Remarks on Type I Blow-Up for the 3D Euler Equations and the 2D Boussinesq Equations

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Abstract
In this paper, we derive kinematic relations for quantities involving the rate of strain tensor and the Hessian of the pressure for solutions of the 3D Euler equations and the 2D Boussinesq equations. Using these kinematic relations, we prove new blow-up criteria and obtain conditions for the absence of type I singularity for these equations. We obtain both global and localized versions of the results. Some of the new blow-up criteria and type I conditions improve previous results of Chae and Constantin (Int Math Res Notices rnab014, 2021).

Keywords Euler equations · Boussinesq equations · Kinematic relations · Blow-up criterion · Type I singularity

Mathematics Subject Classification 35Q31 · 76B03

1 The 3D Euler Equations
1.1 Introduction and Main Results

We consider the homogeneous incompressible Euler equations on \( \mathbb{R}^3 \times [0, T) \).

\[
\begin{align*}
(E) \quad \begin{cases}
  u_t + u \cdot \nabla u &= -\nabla p, \\
  \nabla \cdot u &= 0, \\
  u(x, 0) &= u_0(x)
\end{cases}
\end{align*}
\]

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where \( u(x, t) = (u_1(x, t), u_2(x, t), u_3(x, t)) \) is the fluid velocity and \( p = p(x, t) \) is the scalar pressure. We denote the initial velocity by \( u_0(x) = u(x, 0) \) where \( x \in \mathbb{R}^3 \). For the Cauchy problem of the system (E) with \( u_0 \in W^{2,q}(\mathbb{R}^3), q > 3, \nabla \cdot u_0 = 0 \), there exists a local in time well-posedness result (Kato and Ponce 1988), but the question of the finite time blow-up is a wide open problem. See, e.g., Majda and Bertozzi (2002), Constantin (1994, 2007, 2017) and the references therein for detailed discussions of the problem. For important partial results, we refer Beale et al. (1984); Constantin et al. (1996). See also Kerr (1993), Luo and Hou (2019) and references therein for related numerical works.

We associate with a solution \((u, p)\) of the Euler system (E) the \( \mathbb{R}^3 \times \mathbb{R}^3 \)-valued functions \( S = (S_{ij}) \) and \( P = (P_{ij}) \), where

\[
S_{ij} = \frac{1}{2} (\partial_i u_j + \partial_j u_i), \quad P_{ij} = \partial_i \partial_j p.
\]

For the vorticity \( \omega = \nabla \times u \), we define the direction vectors

\[
\xi = \omega / |\omega|, \quad \zeta = S\xi / |S\xi|,
\]

and the scalar functions

\[
\alpha = S_{ij} \xi_i \xi_j, \quad \rho = P_{ij} \xi_i \xi_j,
\]

where we used the convention of summing over repeated indices. In the case \( \omega(x, t) = 0 \), we set \( \alpha(x, t) = \rho(x, t) = 0 \). These quantities have been introduced previously (Constantin 1994; Majda and Bertozzi 2002; Chae 2007). Note that \( \xi \) is the vorticity direction vector, while \( \zeta \) is the vorticity stretching direction vector. Below we also use the notations \( [f]_+ = \max\{f, 0\} \) and \( [f]_- = \max\{-f, 0\} \).

**Theorem 1.1** Let \((u, p) \in C^1(\mathbb{R}^3 \times (0, T))\) be a solution of the Euler equation (E) with \( u \in C([0, T); W^{2,q}(\mathbb{R}^3)) \), for some \( q > 3 \). If either

(i)

\[
\int_0^T \exp \left( \int_0^t \int_0^s \|[\zeta \cdot P\xi]_-(\tau)\|_{L^\infty} d\tau ds \right) dt < +\infty,
\]

or

\[
\int_0^T \exp \left( \int_0^t \int_0^s \|[S\xi]^2 - 2\alpha^2 - \rho\]_+(\tau)\|_{L^\infty} d\tau ds \right) dt < +\infty,
\]

then \( \limsup_{t \to T} \|u(t)\|_{W^{2,q}} < +\infty \).

(ii) If either

\[
\limsup_{t \to T} (T - t) \|[\zeta \cdot P\xi]_-(t)\|_{L^\infty} < 1,
\]

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or
\[
\limsup_{t \to T} (T - t)^2 \| [S\xi]^2 - 2\alpha^2 - \rho \|_{L^\infty} < 1, \tag{1.3}
\]
then \( \limsup_{t \to T} \| u(t) \|_{W^{2,q}} < +\infty. \)

**Remark 1.1** In Chae and Constantin (2021), we obtained the above theorem with \([\zeta \cdot P\xi]_-\) replaced by \(|P|\), which is the matrix norm of the Hessian of the pressure. Since 
\(|[\zeta \cdot P\xi]_-| \leq |P|\) the above (and the localized version below) improve the results of Theorem 1.1 of Chae and Constantin (2021). Furthermore, the above theorem implies that the dynamical changes of the signs of the scalar quantities \(\zeta \cdot P\xi\) and \(|S\xi|^2 - 2\alpha^2 - \rho\) are important in the phenomena of blow-up/regularity.

