SOME SIMPSON TYPE INEQUALITIES FOR $h$–CONVEX AND
$(\alpha, m)$–CONVEX FUNCTIONS

WENJUN LIU

Abstract. In this paper, we establish some Simpson type inequalities for functions whose third
derivatives in the absolute value are $h$–convex and $(\alpha, m)$–convex, respectively.

1. Introduction

The following inequality is well known in the literature as Simpson’s inequality:

$$
\left| \int_a^b f(t) dt - \frac{b-a}{3} \left[ f(a) + f(b) + 2f\left(\frac{a+b}{2}\right) \right] \right| \leq \frac{1}{2880} \| f^{(4)} \|_\infty (b-a)^5,
$$

where the mapping $f : [a, b] \rightarrow \mathbb{R}$ is supposed to be four time differentiable on the interval
$(a, b)$ and having the fourth derivative bounded on $(a, b)$, that is $\| f^{(4)} \|_\infty = \sup_{x \in (a,b)} |f^{(4)}(x)| < \infty$. This inequality gives an error bound for the classical Simpson quadrature formula, which,
actually, is one of the most used quadrature formulae in practical applications. In recent years,
such inequalities were studied extensively by many researchers and numerous generalizations,
extensions and variants of them appeared in a number of papers (see [1, 5, 6, 10, 11, 12, 17, 19]).

Let us recall definitions of some kinds of convexity as follows.

Definition A. [8] We say that $f : I \rightarrow \mathbb{R}$ is Godunova-Levin function or that $f$ belongs to the
class $Q(I)$ if $f$ is non-negative and for all $x, y \in I$ and $t \in (0, 1)$ we have

$$
f(tx + (1-t)y) \leq \frac{f(x)}{t} + \frac{f(y)}{1-t}.
$$

Definition B. [7] We say that $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ is a $P$–function or that $f$ belongs to the class
$P(I)$ if $f$ is non-negative and for all $x, y \in I$ and $t \in [0, 1]$ we have

$$
f(tx + (1-t)y) \leq f(x) + f(y).
$$

Definition C. [8] Let $s \in (0, 1]$. A function $f : (0, \infty) \rightarrow [0, \infty]$ is said to be $s$–convex in the
second sense if

$$
f(tx + (1-t)y) \leq t^s f(x) + (1-t)^s f(y),
$$

for all $x, y \in (0, b]$ and $t \in [0, 1]$. This class of $s$–convex functions is usually denoted by $K_s^2$.

Definition D. [20] Let $h : J \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a positive function. We say that $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$
is $h$–convex function, or that $f$ belongs to the class $SX(h, I)$, if $f$ is non-negative and for all
$x, y \in I$ and $t \in [0, 1]$ we have

$$
f(tx + (1-t)y) \leq h(t)f(x) + h(1-t)f(y).
$$

If inequality (1.5) is reversed, then $f$ is said to be $h$–concave, i.e. $f \in SV(h, I)$.

Obviously, if $h(t) = t$, then all non-negative convex functions belong to $SX(h, I)$ and all
non-negative concave functions belong to $SV(h, I)$; if $h(t) = \frac{1}{t^4}$, then $SX(h, I) = Q(I)$; if
$h(t) = 1$, then $SX(h, I) \supseteq P(I)$; and if $h(t) = t^s$, where $s \in (0, 1]$, then $SX(h, I) \supseteq K_s^2$. For
recent results concerning $h$–convex functions see [3, 4, 16, 20] and references therein.

2000 Mathematics Subject Classification. 26A51, 26D07, 26D10, 26D15.

Key words and phrases. Simpson type inequality, $h$–convex function, $(\alpha, m)$–convex function.
Theorem C. The function $f : [0, b] \to \mathbb{R}$ is said to be $m$–convex, where $m \in [0, 1]$, if for every $x, y \in [0, b]$ and $t \in [0, 1]$ we have

$$f(tx + m(1-t)y) \leq tf(x) + m(1-t)f(y).$$

Denote by $K_m(b)$ the set of the $m$–convex functions on $[0, b]$ for which $f(0) \leq 0$.

