Poisson structures on the center
of the elliptic algebra $A_{q,p}(\hat{sl}(2)_c)$

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Abstract

It is shown that the elliptic algebra $A_{q,p}(\hat{sl}(2)_c)$ has a non trivial center at the critical level $c = -2$, generalizing the result of Reshetikhin and Semenov-Tian-Shansky for trigonometric algebras. A family of Poisson structures indexed by a non-negative integer $k$ is constructed on this center.

Résumé

On montre que l’algèbre elliptique $A_{q,p}(\hat{sl}(2)_c)$ a un centre non trivial au niveau critique $c = -2$, généralisant le résultat de Reshetikhin et Semenov-Tian-Shansky pour les algèbres trigonométriques. On construit sur ce centre une famille de structures de Poisson indexées par un entier positif $k$. 

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1 Introduction

The concept of deformed Virasoro and $W_N$ algebras has recently arisen in connection with several aspects of integrable systems, both in quantum and statistical mechanics. On the one hand, $q$-deformed Virasoro and $W_N$ algebras of operators were introduced \cite{1,2} as an extension of the Virasoro and $W_N$ algebras identified in the quantum Calogero-Moser model \cite{3}. In the same way as Jack polynomials (eigenfunctions of the quantum Calogero-Moser model) arise as singular vectors of the $W$-deformed $W_N$ algebras, Mac-Donald polynomials (eigenfunctions of the relativistic Ruijsenaars-Schneider model) are singular vectors of the $q$-deformed $W_N$ algebras. The parameter $q$ is connected to the supplementary parameter (speed of light) in Ruijsenaars-Schneider models accounting for relativistic invariance.

On the other hand, it was shown that $q$-deformed Virasoro algebras provided a dynamical symmetry for the Andrew-Baxter-Forrester restriction of RSOS models in statistical mechanics \cite{4}. This model is characterized by a matrix of Boltzmann weights which is an elliptic solution of the Yang-Baxter equation \cite{5} (see also \cite{6} for further developments).

These algebras were shown to arise in fact from a systematic procedure of construction mimicking the already known scheme \cite{8} for Virasoro and $W_N$ algebras: undeformed algebras can be obtained by quantization of a Poisson structure on the center of the enveloping algebra $U(\hat{sl}(N)_c)$ of an affine algebra $\hat{sl}(N)_c$ at the critical level $c = -N$ \cite{8}. This construction is easier to achieve using the bosonized representation \cite{9} of $U(\hat{sl}(N)_c)$, leading to the well-known Miura transformation formula for Virasoro and $W_N$ generators.

The procedure for $q$-deformed Virasoro and $W_N$ algebras is similar. One starts from the quantum affine Lie algebra $U_q(\hat{sl}(N)_c)$ \cite{10,11}. At the critical value $c = -N$, the algebra has a multidimensional center \cite{12} on which a Poisson structure can be defined as limit of the commutator structure \cite{13}. This Poisson structure is the semi-classical limit of the $q$-deformed Virasoro (for $N = 2$) or $W_N$ algebras. Its quantization \cite{14} gives rise to these $q$-deformed algebras. In fact, the construction of classical and quantized algebras was not achieved in \cite{14} by direct computation but using the $q$-deformed bosonization \cite{15} of $U_q(\hat{sl}(N)_c)$ and the corresponding $q$-Miura transformation. Interestingly the $q$-deformed $W_N$ algebras are characterized by elliptic structure coefficients, a fact on which we shall comment at the end: for instance, the $q$-deformed Virasoro algebra is defined by the generating operator $T(z)$ such that

$$f_{1,2}(w/z) T(z) T(w) - f_{1,2}(z/w) T(w) T(z) = \frac{(1 - q)(1 - p/q)}{1 - p} \left( \delta\left(\frac{w}{z^p}\right) - \delta\left(\frac{wp}{z}\right) \right),$$

where

$$f_{1,2}(x) = \frac{1}{1 - x} \frac{\langle x|q, pq^{-1}; p^2\rangle_{\infty}}{\langle x|pq, p^2q^{-1}; p^2\rangle_{\infty}},$$

and as usual

$$\langle x|a_1 \ldots a_k; t\rangle_{\infty} \equiv \prod_{i=1}^{k} \prod_{n=0}^{\infty} (1 - a_i x t^n).$$

The parameters $p$ and $q$ are rewritten as $q = e^h$, $p = e^{h(1-\beta)}$; hence $h$ is the deformation parameter and $\beta$ is the “quantization” parameter, the semi-classical limit $\beta \to 0$ giving back the $q$-deformed Poisson bracket and the limit $h \to 0$ giving back the linear Virasoro algebra.

It is a natural question to extend these constructions to other deformations of affine Lie algebras, in particular to the so-called elliptic quantum algebras introduced by \cite{16} and further studied in
In a second step we examine the limit \( c \rightarrow -2 \) and derive the Poisson structure directly as a limit of this exchange algebra. The direct method is here necessary since no bosonized version à la Wakimoto is get available for this algebra.

