SCHRÖDINGER EQUATIONS, DEFORMATION THEORY AND tt*-GEOMETRY

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ABSTRACT. This is the first of a series of papers to construct the deformation theory of the form Schrödinger equation, which is related to a section-bundle system \((M, g, f)\), where \((M, g)\) is a noncompact complete Kähler manifold with bounded geometry and \(f\) is a holomorphic function defined on \(M\). The deformation of \(f\) will induce the deformation of the corresponding Schrödinger equation. We will prove that the Schrödinger operator has purely discrete spectrum if the section-bundle system \((M, g, f)\) is strongly tame. As the conclusion, we can obtain the Hodge theory, including the Hodge decomposition theorem and the Hard Lefschetz theorem. If the manifold is Stein, then the \(L^2\) cohomology of \(\partial f\) can be computed explicitly and only the middle dimensional homology information is preserved.

If the deformation \(\tau f\) of \(f\) is strong, then we obtain a Hodge bundle over the deformation space. Several operators on the Hodge bundle are defined and they can be assembled into a total connection \(\mathcal{D}\). The integrability of \(\mathcal{D}\) is obtained by proving the Cecotti-Vafa’s equation (\(tt^*\) equation) and the "Fantastic equation" found in this paper, which describes the behavior along the coupling constant \(\tau\) direction. We can also define a family of flat connections and prove they satisfy some interesting identities.

The strong deformation of a strongly tame system includes the following interesting examples:

- Marginal and relevant deformation of \((\mathbb{C}^n, \sqrt{-1} \sum dz_i \wedge \bar{z}_i, W)\), where \(W\) is a nondegenerate quasihomogeneous polynomial, e.g., \(W\) being the quintic polynomials.
- Subdiagram deformation of \((\mathbb{C}^n, \sqrt{-1} \sum \frac{dz_i}{z_i} \wedge \frac{d\bar{z}_i}{\bar{z}_i}, f)\), where \(f\) is a non-degenerate and convenient Laurent polynomials, e.g., \(f = z_1 + \cdots + z_n + \frac{e^{-t}}{z_1 \cdots z_n}\).
- The miniversal deformation of the simple singularities \(A, D, E\) and the parabolic singularities \(X_8, J_9, P_{10}\) in Arnold’s singularity list.

As the simple applications, we can get the mixed Hodge structure, the isomonodromic deformation of o.d.e. with respect to \(\tau\) and the Frobenius manifold structure on the deformation space via a primitive vector which is defined by the oscillating integral along the Lefschetz thimbles.

This work is also the first step attempting to understand the whole Landau-Ginzburg B-model including the higher genus invariants. Our work is mainly based on the pioneer work of Cecotti, Cecotti and Vafa [Ce1, Ce2, CV].

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1. Introduction

1.1. LG model and $tt^*$ geometry: the origin in physics

The $N = 2$ superconformal field theories have important role in the study of string theory since they can be taken as the exactly soluble toy model and meanwhile as the reduction of the higher dimensional quantum field theories. The corresponding lagrangian of the quantum field theory is invariant under the action of the $N = (2, 2)$ superconformal group whose Super Lie algebra consists of three parts: the even part generated by $H, P, M$, the Noether charges for the time translation, the 1-dimensional space translation and the Lorentz rotation; the odd part generated by the supercharges: $Q_+, Q_-, Q_+, \bar{Q}_-$; the Noether charges $F_V, F_A$ of vector $U(1)$ and axial $U(1)$ rotations. They satisfy the following
Schrödinger Equation and $\sigma$-Geometry

The Schrödinger equation and $\sigma$-geometry involve commutation relations (in the sense of super Lie algebra):

\[
\begin{align*}
Q_+^2 &= Q_-^2 = \bar{Q}_+^2 = \bar{Q}_-^2 = 0 \\
[Q_+, \bar{Q}_+] &= H \pm P \\
[\bar{Q}_+, Q_-] &= [Q_+, \bar{Q}_+] = 0 \\
[Q_-, \bar{Q}+] &= [Q_-, \bar{Q}_+] = 0 \\
[iM, Q_+] &= \mp \bar{Q}_+, [iM, \bar{Q}_+] = \mp \bar{Q}_- \\
[iF_+, Q_+] &= -iQ_+, [iF_+, \bar{Q}_+] = i\bar{Q}_+ \\
[iF_A, Q_+] &= \mp iQ_+, [iF_A, \bar{Q}_+] = \pm i\bar{Q}_-.
\end{align*}
\]

Here all the operators acting on the fock space (Hilbert space), and except that $Q_\pm^\dagger = \bar{Q}_\pm$ are conjugate to each other, all other operators are Hermitian. The symmetries will be broken if the Lagrangian does not preserve the corresponding invariance. The Calabi-Yau $\sigma$-model (ref. [Ka] from the view of physicists) viewed as a geometrical realization of the $N = 2$ superconformal algebra has been studied intensively by physicists and mathematicians in the past twenty years. Because of the $\mathbb{Z}_2$ outer automorphism of this algebra, the geometrical realization of this algebra induces the mirror symmetry phenomenon: The A-model theory in one Calabi-Yau manifold $M$ should match the B-model theory of another mirror Calabi-Yau manifold $\tilde{M}$. The A model theory, the B-model theory and their mirror relations have become a main subject in mathematics since 80's (for instance, see the series of publications [Ya]). In each model, there are versions of closed string or open string theory. Here we only talk about theory about closed string. The A model theory has been well-understood so far. It concerns the study of pseudo-holomorphic curves in a symplectic manifold, in another word, studying the moduli problem of the solutions of the following Cauchy-Riemann equation:

\[
\bar{\partial}_J u = 0,
\]

where $u : (\Sigma, j) \to (M, \omega, J)$, $(\Sigma, j)$ is a Riemann surface with complex structure $j$ and $(M, \omega, J)$ is a symplectic manifold with symplectic form $\omega$ and compatible almost complex structure $J$. The virtual fundamental cycle has been constructed (ref. [LiT, FO, Ru]) and one can define the correlation functions (Gromov-Witten invariants) and the generating function and get the A model theory. This model is a topological field theory (or cohomological field theory) in which the observables are represented by the cohomological class of the target manifold $M$. If $M$ is a Kähler manifold, the A model is supposed to be related to the deformation of symplectic structures and the $B$ model is supposed to be related to the deformation of the complex structure. The Kodaira-Spencer theory [Ko] characterized the moduli germ of the complex structure of a compact complex manifold. So far the geometrical construction of the correlation functions for the higher genus ($\geq 2$) has not been finished. The genus 1 part and the holomorphic anomaly equations have been discussed by [BCOV2]. The genus 0 part (or Frobenius manifold structure) was given by Kontsevich-Barannikov construction [BK]. The genus 1 invariants has been defined by H. Fang, Z. Lu and K. Yoshikawa [FLY]. A. Zinger [Zi] and Fang-Lu-Yoshikawa have also solved a conjecture of BCOV about the quintic threefold. A most recent progress has been made by K. Castello and S. Li [CL].

However, the computation of the invariants in Calabi-Yau model is rather difficult. Actually physicists have known another effective way to calculate the invariants much earlier. This is the $N = 2$ SUSY Landau-Ginzburg model. In the paper [Ge], Gepner found a
mysterious correspondence between some of the superconformal models that he had obtained by taking the tensor product of exactly soluble minimal models, and some specific Calabi-Yau manifolds. Further work \cite{CGP1, CGP2, LVW, Mar, GVW} show that it is not an occasional correspondence, and it is really a correspondence between Landau-Ginzburg model and Calabi-Yau model.

The simplest supersymmetric Lagrangian of the Landau-Ginzburg model has the following form

$$\int dz^2 d^4 \theta K(\bar{\Phi}_i, \Phi_i) + \left( \int d^2 z d^2 \theta W(\Phi_i) + c.c. \right),$$

(2)

where the first term is called the $D$-term and the second term is called the $F$-term and is given by a quasi-homogeneous holomorphic function $W$ which is the function of the superfields $\Phi_i$. Like CY $\sigma$ model, LG model has also A model theory and B model theory and the mirror symmetry phenomena between them. A physical description has been given by Witten \cite{Wi4}, Guiffn and Sharpe \cite{GS}. The A model is related to the moduli problem of the following nonlinear Cauchy-Riemann equation:

$$\bar{\partial}u + \frac{\partial W}{\partial u} = 0,$$

where $u : \Sigma \to M$ is a map from the Riemann surface to $M$ (where the target Kähler manifold $M$ is assumed to have $U(1)$ symmetry which is the requirement of supersymmetry (ref. \cite{Del})).

However, this equation is not well-defined on Riemann surface with any genus. For $M = \mathbb{C}^n$, this equation was defined via some line bundle structure (called $W$-structure) on Riemann surface by Witten. This equation for $W = x^r$ case has been used by Witten \cite{Wi1} to construct the virtual cycle on the moduli space of $r$-spin curves, an attempt to generalize his conjecture \cite{Wi} (later solved by Kontsevich and called Witten-Kontsevich theorem) to $A, D, E$ cases. A more general definition has been obtained very recently by Witten coupling with gauge fields \cite{Wi2}. We call this globally defined equation as Witten equation. The moduli problem of Witten equation has been studied by Fan-Jarvis-Ruan and the corresponding cohomological field theory, quantum singularity theory, has been constructed in \cite{FJR2, FJR3}. They also proved the generalized Witten conjecture for $A, D, E$ cases (A case solved earlier by Farber-Shadrin-Zvonkine \cite{FSZ}). The CY/LG correspondence for A model has been checked in some cases by Choido-Ruan \cite{ChR1, ChR2}.

The B side of the LG model has intimate relation with the singularity theory (ref. \cite{AGV}). Singularity theory (or catastrophe theory) studies the germ of the singularity of a holomorphic functions. To get the information of the singularity, one must use the deformation of the given holomorphic function. Based on the deformation parameter space, the topological and geometrical structure of a singularity appear. Similar things happened in physics. Zamolodchikov \cite{Zam} use the deformed massive Conformal field theory to study the conformal one. Like in CY model, the observables, correlations functions and partition function occupy the central position of LG model. In the papers, \cite{CGP1, CGP2, Ce1, Ce2}, the Chiral ring structure has been studied. In particular, in \cite{Ce1, CV}, the $t^r$ equations has been founded which induces an integrable connection on the vacuum bundle. The discovery is remarkable, and we call this equations as Cecotti-Vafa’s equation. A realization of the CV equations is to use the supersymmetric quantum mechanics. Given a holomorphic function $f$, one can obtain the twisted operator $\tilde{\partial}_f = \tilde{\partial} + \partial f \wedge$. Like the
study of $\bar{\partial}$ operator in compact Kähler manifold, the operator $\bar{\partial}_f$ induces a series of operators $\partial_f, \bar{\partial}_f, \Delta_f = \bar{\partial}_f \partial_f + \partial_f \bar{\partial}_f$. This will induce the Hodge theory and Hard Lefschetz theorem as in compact Kähler manifolds. Those ingredients were conjectured by these physicists. In Cecotti and Vafa’s paper, [CV], Page 41, they speculate the existence of a generalized special geometry (compare to the Variation of Hodge structure in compact Kähler manifolds).

However, there is a fundamental problem not solved hidden in their formulation:

**Problem 1.1.** Is the zero spectrum of these Schrödinger operators a discrete spectrum? Since $f$ is a holomorphic function defined on $\mathbb{C}^n$, $\mathbb{C}^n$ is a complete noncompact manifold, it is not necessary that 0 is the discrete spectrum of the minimal self-adjoint extension of the Schrödinger operator $\Delta_f$. It is quite easy to find a counterexample such that zero is a continuous spectrum. In this case, the Hodge decomposition theorem does not hold in the usual form as in the compact Kähler manifold and then all the conclusions based on Hodge decomposition theorem are ineffective.

Furthermore, to get the Cecotti-Vafa’s equation, one has to frequently use taking derivatives with respect to the deformation parameters and sometimes using Stokes integration formulas. To justify all of these operations, one must study carefully the analytic properties of the Schrödinger equation and its solutions.

The Cecotti-Vafa’s equations is very important. It provided and will provide many integrable systems and meanwhile provide the (topological) solutions. On the other hand, the authors in [CFIV] have founded a new supersymmetric index $\text{Tr}(-1)^F e^{-\tau H}$ for a system with Hamiltonian $H$ compared to the Witten index $\text{Tr}(-1)^F e^{-\tau H}$ [Wi] and this index as the function of $\beta$ satisfies an differential equation and has tight connection with the Cecotti-Vafa’s equation. Some examples in [CFIV] show that the index will introduce some integrable systems. The author actually asked the following question:

**Problem 1.2.** Explain why the new index satisfy the Cecotti-Vafa’s equation while satisfies an integrable differential equation (original: coupled integral equations from TBA, thermodynamic Bethe Ansatz, ref. [Zam]) with $\tau$ as the variable?

1.2. tt* geometry: progress in mathematics.

Mathematician began to realize the importance of $tt^*$ equations only ten years later after Cecotti-Vafa’s discovery. In [Het1], C. Hertling found that the $tt^*$ equations and the induced integrable structure are naturally related to the Frobenius manifold structure, which was introduced by B. Dubrovin in the study of topological field theory in the early of 90’s [Du2, Du, Du3]. Such a structure is a geometrical description of some integrable systems, among them, the WDVV equations, the isomonodromic deformation of linear differential systems [Man]. However, such structure was found more earlier, in the earlier of 80’s, by K. Saito in the study of the universal deformation of hypersurface singularities [Sai1, Sai2, Sai3, ST]. The flat structure comes from the existence of the ”primitive forms” called by K. Saito. K. Saito solved the existence for some examples, and the general cases were solved by M. Saito via solving the Riemann-Hilbert problem on $\mathbb{P}^1$ [Sm, Het1].

The key concepts in [Het1] defined by Hertling after he studied Cecotti-Vafa’s work are the TERP structures (or TLEP structures). He compared the trTERP or trTLEP structure with other known structures, including Variation of twistor structure by C. Simpson [Si], the Cecotti-Vafa’s structure, Frobenius type structure and the classical Variation of Hodge structure and mixed Hodge structure. The relation between CV structure and Frobenius type structure has been studied by Sabbah [Sâ]. The combination of CV structure and
Frobenius type structure defined on the holomorphic tangent bundle of a complex manifold will introduce the so called CDV structure [Het1]. Essentially, CDV structure is the Frobenius manifold structure plus an extra compatible real structure defined on the real tangent bundle. A most recent work on these structures has been done by J. Lin [Lin].

In [Het1], Hertling has also constructed a TERP structure for the germ of any hypersurface singularities by using the oscillating integration. Consequently, the primitive forms used to define the Frobenius manifold are just horizontal sections of the connection and meanwhile are the eigenfunctions of the index operator (new supersymmetric index by CFIV) and consists of a one-dimensional space. This avoid to use M. Saito’s existence technique.

Globally, the oscillating integrals and the Fourier-Laplace transformation technique has been used by Sabbah [Sa2, Sa4], Douai [Do] to construct the Frobenius manifold structure for a holomorphic function defined on an affine manifold (with more or less standard metric), in particular, for the non-degenerate convenient Laurent polynomials defined on the algebraic torus. The simplest among them is $z + \frac{c}{\tau}$, which is known to be the LG mirror of the Fano manifold $\mathbb{C}P^1$.

So to my understanding, the $tt^*$ geometry arises from Cecotti-Vafa’s equation and developed to a TERP structure by Hertling. It is a generalization (or a complement) of VHS, twistor structure and Frobenius manifold structure. To obtain a TERP structure from a geometrical structure is a more difficult work.

Because of the essence of the $\mathcal{N}=2$ superconformal algebra, the TERP structure should be a universal property and is possible to be defined on the abstract algebra. This construction has been done by Iritani [Ir] and should also exist on Fan-Jarvis-Ruan’s quantum singularity theory [FJR2] and even other higher dimensional models.

Though some progress has been achieved, in particular, in the study of singularity theory. There are many problems untouched by mathematicians so far. One of the most important side of CV’s equations is its relation to integrable systems. It is unlikely a natural way for algebraic geometry to generate a new integrable system, in particular an integrable partial differential equations and write down the explicit solution and study its asymptotic approximation formula at the degenerate point of the deformation space. The problems arising from physics and mathematics in the study of $tt^*$ geometry are the start point of our present research.

1.3. Our deformation theory of Schrödinger operators.

This paper is the first of a series of papers attempting to understand, explore and complement the geometrical structure concerning the $tt^*$ equations that physicists have discovered. We will construct the mathematical foundation of the deformation theory of the supersymmetric quantum mechanics associated to a tuple $(M, g, f)$. The tuple $(M, g, f)$ consists of a complete noncompact Kähler manifold with metric $g$, which has bounded geometry (see Section for an explanation), and a holomorphic function $f$ defined on $M$. We give a name to it: section-bundle system. The reason is that $(M, g, f)$ is equivalent to the tuple $(M \times \mathbb{C}, g \oplus g_E, s)$, where $s = \tau f$ is the section of the trivial bundle $M \times \mathbb{C}$, where $\tau$ is the coordinate of the fiber $\mathbb{C}$ and $g_E$ is the standard Kähler metric on $\mathbb{C}$. This viewpoint would be very important for the later generalization [FT]. For the section bundle system $(M, g, f)$, one obtain a twist operator $\tilde{\partial}_f = \tilde{\partial} + \partial f \wedge$ acting on the space $\Omega^k(M)$ of smooth $k$-forms and its formal conjugate operator $\tilde{\partial}_f$ with respect to the Kähler metric $g$. So we can define the twisted Laplace operator $\Delta_f = \tilde{\partial}_f^\dagger \tilde{\partial}_f + \tilde{\partial}_f \tilde{\partial}_f^\dagger$. These operators have closed extension to the
space $L^2(\Lambda^k(M))$ of $L^2 k$-forms. Notice that these operators only keep the real grading of forms and do not keep the Hodge grading. We require the restriction of the section-bundle system $(M, g, f)$ to be strongly tame, which will be given in Definition 2.39.

**Definition 1.3.** The section-bundle system is said to be strongly tame, if for any constant $C > 0$, there is

$$|\nabla f|^2 - C|\nabla^2 f| \to \infty, \text{ as } d(x, x_0) \to \infty. \tag{3}$$

Here $d(x, x_0)$ is the distance between the point $x$ and the base point $x_0$.

For strongly tame section-bundle system $(M, g, f)$, we have the fundamental theorem in this paper:

**Theorem 1.4 (Theorem 2.40).** Suppose that $(M, g)$ is a Kähler manifold with bounded geometry. If $\{(M, g), f\}$ is a strongly tame section-bundle system, then the form Laplacian $\Delta f$ has purely discrete spectrum and all the eigenforms form a complete basis of the Hilbert space $L^2(\Lambda^k(M))$.

This theorem can deduce the following conclusions:

**Corollary 1.5 (Theorem 2.42 and Theorem 2.49).** The section-bundle systems $(C^n, i \sum_j dz_j \wedge dz_j, W)$ and $((\mathbb{C}^*)^n, i \sum_j \frac{dw_j}{z_j} \wedge \frac{dw_j}{z_j}, f)$ are strongly tame, if

- $W$ is a non-degenerate quasi-homogeneous polynomial with homogeneous weight 1 and of type $(q_1, \cdots, q_n)$ with all $q_i \leq 1/2$.
- $f$ is a convenient and non-degenerate Laurent polynomial defined on the algebraic torus $(\mathbb{C}^*)^n$.

Therefore, the corresponding form Laplacian has purely discrete spectrum and all the eigenforms form a complete basis of the Hilbert space $L^2(\Lambda^k(M))$.

Note that the conditions for $W$ and $f$ in the above corollary are equivalent to the facts that the hypersurface defined by $\{W = 0\}$ in the weighted projective space and the hypersurface defined by $\{f = 0\}$ in the toric variety are smooth. Therefore, the above theorem says that for any smooth hypersurface in the weighted projective spaces or in the toric varieties there is a corresponding strongly tame section-bundle system such that the form Laplacian has purely discrete spectrum.

Theorem 1.4 is based on an important spectrum theorem, Theorem 2.40 of Kondrat’ev-Shubin [KS] for scalar Schrödinger equation. We can prove that the form Laplacian for strongly tame section-bundle system is the compact perturbation of the scalar Schrödinger operator. So the scalar and the form cases have the same spectrum properties.

Once we have the fundamental spectrum theorem, we can proceed as in the case of compact Kähler manifold. We can obtain the Hodge decomposition theorem, Theorem 2.52 and the Hard Lefschetz theorem, Theorem 2.58-2.60. The proof of the Hard Lefschetz theorem is due to the Kähler-Hodge identities and the $N = 2$ superconformal algebra structure for those twisted operators.

However, unlike in the compact case, we need compute the $L^2$-cohomology, at best in terms of the topological data of the section-bundle system. We have

**Theorem 1.6 (Theorem 2.66).** Let $(M, g)$ be a Kähler stein manifold with bounded geometry and $(M, g, f)$ be strongly tame. If $f$ is a Morse function, then

$$\dim H^k = \begin{cases} 0, & k < n \\ \mu, & k = n. \end{cases} \tag{4}$$
and there is an explicit isomorphism:

\[ i_0 : \mathcal{H}^n \rightarrow \Omega^n(M)/df \wedge \Omega^{n-1}(M). \]  

(5)

Here \( \mu \) is the number of critical points of \( f \).

Then we go to the deformation theory in Section 3. We study the strong deformation (Definition 3.8) of a strongly tame section bundle system \((M, g, f)\). The strong deformation contains the following cases:

- The marginal and relevant deformation of \((\mathbb{C}^n, i \sum_j dz^j \wedge d\bar{z}^j, W)\), where \( W \) is the polynomial as in Corollary 1.5. In particular, including the universal unfolding of the simple singularities \( A_n, D_n, E_6, E_7, E_8 \), and the singularities \( P_8, X_9, J_{10} \) in Arnold’s list [AGV].
- The deformation \( ((\mathbb{C}^*)^n, i \sum_j \frac{dz^j}{t} \wedge \frac{d\bar{z}^j}{t}, \tau f_t) \), where \( f_t = f(z, t) \) is the polynomial as in Corollary 1.5. In particular, this including the deformation: 

\[ z_1 + \cdots + z_n + e^{-t} \]

which is the mirror of \( \mathbb{C}P^n \) and its subdiagram deformation.

Note that the above list contains many important cases:

- corresponds to the deformation theory of smooth hypersurfaces in (weighted) projective space.
- corresponds to the minimal model \( A, D, E \) cases studied in physics and in Givental’s formal Gromov-Witten theory [Gi].
- \( P_8, X_9, J_{10} \) has central charge 1 and corresponds to the elliptic curves and modular forms.
- corresponds to the LG model mirror to the toric Fano varieties in toric geometry (ref. [Bar], [HV]).

All of our conclusions in this paper can be applied to the strong deformation of a strongly tame section-bundle system.

Section 3 is the analytic kernel of the deformation theory, the similar role as in the compact deformation theory in projective space. There are some difference. In our case, we have a fixed background manifold and the deformation takes places when the superpotential function changes and this makes the life easily than in the compact case. On the other hand, since the manifold is non-compact, one must treat all the related quantities in infinite far place, i.e, treating the infinite far boundary. Usually this is very difficult if the potential function has bad behavior at the infinity. In our case, we have maximum principle which controls the asymptotic behavior of the eigenforms, and the Green functions at the infinity. The interior estimate can be obtained via the routine technique from elliptic differential equation of second type. We have the following conclusions:

- Any eigenform of the Schrödinger equation is exponential decaying and the weighted \( C^k \) norms which involving the potential of its higher derivatives are exponential decay. This is given by Proposition 3.44. The similar conclusion hold for Green function which is given by Proposition 3.47.
- The asymptotic behavior of the Green function \( G(z, w) \) as \( z \rightarrow w \) is given by Proposition 3.48.
- The existence and the regularity of the solutions of the non-homogeneous equation are given by Theorem 3.35.

Based on those estimates, we obtain Theorem 3.40 the continuity theorem of the eigenvalues. The stability theorems are given by the differentiability theorems, Theorems 3.51, 3.52 and 3.53 which shows the differentiability of the resolvent operators, projective operators and the Green functions with respect to the deformation parameters. In particular, this
shows that the eigenforms are $C^\infty$ differentiable with respect to the deformation parameter if the deformation is strong deformation.

Therefore Section 2 and 3 has answered the Problem 1.1 in the most interesting cases and meanwhile proved the differentiability and decaying estimate problems after Problem 1.1. This gives a rigorous mathematical foundation for Hodge theory related to the differential geometrical structure of the twisted operators. The decaying estimate of eigenforms also allow us applying the $L^1$-stokes theorem to eigenforms.

Now we go to Section 4. Let $f_\tau(z) := f((t, z)$ be a strong deformation defined on $\mathbb{C}^* \times S \times M$, where $S \subset \mathbb{C}^{\infty}$ is a domain and $\mathbb{C}^* = \mathbb{C} - \{0\}$. Then there is a family of deformation operators $\Delta_f, \partial_f, \bar{\partial}_f, \bar{\partial}_{\bar{f}}, \partial_{\bar{f}}$, which depend on the parameter $(\tau, t) \in \mathbb{C}^* \times S$ and the corresponding space of harmonic forms $\mathcal{H}_{\tau}$ depending on the parameter $(\tau, t)$. Therefore we obtain the Hodge bundle $\mathcal{H}^* \rightarrow \mathbb{C}^* \times S$ which is the subbundle of the trivial complex Hilbert bundle $\Lambda^*_C := L^2 \Lambda^* \times \mathbb{C}^* \times S \rightarrow \mathbb{C} \times S$. We have the ordinary derivatives $\partial_i, \partial_{\bar{i}}, \partial_{\tau}, \partial_{\bar{\tau}}, \partial_i \partial_{\bar{i}}, \partial_{\tau} \partial_{\bar{\tau}}, \partial_i \partial_{\bar{\tau}}, \partial_{\bar{i}} \partial_{\tau}$. The covariant derivatives $D_i,...$ are defined as the projections of the ordinary derivatives to the perpendicular subspace of the space of harmonic forms. The multiplication operators (marginal operators) $f, \partial_f, \partial_{\bar{f}}$ are defined as the unbounded operators acting on the Hilbert bundle and their restrictions to the Hodge bundle are defined as the operator $\mathcal{D}, \mathcal{B}, \mathcal{W}, \mathcal{B}_t$. The metric on $\mathcal{H}$ is naturally obtained by the background metric $g$. The connection $D+\bar{D}$ is the Hermitian connection with respect to $g$. The ordinary real structure $\tau_\mathbb{R}$ is compatible with $D$. The commutation check of those operators induces the first family of equation, 

Cecotti-Vafa’s equations (or $tt^*$ equation) given by Theorem 4.14

\[
D_i B_j = D_j B_i = 0, \quad D_i B_j = D_j B_i, \quad D_i B_j = D_j B_i
\]

\[
[D_i, D_j] = [D_i, D_j] = [B_i, B_j] = [B_i, B_j] = 0
\]

\[
[D_i, D_j] = -[B_i, B_j].
\]

However, we found that this is not the total story and some information has been missed in the previous study of $tt^*$ structure (even in physics literatures), the information coming from the coupling parameter $\tau$. Actually the related operators are $f, \tau D_f, \mathcal{W}_f$ and their conjugates. After a carefully check, we find the second family of equations given by Theorem 4.24 which is called by us as Fantastic equations.

(1) $[D_i, \mathcal{D}_f] + [B_i, \tau D_f] = 0, \quad [D_i, \mathcal{W}_f] + [B_i, \bar{\tau} D_f] = 0$

(2) $[D_i, \mathcal{W}_f] = 0, \quad [D_i, \bar{\mathcal{W}}_f] = 0$

(3) $[\bar{\tau} D_f, B_f] = [\tau D_f, B_f] = 0,$

(4) $[B_i, \mathcal{W}_f] = 0, \quad [B_i, \bar{\mathcal{W}}_f] = 0$

(5) $[\tau D_f, D_f] = -[\mathcal{W}_f, B_f], \quad [\bar{\tau} D_f, D_f] = -[\bar{\mathcal{W}}_f, B_f]$

(6) $[\tau D_f, D_f] = [\bar{\tau} D_f, D_f] = 0$

(7) $[\tau D_f, \mathcal{D}_f] = [\bar{\tau} D_f, \mathcal{W}_f] = 0$

(8) $[\tau D_f, \bar{\tau} D_f] = -[\mathcal{W}_f, \bar{\mathcal{W}}_f]$

In the same way as done in [CC1] [CV], a connection $\mathcal{D}$ combined with all the operators $D_i, \tau D_f, B_i, \mathcal{W}_f$ and their conjugates is obtained, which is a nearly flat connection by Theorem 4.22 i.e., $\mathcal{D}^2 = 0 \mod \mathcal{H}^\perp$. We call it as the Gauss-Manin connection of the bundle $\mathcal{H} \rightarrow \mathbb{C}^* \times S$. To obtain a flat connection, we will transfer the structures on $\mathcal{H}$ to another related bundle $\mathcal{H}_{0,0,p}$ in Section 4.3.
From Section 4.3 one assumes that $M$ is a Kähler stein manifold so that by Theorem 2.66 only the middle dimensional (co)homology information is preserved.

Let $H^0 \to C^* \times S$ be the relative homologic bundle with fiber $H^0_{(r,t)} = H^0_\delta(M, f_{(r,t)}^{-1}, \mathbb{C})$. Then $\mathcal{H}_{\delta,\text{top}} \to C^* \times S$ is the dual bundle of $H^0 \to C^* \times S$ with fiber $\mathcal{H}_{\delta,\text{top}} = H^0_\delta(M, f_{(r,t)}^{-1}, \mathbb{C})$. The intersection pairing, Poincare duality, Witten index, periodic matrix and much concepts have been discussed in this section. We define the bundle homomorphism $\psi_\delta : \mathcal{H}_\delta \to \mathcal{H}_{\delta,\text{top}}$ as

$$[\psi_\delta(\alpha)](\tau, t) = e^{\int_{(\tau, t)} f_{\delta,\tau} \alpha} : S^- \to C^* \times S.$$  

(6)

Similarly, we have the homomorphism $\psi_\delta : \mathcal{H}_\delta \to \mathcal{H}_{\delta,\text{top}}$ as

$$[\psi_\delta(\alpha)](\tau, t) = e^{-\int_{(\tau, t)} f_{\delta,\tau} \alpha} : S^+ \to C^* \times S.$$  

(7)

We can prove the following conclusion.

**Theorem 1.7** (Theorem 4.54). The bundle map $\psi_\delta$ provides an isomorphism between two real Hermitian holomorphic bundles: $(\mathcal{H}_\delta, g, \tau_R)$ and $(\mathcal{H}_{\delta,\text{top}}, \hat{g}, \tau_R)$ and the same for $\psi_\delta$.

Here the metric $\hat{g}$ is defined in Section 4.3. Furthermore, by integrating $S^-_a$ along the corresponding Lefschetz thimble, we can obtain the horizontal section $\Pi_{\delta, a}$. Furthermore, we can prove

**Theorem 1.8** (Theorem 4.61). The correspondence between $(\mathcal{H}_{\delta,\text{top}}, S^-_a, D + \tilde{D}, \hat{\eta}, \tau_R, \delta)$ and $(\mathcal{H}_{\delta,\text{top}}, \Pi_{\delta, a}, \mathcal{D}, \hat{\eta}, \tau_R, \delta)$ is an isomorphism between two real Hermitian bundles with the associated Hermitian connections.

In fact, we obtain the important conclusion:

**Theorem 1.9** (Theorem 4.58). The Cecotti-Vafa equations and the Fantastic equations hold on $\mathcal{H}_{\delta,\text{top}} \to C^* \times S_m$ and the connection $\mathcal{D}$ is flat.

Here $S_m \subset S$ is the set of $(\tau, t)$ such that $f_{(\tau, t)}$ is a Morse function.

A very important fact is that the equation $\mathcal{D} \Pi^- = 0$ and its solutions $\Pi^-$ are given explicitly in terms of the wave functions $\alpha_{\eta}$ of the Schrödinger equation $\Delta_{\delta, \eta} = 0$, where the Schrödinger equation can be defined in a more complicated manifold $M$ except the standard $C^n$ or $(C^*)^n$.

In Section 4.4 we discuss a family of flat connections $\nabla^{G, t}$ by gauge transformation. The gauge transformation is used to change the $(0, 1)$-part of $\mathcal{D}$ into $\tilde{\mathcal{D}}$. The derivatives of $\nabla^{G, t}$ provides another family of flat connections satisfying some interesting identities. The gauge transformation transforms the horizontal sections $\Pi_{\delta, a}$ into the horizontal and holomorphic sections $\Pi^{\mu, \eta}_{\delta, a}$.

So far, by considering the deformation theory of the Schrödinger equations, we have obtained the $tt^*$ geometry that physicists hope to get. $tt^*$ structure can deduce many interesting structures. It induces the mixed Hodge structure in Section 4.5 while inducing the isomonodromic deformation in Section 4.5.3. In Section 4.7 we can obtain the harmonic Higgs bundle structure. In Section 4.6 we will compute the periodic matrix and the primitive vector based on $S_m \subset S_m$. In some situations, e.g., $(C^n, W)$ or $((C^*)^n, f)$, the computation can be simplified. In particular, in $(C^n, W)$ case, the primitive vector is given by the following oscillating integral:

$$\Pi^{\mu, \eta}_{\delta, a} = \Pi^{\mu, \eta}_{\delta, a} T$$

$$\Pi^{\mu, \eta}_{\delta, a} = \tau^{n/2 - 1} e^{-A} \cdot \int_{\mathcal{D}} e^{\tau f + \tilde{f}} dz_1 \wedge \cdots \wedge dz_\nu,$$  

(8)
which is defined in $S^1_m$.

Via this primitive vector, we can construct the Frobenius manifold structure in Section 4.7.

1.4. **Relation to other mathematical branches.**

Naturally, the deformation theory of Schrödinger operators have very tight connections with many mathematical branches. We will describe simply the connections what we knew.

1.4.1. **Singularity theory.**

$tt^*$ geometry was firstly considered by experts studying the singularity theory. The first Frobenius manifold structure for the universal unfolding of a hypersurface singularity was founded by K. Saito and later developed by M. Saito. Though our theory can only treat the universal unfolding of singularity with central charge $c \leq 1$, it deserves to build the detail correspondence. One big difference is about the existence of primitive form (primitive vector). In our case, this is obtained naturally (even given by explicit form) by the period matrix. In M. Saito’s theory, this was obtained by solving Riemann-Hilbert problem on $\mathbb{P}^1$.

For the other hypersurface singularities, the best treatment is to consider the boundary value problem of Schrödinger operators, which will help build the local theory. Hence in some sense, the singularity theory can be described in the framework of differential geometry.

1.4.2. **Deformation theory of hypersurfaces.**

Since the deformation theory of Schrödinger operators can be applied to the defining functions of smooth hypersurface in a (weighted) projective space. It is natural to compare the VHS and period mappings in two sides. The VHS in projective space was proposed by Griffiths, and developed by Deligne, Schmidt, Shrenk,... and many mathematicians. The same comparison can be done for toric hypersurfaces. One should compare with Batyrev’s work [Ba].

1.4.3. **Topological field theory: CY/LG correspondence and mirror symmetry.**

The deformation theory of Schrödinger operators can be seen as the mathematical theory of Landau-Ginzburg B model. One can compare the Frobenius structure with LG A model, Fan-Jarvis-Ruan’s quantum singularity theory to check the mirror symmetry. On the other hand, one can check the LG/CY correspondence: compare with Kontsevich-Barannikov’s Calabi-Yau B model. This correspondence will generalize many results obtained by physicists and generalize the correspondence from the chiral ring structure to Frobenius manifold structure.

In Manin’s book [Ma], some different Frobenius manifold structures were proposed, they are: the isomonodromic deformation of linear differential system, Saito’s Frobenius manifold structure from singularity theory, Kontsevich-Barannikov’s construction on CY manifold and formal Frobenius manifold from quantum cohomology theory. Recently, the Frobenius manifold structure from Fan-Jarvis-Ruan’s quantum singularity theory also appear. Some attempts has been done (ref. [HM], [ChR1, ChR2]) to identify these structures by comparing the initial data. The deformation theory of Schrödinger operators build the rest one corner of the whole theory and provide the hope to build the direct geometrical correspondence between those structures.
1.5. Further problems in the deformation theory.

There are many subsequent problems. We will study the relation of the connection $\mathcal{D}$ with various structures such as CV-structure, TERP-structure, and Saito-Frobenius Structure in our following papers. Also we will discuss the DGBV algebraic structure induced by our deformation theory. After make clear all the relations, we will turn to the computation of the Frobenius manifold structures.

There are also two challenge directions. One is to study the integrable systems and their transformations coming from such a deformation theory. The other is to use the quantization method to capture the information from the deformation theory and try to construct the higher genus invariants for LG B-model.

Finally the deformation theory of Schrödinger type equations should exist not only for the trivial section-bundle system $(\mathbb{M} \times \mathbb{C}, \mathbb{M}, g, \tau_f)$, but also for the other more complicated section-bundle systems. For instance, the model relating the complete intersection CY model.

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2. Differential geometry of Schrödinger operators

2.1. Preliminary of functional analysis.

2.1.1. Min-max principle. We write down the Theorem XIII.1 of [RS]:

Theorem 2.1. Let $H$ be a self adjoint operator of a Hilbert space that is bounded below, i.e., $H \geq C1$ for some $C$. Define

$$\mu_n(H) = \sup_{\psi_1, \ldots, \psi_{n-1}} U_H(\psi_1, \ldots, \psi_{n-1}),$$

where

$$U_H(\psi_1, \ldots, \psi_m) = \inf_{\psi \in D(H) \parallel \psi \parallel = 1 \parallel [\psi, \psi_1, \ldots, \psi_m] \parallel} (\psi, H\psi).$$

where $D(H)$ represent the domain of $H$ and $[\psi, \psi_1, \ldots, \psi_m]$ is the perpendicular complement space of the subspace $[\psi, \psi_1, \ldots, \psi_m]$ generating by $\psi_1, \ldots, \psi_m$.

Then for each $n$, one of the following two conditions must hold:

1. there are $n$ eigenvalues (counting multiplicity) below the bottom of the essential spectrum and $\mu_n(H)$ is the $n$-th eigenvalue (counting multiplicity);
(2) $\mu_n$ is the bottom of the essential spectrum, i.e., $\mu_n = \inf\{\lambda \mid \lambda \in \sigma_{ess}(H)\}$ and in that case $\mu_n = \mu_{n+1} = \mu_{n+2} = \cdots$ and there are at most $n-1$ eigenvalues (counting multiplicity) below $\mu_n$.

2.1.2. Discreteness of spectrum. The following theorem which was the Theorem XIII.64 of [RS] gives the equivalent conditions that the self-adjoint operator $H$ has only discrete spectrum.

**Theorem 2.2.** Let $H$ be a self-adjoint operator that is bounded below. Then the following conditions are equivalent:

1. $(H - \mu)^{-1}$ is compact for some $\mu \in \rho(H)$.
2. $(H - \mu)^{-1}$ is compact for all $\mu \in \rho(H)$.
3. $\{\psi \in D(H) \mid ||\psi|| \leq 1, ||H\psi|| \leq b\}$ is compact for all $b$.
4. $\{\psi \in D(H) \mid ||\psi|| \leq 1, (H\psi, \psi) \leq b\}$ is compact for all $b$.
5. There exists a complete orthonormal basis $\{\varphi_n\}_{n=1}^{\infty}$ in $D(H)$ so that $H\varphi_n = \mu_n\varphi_n$ with $\mu_1 \leq \mu_2 \leq \cdots$ and $\mu_n \to \infty$.
6. $\mu_n(H) \to \infty$ where $\mu_n(H)$ is given by the min-max principle.

2.1.3. Compact perturbation.

**Definition 2.3.** Let $H_0$ be a self-adjoint operator with domain $D(H_0)$ and $H_1$ be a symmetric operator with domain $D(H_0) \subset D(H_1)$. $H_1$ is said to be $H_0$-bounded in form with bound $a$, if there exists a constant $b$ such that for any $\psi \in D(H_0)$ the following holds

$$|\langle H_1\psi, \psi \rangle| \leq a \langle H_0\psi, \psi \rangle + b \langle \psi, \psi \rangle.$$ 

Moreover, if for any $a > 0$, there exists a $b$ such that the above inequality holds, then $H_1$ is said to be $H_0$-compact.

It is known that if $a < 1$, then $H_0 + H_1$ is also self-adjoint operator.

**Theorem 2.4.** Let $H_0$ be a semibounded self-adjoint operator with only discrete spectrum. Let $H_1$ be a symmetric operator which is $H_0$-bounded with bound $a < 1$. Then the self-adjoint operator $H = H_0 + H_1$ has only discrete spectrum. In particular, if $H_1$ is $H_0$-compact, then $H$ has only discrete spectrum.

**Proof.** Since

$$|\langle H_1\psi, \psi \rangle| \leq a \langle H_0\psi, \psi \rangle + b \langle \psi, \psi \rangle,$$

We have for any $\psi \in D(H) = D(H_0)$,

$$(H_1\psi, \psi) \geq (1 - a)(H_0\psi, \psi) - b \langle \psi, \psi \rangle.$$ 

By Min-max principle, Theorem 2.1 we have

$$\mu_n(H) \geq (1 - a)\mu_n(H_0) - b.$$ 

By Theorem 2.2 we know that $\mu_n(H_0) \to \infty$ as $n \to \infty$. Hence

$$\mu_n(H) \to \infty,$$

as $n \to \infty$.

So by Theorem 2.2 again, we know that $H$ has only discrete spectrum.  \hfill $\square$

In 1934, K. Friedrichs [Fr] has proved the following theorem (see Theorem XIII.67 of [RS]):
**Theorem 2.5.** Let \( V \in L^1_{\text{loc}}(\mathbb{R}^n) \) be bounded from below and suppose that \( V(x) \to \infty \), as \( |x| \to \infty \). Then \( H_0 = -\Delta + V \) defined as a sum of quadratic forms is an operator having compact resolvent. In particular, \( H_0 \) has purely discrete spectrum and a complete set of eigenfunctions.

This theorem was generalized in many cases. Here we will use the Theorem of Kondrat’ev-Shubin about the Schrödinger operators on manifolds with bounded geometry.

2.1.4. **Bounded geometry.** Let \( (M, g) \) be a \( n \)-dimensional connected complete Riemannian manifold with metric \( g \). \( (M, g) \) is said to have a bounded geometry, if the following conditions hold:

1. the injectivity radius \( r_0 \) of \( M \) is positive.
2. \( |\nabla^m R| \leq C_m \), where \( \nabla^m R \) is the \( m \)-th covariant derivative of the curvature tensor and \( C_m \) is a constant only depending on \( m \).

**Example 2.6.** Let \( \tilde{M} \) be a compact Kähler manifold, and \( D = \sum_{i=1}^{p} D_i \) a normal crossing divisor in \( \tilde{M} \). Let \( M = \tilde{M} - D \) and \( s_i \) be the defining section of \( D_i \), and \( |\cdot| \) be a Hermitian metric on the associated line bundle \( [D_i] \) with \( |s_i| < 1 \). Then we have the Poincaré metric \( \omega \) associated to \( D \) given by

\[
\omega = C\tilde{\omega} - 2\sum_{i=1}^{n} dd^c \log(-\log|s_i|^2),
\]

for sufficiently large \( C \). Here \( \tilde{\omega} \) is the Kähler form of the manifold \( \tilde{M} \). Locally, the neighborhood near the divisor is given by polydisc of the form \((\Delta^*)^k \times \Delta^{n-k}\) and the classical poincaré metric on \( \Delta^* \) is given by

\[
\omega_{\Delta^*} = \frac{i}{2\pi} \frac{dzd\bar{z}}{(|z|^2 \log(|z|^2))^2}.
\]

However, this metric has zero injective radius. To get a bounded geometrical structure, Cheng-Yau [CY] introduced the local quasi-coordinate system.

For \((\Delta^*, \omega_{\Delta^*})\), the quasi-local coordinate system is defined as follows: for any \( 0 < \eta < 3/4 \), define the coordinate map

\[
\phi_\eta(v) = \exp\left(\frac{(1 + \eta)(v + 1)}{(1 - \eta)(v - 1)}\right), \forall v \in \Delta_{3/4}.
\]

Then \( \cup_{\eta \in (0,1)} \phi_\eta(\Delta_{3/4}) \) covers \( \Delta^* \) and the pull-back of the classical Poincaré metric

\[
\phi^*_\eta(\omega_{\Delta^*}) = \frac{i}{2\pi} \frac{dv \wedge d\bar{v}}{1 - |v|^2}
\]

is independent of \( \eta \). One can check this metric has bounded geometry. For a polydisc \((\Delta^*)^k\), one can take the coordinate system

\[
V_\eta = (\Delta_{3/4})^k \times \Delta^{n-k}, \forall \eta = (\eta_1, \cdots, \eta_k) \in (0, 1)^k.
\]

with coordinate map:

\[
\Phi_\eta(v) = (\phi_{\eta_1}(v), \cdots, \phi_{\eta_k}(v), v_{k+1}, \cdots, v_n), \forall v \in V_\eta.
\]

where \( \phi_\eta \equiv \phi_\eta \), which is defined before. So \( \cup_{\eta \in (0,1)^k} \Phi_\eta(V_\eta) \) gives a quasi-coordinate system such that the pull-back metric \( \Phi^*_\eta(\omega) \) has bounded geometry. Hence via the quasi-coordinate system, \((M, \omega)\) has bounded geometry. The reader can refer [CY] and [Wd1], [Wd2] for the proof.
The Laplace-Beltrami operator on scalar functions on $M$ is defined as
\[
\Delta = 1 \frac{\partial}{\sqrt{g} \, \partial x^i} (\sqrt{g} g^{ij} \frac{\partial u}{\partial x^j}),
\]
where $(x^1, \cdots, x^n)$ are local coordinates, $g = \det(g_{ij})$ and $(g^{ij})$ is the inverse matrix of the metric matrix $(g_{ij})$. Let $V(x)$ be a smooth potential function defined on $M$. The Schrödinger operator is
\[
H_0 = -\Delta + V(x).
\]

**Theorem 2.7.** Assume that $(M, g)$ is a Riemannian manifold of bounded geometry, $H_0 = -\Delta + V$ is the Schrödinger operator with a locally $L^2$-integrable potential $V(x)$ which is semi-bounded:
\[
V(x) \geq -C, \quad x \in E.
\]
There exists $c > 0$, depending only on $(M, g)$ such that the spectrum $\sigma(H)$ consisting of only discrete spectrum if and only if the following condition is satisfied:
\[
\text{(D) For any sequence } \{x_k, k = 1, 2, \cdots \} \subset M \text{ such that } x_k \to \infty \text{ as } k \to \infty, \text{ for any } r < r_0/2 \text{ and any compact subsets } F_k \subset B(x_k, r) \text{ such that } \text{Cap}(F_k) \leq cr^{n-2} \text{ in case } n \geq 3 \text{ and } \text{cap}(F_k) \leq c (\ln \frac{1}{r})^{-1} \text{ in case } n = 2.
\]
\[
\int_{B(x_k, r)/F_k} V(x) \, d\text{vol}_E \to \infty, \text{ as } k \to \infty.
\]
Here Cap$(F_k)$ for $n \geq 3$ means that the harmonic capacity of the set $F_k$ in the normal (geodesic) coordinates centered at $x_k$, and for $n = 2$ it means the same capacity with respect to a ball $B(0, R)$ of fixed radius $R < r_0$.

The following is the definition of the harmonic capacity

**Definition 2.8.** For any compact set $F \subset \Omega$, the harmonic capacity of $F$ with respect to $\Omega$ is
\[
\text{Cap}_\Omega(F) = \left\{ \int_{\Omega} |\nabla u|^2 \, u \in C_0^\infty(\Omega), u = 1 \text{ near } F, 0 \leq u \leq 1 \text{ in } \Omega \right\}
\]

An easy corollary similar to Friedrichs’ theorem is:

**Corollary 2.9.** If the potential $V(x)$ is locally $L^2$-integrable and is semi-bounded:
\[
V(x) \geq -C, \quad x \in E.
\]
Then if $V(x) \to \infty$ as $d(x, x_0) \to \infty$, the spectrum of the Schrödinger operator $H = -\Delta + V$ has only discrete spectrum. Here $x_0$ is fixed base point on $E$.

2.2. Example: Complex harmonic oscillators.

2.2.1. Witten deformation.

Let $f$ be a $C^\infty$ function on an $n$-dimensional Riemannian manifold $M$. Then this forms a simple model of section bundle system $(M \times \mathbb{R}, M, f_r := \tau f)$.

We can choose a local orthonormal basis $\{e^1, \cdots, e^n\}$ of the cotangent bundle around a point $p$ and denote the dual basis as $\{e_1, \cdots, e_n\}$. Define two operators
\[
a^j_k = e^j \wedge, \quad a_k = \iota(e_k),
\]
where $\iota(e_k)$ is the contraction with the vector field $e_k$. The two operators satisfy the commutative relations:
\[
[a^j_k, a^\ell_l] = 0, \quad [a_j, a^\ell_k] = \delta_{jk}, \quad [a_j, a_k] = 0.
\]
Define the operators acting on the space of $p$-forms $\Lambda^p(M)$:
\begin{equation*}
    d_f = d + \tau df \land = e^{-\tau df} \circ d \circ e^{\tau df}, \quad d_f^* = e^{\tau df} \circ d^* \circ e^{-\tau df} = d^* + \iota(\tau df)
\end{equation*}
and the Laplace operator
\begin{equation*}
    \Box_f = d_f^*d_f + d_f^*d_f.
\end{equation*}
The Laplace operator has the following form:
\begin{equation}
    \Box_f = -\Delta + \tau \sum_{k,l} \nabla_l \nabla_k f(a^\dagger_k a_l + a^\dagger_l a_k) + \tau^2 |\nabla f|^2, \quad (16)
\end{equation}
where $\Delta$ is the Laplace-Beltrami operator of $M$.
This is just the Witten deformation Laplacian in Riemannian geometry. In particular, by studying the spectrum behavior of $\Box_f$, one can obtain the Morse inequality (See [Wi]).

2.2.2. real 1-dimensional harmonic oscillator.

The simplest section-bundle system is given by $(\mathbb{R} \times \mathbb{R}, \mathbb{R}, f = \tau tx^2)$ for $\tau > 0$. Now the operator (16) becomes
\begin{equation*}
    -\Delta + 4\tau(a^\dagger a) + 4\tau^2 x^2 : \Lambda^p \to \Lambda^p.
\end{equation*}
For $p = 0$, we have
\begin{equation*}
    -\frac{d^2}{dx^2} + 4\tau^2 x^2 = 2\tau \left( -\frac{d^2}{d(\sqrt{2\tau}x)^2} + \left( \frac{\sqrt{2\tau}x}{\sqrt{2\tau}x} \right)^2 \right)
\end{equation*}
which is precisely the harmonic oscillator and has eigenvalues
\begin{equation*}
    \lambda_k = 2\tau(1 + 2k), \quad k = 0, 1, \cdots,
\end{equation*}
and the corresponding eigenfunctions:
\begin{equation*}
    u_k(\sqrt{2\tau}x) = H_k(\sqrt{2\tau}x)e^{-\frac{x^2}{2\tau}},
\end{equation*}
where $u_k(x) = H_k(x)e^{-\frac{x^2}{2\tau}}$ are the eigenfunctions of the standard Harmonic oscillator
\begin{equation*}
    -\frac{d^2}{dx^2} + x^2
\end{equation*}
and $H_k(x)$ are the Hermitian polynomials.

