Approximating Max-Cut under Graph-MSO Constraints

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Abstract

We consider the max-cut and max-$k$-cut problems under graph-based constraints. Our approach can handle any constraint specified using monadic second-order (MSO) logic on graphs of constant treewidth. We give a $\frac{\alpha}{2}$-approximation algorithm for this class of problems.

Keywords: max cut, approximation algorithm, monadic second-order logic, treewidth, dynamic program

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1. Introduction

This paper considers the classic max-cut problem under a class of graph-based constraints. The max-cut problem is a fundamental combinatorial-optimization problem which has many practical applications (see \cite{1, 2, 3, 4}) as well as strong theoretical results (see \cite{5, 6}). There have also been a number of papers on designing approximation algorithms for constrained max-cut problems (see \cite{7, 8, 9, 10, 11}).

In this paper, we are interested in constraints that are specified by an auxiliary constraint graph. Our main result is a $\frac{\alpha}{2}$-approximation algorithm for max-cut under any graph constraint that can be expressed in monadic second order logic (MSO) (see \cite{12}). This is closely related to a recent result by a subset of the authors; see \cite{13}. The contribution of this paper is in generalizing the class of constraints handled in \cite{13}, making the algorithm design more systematic, and extending the result to the max-$k$-cut setting with $k$ instead of just 2 parts.

In particular, \cite{13} gave a $\frac{\alpha}{2}$-approximation algorithm for max-cut under any graph constraint $S_G$ that has a specific type of dynamic program for optimizing linear objectives. In order to apply this result, one also has to design such a dynamic program separately for each constraint $S_G$, which requires additional constraint-specific work. Indeed, \cite{13} also gave constraint-specific dynamic programs for various graph constraints such as independent set, vertex cover, dominating set and connectivity, all on bounded-treewidth graphs.

In this paper, we bypass the need for constraint-specific dynamic programs by utilizing the language and results from monadic second-order logic. We show that any MSO constraint on a bounded-treewidth graph (defined formally in \cite{12}) admits a dynamic program that satisfies the assumptions needed in \cite{13}. Therefore, we immediately obtain $\frac{\alpha}{2}$-approximation algorithms for max-cut under any MSO graph constraint. We note that MSO constraints capture all the specific graph constraints in \cite{13}, and much more.

We also extend these results to the setting of max-$k$-cut, where we seek to partition the vertices into $k$ parts $\{U_i\}_{i=1}^k$ so as to maximize the weight of edges crossing the partition. In the constrained version, we additionally require each part $U_i$ to satisfy some MSO graph property. We obtain a $\frac{\alpha}{2}$-approximation algorithm even in this setting ($k$ is fixed). This result is a significant generalization over \cite{13} even for $k = 2$, which corresponds to the usual max-cut problem: we now handle constraints on both sides of the cut.

2. Preliminaries

A $k$-partition of vertex set $V$ is a function $h : V \rightarrow [k]$, where the $k$ parts are $U_\alpha = \{v \in V : h(v) = \alpha\}$ for $\alpha \in [k]$. Note that $\bigcup_{\alpha=1}^k U_\alpha = V$ and $U_1, \cdots, U_k$ are disjoint. When we want to refer to the $k$ parts directly, we also use $\{U_\alpha\}_{\alpha=1}^k$ to denote the $k$-partition.

Definition 1 (GCMC). The input to the graph-constrained max-cut (GCMC) problem consists of (i) an $n$-vertex graph $G = (V,E)$ with a graph property which implicitly specifies a collection $S_G$ of vertex $k$-partitions, and (ii) symmetric edge-weights $c : \binom{V}{2} \rightarrow \mathbb{R}_+$. The GCMC problem is to find a $k$-partition in $S_G$ with the maximum weight of crossing edges:

$$\max_{h \in S_G} \sum_{\{u,v\} \in \binom{V}{2}} c(u,v).$$  \hspace{1cm} (1)
Tree Decomposition. Given an undirected graph $G = (V, E)$, a tree decomposition consists of a tree $T = (I, F)$ and a collection of vertex subsets $\{X_i \subseteq V\}_{i \in I}$ such that:

- for each $v \in V$, the nodes $\{i \in I : v \in X_i\}$ are connected in $T$, and
- for each edge $(u, v) \in E$, there is some node $i \in I$ with $u, v \in X_i$.

The width of such a tree decomposition is $\max_{i \in I} \{|X_i| - 1\}$, and the treewidth of $G$ is the smallest width of any tree decomposition for $G$.

We work with “rooted” tree decompositions, also specifying a root node $r \in I$. The depth $d$ of such a tree decomposition is the length of the longest root-leaf path in $T$. The depth of any node $i \in I$ is the length of the $r \leftarrow i$ path in $T$. For any $i \in I$, the set $V_i$ denotes all the vertices at or below node $i$, that is

$$V_i := \bigcup_{k \in \mathcal{T}_i} X_k,$$

where $\mathcal{T}_i := \{k \in I : k \text{ is subtree of } T \text{ rooted at } i\}$.

The following result provides a convenient representation of $T$.

**Theorem 2.1 (Balanced Tree Decomposition; see [13]).** Let $G = (V, E)$ be a graph with tree decomposition $(T = (I, F), \{X_i\}_{i \in I})$ of treewidth $k$. Then $G$ has a rooted tree decomposition $(T' = (I', F'), \{X_i'\}_{i \in I'})$ where $T'$ is a binary tree of depth $2\lceil \log_2 |V| \rceil$ and treewidth at most $3k + 2$. Moreover, for all $i \in I$, there is an $i' \in I'$ such that $X_i = X_i'$. The tree decomposition $T'$ can be found in $O(|V|^2)$ time.

**Definition 2 (CSP instance).** A Constraint Satisfaction Problem (CSP) instance $J = (N, C)$ consists of:

- a set $N$ of boolean variables, and
- a set $C$ of constraints, where each constraint $C_u \in C$ is a $|U|$-ary relation $C_u \subseteq \{0, 1\}^U$ on some subset $U \subseteq N$.

For a vector $x \in \{0, 1\}^N$ and a subset $R$ of variables, we denote by $x_R$ the restriction of $x$ to $R$. A vector $z \in \{0, 1\}^N$ satisfies constraint $C_u \in C$ if $z|_U \subseteq C_U$. We say that $z \in \{0, 1\}^N$ is a feasible assignment for the CSP instance $J$ if $z$ satisfies every constraint $C \in C$. Let $\text{Feas}(J)$ be the set of all feasible assignments of $J$. Finally, $|C| = \sum_{C \in C} |C_U|$ denotes the length of $C$.

**Definition 3 (Constraint graph).** The constraint graph of $J$, denoted $G(J)$, is defined as $G(J) = (N, F)$ where $F = \{(u, v) \mid \exists C \in C \text{ s.t. } (u, v) \subseteq U\}$.

