Approximate Ginzburg-Landau solution for the regular flux-line lattice.
Circular cell method

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A variational model is proposed to describe the magnetic properties of type-II superconductors in the entire field range between $H_{c1}$ and $H_{c2}$ for any values of the Ginzburg-Landau parameter $\kappa > 1/\sqrt{2}$. The hexagonal unit cell of the triangular flux-line lattice is replaced by a circle of the same area, and the periodic solutions to the Ginzburg-Landau equations within this cell are approximated by rotationally symmetric solutions. The Ginzburg-Landau equations are solved by a trial function for the order parameter. The calculated spatial distributions of the order parameter and the magnetic field are compared with the corresponding distributions obtained by numerical solution of the Ginzburg-Landau equations. The comparison reveals good agreement with an accuracy of a few percent for all $\kappa$ values exceeding $\kappa \approx 1$. The model can be extended to anisotropic superconductors when the vortices are directed along one of the principal axes. The reversible magnetization curve is calculated and an analytical formula for the magnetization is proposed. At low fields, the theory reduces to the London approach at $\kappa > 1$, provided that the exact value of $H_{c1}$ is used. At high fields, our model reproduces the main features of the well-known Abrikosov theory. The magnetic field dependences of the reversible magnetization curve are in a good agreement with experimental data on high-$T_c$ superconductors.

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I. INTRODUCTION

The solution to the Ginzburg-Landau (GL) equations found by Abrikosov \cite{1} was used widely \cite{2,3} to study the properties of type-II superconductors at low and high applied magnetic fields $H$, i.e. close to the lower and upper critical fields $H_{c1}$ and $H_{c2}$, respectively. At $H \sim H_{c1}$ the intervortex spacing is much larger than the vortex core size if the GL parameter is large, $\kappa \gg 1$. Therefore in the London model, which is commonly used at low fields \cite{4}, the order parameter in the superconductor is assumed to be constant. In this case, the flux-line lattice (FLL) can be treated as a set of independent vortices: the magnetic flux density is a linear superposition of the contributions of individual vortices, and each contribution coincides with the field of an isolated vortex. The energy of the system is the sum of the self-energies of the vortices and their pairwise interaction \cite{5}. At high fields, the London model looses its applicability because the fraction of the total volume of superconductor occupied by the vortex cores is no longer small \cite{5}. Different approximations were proposed to include the vortex cores and to extend the applicability limits of the London model \cite{6}.

The problem of solving the GL equations for the ideally periodic FLL can be simplified considerably by replacing the hexagonal unit cell of the vortex lattice by a circle of equal area (Wigner-Weisskopf approximation). In this approach both the order parameter and the magnetic flux density within the cell have axial symmetry. The presence of other vortices is taken into account by the boundary condition: the supercurrent density equals zero at the cell boundary. This method was used in Ref. \cite{6}, where an explicit expression for the magnetization in low fields was found, and the results are in good agreement with the predictions of the London model \cite{4,5}. In Refs. \cite{7,8}, the GL equations and the equations of the microscopic theory of superconductivity were solved numerically in the framework of circular cell approaches and it was shown that this approximation not only yields good results at low induction but also at $H \sim H_{c2}$. A similar approach was used in Ref. \cite{9}, where the system of GL equations was reduced to a single equation which can be solved numerically. The Wigner-Weisskopf approximation can be extended even to the case of exotic pairing symmetries, when the order parameter has two complex components \cite{10}.

A numerical method to find the periodic solutions to the GL equations was developed in Refs. \cite{11,12}. This exact method accounts for the actual symmetry of the vortex lattice. It allows calculating the spatial distributions of the magnetic field and the order parameter within the unit cell, the elastic shear modulus of the FLL, and the magnetization for any FLL symmetry and any induction $B$ and GL parameter $\kappa$ with any desired accuracy. However, until recently there was no adequate
approach allowing one to find the magnetization analytically in the entire field range \( H_{c1} < H < H_{c2} \) and to obtain explicit formulas which may be used to analyze experimental data.

