ON DUAL DYNAMIC PROGRAMMING IN SHAPE CONTROL

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ABSTRACT. We propose a new method for analysis of shape optimization problems. The framework of dual dynamic programming is introduced for a solution of the problems. The shape optimization problem for a linear elliptic boundary value problem is formulated in terms of characteristic functions which define the support of control. The optimal solution of such a problem can be obtained by solving the sufficient optimality conditions.

1. Introduction. We consider a classical optimum design problem with the aim to determine the best location of a source in a given bounded domain $\Omega$ in $\mathbb{R}^n$ with smooth boundary $\Gamma = \partial\Omega$. The design problem can be considered as a two level optimization problem. At the first level an optimal control within a set of admissible controls is determined for a given location of the source. At the second level an optimal location of the source in terms of its characteristic function is selected in such a way that the resulting value of the cost functional is the best possible within the set of admissible locations. The problem is considered for an elliptic equation, however the same framework can be used for some evolution equations (compare [5] and [15]). The problem at the second level is nonconvex, which leads to well known difficulties with the solution procedure. In particular, such difficulties and possible relaxation procedures are discussed e.g. in [10], [11] mainly from the point of view of existence of solutions in optimum design problems. The standard techniques in classical optimal control theory are based on the lower semicontinuity of some physical quantity (functional) with respect to control and on the compactness of the set of admissible controls. In the optimum design problem the location of the source is optimized. Thus, the lower semicontinuity of the shape functional is required with respect to some family of sets. So, except for very particular cases, there is no optimal location in optimum design problem (see [3]). That is why among the techniques and tools used to analyze this type of optimum design problems, the methods of relaxation, homogenization and appropriate variational formulations have played a very important role (see e.g. [10], [11]). In the last decade some optimum design problems were investigated for time-dependent state

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equations, including the wave equation or the heat equation (see e.g. [8], [13], [16]). Our aim is also to apply the particular variational method, widely used in the classical optimal control theory, which is the dynamic programming technique. In this way, we need neither relaxation nor homogenization of the problem under investigation. In some cases we are able to characterize the best location of the source explicitly, which is useful for possible applications.

Thus, consider the following problem of optimizing the shape and location of the source \( \omega \subset \Omega \) and its intensity \( v(x), x \in \Omega \) in the following Problem (P):

\[
\begin{align*}
\text{minimize } J(v, \omega) &= \frac{1}{2} \int_{\Omega} (u(x) - z_d(x))^2 \, dx + \frac{1}{2} \int_{\Omega} \chi_{\omega}(x) v(x)^2 \, dx \\
\text{subject to } &-\Delta u(x) = f(x) + v(x) \chi_{\omega}(x) \quad \text{a.e. in } \Omega, \\
|v(x)| &\geq \delta > 0 \quad \text{a.e. in } \Omega, \\
u(x) &= 0 \quad \text{on } \partial \Omega,
\end{align*}
\]

where \( \omega \subset \Omega \) is an open set, such that \( \epsilon_2 \geq \text{vol } \omega \geq \epsilon_1 > 0 \). The numbers \( \delta, \epsilon_1, \epsilon_2 \) are given; \( z_d : \Omega \to \mathbb{R}, f : \Omega \to \mathbb{R} \) are given functions, \( \chi_{\omega} \) is characteristic function of \( \omega \); \( u : \Omega \to \mathbb{R} \) is the state and \( v : \Omega \to \mathbb{R}, v \in L^2(\omega) \) stands for a control. We assume that the functions \( x \to z_d(x), x \to f(x) \) belong to \( L^2(\Omega) \). Let us observe that by standard elliptic theory [6] for each measurable \( \omega \subset \Omega \) and \( v \in L^2(\omega) \) there exists a unique solution \( u \in H^2(\Omega) \) of (1)-(3).

In section 5 we confine ourselves to the case of \( \Omega \) being a ball \( B(0, R) \) in order to be able to characterize the optimal pair \((\bar{\omega}, \bar{v})\) and the minimal value of \( J \). Note, that \( \bar{\omega} \) is an optimal open set in our notation. We call a triplet \((u(\cdot), v(\cdot), \omega)\) "admissible" if it satisfies (1)-(3) and the conditions imposed on \( \omega \). Then the corresponding trajectory \( u(\cdot) \) is said to be admissible and \( \omega \) is an admissible set. The set of all admissible triplets is denoted by \( Ad \).

The problems with unknown characteristic function of a subset \( \omega \) appear in many papers. In [13] the optimum design for two-dimensional wave equation is studied, in [12] an optimal location of the support of the control for one-dimensional wave equation is determined, in [7] and [8] the optimal geometry for controls in stabilization problem is considered. In all mentioned papers different approaches to the design problems have been investigated, and some numerical results are presented. The problem similar to (P), but with all integrals taken over \( \omega \) and without control \( v \), is discussed from the point of view of the existence of an optimal shape in the book [3] and from the geometrical point of view in [9]. The existence of optimum design is essential if we have not at hand any sufficient optimality conditions. From the beginning of the last century, under strong influence of Hilbert, the existence issue became one of the fundamental questions in many branches of mathematics, especially in calculus of variations as well as in its branch, the optimal control theory. Of course, following the existence proof, the next step is derivation of necessary optimality conditions and evaluation of the minimum argument. However, it should be pointed out that for many variational problems the existence of a solution accompanied by some necessary optimality conditions are not sufficient to find the argument of minimum in practice. On the other hand, having in hand a stronger result, i.e., the sufficient optimality conditions for a minimum in a specific problem, replaces the requirement for the existence. In the calculus of variations it was pointed out already by Weierstrass that the most important from practical point
of view for the solution procedure are the so-called sufficient optimality conditions for a relative minimum, i.e. the optimality conditions relative to some possibly smaller set of arguments of functional which is determined by additional (practical) conditions.

