LARGE–$N_C$ METHODS IN KAON PHYSICS

Eduardo de Rafael
CPT, CNRS–Luminy, Marseille

ABSTRACT
This talk reviews recent progress in formulating the dynamics of Kaon Physics, within the framework of the $1/N_C$–expansion in QCD.

1 Introduction
In the Standard Model, the electroweak interactions of hadrons at very low energies can be described by an effective Lagrangian which only has as active degrees of freedom the flavour $SU(3)$ octet of the low–lying pseudoscalar particles. The underlying theory, however, is the gauge theory $SU(3)_C \times SU(2)_L \times U(1)_Y$ which has as dynamical degrees of freedom quarks and gauge fields. Going from these degrees of freedom at high energies to an effective description in terms of mesons at low energies is, in principle, a problem which should be understood
in terms of the evolution of the renormalization group from short–distances to long–distances. Unfortunately, it is difficult to carry out explicitly this evolution because at energies, typically of a few GeV, non–perturbative dynamics like spontaneous chiral symmetry breaking and color confinement sets in.

The suggestion to keep the number of colours \( N_c \) in QCD as a free parameter was first made by G. ’t Hooft \( \text{[1]} \) as a possible way to approach the study of non–perturbative phenomena. Many interesting properties have been proved, which suggest that, indeed, the theory in this limit has the bulk of the non–perturbative properties of the full QCD. The spectrum of the theory in the large–\( N_c \) limit consists of an infinite number of narrow stable meson states which are flavour nonets \( \text{[2]} \). This spectrum looks \textit{a priori} rather different to the real world. The vector and axial–vector spectral functions measured in \( e^+e^- \rightarrow \text{hadrons} \) and in the hadronic \( \tau \)–decay show indeed a richer structure than just a sum of narrow states. There are, however, many instances where one is only interested in observables which are given by weighted integrals of some hadronic spectral functions. Typical examples of that are the coupling constants of the effective chiral Lagrangian of QCD at low energies, as well as the coupling constants of the effective chiral Lagrangian of the electroweak interactions of pseudoscalar particles in the Standard Model. Some of these couplings are the ones needed to understand \( K \)–Physics quantitatively. In these examples the \textit{hadronic world} predicted by large–\( N_c \) QCD provides already a good approximation to the real hadronic spectrum. It is in this sense that I shall show that large–\( N_c \) QCD is a very useful phenomenological approach for understanding non–perturbative QCD physics at low energies; \( K \)–Physics in particular.

2 The Chiral Lagrangian at Low Energies

Typical terms of the chiral Lagrangian at low energies are

\[
\mathcal{L}_{\text{eff}} = \frac{1}{4} F_0^2 \text{tr} \left( D_\mu U D_\mu U^\dagger \right) + L_{10} \text{tr} \left( U^\dagger F_{R\mu
u} U F_{L\mu
u} \right) + \cdots \\
\text{where } U \text{ is a } 3 \times 3 \text{ unitary matrix in flavour space which collects the fields of the low–lying pseudoscalar particles, (the Goldstone fields of spontaneous} \]

\[ e^2 C \text{tr} \left( Q_R U Q_L U^\dagger \right) + \cdots \]

\[
\text{and others.} \]

\[ -e^2 C \frac{V_{ud}}{\sqrt{2}} V_{us}^* g_8 F_0^2 \left( D_\mu U D_\mu U^\dagger \right)_{23} + \cdots, \quad \text{(1)} \]
chiral symmetry breaking.) The first line shows typical terms of the strong interactions in the presence of external currents\(^3\); the second line shows typical terms which appear when photons, \(W'\)s and \(Z'\)s have been integrated out in the presence of the strong interactions. We show under the braces the typical physical processes to which each term contributes. Each term is modulated by a constant: \(F_0^2\), \(L_{10}\), ... \(g_8\), ... which encode the underlying dynamics responsible for the appearance of the corresponding effective term.

There are two crucial observations concerning the relation of these low energy constants to the underlying theory, that I want to emphasize.

