ON THE RANGE OF WEIGHTED PLANAR CAUCHY TRANSFORM

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ABSTRACT. We describe the range of weighted Cauchy transform and its $k$-Bergman projection when action on weighted true poly-Bargmann spaces constituting an orthogonal Hilbertian decomposition of the Hilbert space of Gaussian functions on the complex plane.

1 Introduction

The Cauchy transforms on bounded regular domains and their boundaries are well studied and has been investigated by many authors, see for instance [3, 4, 5, 15, 33, 28, 22, 11, 13, 10] and the references therein. This rich literature is due to their use in solving the $\overline{\partial}$-equation, in developing theory of holomorphic functions [21] and in proving interpolation theorems [27] as well as in providing simple proof of Corona theorem [17]. One of the fundamental examples of weighted Cauchy transform on the whole complex plane is that associated to the Gaussian measure $d\mu = e^{-|z|^2}dxdy$ and defined by

$$C_{\mu}f(z) := \frac{1}{\pi} \int_{\mathbb{C}} \frac{f(\xi)}{z-\xi}d\mu(z) \hspace{1cm} (1.1)$$

on $L^2_{\mu}$, denoting as usual the Hilbert space $L^2(\mathbb{C}, d\mu)$ of all square integrable complex valued functions with respect to the scalar product

$$\langle f, g \rangle := \int_{\mathbb{C}} f(z)\overline{g(z)}d\mu(z).$$

The singular integral operator in (1.1), as operator from $L^2_{\mu}$ into $L^2_{\mu}$, is bounded, compact and belongs to the $p$-Schatten class for every $p > 2$. The spectral properties of $C_{\mu}$ has been investigated in [16, 23]. In [16], Dostanić gave the exact asymptotic behavior of the singular values of $C_{\mu}$ and $PC_{\mu}$, where $P$ denotes the orthogonal projection operator onto the classical Bargmann space $A^2$ constituted of all holomorphic functions belonging to $L^2_{\mu}$. The generalization to the polyanalytic setting was considered by the brothers A. and A. Intissar in [23]. This was possible making use of the Hilbertian orthogonal decomposition

$$L^2_{\mu} = \bigoplus_{n=0}^{\infty} A^2_n$$

where $A^2_n = Ker(\Delta - n)$ are the $L^2$–eigenspaces of the Landau operator $\Delta = \partial_x \partial_x - z \partial_z$ with $A^2_0 = A^2$. The key result in [23] is the explicit action of $C_{\mu}$ in (1.1) on the Itô–Hermite polynomials constituting an orthogonal basis of $L^2_{\mu}$. In fact, the functions $C_{\mu}H_{m,n}$ are given by

$$\langle C_{\mu}H_{m,n}(z) = -e^{-|z|^2}H_{m-1,n}(z; \bar{z}) \rangle \hspace{1cm} (1.2)$$

Moreover, for varying $m = 0, 1, \cdots$, and fixed $n$, (resp.for varying $m = 0, 1, \cdots$, and fixed $m$) they constitute an orthogonal system in $L^2_{\mu}$.

In the present note, we complete the study provided in the afore mentioned papers [16, 23] related to $C_{\mu}$ in $L^2_{\mu}$. Mainly, we provide complete description of which polyanalytic functions on the whole complex plane can be represented as weighted Cauchy integral in

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This leads to the identification of the range of the operator $P_nC_\mu$, where $P_n$ denotes the orthogonal projection on $\mathcal{A}_n^2$ given by

$$P_n f(z) = \frac{1}{\pi} \int_{\mathcal{C}} L_n(|z - \xi|^2) e^{-\xi z} f(\xi) d\mu(\xi). \quad (1.3)$$

To explore these ideas, we begin by reviewing in Section 2 the basic properties of the Itô–Hermite polynomials and their associated weighted poly-Bargmann spaces $\mathcal{A}_n^2$. Our main results on the range of of weighted Cauchy transform and its $k$-Bergman projection on $\mathcal{A}_n^2$ and $L^2_\mu$ are presented and proved in Section 3.

