mu\mu} & M_{\mu\tau} \\
M_{\tau e} & M_{\tau\mu} & M_{\tau\tau}
\end{array} \right) \left( \begin{array}{c}
\nu^c_{eL} \\
\nu^c_{\mu L} \\
\nu^c_{\tau L}
\end{array} \right),
\]

\[ (148) \]

where anti-neutrinos \( \nu^c \) have positive momentum and neutrinos \( \nu \) have negative momentum. Exchanging some rows and some columns, and using the symmetry of the Majorana mass term, this equation can be rewritten as

\[
\frac{i}{\hbar} \frac{d}{dt} \left( \begin{array}{c}
\nu_{eL} \\
\nu_{\mu L} \\
\nu_{\tau L}
\end{array} \right) = \left( \begin{array}{ccc}
-p & 0 & 0 \\
0 & -p & 0 \\
0 & 0 & -p
\end{array} \right) \frac{1}{2} \left( \begin{array}{ccc}
M^*_{ee} & M^*_{e\mu} & M^*_{e\tau} \\
M^*_{\mu e} & M^*_{\mu\mu} & M^*_{\mu\tau} \\
M^*_{\tau e} & M^*_{\tau\mu} & M^*_{\tau\tau}
\end{array} \right) \left( \begin{array}{c}
\nu_{eL} \\
\nu_{\mu L} \\
\nu_{\tau L}
\end{array} \right) .
\]

\[ (149) \]

We can see that this equation is obtained by the exchange \( M \leftrightarrow M^* \) and \( \nu \leftrightarrow \nu^c \) in eq. (137). Therefore, the probabilities of anti-neutrinos are obtained by the exchange \( \nu \leftrightarrow \nu^c \) and \( U \leftrightarrow U^* \) in (139)-(142),

\[
P(\nu^c_{\alpha L} \rightarrow \nu^c_{\alpha L}) = 1 - \sum_j |U_{\alpha j}|^4 \left\{ \frac{m^2_{\alpha}}{E_j} \sin^2(E_j t) \right\}
\]

\[
-2 \sum_{j<k} |U_{\alpha j} U_{\alpha k}|^2 \left\{ 2 \sin^2 \left( \frac{\Delta E_{jk} t}{2} \right) + \frac{E_j E_k - p^2}{E_j E_k} \sin(E_j t) \sin(E_k t) \right\},
\]

\[ (150) \]

\[
P(\nu^e_{\alpha L} \rightarrow \nu^e_{\alpha L}) = -\sum_j |U_{\alpha j} U_{\beta j}|^2 \left\{ \frac{m^2_{\beta}}{E_j} \sin^2(E_j t) \right\}
\]

\[
-2 \sum_{j<k} \text{Re} \left[ U^*_{\alpha j} U_{\beta j} U_{\alpha k} U^*_{\beta k} \right] \left\{ 2 \sin^2 \left( \frac{\Delta E_{jk} t}{2} \right) + \frac{E_j E_k - p^2}{E_j E_k} \sin(E_j t) \sin(E_k t) \right\}
\]

\[
+2 \sum_{j<k} \text{Im} \left[ U^*_{\alpha j} U_{\beta j} U_{\alpha k} U^*_{\beta k} \right] \left\{ \sin(\Delta E_{jk} t) + \frac{E_j - p}{E_j} \cos(E_j t) \sin(E_k t) - \frac{E_j}{E_j} \cos(E_k t) \sin(E_j t) \right\},
\]

\[ (151) \]

\[
P(\nu^e_{\alpha L} \rightarrow \nu_{\alpha L}) = \sum_j |U_{\alpha j} U_{\alpha j}|^2 \frac{m^2_{\beta}}{E_j} \sin^2(E_j t) + 2 \sum_{j<k} \text{Re} \left[ U_{\alpha j} U^*_{\alpha k} U_{\alpha j} U^*_{\alpha k} \right] \frac{m_j m_k}{E_j E_k} \sin(E_j t) \sin(E_k t),
\]

\[ (152) \]

\[
P(\nu^e_{\beta L} \rightarrow \nu_{\beta L}) = \sum_j |U_{\beta j} U_{\beta j}|^2 \frac{m^2_{\beta}}{E_j} \sin^2(E_j t) + 2 \sum_{j<k} \text{Re} \left[ U_{\beta j} U^*_{\beta k} U_{\beta j} U^*_{\beta k} \right] \frac{m_j m_k}{E_j E_k} \sin(E_j t) \sin(E_k t).
\]

