On Cheng’s Eigenvalue Comparison Theorems

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Abstract

We prove Cheng’s eigenvalue comparison theorems [4] for geodesic balls within the cut locus under weaker geometric hypothesis, Theorems (1.1, 3.1, 3.2) and we also show that there are certain geometric rigidity in case of equality of the eigenvalues. This rigidity becomes isometric rigidity under upper sectional curvature bounds or lower Ricci curvature bounds. We construct examples of smooth metrics showing that our results are true extensions of Cheng’s theorem. We also construct a family of complete smooth metrics on \( \mathbb{R}^n \) non-isometric to the constant sectional curvature \( \kappa \) metrics of the simply connected space forms \( \mathbb{M}(\kappa) \) such that the geodesic balls \( B_{\mathbb{R}^n}(r) \) and \( B_{\mathbb{M}(\kappa)}(r) \) have the same first eigenvalue and the geodesic spheres \( \partial B_{\mathbb{R}^n}(s) \) and \( \partial B_{\mathbb{M}(\kappa)}(s) \), \( 0 < s \leq r \), have the same mean curvatures. In the end we construct examples of Riemannian manifolds \( M \) with arbitrary topology with positive fundamental tone \( \lambda^* > 0 \) that generalize Veeravalli’s examples, [6].

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1 Introduction

Let \( M \) be a complete \( n \)-dimensional Riemannian manifold and denote by \( B_M(p, r) \) the geodesic ball with center \( p \) and radius \( r \) and by \( \lambda_1(B_M(p, r)) \) the first Dirichlet eigenvalue of \( B_M(p, r) \). Cheng in [4], using a result of Barta [1], proved that if the sectional curvature of \( M \) is bounded above \( K_M \leq \kappa \) and \( r < \min\{\text{inj}(p), \pi/\sqrt{\kappa}\} \), \( (\pi/\sqrt{\kappa} = \infty \) if \( \kappa \leq 0 \), then \( \lambda_1(B_M(p, r)) \geq \lambda_1(B_{\mathbb{M}(\kappa)}(r)) \), where \( \mathbb{M}(\kappa) \) denote the simply connected space form of constant sectional curvature \( \kappa \). Cheng also in [4] proved that if the Ricci curvature of \( M \) is bounded below \( \text{Ric}_M \geq (n-1)\kappa \) then the reverse inequality \( \lambda_1(B_M(p, r)) \leq \lambda_1(B_{\mathbb{M}(\kappa)}(r)) \) holds for \( r < \text{inj}(p) \). In [5], choosing a suitable test function for the Rayleigh quotient, Cheng improved this later inequality proving that if \( \text{Ric}_M \geq (n-1)\kappa \) then \( \lambda_1(B_M(p, r)) \leq \lambda_1(B_{\mathbb{M}(\kappa)}(r)) \) for every \( r > 0 \),

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with equality holding (for some $r$) if and only if the geodesic balls $B_M(p,r)$ and $B_{M(\kappa)}(r)$ are isometric and $r < \min_j(p)$. That raises the questions of whether it is possible to prove Cheng’s lower eigenvalue inequality beyond the cut locus and show that the geodesic balls are isometric if they have the same first eigenvalue. These questions were addressed in [2] and proven to be true, (under upper sectional curvature bounds), provided that the $(n-1)$-Hausdorff measure $\mathcal{H}^{n-1}(Cut(p) \cap B_M(p,r)) = 0$, where $Cut(p)$ is the cut locus of $p$. In this paper we apply our version of Barta’s theorem (Theorem 2.3) to prove an extension of Cheng’s lower and upper eigenvalue inequalities for geodesic balls within the cut locus (of its center) without sectional or Ricci curvature bounds. These inequalities have a weaker form of geometric rigidity in the equality case and we show with family of examples that this rigidity is all we can expect for.

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To state our result, consider $B_M(p,r) \subset M$ and $B_{M(\kappa)}(r) \subset M(\kappa)$ geodesic balls within the cut locus and let $(t, \theta) \in (0, r) \times S^{n-1}$ be geodesic coordinates for $B_M(p,r)$ and $B_{M(\kappa)}(r)$. Let $H_M(t, \theta)$ and $H_{M(\kappa)}(t, \theta) = H_{M(\kappa)}(t)$ be respectively the mean curvatures of the distance spheres $\partial B_M(p,t)$ and $\partial B_{M(\kappa)}(t)$ at the point $(t, \theta)$ with respect to the unit vector field $-\partial/\partial t$. Our first result is the following theorem.

**Theorem 1.1** If $H_M(s, \theta) \geq H_{M(\kappa)}(s)$, for all $s \in (0, r]$ and all $\theta \in S^{n-1}$ then

$$\lambda_1(B_M(p,r)) \geq \lambda_1(B_{M(\kappa)}(r)).$$

(1)

If $H_M(s, \theta) \leq H_{M(\kappa)}(s)$, for all $s \in (0, r]$ and all $\theta \in S^{n-1}$ then

$$\lambda_1(B_M(p,r)) \leq \lambda_1(B_{M(\kappa)}(r)).$$

(2)

Equality in (1) or (2) holds if and only if $H_M(s, \theta) = H_{M(\kappa)}(s)$, $\forall s \in (0, r]$ and $\forall \theta \in S^{n-1}$.