The following is a localized version of the above theorem.

**Theorem 1.2** Let \((u, p) \in C^1(B(x_0, r) \times (T - r, T))\) be a solution to (E) with \(u \in C([T - r, T); W^{2,q}(B(x_0, r))) \cap L^\infty(T - r, T; L^2(B(x_0, r)))\) for some \(q \in (3, \infty)\). We suppose
\[
\int_{T - r}^T \| u(t) \|_{L^\infty(B(x_0, r))} \, dt < +\infty.
\]
If either

(i)
\[
\int_{T - r}^T \exp \left( \int_0^t \int_0^s \|[\zeta \cdot P\xi]_-\|_{L^\infty(B(x_0, r))} \, d\tau \, ds \right) \, dt < +\infty,
\]
or
\[
\int_{T - r}^T \exp \left( \int_{T - r}^t \int_{T - r}^s \| |S\xi|^2 - 2\alpha^2 - \rho \|_{L^\infty(B(x_0, r))} \, d\tau \, ds \right) \, dt < +\infty,
\]
then for all \(\varepsilon \in (0, r)\) \( \limsup_{t \to T} \| u(t) \|_{W^{2,q}(B(x_0, \varepsilon))} < +\infty. \)

(ii) If (1.2) holds, and if either
\[
\limsup_{t \to T} (T - t)^2 \|[\zeta \cdot P\xi]_-\|_{L^\infty(B(x_0, r))} < 1, \tag{1.4}
\]
or
\[
\limsup_{t \to T} (T - t)^2 \| |S\xi|^2 - 2\alpha^2 - \rho \|_{L^\infty(B(x_0, r))} < 1, \tag{1.5}
\]
then for all \(\varepsilon \in (0, r)\) \( \limsup_{t \to T} \| u(t) \|_{W^{2,q}(B(x_0, \varepsilon))} < +\infty. \)
1.2 Kinematic Relations

We use the particle trajectory mapping \( a \mapsto X(a, t) \) from \( \mathbb{R}^3 \) into \( \mathbb{R}^3 \) generated by \( u = u(x, t) \), which means the solution of the ordinary differential equation,

\[
\begin{cases}
\frac{\partial X(a, t)}{\partial t} = u(X(a, t), t) & \text{on} \ (0, T), \\
X(a, 0) = a \in \mathbb{R}^3.
\end{cases}
\]

The material derivative of \( f = f(x, t) \) is defined by

\[
Dt f := \partial_t f + u \cdot \nabla f.
\]

We note that \((Dt f)(X(a, t), t)) = \partial_t \{ f(X(a, t), t) \} \).

**Proposition 1.1** Let \((u, p)\) be a solution of (E), which belongs to \( C^1(\mathbb{R}^3 \times (0, T))\). We use the above notations. Then, the followings hold true on \( \mathbb{R}^3 \times (0, T) \).

\[
\begin{align*}
&D_t |S\omega| = -\zeta \cdot P\omega, \quad (1.6) \\
&D_t^2 \log |\omega| = |S\xi|^2 - 2\alpha^2 - \rho, \quad (1.7) \\
&(D_t |\omega|)^2 + (|D_t \xi||\omega|)^2 = |S\omega|^2, \quad (1.8) \\
&(D_t |S\omega|)^2 + (|D_t \xi||S\omega|)^2 = |P\omega|^2. \quad (1.9)
\end{align*}
\]

**Remark 1.2** Applying the inequality, \( a_1 + \cdots + a_n \leq \sqrt{n(a_1^2 + \cdots + a_n^2)} \) to equations (1.8), (1.9) and (1.13), respectively, we obtain the following differential inequalities with the coefficients consisting of derivatives of the direction fields \( \xi \) and \( \zeta \),

\[
\begin{align*}
&D_t |\omega| + |D_t \xi||\omega| \leq \sqrt{2}|S\omega|, \quad (1.10) \\
&D_t |S\omega| + |D_t \xi||S\omega| \leq \sqrt{2}|P\omega|, \quad (1.11) \\
&D_t |S\omega| + |D_t \xi|D_t |\omega| + |D_t \xi||D_t \xi||\omega| \leq \sqrt{3}|P\omega|. \quad (1.12)
\end{align*}
\]

For an implication of (1.10) combined with (1.11), in particular, see Remark 1.3.

**Proof of Proposition 1.1** Taking the gradient of (E), we find

\[
D_t \nabla u = -(\nabla u)^2 - P. \quad (1.13)
\]

We observe the decomposition of the matrix,

\[
\nabla u = S + \Omega, \quad \text{where} \quad \Omega_{ij} = \frac{1}{2}(\partial_i u_j - \partial_j u_i) = \frac{1}{2}\epsilon_{ijk} \omega_k.
\]

Here, \( \epsilon_{ijk} \) is the totally skew-symmetric tensor with normalization \( \epsilon_{123} = 1 \). Taking the skew-symmetric part of (1.13), we obtain the vorticity equations

\[
D_t \omega = \omega \cdot \nabla u = S\omega, \quad (1.14)
\]
where we used the fact $\omega_j \partial_j u_i = \omega_j S_{ji} + \frac{1}{2} \epsilon_{jik} \omega_j \omega_k = \omega_j S_{ji}$. Taking the symmetric part of (1.13), on the other hand, we find

$$D_t S = -S^2 + \frac{1}{4} (|\omega|^2 I - \omega \otimes \omega) - P. \quad (1.15)$$

