NON-PERTURBATIVE ANALYSIS OF THE COLEMAN-WEINBERG PHASE TRANSITION

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Abstract

We perform a non-perturbative study of the Coleman-Weinberg phase transition in scalar QED. Our method permits a consistent treatment of the effective potential near the origin, a region not accessible to perturbation theory. As a result, we establish reliably the first order character of the phase transition for arbitrary values of the scalar coupling.

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The seminal Coleman-Weinberg one-loop computation \( [1] \) of the effective potential in scalar quantum electrodynamics revealed the phenomenon of spontaneous symmetry breaking induced by quantum fluctuations. This perturbative calculation indicates a first order transition between the symmetric phase and the one with spontaneous symmetry breaking in dependence on the scalar mass term. This observation has motivated many theoretical investigations, ranging from a lower bound on the Higgs mass in the standard model (for top quark mass below 80 GeV) – the Linde-Weinberg bound \( [2] \) – to speculations about the first order character of the electroweak phase transition at high temperature \( [3] \). The latter could have important consequences for the creation of the baryon asymmetry in the early universe \( [4] \) by providing a new scenario for non-equilibrium conditions at temperatures of the order of the Fermi scale. Another area of applications are models of inflationary cosmology where Coleman-Weinberg type potentials are frequently used \( [5] \).

Strictly speaking, however, the perturbative calculation is only valid for the convex part of the effective scalar potential where all mass terms for fluctuations are positive or zero and the saddlepoint expansion is well defined. The failure of the loop expansion for negative mass terms for the fluctuations manifests itself through an imaginary part in the effective potential. The region in field space relevant for the first order character of the phase transition does not belong to the convex part of the potential and is therefore not directly accessible to perturbation theory. Since the problems are related to the fluctuations of scalar fields and the scalar interactions are small in the critical region, one usually discards the scalar fluctuations and only considers contributions from gauge fields. Indeed, the quartic scalar coupling \( \lambda \) is much smaller than \( e^4 \) (with \( e \) the abelian gauge coupling) in the critical region and the neglect of scalars seems justified at first sight. Unfortunately, this assumption cannot be verified within the standard perturbative framework since a saddle point expansion is crucial for its validity. Furthermore, little is known about the effect of higher-order loop corrections on the weak first order character of the transition.

Recently, a new non-perturbative method for the study of spontaneous symmetry breaking has been proposed. It is based on the concept of the average action \( [6] \) which is the continuum analogue of the blockspin action \( [7] \) in lattice theories. The average action \( \Gamma_k \) is the effective action for averages of fields taken over a (Euclidean) volume \( \sim k^{-4} \). It is obtained by integrating out the quantum fluctuations with
momenta $q^2 > k^2$. The dependence of $\Gamma_k$ on the “average scale” $k$ is governed by an exact non-perturbative evolution equation \cite{8}. It can be shown \cite{8} that the effective average action interpolates between the classical action as defined at some short distance scale $\Lambda^{-1}$ ($\Gamma_\Lambda \equiv S$) and the generating functional of the 1PI Green functions – the usual effective action $\Gamma$ – for $k = 0$ ($\Gamma_0 \equiv \Gamma$). The properties of the phase transition can be read off from the average potential $U_k$ which is a manifestly real quantity for all values of the scalar field. Even for regions where perturbation theory ceases to be valid the non-perturbative evolution equation – which takes the form of a renormalization group-improved one-loop flow equation – still applies. This method seems therefore suitable for an investigation of non-perturbative aspects of the Coleman-Weinberg phase transition.

The concept of the average action was generalized for abelian gauge theories \cite{10}, \cite{11}. The dependence of the effective scalar potential $U_k$ \cite{10} and the gauge coupling \cite{11} on the average scale $k$ was computed in arbitrary dimensions, demonstrating how the new non-perturbative method can successfully cope with the infrared divergences present in the standard perturbative approach. The exact non-perturbative evolution equation for the scale dependence of the effective average action has been adapted to non-abelian \cite{12} and abelian \cite{13} gauge theories. It appeared that the flow equations for the scalar potential and the gauge coupling proposed in \cite{10}, \cite{11} coincide with the exact non-perturbative evolution equation in the limit where the effective action is truncated to include only standard kinetic terms (with wave function renormalization) for the scalar and gauge fields as well as a scalar potential.

