Rayleigh Edge Waves in Two-Dimensional Crystals with Lorentz Forces - from Skyrmion Crystals to Gyroscopic Media

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We investigate, within the framework of linear elasticity theory, edge Rayleigh waves of a two-dimensional elastic solid with broken time-reversal and parity symmetries due to a Berry term. As our prime example, we study the elastic edge wave traveling along the boundary of a two-dimensional skyrmion lattice hosted inside a thin-film chiral magnet. We find that the direction of propagation of the Rayleigh modes is determined not only by the chirality of the thin-film, but also by the Poisson ratio of the crystal. We discover three qualitatively different regions distinguished by the chirality of the low-frequency edge waves, and study their properties. To illustrate the Rayleigh edge waves in real time, we have carried out finite-difference simulations of the model. Apart from skyrmion crystals, our results are also applicable to edge waves of gyroelastic media and screened Wigner crystals in magnetic fields. Our work opens a pathway towards controlled manipulation of elastic signals along boundaries of crystals with broken time-reversal symmetry.

I. INTRODUCTION

Recent years have seen a new surge of excitement around chiral surface waves in hydrodynamics [1–8]. The role of bulk topology for the existence and robustness of such waves has been vigorously investigated [1, 2, 5, 7, 8]. Chiral surface modes have been also recently used as a tool to measure the bulk Hall viscosity of an active fluid [9].

In this work, we study surface waves in two-dimensional crystals which break parity P (spatial reflection) and time-reversal T symmetry, but preserve the combined PT symmetry. In the quantum realm, well-known examples of such systems are two-dimensional thin-film chiral magnets, which host lattices formed out of skyrmion defects [10–13]. Wigner crystals in a magnetic field [14] and Abrikosov vortex lattices in superconductors and rotating superfluids [15]. In the last few years such crystals were also designed in gyroscopic metamaterials [16], [17] and mass-spring networks subject to Coriolis forces [18].

The investigation of waves that propagate along the free surface of an elastic solid, and whose disturbance remains confined to the vicinity of the boundary is an old topic that has been studied in hydrodynamics and solid mechanics. In the quantum realm, well-known examples of such systems are two-dimensional thin-film chiral magnets, which host lattices formed out of skyrmion defects [10–13]. Wigner crystals in a magnetic field [14] and Abrikosov vortex lattices in superconductors and rotating superfluids [15]. In the last few years such crystals were also designed in gyroscopic metamaterials [16], [17] and mass-spring networks subject to Coriolis forces [18].

The general focus of this paper is the investigation of a long-wavelength effective field theory of a two-dimensional skyrmion lattice, where the Cartesian components of the displacement from equilibrium positions \( u^x \) and \( u^y \) are coupled by a Berry term [23, 24]. The displacements are assumed to be small, which allows the framework of linear elasticity to be employed. We find that the behavior of the edge-waves can be tuned by changing the Poisson ratio \( \sigma \) [25]. In fact, we show that there exist three qualitatively distinct phases, captured by the diagram in Fig. 1. The phases are distinguished by the propagation direction of their low-frequency surface waves. In the long-wavelength and low-frequency limit we develop an analytic treatment of these edge waves.

II. SKYRMION CRYSTAL ELASTICITY IN THIN-FILM CHIRAL MAGNETS

It is well-known that an elementary skyrmion defect in a ferromagnet experiences an effective magnetic field \( B \) and the associated Lorentz force because it picks up the Berry phase of \( 2\pi \) whenever encircling a spin 1\( / 2 \) [26]. Moreover, a skyrmion can be characterized by a finite inertial mass \( m \), which was derived in [27] by integrating out fluctuations of its spatial profile. In this paper we study the surface waves of two-dimensional skyrmion lattices present in thin-film chiral magnets such as Fe\(_{80}\)Co\(_{20}\)Si and FeGe [12]. Starting from the continuum theory of [23, 24], the skyrmion dynamics is described by a field theory of coarse-grained elastic variables \( u^i (x) \) with \( i = x, y \), denoting the displacements of skyrmions from their equilibrium positions, see [13] for a pedagogical exposition. The action of the skyrmion displacement field is given by

\[
S [u^i] = \int dt \, d^2x \left[ \frac{\rho}{2} \dot{u}^i \dot{u}^j - \frac{\rho \Omega}{2} \epsilon_{ijk} u^i \dot{u}^j - \mathcal{E}_{el} (u_{ij}) \right],
\]

where the overdot denotes the time derivative, \( \rho \) is the mass density of the skyrmions, \( \Omega = B / m \) is the cyclotron frequency associated with effective magnetic field \( B \), \( u_{ij} = \partial_i u_j \) is the symmetric linearized strain tensor, and \( \mathcal{E}_{el} (u_{ij}) \) the elastic energy density, dictated by the geometry of the crystal. The convention for the completely antisymmetric Levi-Civita symbol is \( \epsilon_{xy} = - \epsilon_{yx} = 1 \) and summation over repeated indices is understood. At low frequencies, the Berry term in Eq. (1), which gives rise to an effective Lorentz force, dominates the first term that encodes Newtonian dynamics. As a result, \( u^x \) and \( u^y \) form a canonically conjugate pair of