Contracting (1.14) with $\omega$, and dividing the both sides by $|\omega|^2$, we have

$$D_t |\omega| = \alpha |\omega|. \quad (1.16)$$

From (1.14) and (1.16), we derive

$$D_t \xi = D_t \omega - \omega \frac{D_t |\omega|}{|\omega|^2} = S\xi - \alpha \xi. \quad (1.17)$$

Applying $D_t$ to (1.14), using (1.15), we find

$$D_t^2 \omega = (D_t S)\omega + S D_t \omega = -S^2 \omega - P \omega + S^2 \omega$$

$$= -P \omega, \quad (1.18)$$

which was the key kinematic relation used in Chae and Constantin (2021). Multiplying (1.18) by $D_t \omega = S\omega$ from the left, we obtain

$$|D_t \omega| D_t |D_t \omega| = \frac{1}{2} D_t |D_t \omega|^2 = D_t \omega \cdot D_t^2 \omega = -S \omega \cdot P \omega.$$

Dividing the both sides by $|D_t \omega| = |S \omega|$, we find

$$D_t |D_t \omega| = D_t |S \omega| = -\xi \cdot P \omega, \quad (1.19)$$

and (1.6) is proved. Now we prove (1.7). Observing $\xi \cdot D_t \xi = 0$, we compute

$$D_t^2 |\omega| = D_t \{\xi \cdot D_t (|\omega|\xi)\} = D_t \xi \cdot D_t \omega + \xi \cdot D_t^2 \omega$$

$$= (S - \alpha I) \xi \cdot S \omega - \xi \cdot P \omega = \left(|S\xi|^2 - \alpha^2 - \rho\right)|\omega|. \quad (1.20)$$

We divide (1.20) by $|\omega|$, then using (1.16), we deduce

$$|S\xi|^2 - \alpha^2 - \rho = \frac{D_t^2 |\omega|}{|\omega|} = D_t \left( \frac{D_t |\omega|}{|\omega|} \right) + \frac{(D_t |\omega|)^2}{|\omega|^2} = D_t^2 \log |\omega| + \alpha^2.$$

Formula (1.7) is proved. Taking the square of (1.17), and multiplying it by $|\omega|^2$, we have

$$|S\omega|^2 = \alpha^2 |\omega|^2 + |D_t \xi|^2 |\omega|^2 = (D_t |\omega|)^2 + |D_t \xi|^2 |\omega|^2, \quad (1.21)$$
and (1.8) is proved. To show (1.9), we compute, using (1.18) and (1.19),

\[
D_t \zeta = \frac{D_t^2 \omega}{|D_t \omega|} - \frac{D_t \omega D_t (|D_t \omega|)}{|D_t \omega|^2} = -\frac{P \omega}{|S \omega|} + \frac{S \omega (\zeta \cdot P \xi) |\omega|}{|S \omega|^2}.
\]

Because \( D_t \zeta \) is perpendicular to \( \zeta \), in view of the fact that \( \zeta \) has unit length, this yields an orthogonal decomposition of \( P \xi \),

\[
P \xi = (\zeta \cdot P \xi) \zeta - |S \xi| D_t \zeta = (\zeta \cdot P \xi) \zeta + \frac{(D_t \xi \cdot P \xi)}{|D_t \xi|^2} D_t \zeta,
\]

which implies

\[
\frac{D_t \zeta}{|D_t \zeta|} \cdot P \xi = -|S \xi| |D_t \zeta|.
\]

The decomposition (1.22), combined with (1.19) and (1.23), implies by the Pythagoras theorem

\[
|P \omega|^2 = (\zeta \cdot P \omega)^2 + \left( \frac{D_t \xi}{|D_t \xi|} \cdot P \omega \right)^2 = (D_t |S \omega|)^2 + |D_t \xi|^2 |S \omega|^2.
\]

The inequality (1.9) follows from this immediately. Substituting (1.21) into (1.24), we have (1.10).

\[\square\]

### 1.3 Proofs of the Main Theorems

In order to prove Theorem 1.1, we shall use the following lemma.

**Lemma 1.1** Let \( \alpha = \alpha(t) \) be a non-decreasing function, and \( \beta = \beta(t) \geq 0 \) on \([a, b]\).

(i) Suppose \( y = y(t) \) satisfies

\[
y(t) \leq \alpha(t) + \int_a^t \beta(\tau) y(\tau) d\tau \quad \forall t \in [a, b].
\]

Then, for all \( t \in (a, b) \) we have

\[
y(t) \leq \alpha(t) \exp\left( \int_a^t \beta(\tau) d\tau \right).
\]
(ii) Here, we assume additionally that \( y(t) \geq 0 \) on \([a, b]\). Suppose \( y(t) \leq \alpha(t) + \int_a^t \int_a^s \beta(\tau) y(\tau) d\tau d\tau \forall t \in [a, b] \).

