FOCK SPACE REPRESENTATIONS FOR THE QUANTUM AFFINE ALGEBRA $U_q(C_2^{(1)})$

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ABSTRACT. We construct the Fock space representations for the quantum affine algebra of type $C_2^{(1)}$ in terms of Young walls. Using this construction, we give a generalized Lascoux-Leclerc-Thibon algorithm for computing the global bases of the basic representations.

1. INTRODUCTION

The theory of crystal bases for the quantum group $U_q(g)$ associated with a symmetrizable Kac-Moody algebra $g$ was developed by Kashiwara [9, 10, 11]. The crystal bases are bases of $U_q(g)$-modules at $q = 0$ and they contain a lot of important information on $U_q(g)$-modules. For example, they have oriented graph structures, called crystal graphs, which behave very nicely under the tensor product. As a consequence, many problems in representation theory are reduced to those in combinatorics. It is one of the most important problems in the theory of crystal bases to give various realizations of crystals. There have been many works on this problem (see for example, [3, 13, 14, 18, 20, 21]). From crystal bases, Kashiwara also recovered true bases of integrable $U_q(g)$-modules, called the global bases, in a canonical way [3]. These bases were proved to be equal to the canonical bases constructed by Lustzig in a geometric way [3, 19]. Recently, using the global bases of the basic representations of the quantum affine algebra of type $A_n^{(1)}$, Lascoux, Leclerc and Thibon discovered that there is a deep connection between the representation theory of the quantum affine algebras and the Hecke algebras [15] (see also [1, 2]).

In this paper, we focus on the quantum affine algebra of type $C_2^{(1)}$. For a dominant integral weight $\Lambda$ of level 1, let $B(\Lambda)$ be the crystal of the basic representation $V(\Lambda)$. In [1], Hong and Kang gave a realization of $B(\Lambda)$ in terms of Young walls associated with $U_q(C_2^{(1)})$. These are made by building colored blocks on a ground state wall $Y_\Lambda$ following certain patterns and rules. Let $Z(\Lambda)$ be the set of proper Young walls on $Y_\Lambda$, and $Y(\Lambda)$ the set of reduced proper Young walls on $Y_\Lambda$. They gave an affine crystal structure on $Z(\Lambda)$, and then showed that $B(\Lambda)$ is isomorphic to the subcrystal $Y(\Lambda)$ of $Z(\Lambda)$.

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Let
\[ F(\Lambda) = \bigoplus_{Y \in \mathcal{Z}(\Lambda)} \mathbb{Q}(q)Y \]
be the \( \mathbb{Q}(q) \)-vector space with a basis \( \mathcal{Z}(\Lambda) \). We give an integrable \( U_q(C_{2}^{(1)}) \)-module structure on \( F(\Lambda) \), the Fock space representation. We show that the crystal of \( F(\Lambda) \) is isomorphic to the abstract affine crystal \( \mathcal{Z}(\Lambda) \) given in [4].

Then, by finding all the maximal vectors in \( \mathcal{Z}(\Lambda) \), we obtain a decomposition of \( F(\Lambda) \) as follows
\[ F(\Lambda) = \bigoplus_{m=0}^{\infty} V(\Lambda - m\delta)^{\oplus p(m)}. \]

From the embedding of \( V(\Lambda) \) into \( F(\Lambda) \), we show that each global basis element \( G(Y) \) (\( Y \in \mathcal{Y}(\Lambda) \)) can be written as a \( \mathcal{Z}[q] \)-linear combination of proper Young walls which are smaller than or equal to \( Y \) with respect to a certain ordering; that is,
\[ G(Y) = Y + \sum_{Z \in \mathcal{Z}(\Lambda)} |Y| \shuffle |Z| G_{Y,Z}(q)Z, \]
where \( G_{Y,Z}(q) \in q\mathbb{Z}[q] \) for \( Y \neq Z \). We also discuss an algorithm for computing the coefficient polynomials \( G_{Y,Z}(q) \) in \( G(Y) \). This kind of algorithm known as Lascoux-Leclerc-Thibon algorithm, was introduced by Lascoux, Leclerc, and Thibon in case of the quantum affine algebra of type \( A_{(1)}^{(1)} \). There are several variants of Lascoux-Leclerc-Thibon algorithm (see [16] for classical type \( A_{n} \), [17] for classical type \( C_{n} \), and [8] for affine types \( A_{2n-1}^{(1)} \), \( A_{2n}^{(2)} \), \( B_{n}^{(1)} \), \( D_{n}^{(1)} \), \( D_{n+1}^{(2)} \)). Our results in this paper are based on the work [8].

2. Quantum affine algebra \( U_q(C_{2}^{(1)}) \)

Let \( I = \{0, 1, 2\} \) be the index set. The generalized Cartan matrix \( A = (a_{ij})_{i,j \in I} \) of affine type \( C_{2}^{(1)} \) and its Dynkin diagram are given by
\[
A = \begin{pmatrix}
2 & -1 & 0 \\
-2 & 2 & -2 \\
0 & -1 & 2
\end{pmatrix}
\]
and 
\[
\begin{array}{cccc}
0 & 1 & 2 \\
\end{array}
\]

Let \( P^\vee = \mathbb{Z}h_0 \oplus \mathbb{Z}h_1 \oplus \mathbb{Z}h_2 \oplus \mathbb{Z}d \) be a free abelian group, called dual weight lattice and set \( \mathfrak{h} = \mathbb{Q} \otimes \mathbb{Z} P^\vee \). For \( i \in I \), we define \( \alpha_i \) and \( \Lambda_i \in \mathfrak{h}^* \) by
\[
\alpha_i(h_j) = a_{ij}, \quad \alpha_i(d) = \delta_{0,i}, \\
\Lambda_i(h_j) = \delta_{ij}, \quad \Lambda_i(d) = 0 \quad (i, j \in I).
\]
The \( \alpha_i \) are called the simple roots and the \( \Lambda_i \) are called the fundamental weights.

Let \( c = h_0 + h_1 + h_2 \) and \( \delta = \alpha_0 + 2\alpha_1 + \alpha_2 \). Then we have \( \alpha_i(c) = 0 \), \( \delta(h_i) = 0 \) for all \( i \in I \) and \( \delta(d) = 1 \). We call \( c \) (resp. \( \delta \)) the canonical central
element (resp. null root). The free abelian group \( P = \mathbb{Z}\Lambda_0 \oplus \mathbb{Z}\Lambda_1 \oplus \mathbb{Z}\Lambda_2 \oplus \mathbb{Z}\delta \) is called the weight lattice.

Let \( q \) be an indeterminate. We denote by \( q^h (h \in P^\vee) \) the basis elements of the group algebra \( \mathbb{Q}(q)[P^\vee] \) with the multiplication \( q^h q^{h'} = q^{h+h'} (h, h' \in P^\vee) \). Set \( q_0 = q_2 = q^2, q_1 = q \) and \( K_i = q^{h_i} (i \in I) \). We will also use the following notations.

\[
[k]_i = \frac{q_i^k - q_i^{-k}}{q_i - q_i^{-1}}, \quad [n]_i! = \prod_{k=1}^{n} [k]_i, \quad \text{and} \quad e^{(n)}_i = e_i^n/[n]_i!, f^{(n)}_i = f_i^n/[n]_i!.
\]

**Definition 2.1.** The quantum affine algebra \( U_q(C^{(1)}_2) \) is the associative algebra with 1 over \( \mathbb{Q}(q) \) generated by the elements \( e_i, f_i (i \in I) \) and \( q^h (h \in P^\vee) \) subject to the following defining relations:

\[
\begin{align*}
q^0 = 1, & \quad q^h q^{h'} = q^{h+h'} (h, h' \in P^\vee), \\
q^h q^{-h} = q^{(h)} e_i, & \quad q^h f_i q^{-h} = q^{(-h)} f_i (h \in P^\vee, i \in I), \\
e_i f_j - f_j e_i = \delta_{i,j} \frac{K_i}{q_i - q_i^{-1}} (i, j \in I), \\
e_1^2 e_1 - (q^2 + q^{-2}) e_0 e_1 e_0 + e_1 e_0^2 = 0, \\
f_0^2 f_1 - (q^2 + q^{-2}) f_0 f_1 f_0 + f_1 f_0^2 = 0, \\
e_1^3 e_0 - (q^2 + 1 + q^{-2}) e_1 e_0 e_1 + (q^2 + 1 + q^{-2}) e_1 e_0 e_0^2 - e_0 e_1^3 = 0, \\
f_0^3 f_0 - (q^2 + 1 + q^{-2}) f_0^2 f_0 f_1 + (q^2 + 1 + q^{-2}) f_0 f_1 f_0 f_0^2 - f_0 f_1^3 = 0, \\
e_1^3 e_2 - (q^2 + 1 + q^{-2}) e_2 e_1 e_2 + (q^2 + 1 + q^{-2}) e_2 e_2 e_1 e_2 - e_2 e_1^3 = 0, \\
f_1^3 f_2 - (q^2 + 1 + q^{-2}) f_1^2 f_2 f_1 + (q^2 + 1 + q^{-2}) f_1 f_2 f_1 f_0 f_1^2 - f_2 f_1^3 = 0, \\
e_2^2 e_1 - (q^2 + q^{-2}) e_2 e_1 e_2 + e_1 e_2^2 = 0, \\
f_2^2 f_1 - (q^2 + q^{-2}) f_2 f_1 f_2 + f_1 f_2^2 = 0, \\
e_0 e_2 = e_2 e_0, \quad f_0 f_2 = f_2 f_0.
\end{align*}
\]

It is also called the quantum affine algebra of type \( C^{(1)}_2 \).

