An Exact Hairy Black Hole Solution for AdS/CFT Superconductors

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We provide an exact hairy black hole solution to an $n + 1$ dimensional gravity-coupled complex scalar field model. The solution has translational invariant horizon and tunable temperatures. Free energy calculations indicate that, there are always temperature ranges in which the hairy black hole is thermodynamically stable against decaying into its no-hair counterpart. Using this solution as an AdS/CFT superconductor model, we get potentially useful critical temperature v.s. dimensions of order parameter operator and conductivity-frequency relations typical of other similar models.

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Studies in applied holographic theories, such as finite temperature AdS/QCD models’ building [1, 2, 3], AdS/CFT superconductor’s description [4, 5], or other quantum phase transitions’ exploration [6, 7, 8] all call for solutions to the scalar field model coupled with gravity,

\[ S = \frac{1}{16\pi G_N} \int dx^{n+1} \sqrt{-g} \left( R - \partial_{\mu} \Psi \partial^{\mu} \Psi - V(|\Psi|) \right) \] (1)
of the form

\[ ds^2 = e^{2A}(-h dt^2 + d\vec{x} \cdot d\vec{x}) + h^{-1} du^2, \quad \Psi = \Psi(u) \] (2)
\[ u \in (0, \infty), \quad A = A(h) \] (3)
\[ \exists \theta_0 > 0, \quad h(u_0) = 0, \quad h'(u_0) \text{ is finite} \] (4)

However, to this day people still have no exact analytical solutions of the desired form, even for the simplest potential $V = m^2|\Psi|^2$. As we know, the best known solutions in this area are [9] and [10]. The former can be derived out from higher dimensional AdS-Schwarzschild solutions through compactification [12]. At the desired dimension its asymptotic is not AdS type. The latter’s horizon has special topologies and the asymptotic is only locally AdS type [13]. Few days before this work is finished, we note[11] in the arxiv, which provides finite temperature solutions to a softwall AdS/QCD model. But its temperature cannot be tuned freely for fixed scalar potentials.

Difficulties to find analytical solutions in this model is due to the non-linearity of the equations of motion, which follows by minimizing the action (1)

\[ \begin{align*}
\bar{t}, xx & : h'' + nh'A' = 0 \\
\bar{t}, uu & : A'' + \frac{1}{n-1} \Psi'^2 = 0 \\
\text{Eins.Eq.} uu & : (n-1)[h'A' + nhA'^2] = h\Psi'^2 - V \\
\text{scalar com} & : 2(\Psi'h e^{nA})' e^{-nA} = \frac{dV}{d|\Psi|}. \tag{7}
\end{align*} \]

Note we will focus on constant phase solutions since any non-trivial phase profile will cause the action deviate from minimal configuration. We checked that the scalar field equation of motion can be derived out from the three components of the Einstein equation. So, only the former three of the above four equations are independent. Of them, the first one can be integrated once to give

\[ A = -\frac{1}{n} \ln h' + \text{const.} \tag{9} \]

Usually given forms of the potential function, even of the simplest quadratic type, to integrate equations (5)–(7) is almost impossible. However, if we ask the question from an inverse direction — given function h of desired asymptotics, can we find out the potential function’s form explicitly? — we may have different gains.

In most cases, when we write down expressions for h with some asymptotics, substitute into eq(9) and get A, into eq(6) and get $\psi'$, integrate and get $\psi(u)$, combining these things into eq(7) and find V(u), we could only find $V(u) - \Psi(u)$ correspondence but not the explicit form of $V(\Psi)$. However, after some trial and errors, we find that the following metric functions and potential can be worked out simultaneously

\[ h[u] = 1 - e^{-n(u-u_0)/\ell} + (n+1)/n \left[ 1 + (n+k) e^{-k(u-u_0)/\ell} / [1 + (n+k)] \right] \] (10)
\[ A[u] = \frac{u}{\ell} - \frac{1}{n} \ln \left[ 1 + e^{-k(u-u_0)/\ell} / [1 + e^{-k(u-u_0)/\ell}] \right] \] (11)
\[ \Psi[u] = \left( \frac{n-1}{n/4} \right)^{\frac{1}{2}} \arctan\left[ e^{-k(u-u_0)/2\ell} \right] \] (12)
\[ V[\Psi] = -\frac{(n-1)}{\ell^2} \left[ \frac{n}{k} + (2 - \frac{k}{n}) \sin^2 \hat{\psi} + \frac{2k}{n} \sin^4 \hat{\psi} \right] + \frac{k^2}{n(2n+k)} \sin^2 \hat{\psi} \tan \frac{2\hat{\psi}}{n} \] (13)
\[ \hat{\psi} = \left( \frac{n/4}{n-1} \right)^{\frac{1}{2}} |\Psi| \] (14)

