A finite Toda representation of the box–ball system with box capacity

Kazuki Maeda

Department of Applied Mathematics and Physics, Graduate School of Informatics, Kyoto University, Kyoto 606-8501, Japan

E-mail: kmaeda@amp.i.kyoto-u.ac.jp

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Abstract

A connection between the finite ultradiscrete Toda lattice and the box–ball system is extended to the case where each box has its own capacity and a carrier has a capacity parameter depending on time. In order to consider this connection, new carrier rules ‘size limit for solitons’ and ‘recovery of balls’, and a concept ‘expansion map’ are introduced. A particular solution to the extended system of a special case is also presented.

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1. Introduction

The box–ball system (BBS), proposed by Takahashi and Satsuma [9], is one of the most important cellular automata obtained from discrete integrable systems through a limiting procedure called ultradiscretization [12]. It is well known that the time evolution of the original BBS is determined by the ultradiscrete KdV (u-KdV) equation

\[ U^{(t+1)}_n = \min \left( 1 - U^{(t)}_n, \sum_{j=-\infty}^{n-1} (U^{(t)}_j - U^{(t+1)}_j) \right), \quad n, t \in \mathbb{Z}, \tag{1} \]

where \( U^{(t)}_n \in \{0, 1\} \) denotes the number of balls in the \( n \)th box at time \( t \). Equation (1) is an ultradiscrete analogue of the discrete KdV equation, and consequently, the original BBS has ultradiscrete soliton solutions. It is also known that the nonautonomous discrete KP (nd-KP) equation with a reduction condition yields a time evolution equation of the BBS with three extensions [1]: box capacity [10], carrier capacity [8] and kind of balls [11].

In this paper, we consider another type of time evolution equations of the BBS first presented by Nagai et al [5]. They discovered that the ultradiscrete Toda (u-Toda) equation on a (non-periodic) finite lattice

\[ Q^{(t+1)}_n = \min \left( E^{(t)}_{n+1} - \sum_{j=0}^{n} Q^{(t)}_j - \sum_{j=0}^{n-1} Q^{(t+1)}_j \right), \tag{2a} \]
which we simply refer to as the finite u-Toda lattice, determines the time evolution of the original BBS. In this equation, the dependent variables $Q_n^{(t)}$ and $E_n^{(t)}$ denote the size of the $n$th soliton at time $t$ and the size of the $n$th empty block at time $t$, respectively. Figure 1 shows an example of the connection between the finite u-Toda lattice and the BBS. We call this representation finite Toda representation of the BBS.

The correspondence between the u-KdV equation and the finite u-Toda equation via the BBS is similar to the Euler–Lagrange correspondence of cellular automaton [3]. This terminology comes from hydrodynamics; the dependent variables of the Euler representation denote the number of particles at each point and the ones of the Lagrange representation denote the position of each particle. According to these definitions, we use the following terms in this paper.

- The Euler representation of the BBS: the equation of the BBS with the variables which denote the number of balls in each box.
- The Lagrange representation of the BBS: the equation of the BBS with the variables which denote the start position of each soliton and each empty block.
- The finite Toda representation of the BBS: the equation of the BBS with the variables which denote the size of each soliton and each empty block.

The u-KdV equation (1) and the finite u-Toda lattice (2) denote the Euler representation and the finite Toda representation of the original BBS, respectively. Additionally, if we know the start position of the first soliton, we can calculate the start positions of all solitons and empty blocks from the values of the variables of the finite Toda representation. In other words, the finite Toda representation and the Lagrange representation can be transformed to each other.

It has not been clarified till now why these two different ultradiscrete equations with a different type of boundary conditions describe the same original BBS. Moreover, the finite Toda representation of the extended BBS is not studied sufficiently. Among the three extensions for the BBS, two types of finite Toda representations have been clarified: Tokihiro et al. [11] discussed the case with several kinds of balls using the finite ultradiscrete hungry Toda lattice; Maeda and Tsujimoto [2] showed that the finite nonautonomous ultradiscrete Toda lattice determines the time evolution of the BBS with a carrier. The main purpose of this paper is to discuss the remaining case; we derive a finite Toda representation of the BBS with box capacity. For this purpose, we use a map from a state of the BBS to a binary sequence, which we call the ‘expansion map’. By using the expansion map, we can define the size of solitons in the BBS with box capacity for all time $t$, especially for interacting solitons. We also consider a finite Toda representation of the BBS with both box capacity and carrier capacity using

$$E_n^{(t+1)} = E_n^{(t)} - Q_{n-1}^{(t+1)} + Q_n^{(t)}, \quad (2b)$$

$$E_0^{(t)} = E_N^{(t)} = +\infty, \quad (2c)$$
new carrier rules ‘size limit for solitons’ and ‘recovery of balls’. Furthermore, we present a particular solution for the fixed box capacity case.

The outline of the rest of this paper is as follows. In section 2, we recall the derivation of the Euler representation of the BBS with a carrier from the 2-reduced nd-KP equation. In addition, we introduce new carrier rules ‘size limit for solitons’ and ‘recovery of balls’, which are used to derive the finite Toda representation of the BBS with a carrier [2] and play an important role in section 4. In section 3, we recall the finite Toda representation of the original BBS and give some remarks. In section 4, we discuss a finite Toda representation of the BBS with variable box capacity. First, we discuss the case of variable box capacity and carrier capacity being variables. In section 5, we give a particular solution to the finite Toda representation of the BBS with fixed box capacity. In section 6, we give concluding remarks.

2. The Euler representation of the BBS with a carrier

The nd-KP equation is given by [13]
\[ (a_n - b_i) f_{n+1}^{k,t+1} + (b_i - c_k) f_{n+1}^{k,t+1} + (c_k - a_n) f_n^{k,t+1} = 0, \quad k, n, t \in \mathbb{Z}. \] (3)

It is shown that an N-soliton solution to the nd-KP equation (3) is presented by

\[ f_n^{k,t} = 1 + \sum_{J \subseteq [0,1,...,N-1]} \left( \prod_{i,j \in J} w_{i,j} \prod_{i,e \neq j} h_{i,n}^{k,t} \right), \]

where \( \xi_i, p_i, q_i, i = 0, 1, \ldots, N - 1 \), are some constants. Now, we impose the 2-reduction with respect to the variable \( k \), i.e. \( f_n^{k+2,t} = f_n^{k,t} \) and \( c_k + 2 = c_k \) for all \( k \in \mathbb{Z} \), and set \( a_n = 1 + \delta_n, b_i = -\mu_i, c_0 = 1 \) and \( c_1 = 0 \). Then, the nd-KP equation (3) reduces to the forms

\[ (1 + \delta_n + \mu_i) f_{n+1}^{0,t+1} f_n^{0,t} = (1 + \mu_i) f_{n+1}^{0,t+1} f_n^{0,t} + \delta_n f_n^{0,t+1} f_n^{0,t+1}, \] (4a)

\[ (1 + \delta_n + \mu_i) f_n^{0,t} f_{n+1}^{0,t+1} = (1 + \delta_n) f_n^{0,t} f_{n+1}^{0,t+1} + \mu_i f_n^{0,t+1} f_{n+1}^{0,t}, \] (4b)

and an N-soliton solution to the reduced equations is given by

\[ f_n^{k,t} = 1 + \sum_{J \subseteq [0,1,...,N-1]} \left( \prod_{i,j \in J} w_{i,j} \prod_{i,e \neq j} h_{i,n}^{k,t} \right), \quad k = 0, 1, \] (5a)

\[ h_{i,n}^{0,t} := \xi_i \prod_{j=0}^{n-1} \frac{1 + \delta_j - p_i}{1 + \mu_j - p_i} \frac{p_i + \mu_j}{p_i + \delta_j - p_i}, \quad h_{i,n}^{1,t} := \frac{1 - p_i}{p_i} h_{i,n}^{0,t}, \] (5b)

\[ w_{i,j} := \left( \frac{p_i - p_j}{1 + p_i - p_j} \right)^2. \] (5c)

