Neutrino tri-bi-maximal mixing from a non-Abelian discrete family symmetry

I. de M. Varzielas∗, S. F. King†, G. G. Ross‡

∗Rudolf Peierls Centre for Theoretical Physics,
University of Oxford, 1 Keble Road, Oxford, OX1 3NP
†CERN,1211 Geneva 23, Switzerland
and
School of Physics and Astronomy,
University of Southampton,
Southampton, SO17 1BJ, U.K.

March 26, 2022

Abstract

The observed neutrino mixing, having a near maximal atmospheric neutrino mixing angle and a large solar mixing angle, is close to tri-bi-maximal. We argue that this structure suggests a family symmetric origin in which the magnitude of the mixing angles are related to the existence of a discrete non-Abelian family symmetry. We construct a model in which the family symmetry is the non-Abelian discrete group \( \Delta(27) \), a subgroup of \( SU(3) \) in which the tri-bi-maximal mixing directly follows from the vacuum structure enforced by the discrete symmetry. In addition to the lepton mixing angles, the model accounts for the observed quark and lepton masses and the CKM matrix. The structure is also consistent with an underlying stage of Grand Unification.

1 Introduction

The observed neutrino oscillation parameters are consistent with a tri-bi-maximal structure [1]:

\[
U_{PMNS} \propto \begin{bmatrix}
-\sqrt{\frac{2}{3}} & \sqrt{\frac{1}{3}} & 0 \\
\sqrt{\frac{1}{6}} & \sqrt{\frac{1}{3}} & \sqrt{\frac{1}{2}} \\
\sqrt{\frac{1}{6}} & \sqrt{\frac{1}{3}} & -\sqrt{\frac{1}{2}}
\end{bmatrix}
\]  

(1)

It has been observed that this simple form might be a hint of an underlying family symmetry, and several models have been constructed that account for this structure of leptonic mixing (e.g. [2]). It is possible to extend the underlying family symmetry to provide a complete description of the complete fermionic structure (e.g. [3]) 1, in which, in contrast to the neutrinos, the quarks have a strongly hierarchical structure with small mixing with Yukawa coupling matrices of the form [4]:

\[
Y^u \propto \begin{bmatrix}
0 & \varepsilon^3_u & O(\varepsilon^3_u) \\
\varepsilon^3_u & \varepsilon^2_u & O(\varepsilon^2_u) \\
O(\varepsilon^3_u) & O(\varepsilon^2_u) & 1
\end{bmatrix}
\]  

(2)

1See [4] for review papers with extensive references on neutrino models
where the expansion parameters are given by
\[ \epsilon_u \approx 0.05, \epsilon_d \approx 0.15. \] (4)

A desirable feature of a complete model of quark and lepton masses and mixing angles is that it should be consistent with an underlying Grand Unified structure, either at the field theory level or at the level of the superstring. The family symmetry models which have been built to achieve this are based on an underlying \( G_f \otimes SO(10) \) structure where the family group \( G_f \) is \( SU(3)_f \). This is very constraining because it requires that all the (left handed) members of a single family should have the same family charge. In this paper we will construct a model based on a non-Abelian discrete family symmetry which preserves the possibility of simple unification by requiring that the discrete symmetry properties of all the members of one family are the same. The discrete non-Abelian group\(^2\) we use is \( \Delta(27) \), the semi-direct product group \( Z_3 \rtimes Z_3' \), which is a subgroup\(^3\) of \( SU(3)_f \). Indeed the dominant terms of the Lagrangian leading to the Yukawa coupling matrices of the form of eq.\(^2\) and eq.\(^4\) are symmetric under \( SU(3)_f \) so much of the structure of the model based on \( SU(3)_f \) is maintained. However the appearance of additional terms allowed by \( Z_3 \times Z_3' \) but not by \( SU(3)_f \) determines the vacuum structure and generates the tri-bi-maximal mixing structure. The choice of the multiplet structure ensures that the model is consistent with a stage of Grand or superstring unification and the resulting model is much simpler than that based on the continuous \( SU(3)_f \) symmetry.