2 The elliptic quantum algebra \( A_{q,p}(\hat{sl}(2)_c) \)

2.1. Consider the \( R \)-matrix of the eight vertex model found by Baxter [19]:

\[
R_{12}(x) = \frac{1}{\mu(x)} \begin{pmatrix}
    a(u) & 0 & 0 & d(u) \\
    0 & b(u) & c(u) & 0 \\
    0 & c(u) & b(u) & 0 \\
    d(u) & 0 & 0 & a(u)
\end{pmatrix}
\]

(2.1)

where the functions \( a(u), b(u), c(u), d(u) \) are given by

\[
a(u) = \frac{\text{snh}(\lambda - u)}{\text{snh}(\lambda)}, \quad b(u) = \frac{\text{snh}(u)}{\text{snh}(\lambda)}, \quad c(u) = 1, \quad d(u) = k \text{snh}(\lambda - u) \text{snh}(u).
\]

(2.2)

The function \( \text{snh}(u) \) is defined by \( \text{snh}(u) = -i \text{sn}(iu) \) where \( \text{sn}(u) \) is Jacobi’s elliptic function with modulus \( k \). If the elliptic integrals are denoted by \( K, K' \) (let \( k^2 = 1 - k^2 \)),

\[
K = \int_0^1 \frac{dx}{\sqrt{(1 - x^2)(1 - k^2 x^2)}} \quad \text{and} \quad K' = \int_0^1 \frac{dx}{\sqrt{(1 - x^2)(1 - k^2 x^2)}},
\]

(2.3)

the functions \( a(u), b(u), c(u), d(u) \) can be seen as functions of the variables

\[
p = \exp\left(-\frac{\pi K'}{K}\right), \quad q = -\exp\left(-\frac{\pi \lambda}{2K}\right), \quad x = \exp\left(\frac{\pi u}{2K}\right).
\]

(2.4)

The normalization factor \( \mu(x) \) is chosen as follows [18]:

\[
\frac{1}{\mu(x)} = \frac{1}{\kappa(x^2)} \frac{(p^2; p^2)_\infty}{(p; p)_\infty^2} \Theta_{p^2}(px^2) \Theta_{p^2}(q^2),
\]

(2.5)

\[
\frac{1}{\kappa(x^2)} = \frac{(q^4 x^{-2}; p, q^4)_\infty (q^2 x^2; p, q^4)_\infty (px^{-2}; p, q^4)_\infty (pq x^2; p, q^4)_\infty}{(q^4 x^2; p, q^4)_\infty (q^2 x^{-2}; p, q^4)_\infty (px^2; p, q^4)_\infty (pq x^{-2}; p, q^4)_\infty},
\]

(2.6)

where one defines the infinite multiple products as usual by

\[
(x; p_1, \ldots, p_m)_\infty = \prod_{n_i \geq 0} (1 - x p_1^{n_1} \ldots p_m^{n_m})
\]

(2.7)

and \( \Theta \) is the Jacobi \( \Theta \) function:

\[
\Theta_{p^2}(x) = (x; p^2)_\infty (p^2 x^{-1}; p^2)_\infty (p^2; p^2)_\infty.
\]

(2.8)

[17] [18]. Defined in a way similar to [12], they use a structure \( R \)-matrix with elliptic dependence associated to the eight-vertex model [19].
Proposition 1 The matrix $R_{12}$, element of $\text{End}(\mathbb{C}^2) \otimes \text{End}(\mathbb{C}^2)$, has the following properties:
- unitarity: $R_{21}(x^{-1})R_{12}(x) = 1$,
- crossing symmetry: $R_{21}(x^{-1})t_i = (\sigma^1 \otimes \mathbb{I})R_{12}(-q^{-1}x)(\sigma^1 \otimes \mathbb{I})$,
- antisymmetry: $R_{12}(-x) = -(\sigma^3 \otimes \mathbb{I})R_{12}(x)(\sigma^3 \otimes \mathbb{I})$,
where $\sigma^1, \sigma^2, \sigma^3$ are the $2 \times 2$ Pauli matrices and $t_i$ denotes the transposition in the space $i$.

The proof is straightforward by direct calculation.

For the definition of the elliptic quantum algebra $\mathcal{A}_{q,p}(\hat{sl}(2)_c)$, we need to use a modified $R$-matrix $R_{12}^+(x)$ defined by
\[ R_{12}^+(x) = \tau(q^{1/2}x^{-1})R_{12}(x), \] (2.9)
where the prefactor $\tau$ is given by
\[ \tau(x) = x^{-1} \frac{(qx^2; q^4)_\infty(q^3x^{-2}; q^4)_\infty}{(qx^2; q^4)_\infty(q^3x^2; q^4)_\infty}. \] (2.10)
Let us stress that the function $\tau$ is periodic: $\tau(x) = \tau(xq^2)$ and $R_{12}^+(x)$ obeys a quasi-periodicity property:
\[ R_{12}^+(p^2x) = (\sigma^1 \otimes \mathbb{I}) \left( R_{21}^+(x^{-1}) \right)^{-1} (\sigma^1 \otimes \mathbb{I}). \] (2.11)
The crossing symmetry and the unitarity property of $R_{12}$ (see Proposition [1]) then allow exchange of inversion and transposition for the matrix $R_{12}^+$ as:
\[ \left( R_{12}^+(x)^{t_2} \right)^{-1} = \left( R_{12}^+(q^2x)^{-1} \right)^{t_2}. \] (2.12)

2.2. The elliptic quantum algebra $\mathcal{A}_{q,p}(\hat{sl}(2)_c)$ has been introduced by [16, 18]. The elliptic quantum algebra $\mathcal{A}_{q,p}(\hat{sl}(2)_c)$ is an algebra of operators $L_{\varepsilon \varepsilon', n}$ such that $(\varepsilon, \varepsilon' = +$ or $-)$

\begin{itemize}
  \item [i)] $L_{\varepsilon \varepsilon', n} = 0$ if $\varepsilon \varepsilon' \neq (-1)^n$,
  \item [ii)] one defines $L_{\varepsilon \varepsilon'}(z) = \sum_{n \in \mathbb{Z}} L_{\varepsilon \varepsilon', n} z^n$,
\end{itemize}
(2.13a-b)
and encapsulate them into a $2 \times 2$ matrix
\[ L = \begin{pmatrix} L_{++} & L_{+-} \\ L_{-+} & L_{--} \end{pmatrix}. \] (2.14)
Note that by virtue of (2.13a-b), $L_{++}(z)$ and $L_{--}(z)$ are even while $L_{+-}(z)$ and $L_{-+}(z)$ are odd functions of $z$.