- The low–energy constants of the Strong Lagrangian, like \(F_0^2\) and \(L_{10}\), are the coefficients of the Taylor expansion of appropriate QCD Green’s Functions. For example, with \(\Pi_{LR}(Q^2)\) the correlation function of a left–current with a right–current in the chiral limit, (where the light quark masses are neglected,) the Taylor expansion

\[
-Q^2\Pi_{LR}(Q^2)|_{Q^2\to 0} = F_0^2 - 4L_{10}Q^2 + \mathcal{O}(Q^4),
\]

defines the constants \(F_0^2\) and \(L_{10}\).

- By contrast, the low–energy constants of the ElectroWeak Lagrangian, like e.g. \(C\) and \(g_8\), are integrals of appropriate QCD Green’s Functions. For example, including the effect of weak neutral currents\(^4\),

\[
C = \frac{3}{32\pi^2} \int_0^\infty dQ^2 \left(1 - \frac{Q^2}{Q^2 + M_Z^2}\right) \left(-Q^2\Pi_{LR}(Q^2)\right).
\]

Their evaluation requires a precise matching of the short–distance and the long–distance contributions of the underlying Green’s functions.

These observations are completely generic, independently of the \(1/N_c\)–expansion. The large–\(N_c\) approximation helps, however, because it restricts the analytic structure of the Green’s functions in general, and \(\Pi_{LR}(Q^2)\) in particular, to be a sum of poles only; e.g., in large–\(N_c\) QCD,

\[
\Pi_{LR}(Q^2) = \sum_V \frac{f_V^2 M_V^2}{Q^2 + M_V^2} - \sum_A \frac{f_A^2 M_A^2}{Q^2 + M_A^2} - \frac{F_0^2}{Q^2},
\]

where the sums are, in principle, extended to an infinite number of states.
There are two types of important restrictions on Green’s functions like \( \Pi_{LR}(Q^2) \). One type follows from the fact that, as already stated above, the Taylor expansion at low euclidean momenta must match the low energy constants of the strong chiral Lagrangian. The other type of constraints follows from the short-distance properties of the underlying Green’s functions which can be evaluated using the operator product expansion (OPE) technology. In the large–\( N_c \) limit, this results in a series of algebraic sum rules which restrict the coupling constants and masses of the hadronic poles.

2.1 The Minimal Hadronic Ansatz Approximation to Large–\( N_c \) QCD

In most cases of interest, the Green’s functions which govern the low–energy constants of the chiral Lagrangian are order parameters of spontaneous chiral symmetry breaking; i.e. they vanish, in the chiral limit, order by order in the perturbative vacuum of QCD. That implies that they have a power fall–out in \( 1/Q^2 \) at large–\( Q^2 \), ( e.g., the function \( \Pi_{LR}(Q^2) \) falls as \( 1/Q^6 \).) That also implies that within a finite radius in the complex \( Q^2 \)–plane, these Green’s functions in the large–\( N_c \) limit, only have a finite number of poles. The minimal hadronic ansatz (MHA) approximation is fixed by the minimum number of poles required to satisfy the OPE constraints. In the case of the \( \Pi_{LR}(Q^2) \) function, the MHA approximation to the large–\( N_c \) expression in Eq. (4) results in the simple function

\[
-Q^2 \Pi_{LR}(Q^2) = F_0^2 \frac{M_V^2 M_A^2}{(Q^2 + M_V^2)(Q^2 + M_A^2)}. \tag{5}
\]

Inserting this in Eq. (3) gives a prediction to the low–energy constant \( C \) which governs the electromagnetic \( \pi^+ + \pi^- \equiv \Delta m_\pi \) mass difference, with the result

\[
\Delta m_\pi = (4.9 \pm 0.4) \text{ MeV}, \quad \text{MHA to Large–} N_c \text{ QCD}, \tag{6}
\]

to be compared with the experimental value

\[
\Delta m_\pi = (4.5936 \pm 0.0005) \text{ MeV}, \quad \text{Particle Data Book}. \tag{7}
\]

The shape of the function in Eq. (5), normalized to its value at \( Q^2 = 0 \), is shown in Fig. 1 below, (the continuous red curve.) Also shown in the same plot is the experimental curve, (the green dotted curve) obtained from the ALEPH
collaboration data\cite{7}; as well as the different shapes predicted by other analytic approaches. Let us comment on them individually.