Definition E. The function $f : [0, b] \to \mathbb{R}$ is said to be $(\alpha, m)$–convex, where $\alpha, m \in [0, 1]^2$, if for all $x, y \in [0, b]$ and $t \in [0, 1]$ we have

$$f(tx + m(1-t)y) \leq t^\alpha f(x) + m(1-t^\alpha)f(y).$$

Denote by $K_\alpha^m(b)$ the class of all $(\alpha, m)$–convex functions on $[0, b]$ for which $f(0) \leq 0$.

If we choose $(\alpha, m) = (1, m)$, it can be easily seen that $(\alpha, m)$–convexity reduces to $m$–convexity and for $(\alpha, m) = (1, 1)$, we have ordinary convex functions on $[0, b]$.

Recently, Özdemir et al. established some Simpson type inequalities for functions whose third derivatives in the absolute value are $m$–convex. In Özdemir et al. established the following inequalities for functions whose third derivatives in the absolute value are $s$–convex in the second sense.

Theorem A. Let $f : I \subset [0, \infty) \to \mathbb{R}$ be a differentiable function on $I^\circ$ such that $f''' \in L_1[a, b]$, where $a, b \in I^\circ$ with $a \leq b$. If $|f'''|\leq q$ is $s$–convex in the second sense on $[a, b]$ for some fixed $s \in [0, 1]$, then

$$\int_a^b f(x)dx - \frac{b-a}{6} \left[ f(a) + 4f \left( \frac{a+b}{2} \right) + f(b) \right] \leq \frac{(b-a)^4}{6} \left[ \frac{2^{4-s}(1+s)(2+s) + 34 + 2^{4+s}(-2 + s) + 11s + s^2}{(1+s)(2+s)(3+s)(4+s)} \right] \left[ |f'''(a)| + |f'''(b)| \right].$$

Theorem B. Let $f : I \subset [0, \infty) \to \mathbb{R}$ be a differentiable function on $I^\circ$ such that $f''' \in L_1[a, b]$, where $a, b \in I^\circ$ with $a \leq b$. If $|f'''|\leq q$ is $s$–convex in the second sense on $[a, b]$ for some fixed $s \in [0, 1]$ and $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, then

$$\int_a^b f(x)dx - \frac{b-a}{6} \left[ f(a) + 4f \left( \frac{a+b}{2} \right) + f(b) \right] \leq \frac{(b-a)^4}{48} \left( \frac{1}{2} \right)^\frac{1}{p} \left( \frac{\Gamma(2p+1)\Gamma(p+1)}{\Gamma(3p+2)} \right)^\frac{1}{p} \left\{ \left[ \frac{1}{2^{s+1}(s+1)} |f'''(a)|^q + \frac{2^{s+1} - 1}{2^{s+1}(s+1)} |f'''(b)|^q \right]^{\frac{1}{q}} \right\}.$$

Theorem C. Suppose that all the assumptions of Theorem are satisfied. Then

$$\int_a^b f(x)dx - \frac{b-a}{6} \left[ f(a) + 4f \left( \frac{a+b}{2} \right) + f(b) \right] \leq \frac{(b-a)^4}{192} \left( \frac{1}{128} \right)^{1-\frac{1}{q}} \times \left\{ \left[ \frac{2^{4-s}}{(3+s)(4+s)} |f'''(a)|^q + \frac{2^{4-s}(34 + 2^{4+s}(-2 + s) + 11s + s^2)}{(1+s)(2+s)(3+s)(4+s)} |f'''(b)|^q \right]^{\frac{1}{q}} \right\}.$$

The main purpose of this paper is to establish some new Simpson type inequalities for functions whose third derivatives in the absolute value are $h$–convex and $(\alpha, m)$–convex, respectively.
2. Simpson type inequalities for \(h\)-convex functions

To prove our main theorems, we need the following identity established in [2]:

**Lemma 1.** Let \(f : I \to \mathbb{R}\) be a function such that \(f'''\) be absolutely continuous on \(I^\circ\), the interior of \(I\). Assume that \(a, b \in I^\circ\), with \(a < b\) and \(f''' \in L_1[a, b]\). Then, the following equality holds:

\[
\int_a^b f(x)dx - \frac{b-a}{6} \left[ f(a) + 4f \left( \frac{a+b}{2} \right) + f(b) \right] = (b-a)^4 \int_0^1 p(t)f'''(ta + (1-t)b)dt,
\]

where

\[
p(t) = \begin{cases} \frac{1}{6} t^2 \left( t - \frac{1}{2} \right), & t \in [0, \frac{1}{2}], \\ \frac{1}{6} (t-1)^2 \left( t - \frac{1}{2} \right), & t \in (\frac{1}{2}, 1]. \end{cases}
\]

Using this lemma, we can obtain the following inequalities for \(h\)-convex functions.