In Ref. [17] Clem proposed a model to solve the GL equations using a trial function for the order parameter (or GL function) \(|\psi|\):

\[
|\psi| = f(r) = \frac{r}{\sqrt{r^2 + \xi^2}},
\]

where \( \xi_v \) is a variational parameter. This model yields an approximate explicit expression for the local magnetic field of an isolated vortex:

\[
h(r) = \frac{1}{\kappa \xi_v K_1(\xi_v)} K_0 \left( \sqrt{r^2 + \xi^2} \right),
\]

where \( K_0 \) are modified Bessel functions. Here and below the following dimensionless variables are used: distance \( r \), magnetic flux density \( h \), and order parameter \( f \) are measured in units of \( \lambda \), \( H/\sqrt{2} \), \( \sqrt{-\alpha/\beta} \), respectively, where \( \lambda \) is the London penetration depth, \( H_c \) is the thermodynamic critical field, and \( \alpha \) and \( \beta \) are the GL coefficients. In this notation, we have \( H_{c2} = \kappa, \Phi_0 = 2\pi/\kappa \), and \( \Phi_0 \) is the magnetic flux quantum. For \( \kappa \gg 1 \) the minimization of the free energy gives \( \xi_v \approx \sqrt{2}/\kappa \) and

\[
H_{c1} = \frac{1}{2\kappa} (\ln \kappa + \varepsilon),
\]

where \( \varepsilon = 0.52 \). The exact value \( \varepsilon = 0.50 \) was calculated in Ref. [18] (see also Ref. [19]) from the numerical solution to the GL equations for an isolated vortex. Note that the lower critical field cannot be found self-consistently in the framework of the London model, because in this approach the magnetic flux density diverges on the vortex axis. Therefore, \( H_{c1} \) should be regarded as a free parameter in the London expression for the magnetization [8]. However, the approximation for \( H_{c1} \) found by cutting off the field of a vortex at a distance equal to the coherence length \( \xi \), is often used in the London approach as well [21]. Note that recently Clem’s trial function (1) was applied to the study of the vortex core structure in superconductors with mixed (\( d+s \)) two-component order parameter [22].

Hao et al. [23] (see also Ref. [24]) extended the model [17] to larger magnetic fields up to \( H_{c2} \) through the linear superposition of the field profiles of individual vortices. In this model, the trial function (1) is multiplied by a second variational parameter \( f_\infty \) to account for the suppression of the order parameter due to the overlapping vortex cores. This model enabled the authors [24] to calculate the magnetization of type-II superconductors in the full range \( H_{c1} < H < H_{c2} \). Their analytical formula is in a good agreement with the well-known Abrikosov high-field result. For the case of low fields, Hao and Clem argued [23] that the London model is quantitatively incorrect (it does not give the correct asymptotics at \( H \to H_{c1} \)) since the contribution of the vortex cores to the total free energy could not be taken into account in this approach. The Clem-Hao model was further extended to include anisotropy [25–29]. This approximation is now widely used for the analysis of the experimental data on magnetization of type II superconductors [27–29].

However, it has been recently shown in Ref. [30] that the Clem-Hao model has some drawbacks. It was argued that the procedure of obtaining the local magnetic flux density by a linear superposition of contributions of individual vortices in the form used in Ref. [23] is valid only at low fields. The application of this approach to the entire field range \( H_{c1} < H < H_{c2} \) leads to an appreciable disagreement between the Clem-Hao model and Abrikosov’s high-field result [30]. In the original papers of Hao et al. [23,25,26], this disagreement was made up by the use of a non-selfconsistent field dependence of the variational parameters. Then, in calculating the magnetic free energy in Ref. [23] the lattice sum was approximated by an integral. As it was shown in Ref. [8], this procedure leads to a noticeable error in the magnetization at low fields. This error and the use of an inaccurate value of \( H_{c1} \) have led the authors [23,25] to the conclusion about the quantitative incorrectness of the London approach at low fields.

Note that in Ref. [21] it was argued too that the Clem-Hao model overestimates the effect of suppression of the order parameter in the vortex cores at small fields.

