Diagonal reflection symmetries, four-zero texture, and trimaximal mixing with predicted $\theta_{13}$ in an $A_4$ symmetric model

Masaki J. S. Yang$^{1}$*

$^{1}$Department of Physics, Saitama University, Shimo-okubo, Sakura-ku, Saitama, 338-8570, Japan

In this paper, we impose a magic symmetry on the neutrino mass matrix $m_{\nu}$ with universal four-zero texture and diagonal reflection symmetries. Due to the magic symmetry, the MNS matrix has trimaximal mixing inevitably. Since the lepton sector has only six free parameters, physical observables of leptons are all determined from the charged leptons masses $m_{e,i}$, the neutrino mass differences $\Delta m^2_{ij}$, and the mixing angle $\theta_{23}$.

This scheme predicts $\sin \theta_{13} = 0.149$, that is almost equal to the latest best fit, as a function of the lepton masses $m_{e,i}$ and the mass differences $\Delta m^2_{ij}$. Moreover, even if the mass matrix has perturbations that break the magic symmetry, the prediction of $\sin \theta_{13}$ is retained with good accuracy for the four-zero texture with diagonal reflection symmetries.

I. INTRODUCTION

To approach the flavor puzzle, a number of flavor structures have been considered. In particular, universal texture [1] that imposes the same flavor structure on quark and leptons is quite appealing in the context of unified theories. For this reason, various universal textures have been considered, such as four-zero texture ($M_f)_{11} = (M_f)_{13,31} = 0$ [2–18] and universal texture zero ($M_f)_{11} = 0$ [19–23].

Meanwhile, a set of generalized CP symmetries (GCPs) [24–47] called diagonal reflection symmetries (DRS) has been proposed [48]. These GCPs remove redundant CP phases and enhance predictive power of flavor textures. By combining with the universal four-zero texture, the mixing angles and CP phases of the CKM and MNS matrices are reproduced well with an accuracy of $10^{-3}$. Since this system has eight parameters in both the quark and lepton sector, it can predict all physical quantities (such as Majorana phases $\alpha_{21}, \alpha_{31}$ and the mass of the lightest neutrino $m_1$) from current observables. As a result, an approximate trimaximal mixing [48–50] that is not assumed in the system is emerged. The trimaximal mixing is associated with a $Z_2$ symmetry called magic symmetry [52]. Therefore, in this paper, we investigate effects of imposing the magic symmetry on the four-zero texture with DRS. In addition, a field theoretical realization of such textures and perturbative breaking of the magic symmetry are discussed.

This paper is organized as follows. The next section gives a review of DRS and four-zero texture. In Sec. 3, we discuss a magic symmetry on the neutrino mass matrix, breaking of the symmetry, and its perturbative effects. In Sec. 4, the seesaw mechanism and a realization of magic symmetry are discussed. The final section is devoted to a summary.

II. DIAGONAL REFLECTION SYMMETRIES AND UNIVERSAL FOUR-ZERO TEXTURE

In this section, we review a previous study [48]. The following four-zero texture of mass matrices reproduce the mixing matrices and mass eigenvalues of fermions,

$$M_u = \begin{pmatrix} 0 & i C_f & 0 \\ -i C_f & B_u & B_u \\ 0 & B_u & A_u \end{pmatrix}, \quad m_{\nu} = \begin{pmatrix} 0 & i \nu & 0 \\ i \nu & B_\nu & B_\nu \\ 0 & B_\nu & A_\nu \end{pmatrix}, \quad M_{d,e} = \begin{pmatrix} 0 & C_{d,e} & 0 \\ C_{d,e} & B_{d,e} & B_{d,e} \\ 0 & B_{d,e} & A_{d,e} \end{pmatrix}, \quad$$ (1)

with real parameters $A_f \sim C_f$ and $a_{\nu} \sim c_{\nu}$. Hermiticity of Yukawa matrices is guaranteed by the parity symmetry in the left-right symmetric model [60–62]. Eq. (1) also has the DRS [48]

$$R M_u^* R = M_u, \quad R m_{\nu}^* R = m_{\nu}, \quad M_{d,e}^* = M_{d,e}, \quad R = \text{diag} (-1, 1, 1). \quad$$ (2)

They are regarded as remnant symmetries such as the magic symmetry [52] of the neutrino mass matrix $m_{\nu}$. For example, these symmetries are realized by vacuum expectation values (vevs) of two scalar fields with different phases.

*Electronic address: yang@krishna.th.phy.saitama-u.ac.jp
The following symmetric matrix always predicts the trimaximal mixing \( [48 \, 63] \). A realization of DRS, magic symmetry, and zero textures is also discussed in Section 4 of this paper.

The flavor mixing matrices are expressed by orthogonal matrices \( O_f \) that diagonalize the mass matrices \( M_f \) and \( m_\nu \) as follows,

\[
V_{\text{CKM}} = O^T_f \begin{pmatrix} -i & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} O_d, \quad U_{\text{MNS}} = O^T_f \begin{pmatrix} +i & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} O_\nu.
\] (3)

This system has eight parameters for both quarks and leptons. These mixing matrices \( [48 \, 63] \) reproduce experiments with an accuracy of \( O(10^{-3}) \). Although this system has an obvious deviation in \( |V_{ub}| \), three-zero texture \( (M_u)_{11} \neq 0 \) with DRS predicts CKM matrices with an accuracy of \( O(10^{-4}) \) \( [63] \).