**Theorem 1.** Let \(h : J \subseteq \mathbb{R} \to \mathbb{R}\) be a non-negative function, and \(f : I \subset [0, \infty) \to \mathbb{R}\) be a differentiable function on \(I^\circ\) such that \(f''' \in L_1[a, b]\), where \(a, b \in I^\circ\) with \(a < b\). If \(|f'''|\) is \(h\)-convex on \([a, b]\), then

\[
\left| \int_a^b f(x)dx - \frac{b-a}{6} \left[ f(a) + 4f \left( \frac{a+b}{2} \right) + f(b) \right] \right| \leq \frac{(b-a)^4}{6} \left( \int_0^1 \frac{1}{6} t^2 \left( t - \frac{1}{2} \right) |f'''(ta + (1-t)b)| dt \right.
\]

\[
+ \int_0^1 \frac{1}{6} (t-1)^2 \left( t - \frac{1}{2} \right) |f'''(ta + (1-t)b)| dt \right)
\]

\[
\leq \frac{(b-a)^4}{6} \left( \int_0^1 t^2 \left( t - \frac{1}{2} \right) (h(t) |f'''(a)| + h(1-t) |f'''(b)|) dt \right.
\]

\[
+ \int_0^1 (t-1)^2 \left( t - \frac{1}{2} \right) (h(t) |f'''(a)| + h(1-t) |f'''(b)|) dt \right)
\]

\[
= \frac{(b-a)^4}{6} \left( \int_0^1 t^2 \left( t - \frac{1}{2} \right) h(t)dt + \int_0^1 (t-1)^2 \left( t - \frac{1}{2} \right) h(t)dt \right) [\|f'''\| + \|f'''\|],
\]

where we have used the fact that

\[
\int_0^\frac{1}{2} t^2 \left( t - \frac{1}{2} \right) h(t)dt + \int_\frac{1}{2}^1 (t-1)^2 \left( t - \frac{1}{2} \right) h(t)dt
\]

\[
= \int_0^\frac{1}{2} t^2 \left( t - \frac{1}{2} \right) h(1-t)dt + \int_\frac{1}{2}^1 (t-1)^2 \left( t - \frac{1}{2} \right) h(1-t)dt
\]

\[
= \int_0^\frac{1}{2} t^2 \left( t - \frac{1}{2} \right) h(t)dt + \int_0^\frac{1}{2} t^2 \left( t - \frac{1}{2} \right) h(1-t)dt.
\]

Hence, the proof of (2.1) is complete. \(\square\)

**Remark 1.** In Theorem 1, if we choose \(h(t) = t^s\), \(s \in (0, 1]\), then (2.1) reduces to (1.6).
**Theorem 2.** Let \( h : J \subseteq \mathbb{R} \to \mathbb{R} \) (\([0, 1] \subseteq J\)) be a non-negative function, and \( f : I \subset [0, \infty) \to \mathbb{R} \) be a differentiable function on \( I^\circ \) such that \( f''' \in L_1[a, b] \), where \( a, b \in I^\circ \) with \( a < b \). If \(|f'''|^{q}\) is \( h\)–convex on \([a, b]\) and \( q > 1 \) with \( \frac{1}{p} + \frac{1}{q} = 1 \), then

\[\left| \int_{a}^{b} f(x) dx - \frac{b-a}{6} \left[ f(a) + 4f \left( \frac{a+b}{2} \right) + f(b) \right] \right| \leq \frac{(b-a)^4}{48} \left( \frac{1}{2} \right)^{\frac{1}{p}} \left( \frac{\Gamma(2p+1)\Gamma(p+1)}{\Gamma(3p+2)} \right)^{\frac{1}{p}} \times \left\{ \left[ \left( \int_{0}^{\frac{1}{2}} h(t)dt \right) |f'''(a)|^q + \left( \int_{0}^{\frac{1}{2}} h(1-t)dt \right) |f'''(b)|^q \right]^{\frac{1}{q}} + \left[ \left( \int_{0}^{\frac{1}{2}} h(1-t)dt \right) |f'''(a)|^q + \left( \int_{0}^{\frac{1}{2}} h(t)dt \right) |f'''(b)|^q \right]^{\frac{1}{q}} \right\} . \tag{2.2} \]