This means that we find exact hairy black hole solutions to the complex scalar field model featured by the potential (13). Obviously, these solutions have tunable temperature and the same singularity structure as the usual AdS-Schwarzschild black holes with spatial-flat horizons. Although the potential function here is not so easily looking, the fact that it is periodical function of the main ar-
gument $\hat{\psi}$ makes us believe that, the probability of finding it in the landscape of M-theory superconducton [14] would not be too small. We know, in string/M theory compactifications, there are many moduli fields which are constrained to take values periodically, e.g. $(0, 2\pi)$.

For small $k$, our potential has similar features as the standard model Higgs field does. The difference is, in the standard model, the Higgs field will uniformly sit at the global minimal of the potential; while the scalar field here has non-trivial spatial profile. From the black hole horizon to infinite, $\Psi$ field tries its best to climb up the local maximal of the potential, see FIG 1. This constitutes a non-trivial hair of the black hole. Asymptotically, $\Psi \to 0$, $V[\Psi] \approx -n(n - 1)\ell^{-2} + m_{\text{eff}}^2 |\Psi|^2$, with

$$m_{\text{eff}}^2 = -(2 - \frac{k}{n})(n - 1)k\ell^{-2}$$

(15)

So near the AdS boundary, our model has little difference from the simple gravity-coupled scalar field theory with quadratic potentials. When $k < 2n$, the effective mass parameter (mass square) of the scalar is negative. But if at the same time

$$n - n\sqrt{1 - \frac{n/4}{n - 1}} < k < n + n\sqrt{1 - \frac{n/4}{n - 1}}$$

(16)

this negative mass square will not lead to instabilities. Because in this case, the Breitenlohner-Freedman bound $m_{\text{eff}}^2\ell^2 > -n^2/4$ is always satisfied. When $k > 2n$, the effective mass square is positive. So for large $k$, $\Psi = 0$ is a meta-stable point of the potential, see also FIG 1.

Since the hair of the black hole breaks symmetry of the model associated with the phase rotation of the scalar field, coupling this scalar field to some $U(1)$ gauge field will give us an ideal model of AdS/CFT superconductors. But before turning to that topic, let us examine thermodynamics of this solution further. Following reference [15], we calculate the entropy, temperature and energy of the solutions as follows

$$S = \frac{V_n}{4G_N}e^{(n-1)\ell} = \frac{V_n}{4G_N}e^{(n-1)u_0/k}, \quad n^- = n - 1$$

(17)

$$T = \frac{e^{\Delta h'}}{4\pi |u_0|} = \frac{n + k}{2\pi n + k}e^{\frac{\Delta h'}{\ell}}, \quad \ell^{-1}$$

(18)

$$E = \frac{V_n}{8\pi G_N} \left\{ [e^{nA}\psi_{AdS,bh}]_{\text{AdS,bh}} - [e^{nA}\psi_{\text{AdS,f}}]_{\text{pure-AdS}} \right\}$$

(19)

= \frac{V_n}{8\pi G_N} \left\{ \frac{n + k}{2n + k} \left( \frac{u_0}{n} - \frac{k}{n} e^{\frac{(n-k)u_0}{u-\infty}} \right) \right\}

By adding boundary term and counter term to the action (11), and make strict holographic renormalization treatment, we will get similar results. It can be checked that, $[dE \neq TdS]_{\text{vary, } u_0}$. This is not a catastrophic. For example, in reference [10] the same question happens. In that reference, violation of the first law is attributed to some chemical potential associated with the scalar particle numbers, i.e. $[dE = TdS + \mu dN]_{\text{vary, } u_0}$. In the current paper, we think the same explanation helps.

