Classification of Differential Calcuti on $U_q(b_+)$, Classical Limits, and Duality

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DAMTP-1998-86
July 18, 1998

Abstract

We give a complete classification of bicovariant first order differential calculi on the quantum enveloping algebra $U_q(b_+)$ which we view as the quantum function algebra $C_q(B_+)$. Here, $b_+$ is the Borel subalgebra of $\mathfrak{sl}_2$. We do the same in the classical limit $q \to 1$ and obtain a one-to-one correspondence in the finite dimensional case. It turns out that the classification is essentially given by finite subsets of the positive integers. We proceed to investigate the classical limit from the dual point of view, i.e. with “function algebra” $U(b_+)$ and “enveloping algebra” $C(B_+)$. In this case there are many more differential calculi than coming from the $q$-deformed setting. As an application, we give the natural intrinsic 4-dimensional calculus of $\kappa$-Minkowski space and the associated formal integral.

1 Introduction

One of the fundamental ingredients in the theory of non-commutative or quantum geometry is the notion of a differential calculus. In the framework of quantum groups the natural notion is that of a bicovariant differential calculus as introduced by Woronowicz [1]. Due to the allowance of non-commutativity the uniqueness of a canonical calculus is lost. It is therefore desirable to classify the possible choices. The most important piece is the space of one-forms or “first order differential calculus” to which we will restrict our attention in the following. (From this point on we will use the term “differential calculus” to denote a bicovariant first order differential calculus).

Much attention has been devoted to the investigation of differential calculi on quantum groups $C_q(G)$ of function algebra type for $G$ a simple Lie group. Natural differential calculi on matrix quantum groups were
obtained by Jurco \cite{2} and Carow-Watamura et al. \cite{3}. A partial classification of calculi of the same dimension as the natural ones was obtained by Schm"{u}dgen and Sch"{u}ler \cite{4}. More recently, a classification theorem for factorisable cosemisimple quantum groups was obtained by Majid \cite{5}, covering the general $C_q(G)$ case. A similar result was obtained later by Baumann and Schmitt \cite{6}. Also, Heckenberger and Schm"{u}dgen \cite{7} gave a complete classification on $C_q(SL(N))$ and $C_q(Sp(N))$.

In contrast, for $G$ not simple or semisimple the differential calculi on $C_q(G)$ are largely unknown. A particularly basic case is the Lie group $B_+$ associated with the Lie algebra $b_+$ generated by two elements $X,H$ with the relation $[H,X] = X$. The quantum enveloping algebra $U_q(b_+)$ is self-dual, i.e. is non-degenerately paired with itself \cite{12}. This has an interesting consequence: $U_q(b_+)$ may be identified with (a certain algebraic model of) $C_q(B_+)$. The differential calculi on this quantum group and on its “classical limits” $C(B_+)$ and $U(b_+)$ will be the main concern of this paper. We pay hereby equal attention to the dual notion of “quantum tangent space”.

In section \secref{2} we obtain the complete classification of differential calculi on $C_q(B_+)$. It turns out that (finite dimensional) differential calculi are characterised by finite subsets $I \subset \mathbb{N}$. These sets determine the decomposition into coirreducible (i.e. not admitting quotients) differential calculi characterised by single integers. For the coirreducible calculi the explicit formulas for the commutation relations and braided derivations are given.

In section \secref{3} we give the complete classification for the classical function algebra $C(B_+)$. It is essentially the same as in the $q$-deformed setting and we stress this by giving an almost one-to-one correspondence of differential calculi to those obtained in the previous section. In contrast, however, the decomposition and coirreducibility properties do not hold at all. (One may even say that they are maximally violated). We give the explicit formulas for those calculi corresponding to coirreducible ones.

More interesting perhaps is the “dual” classical limit. I.e. we view $U(b_+)$ as a quantum function algebra with quantum enveloping algebra $C(B_+)$. This is investigated in section \secref{4}. It turns out that in this setting we have considerably more freedom in choosing a differential calculus since the bicovariance condition becomes much weaker. This shows that this dual classical limit is in a sense “unnatural” as compared to the ordinary classical limit of section \secref{3}. However, we can still establish a correspondence of certain differential calculi to those of section \secref{3}. The decomposition properties are conserved while the coirreducibility properties are not. We give the formulas for the calculi corresponding to coirreducible ones.

Another interesting aspect of viewing $U(b_+)$ as a quantum function algebra is the connection to quantum deformed models of space-time and its symmetries. In particular, the $\kappa$-deformed Minkowski space coming from the $\kappa$-deformed Poincaré algebra \cite{8} is just a simple generalisation of $U(b_+)$. We use this in section \secref{5} to give a natural 4-dimensional differential calculu-
lus. Then we show (in a formal context) that integration is given by the usual Lebesgue integral on $\mathbb{R}^n$ after normal ordering. This is obtained in an intrinsic context different from the standard $\kappa$-Poincaré approach.

A further important motivation for the investigation of differential calculi on $U(b_+)$ and $C(B_+)$ is the relation of those objects to the Planck-scale Hopf algebra [10][11]. This shall be developed elsewhere.

In the remaining parts of this introduction we will specify our conventions and provide preliminaries on the quantum group $U_q(b_+)$, its deformations, and differential calculi.

1.1 Conventions

Throughout, $k$ denotes a field of characteristic 0 and $k(q)$ denotes the field of rational functions in one parameter $q$ over $k$. $k(q)$ is our ground field in the $q$-deformed setting, while $k$ is the ground field in the “classical” settings. Within section 2 one could equally well view $k$ as the ground field with $q \in k^*$ not a root of unity. This point of view is problematic, however, when obtaining “classical limits” as in sections 3 and 4.

The positive integers are denoted by $\mathbb{N}$ while the non-negative integers are denoted by $\mathbb{N}_0$. We define $q$-integers, $q$-factorials and $q$-binomials as follows:

$$[n]_q = \sum_{i=0}^{n-1} q^i \quad [n]_q! = [1]_q[2]_q \cdots [n]_q \quad \left[ \begin{array}{c} n \\ m \end{array} \right]_q = \frac{[n]_q!}{[m]_q![[n-m]]_q}.$$  

For a function of several variables (among them $x$) over $k$ we define

$$(T_{a,x}f)(x) = f(x + a)$$

$$(\nabla_{a,x} f)(x) = \frac{f(x + a) - f(x)}{a}$$

with $a \in k$ and similarly over $k(q)$

$$(Q_{m,x}f)(x) = f(q^m x)$$

$$(\partial_{q,x} f)(x) = \frac{f(x) - f(qx)}{x(1-q)}$$

with $m \in \mathbb{Z}$.

We frequently use the notion of a polynomial in an extended sense. Namely, if we have an algebra with an element $g$ and its inverse $g^{-1}$ (as in $U_q(b_+)$) we will mean by a polynomial in $g, g^{-1}$ a finite power series in $g$ with exponents in $\mathbb{Z}$. The length of such a polynomial is the difference between highest and lowest degree.

