HEIGHT OF SOME AUTOMORPHISMS OF LOCAL FIELDS

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Abstract. In this note, we determine which automorphism subgroups of Aut\(_{(\mathbb{F}_q((x)))}\) are corresponding to \(\mathbb{Z}_p\)-extensions or \(\mathbb{Z}_p \times \mathbb{Z}_p\)-extensions of characteristic 0 fields.

1. Introduction

Let \(k = \mathbb{F}_q\) be a finite field of characteristic \(p > 0\) and let \(L/K\) be a totally ramified abelian extension, where \(K\) is a local field with residue field \(k\). Then \(G = \text{Gal}(L/K)\) has a decreasing filtration by the upper ramification subgroups \(G(r)\), defined for nonnegative \(r \in \mathbb{R}\) (see [10, IV]). Since \(G\) is abelian, \(L/K\) is arithmetically profinite (see [12]). This means that for every \(r \geq 0\) the upper ramification group \(G(r)\) has finite index in \(G\). This allows us to define the Hasse-Herbrand function \(\psi : \mathbb{R}^+ \to \mathbb{R}^+\) where \(\psi_{L/F}(r) = \int_0^r [G : G(t)] dt\) and \(\phi_{L/K}(r) = \psi_{L/K}^{-1}(r)\). The ramification subgroups of \(G\) with the upper numbering are defined by \(G[r] = G(\phi_{L/K}(r))\).

Let \(\text{Aut}_k(k((x)))\) denote the group of continuous automorphisms of \(k((x))\) which induce the identity map on \(k\). A closed abelian subgroup \(G\) of \(\text{Aut}_k(k((x)))\) also has a ramification filtration. The lower ramification subgroups of \(G\) are defined by \(G[\sigma] = \{\sigma \in G : v_x(\sigma(x) - x) \geq r + 1\}\) for \(r \geq 0\). Since \(G[r]\) has finite index in \(G\) for every \(r \geq 0\), the function \(\phi_G : \mathbb{R}^+ \to \mathbb{R}^+\) where \(\phi_G(r) = \int_0^r [G : G(t)]^{-1} dt\) is strictly increasing. We define the ramification subgroups of \(G\) with the upper numbering by \(G(r) = G[\phi_G^{-1}(r)]\).

Wintenberger [11] has shown that the field of norms functor induces an equivalence between a category whose objects are totally ramified abelian \(p\)-adic Lie extensions \(L/K\), where \(K\) is a local field with residue field \(k\), and a category whose objects are pairs \((\mathbb{K}, G)\), where \(\mathbb{K} \simeq k((x))\) and \(G\) is an abelian \(p\)-adic Lie subgroup of \(\text{Aut}_k(\mathbb{K})\). In short, if \(G\) is an abelian \(p\)-adic Lie subgroup of \(\text{Aut}_k(k((x)))\), then there is an abelian \(p\)-adic Lie extensions \(L/K\) corresponding to \((k((x)), G)\) by the equivalence of categories given by the field of norms functor. Moreover, the canonical isomorphism from \(\text{Gal}(L/K)\) onto \(G\) preserves the ramification filtration [6, 11]. This equivalence has been extended to allow \(\text{Gal}(L/K)\) and \(G\) to be arbitrary abelian pro-\(p\) groups by Keating [3]. In the following, we will simply say that \(G\) is corresponding to \(L/K\) if the extension \(L/K\) corresponds to \((k((x)), G)\) by the equivalence of categories given by the field of norms functor.

For \(\sigma \in \text{Aut}_k(k((x)))\), we let

\[i(\sigma) = v_x \left( \frac{\sigma(x)}{x} - 1 \right)\]
Moreover, if $\sigma(x) \equiv x \pmod{x^2}$, then we denote $i_n(\sigma) = i(\sigma^n)$. When $\sigma \in \text{Aut}_k(k((t)))$ has infinite order, the sequence $\{i_n(\sigma)\}$ is strictly increasing and attracts many attentions. In [9] Sen proved that for every $n \in \mathbb{N}$, $i_{n+1}(\sigma) \equiv i_n(\sigma) \pmod{p^{n+1}}$. In [2] Keating determines upper bounds for the $i_n(\sigma)$ in some cases and in [4, 5] the authors improve Keating’s results using Wintenberger’s theory of field of norms [11, 12]. These results are based on the fact that the automorphism subgroups correspond to $\mathbb{Z}_p\times\mathbb{Z}_p$-extensions of characteristic 0 fields in [2, 3, 4] and correspond to $\mathbb{Z}_p\times\mathbb{Z}_p$-extensions of characteristic 0 fields in [5]. In this note, we determine which automorphism subgroups of $\text{Aut}_k(k((t)))$ are corresponding to $\mathbb{Z}_p\times\mathbb{Z}_p$-extensions of characteristic 0 fields. In the following, we will simply say that an extension $L/K$ is of characteristic 0 if the characteristic of $K$ is 0. Likewise, if the characteristic of $K$ is $p$, then we say that the extension $L/K$ is of characteristic $p$.

Motivated by the definition of height of a formal group and height of a $p$-adic dynamical system [7], we have the following definition.

**Definition 1.1.** Let $\sigma \in \text{Aut}_k(k((t)))$ with $\sigma \equiv x \pmod{x^2}$. We say that the **height** of $\sigma$ exists if $\lim_{n \to \infty} i_n(\sigma)/i_{n-1}(\sigma)$ is finite and denote by

$$\text{Height}(\sigma) = \lim_{n \to \infty} \log_p \frac{i_n(\sigma)}{i_{n-1}(\sigma)}.$$ 

Let $G$ be a closed subgroup of $\text{Aut}_k(k((x)))$. Our main result shows that if $G$ is isomorphic to $\mathbb{Z}_p$ then $G$ corresponds to a characteristic 0 field extension if and only if every nonidentity element of $G$ has height 1 and if $G$ is isomorphic to $\mathbb{Z}_p\times\mathbb{Z}_p$, then $G$ corresponds to a characteristic 0 field extension if and only if every nonidentity element of $G$ has height 2.

