FRACTIONAL ELLIPTIC PROBLEM WITH FINITE MANY CRITICAL HARDY–SOBOLEV EXponents

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Abstract. In this paper, we consider the following problem:

\[ (-\Delta)^s u - \frac{\zeta u}{|x|^{2s}} = \sum_{i=1}^k \frac{|u|_{2^*,\theta_i}^{2^*,\theta_i}}{|x|^\theta_i} \quad \text{in} \ R^N, \]

where \( N \geq 3, s \in (0, 1), \zeta \in \left[ 0, 4^s \frac{\Gamma(1 + 2s)}{\Gamma(1 + 2s)} \right), 2^*,\theta_i = \frac{2(N-\theta_i)}{N-2s} \] are the critical Hardy–Sobolev exponents, the parameters \( \theta_i \) satisfy a suitable assumption. Using Morrey space, refinement of Hardy–Sobolev inequality and variational method, we establish the existence of nonnegative solution. Our result generalizes the result obtained by Chen [Electronic J. Differ. Eq. (2018) 1–12 [12]].

1. Introduction

In this paper, we consider the following problem:

\[ (-\Delta)^s u - \frac{\zeta u}{|x|^{2s}} = \sum_{i=1}^k \frac{|u|_{2^*,\theta_i}^{2^*,\theta_i}}{|x|^\theta_i} \quad \text{in} \ R^N, \quad (P) \]

where \( N \geq 3, s \in (0, 1), \zeta \in \left[ 0, 4^s \frac{\Gamma(1 + 2s)}{\Gamma(1 + 2s)} \right), 2^*,\theta_i = \frac{2(N-\theta_i)}{N-2s} \) are the critical Hardy–Sobolev exponents, the parameters \( \theta_i \) satisfy the assumption:

\((H_1)\ 0 < \theta_1 < \cdots < \theta_k < 2s \) (\( k \in \mathbb{N}, \ 2 < k < \infty \)), and \( 2\theta_k - \theta_1 \in (0, 2s) \).

The fractional Laplacian \((-\Delta)^s\) of a function \( u : R^N \rightarrow R \) can be defined as

\[ (-\Delta)^s u = F^{-1}(|\xi|^{2s} F(u)(\xi)), \text{ for all } \xi \in R^N, \]

and for \( u \in C_0^\infty(R^N) \), where \( F(u) \) denotes the Fourier transform of \( u \). The operator \((-\Delta)^s\) in \( R^N \) is a nonlocal pseudo–differential operator taking the form

\[ (-\Delta)^s u(x) = C_{N,s} P.V. \int_{R^N} \frac{u(x) - u(y)}{|x - y|^{N+2s}} dy, \]

where P.V. is the Cauchy principal value and \( C_{N,s} \) is a normalization constant. The fractional power of Laplacian is the infinitesimal generator of Lévy stable diffusion process and arise in anomalous diffusion in plasma, population dynamics, geophysical fluid dynamics, flames propagation, minimal surfaces and game theory (see [3, 8, 17]).

In previous twenty years, the nonlocal elliptic problems have been investigated by many researchers, for example, [5, 27, 28, 29] for subcritical case (Sobolev),
for critical sobolev case, [14, 24, 38, 34] for critical Hardy–Littlewood–Sobolev case. Moreover, a great attention has been devoted to study the existence of solutions for the nonlocal problems with critical Hardy–Sobolev term. Yang [37] studied the following minimizing problem:

\[ H_{0, \theta} := \inf_{u \in D^{s, 2}(\mathbb{R}^N) \setminus \{0\}} \frac{\int_{\mathbb{R}^N} |(-\Delta)^s u(x)|^2 \, dx}{\left( \int_{\mathbb{R}^N} \frac{|u|^{2s \theta}}{|x|^s} \, dx \right)^{\frac{s}{s \theta}}}, \tag{1.1} \]

where \( s \in (0, \frac{N}{2}) \) and \( \theta \in (0, 2s) \). By using Morrey space, refinement of Hardy–Sobolev inequality and variational method, he showed that \( H_{0, \theta} \) is achieved by a positive, radially symmetric and strictly decreasing function.

Ghoussoub and Shakerian [20] investigated the following minimizing problem:

\[ H_{\zeta, \theta} := \inf_{u \in D^{s, 2}(\mathbb{R}^N) \setminus \{0\}} \frac{\int_{\mathbb{R}^N} |(-\Delta)^s u(x)|^2 \, dx - \zeta \int_{\mathbb{R}^N} \frac{|u|^2}{|x|^s} \, dx}{\left( \int_{\mathbb{R}^N} \frac{|u|^{2s \theta}}{|x|^s} \, dx \right)^{\frac{s}{s \theta}}}, \tag{1.2} \]

where \( s \in (0, 1) \), \( N \geq 2s \), \( \theta \in (0, 2s) \) and \( \zeta \in \left(-\infty, 4^* \frac{\Gamma(N+2s)}{\Gamma\left(\frac{N+2s}{2}\right)}\right) \). Applying Ekeland’s variational principle, for \( s \in (0, 1) \), \( \theta \in (0, 2s) \) and \( \zeta \in \left[0, 4^* \frac{\Gamma(N+2s)}{\Gamma\left(\frac{N+2s}{2}\right)}\right) \), they showed that any non–negative minimizer for \( H_{\zeta, \theta} \) is positive, radially symmetric and radially decreasing. Furthermore, they also considered the following problem in [20],

\[ (-\Delta)^s u - \zeta \frac{u}{|x|^{2s}} = \frac{|u|^{2^*_s \theta - 2}}{|x|^{\theta}} + |u|^{2^*_s - 2}u, \quad \text{in} \ \mathbb{R}^N, \tag{1.3} \]

where \( s \in (0, 1) \), \( N \geq 2s \), \( \theta \in (0, 2s) \), \( \zeta \in \left[0, 4^* \frac{\Gamma(N+2s)}{\Gamma\left(\frac{N+2s}{2}\right)}\right) \), and \( 2^*_s = \frac{2N}{N-2s} \) is the critical Sobolev exponent. They combined the \( s \)-harmonic extension with the concentration compactness principle to investigate the existence of solutions for problem (1.3).

