REAL REGULATORS ON SELF-PRODUCTS OF K3 SURFACES

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Abstract. Based on a novel application of an archimedean type pairing to the geometry and deformation theory of K3 surfaces, we construct a regulator indecomposable $K_1$-class on a self-product of a K3 surface. In the Appendix, we explain how this pairing is a special instance of a general pairing on precycles in the equivalence relation defining Bloch’s higher Chow groups.

1. Introduction

Let $X$ be a smooth projective surface. The real regulator map

$$r_{2,1} : \text{CH}^2(X, 1) \to H^{1,1}(X, \mathbb{R}),$$

where $\text{CH}^\bullet(-, \bullet)$ are the higher Chow groups defined in [B], has been extensively studied. The image $\text{Im}(r_{2,1}) \otimes \mathbb{R}$ seems to behave according to the Kodaira dimension $\text{kod}(X)$ of $X$. Our knowledge on the subject suggests the following:

(1) if $\text{kod}(X) \leq 0$, $\text{Im}(r_{2,1}) \otimes \mathbb{R} = H^{1,1}(X, \mathbb{R})$; this is trivial when $\text{kod}(X) < 0$; for $\text{kod}(X) = 0$, a proof is given in [C-L2] for K3 and Abelian surfaces;

(2) if $\text{kod}(X) > 0$, we expect that

$$\text{Im}(r_{2,1}) \otimes \mathbb{R} \cap H^{1,1}_{\text{tr}}(X, \mathbb{R}) = \{0\}$$

for $X$ general, where $H^{1,1}_{\text{tr}}(X, \mathbb{R}) \subset H^{1,1}(X, \mathbb{R})$ is the space of transcendental classes (= orthogonal complement of algebraic classes); this is known in some special cases such as surfaces in $\mathbb{P}^3$ and products of curves [C-L1].

In this paper, we give some evidence that the real regulator

$$r_{3,1} : \text{CH}^3(X \times X, 1) \to H^4(X \times X, \mathbb{R})$$

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on the self product of a smooth projective surface $X$ exhibits a similar pattern of behavior. In particular, we have the following result for a general polarized $K3$ surface.

**Theorem 1.1.** For a general polarized $K3$ surface $(X, L)$,

$$\text{Im}(\mathcal{L}_{3,1}) \otimes \mathbb{R} \neq 0$$

where $\mathcal{L}_{3,1}$ is the reduced real regulator

$$\mathcal{L}_{3,1} : \text{CH}^3(X \times X, 1) \xrightarrow{\tau_{3,1}} H^4(X \times X, \mathbb{R}) \xrightarrow{\text{(projection)}} V(X)$$

and $V(X)$ is the subspace of $H^{1,1}(X, \mathbb{R}) \otimes H^{1,1}(X, \mathbb{R})$ given by

$$V(X) = \{ \omega \in H^{1,1}(X, \mathbb{R}) \otimes H^{1,1}(X, \mathbb{R}) : \text{for all } \gamma \in H^{1,1}(X, \mathbb{R}),$$

$$\omega \wedge (c_1(L) \otimes \gamma) = \omega \wedge (\gamma \otimes c_1(L)) = \omega \wedge [\Delta_X] = 0 \}.$$ 

Here $\Delta_X$ is the diagonal of $X \times X$ and $[Y]$ is the Poincaré dual of $Y$.

One should compare the above theorem with the results in [C-L 3], where it shows that the same regulator map is trivial on a very general product of $K3$ surfaces. The proof of the above theorem is similar to that of the Hodge-$\mathcal{D}$-conjecture for $K3$ surfaces in [C-L 2]. We choose a suitable one-parameter family $W$ of $K3$ surfaces over the unit disk $\Delta = \{ |t| < 1 \}$ and a family of higher Chow cycles $\xi_t \in \text{CH}^3(W_t \times W_t, 1)$. By studying the limit $\xi_0$ of $\xi_t$ as $t \to 0$ and the corresponding limit of the regulator maps $r_{3,1}(\xi_t)$, we are able to deduce (1.3). However, unlike in [C-L 2], where the limit $\xi_0$ is a higher Chow cycle itself, $\xi_0$ here is no longer a cycle in $\text{CH}^3(W_0 \times W_0, 1)$. Needless to say, this makes the computation of $\lim_{t \to 0} r_{3,1}(\xi_t)$ much harder.

In the course of our proof, we discover a natural pairing on $z_{\text{rat}}^*$, which is interesting in its own right. This pairing, though quite easy to define and similar to an archimedean height pairing, to the best of our knowledge, has not been exploited in this situation. An abridged version of this pairing is given in 2.3. In the Appendix (Sec. 3), we show how this pairing is a special instance of a generalized pairing on higher cycles in the equivalence relation defining Bloch’s higher Chow groups ([B]).

We wish to point out that the existence of rational curves on $K3$ surfaces is pivotal to our construction of higher Chow cycles. On the contrary we anticipate the following:

**Conjecture 1.2.** Let $X = X/\mathbb{C} \subset \mathbb{P}^3$ be a very general surface of degree $d \geq 5$. Then the reduced regulator map

$$\mathcal{L}_{3,1} : \text{CH}^3(X \times X, 1) \to H^4_{\text{tr}}(X \times X, \mathbb{R}) \cap H^{2,2}(X \times X),$$

where $\Delta_X$ is the diagonal of $X \times X$ and $[Y]$ is the Poincaré dual of $Y$. 
is zero, where $H^1_{	ext{tr}}(X \times X, \mathbb{R})$ is the space of transcendental cocycles.

The terminology “general” and “very general” means the following. If $W$ is a parameter space of a universal family of projective algebraic manifolds of a given class, then general refers to a point in a nonempty real analytic Zariski open subset of $W$ governed by certain generic properties, whereas very general refers to a point in the countable intersection of nonempty Zariski open subsets.

If one works with the notion of indecomposables as for example defined in [L2], viz., in our case

$$
\text{CH}^3_{\text{ind}}(X \times X, 1) := \text{Coker}(\text{CH}^1(X, 1) \otimes \text{CH}^{-1}(X) \xrightarrow{\cap} \text{CH}^r(X, 1)),
$$

then Theorem 1.1 implies the following:

**Corollary 1.3.** For a very general polarized $K3$ surface $(X, L)$,

$$\text{CH}^3_{\text{ind}}(X \times X, 1) \otimes \mathbb{Q} \neq 0.$$

2. **Real Regulators on Self-products of $K3$ Surfaces**

2.1. **Interesting higher Chow cycles.** The definition of higher Chow groups is given in [B] (also see [EL-V]). For the readers convenience, the definition is also included in the Appendix. Let $C$ and $D$ be two rational curves on $X$, $\Delta_C = (C \times C) \cap \Delta_X$ and $\Delta_D = (D \times D) \cap \Delta_X$. We assume that $C \in |nL|$ and $D \in |mL|$ and fix a point $p \in C \cap D$. Let

(2.1) $\xi = (f_C, C \times C) + (f_D, D \times D) + (f_\Delta, \Delta_X) + \eta \in \text{CH}^3(X \times X, 1)$

where

(2.2) $(f_C) = m\Delta_C - m(p \times C) - m(C \times p)$

(2.3) $(f_D) = n(p \times D) + n(D \times p) - n\Delta_D$

and

(2.4) $(f_\Delta) = n\Delta_D - m\Delta_C$.

It is trivial to find $\eta$ such that $\text{div}(\xi) = 0$. More specifically, since $mC \sim_{\text{rat}} nD$, we can find $g \in \mathbb{C}(X)^*$ such that $(g) = nD - mC$, where $\sim_{\text{rat}}$ is the rational equivalence relation. Then we simply let

(2.5) $\eta = (\pi_1^*g, X \times p) + (\pi_2^*g, p \times X)$

where $\pi_1$ and $\pi_2$ are the projections from $X \times X$ onto the first and second factors respectively.
2.2. Deformation of $\xi$. Next, let us study the deformation of $\xi$ and the corresponding regulator map as $X$ deforms in the moduli space of polarized $K3$’s.

Let $(X, L)$ be a special polarized $K3$ surface and $M \cup N$ be a union of two rational curves on $X$ with the following properties:

1. $\text{rank}_\mathbb{Z} \text{Pic}(X) = 3$;
2. $M$ and $N$ are nodal;
3. $M$ and $N$ meet transversely at $l = M \cdot N \geq 2$ distinct points;
4. $M$ and $N$ are linearly independent in $H^{1,1}(X, \mathbb{Q})$;
5. $M + N \sim_{\text{rat}} kL$ for some integer $k$.

The existence of such $(X, M, N)$ will be proved later.

Now let us consider a general deformation of $X$, i.e., a family of polarized $K3$ surfaces $W$ over the disk $\Gamma \cong \{|t| < 1\}$ with $W_0 = X$. Using the argument in [C1], we see that $M \cup N$ can be deformed to a rational curve on a general fiber $W_t$ by smoothing out all but one intersection between $M$ and $N$. By that we mean there exists a family of rational curves $C \subset W$ over $\Delta$, after a suitable base change, such that $C_0 = M \cup N$ and $C_\nu = M_\nu \cup N_\nu$ after we normalize $C$ by $\nu : C_\nu \to C$; the two components $M_\nu \cong \mathbb{P}^1$ and $N_\nu \cong \mathbb{P}^1$ meet transversely at a point $r_\nu$ and $r = \nu(r_\nu)$ is one of the $l$ intersections in $M \cap N$. So every point in $M \cap N$ except $r$ is “smoothed” by $\nu$. For each choice of $r$, we have a corresponding family $C$ with the above properties. So there are exactly $l$ distinct families of rational curves over $\Gamma$ in $W$ with central fiber $M \cup N$. Let $C$ and $D \subset W$ be two of them.