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where we introduced a small parameter $\varepsilon \equiv (1 + \sqrt{5})/2$ is the golden ratio. The frequency/wavevector space. Contrary to the situation where the latter mode is guaranteed to have the gap $\omega = \Omega$ at $k = 0$ by the Kohn theorem [31]; the system is analogous to a collection of single-species charged particles in a uniform magnetic field that interact through a potential which depends only on their relative distances. In the zero wavevector limit, the polarization of the Kohn mode is circular and its chirality is fixed by the sign of the effective magnetic field $B$. (ii) In the limit of large wavevectors, the Newtonian term dominates over the Berry term in Eq. (3) and we asymptotically recover two linearly dispersing sound modes of a time-reversal invariant two-dimensional solid [32]. In particular, at large momenta, the magnetophonon (5) merges into the transverse ($\omega \equiv (1 + \sqrt{5})/2$) mode with

$$\omega_\pm(k) = \Omega \left[ 1 + \left( \frac{v_1 + v_2}{\Omega^2} \right) k^2 + O \left( \frac{k^4}{\Omega^4} \right) \right].$$

The latter mode is guaranteed to have the gap $\omega = \Omega$ at $k = 0$ by the Kohn theorem [31]; the system is analogous to a collection of single-species charged particles in a uniform magnetic field that interact through a potential which depends only on their relative distances. In the zero wavevector limit, the polarization of the Kohn mode is circular and its chirality is fixed by the sign of the effective magnetic field $B$. (ii) In the limit of large wavevectors, the Newtonian term dominates over the Berry term in Eq. (3) and we asymptotically recover two linearly dispersing sound modes of a time-reversal invariant two-dimensional solid [32]. In particular, at large momenta, the magnetophonon (5) merges into the transverse ($\omega \equiv (1 + \sqrt{5})/2$) mode with $\omega_\pm = \Omega [1 + O(\Omega^2/k^2)]$, while the magnetophonon (6) becomes the longitudinally polarized ($\epsilon_{ij}k_iu_{ij} = 0$) mode with $\omega_+ = \epsilon_{ij}k_iu_{ij} = 0$) mode with $\omega_+ = \epsilon_{ij}k_iu_{ij} = 0$)

The Lagrangian naturally fits into a derivative expansion within the power-counting scheme $u^i \sim O(1)$, $\partial_i \sim O(\varepsilon)$, $\partial_\tau \sim O(\varepsilon^2)$ where we introduced a small parameter $\varepsilon \ll 1$. The difference in the power-counting of temporal and spatial derivatives originates from the soft quadratic dispersion of the magnetophonon. All terms in the Lagrangian (1), except for the Newtonian term $\rho\dot{u}^2/2$, are of order $O(\varepsilon^2)$, defining the leading-order (LO) Lagrangian. On the other hand, the Newtonian term scales as $\varepsilon^2$ and is less relevant at low frequencies, and thus is of the next-to-leading order (NLO). The inclusion of this term allows us to establish the crossover of edge waves...
that exist in the chiral system to the ordinary Rayleigh waves in the absence of a Berry term. We notice here that other NLO terms such as second-order elasticity $\lambda_{ijklmn} \partial_i \partial_j u_k \partial_l \partial_m u_n$ or the dissipationless phonon Hall viscosity $\eta_{ijkl} \partial_i u_j \partial_k \partial_l$ are not included in the present work.

In elastic media, internal stresses and forces are encoded in the stress tensor $T_{ij} = \delta \mathcal{E}_{el}/\delta u_{ij}$ [20, 21]. For a two-dimensional triangular crystal with the elastic energy density (2) the stress tensor is

$$T_{ij} = 4C_1 u_{kk} \delta_{ij} + 4C_2 \tilde{u}_{ij}. \quad (7)$$

### III. RAYLEIGH EDGE MODES

We next turn to the study of exponentially localized Rayleigh waves that propagate on the edge of the skyrmion lattice. The breaking of time-reversal and parity symmetries in (1) due to the Berry term suggests that such modes might be chiral, i.e., propagating only in one direction, see e.g. [34] and [30]. For the sake of simplicity, we consider the skyrmion crystal to fill the lower half-space with $y < 0$. Without loss of generality we also choose $\Omega > 0$ throughout the rest of the paper.