**Proof.** From Lemma \([\text{II}]\) and using the \( s\)–convexity of \(|f'''|^{q}\) and the well-known Hölder’s inequality, we have

\[
\left| \int_{a}^{b} f(x) dx - \frac{b-a}{6} \left[ f(a) + 4f \left( \frac{a+b}{2} \right) + f(b) \right] \right| \\ \leq \frac{(b-a)^4}{6} \left\{ \left( \int_{0}^{\frac{1}{2}} \left( t^2 \left( \frac{1}{2} - t \right) \right)^p dt \right)^{\frac{1}{p}} \left( \int_{0}^{\frac{1}{2}} |f'''(ta + (1-t)b)|^q dt \right)^{\frac{1}{q}} \\ + \left( \int_{0}^{1} \left( t - 1 \right)^2 \left( t - \frac{1}{2} \right)^p dt \right)^{\frac{1}{p}} \left( \int_{0}^{\frac{1}{2}} |f'''(ta + (1-t)b)|^q dt \right)^{\frac{1}{q}} \right\} \\ \leq \frac{(b-a)^4}{6} \left( \frac{\Gamma(2p+1)\Gamma(p+1)}{2^{3p+1}\Gamma(3p+2)} \right)^{\frac{1}{p}} \left\{ \left( \int_{0}^{\frac{1}{2}} \left[ h(t) |f'''(a)|^q + h(1-t) |f'''(b)|^q \right] dt \right)^{\frac{1}{q}} \\ + \left( \int_{0}^{\frac{1}{2}} \left[ h(t) |f'''(a)|^q + h(1-t) |f'''(b)|^q \right] dt \right)^{\frac{1}{q}} \right\} \\ \leq \frac{(b-a)^4}{48} \left( \frac{1}{2} \right)^{\frac{1}{p}} \left( \frac{\Gamma(2p+1)\Gamma(p+1)}{\Gamma(3p+2)} \right)^{\frac{1}{p}} \times \left\{ \left[ \left( \int_{0}^{\frac{1}{2}} h(t)dt \right) |f'''(a)|^q + \left( \int_{0}^{\frac{1}{2}} h(1-t)dt \right) |f'''(b)|^q \right]^{\frac{1}{q}} \\ + \left[ \left( \int_{0}^{\frac{1}{2}} h(t)dt \right) |f'''(a)|^q + \left( \int_{0}^{\frac{1}{2}} h(1-t)dt \right) |f'''(b)|^q \right]^{\frac{1}{q}} \right\} ,
\]

\[\text{where we have used the fact that} \]

\[\int_{0}^{\frac{1}{2}} \left( t^2 \left( \frac{1}{2} - t \right) \right)^p dt = \int_{0}^{1} \left( t - 1 \right)^2 \left( t - \frac{1}{2} \right)^p dt = \frac{\Gamma(2p+1)\Gamma(p+1)}{2^{3p+1}\Gamma(3p+2)} \]

and \( \Gamma \) is the Gamma function. Hence, the proof of (2.2) is complete. \( \square \)

**Remark 2.** In Theorem 2 if we choose \( h(t) = t^s \), \( s \in (0, 1] \), then (2.2) reduces to (1.7).

A different approach leads to the following result.
Theorem 3. Suppose that all the assumptions of Theorem 1 are satisfied. Then
\[
\left| \int_a^b f(x)dx - \frac{b-a}{6} \left[ f(a) + 4f \left( \frac{a+b}{2} \right) + f(b) \right] \right|
\leq \frac{(b-a)^4}{6} \left( \frac{1}{192} \right)^{1-\frac{1}{q}}
\times \left\{ \left[ \left( \int_0^{\frac{1}{2}} t^2 \left( \frac{1}{2} - t \right) |h(t)|^q dt \right) |f'''(a)|^q + \left( \int_0^{\frac{1}{2}} t^2 \left( \frac{1}{2} - t \right) h(1-t)dt \right) |f'''(b)|^q \right]^{\frac{1}{q}} \right\}.
\]

