LETTER TO THE EDITOR

Measurement, Decoherence and Chaos in Quantum Pinball

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Abstract. The effect of introducing measuring devices in a “quantum pinball” system is shown to lead to a chaotic evolution for the particle position as defined in Bohm’s approach to Quantum Mechanics.

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1. Introduction

One of the major problems in the study of the quantum mechanical behaviour of classically chaotic systems, in the context of the orthodox interpretation, is the inapplicability of the usual means of describing the dynamics of any system in terms of well-defined trajectories. In contrast to the orthodox approach to quantum mechanics Bohm’s approach[1, 2, 3] does allow the description of quantum systems in terms of well-defined trajectories and so overcomes this particular limitation. The Bohm trajectories are derived from the wavefunction according to the guidance condition

$$\vec{p} = \Im \left[ \frac{\Psi^\dagger \nabla \Psi}{\Psi^\dagger \Psi} \right]$$

where $\vec{p}$ is the momentum and we assume $\hbar = 1$. In a recent work[5, 6] we used Bohm’s approach to explore the quantum mechanical behaviour of the (classically chaotic) kicked rotor and, notwithstanding the nonlinear quantum potential which arises in this description, found that neither an isolated rotor in a superposition of states nor an otherwise isolated but periodically kicked rotor showed any evidence of chaotic behaviour. The Schrödinger equation is non-mixing (two initially almost identical wavefunctions remain almost identical under the Schrödinger evolution) and this coupled with the fact that Bohm trajectories do not cross in the configuration space of the system, which in this case is one dimensional, makes the conclusion that chaos is not observed not surprising.

Thus far there is little in the extensive quantum chaos literature concerning the effects of measurement on a quantum system. This is probably because, from the orthodox point of view, the effect of measurement on a quantum system is to introduce an inherent randomness of outcomes and this essentially removes the system from the arena of interest for quantum chaology; chaos should arise from the internal dynamics of the system, not as a result of an external randomising influence. The introduction of measuring devices, when they are simply considered as other quantum systems,
enlarges the configuration space of the (now compound) system. Their principle function is to introduce bifurcations in the evolution of the probability density in configuration space which then flows into separate regions. A good measuring device also ensures that the possibility for interference between these different regions in the future is negligible. In common text-book parlance the configuration-space wave then collapses (by some unspecified non-quantum mechanical process) in a random manner into just one of these regions.

In contrast to the usual description, Bohm’s approach has no need for wavepacket collapse. There is a well-defined outcome in each individual measurement since the actual configuration of the compound system is given by a well defined point in configuration space (just as in classical mechanics) and, according to the equations of motion, this point moves into just one the separate regions that develop as the interaction proceeds. Thus a unique correlated state of the measuring device and the measured system is selected. If the device is a good one then, for the practical purposes of calculating the future behaviour of the system, one may neglect all the unoccupied regions of the configuration space and just employ that component of the superposition which is effective in determining the future behaviour. Wave packet collapse to an effective wavefunction is, in Bohm’s theory, just a calculational convenience that one may employ.

At the end of our above-cited paper on the kicked rotor we suggested that, if the rotor were to be subjected to repeated measurements of the angular momentum, one after each kick, then the evolution of the effective wavefunctions associated with different initial conditions could indeed be divergent, leading to chaotic behaviour for the motion of the rotor. For the case of the rotor we have shown that the outcome of a measurement of angular momentum is dependent not only on the initial state of the rotor but also on the initial coordinates of the rotor’s centre of mass within its wavepacket as it enters the Stern-Gerlach apparatus. After the rotor suffers a kick and enters a superposition of angular momentum eigenstates a subsequent measurement will lead to the emergence of a number of wave packets from the exit from the Stern-Gerlach field and, given the fact that the Bohm trajectories may not cross, there will clearly be a corresponding number of bifurcations in the associated trajectories. Each one of these emerging wavepackets is associated with a different angular momentum component eigenstate and since the actual rotor position must, in any individual case, enter just one of the packets as they separate in space, the effective wavefunction (the one which determines the rotor’s behaviour) becomes just one of the eigenstates. In Bohm’s theory we are justified in ignoring the packets not containing the actual rotor centre of mass coordinate for the purposes of calculation providing only that the packets associated with different outcomes remain orthogonal either in virtue of spatial separation or the functioning of some complex recording device with many degrees of freedom. That is, in modern parlance, the alternatives must decohere. On receiving the next kick the effective angular momentum eigenstate wavefunction becomes a superposition of angular momentum eigenstates once more. The values of the coefficients of the eigenstates in this superposition will depend on the rotor’s state before the kick. Consequently the sequences of effective wavefunctions, generated by rotors with arbitrarily close initial positions, in a sequence of kicks followed by angular momentum measurements will in general diverge.