The translational invariance in time and along the horizontal direction motivates the ansatz $u(x, y, t) = e^{i(kx - \omega t)}e^{iy}$ for a solution of (3). The wavevector along the boundary $k$ and the frequency $\omega$ are assumed to be real; confinement near the edge of the system requires the real part of $k$ to be positive.

First, in order to make the following calculation more transparent, we shall focus on the low-frequency limit and drop the NLO Newtonian term $\tilde{u}^2$ in the model (1). The edge ansatz inserted into Eq. (3) results in a characteristic equation for $\kappa$ with two solutions

$$\kappa_{1,2}(k, \omega) = \sqrt{k^2 \pm \frac{\Omega}{\sqrt{v_2(2v_1 + v_2)}} \omega + O(k^2)} \quad (8)$$

The corresponding eigenvectors $u_{1,2}(k, \omega)$ are functions of the wavevector $k$ and frequency $\omega$. Interestingly, here, in contrast to the ordinary Rayleigh construction, both solutions $\kappa_{1,2}$ originate from the single magnetophonon branch.

The general solution with given $k$ and $\omega$ is obtained by forming a linear superposition of $u_{1,2}(k, \omega)$ with two complex constants $a, b$

$$u(x, y, t) = e^{i(kx - \omega t)}(a u_1 e^{i \kappa_1 y} + b u_2 e^{i \kappa_2 y}). \quad (9)$$

Due to the $PT$ symmetry of the model, the dispersion satisfies $\omega(k) = -\omega(-k)$, hence it is sufficient to study only the interval $\omega \geq 0$.

Here we will assume that the crystal is free at the boundary $y = 0$. In this case there are no macroscopic forces acting on it from the outside. Thus there is no flux of linear momentum across the boundary surface at $y = 0$, resulting in the so-called stress-free boundary conditions [35]

$$T_{xy}(x, y = 0) = T_{yx}(x, y = 0) = 0. \quad (10)$$

Substituting the ansatz (9) into the boundary conditions (10) results in the linear system of equations for $a$ and $b$

$$\begin{pmatrix} ik\sigma_1 + \kappa_1 & ik\sigma_2 + \kappa_2 \\ ik + \kappa_1 & ik + \kappa_2 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = 0, \quad (11)$$

where we have introduced the two-dimensional Poisson ratio $\sigma \equiv (2C_1 - C_2)/(2C_1 + C_2)$ and the shorthand $\kappa_{1,2} \equiv u_{1,2}^{ik}/u_{1,2}$. The dispersion relation $\omega(k)$ for the edge waves is obtained from the characteristic equation for the matrix in Eq. (11).

In traditional treatments of elasticity theory, the Poisson ratio was usually assumed to be a positive value [20]; however, in recent years, it was found that elastic systems can be engineered to have a negative Poisson ratio [36, 37] and even more remarkably that such materials actually occur in nature [38]. By now an explosion of research into exotic metamaterials has taken place (for an overview see [39]), which go by the name auxetic materials. These systems have the counter-intuitive property that under uniaxial compression, they contract in the orthogonal direction. In the following we investigate the interval $-1 \leq \sigma \leq 1$, where the elastic system is stable.

Substitution of the two edge modes into (11) yields a dispersion relation $\omega(k)$ of the form

$$\omega(k) = \frac{\alpha}{\sqrt{v_2(2v_1 + v_2)}} k^2 \quad (12)$$

with $\alpha$ being a non-negative and real solution of an unwieldy algebraic equation, which we investigate in detail in Appendix A. This equation does not depend on the magnitude of $k$, but only on its sign. As a consequence, the equations for positive and negative $k$ are in general different, resulting in different solutions $\alpha(\text{sign}(k), \sigma)$, which we will denote by $\alpha_{\pm}(\sigma)$.