If $H$ is a Hopf algebra, then $H^{op}$ will denote the Hopf algebra with the opposite product.
1.2 \( U_q(b_+) \) and its Classical Limits

We recall that, in the framework of quantum groups, the duality between enveloping algebra \( U(g) \) of the Lie algebra and algebra of functions \( C(G) \) on the Lie group carries over to \( q \)-deformations. In the case of \( b_+ \), the \( q \)-deformed enveloping algebra \( U_q(b_+) \) defined over \( k(q) \) as

\[
U_q(b_+) = k(q)(X, g, g^{-1}) \quad \text{with relations}
\]

\[
\begin{align*}
  gg^{-1} &= 1 \\
  Xg &= qgX \\
  \Delta X &= X \otimes 1 + g \otimes X \\
  \Delta g &= g \otimes g \\
  \epsilon(X) &= 0 \\
  \epsilon(g) &= 1 \\
  SX &= g^{-1}X \\
  Sg &= g^{-1}
\end{align*}
\]

is self-dual. Consequently, it may alternatively be viewed as the quantum algebra \( C_q(B_+) \) of functions on the Lie group \( B_+ \) associated with \( b_+ \). It has two classical limits, the enveloping algebra \( U(b_+) \) and the function algebra \( C(B_+) \). The transition to the classical enveloping algebra is achieved by replacing \( q \) by \( e^{-t} \) and \( g \) by \( e^{tH} \) in a formal power series setting in \( t \), introducing a new generator \( H \). Now, all expressions are written in the form \( \sum_j a_j t^j \) and only the lowest order in \( t \) is kept. The transition to the classical function algebra on the other hand is achieved by setting \( q = 1 \). This may be depicted as follows:

\[
\begin{align*}
  U_q(b_+) &\cong C_q(B_+) \\
  q &= e^{-t} \\
  g &= e^{tH} \\
  U(b_+) &\xrightarrow{\text{dual} \cdots} C(B_+)
\end{align*}
\]

The self-duality of \( U_q(b_+) \) is expressed as a pairing \( U_q(b_+) \times U_q(b_+) \rightarrow k \) with itself:

\[
\langle X^n g^m, X^r g^s \rangle = \delta_{n,r} [n]_q! q^{-n(n-1)/2} q^{-ms} \quad \forall n, r \in \mathbb{N}_0, m, s \in \mathbb{Z}
\]

In the classical limit this becomes the pairing \( U(b_+) \times C(B_+) \rightarrow k \)

\[
\langle X^n H^m, X^r s^s \rangle = \delta_{n,r} [n]_s^m \quad \forall n, m, r \in \mathbb{N}_0, s \in \mathbb{Z}
\]

1.3 Differential Calculi and Quantum Tangent Spaces

In this section we recall some facts about differential calculi along the lines of Majid’s treatment in [1].

Following Woronowicz [1], first order bicovariant differential calculi on a quantum group \( A \) (of function algebra type) are in one-to-one correspondence to submodules \( M \) of \( \ker \epsilon \subset A \) in the category \( \mathcal{A} \) of (say) left crossed modules of \( A \) via left multiplication and left adjoint coaction:

\[
a \triangleright v = av \quad \text{Ad}_L(v) = v(1) S v(3) \otimes v(2) \quad \forall a \in A, v \in A
\]
More precisely, given a crossed submodule $M$, the corresponding calculus is given by $\Gamma = \ker \epsilon / M \otimes A$ with $da = \pi(\Delta a - 1 \otimes a)$ ($\pi$ the canonical projection). The right action and coaction on $\Gamma$ are given by the right multiplication and coproduct on $A$, the left action and coaction by the tensor product ones with $\ker \epsilon / M$ as a left crossed module. In all of what follows, “differential calculus” will mean “bicovariant first order differential calculus”.

Alternatively, given in addition a quantum group $H$ dually paired with $A$ (which we might think of as being of enveloping algebra type), we can express the coaction of $A$ on itself as an action of $H^{\text{op}}$ using the pairing:

$$h \triangleright v = \langle h, v_{(1)} S v_{(3)} \rangle v_{(2)} \quad \forall h \in H^{\text{op}}, v \in A$$

Thereby we change from the category of (left) crossed $A$-modules to the category of left modules of the quantum double $A \rhd \bowtie H^{\text{op}}$.

In this picture the pairing between $A$ and $H$ descends to a pairing between $A/\mathbb{k}1$ (which we may identify with $\ker \epsilon \subset A$) and $\ker \epsilon \subset H$. Further quotienting $A/\mathbb{k}1$ by $M$ (viewed in $A/\mathbb{k}1$) leads to a pairing with the subspace $L \subset \ker \epsilon H$ that annihilates $M$. $L$ is called a “quantum tangent space” and is dual to the differential calculus $\Gamma$ generated by $M$ in the sense that $\Gamma \cong \text{Lin}(L, A)$ via

$$A/(\mathbb{k}1 + M) \otimes A \rightarrow \text{Lin}(L, A) \quad v \otimes a \mapsto \langle \cdot, v \rangle a$$

if the pairing between $A/(\mathbb{k}1 + M)$ and $L$ is non-degenerate.

The quantum tangent spaces are obtained directly by dualising the (left) action of the quantum double on $A$ to a (right) action on $H$. Explicitly, this is the adjoint action and the coregular action

$$h \triangleright x = h_{(1)} x S h_{(2)} \quad a \triangleright x = \langle x_{(1)}, a \rangle x_{(2)} \quad \forall h \in H, a \in A^{\text{op}}, x \in A$$

where we have converted the right action to a left action by going from $A \rhd \bowtie H^{\text{op}}$-modules to $H \bowtie \bowtie A^{\text{op}}$-modules. Quantum tangent spaces are subspaces of $\ker \epsilon \subset H$ invariant under the projection of this action to $\ker \epsilon$ via $x \mapsto x - \epsilon(x)1$. Alternatively, the left action of $A^{\text{op}}$ can be converted to a left coaction of $H$ being the comultiplication (with subsequent projection onto $H \otimes \ker \epsilon$).

We can use the evaluation map (2) to define a “braided derivation” on elements of the quantum tangent space via

$$\partial_x : A \rightarrow A \quad \partial_x(a) = da(x) = \langle x, a_{(1)} \rangle a_{(2)} \quad \forall x \in L, a \in A$$

This obeys the braided derivation rule

$$\partial_x(ab) = (\partial_x a)b + a_{(2)} \partial_{a_{(1)} \triangleright x} b \quad \forall x \in L, a \in A$$
Given a right invariant basis \( \{ \eta_i \}_{i \in I} \) of \( \Gamma \) with a dual basis \( \{ \phi_i \}_{i \in I} \) of \( L \) we have

\[
d a = \sum_{i \in I} \eta_i \cdot \partial_i(a) \quad \forall a \in A
\]

where we denote \( \partial_i = \partial_{\phi_i} \). (This can be easily seen to hold by evaluation against \( \phi_i \) \( \forall i \).)

2 Classification on \( C_q(B_+) \) and \( U_q(b_+) \)

In this section we completely classify differential calculi on \( C_q(B_+) \) and, dually, quantum tangent spaces on \( U_q(b_+) \). We start by classifying the relevant crossed modules and then proceed to a detailed description of the calculi.

**Lemma 2.1.** (a) Left crossed \( C_q(B_+) \)-submodules \( M \subseteq C_q(B_+) \) by left multiplication and left adjoint coaction are in one-to-one correspondence to pairs \( (P, I) \) where \( P \in \mathbb{k}(q)[g] \) is a polynomial with \( P(0) = 1 \) and \( I \subseteq \mathbb{N} \) is finite. \( \text{codim } M < \infty \) iff \( P = 1 \). In particular \( \text{codim } M = \sum_{n \in I} n \) if \( P = 1 \).

(b) The finite codimensional maximal \( M \) correspond to the pairs \( (1, \{ n \}) \) with \( n \) the codimension. The infinite codimensional maximal \( M \) are characterised by \( (P, \emptyset) \) with \( P \) irreducible and \( P(g) \neq 1 - q^{-k}g \) for any \( k \in \mathbb{N}_0 \).

(c) Crossed submodules \( M \) of finite codimension are intersections of maximal ones. In particular \( M = \bigcap_{n \in I} M^n \), with \( M^n \) corresponding to \( (1, \{ n \}) \).