In this letter we employ the evolution equations for a non-perturbative study of the phase transition of the abelian Higgs model. We restrict the discussion to four dimensions and vanishing temperature and study the phase transition as a function of the mass term, as in the original papers \cite{1}, \cite{2}. We are mainly interested in the phase structure of the theory and concentrate on the scale-dependent effective scalar potential $U_k(\rho)$, where $\rho = \varphi^* \varphi$. The exact evolution equations for $U_k$ can be inferred from ref. \cite{13}, but are far too complicated to be solved exactly. We therefore proceed to different approximations – all of them non-perturbative and going substantially beyond the one-loop computation. We first consider the limit

\footnote{See ref. \cite{9} for earlier versions of exact renormalization group equations.}

\footnote{See ref. \cite{14} for a different approach.}
where the scalar fluctuations can be neglected and deal with the scalar fluctuations later. Furthermore, we approximate the derivative terms in $\Gamma_k$ by standard kinetic terms for both the scalar field and the gauge field and neglect the wave function renormalization of the scalar field. The flow equation for the scale dependence of the average potential becomes very simple in this case and reads, with $t = \ln(k/\Lambda)$,

$$
\partial_t U''_k(\rho) = -\frac{3}{8\pi^2} e^4 \int_0^\infty dx \ x \partial_t (P(x) + 2e^2 \rho)^{-2}.
$$

(1)

Here primes denote derivatives with respect to $\rho$, $e^2(k)$ is the running abelian charge, and the effective inverse propagator

$$
P(x) = x \left[1 - \exp\left(-\frac{x}{k^2}\right)\right]^{-1}
$$

(2)
is a function of the momentum squared $q^{\mu}q_{\mu} = x$, which contains an effective infrared cutoff $P(0) = k^2$. The partial derivative $\partial_t$ on the r.h.s. of eq. (1) is meant to act only on $P$.

After performing the momentum integration the flow equation (1) becomes a partial differential equation for $U_k$ as a function of the two variables $\rho$ and $k$. It has to be supplemented by an equation for the running of the abelian gauge coupling $e^2(k)$. Since the latter is only logarithmic – the usual one-loop $\beta$ function is a good approximation (cf. eq. (10)) – we may first simplify for constant $e^2$. The evolution equation (1) can then be solved easily in the form

$$
U'''_k(\rho) = \frac{3}{2\pi^2} e^6 \int_0^\infty dx \ x(P + 2e^2 \rho)^{-3}.
$$

(3)

Taking the limit $k \to 0$, we obtain the effective potential (apart from an irrelevant constant)

$$
U_0(\rho) = m_s^2 \rho + \frac{3}{16\pi^2} e^4 \rho^2 \left[\ln\left(\frac{\rho}{\rho_0}\right) - \frac{1}{2}\right],
$$

(4)

where $\rho_0$ is fixed through $U'_0(\rho_0) = 0$. This is exactly the well-known one-loop result of Coleman and Weinberg [1]. The mass term at the origin $m_s^2$ is obtained as an integration constant and may hence be treated as a free parameter. The second free integration constant corresponds to the quartic scalar coupling taken at some appropriate scale, i.e. $U''(\rho = \mu^2)$, or, equivalently, the scale $\rho_0$. As a function of $m_s^2$ the theory indeed exhibits a first order phase transition in this approximation!

Beyond the approximation of constant $e^2$ the logarithmic running of $e^2(k)$ does not alter the qualitative features of the phase transition. A numerical solution of
eq. (4) with running $e^2(k)$ gives only minor quantitative corrections. This also holds if the gauge boson contribution to the wave function renormalization (anomalous dimension) of the scalar field is included (see below). The only remaining difference between the exact evolution equation and the flow equation in the limit of neglected scalar fluctuations concerns then a generalization of the gauge boson kinetic term in $\Gamma_k$ beyond the standard form $\sim F_{\mu\nu}F^{\mu\nu}$ and a similar generalization of the covariant scalar kinetic term. These effects would modify the truncated form $(P(x) + 2e^2\rho)^{-1}$ of the effective gauge boson propagator in the presence of a constant background scalar field $\rho$. In principle, the scale dependence of these modifications can in turn be determined from exact flow equations. In the limit of small gauge couplings they could alternatively be approximated by one-loop expressions; this would allow a determination of the $e^6$-terms in the $\beta$-functions without ever doing a two-loop calculation, in complete analogy to the successful demonstration in pure scalar theories [15]. It is easy to convince oneself that none of these small propagator modifications could possibly alter the first order character of the phase transition indicated by the flow equation (1). The dominant effect $-U''_k(\rho)$ turning negative for small enough $\rho$ as $k$ decreases is independent of the details of the effective propagator. We conclude that in the limit of neglected scalar fluctuations the form of the average potential $U_k$ indeed indicates a first order phase transition. For small $e^2$ the Coleman-Weinberg potential (4) gives an accurate quantitative description, with only minor modifications $\sim e^6$ as compared to a more complete non-perturbative analysis.