Observe that the hypotheses of Theorem 1.1 are implied by an upper sectional curvature bound $K_M \leq \kappa$ and a lower Ricci curvature bound $Ric_M \geq (n-1)\kappa$ respectively. On the other hand we construct examples of smooth metrics on $\mathbb{R}^n = [0, \infty) \times S^{n-1}$ such that the radial sectional curvatures is bounded below $K(x)(\partial t, v) \geq \kappa$ outside a compact set $(x \in \mathbb{R}^n \setminus B_{\mathbb{R}^n}(1))$ but $H_M(s, \theta) \geq H_{M(\kappa)}(s)$, for all $s \in (0, \infty)$ and all $\theta \in S^{n-1}$, see example 4.1. This shows that Theorem 1.1 is a true extension of Cheng's eigenvalue comparison theorem (within the cut locus). The rigidity in case of equality of the eigenvalues, $(H_M(s, \theta) = H_{M(\kappa)}(s), \forall s \in (0, r]$ and $\forall \theta \in S^{n-1}$), implies that the balls $B_M(p,r)$ and $B_{M(\kappa)}(r)$ are isometric if we have that $K_M \leq \kappa$ or $Ric_M \geq (n-1)\kappa$. Moreover, if the metric of $B_M(p,r)$ is expressed in geodesic coordinates by $dt^2 + f^2(t)d\theta^2$, $f(0) = 0$, $f'(0) = 1$, $f(t) > 0$ for $t > 0$ then the rigidity (even without curvature bounds) also implies that the balls $B_M(p,r)$ and $B_{M(\kappa)}(r)$ are isometric, see Remark 4.2. This is the case if the the dimension of $M$ is two. On the other hand we also construct a family of complete smooth metrics $g(\kappa)$ on $\mathbb{R}^n$, $\kappa < 0$ such that $g(\kappa)$ is non isometric to the constant sectional curvature metric of $M(\kappa)$ but the geodesic balls $B_g(\kappa)(r)$, and $B_{M(\kappa)}(r)$ have the same first eigenvalue $\lambda_1(B_{M(\kappa)}(r))$ and their geodesic spheres of same radius have the same mean curvatures, see examples 4.3. These examples show that the rigidity stated in Theorem 1.1 in general is all we can expect without curvature bounds. The proof we present for Theorem 1.1 in fact proves more, we have few generalizations in section 3 (see Theorems 3.1, 3.2). We also generalize Veeravalli’s examples [5], see Theorem 3.3.
2 Preliminaries

A powerful tool to obtain lower bounds for the first Dirichlet eigenvalue of smooth bounded domains in Riemannian manifolds is the following theorem proved by J. Barta in [1].

**Theorem 2.1 (Barta)** Let $\Omega \subset M$ be a domain with compact closure and nonempty smooth boundary $\partial \Omega$. Let $\lambda_1(\Omega)$ be the first Dirichlet eigenvalue of $\Omega$. Let $f \in C^2(\Omega) \cap C^0(\overline{\Omega})$ with $f > 0$ in $\Omega$ and $f|\partial \Omega = 0$. Then

$$\sup_{\Omega} \left( -\frac{\Delta f}{f} \right) \geq \lambda_1(\Omega) \geq \inf_{\Omega} \left( -\frac{\Delta f}{f} \right). \quad (3)$$

**Remark 2.2** The first observation is that to prove the lower inequality in (2) it is necessary only to have that $f > 0$ in $\Omega$. A second observation is that each of the inequalities (3) is strict unless $f$ is a first eigenfunction of $\Omega$. This observation although trivial is essential in the proof of the rigidity statement in Theorem (1.1) and it seems to have passed unobserved by Cheng.

For arbitrary open sets $\Omega$, we proved in [2] the following extension of Barta’s Theorem that gives lower bounds for fundamental tone $\lambda^*(\Omega)$. Recall that the fundamental tone $\lambda^*(\Omega)$ of an open set $\Omega$ is given by

$$\lambda^*(\Omega) = \inf \left\{ \frac{\int_{\Omega} |\nabla f|^2}{\int_{\Omega} f^2}, f \in L^2_{1,0}(\Omega), f \not\equiv 0 \right\},$$

where $L^2_{1,0}(\Omega)$ is the completion of $C^\infty_0(\Omega)$ with respect to the norm $\|\varphi\|^2_\Omega = \int_{\Omega} \varphi^2 + \int_{\Omega} |\nabla \varphi|^2$.

**Theorem 2.3** Let $\Omega \subset M$ be an open subset of Riemannian manifold. Then

$$\lambda^*(\Omega) \geq \sup_{X(\Omega)} \{\inf_{\Omega} (\text{div} X - |X|^2)\}, \quad (4)$$

where $X(\Omega)$ is the set of all vector fields $X$ in $\Omega$ such that $\int_{\Omega} \text{div}(fX) = 0$ for all $f \in C^\infty_0(\Omega)$. If $\Omega$ is a relatively compact open set with smooth boundary then

$$\lambda_1(\Omega) = \sup_{X(\Omega)} \{\inf_{\Omega} (\text{div} X - |X|^2)\}. \quad (5)$$

Both results (Barta’s Theorem and Theorem (2,3)) coincides in bounded domains with smooth boundaries, but the vector field aspect of this version reveal the role of the mean curvatures of the distance spheres in the comparisons of eigenvalues.
2.1 Proof of Theorem 1.1