(2.4)\quad + \left[ \left( \int_0^{\frac{1}{2}} t^2 \left( \frac{1}{2} - t \right) h(1-t)dt \right) |f'''(a)|^q + \left( \int_0^{\frac{1}{2}} t^2 \left( \frac{1}{2} - t \right) h(t)dt \right) |f'''(b)|^q \right]^{\frac{1}{q}} \}

Proof. From Lemma 2 and using the well-known power-mean inequality we have
\[
\left| \int_a^b f(x)dx - \frac{b-a}{6} \left[ f(a) + 4f \left( \frac{a+b}{2} \right) + f(b) \right] \right|
\leq \frac{(b-a)^4}{6} \left\{ \left( \int_0^{\frac{1}{2}} t^2 \left( \frac{1}{2} - t \right) \right)^{1-\frac{1}{q}} \left( \int_0^{\frac{1}{2}} t^2 \left( \frac{1}{2} - t \right) |f'''(ta + (1-t)b)|^q dt \right)^{\frac{1}{q}} \right\}
\]
\[
+ \left( \int_0^{\frac{1}{2}} (t-1)^2 \left( t - \frac{1}{2} \right) dt \right)^{1-\frac{1}{q}} \left( \int_0^{\frac{1}{2}} (t-1)^2 \left( t - \frac{1}{2} \right) |f'''(ta + (1-t)b)|^q dt \right)^{\frac{1}{q}} \}
\]
Since $|f'''|^q$ is $s-$convex, we have
\[
\int_0^{\frac{1}{2}} t^2 \left( \frac{1}{2} - t \right) |f'''(ta + (1-t)b)|^q dt
\]
\[
\leq \int_0^{\frac{1}{2}} t^2 \left( \frac{1}{2} - t \right) (h(t) |f'''(a)|^q + h(1-t) |f'''(b)|^q) dt
\]
\[
= \left( \int_0^{\frac{1}{2}} t^2 \left( \frac{1}{2} - t \right) h(t)dt \right) |f'''(a)|^q + \left( \int_0^{\frac{1}{2}} t^2 \left( \frac{1}{2} - t \right) h(1-t)dt \right) |f'''(b)|^q
\]
and
\[
\int_0^{\frac{1}{2}} (t-1)^2 \left( t - \frac{1}{2} \right) |f'''(ta + (1-t)b)|^q dt
\]
\[
\leq \int_0^{\frac{1}{2}} (t-1)^2 \left( t - \frac{1}{2} \right) (h(t) |f'''(a)|^q + h(1-t) |f'''(b)|^q) dt
\]
\[
= \left( \int_0^{\frac{1}{2}} t^2 \left( \frac{1}{2} - t \right) h(1-t)dt \right) |f'''(a)|^q + \left( \int_0^{\frac{1}{2}} t^2 \left( \frac{1}{2} - t \right) h(t)dt \right) |f'''(b)|^q
\].

Hence, the proof of (2.4) is complete. \(\square\)

Remark 3. In Theorem 3, if we choose $h(t) = t^s$, $s \in (0,1]$, then (2.4) reduces to (1.8).

3. Simpson type inequalities for $(\alpha, m)$-convex functions

We use the following modified identity:

Lemma 2. [14] Lemma 2 Let $f : I \to \mathbb{R}$ be a function such that $f'''$ be absolutely continuous on $I^0$, the interior of $I$. Assume that $a,b \in I^0$, with $a < b$, $m \in (0,1]$ and $f''' \in L^1[a,b]$. Then, the
following equality holds:
\[
\int_a^{mb} f(x)dx - \frac{mb-a}{6} \left[ f(a) + 4f \left( \frac{a+mb}{2} \right) + f(mb) \right]
= (mb-a)^4 \int_0^1 p(t)f'''(ta + m(1-t)b)dt,
\]
where
\[
p(t) = \begin{cases} \frac{1}{6}t^2(t - \frac{1}{2}), & t \in [0, \frac{1}{2}], \\ \frac{1}{6}(t-1)^2(t - \frac{1}{2}), & t \in (\frac{1}{2}, 1]. \end{cases}
\]

Using this lemma, we can obtain the following inequalities for \((\alpha, m)\)-convex functions.