## 2 Preliminaries

The Itô–Hermite polynomials involved in (1.2) are the basic example of bivariate real Hermite polynomials that are not simply a tensor product of univariate real Hermite polynomials. They are due to Itô [26] and play a crucial role in the framework of complex Markov process and constitute an orthogonal basis of $L^2_n$. They have been intensively studied and found several applications in the nonlinear analysis of traveling wave tube amplifiers [8], in spectral theory of some second order differential operators [32, 30, 20], in the study of some special integral transforms [23, 9], in coherent states theory [2, 1], combinatorics [25] and in signal processing [31, 14]. They are defined by their Rodrigues’ formula [26]

$$H_{m,n}(z, \bar{z}) := (-1)^{m+n} e^{z\bar{z}} \partial_z^m \partial_{\bar{z}}^n \left( e^{-|z|^2} \right), \quad (2.1)$$

where $\partial_z$ and $\partial_{\bar{z}}$, as usual, denote the first order partial differential operators

$$\partial_z = \frac{\partial}{\partial z} := \frac{1}{2} \left( \frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right), \quad \partial_{\bar{z}} = \frac{\partial}{\partial \bar{z}} := \frac{1}{2} \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right). \quad (2.2)$$

The hypergeometric representation for $H_{m,n}$ in terms of the Kummer’s function $1F_1$ reads

$$H_{m,n}(z, \bar{z}) = c_{m,n} \frac{z^m \bar{z}^n}{|z|^{2\min(m,n)}} 1F_1 \left( \begin{array}{c} -\min(m, n) \\ m - n + 1 \end{array} \mid |z|^2 \right) \quad (2.3)$$

$$= \begin{cases} \frac{(-1)^m m!}{(n-m)!} z^{m-n} 1F_1 \left( \begin{array}{c} -m \\ n - m + 1 \end{array} \mid |z|^2 \right); & m \leq n \\ \frac{(-1)^n n!}{(m-n)!} \bar{z}^{m-n} 1F_1 \left( \begin{array}{c} -n \\ m - n + 1 \end{array} \mid |z|^2 \right); & m \geq n, \end{cases} \quad (2.4)$$

where $m \wedge n = \min(m, n)$, $m \vee n = \max(m, n)$ and

$$c_{m,n} := \frac{(-1)^{\min(m,n)} \max(m, n)!}{|m - n|!}. \quad (2.5)$$

This last representation is the convenient one for extending $H_{m,n}$ to include negative index, leading to what we call here extended Itô–Hermite functions. For further basic properties of $H_{m,n}$, we refer to [23, 19, 18, 20, 25, 24].

By considering the formal adjoint of $\partial_{\bar{z}}$ in $L^2_\mu$, given by $\partial_{\bar{z}}^* = -\partial_z + z$, we can see that the operator $\Delta = \partial^2 z \partial^2 {\bar{z}} = -\partial_z (\partial_z - z) \partial_{\bar{z}}$ is the usual Landau magnetic Laplacian describing the movement of a single charged particle in the complex plane under the action of an uniform magnetic field applied perpendicularly. The concrete spectral analysis of such Laplacian is well known in the literature, see for instance [7] and the references therein. Thus, the $L^2_\mu$-eigenspaces $\mathcal{A}_n^2 := \ker(\partial_{\bar{z}}^2 - nId)$ form an orthogonal Hilbertian decomposition of $L^2_\mu$. More explicitly, they are characterized as the closed Hilbert subspaces of all
convergent series
\[ f(z) = \sum_{j=0}^{+\infty} H_{j,n}(z,\overline{\alpha})a_{j,n}, \quad \sum_{j=0}^{+\infty} j!|a_{j,n}|^2 < +\infty. \] (2.6)

Moreover, they are exactly the so-called true poly-Bargmann spaces in the terminology of Vasilevski [34] realized as specific closed subspace of \((n+1)\)-polyanalytic \(L^2\)-functions.