As input parameters in the lepton sector, we choose the following eight observables; three charged lepton masses at mass of \( Z \) boson \( m_Z \) \( [64] \),

\[
m_e = 486.570 \text{[keV]}, \quad m_\mu = 102.718 \text{[MeV]}, \quad m_\tau = 1746.17 \text{[MeV]},
\] (4)

three mixing angles \( [65] \),

\[
\sin^2 \theta_{12} = 0.304^{+0.012}_{-0.012}, \quad \sin^2 \theta_{23} = 0.573^{+0.016}_{-0.020}, \quad \sin^2 \theta_{13} = 0.02219^{+0.00062}_{-0.00063},
\] (5)

and the mass-squared differences for the normal ordering \( [65] \)

\[
\Delta m^2_{21} = 74.2^{+2.1}_{-2.0} \text{[meV]^2}, \quad \Delta m^2_{31} = 2517^{+26}_{-28} \text{[meV]^2}.
\] (6)

The errors of neutrino parameters are in the range of 1 \( \sigma \) region. The inverted ordering is excluded because it is inconsistent with the four-zero texture.

A reconstructed neutrino mass matrix is \( [48 \, 63] \)

\[
m^\nu_\nu = \begin{pmatrix} 0 & 8.80 & i \\ 8.80 & 29.6 & 26.3 \\ 0 & 26.3 & 14.1 \end{pmatrix}, \quad M^\nu_\nu \simeq \begin{pmatrix} 0 & \mp \sqrt{m_e m_\mu} & 0 \\ \mp \sqrt{m_e m_\mu} & \pm m_\mu + t^2 m_\tau & t_\tau m_\tau \\ 0 & \mp t_\tau m_\tau & m_\tau \end{pmatrix},
\] (7)

where \( t_\tau \equiv \sin \tau \simeq 0.06 \) is a 23 mixing. The sign of 12 element \( \text{sign}(C_\tau) \) is related to that of 22 element \( \text{sign}(B_e) = \text{sign}(m_{e2}) \) in order to keep the correct sign of the Jarlskog invariant \( J_{\text{MNS}} \) \( [60] \). From a viewpoint of unification, the other solution with \( m_1 \simeq 6.2 \text{[meV]} \) is excluded because it predicts \( (M_\nu)_{22} \simeq m_\tau \).

An absolute value of the MNS matrix is calculated as

\[
|U_{\text{MNS}}| = \begin{pmatrix} 0.8251 & 0.5449 & 0.1490 \\ 0.2755 & 0.6031 & 0.7485 \\ 0.4932 & 0.5825 & 0.6461 \end{pmatrix},
\] (8)

with errors of about \( O(10^{-3}) \) from the best fit values. The predicted MNS matrix has an approximate trimaximal mixing, that is not assumed in the texture Eq. (11). Its cause is explored in the next section.

### III. TRIMAXIMAL MIXING AND MAGIC SYMMETRY

The matrix \( m^\nu_\nu \) \( [7] \) approximately has an eigenvector \( v \sim (1, -1, 1) \) and predicts the trimaximal mixing \( [49 \, 53 \, 54] \). The following symmetric matrix always predicts the trimaximal mixing \( [50 \, 52] \)

\[
m_T = \begin{pmatrix} A & B & C \\ B & D & A + C - D \\ C & A + C - D & B + D - C \end{pmatrix},
\] (9)

with complex parameters \( A \sim D \). A matrix will be called \textit{magic} if the row sums and the column sums are all equal to a number \( \alpha \) \( [52] \). The matrix \( m_T \) satisfies the following \( Z_2 \) symmetry

\[
S_2 m_T S_2^T = m_T, \quad S_2 = \begin{pmatrix} 1/3 & -2/3 & -2/3 \\ -2/3 & 1/3 & -2/3 \\ -2/3 & -2/3 & 1/3 \end{pmatrix}, \quad S_2^2 = I_3.
\] (10)
that is called magic symmetry \[52\].

After a phase redefinition, the mass matrix \( m_\nu T \) \[7\] can approximately be parameterized to a matrix with the (deformed) magic symmetry and zero textures \[67\],

\[
m_\nu T \equiv \begin{pmatrix} 0 & c & 0 \\ c & b + c & b + c \\ 0 & b + c & b \end{pmatrix}.
\] (11)

Here, \( b \) and \( c \) are real parameters. A realization of the texture \[11\] has been discussed in a model with \( A_4 \) symmetry \[67\]. The condition \[10\] for the matrix \( m_\nu T \) \[11\] is deformed to be

\[
\tilde{S}_2 m_\nu T \tilde{S}_2^T = m_\nu T, \quad \tilde{S}_2 = \begin{pmatrix} 1/3 & 2/3 & -2/3 \\ 2/3 & 1/3 & 2/3 \\ -2/3 & -2/3 & 4/3 \end{pmatrix}, \quad \tilde{S}_2 = 13,
\] (12)

that is equivalent to the condition for an eigenvector

\[
m_\nu T \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = -c \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}.
\] (13)

By a proper phase transformation, the generator \( \tilde{S}_2 \) in Eq. \[12\] and the DRS commute. Then, it composes a \( Z_2 \) symmetry only for the neutrinos. Similar observation is found in \( Z_2 \times Z_2 \) symmetry \[68\] \[71\] and trimaximal \( \mu - \tau \) reflection symmetry \[44\].