Let \((t, \theta) \in (0, r] \times S^{n-1}\) be geodesic coordinates for \(B_M(p, r)\) and \(B_{M(\kappa)}(r)\) and \(u : B_{M(\kappa)}(r) \to \mathbb{R}\) be a positive first Dirichlet eigenfunction. It is well known \(u\) is radial function, i.e. \(u(t, \theta) = u(t)\) and \(u'(t) \leq 0\). Observe that \(u(t, \theta) = u(t)\) also defines a smooth function on \(B_M(p, r)\). Now, consider vector fields \(X_1\) on \(B_M(p, r)\) and \(X_2\) on \(B_{M(\kappa)}(r)\) given by

\[
X_1(t, \theta) = -\frac{u'(t)}{u(t)} \cdot \frac{\partial_1}{\partial t}(t, \theta),
\]

\[
X_2(t, \theta) = -\frac{u'(t)}{u(t)} \cdot \frac{\partial_2}{\partial t}(t, \theta).
\]

Here \(\frac{\partial_1}{\partial t}\) and \(\frac{\partial_2}{\partial t}\) are the radial vector fields in \(B_M(p, r)\) and \(B_{M(\kappa)}(r)\) respectively. From now on let us write \(B_M(r)\) instead \(B_M(p, r)\) for simplicity of notation. Now we have that

\[
-\frac{\Delta_M u}{u} = \text{div}_M X_1 - |X_1|^2 = \text{div}_M X_1 - \text{div}_{M(\kappa)} X_2 + |X_2|^2 - |X_1|^2 + \text{div}_{M(\kappa)} X_2 - |X_2|^2
\]

\[
= \text{div}_M X_1 - \text{div}_{M(\kappa)} X_2 - \frac{\Delta_{M(\kappa)} u}{u}
\]

\[
= \text{div}_M X_1 - \text{div}_{M(\kappa)} X_2 + \lambda_1(B_{M(\kappa)}(r)),
\]

(7)

since \(\text{div}_{M(\kappa)} X_2 - |X_2|^2 = -\frac{\Delta_{M(\kappa)} u}{u} = \lambda_1(B_{M(\kappa)}(r))\) and \(|X_1|^2 = |X_2|^2\).

By Theorem (2.1) or (2.3) and by identity (7) we have that

\[
\lambda_1(B_M(r)) \geq \inf_{(t, \theta)} (\text{div}_M X_1 - |X_1|^2) \geq \inf_{(t, \theta)} [\text{div}_M X_1 - \text{div}_{M(\kappa)} X_2] + \lambda_1(B_{M(\kappa)}(r))
\]

(8)

Since \(B_M(r)\) is a smooth domain we can apply Barta’s Theorem and using identity (7) we have that

\[
\lambda_1(B_M(r)) \leq \sup_{(t, \theta)} [\text{div}_M X_1 - |X_1|^2] \leq \sup_{(t, \theta)} [\text{div}_M X_1 - \text{div}_{M(\kappa)} X_2] + \lambda_1(B_{M(\kappa)}(r))
\]

(9)

We will associate the difference \(\text{div}_M X_1 - \text{div}_{M(\kappa)} X_2\) to the mean curvature of the distance spheres through the following well known lemma.

Lemma 2.4 Let \(M \hookrightarrow \overline{M}\) be a smooth hypersurface. Let \(X\) be a smooth vector field on \(\overline{M}\). Then at \(x \in M\) we have that

\[
\text{div}_M X(x) = \text{div}_M X^t(x) - \langle X, \vec{H} \rangle(x) + \langle \nabla_\eta X, \eta \rangle(x),
\]

(10)

where \(X^t\) is the orthogonal projection of \(X\) onto the tangent space \(T_x M\), \(\vec{H}\) is the mean curvature vector of \(M\) at \(x\), \(\nabla\) is the Levi-Civita connection of \(\overline{M}\) and \(\eta \in T_x M^\perp\).
Using this lemma we can compute $\text{div}_M X_1 - \text{div}_{\mathbb{M}(\kappa)} X_2$ at points of $B_M(r)$ and of $B_{\mathbb{M}(\kappa)}(r)$ with the same coordinates $(t, \theta)$.

$$\text{div}_M X_1 - \text{div}_{\mathbb{M}(\kappa)} X_2 = -\langle X_1, \vec{H}_M \rangle_M + \langle X_2, \vec{H}_{\mathbb{M}(\kappa)} \rangle_{\mathbb{M}(\kappa)}$$

$$+ \left\langle \vec{\nabla}_{\partial_1/\partial t} X_1, \frac{\partial_1}{\partial t} \right\rangle_M - \left\langle \vec{\nabla}_{\partial_2/\partial t} X_2, \frac{\partial_2}{\partial t} \right\rangle_{\mathbb{M}(\kappa)}$$

$$= (-u'/u)(H_M - H_{\mathbb{M}(\kappa)}) + (u'/u)' - (u'/u)'$$

(11)

Since

$$\left\langle \vec{\nabla}_{\partial_1/\partial t} X_1, \frac{\partial_1}{\partial t} \right\rangle_M = \left\langle \vec{\nabla}_{\partial_2/\partial t} X_2, \frac{\partial_2}{\partial t} \right\rangle_{\mathbb{M}(\kappa)} = (u'/u)'$$

and $\vec{H}_M = -H_M \cdot \partial_1/\partial t$ and $\vec{H}_{\mathbb{M}(\kappa)} = -H_{\mathbb{M}(\kappa)} \cdot \partial_2/\partial t$. Hence

$$\text{div}_M X_1 - \text{div}_{\mathbb{M}(\kappa)} X_2 = (-u'/u)(H_M - H_{\mathbb{M}(\kappa)}).$$

(12)