\[ (153) \]
Comparing these probabilities with (143)-(146), it turns out that only the sign in (151) is reversed. The oscillation probabilities for $\nu^c$ are also obtained as

$$P(\nu_{\alpha L}^c \to \nu_{\alpha L}^c) = 1 - \sum_j |U_{\alpha j}|^2 \left( \frac{m_j^2}{E_j} \sin^2(E_j t) \right) - 2 \sum_{j<k} |U_{\alpha j} U_{\alpha k}|^2 \left\{ 2 \sin^2 \left( \frac{\Delta E_{jk} t}{2} \right) + \frac{E_j E_k - p^2}{E_j E_k} \sin(E_j t) \sin(E_k t) \right\},$$

$$P(\nu_{\alpha L}^c \to \nu_{\beta L}^c) = - \sum_j |U_{\alpha j} U_{\beta j}|^2 \left( \frac{m_j^2}{E_j} \sin^2(E_j t) \right) - 2 \sum_{j<k} \text{Re} \left[ U_{\alpha j}^* U_{\beta j}^* U_{\alpha k} U_{\beta k} \right] \left\{ 2 \sin^2 \left( \frac{\Delta E_{jk} t}{2} \right) + \frac{E_j E_k - p^2}{E_j E_k} \sin(E_j t) \sin(E_k t) \right\} - 2 \sum_{j<k} \text{Im} \left[ U_{\alpha j}^* U_{\beta j}^* U_{\alpha k} U_{\beta k} \right] \left\{ \sin(\Delta E_{jk} t) + \frac{E_k - P}{E_k} \cos(E_j t) \sin(E_k t) - \frac{E_j - P}{E_j} \cos(E_k t) \sin(E_j t) \right\}.$$

Comparing these probabilities with (153)-(156), the only difference from the corresponding neutrino probabilities is the sign of the term proportional to $\text{Im} \left[ U_{\alpha j}^* U_{\beta j}^* U_{\alpha k} U_{\beta k} \right]$.

VI. RELATION OF OSCILLATION PROBABILITIES

Next, let us investigate the relationship between the CP-conjugate probabilities or the T-conjugate probabilities both in the case for the Dirac and the Majorana neutrinos. In order to do that, we summarize the relation of probabilities. In the Dirac neutrinos, we obtain the following relations about the probabilities without chirality-flip,

$$P(\nu_{\alpha L} \to \nu_{\alpha L}) = P(\nu_{\alpha L}^c \to \nu_{\alpha L}^c) = P(\nu_{\alpha L}^c \to \nu_{\alpha L}^c),$$

(158)

$$P(\nu_{\alpha R} \to \nu_{\alpha R}) = P(\nu_{\alpha R} \to \nu_{\alpha R}) = P(\nu_{\alpha R} \to \nu_{\alpha R}),$$

(159)

$$P(\nu_{\alpha L} \to \nu_{\beta L}) = P(\nu_{\beta L} \to \nu_{\alpha L}) = P(\nu_{\beta L} \to \nu_{\alpha L}),$$

(160)

$$P(\nu_{\beta L} \to \nu_{\beta L}) = P(\nu_{\beta L} \to \nu_{\beta L}) = P(\nu_{\beta L} \to \nu_{\beta L}),$$

(161)

$$P(\nu_{\alpha R} \to \nu_{\alpha R}) = P(\nu_{\beta R} \to \nu_{\alpha R}) = P(\nu_{\beta R} \to \nu_{\alpha R}),$$

(162)

$$P(\nu_{\beta R} \to \nu_{\alpha R}) = P(\nu_{\beta R} \to \nu_{\alpha R}) = P(\nu_{\beta R} \to \nu_{\alpha R}).$$

(163)

The difference of (160) and (161) and the difference of (162) and (163) are both sign of the term proportional to $\text{Im} \left[ U_{\alpha j} U_{\beta j}^* U_{\alpha k} U_{\beta k} \right]$ and $\text{Im} \left[ V_{\alpha j} V_{\beta j}^* V_{\alpha k} V_{\beta k} \right]$ respectively. The probabilities with chirality-flip have the relations,

$$P(\nu_{\alpha L} \to \nu_{\alpha R}) = P(\nu_{\alpha L} \to \nu_{\alpha R}) = P(\nu_{\alpha L} \to \nu_{\alpha R}),$$