**Theorem 4.** Let \(f : I \subset [0, b^*] \to \mathbb{R}\), be a differentiable function on \(I^o\) such that \(f''' \in L_1[a, b]\) where \(a, b \in I\) with \(a < b\), \(b^* > 0\). If \(|f'''|^q\) is \((\alpha, m)\)-convex on \([a, b]\) for \((\alpha, m) \in [0, 1]^2\), \(q > 1\) with \(\frac{1}{p} + \frac{1}{q} = 1\), then

\[
\left| \int_a^{mb} f(x)dx - \frac{mb-a}{6} \left[ f(a) + 4f \left( \frac{a+mb}{2} \right) + f(mb) \right] \right|
\leq \frac{(mb-a)^4}{96} \left( \left( \int_0^{\frac{1}{2}} \left( 1 - t \right)^p dt \right)^\frac{1}{p} \left( \int_0^{\frac{1}{2}} \left| f'''(ta + m(1-t)b) \right|^q dt \right)^\frac{1}{q} + \left( \int_{\frac{1}{2}}^{1} \left( 1 - t \right)^p dt \right)^\frac{1}{p} \left( \int_{\frac{1}{2}}^{1} \left| f'''(ta + m(1-t)b) \right|^q dt \right)^\frac{1}{q} \right).
\]

**Proof.** From Lemma 2 and using Hölder’s inequality we have

\[
\left| \int_a^{mb} f(x)dx - \frac{mb-a}{6} \left[ f(a) + 4f \left( \frac{a+mb}{2} \right) + f(mb) \right] \right|
\leq \frac{(mb-a)^4}{6} \left\{ \left( \int_0^{\frac{1}{2}} \left( 1 - t \right)^p dt \right)^\frac{1}{p} \left( \int_0^{\frac{1}{2}} \left| f'''(ta + m(1-t)b) \right|^q dt \right)^\frac{1}{q} + \left( \int_{\frac{1}{2}}^{1} \left( 1 - t \right)^p dt \right)^\frac{1}{p} \left( \int_{\frac{1}{2}}^{1} \left| f'''(ta + m(1-t)b) \right|^q dt \right)^\frac{1}{q} \right\}.
\]

Due to the \((\alpha, m)\)-convexity of \(|f'''|^q\), we have

\[
\int_0^{\frac{1}{2}} \left| f'''(ta + m(1-t)b) \right|^q dt \leq \int_0^{\frac{1}{2}} \left[ \left| f'''(a) \right|^q + m(1-t^\alpha) \left| f'''(b) \right|^q \right] dt
= \frac{\left| f'''(a) \right|^q + m[2^\alpha(1+\alpha) - 1] \left| f'''(b) \right|^q}{2^{1+\alpha}(1+\alpha)}
\]
and

\[
\int_{\frac{1}{2}}^{1} \left| f'''(ta + m(1-t)b) \right|^q dt \leq \int_{\frac{1}{2}}^{1} \left[ \left| f'''(a) \right|^q + m(1-t^\alpha) \left| f'''(b) \right|^q \right] dt
= \frac{(2^{1+\alpha} - 1) \left| f'''(a) \right|^q + m[2^\alpha(1+\alpha) - (2^{1+\alpha} - 1)] \left| f'''(b) \right|^q}{2^\alpha(1+\alpha)}.
\]

The proof of (3.1) is complete by combining the above inequalities and [23].

