We discuss the propagation of light in the C-metric. We discover that null geodesics admit circular orbits only for a certain family of orbital cones. Explicit analytic formulae are derived for the orbital radius and the corresponding opening angle fixing the cone. Furthermore, we prove that these orbits based on a saddle point in the effective potential are Jacobi unstable. This completes the stability analysis done in previous literature and paves the road for future investigation into light bending in a manifold described by the C-metric.

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**Keywords:** C-metric, geodesic equation, null circular orbits, Jacobi stability analysis
I. INTRODUCTION

The geodesic motion of test particles in a gravitational field is an important field of general relativity since it opens the possibility to test the details of the underlying metric (see e.g. [1]). The case of massless particles is here of special interest as it includes the motion of light [2]. The question of new phenomena arises if we consider a metric which in some limiting case reduces to the Schwarzschild metric (for examples see [3]). Here the fate of the circular orbit, appearing already in the Schwarzschild metric, and issues regarding its stability would deserve an attention. A candidate for such a field of investigation is the C-metric of two black holes which we will briefly introduce below.

The C-metric represents a pair of causally disconnected black holes, each having mass $M$, and accelerating in opposite directions with acceleration parameter $\alpha$ under the action of forces generated by conical singularities [4–6]. Its line element in Boyer-Lindquist coordinates and in geometric units ($G = c = 1$) is given by

$$ds^2 = F(r, \vartheta) \left[ -f(r) dt^2 + \frac{dr^2}{f(r)} + r^2 d\vartheta^2 + r^2 g(\vartheta) \sin^2 \vartheta d\varphi^2 \right]$$

with

$$F(r, \vartheta) = (1 + \alpha r \cos \vartheta)^{-1}, \quad f(r) = (1 - \alpha^2 r^2) \left( 1 - \frac{2M}{r} \right), \quad g(\vartheta) = 1 + 2\alpha M \cos \vartheta,$$

where $\vartheta \in (0, \pi)$, $\varphi \in (-\kappa \pi, \kappa \pi)$, and $r_H < r < r_h$. Here, $r_H = 2M$ represents the Schwarzschild horizon, and $r_h = 1/\alpha$ denotes the acceleration horizon. In order to preserve the spatial ordering of the horizons, we need to assume that $0 < 2\alpha M < 1$. Furthermore, the computation of the Kretschmann invariant for (1) signalizes that the only curvature singularity occurs at $r = 0$ while the horizons of the C-black hole are mere coordinate singularities. It is interesting to observe that for $\alpha \to 0$ the line element (1) goes over into the line element of a Schwarzschild black hole. This fact means that predictions concerning the bending of light in the C-metric should go over for vanishing $\alpha$ into the corresponding predictions in the Schwarzschild metric. In order to remove the conical singularity at $\vartheta = 0$, the parameter $\kappa$ entering in the definition of the range for the angular variable $\varphi$ can be chosen to be [4]

$$\kappa = \frac{1}{1 + 2\alpha M}.$$

According to [4], the conical singularity with constant deficit angle along the half-axis $\vartheta = \pi$ has the interpretation of a semi-infinite cosmic string under tension extending from the source at $r = 0$ to conformal infinity. This allows to interpret (1) as a Schwarzschild-like black hole that is being accelerated along the axis $\vartheta = \pi$ by the action of a force corresponding to the tension in a cosmic string. Note that the range of the rotational coordinate can be restored to its usual value $2\pi$ by means of the rescaling $\varphi = \kappa \varphi$ leading to $\varphi \in (-\pi, \pi)$. In what follows, we will work with the version of the line element (1) obtained after rescaling of the variable $\varphi$, namely

$$ds^2 = -B(r, \vartheta) dt^2 + A(r, \vartheta) dr^2 + C(r, \vartheta) d\vartheta^2 + D(r, \vartheta) d\varphi^2,$$

where

$$B(r, \vartheta) = f(r) F(r, \vartheta), \quad A(r, \vartheta) = \frac{F(r, \vartheta)}{f(r)}, \quad C(r, \vartheta) = \frac{r^2 F(r, \vartheta)}{g(\vartheta)}, \quad D(r, \vartheta) = \kappa^2 r^2 g(\vartheta) F(r, \vartheta) \sin^2 \vartheta$$