The picture is this: two identical rotors, differing only slightly in the values of their initial positions within their identical centre-of-mass wave-packets are subjected to a series of kicks. In the absence of any other interactions the behaviour of the two
rotors will remain closely correlated. The situation is radically different if, between kicks, an angular momentum measurement is carried out. Initially, for a few kick and measurement pairs, the trajectories of the centre of mass of each rotor will be closely correlated and the two sequences of eigenstates resulting from the measurements will be identical. Eventually the center of mass coordinates of the two rotors will be placed either side of one of the bifurcation points that arise on measurement. Then the effective eigenstate for each rotor will be different after the measurement. On the next kick the expansion coefficients of the angular momentum eigenstates will be different and the behaviour of the rotors will no longer be correlated.

2. Quantum Pinball

In order to explore the type of behaviour which arises with a kicked and measured rotor, but in the context of a simple system for which the outcome of the measuring interaction is merely twofold, we consider here a “quantum-pinball system”. The model-system we have in mind is constructed from a series of potential barriers, each with a transmission coefficient of one-half, which are arranged in a typical fairground pinball array (essentially on a triangular lattice within a triangle). A wave packet incident on the first barrier at the apex of the set of pins splits into two equal-sized packets which then propagate on towards two further barriers where the splitting behaviour is repeated. The significant coordinate is clearly perpendicular to the barriers, the motion parallel is unaffected by the barriers.

We shall consider two variants of the pinball, the first is as described so far, but the second has the addition of measuring devices, capable of recording the passage of the particle, in each arm of the network. Before describing the motion of the particle through the whole pinball let us examine the behaviour of a particle as its wavepacket scatters from a single potential barrier.

From the fact that the Bohm-trajectories may not cross we can deduce that all those particles which approach the barrier within the trailing half of the packet must be reflected, whilst those in the front half of the packet must be transmitted.[8] Thus this process, widely held to be inherently random in the usual approach to quantum mechanics, can be given a fully deterministic description in Bohm’s theory. On scattering from the barrier the wavefunction develops into a superposition

\[ \Psi_{\text{inc}} = \Psi_{\text{ref}} + \Psi_{\text{trans}} \]  

(2)
but in each individual case there is a definite outcome as the Bohm trajectory must lead into just one of the packets. Whilst the two packets remain orthogonal, in virtue of their separation in space, only that part of the superposition associated with the actual particle position is active in determining the behaviour of the particle (this we refer to as the effective wavefunction). If the two packets were to be recombined, as happens in the second level of the pinball, then the reflected and transmitted packets would interfere and both would be relevant in determining the particle’s behaviour.

The situation is very different if measuring devices, capable of recording the passage of the particle, are introduced in the transmitted and reflected paths. The wave function develops in the following way

\[ \Psi_{\text{inc}} \Phi_0(1) \Phi_0(2) \rightarrow \Psi_{\text{ref}} \Phi_1(1) \Phi_0(2) + \Psi_{\text{trans}} \Phi_0(1) \Phi_1(2) \]  

(3)
where \( \Phi_1 \) and \( \Phi_0 \) are orthogonal (and non-overlapping in the associated coordinate space) and the number in parenthesis labels the detector. Under these circumstances
if the two beams emerging from the beam splitter are recombined they do not interfere and the behaviour of a particle associated with one part of the superposition remains independent of the other part of the superposition. (The alternatives decohere.) In the configuration space spanned by the particle coordinates and by all the measuring device coordinates the Bohm trajectories do not cross, but in the subspace spanned by just the particle coordinates the trajectories may cross.

With this in mind let us now consider the quantum pinball. In the absence of the measuring devices the incident wavepacket splits at each barrier and subsequently interferes at each further barrier with the packets that have propagated along the various alternate paths to that barrier. Given that the Bohm trajectories in this two-dimensional system cannot cross the order of the trajectories in the initial packet is maintained as the packet propagates through the system and two trajectories that start close together must remain close together. Two such quantum trajectories are shown in figure 1.