One then defines $\mathcal{A}_{q,p}(\hat{gl}(2)_c)$ by imposing the following constraints on $L_{\varepsilon \varepsilon'}(z)$:
\[ R_{12}^+(z/w) L_1(z) L_2(w) = L_2(w) L_1(z) R_{12}^+(z/w), \] (2.15)
where $L_1(z) \equiv L(z) \otimes \mathbb{I}$, $L_2(z) \equiv \mathbb{I} \otimes L(z)$ and $R_{12}^{++}$ is defined by $R_{12}^{++}(x, q, p) \equiv R_{12}^+(x, q, pq^{-2c})$. This shift plays an essential role in establishing the existence of a center at $c = -2$.

Since the $q$-determinant $q$-det $L(z) \equiv L_{++}(q^{-1}z)L_{--}(z) - L_{+-}(q^{-1}z)L_{-+}(z)$ is in the center of
\(A_{q,p}(gl(2)_c)\), it can be “factored out”, being set to the value \(q^{c/2}\) so as to get \(A_{q,p}(sl(2)_c) = A_{q,p}(gl(2)_c)/\langle q-\det L - q^{c/2}\rangle\).

It is useful, both for our next computations and the establishing of the \(p \to 0\) trigonometric limit, to introduce the following two matrices:

\[
L^+(z) \equiv L(q^{c/2}z), \quad L^-(z) \equiv \sigma^1 L(-p^{1/2}z)\sigma^1.
\] (2.16)

They obey coupled constraint relations following from (2.14) and periodicity/unitarity properties of the matrix \(R_{12}^+\):

\[
R_{12}^+(z/w) L_1^+(z) L_2^+(w) = L_2^+(w) L_1^+(z) R_{12}^{+\ast}(z/w),
\]

\[
R_{12}^+(q^{c/2}z/w) L_1^+(z) L_2^+(w) = L_2^+(w) L_1^+(z) R_{12}^{+\ast}(q^{-c/2}z/w).
\] (2.17)

3 The center of the elliptic quantum algebra \(A_{q,p}(sl(2)_c)\)

Theorem 1 At \(c = -2\), the operators generated by

\[
t(z) = Tr(L(z)) = Tr\left(L^+(q^{c/2}z)L^-(z)^{-1}\right)
\] (3.1)

commute with the algebra \(A_{q,p}(sl(2)_c)\).

The formula for \(t(z)\) in the elliptic case is exactly identical to the one in the trigonometric case \([12]\). The proof follows on similar lines and is interesting to work out in more detail since it involves all properties of the \(R\)-matrix and uses tricks to be similarly used later.

Proof:

• step 1: One derives further useful exchange relations from eq. (2.17).

Defining \(\bar{L}^\pm(z) \equiv (L^\pm(z)^{-1})^t\) (this definition is unambiguous at least for such 2 \(\times\) 2 matrices of operators, as can be seen using Borel decomposition), one has:

\[
(R_{12}^+(z/w)^t)^{-1} L_1^+(z) \bar{L}_2^+(w) = \bar{L}_2^+(w) L_1^+(z) (R_{12}^{+\ast}(z/w)^t)^{-1}.
\] (3.2)

This follows from starting with \(R_{12}^+L_1^+L_2^+ = L_2^+L_1^+R_{12}^{+\ast}\), rewriting it as \(L_1^+R_{12}^{+\ast}(L_2^+)^{-1} = (L_2^+)^{-1}R_{12}^{+\ast}L_1^+\), transposing with respect to indices 2 (which is allowed since \(R_{12}\) is a \(c\)-number matrix) to obtain \(L_1^+((L_2^+)^{-1})^t R_{12}^{+\ast}((L_2^+)^{-1})^t L_1^+\) and finally getting (3.2) by multiplying l.h.s. and r.h.s. by suitable inverses of \((R_{12}^+)^t\).

Similarly, one gets

\[
(R_{12}^+(q^{c/2}z/w)^t)^{-1} L_1^+(z) \bar{L}_2^+(w) = \bar{L}_2^+(w) L_1^+(z) (R_{12}^{+\ast}(q^{-c/2}z/w)^t)^{-1},
\] (3.3)
by applying a similar scheme of rewritings to eq. (2.17):

- step 2: To compute \([t(z), L^+(w)]\), one rewrites at \(c = -2\)

\[
t(z) L^+_2(w) = \text{Tr}_1 \left( L^+_1(zq^{-1}) \tilde{L}_1(z)^{t_1} \right) L^+_2(w) = \text{Tr}_1 \left( L^+_1(zq^{-1})^{t_1} \tilde{L}_1(z) L^+_2(w) \right), \tag{3.4}
\]

since one is allowed to exchange transposition under a trace procedure. One then commutes \(L^+_2(w)\) through \(\tilde{L}_1(z)\) using eq. (3.3) as:

\[
t(z) L^+_2(w) = \text{Tr}_1 \left( L^+_1(zq^{-1})^{t_1} (R^+_21(w/qz)^{t_1})^{-1} L^+_2(w) \tilde{L}_1(z) R^+_{21}(qw/z)^{t_1} \right). \tag{3.5}
\]

