CONSTRAINTS ON INTERSECTION FORMS OF SPIN 4-MANIFOLDS BOUNDED BY SEIFERT RATIONAL HOMOLOGY 3 SPHERES IN TERMS OF $\mu$ AND $\kappa$ INVARIANTS.

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Abstract. We give some constraints on intersection forms of spin 4-manifolds bounded by Seifert rational homology 3-spheres in terms of $\mu$-invariant, and compare them with those in terms of $\kappa$-invariant given by Manolescu. Furthermore in case of a Seifert rational homology 3-sphere, we show that the difference between $\kappa$ and $-\overline{\mu}$ is at most 2 (and there is an example where the difference is 2), and they coincide with each other under some extra conditions, including Floer $K_G$ split cases.

Let $Y$ be a closed 3-manifold with spin structure $s$. Then $(Y, s)$ bounds a compact smooth spin 4-manifold $(W, s_W)$, where $s_W$ is a spin structure of $W$ which coincides with $s$ on $Y$. In the case where $Y$ is a Seifert (or more generally a plumbed) rational homology 3-sphere, there is a lift $\overline{\mu}(Y, s)$ of the Rokhlin invariant $\mu(Y, s)$, which is called the $\overline{\mu}$-invariant defined by Neumann ([15]) and Siebenmann ([22]).

In this paper we give the following constraints on the intersection forms of compact spin 4-manifolds bounded by Seifert rational homology 3-spheres in terms of the $\overline{\mu}$ invariant. In this paper we only consider Seifert fibrations over orientable base 2-orbifolds.

**Theorem 1.** Let $(W, s_W)$ be a compact spin 4-manifold bounded by a Seifert rational homology 3-sphere with spin structure $(Y, s)$. Then $\overline{\mu}(Y, s) \equiv \sigma(W)/8 \pmod{2\mathbb{Z}}$ and

$$-b^{-}_{2}(W) + \frac{\sigma(W)}{8} \leq \overline{\mu}(Y, s) \leq b^{+}_{2}(W) + \frac{\sigma(W)}{8}$$

where $b^{+}_{2}(W)$ (resp. $b^{-}_{2}(W)$) is the dimension of the maximal positive definite (resp. negative definite) subspace of the intersection form of $W$, and $\sigma(W)$ is the signature of $W$. 


There are similar constraints on the intersection form of $W$ in terms of the $\kappa$-invariant $\kappa(Y, s)$ of $(Y, s)$ proved by Manolescu (which are valid for general rational homology 3-spheres). In a certain case $\kappa(Y, s)$ coincides with $-\overline{\mu}(Y, s)$, but they are not necessarily the same even in the case of Brieskorn homology 3-spheres ([14]). However by comparing the constraints in terms of $\kappa(Y, s)$ with the estimates given by $\mu(Y, s)$, we obtain the following result. If $Y$ is an integral homology 3-sphere, we denote these invariants simply by $\kappa(Y)$ and $\mu(Y)$ since $Y$ has a unique spin structure.

**Theorem 2.** Let $(Y, s)$ be a Seifert rational homology 3-sphere over a base 2-orbifold of genus 0 with spin structure.

1. We have the following inequality.

\[
0 \leq \kappa(Y, s) + \overline{\mu}(Y, s) \leq 2.
\]

2. If $(Y, s)$ is Floer $K_G$ split, then $\kappa(Y, s) + \overline{\mu}(Y, s) = 0$.

3. We have $0 \leq \kappa(Y, s) + \kappa(-Y, s) \leq 4$ (although $\kappa(Y, s) + \kappa(-Y, s)$ is not necessarily 0).

4. If one of the multiplicities of the singular fibers for the Seifert fibration of $Y$ is even and $\deg Y > 0$, then $\kappa(Y, s) = -\overline{\mu}(Y, s)$.

5. If $Y$ is a Seifert integral homology 3-sphere (with the unique spin structure), $\kappa(Y) + \overline{\mu}(Y) = 0$ or 2. The same claim holds for $-Y$, and hence $\kappa(Y) + \kappa(-Y) = 0$, 2, or 4.

Theorem 2 (4) can be extended to the case when all multiplicities are odd under certain extra conditions on $(Y, s)$ (Remark 5).

1. **The Fukumoto-Furuta Invariant and the Proof of Theorem 1.**

The proof of Theorem 1 is based on the estimates of the Fukumoto-Furuta invariant ([1]) and its relation to the $\overline{\mu}$-invariant of a Seifert rational homology 3-sphere, which we recall in this section.

**Definition 1.** Let $(X, s_X)$ be a compact spin 4-orbifold with spin structure $s_X$ bounded by a rational homology 3-sphere $(Y, s)$ with spin structure. Choose a compact spin 4-manifold $(X', s_{X'})$ with $\partial(X', s_{X'}) = \partial(Y, s)$ and $(

\overline{\mu}(X', s_{X'})$.
(Y, s) and put (Z, sZ) = (X \cup (−X'), sX \cup sX'). Then the Fukumoto-Furuta invariant w(Y, X, sX) is defined to be

\[ w(Y, X, sX) = -\text{ind } D_Z(sZ) + \frac{\sigma(X')}{8} \]

where \( D_Z(sZ) \) is the Dirac operator of (Z, sZ). The value of \( w(Y, X, sX) \) does not depend on the choice of \( X' \).

We note that our sign convention of \( w(Y, X, sX) \) is opposite to the original definition by Fukumoto-Furuta [4] and [5]. Let SingX be the set of all singularities of \( X \). Hereafter we only consider 4-orbifolds with isolated singularities. In such cases the link of every \( x \in \text{Sing}X \) is a spherical 3-manifold with spin structure induced from \( sX \). Then the regular neighborhood of \( x_i \in \text{Sing}X \) in \( X \) is a cone \( cS_i \) over a spherical 3-manifold \( S_i \), and \( X_0 = X \setminus \cup_{x_i \in \text{Sing}X} \text{Int } cS_i \) is a 4-manifold with spin structure \( sX_0 \) induced from \( sX \) satisfying \( \partial X_0 = Y \cup \cup_i (−S_i) \).

**Proposition 1.** [5] The orbifold index theorem shows that

\[ \text{ind } D_Z(sZ) = -\frac{1}{24} \int_Z p_1(Z) + \sum_i \delta^\text{Dir}(S_i, s_{S_i}), \]

\[ \sigma(Z) = \frac{1}{3} \int_Z p_1(Z) + \sum_i \delta^\text{sign}(S_i) \]

where \( p_1(Z) \) is the first Pontrjagin class of \( Z \), \( s_{S_i} \) is the spin structure of \( S_i \) induced from \( s_X \), \( \delta^\text{Dir}(S_i, s_{S_i}) \) and \( \delta^\text{sign}(S_i) \) are the contributions of \( x_i \in \text{Sing}X \) to the corresponding indices. Using these formulae \( w(Y, X, sX) \) is represented as follows.

\[ w(Y, X, sX) = \frac{1}{8}(\sigma(Z) + \sum_{x_i \in \text{Sing}X} \delta_{x_i}) + \frac{\sigma(X')}{8} = \frac{1}{8}(\sigma(X) + \sum_{x_i \in \text{Sing}X} \delta_{x_i}). \]

Here we put \( \delta_{x_i} = -(8\delta^\text{Dir}(S_i, s_{S_i}) + \delta^\text{sign}(S_i)) \).

For later use, we reformulate the contributions of \( \text{Sing}X \) to the above indices in terms of the \( \eta \) invariants ([3]).