In Section 2 we discuss the choice of the non-Abelian discrete group and the multiplet content of the model. Emphasis is put on obtaining a simplified field content and a reduced auxiliary symmetry compared with the \( SU(3)_f \) model in \([3]\). In Section 3 we consider the superpotential terms allowed by the symmetries of the model. Using this we show how the desired vacuum structure arises simply through the appearance of the additional invariants allowed by \( Z_3 \times Z_3' \) but not by \( SU(3)_f \). Section 4 discusses both the Dirac and Majorana mass matrix structure of the model and the resulting pattern of quark, charged lepton and neutrino masses and mixing angles. Finally in Section 5 we present a summary and our conclusions.

2 Field content and symmetries

The symmetry of the model is \( G_f \otimes SU(3) \otimes SU(2) \otimes U(1) \otimes G_f \otimes G \). The additional symmetry group \( G \) is needed to restrict the form of the allowed coupling of the theory and is chosen to be as simple as possible. As discussed above, the family group \( G_f \) is chosen as a non-Abelian discrete group of \( SU(3)_f \) in a manner that preserves the structure of the fermion Yukawa couplings of the associated \( SU(3)_f \) model of \([3]\). This means that \( G_f \) should be a non-Abelian subgroup of \( SU(3)_f \) of sufficient size that it approximates \( SU(3)_f \) in the sense that most of the leading terms responsible for the fermion mass structure in the \( SU(3)_f \) are still the leading terms allowed by \( G_f \) (which being a subgroup, allows further terms which we want to be subleading). The smallest group we have found that achieves this is \( \Delta(27) \), the semi-direct product group \( Z_3 \rtimes Z_3' \). The main change that results from using this smaller symmetry group is the appearance of additional invariants which drive the desired vacuum structure and, because we are no longer dealing with a continuous symmetry, the absence of the associated \( D \)-terms which were very important in determining the vacuum structure in the \( SU(3)_f \) model \([3]\). Due to this, we are able to reduce the total field content of this model, which in turn only requires an additional \( G = U(1) \otimes Z_2 \otimes R \) to control the allowed terms in the superpotential \(^4\) (c.f. \( U(1) \otimes U(1)' \otimes R \) in \([3]\)).

In choosing the representation content of the theory we are guided by the structure of the \( SU(3)_f \) model of \([3]\) which generated a viable form of all quark and lepton masses and mixing. Since \( Z_3 \times Z_3' \) is a discrete subgroup of \( SU(3)_f \) all invariants of \( SU(3)_f \) are invariants of \( Z_3 \times Z_3' \). Using this we can readily arrange that the superpotential terms responsible for fermion masses in the \( SU(3)_f \) model are present in the \( Z_3 \times Z_3' \) model. To implement this we find it convenient to label the representation of the fields of our model by their transformation

\[
Y^d \propto \begin{pmatrix}
0 & 1.5\epsilon_d^3 & 0.4\epsilon_d^2 \\
1.5\epsilon_d^3 & \epsilon_d^2 & 1.3\epsilon_d^2 \\
O(\epsilon_d^3) & O(\epsilon_d^3) & 1
\end{pmatrix}
\] (3)

\(^2\)Such non-Abelian discrete symmetries often occur in compactified string models.
\(^3\)\( Z_3 \times Z_3' \) (where the generators of the distinct \( Z_3 \) don’t commute) is the group \( \Delta(27) \) \([8]\).
\(^4\)\( R \) is an \( R \)- symmetry and for SUSY purposes plays the same role as \( R \)-parity.
properties under the approximate $SU(3)_f$ family group. The Standard Model (SM) fermions $\psi_i, \psi^c_j$ transform as triplets under this group. The transformation properties of such triplets under the $Z_3 \ltimes Z'_3$ discrete group are shown in Table 1. Although the gauge group is just that of the Standard Model it is also instructive, in considering how the model can be embedded in a unified structure, to display the properties of the states under the $SU(4)_P \otimes SU(2)_L \otimes SU(2)_R$ subgroup of $SO(10)$ and this is done in Table 2. We also show in Table 2 the transformation properties under the additional symmetry group $G = U(1) \otimes Z_2 \otimes R$. The transformation properties of the SM Higgs, $H$, responsible for electroweak breaking are also shown in Table 2.