Then, for all \( t \in (a, b) \) we have

\[
y(t) \leq \alpha(t) \exp \left( \int_a^t \int_a^s \beta(\tau) d\tau d\tau \right).
\]

**Proof** In the case (i) from the well-known Gronwall inequality and the assumption of non-decreasing property of \( \alpha \), we have

\[
y(t) \leq \alpha(t) + \int_a^t a(s)\beta(s) \exp \left( \int_s^t \beta(\tau) d\tau \right) ds \leq \alpha(t) + \alpha(t) \int_a^t \beta(s) \exp \left( \int_s^t \beta(\tau) d\tau \right) ds
\]

\[
= \alpha(t) - \alpha(t) \left. \frac{d}{ds} \exp \left( \int_s^t \beta(\tau) d\tau \right) \right|_s^t = \alpha(t) \exp \left( \int_a^t \beta(\tau) d\tau \right) d\tau.
\]

For the case (ii), we observe

\[
y(t) \leq \alpha(t) + \int_a^t \int_0^s \beta(\tau) y(\tau) d\tau d\tau \leq \alpha(t) + \int_a^t \sup_{a \leq \tau < s} y(\tau) \int_a^s \beta(\tau) d\tau d\tau.
\]

Since the function \( t \mapsto \alpha(t) + \int_a^t \sup_{a \leq \tau < s} y(\tau) \int_a^s \beta(\tau) d\tau d\tau \) is non-decreasing on \([a, b]\), setting \( h(t) = \int_a^t \beta(s) ds \) and \( Y(t) = \sup_{a \leq \tau < t} y(\tau) \), we have

\[
Y(t) \leq \alpha(t) + \int_a^t Y(s) h(s) ds.
\]

Applying (i), we obtain

\[
y(t) \leq Y(t) \leq \alpha(t) \exp \left( \int_a^t h(s) ds \right) = \alpha(t) \exp \left( \int_a^t \int_a^s \beta(\tau) d\tau d\tau \right)
\]

for all \( t \in [a, b] \). \( \square \)

**Proof of Theorem 1.1 and Theorem 1.2** We integrate (1.6) along the trajectory for \( t \in [0, s] \) to find

\[
\frac{\partial}{\partial s} [\omega(X(a, s), s)] \leq \frac{\partial}{\partial s} \omega(X(a, s), s) = |(D_s \omega)(X(a, s), s)| = |S \omega(X(a, s), s)|
\]

\[
= |S_0(a) \omega_0(a)| - \int_0^s (\xi \cdot P \xi)(X(a, \tau), \tau) |\omega(X(a, \tau), \tau) d\tau,
\]

from which, after integrating with respect to \( s \) over \([0, t]\), we have
\[ |\omega(X(a, t), t)| \leq |\omega_0(a)| + |S_0(a)\omega_0(a)|t + \int_0^t \int_0^s [\zeta \cdot P_\xi]_-(X(a, \tau), \tau)|\omega(X(a, \tau), \tau)|d\tau ds. \]

Applying Lemma 2.1(ii) to solve this differential inequality, we find
\[ |\omega(X(a, t), t)| \leq (|\omega_0(a)| + |S_0(a)\omega_0(a)|t) \times \exp \left( \int_0^t \int_0^s [\zeta \cdot P_\xi]_-(X(a, \tau), \tau) d\tau ds \right). \] (1.25)

Taking the supremum over \( a \in \mathbb{R}^3 \), and integrating it with respect to \( t \) over \([0, T]\), we find
\[ \int_0^T \|\omega(t)\|_{L^\infty} dt \leq (\|\omega_0\|_{L^\infty} + \|S_0\omega_0\|_{L^\infty} T) \times \int_0^T \exp \left( \int_0^t \int_0^s \|\zeta \cdot P_\xi\|_{L^\infty} d\tau ds \right) dt. \] (1.26)

Integrating (1.7) twice with respect to the time variable over \([0, s]\), we have
\[ |\omega(X(a, t), t)| = |\omega_0(a)| \exp \left( \int_0^t \int_0^s |S_\xi|^2 - 2\alpha^2 - \rho \right)_+(X(a, \tau), \tau) d\tau ds, \] (1.27)
and therefore,
\[ \int_0^T \|\omega(t)\|_{L^\infty} dt \leq \|\omega_0\|_{L^\infty} \int_0^T \exp \left( \int_0^t \int_0^s \|\zeta \cdot P_\xi\|_{L^\infty} d\tau ds \right) dt. \]

Applying the well-known Beale–Kato–Majda criterion (Beale et al. 1984) to (1.26) and (1.27), we obtain the desired conclusion of Theorem 1.1(i). The argument of proof of Theorem 1.1(ii), using the result of (i) is the similar to Chae and Constantin (2021), and we omit it here.