Chen [12] extended the study of problem (1.3) to the following problem:

\[ (-\Delta)^s u - \zeta \frac{u}{|x|^{2s}} = \frac{|u|^{2^*_{\theta_1} - 2}}{|x|^{\theta_1}} + \frac{|u|^{2^*_{\theta_2} - 2}}{|x|^{\theta_2}}, \quad \text{in} \ \mathbb{R}^N, \tag{1.4} \]

where \( N \geq 3 \), \( s \in (0, 1) \), \( \zeta \in \left[0, 4^* \frac{\Gamma(N+2s)}{\Gamma\left(\frac{N+2s}{2}\right)}\right) \), \( \theta_1, \theta_2 \in (0, 2s) \) and \( \theta_1 \neq \theta_2 \). By using concentration compactness principle and mountain pass lemma, they obtain the existence of positive solutions to problem (1.3).

It is worth pointing out that there are many other kinds of problem involving two critical nonlinearities, such as the Laplacian \(-\Delta\) (see [23, 33, 39]), the \( p \)-Laplacian \(-\Delta_p \) (see [16]), the biharmonic operator \( \Delta^2 \) (see [4]), and the fractional operator \((-\Delta)^s\) (see [38, 34]).

A nature and interesting question arises: can we extend the study of problem (1.4) in the finite many critical Hardy–Sobolev exponents?

We answer above question in this paper. By using the refinement of Hardy–Sobolev inequality, Morrey space and Mountain Pass Theorem, we establish the
existence of nontrivial weak solutions of problem \((P)\). The main result of this paper is as follows.

**Theorem 1.1.** Let \(N \geq 3\), \(s \in (0, 1)\), \(\zeta \in \left[0, 4^* \frac{N+2s}{1-2s}\right)\) and \((H_1)\) hold. Then problem \((P)\) has a nonnegative solution.

**Remark 1.1.** Problem \((P)\) is invariant under the weighted dilation
\[ u \mapsto \tau^\frac{N-2s}{2} u(\tau x). \]
Therefore, it is well known that the mountain pass theorem does not yield critical points, but only the Palais–Smale sequences. In this type of situation, it is necessary to show the non–vanishing of Palais–Smale sequences. There are finite many critical Hardy–Sobolev exponents in problem \((P)\), it is difficult to show the non–vanishing of Palais–Smale sequences. In order overcome this difficult, we establish two new inequalities in Lemma 3.2 and Lemma 3.3. By using the inequalities, we show the non–vanishing of Palais–Smale sequences in Lemma 4.4.

**Remark 1.2.** The loss of compactness due to the critical Hardy–Sobolev exponent which makes it difficult to verify the \((PS)_c\) condition, where \(0 < c < c^*\) in Lemma 4.3.

In [12], by using concentration compactness principle, the author verified that the energy functional associated with problem (1.4) satisfied \((PS)_c\) condition. However, there are finite many critical Hardy–Sobolev exponents. Therefore, her method is not available. We overcome this difficult by the refinement of Hardy–Sobolev inequality and Morrey space.

## 2. Preliminaries

Recall that the space \(H^s(\mathbb{R}^N)\) is defined as
\[ H^s(\mathbb{R}^N) = \{ u \in L^2(\mathbb{R}^N) | (-\Delta)^{\frac{s}{2}} u \in L^2(\mathbb{R}^N) \}. \]
This space is endowed with the norm
\[ \|u\|_{H^s}^2 = \|(-\Delta)^{\frac{s}{2}} u\|^2 + \|u\|^2. \]
The space \(D^{s,2}(\mathbb{R}^N)\) is the completion of \(C_0^\infty(\mathbb{R}^N)\) with respect to the norm
\[ \|u\|_{D}^2 = \|(-\Delta)^{\frac{s}{2}} u\|^2. \]
It is well known that \(\Lambda = 4^* \frac{\zeta^2 (N+2s)}{1-2s}\) is the best constant in the Hardy inequality
\[ \Lambda \int_{\mathbb{R}^N} \frac{|u|^2}{|x|^{2s}} dx \leq \|u\|_{D}^2, \text{ for any } u \in D^{s,2}(\mathbb{R}^N). \]

By Hardy inequality and \(\zeta \in [0, \Lambda)\), we derive that
\[ \|u\|_{D}^2 = \|u\|_{H^s}^2 - \zeta \int_{\mathbb{R}^N} \frac{|u|^2}{|x|^{2s}} dx, \]
is an equivalent norm in \(D^{s,2}(\mathbb{R}^N)\), and the following inequalities hold:
\[ \left(1 - \frac{\zeta}{\Lambda}\right) \|u\|_{D}^2 \leq \|u\|_{\zeta}^2 \leq \|u\|_{D}^2. \]
For $s \in (0, 1)$ and $\theta \in (0, 2s)$, we define the best constant:

\begin{equation}
S_s := \inf_{u \in D^{s,2}(\mathbb{R}^N) \setminus \{0\}} \frac{\|u\|_{D}^2}{\left( \int_{\mathbb{R}^N} |u|^{2s} \, dx \right)^{\frac{1}{2}}},
\end{equation}

