Phase retrieval using alternating minimization in a batch setting

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Abstract

This paper considers the problem of phase retrieval, where the goal is to recover a signal $z \in \mathbb{C}^n$ from the observations $y_i = |a_i^* z|, i = 1, 2, \ldots, m$. While many algorithms have been proposed, the alternating minimization algorithm has been one of the most commonly used methods, and it is very simple to implement. Current work [26] has proved that when the observation vectors $\{a_i\}_{i=1}^m$ are sampled from a complex Gaussian distribution $\mathcal{N}(0, I)$, it recovers the underlying signal with a good initialization when $m = O(n)$, or with random initialization when $m = O(n^2)$, and it conjectured that random initialization succeeds with $m = O(n)$.

This work proposes a modified alternating minimization method in a batch setting, and proves that when $m = O(n \log^2 n)$, the proposed algorithm with random initialization recovers the underlying signal with high probability. The proof is based on the observation that after each iteration of alternating minimization, with high probability, the angle between the estimated signal and the underlying signal is reduced.

1 Introduction

This article concerns the phase retrieval problem as follows: let $z \in \mathbb{C}^n$ be an unknown vector, and given $m$ known sensing vectors $\{a_i\}_{i=1}^m \in \mathbb{C}^n$, we have the observations

$$y_i = |a_i^T z|, i = 1, 2, \ldots, m.$$  

Then can we reconstruct $z$ from the observations $\{y_i\}_{i=1}^m$? This problem is motivated from the applications in imaging science, and we refer interested readers to [21] for more detailed discussions on the background in engineering. In particular, this problem has applications in other areas of sciences and engineering as well, as discussed in [5].
Because of the practical ubiquity of the phase retrieval problem, many algorithms and theoretical analysis have been developed for this problem. For example, an interesting recent approach is based on convex relaxation [7, 6, 27], which replaces the non-convex measurements by convex measurements through relaxation. Since the associated optimization problem is convex, it has nice properties such as convergence to the global minimizer, and it has been shown that under some assumptions of the sensing vectors, this method recovers the correct $z$ [4, 14]. However, since these algorithms involves semidefinite programings for $n \times n$ positive semidefinite matrices, the computational cost could be prohibitive when $n$ is large. Recently, several works [1, 13, 15, 16] proposed and analyzed an alternate convex method that uses linear programming instead of semidefinite programming, which is more computational efficient, but the program itself requires an “anchor vector”, which needs to be a good approximate estimation of $z$.

Another line of works are based on Wirtinger flows, i.e., gradient flow in the complex setting [5, 8, 29, 30, 3, 28, 22]. Some theoretical justifications are also provided [5, 22]. However, since the objective functions are nonconvex, these algorithms require careful initializations, which are usually only justified when the measurement vectors follow a very specific model, for example, when the observation vectors $\{a_i\}_{i=1}^m$ are sampled from a complex Gaussian distribution $\mathcal{N}(0, I)$. In addition, there are technical issues in implementation such as choosing step sizes, which makes the implementation slightly more complicated.

To cope with the nonconvexity of the phase retrieval problem, Sun et al. [23] tries to understand the geometric landscape of a nonconvex objective function associated with phase retrieval, and proved that when $m = O(n \log^3 n)$, their cost function has no bad critical point, and as a result, arbitrary initialization is sufficient and a trust-region method (TRM) can be applied to obtain the solution. However, this method is more complicated than the alternate minimization algorithm due to its specific objective function and the associated trust-region method.

The most widely used method is perhaps the alternate minimization algorithm and its variants [12, 10, 11], that is based on alternating projections onto nonconvex sets [2]. This method is very simple to implement and is parameter-free. However, since it is an algorithm for a nonconvex optimization problem, its properties such as convergence is only partially known. Netrapalli et al. [19] studied a resampling version of this algorithm and established its convergence as the number of measurements $m$ goes to infinity when the measurement vectors are independent standard complex normal vectors. Marchesini et al. [18] studied and demonstrated the necessary and sufficient conditions for the local convergence of this algorithm. Recently, Waldspurger [26] showed that when $m \geq Cn$ for sufficiently large $C$, the alternating minimization algorithm succeed with high probability, provided that the algorithm is carefully initialized. In addition, with random initialization the algorithm succeeds with $m \geq Cn^2$. The work also conjectured that the alternate minimizations algorithm with random initialization succeeds with $m > Cn$.

The contribution of this work is to show that a modified version of the
alternating minimization algorithm and random initialization succeeds with high probability when \( m = O(n \log^3 n) \), which partially verifies the conjecture that the alternating minimization algorithm succeeds with high probability when \( m = O(n) \). Compared with the previous methods based on Wirtinger flows and linear programming, the proposed algorithm is more practical since it does not require a good initialization, and compared with previous works that do not depend on good initializations such as semidefinite programming and the analysis of geometric landscape \([23]\), the proposed alternating minimization algorithm is simpler and easier to implement.

The paper is organized as follows. Section 2 presents the algorithm and the main results of the paper, and the proof of a key component, Theorem 2.3 is given in Section 3. We run simulations to verify Theorem 2.3 in Section 4.