In the present paper the framework of dynamic programming together with sufficient optimality conditions (the so-called verification theorem for relative minimum) is proposed for a solution of the optimum design problem. Different approaches are given in [18], [1]. The shape problem is formulated in terms of characteristic functions which define the support of control. There is the lack of convexity in optimization problem (P), since the set of admissible controls given by (2) is nonconvex and the set of admissible characteristic functions \( \chi_\omega \) is nonconvex. Our goal is not the standard analysis of the problem as e.g. in [3], but the explicit solution by application of the sufficient optimality conditions given by dynamic programming. This approach seems to be new and the result obtained is original, to our best knowledge.

We provide a dual dynamic programming approach to control problems (1)-(3). This approach allows us to obtain the sufficient conditions for optimality in problem considered. We believe that the conditions for problems of type (P) in terms of dual dynamic programming, that we formulate here, have not been provided earlier. There are two main difficulties that must be overcome in such problems as (P). The first one consists in the following observation. We have no possibility to perform perturbations of the problem - as it is considered in the fixed set \( \Omega \) with boundary condition (3) - which can be compared to the one-dimensional case given in [2] and [4]. The second one is that we deal with elliptic equation for state and controls (1). The technique we apply is similar to the methods from [5] and [14]. The main idea of the methods from [5] and [14] is that they carry over all objects used in dynamic programming to dual space - space of multipliers (similar to those which appear in the Pontryagin maximum principle). Next, instead of classical value function (which for problem (P) makes no sense), we define an auxiliary function \( V(x,p) \) satisfying the second order partial differential equation of dual dynamic programming (compare [5]). Investigations of the properties of this function lead to an appropriate verification theorem. We introduce also the concept of an optimal dual feedback control and provide new sufficient optimality conditions determined within our framework. Just by using the differential equation of dual dynamic programming (7) (see below) and optimal dual feedback control (section 4) we are able to solve problem (P) completely, i.e. we give effective formulae for \( \omega \) and for \( v(x) \) as well for optimal value of \( J \).

2. Dual dynamic programming approach. In classical dynamic programming (i.e. in one dimensional case) we have a value function \( S(t,s) \) depending on time \( t \) and state variable \( s \). Having possibility to perturb a given point \( (t,s) \) we are able to calculate the full derivative of \( S(t,s) \): \( S_t(t,s) + S_s(t,s)\dot{s} \) and using some properties of the value function we can derive the Hamilton-Jacobi equation. Essential point in that approach is that we can perturb \( S(t,s) \) at each point of the open domain of definition of \( S \). In the case of problem (P) we do not have the possibility to perturb the optimal value of (P). That is why in [5] a new approach to dynamic programming was developed using some ideas of [14]. In [14], instead of considering notions of dynamic programming such as value function \( S(t,s) \) or Hamilton-Jacobi equation in the space \((t,s)\), a new space – the dual space is proposed and new notions of dual
dynamic programming are defined: an auxiliary function, a dual optimal value and a dual Hamilton-Jacobi equation which the auxiliary function should satisfy. The dual space in [14] is, in fact, defined by conjugate (dual) functions (variables) which appear in Pontryagin maximum principle. In turns out that this approach works also in control problems governed by elliptic equations (see [5]). That means: in dual approach to dynamic programming the perturbation of optimal value is not needed – instead we deal with an auxiliary function. However, there is a price to be paid for that, as we have to impose on the auxiliary function some additional condition, called the transversality condition. Our aim is to apply ideas from [5] to the optimum design problem (P). To this end we need to define dual notions in some dual space. Thus let \( P \subset \mathbb{R}^{n+2} \) be an open (dual) set of the variables \((x, p) = (x, y^0, y), x \in \Omega, y^0 < -\frac{1}{2}, |y| \geq 0\). We shall also use the set

\[
P_b = \left\{ (x, p) : x \in \partial \Omega, y^0 < -\frac{1}{2}, |y| \geq 0 \right\}.
\]

Denote by \( W^{2:3}(P) \) the specific Sobolev space of functions of two variables \((x, p)\) having up to the second order weak or generalized derivatives (in the sense of distributions) with respect \( x \) and up to the third order weak derivatives with respect to the variable \( p \). Our notation for the function space is used for the functions depending on the primal variable \( x \), and the dual variable \( p \), the primal and dual variables are independent and the functions in the space \( W^{2:3}(P) \) enjoy different properties with respect to \( x \) and \( p \). Let \( V(x, p) \) of \( W^{2:3}(P) \) be a (auxiliary) function defined on \( P \) and satisfying the following condition:

\[
V(x, p) = y^0 V_{y^0}(x, p) + y V_y(x, p) = p V_p(x, p),
\]

for \((x, p) \in P\). Here, \( V_{y^0}, V_y, \text{ and } V_p \) denote the partial derivatives and the gradient with respect to the dual variables \( y^0, y, \text{ and } p = (y^0, y) \), respectively.