with $F$, $f$, and $g$ defined in (2). The last four decades witnessed a literature proliferation on the C-metric where several different features have been analysed, for instance: singularities and motion [7–9], horizon structure [10–12], spacetime properties at infinity [12–14]. Further studies on the C-metric were undertaken by [15] leading to the discovery that an accelerated black hole has Hawking temperature larger than the Unruh temperature of the accelerated frame. Furthermore, separability of test fields equations on the C-metric background was extensively studied in [16–18]. Concerning geodesic motion in the aforementioned spacetime, [19] studied in detail the geodesic trajectories of time-like particle in the C-metric and in the case of light-like particles, the authors claimed that photon circular orbits are unstable without providing a solid mathematical proof. The reason could be that they base their conclusions on a local maximum in the effective potential. In contrast to this, we find that the circular orbit is due to a saddle point which requires an additional effort to probe into its stability issues. Moreover, [20] referred to [19] regarding the stability analysis of the aforementioned trajectories. Further work on time-like circular orbits and their null limit has been undertaken in [21, 22] where the corresponding stability problem was not studied. In particular, [22] the authors discovered the existence of an invariant plane for the motion of photons but they neither showed that a photon circular orbit is allowed nor they derived an analytical formula for the radius of a circular orbit orbit. Moreover, the stability
problem of motion on the invariant plane is nowhere addressed in the literature which seems to us necessary as the orbit emerges from a saddle point in the effective potential. We fill this gap in the next section where we analyse the (in)stability of the photon circular orbits by constructing the second KCC (Kosambi-Cartan-Chern) invariant \[23–26\] associated to the system of geodesic equations describing the motion of a light-like particle in the C-metric. According to a theorem in \[27, 28\], the circular orbits of null particles are Jacobi stable if and only if the real parts of the eigenvalues of the second KCC invariant are strictly negative everywhere (along the trajectory), and Jacobi unstable, otherwise. The result emerging from our analysis is that such trajectories are Jacobi unstable, and therefore, the stability analysis done in previous literature has been completed. Last but not least, our findings allow to study strong and weak gravitational lensing on the invariant hyperplane of the C-metric by methods analogous to those used in the case of the Schwarzschild-deSitter metric. This analysis is left for future work.

II. GEODESIC EQUATIONS

The relativistic Kepler problem for the C-metric is defined by the geodesic equations \[30\]
\[
\frac{d^2 x^\eta}{d\lambda^2} = -\Gamma^\eta_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}, \quad \Gamma^\eta_{\mu\nu} = \frac{1}{2} g^{\eta\tau} (\partial_\mu g_{\tau\nu} + \partial_\nu g_{\tau\mu} - \partial_\tau g_{\mu\nu})
\] (6)
together with the subsidiary condition
\[
g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} = -\epsilon,
\] (7)
where \(\epsilon = 1\), and \(\epsilon = 0\) corresponds to time-like and light-like geodesics, respectively, and \(\lambda\) is an affine parameter. After the computation of the Christoffel symbols \(\Gamma^\eta_{\mu\nu}\), we find that the equations of motion are represented by the following system of ordinary differential equations
\[
\frac{d^2 t}{d\lambda^2} = -\frac{\partial_r B}{B} \frac{dt}{d\lambda} \frac{dr}{d\lambda} - \frac{\partial_\theta B}{B} \frac{dt}{d\lambda} \frac{d\theta}{d\lambda}
\] (8)
\[
\frac{d^2 r}{d\lambda^2} = -\frac{\partial_r B}{2A} \left(\frac{dt}{d\lambda}\right)^2 - \frac{\partial_\theta A}{2A} \left(\frac{dr}{d\lambda}\right)^2 - \frac{\partial_\theta C}{2A} \left(\frac{d\theta}{d\lambda}\right)^2 + \frac{\partial_r C}{2A} \left(\frac{dr}{d\lambda}\right)^2 + \frac{\partial_\theta D}{2A} \left(\frac{d\phi}{d\lambda}\right)^2,
\] (9)
\[
\frac{d^2 \theta}{d\lambda^2} = -\frac{\partial_\theta B}{2C} \left(\frac{dt}{d\lambda}\right)^2 - \frac{\partial_\theta A}{2C} \left(\frac{dr}{d\lambda}\right)^2 - \frac{\partial_\theta C}{C} \left(\frac{d\theta}{d\lambda}\right)^2 - \frac{\partial_\theta D}{2C} \left(\frac{d\theta}{d\lambda}\right)^2 + \frac{\partial_\theta D}{2C} \left(\frac{d\phi}{d\lambda}\right)^2,
\] (10)
\[
\frac{d^2 \phi}{d\lambda^2} = -\frac{\partial_\theta D}{D} \frac{d\phi}{d\lambda} \left(\frac{d\phi}{d\lambda}\right) - \frac{\partial_\theta D}{D} \frac{d\theta}{d\lambda} \frac{d\phi}{d\lambda}
\] (11)
with functions \(A, B, C,\) and \(D\) defined in \([5]\). Equation \([8]\) can be rewritten as
\[
1 \frac{d}{d\lambda} \left(\frac{B \frac{dt}{d\lambda}}{B} \right) = 0
\] (12)
which integrated leads to
\[
\frac{dt}{d\lambda} = \frac{E}{B}
\] (13)
Since \(B\) is dimensionless and in geometric units \(dt/d\lambda\) is also dimensionless, it follows that the integration constant \(E\) is dimensionless. Furthermore, the C-metric admits a Killing vector \(\partial/\partial t\) \([20]\), and therefore, the aforementioned constant of motion must be related to the energy of the particle. More precisely, \(E\) represents the energy per unit mass \([30]\) and hence, it is dimensionless in geometric units. Similarly, we can integrate equation \([11]\) to obtain
\[
\frac{d\phi}{d\lambda} = \frac{\ell}{D}
\] (14)
Note that the C-metric admits the Killing vector \(\partial/\partial \phi\) \([20]\), and therefore, the integration constant \(\ell\) can be related to the angular momentum of the particle. More precisely, \(\ell\) is the angular momentum per unit mass \([30]\), and in geometric units, it has dimension of length. We point out that the second equation appearing in \([14]\) in \([20]\) and...
corresponding to our (14) should have a factor $C_0^2$ in the denominator. If we substitute (13) and (14) into (1) and (10), we get
\[
\frac{d^2r}{d\lambda^2} = -\frac{\partial_r A}{2A} \left( \frac{dr}{d\lambda} \right)^2 - \frac{\partial_\theta A}{A} \frac{dr}{d\lambda} \frac{d\theta}{d\lambda} + \frac{\partial_r C}{2A} \left( \frac{d\theta}{d\lambda} \right)^2 - \frac{\varepsilon^2}{2} \frac{\partial_r B}{AB^2} + \frac{\ell^2}{2} \frac{\partial_\theta D}{AD^2},
\]
\[
\frac{d^2\theta}{d\lambda^2} = -\frac{\partial_\theta C}{2C} \left( \frac{d\theta}{d\lambda} \right)^2 - \frac{\partial_r C}{C} \frac{dr}{d\lambda} \frac{d\theta}{d\lambda} + \frac{\partial_\theta A}{2C} \left( \frac{dr}{d\lambda} \right)^2 - \frac{\varepsilon^2}{2} \frac{\partial_\theta B}{CB^2} + \frac{\ell^2}{2} \frac{\partial_r D}{CD^2}.
\]