Let us define the dependent variables as

\[ u_n^{(t)} = \frac{f_{n+1}^{0,t+1} f_n^{0,t+1}}{f_n^{0,t+1} f_{n+1}^{0,t+1}}, \quad v_n^{(t)} = (1 + \delta_n + \mu_i) \frac{f_{n+1}^{0,t+1} f_n^{0,t+1}}{f_n^{0,t+1} f_{n+1}^{0,t+1}}, \quad \gamma_n^{(t)} = \frac{f_n^{0,t+1} f_n^{0,t+1}}{f_n^{0,t+1} f_{n+1}^{0,t+1}}, \] (6)
Then, the 2-reduced nd-KP equation (4) yields the equations
\[ u_n^{(t+1)} = \frac{1}{u_n^{(t)}} + (1 + \mu_{t+1}) z_n^{(t+1)}, \quad (7a) \]
\[ z_n^{(t+1)} = \frac{(1 + \delta_n) z_n^{(t+1)} u_{n-1}^{(t)} + \mu_{t+1}}{u_n^{(t+1)}}, \quad (7b) \]
and the identity
\[ u_n^{(t+1)} = u_n^{(t)} z_n^{(t+1)} \]
holds. For positivity, we choose the parameters as \( 0 \leq \delta_n \leq 1 \) and \( 0 \leq \mu_t \leq 1 \) for all \( n, t \in \mathbb{Z} \). When the values of the dependent variables are all positive for all \( n, t \in \mathbb{Z} \), we can ultradiscretize equations (7); putting \( u_n^{(t)} = e^{-U^{(t)}} / \epsilon, \quad \bar{z}_n^{(t)} = e^{-\bar{Z}^{(t)}} / \epsilon, \quad \bar{u}_n^{(t)} = e^{-\bar{U}^{(t)} / \epsilon}, \delta_n = e^{-\Delta_n / \epsilon}, \mu_t = e^{-M_t / \epsilon} \) into (7) and taking a limit \( \epsilon \to +0 \), we obtain the ultradiscrete system
\[ U_n^{(t+1)} = \min (\Delta_n - U_n^{(t)}, \bar{Z}_n^{(t+1)}), \quad (8a) \]
\[ Z_n^{(t+1)} = \min (\bar{Z}_n^{(t+1)} + U_{n-1}^{(t)} - M_{t+1}), \quad (8b) \]
\[ U_n^{(t+1)} = U_n^{(t)} + \bar{Z}_n^{(t+1)} - Z_{n+1}^{(t+1)}, \quad (8c) \]
where \( \Delta_n, M_t \geq 0 \) for all \( n, t \in \mathbb{Z} \). Note that we have used the fundamental formula for the ultradiscretization
\[ \lim_{\epsilon \to +0} -\epsilon \log(e^{-A / \epsilon} + e^{-B / \epsilon}) = \min(A, B). \]

An \( N \)-soliton solution to the ultradiscrete system (8) is obtained as follows. Let us take the constants \( p_i, i = 0, 1, \ldots, N - 1 \), to satisfy the condition \( 0 < p_i < 1 \). Putting \( f_n^{k,t} = e^{-F^{k,t} / \epsilon}, \quad h_i^{k,n} = e^{-H_i^{k,n} / \epsilon}, \quad p_i = e^{-P_i / \epsilon}, \quad \xi_i = e^{-\xi_i / \epsilon}, \quad w_{i,j} = e^{-W_{i,j} / \epsilon} \) into (5) and (6), and taking a limit \( \epsilon \to +0 \), we obtain
\[ U_n^{(t)} = F_n^{0,t+1} - F_n^{0,t+1} + F_n^{1,t+1} - F_n^{1,t+1}, \]
\[ \bar{U}_n^{(t)} = F_n^{0,t} - F_n^{0,t} - F_n^{0,t+1} - F_n^{0,t+1}, \]
\[ Z_n^{(t)} = F_n^{0,t} - F_n^{0,t} + F_n^{1,t} - F_n^{1,t}, \]
\[ F_n^{k,t} = \min \left( 0, \min_{J \subseteq \{0, 1, \ldots, N - 1\} \backslash \{j\}} \left( \sum_{i,j \neq j} W_{i,j} + \sum_{i \neq j} H_i^{k,n} \right) \right), \quad k = 0, 1, \]
\[ H_i^{0,n} = \xi_i - \sum_{j=0}^{n-1} \min(p_i, \Delta_j) + \sum_{j=0}^{t-1} \min(p_i, M_j), \quad H_i^{1,n} = H_{i,n}^{0,n} - P_i, \]
\[ W_{i,j} = 2 \min(p_i, P_j), \]
and \( P_i \geq 0, i = 0, 1, \ldots, N - 1 \).

Let us introduce the time evolution rule of the BBS with the \( n \)th box capacity \( \Delta_n \) and the carrier capacity \( M_{t+1} \) from time \( t \) to \( t + 1 \). We consider the time evolution rule from time \( t \) to \( t + 1 \) as the composition of size limit process and recovery process.
Figure 2. Illustration of the time evolution rule of the BBS with a carrier. The left figure illustrates the size limit process (8a) and (8b) and the right one illustrates the recovery process (8c).

(i) Size limit process: the carrier of balls moves from left ($n = -\infty$) to right ($n = +\infty$). When the carrier passes each box, the carrier gets all balls in the box; if the number of balls exceeds the carrier capacity $M_{t+1}$, the excess balls are removed from the system. At the same time, the carrier puts the balls that the carrier holds into the box as many balls as possible.

(ii) Recovery process: after the size limit process, all the removed balls are recovered to the boxes in which the balls were kept.

Figure 2 illustrates these rules.

Suppose that the dependent variables $U^{(t)}_n$, $\overline{U}^{(t+1)}_n$ and $\overline{Z}^{(t+1)}_n$ denote the following quantities.

- $U^{(t)}_n \in \{0, 1, \ldots, \Delta_n\}$: the number of balls in the $n$th box at time $t$.
- $\overline{U}^{(t+1)}_n \in \{0, 1, \ldots, \Delta_n\}$: the number of balls in the $n$th box after the size limit process from time $t$ to $t+1$.
- $\overline{Z}^{(t+1)}_n \in \{0, 1, \ldots, M_{t+1}\}$: the number of balls in the carrier arriving at the $n$th box in the size limit process from time $t$ to $t+1$.

Then, equations (8) give the time evolution rule: equations (8a) and (8b) define the size limit process

$$\left\{ U^{(t)}_n \right\}_{n=-\infty}^{+\infty} \mapsto \left\{ \overline{U}^{(t+1)}_n \right\}_{n=-\infty}^{+\infty}, \left\{ \overline{Z}^{(t+1)}_n \right\}_{n=-\infty}^{+\infty}$$

and since

$$\overline{U}^{(t+1)}_n + \left( \overline{Z}^{(t+1)}_n + U^{(t)}_n - \min \left( \overline{Z}^{(t+1)}_n + U^{(t)}_n, M_{t+1} \right) \right) = \overline{U}^{(t+1)}_n + \overline{U}^{(t+1)}_n + U^{(t)}_n - \overline{Z}^{(t+1)}_{n+1} - U^{(t+1)}_n = U^{(t)}_n + \overline{Z}^{(t+1)}_n - \overline{Z}^{(t+1)}_{n+1}$$

gives the number of removed balls by the size limit at the $n$th box, equation (8c) defines the recovery process

$$\left\{ \overline{U}^{(t+1)}_n \right\}_{n=-\infty}^{+\infty}, \left\{ \overline{Z}^{(t+1)}_n \right\}_{n=-\infty}^{+\infty} \mapsto \left\{ U^{(t+1)}_n \right\}_{n=-\infty}^{+\infty}.$$
Figure 3 shows an example of a 3-soliton solution to the time evolution equation (8). The parameters are chosen as
\[
\Delta_n = \begin{cases} 
3 & \text{if } n \text{ is even}, \\
5 & \text{if } n \text{ is odd}, 
\end{cases} 
\quad M_t = \begin{cases} 
+\infty & \text{if } t \leq 0, \\
6 & \text{if } t > 0. 
\end{cases}
\]

