TWISTED FUNCTORIALITY IN NONABELIAN HODGE
THEORY IN POSITIVE CHARACTERISTIC

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ABSTRACT. We establish the twisted functoriality in nonabelian Hodge theory
in positive characteristic. As an application, we obtain a purely algebraic proof
of the fact that the pullback of a semistable Higgs bundle with vanishing Chern
classes is again semistable.

1. Introduction

In the classical nonabelian Hodge theory [Sim], one has the following Simpson
correspondence: Let \( X \) be a compact Kähler manifold. There is an equivalence
of categories

\[
C_\text{\text{\text{-}}}^{-1} : \text{HIG}(X) \to \text{MIC}(X),
\]

where \( \text{HIG}(X) \) is the category of polystable Higgs bundles over \( X \) with vanishing
first two Chern classes and \( \text{MIC}(X) \) is the category of semisimple flat bundles
over \( X \). The equivalence is independent of the choice of a background Kähler
metric, and the following functoriality holds: Let \( f : Y \to X \) be a morphism of
compact Kähler manifolds. Then for any \((E, \theta) \in \text{HIG}(X)\), one has a natural
isomorphism in \( \text{MIC}(Y) \)

\[
C_\text{\text{-}}^{-1} f^*(E, \theta) \cong f^* C_\text{\text{-}}^{-1}(E, \theta).
\]

In the nonabelian Hodge theory in positive characteristic [OV], Ogus-Vologodsky
established an analogue of (1.0.1) for derived categories, with the \( W_2 \)-lifting assump-
tion on \( f \) (see Theorem 3.22 [OV]). In a recent preprint [La19], A. Langer
proved the equality (1.0.1) under the assumption that the \( W_2(k) \)-lifting of \( f \) is good
(see Definition 5.1 and Theorem 5.3 in loc. cit.). However, such an assumption
on the lifting of \( f \) is quite restrictive.

Let \( k \) be a perfect field of characteristic \( p > 0 \). Let \( X \) be a smooth variety over \( k \)
and \( D \) a reduced normal crossing divisor in \( X \). One forms the log smooth variety
\( X_{\log} \) whose log structure is the one determined by \( D \). Equip \( k \) and \( W_2(k) \) with
the trivial log structure. Assume that the log morphism \( X_{\log} \to k \) is liftable to
\( W_2(k) \). Choose and then fix such a lifting \( \tilde{X}_{\log} \). Then one has the inverse Cartier
transform \( [8] \) (which is in general not an equivalence of categories without further
condition on the singularities of modules along \( D \))

\[
C_{X_{\log} \subset \tilde{X}_{\log}}^{-1} : \text{HIG}_{\leq p-1}(X_{\log}/k) \to \text{MIC}_{\leq p-1}(X_{\log}/k).
\]

\footnote{Theorem 6.1 [LSYZ] deals with only the case of SNCD. However, a simple étale descent
argument extends the construction to the reduced NCD case.}
Let $Y_\log = (Y, B)$ be a log smooth variety like above, together with a $W_2(k)$-lifting $\tilde{Y}_\log$. Our main result is the following analogue of \[.0.1\] in positive characteristic:

**Theorem 1.1.** Notion as above. Then for any object $(E, \theta) \in \text{HIG}_{\leq p-1}(X_\log/k)$, one has a natural isomorphism

$$C_{Y_\log \subset \tilde{Y}_\log}^{-1} f^\circ(E, \theta) \cong f^* C_{X_\log \subset \tilde{X}_\log}^{-1} (E, \theta),$$

where $f^\circ(E, \theta)$ is the twisted pullback of $(E, \theta)$.

The twisted pullback of $(E, \theta)$ refers to a certain deformation of $f^*(E, \theta)$ along the obstruction class of lifting $f$ over $W_2(k)$. When the obstruction class vanishes, the twisted pullback is just the usual pullback. See §2 for details. Hence, one has the following immediate consequence.

**Corollary 1.2.** Let $f : Y_\log \to X_\log$ be a morphism of log smooth varieties over $k$. Assume $f$ is liftable to $W_2(k)$. Then for any object $(E, \theta) \in \text{HIG}_{\leq p-1}(X_\log/k)$, one has a natural isomorphism in $\text{MIC}_{\leq p-1}(Y'_\log/k)$

$$C_{Y_\log \subset \tilde{Y}_\log}^{-1} f^*(E, \theta) \cong f'^* C_{X_\log \subset \tilde{X}_\log}^{-1} (E, \theta).$$

The notion of twisted pullback and the corresponding twisted functoriality as exhibited in Theorem 1.1 was inspired by the work of Faltings in the $p$-adic Simpson correspondence [Fa]. It is a remarkable fact that char $p$ and $p$-adic Simpson correspondences have many features in common. As an application, we obtain the following result.

**Theorem 1.3.** Let $k$ be an algebraically closed field and $f : (Y, B) \to (X, D)$ a morphism between smooth projective varieties equipped with normal crossing divisors over $k$. Let $(E, \theta)$ be a semistable logarithmic Higgs bundles with vanishing Chern classes over $(X, D)$. If either char$(k) = 0$ or char$(k) = p > 0$, $f$ is $W_2(k)$-liftable and rank$(E) \leq p$, then the logarithmic Higgs bundle $f^*(E, \theta)$ over $(Y, B)$ is also semistable with vanishing Chern classes.

For char$(k) = 0$ and $D = \emptyset$, the result is due to C. Simpson by transcendental means [Sim]. Our approach is to deduce it from the char $p$ statement by mod $p$ reduction and hence is purely algebraic.

