ON THE CALCULATION OF EFFECTIVE ACTIONS
BY STRING METHODS

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ABSTRACT

Strassler’s formulation of the string-derived Bern-Kosower formalism is reconsidered with particular emphasis on effective actions and form factors. Two- and three point form factors in the nonabelian effective action are calculated and compared with those obtained in the heat kernel approach of Barvinsky, Vilkovisky et al. We discuss the Fock-Schwinger gauge and propose a manifestly covariant calculational scheme for one-loop effective actions in gauge theory.

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One of the main differences between string theory and particle theory is the fact that in string theory the full (on-shell) S-matrix can be calculated using the Polyakov path integral, i.e. in first quantization. As string theory reduces to particle theory in the limit of infinite string tension, the same should be possible in principle also for the particle S-matrix. And indeed, by a painstaking analysis of this limit, Bern and Kosower [1] have been able to derive from string theory a new set of Feynman rules for the calculation of one-loop amplitudes in conventional field theory which, though quite different from the conventional ones, appear to be completely equivalent [2]. Those rules are particularly well-suited for calculations in gauge theories, as they combine contributions of different Feynman diagrams into gauge invariant structures [3]. Strassler [4] recently succeeded in deriving essentially the same set of rules without explicit reference to string theory, using one-dimensional path integrals with Green functions adapted to the circle. Those path integrals are the particle theory analogue of the genus one Polyakov path integral. He further demonstrated the advantages of the new formalism for the calculation of both abelian and non-abelian effective actions [5].

In this paper, we will somewhat reformulate this method and apply it to the computation of form factors, i.e. to the organization of the higher derivative terms appearing in the effective action. It will turn out that the two- and three- point form factors for nonabelian gauge theory may be computed quite efficiently. In the three-point case, the calculation will be non-covariant. A manifestly covariant method of calculation will be outlined and applied to the two-point case.

Let us begin with Strassler’s [4] worldline path integral expression for the one-loop effective action of a Dirac spinor of mass \(m\) minimally coupled to an (abelian or non-abelian) background gauge field \(A\):

\[
\Gamma[A] = -2 \int_0^{\infty} \frac{dT}{T}[2\pi E \tau]^2 e^{-\frac{2}{T^2} m^2 T} N \text{Tr} \int DxD\psi \\
\times \exp \left[ - \int_0^T d\tau \left( \left( \frac{1}{2E} \right) \dot{x}^2 + \frac{1}{2} \dot{\psi} \dot{\psi} + igA_{\mu} \dot{x}^\mu - \frac{1}{2} igE \dot{\psi}^\mu F_{\mu\nu} \dot{\psi}^\nu \right) \right].
\]

In this construction, the \(x^\mu(\tau)\)'s are the periodic functions from the circle with circumference \(T\) into \(D\)-dimensional Euclidean spacetime, and the \(\dot{\psi}^\mu(\tau)\)'s their antiperiodic supersymmetric partners, obeying

\[
\{\psi^\mu, \psi^{\nu}\} = g^{\mu\nu}.
\]

\(N\) is the path integral normalization factor,

\[
N^{-1} = \text{Tr} \int DxD\psi \exp \left[ - \int_0^T d\tau \left( \left( \frac{1}{2E} \right) \dot{x}^2 + \frac{1}{2} \dot{\psi} \dot{\psi} \right) \right],
\]

and \(E\) the worldline metric; in the following we set \(E = 2\). Note that our normalization includes the spinor degrees of freedom, which makes for a factor of 4 compared to \([4,5]\). In the nonabelian case path ordering is implied.
Insertions into the free path integral will be contracted using the one-dimensional Green functions adapted to the (anti-)periodicity conditions,

\[
\langle x^\mu(\tau_1)x^\nu(\tau_2) \rangle = -g^{\mu\nu}G_B(\tau_1,\tau_2) = -g^{\mu\nu}\left[ |\tau_1 - \tau_2| - \frac{(\tau_1 - \tau_2)^2}{T} \right],
\]

\[
\langle \psi^\mu(\tau_1)\psi^\nu(\tau_2) \rangle = \frac{g^{\mu\nu}}{2}G_F(\tau_1,\tau_2) = \frac{g^{\mu\nu}}{2}\text{sign}(\tau_1 - \tau_2).
\]

