SURFACES IN 4-MANIFOLDS

RONALD FINTUSHEL AND RONALD J. STERN

Abstract. In this paper we introduce a technique, called rim surgery, which can change a smooth embedding of an orientable surface $\Sigma$ of positive genus and nonnegative self-intersection in a smooth 4-manifold $X$ while leaving the topological embedding unchanged. This is accomplished by replacing the tubular neighborhood of a particular nullhomologous torus in $X$ with $S^1 \times E(K)$, where $E(K)$ is the exterior of a knot $K \subset S^3$. The smooth change can be detected easily for certain pairs $(X, \Sigma)$ called SW-pairs. For example, $(X, \Sigma)$ is an SW-pair if $\Sigma$ is a symplectically and primitively embedded surface with positive genus and nonnegative self-intersection in a simply connected symplectic 4-manifold $X$. We prove the following theorem:

Theorem. Consider any SW-pair $(X, \Sigma)$. For each knot $K \subset S^3$ there is a surface $\Sigma_K \subset X$ such that the pairs $(X, \Sigma_K)$ and $(X, \Sigma)$ are homeomorphic. However, if $K_1$ and $K_2$ are two knots for which there is a diffeomorphism of pairs $(X, \Sigma_{K_1}) \to (X, \Sigma_{K_2})$, then their Alexander polynomials are equal: $\Delta_{K_1}(t) = \Delta_{K_2}(t)$.

1. Introduction

We say that a surface $\Sigma$ is primitively embedded in a simply connected smooth 4-manifold $X$ if $\Sigma$ is smoothly embedded with $\pi_1(X \setminus \Sigma) = 0$. In particular, by Alexander duality, $\Sigma$ must represent a primitive homology class $[\Sigma] \in H_2(X; \mathbb{Z})$. In general, any smoothly embedded (connected) surface $S$ in a simply connected smooth 4-manifold $X$ with $[S] \neq 0$ has the property that the surface $\Sigma$ which represents the homology class $[S] - [E]$ in $X \# \mathbb{CP}^2$ and which is obtained by tubing together the surface $S$ with the exceptional sphere $E$ of $\mathbb{CP}^2$ is primitively embedded (since the surface $\Sigma$ transversally intersects the sphere $E$ in one point).

Given a primitively embedded positive genus surface $\Sigma$ in $X$, in the first part of this paper we shall construct for each knot $K$ in $S^3$ a smoothly embedded surface $\Sigma_K$ in $X$ which is $\Sigma$-compatible; i.e. $[\Sigma] = [\Sigma_K]$ and there is a homeomorphism $(X, \Sigma) \to (X, \Sigma_K)$. This construction will have two properties. The first is that $(X, \Sigma_{\text{unknot}}) = (X, \Sigma)$. The main result of this paper is the second property: under suitable hypotheses on the pair $(X, \Sigma)$, if $K_1$ and $K_2$ are two knots in $S^3$ and if there is a diffeomorphism $(X, \Sigma_{K_1}) \to (X, \Sigma_{K_2})$, then $K_1$ and $K_2$ have the same symmetric Alexander polynomial, i.e. $\Delta_{K_1}(t) = \Delta_{K_2}(t)$. As a special case we show:

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Theorem 1.1. Let $X$ be a simply connected symplectic 4-manifold and $\Sigma$ a symplectically and primitively embedded surface with positive genus and nonnegative self-intersection. If $K_1$ and $K_2$ are knots in $S^3$ and if there is a diffeomorphism of pairs $(X, \Sigma_{K_1}) \to (X, \Sigma_{K_2})$, then $\Delta_{K_1}(t) = \Delta_{K_2}(t)$. Furthermore, if $\Delta_K(t) \neq 1$, then $\Sigma_K$ is not smoothly ambient isotopic to a symplectic submanifold of $X$.

