Multi-objective integer programming: Synergistic parallel approaches

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Abstract

Exactly solving multi-objective integer programming (MOIP) problems is often a very time consuming process, especially for large and complex problems. Parallel computing has the potential to significantly reduce the time taken to solve such problems, but only if suitable algorithms are used. The first of our new algorithms follows a simple technique that demonstrates impressive performance for its design. We then go on to introduce new theory for developing more efficient parallel algorithms. The theory utilises elements of the symmetric group to apply a permutation to the objective functions to assign different workloads, and applies to algorithms that order the objective functions lexicographically. As a result, information and updated bounds can be shared in real time, creating a synergy between threads. We design and implement two algorithms that take advantage of such theory. To properly analyse the running time of our three algorithms, we compare them against two existing algorithms from the literature, and against using multiple threads within our chosen IP solver, CPLEX. This survey of six different parallel algorithms, the first of its kind, demonstrates the advantages of parallel computing. Across all problem types tested, our new algorithms are on par with existing algorithms on smaller cases and massively outperform the competition on larger cases. These new algorithms, and freely available implementations, allows the investigation of complex MOIP problems with four or more objectives.

1 Introduction

In multi-objective integer programming (MOIP), one must consider a range of objective functions. The solution is then the set of all non-dominated vectors, sometimes called the Pareto set. A decision maker can use such a set to compare the various trade-offs that can be made between the objective functions.
The Pareto set can be calculated exactly, or approximated. Approximation techniques include heuristics (or meta-heuristics), swarming (such as those of [14, 22]) and evolutionary (see [3, 21]) algorithms. However, this paper will only consider algorithms which calculate the exact Pareto set, with no omissions or inaccuracies. For an introduction to multi-objective optimisation in general, see [6], and for a very recent and thorough look at exact MOIP algorithms, focusing on branch and bound algorithms, see [20].

This paper looks at parallel multi-objective exact integer optimisation algorithms. Parallel evolutionary algorithms that find approximate solutions have received significant study in the literature, such as in [15, 23], but results on exact parallel algorithms for multi-objective problems are not as widespread. [10] introduce PPM, an algorithm which splits the feasible solution space through a three-stage process. They first find what they call “well-distributed solutions”, and then use these solutions to partition the feasible solution space into regions which can be searched in parallel. [5] then extend this work to create K-PPM, which solves problems with more than two objectives. Being one of the only published algorithms specifically described as a parallel MOIP algorithm, we use it as one of our comparison algorithms.

More recently, [7] demonstrates parallel improvements through problem-specific information, using specifics of their problems to break down the set of feasible solutions into equitable parts. We work with generic optimisation problems, and as such the algorithms of Guo et. al. cannot solve the problems we test against.

Another method of achieving parallelisation is to iteratively find solutions, and use these solutions to split the objective space into smaller parts. Each of these smaller parts can then often be independently searched, as mentioned but not implemented in [1]. This idea, of breaking down the objective space, can also be seen in algorithms that are not necessarily described as parallel, such as in [2, 4, 5, 7, 11]. We implement V-SPLIT from [4] as our second comparison algorithm, as they prove that it reaches the theoretical best-case in terms of integer problems solved, and they show that it is one of the two faster algorithms in the literature, the second being AIRA by [13] which we will also parallelise. V-SPLIT is only a 3-objective algorithm, unlike all other algorithms discussed, so timing results for V-SPLIT are only available on 3-objects.

We present three new algorithms. The first of these calculates the range of values that one of the objective functions may take, and divides this range equally amongst all threads. Unlike existing algorithms, this partitioning takes place before any searching for solutions. Timing results show that running time does improve as more threads are used, and that the performance is at least on par with other algorithms in the literature.

We propose a new parallelisation technique for MOIP algorithms. Threads are given a unique approach to the problem (as determined by an element of the symmetric group $S_n$). This approach, but limited to the biobjective case where the theory is trivial, is used in [14] where it just equally partitions the objective space. We present results which allow the real-time sharing of bounds to all other threads. Even though each thread is solving the same problem, this sharing creates a synergy between the threads. As one thread reduces the required computation for a second thread, the second thread will in turn reduce the required running time for the first thread. This synergy can allow significant performance improvements from parallelisation, especially in problems
where the number of objectives is large. This theory is given as a theoretical background in Section 3 so that it may be used and extended in other parallel algorithms.

We design, implement and test two algorithms based on this theory. These algorithms are compared to the other state-of-the-art algorithms. The results show that the new algorithms perform on par on smaller problems, and outperform the existing algorithms on larger problems. This validates the synergy evident in the theory: we give experimental results that show that the algorithms perform better than existing algorithms across all problems when the thread count is equal to the number of objectives. Even when thread counts are increased beyond this level, we still see our new algorithms scaling well and outperforming the other existing algorithms on all larger problem instances, and performing at a similar level for the smaller problem instances.

We offer our implementations of these algorithms for further use. This opens up many new opportunities to solve new problems in optimisation not only where more variables or more objective functions need to be considered, but also in more time-critical scenarios.

The rest of this paper is organised as follows. Section 2 gives a background and details the notation we use to describe the symmetric group, symmetries and lexicographically constrained MOIP problems. In Section 3 we give the theory that demonstrates the sharing of results between still-running threads. Section 4 describes our new algorithm, and Section 5 discusses some implementation details. The results of our testing are presented and discussed in Section 6. Finally we conclude in Section 7.

2 Background

2.1 Permutations

We will use \( S_n \) to denote the symmetric group on the \( n \) elements \( \{1, 2, \ldots, n\} \). Given a permutation \( s \in S_n \), let \( s(i) \) be the image of \( i \) under \( s \). For example, let \( s = (3, 2, 4, 1) \in S_4 \). Then \( s(1) = 3, s(2) = 2, s(3) = 4 \) and \( s(4) = 1 \).