3. Crystal bases

In this section, we review the crystal basis theory for the quantum affine algebra \( U_q(C^{(1)}_2) \). All the statements and the results in this section hold for a quantized enveloping algebra associated with a symmetrizable Kac-Moody algebra (see [4]). A \( U_q(C^{(1)}_2) \)-module \( M \) is called integrable if

(i) \( M = \bigoplus_{\lambda \in P} M_{\lambda} \) where \( M_{\lambda} = \{ v \in M \mid q^h v = q^{\lambda(h)} v \ \text{for all} \ h \in P^\vee \} \),

(ii) \( M \) is a direct sum of finite dimensional irreducible \( U_q \)-modules, where \( U_i (i \in I) \) is the subalgebra generated by \( e_i, f_i, K_i^{\pm 1} \).
Fix \( i \in I \). An element \( v \in M_\lambda \) may be written uniquely as
\[
v = \sum_{k \geq 0} f_i^{(k)} v_k,
\]
where \( v_k \in \ker e_i \cap M_{\lambda + k\alpha_i} \). We define the endomorphisms \( \tilde{e}_i \) and \( \tilde{f}_i \) on \( M \), called the Kashiwara operators, by
\[
\tilde{e}_i v = \sum_{k \geq 1} f_i^{(k-1)} v_k, \quad \tilde{f}_i v = \sum_{k \geq 0} f_i^{(k+1)} v_k.
\]

Let \( \Lambda_0 = \{ f/g \in \mathbb{Q}(q) \mid f, g \in \mathbb{Q}[q], g(0) \neq 0 \} \) be the localization of \( \mathbb{Q}[q] \) at \( q = 0 \).

**Definition 3.1.** A crystal basis of \( M \) is a pair \( (L, B) \), where
\begin{enumerate}
  \item \( L \) is a free \( \Lambda_0 \)-submodule of \( M \) such that \( M \cong \mathbb{Q}(q) \otimes_{\Lambda_0} L \),
  \item \( B \) is a \( \mathbb{Q} \)-basis of \( L/qL \),
  \item \( L = \bigoplus_{\lambda \in P} L_\lambda \), where \( \lambda = L \cap M_\lambda \),
  \item \( B = \bigcup_{\lambda \in P} B_\lambda \), where \( B_\lambda = B \cap (L_\lambda/qL_\lambda) \),
  \item \( \tilde{e}_i L \subset L, \tilde{f}_i L \subset L \) for all \( i \in I \),
  \item \( \tilde{e}_i B \subset B \cup \{0\}, \tilde{f}_i B \subset B \cup \{0\} \) for all \( i \in I \),
  \item for \( b, b' \in B \), \( \tilde{f}_i b = b' \) if and only if \( b = \tilde{e}_ib' \).
\end{enumerate}

The set \( B \) becomes a colored oriented graph, called the crystal graph, where the arrows are defined by \( b \rightarrow b' \) if and only if \( \tilde{f}_i b = b' \), for \( b, b' \in B \).

For each \( b \in B \) and \( i \in I \), we define \( \varepsilon_i(b) = \max\{ k \geq 0 \mid \tilde{e}_i^k b \in B \} \), \( \varphi_i(b) = \max\{ k \geq 0 \mid \tilde{f}_i^k b \in B \} \). Then we have
\[
\varepsilon_i(b) = \varepsilon_i(h_i, \text{wt}(b)) \quad \text{wt}(\tilde{e}_i b) = \text{wt}(\tilde{e}_i b) + \alpha_i, \quad \text{wt}(\tilde{f}_i b) = \text{wt}(\tilde{f}_i b) - \alpha_i,
\]
\[
\varepsilon_i(\tilde{e}_i b) = \varepsilon_i(b) - 1, \quad \varphi_i(\tilde{e}_i b) = \varphi_i(b) + 1 \quad \text{if } \tilde{e}_i b \in B,
\]
\[
\varepsilon_i(\tilde{f}_i b) = \varepsilon_i(b) + 1, \quad \varphi_i(\tilde{f}_i b) = \varphi_i(b) - 1 \quad \text{if } \tilde{f}_i b \in B.
\]

Set \( P^+ = \{ \lambda \in \mathfrak{h}^* \mid \langle \lambda(h_i) \rangle \in \mathbb{Z}_{\geq 0}, i \in I \} \). For \( \lambda \in P^+ \), let \( V(\lambda) \) be the irreducible highest weight \( U_q(C_2^{(1)}) \)-module with highest weight \( \lambda \) and highest weight vector \( u_\lambda \).

**Theorem 3.2.** [8] Let \( L(\lambda) \) be the free \( \Lambda_0 \)-submodule of \( V(\lambda) \) spanned by the vectors of the form \( \tilde{f}_i \cdots \tilde{f}_i u_\lambda \) \( (i_k \in I, r \in \mathbb{Z}_{\geq 0}) \) and set \( B(\lambda) = \{ \tilde{f}_i \cdots \tilde{f}_i u_\lambda + qL(\lambda) \in L(\lambda)/qL(\lambda) \} \setminus \{0\} \). Then \( (L(\lambda), B(\lambda)) \) is a crystal basis of \( V(\lambda) \), and every crystal basis of \( V(\lambda) \) is isomorphic to \( (L(\lambda), B(\lambda)) \). \( \square \)

There exists an involution \( - \) of \( U_q(C_2^{(1)}) \) as a \( \mathbb{Q} \)-algebra defined by
\[
\tilde{e}_i = e_i, \quad \tilde{f}_i = f_i, \quad \overline{q}^h = q^{-h}, \quad \overline{q} = q^{-1}
\]
for $i \in I$ and $h \in P^\vee$. Set $A = \mathbb{Q}[q, q^{-1}]$. We denote by $U^-_h(C_2^{(1)})$ the $A$-subalgebra of $U_q(C_2^{(1)})$ generated by $f_i^{(m)}$ $(i \in I, n \in \mathbb{Z}_{\geq 0})$. Set $V(\lambda)^A = U^-_h(C_2^{(1)})u_\lambda$.

Theorem 3.3. There exists a unique $A$-basis $G(\lambda) = \{ G(b) \mid b \in B(\lambda) \}$ of $V(\lambda)^A$ such that

$$G(b) \equiv b \mod qL(\lambda) \quad \text{and} \quad \overline{G(b)} = G(b)$$

for all $b \in B(\lambda)$.

The basis $G(\lambda)$ is called the global basis or canonical basis of $V(\lambda)$ associated with the crystal graph $B(\lambda)$.

By extracting the properties of crystal graphs, we can define the notion of abstract crystals [10, 11].