In a complete unified theory, quark and lepton masses will be related. A particular question that arises in such unification is what generates the difference between the down quark and charged lepton masses. In [6] this was done through a variant of the Georgi-Jarlskog mechanism via the introduction of another Higgs field $H_{45}$, which transforms as a 45 of an underlying $SO(10)$ GUT. It has a vacuum expectation value (vev) which breaks $SO(10)$ but leaves the SM gauge group unbroken. In this model we include $H_{45}$ to demonstrate that the model readily Grand Unifies but in practice we only use its vev. This does not necessarily imply that there is an underlying stage of Grand Unification below the string scale but, if not, the underlying theory should provide an alternative explanation for the existence of the pattern of low energy couplings implied by terms involving $H_{45}$.

At this stage there are no terms generating fermion masses and to complete the model it is necessary to break the family symmetry $Z_3 \ltimes Z'_3$ through the introduction of “flavons” that acquire vevs. To reproduce the results of the phenomenologically viable $SU(3)_f$ model we choose a similar but somewhat simplified flavon structure with the $SU(3)_f$ antitriplet fields $\bar{\phi}^i$, $\bar{\phi}^{i}_{23}$ and $\bar{\phi}^{i}_{123}$ and $SU(3)_f$ triplet fields $\phi_i$, $\phi'_i$, as shown in Table 2 and one triplet field for alignment purposes $\phi^A_i$. The transformation properties of these fields under $Z_3 \ltimes Z'_3$ are shown in Table 1. With this choice, as discussed in the next Section, the Yukawa structure of the $SU(3)_f$ model is obtained. One may readily check that the additional terms allowed by the $Z_3 \ltimes Z'_3$ symmetry are subleading in this sector so the phenomenologically acceptable pattern of fermion masses and mixings obtained in [6] is reproduced here if the flavon vacuum structure is as given in [6]. The main difference between the models is the appearance in the potential determining the vacuum structure of additional invariants allowed by $Z_3 \ltimes Z'_3$ and the absence of the $D$–terms associated with a continuous gauge symmetry.

### 3 Symmetry breaking

Following [6] the desired pattern of vevs is given by

\[
\langle \bar{\phi}_{3} \rangle^T = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} a_u & 0 \\ 0 & a_d \end{pmatrix}
\]

\[
\langle \bar{\phi}_{23} \rangle^T = \begin{pmatrix} 0 \\ -b \\ b \end{pmatrix}
\]

\[\text{5Two Higgs are required due to SUSY, represented as } H, \text{ they have the same charges under } G_f \text{ and } G.\]
The former is\[SU(3)_f\]gives rise to two independent quartic invariants under\[\mathbb{Z}_χ\]associated with these higher dimension operators (which can even be the Planck mass\[\Lambda\]), considered here the leading higher order term is of the form\[m^2\phi^i\phi^i\phi^i\phi^i\]not remain true when higher order terms allowed by the discrete family symmetry are included. For the model in one of a finite set of possible minima. However this may only be apparent if higher order terms in the single direction, for example the 3 direction. Due to the underlying discrete symmetry the vev will be quantised symmetric theory, it is not possible in general to rotate the vacuum expectation value of a triplet field to a desired alignment.

Radiative corrections involving superpotential couplings to massive states may drive the mass squared negative\[v^2\leq\Lambda^2\]set radiatively\[\phi\]symmetry. However this does not remain true when higher order terms allowed by the discrete family symmetry are included. For the model considered here the leading higher order term is of the form\[m^2\phi^i(\phi^\dagger φ^j)\]arising as a component of the \([D−]−\text{term}\)\[\chi^1\chi^1\chi^1\chi^1\]. In this we have suppressed the coupling constants and the messenger mass scale (or scales),\[M\], associated with these higher dimension operators (which can even be the Planck mass\[M_P\]). The \([F−]−\text{component}\)\[\chi^1\chi^1\chi^1\chi^1\]drives supersymmetry breaking and\[m_{3/2}\]is the graviton mass\((\phi^i/\phi^j\phi^j\phi^j\phi^j)\). The former is\[SU(3)_f\]symmetric and does not remove the vacuum degeneracy. The second term is not\[SU(3)_f\]