In this paper, we propose a variational model for the description of the regular flux-line lattice in a more consistent fashion as compared to Ref. [23]. Our variational procedure is based on Clem’s trial function (1). However, in contrast to the Clem-Hao model, we do not use the superposition of vortex fields. Instead, we apply the circular cell method and calculate the magnetic flux density directly from the second GL equation. The model enables us to find analytical expressions for the local magnetic field and the order parameter. The results of our variational procedure are compared with the results of the numerical solution of the GL equations. This comparison reveals that the analytical formulas for the spatial distribution of the order parameter and the magnetic field agree with the numerical results to an accuracy of a few percent in a wide range of \( \kappa \approx 1 \) and \( B \). By introducing the effective-mass tensor the theory is extended to include anisotropy when the vortices are directed along one of the principal axes of the crystal. The results for the local order parameter and the magnetic field are then used to calculate the magnetization. The resulting expression for the magnetization is in agreement with the London model in small fields at \( \kappa \gg 1 \), and with the Abrikosov approximation at \( H \sim H_{c2} \). The field dependences of the magnetization found by the variational and numerical approaches practically coincide. At the same time, the difference between the numerical result and the Clem-Hao approach is considerable. We also found the field dependence of the magnetization in the Wigner-Seitz approximation close to \( H_{c2} \), where the GL equations can be linearized [3]. The calculated magnetization curves are
II. THEORETICAL FORMALISM

In the Wigner-Seitz approximation both the order parameter and the magnetic flux density within the cell have axial symmetry. In this case the order parameter $|\psi|$ can be presented as $f(r) \exp(-i\varphi)$, with radius vector $r$ and phase angle $\varphi$. The free energy density of a superconductor can be written as the sum of two contributions: $F = F_{\text{em}} + F_{\text{core}}$. $F_{\text{em}}$ is related to the energies of magnetic field and supercurrent, and $F_{\text{core}}$ to the suppression of $|\psi|$ in the vortex core. It is easy to show that in the framework of the GL theory in the Wigner-Seitz approach $F_{\text{em}}$ and $F_{\text{core}}$ are given by:

$$F_{\text{em}} = \frac{2\pi}{\kappa A_{\text{cell}}} \int_0^R \left[ f^2 \left( a - \frac{1}{\kappa r} \right)^2 + \frac{1}{4} \right] r dr,$$

(4)

$$F_{\text{core}} = \frac{2\pi}{\kappa A_{\text{cell}}} \int_0^R \frac{1}{4} \frac{1}{(1-f)^2} + \frac{1}{\kappa^2} \left( \frac{df}{dr} \right)^2 r dr,$$

(5)

where $a$ is the dimensionless vector potential, $R$ and $A_{\text{cell}} = \pi R^2$ are the cell radius and area, related to the magnetic induction $B$ by $A_{\text{cell}} = 2\pi r B/\kappa$.

The two GL equations can be written as:

$$-\frac{1}{\kappa^2 r} \frac{d}{dr} \left( r \frac{df}{dr} \right) + f^3 - f + f \left( a - \frac{1}{\kappa r} \right)^2 = 0,$$

(6)

$$\frac{d h}{d r} = f^2 \left( a - \frac{1}{\kappa r} \right).$$

(7)

The magnetic field and the vector potential are related by

$$h = \frac{1}{r} \frac{d}{d r} \left( r f a \right).$$

(8)

These equations must be supplemented by the boundary conditions for the magnetic field and the order parameter:

$$h(R) = h_c,$$

(9)

$$f(0) = 0, \quad f'(R) = 0,$$

(10)

$$r f^{-2}(r) \frac{d h}{d r} = -\frac{1}{\kappa} \quad \text{at} \quad r \to 0.$$  

(11)

Condition (11) follows from Eqs. (7) and (8). The system of Eqs. (6)-(11) is much simpler than the similar equations for the hexagonal unit cell. However, even this system can be solved only numerically. Nevertheless, in high and small fields some results can be obtained analytically. At small fields, the approximate solution to the GL equations in the Wigner-Seitz equations found in Ref. Here, the spatial variation of the order parameter at $\kappa \gg 1$ can be neglected when calculating the magnetic flux density. In this case, $h$ can be found analytically from the second GL equation (6) and the boundary conditions (9) and (11). This yields the magnetization:

$$-4\pi M(B) = H_{c1} + \frac{1}{2\kappa} \left[ K_1(R) + \frac{1}{2} \frac{1}{2I_1^2(R)} \right] - B.$$  

(12)

When $H \gg H_{c1}$, the radius of the cell is $R \ll 1$, and Eq. (12) can be expanded in powers of $R$, yielding:

$$-4\pi M(B) = H_{c1} - \frac{1}{4\kappa} \left[ \ln(2\kappa(H-H_{c1})+\sigma) \right],$$

(13)

with $\sigma = 1.3456$. A similar relationship was obtained in the London limit in Ref. for the regular FLL, with $\sigma = 1.3431$ for the triangular lattice.

