A CANONICAL WAY TO DEFORM
A LAGRANGIAN SUBMANIFOLD

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May 22, 1996

Abstract. We derive some important geometric identities for Lagrangian submanifolds immersed in a Kähler manifold and prove that there exists a canonical way to deform a Lagrangian submanifold by a parabolic flow through a family of Lagrangian submanifolds if the ambient space is a Ricci-flat Calabi-Yau manifold.

0. Introduction

Let \( L^n \) be a smooth submanifold, immersed into a Ricci-flat Calabi-Yau manifold \( M^{2n} \) with complex structure \( J \) and metric \( \mathcal{g} \). Let \( \omega = \mathcal{g}(J\cdot,\cdot) \) be the Kählerform on \( M \) and let \( g, \omega \) denote the pullbacks on \( L \). If \( \omega = 0 \), then \( L \) is called Lagrangian. It is an interesting question, if there exists a canonical way to deform an initial Lagrangian submanifold \( L_0 \) through a family of submanifolds \( L_t \) such that each \( L_t \) is also Lagrangian, i.e. \( \frac{d}{dt}\omega = 0 \) and such that the corresponding flow is parabolic. Given a 1-form on a Lagrangian submanifold, one can first use the metric \( g \) to identify this 1-form with a vectorfield and then apply the complex structure which by the Lagrangian condition maps this tangent vector field to a normal vector field. Assuming that we deform the Lagrangian submanifold in this direction one obtains the necessary condition that this 1-form has to be closed in order to maintain the Lagrangian structure. This has been proven in [6] but will also become clear in this paper. However, it is not obvious if this is also sufficient. So if one wants to find such a canonical deformation the first thing one has to make sure is that there exists a canonical, closed 1-form on a Lagrangian submanifold. Surprisingly this is true if the ambient space is Ricci-flat. The corresponding 1-form can be defined in terms of the second fundamental form and we denote this 1-form by “mean curvature form” (see definition below) since the resulting deformation vector field is given by the mean curvature vector which can be defined for arbitrary smooth submanifolds. The organization of the paper is as follows:

In paragraph 1 we state important geometric equalities some of which are analogous to the Gauss-Weingarten, Codazzi equations, etc. and define the mean curvature form. Then we prove that the corresponding flow exists at least on a short time interval and has a unique solution on the moduli space of smooth submanifolds. Finally we derive an evolution inequality for the squared norm of \( \omega \) which
basically follows from the Ricci-flatness of the ambient space and with a parabolic maximum principle we can prove that $\omega$ has to vanish identically if it vanishes at $t = 0$. Thus, if one wants to study the geometry of Lagrangian submanifolds immersed into a Ricci-flat Calabi-Yau manifold this flow provides a tool and there should be no doubt that this is a powerful tool since the corresponding mean curvature flow for hypersurfaces in Riemannian manifolds has already given many results (e.g. [5]).

In paragraph 2 we restate some equations derived in paragraph 1 which in the context of a Lagrangian submanifold become very beautiful and eventually derive the evolution equation for the second fundamental form and the mean curvature form. We also state the evolution equations in the case where $L$ is being deformed by an arbitrary 1-form (formally). As a direct consequence of these evolution equations we finally prove a useful theorem and the paper ends with some questions and remarks.

1. Notations and preliminary results

Let us define the following map:

$$N : T_pL^n \rightarrow (T_pL^n)^\perp, \quad N(v) := (J(v))^\top,$$

where $(T_pL^n)^\perp$ is the normal space of $L$ at $p$ and $^\top$ denotes the projection onto $T_pL^n$. In the forthcoming let us assume that $L$ is a compact submanifold such that $N$ is always an isomorphism between $T_pL$ and $(T_pL)^\perp$. Then we define the following tensor on $L$:

$$h(u, v, w) := -\overline{g}(N(u), \nabla_v w) = \overline{g}(\nabla_v N(u), w)$$

where $\overline{\nabla}$ denotes the covariant derivative on $M$. For a fixed vector $u$ this is the second fundamental form with respect to the normal vector $N(u)$. Assume that $F : L^n \rightarrow M^{2n}$ is an immersion, that $x^i, y^\alpha, i = 1 \ldots n, \alpha = 1 \ldots 2n$ are coordinates for $L^n, M^{2n}$ respectively and set $e_i := \frac{\partial F^\alpha}{\partial x^i} \frac{\partial}{\partial y^\alpha}$, where double indices are always summed from 1 to $n$ or to $2n$ respectively. Furtheron we will always write $\langle u, v \rangle$ instead of $\overline{g}(u, v)$. Then

$$g_{ij} = \overline{g}_{\alpha\beta} \frac{\partial F^\alpha}{\partial x^i} \frac{\partial F^\beta}{\partial x^j}$$

$$N(e_i) = J(e_i) - \omega^l e_l$$

$$\overline{\omega}_{\alpha\beta} = \overline{g}_{\beta\gamma} J^\gamma_{\alpha}$$

$$\overline{\omega}_{\alpha\beta} = -\overline{\omega}_{\beta\alpha}$$

$$\nabla J = 0$$

Since $h(u, \cdot, \cdot)$ is the second fundamental form with respect to $N(u)$, we clearly have

$$h(e_k, e_i, e_j) = h_{kij} = h_{kji}$$

Now we ask: Under what condition is $h_{ijk}$ also symmetric in the first two indices? The answer is
Proposition 1.1. \( h_{kij} = h_{ikj} + \nabla_j \omega_{ik} \)

Proof.

\[
h_{kij} = h_{kji} = -\langle N(e_k), \nabla_{e_j} e_i \rangle = -\langle J(e_k), \nabla_{e_j} e_i \rangle + \omega^l_k \langle e_l, \nabla_{e_j} e_i \rangle \\
= \langle e_k, \nabla_{e_j} J(e_i) \rangle + \omega^l_k \langle e_l, \nabla_{e_j} e_i \rangle \\
= \omega_{ikj} - \langle N(e_i), \nabla_{e_j} e_k \rangle - \omega^l_i \langle e_l, \nabla_{e_j} e_k \rangle + \omega^l_k \langle e_l, \nabla_{e_j} e_i \rangle \\
= h_{ijk} + \omega_{ikj} - \omega^l_i \langle e_l, (\nabla_{e_j} e_k)^T \rangle - \omega^l_k \langle e_l, (\nabla_{e_j} e_i)^T \rangle \\
= h_{ijk} + \nabla_j \omega_{ik}
\]

where we have used that

\[
\langle J(u), J(v) \rangle = \langle u, v \rangle \\
\nabla J = 0 \\
J^2 = -id
\]