## 2. Twisted pullback

We assume our schemes are all noetherian. Let $(R, M)$ be an affine log scheme. Let $f : Y \to X$ be a morphism of log smooth schemes over $R$. Fix an $r \in \mathbb{N}$. Choose and then fix an element $\tau \in \text{Ext}^1(f^*\Omega_X/R, \mathcal{O}_Y)$. The aim of this section is to define the twisted pullback along $\tau$ as a functor

$$\text{TP}_\tau : \text{HIG}_{\leq r}(X/R) \to \text{HIG}_{\leq r}(Y/R),$$

under the following assumption on $r$

**Assumption 2.1.** $r!$ is invertible in $R$. 

Let \( \Omega_{X/R} \) be the sheaf of relative logarithmic Kähler differentials and \( T_{X/R} \) be its \( \mathcal{O}_X \)-dual. They are locally free of rank \( \dim X - \dim R \) by log smoothness. The symmetric algebra \( \text{Sym}^* T_{X/R} = \bigoplus_{k \geq 0} \text{Sym}^k T_{X/R} \) on \( T_{X/R} \) is \( \mathcal{O}_X \)-algebra, and one has the following morphisms of \( \mathcal{O}_X \)-algebras whose composite is the identity:

\[
\mathcal{O}_X \to \text{Sym}^* T_{X/R} \to \mathcal{O}_X.
\]

It defines the zero section of the natural projection \( \Omega_{X/R} \to X \), where we view \( \Omega_{X/R} \) as a vector bundle over \( X \) (see Ex 5.18, Ch. II [Ha]). Set

\[
\mathcal{A}_r := \text{Sym}^* (T_{X/R}) / \text{Sym}^{\geq r+1} (T_{X/R}),
\]

which is nothing but the structure sheaf of the closed subscheme \((r+1)X \) of \( \Omega_{X/R} \) supported along the zero section. In below, we shall use the notations \( \mathcal{A}_r \) and \( \mathcal{O}_{(r+1)X} \) interchangeably. Note as \( \mathcal{O}_X \)-module, \( \mathcal{A}_r = \mathcal{O}_X \oplus T_{X/R} \oplus \cdots \oplus \text{Sym}^r T_{X/R} \). The following lemma is well-known.

**Lemma 2.2.** The category of nilpotent (quasi-)coherent Higgs modules over \( X/R \) of exponent \( \leq r \) is equivalent to the category of (quasi-)coherent \( \mathcal{O}_{(r+1)X} \)-modules.

**Proof.** The natural inclusion \( i : X \to \Omega_{X/R} \) of zero section induces an equivalence of categories between the category of sheaves of abelian groups over \( X \) and the category of sheaves of abelian groups over \( \Omega_{X/R} \) whose support is contained in the zero section. Let \( E \) be a sheaf of abelian groups over \( X \). It has a Higgs module structure if it has

1. a ring homomorphism \( \theta^0 : \mathcal{O}_X \to \text{End}(E) \);
2. an \( \mathcal{O}_X \)-linear homomorphism \( \theta^1 : T_{X/R} \to \text{End}_{\mathcal{O}_X}(E) \).

Since \( \text{Sym}^* T_{X/R} \) is generated by \( T_{X/R} \) as \( \mathcal{O}_X \)-algebra, \( \theta^0 \) and \( \theta^1 \) together extend to a ring homomorphism

\[
\theta^* : \text{Sym}^* T_{X/R} \to \text{End}_{\mathcal{O}_X}(E) \subset \text{End}(E).
\]

If \( \theta^1 \) is nilpotent of exponent \( \leq r \), then \( \text{Sym}^{\geq r+1} (T_{X/R}) \subset \text{Ann}(E) \). Therefore, we obtain an \( \mathcal{A}_r \)-module structure on \( E \). So we obtain a sheaf of \( \mathcal{O}_{(r+1)X} \)-module. As \( E \) is (quasi-)coherent as \( \mathcal{O}_X \)-module, it is (quasi-)coherent as \( \mathcal{O}_{(r+1)X} \)-module. Conversely, for a quasi-coherent \( \mathcal{O}_{\mathcal{A}_r} \)-module \( E \), one obtains a ring homomorphism

\[
\mathcal{A}_r \to \text{End}(E).
\]

Restricting it to the degree zero part, one obtains the \( \mathcal{O}_X \)-module structure on \( E \). While restricting to the degree one component, one obtains a morphism of sheaf of abelian groups

\[
\theta : T_{X/R} \to \text{End}(E), v \mapsto \theta_v := \text{the multiplication by } v.
\]

Since for any \( v \in T_{X/R} \), \( v^{r+1} = 0 \) in \( \mathcal{A}_r \), it follows \( \theta^{r+1} = 0 \), that is the exponent of \( \theta \leq r \). For any \( f \in \mathcal{O}_X, v \in T_{X/R} \) and any \( e \in E \), one verifies that

\[
\theta_v (fe) = \theta_{fv} (e) = f \theta_v (e),
\]

which means that the image of \( \theta \) is contained in \( \text{End}_{\mathcal{O}_X}(E) \). The obtained \( \mathcal{O}_X \)-module is nothing but the pushforward of \( E \) along the composite \((r+1)X \to \Omega_{X/R} \to X \) which is finite. Therefore, \( E \) is (quasi-)coherent as \( \mathcal{O}_X \)-module if it is (quasi-)coherent as \( \mathcal{O}_{(r+1)X} \)-module. \( \square \)
Remark 2.3. An $f^*\Omega_{X/R}$-Higgs module is a pair $(E, \theta)$ where $E$ is an $\mathcal{O}_Y$-module and $\theta : E \to E \otimes f^*\Omega_{X/R}$ is an $\mathcal{O}_Y$-linear morphism satisfying $\theta \wedge \theta = 0$. A modification of the above argument shows that the category of nilpotent (quasi-)coherent $f^*\Omega_{X/R}$-Higgs modules is equivalent to the category of (quasi-)coherent $f^*\mathcal{A}_r$-modules.