**Remark 4.** In Theorem 4, if we choose \(\alpha = 1\), we get the inequality in [14] Theorem 4.
Theorem 5. Let the assumptions of Theorem 4 hold with \( q \geq 1 \). Then
\[
\left\| \int_a^{\text{mb}} f(x)dx - \frac{mb - a}{6} \left[ f(a) + 4f \left( \frac{a + mb}{2} \right) + f(mb) \right] \right\| \leq \frac{(mb - a)^4}{1152} \left\{ \left( \frac{12|f'''(a)|^q + m[2^\alpha(3 + \alpha)(4 + \alpha) - 12]|f'''(b)|^q}{2^\alpha(3 + \alpha)(4 + \alpha)} \right)^{\frac{1}{q}} + \left( \frac{12\alpha^2 + 11\alpha + 34 - 2^{4+\alpha}(2 - \alpha)}{2^\alpha(1 + \alpha)(2 + \alpha)(3 + \alpha)(4 + \alpha)} |f'''(a)|^q \right) \right\}.
\]

(3.2)

Proof. From Lemma 2 using the well known power-mean inequality and \((\alpha, m)\)-convexity of \( |f'''| \), we have
\[
\left\| \int_a^{\text{mb}} f(x)dx - \frac{mb - a}{6} \left[ f(a) + 4f \left( \frac{a + mb}{2} \right) + f(mb) \right] \right\| \leq \frac{(b - a)^4}{6} \left\{ \left( \int_0^{\frac{1}{2}} t^2 \left( \frac{1}{2} - t \right) dt \right)^{1-\frac{1}{q}} \left( \int_0^{\frac{1}{2}} t^2 \left( \frac{1}{2} - t \right) |f'''(ta + m(1-t)b)|^q dt \right)^{\frac{1}{q}} + \left( \int_{\frac{1}{2}}^1 (t - 1)^2 \left( t - \frac{1}{2} \right) dt \right)^{1-\frac{1}{q}} \left( \int_{\frac{1}{2}}^1 (t - 1)^2 \left( t - \frac{1}{2} \right) |f'''(ta + m(1-t)b)|^q dt \right)^{\frac{1}{q}} \right\}.
\]

By using the fact that
\[
\int_0^{\frac{1}{2}} t^2 \left( \frac{1}{2} - t \right) t^{\alpha} dt = \frac{1}{16 \times 2^\alpha(3 + \alpha)(4 + \alpha)},
\]
\[
\int_0^{\frac{1}{2}} t^2 \left( \frac{1}{2} - t \right) (1-t^{\alpha}) dt = \frac{2^\alpha(3 + \alpha)(4 + \alpha) - 12}{192 \times 2^\alpha(3 + \alpha)(4 + \alpha)},
\]
\[
\int_{\frac{1}{2}}^1 (t - 1)^2 \left( t - \frac{1}{2} \right) t^{\alpha} dt = \frac{\alpha^2 + 11\alpha + 34 - 2^{4+\alpha}(2 - \alpha)}{16 \times 2^\alpha(1 + \alpha)(2 + \alpha)(3 + \alpha)(4 + \alpha)}
\]

and
\[
\int_{\frac{1}{2}}^1 (t - 1)^2 \left( t - \frac{1}{2} \right) (1-t^{\alpha}) dt = \frac{2^\alpha(1 + \alpha)(2 + \alpha)(3 + \alpha)(4 + \alpha) - 12[\alpha^2 + 11\alpha + 34 - 2^{4+\alpha}(2 - \alpha)]}{192 \times 2^\alpha(1 + \alpha)(2 + \alpha)(3 + \alpha)(4 + \alpha)},
\]

we obtain
\[
\left\| \int_a^{\text{mb}} f(x)dx - \frac{mb - a}{6} \left[ f(a) + 4f \left( \frac{a + mb}{2} \right) + f(mb) \right] \right\| \leq \frac{(mb - a)^4}{6} \left\{ \left( \frac{12|f'''(a)|^q + m[2^\alpha(3 + \alpha)(4 + \alpha) - 12]|f'''(b)|^q}{192 \times 2^\alpha(3 + \alpha)(4 + \alpha)} \right)^{\frac{1}{q}} + \left( \frac{\alpha^2 + 11\alpha + 34 - 2^{4+\alpha}(2 - \alpha)}{16 \times 2^\alpha(1 + \alpha)(2 + \alpha)(3 + \alpha)(4 + \alpha)} |f'''(a)|^q \right) \right\}. 
\]
\[ +m \left[ \frac{1}{192} - \frac{\alpha^2 + 11\alpha + 34 - 2^{4+\alpha}(2 - \alpha)}{16 \times 2^{\alpha}(1 + \alpha)(2 + \alpha)(3 + \alpha)(4 + \alpha)} \right] \frac{|f'''(b)|}{q} \frac{q}{q} \]  

which implies the desired result. \( \square \)

