High Temperature Phase Transitions in Two-Scalar Theories with Large $N$ Techniques

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Abstract

We consider a theory of a scalar one-component field $\phi$ coupled to a scalar $N$-component field $\chi$. Using large $N$ techniques we calculate the effective potential in the leading order in $1/N$. We show that this is equivalent to a resummation of an infinite subclass of graphs in perturbation theory, which involve fluctuations of the $\chi$ field only. We study the temperature dependence of the expectation value of the $\phi$ field and the resulting first and second order phase transitions.
The original argument of Kirzhnits and Linde [1] for the restoration of the electroweak symmetry at sufficiently high temperatures led to the development of the formalism [2]-[4] for the quantitative study of the behaviour of field theories at finite temperature. An area of application of this formalism is the early universe, where the necessary temperatures for a variety of phase transitions were presumably realized. An example which is relevant for this paper is the transitions which can drive inflation [5]. The electroweak phase transition has attracted a great amount of interest recently [6], due to the possibility that it can create the necessary conditions for the generation of the baryon asymmetry of the universe [7]. The incompatibility of the resulting predictions for the mass of the Higgs boson [8] with the experimental bound has led to extensions of the standard model with additional scalar fields [9]. However, the perturbative approach to high temperature phase transitions, which was developed in refs. [2]-[4] and used in subsequent studies, has been shown to be insufficient for the reliable discussion of critical scalar fields [10]. For a reliable treatment of the phase transition for the $\phi^4$ scalar theory one has to resort either to the renormalization group [10] or other non-perturbative methods, such as large $N$ techniques [11]. Multi-scalar theories can also be problematic in the context of the perturbative approach, and the reliability of the predictions for the high temperature transitions may be questionable. This is the motivation for this work, in which we make use of the $1/N$ expansion in order to study the phase transitions in a model of a scalar one-component field $\phi$ coupled to a scalar $N$-component field $\chi$. We wish to study the effect of the $\chi$ fluctuations at non-zero temperature on the expectation value of $\phi$. We shall show that the leading result in the $1/N$ expansion is equivalent to the resummation of an infinite subclass of perturbative contributions which involve $\chi$ fluctuations. We shall also study the effective potential for $\phi$ at non-zero temperature and the possible phase transitions in dependence on the temperature.

We consider a theory of two real scalar fields: the one-component field $\phi$ and the $N$-component field $\chi$. The classical action is invariant under a $Z_2 \times O(N)$ symmetry and has the form

$$ S[\phi, \chi_a] = \int d^d x \left\{ \frac{1}{2} \partial_\mu \phi \partial^\mu \phi + \frac{1}{2} \partial_\mu \chi^a \partial^\mu \chi_a + \frac{1}{2} M_1^2 \phi^2 + \frac{1}{2} M_2^2 \chi^a \chi_a 
+ \frac{1}{8} \lambda_1 \phi^4 + \frac{1}{8} \lambda_2 (\chi^a \chi_a)^2 + \frac{1}{4} \bar{g} \phi^2 \chi^a \chi_a \right\}, \quad (1) $$

with $a = 0, \ldots, N - 1$. For the time being the dimensionality of the Euclidean space-time $d$ is kept arbitrary. We wish to study the theory in the limit of large $N$. In order to implement the large $N$ approximation we first introduce an auxiliary field $C(x)$ and make use of the identity

$$ \exp \left\{ -\int d^d x \frac{1}{8} \lambda_2 (\chi^a \chi_a)^2 \right\} \sim \int [Dc] \exp \left\{ \int d^d x \left( \frac{1}{2} Nc^2 - \frac{1}{2} \sqrt{N} \lambda_2 \chi^a \chi_a c \right) \right\}. \quad (2) $$

The partition function can now be written as

$$ Z(H, J_a) = \int [D\phi] [D\chi_a] [Dc] \exp \left\{-S'[\phi, \chi, c] + \int d^d x \left( H \phi + J_a \chi^a \right) \right\}, \quad (3) $$
where
\[
S'[\phi, \chi_a, c] = \int d^d x \left\{ \frac{1}{2} \partial_\mu \phi \partial^\mu \phi + \frac{1}{2} \partial_\mu \chi_a \partial^\mu \chi_a + \frac{1}{2} M_1^2 \phi^2 + \frac{1}{2} M_2^2 \chi_a \chi_a 
+ \frac{1}{8} \lambda_1 \phi^4 + \frac{1}{4} \bar{g} \phi^2 \chi_a \chi_a - \frac{1}{2} N c^2 + \frac{1}{2} \sqrt{N} \lambda_2 \chi_a \chi_a c \right\}.
\]