The crucial question in order to firmly establish the first order phase transition concerns, however, the contributions to the effective potential from fluctuations of the scalar field. Are they small as suggested by the small ratio $\lambda/e^2$? In order to give a non-perturbative answer to this question we truncate the average action (with background gauge fixing [16])

$$\Gamma_k = \int \! d^d x \{ Z_{\phi,k}(D_{\mu}\phi)^*D_{\mu}\phi + U_k(\phi^*\phi)$$

$$+ \frac{1}{4}Z_{F,k}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2\alpha}(\partial_{\mu}(A_{\mu} - \bar{A}_{\mu}))^2 \}.$$  

In this approximation and for $\alpha \to 0$ the flow equation for the average potential reads

$$\partial_t U_k(\rho) = 3v^4 \int_0^{\infty} dx \ x \partial_t \ln[P + 2e^2\rho]$$
\begin{align*}
+ v_4 \int_0^\infty dx \, x \partial_t \ln[P + U'_k(\rho) + 2\rho U''_k(\rho)] \\
+ v_4 \int_0^\infty dx \, x \partial_t \ln[P + U'_k(\rho)],
\end{align*}

with \( v_4 = 1/32\pi^2 \). As in (1), the partial derivative \( \partial_t \) acts only on \( P \). We have solved this equation numerically and the corresponding phase diagram is depicted in fig. 1. The basic variables needed in order to describe the shape of the average potential are the location of the minimum \( \rho_0(k) \) (where \( U'_k(\rho_0(k)) = 0 \)) and the quartic coupling related to \( U''_k(\rho_0) \). It is convenient to use dimensionless renormalized parameters, in particular \( e^2(k) \), the renormalized abelian charge, \( \lambda(k) \), the renormalized quartic scalar self-coupling, and \( \kappa(k) \), the dimensionless renormalized location of the potential minimum. Their definitions are as follows:

\begin{align*}
e^2(k) &= Z'_{F,k} e^2(k) \\
\kappa(k) &= k^{-2} Z_{\varphi,k} \rho_0(k) \\
\lambda(k) &= Z^{-2}_{\varphi,k} U''_k(\rho_0),
\end{align*}

where \( Z_{\varphi,k}(Z_{F,k}) \) denotes the wavefunction renormalization of the scalar (gauge) field. In the projection of the phase diagram on the \( \lambda, \kappa \) plane we observe a critical line separating the symmetric phase (SYM) from the phase with spontaneous symmetry breaking (SSB). The other curves represent the flow of \( \kappa(k) \) and \( \lambda(k) \) for initial values \( \kappa(\Lambda) \) and \( \lambda(\Lambda) \) given at a fixed short distance scale \( \Lambda \). (We use \( e^2(\Lambda) = 0.1 \).) The arrows represent the direction of the flow with \( k \to 0 \). Trajectories starting below the critical line end with \( \kappa = 0 \) and, therefore, with the potential minimum at the origin. For trajectories above the critical line \( \kappa \) finally diverges \( \sim k^{-2} \) such that \( \rho_0 \) reaches a constant value corresponding to the square of the vacuum expectation value of \( \varphi \).

For \( \lambda > 10^{-3} \) all quantitative details of the numerical solution can be understood by a simple polynomial expansion of \( U_k \) around \( \rho_0(k) \), setting \( U''_k(\rho_0) = 0 \) and similarly for higher derivatives. In this approximation the evolution equations for the SSB regime read

\begin{align*}
\partial_t e^2 &= \frac{1}{24\pi^2} e^4 \bar{s}_4(2\lambda, 2e^2\kappa) \\
\partial_t \lambda &= \frac{1}{16\pi^2} \lambda^2(9s_2^4(2\lambda\kappa) + 1) + \frac{3}{4\pi^2} e^4 s_2^4(2e^2\kappa) \\
&- \frac{3}{4\pi^2} e^2 \lambda s_4(2e^2\kappa, 2\lambda\kappa) + \frac{1}{4\pi^2} \lambda^3 km_2(2\lambda\kappa, 0)
\end{align*}

\begin{align*}
&+ v_4 \int_0^\infty dx \, x \partial_t \ln[P + U'_k(\rho) + 2\rho U''_k(\rho)] \\
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\end{align*}
\[ \partial_t \kappa = -2\kappa + \frac{1}{16\pi^2} (3s_4^1(2\lambda\kappa) + 1) + \frac{3}{8\pi^2} \frac{e^2}{\lambda} s_1^4(2e^2\kappa) \]
\[ + \frac{3}{8\pi^2} e^2 \kappa s_{1,1}^4(2e^2\kappa, 2\lambda\kappa) - \frac{1}{8\pi^2} \lambda^2 \kappa^2 m_2^4(2\lambda\kappa, 0). \quad (12) \]

(The last two terms in eqs. (11) and (12) arise from the anomalous dimension \( \eta_\varphi = -\partial_t \ln Z_{k,\varphi} \) and correspond to \( 2\eta_\varphi \lambda \) and \( -\eta_\varphi \kappa \), respectively.) The new elements, compared to the standard perturbative one-loop \( \beta \)-functions, are the threshold functions \( s_4^4, s_1^4, s_2^4, s_{1,1}^4 \) and \( m_2^4 \). They describe the effective decoupling of particles with mass much larger than \( k \). Indeed, two different masses play a role in the regime with spontaneous symmetry breaking (SSB regime). The mass of the gauge boson at the scale \( k \) is given by

\[ M^2(k) = 2e^2(k)\rho_0(k) Z_\varphi Z_F^{-1} = 2e^2(k)\kappa(k)k^2 \quad (13) \]

whereas the mass of the Higgs boson corresponds to

\[ m^2(k) = 2\rho_0(k) U_k^m(\rho_0) Z_\varphi^{-1} = 2\lambda(k)\kappa(k)k^2. \quad (14) \]