**Proposition 2.** [26] Let \( g_i \) be a standard metric of \( S_i \) (which is induced from the metric of the constant curvature on the 3-sphere of radius 1 as the universal covering of \( S_i \)). Then we have

\[ \eta^\text{Dir}(S_i, s_{S_i}, g_i) = 2\delta^\text{Dir}(S_i, s_{S_i}), \quad \eta^\text{sign}(S_i, g_i) = \delta^\text{sign}(S_i). \]
Here $\eta^{\text{Dir}}(S_i, s_{S_i}, g_i)$ and $\eta^{\text{sign}}(S_i, g_i)$ are the $\eta$ invariants of the Dirac operator of $(S_i, s_{S_i}, g_i)$ and the signature operator of $(S_i, g_i)$ respectively. We note that the kernel of the Dirac operator of $(S_i, s_{S_i}, g_i)$ is 0 since $g_i$ is a metric of positive scalar curvature.

The value of $w(Y, X, s_X)$ depends on both $Y$ and $X$ in general, but if $Y$ is a Seifert rational homology 3-sphere, it is related to the $\overline{\mu}$ invariant of $Y$ by some particular choice of $X$.

**Definition 2.** Let $P(\Gamma)$ be a plumbed 4-manifold associated with a weighted graph $\Gamma$. We assume that a plumbed 3-manifold $Y = \partial P(\Gamma)$ is a rational homology 3-sphere. Thus $\Gamma$ is a tree such that each vertex $v$ with weight $e(v)$ of $\Gamma$ corresponds to a $D^2$-bundle over $S^2$ with Euler class $e(v)$. Let $\{e_v\}$ be a basis of $H_2(P(\Gamma), \mathbb{Z})$ each of which is represented by a zero section of the $D^2$-bundle over $S^2$ corresponding to $v$. If we fix a spin structure $s_Y$ of $Y = \partial P(\Gamma)$, there exists a unique characteristic element $w_\Gamma = \sum_v \epsilon_v e_v$ satisfying the following conditions.

- $\epsilon_v$ is either 0 or 1.
- The Poincaré dual $PD w_\Gamma \pmod{2} \in H^2(P(\Gamma), \partial P(\Gamma), \mathbb{Z}_2)$ of $w_\Gamma$ is a lift of $w_2(P(\Gamma))$, which is an obstruction to extending $s_Y$ to a spin structure of $P(\Gamma)$.

We denote by $\text{supp}(\Gamma)$ the set $\{v \in \Gamma | \epsilon_v = 1\}$. Then the $\overline{\mu}$ invariant of $(Y, s_Y)$ is defined to be

$$\overline{\mu}(Y, s_Y) = \frac{1}{8}(\sigma(P(\Gamma)) - w_\Gamma \cdot w_\Gamma).$$

The value of $\overline{\mu}(Y, s_Y)$ does not depend on $\Gamma$, although $\Gamma$ satisfying $Y = \partial P(\Gamma)$ is not unique.

**Remark 1.** The $\overline{\mu}$ invariant has the following properties.

1. $\overline{\mu}(Y, s_Y) \equiv \mu(Y, s_Y) \pmod{2\mathbb{Z}}.$ If $Y$ is a plumbed integral homology 3-sphere with the unique spin structure, $\overline{\mu}(Y)$ is an integral lift of $\mu(Y) \in \mathbb{Z}_2$.

2. If $Y$ is a Seifert rational homology 3-sphere (with base 2-orbifold of genus 0), then $Y$ is a plumbed 3-manifold corresponding to a star-shaped graph.
In some cases the $\overline{\mu}$ invariant is related to other spin cobordism invariants. For example, the relations between $\overline{\mu}(Y, s)$ and the correction term $d(Y, s)$ of the Heegaard Floer homology \([13]\) are discussed in \([26]\), \([21]\). If $Y$ is a Seifert integral homology 3-sphere, $-\overline{\mu}(Y)$ coincides with the $\beta$-invariant $\beta(Y)$ defined by Manolescu \([13]\) (Stoffregen \([23]\)), and the same equality holds if $Y$ is a plumbed rational homology 3-sphere corresponding to a graph with at most one bad vertex (Dai \([2]\)). In case of a Seifert integral homology 3-sphere, the relations between $\overline{\mu}(Y)$ and the invariants $\alpha(Y)$, $\gamma(Y)$ and $d(Y)$ are also determined in \([23]\). More generally, $d(Y, s) = -2\overline{\mu}(Y, s)$ for every almost rational plumbed rational homology 3-sphere $Y$ (Dai-Manolescu \([3]\)), where $d(Y, s)$ is the correction term of the involutive Heegaard Floer homology \([8]\). In all above cases, the $\overline{\mu}$ invariant is a spin rational homology cobordism invariant.

The relation between the $\overline{\mu}$ invariant and the Fukumoto-Furuta invariant for a Seifert fibration is discussed in \([4]\), \([5]\), \([19]\), \([24]\), \([25]\). If a Seifert rational homology 3-sphere $Y$ has a fibration over a 2-orbifold $S^2(a_1, \cdots, a_n)$ of genus 0 with singular points of multiplicity $a_i$ (1 $\leq$ $i$ $\leq$ $n$), then $Y$ is represented by Seifert invariants of the form $\{b, (a_1, b_1), \ldots, (a_n, b_n)\}$ satisfying $0 < b_i < a_i$, $\gcd(a_i, b_i) = 1$ (1 $\leq$ $i$ $\leq$ $n$), and the degree of $Y$ is defined to be $\deg Y = b + \sum_{i=1}^{n} b_i/a_i$ (we follow the definition of \([10]\), whose sign convention is opposite to that in \([25]\), where we follow the definition of \([16]\)). In case of a Seifert rational homology 3-sphere, we have the following result.

**Proposition 3.** \([25]^{[1]}\) Let $(Y, s)$ be a Seifert rational homology 3-sphere with spin structure. Then there exist compact spin 4-orbifolds $(X^\pm, s_{X^\pm})$ with $\partial(X^\pm, s_{X^\pm}) = (Y, s)$ satisfying the following conditions.

1. $b_1(X^\pm) = 0$, $b_2^+(X^\pm) \leq 1$, and $b_2^-(X^-) \leq 1$.
2. $w(Y, X^+, s_{X^+}) = w(Y, X^-, s_{X^-}) = \overline{\mu}(Y, s)$.

If $Y$ is spherical, then we can choose $X^\pm$ so that both of them are the cone over $Y$. Suppose that one of the multiplicities of the singular fibers

\[1\]The definitions of $w$ and $\overline{\mu}$ invariants in \([25]\) and \([26]\) are both 8 times those in this paper.
for the Seifert fibration of $Y$ is even. Then we can choose $X^\pm$ so that
$X^+ = X^-$, satisfying $b_2^+(X^\pm) = 1$ and $b_2^-(X^\pm) = 0$ if $\deg Y > 0$, and
$b_2^-(X^\pm) = 0$ and $b_2^+(X^\pm) = 1$ if $\deg Y < 0$.

To construct $X^\pm$, we choose appropriate weighted graphs $\Gamma^\pm$ satisfying
$\partial P(\Gamma^\pm) = Y$ and subgraphs $\Gamma_0^\pm$ of $\Gamma^\pm$ such that $\Gamma_0^\pm$ contain
$\text{supp}(\Gamma^\pm)$ and each of $\Sigma^\pm := \partial P(\Gamma_0^\pm)$ is a union of spherical 3-manifolds
(in fact lens spaces). Then embed $P(\Gamma_0^\pm)$ in $P(\Gamma^\pm)$ and replace their
images with the cones over $S^\pm$ to obtain $X^\pm$. Such $X^\pm$ are constructed
by Saveliev [19] for Seifert integral homology 3-spheres, and the result
is extended to the case of Seifert rational homology 3-spheres in [25].