The effective action is defined as the Legendre transform of the logarithm of the partition function and can be evaluated using standard methods [12, 13]. Without loss of generality we consider expectation values for the fields \(\phi\) and \(\chi_0\). For this reason we set \(J_i = 0\) for \(i = 1, \ldots, N - 1\). A systematic expansion in powers of \(N\) is obtained [14, 11, 15] by treating these expectation values as being \(\mathcal{O}(\sqrt{N})\) and considering couplings \(\mathcal{O}(1/N)\). In practice one uses shifted fields according to
\[
\phi = \sqrt{N} \Phi + \delta \phi \quad \quad \quad \quad c = C + \frac{\delta c}{\sqrt{N}}
\]
\[
\chi_0 = \sqrt{N} X + \delta \chi_0 \quad \quad \quad \quad \chi_i = \delta \chi_i \quad i = 1, \ldots, N - 1
\]
and considers couplings which scale with \(N\) as
\[
\lambda_1 = \frac{\lambda_1}{N}, \quad \lambda_2 = \frac{\lambda_2}{N}, \quad \bar{g} = \frac{g}{N}.
\]
The effective action \(S_{\text{eff}}(\Phi, X, C)\) is calculated as a series in \(1/N\) by evaluating the terms in the loop expansion which are proportional to a given power of \(1/N\). Finally the auxiliary field \(C\) is eliminated by its equation of motion
\[
\frac{\delta S_{\text{eff}}}{\delta C} = 0.
\]

The leading (of order \((1/N)^{-1}\)) contribution to the effective potential is given by the expression
\[
\hat{U}(\rho, \sigma) = \frac{U(\rho, \sigma)}{N} = M_1^2 \rho + \frac{\lambda_1}{2} \rho^2 + M_2^2 \sigma + g \rho \sigma + \sqrt{\lambda_2} C \sigma - \frac{C^2}{2} 
+ \frac{1}{2} \int_\Lambda \frac{d^d q}{2 \pi^d} \ln \left( q^2 + M_2^2 + g \rho + \sqrt{\lambda_2} C \right)
\]
where we have defined
\[
\rho = \frac{\Phi^2}{2}, \quad \quad \sigma = \frac{X^2}{2},
\]
and an ultraviolet cutoff \(\Lambda\) is implied for the momentum integration. The auxiliary field is determined by its equation of motion
\[
\sqrt{\lambda_2} \sigma - C + \frac{1}{2} \int_\Lambda \frac{d^d q}{2 \pi^d} \sqrt{\lambda_2} \left\{ q^2 + M_2^2 + g \rho + \sqrt{\lambda_2} C \right\} = 0.
\]

* This can be achieved through an appropriate rescaling of the fields.
The wave function renormalization does not receive any corrections at this order in $1/N$.
It is convenient to eliminate the auxiliary field from our expressions. For this reason we use the derivatives of $\hat{U}(\rho, \sigma)$ with respect to $\rho$ and $\sigma$. From eqs. (8), (10) we obtain

$$\frac{d\hat{U}}{d\rho} = M_1^2 + \lambda_1 \rho + g \sigma + \frac{g}{2} I_1 \left( \frac{d\hat{U}}{d\sigma} \right)$$  \hspace{1cm} (11)

$$\frac{d\hat{U}}{d\sigma} = M_2^2 + \lambda_2 \sigma + g \rho + \frac{\lambda_2}{2} I_1 \left( \frac{d\hat{U}}{d\rho} \right)$$  \hspace{1cm} (12)

$$\frac{d^2\hat{U}}{d\sigma^2} = \frac{\lambda_2}{1 + \frac{\lambda_2}{2} I_2 \left( \frac{d\hat{U}}{d\sigma} \right)}$$  \hspace{1cm} (13)

$$\frac{d^2\hat{U}}{d\rho d\sigma} = \frac{g}{1 + \frac{\lambda_2}{2} I_2 \left( \frac{d\hat{U}}{d\sigma} \right)}$$  \hspace{1cm} (14)

$$\frac{d^2\hat{U}}{d\rho^2} = \frac{\lambda_1}{\lambda_2} + g \left( \frac{d^2\hat{U}}{d\rho d\sigma} - g \right)$$  \hspace{1cm} (15)

where

$$I_n(w) = \int_{\Lambda} \frac{d^d q}{(2\pi)^d (q^2 + w)^n}. \hspace{1cm} (16)$$

Expressions for higher derivatives of the effective potential can be obtained through differentiation of the above equations. The following equations are satisfied by the derivatives of $\hat{U}(\rho, \sigma)$

$$M_2^2 - \frac{\lambda_2}{g} M_1^2 = \frac{d\hat{U}}{d\rho} - \frac{\lambda_2}{g} \frac{d\hat{U}}{d\sigma} + \left( \frac{\lambda_1 \lambda_2}{g} - g \right) \rho$$  \hspace{1cm} (17)

$$\frac{g}{\lambda_2} = \frac{d^2\hat{U}}{d\rho d\sigma} / \frac{d^2\hat{U}}{d\sigma^2}$$  \hspace{1cm} (18)

$$\left( \frac{\lambda_1 \lambda_2}{g} - g \right) = \left( \frac{d^2\hat{U}}{d\rho^2} \frac{d^2\hat{U}}{d\sigma^2} / \frac{d^2\hat{U}}{d\rho d\sigma} - \frac{d^2\hat{U}}{d\rho d\sigma} \right).$$  \hspace{1cm} (19)