For example, Theorem 1.1 applies to the $K3$ surface where $\Sigma$ is a generic elliptic fiber. It also applies to surfaces of the form $S - E$ in $\mathbb{CP}^2 \# \mathbb{CP}^2$, where $S$ is any positive genus symplectically embedded surface in $\mathbb{CP}^2$.

The outline of this paper is as follows. In §2 we shall construct the surfaces $\Sigma_K$ with $[\Sigma_K] = [\Sigma]$ and show that if $\pi_1(X) = \pi_1(X \setminus \Sigma) = 0$, there is a homeomorphism of $(X, \Sigma)$ with $(X, \Sigma_K)$, i.e. $\Sigma_K$ is $\Sigma$-compatible. We give two descriptions of $\Sigma_K$. One is explicit, while the other describes how to obtain $\Sigma_K$ by removing a tubular neighborhood $T^2 \times D^2$ of a homologically trivial torus in a tubular neighborhood of $\Sigma$ and replacing it with $S^1 \times E(K)$, where $E(K)$ is the exterior of the knot $K$ in $S^3$. This is reminiscent of our construction in [FS] where we performed the same operation on homologically essential tori. There, the Alexander polynomial $\Delta_K(t)$ of $K$ detected a change in the diffeomorphism type of the ambient manifold $X$. Here, we shall show that $\Delta_K(t)$ detects a change in the diffeomorphism type of the embedding of $\Sigma$ in $X$.

If the self-intersection of $\Sigma$ is $n \geq 0$, then in $X_n = X \# n \mathbb{CP}^2$ consider the surface $\Sigma_n = \Sigma - \sum_{j=1}^n E_j$ (resp. $\Sigma_{n,K} = \Sigma_K - \sum_{j=1}^n E_j$) obtained from $\Sigma$ (resp. $\Sigma_K$) by tubing together with the exceptional spheres $E_j$, $j = 1, \ldots, n$, of the copies of $\mathbb{CP}^2$ in $X_n$. If there is a diffeomorphism $H : (X, \Sigma_{K_1}) \to (X, \Sigma_{K_2})$, then there is a diffeomorphism $H_n : (X_n, \Sigma_{n,K_1}) \to (X_n, \Sigma_{n,K_2})$. For each genus $g \geq 1$ we construct in §3 a standard pair $(Y_g, S_g)$, with the properties that $Y_g$ is a Kahler surface, $S_g$ is a primitively embedded genus $g$ Riemann surface in $Y_g$, and the torus used to construct $S_{g,K} = (S_g)_K$ is contained in a cusp neighborhood. Then in §4 we will study SW-pairs, i.e. pairs $(X, \Sigma)$ where $X$ is a smooth simply connected 4-manifold, $\Sigma$ is a primitively embedded genus $g$ surface with self-intersection $n \geq 0$, and the fiber sum of $X_n$ and $Y_g$ along the surfaces $\Sigma_n$ and $S_g$ has a nontrivial Seiberg-Witten invariant $\text{SW}_{X_n \# \Sigma_n = S_g Y_g} \neq 0$. The point here is that the nullhomologous torus used to construct the surface $\Sigma_K$ in $X$ still resides in $X_n \# \Sigma_n = S_g Y_g$ and is now homologically essential and is contained in a cusp neighborhood. It will also follow that if $X$ is a symplectic 4-manifold and $\Sigma$ is a symplectically and primitively embedded surface with nonnegative self-intersection, then $(X, \Sigma)$ is a SW-pair.

In §5 we use in a straightforward fashion the results of [FS] to show that the Alexander polynomial of $K$ distinguishes the $\Sigma_K$ for SW-pairs, and we complete the proof our main theorem:
**Theorem 1.2.** Consider any SW-pair \((X, \Sigma)\). If \(K_1\) and \(K_2\) are two knots in \(S^3\) and if there is a diffeomorphism of pairs \((X, \Sigma_{K_1}) \rightarrow (X, \Sigma_{K_2})\), then \(\Delta_{K_1}(t) = \Delta_{K_2}(t)\).