The proof of Theorem 1 is based on Proposition 3 and the following
orbifold 10/8-theorem.

**Theorem 3.** [4] Let $(Z, s_Z)$ be a closed spin 4-orbifold with spin structure $s_Z$. Then

(1) $\text{ind} D_Z(s_Z) \equiv 0 \pmod{2}$,

(2) $\text{ind} D_Z(s_Z) = 0$ or otherwise

$$1 - b_2^-(Z) \leq \text{ind} D_Z(s_Z) \leq b_2^+(Z) - 1.$$ 

For a compact spin manifold $(W, s_W)$ with $\partial(W, s_W) = (Y, s)$, let
$(Z^\pm, s_{Z^\pm}) = (X^\pm, s_{X^\pm}) \cup (-W, s_W)$ be closed spin 4-orbifolds where $X^\pm$
are the 4-orbifolds chosen in Proposition 3. Then applying Theorem 3
to $(Z^\pm, s_{Z^\pm})$ we have $\overline{\mu}(Y, s) \equiv \frac{\sigma(W)}{8} \pmod{2Z}$ and

$$\overline{\mu}(Y, s) = w(Y, X^-, s_{X^-}) = -\text{ind} D_Z-(s_{Z^-}) + \frac{\sigma(W)}{8} \leq b_2^-(Z^-) - 1 + \frac{\sigma(W)}{8}$$

$$= b_2^-(X^-) + b_2^+(W) - 1 + \frac{\sigma(W)}{8} \leq b_2^+(W) + \frac{\sigma(W)}{8},$$

$$\overline{\mu}(Y, s) = w(Y, X^+, s_{X^+}) = -\text{ind} D_Z+(s_{Z^+}) + \frac{\sigma(W)}{8} \geq 1 - b_2^+(Z^+) + \frac{\sigma(W)}{8}$$

$$= 1 - b_2^-(W) - b_2^+(X^+) + \frac{\sigma(W)}{8} \geq -b_2^-(W) + \frac{\sigma(W)}{8}.$$ 

The above inequalities hold even if $\text{ind} D_Z^\pm(s_{Z^\pm}) = 0$. The second
inequality is also deduced from the first one with respect to $X^\pm$ chosen
for $-Y$ and $-W$, since $\overline{\mu}(-Y) = -\overline{\mu}(Y)$. This completes the proof of
Theorem 1. We will use Proposition 3 to prove Theorem 2 in §3.
Remark 2. In some cases we obtain slightly better estimates than those in Theorem \textup{1} by the above inequalities. If \( Y \) is a spherical 3-manifold \( S \), we can choose \( X^\pm \) so that \( X^+ = X^- = cS \) and hence applying Theorem \textup{3} to \( Z = cS \cup (-W) \) we have either

\[
1 - b_2^-(W) + \frac{\sigma(W)}{8} \leq \overline{\mu}(S, s) \leq b_2^+(W) - 1 + \frac{\sigma(W)}{8}
\]

and \( \overline{\mu}(S, s) \equiv \frac{\sigma(W)}{8} \pmod{2\mathbb{Z}} \), or \( \overline{\mu}(S, s) = \frac{\sigma(W)}{8} \).

If \( Y \) has a fibration over \( S^2(a_1, \ldots, a_n) \) and one of \( a_i \) is even, then we can choose \( X^\pm \) so that \( X^+ = X^- \) as in Proposition \textup{3}. If \( \deg Y > 0 \), \( X^\pm \) satisfies \( b_2^+(X^\pm) = 1 \), \( b_2^-(X^\pm) = 0 \), and hence we have either

\[
-b_2^-(W) + \frac{\sigma(W)}{8} \leq \overline{\mu}(Y, s) \leq b_2^+(W) - 1 + \frac{\sigma(W)}{8}
\]

or \( \overline{\mu}(Y, s) = \frac{\sigma(W)}{8} \). If \( \deg Y < 0 \), then since \( X^\pm \) satisfies \( b_2^+(X^\pm) = 0 \), \( b_2^-(X^\pm) = 1 \), we have either

\[
1 - b_2^-(W) + \frac{\sigma(W)}{8} \leq \overline{\mu}(Y, s) \leq b_2^+(W) + \frac{\sigma(W)}{8}
\]

or \( \overline{\mu}(Y, s) = \frac{\sigma(W)}{8} \). In either case we have \( \overline{\mu}(Y, s) \equiv \frac{\sigma(W)}{8} \pmod{2\mathbb{Z}} \).

If \( Y \) is a Seifert integral homology 3-sphere, then \( Y \) is uniquely determined by \( a_i \)'s up to orientation, and is denoted by \( \Sigma(a_1, \ldots, a_n) \) if \( \deg Y < 0 \). Then \( \Sigma(a_1, \ldots, a_n) \) bounds an associated orbifold disk bundle \( D(a_1, \ldots, a_n) \) over \( S^2(a_1, \ldots, a_n) \). If one of \( a_i \) is even, then \( D(a_1, \ldots, a_n) \) is spin and \( b_2^+(D(a_1, \ldots, a_n)) = 1 \), \( b_2^-(D(a_1, \ldots, a_n)) = 0 \) (\textup{13}). Hence putting \( X^\pm = D^2(a_1, \ldots, a_n) \) we also obtain the same inequality. But in other cases \( D(a_1, \ldots, a_n) \) is not necessarily spin, so we cannot use it.

2. The \( \kappa \) invariant for a rational homology 3-sphere.

For a rational homology 3-sphere, \( K \)-theoretic invariants are defined via the theory of the Pin(2) Seiberg-Witten-Floer stable homotopy type (\textup{11}, \textup{12}) in \textup{6}, \textup{14}. In this section we recall the definition of Manolescu’s \( \kappa \) invariant for a rational homology 3-sphere.
with spin structure \((Y, \mathfrak{s})\) and its properties \cite{14}. Hereafter we put \(G = \text{Pin}(2) = S^1 \cup jS^1\).

**Definition 3.** \cite{14}

1. Let \(\tilde{R}\) (resp. \(\tilde{C}\)) be a real (resp. complex) 1-dimensional \(G\)-representation space where \(G\) acts as multiplication by \(\pm 1\) via the natural projection \(G \to G/S^1 \cong \{\pm 1\}\), and let \(H\) be the set of quaternions on which \(G\) acts as left multiplication. Then the representation ring \(R(G)\) is generated by \(z = 2 - H\) and \(w = 1 - \tilde{C}\) with relations \(w^2 = wz = 2w\).
2. A compact \(G\)-space \(X\) is called a space of type SWF at level \(s\) if the fixed point set \(X^{S^1}\) of the action of \(S^1 \subset G\) on \(X\) is \(G\)-homotopy equivalent to \((\tilde{R}^s)^+\) and \(G\) acts freely on \(X \setminus X^{S^1}\).

**Definition 4.** \cite{14} Let \(X\) be a space of type SWF at level 2 (in which case \(X^{S^1}\) is \(G\) homotopy equivalent to \((\tilde{C}^0)^+\)). Then the inclusion \(i : X^{S^1} \to X\) induces the map \(i^* : \tilde{K}_G(X) \to \tilde{K}_G(X^{S^1})\) whose image is represented as \(J(X) \cdot \beta((\tilde{C}^0)^+)\), where \(J(X)\) is an ideal of \(R(G) \cong \tilde{K}_G(S^0)\) and \(\beta((\tilde{C}^0)^+)\) is the Bott class of \(\tilde{K}_G((\tilde{C}^0)^+)\).