1st construction: For an $r$ satisfying Assumption 2.1, we have a natural morphism:

$$\exp : H^1(Y, f^*T_{X/R}) \to H^1(Y, (f^*\mathcal{A}_r)^*), \tau \mapsto \exp(\tau) = 1 + \tau + \cdots + \frac{\tau^r}{r!},$$

where $(f^*\mathcal{A}_r)^*$ is the unit group of $f^*\mathcal{A}_r$. An element of $f^*\mathcal{A}_r$ is invertible if and only if its image under $f^*\mathcal{A}_r \to \mathcal{O}_Y$ is invertible. So we obtain an $f^*\mathcal{A}_r$-module $\mathcal{F}_r$ of rank one. We introduce an intermediate category $\text{HIG}_{\leq r}(f^*\Omega_{X/R})$, which is the category of nilpotent quasi-coherent $f^*\Omega_{X/R}$-Higgs modules of exponent $\leq r$. We define the functor

$$\text{TP}_r^\mathcal{F} : \text{HIG}_{\leq r}(X/R) \to \text{HIG}_{\leq r}(f^*\Omega_{X/R})$$

as follows: For an $E \in \text{HIG}_{\leq r}(X/R)$, define

$$\text{TP}_r^\mathcal{F}(E) := \mathcal{F}_r \otimes_{f^*\mathcal{A}_r} f^*E$$

as $f^*\mathcal{A}_r$-module. Next, for a morphism $\phi : E_1 \to E_2$ in $\text{HIG}_{\leq r}(X/R)$,

$$\text{TP}_r^\mathcal{F}(\phi) := \text{id} \otimes f^*\phi : \text{TP}_r^\mathcal{F}(E_1) \to \text{TP}_r^\mathcal{F}(E_2)$$

is a morphism of $f^*\mathcal{A}_r$-modules. One has the natural functor from $\text{HIG}_{\leq r}(f^*\Omega_{X/R})$ to $\text{HIG}_{\leq r}(\Omega_{Y/R})$ induced by the differential morphism $f^*\Omega_{X/R} \to \Omega_{Y/R}$. We define the functor $\text{TP}_r^\mathcal{F}$ to be composite of functors

$$\text{HIG}_{\leq r}(X/R) \xrightarrow{\text{TP}_r^\mathcal{F}} \text{HIG}_{\leq r}(f^*\Omega_{X/R}) \to \text{HIG}_{\leq r}(Y/R).$$

This is how Faltings \cite{Fa} defines twisted pullback in the $p$-adic setting, at least for those small $\tau$s.

2nd construction: This is based on the method of exponential twisting \cite{LSZ}, whose basic construction is given as follows:

Step 0: Take an open affine covering $\{U_\alpha\}_{\alpha \in \Lambda}$ of $X$ as well as an open affine covering $\{V_\alpha\}_{\alpha \in \Lambda}$ of $Y$ such that $f : V_\alpha \to U_\alpha$. Let $\{\tau_{\alpha\beta}\}$ be a Cech representative of $\tau$. That is, $\tau_{\alpha\beta} \in \Gamma(V_{\alpha\beta}, f^*T_{X/R})$ satisfying the cocycle relation

$$\tau_{\alpha\beta} = \tau_{\alpha\beta} + \tau_{\beta\gamma}.$$

Step 1: Let $(E, \theta)$ be a nilpotent Higgs module over $X$, whose exponent of nilpotency satisfies Assumption 2.1. For any $\alpha$, set $(E_\alpha, \theta_\alpha) = (E, \theta)|_{U_\alpha}$. Then one forms the various local Higgs modules $\{(f^*E_\alpha, f^*\theta_\alpha)\}$ via the usual pullback.

Step 2: Define

$$G_{\alpha\beta} = \exp(\tau_{\alpha\beta} \cdot f^*\theta) = \sum_{i \geq 0} \frac{(\tau_{\alpha\beta} \cdot f^*\theta)^n}{n!}.$$
The expression makes sense since each term \( \frac{(\tau_{\alpha\beta} f^* \theta)^n}{n!} \) is well defined by assumption. Obviously, \( G_{\alpha\beta} \in \text{Aut}_{\mathcal{O}_Y}(f^*E|_{V_{\alpha\beta}}) \). Because of the cocycle relation, \( \{G_{\alpha\beta}\} \) satisfies the cocycle relation

\[
G_{\alpha\gamma} = G_{\beta\gamma} G_{\alpha\beta}.
\]

Then we use the set of local isomorphism \( \{G_{\alpha\beta}\} \) to glue the local \( \Omega_{Y/R}\)-Higgs modules \( \{(f^*E_\alpha, f^*\theta_\alpha)\} \), to obtain a new Higgs module over \( Y \). The verification details are analogous to §2.2 [LSZ]. It is tedious and routine to verify the glued Higgs module, up to natural isomorphism, is independent of the choice of affine coverings and Cech representatives of \( \tau \). We denote it by \( TP^2_r(E) \). For a morphism \( \phi : E_1 \to E_2 \) of Higgs modules, it is not difficult to see that \( f^*\phi \) induces a morphism \( TP^2_r(\phi) : TP^2_r(E_1) \to TP^2_r(E_2) \).

**Proposition 2.4.** The two functors \( TP^1_r \) and \( TP^2_r \) are naturally isomorphic.

**Proof.** One uses the equivalence in Remark 2.3. It suffices to notice that the element \( \exp(\tau_{\alpha\beta}) \in f^*\mathcal{A}_\tau \) has its image \( G_{\alpha\beta} \) in \( \text{Aut}_{\mathcal{O}_Y}(f^*E|_{V_{\alpha\beta}}) \).

By the above proposition, we set \( TP_r \) to be either of \( TP^i_r \), \( i = 1, 2 \).