Now recall that $(-u'/u) \geq 0$. If $(H_M - H_{\mathbb{M}(\kappa)}) \geq 0$ then (8) and (12) implies (1). Likewise, if $(H_M - H_{\mathbb{M}(\kappa)}) \leq 0$ then (3) and (12) implies (2). To treat the equality case observe that the proof we presented was nothing but giving a suitable positive function $u$ on $B_M(r)$ then applying Barta’s Theorem to find the lower bound for $\inf_{B_M(r)} (\Delta_M u/u) \geq \lambda_1(B_{\mathbb{M}(\kappa)}(r))$. Now, suppose that $\lambda_1(B_M(r)) = \lambda_1(B_{\mathbb{M}(\kappa)}(r))$ then (3) implies that $\lambda_1(B_M(r)) = \inf_{(t,\theta)}(\text{div}_M X_1 - |X_1|^2)$ and $\inf_{(t,\theta)} [\text{div}_M X_1 - \text{div}_{\mathbb{M}(\kappa)} X_2] = 0$. The Remark (2.2) says that the infimum (supremum) in (3) is achieved by a positive function $f$ if and only if the function $f$ is an eigenfunction. Thus $\lambda_1(B_M(r)) = \inf_{(t,\theta)}(\text{div}_M X_1 - |X_1|^2)$ is saying that the function $u : B_M(r) \to \mathbb{R}$ is a positive first eigenfunction of $B_M(r)$, in particular that $\lambda_1(B_M(r)) = \text{div}_M X_1 - |X_1|^2$. From (7) we have that $\text{div}_M X_1 - \text{div}_{\mathbb{M}(\kappa)} X_2 = \lambda_1(B_M(r)) - \lambda_1(B_{\mathbb{M}(\kappa)}(r)) = 0$. On the other hand, $\text{div}_M X_1 - \text{div}_{\mathbb{M}(\kappa)} X_2 = (-u'/u)(H_M - H_{\mathbb{M}(\kappa)})$ and $u'(t) = 0$ if and only if $t = 0$. Therefore we have that $H_M(t, \theta) = H_{\mathbb{M}(\kappa)}(t, \theta)$ for all $t > 0$ and all $\theta$. The equality in (2) is treated in the same way.

3 Generalizations of Theorem 1.1

The first generalization we are going to consider is the following. Let $M$ be a $n$-dimensional complete Riemannian manifold and let $B_M(r) \subset M$ be a geodesic ball within the cut locus. Consider $\mathbb{R}^m = [0, \infty) \times \mathbb{S}^m$ with metric $ds^2 = dt^2 + g^2(t)d\xi^2$, where $g : [0, \infty) \to \mathbb{R}$ is a smooth function satisfying $g(0) = 0$, $g'(0) = 1$, $g(t) > 0$ for $t \in (0, \infty)$. Let $B_{\mathbb{R}^m}(r)$ be a geodesic ball of radius $r$. Let $(t, \xi) \in (0, r) \times \mathbb{S}^{n-1}$ be geodesic coordinates for $B_M(r)$ and $(t, \xi) \in (0, r) \times \mathbb{S}^{m-1}$ be geodesic coordinates for $B_{\mathbb{R}^m}(r)$. Let $H_M(t, \theta)$ and $H_{\mathbb{R}^m}(t, \xi) = H_{\mathbb{R}^m}(t)$ be respectively the mean curvatures of the distance spheres $\partial B_M(t)$ and $\partial B_{\mathbb{R}^m}(t)$ at the points $(t, \theta)$ and $(t, \xi)$ with respect to the unit vector field $-\partial/\partial t$. 5
Theorem 3.1 If $H_M(s, \theta) \geq H_{R^n}(s) = (m - 1)(g'/g)(s), \forall s \in (0, r] \text{ and } \theta \in \mathbb{S}^{n-1}$ then

$$\lambda_1(B_M(r)) \geq \lambda_1(B_{R^n}(r)).$$

If $H_M(s, \theta) \leq H_{R^n}(s) = (m - 1)(g'/g)(s), \forall s \in (0, r] \text{ and } \theta \in \mathbb{S}^{n-1}$ then

$$\lambda_1(B_M(r)) \leq \lambda_1(B_{R^n}(r)).$$

Equality in (13) or (14) holds if and only if $n = m$ and $H_M(s, \theta) = H_{R^n}(s), \forall s \in (0, r] \text{ and } \theta \in \mathbb{S}^{n-1}$.

A positive first eigenfunction $u$ of a geodesic ball $B_{R^n}(r)$ within the cut locus is radial $(u(t, \xi) = u(t))$ and $u'(t) \leq 0$ with $u'(t) = 0 \Leftrightarrow t = 0$. See a proof of that in [3], pages 40-44. Define $v : B_M(r) \to \mathbb{R}$ by $v(t, \theta) = u(t)$ and take vector fields $X_1$ in $B_M(r)$ and $X_2$ in $B_{R^n}(r)$ by

$$X_1(t, \theta) = -\frac{u'(t)}{u(t)} \cdot \frac{\partial}{\partial t}(t, \theta),$$

$$X_2(t, \xi) = -\frac{u'(t)}{u(t)} \cdot \frac{\partial}{\partial \xi}(t, \xi).$$

Proceeding as in the proof of Theorem 1.1

$$-\frac{\Delta_M v}{v}(t, \theta) = \text{div}_M X_1 - |X_1|^2(t, \theta) = \text{div}_M X_1(t, \theta) - \text{div}_{R^n} X_2(t, \xi)$$

$$+ \text{div}_{R^n} X_2(t, \xi) - |X_2|^2(t, \xi)$$

$$+ |X_2|^2(t, \xi) - |X_1|^2(t, \theta).$$

Since we have that $(\text{div}_{R^n} X_2 - |X_2|^2)(t, \xi) = -\frac{\Delta_{R^n} u}{u} = \lambda_1(B_{R^n}(r)), |X_2|^2(t, \xi) - |X_1|^2(t, \theta) = 0$ and $\text{div}X_2(t, \xi) = \text{div}X_2(t)$. Thus we derive that

$$\lambda_1(B_M(r)) \geq \inf_{(t, \theta)}(\text{div}_M X_1 - |X_1|^2) \geq \inf_{(t, \theta)}[\text{div}_M X_1 - \text{div}_{R^n} X_2(t)] + \lambda_1(B_{R^n}(r)).$$