Now, we denote by \( p(x), x \in \Omega, \) the dual trajectory, while \( u(x), x \in \Omega \) stands for the primal trajectory. Let us put

\[
u(x, p) = -V_y(x, p), \text{ for } (x, p) \in P.
\]

Using the function \( u \) it is possible to come back from the dual trajectories \( p(x), x \in \Omega, \) lying in \( P \) to the primal functions \( u(x), x \in \Omega \). How to find \( V_y \) is precisely described below. Further, we confine ourselves only to those admissible trajectories \( u(\cdot), \) for which there exist functions \( p(x) = (y^0, y(x)), (x, p(x)) \in P, y(\cdot) \in L^2(\omega), y(x) = 0, x \in \Omega \setminus \omega, \) such that \( u(x) = u(x, p(x)) \) for \( x \in \Omega \). Thus denote

\[
\text{Ad}_u = \left\{ (u(\cdot), v(\cdot), \omega) \in Ad : \text{there exist } p(x) = (y^0, y(x)), y(\cdot) \in L^2(\omega),
\right.
\]
\[
y(x) = 0, x \in \Omega \setminus \omega, (x, p(x)) \in P, x \in \Omega \text{ and } \psi : \mathbb{R}^n \to \mathbb{R}, \quad (x, y^0, \psi(x)) \in P_0, u(x, y^0, \psi(x)) = 0, p(x) = (y^0, \psi(x)), x \in \partial \Omega
\]
\[
\text{such that } u(x) = u(x, p(x)), x \in \Omega \right\}.
\]

Actually, it means that we are going to study problem (P) possibly in some smaller set \( \text{Ad}_u \), which is determined by the function (5).

Next, we define a dual optimal value \( S_B^u \) for the problem (P) by the formula

\[
S_B^u := \inf_{(u, v, \omega) \in \text{Ad}_u} \left\{ -y^0 \frac{1}{2} \int_{\Omega} (u(x) - z_u(x))^2 dx + \frac{1}{2} \int_{\Omega} \chi_\omega(x) v(x) dx \right\}.
\]
Denote by the symbol $\Delta_x h$ the sum of the second partial derivatives of the function $h : P \rightarrow \mathbb{R}$ with respect to the variable $x_i$, $i = 1, \ldots, n$, i.e.,

$$\Delta_x h(x, p) := \sum_{i=1}^{n} \frac{\partial^2 h}{\partial x_i^2}(x, p)$$

and similarly $\nabla_x h = (h_{x_1}, \ldots, h_{x_n})$. We use also the identity $\Delta_x h = \text{div} \nabla h$. In order to prove the verification theorem we require that the function $V(x, p)$ satisfies the second order partial differential equation in dynamic programming form:

$$\Delta_x V(x, p) = \max\{yf(x) + yv\chi_\omega(x) + y^0\frac{1}{2}(-V_y(x, p) - z_d(x))^2 + y^0\frac{1}{2}\chi_\omega(x)v^2 : |v| \geq \delta, \, \omega \subset \Omega, \, \epsilon_2 \geq \text{vol } \omega \geq \epsilon_1\}, \quad (x, p) \in P$$

with boundary condition

$$V_y(x, p) = 0, \, (x, p) \in P_o.$$  \hfill (8)

Let us note that in (7) maximum is taken instead of supremum since it is attained as it is shown in section 5 (the right hand side of (7) is a concave quadratic function of $v$ which is linear with respect to the characteristic function of $\omega$). We shall not discuss here the question of existence of solution to (7) and satisfying condition (4). We simply assume in verification theorem (in next section) that such a function exists. In section 5 we construct an auxiliary function $V$ satisfying (4) by giving effectively formula for it (see (35)). If we look carefully at the equation (7) then we see that the left hand side of it is an elliptic operator as we deal with elliptic state equation (1), the right hand side is simply the Hamiltonian for our problem (P) considered in dual space $P$. Recall that in one dimensional case (classical) the Hamilton-Jacobi equation is of the form: $S_i(t, s) + H(t, s, S_x(t, s)) = 0$ and its dual form is: $V_i(t, p) + H(t, -V_y(t, p), p) = 0$, where in both cases $H$ is the Hamiltonian of one dimensional optimal control problem, first considered in primal space $(t, s)$ and second in dual space $(t, p)$ (see [14]).

3. A verification theorem. In this section we formulate and prove our main theorem called “verification theorem”, which provides the sufficient optimality conditions for (P) in terms of a solution $V(x, p)$ of the second order partial differential equation of dynamic programming (7).

**Theorem 1.** Assume that there exists a $W^{2,3}(P)$ solution $V(x, p)$ of (7) on $P$ with boundary condition (8) such that (4) holds and let $\bar{u}(x, p) = -V_y(x, p), \, (x, p) \in P$. Let $(\bar{\nu}(\cdot), \bar{\omega}(\cdot), \bar{\omega}) \in Ad_{\bar{\alpha}}$ and $\bar{\nu}(x) = (\bar{\nu}^0, \bar{\nu}(x))$, $\bar{\nu}(\cdot) \in L^2(\bar{\omega})$, $\bar{\nu}(x) = 0$, $x \in \Omega \setminus \bar{\omega}$, $(x, \bar{\nu}(x)) \in P$, be a function such that $\bar{\nu}(x) = -V_y(x, \bar{\nu}(x))$ for $x \in \Omega$. Suppose that

$$\Delta_x V(x, \bar{\nu}(x)) = \bar{y}(x)(f(x) + \chi_\omega(x)\bar{\nu}(x))$$

$$+ \bar{y}^0\frac{1}{2}(-V_y(x, \bar{\nu}(x)) - z_d(x))^2 + \bar{y}^0\chi_\omega(x)\bar{\nu}(x)^2, \quad x \in \Omega.$$  \hfill (9)

Then $(\bar{\nu}(\cdot), \bar{\nu}(\cdot), \bar{\omega})$ is an optimal triplet relative to all $(u(\cdot), v(\cdot), \omega) \in Ad_{\bar{\alpha}}$.