For $p = 1$, if we set $\varphi = u dx$, we have
\begin{equation*}
    \Box_f \varphi = \left( -\frac{d^2}{dx^2}u + 4\tau^2 x^2 u + 4\tau u \right) dx,
\end{equation*}
which has the eigenvalues $\lambda_k + 4\tau = 2\tau(3 + 2k), \quad k = 0, 1, \cdots$, and the corresponding eigenforms are
\begin{equation}
    \varphi_m = u_m dx. \quad (17)
\end{equation}
The smallest eigenform is
\begin{equation}
    \varphi_0 = e^{-\tau x^2} dx. \quad (18)
\end{equation}

In general, one can consider the SBS $(\mathbb{R}^n \times \mathbb{R}^n, f = \tau \sum_j t_j x_j^2)$. The operator (16) becomes
\begin{equation*}
    -\sum_j \frac{d^2}{dx_j^2} + 4\tau \sum_j (t_j a^\dagger_j a_j) + 4\tau^2 \sum_j t_j^2 x_j^2 : \Lambda^p \to \Lambda^p
\end{equation*}
Let \( u_m^j, \lambda_m^j \) be the eigenfunctions and eigenvalues of the \( j \)-th harmonic oscillator \( \frac{d^2}{dx_j^2} + 4\tau^2 r_j^2 \), then the total eigenforms are
\[
\sum_{m=0}^{\infty} u_m^1 \cdots u_m^n \ dx^1 \wedge \cdots \wedge dx^n,
\]
and the eigenvalues are \( \lambda_m^1 + \cdots + \lambda_m^n + 4\tau \sum_j t_j \). These eigenforms form a basis in \( L^2\Lambda^n(\mathbb{R}^n) \). The smallest eigenvalue is \( 6\tau \sum_j t_j \) and the corresponding eigenform is
\[
\phi_0 = e^{-\tau(x_1^2 + \cdots + x_n^2)} dx^1 \wedge \cdots \wedge dx^n. \tag{19}
\]

### 2.2.3. Complex 1-dimensional harmonic oscillator.

Now we consider the section-bundle system \((\mathbb{C} \times \mathbb{C}, \overline{\mathbb{C}}, f_\tau = \tau z^2)\). Then we have the twisted operator \( \hat{\partial}_f = \hat{\partial} + f_\tau \hat{\partial} \), the conjugate \( \overline{f}_\tau \), and the complex Laplace operator (see next section for the computation)
\[
\Delta_f = \overline{f}_\tau \hat{\partial}_f \hat{\partial}_f + \hat{\partial}_f \overline{f}_\tau \hat{\partial}_f = -\frac{\partial^2}{\partial z \partial \bar{z}} + 2(2\tau \bar{a} \bar{\bar{a}} + \bar{\bar{a}} \bar{\bar{a}}) + 4|z|^2|\bar{z}|^2 : \Lambda^p \to \Lambda^p
\]

If \( p = 0 \), then \( \Delta_f u = 0 \) has only zero solution. If \( p = 1 \) and set \( \varphi = u dz + v d\bar{z} \), then we have
\[
\Delta_f \varphi = \left( -\frac{\partial^2 u}{\partial z \partial \bar{z}} + 2\tau v + 4|z|^2|\bar{z}|^2 u \right) dz + \left( -\frac{\partial^2 v}{\partial z \partial \bar{z}} + 2\tau u + 4|z|^2|\bar{z}|^2 v \right) d\bar{z}
\]
Let \( H := -\frac{\partial^2}{\partial z \partial \bar{z}} + 4|z|^2|\bar{z}|^2 \), and define a \( \mathbb{C} \)-linear operator \( \hat{\tau}_v : \Lambda^1 \to \Lambda^1 \) given by
\[
\hat{\tau}_v (udz + v d\bar{z}) = (\tau uz + \bar{\tau} ud\bar{z}).
\]

Then \( \hat{\tau}^2(\varphi) = |\tau|^2 \varphi \), i.e., \( \tau_\mathbb{R} := 1/|\tau| \hat{\tau} \) defines an involution (real structure). Now we can show that
\[
\Delta_f \tau_\mathbb{R} \varphi = \tau_\mathbb{R} \Delta_f \varphi. \tag{20}
\]
This shows that \( \Delta_f \) can be reduced to two operators acting on the two eigenspaces \( E_\mathbb{R} \) of \( \tau_\mathbb{R} \) with respect to the two eigenvalues \( \pm 1 \). \( \varphi = u dz + v d\bar{z} \in E_\mathbb{R} \) iff
\[
\tau v = \pm |\tau| u, \quad \text{and} \quad \bar{\tau} u = \pm |\tau| v.
\]

Therefore the eigenvalue problem of \( \Delta_f \) is reduced to the eigenvalue problem of the following operator
\[
H = 2|\tau| - \frac{\partial^2}{\partial z \partial \bar{z}} + 4|z|^2|\bar{z}|^2 \pm 2|\tau|.
\]
which are the sum of two real 1-dimensional harmonic oscillators. In particular, \( \Delta_f \varphi = 0 \) for \( \varphi \in L^2\Lambda^1 \) iff there exists \( v \in L^2 \) such that
\[
\varphi = -\frac{\tau}{|\tau|} v dz + v d\bar{z} \tag{21}
\]
and \( v \) satisfies
\[
-\frac{\partial^2 v}{\partial z \partial \bar{z}} + 4|z|^2|\bar{z}|^2 v - 2|\tau| v = 0. \tag{22}
\]

The solution space is 1-dimensional and is generated by
\[
\varphi(z) = \left( u_0(2 \sqrt{\tau x}) u_0(2 \sqrt{\tau y}) \right) (-\frac{\tau}{|\tau|} dz + d\bar{z}) = e^{-\tau|z|^2} (-\frac{\tau}{|\tau|} dz + d\bar{z}),
\]
where \( u_0 \) is the smallest eigenfunction of the real 1-dimensional harmonic oscillator.
We have the asymptotic relation as $|\tau| \to \infty$,
\[
(\varphi, \varphi)_{L^2} = \int_{C^0} \varphi \wedge \ast \varphi \sim \frac{C(\frac{z}{|\tau|})}{|\tau|},
\]
where $C(\frac{z}{|\tau|})$ is a constant depending only on the direction $\frac{z}{|\tau|}$.

**Remark 2.10.** Now the Witten index $\text{Tr}(-1)^F e^{-\Delta_{\tau}}$ is 1 and one can also calculate the partition function $\text{Tr} e^{-\Delta_{\tau}} = \sum_{\lambda(\tau)} e^{-\lambda(\tau)}$.

### 2.2.4. Complex $n$-dimensional harmonic oscillator.

Consider the section bundle system $(\mathbb{C}^n \times \mathbb{C}^n, f_\tau = \tau(z_1^2 + \cdots + z_n^2))$. Then the Laplacian is
\[
\Delta_{f_\tau} = \Delta_1 + \cdots + \Delta_n,
\]
where $\Delta_i$ is the complex 1-dimensional harmonic oscillator with respect to $z_i$. It is easy to see that the $L^2$ harmonic $k$-form for $k < n$ is trivial, and the space of $L^2$ harmonic $n$-form is 1-dimensional and is generated by the following element:
\[
\Psi(z_1, \cdots, z_n, \tau) = \varphi(z_1) \wedge \cdots \wedge \varphi(z_n),
\]
where the 1-form $\varphi(\cdot)$ is given by (21).

Similarly, as $|\tau| \to \infty$, we have the asymptotic relation:
\[
(\Psi, \Psi)_{L^2} = \int_{C^0} \Psi \wedge \ast \Psi \sim \frac{C(\frac{z}{|\tau|})}{|\tau|}.
\]

### 2.3. Complete non-compact Kähler manifold with potential function.

Let $(M, g)$ be a complex manifold with a Hermitian metric $g$. Denote by $T = TM, T^* = T^* M, \bar{T} = \overline{T M}$, and $\mathcal{T}^* = T^* M$ the holomorphic tangent bundle, the holomorphic cotangent bundle, the anti-holomorphic tangent bundle and anti-holomorphic cotangent bundles respectively. Let $U$ be a coordinate chart of $M$ having local coordinates $z = (z^1, \cdots, z^n)$. Here we assume that $z_j = x_{2j-1} + ix_{2j}, j = 1, \cdots, n$. So locally, $g = g_{i,j}dz^i d\bar{z}^j$ (by using Einstein summation convention) is a section of the bundle $T^* \otimes \overline{T}^*$. The associated form $\omega = \frac{i}{2}g_{i,j}dz^i \wedge d\bar{z}^j$ and
\[
\frac{\omega^n}{n!} = gd\tau^1 \wedge d\bar{\tau}^2 \cdots \wedge d\bar{\tau}^n = dv_M,
\]
where $g = \det(g_{i,j})$ and $dv_M$ is the real volume element with respect to the Hermitian metric defined on $T_M = T \oplus \bar{T}$.

A $(p, q)$-form is a differential section $\varphi : z \to \{z, \varphi_{i_1, \cdots, i_p, j_1, \cdots, j_q}\}$ of the tensor bundle $\Lambda^{p,q} = (\otimes T^*)^p \otimes (\otimes \overline{T}^*)^q$ such that the components $\varphi_{i_1, \cdots, i_p, j_1, \cdots, j_q}$ are skew-symmetric with respect to $i_1, \cdots, i_p, j_1, \cdots, j_q$. Locally it has the form
\[
\varphi = \frac{1}{p!q!} \sum_{i_1, \cdots, i_p, j_1, \cdots, j_q} \varphi_{i_1, \cdots, i_p, j_1, \cdots, j_q} dz^{i_1} \wedge \cdots \wedge dz^{i_p} \wedge d\bar{z}^{j_1} \wedge \cdots \wedge d\bar{z}^{j_q},
\]
or written as the form without fractional factors
\[
\varphi = \sum_{i_1, \cdots, i_p, j_1, \cdots, j_q} \varphi_{i_1, \cdots, i_p, j_1, \cdots, j_q} dz^{i_1} \wedge \cdots \wedge dz^{i_p} \wedge d\bar{z}^{j_1} \wedge \cdots \wedge d\bar{z}^{j_q}.
\]
Let \((g^{ij})\) be the inverse matrix of the metric matrix \((g_{ji})\) such that \(\sum_{k} g_{kj} g^{kj} = \delta_i^j\) and \(\sum_{k} g_{ik} g^{kj} = \delta_i^j\).

We choose the convention of the form action as follows: if \(\xi = \xi^p \frac{\partial}{\partial x^p} + \xi^q \frac{\partial}{\partial x^q}\) and \(\eta = \eta^p \frac{\partial}{\partial x^p} + \eta^q \frac{\partial}{\partial x^q}\), then

\[
\omega(\xi, \eta) = ig_{ij} dz^i \wedge dz^j (\xi^p \frac{\partial}{\partial z^p} + \xi^q \frac{\partial}{\partial z^q}, \eta^p \frac{\partial}{\partial z^p} + \eta^q \frac{\partial}{\partial z^q}) = ig_{ij}(\xi^i \eta^j - \eta^i \xi^j).
\]

If \(\xi\) is a holomorphic tangent vector, then the length of \(\xi\) is \(|\xi|^2 = g_{ij} \xi^i \bar{\xi}^j\), and the inner product at point \(z\) of two holomorphic tangent vector \(\xi, \eta\) is defined as

\[
(\xi, \eta)(z) = g_{ij} \xi^i \bar{\eta}^j.
\]

Let

\[
\varphi(z) = \frac{1}{p!q!} \sum_{j_1, \ldots, j_q} \varphi_{i_1, \ldots, i_p} \bar{\varphi}_{\bar{j}_1, \ldots, \bar{j}_q} dz^{j_1} \wedge \cdots \wedge dz^{j_q},
\]

\[
\psi(z) = \frac{1}{p!q!} \sum_{\bar{i}_1, \ldots, \bar{i}_p} \psi_{i_1, \ldots, i_p} \bar{\psi}_{\bar{i}_1, \ldots, \bar{i}_q} dz^{\bar{i}_1} \wedge \cdots \wedge dz^{\bar{i}_q},
\]

then the inner product of the two \((p, q)\)-forms is defined as

\[
(\varphi, \psi)(z) = \frac{1}{p!q!} \sum_{i_1, \ldots, i_p} g^{\bar{k}_1 i_1} \cdots g^{\bar{k}_q i_q} \varphi_{i_1, \ldots, i_p} \bar{\psi}_{\bar{k}_1, \ldots, \bar{k}_q}.
\]

Then the \(L^2\) inner product on \(M\) is defined as

\[
(\varphi, \psi) = \int_M (\varphi, \psi)(z) dv_M.
\]

Denote by \(I_p = (i_1, \ldots, i_p)\) the multiple index with length \(p\). Then a \((p, q)\) form \(\varphi\) can be simply written as

\[
\varphi(z) = \frac{1}{p!q!} \sum_{I_p, \bar{I}_q} \varphi_{I_p} \bar{\varphi}_{\bar{I}_q} dz^{i_1} \wedge \cdots \wedge dz^{i_p}.
\]

**Definition 2.11.** Let \(\psi(z) = \frac{1}{p!q!} \sum_{I_p, \bar{I}_q} g_{I_p} \bar{\psi}_{I_p} dz^{i_1} \wedge \cdots \wedge dz^{i_p}\). We can define a \(C\)-linear operator \(\ast : \Omega^{p,q} \rightarrow \Omega^{p,q-p,p}\) (where \(\Omega^{p,q}\) represent the section set of the bundle \(\Lambda^{p,q}\)) as follows:

\[
\ast \psi = (p！(-1)^{p+1}) \sum_{I_p, \bar{I}_q} g_{I_p} \bar{\psi}_{I_p} dz^{i_1} \wedge \cdots \wedge dz^{i_p}.
\]

Here \(g_{I_p} \bar{\psi}_{I_p}\) is the tensor products of the metric tensor \(g_{ij}\):

\[
g_{I_p} \bar{\psi}_{I_p} = g_{i_1 \cdots i_p} \psi_{i_1 \cdots i_p} = \det(g_{ij}).
\]

and

\[
\psi_{I_p} := \sum_{k_1, \ldots, k_p} g^{\bar{k}_1 k_1} \cdots g^{\bar{k}_q k_q} \psi_{k_1 \cdots k_p}.
\]

It is routine to prove the following lemma (See [MK] or [Ko]).

**Proposition 2.12.** The \(\ast\) operator satisfies the properties:

1. \((\varphi, \psi)(z) dv_M = \varphi \wedge \ast \psi\)
2. \(\ast \psi = \ast \bar{\psi}\)
3. \(\ast^2 = (-1)^{p+q}\)
4. \(\psi \wedge \ast \psi = \psi \wedge \ast \bar{\psi}\).
Definition 2.13. Denote by $\partial^\dagger, \tilde{\partial}^\dagger, d^\dagger$ the adjoint operators of $\partial, \tilde{\partial}, d$ with respect to the inner product $(\cdot, \cdot)$, namely we define those adjoint operators by the following identities:

$$(\partial \varphi, \psi) = (\varphi, \partial^\dagger \psi), \quad (\partial \varphi, \psi) = (\varphi, \tilde{\partial}^\dagger \psi), \quad (d \varphi, \psi) = (\varphi, d^\dagger \psi),$$

(32)

for compactly supported form $\varphi$ or $\psi$.

Sometimes it is convenient distinguish the $*$ operator by their domain. The operator $\ast : \Omega^{p,q} \rightarrow \Omega^{p-q,n-p}$ can be written as $\ast^{p,q}$. In this way, the relation $\ast^2 = (-1)^{p+q}$.

If $A$ is a differential operator, we denote by $A^\dagger$ the adjoint operator of $A$ with respect to the metric $(\cdot, \cdot)$. Now the adjoint operator $(\ast^{p,q})^\dagger$ has the identity:

$$(\ast^{p,q})^\dagger = (-1)^{p+q} \ast^{n-q,n-p}, \quad (\ast^{p,q})^\dagger \ast^{p,q} = I$$

(34)

The following result is well-known.

Proposition 2.14.

$\tilde{\partial}^\dagger \psi = - * \partial * \psi, \quad \partial^\dagger \psi = - * \tilde{\partial} * \psi, \quad d^\dagger = - * d * \psi.$

(35)

Proposition 2.15. For $\psi \in \Gamma(\Lambda^{p,q+1})$,

$$(\tilde{\partial}^\dagger \psi)^{\tilde{l}_0\tilde{j}_1\ldots \tilde{j}_s} = (-1)^{p+1} \sum_{k} \left( \frac{\partial}{\partial z^k} + \frac{\partial \log g}{\partial z^k} \right) \psi^{l_k j_1 \ldots j_s}.$$

(36)

One can find this proposition in [MK].

Let $\iota_v$ be the contraction operator contracting with the vector $v$.

Lemma 2.16. We have the properties:

$$(d^\dagger z^\mu \wedge)^\dagger = g^{\alpha\beta} \iota_{\alpha}$$, $$(d^\dagger z^\mu \wedge)^\dagger = g^{\alpha\beta} \iota_{\alpha}$$

(37)

$$(\iota_{\alpha \beta})^\dagger = g^{\alpha\beta} dz^\mu \wedge \wedge, \quad (\iota_{\alpha \beta})^\dagger = g^{\alpha\beta} dz^\mu \wedge \wedge.$$  

(38)

Proof. It suffices to prove the first identity, since the other ones can be induced easily.

Let

$$\varphi = \frac{1}{p! q!} \sum \varphi_{\tilde{l}_0 \ldots \tilde{j}_s} d\tilde{l}_1 \wedge \ldots \wedge d\tilde{j}_s, \quad \psi = \frac{1}{(p+1)! q!} \psi_{\tilde{l}_0 \ldots \tilde{j}_s} d\tilde{l}_0 \wedge \ldots \wedge d\tilde{l}_s.$$  

Then

$$d^\dagger z^\mu \wedge \varphi = \frac{1}{p! q!} \sum \varphi_{\tilde{l}_0 \ldots \tilde{j}_s} d\tilde{l}_1 \wedge \ldots \wedge d\tilde{j}_s \wedge \ldots \wedge d\tilde{l}_s = \frac{1}{(p+1)! q!} \sum (d^\dagger z^\mu \wedge \varphi)_{\tilde{l}_0 \ldots \tilde{j}_s} d\tilde{l}_0 \wedge d\tilde{l}_1 \wedge \ldots \wedge d\tilde{j}_s, \wedge$$

where

$$(d^\dagger z^\mu \wedge \varphi)_{\tilde{l}_0 \ldots \tilde{j}_s} = \sum_{s=0}^{p} (-1)^s \delta_{\tilde{l}_s}^\mu \varphi_{\tilde{l}_0 \ldots \tilde{j}_s}.$$
Lemma 2.17. Therefore, we have

\[
\psi_{i_1 \cdots i_q}(\varphi_{i_1 \cdots i_q}) = \frac{1}{(p + 1)!q!} \sum_{x=0}^{p} (-1)^x g^i_{i_1 \cdots i_q} \varphi_{i_1 \cdots i_q} \bar{g}^i_{i_2 \cdots i_q} \bar{g}^i_{i_3 \cdots i_q} \cdots \bar{g}^i_{i_{p+1} \cdots i_q}.
\]

On the other hand, we have

\[
g^\mu_{i_1 i_2 i_3 \cdots i_q} \psi_{i_1 i_2 i_3 \cdots i_q} = \frac{1}{(p + 1)!q!} \sum_{x=0}^{p} (-1)^x \psi_{i_1 \cdots i_q} \psi_{i_1 \cdots i_q} \psi_{i_1 \cdots i_q} \psi_{i_1 \cdots i_q} \cdots \psi_{i_1 \cdots i_q}.
\]

Therefore, we have

\[
(dz^\mu \wedge \varphi, \psi) = \frac{1}{p!q!} g^\mu_{i_1 i_2 i_3 \cdots i_q} \psi_{i_1 i_2 i_3 \cdots i_q} \psi_{i_1 i_2 i_3 \cdots i_q} \psi_{i_1 i_2 i_3 \cdots i_q} \psi_{i_1 i_2 i_3 \cdots i_q} \cdots \psi_{i_1 i_2 i_3 \cdots i_q}.
\]

In summary, we obtain

\[
(dz^\mu \wedge \varphi, \psi) = (\varphi, g^\mu_{i_1 i_2 i_3 \cdots i_q} \psi_{i_1 i_2 i_3 \cdots i_q} \psi_{i_1 i_2 i_3 \cdots i_q} \psi_{i_1 i_2 i_3 \cdots i_q} \psi_{i_1 i_2 i_3 \cdots i_q} \cdots \psi_{i_1 i_2 i_3 \cdots i_q}).
\]

The following lemma shows the commutation relations of the operators \( t_{\alpha}, dz^\mu, \ldots \) with the \(*\) operator.

Lemma 2.17. 

\[
s^p q^{-1} (dz^\mu) = (-1)^{p+q-1} (dz^\mu) s^p q^{-1}, \quad s^{p-1} q(t_{\alpha}) = (-1)^{p+q+1}(g_{\mu \nu} dz^\mu \wedge) s^{p-1} q,
\]

and the other commutation relations can be obtained by taking complex conjugate or adjoint.

2.3.1. Clifford algebra and \( sl_2(\mathbb{R}) \) Lie algebra.

These operators \( t_{\alpha}, dz^\mu, \ldots \) acting on \( \Omega^{p,q} \) and increase or decrease the degree by 1, we think them as degree 1 operators. We always define our Lie bracket as super Lie bracket, i.e., if \( A \) and \( B \) are two operators,

\[
[A, B] = AB - (-1)^{|A||B|} BA,
\]

where \(|A|\) denotes the degree of \( A \).

It is easy to prove the following formulas which implies the Clifford algebra structure of operators.

Proposition 2.18.

\[
[t_{\alpha}, t_{\beta}] = [t_{\alpha}, t_{\beta}] = [dz^\mu \wedge, dz^\nu \wedge] = [dz^\mu \wedge, dz^\nu \wedge] = 0
\]

\[
[t_{\alpha}, t_{\beta}] = [t_{\alpha}, t_{\beta}] = [dz^\mu \wedge, dz^\nu \wedge] = [dz^\mu \wedge, dz^\nu \wedge] = 0
\]

\[
[t_{\alpha}, t_{\beta}] = [t_{\alpha}, t_{\beta}] = [dz^\mu \wedge, dz^\nu \wedge] = [dz^\mu \wedge, dz^\nu \wedge] = 0
\]

\[
[t_{\alpha}, t_{\beta}] = [t_{\alpha}, t_{\beta}] = [dz^\mu \wedge, dz^\nu \wedge] = [dz^\mu \wedge, dz^\nu \wedge] = 0
\]

\[
[t_{\alpha}, t_{\beta}] = [t_{\alpha}, t_{\beta}] = [dz^\mu \wedge, dz^\nu \wedge] = [dz^\mu \wedge, dz^\nu \wedge] = 0
\]
Define the following operators:

\[ L := ig_{\bar{j}i} dz^i \wedge d\bar{z}^j, \] (43)
\[ \Lambda := ig^{\bar{j}i} \Gamma_{l\bar{j}i}, \] (44)
\[ F \phi^{\rho \eta} = (p + q - n) \phi^{\rho \eta}. \] (45)

Since

\[ \Lambda^\dagger = -ig^{\bar{j}i} \Gamma_{l\bar{j}i}, \] (46)

\[ \Lambda \] is the adjoint operator of \( L \) with respect to the inner product \((\cdot, \cdot)\).

It is obvious that \( L, \Lambda, F \) are real operators and furthermore We have the well-known \( sl_2(\mathbb{R}) \) Lie algebra relation, which can be easily proved using Proposition 2.18.

**Proposition 2.19.**

\[ [\Lambda, L] = F, \ [F, L] = -2L, \ [F, \Lambda] = 2\Lambda. \] (47)

**2.3.2. Connection and curvature.** Now We assume that \((M, g)\) is a Kähler manifold with Kähler metric \( g = g_{ij} dz^i d\bar{z}^j \). Let \( T_{\mathbb{R}} M \) be the real tangent bundle of \( M \) and \( T_{\mathbb{C}} M \) be the complex extension of \( T_{\mathbb{R}} M \). Let \( g \) be the Hermitian metric on \( T_{\mathbb{C}} M \) such that \( g(\frac{\partial}{\partial z^i}, \frac{\partial}{\partial \bar{z}^j}) = g_{ij} \) and \( \nabla \) is the Chern connection defined on \( T_{\mathbb{C}} M \). When restricted to the holomorphic tangent bundle \( T \), we have the holomorphic connection, still denoted by \( \nabla \) such that if \( \xi = \xi^i \frac{\partial}{\partial z^i} \) is a holomorphic vector field, then the covariant derivative is given by

\[ \nabla_j \xi^i = \frac{\partial \xi^i}{\partial z^j} + \sum_k \Gamma^i_{jk} \xi^k, \] (48)

where \( \Gamma^i_{jk} = g^{\bar{l}i} \Gamma_{l\bar{j}k} \). When restricted to \( T \), we have \( \nabla = \bar{\partial} \). So if \( \bar{\eta} = \eta^j \frac{\partial}{\partial \bar{z}^j} \in T \), we have

\[ \nabla_j \eta^j = \bar{\partial}_j \eta^j. \] (49)

We also have the induced connection on the cotangent bundle such that if \( \xi = \xi^i dz^i \) and \( \eta = \eta_i dz^i \), then we have

\[ \nabla_j \xi^i = \partial_j \xi^i - \Gamma^i_{jk} \xi^k, \] (50)
\[ \nabla_j \eta_i = \partial_j \eta_i. \] (51)

We can also define the complex conjugate connection \( \bar{\nabla} \) such that

\[ \bar{\nabla}_j \xi^i = \bar{\partial}_j \xi^i \] (52)
\[ \bar{\nabla}_j \eta_i = \bar{\partial}_j \eta_i. \] (53)

Define

\[ [\nabla_i, \bar{\nabla}_j] = \nabla_i \bar{\nabla}_j - \bar{\nabla}_j \nabla_i. \]

Then it is easy to obtain

**Proposition 2.20.**

\[ [\nabla_i, \bar{\nabla}_j] g_{ik} = -\sum_l \bar{\partial}_j \Gamma^k_{il} g_{lk}. \] (54)
Definition 2.21. Set

\[ R^k_{ij} = -\partial_j \Gamma^k_{ij}. \] (56)

The tensor field \( R^k_{ij} \) is called the curvature operator of the connection \( \nabla \). Define the curvature tensor of \( \nabla \) to be

\[ R_{klij} = g_{kp} R^p_{il}. \] (57)

The Ricci curvature is defined as

\[ R_{ij} = g^{ki} R_{kij}, \] (58)

and the scalar curvature is

\[ R = g^{ki} R_{ik}. \] (59)

Since \( \Gamma^k_{il} = g^{kp} \partial_l g_{ip} \), it is easy to show that

\[ R_{klij} = -\partial_j \partial_i g_{lk} + g^{qr} \partial_j g_{lp} \partial_i g_{qr}, \] (60)

\[ R_{ij} = -\partial_i \partial_j \log g. \] (61)

Here we must use the fact that \( \sum_i \Gamma^i_{il} = \frac{1}{g} \partial_l g \).

It is well-known that the curvature tensor satisfies the symmetric properties:

1. \( R_{kijl} = R_{ljik} = R_{lijk} = R_{lijk} \).

2. \( R_{kijl} = R_{lkij} \).

A straightforward computation shows that

\[ [\nabla_i, \nabla_j] \xi^k = R^k_{ijkl} \xi^l. \] (62)

\[ [\nabla_i, \nabla_j] \xi_k = -R^l_{ikjl} \xi^l, \] (63)

\[ [\nabla_i, \nabla_j] \xi^k = R^l_{ijkl} \xi^l. \] (64)

Note that we have

\[ \overline{R^k_{ijkl}} = R^j_{iklj}. \] (65)

Similarly, we can prove that

\[ [\nabla_i, \nabla_j] \xi_{pq} = R^l_{imj} \xi^m_{pq} - R^m_{ipq} \xi^l_{mj} + R^p_{lqim} \xi^l_{pm}. \] (66)

Proposition 2.22. In Kähler case, for \( \varphi = \varphi_{i_1 \cdots i_p} dz^{i_1} \wedge \cdots \wedge dz^{i_p} \) we have

\[ \partial \varphi = \sum_k \nabla_k \varphi_{i_k \cdots i_p} d z^{i_k} \wedge \cdots \wedge dz^{i_p} \] (66)

\[ \bar{\partial} \varphi = \sum_k \bar{\nabla}_k \varphi_{i_k \cdots i_p} d z^{i_k} \wedge \cdots \wedge dz^{i_p} \] (67)

\[ (\bar{\partial} \varphi)_{i_1 \cdots i_p} = -(-1)^p \sum_{m,n} g^{mn} \nabla_m \varphi_{i_1 \cdots i_p n \cdots i_{p-1}} \] (68)

One can find the proof of the above proposition from [MK].
Theorem 2.24. For any $(p, q)$-form $\varphi = \frac{1}{p!q!} \varphi_{i_1 \cdots i_p \bar{j}_1 \cdots \bar{j}_q} d\bar{z}^{i_1} \wedge \cdots \wedge d\bar{z}^{i_p} \wedge dz^{j_1} \wedge \cdots \wedge dz^{j_q}$, we have

$$(\Delta \varphi)_{i_1 \cdots i_p j_1 \cdots j_q} = - \sum_{\mu \nu} g^{\bar{\nu} \mu} \nabla_\mu \nabla_\nu \varphi_{ij_1 \cdots j_q}$$

$$+ \sum_{l=1}^q \sum_{i=1}^p (-1)^l g^{\bar{\nu} \mu} \left( \sum_{l} R_{\mu j_1 \cdots j_l}^{\bar{\nu} \nu l} \varphi_{i_1 \cdots i_{p-l} \cdots i_p j_1 \cdots j_l} - R_{j_1 \cdots j_l}^{\bar{\nu} \mu \nu} \varphi_{i_1 \cdots i_{p-l} \cdots i_p j_1 \cdots j_l} \right)$$

(70)

Proof. Let $I$ represent the multiple index with length $|I| = p$. Note that

$$(\bar{\partial} \varphi)_{j_1 \cdots j_q} = (-1)^p \sum_{\mu = 0}^q (-1)^\mu \nabla_\mu \varphi_{i_1 \cdots i_p j_1 \cdots j_q}.$$  (71)

Therefore, we have

$$(\bar{\partial} \bar{\partial} \varphi)_{i_1 \cdots i_p j_1 \cdots j_q} = -(-1)^p \sum_{\nu \mu} g^{\bar{\nu} \mu} \nabla_\mu (\bar{\partial} \varphi)_{I j_1 \cdots j_q}$$

$$= - \sum_{\nu \mu} g^{\bar{\nu} \mu} \nabla_\mu \varphi_{ij_1 \cdots j_q} + \sum_{l=1}^q (-1)^l g^{\bar{\nu} \mu} \nabla_\mu \varphi_{i_1 \cdots i_{p-l} \cdots i_p j_1 \cdots j_l}$$

(72)

On the other hand, we have

$$(\bar{\partial} \bar{\partial} \varphi)_{j_1 \cdots j_q} = (-1)^p \sum_{\nu \mu} g^{\bar{\nu} \mu} \nabla_\mu \varphi_{Ij_1 \cdots j_q}$$

$$= \sum_{l=1}^q (-1)^{l+1} \nabla_{j_1} (g^{\bar{\nu} \mu} \nabla_\mu \varphi_{Ij_1 \cdots j_l}.$$  (74)

Hence, we obtain

$$(\Delta \varphi)_{i_1 \cdots i_p j_1 \cdots j_q} = - \sum_{\mu \nu} g^{\bar{\nu} \mu} \nabla_\mu \nabla_\nu \varphi_{i_1 \cdots i_p j_1 \cdots j_q} + \sum_{l=1}^q (-1)^{l+1} g^{\bar{\nu} \mu} \left[ \nabla_\mu, \nabla_{j_1} \right] \varphi_{i_1 \cdots i_{p-l} \cdots i_p j_1 \cdots j_l}$$

$$= - \sum_{\mu \nu} g^{\bar{\nu} \mu} \nabla_\mu \nabla_\nu \varphi_{i_1 \cdots i_p j_1 \cdots j_q}$$

(75)

$$+ \sum_{l=1}^q \sum_{i=1}^p (-1)^l g^{\bar{\nu} \mu} \left( \sum_{l} R_{\mu j_1 \cdots j_l}^{\bar{\nu} \nu l} \varphi_{i_1 \cdots i_{p-l} \cdots i_p j_1 \cdots j_l} - R_{j_1 \cdots j_l}^{\bar{\nu} \mu \nu} \varphi_{i_1 \cdots i_{p-l} \cdots i_p j_1 \cdots j_l} \right)$$

(76)

$$- \sum_{l=1}^q \sum_{i=1}^p (-1)^{l+1} g^{\bar{\nu} \mu} \left( \sum_{m \neq i} R_{\mu j_1 \cdots j_l}^{k_m} \varphi_{i_1 \cdots i_{p-l} \cdots i_p j_1 \cdots j_l} \right).$$

(77)

Since $g^{\bar{\nu} \mu} R_{j_1 \cdots j_l}^{k_m}$ is symmetric about $\bar{\nu}, \bar{k}_m$ indices, but $\varphi_{i_1 \cdots i_{p-l} \cdots i_p j_1 \cdots j_l}$ is antisymmetric about $\bar{\nu}, \bar{k}_m$ indices, we know the last term vanishes. Hence we obtain the result. □

2.3.3. Twisted operators.

Now we assume that $f$ is a holomorphic function defined on the Kähler manifold $(M, g)$. Define the twisted $\bar{\partial}$ operators

$$\bar{\partial} f = \bar{\partial} + \partial f \wedge = \bar{\partial} + f d\bar{z} \wedge,$$

$$\bar{\partial} f = \bar{\partial} + \bar{\partial} f \wedge = \bar{\partial} + \bar{\partial} f \wedge,$$

$$\bar{\partial} f = \bar{\partial} + f \bar{\partial} f \wedge = \bar{\partial} + f \bar{\partial} f \wedge,$$

$$\bar{\partial} f = \bar{\partial} + \bar{\partial} f \wedge = \bar{\partial} + \bar{\partial} f \wedge.$$
where $f_i := \bar{\partial}_i f$. Denote by $\partial_j^\dagger$ and $\bar{\partial}_j^\dagger$ their adjoint operators with respect to the Kähler metric. The Laplace operator is defined as

$$\Delta_f = (\bar{\partial}_j + \partial_j^\dagger)^2.$$  

It is easy to prove the following result.

**Lemma 2.25.**

$$\bar{\partial}(\alpha \wedge * \beta) = \bar{\partial}_f \alpha \wedge * \beta + \alpha \wedge * \bar{\partial}_f * \beta$$  \hspace{1cm} (78)

and so there is

$$\bar{\partial}_j^\dagger = - * \partial_{-f} *$$  

$$\partial_j^\dagger = - * \bar{\partial}_{-f} *.$$  \hspace{1cm} (79)

On the other hand, we know that

$$\bar{\partial}_j^\dagger = \bar{\partial}_j^\dagger + f_\nu g^\nu j_\mu \partial_\mu$$  \hspace{1cm} (80)

$$\partial_j^\dagger = \partial_j^\dagger + f_\nu g^{\nu j}_\mu \bar{\partial}_\mu.$$  \hspace{1cm} (81)

Obviously, $\bar{\partial}_j \partial_j$ are of degree 1 and $\bar{\partial}_j^\dagger, \partial_j^\dagger$ are of degree $-1$, but they don’t preserve the Hodge grading.

**Lemma 2.26.**

$$\bar{\partial}^2_j = \partial^2_j = 0 \hspace{1cm} \bar{\partial}_j \partial_j + \partial_j \bar{\partial}_j = 0$$  \hspace{1cm} (82)

$$\partial^2_j = (\bar{\partial}_j)^2 = 0 \hspace{1cm} \bar{\partial}_j^\dagger \partial_j^\dagger + \partial_j^\dagger \bar{\partial}_j^\dagger = 0.$$  \hspace{1cm} (83)

**Proof.** These identities are the consequence of the commutation relations of operators $\partial, \bar{\partial}, \partial^\dagger, \bar{\partial}^\dagger$ and Lemma 2.18.  \hspace{1cm} $\square$

### 2.3.4. Kähler-Hodge identities

**Proposition 2.27.**

$$[\partial_j, \Lambda] = -i \bar{\partial}_j^\dagger, \quad [\bar{\partial}_j, \Lambda] = i \partial_j^\dagger$$  \hspace{1cm} (84)

$$[\partial_j, L] = -i \partial_j, \quad [\bar{\partial}_j, L] = i \bar{\partial}_j.$$  \hspace{1cm} (85)

**Proof.** It suffices to prove the first identity. The others can be obtained by taking the complex conjugate or the adjoint action.

Now

$$[\partial_j, \Lambda] = \{ \partial + \int d^5 z \wedge i g^{\mu \nu} t_\mu t_\nu \}$$

$$= i \{ \{ \partial, i g^{\mu \nu} t_\mu t_\nu \} + [\int d^5 z \wedge g^{\mu \nu} t_\mu t_\nu] \}$$

The first term

$$[\partial, g^{\mu \nu} t_\mu t_\nu] = \{ \nabla_\mu d^4 z \wedge g^{\mu \nu} t_\mu t_\nu \} = g^{\mu \nu} \nabla_\mu \{ d^4 z \wedge t_\mu - t_\mu \}$$

$$= g^{\mu \nu} \nabla_\mu t_\nu = - \bar{\partial}_j^\dagger.$$  

This is the classical Hodge identity.

The second term

$$[\int d^5 z \wedge g^{\mu \nu} t_\mu t_\nu] = - \int d^5 z g^{\mu \nu} t_\mu \delta_\nu^\mu = - \int d^5 z g^{\mu \nu} \bar{\partial}_\nu = -(f_\mu d^4 z)^\mu.$$  

Combining the two terms, we get

$$[\partial_j, \Lambda] = -i \bar{\partial}_j^\dagger.$$
By the Kähler-Hodge identities, one can easily prove the following conclusions

**Corollary 2.28.**

\[
[\bar{\partial}_j, \partial^j] = 0, \quad [\bar{\partial}_j, \partial^j] = 0 \\
[\partial_j, \partial^j] = \partial_j \partial^j + \bar{\partial}_j \bar{\partial}^j = \Delta_f \\
[\bar{\partial}_j, \bar{\partial}^j] = \bar{\partial}_j \bar{\partial}^j + \partial_j \partial^j = \Delta_f.
\]

**Proposition 2.29.** The (complex) form Laplacian operator $\Delta_f$ has the following local expression:

\[
\Delta_f = \Delta + (g^{i\bar{k}} \nabla_{\bar{f}} t_{i\bar{k}} dz^i \wedge + g^{i\bar{k}} \nabla_{\bar{f}} t_{i\bar{k}} d\bar{z}^j) + |\nabla f|^2. \tag{84}
\]

**Proof.** We have

\[
[\bar{\partial}^j, \partial_j] = [(\bar{\partial} + f_i dz^i \wedge), \bar{\partial} + f_i dz^i] = [\bar{\partial}^j, \partial^j] + [(\nabla_i dz^i), f_i dz^i] + [f_i (d\bar{z}^j \wedge), \nabla_i d\bar{z}^j] + [f_i (d\bar{z}^j \wedge), f_i dz^i].
\]

The second term

\[
[(\nabla_i dz^i), f_i dz^i] = [\nabla_i, f_i] g^{i\bar{k}} t_{i\bar{k}} dz^i \wedge + [d\bar{z}^j, f_i dz^i] f_i \nabla_i \wedge = g^{i\bar{k}} t_{i\bar{k}} d\bar{z}^j \wedge + f_i dz^i \wedge.
\]

Similarly, we can compute out the third term and the fourth term is $|\nabla f|^2$. \hfill \Box

This shows that $\Delta_f = (\bar{\partial}^j + \partial_j)^2 = [\bar{\partial}^j, \partial^j]$ is a real operator and $\Delta_f = [\partial^j, \partial^j]$.

**Remark 2.30.** Notice that the operator $\Delta_f$ does not preserve the Hodge grading, only preserve the grading of the real forms, i.e., $\Delta_f$ is an operator from $\Omega^p$ to $\Omega^p$. Let

\[
\varphi = \sum_{k=0}^{p} \sum_{0 \leq i_1 \leq i_2 \leq \cdots \leq i_n \leq p} \varphi_{i_1 \cdots i_n j_1 \cdots j_{n-k}}.
\]

Then

\[
(\Delta_f \varphi)_{i_1 \cdots i_n j_1 \cdots j_{n-k}} = - \sum_{\mu \nu} g^{i\bar{k}} \nabla_{\bar{\partial}_\nu} \nabla_{\partial^\mu} \varphi_{i_1 \cdots i_n j_1 \cdots j_{n-k}} + [R \circ (\varphi_{i_1 j_{n-k}})]_{i_1 \cdots i_n j_1 \cdots j_{n-k}} \\
+ [L_f \circ (\varphi_{i_1 j_{n-k}})]_{i_1 \cdots i_n j_1 \cdots j_{n-k}} + |\nabla f|^2 \varphi_{i_1 \cdots i_n j_1 \cdots j_{n-k}} \tag{85}
\]

Where at a point $z \in M$, $R, L : \mathcal{A}_p^\mathbb{C} \to \mathcal{A}_p^\mathbb{C}$ are linear maps, and $[\psi]_{i_1 \cdots i_n j_1 \cdots j_{n-k}}$ represent the $(i_1 \cdots i_n j_1 \cdots j_{n-k})$ component of the vector (or $(k, n-k)$-tensor) $\psi$.

Notice that $R$ only depends on the curvature tensor $R_{ijkl}$ and the metric tensor $g_{\mu \nu}$ and $L$ only depends on the metric tensor $g_{\mu \nu}$ and the tensor $(\nabla_k f_l)$.

It is also interesting to discuss the commutation relations of $*$ operators with those twisted operators.

**Proposition 2.31.** The following identities hold ($M$ is only required to be a complex manifold):

\[
* \bar{\partial}^j = (-1)^{p+q} \partial_{-f}* \quad * \partial_j = (-1)^{p+q} \bar{\partial}_{-f}*. \\
* (\bar{\partial}_f) = (-1)^{p+q+1} \bar{\partial}_{-f}* \quad * \partial_f = (-1)^{p+q+1} \partial_{-f}*. \\
* \Delta_f = -\Delta_{-f}.*
\]
In particular, when $M$ is Kähler, there is

$$\ast \Delta f = - \Delta_f \ast$$

**Example 2.32.** Given the complex 2-dimensional section-bundle system $(\mathbb{C}^2, f)$ with the standard Kähler metric, let us give the explicit formula for the twisted Laplacian equation. Because of the $\ast$-action, it suffices to consider the 0, 1, 2-forms.

1. The case that $\psi$ is a function. Then the equation $\Delta_f \psi = 0$ is a complex scalar Schrödinger equation:

$$\Delta \psi + |\nabla f|^2 \psi = 0. \quad (86)$$

2. The case that $\psi$ is a 1-form. Assume that

$$\psi = \psi_1 dz^1 + \psi_2 dz^2 + \psi_3 dz^3 + \psi_4 dz^4.$$

The vector Schrödinger equation has four component equations:

\[
\begin{align*}
\Delta \psi_1 - (f_{11} \psi_1 + f_{12} \psi_2) + |\nabla f|^2 \psi_1 &= 0 \\
\Delta \psi_2 - (f_{21} \psi_1 + f_{22} \psi_2) + |\nabla f|^2 \psi_2 &= 0 \\
\Delta \psi_3 - (f_{31} \psi_1 + f_{32} \psi_2) + |\nabla f|^2 \psi_3 &= 0 \\
\Delta \psi_4 - (f_{41} \psi_1 + f_{42} \psi_2) + |\nabla f|^2 \psi_4 &= 0.
\end{align*}
\]

If we set $\psi = (\psi_1, \psi_2, \psi_3, \psi_4)$, then the action of the linear operator $L_f \circ \cdot$ can be written as the matrix form:

$$L_f \circ \psi = \begin{pmatrix} 0 & 0 & f_{11} & f_{12} \\ 0 & 0 & f_{21} & f_{22} \\ f_{11} & f_{12} & 0 & 0 \\ f_{21} & f_{22} & 0 & 0 \end{pmatrix}.$$  

All the entries of $L_f$ consists of the second order derivatives of $f$.

The case that $\psi$ is a 2-form. $\psi$ has 6 components and has the form:

$$\psi = \psi_{12} dz^1 \wedge dz^2 + \psi_{13} dz^1 \wedge dz^3 + \psi_{14} dz^1 \wedge dz^4 + \psi_{23} dz^2 \wedge dz^3 + \psi_{24} dz^2 \wedge dz^4 + \psi_{34} dz^3 \wedge dz^4.$$  

The vector Schrödinger equation has 6 components:

\[
\begin{align*}
\Delta \psi_{12} - (f_{12} \psi_{12} + f_{13} \psi_{13} + f_{14} \psi_{14}) + |\nabla f|^2 \psi_{12} &= 0 \\
\Delta \psi_{13} - (f_{13} \psi_{13} + f_{14} \psi_{14}) + |\nabla f|^2 \psi_{13} &= 0 \\
\Delta \psi_{14} - (f_{14} \psi_{14}) + |\nabla f|^2 \psi_{14} &= 0 \\
\Delta \psi_{23} - (f_{23} \psi_{23} + f_{24} \psi_{24}) + |\nabla f|^2 \psi_{23} &= 0 \\
\Delta \psi_{24} - (f_{24} \psi_{24}) + |\nabla f|^2 \psi_{24} &= 0 \\
\Delta \psi_{34} - (f_{34} \psi_{34}) + |\nabla f|^2 \psi_{34} &= 0.
\end{align*}
\]

If we set $\psi = (\psi_{12}, \psi_{13}, \psi_{14}, \psi_{23}, \psi_{24}, \psi_{34})$, then the matrix $L_f$ has the form

$$L_f \circ \psi = \begin{pmatrix} 0 & -f_{21} & -f_{22} & f_{11} & f_{12} & 0 \\ -f_{12} & 0 & 0 & 0 & 0 & -f_{11} \\ -f_{22} & 0 & 0 & 0 & 0 & f_{21} \\ f_{11} & 0 & 0 & 0 & 0 & f_{32} \\ f_{21} & 0 & 0 & 0 & 0 & -f_{21} \\ 0 & f_{31} & -f_{11} & f_{11} & -f_{12} & 0 \end{pmatrix}.$$
2.3.5. \( N = 2 \) supersymmetric algebra. Define operators
\[
\begin{align*}
\partial_f &= \partial_f + \bar{\partial}_f, \quad \partial_f^\dagger = -i(\partial_f - \bar{\partial}_f) \\
\partial_f^\dagger &= \partial_f^\dagger + \bar{\partial}_f^\dagger, \quad (\partial_f^\dagger)^2 = i(\partial_f^\dagger - \bar{\partial}_f^\dagger) \\
\square_f &= (\partial_f + \partial_f^\dagger)^2 = [\partial_f, \partial_f^\dagger] = \partial_f \partial_f^\dagger + \partial_f^\dagger \partial_f \\
\square_f^\dagger &= (\partial_f^\dagger + (\partial_f^\dagger)^\dagger)^2 = [\partial_f^\dagger, (\partial_f^\dagger)^\dagger] = \partial_f^\dagger (\partial_f^\dagger)^\dagger + (\partial_f^\dagger)^\dagger \partial_f^\dagger
\end{align*}
\]
Those operators satisfy

**Proposition 2.33.**
\[
\square_f = \square_f^\dagger = 2\Lambda_f \\
[\partial_f, \partial_f^\dagger] = [\partial_f^\dagger, (\partial_f^\dagger)^\dagger] = 0, \quad [\partial_f, (\partial_f^\dagger)^\dagger] = [\partial_f^\dagger, \partial_f] = 0.
\]
*In particular, the Laplacian \( \square_f \) commutes with all above operators.*

For the number operator \( F \), we have

**Proposition 2.34.**
\[
\begin{align*}
[F, \partial_f] &= -\partial_f, \quad [F, \partial_f^\dagger] = \bar{\partial}_f^\dagger \\
[F, \bar{\partial}_f] &= -\bar{\partial}_f, \quad [F, \bar{\partial}_f^\dagger] = \partial_f^\dagger
\end{align*}
\]

Now the operators \( L, \Lambda, F \) and \( \square_f \) generates the even part of a Lie superalgebra, where \( L, \Lambda, F \) generates \( sl(2, \mathbb{R}) \cong spin(3) \) and \( \square_f \) generates \( u(1) \cong so(2) \). The odd operators \( \partial_f \) and \( \bar{\partial}_f^\dagger \) span a spinor representation \( S \), since there hold
\[
\begin{align*}
[L, \partial_f] &= 0, \quad [\Lambda, \partial_f] = i\bar{\partial}_f^\dagger, \quad [F, \partial_f] = -\partial_f \\
[L, \bar{\partial}_f] &= -i\partial_f, \quad [\Lambda, \bar{\partial}_f] = 0, \quad [F, \bar{\partial}_f^\dagger] = \bar{\partial}_f^\dagger \\
[\partial_f, \bar{\partial}_f] &= 0, \quad [\partial_f, \bar{\partial}_f^\dagger] = 0, \quad [\partial_f^\dagger, \bar{\partial}_f^\dagger] = 0
\end{align*}
\]
The dual representation \( S^* \) is given by the operators \( \bar{\partial}_f \) and \( \bar{\partial}_f^\dagger \). The pairing is giving by the relations
\[
[F, \partial_f^\dagger] = \frac{1}{2}[\square_f, [\bar{\partial}_f, \partial_f^\dagger]] = \frac{1}{2}[\square_f]
\]

Hence we obtain a \( spin_{\mathbb{C}}(3) \) supersymmetric algebra.

2.4. **Spectrum theory of Schrödinger operators for differential forms.** *In this section, we always assume that \( (M, g) \) is a complete non-compact manifold with bounded geometry.*

2.4.1. **Sobolev Norms.** Since we have the decomposition
\[
\Lambda^p = \oplus_{\mu + \nu = p} \Lambda^{\mu, \nu},
\]
the \( L^2 \)-inner product is defined as
\[
\langle \cdot, \cdot \rangle = \oplus_{\mu + \nu = p} \langle \cdot, \cdot \rangle_{\mu, \nu}.
\]
In local chart, \( \Lambda^p \) has a basis
\[
\left\{ dz^{i_1} \wedge \cdots \wedge dz^{i_l} \wedge d\bar{z}^{j_1} \wedge \cdots \wedge d\bar{z}^{j_l} \mid 1 \leq i_1 \leq \cdots \leq i_l \leq n, 1 \leq j_1 \leq \cdots \leq j_l \leq n, k + l = p \right\}.
\]
By Pascard’s rule, \( \sum_{k=0}^{p} \binom{n}{k} \binom{n}{p-k} = \binom{2n}{p} \) is just the dimension of \( \Lambda^p \).
For any $\varphi \in \Omega^p$, we can define the Sobolev norm
\[ \|\varphi\|_{k,2} := \left( \sum_{|\alpha|+|\beta| \leq k} \|\nabla^\alpha \bar{\partial}^\beta \varphi\|^2 \right)^{1/2} \] (87)
and the Sobolev space $W^{k,2}(\Lambda^p)$ consisting of all weak differentiable $p$-forms having up to $k$-order $L^2$ integrable derivatives. Let $W^{k,2}_0(\Lambda^p)$ be the closed subspace in $W^{k,2}(\Lambda^p)$ which is the closure of the compactly supported forms. In addition, we denote by $\Omega^k_{\Omega}(M) = \Omega^k(M) \cap L^2(\Lambda^k(M))$ and $\tilde{\Omega}^k_0(M)$ the set of the smoothly compactly supported $k$-forms.

2.4.2. Some analytic theorems on Riemannian manifold with bounded geometry.

The following $L^1$-Stokes theorem was proved by Gaffney [Gal], which is obvious in Euclidean space.

**Theorem 2.35.** Let $M$ be a $n$-dimensional orientable complete Riemannian manifold whose Riemannian tensor is of $C^2$. Let $\gamma$ be a $n-1$-form of class $C^1$ with the property that both $\gamma$ and $d\gamma$ are in $L^1$. Then $\int_M d\gamma = 0$.

**Theorem 2.36** (Density theorem). Let $(M, g)$ be a complete Riemannian manifold with positive injectivity radius and let $k \geq 2$ be an integer. Suppose that there exists a positive constant $C$ such that for any $j = 0, \cdots, k-2$, $|\nabla^j \text{Ricci}| \leq C$. Then for any $p \geq 1$, $W^{k,p}(M) = \tilde{\Omega}^{k,p}(M)$.

**Theorem 2.37** (Sobolev embedding theorem). The Sobolev embeddings are valid for complete manifolds with Ricci curvature bounded from below and positive injectivity radius.

One can find the two theorems and proofs in [He]. The above two theorems can be applied to our case:

**Corollary 2.38.** Let $(M, g)$ be a complete non-compact Kähler manifold with bounded geometry. Then $W^{k,2}(\Lambda^p) = W^{k,2}_0(\Lambda^p)$ for any $k$, and Sobolev embedding theorem holds.