Let $P \subset W$ be a section of $W/\Gamma$ with $P \subset C \cap D$. We will show that $P$ can be chosen such that $p = P_0 \in M \setminus N$. Now let us consider a higher Chow precycle $\xi$ on $Z = W \times_\Gamma W$, whose restriction to a general fiber $W_t \times W_t$ is a class given in (2.1). Here we have $m = n$ with $m$ and $n$ in (2.2)-(2.4). That is, we construct a family version of $\xi$ just as above with $(C, D, P)$ replaced by $(C, D, P)$:

\[ (2.6) \quad \xi = (f_C, C \times_\Gamma C) + (f_D, D \times_\Gamma D) + (f_\Delta, \Delta_W) + \eta \]

Note that we do not necessarily have $\text{div}(\xi) = 0$ although $\text{div}(\xi_t) = 0$ for all $t \neq 0$; $\text{div}(\xi)$ might be supported on some surfaces on the central fiber $Z_0$. It is easy to see that

\[ (2.7) \quad (f_C) \neq \Delta_C - (P \times_\Gamma C) - (C \times_\Gamma P) \]

since otherwise the restriction of $(f_C)$ to $N \times N$ will be $\Delta_N$, which is not rationally equivalent to 0. Hence, unlike the construction on a single $K3$ surface, $f_C$ has zeros or poles outside of $\Delta_C, P \times_\Gamma C$ and $C \times_\Gamma P$; $(f_C)$ is also supported on some surfaces contained in $W_0 \times W_0 = X \times X$. 
We claim that $f_C$ has zeros or poles along $N \times N$. That is, we have

\[(2.8)\quad (f_C) = \Delta_C - (P \times \Gamma C) - (C \times \Gamma P) + \mu(N \times N)\]

for some $\mu \in \mathbb{Z}$. Furthermore we claim that $\mu = 1$, viz.,

\[(2.9)\quad (f_C) = \Delta_C - (P \times \Gamma C) - (C \times \Gamma P) + (N \times N).\]

Note that $f_C$ is a rational function on $C \times \Gamma C$. Or better, we should think of its pullback $f_C^\nu = \nu^* f_C$ on $C^\nu \times \Gamma C^\nu$, where $\nu : C^\nu \to C$ is the normalization of $C$. As described before, the general fiber of $C^\nu$ is a smooth rational curve and hence the central fiber $C^\nu_0$ has to be the union $M^\nu \cup N^\nu$, where $M^\nu$ and $N^\nu$ are the normalizations of $M$ and $N$, respectively, and $M^\nu$ and $N^\nu$ meet transversely at a point $r^\nu$ lying over $r = \nu(r^\nu) \in M \cap N$. To reiterate, the $l$ different choices of $r \in M \cap N$ give arise to $l$ distinct families of rational curves in $W$ with central fiber $M \cup N$. Now on $C^\nu \times \Gamma C^\nu$,

\[(2.10)\quad \Delta_C \not\sim_{\text{rat}} (P^\nu \times \Gamma C^\nu) + (C^\nu \times \Gamma P^\nu)\]

as pointed out in (2.8), where $P^\nu = \nu^{-1}(P)$. On the other hand, the two sides of (2.10) are rationally equivalent when restricted to the general fibers. Therefore, we have

\[(2.11)\quad \Delta_C \sim_{\text{rat}} (P^\nu \times \Gamma C^\nu) + (C^\nu \times \Gamma P^\nu) - \mu(N^\nu \times N^\nu) - \mu_M(M^\nu \times N^\nu) - \mu_N(N^\nu \times M^\nu).\]

When we restrict (2.11) to $M^\nu \times M^\nu$, we have

\[(2.12)\quad \Delta_{M^\nu} \sim_{\text{rat}} (P^\nu_0 \times M^\nu) + (M^\nu \times P^\nu_0) - \mu_M(M^\nu \times r^\nu) - \mu_N(r^\nu \times M^\nu).\]

This forces that $\mu_M = \mu_N = 0$. Similarly, when we restrict it to $M^\nu \times N^\nu$, we have

\[(2.13)\quad - \mu(r^\nu \times N^\nu) + (P^\nu_0 \times N^\nu) \sim_{\text{rat}} 0,\]

and hence $\mu = 1$.

One thing worth noting is that the threefold $C^\nu \times \Gamma C^\nu$ is actually singular. It has a rational double point at $r^\nu \times r^\nu$. The above argument works nevertheless. One may choose to work with a desingularization of $C^\nu \times \Gamma C^\nu$; the same argument works almost without any change.

Similarly, we have

\[(2.14)\quad (f_D) = (P \times \Gamma D) + (D \times \Gamma P) - \Delta_D - (N \times N).\]

We see that the terms $(N \times N)$ cancel each other out. Hence $\xi$ extends to a higher Chow cycle on $Z$. Note that $Z$ is an analytic space; so a higher Chow cycle in this context is likewise regarded as an analytic cycle.
Next, we will attend to the calculation of $r_{3,1}(\xi)(\omega)$ for an algebraic class $\omega \in V(X)$ on the central fiber. Our objective is to show that
\begin{equation}
(2.15) \quad r_{3,1}(\xi)(i_\ast \omega) = r_{3,1}(i^*\xi)(\omega) \neq 0
\end{equation}
for some $\omega \in V(X)$, where $i$ is the inclusion $Z_0 \hookrightarrow Z$ and $i_\ast$ is the Gysin map
\begin{equation}
(2.16) \quad i_\ast : H^{2,2}(Z_0) \to H^{3,3}(Z),
\end{equation}
defined on the level of currents. Note that $H^{3,3}(Z)$ is now Dolbeault (again, as $Z$ is analytic). Actually, we will restrict $\omega$ to a subspace of $V(X)$. Let
\begin{equation}
(2.17) \quad \hat{V}(X) = V(X) \cap [M \times M]^4 \cap [N \times N]^4 \cap H^4_{\text{alg}}(X \times X, \mathbb{Q}),
\end{equation}
where the latter term is the space of algebraic cocycles. Note that the condition $\text{rank}_\mathbb{Z} \text{Pic}(X) = 3$ implies that $\hat{V}(X)$ is nontrivial. We want to show that
\begin{equation}
(2.18) \quad r_{3,1}(\xi)(i_\ast \omega) = r_{3,1}(i^*\xi)(\omega) \neq 0
\end{equation}
for $\omega \neq 0 \in \hat{V}(X)$. Our main technical difficulty to compute $r_{3,1}(\xi)(i_\ast \omega)$ lies in the fact that $f_C$ vanishes on $N \times N$. To overcome this, we blow up $Z$ along $N \times N$. Let $\pi : \tilde{Z} \to Z$ be the blowup with exceptional divisor $E$. Then from (2.6),
\begin{equation}
(2.19) \quad \pi^*\xi = (\tilde{f}_C, \widetilde{C} \times_\Gamma C) + (\tilde{f}_D, \widetilde{D} \times_\Gamma D) + (\tilde{f}_\Delta, \widetilde{\Delta}_W) + \pi^*\eta + \alpha
\end{equation}
where $\widetilde{C} \times_\Gamma C$, $\widetilde{D} \times_\Gamma D$ and $\widetilde{\Delta}_W \subset \tilde{Z}$ are the proper transforms of $C \times_\Gamma C$, $D \times_\Gamma D$ and $\Delta_W$ under $\pi$, respectively, and $\alpha$ is a higher Chow precycle supported on $E$. It is not hard to see that
\begin{equation}
(2.20) \quad \text{div}(\alpha) = -(\widetilde{C} \times_\Gamma C \cap E) + (\widetilde{D} \times_\Gamma D \cap E).
\end{equation}
Both $\widetilde{C} \times_\Gamma C \cap E$ and $\widetilde{D} \times_\Gamma D \cap E$ are rational sections of the $\mathbb{P}^2$ bundle $E$ over $N \times N$. Note that (2.19) and (2.20) does not determine $\pi^*\xi$ uniquely. But as far as the value of
\begin{equation}
(2.21) \quad r_{3,1}(\xi)(i_\ast \omega) = r_{3,1}(\pi^*\xi)(\pi^*i_\ast \omega)
\end{equation}
is concerned, it does not really matter since if we choose $\alpha$ differently, say $\alpha'$, then (using [B]):
\begin{equation}
(2.22) \quad \alpha' - \alpha \in (i_E)_\ast \text{CH}^2(E, 1) \simeq \text{CH}^2(N \times N, 1) \oplus \text{CH}^1(N \times N, 1),
\end{equation}
where $i_E$ is the embedding $E \hookrightarrow \tilde{Z}$. If we assume for the moment that $N$ is smooth, then
\begin{equation}
(2.23) \quad r_{3,1}(\alpha' - \alpha)(\pi^*i_\ast \omega) = 0
\end{equation}
for $\omega \in \hat{V}(X)$ since $\omega \wedge [N \times N] = 0$ for such $\omega$. In summary, the value of $r_{3,1}(\pi^*\xi)(\pi^*i_*\omega)$ is independent of the choice of $\alpha$, once one first passes to a normalization of $N$. This is the crucial observation which enables us to compute the regulators in (2.21) without knowing the exact form of $\pi^*\xi$.

Let $\tilde{Z}_0$ be the proper transform of $Z_0 = W_0 \times W_0$ under the blowup $\pi : \tilde{Z} \to Z$. We have the commutative diagram

$$(2.24) \quad \begin{array}{ccc} Z_0 & \xrightarrow{i} & \tilde{Z} \\ \pi_0 \downarrow & & \pi \downarrow \\ Z_0 & \xrightarrow{i} & Z \end{array}$$

where $\widetilde{i}$ is the inclusion $\tilde{Z}_0 \subset \tilde{Z}$. It is not hard to see that

$$\pi^*i_*\omega = \tilde{i}_*\pi_0^*\omega,$$

using the fact that $\omega \in V(X) \subset [N \times N]^\perp$. Therefore, (2.21) becomes

$$r_{3,1}(\pi^*\xi)(\pi^*i_*\omega) = r_{3,1}(\pi^*\xi)(\tilde{i}_*\pi_0^*\omega) = r_{3,1}(\tilde{i}_*^*\pi^*\xi)(\pi_0^*\omega).$$