As a warm-up, let us reconsider the classical case of the 1-loop effective action for spinor electrodynamics in a constant abelian background [6]. Here choosing an appropriate gauge one may write \[ A \]
spinor electrodynamics in a constant abelian background [6]. Here choosing an appropriate gauge one may write \[ A \]
unit circle. Dots on a variable; in particular,

\[
\dot{G}_B(\tau_1,\tau_2) = \text{sign}(\tau_1 - \tau_2) - \frac{2(\tau_1 - \tau_2)}{T},
\]

\[
\ddot{G}_B(\tau_1,\tau_2) = 2\delta(\tau_1 - \tau_2) - \frac{2}{T}.
\]
The operator traces may be computed using the Fourier bases \( \{ \exp[2\pi ik\tau] \mid k \in \mathbb{Z} \} \) for the periodic and \( \{ \exp[2\pi i(k + \frac{1}{2})\tau] \mid k \in \mathbb{Z} \} \) for the antiperiodic functions. The result is

\[
\begin{align*}
Tr[\dot{G}_B]^{2n} &= 2(-1)^n\zeta(2n)\pi^{-2n}, \\
Tr[G_F]^{2n} &= 2(2^{2n} - 1)(-1)^n\zeta(2n)\pi^{-2n},
\end{align*}
\]

in terms of the Riemann \( \zeta \)– function (traces of odd powers vanish). Finally

\[
\Gamma[A] = -2\int_0^\infty d\tau \frac{d}{d\tau}[4\pi\tau]^{-\frac{D}{2}} e^{-m^2\tau} \exp\left[-\sum_{n=1}^\infty \frac{(g\tau)^{2n}}{n}(2^{2n-1} - 1)\zeta(2n)\pi^{-2n} TrF^{2n}\right].
\]

This is the generating functional for eq. (3.23) in [5] * **. It may be transformed into the standard representation (see e.g. [7])

\[
\Gamma[A] = \frac{1}{8\pi^2} \int_0^\infty ds \frac{d}{ds}[s^{-\frac{D}{2}} e^{-m^2s} \left(g^2ab \frac{\cosh(gas)\cos(gbs)}{\sinh(gas)\sin(gbs)} - \frac{1}{s^2}\right),
\]

where

\[
\begin{align*}
a^2 &= \frac{1}{2} \left[ E^2 - B^2 + \sqrt{(E^2 - B^2)^2 + 4(E \cdot B)^2} \right], \\
b^2 &= \frac{1}{2} \left[ -(E^2 - B^2) + \sqrt{(E^2 - B^2)^2 + 4(E \cdot B)^2} \right],
\end{align*}
\]

by diagonalizing the matrix \( F_{\mu\nu} \), showing

\[
TrF^{2n} = 2\left[(a^2)^n + (-b^2)^n\right],
\]

and using the Taylor expansions of \( \ln[x\cot(x)], \ln[x\coth(x)] \).

After this amusing calculation, let us proceed to our main objective, which is the calculation of form factors in the nonabelian theory. That means we want to calculate (taking the case of a (complex) scalar loop first)

\[
\Gamma[A] = \int_0^\infty \frac{d\tau}{T} [4\pi\tau]^{-\frac{D}{2}} e^{-m^2\tau} N Tr \int Dx \exp\left[-\int_0^\tau d\tau \left(\frac{1}{4}\dot{x}^2 + igA_\mu \dot{x}^\mu\right)\right]
\]

* We believe that the recursion relations derived in [5] for the above traces should be related to the well-known recursion formula for the Bernoulli numbers.