Finally, in §6 we complete the proof of Theorem 1.1 by showing that in the case that \(\Sigma\) is symplectically embedded in \(X\) and \(\Delta_{K}(t) \neq 1\), then \(\Sigma_{K}\) is not smoothly ambient isotopic to a symplectic submanifold of \(X\).

We conclude this introduction with two conjectures. The first conjecture is that, under the hypothesis of Theorem 1.2, there is a diffeomorphism \((X, \Sigma_{K_1}) \rightarrow (X, \Sigma_{K_2})\) if and only if the knots \(K_1\) and \(K_2\) are isotopic. In particular, this conjecture would imply that the study of the equivalence classes of \(\Sigma\)-compatible surfaces under diffeomorphism is at least as complicated as classical knot theory. The second conjecture is a finiteness conjecture: given a symplectic 4-manifold \(X\) and a symplectic submanifold \(\Sigma\), we conjecture that there are only finitely many distinct smooth isotopy classes of symplectic submanifolds \(\Sigma'\) which are topologically isotopic to \(\Sigma\).

2. **The construction of \(\Sigma_{K}\)**

Let \(X\) be a smooth 4-manifold which contains a smoothly embedded surface \(\Sigma\) with genus \(g > 0\). Then there is a diffeomorphism

\[
h : \Sigma \rightarrow T^2 \# \cdots \# T^2 = (T^2 \setminus D^2) \cup (T^2 \setminus (D^2 \cup D^2)) \cup \cdots \cup (T^2 \setminus D^2).
\]

Let \(C \subset \Sigma\) be a curve whose image under \(h\) is the curve \(S^1 \times \{\text{pt}\} \subset T^2 \setminus D^2 = (S^1 \times S^1) \setminus D^2\) in the first \(T^2 \setminus D^2\) summand of \(h(\Sigma)\). Keep in mind that, since there are many such diffeomorphisms \(h\), there are many such curves \(C\). Given a knot \(K\) in \(S^3\) we shall give two different constructions of a surface \(\Sigma_{K,C}\). The first is an explicit construction, while the second shows how to obtain \(\Sigma_{K,C}\) by what we call a rim surgery, a surgical operation on a particular homologically trivial torus in a neighborhood of \(\Sigma\). It is this second construction that will allow us to compute appropriate invariants to distinguish the surfaces \(\Sigma_{K,C}\).

2.1. **An explicit description of \(\Sigma_{K,C}\).** Viewing \(S^1\) as the union of two arcs \(A_1\) and \(A_2\), we have

\[
T^2 \setminus D^2 = (S^1 \times S^1) \setminus D^2 = ((A_1 \cup A_2) \times (A_1 \cup A_2)) \setminus (A_1 \times A_1) = (A_2 \times S^1) \cup (A_1 \times A_2)
\]

with \(h(C) = A_2 \times \{\text{pt}\} \cup A_1 \times \{\text{pt}\}\). Now the normal bundle of \(\Sigma\) in \(X\) when restricted to \(T^2 \setminus D^2 \subset \Sigma\) is trivial, hence it is diffeomorphic to

\[
((A_2 \times S^1) \cup (A_1 \times A_2)) \times D^2 = ((A_2 \times D^2) \times S^1) \cup ((A_1 \times D^2) \times A_2))
\]

Furthermore, under this diffeomorphism, the inclusion

\[
(T^2 \setminus D^2) \times \{0\} \subset (T^2 \setminus D^2) \times D^2
\]
Now tie a knot $K$ in the arc $(A_2 \times \{0\}) \subset (A_2 \times D^2)$ to obtain a knotted arc $A_K$ and to obtain a new punctured torus

$$T_K \setminus D^2 = (A_K \times S^1) \cup ((A_1 \times \{0\}) \times A_2) \subset ((A_2 \times D^2) \times S^1) \cup ((A_1 \times D^2) \times A_2)$$

with

$$\partial(T_K \setminus D^2) = \partial(T \setminus D^2).$$

Then let

$$\Sigma_{K,C} = (T_K \setminus D^2) \cup (T^2 \setminus (D^2 \amalg D^2)) \cup \cdots \cup (T^2 \setminus D^2) \subset N(\Sigma) \subset X.$$