Definition 3.4. An affine crystal is a set $B$ together with the maps $\text{wt} : B \to P$, $\varphi_i, \psi_i : B \to \mathbb{Z} \cup \{-\infty\}$, $\tilde{e}_i, \tilde{f}_i : B \to B \cup \{0\}$ $(i \in I)$ such that for $i \in I$ and $b \in B$,

(i) $\varphi_i(b) = \varepsilon_i(b) + \langle h_i, \text{wt}(b) \rangle$,

(ii) $\text{wt}(\tilde{e}_i b) = \text{wt}(b) + \alpha_i$, $\text{wt}(\tilde{f}_i b) = \text{wt}(b) - \alpha_i$,

(iii) $\varepsilon_i(\tilde{e}_i b) = \varepsilon_i(b) - 1$, $\varphi_i(\tilde{e}_i b) = \varphi_i(b) + 1$ if $\tilde{e}_i b \in B$,

(iv) $\varepsilon_i(\tilde{f}_i b) = \varepsilon_i(b) + 1$, $\varphi_i(\tilde{f}_i b) = \varphi_i(b) - 1$ if $\tilde{f}_i b \in B$,

(v) $\tilde{f}_i b = b'$ if and only if $\tilde{e}_i b' = b$ for $b, b' \in B$,

(vi) $\tilde{e}_i b = \tilde{f}_i b = 0$ if $\varepsilon_i(b) = -\infty$.

The crystal $B(\lambda)$ of $V(\lambda)$ ($\lambda \in P^+$) satisfies the above conditions and it is an affine crystal.

A morphism $\psi : B_1 \to B_2$ of crystals is a map $\psi : B_1 \cup \{0\} \to B_2 \cup \{0\}$ satisfying the conditions:

(i) $\psi(0) = 0$,

(ii) $\text{wt}(\psi(b)) = \text{wt}(b)$, $\varepsilon_i(\psi(b)) = \varepsilon_i(b)$, and $\varphi_i(\psi(b)) = \varphi_i(b)$ if $b \in B_1$ and $\psi(b) \in B_2$,

(iii) $\tilde{f}_i \psi(b) = \psi(b')$ if $b, b' \in B_1$, $\psi(b), \psi(b') \in B_2$ and $\tilde{f}_i b = b'$.

4. Young walls

In this section, we will give a brief review of the results in [4]. The Young walls will be built of two kinds of blocks:

| type | shape | width | thickness | height | volume |
|------|-------|-------|-----------|--------|--------|
| I    | $\square = \square$ | 1     | 1         | $\frac{1}{2}$ | $\frac{1}{2}$ |
| II   | $\square = \square$, $\square = \square$ | 1     | $\frac{1}{2}$ | 1     | $\frac{1}{2}$ |

We also give a coloring of blocks as follows;
Given a dominant integral weight $\Lambda = \Lambda_i$ ($i \in I$) of level 1, that is, $\Lambda(c) = 1$, we fix a frame $Y_{\Lambda}$ called the \textit{ground state wall} of weight $\Lambda$ as follows:

\begin{align*}
Y_{\Lambda_0} &= \begin{array}{c}
\begin{array}{cccc}
0 & 2 & 0 & 2 \\
2 & 0 & 2 & 0 \\
0 & 2 & 0 & 2 \\
0 & 2 & 0 & 2
\end{array}
\end{array} \\
Y_{\Lambda_1} &= \begin{array}{c}
\begin{array}{cccc}
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1
\end{array}
\end{array} \\
Y_{\Lambda_2} &= \begin{array}{c}
\begin{array}{cccc}
2 & 0 & 2 & 0 \\
2 & 0 & 2 & 0 \\
2 & 0 & 2 & 0 \\
2 & 0 & 2 & 0
\end{array}
\end{array}
\end{align*}

On this frame, we build a wall of thickness less than or equal to one unit. The rules for building walls are as follows:

1. The colored blocks should be stacked in columns. No block can be placed on top of a column of half-unit thickness.
2. Except for the right-most column, there should be no free space to the right of any block.
3. The colored blocks should be stacked in a specified pattern which is determined as follows:

Here the shaded blocks in the above patterns are the ones in the ground state walls.
A wall built on $Y_{\Lambda}$ following the above rules is called a Young wall on $Y_{\Lambda}$, for the heights of its columns are weakly decreasing as we proceed from right to left.

**Definition 4.1.**

(1) A column of a Young wall is called a full column if its volume is of an integral value.

(2) A Young wall is said to be proper if none of the full columns have the same heights.

We denote by $Z(\Lambda)$ the set of all proper Young walls on $Y_{\Lambda}$. For $Y \in Z(\Lambda)$, we often write $Y = (y_k)_{k=0}^{\infty} = (\cdots, y_2, y_1, y_0)$ as an infinite sequence of its columns. Let $|y_k|$ be the number of blocks in $y_k$ added to $Y_{\Lambda}$. Then the associated partition is defined to be $|Y| = (|y_k|)_{k=0}^{\infty}$.

**Example 4.2.** We illustrate several examples of proper Young walls. For convenience, we omit the columns of the ground state wall on which no block has been added.

![Young wall examples](image)

**Definition 4.3.** Let $Y$ be a proper Young wall on $Y_{\Lambda}$.

(1) A block of color $i$ (in short, an $i$-block) in $Y$ is called a removable $i$-block if $Y$ remains a proper Young wall after removing the block.

(2) A place in $Y$ is called an admissible $i$-slot if one may add an $i$-block to obtain another proper Young wall.

(3) A column in $Y$ is said to be $i$-removable (resp. $i$-admissible) if there is a removable $i$-block (resp. an admissible $i$-slot) in that column.

We now define the abstract Kashiwara operators $\tilde{E}_i, \tilde{F}_i$ on $Z(\Lambda)$ as follows. Fix $i \in I$ and let $Y = (y_k)_{k=0}^{\infty}$ be a proper Young wall on $Y_{\Lambda}$.

(1) To each column $y_k$ of $Y$, we assign

\[
\begin{cases}
- & \text{if } y_k \text{ is twice } i\text{-removable}, \\
- & \text{if } y_k \text{ is once } i\text{-removable but not } i\text{-admissible}, \\
- & \text{if } y_k \text{ is once } i\text{-removable and once } i\text{-admissible}, \\
+ & \text{if } y_k \text{ is once } i\text{-admissible but not } i\text{-removable}, \\
++ & \text{if } y_k \text{ is twice } i\text{-admissible}, \\
\cdot & \text{otherwise.}
\end{cases}
\]

(2) From this infinite sequence of ±‘s and ·‘s, we cancel out every $(+, -)$-pair to obtain a finite sequence of −‘s followed by +‘s, reading from left to right. This finite sequence $(- \cdots -, + \cdots +)$ is called the $i$-signature of $Y$. 
(3) We define $\tilde{E}_iY$ to be the proper Young wall obtained from $Y$ by removing the $i$-block corresponding to the right-most $-$ in the $i$-signature of $Y$. We define $\tilde{E}_iY = 0$ if there is no $-$ in the $i$-signature of $Y$.

(4) We define $\tilde{F}_iY$ to be the proper Young wall obtained from $Y$ by adding an $i$-block to the column corresponding to the left-most $+$ in the $i$-signature of $Y$. We define $\tilde{F}_iY = 0$ if there is no $+$ in the $i$-signature of $Y$.

Next, we define

$$\text{wt}(Y) = \Lambda - \sum_{i \in I} k_i \alpha_i \in P,$$

$$\varepsilon_i(Y) = \text{the number of $-$'s in the } i\text{-signature of } Y,$$

$$\varphi_i(Y) = \text{the number of $+$'s in the } i\text{-signature of } Y,$$

where $k_i$ denotes the number of $i$-blocks in $Y$ which have been added to $Y$.

**Proposition 4.4.** The set $\mathcal{Z}(\Lambda)$ together with the maps $\text{wt}$, $\varepsilon_i$, $\varphi_i$, $\tilde{E}_i$ and $\tilde{F}_i$ ($i \in I$) becomes an affine crystal.

The part of a column with $a_i$-many $i$-blocks for each $i \in I$ ($a_0 = a_2 = 1, a_1 = 2$) in some cyclic order is called a $\delta$-column. A $\delta$-column in a proper Young wall is removable if it can be removed to yield another proper Young wall.

**Definition 4.5.** A proper Young wall $Y$ is said to be reduced if none of its columns contain a removable $\delta$-column.

**Example 4.6.** Among the proper Young walls given in Example 4.2, the second and the fourth ones are reduced, but the others are not.

Let $\mathcal{Y}(\Lambda) \subset \mathcal{Z}(\Lambda)$ be the set of all reduced proper Young walls on $Y_\Lambda$. Then we have

**Theorem 4.7.** The set $\mathcal{Y}(\Lambda)$ is an affine crystal. Moreover, there exists an affine crystal isomorphism $\mathcal{Y}(\Lambda) \sim \rightarrow \mathcal{B}(\Lambda)$, where $\mathcal{B}(\Lambda)$ is the crystal of the basic representation $V(\Lambda)$.