The threads will be sharing information under specified conditions. One of these conditions is that the permutations must have the same image for the last \( a \) elements, as defined below.

**Definition 1** \((s =_a s')\). Given two elements \( s, s' \in S_n \), if \( s(i) = s'(i) \) for all \((n - a) < i \leq n\), we say that \( s =_a s' \).

For example, if \( s = (4, 1, 2, 3) \) and \( s' = (1, 4, 2, 3) \) then \( s =_2 s' \) as both permutations end with “2,3”.

2.2 Multi objective optimisation

We define a MOIP problem with \( n \) objective functions as \( IP^n = ND_{x \in X} \{f_1(x), \ldots, f_n(x)\} \) where

- each \( f_i(x) \) is a linear function for \( 1 \leq i \leq n \);
- ND is the usual definition of non-dominated; and
- \( X \) is feasible solution space, \( X = \{ x | x \in \mathbb{Z}^c \text{ and } Ax \leq b \} \) for suitable sized matrices \( A \) and \( b \).

In this definition, \( c \) denotes the number of variables in the problem. For conciseness, we assume each objective function is to be minimised, and for more details on multi-objective optimisation we guide the reader to Ehrgott’s book [6].

For a given feasible solution \( x \), we call the associated vector \((f_1(x), f_2(x), \ldots, f_n(x))\) the objective vector of \( x \). Where the feasible solution \( x \) may not be relevant, we may refer to such a vector as simply an objective vector. Our new algorithms deal mostly with objective vectors, specifically the set of objective vectors which correspond exactly to solutions to a MOIP problem. To aid readability we use the term objective solution to refer to an objective vector which corresponds to a solution of a MOIP problem.

The algorithm of Ozlen, Burton and MacRae [13] repeatedly solves constrained lexicographic versions of the given \( IP^n \), which we now define. In Section 3 we define new variants of these which we use for our new algorithms.

**Definition 2** (Constrained lexicographic MOIP). A constrained lexicographic MOIP will be written as \( LIP^n(k, (a_{k+1}, \ldots, a_n)) \). The values \((a_{k+1}, \ldots, a_n)\) are individual upper bounds on the values of \( f_{k+1}(x), \ldots, f_n(x) \). Any objective vector which does not satisfy all these bounds is not feasible for the constrained lexicographic problem. The set of objective vectors to such a problem contains all vectors which are not dominated in the first \( k \) objectives. If two or more objective vectors are identical in their first \( k \) objectives, the final \( n - k \) objectives are considered in lexicographic order and only the objective vector with the smaller objective value is part of the objective solutions.

Recall that the objective solution is the set of objective vectors which correspond to solutions to the problem at hand.

We will define an ordered integer program, written \( OIP \), in Section 3. An \( OIP \) is similar to an \( LIP \), except the ordering is defined by an element of \( S_n \) rather than simple lexicographic ordering. If we let \( e \) be the identity element of \( S_n \), then \( LIP^n(k, (a_{k+1}, \ldots, a_n)) = OIP^n_e(k, (a_{k+1}, \ldots, a_n)) \), so a more formal variant of Definition 2 can be found by taking \( s = e \) in Definition 3.

Technically, these constrained lexicographic problems may more accurately be described as a partially constrained, partially lexicographic problems, but this wording gets cumbersome and is skipped in favour of simply constrained lexicographic.

### 3 Bound sharing

In our parallel algorithms, we assign different tasks to different threads by means of a permutation \( s \in S_n \). To this end, we define a variant of the constrained lexicographic problem, where the objective functions are not ordered lexicographically, but rather are ordered by an element \( s \) of \( S_n \). We call these problems ordered problems, and write them as \( OIP_s \) rather than \( LIP \) to clearly indicate that the objective functions, and bounds on the values of the objective functions, are considered in the order specified by \( s \).

**Definition 3.** Let \( f_1, \ldots, f_n \) be \( n \) linear objective functions for an \( IP^n \) with feasible solution space \( X \), let \( s \in S_n \) and let \( Y \) be the set of objective solutions to
the associated constrained ordered problem $OIP_n^k(k, (a_{s(k+1)}, \ldots, a_{s(n)}))$. Then for any $y \in Y$

1. for any $i$ with $k < i \leq n$, $f_{s(i)}(y) \leq a_{s(i)},$

2. for any $y' \in Y$ with $y' \neq y$, there exists a $j \leq k$ s.t. $f_{s(j)}(y) < f_{s(j)}(y'),$

3. for any $x \in X$ with $f_{s(i)}(x) = f_{s(i)}(y)$ for $i \leq k$, there exists a $j \leq n$ such that for all $j' < j$, $f_{s(j')}(y) = f_{s(j')}(x)$ and $f_{s(j)}(y) < f_{s(j)}(x)$.

In this definition, if there are two objective vectors agree in their first $k$ objectives, the final $n - k$ objectives are considered in lexicographic order, as determined by the permutation $s$.

Lemma 1. If $x$ is an objective solution to $OIP_n^k(k-1, (a_{s(k)}, \ldots, a_{s(n)}))$, then $x$ is an objective solution to $OIP_n^k(k, (a_{s(k+1)}, \ldots, a_{s(n)}))$.

This result follows trivially from Definition 3. The next theorem is a slight variant of Lemma 4.1 from [12] to allow for the different permutations. The original theorem states that if a solution to a $k$ objective problem attains the upper bound on one of these objectives, say $f_i$, then it is also optimal on the $k - 1$-objective problem where we no longer consider $f_i$. Here it is modified to also allow for the permutations which we apply. Recall that $s = n-k$ $s'$ means that the permutations $s$ and $s'$ agree in their final $n - k$ places.