**Proof.** (a) Let \( M \subseteq C_q(B_+) \) be a crossed \( C_q(B_+) \)-submodule by left multiplication and left adjoint coaction and let \( \sum_n X^n P_n(g) \in M \), where \( P_n \) are polynomials in \( g, g^{-1} \) (every element of \( C_q(B_+) \) can be expressed in this form). From the formula for the coaction (3, see appendix) we observe that for all \( n \) and for all \( t \leq n \) the element

\[
X^t P_n(g) \prod_{s=1}^{n-t} (1 - q^{s-n}g)
\]

lies in \( M \). In particular this is true for \( t = n \), meaning that elements of constant degree in \( X \) lie separately in \( M \). It is therefore enough to consider such elements.

Let now \( X^n P(g) \in M \). By left multiplication \( X^n P(g) \) generates any element of the form \( X^k P(g) Q(g) \), where \( k \geq n \) and \( Q \) is any polynomial in \( g, g^{-1} \). (Note that \( Q(q^k g) X^k = X^k Q(g) \).) We see that \( M \) contains the
We see that the integers \( n \) picture. The polynomial \( g \), multiplication with polynomials in \( X \) we arrive at degree zero in \( X \). For any element of the form \( M \) in elements shown does not generate elements of new type.)

Moreover, if \( M \) is generated by \( X^n P(g) \) as a module then these elements generate a basis for \( M \) as a vector space by left multiplication with polynomials in \( g, g^{-1} \). (Observe that the application of the coaction to any of the elements shown does not generate elements of new type.)

Now, let \( M \) be a given crossed submodule. We pick, among the elements in \( M \) of the form \( X^n P(g) \) with \( P \) of minimal length, one with lowest degree in \( X \). Then certainly the elements listed above are in \( M \). Furthermore for any element of the form \( X^k Q(g) \), \( Q \) must contain \( P \) as a factor and for \( k < n \), \( Q \) must contain \( P(g)(1 - q^{1-n}g) \) as a factor. We continue by picking the smallest \( n_2 \), so that \( X^{n_2} P(g)(1 - q^{1-n}g) \in M \). Certainly \( n_2 < n \). Again, for any element of \( X^l Q(g) \) in \( M \) with \( l < n_2 \), we have that \( P(g)(1 - q^{1-n}g)(1 - q^{1-n_2}g) \) divides \( Q(g) \). We proceed by induction, until we arrive at degree zero in \( X \).

We obtain the following elements generating a basis for \( M \) by left multiplication with polynomials in \( g, g^{-1} \) (rename \( n_1 = n \):

\[
\begin{align*}
&X^{n_1+2} P(g) \\
&X^{n_1+1} P(g) \\
&X^{n_1} P(g) \\
&X^{n_1-1} P(g)(1 - q^{1-n_1}g) \\
&X^{n_2} P(g)(1 - q^{1-n_1}g) \\
&X^{n_2-1} P(g)(1 - q^{1-n_1}g)(1 - q^{1-n_2}) \\
&X^{n_3} P(g)(1 - q^{1-n_1}g)(1 - q^{1-n_2}g) \\
&X^{n_3-1} P(g)(1 - q^{1-n_1}g)(1 - q^{1-n_2}g)(1 - q^{1-n_3}) \\
&\vdots \\
&P(g)(1 - q^{1-n_1}g)(1 - q^{1-n_2}g)(1 - q^{1-n_3}g) \ldots (1 - q^{1-n_m}g)
\end{align*}
\]

We see that the integers \( n_1, \ldots, n_m \) uniquely determine the shape of this picture. The polynomial \( P(g) \) on the other hand can be shifted (by \( g \) and
$g^{-1}$) or renormalised. To determine $M$ uniquely we shift and normalise $P$ in such a way that it contains no negative powers and has unit constant coefficient. $P$ can then be viewed as a polynomial in $k(q)[g]$.

We see that the codimension of $M$ is the sum of the lengths of the polynomials in $g$ over all degrees in $X$ in the above picture. Finite codimension corresponds to $P = 1$. In this case the codimension is the sum $n_1 + \ldots + n_m$.

(b) We observe that polynomials of the form $1 - q^jg$ have no common divisors for distinct $j$. Therefore, finite codimensional crossed submodules are maximal if and only if there is just one integer ($m = 1$). Thus, the maximal left crossed submodule of codimension $k$ is generated by $X^k$ and $1 - q^{1-k}g$. For an infinite codimensional crossed submodule we certainly need $m = 0$. Then, the maximality corresponds to irreducibility of $P$.

(c) This is again due to the distinctness of factors $1 - q^jg$.

\begin{corollary}
(a) Left crossed $C_q(B_+)$-submodules $M \subseteq \ker \epsilon \subseteq C_q(B_+)$ are in one-to-one correspondence to pairs $(P, I)$ as in lemma 2.1 with the additional constraint $(1 - g)$ divides $P(g)$ or $1 \in I$. codim $M < \infty$ iff $P = 1$.

In particular codim $M = (\sum_{n \in I} n) - 1$ if $P = 1$.

(b) The finite codimensional maximal $M$ correspond to the pairs $(1, \{1, n\})$ with $n \geq 2$ the codimension. The infinite codimensional maximal $M$ correspond to pairs $(P, \{1\})$ with $P$ irreducible and $P(g) \neq 1 - q^{-k}g$ for any $k \in \mathbb{N}_0$.

(c) Crossed submodules $M$ of finite codimension are intersections of maximal ones. In particular $M = \bigcap_{n \in I} M^n$, with $M^n$ corresponding to $(1, \{1, n\})$.

Proof. First observe that $\sum_n X^n P_n(g) \in \ker \epsilon$ if and only if $(1 - g)$ divides $P_0(g)$. This is to say that that $\ker \epsilon$ is the crossed submodule corresponding to the pair $(1, \{1\})$ in lemma 2.1. We obtain the classification from the one of lemmas 2.1 by intersecting everything with this crossed submodule. In particular, this reduces the codimension by one in the finite codimensional case.

\end{corollary}

\begin{lemma}
(a) Left crossed $U_q(b_+)$-submodules $L \subseteq U_q(b_+)$ via the left adjoint action and left regular coaction are in one-to-one correspondence to the set $3^\mathbb{N} \times 2^\mathbb{N}$. Finite dimensional $L$ are in one-to-one correspondence to finite sets $I \subset \mathbb{N}$ and $\dim L = \sum_{n \in I} n$.

(b) Finite dimensional irreducible $L$ correspond to $\{n\}$ with $n$ the dimension.

(c) Finite dimensional $L$ are direct sums of irreducible ones. In particular $L = \oplus_{n \in I} L^n$ with $L^n$ corresponding to $\{n\}$.

Proof. (a) The action takes the explicit form

\[
g \triangleright X^n g^k = q^{-n} X^n g^k \quad X \triangleright X^n g^k = X^{n+1} g^k (1 - q^{-(n+k)})\]

8
while the coproduct is
\[ \Delta(X^n g^k) = \sum_{r=0}^{n} \binom{n}{r} q^{-r(n-r)} X^{n-r} g^{k+r} \otimes X^r g^k \]
which we view as a left coaction here. Let now \( L \subseteq U_q(\mathfrak{b}_+) \) be a crossed \( U_q(\mathfrak{b}_+) \)-submodule via this action and coaction. For \( \sum_n X^n P_n(g) \in L \) invariance under the action by \( g \) clearly means that \( X^n P_n(g) \in L \forall n \). Then from invariance under the coaction we can conclude that if \( X^n \sum_j a_j g^j \in L \) we must have \( X^n g^j \in L \forall j \). I.e. elements of the form \( X^n g^j \) lie separately in \( L \) and it is sufficient to consider such elements. From the coaction we learn that if \( X^n g^j \in L \) we have \( X^m g^j \in L \forall m \leq n \). The action by \( X \) leads to \( X^n g^j \in L \Rightarrow X^{n+1} g^j \in L \) except if \( n + j = 0 \). The classification is given by the possible choices we have for each power in \( g \). For every positive integer \( j \) we can choose whether or not to include the span of \( \{X^n g^j|\forall n\} \) in \( L \) and for every non-positive integer we can choose to include either the span of \( \{X^n g^j|\forall n\} \) or just \( \{X^n g^j|\forall n \leq -j\} \) or neither. I.e. for positive integers \( \mathbb{N} \) we have two choices while for non-positive (identified with \( \mathbb{N}_0 \)) ones we have three choices.