Let $Y = (y_k)_{k=0}^\infty$ be a proper Young wall in $\mathcal{Z}(\Lambda)$. Let $S$ be an interval in $\mathbb{Z}_{\geq 0}$ which is finite or infinite; i.e. $S = \{k \mid s \leq k < t \}$ for some $0 \leq s < t \leq \infty$. We call $Y' = (y_k)_{k \in S}$ a part of $Y$. If $S$ is infinite; that is, if $Y' = (y_k)_{k=s}^\infty$, then $Y'$ is itself a proper Young wall in $\mathcal{Y}(\Lambda')$ for some $\Lambda'$. If $S$ is finite; that is, if $Y' = (y_{t-1}, \ldots, y_s)$, then $Y'$ is not a proper Young wall, but a finite collection of successive columns in $Y$. Also, by adding or removing blocks only in columns $y_s$ ($s \in S$) of $Y'$, we can extend the notions of admissible $i$-slots, removable $i$-blocks, the $i$-signature, $\varepsilon_i$, and $\varphi_i$ of a part $Y'$ (however, we define $\text{wt}$ only for proper Young walls). The notion of parts will be used when we define the action of $U_q(C_2^{(1)})$ on $\mathcal{Z}(\Lambda)$.

**Example 4.8.** Crystal graph $\mathcal{B}(\Lambda_1)$. 


5. Fock space representation

Given a dominant integral weight $\Lambda = \Lambda_i$ ($i \in I$), we define $\mathcal{F}(\Lambda) = \bigoplus_{Y \in \mathcal{Z}(\Lambda)} \mathbb{Q}(q)Y$ to be the $\mathbb{Q}(q)$-vector space with a basis $\mathcal{Z}(\Lambda)$. In this section, we will define a $U_q(C_2^{(1)})$-module structure on $\mathcal{F}(\Lambda)$, the Fock space representation of $U_q(C_2^{(1)})$. Then we will show that the abstract affine crystal $\mathcal{Z}(\Lambda)$ is isomorphic to the crystal of $\mathcal{F}(\Lambda)$.

Let us define the action of $U_q(C_2^{(1)})$ on $\mathcal{Z}(\Lambda)$. For $Y = (y_k)_{k=0}^{\infty} \in \mathcal{Z}(\Lambda)$ and $q^h$ ($h \in P^\vee$), we define

$$q^hY = q^{\langle h, \text{wt}(Y) \rangle}Y.$$

The actions of $e_i$ and $f_i$ ($i \in I$) are given according to the type of the $i$-block.

**Case 1.** Suppose that $i = 1$ ($q = q_1$).

Let $b$ be a removable 1-block in $y_k$ of $Y$. If the 1-signature of $y_k$ is $--$, or if the 1-signature of $y_k$ is $-$ and there is another 1-block below $b$, we define $Y \nearrow b$ to be the Young wall obtained by removing $b$ from $Y$. If the 1-signature of $y_k$ is $-+$, or if the 1-signature of $y_k$ is $-$ and there is no 1-block below $b$, we define

$$Y \nearrow b = q^{-1}(1 - (-q^2)^{(b)+1})Z,$$
where \( Z \) is the Young wall obtained by removing \( b \) from \( Y \) and \( l(b) \) is the number of \( y_l \)'s with \( l < k \) such that \( |y_l| = |y_k| \). That is, if

\[
Z = \text{Young wall obtained by removing } b \text{ from } Y \text{ and } l(b) \text{ is the number of } y_l \text{'s with } l < k \text{ such that } |y_l| = |y_k|.
\]

In either case, if \( k \geq 1 \), let \( Y_R(b) = (y_l, \ldots, y_0) \) be the part of \( Y \) with finite columns such that \( l \) is the integer satisfying \( |y_k| = |y_{k-1}| = \cdots = |y_{l+1}| < |y_l| \). Set \( R_1(b;Y) = \varphi_1(Y_R(b)) - \varepsilon_1(Y_R(b)) \) if \( k \geq 1 \) and 0 if \( k = 0 \). Then we define

\[
e_1 Y = \sum_b q^{-R_1(b;Y)}(Y \triangleright b),
\]

where \( b \) runs over all removable 1-blocks in \( Y \).

On the other hand, suppose that \( b \) is an admissible 1-slot in \( y_k \) of \( Y \). If the 1-signature of \( y_k \) is ++, or if the 1-signature of \( y_k \) is + and there is no 1-block below \( b \), we define \( Y \triangleright b \) to be the Young wall obtained by adding a 1-block at \( b \). If the 1-signature of \( y_k \) is −+, or if the 1-signature of \( y_k \) is + and there is another 1-block below \( b \), then we define

\[
Y \triangleright b = q^{-1}(1 - (-q^2)^{l(b)+1})Z,
\]

where \( Z \) is the Young wall obtained by adding a 1-block at \( b \) and \( l(b) \) is the number of \( y_l \)'s with \( l > k \) such that \( |y_l| = |y_k| \). That is, if

\[
Y = \begin{cases} \text{Young wall} \quad \text{with } l(b) \text{ number of } y_l \text{'s}, \quad Y_R(b), \end{cases}
\]

then \( Y \triangleright b = q^{-1}(1 - (-q^2)^{l(b)+1})Z \).

In either case, let \( Y_L(b) = (\cdots, y_{l+2}, y_{l+1}) \), where \( l \) is the integer such that \( |y_{l+1}| < |y_l| = |y_{l-1}| = \cdots = |y_k| \), and set \( L_1(b;Y) = \varphi_1(Y_L(b)) - \varepsilon_1(Y_L(b)) \).
FOCK SPACE REPRESENTATION FOR $U_q(C_2^{(1)})$

Then we define

\begin{equation}
(5.3) \quad f_1 Y = \sum_b q^{L_1(b;Y)} (Y \not\nearrow b),
\end{equation}

where $b$ runs over all admissible 1-slots in $Y$.

**Case 2.** Suppose that $i = 0, 2$ ($q^2 = q_i$).

If $b$ is a removable $i$-block in $y_k$ of $Y$, then we define $Y \nearrow b$ to be the Young wall obtained by removing $b$ from $Y$. Consider the following $i$-block $b$ in $y_k$ of $Y$, called a **virtually removable** $i$-block.

In this case, we define $Y \nearrow b$ to be

\begin{align*}
(-q^2)^{l(b)} & \times \quad \ldots \quad b & \quad \text{or} & \quad (-q^2)^{l(b)} & \times \quad \ldots \quad b
\end{align*}

respectively, where $l(b) \geq 1$ is given in the above figure. In either case, if $k \geq 1$, let $Y_R(b) = (y_{k-1}, \ldots, y_0)$. Set $R_i(b; Y) = \varphi_i(Y_R(b)) - \varepsilon_i(Y_R(b))$ if $k \geq 1$, and 0 if $k = 0$. Then we define

\begin{equation}
(5.4) \quad e_i Y = \sum_b q^{-2R_i(b;Y)} (Y \nearrow b),
\end{equation}

where $b$ runs over all removable and virtually removable $i$-blocks in $Y$.

On the other hand, if $b$ is an admissible $i$-slot in $y_k$ of $Y$, then we define $Y \not\nearrow b$ to be the Young wall obtained by adding an $i$-block at $b$. Consider the following $i$-slot $b$ in $y_k$ of $Y$, called a **virtually admissible** $i$-slot.

In this case, we define $Y \not\nearrow b$ to be

\begin{align*}
(-q^2)^{l(b)} & \times \quad \ldots \quad b & \quad \text{or} & \quad (-q^2)^{l(b)} & \times \quad \ldots \quad b
\end{align*}
respectively, where \(l(b) \geq 1\) is given in the above figure. In either case, let \(Y_L(b) = (\cdots, y_{k+2}, y_{k+1})\) and set \(L_i(b; Y) = \varphi_i(Y_L(b)) - \varepsilon_i(Y_L(b))\). Then we define

\[
(5.5) \quad f_i Y = \sum_b q^{2L_i(b; Y)}(Y \vee b),
\]

where \(b\) runs over all admissible and virtually admissible \(i\)-slots in \(Y\).