- step 3: One uses the exchange algebra (2.17) to get a suitable expression for the first three terms in (3.7):

  - transposition with respect to indices 1 gives: \(L^+_1(z)^{t_1} R^+_21(z/w)^{t_1} L^+_2(w) = L^+_2(w) R^+_21(z/w)^{t_1} L^+_1(z)^{t_1}\);
  - unitarity property and redefinition of \(z\) as \(zq^{-1}\) give:

\[
L^+_1(zq^{-1})^{t_1} (R^+_21(qw/z)^{-1})^{t_1} L^+_2(w) = L^+_2(w) (R^+_{21}(qw/z)^{-1})^{t_1} L^+_1(zq^{-1})^{t_1} ; \tag{3.6}
\]

Eq. (2.12) inserted in (3.6) gives (after due exchange of spaces 1 and 2):

\[
L^+_1(zq^{-1})^{t_1} (R^+_21(w/qz)^{t_1})^{-1} L^+_2(w) = L^+_2(w) (R^+_21(w/qz)^{t_1})^{-1} L^+_1(zq^{-1})^{t_1} . \tag{3.7}
\]

- step 4: One now inserts (3.7) into (3.5) to get

\[
t(z) L^+_2(w) = L^+_2(w) \text{Tr}_1 \left( (R^+_21(w/qz)^{t_1})^{-1} L^+_1(zq^{-1})^{t_1} \tilde{L}_1(z) R^+_{21}(qw/z)^{t_1} \right) = L^+_2(w) \text{Tr}_1 \left( (R^+_21(qw/z)^{-1})^{t_1} L^+_1(zq^{-1})^{t_1} \tilde{L}_1(z) R^+_{21}(qw/z)^{t_1} \right) . \tag{3.8}
\]

One now needs to use the fact that under a trace over the space 1 one has \(\text{Tr}_1 \left( R^+_21 Q_1 R'_{21} \right) = \text{Tr}_1 \left( Q_1 R^+_21 R'_{21} \right)\):

\[
t(z) L^+_2(w) = L^+_2(w) \text{Tr}_1 \left( L^+_1(zq^{-1})^{t_1} \tilde{L}_1(z) R^+_21(w/qz)^{t_1} R^+_21(w/qz)^{-1} \tilde{L}_1(z) R^+_21(w/qz)^{t_1} \right) . \tag{3.9}
\]

The last two terms in the right hand side cancel each other, leaving a trivial dependence in space 2 and \(\text{Tr}_1 \left( L^+_1(zq^{-1})^{t_1} \tilde{L}_1(z) \right) \equiv t(z) \) in space 1. This shows the commutation of \(t(z)\) with \(L^+(w)\) and therefore with \(L^-(w) = \sigma^1 L^+(\tilde{w}^q q^{-\tilde{w}} w) \sigma^1\), hence with the full algebra \(A_{q,p}(\tilde{sl}(2)_c)\) at \(c = -2\).

It is very important to notice here that owing to the parity properties of the modes of \(L(z)\) defined above and the identical properties of the modes of \((L(z))^{-1}\) or \(\tilde{L}(z)\), the operator \(t(z)\) is even in \(z\) and has therefore only even modes.

We now study the specific behaviour of the exchange algebra of \(t(z)\) with \(t(w)\) in the neighborhood of \(c = -2\). Notice that we have not proved that \(t(z)\) exhausted the center of \(A_{q,p}(\tilde{sl}(2)_c)\) at \(c = -2\). Hence the exchange algebra may not close on \(t(z), t(w)\) in the neighborhood of \(c = -2\). This will however turn out to be true.
4 Poisson algebra of \( t(z) \)

By virtue of Theorem 1 and by the definition (3.1) of \( t(z) \) itself, the elements \( t(z) \) and \( t(w) \) are mutually commuting at the critical level \( \mathcal{Z} \). This implies a natural Poisson structure on the center \( \mathcal{Z} \): if \( [t(z), t(w)] = (c + 2)\ell(z, w) + o(c + 2) \), then a Poisson bracket on \( \mathcal{Z} \) is yielded by 

\[
\{t(z), t(w)\} = \ell(z, w)_{cr} \text{ (the subscript “cr” meaning that all expressions are taken at } c = -2) .
\]

We now state the main result of the paper:

**Theorem 2** The elements \( t(z) \) form a closed algebra under the natural Poisson bracket on \( \mathcal{Z} \). More precisely, we have (we suppress the subscript “cr” for simplicity)

\[
\{t(z), t(w)\} = -(\ln q) \left( \frac{d}{d(w/z)} \ln \tau(q^{1/2}w/z) - (z/w) \frac{d}{d(z/w)} \ln \tau(q^{1/2}z/w) \right) t(z)t(w) .
\]

**Proof:**

- step 1: one computes the exchange algebra between the operators \( t(z) \) and \( t(w) \). From the definition of the element \( t(z) \), one has

\[
t(z)t(w) = L(z)_{i_1}^1 L(w)_{i_2}^2 = L^+(z)_{i_1}^1 \tilde{L}^-(z)_{i_1}^1 L^+(w)_{i_2}^2 \tilde{L}^-(w)_{i_2}^2 .
\]

The same kind of rewritings of eq. (2.17) done in the first step of the proof of Theorem 1 leads to the following exchange relation between the operators \( \tilde{L}^- \):

\[
R_{12}^+(z/w)^{t_1 t_2} \tilde{L}^-_1 (z) \tilde{L}^-_2 (w) = \tilde{L}^-_2 (w) \tilde{L}^-_1 (z) R_{12}^+(z/w)^{t_1 t_2} .
\]