**Definition 4 (Treewidth of CSP).** The treewidth $\text{tw}(J)$ of a CSP instance $J$ is defined as the treewidth of its constraint graph $G(J)$.

**Definition 5 (CSP extension).** Let $J = (N, C)$ be a CSP instance. We say that $J' = (N', C')$ with $N' \subseteq N'$ is an extension of $J$ if $\text{Feas}(J) = \{z|_{N'} \mid z \in \text{Feas}(J')\}$.

Monadic Second Order Logic. We briefly introduce MSO over graphs. In first-order logic (FO) we have variables for individual vertices/edges (denoted $x, y, \ldots$), equality for variables, quantifiers $\forall, \exists$ ranging over variables, and the standard Boolean connectives $\neg, \land, \lor, \Rightarrow$. MSO is the extension of FO by quantification over sets (denoted $X, Y, \ldots$). Graph MSO has the binary relational symbol $\text{edge}(x, y)$ encoding edges, and traditionally comes in two flavours, MSO$_1$ and MSO$_2$, differing by the objects we are allowed to quantify over: in MSO$_1$ these are the vertices and vertex sets, while in MSO$_2$ we can additionally quantify over edges and edge sets. For example, 3-colorability can be expressed in MSO$_1$ as follows:

$$\exists X_1, X_2, X_3 \left[ \left( \forall x \left( x \in X_1 \lor x \in X_2 \lor x \in X_3 \right) \land \bigwedge_{i=1,2,3} \forall x, y \left( \left( x \not\in X_i \land y \not\in X_i \right) \land \neg \text{edge}(x, y) \right) \right) \right].$$

We remark that MSO$_2$ can express properties that are not MSO$_1$ definable. As an example, consider Hamiltonicity on graph $G = (V, E)$; an equivalent description of a Hamiltonian cycle is that it is a connected 2-factor of a graph:

$$\varphi_{\text{ham}} \equiv \exists F \subseteq E : \varphi_{\text{2-factor}}(F) \land \varphi_{\text{connected}}(F)$$

$$\varphi_{\text{2-factor}}(F) \equiv \left( \forall v \in V : \exists e, f \in F : (e \neq f) \land (v \in e) \land (v \in f) \right)$$

$$\varphi_{\text{connected}}(F) \equiv \neg \exists U, W \subseteq V : (U \cap W = \emptyset) \land \left( (U \cup W = V) \land \neg \exists \{u, v\} \in F : u \in U \land v \in W \right) .$$

We use $\varphi$ to denote an MSO formula and $G = (V, E)$ for the underlying graph. For a formula $\varphi$, we denote by $|\varphi|$ the size (number of symbols) of $\varphi$.

In order to express constraints on $k$-vertex-partitions via MSO, we use MSO formulas $\varphi$ with $k$ free variables $\{U_i\}_{i=1}^k$ where (i) the $U_i$ are enforced to form a partition of the vertex-set $V$, and (ii) each $U_i$ satisfies some individual MSO constraint $\varphi_a$. Because $k$ is constant, the size of the resulting MSO formula is a constant as long as each of the MSO constraints $\varphi_a$ has constant size.

Connecting CSP and MSO. Consider an MSO formula $\varphi$ with $k$ free variables on graph $G$ (as above). For a vector $t \in \{0, 1\}^{k|V|}$, we write $G, t \models \varphi$ if and only if $\varphi$ is satisfied by solution $U_a = \{v \in V : \varphi(a, v) = 1\}$ for $a \in [k]$.

**Definition 6 (CSP$_{\varphi}$ instance).** Let $G$ be a graph and $\varphi$ be an MSO$_2$-formula with $k$ free variables. By CSP$_{\varphi}(G)$ we denote the CSP instance $(N, C)$ with $N = \{\{t(v, a)\} \mid v \in V(G), a \in [k]\}$ and with a single constraint $\{t \mid G, t \models \varphi\}$. 


Observe that Feas(CSP, G) corresponds to the set of feasible assignments of \( \varphi \) on G. Also, the treewidth of \( \text{CSP}_\varphi(G) \) is \( |V| \) which is unbounded. The following result shows that there is an equivalent CSP extension that has constant treewidth.

**Theorem 2.2 (15, Theorem 25).** Let \( G = (V, E) \) be a graph with \( \text{tw}(G) = \tau \) and \( \varphi \) be an MSO-2-formula with \( k \) free variables. Then \( \text{CSP}_\varphi(G) \) has a CSP extension \( J \) with \( \text{tw}(J) \leq f(|\varphi|, \tau) \) and \( |\mathcal{C}| \leq f(|\varphi|, \tau) \cdot |V| \).

To be precise, [15, Theorem 25] speaks of MSO\(_2\) over \( \sigma_2\)-structures, which is equivalent to MSO\(_2\) over graphs; cf. the discussion in [15, Section 2.1].

### 3. Dynamic Program for CSP

In this section we demonstrate that every CSP of bounded treewidth admits a dynamic program that satisfies the assumptions required in [13].

Consider a CSP instance \( J = (V, C) \) with a constraint graph \( G = (V, E) \) of bounded treewidth. Let \((T = (I, F), \{X_i| i \in I \}) \) denote a balanced tree decomposition of \( G \) (from Theorem 2.2). In what follows, we denote the vertex set \( V = [n] = \{1, 2, \ldots, n \} \). Let \( \lambda \) be a symbol denoting an unassigned value. For any \( W \subseteq V \), define the set of configurations of \( W \) as:

\[
\mathcal{K}(W) = \{ (z_1, \ldots, z_n) \in \{0, 1, \lambda \}^V \mid \\
\forall C \in C: (U \subseteq W \implies z_U \in C_U), \\
\forall i \notin W: z_i = \lambda, \forall j \in W: z_j \in \{0, 1\}\}
\]

Let \( k \in \mathcal{K}(W) \) be a configuration and \( v \in V \). Because \( k \) is a vector, \( k(v) \) refers to the \( v \)-th element of \( k \).

**Definition 7 (State Operations).** Let \( U, W \subseteq V \). Let \( k \in \mathcal{K}(U) \) and \( p \in \mathcal{K}(W) \).

- **Configurations** \( k \) and \( p \) are said to be consistent if, for each \( v \in V \), either \( k(v) = p(v) \) or at least one of \( k(v) \) and \( p(v) \) is \( \lambda \).

- If configurations \( k \) and \( p \) are consistent, define

\[
[p \cup k](v) = \begin{cases} 
  p(v), & \text{if } k(v) = \lambda; \\
  k(v), & \text{otherwise}.
\end{cases}
\]

- Define \( k \cap W(v) = \begin{cases} 
  k(v), & \text{if } v \in W; \\
  \lambda, & \text{otherwise}.
\end{cases} \)

We start by defining some useful parameters for the dynamic program.