We will need the following identity

Proposition 1.2.

\[
\nabla_k \nabla_j \omega_{li} - \nabla_l \nabla_j \omega_{ki} = \nabla_j \nabla_i \omega_{lk} + R^s_{ij} \omega_{ks} + R^s_{ijk} \omega_{ls} - R^s_{jkl} \omega_{si}
\]

Proof. This follows from the rule for interchanging derivatives and the fact that \( \omega \) is closed:

\[
\nabla_k \nabla_j \omega_{li} = \nabla_j \nabla_k \omega_{li} + R^s_{ijk} \omega_{ls} + R^s_{ij} \omega_{ks} \\
= -\nabla_j \nabla_k \omega_{li} - \nabla_j \nabla_i \omega_{kl} + R^s_{ij} \omega_{ks} + R^s_{lijk} \omega_{si} \\
= \nabla_l \nabla_j \omega_{ki} + R^s_{ilj} \omega_{ks} - R^s_{jk} \omega_{si} - \nabla_j \nabla_i \omega_{kl} + R^s_{ijk} \omega_{ls}
\]

where we have used the Bianchi identity in the last step.

Remark. If \( v_\alpha, \alpha = 1 \ldots 2n \) is a basis for \( T_p M^{2n} \) and \( a^{\alpha \beta} \) is the inverse of \( a_{\alpha \beta} := \langle v_\alpha, v_\beta \rangle \), then any vector \( w \in T_p M^{2n} \) can be written in the form
\[
w = a^{\alpha \beta} \langle w, v_\alpha \rangle v_\beta.
\]

Now define the following tensor

\[
\eta_{ij} := \langle N(e_i), N(e_j) \rangle = g_{ij} + \omega^l_i \omega_{lj}
\]

This is a positive definite, symmetric tensor as long as \( N \) is an isomorphism. Let us also denote the inverse of \( \eta_{ij} \) by \( \eta^{ij} \). In the forthcoming we will always assume that \( e_i, i = 1 \ldots n \) are not only defined on \( L \) but that we have chosen an extension in a tubular neighborhood of \( L \). All the stated equations like \( \nabla_{e_i} e_j = \ldots \), etc. will mean that this is true if we restrict these terms to the submanifold \( L \). In particular it will always be true that these results are independent of the chosen extension for \( e_i, i = 1 \ldots n \).
Proposition 1.3.

(a) \[ \nabla_{e_k} e_j = \Gamma^n_{kj} e_n - \eta^{mn} h_{mkj} N(e_n) \]

(b) \[ \nabla_l h_{ikj} - \nabla_k h_{ilj} = R^{\rho}_{ijkl} + \eta^{mn} \omega^s_n (h_{mlj} h_{ski} - h_{mkj} h_{sli}) + \eta^{mn} \omega^s_i (h_{mkj} h_{nl} - h_{mlj} h_{nk}) \]

(c) \[ R_{ijkl} = \widetilde{R}_{ijkl} + \eta^{mn} (h_{mkj} h_{nl} - h_{mlj} h_{nk}) \]

where an underlined index means that one has to take the image of this vector under \( N \), e.g. \( \widetilde{R}_{ijkl} = \langle \widetilde{R}(e_k, e_l) e_j, N(e_i) \rangle \)

Proof.

(a): The above remark with \( v_i = e_i, \ v_{i+n} = N(e_i), \ i = 1 \ldots n \) gives

\[ \nabla_{e_k} e_j = g^{\rho \nu} (\nabla_{e_k} e_j, e_\rho) e_\nu + \eta^{mn} (\nabla_{e_k} e_j, N(e_m)) N(e_n) \]
\[ = g^{\rho \nu} (\nabla_{e_k} e_j, e_\rho) e_\nu + \eta^{mn} h_{mkj} N(e_n) \]
\[ = \Gamma^n_{kj} e_n - \eta^{mn} h_{mkj} N(e_n) \]

(b): Let us choose normal coordinates for \( L \) at a fixed point \( p \), then

\[ \Gamma^n_{ij}, \ \nabla_{e_i} e_j, \ [e_i, e_j] = 0, \ \nabla_l h_{ikj} = h_{ikj,l} \]

and

\[ \nabla_l h_{ikj} - \nabla_k h_{ilj} = \langle \nabla_{e_k} N(e_i), \nabla_{e_l} e_j \rangle - \langle \nabla_{e_l} N(e_i), \nabla_{e_k} e_j \rangle + \langle N(e_i), \nabla_{e_k} \nabla_{e_l} e_j - \nabla_{e_l} \nabla_{e_k} e_j \rangle \]
\[ = \langle \nabla_{e_k} N(e_i), \nabla_{e_l} e_j \rangle - \langle \nabla_{e_l} N(e_i), \nabla_{e_k} e_j \rangle + \langle N(e_i), \widetilde{R}(e_k, e_l) e_j \rangle \]

With (a) we get

\[ \nabla_l h_{ikj} - \nabla_k h_{ilj} = \widetilde{R}_{ijkl} - \eta^{mn} (h_{mlj} (\nabla_{e_k} N(e_i), N(e_n)) - h_{mkj} (\nabla_{e_l} N(e_i), N(e_n))) \]

Now we compute

\[ \langle \nabla_{e_k} N(e_i), N(e_n) \rangle = \langle \nabla_{e_k} (J(e_i) - \omega^s_i e_s), N(e_n) \rangle \]
\[ = \langle \nabla_{e_k} J(e_i), N(e_n) \rangle - \omega^s_i \langle \nabla_{e_k} e_s, N(e_n) \rangle \]

since \( \langle e_s, N(e_n) \rangle = 0 \). Then

\[ \langle \nabla_{e_k} N(e_i), N(e_n) \rangle = -\langle \nabla_{e_k} e_i, J(N(e_n)) \rangle + \omega^s_i h_{nks} \]
\[ = \langle \nabla_{e_k} e_i, e_n + \omega^n_s \omega^p_s e_p + \omega^n_s N(e_s) \rangle + \omega^s_i h_{nks} \]
\[ = -\omega^n_s h_{ski} + \omega^s_i h_{nks} \]

Inserting this equality in the above identity gives the desired result.