The exchange relations (2.17), (3.3) and (4.3) and the properties of the matrix \( R_{12} \) given in proposition 1 then allow us to move the matrices \( L^+(w), \tilde{L}^-(w) \) to the left of the matrices \( L^+(z), \tilde{L}^-(z) \). One obtains

\[
t(z)t(w) = \mathcal{Y}(z/w)_{j_1 j_2}^i L(w)_{i_2}^2 L(z)_{i_1}^j ,
\]

where the matrix \( \mathcal{Y}(z/w) \) is given in terms of the matrix \( R_{12}^+ \) by

\[
\mathcal{Y}(z/w) = \left( (R_{12}^+(z/w) R_{12}^+(q^{c+2}w/z)^{-1} R_{12}^+(z/w)^{-1})^{t_2} R_{12}^+(q^{c}z/w)^{t_2} \right)^{t_1} .
\]

Making explicit the dependence of the matrix \( \mathcal{Y}(z/w) \) on the function \( \tau \), one gets

\[
\mathcal{Y}(z/w) = T(z/w) \mathcal{R}(z/w) ,
\]

where the matrix factor \( \mathcal{R}(z/w) \) depends only on the matrix (2.1):

\[
\mathcal{R}(z/w) = \left( (R_{12}(z/w) R_{12}(q^{-c-2}w) R_{12}(w/z))^{t_2} R_{12}(z/q^{c}w)^{t_2} \right)^{t_1} ,
\]

and the numerical prefactor \( T(z/w) \) contains all the \( \tau \) dependence:

\[
T(z/w) = \frac{\tau(z/q^{1/2}w)\tau(w/q^{c+1/2}z)}{\tau(z/q^{-c-3/2}w)\tau(w/q^{1/2}z)} .
\]
One easily checks the nice behaviour of \( T(z/w) \) and \( \mathcal{R}(z/w) \) at \( c = -2 \):

\[
\begin{align*}
T(z/w)_{cr} &= 1 \\
\mathcal{R}(z/w)_{cr} &= \mathbb{I}_2 \otimes \mathbb{I}_2 \\
\implies \mathcal{Y}(z/w)_{cr} &= \mathbb{I}_2 \otimes \mathbb{I}_2 , \quad (4.9)
\end{align*}
\]

making obvious the interest of the factorization \((4.6)\), so that one recovers \( t(z)t(w) = t(w)t(z) \) at the critical level.

\begin{itemize}
\item step 2: One computes the Poisson structure from the exchange algebra \((4.4)\) in the neighborhood of \( c = -2 \). As stressed at the beginning of the section, there is a natural Poisson bracket on \( \mathcal{Z} \). From eqs. \((4.4)\) and \((4.9)\), one writes
\[
t(z) = t(w) = \frac{\mathcal{Y}(z)}{\mathcal{Y}(w)} \frac{d\mathcal{Y}(z)}{dc} \left. \right|_{cr} \frac{T(z/w)}{T(w/z)} \frac{d\mathcal{R}(z)}{dc} \left. \right|_{cr} \mathcal{R}(w/z) \frac{d\mathcal{R}(w)}{dc} \left. \right|_{cr} = \mathbb{I}_2 \otimes \mathbb{I}_2 , \quad (4.10)
\]
\end{itemize}

and therefore
\[
\{t(z), t(w)\} = \left( \frac{d\mathcal{Y}(z)}{dc} \left. \right|_{cr} \right) \left( \frac{d\mathcal{R}(z)}{dc} \left. \right|_{cr} \right) . \quad (4.11)
\]

The main difficulty is of course the calculation of the derivative in \((4.11)\). The equations \((4.6)\) and \((4.9)\) imply
\[
\frac{d\mathcal{Y}(x)}{dc} \left. \right|_{cr} = \frac{d\mathcal{T}(x)}{dc} \left. \right|_{cr} \mathbb{I}_2 \otimes \mathbb{I}_2 + \frac{d\mathcal{R}(x)}{dc} \left. \right|_{cr} . \quad (4.12)
\]

Let us write \( \mathcal{R}(x) \) as
\[
\mathcal{R}(x) = \frac{1}{\mu(xq^2)\mu(xq^{-2})\mu(x^{-1})^2} \mathcal{M}(x) , \quad (4.13)
\]

so that the dependence on the function \( \mu \) is now explicit (note that \( \mathcal{M}(x)_{cr} = \mu(xq^{-2})\mu(x)\mu(x^{-1})^2 \) by virtue of \((4.9)\)). The matrix \( \mathcal{M}(x) \) is given by
\[
\mathcal{M}(x) = \begin{pmatrix}
m_{11} & 0 & 0 & m_{22} \\
0 & m_{12} & m_{21} & 0 \\
0 & m_{21} & m_{12} & 0 \\
m_{22} & 0 & 0 & m_{11}
\end{pmatrix} , \quad (4.14)
\]