2.4.3. Close extension of operators.

The multiplication operator $\bar{\partial} f : \Omega^p \to \Omega^{p+1}$ is a closable operator. Its closure $\bar{\partial} f$ has domain
\[ \text{Dom}(\bar{\partial} f) = \{ \psi \in \Omega^p \cap \Omega^{p+1} \mid \int_M |f| |\nabla \psi|^2 < \infty \}. \]

Similarly, the formal adjoint operator $(d f \wedge)^\dag = \sum g^{ij} \partial_i \partial_j$ is also a closable operator and its closure $(d f \wedge)^\dag$ has domain
\[ \text{Dom}((\bar{\partial} f \wedge)^\dag) = \{ \psi \in \Omega^{p+1} \mid \int_M |\bar{\partial} f | |\nabla \psi|^2 < \infty \}. \]

The operator $(\bar{\partial} f)^\dag = (\bar{\partial} f)^\dag$.

If $\psi \in \Omega^{p+1} \cap \Omega_0^{p+1}(M)$, by Stokes theorem, we have
\[ (\bar{\partial} \psi, \varphi) = (\psi, \bar{\partial} \varphi). \]
Since $\Omega_0^{p+1}(M)$ is a dense set in $L^2$ space, $\bar{\partial}$ as the formal adjoint operator has maximal closed extension $\bar{\partial}_M$ with domain
\[ \text{Dom}(\bar{\partial}_M) = \{ \psi \in L^2(\Lambda^{p+1}(M)) \mid \bar{\partial} f \in L^2(\Lambda^p(M)) \} = W^{1,2}(\Lambda^p(M)). \]
On the other hand, with the Dirichlet boundary condition, \( d \) has the closed extension, the minimal extension \( \tilde{\partial}_M \) with domain \( W^{1,2}_0(\Lambda^p(M)) \). However because of the density theorem, Theorem 2.36 we have
\[
\tilde{\partial}_M = \tilde{\partial}_m.
\]
This means that \( \tilde{\partial} \) has a unique close extension. Similarly, we can prove that the operator \( \tilde{\partial}^i \) has a unique close extension. We drop off the symbol \( \hat{\partial} \) if no confusion occurs.

Now \( \tilde{\partial}_f = \tilde{\partial} + \tilde{\partial} f \wedge \) is an unbounded closable operator, its closure has the domain
\[
\text{Dom}(\tilde{\partial}_f) = \{ \psi \in L^2(\Lambda^p) | \int_M |\tilde{\partial}_f\psi|^2 + |\psi|^2 < \infty \}
\]

In the rest of the paper, we only consider the closure of \( \tilde{\partial}_f \) and \( \tilde{\partial}^i_f \) and don’t think \( \tilde{\partial}_f \) as the sum of two unbounded operators.

The Laplace operator \( \Delta_f = \tilde{\partial}_f \tilde{\partial}^i_f + \tilde{\partial}^i \tilde{\partial}_f \) is lower bounded real symmetric operator. There is an associated closable quadratic form
\[
Q_f(\psi, \varphi) = (\tilde{\partial}_f \psi, \tilde{\partial}^i_f \varphi) + (\tilde{\partial}^i_f \psi, \tilde{\partial}_f \varphi).
\]
By the functional theorem (see [RS0], Theorem VIII.15), the closure of \( Q_f \) uniquely determines an self-adjoint extension \( \Delta_f \) of \( \Delta_f \). We will still use \( \Delta_f \) to represent this self-adjoint extension.

Denote by
\[
\Delta_f = H^0_f + H^1_f,
\]
where
\[
H^0_f := -\sum_{\mu\nu}\tilde{g}^{\mu\nu}\nabla_{\mu}\nabla_{\nu} + |\nabla f|^2, \tag{88}
\]
and
\[
H^1_f := R \circ (\varphi_{h,\tilde{\varphi}}) + L_f \circ (\varphi_{h,\tilde{\varphi}})
\]
In one coordinate chart \( U \), these operators can be viewed as operators acting on the vector-valued \( L^2 \) space \( L^2(U, \mathbb{C}^{\rho(n)}) \), where \( \rho(n) = \binom{2n}{n} \) is the dimension of \( \Lambda^p \).

2.4.4. Tameness of the section-bundle system.

**Definition 2.39.** The section bundle system \( \{(M, g), f\} \) is said to be fundamental tame, if there exists a compact set \( K \) and a constant \( C_0 > 0 \) such that \( |\nabla f| > C_0 \) outside a compact subset \( K \subset M \).

The section-bundle system is said to be strongly tame, if for any constant \( C > 0 \), there is
\[
|\nabla f|^2 - C|\nabla^2 f| \to \infty, \text{ as } d(x, x_0) \to \infty. \tag{90}
\]
Here \( d(x, x_0) \) is the distance between the point \( x \) and the base point \( x_0 \).

Now we have the fundamental theorem of our theory.

**Theorem 2.40.** Suppose that \( (M, g) \) is a Kähler manifold with bounded geometry. If \( \{(M, g), f\} \) is a strongly tame section-bundle system, then the form Laplacian \( \Delta_f \) has purely discrete spectrum and all the eigenforms form a complete basis of the Hilbert space \( L^2(\Lambda^*(M)) \).

**Proof.** Since the section-bundle system is strongly tame, we know that
\[
|\nabla f(x)| \to \infty, \text{ as } d(x, x_0) \to \infty.
\]
Therefore by Corollary 2.39, we know that \( H^0_f \) has purely discrete spectrum. Now we want to prove that \( H^1_f \) is a compact perturbation of \( H^0_f \).
At first, since \((M, g)\) has bounded geometry, there exists a universal constant \(C_R\) such that at point \(x\),
\[ |(R \circ \varphi, \varphi)(x)| \leq C_R(\varphi, \varphi). \]  
(91)

Secondly, for any \(\varepsilon > 0\), we can find a compact set \(K_{\varepsilon}\) such that
\[ |(L_f \circ \varphi, \varphi)(x)| \leq \varepsilon(|\nabla f|^2 \varphi, \varphi)(x) \]  
(92)

hold on \(M - K_{\varepsilon}\).

Since \(f\) and \((M, g)\) are smooth, we can find a constant \(C_{\varepsilon} > 0\) such that for any \(x \in M\),
\[ |(L_f \circ \varphi, \varphi)| \leq \varepsilon(|\nabla f|^2 \varphi, \varphi) + C_{\varepsilon}(\varphi, \varphi). \]  
(93)

Combining the inequalities of \(R\) and \(L_f\), we have
\[ |(H^1_j \varphi, \varphi)| \leq \varepsilon(|\nabla f|^2 \varphi, \varphi) + (C_{\varepsilon} + C_R)(\varphi, \varphi). \]  
(94)

Therefore, we obtain
\[ |(H^1_j \varphi, \varphi)| \leq \varepsilon(H^1_j \varphi, \varphi) + (C_{\varepsilon} + C_R)(\varphi, \varphi). \]  
(95)

Now by Theorem 2.4 we get the final conclusion. □

As the application of our fundamental theorem, Theorem 2.40, we will consider some important examples in the following part.

2.4.5. **Hypersurface section-bundle system** \((\mathbb{C}^{N+1}, W)\).

Let \(W : \mathbb{C}^{N+1} \to \mathbb{C}\) be a quasi-homogeneous polynomial of type \((q_0, \cdots, q_N)\), i.e., for all \(\lambda \in \mathbb{C}\), there is
\[ W(\lambda^{q_0}z_0, \cdots, \lambda^{q_N}z_N) = \lambda W(z_0, \cdots, z_N). \]  
(96)

\(W\) is called non-degenerate, if 0 is its only isolated critical point in \(\mathbb{C}^{N+1}\). Equivalently, if we write
\[ W = \sum_{l=1}^{s} W_l = \sum_{l=1}^{s} c_j \prod_{j=0}^{N} z_j^{a_{lj}}, \]
then the matrix \(A = (a_{lj})_{s \times (N+1)}\) has rank \(N + 1\) and
\[ A \begin{pmatrix} q_0 \\ \vdots \\ q_N \end{pmatrix} = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}. \]

The non-degeneracy of \(W\) also implies that \(\{W = 0\}\) defines a smooth hypersurface in the weighted projective space \(\mathbb{P}^N_{(k_0, \cdots, k_N)}\), where \(q_i = \frac{k_i}{d}, (k_i, d) = 1\).

**Definition 2.41.** Let \(\mathbb{C}^N\) be the canonical Kähler manifold with standard flat metric. Let \(W : \mathbb{C}^{N+1} \to \mathbb{C}\) be a non-degenerate quasi-homogeneous polynomial. The combination \((\mathbb{C}^{N+1}, W)\) is called a hypersurface section-bundle system.

**Theorem 2.42** ([ES], Theorem 3.12). Let \((\mathbb{C}^{N+1}, W)\) be a hypersurface section-bundle system. Suppose that the weights \(q_i \leq \frac{1}{2}\), then the system \((\mathbb{C}^{N+1}, W)\) is strongly tame. Therefore, the form Laplacian \(\Delta_W\) defined on \(\mathbb{C}^{N+1}\) has purely discrete spectrum.
Let $G_j$ be monomials of quasi-homogeneous weight less than 1. Then the following deformation
\[ F(z, t) = W + \sum_j t_j G_j \] (97)
is called by Arnold, Gusein-Zade and Varchenko ([AGV], PP. 416) as the lower deformation of $W$. Physicists call it as the "relative deformation".

**Theorem 2.43.** Let $W$ be a quasi-homogeneous polynomial with homogeneous weight 1 and of type $(q_0, \cdots, q_N)$. Suppose that $q_i \leq 1/2$. Then for any deformation parameter $t$, the section-bundle system $(\mathbb{C}^{N+1}, F(z, t))$ is strongly tame. Therefore, the form Laplacian $\Delta_{F(z,t)}$ defined on $\mathbb{C}^{N+1}$ has purely discrete spectrum.

The proof of Theorem 2.43 is similar to the proof of Theorem 2.42. For the convenience of the reader, we give the proof here.

**Proof.** The proof is based on the following important inequality:

**Lemma 2.44 ([FJR1], Theorem 5.7).** Let $W \in \mathbb{C}[x_1, \ldots, x_N]$ be a non-degenerate, quasi-homogeneous polynomial with weights $q_i := \text{wt}(x_i) < 1$ for each variable $x_i, i = 1, \ldots, N$. Then for any $t$-tuple $(u_1, \ldots, u_N) \in \mathbb{C}^N$ we have
\[ |u_i| \leq C \left( \sum_{j=1}^N |\frac{\partial W}{\partial x_j}(u_1, \ldots, u_N)| + 1 \right)^{\delta_i} \]
where $\delta_i = \frac{q_i}{\min_j (1 - q_j)}$ and the constant $C$ depends only on $W$. If $q_i < 1/2$ for all $i \in \{1, \ldots, N\}$, then $\delta_i \leq 1$ for all $i \in \{1, \ldots, N\}$. If $q_i < 1/2$ for all $i \in \{1, \ldots, N\}$, then $\delta_i < 1$ for all $i \in \{1, \ldots, N\}$.

We set $F(z, t) = W(z) + G(z, t)$. Above all, by Lemma 2.44 we know that
\[ |\partial W| \to \infty, \text{ as } |z| \to \infty. \] (98)

Now let $W_i = c_i \prod z_i^{b_i}$ be a quasi-homogeneous monomial with weight 1 (not necessary a monomial of $W$). By Lemma 2.44 we have for $p \neq q$ (the proof for $p = q$ case is the same),
\[ |\nabla_p \nabla_q W| \leq C |z_p|^{b_p-1} |z_q|^{b_q-1} \prod_{i \neq p, q} |z_i|^{b_i} \leq C \left( \sum_{i=1}^N \left| \frac{\partial W}{\partial z_i}(z_1, \ldots, z_N) \right| + 1 \right)^{\sum b_i \delta_i - \delta_p - \delta_q} \]
\[ = C \left( \sum_{i=1}^N \left| \frac{\partial W}{\partial z_i}(z_1, \ldots, z_N) \right| + 1 \right)^{2 \min_p \min_q (1 - q_p, 1 - q_q)} \leq C \left( \sum_{i=1}^N \left| \frac{\partial W}{\partial z_i}(z_1, \ldots, z_N) \right| + 1 \right)^{2 - 2 \delta_0} \].

Here $\delta_0 = \min_{p,q} (1 - q_p, 1 - q_q)$.

Therefore, we have
\[ |\nabla^2 W| \leq C (|\nabla W|^2 + 1)^{-2 \delta_0} |\nabla W|^2 + C. \] (99)

In particular, we have
\[ |\nabla^2 W| \leq C (|\nabla W|^2 + 1)^{-2 \delta_0} |\nabla W|^2 + C. \] (100)

where $C_0, C_1$ are constants.
We can assume that $G(z, t)$ is a monomial, since its weight is less than 1, there exists polynomial $\hat{G}$ with weight 1 such that the power of each variable of $z_i$ is no less than the corresponding power of $G$. Hence as $|z_i| >> 1$ we have

$$|\nabla G|^2 \leq |\nabla \hat{G}|^2 \leq C \left( \sum_{i=1}^{N} \left| \frac{\partial W}{\partial z_i} (z_1, \ldots, z_N) \right| + 1 \right)^{2-\delta_0} + C \left( |\nabla W|^2 + 1 \right)^{-\delta_0} |\nabla W|^2 + C$$

(101)

$$|\nabla^2 G| \leq |\nabla^2 \hat{G}| \leq C (|\nabla W|^2 + 1)^{-2\delta_0} |\nabla W|^2 + C.$$ 

(102)

Therefore, we have for any $C_0 > 0$,

$$|\nabla F|^2 - C_0 |\nabla^2 F| \geq |\nabla W|^2 + |\nabla G|^2 - 2 |\nabla W||\nabla G| - C |\nabla^2 W| - C |\nabla^2 G|$$

(103)

$$\geq \frac{1}{2} |\nabla W|^2 - C (|\nabla W|^2 + 1)^{-\delta_0} |\nabla W|^2 - C |\nabla^2 W| - C (|\nabla W|^2 + 1)^{-2\delta_0} |\nabla W|^2 - C$$

(104)

This shows that $(\mathbb{C}^{N+1}, F(z, t))$ is a strongly tame section-bundle system, by Theorem 2.40 we get the conclusion. \hfill \Box

**Remark 2.45.** One type of quasi-homogeneous polynomial is extremely interesting in the study of the Laudau-Ginzburg model in Topological field theory. Let

$$W = \sum_{i=1}^{N} C_j \prod_{j=0}^{N} z_{j}^{a_{ij}}.$$ 

(105)

Then the exponents can be written as a $N \times N$ matrix $A = (a_{ij})$. In this case, $W$ is called an invertible singularity. One can transpose the matrix to obtain an exponent matrix $A^T$ and get a singularity $W^T$. $(W, W^T)$ forms a mirror pair in Laudau-Ginzburg model which was observed by Beglund and H"ubsch [BH]. Kreuzer and Skarke [KS] proved that an invertible singularity $W$ is non-degenerate (i.e., rank $A = N$), if and only if it can be written as a sum of (decoupled) invertible polynomials of one of the following three basic types:

- $W_{\text{Fermat}} = z^n$
- $W_{\text{loop}} = z_1^{a_1} z_2 + z_2^{a_2} z_3 + \cdots + z_N^{a_N} z_1$
- $W_{\text{chain}} = z_1^{a_1} z_2 + z_2^{a_2} z_3 + \cdots + z_N^{a_N} z_1$

If we assume that all $a_i \geq 2$, then the weight $q_i \leq 1/2$ and the three type polynomials are all strongly tame.

All $A, D, E$ polynomials, unimodal singularities and etc. are strongly tame according to our definition. The reader can refer to [AGV] and some recent papers [KS, KP].

### 2.4.6. Section bundle system with Laurent polynomial potential.

Let $T$ be the torus $Spec\mathbb{C}[z_1, \ldots, z_n, z_1^{-1}, \ldots, z_n^{-1}] \equiv (\mathbb{C}^*)^n$ and let $f \in \mathbb{C}[z_1, \ldots, z_n, z_1^{-1}, \ldots, z_n^{-1}]$ be a Laurent polynomial having the form

$$f(z_1, \ldots, z_n) = \sum_{\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n} c_{\alpha} z^\alpha.$$ 

(106)

The Newton polyhedron $\Delta = \Delta(f)$ of $f$ is the convex hull of the integral points $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n$. $f$ is said to convenient if 0 is in the interior of the Newton polyhedron.
**Definition 2.46.** Let $\Delta' \subset \Delta$ be an $l$-dimensional face of $\Delta$. Define also the Laurent polynomial with the Newton polyhedron $\Delta'$

$$f^N(z) = \sum_{\alpha' \in \Delta'} c_{\alpha'} z^{\alpha'}.$$  

(107)

For any Laurent polynomial $g$, denote by $g_i$, $1 \leq i \leq n$ the logarithmic derivatives of $g$:

$$g_i(z) = \frac{\partial}{\partial z_i} g(z).$$

**Definition 2.47.** A Laurent polynomial $f$ is called non-degenerate if for every $l$-dimensional edge $\Delta' \subset \Delta (l > 0)$ the polynomial equations

$$f^{N'}(z) = f^N_{1'}(z) = \cdots = f^N_{n'}(z) = 0$$

has no common solutions in $T$.

**Proposition 2.48.** If $f$ is a convenient and non-degenerate Laurent polynomial, then $f$ is strongly tame.

**Proof.** First we prove the sufficiency. $f$ is convenient means that the origin is in the interior of the Newton polyhedron. This is equivalent to say that for any ray $\beta = (\beta_1, \cdots, \beta_n) \in \mathbb{R}^n$, there exists two integer points $\alpha_s$ as the vertices of the Newton polyhedron such that the standard inner products satisfy $\langle \alpha_s, \beta \rangle < 0$ and $\langle \alpha_s, \beta \rangle > 0$.

Do coordinate transformation: let $z_i = e^{t_i}$. Then the metric $g = i \sum_i dt^i \wedge d\bar{t}^i$ and the corresponding Kähler connection is trivial. The function

$$f(z_1, \cdots, z_n) = f(t_1, \cdots, t_n) = \sum_{\alpha} a_{\alpha} e^{\langle \alpha, t \rangle}.$$

Let $F(t) = |\nabla f|^2 - C |\nabla^2 f|$ for any $C > 0$. Then

$$F(t) \geq \sum_i \left| \sum_{\alpha} a_{\alpha} e^{\langle \alpha, t \rangle} \right|^2 - C \sum_{ij} \left| \sum_{\alpha} a_{\alpha} e^{\langle \alpha, t \rangle} \right|^2,$$

(108)

where $\alpha = (\alpha_1, \cdots, \alpha_n)$.

Now taking any ray $t_i = \beta_i |t_i|, \beta_i \in \mathbb{C}, |\beta_i| \leq 1$. Then real part $R(\beta) = (R(\beta_1), \cdots, R(\beta_n))$ defines a line in $\mathbb{R}^n$. Since $f$ is convenient, there exists at least two directions $\alpha_s$ such that $\langle R(\beta), \alpha_s \rangle < 0$ and $\langle R(\beta), \alpha_s \rangle > 0$. We arrange all the vertices $\alpha$ such that

$$\langle R(\beta), \alpha_0 \rangle \leq \langle R(\beta), \alpha_1 \rangle \cdots \leq \langle R(\beta), \alpha_s \rangle \leq \cdots < \langle R(\beta), \alpha_s \rangle,$$

(109)

where $s$ is the number of the vertices of the Newton polyhedron.

Denote by $M(\beta) = \{R(\beta), \alpha_0\}$, $M$ the set of all of $\alpha$ such that $\langle R(\beta), \alpha \rangle = M(\beta)$ and $M^+$ the set of all $\alpha$ such that $0 < \langle R(\beta), \alpha \rangle < M(\beta)$.

Hence we have along the line $\beta$ that

$$\sum_{ij} \left| \sum_{\alpha} a_{\alpha} e^{\langle \alpha, t \rangle} \right|^2 \leq C |e^{M(\beta)t}|^2 \leq C |e^{M(\beta)t}|^2 + C.$$

(110)

On the other hand, we have along the line $\beta$:

$$\sum_{i} \left| \sum_{\alpha} a_{\alpha} e^{\langle \alpha, t \rangle} \right|^2 \geq \sum_{i} \left| \sum_{\alpha \in M} a_{\alpha} e^{\langle \alpha, \beta \rangle t} \right|^2 - C \sum_{\alpha \in M^+} |e^{\langle \alpha, \beta \rangle t}|^2 - C$$

(111)
Claim: for any $\theta \in [0, 2\pi]$, there holds
\[
\sum_i \left| \sum_{\alpha \in M} a_\alpha \alpha_i e^{i \text{Im}(\alpha \beta) \theta} \right| > 0. \tag{112}
\]
If this is true, then we have
\[
\sum_i \left| \sum_{\alpha \in M} a_\alpha \alpha_i e^{i \text{Im}(\alpha \beta) \theta} \right|^2 \geq C_0 |e^{M\beta}||\theta|^2. \tag{113}
\]
Combining (110), (111) and (113) and using Young inequality, we get
\[
F(t) \to \infty,
\]
as $|t| \to \infty$.

To prove the Claim, we prove by contradiction. Suppose that (112) is not true, then there exists a $\theta_0$ such that for any $i = 1, \cdots, n$, there is
\[
\sum_{\alpha \in M} a_\alpha \alpha_i e^{i \text{Im}(\alpha \beta) \theta_0} = 0, \quad i = 1, \cdots, n, \quad (114)
\]
Multiplying the above equality by $\text{Re}(\beta_i)$ and taking the sum, noticing that for any $\alpha \in M$
\[
\sum_i a_\alpha \text{Re}(\beta_i) = M(\beta),
\]
\[
\sum_{\alpha \in M} a_\alpha e^{i \text{Im}(\alpha \beta) \theta_0} = 0. \tag{115}
\]
This contradicts with the fact that $f$ is non-degenerate. Therefore, we proved the Claim.

Now we have the following theorem.

**Theorem 2.49.** Suppose that $f$ is convenient and nondegenerate. Then the form Laplacian $\Delta f$ defined on $T$ has purely discrete spectrum.

**Remark 2.50.** Nondegenerate Laurent polynomials was initially studied by Kouchnirenko [Ko] in singularity theory. Later it becomes a very popular objects in algebraic geometry because of their connection to toric geometry [Ba] and the other algebraic theories (see [CVj] and references there). We knew the concepts of nondegeneracy and convenience from the paper by Sabbah and Douai (see [Sa], [Do], where they constructed the Frobenius structure for such polynomials by algebraic method.

On the other hand, a Laurent polynomial can define an affine hypersurface on the algebraic torus $T$.
\[
Z_f = \{ z \in T | f(z) = 0 \}.
\]
Let $\Delta$ be the Newton polyhedron of $f$. Then the corresponding toric variety $\mathbb{P}_\Delta$ is a compactification of the $n$-dimensional torus $T_\Delta = T$ by algebraic tori $T_{\Delta'}$ for any faces $\Delta' \subset \Delta$. Let $\bar{Z}_f$ be the closure of $Z_f$ in $\mathbb{P}_\Delta$. For any face $\Delta' \subset \Delta$ we have the hypersurface $Z_{f,\Delta'} = \bar{Z}_f \cap T_{\Delta'}$ in $T_{\Delta'}$. Then Batyrev [Ba] gave another geometrical characterization to the nondegeneracy:

A Laurent polynomial $f$ is nondegenerate if and only if $Z_{f\Delta'}$ is a smooth affine subvariety in $T_{\Delta'}$ of codimension 1 for any face $\Delta' \subset \Delta$.

Non-degenerate and convenient Laurent polynomials become important examples in the study of mirror symmetry. For example, the polynomial
\[
f(z_1, \cdots, z_n) = z_1 + \cdots + z_n + \frac{1}{z_1 \cdots z_n}, \tag{116}
\]
is taken as a superpotential in the Laudau-Ginzburg model of topological field theory which is the mirror object of the projective space \( \mathbb{P}^n \). It is well-known that the \( B \)-model of this Laudau-Ginzburg model should coincide with the \( A \)-model, i.e., the Gromov-Witten theory for \( \mathbb{P}^n \).

2.5. \( L^2 \)-cohomology and Hodge decomposition theorem.

2.5.1. \( L^2 \)-cohomology of \( \bar{\partial}_f \) operator.

Since \( \bar{\partial}_f^2 = 0 \), we have the \( \bar{\partial}_f \)-complex for smooth sections,

\[
\cdots \to \Omega^{k-1}(M) \xrightarrow{\bar{\partial}_f} \Omega^k(M) \xrightarrow{\bar{\partial}_f} \Omega^{k+1}(M) \to \cdots.
\]

and for compactly supported forms:

\[
\cdots \to \Omega_0^{k-1}(M) \xrightarrow{\bar{\partial}_f} \Omega_0^k(M) \xrightarrow{\bar{\partial}_f} \Omega_0^{k+1}(M) \to \cdots.
\]

We denote their cohomology groups as \( H_{\bar{\partial}_f}^k(M) \) and \( H^*_{\bar{\partial}_f}(M) \).

We also have the smooth \( L^2 \)-complex:

\[
\cdots \to L^2 \Omega^{k-1}(M) \xrightarrow{\bar{\partial}_f} L^2 \Omega^k(M) \xrightarrow{\bar{\partial}_f} L^2 \Omega^{k+1}(M) \to \cdots.
\]

We denote its cohomology as \( H^*_{\bar{\partial}_f}(M) \).

On the other hand, we can use the closure of \( \bar{\partial}_f \) forms a \( L^2 \)-complex (the composition \( \bar{\partial}_f \circ \bar{\partial}_f = 0 \) is in distribution sense):

\[
\cdots \to L^2 \Lambda^{k-1}(M) \xrightarrow{\bar{\partial}_f} L^2 \Lambda^k(M) \xrightarrow{\bar{\partial}_f} L^2 \Lambda^{k+1}(M) \to \cdots.
\]

We denote its cohomology as \( H^*_{(\bar{\partial}_f,\bar{\partial}_f)}(M) \).

Lemma 2.51. We have the isomorphism of two cohomologies:

\[
H^*_{(\bar{\partial}_f,\bar{\partial}_f)}(M) \cong H^*_{(\bar{\partial}_f,\bar{\partial}_f)}(M) .
\]

Proof. We first prove the following regularity conclusion:

Let \( \psi \in \text{Dom}(\bar{\partial}_f) \) such that \( \bar{\partial}_f \psi = \varphi \) is \( L^2 \)-integrable, then \( \psi \in \Omega^*_{(\bar{\partial}_f)} \).

Since \( \bar{\partial}_f \psi = -\bar{\partial} f \wedge \psi + \varphi \), by interior estimate of \( \bar{\partial} \)-operator (refer to [DS1]), there exists the estimate:

\[
||\psi||_{W^{1,2}(B_r(x_0))} \leq C(||\bar{\partial} f \wedge \psi||_{L^2(B_{2r}(x_0))} + ||\varphi||_{L^2(B_{2r}(x_0))}),
\]

where \( r \leq R, x_0 \in M \) and \( B_r(x_0) \) is the geodesic ball of radius \( r \) in \( M \).

Now using the sobolev embedding theorem \([2,3]\) we know that \( \psi \in L^\alpha(\Lambda^p(B_r(x_0))) \) for \( 2 \leq \alpha \leq \frac{2n}{n-1} \). Here \( n \) is the complex dimension of \( M \). By standard bootstrap argument and Hölder estimate, we know that \( \psi \) is a smooth form in case that \( \varphi \) is smooth.

Now if \( \varphi \in \ker(\bar{\partial}_f) \), then \( \varphi \) is a smooth forms. If \( \varphi = \bar{\partial}_f \psi \), then it lies in the kernel of \( \bar{\partial}_f \) as a \( L^2 \) solution. So \( \varphi \) is automatically smooth by our regularity conclusion. Finally we know that \( \psi \) is a smooth \( L^2 \) form.

Hence we proved the isomorphism. \( \Box \)

By the above lemma, we denote the \( L^2 \)-cohomology simply by \( H^*_{(\bar{\partial}_f,\bar{\partial}_f)}(M) \) which corresponds to either of the two complexes.
2.5.2. Hodge decomposition.

In this part, we always assume that our section bundle system \((M, g, f)\) is strongly tame. If \((M, g, f)\) is strongly tame, then by Theorem 2.40, \(\Delta_f\) has only discrete spectrum in \(L^2\Lambda^*(M)\) space. Let \(\mathcal{H} \subset \text{Dom}(\Delta_f)\) be the subspace of \(\Delta_f\)-harmonic forms. Then we know that \(\dim \mathcal{H} < \infty\).

Let \(E_\mu\) be the eigenspace with respect to the eigenvalue \(\mu\) of \(\Delta_f\), \(P_\mu : L^2\Lambda^k \to E_\mu\) be the projection operators, then we have the spectrum decomposition formulas:

\[
L^2\Lambda^k = \mathcal{H} \oplus \bigoplus_{i=1}^\infty E_{\mu_i}, \\
\Delta_f = \sum_{i} \mu_i P_{\mu_i}.
\]

The Green operator \(G_f\) of \(\Delta_f\) satisfies

\[
G_f \Delta_f + P_0 = \Delta_f G_f + P_0 = I.
\]

This implies the following Hodge-De Rham theorem.

**Theorem 2.52.** There are orthogonal decomposition

\[
L^2\Lambda^k = \mathcal{H}^k \oplus \text{im}(\bar{\partial}_f) \oplus \text{im}(\partial^*_f). \\
\ker(\bar{\partial}_f) = \mathcal{H}^k \oplus \text{im}(\bar{\partial}_f).
\]

In particular, we have the isomorphism

\[
H^*_{(\Omega, \bar{\partial}_f)}(M) \cong \mathcal{H}^*,
\]

where \(\mathcal{H}^k\) means the space of harmonic \(k\)-forms.

**Proof.** For any \(\varphi \in L^2\), we have the decomposition:

\[
\varphi = P_0 \varphi \oplus \bar{\partial}_f \partial_f \varphi \oplus \bar{\partial}_f \partial_f \varphi \oplus \partial_f \bar{\partial}_f \varphi \oplus \varphi,
\]

where \(\oplus\) is the orthogonal direct sum. This gives the decomposition theorem and the decomposition of \(\ker(\bar{\partial}_f)\). Since \(\dim \mathcal{H}^k < \infty\), the image \(\text{im}(\bar{\partial}_f)\) is a closed subspace, and we have the isomorphism

\[
\mathcal{H}^* \cong H^*_{(\Omega, \bar{\partial}_f)}(M)
\]

between closed subspace. \(\square\)

The same conclusions can be obtained for other operators.

**Theorem 2.53.** The following decomposition hold

\[
L^2\Lambda^k = \mathcal{H}^k \oplus \text{im}(\bar{\partial}_f) \oplus \text{im}(\bar{\partial}_f^*) \\
L^2\Lambda^k = \mathcal{H}^k \oplus \text{im} d_f \oplus \text{im} d_f^* \\
L^2\Lambda^k = \mathcal{H}^k \oplus \text{im} d_f^* \oplus \text{im}(d_f^*)^*.
\]

Combined with the above decomposition, we have the five-fold decomposition

\[
L^2\Lambda^k = \mathcal{H}^k \oplus \text{im} d_f d_f^* \oplus \text{im} d_f^* d_f \oplus \text{im} \bar{\partial}_f \partial_f \oplus \text{im} \bar{\partial}_f \partial_f^* \\
= \mathcal{H}^k \oplus \text{im} \bar{\partial}_f \bar{\partial}_f \oplus \text{im} \bar{\partial}_f^* \bar{\partial}_f^* \oplus \text{im} \partial_f \bar{\partial}_f \oplus \text{im} \partial_f \bar{\partial}_f^* \\
= \mathcal{H}^k \oplus \text{im} \bar{\partial}_f \partial_f \oplus \text{im} \bar{\partial}_f \partial_f \oplus \text{im} \bar{\partial}_f \partial_f^* \oplus \text{im} \bar{\partial}_f \partial_f^*.
\]
2.5.3. **Hard Lefschetz theorem.**

**Definition 2.54.** A homogeneous form \( \alpha \in \Lambda^k(\Lambda^{p,q}) \) is called primitive if \( \Lambda \alpha = 0 \). The space of \( k \)-primitive forms is denoted by

\[
\mathcal{P}^k := \bigoplus_{p+q=k} \mathcal{P}^{p,q}.
\]

If \( \alpha \in \mathcal{P}^k \), we can use the formula

\[
\Lambda^r L^r \alpha = \Lambda^{r-1}(\Lambda L^r - L^r \Lambda)\alpha = r(N-k-r+1)\Lambda^{r-1}L^{r-1}\alpha.
\]

to deduce the following conclusions:

**Lemma 2.55.**

1. If \( \alpha \in \mathcal{P}^k \), then \( L^r \alpha = 0 \) for \( r \geq (N+1-k)_+ \), in particular for \( \alpha \in \mathcal{P}^N \), there is \( L\alpha = 0 \).
2. \( \mathcal{P}^k = 0 \) for \( N+1 \leq k \leq 2N \).

**Theorem 2.56.** (Primitive decomposition formula) For every \( \alpha \in \Lambda^k \), there is a unique decomposition

\[
\alpha = \sum_{r \geq (k-N)_+} L^r \alpha_r, \quad \alpha_r \in \mathcal{P}^{k-2r}.
\]

Furthermore, \( \alpha_r = \Phi_{k,r}(L,\Lambda)\alpha \) where \( \Phi_{k,r} \) is noncommutative polynomial in \( L,\Lambda \) with rational coefficients. As a consequence, we have the space decompositions

\[
\Lambda^k = \bigoplus_{r \geq (k-N)_+} L^r \mathcal{P}^{k-2r},
\]
\[
\Lambda^{p,q} = \bigoplus_{r \geq (p+q-N)_+} L^r \mathcal{P}^{p+q-r}.
\]

**Corollary 2.57.** The linear operators

\[
L^{N-k} : \Lambda^k \to \Lambda^{2N-k},
\]
\[
L^{N-p,q} : \Lambda^{p,q} \to \Lambda^{N-p,N-q},
\]

are isomorphisms for \( k \leq N, p + q \leq N \).

The proof of the primitive decomposition formula can be found in [De, GH, CMP].

Since the operators \( \Lambda, L, F \) forms a \( sl_2 \) Lie algebra and commute with the Laplacian operator \( \Delta_f \), we can apply the above primitive decomposition theorem to the space of \( L^2 \) harmonic forms. Therefore in the same way to prove the Hard Lefschetz theorem for compact Kähler manifolds, we can obtain the following result.

**Theorem 2.58** (Hard Lefschetz theorem). Let \( (M, g, f) \) be a strongly tame section-bundle system with complex dimension \( n \). Then we have the isomorphism

\[
L^k : H^{n-k}_{(2,\partial\bar{\partial})} \cong H^{n-k+2}_{(2,\partial\bar{\partial})}, \quad 1 \leq k \leq n.
\]

\[(127)\]

**Definition 2.59.** Let \( n = \dim_C M \). For \( k \leq n \), the primitive \((n-k)\) cohomology, \( PH^{n-k}_{(2,\partial\bar{\partial})}(M) \) is the kernel of \( L^{k+1} \) acting on \( H^{n-k}_{(2,\partial\bar{\partial})}(M) \). Equivalently, it is the kernel of the \( \Lambda \)-operator:

\[
PH^{n-k}_{(2,\partial\bar{\partial})}(M) = \ker(L^{k+1} : H^{n-k}_{(2,\partial\bar{\partial})} \to H^{n-k+2}_{(2,\partial\bar{\partial})})
\]
\[
= \ker(\Lambda : H^{n-k}_{(2,\partial\bar{\partial})} \to H^{n-k-2}_{(2,\partial\bar{\partial})}).
\]

In particular, \( PH^{k}_{(2,\partial\bar{\partial})} = 0 \) for \( k > n \).

Consequently, we have
Lemma 2.62 (Lefschetz’s decomposition theorem). Let \((M, g, f)\) be a strongly tame section-bundle system. Then there is a direct sum decomposition
\[ H^k_{(2,\partial)} = PH^k_{(2,\partial)} \oplus L \cdot PH^{k-2}_{(2,\partial)} \oplus L^2 \cdot PH^{k-4}_{(2,\partial)} \cdots. \] (128)

Remark 2.61. Though the \(\bar{\partial}_f\) is defined in terms of a holomorphic function \(f\), the Lefschetz’s decomposition theorem even holds on real field. The reason is the Laplacian \(\Delta\) section-bundle system. Then there is a direct sum decomposition
\[ \Lambda^k(U) \rightarrow \Lambda^k(U) \rightarrow \cdots \]

2.5.4. Computation of the \(\bar{\partial}_f\) cohomology.

We have defined three cohomology groups:
\[ H^r_{(0,\bar{\partial}_f)}(M), H^r_{(2,\partial)}(M), H^r_{\bar{\partial}_f}(M). \]

We want to discuss their relations. Above all we have the (twisted) poincare lemma:

Lemma 2.62 (\(\bar{\partial}_f\) Poincare lemma). Assume that \(U \subset M\) be a simply connected domain. Let \(\phi\) be a \(\bar{\partial}_f\)-closed k-form on \(U\). Then there exist a \(k-1\) form \(\psi\) and a unique holomorphic \((k,0)\)-form \(\phi\) module \(df \wedge \Omega^{k-1}(U)\) such that
\[ \phi = \phi + \bar{\partial}_f \psi. \]

In particular, for the case \(k < n\) and the case \(k = n\) with the condition \(df \neq 0\) on \(U\), \(\phi \equiv 0\); if \(df = 0\) at some points on \(U\), then \(\phi \neq 0\).

Proof. Assume \(\phi\) has the following decomposition
\[ \phi = \sum_p \phi^{p,k-p}. \]

Then \(\bar{\partial}_f \phi = 0\) decompose into
\[ \bar{\partial}_f \phi^{p,k-p} + \bar{\partial}_f \wedge \phi^{p-1,k-p+1} = 0, 0 \leq p \leq k. \]

By the Dolbeault lemma, there exists a \((0, k-1)\) form \(\psi^{0,k-1}\) such that \(\phi^{0,k} = \bar{\partial}_f \psi^{0,k-1}\). Then we have
\[ \begin{align*}
0 &= \bar{\partial}_f \psi^{1,k-1} + \bar{\partial}_f \wedge \psi^{0,k-1} \\
&= \bar{\partial}_f [\psi^{1,k-1} - \bar{\partial}_f \wedge \psi^{0,k-1}] \\
\end{align*} \]

By Dolbeault lemma again, we can obtain \(\psi^{1,k-2}\) such that
\[ \psi^{1,k-1} = \bar{\partial}_f \psi^{1,k-2} + \bar{\partial}_f \wedge \psi^{0,k-1}. \]

By induction, once we obtain \(\psi^{p-1,k-p}\) we can find \(\psi^{p,k-p}\) such that
\[ \psi^{p,k-p} = \bar{\partial}_f \psi^{p,k-p-1} + \bar{\partial}_f \wedge \psi^{p-1,k-p}. \]

In particular, we arrive at the equation
\[ \bar{\partial}_f [\psi^{0,0} - df \wedge \psi^{k-1,0}] = 0. \]

Setting \(\phi = \psi^{0,0} - df \wedge \psi^{k-1,0}\), which is holomorphic, and defining \(\psi = \sum_p \psi^{p,k-p-1}\), we obtain:
\[ \phi = \phi + \bar{\partial}_f \psi. \]

Consider the Koszul complex \(K(df, U)\)
\[ 0 \rightarrow \mathcal{O}_U \xrightarrow{df} \cdots \xrightarrow{df} \Lambda^{k-1,0}(U) \xrightarrow{df} \Lambda^{k,0}(U) \rightarrow \cdots \Lambda^n(U) \rightarrow 0, \]
the division lemma of De Rham (see [Ku], PP. 18) shows that the cohomology groups
\[ H^*(K(df, U)) \equiv \begin{cases} \Omega^p(\Omega^d U) & \text{if } k < n, \\ \Omega^p(\Omega^d U \cap df) & \text{if } k = n \end{cases} \]
(129)
So if \( k < n \), there exists a holomorphic form \( \hat{\phi} \) such that \( \phi = df \land \hat{\phi} = \bar{\partial}f(\hat{\phi}) \) and \( \psi = \bar{\partial}f(\hat{\phi} + \psi) \). \( k = n \) case can be considered similarly. Hence we get the conclusion. \( \square \)

Now we turn to the global computation of the complex \( (\Omega^p(M), \bar{\partial}_f = \bar{\partial} + df \land) \). This complex can be viewed as the total complex of the double complex \( (\Omega^{*,*}(M), \bar{\partial}, df \land) \).

Obviously we have the relations
\[ \bar{\partial}_f^2 = (df \land)^2 = [\bar{\partial}, df \land] = 0. \]
There are two filtrations on \((\Lambda^k, \bar{\partial}_f)\) given by
\[ 'F^p \Lambda^k = \oplus_{q+k \geq p} \Lambda^p \Lambda^q, \quad ''F^q \Lambda^k = \oplus_{p+q \geq k} Q^p Q^q \]
So there are two spectral sequences, \( 'E_r \) and \( ''E_r \), both abutting to \( H^*((\Lambda^*, \bar{\partial}_f)) \) = \( H^* \).
One can obtain the result
\[ \begin{cases} 'E^p_q \equiv H^p_{\bar{\partial}}(H^{p+q}_0(\Lambda^{*,*})) \\ ''E^p_q \equiv H^p_{\bar{\partial}}(H^{p+q}_0(\Lambda^{*,*})) \end{cases} \]

The second identity is the \( q \)-th cohomology group of the following complex:
\[ \cdots \rightarrow H^p_0(\Lambda^{*,*}) \xrightarrow{df \land} H^p_0(\Lambda^{*,*}) \xrightarrow{df \land} \cdots, \]
which is equivalent to
\[ \cdots \rightarrow H^p_0(M) \xrightarrow{df \land} H^{p+1}_0(M) \xrightarrow{df \land} \cdots. \]

We can obtain some explicit result under some requirement to the manifold \( M \). If \( M \) is a stein manifold, then there is
\[ H^p_0(M) = 0, \quad \forall p > 0. \]
(131)

This is true for punctured polydiscs \((\mathbb{C}^*)^k \times \mathbb{C}^l\).

In this case, we get the Koszul complex:
\[ \cdots \rightarrow \Omega^q(M) \xrightarrow{df \land} \Omega^{q+1}(M) \xrightarrow{df \land} \cdots. \]
(132)

**Theorem 2.63.** Suppose that \( M \) is a complete noncompact stein manifold, then we have
\[ H^k_{\bar{\partial}_f} = \oplus_{0 \leq p \leq k} G^p H^k_{\bar{\partial}_f} = \oplus_{0 \leq p \leq k} H^{k-p}_{\bar{\partial}_f}(\Omega^p) 
\]
\[ = \begin{cases} 0, & \text{if } k < n \\ \Omega^p/df \land \Omega^{p-1}, & \text{if } k = n, \end{cases} \]

In particular, we have the two consequences:

1. If \((\mathbb{C}^{n+1}, W)\) is a section-bundle system with the potential being a non-degenerate quasihomogeneous polynomial, there is isomorphism
\[ H^k_{\bar{\partial}_f} = \begin{cases} 0, & \text{if } k < n \\ \mathbb{C}[z_0, \ldots, z_n]/J_W, & \text{if } k = n, \end{cases} \]
(133)

Here \( J_W \) is the Jacobi ideal of \( W \).
(2) if \((T^n, f)\) is a section-bundle system with non degenerate and convenient Laurent polynomial \(f\), then

\[
H^k_{\partial_f} = \begin{cases} 
0, & \text{if } k < n \\
\mathbb{C}[z_1, \ldots, z_n, z_1^{-1}, \ldots, z_n^{-1}]/J_f, & \text{if } k = n.
\end{cases}
\]

There are two natural maps:

\[i_1 : H^*_{\partial_f}(M) \to H^*_{\partial_f(\Omega^n, f)}(M), \quad i_2 : H^*_{\partial_f(\Omega^n, f)}(M) \to H^*_{\partial_f}(M)\]

**Proposition 2.64.** The maps

\[i_1 : H^*_{(0,\partial_f)}(M) \to H^*_{(\partial_f, 2)}(M), \quad j := i_2 \circ i_1 : H^*_{(0,\partial_f)}(M) \to H^*_{\partial_f}(M)\]

are injective.

**Proof.** It suffices to prove that \(i_2 \circ i_1 = j : H^*_{0,\partial_f}(M) \to H^*_{\partial_f}(M)\) is injective. Let \(\varphi\) be a class in \(H^k_{0,\partial_f}(M)\). Suppose that the image \(j(\varphi)\) is zero in \(H^k_{\partial_f}(M)\), i.e., there exists a smooth \(k - 1\) form \(\tilde{\psi}\) such that

\[\varphi = \partial_f \tilde{\psi}.\]

Notice that \(\partial_f \tilde{\psi} \equiv 0\) near the infinity place. Hence there exists a \((k - 2)\) form \(\eta\) such that \(\tilde{\psi} - \partial_f \eta\) vanishes near the infinity place. This fact is true since the cohomology groups \(H^*_\partial\) are trivial when restricted to the infinite far place (fundamental tame condition). This shows that \(i_1\) is injective.

\[\square\]

**Remark 2.65.** The map \(i_2 : H^*_{(\partial_f, 2)} \to H^*_{\partial_f}(M)\) need not be injective or surjective. Since \(\mathcal{H}^n \cong H^*_{(\partial_f, 2)}(M)\) and \(H^*_{\partial_f}(M) \cong \Omega^n / d\Omega^{n-1}\), we can define the map \(i_2\) by the following map \(i_{0\partial} : \mathcal{H}^n \to \Omega^n / d\Omega^{n-1}\) if \(M\) is simply-connected: let \(\alpha \in \mathcal{H}^n\), then by Poincare lemma [2.62] there exists a unique holomorphic \(n\)-form \(i_{0\partial}(\alpha) \in \Omega^n(M) / d\Omega^{n-1}(M)\) and a smooth \((n - 1)\) form \(\tilde{\psi}\) such that

\[\alpha = i_{0\partial}(\alpha) + \partial_f \tilde{\psi}.
\]

We can define another map \(i_{\partial0} : \Omega^n / d\Omega^{n-1} \to \mathcal{H}^n\) as follows. Let \(g \in \Omega^n(M) / d\Omega^{n-1}(M)\). Since the Koszul complex \(K(d\Omega, U)\) over a small neighborhood \(U\) of \(\infty\) is exact, there exists a holomorphic \(n - 1\) form \(\tilde{g}\) such that \(g = d\tilde{f} \wedge \tilde{g} = \partial_f(\tilde{g})\) on \(U\). Take a cut-off function \(\chi\) which is 1 on a smaller domain of \(U\) and with compact support in \(U\), then \(g - \partial_f(\chi \tilde{g})\) is a compactly supported \(n\)-form on \(M\), then we define \(i_{\partial0}(g) = P_0(g - \partial_f(\chi \tilde{g}))\). This definition is independent of the choice of \(\chi, \tilde{g}\). If there is another pair \(\tilde{g} \tilde{g}_0\) such that \(g - \partial_f(\chi \tilde{g}_0)\) is compactly supported, then \(\partial_f(\chi \tilde{g}_0) - \partial_f(\chi \tilde{g})\) is also compactly supported, and then \(P_0[\partial_f(\chi \tilde{g}_0) - \partial_f(\chi \tilde{g})] = 0\).

**Theorem 2.66.** Let \((M, g)\) be a Kähler manifold with bounded geometry and \((M, g, f)\) be strongly tame. If \(f\) is a Morse function, then

\[
\dim \mathcal{H}^k = \begin{cases} 
0, & \text{if } k < n \\
\mu, & \text{if } k = n.
\end{cases}
\]

and there is an explicit isomorphisms:

\[i_{0\partial} : \mathcal{H}^n \to \Omega^n(M) / d\Omega^{n-1}(M).
\]
Proof. Let \( f_i = \tau f, \tau \in \mathbb{C} \) and \( p_1, \cdots, p_n \) be the critical points of \( f \) or \( f_i \). By Witten-Morse-Smale theory, there is an isomorphism between the Witten complex \((L^2(\Omega^*(M)), df, \partial)\) and the Morse-Smale complex \((C^*(M, 2Re(f)), \partial)\) given by the non-degenerate critical points and the negative gradient flow of \( 2Re(f) \). Since all the non degenerate critical points of \( 2Re(f) \) has Morse index \( n \), so only the middle dimensional homology group is non-trivial and with dimension \( \mu \). By our Hodge theorem and the isomorphism \( H_{(2),\overline{\partial}}^*(\mathbb{C}) \cong H_{(2),\overline{\partial}}^*(L^2(\Omega^*(M)), df, \partial) \cong H_{(2),\overline{\partial}}^*(C^*(M, 2Re(f)), \partial) \), we obtain the conclusion. We get an indirect isomorphism to the \( C \)-space \( \Omega^p/df \wedge \Omega^{q-1} \). Given an order of those critical points. This order is the same for any \( \tau \). As \( |\tau| \to \infty \), the mass of the harmonic \( n \)-forms will concentrate at the critical points \( p_i \). For large \( \tau \), the map \( i_{0b} \) is isomorphic and provides the explicit isomorphism between \( \mathcal{H}^n \) and \( \Omega^p(M)/df \wedge \Omega^{q-1}(M) \). □

By Theorem 2.66 we know that the singular behavior of \( \mathcal{H} \) can only happen when \( f \) is a degenerate holomorphic function.

2.6. \( \mathbb{Z}_2 \)-symmetries, orbifoldizing and splitting.

2.6.1. \( \mathbb{Z}_2 \)-symmetries.

In compact Kähler manifold, the standard complex conjugate \( \overline{\tau} \) and the \(*\) operator provides the \( \mathbb{Z}_2 \) symmetries to the Hodge theory. In our case, if the section-bundle system \((M, g, f)\) is Kähler, the twisted Laplace operator \( \Delta_f \) is real. Therefore the complex conjugate \( \overline{\tau} \) provides the \( \mathbb{Z}_2 \) symmetry to the space \( \mathcal{H}^* \) of harmonic forms. However, the \(*\) operator does not communicate with \( \Delta_f \) operator, instead we have

\[
*\Delta_f = -\Delta_{-f} *.
\]

Denote by \( \mathcal{H}^*_{\overline{\tau}} \) the space of \( \Delta_f \)-harmonic forms and \( \mathcal{H}^*_{\tau} \) the space of \( \Delta_{-f} \)-harmonic forms. Therefore if we want our theory keep the \( \mathbb{Z}_2 \)-symmetry of \(*\)-operator, we should consider the total space:

\[
\mathcal{H}^*_{\text{tot}} := \mathcal{H}^*_{\overline{\tau}} \oplus \mathcal{H}^*_{\tau}.
\]

The Riemannian-Hodge bilinear relation on \( \mathcal{H}^*_{\text{tot}} \) is given by

\[
\langle \phi, \psi \rangle = \int_M \phi \wedge \psi \wedge \omega^{n-k}, \forall \phi, \psi \in \mathcal{H}^*_{\text{tot}}.
\]

(137)

it is \((-1)^k\) symmetric and satisfies the relation:

\[
g(\phi, \psi) = \langle \phi, *\overline{\psi} \rangle.
\]

(138)

Therefore, the bilinear relation naturally satisfies the positivity for any primitive harmonic forms in \( \mathcal{H}^*_{\text{tot}} \).

We have the analogous \( \mathbb{Z}_2 \) symmetry relation between cohomology groups. In addition to the operator \( \overline{\partial}^\tau_f \) and its complex, we can also consider the complex \((\Omega^p, \partial_f)\) and the cohomology \( H^*_{\partial_f} \). They have the dual relation because of the action of the \(*\) operator. * operator gives the following commutation relation:

\[
*\overline{\partial}^\tau_f \equiv (-1)^{p+q}\partial_{-f} *.
\]

Therefore we have the following conclusion.

Proposition 2.67. We have the isomorphisms

\[
\mathcal{H}^*_{\overline{\tau}} \cong H^*_{(2),\overline{\partial}_f} \cong H^*_{(2),\partial_f} \cong \mathcal{H}^*_{\tau}.
\]

(139)
2.6.2. orbifoldizing and splitting.