Theorem 1. Let $s$ be an element of $S_n$, let $Y$ be the set of objective solutions to $OIP_n^k(k-1, (a_{s(k)}, \ldots, a_{s(n)}))$, let $a = \max\{f_{s(k)}(y) | y \in Y\}$, and let $Y'$ be the set of objective solutions to $OIP_n^k(k, (a_{s(k+1)}, \ldots, a_{s(n)}))$. Then for any $y' \in Y'$, either

1. $y' \in Y$, or
2. $f_{s(k)}(y') > a_{s(k)}$, or
3. $f_{s(k)}(y') < a$.

Proof. This holds trivially if either $y' \in Y$ or $f_{s(k)}(y') > a_{s(k)}$ so assume $y' \notin Y$ and $f_{s(k)}(y') \leq a_{s(k)}$. Then as $y'$ is dominated in $Y$, let $y \in Y$ be an element that dominates $y'$ in $Y$. Let $i$ be the smallest integer such that $f_{s(i)}(y') < f_{s(i)}(y)$. There must be such an $i$ as otherwise $y$ would also dominate $y'$ in $Y'$. We will take cases on $i$.

If $i > k$ then $y$ and $y'$ obtain equal values for the first $k$ objectives. However, the last $n - k$ objectives are considered in lexicographic order, as determined by $s$. As $y$ and $y'$ are both feasible for $OIP_n^k(k, (a_{s(k+1)}, \ldots, a_{s(n)}))$, there must be some $j$ such that for $j' < j$, $f_{s(j')}(y') = f_{s(j')}(y)$, and $f_{s(j)}(y') < f_{s(j)}(y)$. However, both $y$ and $y'$ are also feasible for $OIP_n^k(k-1, (a_{s(k)}, \ldots, a_{s(n)}))$, and $s = n-k$ $s'$. Then by the same argument we must find an $i$ such that for $i' < i$, $f_{s(i')}(y) = f_{s(i')}(y)$, and $f_{s(i)}(y) < f_{s(i)}(y')$. This is clearly a contradiction.

If $i < k$, then clearly $y$ cannot dominate $y'$ in $Y$, leading to a contradiction. Lastly, if $i = k$ then $f_{s(k)}(y') < f_{s(k)}(y) \leq a$. \qed
Both problems in this theorem do have identical bounds for their final \(n - k\) places; this is not a typographical mistake. This identity between bounds is exactly why threads can share data, and forms the basis of our algorithm.

Before giving our algorithm, we define exactly what it means to have found all objective solutions above some bounds.

**Definition 4.** Given a problem \(P = \text{OIP}^n_k(\langle a_{s(k+1)}, \ldots, a_{s(n)} \rangle)\), we will say that a thread \(t\) has found all solutions above \(P\) if, for all \(j > k\), \(t\) has determined all solutions \(x\) to \(\text{OIP}^n_j(\langle a_{s(j+1)}, \ldots, a_{s(n)} \rangle)\) which also satisfy \(f_{s(j)}(x) > a_{s(j)}\).

Our new work here explains exactly when threads are able to share these updated bounds to other threads. We first present a simplified version of Theorem 2 to help introduce the reader to our approach.

**Lemma 2.** Let \(w\) represent a thread which

1. is currently solving \(P = \text{OIP}^n_{k-1}(\langle a_{s(n)} \rangle)\), and
2. has found all solutions above \(P\).

Then all solutions to the original IP with \(f_{s(n)}(x') > a_{s(n)}\) are known.

This lemma is also given in [16], and the proof of this lemma follows trivially from Definition 4. The lemma says that if a thread is solving \(\text{OIP}^n_{k-1}(\langle a_{s(n)} \rangle)\), and has found all solutions above this problem, then any other thread can also ignore any solution \(x\) for which \(f_{s(n)}(x) > a_{s(n)}\). Other threads will be using other permutations, so the bound on \(f_{s(n)}\) may not be the “last” bound for other threads. This sharing of bounds across many objective functions can create a synergy between threads, where one thread can supply a bound to other threads, which in turn means that those threads also find new bounds faster and these new bounds can be shared back to the original thread.

As mentioned, the above lemma is actually a simplified version of our result, and only shares the bounds on the “last” objective function. Theorem 2 is a more general result, describing exactly when bounds on any objective functions may be shared between threads. The theorem states that if two threads agree, in both permutations and bounds, in their last \(j\) positions, then the bound that a given thread has on objective \(n - j\) i.e., the objective just “before” the last \(j\) can be shared to the other thread, and vice-versa. Lemma 2 allows the bound on the last objective to be shared globally i.e., all threads can use the bound. In comparison Theorem 2 describes the sharing of bounds on any objective, but does place restrictions on which other threads can use this bound.

We first give two examples of the usage of this sharing, before giving the theorem and proof below.

**Example 1.** Let \(s_1 = (5, 1, 4, 2, 3)\) and \(s_2 = (1, 4, 5, 2, 3)\), and let \(P_1 = \text{OIP}^5_{s_1}(2, (13, 15, 18))\) and let \(P_2 = \text{OIP}^5_{s_2}(2, (8, 15, 14))\). Note that \((5, 1, 4, 2, 3) =_{2} (1, 4, 5, 2, 3)\). That is, \(s_1\) and \(s_2\) have the same elements in the final two positions of each permutation. Since \(P_1\) and \(P_2\) do not have the same bounds on \(f_{s_1(5)}\), we have to take \(j = 0\) in Theorem 2. This means that the bound on \(f_{s_2(5)}\) from \(P_2\) can be shared to \(P_1\). The end result is that thread running \(P_1\) can immediately set the bound on \(f_{s_1(5)}\) to 14, so the new version of \(P_1\) to be solved is \(P_1' = \text{OIP}^5_{s_1}(2, (13, 15, 14))\).