The particle trajectories are very different if we include the measuring devices. The splitting behaviour at the first barrier remains the same as in the case discussed above, but subsequently the system evolves in a very different manner. The presence of the measuring devices ensures that there is no interference between any of the packets that propagate along different paths in the pinball apparatus. These alternative paths may then be said to form a set of consistent histories. Consequently the behaviour of the trajectories at each barrier is the same as their behaviour at the first barrier: the packet splits at the centre, those in front are transmitted whilst those behind are reflected. Which path a given particle takes through the pinball is determined at the outset by its position within the initial packet. Every time the particle scatters from a barrier its position with respect to the centre of the emerging packet is different. In fact a simple calculation shows that the sequence of positions of the particle in the wave packet, as the particle scatters from successive barriers, is given by

\[ x_{n+1} = 2x_n \mod 1 \]  

This behaviour of the particle coordinate is identical to that discussed by Goldstein, Dürr and Zanghì in the context of a particle oscillating back and forth in a double-well potential and subjected to repeated position measurements. In our example of chaotic Bohm motion, the position of the particle relative to the centre of the wave packet, as it scatters from the barriers, determines the path taken through the pinball system. We can think of the particle position in the wavepacket as an internal coordinate and the centre of the packet actually containing the particle as an external or macroscopic coordinate. Since the internal coordinate behaves chaotically according to equation (4) the external coordinate follows an apparently random walk through the pinball. If the system is set up twice, with slightly different initial internal coordinate in each case, the corresponding motions of the wave packets through the pinball soon become totally uncorrelated, as the iterations of equation (4) quickly diverge. Two such trajectories are shown in figure 2. In practise of course it is not experimentally possible to set the system up with the same wave packet but different initial particle position within the packet, any attempt to determine the particle position will also alter its wave function.

When the particle in the pinball is not measured we see that the Bohm trajectories are very different to those of a classical particle in such a pinball device. With the measuring devices in place the trajectories are similar to those of a classical particle. In the WKB limit (in which the pins are smoothly varying potentials in the distance
of one wavelength) and when the wavepacket is small compared to the size of the
pins one would expect the whole wavepacket to follow a single path through the
pinball. In this case the Bohm trajectories would not diverge, they would all be
grouped around the same path. As the packet spreads it would begin to split on
scattering from the pins. If no measuring devices are present on the paths the wave-
like behaviour discussed above will develop, with the measuring devices in place the
internal coordinate will become chaotic and the packet trajectory will become random.
However, if the measuring devices also reconstitute the original packet dimensions on
each occasion (thus preventing the spreading of the packet from allowing different
paths through the pinball developing) the packet will continue to follow a classical
path, as will the Bohm trajectories.

3. Conclusion

Since measurements are held to be inherently random processes in orthodox
interpretations of quantum theory the study of their effects generally falls outside
the arena of interest for quantum chaologists. In Bohm's approach measurements are
deterministic dynamical processes and their effect on the evolution of the coordinates
of a system is a legitimate case for study in the search for quantum chaos. Here, in
accordance with the results of D"urr et.al., we have shown that quantum chaos will
arise in simple quantum systems, such as the kicked-rotor or pinball systems, when
they are subjected to repeated measurements.

The quantum kicked-rotor when completely isolated does not show chaotic motion,
so how does chaos arise in the classical limit? One answer is that classical and hence
chaotic behaviour in the motion of kicked rotor will develop when it is allowed to
interact with certain types of environment (here a set of measuring devices, which may
themselves mimic a more complicated interaction with an more general environment
capable of storing information about the rotor’s state) in such a way that the various
alternatives that arise in the time-development of the system’s wave function decohere.

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Figure captions

**Figure 1.** Two Bohm trajectories in the quantum pinball with slightly different initial internal coordinates, in the absence of measuring devices. There is no divergence.

**Figure 2.** Two Bohm trajectories in the quantum pinball started with slightly different initial internal coordinates, with the measuring devices in place. The trajectories diverge as the internal coordinate executes completely chaotic motion.
Position vs. Iteration graph with two lines labeled 'fort.1' and 'fort.2'.