We show the numerical solution of $\alpha_{\pm}(\sigma)$ in Fig. 2. As the value of $\sigma$ is varied, one finds three qualitatively different regimes. For $\sigma < 0$ only the $\alpha_-$ branch exists: edge waves can only propagate towards left, while propagation to the right is forbidden. We find analytically in Appendix A that for $\sigma > \varphi^{-1} = (\sqrt{5} - 1)/2$, i.e. the inverse golden ratio, the edge waves are once again chiral, but with propagation in the opposite direction. In the interval $0 \leq \sigma \leq \varphi^{-1}$ both branches of $\alpha_{\pm}$ exist and consequently edge waves can propagate in both directions. The dispersion of the surface waves is generically asymmetric, since in general $\alpha_+ \neq \alpha_-$. However, it is clear from Fig. 2 that at the point $\sigma = 1/3$ [40] the spectrum is symmetric, see Appendix A for the analytical justification, where we also determine the value $\alpha_{\pm} = 2\sqrt{2}/3$.

In the presence of the sub-leading Newtonian term we solved the edge problem numerically. The resulting spectrum is sketched in Fig. 1. The inclusion of the Newtonian term results in propagation in a forbidden direction for momentum and frequency larger than a critical value $k_{\text{crit}}$ and $\omega_{g} \equiv \omega(k_{\text{crit}})$. We checked that ordinary Rayleigh waves are recovered for $\omega \gg \omega_g$.

In order to illustrate our findings, we have carried out finite-difference simulations of dynamics encoded by (3) subject to
FIG. 2: For small momenta the dispersion relation has the quadratic form $\omega = \alpha \sqrt{v_2(2v_1 + v_2)} / \Omega k^2$. When only one of the branches $\alpha_{\pm}$ exists, the edge wave propagates unidirectionally. This happens for $\sigma < 0$ and $\sigma > \varphi^{-1}$, see Appendix A.

![Image](image.png)

FIG. 3: Edge excitations as seen in finite-difference simulations [41]. In all three plots, the system was displaced in a small region near the boundary at $x = 0$ and evolved over time. The three values of $\sigma$ are representative of the three regimes shown in Fig. 1. To guide the eye, we colored in magenta the grid points that have large amplitude defined by a threshold value.

![Image](image.png)

In order to investigate the transition between the three regimes, we studied the magnitude of the frequency gap $\omega_g$ as a function of the Poisson ratio $\sigma$. The result is displayed in Fig. 4. The figure demonstrates that the non-chiral regime ($\omega_g = 0$) exists inside a finite interval $\sigma_1 < \sigma < \sigma_2$. This implies that the gap vanishes in a non-analytic way, reminiscent of the behavior of an order parameter near a continuous phase-transition. Indeed, we find that the gap $\omega_g$ vanishes linearly near the critical ratios $\sigma_1 = 0$ and $\sigma_2 = \varphi^{-1}$, see the insets of Fig. 4.

A particularly simple case of surface modes is found in the limit where the compressional modulus $C_1$ vanishes, i.e. for $\sigma = -1$. In the time-reversal invariant setting, this maximally auxetic problem emerges in the twisted Kagome lattice [42]. We find for our system that at $\sigma = -1$ edge modes exist and the frequency spectrum is a flat band. This implies that once a deformation is introduced at the edge of the system, it does not propagate but remains there forever frozen. Such excitations have been studied in the literature [42–44] and are known as floppy modes. It is interesting to note that these solutions have a hidden holomorphicity property related to the fact that, when $\sigma = -1$, the boundary conditions (10) become the Cauchy-Riemann equations for the field $u_x + i u_y$, see Appendix C for more details.

IV. CONCLUSIONS AND OUTLOOK

We analyzed Rayleigh edge waves that travel on the edge of two-dimensional crystals in the presence of Lorentz forces and mapped out how their propagation direction depends on the Poisson ratio, see Fig. 1. The existence of these waves is not protected by topology, but rather originates from spontaneously broken translational symmetry. In addition to skyrmion crystals, we expect our findings to be directly applicable to boundary excitations ofscreened Wigner crystals in an external magnetic field [30]. Moreover, our re-
sults shed new light on elastic gyroscopic systems [45], where
dge modes are currently under active investigation [46, 47].
Our work indicates that in all these systems the chirality of
Rayleigh edge waves can be controlled by changing the elas-
tic properties of the medium.

Extensions of this study to Abrikosov vortex crystals in
superconductors and superfluids [15] are non-trivial exciting
frontiers. It would be also intriguing to generalize this work
and investigate edge excitations in two-dimensional crys-
tals, where time-reversal breaking originates from a different
mechanism, such as for example the odd elasticity discovered
in [48, 49].