- The suggestion to use large–$N_c$ QCD combined with lowest order $\chi$PT loops, was first proposed by Bardeen, Buras and Gérard in a series of seminal papers\cite{1}. The same approach has been applied by the Dortmund group\cite{11}, in particular to the evaluation of $\epsilon'/\epsilon$. In this approach the hadronic ansatz to the Green’s functions consists of Goldstone poles only. The predicted hadronic shape, as shown by the BBG, HKPSB line (black dotted), is a constant.

- The Trieste group evaluate the relevant Green’s functions using the constituent chiral quark model ($C\chi$QM)\cite{12,13,14}. They have obtained a

\footnote{See refs.\cite{8,9,10} and references therein.}
long list of predictions\textsuperscript{14}, in particular $\epsilon'/\epsilon$. The model gives an educated first guess of the low–$Q^2$ behaviour of the Green’s functions, as one can judge from the $C\chi$QM–curve (green dashed), but it fails to reproduce the short–distance QCD–behaviour.

• The extended Nambu–Jona-Lasinio (ENJL) model\textsuperscript{15} has a better low–energy behaviour, as the ENJL–curve (blue dot–dashed) shows, but it fails to reproduce the short–distance behaviour of the OPE in QCD. Arguments to do the matching to short–distance QCD have been forcefully elaborated in refs.\textsuperscript{16}, which also have made a lot of predictions; a large value for $\epsilon'/\epsilon$ in particular.

In view of the difficulties that these analytic approaches have in reproducing the shape of the simplest Green’s function one can think of, it is difficult to attribute more than a qualitative significance to their “predictions”; $\epsilon'/\epsilon$ in particular, which requires the interplay of several other Green’s functions much more complex than $\Pi_{LR}(Q^2)$.

3 Applications

The large–$N_c$ approach that we propose in order to compute a specific coupling of the chiral electroweak Lagrangian consists of the following steps:

1. Identify the underlying QCD Green’s functions.

2. Work out the short–distance behaviour and the long–distance behaviour of the relevant Green’s functions.

3. Make a large–$N_c$ ansatz for the underlying Green’s functions.

We have tested this approach with the calculation of a few low–energy observables:

• The electroweak $\Delta m_\pi$ mass difference\textsuperscript{17} which we have already discussed.

\textsuperscript{2}For a review, see e.g. ref.\textsuperscript{18} where earlier references can be found.
• The hadronic vacuum polarization contribution to the anomalous magnetic moment of the muon $a_\mu$ \cite{18}, with the result

$$a_\mu|_{\text{HVP}} = (5.7 \pm 1.7) \times 10^{-8},$$  \hspace{1cm} (8)

to be compared with an average of recent phenomenological determinations \cite{3}

$$a_\mu|_{\text{HVP}} = (6.949 \pm 0.064) \times 10^{-8}. \hspace{1cm} (9)$$

• The $\pi^0 \rightarrow e^+e^-$ and $\eta \rightarrow \mu^+\mu^-$ decay rates \cite{20}, with the results shown in Table 1 below, where

$$R(P \rightarrow l^+l^-) = \frac{\Gamma(P \rightarrow l^+l^-)}{\Gamma(P \rightarrow \gamma\gamma)}. \hspace{1cm} (10)$$

| Branching Ratio | Large–$N_c$ Approach | Experiment |
|-----------------|----------------------|------------|
| $R(\pi^0 \rightarrow e^+e^-) \times 10^8$ | $6.2 \pm 0.3$ | $6.28 \pm 0.55$ |
| $R(\eta \rightarrow \mu^+\mu^-) \times 10^9$ | $1.4 \pm 0.2$ | $1.47 \pm 0.20$ |
| $R(\eta \rightarrow e^+e^-) \times 10^8$ | $1.15 \pm 0.05$ | $< 1.8 \times 10^4$ |

These successful predictions have encouraged us to start a project of a systematic analysis of non–leptonic $K$–decays within this large–$N_c$ approach. So far we have completed two calculations of $K$–matrix elements within this large–$N_c$ approach, which we next discuss.