** Analogous methods have been already used by Metsaev and Tseytlin [15] to obtain a generalized Schwinger formula for the bosonic string and, by taking the infinite string tension limit of this generalization, for YM theory.
up to a fixed power in the background field, but to all orders in the covariant derivatives $D_\mu = \partial_\mu - igA_\mu$ of the field (for the history of this subject and for possible physical applications see ref. [8] and references therein). To begin with, we introduce the loop center of mass $x_0$, writing

$$x^\mu(\tau) = x_0^\mu + y^\mu(\tau)$$

with

$$\int_0^T d\tau y^\mu(\tau) = 0,$$

and extract the integral over the center of mass from the path integral:

$$\int D x = \int dD x_0 \int D y.$$ (15)

Next we Taylor expand the interaction part $\dot{x}^\mu A_\mu(x(\tau))$ with respect to $x_0$, use $\dot{x}^\mu = \dot{y}^\mu$ to write

$$\dot{x}^\mu A_\mu(x) = \dot{y}^\mu e^{y\partial} A_\mu(x_0)$$

and expand the path-ordered interaction exponential to get

$$\Gamma[A] = Tr \int_0^\infty \frac{dT}{T} \frac{dD}{4\pi T} e^{-m^2T} \int dD x_0 \sum_{n=0}^\infty \frac{(-ig)^n}{n!} T \int_0^{T_1=T} d\tau_2 \int_0^{\tau_2} d\tau_3 \ldots \int_0^{\tau_{n-1}} d\tau_n$$

$$\times \mathcal{N} \int D y \dot{y}^{\mu_1}(\tau_1) e^{y(\tau_1)\partial(1)} A^{(1)}_{\mu_1}(x_0) \ldots \dot{y}^{\mu_n}(\tau_n) e^{y(\tau_n)\partial(\mu)} A^{(n)}(x_0) \exp \left[ - \int_0^T d\tau \frac{\dot{y}^2}{4} \right].$$ (17)

We have labeled the background fields $A_{\mu_1}, \ldots, A_{\mu_n}$, and the first $\tau$–integration has been eliminated by using the freedom of choosing the point 0 somewhere on the loop. Instead of restricting the colour trace by fixing the cyclic order, we have introduced an explicit factor of $\frac{1}{n}$. The single terms in this expansion may now be computed by Wick contractions in the one-dimensional worldline field theory, using the above bosonic Green function on the circle and formulas familiar from string theory,

$$\langle e^{y(\tau_1)\partial(1)} e^{y(\tau_2)\partial(2)} \rangle = e^{-G_B(\tau_1, \tau_2)\partial(1)\partial(2)},$$

$$\langle \dot{y}^{\mu}(\tau_1) e^{y(\tau_2)\partial(2)} \rangle = -\dot{G}_B(\tau_1, \tau_2)\partial^{\mu}.$$ (18)

For the term quadratic in $A_\mu$, this leads to a single $\tau$–integration

$$T \int_0^T d\tau_2 \left[ \tilde{G}_B(T, \tau_2) g^{\mu_1\mu_2} - \dot{G}_B(T, \tau_2)^2 \partial^{\mu_1}(1) \partial^{\mu_2} \right] e^{-G_B(\tau_1, \tau_2)\partial(1)\partial(2)} A^{(1)}_{\mu_1} A^{(2)}_{\mu_2},$$ (19)
which using the scaling properties of $G_B, \dot{G}_B, \ddot{G}_B$ and partial integration with respect to $\tau_2$ may be transformed into

$$T^2 \int_0^1 du_2 \dot{G}_B(1, u_2)^2 \left[ g^{\mu_1 \mu_2} \partial_{(1)} \partial_{(2)} - \partial_{(2)}^{\mu_1} \partial_{(1)}^{\mu_2} \right] e^{-T G_B(1, u_2) \partial_{(1)} \partial_{(2)} A^{(1)}_{\mu_1} A^{(2)}_{\mu_2}}, \quad (20)$$