where $S_s$ is attained in $\mathbb{R}^N$. For $s \in (0, 1)$, $\theta \in (0, 2s)$ and $\zeta \in [0, \Lambda)$, we define the best constant:

\begin{equation}
H_{\zeta, \theta} := \inf_{u \in D^{s,2}(\mathbb{R}^N) \setminus \{0\}} \frac{\|u\|_{D}^2 - \zeta \int_{\mathbb{R}^N} \frac{|u|^{2s}}{|x|^\theta} \, dx}{\left( \int_{\mathbb{R}^N} \frac{|u|^{2s}}{|x|^\theta} \, dx \right)^{\frac{1}{2}}},
\end{equation}

where $H_{\zeta, \theta}$ is attained in $\mathbb{R}^N$ (see [20]). A measurable function $u : \mathbb{R}^N \to \mathbb{R}$ belongs to the Morrey space $\|u\|_{L^p, \varpi}(\mathbb{R}^N)$ with $p \in [1, \infty)$ and $\varpi \in (0, N]$ if and only if

$$
\|u\|_{L^p, \varpi}(\mathbb{R}^N) = \sup_{R > 0, x \in \mathbb{R}^N} R^{\varpi - N} \int_{B(x, R)} |u(y)|^p \, dy < \infty.
$$

**Lemma 2.1.** [26, Theorem 1] For $s \in (0, \frac{N}{2s})$, there exists $C_1 > 0$ such that for $\iota$ and $\vartheta$ satisfying $\frac{2}{2s} \leq \iota < 1$, $1 \leq \vartheta < 2^*_s = \frac{2N}{N - 2s}$, we have

$$
\left( \int_{\mathbb{R}^N} |u|^{2\iota} \, dx \right)^{\frac{1}{\iota}} \leq C_1 \|u\|_{D}^{1-\iota} \|u\|_{L^{\frac{2(N-2\iota)}{N-2s}}(\mathbb{R}^N)},
$$

for any $u \in D^{s,2}(\mathbb{R}^N)$.

We introduce the energy functional associated to problem (P) by

$$
I(u) = \frac{1}{2} \|u\|_2^2 - \sum_{i=1}^{k} \frac{1}{2^*_{s,\vartheta_i}} \int_{\mathbb{R}^N} \frac{|u|^{2*_{s,\vartheta_i}}}{|x|^\vartheta_i} \, dx.
$$

The Nehari manifold associated with problem (P), which is defined by

$$
\mathcal{N} = \{u \in D^{s,2}(\mathbb{R}^N) | \langle I'(u), u \rangle = 0, \ u \neq 0 \},
$$

and

$$
c_0 = \inf_{u \in \mathcal{N}} I(u), \ c_1 = \inf_{u \in D^{s,2}(\mathbb{R}^N)} \max_{t \geq 0} I(tu) \text{ and } c = \inf_{\Upsilon \in \Gamma} \max_{t \in [0,1]} I(\Upsilon(t)),
$$

where $\Gamma = \{ \Upsilon \in C([0, 1], D^{s,2}(\mathbb{R}^N)) : \Upsilon(0) = 0, I(\Upsilon(1)) < 0 \}$.

### 3. Some key inequalities

In this section, we show some key inequalities.

In [37, 38], the authors obtained the refinement of Hardy–Sobolev inequality. However, their parameter $\tilde{\vartheta}$ satisfying (see [37, Theorem 1])

$$
1 \leq \tilde{\vartheta} < 2^*_s.
$$

It is easy to see that

$$
2^*_{s,\vartheta} = \frac{2(N - \vartheta)}{N - 2s} < \frac{2N}{N - 2s} = 2^*_s,
$$

for $s \in (0, \frac{N}{2})$ and $\theta \in (0, 2s)$. It is natural to ask the case of $\tilde{\vartheta} \in [2^*_s, 2^*_s \vartheta]$. In next lemma, we extend the parameter $\tilde{\vartheta}$ from $[1, 2^*_s]$ to $[1, 2^*_s]$. 
Lemma 3.1. [Refinement of Hardy–Sobolev inequality] For \( s \in (0, \frac{N}{2}) \) and \( \theta \in (0,2s) \), there exists \( C_2 > 0 \) such that for \( \nu \) and \( \vartheta \) satisfying \( \frac{\nu}{2s} \leq \nu < 1 \), \( 1 \leq \vartheta < 2^* \), we have

\[
\left( \int_{\mathbb{R}^N} \frac{|u|^{2s,\vartheta}}{|x|^\vartheta} \, dx \right)^{\frac{1}{2s,\vartheta}} \leq C_2 \| u\|_D^{\frac{\theta(N-2s)+\vartheta(N-2s-\theta)}{2\theta(N-2s)}} \| u\|_L^{N(1-\theta)(2s-\theta)} \| \cdot \|_2^{\frac{N_0}{2s}} (\mathbb{R}^N),
\]

for any \( u \in D^{s,2}(\mathbb{R}^N) \).