It is interesting to interpret the expressions (11)-(15) in terms of perturbation theory. The first two include the classical contributions to the mass terms, as well as the leading quantum corrections coming from the summation of an infinite series of “daisy” and “super-daisy” graphs [2]. A typical example of “super-daisy” corrections is presented in fig. 1a. These leading corrections involve only the “Goldstone” fields $\chi_i$, whose mass is equal to $d\hat{U}/d\sigma$. The next three expressions incorporate the leading corrections to the quartic couplings. Eqs. (13)-(15) can be expanded in a power series of the bare couplings $\lambda_1, \lambda_2, g$. The usual infinite series of “chain” graphs is reproduced, with each chain composed of one loop graphs involving two full $\chi_i$ field propagators. The form of these corrections is shown in fig. 1b, where the black circles denote $\chi_i$ field propagators.
incorporating the “super-daisy” corrections. We can see, therefore, how the leading result in $1/N$ can be interpreted as a resummation of an infinite subclass of diagrams of perturbation theory. This subclass is dominant for large $N$, but it is not sufficient for the discussion of certain aspects of the theory. For example, since the fluctuations of the “radial” fields $\phi, \chi_0$ are not taken into account at this order in $1/N$, we do not expect to obtain a convex effective potential. Also, in the case of a second order phase transition, we do not expect to observe non-trivial behavior when the “radial” fields become critical. Instead we expect mean field behavior associated with these fields.

From this point on we concentrate on the four-dimensional theory. We first define the zero temperature theory in the phase with spontaneous symmetry breaking. We are interested in studying the effect of the fluctuations of the $\chi$ fields on the expectation value of $\phi$. For this purpose we choose a pattern of symmetry breaking which corresponds to a choice for the minimum of the potential $(\rho_0, \sigma_0)$ such that $(\rho_0 \neq 0, \sigma_0 = 0)$. This choice preserves the $O(N)$ symmetry of the $\chi$ fields while breaking the $Z_2$ associated with the $\phi$ field. At the classical level, a sufficient condition for the potential to have a single minimum at $(\rho_0 \neq 0, \sigma_0 = 0)$ is

$$\lambda_1^2 > g \rho_0.$$

The integrals $I_n$ defined in eq. (16) have been discussed extensively in the literature [2, 11] and we simply quote the results which are relevant to our investigation. For $d = 4$ $I_1(w)$ is given by

$$I_1(w) = \frac{\Lambda^2}{16\pi^2} + \frac{w}{16\pi^2} \ln \left( \frac{w}{\Lambda^2} \right),$$

where we have assumed $\Lambda^2 \gg w$. We recognize the quadratic and logarithmic divergences of the four-dimensional theory. The rest of $I_n$ can be obtained through differentiation with respect to $w$. At any point $\rho$ the renormalized theory can be parametrized in terms of the masses of the $\phi$ and $\chi$ fields

$$m_{1R}^2 = \frac{d^2 \hat{U}}{d\rho^2}(\rho, 0) + 2\rho \frac{d^3 \hat{U}}{d\rho^3}(\rho, 0),$$

$$m_{2R}^2 = \frac{d^2 \hat{U}}{d\sigma^2}(\rho, 0),$$

respectively, and the quartic couplings

$$\lambda_{1R} = \frac{d^2 \hat{U}}{d\rho^2}(\rho, 0), \quad \lambda_{2R} = \frac{d^2 \hat{U}}{d\sigma^2}(\rho, 0), \quad g_R = \frac{d^2 \hat{U}}{d\rho d\sigma}(\rho, 0).$$

It is more convenient to use $d\hat{U}/d\rho$ instead of $m_{1R}^2$ for the parametrization of the effective potential. The two are related through eqs. (21). From eqs. (11)-(15), (20) we obtain

$$\frac{d\hat{U}}{d\rho} = M_1^2 + \lambda_1 \rho + \frac{g}{32\pi^2} \Lambda^2 + \frac{g}{32\pi^2} m_{2R}^2 \ln \left( \frac{m_{2R}^2}{\Lambda^2} \right),$$

$$m_{2R}^2 = M_2^2 + g \rho + \frac{\lambda_2}{32\pi^2} \Lambda^2 + \frac{\lambda_2}{32\pi^2} m_{2R}^2 \ln \left( \frac{m_{2R}^2}{\Lambda^2} \right).$$
\[
\lambda_{2R} = \lambda_2 \left[ 1 - \frac{\lambda_2}{32\pi^2} \ln \left( \frac{m_{2R}^2}{\Lambda^2} \right) \right]^{-1}
\]

(26)

\[
g_R = g \left[ 1 - \frac{\lambda_2}{32\pi^2} \ln \left( \frac{m_{2R}^2}{\Lambda^2} \right) \right]^{-1}
\]

(27)

\[
\lambda_{1R} = \lambda_1 + \frac{g}{\lambda_2} (g_R - g)
\]

(28)

where \( \tilde{\Lambda}^2 = \Lambda^2/e \). The above equations uniquely specify the renormalized parameters of the theory in terms of the bare ones. They indicate how the ultraviolet divergences can be absorbed in the definitions of the renormalized mass terms and couplings. We also point out that the theory is well behaved in the infrared, since the infrared logarithmic singularities are cut off by the mass of the \( \chi \) fields. The presence of the logarithms can give rise to the Coleman-Weinberg mechanism for radiative symmetry breaking \[12\]. If \( m_{2R}^2 \) is sufficiently small the logarithm in eq. (24) gives a negative contribution to \( d\tilde{U}/d\rho \) which can lead to symmetry breaking. We shall return to this point after removing the ultraviolet divergences from our expressions.