Clearly, the finite dimensional \( L \) are those where we choose only to include finitely many powers of \( g \) and also only finitely many powers of \( X \). The latter is only possible for the non-positive powers of \( g \). By identifying positive integers \( n \) with powers \( 1 - n \) of \( g \), we obtain a classification by finite subsets of \( \mathbb{N} \).

(b) Irreducibility clearly corresponds to just including one power of \( g \) in the finite dimensional case.

(c) The decomposition property is obvious from the discussion.

\[ \square \]

**Corollary 2.4.** (a) Left crossed \( U_q(\mathfrak{b}_+) \)-submodules \( L \subseteq \ker \epsilon \subseteq U_q(\mathfrak{b}_+) \) via the left adjoint action and left regular coaction (with subsequent projection to \( \ker \epsilon \) via \( x \mapsto x - \epsilon(x)1 \)) are in one-to-one correspondence to the set \( 3^\mathbb{N} \times 2^{\mathbb{N}_0} \). Finite dimensional \( L \) are in one-to-one correspondence to finite sets \( I \subseteq \mathbb{N} \setminus \{1\} \) and \( \dim L = \sum_{n \in I} n \).

(b) Finite dimensional irreducible \( L \) correspond to \( \{n\} \) with \( n \geq 2 \) the dimension.

(c) Finite dimensional \( L \) are direct sums of irreducible ones. In particular \( L = \oplus_{n \in I} L^n \) with \( L^n \) corresponding to \( \{n\} \).

**Proof.** Only a small modification of lemma 2.3 is necessary. Elements of the form \( P(g) \) are replaced by elements of the form \( P(g) - P(1) \). Monomials with non-vanishing degree in \( X \) are unchanged. The choices for elements of degree 0 in \( g \) are reduced to either including the span of \( \{X^k|\forall k > 0\} \) in the crossed submodule or not. In particular, the crossed submodule characterised by \( \{1\} \) in lemma 2.3 is projected out. \[ \square \]
Differential calculi in the original sense of Woronowicz are classified by corollary 2.2 while from the quantum tangent space point of view the classification is given by corollary 2.4. In the finite dimensional case the duality is strict in the sense of a one-to-one correspondence. The infinite dimensional case on the other hand depends strongly on the algebraic models we use for the function or enveloping algebras. It is therefore not surprising that in the present purely algebraic context the classifications are quite different in this case. We will restrict ourselves to the finite dimensional case in the following description of the differential calculi.

Theorem 2.5. (a) Finite dimensional differential calculi \( \Gamma \) on \( C_q(B_+) \) and corresponding quantum tangent spaces \( L \) on \( U_q(b_+) \) are in one-to-one correspondence to finite sets \( I \subset \mathbb{N} \setminus \{1\} \). In particular \( \dim \Gamma = \dim L = \sum_{n \in I} n \).

(b) Coirreducible \( \Gamma \) and irreducible \( L \) correspond to \( \{n\} \) with \( n \geq 2 \) the dimension. Such a \( \Gamma \) has a right invariant basis \( \eta_0, \ldots, \eta_{n-1} \) so that the relations

\[
\begin{align*}
\text{d}X &= \eta_1 + (q^{n-1} - 1)\eta_0 X \\
[\alpha, \eta_0] &= \text{d} \alpha \quad \forall \alpha \in C_q(B_+) \\
[g, \eta_i]_{q^{n-1-i}} &= 0 \quad \forall i \\
[X, \eta_i]_{q^{n-1-i}} &= \begin{cases} 
\eta_{i+1} & \text{if } i < n - 1 \\
0 & \text{if } i = n - 1
\end{cases}
\end{align*}
\]

hold, where \( [a, b]_p := ab - pba \). By choosing the dual basis on the corresponding irreducible \( L \) we obtain the braided derivations

\[
\partial: f : = : Q_{n-1-i,g}Q_{n-1-i,X} \frac{1}{[i]_q!} (\partial_q X)^i : \quad \forall i \geq 1
\]

\[
\partial_0: f : = : Q_{n-1,g}Q_{n-1,X} f - f :
\]

for \( f \in k(q)[X, g, g^{-1}] \) with normal ordering \( k(q)[X, g, g^{-1}] \to C_q(B_+) \) given by \( g^n X^m \mapsto g^n X^m \).

(c) Finite dimensional \( \Gamma \) and \( L \) decompose into direct sums of coirreducible respectively irreducible ones. In particular \( \Gamma = \bigoplus_{n \in I} \Gamma^n \) and \( L = \bigoplus_{n \in I} L^n \) with \( \Gamma^n \) and \( L^n \) corresponding to \( \{n\} \).

Proof. (a) We observe that the classifications of lemma 2.1 and lemma 2.3 or corollary 2.2 and corollary 2.4 are dual to each other in the finite (co)dimensional case. More precisely, for \( I \subset \mathbb{N} \) finite the crossed submodule \( M \) corresponding to \( (1, I) \) in lemma 2.1 is the annihilator of the crossed submodule \( L \) corresponding to \( I \) in lemma 2.3 and vice versa. \( C_q(B_+)/M \) and \( L \) are dual spaces with the induced pairing. For \( I \subset \mathbb{N} \setminus \{1\} \) finite this descends to \( M \) corresponding to \( (1, I \cup \{1\}) \) in corollary 2.3 and \( L \) corresponding to \( I \) in corollary 2.4. For the dimension of \( \Gamma \) observe \( \dim \Gamma = \dim \ker \epsilon / M = \text{codim} \, M \).
(b) Coirreducibility (having no proper quotient) of $\Gamma$ clearly corresponds to maximality of $M$. The statement then follows from parts (b) of corollaries 2.2 and 2.4. The formulas are obtained by choosing the basis $\eta_0, \ldots, \eta_{n-1}$ of $\ker \epsilon / M$ as the equivalence classes of

$$(g - 1)/(q^{n-1} - 1), X, \ldots, X^{n-1}$$

The dual basis of $L$ is then given by

$$g^{1-n} - 1, Xg^{1-n}, \ldots, q^{k(k-1)} \frac{1}{[k]_q!} X^k g^{1-n}, \ldots, q^{(n-1)(n-2)} \frac{1}{[n-1]_q!} X^{n-1} g^{1-n}$$

(c) The statement follows from corollaries 2.2 and 2.4 parts (c) with the observation

$$\ker \epsilon / M = \ker \epsilon / \bigcap_{n \in I} M^n = \oplus_{n \in I} \ker \epsilon / M^n$$

\[\square\]

**Corollary 2.6.** There is precisely one differential calculus on $C_q(B_+)$ which is natural in the sense that it has dimension 2. It is coirreducible and obeys the relations

$$[g, d X] = 0 \quad [g, d g]_q = 0 \quad [X, d X]_q = 0 \quad [X, d g]_q = (q - 1)(d X)g$$

with $[a, b]_q := ab - qba$. In particular we have

$$d : f := d g : \partial_{q,g} f + d X : \partial_{q,X} f : \quad \forall f \in \mathbb{k}(q)[X, g, g^{-1}]$$

**Proof.** This is a special case of theorem 2.3. The formulas follow from (b) with $n = 2$. \[\square\]

### 3 Classification in the Classical Limit

In this section we give the complete classification of differential calculi and quantum tangent spaces in the classical case of $C(B_+)$ along the lines of the previous section. We pay particular attention to the relation to the $q$-deformed setting.