**Example 5.1.**

\[
(1) \quad \begin{array}{|c|c|c|c|c|c|}
\hline
& & & 1 & 1 & 1 \\
\hline
1 & 1 & 1 & 1 & 1 & 1 \\
\hline
1 & 1 & 1 & 1 & 1 & 1 \\
\hline
\end{array} = \begin{array}{|c|c|c|c|c|c|}
\hline
& & & q & 1 & 1 \\
\hline
1 & 1 & 1 & 1 & 1 & 1 \\
\hline
1 & 1 & 1 & 1 & 1 & 1 \\
\hline
\end{array} + q(1 + q^6) \begin{array}{|c|c|c|c|c|c|}
\hline
& & & 2 & 0 & 0 \\
\hline
1 & 1 & 1 & 1 & 1 & 1 \\
\hline
1 & 1 & 1 & 1 & 1 & 1 \\
\hline
\end{array} + \begin{array}{|c|c|c|c|c|c|}
\hline
& & & 2 & 0 & 0 \\
\hline
1 & 1 & 1 & 1 & 1 & 1 \\
\hline
1 & 1 & 1 & 1 & 1 & 1 \\
\hline
\end{array}
\]

\[
(2) \quad \begin{array}{|c|c|c|c|c|c|}
\hline
& & & 1 & 2 & 0 \\
\hline
1 & 1 & 1 & 1 & 1 & 1 \\
\hline
1 & 1 & 1 & 1 & 1 & 1 \\
\hline
\end{array} = \begin{array}{|c|c|c|c|c|c|}
\hline
& & & q & 2 & 0 \\
\hline
1 & 1 & 1 & 1 & 1 & 1 \\
\hline
1 & 1 & 1 & 1 & 1 & 1 \\
\hline
\end{array} + q^8 \begin{array}{|c|c|c|c|c|c|}
\hline
& & & 2 & 0 & 0 \\
\hline
1 & 1 & 1 & 1 & 1 & 1 \\
\hline
1 & 1 & 1 & 1 & 1 & 1 \\
\hline
\end{array} + q^4 \begin{array}{|c|c|c|c|c|c|}
\hline
& & & 2 & 0 & 0 \\
\hline
1 & 1 & 1 & 1 & 1 & 1 \\
\hline
1 & 1 & 1 & 1 & 1 & 1 \\
\hline
\end{array}
\]

**Theorem 5.2.** \(\mathcal{F}(\Lambda)\) is an integrable \(U_q(C_2^{(1)})\)-module.

**Proof.** First, it follows directly from the definition of the actions of \(U_q(C_2^{(1)})\) that

\[
(5.6) \quad \begin{align*}
q^h q^h' Y &= q^{h+h'} Y, \\
q^h e_i q^{-h} Y &= q^{\langle h, \alpha_i \rangle} e_i Y, \\
q^h f_i q^{-h} Y &= q^{-\langle h, \alpha_i \rangle} f_i Y
\end{align*}
\]

for \(Y \in \mathcal{Z}(\Lambda)\), \(i \in I\) and \(h, h' \in P^\vee\). Since \(e_i\) and \(f_i\) (\(i \in I\)) act locally nilpotently on \(\mathcal{F}(\Lambda)\), if we show that

\[
(5.7) \quad [e_i, f_j] Y = \delta_{ij} K_i - K_i^{-1} \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}} Y
\]

for \(Y \in \mathcal{Z}(\Lambda)\) and \(i, j \in I\), then the Serre relations will follow from Proposition B.1 in \[12\]. The verification of \((5.7)\) can be obtained by modifying the arguments in \[8\]. □
Let \( \mathcal{L}(\Lambda) = \bigoplus_{Y \in \mathcal{Z}(\Lambda)} \mathbb{A}_0 Y \). Then we also obtain

**Theorem 5.3.** The pair \((\mathcal{L}(\Lambda), \mathcal{Z}(\Lambda))\) is a crystal basis of \( \mathcal{F}(\Lambda) \) and the crystal of \( \mathcal{F}(\Lambda) \) is isomorphic to the affine crystal \( \mathcal{Z}(\Lambda) \) defined in Section 4.

**Proof.** Let us give a sketch of the proof (see [8] for a detailed argument). It is clear that \((\mathcal{L}(\Lambda), \mathcal{Z}(\Lambda))\) satisfies the first four conditions in Definition 3.2. Fix \( i \in I \). For each \( Y \in \mathcal{Z}(\Lambda) \), we can find a subset \( B_{Y,i} \) of \( \mathcal{Z}(\Lambda) \) containing \( Y \) such that

1. \( V_{Y,i} = \bigoplus_{Z \in B_{Y,i}} \mathbb{Q}(q)Z \) is a \( U_{(i)} \)-submodule of \( \mathcal{F}(\Lambda) \),
2. \( (L_{Y,i}, B_{Y,i}) \) is a crystal basis of \( V_{Y,i} \) where \( L_{Y,i} = \bigoplus_{Z \in B_{Y,i}} \mathbb{A}_0 Z \),
3. \( \tilde{e}_i Z = \tilde{E}_i Z \) and \( \tilde{f}_i Z = \tilde{F}_i Z \) for all \( Z \in B_{Y,i} \).

From (i) and (ii), it follows that \((\mathcal{L}(\Lambda), \mathcal{Z}(\Lambda))\) satisfies the rest three conditions in Definition 3.2. The condition (iii) implies that the Kashiwara operators on \( \mathcal{Z}(\Lambda) \) induced by the \( U_q(C_2^{(1)}) \) action on \( \mathcal{F}(\Lambda) \) coincide with the abstract Kashiwara operators on \( \mathcal{Z}(\Lambda) \). Therefore, the crystal of \( \mathcal{F}(\Lambda) \) is isomorphic to the abstract affine crystal \( \mathcal{Z}(\Lambda) \) defined in Section 4. \( \square \)

**Corollary 5.4.**

\[
\mathcal{F}(\Lambda) = \bigoplus_{m=0}^{\infty} V(\Lambda - m\delta)^{\otimes p(m)},
\]

where \( p(m) \) is the number of partitions of \( m \).

**Proof.** We will show that

1. the weight of each maximal vector in \( \mathcal{Z}(\Lambda) \) is of the form \( \Lambda - m\delta \) for some \( m \geq 0 \),
2. there exists a bijection between the set of partitions of \( m \) \( (m \geq 0) \) and the set of maximal vectors in \( \mathcal{Z}(\Lambda) \) with weight \( \Lambda - m\delta \).

Let \( Y = (y_k)_{k=0}^{\infty} \in \mathcal{Z}(\Lambda) \) be a maximal vector, that is, \( \tilde{E}_i Y = 0 \) for all \( i \in I \). Suppose that \( Y \) is the ground state wall \( Y_\Lambda \). Since \( \text{wt}(Y_\Lambda) = \Lambda \) and \( \mathcal{Z}(\Lambda)_\Lambda = \{ Y_\Lambda \} \), the multiplicity of \( V(\Lambda) \) in \( \mathcal{F}(\Lambda) \) is 1. Now, we assume that \( Y \neq Y_\Lambda \). Let \( l \) be the largest integer such that \( |y_l| \neq 0 \). Suppose that \( Y' = (y_k)_{k=l+1}^{\infty} \in \mathcal{Y}(\Lambda_j) \) for some \( j \in I \).

**Case 1.** \( j = 1 \)

We see from the pattern for \( \mathcal{Z}(\Lambda) \) that \( \Lambda = \Lambda_1 \). By the maximality of \( Y \), the 1-signature of \( y_{l+1} \) is + and the 1-signature of \( y_l \) is – or –+. Hence \( y_l \) is obtained by adding some \( \delta \)-columns to the ground state wall. If \( l \neq 0 \), let \( l' \) be the largest integer such that \( l' < l \) and \( |y_{l'}| > |y_l| \). Again by the maximality of \( Y' \), the 1-signature of \( y_{l'} \) is – or –+, which means that \( y_{l'} \) is also obtained by adding some \( \delta \)-columns to the ground state wall. Repeating the above argument from left to right, we conclude that for \( 0 \leq k \leq l \), the total volume of the blocks added on the \( k \)th column is \( 2m_k \) for some \( m_k \geq 1 \). Hence, \((m_0, \ldots, m_l)\) forms a partition, and \( \text{wt}(Y) = \Lambda - (\sum_{k=0}^{l} m_k)\delta \).
Case 2. \( j = 0, 2 \)
We may assume that \( j = 0 \). By the maximality of \( Y \), we observe that the \( y_{l+1} \) is 0-admissible and \( y_{l} \) is 0-removable but not 2-removable. Hence \( y_{l} \) is obtained by adding some \( \delta \)-columns to the ground-state wall. If \( l \neq 0 \), let \( l' \) be the largest integer such that \( l' < l \) and \( |y_{l'}| > |y_{l}| \). If \( y_{l'+1} \) is 0-admissible, then by the maximality of \( Y \), \( y_{l'} \) is 0-removable but not 2-removable. On the other hand, if \( y_{l'+1} \) is 2-admissible, then by the maximality of \( Y \), \( y_{l'} \) is 2-removable but not 0-removable. As in Case 1, by repeating the above argument, we conclude that for \( 0 \leq k \leq l \), the total volume of the blocks added on the \( k \)th column is \( 2m_{k} \) for some \( m_{k} \geq 1 \). Hence, \( (m_{0}, m_{1}, \cdots, m_{l}) \) forms a partition and \( \text{wt}(Y) = \Lambda - \left( \sum_{k=0}^{l} m_{k} \right) \delta \).