The physical masses of these particles (as defined by the two-point function at zero momentum) are obtained from eqs. (13),(14) in the limit \( k \to 0 \). The threshold functions are all normalized to one in the limit of vanishing arguments. For \( \kappa \to 0 \) the evolution equations (10),(11) therefore reduce to the standard perturbative renormalization group equations. On the other hand, the threshold functions vanish for large arguments \( M^2/k^2 \) or \( m^2/k^2 \). Their precise form depends on the precise form of the infrared cutoff function \( R_k(x) \) (for details see ref. [17]). With \( R_k(x) = Z_{k,\varphi,k} \exp(-\frac{\varphi}{k^2})[1 - \exp(-\frac{\varphi}{k^2})]^{-1} \) they read for large values of the argument \( (w \gg 1) \)

\[ [17] s_n^4(w) = 6n\zeta(3)w^{-(n+1)} - 12n(n + 1)(\zeta(3) + \zeta(4))w^{-(n+2)} + \mathcal{O}(w^{-(n+3)}) \quad (15) \]

and similarly for the other threshold functions. The appearance of threshold functions in the \( \beta \)-functions is physically expected. Particles with masses much larger than \( k \) should not influence the running of couplings with the infrared cutoff scale \( k \). Their mass already acts as an effective infrared cutoff and the variation of a second, much smaller cutoff \( k \) should only have a small effect.
The second new ingredient is the evolution equation for $\kappa$. It is closely related to the “quadratic renormalization” of $\rho_0$, or, equivalently, the scalar mass term. It describes within a renormalization group framework the physics related to the “quadratic divergences” appearing in perturbation theory. Such an equation is necessary in order to give a meaningful description of the threshold effects since those must depend on the ratio of the renormalized ($k$-dependent) mass over $k$. Although well known in practice in the Wilson approach to renormalization (for example lattice studies), the effect of quadratic renormalization is missing in many versions of the renormalization group equations. This happens typically if there is no independent running scale to which the masses can be compared, as for example in the Coleman-Weinberg analysis of the effective potential were both the gauge boson mass and the running scale are given by the “classical” or “background” field $\varphi$.

In our approach the renormalization group equation for the $k$-dependent minimum of the average potential arises naturally from the minimum condition

$$\frac{d}{dt}U'_k(\rho_0(k)) = \frac{\partial}{\partial t}U'_k(\rho_0) + U''_k(\rho_0) \frac{\partial \rho_0}{\partial t} = 0$$

and the evolution equation for $\partial_t U'_k$ which involves terms $\sim e^2 k^2$ and $\lambda k^2$. This also explains the perhaps somewhat surprising negative power of $\lambda$ in the third term of eq. (12): For very small $\lambda$ even a moderate change in $U'_k(\partial_t U'_k \sim e^2 k^2)$ induces a large change in the location of the minimum.

Given a set of initial values $e^2(\Lambda), \kappa(\Lambda), \lambda(\Lambda)$ at some short-distance scale $k = \Lambda$, the evolution equations describe how the various couplings run in dependence on $k$. The solution for $k \to 0$ gives the vacuum expectation value $\rho_0(k = 0)$ as well as the renormalized gauge coupling and quartic coupling at zero momentum. The value of $\rho_0(k = 0)$ determines whether the theory is in the symmetric phase ($\rho_0(0) = 0$) or in the SSB phase ($\rho_0(0) > 0$). For a theory in the symmetric phase $\kappa$ reaches zero at some scale $k_s > 0$. Once $\kappa(k_s) = 0$, the evolution has to be continued using equations appropriate for the symmetric regime. Since most of the physical properties can be derived from the equations in the SSB regime, we will not present these equations explicitly here and refer the reader to ref. [10]. For a theory in the SSB phase $\kappa$ remains positive for all values of $k$ and eqs. (10)-(12) are sufficient.