Remark 1. Eliminating the variable \( \overline{U}_t^{(t+1)} \) from equations (8a) and (8b), we have the relation
\[
-Z_{n+1}^{(t+1)} = \min (\Delta_n - U_n^{(t)}, Z_n^{(t+1)}) - \min (Z_{n+1}^{(t+1)} + U_n^{(t)}, M_{t+1}). 
\tag{9}
\]

From equation (8c), the relation
\[
Z_n^{(t+1)} = \sum_{j=-\infty}^{n-1} (U_j^{(t)} - U_j^{(t+1)}) 
\tag{10}
\]
also holds. Substituting (10) into (9), we obtain the equation
\[
U_n^{(t+1)} = \min \left( \Delta_n - U_n^{(t)}, \sum_{j=-\infty}^{n-1} (U_j^{(t)} - U_j^{(t+1)}) \right) 
+ \max \left( 0, \sum_{j=-\infty}^{n} U_j^{(t)} - \sum_{j=-\infty}^{n-1} U_j^{(t+1)} - M_{t+1} \right), 
\tag{11}
\]
where we have used the formula
\[
-\min(-A, -B) = \max(A, B). 
\]
Equation (11) has the same form as of the time evolution equation of the ‘BBS with a carrier’ presented by Takahashi and Matsukidaira [8].

If we choose \( M_{t+1} = +\infty \), then (11) yields
\[
U_n^{(t+1)} = \min \left( \Delta_n - U_n^{(t)}, \sum_{j=-\infty}^{n-1} (U_j^{(t)} - U_j^{(t+1)}) \right), 
\tag{12}
\]
which is the nonautonomous \( u \)-KdV equation. In addition, (8b) yields
\[
\overline{Z}_n^{(t+1)} = \overline{Z}_{n+1}^{(t+1)} + U_n^{(t)} - \overline{U}_n^{(t+1)}, 
\]
and comparing this relation with (8c), we have the relation \( U_n^{(t+1)} = \overline{U}_n^{(t+1)} \).
3. The finite Toda representation of the original BBS

We recall the relation between the finite u-Toda lattice and the original BBS. The bilinear form of the discrete Toda lattice is given by

$$\tau_n^{(t+1)} - \tau_n^{(t+1)} = q_n^{(t)} q_{n+1}^{(t)} + \tau_n^{(t)} \tau_n^{(t)} , \quad n, t \in \mathbb{Z} .$$

Let us introduce the dependent variables

$$q_n^{(t)} = \frac{r_n^{(t)} r_{n+1}^{(t)}}{r_n^{(t)} r_{n+1}^{(t)}}, \quad e_n^{(t)} = \frac{r_n^{(t)} r_{n-1}^{(t)}}{r_n^{(t)} r_{n+1}^{(t)}}, \quad d_n^{(t)} = \frac{e_n^{(t)} e_{n+1}^{(t)}}{e_n^{(t)} e_{n-1}^{(t)}} .$$

Then, (13) yields the equation

$$q_n^{(t+1)} = e_{n+1}^{(t+1)} + d_n^{(t+1)} , \quad (14a)$$

and the identities

$$e_n^{(t+1)} = e_n^{(t)} e_{n+1}^{(t)} - d_n^{(t)} q_{n+1}^{(t)} , \quad d_n^{(t+1)} = q_{n-1}^{(t)} d_n^{(t)} - q_n^{(t)} q_{n+1}^{(t)} . \quad (14b)$$

Putting $q_n^{(t)} = e^{-E^t_n/\epsilon}$, $e_n^{(t)} = e^{E^t_n/\epsilon}$, and taking a limit $\epsilon \to +0$, we obtain the u-Toda lattice

$$Q_n^{(t+1)} = \min\left( E_n^{(t)} , Q_n^{(t+1)} \right) , \quad (15a)$$

$$E_n^{(t+1)} = E_n^{(t)} - Q_n^{(t+1)} + Q_n^{(t)} , \quad (15b)$$

$$D_n^{(t+1)} = D_{n-1}^{(t+1)} - Q_n^{(t+1)} + Q_n^{(t)} . \quad (15c)$$

Furthermore, we impose the terminating condition for discussing the finite Toda representation

$$E_0^{(t)} = E_0^{(t)} = +\infty , \quad D_0^{(t+1)} = Q_0^{(t)} , \quad (15d)$$

where $N$ is a positive integer, which denotes the number of solitons in the original BBS.

Let the variables $Q_n^{(t)}$, $E_n^{(t)}$ and $D_n^{(t+1)}$, respectively, denote the following quantities of the original BBS.

- $Q_n^{(t)}$: the size of the $n$th soliton at time $t (n = 0, 1, \ldots, N - 1)$.
- $E_n^{(t)}$: the size of the $n$th empty block, namely, the distance between the $(n - 1)$th soliton and the $n$th one at time $t (n = 1, 2, \ldots, N - 1)$.
- $D_n^{(t+1)}$: the number of balls in the carrier after getting $Q_n^{(t)}$ balls ($n = 0, 1, \ldots, N - 1$).

Then, the next theorem gives a fundamental result on the connection between the finite u-Toda lattice and the BBS.

**Theorem 1** (Nagai et al [5]). *The finite u-Toda lattice (15) determines the time evolution of the original BBS.*

**Remark 2.** Conventionally, the d-Toda lattice (qd-type) has been written in the form

$$q_n^{(t+1)} + e_n^{(t+1)} = q_n^{(t)} + e_{n+1}^{(t)} , \quad q_n^{(t)} e_{n+1}^{(t)} = q_n^{(t)} e_{n+1}^{(t)} ,$$

or

$$q_n^{(t+1)} = q_n^{(t)} + e_n^{(t+1)} + e_n^{(t)} , \quad q_n^{(t)} e_{n+1}^{(t)} = q_n^{(t)} e_{n+1}^{(t)} , \quad (16a)$$

$$e_n^{(t+1)} = e_n^{(t)} + q_n^{(t+1)} - q_n^{(t)} , \quad (16b)$$

which we cannot ultradiscretize directly due to the ‘negative problem’. Equations (16) are called Rutishauser’s qd algorithm in numerical algorithms [6]. On the other hand, equations

\[7\]
Figure 4. The Lagrange representation and finite Toda representation of the BBS.

\[(14)\] are called the dqd algorithm. The dqd algorithm is the subtraction-free form of the qd algorithm and computes matrix eigenvalues or singular values more accurately than the qd algorithm.

Suppose the finite lattice condition
\[
e^{(t)}_0 = \sum_{j=0}^{n-1} q^{(t)}_j = \sum_{j=0}^{n-1} q^{(t+1)}_j,
\]
\[(16a)\]

Nagai et al [5] rewrote \[(16a)\] using \[(16b)\] as
\[
q^{(t+1)}_n = q^{(t)}_n - e^{(t+1)}_n,
\]
\[(16b)\]

Then, they could ultradiscretize the finite Toda lattice:
\[
Q^{(t+1)}_n = \min \left( E^{(t)}_{n+1} - \sum_{j=0}^{n} q^{(t)}_j - \sum_{j=0}^{n-1} q^{(t+1)}_j \right),
\]
\[(17a)\]

\[
E^{(t+1)}_n = E^{(t)}_n - q^{(t+1)}_n + Q^{(t)}_n,
\]
\[(17b)\]

\[
E^{(t)}_0 = E^{(t)}_N = +\infty.
\]
\[(17c)\]

On the other hand, by introducing an auxiliary variable
\[
d^{(t+1)}_n := d^{(t)}_n - q^{(t+1)}_n = q^{(t+1)}_n - e^{(t+1)}_n,
\]
we can ultradiscretize the finite d-Toda lattice directly without the negative problem and obtain the finite u-Toda lattice of the dqd form \[(15)\]. From the viewpoint of the BBS, the variable \(D^{(t+1)}_n\) denotes the number of balls in the carrier. Therefore, the finite u-Toda lattice of the dqd form \[(15)\] is important to consider the finite Toda representation of the BBS with a carrier.