If $M$ is a Stein manifold and $f$ is fundamental tame, then only the middle dimensional cohomology groups of $H^*_{(2), \partial_j}$ and $H^*_{\partial_j}(M)$ are not trivial. Hence they have no ring structures when compared to the cohomology ring defined on a closed manifold. To compensate this default in LG/CY correspondence, Intriligator and Vafa [IV] used the orbifoldizing method. Such idea has been developed by R. Kaufmann to the abstract algebraic structures [Kau]. Their constructions only provided the state space structure and grading information. The orbifoldized singularity theory for a non-degenerate quasi-homogeneous singularities has been recently constructed by M. Krawitz [Kr]. The state space correspondence between LG A and B model, and between CY model and LG A model for the invertible $\gamma$-twistor sector of $(C^n, W)$ as the $\gamma$-twistor sector of $(C^n, W)$. Note that by Lemma 3.2.1 of [FJR2], 0 is the only singularity of $W$. On each twister sector we can study the deformation theory of the Schrödinger operator associated to $W$. What extra information can be extracted and the comparison with other orbifolding process is an interesting problem.

Except the orbifoldizing operation, there is another simple phenomenon, i.e., the splitting of the variables of the Schrödinger operators due to the splitting of the section-bundle system $(\mathbb{C}^n, W = W_1 + W_2) = (\mathbb{C}^{n_1}, W_1) \times (\mathbb{C}^{n_2}, W_2)$. In this case, the harmonic form of the total space is the product of two harmonic forms of lower dimensional Schrödinger systems. Note that such decomposition can’t be done for projective hypersurface. An example is given by the Fermat polynomials $z_1^{n_1} + \cdots + z_n^{n_2}$. Therefore such splitting at the conformal point of the deformation parameter space will be very helpful to the computation of the topological quantities of the corresponding projective hypersurfaces.

3. Deformation theory

3.1. Deformation of superpotential.

A superpotential of a schrödinger system is a holomorphic function $f : M \to \mathbb{C}$ defined on a (non-compact) complete complex manifold $M$ with dimension $n$. The deformation of $f$ will induce the deformation of the Schrödinger equation. In this section, we will consider the deformation of the potential and define the so-called strong deformation which is required in this paper.

3.1.1. Milnor numbers.

Let $z \in M$ be an isolated critical point of $f$, and $B(z)$ be a ball centered at $z$ such that $z$ is the only critical point of $f$ at $B(z)$. Then it is well-known (or see [Mil]) that the topological
degree of the map:

\[ \partial B \to S^{2n-1} : z \to \frac{\partial f}{|\partial f|} \]
equals to the Milnor number of \( f \) at \( z \),

\[ \mu_z(f) := \dim \mathcal{O}/J_f. \]

We can also define the global Milnor number of \( f \) on an open domain \( U \).

**Definition 3.1.** Let \( \partial U \) be the boundary of \( U \) which is assumed to be a smooth compact \( 2n - 1 \)-dimensional manifold and \( \partial f|_{\partial U} \neq 0 \). Then the Milnor number of \( f \) on \( U \) is defined to be the topological degree of the map

\[ \partial U \to S^{2n-1} : z \to \frac{\partial f}{|\partial f|}. \]

We denote the Milnor number of \( f \) in \( U \) by \( \mu_U(f) \).

In particular, if \( z \) is not a critical point of \( f \), we define the Milnor number at \( z \) to be zero. If \( \partial f \) has positive dimensional zero locus in \( U \), then we define \( \mu_U(f) = \infty \).

The proof of the following conclusions about the Milnor number can be found in Appendix B of [Mi].

**Theorem 3.2.** Let \( U \subset M \) be an open set and \( f : U \to M \) be a holomorphic function. Let \( V \subset U \) be an open subset with compact closure in \( U \) with the boundary \( \partial V \) being a smooth compact manifold, and such that \( \partial f|_{\partial V} \neq 0 \), then \( f \) has only finitely many isolated critical points \( p_1, \ldots, p_l \) in \( V \) and has the identities:

\[ \mu_V(f) = \sum_{j=1}^{l} \mu_{p_j}(f). \]

(141)

Furthermore, if \( f(z,t) \) is a holomorphic function defined on \( U \times [0, 1] \) such that \( \partial_z f(z,t) \neq 0 \) on \( \partial V \times [0, 1] \), then

\[ \mu_V(f(z,0)) = \mu_V(f(z,1)). \]

(142)

We follow Broughton’s definition (see [Br]) of a tame holomorphic function.

**Definition 3.3.** A holomorphic function \( f \) is called tame if there is a compact neighborhood \( U \) of the critical points of \( f \) such that \( \|\partial f\| \) is bounded away from 0 on \( M - U \).

For tame holomorphic function \( f \) we can define the global Milnor number \( \mu(f) \) on \( \mathbb{C}^m \) and certainly we have

\[ \mu(f) < \infty. \]

(143)

3.1.2. **Strong deformation.**

**Definition 3.4.** Let \( f : M \to \mathbb{C} \) be a holomorphic function defined on a (non-compact) complete complex manifold and \( S \subset \mathbb{C}^m \) be a (open or closed) domain. A deformation of \( f \) on \( S \) is a holomorphic function \( F(x,t) \),

\[ F : M \times S \to \mathbb{C} \]
such that \( F(x,0) = f \).
Definition 3.5. Let \( F'(x, \lambda') : M \times S' \to \mathbb{C} \) be an another deformation of \( f : M \to \mathbb{C} \). \( F' \) is said to be embedded in the deformation \( F(z, \lambda) : M \times S \to \mathbb{C} \), if there is a biholomorphic map \( g(z, \lambda) : M \to M \) for any \( \lambda \in S \) and an holomorphic submersion \( \varphi : S \to S' \) such that

\[
F'(x, \lambda') = F(g(z, \lambda), \varphi(\lambda)).
\] (144)

If the submersion is a diffeomorphism, then we say that the two deformations are equivalent. A deformation \( F \) is called a maximum deformation, if it can’t be embedded into another different deformation.

Remark 3.6. In singularity theory, it is well known that there is a versal deformation of a singularity (a germ of function with isolated critical point). In our case, the deformation is global, we don’t know if there exists a versal deformation. It relates to the automorphism group of \( M \).

Proposition 3.7. Let \( F : M \times S \to \mathbb{C} \) be a deformation of \( f \). Suppose that for any \( t \in S \), \( f_t \) is a tame function. Then the function \( \mu(f_t) \) is lower semicontinuous in \( S \).

Proof. This is because the small continuous perturbation of \( f \) will not change the number of critical points contained in a compact set \( K \) of the manifold \( M \). However the perturbation may generates new critical points of \( f_t \) outside \( K \). Therefore \( \mu(f_t) \) is only lower semicontinuous.

Definition 3.8. Let \( F : M \times S \to \mathbb{C} \) be a deformation of \( f : M \to \mathbb{C} \). It is called a strong deformation of \( f \) on \( S \subset \mathbb{C}^m \), if the following two conditions hold (denote by \( f_t = F(\cdot, t) \)):

1. \( \sup_{t \in S} \mu(f_t) < \infty \).
2. For any \( t \in S \), \( f_t \) is strongly tame.
3. For any \( t \in S \), \( \Delta_t := \Delta f_t \) have common domains in the space of \( L^2 \) forms.

The third condition is a technique condition. Usually we consider the deformation of the form:

\[
f_t = f + \sum_{i=1}^m t_i g_i. \tag{145}
\]

The following lemma grantees the condition (3) in Definition 3.8.

Lemma 3.9. Let \( f_t \) be a deformation of a strongly tame holomorphic function \( f \) having the form \((145)\), and satisfy: for any \( C > 0 \), \( |\nabla f|^2 - C|\nabla g| \to \infty \) as \( d(z, z_0) \to \infty \) and there exists \( C_0 \) such that \( |\nabla g| \leq C_0|\nabla f| \) near infinity. Then \( f_t \) is a strong deformation for small \( t = (t_1, \cdots, t_m) \).

Proof. It suffices to prove the condition (3) in the definition 3.8. We know that \( \varphi \in \text{Dom}(\Lambda_0) \) if and only if the graph norm

\[
\|\varphi\|^2_{g,0} := \int_M |\tilde{\partial}_j \varphi|^2 + |\tilde{\partial}_i \varphi|^2 + |\varphi|^2 < \infty.
\]

By \( L^1 \) Stokes theorem, the above inequality is equivalent to the following inequality

\[
\int_M (\Delta_\partial \varphi, \varphi) + (L_f(\varphi) + |\nabla f|^2 \varphi, \varphi) + (\varphi, \varphi) < \infty. \tag{146}
\]

Since \( (M, g, f) \) is strongly tame, the above inequality shows the equivalence of the graph norm \( \|\cdot\|_{g,0} \) and the following norm:

\[
\int_M (\Delta_\partial \varphi, \varphi) + (|\nabla f|^2 + 1)|\varphi|^2 < \infty. \tag{147}
\]
Since for any $C > 0$, $|\nabla f|^2 - C|\nabla g_i| \to \infty$ as $d(z, z_0) \to \infty$ and there exists $C_0$ such that $|\nabla g_i| \leq C_0|\nabla f|$ near infinity, (147) is equivalent to

$$\int_M (\Delta \phi, \phi) + (|\nabla f|^2 + 1)|\phi|^2 < \infty. \quad (148)$$

which is equivalent to the graph norm of $\Delta$:

$$\|\phi\|_{g, t}^2 := \int_M |\bar{\partial} f \phi|^2 + |\bar{\partial}^t \phi|^2. \quad (151)$$

In particular, we take $G_1 = 1$, the constant map.
Following physics’ notion (see for example, [KTS]), we can distinguish the role of each monomials.

**Definition 3.12.** Assume that $W$ has weight 1. We can think of the deformation of $F$ as a quasihomogeneous polynomial of $(z, t)$. Then each deformation parameter $t_i$ is called:

1. relevant, if the weight of $t_i$ is positive;
2. marginal, if the weight of $t_i$ is zero.
3. irrelevant, if the weight of $t_i$ is negative.

It is interesting to check the miniversal deformation of the singularities on Arnold’s classification table [AGV]. Here we consider the simple singularities and the unimodal singularities.

1. Simple singularities includes the following $A, D, E$ singularities:
   - $A_n$: $W = x^{n+1}, n \geq 1$;
   - $D_n$: $W = x^n + xy^2, n \geq 4$;
   - $E_6$: $W = x^3 + y^4$;
   - $E_7$: $W = x^3 + xy^3$;
   - $E_8$: $W = x^3 + y^5$;

The miniversal deformations of those singularities are relative, therefore by Theorem 2.43 such deformations are strong deformation according to our definition.

2. Unimodal singularities. Such type singularities includes the three big groups:
   1. Relevant, if the weight of $a$ is positive; $\frac{1}{r} + \frac{1}{q} + \frac{1}{p} < 1, a \neq 0$.
   2. Marginal, if the weight of $a$ is zero.
   3. Irrelevant, if the weight of $a$ is negative.

If we take the parameter $a$ as the deformation parameter, then $a$ is a marginal variable. The other parameters appeared in the miniversal deformation are relative variables. Therefore by Theorem 2.43 the miniversal deformation is strong deformation.

(iii) 14 Exceptional singularities.

$$E_{12}: x^3 + y^7 + axy^5, E_{13}: x^3 + xy^8 + ay^8;$$

$$E_{14}: x^3y^8 + axy^6, Z_{11}: x^3y + y^5 + axy^5;$$

$$Z_{12}: x^3y + xy^4 + ax^2y^3, Z_{13}: x^3y + y^6 + axy^5;$$

$$W_{12}: x^4 + y^5 + ax^2y^3, W_{13}: x^4 + xy^4 + ax^2y^3;$$

$$Q_{10}: x^3 + y^4 + yz^2 + axy^3, Q_{11}: x^3 + y^2z + xz^3 + az^5;$$

$$Q_{12}: x^3 + y^5 + yz^2 + axy^4, S_{11}: x^4 + y^2z + xz^2 + axz^2;$$

$$S_{12}: x^3y + y^2z + xz^3 + ax^2z, U_{12}: x^3 + y^3 + z^4 + axyz^2.$$
So in this case the behavior of the functions \(|\nabla W(a)|^2 - C|\nabla dW(a)|\) at the infinity is much influenced by the irrelevant term and the method of Theorem 2.42 can’t be used. So in this case whether 0 is the discrete spectrum of the Schrödinger equations are not clear yet.

Therefore we reach our conclusion:

**Theorem 3.13.** The miniversal deformation of the simple singularities \(A_n, D_{n+1}, E_6, E_7, E_8\) and the unimodal singularities \(P_8, X_9, J_{10}\), and the deformation of \(T_{p,q,r}(a)\) are all strong deformations.

**3.1.4. Marginal deformation of nondegenerate quasihomogeneous polynomials.**

There is a particular interesting deformation of a non-degenerate quasihomogeneous polynomial with given type. Actually \(P_8, X_9, J_{10}\) are such deformations. Such deformation has a global \(\mathbb{C}\) action, hence the marginal deformation induces the complex deformation of the hypersurface in projective spaces.

Let us simply recall some basic results in the deformation theory of compact complex manifolds. Let \(X_0\) be a compact complex manifold. Then a smooth deformation of \(X_0\) is a fibration \(X \to (S, 0)\), where \(S\) is the deformation space and the fiber at 0 is just \(X_0\). Two deformation \(X \to (S, 0)\) and \((Y \to (T, 0))\) is called equivalent if there is a fiber-preserving biholomorphic map between \(X\) and \(Y\). If \((T, 0) \to (S, 0)\) is a holomorphic map, then we can have the induced deformation \(Y \to (T, 0)\) which is the pull-back of the fiber \(X \to (S, 0)\).

**Definition 3.14.** A deformation of \(X_0\) is called complete if any other deformation of \(X_0\) can be induced from it. If the inducing map is unique, we call the deformation universal. If only the derivative at the base point is unique it is called versal. Versal families are unique up to isomorphism.

A local deformation of complex structure in a small neighborhood \(U_i\) is given by a holomorphic vector field \(v_i\). The difference \(v_i - v_j\) in \(U_i \cap U_j\) defines a Čech cocycle \(\theta(v)\) with value in the sheaf \(\Theta_{X_0}\) of germs of holomorphic vector fields on the fiber \(X_0\). Therefore each deformation \(v\) of \(X_0\) gives a KS (Kodaira-Spencer) class by the KS map:

\[
\rho : T_{S,0} \to \theta(v) \in H^1(X_0, \Theta_{X_0}).
\]

We have the Kodaira’s completeness theorem (see for example, [Ka] or [CMP]).

**Theorem 3.15.**

1. A smooth family of compact complex manifolds with a surjective KS map is complete at the base point. In particular, if the KS map is a bijection, the family is versal.
2. If \(H^2(X, \Theta_X) = 0\). Then a versal deformation for \(X_0\) exists whose KS map is an isomorphism.
3. If moreover \(H^0(X, \Theta_X) = 0\), this versal deformation is universal.

However, the existence of the versal deformation did not known until Kuranish’s theorem [Kur] after generalizing the deformation notion to analytic space:

**Theorem 3.16.** For any compact complex manifold \(X_0\) there exists a versal deformation with a bijective KS map. If \(H^0(X_0, \Theta_{X_0}) = 0\), such a deformation can be chosen universal. The base \(S\) of the deformation is the zero locus of the Kuranishi map \(K : H^1(X_0, \Theta_{X_0}) \to H^2(X_0, \Theta_{X_0})\) which satisfies \(dK(0) = 0\).
In particular, if $K \equiv 0$ (for example, $H^2(X_0, \Theta_{X_0}) = 0$), then the Kuranishi space $S$ is smooth.

The deformation theory of Calabi-Yau manifolds are in particular interesting because its importance in string theory. The following theorem due to Tian \cite{Tian} says that there is no obstruction for CY manifold:

**Theorem 3.17.** If $X_0$ is a compact Kähler CY manifold, then the local universal deformation space of $X_0$ is isomorphic to an open set in $H^1(X_0, \Theta_{X_0})$.

Now if $X_0$ is a 3-dimensional compact Kähler CY manifold, then by serre duality, there is

$$H^2(X_0, \Theta_{X_0}) \cong H^1(X_0, \Omega^1(K_{X_0})) \cong H^1(X_0, \Omega^1) \neq 0.$$  

This says even the obstruction space $H^2(X_0, \Theta_{X_0})$ is not zero, the Kuranishi space is still smooth.

The deformation theory of a smooth hypersurface in projective space can also be described clearly.

Let $\mathbb{P}^{n-1}$ be the projective space with homogeneous coordinates $[z_1, \cdots, z_n]$. Let $f$ be a nondegenerate homogeneous polynomial with degree $d$, then $f$ defines a smooth hypersurface $X = \{f = 0\}$ in the projective space. Let $S^d$ be the parameter space for the tautological family of degree $d$ hypersurface in $\mathbb{P}^{n-1}$. So in general the deformation has the form:

$$f_t(z) = t_1 z_1^d + t_2 z_2^{d-1}z_2 + \cdots + t_n z_n^d. \quad (152)$$

The number $\mu$ of the monomial $z_1^{k_1} \cdots z_n^{k_n}$ of degree $k_1 + \cdots + k_n = d$ is

$$\mu = \binom{n - 1 + d}{d} \quad (153)$$

The tangent space at $X_0$ is

$$T = T_{X_0} \cong S^d / \mathbb{C}f,$$

On the other hand, the group $GL(n)$ acts on the homogeneous polynomials and the tangent space of the orbit space is $J_f^d / \mathbb{C}f$, where $J_f^d$ is the ideal generated by these polynomials $z_j \frac{\partial f}{\partial z_j}, i, j = 1, \cdots, n$. In fact, we can choose a one-parameter transformation $g_t: z_j \to z_j + tz_i$ and fix all other variables. Then

$$\frac{d}{dt} |_{t=0} (g_t \circ f) = z_i \cdot \frac{\partial f}{\partial z_j},$$

which shows that the tangent space of the $GL(n)$ orbit at $f$ is given by $J_f^d / \mathbb{C}f$. Therefore, we have the isomorphism:

$$T_0 S \cong S^d / J_f^d. \quad (154)$$

Let $\nu_{X_0/\mathbb{P}^{n-1}}$ be the normal bundle of $X_0$ in $\mathbb{P}^n$. Then the infinitesimal deformation of $X_0$ in $\mathbb{P}^{n-1}$ is classified by $H^0(X_0, \nu_{X_0/\mathbb{P}^{n-1}})$. For any $\nu \in T_{S,0}$, we can define a characteristic map $\sigma: T_{S,0} \to H^0(X_0, \nu_{X_0/\mathbb{P}^{n-1}})$ such that

$$T_0 S \cong S^d / J_f \cong H^0(\nu_{X_0/\mathbb{P}^{n-1}}). \quad (155)$$

Let

$$\delta: H^0(\nu_{X_0/\mathbb{P}^{n-1}}) \to H^1(\Theta_{X_0})$$

be the coboundary map defined by the exact sequence of sheaves:

$$0 \to \Theta_{X_0} \to \Theta_{\mathbb{P}^{n-1}|X_0} \to \nu_{X_0/\mathbb{P}^{n-1}} \to 0.$$  

Then it is easy to see the KS map $K$ is the combination map $\delta \circ \sigma$. The map $K$ sends the element $G \in S^d$ to $f + tG$. 

On the other hand, we have the exact sequence:

$$0 \to \Theta_{\mathbb{P}^{n-1}}(-d) \to \Theta_{\mathbb{P}^{n-1}} \to \Theta_{\mathbb{P}^{n-1}}|_{X_0} \to 0. \quad (156)$$

By Bott vanishing theorem, we have

$$H^1(\Theta_{\mathbb{P}^{n-1}}) \cong H^{n-2}(\Omega^1_{\mathbb{P}^{n-1}}(-n)) = 0,$$

and

$$H^2(\Theta_{\mathbb{P}^{n-1}}(-d)) = 0,$$

unless \( n = 4, d = 4 \). By exact sequence of the cohomology group, we have

$$H^1(\Theta_{\mathbb{P}^{n-1}}|_{X_0}) = 0,$$

unless \( n = 4, d = 4 \). This shows that the coboundary map \( \delta \) is surjective and so the KS map is surjective. Since the map

$$S^d/J_f \to H^1(X_0, \Theta_{X_0}) \quad (157)$$

is an isomorphism, by Kodaira completeness theorem this provides a versal family. Since the automorphism group \( H^0(X_0, \Theta_{X_0}) = 0 \), this is also a universal deformation. In summary, we have

**Theorem 3.18.** Assume that \( n \geq 4, d \geq 3 \). Except the case \( n = 4, d = 4 \), the universal deformation of a smooth hypersurface in \( \mathbb{P}^{n-1} \) with degree \( d \) is given by the isomorphism:

$$S^d/J_f \to H^1(X_0, \Theta_{X_0}). \quad (158)$$

The dimension of the moduli space \( S \) is

$$\mu = \left( \frac{n - 1 + d}{d} \right) - n^2. \quad (159)$$

**Example 3.19.** Consider the moduli space of elliptic curves, then the dimension is

$$\left( \frac{2 + 3}{3} \right) - 3^2 = 1.$$

**Example 3.20.** Consider the moduli space of the quintic polynomials in \( \mathbb{P}^4 \), then the dimension is

$$\left( \frac{4 + 5}{5} \right) - 5^2 = 101.$$

Our analysis of the moduli of the deformation space implies the following conclusion about the number of modules:

**Theorem 3.21** *(LG/CY correspondence between moduli numbers)*. Let \( f \) be a smooth hypersurface of degree \( d \) in the projective space \( \mathbb{P}^{n-1} \). If \( n \geq 4 \) and \( d \geq 3 \) but except the case \( n = 4, d = 4 \), then the dimension of its deformation space equals to the number of marginal deformations in the universal unfolding of the singularity \( f \).

**Proof.** We know that the number of moduli of the hypersurface of degree \( d \) in the projective space \( \mathbb{P}^{n-1} \) is

$$\mu = \left( \frac{n - 1 + d}{d} \right) - n^2. \quad (160)$$

On the other hand, as singularity germ \( f \) is holomorphic equivalent to its normal form, the homogeneous Fermat polynomials \( f_0 = z_1^d + \cdots + z_n^d \). Its Milnor algebra at 0 is generated by the polynomials \( z_{k_1} \cdots z_{k_n} \), \( 0 \leq k_i \leq d - 2, i = 1, \ldots, n \). So we need to compute the number of polynomials \( z_{k_1}^{k_1} \cdots z_{k_n}^{k_n} \) with \( k_1 + \cdots + k_n = d \) in the Milnor algebra. Imagine that we have
$d$ white balls and $n-1$ red balls in one line, and then in total we have $n-1+d$ balls in one line and order them from 1 to $n-1+d$. If we ignore the colors of the balls, then there are
\[
\binom{n-1+d}{n-1}
\]
ways to choose $n-1$ balls. This is equivalent to insert $n-1$ red balls into the $d$ white balls. However, we must rule out the choices with respect to the decomposition $d = d + 0$ and $d = (d - 1) + 1$. For $d = d + 0$, there are $n$ ways to insert the $d$ white balls together into the two neighboring red balls. For $d = (d - 1) + 1$, there are $n(n-1)$ ways to put $d - 1$ white balls together between two neighboring red balls and the rest one white ball to other gap. So we have
\[
\binom{n-1+d}{n-1} - n - n(n-1)
\]
marginal deformations in the universal unfolding of the singularity $z_1^d + \cdots + z_n^d$. This proves the conclusion.

**Remark 3.22.** For any $n, d$, the number of marginal deformations in the universal unfolding of $f = z_1^d + \cdots + z_n^d$ is given by (161). In case $n = 4, d = 4$ which corresponds to $K3$ surface in $\mathbb{P}^3$, the moduli of the complex structure is $H^1(X, \Theta_X) = 20$ (ref. [Ko], Page 247). The reason is that one dimensional deformation is non-algebraic (is transcendental). The number of the marginal deformations in the universal unfolding of $f$ is 19. The moduli of the quadratic surface has been given in some formula of [Ko] and the dimension of the Milnor number of the singularity $f$ is 1.

### 3.1.5. Deformation of nondegenerate and convenient Laurent polynomials.

S. Barannikov [Bar] has studied the Frobenius manifold associated to a special Laurent polynomial and compare it with the quantum cohomology of $\mathbb{C}P^n$. Later Douai and Sabbah ([Da], [DS1], [DS2], [Sa]) have studied the deformation theory for general nondegenerate and convenient Laurent polynomials defined on the algebraic torus $(\mathbb{C}^*)^n$. They can construct the Frobenius manifold structure based on this deformation. We will show the deformations they considered are actually strong deformation in our notation.

Let $f$ be a convenient and nondegenerate Laurent polynomial. The $\mathbb{C}$-vector space $Q_f = \mathbb{C}[z_1, \ldots, z_n, z_1^{-1}, \ldots, z_n^{-1}] / J_f$ is finite dimensional and its dimension $\mu(f)$ is the sum of the Milnor numbers of $f$ at each critical point. Let $\{g_i, i = 1, \ldots, r\}$ be the set of monomials such that their corresponding lattice points are contained in the interior of the Newton polyhedron of $f$ and injects to the Jacobi space $Q_f$. Then the deformation is
\[
F(z, t) = f(z) + \sum_{j=1}^r t_j g_j.
\]
We call it as the subdiagram deformation. It was shown in PP. 23 of [Sa] that for any $t = (t_1, \ldots, t_r) \in \mathbb{C}^r$ the Laurent polynomial $F(z, t)$ is convenient and nondegenerate. Therefore by Proposition 2.48 we have the following result.

**Theorem 3.23.** Let $f$ be a convenient and nondegenerate Laurent polynomial defined on the algebraic torus $(\mathbb{C}^*)^n$. Then the deformation $F(z, t)$ with base $\mathbb{C}^r$ defined by identity (162) is a strong deformation.
3.2. A priori estimate, existence and regularity. In this section, we always keep the following basic assumptions:

- \((M, g, f)\) is a strongly tame section-bundle system.
- \(\lambda_0\) is not a spectrum point of \(\Delta_f\), and the section \(\psi \in \text{Dom}(\Delta_f)\) satisfies the equation \((\lambda_0 - \Delta_f)\psi = \varphi\).

Since \((M, g, f)\) is supposed to be strongly tame, there exists a compact set \(K_{\lambda_0} \subset M\) such that for any \(z \notin K_{\lambda_0}\), the following inequality holds

\[
2\lambda_0 + \frac{1}{2} |\nabla f|^2 \leq 2|L(\nabla \varphi)| + |\nabla f|^2 \leq 2|\nabla f|^2. \tag{163}
\]

where \(L(\nabla \varphi) := g^{i\bar{j}} \nabla_i \varphi_{\bar{j}} \cdot dz^i \wedge \text{ and } L_f(\cdot) = L(\nabla \varphi) \cdot + L(\nabla \partial \varphi).

We have the global energy estimate.

**Lemma 3.24.** Under the basic assumption in this section, the following inequality holds

\[
\int_M |\tilde{\partial} \psi|^2 + |\bar{\partial} \psi|^2 + |\nabla f|^2 |\psi|^2 \leq C \int_M |\psi|^2, \tag{164}
\]

where \(C\) depends on \((M, g, f), \lambda_0\) and the distance of \(\lambda_0\) to the spectrum of \(\Delta_f\).

**Proof.** We have

\[
(\Delta_f \psi, \psi)_{L^2} = (\lambda_0 \psi, \psi)_{L^2} - (\varphi, \psi)_{L^2}.
\]

Since \(\Delta_f = \Delta_0 + L(\nabla \varphi) \circ + L(\nabla \varphi) \circ + |\nabla f|^2\), we have

\[
\int_M |\tilde{\partial} \psi|^2 + |\bar{\partial} \psi|^2 + |\nabla f|^2 |\psi|^2 + \int_M 2\text{Re}(L(\nabla \varphi) \circ \psi, \psi) \leq (\lambda_0 + 1) ||\psi||_{L^2} + ||\varphi||^2_{L^2}.
\]

Notice that \((163)\) holds for \(z \in M - K_{\lambda_0}\), so plus the interior integration over \(K_{\lambda_0}\) we have the estimate:

\[
\int_M |\tilde{\partial} \psi|^2 + |\bar{\partial} \psi|^2 + \frac{1}{2} |\nabla f|^2 |\psi|^2 \leq (\lambda_0 + 2)||R_{K_{\lambda_0}}(\Delta_f) \psi||_{L^2} + ||\varphi||_{L^2} \leq C||\psi||^2_{L^2(M)}.
\]

So we proved the conclusion. \(\square\)

The following lemma gives the local energy estimate near the infinite far place:

**Lemma 3.25.** Suppose that the basic assumption in this section holds. Then for any ball \(B_R(z_0) \cap K_{\lambda_0} = \emptyset\), we have

\[
\int_{B_R(z_0)} |\nabla \psi|^2 + |\nabla f|^2 |\psi|^2 \leq C \int_{B_R(z_0)} |\psi|^2 + |\varphi|^2, \tag{165}
\]

where \(C\) only depends on the geometry of \((M, g, f)\) and \(\lambda_0\).

**Proof.** Let \(\chi(z)\) be a smooth cut-off function with support in \(B_R(z_0)\) and equals 1 on \(B_{\frac{3}{2}}(z_0)\). By the equation of \(\psi\), we obtain

\[
\int_M \lambda_0(\psi, \chi^2 \psi) - (\Delta_f \psi, \chi^2 \psi) = \int_M (\varphi, \psi) \chi^2.
\]

Note that \(\Delta_f = \Delta_0 + L(\nabla \varphi) + L(\nabla \varphi) + |\nabla f|^2\), replacing it into the above inequality, there is

\[
\int_{B_R} \chi^2 |\nabla \psi|^2 + |\nabla f|^2 |\psi|^2 + 2\text{Re}(L(\nabla \varphi) \circ \psi, \psi) \chi^2 - \lambda_0 \chi^2 |\psi|^2 = - \int_{B_R} (\varphi, \psi) \chi^2 - \int_{B_R} (\nabla \psi, 2\chi \nabla \chi \psi).
\]
So we have
\[ \int_0^{R^2} |\nabla^2 \psi|^2 + |(\nabla f)^2 - 2|L(\nabla \partial f)| - \lambda_0)|\psi|^2 \leq \int |\varphi||\nabla^2 \chi + |\nabla \chi| |\psi|. \]

By Cauchy inequality, there is
\[ \int_{B_R} |\nabla^2 \psi|^2 + |(\nabla f)^2 - 2|L(\nabla \partial f)| - \lambda_0 - \epsilon)|\psi|^2 \leq C_\epsilon \int_{B_R} (|\varphi|^2 + |\nabla \chi|^2 |\psi|^2). \]

So we get the conclusion.

The following lemma gives the weak maximum principle near the infinite far place:

**Lemma 3.26.** Suppose that the basic assumption in this section holds. If \( B_R(z_0) \cap K_{J_0} = \emptyset \), then
\[ \sup_{B_{\frac{R}{2}}} |\psi(z)| \leq C(|\varphi|_{L^2} + |\varphi|_{L^2(M)}), \]

where \( C \) only depends on \((M, g, f)\) and \( \lambda_0 \).

**Proof.** We have the identity
\[ \Delta |\psi|^2 = \Delta \psi \cdot \psi + (\psi, \Delta \psi) - 2|\nabla \psi|^2 \]
\[ = \psi \Delta |\psi|^2 - 2|\nabla |\psi|^2| - 4Re(L_f \circ (\psi, \psi) + 2\lambda_0 |\psi|^2) - 2Re(\psi, \psi), \]

Hence if the point \( z \not\in K_{J_0} \), we have
\[ \Delta |\psi|^2 + 2|\nabla |\psi|^2| \leq 2|\varphi||\psi|, \]

and then in weak sense that
\[ \Delta |\psi| \leq |\varphi|. \]

By weak maximum principle (see [HL, GT]), if \( B_R(z_0) \cap K_{J_0} = \emptyset \), then
\[ \sup_{B_{\frac{R}{2}}} |\psi(z)| \leq C(|\varphi|_{L^2} + |\varphi|_{L^2(M)}). \]

**Corollary 3.27.** Suppose that the basic assumption in this section holds. Then as \( d(z, z_0) \to \infty \),
\[ |\psi(z)| \to 0. \]

and the following inequality holds:
\[ \sup_{M - K_{J_0}} |\psi(z)| \leq C|\varphi|_{L^2(M)}, \]

where \( C \) only depends on \((M, g, f)\) and \( \lambda_0 \).

**Proof.** Since \( \psi, \varphi \in L^2(M) \), for any \( \epsilon > 0 \), there exists a \( R_0 > 0 \) such that
\[ |\varphi|_{L^2(M_{R_0})} + |\varphi|_{L^2(M_{R_0})} \leq \epsilon, \]

where \( M_{R_0} := \{ z \in M | d(z, z_0) > R_0 \} \). By Lemma 3.26 there is
\[ \sup_{B_{\frac{R}{2}}} |\psi(z)| \leq C\epsilon. \]

This shows that \( |\psi(z)| \to 0 \) and the second conclusion holds naturally.
Now we want to estimate the higher order derivatives of $\psi$. We define a subspace in $L^2(M)$ space, $W^{k,2}(M)$, whose norm is given by

$$
\|\varphi\|_{W^{k,2}(M)} = \left(\sum_{l,j\leq k} \int_M |D_l^j \varphi|^2 dV_M\right)^{1/2}.
$$

(168)

Here the first order operator $D$ represents $\bar{\partial}_f$ or $\bar{\partial}_f^j$. We know that if $(M, g, f)$ is strongly tame and $k = 1$, then the norm $\|\cdots\|_{W^{2,2}(M)}$ is equivalent to the following norm:

$$
\left(\int_M |\nabla \varphi|^2 + (|\nabla f|^2 + 1) dV_M\right)^{1/2}.
$$

Theorem 3.28. Suppose that the basic assumption in this section holds. Furthermore assume that $\varphi \in W^{k,2}_f(B_R(0))$, where $B_R(0) \cap K_0 = \emptyset$. Then there exists a constant $C_k$ depending only on the geometry of $(M, g, f)$ and $\lambda_0$, $k$ such that

$$
\sum_{l,j\leq k} \int_{B_R(0)} |\nabla D_l^j \varphi|^2 \leq C_k \left(\|\varphi\|_{W^{k+2,2}(B_R(0))}^2 + \|\psi\|_{L^2(B_R(0))}^2\right).
$$

(169)

Proof. We consider $k = 1$ case in detail. Notice that

$$(\lambda_0 - \Delta_f) \bar{\partial}_f \psi = \bar{\partial}_f \varphi, \quad (\lambda_0 - \Delta_f) \bar{\partial}_f^j \psi = \bar{\partial}_f^j \varphi.
$$

Replacing $\bar{\partial}_f \psi$ and $\bar{\partial}_f^j \varphi$ into the inequality in Lemma 3.25 respectively, we get two inequalities and then sum them to get

$$
\int_{B_R(0)} |\nabla \bar{\partial}_f \psi|^2 + |\nabla \bar{\partial}_f^j \psi|^2 + |\nabla f|^2 (|\bar{\partial}_f \psi|^2 + |\bar{\partial}_f^j \psi|^2) \leq C \int_{B_{2R}} \left(\|\varphi\|_{W^{k+2,2}(B_R(0))}^2 + \|\varphi\|_{L^2(B_R(0))}^2\right).
$$

Applying the strongly tame condition to the left hand side and applying Lemma 3.25 to the right hand side, we obtain

$$
\int_{B_R(0)} |\nabla \bar{\partial}_f \psi|^2 + |\nabla \bar{\partial}_f^j \psi|^2 + |\nabla f|^2 (|\bar{\partial}_f \psi|^2 + |\bar{\partial}_f^j \psi|^2) \leq C \left(\|\varphi\|_{W^{k+2,2}(B_R(0))}^2 + \|\varphi\|_{L^2(B_R(0))}^2\right).
$$

(170)

$k \geq 2$ cases can be obtained by recursion method.

Now we can get the uniform estimate for the derivatives of $\psi$:

Theorem 3.29. Suppose that the basic assumption in this section holds and assume that $\varphi \in W^{k+2}_f(M)$. If $B_R(0) \cap K_0 = \emptyset$, then there exists a constant $C_k$ such that for any $l, |l| \leq k + 1$ the following holds:

$$
\sup_{B_R^2(0)} |D^l \psi| \leq C_k \left(\|\varphi\|_{W^{k+2,2}(B_R(0))} + \|\varphi\|_{L^2(B_R(0))}\right).
$$

(171)

where $C$ only depends on the geometry of $(M, g, f)$ and $\lambda_0$, $k$.

Proof. $D^l \psi$ satisfies

$$(\lambda_0 - \Delta_f) D^l \psi = D^l \varphi.
$$

By Lemma 3.26, we have

$$
\sup_{B_R^2(0)} |D^l \psi| \leq C_k \left(||D^l \varphi||_{L^2(B_{2R}(0))} + ||D^l \varphi||_{L^2(B_{2R}(0))}\right).
$$
By Theorem 3.28 and the tameness of \((M, g, f)\), we have the estimate
\[
\|D'\psi\|_{L^2(\Omega_{\epsilon_0})}^2 \leq C \int |\tilde{\partial}_f D^{-1}\psi|^2 + |\tilde{\partial}_f^\dagger D^{-1}\psi|^2 \\
\leq C \int_{\Omega_{\epsilon_0}} |\nabla D^{-1}\psi|^2 + |D^{-1}\psi|\nu_f|^2 \\
\leq C_{k-1} \left( \|\psi\|_{W^{k}^2(\Omega_{\epsilon_0})}^2 + \|\psi\|_{L^2(\Omega_{\epsilon_0})}^2 \right).
\]
Therefore, we have the estimate
\[
\sup_{B_{\epsilon_0}^\dagger(z_0)} |D'\psi| \leq C_k \left( \|\psi\|_{W^{k}^2(\Omega_{\epsilon_0})} + \|\psi\|_{L^2(\Omega_{\epsilon_0})} \right). \tag{172}
\]

Corollary 3.30. Suppose that the basic assumption in this section holds and assume that \(\varphi \in W^{k,2}_f(M)\). Then for any \(l, |l| \leq k\),
\[
|D'\psi(z)| \to 0, \text{ as } |z| \to \infty.
\]
Furthermore, we have the estimate
\[
\sup_{M-K_{\lambda_0}} |D'\psi(z)| \leq C_k \|\varphi\|_{W^{k,2}_f(M)}, \tag{173}
\]
where \(C_k\) only depends on the geometry of \((M, g, f)\) and \(\lambda_0, k\).

3.2.1. Equivalence of the norms. Here we will discuss the relations between \(D'\) and \(\nabla\).

We have four basic operators \(\tilde{\partial}, \tilde{\partial}, \partial f \wedge, (\partial f \wedge)^\dagger\). They satisfy the following commutation relations
\[
[\tilde{\partial}^i, \partial f \wedge] = \gamma^{\mu
u} \nabla_{\nu} f_{t_0} d\epsilon^\mu \wedge = f^{\mu}_s t_{0, s} \mu \wedge \\
[\tilde{\partial}, (\partial f \wedge)^\dagger] = \gamma^{\mu\nu} d\epsilon^\mu \wedge, \\
[\tilde{\partial}, \partial f \wedge] = 0, [\tilde{\partial}^i, (\partial f \wedge)^\dagger] = 0.
\]
The first order differential operators of \(D'\) have two which have the formulas:
\[
\tilde{\partial}_f = \tilde{\partial} + \partial f \wedge = \tilde{\partial} + f_a d\epsilon^a \wedge, \tilde{\partial}^i_f = \tilde{\partial}^i + f_a t_{0, a}.
\]
The second order operators have two which have the formulas:
\[
\tilde{\partial}_f \tilde{\partial}_f = \tilde{\partial}^i \tilde{\partial} + f^{\mu}_a t_{0, a} \mu - f_a d\epsilon^a \wedge \tilde{\partial}^i + f^{\mu}_a t_{0, a} \mu \wedge + f_t a t_{0, a} \cdot (f_a d\epsilon^a) \\
\tilde{\partial}^i \tilde{\partial}_f = \tilde{\partial}^i \tilde{\partial} + f_a d\epsilon^a \wedge \tilde{\partial}^i - f_a t_{0, a} \mu + f^{\mu}_a t_{0, a} \mu \wedge + f_a f^s_a d\epsilon^a \wedge t_{0, a}.
\]
The higher order differential operators have the four types:
\[
\Delta_f, \Delta^i_f, \tilde{\partial}^i \tilde{\partial} f, \tilde{\partial}^i f, \Delta^i_f, \Gamma^i_f.
\]
Here
\[
\Delta_f = \Delta_{\tilde{\partial}} + f^{\mu}_a t_{0, a} \mu \wedge + f_t a t_{0, a} \mu \wedge + |\nabla f|^2.
\]
Each operator then has the types \(D_1, D_2, D_1 \cdots D_1\) or \(D_1 D_2 D_1 \cdots D_2\), where \(D_1\) can represent \(\tilde{\partial}_f\) or \(\tilde{\partial}^i_f\) and \(D_2\) then represents the rest one. We need move all the terms \(f_a d\epsilon^a \wedge\) or \(f^s a t_{0, a}\) from the sequence to the left hand side. When commuting with \(\tilde{\partial}\) or \(\tilde{\partial}^i\), a higher order derivatives of \(f\) will generate. Hence finally, all the terms in \(D_1 D_2 D_1 \cdots D_1\) have the form:
\[
c_{ij} \nabla^k f \nabla^{j+1}, 0 \leq k \leq s,
\]
where $s$ is the length of $D_1D_2D_3\cdots D_l$ and $\nabla^k = \nabla^{i_1}\cdots \nabla^{i_k}, i_k \in \{1, \cdots , n, \bar{1}, \cdots , \bar{n}\}$.

If the length is $2s$, then the highest order derivatives is permutation of $\partial, \bar{\partial}$’s and the 0-th order term is

$$|\nabla f|^{2s} - \sum_{|I| = 2s} C_I \nabla^I f : L,$$

where $L$ is a multiplication operator preserving the degree of the forms.

**Definition 3.31.** Let $(M, g, f)$ be a strongly tame section-bundle system. If for any $s = 1, 2, \cdots$, and for any $C > 0$, the following relations hold:

$$|\nabla f|^{2s} - C \sum_{|I| = 2s} |\nabla^I f| \to \infty, \text{ as } d(z, z_0) \to \infty, \quad (174)$$

then $f$ is said to be strongly regular tame and $(M, g, f)$ is said to be a strongly regular tame section-bundle system.

**Example 3.32.** Let $W$ be a nondegenerate quasi-homogeneous polynomial, then $(\mathbb{C}^N, W)$ with the standard Kähler metric is a strongly regular tame section-bundle system.

**Example 3.33.** Let $f$ be a nondegenerate and convenient Laurent polynomial defined on the algebraic torus $T$, then $(T, f)$ with the standard Kähler metric is a strongly regular tame section-bundle system.

**Proposition 3.34.** Let $(M, g, f)$ be a strongly regular tame section-bundle system. Then the norm $\| \cdot \|_{W^{2s}}$ is equivalent to the following norm

$$\left( \| \cdot \|_{W^{2s}}^2 + \| (\nabla f + 1) \cdot \nabla^I u \|_{L^2}^2 \right)^{1/2}.$$

**Proof.** Apply pointwise interpolation theorem to derivatives with middle order. □

**3.2.2. Existence and regularity.** We know that if $\lambda_0$ is not a spectrum point of $\Delta_f$, then $(\lambda_0 - \Delta_f)\psi = \varphi$ has a unique solution in $W^{1,2}_f(M)$ if $\varphi \in L^2(M)$. This is a weak solution of the Schrödinger equation. The existence of the weak solution is equivalent to the application of the Lax-Milgram theorem for quadratic form which is coercive and has positive lower bound. If $\varphi \in W^{1,2}_f(M)$, then $\psi \in W^{k+1,2}_f(M)$. We can define a function class $\mathcal{S}_f$ consisting of the smooth function $u$ such that for any $I, J, |I|, |J| = 0, 1, \cdots$, the following holds:

$$\sup_M |\nabla^I f| \nabla^J u | < \infty.$$

If $f$ is a polynomial, then $\mathcal{S} \subset \mathcal{S}_f$, the function space of rapid decrease. In particular, the function space of compact support $C_0(M) \subset \mathcal{S}_f$ for any holomorphic function $f$.

We have the existence and regularity theorem:

**Theorem 3.35.** Let $(M, g, f)$ be a strongly tame section-bundle system and assume that $\lambda_0$ is not a spectrum point of $\Delta_f$. If $\varphi \in \mathcal{S}_f$, then the equation

$$(\lambda_0 - \Delta_f)\psi = \varphi$$

has a unique solution in $W^{k,2}_f(M)$ for any $k$. 

3.3. Continuity of the spectrum. Above all, we cite some conclusions about unbounded self-adjoint operators in functional analysis. The reader can refer to Theorem VIII 20, Theorem 23 and Theorem 25 of the book [RS] to find the proof of Proposition 3.37.

**Definition 3.36.** Let \( A_n, n = 1, \ldots, \) and \( A \) be self-adjoint operators. Then \( A_n \) is said to converge to \( A \) in the norm resolvent sense if their resolvent \( R(A_n) \rightarrow R(A) \) in norm for all \( \lambda \) with \( \text{im} \lambda \neq 0 \). \( A_n \rightarrow A \) is said to converge in strong resolvent sense, if the resolvent \( R(A_n) \rightarrow R(A) \) strongly for all \( \lambda \) with \( \text{im} \lambda \neq 0 \).

Here we only use the norm resolvent convergence.

**Proposition 3.37.** The following conclusions hold:

1. Let \( \{A_n\}_{n=1}^{\infty} \) and \( A \) be self-adjoint operators with a common domain \( D \) and norm \( \| \cdot \|_D \) with \( \|A\varphi\| + \|\varphi\| \). If

\[
\sup_{\|\varphi\|=1} \|A_n\varphi - A\varphi\| \rightarrow 0,
\]

then \( A_n \rightarrow A \) in the norm resolvent sense.

2. If \( A_n \rightarrow A \) in the norm resolvent sense and \( f \) is a continuous function on \( \mathbb{R} \) vanishing at \( \infty \), then \( \|f(A_n) - f(A)\| \rightarrow 0 \).

3. Let the interval \( I = [a, b] \subset \mathbb{R} \) has no intersection with the spectrum \( \sigma(A) \) of \( A \), then for large \( n \) the projection \( P_I(A_n) \) is well-defined and satisfies

\[
\|P_I(A_n) - P_I(A)\| \rightarrow 0.
\]

Consider the strong deformation with the form

\[
f_t = f + \sum_i t_i g_i.
\]

**Lemma 3.38.** Let \( \varphi \in \text{Dom}(\Delta_0) \) and \( \|\varphi\|_{L^2} = 1 \), then

\[
\|\Delta \varphi - \Delta_0 \varphi\|_{L^2} \rightarrow 0.
\]

**Proof.** Since \( \Delta_t - \Delta_0 \) is a symmetric operator, it suffices to prove that

\[
|(\Delta_t - \Delta_0)\varphi, \varphi| \rightarrow 0,
\]

for any \( \varphi \) such that \( \|\varphi\|_{L^2} < \infty \). We have

\[
|\langle \Delta_t - \Delta_0 \varphi, \varphi \rangle| = |\langle L(\nabla g_t)(\varphi), \varphi(t) \rangle + |t|^2\|\nabla g_t\|^2\|\varphi\|^2 + 2t\text{Re}(\nabla f \varphi, \nabla g_t)\varphi| \\
\leq |t|\|\nabla f\varphi\|^2 \leq |t|\|\varphi\|_{L^2}.
\]

Thus

\[
\|\Delta_t \varphi - \Delta_0 \varphi\|_{L^2} \rightarrow 0.
\]

\(\square\)

Therefore by 3.38 and Proposition 3.37 we have the corollary:

**Corollary 3.39.** Let the interval \( I = [a, b] \subset \mathbb{R} \) has no intersection with the spectrum \( \sigma(\Delta_f) \) of \( \Delta_f \), then there exists a constant \( \delta > 0 \), if \( |t| < \delta \), then

\[
\dim \text{im} P_I(\Delta_t) = \dim \text{im} P_I(\Delta_0).
\]

By Theorem 2.2 we can list all the eigenvalues of \( \Delta_t \) in the following order:

\[
0 = \lambda_1(t) \leq \lambda_2(t) \leq \cdots \lambda_k(t) \leq \cdots \rightarrow \infty.
\]

We have the continuity theorem of eigenvalues:

**Theorem 3.40.** \( \lambda_k(t) \) is a continuous function for \( t \in S \).
Proof. This theorem can be proved using the induction method with respect to \( k \). The only required fact is Lemma 3.38. The reader can refer to Theorem 7.2 of \[Ko\] for a description of the proof. \( \square \)

Let \( P_{0,t} \) be the projection operator from the \( L^2 \) space to the space of \( \Delta_f \)-harmonic forms. By using Lemma 3.38, we can easily prove the following result:

**Theorem 3.41.** \( \dim P_{0,t} \) is uppersemicontinuous in \( t \in S \).

3.4. **Estimate of the eigenforms and the Green function.** We use the maximum principle of the scalar Laplace operator to build the decay estimate of the eigenforms of \( \Delta_f \).

Consider the fundamental solution of the following linear scalar equation:

\[
(\Delta + a^2)E(z) = \delta(z),
\]

where \( k \) is a given nonzero constant.

**Lemma 3.42.** Let \((M, g)\) be a \( n \)-dimensional complete non-compact Riemannian manifold with bounded geometry. Then the fundamental solution \( E(x) \) of the operator exists and is unique. It has the approximating estimate as \( d(x, x_0) \rightarrow \infty \):

\[
E_a(x) = c(d(x, x_0)^{(n-1)/2})e^{-ad(x, x_0)}(1 + o(1)).
\]

Here \( d(x, x_0) \) is the distance function from the point \( x \) to \( x_0 \).

**Proof.** If \( M \) is the standard Euclidean space, then the function \((d(x, x_0))^{(n-1)/2}e^{-ad(x, x_0)}\) is the fundamental solution of the equation (175). If \( M \) is the complete Riemannian manifold with bounded geometry, the conclusion is obtained by using comparison principle. \( \square \)

**Theorem 3.43.** Let \((M, g, f)\) be a strongly tame section-bundle system and \( \varphi \) is an eigenform of \( \Delta_f \) corresponding to the eigenvalue \( \lambda \). Then there exists a constant \( C \), for any \( a > 0 \) there is:

\[
|\varphi| \leq Ce^{-ad(z_0)},
\]

where \( z_0 \) is an arbitrarily given base point on \( M \).

**Proof.** Since \( \varphi \) is an eigenform of the self-adjoint operator \( \Delta_f \), it has \( W^{2,2}_{loc} \) smoothness. By \( L^p \) and Schauder theory of elliptic operators, \( \varphi \) is \( C^\infty \).

The Laplace operator on \( M \) is given by

\[
\Delta = -g^{\mu\nu}\nabla_\mu \nabla_\nu.
\]

Let \( \varphi \) be a eigenform of \( \Delta_f \) with eigenvalue \( \lambda \), i.e.,

\[
\lambda \varphi = \Delta_f \varphi = \Delta \varphi + L_f \circ (\varphi) + |\nabla f|^2 \varphi.
\]

Then We have

\[
\Delta|\varphi|^2 = (\Delta \varphi, \varphi) + (\varphi, \Delta \varphi) - 2|\nabla \varphi|^2
= (2\lambda - |\nabla f|^2)|\varphi|^2 - 2|\nabla \varphi|^2 + (-|\nabla f|^2)|\varphi|^2 - 2Re(L_f \circ (\varphi), \varphi)),
\]

or for any \( a > 0 \) there is

\[
(\lambda + a^2)|\varphi|^2 = (a^2 + 2\lambda - |\nabla f|^2)|\varphi|^2 - 2|\nabla \varphi|^2 + (-|\nabla f|^2)|\varphi|^2 - 2Re(L_f \circ (\varphi), \varphi)).
\]

There exists a constant \( R \) depending only on \( M, a \) and the geometry of the section bundle system \((M, g, f)\) such that outside the ball \( B_R(z_0) \) the following inequality holds

\[
(\lambda + a^2)(|\varphi|^2 - ME_0(z)) \leq 0.
\]
Now we can choose \( M \) large enough such that on \( \partial B_R(z_0) \), there is

\[ |\varphi|^2(z) - ME_a(z) \leq 0. \]

Using the maximum principle in Proposition 3.45, we have the following estimate.