This example can be followed on to the next example.
**Example 2.** Take $s_1 = (5, 1, 4, 2, 3)$ and $s_2 = (1, 4, 5, 2, 3)$ again, and let $P_1 = OIP^n_{s_1}(2, (13, 15, 14))$ and $P_2 = OIP^n_{s_2}(2, (8, 15, 14))$. Again, $s_1 =_2 s_2$. Now $P_1$ and $P_2$ agree on bounds $a_{s(4)}$ and $a_{s(5)}$, so we take $j = 2$ in Theorem 2. That means that the bound on objective $f_{s_1}(3)$ from $P'_1$ can be given to the thread solving $P_2$, and the bound on objective $f_{s_2}(3)$ from $P_2$ can be shared to the thread solving $P_1$. More specifically, as $s_2(3) = 5$, the thread solving $P'_1$ can use $f_5(x) \leq 8$ as an upper bound for any new solutions, and as $s_1(3) = 4$, the thread solving $P_2$ can use $f_4(x) \leq 13$ as an upper bound on for any new solutions. These upper bounds apply even though $P'_1$ would otherwise not have any bound on $f_5$, and that if such a bound makes the problem infeasible then there are no new solutions to $P'_1$ which have not been found by $P_2$.

We now give the exact theorem and proof.

**Theorem 2.** Let $t$ represent a thread which

1. is currently solving $P = OIP^n_{s_t}(k - 1, (a_{s_t(k)}, \ldots, a_{s_t(n)}))$, and
2. has found all solutions above $P$.

For any other thread $t'$ which is currently solving $P' = OIP^n_{s'}(k', (a_{s'(k'+1)}, \ldots, a_{s'(n)}))$, and for any integer $j \geq 0$ such that all the following hold

1. $j < n - k$,
2. $j < n - k'$,
3. $s =_{n-j} s'$, and
4. $a_{s(n-i)} = a'_{s'(n-i)}$ for $0 \leq i < j$,

all solutions $x'$ to $P'$ with $f_{s'(n-j)}(x') \geq a_{s(n-j)}$ are known.

**Proof.** Let $x'$ be a solution to $P'$. Then by Lemma 1 $x'$ is also a solution to $OIP^n_{s_t}(n - j, (a'_{s'(n-j+1)}, \ldots, a_{s'(n)}))$. However by the conditions in this theorem, this problem is identical to $OIP^n_{s_t}(n - j, (a_{s(n-j+1)}, \ldots, a_{s(n)}))$, and by the definition of all solutions above $P$, $t$ has found all solutions to $OIP^n_{s_t}(n - j, (a_{s(n-j+1)}, \ldots, a_{s(n)}))$ with $f_{s'(n-j)}(x') \geq a_{s(n-j)}$. 

We can recover Lemma 2 from this theorem by letting $j = 0$.

4 New algorithms

4.1 Efficient Parallel Projection (EPP)

The objective space for a MOIP can be envisioned as a $k$-dimensional vector space, where each dimension represents one objective function. The Efficient Projection Partitioning (EPP) algorithm projects the whole solution space down to one dimension. Given an objective vector $x = (x_1, \ldots, x_n)$, the projection is achieved through the $n$-th projection map $\text{proj}_n(x) = x_n$. That is, the objective space is partitioned by only considering the values attained by one objective function. First we need the following lemma.
Lemma 3. For \( n > 1 \), an objective solution to a MOIP with objective functions \( f_1, \ldots, f_n \) that achieves a maximum value on \( f_n \) is also a solution to a MOIP on the same feasible solution space but restricted to the objective functions \( f_1, \ldots, f_{n-1} \).

This is a well known lemma; for recent proofs see e.g. Lemma 4.1 in [12] or Theorem 2 in [5]. EPP first calculates all solutions on the first \( n-1 \) objective functions recursively, with the solution when \( n = 1 \) being trivial. The set of objective solutions on \( n-1 \) objectives, along with the above Lemma, is used to determine the maximum value on the \( n \)-th objective; the minimum is found by simple integer programming. This gives a range of values which \( f_n \) can take, which is divided up equally amongst all threads.

Algorithm 1: The Efficient Projection Partitioning (EPP) algorithm.

Data: The MOIP \( IP^m \) on \( n \) objective functions, and an integer \( T \) representing number of threads to use.

Result: The non-dominated solutions.

if \( n = 1 \) then
  Solve the single-objective problem and return the solution
else
  Let \( X \) be the feasible solution space for this problem. Let \( IP^{n-1} \) be this same problem restricted to the first \( n-1 \) objective functions.
  Calculate the solutions \( Y \) to \( IP^{n-1} \) using Algorithm 1
  Let \( U = \max \{ f_n(y) | y \in Y \} \)
  Let \( L = \min \{ f_n(y) | y \in X \} \)
  Let \( step = \frac{U - L}{T - 1} \)
  for \( t \in \{0, \ldots, T - 1\} \) do
    Let \( l = L + t \times step \)
    Let \( u = l + step \)
    Start a MOIP solver in a new thread to find all solutions \( y \) satisfying \( l < f_k(y) \leq u \).
  Return the union of the results from all threads started

4.2 CLUSTER and SPREAD

We next introduce the two algorithms CLUSTER and SPREAD, which apply our permutation parallelisation technique to the algorithm of [13]. First, Algorithm 2 is the algorithm which will initialise and launch all sub-problems. The initialisation process lets each thread determine which other threads it might be sending information to, and from which threads it might be receiving information. Each parallel thread will be running Algorithm 3 where new solutions will be found and new bounds will be calculated and shared.