2 Algorithm and Main Results

The alternating minimization method is one of the earliest methods that was introduced for phase retrieval problems \([12, 10, 11]\), and it is based on alternating projections onto nonconvex sets \([2]\). In particular, its goal is to find a vector in \( \mathbb{C}^m \) such that it lies in both the set \( S = \text{range}(A) \in \mathbb{C}^m \) and the set of correct amplitude \( \{ w \in \mathbb{C}^m : |w_i| = y_i \} \). For this purpose, the algorithm picks an initial guess in \( \mathbb{C}^m \), and alternatively projects it onto both sets. Let \( A \in \mathbb{C}^{m \times n} \) be a matrix with columns given by \( a_1, a_2, \ldots, a_m \), then the projections \( P_S, P_A : \mathbb{C}^m \to \mathbb{C}^m \) can be defined by

\[
P_S(w) = A(A^T A)^{-1} A^T w, \quad [P_A(w)]_i = \frac{w_i}{|w_i|},
\]

and the alternating minimization algorithm is given by

\[
w^{(k+1)} = P_S P_A w^{(k)}. \tag{1}
\]

In fact, the alternating minimization method can be explicitly written down in terms of \( \{a_i\}_{i=1}^m \) as follows. Writing \( w^{(k)} = Ax^{(k)} \) and let \( e_i \in \mathbb{C}^m \) be the indicator vector of the \( i \)-th coordinate, then the update formula is

\[
Ax^{(k+1)} = A(A^T A)^{-1} A^T \left( \sum_{i=1}^m a_i^T z \frac{a_i^T x^{(k)}}{|a_i^T x^{(k)}|} e_i \right) = A(A^T A)^{-1} \left( \sum_{i=1}^m \frac{|a_i^T z|}{|a_i^T x^{(k)}|} a_i a_i^T x^{(k)} a_i \right),
\]

which implies

\[
x^{(k+1)} = (A^T A)^{-1} \left( \sum_{i=1}^m \frac{|a_i^T z|}{|a_i^T x^{(k)}|} a_i a_i^T x^{(k)} \right), \tag{2}
\]

Define

\[
g_i(x) = \frac{|a_i^T z|}{|a_i^T x|} a_i a_i^T x, \quad g(x) = \sum_{i=1}^m g_i(x), \quad T(x) = (A^T A)^{-1} g(x)
\]

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then the algorithm [2] can be written as

\[ x^{(k+1)} = T(x^{(k)}). \]

In this work, we will consider the algorithm [2] in a batch setting. Similar to AltMinPhase [19], we divide the sampling vectors \( \{a_i\}_{i=1}^m \) (the rows of the matrix \( A \)) and corresponding observations \( \{y_i\}_{i=1}^m \) into \( B \) disjoint blocks \( (y^{(1)}, A^{(1)}), \ldots, (y^{(B)}, A^{(B)}) \) of roughly equal size, and perform alternating minimization [1] to the disjoint blocks cyclically. The procedure is summarized as Algorithm 1, where \( T^{(k)} \) represents the alternating minimization operator with the \( k \)-th block \( (y^{(k)}, A^{(k)}) \). We remark that while it is similar to AltMinPhase, this algorithm uses partitions cyclically, rather than only using each partition once. As a result, it only requires finite observations to estimate \( z \) exactly, which is different than the method in [19].

**Algorithm 1** Alternating minimization in a batch setting

**Input:** The sampling vectors \( A \in \mathbb{C}^{m \times n} \) and corresponding observations \( y \in \mathbb{C}^m \) partitioned into \( B \) disjoint blocks \( (y^{(1)}, A^{(1)}), \ldots, (y^{(B)}, A^{(B)}) \) of roughly equal size.

**Output:** An estimator of the underlying signal \( z \).

**Steps:**
1. Let \( x^{(0)} \) be a random unit vector in \( \mathbb{C}^n \), \( k = 0 \).
2. Repeat
3. \( x^{(k+1)} \leftarrow T^{(\text{mod}(k, B)+1)}x^{(k)} \), \( k = k + 1 \)
4. Until Convergence
**Output:** \( \lim_{k \to \infty} x^{(k)} \).

Before we state our main result, we present an auxiliary function \( h(\theta) \) and its related properties as follows. Since its proof is rather complicated and is independent with respect to the other parts of this work, we defer it to the Appendix.

**Lemma 2.1.** Let \( a_1 \) and \( a_2 \) be two complex variables independently sampled from a complex Gaussian distribution \( N(0,1) \). Let \( h(\theta) = \mathbb{E}_{a_1,a_2 \sim N(0,1)} |a_1| |a_1 \sin \theta + a_2 \cos \theta| \), then there exists \( c > 0 \) such that for all \( 0 < \theta < \pi/2 \), \( h'(\theta) \geq c \min(\theta, \pi/2 - \theta) \). In addition, there exists \( c' > 0 \) such that \( \min_{0 \leq \theta < \pi/2} h(\theta) < c' \).

To visualize Lemma 2.1, we randomly reproduce \( 10^6 \) samples of \( (a_1, a_2) \), calculate the average values of \( h(\theta) \) and \( h'(\theta) \) and plot them in Figure 1. The right figure verifies that Lemma 2.1 holds. We remark that if \( a_1 \) and \( a_2 \) are sampled from real Gaussian distribution \( N(0,1) \), then \( h(\theta) = \frac{1}{\pi} |2\theta \sin \theta + 2 \cos \theta| \), but in the complex setting, the calculation is more complicated and there is no known explicit formula.

Based on Lemma 2.1, we have the following result for Algorithm 1.
Theorem 2.2. There exists $C_0, C'_0, C''_0, C_1, C_2$ such that when $n > C'_0$, $m > CC'_0n \log^2 n$ and $B = C \log n$, then with probability at least $1 - 2 / \log n - BC_1 \exp(-C_2 m / B)$, Algorithm 1 recovers the underlying $z$.