There exists a constant \( C \) which is not a spectrum point of \( M \) and \( B_R(z_0) \) such that

\[ |\varphi| \leq C_0 e^{-ad(z,z_0)}. \]

By the continuity theorem of the spectrum, Theorem 3.40, we have:

\[ \lambda_k(t) \leq \lambda_0 + \rho(t). \]

**Corollary 3.44.** Suppose that \((M, g, f)\) is a strong deformation of the section-bundle system \((M, g, f)\) in parameter space \( S \). Then for any eigenform \( \varphi(t) \) of \( \lambda(t) \), there exists a constant \( C_0 \) only depending on \((M, g, f)\) and \( \lambda_0 \) such that

\[ |\varphi| \leq C_0 e^{-ad(z,z_0)}. \]

3.4.1. **Estimate of the higher order derivatives.**

Let \((\varphi, \lambda)\) be a solution of the eigenvalue problem \((\lambda - \Delta_f)\varphi = 0\). Choose a point \( \lambda_0 \in \mathbb{R} \) which is not a spectrum point of \( \Delta_f \). Then we have

\[ (\lambda - \Delta_f)\varphi = (\lambda - \lambda_0)\varphi. \]  

By bootstrap argument, we know that \( \varphi \in W^{k,2}_f(M) \) for any \( k \). By Theorem 3.28,3.29 we have the following estimate.

**Proposition 3.45.** Let \( K_{a,b} \) be the domain in Lemma 3.23 and \( B_0(z) \cap K_{a,b} = \emptyset \). Then there exists a constant \( C_k \) depending only on the geometry of \((M, g, f)\) and \( \lambda_0, \lambda, k \) and any \( a > 0 \) such that

\[ \sum_{j+|I|=k} \int_{B_0(z_0)} |\nabla D^I_\varphi|^2 |\nabla f|^2 \leq C_k e^{-ad(z,z_0) - R}, \]

and the pointwise estimate:

\[ |D^I_\varphi(z)| \leq C_k e^{-ad(z,z_0)}. \]

**Proof.** Since for any \( I, |I| = 0, 1, \cdots, D^I_\varphi \) is a solution of the equation (178), we can use the local energy estimate, Lemma 3.23 to get the control of the local \( W^{k,2}_f \) norm by the local \( L^2 \) norm of \( \varphi \). Replacing this estimate into Theorem 3.28,3.29 we get the control by the local \( L^2 \) norm. Finally we use Theorem 3.43 estimate to get the decay estimate.

Similarly, we have the uniform decay estimate for the strong deformation.

**Corollary 3.46.** Suppose that \((M, g, f)\) is a strong deformation of the section-bundle system \((M, g, f)\) in parameter space \( S \). Then for any eigenform \( \lambda(t) \) of \( \lambda(t) \), there exists a constant \( C_k \) only depending on \((M, g, f)\), \( \lambda_0 \) and any \( a > 0 \) such that the following estimate hold:

\[ \sum_{j+|I|=k} \int_{B_0(z_0)} |\nabla D^I_\varphi|^2 |\nabla f|^2 \leq C_k e^{-ad(z,z_0) - R} \]

\[ |D^I_\varphi(t, z)| \leq C_k e^{-ad(z,z_0)}, \text{ for any } I, |I| \leq k. \]

3.4.2. **Estimate of the Green function.** Let \( G(z, w) \) be the Green function of \( \Delta_f \), i.e, it satisfies

\[ (\Delta_f)G(z, w) = \delta(z - w). \]

In the domain \( \{z, w\} \in M \times M | dist(z, w) \geq 1 \) \( G(z, w) \) is a harmonic form and so has the following decay estimate:
Proposition 3.47. For any \((z, w) \in M \times M\) such that the distance \(d(z, w)\) is large enough, there exists a constant \(C_0\) only depending on \((M, g, f)\), \(\lambda_0\) and any \(a > 0\) such that the following estimate hold:

\[
|D_l G(z, w)| \leq C_k e^{-a d(z, w)}, \quad \text{for any } |l| \leq k.
\]

In particular, if \(f\) is strongly regular tame, then

\[
|\nabla^2_l G(z, w)| + |\nabla f(z)|^{2|\ell|} |G(z, w)| \leq C_k e^{-a d(z, w)}, \quad \text{for any } |l| \leq k.
\]

The diagonal \(\mathcal{D} \subset M \times M\) is the singular set of \(G(z, w)\). The asymptotic property of \(G(z, w)\) as \(d(z, w) \to 0\) is given by the following result

Proposition 3.48. As \(z \to w\), there holds

\[
G(z, w) \to \begin{cases} c_0 d(z, w)^{-(2n-2)}, & n \geq 2 \\
c \log(d(z, w)), & n = 1. \end{cases}
\]  

(181)

Proof. By the work of Malgrange [Ma] and Li-Tam [LT], there exists a bounded symmetric Green’s function \(E_0(x, y)\) on any complete Riemannian manifold, i.e., \(E_0(x, y)\) satisfies:

\begin{itemize}
  \item \(E_0(x, y) = E_0(y, x)\);
  \item \((\Delta_0) E_0(x, y) = \delta(x - y),\) \((\Delta_0) E_0(y, x) = \delta(y - x).\)
  \item \(E_0(x, y)\) is bounded in any domain in \(M \times M\) which has positive distance to the diagonal \(\mathcal{D}\).
\end{itemize}

Here \(\Delta_0 = -\frac{1}{\sqrt{g}} \sqrt{g} \frac{\partial}{\partial x} (\sqrt{g} g^{ij} \frac{\partial}{\partial x_j})\) is the scalar Laplacian operator.

Since our operators are all real, we will use real coordinate in the following proof and denote the points \(z, w\) by \(x, y\). Notice that the real dimension of \(M\) is \(2n\).

Now the twisted Laplacian of \(k\)-form has the expression:

\[
\Delta_f = \Delta_0 + L_f + |
\nabla f|^2 = \Delta_0 + (R + L_f + |
\nabla f|^2).
\]

Define

\[
G_0(x, y) = \sum_{l, l_0 + |l| = k} E_0(x, y) dz^l \wedge d\zeta^l,
\]

Then the action of the scalar Laplacian is

\[
\Delta_0 G_0(x, y) = \delta(x - y),
\]

where \(\delta(x - y)\) is viewed as a vector.

Define \(F = (R + L_f + |
\nabla f|^2).\) Then the Green function \(G(x, y)\) of \(\Delta_f\) satisfies

\[
(\Delta_0 + F) G(x, y) = \delta(x - y).
\]

Assume that \(G(x, y) = G_0(x, y) + R_0(x, y)\), then we get the equation of \(R_0(x, y)\):

\[
\Delta_f R_0(x, y) = -F \circ G_0(x, y).
\]

(182)

Since the injective radius has positive lower bound \(\rho_0\), we can take normal coordinates system around \(y\). Locally we can take the Taylor expansion of \(F\):

\[
F(x, y) = \sum_{j=1}^{2n-2} \frac{F^{(j)}(y)}{j!} (x - y)^j + F_{2n-1}(x, y).
\]

We also write \(R_0(x, y)\) as a linear combination of \(2n - 1\) unknown vector-valued functions:

\[
R_0(x, y) = \sum_{j=1}^{2n-2} R_j(x, y) + R_{2n-1}(x, y).
\]
Consider the $2n - 2$ Dirichlet boundary value problems:

$$\begin{cases}
\Delta_j R_j(x, y) = -\frac{F^{(1)}(y)}{y} (x - y)^j G_0(x, y) \\
R_j(x, y) = 0, \text{ on } \partial B_{\rho_0}(y).
\end{cases} \tag{183}$$

The key point here is that $(x - y)^j G_0(x, y)$ is at least $L^{1+\epsilon}$ integrable for some $\epsilon > 0$. In fact, we have

$$(x - y)^j G_0(x, y) \in L^q,$$

where

$$\begin{cases}
1 < q < \frac{2n}{2n - 2 - j}, & n > 1 \\
1 < q < \infty, & n = 1.
\end{cases}$$

By $L^p$ estimate, we know that $R_j(x, y) \in W^{2,q}$, $j = 1, \cdots, 2n - 2$, whose singularity is weaker than $G_0(x, y)$. The rest equation is

$$\Delta_j R_{2n-1}(x, y) = F_{2n-1} G_0(x, y), \tag{184}$$

where the right hand side is a continuous function. So $R_j(2n - 1)$ is at least $C^1$ differentiable. Hence we obtain the decomposition of $G(x, y)$:

$$G(x, y) = G_0(x, y) + R_1(x, y) + \cdots + R_{2n-2}(x, y) + R_{2n-1}(x, y), \tag{185}$$

where the highest singularity comes from $G_0(x, y)$. So we proved our conclusion. □

3.5. Stability. Let $\Delta_t$ be a strong deformation of $\Delta_0$ for $t \in S$. In this part, we want to construct the differentiability of the Projection operator $P_t(\Delta_t)$ and the Green operator $G_t$.

**Definition 3.49.** Let $\mathcal{L}_t$ be a family of linear operators acting on $L^2(\Lambda^*(M))$. If for any section $\psi_t(z) := \psi(t, z)$ and any space derivatives $\partial^l |l| = 0, 1, \cdots$, the section $\partial^l \psi_t(z)$ is $C^r$ differentiable with respect to the variables $(t, z)$, then the section $\psi_t(z)$ is said to be $C^r$ differentiable. If $\psi_t$ is $C^r$ differentiable section with compact support in $z$ direction for each $t$ and the section $\mathcal{L}_t \psi_t$ is $C^r$ differentiable with respect to $t, z$, then $\mathcal{L}_t$ is called $C^r$ differentiable.

Since $S$ is assumed to be a compact neighborhood around $t = 0$, we can choose $\lambda_0$ belonging to the regular point set of any $\Delta_t$ such that $R_{\lambda_0}(\Delta_t) = (\lambda_0 - \Delta_t)^{-1}$ exists. Therefore, the differentiability of $\Delta_t$ is equivalent to the differentiability of $(\lambda_0 - \Delta_t)$. We have the estimate:

$$||R_{\lambda_0}(\Delta_t)\varphi||_{L^2(M)} \leq C||\varphi||_{L^2(M)}, \forall \varphi \in L^2. \tag{186}$$

We need the following computation:

**Lemma 3.50.** Let $f_t = f + \sum t_i g_i$. Then we have

$$\begin{align*}
\partial_t \Delta_t &= L(\nabla g_i) \circ + \nabla f \cdot \nabla g_i + t_i |\nabla g_i|^2 \\
\partial_t \partial_t \Delta_t &= \delta_{ij} |\nabla g_i|^2 \\
\partial_t \partial_t \Delta_t &= 0 \\
\partial_t \Delta_t &= L(\nabla g_i) \circ + \nabla g_i \cdot \nabla f + t_i |\nabla g_i|^2.
\end{align*} \tag{187}$$

These are all 0 order multiplication operators. Since $f_t$ is a strong deformation, we have pointwise estimate:

$$||\partial_t \Delta_t||, ||\partial_t \partial_t \Delta_t||, ||\partial_t \partial_t \Delta_t|| \leq C(|\nabla f|^2 + 1), \tag{188}$$

where $C$ depends only on $M, g, f, g_i$.

**Theorem 3.51.** Suppose that $\Delta_t$ is a strong deformation of $\Delta_0$ in $S$. Then the resolvent $R_{\lambda_0}(\Delta_t)$ is $C^\infty$ differentiable in $S$. 
Proof. Let \((\lambda_0 - \Delta_t)\psi_t = \varphi_t\), where \(\varphi_t\) is \(C^r\) differentiable in \((t, z)\) and for each \(t\) \(\varphi_t\) is a compactly supported section. We want to prove that \(\psi_t\) is \(C^r\) differentiable in \((t, z)\). We prove by induction in \(r\) and firstly prove the continuity of \(\psi_t\).

Let \(\Omega' \subset \subset \Omega \subset M\) be a compact domain. Then by the apriori estimate of the elliptic operators, we have

\[
\|\psi_t\|_{W^{3,2}(\Omega')} \leq C \left(\|\lambda_0 - \Delta_t\|_{W^{3,2}(\Omega)} + \|\psi_t\|_{L^2(\Omega)}\right),
\]

where \(C\) only depends on \((M, g, f), m\) and the distance between \(\partial \Omega\) and \(\partial \Omega'\).

Since \(\lambda_0 - \Delta_t\) is an isomorphism, we have

\[
\|\psi_t\|_{L^2(\Omega)} \leq \|\psi_t\|_{L^2(M)} \leq C\|\lambda_0 - \Delta_t\|_{L^2(M)}. \tag{190}
\]

Therefore, we have estimate

\[
\|\psi_t\|_{W^{3,2}(\Omega')} \leq C\|\lambda_0 - \Delta_t\|_{L^2(M)} \tag{191}
\]

By (191), we have

\[
\|\psi_t - \psi_s\|_{W^{3,2}(\Omega')} \leq C \left(\|\lambda_0 - \Delta_t\|_{L^2(M)}\|\psi_t - \psi_s\|_{L^2(M)} \right)
\]

\[
\leq C \left(\|\lambda_0 - \Delta_t\|_{L^2(M)} + \|\Delta_t - \Delta_s\|_{L^2(M)} \right)
\]

\[
\leq C \left(\|\lambda_0 - \Delta_t\|_{L^2(M)} + \|\Delta_t - \Delta_s\|_{L^2(M)} \right). \tag{192}
\]

Let \(t \to s\). The first term vanishes, since for any \(I, |I| \leq m\),

\[
|\nabla^I \varphi_t - \nabla^I \varphi_s| \to 0,
\]

because of the \(C^0\) continuity of \(\varphi_t\) with respect to \(t\). The second term vanishes because the coefficients of the operator \(\Delta_t\) depends on \(t\) continuously (even smoothly). The third term vanish because of the dominant convergence theorem. Now the last term

\[
\|\Delta_t - \Delta_s\|_{L^2(M)} \to 0, \text{ as } t \to s.
\]

Here we used the estimate (188), the global estimate from Lemma 4.28 and the fact that \(\varphi\) has compact support.

Now by sobolev embedding theorem on \(\Omega'\), if \(m\) satisfies \(m > r - 2 + \frac{8}{n}\), then for any \(I, |I| \leq r\) there is

\[
|\nabla^I \psi_t(z) - \nabla^I \psi_s(z)| = o(1).
\]

Therefore we proved the \(C^0\) continuity of \(R_{\lambda_0}(\Delta_t)\).

Now consider the \(C^r\) continuity. Take a mollifier \(\rho_\varepsilon(t)\) such that its support lies in a very small neighborhood of \(t = 0\) and it tends to \(\delta\) function as \(\varepsilon \to 0\). For any section \(\psi_t(z)\), we obtain a smooth function \(\psi^\varepsilon_t(z)\) with respect to \(t\). \(\psi^\varepsilon\) is called the mollification of \(\psi\) in \(t\) direction. Then we have the equation:

\[
(\lambda_0 - \Delta_t)\psi_t^\varepsilon = \varphi_t^\varepsilon.
\]

Take the derivative \(\partial_t\), we have

\[
(\lambda_0 - \Delta_t)\partial_t\psi_t^\varepsilon = \partial_t\varphi_t^\varepsilon + (\partial_t\Delta_t)\psi_t^\varepsilon. \tag{193}
\]

By (191), there is

\[
\|\partial_t\psi_t^\varepsilon\|_{W^{3,2}(\Omega')} \leq C \left(\|\partial_t\varphi_t^\varepsilon + (\partial_t\Delta_t)\psi_t^\varepsilon\|_{W^{3,2}(\Omega')} + \|\partial_t\varphi_t^\varepsilon + (\partial_t\Delta_t)\psi_t^\varepsilon\|_{L^2(M)}\right)
\]

\[
\leq C \left(\|\partial_t\varphi_t^\varepsilon\|_{W^{3,2}(\Omega')} + \|\psi_t^\varepsilon\|_{W^{3,2}(\Omega')} + \|\partial_t\varphi_t^\varepsilon\|_{L^2(M)} + \|\partial_t\Delta_t\psi_t^\varepsilon\|_{L^2(M)}\right).
\]

The last term is controlled by the inequality:

\[
\|\partial_t\Delta_t\psi_t^\varepsilon\|_{L^2(M)} \leq C\|((\nabla f)^2 + 1)\psi_t\|_{L^2(M)} \leq C\|\varphi\|_{W^{3,2}(M)}.
\]
By Sobolev embedding theorem and the property of mollification, we known that \( \partial_t \psi_t \) exists.

To prove its continuity, we want to get a uniform control of the term \( |\nabla^l (\partial_t \psi_t) - \nabla^l (\partial_t \psi_s)| \) in \( \Omega' \).

This term is still controlled by the Sobolev norm on \( \Omega \). by (192) we have

\[
\begin{align*}
&\| \partial_t \psi_t - \partial_t \psi_s \|_{W^{l-2,2}(\Omega')} \\
\leq &\| \partial_t \psi_t + (\partial_t \Delta \psi_t - \partial_t \psi_s - (\partial_t \Delta_t) \psi_t) \|_{W^{l-2,2}(\Omega)} \\
+ &\| (\Delta_t - \Delta_s) \partial_t \psi_t + (\partial_t \Delta_t) \psi_t - (\partial_t \Delta_s) \psi_t \|_{L^2(M)} \\
+ &\| (\Delta_t - \Delta_s) \partial_t \psi_s \|_{L^2(M)}.
\end{align*}
\]

Let \( t \to s \), then the first and the second term vanish obviously, because of the continuity of \( \partial_t \psi_t, \psi_t \) and the coefficients of \( \Delta_t \). Since

\[
\| (\partial_t \Delta_t) \psi_t \|_{L^2(M)} \leq C \| |\nabla f|^2 + 1 \| \psi_t \|_{L^2(M)} \leq C \| \psi_t \|_{W^{l-2,2}(M)},
\]

we can use the dominant convergence theorem to deduce that the third term tends to zero as \( t \to s \).

For the last term we have

\[
\| (\Delta_t - \Delta_s) \partial_t \psi_s \|_{L^2(M)} \leq C |t - s| \| |\nabla f|^2 + 1 \| \partial_t \psi_s \| \leq C |t - s| \| \psi_s \|_{W^{l-2,2}(M)} \to 0,
\]

as \( t \to s \). So up to now, we have proved the \( C^1 \) smoothness.

After proving the \( C^1 \) smoothness, we can consider the equation satisfied by the higher order derivatives. By the same way but tedious computation, we can prove any \( C^r \) smoothness of \( R_{2\nu}(\Delta_t) \).

**Theorem 3.52.** Let \( I = [a, b] \subset \mathbb{R} \) and \( a, b \) is not the spectrum point of \( \Delta_0 \). Then there exists a \( \delta > 0 \) such that for any \( t, |t - t_0| < \delta \), the projection \( E_t(I) \) with respect to \( I \) is well-defined and \( C^\infty \) differentiable.

**Proof.** Let \( \Sigma \) be a simple closed curve in \( \mathbb{C} \) containing \( I \) and intersects the real axis only at \( a, b \) points. Then

\[
E_{2\nu}(I) = \int_{\Sigma} R_{2\nu}(\Delta_0) d\lambda.
\]

Since the spectrum of \( \Delta_t \) is continuous, there is a \( \delta > 0 \) such that for any \( t, |t - t_0| < \delta \), the interval \( I \) has no intersection points with the spectrum of \( \Delta_t \). Hence for any \( \lambda \in \Sigma \), the resolvent \( R_{2\nu}(\Delta_t) \) is well-defined and is \( C^\infty \) differentiable with respect to \( t \). Therefore the projection \( E_t(I) \) is \( C^\infty \) differentiable.

**Theorem 3.53.** Let \( G_t \) be the Green function of \( \Delta_t \) with parameter \( t \in S \). If \( (M, g, f) \) is a strong deformation of \( (M, g, f) \) on \( S \), then \( G_t \) is a \( C^\infty \) differentiable operator.

**Proof.** Since \( S \) is compact, there exists a point \( \lambda_0 \in \mathbb{R} \) such that \( \lambda_0 \) is less than the first nonzero eigenvalues of all of \( \Delta_t, t \in S \). Let \( \Sigma \) be a simple half closed curve in \( \mathbb{C} \) containing \([\lambda_0, +\infty)\) and intersects the real axis only at \( \lambda_0 \). Then

\[
G_t = \int_{\Sigma} \frac{1}{\lambda - \lambda_0} R_t(\lambda) d\lambda.
\]

Note that the integration is absolutely uniform convergent, so the differentiability of \( R_t(\lambda) \) for \( t \in S \) implies the \( C^\infty \) differentiability of \( G_t \).
4. Hilbert bundle and Hodge bundle.

Let $f_t(z) := f_t(z)$ be a family of holomorphic functions defined on $\mathbb{C}^* \times S \times M$, where $S \subset \mathbb{C}^n$ is a domain and having coordinates $t = (t_1, \cdots, t_m)$ and $\mathbb{C}^* \equiv \mathbb{C} \setminus \{0\}$. Then we have a family of operators $\Delta_f, \bar{\partial}_f, \partial_f, \bar{\partial}^f, \partial^f$, which depend on the parameter $(\tau, t) \in \mathbb{C}^* \times S$. So the space of harmonic forms $\mathcal{H}^*_{\tau,t}$ also depends on the parameter $(\tau, t)$.

We still assume that $(M, g)$ is a Kähler manifold with bounded geometry.

We have the trivial complex Hilbert bundle $\Lambda^\ast_{\tau} := L^2\Lambda^\ast \times \mathbb{C}^* \times S \to \mathbb{C} \times S$ which is a graded bundle. The Hermitian metric on the Hilbert bundle $\Lambda^\ast_{\tau}$ is the usual one on the fiber:

$$(\alpha, \beta)_{L^2} = \int_M \alpha \wedge \ast \beta,$$

which is independent of $(\tau, t)$.

Fix $(\tau_0, t_0) \in S$ and let $(\tau, t)$ be a point in a small neighborhood $U_0$ of $(\tau_0, t_0)$. For any $(\tau, t) \in U_0$, we can choose uniformly bounded eigenforms $[\alpha_\tau(t, \tau, t)]_{\alpha=1}^\infty$ of $\Delta_f$ such that they form a complete basis of $L^2\Lambda^\ast(M)$ and the corresponding eigenvalues have the order:

$$0 = \lambda_0 \leq \lambda_1 \leq \cdots.$$

Set

$$g_{ab}(\tau, t) = g(\alpha_a, \alpha_b),$$

which satisfies $g_{ab} = \overline{g_{ba}}$. We have the metric tensor

$$g = g_{ab} \alpha^a \otimes \alpha^b = \overline{g}^b \alpha_a \otimes \alpha_b,$$

where $\{\alpha^a\}$ is the dual basis in the dual Hilbert bundle and $\{g^b \}$ is the inverse operator defined by the dual metric on the dual Hilbert bundle.

If $\alpha = \sum_a f_a \alpha_a, \beta = \sum_b h_b \beta_b$, then

$$g(\alpha, \beta) = \sum_{ab} f_ag_{ab}h_b = (g_{ab}h_a)(\alpha).$$

Hence $g$ can be viewed as a bounded operator mapping the section of the Hilbert bundle to its dual bundle.

There are some unbounded $\mathbb{C}$-linear operator acting on the Hilbert bundle. The first is the differentiation. Since $\partial_\tau \alpha_a := \frac{\partial}{\partial t_\tau} \alpha_a(t)$ is an exponential decay function by Theorem 3.44 we can define the tensors $\Gamma_i$ and $\Gamma_i$ as follows (infinite dimensional case)

$$(\Gamma_i)_{ab} = g(\partial_\tau \alpha_a, \alpha_b) = g_{\overline{a} \overline{b}}(\Gamma_i)_{a}^c$$

$$(\Gamma_i)_{ab} = g(\bar{\partial}_\tau \alpha_a, \alpha_b) = g_{\bar{a} \bar{b}}(\Gamma_i)_{a}^c,$$

and the covariant derivative is defined as

$$D_i := \partial_i - \Gamma_i, \quad D_i := \bar{\partial}_i - \Gamma_i.$$

Now let $\{\alpha'_a\}$ be another basis on the same fiber, and

$$S := S_{a'b},$$

is the transformation bounded operator, it is easy to see that

$$\Gamma = S^{-1} \cdot \Gamma' \cdot S + S^{-1} \partial_\beta S.$$
Hence $D_i, D_j$ is really a covariant derivative. Define the connection
\[ D = \sum_i (dt^i D_i + d\bar{t}^i D_i). \]  
(194)

$D$ is a Hermitian connection with respect to the bundle metric $g$, since it is easy to prove that
\[ D_ig_{\bar{a}b} = D_{\bar{a}}g_{ib} = 0. \]

There is another operator $\tau \partial_s$ acting on the Hilbert bundle. We define
\[ (\Gamma_{i})_{\bar{a}b} = g(\tau \partial_s \alpha_a, \alpha_b). \]  
(195)

Similarly, we have $\bar{\tau} \partial_s$.

There is a special element given by the isomorphism of Theorem 2.66 which we assume
\[ \alpha_1 = 1 + \bar{\partial} / R_1, \]  
(196)

which $R_1$ is a smooth $n-1$ form has at most polynomial growth. Partial Christoffel symbols involving $\alpha_1$ will vanish which is given by the following proposition:

**Proposition 4.1.** We have
\[ (\Gamma_{i})_{\bar{a}b} = (\Gamma_{i})_{\bar{b}a} = (\Gamma_{i})_{\bar{a}1} = (\Gamma_{i})_{\bar{b}1} = 0, \forall i, b. \]  
(197)

**Proof.** We have
\[ (\Gamma_{i})_{\bar{a}b} = g(\partial_i \alpha_a, \alpha_b) = g(\bar{\partial}_i (\partial_s R_1), \alpha_b) = 0, \forall b, \]
and
\[ (\Gamma_{i})_{\bar{b}a} = (\Gamma_{i})_{1\bar{b}a} = 0. \]

The others can be obtained similarly. \( \square \)

**Remark 4.2.** If there is a projection operator defined on the Hilbert bundle $\Lambda^*_C$ with is compatible with all $\Delta_s$, then the Hilbert bundle can be split into two parts and the connection $D$ can be restricted to the connection on each subbundles. In particular, $D$ can be restricted to the holomorphic subbundle, the real bundle and the Hodge bundle which we will defined subsequently.

4.1.1. **Real structure and Maslov index.** The canonical real structure in $L^2\Lambda^*$ is given by the complex conjugate, which we denote by $\tau_R$. Let $\{\alpha_a\}_{a=1}^\infty$ be a basis of $L^2\Lambda^*$ and then $\alpha_a := \overline{\alpha_a} = \tau_R \cdot \alpha_a$ is also a basis of $L^2\Lambda^*$. $\tau_R \cdot \alpha_a$ can be represented by the linear combination of $\{\alpha_a\}$:
\[ \tau_R \cdot \alpha_a = M^b_a \alpha_b = M^b_a \alpha_b, \]  
(198)

Similarly,
\[ \tau_R \cdot \alpha_{\bar{a}} = M^b_{\bar{a}} \alpha_{\bar{b}} = M^b_{\bar{a}} \alpha_{\bar{b}}. \]

Obviously we have
\[ M^b_a = M^b_{\bar{a}} = \delta^b_a, \quad M^b_{\bar{a}} = M^b_a. \]

The anti-linear map $\tau_R$ can be written as $\tau_R = M^b_a \alpha_a \otimes \alpha^b$. It is easy to prove that
\[ \eta_{b\bar{c}} = g(\alpha_b, \tau_R \cdot \alpha_{\bar{c}}) = g_{b\bar{c}} M^b_{\bar{a}}. \]  
(199)

If $\alpha_a$ and $\alpha_{\bar{b}}$ are two forms, then we have
\[ \eta_{b\bar{c}} = \eta_{bc}. \]  
(200)

If $\{\alpha_a(t)\}$ is a real basis, then $M \equiv I$. This shows that
\[ DM \equiv 0, \]
i.e., the matrix $M$ is a parallel tensor.

Assume that there is a spectrum gap for Laplacians $\Delta_{\ell}$ on $S$, i.e., there is an open interval $(\mu, \nu)$ such that its intersection with any spectrum of $\Delta_{\ell}$ is empty. In this case, the bundle $\Lambda_{\mathbb{C}}$ is split into the direct sum of two bundles. We denote the bundle with finite rank $\mu_{t}$ by $\Lambda_{\mathbb{C}}^{\mu_{t}}$.

Fix the parameter $t$ and take a real orthonormal basis $\{\alpha_{a}(t_{0})\}_{a=1}^{\mu_{t}}$ of the fiber space $\Lambda_{\mathbb{C}}^{\mu_{t}}(t_{0})$ at the point $t_{0} \in S$. Denote by $\Lambda_{\mathbb{R}}$ the real space generated by this basis. Using the parallel transportation defined by $D$ and a path $l_{t_{0}}$ connecting $t$ and $t_{0}$, one can obtain the unitary operator $P_{l_{t_{0}}}: \Lambda_{\mathbb{C}}(t_{0}) \to \Lambda_{\mathbb{C}}(t)$. $P_{l_{t_{0}}} (\Lambda_{\mathbb{R}})$ defines a totally real Hilbert space in $\Lambda_{\mathbb{C}}^{\mu_{t}}(t) \cong \mathbb{C}^{\mu_{t}}$.

Let $\mathbb{R}(\mu_{t})$ be the set of totally real subspaces. Then it is a homogeneous space

$$\mathbb{R}(\mu_{t}) = \text{GL}(\mu_{t}, \mathbb{C})/\text{GL}(\mu_{t}, \mathbb{R}).$$

Define

$$\overline{\mathbb{R}}(\mu_{t}) = \{ A \in \text{GL}(\mu_{t}, \mathbb{C}) | A\overline{A} = I \}.$$

Then we have the following lemma.

**Lemma 4.3 (FODD).** The map

$$L : \mathbb{R}(\mu_{t}) \to \overline{\mathbb{R}}(\mu_{t}); \quad A \cdot \mathbb{R} \to A^{-1} \overline{A}$$

is a diffeomorphism with respect to the obvious smooth structure on $\mathbb{R}(\mu_{t})$ and $\overline{\mathbb{R}}(\mu_{t})$.

**Definition 4.4** (Generalized Maslov index). Let $l : S^{1} \to \mathbb{R}(\mu_{t})$ be a loop such that at point $t$, there is

$$l(t) = P_{l_{t_{0}}} (\Lambda_{\mathbb{R}}).$$

Then the generalized Maslov index $m(l)$ is defined to be the winding number

$$\det oL \circ l : S^{1} \to \mathbb{C} \setminus \{0\}.$$

If we write the real structure matrix $M$ in terms of the parallel orthonormal frame $\{\alpha_{a}(t)\}_{a=1}^{\mu_{t}}$, then

$$m(l) = \deg \circ \det M. \quad (201)$$

The generalized Maslov index defined here satisfies the properties that an usual Maslov index should have.

Let $\mathcal{H} \subset \Lambda_{\mathbb{C}}$ be the Hodge bundle over $\mathbb{C}^{*} \times S$, and each fiber at $(\tau, t) \in \mathbb{C}^{*} \times S$ is the space of all harmonic $n$ forms of the Laplacian $\Delta_{\ell}$.

Suppose that the primitive forms $\{\alpha_{1}(\tau, t), \cdots, \alpha_{\mu}(\tau, t)\}$ give a local frame of $\mathcal{H}$ near a point $(\tau_{0}, t_{0}) \in \mathbb{C}^{*} \times S$ which is the restriction of a frame in $\Lambda_{\mathbb{C}}$. Then we have the identities:

$$\tilde{\partial}_{f} \alpha_{a}(\tau, t) = \tilde{\partial} \tilde{\partial} \alpha_{a}(\tau, t) = 0, \quad a = 1, \cdots, \mu. \quad (202)$$

Since $\Lambda \alpha_{a}(\tau, t) = 0$, by Hodge identity $\frac{1}{4}[\partial_{f}, \Lambda] = \tilde{\partial} \tilde{\partial}$ we know that $\Lambda \partial_{f} \alpha_{a}(\tau, t) = 0$. Since there is no primitive $n + 1$ form, this shows that

$$\partial_{f} \alpha_{a}(\tau, t) = 0. \quad (203)$$

Similarly, we can prove that $\partial_{f} \alpha_{a}(\tau, t) = 0$.

In summary, we have the important properties of the frame of primitive harmonic $n$-forms in $\mathcal{H}$.

**Proposition 4.5.** Suppose that $\{\alpha_{1}(\tau, t), \cdots, \alpha_{\mu}(\tau, t)\}$ is a local frame of $\mathcal{H}$ consisting of the primitive harmonic $n$-forms, then we have the identities

$$\tilde{\partial}_{f} \alpha_{a}(\tau, t) = \tilde{\partial} \tilde{\partial}_{f} \alpha_{a}(\tau, t) = \partial_{f} \alpha_{a}(\tau, t) = \tilde{\partial} \tilde{\partial}_{f} \alpha_{a}(\tau, t) = 0, \quad a = 1, \cdots, \mu. \quad (204)$$
4.2. Cecotti-Vafa’s equation and "Fantastic" equation.

4.2.1. Cecotti-Vafa’s equations.

Now we fix the parameter $\tau$ in the following discussion and denote by $\partial_i f = \partial_i f$, the derivative of $f$, with respect to $t_i$. Later we always use $i, j, k, \ldots$ to represent the terms with respect to the parameter $t^i, t^j, t^k$ and etc. In the following part, we will simply write $f_{\tau, t} = \tau f_i$ as $f$ if there is no danger of confusion. It is easy to obtain the following lemma:

**Lemma 4.6.**

\[
\begin{align*}
[\partial_i, \partial_j] &= \partial_i(\partial_j f) \wedge, & \quad [\partial_i, \bar{\partial}_j] &= 0, \\
[\partial_i, \bar{\partial}_j] &= 0, & \quad [\bar{\partial}_i, \bar{\partial}_j] &= \partial_i f_a (dz^a \wedge)^{\bar{\partial}} = \partial_i f_a g^{ab} \iota_{\bar{s}_b}.
\end{align*}
\]

**Lemma 4.7.** By our stability analysis, the Projection operator $P_0$ and the Green operator $G = G_f$ are smooth with respect to $\partial_i, \bar{\partial}_i$. We have the commutation relations:

\[
\begin{align*}
\{[\partial_i, \Delta_f] &= [\partial_i(\partial_j f) \wedge, \bar{\partial}]_f], & \quad [\bar{\partial}_i, \Delta_f] &= [\bar{\partial}_i(\partial_j f) \wedge, \partial]_f], \\
\{[\partial_i, G] &= -G(\partial_i P_0 + [\partial_i, \Delta_f] \cdot G), & \quad [\bar{\partial}_i, G] &= -G(\bar{\partial}_i P_0 + [\bar{\partial}_i, \Delta_f] \cdot G).
\end{align*}
\]

When restricted to the space of primitive harmonic forms, we have

\[
[\partial_i, P_0] = G \cdot \Delta_f \cdot \partial_i, \quad [\bar{\partial}_i, P_0] = G \cdot \Delta_f \cdot \bar{\partial}_i
\]

and

\[
\bar{\partial}_i G = -G(\partial_i + [\partial_i, \Delta_f] \cdot G).
\]

**Proof.** It suffices to prove the identities related to the derivative $\partial_i$.

We have

\[
[\partial_i, \Delta_f] = [\partial_i, [\bar{\partial}_j, \partial_j]_f]]
\]

\[
=[\partial_i, \bar{\partial}_j]_f + [[\partial_i, \bar{\partial}_j]_f, \partial_j] = [\partial(\partial_i f) \wedge, \partial]_f.
\]

This proves the first identity.

Differentiating the formula:

\[
P_0 + \Delta_f G = I,
\]

we have

\[
\bar{\partial}_i P_0 + [\partial_i, \Delta_f] \cdot G + \Delta_f \cdot \bar{\partial}_i G = 0,
\]

which gives

\[
\bar{\partial}_i G = [\partial_i, G] = -G(\partial_i P_0 + [\partial_i, \Delta_f] G).
\]

Let $\alpha_a$ be a primitive harmonic form. Take the derivative to the identity $P_0 \alpha_a = \alpha_a$, we have

\[
[\partial_i, P_0] \alpha_a = (I - P_0) \partial_i \alpha_a = \Delta_f G(\partial_i \alpha_a).
\]

So we are done.

**Definition 4.8.** Define tensors $B_i, \bar{B}_i := B_j$ with the components:

\[
(B_i)_{ab} = g((\partial_j f) \alpha_a, \alpha_b), \quad (\bar{B}_i)_{ab} = g((\bar{\partial}_j f) \alpha_a, \alpha_b),
\]

and the matrix-valued 1-forms:

\[
B = \sum_{j=1}^\mu B_{i} dt^j, \quad \bar{B} = \sum_{j=1}^\mu B_{i} dt^j.
\]
Remark 4.9. It is easy to see that the matrix $\bar{B}_i$ is the complex conjugate transpose of $B_i$, i.e., in local coordinates, $(\bar{B}_i)_{ab} = (B_i)_{ba}$. So in some references, people write $\bar{B} = B^\dagger$, i.e., $(\cdot)^\dagger = (\cdot)^T$, the complex transpose of matrix. Here the reader should be careful that if $\dagger$ used for matrix it has different meaning to $\dagger$ used for differential operators. Also the short line over the head of $\bar{B}$ does not mean the complex conjugate of the matrix $B$, instead we take $\bar{B}$ as an independent matrix from $B$. If we want to consider the relations with the matrix $B$, one should use $B^\dagger = \bar{B}$.

We have the formula:

Lemma 4.10. For any $i = 1, \cdots, m$, $a = 1, \cdots, \mu$, we have

$$
\begin{aligned}
(\partial f)_{\alpha a} &= (B_i \gamma_a [\partial f]_{\alpha a} + \bar{\partial} f(\gamma_i)_{\alpha a}) \\
(\bar{\partial} f)_{\alpha a} &= (\bar{B}_i \gamma_a [\bar{\partial} f]_{\alpha a} + \bar{\bar{\partial}} f(\bar{\gamma}_i)_{\alpha a}),
\end{aligned}
$$

(210)

where

$$
\begin{aligned}
(\gamma_i)_a &= \bar{\partial} f \cdot G \cdot [\partial f]_{\alpha a} \\
(\bar{\gamma}_i)_a &= \bar{\bar{\partial}} f \cdot G \cdot [\bar{\partial} f]_{\alpha a}.
\end{aligned}
$$

(211)

Here $G$ is the Green function.

Proof. Using Hodge decomposition formula, we have

$$(\partial f)_{\alpha a} = P_0 [(\partial f)_{\alpha a} + \Delta f \cdot G \cdot (\partial f)_{\alpha a}]$$

$$= (\bar{B}_i \gamma_a [\partial f]_{\alpha a} + \bar{\partial} f(\gamma_i)_{\alpha a}).$$

Set

$$(\gamma_i)_a = \bar{\partial} f \cdot G \cdot (\partial f)_{\alpha a},$$

then we get the first identity. The second one can be proved in the same way. \(\square\)

Lemma 4.11. For any $i = 1, \cdots, m$, $a = 1, \cdots, \mu$, we have

$$
\begin{aligned}
D_i a_\alpha &= \partial f(\gamma_i)_{\alpha a} \\
D_\bar{\gamma} a_\alpha &= \bar{\partial} f(\bar{\gamma}_i)_{\alpha a},
\end{aligned}
$$

(212)

Proof. $a_\alpha(t)$ satisfies the equation:

$$
\Delta f \cdot a_\alpha(t) = 0.
$$

Taking derivative $\partial_t$ to the above equality, we have

$$
\Delta_f (\partial_t a_\alpha) + [\bar{\partial}(\partial f) \land \bar{\partial} - \partial(\bar{\partial} f)] a_\alpha = 0.
$$

Hence

$$
\Delta_f (\partial_t a_\alpha) = -\bar{\partial}^i_j(\partial_j f (\partial_i a_\alpha)) = \partial_f \bar{\partial}^i_j(\partial_j f) a_\alpha,
$$

where we used the fact that $\partial(f a_\alpha) = 0$. Therefore,

$$
D_i a_\alpha = \partial_t a_\alpha - \Gamma_i a_\alpha = \partial_f \bar{\partial}^i_j(\partial_j f) a_\alpha.
$$

The other identity can be proved similarly. \(\square\)

Some special data can be easily obtained in the following proposition.

Proposition 4.12. We have

$$
(B_1 \gamma_a)^\dagger = \tau \delta_a^\dagger, \quad (\bar{B}_1 \gamma_a)^\dagger = \bar{\tau} \delta_a^\dagger
$$

(213)

$$
(\gamma_1)_a = (\bar{\gamma}_1)_a = 0, \forall a = 1, \cdots, \mu
$$

(214)

$$
D_1 a_\alpha = D_\bar{\gamma} a_\alpha = 0, \forall a = 1, \cdots, \mu
$$

(215)
Remark 4.13. We have defined the operator $B_i$, which acts on $\alpha_a$ as $B_i \cdot \alpha_a = (B_i)_a^b \alpha_b$. On the other hand, $(B_i)_{\bar{a}b} = g(B_i \cdot \alpha_a, \alpha_b) = (B_i)_{\bar{a}}^b \alpha_b$. Hence, we have used the convention that the action of the operator $B_i$ is a matrix multiplication by the matrix $(B_i)_a^b$, where $a$ is the row index and $b$ is the column index. In this convention, if $B_i, B_j$ are operators, then

$$[B_i, B_j] \cdot \alpha_a = B_i((B_j)_a^b \alpha_b) - B_j((B_i)_a^b \alpha_b) = (B_i)_a^b (B_j)_c^d \alpha_c - (B_j)_a^b (B_i)_c^d \alpha_c$$

$$= (B_i \cdot B_j - B_i \cdot B_j)_{\bar{a}}^b \alpha_b = [B_i, B_j] \alpha_a = -[B_i, B_j] \alpha_a.$$  

The front $[B_i, B_j]$ is understood as operator action and the rear Lie algebra $-[B_i, B_j]$ is understood as the matrix multiplication. We will use the later expression in our formulas.

Theorem 4.14. The connection $D$ and the operators $B_i$ satisfy the following Cecotti-Vafa’s equations on $\mathcal{H}$, i.e., after \mod $\mathcal{H}^\perp$,

$$D_i B_j = D_i B_j = 0, \quad D_i B_j = D_i B_j, \quad D_i B_j = D_i B_j$$

$$[D_i, D_j] = [D_i, D_j] = [B_i, B_j] = [B_i, B_j] = 0$$

$$[D_i, D_j] = -[B_i, B_j].$$

Proof. Take the covariant derivatives to $B_j$:

$$D_j(B_j)_{\bar{a}b} = \partial_j(B_j)_{\bar{a}b} - (\Gamma^k_j(B_j)_{\bar{a}\bar{k}} - (\Gamma^k_j(B_j)_{\bar{a}k})_{\bar{b}b}$$

Since

$$\partial_j(B_j)_{\bar{a}b} = g(\partial_j \partial_j f \alpha_a, \alpha_b) + g(\partial_j f \alpha_a, \alpha_b) + g(\partial_j f \alpha_a, \alpha_b)$$

$$= g(\partial_j \partial_j f \alpha_a, \alpha_b) + (\Gamma^k_j(B_j)_{\bar{a}\bar{k}} + (\Gamma^k_j(B_j)_{\bar{a}k})_{\bar{b}b}$$

$$+ g(\partial_j f \partial_j f \alpha_a, \alpha_b) + g(\partial_j f \alpha_a, \alpha_b) + g(\partial_j f \alpha_a, \alpha_b) + g(\partial_j f \alpha_a, \alpha_b) + g(\partial_j f \alpha_a, \alpha_b) + g(\partial_j f \alpha_a, \alpha_b)$$

$$= g(\partial_j \partial_j f \alpha_a, \alpha_b) + (\Gamma^k_j(B_j)_{\bar{a}\bar{k}} + (\Gamma^k_j(B_j)_{\bar{a}k})_{\bar{b}b}$$

$$- g(\partial_j f \alpha_a, \alpha_b) = - g(\partial_j f \alpha_a, \alpha_b) - g(\partial_j f \alpha_a, \alpha_b)$$

We know that

$$D_i(B_i)_{\bar{a}b} = g(\partial_i \partial_i f \alpha_a, \alpha_b)$$

$$- g(\partial_i f \alpha_a, \alpha_b) - g(\partial_i f \alpha_a, \alpha_b)$$

By symmetry we have

$$D_i B_j = D_j B_i.$$  

Take the complex conjugate transpose, we have $D_i B_j = \bar{D}_j \bar{B}_i$ or $D_i B_j = D_j B_i$.

We have

$$g((\partial_j f)\partial_j f \alpha_a, \alpha_b) = g((\partial_j f)(B_j)_{\bar{a}}^b \alpha_b)$$

$$= (B_j)_{\bar{a}}^b (B_j)_{\bar{a}}^b + g(\partial_j f \partial_j f \alpha_a, \alpha_b)$$

$$= (B_j)_{\bar{a}}^b (B_j)_{\bar{a}}^b,$$

which induces that

$$B_i B_j = B_j B_i.$$
We have the identity:
\[ D_t(B_j)_{\bar{a}b} = \partial_t(B_j)_{\bar{a}b} - (\Gamma^i_j)_b(B_j)_{\bar{a}\bar{b}} - (\Gamma^i_j)_b(B_j)_{a\bar{b}}. \]

Compute the first term:
\[
\partial_t(B_j)_{\bar{a}b} = \partial_t g((\partial f)_a\alpha_a, \alpha_b) + g((\partial f)_a\alpha_a, \partial \alpha_b) \\
= (\Gamma^i_j)_b(B_j)_{\bar{a}b} + (\Gamma^i_j)_b(B_j)_{a\bar{b}}.
\]

Hence we proved that
\[ D_t(B_j) = 0. \]

By Lemma 4.11, we have the following computation:
\[
D_t(\tilde{D}_j\alpha_a) = \partial_t(\tilde{\partial}_j(\tilde{\gamma}_j)_a) - (\Gamma^i_j)_b(\tilde{\partial}_j(\tilde{\gamma}_j)_b) \\
= \tilde{\partial}_j(D_t(\tilde{\gamma}_j)_a) + \partial(\partial f) \wedge (\tilde{\gamma}_j)_a \\
= \tilde{\partial}_j(D_t(\tilde{\gamma}_j)_a) + \partial((\partial f)(\tilde{\gamma}_j)_a) - (\partial f)\partial(\tilde{\gamma}_j)_a \\
= \tilde{\partial}_j(D_t(\tilde{\gamma}_j)_a) + \partial((\partial f)(\tilde{\gamma}_j)_a) - (\partial f)((\partial f)\alpha_a - (\tilde{\partial}_j)_b\alpha_b - (\tilde{\partial}_f)\wedge (\tilde{\gamma}_j)_a) \\
= \tilde{\partial}_j(D_t(\tilde{\gamma}_j)_a) + \partial((\partial f)(\tilde{\gamma}_j)_a) + (\partial f)(\tilde{\partial}_j)_b\alpha_b - (\partial_f)\tilde{\partial}_f\alpha_a.
\]

Similarly, we have
\[
\tilde{\partial}_j(D_t\alpha_a) = \partial f((\partial f)(\tilde{\gamma}_j)_a) - (\partial f)((\partial f)\alpha_a + (\tilde{\partial}_f)\tilde{\partial}_f\alpha_b).
\]

Hence we have
\[
[D_t, \tilde{D}_j]\alpha_a = \tilde{\partial}_j(D_t(\tilde{\gamma}_j)_a) - \partial f((\partial f)(\tilde{\gamma}_j)_a) + \partial f((\partial f)(\tilde{\gamma}_j)_a) - \tilde{\partial}_j((\partial f)(\tilde{\gamma}_j)_a) + (\partial f)(\tilde{\partial}_j)_b\alpha_b - (\tilde{\partial}_f)\tilde{\partial}_f\alpha_a \\
= \tilde{\partial}_j(D_t(\tilde{\gamma}_j)_a) - (\tilde{\partial}_f)(\tilde{\gamma}_j)_a - \partial f(D_t(\tilde{\gamma}_j)_a) + [\tilde{\partial}_f(\tilde{\partial}_f)\tilde{\partial}_f(\tilde{\gamma}_j)_a] \\
= \tilde{\partial}_j(D_t(\tilde{\gamma}_j)_a) - (\tilde{\partial}_f)(\tilde{\gamma}_j)_a + (\tilde{\partial}_f)(\tilde{\partial}_f)(\tilde{\gamma}_j)_a - [\tilde{\partial}_f(\tilde{\partial}_f)(\tilde{\gamma}_j)_a] \\
+ (\tilde{\partial}_f)(\tilde{\partial}_f)(\tilde{\gamma}_j)_a - (\tilde{\partial}_f)(\tilde{\partial}_f)(\tilde{\gamma}_j)_a.
\]

Acting by the projection $P_0$, we obtain
\[ [D_t, \tilde{D}_j]\alpha_a = [-B_t, \tilde{B}_j]\alpha_a. \]

The other identities $[D_t, D_j] = 0$ and etc. can be easily seen to be true when in view of the above proof.

**Definition 4.15.** Let $t = (t_1, \ldots, t_m)$ be the local holomorphic coordinates of the deformation parameter space $S$. Define the sections $D, \tilde{D}, B, \tilde{B} \in \Omega^1(S) \otimes \Gamma(\text{End}(\mathcal{H})$) as:
\[
D = \sum_{i=1}^m dt^i D_i, \quad \tilde{D} = \sum_{i=1}^m dt^i \tilde{D}_i \\
B = \sum_{i=1}^m dt^i B_i, \quad \tilde{B} = \sum_{i=1}^m dt^i \tilde{B}_i.
\]

Using the above definitions, the $\pi^* \pi$ equation can be written in the following forms.
Corollary 4.16.

1. \([D, B] = DB = 0, [\bar{D}, B] = B\bar{D} = [D, \bar{B}] = D\bar{B} = 0\)
2. \(D^2 = \frac{1}{2}[D, D] = 0, (\bar{D})^2 = \frac{1}{2}[\bar{D}, \bar{D}] = 0\)
3. \(B \wedge B = 0, B \wedge \bar{B} = 0\)
4. \([D, \bar{D}] = -[B, \bar{B}]\).

4.2.2. "Fantastic" equations.
Now we allow the change of the parameter \(\tau\) and consider the action of the operator \(\tau \partial_\tau\). We will continue to use the simple notation \(f = f_\tau\).