This system of coupled non-linear differential equations still looks quite complicated. It allows, however, for analytical solutions in some approximations which make the main characteristics easy to understand.
First we discuss the linear regime defined by $2\lambda \kappa \ll 1$, $2e^2\kappa \ll 1$. All masses are then much smaller than the scale $k$ and we should recover the results of perturbation theory. The evolution equations simplify considerably:

$$\partial_t e^2 = \frac{4}{3} v_4 e^4$$

(17)

$$\partial_t \lambda = 24v_4 \left( \frac{5}{6} \lambda^2 - \lambda e^2 + e^4 \right)$$

(18)

$$\partial_t \kappa = -2\kappa + 8v_4 - 12v_4\lambda\kappa + 12v_4 \frac{e^2}{\lambda} - 24v_4 \frac{e^4\kappa}{\lambda} + 12v_4 e^2\kappa.$$  (19)

Equations (17), (18) coincide with the standard one-loop result in perturbation theory. The third and fifth term of eq. (18) are obtained through the expansion of $s_4^1(w)$ for small $w$: $s_4^1(w) = 1 - w + O(w^2)$. Combining eqs. (18) and (19) one can obtain the anomalous mass dimension of the scalar field in one-loop order. The running of the gauge coupling

$$e^2(k) = \frac{e^2(\Lambda)}{1 + \frac{4}{3} v_4 e^2(\Lambda) \ln \frac{k}{k_c}}$$

(20)

is a small logarithmic effect and will be neglected in the following ($e^2(k) = e^2$). There is no fixpoint for the running of $\lambda$ (or for the evolution of the ratio $\lambda/e^2$) and the positive $\beta$ function drives $\lambda$ always towards zero. The interesting physics which is associated with the first order character of the Coleman-Weinberg-type phase transition occurs for values of the quartic coupling $\lambda$ much smaller than $e^2$, typically $\lambda \sim v_4 e^4$. For $\lambda \ll e^2$ we can neglect the terms $\sim \lambda^2$ and $e^2\lambda$ in eq. (18). The running of $\lambda$ is then easily found

$$\lambda(k) = 24v_4 e^4 \ln \left( \frac{k}{k_c} \right)$$

(21)

$$k_c = \Lambda \exp \left( -\frac{\lambda(\Lambda)}{24v_4 e^4} \right).$$

(22)

According to eq. (18) $\lambda$ reaches zero at some scale $k_c \neq 0$ and becomes negative for $k < k_c$. To solve eq. (13) for $\kappa$ in the limit $|\lambda| \ll e^2$ we observe the existence of an UV-stable fixpoint for $\lambda\kappa$:

$$\partial_t (\lambda\kappa) = -2 [\lambda\kappa - (\lambda\kappa)_*]$$

(23)

$$(\lambda\kappa)_* = 6v_4 e^2$$

(24)
This fixpoint is represented by the dashed line in the phase diagram in fig. 1. Combining the solution of eq. (23) with eq. (21) we find

\[
\kappa(k) = \frac{1}{4e^2} \frac{1 + \Delta_\Lambda \kappa^2}{\ln(k/k_c)}
\]

\[
\Delta_\Lambda = \frac{\lambda(\Lambda)\kappa(\Lambda)}{(\lambda\kappa)_s} - 1.
\] (25)

We have parametrized the initial value at \( \Lambda \) such that \( \Delta_\Lambda \) measures the deviation of \( \lambda\kappa \) from its fixpoint value of eq. (24) at the scale \( \Lambda \).

Due to the running of \( \lambda \) there is no fixpoint in \( \kappa \). For \( \Delta_\Lambda = 0 \) one sees that \( \kappa \) diverges for \( k \to k_c \). On the other hand, for sufficiently negative \( \Delta_\Lambda \), \( \kappa(k) \) will vanish at some scale \( k_s > k_c \) and the evolution will continue in the symmetric regime. For \( k < k_s \) we replace the parameter \( \kappa(k) \) by

\[
m_s^2(k) = Z_{\phi,k}^{-1} U_k''(0)
\] (26)

and use evolution equations adapted to the symmetric regime. In particular, the scale dependence of the mass term simplifies for \( |\lambda| \ll e^2 \) to

\[
\partial_t m_s^2 = -\frac{3}{8\pi^2} e^2 k^2.
\] (27)

The mass term at \( k = 0 \) is therefore connected to \( k_s \) by

\[
m_s^2 = \frac{3}{16\pi^2} e^2 k_s^2.
\] (28)

There is a critical value

\[
\Delta_{\Lambda_c} = -\frac{k_c^2}{\Lambda^2}
\] (29)

separating these two different behaviours. For \( \Delta_\Lambda > \Delta_{\Lambda_c} \) we see that \( \kappa(k \to k_c) \) diverges, whereas for \( \Delta_\Lambda < \Delta_{\Lambda_c} \) we end in the symmetric phase. For \( \Delta_\Lambda = \Delta_{\Lambda_c} \) we find that \( \kappa(k \to k_c) \) reaches a finite non-zero value

\[
\kappa_c = \frac{1}{2e^2}.
\] (30)