We define the renormalized theory at zero temperature in the phase with spontaneous symmetry breaking in terms of the minimum of the effective potential \( \rho_0 \) (where \( d\tilde{U}/d\rho (\rho_0, 0) = 0 \)) and the parameters \( m_{2R0}^2 = d\tilde{U}/d\sigma (\rho_0, 0) \), \( \lambda_{1R0} = d^2\tilde{U}/dp^2 (\rho_0, 0) \), \( \lambda_{2R0} = d^2\tilde{U}/d\sigma^2 (\rho_0, 0) \), \( g_{R0} = d^2\tilde{U}/d\rho (\rho_0, 0) \). By making use of eqs. (24)-(28) we can relate the parameters at any other point \( \rho \neq \rho_0 \) to those at the minimum. We find

\[
\frac{d\tilde{U}}{d\rho} = \lambda_{1R0} (\rho - \rho_0) + \frac{g_{R0}}{32\pi^2} \left\{ -m_{2R}^2 + m_{2R0}^2 + m_{2R}^2 \ln \left( \frac{m_{2R}^2}{m_{2R0}^2} \right) \right\}
\]

(29)

\[
m_{2R}^2 = m_{2R0}^2 + g_{R0} (\rho - \rho_0) + \frac{\lambda_{2R0}}{32\pi^2} \left\{ -m_{2R}^2 + m_{2R0}^2 + m_{2R}^2 \ln \left( \frac{m_{2R}^2}{m_{2R0}^2} \right) \right\}
\]

(30)

\[
\lambda_{2R} = \lambda_{2R0} \left[ 1 - \frac{\lambda_{2R0}}{32\pi^2} \ln \left( \frac{m_{2R}^2}{m_{2R0}^2} \right) \right]^{-1}
\]

(31)

\[
g_R = g_{R0} \left[ 1 - \frac{\lambda_{2R0}}{32\pi^2} \ln \left( \frac{m_{2R}^2}{m_{2R0}^2} \right) \right]^{-1}
\]

(32)

\[
\lambda_{1R} = \lambda_{1R0} + \frac{g_{R0} g_R}{32\pi^2} \ln \left( \frac{m_{2R}^2}{m_{2R0}^2} \right)
\]

(33)

The absence of the ultraviolet cutoff from the above relations is a manifestation of the renormalizability of the theory. In order to guarantee that the mass term \( m_{2R}^2 \) remains positive at any point \( (\rho, 0) \) we require that \( m_{2R0}^2 \geq g_{R0} \rho_0 \). We can now see how the Coleman-Weinberg mechanism for symmetry breaking arises. For a choice of parameters \( m_{2R0}^2 = g_{R0} \rho_0 \), \( \lambda_{2R0} = 0 \), \( \lambda_{1R0} = g_{R0}^2/32\pi^2 \) the effective potential is given by

\[
\frac{d\tilde{U}}{d\rho} = \frac{g_{R0}^2}{32\pi^2} \rho \ln \left( \frac{\rho}{\rho_0} \right)
\]

(34)
It is clear that in this region of parameter space the breaking of the symmetry is driven by the logarithm arising from the radiative corrections. At non-zero temperature, we expect first order phase transitions to appear for theories with radiative symmetry breaking.

In order to extend our discussion to the non-zero temperature problem we only need to recall that, in Euclidean formalism, non-zero temperature \( T \) results in periodic boundary conditions in the time direction (for bosonic fields), with periodicity \( 1/T \). This leads to a discrete spectrum for the zero component of the momentum \( q_0 \) and replaces the integration over \( q_0 \) by summation over the discrete spectrum. As a result, eqs. (11)-(15) remain valid (with \( d = 4 \)), but eq. (16) is replaced by

\[
I_n(w, T) = T \sum_m \int_\Lambda \frac{d^3q}{(2\pi)^3} \frac{1}{(q^2 + 4\pi^2 m^2 T^2 + w)^n}. \tag{35}
\]

We can separate the non-zero temperature contribution to the above expression by defining

\[
I_n(w, T) = I_n(w) + \Delta I_n(w, T), \tag{36}
\]

with \( I_n(w) \) given by eq. (16) (with \( d = 4 \)). \( \Delta I_1(w, T) \) has been evaluated elsewhere \([2]\) and reads

\[
\Delta I_1(w, T) = T^2 \left\{ \frac{\pi^2}{2\pi^2} \left[ \frac{\pi^2}{6} - \frac{\pi}{2} \sqrt{\tilde{w}} - \frac{1}{8} \tilde{w} \ln \left( \frac{\tilde{w}}{\tilde{c}^2} \right) \right] \right\} \tag{37}
\]

where \( \tilde{c} = \exp \left( \frac{1}{2} + \ln(4\pi) - \gamma \right) \approx 11.6. \)