(164)

$$P(\nu_{\alpha L} \to \nu_{\beta R}) = P(\nu_{\alpha L} \to \nu_{\beta R}) = P(\nu_{\alpha L} \to \nu_{\beta R}),$$

(165)

$$P(\nu_{\beta R} \to \nu_{\alpha L}) = P(\nu_{\beta R} \to \nu_{\alpha L}) = P(\nu_{\beta R} \to \nu_{\alpha L}^c),$$

(166)

Next, the relations on the Majorana neutrino oscillation probabilities without chirality-flip are given by

$$P(\nu_{\alpha L} \to \nu_{\alpha L}) = P(\nu_{\alpha L} \to \nu_{\alpha L}) = P(\nu_{\alpha L} \to \nu_{\alpha L}) = P(\nu_{\alpha L} \to \nu_{\alpha L}),$$

(167)

$$P(\nu_{\alpha L} \to \nu_{\beta L}) = P(\nu_{\beta L} \to \nu_{\alpha L}) = P(\nu_{\beta L} \to \nu_{\alpha L}) = P(\nu_{\beta L} \to \nu_{\beta L}),$$

(168)

$$P(\nu_{\beta L} \to \nu_{\alpha L}) = P(\nu_{\beta L} \to \nu_{\beta L}) = P(\nu_{\beta L} \to \nu_{\beta L}^c).$$

(169)
The difference between \(168\) and \(169\) is also the sign of the term proportional to \(\text{Im} [U_{\alpha j} U_{\beta j}^* U_{\alpha k}^* U_{\beta k}]\). About the oscillation probabilities with chirality-flip, we have the following relations,

\[
P(\nu_{\alpha L} \rightarrow \nu_{\beta L}) = P(\nu_{\alpha L}^c \rightarrow \nu_{\beta L}) = P(\nu_{\alpha L}^c \rightarrow \nu_{\alpha L}) = P(\nu_{\alpha L} \rightarrow \nu_{\alpha L}^c),
\]

\[
P(\nu_{\alpha R} \rightarrow \nu_{\beta R}) = P(\nu_{\alpha R}^c \rightarrow \nu_{\beta R}) = P(\nu_{\alpha R}^c \rightarrow \nu_{\alpha R}) = P(\nu_{\alpha R} \rightarrow \nu_{\alpha R}^c).
\]

Next, let us present the differences between CP-conjugate probabilities, T-conjugate probabilities and CPT-conjugate probabilities. They are obtained by the following replacement in an original probability,

\[
\text{CP conjugate : } U \leftrightarrow U^*, V \leftrightarrow V^*, \nu \leftrightarrow \nu^c,
\]

\[
\text{T conjugate : } \alpha \leftrightarrow \beta,
\]

\[
\text{CPT conjugate : } U \leftrightarrow U^*, V \leftrightarrow V^*, \nu \leftrightarrow \nu^c, \alpha \leftrightarrow \beta
\]

(173)

First, in the case of the Dirac neutrinos, they are respectively given by

\[
\Delta P_{\text{CP}}(\nu_{\alpha L} \rightarrow \nu_{\beta L}) = P(\nu_{\alpha L} \rightarrow \nu_{\beta L}) - P(\nu_{\alpha L}^c \rightarrow \nu_{\beta L})
\]

\[
= -4 \sum_{j<k} \text{Im} [U_{\alpha j} U_{\beta j}^* U_{\alpha k}^* U_{\beta k}] \left\{ \sin(\Delta E_{jk} t) + \frac{E_{k} - p}{E_{k}} \cos(E_{k} t) \sin(E_{j} t) - \frac{E_{j} - p}{E_{j}} \cos(E_{j} t) \sin(E_{k} t) \right\}
\]

(174)

\[
\Delta P_{\text{CP}}(\nu_{\alpha R} \rightarrow \nu_{\beta R}) = P(\nu_{\alpha R} \rightarrow \nu_{\beta R}) - P(\nu_{\alpha R}^c \rightarrow \nu_{\beta R}) = 0,
\]

\[
\Delta P_{\text{T}}(\nu_{\alpha L} \rightarrow \nu_{\beta L}) = P(\nu_{\alpha L} \rightarrow \nu_{\beta L}) - P(\nu_{\alpha L} \rightarrow \nu_{\beta L}) = 0
\]