The proof of our result is based on the following straightforward consequence of Theorem 4 of [8].

**Lemma 1.2.** Let $L/K$ be an abelian extension and let $G$ denote the Galois group $\text{Gal}(L/K)$.

1. If $K$ is of characteristic $p$, then the mapping $\sigma \to \sigma^p$ maps $G(n)$ into $G(pn)$, for all $n \in \mathbb{N}$.
2. If $K$ is of characteristic 0 with absolute ramification index $e$, then the mapping $\sigma \to \sigma^p$ induces a homomorphism which maps $G(n)/G(n+1)$ onto $G(n+e)/G(n+e+1)$, for all $n$ large enough.

We remark that since $G$ is abelian, every upper ramification break $u$ (i.e. $G(u) \supseteq G(u+\epsilon)$, $\forall \epsilon > 0$) is an integer (see for instance [10, V]). Therefore, we can apply Lemma 1.2 to the case where $n$ is an upper ramification break of $G$. Moreover, if $K$ is of characteristic 0 and $\text{Gal}(L/K)$ is a pro-$p$ group, then Lemma 1.2(2) shows that the mapping $\sigma \to \sigma^p$ maps $G(n)$ onto $G(n+e)$, for $n$ sufficiently large. Therefore, in this case, if there is no nontrivial $p$-torsion element in $G$, then the mapping $\sigma \to \sigma^p$ induces an isomorphism between $G(n)/G(n+1)$ and $G(n+e)/G(n+e+1)$, for all $n$ large enough. In particular, if $n$ is large enough and $n$ is an upper ramification break of $G$, then $n+e$ is also an upper ramification break of $G$.

2. $\mathbb{Z}_p$-extensions

Given $\sigma \in \text{Aut}_k(k((t)))$ with $\sigma \equiv x \pmod{x^2}$, write $\lim_{n \to \infty} (i_n(\sigma)/p^n) = (p/(p-1))e$. It is well-known that either $e$ is a positive integer or $e = \infty$ (see
for instance [13]. Moreover, $e$ is a positive integer if and only if the field extension $E/F$ corresponding to the closed subgroup generated by $\sigma$ is of characteristic 0. In fact, in this case, $e$ is the absolute ramification index of $F$. If $e$ is finite, then it’s clear that $\lim_{n \to \infty} (i_n(\sigma)/i_{n-1}(\sigma)) = p$. That is $\text{Height}(\sigma) = 1$. In this section, we will show that the converse is also true. Thus, for the case $\sigma \in \text{Aut}_k(k((x)))$ with $\text{Height}(\sigma) = 1$, the closed cyclic group generated by $\sigma$ corresponds to a $\mathbb{Z}_p$-extension of characteristic 0 field.

We prove this by contradiction. Suppose that the corresponding $\mathbb{Z}_p$-extension is of characteristic $p$. Then it is also true that the $\mathbb{Z}_p$-extension corresponding to the closed subgroup $H$ generated by $\sigma^{p^n}$ is of characteristic $p$. By considering the ramification groups of $H$, we have $\sigma^{p^n} \in H[i_n(\sigma)] \setminus H[i_n(\sigma) + \epsilon]$ and $\sigma^{p^{n+1}} \in H[i_{n+1}(\sigma)] \setminus H[i_{n+1}(\sigma) + \epsilon], \forall \epsilon > 0$. Therefore $\phi_H(i_n(\sigma))$ and $\phi_H(i_{n+1}(\sigma))$ are upper ramification breaks of $H$ and hence we can apply Lemma 1 (2) to get $\sigma^{p^{n+1}} \in H(p\phi_H(i_n(\sigma)))$. In other words,

$$\phi_H(i_{n+1}(\sigma)) = i_n(\sigma) + \frac{i_{n+1}(\sigma) - i_n(\sigma)}{p} \geq p \phi_H(i_n(\sigma)) = p i_n(\sigma), \forall n \in \mathbb{N}.$$  

This says that

$$i_{n+1}(\sigma) \geq (p^2 - p + 1)i_n(\sigma), \forall n \in \mathbb{N},$$

and hence contradicts to the assumption that $\lim_{n \to \infty} \frac{i_n(\sigma)}{i_{n-1}(\sigma)} = p$.

Conversely, suppose that $G$ is corresponding to a characteristic 0 field extension $E/F$ with $e = v_F(p)$ being the absolute ramification index of $F$. By the definition of lower ramification group, for every $n \in \mathbb{N}$, $\sigma^{p^n} \in G[i_n(\sigma)] \setminus G[i_n(\sigma) + \epsilon]$ and $\sigma^{p^{n+1}} \in G[i_{n+1}(\sigma)] \setminus G[i_{n+1}(\sigma) + \epsilon], \forall \epsilon > 0$. On the other hand, by Lemma 1 (2) and the remark following it, when $n$ is large enough if we let $u = \phi_G(i_n(\sigma))$, then $u + e$ is an upper ramification break of $G$. Moreover, since $\sigma^{p^n} \in G[i_n(\sigma)] = G(u)$, $\sigma^{p^{n+1}} \in G(u)^p = \{g^p : g \in G(u)\} = G(u + e)$. In other words, $u + e = \phi_G(i_n(\sigma)) + e = \phi_G(i_{n+1}(\sigma))$, and hence