### 2.2. A description of $\Sigma_{K,C}$ via rim surgery.
 Keeping the notation above, we first recall how, via a 3-manifold surgery, we can tie a knot $K$ in the arc $(A_2 \times \{0\}) \subset (A_2 \times D^2)$. In short, we just remove a small tubular neighborhood in $A_2 \times D^2$ of a pushed-in copy $\gamma$ of the meridional circle $\{0\} \times S^1 \subset A_2 \times D^2$ and sew in the exterior of the knot $K$ in $S^3$ so that the meridian of $K$ is identified with $\gamma$. This has the effect of tying a knot in the arc $A_2 \times \{0\} \subset A_2 \times D^2$. More specifically, consider the standard embedding of the solid torus $A = (A_1 \cup A_2) \times D^2 = S^1 \times D^2$ in $S^3$ with complementary solid torus $B = D^2 \times S^1$ with core $C' = \{0\} \times S^1 \subset D^2 \times S^1$. In $A \setminus C = (S^1 \times D^2) \setminus C = S^1 \times S^1 \times (0, 1] = (A_1 \cup A_2) \times S^1 \times (0, 1]$, consider the circle $\gamma = \{t\} \times S^1 \times \{\frac{1}{2}\}$, with $t \in A_2$, and with tubular neighborhood $N(\gamma) \subset A \setminus C$. The curve $\gamma$ is isotopic in $S^3 \setminus C$ to the core $C'$ of $B$. We denote by $\gamma'$ the curve $\gamma$ pushed off into $\partial N(\gamma)$ so that the linking number in $S^3$ of $\gamma$ and $\gamma'$ is zero. For later reference, note that $D = (A \setminus N(\gamma)) \cup B$ is again diffeomorphic to a solid torus. (It is the exterior of the unknot $\gamma \subset A \subset S^3$.) The core of $D$ is isotopic (in $D$) to $C$.

Let $M_K$ be the 3-manifold obtained by performing 0-framed surgery on $K$. Then the meridian $m$ of $K$ is a circle in $M_K$ and has a canonical framing in $M_K$; we denote a tubular neighborhood of $m$ in $M_K$ by $m \times D^2$. Let $S_K$ denote the 3-manifold

$$S_K = (A \setminus N(\gamma)) \cup (M_K \setminus (m \times D^2)).$$

The two pieces are glued together so as to identify $\gamma'$ with $m$. In other words, we remove $N(\gamma)$ and sew in the exterior $E(K)$ of the knot $K$ in $S^3$. Note that the core $C$ of the solid torus $A$ is untouched by this operation, so $C \subset S_K$. Also, the boundary $\partial A$ of $A$ and the set $G = A_1 \times D^2 \subset (A_1 \cup A_2) \times D^2 \subset A$ remain untouched and thus can be viewed as subsets of $S_K$.

**Lemma 2.1.** There is a diffeomorphism $h : S_K \to A$ which is the identity on $G$ and on the boundary. Furthermore, $h(C)$ is the knotted core $K \subset A$. 
Proof. In $S^3 = A \cup B$, the above operation replaces a tubular neighborhood of the unknot $\gamma \subset A \subset S^3$ with the exterior $E(K)$ of the knot $K$ in $S^3$. Thus there is a diffeomorphism $h : E(K) \cup D \to A \cup B = S^3$ sending the core circle of $D$ to the knot $K$. Now $C' \subset B \subset E(K) \cup D$ is unknotted, since in $D$, the curve $C'$ is isotopic to $\gamma'$, which bounds a disk. Thus $S_K$, which is the complement of a tubular neighborhood of $C'$, is an unknotted solid torus in $S^3 = E(K) \cup D$. Furthermore, as we have noted above, $C$ is isotopic to the core of $D$; so $C \subset S_K$ is the knot $K$. Thus there is a diffeomorphism $h : S_K \to S^1 \times D^2$ which is the identity on the boundary. After an isotopy rel boundary we can arrange that $h(G) = G$. 