At this point a couple of remarks are in order. After a careful scrutiny of equation (16) in [20], we noticed there that the corresponding coefficient multiplying the term $r^2$ has a spurious extra factor 2. Furthermore, the coefficient going together with the term $\dot{r}\dot{\theta}$ should have $P$ replaced by $r$. Note that the presence of $P$ in the denominator of the term containing $\dot{r}\dot{\theta}$ in (16) would cause this term to have different dimension as all other terms in (16). The expressions for the coefficient functions entering in (15) and (16) can be found in the Appendix. Differently as in the Schwarzschild case, due to the lack of spherical symmetry equation (16) cannot be solved without loss of generality by assuming that the motion of the particle takes place on the equatorial plane, i.e. $\vartheta = \pi/2$. Furthermore, the constraint equation (17) can be expressed with the help of (13) and (14) as
\[
A \left( \frac{dr}{d\lambda} \right)^2 + C \left( \frac{d\theta}{d\lambda} \right)^2 = -\epsilon + \frac{\varepsilon^2}{B} - \frac{\ell^2}{D}.
\]

If we multiply (17) by a factor 1/2 so that in the limit of vanishing $\alpha$ equation (17) reproduces correctly equation (25.26) in [30] for the Schwarzschild case, and introduce the effective potential
\[
U_{eff}(r, \vartheta) = \frac{1}{2} \left( \epsilon B + \frac{\ell^2 B}{D} \right),
\]
we can rewrite equation (17) in a more suitable form to study the motion of a particle in the C-metric, namely
\[
\frac{AB}{2} \left( \frac{dr}{d\lambda} \right)^2 + \frac{BC}{2} \left( \frac{d\theta}{d\lambda} \right)^2 + U_{eff} = E,
\]
where we set $E = \varepsilon^2/2$ and
\[
AB = F^2, \quad BC = r^2 F^2 \frac{f}{g}.
\]

Notice that $A$, $B$, and $C$ are positive definite and therefore $E - U_{eff} \geq 0$ as in classical mechanics. The equality, $E = U_{eff}$, corresponds to a circular orbit and a critical point of the effective potential. Before proceeding to study light bending in the C-metric, we recall that in the case of null geodesics we have $\epsilon = 0$ and therefore, the effective potential takes on the more simpler form
\[
\mathcal{U}(r, \vartheta) = \frac{\ell^2 B}{2D}.
\]

At this point a couple of remarks are in order. First of all, the effective potential $V_{eff}$ introduced in [20] in equation (19) is related to our potential as follows
\[
2U_{eff}(r, \vartheta) = V_{eff}^2(r, \vartheta).
\]

It is straightforward to verify that the effective potential $V_{eff} = \sqrt{2U_{eff}}$ defined by [20] does not reproduce in the limit $\alpha \to 0^+$ and for $\vartheta = \pi/2$ the effective potential for a particle in the Schwarzschild metric which is given in geometric units by [30 32]
\[
V_{eff,Sch}(r) = \left\{ \begin{array}{ll}
-\frac{M}{\ell^2} + \frac{\ell^2 - M^2 \ell^2}{r^2} & \text{, } m \neq 0 \\
-\frac{2M^2}{\ell^2} - \frac{M^2 \ell^2}{r^2} & \text{, } m = 0
\end{array} \right.,
\]
where $m$ denotes the mass of the particle. Furthermore, in the Schwarzschild case and for $\epsilon = 0$, the critical point of the effective potential coincides with the critical point of the corresponding geodesic equation (see Appendix [3]). In Section III we will show that the same continues to be true also in the case of the C-metric. We conclude this part by
expressing the geodesic equations (15) and (16) subject to the constraint (19) in a form that simplifies considerably the analysis of the critical point(s) arising from the dynamical system associated to the new system of equations and the study of the Jacobian (in)stability of the circular orbits. To this purpose, we replace (19) into (15) and (16) to obtain