At high fields, the magnetization is given by:

$$M = \frac{H-H_{c2}}{4\pi \beta_A (2\kappa^2 - 1)},$$

(14)

where $\beta_A$ depends only on the symmetry of the FLL:

$$\beta_A = \frac{\int f^4 d^2 r}{\left[ \int f^2 d^2 r \right]^2}.$$

(15)

Here the integrals are taken over the area of the unit cell. Let us find the value of $\beta_A$ for the circular cell. Near $H_{c2}$, the magnetic field undergoes only slight spatial variation. The vector potential in this case is $a \approx Br/2$. The order parameter is small at $H \sim H_{c2}$, so the first GL equation (6) can be linearized. The resulting equation has the analytical solution:

$$f(r) = s \exp \left( -\frac{kr B}{2} \right) \Phi \left( \frac{B - \kappa}{2}, \frac{kr B}{2} \right),$$

(16)

where $\Phi$ is the Kummer function. Factor $s$ depends on the nonlinear term in Eq. (6) but it does not affect the value of $\beta_A$. Using Eqs. (15) and (16), we find $\beta_A = 1.1576$ for the regular FLL, this value is close to $\beta_A = 1.1596$ calculated in Ref. for the triangular lattice.

Thus, in both limits of low and high fields, the magnetization in the Wigner-Seitz approximation is in good agreement with that for the regular triangular FLL, which has the lowest energy. In the next section, we propose a variational model to solve the GL equations in the whole field range between $H_{c1}$ and $H_{c2}$ at any $\kappa > 1/\sqrt{2}$ with good accuracy.
III. Variational Procedure

Using the trial function (1) multiplied by a variational prefactor, \( f(r) = f_\infty r / \sqrt{r^2 + \xi^2} \), allows us to solve the second GL equation (7) analytically within the Wigner-Seitz cell:

\[
h(r) = u_0 f_\infty \sqrt{r^2 + \xi^2} + v K_0 f_\infty \sqrt{r^2 + \xi^2},
\]

where \( u \) and \( v \) can be found from the boundary conditions (9) and (11):

\[
u = \frac{f_\infty}{\kappa \xi_v} \frac{K_1(f_\infty \rho)}{K_1(f_\infty \xi_v) I_1(f_\infty \rho) - I_1(f_\infty \xi_v) K_1(f_\infty \rho)},
\]

and we introduced the notation \( \rho = \sqrt{R^2 + \xi^2} \). Note that, rigorously speaking, the Clem trial function (1) does not meet the condition (10) since its derivative can not be equal to zero at the cell boundary. However, \( df/dr \) is small at \( r = R \) and the comparison between the results of variational and numerical methods demonstrates good accuracy of the approach.

The values of variational parameters should be found by minimization of the total free energy density \( F = F_{\text{em}} + F_{\text{core}} \). Using Eqs. (7) and (4), it is possible to obtain the following expression for the magnetic energy density: \( F_{\text{em}} = B h(0) \). Taking into account Eqs. (17)-(19), we get:

\[
F_{\text{em}} = \frac{B f_\infty}{\kappa \xi_v} \times
\]

\[
\times \frac{K_0(f_\infty \xi_v) I_1(f_\infty \rho) - I_0(f_\infty \xi_v) K_1(f_\infty \rho)}{K_1(f_\infty \xi_v) I_1(f_\infty \rho) - I_1(f_\infty \xi_v) K_1(f_\infty \rho)}
\]

The energy related to the spatial variation of the order parameter in the vortex core is found from Eq. (5) by a straightforward integration:

\[
F_{\text{core}} = \frac{1}{2} (1 - f_\infty^2)^2 + \frac{B_0(\kappa \xi_v)^2 f_\infty^2 (1 - f_\infty^2)}{2 + B_0(\kappa \xi_v)^2} + \frac{B f_\infty^2 (1 + B_0(\kappa \xi_v)^2)}{\kappa (2 + B_0(\kappa \xi_v)^2)^2}.
\]