To obtain $\Sigma_{K,C}$ we cross everything with $S^1$; i.e. remove the neighborhood $N(\gamma) \times S^1 \subset (A_2 \times D^2) \times S^1 \subset N(\Sigma)$ of the (nullhomologous) torus $\gamma \times S^1 \subset (A_2 \times D^2) \times S^1 \subset N(\Sigma)$ and sew in $E(K) \times S^1$ as above on the $E(K)$ factor and the identity on the $S^1$ factor. We refer to this as a rim surgery on $\Sigma$. Notice that this construction does not change the ambient manifold $X$. Except where it is absolutely necessary to keep track of the curve $C$, we shall suppress it from our notation and abbreviate $\Sigma_{K,C}$ as $\Sigma_K$.

2.3. The complement of $\Sigma_K$. From the construction, it is clear that if the complement of $\Sigma$ in $X$ is simply connected, then so is the complement of $\Sigma_K$ in $X$, since the meridian of the knot (which is identified with the boundary of the normal fiber to $\Sigma$) normally generates the fundamental group of the exterior of $K$. Now there is a map $f : E(K) \to B \cong D^2 \times S^1$ which induces isomorphisms on homology and restricts to a homeomorphism $\partial E(K) \to \partial B$ taking the class of a meridian to $[\{pt\} \times S^1]$ and the class of a longitude to $[\partial D^2 \times \{pt\}]$. The map $f \times \text{id}_{S^1}$ on $E(K) \times S^1$ extends via the identity to a homotopy equivalence $X \setminus N(\Sigma_K) \to X \setminus N(\Sigma)$ which restricts to a homeomorphism $\partial N(\Sigma_K) \to \partial N(\Sigma)$. Then topological surgery guarantees the existence of a homeomorphism $h : (X, \Sigma) \to (X, \Sigma_K)$.

If $\pi_1(X \setminus \Sigma) \neq 0$, it is not clear when $X \setminus \Sigma_K$ is homeomorphic (or even homotopy equivalent) to $X \setminus \Sigma_K$. We avoid such issues in this paper and only deal with the case where $\pi_1(X \setminus \Sigma) = 0$. However, as already noted; the surface $\Sigma - E$ in $X \# \overline{\mathbb{CP}}^2$ obtained by tubing together the surface $\Sigma$ with the exceptional sphere $E$ of $\overline{\mathbb{CP}}^2$ is primitively embedded; so there is a homeomorphism $h : (X \# \overline{\mathbb{CP}}^2, \Sigma - E) \to (X \# \overline{\mathbb{CP}}^2, \Sigma_K - E)$. In summary:

**Theorem 2.2.** Let $X$ be a simply connected smooth 4-manifold with a primitively embedded surface $\Sigma$. Then for each knot $K$ in $S^3$, the above construction produces a $\Sigma$-compatible surface $\Sigma_K$.

3. The standard pair $(Y_g, S_g)$

Let $g > 0$. In this section we shall construct a simply connected smooth 4-manifold $Y_g$ and a primitive embedding of $S_g$, the surface of genus $g$, in $Y_g$ such that the torus used in
the previous section to construct the $S_g$-compatible embedding $(S_g)_K = S_{g,K}$ is contained in a cusp neighborhood.