The proof of Theorem 1.2, using the key pointwise estimates of the vorticity along the trajectories, (1.25) and (1.26) is similar to the corresponding ones in Chae and Constantin (2021), and we do not repeat it here.

\[ \square \]

Remark 1.3 The linear differential inequalities (1.10) and (1.11) along the trajectory can be solved as
\[ |\omega(X(a, t), t)| \leq |\omega_0(a)| e^{-\int_0^t |D_\tau \xi(X(a, s), s)| ds} \]
\[ + \sqrt{2} \int_0^t |S_\omega(X(a, s), s)| e^{-\int_s^t |D_\tau \xi(X(a, \tau), \tau)| d\tau} ds, \] (1.28)
and
\[
|S\omega(X(a, t), t)| \leq |S_0\omega_0(a)|e^{-\int_0^t |D_t\zeta(X(a,s), s)|ds} \\
+ \sqrt{2} \int_0^t |P\xi(X(a, s), s)||\omega(X(a, s), s)|e^{-\int_0^s |D_t\zeta(X(a,\tau), \tau)|d\tau} ds,
\]
respectively. Parenthetically one can also use (1.10) to deduce
\[
D_t|S\omega| + |D_t\xi||D_t\xi||\omega| \leq \sqrt{2}|P\omega|,
\]
and then,
\[
|S\omega(X(a, t), t)| \leq |S_0\omega_0(a)|e^{-\int_0^t |D_t\zeta(X(a,s), s)||D_t\xi(X(a,s), s)|ds} \\
+ \sqrt{2} \int_0^t |P\xi(X(a, s), s)||\omega(X(a, s), s)|e^{-\int_0^s |D_t\zeta(X(a,\tau), \tau)||D_t\xi(X(a,\tau), \tau)|d\tau} ds
\]
instead of (1.29). Inserting (1.29) into (1.28), we find
\[
|\omega(X(a, t), t)| \leq |\omega_0(a)| + \sqrt{2}|S_0(a)\omega_0(a)| \int_0^t e^{-\int_0^s |D_t\zeta|d\tau} e^{-\int_0^s |D_t\xi|d\tau} ds \\
+ 2 \int_0^t \int_0^s |P\xi(X(a, \sigma), \sigma)||\omega(X(a, \sigma), \sigma)|e^{-\int_0^\sigma |D_t\zeta|d\tau} e^{-\int_0^\sigma |D_t\xi|d\tau} d\sigma ds.
\]
Applying Lemma 1.1 to (1.30), we obtain
\[
|\omega(X(a, t), t)| \leq \left(|\omega_0(a)| + \sqrt{2}|S_0(a)\omega_0(a)|t\right) \\
\times \exp\left(2 \int_0^t \int_0^s |P\xi(X(a, \sigma), \sigma)|e^{-\int_0^\sigma |D_t\zeta(X(a,\tau), \tau)|d\tau} e^{-\int_0^\sigma |D_t\xi(X(a,\tau), \tau)|d\tau} d\sigma ds\right).
\]
Since the quantities expressing the magnitudes of the changes of the two direction vectors $D_t\xi$ and $D_t\zeta$ contribute to the integral in the right-hand side of (1.31) through factors like $e^{-\int_0^s |D_t\zeta(X(a,\tau), \tau)|d\tau}$, they appear to have a desingularizing effect for the vorticity. We do not know, however, a way to exploit this effect in the blow-up criterion and the absence of the type I blow-up. If we ignore the factor $e^{-\int_0^s |D_t\zeta(X(a,\tau), \tau)|d\tau}$ in (1.31), taking supremum over $a \in \mathbb{R}^3$, and integrating over $t \in [0, T]$, then we have an estimate
\[
\int_0^T \|\omega(t)\|_{L^\infty} dt \leq \left(\|\omega_0\|_{L^\infty} + \sqrt{2}\|S_0\omega_0\|_{L^\infty} T\right) \times \\
\times \int_0^T \exp\left(2 \int_0^t \int_0^s \|P\xi(\tau)\|_{L^\infty} d\tau ds\right) dt
\]
which yields a blow-up criterion weaker than (1.1).
2 The 2D Boussinesq Equations

Here we are concerned with the homogeneous incompressible Boussinesq equation on $\mathbb{R}^2$.

\[
\begin{align*}
(B) \quad & \begin{cases} 
    u_t + u \cdot \nabla u = -\nabla p + \theta e_2, \\
    \theta_t + u \cdot \nabla \theta = 0, \\
    \nabla \cdot u = 0,
\end{cases}
\end{align*}
\]

where $u(x, t) = (u_1(x, t), u_2(x, t))$ is the fluid velocity and $p = p(x, t)$ is the pressure, and $\theta = \theta(x, t)$ is the temperature. Let $u_0(x) = u(x, 0), \theta_0(x, 0)$ be the initial data of the system (B). The local well-posedness for the Boussinesq system for $(u_0, \theta_0) \in W^{2, q}(\mathbb{R}^2), q > 2$, is well known (see, e.g., Chae and Nam 1997), but the question of finite time blow-up is a wide open problem similarly to the case of the 3D Euler equations. It is also well known that there exists a strong similarity between (B) and the axisymmetric solution of the 3D Euler equations (see, e.g., Majda and Bertozzi 2002).