**Definition 8.** For each node \( i \in I \) with children nodes \( \{j, j'\} \), we associate the following:

1. state space \( \Sigma_i = \mathcal{K}(X_i) \).
2. for each \( \sigma \in \Sigma_i \), there is a collection of partial solutions.

\[
\mathcal{H}_{i, \sigma} := \{ k \in \mathcal{K}(V) \mid k \cap X_i = \sigma \}.
\]

3. for each \( \sigma \in \Sigma_i \), there is a collection of valid combinations of children states

\[
\mathcal{F}_{i, \sigma} = \{ (\sigma_j, \sigma_{j'}) \in \Sigma_j \times \Sigma_{j'} \mid (\sigma_j \cap X_i) = (\sigma \cap X_j) \text{ and } (\sigma_{j'} \cap X_i) = (\sigma \cap X_{j'}) \}.
\]

In words, (a) \( \Sigma_i \) is just the set of configurations for the vertices \( X_i \) in node \( i \), (b) \( \mathcal{H}_{i, \sigma} \) are those configurations for the vertices \( V_i \) in the subtree rooted at \( i \) that are consistent with \( \sigma \), (c) \( \mathcal{F}_{i, \sigma} \) are those pairs of states at the children \( \{j, j'\} \) that agree with \( \sigma \) on the intersections \( X_j \cap X_i \) and \( X_{j'} \cap X_i \) respectively.

**Theorem 3.1 (Dynamic Program for CSP).** Let \((T = (I, F), \{X_i| i \in I \}) \) be a tree decomposition of a CSP instance \((V, C)\) of bounded treewidth. Then \( \Sigma_i, \mathcal{F}_{i, \sigma} \) and \( \mathcal{H}_{i, \sigma} \) from Definition 8 satisfy the conditions:

1. (bounded state space) \( \Sigma_i \) and \( \mathcal{F}_{i, \sigma} \) are all bounded by a constant, that is, \( \max_{\sigma \in \Sigma_i} |\mathcal{H}_{i, \sigma}| = O(1) \).

2. (required state) For each \( i \in I \) and \( \sigma \in \Sigma_i \), the intersection with \( X_i \) of every vector in \( \mathcal{H}_{i, \sigma} \) is the same, in particular \( h \cap X_i = \sigma \) for all \( h \in \mathcal{H}_{i, \sigma} \).

3. By condition 2, for any leaf \( \ell \in I \) and \( \sigma \in \Sigma_\ell \), we have \( \mathcal{H}_{\ell, \sigma} = \{\sigma\} \) or \( \emptyset \).

4. (subproblem) For each non-leaf node \( i \in I \) with children \( \{j, j'\} \) and \( \sigma \in \Sigma_i \),

\[
\mathcal{H}_{i, \sigma} = \{ \sigma \cup h_j \cup h_{j'} \mid h_j \in \mathcal{H}_{j, w_j}, \ h_{j'} \in \mathcal{H}_{j', w_{j'}}, \ (w_j, w_{j'}) \in \mathcal{F}_{i, \sigma} \}.
\]

5. (feasible subsets) At the root node \( r \), we have Feas(V, C) = \( \bigcup_{\sigma \in \Sigma_r} \mathcal{H}_{r, \sigma} \).

**Proof.** Let \( q = O(1) \) denote the treewidth of \( T \). We now prove each of the claimed properties.

**Bounded state space.** Because \(|X_i| \leq q + 1\), we have \( |\Sigma_i| = |\mathcal{K}(X_i)| \leq 3^{q+1} = O(1) \) and \( |\mathcal{F}_{i, \sigma}| \leq |\Sigma_j \times \Sigma_{j'}| \leq (3^{q+1})^2 = O(1) \).

**Required state.** This holds immediately by definition of \( \mathcal{H}_{i, \sigma} \) in Definition 8.

**Subproblem.** We first prove the “\( \subseteq \)” inclusion of the statement. Consider any \( h \in \mathcal{H}_{i, \sigma} \subseteq \mathcal{K}(V_i) \). Let \( h_j = h \cap V_j, w_j = h \cap X_j \) and analogously for \( j' \). Observe that for \( U \subseteq V \subseteq V \) we have that \( k \in \mathcal{K}(V) \implies k \cap U \in \mathcal{K}(U) \). By this observation, \( h_j \in \mathcal{K}(V_j) \). Moreover, \( h_j \cap X_{j'} = h \cap X_{j'} = w_{j'} \), which implies \( h_{j'} \in \mathcal{H}_{j', w_{j'}} \). Again, the same applies for \( j' \) and we have \( h_{j'} \in \mathcal{H}_{j', w_{j'}} \). Finally, note that \( w_j \cap X_i = h \cap X_j \cap X_i = (h \cap X_j) \cap X_i = \sigma \cap X_i \) and similarly \( w_{j'} \cap X_i = \sigma \cap X_{j'} \). So we have \( (w_j, w_{j'}) \in \mathcal{F}_{i, \sigma} \).

Now, we prove the “\( \supseteq \)” inclusion of the statement. Consider any two partial solutions \( h_j \in \mathcal{H}_{j, w_j} \) and \( h_{j'} \in \mathcal{H}_{j', w_{j'}} \) with \( (w_j, w_{j'}) \in \mathcal{F}_{i, \sigma} \). Note that \( h_j \) and \( \sigma \) (similarly \( h_{j'} \) and \( \sigma \)) are consistent by definition of \( \mathcal{F}_{i, \sigma} \). We now claim that \( h_j \) and \( h_{j'} \) are also consistent:
take any $v \in V$ with both $h_j(v), h_f(v) \neq \lambda$, then we must have $v \in V_j \cap V_f \subseteq X_j \cap X_f$ as $X_j$ is a vertex separator, and so $h_j(v) = \sigma(v) = h_f(v)$ by definition of $F_{i,r}$. Because $\sigma, h_f$ and $h_j$ are mutually consistent, $h = \sigma \cup h_j \cup h_f$ is well-defined. It is clear from the above arguments that $h \cap X_i = \sigma$. In order to show $h \in \mathcal{H}_{i,r}$ we now only need $h \in \mathcal{K}(V)$, that is, $h$ does not violate any constraint that is contained in $V_i$. For contradiction assume that there is such a violated constraint $C_S$ with $S \subseteq V_i$. Then $S$ induces a clique in the constraint graph $G$ and thus there must exist a node $k$ among the descendants of $i$ such that $S \subseteq V_k$. But $k$ cannot be in the subtree rooted in $j$ or $j'$, because then $C_S$ would have been violated already in $h_j$ or $h_f$, and also it cannot be that $i = k$, because then $C_S$ would be violated in $\sigma$, a contradiction.