**Proposition 2.5.** The functor \( TP_r \) has the following properties:

(i) it preserves rank;

(ii) it preserves direct sum;

(iii) Let \( E_i, i = 1, 2 \) be two nilpotent Higgs modules over \( X/R \) whose exponents of nilpotency satisfies \( (r_1 + r_2)! \) being invertible in \( \mathcal{R} \). Then there is a canonical isomorphism of Higgs modules over \( Y \):

\[
TP_r(E_1 \otimes E_2) \cong TP_r(E_1) \otimes TP_r(E_2).
\]

**Proof.** The first two properties are obvious. To approach (iii), one uses the second construction. Note that when \( r = 0 \), it is nothing but the fact \( f^*(E_1 \otimes E_2) = f^*E_1 \otimes f^*E_2 \). When the exponents \( r_i, i = 1, 2 \) satisfies the condition, one computes that

\[
\exp(\tau \cdot (f^*\theta_1 \otimes id + id \otimes f^*\theta_2)) = \exp(\tau \cdot f^*\theta_1 \otimes id) \exp(id \otimes \tau \cdot f^*\theta_2),
\]

using the equality

\[
(f^*\theta_1 \otimes id)(id \otimes f^*\theta_2) = (id \otimes f^*\theta_2)(f^*\theta_1 \otimes id) = f^*\theta_1 \otimes f^*\theta_2.
\]

To conclude this section, we shall point out that there is one closely related construction that works for all \( r \in \mathbb{N} \).

**3rd construction:** Note that the element \( \tau \in \text{Ext}^1(f^*\Omega_X, \Omega_Y) \cong \text{Ext}^1(\mathcal{O}_Y, f^*T_{X/R}) \) corresponds to an extension of \( \mathcal{O}_Y \)-modules

\[
0 \to f^*T_{X/R} \to \mathcal{E}_r \xrightarrow{pr} \mathcal{O}_Y \to 0.
\]

Notice that \( \mathcal{E}_r \) admits a natural \( f^*\text{Sym}^*T_{X/R} \)-module structure: In degree zero, this is \( \mathcal{O}_Y \)-structure; in degree one,

\[
f^*T_{X/R} \otimes_{\mathcal{O}_Y} \mathcal{E}_r \xrightarrow{id \otimes pr} f^*T_{X/R} \otimes_{\mathcal{O}_Y} \mathcal{O}_Y = f^*T_{X/R} \subset \mathcal{E}_r,
\]
and therefore $\theta : f^*T_{X/R} \rightarrow \text{End}_{\mathcal{O}_Y}(\mathcal{E}_r)$. By construction, $\theta \neq 0$ but $\theta^2 = 0$. For any $r \in \mathbb{N}$, set

$$\mathcal{E}^r := \text{Sym}^r \mathcal{E}_r.$$  

The proof of the next lemma is straightforward.

**Lemma 2.6.** For any $r \in \mathbb{N}$, $\mathcal{E}^r$ is a nilpotent $f^*\Omega_{X/R}$-Higgs bundle of exponent $r$. It admits a filtration $F^*$ of $f^*\Omega_{X/R}$-Higgs subbundles:

$$\mathcal{E}^r = F^0 \supset F^1 \supset \cdots \supset F^r \supset 0,$$

whose associated graded $\text{Gr}_{F^*} \mathcal{E}^r$ is naturally isomorphic to $f^*\mathcal{A}_r$. When $\tau = 0$, $\mathcal{E}^r = f^*\mathcal{A}_r$ as $f^*\Omega_{X/R}$-Higgs bundle.

By the lemma, $\mathcal{E}^r$ is an $f^*\mathcal{A}_r$-module of rank one. Therefore, one may replace the tensor module in the definition of $\text{TP}^\mathcal{E}^r_\tau$ with $\mathcal{E}^r$. This defines a new functor $\text{TP}^\mathcal{E}^r_\tau$ and hence the third twisted pullback functor $\text{TP}^3_\tau$.

**Remark 2.7.** When one is interested only in coherent objects, one may drop the nilpotent condition in the construction. This is because by Cayley-Hamilton, there is an element of form $v^r - a_1v^{r-1} + \cdots + (-1)^r a_r \in \text{Sym}^r T_{X/R}$ annihilating $E$, so that $\text{Sym}^r T_{X/R}$-module structure on $E$ factors though $\text{Sym}^r T_{X/R} \rightarrow \mathcal{A}_r$.

We record the following statement for further study.

**Proposition 2.8.** Assume $r \in \mathbb{N}$ satisfy Assumption 2.1. Then as $f^*\mathcal{A}_r$-modules,

(i) $F^r_\tau \cong \mathcal{E}^r_\tau$ for $r \leq 1$;

(ii) $F^r_\tau \not\cong \mathcal{E}^r_\tau$ for $r > 1$.

**Proof.** Obviously, $F^0_\tau \cong \mathcal{E}^0_\tau \cong \mathcal{O}_Y$. Assume $r \geq 1$. We illustrate our proof by looking at the case of $X/R$ being a relative curve. We describe $\mathcal{E}^r$ in terms of local data: Take $r = 1$ first. Let $U_\alpha$ be an open subset of $X$ with $\partial_\alpha$ a local basis of $\Gamma(U_\alpha, T_{X/R})$. Assume that $V_\alpha$ to be an open subset of $Y$ such that $f : V_\alpha \rightarrow U_\alpha$. We may assume the gluing functions between two different local basis are identity. Let $\{\tau_{\alpha \beta}\}$ be a Cech representative of $\tau$. Write $\tau_{\alpha \beta} = a_{\alpha \beta} f^*\partial_\beta$. Then $\mathcal{E}_r$ is the $\mathcal{O}_Y$-module obtained by gluing $\{\mathcal{O}_{V_\alpha} \oplus f^* T_{U_\alpha/R}\}$ via the following gluing matrix:

$$\begin{pmatrix}
1 \\
\frac{1}{f^*\partial_\alpha}
\end{pmatrix} = \begin{pmatrix}
1 & a_{\alpha \beta} \\
0 & 1
\end{pmatrix} \cdot \begin{pmatrix}
1 \\
\frac{1}{f^*\partial_\beta}
\end{pmatrix}.$$

