BIG SLICES VERSUS BIG RELATIVELY WEAKLY OPEN SUBSETS IN BANACH SPACES

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ABSTRACT. We study the unknown differences between the size of slices and relatively weakly open subsets of the unit ball in Banach spaces. We show that every Banach space containing $c_0$ isomorphically satisfies that every slice of its unit ball has diameter 2 so that its unit ball contains nonempty relatively weakly open subsets with diameter arbitrarily small, which answer an open question and stresses the differences between the size of slices and relatively weakly open subsets of the unit ball of Banach spaces.

1. INTRODUCTION

The well known Radon-Nikodym (RNP) property in Banach spaces is characterized by the existence of slices with diameter arbitrarily small in every closed and bounded subset of the space. Similarly, a Banach space $X$ has the point of continuity property (PCP) if every nonempty closed and bounded subset of $X$ has relatively weakly open subsets with diameter arbitrarily small. We refer to [5], [9] and [10] for background about RNP and PCP. It is clear then that RNP implies PCP, however there are Banach spaces satisfying PCP and failing RNP [6]. In the last years, one can find what we can call the big slice phenomena, that is, examples of Banach spaces where every slice or every nonempty relatively weakly open subset of its unit ball has diameter 2, a property extremely opposite to RNP or PCP. These examples include infinite-dimensional uniform algebras [13], infinite-dimensional $C^*$-algebras [4], infinite-dimensional M-embedded spaces [12], Banach spaces with the Daugavet property [15], etc. Also, it is known [7, Lemma I.1.3] that all these spaces have extremely rough dual norm, which is a property extremely opposite to the Fréchet differentiability. The big slice phenomena probably started in the paper of O. Nygaard and D. Werner [13], but after discover many examples with this phenomena and the connections with other well known geometrical properties, like Daugavet property or

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extreme roughness, the phenomena has now its own life and new geometrical properties in Banach spaces have appeared, the slice diameter 2 property and the diameter 2 property. Recall now the precise definitions.

Given a Banach space $X$, we say that $X$ has the slice diameter 2 property (SD2P) if every slice of the unit ball of $X$ has diameter 2. Similarly, we say that $X$ has the diameter 2 property (D2P) if every nonempty relatively weakly open subset of the unit ball of $X$ has diameter 2.

With the above definitions it is clear that SD2P implies non-RNP and D2P implies non-PCP. As RNP and PCP are isomorphic properties, that is, they are independent of the equivalent norm considered in the space, one can see SD2P and D2P like the extremely opposite geometrical properties to RNP or PCP, since SD2P and D2P are not independent of the norm considered in the space.

From the definitions, one deduce that D2P implies SD2P. In fact all known Banach spaces with the SD2P up to now also satisfying D2P. It is then an open problem whether these two properties are in fact different. As RNP and PCP are different properties, it is natural thinking that SD2P and D2P are also different properties. However, the well known example of Banach space with PCP and failing RNP is $B$, the natural predual of James tree space $JT$, constructed in [11], and it is proved in the paper by W. Schachermayer, A. Sersouri and E. Werner [14] that $B$ fails the SD2P. So the natural candidate to example of Banach space with SD2P and failing D2P doesn’t work.

The aim of this note is to prove the existence of a Banach space satisfying SD2P and failing D2P, which answers by the negative an open problem stated firstly in [1]. In fact, much more can be shown. We prove in Theorem 2.4 that every Banach space containing isomorphically $c_0$, the classical Banach space of null sequences with the sup norm, can be equivalently renormed satisfying SD2P and so that its unit ball contains nonempty relatively weakly open subsets with diameter arbitrarily small. As a consequence, every Banach space containing isomorphic copies of $c_0$ can be equivalently renormed satisfying SD2P and failing D2P. For this, we first construct in Proposition 2.1 and Theorem 2.2 a closed, bounded, convex and symmetric subset of $c$, the Banach space of convergent sequences with the sup norm, so that every slice of $K$ has diameter 2 and its unit ball contains nonempty relatively weakly open subsets with diameter arbitrarily small. Finally, we get in Corollary 2.6 that the $\ell_p$-sum of Banach spaces satisfying SD2P and failing D2P also satisfies SD2P and fails D2P.

We pass now to introduce some notation. For a Banach space $X$, $X^*$ denotes the topological dual of $X$, $B_X$ and $S_X$ stand for the closed unit ball and unit sphere of $X$, respectively, and $w$ denotes the weak topology in $X$. We consider only real Banach spaces. A slice of a set $C$ in $X$ is a set of $X$ given by

$$S = \{ x \in C : x^*(x) > \sup x^*(C) - \alpha \}$$

where $x^* \in X^*$ and $\alpha < \sup x^*(C)$. 

Recall that a slice of $B_X$ is a nonempty relatively weakly open subset of $B_X$ and the family
\[
\{ \{ x \in B_X : |x^*_i(x - x_0)| < \varepsilon, 1 \leq i \leq n \} : n \in \mathbb{N}, x_1^*, \ldots, x_n^* \in X^* \}
\]
is a basis of relatively weakly open neighborhoods of $x_0 \in B_X$. So every relatively weakly open subset of $B_X$ has nonempty intersection with $S_X$, whenever $X$ has infinite dimension.

$\mathbb{N}^{<\omega}$ stands for the set of all ordered finite sequences of positive integers and denotes by $0$ the empty sequence. If $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^{<\omega}$, we define the length of $\alpha$ by $|\alpha| = n$ and $|0| = 0$. Also we use the natural order in $\mathbb{N}^{<\omega}$ given by:
\[
\alpha \leq \beta \text{ if } |\alpha| \leq |\beta| \text{ and } \alpha_i = \beta_i \forall i \in \{1, \ldots, |\alpha|\}.
\]

Also we do $0 \leq \alpha \forall \alpha \in \mathbb{N}^{<\omega}$.