Thus the system of equations (17)-(19) exhibits a bifurcation phenomenon at the critical value \( \Delta_{\Lambda_c} \): No finite non-vanishing value of \( \kappa(k_c) \) except \( 1/2e^2 \) can be reached.
In the region of initial values $\Delta_\Lambda < 2\Delta_\Lambda_c$ the linear approximation of eqs. (17)-(19) remains always valid. In this region the mass term is bounded from below
\[ m^2_s > \frac{3}{8\pi^2} e^2 k_c^2 = \frac{3}{8\pi^2} e^2 \Lambda^2 \exp \left( -\frac{\lambda(\Lambda)}{24v_4 e^4} \right). \]  
(31)

In the SSB phase $\kappa$ always diverges and there must be some regime where the linear approximation breaks down. A quick inspection of the full equations (11)-(12) shows, however, that for $\lambda > \frac{3}{8\pi^2} e^4$ the threshold functions only imply a stop of the running of $\lambda$ and $\rho_0$ for $2e^2\kappa > 1$. We can therefore approximate
\[ \frac{\partial \lambda}{\partial t} = 0 \quad \text{for} \quad \begin{cases} 2e^2\kappa > 1 \\ \lambda > \frac{3}{8\pi^2} e^4. \end{cases} \]  
(32)

For $\Delta_\Lambda > 0$ we find that $2e^2\kappa$ becomes larger than one at a scale $k_F$ where $\ln(k_F^2/k_c^2) > 1, \lambda(k_F) > \frac{3}{8\pi^2} e^4$. For this region there is a lower bound on the vacuum expectation value and the Higgs mass
\[ \rho_0(k = 0) > \frac{\exp(1)}{2e^2} k_c^2 = \frac{\exp(1)}{2e^2} \Lambda^2 \exp \left( -\frac{\lambda(\Lambda)}{24v_4 e^4} \right) \]
\[ m^2 > \frac{3\exp(1)}{8\pi^2} e^2 k_c^2. \]  
(33)

In order to make a quantitative comparison with the perturbative Coleman-Weinberg analysis [1] for $\lambda = O(e^4)$ we want to identify the value $\kappa(\Lambda)$ which leads to the “massless theory”, i.e. $m^2_s(k = 0) = 0$. For a quartic potential at the short distance scale $\Lambda$ (and $Z_{\varphi,\Lambda} = 1$) we observe
\[ m^2_s(\Lambda) = -\lambda(\Lambda)\kappa(\Lambda)\Lambda^2. \]  
(34)

Using the flow equations (23) and (27) allows us to express the initial deviation $\Delta_\Lambda$ from the $(\lambda\kappa)$-fixpoint (24) in terms of the mass term at the origin $m^2_s$ for $k = 0$:
\[ \Delta_\Lambda = -\frac{1}{6v_4 e^2} \frac{m^2_s(k = 0)}{\Lambda^2}. \]  
(35)

Therefore, the “massless theory” corresponds to $\Delta_\Lambda = 0$. Using the linear equations (21), (27) as long as $2e^2\kappa < 1$ and eq. (12) for $2e^2\kappa > 1$ we deduce for $m^2_s = 0$ the mass relation (for $k = 0$)
\[ \frac{m^2}{M^2} = \frac{\lambda(k_b)}{e^2} = \frac{3}{8\pi^2} e^2, \]  
(36)
where $k_b$ is determined by $2e^2\kappa(k_b) = 1$. Thus, in this approximation the well-known result of Coleman and Weinberg \cite{1} is recovered.

We conclude that except for the immediate vicinity of $\Delta_{\Lambda_c}(2\Delta_{\Lambda_c} < \Delta_{\Lambda} < 0)$ the system of equations (10)-(12) reproduces the results of the Coleman-Weinberg analysis, including the effect of dimensional transmutation (cf. eqs. (31), (33) and (36)). It is tempting to think that the bifurcation phenomenon observed in the linear approximation corresponds to a first order transition, leading to mass bounds somewhat weaker than (31) and (33). We observe that the bifurcation occurs for values $\lambda \ll e^4$. If the effect of scalar fluctuations could be neglected for very small $\lambda$, we may use eq. (1) to establish the first order character of the phase transition. Using the system of flow equations (10)-(12) for $\lambda > e^4$ and eq. (1) for $\lambda < e^4$ would then give a detailed non-perturbative quantitative picture of the Coleman-Weinberg phase transition.