(175)

\[
\Delta P_{\text{T}}(\nu_{\alpha R} \rightarrow \nu_{\beta R}) = P(\nu_{\alpha R} \rightarrow \nu_{\beta R}) - P(\nu_{\alpha R} \rightarrow \nu_{\alpha R}) = 0,
\]

\[
\Delta P_{\text{CPT}}(\nu_{\alpha L} \rightarrow \nu_{\beta L}) = P(\nu_{\alpha L} \rightarrow \nu_{\beta L}) - P(\nu_{\beta L} \rightarrow \nu_{\alpha L}) = 0
\]

(176)

\[
\Delta P_{\text{CPT}}(\nu_{\alpha R} \rightarrow \nu_{\beta R}) = P(\nu_{\alpha R} \rightarrow \nu_{\beta R}) - P(\nu_{\beta R} \rightarrow \nu_{\alpha R}) = 0
\]

(177)

Second, in the case of the Majorana neutrinos, they are respectively given by

\[
\Delta P_{\text{CP}}(\nu_{\alpha L} \rightarrow \nu_{\beta L}) = P(\nu_{\alpha L} \rightarrow \nu_{\beta L}) - P(\nu_{\alpha L}^c \rightarrow \nu_{\beta L})
\]

\[
= -4 \sum_{j<k} \text{Im} [U_{\alpha j} U_{\beta j}^* U_{\alpha k}^* U_{\beta k}] \left\{ \sin(\Delta E_{jk} t) + \frac{E_{k} - p}{E_{k}} \cos(E_{k} t) \sin(E_{j} t) - \frac{E_{j} - p}{E_{j}} \cos(E_{j} t) \sin(E_{k} t) \right\}
\]

(183)

\[
\Delta P_{\text{CP}}(\nu_{\alpha L} \rightarrow \nu_{\beta L}) = P(\nu_{\alpha L} \rightarrow \nu_{\beta L}) - P(\nu_{\beta L} \rightarrow \nu_{\alpha L}) = 0,
\]

\[
\Delta P_{\text{T}}(\nu_{\alpha L} \rightarrow \nu_{\beta L}) = P(\nu_{\alpha L} \rightarrow \nu_{\beta L}) - P(\nu_{\beta L} \rightarrow \nu_{\alpha L}) = 0
\]

(184)

\[
\Delta P_{\text{CPT}}(\nu_{\alpha L} \rightarrow \nu_{\beta L}) = P(\nu_{\alpha L} \rightarrow \nu_{\beta L}) - P(\nu_{\beta L} \rightarrow \nu_{\alpha L}) = 0
\]

(185)

These results hold even when we extend to n-generations. In the case of both Dirac and Majorana neutrinos, there
is no direct CP and T violation related to the new CP phases in vacuum. Namely, the differences between the original and the CP or T-conjugate probabilities vanish in the oscillations with chirality-flip. This means that we cannot explain the reason for the existence of matter in the universe by neutrino oscillations in vacuum even if neutrinos are the Majorana particles.

VII. COMPARISON OF CONVENTIONAL RESULT AND NEW RESULT FOR MAJORANA NEUTRINOS

In this section, we review the previous results [28-33] on the probabilities for $\nu \leftrightarrow \nu^c$ oscillations and compare with our results. The amplitudes in previous papers are given by

\[ A(\nu_{\alpha L} \rightarrow \nu'_{\beta L}) = \sum_j \left[ U_{\alpha j}^* U_{\beta j} \frac{m_j}{E_j} e^{-iE_j t} \right] K, \quad (189) \]
\[ A(\nu^c_{\alpha L} \rightarrow \nu_{\beta L}) = \sum_j \left[ U_{\alpha j} U_{\beta j}^* \frac{m_j}{E_j} e^{-iE_j t} \right] \bar{K}, \quad (190) \]

where $K$ and $\bar{K}$ are the kinematical factors independent of the index $j$ (and satisfying $|K| = |ar{K}|$). On the other hand, our results are

\[ A(\nu_{\alpha L} \rightarrow \nu^c_{\beta L}) = \sum_j U_{\alpha j}^* U_{\beta j} \frac{m_j}{E_j} (e^{-iE_j t} - e^{-iE_j t}) = -i \sum_j U_{\alpha j}^* U_{\beta j} \frac{m_j}{E_j} \sin(E_j t), \quad (191) \]
\[ A(\nu^c_{\alpha L} \rightarrow \nu_{\beta L}) = \sum_j U_{\alpha j} U_{\beta j}^* \frac{m_j}{E_j} (e^{iE_j t} - e^{-iE_j t}) = i \sum_j U_{\alpha j} U_{\beta j} \frac{m_j}{E_j} \sin(E_j t). \quad (192) \]