In Algorithm 2 the method for selecting permutations is not specified. We devise two ways of selecting permutations, which in turn create the two algorithms which we call CLUSTER and SPREAD. CLUSTER assigns permutations to maximise \( i \) where \( s =_s s' \) for all selected \( s \) and \( s' \). For instance, we could assign \( (1, 2, 3, 4, 5), (2, 1, 3, 4, 5), (1, 3, 2, 4, 5), (3, 1, 2, 4, 5), (2, 3, 1, 4, 5) \) and \( (3, 2, 1, 4, 5) \) to six threads solving a 5-objective problem. In other words, all of these have 4 and 5 as their final two elements, and thus can share updates on their third objectives. These six threads would be sharing updated bounds on
deeper levels of the recursion, meaning the algorithms will share bounds more often. This reduces the time between the determination of a new bound, and when threads can use the new bound, potentially minimising the amount of redundant work completed. As a downside, though, these bounds might not be shareable with all other threads.

The second option, which we call SPREAD, assigns permutations to minimise $i$ where $s = s'$ for all selected $s$ and $s'$. For instance, this could mean assigning $(1,2,3,4,5), (2,3,4,5,1), (3,4,5,1,2), (4,5,1,2,3), (5,1,2,3,4)$ and $(2,3,4,1,5)$ to six threads solving a 5-objective problem. All five objectives occur as a “fifth” objective in some thread, so every thread will be able to update bounds on every objective. The sharing of these bounds would mainly happen at the higher level of recursion, i.e. not as often, but the bounds will be shared to more threads. We discuss in Section 6 how different selection methods can impact the running time of the algorithm.

Algorithm 2: Our new parallel algorithm. This particular algorithm will set up each thread with an appropriately selected permutation $s$. The actual work is done in Algorithm 3 which is called from this algorithm.

**Data:** The problem $H^m$, and $t$ representing the number of threads to use

**Result:** $ND$: The non-dominated solutions

**begin**

Let $L$ be a list of thread details, to be used to tell threads where they are sharing information

for $i \in \{1, \ldots, t\}$ do

Create a thread $w$

Select a permutation $s \in S_n$

Create the problem $P_t = OIP^n_s(n, ())$

Store the details of this thread in $L$

for Each element $l$ in $L$ do

Launch Algorithm 3 with the corresponding problem $P_t = OIP^n_s(n, ())$ taken from $l$, as well as a copy of $L$

Wait for all threads to complete

Let $ND = \bigcup_t \{ \text{solutions to } P_t \}$

In Theorem 1, we define $\hat{a}$ to be the maximum value of $f_{s(k)}(y)$ for any solution $y$. To allow each thread to apply Theorem 1 we must therefore share not only updated bounds, but the maximum value of $f_{s(k)}$ that is attained. Theorem 1 then trivially verifies correctness of this algorithm.

5 Implementation details

The implementation of this new algorithm is based on AIRA as used in [16]. The availability of the source code sped up the implementation process. The implementation is in C++11, and uses the shared memory and threading features of the Standard Template Library to handle all thread creation and data sharing. The code is published on Github [19], and test cases are also provided [17, 18] for others to utilise.
Algorithm 3: This algorithm calculates actual solutions to the problem at hand. The set-up for this algorithm is performed by Algorithm 2.

Data: The problem $OIP^n_{\mathbf{a}}(k, (a_{s(k+1)}, \ldots, a_{s(n)}))$, and the details of all other threads solving the same original problem $IP^n$.

Result: $ND_k$, the non-dominated solutions.

begin
Set $ND_k = \emptyset$.
if a relaxation of this problem is already solved and each solution to said relaxation satisfies the current bounds then
Let $ND_k$ be this set of solutions
else
if $k = 1$ then
Solve the single-objective problem.
if the problem is feasible, with solution $x$ then
Set $ND_k = \{x\}$
else
Let $a_{s(k)} = \infty$
From $OIP^n_{\mathbf{a}}(k, (a_{s(k+1)}, \ldots, a_{s(n)}))$, create $P = OIP^n_{\mathbf{a}}(k - 1, (a_{s(k)}, a_{s(k+1)}, \ldots, a_{s(n)}))$.
Solve $P$ using this algorithm
while $P$ is feasible do
Let $Y$ be the solutions to $P$, as determined by this algorithm
Let $ND_k = ND_k \cup Y$
Let $a_{s(k)} = \max \{a_{s(k)}, \max \{f_k(x) | x \in Y\}\}$
for Each thread $w$ with corresponding permutation $s'$ do
Use Theorem 2 to update the bounds on $P$
if $s = n-k$ then
if $w$ has found a higher value for $a_{s(k)}$ then
Update $a_{s(k)}$
Update $P$ with the new value of $a_{s(k)}$
Solve $P$ using this algorithm
endfor
endwhile
endif
endif
end

5.1 Comparison algorithms

We compare the running time of both variants of our algorithm against the following algorithms. AIRA \cite{13} is a state of the art MOIP solver, which uses IBM ILOG CPLEX as a single-objective IP solver internally. In recent results \cite{4}, AIRA was shown to be one of two algorithms to outperform all others, with the second being V-SPLIT which we discuss below. One very simple method of parallelising AIRA, or indeed most MOIP algorithms, is to allow the IP solver to utilise more threads. This technique was also seen in \cite{1}, and we call such an improvement CPLEX. We do not expect that CPLEX will be competitive in this setting, as CPLEX would not understand the whole MOIP problem. Instead, these numbers display the significant improvements that can be achieved by designing algorithms to suit parallelisation.