3.1 The $B_K$–Factor of $K^0 – \bar{K}^0$ Mixing

The factor in question is conventionally defined by the matrix element of the four–quark operator $Q_{\Delta S=2}(x) = (\bar{s}_L\gamma^\mu d_L)(\bar{s}_L\gamma^\mu d_L)(x)$:

$$\langle \bar{K}^0|Q_{\Delta S=2}(0)|K^0 \rangle = \frac{4}{3} f_K^2 M_K^2 B_K(\mu). \hspace{1cm} (11)$$

To lowest order in the chiral expansion the operator $Q_{\Delta S=2}(x)$ bosonizes into a term of $O(p^2)$

$$Q_{\Delta S=2}(x) \Rightarrow -\frac{F_4^2}{4} g_{\Delta S=2}(\mu) \left[ (D_\mu U^\dagger) U \right]_{23} \left[ (D_\mu U^\dagger) U \right]_{23}, \hspace{1cm} (12)$$

\cite{3}See e.g. Prades’s talk at KAON2001 \cite{19} and references therein.
with \(g_{\Delta S=2}(\mu)\) a low energy constant, to be determined, which is a function of the renormalization scale \(\mu\) of the Wilson coefficient \(C(\mu)\) which modulates the operator \(Q_{\Delta S=2}(x)\) in the four–quark effective Lagrangian. The coupling constant \(g_{\Delta S=2}(\mu)\), which has to be evaluated in the same renormalization scheme as the Wilson coefficient \(C(\mu)\) has been calculated, is given by an integral \(\text{21)}\) conceptually similar to the one which determines the electroweak constant \(C\) in Eq. (3). The invariant \(\hat{B}_K\) defined as

\[
\hat{B}_K = \frac{3}{4} C(\mu) \times g_{\Delta S=2}(\mu),
\]

(13)
can then be evaluated, with no free parameters, with the result \(\text{21)}\)

\[
\hat{B}_K = 0.38 \pm 0.11. \tag{14}
\]

When comparing this result to other determinations, specially in lattice QCD, it should be realized that the unfactorized contribution to this result is the one in the chiral limit. It is possible, in principle, to calculate chiral corrections within the same large–\(N_c\) approach, but this has not yet been done.

The result in Eq. (14) is compatible with the old current algebra prediction \(\text{22)}\) which, to lowest order in chiral perturbation theory, relates the \(B_K\) factor to the \(K^+ \to \pi^+\pi^0\) decay rate. In fact, our calculation of the \(B_K\) factor can be viewed as a successful prediction of the \(K^+ \to \pi^+\pi^0\) decay rate!

As discussed in \(\text{23)}\) the bosonization of the four–quark operator \(Q_{\Delta S=2}\) and the bosonization of the operator \(Q_2-Q_1\) which generates \(\Delta I = 1/2\) transitions are related to each other in the combined chiral limit and next–to–leading order in the \(1/N_c\)–expansion. Lowering the value of \(\hat{B}_K\) from the leading large–\(N_c\) prediction \(\hat{B}_K = 3/4\) to the result in Eq. (14) is correlated with an increase of the coupling constant \(g_8\) in the lowest order effective chiral Lagrangian, (see Eq. (1)), which generates \(\Delta I = 1/2\) transitions, and provides a first step towards a quantitative understanding of the dynamical origin of the \(\Delta I = 1/2\) rule.
3.2 ElectroWeak Four–Quark Operators