Lemma 4.17. We have the commutation relations:

\[
[\tau \partial_\tau, \partial_\tau f] = 0, [\bar{\tau} \partial_\bar{\tau}, \bar{\partial}_f] = 0 \tag{216}
\]

\[
[\tau \partial_\tau, \bar{\partial}_f] = \partial f \wedge, [\bar{\tau} \partial_\bar{\tau}, \partial_f] = \bar{\partial}_f \wedge \tag{217}
\]

\[
[\tau \partial_\tau, \bar{\partial}_f] = 0, [\bar{\tau} \partial_\bar{\tau}, \partial_f] = 0 \tag{218}
\]

\[
[\bar{\tau} \partial_\bar{\tau}, \partial_f] = (\partial f \wedge)^1, [\tau \partial_\tau, \partial_f] = (\bar{\partial}_f \wedge)^1. \tag{219}
\]

Proof. A straightforward computation. \(\Box\)

Lemma 4.18. The commutation relations hold:

\[
[\tau \partial_\tau, \Delta_f] = [\partial f \wedge, \bar{\partial}_f] = \nabla^\tau(\partial f) \wedge \iota_\partial + |\partial f|^2 \tag{220}
\]

\[
[\tau \partial_\bar{\tau}, \Delta_f] = [\bar{\partial}_f \wedge, \partial_f] = \nabla^{\bar{\tau}}(\bar{\partial}_f) \wedge \iota_\bar{\partial} + |\partial f|^2. \tag{221}
\]

Proof. The result is due to the equality we proved before:

\[
[\partial f \wedge, \bar{\partial}_f] = g^{\bar{\mu}\nu}\nabla_\mu(\partial_\tau f)dz^\nu \wedge \iota_\bar{\partial}. \tag{222}
\]

\(\Box\)

By Lemma 4.18 we have

\[
\Delta_f(\tau \partial_\tau a_\alpha) = -[\partial f \wedge, \bar{\partial}_f]a_\alpha = -\bar{\partial}_f(\partial f \wedge a_\alpha) = \partial_f\bar{\partial}_f(\partial f a_\alpha). \tag{223}
\]

By Hodge decomposition formula, there exists a matrix \(\Gamma_\tau = \Gamma_\tau^a_a\) such that

\[
(\tau \partial_\tau a_\alpha) = \Gamma_\tau^a_a a_\alpha + G \cdot \partial_f \bar{\partial}_f(f a_\alpha) = \Gamma_\tau^a_a a_\alpha + \partial_f \bar{\partial}_f G \cdot (f a_\alpha). \tag{224}
\]

Similarly, there exists a matrix \(\bar{\Gamma}_\bar{\tau} \equiv \Gamma_\bar{\tau}\) such that

\[
(\bar{\tau} \partial_\bar{\tau} a_\alpha) = \Gamma_\bar{\tau}^b_b a_\alpha + \partial_f \bar{\partial}_f G \cdot (f a_\alpha). \tag{225}
\]

Definition 4.19. Define two operators:

\[
D_\tau = \partial_\tau - \frac{1}{\tau} \Gamma_\tau, D_\tau = \partial_\tau - \frac{1}{\tau} \bar{\Gamma}_\bar{\tau}. \tag{226}
\]

So we have the formula by the above discussions:

Lemma 4.20.

\[
\tau D_\tau a_\alpha = \partial_f(\gamma_\tau)a_\alpha \tag{226}
\]

\[
\bar{\tau} D_\bar{\tau} a_\alpha = \bar{\partial}_f(\bar{\gamma}_\bar{\tau})a_\alpha. \tag{227}
\]
where

\[(\gamma_{\tau})_{ab} = \tilde{\partial} G \cdot (f \alpha_a)\]  \hspace{1cm} (228) \\
\[(\overline{\gamma}_{\tau})_{ab} = \partial G \cdot (f \alpha_a).\]  \hspace{1cm} (229)

Since we have already defined the covariant derivatives $D_i, D_t$ along the deformation direction in addition to $\tau$-direction, now we get the covariant derivatives $D$ along any direction and the total differentials are defined as:

\[D = \sum_i d\hat{t} D_i + d\tau D_{\tau}, \quad \tilde{D} = \sum_i d\hat{t} D_i + d\tau D_{\tau}.\]  \hspace{1cm} (230)

**Lemma 4.21.** The operator $D$ is a metric connection with respect to $g$.

**Proof.** We have the covariant derivative computation:

\[D\tau g_{ab} = \partial \tau g_{ab} - \frac{\Gamma^c_{ab}}{\tau} g_{cb} - \frac{\Gamma^c_{ba}}{\tau} g_{ac}\]

\[= g(D\tau \alpha_a + \frac{\Gamma^b}{\tau} \cdot \alpha_a, \alpha_b) + g(\alpha_a, D\tau \alpha_b + \frac{\Gamma^b}{\tau} \cdot \alpha_b) - g(\frac{\Gamma^c}{\tau} \cdot \alpha_a, \alpha_b) - g(\alpha_a, \frac{\Gamma^c}{\tau} \cdot \alpha_b)\]

\[= 0.\]

Similarly, we can prove $D\tau g_{ab} = 0$.

Hence including the result from the first $\eta^*$ equations, we know that $D$ is a metric connection with respect to $g$. \hfill \Box

Analogous to the relation between $D_i$ and $B_i$, we have operators $\mathcal{U}_r, \overline{\mathcal{U}}_r$ corresponding to $D_r, D_{\tau}$: which are defined as the fibration multiplication by $f$ and $\tilde{f}$.

Define

\[\mathcal{U}_{\tau ab} := g(f \cdot \alpha_a, \alpha_b), \quad \mathcal{U}_r = \mathcal{U}_{\tau ab} \alpha^a \otimes \alpha^b = \mathcal{U}_{\tau} \alpha^a \otimes \alpha^b = \mathcal{U}_{\tau} \alpha^a \otimes \alpha^b.\]  \hspace{1cm} (231)

Here $\mathcal{U}_{\tau ab} = \mathcal{U}_{\tau} M_{\tilde{a} \tilde{b}}$, where $M$ is the matrix representation of the real structure $\tau_R$. It is easy to see that

\[\mathcal{U}_{\tau ab} = \eta(\alpha_a, \alpha_b) = \eta(\alpha_a, f \alpha_b) = \mathcal{U}_{\tau ba},\]  \hspace{1cm} (232)

i.e., $\mathcal{U}_r = \mathcal{U}_r^*$. Similarly, we can define $(\overline{\mathcal{U}}_{\tau})_{ab} = g(\tilde{f} \alpha_a, \alpha_b)$ and the corresponding operator $\overline{\mathcal{U}}_r$.

**Lemma 4.22.** We have relations:

\[f \alpha_a = \mathcal{U}_r \cdot \alpha_a + \tilde{\partial} f (\gamma_{\tau})_a\]  \hspace{1cm} (233) \\
\[\tilde{f} \alpha_a = \overline{\mathcal{U}}_r \cdot \alpha_a + \tilde{\partial} \tilde{f} (\gamma_{\tau})_a.\]  \hspace{1cm} (234)

**Proof.** It is a straightforward computation: it suffices to prove the first identity,

\[\tilde{\partial} \tilde{f} (\gamma_{\tau})_a = \tilde{\partial} \tilde{f} \cdot \tilde{\partial} G \cdot (f \alpha_a) = \Delta G(f \alpha_a) = (f - \mathcal{U}_r) \cdot \alpha_a,

where we used the fact that $\tilde{\partial} \tilde{f} (f \alpha_a) = 0$. \hfill \Box

**Lemma 4.23.** The following identity holds:

\[B = B^*, \quad \mathcal{U}_r = \mathcal{U}_r^*.\]  \hspace{1cm} (235)

**Proof.** Since $\eta$ is a real symmetric bilinear form, and the multiplication operator $\partial_1 f$ can be inserted equivalently in the two variable positions, so $B = B^*$. Similarly, we have $\mathcal{U}_r = \mathcal{U}_r^*$. \hfill \Box
Theorem 4.24. The operators $D_i, D_τ, B, \mathcal{W}_τ$ satisfy the following relations (We call it as "Fantastic" equations) on $\mathcal{H}$, i.e., after mod $\mathcal{H}^\perp$.

(1) $[D_i, \mathcal{W}_τ] + [B_i, τ D_τ] = 0$, $[D_i, \mathcal{W}_τ] + [B_i, \mathcal{W}_τ] = 0$
(2) $[D_i, \mathcal{W}_τ] = 0$, $[D_i, \mathcal{W}_τ] = 0$
(3) $[τ D_τ, B_i] = [τ D_τ, B_i] = 0$
(4) $[B_i, \mathcal{W}_τ] = 0$, $[B_i, \mathcal{W}_τ] = 0$
(5) $[τ D_τ, D_i] = -[τ D_τ, B_i]$, $[τ D_τ, D_i] = -[τ D_τ, B_i]$
(6) $[τ D_τ, D_i] = [τ D_τ, D_i] = 0$
(7) $[τ D_τ, \mathcal{W}_τ] = [τ D_τ, \mathcal{W}_τ] = 0$
(8) $[τ D_τ, \mathcal{W}_τ] = -[τ D_τ, \mathcal{W}_τ]$

Proof. (1) To prove the identities in the first row, it suffices to prove the first one. At first we have the computation

$$\eta(D a_α, f a_b)$$

$$= \eta(f \partial_i (\gamma_i) a_α, a_b) = -\eta(f \cdot \bar{\partial}_i f G(\partial_i f) a_α, a_b)$$

$$= -\eta(G \cdot (\partial f \wedge a_α), \partial_i f a_α) = -\eta(G \cdot (\partial f \wedge a_α), \partial f \wedge a_b),$$

and

$$\eta(\partial f a_δ, τ D_α a_β)$$

$$= \eta(\partial f a_β, \partial f((\gamma_i) a_β)) = \eta(\partial f \bar{\partial}_i f G f a_α, (\partial f a_β)$$

$$= -\eta(G(\partial f \wedge a_α), \partial(\partial f) \wedge a_b).$$

Then we have

$$[[B_i, τ D_τ] + [D_i, \mathcal{W}_τ]_{ab} = -(τ D_τ)(B_i)_{ab} + D_i \mathcal{W}_{rab}$$

$$= - (B_i)_{ab} - \eta(\partial f a_β, (τ D_τ a_α)) - \eta((\partial f) a_α, (τ D_τ a_β))$$

$$+ (B_i)_{ab} + \eta(D a_δ, f a_b) + \eta(f a_δ, D a_β)$$

$$= \eta(G(\partial f \wedge a_α), \partial(\partial f) \wedge a_b) + \eta(G(\partial f \wedge a_β), \partial(\partial f) \wedge a_α)$$

$$- \eta(G(\partial f \wedge a_α), \partial f \wedge a_b) - \eta(G(\partial f \wedge a_β), \partial f \wedge a_α)$$

$$= 0.$$

(2) To prove the second row, it suffices to prove that $[D_i, \mathcal{W}_τ] = 0$. We have

$$D_i(\mathcal{W}_{rab}) = \partial_i \mathcal{W}_{rab} - (\Gamma_i)_{a_δ} \mathcal{W}_{rcb} - (\Gamma_i)_{b_δ} \mathcal{W}_{rac}$$

$$= \eta(\partial a_δ, f a_β) + (\Gamma_i)_{a_δ} \mathcal{W}_{rcb} - (\Gamma_i)_{b_δ} \mathcal{W}_{rac} = 0.$$

Here we used the fact that

$$\eta(\partial f \cdot \partial a_α, a_β) = \eta(\partial f (\gamma_i) a_α, a_β) + \eta(\partial f (\gamma_i) a_δ, a_β) = (\Gamma_i)_{a_δ} \mathcal{W}_{rcb} + 0.$$

(3) To prove the second row, it suffices to prove that $[D_τ, B_i] = 0$. We have

$$τ D_τ(B_i_{ab}) = τ \partial_i (B_i_{ab}) - (\Gamma_i)_{a_δ} B_i_{rb} - (\Gamma_i)_{b_δ} B_i_{rc}$$

$$= \eta(\partial f \tau a_δ, a_β) + (\Gamma_i)_{a_δ} B_i_{rb} - (\Gamma_i)_{b_δ} B_i_{rc} = 0.$$
Here we used the fact that
\[ \eta(\bar{\partial}_j f \bar{\partial}_a a_a, a_b) = \eta(\bar{\partial}_j f (\Gamma_\tau \bar{\tau} a_a, a_b) + \eta(\bar{\partial}_j f \bar{\partial}_j G \cdot \partial_f (f a_a), a_b) = (\Gamma_\tau \bar{\tau} B_{a b} + 0. \]

(4) Since \( \mathcal{Z}_r = \mathcal{Z}_r^* \) and \( \mathcal{B} = \mathcal{B}^* \), it suffices to prove that
\[ B^* \mathcal{Z}_r = (B^* \mathcal{Z}_r)^*. \]

Now we have
\[ \eta(B_i \cdot \mathcal{Z}_r a_a, a_b) = \eta(B_i \cdot (\bar{\partial} f a_a - \partial_j (\bar{\gamma})_a), a_b) \]
\[ = \eta(B_i \cdot f a_a - \partial_j (\bar{\gamma})_a, f a_b) \]
\[ = \eta(f a_a, a_b) = \eta(a_a, (B_i \cdot \mathcal{Z}_r) a_b) \]

So we get the proof of the fourth row.

(5) Prove the 5th row. For simplicity we can take \( a_a \) be the real basis. We have
\[ \eta([\tau D_\tau, D_1] a_a, a_b) \]
\[ = \eta(\tau D_\tau D_1 a_a, a_b) - \eta(D_\tau D_\tau a_a, a_b) \]
\[ = - \eta(D_\tau D_\tau a_a, a_b) + \eta(\tau D_\tau D_1 a_a, a_b) \]
\[ = - \eta(\bar{\partial}_j (\bar{\gamma})_a, \partial_j (\gamma)_b) + \eta(\partial_j (\gamma)_a, \bar{\partial}_j (\bar{\gamma})_b) \]
\[ = - g(\bar{\partial}_j (\bar{\gamma})_a, \partial_j (\gamma)_b) + g(\partial_j (\gamma)_a, \bar{\partial}_j (\bar{\gamma})_b) \]
\[ = - g(\bar{\partial}_j \bar{\partial}_j G(\bar{\partial}_f a_a, \bar{\partial}_j G(\bar{\gamma})_a)) + g(\bar{\partial}_f \partial_j G(a_a), \bar{\partial}_j G(\partial_f a_a)) \]
\[ = - g((I - P_0)(\bar{\partial}_f a_a) + g((I - P_0)(\bar{\partial}_f a_a) \]
\[ = - \eta(\bar{\partial}_f a_a, (I - P_0)(\bar{\partial}_f a_a)) \]
\[ = \eta(B_i a_a, a_b) = \eta((\mathcal{Z}_r, B_i) \cdot a_a, a_b) \]
\[ = \eta(-[(\mathcal{Z}_r, B_i)] a_a, a_b) \]

We have used the facts that \( \mathcal{Z}_r^* = \mathcal{Z}_r, B_i^* = B_i \) for the last second equality. By Remark [4,13] the action of the operator \( [\mathcal{Z}_r, B_i] \) is given by the matrix multiplication \(-[(\mathcal{Z}_r, B_i)] \).

(6) To prove the 6th row, we have
\[ \eta([\tau D_\tau, D_1] a_a, a_b) \]
\[ = \eta(\tau D_\tau D_\tau a_a, a_b) - \eta(D_\tau D_\tau a_a, a_b) \]
\[ = - \eta(D_\tau D_\tau a_a, a_b) + \eta(\tau D_\tau D_\tau a_a, a_b) \]
\[ = - \eta(\bar{\partial}_j (\gamma)_a, \partial_j (\gamma)_b) + \eta(\partial_j (\gamma)_a, \bar{\partial}_j (\gamma)_b) \]
\[ = - g(\bar{\partial}_j (\gamma)_a, \partial_j (\gamma)_b) + g(\partial_j (\gamma)_a, \bar{\partial}_j (\gamma)_b) \]
\[ = - g(\bar{\partial}_j \bar{\partial}_j G(\partial_f a_a, \gamma)_b) + g(\partial_j \partial_j G(\partial_f a_a, \bar{\gamma})_b) = 0. \]
(7) Prove the 7th row. We have
\[(\tau D_{\tau}, \tilde{Z}_{\tau})_{ab} = \eta(f \tau D_{\tau}a_{\tau}, a_{b}) + \eta(f_a, \tau D_{\tau}a_{b})\]
\[= \eta(f \tilde{\gamma}_a, a_{b}) + \eta(f_a, \tilde{\gamma}_a)\]
\[= 0.\]

(8) Take \(a_{\tau}\) to be real basis, we have
\[\eta((\tau D_{\tau}, \tilde{Z}_{\tau})_{a_{\tau}, a_{b}})\]
\[= g((\tau D_{\tau}, \tilde{Z}_{\tau})_{a_{\tau}, a_{b}})\]
\[= - g(\tilde{\gamma}_a, \tilde{\gamma}_a) + g(\tau D_{\tau}a_{\tau}, \tau D_{\tau}a_{b})\]
\[= - g(\tilde{\gamma}_a, \tilde{\gamma}_a) + g(\tilde{\gamma}_a, \tilde{\gamma}_a)\]
\[= 0.\]

\[\Box\]

4.2.3. Connection \(\mathcal{D}\) on the Hodge bundle \(\mathcal{H}^\perp\).

Define the covariant derivatives \(\nabla_i = D_i - B_i, \tilde{\nabla}_i = D_i - B_i\) and the corresponding connections:
\[\nabla_i : \mathcal{H} \to \mathcal{H} \otimes \Lambda^1 (T^*S), \nabla_i : \mathcal{H} \to \mathcal{H} \otimes \Lambda^2 (T^*S)\]
by
\[\nabla_i = \sum d\tau^i \wedge \nabla_i, \tilde{\nabla}_i = \nabla_i = \sum d\tilde{\tau}^i \wedge \nabla_i,\]
\(\nabla_i, \nabla_i\) are the (1, 0) and (0, 1) parts of the following connection for fixed \(\tau\):
\[\mathcal{D}_i = \nabla_i + \tilde{\nabla}_i : \mathcal{H} \to \mathcal{H} \otimes T^*_\tau S.\] (237)

Now the Cecotti-Vafa’s equations is equivalent to the following identities, after \(\mod \mathcal{H}^\perp\),
\[\nabla_i^2 = \nabla_i^2 = [\nabla_i, \nabla_i] = 0.\] (238)

Define the covariant derivatives along the \(\tau\)-direction:
\[\nabla_{\tau} = D_{\tau} - \frac{\mathcal{U}}{\tau}, \nabla_{\tilde{\tau}} = D_{\tilde{\tau}} - \frac{\tilde{\mathcal{U}}}{\tilde{\tau}}.\] (239)
and
\[\nabla_\tau = d\tau \wedge \nabla_\tau, \nabla_{\tilde{\tau}} = d\tilde{\tau} \wedge \nabla_{\tilde{\tau}}.\] (240)

Here we use the same notation to denote the differential and its derivatives along the \(\tau\) direction. We also define the 1-forms
\[\mathcal{U} = \mathcal{U}_i d\tau, \tilde{\mathcal{U}} = \tilde{\mathcal{U}}_i d\tilde{\tau}.\] (241)

The connection \(\mathcal{D}_\tau\) is defined as
\[\mathcal{D}_\tau = \nabla_\tau + \nabla_{\tilde{\tau}}.\] (242)
The total connection $\mathcal{D}$ is defined on the bundle $\mathcal{H} \to \mathbb{C}^* \times S$ and given by
\[
\nabla := \nabla_t + \nabla_\tau, \quad \bar{\nabla} := \nabla_t + \nabla_\tau, \quad \mathcal{D} = \nabla + \bar{\nabla}.
\] (243)

**Theorem 4.25.** $\mathcal{D}$ is a nearly flat connection of the Hodge bundle $\mathcal{H} \to \mathbb{C}^* \times S$, i.e., $\mathcal{D}^2 = 0$, mod $\mathcal{H}^\perp$ or equivalently
\[
\nabla^2 = \bar{\nabla}^2 = [\nabla, \bar{\nabla}] = 0, \quad \text{mod} \quad \mathcal{H}^\perp.
\] (244)

**Proof.** When expanding the components of $\mathcal{D}$, $\mathcal{D}^2 = 0$ is the conclusion of the identities appeared in the Cecotti-Vafa’s equations and the Fantastical equations. We can prove $[\nabla, \bar{\nabla}] = 0$ as an example. We have
\[
[\nabla, \bar{\nabla}] = [(D_t + D_\tau) - (B + \frac{\mathcal{U}}{\tau}), (D_t + D_\tau) - (\bar{B} + \frac{\mathcal{W}}{\bar{\tau}})]
\]
\[
= - [D_t + D_\tau, \bar{B} + \frac{\mathcal{W}}{\bar{\tau}}] + \left( [D_t, D_\tau] + D_t + D_\tau + [B, \frac{\mathcal{W}}{\tau}, B] + [B, \frac{\mathcal{W}}{\tau}, \bar{B}] + [\frac{\mathcal{W}}{\tau}, \frac{\mathcal{W}}{\bar{\tau}}] \right).
\]
The first term vanishes, since we have $[D_t, \bar{B}] = 0$ by CV equation (1); $[D_t, \mathcal{W}] = 0$ by Fantastical equation (7); $[D_t, \bar{W}] = 0$ by Fantastical equation (2); $[D_t, \bar{B}] = 0$ by Fantastical equation (3). The third term vanishes due to the same reason. The second term is
\[
[D_t, D_t] + [D_t, D_\tau] + [D_\tau, D_t] + [D_\tau, D_\tau] + [B, \bar{B}] + [B, \frac{\mathcal{W}}{\tau}, B] + [\frac{\mathcal{W}}{\tau}, \bar{B}] + [\frac{\mathcal{W}}{\tau}, \frac{\mathcal{W}}{\bar{\tau}}].
\]
By CV equation (4), $[D_t, D_t] + [B, \frac{\mathcal{W}}{\tau}] = 0$; By Fantastical equation (5), we have
\[
[D_t, D_t] + [B, \frac{\mathcal{W}}{\tau}] = \left( [D_t, D_\tau] + [B_t, \frac{\mathcal{W}_t}{\tau}] \right) dt^\tau \wedge d\bar{\tau} = 0,
\]
and
\[
[D_\tau, D_t] + [\frac{\mathcal{W}}{\tau}, B_t] = \left( [D_\tau, D_t] + [B_t, \frac{\mathcal{W}_t}{\tau}] \right) dt^\tau \wedge d\tau = 0.
\]
By Fantastical equation (8), we have
\[
[D_\tau, D_\tau] + [\frac{\mathcal{W}}{\tau}, \frac{\mathcal{W}}{\bar{\tau}}] = \left( [D_\tau, D_\tau] + [\frac{\mathcal{W}_t}{\tau}, \frac{\mathcal{W}_t}{\bar{\tau}}] \right) dt^\tau \wedge d\bar{\tau} = 0.
\]
Therefore, we proved $[\nabla, \bar{\nabla}] = 0$. \hfill \Box

4.2.4. **Asymptotic estimates.**

In this part, we will use the Witten-Helffer-Sjöstrand method to estimate the growth order of the harmonic $n$ forms $\alpha$, and the operators $B_t, \mathcal{U}_t$ with respect to $\tau$.

Assume that $f_\tau(t)$ is a Morse function with $\mu$ non-degenerate critical points $\{p_i\}$. Let $\alpha_n(t, \tau)$ be a primitive harmonic $n$-form, then $\alpha_n(t, \tau)$ also satisfies the equation:
\[
\square_{f_\tau} \alpha = 0,
\]
where $\square_{f_\tau} = d_{2 \text{Re} f_\tau} \circ d_{1 \text{Re} f_\tau} + d_{1 \text{Re} f_\tau} \circ d_{2 \text{Re} f_\tau}$, where $d_{2 \text{Re} f_\tau} = d + d_{\text{Re} f_\tau}$, is the Witten deformation operator. Let $\{D_{2\epsilon}(p_\tau); z_1, \cdots, z_n\}$ be a good coordinate disc having radius $2\epsilon$, centered at $p_\tau$ such that it has flat metric and $f_\tau$ has the form
\[
f(\tau, t) = \tau(z_1^2 + \cdots + z_n^2).
\]
Choose a smooth function $\gamma_\epsilon(z)$ satisfying:

$$
\gamma_\epsilon(z) = \begin{cases} 
1 & |z| \leq \epsilon \\
0 & |z| \geq 2\epsilon.
\end{cases}
$$

Denote by

$$
\tilde{\varphi}_\tau^a(z) = C_a \tau^{n/2} e^{-\tau \sum |z|^2},
$$

where $C_a$ is chosen such that

$$
\|\tilde{\varphi}_\tau^a\|_{L^2(C^\tau)} = 1.
$$

Define

$$
\tilde{\varphi}_\tau^a(z) = \frac{\gamma_\epsilon(z) \varphi_o^a(z)}{\|\gamma_\epsilon(z) \varphi_o^a(z)\|_{L^2(C^\tau)}},
$$

then $\tilde{\varphi}_\tau^a \in \Omega^a_\tau(M)$ and satisfies

$$
\|\tilde{\varphi}_\tau^a\|_{L^2(M)} = 1.
$$

As first, we have spectrum gap theorem essentially due to Witten (here we consider $L^2$ integrable form).

**Theorem 4.26 (Witten).** There exist constants $C_1, C_2, C_3$ and $T_0$ depending only on $(M, g, f)$ so that if $|\tau| > T_0$, then

$$
\text{Spec}(\Box_{2 \text{Re}(f)}) \cap (C_1 e^{-C_2 |\tau|}, C_3 |\tau|) = \emptyset.
$$

Hence as $|\tau| \to \infty$, we have the decomposition

$$
(L^2(\Omega^\tau(M), d_2 \text{Re}(f))) = (L^2(\Omega^\tau(M))_\text{sm}, d_2 \text{Re}(f)) \oplus (L^2(\Omega^\tau(M))_\text{crit}, d_2 \text{Re}(f)).
$$

Note that the Morse function $2 \text{Re}(f)$ is very special, since all the $\mu$ critical points have Morse indices $n$.

As $|\tau| \to \infty$, the form $\alpha_a$ will concentrate near the critical points. Without loss of generality, we can assume that $\alpha_a$ concentrates at $p_a$ and satisfy

$$
\|\alpha_a\|_{L^2(M)} = 1.
$$

By Theorem 3.1, Proposition 3.3 of [HS], Theorem 8.30 of [BZ] and our asymptotic estimate near the infinity far place, we can obtain the following asymptotic estimate:

**Theorem 4.27.** There exists $\epsilon > 0$, $T_0$ and $C$ so that for $|\tau| > T_0$ and any critical point $p_a$, there is

$$
\sup_{z \in M \setminus D_{2\epsilon}(p_a)} |\alpha_a(\tau, z)| \leq C e^{-\epsilon |\tau|},
$$

and

$$
|\alpha_a(\tau, z) - \tilde{\varphi}_\tau^a(z)| \leq C \frac{1}{|\tau|} \in D_{2\epsilon}(p_a) \cap W^-(p_a).
$$

Here $W^-(p_a)$ is the unstable manifold of the critical point $p_a$ with respect to the gradient flow of $2 \text{Re}(f)$.

Applying the above theorem, it is easy to obtain the following conclusion.

**Corollary 4.28.** Let $|\alpha_a|$ be a local unit frame of the Hodge bundle $\mathcal{H}$ at the base point $(\tau, t) \in C^* \times S_m$. Then if $|\tau|$ is large enough, we have the asymptotic formula:

$$
\mathcal{L}_\tau = \tau \mathcal{L}^\tau, \ B_\tau = \tau B^\tau, \ A^\tau = \tau A^\tau \ 
$$

where $|\mathcal{L}^\tau|, |B^\tau|, |\mathcal{L}^\tau|, |B^\tau|, |A^\tau| \leq C$ and $C$ is a constant depending only on $M, g, f, t$ but not on $\tau$. 

4.3. Wall-crossing phenomenon and duality of Gauss-Manin connections.

Note that the action of the connections on the Hodge bundle is not closed, and the Cecotti-Vafa and Fantastic equations only hold up to a term in the perpendicular subspace $H^\perp$. In this section, we want to construct the bundle $H_{0,\top}$ such that the action of the connections is closed and the Cecotti-Vafa and Fantastic equations really hold on $H_{0,\top}$. This is done by considering the integration of the middle dimensional harmonic forms over the Lefschetz thimble. So at first we study the dual homology structure.

At the moment, we assume that the deformation space $S$ is a open domain with constant global Milnor number on $M$.

Let $S_m(\tau) \subset S$ be the set of $t \in S$ such that all the critical points of $f_{(t,\tau)}$ are Morse critical points. Obviously $S_m(\tau)$ is independent of $\tau$. We can stratify the space $S_m(\tau)$ in the following way. Let $S_r(\tau) := S^0_m(\tau) \subset S_m(\tau)$ be the set of all $t$ in $S_m(\tau)$ such that all the critical values have different imaginary parts. We call the point $t$ in $S_r(\tau)$ as a regular point. Let $S^k_m(\tau) \subset S$, $k \leq \mu$, be the set of all $t$ in $S_m(\tau)$ such that there exist at most $k + 1$ critical values of $f_{(t,\tau)}$ have the same imaginary parts.

We have the following result and the proof is similar to Theorem 3.1.4 of [FJR3].

**Theorem 4.29.**

1. $S_m(\tau)$ is a path connected dense open set in $S$.

2. $S_m(\tau) - S_r(\tau)$ is a union of real hypersurfaces and separates the set $S_r(\tau)$ into a system of chambers.

3. $S^1_m(\tau) \subset S_m(\tau)$ is a path connected dense open set in $S$, and $S^2_m(\tau) - S^1_m(\tau)$ is a real codimension 2 real analytic set in $S_m(\tau)$.

4. There is a decomposition $S = S^1_m(\tau) \cup (S - S^1_m(\tau))$, where $S - S^1_m(\tau)$ is a real codimension 2 real analytic set in $S$.

Define the following subsets in $C^* \times S$:

$$S_m = \cup_{r \in C^*} S_m(\tau), \quad S^k_m = \cup_{r \in C^*} S^k_m(\tau), \quad k = 0, 1, \cdots, \mu.$$  

**Corollary 4.30.** Theorem 4.29 holds for $S_m, S^k_m, S_r$ in $C^* \times S$ instead of $S_m(\tau), S^k_m(\tau), S_r(\tau)$.

**Example 4.31.** Let $f_{t,\tau} = \tau f_t = \tau(x^3 + t_1 x + t_2)$, where $f_t$ is the miniversal deformation of $f = x^3$. Then $f_t$ has two critical points $x_1(t), x_2(t)$ for any $t$, which is given by $x^3 = -\frac{1}{\tau}$.

The walls are given by the hypersurface:

$$0 = \text{im}(f_t(x_1(t)) - f_t(x_2(t))) = \text{im}((x_1 - x_2)(x_1^2 + x_2^2 + x_1 x_2 + t_1)), $$

which is equivalent to

$$\text{im}(\sqrt{-1} t_1^2) = 0.$$  

Let $t_1 = r_1 e^{i\theta}$, then the equation of the walls in $(t_1, t_2)$ space are given by

$$\theta_1 = \frac{2\pi k}{3}, \quad k = 0, 1, 2.$$  

So if $\tau = 1$, we have 3 walls separating the $t = (t_1, t_2)$ space and intersecting at 0. Note that the equations of the walls are independent of $t_2$.

For the total space $C^* \times S$, we have the wall equation:

$$\text{im} \tau(\sqrt{-1} t_1^2) = 0,$$

which is the rotation of the walls at $\tau = 1$ along the $\tau$ direction.
Example 4.32. Let $p \geq 3$ and $f(x) = f(x_1, \cdots, x_n) = x_1^p + \cdots + x_n^p$. Then the residue classes of monomials $[x = x_1^i \cdots x_n^i, 0 \leq v_i \leq p_i - 2, i = 1 \cdots n]$ form the basis of the Milnor ring $\mathbb{C}[x]/(\partial_x f)$. The moduli space of marginal deformation and relative deformation forms our $S$, and the marginal deformation forms part of $S - S_m(\tau)$ and is a subspace of codimension greater than 1. A typical example is $f(t) = x_1^3 + x_2^3 + x_3^3$, and $S$ is given by the miniversal deformation $f(t) = x_1^3 + x_2^3 + x_3^3 + t_8 x_1 x_2 x_3$. For simplicity, we use $S_m, S_m^k, S_r$ to replace the notations $S_m(\tau), S_m^k(\tau), S_r(\tau)$ in Sections 4.3.1, Section 4.5.2.

4.3.1. Relative homology and Lefschetz thimble.

Let $(M, g)$ be a Stein manifold and $(\tau, t) \in S_m \subset S$, i.e., $f(\tau, t)$ is a tame holomorphic Morse function. Since $f(\tau, t) = \tau f_t$, so if $f_t \in S_m$ then for any $\tau \in \mathbb{C}^*$, $f(\tau, t) \in S_m$. So $f(\tau, t)$ has finitely many critical points on $M$. We think $(M, g)$ as a real Riemannian manifold and consider the negative gradient system generated by the dual vector field of $2 \text{Re}(f(\tau, t))$. Let $\alpha > 0$ and define the sets $f_{(\tau, t)}^{\geq \alpha} := \{z \in M : 2 \text{Re}(f(\tau, t)(z)) \geq \alpha\}$, $f_{(\tau, t)}^{< \alpha} := \{z \in M : 2 \text{Re}(f(\tau, t)(z)) \leq -\alpha\}$.

By Morse theory, we know that if there is no critical values of $2 \text{Re}(f(\tau, t))$ between $[\alpha, \beta]$, then the set $f_{(\tau, t)}^{\geq \alpha}$ is the deformation kernel of $f_{(\tau, t)}^{\beta}$ by the flow generated by the vector field $\nabla f_{(\tau, t)}$.

So if $\alpha$ is large enough, there is no critical points in $f_{(\tau, t)}^{< \alpha}$ and each $f_{(\tau, t)}^{< \alpha}$ has the same homotopy type. We denote the equivalence class by $f_{(\tau, t)}^{(\beta)}$. Similarly, we can define $f_{(\tau, t)}^{(\alpha)}$ for $f_{(\tau, t)}^{> \alpha}$ if $\alpha$ is large enough. We have the relative homology group $H_i(M, f_{(\tau, t)}^{\infty}, \mathbb{Z})$ and $H_i(M, f_{(\tau, t)}^{\infty}, \mathbb{Z})$. We can perturb $f(\tau, t)$ a little bit such that $(\tau, t) \in S_r$. Then we have the relative homology group $H_i(M, f_{(\tau, t)}^{\infty}, \mathbb{Z})$. Since $f(\tau, t)$ is a holomorphic Morse function, its real part $\text{Re}(f(\tau, t))$ is a real Morse function with Morse index $n$. We can define the unstable manifold at a critical point $z_0$ of $f(\tau, t)$ (or $\text{Re}(f(\tau, t))$):

$\mathcal{W}^{-}_{a} := \{z \in M : z \cdot s \rightarrow z_0, \text{as } s \rightarrow -\infty\}$,

where $z \cdot s$ represents the action of the flow at time $s$. Similarly, we can define the stable manifold at $z_0$:

$\mathcal{W}^{+}_{a} := \{z \in M : z \cdot s \rightarrow z_0, \text{as } s \rightarrow +\infty\}$.

Definition 4.33. We call $\mathcal{W}^{+}_{a}$ as the positive or negative Lefschetz thimble of $f(\tau, t)$ at the critical point $z_0$.

We have the following theorem, which should be known (but we can’t find the appropriate references for general $M$).

Theorem 4.34. Suppose that $f$ is a holomorphic Morse function defined on the complex manifold $M$ which is fundamental tame, then only the middle dimensional relative homology $H_i(M, f^{\infty}, \mathbb{Z})$ is nontrivial and it is a free abelian group and is generated by the Lefschetz thimble $\{\mathcal{W}^{a}_a, a = 1, \cdots, \mu\}$, where $\mu$ is the number of critical points of $f$ or Milnor number.

Proof. We can construct the relative Morse complex $(C_k, \partial)$, where each group $C_k = \oplus \mathbb{Z}k_a$ is a free abelian group and generated by the critical points $k_a$ with index $k$. Now only the middle dimensional homology group is nontrivial. Hence we get the conclusion. □
Remark 4.35. There is a natural map
\[ \partial : H_n(M, f^{-\alpha}, \mathbb{Z}) \to H_{n-1}(f^{-1}(-\alpha), \mathbb{Z}) \]
given by taking the boundary of \( \mathcal{C}_a^- \). In fact, we have
\[ \partial \mathcal{C}_a^- = \text{Re}(f)^{-1}(-\alpha) \cap \mathcal{C}_a^- . \]
is a real \( n - 1 \) cycle in \( f^{-1}(-\alpha) \). However, in general we don’t know if this map is surjective or injective because of the topology of \( M \). In the case that \( M = \mathbb{C}^n \), this map is an isomorphism and \( \partial \mathcal{C}_a^- \) are nontrivial cycles and are called the vanishing cycles related to the critical points \( z_a \).

If the perturbation \( t \) is small and preserve the tameness, then the set \( f_t^{-\varepsilon-\alpha} \) is homotopic to \( f_t^{-\varepsilon-\alpha} \) for large \( \varepsilon \). Therefore we have the isomorphism even if \( f \) is not a Morse function:
\[ H_*(M, f^{-\alpha}, \mathbb{Z}) \cong H_*(M, f_t^{-\alpha}, \mathbb{Z}). \]
(254)

If \( (\tau, t) \in S_m \), we can define an intersection pairing:
\[ \langle \cdot, \cdot \rangle : H_*(M, f_t^{-\alpha}, \mathbb{Z}) \times H_*(M, f_t^{-\alpha}, \mathbb{Z}) \to \mathbb{Z} \]
by choosing two special basis in the two relative homology groups.

Let \( \mathcal{C}_a^- \) and \( \mathcal{C}_b^+ \) be the unstable and stable (smooth) manifold of the critical point \( \kappa_a \), then \( \mathcal{C}_a^- \) and \( \mathcal{C}_b^+ \) form the two basis and we define
\[ \langle \mathcal{C}_a^-, \mathcal{C}_b^+ \rangle = I_{a^-b^+} = \begin{cases} 1 & a = b \\ 0 & a \neq b. \end{cases} \]

Similarly, one can define the intersection number
\[ I_{a^-b^+} = (-1)^a \begin{cases} 1 & a = b \\ 0 & a \neq b. \end{cases} \]
and the intersection pairing:
\[ \langle \cdot, \cdot \rangle : H_*(M, f_t^{-\alpha}, \mathbb{Z}) \times H_*(M, f_t^{-\alpha}, \mathbb{Z}) \to \mathbb{Z}. \]

Definition 4.36. Since for any \( t \in S \), \( f_{(\tau, t)} \) is assumed to be fundamental tame and has the same Milnor number \( \mu \). We obtain a \( \mathbb{Z} \)-coefficient local system \( H^\partial \) over \( \mathbb{C}^* \times S \) whose fiber at \( (\tau, t) \in \mathbb{C}^* \times S \) is \( H^\partial_{(\tau, t)} = H_{n}(M, f_t^{-\alpha}, \mathbb{Z}) \). Similarly, we can obtain the dual local system \( H^\partial \) over \( \mathbb{C}^* \times S \) with fiber \( H^\partial_{(\tau, t)} = H_n(M, f_t^{-\alpha}, \mathbb{Z}) \). Define the corresponding bundle as \( H^\partial = H^\partial \otimes \mathbb{C} \) and \( H^\partial = H^\partial \otimes \mathbb{C} \). The total space is defined as \( H_{\text{tot}} = H^\partial \oplus H^\partial \). We call such bundles as the relative homological Milnor fibrations.

Note that the fibers have the isomorphism:
\[ H^\partial_{(\tau, t)} \cong H^\partial_{(-\tau, t)} \]
and the intersection pairing \( I \) can be thank as the following pairing:
\[ I : H^\partial_{(\tau, t)} \times H^\partial_{(-\tau, t)} \to \mathbb{Z}. \]

For \( (\tau, t) \in S_m \), we can order each critical point \( \kappa_a \) in the way that \( a < b \) if and only if \( \text{im}(\sigma_a) < \text{im}(\sigma_b) \), where \( \sigma_a \) is the critical value corresponding to the critical point \( \kappa_a \). Therefore we get the order of the Lefschetz thimbles \( H^\partial_{a}(\tau, t) \) in the fiber \( H^\partial_{(\tau, t)} \) and \( H^\partial_{(\tau, t)} \).

The \( \mathbb{Z} \)-lattices of the relative homological Milnor fibrations give the flat connection \( \mathcal{D}_{\text{top}} \) of the corresponding bundle which is called the topological Gauss-Manin connection. Let
Theorem 4.37 (Wall-crossing formula). Let \( \{\kappa_a(\pm 1)\}_{a=1}^n \) be the set of the ordered critical points at \( s = \pm 1 \). Assume without loss of generality that \( \kappa_a(\pm 1) = \kappa_a \) is fixed for \( a \neq i \), \( \kappa_i(s = \pm 1) = \kappa_i(\pm 1) \) and \( \text{im}(\sigma_i(0)) = \text{im}(\sigma_{i+1}) \).

If the perturbation satisfies \( \text{Re}(\sigma_i(s)) < \text{Re}(\sigma_{i+1}) \), we have the left transformation

\[
\begin{align*}
\mathcal{C}_a^- (+) &= \mathcal{C}_a^- (-), \\
\mathcal{C}_i^- (+) &= \mathcal{C}_{i+1}^- (-) + I_{\tau, (i+1)}^* \mathcal{C}_i^- (-), \\
\mathcal{C}_{i+1}^+ (+) &= \mathcal{C}_i^+ (-)
\end{align*}
\]

(255)

where \( I_{\tau, (i+1)}^* = \#(\mathcal{C}_i^+ (s = 0) \cap \mathcal{C}_{i+1}^-) \) is the intersection number of the stable manifold of the critical point \( \kappa_i(s = 0) \) with the unstable manifold of the critical point \( \kappa_{i+1} \).

If the perturbation satisfies \( \text{Re}(\sigma_i(s)) > \text{Re}(\sigma_{i+1}) \), we have the right transformation

\[
\begin{align*}
\mathcal{C}_a^- (+) &= \mathcal{C}_a^- (-), \\
\mathcal{C}_i^- (+) &= \mathcal{C}_i^- (-), \\
\mathcal{C}_{i+1}^+ (+) &= \mathcal{C}_{i+1}^+ (-) + I_{\tau, (i+1)}^* \mathcal{C}_i^- (-)
\end{align*}
\]

(256)

Proof. The proof is similar to the case \( M = \mathbb{C}^n \). The reader can see [13].

Similarly, one can get the transformation formula of the dual Lefschetz thimbles in \( H^0 \).

Since the connection \( \mathcal{D}_{top} \) is flat, the parallel transportation along a closed loop \( \gamma \in \pi_1(S_m, (\tau_0, t_0)) \) defines the monodromy action \( h_\gamma^* \) on the fiber \( H^0(S_m, (\tau_0, t_0)) \). The action is independent on the representative element in its homotopy class \( [\gamma] \) in \( \pi_1(S_m, (\tau_0, t_0)) \). Thus we have the monodromy representation:

\[ T : \pi_1(S_m, (\tau_0, t_0)) \to \text{Aut} H^0(S_m, (\tau_0, t_0)) : [\gamma] \to h_\gamma^* \]

Fix a basis in \( H^0(S_m, (\tau_0, t_0)) \) and let it moves along a loop in \( S_m \). When the loop crosses the wall of each chamber, the basis will be changed because the two vicinal critical points will exchange instantons. The explicit representation is given by the Wall-crossing formula, Theorem 4.37. Therefore, it is easy to see the following conclusion holds:

Proposition 4.38. Let \( l_{t_0} = \{(e^{i\theta} \tau_0, t_0) | \theta \in [0, 2\pi] \} \) be a loop in \( S_m \). Then the isomorphism \( h_\gamma^* \in \text{Aut}(H^0(S_m, (\tau_0, t_0))) \) induced by the parallel transportation by \( \mathcal{D}_{top} \) along \( l \) is the same as the monodromy transformation \( T \).

4.3.2. Witten index and intersection matrix

Let \( (\tau_0, t_0) \in S_\tau \) be the base point and \( \{\mathcal{C}_a^-(0)\} \) be a ordered basis of \( H^0_\tau \). The special path \( l_{t_0} = \{(e^{i\theta} \tau_0, t_0) | \theta \in [0, \pi] \} \subset S_m \) maps \( f_{\tau} \) to \( -f_{\tau} \) and induces the action

\[ I_{W} : H^0(S_m, (\tau_0, t_0)) \to H^0(-\tau_0, t_0) \equiv H^0(S_m, (\tau_0, t_0)), \]

\( I_{W} \) is a reflection about the origin in the image plane \( \mathbb{C} \). The base transformation is given by

\[ \mathcal{C}_{\mu-a}^+(0) = (I_{W})(\mathcal{C}_a^- (0)) = \mathcal{C}_a^-(\pi) = (I_{W})(0) = \mathcal{C}_a^- (0), \]

(257)
where

\[(I_W^-)_{ba} := \begin{cases} \#(\mathcal{C}_b^- \cap \mathcal{C}_a^+), & \text{if } \text{im } f(p_a) < \text{im } f(p_b) \\ 1, & \text{if } \mathcal{C}_a = \mathcal{C}_b^+ \\ 0, & \text{if } \text{im } f(p_a) > \text{im } f(p_b) \end{cases} \]

Similarly, the loop induces another map:

\[I_W^+ : H^\oplus_{(\tau_0, \theta_0)} \rightarrow H^\oplus_{(-\tau_0, \theta_0)} \cong H^\oplus_{(\tau_0, \theta_0)},\]

and the base transformation is given by

\[\mathcal{C}_{\mu-a}(0) = \mathcal{C}_a^+(\pi) = (I_W^+)_{ba}\mathcal{C}_a^+(0), \tag{258}\]

where

\[(I_W^+)_{ba} := \begin{cases} \#(\mathcal{C}_b^- \cap \mathcal{C}_a^+), & \text{if } \text{im } f(p_a) > \text{im } f(p_b) \\ (-1)^a, & \text{if } \mathcal{C}_a = \mathcal{C}_b^+ \\ 0, & \text{if } \text{im } f(p_a) < \text{im } f(p_b) \end{cases} \]

**Definition 4.39.** We call the transformation group \(I_W^\circ \in \text{End}(H^\oplus_{(\tau_0, \theta_0)}, H^\oplus_{(\tau_0, \theta_0)} \cong H^\oplus_{(\tau_0, \theta_0)})\) (and \(I_W^-\)) as the Witten map and the corresponding matrices as the Witten indices.

**Proposition 4.40.** Given base point \((\tau_0, \theta_0) \in S\), the monodromy transformation \(T\) and the Witten maps have the following relations:

\[T = (I_W^-)^2, \quad T = (I_W^+)^2, \quad I_W^\circ = ((I_W^\circ)^{-1})^T. \tag{259}\]

**Proof.** The front two identities are obvious since the path defining the Witten maps is half the loop defining the monodromy transformation. The third one is true since we have the pairing \(\#(\mathcal{C}_a(t) \cap \mathcal{C}_b^-(t)) = \delta_{ab}\). \qed

**Lemma 4.41.** Let \(P\) be a parallel transformation along a path \(\gamma(\tau)\) from the fiber \(H^\oplus_{(\tau_0, \theta_0)}\) to \(H^\oplus_{(\tau_0, \theta_0)}\) (or from \(H^\oplus_{(\tau_0, \theta_0)}\) to \(H^\oplus_{(\tau_0, \theta_0)}\)) across a wall, then we have \(P \circ I_W^\circ = I_W^\circ \circ P\).

**Proof.** The deformation is given by \(e^{\theta} f(\tau), \theta \in [0, \pi], \tau \in [-1, +1]\) and the corresponding deformation parameters lie on \(S_1\). It is routine to construct a homotopy exchange \(I_W^\circ \circ P\) and \(P \circ I_W^\circ\). \qed

**Definition 4.42.** We can also define a non-degenerate symmetric bilinear form in \(H^\oplus_{(\tau, t)}\) for any \((\tau, t) \in S\), as follows: fix a basis \(\{\mathcal{C}_a^+, \mathcal{C}_b^-\}_{a=1}^m\), and define

\[I_W(\mathcal{C}_b^+, \mathcal{C}_a^-) := \langle \mathcal{C}_b^+, I_W^\circ(\mathcal{C}_a^-) \rangle \tag{260}\]

Similarly, we can define the non-degenerate symmetric bilinear form \(I_W^\circ\) in \(H^\oplus_{(\tau, t)}\) via \(I_W^\circ\).

Via Lemma 4.41, it is easy to get the following result.

**Proposition 4.43.** The non-degenerate bilinear form \(I_W\) is parallel with respect to the topological Gauss-Manin connection \(\mathcal{D}_{\text{top}}\). Therefore, \(I_W\) is well-defined on \(S_1\) which is independent of the choice of the basis.

**Remark 4.44.** We can extend the intersection pairing \(I_W, \langle \cdot, \cdot \rangle\) to the whole space \(\mathbb{C}^* \times S\) by using isomorphism \(\mathcal{D}_{\text{top}}\) to identify them to the nearby perturbed quantities and then show they are independent of the perturbation. Some times we will not distinguish the notation \(I_W\) to represent the intersection pairing \(I_W\) for the same homology group and the pairing \(\langle \cdot, \cdot \rangle\) between two homology groups.

**Remark 4.45.** The intersection form \(\langle \cdot, \cdot \rangle\) gives a symplectic structure on the bundle \(\mathcal{M}_{\text{tot}} \rightarrow S_1\).
4.3.3. **Poincaré duality.**

The intersection number can be expressed as the integration of the dual forms:

\[(I_W)_{a \cdot b^*} = \langle \Omega_a^-, \Omega_b^+ \rangle = \#(\Omega_a^- \cap \Omega_b^+) = \int_{\mathcal{C}_a} PD(\Omega_b^+),\]  

(261)

where \( PD : H_\alpha(M, f_\infty^{\pm}, \mathbb{R}) \to H^\alpha(M, f_\infty^{\pm}, \mathbb{R}) \) is the Poincaré dual operator. Sometimes we also denote the integration as the intersection form:

\[\int_{\mathcal{C}_a} PD(\Omega_b^+) =: \langle \Omega_a^-, PD(\Omega_b^+) \rangle.\]  

(262)

For simplicity, we denote \( f := f_{(\tau, t)} \) in this part of discussion.

**Definition 4.46.** Define \( \mathcal{H}_{(\tau, t)}^{(t)} := H^\alpha(M, f_\infty^{\pm}, \mathbb{R}) \) and \( \mathcal{H}_{\phi, (\tau, t)}^{(t)} := H^\alpha(M, f_\infty^{\pm}, \mathbb{R}) \). Let \( \mathcal{H}_{(\theta, \phi, \tau, t)} \) and \( \mathcal{H}_{(\phi, \tau, t)} \) be the bundles with fiber at \((\tau, t)\) to be \( \mathcal{H}_{(\tau, t)}^{(t)} \) and \( \mathcal{H}_{(\tau, t)}^{(t)} \) respectively. Obvi-ously, we have

\[\mathcal{H}_{(\alpha, \phi, \tau, t)} = \mathcal{H}_{(\tau, t)}^{(t)} = \mathcal{H}_{(\tau, t)}^{(t)}.\]

There is an intersection pairing between \( \mathcal{H}_{(\tau, t)}^{(t)} \) and \( \mathcal{H}_{(\tau, t)}^{(t)} \) induced by the intersection pairing \( I_W \) of the relative homology groups.

We want to represent those cohomology classes of \( \mathcal{H}_{(\tau, t)} \) and the pairing in differential forms and the corresponding integration.

Let \( \{ \alpha_c, c = 1, \cdots, \mu \} \) be a local frame of \( \mathcal{H}_{(\tau, t)}^{(t)} \) consisting of the \( L^2 \) harmonic \( n \)-forms. Then \( \alpha_c \) are primitive forms and satisfy

\[\hat{\partial}_f \alpha_c = 0, \; \partial_f \alpha_c = 0.\]

**Lemma 4.47.** Let \( S_c^- = e^{(f + \tilde{f})} \alpha_c \) and \( S_c^+ = e^{-(f + \tilde{f})} \alpha_c \). Then \( S_c^- \) and \( S_c^+ \) are \( d \)-closed \( n \)-forms on \( M \).

**Proof.** We have

\[d(e^{(f + \tilde{f})} \alpha_c) = e^{(f + \tilde{f})} [d \alpha_c + d \tilde{f} + df] = 0.\]

On the other hand, since \( \ast \alpha_c \) are closed \( \partial_{-f} \) and \( \partial_{-f} \) forms, \( S_c^+ \) are also closed \( d \)-forms. \( \square \)

We have

\[\eta_{cd} = \int_M \alpha_c \wedge \ast \alpha_d = \int_M S_c^- \wedge S_d^+.\]

Define

\[\Pi_{ac} = \int_{\mathcal{C}_a} S_c^- , \; \Pi_{bd} = (-1)^n \int_{\mathcal{C}_b} S_d^+ .\]

Assume that

\[PD(\Omega_a^+_{\tau}) = \sum_d c_{ad} S_d^- , \; PD(\Omega_b^-_{\tau}) = \sum_d c_{bd} S_d^+.\]

Then we have

\[\Pi_{bd} = (-1)^n \int_M PD(\Omega_a^+_{\tau}) \wedge S_d^- = (-1)^n \sum_k \int_M c_{bk} S_k^- \wedge S_d^- = (-1)^n \sum_k \eta_{bd} c_{bk} .\]

and similarly

\[\Pi_{ac} = \sum_k \eta_{ak} c_{ak} .\]
So we obtain
\[ c^+_{bk} = (-1)^n \sum_d \Pi^+_d t^d c^-_{a_k} = \sum_d \Pi^-_a t^d. \]

Thus we can prove that
\[ (I_W)_{a,b} = \#(\mathcal{E}^-_a \cap \mathcal{E}^+_b) = \int_{\mathcal{E}^-_a} PD(\mathcal{E}^+_b) \]
\[ = (-1)^n \sum_{d,e} \int_M c^+_{ad} S^+_d \wedge c^-_{be} S^-_e = \sum_{ld} \Pi^-_a t^d \Pi^+_d \]

**Proposition 4.48.** The real structure \( \tau_R = M \), and the matrices \( I_W, \Pi^-, \Pi^+, \eta \) have the following relations for \( (\tau, t) \in \mathbb{C}^* \times S_r \):
\[ I_W = \Pi^-(\tau, t) \cdot \eta^{-1} \cdot \Pi^+(\tau, t), \quad M = \Pi \Pi^{-1}, \quad g = \eta \cdot M. \] (263)

**Corollary 4.49.** The matrices \( \Pi^+(\tau, t), (\tau, t) \in \mathbb{C}^* \times S_r \), are nondegenerate matrices, hence the spaces \( \mathcal{H}^{(\tau, t)}_{\oplus, \top}, \mathcal{H}^{(\tau, t)}_{\ominus, \top} \) are generated by the \( d \)-closed forms \( S_c^- \)'s and \( S_c^+ \)'s.