For a solution $(u, p, \theta)$ of the system (B), let us introduce the $\mathbb{R}^{2 \times 2}$-valued functions $U = (\partial_i u_j) \text{ and } P = (\partial_i \partial_j p)$. For the vector filed $\nabla \perp \theta = (-\partial_2 \theta, \partial_1 \theta)$, we define the direction vectors

\[ \xi = \nabla \perp \theta / |\nabla \perp \theta|, \quad \zeta = U \nabla \perp \theta / |U \nabla \perp \theta|, \]

and the scalar functions

\[ \alpha = \xi \cdot U \xi, \quad \rho = \xi \cdot P \xi. \]

**Theorem 2.1** Let $(u, p) \in C^1(\mathbb{R}^2 \times (0, T))$ be a solution of the Boussinesq equation (B) with $u \in C([0, T); W^{2, q}(\mathbb{R}^2))$, for some $q > 2$. If either

(i)

\[ \int_0^T (T - t) \exp \left( \int_0^t \int_0^s \| [\xi \cdot P \xi]_-(\tau) \|_{L^\infty} d\tau ds \right) dt < +\infty, \quad (2.1) \]

or

\[ \int_0^T (T - t) \exp \left( \int_0^t \int_0^s \| [U \xi]^2 - 2\alpha^2 - \rho]_+ (\tau) \|_{L^\infty} d\tau ds \right) dt < +\infty, \quad (2.2) \]

then $\lim \sup_{t \to T} \| u(t) \|_{W^{2, q}} < +\infty$.

(ii) If either

\[ \lim \sup_{t \to T} (T - t)^2 \| [\xi \cdot P \xi]_-(t) \|_{L^\infty} < 2, \quad (2.3) \]
or
\[
\limsup_{t \to T} (T - t)^2 \|[U\xi]_2^2 - 2\alpha^2 - \rho \|_{L^\infty} < 2,
\] (2.4)

then \( \limsup_{t \to T} \|u(t)\|_{W^{2,q}} < +\infty \).

**Remark 2.1** Note the relaxed smallness condition of for the nonexistence of type I blow-up in (2.3) and (2.4) compared to (1.2) and (1.3), respectively, in the case of 3D Euler equations. This is due to the extra factor, \((T - t)\) in (2.1) and (2.2), which originate from the non-blow-up criterion, \(\int_0^T (T - t) \|\nabla \theta(t)\|_{L^\infty} dt < +\infty\) in Chae and Wolf (2019a, Theorem 1.2 (ii)).

The following is a localized version of the above theorem.

**Theorem 2.2** Let \((u, p) \in C^1(B(x_0, r) \times (T - r, T))\) be a solution to (E) with \(u \in C([T - r, T); W^{2,q}(B(x_0, r))) \cap L^\infty(T - r, T; L^2(B(x_0, r)))\) for some \(q \in (2, \infty)\). Let us assume
\[
\int_{T - r}^T \|u(t)\|_{L^\infty(B(x_0, r))} dt < +\infty.
\]

If either

(i) \[
\int_{T - r}^T (T - t) \exp \left( \int_0^t \int_0^s \|[\zeta \cdot P\xi]_-(\tau)\|_{L^\infty(B(x_0, r))} d\tau ds \right) dt < +\infty, \tag{2.5}
\]

or
\[
\int_{T - r}^T (T - t) \exp \left( \int_0^t \int_{T - r}^T \|[U\xi]_2^2 - 2\alpha^2 - \rho \|_{L^\infty(B(x_0, r))} d\tau ds \right) dt < +\infty, \tag{2.6}
\]

then for all \(\varepsilon \in (0, r)\) \( \limsup_{t \nearrow T} \|u(t)\|_{W^{2,q}(B(x_0, \varepsilon))} < +\infty \).

(ii) If either
\[
\limsup_{t \to T} (T - t)^2 \|[\zeta \cdot P\xi]_-(t)\|_{L^\infty(B(x_0, r))} < 2, \tag{2.7}
\]

or
\[
\limsup_{t \to T} (T - t)^2 \|[U\xi]_2^2 - 2\alpha^2 - \rho \|_{L^\infty(B(x_0, r))} < 2, \tag{2.8}
\]

then for all \(\varepsilon \in (0, r)\) \( \limsup_{t \nearrow T} \|u(t)\|_{W^{2,q}(B(x_0, \varepsilon))} < +\infty \).
Remark 2.2 Similarly to Remark 2.1, we also note here relaxed smallness condition of for the nonexistence of type I blow-up in (2.7) and (2.8) compared to (1.4) and (1.5), respectively. This is due to the extra factor, $(T-t)$ in (2.5) and (2.6), which are from the local version of the non-blow-up criterion, \( \int_0^T (T-t) \| \nabla \perp \theta(t) \|_{L^\infty(B(x_0, r))} \, dt < +\infty \) in Chae and Wolf (2019b, Theorem 2.1).

2.1 Kinematic Relations

Proposition 2.1 Let \((u, p, \theta)\) be a solution of (B), which belongs to \(C^1(\mathbb{R}^2 \times (0, T))\).

