Integrability of the twistor space for a hypercomplex manifold.

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Introduction

A hyperkähler manifold is by definition a Riemannian manifold equipped with a smooth parallel action of the algebra of quaternions on its tangent bundle. Hyperkähler manifolds were introduced by Calabi in [C] and have since been the subject of much research. They have been shown to possess many remarkable properties and a rich inner structure. (See [Bes], [HKLR] for an overview.) In particular, there is a so-called twistor space associated to every hyperkähler manifold. The twistor space is a complex manifold equipped with some additional structures, and many of the differential-geometric properties of a hyperkähler manifold can be described in terms of holomorphic properties of its twistor space.

Some of the properties of hyperkähler manifolds do not actually depend on the Riemannian metric but only on the quaternion action. In particular, for every manifold equipped with a smooth action of quaternions (or, for brevity, a quaternionic manifold) one can construct an almost complex manifold which becomes the twistor space in the hyperkähler case. Thus it would be very convenient to have a notion of “a hyperkähler manifold without a metric”, in the same sense as a complex manifold is a Kähler manifold without a metric. The analogy with the Kähler case suggests that this would require a certain integrability condition on the quaternionic action, automatic in the Riemannian case. One version of such a condition was suggested in [Bes], but the resulting notion of an integrable quaternionic manifold is too restrictive and excludes many interesting examples.

A more convenient notion is that of a hypercomplex manifold (see [B]). By definition a hypercomplex manifold is a smooth manifold equipped with two integrable anticommuting almost complex structures. (Note that two anticommuting almost complex structures induce an action of the whole
quaternion algebra on the tangent bundle, and their integrability is in fact a condition on the resulting quaternionic manifold.) In this paper we show that this condition is in fact equivalent to the integrability of the almost complex twistor space associated to the quaternionic manifold in question.

Here is a brief outline of the paper. In Section 1 we give the necessary definitions and formulate the result (Theorem 1). In Section 2 we describe a linear-algebraic construction somewhat analogous to the Borel-Weyl localization of finite-dimensional representations of a reductive group. In Section 3 we use this version of localization to prove Theorem 1. The paper is essentially self-contained and does not require any prior knowledge of the theory of hyperkähler manifolds.

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1 Preliminaries.

1.1 Let $\mathbb{H}$ be the algebra of quaternions.

**Definition 1.1** A quaternionic manifold is a smooth manifold $M$ equipped with a smooth action of the algebra $\mathbb{H}$ on the tangent bundle $\Theta(M)$ to $M$.

Let $M$ be a quaternionic manifold. Every algebra embedding $I : \mathbb{C} \to \mathbb{H}$ defines by restriction an almost complex structure on $M$. Call it the induced almost complex structure and denote it by $M_I$.

1.2 The set Maps$(\mathbb{C}, \mathbb{H})$ of all algebra embeddings $\mathbb{C} \to \mathbb{H}$ can be given a natural structure of a complex manifold as follows. The algebra $\mathbb{H} \otimes \mathbb{R} \mathbb{C}$ is naturally isomorphic to the $2 \times 2$-matrix algebra over $\mathbb{C}$. Every algebra embedding $I : \mathbb{C} \to \mathbb{H}$ defines a structure of a 2-dimensional vector space $\mathbb{H}_I$ on $\mathbb{H}$ by means of left multiplication by $I(\mathbb{C})$. It also defines a 1-dimensional subspace $I(\mathbb{C}) \subset \mathbb{H}_I$. The action of $\mathbb{H}$ on itself by right multiplication preserves the complex structure $\mathbb{H}_I$ and extends therefore to an action of the matrix algebra $\mathbb{H} \otimes \mathbb{R} \mathbb{C}$.