The difference of our result from the previous result is in the negative energy part proportional to $e^{iE_j t}$. In the case that we calculate the oscillation probabilities based on the Dirac equation, $\nu$ and $\nu^c$ are included in the same multiplet and a state of the neutrino is represented as the linear combination of both positive and negative energy parts.

Next, we compare the oscillation probabilities. The probabilities presented in the previous papers are given by

\[ P(\nu_{\alpha L} \rightarrow \nu^c_{\beta L}) = \frac{|K|^2}{E^2} \left[ \sum_j \left| m_j U_{\alpha j} U_{\beta j}^* \right|^2 + 2 \sum_{j<k} m_j m_k \Re \left( U_{\alpha j}^* U_{\beta j}^* U_{\alpha k} U_{\beta k} e^{-i\Delta E_{jk} t} \right) \right] \]
\[ = \frac{|K|^2}{E^2} \left[ \sum_j \left| m_j U_{\alpha j} U_{\beta j}^* \right|^2 + 2 \sum_{j<k} m_j m_k \left\{ \Re \left( U_{\alpha j}^* U_{\beta j} U_{\alpha k}^* U_{\beta k} \right) \cos(\Delta E_{jk} t) - \Im \left( U_{\alpha j}^* U_{\beta j} U_{\alpha k}^* U_{\beta k} \right) \sin(\Delta E_{jk} t) \right\} \right], \quad (193) \]
\[ P(\nu^c_{\alpha L} \rightarrow \nu_{\beta L}) = \frac{K^2}{E^2} \left[ \sum_j \left| m_j U_{\alpha j} U_{\beta j} \right|^2 + 2 \sum_{j<k} m_j m_k \Re \left( U_{\alpha j} U_{\beta j}^* U_{\alpha k} U_{\beta k}^* e^{-i\Delta E_{jk} t} \right) \right] \]
\[ = \frac{K^2}{E^2} \left[ \sum_j \left| m_j U_{\alpha j} U_{\beta j} \right|^2 + 2 \sum_{j<k} m_j m_k \left\{ \Re \left( U_{\alpha j} U_{\beta j} U_{\alpha k}^* U_{\beta k}^* \right) \sin^2 \left( \frac{\Delta E_{jk} t}{2} \right) + \Im \left( U_{\alpha j} U_{\beta j} U_{\alpha k}^* U_{\beta k}^* \right) \sin(\Delta E_{jk} t) \right\} \right], \quad (194) \]

where

\[ |\langle m_{\alpha \beta} \rangle|^2 = \sum_j \left| m_j U_{\alpha j} U_{\beta j} \right|^2 \]

is effective mass of the Majorana neutrinos. Accordingly, there is a difference between CP-conjugate probabilities,

\[ P(\nu_{\alpha L} \rightarrow \nu^c_{\beta L}) - P(\nu^c_{\alpha L} \rightarrow \nu_{\beta L}) = \frac{|K|^2}{E^2} \left[ -4 \sum_{j<k} m_j m_k \Im \left( U_{\alpha j} U_{\beta j} U_{\alpha k} U_{\beta k}^* \right) \sin(\Delta E_{jk} t) \right]. \quad (196) \]
On the contrary, in this paper, we have the same probability for CP-conjugate probabilities as

\[ P(\nu_{\alpha L} \rightarrow \nu_{\beta L}') = P(\nu_{\alpha L}^c \rightarrow \nu_{\beta L}) = \left| \sum_j \frac{m_j}{E_j} U_{\alpha j} U_{\beta j} \sin(E_j t) \right|^2 \approx \frac{1}{2E^2} |m_{\alpha\beta}|^2, \quad (197) \]

where the last term is obtained by the averaging the sine term. Then, the difference between CP-conjugate probabilities becomes

\[ P(\nu_{\alpha L} \rightarrow \nu_{\beta L}') - P(\nu_{\alpha L}^c \rightarrow \nu_{\beta L}) = 0. \quad (198) \]

Therefore, we found that the CP violation due to the Majorana CP phase does not appear even if we consider \( \nu \leftrightarrow \nu^c \) oscillations with different flavor.