The mass matrix \( m_\nu T \) is exactly diagonalized by so-called \( \nu TBM \) mixing \[72\] \[73\], which is a combination of the tri-bi-maximal \[74\] and a 13 mixing

\[
O_{13}^T U_{TBM}^T m_\nu T U_{TBM} O_{13} = m_{\nu T}^{\text{diag}},
\] (14)

where

\[
U_{TBM} = \begin{pmatrix} \sqrt{2} \sqrt{3} & \frac{1}{\sqrt{3}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ \sqrt{\frac{1}{6}} & \sqrt{\frac{1}{3}} & \sqrt{\frac{1}{2}} \end{pmatrix}, \quad O_{13} = \begin{pmatrix} \cos \phi_{13} & 0 & \sin \phi_{13} \\ 0 & 1 & 0 \\ -\sin \phi_{13} & 0 & \cos \phi_{13} \end{pmatrix}, \quad \tan 2\phi_{13} = \frac{\sqrt{3} c}{2b + c},
\] (15)

and

\[
m_{\nu T}^{\text{diag}} = \text{diag}(b + c - \sqrt{b^2 + bc + c^2}, -c, b + c + \sqrt{b^2 + bc + c^2}).
\] (16)

From Eq. \[10\], the two mass differences \( \Delta m_{ij}^2 \) are written by \( b \) and \( c \),

\[
\Delta m_{31}^2 = 4(b + c)\sqrt{b^2 + bc + c^2}, \quad \Delta m_{21}^2 = (b + c)(2\sqrt{b^2 + bc + c^2} - 2b - c).
\] (17)

Conversely, the parameters \( b \) and \( c \) are determined from the best fits of \( \Delta m_{13}^2 \) \[9\] as

\[(c, b) = (9.19, 17.5) \text{[meV]}, \quad \text{or} \quad (-11.3, 33.0) \text{[meV]}.
\] (19)

We exclude the second solution with \( c < 0 \), because it corresponds to the solution \( m_1 = 6.2 \text{[meV]} \) in Eq. \[7\] that requires \( (M_\nu)_{22} \simeq m_\tau \).

A reconstructed mass matrix with phases is

\[
m_\nu T = \begin{pmatrix} 0 & 9.19 i & 0 \\ 9.19 i & 26.7 & 26.7 \\ 0 & 26.7 & 17.5 \end{pmatrix} \text{[meV]} = \begin{pmatrix} 0 & i m_2 & 0 \\ i m_2 & \frac{m_3 + m_1}{2} & \frac{m_3 + m_1}{2} \\ 0 & \frac{m_3 + m_1}{2} & \frac{m_3 + m_1}{2} - m_2 \end{pmatrix}.
\] (20)

Mass eigenvalues \[10\] are found to be

\[
m_{\nu T}^{\text{diag}} = \text{diag}(3.21, 9.19, 50.3) \text{[meV]}.
\] (21)
The MNS matrix \( \mathbf{U}_{\text{MNS}} \) is approximately expressed as

\[
\mathbf{U}_{\text{MNS}} = O^T \begin{pmatrix} +i & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} O_{\nu}
\]

\[
\simeq \begin{pmatrix} \pm \sqrt{m_{e}/m_{\mu}} & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} +i & 0 & 0 \\ 0 & c_\tau & s_\tau \\ 0 & -s_\tau & c_\tau \end{pmatrix} U_{\text{TBM}} O_{13} \begin{pmatrix} \pm 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},
\]

where \( s_\tau \equiv \sin \tau, \ c_\tau \equiv \cos \tau \). The symbol \( \pm \) denotes the sign of \( C_\tau \) (7). To retain the correct sign of the Jarlskog invariant, a diagonal phase matrix \( \text{diag}(\pm 1, 1, 1) \) is added. By neglecting a small parameter \( s_\tau \), \( U_{e3} \) is predicted as a function of the lepton masses \( m_{e,\mu} \) and the mass differences \( \Delta m_{21}^2 \):

\[
U_{e3} \simeq \sqrt{\frac{m_{e}}{2m_{\mu}}} c_{\phi_{13}} + \sqrt{\frac{m_{e}}{6m_{\mu}}} \phi_{13} + i \sqrt{\frac{2}{3}} s_{\phi_{13}}, \quad |U_{e3}| \simeq 0.149,
\]

\[
\sin \phi_{13} = \sqrt{\frac{1}{2} - \frac{2b + c}{4b^2 + bc + c^2}}.
\]