We conclude this section by noticing that \(A_n^2\) is a reproducing kernel Hilbert space and the \(\{H_{m,n}; m = 0, 1, 2, \cdots\}\) is an orthogonal system of it. Thus, the expansion series of its reproducing kernel function \(w \mapsto K_n(z,w) = \overline{K_n(w,z)} \in A_n^2\) in terms of the Itô–Hermite polynomials reads

\[ K_n(z,w) = \sum_{m=0}^{+\infty} \frac{H_{m,n}(z,\overline{\alpha})H_{n,m}(w,\overline{\alpha})}{\sqrt{\pi m!n!}}, \quad (2.7) \]

since \(H_{m,n}(z,\overline{\alpha})/\sqrt{\pi m!n!}\) form an orthonormal basis of \(A_n^2\), while the closed expression of \(K_n\) in terms of the Laguerre polynomials is given by [7]

\[ K_n(z,w) = \frac{e^{zw}}{\pi} L_n(|z-w|^2), \quad (2.8) \]

so that the orthogonal projection \(f \mapsto P_nf\) of \(L^2(\mathbb{C},d\mu)\) onto \(A_n^2\) reads \(P_nf(w) = \langle K_n(\cdot,w), f \rangle\).

3 The range and the null space of \(C_\mu\)

We begin by noticing that (1.2) shows clearly that for every \(m,n = 0, 1, 2, \cdots\), the image \(\psi_{m,n} := C_\mu H_{m,n}\) belongs to \(L^2_\mu\). Therefore, the functions \(\psi_{m,n}\) can be expanded, according to the orthogonal decomposition of \(L^2_\mu\) in terms of \(A_n^2\), as

\[ \hat{\psi}_{m,n} = \sum_{n=0}^{+\infty} \hat{\psi}_{m,n} \]

with the component \(\hat{\psi}_{m,n}\) are given by \(\hat{\psi}_{m,n} := P_n\psi_{m,n} \in A_n^2\). Accordingly, our first aim below is to look for the explicit expression of \(\hat{\psi}_{m,n}\). To this end, we use \(\varepsilon_p\) to mean 1 for nonnegative integer \(p\) and 0 otherwise.

**Proposition 3.1.** We have

\[ P_n(C_\mu H_{j,k})(z) = \varepsilon_{n+j-k-1} \frac{(-1)^{n+k}(n+j-1)!}{2^{n+j}n!(n+j-k-1)!} H_{n+j-k-1,n}(z,\overline{\alpha}). \quad (3.1) \]

**Proof.** From the expansion series of the reproducing kernel \(K_n\) of \(A_n^2\) in terms of the Itô–Hermite polynomials given through (2.7) and the explicit expression of \(\psi_{j,k} = C_\mu H_{j,k}\) given through (1.2), it follows

\[ P_n(\psi_{j,k})(z) = -\sum_{m=0}^{+\infty} \frac{H_{m,n}(z,\overline{\alpha})}{\sqrt{\pi m!n!}} \int_\mathbb{C} H_{n,m}(w,\overline{\alpha})H_{j-1,k}(w,\overline{\alpha})e^{-|w|^2}d\lambda(w). \]

The hypergeometric representation of \(H_{j,k}\) infers

\[ P_n(\psi_{j,k})(z) = \sum_{m=0}^{+\infty} H_{m,n}(z,\overline{\alpha})J_{m,n,j,k} \]

where we have set

\[ J_{m,n,j,k} := -\frac{C_{m,n}C_{j,k}}{\pi m!n!} \int_\mathbb{C} \frac{w^{m-n}w^{j-1}w^{m-j}w^{j-k}R_{m+1,n,j,k}(|w|^2)e^{-|w|^2}d\lambda(w)}{|w|^{2(m+n+(j-1)\wedge k)}}, \]

with \(\wedge\) standing for the minimum of the two arguments.
and

\[ R_{m,n,j,k}(t) := \binom{-m-1, n}{|m-1|, n+n+1, k+1} 1 F_1 \left( \begin{array}{l}
-(m-1) \wedge n \\
|m-1-n|+1 \\
n, k+1 \\
|j-1-k|+1 \\
t
\end{array} \right). \]  