1. There exists \(k \geq 0\) such that both \(w^k\) and \(z^k\) belong to \(J(X)\), so we can define \(k(X) = \min\{k \geq 0 \mid \text{there exists } x \in J(X) \text{ such that } wx = z^k w\}\).
2. \(X\) is called \(K_G\) split if \(J(X) = (z^k)\) for some \(k \geq 0\).

**Lemma 1.** \cite{14}

1. Let \(f : X \to X'\) is a \(G\) equivariant map between two spaces of type SWF at level 2\(t\), and assume that \(f^{S^1} : X^{S^1} \to X'^{S^1}\) is a \(G\) homotopy equivalence, where \(f^{S^1}\) is the restriction of \(f\) to \(X^{S^1}\). Then \(k(X) \leq k(X')\).
2. Let \(X\) and \(X'\) be spaces of type SWF at level 2\(t\) and 2\(t'\) respectively and \(t < t'\). Let \(f : X \to X'\) be a \(G\) equivariant map such that \(f^G : X^G \to X'^G\) is a homotopy equivalence, where \(f^G\) is

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\(^2\) Theorems \cite{14, 5} and Corollary \cite{14} below are described for integral homology 3-spheres in \cite{14}. But as is pointed out in \cite{14}, these results are easily generalized to the case of rational homology 3-spheres. We give the outline of their proofs below.
the restriction of \( f \) to the fixed point set of the \( G \) action on \( X \).
Then \( k(X) + t \leq k(X') + t' \). Furthermore, if \( X \) is \( K_G \) split, then \( k(X) + t + 1 \leq k(X') + t' \).

A space of type SWF associated with a rational homology 3-sphere with spin structure \((Y, s)\) is constructed as follows. We fix a Riemann metric \( g \) of \( Y \). Let \( V = i \ker d^* \oplus \Gamma(S) \) where \( i \ker d^* \subset i\Omega^1(Y) \) and \( \Gamma(S) \) is the space of sections of the Spinor bundle \( S \) over \( Y \) associated with \( s \). Then the action of \( G \) on \( V \) is defined as

\[
e^{i\theta}(a, \phi) = (a, e^{i\theta} \phi), \quad j(a, \phi) = (-a, j\phi),
\]

and the gradient of the Chern-Simons-Dirac functional \( \nabla CSD \) over \( V \) is represented as a sum of a \( G \) equivariant linear operator \( \ell \) and a compact operator \( c \). Let \( V^\mu_r \) be the finite dimensional subspace of \( V \) spanned by eigenvectors corresponding to the eigenvalues of \( \ell \) belonging to \((\tau, \mu]\). Then \( V^\mu_r \) is a direct sum of \( V^\mu_r(\tilde{R}) = \tilde{R}^s \) and \( V^\mu_r(H) = H^t \) for some \( s, t \). Manolescu defined a finite dimensional approximation of \( \nabla CSD \) over \( V^\nu_v \) and corresponding flows (approximated Seiberg-Witten flows) for a large \( \nu > 0 \) and showed that a set \( S_\nu \) of critical points and flows between them in a large ball centered at the origin in \( V^\nu_v \) forms an isolated invariant set. Thus a \( G \) equivariant Conley index \( I_\nu \) of \( S_\nu \) is defined up to \( G \) homotopy equivalence (\cite{11}, \cite{13}, \cite{14}).

**Proposition 4.** \cite{13}, \cite{14}

1. There exists a unique critical point \( \Theta \) of the above flow corresponding to the unique isolated and nondegenerate reducible critical point of \( \nabla CSD \) (which is perturbed if necessary). The Conley index \( I(\Theta) \) of \( \Theta \) is \((V^0_{-\nu})^+\).
2. \( I_\nu \) is a space of type SWF at level \( \dim V^\nu_v(\tilde{R}) \).

**Definition 5.** \cite{13} For \((Y, s)\) choose a compact 4-manifold with spin structure \((W, s_W)\) bounded by \((Y, s)\) and a metric \( g_W \) of \( W \) extending the given metric \( g \) of \( Y \). Then define \( n(Y, s, g) \) to be

\[
n(Y, s, g) = \ind_C D_W(s_W) + \frac{\sigma(W)}{8} \in \frac{1}{8} \mathbb{Z}
\]

where \( \ind_C D_W(s_W) \) is the Atiyah-Patodi-Singer index of the Dirac operator of \((W, s_W)\) associated with \( g_W \). The value of \( n(Y, s, g) \) is
independent of the choice of \((W, s_W)\). If \(Y\) is an integral homology 3-sphere, \(n(Y, s, g) \in \mathbb{Z}\) since \(\sigma(W)\) is divisible by 8.

**Definition 6.** [11] A Seiberg-Witten Floer stable homotopy type \(\text{SWF}(Y, s)\) of \((Y, s)\) is a formal desuspension of \(I_\nu\) (up to stable equivalence) defined as follows: If \(\dim V^0_\nu(\hat{R})\) is even, then \(V^0_\nu = \hat{R}^{2k} \oplus H^\ell\) for some \(k, \ell\) and

\[
\text{SWF}(Y, s) = \Sigma^{-(k\hat{C} + (\ell + \frac{1}{2} n(Y, s, g))H)} I_\nu,
\]

while if \(\dim V^0_\nu(\hat{R})\) is odd, then \(V^0_\nu = \hat{R}^{2k-1} \oplus H^\ell\) for some \(k, \ell\) and

\[
\text{SWF}(Y, s) = \Sigma^{-(k\hat{C} + (\ell + \frac{1}{2} n(Y, s, g))H)} (\Sigma \hat{R} I_\nu).
\]

More precisely, \(\text{SWF}(Y, s)\) is an object of the \(G\) equivariant graded suspension category \(\mathfrak{C}\).

**Definition 7.** [14] The \(\kappa\) invariant of \((Y, s)\) is defined to be

\[
\kappa(Y, s) = 2k(\text{SWF}(Y, s))
\]

\[
= \begin{cases} 
2k(I_\nu) - 2 \dim_H V^0_\nu(H) - n(Y, s, g) & \text{if } I_\nu \text{ is at even level,} \\
2k(\Sigma \hat{R} I_\nu) - 2 \dim_H V^0_\nu(H) - n(Y, s, g) & \text{if } I_\nu \text{ is at odd level.}
\end{cases}
\]

**Proposition 5.** [14]

1. \(\kappa(Y, s) \equiv \pi(Y, s) \pmod{2\mathbb{Z}}\). If \((Y, s)\) is an integral homology 3-sphere with the unique spin structure, then \(\kappa(Y, s)\) is an integral lift of \(\pi(Y, s) \in \mathbb{Z}_2\).

2. \(\kappa(Y, s)\) is a spin rational homology cobordism invariant.

For a spin 4-manifold \((W, s_W)\) bounded by \((Y, s)\), ker \(\mathcal{D}_W(s_W)\) and coker \(\mathcal{D}_W(s_W)\) are vector spaces over \(H\), and \(\text{ind}_C \mathcal{D}_W(s_W) = 2 \text{ind}_H \mathcal{D}_W(s_W)\). It follows that \(\kappa(Y, s) \equiv n(Y, s, g) \equiv \sigma(W)/8 \equiv \mu(Y, s) \pmod{2\mathbb{Z}}\). The second claim in Proposition 5 is proved by applying Theorem 4 below to a spin rational homology cobordism between rational homology 3-spheres.