The proof of Theorem 2.2 depends on Theorem 2.3, which states that the angle between $T(x)$ and $z$ is smaller than $\pi / 2$ with high probability. We remark that in the following statements and proofs, we use $c, c', C, C', C_1, C_2$ to denote any fixed constants as $m, n \to \infty$. Depending on the context, they might denote different values in different equations and expressions.

Theorem 2.3. For any fixed $x \in \mathbb{C}^n$, Let $\theta(x) = \sin^{-1}(|x^* z| / \|x\| \|z\|)$, then there exists $C''_0 > 0$ such that if $\theta(x) > \frac{\pi}{2} - \epsilon$, then with probability at least $1 - 2 / \log^2 n$, 

$$
\theta(T(x)) > \left(1 - \frac{1}{\log n}\right) \left(\theta(x) + \tan^{-1}\left(\frac{h'(\theta(x))}{h(\theta(x))}\right)\right).
$$

Combining this result with the following two lemmas, we proved Theorem 2.2. Lemma 2.5 is a restatement of [26, Theorem 3.1]. While the original statement measure the distance between $x$ and $z$ by $\min_{\phi \in \mathbb{R}} \|e^{i\phi} x - z\|$, applying the fact that the operator $T(x)$ is independent of the magnitude of $x$ and only depends on $x / \|x\|$, we can use $\pi / 2 - \theta(x)$ to measure the distance between $x$ and $z$ instead.

Lemma 2.4. With probability $1 - C / \log n - \exp(-Cn)$, $\theta(x^{(0)}) = \sin^{-1}(|x^{(0)*} z| / \|x^{(0)}\| \|z\|) > \sin^{-1}(1 / (2 \log n \sqrt{n})).$

Lemma 2.5 (Theorem 3.1 in [26]). There exists $\epsilon, C_1, C_2, M > 0$ and $0 < \delta < 1$ such that if $m \geq Mn$, then with probability $1 - C_1 \exp(-C_2 m)$, for any $x$ such that $\theta(x) > \frac{\pi}{2} - \epsilon$, then 

$$
\frac{\pi}{2} - \theta(T(x)) \leq \delta \left(\frac{\pi}{2} - \theta(x)\right).
$$

Proof of Lemma 2.4. WLOG assume $z = (1, 0, \cdots, 0)$, then $|x^{(0)*} z| = |x^{(0)}_1| / \|x^{(0)}\|$. Using Hanson-Wright inequality [20] with $\|x^{(0)}\|^2 = x^{(0)*} x^{(0)}$, we have that
with probability $1 - \exp(-Cn)$, $\|x^{(0)}\| < 2\sqrt{n}$. In addition, with probability at least $1 - C/\log n$, $|x_1^{(0)}| > 1/\log n$. Combing these two observations, Lemma 2.4 is proved.

**Proof of Theorem 2.2** Applying Lemma 2.5 to all $B$ batches of the measurements and observations, as long as $m/B > Mn$ and $BC_1 \exp(-C_2m/B) < 1$ (which holds asymptotically as $n \to \infty$ when $C_0 > 1$), it is sufficient to show that $\theta(x^{(i)}) > \frac{\pi}{2} - \epsilon$ for some $0 \leq i \leq B$.

If for all $0 \leq i < B$, $\theta(x^{(i)}) < \frac{\pi}{2} - \epsilon$, then Lemma 2.1 implies that there exists $c_\epsilon > 0$ such that $\tan^{-1}\frac{h'(\theta(x^{(i)}))}{h(\theta(x^{(i)}))} > c_\epsilon \theta(x^{(i)})$ for all $0 \leq i < B$. Then when $m/B > CC_0' n \log^3 n$, each patch has $C_0' \log^2 n$ observations. As a result, Theorem 2.3 implies that with probability $1 - 2B/\log n = 1 - 2C/\log n$,

$$\theta(x^{(B)}) > \left[(1 + c_\epsilon \left(1 - \frac{1}{\log n}\right)^B\right] \theta(x^{(0)}). \quad (4)$$

Choose $C = \frac{1}{\log(1 + c_\epsilon / 2)}$, then there exists $C_0$ such that for $n > C_0'$, $(1 + c_\epsilon \left(1 - \frac{1}{\log n}\right)) > 1 + c_\epsilon / 2$, and it can be shown that

$$\left[(1 + c_\epsilon \left(1 - \frac{1}{\log n}\right)^C\log n \sin^{-1}\left(\frac{1}{2\log n \sqrt{n}}\right) > \frac{\pi}{2},$$

then this is a contradiction to (4), Lemma 2.4 and the fact that $\theta(x^{(B)})$ cannot be larger than $\pi/2$. Therefore, there exist $0 \leq i < B$ such that $\theta(x^{(i)}) > \frac{\pi}{2} - \epsilon$, and Theorem 2.2 is proved.

### 2.1 Discussion

Theorem 2.2 has several interesting connections with some previous analysis of the alternating minimization algorithm. First of all, it complements the analysis of AltMinPhase in [19]. While it is one of the first theoretical guarantees for the alternating minimization algorithm, it is unclear that how we will divide the samples in distinct blocks such as how to choose the number of size of blocks. In comparison, Theorem 2.2 gives a suggested number of blocks to use in practice.

In addition, Theorem 2.3 also shows that when the size of each block is in the order of $O(n)$ up to a logarithmic factor, then the algorithm would improve the estimation of $z$ after one iteration of alternating minimization, in the sense that it decreases the angle between $z$ and the estimator.