where $G_B$ now refers to the unit circle, $G_B(1, u_2) = u_2(1 - u_2)$. But

$$\left[ g^{\mu_1 \mu_2} \partial_{(1)} \partial_{(2)} - \partial_{(2)}^{\mu_1} \partial_{(1)}^{\mu_2} \right] A^{(1)}_{\mu_1} A^{(2)}_{\mu_2}$$

is just the "abelian" part of $\frac{1}{2} F^{(1)}_{\mu \nu} F^{(2) \mu \nu}$, and we need not see more of this tensor to determine its form factor: Anticipating covariantization and performing further partial integrations both with respect to $u_2$ (to get rid of the $\dot{G}_B$) and to $x_0$ (yielding $\partial_{(1)} \partial_{(2)} = -\partial_{(2)}^2$) to get an expression standardized in the sense of [8], we can already predict the following result for the full two-point one-loop effective action:

$$\Gamma_2[A] = Tr \int_0^\infty \frac{dT}{T} \frac{d^D x_0 F_{\mu \nu}(x_0) F^{scal}_2(\xi) F^{\mu \nu}(x_0)}{4 \pi T} - \frac{\xi}{2^2 T^2 (g^2)^2} \int d^D x_0 F_{\mu \nu}(x_0) F^{\mu \nu}(x_0) \int_0^1 du e^{-u(1-u)\xi}$$

$$\xi = -TD \mu \nu.$$

This is the same expression as has been reached by [8] using covariant perturbation theory.

Before proceeding to the three-point case, let us say something about the systematics of the calculation in general. Every $\dot{y}(\tau_i)$ has to be contracted, either with another $\dot{y}(\tau_k)$, or with an $e^{y(\tau_i) \partial_{(i)}}$. The most basic case is if there are only contractions of the second kind, as the result will be just one or a product of several closed chains of $G_B$'s,

$$\dot{G}_B(\tau_{i_1}, \tau_{i_2}) \dot{G}_B(\tau_{i_2}, \tau_{i_3}) \ldots \dot{G}_B(\tau_{i_n}, \tau_{i_1}) \partial_{(i_1)}^{\mu_{i_1}} \ldots \partial_{(i_n)}^{\mu_{i_n}}, \quad (23)$$

multiplied by a universal factor of

$$exp \left[ -\sum_{i<k} G_B(\tau_i, \tau_k) \partial_{(i)} \partial_{(k)} \right] \quad (24)$$

from the contraction of the exponentials among themselves.

Contractions of the first kind would result in $\dot{G}_B$'s, which might be reverted into $\dot{G}_B$'s by partial integrations later (this problem has already been discussed at length in refs.
[1-5]). From the systematic point of view, however, we prefer to eliminate the $\hat{G}_B$’s before they arise: Before contracting a $\dot{y}(\tau_i)$ with a $\dot{y}(\tau_k)$, one performs a partial integration in $\tau_k$. This leaves a new $\dot{y}(\tau_k)$ after contraction of the $\dot{y}(\tau_i)$ in the main part of the partial integration, and hence the $\langle \dot{y}\dot{y} \rangle$ chain construction works in this case, too. But there are further terms both from the boundary of the $\tau_k$ integration and from differentiating the $\tau_k$–dependent boundary of $\tau_{k+1}$.