$$e = \phi_G(i_{n+1}(\sigma)) - \phi_G(i_n(\sigma)) = \frac{1}{p^{n+1}}(i_{n+1}(\sigma) - i_n(\sigma))$$

because $G$ is isomorphic to $\mathbb{Z}_p$. This is true for all $n$ large enough. Therefore, we conclude that there exists $m \in \mathbb{N}$ such that for all $n > m$,

$$i_n(\sigma) = i_m(\sigma) + \sum_{j=m+1}^{n} (i_j(\sigma) - i_{j-1}(\sigma))$$

$$= i_m(\sigma) + e(p^{n+1} + \cdots + p^m)$$

$$= i_m(\sigma) + \frac{ep}{p-1}(p^n - p^m).$$

This shows

$$\lim_{n \to \infty} \frac{i_n(\sigma)}{p^n} = \frac{ep}{p-1}.$$  

We summarize this result as the following.

**Theorem 2.1.** Suppose that $G \subseteq \text{Aut}_k(k((x)))$ is a closed subgroup generated by $\sigma$ which is isomorphic to $\mathbb{Z}_p$. Then the following are equivalent:

1. $\lim_{n \to \infty} \frac{i_n(\sigma)}{i_{n-1}(\sigma)} = p$
Therefore, \( \tau \) since Lemma 1.2 (1) says \( i \) and (3) by any nonidentity element \( \tau \in G \). This is because the closed subgroup of \( G \) generated by \( \tau \) is a finite index subgroup. In other words, we shows that the \( \mathbb{Z}_p \)-extension corresponding to \( G \) is of characteristic 0 if and only if every nonidentity element \( \tau \in G \) has \( \text{Height}(\tau) = 1 \).

3. \( \mathbb{Z}_p \times \mathbb{Z}_p \)-extensions

In this section we extend the result of the previous section to the case that \( G \subseteq \text{Aut}_k(k((x))) \) is isomorphic to \( \mathbb{Z}_p \times \mathbb{Z}_p \). In this case we show that every nonidentity element of \( G \) has height 2 if and only if the \( \mathbb{Z}_p \times \mathbb{Z}_p \)-extension corresponding to \( G \) is of characteristic 0.

Let \( G \) be a closed subgroup of \( \text{Aut}_k(k((x))) \) which is isomorphic to \( \mathbb{Z}_p \times \mathbb{Z}_p \) and suppose that for every nonidentity element \( \sigma \in G \) we have \( \lim_{n \to \infty} i_n(\sigma)/i_{n-1}(\sigma) = p^2 \). Again, we use method of contradiction to show that the \( \mathbb{Z}_p \times \mathbb{Z}_p \)-extension corresponding to \( G \) is of characteristic 0. First, suppose that the corresponding \( \mathbb{Z}_p \times \mathbb{Z}_p \)-extension is of characteristic \( p \). Then for any two linearly independent elements \( \sigma, \tau \in G \), since \( \langle \sigma, \tau \rangle \) is a finite index subgroup of \( G \) (we use \( \langle \sigma, \tau \rangle \) to denote the closed subgroup of \( G \) generated by \( \sigma \) and \( \tau \)), the field extension corresponding to \( \langle \sigma, \tau \rangle \) is also a characteristic \( p \) field extension. Similarly, for \( m, n \in \mathbb{N} \), the \( \mathbb{Z}_p \times \mathbb{Z}_p \) extension corresponding to \( \langle \sigma^p, \tau^p \rangle \) is also of characteristic \( p \). We consider several cases.

For a given nonidentity \( \sigma \in G \), we first suppose that for every \( N \in \mathbb{N} \), there exist \( n, m > N \) and \( \tau \) of \( G \) such that \( i_n(\sigma) < i_m(\tau) < i_{m+1}(\tau) \leq i_{n+1}(\sigma) \). Notice that the field extension corresponding to the closed subgroup \( H = \langle \sigma^p, \tau^p \rangle \) is also of characteristic \( p \). By considering the lower ramification subgroups of \( H \), we have

\[
H[i_n(\sigma)] = \langle \sigma^p, \tau^p \rangle \supseteq H[i_n(\sigma) + 1] = \cdots = H[i_m(\tau)] = \langle \sigma^{p+1}, \tau^p \rangle \\
\supseteq H[i_m(\tau) + 1] = \cdots = H[i_{m+1}(\tau)] = \langle \sigma^{p+1}, \tau^{p+1} \rangle.
\]

Therefore,

\[
\phi_H(i_m(\tau)) = i_n(\sigma) + \frac{i_m(\tau) - i_n(\sigma)}{p}
\]

and

\[
\phi_H(i_{m+1}(\tau)) = i_n(\sigma) + \frac{i_m(\tau) - i_n(\sigma)}{p} + \frac{i_{m+1}(\tau) - i_m(\tau)}{p^2}.
\]

Since \( \tau^p \in H[i_m(\tau)] = H(\phi_H(i_m(\tau))) \) and

\[
\tau^{p+1} \in H(\phi_H(i_{m+1}(\tau))) \setminus H(\phi_H(i_{m+1}(\tau)) + \epsilon), \forall \epsilon > 0,
\]

Lemma 1.2 (1) says

\[
i_n(\sigma) + \frac{i_m(\tau) - i_n(\sigma)}{p} + \frac{i_{m+1}(\tau) - i_m(\tau)}{p^2} \geq p(i_n(\sigma) + \frac{i_m(\tau) - i_n(\sigma)}{p}).
\]

Therefore by \( i_n(\tau) \geq i_{m+1}(\tau) \) and \( i_m(\tau) > i_n(\sigma) \), we have

\[
i_{n+1}(\sigma) \geq (p^2 - p + 1)i_n(\tau) + (p^3 - 2p^2 + p)i_n(\sigma) > (p^3 - p^2 + 1)i_n(\sigma).
\]
Since for every $N \in \mathbb{N}$, this is true for some $n > N$, it contradicts to the assumption that $\lim_{n \to \infty} \frac{i_n(\sigma)}{i_{n+1}(\sigma)} = p^2$.