**Definition 4.50.** The matrix \( \Pi^+(\tau, t), (\tau, t) \in \mathbb{C}^* \times S_r \), are called the periodic matrices.

**Definition 4.51.** Note that \( \mathcal{H}_\oplus \) is the Hodge bundle with fiber at \( (\tau, t) \) to be the space of \( \Delta_{f,\alpha} \)-harmonic \( n \)-forms. We define the bundle isomorphism \( \psi_\oplus : \mathcal{H}_\oplus \to \mathcal{H}_{\ominus, \top} \) as
\[ [\psi_\oplus(\alpha)](\tau, t) = e^{\ell_f + \bar{f}_1 \alpha}(\tau, t), \quad (\tau, t) \in \mathbb{C}^* \times S. \] (264)

Similarly, we have the isomorphism \( \psi_\ominus : \mathcal{H}_\ominus \to \mathcal{H}_{\ominus, \top} \) as
\[ [\psi_\ominus(\alpha)](\tau, t) = e^{-\bar{f}_1 - \ell_f \alpha}(\tau, t), \quad (\tau, t) \in \mathbb{C}^* \times S. \] (265)

Define the operator \( \hat{\ast} : \mathcal{H}_{\ominus, \top} \to \mathcal{H}_{\ominus, \top} \) as
\[ \hat{\ast}(S^-) = \hat{\ast}(e^{\ell_f + \bar{f}_1} \ast \alpha). \]

Note that \( \ast \alpha(\tau, t) \) is a \( \Delta_{-f} \)-harmonic form and lies in \( \mathcal{H}^{(\tau, t)}_{\ominus, \top} \equiv \mathcal{H}^{(-\tau, t)}_{\ominus, \top} \).

**Proposition 4.52.** The following relations hold:
1. \( z^2 = \ast^2 = (-1)^n \), (266)
2. the diagram below commutes.

\[ \begin{array}{c}
\mathcal{H}_\oplus \xrightarrow{\psi_\oplus} \mathcal{H}_{\ominus, \top} \\
\downarrow \ast \quad \downarrow \hat{\ast} \\
\mathcal{H}_\ominus \xrightarrow{\psi_\ominus} \mathcal{H}_{\ominus, \top}
\end{array} \]

**Definition 4.53.** Define the pairing \( \hat{\eta} \) in \( \mathcal{H}_{\ominus, \top} \) as
\[ \hat{\eta}(S_1^-, S_2^-) = (S_1^-, \hat{\ast} S_2^-). \] (267)

Plus the real structure \( \tau_R \), we can define a Hermitian metric on \( \mathcal{H}_{\ominus, \top} \) as
\[ \hat{g}(S_1^-, S_2^-) = \hat{\eta}(S_1^-, \tau_R \cdot S_2^-). \] (268)

**Theorem 4.54.** The bundle map \( \psi_\oplus \) provides an isomorphism between two real Hermitian holomorphic bundles: \( (\mathcal{H}_\oplus, g, \tau_R) \) and \( (\mathcal{H}_{\ominus, \top}, \hat{g}, \tau_R) \) and the same for \( \psi_\ominus \).
Proof. Since the function $e^{f+\bar{f}}$ is a real function, the isomorphism $\psi_\circ$ preserves the real structure. The other facts are easy to see. Note that if $\tilde\partial$ defines the holomorphic structure on $\mathcal{H}_\circ\text{top}$, the pull-back operator $\psi_\circ^* (\tilde\partial) = e^{-f-\bar{f}} \cdot \tilde\partial \cdot e^{f+\bar{f}} = \tilde\partial + \bar{\partial f}$ defines the holomorphic structure on $\mathcal{H}_\circ$. □

Remark 4.55. The operator $\hat{\partial}$ does not correspond to the Witten map $I^*_W$, since there is $(I^*_W)^2 = T$, the monodromy transformation.

4.3.4. Flat connection and horizontal sections of bundle $\mathcal{H}_\circ\text{top}$.

Fix $(\tau_0, t_0) \in S_r$, and let $(\tau, t) \in S_r, f := f(\tau, t) = \tau f_t$ and $\mathcal{O}_c^\circ (\tau, t) \in H_0(M, f_{(\tau', t')}, \mathbb{Z})$.

We know that the $n$ form $S^- a = S^- (\tau, t) = e^{f(\tau, t) - H(\tau, t)\alpha_a}$ is d-closed. Now using the formulas

$$D_a \alpha_a = \bar{\partial} f (\gamma_{\alpha_a}), (\gamma_{\alpha_a}) = \bar{\partial} f (\gamma_{\alpha_a}) + \bar{\partial} (f \alpha_a),$$

we have

$$D_a S^- a = \bar{\partial} (e^{f+\bar{f}} \alpha_a) - \Gamma_a (e^{f+\bar{f}} \alpha_a)$$

$$= e^{f+\bar{f}} (\bar{\partial} f \alpha_a + D_a \alpha_a)$$

$$= (B_{\alpha_a}) e^{f+\bar{f}} \alpha_a + e^{f+\bar{f}} (\bar{\partial} f (\gamma_{\alpha_a}) + \bar{\partial} f (\gamma_{\alpha_a}))$$

$$= B_{\alpha_a} S^- a + d(e^{f+\bar{f}} (\gamma_{\alpha_a})).$$

This is equivalent to

$$\nabla_i S^- a = d(e^{f+\bar{f}} (\gamma_{\alpha_a})).$$

Similarly, we have

$$\nabla_i S^- a = d(e^{f+\bar{f}} (\gamma_{\alpha_a})).$$

On the other hand, we can consider the covariant derivative along $\tau$ direction. We knew that

$$D_{\tau} \alpha_a = \bar{\partial} f (\gamma_{\alpha_a})_{\tau}, (\gamma_{\alpha_a})_{\tau} = \bar{\partial} f (f \alpha_a).$$

So

$$D_{\tau} S^- a = e^{f+\bar{f}} (\bar{\partial} f \alpha_a + D_{\tau} \alpha_a)$$

$$= e^{f+\bar{f}} \frac{1}{\tau} (\bar{\partial} f \alpha_a + \bar{\partial} f (\gamma_{\alpha_a})_{\tau} + \bar{\partial} f (\gamma_{\alpha_a})_{\tau}),$$

which is equivalent to

$$\nabla_{\tau} S^- a = \frac{1}{\tau} d(e^{f+\bar{f}} (\gamma_{\alpha_a})_{\tau}).$$

Similarly, we have

$$\nabla_{\tau} S^- a = \frac{1}{\tau} d(e^{f+\bar{f}} (\gamma_{\alpha_a})_{\tau}).$$

In summary, we have

**Lemma 4.56.**

$$\nabla_i S^- a = d(e^{f+\bar{f}} (\gamma_{\alpha_a})_{\tau}), \nabla_i S^- a = d(e^{f+\bar{f}} (\gamma_{\alpha_a})_{\tau})$$

$$\nabla_{\tau} S^- a = \frac{1}{\tau} d(e^{f+\bar{f}} (\gamma_{\alpha_a})_{\tau}), \nabla_{\tau} S^- a = \frac{1}{\tau} d(e^{f+\bar{f}} (\gamma_{\alpha_a})_{\tau}).$$
Theorem 4.58. and is the fundamental solution matrix of the equation:

\[ C \exists \]

where the matrix \( \int \) vectors in Theorem 4.24 to the frame consisting of the sections defined before.

Theorem 4.57. have the following result by the above analysis.

Lemma 4.59. Assume that \( (\tau_0, t_0) \in S_\tau \) and \( c, \tau \), \( f = 1, \cdots, a \) be the dual cycle of \( S_\tau \). For any \( a = 1, \cdots, \mu \), we have

\[ d(\mathcal{C}_f^-(\tau, t), S_a^-(\tau, t)) = 0 = \mathcal{C}_f^-(\tau, t), \mathcal{D}(S_a^-(\tau, t)). \] (279)

Proof. Since \( \mathcal{C}_f^-(\tau, t) \) be the dual cycle, we have,

\[ \langle \mathcal{C}_f^-(\tau, t), S_a^-(\tau, t) \rangle = \delta_{fa}, \]

and so

\[ d(\mathcal{C}_f^-(\tau, t), S_a^-(\tau, t)) = 0. \]

On the other hand, we have

\[ \langle \mathcal{C}_f^-(\tau, t), \mathcal{D}(S_a^-(\tau, t)) \rangle = 0, \]

and the conclusion is proved. \( \Box \)

Theorem 4.60. The connection \( \mathcal{D} \) can be identified with the topological Gauss-Manin connection \( \mathcal{D}_{\text{top}} \) over \( S_\tau \) of the dual bundle \( \mathcal{H}_{\text{top}} \) of \( H^\tau \).
Proof. Since any local section \( S(t) \) of \( \mathcal{H}_{0,\top} \) can be written as the linear combination of the product terms \( g(\tau,t)S_{\bar{a}}(\tau,t) \), by Lemma 4.59 it is easy to get
\[
d(\mathcal{C}_{\bar{a}}(\tau,t), S(\tau,t)) = \langle \mathcal{C}_{\bar{a}}(\tau,t), \mathcal{D}S(\tau,t) \rangle. \tag{280}
\]
Now if \( \mathcal{C}_{\bar{a}}(\tau,t) \) is a horizontal section w.r.t. \( \mathcal{D}_{\top} \) of the bundle \( H^0 \) and \( S_{\bar{a}} \) are the dual forms, then we can prove in the same way as in Lemma 4.59 that
\[
d(\mathcal{C}_{\bar{a}}(\tau,t), S_{\bar{a}}(\tau,t)) = \langle \mathcal{D}_{\top} \mathcal{C}_{\bar{a}}(\tau,t), S_{\bar{a}}(\tau,t) \rangle, \tag{281}
\]
and furthermore, the above equality holds for any section \( \mathcal{C}_{\bar{a}}(\tau,t) \) of \( H^0 \). Combining the two equalities (280) and (281), we actually obtain
\[
d(\mathcal{C}(\tau,t), S(\tau,t)) = \langle \mathcal{D}_{\top} \mathcal{C}(\tau,t), S(\tau,t) \rangle + \langle \mathcal{C}(\tau,t), \mathcal{D}S(\tau,t) \rangle.
\]
for any local sections \( \mathcal{C}(\tau,t) \) of \( H^0 \) and \( S(\tau,t) \) of \( \mathcal{H}_{0,\top} \). This proves the conclusion. \( \square \)

4.3.6. Holomorphic structure on \( \mathcal{H}_{0,\top} \).

Since the \( (0,1) \) part of \( \mathcal{D} \) is integrable, by the celebrated Koszul-Margrange integrability theorem (see [DK], Chapter 2), we know that \( \mathcal{H} \) has a holomorphic structure and the holomorphic structure is determined by the \( (0,1) \) part \( \mathcal{D}^{0,1} \) of \( \mathcal{D} \).

The identity \( [D_r, D_j] = -[B_r, B_j] \) shows that the curvature \( D_r := D \) along the deformation direction is
\[
F_{ij}^{D} = -[B_i, B_j]. \tag{282}
\]
Hence usually \( D \) is not a flat connection on the Hodge bundle \( \mathcal{H} \). Similarly the equation (8) and (5) in Fantastic equations show that \( D_r \) is not flat and the \( t \)-direction and the \( \tau \) direction has nontrivial curvature.

4.3.7. Correspondence.

We can think of the family of the closure of each chamber of \( S_r \) in \( S^1_m \) as an atlas covering \( S^1_m \). If we fix a basis \( \{\mathcal{C}_{\bar{a}}\} \) of \( H^0 \) in this chamber, the horizontal sections \( \Pi_{\bar{a}} \) whose component \( \Pi_{\bar{a}} \) are the integration of \( S_{\bar{a}} \) over the fixed cycle \( \mathcal{C}_{\bar{a}} \) are well defined over the closure of each chamber and form a trivialization of the bundle \( \mathcal{H}_{0,\top} \). The transformation between two chambers is given by the Picard-Lefschetz transformation formula.

Now the bundle \( \mathcal{H}_{0,\top} \) equipped with the data \( (S_{\bar{a}}^+, D + \bar{D}, \hat{\eta}, \tau_R, \hat{\mathbf{t}}) \) can be identified with the bundle \( \mathcal{H}_{0,\top} \) with the data \( (\Pi_{\bar{a}}, \mathcal{D}, \hat{\eta}, \tau_R, \hat{\mathbf{t}}) \). The correspondence is given by
\[
\begin{align*}
S_{\bar{a}}^+ &\Leftrightarrow \Pi^+_a, \\
D + \bar{D} &\Leftrightarrow \mathcal{D} = \partial - (B + \frac{\partial}{\tau}) + \bar{\partial} - (\bar{B} + \bar{\partial}), \\
\tau_R &\Leftrightarrow \tau_R, \\
\hat{\mathbf{t}}S_{\bar{a}}^+ &\Leftrightarrow \hat{\eta}\Pi^+_a := \Pi^+_a, \\
\hat{\eta}(S_{\bar{a}}^+, S_{\bar{b}}^+) &\Leftrightarrow \hat{\eta}(\Pi^+_a, \Pi^+_b) := (\Pi^+_a, \Pi^+_b)_{TT} := (\Pi^+_a)T \cdot (I^T_W)^{-1} \cdot (\Pi^+_b).
\end{align*}
\]
If \( \alpha_a, \alpha_b \) correspond to \( S_{\bar{a}}^+, S_{\bar{b}}^+ \) and to \( \Pi^+_a, \Pi^+_b \), then it is easy to check that
\[
\hat{\eta}(\Pi^+_a, \Pi^+_b) = \hat{\eta}(S_{\bar{a}}^+, S_{\bar{b}}^+) = \eta(\alpha_a, \alpha_b). \tag{283}
\]
We also have the formula for the pairings:
\[
\langle S_{\bar{a}}^+, S_{\bar{b}}^+ \rangle = \langle \Pi^+_a, \Pi^+_b \rangle_{\mathbf{t}}. \tag{284}
\]
There is analogous result for \( \mathcal{H}_{0,\top} \).

Therefore, we have the following conclusion:
Theorem 4.61. The correspondence between \((\mathcal{H}_{\Omega^0}, S_{\alpha}^{-}, D + \bar{\partial}, \eta, \tau_R, \hat{\omega})\) and \((\mathcal{H}_{\Omega^0}, \Pi_{\alpha}^{-}, \mathcal{D}, \tilde{\eta}, \tau_R, \hat{\omega})\) is an isomorphism between two real Hermitian bundles with the associated Hermitian connections.

Remark 4.62. The data \((\mathcal{H}_{\Omega^0}, \Pi_{\alpha}^{-}, \mathcal{D}, \tilde{\eta}, \tau_R, \hat{\omega})\) are well-defined on \(S_{\alpha}^{-} \subset S\). However, since the fiber \(\mathcal{H}_{\Omega^0}\) at \((\tau, t)\) is the dual space of \(H^0\), the bundle \(\mathcal{H}_{\Omega^0}\) can be naturally extended over \(S\) by taking the dual space of \(H^0\) for \((\tau, t) \in S - S_{m}^{1}\).

4.4. Deformation of flat connections. At first we discuss the deformation of flat connections of the bundle \(\mathcal{H}_{\Omega^0}\) and then the deformation can be transferred to the bundle \(\mathcal{H}_{\Omega^0}\) with base connection \(\mathcal{D}\). All the conclusions hold if we replace the connection \(D'\) by \(\bar{\partial}\).

4.4.1. Holomorphic frame of \(D\).

By Remark 2.65, if \((\tau, t) \in S_{m}\), then we can use the isomorphism \(i_{\Omega^0} : \mathcal{H}_{\Omega} \cong \Omega^0/df \wedge \Omega^n\) to construct a basis \(\alpha_a(\tau, t)\) of \(\mathcal{H}^{(\tau)}\) such that
\[
\alpha_a = i_{\Omega^0}(\alpha_a) + \bar{\partial}_j R_{a},
\]
where \(i_{\Omega^0}(\alpha_a)\) is holomorphic and uniquely determined by \(\alpha_a\). We call this basis as a holomorphic frame of the connection \(D\). Obviously, we have
\[
\bar{\partial}_j \alpha_a = \bar{\partial}_j (\partial_i R_{a}), \quad \bar{\partial}_i \alpha_a = \bar{\partial}_i (\partial_j R_{a}).
\]
This shows that for holomorphic frame of \(D\), the following Christoffel symbols vanish:
\[
(\Gamma_i)_{ab} = (\Gamma_i)_{ab} = (\Gamma_j)_{ab} = 0.
\]
So the connection \(D\) has the form \(D = D' + \bar{\partial}\) in a holomorphic frame \(\{\alpha_a\}\) of the Hodge bundle \(\mathcal{H}\). The Cauchy-Riemann operator \(\bar{\partial}\) can be decomposed into the sum along \(\tau\) and \(t\) directions: \(\bar{\partial} = \bar{\partial}_\tau + \bar{\partial}_t\). In this frame, we have
\[
\mathcal{D} = D' + \bar{\partial} - B - \frac{\hat{\omega}}{\tau} - \hat{\omega} - \frac{\hat{\omega}}{\tau}.
\]
Correspondingly we have the expressions on the topological bundle \(\mathcal{H}_{\Omega,\top}\). The Cecotti-Vafa’s equations and Fantastic equations also hold in this holomorphic frame.

4.4.2. Gauge Transformation.

In holomorphic frame, Equation (1) in Cecotti-Vafa’s equation and Equation (1) in the Fantastic equations have the following form
\[
\begin{cases}
\bar{\partial}_j B = 0 \\
\bar{\partial}_t B = -\frac{1}{\tau} \bar{\partial}_i \hat{\omega}.
\end{cases}
\]

Proposition 4.63. The following matrix equation:
\[
\begin{cases}
\bar{\partial} A = (B + \frac{\hat{\omega}}{\tau}) \\
D' A = 0
\end{cases}
\]
has a local solution \(A \in \text{End}(\mathcal{H})\) in \(\mathbb{C}^* \times S\) satisfying the initial condition at a point \(p_0 = (\tau_0, t_0)\):
\[
A(p_0) = \text{Id}.
\]
and furthermore, \(A\) can be chosen to be symmetric w.r.t. the metric \(\eta\), i.e., \(A = A^*\).

Remark 4.64. For any number \(k_0 \in \mathbb{C}\), \(A + k_0 \text{Id}\) is also a solution of (289).
**Proof.** Since the R.H.S. of the first equation satisfies the integrable condition:

\[ \tilde{\partial}(\tilde{B} + \frac{\mathcal{U}}{\tau}) = \tilde{\partial}_{\tau} \tilde{\partial} \tilde{B} + \frac{1}{\tau} \tilde{\partial}_{\tau} \tilde{\partial} \mathcal{U} = 0, \]

where we used the equation (288). Plus the second gauge equation and by Cauchy-Kovalevski theorem, we have the conclusion.

Due to fact that the connections \( D', \tilde{\partial} \) are metric connection w.r.t. \( \eta \) or \( g \) and the operator \( \tilde{B} + \frac{\mathcal{U}}{\tau} \) is symmetric w.r.t \( \eta \), we can choose the solution \( A \) to be symmetric. \( \square \)

**Lemma 4.65.** The solution \( A \) in Proposition 4.63 satisfies the commutation relations:

\[ [\tilde{\partial} A, A] = [\tilde{B} + \frac{\mathcal{U}}{\tau}, A] = 0. \] (291)

**Proof.** By the CV and Fantastic equations, we can easily check that

\[ \tilde{\partial}([\tilde{B} + \frac{\mathcal{U}}{\tau}, A]) = D'([\tilde{B} + \frac{\mathcal{U}}{\tau}, A]) = 0. \]

Hence \([\tilde{B} + \frac{\mathcal{U}}{\tau}, A]\) is a constant, but by normalization condition \( A(p_0) = \text{Id} \), we know that it must be zero. \( \square \)

**Corollary 4.66.** Let \( A^\dagger \) be the conjugate operator of \( A \) with respect to the metric \( g \). Then we have

\[ \begin{cases} D'A^\dagger = B + \frac{\mathcal{U}}{\tau} \\ \tilde{\partial}A^\dagger = 0 \end{cases} \] (292)

and

\[ (A^\dagger)^* = A^*, \quad [A^\dagger, B + \frac{\mathcal{U}}{\tau}] = 0. \] (293)

**Proof.** Taking the conjugate of Equation (289), we have

\[ (\tilde{\partial}A)^\dagger = (\tilde{B} + \frac{\mathcal{U}}{\tau})^\dagger = B + \frac{\mathcal{U}}{\tau}. \]

Compute the matrix \((\tilde{\partial}A)^\dagger\). We have

\[ g((\tilde{\partial}A)^\dagger \alpha_a, \alpha_b) = g(\tilde{\partial}A \alpha_b, \alpha_a) \]

\[ = g(\tilde{\partial}A \alpha_a, \alpha_b) - g(A^\dagger \alpha_a, \tilde{\partial} \alpha_b) \]

\[ = \partial g(A^\dagger \alpha_a, \alpha_b) - g(D' \alpha_a, A \alpha_b) - \left( \partial g(A^\dagger \alpha_a, \alpha_b) - g(D' \alpha_a, A \alpha_b) \right) \]

\[ = g(D' \alpha_a, A \alpha_b) - A^\dagger (D' \alpha_a, \alpha_b) \]

\[ = g(D', A^\dagger \alpha_a, \alpha_b). \]

Therefore, we have

\[ D'A^\dagger = B + \frac{\mathcal{U}}{\tau}. \]

Similarly, we can prove the other equalities. \( \square \)

**Proposition 4.67.** Define the gauge transformation \( S = e^A \). Then

\[ S^{-1} \varphi S = \nabla^G + \tilde{\partial}, \] (294)

where

\[ \nabla^G = D' - e^{-A}(B + \frac{\mathcal{U}}{\tau})e^A = e^{-A}(D' - B - \frac{\mathcal{U}}{\tau})e^A. \] (295)

is a \((1,0)\)-form satisfying:

\[ (\nabla^G)^2 = 0, \quad [\tilde{\partial}, \nabla^G] = 0. \] (296)
Proof. The direct computation shows that

\[ S^{-1} \partial S = D' + S^{-1} D'S - S^{-1}(B + \frac{W}{\tau})S + \tilde{\partial} + S^{-1}\{\tilde{\partial}S - (\tilde{B} + \frac{W}{\tau})S\} \]

\[ = \tilde{\partial} + D' + e^{-\Lambda} D'e^\Lambda - e^{-\Lambda}(B + \frac{W}{\tau})e^\Lambda + e^{-\Lambda}(\tilde{\partial}e^\Lambda - (\tilde{B} + \frac{W}{\tau})e^\Lambda) \]

\[ = \tilde{\partial} + D' - e^{-\Lambda}(B + \frac{W}{\tau})e^\Lambda. \]

In the above computation, we have implicitly used the commutation relation \([\tilde{\partial}, A] = 0\) which was proved in Lemma 4.65. This commutation relation will be used frequently in the later computation without saying.

The other conclusions are the corollary of the fact \(\mathcal{G}^2 = 0\). \(\square\)

4.4.3. **One parameter transformation group of flat connections.**

Now we consider a family of connections: for \(s \in \mathbb{R}\),

\[ \nabla^{G,s} = sD' - e^{-s\Lambda}(B + \frac{W}{\tau})e^\Lambda. \tag{297} \]

**Lemma 4.68.** \(\nabla^{G,s}\) satisfies:

\[ (\nabla^{G,s})^2 = 0, \quad [\tilde{\partial}, \nabla^{G,s}] = 0, \quad [D', \nabla^{G,s}] = 0. \tag{298} \]

**Proof.** The first equality is obvious. For the second one, we have the computation:

\[ [\tilde{\partial}, \nabla^{G,s}] = [\tilde{\partial}, e^{-s\Lambda}(sD' - \frac{W}{\tau})e^\Lambda] \]

\[ = [\tilde{\partial}, e^{-s\Lambda}](sD' - \frac{W}{\tau})e^\Lambda + e^{-s\Lambda}[\tilde{\partial}, (sD' - \frac{W}{\tau})e^\Lambda] \]

\[ = [\tilde{\partial}, e^{-s\Lambda}](sD' - \frac{W}{\tau})e^\Lambda + e^{-s\Lambda}[\tilde{\partial}, (sD' - \frac{W}{\tau})]e^\Lambda - e^{-s\Lambda}(sD' - \frac{W}{\tau})[\tilde{\partial}, e^\Lambda] \]

\[ = e^{-s\Lambda}\left(-s\tilde{\partial}[\tilde{B} + \frac{W}{\tau}, D'] + s\left(\tilde{\partial}D' + [\tilde{B} + \frac{W}{\tau}, B] + \frac{W}{\tau}\right) - [\tilde{\partial}, B + \frac{W}{\tau}]\right)e^\Lambda \]

\[ = 0 \]

We have used CV equations and Fantastic equations in the last equality.

For the third one, we use the relation \([D', B + \frac{W}{\tau}] = 0\) and the fact that \(D' A = 0\), we have

\[ [D', \nabla^{G,s}] = e^{s\Lambda}[D', sD' - \frac{W}{\tau}]e^{-s\Lambda} = 0. \]

So we are done. \(\square\)

Take the derivative with respect to \(s\), one has:

\[ \nabla^{G,s} := \frac{\partial}{\partial s} \nabla^{G,s} = e^{-s\Lambda}[D' + [A, B] + \frac{W}{\tau}]e^\Lambda. \tag{299} \]

Note that

\[ \nabla^\eta := \nabla^{G,0} = D' + [A, B + \frac{W}{\tau}]. \tag{300} \]

is a \((1, 0)\)-type connection.

Using this connection, the other connections at time \(s\) can be represented by means of the adjoint action and the corresponding integrals.
Proposition 4.69. We have the formulas:
\[ \nabla^\eta_s = Ad(e^{-\eta A})(\nabla^\eta), \]
\[ \nabla^{G,s} = -(B + \frac{\eta}{\tau}) + \int_0^s (e^{-\eta \text{Ad}(A)})(\nabla^\eta)ds. \]
and
\[ \nabla^\eta = D' - D'(A^\dagger, A), \]
\[ \nabla^{G,s} = sD' - D'(e^{-\eta \text{Ad}(A^\dagger)}). \]

Theorem 4.70. The connection \( \nabla^\eta \) satisfies the following identities:
\[ [\bar{\partial}, \nabla^\eta] = 0, \quad \nabla^\eta \eta = 0, \quad \bar{\partial} \eta = 0, \quad \nabla^\eta(B + \frac{\eta}{\tau}) = 0, \quad (\nabla^\eta)^2 = 0 \]
\[ \nabla^\eta A^\dagger - [B + \frac{\eta}{\tau}, [A^\dagger, A]] - (B + \frac{\eta}{\tau}) = 0 \]
\[ \frac{d}{ds} \nabla^{G,s} = -[\nabla^{G,s}, \frac{d}{ds} \nabla^{G,s}], \]
\[ \frac{d}{ds} \nabla^\eta = -[\nabla^\eta, \frac{d}{ds} \nabla^\eta]. \]

Proof. Since \( g \) and the real structure is parallel to the connection \( D \), we have \( (D' + \bar{\partial}) \eta = 0 \), which is equivalent to \( D' \eta = 0, \bar{\partial} \eta = 0 \).

For each \( s \), we have
\[ (\nabla^{G,s})^2 = 0, \quad [\bar{\partial}, \nabla^{G,s}] = 0. \]

Take the derivative to the above identities at \( s = 0 \), there is
\[ \nabla^\eta(B + \frac{\eta}{\tau}) = 0, \quad [\bar{\partial}, \nabla^\eta] = 0. \]

Since
\[ [A, B + \frac{\eta}{\tau}]^* = [B^* + \frac{\eta}{\tau} A^*] = [B + \frac{\eta}{\tau}, A] = -[A, B + \frac{\eta}{\tau}], \]
this together with the fact \( D' \eta = 0 \) shows that \( \nabla^\eta \eta = 0 \).

Finally, we want to prove \( (\nabla^\eta)^2 = 0 \). We take the derivatives twice to \( (\nabla^{G,s})^2 = 0 \), then
\[ \frac{d}{ds} \nabla^{G,s} = -[\nabla^{G,s}, \frac{d}{ds} \nabla^{G,s}], \]
\[ \frac{d}{ds} \nabla^\eta = -[\nabla^\eta, \frac{d}{ds} \nabla^\eta]. \]

Take \( s = 0 \), then we have
\[ (\nabla^\eta)^2 = -[B + \frac{\eta}{\tau}, [A, \nabla^\eta]]. \]

Compute the right hand side,
\[ -[B + \frac{\eta}{\tau}, [A, \nabla^\eta]] = -[[B + \frac{\eta}{\tau}, A], \nabla^\eta] + [[B + \frac{\eta}{\tau}, \nabla^\eta], A] \]
\[ = [\nabla^\eta, \nabla^\eta] = 2(\nabla^\eta)^2, \]
where we used the facts that \( [D', \nabla^\eta] = 0 = \nabla^\eta(B + \frac{\eta}{\tau}). \) Replacing the above equality into (308), we obtain
\[ (\nabla^\eta)^2 = 0. \]

To prove Equation (306), we use Corollary 4.66 and Proposition 4.69 as follows. Since
\[ \mathcal{D}^{1.0} = \nabla^\eta - [A, B + \frac{\eta}{\tau}] - (B + \frac{\eta}{\tau}), \]
we have
\[ \mathcal{D}^{1.0} A^\dagger = D' A^\dagger - [B + \frac{\eta}{\tau}, A^\dagger] = B + \frac{\eta}{\tau} - [B + \frac{\eta}{\tau}, A^\dagger]. \]
Hence
\[ [\nabla^g - [A, B + \frac{\mathcal{U}}{\tau}] - (B + \frac{\mathcal{U}}{\tau}), A^1] = B + \frac{\mathcal{U}}{\tau} - [B + \frac{\mathcal{U}}{\tau}, A^1]. \]
This gives Equation (306).

Define
\[ \eta' = (e^{-sA})^* \eta = \eta(e^{sA}, e^{sA}), \quad B^* = \text{Ad}(e^{-sA})(B), \quad \mathcal{U}^* = \text{Ad}(e^{-sA})(\mathcal{U}). \] (309)

It is easy to see the following result hold:

**Proposition 4.71.** For any \( s \in \mathbb{R} \), there is
\[ [\bar{\partial}, \nabla^g] = 0, \quad \nabla^g \eta^t = 0, \quad (\bar{\partial} - s(\bar{B} + \frac{\bar{\mathcal{U}}}{\tau}))\eta^t = 0, \quad \nabla^g(B^t + \frac{\mathcal{U}^t}{\tau}) = 0, \quad (\nabla^g)^2 = 0. \] (310)

**Proof.** The second, fourth and fifth equalities are the result of composite computation. The third one is easy to see. The first one comes from the identity:
\[ [D', \bar{\partial}] = -[B + \frac{\mathcal{U}}{\tau}, \bar{B} + \frac{\bar{\mathcal{U}}}{\tau}]. \]

\[ \square \]

4.4.4. **Holomorphic structure of** \( (\mathcal{H}_{\text{top}}, \mathcal{D}) \). 

Now we can do gauge transformation to the connection \( \mathcal{D} \) of \( \mathcal{H}_{\text{top}} \) by replacing the frame \( \Pi'^a \) by \( \Pi^{-A}_a := e^{-A} \cdot \Pi'^a \). In this frame, the connection has the decomposition:
\[ \mathcal{D} = \nabla + \bar{\nabla} = \nabla^G + \bar{\partial}. \]
and \( e^{-A} \cdot \Pi^{-A}_a, a = 1, \cdots, \mu \) are the solutions of the following equations:
\[ \nabla^G y(t) = 0, \quad \bar{\nabla} y(t) = 0. \] (311)
The fundamental solution matrix is \( \Pi^{-A} := e^{-A} \cdot \Pi^{-A} = (e^{-A} \cdot \Pi^{-A}_1, \cdots, e^{-A} \cdot \Pi^{-A}_\mu) \).

In summary, we have the following conclusion:

**Theorem 4.72.** Given a holomorphic frame \( \{\alpha_a, a = 1, \cdots, \mu\} \) of the Hodge bundle \( \mathcal{H} \) with respect to the connection \( \mathcal{D} + \bar{\mathcal{D}} \), there is the correspondent frame \( \{S^{-A}_a = e^{f^+} \alpha_a, a = 1, \cdots, \mu\} \) of \( \mathcal{H}_{\text{top}} \). The integration of \( S^{-A}_a \) over the Lefschetz thimbles induces the horizontal frame \( \{\Pi^{-A}_a, a = 1, \cdots, \mu\} \) of the bundle \( (\mathcal{H}_{\text{top}}, \mathcal{D}) \). The frame \( \{\Pi^{-A}_a, a = 1, \cdots, \mu\} \) is the holomorphic and horizontal frame of the bundle \( (\mathcal{H}_{\text{top}}, e^{-A} \cdot \mathcal{D}, e^{-A} = \nabla^G + \bar{\partial}) \). In particular, the fundamental solution matrix \( \Pi^{-A}(\tau, t) \) gives a holomorphic mapping
\[ \Pi^{-A} : S_{\tau} \to GL(\mu, \mathbb{C}). \] (312)

4.5. **Monodromy, Picard-Fuchs equation and mixed Hodge structure.**

This section will discuss the global behavior of the bundle \( \mathcal{H}_{\text{top}} \) with flat connection \( \mathcal{D} \) over the deformation space \( \mathbb{C}^* \times S \). There are two ways to look at this bundle: The first way is that we can fix the parameter \( \tau \in \mathbb{C}^* \) and discuss the restricted bundle \( \mathcal{H}_{\text{top}} \to S \) with the corresponding structure; the second way is that we can fix the deformation parameter \( t \in S \) and consider the bundle \( \mathcal{H}_{\text{top}} \to \mathbb{C}^* \). The second way is naturally related to the meromorphic bundle over \( \mathbb{P}^1 \) and the Riemann-Hilbert-Birkhoff problem. All we want to understand is the monodromy operator, which will give us the grading of the bundle, i.e., the mixed Hodge structure of \( \mathcal{H}_{\text{top}} \to \mathbb{C}^* \times S \).

Above all we take the first consideration and fix the parameter \( \tau \in \mathbb{C}^* \) and consider the restricted bundle \( \mathcal{H}_{\text{top}} \to S \).
Let \( i_1 : \mathbb{C}^* \hookrightarrow \mathbb{C}^* \times S, \ i_2 : S \hookrightarrow \mathbb{C}^* \times S \) be the inclusion map. We consider firstly the pull-back structure of the bundle \( \mathcal{H}_{\text{top}} \rightarrow \mathbb{C}^* \times S \) to \( S \) by \( i_2 \). So we fix \( \tau = \tau_0 \) and denote by \( f_i := f_{i(\tau_0,0)} \).

Now \( \mathcal{H}_{\text{top}} \rightarrow S \) is a flat bundle and the integrable connection \( \mathcal{D} \) is the natural extension of the topological Gauss-Manin connection \( \mathcal{D}_{\text{top}} \) defined on \( \mathcal{H}_{\text{top}} \rightarrow S^1_m \). The singular complimentary set \( S - S^1_m \) is of complex codimension 1. We want to study the limit behavior of the geometrical structure around the singular set.

Note that the set \( S - S^1_m \) is the disjoint union of two sets \( S - S_m \) and \( S_m - S^1_m \). The front set contains more singular points \( t \) that \( f_i \) has degenerate critical points. The rear set contains the Morse critical points which are the intersection points of "two walls".

Because the monodromy transformation \( T \) is the same when restricted to the complex 1-dimensional space \( l \) corresponding to one deformation direction except the direction of constant term in \( f_i \) (See Theorem 4.43 of [Zo]). We can only consider the deformation along this direction.

Since we are discussing the local deformation, we can assume without loss of generality that \( 0 \in S - S^1_m \) and let \( f_i = f_{i0} + tz_1 \) (otherwise, we can do a linear coordinate transformation). Choose a regular point \( t_0 \) as the base point on \( l \). Since \( f_i \) is fundamentally tame on \( M \), we can choose a loop \( \gamma \subset S^1_m \cap l \) based on \( t_0 \) such that the interior of \( f^{-1}_i(\gamma) \) contains the whole critical points of \( f_i \) for small \( t \in l \).

Since \( t_0 \) is regular, we can choose integral basis in \( \mathcal{H}_{\text{top},h} \), i.e., the dual pairing of the elements in this basis with the Lefschetz thimbles are integers. In the integral basis, the monodromy matrix \( T = (h^* \gamma)^{-1} \) has integer entries.

We can also choose the integral basis to be holomorphic. So in holomorphic frame, the period matrix is holomorphic and satisfies the following equation:

\[
\nabla^G \Pi^{-h} = (\frac{d}{dt} + \Gamma(t))\Pi^{-h} = 0.
\]

Consider the equation satisfied by the parallel section \( \phi(t) \),

\[
\frac{d\phi}{dt} + \Gamma(t)\phi = 0.
\]

Because of the instanton exchange phenomenon, the solutions are multiple valued sections. One of the fundamental matrix of solutions of this system is the periodic matrix, since the period matrix \( \Pi^{-h} \) is non degenerate.

The matrix representation of the monodromy operator \( T \) is uniquely determined by a basis in \( \mathcal{H}_{\text{top},h} \). Now we fix a basis of \( \mathcal{H}_{\text{top},h} \) and write

\[ T = e^{2\pi i R}. \]

Notice that the matrix function \( i^R \) has the same monodromy of the fundamental matrix \( \Pi^{-h} \) of solutions. So

\[ \Pi^{-h} = Z(t)i^R, \]

such that \( Z(t) \) is a single valued function (with possible singularities at \( t \)) on \( S_m \). Let \( C \) be a constant matrix, then we have

\[ \Pi^{-h} C = Z(t)Cf^{-1}RC. \]

Therefore, if we choose the suitable initial basis \( C \) at \( t_0 \), the matrix \( R \) can be reduced to the Jordan normal form. At first, this Jordan norm form can be divided into a big block diagonal matrix and the number of the big blocks is exactly the number of isolated critical points, i.e., the global monodromy matrix can be reduced to the local monodromy group \( T_i \) around the critical point \( z_i \). Secondly the local monodromy group corresponding to a
critical point is still a block matrix and each block is a Jordan standard block (still denoted by) $R_i$ with all the diagonal entries to be $\alpha_i$, the entries in the lower diagonal are 1 and other entries vanish. Let $\alpha_{ii}$ be the eigenvalue of one Jordan block $R_{ii}$ of the local monodromy group $T_i$. Then
\[ R_{ii} = \alpha_{ii} I + N_{ii}, \]
where $N_{ii}$ is the corresponding nilpotent matrix. In summary, for a local monodromy $T_i$, there is the matrix decomposition:
\[ T_i = e^{2\pi i R_i} = \lambda_i e^{2\pi i N_i} = \lambda_i T_i^u, \tag{315} \]
where $\lambda_i = \text{diag}(e^{2\pi i \alpha_{i1}}, \ldots, e^{2\pi i \alpha_{in}})$ is the eigenvalue of $T_i$ and $T_i^u$ is the unipotent part of the monodromy:
\[ N_i = \frac{1}{2\pi i} \log T_i^u. \tag{316} \]

**Theorem 4.73** (Regularity theorem). The periodic matrix is uniformly bounded near the critical points $t_i \in l$. In particular, if $0 \in S - S_m$, then the periodic matrix is uniformly bounded near $0 \in l$.

**Proof.** We can assume that the isolated critical point $t_i$ is the origin. Due to the decay property of the harmonic forms and the tameness of $f_0$, it is easy to see that the periodic matrix $\Pi_{\mathcal{H}}$ is uniformly bounded for small $t$. □

Once we have the regularity theorem, the connection matrix of $\nabla^G$ has the decomposition under a (possible meromorphic) gauge transformation at $t_i$:
\[ \Gamma_i(t) = \Gamma_0(t) + \Gamma_{i1}(t), \]
where $\Gamma_{i1}$ is a constant matrix. This fact is based on the singularity study of the coefficient matrix $\Gamma_i(t)$ and the fundamental matrix of solutions of the Picard-Fuchs type equation. The detail proof can be found in section 7.3-7.7 of [Ku].

Therefore we can define the residue of the connection $\nabla^G$ at $t_i$ as
\[ \Gamma_{i0} = \text{res}_{t_i} \nabla^G. \]

The proof of the following two theorems are the same to that in [Ku].

**Theorem 4.74.** The local monodromy group $T_i$ around $t_i$ can be extended to the point $t_i$ such that $\lim_{t \to t_i} T_i(t) = T_{i,\text{inf}}$, where
\[ T_{i,\text{inf}} = e^{-2\pi i \text{res}_{t_i} \nabla^G}. \tag{317} \]

Here $T_{i,\text{inf}}$ is called the infinitesimal monodromy transformation at $t_i$.

For local monodromy transformation $T_i$ around a point $t_i$ such that $f_i$ has only isolated critical point, we have the following monodromy theorem.

**Theorem 4.75.**

1. $T_i$ can be reduced to the Jordan normal form, and the eigenvalue of each Jordan block is the root of unity, i.e., there is an integer $N$ such that $T_i^N$ is unipotent.
2. The size of the Jordan block does not exceed $n + 1$, i.e., $(T_i^N - I)_{n+1} = 0$.

**Proof.** The proof is the same to the monodromy theorems appeared in the deformation theories, for instance, the deformation theory of projective varieties or the deformation theory of singularities (see the comment in [Ku]). The key point is that the monodromy matrix is unimodular integral matrix. □
Remark 4.76. The local monodromy transformation $T_i$ and the infinitesimal monodromy transformation $T_{i,\text{inf}}$ have the same eigenvalues, since the characteristic polynomial of $T_i$ has integer coefficients and does not change as the base point $t_0$ tending to $t_i$.

Now we study the action of the local or infinitesimal monodromy groups.

Lemma 4.77. Let $C_h$ be a chamber in $S$ and $t_0 \in C_h$. Then

$T_{h,\text{inf}} = 1d.$

Proof. Since $C_h$ is open, there is open set $U_0 \ni t_0$. It suffices to prove that for any loop $\gamma \subset U_0$, the monodromy transformation along $\gamma$ is the identity.

Fix a basis $\{\pi_\tau(0), \cdots, \pi_\mu(0)\}$ of $\mathcal{H}_{\top}(\gamma, t_0)$ and assume that $\{\pi_\tau(\tau), \cdots, \pi_\mu(\tau)\}$ is the parallel transportation by connection $\nabla^G$ along the loop $\gamma$, where $\tau \in [0, 1]$. Then the monodromy group is given by the matrix $\tilde{\gamma}(\pi_\tau(0), \pi_\tau(1))$. If $\pi_\tau(0)$ is chosen to be the dual cocycle of the Lefschetz thimble $\mathcal{H}(\gamma, t_0)$, $a = 1, \cdots, \mu$, then

$\tilde{\gamma}(\pi_\tau(0), \pi_\tau(1)) = \tilde{\gamma}(s_{\tau}^0(0), s_{\tau}^1(1)) = \int_{\mathcal{H}(\gamma, t_0)} s_{\tau}^1(1).$

Since the loop $\gamma$ is in the chamber $C_h$, the moving of the Lefschetz thimble will not produce instantons, i.e., the Lefschetz thimble keep their relative homological classes. Therefore, we have

$$\int_{\mathcal{H}(\gamma, t_0)} s_{\tau}^1(1) = \int_{\mathcal{H}(\gamma, t_0)} s_{\tau}^1(1) = \delta_{ij}.$$  

This proves the result. \qed

The proof of the above lemma also shows that

Corollary 4.78. If $t_0$ is a point on the wall of chambers, then the infinitesimal monodromy group $T_{h,\text{inf}}$ is nontrivial.

4.5.1. Mixed Hodge structure.

Now fix a point $t_0 \in S^1_m$ and let $T \in \text{Aut}(\mathcal{H}_h)$ be the monodromy transformation. By monodromy theorem, we can choose a special horizontal basis of $\mathcal{H}_{\top}(\gamma, t_0)$ such that $T$ can be expressed as a Jordan normal matrix:

$$T = \begin{pmatrix} J_{\mu_1} & 0 & \cdots & 0 \\ 0 & \ddots & \cdots & \vdots \\ \vdots & \cdots & \ddots & \vdots \\ 0 & \cdots & 1 & J_{\mu_n} \end{pmatrix}, \text{ where } J_{\mu_j} = \begin{pmatrix} \beta_{\mu_1} & \cdots & 0 \\ 1 & \ddots & \vdots \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 & \beta_{\mu_n} \end{pmatrix}$$

and $\mu_1 + \cdots + \mu_n = n$.

So we have the decomposition $T = e^{2\pi i T_{\text{inf}}} = e^{2\pi i \beta} e^{2\pi i N} = T^T_{\text{inf}}$, where $\beta = \text{diag}(\beta_{\mu_1}, \cdots, \beta_{\mu_n})$, $\lambda_i = e^{2\pi i \beta}$ is the eigenvalue of $T$, and

$$N = \frac{1}{2\pi i} \log T_{\text{inf}}$$

is nilpotent matrix. Here the matrix $N$ is uniquely determined by $T$ and the matrix $\beta$ is uniquely determined modulo a scalar matrix.

Now the parallel transportation of the connection $\nabla^G$ gives a splitting of the bundle over $S^1_m$:

$$\mathcal{H}_{\top}(\gamma, t_0) = \mathcal{H}_{\delta}^{j} \text{ if } V_i,$$
where the subbundles \( V_i \) correspond to the Jordan block \( J_i \) and is generated by the set of sections:

\[
V_i = \{ \pi \mid (J_i - \mu)\pi = 0 \}.
\]  

(318)

Therefore for any horizontal section \( \pi^- \) of \( \mathcal{H}_{0,\text{top}} \), we have the decomposition:

\[
\pi^- = \sum_i \pi_i^- \in V_i.
\]

(320)

If \( \pi^- \) is a section of any subbundle \( V_i \), then \( \pi^- \) is said to be homogeneous and is denoted by \( \pi_i^- \). Its spectrum with respect to \( \beta \) is defined as \( \beta(\pi_i^-) \) equals to some \( \beta_m \). We can shift the matrix \( \beta \) by a scalar matrix such that its minimum spectrum point lies in \([0, 1]\) and denote by \( \hat{\beta} \) the maximum spectrum point. Let \( \text{Sp}(\beta) \) be the spectrum set of the matrix \( \beta \).

On the other hand, we can consider the monodromy transformation of the bundle \( \mathcal{H}_{0,\text{top}} \rightarrow S^1 \), which is the inverse of \( T \). The corresponding matrices are \(-\beta, -N\). Now we consider the total space \( \mathcal{H}_{\text{tot},\text{top}} = \mathcal{H}_{0,\text{top}} \otimes \mathcal{H}_{0,\text{top}} \) and it has a natural Hodge filtration defined as follows. At first, we define the degree of each homogeneous element \( \pi = \pi_-(\pi_+) \) in \( \mathcal{H}_{0,\text{top}} \) or \( \mathcal{H}_{0,\text{top}} \) by shifting its spectrum by \( \hat{\beta} \), i.e.,

\[
\text{deg}(\pi) = \beta(\pi) + \hat{\beta}.
\]  

(319)

Hence the degree of any homogeneous element lies in the interval \([0, 2\hat{\beta}]\) and we denote by \( \Lambda_{\text{deg}} \) the set of degrees of all homogeneous elements.

**Definition 4.79.** The Hodge filtration \( F^p, \mathcal{H}_{\text{tot},\text{top}} \) \( p \in \Lambda_{\text{deg}} \) is a subbundle of \( \mathcal{H}_{\text{tot},\text{top}} \) which is generated by all the homogeneous horizontal sections \( \pi^h \) with \( \text{deg}(\pi^h) \leq 2\hat{\beta} - p \). Therefore, we obtain a decreasing filtration of \( \mathcal{H}_{\text{tot},\text{top}} \):

\[
0 \subset F^{2\hat{\beta}} \subset \cdots \subset F^0 = \mathcal{H}.
\]  

(320)

With the real operator \( \tau_R \), the lattice \( H_Z \) generated by the integral horizontal sections of \( \mathcal{H}_{\text{tot},\text{top}} \), the triple \( (H_Z, \tau_R, F^* \mathcal{H}_{\text{tot},\text{top}}) \) is called forming a pure Hodge structure with weight \( \Lambda_{\text{deg}} \).

**Remark 4.80.** In Griffiths’ work constructing the variation of polarized Hodge structure of smooth projective varieties, the Weil operator has the crucial role which gives an auto-morphism of the Hodge bundle with the fixed type. In our case the corresponding operator is \( \hat{\pi} \) operator, which does not preserve the bundle \( \mathcal{H}_{0,\text{top}} \), but exchange \( \mathcal{H}_{0,\text{top}} \) and \( \mathcal{H}_{0,\text{top}} \). This is why we take account the total space \( \mathcal{H}_{\text{tot},\text{top}} \) (or correspondingly \( \mathcal{H}_{\text{tot}} \) of Hodge bundle).

Since our construction used horizontal sections of \( \mathcal{H}_{\text{tot},\text{top}} \), the filtration naturally satisfies the Griffiths’ transversality condition: \( \nabla^G F^p \subset F^{p-1} \otimes \Omega^1(S^1) \). The pairing \( \langle \phi, \psi \rangle_\tau \) is \((-1)^n\) symmetric, non-degenerate and satisfies the positivity:

\[
\langle \phi, \hat{\pi} \tau_R(\phi) \rangle_\tau \geq 0.
\]

By the duality of the Gauss-Manin connection \( \mathcal{D} \) with the topological Gauss-Manin connection \( \mathcal{D}_{\text{top}} \), we know that \( \mathcal{D} \) is the metric connection with respect to the nondegenerate form \( \langle \cdot, \cdot \rangle_\tau \). We actually proved the following conclusion:

**Theorem 4.81.** The triple \( (H_Z, \mathcal{D}, \langle \cdot, \cdot \rangle, F^* \mathcal{H}_{\text{ind}}) \) forms a polarized variation of Hodge structure on \( S_\mu \) with weight \( \Lambda_{\text{deg}} \).

Since \( \nabla^G \) preserves the degree of a homogeneous section, it commutes with \( T_\mu \) and then with \( T_\nu \). So the commutation of the nilpotent operator \( N \) and \( \nabla^G \) allows us define a weight
filtration $W_\bullet$ of the bundle $\mathcal{H}_{\text{tot,top}}$ over $S^1_m$ by the action of the monodromy group such that

(i) $N(W_i) \subset W_{i-2}$,

(ii) $N' : Gr_r \to Gr_{r-1}$ is an isomorphism for $r$.

The following conclusion is obvious:

**Theorem 4.82.** $(H_Z, W_\bullet, F^\bullet)$ forms a mixed Hodge structure of $\mathcal{H}_{\text{tot,top}}$ over $S^1_m$.

Sometimes we are only interested in a local deformation space of $S$, for example, in the chamber $C_h$. When considering the local cases, the local monodromy group will become smaller. In particular, if we consider the deformation in the chamber $C_h$, the restricted Hodge bundle is semi-simple and can be decomposed as the direct sum of 1-dimensional line bundles. In this case, the weight filtration is trivial.