The classical limit $C(B_+)$ of the quantum group $C_q(B_+)$ is simply obtained by substituting the parameter $q$ with 1. The classification of left crossed submodules in part (a) of lemma 2.1 remains unchanged, as one may check by going through the proof. In particular, we get a correspondence of crossed modules in the $q$-deformed setting with crossed modules in the classical setting as a map of pairs $(P, I) \mapsto (P, I)$ that converts polynomials $\mathbb{k}(q)[g]$ to polynomials $\mathbb{k}[g]$ (if defined) and leaves sets $I$ unchanged.
This is one-to-one in the finite dimensional case. However, we did use the distinctness of powers of \( q \) in part (b) and (c) of lemma 2.1 and have to account for changing this. The only place where we used it, was in observing that factors \( 1 - q^j g \) have no common divisors for distinct \( j \). This was crucial to conclude the maximality (b) of certain finite codimensional crossed submodules and the intersection property (c). Now, all those factors become \( 1 - g \).

**Corollary 3.1.** (a) Left crossed \( C(B_+) \)-submodules \( M \subseteq C(B_+) \) by left multiplication and left adjoint coaction are in one-to-one correspondence to pairs \( (P, I) \) where \( P \in k[g] \) is a polynomial with \( P(0) = 1 \) and \( I \subset \mathbb{N} \) is finite. \( \text{codim } M < \infty \) iff \( P = 1 \). In particular \( \text{codim } M = \sum_{n \in I} n \) if \( P = 1 \).

(b) The infinite codimensional maximal \( M \) are characterised by \( (P, \emptyset) \) with \( P \) irreducible and \( P(g) \neq 1 - g \) for any \( k \in \mathbb{N}_0 \).

In the restriction to \( \ker \epsilon \subseteq C(B_+) \) corresponding to corollary 2.2 we observe another difference to the \( q \)-deformed setting. Since the condition for a crossed submodule to lie in \( \ker \epsilon \) is exactly to have factors \( 1 - g \) in the \( X \)-free monomials this condition may now be satisfied more easily. If the characterising polynomial does not contain this factor it is now sufficient to have just any non-empty characterising integer set \( I \) and it need not contain 1. Consequently, the map \( (P, I) \mapsto (P, I) \) does not reach all crossed submodules now.

**Corollary 3.2.** (a) Left crossed \( C(B_+) \)-submodules \( M \subseteq \ker \epsilon \subseteq C(B_+) \) are in one-to-one correspondence to pairs \( (P, I) \) as in corollary 3.1 with the additional constraint \( (1 - g) \) divides \( P(g) \) or \( I \) non-empty. \( \text{dim } L < \infty \) iff \( P = 1 \). In particular \( \text{dim } L = (\sum_{n \in I} n) - 1 \) if \( P = 1 \).

(b) The infinite codimensional maximal \( M \) correspond to pairs \( (P, \{1\}) \) with \( P \) irreducible and \( P(g) \neq 1 - g \).

Let us now turn to quantum tangent spaces on \( U(b_+) \). Here, the process to go from the \( q \)-deformed setting to the classical one is not quite so straightforward.

**Lemma 3.3.** Proper left crossed \( U(b_+) \)-submodules \( L \subseteq U(b_+) \) via the left adjoint action and left regular coaction are in one-to-one correspondence to pairs \( (l, I) \) with \( l \in \mathbb{N}_0 \) and \( I \subset \mathbb{N} \) finite. \( \text{dim } L < \infty \) iff \( l = 0 \). In particular \( \text{dim } L = \sum_{n \in I} n \) if \( l = 0 \).

**Proof.** The left adjoint action takes the form

\[
X \triangleright X^n H^m = X^{n+1}(H^m - (H + 1)^m) \quad H \triangleright X^n H^m = nX^n H^m
\]

while the coaction is

\[
\triangle(X^n H^m) = \sum_{i=1}^{n} \sum_{j=1}^{m} \binom{n}{i} \binom{m}{j} X^i H^j \otimes X^{n-1} H^{m-j}
\]
Let \( L \) be a crossed submodule invariant under the action and coaction. The (repeated) action of \( H \) separates elements by degree in \( X \). It is therefore sufficient to consider elements of the form \( X^n P(H) \), where \( P \) is a polynomial. By acting with \( X \) on an element \( X^n P(H) \) we obtain \( X^{n+1}(P(H) - P(H+1)) \). Subsequently applying the coaction and projecting on the left hand side of the tensor product onto \( X \) (in the basis \( X^i H^j \) of \( U(\mathfrak{b}_+) \)) leads to the element \( X^n(P(H) - P(H+1)) \). Now the degree of \( P(H) - P(H+1) \) is exactly the degree of \( P(H) \) minus 1. Thus we have polynomials \( X^n P_i(H) \) of any degree \( i = \deg(P_i) \leq \deg(P) \) in \( L \) by induction. In particular, \( X^n H^m \in L \) for all \( m \leq \deg(P) \). It is therefore sufficient to consider elements of the form \( X^n H^m \).

Given such an element, the coaction generates all elements of the form \( X^i H^j \) with \( i \leq n, j \leq m \).

For given \( n \), the characterising datum is the maximal \( m \) so that \( X^n H^m \in L \). Due to the coaction this cannot decrease with decreasing \( n \) and due to the action of \( X \) this can decrease at most by 1 when increasing \( n \) by 1. This leads to the classification given. For \( l \in N_0 \) and \( I \subset N \) finite, the corresponding crossed submodule is generated by

\[
X^{n_m-1}H^{l+m-1}, X^{n_m+n_{m-1}-1}H^{l+m-2}, \ldots, X^{(\sum_i n_i)-1}H^l
\]

and

\[
X^{(\sum_i n_i)+k}H^{l-1} \quad \forall k \geq 0 \quad \text{if} \quad l > 0
\]

as a crossed module. \( \square \)

For the transition from the \( q \)-deformed (lemma 2.3) to the classical case we observe that the space spanned by \( g^{s_1}, \ldots, g^{s_m} \) with \( m \) different integers \( s_i \in \mathbb{Z} \) maps to the space spanned by \( 1, H, \ldots, H^{m-1} \) in the prescription of the classical limit (as described in section 2.2). I.e. the classical crossed submodule characterised by an integer \( l \) and a finite set \( I \subset \mathbb{N} \) comes from a crossed submodule characterised by this same \( I \) and additionally \( l \) other integers \( j \in \mathbb{Z} \) for which \( X^j g^{1-j} \) is included. In particular, we have a one-to-one correspondence in the finite dimensional case.

To formulate the analogue of corollary 2.4 for the classical case is essentially straightforward now. However, as for \( C(\mathfrak{b}_+) \), we obtain more crossed submodules than those from the \( q \)-deformed setting. This is due to the degeneracy introduced by forgetting the powers of \( g \) and just retaining the number of different powers.

**Corollary 3.4.** (a) Proper left crossed \( U(\mathfrak{b}_+) \)-submodules \( L \subset \ker \epsilon \subset U(\mathfrak{b}_+) \) via the left adjoint action and left regular coaction (with subsequent projection to \( \ker \epsilon \) via \( x \mapsto x - \epsilon(x)1 \)) are in one-to-one correspondence to pairs \((l, I)\) with \( l \in N_0 \) and \( I \subset N \) finite where \( l \neq 0 \) or \( I \neq \emptyset \). \( \dim L < \infty \) iff \( l = 0 \). In particular \( \dim L = (\sum_{n \in I} n) - 1 \) if \( l = 0 \).