Our work also partially answers the conjecture from the work [19] that when the initialization is randomly chosen and $m = O(n)$, the alternating minimization algorithm succeeds with high probability. In comparison, we proved that the alternating minimization algorithm in a batch setting succeeds with $m = O(n \log^2 n)$, which is an improvement from the estimation $m = O(n^2)$ in [19] (though we remark that the result in [19] is for the non-batch setting).
An interesting observation from [19] is the existence of stationary points when \( m < O(n^2) \). In comparison, Theorem 2.2 shows that the proposed algorithm could avoid these stationary points from random initialization. In this sense, Theorem 2.2 is very different from most existing theoretical guarantees for phase retrieval, which are based on the observations that there is no stationary point, or there is no stationary point within a neighborhood of \( z \).

We also emphasize the universality of random initialization. While alternating minimization algorithms (and many other algorithms) has been shown to succeed with \( m = O(n) \), when there is a good initialization in the sense that it is not too far away from \( z \). For example, [19] uses the top eigenvector of \( \sum_{i=1}^{m} |a_i^* z|^2 a_i a_i^* \), and [8] applies a similar estimator with a thresholding-based scheme by using the top eigenvector of

\[
\sum_{i=1}^{m} |a_i^* z|^2 a_i a_i^* \leq \frac{9}{m} \sum_{j=1}^{m} |a_i^* z|^2
\]

and a similar scheme is also used in [3]. In addition, [28] introduces an an orthogonality-promoting initialization that is obtained with a few simple power iterations, aiming to resolve the issue when the distribution of \( a_i \) is heavy-tailed. However, their theories for proving that it is a good initialization usually only holds when \( \{a_i \}_{i=1}^{m} \) are i.i.d. sampled from \( N(0, I) \). In comparison, random initialization is more universal and can be generalized to the setting \( \{a_i \}_{i=1}^{m} \) are i.i.d. sampled from \( N(0, \Sigma) \). For this case, if we let \( \tilde{a}_i = \Sigma^{-\frac{1}{2}} a_i \), \( \tilde{x}^{(k)} = \Sigma^{\frac{1}{2}} x^{(k)} \), and \( \tilde{z} = \Sigma^{\frac{1}{2}} z \), then the update formula (2) reduces to the setting where \( \tilde{a}_i \sim N(0, I) \). This suggested that as long as \( \Sigma \) is well-conditioned, Theorem 2.2 can still be generalized to this setting. In particular, based on this observation, Theorem 2.2 can be generalized to the setting \( a_i \sim N(0, \Sigma) \) as follows.

**Corollary 2.6.** Assuming that \( a_i \sim N(0, \Sigma) \), then the statement in Theorem 2.2 holds as long as there exists \( c' > 0 \) such that \( \frac{n}{\text{tr}(\Sigma)} \|z\| > c' \).

At last, we emphasize that Theorem 2.2 does not apply to the standard alternating minimization algorithm (i.e., not in a batch setting). The reason is that, the probabilistic estimation in Theorem 2.3 only holds for a fixed \( x \) that is independent of \( A \). However, in the standard alternating minimization algorithm, \( x^{(k)} \) for \( k > 1 \) depends on \( A \), and Theorem 2.3 cannot be used to estimate \( \theta(x^{(k+1)}) \). In comparison, Theorem 3.1 in [26] holds for all \( x \) as long as \( x^{(k)} \) is sufficiently close to \( z \). It is unclear that how we can find a method generalized Theorem 2.2 to the standard alternating minimization algorithm, by “decoupling” the dependence of \( x^{(k)} \) and \( A \). This is an open question and we consider it as an interesting future direction.

### 3 Proof of Theorem 2.3

One of the key element of this work is Theorem 2.3 which describes the performance of the alternating minimization in each iteration. To prove Theorem 2.3, we first present a few lemmas.
Lemma 3.1. WLOG assume that \( \|x\| = \|z\| = 1 \), and \( \theta = \theta(x) = \sin^{-1}(|x^* z|) \). Let \( \eta \in [0, 2\pi) \) and \( w \) be chosen such that \( \|w\| = 1 \), \( w \perp z \) (i.e., \( w^* z = 0 \)), and \( x = \sin(\theta)z \exp(i\eta) + \cos(\theta)w \). Then

\[
\mathbb{E} g_i(x) = h(\theta)x + h'(\theta)d,
\]

where \( d = \cos(\theta)z \exp(i\eta) - \sin(\theta)w \).

Proof of Lemma 3.1. The proof is based on the observation that \( g_i(x) \) is the derivative of \( |a^*_i x| |a^*_i z| \). In particular, this work defines the derivatives of real valued functions over complex variables as follows: \( \nabla f(x) \) is chosen such that

\[
f(x + \Delta x) = f(x) + \Re(\nabla f(x)^* \Delta x) + o(|\Delta x|).
\]

Then we can define \( G(x) = \sum_{i=1}^n G_i(x) \) with \( G_i(x) = |a^*_i x| \). Then we have \( g_i(x) = \nabla G_i(x) \) and \( g(x) = \nabla G(x) \).