To this end, consider the $(2, 2g + 1)$-torus knot $T(2, 2g + 1)$. It is a fibered knot whose fiber is a punctured genus $g$ surface and whose monodromy $t'$ is periodic of order $4g + 2$. If we attach a 2-handle to $\partial B^4$ along $T(2, 2g + 1)$ with framing 0, we obtain a manifold $C(g)$ which fibers over the 2-disk with generic fiber a Riemann surface $S_g$ of genus $g$ and whose monodromy map $t$, induced from $t'$, is periodic of order $4g + 2$. If we attach a 2-handle to $\partial B^4$ along $T(2, 2g + 1)$ with framing 0, we obtain a manifold $C(g)$ which fibers over the 2-disk with generic fiber a Riemann surface $S_g$ of genus $g$ and whose monodromy map $t$, induced from $t'$, is periodic of order $4g + 2$. The singular fiber is the topologically (non-locally flatly) embedded sphere obtained from the cone in $B^4$ on the torus knot $T(2, 2g + 1)$ union the core of the 2-handle. Now consider the fibration over the punctured 2-sphere obtained from gluing together $4g + 2$ such neighborhoods $C(g)$ along a neighborhood of a fiber in the boundary of $C(g)$. This is a complex surface, and the monodromy is trivial around a loop which contains in its interior the images of all the singular fibers. Thus we may compactify this manifold to obtain a complex surface $Y_g$ which is holomorphically fibered over $S^2$. For example, $Y_1$ is the rational elliptic surface $\mathbb{CP}^2 \# 9 \overline{\mathbb{CP}}^2$. In fact, $Y_g$ is just the Milnor fiber of the Brieskorn singularity $\Sigma(2, 2g + 1, 4g + 1)$ union a generalized nucleus consisting of the 4-manifold obtained as the trace of the 0-framed surgery on $T(2, 2g + 1)$ and a $-1$ surgery on a meridian [Fu]. The fibration $\pi : Y_g \to S^2$ has a holomorphic section which is a sphere $\Lambda$ of self-intersection $-1$ (the sphere obtained by the $-1$-surgery above (cf. [Fu]). This proves that $\pi_1(Y_g \setminus S_g) = 0$; so $S_g$ is a primitively embedded surface with self-intersection 0.

Let $T$ denote the torus in $S_g \times D^2$ on which we perform a rim surgery in order to obtain the surface $S_{g,K}$. We wish to see that $T$ lies in a cusp neighborhood. A cusp neighborhood is nothing more than the regular neighborhood of a torus together with two vanishing cycles, one for each generating circle in the torus. The torus $T$ has the form $T = \gamma \times \tau$ where $\tau$ is a closed curve on $S_g$ and $\gamma = \{ pt \} \times (\{ \frac{1}{2} \} \times \partial D^2)$. The curve $\tau$ is one of the generating circles for $H_1(S_g; \mathbb{Z})$ with a dual circle $\sigma$. The curve $\gamma$ spans a $-1$-disk contained in $\Lambda$. The curve $\tau$ degenerates to a point on the singular fiber in $C(g)$. Thus we see both required vanishing cycles.

4. SW-pairs

Recall that the Seiberg-Witten invariant $SW_X$ of a smooth closed oriented 4-manifold $X$ with $b^+ > 1$ is an integer valued function which is defined on the set of $spin^c$ structures over $X$, (cf. [W]). In case $H_1(X; \mathbb{Z})$ has no 2-torsion, there is a natural identification of the $spin^c$ structures of $X$ with the characteristic elements of $H^2(X; \mathbb{Z})$. In this case we view the Seiberg-Witten invariant as

$$SW_X : \{ k \in H^2(X, \mathbb{Z}) | k \equiv w_2(TX) \pmod{2} \} \to \mathbb{Z}.$$
The Seiberg-Witten invariant $SW_X$ is a smooth invariant whose sign depends on an orientation of $H^0(X; \mathbb{R}) \otimes \det H^2(X; \mathbb{R}) \otimes \det H^1(X; \mathbb{R})$. If $SW_X(\beta) \neq 0$, then we call $\beta$ a basic class of $X$. It is a fundamental fact that the set of basic classes is finite. If $\beta$ is a basic class, then so is $-\beta$ with

$$SW_X(-\beta) = (-1)^{(e+\text{sign})(X)/4} SW_X(\beta)$$

where $e(X)$ is the Euler number and $\text{sign}(X)$ is the signature of $X$.