As in the \( q \)-deformed setting, we give a description of the finite dimensional differential calculi where we have a strict duality to quantum tangent spaces.
Proposition 3.5. (a) Finite dimensional differential calculi $\Gamma$ on $C(B_+)$ and finite dimensional quantum tangent spaces $L$ on $U(b_+)$ are in one-to-one correspondence to non-empty finite sets $I \subset \mathbb{N}$. In particular $\dim \Gamma = \dim L = \left( \sum_{n \in I} n \right) - 1$.

The $\Gamma$ with $1 \in \mathbb{N}$ are in one-to-one correspondence to the finite dimensional calculi and quantum tangent spaces of the $q$-deformed setting (theorem 2.5(a)).

(b) The differential calculus $\Gamma$ of dimension $n \geq 2$ corresponding to the coirreducible one of $C_q(B_+)$ (theorem 2.5(b)) has a right invariant basis $\eta_0, \ldots, \eta_{n-1}$ so that

$$dX = \eta_1 + \eta_0 X \quad dg = \eta_0 g$$

$$[g, \eta_i] = 0 \quad [X, \eta_i] = \begin{cases} 0 & \text{if } i = 0 \text{ or } i = n - 1 \\ \eta_{i+1} & \text{if } 0 < i < n - 1 \end{cases}$$

hold. The braided derivations obtained from the dual basis of the corresponding $L$ are given by

$$\partial_i f = \frac{1}{i!} \left( \frac{\partial}{\partial X} \right)^i f \quad \forall i \geq 1$$

$$\partial_0 f = \left( X \frac{\partial}{X} + g \frac{\partial}{g} \right) f$$

for $f \in C(B_+)$. 

(c) The differential calculus of dimension $n - 1$ corresponding to the one in (b) with $1$ removed from the characterising set is the same as the one above, except that we set $\eta_0 = 0$ and $\partial_0 = 0$.

Proof. (a) We observe that the classifications of corollary 3.1 and lemma 3.3 or corollary 3.2 and corollary 3.4 are dual to each other in the finite (co)dimensional case. More precisely, for $I \subset \mathbb{N}$ finite the crossed submodule $M$ corresponding to $(1, I)$ in corollary 3.1 is the annihilator of the crossed submodule $L$ corresponding to $(0, I)$ in lemma 3.3 and vice versa. $C(B_+)/M$ and $L$ are dual spaces with the induced pairing. For non-empty $I$ this descends to $M$ corresponding to $(1, I)$ in corollary 3.2 and $L$ corresponding to $(0, I)$ in corollary 3.4. For the dimension of $\Gamma$ note $\dim \Gamma = \dim \ker \epsilon / M = \text{codim } M$.

(b) For $I = \{1, n\}$ we choose in $\ker \epsilon \subset C(B_+)$ the basis $\eta_0, \ldots, \eta_{n-1}$ as the equivalence classes of $g - 1, X, \ldots, X^{n-1}$. The dual basis in $L$ is then $H, X, \ldots, \frac{1}{k!} X^k, \ldots, \frac{1}{(n-1)!} X^{n-1}$. This leads to the formulas given.

(c) For $I = \{n\}$ we get the same as in (b) except that $\eta_0$ and $\partial_0$ disappear.

The classical commutative calculus is the special case of (b) with $n = 2$. It is the only calculus of dimension 2 with $dg \neq 0$. Note that it is not coirreducible.
4 The Dual Classical Limit

We proceed in this section to the more interesting point of view where we consider the classical algebras, but with their roles interchanged. I.e. we view $U(b_+)$ as the “function algebra” and $C(B_+)$ as the “enveloping algebra”.

Due to the self-duality of $U_q(b_+)$, we can again view the differential calculi and quantum tangent spaces as classical limits of the $q$-deformed setting investigated in section 2.

In this dual setting the bicovariance constraint for differential calculi becomes much weaker. In particular, the adjoint action on a classical function algebra is trivial due to commutativity and the adjoint coaction on a classical enveloping algebra is trivial due to cocommutativity. In effect, the correspondence with the $q$-deformed setting is much weaker than in the ordinary case of section 3. There are much more differential calculi and quantum tangent spaces than in the $q$-deformed setting.

We will not attempt to classify all of them in the following but essentially contend ourselves with those objects coming from the $q$-deformed setting.

**Lemma 4.1.** Left $C(B_+)$-subcomodules $\subseteq C(B_+)$ via the left regular coaction are $\mathbb{Z}$-graded subspaces of $C(B_+)$ with $|X^n g^m| = n + m$, stable under formal derivation in $X$.

By choosing any ordering in $C_q(B_+)$, left crossed submodules via left regular action and adjoint coaction are in one-to-one correspondence to certain subcomodules of $C(B_+)$ by setting $q = 1$. Direct sums correspond to direct sums.

This descends to $\ker \epsilon \subseteq C(B_+)$ by the projection $x \mapsto x - \epsilon(x)1$.

**Proof.** The coproduct on $C(B_+)$ is

$$\Delta(X^n g^k) = \sum_{r=0}^{n} \binom{n}{r} X^{n-r} g^{k+r} \otimes X^r g^k$$

which we view as a left coaction. Projecting on the left hand side of the tensor product onto $g^l$ in a basis $X^n g^k$, we observe that coacting on an element $\sum_{n,k} a_{n,k} X^n g^k$ we obtain elements $\sum_{n} a_{n,l-n} X^n g^{l-n}$ for all $l$. I.e. elements of the form $\sum_{n} b_n X^n g^{l-n}$ lie separately in a submodule and it is sufficient to consider such elements. Writing the coaction on such an element as

$$\sum_{t} \frac{1}{t!} X^t g^{l-t} \otimes \sum_{n} b_n \frac{n!}{(n-t)!} X^{n-t} g^{l-n}$$

we see that the coaction generates all formal derivatives in $X$ of this element. This gives us the classification: $C(B_+)$-subcomodules $\subseteq C(B_+)$ under the left regular coaction are $\mathbb{Z}$-graded subspaces with $|X^n g^m| = n + m$, stable under formal derivation in $X$ given by $X^n g^m \mapsto nX^{n-1} g^m$. 


The correspondence with the $C_q(B_+)$ case follows from the trivial observation that the coproduct of $C(B_+)$ is the same as that of $C_q(B_+)$ with $q = 1$.

The restriction to $\ker \epsilon$ is straightforward.

**Lemma 4.2.** The process of obtaining the classical limit $U(b_+)$ from $U_q(b_+)$ is well defined for subspaces and sends crossed $U_q(b_+)$-submodules $\subseteq U_q(b_+)$ by regular action and adjoint coaction to $U(b_+)$-submodules $\subseteq U(b_+)$ by regular action. This map is injective in the finite codimensional case. Intersections and codimensions are preserved in this case.

This descends to $\ker \epsilon$.

**Proof.** To obtain the classical limit of a left ideal it is enough to apply the limiting process (as described in section 1.2) to the module generators (We can forget the additional comodule structure). On the one hand, any element generated by left multiplication with polynomials in $g$ corresponds to some element generated by left multiplication with a polynomial in $H$, that is, there will be no more generators in the classical setting. On the other hand, left multiplication by a polynomial in $H$ comes from left multiplication by the same polynomial in $g - 1$, that is, there will be no fewer generators.

The maximal left crossed $U_q(b_+)$-submodule $\subseteq U_q(b_+)$ by left multiplication and adjoint coaction of codimension $n$ ($n \geq 1$) is generated as a left ideal by $\{1 - q^{1-n}g, X^n\}$ (see lemma 2.1). Applying the limiting process to this leads to the left ideal of $U(b_+)$ (which is not maximal for $n \neq 1$) generated by $\{H + n - 1, X^n\}$ having also codimension $n$.