In addition, we can calculate \( \mathbb{E} G_i(x) \). Since the expectation is invariant to unitary transformations of \( x \) and \( zm \) WLOG we may assume \( z = (1, 0, \cdots, 0) \) and \( x = (\sin(\theta), \cos(\theta), 0, \cdots, 0) \). Then it is clear that

\[
\mathbb{E}[G_i(x)] = \mathbb{E}_{(a_1, a_2) \sim N(0,1)} \left[ |a_1| a_1 \sin \theta + a_2 \cos \theta \right] = h(\theta).
\]

Since \( \mathbb{E}[G_i(x)] \) only depends on the \( \theta(x) = \cos^{-1}(|x^* z|) \) and \( \|x\| \), its derivative is only nonzero at two directions: \( x \) and the direction where \( \theta(x) \) changes most. Since the function is linear: \( G_i(x + t\alpha) = (1 + t)G_i(x) \), we have

\[
x^* \nabla \mathbb{E}[G_i(x)] = \mathbb{E}[G_i(x)].
\]

The direction where \( \theta(x) \) changes most is given by \( d \) where \( \theta(x + td) = \theta(x) + t + O(t^2) \). Since \( \|x + td\| = \|x\| + O(t^2) \), we have

\[
d^* \nabla \mathbb{E}[G_i(x)] = \frac{d}{d\theta} h(\theta(x)).
\]

Combining above observations together, Lemma 3.1 is proved.

Lemma 3.2.

\[
\left\| \mathbb{E}T(x) - m \mathbb{E} \left( \frac{1}{1 + \text{tr}(\Sigma^{-1})} \Sigma^{-1} \right) \mathbb{E} g_i(x) \right\| < C \quad (5)
\]

\[
\left\| z^* \mathbb{E}T(x) - m z^* \mathbb{E} \left( \frac{1}{1 + \text{tr}(\Sigma^{-1})} \Sigma^{-1} \right) \mathbb{E} g_i(x) \right\| < C/\sqrt{m} \quad (6)
\]

Lemma 3.3.

\[
\text{Var}[e^*_i T(x)] < C/m, \text{Var}[z^* T(x)] < C/m.
\]
To prove them, we apply the fact from [24] Theorem 1.1 that for any for any 
\( n \times n \) complex gaussian matrix \( A \),
\[
\Pr \left( \sigma_n (A) \leq t \sqrt{n} \right) < t. \tag{7}
\]
Therefore, for any \( m \times n \) complex gaussian matrix \( A \), we can consider its \( \left[ \frac{n}{m} \right] \)
indipendent submatrices of size \( n \times n \) to obtain
\[
\Pr \left( \sigma_n (A) \leq t \sqrt{n} \right) < t^{\left( \frac{n}{m} \right)}, \tag{8}
\]
and for any fixed matrix \( \Sigma \), Hanson-Wright inequality [20] implies
\[
\Pr \left( |a_i^* \Sigma a_i - \text{tr}(\Sigma)| > t \right) \leq 2 \exp \left[ -c \min \left( \frac{t^2}{\| \Sigma \|_{\text{tr}}^2}, \frac{t}{\sqrt{m}} \right) \right].
\]
Proof of Lemma 3.1. Applying
\[
T(x) = \Sigma^{-1} \sum_{i=1}^{m} g_i(x) = \sum_{i=1}^{m} \Sigma_i^{-1} - \Sigma_i^{-1} a_i a_i^T \Sigma_i^{-1} g_i(x) = \sum_{i=1}^{m} \frac{1}{1 + a_i^* \Sigma_i^{-1} a_i} \Sigma_i^{-1} g_i(x),
\]
we have
\[
\left\| \mathbb{E} T(x) - \sum_{i=1}^{m} \mathbb{E} \left( \frac{1}{1 + \text{tr}(\Sigma_i^{-1})} \right) \mathbb{E} g_i(x) \right\| = \left\| \mathbb{E} \left( \frac{1}{1 + a_i^* \Sigma_i^{-1} a_i} - \frac{1}{1 + \text{tr}(\Sigma_i^{-1})} \right) \Sigma_i^{-1} g_i(x) \right\|
\]
\[
\leq \mathbb{E} \left( \left\| \left( \frac{1}{1 + a_i^* \Sigma_i^{-1} a_i} - \frac{1}{1 + \text{tr}(\Sigma_i^{-1})} \right) \Sigma_i^{-1} g_i(x) \right\| \right) \tag{9}
\]
Since
\[
\Pr \left( \left| \frac{1}{1 + a_i^* \Sigma_i^{-1} a_i} - \frac{1}{1 + \text{tr}(\Sigma_i^{-1})} \right| > \frac{t}{\sqrt{m}} \right) \leq \Pr \left( |a_i^* \Sigma_i^{-1} a_i - \text{tr}(\Sigma_i^{-1})| > \frac{t}{\sqrt{m}} \right)
\]
\[
\leq \exp \left[ -c \min \left( \frac{t^2}{m \| \Sigma_i^{-1} \|_{\text{tr}}^2}, \frac{t}{\sqrt{m} \| \Sigma_i^{-1} \|_{\text{tr}}} \right) \right] \leq \exp \left[ -c \min \left( \frac{t^2}{nm \| \Sigma_i^{-1} \|_{\bullet}}, \frac{t}{\sqrt{m} \| \Sigma_i^{-1} \|_{\bullet}} \right) \right]
\]
and \(8\) implies that
\[
\Pr \left( \| \Sigma_i^{-1} \|_{\bullet} < \frac{t}{m-1} \right) < \left( \frac{n}{m-1} \right)^{\frac{1}{2} \left( \frac{m-1}{n} \right)} \tag{10}
\]
so
\[
\Pr \left( \left| \frac{1}{1 + a_i^* \Sigma_i^{-1} a_i} - \frac{1}{1 + \text{tr}(\Sigma_i^{-1})} \right| > \sqrt{\frac{t}{m}} \right) \leq \exp \left[ -c \min \left( \frac{(m-1)^2 t}{nm}, (m-1) \sqrt{t} \right) \right] + \left( \frac{n}{m-1} \right)^{\frac{1}{2} \left( \frac{m-1}{n} \right)} \tag{11}
\]
In addition, with probability \(1 - 2 \exp(-t^2/2), |a_i^T z| < t\) and with probability \(1 - \exp(-tn^2), \|a_i\| < t \sqrt{n}\), which means that with probability \(1 - 2 \exp(-t^2/2) - \exp(-tn^2), \|g_i(x)\| < t^2 \sqrt{n}\). So combining it with \(9\) and \(10\), we have \(7\):
\[
\left\| \mathbb{E} T(x) - \sum_{i=1}^{m} \mathbb{E} \left( \frac{1}{1 + \text{tr}(\Sigma_i^{-1})} \right) \mathbb{E} g_i(x) \right\| < C.
\]
\(6\) can be proved similarly. \(\square\)
Proof of Lemma 3.3. Applying the Tensorization of variance theorem [25 Theorem 2.3], denote the variance when \( \{a_i\}_{i \neq j} \) are fixed by