Conversely, for a given partition \( (m_{k})_{k=0}^{\infty} \) of \( m \geq 0 \), we can find a unique \( Y = (y_{k})_{k=0}^{\infty} \in \mathcal{Z}(\Lambda) \) such that \( y_{k} \) is obtained by adding \( m_{k} \) many \( \delta \)-columns to the \( k \)th column of the ground state wall \( Y_{\Lambda} \) (hence the total volume of the blocks added to the \( k \)th column is \( 2m_{k} \)). It is easy to check that \( Y \) is a maximal vector with \( \text{wt}(Y) = \Lambda - m\delta \).

Example 5.5. In \( \mathcal{Y}(\Lambda_{1}) \), the maximal vector of weight \( \Lambda - 4\delta \) corresponding to the partition \((2, 1, 1)\) is

In this section, we will describe an algorithm for computing the global basis for the basic representation \( V(\Lambda) \) where \( \Lambda = \Lambda_{i} \) \((i \in I)\). We have seen that there exists an embedding \( V(\Lambda) \simeq \mathcal{F}(\Lambda) \), and that \( \mathcal{Y}(\Lambda) \) is a crystal of \( V(\Lambda) \). By Theorem 3.3, there exists an \( A \)-basis \( G(\Lambda) = \{G(Y) \mid Y \in \mathcal{Y}(\Lambda)\} \) of \( V(\Lambda)^{A} \). For each \( Y \in \mathcal{Y}(\Lambda) \), the global basis element \( G(Y) \) can be written as an \( A \)-linear combination of proper Young walls in \( \mathcal{Z}(\Lambda) \). Hence, our algorithm is to compute the coefficients of proper Young walls in each global basis element \( G(Y) \). We will follow the arguments in [8].

We start with certain orderings. For \( Y = (y_{k})_{k=0}^{\infty} \) and \( Z = (z_{k})_{k=0}^{\infty} \) in \( \mathcal{Z}(\Lambda) \), consider their associated partitions \( |Y| \) and \( |Z| \). We define \( |Y| \triangleright |Z| \) if and only if \( \sum_{k=l}^{\infty} |y_{k}| \geq \sum_{k=l}^{\infty} |z_{k}| \) for all \( l \geq 0 \). Note that it is not a partial ordering on \( \mathcal{Z}(\Lambda) \) since there exist \( Y \neq Z \) in \( \mathcal{Z}(\Lambda) \) such that \( |Y| = |Z| \). We also define \( |Y| > |Z| \) if \( |y_{k}| > |z_{k}| \) where \( k \) is the largest integer such that \( |y_{k}| \neq |z_{k}| \). Note that \( |Y| \triangleright |Z| \) implies \( |Y| \geq |Z| \). Next, on the set of the proper Young walls with the same associated partitions, we fix an arbitrary
total ordering $\succ$. Then we define a total ordering $\succ$ on $\mathcal{Z}(\Lambda)$ as follows:

$$\text{(6.1)} \quad Y \succ Z \iff (|Y| > |Z|) \text{ or } (|Y| = |Z| \text{ and } Y \succ Z).$$

For $Y \in \mathcal{Z}(\Lambda)$, we write

$$\text{(6.2)} \quad f^{(r)}_i = \sum_{Z \in \mathcal{Z}(\Lambda) \atop \text{wt}(Z) = \text{wt}(Y) - \alpha_i} Q_{Y,Z}(q),$$

where $i \in I$, $r \geq 1$, and $Q_{Y,Z}(q) \in \mathbb{Q}(q)$. If $Z = (z_k)_{k=0}^\infty \in \mathcal{Z}(\Lambda)$ satisfies $Q_{Y,Z}(q) \neq 0$, we can find a unique sequence of proper Young walls $Y = Y_0, Y_1, \ldots, Y_r = Z$ such that

(i) $\lambda_{k+1} Y_{k+1} = Y_k \not\succ b_{k+1}$ for a (virtually) admissible $i$-slot $b_{k+1}$ of $Y_k$ and $\lambda_{k+1} \in \mathbb{Z}[q, q^{-1}]$,

(ii) $b_{k+1}$ is placed on $b_k$ or to the right of $b_k$.

For each $k$, let $Q_{Y_k, Y_{k+1}}(q)$ be the coefficient of $Y_{k+1}$ in the expression of $f_i Y_k$. We define

$$\text{(6.3)} \quad Q^0_{Y,Z}(q) = \prod_{k=0}^{r-1} Q_{Y_k, Y_{k+1}}(q) \in \mathbb{Z}[q, q^{-1}].$$

Suppose that $i = 0, 2$ (that is, $i$-blocks are of type II). Then by induction on $r$, we have

$$\text{(6.4)} \quad Q_{Y,Z}(q) = Q^0_{Y,Z}(q) q^{\sigma(2)} \in \mathbb{Z}[q, q^{-1}].$$

Suppose that $i = 1$ (that is, $i$-blocks are of type I). Let us assume that $b_k$ is located in the $i_k$th column of $Y_{k-1}$ ($1 \leq k \leq r$). Note that each $b_k$ ($1 \leq k \leq r$) can be viewed as a block (not necessarily removable) in $Z$. Set

$$\text{(6.5)} \quad J_1 = \{ k \mid b_{k-1} \text{ is beneath } b_k \},$$

$$J_2 = \{ k \mid \text{there exists a 1-block } (\neq b_{k-1}) \text{ beneath } b_k \},$$

$$J_3 = \{ k \mid \text{there exists no 1-block on and beneath } b_k \text{ in } Z \},$$

$$S = \{ k \in J_2 \mid k - 1 \in J_3 \text{ and } |z_{i_k-1}| = |z_{i_k}| - 1 \}.$$ 

Put $n_j = |J_j|$ ($j = 1, 2, 3$), and $\mu_k = q \lambda_k$ ($k \in S$). Note that $2n_1 + n_2 + n_3 = r$. By induction on $r$, we have

$$\text{(6.6)} \quad Q_{Y,Z}(q) = Q^0_{Y,Z}(q) q^{\sigma(n_1, n_2, n_3)} \prod_{k \in S} \mu_k,$$

where $\sigma(n_1, n_2, n_3) = 4(\binom{n_2}{2}) + (\binom{n_3}{2}) + 2n_1(n_2 + n_3) + n_2 n_3$ (see [8]). Note that $[2]^{n_1} \prod_{k \in S} \mu_k$ divides $Q^0_{Y,Z}(q)$, which implies that $Q_{Y,Z}(q) \in \mathbb{Z}[q, q^{-1}]$.

Therefore, in both cases, $Q_{Y,Z}(q)$ is a Laurent polynomial with integral coefficients.

For $Y \in \mathcal{Z}(\Lambda)$, let $b$ be a block in $Y$. We define the coordinate of $b$ to be the pair $(k, l)$ if $b$ is located in the $k$th column of $Y$ and the maximal number of unit cubes lying below $b$ is $l$. Note that a block in $Y$ is not
uniquely determined by its coordinates since two different blocks, which form a unit cube, have the same coordinate.

For a given coordinate $c = (k,l) \ (k, l \geq 0)$, we define the ladder at $c$ to be the finite sequence of coordinates as follows:

$$c = (k,l), (k - 1, l + 2), (k - 2, l + 4), \ldots, (0, l + 2k).$$

For $Y \in \mathcal{Y}(\Lambda)$, let $y_k$ be the left-most column in $Y$ such that $|y_k| \neq 0$. Choose an $i$-block $b$ placed on top of $y_k$ with a coordinate $c = (k,l)$. If the $i$-block is of type II and there is another block of type II on top of $y_l$, we choose the block at the front. Let $L_c$ be the ladder at $c$. Then it is the left-most ladder having nontrivial intersection with $Y$. We define $\overline{Y}$ to be the proper Young wall which is obtained by removing all the blocks in $Y$, which are contained $L_c$. It is easy to see that $\overline{Y}$ is also reduced.