As $\mathbb{N}^{<\omega}$ is a countable set we can construct a bijective map $\phi : \mathbb{N}^{<\omega} \to \mathbb{N}$ so that $\phi(0) = 1$ and $\phi(\alpha) \leq \phi(\beta)$ whenever $\alpha \leq \beta \in \mathbb{N}^{<\omega}$ and $\phi(\alpha, j) \leq \phi(\alpha, k)$ for every $\alpha \in \mathbb{N}^{<\omega}$ and $j \leq k \in \mathbb{N}$. Indeed, consider $\{p_n\}$ an enumeration of prime positive integers numbers and define the bijective map $\phi_0 : \mathbb{N}^{<\omega} \to \mathbb{N}$ given by $\phi_0(\alpha_1, \ldots, \alpha_k) = p_{\alpha_1} \cdot \cdots \cdot p_{\alpha_k}$. Now take an strictly increasing map $\phi_1 : \phi_0(\mathbb{N}^{<\omega}) \to \mathbb{N}$ and put $\phi = \phi_0 \circ \phi_1$. Then $\phi$ satisfies the desired properties. Observe that, from the above construction, $\{\phi(\alpha, j)\}^j$ is a strictly increasing sequence for every $\alpha \in \mathbb{N}^{<\omega}$.

\section*{2. Main result}

We begin constructing a subset $A$ of $c$, the space of convergent scalar sequences with the sup norm. For this, $\{e_\alpha\}$ and $\{e^*_\alpha\}$ stand for the usual basis and the sequence of biorthogonal functionals of $c_0$, the space of null scalars sequences with the sup norm. Define for every $\alpha \in \mathbb{N}^{<\omega}$, $e_\alpha = (e_\phi(\alpha) \in c, e^*_\alpha =: e^*_\phi(\alpha) \in c^* \text{ and } x_\alpha \in c \text{ by } x_\alpha(i) = 1 \text{ if } \phi^{-1}(i) \leq \alpha \text{ and } x_\alpha(i) = -1 \text{ in otherwise. \ It is clear that } x_\alpha \in S_c \text{ for every } \alpha \in \mathbb{N}^{<\omega}. \text{ Note that if } \alpha, \beta \in \mathbb{N}^{<\omega} \text{ are incomparable then } \|x_\alpha - x_\beta\|_c = 2.\n
Define $A = \{x_\alpha : \alpha \in \mathbb{N}^{<\omega}\}$, which is a subset of the unit sphere of $c$ and $K = \overline{co}(A \cup -A)$ which is a closed, convex and symmetric subset of $B_c$ with diameter 2. Throughout this note, the aforementioned elements $e_\alpha, e^*_\alpha, x_\alpha$ and the sets $A$ and $K$ will be used without previous notice.

The first step is to prove that every slice of $K$ has diameter 2.

\begin{proposition}
\textbf{Proposition 2.1.} Every slice of $K$ has diameter 2, as a subset of $c$.
\end{proposition}

\begin{proof}
Pick $x^* \in S_{c^*}$, $\lambda < \sup x^*(K)$ and put $S = \{ x \in K : x^*(x) > \sup x^*(K) - \lambda \}$. As $S$ is a slice of $K$ and $K = \overline{co}(A \cup -A)$, we deduce that $S$ meets $A$ or $S$ meets $-A$. From the symmetry of $K$ we can assume that $S \cap A \neq \emptyset$. Then there is $\alpha \in \mathbb{N}^{<\omega}$ such that $x_\alpha \in S$. Pick $j \in \mathbb{N}$. Then $x(j, \alpha)$ is an element in $A$ given by $x(j, \alpha)(i) = 1$ if $\phi^{-1}(i) \leq (\alpha, j)$ and $x(j, \alpha)(i) = -1$
in otherwise. Hence \( \{ x_{(\alpha,j)} \}_j \) is a sequence in \( A \subset K \) weakly convergent to \( x_\alpha \). So there is \( j \in \mathbb{N} \) such that \( x_{(\alpha,j)} \in S \) and 
\[
diam(S) \geq \| x_{(\alpha,j)} - x_\alpha \| \geq \| x_{(\alpha,j)}(\phi((\alpha,j))) - x_\alpha(\phi((\alpha,j))) \| = |1 - (-1)| = 2.
\]
Recalling that \( K \) has diameter 2, we deduce that \( S \) has diameter 2, being \( S \) any slice of \( K \). ■

Now, we prove that \( K \), as a subset of \( c \) has relatively weakly open subsets with diameter arbitrarily small.

**Proposition 2.2.** Given \( n \in \mathbb{N} \) and \( \rho > 0 \) with \( \rho < \frac{1}{n(32n - 21)} \), one has that 
\[
diam(W_n) < \frac{9}{n},
\]
where \( W_n \) is the relative weak open subset of \( K \subset c \) given by 
\[
W_n = \{ x \in K : e^*_{(0,i)}(x) > \frac{1}{n} - 1 - \rho, \ 1 \leq i \leq n, \ \lim_n x(n) < -1 + \rho \}.
\]

**Proof.** First of all, check that 
\[
x_0 = \sum_{i=1}^n \frac{x_{(0,i)}}{n} \in W_n.
\]
For this note that \( \lim_n x_{(0,i)}(n) = -1 \) and then \( \lim_n x_0(n) = -1 < -1 + \rho \). Furthermore \( x_0 \) is a convex combination of elements of \( A \subset K \) and so \( x_0 \in K \). Finally, for \( 1 \leq j \leq n \), one has 
\[
e^*_{(0,j)}(x_0) = \sum_{i=1}^n \frac{1}{n} e^*_{(0,i)}(x_{(0,i)}) + \frac{1}{n} e^*_{(0,j)}(x_{(0,j)}) = - \frac{n - 1}{n} + \frac{1}{n} > \frac{1}{n} - 1 - \rho.
\]