The effective potential is now temperature dependent. Similarly to the zero temperature case, at any point \( \rho \) we define the (temperature dependent) masses of the \( \phi \) and \( \chi \) fields

\[
m_{1R}^2(T) = \frac{dU}{d\rho}(\rho, 0, T) + 2\rho \frac{d^2U}{d\rho^2}(\rho, 0, T) \tag{39}
\]

\[
m_{2R}^2(T) = \frac{d\hat{U}}{d\sigma}(\rho, 0, T) \tag{40}
\]

respectively, and the quartic couplings

\[
\lambda_{1R}(T) = \frac{d^2\hat{U}}{d\rho^2}(\rho, 0, T), \quad \lambda_{2R}(T) = \frac{d^2\hat{U}}{d\sigma^2}(\rho, 0, T), \quad g_{1R}(T) = \frac{d^2\hat{U}}{d\rho d\sigma}(\rho, 0, T). \tag{41}
\]

Again for convenience we use \( d\hat{U}/d\rho(T) \) instead of \( m_{1R}^2(T) \). Eqs. (24)-(28) are replaced by

\[
\left\{ \frac{d\hat{U}}{d\rho}(T) = M^2_1 + \lambda_1 \rho + \frac{g}{32\pi^2} \Lambda^2 + \frac{g}{2} \left\{ \frac{1}{16\pi^2} m_{2R}^2(T) \ln \left( \frac{m_{2R}^2(T)}{\Lambda^2} \right) + \Delta I_1 \left( m_{2R}^2(T), T \right) \right\} \right\}
\]
\[m_{2R}^2(T) = M_2^2 + g\rho + \frac{\lambda_2}{32\pi^2}\Lambda^2 + \frac{\lambda_2}{2} \left \{ \frac{1}{16\pi^2} m_{2R}^2(T) \ln \left( \frac{m_{2R}^2(T)}{\Lambda^2} \right) + \Delta I_1 \left( m_{2R}^2(T), T \right) \right \} \] (42)

\[\lambda_{2R}(T) = \lambda_2 \left[ 1 - \frac{\lambda_2}{2} \left \{ \frac{1}{16\pi^2} \ln \left( \frac{m_{2R}^2(T)}{\Lambda^2} \right) - \Delta I_2 \left( m_{2R}^2(T), T \right) \right \} \right]^{-1} \] (43)

\[g_R(T) = g \left[ 1 - \frac{\lambda_2}{2} \left \{ \frac{1}{16\pi^2} \ln \left( \frac{m_{2R}^2(T)}{\Lambda^2} \right) - \Delta I_2 \left( m_{2R}^2(T), T \right) \right \} \right]^{-1} \] (44)

\[\lambda_{1R}(T) = \lambda_1 + \frac{g}{\lambda_2} (g_R(T) - g). \] (45)

The final step is to relate the temperature dependent renormalized parameters at a point \(\rho\) to the parameters at the minimum of the zero temperature effective potential \(\rho_0\). The calculation is straightforward and we find

\[\frac{d\tilde{U}}{d\rho}(T) = \lambda_{1R0}(\rho - \rho_0) + \frac{g_{R0}}{32\pi^2} \left \{ -m_{2R}^2(T) + m_{2R0}^2 + m_{2R}^2(T) \ln \left( \frac{m_{2R}^2(T)}{m_{2R0}^2} \right) \right \} \] (47)

\[+ \frac{g_{R0}}{2} \Delta I_1 \left( m_{2R}^2(T), T \right) \]

\[m_{2R}^2(T) = m_{2R0}^2 + g_{R0}(\rho - \rho_0) + \frac{\lambda_{2R0}}{32\pi^2} \left \{ -m_{2R}^2(T) + m_{2R0}^2 + m_{2R}^2(T) \ln \left( \frac{m_{2R}^2(T)}{m_{2R0}^2} \right) \right \} \] (48)

\[+ \frac{\lambda_{2R0}}{2} \Delta I_1 \left( m_{2R}^2(T), T \right) \]

\[\lambda_{2R}(T) = \lambda_{2R0} \left[ 1 - \frac{\lambda_{2R0}}{32\pi^2} \ln \left( \frac{m_{2R}^2(T)}{m_{2R0}^2} \right) + \frac{\lambda_{2R0}}{2} \Delta I_2 \left( m_{2R}^2(T), T \right) \right]^{-1} \] (49)

\[g(R) = g_{R0} \left[ 1 - \frac{\lambda_{2R0}}{32\pi^2} \ln \left( \frac{m_{2R}^2(T)}{m_{2R0}^2} \right) + \frac{\lambda_{2R0}}{2} \Delta I_2 \left( m_{2R}^2(T), T \right) \right]^{-1} \] (50)

\[\lambda_{1R}(T) = \lambda_{1R0} + \frac{g_{R0} g_R(T)}{2} \left \{ \frac{1}{16\pi^2} \ln \left( \frac{m_{2R}^2(T)}{m_{2R0}^2} \right) - \Delta I_2 \left( m_{2R}^2(T), T \right) \right \} \] (51)

Eqs. (47)-(51) are the master equations for the study of the behavior of the theory at non-zero temperature. For a given zero temperature renormalized theory, as specified by the parameters \(\rho_0, m_{2R0}, \lambda_{1R0}, \lambda_{2R0}, g_{R0}\), they encode all the information (in leading order in \(1/N\)) related to phase change and metastability at non-zero temperature.