Now suppose that for every $N \in \mathbb{N}$, there exist $n, m > N$ and $\tau$ of $G$ such that $i_n(\sigma) < i_m(\tau) < i_{n+1}(\sigma) < i_{n+2}(\sigma) \leq i_{m+1}(\tau)$. Notice that the field extension corresponding to the closed subgroup $H = \langle \sigma^{p^2}, \tau^{p^m} \rangle$ is also of characteristic $p$. By considering the lower ramification subgroups of $H$, we have

$$H[i_n(\sigma)] = \langle \sigma^{p^2}, \tau^{p^m} \rangle \supseteq H[i_n(\sigma) + 1] = \cdots = H[i_m(\tau)] = \langle \sigma^{p^{n+1}}, \tau^{p^m} \rangle$$

$$\supseteq H[i_{n+1}(\sigma)] = \cdots = H[i_{n+2}(\sigma)] = \langle \sigma^{p^{n+2}}, \tau^{p^{m+1}} \rangle.$$

Therefore

$$\phi_H(i_{n+2}(\sigma)) = \phi_H(i_{n+1}(\sigma)) + \frac{i_{n+2}(\sigma) - i_{n+1}(\sigma)}{p^3}.$$  

By Lemma 1.2 (1), $\phi_H(i_{n+1}(\sigma)) \geq p \phi_H(i_n(\sigma)) = p i_n(\sigma)$ and since $\sigma^{p+2} \notin H[i_{n+2}(\sigma) + \epsilon], \forall \epsilon > 0$, we have $\phi_H(i_{n+2}(\sigma)) \geq p \phi_H(i_{n+1}(\sigma))$. This implies

$$\frac{i_{n+2}(\sigma) - i_{n+1}(\sigma)}{p^3} \geq (p-1)\phi_H(i_{n+1}(\sigma)) \geq (p-1)p i_n(\sigma),$$

and hence

$$i_{n+2}(\sigma) \geq (p^5 - p^4)i_n(\sigma) + i_{n+1}(\sigma).$$

Since for every $N \in \mathbb{N}$, this is true for some $n > N$, it contradicts to the assumption that $\lim_{n \to \infty} \frac{i_n(\sigma)}{i_{n+1}(\sigma)} = p^2$.

Now we only have the following two cases to consider:

1. There exists $N$ such that there is neither $m, n > N$ nor any nonidentity $\tau \in G$ such that $i_n(\sigma) < i_m(\tau) < i_{n+1}(\sigma)$.
2. There exists $m, n \in \mathbb{N}$ and a nonidentity $\tau \in G$ such that $i_{n+j}(\sigma) < i_{m+j}(\tau) < i_{n+j+1}(\sigma)$, for all $j \in \mathbb{N}$.

For the case (1), there exists $N \in \mathbb{N}$ such that

$$G[i_n(\sigma)] \supseteq G[i_n(\sigma) + 1] = \cdots = G[i_{n+1}(\sigma)], \forall n > N.$$  

Now let $H = G[i_n(\sigma)]$. Then by the contrapositive assumption, the field extension corresponding to $H$ is also of characteristic $p$. Since $\phi_H(i_{n+1}(\sigma)) = i_n(\sigma) + (1/p^2)(i_{n+1}(\sigma) - i_n(\sigma))$, again by Lemma 1.2 (1) we have $i_n(\sigma) + (1/p^2)(i_{n+1}(\sigma) - i_n(\sigma)) \geq p i_n(\sigma)$ and hence

$$i_{n+1}(\sigma) \geq (p^3 - p^2 + 1)i_n(\sigma) \geq (p^2 + 1)i_n(\sigma).$$

This is true for all $n > N$, and hence it contradicts to the assumption that $\text{Height}(\sigma) = 2$.

For the case (2), for every $j \in \mathbb{N}$, let $H = \langle \sigma^{p^{n+j}}, \tau^{p^{m+j}} \rangle$ and by considering the ramification subgroups of $H$, we have

$$\phi_H(i_{n+j+1}(\sigma)) = i_{n+j}(\sigma) + \frac{i_{m+j}(\tau) - i_{n+j}(\sigma)}{p} + \frac{i_{n+j+1}(\sigma) - i_{m+j}(\tau)}{p^2}.$$  