Likewise, we can derive

$$\lambda_1(B_M(r)) \leq \sup_{(t, \theta)}(\text{div}_M X_1 - |X_1|^2) \leq \sup_{(t, \theta)}[\text{div}_M X_1 - \text{div}_{R^n} X_2(t)] + \lambda_1(B_{R^n}(r)).$$

Then applying Lemma 2.4 we have that

$$\text{div}_M X_1(t, \theta) - \text{div}_{R^n} X_2(t) = -\frac{u'(t)}{u(t)}(H_M(t, \theta) - H_{R^n}(t))$$
If $H_M(s, \theta) \geq H_{\mathbb{R}^m}(s), \forall s \in (0, r]$ and $\theta \in S^{n-1}$ then $\lambda_1(B_M(r)) \geq \lambda_1(B_{\mathbb{R}^m}(r))$. On the other hand if $H_M(s, \theta) \leq H_{\mathbb{R}^m}(s), \forall s \in (0, r]$ and $\theta \in S^{n-1}$ then $\lambda_1(B_M(r)) \leq \lambda_1(B_{\mathbb{R}^m}(r))$. In case that $\lambda_1(B_M(r)) = \lambda_1(B_{\mathbb{R}^m}(r))$ we have by (1.7) that $\lambda_1(B_M(r)) = \text{div}_M X_1 - |X_1|^2$ and $\text{div}_M X_1(s, \theta) - \text{div}_{\mathbb{R}^m} X_2(s) = 0$ for all $s \in (0, r]$ and $\theta \in S^{n-1}$. Thus by Remark (2.22) the function $v$ is a positive eigenfunction of $B_M(r)$ and $H_M(s, \theta) = H_{\mathbb{R}^m}(s)$ for all $s \leq r, \theta \in S^{n-1}$.

To prove that $m = n$ we proceed as follows. Let $p$ be the center of the ball $B_M(r)$. For fixed $\theta \in S^{n-1} \subset T_pM$, let $\tau_t$ denote parallel translation by $t$ units along the unique minimal geodesic $\gamma_\theta$ satisfying $\gamma_\theta(0) = p$ and $\gamma_\theta'(0) = \theta$. For $\eta \in T_pM$ set $R_{\eta} = \tau_{-t}\{R(\gamma_\theta'(t), \tau_t\eta)\gamma_\theta(t)\}$, where $R$ is the Riemannian curvature tensor and set $A(t, \theta)$ the path of linear transformations of $\theta^\perp$ satisfying $A'' + RA = 0$ with initial conditions $A(0, \theta) = 0, A'(0, \theta) = I$. The Riemannian metric of $M$ on the geodesic ball $B_M(r)$ is expressed by $ds^2(\exp t\theta) = dt^2 + |A(t, \theta)d\theta|^2$. Set $\sqrt{G}(t, \theta) = \text{det} A(t, \theta)$. The mean curvature $H_M(t, \theta)$ of the geodesic sphere $\partial B_M(t)$ at a point $(t, \theta)$ (with respect to $-\partial/\partial t$) is given by $\sqrt{G}(t, \theta)$.

Moreover for small $t$ we have the Taylor expansions $\sqrt{G}(t, \theta) = t^{n-1}(1 - t^2Ric(\theta, \theta)/6 + O(t^3))$. See [3], pages 316-317. Thus,

$$\frac{\sqrt{G}(t, \theta)}{\sqrt{G}(t, \theta)} = \frac{(n - 1) - (n + 1)t^2Ric(\theta, \theta)/6 + O(t^3)}{t(1 - t^2Ric(\theta, \theta)/6 + O(t^3))}$$

(19)

On the other hand the metric of $B_{\mathbb{R}^m}(r)$ is given by $dt^2 + g^2(t)d\xi^2$, where $g(0) = 0, g'(0) = 1$. The mean curvature $H_{\mathbb{R}^m}(t, \xi)$ of the geodesic sphere $\partial B_{\mathbb{R}^m}(t)$ at a point $(t, \xi)$ is given by $(m - 1)\frac{g'(t)}{g(t)}$. The Taylor expansion of $g$ is given by $g(t) = t + g''(0)t^2/2 + O(t^3)$. Therefore,

$$\frac{(m - 1)g'(t)}{g(t)} = (m - 1)\frac{1 + g''(0)t + O(t^2)}{t(1 + g''(0)t/2 + O(t^2))}$$

(20)

Now, we have that $H_M(t, \theta) = H_{\mathbb{R}^m}(t)$ for all $t \in (0, r]$. Then

$$\frac{(n - 1) - (n + 1)t^2Ric(\theta, \theta)/6 + O(t^3)}{(1 - t^2Ric(\theta, \theta)/6 + O(t^3))} = (m - 1)\frac{1 + g''(0)t + O(t^2)}{(1 + g''(0)t/2 + O(t^2))}$$

(21)

Letting $t \to 0$ we have that $n = m$.