Remark 3. Here, we remark on the Lagrange representation of the BBS, which is also a terminology from hydrodynamics as the Euler representation; the dependent variables of the Lagrange representation denote the position of solitons. Let the variables \(X^{(t)}_n\) and \(Y^{(t)}_n\) denote the start position of the \(n\)th soliton and the one of the \(n\)th empty block at time \(t\), respectively (see figure 4). Then, the Lagrange representation of the BBS is given by \[4\]
\[
X^{(t+1)}_n = Y^{(t)}_{n+1},
\]
\[(18a)\]
\[ Y_n^{(t+1)} = Y_n^{(t)} + \min \left( X_n^{(t)} - Y_n^{(t)}, \sum_{j=1}^{n} (Y_j^{(t)} - X_j^{(t)}) - \sum_{j=1}^{n-1} (Y_j^{(t+1)} - X_j^{(t+1)}) \right), \quad (18b) \]

\[ Y_0^{(t)} = -\infty, \quad X_N^{(t)} = +\infty. \quad (18c) \]

Relations between these variables and the variables of the finite u-Toda lattice (17) are given by

\[ X_n^{(t)} = Y_n^{(t)} + E_n^{(t)}, \quad Y_n^{(t)} = X_{n-1}^{(t)} + Q_{n-1}. \quad (19) \]

We can readily show that (18) and (19) yield the finite u-Toda lattice (17). Conversely, we can calculate the values of \( \{X_n^{(t)}\}_{n=1}^{N} \) and \( \{Y_n^{(t)}\}_{n=1}^{N} \) from the values of \( X_0^{(t)} \), \( Q_0^{(t)} \), \( E_0^{(t)} \), \( E_1^{(t)} \), and \( E_{N-1}^{(t)} \). In other words, the finite u-Toda lattice (17) and the equation \( X_0^{(t+1)} = X_0^{(t)} + Q_0^{(t)} \), which is obtained from (18a) and (19), uniquely determine the time evolution of the BBS.

4. Extension of the finite Toda representation to the case of variable box capacity \( \Delta_n \) and variable carrier capacity \( M_t \)

In previous studies, the finite Toda representation is considered only for the BBS with box capacity 1. In this section, we extend the finite Toda representation to the case in which each box has its own capacity \( \Delta_n \). First, we consider the case of carrier capacity \( M_t = +\infty \). The Euler representation of this case is given by (12).

We first define the size of solitons and the one of empty blocks for the BBS with variable box capacity \( \Delta_n \) at an any time \( t \). For this purpose, we refer to the work of Takahashi and Satsuma [10]. They analyzed the BBS with the fixed box capacity \( \Delta \) using a map from a state of box capacity \( \Delta \) to a binary sequence. We generalize this map for the case of variable box capacity \( \Delta_n \).

Suppose that a state of the Euler representation (12) \( \{U_n^{(t)}\}_{n=-\infty}^{+\infty} \), such that \( U_n^{(t)} \in [0, 1, \ldots, \Delta_n] \) is given. We assume that, for simplicity, \( U_n^{(t)} = 0 \) for \( n < 0 \). Let us define a map \( \{U_n^{(t)}\}_{n=-\infty}^{+\infty} \rightarrow \{V_n^{(t)}\}_{n=-\infty}^{+\infty} \), where \( V_n^{(t)} \in [0, 1] \), as follows.

1. \( V_n^{(t)} = 0 \) for \( n < 0 \).
2. Let \( s_0 = 0 \) and \( s_n = \sum_{j=0}^{n-1} \Delta_j \) for \( n = 1, 2, \ldots \). From \( n = 0 \) to \( +\infty \), if \( V_n^{(t)} = 1 \), then
   
   \[
   V_{s_n}^{(t)} = V_{s_n+1}^{(t)} = \cdots = V_{s_n+U_n^{(t)}-1}^{(t)} = 1, \\
   V_{s_n+U_n^{(t)}}^{(t)} = V_{s_n+U_n^{(t)}+1}^{(t)} = \cdots = V_{s_n+\Delta_n-1}^{(t)} = 0; \\
   
   \]

   otherwise,
   
   \[
   V_{s_n}^{(t)} = V_{s_n+1}^{(t)} = \cdots = V_{s_n-U_n^{(t)}-1+\Delta_n}^{(t)} = 0, \\
   V_{s_n-U_n^{(t)}+\Delta_n}^{(t)} = V_{s_n-U_n^{(t)}+\Delta_n+1}^{(t)} = \cdots = V_{s_n+\Delta_n-1}^{(t)} = 1. \\
   
   \]

Note that the relation \( U_{n}^{(t)} = \sum_{j=s_n}^{s_n+\Delta_n-1} V_{n}^{(t)} \) holds.

We refer the \( j \)th number \( V_j^{(t)} \) in the binary sequence as the \( j \)th segment. By using this map, the \( n \)th box is expanded to the block composed from the \( s_n \)th to \( (s_n + \Delta_n - 1) \)th segments. We call this map expansion map from a state of the BBS with the variable box capacity \( \Delta_n \) to a binary sequence.
Figures 5 and 6 show examples of the expansion map. As shown in figure 6, the expansion map enables us to define the size of the $Q_n(t)$ and the one of the $E_n(t)$ for the BBS with the box capacity $\Delta_n$ at any time $t$ in the same way as for the BBS with the box capacity 1. Let $D_n^{(t+1)}$ denote the number of balls that the carrier holds after getting $Q_n(t)$ balls and $\Lambda_n^{(t)}$ denote the capacity of the box that contains the beginning (leftmost) segment of the $n$th empty block. Then, we arrive at the following theorem.

**Theorem 2.** Let the variables $Q_n^{(t)}$, $E_n^{(t)}$ and $D_n^{(t+1)}$ denote the quantities of the BBS as explained in the previous section. Then, the time evolution of the BBS with the variable box capacity $\Delta_n$ is given by

$$Q_n^{(t+1)} = \min \left( E_n^{(t)} - \max(0, \Lambda_n^{(t)} - D_n^{(t+1)}), D_n^{(t+1)} \right),$$  (20a)

$$E_n^{(t+1)} = E_n^{(t)} - Q_n^{(t+1)} + Q_n^{(t)} - \max(0, \Lambda_n^{(t)} - D_n^{(t+1)}) + \max(0, \Lambda_n^{(t)} - D_n^{(t+1)}),$$  (20b)

$$D_n^{(t+1)} = D_n^{(t+1)} - Q_n^{(t+1)} + Q_n^{(t)},$$  (20c)

$$E_0^{(t)} = E_0^{(t)} = +\infty, \quad D_0^{(t+1)} = Q_0^{(t)}.$$  (20d)

We note that, from (20a), $Q_n^{(t+1)} \leq D_n^{(t+1)}$ holds for all $n$ and $t$. Since the size of the $n$th soliton $Q_n^{(t)}$ should be positive for all $n$ and $t$, from (20c), the inequality $D_n^{(t+1)} \geq 1$ holds for all $n$ and $t$. Thus, all the terms max $0, \Lambda_n^{(t)} - D_n^{(t+1)}$ are equal to zero when $\Lambda_n^{(t)} = 1$ for all $n$ and $t$, the case of the original BBS. In this case, equations (20) reduce to the finite Toda...
representation of the original BBS (15). Hence, we can say that the ultradiscrete system (20) is a generalization of the finite u-Toda lattice (15).