| Field | SU(3)_f | SU(4)_PS | SU(2)_L | SU(2)_R | R | U(1) | \(Z_2\) |
|-------|---------|----------|---------|---------|---|------|---------|
| \(ψ\) | 3       | 4        | 2       | 1       | 1 | 0    | 1       |
| \(ψ^c\) | 3       | 4        | 1       | 2       | 1 | 0    | 1       |
| \(θ\) | 3       | 4        | 1       | 2       | 0 | 0    | −1      |
| \(H\) | 1       | 1        | 2       | 2       | 0 | 0    | 1       |
| \(H_{45}\) | 1     | 15       | 1       | 3       | 0 | 2    | 1       |
| \(φ_{123}\) | 3     | 1        | 1       | 1       | 0 | −1   | 1       |
| \(φ_3\) | 3       | 1        | 1       | 1       | 0 | 3    | 1       |
| \(φ_1\) | 3       | 1        | 1       | 1       | 0 | −4   | −1      |
| \(φ_3\) | 3       | 1        | 1       | 3 ⋅ 1   | 0 | 0    | −1      |
| \(φ_{23}\) | 3     | 1        | 1       | 1       | 0 | −1   | −1      |
| \(φ_{123}\) | 3     | 1        | 1       | 1       | 0 | 1    | −1      |

Table 2: Symmetries and Charges

\[
\langle φ_{123}\rangle \propto \langle \overline{φ}_{123}\rangle^T = \begin{pmatrix} c \\ c \\ c \end{pmatrix}
\]

(7)

\[
\langle φ_1\rangle \propto \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}
\]

(8)

\[
\langle θ\rangle \propto \langle φ_3\rangle \propto \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}
\]

(9)

where the \(SU(2)_R\) structure of \(\langle \overline{φ}_3\rangle\) has been displayed.

The alignment of these vevs can proceed in various ways. By including additional driving fields in the manner discussed in [10] one can arrange their \(F−\text{terms}\) give a scalar potential whose minimum has the desired vacuum alignment. Here however we show that an even simpler mechanism involving \(D−\text{terms}\) only achieves the desired alignment.

To understand how this vacuum alignment works note that, unlike the case for the continuous \(SU(3)_f\) symmetric theory, it is not possible in general to rotate the vacuum expectation value of a triplet field to a single direction, for example the 3 direction. Due to the underlying discrete symmetry the vev will be quantised in one of a finite set of possible minima. However this may only be apparent if higher order terms in the potential are included for the lower order terms may have the enhanced \(SU(3)_f\) symmetry.

To make this more explicit, consider a general \(SU(3)_f\) triplet field \(φ_i\). It will have a SUSY breaking soft mass term in the Lagrangian of the form\[m^2\phi^i\phi^i\]which is invariant under the approximate \(SU(3)_f\) symmetry. Radiative corrections involving superpotential couplings to massive states may drive the mass squared negative at some scale \(Λ\) triggering a vev for the field \(φ_i\), \([\phi^\dagger φ_i]=v^2\), with \(v^2\leqΛ^2\) set radiatively\[6\]. At this stage the vev of \(φ\) can always be rotated to the 3 direction using the approximate \(SU(3)_f\) symmetry. However this does not remain true when higher order terms allowed by the discrete family symmetry are included. For the model considered here the leading higher order term is of the form\[m^2\phi^i(\phi^\dagger φ^j)\]arising as a component of the \([D−]−\text{term}\). In this we have suppressed the coupling constants and the messenger mass scale (or scales), \(M\), associated with these higher dimension operators (which can even be the Planck mass\(M_P\)). The \([F−]−\text{component}\)\[\chi^1\chi^1\chi^1\chi^1\]drives supersymmetry breaking and\[m_{3/2}\]is the graviton mass\((\phi^i/\phi^j\phi^j\phi^j\phi^j)\). The former is\(SU(3)_f\) symmetric and does not remove the vacuum degeneracy. The second term is not\(SU(3)_f\)