In order to identify the regions of parameter space which lead to high temperature phase transitions of different order, it is instructive to study eqs. (47)-(51) analytically in some limiting cases. For \(\lambda_{2R0} = 0\), \(m_{2R0}^2 = g_{R0}\rho_0\), eq. (48) gives for the renormalized \(\chi\) field mass

\[m_{2R}^2(T) = g_{R0}\rho. \] (52)
As a result the radiative contributions of the $\chi$ fields to the effective potential involve a strong $\rho$ dependence. We find

$$\frac{d\hat{U}}{d\rho}(T) = \left(\lambda_{1R0} - \frac{g_{R0}^2}{32\pi^2}\right)(\rho - \rho_0) + \frac{g_{R0}^2}{32\pi^2}\rho \ln \left(\frac{\rho}{\rho_0}\right) + \frac{g_{R0}}{2}\Delta I_1(g_{R0}\rho, T)$$  \hspace{1cm} (53)

Let us consider first the case $\lambda_{1R0} = \frac{g_{R0}^2}{32\pi^2}$, which corresponds to radiative symmetry breaking. The logarithmic term gives a negative contribution to $d\hat{U}/d\rho(T)$. Its effect is compensated by the positive high temperature contribution $\propto \Delta I_1(g_{R0}\rho, T)$ which increases with temperature. There are two points of zero $d\hat{U}/d\rho(T)$, corresponding to the minimum of the potential at non-zero $\rho_0(T)$ and the maximum of the barrier separating it from another minimum at zero. For sufficiently high temperature the minimum at zero becomes the absolute minimum of the potential and the secondary one eventually disappears. The behaviour is characteristic of a first order phase transition \[4, 16\]. The size of the discontinuity of the order parameter is set by the logarithmic term and is

$$\delta \rho = \mathcal{O}(\rho_0).$$  \hspace{1cm} (54)

This classifies the transition as a strongly first order one. The high temperature expansion of eq. (38) for $\Delta I_1(g_{R0}\rho, T)$ is not adequate for the study of this case, since $g_{R0}\delta \rho/T_{cr}^2$ can be estimated to be larger than one. In the opposite limit $\lambda_{1R0} \gg \frac{g_{R0}^2}{32\pi^2}$, the logarithmic term is a minor correction. Making use of the high temperature expansion of eq. (38), we rewrite eq. (53) as

$$\frac{d\hat{U}}{d\rho}(T) = \lambda_{1R0}(\rho - \rho_0) + \frac{g_{R0}}{24}T^2 - \frac{g_{R0}^3}{8\pi}T\sqrt{\rho} ...$$  \hspace{1cm} (55)

Taking into account the dominant contribution $\propto T^2$ would lead to a prediction for a second order phase transition. However, the second term in the high temperature expansion gives again a negative contribution to $d\hat{U}/d\rho(T)$, which results in a weakly first order transition \[4\]. The critical temperature is $T_{cr}/\rho_0 \simeq 24\lambda_{1R0}/g_{R0}$ and the discontinuity in $\rho$ is much smaller than $\rho_0$

$$\delta \rho = \mathcal{O}\left(\frac{g_{R0}^2}{32\pi^2\lambda_{1R0}}\rho_0\right),$$  \hspace{1cm} (56)

justifying the use of the high temperature expansion. For $\lambda_{2R0} = 0$, $m_{2R0}^2 = g_{R0}\rho_0$, the phase transition remains first order for arbitrarily large values of $\lambda_{1R0}$. Even though the picture seems consistent at this level of the $1/N$ expansion, serious complications appear at higher orders. From dimensional analysis it is expected that multi-loop corrections to the effective potential which involve the $\phi$ field (and which have not been taken into account by the $1/N$ expansion so far) are proportional to powers of $\lambda_{1R0}T/\sqrt{m_{1R0}^2(T)}$, with the mass term defined in eq. (39). These contributions are divergent near the critical temperature when the discontinuity in the order parameter and the mass term approach zero. As a result they overwhelm and, therefore, invalidate the leading order
result for weakly enough first order transitions. An adequate treatment of the infrared problem must control the physics associated with the fluctuations of the “radial” field $\phi$. This was done in ref. [10] through the renormalization group approach for the $O(N)$-symmetric scalar theory. The $N = 1$ theory was shown to have a second order transition, with an effective three-dimensional critical behaviour. For our model it is reasonable to expect that the fluctuations of the $\phi$ field can affect the order of a transition, when this is predicted to be weakly first order by the leading $1/N$ calculation. As a result the first order character of the transition is not reliably established by the leading $1/N$ result for large $\lambda_{1R0}$, even for $\lambda_{2R0} = 0$, $m_{2R0}^2 = g_{R0}\rho_0$. Also, deviations from the above values for $\lambda_{2R0}$ and $m_{2R0}^2$ lead to predictions of second order phase transitions for sufficiently large $\lambda_{1R0}$, even within the leading $1/N$ calculation. This becomes apparent through the numerical study of the effective potential.