\[ \frac{d^2r}{d\lambda^2} + \left( \partial_r \ln \sqrt{AC} \right) \left( \frac{dr}{d\lambda} \right)^2 + \left( \partial_\theta \ln A \right) \frac{dr}{d\lambda} \frac{d\theta}{d\lambda} + \frac{E}{AB} \partial_r \ln \frac{B}{C} = 0, \]  

(24)

\[ \frac{d^2\theta}{d\lambda^2} + \left( \partial_\theta \ln \sqrt{AC} \right) \left( \frac{d\theta}{d\lambda} \right)^2 + \left( \partial_r \ln C \right) \frac{dr}{d\lambda} \frac{d\theta}{d\lambda} + \frac{\ell^2}{2CD} \partial_\theta \ln \frac{A}{D} = 0. \]  

(25)

### III. NULL CIRCULAR ORBITS

We start by investigating the critical points of the effective potential (21). First of all, the condition \( \partial_r \mathfrak{V} = 0 \) gives rise to the quadratic equation

\[ \alpha^2 M r^2 + r - 3M = 0. \]  

(26)

It is comforting to see that in the limit of vanishing \( \alpha \), equation (26) has two coinciding roots reproducing the radius of the photon sphere \( r_\gamma = 3M \) for a Schwarzschild black hole. Since the radial variable \( r \) belongs to the interval \( (2M, 1/\alpha) \), the only acceptable root for (26) is

\[ r_c = \frac{6M}{1 + \sqrt{1 + 12\alpha^2 M^2}}. \]  

(27)

From (27), we immediately see that \( r_c < r_\gamma \). This implies that a null circular orbit for a photon in the C-metric takes place at a closer distance to the event horizon than in the Schwarzschild case. Furthermore, if we require that \( \partial_\theta \mathfrak{V} = 0 \), we end up with the equation

\[ 3\alpha M \cos^2 \vartheta + \cos \vartheta - \alpha M = 0. \]  

(28)

The root

\[ \cos \vartheta_1 = -\frac{1 + \sqrt{1 + 12\alpha^2 M^2}}{6\alpha M} = \frac{2\alpha M}{1 - \sqrt{1 + 12\alpha^2 M^2}} \]  

(29)

must be disregarded because we would have \( \cos \vartheta_1 < -1 \) for \( \alpha M \in (0, 1/2) \). Hence, the only admissible solution to (28) is

\[ \vartheta_c = \arccos \left( \frac{2\alpha M}{1 + \sqrt{1 + 12\alpha^2 M^2}} \right). \]  

(30)
Since \( \alpha M \in (0, 1/2) \) and \( \theta_c \) is monotonically decreasing in the variable \( \alpha M \), it is straightforward to verify that the angular variable associated to the critical points cannot take on any value from 0 to \( \pi \) but it is constrained to the range \((\theta_{c,\min}, \pi/2)\) with \( \theta_{c,\min} = \arccos(1/3) \approx 70.52^\circ \). Concerning the classification of these critical points, if we compute the determinant \( \Delta \) of the Hessian matrix associated to our effective potential and we evaluate it at the critical point \((r_c, \theta_c)\) found above, it turns out that

\[
\Delta(r_c, \theta_c) = -\frac{\ell^4}{139968M^6\kappa^4} \frac{(1 + \tau)^7 S(x)}{(1 + \tau + 4x^2)^3 T(x)}
\]

(31)

with \( x := \alpha M, \tau = \tau(x) := \sqrt{1 + 12x^2} \) and

\[
S(x) := (1728\tau + 8640)x^{10} + (1008\tau - 720)x^8 - (492\tau + 828)x^6 + (25 - 35\tau)x^4 + (13\tau + 19)x^2 + 1 + \tau,
\]

(32)

\[
T(x) := 32x^8 + (32\tau + 176)x^6 + (48\tau + 114)x^4 + (14\tau + 20)x^2 + 1 + \tau.
\]

(33)

Since \( x \in (0, 1/2) \) the term \( T(x) \) in \( (31) \) is always positive and therefore, the sign of \( (31) \) is controlled by term \( S(x) \). As it can be evinced from Fig. 1, the function \( S(x) \) is always positive on the interval \((0, 1/2)\), and therefore, \( \Delta(r_c, \theta_c) < 0 \). This means that the critical point \((r_c, \theta_c)\) of the effective potential is a saddle point. As a check of the validity of our computation, we computed the value of the effective potential at the saddle point, namely

\[
\mathfrak{H}(r_c, \theta_c) = \frac{\ell^2[216x^6 - 54x^2 + \tau(1 - 6x^2) + 1]}{27M^2\kappa^2(4x^2 + \tau + 1)}, \quad x = \alpha M, \quad \tau = \tau(x) = \sqrt{1 + 12x^2},
\]