These are the four–quark operators generated by the so called electroweak Penguin like diagrams

\[ \mathcal{L} \Rightarrow \cdots C_7(\mu)Q_7 + C_8(\mu)Q_8, \]

with

\[ Q_7 = 6(\bar{s}_L\gamma^\mu d_L) \sum_{q=u,d,s} e_q(\bar{q}_R\gamma^\mu q_R) \quad \text{and} \quad Q_8 = -12 \sum_{q=u,d,s} e_q(\bar{s}_L q_R) (\bar{q}_R d_L), \]

and \( C_7(\mu), C_8(\mu) \) their corresponding Wilson coefficients. They generate a term of \( O(p^0) \) in the effective chiral Lagrangian, therefore, the matrix elements of these operators, although suppressed by an \( e^2 \) factor, are chirally enhanced. Furthermore, the Wilson coefficient \( C_8 \) has a large imaginary part, which makes the matrix elements of the \( Q_8 \) operator to be particularly important in the evaluation of \( \epsilon'/\epsilon \).

Within the large–\( N_c \) framework, the bosonization of these operators produce matrix elements with the following counting

\[ \langle Q_7 \rangle|_{O(p^0)} = O(N_c) + O(N_c^0) \quad \text{and} \quad \langle Q_8 \rangle|_{O(p^0)} = O(N_c^2) + O(N_c^0), \]

Zweig suppressed

A first estimate of the underlined contributions was made in ref. [26]. The inclusion of final state interaction effects based on the leading large–\( N_c \) determination of \( \langle Q_8 \rangle \) (and \( \langle Q_6 \rangle \)) in connection with a phenomenological determination of \( \epsilon'/\epsilon \), has been recently discussed in [27].

The bosonization of the \( Q_7 \) operator to \( O(p^0) \) in the chiral expansion and to \( O(N_c) \) is very similar to the calculation of the \( Z \)–contribution to the coupling constant \( C \) in Eq. (3). An evaluation which also takes into account the renormalization scheme dependence has been recently made in [28].

The contribution of \( O(N_c^0) \) to \( \langle Q_8 \rangle|_{O(p^0)} \) is Zweig suppressed. It involves the sector of scalar (pseudoscalar) Green’s functions where it is hinted from various phenomenological sources that the restriction to just the leading large–\( N_c \) contribution may not always be a good approximation. Fortunately, as first

\[ ^{4}\text{See e.g., Buras lectures [24]} \]
pointed out in Ref. 31, independently of large-$N_c$ considerations, the bosonization of the $Q_8$ operator to $O(p^0)$ in the chiral expansion can be related to the four-quark condensate $\langle O_2 \rangle \equiv \langle 0| (\bar{s}_L s_R) (\bar{d}_R d_L) |0 \rangle$ by current algebra Ward identities, the same four-quark condensate which also appears in the OPE of the $\Pi_{LR}(Q^2)$ function discussed above. The crucial observation, here, is that large-$N_c$ QCD gives a rather good description of the $\Pi_{LR}(Q^2)$–function, as we have seen earlier; in particular it implies that \[ \lim_{Q^2 \to \infty} (-Q^2 \Pi_{LR}(Q^2)) Q^4 = \sum_V f_V^2 M_V^6 - \sum_A f_A^2 M_A^6. \] (18)
This results in a determination of the matrix elements of $\langle Q_8 \rangle |O(p^0)\rangle$ which does not require the separate knowledge of the Zweig suppressed $O(N_c^0)$ term in Eq. (17).

The numerical results we get for the matrix elements
\[ M_{7,8} \equiv \langle (\pi\pi)_{J=2} | Q_{7,8} | K^0 \rangle \] 2 GeV \] at the renormalization scale $\mu = 2$ GeV in the two schemes NDR and HV and in units of GeV$^3$ are shown in Table 2 below, (the first line.)