Another generalization of Theorem (1.1) is obtained considering the incomplete cone over an $(n - 1)$-dimensional compact Riemannian manifold $(N, dh^2)$. The incomplete cone $C_f(N)$ over $N$ is the Riemannian space $C(N) = (0, \infty) \times N$ with metric $ds^2 = dt^2 + f^2(t, x) dh^2$, where $f : [0, \infty) \times N \to \mathbb{R}$ is a smooth function satisfying $f(0, x) = 0, f'(0, x) = 1, f(t, x) > 0$ for all $t > 0$. The completed cone $C_f(N) = C_f(N) \cup \{p\}, p = \{0\} \times N$. The Euclidean space $\mathbb{R}^m$ with metric $ds^2 = dt^2 + g^2(t)d\theta^2$ is the completed cone $\overline{C_f(S^{m-1})}$. The next theorem compares the fundamental tone $\lambda^*(C_f(N)(r))$ of the the trunked cone $C_f(N)(r) = (0, r) \times N$ with the lowest Dirichlet eigenvalue $\lambda_1(B_{\mathbb{R}^m}(r))$ of the geodesic ball $B_{\mathbb{R}^m}(r)$.
Theorem 3.2 Let $C_f(N)$ be a incomplete cone over a compact $(n-1)$-dimensional Riemannian manifold $(N, dh^2)$ and $\mathbb{R}^m$ with metric $ds^2 = dt^2 + g^2(t)d\theta^2$. If
\[(n-1)(f'/f)(t, x) \geq (m-1)(g'/g)(t),\]for all $x \in N$ and all $t \in (0, r) \text{ where}'$ means the derivative with respect to the variable $t$. Then
\[\lambda^*(C_f(N)(r)) \geq \lambda_1(B_{\mathbb{R}^n}(r))\]If (22) holds for all $t > 0$ then letting $r \to \infty$ we have that
\[\lambda^*(C_f(N)) \geq \lambda^*(\mathbb{R}^m)\]
The proof of Theorem (3.2) is similar to the proof of Theorem (1.1). We take $u$ to be a positive first Dirichlet eigenfunction of $B_{\mathbb{R}^n}(r)$ and consider the vector fields $X_1(t, x) = -\frac{u'}{u}(t) \cdot \frac{\partial_1}{\partial t}(t, x)$ and $X_2(t, \theta) = -\frac{u'}{u}(t) \cdot \frac{\partial_2}{\partial t}(t, \theta)$. Thus we have by Theorem (2.3) that
\[\lambda^*(C_f(N)(r)) \geq \inf[\text{div}X_1 - |X_1|^2] = \inf[\text{div}X_1 - \text{div}X_2] + \lambda_1(B_{\mathbb{R}^n}(r)).\]Observe that the slice $t \times N$, $\{t\} \in (0, r)$ is a smooth hypersurface of $C_f(N)(r)$ thus we may apply Lemma (2.4) to obtain that $\text{div}X_1(t, x) - \text{div}X_2(t, \theta) = (n-1)f'(t, x) - (m-1)f'(t, \theta) \geq 0$. This together with (23) proves (24).

These ideas used in the proofs of theorems (1.1, 3.1, 3.2) can be used to obtain examples of Riemannian manifolds $M$ with arbitrary fundamental groups and variable sectional curvatures and with positive fundamental tone $\lambda^*(M) > 0$. For instance, let $M = \mathbb{R}^m \times N$ with the metric $ds^2 = dt^2 + f^2(t, \theta)d\theta^2 + g^2(t, \theta)dh^2$ where $(N, dh^2)$ is a complete $n$-dimensional Riemannian manifold and $f, g : \mathbb{R}^n \to [0, \infty)$ are smooth functions, $f$ satisfying $f(0, \theta) = 0$, $f'(0, \theta) = 1$ and $f(t, \theta) > 0$ for $t > 0$ and $\theta \in S^{m-1}$, $g(t, \theta) > 0$ for all $(t, \theta) \in [0, \infty) \times S^{m-1}$. Let $\Omega = B_{\mathbb{R}^m}(r) \times W \subset M$ where $B_{\mathbb{R}^m}(r) \subset \mathbb{R}^m$ is a ball with radius $r$ nd $W \subset N$ is a domain with compact closure and smooth boundary $\partial W$ (possibly empty). Let $B_{M^1(\kappa)}(r) \subset M^1(\kappa)$ be a geodesic ball of radius $r$ in the simply connected $l$-dimensional space form of constant sectional curvature $\kappa$ with metric $dt^2 + S^2_k(t)d\theta^2$.

Theorem 3.3 If $(m-1)f'/f(t, \theta) + n\frac{g'}{g}(t, \theta) \geq (l-1)\frac{S^l_k}{S_k}(t)$ for all $t \in [0, r]$ and $\theta \in S^{m-1}$, then
\[\lambda_1(\Omega) \geq \lambda_1(B_{M^1(\kappa)}(r)) + \inf_{(t, \theta) \in \Omega} \frac{1}{g^2} \cdot \lambda_1(W).\]If $r = \infty$ and letting $W = N$ we have that
\[\lambda^*(M) \geq (l-1)^2\kappa^2/4 + \inf_{t, \theta} \frac{1}{g^2} \cdot \lambda^*(N).\]
If \((m-1) \frac{f'}{f}(t, \theta) + n \frac{g'}{g}(t, \theta) \leq (l-1) \frac{S_k'}{S_k} (t)\) for all \(t \in [0, r]\) and \(\theta \in S^{m-1}\), then

\[
\lambda_1(\Omega) \leq \lambda_1(B_{\mathbb{M}^1(\kappa)}) + \sup_{(t, \theta) \in \Omega} \left[ \frac{1}{g^2} \right] \cdot \lambda_1(W). \tag{26}
\]