(34)

and we verified that in the limit of vanishing \( \alpha \), it reproduces correctly the value \( \ell^2/(54M^2) \) obtained from \((25, 27)\) in \( \text{Eq.}\ 30 \) for the maximum of the effective potential computed at the radius of the photon sphere. Furthermore, \( \text{Eq.}\ 34 \) attains its maximum for \( \alpha = 0 \) (Schwarzschild case), and then, it decreases monotonically to zero as \( \alpha M \) approaches the value \( 1/2 \). As in the Schwarzschild case (see Appendix \( \text{B} \)), the critical point of the effective potential for massless particles in the C-metric coincides with the critical point of the dynamical system represented by \((24, 25)\) and \((26)\). To see that, let \( r = r_0 \) and \( \theta = \theta_0 \) be the coordinates of a candidate critical point for the system \((24, 25)\) and \((26)\). Then, the latter simplifies to the following system of equations

\[
\partial_r \left( \frac{B}{C} \right) \big|_{(r_0, \theta_0)} = 0, \quad \partial_{\theta} \left( \frac{A}{D} \right) \big|_{(r_0, \theta_0)} = 0.
\]

(35)

By means of \( \text{Eq.}\ 14 \) the above equations reduce to

\[
\frac{d}{dr} \left( \frac{f(r)}{r^2} \right) \bigg|_{r=r_0} = 0, \quad \frac{d}{d\theta} \left( \frac{1}{g \sin^2 \theta} \right) \bigg|_{\theta=\theta_0} = 0.
\]

(36)

At this point, a straightforward computation shows that the first equation in \( \text{Eq.}\ 14 \) reduces to the quadratic equation \( \text{Eq.}\ 30 \) and therefore, we conclude that \( r_0 = r_c \) with \( r_c \) given by \( \text{Eq.}\ 27 \). Concerning the second equation in \( \text{Eq.}\ 36 \), it can be easily seen that its r.h.s is equivalent to

\[
\frac{d}{d\theta} \left( \frac{1}{g \sin^2 \theta} \right) = -\frac{2(3\alpha M \cos^2 \theta + \cos \theta - \alpha M)}{\sin^2 \theta (1 + 2\alpha M \cos \theta)^2}
\]

(37)

and hence, it will vanish when \( \text{Eq.}\ 28 \) is satisfied. At this point, we can perform the same analysis done to study the roots of \( \text{Eq.}\ 28 \) and it follows that \( \theta_0 = \theta_c \). We conclude this section with the observation that the impact parameter \( b = \ell/\mathcal{E} \) associated to the circular orbit \( r = r_c \), taking place on the cone with opening angle \( \theta = \theta_c \) must be tuned as follows

\[
b_c = \kappa r_c \sin \theta_c \sqrt{\frac{g(\theta_c)}{f(r_c)}}.
\]

(38)

As a validity check of the above result we verified that in the limit of vanishing acceleration, \( b_c \) goes over to the value \( 3\sqrt{3}M \) of the critical impact parameter in the Schwarzschild metric. This can be easily seen by constructing an expansion of \( \text{Eq.}\ 38 \) in the parameter \( x = \alpha M \) around \( x = 0 \) which leads to

\[
b_c = 3\sqrt{3}M - 6\sqrt{3}M x + 27\sqrt{3}M x^2 + O(x^3).
\]

(39)

Similarly as in Schwarzschild, we expect that
FIG. 2: Plot of the critical impact parameter per unit black hole mass $b_c/M$ defined in (38) as a function of $x = \alpha M$ on the interval $(0, 1/2)$. Note that $b_c/M$ reduces to the corresponding Schwarzschild value $3\sqrt{3}$ for alpha $x = 0$ and it exhibits a minimum at $x_m = \frac{1}{2} - \frac{\sqrt{2}}{4}$ where $b_c/M = 4\sqrt{3}(3 + 2\sqrt{2})/9$.

1. if $b < b_c$ the photon is doomed to be absorbed by the black hole;

2. if $b > b_c$ the photon is able to escape the gravitational pull and it will eventually reach the acceleration horizon.

In particular, if $b \gg b_c$, i.e. the distance of closest approach is much larger than the radius $r_c$ of the photon circular orbit on the invariant cone, we expect weak gravitational lensing. In the case $b \approx b_c$, we are instead in the regime of strong gravitational lensing. In this case, the photon can orbit several times around the black hole before it flies off.

The analysis of the strong/weak gravitational lensing on the invariant cones of the C-metric will be done in an upcoming paper.