For a given set of parameters of the zero temperature theory, we have solved eqs. (47) and (48) for $d\hat{U}/d\rho(T)$, using the full expression (37) for $\Delta I_1(m_{2R}^2(T), T)$. The effective potential is obtained through numerical integration of the result. Two typical examples are presented in fig. 2, where $\hat{U}(\rho, T)$ is plotted for various temperatures. (From this point on we omit the $\sigma$ dependence of the potential in our notation, since always $\sigma = 0$.) In the first example the zero temperature parameters are $m_{2R0}/\rho_0 = 1.55$, $\lambda_{1R0} = 10^{-2}$, $\lambda_{2R0} = 10^{-3}$, $g_{R0} = 1.5$ and a strongly first order transition is observed. In the second $\lambda_{1R0} = 5 \times 10^{-2}$ while the other parameters have the same values. The strength of the first order transition is clearly diminished. The effect of increasing $\lambda_{1R0}$ on the strength of the first order transition is more obvious in fig. 3, where we plot the location of the minimum of the potential as a function of temperature, for $m_{2R0}/\rho_0 = 1.55$, $\lambda_{2R0} = 10^{-3}$, $g_{R0} = 1.5$ and various values of $\lambda_{1R0}$. The solid lines indicate the location of the minimum as long as it corresponds to the true vacuum of the potential. The dashed ones indicate that a deeper minimum has appeared at zero, while we are still following the location of a false vacuum. At the point where the solid and dashed lines meet the two minima have equal depth. It is clear that the discontinuity in $\rho$ diminishes with increasing $\lambda_{1R0}$. The line for $\lambda_{1R0} = 1.0$ corresponds to a second order phase transition. This is due to the fact that the other zero temperature parameters deviate slightly from the values $\lambda_{2R0} = 0$, $m_{2R0}^2 = g_{R0}\rho_0$ which were considered in the analytical discussion. In figs. 4 and 5 we demonstrate the effect of the other parameters of the zero temperature theory on the strength of the first order transition. In fig. 4 it is shown that increasing $m_{2R0}^2$ reduces the strength of the first order transition, which eventually becomes second order. Similarly, fig. 5 shows that larger self-interactions for the $\chi$ fields result in more weakly first order transitions, which again turn second order. We mention at this point, that the omission of the $\phi$ field fluctuations at this order in $1/N$ results in mean field behaviour for the second order transitions of the theory. If the temperature dependence of $m_{1R0}^2(T)$ in the symmetric phase is parametrized as $m_{1R0}^2(T) = d\hat{U}/d\rho(0, T) \propto (T - T_c)^{2\nu}$, the numerical solution of eqs. (17), (48) gives $\nu = 0.5$ very close to the critical temperature.

We conclude that a first order transition is obtained when the temperature dependent mass $m_{2R}^2(T)$ of the $\chi$ fields has a strong dependence on $\rho$. This is achieved for the
zero temperature parameter range \( m_{2R0}^2 \simeq g_{R0}\rho_0, \lambda_{2R0} \simeq 0 \). Then the strength of the first order transition is maximum for \( \lambda_{1R0} \simeq g_{R0}^2/32\pi^2 \) and diminishes for increasing \( \lambda_{1R0} \). Deviations from the above range of parameters reduce the dependence of \( m_{2R}(T) \) on \( \rho \) and consequently the strength of the transition, which eventually becomes second order. Only for \( m_{2R0}^2 = g_{R0}\rho_0, \lambda_{2R0} = 0 \) the phase transition is predicted to be first order for arbitrarily large \( \lambda_{1R0} \). However, for the choice of parameters for which weakly first order transitions are predicted, the \( \phi \) field fluctuations (which are not taken into account by the leading \( 1/N \) calculation) become important, as indicated by divergent contributions at higher orders of the \( 1/N \) expansion. Studies based on the renormalization group approach \([10, 11]\) indicate that the incorporation of these fluctuations leads to second order transitions. As a result the nature of these transitions cannot be firmly established in the context of the \( 1/N \) expansion. The renormalization group becomes an indispensable tool for the resolution of these open questions. Studies of two-scalar models with the use of the renormalization group are presented in refs. \([17]\).

**References**

[1] D.A. Kirzhnits and A.D. Linde, Phys. Lett. B 42, 471 (1972).

[2] L. Dolan and R. Jackiw, Phys. Rev. D 9, 3320 (1974).

[3] S. Weinberg, Phys. Rev. D 9, 3357 (1974).

[4] D.A. Kirzhnits and A.D. Linde, JETP 40, 628 (1974); Ann. Phys. 101, 195 (1976).