The field dependence of the variational parameters is calculated numerically by minimization of the total free energy \( F(B, \kappa, \xi_v, f_\infty) \) with respect to \( \xi_v \) and \( f_\infty \). This dependence can be approximated by the following analytical expressions with an accuracy of about 0.5%:

\[
\xi_v(B, \kappa) = \xi_v0 \times
\]

\[
\times \left( 1 - 4.3 \left( 1 - \frac{B}{1.05 \kappa} \right)^6.3 \left( \frac{B}{\kappa} \right) \right)^{1/2}
\]

\[
\times \left( 1 - 0.56 \left( \frac{B}{\kappa} \right)^{9.9} \right)^{1/2}.
\]

\[
f_\infty(B, \kappa) = \left( 1 - \frac{B^2}{2.8 \kappa^2} \right) \times \left( 1 - \left( \frac{B}{B_0} \right)^4 \right)^{1/2}
\]

\[
\times \left( 1 + 1.7B \kappa \left( 1 - \frac{1.4B}{\kappa} \right)^2 \right)^{1/2},
\]

where \( t = 0.985 \), \( \xi_v0 \) is the value of \( \xi_v \) at \( B = 0 \). The latter can be calculated from the condition \( df/d\xi_v = 0 \) at \( B = 0 \):

\[
\kappa \xi_v0 = \sqrt{2} \left[ 1 - \frac{K_0^2(\xi_v0)}{K_1^2(\xi_v0)} \right].
\]

When \( \kappa \gg 1 \), Eq. (24) has the solution \( \xi_v0 = \sqrt{2} / \kappa (\sqrt{2} \xi_v \) in dimensional variables). Eqs. (1), (17)-(19), (23), and (24) give the distributions of the order parameter and the magnetic field within the Wigner-Seitz cell.

\[\text{FIG. 1. The spatial distribution of the dimensionless order parameter in the unit cell of the flux-line lattice at different magnetic inductions } B = 0.1 H_{c2}, B = 0.5 H_{c2}, B = 0.8 H_{c2} \text{ for } \kappa = 10. \text{ The solid lines correspond to the variational calculations in the Wigner-Seitz approximation. The dashed lines correspond to the numerical solution of the Ginzburg-Landau equations for the triangular lattice (nearest neighbor vortices are in the plane of the graph). The distance is measured in units of the intervortex spacing } d \text{ in the triangular lattice.} \]

The upper critical field is determined as the field at which the order parameter in the superconductor becomes equal to zero. As can be seen from Eq. (23), one has \( f = 0 \) at \( H = 0.985 \kappa \). Thus, the difference between the exact \( H_{c2} = \kappa \) and its calculated value is about
1.5%. This result is quite natural, since variational procedures in general give only approximate solutions to the GL equations. Similarly, the Clem value of $H_{c2}$ slightly differs from the numerically calculated one.

Now we compare the obtained results for the order parameter and the magnetic field with the similar distributions computed for the triangular lattice by the numerical method proposed in Ref. [16]. The dependence of the order parameter on the distance from the cell center is shown in Fig. 1 for different values of the magnetic induction at $\kappa = 10$. The spatial distribution of the order parameter in the triangular lattice along the nearest neighbor direction is also shown in Fig. 1. The results of our variational calculations are close to the numerical ones at any values of the magnetic induction. The difference does not exceed several percent. Such an accuracy of our approach remains in a wide range of $\kappa \gtrsim 1$. The magnetic flux density as a function of the distance from the vortex axis is shown in Fig. 2 at $\kappa = 10$, $B = 0.5H_{c2}$. There is good agreement between the variational and numerical results for $h$. In Fig. 3, the spatial average of the order parameter squared, $\omega = \langle |\psi|^2 \rangle$, is plotted as a function of the magnetic induction at $\kappa = 100$. The comparison with the numerical result shows that the deviation of variationally-calculated $\omega$ from the exact dependence does not exceed one percent in small and intermediate fields. Near $H_{c2}$ this deviation increases due to a small difference between the calculated $H_{c2}$ and the exact value.