As in [FS] we need to view the Seiberg-Witten invariant as a Laurent polynomial. To do this, let $\{\pm \beta_1, \ldots, \pm \beta_n\}$ be the set of nonzero basic classes for $X$. We may then view the Seiberg-Witten invariant of $X$ as the 'symmetric' Laurent polynomial

$$SW_X = b_0 + \sum_{j=1}^{n} b_j (t_j + (-1)^{(e+\text{sign})(X)/4} t_j^{-1})$$

where $b_0 = SW_X(0)$, $b_j = SW_X(\beta_j)$ and $t_j = \exp(\beta_j)$.

Now let $\Sigma$ be genus $g > 0$ primitively embedded surface in the simply connected 4-manifold $X$. If the self-intersection of $\Sigma$ is $n \geq 0$, then in $X_n = X \# n \mathbb{CP}^2$, consider the surface $\Sigma_n = \Sigma - \sum_{j=1}^{n} E_j$ (resp. $\Sigma_{n,K} = \Sigma_K - \sum_{j=1}^{n} E_j$) obtained from $\Sigma$ (resp. $\Sigma_K$) by tubing together with the exceptional spheres $E_j$, $j = 1, \ldots, n$, of the $\mathbb{CP}^2$ in $X_n$. Note that the fiber sum $X_n \# \Sigma_{n=S_g Y_g}$ of $X_n$ and $Y_g$ along the surfaces $\Sigma_n$ and $S_g$ has $b^+ > 1$. An SW-pair is such a pair $(X, \Sigma)$ which satisfies the property that the Seiberg-Witten invariant $SW_{X_n \# \Sigma_{n=S_g Y_g}} \neq 0$.

As we have pointed out earlier, there are several curves $C$ that can be used to construct the surfaces $\Sigma_K, C$, and there are potentially several different fiber sums that can be performed in the construction of $X_n \# \Sigma_{n=S_g Y_g}$. We pin down our choice of $C$ by declaring it to be the image of the curve $\sigma$ from §3 under the diffeomorphism used in the construction of the fiber sum. A simple Mayer-Vietoris argument shows that in $X_n \# \Sigma_{n=S_g Y_g}$ the rim torus (equivalently $\gamma \times \tau$) becomes homologically essential and is contained in a cusp neighborhood. Thus our results from [FS] apply.

5. SW-pairs and the Alexander polynomial

We are now in a position to prove our main theorem:

**Theorem 1.2.** Consider any SW-pair $(X, \Sigma)$. If $K_1$ and $K_2$ are two knots in $S^3$ and if there is a diffeomorphism of pairs $(X, \Sigma_{K_1}) \to (X, \Sigma_{K_2})$, then $\Delta_{K_1}(t) = \Delta_{K_2}(t)$.

**Proof.** With notation as above, we have a diffeomorphism $(X_n, \Sigma_{n,K_1}) \to (X_n, \Sigma_{n,K_2})$. Then there is a diffeomorphism

$$Z_1 = X_n \# \Sigma_{n,K_1=S_g Y_g} \to Z_2 = X_n \# \Sigma_{n,K_2=S_g Y_g}.$$
It follows from \( \text{FS} \) that \( \text{SW}_{Z_i} = \text{SW}_{X_n \# \Sigma_n = s_g Y_g} \cdot \Delta_{K_1}(t) \) for \( t = \exp(2[T]) \), where \( T \) denotes the rim torus. Since \( (X, \Sigma) \) is a SW-pair, and since \( [T] \neq 0 \) in \( H_2(Z_i; \mathbb{Z}) \) we must have \( \Delta_{K_1}(t) = \Delta_{K_2}(t) \).

6. Rim surgery on symplectically embedded surfaces

We conclude with a proof of our claim of the introduction.