(c): This is only a reformulation of the Gauss curvature equations.
Lemma 1.4.
\[ \nabla_l h_{kij} - \nabla_k h_{lij} = R_{ijkl} + \omega_l^s R_{sijk} + \eta_{mn} \omega_n^s (h_{mlj} h_{ski} - h_{mkj} h_{sli}) \]

Proof. Using Proposition 1.1 we obtain
\[ \nabla_l h_{kij} - \nabla_k h_{lij} = \nabla_l (h_{ikj} + \nabla_j \omega_{ik}) - \nabla_k (h_{ilj} + \nabla_j \omega_{il}) \]
\[ = \nabla_l h_{ikj} - \nabla_k h_{ilj} + \nabla_k \nabla_j \omega_{li} - \nabla_l \nabla_j \omega_{ki} \]
and with Proposition 1.2 and 1.3 (b)
\[ \nabla_l h_{kij} - \nabla_k h_{lij} = R_{ijkl} + \eta_{mn} \omega_n^s (h_{mlj} h_{ski} - h_{mkj} h_{sli}) \]
\[ + \eta_{mn} \omega_i^s (h_{mkj} h_{nls} - h_{mlj} h_{nks}) + \nabla_j \nabla_i \omega_{lk} + R_{sijl} \omega_{ki} - R_{sijkl} \omega_{si} \]
and by the Gauss curvature equations we have
\[ \eta_{mn} \omega_i^s (h_{mkj} h_{nls} - h_{mlj} h_{nks}) - R_{sijkl} \omega_{si} = \omega_i^s R_{sijkl} \]
which proves the lemma.

Definition 1.5.
\[ H := H_i dx^i := g_{kl} h_{ikl} dx^i \]
is called the mean curvature form on \( L \).

Remark. Using \( \eta^i_j \) to raise indices we can identify this 1-form with the vector field \( \eta^m \omega_n^s (h_{mlj} h_{ski} - h_{mkj} h_{sli}) \) and \( N \) maps this vector field to the outward pointing mean curvature vector field on \( L \) (the notion “outward pointing” does not really make sense in arbitrary codimensions but we will denote this vector by outward pointing since the construction is analogous to the construction of the outward pointing mean curvature vector in codimension 1). Note that if \( L \) is Lagrangian then \( \eta_{ij} = g_{ij} \).

We are now going to prove that the following evolution exists at least on some short time interval and is unique if \( L \) is compact. Therefore assume that \( L \) is a compact manifold smoothly immersed into a Ricci-flat Calabi-Yau manifold by an immersion \( F \) and assume that \( \eta_{ij} \) is positive definite everywhere on \( L \). We want to find a smooth family of immersions \( F_t \) such that
\[ \frac{d}{dt} F_t = -\eta^m \omega_n^s (h_{mlj} h_{ski} - h_{mkj} h_{sli}) \]

Proposition 1.6. \((*)\) has a unique solution on a short time interval.

Proof. The proof is formally the same as for the mean curvature flow in codimension 1. We will use Hamilton’s existence theorem for evolution equations with an integrability condition [2]. Let \( \pi_N \) be the projection on the normal bundle and \( \pi_L \) be the projection on the tangent bundle of \( L \). Here we have
\[ \frac{d}{dt} F = E(F) = -\eta^m \omega_n^s (h_{mlj} h_{ski} - h_{mkj} h_{sli}) \]
On the symbol level we get
\[ \tilde{g}_{ij} = g_{ij} \frac{\partial F^\alpha}{\partial x^i} \tilde{F}^\beta \xi_j + \frac{\partial F^\beta}{\partial x^j} \tilde{F}^\alpha \xi_i \]
\[ \tilde{\Gamma}^i_l = \frac{1}{2} \bar{g}^{kl} (\bar{g}_{ik} \xi_j + \bar{g}_{jk} \xi_i - \bar{g}_{ij} \xi_k) \]
which combine to

\[ \tilde{\Gamma}_{ij}^l = g^{kl} \tilde{g}_{\beta\gamma} \frac{\partial F^\beta}{\partial x^k} \tilde{F}^\gamma \xi^j \]

If we multiply 1.3 (a) with \( g^{kj} \) we see that the symbol is given by

\[
\sigma DE(F) \tilde{F}^\alpha(\xi) = g^{ij}(\xi^i \xi^j \tilde{F}^\alpha - g^{kl} \tilde{g}_{\beta\gamma} \frac{\partial F^\beta}{\partial x^k} \tilde{F}^\gamma \xi^i \xi^j \frac{\partial F^\alpha}{\partial x^l})
\]

\[
= |\xi|_g^2 (\tilde{F}^\alpha - \pi_L(\tilde{F})^\alpha) = |\xi|_g^2 \pi_N(\tilde{F})^\alpha
\]

Therefore we are done if we use \( \pi_L(\tilde{F}) = 0 \) as our integrability condition.

We will need the evolution equations for \( g_{ij} \) and \( \omega_{ij} \).

**Lemma 1.7.**

(a) \[ \frac{d}{dt} g_{ij} = -2\eta^{kl} H_k h_{lij} \]

(b) \[ \frac{d}{dt} d\mu = -\eta^{mn} H_m H_n d\mu \]

(c) \[ \frac{d}{dt} \omega = dH \]

where \( d\mu \) is the volume form on \( L_t \).