\[6\text{The radiative corrections to the soft mass term depend on the details of the underlying theory at the string or unification scale.}\]
symmetric and does lead to an unique vacuum state. For the case that the coefficient of $m^2_{3/2}(\phi_i^2 \phi_j \phi_k^2)$ is positive the minimum corresponds to the vev $T<\phi_i > T= \nu(1,1,1)/\sqrt{3}$ (c.f. eq. 6). For the case the coefficient is negative, the vev has the form $<\phi_i > T= \nu(0,0,1)$ (c.f. eq. 3). Thus we see that, in contrast to the continuous symmetry case, the discrete non-Abelian symmetry leads to a finite number of candidate vacuum states. Which one is chosen depends on the sign of the higher dimension term which in turn depends on the details of the underlying theory. In this paper we do not attempt to construct the full theory and so cannot determine this sign. What we will demonstrate, however, is that one of the finite number of candidate vacua does have the correct properties to generate a viable theory of fermion masses and mixings.

The vacuum alignment needed for this model can now readily be obtained. Suppose that a combination of radiative corrections and the $U(1)$ D-term drive $m^2_{\phi_{123}}$, $m^2_{\phi_1}$ and $m^2_{\phi_3}$ negative close to the messenger scale, $\Lambda_{\phi_{123},\phi_1,\phi_3} \lesssim M$. The symmetries of the model ensure that the leading terms fixing their vacuum structure are of the form $m^2_{3/2}(\phi_i^1 \phi_j \phi_k^1 \phi_1)$, $m^2_{3/2}(\phi_i \phi_1^i \phi_1^j)$, $m^2_{3/2}(\phi_i^1 \phi_1 \phi_1^j \phi_3)$, plus similar terms involving $\phi_3$. Provided the unmixed terms of the form of the first two terms dominate the vevs will be determined by the signs of these terms. If the quartic term involving $\phi_{123}$ is positive $\phi_{123}$ will acquire a vev in the $(1,1,1)$ direction as in eq. 7. If the quartic term involving $\phi_1$ is negative $\phi_1$ will acquire a vev in the $(1,0,0)$ direction as in eq. 8 where the non zero entry just defines the 1 direction. Finally if the quartic term involving $\phi_3$ is also negative it will acquire a vev with a single non-zero entry but the position of this entry will depend on the leading $D-$term resolving this ambiguity. If the term $m^2_{3/2}(\phi_i^1 \phi_1^j \phi_3)$ dominates and has positive coefficient it will force the vevs of these fields to be orthogonal and so $\phi_3$ has a vev in the $(0,0,1)$ direction, c.f. eq. 8, where again the non zero entry just defines the 3 direction. In a similar manner it is straightforward to see how the fields $\phi_i$ and $\theta$ align along the $(0,0,1)$ direction if the quartic terms $m^2_{3/2}(\phi_i \phi_3 \phi_1^j \phi_1)$ and $m^2_{3/2}(\phi_i \phi_1 \phi_1^j \phi_3)$ dominate and have negative coefficients. The scale of their vevs is determined by the scale at which their soft mass squared become negative (the direction of $\langle \phi_i \rangle$ is not very relevant, but with the above terms similar to $\theta$ it can take the form in eq. 8 and we take it to be so for simplicity).