**Proof.** Our proof starts with the observation that from transversality condition (4), we see that for $x \in \Omega$,

$$\Delta_x V(x, p(x)) = \bar{y}^0\Delta_x V_{y^0}(x, p(x)) + y(x)\Delta_x V_y(x, p(x))$$

\hfill (10)
Hence, (7) and (12) imply conditions for optimality which follow from the verification theorem. Optimal dual feedback control. We present appropriate definitions and the sufficient conditions for optimality which follow from the verification theorem. We conclude from (10)–(13) that

\[ W(x, p(x)) = -\Delta x V_p(x, p(x)) + y(x)(f(x) + \chi_\omega(x)v(x)) + \frac{1}{2} \left( -V_p(x, p(x)) - z_d(x) \right)^2 + \frac{1}{2} \chi_\omega(x)v(x)^2, \quad x \in \Omega. \]

Hence, (7) and (12) imply

\[ W(x, p(x)) \leq 0 \quad \text{for} \quad x \in \Omega \] (13)

and finally, after integrating (13) and applying (3), we have

\[-\int_{\Omega} \nabla V_p(x, p(x)) dx \leq -\frac{1}{2} \int_{\Omega} \left( -V_p(x, p(x)) - z_d(x) \right)^2 + \chi_\omega(x)v(x)^2 \] (14)

Thus from (14) and the Green formula it follows that

\[-\int_{\partial \Omega} \nabla V_p(s, \psi(s))
\nu(s)ds \leq -\frac{1}{2} \int_{\Omega} \left( -V_p(x, p(x)) - z_d(x) \right)^2 + \chi_\omega(x)v(x)^2 \] (15)

where \( \nu(\cdot) \) is the exterior unit normal vector to \( \partial \Omega \). In the same manner applying (9) and (3) we have

\[-\int_{\partial \Omega} \nabla V_p(s, \psi(s))\nu(s)ds = -\frac{1}{2} \int_{\Omega} \left[ (u(x) - z_d(x))^2 + \chi_\omega(x)v(x)^2 \right] dx. \]

(16)

Combining (15) with (16) gives

\[-\frac{1}{2} \int_{\Omega} \left[ (u(x) - z_d(x))^2 + (v(x)\chi_\omega(x))^2 \right] dx \leq -\frac{1}{2} \int_{\Omega} \left[ (u(x) - z_d(x))^2 + (v(x)\chi_\omega(x))^2 \right] dx \]

which completes the proof. \( \square \)

4. Optimal dual feedback control. This section is devoted to the notion of an optimal dual feedback control. We present appropriate definitions and the sufficient conditions for optimality which follow from the verification theorem.

Define in \( P \) a function \( v(\cdot, \cdot) \) with values in \( \{ v \in \mathbb{R} : |v| \geq \delta > 0 \} \). For each fixed \( x \in \Omega \) define in \( P \):

- for \( \left| \frac{y}{y} \right| \geq \frac{\delta}{2} \) a function \( \chi(\cdot, \cdot) \) which to those \( (x, \rho) \in P \) assigns an admissible set \( \omega \subset \Omega, x \in \omega, \) such that \( \chi(x, p) = \chi_\omega(x) \),
- for \( \left| \frac{y}{y} \right| < \frac{\delta}{2} \) a function \( \chi(\cdot, \cdot) \) which to those \( (x, p) \in P \) assigns an admissible set \( \omega \subset \Omega, x \notin \omega, \) such that \( \chi(x, p) = \chi_\omega(x) \),
i.e., for a given $p$ only one $\omega$ is assigned. This means that $\chi(x, p) = 0$ for $\left| \frac{x}{p} \right| < \frac{\delta}{2}$.

**Definition 2.** Let the functions $v(\cdot, \cdot)$ and $\chi(\cdot, \cdot)$ defined above be given. A pair $(v, \chi)$ is called a dual feedback control, if there is a solution $u(x, p)$, $(x, p) \in P$, of the partial differential equation

$$-\Delta_x u = f(x) + \chi(x, p)v(x, p)$$

defining nonempty set $\text{Ad}_u$.

**Definition 3.** A dual feedback control $(\nabla(x, p), \bar{\chi}(x, p))$, $(x, p) \in P$, is called an optimal dual feedback control, if the following conditions are verified. There exist the functions $\bar{v}(\cdot, \cdot) \in L^2(\bar{\omega})$, $\bar{y}(\cdot) = 0$, $x \in \Omega \setminus \bar{\omega}$, $(x, \bar{p}(x)) \in P$, such that there exist a triplet $(\bar{u}(\cdot, \cdot), \bar{v}(\cdot), \bar{\omega}) \in \text{Ad}_u$ satisfying

$$\bar{u}(x) = \bar{u}(x, \bar{p}(x)), \quad \bar{v}(x) = \nabla(x, \bar{p}(x)), \quad \bar{\chi}(x, x) = \bar{\chi}(x, \bar{p}(x)).$$

In addition, the optimal value $S_D^\pi$ (see (6)) is defined by the triplet $(\bar{u}(\cdot, \cdot), \bar{v}(\cdot), \bar{\omega})$:

$$S_D^\pi = -\bar{y}^0 \frac{1}{2} \int_{\Omega} (\bar{u}(x) - z_d(x))^2 + \bar{\chi}(x, x)\bar{v}(x)^2 dx$$

and there is $V(x, p)$, $(x, p) \in P$ belonging to $W^{2,3}(P)$, satisfying (4), such that $\Delta_x V^p(\cdot, \bar{\pi}(\cdot)) \in L^2(\Omega)$ and

$$\bar{y}^0 \int_{\Omega} \Delta_x V^p(s, \bar{\pi}(s)) ds = -S_D^\pi,$$

$V_p(x, p) = -\bar{v}(x, p)$.