If \(r = \infty\), and letting \(W = N\) we have that

\[
\lambda^*(M) \leq (l-1)^2 \kappa^2 / 4 + \sup_{(t, \theta)} \left[ \frac{1}{g^2} \right] \cdot \lambda^*(N)
\]

Choose a positive function \(\psi : \Omega \rightarrow \mathbb{R}\) given by \(\psi(t, \theta, x) = u(t) \cdot \xi(x)\) where \(u\) and \(\xi\) are positive eigenfunctions of \(B_{\mathbb{M}^1(\kappa)}(r)\) and \(W\) respectively, i.e. \(u\) satisfies the differential equation

\[
\frac{\partial^2 u}{\partial t^2}(t) + (l-1) \frac{S_k'}{S_k}(t) \frac{\partial u}{\partial t}(t) + \lambda_1(B_{\mathbb{M}^1(\kappa)}(r)) u(t) = 0 \tag{27}
\]

with \(u(0) = 1\), \(u'(0) = 0\) and \(\xi : W \rightarrow \mathbb{R}\) satisfies \(\Delta_{dh^2} \xi + \lambda_1(W) \xi = 0\) in \(W\) and \(\xi \partial W = 0\). It is clear that \(\psi \in C^2(\Omega) \cap C^0(\Omega^2)\) with \(\psi > 0\) in \(\Omega\) and \(\psi \partial \Omega = 0\). The Laplace operator of \(ds^2\) is written in geodesic coordinates is given by

\[
\Delta_{ds^2} = \frac{\partial^2}{\partial t^2} + \left[ (m-1) \frac{1}{f} \frac{\partial f}{\partial t} + n \frac{1}{g} \frac{\partial g}{\partial t} \right] \frac{\partial}{\partial t} + \frac{m-3}{f^3} \frac{g^2}{d\theta^2} (\nabla_{dh^2} f, \nabla_{dh^2} g, \nabla_{dh^2} \cdot) \tag{28}
\]

where \(\nabla_{dh^2}\) and \(\Delta_{dh^2}\) are respectively the gradients and the Laplacian of \(S^{m-1}\) and \(\nabla_{dh^2}\) and \(\Delta_{dh^2}\) are respectively the gradients and the Laplacian of \(N\). Computing \(-\Delta_{ds^2} \psi / \psi\) we have,

\[
-\frac{\Delta_{ds^2} \psi}{\psi} = -\frac{u''}{u} + (m-1) \frac{f'}{f} \frac{u'}{u} + n \frac{g'}{g} \frac{u'}{u} - \frac{1}{g^2} \frac{\Delta_{dh^2} \xi}{\xi} = \lambda_1(B_{\mathbb{M}^1(\kappa)}(r)) - \frac{u'}{u} \left( (m-1) \frac{f'}{f} + n \frac{g'}{g} - (s-1) \frac{S_k'}{S_k} \right) + \frac{1}{g^2} \lambda_1(W) \tag{29}
\]

If \((m-1) \frac{f'}{f} + n \frac{g'}{g} \geq (s-1) \frac{S_k'}{S_k}\) then from \(29\)

\[
\inf(-\frac{\Delta \psi}{\psi}) \geq \lambda_1(B_{\mathbb{M}^1(\kappa)}(r)) + \inf \frac{1}{g^2} \lambda_1(W).
\]

If \((m-1) \frac{f'}{f} + n \frac{g'}{g} - (s-1) \frac{S_k'}{S_k} \leq 0\) then

\[
\sup(-\frac{\Delta \psi}{\psi}) \leq \lambda_1(B_{\mathbb{M}^1(\kappa)}(r)) + \sup \frac{1}{g^2} \lambda_1(W).
\]

Since \(-(u'/u) \geq 0\).
4 Examples

In this section we construct examples of metrics showing certain aspects of Cheng’s eigenvalue comparison theorem. In this first example we construct a family of metrics on $\mathbb{R}^n$ with radial sectional $K_{\mathbb{R}^n} > \kappa$ outside a compact set and such that the mean curvatures of the distance spheres satisfy $H_{\mathbb{R}^n}(t, \theta) \geq H_{\mathbb{M}^n(\kappa)}(t, \theta) = (n-1)(S'_\kappa/S_\kappa)(t)$.

Example 4.1 Let $\mathbb{R}^n = [0, \infty) \times S^{n-1}$ with the metric $ds^2 = dt^2 + f^2(t)d\theta^2$, $f(0) = 0$, $f'(0) = 1$. Set $\psi(t) = (-f'S_\kappa + fS'_\kappa)(t)$, where $'$ means differentiation with respect to $t$ and $S_\kappa$ is given by

$$S_\kappa(t) = \begin{cases} 
\sinh(\sqrt{-\kappa}t)/\sqrt{-\kappa} & \text{if } \kappa = -k^2 \\
t & \text{if } \kappa = 0, \quad C_\kappa(t) = S'_\kappa(t) \\
\sin(\sqrt{\kappa}t)/\sqrt{\kappa} & \text{if } \kappa = k^2 
\end{cases}$$