IV. JACOBI STABILITY ANALYSIS OF THE CIRCULAR ORBITS

In this section, we dwell with the problem of determining whether or not the class of circular orbits we previously found are Jacobi stable. To study the Jacobi (in)stability of photon circular orbits with radius $r_c$ and occurring on a cone with opening angle $\vartheta = \vartheta_c$, we shall use the KCC theory which represents a powerful mathematical method for the analysis of dynamical systems \[23–26, 33\]. To this purpose, we consider null particles and rewrite (24) and (25) as a dynamical system of the form

$$\frac{d^2x^i}{d\lambda^2} + g^i(x^1, x^2, y^1, y^2) = 0$$

where

$$g^1(x^1, x^2, y^1, y^2) = [\partial_1 \ln \sqrt{AC}] (y^1)^2 + (\partial_2 \ln A) y^1 y^2 + \frac{E}{AB} \partial_1 \ln \frac{B}{C},$$

$$g^2(x^1, x^2, y^1, y^2) = [\partial_2 \ln \sqrt{AC}] (y^2)^2 + (\partial_1 \ln C) y^1 y^2 + \frac{\ell^2}{2CD} \partial_2 \ln \frac{A}{D}$$

with $x^1 := r$, $x^2 := \vartheta$, and $y^i = dx^i/d\lambda$ for $i = 1, 2$. Furthermore, we make the reasonable assumption that $g^1$ and $g^2$ are smooth functions in a neighbourhood of the initial condition $(x_{10}, x_{20}, y_{10}, y_{20}, \lambda_c) = (r_c, \vartheta_c, 0, 0, \lambda_c) \in \mathbb{R}^5$. In this approach, it is possible to describe the evolution of a dynamical system in geometric terms, by considering it as a geodesic in a Finsler space \[33\]. More precisely, if we perturb the geodesic trajectories of the system (40) into neighbouring ones according to

$$\tilde{x}^i(\lambda) = x^i(\lambda) + \eta \xi^i(\lambda),$$

(43)
where \( |\eta| \) is a small parameter, and \( \xi^i(\lambda) \) represents the components of some contravariant vector field defined along the geodesic trajectory \( x^i(\lambda) \), the equation governing the perturbative part in \( \eta \) can be obtained by replacing first \( \epsilon \) into \( \epsilon_0 \) and by letting then \( \eta \to 0 \). This procedure leads to the equation

\[
\frac{d^2 \xi^i}{d\lambda^2} + 2N^i_j \frac{d\xi^j}{d\lambda} + \frac{\partial g^i}{\partial x^j} \xi^j = 0,
\]

where

\[
N^i_j = \frac{1}{2} \frac{\partial g^i}{\partial y^j}
\]

defines the coefficients of a non-linear connection \( N \) on the tangent bundle. The latter also enters the definition of the KCC covariant differential, namely

\[
\frac{D\sigma^i}{d\lambda} = \frac{d\sigma^i}{d\lambda} + N^i_j \sigma^j
\]

with \( \sigma = \sigma^i \partial / \partial x^i \) some contravariant vector field. Furthermore, equation (44) can be written in terms of (40) in the covariant form

\[
\frac{D^2 \xi^i}{d\lambda^2} = P^i_j \xi^i.
\]

Equation (47) is called the Jacobi equation or the variation equation associated to the system (40). Proceeding as in [33], five geometrical invariants can be obtained for the system (40). However, only the second invariant controls the Jacobi (in)stability of the system, namely the tensor

\[
P^i_j = - \frac{\partial g^i}{\partial x^j} - g^{i\nu} G^\nu_{\rho j} + y^r \frac{\partial N^i_j}{\partial x^r} + N^i_j N^i_j + \frac{\partial N^i_j}{\partial \lambda}, \quad G^\nu_{\rho j} = \frac{\partial N^i_j}{\partial y^\rho}
\]

where \( G^\nu_{\rho j} \) is called the Berwald connection [34, 37]. Note that the term \( \partial N^i_j / \partial \lambda \) in (48) will not contribute because the system (40) is autonomous in the variable \( \lambda \). More precisely, if \( \xi^i = v(\lambda) \nu^i \) is a Jacobi field with speed \( v \) along the geodesic \( x^i(\lambda) \) where \( \nu^i \) is the unit normal vector field, then the Jacobi field equation (47) can be represented in the scalar form as [30]

\[
\frac{d^2 v}{d\lambda^2} + Kv = 0
\]

with \( K \) denoting the flag curvature of the manifold. The sign of \( K \) governs the geodesic trajectories, that is if \( K < 0 \), then such trajectories disperse, i.e. they are Jacobi unstable, and otherwise, if \( K > 0 \), they tend to focus together, i.e. they are Jacobi stable. Instead of studying the Jacobi (in)stability of the circular orbits by analyzing the sign of \( K \), it turns out that it is more convenient to use the following result: an integral curve \( \gamma \) of (10) is Jacobi stable if and only if the real parts of the eigenvalues of the second KCC invariant \( P^i_j \) are strictly negative everywhere along \( \gamma \), and Jacobi unstable otherwise. We refer to [27, 28, 33] for the proof of this statement. In order to apply this result, let us introduce the matrix

\[
P := \begin{pmatrix} P^1_1 & P^1_2 \\ P^2_1 & P^2_2 \end{pmatrix}
\]

evaluated at a circular orbit with \( x^1 = r_c \) and \( x^2 = \vartheta_c \) given by (27) and (50), respectively. Then, the associated characteristic equation is

\[
\det \begin{pmatrix} P^1_1(r_c, \vartheta_c) - \lambda & P^1_2(r_c, \vartheta_c) \\ P^2_1(r_c, \vartheta_c) & P^2_2(r_c, \vartheta_c) - \lambda \end{pmatrix} = 0.
\]