When we restrict $\pi^*\xi$ (see (2.19)) to $\tilde{Z}_0$, we have

$$\tilde{i}_*^*\pi^*\xi = (c_{M \times M}, \widetilde{M \times M}) + (c_\Delta, \widetilde{\Delta}_W) + \tilde{i}_*^*\pi^*\eta$$

$$+ (\phi_{M \times N}, \widetilde{M \times N}) + (\phi_{N \times M}, \widetilde{N \times M}) + \tilde{i}_*^*\alpha,$$

where $c_\bullet$ are nonzero constants, $\widetilde{M \times M}, \widetilde{M \times N}$ and $\widetilde{N \times M}$ are proper transforms of $M \times M$, $M \times N$ and $N \times M$, and $\phi_{M \times N}$ and $\phi_{N \times M}$ are rational functions on $\widetilde{M \times N}$ and $\widetilde{N \times M}$, respectively. Note that $\widetilde{M \times N}$ and $\widetilde{N \times M}$ are blowups of $M \times N$ and $N \times M$ along $N \times N$ and hence

$$\widetilde{M \times N} \cong M \times N \text{ and } \widetilde{N \times M} \cong N \times M,$$

using the assumption that $M$ and $N$ meet transversally. So we may write $M \times N$ and $N \times M$ for $\widetilde{M \times N}$ and $\widetilde{N \times M}$ wherever no confusion is possible. It is easy to see that

$$r_{3,1}(\tilde{i}_*^*\pi^*\eta)(\pi_0^*\omega) = 0.$$

Also the contributions of the first two terms on the RHS of (2.27) are zero as well since

$$\omega \wedge [M \times M] = \omega \wedge [\Delta_W] = 0.$$
So all the nontrivial contributions to \( r_{3,1}(\tilde{i}^* \pi^* \xi)(\pi_0^* \omega) \) come from the last three terms of (2.27). It follows from (2.13) that

\[
(\phi_{M \times N}) = (r_1 \times N) - (r_2 \times N),
\]

where \( r_1 \) and \( r_2 \) are two distinct points among the intersection \( M \cap N \). Similarly,

\[
(\phi_{N \times M}) = (N \times r_1) - (N \times r_2).
\]

More precisely, there exists a rational function \( \phi \in \mathbb{C}(M)^* \) such that

\[
(\phi) = r_1 - r_2, \quad \phi_{M \times N} = (\rho_1)^* \phi, \quad \text{and} \quad \phi_{N \times M} = (\rho_2)^* \phi
\]

where \( \rho_1 \) and \( \rho_2 \) are the projections \( M \times N \to M \) and \( N \times M \to M \), respectively. It remains to figure out how to compute the term

\[
(2.34) \quad r_{3,1}(\tilde{i}^* \alpha)(\pi_0^* \omega).
\]

Once that is done, we will know exactly how to compute \( r_{3,1}(\tilde{i}^* \pi^* \xi)(\pi_0^* \omega) \). Then by an appropriate choice of \( \omega \), we will arrive at (2.18).

Obviously, \( \pi_0 : \tilde{Z}_0 \to Z_0 \) is the blowup of \( Z_0 \) along \( N \times N \) with exception divisor \( E_0 = E \cap \tilde{Z}_0 \). Note that \( E_0 \) is a \( \mathbb{P}^1 \) bundle over \( N \times N \). Since \( \tilde{i}^* \alpha \) is supported on \( E_0 \), it can be regarded as a higher Chow precycle on \( E_0 \). More precisely, let \( \nu : E_0^\nu \to \tilde{Z}_0 \) be the normalization of \( E_0 \). There exists a higher Chow precycle

\[
(2.35) \quad \beta \in C_{\text{pre}}^2(E_0^\nu, 1)
\]

such that \( \nu_* \beta = \tilde{i}^* \alpha \). Then

\[
(2.36) \quad r_{3,1}(\tilde{i}^* \alpha)(\pi_0^* \omega) = r_{2,1}(\beta)(\nu^* \pi_0^* \omega)
\]

where

\[
(2.37) \quad \text{div}(\beta) = -\nu^* \tilde{i}^* (C \times_{\Gamma} C) + \nu^* \tilde{i}^* (D \times_{\Gamma} D).
\]

2.3. Pairing on \( z^*_{\text{rat}} \). The RHS of (2.36) can be put in a more general context as follows. Let \( Y \) be a smooth projective variety of pure dimension \( n \). Let \( z^k_{\text{rat}}(Y) \) be the subgroup of the algebraic cycles \( z^k(Y) \) of codimension \( k \) on \( Y \), that are rationally equivalent to zero. Note that for every \( \eta \in z^k_{\text{rat}}(Y) \), there exists \( \beta \in C_{\text{pre}}^k(Y, 1) \) with

\[
(2.38) \quad \text{div}(\beta) = \eta.
\]

We have a pairing

\[
(2.39) \quad \langle \ , \rangle : z^k_{\text{rat}}(Y) \times z^{n+1-k}_{\text{rat}}(Y) \to \mathbb{R}
\]

given by

\[
(2.40) \quad \langle \eta, \varepsilon \rangle = r_{k,1}(\beta)(\varepsilon) = \sum_{D_i \cap \varepsilon} \log |f_i|
\]
where \( \eta \in z^k_{rat}(Y) \), \( \varepsilon \in z^{n+1-k}_{rat}(Y) \) and \( \beta = \sum (f_i, D_i) \) is a higher Chow precycle satisfying (2.38). Note that the RHS of (2.40) is only well-defined for \( |\eta| \cap |\varepsilon| = \emptyset \). A less obvious statement however, is that \( \beta \) can be chosen so that \( D_i \cap \varepsilon \) is a proper intersection for all \( i \) (hence is a zero-cycle). Although we explain this in detail in the Appendix, the existence of \( \beta \) can be deduced from [B] (Lemma (4.2)), together with a standard norm argument. It is easy to see that the pairing is well-defined, i.e., it is independent of the exact choice of \( \beta \), since if \( \text{div}(\beta - \beta') = 0 \), then \( \beta - \beta' \) is a higher Chow cycle and hence

\[
(2.41) \quad r_{k,1}(\beta - \beta')(\varepsilon) = 0
\]

as \( \varepsilon \sim_{\text{rat}} 0 \). It is also obvious that this pairing extends naturally to \( z^*_{rat}(Y) \otimes \mathbb{Q} \). The projection formula holds trivially from the definition. That is, we have

**Proposition 2.1.** Let \( \pi : X \rightarrow Y \) be a flat surjective morphism between two smooth projective varieties \( X \) and \( Y \). Then \( \langle \eta, \pi^*\varepsilon \rangle = \langle \pi_*\eta, \varepsilon \rangle \) for all \( \eta \in z^k_{rat}(X) \) and \( \varepsilon \in z^{m-k+1}_{rat}(Y) \) with \( |\pi_*\eta| \cap |\varepsilon| = \emptyset \), where \( m = \dim X \).

A little less obvious fact is that this pairing is symmetric. That is, it has the following property which we will call the *reciprocity property* of the pairing (2.39).

**Proposition 2.2.** For all \( \eta \in z^k_{rat}(Y) \) and \( \varepsilon \in z^{n+1-k}_{rat}(Y) \) with \( |\eta| \cap |\varepsilon| = \emptyset \), \( \langle \eta, \varepsilon \rangle = \langle \varepsilon, \eta \rangle \).

**Proof.** This can be deduced from Theorem 3.2 on a generalized pairing, in the Appendix; however it is instructive to give a direct proof of this. Let \( (f, D) \) and \( (g, E) \) be the higher Chow precycles such that \( \eta = \text{div}(f) \) and \( \varepsilon = \text{div}(g) \). Again, by Lemma (4.2) in [B], we can assume that with regard to the pairs \( (f, D), (g, E) \), everything is in “general” position. For notational simplicity, let us assume that \( D \) and \( E \) are irreducible and meet properly along an irreducible curve \( C \). Let

\[
(2.42) \quad f_{C} := f|_{C} \in \mathbb{C}(C)^* \quad g_{C} := g|_{C} \in \mathbb{C}(C)^*
\]

For every point \( p \in C \), put

\[
(2.43) \quad T_p(f_c, g_c) = (-1)^{\nu_p(f_c)}\nu_p(g_c) \left( \frac{f_c^{\nu_p(g_c)}}{g_c^{\nu_p(f_c)}} \right)_p.
\]
where \( \nu_p(h) \) is the vanishing order of a function \( h \) at \( p \). Since \( |\varepsilon| \cap |\eta| = \emptyset \), it follows that

\[
T_p\{f_c, g_c\} = \begin{cases} 
  f_{c\nu_p(g_c)}(p) & \text{if } \nu_p(g_c) \neq 0 \\
  g_{c\nu_p(f_c)}(p) & \text{if } \nu_p(f_c) \neq 0 \\
  1 & \text{otherwise}
\end{cases}
\]

Then it is a consequence of Weil reciprocity:

\[
\prod_{p \in C} T_p\{f_c, g_c\} = 1.
\]

that

\[
\int_{D \cap \text{div}(g)} \log |f| = \int_{E \cap \text{div}(f)} \log |g|.
\]

Obviously, this is equivalent to \( \langle \eta, \varepsilon \rangle = \langle \varepsilon, \eta \rangle \).

In addition, this pairing is also nondegenerate.

**Proposition 2.3.** If \( \langle \eta, \varepsilon \rangle = 0 \) for all \( \varepsilon \in z_{\text{rat}}^{n+1-k}(Y) \) with \( |\eta| \cap |\varepsilon| = \emptyset \), then \( \eta = 0 \).

**Proof.** Let \((f, D)\) and \((g, E)\) be the higher Chow cycles such that \( \eta = \text{div}(f) \) and \( \varepsilon = \text{div}(g) \). Assume to the contrary that \( \eta \neq 0 \) and choose \( E \) such that

\[
E \cap \eta = \sum_{i=1}^{N} (p_i - q_i), \quad \{p_1, \ldots, p_N\} \cap \{q_1, \ldots, q_N\} = \emptyset.
\]

So it suffices to find \( g \in C(E)^* \) such that

\[
\log \left| \prod_{i=1}^{N} \frac{g(p_i)}{g(q_i)} \right| = \sum_{i=1}^{N} \left[ \log |g(p_i)| - \log |g(q_i)| \right] \neq 0.
\]

This is obvious for \( E = \mathbb{P}^1 \). For arbitrary \( E \), it is enough to take a general projection \( E \to \mathbb{P}^1 \).