Proof. By using Hölder inequality and fractional Hardy inequality, we obtain

\[
\int_{\mathbb{R}^N} \frac{|u|^{2s,\vartheta}}{|x|^\vartheta} \, dx = \int_{\mathbb{R}^N} \frac{|u|^{\frac{\vartheta}{2}}}{|x|^\vartheta} \cdot |u|^{\frac{2(N-\vartheta)}{N-2s}} \, dx
\]

\[
\leq \left( \int_{\mathbb{R}^N} \frac{|u|^{\frac{\vartheta}{2}}}{|x|^\vartheta} \, dx \right)^{\frac{\vartheta}{2}} \left( \int_{\mathbb{R}^N} |u|^{\frac{(2s-\vartheta)N}{(N-2s)2s}} \, dx \right)^{1-\frac{\vartheta}{2}}
\]

(3.1)

Combining (3.1) and Lemma 2.1, we have

\[
\left( \int_{\mathbb{R}^N} \frac{|u|^{2s,\vartheta}}{|x|^\vartheta} \, dx \right)^{\frac{1}{2s,\vartheta}} \leq \left( \frac{1}{A} \right)^{\frac{\theta(N-2s)}{2(\nu+N-2s)}} \| u\|_D^{\frac{\theta(N-2s)}{2(\nu+N-2s)}} \| u\|_L^{\frac{N(2s-\vartheta)}{N(N-2s)}} (\mathbb{R}^N)
\]

\[
\leq \left( \frac{1}{A} \right)^{\frac{\theta(N-2s)}{2(\nu+N-2s)}} \| u\|_D^{\frac{\theta(N-2s)}{2(\nu+N-2s)}} \left( C_1 \| u\|_D \| u\|_{L^{2^*(\nu+N-2s)}} \right)^{1-\nu} \| u\|_L^{\frac{N(2s-\vartheta)}{N(N-2s)}} (\mathbb{R}^N)
\]

\[
= C_2 \| u\|_D \| u\|_L^{\frac{N(1-\theta)(2s-\theta)}{2(N-2s)}} (\mathbb{R}^N),
\]

\[
\square
\]

Lemma 3.2. Let \( s \in (0, \frac{N}{2}) \) and \( 0 < \theta < \tilde{\theta} < 2s \). Then the inequality

\[
\int_{\mathbb{R}^N} \frac{|u|^{2s,\vartheta}}{|x|^\vartheta} \, dx \leq \left( \int_{\mathbb{R}^N} \frac{|u|^{2s,\vartheta}}{|x|^\vartheta} \, dx \right)^{\frac{\theta}{\tilde{\theta}}} \left( \int_{\mathbb{R}^N} |u|^{2s} \, dx \right)^{\frac{\tilde{\theta}-\theta}{\tilde{\theta}}},
\]

holds for all \( u \in D^{s,2}(\mathbb{R}^N) \).
Since 0 < \theta < 2s, we obtain

\begin{align*}
\int_{\mathbb{R}^N} \frac{|u|^{2^*_\theta}}{|x|^{\theta}} \, dx &= \int_{\mathbb{R}^N} \frac{|u|^{\frac{\theta}{2}} \frac{2^*(N-\theta)}{N-2\theta}}{|x|^{\theta}} \cdot |u|^{\frac{\theta}{2}} \frac{2^*(N-\theta)}{N-2\theta} \, dx \\
&\leq \left( \int_{\mathbb{R}^N} \frac{|u|^{\frac{\theta}{2}} \frac{2^*(N-\theta)}{N-2\theta}}{|x|^{\theta}} \, dx \right)^{\frac{\theta}{2}} \left( \int_{\mathbb{R}^N} |u|^{\frac{\theta}{2}} \frac{2^*(N-\theta)}{N-2\theta} \, dx \right)^{\frac{\theta}{2}} \\
&= \left( \int_{\mathbb{R}^N} \frac{|u|^{2^*(N-\theta)}}{|x|^{\theta}} \, dx \right)^{\frac{\theta}{2}} \left( \int_{\mathbb{R}^N} |u|^{\frac{2^*(N-\theta)}{N-2\theta}} \, dx \right)^{\frac{\theta}{2}}.
\end{align*}

\textbf{Lemma 3.3.} Let \( s \in (0, \frac{N}{2}) \), \( 0 < \bar{\theta} < \theta < 2s \) and \( 2\theta - \bar{\theta} < 2s \). Then the inequality

\begin{align*}
\int_{\mathbb{R}^N} \frac{|u|^{2^*_\theta}}{|x|^{\theta}} \, dx \leq \left( \int_{\mathbb{R}^N} \frac{|u|^{2^*_\theta}}{|x|^{\theta}} \, dx \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}^N} |u|^{2^*(2\theta - \bar{\theta})} \, dx \right)^{\frac{1}{2}}
\end{align*}

holds for all \( u \in D^{s,2}(\mathbb{R}^N) \).

\textbf{Proof.} For any \( u \in D^{s,2}(\mathbb{R}^N) \), by using Hölder inequality and 0 < \bar{\theta} < \theta < 2s, we obtain

\begin{align*}
\int_{\mathbb{R}^N} \frac{|u|^{2^*_\theta}}{|x|^{\theta}} \, dx &= \int_{\mathbb{R}^N} \frac{|u|^{\frac{\theta}{2}} \frac{2^*(N-\theta)}{N-2\theta}}{|x|^{\theta}} \cdot |u|^{\frac{\theta}{2}} \frac{2^*(N-\theta)}{N-2\theta} \, dx \\
&\leq \left( \int_{\mathbb{R}^N} \frac{|u|^{\frac{\theta}{2}} \frac{2^*(N-\theta)}{N-2\theta}}{|x|^{\theta}} \, dx \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}^N} |u|^{\frac{2^*(N-\theta)}{N-2\theta}} \, dx \right)^{\frac{1}{2}}.
\end{align*}

Since 0 < 2\theta - \bar{\theta} < 2s, we get

\begin{align*}
\int_{\mathbb{R}^N} \frac{|u|^{2^*_\theta}}{|x|^{\theta}} \, dx \leq \left( \int_{\mathbb{R}^N} \frac{|u|^{2^*_\theta}}{|x|^{\theta}} \, dx \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}^N} |u|^{2^*(2\theta - \bar{\theta})} \, dx \right)^{\frac{1}{2}}.
\end{align*} \qedhere

4. The proof of Theorem 1.1

In this section, we show the existence of nonnegative solution of problems (P). We prove some properties of the Nehari manifold associated with problem (P).