Again, by the contrapositive assumption, the field extension corresponding to $H$ is of characteristic $p$, and hence by Lemma 1.2 (1)

$$\phi_H(i_{n+j+1}(\sigma)) \geq p \phi_H(i_{n+j}(\sigma)) = p i_{n+j}(\sigma).$$
This contradicts to the assumption that \( \lim_{j \to \infty} i_{n+j+1}(\sigma) = p^2 \), for every 1 > \( \epsilon > 0 \), there exists \( j \) large enough such that \( p^2 - \epsilon < \lim_{j \to \infty} i_{n+j+1}(\sigma) < p^2 + \epsilon \). Similarly, \( p^2 - \epsilon < \lim_{j \to \infty} i_{n+j+1}(\tau)/i_{n+j}(\tau) < p^2 + \epsilon \). Hence, we can have either \( i_{n+j}(\tau) < (p+\epsilon)i_{n+j}(\sigma) \) or \( i_{n+j}(\sigma) < (p+\epsilon)i_{n+j}(\tau) \). Otherwise \( i_{n+j}(\tau) \geq (p+\epsilon)i_{n+j}(\sigma) \) and \( i_{n+j+1}(\sigma) \geq (p+\epsilon)i_{n+j}(\tau) \), imply \( i_{n+j+1}(\sigma) \geq (p+\epsilon)^2i_{n+j}(\sigma) > (p^2 + \epsilon)i_{n+j}(\sigma) \). Without lose of generality (switching \( \sigma \) and \( \tau \) if necessary), for every \( N \in \mathbb{N} \) and 1 > \( \epsilon > 0 \), we can find \( j > N \) such that \( i_{n+j}(\tau) < (p+\epsilon)i_{n+j}(\sigma) \) and hence by Equation (3.1), we get
\[
i_{n+j+1}(\sigma) \geq (p^3 - p^2 + p)i_{n+j}(\sigma) - (p - 1)(p+\epsilon)i_{n+j}(\sigma).
\]
Thus
\[
i_{n+j+1}(\sigma) \geq (p^3 - 2p^2 + (2-\epsilon)p + \epsilon)i_{n+j}(\sigma).
\]
This contradicts to the assumption that \( \lim_{j \to \infty} i_{n+j+1}(\sigma) = p^2 \), for \( p \geq 3 \).

For the case \( p = 2 \), considering the ramification subgroups
\[
H[i_{n+j}(\sigma)] \supseteq H[i_{m+j}(\tau)] \supseteq H[i_{n+j+1}(\sigma)] \supseteq H[i_{m+j+1}(\tau)] \supseteq H[i_{n+j+2}(\sigma)],
\]
we have
\[
\phi_H(i_{n+j+2}(\sigma)) = i_{n+j}(\sigma) + \frac{i_{m+j}(\tau) - i_{n+j}(\sigma)}{2} + \frac{i_{n+j+1}(\sigma) - i_{m+j}(\tau)}{4} + \frac{i_{m+j+1}(\tau) - i_{n+j+1}(\sigma)}{8} + \frac{i_{n+j+2}(\sigma) - i_{m+j+1}(\tau)}{16}.
\]
Again by the assumption that the corresponding field extension is of characteristic 2, we have \( \phi_H(i_{n+j+2}(\sigma)) \geq 2\phi_H(i_{n+j+1}(\sigma)) \) and deduce that
\[
i_{n+j+2}(\sigma) \geq 8i_{n+j}(\sigma) + 4i_{m+j}(\tau) + 6i_{n+j+1}(\sigma) - i_{m+j+1}(\tau).
\]
Again, without lose of generality, for every \( N \in \mathbb{N} \) and 1 > \( \epsilon > 0 \), we can assume there exists \( j > N \) such that \( i_{n+j+1}(\sigma) > (4-\epsilon)i_{n+j}(\sigma) \), \( i_{m+j+1}(\tau) < (4+\epsilon)i_{m+j}(\tau) \) and \( i_{m+j}(\tau) < (2+\epsilon)i_{n+j}(\sigma) \). Therefore, by using \( i_{m+j}(\tau) > i_{n+j}(\sigma) \), we get
\[
i_{n+j+2}(\sigma) > (28 - 12\epsilon - 2\epsilon^2)i_{n+j}(\sigma).
\]
This contradicts to the assumption that \( \lim_{j \to \infty} \frac{i_{n+j+2}(\sigma)}{i_{n+j}(\sigma)} = 2^4 \). We complete the proof of showing that if every nonidentity element of \( G \) is of height 2, then the \( \mathbb{Z}_p \times \mathbb{Z}_p \)-extension corresponding to \( G \) is of characteristic 0.

Conversely, suppose the field extension \( E/F \) corresponding to \( G \) is of characteristic 0 with \( \epsilon = v_F(p) \) being the absolute ramification index of \( F \). Then since there is no \( p \)-torsion element in \( G \), by Lemma [1.2(2) and the remark following it, there exists an \( N \) such that the raise to \( p \)-th power map \( G(u)/G(u+\epsilon) \to G(u+\epsilon)/G(u+\epsilon+\epsilon) \) is an isomorphism for all \( u > N \). In other words, there exists \( N \in \mathbb{N} \) such that for every upper ramification break \( u > N \), \([G(u) : G(u+\epsilon)]\) is either always 2 or always 1. For simplicity, we call the former depth 2 case and the latter depth 1 case.

For depth 2 case, it means that for every \( \sigma, \tau \in G \), there exists \( n, m \in \mathbb{N} \) such that \( i_{n+j}(\sigma) = i_{m+j}(\tau) \) for all \( j \in \mathbb{N} \). Therefore, for every \( \sigma \in G \), we choose another \( \tau \in G \) so that \( G[i_{n}(\sigma)] = G[u_1] = \langle \sigma^{p^n}, \tau^{p^n} \rangle \), \( G[i_{n+1}(\sigma)] = G[u_2] = \langle \sigma^{p^{n+1}}, \tau^{p^{n+1}} \rangle \), where \( u_1, u_2 \) are upper ramification breaks and \( G(u)^p = G(u+\epsilon) \) for all \( u \geq u_1 \). Let \( G' \) be the closed subgroup of \( G \) generated by \( \sigma \) and \( \tau \). It is clear that \( G' \) is of finite index over \( G \) and hence the field extension \( E/F' \) corresponding to \( G' \).
is also of characteristic 0. Let $e'$ be the absolute ramification index of $F'$. Since $G'[i] = G[i] \cap G'$, we get $\phi_{G'}(i_{n+1}(\sigma)) - \phi_{G'}(i_n(\sigma)) = (i_{n+1}(\sigma) - i_n(\sigma))/p^{m+2} = e'$. Inductively, we have