The last free parameter \( s_\tau \) is determined from 23 and 33 elements of \( \mathbf{U}_{\text{MNS}} \) that have relatively good accuracy.

\[
|(U_{\text{MNS}})_{23}|/(|U_{\text{MNS}}|)_{33} = \tan \theta_{23} \Rightarrow s_\tau \simeq 0.0259.
\]

Therefore, \( \theta_{12} \) is predicted by taking \( \theta_{23} \) as an input parameter;

\[
\sin^2 \theta'_{12} = \frac{|(U_{\text{MNS}})_{12}|^2}{1 - |(U_{\text{MNS}})_{13}|^2} = 0.341, \quad \sin \theta'_{12} = 0.584.
\]

This is (barely) in the 3 \( \sigma \) region of the best fit value (5). Reconstructed \( \mathbf{U}_{\text{MNS}} \) is found to be

\[
|\mathbf{U}_{\text{MNS}}| = \begin{pmatrix} 0.803 & 0.577 & 0.149 \\ 0.300 & 0.592 & 0.748 \\ 0.516 & 0.562 & 0.646 \end{pmatrix}.
\]

All absolute values of components are within 3\( \sigma \) range of global fit. Moreover, the value of \( U_{e3} \) is very closer to the best fit value, \( \sin 8.57^\circ = 0.1490 \). Errors from the best fit are only about \( O(10^{-2}) \).

\[
|\mathbf{U}_{\text{MNS}}^{\text{best}}| - |\mathbf{U}_{\text{MNS}}| = \begin{pmatrix} -0.022 & 0.033 & 0.000 \\ 0.027 & -0.013 & 0.000 \\ 0.021 & -0.018 & 0.000 \end{pmatrix}.
\]

A. Breaking of the magic symmetry

The four-zero texture with DRS and trimaximal condition predicts somewhat too large \( \theta'_{12} \) (27). To fix this discrepancy, here we parameterize breakings of the magic symmetry. A deviation of the reconstructed mass matrix (20) from the best fit (7) is

\[
\Delta m_\nu = m_\nu^r - m_{\nu T}^r = - \begin{pmatrix} 0 & 0.39i & 0 \\ 0.39i & -2.9 & 0.04 \\ 0 & 0.04 & 3.4 \end{pmatrix}.
\]

Thus, for \( m_{\nu T} \) (11), we define breaking parameters \( \delta \) and \( \epsilon \) as follows;

\[
\delta m_\nu = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \delta & 0 \\ 0 & 0 & \epsilon \end{pmatrix}, \quad m_\nu^r = m_{\nu T} + \delta m_\nu = \begin{pmatrix} 0 & c & 0 \\ c & b + c + \delta & b + c \\ 0 & b + \epsilon & b + \epsilon \end{pmatrix}.
\]
Since $m'_{
u}$ in Eq. (31) has four parameters, it describes four-zero texture with DRS (1) without loss of generality.

Next, we will survey effects of perturbations $\delta$ and $\epsilon$ in the diagonalization of $m'_{\nu}$. Defining a matrix $O = U_{\text{TBM}} O_{13}$ in Eqs. (14) and (15), we obtain

$$O^T m_{\nu} O = m_{\nu}^{\text{diag}}.$$  \hspace{1cm} (32)

Let $\delta O$ and $\delta m_{\nu}^{\text{diag}}$ be perturbative corrections to the orthogonal and eigenvalue matrices. The diagonalization of the full mass matrix $m_{\nu}'$ is written as

$$(O^T + \delta O^T)(m_{\nu}O + \delta m_{\nu})O + \delta O^T m_{\nu}O = m_{\nu}^{\text{diag}} + \delta m_{\nu}^{\text{diag}}.$$  \hspace{1cm} (33)

By subtracting Eq. (32) from Eq. (33), an expression for the first-order perturbation is found to be

$$O^T m_{\nu} O + O^T \delta m_{\nu} O + \delta O^T m_{\nu} O = \delta m_{\nu}^{\text{diag}}.$$  \hspace{1cm} (34)

The characteristic equation $m_{\nu} O = O m_{\nu}^{\text{diag}}$ from Eq. (32) and the orthogonality relation $\delta O^T O + O^T \delta O = 0$ lead to

$$m_{\nu}^{\text{diag}} O^T \delta O + O^T \delta m_{\nu} O - O^T \delta O m_{\nu}^{\text{diag}} = \delta m_{\nu}^{\text{diag}}.$$  \hspace{1cm} (35)

For diagonal elements, it is rewritten as

$$(\delta m_{\nu}^{\text{diag}})_{ii} = (O^T \delta m_{\nu} O)_{ii},$$  \hspace{1cm} (36)

and for off-diagonal elements,

$$(O^T \delta O)_{ij} = -\frac{(O^T \delta m_{\nu} O)_{ij}}{m_i - m_j}.$$  \hspace{1cm} (37)

They are equivalent to the usual perturbative relations in quantum mechanics. Since $O^T (O + \delta O) = 1 + O^T \delta O$ holds, Eq. (37) represents a perturbative rotation in the diagonalized basis.