\textbf{Lemma 4.1.} Assume that the assumptions of Theorem 1.1 hold. Then

\[ c_0 = \inf_{u \in \mathcal{N}} I(u) > 0. \]

\textbf{Proof.} We divide our proof into two steps.

\textbf{Step 1.} We claim that any limit point of a sequence in \( \mathcal{N} \) is different from zero.

According to \( \langle I'(u), u \rangle = 0 \) and (2.2), for any \( u \in \mathcal{N} \), we obtain

\[ 0 = \langle I'(u), u \rangle \geq \|u\|_2^2 - \sum_{i=1}^{k} \frac{1}{H_{\xi,\delta_i}^N} \|u\|^{2^*_{\theta_i}}. \]
From above expression, we have

\[(4.1) \quad \|u\|_\zeta^2 \leq \sum_{i=1}^{k} \frac{1}{\|u\|_{\zeta, \theta_i}^{2s, \theta_i}}.\]

Set

\[\kappa := \sum_{i=1}^{k} \frac{1}{\|u\|_{\zeta, \theta_i}^{2s, \theta_i}}.\]

Applying (2.2), we get

\[0 < \kappa < \infty.\]

From (H1), we know

\[2 < 2^{s, \theta_k} < \cdots < 2^{s, \theta_i} < 2^{s}.\]

Now the proof of Step 1 is divided into two cases: (i) \(\|u\|_\zeta \geq 1\); (ii) \(\|u\|_\zeta < 1\).

**Case (i) \(\|u\|_\zeta \geq 1\).** From (4.1), we have

\[\|u\|_\zeta^2 \leq \kappa \|u\|_{\zeta, \theta_1}^{2s, \theta_1},\]

which implies that

\[(4.2) \quad \|u\|_\zeta \geq \kappa^{\frac{1}{2s, \theta_1}}.\]

**Case (ii) \(\|u\|_\zeta < 1\).** From (4.1), we know

\[(4.3) \quad \|u\|_\zeta \geq \kappa^{\frac{1}{2s, \theta_k}}.\]

Combining (4.2) and (4.3), we deduce that

\[(4.4) \quad \|u\|_\zeta \geq \begin{cases} \kappa^{\frac{1}{2s, \theta_1}}, & \kappa < 1, \\ \kappa^{\frac{1}{2s, \theta_k}}, & \kappa \geq 1. \end{cases}\]

Hence, we know that any limit point of a sequence in \(\mathcal{N}\) is different from zero.

**Step 2.** Now, we claim that \(I\) is bounded from below on \(\mathcal{N}\). For any \(u \in \mathcal{N}\), by using (4.4), we get

\[I(u) \geq \left(1 - \frac{1}{2s, \theta_k} \right) \|u\|_\zeta^2 \geq \begin{cases} \left(\frac{1}{2} - \frac{1}{2s, \theta_k} \right) \kappa^{\frac{2}{2s, \theta_1}}, & \kappa \leq 1, \\ \left(\frac{1}{2} - \frac{1}{2s, \theta_k} \right) \kappa^{\frac{2}{2s, \theta_k}}, & \kappa > 1. \end{cases}\]

Therefore, \(I\) is bounded from below on \(\mathcal{N}\), and \(c_0 > 0\). \(\square\)

**Lemma 4.2.** Assume that the assumptions of Theorem 1.1 hold. Then

(i) for each \(u \in D^{s,2}(\mathbb{R}^N) \setminus \{0\}\), there exists a unique \(t_u > 0\) such that \(t_u u \in \mathcal{N}\);

(ii) \(c_0 = c_1 = c > 0\).

**Proof.** The proof is standard, so we sketch it. Further details can be derived as in the proofs of Theorem 4.1 and 4.2 in [36]. We omit it. \(\square\)

We show that the functional \(I\) satisfies the Mountain–Pass geometry, and estimate the Mountain–Pass levels.
Lemma 4.3. Assume that the assumptions of Theorem 1.1 hold. Then there exists a \((PS)_c\) sequence of \(I\) at level \(c\), where

\[
0 < c < c^* = \min \left\{ \frac{2s - \theta_1}{2(N - \theta_1)} H_{S, \theta_1}^{N - \theta_1}, \ldots, \frac{2s - \theta_k}{2(N - \theta_k)} H_{S, \theta_k}^{N - \theta_k} \right\}.
\]

Proof. The proof is standard, so we sketch it. Further details can be derived as in the proofs of Theorem 2 in [16], we omit it. \(\square\)

The following result implies the non–vanishing of \((PS)_c\) sequence.

Lemma 4.4. Assume that the assumptions of Theorem 1.1 hold. Let \(\{u_n\}\) be a \((PS)_c\) sequence of \(I\) with \(c \in (0, c^*)\), then

\[
\lim_{n \to \infty} \int_{\mathbb{R}^N} |u_n|^{2s, \theta_i} |x|^{\theta_i} \, dx > 0, (i = 1, \ldots, k).
\]

Proof. It is easy to see that \(\{u_n\}\) is uniformly bounded in \(D^{s,2}(\mathbb{R}^N)\). The proof of this Lemma is divided into three cases:

1. \(\lim_{n \to \infty} \int_{\mathbb{R}^N} |u_n|^{2s, \theta_1} |x|^{\theta_1} \, dx > 0;\)
2. \(\lim_{n \to \infty} \int_{\mathbb{R}^N} |u_n|^{2s, \theta_k} |x|^{\theta_k} \, dx > 0;\)
3. \(\lim_{n \to \infty} \int_{\mathbb{R}^N} |u_n|^{2s, \theta_j} |x|^{\theta_j} \, dx > 0, (j = 2, \ldots, k - 1);\)