$$\frac{i_{n+j}(\sigma) - i_{n+j-1}(\sigma)}{p^{n+m+2j}} = e'.$$

This shows

$$\frac{i_{n+1}(\sigma) - i_n(\sigma)}{i_n(\sigma) - i_{n-1}(\sigma)} = p^2$$

for all $n$ large enough. Moreover, since

$$i_{n+j}(\sigma) = i_n(\sigma) + (i_{n+1}(\sigma) - i_n(\sigma)) + \cdots + (i_{n+j}(\sigma) - i_{n+j-1}(\sigma)),$$

we have

$$i_{n+j}(\sigma) = i_n(\sigma) + \frac{p^{2m+ne'} - 1}{p^2 - 1}(p^{2j} - 1)$$

and hence the limit $\lim_{n \to \infty} \frac{i_n(\sigma)}{p^{2n}}$ exists.

For depth 1 case, it means that for every $\sigma \in G$, there exists $\tau \in G$ and $n, m \in \mathbb{N}$ such that $i_{n+j}(\sigma) < i_{n+j}(\tau) < i_{n+j+1}(\sigma)$ for all $j \in \mathbb{N}$. Therefore, for every $\sigma \in G$, we choose another $\tau \in G$ satisfying this condition so that $G[i_{n+j}(\sigma)] = G(u_1) = \langle \sigma^{p^n}, \tau^{p^m} \rangle, G[i_{n}(\tau)] = G(u_2) = \langle \sigma^{p^{n+1}}, \tau^{p^m} \rangle$ and $G[i_{n+1}(\sigma)] = G(u_3) = \langle \sigma^{p^{n+1}}, \tau^{p^{n+1}} \rangle$, where $u_1, u_2, u_3$ are three consecutive upper ramification breaks and $G(u)^p = G(u + e)$ for all $u \geq u_1$. Again, let $G' = \langle \sigma, \tau \rangle$ and let $e'$ be the absolute ramification index of $F'$. We have

$$\phi_{G'}(i_{n+1}(\sigma)) - \phi_{G'}(i_n(\sigma)) = \frac{i_{m}(\tau) - i_n(\sigma)}{p^{n+m+1}} + \frac{i_{n+1}(\sigma) - i_{m}(\tau)}{p^{n+m+2}} = e'.$$

Similarly,

$$\frac{i_{n+1}(\sigma) - i_{m}(\tau)}{p^{n+m+2}} + \frac{i_{m+1}(\sigma) - i_{n+1}(\sigma)}{p^{n+m+3}} = e'.$$

Inductively, we have

$$\frac{i_{m}(\tau) - i_n(\sigma)}{p^{n+m+1}} = \frac{i_{n+j}(\tau) - i_{n+j}(\sigma)}{p^{n+m+1+2j}}, \forall j \in \mathbb{N}.$$  

Similarly, we can get

$$\frac{i_{n+1}(\sigma) - i_{m}(\tau)}{p^{n+m+2}} = \frac{i_{n+1+j}(\sigma) - i_{m+j}(\tau)}{p^{n+m+2+2j}}, \forall j \in \mathbb{N}.$$  

This shows

$$\frac{i_{n+1}(\sigma) - i_n(\sigma)}{i_n(\sigma) - i_{n-1}(\sigma)} = p^2$$

for all $n$ large enough and we also get the limit $\lim_{n \to \infty} \frac{i_n(\sigma)}{p^{2n}}$ exists.

We summarize this result as the following.

**Theorem 3.1.** Suppose that $G \subseteq \text{Aut}_k(k((x)))$ is a closed subgroup which is isomorphic to $\mathbb{Z}_p \times \mathbb{Z}_p$. Then the following are equivalent:

1. For every nonidentity $\sigma \in G$, Height$(\sigma) = 2$.
2. For every nonidentity $\sigma \in G$, the sequence $\{i_n(\sigma)/p^{2n}\}_n$ converges.
3. For every nonidentity $\sigma \in G$, $i_{n+1}(\sigma) - i_n(\sigma)/i_n(\sigma) - i_{n-1}(\sigma) = p^2$ for all $n$ sufficiently large.
4. The $\mathbb{Z}_p \times \mathbb{Z}_p$-extension corresponding to $G$ is of characteristic 0.
Remark 3.2. It is reasonable to extend Theorem 3.1 to the case that $G$ is a closed subgroup of $\text{Aut}_k(k((x)))$ which is isomorphic to a free $\mathbb{Z}_p$-module of rank $n > 2$. Our method seems not applicable to show that if every nonidentity element of $G$ has height $n$, then $G$ corresponds to an extension of characteristic 0. However, in [1], we use different approach to show that the corresponding statements (3) and (4) are equivalent.