(3.2)

By passing to polar coordinates, the expression of \( J_{m,n,j,k} \) reduces to

\[ J_{m,n,j,k} = -\frac{2c_n c_j c_{j-1}}{m! n!} \delta_{n,j,k} \int_0^\infty \frac{e^{r(n+m+j-k+1)}}{r^{(m+n+j-k+1)}} r^{n+j-k-1} R_{m,n,j,k} \left( r^2 \right) e^{-2r} dr. \]

Therefore, the only nonzero term in such expansion of \( P_n(C_{\mu} H_{j,k}) \), when \( n + j \geq k + 1 \), corresponds to the special case \( m = n + j - k - 1 \). Otherwise, \( P_n(C_{\mu} H_{j,k}) = 0 \). Thus, we have

\[ P_n(\psi_{j,k}) = \varepsilon_{n+j-k-1} J_{n+j-k-1,n,j,k} H_{n+j-k-1,n} (z, \bar{z}). \]  

(3.3)

The occurring integrals are clearly convergent. For the explicit computation of \( J_{n+j-k-1,n,j,k} \) can be handled by distinguishing two cases \( j \geq k+1 \) and \( j \leq k+1 \), and making appeal to the integral formula for the product of two confluent hypergeometric function [29, p. 293], and the Gauss’s theorem giving the special value of the Gauss hypergeometric function \( 2 F_1 \) at \( x = 1 \). Indeed, we have

\[ J_{n+j-k-1,n,j,k} = -\frac{c_{n+j-k-1,n} c_{j-1,k}}{(n+j-k-1)! n!} \int_0^\infty t^{j-k-1} R_{n+j-k,n,j,k}(t) e^{-2t} dt \]

\[ = -\frac{c_{n+j-k-1,n} c_{j-1,k}}{(n+j-k-1)! n!} \frac{\Gamma(|k+1-j|+1)}{2^{n+j-k-1} |n+j-k+1|+1} \]

\[ \times 2 F_1 \left( \begin{array}{l}
-(n+j-k-1) \wedge n, -(j-1) \wedge k \\
|k+1-j|+1 \\
\end{array} \right) \]

\[ = \frac{(-1)^{n+k+1} \Gamma(j+n)}{2^{n+j} n! \Gamma(-k+j+n)}. \]

Finally, inserting this in (3.3) yields (3.1). \( \square \)

The following result describes the range \( R^\ell_n = P_n C_{\mu}(\mathbb{A}^2_\ell) \) of the restriction of \( P_n C_{\mu} \) to the true poly-Bargmann space \( \mathbb{A}^2_\ell \) for given nonnegative integer \( \ell \). We also consider \( \tilde{R}^\ell_n = P_n C_{\mu}(\tilde{A}^2_\ell) \), where \( \tilde{A}^2_\ell := \text{Span}(\mathbb{H}_{1,n}; n = 0, 1, 2, \ldots) \).

**Theorem 3.2.** The following assertions hold true:

i) The space \( R^\ell_n \) is an infinite vector space spanned by \( H_{n+j-\ell-1,n}; j \geq \max(0, \ell+1-n) \).

ii) The space \( \tilde{R}^\ell_n \) is trivial, \( \tilde{R}^\ell_n = \{0\} \).

iii) For \( \ell + n > 0 \), \( \tilde{R}^\ell_n \) is a finite dimensional vector space of dimension \( n + \ell \). An orthogonal basis of \( \tilde{R}^\ell_n \) is given by \( H_{k,n}(z, \bar{z}); k = 0, 1, \ldots, n + \ell - 1 \).