**Proof.** Let us consider the general \( \Lambda^{(j)} \geq 1 \) case. As we mentioned earlier, (20) has the additional terms max \( 0, \Lambda^{(j)}_{n+1} - D^{(j+1)}_n \), which do not appear in the case of box capacity 1 (15). Hence, we shall investigate the role of the terms max \( 0, \Lambda^{(j)}_{n+1} - D^{(j+1)}_n \).

Let us consider the time evolution of the BBS with the box capacity \( \Delta_n \) from time \( t \) to \( t+1 \). Assume that \( Q^{(j+1)}_n, j = 0, 1, \ldots, n-1 \), and \( E^{(j+1)}_n, j = 1, 2, \ldots, n-1 \), are given (see figure 7). Let \( m \) be the index of the box that contains the leftmost segment of the \( n+1 \)th empty block at time \( t \). Then, the capacity of the \( m \)th box \( \Delta_m \) is equal to \( \Lambda^{(j)}_{n+1} \) by definition. Moreover, the relation

\[
D^{(j+1)}_n = \sum_{j=0}^{n-1} Q^{(j+1)}_j - \sum_{j=0}^{n-1} Q^{(j+1)}_j + \sum_{j=-\infty}^{m-1} (U^{(j)}_j - U^{(j+1)}_j) + U^{(j+1)}_m,
\]

where \( U^{(j)}_j \) denotes the number of balls in the \( j \)th box at time \( t \), also holds by definition. Hence, we can calculate the quantity \( U^{(j+1)}_m \) by the nu-KdV equation (12):

\[
U^{(j+1)}_m = \min \left( \Delta_m - U^{(j)}_m, \sum_{j=-\infty}^{m-1} (U^{(j)}_j - U^{(j+1)}_j) \right)
= \min \left( \Lambda^{(j)}_n - U^{(j)}_m, D^{(j+1)}_n - U^{(j)}_m \right).
\]

Then, we obtain the relation

\[
\Delta_m - U^{(j)}_m - U^{(j+1)}_m = \Lambda^{(j)}_n - U^{(j)}_m - \min \left( \Lambda^{(j)}_n - U^{(j)}_m, D^{(j+1)}_n - U^{(j)}_m \right)
= - \min \left( 0, D^{(j+1)}_n - \Lambda^{(j)}_n \right)
= \max \left( 0, \Lambda^{(j)}_n + D^{(j+1)}_n \right),
\]

where we have used the identity \(- \min(-A, -B) = \max(A, B)\). This relation implies that the term max \( 0, \Lambda^{(j)}_n + D^{(j+1)}_n \) denotes the size of interspace inserted between the \( n \)th soliton at time \( t \) and the \( n+1 \)th at time \( t + 1 \).

Once we note the role of the terms max \( 0, \Lambda^{(j)}_n + D^{(j+1)}_n \), we can clarify the meaning of equations (20a) and (20b). Since the term \( E^{(j+1)}_n = \max \left( 0, \Lambda^{(j)}_n + D^{(j+1)}_n \right) \) in (20a) denotes the difference between the size of the inserted space and the one of the \( n+1 \)th empty block,
\(Q_n^{(t)}\) should be determined by (20a). Similarly, \(E_n^{(t)}\) should be determined by (20b). It is also true for \(n = 0\) and 1; then, the proof is completed by induction.

Next, we construct the finite Toda representation of the BBS with both box capacity and carrier capacity from two time evolution maps: the size limit map and the recovery map. This is the same as for the construction of the Euler representation explained in section 2. Figure 8 shows an example.

In the next theorem, we use the following notations.
- \(Q_n^{(t)}\), \(E_n^{(t)}\): the size of the \(n\)th soliton and the one of the \(n\)th empty block at time \(t\), respectively.
- \(\overline{Q}_n^{(t)}\), \(\overline{E}_n^{(t)}\): the size of the \(n\)th soliton and the one of the \(n\)th empty block after the size limit process from time \(t\) to \(t + 1\).
- \(\overline{C}_n^{(t+1)}\), \(\overline{D}_n^{(t+1)}\): some quantities that will be explained in the proof of the next theorem in detail.
- \(K_n^{(t)}\), \(\Lambda_n^{(t)}\): the capacity of the box which contains the leftmost segment of the \(n\)th soliton and the one of the \(n\)th empty block at time \(t\), respectively.

**Theorem 3.** Let the variables \(Q_n^{(t)}\), \(E_n^{(t)}\), \(\overline{Q}_n^{(t)}\), \(\overline{E}_n^{(t)}\), \(\overline{C}_n^{(t+1)}\) and \(\overline{D}_n^{(t+1)}\) denote the quantities of the BBS as explained earlier. Then, the size evolution of the BBS with the box capacities \(K_n^{(t)}\) and \(\Lambda_n^{(t)}\), and the carrier capacity \(M_{t+1}\) is given by

\[
\overline{Q}_n^{(t+1)} = \min \left(\overline{Q}_n^{(t)} - \max \left(0, \Lambda_n^{(t)} - \overline{D}_n^{(t+1)}\right), \overline{E}_n^{(t+1)}\right), \quad (21a)
\]

\[
\overline{E}_n^{(t+1)} = \overline{E}_n^{(t)} - \overline{Q}_n^{(t)} + Q_n^{(t)} - \max \left(0, \Lambda_n^{(t)} - \overline{D}_n^{(t+1)}\right) + \max \left(0, \Lambda_n^{(t)} - \overline{D}_n^{(t+1)}\right), \quad (21b)
\]

\[
\overline{C}_n^{(t+1)} = \min \left(\overline{D}_n^{(t+1)} - \overline{Q}_n^{(t+1)} + K_n^{(t)}, M_{t+1}\right), \quad (21c)
\]

\[
\overline{D}_n^{(t+1)} = \min \left(\overline{C}_n^{(t+1)} + Q_n^{(t)} - K_n^{(t)}, M_{t+1}\right), \quad (21d)
\]

\[
Q_n^{(t+1)} = Q_n^{(t)} + \overline{C}_n^{(t+1)} - \overline{C}_n^{(t+1)} - K_n^{(t)} + K_n^{(t+1)}, \quad (21e)
\]

\[
E_n^{(t+1)} = \overline{E}_n^{(t)} + \overline{Q}_n^{(t+1)} - Q_n^{(t)} - \overline{D}_n^{(t+1)} + \overline{D}_n^{(t+1)}, \quad (21f)
\]

\[
E_0^{(t)} = E_0^{(t)} = \overline{E}_N^{(t)} = +\infty, \quad \overline{C}_0^{(t+1)} = K_0^{(t)}, \quad (21g)
\]

where the carrier capacity \(M_{t+1}\) must be chosen to satisfy the condition \(K_n^{(t)} \leq M_{t+1}\) for all \(n\) and \(t\).
When the quantities \( \{Q_n^{(t)}\}_{n=0}^{N-1} \) and \( \{E_n^{(t)}\}_{n=0}^{N-1} \) are given, we can calculate \( D_n^{(t+1)} \) using (21d) and (21g). Next, we can calculate \( D_n^{(t+1)} \) by (21a), \( C_n^{(t+1)} \) by (21c) and \( D_n^{(t+1)} \) by (21d). In a repetitive manner, we can obtain the quantities \( \{D_n^{(t+1)}\}_{n=0}^{N-1}, \{C_n^{(t+1)}\}_{n=0}^{N-1} \) and \( \{E_n^{(t+1)}\}_{n=0}^{N-1} \). Finally, we can calculate the quantities \( \{C_n^{(t+1)}\}_{n=0}^{N-1}, \{Q_n^{(t+1)}\}_{n=0}^{N-1} \) and \( \{E_n^{(t+1)}\}_{n=0}^{N-1} \) by (21b), (21e) and (21f), respectively. Hence, the time evolution is determined by (21).