Let $\tilde{I} \subset \mathbb{H} \otimes \mathbb{R} \mathbb{C}$ be the annihilator of $I(\mathbb{C}) \subset \mathbb{H}_I$. The ideal $\tilde{I} \subset \mathbb{H}$ is a maximal right ideal, moreover, we have $\mathbb{H}_I = \mathbb{H} \otimes \mathbb{C}/\tilde{I}$. It is easy to check that every maximal right ideal of $\mathbb{H} \otimes \mathbb{R} \mathbb{C}$ can be obtained in this way. This establishes a bijection between Maps$(\mathbb{C}, \mathbb{H})$ and the set of maximal right
ideals in $\mathbb{H} \otimes_{\mathbb{R}} \mathbb{C}$. It is well-known that this set coincides with the complex projective line $\mathbb{C}P^1$.

1.3 Let now $M$ be a quaternionic manifold, and let $X = M \times \mathbb{C}P^1$ be the product of $M$ with the smooth manifold underlying $\mathbb{C}P^1$. For every point $x = m \times I \in M \times \mathbb{C}P^1$ the tangent bundle $T_xX$ decomposes canonically as $T_xX = T_mM \oplus T_I\mathbb{C}P^1$. Define an endomorphism $I : T_I\mathbb{C}P^1 \to T_I\mathbb{C}P^1$ as follows: it acts as the usual complex structure map on $T_I\mathbb{C}P^1$, and on $T_mM$ it acts as the induced complex structure map $I : T_mM \to T_mM$ corresponding to $I \in \mathbb{C}P^1 \cong \text{Maps}(\mathbb{C}, \mathbb{H})$. The map $I$ obviously depends smoothly on the point $x \in X$ and satisfies $I^2 = -1$. Therefore, it defines an almost complex structure on the smooth manifold $X$.

**Definition 1.2** The almost complex manifold $X = M \times \mathbb{C}P^1$ is called the **twistor space** of the quaternionic manifold $M$.

1.4 The twistor space $X$ has the following obvious properties.

i The canonical projection $\pi : X \to \mathbb{C}P^1$ is compatible with the almost complex structures.

ii For every point $m \in M$ the embedding $\tilde{m} = m \times \text{id} : \mathbb{C}P^1 \to M \times \mathbb{C}P^1 = X$ is compatible with the almost complex structures.

The embedding $\tilde{m} : \mathbb{C}P^1 \to X$ will be called the **twistor line** corresponding to the point $m \in M$.

1.5 The goal of this paper is to prove the following theorem.

**Theorem 1** Let $M$ be a quaternionic manifold, and let $X$ be its twistor space. The following conditions are equivalent:

i For two algebra maps $I, J : \mathbb{C} \to \mathbb{H}$ such that $I \neq J$ and $\overline{T} \neq J$ the induced almost complex structures $M_I, M_J$ on $M$ are integrable.

ii For every algebra map $I : \mathbb{C} \to \mathbb{H}$ the induced almost complex structure $M_I$ on $M$ is integrable.

iii The almost complex structure on $X$ is integrable.

The quaternionic manifold satisfying (i) is called **hypercomplex**. Note that (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i) is obvious, so it suffices to prove (i) $\Rightarrow$ (iii).
2 Localization of $\mathbb{H}$-modules.

2.1 We begin with some linear algebra.

**Definition 2.1** A *quaternionic vector space* $V$ is a left module over the algebra $\mathbb{H}$.

Let $V$ be a quaternionic vector space. Every algebra map $I : \mathbb{C} \to \mathbb{H}$ defines by restriction a complex vector space structure on $V$, which we will denote by $V_I$. Note that $\mathbb{H}$ is naturally a left module over itself. The associated 2-dimensional complex vector space $\mathbb{H}_I$ is the same as in 1.2, and we have $V_I = \mathbb{H}_I \otimes \mathbb{H} V$.

2.2 Let SB be the Severi-Brauer variety associated to the algebra $\mathbb{H}$, that is, the variety of maximal right ideals in $\mathbb{H}$. By definition SB is a real algebraic variety, an $\mathbb{R}$-twisted form of the complex projective line $\mathbb{C}P^1$. It is also equipped with a canonical maximal right ideal $\mathcal{I} \subset \mathbb{H} \otimes \mathcal{O}$ in the flat coherent algebra sheaf $\mathbb{H} \otimes \mathcal{O}$ on SB.