Altogether we obtain for one contraction step of the first kind a factor of

$$-\hat{G}_B(\tau_i, \tau_k)g^{\mu_i\mu_k}\left[-\dot{y}(\tau_k)\partial_x^{(k)} + \delta(\tau_k - \tau_{k-1}) - \delta(\tau_k) - \delta(\tau_k - \tau_{k+1}) \right]. \quad (25)$$

Whereas the first term of this expression is part of a continuing chain, the other ones may be represented pictorially as the end of a $\langle \dot{y}\dot{y} \rangle$ chain at $\tau_k \neq \tau_i$, if contraction has started with $\dot{y}(\tau_i)$. As one can infer already from the two-point calculation, it is useful to always combine the first term with the double contraction

$$\langle \dot{y}(\tau_i)e^{y(\tau_k)\partial_{(k)}}\rangle \langle e^{y(\tau_i)\partial_{(1)}}\dot{y}(\tau_k) \rangle.$$ 

The result of one particular total contraction will be a product of both open and closed chains of $\hat{G}_B$’s, where open chains start freely and may but need not necessarily end on other chains (closed or open).

Treating the three-point path integral in this way, we obtain (after the usual rescaling $\tau = Tu$) the $u$–integrals

$$\int_0^1 du_2 \int_0^{u_2} du_3 \left\{ \left[ (g^{\lambda\mu}\partial_{(1)}\partial_{(2)} - \partial_{(2)}\partial_{(1)}^\mu)\hat{G}_B^{(2)}(\tau_i, \tau_k) + \partial_{(2)}^\nu \hat{G}_B \right] + \text{cyclic terms} \right\}
- \hat{G}_B^{12}\hat{G}_B^{13}\hat{G}_B^{23}\left[ g^{\lambda\mu}(\partial_{(1)}^\nu \partial_{(2)}\partial_{(3)} - \partial_{(2)}^\nu \partial_{(1)}\partial_{(3)}) + \text{c. t.} + \partial_{(3)}^\lambda \partial_{(1)}^\mu \partial_{(2)}^\nu - \partial_{(2)}^\lambda \partial_{(3)}^\mu \partial_{(1)}^\nu \right] \}
\times \text{exp} \left[ -\hat{G}_B^{12}\partial_{(1)}\partial_{(2)} - \hat{G}_B^{13}\partial_{(1)}\partial_{(3)} - \hat{G}_B^{23}\partial_{(2)}\partial_{(3)} \right] A^{(1)}(x_0)\nu_\mu(x_0)A^{(3)}(x_0)
+ \text{boundary terms} \right), \quad (26)$$

where ”boundary terms” stands for all terms involving one of the $\delta$ – functions in the contraction formula (25). Those terms reduce to a single $u$ – integration and may be written as functions of $f(\partial_{(1)}^2)$, $f(\partial_{(2)}^2)$ and $f(\partial_{(3)}^2)$, where $f$ is the basic two-point function defined in eqs.(22) (part of them in fact does not contribute to the three-point form factors, but to the covariantization of the two-point form factor). According to the analysis of [8], on the three-point level there are – up to powers of covariant derivatives, Bianchi identities and partial integrations in $x_0$ – two independent invariant tensor structures, which may be chosen as

* ($\delta$ – functions with undefined $\tau_{k\pm 1}$ should be skipped)
\[ T_1 = \text{Tr} F^\lambda \alpha F_\alpha^\nu F_\mu^\mu, \]
\[ T_2 = \text{Tr} F^\alpha \beta D^\mu F_\mu^\alpha D^\nu F_\nu^\beta. \]  

The terms in the second square bracket of formula (26) combine to produce simply the "abelian" part of \(-T_1\), while the terms in the first square bracket contribute to both \(T_1\) and \(T_2\). Collecting all contributions to \(T_1\) we obtain for its coefficient the \(u\) – integral

\[
\int_0^1 du_2 \int_0^{u_2} du_3 \exp \left[ -G_{B12} \partial_{(1)} \partial_{(2)} - G_{B13} \partial_{(1)} \partial_{(3)} - G_{B23} \partial_{(2)} \partial_{(3)} \right] \left\{ \dot{G}_{B12} \dot{G}_{B13} \dot{G}_{B23} \right. \\
- \frac{1}{2} \left[ G_{B12}^2 (-\dot{G}_{B13} + \dot{G}_{B23}) + G_{B13}^2 (\dot{G}_{B12} + \dot{G}_{B23}) + G_{B23}^2 (\dot{G}_{B12} - \dot{G}_{B13}) \right].
\]