Now, we formulate and prove the sufficient optimality conditions for the existence of an optimal dual feedback control, again in terms of the auxiliary function $V(x, p)$.

**Theorem 4.** Let $(\nabla(x, p), \bar{\chi}(x, p))$ be a dual feedback control in $P$ and $\bar{u}(x, p)$, $(x, p) \in P$, be defined according to Definition 2. Suppose that there exists a $W^{2,3}(P)$ solution $V(x, p)$ of (7) on $P$ such that (4) holds and that

$$V_p(x, p) = -\bar{v}(x, p) \quad \text{for} \quad (x, p) \in P. \tag{17}$$

Let $\bar{\pi}(x) = (\bar{\gamma}, \bar{y}(x)), \bar{y}(\cdot) \in L^2(\bar{\omega})$, $\bar{y}(x) = 0$, $x \in \Omega \setminus \bar{\omega}$, $(x, \bar{p}(x)) \in P$, be a function such that there is a triplet $(\bar{\pi}(\cdot), \bar{\pi}(\cdot), \bar{\omega}) \in \text{Ad}_u$ and $\bar{v}(x) = \bar{u}(x, \bar{p}(x))$, $\bar{v}(x) = \nabla(x, \bar{p}(x))$, $\bar{\chi}(x, x) = \bar{\chi}(x, \bar{p}(x))$, $x \in \Omega$. Furthermore, assume that:

$$\bar{y}^0 \int_{\Omega} \text{div}\nabla V^p(s, \bar{\pi}(s)) ds = \bar{y}^0 \frac{1}{2} \int_{\Omega} [\left\{ (\bar{u}(x, \bar{p}(x)) - z_d(x))^2 + \bar{\chi}(x, \bar{p}(x))\nabla(x, \bar{p}(x))^2 \right\} dx]. \tag{18}$$

Then $(\nabla(x, p), \bar{\chi}(x, p))$, $(x, p) \in P$, is an optimal dual feedback control.

**Proof.** Take any function $p(x) = (\bar{\gamma}, y(x)), y(\cdot) \in L^2(\omega)$, $y(x) = 0$, $x \in \Omega \setminus \omega$, $(x, p(x)) \in P$, such that there is a triplet $(u(\cdot), v(\cdot), \omega) \in \text{Ad}_u$ and $u(x) = \bar{u}(x, p(x))$, $v(x) = \nabla(x, p(x))$, $\chi(x, x) = \bar{\chi}(x, p(x))$, $x \in \Omega$. By (17), it follows that $u(x) = -V_p(x, p(x))$ for $x \in \Omega$. In the same way, as in the proof of Theorem 1, equation (18) gives

$$-\bar{y}^0 \frac{1}{2} \int_{\Omega} [\left\{ (\bar{u}(x, \bar{p}(x)) - z_d(x))^2 + \bar{\chi}(x, \bar{p}(x))\nabla(x, \bar{p}(x))^2 \right\} dx \leq -\bar{y}^0 \frac{1}{2} \int_{\Omega} [\left\{ (\bar{u}(x, \bar{p}(x)) - z_d(x))^2 + \bar{\chi}(x, \bar{p}(x))\nabla(x, \bar{p}(x))^2 \right\} dx. \tag{19}$$
We conclude from (19) that

$$S_\Omega \Pi = -\gamma \frac{1}{2} \int_\Omega \left[ (\overline{u}(x, \bar{p}(x)) - z_d(x))^2 + \bar{\chi}(x, \bar{p}(x))\nabla(x, \bar{p}(x))^2 \right] dx \quad (20)$$

and it is sufficient to show that \((\nabla(x, p), \bar{\chi}(x, p))\) is an optimal dual feedback control, by Theorem 1 and Definition 2.

5. Solution of the problem (P). From now on we assume that \(\Omega\) is a ball \(B(0, R)\) of radius \(R > \sqrt{\frac{2}{\pi}} \Gamma(n/2 + 1)\) (\(\Gamma(\cdot)\) is defined below in (22)) and for convenience with center at the origin 0. Thus we shall look for admissible \(\omega\) from the ball \(B(0, R)\). For the sake of simplicity, we shall still write \(\Omega\) instead of \(B(0, R)\). Writing \(\omega \subset \Omega\) we always assume in this section that \(\omega\) is admissible. We choose the ball \(B(0, R)\) for further investigation as in this case equation (1) has explicit solution

$$u(x) = -\int_\Omega G(x, s)f(s)ds - \int_\Omega G(x, s)v(s)\chi_s(s)ds.$$  

We recall the explicit formula for Green’s function \(G(x, s)\):

$$G(x, s) = \begin{cases} \Gamma(|x - s|) - \Gamma\left(\frac{|s|}{R} |x - \bar{s}|\right), & \text{for } s \neq 0, \\ \Gamma(|x|) - \Gamma(R), & \text{for } s = 0, \end{cases} \quad (21)$$