**Theorem 4.** [14]

Let \((Y_i, s_i)\) \((i = 0, 1)\) be rational homology 3-spheres with spin structures and \((W, s_W)\) be a spin cobordism with \(b_1(W) = 0\) from \((Y_0, s_0)\) to
(Y_1, s_1). Then

\[-b^-(W) + \sigma(W) \frac{\sigma(W)}{8} - 1 \leq \kappa(Y_0, s_0) - \kappa(Y_1, s_1) \leq b^+(W) + \sigma(W) \frac{\sigma(W)}{8} + 1.\]

The first term of the above inequality can be replaced by \(-b^-(W) + \sigma(W) \frac{\sigma(W)}{8}\) if \(b^-(W)\) is even, and the last term can be replaced by \(b^+(W) + \sigma(W) \frac{\sigma(W)}{8}\) if \(b^+(W)\) is even.

Definition 8. [14] \((Y, s)\) is called Floer \(K_G\) split if \(I_\nu\) (resp. \(\Sigma R I_\nu\)) is \(K_G\) split when \(I_\nu\) is at even level (resp. at odd level), where \(I_\nu\) is the Conley index that appears in the definition of \(\kappa(Y, s)\).

If one of \((Y_i, s_i)\) is Floer \(K_G\) split, the inequalities in Theorem 4 are slightly improved.

Theorem 5. [14] Let \((W, s_W)\) be a spin cobordism as in Theorem 4.

(1) If \((Y_0, s_0)\) is Floer \(K_G\) split, then

\[
\kappa(Y_0, s_0) - \kappa(Y_1, s_1) \leq \begin{cases} b^+(W) + \sigma(W) \frac{\sigma(W)}{8} - 1 & \text{if } b^+(W) \text{ is odd,} \\ b^+(W) + \sigma(W) \frac{\sigma(W)}{8} - 2 & \text{if } b^+(W) \text{ is even and } b^+(W) > 0. \end{cases}
\]

(2) If \((Y_1, s_1)\) is Floer \(K_G\) split, then

\[
\kappa(Y_1, s_1) - \kappa(Y_0, s_0) \leq \begin{cases} b^-(W) - \sigma(W) \frac{\sigma(W)}{8} - 1 & \text{if } b^-(W) \text{ is odd,} \\ b^-(W) - \sigma(W) \frac{\sigma(W)}{8} - 2 & \text{if } b^-(W) \text{ is even and } b^-(W) > 0. \end{cases}
\]

Since \(S^3\) with the unique spin structure is Floer \(K_G\) split and \(\kappa(S^3) = 0\), we have the following corollary.

Corollary 1. [14]

Let \((Y, s)\) be a rational homology 3-sphere with spin structure and \((W, s_W)\) be a compact spin 4-manifold with \(b_1(W) = 0\) and \(\partial(W, s_W) = (Y, s)\).

(1)

\[
-\kappa(Y, s) \leq \begin{cases} b^+(W) + \sigma(W) \frac{\sigma(W)}{8} - 1 & \text{if } b^+(W) \text{ is odd,} \\ b^+(W) + \sigma(W) \frac{\sigma(W)}{8} - 2 & \text{if } b^+(W) \text{ is even and } b^+(W) > 0. \end{cases}
\]
(2) Furthermore if \((Y, s)\) is Floer \(K_G\) split, then we have
\[
\kappa(Y, s) \leq \begin{cases} 
  b_2(W) - \frac{\sigma(W)}{8} - 1 & \text{if } b_2^-(W) \text{ is odd}, \\
  b_2(W) - \frac{\sigma(W)}{8} - 2 & \text{if } b_2^-(W) \text{ is even and } b_2^-(W) > 0.
\end{cases}
\]

**Remark 3.** Theorem 5 does not hold for a cobordism \(W\) with \(b_2^+(W) = 0\) or \(b_2^-(W) = 0\). Likewise for a spin 4-manifold \((W, s_W)\) bounded by \((Y, s)\) with \(b_2^+(W) = 0\) (resp. \(b_2^- (W) = 0\)), we only have the inequality
\[-\kappa(Y, s) \leq b_2^-(W) + \frac{\sigma(W)}{8}\]
(resp. \(\kappa(Y, s) \leq b_2^-(W) - \frac{\sigma(W)}{8}\)) as in the case where \(W = S^3 \times [0, 1]\) or \(W = D^4\).

The proofs of Theorems 4 and 5 are based on the finite dimensional approximation of the Seiberg-Witten map for a cobordism \(W\) with certain boundary conditions developed in [11]. Let \((V_i)_{i=0}^N\) be the finite dimensional vector spaces used in the definitions of the approximated Seiberg-Witten flows for \((Y_i, s_i)\), and \(I_i\) be the Conley indices \(I_i\) used in the definitions of \(SWF(Y_i, s_i)\) \((i = 0, 1)\). Then we obtain a \(G\) equivariant map \(f: Z_0 \to Z_1\) where \(Z_0\) and \(Z_1\) are defined (up to stable equivalence) as follows ([14]).

\[
Z_0 = (H^{N + \text{ind}_H D_W(s_W) - \dim(V_0)_{\nu}(H)} + (\tilde{R}^{M - \dim_R(V_0)_{\nu}(\tilde{R})})^+ \wedge I_0
\]
\[
Z_1 = (H^{N - \dim_H(V_1)_{\nu}(H)} + (\tilde{R}^{M - \dim_R(V_1)_{\nu}(\tilde{R}) + b_2^+(W)})^+ \wedge I_1
\]
for some \(M, N\) (we may assume that \(M\) is even). Here \(Z_0\) and \(Z_1\) are spaces of type \(\text{SWF}\) at level \(M\) and \(M + b_2^+(W)\) respectively, and the restriction of \(f\) to the \(S^1\) fixed point sets induces the map

\[
(f^{S^1}: Z_0^{S^1} \cong S^M \to Z_1^{S^1} = (\tilde{R}^{b_2^+(W)})^+ \wedge S^M
\]
which is a \(G\) homotopy equivalence if \(b_2^+(W) = 0\). Furthermore the restriction of \(f\) to the \(G\) fixed point sets induces a \(G\) homotopy equivalence \(f^G: Z_0^G \to Z_1^G\). Hence if \(b_2^+(W)\) is even, by Lemma 4 we obtain

\[
k(Z_0) \leq k(Z_1) + \frac{1}{2} b_2^+(W).
\]

If \(I_i\) is at odd level, then \(I_i\) and \(M\) in the description of \(Z_i\) are replaced with \(\Sigma \tilde{R} I_i\) and \(M - 1\) respectively. Then we have

\[
k(Z_0) = \text{ind}_H D_W(s_W) - \dim_H(V_0)_{\nu}(H) + k(I_0) + N
\]
\[
k(Z_1) = - \dim_H(V_1)_{\nu}(H) + k(I_1) + N
\]
where \( I'_i = I_i \) (resp. \( \Sigma \hat{\mathbf{R}} I_i \)) if \( I_i \) is at even level (resp. odd level), since \( k(\Sigma \hat{\mathbf{C}} X) = k(X) \) and \( k(\Sigma \hat{\mathbf{H}} X) = k(X) + 1 \) for any space \( X \) of type SWF at even level \([14]\). It follows from the above inequality and the definition of \( \kappa(Y_i, s_i) \) that

\[
\kappa(Y_0, s_0) - \kappa(Y_1, s_1) \leq b_2^+(W) - \text{ind}_C \mathcal{D}_W(s_W) + n(Y_1, s_1, g_1) - n(Y_0, s_0, g_0)
\]

\[
= b_2^+(W) + \frac{\sigma(W)}{8}
\]

since \( n(Y_1, s_1, g_1) - n(Y_0, s_0, g_0) = \text{ind}_C \mathcal{D}_W(s_W) + \frac{\sigma(W)}{8} \) (Here we assume that \( \ker D_Y(s_i) = 0 \) by using the perturbed Chern-Simons-Dirac functional if necessary, which ensures that \( n(-Y_i, s_i, g_i) = -n(Y_i, s_i, g_i) \).) If \( b_2^+(W) \) is odd, by using \( W \# S^2 \times S^2 \) instead of \( W \) we deduce

\[
\kappa(Y_0, s_0) - \kappa(Y_1, s_1) \leq b_2^+(W) + \frac{\sigma(W)}{8} + 1.
\]