The second comparison algorithm is K-PPM, as described in \cite{5}. This one of the only recent general MOIP algorithms that is specifically described as being parallel. K-PPM utilises a 3-step process to create a number of sub-problems. The first phase calculates the ideal and nadir points of the given problem by recursively solving smaller problems. This does have a cost, one that the authors of K-PPM discussed in \cite{5}. We chose to implement K-PPM exactly as they described it, so as to not complicate the results. These ideal and nadir points are used in the second phase to calculate some well-distributed solutions, which in turn are used to partition the solution space. This partitioning of the solution space creates a number of sub-problems. Each of these can be solved in parallel by either a generic serial MOIP solver, or potentially a specialised solver. We chose to use AIRA as the generic MOIP solver for K-PPM, as it is a modern and open source generic MOIP solver, and being very similar to Algorithm 2 this will reduce any differences caused by the MOIP solver chosen and will instead allow us to highlight the differences due to our new parallelisation technique.

The final comparison algorithm is V-SPLIT, as described in \cite{4}. V-SPLIT follows a common approach in MOIP algorithms, where new objective solutions are found and then used to reduce and/or partition the objective space. Other recent algorithms that take such an approach can be found in \cite{2, 11, 8, 9}. V-SPLIT, as given in \cite{4}, is only suitable for 3-objective problems. However we still use V-SPLIT as the only recent comparison between sequential exact MOIP algorithms \cite{4} showed that V-SPLIT and AIRA are the two leading competitors in the field. Two variants of V-SPLIT were presented in \cite{4}, using either a weighted Tchebycheff or an $\epsilon$-constraint method to find individual solutions. While the $\epsilon$-constraint method was faster, the authors of \cite{4} also point out that this method does not allow the arbitrary selection of the next sub-space to be searched. As a parallel implementation of V-SPLIT would require the simultaneous searching of multiple sub-spaces, we implement a parallel version of the weighted Tchebycheff V-SPLIT algorithm. Note that V-SPLIT is specifically designed for 3-objective problems, and as such we can only test it on 3-objective problems.

5.2 Execution environment

We ran our implementation on the Raijin, a supercomputer run by the National Computing Infrastructure in Australia, which utilises Intel Xeon E5-2670 CPUs running at 2.60GHz. Our code was compiled with GCC 4.9.0, and we
used CPLEX 12.7.0 as our single objective IP solver. We ran each algorithm over a series of randomly generated 3- and 4-objective assignment, knapsack and travelling salesperson problems. Five different instances of each type of problem were generated for a series of problem sizes. We tested each algorithm with 3 and 6 threads for 3-objective problems, and 4, 8, and 12 threads for 4-objective problems. For the 3-objective problems, SPREAD and CLUSTER with 6 threads are equivalent, as there are only 6 permutations in $S_3$. We also ran the non-parallel AIRA on all of these problems to see whether the parallel algorithms actually improved the running time. We have excluded the running times for problems which were solved very quickly (under one second) as well as problems which did not complete in the given time limits (48 hours) across all algorithms. Additionally, we do not include running times for CPLEX on 12 threads as it had already showed minimal improvement moving to 8 threads. The test problems are available for further use at [17][18].
| Threads | AIRA   | CPLEX | K-PPM | EPP   | V-SPLIT | SPREAD | CLUSTER |
|---------|--------|-------|-------|-------|---------|--------|---------|
|         | 1  | 3 | 6 | 3 | 6 | 3 | 6 | 3 | 6 | 3 | 6 |
| AP10    | 18.61 | 11.09 | 11.01 | 15.66 | 9.53 | 9.39 | 6.37 | 9.24 | 4.56 | 7.92 | 8.00 | 4.93 |
| AP15    | 100 | 61.82 | 58.74 | 72.56 | 48.28 | 47.94 | 29.52 | 45.06 | 22.21 | 42.78 | 43.40 | 24.76 |
| AP20    | 405 | 272 | 251 | 290 | 204 | 202 | 122 | 183 | 90.51 | 176 | 181 | 93.15 |
| AP25    | 1085 | 755 | 681 | 792 | 566 | 535 | 315 | 524 | 260 | 468 | 479 | 233 |
| AP30    | 2706 | 2743 | 1713 | 1818 | 1399 | 1396 | 801 | 1463 | 713 | 1185 | 1189 | 582 |
| AP40    | 7051 | 6330 | 4624 | 4998 | 4056 | 4016 | 2183 | 4879 | 2445 | 3104 | 3148 | 1530 |
| KP50    | 38.62 | 38.64 | 38.90 | 34.98 | 21.69 | 21.78 | 13.62 | 14.23 | 6.99 | 17.37 | 17.98 | 11.31 |
| KP75    | 237 | 186 | 179 | 222 | 148 | 137 | 76.44 | 123 | 59.49 | 104 | 104 | 62.80 |
| KP100   | 667 | 491 | 461 | 572 | 454 | 412 | 259 | 472 | 219 | 309 | 309 | 177 |
| KP125   | 2063 | 1255 | 1102 | 1644 | 1189 | 1151 | 727 | 1488 | 727 | 866 | 869 | 524 |
| KP150   | 3338 | 4223 | 1883 | 2524 | 2284 | 1816 | 1099 | 3452 | 1585 | 1467 | 1503 | 802 |
| KP200   | 18643 | 9118 | 8331 | 13176 | 11087 | 9946 | 6897 | 11408 | 5790 | 8120 | 8273 | 3788 |
| TSP10   | 7.13 | 5.88 | 5.26 | 7.54 | 4.66 | 4.04 | 2.62 | 3.39 | 1.60 | 3.20 | 3.21 | 2.51 |
| TSP12   | 14.21 | 12.41 | 10.45 | 13.65 | 9.31 | 8.85 | 5.29 | 7.09 | 3.38 | 6.72 | 6.73 | 4.86 |
| TSP15   | 164 | 143 | 111 | 129 | 88.78 | 93.99 | 58.63 | 76.67 | 36.72 | 72.02 | 72.18 | 51.75 |
| TSP20   | 1026 | 748 | 752 | 641 | 467 | 569 | 323 | 553 | 266 | 436 | 434 | 283 |
| TSP30   | 12213 | 9413 | 8451 | 8242 | 6058 | 6340 | 3529 | 13021 | 6438 | 5147 | 5110 | 2890 |