Under the assumption for $r$, $\mathcal{E}^r$ is obtained by gluing $\{f^* \mathcal{A}_r|_{U_\alpha} = \mathcal{O}_{V_\alpha} \oplus f^* T_{U_\alpha/R} \oplus \cdots \oplus f^* T_{U_\alpha/R}^{\otimes r}\}$ via the gluing matrix:

$$\begin{pmatrix}
\frac{1}{f^*\partial_\alpha} \\
\frac{1}{f^*\partial_\alpha} \\
\vdots \\
0 \\
0 \\
0
\end{pmatrix} = \begin{pmatrix}
1 & a_{\alpha \beta} & a_{\alpha \beta}^2 & \cdots & a_{\alpha \beta}^r \\
0 & 1 & a_{\alpha \beta} & \cdots & a_{\alpha \beta}^{r-1} \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & 1 & a_{\alpha \beta} \\
0 & 0 & \cdots & 1 & 1
\end{pmatrix} \cdot \begin{pmatrix}
\frac{1}{f^*\partial_\beta} \\
\frac{1}{f^*\partial_\beta} \\
\vdots \\
\frac{1}{f^*\partial_\beta} \\
\frac{1}{f^*\partial_\beta}
\end{pmatrix}.$$
As comparison, $\mathcal{F}_r^\tau$ is obtained by gluing $\{f^*\mathcal{A}_r|_{U_\alpha}\}$ via the following transition functions

$$
\begin{pmatrix}
1 \\
\frac{1}{f^*\partial_\alpha} \\
\vdots \\
\frac{1}{f^*\partial_\alpha^{r-1}} \\
\frac{1}{f^*\partial_\alpha^r}
\end{pmatrix}
= \begin{pmatrix}
1 & a_{\alpha\beta} & \frac{a_{\alpha\beta}^2}{2!} & \cdots & \frac{a_{\alpha\beta}^r}{r!} \\
0 & 1 & a_{\alpha\beta} & \cdots & \frac{a_{\alpha\beta}^{r-1}}{(r-1)!} \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & 1 & a_{\alpha\beta} \\
0 & 0 & \cdots & 0 & 1
\end{pmatrix}
\begin{pmatrix}
1 \\
\frac{1}{f^*\partial_\beta} \\
\vdots \\
\frac{1}{f^*\partial_\beta^{r-1}} \\
\frac{1}{f^*\partial_\beta^r}
\end{pmatrix}.
$$

Therefore, $\mathcal{F}_r^\tau$ and $\mathcal{E}_r^\tau$ are isomorphic as $\mathcal{O}_Y$-modules. However, when $r \geq 2$, the Higgs structures of these two bundles differ: For $\mathcal{F}_r^\tau$, the Higgs field along $\partial_\alpha$ is given by

$$
\theta_{\partial_\alpha} \begin{pmatrix}
1 \\
\frac{1}{\partial_\alpha} \\
\vdots \\
\frac{1}{\partial_\alpha^{r-1}} \\
\frac{1}{\partial_\alpha^r}
\end{pmatrix}
= \begin{pmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & 1 & 0 \\
0 & 0 & \cdots & 0 & 0
\end{pmatrix}
\begin{pmatrix}
1 \\
\frac{1}{\partial_\alpha} \\
\vdots \\
\frac{1}{\partial_\alpha^{r-1}} \\
\frac{1}{\partial_\alpha^r}
\end{pmatrix},
$$

while for Higgs field action for $\mathcal{E}_r^\tau$ is given by

$$
\theta_{\partial_\alpha} \begin{pmatrix}
1 \\
\frac{1}{f^*\partial_\alpha} \\
\vdots \\
\frac{1}{f^*\partial_\alpha^{r-1}} \\
\frac{1}{f^*\partial_\alpha^r}
\end{pmatrix}
= \begin{pmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & 1 & 0 \\
0 & 0 & \cdots & 0 & 0
\end{pmatrix}
\begin{pmatrix}
1 \\
\frac{1}{f^*\partial_\alpha} \\
\vdots \\
\frac{1}{f^*\partial_\alpha^{r-1}} \\
\frac{1}{f^*\partial_\alpha^r}
\end{pmatrix}.
$$

\[\square\]

3. Twisted functoriality

Now we come back to the setting in §1. First we make the following

**Definition 3.1.** Let $k$, $f : Y_{\log} \to X_{\log}$ and $\tilde{X}_{\log}, \tilde{Y}_{\log}$ be as in §1. For a Higgs module $(E, \theta) \in \text{HIG}_{\leq p-1}(X_{\log}/k)$. Then the twisted pullback $f^<>(E, \theta)$ is defined to be $\text{TP}_{ob(f)}(E, \theta)$, where $ob(f)$ is the obstruction class of lifting $f$ to a morphism $\tilde{Y}_{\log} \to \tilde{X}_{\log}$ over $W_2(k)$.

Assume that $f$ admits a $W_2(k)$-lifting $\tilde{f}$. In Langer’s proof of functoriality Theorem 5.3 [?], the existence of local logarithmic Frobenius liftings $F_{\tilde{X}_{\log}}$ and $F_{\tilde{Y}_{\log}}$ such that $F_{\tilde{X}_{\log}} \circ \tilde{f} = \tilde{f} \circ F_{\tilde{Y}_{\log}}$ is crucial-this is where the condition of $\tilde{f}$ being good enters. However, one notices that any local logarithmic Frobenius liftings on $\tilde{X}$ and $\tilde{Y}$ commute with $\tilde{f}$ up to homotopy. A heuristic reasoning shows that this homotopy should be intertwined with the homotopies caused by local logarithmic Frobenius liftings of both $\tilde{X}$ and $\tilde{Y}$, as well as the one caused by local liftings of the morphism (no $W_2$-lifting on $f$ is assumed any more). Turning this soft homotopy argument into exact differential calculus in positive characteristic yields the proof for the claimed twisted functoriality.
To start with proof of Theorem \[\text{[1.1]}\] we take an étale covering \[\mathcal{X} = \coprod_i X_i \to X\] with \(X_i\) affine and the pullback of \(D\) along each \(X_i \to X\) simple normal crossing. Then we take an étale covering \(\pi : \mathcal{Y} = \coprod_i Y_i \to Y\) with similar properties and \(f\) restricts to a local morphism \(f_i : Y_{i,\log} \to X_{i,\log}\) for each \(i\). For each \(i\), we choose logarithmic Frobenius lifting over \(W_2(k)\)

\[
F_{X_{i,\log}} : \bar{X}_{i,\log} \to \bar{X}_{i,\log}, \quad F_{Y_{i,\log}} : \bar{Y}_{i,\log} \to \bar{Y}_{i,\log},
\]

and also a \(W_2(k)\)-lift \(\tilde{f}_i : \bar{Y}_{i,\log} \to \bar{X}_{i,\log}\). Such local lifts exist. Set

\[
(V_1, \nabla_1) = C^{-1}_{\log}(\varphi \circ (E, \theta), \quad (V_2, \nabla_2) = f^* C^{-1}_{\log}(E, \theta).
\]