Let $V$ be a quaternionic vector space. Consider the flat coherent sheaf $V \otimes \mathcal{O}$ on SB of right $\mathbb{H} \otimes \mathcal{O}$-modules, and let

$$V_{\text{loc}} = V \otimes \mathcal{O} / \mathcal{I} : V \otimes \mathcal{O}$$

be its quotient by the right ideal $\mathcal{I}$. Call the sheaf $V_{\text{loc}}$ the *localization* of the quaternionic vector space $V$. The localization is functorial in $V$ and gives a full embedding of the category of quaternionic vector spaces into the category of flat coherent sheaves on SB.

Say that a flat coherent sheaf $\mathcal{E}$ on SB is of *weight* $k$ if the sheaf $\mathcal{E} \otimes \mathbb{C}$ on the complex projective line $\mathbb{C}P^1 \cong \text{SB} \otimes \mathbb{C}$ is isomorphic to a sum of several copies of the line bundle $\mathcal{O}(k)$. The essential image of the localization functor is the full subcategory of flat coherent sheaves of weight 1.

2.3 The set $\mathbb{C}P^1 \cong \text{SB}(\mathbb{C})$ of $\mathbb{C}$-valued points of the variety SB was canonically identified in 1.2 with the set $\text{Maps}(\mathbb{C}, \mathbb{H})$ of algebra maps $\mathbb{C} \to \mathbb{H}$. Let $I : \mathbb{C} \to \mathbb{H}$ be an algebra map, and let $\hat{I} \in \mathbb{C}P^1$ be the corresponding $\mathbb{C}$-valued point of the variety SB or, equivalently, the maximal right ideal in the algebra $\mathbb{H} \otimes \mathbb{C}$.

**Lemma 2** Consider a quaternionic vector space $V$, and let $V_{\text{loc}}$ be its localization. The fiber of the sheaf $V_{\text{loc}}$ at the point $\hat{I} \in \mathbb{C}P^1$ is canonically isomorphic to the vector space $V_I$. 
Proof. Indeed,

\[ V_I \cong \mathbb{H}_I \otimes_{\mathbb{H}} V \cong \mathbb{H}_I \otimes_{\mathbb{H}} \mathbb{C} V \otimes \mathbb{C} \cong V \otimes \mathbb{C} / \tilde{I} \cdot V \otimes \mathbb{C} = V_{\text{loc}}|_{\tilde{I}}, \]

and all the isomorphisms are canonical. \( \square \)

2.4 Consider now the set \( \mathbb{C}P^1 \cong \text{Maps}(\mathbb{C}, \mathbb{H}) \) as the smooth complex-analytic variety, and let \( \mathcal{V} \) be the trivial bundle on \( \mathbb{C}P^1 \) with the fiber \( V \otimes_{\mathbb{R}} \mathbb{C} \). Since \( \mathcal{V} \) is trivial, we have a canonical holomorphic structure operator \( \tilde{\partial} : \mathcal{V} \to \mathcal{A}^{0,1}(\mathcal{V}) \) from \( \mathcal{V} \) to the bundle \( \mathcal{A}^{0,1}(\mathcal{V}) \) of \( \mathcal{V} \)-valued \((0,1)\)-forms on \( \mathbb{C}P^1 \).

The action of \( \mathbb{H} \) on \( V \) induces an operator \( \mathcal{I} : \mathcal{V} \to \mathcal{V} \) which acts as \( I(\sqrt{-1}) \) on the fiber \( V \) of \( \mathcal{V} \) at a point \( I \in \text{Maps}(\mathbb{C}, \mathbb{H}) \). The operator \( \mathcal{I} \) obviously depends smoothly on the point \( I \). It satisfies \( I^2 = -1 \) and induces therefore a smooth “Hodge type” decomposition \( \mathcal{V} = \mathcal{V}^{1,0} \oplus \mathcal{V}^{0,1} \).