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Appendix A: Analytic values of $\sigma_1$ and $\sigma_2$, edge-wave
dispersions at $\sigma = 1/3$, $\sigma = \sigma_{1,2}$ and floppy modes

Insertion of the edge-wave ansatz into the stress-free boundary
condition (11) results in an equation for $\alpha_{\pm}$:

$$2(\sigma-1)\sigma^2(\alpha-(\sqrt{2-2\alpha^2+2\sqrt{1-\alpha^2}})\sqrt{2-2\alpha^2+2\sqrt{1-\alpha^2}})\equiv 0 \quad (A1)$$

$$2(\sigma-1)\sigma^2(\alpha-(\sqrt{2-2\alpha^2+2\sqrt{1-\alpha^2}})\sqrt{2-2\alpha^2+2\sqrt{1-\alpha^2}})\equiv 0 \quad (A2)$$

We are considering non-negative values of $\omega$, thus $\alpha_{\pm} \geq 0$. The form of the spectrum (12) yields for $\kappa$ given by (8) the values

$$\kappa_{1,2} = \sqrt{1 \pm \alpha} \left| k \right|.$$  \hspace{1cm} (A3)

In order to have both $\kappa_{1,2}$ real, the condition $\alpha \leq 1$ must be satisfied. The analytic values of $\sigma_1$ and $\sigma_2$ can be found by imposing
these limits. We first note that for $\alpha_+ \to 1$ one finds $\sigma \to 0$ using Eq. (A1), and thus

$$\sigma_1 = 0. \quad \text{(A4)}$$

The value $\sigma_2$ is obtained by letting $\alpha_- \to 1$ in Eq. (A2). In this limit that equation reduces to

$$\sigma_2 \sqrt{1 - \sigma_2} - 1 + \sigma_2 = 0 \quad \text{(A5)}$$

with solution

$$\sigma_2 = \frac{\sqrt{5} - 1}{2} \equiv \varphi^{-1}, \quad \text{(A6)}$$

which is the inverse of the golden ratio $\varphi$.

1. Symmetric point $\sigma = 1/3$

When $\sigma = 1/3$, both equations (A1) and (A2) reduce to the same form, thus $\alpha_+ = \alpha_-$. The equation that is satisfied by $\alpha_{\pm}$
is

$$\left(3\sqrt{3} - 3\alpha^2 + 3\sqrt{1 - \alpha} - 3\sqrt{\alpha + 1} + \sqrt{3} \right) \alpha^2 + 8 \left(\sqrt{1 - \alpha} + \sqrt{\alpha + 1} \right) \alpha + 8 \left(\sqrt{1 - \alpha} - \sqrt{\alpha + 1} \right) = 0. \quad \text{(A7)}$$

It is straightforward to verify that the only admissible solution is

$$\alpha_{\pm} = \frac{2\sqrt{2}}{3}. \quad \text{(A8)}$$

Thus the long-wavelength edge-wave dispersion at $\sigma = 1/3$ takes on the particularly simple form

$$\omega = \frac{2\sqrt{2} \nu_2}{3 \Omega} k^2. \quad \text{(A9)}$$
2. Asymptotic behaviour of $\alpha_{\pm}$ at $\sigma = \pm 1$

When $\sigma \to 1^-$ the value of the corresponding $\alpha_+$ tends to 0. By setting

$$\sigma = 1 - \epsilon$$  \hspace{1cm} (A10)
$$\alpha = \delta$$  \hspace{1cm} (A11)

in equation (A1) and expanding in small $\delta, \epsilon$, we arrive at the equation

$$2\delta \epsilon - \sqrt{2\delta^2 \epsilon} = 0$$  \hspace{1cm} (A12)

which has the solution $\delta = \sqrt{2} \sqrt{\epsilon}$. This yields the asymptotic

$$\alpha_+ \sim \sqrt{2} \sqrt{1 - \sigma} \text{ as } \sigma \to 1^-$$  \hspace{1cm} (A13)

and as a consequence

$$\omega \sim \frac{2\nu_2}{\Omega} k^2 \text{ as } \sigma \to 1^-.$$  \hspace{1cm} (A14)

When $\sigma \to -1^+$, the value of $\alpha_-$ tends to 0. Here we set

$$\sigma = -1 + \epsilon$$  \hspace{1cm} (A15)
$$\alpha = \delta$$  \hspace{1cm} (A16)

and upon expanding (A2) we find $\delta = 3\epsilon/2$ and thus

$$\alpha_- \sim \frac{3}{2} (1 + \sigma) \text{ as } \sigma \to -1^+$$  \hspace{1cm} (A17)

and therefore

$$\omega \sim \frac{3\nu_2}{2\Omega} (1 + \sigma) k^2 \text{ as } \sigma \to -1^+.$$  \hspace{1cm} (A18)

Since $\sigma \to -1^+$ is equivalent to $\nu_1 \to 0$, we can also write this asymptotic relation as

$$\omega \sim \frac{6\nu_2}{\Omega} k^2 \text{ as } \nu_1 \to 0.$$  \hspace{1cm} (A19)

In both limits, $\sigma \to -1^+$ and $\sigma \to 1^-$, the spectra become flat. Such flat spectra are associated with excitations called floppy modes.