the entries of which depending only on the elliptic functions \( a(u), b(u), c(u), d(u) \) by
\[
\begin{align*}
m_{11} &= a(xq^c)a(xq^{-c-2})a(x^{-1})^2 + 2a(xq^c)a(x^{-1})d(xq^{-c-2})d(x^{-1}) + a(xq^c)a(xq^{-c-2})d(x^{-1})^2 \\
&\quad -2b(xq^{-c-2})b(x) + b(x)^2 + 1 \quad (4.15a)
m_{22} &= a(xq^{-c-2})a(x^{-1})^2 + 2a(x^{-1})d(xq^{-c-2})d(x^{-1}) + a(xq^{-c-2})d(x^{-1})^2 \\
&\quad -2a(xq^c)b(xq^{-c-2})b(x) + a(xq^c)b(x)^2 + a(xq^c) \quad (4.15b)
m_{12} &= a(x^{-1})^2d(xq^c)d(xq^{-c-2}) + 2a(xq^{-c-2})a(x^{-1})d(xq^c)d(x^{-1}) + b(xq^c)b(xq^{-c-2})b(x)^2 \\
&\quad -2b(xq^c)b(x) + b(xq^c)b(xq^{-c-2}) + d(xq^c)d(xq^{-c-2})d(x^{-1})^2 \quad (4.15c)
m_{21} &= a(x^{-1})^2b(xq^c)d(xq^{-c-2}) + 2a(xq^{-c-2})a(x^{-1})b(xq^c)d(x^{-1}) + b(xq^c)b(xq^{-c-2})b(x)^2 \\
&\quad -2b(x)d(xq^c) + b(xq^{-c-2})d(xq^c) + b(xq^c)d(xq^{-c-2})d(x^{-1})^2 \quad (4.15d)
\end{align*}
\]
Using the formula
\[ \left. \frac{d}{dc} f(xq^{2\alpha(c+2)}) \right|_{c=0} = \pm (\ln q) x \frac{d}{dx} f(xq^{2\alpha}) = \pm (2iK - \lambda) \frac{d}{du} f(u - 2\alpha \lambda), \] (4.16)
one shows after a long calculation, using various tricks in elliptic functions theory, that at \( c = -2 \)the derivatives of the diagonal entries of \( \mathcal{M} \) are equal while the derivatives of the off-diagonal entriesidentically vanish, namely:
\[ \left. \frac{d}{dc} m_{11} \right|_{c=0} = \left. \frac{d}{dc} m_{12} \right|_{c=0} = (2iK - \lambda) \left( (1 - b(u + \lambda)^2) \frac{d}{du} b(u)^2 - (1 - b(u)^2) \frac{d}{du} b(u + \lambda)^2 \right), \] (4.17)
\[ \left. \frac{d}{dc} m_{21} \right|_{c=0} = \left. \frac{d}{dc} m_{22} \right|_{c=0} = 0. \] (4.18)
In particular the following relations are essential in this derivation:
\[ \frac{\text{snh}(a-u)\text{snh}(a-v) - \text{snh}(u)\text{snh}(v)}{1 - k^2 \text{snh}(u)\text{snh}(v)\text{snh}(a-u)\text{snh}(a-v)} = \text{snh}(a-u-v)\text{snh}(a), \]
\[ \frac{\text{snh}(u)\text{snh}(a-u) - \text{snh}(v)\text{snh}(a-v)}{\text{snh}(a-u) - \text{snh}(a-v)} = \frac{\text{snh}(a-u-v)}{\text{snh}(a)}, \]
\[ \frac{\text{snh}(u)\text{snh}'(v) - \text{snh}(v)\text{snh}'(u)}{\text{snh}^2(u) - \text{snh}^2(v)} = \frac{1}{\text{snh}(u+v)}. \]
Moreover, one has \( \mu(x)\mu(x^{-1}) = 1 - b(u)^2 \) and \( \frac{\mu(x)}{\mu(xq^{-2})} = \frac{1 - b(u)^2}{1 - b(u + \lambda)^2} \), which imply
\[ \left. \frac{d}{dc} \left( \frac{1}{\mu(xq^c)\mu(xq^{-c-2})\mu(x^{-1})^2} \right) \right|_{c=0} = (2iK - \lambda) \frac{1}{\mu(x)\mu(x-1)^2} \frac{d}{du} \left( \frac{\mu(x)}{\mu(xq^{-2})} \right) \]
\[ = -(2iK - \lambda) \frac{(1 - b(u + \lambda)^2) \frac{d}{du} b(u)^2 - (1 - b(u)^2) \frac{d}{du} b(u + \lambda)^2}{(1 - b(u)^2) (1 - b(u + \lambda)^2)^2}. \] (4.19)
Therefore, at the critical level, one has
\[ \left. \frac{d}{dc} \mathcal{R}(x) \right|_{c=0} = \mathcal{M}(x) \left. \frac{1}{\mu(xq^c)\mu(xq^{-c-2})\mu(x^{-1})^2} \frac{d}{dc} \mathcal{M}(x) \right|_{c=0} = 0. \] (4.20)
Finally, the derivative of the term \( T(x) \) is given by (using the same trick (4.16))
\[ \left. \frac{d}{dc} T(x) \right|_{c=0} = -(\ln q) \left( x^{-1} \frac{d}{dx} \ln \tau(x^{-1} q^{1/2}) - x \frac{d}{dx} \ln \tau(x q^{1/2}) \right). \] (4.21)
Now, from eqs. (4.12) and (4.20), \( \frac{d}{dc} \mathcal{Y}(x) \) at the critical level is proportional to the identity matrix.The formula (4.13) of Theorem 2 then immediately follows taking \( x = z/w \).

The structure of the Poisson bracket therefore derives wholly from the prefactor in the \( \mathcal{Y} \) matrix (4.8), a fact to be kept in mind. As a consequence, any dependence in \( p \) is absent in the Poissonbracket structure function.