defined for all \(x, s \in B(0, R)\), where \(\bar{s} = \frac{R^2}{|s|^2} s\), for \(s \neq 0\) and \(\bar{s} = \infty\) for \(s = 0\). Here we denote

$$\Gamma(x - s) = \Gamma\left(|x - s|\right) = \begin{cases} \frac{1}{n(2 - n)\omega_n} |x - s|^{2-n}, & \text{for } n > 2, \\ \frac{1}{2\pi} \log |x - s|, & \text{for } n = 2, \end{cases} \quad (22)$$

where \(\omega_n\) stands for the volume of unit ball in \(\mathbb{R}^n\). In general case we only know that the Green’s function exists (see e.g. [6]). Thus we are able to derive a solution \(V(\cdot, p)\) in \(\Omega\) explicitly.

In order to solve the problem (P) we apply Theorems 1, 4. Thus, we need first to evaluate the dual feedback controls \(\nabla(x, p)\) and \(\chi(x, p)\) from equation (7) and then to establish an explicit formula for \(u(x, p)\) in terms of the dual feedback. Next we have to solve equation (7) explicitly. As the last part, we find the function \(\bar{p}(x) = (\bar{p}, \bar{y}(x))\), \(\bar{y}(\cdot) \in L^2(\omega)\) satisfying (9).

Now we formulate the main theorem of the paper, i.e. we describe the solution to problem (P).

**Theorem 5.** For problem (P) there exists optimal triplet \((\bar{u}(\cdot), \bar{v}(\cdot), \bar{\omega})\). The set \(\bar{\omega}\) satisfies:

$$\min\left\{ \int_\Omega \left( \int_\omega G(x, s)f(s)ds + \int_\omega G(x, s)yds + z_d(x) \right)^2 dx : |y| > \delta, \ \omega \subset \Omega, \epsilon_2 \geq vol \omega \geq \epsilon_1 \right\} = \int_\Omega \left( \int_\omega G(x, s)f(s)ds + \int_\omega G(x, s)yds + z_d(x) \right)^2 dx$$
with the optimal control  \( \tilde{v}(x) = \tilde{y}(x), x \in \omega \), where

\[
\tilde{y}(x) = \begin{cases} 
- \frac{\int_{\Omega} G(x, s)f(s)ds + z_d(x)}{\int_{\Omega} G(x, s)ds} & \text{if } \left| \frac{\int_{\Omega} G(x, s)f(s)ds + z_d(x)}{\int_{\Omega} G(x, s)ds} \right| > \delta, \\
\delta \cdot \text{sign} \left( -\frac{\int_{\Omega} G(x, s)f(s)ds + z_d(x)}{\int_{\Omega} G(x, s)ds} \right) & \text{otherwise}.
\end{cases}
\]

and the state

\[
\bar{u}(x) = -\int_{\Omega} G(x, s)f(s)ds - \int_{\Omega} G(x, s)\tilde{y}(s)ds,
\]

with the minimal value of \( J \) given by

\[
J(\bar{v}, \bar{\omega}) = \frac{1}{2} \int_{\bar{\omega}} \bar{y}^2(s)ds + \frac{1}{2} \int_{\Omega} \left( \int_{\Omega} G(x, s)f(s)ds + \int_{\Omega} G(x, s)\tilde{y}(s)ds + z_d(x) \right)^2 dx.
\]

**Remark 6.** The shape of \( \bar{\omega} \) depends on the functions \( f(\cdot) \) and \( z_d(\cdot) \) and can be calculated numerically (for a fixed space dimension \( n \)).

**Proof.** Accordingly to Theorem 1, first, we have to solve equation (7):

\[
\Delta_x V(x, p) = \max \left\{ y(f(x) + v\chi_\omega(x)) + y^0\frac{1}{2}(-V_y(x, p) - z_d(x))^2 \right. \\
+ y^0\frac{1}{2}v^2\chi_\omega(x) : |v| \geq \delta, \omega \subset \Omega, \epsilon_2 \geq \text{vol } \omega \geq \epsilon_1 \left\}, \quad (x, p) \in P.
\]

The evaluation of the “max” with respect to \( v \) in (23) (quadratic concave function) allows us to rewrite this equation in the form, for \( (x, p) \in P \) and \( |y/y^0| > \delta/2 \),

\[
\Delta_x V(x, p) = yf(x) + y^0\frac{1}{2}(-V_y(x, p) - z_d(x))^2 \\
+ \max \left\{ -\frac{1}{2}y^2\chi_\omega(x) : \omega \subset \Omega, \epsilon_2 \geq \text{vol } \omega \geq \epsilon_1 \right\}
\]

and in the form, for \( (x, p) \in P \) and \( |y/y^0| \leq \delta/2 \),

\[
\Delta_x V(x, p) = yf(x) + y^0\frac{1}{2}(-V_y(x, p) - z_d(x))^2.
\]