To compare this coefficient with the results of [8], we again standardize by partially integrating both with respect to \(u\) and \(x_0\), introduce

\[ \xi_i = -TD^{(i) \mu} D_\mu^{(i)}, \]

and performe the change of variables

\[ \alpha_1 = u_2 - u_3 \]
\[ \alpha_2 = 1 - (u_1 - u_3) \]
\[ \alpha_3 = u_1 - u_2, \]

leading to

\[
\Gamma_{T_1}[A] = \int_0^\infty \frac{dT}{T} [4\pi T]^{-\frac{D}{2}} e^{-m^2 T} T^3 (-ig^3) \int d^D x_0 F_{T_1}^{\text{scal}}(\xi_1, \xi_2, \xi_3) \text{Tr} F^\lambda \alpha F_\alpha^\nu F_\mu^\mu(x_0),
\]

\[
F_{T_1}^{\text{scal}}(\xi_1, \xi_2, \xi_3) = \frac{4}{3\Delta^2} \left[ \Sigma (\Delta + \Xi) + 12 \Xi + 8 \frac{\Xi^2}{\Delta} \right] F(\xi_1, \xi_2, \xi_3) + \text{boundary terms}.
\]

Here we have defined

\[ \Sigma = \xi_1 + \xi_2 + \xi_3 \]
\[ \Xi = \xi_1 \xi_2 \xi_3 \]
\[ \Delta = \xi_1^2 + \xi_2^2 + \xi_3^2 - 2\xi_1 \xi_2 - 2\xi_1 \xi_3 - 2\xi_2 \xi_3 \]

and the basic three-point function

\[
F(\xi_1, \xi_2, \xi_3) = \int_{\alpha \geq 0} d^3 \alpha \delta(1 - \alpha_1 - \alpha_2 - \alpha_3) \exp[-\alpha_1 \alpha_2 \xi_3 - \alpha_2 \alpha_3 \xi_1 - \alpha_3 \alpha_1 \xi_2] \]
originating from the universal contraction of exponentials alone. Again we have assumed
that the higher order calculations will produce the missing "nonabelian" parts of the tensor
$T_1$, as should be granted by the consistency of the Bern-Kosower formalism. Expression
(30) for the form factor $F^{scal}_{T_1}$ of $T_1$ now may be easily identified with the one found in [8]
by covariant perturbation theory *.