The computation of the entries of the matrix \( P \) can be optimized if we observe that the connection \( N \) vanish along a circular orbit with \( x^1 = r_c \) and \( x^2 = \vartheta_c \). This is due to the fact that the functions \( g^i \) given by (11) and (12) are quadratic in the variables \( y^1 \) and \( y^2 \), and hence, the terms \( N^i_j \) defined through (15) contain only linear combinations of \( y^1 \) and \( y^2 \) but \( g^i = dx^i/d\lambda = 0 \) along a circular orbit with \( x^1 = r_c \) and \( x^2 = \vartheta_c \). Similarly, terms of the form \( y^i(\partial N^i_j / \partial x^r) \) must also vanish when evaluated along the circular trajectories since \( y^i = 0 \) there. Hence, the only
FIG. 3: Plot of the functions $\Omega_1$ (thin solid line) and $\Omega_2$ (thick solid line) defined in (57) as a functions of $x = \alpha M$ on the interval $(0, 1/2)$. The function $\Omega_1$ is always negative on $(0, 1/2)$.

terms contributing to the computation of the eigenvalues are the first two terms on the l.h.s. of (48). Let $x = \alpha M$ and $\rho = r/M$. After a lengthy but straightforward computation where we made use of (36) we find that

$$P_1^1(\rho_c, \vartheta_c) = -\frac{E}{M^2} \left[ \frac{\rho^2}{f F^2} \frac{d^2}{d\rho^2} \left( \frac{f}{\rho^2} \right) \right]_{(\rho_c, \vartheta_c)},$$

$$P_2^2(\rho_c, \vartheta_c) = - \left( \frac{L}{M} \right)^2 \left[ \frac{g}{2\kappa^2 \rho^4 F^2} \frac{d^2}{d\vartheta^2} \left( \frac{1}{g \sin^2 \vartheta} \right) \right]_{(\rho_c, \vartheta_c)},$$

$$P_1^2(\rho_c, \vartheta_c) = \frac{E}{M} \left[ \frac{3\rho^2}{2f F^3} (\partial_\vartheta F) \frac{d}{d\rho} \left( \frac{f}{\rho^2} \right) \right]_{(\rho_c, \vartheta_c)} = 0,$$

$$P_2^1(\rho_c, \vartheta_c) = - \frac{L^2}{M^3} \left\{ \frac{1}{2\kappa^2} [\partial_\rho \left( \frac{g}{\rho^4 F^2} \right) + \frac{g}{2\rho^6 F^3} \partial_\rho (\rho^2 F) \frac{d}{d\vartheta} \left( \frac{1}{g \sin^2 \vartheta} \right) ] \right\}_{(\rho_c, \vartheta_c)} = 0$$

with $L := \ell/M$. This implies that the zeroes of the characteristic equation (51) are real and given by

$$\lambda_1 = -\frac{E}{M^2} \Omega_1(\rho_c, \vartheta_c), \quad \lambda_2 = -\frac{L^2}{M^2} \Omega_2(\rho_c, \vartheta_c),$$

where

$$\Omega_1(\rho_c, \vartheta_c) = \left[ \frac{\rho^2}{f F^2} \frac{d^2}{d\rho^2} \left( \frac{f}{\rho^2} \right) \right]_{(\rho_c, \vartheta_0)}, \quad \Omega_2(\rho_c, \vartheta_c) = \left[ \frac{g}{2\kappa^2 \rho^4 F^2} \frac{d^2}{d\vartheta^2} \left( \frac{1}{g \sin^2 \vartheta} \right) \right]_{(\rho_c, \vartheta_0)}.$$  

From Fig. 3 we immediately see that $\Omega_1$ is always negative for $x \in (0, 1/2)$ and therefore, the eigenvalue $\lambda_1$ is always strictly positive. As a consequence, photon circular orbits with radius $r_c$ given by (27) occurring on the cones $\vartheta = \vartheta_c$ with $\vartheta_0 \in (\pi/2, \pi)$ given by (30) are Jacobi unstable.

V. CONCLUSIONS

Light bending and possible bound states of light are genuine effects of general relativity. Whereas light bending has been studied and even observed in a variety of situations, bound orbits of massless particles are a rare case and deserve a special attention (see e.g. [37] and references therein). In general, it is known that a local maximum in the effective potential corresponds to an unstable circular orbit of light. However, a saddle point presents a more challenging problem and needs to be examined from case to case. In the C-metric of two black hole we indeed find such a saddle point. We perform a Jacobi analysis to probe into the stability issues of the circular orbit related to the saddle point. A careful lengthy analysis reveals that the orbit is unstable. The Jacobi method which we have used here might find a wider application in future investigations.
Appendix A: Coefficient functions appearing in \((15)\) and \((16)\)