[5] A.H. Guth, Phys. Rev. D 23, 347 (1981); A.D. Linde, Phys. Lett. B 108, 389 (1982); A. Albrecht and P.J. Steinhardt, Phys. Rev. Lett. 48, 1220 (1982).

[6] See the Proceedings of the NATO Advanced Research Workshop: Electroweak physics and the early universe, Sintra, 1994 (Plenum Press) and references therein.

[7] V.A. Kuzmin, V.A. Rubakov, and M.E. Shaposhnikov, Phys. Lett. B 155, 36 (1985); M.E. Shaposhnikov, Nucl. Phys. B 287, 757 (1987); ibid 299, 797 (1988).

[8] A.I. Bochkarev and M.E. Shaposhnikov, Mod. Phys. Lett. A 2, 417 (1987).

[9] A.I. Bochkarev, S.V. Kuzmin and M.E. Shaposhnikov, Phys. Lett. B 244, 275 (1990); Phys. Rev. D 43, 369 (1991); N. Turok and J. Zadrozny, Nucl. Phys. B 369, 729 (1992); S. Myint, Phys. Lett. B 287, 325 (1992); J.R. Espinosa, M. Quiros and F. Zwirner, Phys. Lett. B 307, 106 (1993).

[10] N. Tetradis and C. Wetterich, Nucl. Phys. B. 398, 659 (1993); preprint DESY-93-094, HD-THEP-93-28, to appear in Nucl. Phys. B; Int. J. Mod. Phys. A 9, 4029 (1994).

[11] M. Reuter, N. Tetradis and C. Wetterich, Nucl. Phys. B 401, 567 (1993).
[12] S. Coleman and E. Weinberg, Phys. Rev. D 7, 1888 (1973).

[13] R. Jackiw, Phys. Rev. D 9, 1686 (1974).

[14] H.J. Schnitzer, Phys. Rev. D 10, 1800 (1974); S. Coleman, R. Jackiw and H.D. Politzer, Phys. Rev. D 10, 2491 (1974); R.G. Root, Phys. Rev. D 10 3322 (1974).

[15] V. Jain, Nucl. Phys. B 394, 707 (1993); H. Meyer-Ortmanns and A. Patkos, Phys. Lett. B 297, 331 (1992).

[16] J. Iliopoulos and N. Papanicolaou, Nucl. Phys. B 111, 209 (1976); A. Guth and E. Weinberg, Phys. Rev. Lett. 45, 1131 (1980); E. Witten, Nucl. Phys. B 177, 477 (1981); M. Sher, Phys. Rep. 179, 273 (1989).

[17] S. Bornholdt, N. Tetradis and C. Wetterich, preprints OUTP-94-13 P, HD-THEP-94-28, and OUTP-94-14 P, HD-THEP-94-15; V. Jain and A. Papadopoulos, Phys. Lett. B 314, 95 (1993); M. Alford and J. March-Russell, Nucl. Phys. B 417, 527 (1994).

Figures

Fig. 1 a) A typical example of the “super-daisy” corrections summed by eqs. (11)-(12).

b) An example of the “chain” graphs summed by eqs. (13)-(15). Black circles denote full $\chi_i$ field propagators.

Fig. 2 $\hat{U}(\rho, T)$ at increasing temperatures ($T_5 > T_4 > T_3 > T_2 > T_1$), for zero temperature parameters:

a) $m_{2R0}/\rho_0 = 1.55$, $\lambda_{1R0} = 10^{-2}$, $\lambda_{2R0} = 10^{-3}$, $g_{R0} = 1.5$.

b) $m_{2R0}/\rho_0 = 1.55$, $\lambda_{1R0} = 5 \times 10^{-2}$, $\lambda_{2R0} = 10^{-3}$, $g_{R0} = 1.5$.

Fig. 3 Position of the potential minimum against temperature for zero temperature parameters: $m_{2R0}/\rho_0 = 1.55$, $\lambda_{2R0} = 10^{-3}$, $g_{R0} = 1.5$, and $\lambda_{1R0} = 0.01, 0.05, 0.2, 0.5, 1$. Solid lines indicate positions of true vacua, dashed lines indicate positions of false vacua. The circles indicate two minima of equal depth.

Fig. 4 Position of the potential minimum against temperature for zero temperature parameters: $\lambda_{1R0} = 10^{-2}$, $\lambda_{2R0} = 10^{-3}$, $g_{R0} = 1.5$, and $m_{2R0}/\rho_0 = 1.55$, 2.0, 2.75, 3.5, 4.5. Solid lines indicate positions of true vacua, dashed lines indicate positions of false vacua. The circles indicate two minima of equal depth.

Fig. 5 Position of the potential minimum against temperature for zero temperature parameters: $m_{2R0}/\rho_0 = 1.55 \times 10^{-1}$, $\lambda_{1R0} = 10^{-3}$, $g_{R0} = 1.5 \times 10^{-1}$, and $\lambda_{2R0} = 0.1, 0.5, 1, 1.75, 2.5$. Solid lines indicate positions of true vacua, dashed lines indicate positions of false vacua. The circles indicate two minima of equal depth.
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