Now let us go back to \((2.36)\). Its RHS can be interpreted as the pairing

\[
r_{2,1}(\beta)(\nu^*\pi_0^*\omega) = \langle \text{div}(\beta), \nu^*\pi_0^*\varepsilon \rangle
\]

where \( \text{div}(\beta) \) is given in \((2.37)\) and \( \varepsilon \in z^2(Z_0) \) is an algebraic cycle with \( [\varepsilon] = \omega \). It is easy to see that \( \nu^*\pi_0^*\varepsilon \in z_{\text{rat}}^2(E_0^c) \) for \( \varepsilon \) with \( [\varepsilon] = \omega \) since

\[
\omega \wedge [N \times N] = 0.
\]
For a general choice of $\varepsilon$, we clearly have $|\text{div}(\beta)| \cap |\nu^*\pi_0^*\varepsilon| = \emptyset$. Let $N^\nu \cong \mathbb{P}^1$ be the normalization of $N$. Then $E_0^\nu$ is a $\mathbb{P}^1$ bundle over $N^\nu \times N^\nu$. We have the commutative diagram

\[
\begin{array}{ccc}
E_0^\nu & \xrightarrow{\nu} & \tilde{Z}_0 \\
\downarrow{\pi_N} & & \downarrow{\pi_0} \\
N^\nu \times N^\nu & \xrightarrow{\nu_N} & Z_0
\end{array}
\]

Hence, $\nu^*\pi_0^*\varepsilon = \pi_N^*\nu_N^*\varepsilon$ and

\[
\langle \text{div}(\beta), \nu^*\pi_0^*\varepsilon \rangle = \langle \text{div}(\beta), \pi_N^*\nu_N^*\varepsilon \rangle = \langle (\pi_N)_*\text{div}(\beta), \nu_N^*\varepsilon \rangle
\]

for all $\varepsilon \in z^2(Z_0)$ with $[\varepsilon] \in \hat{\nu}(X)$ and $|\text{div}(\beta)| \cap |\pi_N^*\nu_N^*\varepsilon| = \emptyset$. Since $\text{div}(\hat{\imath}^*\pi_N^*\xi) = 0$, it follows from (2.31) and (2.32) that

\[
\langle (\pi_N)_*\text{div}(\beta), \nu_N^*\varepsilon \rangle = -(r_1^\nu \times N^\nu) - (N^\nu \times r_1^\nu) + (r_2^\nu \times N^\nu) + (N^\nu \times r_2^\nu),
\]

where $r_1^\nu$ and $r_2^\nu$ are the points over $r_1$ and $r_2$, respectively, under the normalization $\hat{N}^\nu \to N$. Combining this with (2.33), we have

\[
r_{3,1}(\hat{\imath}^*\pi_N^*\xi) = \int_{\varepsilon \cap N} \log |\phi| + \int_{\varepsilon \cap N \times M} \log |\phi| = \int_{(p_1)_{\varepsilon}}\log |\phi| + \int_{(p_2)_{\varepsilon}}\log |\phi|
\]

(2.54)

where $p_1$ and $p_2$ are two projections $N \times N \to N$ and $\varphi \in \mathbb{C}(N)^*$ is a rational function with $(\varphi) = r_2 - r_1$. Here by $(\rho_1)_{\varepsilon}$ we really mean $(\rho_1)_{\varepsilon}(\varepsilon \cap M \times N)$. It remains to find appropriate $\varepsilon$ such that the RHS of (2.54) is nonzero.

2.4. **Choice of $\varepsilon$.** Consider $\varepsilon = \gamma \otimes \delta$, where $\gamma, \delta \in z^2(X)$ satisfy

\[
[\gamma] \wedge [M] = [\gamma] \wedge [N] = 0
\]

and

\[
[\delta] \wedge [M + N] = [\gamma] \wedge [\delta] = 0.
\]

Obviously, $[\varepsilon] \in \hat{\nu}(X)$ for such $\gamma$ and $\delta$. By (2.55),

\[
(\rho_2)_{\varepsilon} = \deg(\gamma \cdot N)(\delta \cdot M) = 0
\]

and

\[
(\rho_2)_{\varepsilon} = \deg(\gamma \cdot N)(\delta \cdot N) = 0.
\]
Therefore,
\[
\begin{align*}
  r_{3,1}(\tilde{\iota}^*\pi^*\xi)(\pi_0^*\varepsilon) &= \text{deg}(\delta \cdot N) \left( \int_{\gamma \cap M} \log |\phi| + \int_{\gamma \cap N} \log |\varphi| \right) \\
  &= \text{deg}(\delta \cdot N) \left( \langle r_1 - r_2, \gamma \rangle_M + \langle r_2 - r_1, \gamma \rangle_N \right)
\end{align*}
\]  

(2.59)

where we use $\langle \ , \ \rangle_M$ and $\langle \ , \ \rangle_N$ for the pairings:

\[
\langle \ , \ \rangle_M : z_{\text{rat}}^1(M) \times z_{\text{rat}}^1(M) \rightarrow \mathbb{R}
\]

and

\[
\langle \ , \ \rangle_N : z_{\text{rat}}^1(N) \times z_{\text{rat}}^1(N) \rightarrow \mathbb{R}
\]

(2.60)

respectively. Note that for every $\gamma$ satisfying (2.55), we can always find $\delta$ satisfying (2.56) and $[\delta] \wedge [N] \neq 0$ by dimension counting. Therefore, $\text{deg}(\delta \cdot N) \neq 0$ and it suffices to find $\gamma$ satisfying (2.55) and

\[
\langle r_1 - r_2, \gamma \rangle_M + \langle r_2 - r_1, \gamma \rangle_N \neq 0.
\]

(2.62)

As a side note, we see that the LHS of (2.62) vanishes for $\gamma \in z_{\text{rat}}^1(X)$ and $r_1, r_2 \not\in |\gamma|$; if $\gamma = (g)$ for some $g \in \mathbb{C}(X)^*$ and $g(r_1)g(r_2) \neq 0$, then

\[
\begin{align*}
  \langle r_1 - r_2, \gamma \rangle_M + \langle r_2 - r_1, \gamma \rangle_N \\
  &= \langle \gamma, r_1 - r_2 \rangle_M + \langle \gamma, r_2 - r_1 \rangle_N \\
  &= (\log |g(r_1)| - \log |g(r_2)|) + (\log |g(r_2)| - \log |g(r_1)|) = 0.
\end{align*}
\]

(2.63)

This is consistent with the fact that $r_{3,1}(\tilde{\iota}^*\pi^*\xi)(\pi_0^*\varepsilon) = 0$ for $[\varepsilon] = 0$. Let

\[
\mu : M^\nu \cup_{\{r_1, r_2\}} N^\nu \rightarrow X
\]

(2.64)

be a partial normalization of $M \cup N$ that normalizes every singularity of $M \cup N$ except $r_1$ and $r_2$. That is, $M^\nu \cup_{\{r_1, r_2\}} N^\nu$ is the union of $M^\nu \cong \mathbb{P}^1$ and $N^\nu \cong \mathbb{P}^1$ meeting transversely at two points which we still denote by $r_1$ and $r_2$.

Consider $\text{Pic}^{0,0}(M^\nu \cup_{\{r_1, r_2\}} N^\nu)$, which is the Picard group of Cartier divisors (line bundles) on $M^\nu \cup_{\{r_1, r_2\}} N^\nu$ whose degrees are zero when restricted to $M^\nu$ and $N^\nu$, respectively. We have a well-defined map

\[
\text{Pic}^{0,0}(M^\nu \cup_{\{r_1, r_2\}} N^\nu) \rightarrow \mathbb{C}^*
\]

(2.65)

sending

\[
\gamma \rightarrow \left( \frac{\phi_{M,\gamma}(r_1)}{\phi_{M,\gamma}(r_2)} \right) / \left( \frac{\phi_{N,\gamma}(r_1)}{\phi_{N,\gamma}(r_2)} \right)
\]

(2.66)
where $\phi_{M,\gamma} \in \mathbb{C}(M^\nu)^*$ and $\phi_{N,\gamma} \in \mathbb{C}(N^\nu)^*$ are rational functions such that

$\gamma = (\phi_{M,\gamma}) + (\phi_{N,\gamma})$.

It is well known that $\text{Pic}^{0,0}(M^\nu \cup \{r_1, r_2\}, N^\nu) \cong \mathbb{C}^*$. Actually, (2.65) gives such an isomorphism. We have the natural pullback map

$\mu^* : M^\perp \cap N^\perp \to \text{Pic}^{0,0}(M^\nu \cup \{r_1, r_2\}, N^\nu)$

where

$M^\perp \cap N^\perp = \{\gamma \in \text{Pic}(X) : \gamma \cdot M = \gamma \cdot N = 0\}$.

Combining this map with (2.65), we have the map

$h : M^\perp \cap N^\perp \xrightarrow{\mu^*} \text{Pic}^{0,0}(M^\nu \cup \{r_1, r_2\}, N^\nu) \xrightarrow{\sim} \mathbb{C}^*$

for which it is easy to see that

$\log |h(\gamma)| = \langle r_1 - r_2, \gamma \rangle_M + \langle r_2 - r_1, \gamma \rangle_N$.

Therefore, it comes down to find $\gamma \in M^\perp \cap N^\perp$ such that $\log |h(\gamma)| \neq 0$. We will show such $\gamma$ exists for a general deformation of $(X, M, N)$. More precisely, we expect the following to be true.

**Conjecture 2.4.** Let $X$ be a $K3$ surface and $M$ and $N$ are two rational curves on $X$ that meets transversely along at least two distinct points $r_1$ and $r_2$. Suppose that $M^\perp \cap N^\perp \neq 0$. Let $(X', \mathcal{M}, \mathcal{N})$ be a general deformation of $(X, M, N)$. That is, $X'$ is a family of $K3$ surfaces over a quasi-projective curve $\Gamma$ and $\mathcal{M}$ and $\mathcal{N} \subset X'$ are two families of rational curves over $\Gamma$ with $(X_0, \mathcal{M}_0, \mathcal{N}_0) = (X, M, N)$. Let

$\mu^* : M^\perp \cap N^\perp \to \text{Pic}^{0,0}(M^\nu \cup \{r_1, r_2\}, N^\nu)$

be the family version of the map (2.70), where we fix two sections $R_1$ and $R_2$ of $X'/\Gamma$ with $R_i \subset \mathcal{M} \cap \mathcal{N}$ and $R_i \cap X_0 = r_i$ for $i = 1, 2$. Then $\mathbb{C}^*$ is dominated by one of the components of $M^\perp \cap N^\perp$ under the map $h$.

We state the above as a conjecture since we won’t be able to prove it in full generality. However, we only need to prove it for some special triples $(X, M, N)$ anyway. So now we come to the geometric side of the problem, i.e., to find such $(X, M, N)$ with the required properties at the very beginning of our construction and for which Conjecture 2.4 holds.
2.5. Construction of \((X, M, N)\). Let \(X\) be a \(K3\) surface with Picard lattice

\[
\begin{pmatrix}
2m & 1 & 1 \\
1 & -2 & 2 \\
1 & 2 & -2
\end{pmatrix}
\]

(2.73)

That is, \(\text{Pic}(X)\) is generated by \(E_1, E_2\) and \(G\) with \(E_1^2 = -2, E_1 \cdot E_2 = 2, E_1 \cdot G = 1\) and \(G^2 = 2m\), where \(m = -1\) or 0. The pencil \(|E_1 + E_2|\) realizes \(X\) as an elliptic fibration \(X \rightarrow \mathbb{P}^1\). It has exactly 23 singular fibers; one of them is \(E_1 \cup E_2\) with two smooth rational curves \(E_1\) and \(E_2\) meeting transversely at two points; the other 22 singular fibers are rational curves each with exactly one node. Such \(K3\) surface can be polarized by the very ample divisor \(G + k(E_1 + E_2)\) which results in a \(K3\) surface of genus \(2k + m + 1\). By choosing different combinations of \((m, k)\), \((X, G + k(E_1 + E_2))\) lies on every irreducible component of the moduli space of polarized \(K3\) surfaces. In other word, every polarized \(K3\) surfaces can be deformed to \((X, G + k(E_1 + E_2))\) for some \(m\) and \(k\).