In below, we exhibit an isomorphism between \((V_1, \nabla_1), \ i = 1, 2\) after pulling back to the étale covering \(\mathcal{Y}\) which satisfies the descent condition. The whole proof is therefore divided into two steps.

**Step 1: Isomorphism over \(\mathcal{Y}\)**

As \(\mathcal{Y}\) is a disjoint union of open affine log schemes \(\{Y_{i,\log}\}\)s, it suffices to construct an isomorphism for each open affine. In the foregoing argument, we drop out the subscript \(i\) everywhere. Notice first that the two morphisms \(\tilde{f}^* \circ F_{\bar{X}_{\log}}^*\) and \(F_{\bar{Y}_{\log}}^* \circ \tilde{f}^*\) coincide after reduction modulo \(p\). Thus, it defines an element

\[
\nu_f \in \text{Hom}_{\mathcal{X}}(\Omega_{X_{\log}/k}, \mathcal{O}_Y)
\]

such that

\[
\nu_f \circ d = \frac{1}{p}(F_{\bar{Y}_{\log}}^* \circ \tilde{f}^* - \tilde{f}^* \circ F_{\bar{X}_{\log}}^*).
\]

So we get \(\nu_f \cdot \theta \in \Gamma(Y, \text{End}_{\mathcal{O}_Y}(g^*E))\), where \(g = F_X \circ f = f \circ F_Y\).

**Lemma 3.2.** \(\exp(\nu_f \cdot \theta)\) defines an isomorphism \((V_1, \nabla_1) \to (V_2, \nabla_2)\). That is, there is a commutative diagram:

\[
\begin{array}{ccc}
V_1 & \xrightarrow{\exp(\nu_f \cdot \theta)} & V_2 \\
\nabla_1 & \downarrow & \nabla_2 \\
V_1 \otimes \Omega_{Y_{\log}/k} & \xrightarrow{\exp(\nu_f \cdot \theta) \otimes \text{id}} & V_2 \otimes \Omega_{Y_{\log}/k}.
\end{array}
\]

**Proof.** Recall that over \(Y\), \(V_1 = V_2 = g^*E\). So \(\exp(\nu_f \cdot E)\) defines an isomorphism from \(V_1\) to \(V_2\). Moreover, the connections are given by

\[
\nabla_1 = \nabla_{\text{can}} + (id \otimes \frac{dF_{\bar{Y}_{\log}}}{p})(F_{\bar{Y}_{\log}}^*f^*\theta),
\]

and respectively by

\[
\nabla_2 = f^*(\nabla_{\text{can}} + (id \otimes \frac{dF_{\bar{X}_{\log}}}{p})(F_{\bar{X}_{\log}}^*\theta)).
\]

Now we are going to check the commutativity of the above diagram. Take a local section \(e \in E\). Then

\[
\exp(\nu_f \cdot \theta) \otimes \text{id} \circ \nabla_1(g^*e) = \exp(\nu_f \cdot \theta)(id \otimes \frac{dF_{\bar{Y}_{\log}}}{p})(F_{\bar{Y}_{\log}}^*f^*(\theta(e))).
\]
On the other hand, $\nabla_2 \circ \exp(\nu_f \cdot \theta)(e)$ equals

$$\exp(\nu_f \cdot \theta) d(\nu_f \cdot \theta)(g^* e) + \exp(\nu_f \cdot \theta)(f^* (\text{id} \otimes \frac{dF_{\tilde{Y}_{\log}}}{p})(F_{\tilde{X}}^*(\theta)(e)))$$

We take a system of local coordinates $\{x_i\}$ for $\tilde{X}$ and use the same notion for its reduction modulo $p$. Write $\theta = \sum_i \theta_i d x_i$, and $\nu_f = \sum_i u_i \partial_{x_i}$ with $u_i \in \mathcal{O}_Y$. Thus

$$d(\nu_f \cdot \theta) = d\left(\sum_i g^* \theta_i \cdot u_i\right).$$

As $d$ is $\mathcal{O}_X$-linear, it equals

$$\sum_i g^* \theta_i \cdot d u_i = \sum_i g^* \theta_i \cdot d\left(\frac{(F_{\tilde{Y}_{\log}}^* \circ \tilde{f}^* - \tilde{f}^* \circ F_{\tilde{X}_{\log}}^*)(x_i)}{p}\right).$$

So $d(\nu_f \cdot \theta)(g^* e) = \sum_i g^* \theta_i(e) \cdot \frac{(F_{\tilde{Y}_{\log}}^* \circ \tilde{f}^* - \tilde{f}^* \circ F_{\tilde{X}_{\log}}^*)(x_i)}{p}$. On the other hand,

$$(\text{id} \otimes \frac{dF_{\tilde{Y}_{\log}}}{p})(F_{\tilde{Y}}^* f^* \theta(e)) = \sum_i g^* \theta_i(e) \cdot \frac{i d \otimes dF_{\tilde{Y}_{\log}}}{p}(F_{\tilde{Y}}^* f^* (dx_i))$$

$$= \sum_i g^* \theta_i(e) \cdot \frac{d(F_{\tilde{Y}_{\log}}^* \tilde{f}^*(x_i))}{p},$$

and similarly,

$$f^* (\text{id} \otimes \frac{dF_{\tilde{X}_{\log}}}{p})(F_{\tilde{X}}^* \theta(e)) = \sum_i g^* \theta_i(e) \cdot \frac{\tilde{f}^* d(F_{\tilde{X}_{\log}}^*(x_i))}{p}$$

$$= \sum_i g^* \theta_i(e) \cdot \frac{d(\tilde{f}^* F_{\tilde{X}_{\log}}^*(x_i))}{p}. $$