The Christoffel symbols for the metric \((4)\) and the coefficient functions in \((15)\) and \((16)\) have been computed using Maple 18. The non vanishing Christoffel symbols are

\[
\Gamma_{tt} = \frac{\partial_t B}{2B}, \quad \Gamma_{t\theta} = \frac{\partial_t B}{2B}, \quad \Gamma_{\phi\theta} = \frac{\partial_\phi D}{2D}, \quad \Gamma_{\phi\rho} = \frac{\partial_\phi D}{2D}, \quad \Gamma_{rr} = \frac{\partial_r B}{2A}, \quad \Gamma_{r\phi} = \frac{\partial_\phi A}{2A}, \quad \Gamma_{\phi\phi} = \frac{\partial_\phi D}{2C}, \quad \Gamma_{\phi\rho} = \frac{\partial_\rho D}{2C}, \quad \Gamma_{\rho\phi} = \frac{\partial_\phi C}{2C}, \quad \Gamma_{\rho\rho} = \frac{\partial_\rho C}{2C}, \quad \Gamma_{\rho\phi} = \frac{\partial_\phi C}{2C}, \quad \Gamma_{\phi\rho} = \frac{\partial_\rho C}{2C}. \tag{A1}
\]

Moreover, we have

\[
-\frac{\partial_r A}{2A} = \frac{f'}{2f} + \alpha \sqrt{F} \cos \vartheta, \quad \frac{\partial_r C}{2A} = \frac{f' \sqrt{F}}{g}, \quad -\frac{\partial_{\theta} A}{A} = -2 \alpha r \sqrt{F} \sin \vartheta, \tag{A4}
\]

\[
\frac{\partial_{\theta} C}{2C} = \frac{g'}{2g} - \alpha \vartheta \sqrt{F} \sin \vartheta, \quad \frac{\partial_{\theta} A}{2C} = \frac{\alpha g \sin \vartheta}{r f} \sqrt{F}, \quad -\frac{\partial_{C}}{C} = -\frac{2}{r} \sqrt{F}, \tag{A5}
\]

and

\[
-\frac{\varepsilon^2}{2} \frac{\partial_r B}{AB^2} + \frac{\varepsilon^2}{2} \frac{\partial_\rho D}{AD^2} = -\frac{1}{F \sqrt{F}} \left[ \frac{\varepsilon^2}{F} \left( \frac{f'}{2f} - \alpha \sqrt{F} \cos \vartheta \right) - \frac{\varepsilon^2}{F} \frac{\kappa^2}{\sqrt{g} \sin^2 \vartheta} \right], \tag{A6}
\]

\[
-\frac{\varepsilon^2}{2} \frac{\partial_\rho B}{CB^2} + \frac{\varepsilon^2}{2} \frac{\partial_\rho D}{CD^2} = -\frac{g \sin \vartheta}{F \sqrt{F}} \left[ \frac{\alpha \varepsilon^2}{\kappa^2 g \sin^2 \vartheta} - \frac{\varepsilon^2}{F} \frac{1}{r^4 F} \left( \frac{g'}{2g \sin \vartheta} + \frac{\cos \vartheta}{\sin^2 \vartheta} + \frac{\alpha}{r^3} \right) \right], \tag{A7}
\]

where prime denotes differentiation with respect to \(r\) and dot stands for differentiation with respect to \(\vartheta\).

Appendix B: Critical points for the Schwarzschild metric

First of all, it is straightforward to verify that the effective potential \((23)\) in the case of vanishing mass has a maximum at \(r_\gamma = 3M\) representing an unstable photon circular orbit. Next, we take another approach to get the radius of the photon sphere. Namely, we compute the critical point(s) of the geodesic equations \((15)\) and \((16)\) adapted to the Schwarzschild case, we impose that such point(s) also satisfy the corresponding constraint equation, and we show that there is only one critical point and it must coincide with the radius of the photon sphere. We start by observing that in the Schwarzschild case equation \((10)\) reduces to a trivial identity because of the spherical symmetry of the manifold. Hence, if \((r_k, \vartheta_k)\) denotes a critical point of the Schwarzschild geodesic equation where \(\vartheta_k\) can take on any value on the interval \([0, \pi]\), equation \((15)\) becomes

\[
-\frac{\varepsilon^2}{2} \frac{\tilde{f}'(r_k)}{\tilde{f}(r_k)} + \frac{\varepsilon^2}{2} \frac{\tilde{f}(r_k)}{r_k^2} = 0, \quad \tilde{f}(r) = 1 - \frac{2M}{r}. \tag{B1}
\]

Moreover, in the Schwarzschild case the constraint equation \((19)\) evaluated at \(r = r_k\) gives

\[
\varepsilon^2 \frac{\tilde{f}'(r_k)}{r_k^2} = \varepsilon^2. \tag{B2}
\]

We replace \((B2)\) into \((B1)\) to obtain the equation

\[
\frac{r_k - 3M}{r_k(r_k - 2M)} = 0 \tag{B3}
\]

from which it follows that \(r_k = r_\gamma\).

[1] Steven Weinberg, “Gravitation and Cosmology”, John Wiley and Sons, New York 1972