**Proof.** From now on let us always assume that we have chosen normal coordinates at a fixed point \( p \in L \) with respect to the metric \( g_{ij} \) such that in addition \( e_i, i = 1, \ldots, n \) are eigenvectors for \( \eta_{ij} \), i.e. \( \eta_{ij} \) becomes diagonal in this coordinate frame. We will also assume that we have chosen normal coordinates at \( p \) for the ambient manifold \( M \). Under this assumptions it is easy to see that

\[
\frac{d}{dt} g_{ij} = \frac{d}{dt} \left( g_{ij} \frac{\partial F^\alpha}{\partial x^i} \frac{\partial F^\beta}{\partial x^j} \right) = \frac{g_{ij}}{\partial x^i} \left( \frac{dF^\alpha}{dt} \right) \frac{\partial F^\beta}{\partial x^j} + \frac{1}{\partial x^j} \left( \frac{dF^\beta}{dt} \right) \frac{\partial F^\alpha}{\partial x^i}
\]

\[
= -\eta^{mn} H_m (\nabla e_i N(e_n), e_j) - \eta^{mn} H_m (\nabla e_j N(e_n), e_i)
\]

\[
= -2\eta^{mn} H_m h_{lij}
\]

which gives (a) and (b). For \( \omega \) we can proceed similarly

\[
\frac{d}{dt} \omega_{ij} = \frac{d}{dt} \left( \overline{\omega}_{ij} \frac{\partial F^\alpha}{\partial x^i} \frac{\partial F^\beta}{\partial x^j} \right)
\]

\[
= \overline{\omega}_{ij} \frac{dF^\gamma}{dt} \frac{\partial F^\alpha}{\partial x^i} \frac{\partial F^\beta}{\partial x^j} + \overline{\omega}_{ij} \frac{\partial}{\partial x^i} \left( \frac{dF^\alpha}{dt} \right) \frac{\partial F^\beta}{\partial x^j} + \overline{\omega}_{ij} \frac{\partial}{\partial x^j} \left( \frac{dF^\beta}{dt} \right) \frac{\partial F^\alpha}{\partial x^i}
\]

Then we use the closedness of \( \overline{\omega} \) to obtain

\[
\frac{d}{dt} \omega_{ij} = -\overline{\omega}_{ij} \frac{dF^\gamma}{dt} \frac{\partial F^\alpha}{\partial x^i} \frac{\partial F^\beta}{\partial x^j} - \overline{\omega}_{ij} \frac{\partial}{\partial x^i} \left( \frac{dF^\alpha}{dt} \right) \frac{\partial F^\beta}{\partial x^j} + \overline{\omega}_{ij} \frac{\partial}{\partial x^j} \left( \frac{dF^\beta}{dt} \right) \frac{\partial F^\alpha}{\partial x^i}
\]

\[
= \frac{\partial}{\partial x^i} \left( \overline{\omega}_{ij} \frac{dF^\gamma}{dt} \frac{\partial F^\beta}{\partial x^j} \right) - \overline{\omega}_{ij} \frac{\partial}{\partial x^j} \left( \frac{dF^\beta}{dt} \right) \frac{\partial F^\alpha}{\partial x^i}
\]

\[
= \frac{\partial}{\partial x^i} \left( \overline{\omega}_{ij} \frac{dF^\gamma}{dt} \frac{\partial F^\beta}{\partial x^j} \right) - \overline{\omega}_{ij} \frac{\partial}{\partial x^j} \left( \frac{dF^\beta}{dt} \right) \frac{\partial F^\alpha}{\partial x^i}
\]

\[
= \frac{\partial}{\partial x^i} \left( \overline{\omega}_{ij} \frac{dF^\gamma}{dt} \frac{\partial F^\beta}{\partial x^j} \right) - \overline{\omega}_{ij} \frac{\partial}{\partial x^j} \left( \frac{dF^\beta}{dt} \right) \frac{\partial F^\alpha}{\partial x^i}
\]
by the antisymmetry of $\mathcal{F}$. Finally

\[ \mathcal{F}\left(\frac{dF}{dt}, \frac{\partial F}{\partial x^i}\right) = -\eta^{kl} H_k \mathcal{F}(N(e_t), e_i) = -\eta^{kl} H_k \mathcal{F}(N(e_l), e_i) = \eta^{kl} H_k (e_l + \omega_i \omega_{mi}) = \eta^{kl} H_k \eta_{li} = H_i \]

and therefore

\[ \frac{d}{dt} \omega_{ij} = H_{j,i} - H_{i,j} \]

which proves (c).

**Proposition 1.8.** For each compact time interval $[0, T]$ on which a smooth solution of (*') exists and the flow is well defined, i.e. where $N$ is an isomorphism we can find a positive constant such that

\[ \frac{d}{dt} |\omega|^2 \leq \Delta |\omega|^2 + c|\omega|^2 \]

**Proof.** First we will need the evolution equation for $|\omega|^2 := g^{ik} g^{jl} \omega_{ij} \omega_{kl}$

\[ \frac{d}{dt} |\omega|^2 = -2g^{in} g^{km} g^{jl} \omega_{ij} \omega_{kl} \frac{d}{dt} g_{mn} + 2g^{ik} g^{jl} \omega_{ij} \frac{d}{dt} \omega_{kl} \]

Using Lemma 1.4 and 1.7 we obtain

\[ \frac{d}{dt} |\omega|^2 = 4g^{im} g^{kn} g^{jl} \omega_{ij} \omega_{kl} \eta^{st} H_s h_{tmn} + 2g^{ik} g^{jl} \omega_{ij} (\nabla_k H_l - \nabla_l H_k) \]

\[ = 4g^{im} \omega^{kn} \omega^{jl} \eta^{st} H_s h_{tmn} + 2g^{ik} g^{jl} \omega_{ij} g^{pq} \left( R_{g_{lk}} + \nabla_q \nabla_p \omega_{kl} + \omega_p R_{g_{lk}} \right) + \omega^{sl} R_{g_{pkq}} + \omega^{lk} R_{g_{pk}} + \eta^{mn} \omega^{s} (h_{mk} h_{slp} - h_{ml} h_{skp}) \]

\[ = 4\omega^{sl} \omega^{mk} \eta^{st} H_s h_{tmn} + 2\omega^{lk} R_{g_{pk}} + \Delta |\omega|^2 - 2|\nabla_i \omega_{kl}|^2 \]

\[ + 2\omega^{lk} \omega^{pm} R_{g_{pk}} + 4\omega^{lk} \omega^{st} \omega_{kl} \omega_{st} + \omega^{lk} \eta^{mn} \omega^{s} (h_{mk} h_{slp} - h_{ml} h_{skp}) \]