\[
\text{Var}(z^* T(x)),
\]
then we have

\[
\text{Var}(z^* T(x)) \leq \mathbb{E} \sum_{j=1}^{m} [\text{Var}_j(z^* T(x))].
\]

(11)

Then

\[
\text{Var}_j(z^* T(x)) \leq \left|z^* \Sigma_j^{-1} \sum_{i=1}^{m} a_i(x) - z^* \Sigma_j^{-1} \sum_{i=1, i \neq j}^{m} g_i(x)\right|^2
\]

\[
\leq 2 \left|z^* \Sigma_j^{-1} a_j \frac{a_j^T z}{a_j^T x} a_j^T x\right|^2 + 2 \left|z^* \Sigma_j^{-1} a_j a_j^T \Sigma_j^{-1} \sum_{i=1, i \neq j}^{m} g_i(x)\right|^2
\]

\[
\leq 8 \left|z^* \Sigma_j^{-1} a_j a_j^T z\right|^2 + 2 \left|z^* \Sigma_j^{-1} a_j a_j^T \Sigma_j^{-1} \sum_{i=1, i \neq j}^{m} g_i(x)\right|^2,
\]

which is smaller than

\[
t^2 \left\{ 8 \left|z^* \Sigma_j^{-1} a_j^T z\right|^2 + 2 \left|z^* \Sigma_j^{-1} \sum_{i=1, i \neq j}^{m} g_i(x)\right|^2 \right\}.
\]

Combining it with the estimation of \( ||\Sigma_j^{-1}|| \) in [10] and the estimation of \( || \sum_{i \neq j} g_i(x) || \) in Lemma 3.4 (note that the estimation of \( || \sum_{i \neq j} g_i(x) || \) is similar to the estimation of \( || g(x) || \)), we have

\[
\mathbb{E} \text{Var}_j(z^* T(x)) < C/m^2.
\]

Lemma 3.3 then follows from (11). □

**Lemma 3.4.** There exists \( C > 0 \) such that

\[
\Pr(||g(x)|| > Ct) < \exp(-t^2).
\]

**Proof.** Let \( P_{\text{Sp}(x,z)^\perp} \in \mathbb{C}^{n \times n - 2} \) be a projector matrix to the \( n - 2 \)-dimensional subspace orthogonal to \( \text{Sp}(x,z) \), then

\[
P_{\text{Sp}(x,z)^\perp} g(x) = \sum_{i=1}^{m} \frac{a_i^T z}{|a_i^T x|} a_i^T x P_{\text{Sp}(x,z)^\perp} a_i,
\]

10
where \( P_{\text{Sp}(x,z)} a_i \in \mathbb{C}^{n-2} \) is i.i.d. sampled from \( N(0, I) \) and is independent with respect to \( |a_i^T x| \). As a result, \( P_{\text{Sp}(x,z)} g(x) \in \mathbb{C}^{n-2} \) is a vector that is element-wisely i.i.d. sampled from \( N(0, \sum_{i=1}^{m} |a_i^T z|^2) \).

Applying Hansen-Wright inequality [20], we have

\[
\text{Pr}(\| P_{\text{Sp}(x,z)} g(x) \|^2 > 2tn \sum_{i=1}^{n} |a_i^T z|^2) < \exp(-Cn^2),
\]

and

\[
\| P_{\text{Sp}(x,z)} g(x) \| \leq \sum_{i=1}^{n} \| P_{\text{Sp}(x,z)} a_i a_i^* z \| \leq \sum_{i=1}^{n} \| P_{\text{Sp}(x,z)} a_i \|^2.
\]

In addition, Berstein’s inequality implies that there exists \( C > 0 \) such that

\[
\text{Pr}(\sum_{i=1}^{n} |a_i^* z|^2 > Ct) < \exp(-t^2), \quad \text{Pr}(\sum_{i=1}^{n} \| P_{\text{Sp}(x,z)} a_i \|^2 > Ct) < \exp(-t^2).
\]

Combining these estimations together with

\[
\| g(x) \| \leq \| P_{\text{Sp}(x,z)} g(x) \| + \| P_{\text{Sp}(x,z)^*} g(x) \|,
\]

the lemma is proved. \( \square \)