In particular, a correction to the 13 element is

$$(\delta O)_{13} = -\sum_{i=1}^{2} O_{1i} (O^T \delta m_{\nu} O)_{i3}.$$  \hspace{1cm} (38)

By neglecting $m_1$ and $m_2$, this equation can be transformed as

$$(\delta O)_{13} \simeq \frac{1}{m_3} \sum_{i=1}^{2} O_{1i} (O^T \delta m_{\nu} O)_{i3}$$  \hspace{1cm} (39)

$$= \frac{1}{m_3} [(O^T \delta m_{\nu} O)_{13} - O_{13} (O^T \delta m_{\nu} O)_{33}].$$  \hspace{1cm} (40)

Since the first term vanishes $(\delta m_{\nu})_{ii} = 0$, from Eq. (31), a correction to $\sin \theta_{13}$ eventually becomes

$$(\delta O)_{13} \simeq \frac{[O_{13} (O^T \delta m_{\nu} O)_{33}]}{m_3} \simeq -\sin \theta_{13} \frac{\delta + \epsilon}{2 m_3}.$$  \hspace{1cm} (41)

We used $O \simeq U_{\text{TBM}}$ in the last equality. Since $\delta \simeq -\epsilon$ holds in the best fit (31), the first-order perturbations for $O_{13}$ is very small. Thus, the prediction of $\sin \theta_{13}$ (21) is approximately retained for the best fit of $m_{\nu}$ (1) with $b = (m_{\nu})_{23} - (m_{\nu})_{12}$ and $c = (m_{\nu})_{12}$.

For comparable values of $\delta, \epsilon \sim 3$ [meV] and $m_3 \sim 50$ [meV], an error to $\sin \theta_{13}$ is estimated as

$$|\delta O_{13}| \sim 0.0045, \frac{\delta |\sin \theta_{13}|}{\sin \theta_{13}} \sim 0.03.$$  \hspace{1cm} (42)

Therefore, even if the four-zero texture with DRS has perturbations that break the magic symmetry, it predicts the correct $\sin \theta_{13}$ with good accuracy. This result comes from the fact that $m_3$ is the largest eigenvalue and $\sin \theta_{13}$ is relatively small.
IV. TYPE-I SEESAW MECHANISM AND REALIZATION OF MAGIC SYMMETRY

In a model with the type-I seesaw mechanism \([75–77]\), the following matrices \(Y_\nu\) and \(\tilde{m}_\nu\) that have the same forms as Eq. (1) and (20)

\[
Y_\nu = \begin{pmatrix}
0 & i C_\nu & 0 \\
-i C_\nu & B_\nu & B_\nu \\
0 & B_\nu & A_\nu
\end{pmatrix}, \quad \tilde{m}_\nu = \begin{pmatrix}
0 & i c & 0 \\
i c & b + c & b + c \\
0 & b + c & b
\end{pmatrix},
\]

predict a four-zero texture for the Majorana mass matrix of right-handed neutrinos \(M_R\) \([7, 9]\):

\[
M_R = \frac{v^2}{2} Y_\nu^{-1} \tilde{m}_\nu Y_\nu^T \tag{43}
\]

Moreover, the matrix \(M_R\) also has the DRS (2), \(R M_R^\dagger R = M_R\). A hierarchical \(Y_\nu\) with \(A_\nu \gg B_\nu, \tilde{B}_\nu \gg C_\nu\) yields a strongly hierarchical \(M_R\).

\[
M_R \sim \frac{v^2}{2} \begin{pmatrix}
0 & i C^2 & 0 \\
i C^2 & \frac{B^2}{b} & \frac{A_\nu B_\nu}{b^2} \\
0 & \frac{A_\nu B_\nu}{b^2} & \frac{A^2_\nu}{b}
\end{pmatrix}. \tag{46}
\]

However, from Eq. (46), it seems difficult to derive a condition for the magic symmetry.

The mass matrix \(\tilde{m}_\nu\) \([13]\) has the following texture and symmetry,

- **four-zero texture**,  
- **diagonal reflection symmetry**,  
- **(deformed) magic symmetry**,  

that are retained in the type-I seesaw mechanism. By imposing all these conditions on the Yukawa matrix \(Y_\nu\), the matrix \(M_R\) exhibits the same properties and is written by two parameters,

\[
Y_\nu = \begin{pmatrix}
0 & i C_\nu \\
-i C_\nu & C_\nu + B_\nu & C_\nu + B_\nu \\
0 & C_\nu + B_\nu & B_\nu
\end{pmatrix}, \quad M_R = \begin{pmatrix}
0 & -i C_R \\
-i C_R & C_R + B_R & C_R + B_R \\
0 & C_R + B_R & B_R
\end{pmatrix}. \tag{47}
\]