For a numerical study of the evolution equation (6) for $\lambda \ll e^4$ we have to consider potentials which may have more than one local minimum. For this reason we use for our numerical investigation an approximation of $U_k$ as a quartic polynomial in $\rho$, i.e. we include couplings up to $\varphi^8$. As long as $U_k$ exhibits a local “asymmetric minimum” ($\lambda > 0, \kappa > 0$), the effective average potential is parametrized in terms of the four variables $\lambda, \kappa, m_s^2/k^2$, and $\lambda_s$, where $\lambda_s(k) = Z_{\varphi,k}U_k''(\rho = 0)$ denotes the renormalized quartic scalar self-coupling at the origin. This parametrization is best suited for the study of a (weak) first order phase transition: We expect that the system near the phase transition develops a small (positive) mass term at the origin. Therefore, it is most important that all the relevant quantities like masses and quartic couplings are followed by their appropriate evolution equation. Higher couplings like $U_k'''(\rho_0)$ appearing on the r.h.s. of the evolution equation for $\kappa$ and $\lambda$ are determined as functions of the four variables using the polynomial approximation for $U_k$. Our parametrization yields information on the global structure of the effective potential since it contains information on the symmetric extremum $(m_s^2, \lambda_s)$ as well as on the asymmetric one ($\kappa, \lambda$). When the broken phase disappears, ($\lambda \leq 0$ or $\kappa = 0$), the evolution is continued in the symmetric regime with the corresponding evolution equations. A more detailed account on this method and some generalisations will be given elsewhere \cite{19}.

\footnote{The two systems of flow equations have a smooth overlap for small $e^2$.}
The phase diagram in the range of small values of $\lambda(k)$ is shown in fig. 2. We have specified the initial conditions at some fixed short distance scale $\Lambda$ by the requirement that the classical potential $U_\Lambda$ is quartic in the fields, $U(\rho) = \frac{1}{2} \lambda(\Lambda)(\rho - \rho_0(\Lambda))^2$. For all trajectories, we fix $e^2(\Lambda)$ and $\lambda(\Lambda)$. The phase diagram is then obtained through the variation of $\kappa(\Lambda)$ which serves as a free parameter.

The phase transition is clearly of the weak first order type and we observe three characteristic regions in the $(\lambda, \kappa)$ plane: In region I the average potential has only one minimum. (This means $m_\Phi^2 < 0$ for $\kappa > 0$.) For trajectories remaining within region I this holds for all values of $k$. Once $\kappa(k_s)$ becomes zero, region I should be continued in the symmetric regime with the only minimum now at the origin. The behaviour of trajectories remaining within region I is very similar to a second order phase transition. In particular this concerns the values of $\lambda$ shown in fig. 1. For not too small values of $\lambda(\Lambda)$ the phase transition is therefore “almost of the second order”. The weak first order character of the transition is related to trajectories so close to the critical line that they cross the dashed boundary between regions I and II. For regions II and III the average potential has two local minima. The “asymmetric minimum” occurs for $\kappa > 0$ and is the absolute minimum in region II. The “symmetric minimum” is at the origin. This becomes the lowest minimum in region III. The (dashed) borderline separating I and II corresponds to $m_\Phi^2 = 0$. Therefore the “massless theory” (the Coleman-Weinberg case) is described by the trajectory A for which $m_\Phi^2(k = 0) = 0$. On this trajectory the mass relation (36) is found to coincide with the numerical result within 4%. The regions II and III are separated by the dashed-dotted line. This borderline describes the effective potential with degenerate minima $U_k(0) = U_k(\varphi_0)$. The trajectory with initial value $\kappa_{\text{crit}}$ is running asymptotically towards a potential with degenerate minima and corresponds to the critical line of the phase diagram. All trajectories to the right of B with $\kappa(\Lambda) > \kappa_{\text{crit}}$ belong to the phase with spontaneous symmetry breaking, whereas all trajectories to the left with $\kappa(\Lambda) < \kappa_{\text{crit}}$ will end up for $k \to 0$ with lowest minimum at the origin and therefore belong to the symmetric phase. The trajectory C is the spinodal line of the model. It describes, similarly to A, an asymptotically massless theory but now for scalar fields at the asymmetric local minimum. The first order part of the phase diagram can only be seen in the small interval of initial values $\kappa_\Lambda$ with $\kappa_\Lambda^B < \kappa_\Lambda < \kappa_\Lambda^A$. This corresponds to $\lambda$ smaller than approximately
\frac{1}{32\pi} e^4 \text{ in the final part of the running.}