3. Floppy modes at $\sigma = -1$

As discussed in the main text, for $\sigma = -1$ the system supports floppy modes. Setting $\sigma = -1$ and inserting the edge-wave ansatz (9) into the equations of motion (3) produces two modes with $\kappa_{\pm} = \sqrt{k^2 - (\omega^2 \pm \omega\Omega)/v_2}$ and circular polarizations $\epsilon_{\pm} = \pm i$. The boundary conditions (10) enforce $\omega = 0$ or $\omega = \Omega$ and $\kappa_{\pm} = |k|$. The latter is automatically satisfied for the $\omega = 0$ bulk mode. But, for $\omega = \Omega$, the $\kappa_+$-mode violates this condition.

For $\omega = 0$ we find the floppy mode

$$u = (-i \text{ sign}(k), 1)^T e^{ikx + |k|y},$$  \hspace{1cm} (A20)

while the time-dependent solution with $\omega = \Omega$ only exists for $k > 0$ and has the form

$$u = (-i, 1)^T e^{i(kx - \Omega t) + ky}.$$  \hspace{1cm} (A21)

We assumed above that $\Omega > 0$. If, instead, $\Omega < 0$, then the time-dependent solution has the frequency $\omega = -\Omega$. This change of sign modifies the sign of the allowed $k$ values in Eq. (A21) and thereby reverses the direction of propagation.
Appendix B: Finite-Difference Solution of the Equations of Motion

In the main part of the paper, we displayed snapshots of Rayleigh waves propagating along the boundary of a square-grid system. These snapshots are obtained from finite-difference simulations of the partial differential equation (3) subject to the boundary conditions (10). To this end, we discretize space by introducing a quadratic grid. Along the vertical sides of the square, we use periodic boundary conditions. In the horizontal direction, where we observe surface waves, we use the free boundary conditions (10). In the absence of the magnetic field term, our bulk equations of motion are reduced to those considered in the classic finite-difference treatment of Kelly et al.[50], where an explicit scheme was introduced. We employ the same discretization, but treat the magnetic field term exactly. The equations of motion are discretized after rewriting them as first-order equations in time, by introducing the velocity fields $w_x = \dot{u}_x$ and $w_y = \dot{u}_y$:

$$[w_x]^n_{l+1} = \cos(h\Omega)[w_x]_l^n + \sin(h\Omega)[w_y]_l^n + h[F_x]_l^n,$$

$$[w_y]^n_{l+1} = \cos(h\Omega)[w_y]_l^n - \sin(h\Omega)[w_x]_l^n + h[F_y]_l^n,$$

$$[u_x]^n_{l+1} = [u_x]_l^n + h[w_x]_l^n,$$

$$[u_y]^n_{l+1} = [u_y]_l^n + h[w_y]_l^n,$$

where the subscript $l$ is the time index, $h$ is the discrete time step and $F_x$ and $F_y$ are the centrally discretized elastic forces. The magnetic field discretization is exact in the absence of elastic forces.

Since our focus is on Rayleigh waves, the boundary is particularly important. We use a stable finite-difference scheme that was invented by Vidale and Clayton [51] for the study of surface waves. In their method, an auxiliary horizontal layer is added to the last grid layer, and the actual free surface is considered to be in-between these two layers. The time evolution in all but the boundary conditions, which are

$$\partial_y u_x + \partial_x u_y = 0,$$

$$\sigma \partial_x u_x + \partial_y u_y = 0.$$

These conditions have to be imposed at the free surface, which is obtained from the last two layers by averaging. Discretizing these equations using central differences yields

$$u_x^n - u_x^0 + \frac{1}{2} \left[ (u_y^0 + u_x^1) - (u_y^0 - u_x^1) \right] = 0,$$

$$\sigma \left[ \frac{u_x^0 + u_x^1}{2} - \frac{u_x^0 - u_x^1}{2} \right] + (u_y^1 - u_y^0) = 0,$$

where the indices 0 and 1 denote the last and penultimate horizontal layers, respectively. These equations have to be solved in order to find the $u_x^0$ and $u_y^0$. We can cast these equations as matrix equations by introducing the tridiagonal matrix $T$ with components $T_{nm} = \delta_{n,m-1} - \delta_{n,m+1}$ and forming vectors $u_x^0$ and $u_y^0$ out of the displacements:

$$\frac{1}{4} T u_y^0 - u_x^0 = -\frac{1}{4} T u_y^1 - u_x^1,$$

$$\frac{\sigma}{4} T u_x^0 - u_y^0 = -\frac{\sigma}{4} T u_x^1 - u_y^1.$$

(B1)