The energy definitions (19) involve regularizations in which an infinite part was subtracted. However, for $k < n$ models, this regularization may subtract too much so that the resulting energy becomes negative. For $k > n$ there are contrary questions. Tuning the subtractions appropriately (assure positivity of the energy but hold the $k$ relevant part in the finite part), we will get

$$E = \frac{V_n}{8\pi G_N} \left\{ \frac{n + k}{2n + k} \left( \frac{u_0}{n} - \frac{k}{n} e^{\frac{(n-k)u_0}{u-\infty}} \right) \right\}$$

(20)

As results, the free energy $F = E - TS$ of system reads

$$F = \frac{V_n}{8\pi G_N} \left\{ \frac{n + k}{2n + k} \left( \frac{u_0}{n} - \frac{k}{n} e^{\frac{(n-k)u_0}{u-\infty}} \right) \right\}$$

(21)

Comparing this result with that of the no-hair counterpart, we see that, for $k < n$ case, referring to the left part of figure 2 the hairy black hole at lower temperatures has lower free energy than their no-hair partners so the former is favored thermodynamically at low temperatures; for $k > n$ case, referring to the right part the same figure, the hairy black hole at higher temperatures have lower free energy so is favored thermodynamically at high temperatures.
For $k < n$ models, if at high temperatures the system lies at the no-hair phase in which it is symmetric with respect to the $U(1)$ phase rotation of the scalar field, as temperature lowers, it will undergo a phase transition and break the symmetry spontaneously, i.e. the scalar field attains non-trivial profile with specific phases. The phase transition temperature depends on $n$ and $k$. Figure 2 displays this dependence explicitly, from which we easily see that increasing $|m_{\text{eff}}|$ will increase the phase transition temperature. When translated into the dual field theories, this may be a potentially useful result for looking for superconductors with higher critical temperatures. Of course it should be noted that, for looking for superconductors with higher critical temperatures. This is contrary to the actual case of the real superconductors. So in the following explorations, we will neglect this case.

To build superconductor models in holographic languages, we couple our scalar field to some $U(1)$ gauge field

$$S = \frac{1}{16\pi G_N} \int dx^{n+1} \sqrt{-g} \left( R - \frac{1}{4} F_{\mu\nu}^2 - (\partial_\mu - i q A_\mu) \Psi (\partial^\mu + i q A^\mu) \Psi^* - V(\Psi) \right)$$

By previous analysis, we know the scalar field in $k < n$ models will condensate and break the local $U(1)$ symmetry of the system spontaneously at low temperatures. As results, materials described by the dual conformal (approximately) field theory will go into superconducting phase. At this phase, no direct current is allowed in the material. But we can use some small electro-magnetic field to perturb the material and measure its response. By AdS/CFT correspondence, this means that we perturb the background geometry\footnote{We thank very much to Dr/Professor C. P. Herzog to provide us his notebook on numerical details of the work \cite{5}, from which we learned this technique.} by small, but uniform gauge field $A_m dx^m$

$$A = e^{-i\omega t} a_x(u) dx,$$

Note we do not include backreaction of the background geometry. For this perturbation, the linearized Maxwell equation reads

$$a''_x + \frac{a'_x}{(n-2)} A' + \frac{h'}{h} a_x + \frac{\omega^2}{2h^2e^{2A}} - \frac{2q^2|\Psi|^2}{h} = 0$$

Near horizon of the black hole, causality requires the perturbing field $a_x$ satisfy falling boundary condition,

$$a_x \propto e^{-i\omega t/(h_0 e^{A_0})} \ln(u-u_0).$$

While in the asymptotically infinite region, from the background field expressions\footnote{We thank very much to Dr/Professor C. P. Herzog to provide us his notebook on numerical details of the work \cite{5}, from which we learned this technique.} and the perturbation eq(24) we can show that

$$a_x = a_x(0) + a_x(1) e^{-(n-2)u} + \cdots.\tag{26}$$

According to the AdS/CFT dictionary, in the dual CFT side, $a_x(0)$ and $a_x(1)$ correspond to external field strength and the resulting current respectively, so the conductivity there can be calculated through