**Feasible subsets.** Clearly, the set $\mathcal{F}(V, C)$ of feasible CSP solutions is equal to $\mathcal{K}(V)$. Because $\mathcal{H}_{i,r}$ is those $k \in \mathcal{K}(V)$ with $k \cap X_i = \sigma$, the claim follows. □

We note that Theorem 3.1 proves Assumption 1 in [13]. To clarify the comparison, Assumption 1 is:

**Assumption 1 (Assumption 1 in [13]).** Let $(\mathcal{F} = (I, \mathcal{F}), \{X_i | i \in I\})$ be any tree decomposition. Then there exist $\Sigma_i, \mathcal{F}_{i,r}$ and $\mathcal{H}_{i,r}$ (see Definition 5) that satisfy the following conditions:

1. (bounded state space) $\Sigma_i$ and $\mathcal{F}_{i,r}$ are all bounded by constant, that is, $\max |\Sigma_i| = O(1)$ and $\max |\mathcal{F}_{i,r}| = O(1)$.
2. (required state) For each $i \in I$ and $\sigma \in \Sigma_i$, the intersection with $X_i$ of every set in $\mathcal{H}_{i,r}$ is the same, denoted $X_{i,\sigma}$, that is $S \cap X_i = X_{i,\sigma}$ for all $S \in \mathcal{H}_{i,r}$.
3. By condition 2, for any leaf $r$ in $I$ and $\sigma \in \Sigma_i$, we have $\mathcal{H}_{i,r} = \{X_{i,\sigma}\}$ or $\emptyset$.
4. (subproblem) For each non-leaf node $i \in I$ with children $\{j, j'\}$ and $\sigma \in \Sigma_i$,

$$\mathcal{H}_{i,\sigma} = \left\{ X_{i,\sigma} \cup S_j \cup S_{j'} : S_j \in \mathcal{H}_{j,w_j}, S_{j'} \in \mathcal{H}_{j', w_{j'}}, (w_j, w_{j'}) \in \mathcal{F}_{i,r} \right\}.$$

5. (feasible subsets) At the root node $r$, we have $\mathcal{S}_G = \bigcup_{\sigma \in \Sigma_r} \mathcal{H}_{r,\sigma}$.

Assumption 1 is used in the main result of [13], which is restated below.

**Theorem 3.2 (Theorem 4 in [13]).** Consider any instance of the GCMC problem on a bounded-treewidth graph $G$. If the graph constraint $\mathcal{S}_G$ satisfies Assumption 1 then we obtain a $\frac{1}{2}$-approximation algorithm.

We will use this result in Section 4 but we will modify its proof slightly in Section 5 for max-k-cut.

### 4. The Max-Cut Setting

Here, we consider the GCMC problem when $k = 2$ and there is a constraint $\mathcal{S}_G$ for only one side of the cut. We show that the above dynamic-program structure can be combined with [13] to obtain a $\frac{1}{2}$-approximation algorithm.

Formally, there is an MSO formula $\varphi$ with one free variable defined on graph $G = (V, E)$ of bounded treewidth. The feasible vertex subsets $\mathcal{S}_G$ are those $\subseteq V$ that satisfy $\varphi$. There is also a symmetric weight function $c : \left\{ \frac{1}{2} \right\} \rightarrow \mathbb{R}_+$. We are interested in the following problem (GMC$_G$).

$$\max_{X \in \mathcal{S}_G} \sum_{u \in X, v \notin X} c(u, v). \quad (2)$$

We note that this is precisely the setting of [13].

**Theorem 4.1.** There is a $\frac{1}{2}$-approximation algorithm for GMC$_G$ when the constraint $\mathcal{S}_G$ is given by any MSO formula on a bounded-treewidth graph.

**Proof.** The proof uses Theorem 3.2 from [13] as a black-box. Note that the constraint $\mathcal{S}_G$ corresponds to feasible assignments to CSP$_G(G)$ as in Definition 6. Consider the CSP extension $\varphi$ obtained after applying Theorem 4.2 to CSP$_G(G)$. Then $\varphi$ has variables $V' \supseteq V$ and bounded treewidth. We obtain an extended weight function $c : \left\{ \frac{1}{2} \right\} \rightarrow \mathbb{R}_+$ from $c$ by setting $c'(u, v) = c(u, v)$ if $u, v \in V$ and $c'(u, v) = 0$ otherwise. We now consider a new instance of GMC$_G$ on vertices $V'$ and constraint $\varphi$. Due to the bounded-treewidth property of $\varphi$, we can apply Theorem 3.1 which proves that Assumption 1 is satisfied by the dynamic program in Definition 5. Combined with Theorem 4.2, we obtain the claimed result. □

### 5. The Max-k-Cut Setting

In this section, we generalize the setting to any constant $k$, i.e., problem 1. Recall the formal definition from [2]. Here the graph property $\mathcal{S}_G$ is expressed as an MSO formula with $k$ free variables on graph $G$. Our main result is the following:

**Theorem 5.1.** There is a $\frac{1}{2}$-approximation algorithm for any GCMC instance with constant $k$ when the constraint $\mathcal{S}_G$ is given by any MSO formula on a bounded-treewidth graph.

**Remark 1.** The complexity of Theorem 5.1 in terms of the treewidth $\tau$, length $|\varphi|$ of $\varphi$, depth $d$ of a tree decomposition of $G$, and maximum degree $r$ of a tree decomposition of $G$, is $s^d$, where $s$ is the number of states of the dynamic program, namely $f(|\varphi|, \tau)$ for $f$ from Theorem 2.2. From the perspective of parameterized complexity [16] our algorithm is an XP algorithm parameterized by $\tau$, i.e., it has runtime $n^{O(\tau)}$ for some computable function $g$. 

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Let \( G = (V, E) \) be the input graph (assumed to have bounded treewidth) and \( \varphi \) be any MSO formula with \( k \) free variables. Recall the CSP instance \( CS_{\mathcal{P}}(G) \) on variables \( \{y(v, \alpha) : v \in V, \alpha \in [k]\} \) from Definition 3.1. Feasible solutions to \( CS_{\mathcal{P}}(G) \) correspond to feasible \( k \)-partitions in \( \mathcal{S}_G \). Now consider the CSP extension \( \bar{\varphi} \) obtained after applying Theorem 2.3 to \( CS_{\mathcal{P}}(G) \). Note that \( \bar{\varphi} \) is defined on variables \( V' \supseteq \{(v, \alpha) : v \in V, \alpha \in [k]\} \) and has bounded treewidth. Let \( \mathcal{T} \) denote the tree decomposition for \( \bar{\varphi} \). Below we utilize the dynamic program from Definition 8 applied to \( \bar{\varphi} \): recall the quantities \( \Sigma, \mathcal{F}_{\tau, \sigma} \) etc. We will also refer to the variables in \( V' \) as vertices, especially when referring to the tree decomposition \( \mathcal{T} \); note that these are different from the vertices \( V \) in the original graph \( G \).