Such \(X\) can be explicitly constructed as a double cover of \(S = \mathbb{P}^1 \times \mathbb{P}^1\) or \(\mathbb{F}_1\) ramified over a smooth curve \(R \in |-2K_S|\), where \(K_S\) is the canonical divisor of \(S\). If we take \(R\) to be a general member of the linear system, \(\text{Pic}(X)\) has only rank 2. So \(R\) has to be special. It is not hard to see that \(R\) has the following property.

Let \(\text{Pic}(S)\) be generated by \(C\) and \(F\) with \(C^2 = m, C \cdot F = 1\) and \(F^2 = 0\). Then there exists a curve \(E \in |F|\) such that \(E\) is tangent to \(R\) at two points each with multiplicity 2, i.e., \(E\) is a bitangent of \(R\). It is easy to see that \(\pi^{-1}(E) = E_1 \cup E_2\) and \(X\) has Picard lattice (2.73) with \(G = \pi^*C\), where \(\pi\) is the double covering map \(X \rightarrow S\). We can say a lot about \(X\) through this representation of \(X\).

We choose \(M\) and \(N\) to be two rational curves in the linear series \(|E_1 + E_2|\) and \(|G + (k - 1)(E_1 + E_2)|\), respectively. First, we need to verify the following.

**Proposition 2.5.** Let \(X\) be a general \(K3\) surface with Picard lattice (2.73). Then for every \(k \geq 1\), there exists rational curves

\[
M \in |E_1 + E_2| \quad \text{and} \quad N \in |G + (k - 1)(E_1 + E_2)|
\]

(2.74)

such that both \(M\) and \(N\) are nodal and they meet transversely at two points.

**Proof.** We basically follow the same idea in [C1]. Every \(K3\) surface can be degenerated to a union of two rational surfaces. In this case, \(X\) is not arbitrarily general but we can still degenerate it to a union
of rational surfaces. That is, there exists a family of surfaces $\mathcal{X}$ over a smooth projective curve $\Gamma$ whose general fibers are smooth $K3$ surfaces with Picard lattice $(2.73)$ and whose fiber $\mathcal{X}_0 = S_1 \cup S_2$ over a fixed point $0 \in \Gamma$ is a union of two surfaces with $S_1, S_2 \cong \mathbb{P}^1 \times \mathbb{P}^1$ if $m = 0$ and $S_1, S_2 \cong \mathbb{F}_1$ if $m = -1$.

Just as in $[C1]$, the two smooth rational surfaces $S_1$ and $S_2$ meet transversely along a smooth elliptic curve $D \in |-K_{S_i}|$ and the threefold $\mathcal{X}$ has 16 rational double points $p_1, p_2, ..., p_{16}$ lying on $D$. The tuple $(S_1, S_2, D, p_1, p_2, ..., p_{16})$ has the following properties:

- $S_1 \cup S_2$ is projective and polarized with ample divisor $C + kF$; this is true if and only if

\[(2.75) \quad \mathcal{O}_{S_1}(C + kF) \Big|_D = \mathcal{O}_{S_2}(C + kF) \Big|_D \]

where $C$ and $F$ are the generators of $\text{Pic}(S_i)$ given as before;

- the 16 rational double points $p_1, p_2, ..., p_{16}$ satisfies

\[(2.76) \quad \mathcal{O}_D(p_1 + p_2 + ... + p_{16}) = \mathcal{O}_D(-K_{S_1} - K_{S_2}). \]

The above properties are shared by all $S_1 \cup S_2$’s as degenerations of general $K3$ surfaces. In this case, the general fibers of $\mathcal{X}$ are special $K3$ surfaces. So $S_1 \cup S_2$ has the following additional properties:

- $\text{Pic}(S_1 \cup S_2)$ has rank 2 and is generated by $C$ and $F$, i.e.,

\[(2.77) \quad \mathcal{O}_{S_1}(C) \Big|_D = \mathcal{O}_{S_2}(C) \Big|_D \quad \text{and} \quad \mathcal{O}_{S_1}(F) \Big|_D = \mathcal{O}_{S_2}(F) \Big|_D \quad ; \]

- two of the 16 points $p_1, p_2, ..., p_{16}$, say $p_1$ and $p_2$, satisfy

\[(2.78) \quad \mathcal{O}_D(p_1 + p_2) = \mathcal{O}_D(F). \]

It is clear that the divisor $C$ on $S_1 \cup S_2$ deforms to $G$ on the general fibers. By $(2.78)$, there is a unique curve $J_i \in |F|$ on $S_i$ that passes through $p_1$ and $p_2$ for $i = 1, 2$. Using the deformational arguments in $[C1]$, we see that $J_i$ deforms to $E_i$ on the general fibers.

Let $M = M_1 \cup M_2$ be a curve in the pencil $|F|$ that passes through one of the 16 double points other than $p_1$ and $p_2$, say $p_3 \in M_i$ for $i = 1, 2$. Again using the arguments in $[C1]$, $M$ deforms to a nodal rational curve in $|E_1 + E_2|$ on a general fiber.

Next, let $N = N_1 \cup N_2$ be a curve in the linear series $|C + (k - 1)F|$, where $N_i \subset S_i$ is an irreducible curve in $|C + (k - 1)F|$ that meets $D$ only at one point. By $[C1]$, $N$ can be deformed to a nodal rational curve on a general fiber.
Since $p_3$ is a general point on $D$, it is easy to see that $M$ and $N$ meet transversely. The same is true for a deformation of $M$ and $N$. We are done. \hfill $\square$

Let $M \cap N = \{r_1, r_2\}$ and $E_i \cap N = q_i$ for $i = 1, 2$. Obviously, \begin{equation}
\gamma = E_1 - E_2 \in M^\perp \cap N^\perp
\end{equation}
and $h(\gamma)$ is exactly the cross ratio of the four points $r_1, r_2, q_1, q_2$ lying on $N^\nu \cong \mathbb{P}^1$. So Conjecture 2.4 holds for $(X, M, N)$ if we can show that the moduli of $(r_1, r_2, q_1, q_2)$, as four points on $\mathbb{P}^1$, varies as $(X, M, N)$ deforms.

**Proposition 2.6.** Conjecture 2.4 holds for $(X, M, N)$ constructed above.

**Proof.** We use the same degeneration of $(X, M, N)$ as in the proof of Proposition 2.5. Let $\mathcal{X}$ be the family of surfaces constructed there. After a base change if necessary, we have two families of rational curves $\mathcal{M}$ and $\mathcal{N} \subset \mathcal{X}$ over $\Gamma$ whose central fibers $\mathcal{M}_0 = M$ and $\mathcal{N}_0 = N$ are the curves constructed there. Also we have two families of rational curves $\mathcal{E}_1$ and $\mathcal{E}_2 \subset \mathcal{X}$ over $\Gamma$ with $\mathcal{E}_i \cap \mathcal{X}_0 = J_i$. And we have two distinct sections $R_i$ and $R_2$ of $\mathcal{X}/\Gamma$ with $R_i \subset \mathcal{M} \cap \mathcal{N}$. Let $Q_i = \mathcal{E}_i \cap \mathcal{N}$ for $i = 1, 2$. Now we have a map \begin{equation}
\lambda : \Gamma \to \overline{M}_{0,4} \cong \mathbb{P}^1
\end{equation}
sending \begin{equation}
t \in \Gamma \xrightarrow{\lambda} (\mathcal{N}_t^\nu, R_1 \cap \mathcal{N}_t, R_2 \cap \mathcal{N}_t, Q_1 \cap \mathcal{N}_t, Q_2 \cap \mathcal{N}_t)
\end{equation}
where $M_{0,4}$ is the moduli space of $\mathbb{P}^1$ with four marked points and $\overline{M}_{0,4}$ is its stable closure. It suffices to show that $\lambda$ is dominant. This is more or less obvious from the construction of $N$. Since $N = N_1 \cup N_2$ and the four points $R_i \cap \mathcal{X}_0$ and $Q_i \cap \mathcal{X}_0$ have two on each component $N_i$, $\lambda(0)$ must belong to $\overline{M}_{0,4} \setminus M_{0,4}$. On the other hand, $\lambda(t) \in M_{0,4}$ for $t$ general. So $\lambda$ is nonconstant and hence dominant. \hfill $\square$

Finally, we need to verify the following.

**Proposition 2.7.** Let $W$ be a family of polarized $K3$ surfaces over the disk $\Gamma \cong \{|t| < 1\}$ whose general fibers are $K3$ surfaces with Picard rank 1 and whose central fiber $W_0 = X$ is a $K3$ surface with Picard lattice $(2.73)$. Suppose that the Kodaira-Spencer class associated to $W$ is nonzero. Let $\mathcal{C}$ and $\mathcal{D} \subset W$ be two distinct families of rational curves over $\Gamma$ with $\mathcal{C}_0 = \mathcal{D}_0 = M \cup N$, where $M$ and $N$ are rational curves on $X$ given as above. Then there exists a section $P \subset W$ of $W/\Gamma$ such that $P \subset \mathcal{C} \cap \mathcal{D}$ and $P_0 \in M \setminus N$. 
Proof. By Proposition 2.5, $M$ and $N$ meet transversely at two points $r_1$ and $r_2$. Note that $M$ is a rational curve with one node $p$. Clearly, $p \neq r_i$ for $i = 1, 2$. Actually, we will show that $P$ can be chosen such that $P_0 = p$.

The two families $C$ and $D$ are two deformations of the union $M \cup N$, each smoothing out one of $r_i$. WLOG, suppose that $C$ smooths out $r_2$ and $D$ smooths out $r_1$. By that we mean there are two sections $R_1$ and $R_2$ of $W/\Gamma$ such that $R_i \cap W_0 = r_i$ and $R_i \cap W_t$ is a node of $C_t$ if $i = 2$ and a node of $D_t$ if $i = 1$.