Now it can be easily checked that each absolute term which depends quadratically on $\omega$ can be bounded above by a term $c|\omega|^2$ since $L \times [0, T]$ is compact, e.g. in normal coordinates we see that this is true for a term of the form $2\omega^{sl} \omega^{mk} \omega_{sm}$:

\[ 2\omega^{sl} \omega^{mk} \omega_{sm} = 2 \sum_{s, l} (\omega_{sl} \sum_m \omega_{ml} a_{sm}) \leq |\omega|^2 + \sum_{s, l} (\sum_m \omega_{ml} a_{sm})^2 \]

\[ \leq |\omega|^2 + n \sum_{s, l} (\sum_m \omega_{ml})^2 (a_{sm})^2 \]

\[ \leq (1 + n|a|^2)|\omega|^2 \]

These constants depend on the dimension $n$, on $\eta_{ij}$, $h_{ijk}$, $T$ and the Riemann curvatures. This gives

\[ \frac{d}{dt} |\omega|^2 \leq 4\Delta |\omega|^2 + 2c|\omega|^2 + 2\omega^{lk} R_{g_{pk}} \]

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The only thing which we have to prove is that the last term in the above inequality can be bounded by a term of the form $c|\omega|^2$. This will follow from the Ricci flatness of the ambient space. Let us assume that $v_i$ is an orthonormal basis of eigenvectors for $\eta_{ij}$ and that, at a fixed point $p \in L$ we have chosen normal coordinates for $L$ such that $e_i = v_i$ and that we have also chosen normal coordinates for $M$ such that the corresponding coordinate frame is given by $e_i, N(e_i)$. Then the metric $\overline{g}^{\alpha\beta}$ at this point is given by

$$\overline{g}^{\alpha\beta} = \begin{pmatrix} g^{ij} & 0 \\ 0 & \eta^{ij} \end{pmatrix} = \begin{pmatrix} 1 & \cdots & \frac{1}{1-a_1} & \cdots & \frac{1}{1-a_n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{1}{1-a_1} & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{1}{1-a_n} & \cdots & \cdots & \cdots & \cdots \end{pmatrix}$$

where $a_i := \sum_{i=1}^{n} \omega_{il}^2$. For the Ricci curvature we obtain

$$0 = \overline{\text{Ric}}(Y, X) = \sum_{i=1}^{n} \langle \overline{\text{R}}(X, e_i) e_i, Y \rangle + \sum_{i=1}^{n} \frac{1}{1-a_i} \langle \overline{\text{R}}(X, N(e_i)) N(e_i), Y \rangle$$

Using the Kähler identity we see that this is equal to

$$0 = \sum_{i=1}^{n} \langle \overline{\text{R}}(X, e_i) J(e_i), J(Y) \rangle + \sum_{i=1}^{n} \frac{1}{1-a_i} \langle \overline{\text{R}}(X, N(e_i)) J(N(e_i)), J(Y) \rangle$$

$$= \sum_{i=1}^{n} \langle \overline{\text{R}}(X, e_i) N(e_i), J(Y) \rangle + \sum_{i, l=1}^{n} \omega_{il} \langle \overline{\text{R}}(X, e_i) e_l, J(Y) \rangle$$

$$- \sum_{i=1}^{n} \langle \overline{\text{R}}(X, N(e_i)) e_i, J(Y) \rangle - \sum_{i, l=1}^{n} \omega_{il} \langle \overline{\text{R}}(X, N(e_i)) e_l, J(Y) \rangle$$

$$+ \sum_{i=1}^{n} \frac{a_i}{1-a_i} \langle \overline{\text{R}}(X, N(e_i)) J(N(e_i)), J(Y) \rangle$$

Now we use the Bianchi identity and see that

$$\sum_{i=1}^{n} \langle \overline{\text{R}}(e_i, N(e_i)) X, J(Y) \rangle = \sum_{i, m=1}^{n} \omega_{im} \langle \overline{\text{R}}(X, e_i) e_m, J(Y) \rangle$$

$$- \sum_{i, m=1}^{n} \omega_{im} \langle \overline{\text{R}}(X, N(e_i)) J(e_m), J(Y) \rangle$$

$$+ \sum_{i=1}^{n} \frac{a_i}{1-a_i} \langle \overline{\text{R}}(X, N(e_i)) J(N(e_i)), J(Y) \rangle$$

If we choose $X = e_l, Y = -J(e_k)$ then we get

$$2\omega^{kl} \overline{\text{R}}^p_{l k} = 2 \sum_{i, k, l, m=1}^{n} \omega_{kl} \omega_{im} \overline{\text{R}}_{k m l i} - 2 \sum_{i, k, l, m=1}^{n} \omega_{kl} \omega_{im} \langle \overline{\text{R}}(e_l, N(e_i)) J(e_m), e_k \rangle$$

$$+ 2 \sum_{i, k, l, m=1}^{n} \omega_{kl} \frac{a_i}{1-a_i} \langle \overline{\text{R}}(e_l, N(e_i)) J(N(e_i)), e_k \rangle$$
and since $a_i \leq |\omega|^2$ and $\frac{1}{1-a_i}$ is bounded as long as $N$ is an isomorphism, we can bound each of these terms by a constant multiple of $|\omega|^2$. That eventually proves Proposition 1.8.

**Theorem 1.9.** If $L$ is a closed manifold, smoothly immersed as a Lagrangian submanifold into a Ricci-flat Calabi-Yau manifold, then equation (*) has a unique, smooth solution at least on some short time interval and as long as the solution exists all submanifolds $L_t$ are Lagrangian submanifolds.