Claim 1. Let \( \{U_i\}_{i=1}^k \) be a \( k \)-partition satisfying \( \mathcal{S}_G \).

There is a collection of sets \( \{b[i] \in \Sigma_{i \leq t} \} \) such that:

- for each node \( i \in I \) with children \( j \) and \( j' \), \( \{b[j], b[j']\} \in \mathcal{F}_{i, b[i]} \),
- for each leaf \( t \), \( H_{\mathcal{L}, b[t]} \neq \emptyset \),
- \( U_i = \{v \in V : b_T((v, \alpha)) = 1\} \) for all \( \alpha \in [k] \), where \( b_T = \bigcup_{t \in T} b[t] \).

Moreover, for any vertex \( (v, \alpha) \in V' \), \( \bar{\varphi} \) denotes the highest node in \( \mathcal{T} \) containing \( (v, \alpha) \) then we have:
\[ v \in U_a \text{ if and only if } b[\bar{\varphi}](v, \alpha) = 1. \]

Proof. By definition of CSP \( \varphi \), we know that it has some feasible solution \( t \in \{0, 1\}^V \) where \( U_a = \{v \in V : t(v, \alpha) = 1\} \) for all \( \alpha \in [k] \). Now, using Theorem 3.1 we have \( t \in \bigcup_{i \in \Sigma} \mathcal{H}_{\mathcal{L}} \). We define the states \( b[i] \) in a top-down manner. We will also define an associated vector \( t_i \in \mathcal{H}_{\mathcal{L}, b[i]} \) at each node \( i \). At the root, we set \( b[r] = \sigma \) such that \( t_i \in \mathcal{H}_{\mathcal{L}} \); this is well-defined because \( t \in \bigcup_{i \in \Sigma} \mathcal{H}_{\mathcal{L}} \). We also set \( t_r = t \). Having set \( b[i] \) and \( t_i \in \mathcal{H}_{\mathcal{L}, b[i]} \) for any node \( i \) with children \( j \) and \( j' \), we use Theorem 3.1 to write \( h_i = b[i] \cup h_j \cup h_{j'} \), where \( h_j \in \mathcal{H}_{\mathcal{L}, b[j]} \) and \( h_{j'} \in \mathcal{H}_{\mathcal{L}, b[j']} \). Then we set \( b[j] = w_j, t_j = h_j \) and \( b[j'] = w_{j'}, t_{j'} = h_{j'} \) for the children of node \( i \).

The first condition in the claim is immediate from the definition of states \( b[i] \). By induction on the depth of node \( i \), we obtain \( t_i \in \mathcal{H}_{\mathcal{L}, b[i]} \) for each node \( i \). This implies that \( \mathcal{H}_{\mathcal{L}, b[i]} \neq \emptyset \) for each leaf \( t \), which proves the second condition; moreover, by Theorem 3.1, we have \( t_r = t[0] \). Now, by definition of the vectors \( t_i \), we obtain \( t = \bigcup_{i \in \mathcal{S}} b[i] = b_T \) which, combined with the third condition in the claim.

Because \( t = \bigcup_{i \in \mathcal{S}} b[i] \), it is clear that if \( b[\bar{\varphi}](v, \alpha) = 1 \) then \( v \in U_a \). In the other direction, suppose \( b[\bar{\varphi}](v, \alpha) \neq 1 \): we will show \( v \not\in U_a \). Since \( \bar{\varphi} \) is the highest node containing \((v, \alpha)\), it suffices to show that \( \bar{\varphi}(v, \alpha) \neq 1 \). But this follows directly from Theorem 3.1 because \( \bar{\varphi} \in \mathcal{H}_{\mathcal{L}, \bar{\varphi}(v, \alpha)} \), \((v, \alpha) \in X_{\bar{\varphi}} \) and \( b[\bar{\varphi}](v, \alpha) \neq 1 \).  

LP relaxation for Max-k-Cut. We start with some additional notation related to the tree decomposition \( \mathcal{T} \) (from Theorem 2.3) and the dynamic program for CSP (from Theorem 3.1).

- For any node \( i \in I, T_i \) is the set consisting of (1) all nodes \( N \) on the \( r-i \) path in \( \mathcal{T} \), and (2) children of all nodes in \( N \) \( \{i\} \).
- \( \mathcal{P} \) is the collection of all node subsets \( J \) such that \( J \subseteq T_{\ell}, U_{\ell} \) for some pair of leaf-nodes \( \ell_1, \ell_2 \).
- \( s[i] \in \Sigma \) denotes a state at node \( i \). Moreover, for any subset of nodes \( N \subseteq I \), we use the shorthand \( s[N] := (s[k] : k \in N) \).
- \( a[i] \in \Sigma \) denotes a state at node \( i \) chosen by the algorithm. Similar to \( s[N] \), for any subset \( N \subseteq I \) of nodes, \( a[N] := (a[k] : k \in N) \).
- \( \overline{\alpha} \in I \) denotes the highest tree-decomposition node containing vertex \((v, \alpha) \in V' \).

The LP (see Figure 1) that we use here is a generalization of that in [13]. The variables are \( y(s[N]) \) for all \( \{s[k] \in \Sigma \mid k \in N \} \) and \( \mathcal{N} \in \mathcal{P} \). Variable \( y(s[N]) \) corresponds to the probability of the joint event that the solution (in \( \mathcal{S}_G \)) “induces” state \( s[k] \) at each node \( k \in \mathcal{N} \). Variable \( z_{\mathcal{N}} \) corresponds to the probability that edge \((u, v) \in E \) is cut by part \( \alpha \) of the \( k \)-partition.

In constraint (6), we use \( j \) and \( j' \) to denote the two children of node \( i \) in \( I \). We note that constraints (4)-(8) which utilize the dynamic-program structure, are identical to the constraints (4)-(8) in the LP from [13]. This allows us to essentially reuse many of the claims proved in [13], which are stated below.

Claim 2. Let \( y \) be feasible to (LP). For any node \( i \in I \) with children \( j \) and \( j' \) and \( s[k] \in \Sigma \) for all \( k \in T_i \),

\[ y(s[T_i]) = \sum_{j: j \in \mathcal{S}} \sum_{j': j' \in \mathcal{S}} y(s[T_i \cup (j, j')]). \]

Proof. Note that \( T_i \cup (j, j') \subseteq T_i \) for any leaf node \( t \) in the subtree below \( i \). So \( T_i \cup (j, j') \in \mathcal{P} \) and the variables \( y(s[T_i \cup (j, j')]) \) are well-defined. The claim follows by two applications of (4).

Lemma 1. (LP) has a polynomial number of variables and constraints.