We blow up $W$ along $M$. The same technique was used in [C2] and [C-L2]. Let $\pi : \tilde{W} \to W$ be the blowup with exceptional divisor $Q$. Let $\tilde{C}, \tilde{D}, \tilde{X}$ and $\tilde{N}$ be the proper transforms of $C, D, X$ and $N$ under $\pi$, respectively. The central fiber $\tilde{W}_0$ is the union of two surfaces $\tilde{X}$ and $Q$ which meet along a curve $\tilde{M} \cong M$. Clearly, $\tilde{N} \subset \tilde{X}$ meets $\tilde{M}$ transversely at two points $\tilde{r}_1$ and $\tilde{r}_2$ over $r_1$ and $r_2$. By [C2], we see that $\pi : Q \to M$ is a $\mathbb{P}^1$ bundle and the normalization $Q' \to Q$ is $\mathbb{P}^1 \times \mathbb{P}^1$. And the threefold $\tilde{W}$ has a rational double point $q \in Q \setminus \tilde{M}$ over $p \in M$. Using the techniques in [C2], we see that the curve $C = \tilde{C} \cap Q$ is a section of $Q/M$ satisfying

- $C$ meets $\tilde{M}$ transversely at $\tilde{r}_1$;
- $C \cap \pi^{-1}(p) = q$.

Clearly, these two conditions determine $C$ uniquely. Similarly, $D = \tilde{D} \cap Q$ is a section of $Q/M$ satisfying

- $D$ meets $\tilde{M}$ transversely at $\tilde{r}_2$;
- $D \cap \pi^{-1}(p) = q$.

Therefore, $C \neq D$ and hence $\tilde{C}_0$ and $\tilde{D}_0$ meet properly at $q$. Consequently, there is a section $\tilde{P}$ of $\tilde{W}/\Gamma$ and hence a section $P$ of $W/\Gamma$ such that $P \subset C \cap D$ and $P(0) = p$. Indeed, there are exactly two such sections since the preimages of $q$ under the normalization $Q' \to Q$ consist of two points. \hfill \square

Now we can conclude that for a general polarized $K3$ surface $(X, L)$, the image $\text{Im}(r_{3,1}) \otimes \mathbb{R}$ is nontrivial.

3. Appendix: A generalized archimedean pairing

In this section, and for each $m \geq 0$, we construct a pairing on the cycle level, involving the equivalence relation in the definition of Bloch’s higher Chow groups $\text{CH}^r(X, m)$ defined below. The case when $m = 0$ has already been defined in [2.3] and the nature of this pairing is more
akin to the archimedean height pairing defined in the literature. Although we have only used this pairing in the special instance when $m = 0$, a general construction of this pairing for all $m$ is in order. We first recall that two subvarieties $V_1$, $V_2$ of a given variety intersect properly if $\text{codim}\{V_1 \cap V_2\} \geq \text{codim} V_1 + \text{codim} V_2$. This notion naturally extends to algebraic cycles.

(i) **Higher Chow groups.** Let $W/\mathbb{C}$ a quasi-projective variety. Put $z^r(W) = \text{free abelian group generated by subvarieties of codimension } r$ in $W$. Consider the $m$-simplex:

$$\Delta^m = \text{Spec}\left\{ \frac{\mathbb{C}[t_0, \ldots, t_m]}{(1 - \sum_{j=0}^m t_j)} \right\} \simeq \mathbb{C}^m.$$

We set

$$z^r(W, m) = \left\{ \xi \in z^k(W \times \Delta^m) \mid \text{\xi meets all faces } \{ t_{i_1} = \cdots = t_{i_\ell} = 0, \ \ell \geq 1 \} \text{ properly} \right\}.$$

Note that $z^r(W, 0) = z^r(W)$. Now set $\partial_j : z^r(W, m) \to z^r(W, m - 1)$, the restriction to $j$-th face given by $t_j = 0$. The boundary map $\partial = \sum_{j=0}^m (-1)^j \partial_j : z^k(W, m) \to z^k(W, m - 1)$, satisfies $\partial^2 = 0$.

**Definition 3.1.** ($\mathbb{B}$) $CH^\bullet(W, \bullet) = \text{homology of } \{ z^\bullet(W, \bullet), \partial \}$. We put $CH^k(W) := CH^k(W, 0)$.

(ii) **Cubical version.** Let $\Box^m := (\mathbb{P}^1 \setminus \{1\})^m$ with coordinates $z_i$ and $2^m$ codimension one faces obtained by setting $z_i = 0, \infty$, and boundary maps $\partial = \sum (-1)^{i-1} (\partial_{i}^0 - \partial_{i}^\infty)$, where $\partial_{i}^0$, $\partial_{i}^\infty$ denote the restriction maps to the faces $z_i = 0, z_i = \infty$ respectively. The rest of the definition is completely analogous for $z^r(X, m) \subset z^r(X \times \Box^m)$, except that one has to quotient out by the subgroup $z^r_{\text{deg}}(X, m) \subset z^r(X, m)$ of degenerate cycles obtained via pullbacks $\text{Pr}_j^* : z^r(X, m - 1) \to z^r(X, m)$, $\text{Pr}_j : X \times \Box^m \to X \times \Box^{m-1}$ the $j$-th canonical projection. It is known that both complexes are quasi-isomorphic.

In this section we will adopt the cubical version of $CH^\bullet(W, \bullet)$. The intersection product for cycles in the cubical version, is easy to define. On the level of cycles, and in $W \times W \times \Box^{m+n}$, one has

$$z^r(W, m) \times z^k(W, n) \to z^{r+k}(W \times W, m + n);$$

however the pullback along the diagonal

$$z^{r+k}(W \times W, m + n) \to z^{r+k}(W, m + n),$$
is not well-defined, even for smooth $W$. In particular, for smooth $W$, the issue of when an intersection product is defined, which is a general position statement involving proper intersections, has to be addressed since we will be working on the level of cycles. On the level of Chow groups, Bloch’s Lemma 3.6 below (adapted to the cubical situation) guarantees a pullback for smooth $W$:

$$\text{CH}^\bullet(W \times W, \bullet) \to \text{CH}^\bullet(W, \bullet),$$

and hence an intersection product for smooth $W$. Let $X$ be a projective algebraic manifold of dimension $d$, and let $z^r_{\text{rat}}(X, m) := \partial(z^r(X, m + 1)) \subset z^r(X, m)$ be the equivalence relation subgroup defining the higher Chow groups $\text{CH}^r(X, m)$. Now introduce

$$\Xi^0(r, m, X) = \{ (\xi_1, \xi_2) \in z^r_{\text{rat}}(X, m) \times z^{d-r+m+1}_{\text{rat}}(X, m) \mid |\xi_1| \cap |\xi_2| = \emptyset \},$$

$$\Xi^+(r, m, X) = \left\{ (\xi_1, \xi_2) \in \Xi^0(r, m, X) \mid \begin{array}{l} \xi_1 = \partial \xi'_1, \text{ where} \\
\xi'_1 \cap \xi_2 \text{ is defined} \\
in z^{d+m+1}_{\text{rat}}(X, 2m+1) \end{array} \right\},$$

$$\Xi^-(r, m, X) = \left\{ (\xi_1, \xi_2) \in \Xi^0(r, m, X) \mid \begin{array}{l} \xi_2 = \partial \xi'_2, \text{ where} \\
\xi_1 \cap \xi'_2 \text{ is defined} \\
in z^{d+m+1}_{\text{rat}}(X, 2m+1) \end{array} \right\},$$

$$\Xi(r, m, X) = \Xi^+(r, m, X) \cap \Xi^-(r, m, X).$$

Let $R(p) = R(2\pi i)^p$. Note that $\mathbb{C} = R(p) \oplus R(p - 1)$, and hence there is a projection $\pi_p : \mathbb{C} \to R(p)$.

**Theorem 3.2.** There are natural pairings

$$\langle \cdot, \cdot \rangle^+_m : \Xi^+(r, m, X) \to \mathbb{R}(m),$$

$$\langle \cdot, \cdot \rangle^-_m : \Xi^-(r, m, X) \to \mathbb{R}(m),$$

which satisfy the following:

(i) On $\Xi(r, m, X)$,

$$\langle \cdot, \cdot \rangle^+_m = (-1)^m \langle \cdot, \cdot \rangle^-_m.$$

(ii) (Bilinearity) If $(\xi_1^{(1)}, \xi_2)$, $(\xi_1^{(2)}, \xi_2) \in \Xi^+(r, m, X)$, then

$$\langle \xi_1^{(1)} + \xi_1^{(2)}, \xi_2 \rangle^+_m = \langle \xi_1^{(1)} + \xi_1^{(2)} \rangle^+_m + \langle \xi_1^{(2)} \rangle^+_m.$$

If $(\xi_1, \xi_2^{(1)})$, $(\xi_1, \xi_2^{(2)}) \in \Xi^-(r, m, X)$, then

$$\langle \xi_1, \xi_2^{(1)} + \xi_2^{(2)} \rangle^-_m = \langle \xi_1, \xi_2^{(1)} \rangle^-_m + \langle \xi_2^{(2)} \rangle^-_m.$$

(iii) (Projection formula) Let $\pi : X \to Y$ be a flat surjective morphism between two smooth projective varieties, with $\dim X = d$. Then

$$\langle \xi_1, \pi^* \xi_2 \rangle^\pm = \langle \pi_\xi, \xi_1, \xi_2 \rangle^\pm \text{ for all } \xi_1 \in z^r_{\text{rat}}(X, m) \text{ and } \xi_2 \in z^{d-r+m+1}_{\text{rat}}(Y, m)$$

with $(\pi_\xi, \xi_1, \xi_2) \in \Xi^\pm(r + s - d, m, Y)$, where $s := \dim Y$. 