We obtain the other inequalities in Theorem 4 by applying the above argument to the cobordism \((-W, s_W)\) from \((Y_1, s_1)\) to \((Y_0, s_0)\). If \((Y_0, s_0)\) is Floer \( K_G \) split and \( b_2^+(W) \) is even and nonzero, then Lemma 1 shows that \( k(Z_0) + 1 \leq k(Z_1) + \frac{b_2^+(W)}{2} \), from which we deduce

\[
\kappa(Y_0, s_0) - \kappa(Y_1, s_1) \leq b_2^+(W) - 2 - \text{ind}_C \mathcal{D}_W(s_W)
\]

\[
+ n(Y_1, s_1, g_1) - n(Y_0, s_0, g_0) = b_2^+(W) + \frac{\sigma(W)}{8} - 2.
\]

We obtain the other inequalities in Theorem 5 by replacing \( W \) with \( W \# S^2 \times S^2 \) or \(-W\) as before.

3. Proof of Theorem 2

To prove Theorem 2 we first consider the relation between the \( \kappa \) and the \( \overline{\mu} \) invariants of a spherical 3-manifold.

**Proposition 6.** \([14]\) Let \((Y, s)\) be a rational homology 3-sphere with spin structure that admits a metric \( g \) of positive scalar curvature. Then \( \kappa(Y, s) = -n(Y, s, g) \) and \((Y, s)\) is Floer \( K_G \) split.

In the above case, the critical point set of the approximated Seiberg-Witten flow on \( V \) only consists of the unique reducible element. Hence we have \( I_\nu = (V_\nu^0)^+ \) and \( \kappa(Y, s) = k(S^0) - n(Y, s, g) = -n(Y, s, g) \).
Furthermore the ideal $\mathfrak{J}(I_v)$ (or $\mathfrak{J}(\Sigma R I_v)$ if $I_v$ is at odd level) is generated by $z^k$ for some $k$, since $\mathfrak{J}(\Sigma \mathcal{C}X) = \mathfrak{J}(X)$ and $\mathfrak{J}(\Sigma \mathcal{H}X) = z\mathfrak{J}(X)$ for a space $X$ of type SWF at even level \( \{14\} \). Hence $(Y, s)$ is Floer $K_G$ split.

For a rational homology 3-sphere $(Y, s)$ with metric $g$, let $(W, s_W)$ be a compact spin 4-manifold bounded by $(Y, s)$ and $g_W$ a metric of $W$ extending $g$. Then the index of the Dirac operator $\mathcal{D}_W(s_W)$ of $(W, s_W)$ associated with $g_W$ and the signature of $W$ are described as

$$\text{ind}_C \mathcal{D}_W(s_W) = -\frac{1}{24} \int_W p_1 - \frac{1}{2} (\dim \ker \mathcal{D}_Y(s) + \eta^\text{Dir}(Y, s, g)),$$

$$\sigma(W) = \frac{1}{3} \int_W p_1 - \eta^\text{sign}(Y, g).$$

It follows that

$$n(Y, s, g) = -\frac{1}{8} (4 \dim \ker \mathcal{D}_Y(s) + 4\eta^\text{Dir}(Y, s, g) + \eta^\text{sign}(Y, g)).$$

If $g$ is a metric of positive scalar curvature, $\dim \ker \mathcal{D}_Y(s) = 0$. For a spherical 3-manifold with spin structure $(S, s)$ equipped with the standard metric $g$, we have the following equation as a special case of Proposition 3, which is deduced from Proposition 1 and Proposition 2.

$$\mu(S, s) = w(S, cS, s_{cS}) = -\frac{1}{8} (4\eta^\text{Dir}(S, s, g) + \eta^\text{sign}(S, g))$$

where $s_{cS}$ is a spin structure on the cone $cS$ over $S$ extending $s$. It follows from (1), (2) and Proposition 6 that $\kappa(S, s) = -\mu(S, s)$.

**Remark 4.** To describe the above equation we assume that the Clifford multiplication $c$ of the volume form $\text{vol}_Y$ of $Y$ satisfies $c(\text{vol}_Y) = -1$ as in \( \{17\} \). If we define $c$ so that it satisfies $c(\text{vol}_Y) = 1$ as in \( \{11\} \), \( \{13\} \), \( \{14\} \), the sign of $\eta^\text{Dir}(Y, s, g)$ should be changed and the first equation in Proposition 2 should be replaced with $\eta^\text{Dir}(S, s, g) = -2\delta^\text{Dir}(S, s)$.

Suppose that a rational homology 3-sphere with spin structure $(Y, s)$ bounds a compact spin 4-orbifold $(X, s_X)$ with $s_X|_Y = s$. Let $(S_i, s_i)$ \((i = 1, \ldots, n)\) be spherical 3-manifolds, which are the links of all isolated singularities of $X$, where a spin structure $s_i$ is induced from $s_X$. Putting $X_0 = X \setminus \bigcup_{i=1}^n \text{Int } cS_i$, we have a compact spin 4-manifold with $\partial(X_0, s_{X_0}) = (Y, s) \cup \bigcup_{i=1}^n (-S_i, s_i)$ whose spin structure $s_{X_0}$ is induced
from \( \mathfrak{s}_X \). Furthermore \((X_0, \mathfrak{s}_{X_0})\) contains a spin submanifold \( W \) of codimension 0 with \( \partial W = \bigsqcup_{i=1}^n S_i \cup \bigsqcup_{i=1}^m (-S_i) \) which is constructed from the collars of \( S_i \) by attaching 1-handles in \( X_0 \). Let \( X'_0 = X_0 \setminus \text{Int} \ W \) and \( \mathfrak{s}_{X'_0} \) be a spin structure induced from \( \mathfrak{s}_X \). Then \((X'_0, \mathfrak{s}_{X'_0})\) is a spin cobordism from \((S_0, \mathfrak{s}_0) = (\bigsqcup_{i=1}^n S_i, \bigsqcup_{i=1}^m \mathfrak{s}_i)\) to \((Y, \mathfrak{s})\). We note that \( S_0 \) admits a metric \( g_0 \) of positive scalar curvature ([7], [20]). Thus if we choose a metric \( g_W \) on \( W \) extending \( g_0 \) and the standard metrics \( g_i \) of \( S_i \) \((i \geq 1)\), we obtain a \( G \) equivariant map \( f : Z_0 \to Z_1 \) for the cobordism \( W \) from \( \cup_{i=1}^n S_i \) to \( S_0 \) of the following form (up to stabilization) by a procedure similar to those in [13, 14] since the Conley index for a disjoint union of \( S_i \) is the smash product of the Conley indices of \( S_i \) ([13]).