4. Ramification Index

Suppose that $G$ is isomorphic to $\mathbb{Z}_p$ with generator $\sigma$ and is corresponding to a $\mathbb{Z}_p$-extension $E/F$ of characteristic 0. Then we can use the limit $\lim_{n \to \infty} i_n(\sigma)/p^n$ to determine the absolute ramification index $e$ of $F$. In fact, by (2.1) we have $e = \frac{p-1}{p} \lim_{n \to \infty} i_n(\sigma)/p^n$. For the case that $G$ is isomorphic to $\mathbb{Z}_p \times \mathbb{Z}_p$ with $G = \langle \sigma, \tau \rangle$ and is corresponding to a $\mathbb{Z}_p \times \mathbb{Z}_p$-extension of characteristic 0, both limits $\lim_{n \to \infty} i_n(\sigma)/p^{2n}$ and $\lim_{n \to \infty} i_n(\tau)/p^{2n}$ exist, so it is interesting to know whether it is possible to determine the absolute ramification index by purely using the limits $\lim_{n \to \infty} i_n(\sigma)/p^{2n}$ and $\lim_{n \to \infty} i_n(\tau)/p^{2n}$.

For the case of depth 2, if $G(N)$ is generated by $\sigma^{p^n}, \tau^{p^n}$ and $G(u) = G(u + e)$ for $u \geq N$, then as indicated above

$$i_{n+j}(\sigma) = i_n(\sigma) + \frac{p^{2+n}e}{p^2 - 1} (p^{2j} - 1), \forall j \in \mathbb{N},$$

and hence

$$\lim_{j \to \infty} \frac{i_j(\sigma)}{p^{2j}} = \frac{p^2}{p^2 - 1} p^{m-n} e.$$

Similarly,

$$\lim_{j \to \infty} \frac{i_j(\tau)}{p^{2j}} = \frac{p^2}{p^2 - 1} p^{n-m} e.$$

Therefore, we have

$$e = \frac{p^2}{p^2 - 1} \left( \lim_{j \to \infty} \frac{i_j(\sigma)}{p^{2j}} \right) \left( \lim_{j \to \infty} \frac{i_j(\tau)}{p^{2j}} \right).$$

Notice that in this case, since $i_{n+j}(\sigma) = i_{m+j}(\tau)$ for all $j \in \mathbb{N}$, if we set

$$\lim_{j \to \infty} \frac{i_j(\sigma)}{p^{2j}} = \gamma_1, \lim_{j \to \infty} \frac{i_j(\tau)}{p^{2j}} = \gamma_2,$$

then $\gamma_1/\gamma_2 = p^{2(m-n)}$. In other words, $\log_p \gamma_1 - \log_p \gamma_2$ must be an even number.

Example 4.1. For an odd prime $p$, let $F = \mathbb{Q}_p(\zeta)$ be the unramified extension of degree 2 over $\mathbb{Q}_p$ with $\zeta$ being a unit in $\mathcal{O}_F$. Consider the Lubin-Tate formal group over $\mathcal{O}_F$ constructed by $[p](x) = px + x^{p^2}$. For $a \in \mathcal{O}_p$, let $[a](x) \in \mathcal{O}_F[[x]]$ be the automorphism of the Lubin-Tate formal group with leading coefficient $a$ and we denote its reduction by $\sigma_a \in \mathbb{F}_p[[x]]$. For $a \in \mathcal{O}_p$ with $v_F(a - 1) = r$, it is well-known that $i_n(\sigma_a) = p^{2(r+n)} - 1$ and hence we have $\lim_{n \to \infty} \frac{i_n(\sigma_a)}{p^{2n}} = p^{2r}$. For the case that $a = 1 + p$ and $\beta = 1 + \zeta p$, we know that the closed subgroup generated by $\sigma_a, \sigma_\beta$ corresponds to an extension $N/M$ where $M$ is the extension of $F$ generated
by the $p$-torsion elements, i.e., elements that satisfy $|p|(x) = 0$. Therefore, we have the ramification index of $M$ over $\mathbb{Q}_p$ is $p^2 - 1$ which is equal to

$$\frac{p^2 - 1}{p^2} \sqrt{\lim_{j \to \infty} \frac{i_j(\sigma_\alpha)}{p^{2j}}} \sqrt{\lim_{j \to \infty} \frac{i_j(\sigma_\beta)}{p^{2j}}}.$$  

Similarly, for the extension $N/M'$ corresponding the closed subgroup $G'$ generated by $\alpha, \beta^p$, we have the ramification index of $M'$ over $\mathbb{Q}_p$ is $(p^2 - 1)p$. Notice that $G'$ is also generated by $\sigma_{\alpha}, \sigma_{\beta'}$ where $\beta' = \alpha\beta^p$, but the ramification index of $M'$ over $\mathbb{Q}_p$ is not equal to

$$\frac{p^2 - 1}{p^2} \sqrt{\lim_{j \to \infty} \frac{i_j(\sigma_\alpha)}{p^{2j}}} \sqrt{\lim_{j \to \infty} \frac{i_j(\sigma_{\beta'})}{p^{2j}}} = p^2 - 1.$$  

This is because the ramification subgroup of $G(u)$ is not of the form $\langle \sigma^{p^n}, \tau^{p^m} \rangle$ when $u$ is large enough.