(iv) (Reciprocity) \( \langle \xi_1, \xi_2 \rangle_m = (-1)^m \langle \xi_2, \xi_1 \rangle_m \), where

\[
\langle \ , \ \rangle_m := \langle \ , \ \rangle_{\Xi(r,m,X)} = (-1)^m \langle \ , \ \rangle_{\Xi(r,m,X)}.
\]

**Proof.** We first recall the definition of real Deligne cohomology. Let \( D^\bullet_X \) be the sheaf of complex-valued currents acting on \( C^\infty \) complex-valued compactly supported \((2d - \bullet)\)-forms, where we recall \( \dim X = d \). One has a decomposition into Hodge type:

\[
D^\bullet_X = \bigoplus_{p+q=\bullet} D^{p,q}_X,
\]

where \( D^{p,q}_X \) acts on \((d-p, d-q)\) forms, with Hodge filtration,

\[
F^r D^\bullet_X = \bigoplus_{p+q=\bullet, p \geq r} D^{p,q}_X.
\]

One has filtered quasi-isomorphism of complexes,

\[
(F^r) \Omega^\bullet_X \hookrightarrow (F^r) E^\bullet_X \hookrightarrow (F^r) D^\bullet_X,
\]

where \( E^\bullet_X \) (resp. \( \Omega^\bullet_X \)) is the sheaf complex of germs of complex-valued \( C^\infty \) (resp. holomorphic) forms on \( X \). Let us put \( E^\bullet_X, R := \text{sheaf complex of germs of real } C^\infty \) forms, and likewise \( D^\bullet_X, R \) the sheaf complex of real-valued currents. We define \( D^{\bullet, R}_X(p) = D^{\bullet, R}_X \otimes R(p), E^{\bullet, R}_X(p) = E^{\bullet, R}_X \otimes R(p) \). The global sections of a given sheaf \( S \) over \( X \) will be denoted by \( S(X) \). Next, for a morphism of complexes \( \lambda : A^\bullet \rightarrow C^\bullet \), we recall the cone complex:

\[
\text{Cone}(A^\bullet \xrightarrow{\lambda} B^\bullet) = A^\bullet[1] \oplus B^\bullet,
\]

with differential

\[
\delta_D : A^{q+1} \oplus B^q \rightarrow A^{q+2} \oplus B^{q+1}, \quad (a, b) \xmapsto{\delta_D} (-da, \lambda(a) + db).
\]

**Definition 3.3.** The real Deligne chomology of \( X \) is given by

\[
H^i_D(X, \mathbb{R}(j)) := H^i(\text{Cone}(F^j D^\bullet_X(X) \xrightarrow{-\pi_{j-1}} D^{\bullet, R}_X(j-1)(X))[1])[-1]).
\]

We now recall the description of the real regulator \( R_{r,m,X} : \text{CH}^r(X, m) \rightarrow H^{2r-m}_D(X, \mathbb{R}(r)) \),

(see [Go], as well as [K], [KLM]). For nonzero rational functions \( \{f_1, \ldots, f_m\} \) defined on a complex variety, we introduce the real current \( 2\pi i R_m \), where (Go):

\[
R_m(f_1, \ldots, f_m) :=
\]

\[
(2\pi i)^{-m} \text{Alt}_m \sum_{j \geq 0} \frac{1}{(2j+1)!(m-2j-1)!} \log |f_1|d\log |f_2|\wedge \cdots \wedge \log |f_m|d\log |f_2|\wedge \cdots \wedge \log |f_m|
\]
\[
\cdots \wedge d \log |f_{2j+1}| \wedge d \arg(f_{2j+2}) \wedge \cdots \wedge d \arg(f_m),
\]
and where
\[
\text{Alt}_m F(x_1, \ldots, x_m) := \sum_{\sigma \in S_m} (-1)^{|\sigma|} F(x_{\sigma(1)}, \ldots, x_{\sigma(m)}),
\]
is the alternating operation, and \(S_m\) is the group of permutations on \(m\) letters. Consider \(\square^m\) with affine coordinates \((z_1, \ldots, z_m)\), and introduce the operators
\[
R_{\square} := R_m(z_1, \ldots, z_m), \quad \Omega_{\square} = \sum_{j=1}^m d \log z_j,
\]
For a \(\xi \in z^r(X, m)\) we consider the currents on \(X\):
\[
R_m(\xi) := \int_{\xi} \Pr_{\square}^*(R_{\square}) \wedge \Pr_X^*(-), \quad \Omega_m(\xi) := \int_{\xi} \Pr_{\square}^*(\Omega_{\square}) \wedge \Pr_X^*(-).
\]
It is easy to check that
\[
(R_m(\xi), \Omega_m(\xi)) = (0, 0) \quad \text{for} \quad \xi \in z^r_{\text{dgt}}(X, m),
\]
and that \((2\pi i)^m dR_{\square} = \pi_{m-1}(\Omega_{\square})\) as forms (or as currents acting on forms compactly supported away from the \(2^m\) faces of \(\square^m\)). Then up to a twist, \(R_{r,m,X}\) is induced by:
\[
\xi \in z^r(X, m) \mapsto (\Omega_m(\xi), (2\pi i)^m R_m(\xi)),
\]
(see [K] or [KLM], where it follows that \((2\pi i)^m dR_m(\xi) = \pi_{m-1}(\Omega_m(\xi))\) if \(\partial \xi = 0\). For \(m = 0\), note that \((\Omega_0(\xi), (2\pi i)^m R_0(\xi)) = (1_\xi, 0)\), where \(1_\xi\) defines the current on \(X\) given by integration over \(\xi\).

Now to the proof of the theorem. For simplicity, we will assume given \((\xi_1, \xi_2) \in \Xi(r, m, X)\). By definition, this implies that
\[
\xi_1 \cap \xi_2, \xi_1 \cap \xi_2 \in z^{d+m+1}(X, 2m + 1),
\]
which is important in ensuring that the integrals given below converge. The prescription for the following pairings is based on the formalism of a cup product operation on Deligne complexes. Namely we define:
\[
\langle \xi_1, \xi_2 \rangle^+_m := (2\pi i) \left[ \int_{\xi_1 \cap \xi_2} R_{m+1}(\xi_1') \wedge \pi_m(\Omega_m(\xi_2)) +\right. \\
\left. (-1)^{m+1} \int_{\xi_2 \cap \xi_1} \pi_{m+1}(\Omega_{m+1}(\xi_1')) \wedge R_m(\xi_2) \right],
\]
\[
= (2\pi i) \left[ \int_{\xi_1 \cap \xi_2} R_{m+1}(\xi_1') \wedge \pi_m(\Omega_m(\xi_2)) \right],
\]
using the fact that \( \dim |\xi_1 \cap \xi_2| \leq m \) and that \( \Omega_{m+1}(\xi'_1) \) is a holomorphic current involving \( m+1 \) holomorphic differentials. Likewise,

\[
\langle \xi_1, \xi_2 \rangle^-_m := (2\pi i) \int_{\xi_1 \cap \xi_2'} R_{m}(\xi_1) \wedge \pi_m(\Omega_{m+1}(\xi'_2)) + (-1)^m \int_{\xi_2' \cap \xi_1} \pi_m(\Omega_m(\xi_1)) \wedge R_{m+1}(\xi'_2),
\]

\[
= (2\pi i)(-1)^m \int_{\xi_2' \cap \xi_1} \pi_m(\Omega_m(\xi_1)) \wedge R_{m+1}(\xi'_2).
\]

The interpretation of these integrals is fairly clear. For instance

\[
\int_{\xi_1 \cap \xi_2} R_{m+1}(\xi'_1) \wedge \pi_m(\Omega_m(\xi_2))
\]

means the following: In the product

\[
X \times X \times \square^{m+1} \times \square^m,
\]

let

\[
\delta : X \times \square^{m+1} \times \square^m \rightarrow X \times X \times \square^{m+1} \times \square^m,
\]

be induced from the diagonal embedding \( X \hookrightarrow X \times X \), together with the identity map on \( \square^{m+1} \times \square^m \), and \( \Pr_j, j = 1, 2, 3, 4 \), the canonical projections. Then

\[
\int_{\xi_1 \cap \xi_2} R_{m+1}(\xi'_1) \wedge \pi_m(\Omega_m(\xi_2)) = \int_{\xi_1 \cap \xi_2} \delta^* \left[ \Pr_3^* (R_{\square^{m+1}}) \wedge \Pr_4^* (\pi_m(\Omega_m)) \right].
\]

The relationship between \( \langle , \rangle^+_m \) and \( \langle , \rangle^-_m \) is precisely where reciprocity comes into play. Observe that

\[
\partial(\xi_1 \cap \xi_2) = \xi_1 \cap \xi_2' + (-1)^{m+1} \xi'_1 \cap \xi_2,
\]

and by \[\text{[B]},\]

\[
\xi_1 \cap \xi_2' = (-1)^{m(m+1)} \xi'_2 \cap \xi_1 = \xi'_2 \cap \xi_1.
\]

Next,

\[
\partial(\xi'_1 \cap \xi'_2) \mapsto 0 \in H^{2d+1}(X, \mathbb{R}(d + m + 1)),
\]

and together with the fact that \( \max \{ \dim |\xi'_1 \cap \xi_2|, \dim |\xi'_1 \cap \xi'_2| \} \leq m \), and that \( \Omega_{m+1}(\xi'_1) \cup \Omega_m(\xi_2), \Omega_m(\xi_1) \cup \Omega_{m+1}(\xi'_2) \) are currents involving \( 2m + 1 \) holomorphic differentials, hence

\[
\Omega_{m+1}(\xi'_1) \cup \Omega_m(\xi_2) = 0 = \Omega_m(\xi_1) \cup \Omega_{m+1}(\xi'_2),
\]

we arrive at

\[
(-1)^m \langle \xi_2, \xi_1 \rangle^+_m := (2\pi i) \int_{\xi_2' \cap \xi_1} R_{m+1}(\xi'_2) \wedge \pi_m(\Omega_m(\xi_1))
\]

\[
= \langle \xi_1, \xi_2 \rangle^-_m.
\]
I.e.

\[ \langle \xi_1, \xi_2 \rangle_m^+ = (-1)^m \langle \xi_2, \xi_1 \rangle_m^+ = (-1)^m \langle \xi_1, \xi_2 \rangle_m^- . \]

The formula for \( \langle \xi_1, \xi_2 \rangle_m^+ \) above essentially involves the product structure for a complex of forms defining real Deligne cohomology (see [EV]), translated in the terminology of our special class of currents above. Similar ideas have been worked out in [L1]. If either \( \partial \xi_1' = 0 \) or \( \partial \xi_2' = 0 \), then this reduces to a cup product in Deligne cohomology of the Beilinson (real) regulator of a higher Chow cycle, together with one which is nullhomologous (in Deligne cohomology), which is zero. Hence this pairing does not depend on the choices of \( \xi_j' \). The theorem essentially follows from this. □

Now recall that

\[ \langle \cdot, \cdot \rangle_m := \langle \cdot, \cdot \rangle_m^+ \big|_{\Xi(r,m,X)} = (-1)^m \langle \cdot, \cdot \rangle_m^- \big|_{\Xi(r,m,X)} . \]

It is natural to pose the following nondegeneracy type question.