| METHOD              | $M_7$(NDR) | $M_7$(HV) | $M_8$(NDR) | $M_8$(HV) |
|---------------------|------------|-----------|------------|-----------|
| Large–$N_c$ Approach|            |           |            |           |
| Ref. 28             | 0.11 ± 0.03| 0.67 ± 0.20| 3.5 ± 1.1  | 3.5 ± 1.1 |
| Lattice QCD         |            |           |            |           |
| Ref. 30             | 0.11 ± 0.04| 0.18 ± 0.06| 0.51 ± 0.10| 0.62 ± 0.12|
| Dispersive Approach |            |           |            |           |
| Ref. 31             | 0.22 ± 0.05|           | 1.3 ± 0.3  |           |
| Ref. 34             | 0.35 ± 0.10|           | 2.7 ± 0.6  |           |
| Ref. 32             | 0.18 ± 0.12| 0.50 ± 0.06| 2.13 ± 0.85| 2.44 ± 0.86|
| Ref. 35             | 0.16 ± 0.10| 0.49 ± 0.07| 2.22 ± 0.67| 2.46 ± 0.70|

Also shown in the same table are other evaluations of matrix elements with which we can compare scheme dependences explicitly 5. Several remarks are

5There is a new "dispersive determination" in the literature 34) since the KAON2001 conference, but it is controversial as yet; this is why we do not include it in the Table.
in order

- Our evaluations of $M_7$ do not include the terms of $O(\alpha_s^2)$ because, as pointed out in Ref. 28), the available results in the literature 29) were not calculated in the right basis of four-quark operators needed here.

- We find that our results for $M_7$ are in very good agreement with the lattice results in the NDR scheme, but not in the HV scheme. This disagreement is, very likely, correlated with the strong discrepancy we have with the lattice result for $M_8$(NDR).

- The recent revised dispersive approach results 33, 35), which now include the effect of higher terms in the OPE, are in agreement, within errors, with the large-$N_c$ approach results. In fact, the agreement improves further if the new $O(\alpha_s^2)$ corrections, which have now been calculated in the right basis 35), are also incorporated in the large-$N_c$ approach.

- Both the revised dispersive approach results and the large-$N_c$ approach results for $M_8$ are higher than the lattice results. The discrepancy may originate in the fact that, for reasonable values of $\langle \bar{\psi}\psi \rangle$, most of the contribution to $M_8$ appears to come from an OZI-violating Green’s function which is something inaccessible in the quenched approximation at which the lattice results, so far, have been obtained.

4 Acknowledgements

I wish to thank Marc Knecht, Santi Peris, Michel Perrottet, and Toni Pich for enjoyable collaborations on the various topics reported here.

References

1. G. ’t Hooft, Nucl. Phys. B72 (1974) 461; B73 (1974) 461.
2. E. Witten, Nucl. Phys. B160 (1979) 57.
3. J. Gasser and H. Leutwyler, Nucl. Phys. B250 (1985) 465.
4. M. Knecht, S. Peris and E. de Rafael, Phys. Lett. B443 (1998) 255.
5. M. Knecht and E. de Rafael, Phys. Lett. B424 (1998) 355.
6. Review of Particle Physics, Eur. Phys. J. C15 (2000) 1.

7. ALEPH Collaboration, R. Barate et al, Z. Phys. C76 (1997) 15; ibid Eur. Phys. J. C4 (1998) 409.

8. A. Buras, The 1/N approach to nonleptonic weak interactions, in CP violation, ed. C. Jarlskog, World Scientific, Singapore, 1998.

9. W. Bardeen, Weak decay amplitudes in large $N_c$ QCD, in Proc. of Ringberg Workshop, Nucl. Phys.B (Proc. Suppl.) 7A (1989) 149.

10. J.-M. Gérard, Acta Phys. Pol. B21 (1990) 257.

11. T. Hambye, G.O. Köhler, E.A. Paschos, P.H. Soldan and W.A. Bardeen, Phys. Rev. D58 (1998) 014017, and references therein.