Now let us return to the case of the spinor loop, eq. (1). At the two-point level, there
is one additional contraction to compute compared to the scalar case, namely
\[
\langle \psi^\mu_1 (\tau_1) e^{y(\tau_1)} \partial_1 F^{(1)}_{\mu_1 \nu_1} (x_0) \psi_{\mu_1} (\tau_1) \psi_{\mu_2} (\tau_2) e^{y(\tau_2) \partial_2} F^{(2)}_{\mu_2 \nu_2} (x_0) \psi_{\mu_2} (\tau_2) \rangle, \tag{33}
\]
giving a $u$– integral
\[
\frac{1}{4} T^2 \int_0^1 du G_F (1, u_2)^2 [g_{\mu_1 \nu_2} g_{\nu_1 \mu_2} - g_{\mu_1 \mu_2} g_{\nu_1 \nu_2}] e^{-T G_B (u, u_2)} \partial_1 F^{(1)}_{\mu_1 \nu_1} F^{(2)}_{\mu_2 \nu_2}. \tag{34}
\]
Using $G_F (1, u_2) = 1$ and taking the global factor of $-2$ in (1) into account, we obtain the
following relationship between the scalar and spinor two-point form factors:
\[
F^{spin}_2 (\xi) = -2 [F^{scal}_2 (\xi) - \frac{1}{4} f (\xi)]. \tag{35}
\]
In the three-point case, we have two additional contraction possibilities: The first one is
a contraction of
\[
\langle \psi^\mu_1 (y) \psi_{\mu_2} (y \partial_2) F^{(1)}_{\mu_1 \nu_1} \psi_{\mu_2} (\tau_2) e^{y(\tau_2) \partial_2} F^{(2)}_{\mu_2 \nu_2} (x_0) \psi_{\mu_2} (\tau_2) \rangle, \tag{36}
\]
and it simply produces a contribution of $\frac{1}{3} F (\xi_1, \xi_2, \xi_3)$ to $T_1$. The second one,
\[
\langle \psi^\mu_1 (y \partial_1) F^{(1)}_{\mu_1 \nu_1} \psi_{\mu_2} (\tau_2) e^{y(\tau_2) \partial_2} F^{(2)}_{\mu_2 \nu_2} (x_0) \psi_{\mu_2} (\tau_2) \rangle, \tag{37}
\]
contributes to $T_1$ and $T_2$, with a contribution to $T_1$ of $-\frac{1}{3} F (\xi_1, \xi_2, \xi_3)$ and thus cancelling
the first one. The final result for the $T_1$– form factor for spinor loops becomes simply
\[
F^{spin}_{T_1} (\xi_1, \xi_2, \xi_3) = -2 F^{scal}_{T_1} (\xi_1, \xi_2, \xi_3). \tag{38}
\]
The form factors for $T_2$ will be treated in a more detailed publication [10], as well as
those of the cyclic four-point tensor. Now let us rather sketch briefly how this type of
calculations might be done in a manifestly covariant way.

For fixed $x_0$ the Fock–Schwinger gauge [11]
\[
y^\mu A^a_\mu (x_0 + y (\tau)) = 0 \tag{39}
\]
* The fact that superparticle path integrals can be an efficient alternative to heat kernel
methods is well-known from the calculation of index densities [9].
is very convenient for writing down manifestly gauge-invariant expressions for the effective action. In this gauge,

\[ A^\mu_a(x_0 + y) = \int_0^1 d\eta \eta F^a_{\rho \mu}(x_0 + \eta y) y^\rho. \] (40)

Following the proof of [11], \( F^a_{\rho \mu} \) can be covariantly Taylor–expanded as

\[ F^a_{\rho \mu}(x_0 + \eta y) = (e^{\eta y \cdot D})^{ab} F^b_{\rho \mu}(x_0) \] (41)

leading to

\[ A^\mu(x_0 + y) = \int_0^1 d\eta \eta y^\rho e^{\eta y \cdot D} F_{\rho \mu}(x_0) \]

\[ = \frac{1}{2} y^\rho F_{\rho \mu} + \frac{1}{3} y^\nu y^\rho D_{\nu \mu} F_{\rho \mu} + ... \] (42)

Expanding again our starting expression (12) for the scalar effective action in powers of \( g \) and using the Fock-Schwinger gauge relation (42), we arrive at

\[ \Gamma[F] = \int_0^\infty \frac{dT}{T} [4\pi T]^{-\frac{n}{2}} e^{-m^2 T} \int d^D x_0 \sum_{n=0}^{\infty} \frac{(-ig)^n}{n!} T \int_0^{\tau_1 = T} d\tau_2 \int_0^{\tau_2} d\tau_3 \ldots \int_0^{\tau_{n-1}} d\tau_n \]

\[ \times \int_0^1 d\eta_1 \eta_1 \ldots \int_0^1 d\eta_n \eta_n \frac{1}{\eta_1 \ldots \eta_n} \frac{\partial}{\partial D^{(1)}_{\rho_1}} \ldots \frac{\partial}{\partial D^{(n)}_{\rho_n}} \]

\[ \mathcal{N} \int D(\dot{y}^{\mu_1}(\tau_1) e^{\eta_1 y(\tau_1) \cdot D^{(1)}} \ldots \dot{y}^{\mu_n}(\tau_n) e^{\eta_n y(\tau_n) \cdot D^{(n)}} \exp \left[-\int_0^T d\tau \dot{\gamma}^2 \right] \]