If \( M_{t+1} = +\infty \), then (21c) and (21d) reduce to \( C_n^{(t+1)} = D_n^{(t+1)} = D_n^{(t+1)} + K_n^{(t)} \) and \( D_n^{(t+1)} = C_n^{(t)} + Q_n^{(t)} - K_n^{(t)} \), respectively. Thus, we have the equation \( D_n^{(t+1)} = D_n^{(t+1)} + Q_n^{(t)} \), and substituting them into (21e) and (21f), we obtain \( Q_n^{(t+1)} = C_n^{(t)} \) and \( E_n^{(t+1)} = E_n^{(t+1)} \). Hence, in this case, the ultradiscrete system (21) reduces to the system (20). We can therefore say that the system (21) is a generalization of the system (20).

**Proof.** Let us show that equations (21a)–(21d) describe the size limit process and (21e)–(21f) describe the recovery process.

First, we consider the size limit process. Equations (21a) and (21b) have the same forms as of (20a) and (20b). Thus, we shall investigate the variables \( C_n^{(t+1)} \) and \( D_n^{(t+1)} \), which are defined by (21c) and (21d). Suppose that the carrier capacity is chosen as \( K_n^{(t)} \leq M_{t+1} < \infty \), \( Q_n^{(t)} = 0, 1, \ldots, N-1 \), and \( E_n^{(t)} = 1, 2, \ldots, \), are given, and the quantity \( D_n^{(t+1)} \) denotes the number of balls that the carrier holds after getting \( Q_n^{(t)} \) balls from boxes and restricting the number of balls in the carrier to \( M_{t+1} \) balls. Since the inequality \( D_n^{(t+1)} \leq D_n^{(t+1)} \) holds from (21a), it is sufficient to consider the following two cases: the case in which the carrier drops off all balls temporarily \( D_n^{(t+1)} - D_n^{(t+1)} = 0 \) and the case in which the carrier has balls just before getting \( Q_n^{(t)} \) balls \( D_n^{(t+1)} - D_n^{(t+1)} > 0 \).

(i) If \( D_n^{(t+1)} - D_n^{(t+1)} = 0 \), then \( C_n^{(t+1)} = \min(D_n^{(t+1)} - Q_n^{(t+1)} + K_n^{(t)}, M_{t+1}) = K_n^{(t)} \) holds from the assumption. We should note that, in this case, the number of balls that the carrier holds is zero temporarily before getting \( Q_n^{(t)} \) balls. Thus, from (21d), we have \( D_n^{(t+1)} = \min(Q_n^{(t)}, M_{t+1}) \), which indicates that the quantity \( D_n^{(t+1)} \) is again the number of balls that the carrier holds after getting \( Q_n^{(t)} \) balls and restricting the number of balls in the carrier to \( M_{t+1} \) balls.

(ii) The case of \( D_n^{(t+1)} - D_n^{(t+1)} > 0 \). Let \( m \) be the index of the box that contains the leftmost segment of the \( n \)th soliton at time \( t \). Under the assumption, in the terms of the variables of the Euler representation (8), \( D_n^{(t+1)} - D_n^{(t+1)} > 0 \) implies that \( C_n^{(t+1)} = \min(D_n^{(t+1)} - Q_n^{(t+1)} + K_n^{(t)}, M_{t+1}) = K_n^{(t)} \) should hold (see figure 9). Now, \( D_n^{(t+1)} - Q_n^{(t+1)} + K_n^{(t)} \) and \( D_n^{(t+1)} = \min(C_n^{(t+1)} + (Q_n^{(t)} - U_n^{(t)}) - U_n^{(t)}, M_{t+1}) \), respectively. Therefore, the quantity \( D_n^{(t+1)} \) is again the number of balls that the carrier holds after getting \( Q_n^{(t)} \) balls and restricting the number of balls in the carrier to \( M_{t+1} \) balls. We can summarize the change of the number of balls that the carrier holds as table 1.

Thus, together with the proof of theorem 3, it is proved that (21a)–(21d) describe the size limit process by induction.

Furthermore, since the number of balls removed by the size limit process is given by \( (D_n^{(t+1)} - D_n^{(t+1)} + K_n^{(t)}) - C_n^{(t)} \) and \( (E_n^{(t+1)} + Q_n^{(t)} - K_n^{(t)}) - D_n^{(t+1)} \), we obtain the equations...
of the recovery process

\[
Q_n^{(t+1)} = Q_n^{(t)} + K_n^{(t)} - D_n^{(t)} + \bar{Q}_{n+1}^{(t+1)} - \bar{K}_{n+1}^{(t+1)}
\]

\[
L_n^{(t+1)} = \bar{L}_n^{(t+1)} - Q_n^{(t)} - K_n^{(t)} + D_n^{(t+1)} + \bar{L}_{n-1}^{(t+1)} - \bar{K}_{n-1}^{(t+1)}
\]

which lead to equations (21e) and (21f), and the proof is complete. □

**Remark 4.** Furthermore, the variables \(X_0^{(t)}\) and \(\bar{X}_0^{(t)}\), which denote the indices of the leftmost segment of the 0th soliton at time \(t\), satisfy the equations

\[
X_0^{(t+1)} = X_0^{(t)} + Q_0^{(t)} + \text{max} \left( 0, \lambda_1^{(t)} - D_0^{(t+1)} \right)
\]

\[
\bar{X}_0^{(t+1)} = \bar{X}_0^{(t)} - Q_0^{(t)} + D_0^{(t+1)} = X_0^{(t)} + \text{max} \left( D_0^{(t+1)}, \lambda_1^{(t)} \right)
\]
5. Particular solution for the fixed box capacity case

In this section, we discuss a particular solution to the ultradiscrete system (21) with a special condition: all boxes have the constant capacity \( \Delta \).

Let us consider the bilinear equations

\[
\begin{align*}
\tau_{n+1}^{0,0} \tau_n^{0,1} &= \tau_{n+1}^{0,0} \tau_n^{0,1} + \tau_{n+1}^{0,1} \tau_n^{0,1}, \quad (23a) \\
\delta \tau_{n+1}^{0,0} \tau_n &= (\delta - \mu_1) \tau_{n+1}^{0,0} \tau_n + \mu_1 \tau_{n+1}^{0,1} \tau_n^{0,0}, \quad (23b) \\
\tau_{n+1}^{0,0} \tau_n^{0,1} &= \tau_{n+1}^{0,0} \tau_n^{0,1} + \mu_1 \tau_{n+1}^{0,1} \tau_n^{0,0}, \quad (23c) \\
(\delta - \mu_1) \tau_{n+1}^{0,0} \tau_n^{0,1} + \tau_{n+1}^{0,0} \tau_n^{0,1} &= \tau_{n+1}^{0,0} \tau_n^{0,1} + \tau_{n+1}^{0,1} \tau_n^{0,1}, \quad (23d)
\end{align*}
\]

where \( \delta \) is a constant and \( \mu_1 \) is a parameter depending on \( t \). We introduce the dependent variables

\[
\begin{align*}
q_n^{(t)} &= \frac{\tau_n^{0,0} \tau_n^{0,1}}{\tau_{n+1}^{0,1} \tau_n^{0,1}}, \\
e_n^{(t)} &= \delta \frac{\tau_n^{0,0} \tau_n^{0,1}}{\tau_{n+1}^{0,1} \tau_n^{0,1}}, \\
\varepsilon_n^{(t)} &= \delta (\delta - \mu_1) \frac{\tau_n^{0,0} \tau_n^{0,1}}{\tau_{n+1}^{0,1} \tau_n^{0,1}}, \\
\nu_n^{(t)} &= \delta \frac{\tau_n^{0,0} \tau_n^{0,1}}{\tau_{n+1}^{0,1} \tau_n^{0,1}}.
\end{align*}
\]