Let us note, that from (24) we can also infer, because the expression \(-\frac{1}{2}y^2/y^0\) is positive (recall that \( y^0 < -1/2 \)), that the following relation is valid

\[
\max \left\{ -\frac{1}{2}y^2\chi_\omega(x) : \omega \subset \Omega, \epsilon_2 \geq \text{vol } \omega \geq \epsilon_1 \right\}
\]

\[
= -\frac{1}{2}y^2 \max \{ \chi_\omega(x) : \omega \subset \Omega, \epsilon_2 \geq \text{vol } \omega \geq \epsilon_1 \}
\]

with subset \( \omega \) depending on \( x \) and independent of \( p \). From the above, we infer that the dual feedback control \((v(x, p), \chi(x, p))\) has the form

\[
v(x, p) = \bar{v}(p) = \begin{cases} 
-\frac{y}{y^0} & \text{for } |y/y^0| > \delta, \quad (x, p) \in P, \\
\delta & \text{for } |y/y^0| \leq \delta, \quad (x, p) \in P,
\end{cases}
\]

\[
\chi(x, p) = \begin{cases} 
1 = \bar{\chi}(x) & |y/y^0| \geq \delta/2, \\
0, & \text{otherwise}.
\end{cases}
\]
Let us consider different expression depending on \( \omega \). Now we assign to each \( p \), with \( |y/y^0| \geq \delta/2 \), an admissible set \( \omega \) in the way described below. It is clear that for each \( p \)

\[
\min \{ \int_\Omega \left( \int_\Omega G(x, s)f(s)ds + \int_\omega G(x, s)\bar{\nu}(p)ds + z_d(x) \right)^2 dx : \epsilon_2 \geq \text{vol} \ \omega \geq \epsilon_1 \}
\]

is attained for some \( \omega(p) \) with its shape dependent on the functions \( f(\cdot) \) and \( z_d(\cdot) \). Really, let us observe that \( \omega \) is attained for some \( \omega(p) \) with its shape dependent on the functions \( f(\cdot) \) and \( z_d(\cdot) \). Thus there is a minimizing sequence \( \{w(\cdot, \omega)\} \subset W \) such that

\[
\lim \inf_{k \to \infty} I(w(\cdot, \omega_k)) = \inf_{w \in W} I(w(\cdot, \omega)).
\]

Let us note that all solutions of (30), with \( \omega \subset \Omega, \epsilon_2 \geq \text{vol} \ \omega \geq \epsilon_1 \), are bounded and have bounded their gradients (see e.g. [6]). Thus the set \( W \) is strongly compact in \( L^2(\Omega) \) and closed in the topology of pointwise convergence. We only justify the last assertion. To this effect let us note that for each fixed \( x \in \Omega, w(x, \cdot) \) defined by (29) is absolutely continuous measure with respect to Lebesgue measure in \( \Omega \). Thus, if we take a sequence \( \omega_l \subset \Omega, \epsilon_2 \geq \text{vol} \ \omega \geq \epsilon_1, l = 1, 2, \ldots \) such that \( \lim_{l \to \infty} w(x, \omega_l) \) exists and is equal to some \( w(x, \omega_0) \) then \( \epsilon_2 \geq \text{vol} \ \omega_0 \geq \epsilon_1 \) and hence \( w(\cdot, \omega_0) \in W \). Therefore the sequence \( \{w(\cdot, \omega_k)\} \) has a subsequence which is pointwise convergent to some \( w(\cdot, \omega(p)) \), \( \epsilon_2 \geq \text{vol} \ \omega(p) \geq \epsilon_1 \) and

\[
\lim \inf_{k \to \infty} I(w(\cdot, \omega_k)) = I(w(\cdot, \omega(p))).
\]

Thus \( \chi(x, p) = \chi_{\omega(p)}(x) \) for \( (x, p) \in P, |y/y^0| \geq \delta/2 \). According to (4) and boundary condition (8) a solution to equation (23) has to satisfy:

\[
V(x, p) = yV_y(x, p) + y^0V_{y^0}(x, p), \quad (x, p) \in P; \quad V_y(x, p) = 0, \ (x, p) \in P_b. \quad (31)
\]

Applying known formula for the solution of Poisson’s equation in a ball [6] we have for the equation (with \( p \) as a parameter)

\[
-\Delta_x u(x, p) = f(x) + \chi(x, p)v(x, p), \ (x, p) \in P; \quad u(x, p) = 0, \ (x, p) \in P_b
\]

with \( v(x, p) \) and \( \chi(x, p) \) as in (28), i.e.,

\[
-\Delta_x u(x, p) = f(x) + \bar{\nu}(p)\chi(x, p), \ u(x, p) = 0 \text{ on } P_b, \quad (32)
\]
a solution
\[ u(x, p) = -\int_{\Omega} G(x, s)f(s)ds - \int_{\Omega} G(x, s)\bar{v}(p)\chi(s, p)ds. \] (33)

Here \( G(x, s) \) is the Green’s function for \( \Omega \) defined in (21), having the following properties: \( G(x, s) = G(s, x), \) \( G(x, s) \leq 0 \) for \( x, s \in \Omega, \) \( G(x, s) = 0, \) \( x \in \partial\Omega, s \in \Omega. \)

From the relation \( V_y(x, p) = -u(x, p) \) we get the first element of the right hand side of (31). To find the second element let us notice that \( V_y(x, p) \) must be defined by our cost function with suitable \( u(x, p), v(x, p) \) and \( \chi(x, p). \) Therefore, let us assume that for \((x, p) \in P\)
\[ V_y(x, p) = \frac{1}{2} \int_{\Omega} G(x, s)(u(s, p) + z_d(s))^2 ds + \frac{1}{2} \int_{\Omega} G(x, s)\bar{v}(p)\chi(s, p)ds, \] (34)

where \( u(s, p) \) is defined by (33). Then by the above and (31) we have, for \((x, p) \in P:\)
\[ V(x, p) = y_{\partial} \left[ \int_{\Omega} G(x, s)f(s)ds + \int_{\Omega} G(x, s)\bar{v}(p)\chi(s, p)ds \right] \]
\[ + y^0 \left[ \frac{1}{2} \int_{\Omega} G(x, s)(u(s, p) + z_d(s))^2 ds + \frac{1}{2} \int_{\Omega} G(x, s)\bar{v}(p)^2\chi(s, p)ds \right] ds. \] (35)