**Proof.** Existence and uniqueness follows from Proposition 1.6. To prove that this flow preserves the Lagrangian structure we fix a time interval $[0, T]$ and look at the function $f_\epsilon := |\omega|^2 - \epsilon e^{2ct}$, where $\epsilon$ is an arbitrary, positive constant and $c$ is as in Proposition 1.8. Obviously $f_\epsilon(0) = -\epsilon < 0$. On the other hand we obtain from 1.8 that

$$\frac{d}{dt} f_\epsilon \leq \Delta f_\epsilon + cf_\epsilon - c\epsilon e^{2ct} < \Delta f_\epsilon + cf_\epsilon$$

Thus by the parabolic maximum principle we get that $f_\epsilon < 0$ for all $t \in [0, T]$ and all positive $\epsilon$. Then we let $\epsilon$ tend to zero and obtain the inequality $|\omega|^2 \leq 0$ on $[0, T]$. Since also $|\omega|^2 \geq 0$ this shows that $|\omega|^2$ has to vanish identically and thus $\omega \equiv 0$. This proves the theorem.

2. THE LAGRANGIAN CASE

We turn our attention to the Lagrangian submanifolds. Since $\omega \equiv 0$, $\eta_{ij} = g_{ij}$, $N = J$ we can restate the equations in 1.1 and 1.3 which become very beautiful

**Proposition 2.1.**

(a) $h_{ijk} = h_{jik} = h_{jki}$

(b) $\frac{\partial^2 F^\gamma}{\partial x^k \partial x^j} - \Gamma^m_{kj} \frac{\partial F^\gamma}{\partial x^m} + \Gamma^m_{\alpha \beta} \frac{\partial F^\alpha}{\partial x^k} \frac{\partial F^\beta}{\partial x^j} = -g^{mn} h_{mkj} J_{\beta}^\gamma \frac{\partial F^\beta}{\partial x^n}$

(c) $\nabla_l h_{kji} - \nabla_k h_{lji} = \overline{R}_{l|kji}$

(d) $R_{ijkl} = \overline{R}_{ijkl} + g^{mn}(h_{mkj} h_{njl} - h_{mjl} h_{njk})$

These equations are true for any Lagrangian submanifold. If in addition $L$ is immersed into a Ricci-flat Calabi-Yau manifold then we also have the important identity

(e) $dH = 0$

In the forthcoming we will set $\nu_\alpha := \nu_\alpha, \partial_\alpha := \partial_\alpha, e_\beta := E_\beta, \partial_\beta$, and $e_\alpha := \partial_\alpha F^\alpha$.
**Proposition 2.2.**

(a) \[ h_{jkl} = \bar{\omega}_{\alpha\beta} e^\alpha_{jk} e^\beta_{l} + \bar{\omega}_{\delta\gamma} \Gamma^\delta_{\alpha\beta} e^\alpha_{j} e^\beta_{k} e^\gamma_{l} \]

(b) \[ \frac{\partial \nu^\gamma_s}{\partial x^i} = \Gamma^l_{is} \nu^\gamma_l + h^s_t e^\gamma_l e^\alpha_{j} e^\beta_{k} e^\gamma_{l} \]

**Proof.** (a) can be obtained by multiplying 2.1 (b) with $\bar{\Gamma}_{\gamma\delta} \nu^\delta_l$. For (b) we calculate

\[ \nabla_{e_i} \nu_s = (\nabla_{e_i} e_s, e_k) \nu_t g^{kl} + (\nabla_{e_i}, e_k) \nu_t g^{kl} - (e_s, \nabla_{e_i} e_k) \nu_t g^{kl} = (\nabla_{e_i}, e_k) \nu_t g^{kl} + h_{sik} \nu_t g^{kl} = \Gamma^l_{is} \nu_l + h^s_t e^l_{e_l} \]

On the other hand we have

\[ \nabla_{e_i} \nu_s = e^\alpha_i \nabla_{\alpha} (\nu^\gamma_s e^\beta_l \partial \partial y^\gamma) = (\frac{\partial \nu^\gamma_s}{\partial x^i} + e^\alpha_i \nu^\gamma_s \Gamma^\gamma_{\alpha\beta}) \frac{\partial}{\partial y^\gamma} \]

Both equations together imply (b).

The evolution equation now takes the form

\[ (*) \quad \frac{d}{dt} F^\alpha = -g^{mn} H_m J^\alpha_{\beta} \frac{\partial F^\beta}{\partial x^n} = -H^n \nu^\alpha_n \]

We also want to restate Lemma 1.7

**Lemma 2.3.**

(a) \[ \frac{d}{dt} g_{ij} = -2H^l h_{lij} \]

(b) \[ \frac{d}{dt} d\mu = -H^l H_l d\mu \]

(c) \[ \frac{d}{dt} \omega_{ij} = 0 \]

Now we derive the evolution equation for $h_{jkl}$. The best way to do this is to assume that we have chosen double normal coordinates at a given fixed point $p \in L$, i.e. normal coordinates for $L$ and $M$. With Proposition 2.2 (a) and (*) we obtain

\[ \frac{d}{dt} h_{jkl} = -\bar{\omega}_{\alpha\beta} \frac{\partial^2}{\partial x^i \partial x^j} (H^n \nu^\alpha) e^\beta_{k} e^\gamma_{l} - \bar{\omega}_{\alpha\beta} e^\alpha_{i} \frac{\partial}{\partial x^j} (H^n \nu^\beta) - \bar{\omega}_{\delta\gamma} \Gamma^\delta_{\alpha\beta} \frac{\partial}{\partial x^j} (H^n \nu^\gamma) - \bar{\omega}_{\delta\gamma} e^\alpha_{i} e^\beta_{j} e^\gamma_{l} \]
since all other derivatives vanish in normal coordinates. Let us denote these three terms by \( A, B \) and \( C \). For \( A \) we get

\[
-\omega_{\alpha\beta}\frac{\partial^2}{\partial x^j \partial x^k} (H^n \nu_{i}^\alpha) e_i^\beta = H^n \partial^2 \omega_{\alpha\beta} + \omega_{\beta\alpha} (H^n \nu_{i,\alpha}^{\alpha} + H^n \nu_{i,\alpha}^{\beta}) e_i^\beta + \omega_{\beta\alpha} H^n (\nu_{n,j}^{\alpha} e_i^\beta)
\]