Then \( W_n \) is nonempty. In order to prove that \( diam(W_n) < \frac{9}{n} \), it is enough to see that \( diam(W_n \cap co(A \cup -A)) < \frac{9}{n} \). For this, pick arbitraries \( x, x' \in co(A \cup -A) \), hence there are \( \lambda, \lambda' \in (0,1] \), \( a, a' \), \( -b, -b' \in co(A) \) such that \( x = \lambda a + (1 - \lambda)b \) and \( x' = \lambda' a' + (1 - \lambda')b' \). Now \( \lim_n a(n) = \lim_n a'(n) = -1 \) and \( \lim_n b(n) = \lim_n b'(n) = 1 \). As \( x \in W_n \), we have that \( \lim_n \lambda a(n) + (1 - \lambda)b(n) < -1 + \rho \) and then we get that 
\[
2(1 - \lambda) < \rho.
\]
Similarly we get that 
\[
2(1 - \lambda') < \rho,
\]
and so 
\[
|\lambda - \lambda'| < \rho/2.
\]
For \( i \in \{1, \ldots, n\} \) one has, taking into account 2.1 and 2.2 and the fact \( x = \lambda a + (1 - \lambda)b \in W_n \) that 
\[
e^*_{(0,i)}(a) > \frac{\frac{1}{n} - 1 - \rho - (1 - \lambda)e^*_{(0,i)}(b)}{\lambda} > \frac{\frac{1}{n} - 1 - \rho - \rho/2}{\lambda}.
\]
It follows that 
\[
e^*_{(0,i)}(a) > (\frac{\frac{1}{n} - 1 - \frac{3}{2} \rho}{2})(1 - \frac{\rho}{2})^{-1}.
\]
Similarly, one gets that

\[ e^*_i(a') > \left( \frac{1}{n} - 1 - \frac{3\rho}{2} \right)(1 - \frac{\rho}{2})^{-1}. \tag{2.5} \]

Now, applying 2.1, 2.2 and 2.3 we have that

\[ \|x - x'\| \leq \|la - \lambda'\| + \|1 - \lambda\|b - (1 - \lambda')b'\| \leq \|la - \lambda\|a' + \rho \leq \lambda\|a - a'\| + \|\lambda - \lambda'\| + \rho \leq \|a - a'\| + \frac{3\rho}{2}. \tag{2.6} \]

Now, our goal is estimate \(\|a - a'\|\). For this put \(a = \sum_{j=1}^{p} \lambda_j x_{\alpha_j}\) and \(a' = \sum_{j=1}^{q} \beta_j x_{\alpha'_j}\), where \(p, q \in \mathbb{N}, \lambda_j, \beta_j > 0, \sum_{j=1}^{p} \lambda_j = \sum_{j=1}^{q} \beta_j = 1\) and \(x_{\alpha_j}, x_{\alpha'_j} \in A\). Denotes by \(1\) the sequence in \(c\) with all its coordinates equal 1, then \(\|a - a'\| = \|a + 1 - (a' + 1)\|\). Now

\[ a + 1 = \sum_{j=1}^{p} \lambda_j x_{\alpha_j} + 1 = \sum_{j=1}^{p} \lambda_j(x_{\alpha_j} + 1) = \sum_{j=1}^{p} \lambda_j \hat{x}_{\alpha_j}, \]

where \(\hat{x}_{\alpha_j}\) is the element in \(c\) given by \(\hat{x}_{\alpha_j}(i) = 2\) if \(\phi^{-1}(i) \leq \alpha_j\) and \(\hat{x}_{\alpha_j}(i) = 0\) in otherwise. Similarly \(a' + 1 = \sum_{j=1}^{q} \beta_j \hat{x}_{\alpha'_j}\), where \(\hat{x}_{\alpha'_j}\) is the element in \(c\) given by \(\hat{x}_{\alpha'_j}(i) = 2\) if \(\phi^{-1}(i) \leq \alpha'_j\) and \(\hat{x}_{\alpha'_j}(i) = 0\) in otherwise.

Now we have from 2.4 and 2.5 that

\[ e^*_i(\hat{a}), \ e^*_i(\hat{a}') > \left( \frac{1}{n} - 2\rho \right)(1 - \frac{\rho}{2})^{-1} \quad \forall \ 1 \leq i \leq n, \]

where \(\hat{a} = a + 1\) and \(\hat{a}' = a' + 1\).

For every \(i \in \{1, \ldots, n\}\) we define now

\[ A_i = \{ j \in \{1, \ldots, p\} : \alpha_j \geq (0, i) \}, \ A'_i = \{ j \in \{1, \ldots, q\} : \alpha'_j \geq (0, i) \}. \]

If \(i \neq k\) then \(\alpha_i\) and \(\alpha_k\) are incomparable and also \(\alpha'_i\) and \(\alpha'_k\) are incomparable, hence \(A_i \cap A_k = \emptyset\) and \(A'_i \cap A'_k = \emptyset\).