Lemma 3 The quotient \( \mathcal{V}^{1,0} \) is compatible with the holomorphic structure \( \tilde{\partial} \) on \( \mathcal{V} \). In other words, there exists a unique holomorphic structure operator \( \tilde{\partial} : \mathcal{V}^{1,0} \to \mathcal{A}^{0,1}(\mathcal{V}^{1,0}) \) making the diagram

\[
\begin{array}{ccc}
\mathcal{V} & \longrightarrow & \mathcal{V}^{1,0} \\
\tilde{\partial} \downarrow & & \tilde{\partial} \downarrow \\
\mathcal{A}^{0,1}(\mathcal{V}) & \longrightarrow & \mathcal{A}^{0,1}(\mathcal{V}^{1,0})
\end{array}
\]

commutative.

Proof. This follows directly from Lemma 2 by the usual correspondence between flat coherent sheaves and holomorphic bundles on the underlying complex-analytic variety. \( \square \)

3 Proof of the theorem.

3.1 Let \( Z \) be a smooth almost complex manifold. Let \( \mathcal{A}^1(Z, \mathbb{C}) \) be the complexified cotangent bundle to \( Z \), and let \( \mathcal{A}(Z, \mathbb{C}) \) be its exterior algebra. The almost complex structure on \( Z \) induces the Hodge type decomposition \( \mathcal{A}^i(Z, \mathbb{C}) = \bigoplus_{p+q=i} \mathcal{A}^{p,q}(Z) \). Recall that the Nijenhuis tensor \( N \) of the almost complex manifold \( Z \) is the composition

\[ N = P \circ d_Z \circ i : \mathcal{A}^{1,0}(Z) \to \mathcal{A}^1(Z, \mathbb{C}) \to \mathcal{A}^2(Z, \mathbb{C}) \to \mathcal{A}^{0,2}(Z), \]
where $d_Z$ is the de Rham differential, $i : \mathcal{A}^{0,0}(Z) \to \mathcal{A}(Z, \mathbb{C})$ is the canonical embedding, and $P : \mathcal{A}(Z, \mathbb{C}) \to \mathcal{A}^{0,0}(Z)$ is the canonical projection.

Recall also that the almost complex manifold $Z$ is called integrable if its Nijenhuis tensor $N_Z : \mathcal{A}^{1,0}(Z) \to \mathcal{A}^{0,2}(Z)$ vanishes.

**3.2** We can now begin the proof of Theorem 1. First we will prove a sequence of preliminary lemmas. Let $M$ be a smooth quaternionic manifold, and let $X$ be its twistor space. Since by definition $X = M \times \mathbb{C}P^1$ as a smooth manifold, the cotangent bundle $\mathcal{A}^{1}(X)$ decomposes canonically as

$$\mathcal{A}^{1}(X) = \sigma^* \mathcal{A}^{1}(M) \oplus \pi^* \mathcal{A}^{1}(\mathbb{C}P^1),$$

(3.1)

where $\sigma : X \to M$, $\pi : X \to \mathbb{C}P^1$ are the canonical projections.

The almost complex structure $\mathcal{I}$ on $X$ preserves the decomposition (3.1). Therefore (3.1) induces decompositions

$$\mathcal{A}^{1,0}(X) = \mathcal{A}^{1,0}_{M}(X) \oplus \mathcal{A}^{1,0}_{\mathbb{C}P^1}(X),$$

$$\mathcal{A}^{0,2}(X) = \mathcal{A}^{0,2}_{M}(X) \oplus \mathcal{A}^{0,2}_{\mathbb{C}P^1}(X),$$

and, consequently, a decomposition

$$\mathcal{A}^{0,2}(X) = \left(\mathcal{A}^{0,1}_{M}(X) \otimes \mathcal{A}^{0,1}_{\mathbb{C}P^1}(X)\right) \oplus \mathcal{A}^{0,2}_{M}(X).$$

(3.2)

(Note that $\mathcal{A}^{0,1}_{\mathbb{C}P^1}(X)$ is of rank 1, therefore $\mathcal{A}^{0,2}_{\mathbb{C}P^1}(X)$ vanishes). Moreover, since the projection $\pi : X \to \mathbb{C}P^1$ is compatible with the almost complex structures, we have canonical isomorphisms

$$\mathcal{A}^{p,q}_{\mathbb{C}P^1}(X) \cong \pi^* \mathcal{A}^{p,q}(\mathbb{C}P^1).$$

**3.3** Let $N_X : \mathcal{A}^{1,0} \to \mathcal{A}^{0,2}(X)$ be the Nijenhuis tensor of the almost complex manifold $X$. We begin with the following.