The relative alignment of the remaining terms follows in a similar manner. Consider the field $\phi_{123}$ with a soft mass squared becoming negative at a scale $b < v$. For $\phi_{123}$ we want the dominant term aligning its vev to be $m^2_{3/2}(\phi_i \phi_3 \phi_1 \phi_1^j \phi_1)$, with positive coefficient. It will then acquire a vev orthogonal to that of $\phi_{123}$. The choice of the particular orthogonal direction will be determined by terms like $m^2_{3/2}(\phi_i \phi_1 \phi_1^j \phi_3)$ or $m^2_{3/2}(\phi_i \phi_2 \phi_3 \phi_3)$. If the latter dominates with a positive coefficient, it will drive $\langle \phi_{123} \rangle$ orthogonal to $\phi_1$ - the form given in eq. 8. Finally consider the field $\phi_{123}$ with a soft mass squared becoming negative at a scale $c \ll v$. The leading terms determining its vacuum alignment are $m^2_{3/2}(\phi_i \phi_3 \phi_1 \phi_1^j \phi_1)$ and $m^2_{3/2}(\phi_i \phi_1 \phi_1^j \phi_3)$. If the latter dominates with a negative coefficient, $\phi_{123}$ will be aligned in the same direction as $\phi_{123}$ and have the form given in eq. 7. Note that the term involving $\phi_{123}$ is accidental in the sense that it is dependant on the $U(1)$ assignments of the field.

In summary, we have shown that higher order $D-$terms constrained by the discrete family symmetry lead to a discrete number of possible vacuum states. Which one is the vacuum state depends on the coefficients of these higher order terms which are determined by the underlying unified GUT or string theory. Our analysis has shown that the vacuum structure needed for a viable theory of fermion masses can readily emerge from this discrete set of states.

\footnote{In general, the phases are different for each entry of this vev. For simplicity we omit them, as they don’t affect the results.}
4 The mass matrix structure

4.1 Yukawa terms

We turn now to the structure of the quark and lepton mass matrices. The leading Yukawa terms allowed by the symmetries are:

\[ P_Y \sim \frac{1}{M^2} \tilde{\phi}_3 \psi_i \phi^c_j \psi_j H \]  
\[ + \frac{1}{M^3} \tilde{\phi}_{23} \psi_i \phi^c_j \psi_j H H_{45} \]  
\[ + \frac{1}{M^2} \tilde{\phi}_{123} \psi_i \phi^c_j \psi_j H \]  
\[ + \frac{1}{M^5} \phi_{123} \psi_i \phi^c_j \psi_j H_{45} \phi_{123} \phi_{123} \phi_{123} \phi_{123} \phi_{123} \]

Although of a slightly different from from that in [6] these terms realize the same mass structure and we refer the reader to [6] for the details. It gives a phenomenologically consistent description of all the quark masses and mixing angles and the charged lepton masses, generating their hierarchical structure through an expansion in the family symmetry breaking parameters. The main differences in the way this is achieved lies in eqs. (14, 15, 16). The terms in eqs. (14,15) account for the observed $O(\epsilon_3^3)$ difference in the 12, 21 and 13, 31 entries of the down-type quark mass matrix (c.f. eq.(3)) [5].

The term in eq.(16) is undesirable, but allowed by the symmetries nonetheless. Naively, one expects it would contribute to the 11 element at $O(\epsilon_4^4)$ giving unwanted corrections to the phenomenologically successful Gatto-Sartori-Tonin relation [11] which results if the 11 entry is less than this order [6]. Fortunately, this texture zero is preserved at that order, as the vevs of $\phi_3$ and $\bar{\phi}_3$ are slightly smaller than the relevant messenger mass scales, and in the eq.(16) there are four such fields, suppressing the term sufficiently. As such, the desired small magnitude of this term can be maintained while keeping the dimensionless coefficients in front of all the allowed Yukawa terms as $O(1)$.

4.2 Majorana terms

The leading terms that contribute to the right-handed neutrino Majorana masses are:

\[ P_M \sim \frac{1}{M} \theta^i \psi_i \theta^j \psi_j \]  
\[ + \frac{1}{M^2} \phi^c_{23} \psi^c_i \phi^c_{23} \psi^c_j \theta^k \phi_{123} \theta^l \phi_{123} \]  
\[ + \frac{1}{M^5} \phi^c_{123} \psi^c_i \phi^c_{123} \psi^c_j \theta^k \phi_{123} \theta^l \phi_{123} \]

Note that these terms are different from those in [13] and lead to a different form for the ratios of the Majorana masses. The vev of $\phi_3$ controls the hierarchy between $M_1$ (given essentially by eq.(19)) and $M_2$ (from eq. [18]).