**Theorem 1.1.** Let \( X \) be a simply connected symplectic 4-manifold and \( \Sigma \) a symplectically and primitively embedded surface with positive genus and nonnegative self-intersection. If \( K_1 \) and \( K_2 \) are knots in \( S^3 \) and if there is a diffeomorphism of pairs \( (X, \Sigma_{K_1}) \rightarrow (X, \Sigma_{K_2}) \), then \( \Delta_{K_1}(t) = \Delta_{K_2}(t) \). Furthermore, if \( \Delta_{K}(t) \neq 1 \), then \( \Sigma_K \) is not smoothly ambient isotopic to a symplectic submanifold of \( X \).

**Proof.** Since \( \Sigma \) and \( S_g \) are symplectic submanifolds of \( X \) and \( Y_g \), the fiber sum \( X_n \# \Sigma_n = S_g Y_g \) is also a symplectic manifold \( [G] \). Thus \( \text{SW}_{X_n \# \Sigma_n = S_g Y_g} \neq 0 \) \([I]\); so \( (X, \Sigma) \) forms an SW-pair. This proves the first statement of the theorem.

Next, suppose that \( \Sigma_K \) is smoothly ambient isotopic to a symplectic submanifold \( \Sigma' \) of \( X \). This isotopy carries the rim torus \( T \) to a rim torus \( T' \) of \( \Sigma' \). We have

\[
\text{SW}_{X_n \# \Sigma_n = S_g Y_g} = \text{SW}_{X_n \# \Sigma_n = \Sigma_n, K = s_g Y_g} = \text{SW}_{X_n \# \Sigma_n = S_g Y_g} \cdot \Delta_K(t)
\]

with \( t = \exp(2[T']) \) when this expression is viewed as \( \text{SW}_{X_n \# \Sigma_n = S_g Y_g} \). As above, \( [T'] \neq 0 \) in \( H_2(X_n \# \Sigma_n = S_g Y_g; \mathbb{Z}) \).

Symplectic forms \( \omega_X \) on \( X_n \) (with respect to which \( \Sigma_n' \) is symplectic) and \( \omega_Y \) on \( Y_g \) induce a symplectic form \( \omega \) on the symplectic fiber sum \( X_n \# \Sigma_n = S_g Y_g \) which agrees with \( \omega_X \) and \( \omega_Y \) away from the region where the manifolds are glued together. In particular, since \( T' \) is nullhomologous in \( X_n \), we have \( \langle \omega, T' \rangle = \langle \omega_X, T' \rangle = 0 \). Now \([I]\) implies that the basic classes of \( X_n \# \Sigma_n = S_g Y_g \) are exactly the classes \( b + 2mT' \) where \( b \) is a basic class of \( X_n \# \Sigma_n = S_g Y_g \) and \( m \) has a nonzero coefficient in \( \Delta_K(t) \). Thus the basic classes of \( X_n \# \Sigma_n = S_g Y_g \) can be grouped into collections \( \mathcal{C}_b = \{ b + 2mT' \} \), and if \( \Delta_K(t) \neq 1 \) then each \( \mathcal{C}_b \) contains more than one basic class. Note, however, that \( \langle \omega, b + 2mT' \rangle = \langle \omega, b \rangle \). Now Taubes has shown \([I2]\) that the canonical class \( \kappa \) of a symplectic manifold with \( b^+ > 1 \) is the basic class which is characterized by the condition \( \langle \omega, \kappa \rangle > \langle \omega, b' \rangle \) for any other basic class \( b' \). But this is impossible for \( X_n \# \Sigma_n = S_g Y_g \) since each \( \mathcal{C}_b \) contains more than one class. \( \square \)

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DEPARTMENT OF MATHEMATICS, MICHIGAN STATE UNIVERSITY
EAST LANSING, MICHIGAN 48824

E-mail address: ronfint@math.msu.edu

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA
IRVINE, CALIFORNIA 92697

E-mail address: rstern@math.uci.edu