\[
Z_0 = (\mathbf{H}^{N+\text{ind}_H \mathcal{D}_W(\mathfrak{s}_W)} - \sum_{i=1}^n \dim_H (V_i)^0_{-\nu}(\mathbf{H})) + \wedge (\mathbf{R}^{M - \sum_{i=1}^n \dim_R (V_i)^0_{-\nu}(\mathbf{R})) + \wedge (\sum_{i=1}^n I_i)
\]

\[
Z_1 = (\mathbf{H}^{N-\text{dim}_H (V_0)^0_{-\nu}(\mathbf{H}))} + \wedge (\mathbf{R}^{M - \dim_R (V_0)^0_{-\nu}(\mathbf{R}) + b_2^+(W)) + \wedge I_0
\]

where \((V_i)^0_{-\nu}\) and \(I_i\) are the vector space and the Conley index for \( S_i\) defined as above \((0 \leq i \leq n)\). Since \( I_i = (V_i)^0_{-\nu}\) and \( b_2^+(W) = 0\), \( Z_0 \) and \( Z_1\) are of type SWF at level \( M \) (which can be assumed to be even), \( f : Z_0^s \to Z_1^s \) is a \( G \) homotopy equivalence. Furthermore \( k(Z_0) = N + \text{ind}_H \mathcal{D}_W(\mathfrak{s}_W) \) and \( k(Z_1) = N\). Hence applying Lemma 1 to \( f \) we deduce that \( \text{ind}_H \mathcal{D}_W(\mathfrak{s}_W) \leq 0 \). We note that since \( \sigma(W) = 0 \), we have

\[
\text{ind}_C \mathcal{D}_W(\mathfrak{s}_W) = -\frac{1}{24} \int_W p_1 - \frac{1}{2} (\eta^{\text{Dir}} (S_0, \mathfrak{s}_0, g_0) + \sum_{i=1}^n \eta^{\text{Dir}} (-S_i, \mathfrak{s}_i, g_i))
\]

\[
= -\frac{1}{8} (4\eta^{\text{Dir}} (S_0, \mathfrak{s}_0, g_0) + \eta^{\text{sign}} (S_0, g_0) + \sum_{i=1}^n (\eta^{\text{Dir}} (-S_i, \mathfrak{s}_i, g_i) + \eta^{\text{sign}} (-S_i, g_i)))
\]

\[
= n(S_0, \mathfrak{s}_0, g_0) - \sum_{i=1}^n n(S_i, \mathfrak{s}_i, g_i)
\]

By a similar map defined for a cobordism \(-W\) from \( S_0 \) to \( \cup_{i=1}^n S_i \), we have \( \text{ind}_H \mathcal{D}_{(-W)}(\mathfrak{s}_W) = -\text{ind}_H \mathcal{D}_W(\mathfrak{s}_W) \leq 0 \) since \( b_2^+(W) = b_2^+(W) = 0 \). It follows that \( \text{ind}_C \mathcal{D}_W(\mathfrak{s}_W) = 2 \text{ind}_H \mathcal{D}_W(\mathfrak{s}_W) = 0 \) and

\[
\kappa(S_0, \mathfrak{s}_0) = -n(S_0, \mathfrak{s}_0, g_0) = -\sum_{i=1}^n n(S_i, \mathfrak{s}_i, g_i) = -\sum_{i=1}^n \mu(S_i, \mathfrak{s}_i).
\]
Hereafter we assume that \((Y, s)\) is a Seifert rational homology 3-sphere with spin structure and choose the spin 4-orbifolds \((X^\pm, s_{X^\pm})\) bounded by \((Y, s)\) satisfying the conditions in Proposition 3. Let \((S^\pm_i, s^\pm_i)\) \((i = 1, \ldots, n^\pm)\) be the spherical 3-manifolds which are the links of the isolated singularities of \(X^\pm\), where \(s^\pm_i\) is induced from \(s_{X^\pm}\). Subtracting all the cones \(cS^\pm_i\) and the spin 4-manifold of codimension 0, which is a spin cobordism from \(\bigcup_{i=1}^{n^\pm}(S^\pm_i, s^\pm_i)\) to \((S^\pm_0, s^\pm_0) = (\sharp_{i=1}^{n^\pm}S^\pm_i, \sharp_{i=1}^{n^\pm}s^\pm_i)\) constructed as above from \(X^\pm\), we obtain compact spin cobordisms \((X^\pm_0, s_{X^\pm_0})\) from \((S^\pm_0, s^\pm_0)\) to \((Y, s)\) satisfying

\[
b_1(X^\pm_0) = b_1(X^\pm) = 0, \ b_2^+(X^\pm_0) = b_2^+(X^\pm) \leq 1, \ b_2^-(X^-_0) = b_2^-(X^-) \leq 1.
\]

Since \((S^\pm_0, s^\pm_0)\) is Floer \(K_G\) split, we have the following inequalities by applying Theorem 4 and Theorem 5.

\[
\begin{align*}
(4) \quad -b_2^-(X^\pm_0) + \frac{\sigma(X^\pm_0)}{8} - 1 \leq \kappa(S^\pm_0, s^\pm_0) - \kappa(Y, s) \\
\leq b_2^+(X^\pm_0) + \frac{\sigma(X^\pm_0)}{8} - 1
\end{align*}
\]

unless \(b_2^+(X^\pm_0) = 0\). If \(b_2^+(X^\pm_0) = 0\), we replace \(X^\pm_0\) with \(X^\pm_0 \sharp S^2 \times S^2\) to obtain the inequality \(\kappa(S^+, s_{S^+}) - \kappa(Y, s) \leq \sigma(X^+_0)/8\) by applying Theorem 5. If \(b_2^-(X^-_0)\) is even (in fact 0), then the first inequality of (4) for \(S^-\) can be replaced with

\[
(5) \quad -b_2^-(X^-_0) + \frac{\sigma(X^-_0)}{8} \leq \kappa(S^-, s_{S^-}) - \kappa(Y, s).
\]

On the other hand it follows from Propositions 1, 2, 3 and the equations (1), (2), (3) that

\[
(6) \quad \overline{\nu}(Y, s) = w(Y, X^\pm, s_{X^\pm})
\]

\[
= \frac{1}{8}(\sigma(X^\pm_0) - \sum_{i=1}^{n}(4\eta^{\text{Dir}}(S^\pm_i, s^\pm_i, g^\pm_i) + \eta^{\text{sign}}(S^\pm_i, g^\pm_i)))
\]

\[
= \frac{1}{8}\sigma(X^\pm_0) + \sum_{i=1}^{n} n(S^\pm_i, s^\pm_i, g^\pm_i) = \frac{1}{8}\sigma(X^\pm_0) - \kappa(S^\pm_0, s^\pm_0)
\]

Therefore by subtracting (6) from (4), we obtain
\( -2 \leq -b^+_2(X_0^-) - 1 \leq -\kappa(Y, s) - \overline{\mu}(Y, s) \leq b^+_2(X_0^+) - 1 \leq 0 \)

if \( b^+_2(X_0^+) = 1 \). We replace \( X_0^+ \) with \( X_0^+ \# S^2 \times S^2 \) or \( X_0^+ \# 2S^2 \times S^2 \) if \( b^+_2(X_0^+) = 0 \) to obtain the last inequality by applying Theorem 5. If \( Y \) is a Seifert integral homology 3-sphere (with the unique spin structure), \( \kappa(Y) + \overline{\mu}(Y) = 0 \) or \( 2 \) since both \( \kappa(Y) \) and \( \overline{\mu}(Y) \) are integers and the integral lifts of the Rokhlin invariant.

Suppose that a Seifert rational homology 3-sphere \((Y, s)\) is Floer \( K_G \) split. Then by Theorem 5 we obtain

\[ 0 \leq -b^+_2(X_0^-) + 1 \leq -\kappa(Y, s) - \overline{\mu}(Y, s) \]

if \( b^-_2(X_0^-) = 1 \) in addition to (7). We obtain the same estimate by (5) or applying Theorem 5 to \( X_0^- \# S^2 \times S^2 \) or \( X_0^- \# 2S^2 \times S^2 \) if \( b^-_2(X_0^-) = 0 \). Thus we have \( \kappa(Y, s) + \overline{\mu}(Y, s) = 0 \).