Another difference is the probability at zero-distance. In the previous papers \[28–33\], it has been pointed out the zero-distance effect. Namely, if we take the limit of \( t \rightarrow 0 \) in eq. (193), the probability has non-zero value,

\[ P(\nu_{\alpha L} \rightarrow \nu_{\beta L}') = \frac{|K|^2}{E^2} |\langle m \rangle_{\alpha\beta}|^2. \quad (199) \]

However, there is no zero-distance effect in our result from eq. (197).

### VIII. SUMMARY

In three generations, we have derived the exact neutrino oscillation probabilities relativistically by using the Dirac equation. The results obtained in the three generations can be extended to the case of \( n \) generations. We have calculated various oscillation probabilities both in the Dirac neutrinos and the Majorana neutrinos. These probabilities can be calculated by the same formulation and can be understood in a unified way. The oscillation probabilities about the Dirac neutrinos derived in this paper are classified as

- the probabilities from left-handed neutrino with negative helicity \( \nu_{\alpha L} \) to other neutrinos
- the probabilities from left-handed neutrino with positive helicity \( \nu_{\alpha L}' \) to other neutrinos
- the probabilities from right-handed neutrino with positive helicity \( \nu_{\alpha R} \) to other neutrinos
- the probabilities from right-handed neutrino with negative helicity \( \nu_{\alpha R}' \) to other neutrinos
- the probabilities from right-handed anti-neutrino with positive helicity \( \nu_{\alpha L}' \) to other neutrinos
- the probabilities from right-handed anti-neutrino with negative helicity \( \nu_{\alpha R}' \) to other neutrinos
- the probabilities from left-handed anti-neutrino with negative helicity \( \nu_{\alpha L}' \) to other neutrinos
- the probabilities from left-handed anti-neutrino with positive helicity \( \nu_{\alpha R}' \) to other neutrinos

In these probabilities, both oscillations with and without chirality-flip are included. About the Majorana neutrinos, the probabilities are classified as

- the probabilities from left-handed neutrino with negative helicity \( \nu_{\alpha L} \) to other neutrinos
- the probabilities from left-handed neutrino with positive helicity \( \nu_{\alpha L}' \) to other neutrinos
- the probabilities from right-handed anti-neutrino with positive helicity \( \nu_{\alpha L}' \) to other neutrinos
- the probabilities from right-handed anti-neutrino with negative helicity \( \nu_{\alpha R}' \) to other neutrinos

In these probabilities, the oscillations between neutrinos and anti-neutrinos are included. These probabilities are not independent but related to each other.

As neutrinos have finite mass, there are two components for each chirality corresponding to positive and negative helicities. We have shown that the probability is different for each component even if neutrinos have the same chirality. We have also shown the probabilities depend on not only the mass squared differences but also the absolute masses of neutrinos. Besides, the new CP phases appear in the probabilities of oscillations with chirality-flip. These new CP phases are equivalent to the Majorana CP phases in the case of Majorana neutrinos. We have also investigated the CP dependence of oscillation probabilities in vacuum and counted the number of the CP phases in \( n \) generations.

In the case of Majorana neutrinos, there is no direct CP violation in \( \nu_\alpha \leftrightarrow \nu_\beta \) oscillations even if the flavors, \( \alpha \) and \( \beta \), are different as in the same as two generations \[33\]. In other words, the difference between the CP-conjugate probabilities \( P(\nu_{\alpha L} \rightarrow \nu_{\beta L}') - P(\nu_{\alpha L}^c \rightarrow \nu_{\beta L}) \) vanishes. Although there is only indirect CP violation, we obtain the information of the new CP phases through both cosine and sine terms. So, we can determine the value of the CP phases. Furthermore, it has been said that the zero-distance effect appears in the oscillations between neutrinos and anti-neutrinos with different flavors in the Majorana neutrino case. However, we have shown that the zero-distance effect does not appear in our formulation. These are different from the results written in previous papers \[28–33\].
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