**Proof.** Starting from the expansion in (2.6) for for given \( f \in L^2_{\mu} \), we can write

\[ P_n C_{\mu} f(z) = \sum_{j,k=0}^{\infty} P_n(C_{\mu} H_{j,k})(z) \alpha_{j,k}. \]

Thus, by rewriting the double summation as

\[ \sum_{j,k=0}^{\infty} = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} + \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} + \sum_{j=0}^{\infty} \sum_{k=0}^{\infty}, \]

we get from Proposition (3.1)

\[ P_n C_{\mu} f(z) = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} (-1)^{n+k+1} \frac{(j+n)}{2^{n+j} n! \Gamma(n+j-k)} H_{n+j-k-1,n}(z, \bar{z}) \alpha_{j,k}. \]  

(3.4)
So that for $f \in \mathcal{A}^{2}_{\ell}$, we have $\alpha_{j,k} = 0$ for every $k \neq \ell$ and the previous expression reduces further to
\[
P_{\mu}C_{\mu}f(z) = \sum_{j=0}^{\infty} \frac{(-1)^{n+\ell+1}\Gamma(j+n)}{2^{n+jn!\Gamma(n+j-\ell)}} H_{n+j-\ell-1,n}(z, \overline{z})\alpha_{j,\ell}.
\]
This proves (i).

Similarly, for the case of $f \in \mathcal{A}^{2}_{\ell}$, we have $\alpha_{j,k} = 0$ for every $j \neq \ell$, so that (3.5) gives rise to $P_{\mu}C_{\mu}f(z) = 0$ when $\ell = n = 0$, and therefore
\[
P_{\mu}C_{\mu}f(z) = \sum_{k=0}^{n+\ell-1} \frac{(-1)^{n+k+1}\Gamma(\ell+n)}{2^{n+kn!\Gamma(n+k-\ell)}} H_{n+\ell-1,n}(z, \overline{z})\alpha_{\ell,k}
\]
whenever $n + \ell > 0$. For any $\ell$ such that $n + \ell > 0$, the $k$ such that $k \leq n + \ell - 1$ satisfy $n + \ell - 1 - k \geq 0$. \hfill $\Box$

**Remark 3.3.** The spaces $R^{\ell}_{n}$ (resp. $\tilde{R}^{\ell}_{n}$); $n, \ell = 0, 1, 2, \cdots$, are pairwise orthogonal with respect to $n$, independently of $\ell$, and form a decreasing (resp. increasing) sequence of in $\ell$, for fixed $n$. Indeed, we have $R^{\ell}_{n} \supset R^{\ell+1}_{n}$ and $\tilde{R}^{\ell}_{n} \subset \tilde{R}^{\ell+1}_{n}$.

The characterization of the full range of $C_{\mu}$ on $L^{2}_{\mu}$ by means of their $k$-Bergman projections $P_{\mu}$ requires further investigations. However, by considering the spaces
\[
E^{+}_{j} = \text{span}\{\psi_{n,n+j}; n = 0, 1, 2, \cdots\}
\]
and
\[
E^{-}_{j} = \text{span}\{\psi_{n+j,n}; n = 0, 1, 2, \cdots\},
\]
for given nonnegative integer $j$, we can prove the spaces $E_{\ell}$, for varying integer $\ell = \cdots, -2, -1, 0, 1, 2, \cdots$, defined by $E_{\ell} = E^{+}_{|\ell|}$ for $\ell \geq 0$ and $E^{\ell}_{\ell} = E^{+}_{|\ell|}$ when $\ell < 0$, form an orthogonal Hilbertian decomposition of the range of the weighted Cauchy transform $C_{\mu}$ in $L^{2}_{\mu}$.