More generally, the picture given for arbitrary finite codimensional left crossed modules of $U_q(b_+)$ in terms of generators with respect to polynomials in $g, g^{-1}$ in lemma 2.1 carries over by replacing factors $1 - q^{1-n}g$ with factors $H + n - 1$ leading to generators with respect to polynomials in $H$. In particular, intersections go to intersections since the distinctness of the factors for different $n$ is conserved.

The restriction to $\ker \epsilon$ is straightforward.

We are now in a position to give a detailed description of the differential calculi induced from the $q$-deformed setting by the limiting process.

**Proposition 4.3.** (a) Certain finite dimensional differential calculi $\Gamma$ on $U(b_+)$ and quantum tangent spaces $L$ on $C(B_+)$ are in one-to-one correspondence to finite dimensional differential calculi on $U_q(b_+)$ and quantum tangent spaces on $C_q(B_+)$. Intersections correspond to intersections.

(b) In particular, $\Gamma$ and $L$ corresponding to coirreducible differential calculi on $U_q(b_+)$ and irreducible quantum tangent spaces on $C_q(B_+)$ via the limiting process are given as follows: $\Gamma$ has a right invariant basis

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\( \eta_0, \ldots, \eta_n \) so that
\[
\begin{align*}
\mathrm{d} X &= \eta_1 \\
\mathrm{d} H &= (1 - n) \eta_0
\end{align*}
\]
\[
[H, \eta_i] = (1 - n + i) \eta_i \quad \forall i
\]
\[
[X, \eta_i] = \begin{cases} 
\eta_{i+1} & \text{if } i < n - 1 \\
0 & \text{if } i = n - 1
\end{cases}
\]
holds. The braided derivations corresponding to the dual basis of \( L \) are given by
\[
\partial_i : f := : T_{1-n+i,H} \frac{1}{i!} \left( \frac{\partial}{\partial X} \right)^i f : \quad \forall i \geq 1
\]
\[
\partial_0 : f := : T_{1-n,H} f - f :
\]
for \( f \in \mathbb{k}[X, H] \) with the normal ordering \( \mathbb{k}[X, H] \to U(b_+) \) via \( H^n X^m \mapsto H^n X^m \).

Proof. (a) The strict duality between \( C(B_+)-\)subcomodules \( L \subseteq \ker \epsilon \) given by lemma 4.1 and corollary 2.4 and \( U(b_+)-\)modules \( U(b_+)/(\mathbb{k}1 + M) \) with \( M \) given by lemma 4.2 and corollary 2.2 can be checked explicitly. It is essentially due to mutual annihilation of factors \( H+k \) in \( U(b_+) \) with elements \( g^k \) in \( C(B_+) \).

(b) \( L \) is generated by \( \{ g^{1-n} - 1, X g^{1-n}, \ldots, X^{n-1} g^{1-n} \} \) and \( M \) is generated by \( \{ H(H + n - 1), X(H + n - 1), X^n \} \). The formulas are obtained by denoting with \( \eta_0, \ldots, \eta_{n-1} \) the equivalence classes of \( H/(1-n), X, \ldots, X^{n-1} \) in \( U(b_+)/(\mathbb{k}1 + M) \). The dual basis of \( L \) is then
\[
g^{1-n} - 1, X g^{1-n}, \ldots, \frac{1}{(n-1)!} X^{n-1} g^{1-n}
\]

In contrast to the \( q \)-deformed setting and to the usual classical setting the many freedoms in choosing a calculus leave us with many 2-dimensional calculi. It is not obvious which one we should consider to be the “natural” one. Let us first look at the 2-dimensional calculus coming from the \( q \)-deformed setting as described in (b). The relations become
\[
[d H, a] = da \quad [d X, a] = 0 \quad \forall a \in U(b_+)
\]
\[
\begin{align*}
\mathrm{d} : f := d H : \nabla_{1,H} f : + d X : \frac{\partial}{\partial X} f :
\end{align*}
\]
for \( f \in \mathbb{k}[X, H] \).

We might want to consider calculi which are closer to the classical theory in the sense that derivatives are not finite differences but usual derivatives. Let us therefore demand
\[
\begin{align*}
\mathrm{d} P(H) &= d H \frac{\partial}{\partial H} P(H) \quad \text{and} \quad \mathrm{d} P(X) = d X \frac{\partial}{\partial X} P(X)
\end{align*}
\]
for polynomials \( P \) and \( d X \neq 0 \) and \( d H \neq 0 \).
Proposition 4.4. There is precisely one differential calculus of dimension 2 meeting these conditions. It obeys the relations

\[ [a, dH] = 0 \quad [X, dX] = 0 \quad [H, dX] = dX \]

\[ d : f = d H : \frac{\partial}{\partial H} f + d X : \frac{\partial}{\partial X} f : \]

where the normal ordering \( \mathbb{k}[X, H] \to U(b_+) \) is given by \( X \| H \mapsto U(b^+_{\mathbb{k}}) \).

Proof. Let \( M \) be the left ideal corresponding to the calculus. It is easy to see that for a primitive element \( a \) the classical derivation condition corresponds to \( a^2 \in M \) and \( a \notin M \). In our case \( X^2, H^2 \in M \). If we take the ideal generated from these two elements we obtain an ideal of \( \ker \epsilon \) of codimension 3. Now, it is sufficient without loss of generality to add a generator of the form \( \alpha H + \beta X + \gamma X H \). \( \alpha \) and \( \beta \) must then be zero in order not to generate \( X \) or \( H \) in \( M \). I.e. \( M \) is generated by \( H^2, XH, X^2 \). The relations stated follow. \( \square \)

5 Remarks on \( \kappa \)-Minkowski Space and Integration

There is a straightforward generalisation of \( U(b_-) \). Let us define the Lie algebra \( b_{n+} \) as generated by \( x_0, \ldots, x_{n-1} \) with relations

\[ [x_0, x_i] = x_i \quad [x_i, x_j] = 0 \quad \forall i, j \geq 1 \]

Its enveloping algebra \( U(b_{n+}) \) is nothing but (rescaled) \( \kappa \)-Minkowski space as introduced in \( [9] \). In this section we make some remarks about its intrinsic geometry.

We have an injective Lie algebra homomorphism \( b_{n+} \to b_+ \) given by \( x_0 \mapsto H \) and \( x_i \mapsto X \). This is an isomorphism for \( n = 2 \). The injective Lie algebra homomorphism extends to an injective homomorphism of enveloping algebras \( U(b_+) \to U(b_{n+}) \) in the obvious way. This gives rise to an injective map from the set of submodules of \( U(b_+) \) to the set of submodules of \( U(b_{n+}) \) by taking the pre-image. In particular this induces an injective map from the set of differential calculi on \( U(b_+) \) to the set of differential calculi on \( U(b_{n+}) \) which are invariant under permutations of the \( x_i i \geq 1 \).

Corollary 5.1. There is a natural \( n \)-dimensional differential calculus on \( U(b_{n+}) \) induced from the one considered in proposition [4.4]. It obeys the relations

\[ [a, dx_0] = 0 \quad \forall a \in U(b_{n+}) \quad [x_i, dx_j] = 0 \quad [x_0, dx_i] = dx_i \quad \forall i, j \geq 1 \]

\[ d : f = \sum_{\mu=0}^{n-1} dx_{\mu} : \frac{\partial}{\partial x_{\mu}} f : \]
where the normal ordering is given by
\[ k[x_0, \ldots, x_{n-1}] \to U(b_{n+}) \text{ via } x_{n-1}^{m_{n-1}} \cdots x_0^{m_0} \to x_{n-1}^{m_{n-1}} \cdots x_0^{m_0} \]

Proof. The calculus is obtained from the ideal generated by
\[ x_0^2, x_ix_j, x_ix_0 \quad \forall i, j \geq 1 \]
being the pre-image of \( X^2, XH, X^2 \) in \( U(b_+) \).