Thus, the results of our variational approach agree well with the exact numerical solution to the GL equations. In the next section we shall apply this approach to the calculation of the magnetization curve.

\begin{figure}[h]
\centering
\includegraphics[width=0.45\textwidth]{fig2}
\caption{The spatial distribution of the dimensionless magnetic field in the unit cell of flux-line lattice at the magnetic induction $B = 0.5H_{c2}$ for $\kappa = 10$. The solid line corresponds to the variational calculations in the Wigner-Seitz approximation. The dashed line corresponds to the numerical solution.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=0.45\textwidth]{fig3}
\caption{Averaged order parameter squared $\omega$ vs $B$ in dimensionless units for $\kappa = 100$. The solid line corresponds to the variational calculations and the dot line to the numerical solution.}
\end{figure}

The anisotropy in the GL functional can be taken into account by introducing the phenomenological effective-mass tensor $m_j$ ($j = 1, 2, 3$), where $m_j$ are the effective masses in the direction of the principal axes $x_j$. It was shown in Ref. [18] that the Ginzburg-Landau equations can be transformed to isotropic form by a simple transformation if $\kappa$ is replaced by $\kappa^* = \kappa p_1^{-1/2}$, where $p_1 = m_1/\sqrt{m_1m_2m_3}$, (the vortices are directed along the $x_1$ axis). Thus, in this case, the Wigner-Seitz approximation can be used. The Wigner-Seitz cell has an elliptic shape and can be transformed to the circular cell by a scaling transformation: the distance along the $x_j$ axis should be normalized by $m_1m_2m_3/\sqrt{m_1m_2m_3}$. The magnetization of the anisotropic superconductor can be found from the magnetization of the isotropic one by replacing the GL parameter $\kappa$ by $\kappa^*$.

\section*{IV. MAGNETIZATION}

The magnetization is defined by the well-known relationship:

$$-4\pi M = H - B. \quad (25)$$

It can be calculated by two equivalent methods if the exact solution to the GL equations is known. First, the magnetic field $H$ may be calculated by minimization of the Gibbs free energy $G = F - 2BH$:

$$H = \frac{1}{2} \frac{\partial F}{\partial B}. \quad (26)$$

This derivative was calculated, e.g. in Refs. [8,23]. The second approach uses the virial theorem for the flux-line lattice, which was proven in Ref. [34], namely the applied magnetic field $H$ can be found from the local magnetic field $h$ and the order parameter $f$ as...
\[ H = \frac{1}{2BA_{\text{cell}}} \int (f^2 - f^4 + 2h^2) \, d^2r, \quad (27) \]

where the integral is taken over the area of the unit cell. Both methods are equivalent if the exact solutions \( f \) and \( h \) are used [24].

The variational model gives the spatial distributions of the order parameter and the magnetic field within the cell, which are close to the exact results. Let us find the magnetization by means of both methods. According to Eq. (26) the magnetization \( M \) is:

\[-4\pi M = -B + \frac{f_\infty}{\kappa \xi_v} \times \]
\[ \times K_0(f_\infty \xi_v)I_1(f_\infty \rho) + I_0(f_\infty \xi_v)K_1(f_\infty \rho) + \]
\[ K_1(f_\infty \xi_v)I_1(f_\infty \rho) - I_1(f_\infty \xi_v)K_1(f_\infty \rho) + \]
\[ + \frac{1}{2B\kappa^2 \xi_v^3} \left\{ K_1(f_\infty \xi_v)I_1(f_\infty \rho) - \right. \]
\[ - I_1(f_\infty \xi_v)K_1(f_\infty \rho) \right\}^{-2} + \frac{f_\infty^2 (2 + 3B\kappa \xi_v^2)}{2\kappa (2 + B\kappa \xi_v^2)^3} + \]
\[ + \frac{\kappa^2 f_\infty^2 \xi_v^2}{2} \left\{ 1 - f_\infty^2 \ln \left[ \frac{2}{B\kappa \xi_v^2} + 1 \right] + \right. \]
\[ + \frac{f_\infty^2 (2 + 3B\kappa \xi_v^2)}{2\kappa (2 + B\kappa \xi_v^2)^3} - \frac{f_\infty^2 (2 + 3B\kappa \xi_v^2)}{2\kappa (2 + B\kappa \xi_v^2)^3} \}. \quad (28)\]

Thus, in the former case the dependence of the magnetization on the magnetic induction \( B \) is given by Eqs. (28) and (22)-(24). The dependence of \( H \) on \( B \) is given by Eq. (25). Thus, we find the implicit function \( M(H) \).