$$\sigma = \frac{J_x}{E_x} = -\frac{i}{\omega} a_x(1)$$

In the normal phase $\Psi = 0$, eq(24) can be changed into

$$\frac{d^2}{du^2} a_x + \omega^2 e^{(2n-6)A} a_x = 0, \quad du_x = du/[he^{-(n-2)A}]$$

where $h$ and $A$ should be given by letting $k = 0$ in eqs\footnote{We thank very much to Dr/Professor C. P. Herzog to provide us his notebook on numerical details of the work \cite{5}, from which we learned this technique.}. For $n = 3$, this equation can be solved exactly

$$a_x = e^{-iu_x}, \quad u_x = \int_0^{\infty} \frac{de^{-(u-u_0)}}{1-e^{-3(u-u_0)}}.$$\tag{29}

This solution is selected from two possibilities by the falling boundary condition (25). Expanding it into the form of (26) around $u \to \infty$, we find that for $n = 3$ case, the normal phase conductivity reads

$$\sigma(\omega) = 1.$$\tag{30}

For $n \geq 4$, we do not know how to solve eq(28) analytically. But numerics tell us that, $\sigma(\omega) \propto \omega^{n-3}$ as $\omega \to \infty$.

In the superconduction phase, $\Psi \neq 0$. In this phase even for $n = 3$, eq(24) cannot be solved analytically. But we can solve it numerically and get the conductivity approximately as

$$\sigma = \frac{i a_x'}{\omega a_x} \bigg|_{n-2 \to \infty}$$

Figure 4 displays our numerical results of conductivities for some specific $n, k$ and $q$ values. Two features of the figure should be noted here. The first is, there is a $\delta$ function peak in the real part of the conductivities as the result of Kramers-Kronig relation

$$\text{Im}[\sigma(\omega)] = -\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\text{Re}[\sigma(\tilde{\omega})]}{\tilde{\omega}-\omega} d\tilde{\omega}.$$\tag{32}
and the fact that imaginary part of the conductivity has a pole of the form $\text{Im}[\sigma]\sim \frac{1}{\omega}$, This is just the usual definition of superconductivity, i.e. infinite DC conductivity. Note this infinity cannot be attributed to translation invariance, since in the calculation we fixed the background geometry. As results, the translation invariance is broken implicitly, see reference $[2]$. The second is that, conductivities of different dimensional models behave hierarchically, $\sigma_3 \ll \sigma_4/T \ll \sigma_5/T^2$. Note to plot the conductivity curves of different dimensional models in the same figure and make them look clear enough, we multiply different numerical factors on the conductivities. On the first feature, our $n=3$ result coincides with those based on AdS abelian-higgs models, such as $[3, 18, 19, 20, 21]$; our $n=4$ result is similar to those based on D3/D7 brane models $[22]$, see also reference $[10]$. On the second feature, our results for 4 dimensional models are similar to that of reference $[10]$ so can be supported by that paper.

A little summary: we construct exact analytical solutions to an $n+1$ dimensional gravity-coupled complex scalar field model. When the scalar field in this model is coupled to $U(1)$ gauge field, the resulting system provide holographic descriptions for superconductors. We calculate conductivities in this model and find results typical of other holographic superconductors. But the superconduction mechanism in our model is not totally the same as that proposed by Gubser $[4]$, in which the scalar field condensates due to it’s coupling with the gauge field. Our scalar field condensates due to its self-interaction, the potentials of which have the typical Mexican hat shape. It is worth to note that existence of hairy black hole solutions as we provide in this paper does not violate the no hair theorem $[23]$ refined by Hertogs. The main reason is that our solutions do not satisfy the positive energy condition. Few months after the first version of this paper, reference $[24]$ and $[25]$ appears on arxiv. $[24]$ used method similar to us and construct more hairy black hole solutions. While $[25]$ proposed a new version of no hair theorem in which the existence of our solutions can be understood naturally.

![Graph]

**FIG. 4:** Conductivity-frequency dependence for $n = 3$ (red), 4 (green) and 5 (blue) models at each own’s superconduction critical temperature $T = T_c$. All three curves has $k = 2$, $q = 1f^{-1}$. The conductivity of $n = 4$ model has been scaled a factor of 1/30, while that of the $n = 5$ models is scaled a factor of 1/900.

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