**Proof of Theorem 2.3.** Applying the Chebyshev’s inequality to and Lemma 3.3, we have that with probability at least \( 1 - 2/\log 2 n \), we have

\[
\| T(x) - \mathbb{E} T(x) \| < C \sqrt{n} \log n/m, \quad \| z^* T(x) - z^* \mathbb{E} T(x) \| < C \log n/m.
\]

Applying [9],

\[
\text{Pr}(\sqrt{n} + t \geq \sigma_1(\Sigma)) > 1 - 2 \exp(-t^2/2),
\]

so \( \mathbb{E}\left( \frac{1}{1+\text{tr}(\Sigma^{-1})} \Sigma^{-1} \right) \) is a scalar matrix with each diagonal elements larger than \( c/m \). In addition, Lemma 3.1 implies that \( \| \mathbb{E} g_i(x) \| > c/n \). Combining it with (13) and Lemma 3.2

\[
\theta(T(x)) = \frac{z^* T(x)}{\| T(x) \|} \geq \frac{c \theta(x) + \tan^{-1} \frac{h'(\theta(x))}{h(\theta(x))} - C \log n/m}{c + C \log n \sqrt{m}/m}.
\]

Applying the assumptions that \( \theta(x) > 2/\log n \sqrt{n} \) and \( m > C_0 n \log^2 n \), and recall that \( h'(x) > 0 \) from Lemma 2.1 Theorem 2.3 is proved. \( \square \)
Figure 2: Comparison between the predicted and the empirical value of $\theta(T(x))$, with various settings of $(n, m)$. 
4 Simulations

This section aims to verify the result in Theorem 2.3. In particular, we would like to investigate whether empirically, \( \theta(x) \) and \( \theta(T(x)) \) has the relation predicted by Theorem 2.3 and its proof:

\[
\theta(T(x)) \approx \theta(x) + \tan^{-1} \frac{h'(\theta(x))}{h(\theta(x))}.
\] (14)

For this purpose, we run simulations and compared the empirically observed \( \theta(T(x)) \) and the predicted values. We run two simulations with different settings of \( n, m \). For each setting and each \( \theta(x) \), we repeat the alternating minimization algorithm randomly by 1000 times, and visualized the 10\%, 50\%, 90\% quantile of the observed \( \theta(T(x)) \) in Figure 2 as well as the predicted value in (14). The figure clearly indicates that our predicted value is close to the empirical values, and as a result, \( T(\theta(x)) > \theta(x) \) with high probability as long as \( \theta(x) \) is not too small, which means that with high probability, the alternating minimization algorithm monotonically reduces the angle between the estimated and the underlying signal. In addition, the variance of the distribution of \( \theta(T(x)) \) is shown to be in the order of \( 1/\sqrt{m} \).

5 Summary and Future Directions

This work analyzes the performance of the alternating minimization algorithm for phase retrieval. Theoretical analysis shows that the angle between the current iteration and the underlying signal is reduced at each iteration with high probability. Based on it, it proved that alternating minimization in a batch setting with random initialization can recover the underlying signal as long as \( m = O(n \log C_0 n) \) for some \( C_0 > 0 \).

A future direction is the analysis of standard alternating minimization without the batch setting. Current work only analyzes the performance of phase retrieval per iteration, as discussed at the end of Section 2.1, it does not apply to the standard alternating minimization algorithm. We hope to find a way to uncouple the correlation between \( x^{(k)} \) and \( A \), to prove the conjecture that alternating minimization algorithm succeeds without the batch setting. It is also interesting to improve the probabilistic estimation in this work, for example, removing the logarithmic factors in the current estimation \( m = O(n \log^3 n) \).

6 Appendix

Proof of Lemma 2.1. Write it in terms of real variables, we have

\[
h(\theta) = \mathbb{E}_{a_1,a_2,b_1,b_2 \sim N(0,1)} \sqrt{a_1^2 + b_1^2} \sqrt{(a_1 \sin \theta + a_2 \cos \theta)^2 + (b_1 \sin \theta + b_2 \cos \theta)^2}
\]
Using \((\sqrt{f(x)})'' = (\frac{1}{2} f(x)^{-1/2} f'(x))' = \frac{1}{2} f(x)^{-1/2} f''(x) - \frac{1}{4} f(x)^{-3/2} f'(x)^2\) and

\[
[(a_1 \sin \theta + a_2 \cos \theta)^2 + (b_1 \sin \theta + b_2 \cos \theta)^2]' \]

\begin{align*}
= & 2(a_1^2 - a_2^2 + b_2^2) \sin \theta \cos \theta + 2(a_1 a_2 + b_1 b_2)(\cos^2 \theta - \sin^2 \theta) \\
= & 2(a_1^2 - a_2^2 + b_1^2 - b_2^2)(\cos^2 \theta - \sin^2 \theta) - 8(a_1 a_2 + b_1 b_2) \cos \theta \sin \theta.
\end{align*}

So

\[
h''(\theta) = \mathbb{E} \frac{\sqrt{a_1^2 + b_1^2}}{|f(\theta)|^{\frac{3}{2}}} \left[ f(\theta) (a_1^2 - a_2^2 + b_1^2 - b_2^2)(\cos^2 \theta - \sin^2 \theta) - 4(a_1 a_2 + b_1 b_2) \cos \theta \sin \theta \right]
\]

and

\[
h''(0) = \mathbb{E} \frac{\sqrt{a_1^2 + b_1^2}}{|a_2^2 + b_2^2|^{\frac{3}{2}}} \left[ a_2^2 + b_2^2 (a_1^2 + b_1^2 - a_2^2 - b_2^2) - (a_1 a_2 + b_1 b_2)^2 \right].
\]