We easily verify that \( V \) defined by (35) satisfies equation (24), and that relation (4) is also satisfied. Thus we are ready to define the set \( Ad_u \) of admissible triplets:
\[ Ad_u = \{(u(\cdot), v(\cdot), \omega) \in Ad : \text{there is } p(x) = (y^0, y(x)), y(\cdot) \in L^2(\omega), y(x) = 0, x \in \Omega \setminus \omega, (x, p(x)) \in P \text{ and } \psi : \mathbb{R}^n \rightarrow \mathbb{R}, \]
\[ (x, y^0, \psi(x)) \in P_b, u(x, y^0, \psi(x)) = 0, p(x) = (y^0, \psi(x)), x \in \partial\Omega \]
\[ \text{such that } u(x) = u(x, p(x)), x \in \Omega, \]

where \( u(x, p) \) is defined by (32). We see that for this \( u, \) for each \( p(x) = (y^0, y(x)), y(\cdot) \in L^2(\omega), y(x) = 0, x \in \Omega \setminus \omega, \) there exists a solution \( u \) of (32) and \( u(x) = 0, x \in \partial\Omega. \) Hence we infer that for \( u \) defined by (32) \( Ad_u = Ad. \) According to verification theorem the next step is to find a function \( \bar{v}(x) = (\bar{y}^0, \bar{y}(x)), \bar{y}(\cdot) \in L^2(\omega), \bar{y}(x) = 0, \]
\[ x \in \Omega \setminus \omega \text{ satisfying (23) for defined above function } V. \] To this effect we can assume that \( \bar{y}^0 = -1 \) and find \( \bar{y} \) minimizing the functional
\[ (-1, y) \rightarrow \int_{\Omega} \left( \int G(x, s)f(s)ds + \int_{\omega((1, y))} G(x, s)\bar{v}(-1, y)ds + z_d(x) \right) dx. \]

In a similar way, as above, one can prove (taking into account the form of \( \bar{v} \)) that such a \( \bar{y} \) exists and is defined, for \( x \in \omega((-1, \bar{y})) \), by the following function
\[ \bar{y}(x) = \begin{cases} \frac{-\int_{\omega((-1, \bar{y}))} G(x, s)f(s)ds + z_d(x)}{\int_{\omega((-1, \bar{y}))} G(x, s)ds} & \text{if } \left| \frac{\int_{\omega((-1, \bar{y}))} G(x, s)f(s)ds + z_d(x)}{\int_{\omega((-1, \bar{y}))} G(x, s)ds} \right| > \delta, \\ \delta \cdot \text{sign} \left( \frac{-\int_{\omega((-1, \bar{y}))} G(x, s)f(s)ds + z_d(x)}{\int_{\omega((-1, \bar{y}))} G(x, s)ds} \right) & \text{otherwise.} \end{cases} \]

Next note that the function \((-1, \bar{y}(x)), x \in \omega((-1, \bar{y})) \) satisfies (26) with \( V \) defined by (35) and \( \bar{v}((-1, \bar{y}(x))) = \bar{y}(x). \) From (33) we see that for \( \omega((-1, \bar{y})) \) and \( \bar{v}, \) the function \( u(x, (-1, \bar{y}(x))) \) takes the form
\[ u(x, (-1, \bar{y})) = -\int_{\Omega} G(x, s)f(s)ds - \int_{\omega((-1, \bar{y}))} G(x, s)\bar{v}(s)ds \]
and the cost functional $J$ takes the form

$$J(\tilde{y}, \omega((-1, \tilde{y}))) = \frac{1}{2} \int_{\omega((-1, \tilde{y}))} \tilde{y}^2(s)ds$$

$$+ \frac{1}{2} \int_{\Omega} \left( \int_{\Omega} G(x,s)f(s)ds + \int_{\omega((-1, \tilde{y}))} G(x,s)\tilde{y}(s)ds + z_d(x) \right)^2 dx .$$

Thus, in this case the triplet: $\tilde{u}(x) = u(x, (-1, \tilde{y}(x)))$, $\tilde{v}(x) = \bar{v}((-1, \tilde{y}(x))) = \tilde{y}(x)$, $\omega((-1, \tilde{y}))$ is optimal and

$$\frac{1}{2} \int_{\Omega} \left( - \int_{\Omega} G(x,s)f(s)ds - \int_{\omega((-1, \tilde{y}))} G(x,s)\tilde{y}(s)ds - z_d(x) \right)^2 dx + \frac{1}{2} \int_{\omega((-1, \tilde{y}))} \tilde{y}^2(s)ds$$

is the optimal value of the shape functional functional $J$. Thus we have proved the theorem and solved our problem (P) in the ball $B(0, R)$. \hfill \Box

Remark 7. The dynamic programming by construction furnishes the sufficient optimality conditions, therefore it is a powerful tool for solution of optimization problems which enjoy the special structure. In the paper a model problem is considered, and its solution is characterized. However, the problem is of the specific form, the unknown shape is defined by characteristic functions of admissible subsets in $\Omega$. In a subsequent paper we are going to apply dynamic programming technique to more general shape optimization problems.

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