Now we have that from 2.7 that

\[ \sum_{j \in A_i} \lambda_j > \left( \frac{1}{n} - 2\rho \right)(1 - \frac{\rho}{2})^{-1}, \quad \sum_{j \in A'_i} \beta_j > \left( \frac{1}{n} - 2\rho \right)(1 - \frac{\rho}{2})^{-1}. \tag{2.8} \]

Then

\[ 1 = \sum_{j=1}^{p} \lambda_j = \sum_{j \in \cup_{i=1}^{n} A_i} \lambda_j + \sum_{j \in (\cup_{i=1}^{n} A_i)^c} \lambda_j = \sum_{i=1}^{n} \sum_{j \in A_i} \lambda_j - \sum_{j \in (\cup_{i=1}^{n} A_i)^c} \lambda_j, \]

and we deduce from 2.8 that, for every \(k \in \{1, \ldots, n\}\),

\[ \sum_{j \in A_k} \lambda_j = \sum_{i=1}^{n} \sum_{j \neq k} \lambda_j - \sum_{j \in (\cup_{i=1}^{n} A_i)^c} \lambda_j < 1 - \sum_{i=1}^{n} \frac{1}{n} - \frac{3\rho}{2} = 1 - (n - 1)\left( \frac{1}{n} - 2\rho \right)(1 - \frac{\rho}{2})^{-1}. \tag{2.9} \]
Similarly
\begin{equation}
\sum_{j \in A_k'} \beta_j < 1 - (n - 1)\left(\frac{1}{n} - 2\rho\right)(1 - \frac{\rho}{2})^{-1}.
\end{equation}

Also, from (2.8)
\begin{equation}
\sum_{j \in (\bigcup_{i=1}^n A_i)^c} \lambda_j, \sum_{j \in (\bigcup_{i=1}^n A_i)^c} \beta_j < (2n - \frac{1}{2})\rho(1 - \frac{\rho}{2})^{-1}.
\end{equation}

Observe that the vectors \(\sum_{j \in A_i} \lambda_j \hat{x}_{a_j} - \sum_{j \in A_i'} \beta_j \hat{x}_{a_j'}\) have disjoint supports for coordinates \(k > 1\) and \(1 \leq i \leq n\), and so
\begin{equation}
\max_{k > 1}\left|\left(\sum_{i=1}^n \left(\sum_{j \in A_i} \lambda_j \hat{x}_{a_j} - \sum_{j \in A_i'} \beta_j \hat{x}_{a_j'}\right)(k)\right)\right| \leq 2\max_{1 \leq i \leq n}\left\{\sum_{j \in A_i} \lambda_j + \sum_{j \in A_i'} \beta_j\right\}
\end{equation}
since \(\|\hat{x}_{a_j}\| = \|\hat{x}_{a_j'}\| = 2\).

Now, applying (2.11) and (2.12) we get
\[
\|\hat{a} - \hat{a}'\| = \left|\sum_{j = 1}^p \lambda_j \hat{x}_{a_j} - \sum_{j = 1}^q \beta_j \hat{x}_{a_j'}\right| = 
\]
\[
\max\left\{\left|\left(\sum_{j = 1}^p \lambda_j \hat{x}_{a_j} - \sum_{j = 1}^q \beta_j \hat{x}_{a_j'}\right)(k)\right| : k \in \mathbb{N}\right\} = 
\]
\[
\max\left\{\left|\left(\sum_{j = 1}^p \lambda_j \hat{x}_{a_j} - \sum_{j = 1}^q \beta_j \hat{x}_{a_j'}\right)(\phi(0))\right|, \left|\left(\sum_{j = 1}^p \lambda_j \hat{x}_{a_j} - \sum_{j = 1}^q \beta_j \hat{x}_{a_j'}\right)(k)\right| : k > 1\right\} = 
\]
\[
\max\left\{0, \left|\left(\sum_{j = 1}^p \lambda_j \hat{x}_{a_j} - \sum_{j = 1}^q \beta_j \hat{x}_{a_j'}\right)(k)\right| : k > 1\right\} \leq 
\]
\[
\max\left\{\left|\left(\sum_{i = 1}^n \left(\sum_{j \in A_i} \lambda_j \hat{x}_{a_j} - \sum_{j \in A_i'} \beta_j \hat{x}_{a_j'}\right)(k)\right) + \left(\sum_{j \in (\bigcup_{i=1}^n A_i)^c} \lambda_j \hat{x}_{a_j}\right)(k) - \left(\sum_{j \in (\bigcup_{i=1}^n A_i)^c} \beta_j \hat{x}_{a_j'}\right)(k)\right| : k > 1\right\} \leq 
\]
\[
\max\left\{\left|\left(\sum_{i = 1}^n \left(\sum_{j \in A_i} \lambda_j \hat{x}_{a_j} - \sum_{j \in A_i'} \beta_j \hat{x}_{a_j'}\right)(k)\right)\right| : k > 1\right\} + 2\left(\sum_{j \in (\bigcup_{i=1}^n A_i)^c} \lambda_j + \sum_{j \in (\bigcup_{i=1}^n A_i)^c} \beta_j\right) \leq 
\]
\[
2\max_{1 \leq i \leq n}\left\{\sum_{j \in A_i} \lambda_j + \sum_{j \in A_i'} \beta_j\right\} + 4(2n - \frac{1}{2})\rho(1 - \frac{\rho}{2})^{-1} \leq 
\]
4(1 - (n - 1)(\frac{1}{n} - \frac{2\rho}{2})(1 - \frac{\rho}{2})^{-1}) + 4(2n - \frac{1}{2})\rho(1 - \frac{\rho}{2})^{-1} = \frac{4(\frac{1}{n} + (4n - 3\rho)(1 - \frac{\rho}{2})^{-1}}{

Finally, we conclude from 2.6 and the above estimation that

\|x - x'\| \leq \|a - a'\| + 3\rho/2 \leq \frac{8}{n} + (32n - 21)\rho < \frac{9}{n}

since \(\rho < \frac{1}{n(32n-21)}\). Hence we have proved that diam\(W_n\) < \(\frac{2}{n}\).