Note also that although we have not (and will not here) shown that the generators \( t(z) \) generatethe whole center at \( c = -2 \), we have nonetheless proved that their exchange algebra did generate aclosed Poisson algebra for the corresponding set of classical variables.
5 Explicit Poisson structures

From the equation (4.1) and the periodicity property of the function \( \tau(x) \), one gets easily

\[
\{ t(z), t(w) \} = -(2 \ln q) \sum_{n \geq 0} \left( \frac{2x^2q^{4n+2}}{1-x^2q^{4n+2}} - \frac{2x^{-2}q^{4n+2}}{1-x^{-2}q^{4n+2}} \right) + \sum_{n > 0} \left( -\frac{2x^2q^{4n}}{1-x^2q^{4n}} + \frac{2x^{-2}q^{4n}}{1-x^{-2}q^{4n}} - \frac{x^2}{1-x^2} + \frac{x^{-2}}{1-x^{-2}} \right) t(z) t(w), \tag{5.1}
\]

where \( x = z/w \). Interpretation of the formula (5.1) must now be given in terms of the modes of \( t(z) \), defined in the sense of generating function:

\[
t_n = \oint_{C} \frac{dz}{2\pi iz} z^{-n} t(z). \tag{5.2}
\]

The structure function \( f(z/w) \) which defines the Poisson bracket (5.1) has singularities which require a careful analysis of the procedure to get Poisson structures for \( \{ t_n, n \in \mathbb{Z} \} \). It is periodic with period \( q^2 \) and has simple poles at \( z/w = \pm q^k \) for \( k \in \mathbb{Z} \). In particular it is singular at \( z/w = \pm 1 \).

As a consequence, the expected definition of the Poisson structure \( \{ t_n, t_m \} \) as a double contour integral of (5.1) must be made more precise. Deformation of, say, the \( w \)-contour while the \( z \)-contour is kept fixed may induce the crossing of singularities of \( f(z/w) \) which in turn modifies the computed value of the Poisson bracket. One can equivalently say that the explicit evaluation of

\[
\oint_{2\pi iz} \oint_{2\pi iw} f(z/w) t(z) t(w) \]

requires an expansion of \( f(z/w) \) in series of \( z/w \), the form of which in turn depends explicitly upon the value of \( |z/w| \) being in a particular interval \([q^k, q^{k+1}]\) for some \( k \in \mathbb{Z} \).

Moreover, the singularity at \( z/w = \pm 1 \) implies that one cannot identify \( \oint_{C_1} \frac{dz}{2\pi iz} \oint_{C_2} \frac{dw}{2\pi i w} \) with the permuted double integral \( \oint_{C_2} \frac{dz}{2\pi iz} \oint_{C_1} \frac{dw}{2\pi i w} \) since by deformation of \( C_1 \to C_2 \) and \( C_2 \to C_1 \), the contours necessarily cross at some point. The immediate consequence is that the quantity

\[
\oint_{C_2} \frac{dz}{2\pi iz} \oint_{C_1} \frac{dw}{2\pi i w} z^{-n} w^{-m} f(z/w) t(z) t(w) \]

is not antisymmetric under the exchange \( n \leftrightarrow m \) and cannot be taken as a Poisson bracket.

This leads us to define the Poisson bracket as:

**Definition 1**

\[
\{ t_n, t_m \} = \frac{1}{2} \left( \oint_{C_1} \frac{dz}{2\pi iz} \oint_{C_2} \frac{dw}{2\pi i w} + \oint_{C_2} \frac{dz}{2\pi iz} \oint_{C_1} \frac{dw}{2\pi i w} \right) z^{-n} w^{-m} f(z/w) t(z) t(w). \tag{5.3}
\]

Such a procedure guarantees the antisymmetry of the postulated Poisson structure due to the property \( f(z/w) = -f(w/z) \). Remark that this definition is a semi-classical limit of a well-known procedure in two-dimensional field theory by which one computes commutators of field modes out of radial-ordered contour integrals of their operator product expansions. Here the relevant exchange algebra is given by (4.4)-(4.5).

The problem of contour deformation crossing the singularities at \( z/w = \pm q^k \) where \( k \neq 0 \) and modifying the Poisson bracket is solved as follows. To simplify the notations we now choose the contours \( C_1 \) and \( C_2 \) to be circles of radii \( R_1 \) and \( R_2 \) respectively. We now state:
Proposition 2 For any $k \in \mathbb{Z}^+$ such that $R_1/R_2 \in [q^{\pm k}, q^{\pm (k+1)}]$, Definition 4 defines a consistent Poisson bracket whose specific form depends on $k$.

By “consistent” we mean “antisymmetric and obeying the Jacobi identity”. To prove Proposition 2 we now evaluate the postulated Poisson brackets.

- case $k = 0$: The first four terms in (5.3) can be expanded in convergent power series of $x^\pm s$ without further ado since they have no singularity in this region and $x \in [q, q^{-1}]$ ($|q| < 1$), hence the expansion parameters $x^\pm 2q^{4n+2}$ ($n \geq 0$), $x^\pm 2q^{4n}$ ($n > 0$) are smaller than 1. Their contribution to the Poisson structure is

$$\{t_n, t_m\}_{k=0} = -4 \ln q \oint_{C_1} \frac{dz}{2\pi i z} \oint_{C_2} \frac{dw}{2\pi i w} \sum_{s>0} \left( \frac{q^{4s} - q^{2s}}{1 - q^{4s}} x^{-2s} + \frac{q^{2s} - q^{4s}}{1 - q^{4s}} x^{2s} \right) z^{-n} w^{-m} t(z) t(w),$$