**Theorem 3.4.** We have
\[
C_{\mu}(L^{2}_{\mu}) = \bigoplus_{\ell \in \mathbb{Z}} E_{\ell}.
\]

**Proof.** We begin by showing that the spaces $E_{\ell}$ are mutually orthogonal in $L^{2}_{\mu}$. By direct computation, using (1.2), (2.3) and Fubini’s theorem, infers
\[
\langle \psi_{m,n}, \psi_{j,k} \rangle = \int_{\mathbb{C}} \overline{\psi_{m,n}}(z)\psi_{j,k}(z)e^{-|z|^{2}}d\lambda(z)
\]
\[
= \frac{c_{m-1,n}c_{j-1,k}}{2} A_{m,n,j,k} \int_{0}^{\infty} \frac{t^{m+n-1}}{\min(m-1,n+\min(j-1,k))} R_{m,n,j,k}(t)e^{-3t}dt,
\]
where $c_{m,n}$ and $R_{m,n,j,k}$ are as in (2.5) and (3.2), respectively. This shows that the orthogonality of the system $(\psi_{m,n})_{m,n}$ in $L^{2}_{\mu}$ is equivalent to the nullity of the angular part given by $A_{m,n,j,k} = 2\pi \delta_{m-j-k,0}$. Thus, for $m - j = \ell \neq n - k = \ell'$, we have $\langle \psi_{m,n}, \psi_{j,k} \rangle_{H} = 0$. This proves in particular that the $E_{\ell}$; $\ell \in \mathbb{Z}$, form an orthogonal sequence in $L^{2}_{\mu}$. Subsequently, from general theory of functional analysis, the orthogonal sum $\bigoplus_{\ell \in \mathbb{Z}} E_{\ell}$ is a closed subspace of $L^{2}_{\mu}$. The inverse inclusion is clear. \hfill $\Box$

**References**

[1] Ali S.T., Bagarello F., Gazeau J-P., Quantizations from reproducing kernel spaces. Ann. Physics 332 (2013), 127–142.

[2] Ali S.T., Bagarello F., Honnouvo G., Modular structures on trace class operators and applications to Landau levels, J. Phys. A, 43, no. 10 (2010) 105202, 17 pp.

[3] Anderson J.M., Hinkkanen A., The Cauchy transform on bounded domains. Proc. Amer. Math. Soc. 107 (1989) no. 1, 179–185.
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[4] Anderson J.M., Khavinson D., Lomonosov V., Spectral Properties of Some Integral Operators Arising in Potential Theory. The Quarterly Journal of Mathematics, 43(4) (1992) 387-407.

[5] Arazy J., Khavinson, D., Spectral estimates of Cauchy’s transform in $L^2(\mathbb{C})$. Integral Equations and Operator Theory, 15(6) (1992) 901-919.

[6] Askour N., Intissar A., Mouayn Z., Espaces de Bargmann généralisés et formules explicites pour leurs noyaux reproduisants. C. R. Acad. Sci. Paris Sér. I Math. 325 (1997), no. 7, 707–712.

[7] Askour N., Intissar A., Mouayn Z., Explicit formulas for reproducing kernels of generalized Bargmann spaces of $C^\infty$. J. Math. Phys. 41 (2000) no. 5, 3057–3067.

[8] Barrett M.J., Nonlinear analysis of travelling wave tube amplifiers using complex Hermite polynomials. Preprint 1990.

[9] Benahmadi A., El Hamyani A., Ghanmi A., $S$–polyregular Bargmann spaces. Adv. Appl. Clifford Algebr. 29 (2019) no. 4. Paper No. 84, 30 pp.

[10] Bell S.R., The Cauchy transform, potential theory and conformal mapping. Second edition. Chapman & Hall/CRC, Boca Raton, FL, 2016.

[11] Brennan J.E., The Cauchy integral and certain of its applications, J. Contemp. Math. Anal., 39 (2004) 2-49.

[12] Brychkov Y.A, Handbook of Special Functions: Derivatives, Integrals, Series and Other Formulas. CRC Press, Boca Raton, FL, 2008.