Using the fact that when \(a_1^2 + b_1^2\) and \(a_2^2 + b_2^2\) are fixed then \(\mathbb{E}[a_1 a_2 + b_1 b_2]^2 = \frac{1}{2}[a_1^2 + b_1^2][a_2^2 + b_2^2]\), we have

\[
h''(0) = \mathbb{E} \frac{\sqrt{a_1^2 + b_1^2}}{|a_2^2 + b_2^2|^{\frac{3}{2}}} \left[ \frac{1}{2} [a_2^2 + b_2^2] [a_1^2 + b_1^2] - [a_2^2 + b_2^2]^2 \right]
\]

\[
= \mathbb{E} \frac{1}{2} [a_2^2 + b_2^2]^{-\frac{3}{2}} [a_1^2 + b_1^2]^{\frac{3}{2}} - \sqrt{a_1^2 + b_1^2} \sqrt{a_2^2 + b_2^2}.
\]

Applying

\[
\mathbb{E}(a_1^2 + b_1^2)^k = \frac{1}{\pi} \int_{x,y} (x^2+y^2)^k e^{-x^2-y^2} \, dx \, dy = 2 \int_0^\infty r^{2k+1} e^{-r^2} \, dr = \int_2^\infty z^k e^{-z} \, dz = \Gamma(k+1),
\]

\(h''(0) = \frac{1}{2} \Gamma\left(\frac{1}{2}\right) \Gamma\left(\frac{5}{2}\right) - \Gamma\left(\frac{3}{2}\right)^2 = \frac{\pi}{8} > 0\). Using the fact that

\[
h''(\varphi) = \frac{d}{d\theta} \mathbb{E} \sqrt{(-a_1 \sin \varphi + a_2 \cos \varphi)^2 + (-b_1 \sin \varphi + b_2 \cos \varphi)^2} \sqrt{(a_1 \sin \theta + a_2 \cos \theta)^2 + (b_1 \sin \theta + b_2 \cos \theta)^2} \bigg|_{\theta = \varphi}
\]

and applying the same proof we have

\[
h''(\varphi) = \mathbb{E} \sqrt{(-a_1 \sin \varphi + a_2 \cos \varphi)^2 + (-b_1 \sin \varphi + b_2 \cos \varphi)^2} \frac{[a_2^2 + b_2^2][a_1^2 + b_1^2 - a_2^2 - b_2^2] - [a_1 a_2 + b_1 b_2]^2}{[a_2^2 + b_2^2]^{\frac{3}{2}}}
\]

and as a special case,

\[
h''\left(\frac{\pi}{2}\right) = \mathbb{E} \frac{1}{2} [a_1^2 + b_1^2] - [a_2^2 + b_2^2] = -\frac{1}{2} \Gamma(2) = -1.
\]
Next we will show that $h''(\theta)$ is well-defined and Lipschitz continuous. In fact, applying (15) and the fact that $(-a_1 \sin \phi_1 + a_2 \cos \phi_1)^2 - (-a_1 \sin \phi_2 + a_2 \cos \phi_2)^2 < |\phi_1 - \phi_2|^2(a_1^2 + a_2^2)$,

$$|h''(\phi_1) - h''(\phi_2)| \leq \mathbb{E}|\phi_1 - \phi_2| \sqrt{\frac{a_1^2 + b_1^2 + \sqrt{a_2^2 + b_2^2}}{a_2^2 + b_2^2}} \left[\frac{3}{2} [a_1^2 + b_1^2] |a_1^2 + b_1^2| + [a_2^2 + b_2^2]^2 + [a_1 a_2 + b_1 b_2]^2\right].$$

Then we obtain the Lipschitz continuity of $\varphi''(\theta)$ with Lipschitz factor given by

$$L = \mathbb{E} \sqrt{\frac{a_1^2 + b_1^2 + \sqrt{a_2^2 + b_2^2}}{a_2^2 + b_2^2}} \left[\frac{3}{2} [a_1^2 + b_1^2] |a_1^2 + b_1^2| + [a_2^2 + b_2^2]^2\right] = \frac{3}{2} \Gamma \left(\frac{1}{2}\right) \Gamma \left(\frac{5}{2}\right) + \Gamma(2).$$

Then to prove for all $0 < \theta < \pi/2$, $h'(\theta) \geq \epsilon \min(\theta, \pi/2 - \theta)$, it is sufficient to verify that $\min_{0 < \theta < \pi/2} h'(\theta) > 0$. Note that $h'(\theta)$ is also Lipschitz continuous with a Lipschitz factor $1 + \pi L$, it can be verified by numerically by checking $\theta = \frac{\pi L}{\pi L} + \delta, \frac{\pi L}{\pi L} + 2\delta, \cdots, \frac{\pi}{2} - \frac{\pi}{2\pi L}$ and show that at these locations, $h'(\theta) > \delta/(1 + \pi L)$.

Based on the Lipschitz continuity of $h''(\theta)$ it is easy to obtain the Lipschitz continuity of $h'$ and $h$ in $[0, \pi/2]$. As a result, there exists $c' > 0$ such that $\min_{0 \leq \theta \leq \pi/2} h(\theta) < c'$.

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