The above results find a closed, bounded, convex and symmetric subset \(K\) of \(B_c\) satisfying that every slice of \(K\) has diameter 2 and \(K\) contains nonempty relatively weakly open sets with diameter arbitrarily small. Our next goal is showing how one can get a Banach space whose unit ball behaves like \(K\) with respect the size of slices and relatively weakly open subsets. For this we need the following

**Lemma 2.3.** Let \(X\) be a Banach space containing an isomorphic copy of \(c_0\). Then there is an equivalent norm \(||\cdot||\) in \(X\) satisfying that \((X, ||\cdot||)\) contains an isomorphic copy of \(c\) and for every \(x \in B_{(X, ||\cdot||)}\) there are sequences \(\{x_n\}\), \(\{y_n\} \in B_{(X, ||\cdot||)}\) weakly convergent to \(x\) such that \(||x_n - y_n|| = 2\) for every \(n \in \mathbb{N}\). In fact, \(x_n = x + (1 - \alpha_n)e_n\) and \(y_n = x - (1 + \alpha_n)e_n\) for some scalars sequence \(\{\alpha_n\}\) with \(|\alpha_n| \leq 1\) for every \(n\).

**Proof.** As \(X\) contains isomorphic copies of \(c\), we can assume that \(c\) is, in fact, an isometric subspace of \(X\). Then for every \(Y\) separable subspace of \(X\) containing \(c\), there is a linear and continuous projection \(P_Y : Y \to c\) with \(||P|| \leq 8\). Indeed, let us consider the onto linear isomorphism \(T : c \to c_0\) given by \(T(x)(1) = \frac{1}{2}\lim_n x(n)\) and \(T(x)(n) = \frac{1}{2}(x(n) - \lim_n x(n))\) for every \(n \geq 1\). Note that \(||T|| = 1\) and \(||T^{-1}|| = 4\). Now, following [8, Th. 5.14], we get the desired projection \(P_Y\) with \(||P_Y|| \leq 2||T^{-1}|| = 8\).

Let \(\mathcal{Y}\) be the family of subspaces \(Y\) of \(X\) containing \(c\) such that \(c\) has finite codimension in \(Y\). Consider the filter basis \(\mathcal{Y}\) given by \(\{Y \in \mathcal{Y} : Y_0 \subset Y\}\), where \(Y_0 \in \mathcal{Y}\) and call \(\mathcal{U}\) the ultrafilter containing the generated filter by the above filter basis.

For every \(Y \in \mathcal{Y}\), we define a new norm in \(X\) given by

\[\|x\|_Y := \max\{\|P_Y(x)\|, \|x - P_Y(x)\|\}.\]

Finally, we define the norm on \(X\) given by \(||x|| := \lim_{\mathcal{U}} ||x||_Y\). Observe that \(\frac{1}{8}\|x\| \leq ||x|| \leq 3\|x\|\) for every \(x \in X\) and so \(||\cdot||\) is an equivalent norm in \(X\) such that \(||x|| = \|x\|_\infty\) for every \(x \in c\), where \(||\cdot||_\infty\) is the sup norm in \(c\). Hence \((X, ||\cdot||)\) contains an isometric copy of \(c\).
Pick \(x_0 \in B_{|| \cdot ||}(X)\). In order to prove the remaining statement let \(\{e_n\}\) and \(\{e^*_n\}\) the usual basis of \(c_0\) and the biorthogonal functionals sequence, respectively.

Chose \(\lambda \in \mathbb{R}\) and \(n \in \mathbb{N}\). For every \(Y \in \mathcal{Y}\) with \(x_0 \in Y\) we have that
\[
||x_0 + \lambda e_n||_Y = \max\{||P_Y(x_0) + \lambda e_n||, ||x_0 - P_Y(x_0)||\} =
\]
\[
\max\{||\lambda + e^*_n(P_Y(x_0))||, ||P_Y(x_0) - e^*_n(P_Y(x_0))e_n||, ||x_0 - P_Y(x_0)||\}.
\]

Call \(\beta_n = \lim\{||P_Y(x_0) - e^*_n(P_Y(x_0))e_n||, ||x_0 - P_Y(x_0)||\}\) and \(\alpha_n = \lim\{||\lambda + \alpha_n||, \beta_n\}\). Note that \(|\alpha_n| \leq 1\) and \(\beta_n \leq 1\) since \(||x_0|| \leq 1\).

Doing \(x_n := x_0 + (1 - \alpha_n)e_n\) and \(y_n := x_0 - (1 + \alpha_n)e_n\) for every \(n\), we get that \(x_n, y_n \in B_{|| \cdot ||}(X)\). Finally, it is clear that \(\{x_n\}\) and \(\{y_n\}\) are weakly convergent sequences to \(x_0\) and \(||x_n - y_n|| = 2\) for every \(n \in \mathbb{N}\).

**Theorem 2.4.** Let \(X\) be a Banach space containing an isomorphic copy of \(c_0\). Then there is an equivalent norm in \(X\) such that:

i) Every slice of new unit ball of \(X\) has diameter 2 for the new equivalent norm.

ii) There are nonempty relatively weakly open subsets of the new unit ball in \(X\) with diameter arbitrarily small for the new equivalent norm.

**Proof.** From the above lemma, we can assume that \(X\) contains an isometric copy of \(c\) and for every \(x \in B_X\) there are sequences \(\{x_n\}, \{y_n\} \in B_X\) weakly convergent to \(x\) such that \(||x_n - y_n|| = 2\) for every \(n \in \mathbb{N}\).

Fix \(0 < \varepsilon < 1\) and consider in \(X\) the equivalent norm \(|| \cdot ||\) whose unit ball is \(B_{\varepsilon} = \overline{r_{\varepsilon}}(A \cup -A \cup \{1 - \varepsilon\}B_X + \varepsilon B_{c_0})\). Then we have \(||x||_{\varepsilon} \leq \frac{1}{1 - \varepsilon} ||x||\) for every \(x \in X\) and \(||x|| = ||x||_{\infty}\) for every \(x \in c\).