The radial sectional curvature of $\mathbb{R}^n, ds^2$ is bounded above by $\kappa$ if and only if $\psi'_\kappa(t) \leq 0$. The mean curvatures of $\partial B_{\mathbb{R}^n}(t)$ and $\partial B_{\mathbb{M}^n(\kappa)}(t)$ satisfies $H_{\mathbb{R}^n}(t, \theta) \geq H_{\mathbb{M}^n(\kappa)}(t)$ if and only if $\psi_\kappa(t) \leq 0$. From $\psi_\kappa(t) = (-f'S_\kappa + fS'_\kappa)(t)$ we have that $\psi_\kappa(0) = \psi'_\kappa(0) = 0$. Solving the differential equation we have

$$f(t) = S_\kappa(t) + S'_\kappa(t) \int_0^t \psi_\kappa(s)/S_\kappa(s)ds$$

Let $\psi_\kappa : [0, \infty) \to \mathbb{R}$ be a smooth function satisfying $\psi(0) = \psi'_\kappa(0) = 0$, $\psi(t) \leq 0$, $\psi'_\kappa(t) > 0$ for $t > 1$ and $|\int_0^t \psi_\kappa(s)/S_\kappa(s)ds| < \infty$. This yields a metric $ds^2_f = dt^2 + f^2(t)d\theta^2$ with sectional curvature $K_{\mathbb{R}^n} > \kappa$ outside a compact set and such that the mean curvatures of the distance spheres satisfy $(n-1)(f'/f)(t) = H_{\mathbb{R}^n}(t, \theta) \geq H_{\mathbb{M}^n(\kappa)}(t, \theta) = (n-1)(S'_\kappa/S_\kappa)(t)$.

Remark 4.2 If the metric of $M$ is expressed by $dt^2 + f^2(t)d\theta^2$ then $H_M(s, \theta) = H_{\mathbb{M}(\kappa)}(s)$ for all $s \in (0, r]$ and all $\theta \in S^{n-1}$ implies that $B_M(r)$ is isometric to $B_{\mathbb{M}(\kappa)}(r)$. Because the equality $H_M(s, \theta) = H_{\mathbb{M}(\kappa)}(s)$ for all $s \in (0, r]$ and all $\theta$ is equivalent to have $\psi_\kappa(s) = 0$, $s \in [0, r]$ but this would imply that $f(s) = S_\kappa(s)$, $s \in [0, r]$.

The next example shows that the rigidity in Theorem (1.1) is all we can expect without curvature bounds.

Example 4.3 For every $\kappa \in \mathbb{R}$, consider the metric $g = g(\kappa)$ on $M = [0, a] \times S^{n-1}$, where $a = \infty$ if $k \leq 0$ and $a = \pi/\sqrt{\kappa}$ if $\kappa > 0$, given in geodesic coordinates by the matrix $g_{11}(t, \theta) = 1$, $g_{22}(t, \theta) = (S_\kappa^2(t)/t^2) \cdot \theta_{22}$, $g_{33}(t, \theta) = t^2 \cdot \theta_{33}$, $g_{ij}(t, \theta) = S_\kappa^2(t) \cdot \theta_{ij}$, $i \geq 4$, $g_{ij}(t, \theta) = 0$ if $i \neq j$, where $d\theta_{ij} = (\theta_{ij})$ is the canonical metric of $S^{n-1}(1)$. This metric $g(\kappa)$ is smooth if $\kappa \leq 0$. If $\kappa > 0$ the metric $g(\kappa)$ is smooth except at $(\pi, \theta)$. Let $h = h(\kappa)$ be the metric of constant sectional curvature of $\mathbb{M}(\kappa)$ given by the matrix $h_{11} = 1$, $h_{ii} = S_\kappa^2(t) \cdot \theta_{ii}$, $i \geq 2$. For $\kappa \neq 0$, $g(\kappa)$ is not isometric to $h(\kappa)$. Let $\Delta_g$ and $\Delta_h$ denote the Laplace operator of these two metrics.
written in geodesic coordinates. They are given by

\[ \Delta_g = \frac{\partial^2}{\partial t^2} + (n-1) \frac{C_\kappa}{S_\kappa} \frac{\partial}{\partial t} + \frac{t^2}{S_\kappa^4} \frac{\partial}{\partial \theta_2} + \frac{1}{t^2} \frac{\partial}{\partial \theta_3} + \sum_{i=4}^{n} \frac{1}{S_\kappa^2} \frac{\partial}{\partial \theta_i}. \]

\[ \Delta_h = \frac{\partial^2}{\partial t^2} + (n-1) \frac{C_\kappa}{S_\kappa} \frac{\partial}{\partial t} + \sum_{i=2}^{n} \frac{1}{S_\kappa^2} \frac{\partial}{\partial \theta_i}. \]

We have that the geodesic spheres \( \partial B_M(s) \) and \( \partial B_{M(\kappa)}(s) \) have the same mean curvature \( H_M(s) = H_{M(\kappa)}(s) = (n-1)(C_\kappa/S_\kappa)(s), s \in (0, r]. \) And the geodesic balls \( B_M(r) \) and \( B_{M(\kappa)}(r) \) have the same first eigenvalue. For if \( u \) be a first Dirichlet eigenfunction of the geodesic ball \( B_{M(\kappa)}(r) \), if \( \kappa > 0 \) suppose that that \( r < \pi/\sqrt{\kappa}. \) Thus \( \Delta_h u + \lambda_1(B_{M(\kappa)}(r))u = 0 \) in \( B_{M(\kappa)}(r) \) and \( u = 0 \) on \( \partial B_{M(\kappa)}(r) \). Since \( u \) is radial we have that \( \Delta_h u(t) = \Delta_g u(t) = -\lambda_1(B_{M(\kappa)}(r))u(t) \). This shows that \( u(t) \) is a first Dirichlet eigenfunction of the geodesic ball \( B_M(r) \) with same eigenvalue \( \lambda_1(B_{M(\kappa)}(r)). \)

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