Table 1: Running times of each algorithm and various thread counts on a number of assignment, knapsack and travelling salesman problems with three objectives. As CLUSTER and SPREAD are identical when using 6 threads, we only list the running times under the column CLUSTER.
| Threads | AIRA  | CPLEX | K-PPM | EPP | CLUSTER | SPREAD |
|---------|-------|-------|-------|-----|---------|--------|
|         | 1     | 4     | 8     | 4   | 8       | 12     |
|         | 4     | 8     | 12    | 4   | 8       | 12     |
|         | 4     | 8     | 12    | 4   | 8       | 12     |
|         | 4     | 8     | 12    | 4   | 8       | 12     |
|         | 4     | 8     | 12    | 4   | 8       | 12     |

Table 2: Running times of each algorithm and various thread counts on a number of assignment, knapsack and travelling salesman problems with four objectives.
6 Discussion

We see that CPLEX does gain some, but not much, improvement with parallelisation. In particular, going from one to three (or four) threads does seem slightly useful, but stepping beyond this is less effective, especially for the smaller problems. This is consistent with expectation for this approach.

K-PPM does improve as more threads are introduced, performing better than CPLEX. However, it is in turn beaten by the remaining four algorithms. For smaller problems, we see that V-SPLIT marginally outperforms EPP. In [4] the largest problem solved was a knapsack problem with 50 objects, equivalent to our smallest knapsack problem. Our timing results do then correlate with those from [4]. However as the problems get bigger, EPP performs better than V-SPLIT. It is surprising that EPP should be competitive, as initially EPP seems like a very basic parallel algorithm.

Both CLUSTER and SPREAD display performance improvements as more threads are utilised, and for the larger problems solved both outperform all other algorithms. On 3-objective problems, and with 3 threads, both CLUSTER and SPREAD appear to perform similarly. This is not too surprising, as there are only 6 possible permutations to choose from, so the difference is not as evident. Clearly as \(|S_3| = 6\) SPREAD and CLUSTER are identical on 6 threads and so only one column is shown.

The difference between CLUSTER and SPREAD becomes more evident on 4-objective problems, where we only choose 4 (or 8 or 12) of the possible 24 permutations. On a significant proportion of our test cases we see that SPREAD beats EPP, but EPP beats CLUSTER. An analysis of the running of CLUSTER and SPREAD gives one possible explanation for the difference in running times between the two. When solving a biobjective problem (such as \(OIP(2, (a_3, \ldots, a_n))\), which gets solved repeatedly), the algorithms often only find one or two new solutions i.e., solutions which aren’t found via a relaxation. However, if two threads are attempting to solve a biobjective problem from two different permutations, there is only ever a performance increase if they can solve for different solutions, which requires at least 3 new solutions in each biobjective problem. This was very rare in our randomly-generated problems. It is definitely plausible that there exist problems where each new biobjective problem has numerous solutions, and in these cases we believe that the CLUSTER algorithm may perform better, but we are not aware of any research into finding such problems.

7 Conclusion

We demonstrate a new paradigm for approaching parallelisation in multi-objective optimisation problems. By utilising different permutations of objective functions, our new theory presents many different directions from which a MOIP problem can be solved. This allows parallel algorithms to start searching almost immediately for solutions to the problem, rather than spending time trying to find an equitable split of the search space. The threads are also able to communicate in real time, and this communication creates a synergy where each thread can reduce the running time of all other threads, which in turn can speed up the first thread.
We give the first comparative look at the running time of exact MOIP algorithms in parallel. This shows that even some seemingly sequential algorithms such as V-SPLIT can benefit from parallelisation. We also introduce three of our own new parallel algorithms, along with implementations. All three new algorithms perform competitively on the smaller test cases, and on larger test cases we significantly outperform existing results. This may prompt more study into larger and more complex MOIP problems, problems which until now may have been impractical to solve.

Two of our new algorithms utilise the synergistic theory we present. One of these, SPREAD, significantly outperforms all other algorithms on the larger test cases, including the other synergistic algorithm CLUSTER. The difference between SPREAD and CLUSTER is how permutations are chosen. It may be useful to further study how this choice may affect the running time of our algorithms, especially as it relates to specific MOIP problems. The extension of EPP to projections to two or more dimensions may also prove useful in scenarios where many threads are available.

The publication of our implementations as well as our algorithms allows the easier comparison of the running time of exact MOIP algorithms, and will hopefully spur further research and development in this field.

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A Examples

The first two examples are detailed walk throughs of solving constrained lexicographic problems.