(5.4)

where $C_1, C_2$ can be arbitrarily deformed once the contour integrals are evaluated by residue method. The last two terms have poles at $z/w = \pm 1$ and require two distinct expansions depending whether $R_1 > R_2$ or $R_2 > R_1$. Their contribution can then be reintroduced in the formal series expansion (5.4) to give:

$$\{t_n, t_m\}_{k=0} = -2 \ln q \oint_{C_1} \frac{dz}{2\pi i z} \oint_{C_2} \frac{dw}{2\pi i w} \sum_{s \in \mathbb{Z}} \frac{q^s - q^{-s}}{q^s + q^{-s}} \left( \frac{z}{w} \right)^{2s} z^{-n} w^{-m} t(w) t(z),$$

(5.5)

which, evaluated by residues, gives an unambiguous answer. Remarkably this is the (not centrally extended) Poisson bracket structure found in the trigonometric (albeit with elliptic prefactor) case using a completely different approach [13]. This identification requires the use of the parity of $t(z)$ to redefine modes as $\tilde{t}_n \equiv \oint \frac{dz}{2\pi i z} z^{2n} t(z)$ in which case (5.3) is exactly formula (9.4) in [13]. It is indeed a consistent Poisson structure and its identification establishes a connection between the elliptic and trigonometric $q$-deformed algebras, to be commented at the end.

- case $k \neq 0$: The Poisson bracket (5.3) can be evaluated as the sum of the $k = 0$ bracket (5.3) and the contributions to (5.3) of the poles at $z/w = \pm q^{kn}$ where $n = 1, \ldots, k$, from the deformation of the contour $C_1$ (for instance, keeping $C_2$ fixed). This contribution can be evaluated step by step. The deformation of $C_1$ from sector $k - 1$ to sector $k$ adds to (5.3) the contribution from the poles $z/w = \pm q^{\pm k}$. This contribution is given by a contour integral around each pole, with an overall sign depending on the parity of $k$. It reads as:

$$- (-1)^k \ln q \oint_{C_1} \frac{dz}{2\pi i z} 2t(z) t(zq^k) z^{-n-m} q^{-km} + (-1)^k \ln q \oint_{C_2} \frac{dz}{2\pi i z} 2t(z) t(zq^{-k}) z^{-n-m} q^{km}$$

(5.6)

or equivalently, defining the distribution $\delta(u) = \sum_{n \in \mathbb{Z}} u^n$:

$$- (-1)^k \ln q \oint_{C_1} \frac{dz}{2\pi i z} \oint_{C_2} \frac{dw}{2\pi i w} t(z) t(w) z^{-n} w^{-m} \left( \delta\left(\frac{w}{zq^k}\right) - \delta\left(\frac{wq^k}{z}\right) + \delta\left(\frac{w}{zq^k}\right) - \delta\left(\frac{wq^k}{z}\right) \right),$$

(5.7)

where, as in (5.3), the integrals are evaluated term by term using residue formula and therefore the relative position of the contours $C_1$ and $C_2$ is not relevant. Again, factorization of $\delta(zq^k/w) + \delta(-zq^k/w)$ is made possible by the parity properties of $t(z) \equiv t(-z)$. Interestingly enough, when
one plugs back (5.4) into (5.3) using the formal series definition for the δ distribution, one gets a compact formula very similar to (5.5):

\[
\{t_n, t_m\}_k = (-1)^{k+1} 2 \ln q \oint_{C_1} \frac{dz}{2\pi i z} \oint_{C_2} \frac{dw}{2\pi i w} \sum_{s \in \mathbb{Z}} \frac{q^{(2k+1)s} - q^{-(2k+1)s}}{q^s + q^{-s}} \left(\frac{z}{w}\right)^{2s} z^{-n} w^{-m} t(w) t(z),
\] (5.8)

which we now take as the final generic definition of the Poisson brackets deduced from (5.1). One indeed shows easily that for all \(k\), (5.8) obeys Jacobi identity and is antisymmetric.

6 Conclusion

We have obtained a family, indexed by an integer \(k\), of consistent Poisson brackets defined on the center \(\mathcal{Z}\) at \(c = -2\) of \(A_{q,p}(\widehat{sl}(2)_c)\). The form of these Poisson brackets is similar to the form of Poisson bracket obtained by [13].

We would like to expand a little more on this similarity. It is difficult to take a direct limit \(p \to 0\) of our algebra to get the trigonometric \(q\)-deformed algebra, since this limit is highly singular [18]. This seems to preclude a direct derivation from our algebra to the one obtained in [13].

However we think that fruitful comparison can already be made at this point.

The occurrence of an elliptic Poisson algebra in [13] starting from a trigonometric \(R\)-matrix is probably connected to the introduction of an explicit elliptic prefactor in the initially trigonometric \(R\)-matrix. Our computation has indeed shown that the elliptic Poisson brackets entirely arise from the \(\tau\)-prefactor in our \(R\)-matrix. A similar phenomenon might then occur in [13].

Two apparent discrepancies between the two structures can also be understood. Frenkel and Reshetikhin obtain one single Poisson bracket structure, which seems to correspond more closely to our Poisson bracket at \(k = 1\), but with a purely central \(\delta\)-term instead of the \(t(z)t(zq)\) term in (5.7). Remember however that the derivation in [13] uses an explicit representation of the trigonometric \(q\)-deformed algebra in terms of free quasi-bosons. It is possible that this particular representation, combined with the non trivial procedure leading from full elliptic to trigonometric algebra, leads to a degeneracy of such terms as \(t(z)t(zq)\) and contains implicitly the restriction to the \(k = 1\) bracket. Provided that the limit procedure \(p \to 0\) be better understood, this indicates a scheme through which the results in [13] could be connected to our general setting.

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