For the case of depth 1, if $G(u)$ is generated by $\sigma^{p^n}, \tau^{p^m}$ and $G(u')^p = G(u' + e)$ for $u' \geq u$, then as indicated above

$$\frac{i_{m+j}(\tau) - i_{n+j}(\sigma)}{p^{n+m+1+2j}} + \frac{i_{n+1+j}(\sigma) - i_{m+j}(\tau)}{p^{n+m+2+2j}} = e, \forall j \in \mathbb{N}.$$  

If we set

$$\lim_{j \to \infty} \frac{i_j(\sigma)}{p^{2j}} = \gamma_1, \quad \lim_{j \to \infty} \frac{i_j(\tau)}{p^{2j}} = \gamma_2,$$

then

$$e = \frac{p - 1}{p^{m-n+1}} \gamma_1 + \frac{p - 1}{p^{m-n+2}} \gamma_2.$$  

Moreover, without lose of generality we assume that $i_{n+j}(\sigma) < i_{m+j}(\tau) < i_{n+j+1}(\sigma)$ for all $j \in \mathbb{N}$. Diving by $p^{2(n+j)}$ and taking limits, we get

$$\gamma_1 \leq p^{2(m-n)} \gamma_2 \leq p^2 \gamma_1.$$  

However, $(i_{m+j}(\tau) - i_{n+j}(\sigma))/p^{n+m+1+2j}$ is a nonzero constant $c$ for all $j \in \mathbb{N}$ (by (3.3)). Taking limits, we get

$$p^{2(m-n)} \gamma_2 - \gamma_1 = p^{m-n+1} c \neq 0.$$  

Similarly, $p^{2(m-n)} \gamma_2 \neq p^2 \gamma_1$. In other words, $\log_p \gamma_1 - \log_p \gamma_2$ cannot be an even number and $m - n$ is the unique integer between $(1/2)(\log_p \gamma_1 - \log_p \gamma_2)$ and $1 + (1/2)(\log_p \gamma_1 - \log_p \gamma_2)$.

**Example 4.2.** For an odd prime $p$, let $F = \mathbb{Q}_p(\pi)$ be the totally ramified extension of degree 2 over $\mathbb{Q}_p$ with $\pi$ being a prime element in $\mathcal{O}_F$. Consider the Lubin-Tate formal group over $\mathcal{O}_F$ constructed by $[\pi](x) = \pi x + x^p$. For $\alpha \in \mathcal{O}_F^*$, let $[\alpha](x) \in \mathcal{O}_F[[x]]$ be the automorphism of the Lubin-Tate formal group with leading coefficient $\alpha$ and we denote its reduction by $\sigma_\alpha \in \mathbb{F}_p[[x]]$. For $\alpha \in \mathcal{O}_F'$ with $v_F(\alpha - 1) = r$, it is well-known that $i_n(\sigma_\alpha) = p^{(r+2n)} - 1$ and hence we have $\lim_{j \to \infty} \frac{i_j(\sigma_\alpha)}{p^{2j}} = p^r$. For the case that $\alpha = 1 + \pi$ and $\beta = 1 + \pi^2$, consider $G$ being the closed subgroup generated by $\sigma_\alpha, \sigma_\beta$. We have

$$\gamma_1 = \lim_{j \to \infty} \frac{i_j(\sigma_\alpha)}{p^{2j}} = p, \quad \gamma_2 = \lim_{j \to \infty} \frac{i_j(\sigma_\beta)}{p^{2j}} = p^2.$$
Moreover, $p - 1$ is the first upper ramification break of $G$ where $G[p - 1] = G(p - 1)$ is generated by $\sigma_\alpha, \sigma_\beta$ and $G(u^p) = G(u + e)$ for $u \geq p - 1$. Notice that $m - n = 1 - 1 = 0$ is the only integer between $(1/2)(\log_p \gamma_1 - \log_p \gamma_2) = -1/2$ and $1 + (-1/2)$. On the other hand, $G$ corresponds to an extension $N/M$ where $M$ is the extension of $F$ generated by the $\pi$-torsion elements, i.e. elements that satisfy $[\pi](x) = 0$. Therefore, we have the ramification index of $M$ over $F$ is $p - 1$ and hence the ramification index of $M$ over $\mathbb{Q}_p$ is $2(p - 1)$ which is equal to

$$\frac{p - 1}{p^{1-1+2p^2}} + \frac{p - 1}{p^{1-1+2p^2}}.$$

Similarly, for the extension $N/M'$ corresponding the closed subgroup $G'$ generated by $\alpha, \beta^p$, we have the ramification index of $M'$ over $\mathbb{Q}_p$ is $2(p - 1)p$.

We summarize our result as the following.

**Theorem 4.4.** Let $G$ be a closed subgroup of $\text{Aut}_k(k((x)))$ which is isomorphic to $\mathbb{Z}_p \times \mathbb{Z}_p$. Suppose $G$ corresponds to an extension of characteristic 0. Suppose further that $G = \langle \sigma, \tau \rangle$, $G(u) = \langle \sigma^p, \tau^p \rangle$ and $G(u^p) = G(u + e)$ for all $u' \geq u$. Let

$$\gamma_1 = \lim_{j \to \infty} \frac{i_j(\sigma)}{p^{2j}}, \gamma_2 = \lim_{j \to \infty} \frac{i_j(\tau)}{p^{2j}}.$$

(1) If $\log_p(\gamma_1/\gamma_2)$ is an even number, then $G$ is of depth 2 and

$$e = \frac{p - 1}{p^2} \sqrt{\gamma_1 \gamma_2}.$$

(2) If $\log_p(\gamma_1/\gamma_2)$ is not an even number, then $G$ is of depth 1. Furthermore, let $a$ be the unique integer between $(1/2)\log_p(\gamma_1/\gamma_2)$ and $1 + (1/2)\log_p(\gamma_1/\gamma_2)$. Then

$$e = \frac{p - 1}{p^{a+1}} \gamma_1 + \frac{p - 1}{p^{2-a}} \gamma_2.$$

**Remark 4.4.** In the case of depth 2, let $a$ be the integer $(1/2)\log_p(\gamma_1/\gamma_2)$. Then we have

$$\frac{p - 1}{p^2} \sqrt{\gamma_1 \gamma_2} = \frac{p - 1}{p^{a+1}} \gamma_1 + \frac{p - 1}{p^{2-a}} \gamma_2.$$

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