Proof. There are \( \binom{k}{2} \cdot k = O(kn^2) \) variables \( z_{\mathcal{N}} \). Because the tree is binary, we have \( |T_i| \leq 2d \) for any node \( i \), where \( d = O(\log n) \) is the depth of the tree decomposition. Moreover there are only \( O(n^2) \) pairs of leaves as there are \( O(n) \) leaf nodes. For each pair \( \ell_1, \ell_2 \) of leaves, we have \( |T_{\ell_1} \cup T_{\ell_2}| \leq 4d \). Thus \( |\mathcal{P}| \leq O(n^2) \cdot 2^{4d} \) poly(n). By Theorem 3.1, we have \( \max |\mathcal{H}_{\mathcal{L}}| = \Theta(1) \), so the number of \( y \)-variables is at most \( |\mathcal{P}| \cdot (\max |\mathcal{H}_{\mathcal{L}}|)^{\max d} = \text{poly}(n) \). This shows that (LP) has polynomial size and can be solved optimally in polynomial time. Finally, it is clear that the rounding algorithm runs in polynomial time.
maximize \[
\frac{1}{2} \sum_{(u,v) \in E} c_{uv} \sum_{a=1}^{k} z_{ava} \quad \text{(LP)}
\]
\[
z_{ava} = \sum_{s(\{u,v\})} y(s[\{u,v\}]),
\]
\[
\forall [u,v] \in \binom{V}{2}, \forall a \in [k];
\]
\[
y(s[N]) = \sum_{s([i]) \in \Sigma_i} y(s[N \cup \{i\}],
\]
\[
\forall s[k] \in \Sigma_k, \forall k \in N, \forall N \in \mathcal{P}, \forall i \in N \cup \{i\} \in \mathcal{P};
\]
\[
\sum_{s[r] \in \Sigma_r} y(s[r]) = 1;
\]
\[
y(s[i]) = \begin{cases} 1, & \text{if } s[i] = b[i] \text{ for all } i \in N; \\ 0, & \text{otherwise.} \end{cases}
\]

Figure 1: The LP formulation

Lemma 2. \((\text{LP})\) is a valid relaxation of GCMC.

Proof. Let \(k\)-partition \(\{U_a\}_{a=1}^k\) be any feasible solution to \(S_G\). Let \(\{b[i]\}_{i \in I}\) denote the states given by Claim 1 corresponding to \(\{U_a\}_{a=1}^k\). For any subset \(N \in \mathcal{P}\) of nodes, and for all \(\{s[i]\} \in \Sigma_{[i]}\), set
\[
y(s[N]) = \begin{cases} 1, & \text{if } s[i] = b[i] \text{ for all } i \in N; \\ 0, & \text{otherwise.} \end{cases}
\]

Clearly constraints (4) and (8) are satisfied. By the first property in Claim 1, constraint (6) is satisfied. And by the second property in Claim 1, constraint (7) is also satisfied. The last property in Claim 1 implies that \(v \in U_a \iff b(\{v\}) = 1\) for any vertex \(v \in V\). So any edge \((u,v)\) is cut by \(U_a\) exactly when \(b(\{u,v\}) = 1\). Using the setting of variable \(z_{ava}\) in (3) it follows that \(z_{ava}\) is exactly the indicator of edge \((u,v)\) being cut by \(U_a\). Finally, the objective value is exactly the total weight of edges cut by the \(k\)-partition \(\{U_a\}_{a=1}^k\) where the coefficient \(\frac{1}{2}\) comes from the fact that that summation counts each cut-edge twice. Thus \((\text{LP})\) is a valid relaxation.

Rounding Algorithm. This is a top-down procedure, exactly as in [13]. We start with the root node \(r \in I\). Here \(y(s[r]) : s[r] \in \Sigma_r\) defines a probability distribution over the states of \(r\). We sample a state \(s[r] \in \Sigma_r\) from this distribution. Then we continue top-down: for any node \(i \in I\), given the chosen states \(s[a]\) at each \(k \in T_i\), we sample states for both children of \(i\) simultaneously from their joint distribution given at node \(i\). Our algorithm is formally described in Algorithm 1.

\textbf{Algorithm 1: Rounding Algorithm for \((\text{LP})\)}

\textbf{Input} : Optimal solution of \((\text{LP})\)
\textbf{Output} : A vertex partition of \(V\) in \(S_G\).

1. Sample a state \(a[i]\) at the root node by distribution \(y(s[r])\).
2. Do process all nodes \(i\) in \(T\) in order of increasing depth:
   a. Sample states \(a[j], a[j']\) for the children of node \(i\) by joint distribution
   \[
   \text{Pr}[a[j] = s[j] \text{ and } a[j'] = s[j']] = \frac{y(s[T_i \cup \{j, j'\})]}{y(s[T_i])},
   \]
   where \(s[T_i] = a[T_i]\).
3. Do process all nodes \(i\) in \(T\) in order of decreasing depth:
   a. \(h_i = a[i] \cup h_j \cup h_f\) where \(j, j'\) are the children of \(i\).
4. Set \(U_a = \{v \in V : h_i((v,a)) = 1\}\) for all \(a \in [k]\).
5. return \(k\)-partition \(\{U_a\}_{a=1}^k\).

Lemma 3. The algorithm’s solution \(\{U_a\}_{a=1}^k\) is always feasible.

Proof. Note that the distributions used in Step 1 and Step 3 are well-defined due to Claim 2 so the states \(a[i]\) are well-defined. Moreover, by the choice of these distributions, for each node \(i\), \(y(a[T_i]) > 0\).

We now show that for any node \(i \in I\) with children \(j, j'\) we have \((a[j], a[j']) \in \mathcal{F}\).

For the inductive step, consider node \(i \in I\) with children \(j, j'\) where \(h_i \in \mathcal{H}_{\text{adj}}\).

We show by induction that for each node \(i \in I\), \(h_i \in \mathcal{H}_{\text{adj}}\). The base case is when \(i\) is a leaf. In this case, due to constraint (7) and the fact that \(y(a[T_i]) > 0\) we know that \(\mathcal{H}_{\text{adj}} \neq \emptyset\). So \(h_i = a[i] \in \mathcal{H}_{\text{adj}}\) by Theorem 3.113. For the inductive step, consider node \(i \in I\) with children \(j, j'\) where \(h_i \in \mathcal{H}_{\text{adj}}\) and \(h_f \in \mathcal{H}_{\text{adj}}\).

Moreover, from the property above, \((a[j], a[j']) \in \mathcal{F}\). Using Theorem 3.113 we have \(h_i = a[i] \cup h_j \cup h_f \in \mathcal{H}_{\text{adj}}\).

Using Theorem 3.113, we obtain that \(h_i \in \mathcal{F}\). Now let \(h'\) denote the restriction of \(h_i\) to the variables \((v, a) : v \in V, a \in [k]\).