Case 1. Suppose on the contrary that

\[
\lim_{n \to \infty} \int_{\mathbb{R}^N} |u_n|^{2s, \theta_1} |x|^{\theta_1} \, dx = 0.
\]

From \((H_1)\), we know

\[
0 < 2\theta_2 - \theta_1 < \cdots < 2\theta_k - \theta_1 < 2s.
\]

Since \(\{u_n\}\) is uniformly bounded in \(D^{s,2}(\mathbb{R}^N)\), there exists a constant \(0 < C < \infty\) such that \(\|u_n\|_D \leq C\). Applying (4.6) and (2.2), we obtain

\[
\lim_{n \to \infty} \int_{\mathbb{R}^N} |u_n|^{2s, \theta_i} |x|^{2\theta_i - \theta_1} \, dx \leq C, (i = 2, \ldots, k).
\]

According to Lemma 3.3, (4.5) and (4.7), we obtain

\[
\lim_{n \to \infty} \int_{\mathbb{R}^N} |u_n|^{2s, \theta_i} \, dx = 0 (i = 2, \ldots, k).
\]

By using (4.5), (4.8) and the definition of \((PS)_c\) sequence, we obtain

\[
c + o(1) = \frac{1}{2} \|u_n\|^2,\]

and

\[
o(1) = \|u_n\|^2.
\]

These yield \(c = 0\). This contradicts with \(c > 0\).
Case 2. Suppose on the contrary that

$$\lim_{n \to \infty} \int_{\mathbb{R}^N} \frac{|u_n|^{2^*_s \theta_k}}{|x|^\theta_k} \, dx = 0. \tag{4.9}$$

By using (2.1) and $$\|u_n\|_D \leq C$$, we have

$$\lim_{n \to \infty} \int_{\mathbb{R}^N} |u_n|^{2^*_s} \, dx \leq C. \tag{4.10}$$

Applying ($H_1$), Lemma 3.2, (4.9) and (4.10), we obtain

$$\lim_{n \to \infty} \int_{\mathbb{R}^N} \frac{|u_n|^{2^*_s \theta_i}}{|x|^\theta_i} \, dx \leq \left( \lim_{n \to \infty} \int_{\mathbb{R}^N} \frac{|u_n|^{2^*_s \theta_k}}{|x|^\theta_k} \, dx \right)^{\frac{\theta_i}{\theta_k}} \left( \lim_{n \to \infty} \int_{\mathbb{R}^N} |u_n|^{2^*_s} \, dx \right)^{\frac{\theta_k - \theta_i}{\theta_k}} = 0 \quad (i = 1, \ldots, k - 1). \tag{4.11}$$

By using (4.9), (4.11) and the definition of ($PS_c$) sequence, similar to Case 1, we get

$$c = 0.$$  This is a contradiction.

Case 3. Set $$j \in [2, k - 1]$$. Suppose on the contrary that

$$\lim_{n \to \infty} \int_{\mathbb{R}^N} \frac{|u_n|^{2^*_s \theta_j}}{|x|^\theta_j} \, dx = 0. \tag{4.12}$$

From ($H_1$), we know

$$0 < 2\theta_{j+1} - \theta_j < \cdots < 2\theta_k - \theta_j < 2\theta_k - \theta_1 < 2s.$$  Similar to (4.8), we obtain

$$\lim_{n \to \infty} \int_{\mathbb{R}^N} \frac{|u_n|^{2^*_s \theta_i}}{|x|^\theta_i} \, dx = 0 \quad (i = j + 1, \ldots, k).$$

Similar to (4.11), we have

$$\lim_{n \to \infty} \int_{\mathbb{R}^N} \frac{|u_n|^{2^*_s \theta_i}}{|x|^\theta_i} \, dx = 0 \quad (i = 1, \ldots, j - 1).$$

Similar to Case 1, we get $$c = 0$$. This is a contradiction.  \(\square\)

The proof of Theorem 1.1: We divide our proof into five steps.

Step 1. Since $$\{u_n\}$$ is a bounded sequence in $$D^{s,2}(\mathbb{R}^N)$$, up to a subsequence, we assume that

$$u_n \rightharpoonup u, \text{ in } D^{s,2}(\mathbb{R}^N), \quad u_n \to u, \text{ a.e. in } \mathbb{R}^N,$$

$$u_n \to u, \text{ in } L^r_{loc}(\mathbb{R}^N) \text{ for all } r \in [1, 2^*_s).$$

According to Lemma 3.1, and Lemma 4.4, there exists $$C > 0$$ such that

$$\|u_n\|_{L^{2,N-2s}(\mathbb{R}^N)} \geq C > 0.$$  On the other hand, since the sequence is bounded in $$D^{s,2}(\mathbb{R}^N)$$ and $$D^{s,2}(\mathbb{R}^N) \hookrightarrow L^{2^*_s}(\mathbb{R}^N) \hookrightarrow L^{2,N-2s}(\mathbb{R}^N)$$, we have

$$\|u_n\|_{L^{2,N-2s}(\mathbb{R}^N)} \leq C,$$
for some $C > 0$ independent of $n$. Hence, there exists a positive constant which we denote again by $C$ such that for any $n$ we obtain

$$C \leqslant \|u_n\|_{L^{2, N-2s}(\mathbb{R}^N)} \leqslant C^{-1}.$$  

So we may find $\sigma_n > 0$ and $x_n \in \mathbb{R}^N$ such that

$$\frac{1}{\sigma_n^2} \int_{B(x_n, \sigma_n)} |u_n(y)|^2 \, dy \geqslant \|u_n\|_{L^{2, N-2s}(\mathbb{R}^N)}^2 - \frac{C}{2n} \geqslant C_3 > 0.$$  