Then, (23d) yields the relation

\[
1 + \delta^{-1} d_n^{(t)} = (\delta - \mu_1) \frac{\tau_n^{0,0} \tau_n^{0,1}}{\tau_{n+1}^{0,1} \tau_n^{0,1}}.
\]

Furthermore, (23a)–(23c) yield the equations

\[
\begin{align*}
q_n^{(t+1)} &= e_n^{(t)} (1 + \delta^{-1} q_n^{(t+1)}) + \nu_n^{(t+1)}, \quad (24a) \\
\nu_n^{(t+1)} &= (\delta - \mu_{t+1}) \frac{\nu_n^{(t+1)}}{q_n^{(t+1)}} + \mu_{t+1}, \quad (24b) \\
\nu_n^{(t+1)} &= \frac{\nu_n^{(t+1)}}{q_n^{(t+1)}} (1 + \delta^{-1} \nu_n^{(t+1)}) + \mu_{t+1} + \frac{\mu_{t+1}}{1 - \delta^{-1} \mu_{t+1}}, \quad (24c)
\end{align*}
\]

and the identities

\[
\begin{align*}
\varepsilon_n^{(t+1)} &= e_n^{(t)} q_n^{(t)} (1 + \delta^{-1} \varepsilon_n^{(t+1)}) + \varepsilon_n^{(t+1)}, \quad (24d) \\
q_n^{(t+1)} &= q_n^{(t)} \frac{\varepsilon_n^{(t+1)}}{\varepsilon_n^{(t+1)}} (1 + \delta^{-1} q_n^{(t+1)}), \quad (24e) \\
e_n^{(t+1)} &= e_n^{(t)} \frac{\varepsilon_n^{(t+1)}}{\varepsilon_n^{(t+1)}} (1 + \delta^{-1} e_n^{(t+1)}), \quad (24f)
\end{align*}
\]

hold. In addition, we impose the finite lattice condition

\[
e_0^{(t)} = e_0^{(t)} = \varepsilon_0^{(t)} = \varepsilon_0^{(t)} = 0.
\]
In the bilinear equations (23), this condition implies
\[ \tau_{k,t}^{n} = \tau_{N+1}^{k,t} = \tau_{N+1}^{k,t} = \tau_{N+1}^{k,t} = 0. \]

We assume that the constant \( \delta \) and parameter \( \mu_t \) satisfy the condition \( 0 < \mu_t < \delta \) for all \( t \in \mathbb{Z} \). Then, putting \( q_n^{(t)} = e^{-\sum_{i=1}^{t} i^t \tau_n^{(t)}}, \quad e_n^{(t)} = e^{-\sum_{i=1}^{t} i^t \tau_n^{(t)}}, \quad d_n^{(t)} = e^{-\sum_{i=1}^{t} i^t \tau_n^{(t)}}, \quad \delta_n = e^{-\sum_{i=1}^{t} i^t \tau_n^{(t)}}, \]
\( \sigma_n^{(t)} = e^{-\sum_{i=1}^{t} i^t \tau_n^{(t)}}, \quad \delta_n = e^{-\sum_{i=1}^{t} i^t \tau_n^{(t)}} \), we obtain the ultradiscrete system (21) with the condition \( K_n^{(t)} = \Lambda_n^{(t)} = \Delta \leq M_{i+1} \) for all \( n,t \in \mathbb{Z} \).

The following theorem is proved by using a determinant identity called the Plücker relation.

**Theorem 4.** A particular solution to the bilinear equations (23) with the semi-infinite lattice condition \( \tau_{k,t}^{n} = \tau_{N+1}^{n} = 0 \) for all \( k,t \in \mathbb{Z} \) is given by the Hankel determinants
\[ \tau_{n}^{k,t} = \begin{cases} 0 & \text{if } n < 0, \\ 1 & \text{if } n = 0, \\ \left| \xi_{k+i+j}^{(t)} \right|_{0 \leq i,j \leq n-1} & \text{if } n > 0, \end{cases} \tag{25a} \]
\[ \tau_{n}^{k,t} = \begin{cases} 0 & \text{if } n < 0, \\ 1 & \text{if } n = 0, \\ \left| \xi_{k+i+j}^{(t)} \right|_{0 \leq i,j \leq n-1} & \text{if } n > 0, \end{cases} \tag{25b} \]
where \( \xi_n^{(t)} \) and \( \bar{\xi}_n^{(t)} \) are the arbitrary functions satisfying the dispersion relation
\[ \bar{\xi}_{n}^{(t+1)} = -\delta \bar{\xi}_{n+1}^{(t)} + \xi_{n}^{(t)} = (\mu_t - \delta)\bar{\xi}_{n+1}^{(t)} + \bar{\xi}_{n}^{(t)}, \quad n = 0, 1, \ldots . \tag{26} \]

Hereafter, we choose the arbitrary functions as
\[ \xi_n^{(t)} = \sum_{i=0}^{N-1} \eta_{i}^{(t)} (p_i + \delta)^{t}, \quad \bar{\xi}_n^{(t)} = \sum_{i=0}^{N-1} \eta_{i}^{(t-1)} (p_i + \delta)^{t+i+1}, \quad \eta_i^{(t)} := \frac{w_t \prod_{j=0}^{t} (p_i + \mu_j)}{(p_i + \delta)^{t+i+1}}, \tag{27} \]
where \( p_i \) and \( w_i, i = 0, 1, \ldots , N-1 \), are some constants. Then, the dispersion relation (26) is satisfied, and the finite lattice condition \( \tau_{k,t}^{n} = \tau_{N+1}^{n} = \tau_{N+1}^{n} = 0 \) holds for all \( k,t \in \mathbb{Z} \).

Substituting (27) into (25) and expanding the Hankel determinants using the Cauchy–Binet formula, we obtain
\[ \tau_{n}^{k,t} = \sum_{0 \leq r_s < r_l \leq 1} \left( \prod_{0 \leq i,j \leq n-1} \left( \frac{p_i - p_l}{(p_l + \delta)(p_r + \delta)} \right)^2 \prod_{i=0}^{n-1} w_t \prod_{j=0}^{t} (p_i + \mu_j) \right), \]
\[ \tau_{n}^{k,t} = \sum_{0 \leq r_s < r_l \leq 1} \left( \prod_{0 \leq i,j \leq n-1} \left( \frac{p_i - p_l}{(p_l + \delta)(p_r + \delta)} \right)^2 \prod_{i=0}^{n-1} w_t \prod_{j=0}^{t} (p_i + \mu_j) \right), \]
for \( n = 1, 2, \ldots , N \). These expressions can be ultradiscretized directly: putting \( p_n = e^{-\sum_{i=1}^{t} i^t \tau_n^{(t)}}, \quad w_n = e^{-\sum_{i=1}^{t} i^t \tau_n^{(t)}}, \quad \tau_{n}^{k,t} = e^{-\sum_{i=1}^{t} i^t \tau_n^{(t)}}, \quad \tau_{n}^{k,t} = e^{-\sum_{i=1}^{t} i^t \tau_n^{(t)}}, \) and taking a limit \( \epsilon \to +0 \), we obtain the next theorem.