8We take a symmetric form for the mass matrices as would be expected if there is an underlying $SO(10)$ GUT [2]
It is set by radiative breaking to lie close to the scale of $\sqrt{\bar{\phi}_{23}}$, such that after seesaw we can fit the ratio of the neutrino squared mass differences $\frac{\Delta m^2}{\Delta m^2_{\odot}}$. The hierarchy between the lightest Majorana mass $M_1$, and the heaviest, $M_3$ is

$$\frac{M_1}{M_3} \approx \epsilon^4 \frac{M_4^4}{M_{\nu_R}^4}$$  \hspace{1cm} (20)$$

where $M_d$ is the mass of the messenger responsible for the down quark mass (for details on the messenger sector, we again refer the reader to [6]).

For a viable pattern of neutrino mixing we need to ensure that the hierarchy in eq.(20) is sufficiently strong to suppress the contribution from $\nu_3$ exchange which would otherwise give an unacceptably large $\nu_\tau$ component in the atmospheric (and/or solar) neutrino eigenstates. This requirement on the Majorana hierarchy puts a lower bound on the mass of corresponding right-handed neutrino messenger, as is clear from eq.(20). The light neutrino eigenstates also have an hierarchical mass structure so the heaviest of the light effective neutrinos has a mass given approximately by $\sqrt{\Delta m^2_{\odot}}$. Using this, together with eq.(20), we find

$$M_3 \approx \epsilon^2 (\langle H \rangle^2 \frac{M_4^4}{M_{\nu_R}^4} \Delta m^2_{\odot} 2^{-\frac{1}{2}} \approx 10^{13} \frac{M_4^4}{M_{\nu_R}^4} \text{GeV}$$  \hspace{1cm} (21)$$

where $M_{\nu_R}$ is the mass of the messenger responsible for the Dirac neutrino mass.

The final structure of neutrino mixing is very similar to the one in [6], and generates the same predictions for the neutrino mixing angles. The leptonic mixing angles are obtained after taking into account the (small) effect of the charged leptons, yielding nearly tri-bi-maximal mixing [12]:

$$\sin^2 \theta_{12} = \frac{1}{3} \pm 0.052$$  \hspace{1cm} (22)$$

$$\sin^2 \theta_{23} = \frac{1}{2} \pm 0.061$$  \hspace{1cm} (23)$$

$$\sin^2 \theta_{13} = 0.0028$$  \hspace{1cm} (24)$$

This leads to the prediction for the reactor angle of $\theta_{13} \approx \theta_C/(3\sqrt{2}) \approx 3^\circ$, where $\theta_C$ is the Cabibbo angle, i.e. the prediction is a factor of 3 smaller than the Cabibbo angle due to the Georgi-Jarlskog factor, and also a factor of $\sqrt{2}$ smaller due to commutation through the maximal atmospheric angle. Also $\theta_{12}$ can be related to $\theta_{13}$ and the CP violating phase $\delta$, via the so called neutrino sum rule first derived by one of us in [3]:

$$\theta_{12} + \theta_{13} \cos(\delta - \pi) \approx 35.26^\circ.$$  \hspace{1cm} (25)$$

The above predictions were first shown to follow from the charged lepton corrections to tri-bi-maximal mixing in the $SO(3)$ model proposed by one of us in [3] and later shown to be applicable to a class of models in [12], including the present model discussed here and in [6].\hspace{1cm}10

5 Summary and conclusions

We have constructed a complete theory of fermion masses and mixings based on the spontaneous breaking of the discrete non-Abelian symmetry group $Z_3 \ltimes Z'_3$. The model is constructed in a manner consistent with an underlying Grand Unified symmetry with all the members of a family of fermions having the same symmetry properties under the family symmetry group. Many of the properties of the model rely on the approximate $SU(3)_f$ symmetry that the discrete group possesses and the model is very close to the continuous $SU(3)_f$ family symmetry model of reference [6]. The main difference is a significant simplification in the vacuum alignment mechanism in which the near tri-bi-maximal mixing in the lepton sector directly follows from the

\hspace{1cm}9This is different from the $SU(3)_f$ model [10] which predicted the ratio $\frac{M_1}{M_2}$ to be associated with the expansion parameter $\epsilon_d$ that was set by the quark sector, and was consistent with the experimentally measured value $\frac{\Delta m^2_{\odot}}{\Delta m^2_{\odot}}$.