This completes the proof.

\[ \square \]

**Step 2: Descent condition**

In Step 1, we have constructed an isomorphism $\exp(\nu_f \cdot \theta) : \pi^*(V_1, \nabla_1) \to \pi^*(V_2, \nabla_2)$ whose restriction to $Y_{\text{log}}$ is given by $\exp(\nu_f \cdot \theta)$. Let $p_i : Y \times_Y Y \to Y, i = 1, 2$ be two projections. In below, we show that

$$p_1^*(\exp(\nu_f \cdot \theta)) = p_2^*(\exp(\nu_f \cdot \theta)).$$

The obstruction class $ob(F_X)$ (resp. $ob(F_Y)$ and $ob(f)$) of lifting $F_X$ (resp. $F_Y$ and $f$) over $W_2$ has its Cech representative landing in $\Gamma(X_{ij}, F_X^* T_{X_{\text{log}}/k})$ (resp. $\Gamma(Y_{ij}, F_Y^* T_{Y_{\text{log}}/k})$ and $\Gamma(Y_{ij}, f^* T_{X_{\text{log}}/k})$). We have the following natural maps:

$$f^* : H^1(X, F_X^* T_{X_{\text{log}}/k}) \to H^1(Y, f^* F_X^* T_{X_{\text{log}}/k} = H^1(Y, g^* T_{X_{\text{log}}/k}),$$

$$F_Y^* : H^1(Y, f^* T_{X_{\text{log}}/k}) \to H^1(Y, g^* T_{X_{\text{log}}/k}),$$

and

$$f_* : H^1(Y, F_Y^* T_{Y_{\text{log}}/k}) \to H^1(Y, F_Y^* f^* T_{X_{\text{log}}/k}) = H^1(Y, g^* T_{X_{\text{log}}/k}),$$

$$f_* : H^1(Y, F_Y^* T_{Y_{\text{log}}/k}) \to H^1(Y, F_Y^* f^* T_{X_{\text{log}}/k}) = H^1(Y, g^* T_{X_{\text{log}}/k},$$

$$f_* : H^1(Y, F_Y^* T_{Y_{\text{log}}/k}) \to H^1(Y, F_Y^* f^* T_{X_{\text{log}}/k}) = H^1(Y, g^* T_{X_{\text{log}}/k}).$$
which is induced by $f_* : T_{Y_{ij}/k} \to f^*T_{X_{ij}/k}$.

**Lemma 3.3.** One has an equality in $\Gamma(Y_{ij}, g^*T_{X_{ij}/k})$, where $Y_{ij} = Y_i \times_Y Y_j$:

$$\nu_{f_i} - \nu_{f_j} = ob(F_Y)_{ij} + ob(f)_{ij} - ob(F_X)_{ij}$$

where we understand the obstruction classes as their images via the natural morphisms. Consequently, there is an equality in $H^1(Y, g^*T_{X_{ij}/k})$:

$$[\nu_{f_i} - \nu_{f_j}] = ob(F_Y) + ob(f) - ob(F_X).$$

**Proof.** First, we observe the following identity

$$\frac{1}{p} (F_{Y_{ij, log}}^* \circ \tilde{f}_i^* - F_{Y_{ij, log}}^* \circ \tilde{f}_j^*) = \frac{1}{p} [(F_{Y_{ij, log}}^* - F_{Y_{ij, log}}^* \circ \tilde{f}_i^*) \circ \tilde{f}_j^*] + \frac{1}{p} [F_{Y_{ij, log}}^* \circ (\tilde{f}_i^* - \tilde{f}_j^*)]$$

$$= \frac{F_{Y_{ij, log}}^* - F_{Y_{ij, log}}^* \circ \tilde{f}_i^*}{p} \circ \tilde{f}_j^* + F_{Y_j}^* \circ \frac{\tilde{f}_i^* - \tilde{f}_j^*}{p}$$

It follows that

$$(\nu_{f_i} - \nu_{f_j}) \circ d = \frac{1}{p} (F_{Y_{ij, log}}^* \circ \tilde{f}_i^* - F_{Y_{ij, log}}^* \circ \tilde{f}_j^*) - \frac{1}{p} (\tilde{f}_i^* \circ F_{X_{ij, log}} - \tilde{f}_j^* \circ F_{X_{ij, log}})$$

$$= \frac{F_{Y_{ij, log}}^* - F_{Y_{ij, log}}^* \circ \tilde{f}_i^*}{p} \circ \tilde{f}_j^* + F_{Y_j}^* \circ \frac{\tilde{f}_i^* - \tilde{f}_j^*}{p} - \frac{F_{X_{ij, log}}^* - F_{X_{ij, log}}^* \circ \tilde{f}_i^*}{p} \circ \tilde{f}_j^*$$

$$= ob(F_Y)_{ij} \circ f_i^* \circ d + F_{Y_j}^* \circ ob(f)_{ij} \circ d - f_j^* \circ ob(F_X)_{ij} \circ d$$

In the second equality, the last term vanishes because

$$\frac{\tilde{f}_i^* - \tilde{f}_j^*}{p} \circ F_{X_i}^* = ob(f) \circ (dF_{X_i}^*) = 0.$$ 