In this case, the light neutrino mass \(\tilde{m}_\nu\) is obtained as

\[
\tilde{m}_\nu = \frac{v^2}{2} Y_\nu M_R^{-1} Y_\nu^T \tag{48}
\]

\[
\tilde{m}_\nu = \frac{v^2}{2} \begin{pmatrix}
0 & \frac{i C^2}{C_R} \\
\frac{i C^2}{C_R} & \frac{B^2}{b} + \frac{B^2}{B_R} \\
0 & \frac{B^2}{b} + \frac{B^2}{B_R}
\end{pmatrix}, \tag{49}
\]

with \(v = 246\) [GeV]. Indeed this \(\tilde{m}_\nu\) is magic, diagonal reflection symmetric, and has four-zero texture. The parameters of \(M_R\) are concisely expressed by that of \(\tilde{m}_\nu\) and \(Y_\nu\) as

\[
B_R = \frac{v^2 B^2}{2}, \quad C_R = \frac{v^2 C^2}{2}. \tag{50}
\]

Therefore, in this scheme, the neutrino sector \((\tilde{m}_\nu, Y_\nu, \text{and} M_R)\) has only two free parameters.

Diagonalization of \(Y_\nu\) and \(M_R\) have the same form to that of \(m_{\nu,T}\), Eq. (14) and (15). Therefore, mass eigenvalues of \(M_R\) can be written in the same way as Eq. (16),

\[
M_R^{\text{diag}} = \text{diag}(B_R + C_R - \sqrt{B_R^2 + B_R C_R + C_R^2}, -C_R, B_R + C_R + \sqrt{B_R^2 + B_R C_R + C_R^2}). \tag{51}
\]
A. Realization of symmetries and texture

In order to justify the symmetries assumed above, we consider a partial compositeness-like realization by the two Higgs doublet model (2HDM) with $A_4$ flavor symmetry. The $A_4$ flavor symmetry [78, 79] has been studied in a wide range of models [78–108]. The group consists of the following generators $S$ and $T$:

$$S^2 = T^3 = (ST)^3 = 1. \quad (52)$$

There are four irreducible representations $1, 1', 1''$ and $3$. For the 3 dimensional representation, $S$ and $T$ are usually taken as follows

$$S = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad T = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (53)$$

The basic idea of the partial compositeness [109–116] is that the SM fields at low energy are the mixed states between elemental (massless) fields and composite (massive) fields, like $\rho - \gamma$ mixing. Flavor structures are induced from mixings between massive and massless fermions with the same quantum numbers.

Let us consider a model with $A_4$ flavor symmetry and field contents as in Table 1. $U(1)_{PQ}$ is a chiral symmetry that distinguishes several particle species, and GCP is a generalized CP symmetry. Since the GCP charge of a field with nontrivial transformations under $A_4$ is 1, we do not need to consider the consistency conditions [30, 31]. $H$ and $H_2$ are two Higgs doublets with different chiral charges, $L, E, L', N', E'$ are fields with heavy masses and the same charges as the corresponding SM fields. $\varphi, \varphi'$, and $\Delta$ are flavons with 3 and 1$'$ representation, $\Phi$ is a scalar field that breaks $U(1)_{PQ}$.

| \[SU(2)_L \quad U(1)_Y \quad A_4 \quad U(1)_{PQ} \quad GCP\] | \[l_{Li}\] | 2 | -1/2 | 3 | 0 | 1
| \[\nu_{Ri}\] | 1 | 0 | 3 | 0 | 1
| \[e_{Ri}\] | 1 | -1 | 3 | -2 | 1
| \[H\] | 2 | 1/2 | 1 | 0 | 1
| \[H_2\] | 2 | 1/2 | 1 | 2 | -1
| \[L_{(L,R)}\] | 2 | -1/2 | 1 | -1 | 1
| \[E_{(L,R)}\] | 1 | -1 | 1 | -1 | 1
| \[N'_{(L,R)}\] | 1 | 0 | 1' | 0 | 1
| \[L'_{L}\] | 2 | -1/2 | 1' | 0 | 1
| \[L'_{R}\] | 2 | -1/2 | 1'' | 0 | 1
| \[E'_{L}\] | 1 | -1 | 1' | -2 | 1
| \[E'_{R}\] | 1 | -1 | 1' | 0 | 1
| \[\varphi\] | 1 | 1 | 3 | 1 | 1
| \[\varphi'\] | 1 | 1 | 3 | 0 | 1
| \[\Delta\] | 1 | 1 | 1' | 0 | 1
| \[\Phi\] | 1 | 1 | 1 | -2 | 1

TABLE I: Charge assignments of fields under gauge, flavor, and GCP symmetries.