Within the second approach, the integral (27) can be calculated only numerically when \( f \) and \( h \) are defined by Eqs. (1) and (17). Our calculations show that not only the values of \( H \) found by both methods coincide, but also the values of \( M \), which is usually much smaller than \( H \), are practically indistinguishable. The difference between them is much less than one percent at any induction and \( \kappa > 1/\sqrt{2} \). Note that the result for \( M \) in the second approach is very sensitive to the perturbations of \( f(r) \) and \( h(r) \). For example, if one puts \( f_\infty = 1 \) near \( H_{c1} \) and minimizes the free energy only with respect to \( \xi_v \) this does not change \( f(r) \) and \( h(r) \) considerably. However, this procedure would lead to an appreciable change of \( M(H) \) when using Eq. (27), while according to Eq. (26) the magnetization practically remains the same. Below, we shall use Eq. (28) for the magnetization.

At low fields the variational parameters (22) and (23) may be considered as constants independent of \( B \) when \( \kappa \gg 1 \). In this case, Eq. (28) can be expressed as a power series in terms of \( \xi_v \). As a result, it is possible to obtain Eq. (12) with \( H_{c1} \) given by Eq. (3) at \( \varepsilon = 0.52 \). Thus, in small fields the model reduces to the London approximation provided that the variationally-calculated value of \( H_{c1} \) is used, which is practically indistinguishable from the exact \( H_{c1} \). Actually, the use of the exact \( H_{c1} \) in small fields is equivalent to taking into account the effect of vortex cores. The field dependence of the magnetization (28) is shown in Fig. 4 for \( \kappa = 100 \). The magnetization curves corresponding to the London and Abrikosov approximations are also plotted. At low fields, the magnetization practically coincides with the results of the London approach, the difference between them does not exceed 0.5%. At \( H \approx H_{c2} \), the behavior of the magnetization is in good agreement with the Abrikosov high-field result (14). Although near \( H_{c2} \) both curves are close to each other, the error of our approximation is not so small as in low fields because of the slight deviation of the calculated \( H_{c2} \) from the exact value. For example, at \( H = 0.8H_{c2} \), \( \kappa = 100 \) the error of our variational procedure is about 5%. In order to improve the accuracy near \( H_{c2} \), one may put the constant \( t \) in Eq. (23) to be equal to 1. As a result, the difference between the variational and the Abrikosov \( M(H) \) curves decreases in the vicinity of \( H_{c2} \), whereas at low and intermediate fields the magnetization does not change.

In the inset of Fig. 4, we compare the calculated dependence \(-4\pi M(H)\) with that found in Ref. [30], where the Clem-Hao approach of superposition of vortex fields [23] was used. It is clearly seen that this approach is valid only at small fields, and its use leads to an appreciable error in the magnetization even in the intermediate field range. An additional error in \( M \) arises in small fields due to the approximate replacement of the lattice sums by integrals [23] in calculating the magnetic free energy; for more details see Ref. [30].

\[ \text{FIG. 4. Calculated } -4\pi M(H) \text{ using the variational method for } \kappa = 100 \text{ (solid line). Also shown are the London dependence } -4\pi M(H) \text{ (13) with the exact value of } H_{c1} \text{ (3) (dashed line) and the Abrikosov high-field result } (16) \text{ (dot line). The inset refers to the magnetization found within the framework of the variational model (solid line) and the result of calculations } [30] \text{ for the magnetization in the Clem-Hao approximation } [23] \text{ (dot line). The difference between these curves arises due to the superposition of vortex fields used in the Clem-Hao model } [23]. \text{ Dimensionless variables are used.} \]
with the results of the London (at \( \kappa \gg 1 \)) and the Abrikosov approximations, respectively. As we found above, the results of the variational approach are in agreement with these approximations. In the intermediate field range, where the London and the Abrikosov approaches are not applicable, the difference between the values of the magnetization calculated by numerical and variational methods is not bigger than 1\% in a wide range of \( \kappa \gg 1 \) values. Thus, our results for the magnetization appear to be a good approximation to the exact numerical solution of the GL equations at \( \kappa \gg 1 \).