**Remark 4.83.** Once we obtained a polarized VHS, one can define the period domain in the flag manifold and study the behavior of the period mapping. For instance, since the transversality holds the period mapping is a holomorphic map from $S_r$ to the period domain. We can also study the limit Mixed Hodge structure and etc.

Therefore the concepts and the structures of VHS, period mapping and period domain has been generalized to much broad area beyond that has already been seen in complex geometry.

After that a natural problem is to compare the HS we built here with those already well-known. We will consider those problems in the forthcoming papers.

4.5.2. **Quasi-homogeneous case.**

Since the monodromy transformation is given by the Picard-Lefschetz transformation formula for the Lefschetz thimbles, which is determined by the topology of the manifold $M$, there is no general formula for the monodromy transformation. However, if $M = \mathbb{C}^n$, then the Picard-Lefschetz transformation formula is the same to that for the vanishing cycles (see [E]). We have the following conclusion:

**Proposition 4.84.** Let $f$ be a holomorphic function on $\mathbb{C}^n$ with isolated critical point and $(\mathbb{C}^n, f_t)$ be a strong deformation of the holomorphic function $f$ based on $S$ such that the global Milnor number of each $f_t$ equals to that of $f$. Then the global monodromy group of the corresponding Hodge bundle is the monodromy group of the singularity $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$.

**Remark 4.85.** The miniversal deformation of the simple singularities $A_n, D_n, E_6, E_7, E_8$, the hyperbolic singularities $T_{p,q,r}$ and the parabolic singularities $P_8, X_9, J_{10}$ in Arnold’s list satisfy the requirement in Proposition 4.84.

If the deformation of a non-degenerate quasi-homogeneous polynomial $W$ satisfying the requirements in Proposition 4.84, we have the following properties (see PP. 29 of [Ku]):

**Proposition 4.86.** The monodromy group $T$ on the deformation space $S$ is semi-simple and its eigenvalues are

$$\lambda_I = e^{2\pi i \alpha_I}, I = (i_1, \cdots, i_n) \in A, \alpha_I = \sum_{j=1}^n q_j(i_j + 1).$$

Here $A$ is the index set of the local algebra $Q_{W,0}$, and $\alpha_I$ is actually the degree of the element $\bar{z}^I d\bar{z}^1 \wedge \cdots \wedge d\bar{z}^n = \bar{z}^{i_1} \cdots \bar{z}^{i_n} d\bar{z}^1 \wedge \cdots \wedge d\bar{z}^n$. 
In this case the monodromy operator is semi-simple and the weight filtration is trivial. We can construct the filtration \( F^* \) for the bundle \( \mathcal{H}_{\text{tot,lop}} \), which is related to the Schrödinger operator \( \Delta_y \) as before. The weight is the set \( \Lambda_{\text{deg}} \) with the maximum degree \( 2c = 2 \sum_i (1 - 2q_i) \). If \( c \) is an integer, then there is an nonempty subset \( \Lambda' \subset \Lambda_{\text{deg}} \) which contains all integral degrees. The information of the filtration \( F^p, p \in \Lambda' \) was conjectured can be identified with the Hodge structure on the hypersurface \( W = 0 \) in the projective space \( \mathbb{P}^{n-1} \) (see [Ce1]). This is just the B side of Gepner’s idea [Ge] about LG/CY correspondence. We will also check this point in our future study.

4.5.3. A family of flat bundles over \( \mathbb{C}^* \). Now we consider the inclusion \( i_t := i_1 : \mathbb{C}^* \hookrightarrow \mathbb{C}^* \times S^1_m \) and the pull-back structure \( (i_t^* \mathcal{H}_{\text{tot,lop}} \to \mathbb{C}^* \times \mathbb{C}^*) \). \( i_t^* \mathcal{H} \) is the induced connection and has the splitting:

\[
\tau^* \mathcal{F} = \tau^* \mathcal{G} + \tau^* \mathcal{F}.
\]

Since \( \nabla^\mathcal{G} = \partial - e^{-\text{ad}A} (B + \frac{\partial_\tau \partial}{\tau} d\tau) \), we have

\[
\nabla^{\mathcal{G}} = \partial_x - \frac{e^{-\text{ad}A(x, \tau)}(\mathcal{H})}{\tau} d\tau.
\]

Now the horizontal section \( y(\tau, t) \) satisfies the above complex o.d.e:

\[
\frac{dy(\tau, t)}{d\tau} = \left( \frac{e^{-\text{ad}A(x, \tau)}(\mathcal{H})}{\tau} \right) y.
\]

Since we are interested in the case of \( \tau \to \infty \), we change the coordinate: \( \tau = 1/s \). So \( \partial_x = -s^2 \partial_x \) and the equation (322) becomes

\[
\frac{dy(\tau, s)}{ds} = -\left( \frac{e^{-\text{ad}A(x, \tau)}(\mathcal{H})}{s} \right) y.
\]

Combining with Theorem 4.72 we have the following interesting conclusion.

**Theorem 4.87.** The \( \mu \) vectors \( \Pi^1_{\text{top}}(1/s, t), a = 1, \cdots, \mu \) satisfy simultaneously the following two equations:

\[
\begin{cases}
\nabla^\mathcal{G} y = 0 \\
\frac{dy(\tau, t)}{ds} = -\left( \frac{e^{-\text{ad}A(x, \tau)}(\mathcal{H})}{s} \right) y.
\end{cases}
\]

Now for fixed \( t \in S^1_m(\tau) \), \( \tau_f \) has \( \mu \) nondegenerate critical points and it is possible that for some \( \tau \) there are 3 critical values having the same imaginary parts. The solutions \( \Pi^1_{\text{top}}(\tau, t) \) are not well-defined at those points and there is monodromy around these points. Therefore the equation (323) has more singular points except \( 0, \infty \). Equation (323) gives a monodromy deformation of o.d.e. The study of such o.d.e. is related to the Riemann-Hilbert problem, Painlevé equations and other integrable systems.

4.6. **Period matrix, Primitive vector and flat coordinate systems.**

4.6.1. **Computation of the period matrix.**

At the first glance, it seems that the computation of the period matrix needs to solve the Schrödinger equation, which is almost an impossible task. However, Under some mild assumption to the section bundle system \( (M, g, f) \) and its deformation, we can give the explicit formula of the period matrix, equivalently this gives the formula of the horizontal sections, which is the solution of the Gauss-Manin connection.

Let \( (M, g, f) \) be a strongly tame section-bundle system and \( M \) be stein. Let \( (\tau, t) \in S \), and consider the deformation \( f_{(\tau, t)} = \tau f_t \).
Remember that we have by Theorem 2.66 an isomorphism
\[ i_{0*} : \mathcal{F} \to \mathcal{F}_{hol} = \Omega^n / df_{(r,)} \wedge \Omega^{n-1}, \]
given by
\[ \alpha_a = i_{0*}(\alpha_a) + \tilde{\partial}_{(r,)} R_a, \] (325)
where \( R_a \) is a smooth \((n-1)\)-form.

At first we prove an important lemma.

**Lemma 4.88.** Let \( \{\alpha_a, a = 1, \cdots, \mu\} \) be a frame of the Hodge bundle \( \mathcal{F} \) over an open neighborhood \( U \). Let \( R \) be a smooth \( n-1 \) form in \( U \). If at any point \((\tau, t) \in U \), there is
\[ \sum_a \int_M |R| \cdot |\alpha_a| < \infty, \] (326)
then
\[ \int_{\mathcal{E}_0} e^{f+\tilde{f}} \tilde{\partial}_f R = \int_{\mathcal{E}_0} e^{f+\tilde{f}} \tilde{\partial}_f R = 0. \] (327)

In particular, this is true if \( R \) has only polynomial growth.

**Proof.** It suffices to prove the first identity. We have
\[
\int_{\mathcal{E}_0} e^{f+\tilde{f}} \tilde{\partial}_f R = \int_M PD(\mathcal{E}_0^-) \wedge e^{f+\tilde{f}} \tilde{\partial}_f R \\
= \int_M c_{\alpha} S_{\alpha}^+ \wedge e^{f+\tilde{f}} \tilde{\partial}_f R = \int_M c_{\alpha} e^{-f-\tilde{f}} \bar{\alpha}_{\epsilon} \wedge e^{f+\tilde{f}} \tilde{\partial}_f R \\
= \pm \int_M c_{\alpha} (\tilde{\partial}_f R, \bar{\alpha}_{\epsilon}) = 0.
\]

Here we have used the \( L^1 \) Stokes theorem and the fact that \( \tilde{\partial}_f \alpha_{\epsilon} = 0 \). \( \square \)

**Proposition 4.89.** If at any \((\tau, t)\), there is
\[ \sum_{a,b} \left( \int \alpha_a \cdot |R_b| + \int_M |R_a| \cdot |\alpha_a| \right) < \infty, \]
then we have
\[ \Pi_{b}^{-h} = (\Pi_{1b}^{-h}, \cdots, \Pi_{\mu b}^{-h})^T, \quad \Pi_{b}^{-h} = (\Pi_{1b}^{-h}, \cdots, \Pi_{\mu b}^{-h})^T, \] (328)
and
\[ \Pi_{ab}^{-h} = e^{-A} \cdot \int_{\mathcal{E}_0} e^{f+\tilde{f}} i_{0*}(\alpha_b), \quad \Pi_{ab}^{-h} = (-1)^f e^{A} \cdot \int_{\mathcal{E}_0} e^{-f-\tilde{f}} \bar{\alpha}_{\epsilon} \cdot i_{0*}(\alpha_b). \] (329)

In particular, this is true if \( f \) is a polynomial on \( M \).

**Proof.** By Equation (325), we have
\[
\int_{\mathcal{E}_0} e^{f+\tilde{f}} \alpha_b = \int_{\mathcal{E}_0} e^{f+\tilde{f}} i_{0*}(\alpha_b) + \int_{\mathcal{E}_0} e^{f+\tilde{f}} \tilde{\partial}_f R_b \\
= \int_{\mathcal{E}_0} e^{f+\tilde{f}} i_{0*}(\alpha_b).
\]

Similarly, we can prove the formula for \( \Pi_{ab}^{-h} \).

If \( f \) is polynomial, then \( f_{(r,0)} \) is polynomial and the generators in \( \Omega^\alpha(M) / df_{(r,)} \wedge \Omega^{\alpha-1}(M) \) has only polynomial growth and the function \( R_a \) can be chosen has only polynomial growth. \( \square \)
Choose \( \{ \alpha_s \} \) to be a unit frame of the Hodge bundle \( \mathcal{H} \). We can apply the above formula to the strong tame deformation of a nondegenerate quasi-homogeneous polynomial.

**Corollary 4.90.** Let \((\mathbb{C}^n, W)\) be a strongly tame section-bundle system with \( W \) a nondegenerate quasihomogeneous polynomial. Let \( W_t, t = (t_1, \cdots, t_m) \) be a strong deformation of \( W \) such that \( \partial_t W, i = 1, \cdots, m \) form a \( m \)-dimensional \( \mathbb{C} \)-linear vector space in \( \Omega^n/df \wedge \Omega^{n-1} \), then the periodic matrix have the following expression:

\[
\Pi^{-h}_i = (\Pi^{h}_{ii}, \cdots, \Pi^{h}_{im}) \quad \text{and} \quad \Pi^{h}_i = (\Pi^{h}_{ii}, \cdots, \Pi^{h}_{im}),
\]

and

\[
\Pi^{h}_{ai} = \tau^{n/2} e^{-A} \int_{\mathbb{C}^n} e^{\tau W + \tau \bar{W}} \partial_t W dz_1 \wedge \cdots \wedge dz_n,
\]

\[
\Pi^{h}_{ai} = (-1)^a \tau^{n/2} e^{A} \int_{\mathbb{C}^n} e^{-\tau W + \tau \bar{W}} \partial_t W dz_1 \wedge \cdots \wedge dz_n.
\]

for \( 1 \leq a \leq m, 1 \leq i \leq m \).

**Proof.** By Theorem 4.27 the growth order of \( \alpha_i \) near the critical point is \( n/2 \) as \( |\tau| \to \infty \), hence we have \( \mathbb{I}_{ik}(\alpha_i) = \tau^{n/2} \partial_t W, \) which has only polynomial growth, and since \( \alpha_i(\tau, t) \) is exponentially decaying, we have

\[
\sum_{i,b} |(\tau^{n/2} \partial_t W, \alpha_i)|_2 < \infty.
\]

So we are done. \( \square \)

Similarly, we have the following conclusion:

**Corollary 4.91.** Let \( ((\mathbb{C}^*)^n, \frac{dz_1}{z_1} \wedge \cdots \wedge \frac{dz_n}{z_n}, \tau f_i) \) be a strong deformation of nondegenerate and convenient Laurent polynomials. Let \( \partial_t f, i = 1, \cdots, m \) form a \( m \)-dimensional \( \mathbb{C} \)-linear vector space in \( \Omega^n/df \wedge \Omega^{n-1} \), then the periodic matrix have the following expression:

\[
\Pi^{-h}_i = (\Pi^{h}_{ii}, \cdots, \Pi^{h}_{im}) \quad \text{and} \quad \Pi^{h}_i = (\Pi^{h}_{ii}, \cdots, \Pi^{h}_{im}),
\]

and

\[
\Pi^{h}_{ai} = \tau^{n/2} e^{-A} \int_{\mathbb{C}^n} e^{\tau f + \tau \bar{f}} \partial_t f \frac{dz_1}{z_1} \wedge \cdots \wedge \frac{dz_n}{z_n},
\]

\[
\Pi^{h}_{ai} = (-1)^a \tau^{n/2} e^{A} \int_{\mathbb{C}^n} e^{-\tau f + \tau \bar{f}} \partial_t f \frac{dz_1}{z_1} \wedge \cdots \wedge \frac{dz_n}{z_n}.
\]

for \( 1 \leq a \leq m, 1 \leq i \leq m \).

### 4.6.2. Primitive vector

Assume that the set of \( \{ \partial_t f(\tau, t), j = 1, \cdots, m \} \), where \( m = \dim S \), forms a partial basis of the algebra in \( \Omega^n(M)/df(\tau, t)\Omega^{n-1}(M) \) and corresponds to the set \( \{ \alpha_j, j = 1, \cdots, m \} \) of holomorphic frame of harmonic \( n \) forms. Then the \( m \) vector \( \Pi^{-h}_j, j = 1, \cdots, \mu \) can be expressed by the vector \( \Pi^{-h}_1 \) as below:

**Proposition 4.92.**

\[
\Pi^{-h}_j = \partial_j \Pi^{-h}_1
\]

\[
\tau \partial_1 \Pi^{-h}_1 = e^{-ad(A)}(\Omega^t) \cdot \Pi^{-h}_1
\]
where
\[ \Pi^{-\circ}_1 = (\Pi_{11}^{-\circ}, \ldots, \Pi_{\mu \mu}^{-\circ})^T \]
\[ \Pi^{-\circ}_{a1} = \tau n^{-1} e^{-A} \int_{\mathbb{C}^n} e^{\tau f + t \bar{f}} dz_1 \wedge \cdots \wedge dz_n. \tag{338} \]

**Proof.** By Proposition 4.89 and the fact that \( \partial A = 0 \), we have
\[ \Pi_{11}^{-\circ} = e^{-A} \int_{\mathbb{C}^n} e^{\tau f + t \bar{f}} \partial_j f dz_1 \wedge \cdots \wedge dz_n \]
\[ = e^{-A} \cdot \partial_j \int_{\mathbb{C}^n} e^{\tau f + t \bar{f}} = \partial_j e^{-A} \cdot \frac{1}{\tau} \Pi_{11}^{-\circ} = \partial_j \Pi_{11}^{-\circ}. \]

Similarly, using the fact that \( \partial_i A = 0 \) we can prove the second identity. \( \square \)

Analogously, the \( m \) vector \( \Pi_1^{+\circ}, j = 1, \cdots, \mu \) can be expressed as the derivatives of another vector \( \Pi_1^{+\circ} \):
\[ \Pi_1^{+\circ} = \partial_1 \Pi_1^{+\circ}, \tag{339} \]
where
\[ \Pi_1^{+\circ} = (\Pi_{11}^{+\circ}, \ldots, \Pi_{\mu \mu}^{+\circ})^T \]
\[ \Pi_{a1}^{+\circ} = (-1)^{n^a/2 - 1} \int_{\mathbb{C}^n} e^{-\tau f - t \bar{f}} \bar{e} dz_1 \wedge \cdots \wedge dz_n. \tag{340} \]

In particular, we have the identity:
\[ \Pi_1^{+\circ} = \frac{1}{\tau} \Pi_1^{-\circ}, \Pi_1^{+\circ} = \frac{1}{\tau} \Pi_1^{+\circ} \tag{341} \]
Furthermore, if \( f(x) \) is a universal deformation of \( \tau f \), then all the horizontal sections \( \Pi_a^{-\circ}(\Pi_1^{+\circ}), a = 1, \cdots, \mu \) are generated by the section \( \Pi_1^{-\circ}(\Pi_1^{+\circ}) \).

**Definition 4.93.** Assume that \( f(x) \) is a universal deformation of \( \tau f \). The vector \( \Pi_1^{+\circ}(\Pi_1^{+\circ}) \) is called the primitive vector (imitating Saito’s notation \([Sai1]\)), since the period matrices can be obtained by taking the Jacobi matrix of the map \( \Pi_1^{-\circ} \) or \( \Pi_1^{+\circ} \):
\[ \Pi^{-\circ} = \frac{\partial(\Pi_{11}^{-\circ}, \ldots, \Pi_{\mu \mu}^{-\circ})}{\partial(t_1, \ldots, t_\mu)} , \quad \Pi^{+\circ} = \frac{\partial(\Pi_{11}^{+\circ}, \ldots, \Pi_{\mu \mu}^{+\circ})}{\partial(t_1, \ldots, t_\mu)}. \tag{342} \]

The primitive vector \( \Pi_1^{+\circ} \) gives a \( C^\infty \) multiple-value mapping:
\[ \Pi_1^{+\circ} : S^1_m \rightarrow \mathbb{C}^\mu. \tag{343} \]

Since \( \Pi_1^{+\circ} \) is independent, which can be obtained by \( \Pi_1^{-\circ} \), it suffices to consider \( \Pi_1^{-\circ} \). For convenience, we denote by \( \Pi_1 := \Pi_1^{-\circ} \).

If we identify the holomorphic frame of \( \mathcal{H}_{\text{hol}} \) with the frame of the holomorphic tangent bundle of \( S \) and use the pull-back metric of the metric \( \tilde{g} \) on \( \mathcal{H}_{\text{hol}} \), then locally the primitive vector gives a totally geodesic embedding into the Euclidean space \( \mathbb{C}^\mu \), i.e.,
\[ \mathcal{D} d\Pi_1 = 0. \tag{344} \]

Globally, we can obtain a map:
\[ \Pi_1 : S^1_m \rightarrow \mathbb{C}^\mu/T, \]
where \( T \) is the monodromy transformation with respect to our background frame on \( \mathcal{H}_{\text{hol}} \).

**Remark 4.94.** An example given by \( z_1^2 + z_2^5 + z_3^2 z_4^2 \) (singularity of type \( T_{2,5,5} \)), Page 29 of [Ku1] shows that the monodromy \( T \) can have infinite order.
4.6.3. **Flat coordinate system.**

At first we give a lemma on the harmonic coordinate system of a smooth manifold $M$ with connection $\nabla$.

**Definition 4.95.** Let $(x^1, \cdots, x^n)$ be a local coordinate system of a chart $U \subset M$. If there exist $n$ smooth functions $u^i = u^i(x^1, \cdots, x^n)$ such that the Jacobi matrix $J = \frac{\partial (u^1, \cdots, u^n)}{\partial (x^1, \cdots, x^n)}$ does not vanish on $U$ and satisfy

$$\nabla du^i = 0, \quad i = 1, \cdots, n,$$

(345)

then $(u^1, \cdots, u^n)$ is called a harmonic coordinate system.

The specialty of the harmonic coordinate system is due to the following known result in geometrical analysis (ref.[Jo]):

**Proposition 4.96.** In harmonic coordinate system, the Christoffel symbols $\Gamma^i_{jk}$ vanishes and the covariant derivative $\nabla_i = \partial_i$.

**Proof.** A simple proof. $\nabla du^a = 0$ is equivalent to the identities:

$$\frac{\partial}{\partial t^j} \left( \frac{\partial u^a}{\partial t^i} \right) - \Gamma^i_{jk} \left( \frac{\partial u^a}{\partial t^i} \right) = 0, \quad \forall a, j, k. \quad (346)$$

The Christoffel symbols under the coordinate system $\{u^a\}$ is

$$\nabla \frac{\partial}{\partial u^a} = \frac{\partial}{\partial t^i} \left( \frac{\partial u^p}{\partial t^i} \right) \Gamma^i_{ij} \frac{\partial}{\partial t^p} = 0.$$

\[\square\]

**Theorem 4.97.** The primitive form is a biholomorphic map from each chamber of $S^1_m$ to its image in $\mathbb{C}^\mu$.

**Proof.** The complex Jacobian of $\Pi_1$:

$$\frac{\partial (\Pi_1, \cdots, \Pi_{\mu})}{\partial (t^1, \cdots, t^\mu)}$$

is just the period matrix $(\Pi^{-h}_{ab})$, which is nondegenerate by the fact that

$$I_W = \Pi^{-1} \cdot \eta^{-1} \cdot \Pi^+.$$

Hence the primitive vector is a biholomorphic map from each chamber of $S^1_m$ to its image in $\mathbb{C}^\mu$. \[\square\]

By the above lemma, we can define the new coordinates in each chamber of $S^1_m$: $u = (u^1, \cdots, u^\mu) = \Pi_1$.

Now the complex Jacobian $J$ of the coordinate transformation is the period matrix $\Pi^{-h}_{ab}$ which is holomorphic in $t$ and the real Jacobian has the block diagonal matrix:

$$\begin{pmatrix}
\Pi^{-h}_{ab} & 0 \\
0 & \Pi^{-h}_{ab}
\end{pmatrix}$$

Hence in the $u$-coordinate system, the connection matrix of $\nabla^G$ vanishes and $\nabla^G = \partial$.

**Definition 4.98.** The local holomorphic coordinate systems $u = (u^1, \cdots, u^\mu)$ is called the flat coordinate system with respect to the connection $\nabla^G$. In fact, we have constructed a family of flat coordinate systems depending on $\tau \in \mathbb{C}^\mu$.

4.7. **Harmonic Higgs bundle and Frobenius manifold.**
4.7.1. Harmonic Higgs bundle.

We will show that the bundle \((\mathcal{H}_{\text{top}}, \mathcal{D})\) satisfying the Cecotti-Vafa’s equations and “fantastic” equations will induce many interesting structures, for example, the structure of harmonic Higgs bundle.

The definition of the harmonic Higgs bundle can be found in Appendix [A].

Now we know that the data \((\mathcal{H}_{\text{top}}, \hat{g}, \tau_{t})\) forms a real Hermitian holomorphic bundle. On the other hand, we have

**Theorem 4.99.** The triple \((\mathcal{H}_{\text{top}}, \hat{g}, B + \frac{\partial}{\partial t})\) forms a harmonic Higgs bundle over \(\mathbb{C}^* \times S\). Moreover, if we think \(t\) as the parameter, then \((\mathcal{H}_{\text{top}}, \hat{g}, B)\) forms a family of harmonic Higgs bundle over \(S\).

**Proof.** At first we know that \(\mathcal{H}_{\text{top}}\) has a holomorphic structure with the Hermitian metric \(\hat{g}\) and \(\mathcal{D} = \mathcal{D}^{1,0} + \mathcal{D}^{0,1}\) is a Chern connection with respect to \(\hat{g}\). Here \(\mathcal{D}^{0,1}\) gives the holomorphic structure on \(\mathcal{H}_{\text{top}}\) and the holomorphic parallel frame is given by \(\Pi_{a}^{\tau} = \tau_{a}, a = 1, \cdots, \mu\). Secondly, \(B + \frac{\partial}{\partial t} : \mathcal{H}_{\text{top}} \to \Omega^{1}(\mathbb{C}^* \times S) \otimes \mathcal{H}_{\text{top}}\) is holomorphic (i.e., \(\hat{g}(B + \frac{\partial}{\partial t}) = 0\)) and satisfies the equation \((B + \frac{\partial}{\partial t}) \wedge (B + \frac{\partial}{\partial t}) = 0\). So \(B + \frac{\partial}{\partial t}\) is a Higgs field on \(\mathcal{H}_{\text{top}}\). Finally the connection \(\mathcal{D}\) is a flat connection, which is the conclusion of the Cecotti-Vafa’s equations and the “fantastic equations”. Therefore \((\mathcal{H}_{\text{top}}, \hat{g}, B + \frac{\partial}{\partial t})\) is a harmonic Higgs bundle.  

\(\square\)

4.7.2. Constructing the Frobenius manifold structure via primitive vector.

Now we assume that the deformation \(f_{(\tau,t)}\) satisfies the “SUBBUNLE” condition:

1. The elements \(i_{g_{0}}(\partial_{j}f_{(\tau,t)}) , j = 1, \cdots , m\) generates a subbundle \(\mathcal{H}_{m}\) of \(\mathcal{H}\).
2. \(i_{g_{0}}(\partial_{1}f_{(\tau,t)}) = i_{g_{0}}(\tau dz_{1} \wedge \cdots \wedge dz_{n}) = \tau^{-1/2}A_{1}\).
3. There is a perpendicular decomposition: \(\mathcal{H} = \mathcal{H}_{m} \oplus \mathcal{H}_{m}^{\perp}\) with respect to \(g\) such that \(\mathcal{H}_{m}\) is the invariant subspace of \(\mathcal{H}\) under the action of all \(\partial_{j}f_{(\tau,t)}\).

If the strong deformation satisfies the above splitting condition, then we can restrict our discussion on \(\mathcal{H}_{m}\). In this case, the number of the deformation parameters equals the dimension of the space \(\mathcal{H}_{m}\) of the \(n\)-harmonic forms. So without loss of generality, we assume that \(m = \mu\) and \(\mathcal{H}_{m} = \mathcal{H}\).

In this part, we set \(\tau = 1\). For simplicity, we ignore the discussion of the Euler field and leave the related topic in future paper.

We want to construct the Frobenius manifold structure on the holomorphic tangent space \(TS_{\nu}\) with flat coordinates \((u_{1}, \cdots , u_{\nu})\) via the primitive vector. Let us recall the definition of Frobenius manifold which was introduced by Dubrovin [Du]. We follow the description of Manin [Ma] (with some change of the notations).

**Definition 4.100.** Let \(S\) be a complex manifold. Consider a triple \((TS, g, A)\) consisting of an affine flat structure, a metric and a \((0, 3)\) type symmetric tensor \(A\) such that the following structures hold:

1. Define an \(O_{S}\)-linear symmetric multiplication \(\circ = \circ_{A,g}\) on \(TS\) by:
   \[ A(X, Y, Z) = g(X \circ Y, Z) = g(X, Y \circ Z). \quad (347) \]
   If (347) holds, then we say that the metric is invariant with respect to the multiplication. \((TS, g, A, \circ)\) is called a pre-Frobenius manifold.

2. If in addition, for any point on \(S\) there is a potential function \(F_{0}\) defined locally such that for any flat local tangent fields \(X, Y, Z\)
   \[ A(X, Y, Z) = (XYZ)F_{0}, \quad (348) \]
   then the pre-Frobenius manifold is called potential.
(3) A pre-Frobenius manifold is called associative, if the multiplication $\circ$ is associative.
(4) A pre-Frobenius manifold is called Frobenius, if it is simultaneously potential and associative.

We still assume that the strong deformation satisfies the requirement in last section.
We have the bundle $\mathcal{H}_{\text{top}}$, defined over $\mathbb{C}^* \times S^1_m$. By Theorem 4.72 and Theorem 4.99, the bundle $\mathcal{H}_{\text{top}}$ has the following structures:

1. non-degenerate bilinear symmetric form $\hat{\eta}$, real structure $\tau_R$ and a corresponding Hermitian metric $\hat{g} = \hat{\eta}(\cdot, \tau_R \cdot)$;
2. Gauss-Manin connection $\mathcal{D} = \nabla^{\mathcal{G}} + \hat{\partial}$ which is the Chern connection of $(\mathcal{H}_{\text{top}}, \hat{g})$.
3. Flat structure given by the horizontal (multivalued) sections $(\hat{s}_a^\top, a = 1, \cdots, \mu)$ or $(\Pi_{a}, a = 1, \cdots, \mu)$.
4. Higgs field $B = B dt^i = (B_{ab}) dt^a \otimes dt^b$ which is holomorphic.

In particular, the pairing $\hat{\eta}$ and the metric $\hat{g}$ is holomorphic, since we have the relation $\hat{\eta}_{ab} = \hat{\eta}(\hat{s}_a^\top, \hat{s}_b^\top) = \eta(a_\alpha, \alpha_b) = \eta_{ab}$, and the later is holomorphic. Since the real structure commutes with $\hat{\partial}$, so $\hat{g}$ is holomorphic also.

Note that $B$ has the following representation:

$$
(B_{ab}) = \hat{\eta}(\partial_i f, \hat{s}_a^\top, \hat{s}_b^\top) = \eta((\partial_i f) \alpha_a, \alpha_b) = \int_M (\partial_i f) \alpha_a \wedge *\alpha_b.
$$

By Theorem 4.97, the primitive vector $\Pi_1$ is a non-singular holomorphic section of $\mathcal{H}_{\text{top}}$:

$$
\Pi_1 : S^1_m \to \mathcal{H}.
$$

We can use it to shift all the structures on $\mathcal{H}_{\text{top}}$ to the holomorphic tangent bundle $TS^1_m$. This is done by defining the map:

$$
\hat{\Pi}_1 : TS^1_m \to \mathcal{H}_{\text{top}},
$$

given by

$$
X \mapsto \iota_X(B) \cdot \Pi_1,
$$

where $\iota_X$ means the contraction with $X$.

**Proposition 4.101.** If we choose $\Pi_j = \partial_j \Pi_1 = \Pi_{a, j}^{-1}$ as the frame on the bundle $\mathcal{H}_{\text{top}}$, then we have

$$
\hat{\Pi}_1(\partial_j) = \Pi_j, j = 1, \cdots, \mu.
$$

**Proof.** We have

$$
\hat{\Pi}_1(\partial_j) = (B_{i}^{aj}) \Pi_a = \hat{\eta}^{j\alpha} \hat{\eta}((\partial_j f) \hat{s}_a^\top, \hat{s}_j^\top) \Pi_a
$$

$$
= \hat{\eta}^{j\alpha} \hat{\eta}(\hat{s}_j^\top, \hat{s}_a^\top) \Pi_a = \hat{\eta}^{j\alpha} \hat{\Pi}_1 \Pi_a = \Pi_j.
$$

So we can define the structures on the tangent bundle $TS^1_m$:

$$
\nabla^{\text{tan}} = \hat{\Pi}_1(\nabla^{\mathcal{G}}), \ \eta^{\text{tan}} = \hat{\Pi}_1(\hat{\eta}), \ e := (\hat{\Pi}_1)^{-1}(\Pi_1) = \partial_1.
$$

**Lemma 4.102.** The following conclusions hold:

1. $\nabla^{\text{tan}}$ is a torsion-free, and flat connection and satisfies: $\nabla^{\text{tan}} \eta^{\text{tan}} = 0$.
2. $\nabla^{\text{tan}} e = 0$. 
Proof. We have
\[(\nabla^{\text{tan}})^2 \hat{\Pi}_1(\nabla^G \circ \nabla^G) = 0, \nabla^{\text{tan}} \eta^{\text{tan}} = \hat{\Pi}_1(\nabla^G \delta) = 0, \nabla^{\text{tan}} e = \hat{\Pi}_1(\nabla^G \Pi_1) = 0.\]

On the other hand, we have
\[
\nabla^{\text{tan}} \partial_k - \nabla^{\text{tan}} \partial_j = \nabla^G_{\partial_k}(\hat{\Pi}_1(\partial_j)) - \nabla^G_{\partial_j}(\hat{\Pi}_1(\partial_k))
\]
\[= \nabla^G_{\partial_j}(B_k \cdot \Pi_1) - \nabla^G_{\partial_k}(B_j \cdot \Pi_1) = \iota_{\partial_j, \partial_k}(\nabla^G B) \cdot \Pi_1 + 0 = 0.\]

This proves that \(\nabla^{\text{tan}}\) is torsion-free. \(\Box\)

Now we define the multiplication in the tangent bundle \(TS^1_m\):
\[
\hat{\Pi}_1(X \circ Y) := (\iota_X B)\hat{\Pi}_1(Y) = (\iota_X B)(\iota_Y B)\Pi_1. \tag{352}
\]

**Lemma 4.103.** The following conclusions hold:

1. The multiplication \(\circ\) is commutative: \(X \circ Y = Y \circ X\).
2. \(e\) is the unit field such that \(e \circ X = X\).
3. The multiplication is compatible with the metric \(\eta^{\text{tan}}\):
   
   - \(\eta^{\text{tan}}(X \circ Y, Z) = \eta^{\text{tan}}(X, Y \circ Z)\).
   - \(\eta^{\text{tan}}(X \circ Y, e) = \eta^{\text{tan}}(X, Y)\).

**Proof.** The commutativity of \(\circ\) is due to the fact that \(B \wedge B = 0\). We have \(\iota_{\partial_i} B = ((B_i)_{ab}) = (\partial_i) = \text{Id}\), which shows that \(e \circ X = X\). The conclusion (3) is easily obtained by using the\(B = B^*\) with respect to \(\eta\) and hence \(\hat{\delta}\). \(\Box\)

By the above analysis, we know that \(e = \partial_1\) is an unit field which is parallel with respect to \(\nabla^{\text{tan}}\) and is the unit of the multiplication \(\circ\).

**Lemma 4.104.** \((TS^1_m, \eta^{\text{tan}}, B, \circ)\) has pre-Frobenius manifold structure.

Furthermore, we have

**Theorem 4.105.** \((TS^1_m, \eta^{\text{tan}}, B, \circ)\) is potential and associative. Hence it has Frobenius manifold structure.

**Proof.** To prove the associativity, we need to prove that
\[
(\partial_i \circ \partial_j) \circ \partial_k = \partial_i \circ (\partial_j \circ \partial_k),
\]
which is equivalent to the identity:
\[
B^e_{ij} B^c_{ek} = B^e_{jk} B^c_{ei}. \tag{353}
\]

This is just the commutation relation \([B_i, B_k] = 0\). Hence we proved the associativity.

We can lower down the upper indices in (353) to obtain the following equivalent formula:
\[
B_{ijc} \eta^e B_{ek} = B_{jkc} \eta^e B_{ec}. \tag{354}
\]

Now we want to show that \(B\) is actually an integrable tensor.

The identity \(\nabla B = 0\) in flat coordinates \([u_1(\tau, t), \cdots, u_p(\tau, t)]\) has the form
\[
\partial_{\tau} B_{abcd} = \partial_{\tau} B_{abcd}.
\]

This is equivalent to
\[
\delta(B_{abcd} \partial^a) = 0.
\]

By Poincare lemma, there is a matrix \(\hat{A}_{cd}\) such that
\[
B_{abcd} \partial^a = \partial_j (\hat{A}_{cd}).
\]
Since $B_{acd}$ is symmetric in $c$ and $d$, we can define the symmetric tensor

$$A_{cd} = \frac{\hat{A}_{cd} + \hat{A}_{dc}}{2}.$$ 

$A_{cd}$ also satisfies

$$B_{acd} du^a = \partial_a (A_{cd}).$$

Since $B_{acd} = B_{cad}$, we have

$$\partial_d (A_{cd}) = \partial_c (A_{ad}).$$

This is again equivalent to

$$\partial (A_{ad} du^a) = 0.$$ 

By Poincare lemma again, there exist $\Phi_d$ such that

$$A_{ad} du^a = \partial (\Phi_d)$$

or $A_{ad} = \partial_a \Phi_d$.

Since $A_{ad} = A_{da}$, we have

$$\partial_a \Phi_d = \partial_d \Phi_a.$$ 

This shows that $\Phi_d du^a$ is exact, and by Poincare lemma again there exists a function $F_0$ such that

$$\partial_a F_0 = \Phi_a.$$ 

By our induction, we know that

$$B_{acd} = \partial_d \partial_c \partial_a F_0.$$ 

Hence the pre-Frobenius manifold is potential.

So we have proved that $(T S, \eta, B, \circ)$ is a Frobenius manifold. \hfill $\square$

Now the identity (354) is just the WDVV equation.

$$\partial \partial \partial F_0 = \partial \partial \partial F_0,$$

(356)

**Definition A.1** (Harmonic Higgs bundle). Let $E$ be a holomorphic bundle on a complex manifold $M$, which is equipped with a Hermitian non-degenerate sesquilinear form $\eta$ (not necessary positive). We say that $(E, h)$ is a Hermitian holomorphic bundle. $B$ is said to be a Higgs field if $B$ is a holomorphic $O_M$-linear morphism $E \to \Omega^1_M \otimes O_M E$ satisfying

$$B \wedge B = 0; \quad (357)$$

$$A_{ad} du^a = \partial (\Phi_d)$$

or $A_{ad} = \partial_a \Phi_d$.

By our induction, we know that

$$B_{acd} = \partial_d \partial_c \partial_a F_0.$$ 

Hence the pre-Frobenius manifold is potential.

So we have proved that $(T S, \eta, B, \circ)$ is a Frobenius manifold. \hfill $\square$

Now the identity (354) is just the WDVV equation.

$$\partial \partial \partial F_0 = \partial \partial \partial F_0,$$

(356)

**APPENDIX A. HARMONIC HIGGS BUNDLE**

**Definition A.2** (real structure). Let $(E, h, B)$ be a Hermitian holomorphic bundle with Higgs field $B$. Suppose that $D$ is the Chern connection with respect to $h$ such that $D = D' + d''$. If the connection $\bar{\partial} = D + B + B'$ is flat, then we say that $(E, h, B)$ is a harmonic Higgs bundle. The integrability is equivalent to the following $tt^*$-equations:

$$\partial \partial \partial F_0 = \partial \partial \partial F_0,$$

(356)

Here $B'$ means the $h$-adjoint of $B$.

**Definition A.2** (real structure). Let $(E, h)$ be a Hermitian holomorphic bundle, an anti-linear operator $\tau_R : E \to E$ is called a real structure if the following conditions hold:

1. $\tau_R^2 = I$;
2. $h(\tau_R \cdot, \tau_R \cdot) = h(\cdot, \cdot)$;
Obviously $h$ is real on $E_{\mathbb{R}} = \text{ker}(\tau_{\mathbb{R}} - I)$.

One can easily show the following relations:

$$D(\tau_{\mathbb{R}}) = 0 \iff D' \tau_{\mathbb{R}} = \tau_{\mathbb{R}} d'' \iff \tau_{\mathbb{R}} D' = d'' \tau_{\mathbb{R}}.$$  

Using the real structure $\tau_{\mathbb{R}}$, one can define a symmetric quadratic form $g$ on the bundle $E$:

$$g(u, v) := h(u, \tau_{\mathbb{R}} v). \quad (3.60)$$

Since $D$ is compatible with $h$ and $\tau_{\mathbb{R}}$, it is easy to show that $g$ is holomorphic and $D'$ is compatible with $\tau_{\mathbb{R}}$, i.e.

$$D(g) = 0. \quad (3.61)$$

Let $P$ be a bundle operator, then we can define $P^* = \tau_{\mathbb{R}} P^\dagger \tau_{\mathbb{R}}$, which is just the $g$-adjoint of $P$.

**Definition A.3.** A harmonic Higgs bundle $(E, h, B)$ with a real structure $\tau_{\mathbb{R}}$ is called a real harmonic Higgs bundle, and denoted by $(E, h, B, \tau_{\mathbb{R}})$. The tuple $(E, h, \tau_{\mathbb{R}})$ is called a real Hermitian holomorphic bundle.

**Appendix B. Abstract variation Hodge structure and Period domain**

The context of this part is somehow standard and can be found in many books, for example, [CMP], [Ku] and etc.

**B.0.3. Pure Hodge structure.**

**Definition B.1.** A pure Hodge structure (HS) of weight $k$ is a pair $(H_\mathbb{Z}, F^\bullet, I)$, where $H_\mathbb{Z}$ is a lattice with finite rank, $I$ is an involution acting on $H = H_\mathbb{Z} \otimes \mathbb{C}$ and $F^\bullet$ is a decreasing filtration of $H$:

$$0 = F^k \subset \cdots \subset F^0 = H_\mathbb{C},$$

such that

$$H = F^p \oplus \tau(F^{k-p+1})$$

for all $p \in \mathbb{Z}$.

The HS of weight $k$ defines a Hodge decomposition by:

$$H^{p,q} = F^p \cap \tau(F^q),$$

and

$$F^p = \oplus_{r \geq p} H^{r,k-r} = H^{p,n-p} \oplus H^{p+1,n-p-1} \oplus \cdots \oplus H^{k,0},$$

i.e.,

$$F^k = H^{k,0}, \quad F^{k-1} = H^{k,0} \oplus H^{k-1,1}, \quad \cdots, \quad F^0 = H.$$

**Definition B.2.** $h^{p,q} = \dim_{\mathbb{C}} H^{p,q}$ are called the Hodge numbers with respect to the pure HS.
B.0.4. **Polarized HSs.**

**Definition B.3.** A polarized HS of weight $k$ is an HS $(H_Z, F^*, I)$ together with a non-degenerate integral bilinear form,

$$\eta : H_Z \times H_Z \to \mathbb{Z},$$

symmetric, if $k$ is even and antisymmetric, if $k$ is odd, and a $\mathbb{R}$-linear transformation $C : H \to H$ (called the generalized Weil operator) such that the Riemannian bilinear relations hold:

$$\eta(F^p, F^{k-p+1}) = 0,$$

$$\eta(C\varphi, \overline{\varphi}) > 0.$$

The bilinear form $\eta$ is called the polarization of the HS $F^*.$

**Example B.4.** The deformation of the compact complex manifolds provide the pure Hodge structure.

B.0.5. **Mixed Hodge structure.**

**Definition B.5.** A mixed Hodge structure (MHS) is a triple $(H_Z, W_\ast, F^\ast)$, where $H_Z$ is a lattice, $W_\ast$ is a finite increasing filtration on $H_Z = H_Z \otimes \mathbb{Q} \otimes \mathbb{C}$ and $F^\ast$ is a finite decreasing filtration on $H = H_Z$. Denote by $W_\ast = W_\ast Q$. Then the two filtrations should induce a HS of weight $k$ on the quotient $Gr_k H := Gr_k W := W_k / W_{k-1}$. Here $F^i Gr_k W$ is the image of $F^i \cap W_k$ in $Gr_k W$, i.e.,

$$F^i Gr_k W = (W_k \cap F^i + W_{k-1}) / W_{k-1}.$$

$F^\ast$ is called the Hodge filtration and $W_\ast$ the weight filtration.

The compatible condition of $W_\ast$ and $F^\ast$ is equivalent to the decomposition for any $k, l$:

$$Gr_k H = F^l Gr_k H \oplus I(F^{k-l} Gr_k H).$$

**Definition B.6.** Let $H^{p,q} = F^p Gr_{p+q} H$, then $Gr_k = \oplus_{p+q} H^{p,q}$. The numbers $h^{p,q}$ are called the Hodge numbers of the MHS $(W_\ast, F^\ast)$.

If there are two MHSs on spaces $H$ and $H'$, then the space of linear mappings Hom$(H, H')$ is endowed with a natural MHS,

$$W_\ast \text{Hom}(H, H') = \{ \varphi : H_Z \to H'_Z | \varphi(W_k) \subset W'_{k+a}, \forall k \}$$

$$F^l \text{Hom}(H, H') = \{ \varphi : H \to H' | \varphi(F^i) \subset (F')^{l+b}, \forall l \}.$$

It is easy to see that

$$\text{Hom}(H, H')^{p,q} = \oplus \text{Hom}(H^{p,q}, (H')^{p+q,a+b}).$$

A morphism $\varphi, H \to H'$ is called a MHS morphism of type $(a, a)$ if

$$\varphi(W_k) \subset W'_{k+2a}, \varphi(F^i) \subset (F')^{i+a}.$$

$\varphi$ is called an MHS morphism if $a = 0$. An important property of a MHS morphism is their compatibility with the two filtrations, i.e.,

$$\varphi(W_k) = W'_{k+2a} \cap \text{im} \varphi, \varphi(F^i) = (F')^{i+a} \cap \text{im} \varphi(H).$$

Therefore the morphism of MHS can be applied to the morphism of complexes of MHS.
B.0.6. Weight filtration of a nilpotent operator.

Let \( N : H \to H \) be a nilpotent operator on a vector space \( H, N^{k+1} = 0 \). Then there exists a unique increasing filtration \( W_* \) on \( H \):

\[
\cdots \subset W_{-1} \subset W_0 \subset W_1 \subset \cdots,
\]

such that

(i) \( N(W_i) \subset W_{i-2} \),

(ii) \( N' : Gr_r \to Gr_{r-r} \) is an isomorphism for \( r \).

This filtration is called the weight filtration of a nilpotent operator \( N \) (with center 0).

The filtration can be constructed as follows. Take a basis \( u_i \) such that \( N \) has the Jordan normal form and \( u_i \) satisfies

\[
\begin{pmatrix}
0 & \cdots & 0 \\
1 & \ddots & \\
& \ddots & \\
0 & \cdots & 1
\end{pmatrix}, \quad Nu_1 = u_2, \cdots, Bu_{q-1} = u_q, Nu_q = 0.
\]

Number basis for each block such that the numbers are symmetric relative to 0 and \( N \) decreases the index by 2, i.e., \( Nu_i = u_{i-2} \),

(i) if \( q = 2k + 1 \) is odd, then

\( u_{2k}, \cdots, u_2, u_0, u_{-2}, \cdots, u_{-2k} \).

(ii) if \( q = 2k \) is even, then

\( u_{2k-1}, \cdots, u_{-1}, u_1, \cdots, u_{-(2k-1)} \).

Now \( W_k \) is defined as the subspace generated by vectors of weight \( \leq k \).

We can shift the index of the weight filtration \( W_* \) such that it is symmetric about the index \( n \): define

\( W[-n]_k := W_{k-n} \).

So we get a new weight filtration \( W[-n] \) satisfying:

\( N' : Gr_{r-r} \to Gr_{n-r} \) is an isomorphism for any \( r \).

Let \( H = H/\ker(N) \) and \( N \) is the restriction of \( N \) to \( \overline{H} \). Then we have an induced weight filtration by \( N : \overline{W}_* \). On the other hand, we can directly restrict the filtration \( W_*[-n] \) to \( H \) to get \( W_*[-n] \). It is easy to see that the following relation holds:

\[
W_*[-n] = \overline{W}[-n-1]_*.
\] (362)

B.0.7. Variation of HS.

**Definition B.7.** A (real) variation of Hodge structure consists of a triple \((H, \nabla, F^*)\), where \( H \) is a real vector space on a complex base manifold \( S \), \( \nabla \) is a flat connection on \( H \), and \( F^* \) is a filtration of \( H_C := H \otimes \mathbb{C} \) by holomorphic subbundle which gives the Hodge structure in each fiber and which satisfies the following Griffiths’s transversality condition: the \((1, 0)\) part \( \nabla' \) of the connection \( \nabla \) satisfies

\[
\nabla'(F^i) \subset F^{i-1} \otimes \Omega^{1,0}(S).
\] (363)

Here the HS on each fiber need not be obtained from an integral lattice. However, in most geometrical cases the HS on each fiber comes from a locally constant system of free \( \mathbb{Z} \)-modules.
Definition B.8. A polarization of the VHS $H$ consists of an underlying rational structure $E = E_Q \otimes \mathbb{R}$ and a nondegenerate bilinear form

$$\eta : E_Q \times E_Q \to \mathbb{Q},$$

where $\mathbb{Q}$ is a trivial $\mathbb{Q}$-valued local system on $S$.

This form $\eta$ must satisfy the following properties:

1. $\eta$ induces a polarization on each fiber;
2. $\eta$ is flat with respect to the flat connection $\nabla$, i.e.,

$$d\eta(s, s') = \eta(\nabla s, s') + \eta(s, \nabla s').$$

B.0.8. Period mapping and period domain. Now a polarized VHS defines a monodromy representation:

$$\rho : \pi_1(S, o) \to G_Q = \text{Aut}((E_0)_Q, \eta),$$

where $o$ is a base point. Its image

$$\Gamma = \rho(\pi_1(S, o)) \subset G_Q$$

is called the monodromy group.

Definition B.9. Giving an integer $k$, a lattice $H_\mathbb{Z}$ and a bilinear form $\eta$ (on the lattice) and positive integers $h, p, q$ (which is the dimension of $H_{p,q}$). The space $D$ consists of all polarized Hodge structures $H_{\mathbb{Z}}, H_{p,q}, \eta$ with dimension $\dim H_{p,q} = h, p, q$ is called the period domain.

The period domain can be constructed explicitly. The filtration $F^*$ with fixed dimension $f^i = \dim F^i$ is a flag variety and forms a subvariety of a product of Grassmannian varieties with the natural complex structure. If the filtration satisfies the first Riemann relation:

$$\eta(F^p, F^{k-p+1}) = 0,$$

then only the half filtration is free: $F^k \subset \cdots \subset F^v, v = [k + 1/2]$. Therefore all such filtrations satisfying the first Riemann relation forms a variety $\tilde{D}$ in the Grassmannian manifold associated to the complexified space $H_{\mathbb{C}}$ and fixed dimension of subspaces.

The second bilinear relation is a positive condition which shows that all such filtrations satisfying the first and the second Riemann relations form an open subvariety $D$ in $\tilde{D}$.

The variety $\tilde{D}$ and $D$ are actually homogeneous spaces. Let $G_{\mathbb{C}} = \text{Aut}(H, \eta)$ be the group of linear automorphisms of the space $H$ preserving the form $\eta$. $G_{\mathbb{C}}$ acts on $\tilde{D}$ transitively. If $B$ is an isotropic subgroup at a fixed point (a fixed filtration), then

$$\tilde{D} = G_{\mathbb{C}}/B.$$

In a suitable basis, $B$ consists of the block-triangular matrices. such a group is called a parabolic subgroup. It can be shown that $\tilde{D}$ is smooth.

Take the subgroup $G_{\mathbb{R}} = \text{Aut}(H_\mathbb{R}, \eta) \subset G_{\mathbb{C}}$ and its subgroup $K = B \cap G_{\mathbb{R}}$ ($K$ is compact, since the restriction to the quotient spaces $F^p/F^{p+1}$ are orthogonal groups). We have

$$D = G_{\mathbb{R}}/K.$$

The following proposition can be found in PP. 143 of [CMP].

Proposition B.10. Let $D = G_{\mathbb{R}}/K$ be a period domain for weight $k$ Hodge structures with Hodge numbers $h^{p,q}, p = 0, \cdots, k$.

1. If $k = 2m + 1$, the form $\eta$ is antisymmetric and then $G = Sp(n, \mathbb{R})$, where $n = \dim_{\mathbb{C}} H_{\mathbb{C}}$, and $K = \prod_{p \leq m} U(h^{p,q})$. 


(2) If $k = 2m$, the form $\eta$ is symmetric and $G = SO(s, t)$, where $s$ is the sum of the Hodge numbers $h^{p,q}$ with $p$ even, while $t$ is the sum of the remaining Hodge numbers. Here we have $K = \prod_{p < k} U(h^{p,q} \times SO(h^{m,m}))$.

**Definition B.11.** Let $(H, \nabla, F^*)$ be a VHS over a complex manifold $S$ (or analytic space). Then each fiber of the bundle $H$ gives a point in the period domain $D = G/K$. Hence this gives a mapping

$$\tilde{\varphi} : S \to D,$$

which is a multivalued mapping because of the action of the monodromy group $\Gamma$. Since $\Gamma$ is discrete, we can push down $\tilde{\varphi}$ to get a singular valued mapping:

$$\varphi : S \to D/\Gamma.$$

$\varphi$ is called the period mapping. Here the space $D/\Gamma$ becomes a complex orbifold.

The following result is well-known.

**Theorem B.12 (Griffiths Transversally Theorem).** Let $(H, \nabla, F^*)$ be a VHS. Then the sub-bundle $F^p$ and the period mapping $\varphi$ are holomorphic.

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