We use the above notations. Then, the followings hold true on \(\mathbb{R}^2 \times (0, T)\).

\[
\begin{align*}
D_t |U \nabla \perp \theta| &= -\zeta \cdot P \nabla \perp \theta, \\
D_t^2 \log |\nabla \perp \theta| &= |U \xi|^2 - 2\alpha^2 - \rho, \\
(D_t|U \nabla \perp \theta|)^2 + (D_t|U \nabla \perp \theta|)^2 &= |U \nabla \perp \theta|^2, \\
(D_t|U \nabla \perp \theta|)^2 + (D_t|U \nabla \perp \theta|)^2 &= |P \nabla \perp \theta|^2, \\
(D_t|U \nabla \perp \theta|)^2 + (D_t|U \nabla \perp \theta|)^2 + (D_t|D_t \nabla \perp \theta|)^2 &= |P \nabla \perp \theta|^2.
\end{align*}
\]

Remark 2.3 Although the above results look similar to those in Proposition 1.1, we have essentially different features, since we do not use the symmetric part of \(U\) and because there exists no relation between \(\nabla \perp \theta\) and the skew-symmetric part of \(U\).

Proof of Proposition 2.1 Taking \(\nabla\) on the first equation of (B), we find

\[
D_t U + U^2 = -P + \nabla(\theta e_2),
\]

Taking \(\nabla \perp\) on the second equation of (B),

\[
D_t \nabla \perp \theta = U \nabla \perp \theta.
\]

Let us compute

\[
D_t^2 \nabla \perp \theta = D_t U \nabla \perp \theta + U D_t \nabla \perp \theta \\
= -U^2 \nabla \perp \theta - P \nabla \perp \theta + U^2 \nabla \perp \theta + \nabla \perp \theta \cdot \nabla(\theta e_2) \\
= -P \nabla \perp \theta,
\]

where we used the fact

\[
\nabla \perp \theta \cdot \nabla(\theta e_2) = 0.
\]

We multiply (2.16) by \(D_t \nabla \perp \theta\) to have

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\[
|D_t \nabla \perp \theta| D_t \nabla \perp \theta = \frac{1}{2} D_t \left( |D_t \nabla \perp \theta|^2 \right) = D_t \nabla \perp \theta \cdot D_t^2 \nabla \perp \theta \\
= -U \nabla \perp \theta \cdot P \nabla \perp \theta.
\]

Dividing the both sides by \(|D_t \nabla \perp \theta| = |U \nabla \perp \theta|\), we find
\[
D_t |D_t \nabla \perp \theta| = D_t |U \nabla \perp \theta| = -\xi \cdot P \xi |\nabla \perp \theta|,
\]
and (2.9) is proved. Multiplying (2.15) by \(\nabla \perp \theta\), we deduce
\[
D_t |\nabla \perp \theta| = \alpha |\nabla \perp \theta|. \tag{2.18}
\]

Using (2.15) and (2.18), we compute
\[
D_t \xi = \frac{D_t \nabla \perp \theta}{|\nabla \perp \theta|} - \frac{\nabla \perp \theta D_t |\nabla \perp \theta|}{|\nabla \perp \theta|^2} = U \xi - \alpha \xi. \tag{2.19}
\]
This can be viewed as an orthogonal decomposition of \(U \xi\),
\[
U \xi = \alpha \xi + D_t \xi = \alpha \xi + \frac{D_t \xi \cdot U \xi}{|D_t \xi|^2} D_t \xi,
\]
which shows
\[
D_t \xi \cdot U \xi = |D_t \xi|^2 = |U \xi|^2 - \alpha^2 = |U \xi|^2 - \frac{(D_t |\nabla \perp \theta|)^2}{|\nabla \perp \theta|^2}. \tag{2.20}
\]

Multiplying the both sides of (2.20) by \(|\nabla \perp \theta|^2\), formula (2.11) follows immediately.

Using (2.14) and (2.19), we compute
\[
D_t^2 \log |\nabla \perp \theta| = D_t \alpha = D_t (\xi \cdot U \xi) \\
= D_t \xi \cdot U \xi + \xi \cdot D_t U \xi + \xi \cdot U D_t \xi \\
= (U \xi - \alpha \xi) \cdot U \xi + \xi \cdot (-U^2 - P) \xi + \xi \cdot U (U \xi - \alpha \xi) \\
= |U \xi|^2 - 2\alpha^2 - \rho,
\]
where we used \(\xi \cdot \nabla (\theta e_2) = 0\), which follows from (2.17). Formula (2.10) is proved.