Next we discuss the case of small \( \kappa \) values. In Fig. 5, the field dependences of the magnetization are plotted for several small \( \kappa \) values. The solid and dotted lines correspond to the variational and numerical calculations, respectively. The agreement between these results is good at \( \kappa \gtrsim 1 \). At smaller \( \kappa \) values the variational and the exact numerical results differ near the lower critical field. In this case, the intervortex distance is of the order of the coherence length almost in the entire field range, and the variational approach based on an appropriate trial function for the order parameter in the circular Wigner-Seitz cell may lead to some deviation from the exact solution.

In the inset of Fig. 5 we compare the magnetization calculated by means of our variational procedure and by the Clem-Hao model at \( \kappa = 1 \). It is seen that also for small (even as for large \( \kappa \), see above) the calculation method proposed in Ref. [23] leads to an inaccurate magnetization.

Our formulas for the magnetization may be used for the analysis of experimental data. In Fig. 6 the calculated magnetization curves are compared with the measured magnetization of YBa\(_2\)Cu\(_4\)O\(_8\) polycrystals [28] and Nd\(_{1.85}\)Ce\(_{0.15}\)CuO\(_{4-\delta}\) single crystals [29]. In these papers, the magnetization curves at different temperatures were analyzed and reduced to the dimensionless form based on the Clem-Hao formulas with non-selfconsistent field dependences of the variational parameters [23]. The resulting magnetization curve is close to the Abrikosov high-field result and to our variational dependence in the intermediate field range. The \( \kappa \) values obtained in Refs. [28,29] were \( \kappa = 70 \) for YBa\(_2\)Cu\(_4\)O\(_8\) and \( \kappa = 80 \) for Nd\(_{1.85}\)Ce\(_{0.15}\)CuO\(_{4-\delta}\).

In Fig. 6 we compare these experimental curves with our variational result. The circles and triangles give the experimental data for YBa\(_2\)Cu\(_4\)O\(_8\) [28] and Nd\(_{1.85}\)Ce\(_{0.15}\)CuO\(_{4-\delta}\) [29], respectively. The inset shows the magnetization calculated by our variational model and plotted versus the logarithm of the applied field \( H \) at \( \kappa = 100 \).

**FIG. 6.** The field dependence of magnetization in dimensionless units. The solid and dotted lines show the theoretical variational dependences at \( \kappa = 70 \) and \( \kappa = 80 \). The circles and triangles give the experimental data for YBa\(_2\)Cu\(_4\)O\(_8\) [28] and Nd\(_{1.85}\)Ce\(_{0.15}\)CuO\(_{4-\delta}\) [29], respectively. The inset shows the magnetization calculated by our variational model and plotted versus the logarithm of the applied field \( H \) at \( \kappa = 100 \).

**V. CONCLUSIONS**

We proposed an approximate method to solve the Ginzburg-Landau equations for the regular flux-line lattice at any values of the magnetic induction and the Ginzburg-Landau parameter \( \kappa > 1/\sqrt{2} \). The Wigner-Seitz approximation is used, and the hexagonal unit cell of the vortex lattice is replaced by a circle with the same area. Our model is based on Clem’s trial function for the order parameter. The use of this function allows us to find the magnetic flux density self-consistently from the second Ginzburg-Landau equation. The comparison
between the variational results and the results of exact numerical solution of the Ginzburg-Landau equations reveals good accuracy of our approach: the difference between the spatial distributions of the order parameter and the magnetic field does not exceed several percent. Such accuracy remains in a wide range of values of the magnetic induction and $\kappa \gg 1$. The method is applied to the calculation of the field dependence of the reversible magnetization. An analytical expression for the magnetization is proposed. At low fields, the obtained dependence agrees with the predictions of London theory and the calculation of the field dependence of the reversible magnetization is proposed. At high fields, it is in good agreement with the Abrikosov result. It is shown that the values of the magnetization calculated within the framework of our variational model and of the numerical method of solution of the Ginzburg-Landau equations are practically indistinguishable, especially in small and intermediate magnetic fields at $\kappa \gtrsim 1$. Our model yields the limits of the Clem-Hao model for the magnetization. The presented analytical formulas for the magnetization may be used to analyze experimental data. As an illustration we compared the experimental and calculated magnetization curves for different high-$T_c$ superconductors (YBa$_2$Cu$_3$O$_{6-}\delta$) and found good agreement between theory and experiment.

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