Let $\bar{u}_n(x) = \sigma_n^{\frac{N-2s}{4}} u_n(x_n + \sigma_n x)$. We may readily verify that

$$\tilde{I}(\bar{u}_n) = I(u_n) \to c$$  

and $\tilde{I}(\bar{u}_n) \to 0$ as $n \to \infty$,

where

$$\tilde{I}(\bar{u}_n) = \frac{1}{2} \|\bar{u}_n\|_D^2 - \frac{1}{2} \int_{\mathbb{R}^N} \frac{|\bar{u}_n|^2}{|x + \frac{x_n}{\sigma_n}|^{2s}} \, dx - \sum_{i=1}^k \frac{1}{2s, \theta_i} \int_{\mathbb{R}^N} \frac{|\bar{u}_n|^{2s, \theta_i}}{|x + \frac{x_n}{\sigma_n}|^{\theta_i}} \, dx.$$  

Now, for all $\varphi \in D^{s, 2}(\mathbb{R}^N)$, we obtain

$$\langle \tilde{I}'(\bar{u}_n), \varphi \rangle = \langle I'(u_n), \varphi \rangle \leqslant \|I'(u_n)\|_{D^{-1}} \|\varphi\|_D$$  

and $\|\varphi\|_D = \|\varphi\|$, we get

$$\tilde{I}(\bar{u}_n) \to 0$$  

as $n \to \infty$.

Thus there exists $\bar{u}$ such that

$$u_n \rightharpoonup \bar{u}, \text{ in } D^{s, 2}(\mathbb{R}^N), \quad u_n \to \bar{u}, \text{ a.e. in } \mathbb{R}^N,$$  

$$u_n \to \bar{u}, \text{ in } L^r_{\text{loc}}(\mathbb{R}^N) \text{ for all } r \in [1, 2^*_s).$$

Then

$$\int_{B(0, 1)} |\bar{u}_n(y)|^2 \, dy = \frac{1}{\sigma_n^2} \int_{B(x_n, \sigma_n)} |u_n(y)|^2 \, dy \geqslant C_3 > 0.$$  

As a result, $\bar{u} \neq 0$.

**Step 2.** Now, we claim that $\left\{\frac{\bar{u}_n}{\sigma_n}\right\}$ is bounded. If $\frac{\bar{u}_n}{\sigma_n} \to \infty$, then for any $\varphi \in D^{s, 2}(\mathbb{R}^N)$, we get

$$\lim_{n \to \infty} \int_{\mathbb{R}^N} \frac{\bar{u}_n \varphi}{|x + \frac{x_n}{\sigma_n}|^{2s}} \, dx = 0$$  

and

$$\lim_{n \to \infty} \int_{\mathbb{R}^N} \frac{|\bar{u}_n|^{2s, \theta_i}}{|x + \frac{x_n}{\sigma_n}|^{\theta_i}} \, dx = 0.$$  

Since $\bar{u}_n \rightharpoonup \bar{u}$ weakly in $D^{s, 2}(\mathbb{R}^N)$, we know

$$\lim_{n \to \infty} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(\bar{u}_n(x) - \bar{u}_n(y))(\varphi(x) - \varphi(y))}{|x - y|^{N+2s}} \, dx \, dy = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(\bar{u}(x) - \bar{u}(y))(\varphi(x) - \varphi(y))}{|x - y|^{N+2s}} \, dx \, dy.$$  

Applying $\lim_{n \to \infty} \langle \tilde{I}'(\bar{u}_n), \varphi \rangle \to 0$, (4.13) and (4.14), we have

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(\bar{u}(x) - \bar{u}(y))(\varphi(x) - \varphi(y))}{|x - y|^{N+2s}} \, dx \, dy = 0.$$
Let \( \varphi = \tilde{u} \). Then \( \|\tilde{u}\|_D = 0 \), so \( u(x) = u(y) \) a.e. \((x, y) \in \mathbb{R}^N \times \mathbb{R}^N\), that is \( u = a \in \mathbb{R} \) a.e. in \( \mathbb{R}^N \) (see [5, Page 27, Line 12]).

If \( a \neq 0 \), then 0 < \( \|\tilde{u}\|_{L^2(\mathbb{R}^N)} \leq 4 \) \( \|\tilde{u}\|_D = 0 \), which is a contradiction.

If \( a = 0 \), then it contradicts with \( \tilde{u} \neq 0 \).

Hence, \( \{\frac{\tilde{x}_n}{\tilde{a}_n}\} \) is bounded.

**Step 3.** In this step, we study another \((PS)_c\) sequence of \( I \). Let \( \tilde{u}_n(x) = \sigma_n \tilde{x}_n u_n(\sigma_n x) \). Then we can verify that

\[
I(\tilde{u}_n) = I(u_n) \to c, \quad I'(\tilde{u}_n) \to 0 \quad \text{as} \quad n \to \infty.
\]

Arguing as before, we have

\( \tilde{u}_n \to \tilde{u} \) in \( D^{1,2}(\mathbb{R}^N) \), \( \tilde{u}_n \to \tilde{u} \) a.e. in \( \mathbb{R}^N \),

\( \tilde{u}_n \to \tilde{u} \) in \( L^2_{loc}(\mathbb{R}^N) \) for all \( r \in [1, 2^*_s) \).