In conclusion, the weak first order character of the phase transition has now been established on firm non-perturbative grounds. In particular, we emphasize that our non-perturbative evolution equation \((\mathbf{6})\) is valid for all values of \(\rho\), in contrast to the perturbative treatment which becomes unreliable around the origin, as pointed out in the original work \([1]\). The average potential is always real. The perturbative imaginary part has to be considered as a pure artefact of perturbation theory. The average potential becomes convex for \(k \rightarrow 0\) as can be seen on very general grounds \([6], [8]\). This approach to convexity is described by the exact evolution equation, but any polynomial approximation for the potential breaks down in the SSB phase\([7]\). We are not interested here in the physics related to the approach to convexity and stop the running of \(U_k\) once \(k^2\) becomes of the order of the curvature of the potential at a local maximum. At these values of \(k\) the masses \(m_s^2(k), m^2(k)\) or \(M^2(k)\) have already reached almost constant values (and similarly for the interactions), and we use in practice these values instead of the values for \(k = 0\). We can then proceed to compute the physical photon mass \(M^2\) (which is proportional to the vacuum expectation value \(\rho_0\)), the scalar mass in the asymmetric minimum \(m^2\), and at the origin \(m_s^2\) as a function of the initial value \(\kappa(\Lambda)\). As discussed before we may use \(m_s^2\) to label the distance from the phase transition. In fig. 3 we have plotted the various masses as a function of \(m_s^2\) and indicated values corresponding to the curves A, B and C in fig. 2. The mass ratio \(m^2/M^2\) is given in fig. 4 together with the Coleman-Weinberg prediction \((36)\) for \(m_s^2 = 0\). The visible small difference between our curve and the prediction \((36)\) is actually not a real effect due to the scalar fluctuations neglected for eq. \((36)\) but rather an artefact of the polynomial approximation to the potential. This can be seen by using the numerical solution with polynomial approximation, where the scalar fluctuations are discarded. The resulting curve for \(m^2/M^2\) is essentially identical to fig. 4. We conclude that the effects of scalar fluctuations are really tiny for the small values of \(\lambda\) relevant for the first order character of the transition. Our non-perturbative approach permits a very precise quantitative understanding of all physically interesting masses and couplings as well as of the complete shape of the effective potential. This is achieved through the combination of the numerical solution of eq. \((\mathbf{6})\) in the range of \(\lambda(k) > 10e^4\)

\footnote{For a discussion of the approach to convexity for the pure scalar theory see ref. \([18]\).}
(cf. fig.1) with the analytical solution \( \lambda(k) < 10e^4 \).

In summary, we find a first order phase transition for scalar QED for arbitrary values of the quartic scalar coupling \( \lambda \). The mass gap between the symmetric phase and the one with spontaneous symmetry breaking is exponentially small (except for tiny values of \( \lambda \)). This is an aspect of dimensional transmutation [1] due to the logarithmic running of \( \lambda \). In the vicinity of the phase transition the running quartic scalar coupling is always very small. As a result, we find that the one-loop calculation of the effective potential in the abelian Higgs model [1] gives a rather accurate description of all physical quantities of interest in the vicinity of the phase transition. Our non-perturbative method permits (for the first time) a consistent treatment of the region around \( \varphi = 0 \) in the phase with spontaneous symmetry breaking. Comparison with our non-perturbative results gives a simple prescription of how to deal, in this particular situation, with the effects of scalar fluctuations in the one-loop calculation: The unphysical imaginary part should be discarded or the scalar fluctuations should simply be omitted, in the range of \( \varphi \) for which a negative mass squared for the scalar fluctuations is predicted by the one-loop calculation. The one-loop potential computed with this prescription is very close to the average potential \( k_{cv} \) which is given by \( k_{cv}^2 \approx 2m_{max}^2 \), with \( m_{max}^2 = -U'(\rho_{max}) \) corresponding to the curvature at the local maximum of the potential.

More generally, the effective potential \( U_0 \) is not the most appropriate quantity for a description of spontaneous symmetry breaking. On general grounds \( U_0 \) is convex and therefore more subtle to handle in the phase with spontaneous symmetry breaking. On the other hand, the average potential \( U_k \) is not convex and the quantum effects leading to convexity of the effective potential \( U_0 \) only set in for \( k < k_{cv} \) [18]. As long as the relevant physical distances are smaller than \( k_{cv}^{-1} \) the fluctuations leading to convexity can be neglected. This also holds for all properties related to the true vacuum (corresponding to the absolute minimum of the potential), such as the masses of physical particles and their interactions. We should mention, however, that the flattening of the potential due to the approach to convexity may play a role in certain cosmological scenarios of inflation.

The present work has also permitted precise quantitative tests for analytical and numerical approximate solutions of the non-perturbative evolution equations. These evolution equations remain valid in three dimensions or for a non-vanishing
temperature where perturbation theory fails due to severe infrared problems. In
the presence of large scalar couplings the one-loop approximation neglecting scalar
fields is not expected to give a qualitatively correct description. The present work is
therefore an excellent starting point for a non-perturbative treatment of the three-
dimensional abelian Higgs model – which should describe the phase transition of
superconductors – or of the high temperature phase transition in four-dimensional
scalar quantum electrodynamics.

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Figure Captions

Fig. 1: The phase diagram of scalar QED. The thick line separates the symmetric phase (SYM) from the phase with spontaneous symmetry breaking (SSB).

Fig. 2: The phase diagram for small $\lambda$. We indicate the “massless” (A), critical (B), and spinodal (C) trajectory.

Fig. 3: The mass of the gauge field $M$ as a function of the scalar mass term at the origin $m^2_s$. We also indicate the scalar mass $m$ for spontaneous symmetry breaking. All masses are given in units of the cutoff $\Lambda$.

Fig. 4: The ratio of the Higgs-boson to gauge-boson mass as a function of $m^2_s$. The star denotes the Coleman-Weinberg result.
This figure "fig1-1.png" is available in "png" format from:

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