The right-hand sides are given. Solving the first equation for $u_x^0$ and inserting it into the second, we find an equation for $u_y^0$ alone:

$$\left( I - \frac{\sigma}{16} T^2 \right) u_y^0 = \frac{\sigma}{16} T^2 u_y^1 + \frac{\sigma}{2} T u_x^1 + u_y^1.$$

The first step is to solve this matrix equation for $u_y^0$; in the second step, one finds $u_x^0$ by using equation (B1):

$$u_x^0 = \frac{1}{4} T u_y^0 + \frac{1}{4} T u_y^1 + u_x^1.$$

(B3)

The matrix equation in the first step is of the form

$$(I + a T^2) x = b.$$  (B4)
As noted in [51] this matrix is pentadiagonal and can be solved by methods similar to those for tridiagonal matrices [52]. Written out, this matrix equation becomes
\[
ax_{n-2} + (1 - 2a)x_n + ax_{n+2} = b_n,
\]
which is a three-term recursion relation that only connects the even/odd indexed terms. It can be solved by making a two-term recursion ansatz
\[
x_n = A_n x_{n+2} + B_n.
\]
(B5)

We use this to eliminate \(x_{n+2}\) in the three-term recursion, which results in
\[
x_n = -\frac{a}{1 - 2a + aA_{n-2}} x_{n-2} + \frac{b_n - aB_{n-2}}{1 - 2a + aA_{n-2}}.
\]
Comparing this with the two-term ansatz, we find:
\[
A_n = -\frac{a}{1 - 2a + aA_{n-2}}
\]
(B6)
\[
B_n = \frac{b_n - aB_{n-2}}{1 - 2a + aA_{n-2}}
\]
(B7)

Let us assume that the initial conditions \(x_0, x_1\) are specified and put \(A_0 = 0, B_0 = x_0\) and \(A_1 = 0, B_1 = x_1\). Then we can determine from (B6)-(B7) all the remaining \(A_n, B_n\). Next we take the initial condition on \(x_N\) and use (B5) to determine all the \(x_i\) between \(i = 1\) and \(i = N - 1\), thereby solving the inversion problem (B4). In our particular problem (B3) we have
\[
a = \frac{\sigma}{16}
\]
\[
b_n = \left[ \frac{\sigma}{16} T^2 u_1 + \frac{\sigma}{2} T \bar{u}_2 + u'_y \right]_n.
\]

A full update step consists of a bulk update followed by the boundary updates of auxiliary layers (on the top and bottom of the square).

Appendix C: Complex Formulation of the Equations of Elasticity and Holomorphicity at \(\sigma = -1\)

When the Poisson ratio takes on the value \(\sigma = -1\), the edge-wave solutions have a hidden property. To see this we reformulate the elasticity equations in complex form by combining the real strain components \(u_x\) and \(u_y\) into one complex field \(\psi \equiv u_x + iu_y\). The equations of motion (3) are
\[
\begin{align*}
\ddot{u}_x + \Omega \dot{u}_y - 2v_1 \partial_x (\partial_x u_x + \partial_y u_y) - v_2 (\partial_x^2 + \partial_y^2) u_x &= 0 \\
\ddot{u}_y - \Omega \dot{u}_x - 2v_1 \partial_y (\partial_x u_x + \partial_y u_y) - v_2 (\partial_x^2 + \partial_y^2) u_y &= 0
\end{align*}
\]
and by multiplying the second equation by \(i\) and adding it to the first, we obtain
\[
\dddot{\psi} - i\Omega \ddot{\psi} - 4v_1 (\partial_x^2 \ddot{\psi} + \partial_x \partial_z \dot{\psi}) - 4v_2 \partial_z \dot{\psi} \ddot{\psi} = 0,
\]
(C1)
where we introduced the complex derivatives \(\partial_z \equiv (\partial_x - i\partial_y)/2\) and \(\partial_z \equiv (\partial_x + i\partial_y)/2\). The boundary conditions (10) in real space read
\[
\begin{align*}
\partial_x u_y + \partial_y u_x &= 0 \\
\sigma \partial_x u_x + \partial_y u_y &= 0.
\end{align*}
\]
(C2)
(C3)