In order to prove ii), fix \(\gamma > 0\). Pick \(n \in \mathbb{N}\) with \(18 < n(1 - \varepsilon)\gamma\) and choose \(\rho\) such that \(0 < \rho < \frac{1}{n(2n - 21)}\), \(2\rho < \gamma\) and \(2\rho < \varepsilon\). Consider the relative weak open subset of \(K\) given by
\[
W_n = \{x \in K : e^*_n(x) > \frac{1}{n} - 1 - \rho, 1 \leq n \leq n, \lim_{n} x(n) < -1 + \rho\}.
\]

From Proposition 2.2, \(W_n \neq \emptyset\) and \(\text{diam}_{|| \cdot ||}(W_n) \leq 9/n\).

Now, we define
\[
W = \{x \in B_{\varepsilon} : e^*_n(x) > \frac{1}{n} - 1 - \frac{1}{2} \rho, 1 \leq n \leq n, \lim_{n} x(n) < -1 + \rho^2\},
\]
where \(e^*_n\) and \(\lim_n\) denote the Hanh-Banach extensions to \(X\) of the corresponding functionals on \(c\). It is clear that \(||e^*_n||_{\varepsilon} = ||e^*_n|| = 1\) for every \(i = 1, \ldots, n\) and \(||\lim_n||_{\varepsilon} = ||\lim_n|| = 1\).

We prove that \(x_0 = \sum_{i=1}^{n} \frac{x_{(0, i)}}{n} \in W\). For this note that \(\lim_n x_{(0, i)}(n) = -1\) and then \(\lim_n x_0(n) = -1 < -1 + \rho^2\). Furthermore \(x_0\) is a convex combination of elements of \(A\) and so \(x_0 \in B_{\varepsilon}\). Finally, for \(1 \leq j \leq n\), one
has that

\[ e^*_0(x_0) = \sum_{i=1}^{n} \frac{1}{n} e^*_0(x_{(i)}(x_0)) + \frac{1}{n} e^*_0(x_{(0,j)}(x_0)) = -\frac{n-1}{n} + \frac{1}{n} > -1 - \frac{1}{2} \rho. \]

Then \( W \) is a nonempty relative weak open subset of \( B_\varepsilon \). In order to estimate the diameter of \( W \), it is enough compute the diameter of \( W \cap co(A \cup -A \cup [(1-\varepsilon)B_X + \varepsilon B_{co}]) \). Furthermore, \( co(A \cup -A \cup [(1-\varepsilon)B_X + \varepsilon B_{co}]) = co(co(A) \cup co(-A) \cup [(1-\varepsilon)B_X + \varepsilon B_{co}]) \). So, given \( x \in W \), we can assume that \( x = \lambda_1 a + \lambda_2 (-b) + \lambda_3 [(1-\varepsilon) x_0 + \varepsilon y_0] \), where \( \lambda_i \in [0,1] \) with \( \sum_{i=1}^{3} \lambda_i = 1 \) and \( a, b \in co(A) \), \( x_0 \in B_X \), and \( y_0 \in B_{co} \). Since \( x \in W \), we have that \( \lim_n(x) < -1 + \rho^2 \), and hence,

\[-\lambda_1 + \lambda_2 + \lambda_3 (1-\varepsilon) \lim_n(x_0) = -\lambda_1 + \lambda_2 + \lambda_3 \lim_n((1-\varepsilon)x_0 + \varepsilon y_0) < -1 + \rho^2.\]

Note that \( -1 \leq \lim_n(x_0) \). This implies that

\[ 2\lambda_2 + \lambda_3 \varepsilon - 1 = -\lambda_1 + \lambda_2 - \lambda_3 (1-\varepsilon) < -1 + \rho^2. \]

Since \( 2\rho < \varepsilon \), and so \( \lambda_2 + \lambda_3 < \frac{1}{2} \rho \). As a consequence we get that \( \lambda_1 > 1 - \frac{\rho}{2} \).

If \( i \in \{1, \ldots, n\} \) then

\[ \frac{1}{n} - 1 - \frac{1}{2} \rho < e^*_0(x_{(i)}(x) = \lambda_1 e^*_0(a) + \lambda_2 e^*_0(-b) + \lambda_3 e^*_0(1-\varepsilon)x_0 + \varepsilon y_0]. \]

Since \( \|e^*_0(x_{(i)}(x) \leq 1 \) and \( -b, (1-\varepsilon)x_0 + \varepsilon y_0 \in B_\varepsilon \), we have that \( e^*_0(-b) \leq 1 \) and \( e^*_0(1-\varepsilon)x_0 + \varepsilon y_0 \leq 1. \) It follow that

\[ \lambda_1 e^*_0(a) + \lambda_2 e^*_0(-b) + \lambda_3 e^*_0(1-\varepsilon)x_0 + \varepsilon y_0] \leq \lambda_1 e^*_0(a) + \lambda_2 + \lambda_3 \leq \lambda_1 e^*_0(a) + \frac{1}{2} \rho. \]

We deduce that

\[ e^*_0(\lambda_1 a) > \frac{1}{n} - 1 - \rho \]

for \( 1 \leq i \leq n. \) On the other hand, we have that

\[ \lim_n(\lambda_1 a) = -\lambda_1 < -1 + \rho - \frac{1}{2} \rho < -1 + \rho, \]

and we conclude that \( \lambda_1 a \in W_n \).