Using (2.16) and (2.9), we compute
\[
D_t \xi = \frac{D_t^2 \nabla \perp \theta}{|D_t \nabla \perp \theta|} - \frac{D_t \nabla \perp \theta D_t (|D_t \nabla \perp \theta|)}{|D_t \nabla \perp \theta|^2} \\
= -\frac{P \nabla \perp \theta}{|U \nabla \perp \theta|} + \frac{U \nabla \perp \theta (\xi \cdot P \xi |\nabla \perp \theta|)}{|U \nabla \perp \theta|^2} \\
= -\frac{P \xi + (\xi \cdot P \xi) \xi}{|U \xi|}. \tag{2.21}
\]
Formula (2.21) yields an orthogonal decomposition of $P\xi$,

$$P\xi = (\zeta \cdot P\xi)\zeta - |U\xi|D_t\zeta = (\zeta \cdot P\xi)\zeta + \frac{(D_t\zeta \cdot P\xi)}{|D_t\zeta|^2} D_t\zeta,$$

which implies

$$\frac{D_t\zeta \cdot P\xi}{|D_t\zeta|} = -|U\xi||D_t\zeta|.$$

The decomposition (2.22) also implies by the Pythagoras theorem, and then using (2.9) and (2.23),

$$|P\nabla^\perp\theta|^2 = (\zeta \cdot P\nabla^\perp\theta)^2 + \left(\frac{D_t\zeta \cdot P\nabla^\perp\theta}{|D_t\zeta|}\right)^2$$

$$= (D_t|U\nabla^\perp\theta|)^2 + |D_t\zeta|^2|U\nabla^\perp\theta|^2,$$

which verifies (2.12). Inserting the expression of $|U\nabla^\perp\theta|^2$ in (2.12) into (2.13), we obtain (2.14).

### 2.2 Proof of the Main Results

**Proof of Theorem 2.1 and Theorem 2.2** The proof is similar to the case of 3D Euler equations. The main difference is that here we start from the kinematic relations of the Boussinesq equations in Proposition 2.1. Integrating (2.9) along the trajectory for $t \in [0, s]$, we obtain

$$\frac{\partial}{\partial s} |\nabla^\perp\theta(X(a, s), s)| \leq \left| \frac{\partial}{\partial s} \nabla^\perp\theta(X(a, s), s) \right|$$

$$= |(D_s\nabla^\perp\theta)(X(a, s), s)| = |U\nabla^\perp\theta(X(a, s), s)|$$

$$= |S_0(a)\omega_0(a)| - \int_0^t (\zeta \cdot P\xi)(X(a, \tau), \tau)|\omega(X(a, \tau), \tau)|d\tau.$$

After integrating this again with respect to $s$ over $[0, t]$, we find

$$|\nabla^\perp\theta(X(a, t), t)| \leq |\nabla^\perp\theta_0(a)| + |\nabla^\perp\theta_0(a) \cdot \nabla u_0(a)|t$$

$$+ \int_0^t \int_0^s (\zeta \cdot P\xi)_-(X(a, \tau), \tau)|\nabla^\perp\theta(X(a, \tau), \tau)|d\tau ds.$$

Thanks to Lemma 2.1(ii), we find

$$|\nabla^\perp\theta(X(a, t), t)| \leq (|\nabla^\perp\theta_0(a)| + |\nabla^\perp\theta_0 \cdot \nabla u_0(a)(a)|t) \times$$

$$\times \exp \left( \int_0^t \int_0^s (\zeta \cdot P\xi)_-(X(a, \tau), \tau)d\tau ds \right).$$  

(2.24)
Taking the supremum over \( a \in \mathbb{R}^2 \), and integrating it with respect to \( t \) over \([0, T]\) after multiplying by \( T - t \), we find

\[
\int_0^T (T - t) \| \nabla \perp \theta(t) \|_{L^\infty} \, dt \leq (\| \nabla \perp \theta_0(a) \|_{L^\infty} + \| \nabla \perp \theta_0 \cdot \nabla u_0 \|_{L^\infty} T) \times \\
\times \int_0^T (T - t) \exp \left( \int_0^t \int_0^s \| [\xi \cdot \mathcal{P} \xi](\tau) \|_{L^\infty} \, d\tau \, ds \right) \, dt. \tag{2.25}
\]

Integrating (2.10) twice with respect to the time variable over \([0, s]\), we have

\[
|\nabla \perp \theta(X(a, t), t)| \leq |\nabla \perp \theta_0(a)| \exp \left( \int_0^t \int_0^s |U\xi X_2 - 2a^2 - \rho|^+ (X(a, \tau), \tau) \, d\tau \, ds \right), \tag{2.26}
\]

and from which we also deduce

\[
\int_0^T (T - t) \| \nabla \perp \theta(t) \|_{L^\infty} \, dt \leq \| \nabla \perp \theta_0 \|_{L^\infty} \times \\
\times \int_0^T (T - t) \exp \left( \int_0^t \int_0^s \| [|U\xi|^2 - 2a^2 - \rho|^+ ]^+ (\tau) \|_{L^\infty} \, d\tau \, ds \right) \, dt. \tag{2.27}
\]

Applying the blow-up criterion of Chae and Wolf (2019a, Theorem 1.2 (ii)) to (2.25) and (2.27), we obtain the desired conclusion of Theorem 2.1(i). The proof of Theorem 2.1(ii), using the result of (i) is the similar to the one in Chae and Constantin (2021).

The proof of Theorem 2.2, using the key estimates (2.24) and (2.26) is also similar to the corresponding ones in Chae and Constantin (2021). The essential point here is that we apply the local version of the blow-up criterion (Chae and Wolf 2019b, Theorem 2.1).

\[\Box\]

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**Declarations**

**Conflict of interest** The authors declare that they have no conflict of interest.

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