**Theorem 5.** A particular solution to the ultradiscrete system (21) with the condition \( K_n^{(t)} = \Lambda_n^{(t)} = \Delta \leq M_{i+1}^{(t)} \) for all \( n,t \in \mathbb{Z} \) is given by
\[ Q_n^{(t)} = T_n^{(t+1)} + T_n^{(t+1)} - T_n^{(t+1)}, \quad \bar{Q}_n^{(t)} = T_n^{(t)} - T_n^{(t)} + T_n^{(t+1)}, \]
\[ E_n^{(t)} = T_n^{(t)} - T_n^{(t)} - T_n^{(t+1)} + 2\Delta, \quad \bar{E}_n^{(t)} = T_n^{(t)} - T_n^{(t)} + T_n^{(t+1)} - T_n^{(t+1)} + 2\Delta, \]
In fact, these variables have
\( q_{n,t} = T_{n+1}^{0,t} - T_{n+1}^{1,t-1} + T_{n}^{1,t} - T_{n}^{0,t-1} + \Delta, \quad T_{n+1}^{0,t} = T_{n}^{0,t} - T_{n}^{1,t-1} + T_{n+1}^{1,t} - T_{n+1}^{2,t-1}, \)

\[
T_{n}^{i,j} = \min_{0 \leq n, r, t, \leq \infty, n \leq N-1} \left( \sum_{i=0}^{n-1} \left( W_{r} + 2(n - 1 - i)P_{r} \right) \right), \quad n = 1, 2, \ldots, N, \]

\[
T_{n}^{i,j} = \min_{0 \leq n, r, t, \leq \infty, n \leq N-1} \left( \sum_{i=0}^{n-1} \left( W_{r} + 2(n - 1 - i)P_{r} \right) \right), \quad n = 1, 2, \ldots, N, \]

\[
T_{N+1}^{0,1} = T_{N+1}^{1,0} = T_{n+1}^{0,1} = +\infty, \quad T_{0}^{0,1} = T_{n+1}^{1,0} = 0, \]

where \( P_{r} \) and \( W_{r} \), \( i = 0, 1, \ldots, N - 1 \), are some constants satisfying \( P_{0} \leq P_{1} \leq \cdots \leq P_{N-1} \).

\textbf{Remark 5.} There exists a Bäcklund transformation from the discrete system (24) to the nonautonomous discrete Toda (nd-Toda) lattice:

\[
q_{n,t}^{(t)} = \frac{\delta^{-1}q_{n,t}^{(t)}}{(1 + \delta^{-1}q_{n,t}^{(t)}) (1 + \delta^{-1}e_{n,t}^{(t)})}, \quad e_{n,t}^{(t)} = \frac{\delta^{-1}e_{n,t}^{(t)}}{(1 + \delta^{-1}q_{n,t}^{(t)}) (1 + \delta^{-1}e_{n,t}^{(t)})},
\]

\[
q_{n,t}^{(t)} = \frac{\delta^{-1}q_{n,t}^{(t)}}{(\delta - \mu_{t})(1 + \delta^{-1}q_{n,t}^{(t)}) (1 + \delta^{-1}e_{n,t}^{(t)})}, \quad e_{n,t}^{(t)} = \frac{\delta^{-1}e_{n,t}^{(t)}}{(\delta - \mu_{t})(1 + \delta^{-1}q_{n,t}^{(t)}) (1 + \delta^{-1}e_{n,t}^{(t)})}.
\]

In fact, these variables have \( \tau \)-function expressions

\[
q_{n,t}^{(t)} = \frac{\delta^{-1}q_{n,t}^{(t)}}{(1 + \delta^{-1}q_{n,t}^{(t)}) (1 + \delta^{-1}e_{n,t}^{(t)})}, \quad e_{n,t}^{(t)} = \frac{\delta^{-1}e_{n,t}^{(t)}}{(1 + \delta^{-1}q_{n,t}^{(t)}) (1 + \delta^{-1}e_{n,t}^{(t)})},
\]

\[
q_{n,t}^{(t)} = \frac{\delta^{-1}q_{n,t}^{(t)}}{(\delta - \mu_{t})(1 + \delta^{-1}q_{n,t}^{(t)}) (1 + \delta^{-1}e_{n,t}^{(t)})}, \quad e_{n,t}^{(t)} = \frac{\delta^{-1}e_{n,t}^{(t)}}{(\delta - \mu_{t})(1 + \delta^{-1}q_{n,t}^{(t)}) (1 + \delta^{-1}e_{n,t}^{(t)})}.
\]

Since the bilinear equations

\[
T_{n+1}^{0,1} = \delta (\mu_{t}) T_{n+1}^{0,1} + T_{n}^{0,t}, \quad \delta T_{n+1}^{0,1} = (\delta - \mu_{t}) T_{n+1}^{0,1} + \mu_{t} T_{n+1}^{0,1},
\]

hold (these are proved using the Plücker relation), we have the equations

\[
q_{n,t}^{(t+1)} = e_{n+1,t}^{(t+1)} + \bar{q}_{n,t}^{(t+1)}, \quad \bar{q}_{n,t}^{(t+1)} = \bar{q}_{n+1,t}^{(t+1)} \frac{q_{n,t}^{(t)}}{q_{n+1,t}^{(t+1)}} + \sigma_{t+1},
\]

where

\[
\bar{q}_{n,t}^{(t)} := (\delta - \mu_{t})^{-1} \frac{T_{n+1}^{0,t}}{T_{n}^{0,t}}, \quad \sigma_{t} := \delta^{-1} \mu_{t}/(\delta - \mu_{t}).
\]

Additionally, we have the identities

\[
q_{n,t}^{(t+1)} = e_{n,t}^{(t)} \frac{q_{n,t}^{(t)}}{\bar{q}_{n,t}^{(t+1)}}, \quad q_{n+1,t}^{(t+1)} = \bar{q}_{n+1,t}^{(t+1)} \frac{q_{n,t}^{(t)}}{\bar{q}_{n,t}^{(t+1)}}, \quad e_{n,t}^{(t)} = \bar{q}_{n,t}^{(t)} \frac{q_{n,t}^{(t)}}{e_{n+1,t}^{(t+1)}} + \bar{q}_{n+1,t}^{(t+1)} \frac{q_{n,t}^{(t)}}{e_{n,t}^{(t+1)}}.
\]

Eliminating \( \bar{q}_{n,t}^{(t+1)} \) from these equations, we obtain the modified nd-Toda lattice \( [2] \)

\[
q_{n,t}^{(t+1)} + e_{n+1,t}^{(t+1)} = q_{n,t}^{(t)} + e_{n,t+1}^{(t)} + \sigma_{t+1}, \quad \bar{q}_{n,t}^{(t+1)} e_{n,t}^{(t)} = q_{n,t}^{(t)} e_{n,t}^{(t)}.
\]
and the finite lattice condition is given by
\[ e^{(t)}_{0} = e^{(t)}_{N} = e^{(t)}_{0} = e^{(t)}_{N} = 0. \]

6. Concluding remarks

In this paper, we have derived the finite Toda representation of the BBS with box capacity by introducing the expansion map from a state of the BBS to a binary sequence. Furthermore, we have given a particular solution for the fixed box capacity case. Hence, we can say that the ultradiscrete system (21) is integrable if the parameters \( K^{(t)}_{n} \) and \( \Lambda^{(t)}_{n} \) are chosen as constants. Since there is a connection between the ultradiscrete system (21) and the BBS with variable box capacity, which is integrable, we expect that the ultradiscrete system (21) of the variable box capacity case is also integrable and a discrete system derived through the inverse ultradiscretization has determinant solutions. This problem is left for future research.

In the proof of theorem 3, the variables \( C^{(t+1)}_{n} \) and \( D^{(t+1)}_{n} \) have played important roles; these variables denote the number of balls that the carrier holds. Moreover, these variables correspond to the variables that are introduced to remove subtractions in the discrete equations (see remark 2). This result gives us a guideline for the ultradiscretization of Toda-type integrable systems and making connections between these systems and BBSs.

In 2000, Spiridonov and Zhedanov [7] proposed a Toda-type nonautonomous integrable system called the \( R_{II} \) chain, which is derived as the compatibility conditions of spectral transformations for some biorthogonal rational functions. By using the techniques developed in this paper, we will be able to ultradiscretize the \( R_{II} \) chain and consider a corresponding BBS.

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