Then, using the CSP extension result (Theorem 2.23), we obtain that \(h'\) is feasible for \(CS_{\mathcal{P}}(G)\). In other words, the \(k\)-partition \(\{U_a\}_{a=1}^k\) satisfies \(S_G\).
Claim 3. For any node \( i \) and states \( s[k] \in \Sigma_k \) for all \( k \in T_i \), the rounding algorithm satisfies \( Pr[a[T_i] = s[T_i]] = y(s[T_i]) \).

Proof. We proceed by induction on the depth of node \( i \). It is clearly true when \( i = r \), i.e., \( T_i = \{r\} \). Assuming the statement is true for node \( i \), we will prove it for \( i \)'s children. Let \( j, j' \) be the children nodes of \( i \); note that \( T_j = T_{j'} = T_i \cup \{j, j'\} \). Then using (9), we have

\[
Pr[a(T_i) = s(T_i) \mid a(T_i) = s(T_i)] = \frac{y(s(T_i) \cup \{j, j'\})}{y(s(T_i))}.
\]

Combined with \( Pr[a(T_i) = s(T_i)] = y(s(T_i)) \) we obtain \( Pr[a(T_i) = s(T_i)] = y(s(T_j)) \) as desired.

Lemma 4. Consider any \( u, v \in V \) and \( \alpha \in [k] \) such that \( u \neq v \) in \( T_u \). Then the probability that edge \((u, v)\) is cut by \( U_\alpha = \{v \in V : h_i((v, \alpha)) = 1\} \) is \( z_{uv} \).

Proof. Applying Claim 3 with node \( i \) as for any \( \{s[k] \in \Sigma_k : k \in T_i \} \), we have \( Pr[a[T_i] = s[T_i]] = y(s[T_i]) \). Let \( D_{uv} = \{s[uv] \in \Sigma_{uv} : s[uv]|(u, \alpha) = 1\} \). Similarly \( D_{uv} = \{v[uv] \in \Sigma_{uv} : v[uv]|(v, \alpha) = 1\} \). Because \( u \neq v \), we have

\[
Pr[u \in U_\alpha, v \notin U_\alpha] = \sum_s \sum_{v \in \alpha} \sum_{u \in \alpha} \sum_{s[uv]} y(s[uv]) = \sum_{v \in \alpha} \sum_{u \in \alpha} \sum_{s[uv]} y(s[uv]).
\]

The last equality above is by repeated application of LP constraint (4) where we use \( T_{uv} \in \mathcal{P} \). Similarly,

\[
Pr[u \notin U_\alpha, v \in U_\alpha] = \sum_{v \in \alpha} \sum_{u \in \alpha} \sum_{s[uv]} y(s[uv]).
\]

which combined with constraint (3) implies \( Pr[[u, v] \cap U_\alpha] = 1 \) \( z_{uv} \).

Lemma 5. Consider any \( u, v \in V \) and \( \alpha \in [k] \) such that \( u \neq v \) in \( T_u \) and \( v \neq u \) in \( T_v \). Then the probability that edge \((u, v)\) is cut by \( U_\alpha = \{v \in V : h_i((v, \alpha)) = 1\} \) is at least \( z_{uv}/2 \).

Proof. We first state a useful observation.

Observation 1 (Observation 1 in [13]). Let \( X, Y \) be two jointly distributed \([0, 1]\) random variables. Then \( Pr(X = 1)Pr(Y = 0) + Pr(X = 0)Pr(Y = 1) \geq \frac{1}{2} Pr(X = 1, Y = 0) \).

Now we start to prove Lemma 5. In order to simplify notation, we define:

\[
z_{uv} = \sum_{s[uv] \in D_{uv}} y(s[uv]),
\]

\[
z_{uv} = \sum_{s[uv] \in D_{uv}} y(s[uv]).
\]

Note that \( z_{uv} = z_{uv}^+ + z_{uv}^- \).

Let \( D_{uv} = \{s[uv] \in \Sigma_{uv} : s[uv]|(u, \alpha) = 1\} \) and \( D_{uv} = \{s[uv] \in \Sigma_{uv} : s[uv]|(v, \alpha) = 1\} \). Let \( \|u\| \) denote the least common ancestor of nodes \( u \) and \( v \), and \( \{j, j'\} \) the two children of \( i \). Note that \( T_j = T_{j'} = T_i \cup \{j, j'\} \) and \( T_{uv}, T_{uv} \subseteq T_j \). Because \( u \neq v \) and \( v \neq u \), both \( u \) and \( v \) are strictly below \( j \) and \( j' \) (respectively) in the tree decomposition.

For any choice of states \( \{s[k] \in \Sigma_k : k \in T_j \} \), define:

\[
z_{uv}^+(s[T_j]) = \sum_{s[uv] \in D_{uv}} y(s[T_j] \cup \{s[uv]\}) \]

and similarly \( z_{uv}^-(s[T_j]) \).

In the rest of the proof, we fix states \( \{s[k] \in \Sigma_k : k \in T_j \} \) and condition on the event \( E \) that \( a[T_j] = s[T_j] \). We will show \( Pr[[u, v] \cap U_\alpha = 1 \mid E] \)

\[
\geq \frac{1}{2} \left( z_{uv}^+(s[T_j]) + z_{uv}^-(s[T_j]) \right).
\]

By taking expectation over the conditioning \( E \), this would imply Lemma 5.

We now define the following indicator random variables (conditioned on \( E \)).

\[
I_{uv} = \begin{cases} 0 & \text{if } a[uv] \notin D_{uv} \\ 1 & \text{if } a[uv] \in D_{uv} \end{cases}
\]

Observe that \( I_{uv} \) and \( I_{uv} \) are independent because \( u \neq v \) and \( v \neq u \) appear in distinct subtrees under node \( i \). So,

\[
Pr[[u, v] \cap U_\alpha = 1 \mid E] = Pr[I_{uv} = 1] \cdot Pr[I_{uv} = 0] + Pr[I_{uv} = 0] \cdot Pr[I_{uv} = 1] \]

(11)

For any \( s[k] \in \Sigma_k \) for \( k \in T_{uv} \setminus T_j \), we have by Claim 3 and \( T_j \subseteq T_{uv} \) that

\[
Pr[a[T_{uv}] = s[T_{uv}] \mid a[T_j] = s[T_j]] = \frac{y(s[T_{uv}])}{y(s[T_j])}.
\]

Therefore \( Pr[I_{uv} = 1] \) equals

\[
\sum_{s[uv] \in D_{uv}} \sum_{s[T_j] \in \Sigma_{uv}} \frac{y(s[T_j] \cup \{s[uv]\})}{y(s[T_j])}.
\]