Let us try to push the analogy with the commutative case further and take a look at the notion of integration. The natural way to encode the condition of translation invariance from the classical context in the quantum group context is given by the condition
\[ (\int \otimes \text{id}) \circ \triangle a = 1 \int a \quad \forall a \in A \]
which defines a right integral on a quantum group \( A \). (Correspondingly, we have the notion of a left integral.) Let us formulate a slightly weaker version of this equation in the context of a Hopf algebra \( H \) dually paired with \( A \). We write
\[ \int (h - \epsilon(h)) \triangleright a = 0 \quad \forall h \in H, a \in A \]
where the action of \( H \) on \( A \) is the coregular action \( h \triangleright a = a_{(1)}(a_{(2)}, h) \) given by the pairing.

In the present context we set \( A = U(b_{n+}) \) and \( H = C(B_{n+}) \). We define the latter as a generalisation of \( C(B_+) \) with commuting generators \( g, p_1, \ldots, p_{n-1} \) and coproducts
\[ \Delta p_i = p_i \otimes 1 + g \otimes p_i \quad \Delta g = g \otimes g \]
This can be identified (upon rescaling) as the momentum sector of the full \( \kappa \)-Poincaré algebra (with \( g = e^{p_0} \)). The pairing is the natural extension of \([13]\):
\[ \langle x_{n-1}^{m_{n-1}} \cdots x_1^{m_1} x_0^{m_0}, p_{n-1}^{r_{n-1}} \cdots p_1^{r_1} g^{s} \rangle = \delta_{m_{n-1}, r_{n-1}} \cdots \delta_{m_1, r_1} m_{n-1}! \cdots m_1! s^k \]
The resulting coregular action is conveniently expressed as (see also [13])
\[ p_i \triangleright : f : = : \frac{\partial}{\partial x_i} f : \quad g \triangleright : f : = : T_{1,x_0} f : \]
with \( f \in \mathbb{k}[x_0, \ldots, x_{n-1}] \). Due to cocommutativity, the notions of left and right integral coincide. The invariance conditions for integration become
\[ \int : \frac{\partial}{\partial x_i} f : = 0 \quad \forall i \in \{1, \ldots, n-1\} \quad \text{and} \quad \int : \nabla_{1,x_0} f : = 0 \]
The condition on the left is familiar and states the invariance under infinitesimal translations in the $x_i$. The condition on the right states the invariance under integer translations in $x_0$. However, we should remember that we use a certain algebraic model of $C(B_{n+})$. We might add, for example, a generator $p_0$ to $C(B_{n+})$ that is dual to $x_0$ and behaves as the “logarithm” of $g$, i.e. acts as an infinitesimal translation in $x_0$. We then have the condition of infinitesimal translation invariance

$$\int : \frac{\partial}{\partial x_\mu} f : = 0$$

for all $\mu \in \{0, 1, \ldots, n - 1\}$.

In the present purely algebraic context these conditions do not make much sense. In fact they would force the integral to be zero on the whole algebra. This is not surprising, since we are dealing only with polynomial functions which would not be integrable in the classical case either. In contrast, if we had for example the algebra of smooth functions in two real variables, the conditions just characterise the usual Lesbegue integral (up to normalisation). Let us assume $k = \mathbb{R}$ and suppose that we have extended the normal ordering vector space isomorphism $\mathbb{R}[x_0, \ldots, x_{n-1}] \cong U(b_{n+})$ to a vector space isomorphism of some sufficiently large class of functions on $\mathbb{R}^n$ with a suitable completion $\hat{U}(b_{n+})$ in a functional analytic framework (embedding $U(b_{n+})$ in some operator algebra on a Hilbert space). It is then natural to define the integration on $\hat{U}(b_{n+})$ by

$$\int : f : = \int_{\mathbb{R}^n} f \, dx_0 \cdots dx_{n-1}$$

where the right hand side is just the usual Lesbegue integral in $n$ real variables $x_0, \ldots, x_{n-1}$. This integral is unique (up to normalisation) in satisfying the covariance condition since, as we have seen, these correspond just to the usual translation invariance in the classical case via normal ordering, for which the Lesbegue integral is the unique solution. It is also the $q \to 1$ limit of the translation invariant integral on $U_q(b_{n+})$ obtained in [14].

We see that the natural differential calculus in corollary 5.1 is compatible with this integration in that the appearing braided derivations are exactly the actions of the translation generators $p_\mu$. However, we should stress that this calculus is not covariant under the full $\kappa$-Poincaré algebra, since it was shown in [15] that in $n = 4$ there is no such calculus of dimension 4. Our results therefore indicate a new intrinsic approach to $\kappa$-Minkowski space that allows a bicovariant differential calculus of dimension 4 and a unique translation invariant integral by normal ordering and Lesbegue integration.
Acknowledgements

I would like to thank S. Majid for proposing this project, and for fruitful discussions during the preparation of this paper.

Appendix: The adjoint coaction on $U_q(b_+)$

The coproduct on $X^n$ is

$$\Delta(X^n) = \sum_{r=0}^{n} \left[ \begin{array}{c} n \\ r \end{array} \right]_q g^r X^{n-r} \otimes X^r$$

$$(\text{id} \otimes \Delta) \Delta(X^n) = \sum_{r=0}^{n} \sum_{i=0}^{r} \left[ \begin{array}{c} n \\ r \\ i \end{array} \right] q^i g^r X^{n-r} \otimes g^i X^{r-i} \otimes X^i$$

From this we get

$$\text{Ad}_L(X^n) = \sum_{r=0}^{n} \sum_{s=0}^{r} \left[ \begin{array}{c} n \\ r \\ s \end{array} \right]_q g^r X^{n-r} s X^s \otimes g^s X^{r-s}$$

$$= \sum_{r=0}^{n} \sum_{s=0}^{r} \left[ \begin{array}{c} n \\ r \\ s \end{array} \right]_q g^r X^{n-r} (-g^{-1}X)^s \otimes g^s X^{r-s}$$

$$= \sum_{t=0}^{n} \sum_{s=0}^{n-t} \left[ \begin{array}{c} n \\ t \\ s \end{array} \right]_q g^{t+s} X^{n-t-s} (-g^{-1}X)^s \otimes g^s X^t$$

$$= \sum_{t=0}^{n} \sum_{s=0}^{n-t} \left[ \begin{array}{c} n \\ t \\ s \end{array} \right]_q g^{t+s} X^{n-t-s} (-g^{-1}X)^s \otimes g^s X^t$$

$$= \sum_{t=0}^{n} \left[ \begin{array}{c} n \\ t \end{array} \right]_q g^t X^{n-t} \otimes X^t \sum_{s=0}^{n-t} \left[ \begin{array}{c} n-t \\ s \end{array} \right]_q q^{s(s+1)/2} (-q^{-n}g)^s$$

$$= \sum_{t=0}^{n} \left[ \begin{array}{c} n \\ t \end{array} \right]_q g^t X^{n-t} \otimes X^t \prod_{u=1}^{n-t} (1 - q^{u-n}g)$$

where we have used

$$\sum_{i=0}^{n} \left[ \begin{array}{c} n \\ i \end{array} \right]_q q^{i(i+1)/2} x^i = \prod_{j=1}^{n} (1 + q^j x)$$

which can be easily checked by induction. Using the property

$$\text{Ad}_L(\alpha g^n) = \text{Ad}_L(\alpha)(1 \otimes g^n) \quad \forall n \in \mathbb{Z}$$

we obtain for any polynomial $P$ in $g, g^{-1}$:

$$\text{Ad}_L(X^n P(g)) = \sum_{t=0}^{n} \left[ \begin{array}{c} n \\ t \end{array} \right]_q g^t X^{n-t} \otimes X^t P(g) \prod_{u=1}^{n-t} (1 - q^{u-n}g) \quad (3)$$
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