\[ F_{\rho_1 \mu_1}^{a_1} \ldots F_{\rho_n \mu_n}^{a_n} \operatorname{Tr}(T^{a_1} \ldots T^{a_n}), \] (43)

where the \( D^{(i)} \) act on \( F_{\rho_i \mu_i}^{a_i} \) and in this sense commute in the prefactor. \( \frac{1}{\eta_i} \frac{\partial}{\partial D^{(i)}_{\rho_i}} \) creates the \( y^{\rho_i} \) of relation (42). Note that the contraction of a certain \( y(\tau_1) \) in \( e^{\eta_1 y(\tau_1) \cdot D^{(1)}} \) with \( y(\tau_k)(k \neq i) \) and the commutation of \( y(\tau_i) D^{(i)} \) factors in the exponential are noncommuting operations in the nonabelian case.

The systematics of the contractions are the same as before. However, the contraction of the exponentials now leads to

\[ \exp \left[-\sum_{i<k} \eta_i \eta_k G_{B}(\tau_i, \tau_k) D^{(i)} D^{(k)} \right], \] (44)

which is symbolical in the case of nonabelian gauge theories: Each polynomial in \( D^{(i)}_{\mu} \) for fixed \( i \) has to be written in all possible orderings, and the resulting expression has to be normalized by the number of possible orderings. This is also the way further factors \( D^{(i)} \) have to be handled.
In nonabelian gauge theory the ordering of $T^a$ matrices and of the various covariant $D^{(i)}$ acting on the $F_{\mu\nu}$ in the ith position requires reduction to a set of standard invariants using $D$-commutators (leading to further $F_{\mu\nu}$’s) and Bianchi identities, an admittedly painful procedure.

Formfactors of $n$ fields $F_{\mu\nu}$ in the covariant formalism just described contain one further integration $\eta_i$ per field. In the case of two fields $F_{\mu\nu}$ we have checked that the additional integrations can be performed and that one arrives at the $\alpha$ – form of ref. [8]. In the case with three $F_{\mu\nu}$ this seems to be a more serious problem and is under consideration.

For application in QCD and in the standard electroweak theory of course besides the complex scalar fields we need gauge fields and fermionic fields in the loop and also Higgs fields in the background. In this context the above representation of the fermions by spinning superparticles in the loop [12] appears to be very economical. However if we want to calculate fluctuation determinants, e.g. for a discussion of radiative corrections to instantons, sphalerons, bounce identities, a different technique seems to have advantages: To calculate the bilinear part of the quantum Lagrangian in a gauge field – Higgs fields background in a convenient quantum gauge (for the background field we can have a separate gauge). The ’t Hooft-Feynman background gauge seems to be most practical - see ref. [13,14]. The corresponding set of operators has the form $-D^2(A) + V(\phi, F_{\mu\nu})$, with $D(A)$ being some collection of covariant derivatives in some gauge group representations and $V$ a general potential matrix built out of background scalar field $\phi$, field strength $F_{\mu\nu}$, and covariant derivatives thereof (in the case of the electroweak theory the main matrix operator depends on $3 \times 3 + 4 = 13$ fields [13]). $\text{LogDet}(-D^2 + V)$ is then evaluated with the methods discussed above. Proceeding as in the pure gauge case, there will be also factors $e^{y(u_i).D^{(i)}}V^{(i)}(\phi, F_{\mu\nu})$ in the trace. The resulting expressions are lengthy but straightforward to write down and to evaluate for a finite number of $V$ and $D$, if a convention for a set of independent operators is fixed. This procedure does not have the cancellations of that of ref. [13] with plane wave insertions in $\text{TrLog}$, and it is within the range of the method to extend their results beyond the third order * (or beyond the 6th order in the case without background field [14]).

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