12. A. Manohar and H. Georgi, Nucl. Phys. B234 (1984) 189.

13. D. Espriu, E. de Rafael and J. Taron, Nucl. Phys. B345 (1990) 22.

14. A. Pich and E. de Rafael, Nucl. Phys. B358 (1991) 311.

15. S. Bertolini, M. Fabbrichesi and J.O. Egg, Rev. Mod. Phys. 72 (2000) 65 and references therein.

16. J. Bijnens, Phys. Rep. 265(6) (1996) 369.

17. J. Bijnens and J. Prades, JHEP 9901 (1999) 023; ibid 0001 (2000) 002; ibid 0006 (2000) 035.

18. M. Perrottet and E. de Rafael, unpublished.

19. J. Prades, hep-ph/0108192

20. M. Knecht, S. Peris, M. Perrottet and E. de Rafael, Phys. Rev. Lett. 83 (1999) 5230.

21. S. Peris and E. de Rafael, Phys. Lett. B490 (2000) 213, erratum hep-ph/0006146 v3.

22. J.F. Donoghue, E. Golowich and B.R. Holstein, Phys. Lett. B119 (1982) 412.
23. A. Pich and E. de Rafael, Phys. Lett. B374 (1996) 186.
24. A.J. Buras, *Weak Hamiltonian, CP Violation and Rare Decays*, in les Houches Session LXVIII, North Holland, 1999.
25. J. Bijnens and M. Wise, Phys. Lett. B137 (1984) 245.
26. M. Knecht, S. Peris and E. de Rafael, Phys. Lett. B457 (1999) 227.
27. E. Pallante, A. Pich and I. Scimemi, [hep-ph/0105011](https://arxiv.org/abs/hep-ph/0105011) and references therein.
28. M. Knecht, S. Peris and E. de Rafael, Phys. Lett. B508 (2001) 117.
29. L.V. Lanin, V.P. Spiridonov and K. G. Chetyrkin, Sov. J. Nucl. Phys. 44 (1986) 896.
30. A. Donini et al, Phys. Lett. B470 (1999) 233.
31. J.F. Donoghue and E. Golowich, Phys. Lett. B478 (2000) 172.
32. S. Narison, Nucl. Phys. B593 (2001) 3; [hep-ph/0012019](https://arxiv.org/abs/hep-ph/0012019) and private communication.
33. E. Golowich, Talk at the EPS-HEP-2001 Conference, Budapest.
34. J. Bijnens, E. Gámiz and J. Prades, [hep-ph/0108240](https://arxiv.org/abs/hep-ph/0108240) submitted to JHEP.
35. V. Cirigliano, J. Donoghue, E. Golowich and K. Maltman, [hep-ph/0109113](https://arxiv.org/abs/hep-ph/0109113) v1.
Discussion

Martinelli - Don’t you think that your results for the $M_8$ matrix elements could be affected by chiral corrections?

de Rafael - There are chiral corrections, of course, which so far have not been evaluated; but the results that you reported in your talk, showing that the lowest order chiral Ward identities for these matrix elements are well reproduced in lattice QCD, are an indication that the chiral corrections should be rather small in this case.

Isidori - Does your calculation of the $P \to \gamma^*\gamma^*$ amplitude, used to predict $P \to ll$ widths, reproduce well the available data on the $P \to \gamma^*\gamma$ form factor?

de Rafael - Yes, but I suggest you to discuss this issue with Andreas Nyffeler, who has recently performed a detailed study of this subject with Marc Knecht in hep-ph/0106034.

Roberts - Could your large–$N_c$ methods be applied to a calculation of the hadronic light–by–light contribution to $g_\mu - 2$?

de Rafael - Yes, and in fact we have been discussing quite a lot about this in Marseille. We think that it should be possible to do a rather clean evaluation of the contribution which comes from two $\langle PVV \rangle$ three–point functions.

Buras - I am puzzled by the fact that your hadronic integrals go all the way to infinity.

de Rafael - This is due to the fact that we both are using a renormalization scheme based on dimensional regularization: you at short–distances and we at long–distances. We both depend on a scale $\mu$ which has to cancel in the matching. If we were doing a cut–off renormalization, the long–distance integrals would then be cut at a certain scale $\Lambda$ indeed; but then you would have to redo all the short–distance analyses in the same cut–off renormalization which, as Donoghue pointed out, would introduce operators of higher dimension at short–distances.