**Remark 5.** In Theorem 5, if we choose \( \alpha = 1 \), we have the inequality in [14, Theorem 5].

**References**

[1] M. Alomari and M. Darus, On some inequalities of Simpson-type via quasi-convex functions and applications, Transylv. J. Math. Mech. 2 (2010), no. 1, 15–24.
[2] M. Alomari and S. Hussain, Two inequalities of Simpson type for quasi-convex functions and applications, Appl. Math. E-Notes 11 (2011), 110–117.
[3] M. Bombardelli and S. Varošanec, Properties of \( h \)-convex functions related to the Hermite-Hadamard-Fejér inequalities, Comput. Math. Appl. 58 (2009), no. 9, 1869–1877.
[4] P. Burai and A. Házy, On approximately \( h \)-convex functions, J. Convex Anal. 18 (2011), no. 2, 447–454.
[5] S. S. Dragomir, On Simpson’s quadrature formula for mappings of bounded variation and applications, Tamkang J. Math. 30 (1999), no. 1, 53–58.
[6] S. S. Dragomir, R. P. Agarwal and P. Cerone, On Simpson’s inequality and applications, J. Inequal. Appl. 5 (2000), no. 6, 533–579.
[7] S. S. Dragomir, J. Pečarić and L. E. Persson, Some inequalities of Hadamard type, Soochow J. Math. 21 (1995), no. 3, 335–341.
[8] E. K. Godunova and V. I. Levin, Inequalities for functions of a broad class that contains convex, monotone and some other forms of functions, in *Numerical mathematics and mathematical physics (Russian)*, 138–142, 166, Moskov. Gos. Ped. Inst., Moscow.
[9] H. Hudzik and L. Maligranda, Some remarks on \( s \)-convex functions, Aequationes Math. 48 (1994), no. 1, 100–111.
[10] V. N. Huy and Q. -A. Ngô, New inequalities of Simpson-like type involving \( n \) knots and the \( m \)th derivative, Math. Comput. Modelling 52 (2010), no. 3-4, 522–528.
[11] Z. Liu, An inequality of Simpson type, Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci. 461 (2005), no. 2059, 2155–2158.
[12] Z. Liu, Some sharp modified Simpson type inequalities and applications, Vietnam J. Math. 39 (2011), no. 2, 135–144.
[13] V.G. Miheşan, A generalization of the convexity, Seminar of Functional Equations, Approx. and Convex, Cluj-Napoca (Romania) (1999).
[14] M. E. Özdemir, M. Avci and H. Kavurmaci, Simpson type inequalities for \( m \)-convex functions, arXiv: 1112.3559v1 [math.FA].
[15] M. E. Özdemir, M. Avci and H. Kavurmaci, Simpson type inequalities for functions whose third derivatives in the absolute value are \( s \)-convex and \( s \)-concave functions, arXiv:1206.1193v1 [math.CA].
[16] M. Z. Sarıkaya, A. Şahin and H. Yıldırım, On some Hadamard-type inequalities for \( h \)-convex functions, J. Math. Inequal. 2 (2008), no. 3, 335–341.
[17] M. Z. Sarıkaya, E. Set and M. E. Özdemir, On new inequalities of Simpson’s type for \( s \)-convex functions, Comput. Math. Appl. 60 (2010), no. 8, 2191–2199.
[18] G. H. Toader, Some generalisations of the convexity, Proc. Colloq. Approx. Optim. (1984), 329–338.
[19] K.-L. Tseng, G.-S. Yang and S. S. Dragomir, On weighted Simpson type inequalities and applications, J. Math. Inequal. 1 (2007), no. 1, 13–22.
[20] S. Varošanec, On \( h \)-convexity, J. Math. Anal. Appl. 326 (2007), no. 1, 303–311.

College of Mathematics and Statistics, Nanjing University of Information Science and Technology, Nanjing 210044, China

E-mail address: wjliu.cn@gmail.com