**Example 6.1.** Let $Y$ be a reduced proper Young wall given in the following figure. Then $\overline{Y}$ is obtained by removing all the blocks in the ladder $L_{(2,0)} : (2,0), (1, 2), (0, 4)$.

Furthermore, if $Y$ is a reduced proper Young wall and $\overline{Y}$ is obtained by removing $r$ many $i$-blocks from $Y$, then the coefficient of $Y$ in $j_i^{(r)} \overline{Y}$ is 1 by (6.4) and (6.6).

Let $Y$ be a proper Young wall in $\mathcal{Z}(\Lambda)$. Let $L$ be a ladder such that there exists at least one block in $Y$ whose coordinate is in $L$. We denote $Y \cap L$ by the set of all the blocks in $Y$ whose coordinates are in $L$. Suppose that there are $r$ many $i$-blocks in $Y \cap L$ for some $r \geq 0$ and $i \in I$. Move these $i$-blocks to the first $r$ many $i$-slots in $L$ from the bottom. Repeat this procedure ladder by ladder until no block can be moved downward along a ladder. Then we obtain another proper Young wall $Y^R$, which we call the reduced form of $Y$. By definition, $Y^R$ is a reduced proper Young wall and $|Y^R| \geq |Y|$, where the equality holds if and only if $Y$ is reduced.

**Example 6.2.**
If \( Y = \begin{pmatrix} 0 & 0 & 2 \\ 1 & 1 & 2 \\ 1 & 1 & 2 \\ 0 & 2 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 2 & 1 \\ 1 & 1 & 1 \end{pmatrix} \), then \( Y^R = \begin{pmatrix} 0 & 0 & 2 \\ 1 & 1 & 2 \\ 1 & 1 & 2 \\ 0 & 2 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 2 & 1 \\ 1 & 1 & 1 \end{pmatrix} \).

Now, we are in a position to describe the algorithm. First, we will construct an \( A \)-basis \( A(\Lambda) = \{ A(Y) \mid Y \in Y(\Lambda) \} \) for \( V(\Lambda)^A \), which is invariant under the involution \( - \).

For \( Y \in Y(\Lambda) \), there exists a unique sequence of reduced proper Young walls \( \{ Y_k \}_{k=0}^N \) such that \( Y_0 = Y \), \( Y_{k+1} = Y_k \) \((0 \leq k < N)\), \( Y_N = Y_\Lambda \). Suppose that \( \text{wt}(Y_k) = \text{wt}(Y_{k-1}) + r_k \alpha_{i_k} \) \((1 \leq k \leq N)\). We define

(6.7) \[ A(Y) = f_{i_1}^{(r_1)} \cdots f_{i_N}^{(r_N)} Y_\Lambda \in V(\Lambda)^A. \]

It is clear that \( A(Y) = A(Y) \). We write

(6.8) \[ A(Y) = \sum_{Z \in Z(\Lambda)} A_{Y,Z}(q) Z, \]

where \( A_{Y,Z}(q) \in \mathbb{Z}[q, q^{-1}] \) (see (6.4) and (6.6)).

**Proposition 6.3.** Let \( Y \) be a reduced proper Young wall. For \( Z \in Z(\Lambda) \), we have

(a) if \( A_{Y,Z}(q) \neq 0 \), then \( |Y| \succeq |Z^R| \) and \( \text{wt}(Y) = \text{wt}(Z) \);
(b) if \( A_{Y,Z}(q) \neq 0 \) and \( |Y| = |Z| \), then \( Y = Z \) and \( A_{Y,Y}(q) = 1 \);
(c) \( A(\Lambda) \) is a \( \mathbb{Q}(q) \)-basis of \( V(\Lambda) \).

To prove this, we need the following technical lemma.

**Lemma 6.4.** Let \( Y \in Y(\Lambda) \) and \( Z \in Z(\Lambda) \) be such that \( |Y| \succeq |Z^R| \). Suppose that \( \text{wt}(\overline{Y}) = \text{wt}(Y) + r \alpha_i \) for some \( i \in I \) and \( r \geq 1 \). Then, for each \( W \in Z(\Lambda) \) occurring in the expansion of \( f_i^{(r)} Z \), we have

(a) \( |Y| \succeq |W^R| \);
(b) if \( |Y| = |W| \), then \( |\overline{Y}| = |Z| \) and \( Z \) is reduced;
(c) if \( |Y| = |W| \) and \( \overline{Y} = Z \), then \( Y = W \). \( \square \)

**Proof of Proposition 6.3.** We will use induction on \( l \), the number of blocks in \( Y \) which have been added to \( Y_\Lambda \). If \( l = 1 \), it is clear. Suppose that \( l > 1 \), and (a) and (b) hold for \( l' < l \). If \( A(Y) = f_{i_1}^{(r_1)} \cdots f_{i_N}^{(r_N)} Y_\Lambda \) for some
$N \geq 1$, then we have

\begin{equation}
A(Y) = f^{(r_1)}_{i_1} A(Y) = \sum_{|Y| \geq |Z|} A_{Y,Z}(q) f^{(r_1)}_{i_1} Z
\end{equation}

= \sum_{|Y| \geq |Z|} A_{Y,Z}(q) \left( \sum_{|Y| \geq |W|} Q_{Z,W}(q) W \right)

by Lemma 6.4 (a)

= \sum_{|Y| \geq |W|} \left( \sum_{|Y| \geq |Z|} A_{Y,Z}(q) Q_{Z,W}(q) \right) W.

We have

\begin{equation}
A_{Y,W}(q) = \sum_{|Y| \geq |Z|} A_{Y,Z}(q) Q_{Z,W}(q) \in \mathbb{Z}[q, q^{-1}],
\end{equation}

and $A_{Y,W}(q) = 0$ unless $|Y| \geq |W|$ and $\text{wt}(Y) = \text{wt}(W)$. If $A_{Y,W}(q) \neq 0$ and $|Y| = |W|$, then Lemma 6.4 (b) and (6.10) imply that $|Z| = |Y|$ and $A_{Y,Z}(q) \neq 0$. Hence, $Z = Y$ by induction hypothesis. Finally, we have $Y = W$ by Lemma 6.4 (c), and then $A_{Y,Y}(q) = A_{Y,Y}(q) Q_{Y,Y}(q) = 1$, which completes the induction argument.

By (a) and (b), $A(\Lambda)$ is linearly independent over $\mathbb{Q}(q)$. Since $\dim V(\Lambda)_\lambda = |Y(\Lambda)_\lambda|$ for all $\lambda \leq \Lambda$, $A(\Lambda)$ is a $\mathbb{Q}(q)$-basis of $V(\Lambda)$. This proves (c).

For $Y \in Y(\Lambda)$, we write

\begin{equation}
G(Y) = \sum_{Z \in Z(\Lambda)} G_{Y,Z}(q) Z \in V(\Lambda)^A
\end{equation}

where $G_{Y,Z} \in \mathbb{Q}(q)$. Note that the coefficient $G_{Y,Z}(q)$ satisfies

(i) $G_{Y,Z}(q) \in \mathbb{Q}[q]$,
(ii) $G_{Y,Z}(q) \in q \mathbb{Q}[q]$ unless $Y = Z$,
(iii) $G_{Y,Y}(q) = 1$.