**Lemma 4** The restriction of the Nijenhuis tensor $N_X$ to the subbundle $\pi^* \mathcal{A}^{1,0}(\mathbb{C}P^1) \cong \mathcal{A}^{1,0}_{\mathbb{C}P^1}(X) \subset \mathcal{A}^{1,0}(X)$ vanishes.

**Proof.** Indeed, since the map $\pi : X \to \mathbb{C}P^1$ is compatible with the almost complex structures, the diagram

$$\begin{array}{ccc}
\pi^* \mathcal{A}^{1,0}(\mathbb{C}P^1) & \xrightarrow{\mathcal{N}} & \mathcal{A}^{1,0}(X) \\
\downarrow & & \downarrow N_X \\
\pi^* \mathcal{A}^{0,2}(\mathbb{C}P^1) & \xrightarrow{\mathcal{N}} & \mathcal{A}^{0,2}(X)
\end{array}$$
is commutative, and \( A^{0,2}(\mathbb{C}P^1) \) vanishes. □

Therefore the Nijenhuis tensor \( N_X \) factors through a map

\[
N_X : A^{1,0}_M(X) \to A^{0,2}(X).
\]

3.4 Let now \( N_X = N_1 + N_2 \) be the decomposition of the Nijenhuis tensor with respect to \((3.2)\), so that \( N_1 \) is a map

\[
N_1 : A^{1,0}_M(X) \to A^{0,1}(X) \otimes \pi^* A^{0,1}(\mathbb{C}P^1),
\]

and \( N_2 \) is a map \( N_2 : A^{1,0}_M(X) \to A^{0,2}_M(X) \).

**Lemma 5** The component \( N_1 \) of the Nijenhuis tensor \( N_X \) vanishes.

**Proof.** It suffices to prove that for every point \( m \in M \) the restriction \( \tilde{m}^* N_1 \) of \( N_1 \) onto the corresponding twistor line \( \tilde{m} : \mathbb{C}P^1 \to X \) vanishes. Consider a point \( m \in M \). Let \( i : \tilde{m}^* A^{1,0}_M(X) \to \tilde{m}^* A^{1}_M(X, \mathbb{C}) \) be the canonical embedding, and let

\[
P : A^{0,1}(\mathbb{C}P^1) \otimes \tilde{m}^* A^{1}_M(X, \mathbb{C}) \to A^{0,1}(\mathbb{C}P^1) \otimes \tilde{m}^* A^{0,1}_M(X)
\]

be the canonical projection. Since the twistor line \( \tilde{m} : \mathbb{C}P^1 \to X \) is compatible with the almost complex structures, we have \( \tilde{m}^* \pi^* A^{0,1}(\mathbb{C}P^1) \cong A^{0,1}(\mathbb{C}P^1) \), and

\[
\tilde{m}^* N_1 = P \circ \tilde{\partial} \circ i : \tilde{m}^* A^{1,0}_M(X) \to \tilde{m}^* A^{1}_M(X, \mathbb{C}) \to \tilde{m}^* A^{1}_M(X, \mathbb{C}) \otimes A^{0,1}(\mathbb{C}P^1) \to \tilde{m}^* A^{0,1}_M(X) \otimes A^{0,1}(\mathbb{C}P^1),
\]

where \( \tilde{\partial} : \tilde{m}^* A^{1}_M(X, \mathbb{C}) \to \tilde{m}^* A^{1}_M(X, \mathbb{C}) \otimes A^{0,1}(\mathbb{C}P^1) \) is the trivial holomorphic structure operator on the constant bundle \( \tilde{m}^* A^{1}_M(X, \mathbb{C}) \).