The last equality follows by repeatedly using LP constraint (4) and the fact that \( T_{uv} \in \mathcal{P} \). Furthermore, note that \( T_j \cup \{\overline{uv}, \overline{uv}\} \) is \( T_{uv} \in \mathcal{P} \); again by constraint (4),

\[
Pr[I_{uv} = 1] = \sum_{s[uv] \in D_{uv}} \sum_{s[T_j] \in \Sigma_{uv}} \frac{y(s[T_j] \cup \{s[uv]\})}{y(s[T_j])} + z_{uv}^+(s[T_j])
\]
Similarly,
\[
\Pr[I_{uv} = 1] = \frac{\sum_{s : s \in D_{uv}} \frac{\gamma(s(T \cup \{uv\}))}{\gamma(s(T))}}{\sum_{s : s \in D_{uv}} \frac{\gamma(s(T \cup \{uv\}))}{\gamma(s(T))}} \equiv \sum_{s : s \in D_{uv}} \frac{\gamma(s(T \cup \{uv\}))}{\gamma(s(T))} + z_{uv}(s(T)).
\]
\[
\Pr[I_{uv} = 0] = \frac{\sum_{s : s \in D_{uv}} \frac{\gamma(s(T \cup \{uv\}))}{\gamma(s(T))}}{\sum_{s : s \in D_{uv}} \frac{\gamma(s(T \cup \{uv\}))}{\gamma(s(T))}} \equiv \sum_{s : s \in D_{uv}} \frac{\gamma(s(T \cup \{uv\}))}{\gamma(s(T))} + z_{uv}(s(T)).
\]

Now define \(0, 1\) random variables \(X\) and \(Y\) jointly distributed as:

| \(X\) | \(Y = 0\) | \(Y = 1\) |
|------|--------|--------|
| \(X = 0\) | \(\Pr[I_{uv} = 0] - \sum_{s : s \in D_{uv}} \frac{\gamma(s(T \cup \{uv\}))}{\gamma(s(T))}\) | \(\sum_{s : s \in D_{uv}} \frac{\gamma(s(T \cup \{uv\}))}{\gamma(s(T))}\) |
| \(X = 1\) | \(\sum_{s : s \in D_{uv}} \frac{\gamma(s(T \cup \{uv\}))}{\gamma(s(T))}\) | \(\Pr[I_{uv} = 1] - \sum_{s : s \in D_{uv}} \frac{\gamma(s(T \cup \{uv\}))}{\gamma(s(T))}\) |

Note that \(\Pr[X = 1] = \Pr[I_{uv} = 1] = 1\) and \(\Pr[Y = 1] = \Pr[I_{uv} = 1] - \sum_{s : s \in D_{uv}} \frac{\gamma(s(T \cup \{uv\}))}{\gamma(s(T))} + z_{uv}(s(T)) = \Pr[I_{uv} = 1].\)

So, applying Observation 1 and using (11) we have \(\Pr[[u, v] \cap U_{\alpha} = 1 | \mathcal{E}]\) is at least

\[
\frac{1}{2} \left( \Pr[X = 0, Y = 1] + \Pr[X = 1, Y = 0] \right),
\]

which implies (10).

**Lemma 6.** For any \(u, v \in V\), the probability that edge \((u, v)\) is cut by the \(k\)-partition \(\{U_{\alpha}\}_{\alpha=1}^{k}\) is at least \(\frac{1}{2} \sum_{\alpha=1}^{k} z_{uv}^\alpha\).

**Proof.** Edge \((u, v)\) is cut by \(\{U_{\alpha}\}_{\alpha=1}^{k}\) if and only if \(u \in U_{\alpha}\) and \(v \in U_{\beta}\) for some \(\alpha \neq \beta\). Enumerating all partition parts and applying Lemmas 3 and 4 we get that the probability is at least \(\frac{1}{2} \sum_{\alpha=1}^{k} z_{uv}^\alpha\). The extra factor of \(\frac{1}{2}\) is because any cut edge \((u, v)\) is cut by the partition twice: by the parts containing \(u\) and \(v\).

From Lemmas 2, 3 and 6 we obtain Theorem 5.1

**6. Applications**

We claim that MSO can be powerful enough to model various graph properties; to that end, consider the following formulae, meant to model that a set \(S\) is a vertex cover, an independent set, a dominating set, and a connected set, respectively:

\[
\varphi_{vc}(S) \equiv \forall [u, v] \in E : (u \in S) \lor (v \in S)
\]

\[
\varphi_{is}(S) \equiv \forall [u, v] \in E : \neg ((u \in S) \land (v \in S))
\]

\[
\varphi_{ds}(S) \equiv \forall v \in V : \exists u \in S : (v \notin S) \rightarrow [u, v] \in E
\]

\[
\varphi_{conn}(S) \equiv \neg \big[ \exists U, V \subseteq S : U \cap V = \emptyset \land U \cup V = S \land \neg (\exists (u, v) \in E : u \in U \land v \in V) \big]
\]

We argue as follows: \(\varphi_{vc}\) is true if every edge has at least one endpoint in \(S\); \(\varphi_{is}\) is true if every edge does not have both endpoints in \(S\); \(\varphi_{ds}\) is true if for each vertex \(v\) not in \(S\) there is a neighbor \(u\) in \(S\); finally, \(\varphi_{conn}\) is true if there does not exist a partition \(U, V\) of \(S\) with an edge going between \(U\) and \(V\).

We also show how to handle the precedence constraint. Let \(G\) be a directed graph; we require \(S\) to satisfy that, for each arc \((u, v) \in E\), either \(v \notin S\) or \(u, v \in S\). This can be handled directly with CSP constraints: we have a binary variable for each vertex with the value 1 indicating that a vertex is selected for \(S\); then, for each arc \((u, v) \in E\), we have a constraint \(C_{(u,v)} = \{(1,0), (0,0), (1,1)\}\).

It is known that many other properties are expressible in MSO, such as that \(S\) is \(k\)-colorable, \(k\)-connected (both for fixed \(k \in \mathbb{N}\)), planar, Hamiltonian, chordal, a tree, not containing a list of graphs as minors, etc. It is also known how to encode directed graphs into undirected graphs in an “MSO-friendly” way [12], which allows the expression of various properties of directed graphs. Our results also extend to so-called counting MSO, where we additionally have a predicate of the form \(|X| = p \mod q\) for a fixed integer \(q \in \mathbb{N}\).

**7. Conclusions**

In this paper we obtained \(\frac{1}{2}\)-approximation algorithms for graph-MSO-constrained max-\(k\)-cut problems, where the constraint graph has bounded tree-width. This work generalizes the class of constraints handled in [13] and extends the result to the setting of max-\(k\)-cut. Getting an approximation ratio better than \(\frac{1}{2}\) for any of these problems is an interesting question, even for a specific MSO-constraint. Regarding Remark 1 could our algorithm be improved to an FPT algorithm (runtime \(g(\tau)n^{O(1)}\) for some function \(g\))? If not, is there an FPT algorithm parameterized by the (more restrictive) tree-depth of \(G\)?

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