□

Now we turn the above equality into an equality required in the descent condition. The transition function of $V_1$ is given by

$$a_{ij} := \exp(ob(F_Y)_{ij} \cdot f_i^* \theta) \cdot \exp(F_{Y_j}^*(ob(f)_{ij} \cdot \theta)),$$

while the transition function for $V_2$ is given by

$$b_{ij} := f_j^* \exp((ob(F_X)_{ij} \cdot \theta)).$$

Then Lemma 3.3 implies the commutativity of the following diagram over $Y_{ij}$:

$$\begin{align*}
V_1|_{Y_i} \xrightarrow{\exp (\nu_{f_i} \theta)} & \quad V_2|_{Y_i} \\
\downarrow a_{ij} & \quad \downarrow b_{ij} \\
V_1|_{Y_j} \xrightarrow{\exp (\nu_{f_j} \theta)} & \quad V_2|_{Y_j}.
\end{align*}$$

The commutativity is nothing but the descent condition for the isomorphism $\exp(\nu_f \cdot \theta)$. So we are done.
4. Semistability under pullback

Semistability is not always preserved under pullback. After all, semistability refers to some given ample line bundle and an ample line bundle does not necessarily pull back to an ample line bundle. Even worse, in the positive characteristic case, there are well-known examples of semistable vector bundles over curves which pull back to unstable bundles under Frobenius morphism.

For a polystable Higgs bundle with vanishing Chern classes in characteristic zero, this is handled by the existence of Higgs-Yang-Mills metric—it is a harmonic bundle by this case and harmonic bundles pulls back to harmonic bundles. Consequently, the pullback of a polystable Higgs bundle with vanishing Chern classes is again polystable with vanishing Chern classes. For a semistable Higgs bundle with vanishing Chern classes, one takes a Jordan-Hölder filtration of the Higgs bundle and the semistability of the pullback follows from that of the polystable case. In the following, we provide a purely algebraic approach to the semistable case. We proceed to the proof of Theorem 1.3.

**Proof.** We resume the notations of Theorem 1.3. Since taking Chern class commutes with pullback, the statement about vanishing Chern classes of the pullback is trivial. We focus on the semistability below. In the following discussion, we choose and then fix an arbitrary ample line bundle \( L \) (resp. \( M \)) over \( X \) (resp. \( Y \)). We consider first the characteristic zero setting. Fix a \( W_2(k) \)-lifting \( \tilde{f} : \tilde{Y}_{\log} = (\tilde{Y}, \tilde{B}) \to X_{\log} = (X, D) \). First, we observe that the proof of Theorem A.4 [LSZ] works verbatim for a semistable logarithmic Higgs bundle, so that there exists a filtration \( Fil_{-1} \) on \( E \) such that \( Gr_{Fil_{-1}}(E, \theta) \) is semistable. Now applying [LSZ Theorem A.1], [La14] Theorem 5.12 to the nilpotent semistable Higgs bundle \( Gr_{Fil_{-1}}(E, \theta) \), we obtain a flow of the following form:

\[
\begin{align*}
(E, \theta) \quad & \xrightarrow{Gr_{Fil_{-1}}} \quad (H_0, \nabla_0) \\
& \quad \xrightarrow{Gr_{Fil_0}} (H_1, \nabla_1) \\
& \quad \cdots
\end{align*}
\]

in which each Higgs term in bottom is semistable. Next, because of Corollary 1.2, we may obtain the pullback flow as follows:

\[
\begin{align*}
f^*(E, \theta) \quad & \xrightarrow{Gr_{Fil_{-1}}} \quad f^*(H_0, \nabla_0) \\
f^*(E_0, \theta_0) \quad & \xrightarrow{Gr_{Fil_0}} f^*(H_1, \nabla_1) \\
f^*(E_1, \theta_1) \quad & \cdots
\end{align*}
\]

Now as the Higgs terms \((E_i, \theta_i)\)'s in the first flow are semistable of the same rank and of vanishing Chern classes, the set \( \{(E_i, \theta_i)\}_{i\geq0} \) form a bounded family. So the set \( \{f^*(E_i, \theta_i)\}_{i\geq0} \) also forms a bounded family. In particular, the degrees of subsheaves in \( \{f^*E_i\}_{i\geq0} \) have an upper bound \( N \). Suppose \( f^*(E, \theta) \) is unstable, that is, there exists a saturated Higgs subsheaf \((F, \eta)\) of positive degree \( d \) in \( f^*(E, \theta) \). Then \( Gr_{Fil_{-1}}(F, \eta) \subset f^*(E_0, \theta_0) \) is a Higgs subsheaf of degree...
d. It implies that \( \text{Gr}_{f^*F_{\text{dots}}} \circ C^{-1}_{Y_{\text{log}}} (F, \eta) \subset f^*(E_1, \theta_1) \) is of degree \( pd \). Iterating this process, one obtains a subsheaf in \( f^*(E_1, \theta_1) \) whose degree exceeds \( N \). Contradiction. Therefore, \( f^*(E, \theta) \) is semistable.

Now we turn to the char zero case. By the standard spread-out technique, there is a regular scheme \( S \) of finite type over \( \mathbb{Z} \), and an \( S \)-morphism \( f : (Y, B) \to (X, D) \) and an \( S \)-relative logarithmic Higgs bundle \( (E, \Theta) \) over \( (X, D) \), together with a \( k \)-rational point in \( S \) such that \{ \( f : (Y, B) \to (X, D), (E, \theta) \) \} pull back to \{ \( f : (Y, B) \to (X, D), (E, \theta) \) \}. In above, we may assume that \( X \) (resp. \( Y \)) is smooth projective over \( S \) and \( D \) (resp. \( B \)) is an \( S \)-relative normal crossing divisor in \( X \) (resp. \( Y \)). For a geometrically closed point \( s \in S \) and a \( W_2(k(s)) \)-lifting \( \tilde{s} \to S \), we obtain a family \( f_s : (Y, B)_s \to (X, D)_s \) over \( k(s) \) which is \( W_2 \)-liftable. Once taking an \( s \in S \) such that \( \text{char}(k(s)) \geq \text{rank}(E_s) = \text{rank}(E) \), we are in the previous char \( p \) setting. Hence it follows that \( f^*_s(E, \Theta)_s \) is semistable. From this, it follows immediately that \( f^*(E, \theta) \) is also semistable.

\[ \square \]

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