Let \( V = T_m M \) be the tangent space to the manifold \( M \) at the point \( m \). Since \( M \) is quaternionic, \( V \) is canonically a quaternionic vector space. Let \( V \) and \( V^{1,0} \) be as in Lemma 3, and let \( \nu^* \) and \( (V^{1,0})^* \) be the dual bundles on \( \mathbb{C}P^1 \). We have canonical bundle isomorphisms

\[
\nu^* \cong \tilde{m}^* A^{1}_M(X, \mathbb{C}) \quad \quad (V^{1,0})^* \cong \tilde{m}^* A^{1,0}_M(X)
\]

compatible with the natural embeddings. By the statement dual to Lemma 3, there exists a holomorphic structure operator \( \tilde{\partial} : \tilde{m}^* A^{1,0}_M(X) \to \tilde{m}^* A^{1,0}_M(X) \otimes A^{0,1}(\mathbb{C}P^1) \) making the diagram

\[
\begin{array}{ccc}
\tilde{m}^* A^{1,0}_M(X) & \xrightarrow{i} & \tilde{m}^* A^{1}_M(X, \mathbb{C}) \\
\tilde{\partial} \downarrow & & \tilde{\partial} \downarrow \\
\tilde{m}^* A^{1,0}_M(X) \otimes A^{0,1}(\mathbb{C}P^1) & \xrightarrow{i \otimes \text{id}} & \tilde{m}^* A^{1}_M(X, \mathbb{C}) \otimes A^{0,1}(\mathbb{C}P^1)
\end{array}
\]
commutative. Therefore $N_1 = P \circ \partial \circ i = P \circ (i \otimes \text{id}) \circ \bar{\partial}$. But $P \circ (i \otimes \text{id}) = 0$, hence $N_1$ vanishes. □

3.5 We can now prove Theorem [1]. As we have already proved, the Nijenhuis tensor $N_X$ of the twistor space $X$ reduces to a bundle map

$$N_X : \mathcal{A}^1_{M}^{0}(X) \to \mathcal{A}^0_{M}^{2}(X).$$

This map vanishes identically if and only if for every point $m \in M$ the restriction $\tilde{m}^*N_X$ of $N_X$ to the twistor line $\tilde{m} : \mathbb{C}P^1 \to X$ vanishes.

Consider a point $m \in M$. By Lemma [3] the restriction $\tilde{m}^*\mathcal{A}^1_{M}^{0}(X)$ carries a natural holomorphic structure, and it is a holomorphic bundle of weight $-1$ with respect to this structure (in the sense of [2.2]). Consequently, the bundle $\tilde{m}^*\mathcal{A}^0_{M}^{2}(X)$ is a holomorphic bundle of weight 2. Moreover, the Nijenhuis tensor

$$\tilde{m}^*N_X = P \circ d \circ i : \tilde{m}^*\mathcal{A}^1_{M}^{0}(X) \to \mathcal{A}^0_{M}^{2}(X)$$

is a holomorphic bundle map.

For every algebra map $I \in \text{Maps}(\mathbb{C}, \mathbb{H}) \cong \mathbb{C}P^1$, the restriction of the Nijenhuis tensor $N_X$ to a fiber $M \times I \in M \times \mathbb{C}P^1 = X$ of the projection $\pi : X \to \mathbb{C}P^1$ is the Nijenhuis tensor for the induced almost complex structure $M_I$ on the manifold $M$. Assume that Theorem [1] (i) holds. Then at least four distinct induced almost complex structures on $M$ corresponding to $I, J, T, \mathcal{J} \in \text{Maps}(\mathbb{C}, \mathbb{H})$ are integrable. Consequently, the Nijenhuis tensor $N_X$ vanishes identically on fibers of the projection $\pi : X \to \mathbb{C}P^1$ over at least four distinct points of $\mathbb{C}P^1$. Therefore the restriction $\tilde{m}^*N_X$ has at least four distinct zeroes. But as we have proved, $\tilde{m}^*N_X$ is a holomorphic map from a bundle of weight $-1$ to a bundle of weight 2. Therefore it vanishes identically. Hence the almost complex manifold $X$ is integrable, which finishes the proof of Theorem [1].

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