\hspace{1cm}10Note that the prediction for $\theta_{13}$ in [9] has been corrected here.
non-Abelian discrete group. In addition to the prediction of near tri-bi-maximal mixing the model preserves
the Gatto-Sartori-Tonin [11] relation between the light quark masses and the Cabibbo mixing angle, and can
accommodate the GUT relations between the down quark and lepton masses. It also provides an explanation
for the hierarchy of quark masses and mixing angles in terms of an expansion in powers of a family symmetry
breaking parameter.

Acknowledgments

We are grateful to Michal Malinsky for pointing out an error in the vacuum alignment discussion in the original
version of this paper.

The work of I. de M. Varzielas was supported by FCT under the grant SFRH/BD/12218/2003.

This work was partially supported by the EC 6th Framework Programme MRTN-CT-2004-503369.

References

[1] L. Wolfenstein, Phys. Rev. D 18 (1978) 958; P. F. Harrison, D. H. Perkins and W. G. Scott, Phys.
Lett. B 530 (2002) 167 [arXiv:hep-ph/0202074]; P. F. Harrison and W. G. Scott, Phys. Lett. B 535
(2002) 163 [arXiv:hep-ph/0203209]; P. F. Harrison and W. G. Scott, Phys. Lett. B 557 (2003) 76
[arXiv:hep-ph/0302025]; C.I.Low and R.R.Volkas, hep-ph/0305243.

[2] W. Grimus and L. Lavoura, JHEP 0601, 018 (2006) [arXiv:hep-ph/0509239]; E. Ma,
arXiv:hep-ph/0511133 G. Altarelli and F. Feruglio, Nucl. Phys. B 741, 215 (2006) [arXiv:hep-ph/0512103];
N. Haba, A. Watanabe and K. Yoshioka, arXiv:hep-ph/0603116 R. N. Mohapatra, S. Nasri and H. B. Yu,
arXiv:hep-ph/0605020.

[3] K. S. Babu and J. Kubo, Phys. Rev. D 71 (2005) 056006 [arXiv:hep-ph/0411226]; S. F. King, JHEP 0508,
105 (2005) [arXiv:hep-ph/0506297].

[4] S. F. King, Rept. Prog. Phys. 67 (2004) 010 [arXiv:hep-ph/0310204]; A. Zee, Phys. Lett. B 630 (2005)
58 [arXiv:hep-ph/0508278]; R. N. Mohapatra et al., [arXiv:hep-ph/0510213]; R. N. Mohapatra and
A. Y. Smirnov, arXiv:hep-ph/0603118.

[5] R. G. Roberts, A. Romanino, G. G. Ross and L. Velasco-Sevilla, Nucl. Phys. B 615, 358 (2001)
arXiv:hep-ph/0104088.

[6] I. de Medeiros Varzielas and G. G. Ross, arXiv:hep-ph/0507176.

[7] S. F. King and G. G. Ross, Phys. Lett. B 520 (2001) 243 [arXiv:hep-ph/0108112]; S. F. King and G. G. Ross,
Phys. Lett. B 574 (2003) 239 [arXiv:hep-ph/0307190].

[8] W. M. Fairbairn, T. Fulton, and W. H. Klink Journal of Mathematical Physics – August 1964 – Volume
5, Issue 8, pp. 1038-1051.

[9] H. Georgi and C. Jarlskog, Phys. Lett. B 86 (1979) 297.

[10] I. de Medeiros Varzielas, S. F. King and G. G. Ross, arXiv:hep-ph/0512313.

[11] R. Gatto, G. Sartori and M. Tonin, Phys. Lett. B 28 (1968) 128.

[12] S. Antusch and S. F. King, Phys. Lett. B 631 (2005) 42 arXiv:hep-ph/0508044.