Example 3 (Calculating $L\!\!IP^3(2,(52))$). Consider the following set of objective vectors

| $f_1$ | $f_2$ | $f_3$ |
|------|------|------|
| 50   | 24   | 44   |
| 46   | 41   | 41   |
| 37   | 46   | 37   |
| 37   | 44   | 42   |
| 32   | 39   | 54   |

The value (52) in the definition of the problem says that we are only interested in objective vectors which satisfy $f_3 \leq 52$. This immediately rules out $(32, 39, 54)$, and we no longer use this objective vector for any domination tests, leaving us with the following.

| $f_1$ | $f_2$ | $f_3$ |
|------|------|------|
| 50   | 24   | 44   |
| 46   | 41   | 41   |
| 37   | 46   | 37   |
| 37   | 44   | 42   |

Next, the 2 indicates that we want to discard any objective vector which is dominated in its first two objective values by some other objective vector which we have not discarded. This is represented in the table by the columns to the left of the vertical line. We see that $(37, 46, 37)$ is dominated over the first two
objectives by (37, 44, 42). Even though 37 < 42, we discard (37, 46, 37) as we only consider the first two objectives. All remaining objective vectors are non-dominated in their first two objective values, so we are done and the objective solutions to $LIP^n(2, (52) are \{(50, 24, 44), (46, 41, 41), (37, 44, 42)\}.

Example 4 (Calculating $LIP^3(1, (48, 43))$). Again we are working from

\[
\begin{array}{ccc}
 f_1 & f_2 & f_3 \\
 (50) & 24 & 44 \\
 (46) & 41 & 41 \\
 (37) & 46 & 37 \\
 (37) & 44 & 42 \\
 (32) & 39 & 54.
\end{array}
\]

We can immediately discard (50, 24, 44) and (32, 39, 54) from the given upper bounds (48, 43), leaving

\[
\begin{array}{ccc}
 f_1 & f_2 & f_3 \\
 (46) & 41 & 41 \\
 (37) & 46 & 37 \\
 (37) & 44 & 42.
\end{array}
\]

We next consider dominance in the first objective only, letting us discard (46, 41, 41). This leaves us with

\[
\begin{array}{ccc}
 f_1 & f_2 & f_3 \\
 (37) & 46 & 37 \\
 (37) & 44 & 42.
\end{array}
\]

These are equal in their first objective, so neither dominates the other. We then consider the final two objective functions in lexicographic order. That is, we consider $f_2$ before $f_3$ and so-on. As 44 < 46, we discard (37, 46, 37) and the objective solution to $LIP^3(1, (48, 43))$ is \{(37, 44, 42)\}.

We now show how different permutations $s$ affect the ordered variants, ordered lexicographic problems.

Example 5 (Calculating $OIP^3_{(2, 1, 3)}(1, (48, 43))$). We work from the same initial objective vectors set as the earlier examples.

\[
\begin{array}{ccc}
 f_1 & f_2 & f_3 \\
 (50) & 24 & 44 \\
 (46) & 41 & 41 \\
 (37) & 46 & 37 \\
 (37) & 44 & 42 \\
 (32) & 39 & 54.
\end{array}
\]

To aid our understanding of how the permutation affects the problem, however, we rearrange the columns according to $s$ to give

\[
\begin{array}{ccc}
 f_2 & f_1 & f_3 \\
 (24) & 50 & 44 \\
 (41) & 46 & 41 \\
 (46) & 37 & 37 \\
 (44) & 37 & 42 \\
 (39) & 32 & 54.
\end{array}
\]
We now demonstrate which of these correspond to solutions of $OIP_{(s,1,3)}^3(1,(48,43))$.

First, we discard objective vectors that break the given bounds. As $s(2) = 1$, we discard objective vectors with $f_1 > 48$. The 48 refers to an upper bound on $f_1$ due to the permutation $s$. This causes us to discard $(50,24,44)$ (which appears as $(24,50,44)$ in the above table as we re-ordered the columns). Also, as $s(3) = 3$, we discard objective vectors with $f_3 > 52$. That is, we once again discard $(32,39,54)$.

\[
\begin{array}{ccc}
    f_2 & f_1 & f_3 \\
    41 & 46 & 41 \\
    46 & 37 & 37 \\
    44 & 37 & 42 \\
\end{array}
\]

We now consider dominance on objective $f_2$. We use $f_2$ as $s(1) = 2$, and see that $(46,41,41)$ is the unique solution to attain a minimum on $f_2$. Our objective solution is \{$(46,41,41)$\}.

**Example 6** (Calculating $OIP_{(1,3,2)}^3(1,(51,50))$). We work from the same initial objective vectors, and again permute the columns according to $s$.

\[
\begin{array}{ccc}
    f_1 & f_3 & f_2 \\
    50 & 44 & 24 \\
    46 & 41 & 41 \\
    37 & 37 & 46 \\
    37 & 42 & 44 \\
    32 & 54 & 39 \\
\end{array}
\]

As $s(2) = 3$, we discard objective vectors that don’t satisfy $f_3 < 51$, that is $(32,39,54)$. And as $s(3) = 2$, we discard objective vectors that don’t satisfy $f_2 < 50$, but all objective vectors satisfy this bound. Next we consider dominance across objective $s(1) = 1$.

\[
\begin{array}{ccc}
    f_1 & f_3 & f_2 \\
    50 & 44 & 24 \\
    46 & 41 & 41 \\
    37 & 37 & 46 \\
    37 & 42 & 44 \\
\end{array}
\]

Once again we are left with $(37,46,36)$ and $(37,44,42)$, which are equal in their first objective.

\[
\begin{array}{ccc}
    f_1 & f_3 & f_2 \\
    37 & 37 & 46 \\
    37 & 42 & 44 \\
\end{array}
\]

However, we now consider the remaining two objectives in the order prescribed by $s$. As $s(2) = 3$, we consider values of $f_3$ next and as $36 < 42$, we discard $(37,44,42)$. Therefore the objective solution to $OIP_{(1,3,2)}^3(1,(51,50))$ is \{$(37,46,36)$\}.

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