Similar to the simplified two-site description of composite Higgs model [110], the most general Lagrangian under
the symmetry imposed on the model is divided into three parts:

\[
\mathcal{L}_{\text{heavy}} = \bar{L}(i\gamma^\mu - M_L)\gamma^\nu L + \bar{E}(i\gamma^\mu - M_E)\gamma^\nu E + \mathcal{N}^\dagger(i\gamma^\mu - M_{N'})\gamma^\nu N' + \bar{L}'i\gamma^\mu L' + \bar{E}'i\gamma^\mu E'
\]

\[
- (Y^E_L)L_H E_R + Y^N L_H H_N' + Y^{E'}_L L_H H_R' + \bar{Y}^F \bar{L}_H \bar{H} L_R + \text{h.c.)}
\]

\[
- \left( y_L L_H L_R' \Delta^e + y_{E'} E_L E_R' \Phi + \frac{y_{NL}}{2} \Delta^e N_L' N_L' + \frac{y_{NR}}{2} \Delta^e N_R' N_R' + \text{h.c.} \right),
\]

\[
\mathcal{L}_{\text{light}} = \bar{L}_L \gamma^\mu \gamma^\nu L_L + \bar{\nu}_R \gamma^\mu \gamma^\nu \nu_R + \bar{e}_R \gamma^\mu \gamma^\nu \nu_R,
\]

\[
H_{\text{mixing}} = \left( \lambda^L \bar{L}_L R L \phi_i + \lambda^E \bar{E}_L e_R \phi_i \right)
\]

\[
+ \lambda^{L'} \bar{L}_L L'_R P_{i j} \phi_j + \lambda^N \bar{N}_L \nu_R P_{i j} \phi_j + \lambda^{E'} \bar{E}_L e_R P_{i j} \phi_j + \text{h.c.},
\]

where \( \bar{H} \equiv i\sigma^2 H^* \) is the conjugate field of the Higgs doublet \( H \) and

\[
P = \begin{pmatrix}
1 & 0 & 0 \\
0 & \omega & 0 \\
0 & 0 & \omega^2
\end{pmatrix}.
\]

A combination such as \( \nu_R \phi_j \) constitutes a \( 1' \) representation.

The scalar fields are assumed to have real vevs

\[
\langle \phi \rangle = V_\phi \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \quad \langle \phi' \rangle = V_{\phi'} \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \quad \langle \Delta \rangle = V_\Delta, \quad \langle \Phi \rangle = V_\Phi, \quad \langle H \rangle = V_H, \quad \langle H_2 \rangle = V_{H_2},
\]

where \( V_\chi \in \mathbb{R} \) for \( \chi = \phi, \phi', \Delta, \Phi, H \) and \( H_2 \). The vevs of \( \phi, \phi', \Delta, \) and \( \Phi \) break \( A_4 \) and \( U(1)_{PQ} \), but retain GCP symmetry. For the Higgs doublets \( H \) and \( H_2 \), the reality of vevs are achieved over a wide range of parameters. According to the spirit of partial compositeness, the masses of heavy fields are assumed to be larger than those of flavon’s vevs, \( M_F, \langle \Delta \rangle, \langle \Phi \rangle \gg \lambda_F \langle \phi \rangle \). Due to this, the linear mixing terms \( H_{\text{mixing}} \) induce mass terms between massive and massless fields.

When the heavy fields are integrated out, the SM interactions at low energy are represented by seesaw-like formulae

\[
y_e = y_{e0}1_3 r_e + \langle \phi \rangle \lambda^L M^{-1}_L Y^E M^{-1}_E \lambda^L \langle \phi' \rangle + \langle \phi' \rangle P^* \lambda^{L'} (y_L \langle \Delta \rangle)^{-1} Y^{E'} \langle y_{E'} \langle \Phi \rangle \rangle^{-1} \lambda^{E'} \langle \phi' \rangle^T,
\]

\[
y_{e'} = y_{e0}1_3 + \langle \phi' \rangle P^* \lambda^L (y_{L'} \langle \Delta \rangle)^{-1} Y^{E'} M^{-1}_N \lambda^N \langle \phi' \rangle^T,
\]

\[
m_r = m_{r0}1_3 + \langle \phi' \rangle P^* \langle y_{NL} \langle \Delta \rangle \rangle^{-1} \lambda^N P^* \langle \phi' \rangle^T.
\]

Here, \( r_e = \langle H_2 \rangle / \langle H \rangle \) is a factor that takes into account the ratio of the vevs of the two Higgs doublets. The term with \( Y^{E'} \) in Eq. (63) does not contribute SM matrices in the first order of \( \lambda^L \langle \phi \rangle / M_F \). Figure 1 shows diagramatic explanations of Eq. (63).

![FIG. 1: A diagrammatic description of processes that generate the flavor structure of \( y_e \).](image)

\[\text{In order not to change the eigenstates of } N'_{L,R} \text{ significantly, the lepton number violating (LNV) parameters } y_{NL}(\Delta) \text{ and } y_{NR}(\Delta) \text{ are assumed as } M_{N'} \gg y_{NL}(\Delta), y_{NR}(\Delta) \text{ in Eq. (63)}. \text{ However, even if this does not hold and both } y_{NL} \text{ and } y_{NR} \text{ contribute to LNV, the final result remains the same because the flavor structure is only generated by } \langle \phi \rangle \text{ and } \langle \phi' \rangle.\]
The flavor structure by terms with flavons are
\[ \langle \varphi \rangle \otimes \langle \varphi \rangle^T = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \quad \langle \varphi' \rangle \otimes \langle \varphi' \rangle^T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega \end{pmatrix}. \] (66)

The first term is the democratic matrix, that have been well discussed for a long time
\[ [117–136]. \]