Using 2.2 (b) we see that this simplifies to

\[
A = H^n \partial^2 \omega_{\alpha\beta} + \omega_{\beta\alpha} H^n \nu_{n,jk}^{\alpha} e_i^\beta
\]

\[
= H^n \partial^2 \omega_{\alpha\beta} + \omega_{\beta\alpha} H^n \nu_{n,jk}^{\alpha} (\Gamma^m_{jn} \nu_m^\alpha + h_{nj} m e_m^\alpha - e_j^\gamma \nu_n^\delta \Gamma^x_{\gamma\delta},k)
\]

\[
= H^n \partial^2 \omega_{\alpha\beta} + \omega_{\beta\alpha} H^n \nu_{n,jk}^{\alpha} (\Gamma^m_{jn,k} \nu_m^\alpha - h_{nj} m h_{mk} e_m^\alpha - \Gamma^\gamma_{\delta\eta} e_k^\gamma e_l^\eta)
\]

again since all other derivatives vanish and because of \( \omega_{\alpha\beta} e_i^\alpha e_j^\beta = 0 \). Finally

\[
A = H^n \partial^2 \omega_{\alpha\beta} + \omega_{\beta\alpha} H^n \nu_{n,jk}^{\alpha} e_i^\beta
\]

\[
= H^n \partial^2 \omega_{\alpha\beta} + \omega_{\beta\alpha} H^n \nu_{n,jk}^{\alpha} (\Gamma^m_{jn} \nu_m^\alpha + h_{nj} m e_m^\alpha - e_j^\gamma \nu_n^\delta \Gamma^x_{\gamma\delta},k)
\]

\[
= H^n \partial^2 \omega_{\alpha\beta} + \omega_{\beta\alpha} H^n \nu_{n,jk}^{\alpha} (\Gamma^m_{jn,k} \nu_m^\alpha - h_{nj} m h_{mk} e_m^\alpha - \Gamma^\gamma_{\delta\eta} e_k^\gamma e_l^\eta)
\]

since in normal coordinates \( \nabla_k \nabla_j H^n = (\nabla_j H^n)_{,k} \). For \( B \) we obtain

\[
B = -\omega_{\alpha\beta} e_j^\alpha \frac{\partial}{\partial x^j} (H^n \nu_{i}^\beta) = \omega_{\alpha\beta} h^s_{jk} \nu_{i}^\alpha (H^n \nu_{i}^\beta + H^n h_{nl} m e_m^\beta) = -H^n h^s_{jk} h_{lns}
\]

where we have used 2.1 (b) and 2.2 (b). Now \( \omega_{\beta\alpha} e_i^\beta = \overline{g}_{\alpha\beta} \nu_{i}^\beta \). Combining \( A, B \) and \( C \) gives

\[
A + B + C = \nabla_k \nabla_j H_l - H^n (h_{nj} m h_{mk} + h_{nl} m h_{mk})
\]

\[
- H^n (\overline{g}_{\alpha\beta} \Gamma^\gamma_{\delta\eta} \nu_{i}^\gamma \nu_{i}^\delta \nu_{i}^\eta e_j^\gamma - \overline{g}_{\delta\gamma} \Gamma^\gamma_{\alpha\beta} \nu_{i}^\gamma \nu_{i}^\alpha \nu_{i}^\beta e_k^\gamma)
\]

After rearranging indices in the last term we see that this term equals

\[
-H^n (\overline{g}_{\alpha\beta} \Gamma^\gamma_{\delta\eta} \nu_{i}^\gamma \nu_{i}^\delta \nu_{i}^\eta e_j^\gamma - \overline{g}_{\delta\gamma} \Gamma^\gamma_{\alpha\beta} \nu_{i}^\gamma \nu_{i}^\alpha \nu_{i}^\beta e_k^\gamma) = H^n \overline{g}_{\alpha\beta} (\Gamma^\gamma_{\gamma\delta\eta} - \Gamma^\gamma_{\delta\gamma\eta}) \nu_{i}^\gamma \nu_{i}^\alpha \nu_{i}^\beta e_k^\gamma
\]

But since we have chosen normal coordinates this is exactly

\[
H^n \overline{g}_{\alpha\beta} (\Gamma^\gamma_{\gamma\delta\eta} - \Gamma^\gamma_{\delta\gamma\eta}) \nu_{i}^\gamma \nu_{i}^\alpha \nu_{i}^\beta e_k^\gamma = H^n \overline{R}_{\alpha\beta} (\Gamma^\gamma_{\gamma\delta\eta} - \Gamma^\gamma_{\delta\gamma\eta}) \nu_{i}^\gamma \nu_{i}^\alpha \nu_{i}^\beta e_k^\gamma = H^n \overline{R}_{\alpha\beta}
\]

Therefore we obtain the result

**Lemma 2.4.**

\[
\frac{d}{dt} h_{jkl} = \nabla_k \nabla_j H_l - H^n (h_{nj} m h_{mk} + h_{nl} m h_{mk}) + H^n \overline{R}_{\alpha\beta}
\]

Note that the RHS of 2.4 is a symmetric three tensor since this is true for \( h_{jkl} \). This can also be proved directly by using the rule for interchanging derivatives, the Gauss curvature equations, the first Bianchi identity and the Kähler identity.

If we assume that \( \theta \) is being deformed by a different 1-form \( \theta \) then it can be easily checked that the same calculations as above give the general evolution equations.
Lemma 2.5.

(a) \[ \frac{d}{dt} g_{ij} = -2\theta^i h_{lij} \]

(b) \[ \frac{d}{dt} d\mu = -\theta^i H_i d\mu \]

c) \[ \frac{d}{dt} \omega = d\theta \]

d) \[ \frac{d}{dt} h_{jkl} = \nabla_k \nabla_j \theta_l - \theta^n (h^m_{nj} h_{mkl} + h^m_{nl} h_{mkj}) + \theta^n \mathcal{R}^n_{mkjl} \]

Let \( d^\dagger \) denote the negative adjoint to \( d \), i.e. \( d^\dagger \theta = \nabla_i \theta^i = g^{ij} \nabla_i \theta_j \). We want to calculate the time derivative of the mean curvature form.