Finally, given \( x, x' \in W \), we can assume that

\[ x = \lambda_1 a + \lambda_2 (-b) + \lambda_3 [(1-\varepsilon)x_0 + \varepsilon y_0], \]

where \( \lambda_i, \lambda_i' \in [0,1] \) with \( \sum_{i=1}^{3} \lambda_i = \sum_{i=1}^{3} \lambda_i' = 1 \), and \( a, b, a', b' \in co(A) \), \( x_0, x_0' \in B_X \) and \( y_0, y_0' \in B_{co} \). We have that

\[ \|x - x'\|_\varepsilon \leq \|\lambda_1 a - \lambda_1 a'\|_\varepsilon + \lambda_2 + \lambda_3 + \lambda_3' < \|\lambda_1 a - \lambda_1 a'\|_\varepsilon + \rho. \]

We recall that, \( \|x\| \leq \frac{1}{1-\varepsilon} \|x\| \) for every \( x \in X \) and \( \|x\| = \|x\|_\infty \) for every \( x \in c \), and that \( \lambda_1 a, \lambda_1 a' \in W_n \). Then

\[ \|x - x'\|_\varepsilon \leq \frac{1}{1-\varepsilon} \|\lambda_1 a - \lambda_1 a'\|_\infty + \rho \leq \frac{9}{n(1-\varepsilon)} + \rho \leq \gamma. \]
Hence $\text{diam}_{\|\cdot\|_\varepsilon}(W) \leq \gamma$.

In order to prove i), note that $B_\varepsilon \subset B_X$ and so $\|x\|_\varepsilon \geq \|x\|$ for every $x \in X$.

Pick now $f \in X^*$, $\|f\|_\varepsilon = 1$ and $\beta > 0$ and consider the slice

$$S = \{x \in B_\varepsilon : f(x) > 1 - \beta\}.$$ 

Hence there are $a \in A \cup -A$ or $(1 - \varepsilon)x_0 + \varepsilon y_0 \in (1 - \varepsilon)B_X + \varepsilon B_{c_0}$ such that $a \in S$ or $(1 - \varepsilon)x_0 + \varepsilon y_0 \in S$.

From the symmetry of $A \cup -A$, we can assume that $a \in A$, so there is $\alpha \in \mathbb{N}^{<\omega}$ such that $a = x_\alpha$. We recall that $x_{(\alpha,j)}(k) = 1$ if $\phi^{-1}(k) \leq (\alpha,j)$ and $x_{(\alpha,j)}(k) = -1$ in otherwise, then $\{x_{(\alpha,j)}\}_j$ is a weakly convergent sequence to $x_\alpha$. Hence we can choose $j$ so that $x_{(\alpha,j)} \in S$. Note that $x_{(\alpha,j)} - x_\alpha = 2e_{(\alpha,j)}$, then $2 = \|2e_{(\alpha,j)}\|_\infty = \|x_{(\alpha,j)} - x_\alpha\| \leq \|x_{(\alpha,j)} - x_\alpha\|_\varepsilon$. It follow that $\text{diam}_{\|\cdot\|_\varepsilon}(S) = 2$.

In the case that there is $x_0 \in B_X$ and $y_0 \in B_{c_0}$ such that

$$(1 - \varepsilon)x_0 + \varepsilon y_0 \in S,$$

as $S$ is a norm open set, we can assume that $y_0$ has finite support. From the above lemma, there is a scalars sequence $\{t_j\}$ with $|t_j| \leq 1$ for every $j$ such that, putting $x_j = x_0 + (1 - t_j)e_j$ and $y_j = x_0 - (1 + t_j)e_j$ for every $j$, we have that $\{x_j\}$ and $\{y_j\}$ are weakly convergent sequences in $B_X$ to $x_0$. We put $j_0$ such that $e_j^*(y_0) = 0$ for every $j \geq j_0$, then $y_0 + e_j, y_0 - e_j \in B_{c_0}$ for every $j \geq j_0$.

So it follows that $\{(1 - \varepsilon)x_j + \varepsilon(y_0 + e_j)\}_{j \geq j_0}$ and $\{(1 - \varepsilon)y_j + \varepsilon(y_0 - e_j)\}_{j \geq j_0}$ are sequences in $(1 - \varepsilon)B_\varepsilon + \varepsilon B_{c_0} \subset B_{\varepsilon}$ weakly convergent to $(1 - \varepsilon)x_0 + \varepsilon y_0$. Hence we can chose $j$ so that $(1 - \varepsilon)x_j + \varepsilon(y_0 + e_j), (1 - \varepsilon)y_j + \varepsilon(y_0 - e_j) \in S$. Then

$$\|[(1 - \varepsilon)x_j + \varepsilon(y_0 + e_j)] - [(1 - \varepsilon)y_j + \varepsilon(y_0 - e_j)]\|_\varepsilon = \|2(1 - \varepsilon)e_j + 2\varepsilon e_j\|_\varepsilon = \|2e_j\|_\varepsilon \geq \|2e_j\| = \|2e_j\|_\infty = 2,$$

and $\text{diam}_{\|\cdot\|_\varepsilon}(S) = 2$. \hfill \blacksquare

As a consequence of the above result we have the following corollary, which answers by the negative the problem about the equivalence between SD2P and D2P.

**Corollary 2.5.** Every Banach space containing an isomorphic copy of $c_0$ can be equivalently renormed satisfying SD2P and failing D2P.

Our final result establishes a stability property of Banach spaces with SD2P and failing D2P by $\ell_2$-sums.

**Corollary 2.6.** The $\ell_2$-sum of Banach spaces with SD2P and failing D2P also has SD2P and fails D2P.

The proof of the above Corollary is an immediate consequence of the next lemma, which give us the stability of SD2P and small weak open subsets for $\ell_2$-sums. In fact, this stability is also true for $\ell_p$-sums, whenever $1 \leq p < \infty$. 
Lemma 2.7. Let \( \{X_n\} \) be a sequence of Banach spaces and let \( \{\varepsilon_n\} \) be a sequence of positive real numbers with \( \lim n \varepsilon_n = 0 \). Assume that, for every \( n \in \mathbb{N} \), \( X_n \) has SD2P and \( B_{X_n} \) contains a nonempty relatively weakly open subset with diameter less than \( \varepsilon_n \). Then \( \ell_2 - \bigoplus_n X_n \) has the SD2P and the unit ball of \( \ell_2 - \bigoplus_n X_n \) contains nonempty relatively weakly open subsets with diameter arbitrarily small.