Since \( \{\frac{\tilde{x}_n}{\tilde{a}_n}\} \) is bounded, there exists \( \tilde{R} > 0 \) such that

\[
\int_{B(0, \tilde{R})} |\tilde{u}_n(y)|^2 \, dy > \int_{B(\frac{\tilde{x}_n}{\tilde{a}_n}, 1)} |\tilde{u}_n(y)|^2 \, dy = \frac{1}{\sigma_n^2} \int_{B(x_n, \sigma_n)} |u_n(y)|^2 \, dy \geq C_3 > 0.
\]

As a result, \( \tilde{u} \neq 0 \).

**Step 4.** In this step, we show \( \tilde{u}_n \to \tilde{u} \) strongly in \( D^{s,2}(\mathbb{R}^N) \). It is easy to see that

\[
\langle I'(\tilde{u}), \varphi \rangle = 0.
\]

Combining (4.15) and \( \tilde{u} \neq 0 \), we get \( \tilde{u} \in \mathcal{N} \). Set

\[
K(u) = \sum_{i=1}^k \left( \frac{1}{2} - \frac{1}{2s_i} \right) \int_{\mathbb{R}^N} |u|^2 \frac{s_i}{|x|^{s_i}} \, dx.
\]

Applying Lemma 4.2, Brézis–Lieb lemma, \( \tilde{u} \in \mathcal{N} \) and Lemma 4.1, we obtain

\[
c_0 = c = I(\tilde{u}_n) - \frac{1}{2} \langle I'(\tilde{u}_n), \tilde{u}_n \rangle
\]

\[
I(\tilde{u}) - \frac{1}{2} \langle I'(\tilde{u}), \tilde{u} \rangle = I(\tilde{u}) \geq c_0.
\]

Therefore, the inequalities above have to be equalities. We know

\[
\lim_{n \to \infty} K(\tilde{u}_n) = K(\tilde{u}).
\]

By using Brézis–Lieb lemma again, we have

\[
\lim_{n \to \infty} K(\tilde{u}_n) - \lim_{n \to \infty} K(\tilde{u}_n - \tilde{u}) = K(\tilde{u}) + o(1).
\]

Hence, we deduce that

\[
\lim_{n \to \infty} K(\tilde{u}_n - \tilde{u}) = 0,
\]

which implies that

\[
\lim_{n \to \infty} \int_{\mathbb{R}^N} \frac{|\tilde{u}_n - \tilde{u}|^{2s_i}}{|x|^{s_i}} \, dx = 0, \quad \text{for all} \ i = 1, \ldots, k.
\]
According to $\langle \hat{I}'(\hat{u}_n), \hat{u}_n \rangle = o(1)$, $\langle \hat{I}'(\hat{u}), \hat{u} \rangle = 0$ and Brézis–Lieb lemma, we obtain

$$o(1) = \langle \hat{I}'(\hat{u}_n), \hat{u}_n \rangle - \langle \hat{I}'(\hat{u}), \hat{u} \rangle$$

$$= \|\hat{u}_n - \hat{u}\|_\zeta^2 \sum_{i=1}^k \int_{\mathbb{R}^N} \frac{|\hat{u}_n - \hat{u}|^{2^*_s, \eta_i}}{|x|^{\eta_i}} \, dx + o(1),$$

which implies that

$$\lim_{n \to \infty} \|\hat{u}_n - \hat{u}\|_\zeta^2 = \lim_{n \to \infty} \int_{\mathbb{R}^N} \frac{|\hat{u}_n - \hat{u}|^{2^*_s, \eta_i}}{|x|^{\eta_i}} \, dx + o(1).$$

Combining (4.17) and (4.18), we get

$$\lim_{n \to \infty} \|\hat{u}_n - \hat{u}\|_\zeta^2 = 0.$$

Since $\hat{u} \not\equiv 0$, we know that $\hat{u}_n \to \hat{u}$ strongly in $D^{s,2}(\mathbb{R}^N)$.

**Step 5.** By using (4.16) again, we know $\hat{I}(\hat{u}) = c$, which means that $\hat{u}$ is a nontrivial solution of problem (P) at the energy level $c$. We have

$$0 = \langle \hat{I}'(\hat{u}), \hat{u} \rangle$$

$$= \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(\hat{u}(x) - \hat{u}(y))(\hat{u}^-(x)-\hat{u}^-(y))}{|x-y|^{N+2s}} \, dx \, dy - \zeta \int_{\mathbb{R}^N} \frac{\hat{u} \hat{u}^-}{|x|^{2s}} \, dx$$

$$- \sum_{i=1}^k \int_{\mathbb{R}^N} \frac{|\hat{u}|^{2^*_s, \eta_i} - 2 \hat{u} \hat{u}^-}{|x|^{\eta_i}} \, dx,$$

where $\hat{u}^- = \max\{0, -\hat{u}\}$. For a.e. $x, y \in \mathbb{R}^N$, we obtain

$$(\hat{u}(x) - \hat{u}(y))(\hat{u}^-(x)-\hat{u}^-(y)) \leq -|\hat{u}^-(x)-\hat{u}^-(y)|^2.$$

Then, we get

$$0 \leq -\|\hat{u}^-\|_D^2 - \zeta \int_{\mathbb{R}^N} \frac{|\hat{u}^-|^2}{|x|^{2s}} \, dx - \sum_{i=1}^k \int_{\mathbb{R}^N} \frac{|\hat{u}^-|^{2^*_s, \eta_i}}{|x|^{\eta_i}} \, dx \leq -\|\hat{u}^-\|_D^2.$$

Thus, $\|\hat{u}^-\|_D^2 = 0$. Hence, we can choose $\hat{u} \geq 0$. \qed

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