For a general rational homology 3-sphere with spin structure \((Y, s)\), it is proved that \( \kappa(Y, s) + \kappa(-Y, s) \geq 0 \) \([14]\). Note that \( \kappa(Y, s) \) is not necessarily the same as \(-\kappa(-Y, s)\), while we have \( \overline{\mu}(-Y, s) = -\overline{\mu}(Y, s) \).

If \( Y \) is a Seifert rational homology 3-sphere, the above result shows that \( 0 \leq \kappa(Y, s) + \overline{\mu}(Y, s) \leq 2 \), \( 0 \leq \kappa(-Y, s) - \overline{\mu}(Y, s) \leq 2 \), and hence \( 0 \leq \kappa(Y, s) + \kappa(-Y, s) \leq 4 \). If \( Y \) is a Seifert integral homology 3-sphere, we also have \( \kappa(-Y) + \overline{\mu}(-Y) = \kappa(-Y) - \overline{\mu}(Y) = 0 \) or \( 2 \). It follows that \( \kappa(Y) + \kappa(-Y) = 0, 2, \) or \( 4 \). If \( Y \) has a fibration such that one of the multiplicities is even and has a positive degree, then we apply the above inequalities for \( Y \) to \( X^- = X^+ \) satisfying \( b^-_2(X^+) = 0 \) and \( b^+_2(X^+) = 1 \), which is chosen in Proposition 3. It follows from (5), (6), (7) that \( \kappa(Y, s) = -\overline{\mu}(Y, s) \).

This completes the proof of Theorem 2.

**Example 1.** We give a list of the values of \( \kappa, \overline{\mu} \) together with \( \beta \) and \( d \) of some Brieskorn homology 3-spheres. The computations of the \( \kappa \) invariants below are due to Manolescu \([14]\). It is pointed out in \([14]\) that \( \pm \Sigma(2, 3, 12n + 1) \) and \( \pm \Sigma(2, 3, 12n + 5) \) are Floer \( K_G \) split, while \( \pm \Sigma(2, 3, 12n - 1) \) and \( \pm \Sigma(2, 3, 12n - 5) \) are not.
\[
\begin{array}{cccc}
\kappa & \bar{\mu} & \beta & d \\
\Sigma(2,3,12n-1) & 2 & 0 & 0 & 0 \\
-\Sigma(2,3,12n-1) & 0 & 0 & 0 & 0 \\
\Sigma(2,3,12n+1) & 0 & 0 & 0 & 0 \\
-\Sigma(2,3,12n+1) & 0 & 0 & 0 & 0 \\
\Sigma(2,3,12n-5) & 1 & 1 & -1 & -2 \\
-\Sigma(2,3,12n-5) & 1 & -1 & 1 & 2 \\
\Sigma(2,3,12n+5) & 1 & -1 & 1 & 2 \\
-\Sigma(2,3,12n+5) & -1 & 1 & -1 & -2 \\
\end{array}
\]

Remark 5. If \( Y \) is a Seifert fibration of the form \( \{b, (a_1, b_1), \ldots, (a_n, b_n)\} \), then \( Y \) is represented by a framed link \( \bigcup_{i=0}^{n} K_i \), where \( K_0 \) is an unknot with framing \( b \) and \( K_i \) \( (i \geq 1) \) is a meridian of \( K_0 \) with framing \(-b_i/a_i\). Let \( h \) be a meridian of \( K_0 \) (a general fiber of \( Y \)) and \( g_i \) be a meridian of \( K_i \) \( (i \geq 1) \). Then a spin structure \( s \) on \( Y \) corresponds to a homomorphism \( c : H_1(S^3 \setminus \bigcup_{i=0}^{n} K_i, \mathbb{Z}) \to \mathbb{Z}_2 \) satisfying

\[
\sum_{i=1}^{n} c(g_i) + bc(h) \equiv b \pmod{2}
\]

[25]. The description of the Seifert invariants in [25] is different from that of this paper, but the above condition is still valid.) Suppose that all \( a_i \) are odd and \( \sum_{i=1}^{n} b_i \equiv b \pmod{2} \) (such a case only occurs when \( |H_1(Y, \mathbb{Z})| \) is even). Then we have a spin structure \( s \) on \( Y \) corresponding to \( c \) that satisfies \( c(h) \equiv 0, c(g_i) \equiv b_i \pmod{2} \). In this case \( X^\pm \) in Proposition 3 can be also chosen so that \( X^+ = X^- \) satisfying \( b_2^+(X^\pm) = 1 \) and \( b_2^-(X^\pm) = 0 \) if \( \deg Y > 0 \), and \( b_2^-(X^\pm) = 1 \) and \( b_2^+(X^\pm) = 0 \) if \( \deg Y < 0 \) (25). Thus it follows from the proof of Theorem 2 that \( \kappa(Y, s) = -\bar{\mu}(Y, s) \) if \( \deg Y > 0 \).

Remark 6. If a Seifert rational homology 3-sphere with spin structure \( (Y, s) \) bounds a negative definite spin 4-manifold \( W \), the inequalities in Theorem 1 coincide with those in [26]. Furthermore for a Seifert integral homology 3-sphere, the second inequality in Theorem 1 coincides with those in terms of the \( \beta \) invariant (13), and in terms of
the $d$ invariant ([8]) under the relations $\beta(Y) = -\overline{\mu}(Y)$ ([2], [23]) and $d(Y) = -2\overline{\nu}(Y)$ ([3]).

**Remark 7.** We have $\kappa(Y) + \overline{\nu}(Y) = 0$ or 2 for a Seifert integral homology 3-sphere $Y$ by Theorem 2. If $\kappa(Y) = -\overline{\mu}(Y)$, then Corollary 1 shows that $\overline{\nu}(Y) \leq b_2^+(W) + \frac{\sigma(W)}{8} - 1$ for a compact spin 4-manifold $W$ bounded by $Y$, which give the better estimate than Theorem 1. If $\kappa(Y) = 2 - \overline{\nu}(Y)$, the estimate in Theorem 1 is slightly better than or the same as that of $\overline{\nu}(Y)$ given in Corollary 1.

J.Lin [9] proves other constraints on the intersection forms of spin 4-manifolds bounded by homology 3-spheres in terms of KO invariants, which give estimates better than those in Theorem 1 and Theorems 4, 5 in some cases. In [9] several estimates for spin 4-manifolds bounded by certain Brieskorn homology 3-spheres are also given by considering the closed spin 4-orbifolds as in [4]. We obtain similar estimates by using Proposition 3 as follows. Suppose that a Seifert rational homology 3-sphere $Y$ (with spin structure) bounds a spin 4-manifold $W$. Let $Z = X^- \cup (-W)$ be a closed spin 4-orbifold, where $X^-$ is a spin 4-orbifold bounded by $Y$ chosen in Proposition 3. Then we have a $G$ equivariant map

$$f : (H^{ind_\mathcal{D}_-Z})^+ \to (\tilde{\mathbb{R}}^{b_2^+(Z)})^+$$

whose restriction to the $G$-fixed point set is a homotopy equivalence (we choose $Z$ according to our sign convention). It is proved in [9] that there is no such map of the form $f : (H^{4\ell})^+ \to (\tilde{\mathbb{R}}^{8\ell+2})^+$ if $\ell > 0$. Note that $b^-(X^-) \leq 1$ and $\text{ind}_\mathcal{D}_-Z = \overline{\nu}(Y) - \frac{\sigma(W)}{8}$. It follows that if $\overline{\nu}(Y) - \frac{\sigma(W)}{8} > 0$ and divisible by 8, then

$$\overline{\nu}(Y) \leq b_2^+(W) + \frac{\sigma(W)}{8} - 2,$$

which gives a better estimate than those by Theorem 1 or Corollary 1.

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