Proof. For every \( n \in \mathbb{N} \) let \( U_n \) be a nonempty relatively weakly open subset of \( B_{X_n} \) with diameter less than \( \varepsilon_n \). Then \( \|x\| > 1 - \varepsilon_n \) for every \( x \in U_n \).

Call \( Z = \ell_2 - \bigoplus_n X_n \) and define, for \( m \in \mathbb{N} \), \( V_m = \{x_n \in B_Z : x_m \in U_m\} \). Now \( V_m \) is a relative weak open subset of \( B_Z \). For fixed \( m \), as \( \|x_m\| > 1 - \varepsilon_m \) we have that \( \sum_{n=1}^{\infty} \|x_n\|^2 \leq 1 - (1 - \varepsilon_m)^2 \) for every \( \{x_n\} \in V_m \).

Pick \( \{x_n\}, \{y_n\} \in V_m \). So \( \|x_m - y_m\| < \varepsilon_m \), since \( \text{diam}(U_m) < \varepsilon_m \) and then

\[
\|\{x_n\} - \{y_n\}\|^2 = \sum_{n=1}^{\infty} \|x_n - y_n\|^2 = \\
\sum_{n=1}^{\infty} \|x_n - y_n\|^2 + \|x_m - y_m\|^2 \\
\sum_{n=1}^{\infty} (\|x_n\| + \|y_n\|)^2 + \|x_m - y_m\|^2 < \\
\sum_{n=1}^{\infty} \|x_n\|^2 + \sum_{n=1}^{\infty} \|y_n\|^2 + 2 \sum_{n=1}^{\infty} \|x_n\|\|y_n\| + \varepsilon_m^2 \leq \frac{2(1 - (1 - \varepsilon_m)^2)}{2} + 2(\sum_{n=1}^{\infty} \|x_n\|^2)^{1/2}(\sum_{n=1}^{\infty} \|y_n\|^2)^{1/2} + \varepsilon_m^2 \leq \frac{4(1 - (1 - \varepsilon_m)^2)}{2} + \varepsilon_m^2,
\]

and \( \text{diam}(V_m) \leq (4(1 - (1 - \varepsilon_m)^2) + \varepsilon_m)^{1/2} \). As \( \lim_m \varepsilon_m = 0 \), we conclude that \( B_Z \) has nonempty relatively weakly open subsets with diameter arbitrarily small.

We pass now to prove that \( Z \) has SD2P. Take \( f \in S_Z \), \( 0 < \alpha < 1 \) and consider an arbitrary slice of \( B_Z \)

\[
S = \{z \in B_Z : f(z) > 1 - \alpha\}.
\]

Pick \( z_0 \in S_Z \cap S \), then choose \( 0 < \varepsilon < \alpha \) so that \( f(z_0) > 1 - \alpha + \varepsilon \).

We denotes by \( P_k \) the projection of \( Z \) onto \( \ell_2 - \bigoplus_{i=1}^k X_i \), which is a norm one projection for every \( n \in \mathbb{N} \). As \( f(z_0) > 1 - \alpha + \varepsilon \), there is \( k \in \mathbb{N} \) such that \( P_k^*(f)(P_k(z_0)) > 1 - \alpha + \varepsilon \), where \( P_k^* \) denotes the transposed projection of \( P_k \).

Consider the slice of the unit ball in \( Y = \ell_2 - \bigoplus_{i=1}^k X_i \) given by \( T = \{y \in B_Y : P_k^*(f)(y) > 1 - \alpha + \varepsilon\} \). In order to prove that \( \text{diam}(S) = 2 \), fix \( \rho > 0 \) and take \( y_1, y_2 \in B_Y \) such that \( \|y_1 - y_2\| > 2 - \rho \). This is possible, because it is known that the finite \( \ell_2 \)-sum of Banach spaces with SD2P has
too SD2P [2, Theorem 2.4]. Now we see $y_1, y_2$ as elements in $Z$, via the natural isometric embedding of $Y$ into $Z$, and we have that $y_1, y_2 \in S$ with $\|y_1 - y_2\|_Z > 2 - \rho$, hence $\text{diam}(S) \geq 2 - \rho$. As $\rho$ was arbitrary, we conclude that $\text{diam}(S) = 2$. \newline

Now the proof of Corollary 2.6 is complete.

It would be interesting to know if there is some Banach space with PCP and SD2P.

On the other hand, there is a stronger property than D2P, the strong diameter two property: a Banach space $X$ satisfies the strong diameter two property (strong D2P) if every convex combination of slices in the unit ball of $X$ has diameter 2. Its clear that strong D2P implies D2P, and it is known that these two properties are in fact different [2]. Indeed, in [2, Theorem 3.2] is proved that $c_0 \oplus 2c_0$ is a Banach space with D2P and failing strong D2P. The failure of strong D2P of $c_0 \oplus 2c_0$ is shown finding out an average of two slices in the unit ball with diameter strictly less than 2, however the unit ball of $c_0 \oplus 2c_0$ has no convex combination of slices with diameter arbitrarily small. In fact, it is not difficult to check that the diameter of any convex combination of slices in the unit ball of $c_0 \oplus 2c_0$ is at least 1. So it would be interesting to know if there is some Banach space with D2P and so that its unit ball contains convex combinations of slices with diameter arbitrarily small.

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