Combining all the flavor-independent coefficients, we obtain a complex symmetric matrix for Yukawa of leptons;
\[ y_e = a_e \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} + b_e \begin{pmatrix} 0 & 0 & 0 \\ 0 & \omega & -1 \\ 0 & -1 & \omega^2 \end{pmatrix} + c_e \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \] (67)

On the other hand, since there is no contribution of the democratic matrix to the neutrinos,
\[ y_{\nu}, m_R = b_{\nu,R}^* \begin{pmatrix} 0 & 0 & 0 \\ 0 & \omega & -1 \\ 0 & -1 & \omega^2 \end{pmatrix} + c_{\nu,R}^* \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \] (68)

By the following basis transformation, these three mass matrices are converted to the four-zero texture
\[ U^T y_e U = \begin{pmatrix} 0 & c_e & 0 \\ c_e & -b_e & b_e \\ 0 & b_e & 3a_e - b_e + c_e \end{pmatrix}, \quad U^T (y_{\nu}, m_R) U = \begin{pmatrix} 0 & c_{\nu,R} & 0 \\ c_{\nu,R} & -b_{\nu,R} & b_{\nu,R} \\ 0 & b_{\nu,R} & -b_{\nu,R} + c_{\nu,R} \end{pmatrix}, \] (69)

where
\[ U = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ \omega & \omega^2 & 1 \\ \omega^2 & \omega & 1 \end{pmatrix}. \] (70)

For \( y_e \) and \( M_R \), a \( Z_2 \) symmetry due to \( S \) in Eq. [63] remains unbroken.
\[ S(y_{\nu}, m_R) S = (y_{\nu}, m_R). \] (71)

This \( Z_2 \) is changed to magic symmetry [10] by the basis transformation.
\[ -U^T S U = U^T \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} U = \begin{pmatrix} \frac{1}{3} & -\frac{2}{3} & -\frac{2}{3} \\ -\frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \\ -\frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \end{pmatrix} = S_2. \] (72)

Finally, since a nontrivial GCP charge is imposed only on \( H_2 \) in Table 1, the GCP invariance restricts complex phases of couplings as
\[ c_e^* = (-1)c_e, \quad a_e^* = a_e, \quad b_e^* = b_e, \quad b_f^* = b_f, \quad c_f^* = c_f, \quad \text{for } f = \nu, R. \] (73)

Among these parameters, only \( c_e \) is purely imaginary, and all other \( a_e, b_e, b_{\nu,R}, c_R \) are real. In this basis, the GCP is broken by the real vev \( \langle H_2 \rangle \). By redefining the phases and parameters, Eq. (69) results in the four-zero texture with DRS and magic symmetry [10] and [17]. The hierarchy of lepton masses requires that a tiny value of \( c_e \). The term with \( c_e \) is forbidden by the \( U(1)_{PQ} \) symmetry if \( H_2 \) does not exist. Thus, for example, \( c_e \) could be made naturally small by using heavy 3-representation fermions instead of \( H_2 \).

V. SUMMARY

In this paper, we impose a magic symmetry on the neutrino mass matrix \( m_\nu \) with universal four-zero texture and diagonal reflection symmetries. Due to the magic symmetry, the MNS matrix has inevitably trimaximal mixing.
Since free parameters are reduced by two by fixing the eigenvector, the lepton sector has only six free parameters. Therefore, physical observables of leptons are all determined from the charged leptons masses $m_{e\tau}$, the neutrino mass differences $\Delta m_{31}^2$, and the mixing angle $\theta_{13}$.

This scheme predicts $\sin \theta_{13} = 0.149$, that is almost equal to the best fit, as a function of the lepton masses $m_{e\mu}$ and the mass differences $\Delta m_{31}^2$. Moreover, even if the mass matrix has perturbations that break the magic symmetry, this prediction of $\sin \theta_{13}$ is retained with good accuracy for the four-zero texture with DRS.

The diagonal reflection symmetries, four-zero texture, and magic symmetry are all seesaw-invariant. Therefore, the imposition of these conditions on the neutrino Yukawa matrix $Y_\nu$ leads to the same structure for the mass of right-handed neutrinos $M_R$. In this case, the neutrino sector has only two parameters, and $m_\nu$ is concisely represented from parameters of $Y_\nu$ and $M_R$ by the type-1 seesaw mechanism.

For the justification of the assumed symmetries, we considered a partial compositeness-like realization by the two Higgs doublet model with $A_4$ flavor symmetry. By a vev of a 3 representation flavon $\varphi \propto (1,1,1)$, the democratic texture emerges only in the electron-type Yukawa matrix. The four-zero texture appears by proper basis transformation. Since a contribution of democratic texture does not exist in the neutrino sector, a $Z_2$ subgroup of $A_4$ is preserved. This remnant symmetry is identified with the magic symmetry by the basis transformation.

**Acknowledgement**

This study is financially supported by JSPS Grants-in-Aid for Scientific Research No. 18H01210, No. 20K14459, and MEXT KAKENHI Grant No. 18H05543.