[13] Cima J. A., Matheson A., AThESON, Ross W.T., The Cauchy Transform, Surveys and Monographs of the AMS 125, AMS, Providence, RI, 2006.

[14] Dallinger R., Ruotsalainen H., Wichman R., Rupp M., Adaptive pre-distortion techniques based on orthogonal polynomials. In Conference Record of the 44th Asilomar Conference on Signals, Systems and Computers, IEEE (2010) pp 1945-1950.

[15] Dostanić M.R., The properties of the Cauchy transform on a bounded domain, Journal of the Operator Theory 36 (1996) 233-247

[16] Dostanić M.R., Spectral properties of the Cauchy transform in $L^2(\mathbb{C},d\mu)$. Q. J. Math. 51 (2000), no. 3, 307–312.

[17] Gamelin T., Wolff’s proof of the corona theorem. Israel J. Math. 37 (1980) 113–119.

[18] Ghanmi A., A class of generalized complex Hermite polynomials. J. Math. Anal. App. 340 (2008), 1395-1406.

[19] Ghanmi A., Operational formulae for the complex Hermite polynomials $H_{p,q}(z,\bar{z})$. Integral Transforms Spec. Funct. 2013; 24 (11):884-895.

[20] Ghanmi A., Mehler’s formulas for the univariate complex Hermite polynomials and applications. Math. Methods Appl. Sci.

[21] Hörmander L., An introduction to complex analysis in several variables. Third edition. North-Holland Mathematical Library, 7. North-Holland Publishing Co., Amsterdam, 1990.

[22] Hruscev S.V., Vinogradov S.A., Inner functions and multipliers of Cauchy type integrals, Ark. Mat., 19 (1981) 23-42.

[23] Intissar.A, Intissar.A, Spectral properties of the Cauchy transform on $L^2(\mathbb{C},e^{-|z|^2}d\lambda(z))$. J. Math. anal. Appl. 313, no 2 (2006) 400-418.

[24] Ismail M.E.H., Analytic properties of complex Hermite polynomials. Trans. Amer. Math. Soc. 368 (2016), no. 2, 1189-1210.

[25] Ismail M.E.H., Simeonov P., Complex Hermite polynomials: their combinatorics and integral operators. Proc. Amer. Math. Soc. 143 (2015), no. 4, 1397–1410.

[26] Itô K., Complex multiple Wiener integral. Jap. J. Math., 22 (1952) 63-86.

[27] Jones P. W., $L^\infty$ estimates of the $\partial$-problem in a half-space. Acta Math. 150 (1983) 137-152.

[28] Kalaj D., Cauchy transform and Poisson’s equation. Advances in Mathematics, 231(1) (2012) 213-242.

[29] Magnus W., Oberhettinger F., Soni R.P., Formulas and Theorems in the Special Functions of Mathematical Physics. Springer -Verlag, Berlin, 1966

[30] Matsumoto H., Ueki N., Spectral analysis of Schrödinger operators with magnetic fields. J. Funct. Anal. (1) 140 (1996) 218–255.

[31] Raich R., Zhou G., Orthogonal polynomials for complex Gaussian processes. IEEE Trans. Signal Process., vol. 52 (2004) no. 10, pp. 2788-2797.

[32] Shigekawa I., Eigenvalue problems of Schrödinger operator with magnetic field on compact Riemannian manifold, J. Funct. Anal. 75 (1987) 92-127.

[33] Tolsa X., $L^2$-boundedness of the Cauchy integral operator for continuous measures, Duke Mathematical Journal 98 (2) (1999), 269-304

[34] Vasilevski N.L., Poly-Fock spaces. Oper. Theory, Adv. App. 117 (2000) 371–386.

[35] Wünsche A., Laguerre 2D-functions and their application in quantum optics, J. Phys. A 31 (1998), no. 40, 8267-8287

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