Consider the following three matrices indexed by $Y(\Lambda)$ where the indices are decreasing with respect to the total ordering $>$;

\begin{equation}
G = (G_{Y,Z}(q)), \quad H = (H_{Y,W}(q)), \quad A = (A_{W,Z}(q)),
\end{equation}

where $H$ is the transition matrix from $A(\Lambda)$ to $G(\Lambda)$ as $\mathbb{Q}(q)$-bases of $V(\Lambda)$. We have $G = HA$. By the $-$ invariance of $G(\Lambda)$ and $A(\Lambda)$, $H$ is also invariant under the involution $-$. Since $A$ is a unipotent matrix with entries in $A$, $H$ is a matrix with entries in $A$, which yields:

**Proposition 6.5.** $A(\Lambda)$ is an $A$-basis of $V(\Lambda)^A$.  \qed
Also by the argument in [13], $H$ is a unipotent matrix. Hence for each $Y \in \mathcal{Y}(\Lambda \lambda)$ $(\lambda \leq \Lambda)$, $A(Y)$ can be expressed uniquely as follows;

\begin{equation}
A(Y) = G(Y) + \sum_{Z \in \mathcal{Y}(\Lambda \lambda) \text{ such that } Y > Z} \gamma_{Y,Z}(q)G(Z),
\end{equation}

for some $\gamma_{Y,Z}(q) \in \mathbb{Q}[q,q^{-1}]$ such that $\gamma_{Y,Z}(q) = \gamma_{Y,Z}(q^{-1})$. If $Y$ is the minimal element in $\mathcal{Y}(\Lambda \lambda)$, then $A(Y) = G(Y)$. Suppose that $Y$ is not minimal and $G(Y')$ are given for $Y' \in \mathcal{Y}(\Lambda \lambda)$ such that $Y' < Y$. Then $\gamma_{Y,Y'}(q)$ are determined inductively as follows;

1. if $Y'$ is the maximal one such that $Y > Y'$ and $A_{Y,Y'}(q) = \sum_{i=-r'} a_i q^{-i}$, then $\gamma_{Y,Y'}(q) = \sum_{i=1}^{r'} a_i (q^i + q^{-i}) + a_0$.
2. if the coefficient of $Y'$ in $A(Y) - \sum_{Y > Z > Y'} \gamma_{Y,Z}(q)G(Z)$ is given by $\sum_{i=-r'} a_i q^{-i}$, then $\gamma_{Y,Y'}(q) = \sum_{i=1}^{r'} a_i (q^i + q^{-i}) + a_0$.

To summarize, we have

**Theorem 6.6.** For a reduced proper Young wall $Y \in \mathcal{Y}(\Lambda \lambda)$ $(\lambda \leq \Lambda)$, the corresponding global basis element is of the following form;

\begin{equation}
G(Y) = Y + \sum_{Z \in \mathcal{Z}(\Lambda \lambda) \text{ such that } |Y| > |Z|} G_{Y,Z}(q)Z,
\end{equation}

where $G_{Y,Z}(q) \in q\mathbb{Z}[q]$ for $Y \neq Z$.

**Example 6.7.** In the following, we list $G(Y)$, where $Y$ is the reduced proper Young wall in $\mathcal{Y}(\Lambda_1)$ (Example 4.8). Set $k = \sum_{i \in I} k_i$ where $\text{wt}(Y) = \Lambda_1 - \sum_{i \in I} k_i \alpha_i$

1. $k = 1$
   \[ G( \begin{array}{c} 1 \\ 1 \end{array} ) = f_1 Y_{\Lambda_1} = \begin{array}{c} 1 \\ 1 \end{array} \]

2. $k = 2$
   \[ G( \begin{array}{c} 1 \\ 1 \\ 1 \end{array} ) = f_0 f_1 Y_{\Lambda_1} = \begin{array}{c} 0 \\ 0 \\ 1 \end{array} \]
   \[ G( \begin{array}{c} 1 \\ 1 \\ 2 \end{array} ) = f_2 f_1 Y_{\Lambda_1} = \begin{array}{c} 2 \\ 2 \\ 1 \end{array} \]

3. $k = 3$
   \[ G( \begin{array}{c} 1 \\ 1 \\ 2 \\ 1 \end{array} ) = f_0 f_2 f_1 Y_{\Lambda_1} = \begin{array}{c} 0 \\ 0 \\ 3 \\ 1 \end{array} \]
Example 6.8. In the previous example, we have seen that $G(Y) = A(Y)$. But this does not always hold for all $Y \in \mathcal{Y}(\Lambda)$. Furthermore, the coefficient polynomial $G_{Y,Z}(q)$ do not always have non-negative integral coefficients. Observe that
FOCK SPACE REPRESENTATION FOR $U_q(C^{(1)}_2)$

\[ A(\begin{array}{cccc}
1 & 2 & 1 & 0 \\
0 & 2 & 1 & 0 \\
1 & 2 & 1 & 0 \\
0 & 2 & 1 & 0 \\
\end{array}) = f_1 f_0 f_1 f_2 f_1 f_0 f_2 Y_{\Lambda_2} \]

\[ = \begin{array}{cccc}
1 & 1 & 1 & 0 \\
0 & 1 & 1 & 0 \\
2 & 0 & 2 & 0 \\
2 & 0 & 2 & 0 \\
\end{array} + q(1+q^6) \begin{array}{cccc}
1 & 1 & 1 & 0 \\
0 & 1 & 1 & 0 \\
2 & 0 & 2 & 0 \\
2 & 0 & 2 & 0 \\
\end{array} \]

\[ + q^2(1+q^2)(1-q^4) \begin{array}{cccc}
1 & 1 & 1 & 0 \\
0 & 1 & 1 & 0 \\
2 & 0 & 2 & 0 \\
2 & 0 & 2 & 0 \\
\end{array} + (1+q^2)^2 \begin{array}{cccc}
1 & 1 & 1 & 0 \\
0 & 1 & 1 & 0 \\
2 & 0 & 2 & 0 \\
2 & 0 & 2 & 0 \\
\end{array} + q(1+q^2) \begin{array}{cccc}
1 & 1 & 1 & 0 \\
0 & 1 & 1 & 0 \\
2 & 0 & 2 & 0 \\
2 & 0 & 2 & 0 \\
\end{array}. \]

On the other hand, we have

\[ A(\begin{array}{cccc}
1 & 1 & 1 & 0 \\
0 & 1 & 1 & 0 \\
1 & 1 & 1 & 0 \\
0 & 1 & 1 & 0 \\
\end{array}) = f_1^{(2)} f_2 f_0^{(2)} f_1^{(2)} f_2 Y_{\Lambda_2} \]

\[ = \begin{array}{cccc}
1 & 1 & 1 & 0 \\
0 & 1 & 1 & 0 \\
1 & 1 & 1 & 0 \\
0 & 1 & 1 & 0 \\
\end{array} + q \begin{array}{cccc}
1 & 1 & 1 & 0 \\
0 & 1 & 1 & 0 \\
1 & 1 & 1 & 0 \\
0 & 1 & 1 & 0 \\
\end{array} + q^4 \begin{array}{cccc}
1 & 1 & 1 & 0 \\
0 & 1 & 1 & 0 \\
1 & 1 & 1 & 0 \\
0 & 1 & 1 & 0 \\
\end{array} = G(\begin{array}{cccc}
1 & 1 & 1 & 0 \\
0 & 1 & 1 & 0 \\
1 & 1 & 1 & 0 \\
0 & 1 & 1 & 0 \\
\end{array}). \]

Therefore, we obtain

\[ G(\begin{array}{cccc}
1 & 1 & 1 & 0 \\
0 & 1 & 1 & 0 \\
1 & 1 & 1 & 0 \\
0 & 1 & 1 & 0 \\
\end{array}) = A(\begin{array}{cccc}
1 & 1 & 1 & 0 \\
0 & 1 & 1 & 0 \\
1 & 1 & 1 & 0 \\
0 & 1 & 1 & 0 \\
\end{array}) - G(\begin{array}{cccc}
1 & 1 & 1 & 0 \\
0 & 1 & 1 & 0 \\
1 & 1 & 1 & 0 \\
0 & 1 & 1 & 0 \\
\end{array}) \]

\[ = \begin{array}{cccc}
1 & 1 & 1 & 0 \\
0 & 1 & 1 & 0 \\
1 & 1 & 1 & 0 \\
0 & 1 & 1 & 0 \\
\end{array} + q(1+q^6) \begin{array}{cccc}
1 & 1 & 1 & 0 \\
0 & 1 & 1 & 0 \\
1 & 1 & 1 & 0 \\
0 & 1 & 1 & 0 \\
\end{array} \]

\[ + q^2(1+q^2)(1-q^4) \begin{array}{cccc}
1 & 1 & 1 & 0 \\
0 & 1 & 1 & 0 \\
1 & 1 & 1 & 0 \\
0 & 1 & 1 & 0 \\
\end{array} + (1+q^2)^2 \begin{array}{cccc}
1 & 1 & 1 & 0 \\
0 & 1 & 1 & 0 \\
1 & 1 & 1 & 0 \\
0 & 1 & 1 & 0 \\
\end{array} + q(1+q^2) \begin{array}{cccc}
1 & 1 & 1 & 0 \\
0 & 1 & 1 & 0 \\
1 & 1 & 1 & 0 \\
0 & 1 & 1 & 0 \\
\end{array}. \]
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