**Question 3.4.** Suppose that \( \langle \xi_1, \xi_2 \rangle_m = 0 \) for all \( \xi_2 \) with \( (\xi_1, \xi_2) \in \Xi(r,m,X) \). Is it the case that \( \xi_1 \in z^r_dgt(X,m) \)? Or if that is too strong, a possibly weaker statement could be \( \Omega_m(\xi_1) = 0 \) (as a current on \( X \))? In the case \( m = 0 \), Proposition 2.3 answers the above question definitively in the affirmative. For our next result, and as a partial answer to the above question, the reader can consult [El-V] (p. 187) for the business of a graph map.

**Proposition 3.5.** Let \( \xi \in z^r(X,m) \) be represented as the graph of a cycle of the form:

\[ \xi_1 = \sum_{\alpha \in I} \left( \{ f_{1,\alpha}, \ldots, f_{m,\alpha} \}, D_\alpha \right), \]

where \( \{ f_{1,\alpha}, \ldots, f_{m,\alpha} \} \in \mathbb{C}(D_\alpha)^* \), and where \( \{ D_\alpha, \alpha \in I \} \) are distinct irreducible subvarieties with \( \text{codim}_X D_\alpha = r - m \). Then

\[ \langle \xi_1, \xi_2 \rangle_m = 0 \text{ for all } \xi_2 \text{ with } (\xi_1, \xi_2) \in \Xi(r,m,X) \Rightarrow \Omega_m(\xi_1) = 0 . \]

**Proof.** Let us assume that \( \Omega_m(\xi_1) \neq 0 \). This is equivalent to saying \( \pi_m(\Omega_m(\xi_1)) \neq 0 \). Then for some \( 0 \in I \),

\[ \int_{D_0} \pi_m \left[ \bigwedge_1^m d \log f_{0,j} \right] \wedge (-) \neq 0 . \]

Now let \( E \subset X \) be a general choice of irreducible subvariety with \( \text{codim}_X E = d - r \), and let \( \{ g_1, \ldots, g_{m+1} \} \in \mathbb{C}(E)^* \). By distinctness of the \( \{ D_\alpha, \alpha \in I \} \), we can choose \( g_1 \) such that \( g_1 \big|_{D_\alpha} \equiv 1 \) for all \( \alpha \neq 0 \).
If we put $\xi'_2$ to correspond to the graph of $(\{g_1, \ldots, g_{m+1}, E\}$, then using the dictionary
\[
(\{f_0, 1, \ldots, f_0, m, D_0\} \leftrightarrow (\{f_1, \ldots, f_m\}; D),
\]
we have
\[
(3.1)
\]
\[
(2 \pi i)^{-1} \langle \xi_1, \xi_2 \rangle_m = \pm \int_{D \cap E} \pi_m(\Omega_m(f_1, \ldots, f_m)) \wedge R_{m+1}(g_1, \ldots, g_{m+1}).
\]
Next, by replacing $g_1$ by
\[
g_{1, \varepsilon} := 1 + (h - 1) \left( \frac{(g_1 - 1)}{(g_1 - 1) + \varepsilon} \right),
\]
where $h$ is any rational function and $\varepsilon > 0$, and letting $\varepsilon \to 0^+$, it follows that we can reduce to the situation of an arbitrary choice of $\{g_1, \ldots, g_{m+1}\}$ in equation (3.1). In particular, if we consider
\[
F := (f_1, \ldots, f_m) : D \cap E \to (\mathbb{P}^1)^{x_m},
\]
then
\[
\pi_m(\Omega_m(f_1, \ldots, f_m)) = F^\ast(\pi_m(\Omega_m(z_1, \ldots, z_m))).
\]
Thus equation (3.1) and the non-triviality of $\langle \xi_1, \xi_2 \rangle_m$ reduces to showing the non-triviality of
\[
\int_{\mathbb{C}^m} \pi_m(\Omega_m(z_1, \ldots, z_m)) \wedge R_{m+1}(w_1, \ldots, w_{m+1}),
\]
for a choice of functions $\{w_1, \ldots, w_{m+1}\}$ of $z := (z_1, \ldots, z_m)$. That one can find such functions is not difficult, and left to the reader. \qed

It is instructive to work out a couple example cases.

Case $m = 0$: We recover the pairing in 2.3. We first recall the following key result.

Lemma 3.6. ([13], Lemma (4.2)) Let $Y$ be a smooth, quasi-projective $k$-variety and let $y = \{Y_i\}$ be a finite collection of closed subvarieties. Then the inclusion $z^*_Y(Y, \bullet) \subset z^*(Y, \bullet)$ is a quasi-isomorphism.

Here $z^*_y(Y, \bullet) \subset z^*(Y, \bullet)$ is the subcomplex of cycles that meet $y \times \square$ properly. In the case $m = 0$, we have $\Xi^0(r, 0, X) \subset z^r_{\text{rat}}(X) \times z^d_{\text{rat}+1}(X)$. Let $(\xi_1, \xi_2) \in z^r_{\text{rat}}(X) \times z^d_{\text{rat}+1}(X)$. By considering the cases where $y = |\xi_j|$, $j = 1, 2$, it follows that
\[
\Xi^0(r, 0, X) = \Xi^+(r, 0, X) = \Xi^-(r, 0, X) = \Xi(r, 0, X),
\]
and thus we have a pairing
\[ \langle \xi_1, \xi_2 \rangle := \langle \xi_1, \xi_2 \rangle_0 : z^r_{\text{rat}}(X) \times z^{d-r+1}_{\text{rat}}(X) \to \mathbb{R}, \]
defined for all pairs \((\xi_1, \xi_2)\) where \(|\xi_1| \cap |\xi_2| = \emptyset\). Let \(\xi_1 := \text{div}(f, D) \in z^r_{\text{rat}}(X, 0), \xi_2 := \text{div}(g, E) \in z^{d-r+1}_{\text{rat}}(X, 0)\) be given. In this case \(D\) and \(E\) are irreducible subvarieties of \(X\) of \(\text{codim}_X D = r - 1\) and \(\text{codim}_X E = d - r\), and \(f \in \mathbb{C}(D)^*, g \in \mathbb{C}(E)^*\). Then
\[ \langle \xi_1, \xi_2 \rangle_0 := \int_{D \cap E_j} \log |f| = \int_{E \cap \xi_1} \log |g| =: \langle \xi_2, \xi_1 \rangle_0. \]
The reader can check that Proposition 2.3 can be deduced from Proposition 3.5.

Case \(m = 1\): Let
\[ \xi_1 := T(\{f_1, f_2\}, D) \in z^r_{\text{rat}}(X, 1), \]
and
\[ \xi_2 := T(\{g_1, g_2\}, E) \in z^{d-r+2}_{\text{rat}}(X, 1) \]
be given, where \(T\) is the Tame symbol. In this case \(D\) and \(E\) are irreducible subvarieties of \(X\) of \(\text{codim}_X D = r - 2\) and \(\text{codim}_X E = d - r\), and \(f_j \in \mathbb{C}(D)^*, g_j \in \mathbb{C}(E)^*, j = 1, 2\). Note that
\[ \xi_1 := \sum_{\text{cd}_D, D_i=1} (-1)^{\nu_D(f_i)\nu_D(f_2)} \frac{\nu_{D_i}(f_1)}{\nu_{D_i}(f_2)} D_i, \]
\[ \xi_2 := \sum_{\text{cd}_E, E_j=1} (-1)^{\nu_E(g_1)\nu_E(g_2)} \frac{\nu_{E_j}(g_1)}{\nu_{E_j}(g_2)} E_j. \]
Then
\[ 2\pi \langle \xi_1, \xi_2 \rangle_1 = \sum_j \int_{D \cap E_j} \left[ \log |f_1| d \arg(f_2) - \log |f_2| d \arg(f_1) \right] \wedge \pi_1 \left[ d \log \frac{\nu_{E_j}(g_2)}{\nu_{E_j}(g_1)} \right]. \]

REFERENCES

[B] S. Bloch, Algebraic cycles and higher K-theory, Advances in Math. 61 (1986), 267-304.

[B-G] The moving lemma for higher Chow groups, J. Algebr. Geom. 3(3) (1994), 493-535.

[C1] X. Chen, Rational Curves on K3 Surfaces, J. Alg. Geom. 8 (1999), 245-278. Also preprint [math.AG/9804075]

[C2] X. Chen, A simple proof that rational curves on K3 are nodal, Math. Ann. 324 (2002), no. 1, 71-104.

[C-L] X. Chen and J. D. Lewis, Noether-Lefschetz for K_1 of a certain class of surfaces, Bol. Soc. Mat. Mexicana 10(3) (2004), 29-41.
[C-L2] The Hodge-D-conjecture for $K3$ and Abelian surfaces, *J. Alg. Geom.* 14 (2005), 213-240.

[C-L3] The real regulator for a product of $K3$ surfaces, in *Mirror Symmetry V, Proceedings of the BIRS conference in Banff, Alberta* (Edited by S.-T. Yau, N. Yui and J. D. Lewis), AMS/IP Studies in Advanced Mathematics, Volume 38, (2006), 271-283.

[El-V] P. Elbaz-Vincent, A short introduction to higher Chow groups, in *Transcendental Aspects of Algebraic Cycles*, Proceedings of the Grenoble Summer School, 2001, Edited by S. Müller-Stach and C. Peters, London Mathematical Society Lecture Note Series 313, Cambridge University Press (2004), 171-196.

[EV] H. Esnault and E. Viehweg, Deligne-Beilinson cohomology, in *Beilinson’s conjectures on special values of $L$-functions*, (Academic Press, Boston, 1988), 43-92.

[Go] A. Goncharov, Chow polylogarithms and regulators, *Mathematical Research Letters* 2 (1995), 95-112.

[K] M. Kerr, Geometric construction of Regulator currents with applications to algebraic cycles, Thesis, Princeton University (2003).

[KLM] M. Kerr, J. D. Lewis, S. Müller-Stach, The Abel-Jacobi map for higher Chow groups, *Compositio Math.* 142 (2006), 374-396.

[L1] J. D. Lewis, Real regulators on Milnor complexes, *K-Theory* 25 (2002), 277-298.

[L2] A note on indecomposable motivic cohomology classes, *J. reine angew. Math.* 485, (1997), 161-172.

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