Multiplying the second equation by \(i\) and adding it to the first, we obtain the boundary conditions in complex form
\[
(3 - \sigma) \partial_z \psi = (1 + \sigma) (\partial_z \ddot{\psi} + \partial_z \ddot{\psi} + \partial_z \dot{\psi}).
\]
(C4)

At \(\sigma = -1\) the boundary conditions (C2) and (C3) are the Cauchy-Riemann equations for the real and imaginary parts of \(\psi\) at \(y = 0\). In addition, the real parts of the modes (A20) and (A21) give rise to \(\psi\)'s that are holomorphic functions of the complex variable \(z \equiv x + iy\) in the bulk. In particular, the time-independent mode yields \(\psi = i \exp(-i|k|z)\), while the time-dependent mode is \(\psi = i \exp(-ikz + i\Omega t)\). Using a conformal transformation we can map these edge-modes, which are localized near the boundary of the complex half-plane, onto edge-waves that propagate along the boundary of an arbitrarily shaped region. In other words, the transformed solutions will satisfy the boundary conditions on the new edge and solve the bulk (Laplace) equations of motion [53].
FIG. 5: Numerical solution of the edge wave dispersion (red) for equal elastic moduli $C_1 = C_2$. For comparison, also the dispersions relations of the bulk modes $\omega_-$ (blue) and $\omega_+$ (yellow) are plotted.

Appendix D: Symmetric Edge Spectrum

In Appendix A, we have shown that a symmetric spectrum of edge excitations emerges for the value of Poisson ratio $\sigma = 1/3$, i.e. for equal elastic moduli, $C_1 = C_2 \equiv C$. Hereby we show that this property holds even at NLO, see Figure 5.

The proof is based on the characteristic equation $d(k, \omega) = 0$ of the matrix appearing in the boundary conditions (11).

The expressions are cumbersome and it turns out to be more convenient to study the parity property of the auxiliary function $\tilde{d}(k, \omega)$ instead of the characteristic polynomial $d(k, \omega)$.

In terms of the inverse decay lengths $\kappa_{\pm}(k, \omega)$ and the polarizations $\epsilon_{\pm}(k, \omega)$ the auxiliary function takes the following form

$$\tilde{d}(k, \omega) = \left( k^2 + 3\kappa_{-}\kappa_{+} \right) + (\kappa_{+} - \kappa_{-}) \left[ \frac{\epsilon_{+} - \epsilon_{-} - 3}{\epsilon_{+} - \epsilon_{-}} (ik) \right]. \quad (D1)$$

We will argue that the auxiliary function $\tilde{d}$ is an even function of the wavevector $k$. First, after introducing $2C/\rho \equiv v$, we notice that

$$\kappa_{\pm}(k, \omega) = \sqrt{k^2 - \left( 2\omega^2 \pm \omega\sqrt{3\Omega^2 + \omega^2} \right) / 3v} \quad (D2)$$

are even functions of $k$. As a result, all the functions outside the square brackets in (D1) are even. Polarization functions have no parity symmetry

$$\epsilon_{\pm}(k, \omega) = -i \frac{3\Omega^2 \omega + 2k\sqrt{3v \left( -2\omega^2 \pm \omega\sqrt{3\Omega^2 \omega^2 + \omega^2} + 3vk^2 \right)}}{\omega^2 - \sqrt{3\Omega^2 \omega^2 + \omega^2} - 6vk^2},$$

however, together with $(ik)$, they lead to a function inside the square brackets

$$g(k, \omega) = \frac{-k^2}{\sqrt{\omega^2 \Omega^2 + \omega^4 / 3}} \left[ \sqrt{-v \left( 2\omega^2 + \sqrt{3\omega^2 \Omega^2 + \omega^2} - 3vk^2 \right)} + \sqrt{-v \left( 2\omega^2 - \sqrt{3\omega^2 \Omega^2 + \omega^2} - 3vk^2 \right)} \right] \quad (D3)$$

which is manifestly even under the change of sign of the wavevector. This proves that $\tilde{d}(k, \omega) = \tilde{d}(-k, \omega)$ therefore the edge-wave spectrum at $\sigma = 1/3$ is symmetric, $\omega(k) = \omega(-k)$.

[1] P. Delplace, J. B. Marston, and A. Venaille, Science 358, 1075 (2017).
[2] S. Shankar, M. J. Bowick, and M. C. Marchetti, Phys. Rev. X (2017).