Lemma 2.6.

\[ \frac{d}{dt} H = dd^\dagger \theta \]

Proof. Using 2.5 (a) and (d) we see that

\[ \frac{d}{dt} H_k = \frac{d}{dt} (g^i h_{ijkl}) = 2g^{jm} g^{ln} \theta^s h_{smn} h_{ijkl} + \nabla_k d^\dagger \theta - 2\theta^n h^m_{nj} h_{mkkl} + \theta^n \mathcal{R}^n_{mkkl} \]

The last term is a Ricci curvature and vanishes and two terms cancel which gives

\[ \frac{d}{dt} H_k = \nabla_k d^\dagger \theta . \]

\( L \) is called special Lagrangian, if it is Lagrangian and calibrated with respect to the real part of the Calabi-Yau form. Since calibrated manifolds are minimal we obtain as a direct consequence of 2.5 (c) and 2.6:

Corollary 2.7. If \( L \) is a family of special Lagrangian surfaces evolving under \( \theta \) then \( \theta \) has to be closed and coclosed, i.e. harmonic.

Let us finally prove the following identity which might be useful for further investigations of the mean curvature flow.

Proposition 2.8.

\[ \nabla_i \nabla_j h_{rs k} = \nabla_r \nabla_s h_{ijk} + \nabla_i \mathcal{R}_{jrsk} + \nabla_r \mathcal{R}_{isjk} \]

\[ + h^n_{sk} R_{njri} + h^n_{jk} R_{nsri} + h^n_{js} R_{nkri} \]

\[ + h^n_{is} R_{nkrj} + h^n_{ik} R_{nsrj} + h^n_{rj} R_{nsi} + h^n_{nr} R_{nks} \]

\[ - h^n_{ir} R_{njsk} + h^n_{ij} R_{nrks} - h^n_{rs} R_{mikj} + h^n_{ri} R_{msk} \]

\[ + h^n_{sk} (h_{nr} h_{mji} - h_{ni} h_{mjr}) + h^n_{jk} (h_{nr} h_{msi} - h_{ni} h_{msr}) \]

\[ + h^n_{kn} (h_{mr} h_{msj} - h_{mj} h_{msr}) \]

\[ . \]
Proof. Using 2.1 (c) and the rule for interchanging derivatives we obtain

\[ \nabla_i \nabla_j h_{rsk} = \nabla_r \nabla_i h_{jsk} + \nabla_i \overline{R}_{jrsk} + h^n_{sk} R_{njri} + h^n_{jk} R_{nsri} + h^n_{js} R_{nkri} \]

\[ = \nabla_r \nabla_s h_{jik} + \nabla_i \overline{R}_{jrsk} + \nabla_r \overline{R}_{isjk} + h^n_{sk} R_{njri} + h^n_{jk} R_{nsri} + h^n_{js} R_{nkri} \]

Then the result follows from the Gauss curvature equations and the fact that

\[ \nabla_i R_{jrsk} = \nabla_i R_{jrsk} - h^n_{ij} \overline{R}_{nrks} + h^n_{ri} \overline{R}_{njks} \]

\[ + h^n_{is} \overline{R}_{nkjr} - h^n_{ik} \overline{R}_{nsjr} \]

**Theorem 2.9.** Assume that \( L_t = F_t(L) \) is compact, orientable and evolves under the mean curvature flow and that \( x_0 \) is an arbitrary but fixed point on \( L \).

(a) There exists a unique smooth family of functions \( \phi_t \), smoothly depending on time such that

\[ H_t(x) = H_0(x) + d \left( \int_0^t \Delta_\tau \phi_\tau(x) d\tau \right) \]

\[ \Delta_t (\phi_t - \int_0^t \Delta_\tau \phi_\tau(x) d\tau) = d_\tau^t H_0 \]

\[ \phi_t(x_0) = 0 \]

in particular the cohomology class of \( H \) does not change.

(b) If the first Betti number of \( L \) vanishes, then there exists a unique smooth family of functions \( \phi_t \) such that

\[ H_t = d\phi_t \]

\[ \frac{d}{dt} \phi_t = \Delta_t \phi_t \]

\[ \min_L \phi_0 = 0 \]

Proof. Define the form \( \tilde{H}_t(x) := H_0(x) + d \left( \int_0^t \overline{d}_H \phi_\tau(x) d\tau \right) \) where we integrate pointwise. This form surely exists, since \( \overline{d}_H \phi_t \) is smooth. For the time derivative we obtain

\[ \frac{d}{dt} \tilde{H}_t = d \overline{d}_H t H_t = \frac{d}{dt} H_t \]

and since \( \tilde{H}_0 = H_0 \) we conclude \( \tilde{H}_t = H_t \). Now we use the decomposition theorem and can express \( H_t \) as a unique sum \( H_t = \psi_t + d\phi_t \), where \( \overline{d}_H \psi_t = 0 \), \( \phi_t(x_0) = 0 \) and \( \psi_t, \phi_t \) are smooth. Then \( \overline{d}_H t H_t = \overline{d}_H t \phi_t = \Delta_t \phi_t \). This proves (a), (b) is a direct consequence of (a) since then the harmonic part \( \psi_t \equiv 0 \) and therefore

\[ H_t - H_0 = d(\phi_t - \phi_0) = d \left( \int_0^t \Delta_\tau \phi_\tau(x) d\tau \right) \]

This implies

\[ d \left( \int_0^t \frac{d}{dt} \phi_\tau - \Delta_\tau \phi_\tau(x) d\tau \right) = 0 \]
which means that there exists a smooth function \( f(t) \) such that \( \frac{d}{dt} \phi_t - \Delta_t \phi_t = f(t) \). Now define \( \tilde{\phi}_t := \phi_t - \int_0^t f(\tau) d\tau - \min_L \phi_0 \). This function has all the desired properties. If \( \phi_t, \tilde{\phi}_t \) are two functions with the same properties then \( d(\phi_t - \tilde{\phi}_t) = 0 \) and consequently there exists a function \( f(t) \) such that \( \phi_t = \tilde{\phi}_t + f(t) \). Since \( \frac{d}{dt}(\phi_t - \tilde{\phi}_t) = \Delta_t (\phi_t - \tilde{\phi}_t) = 0 \) we conclude that this function is a constant \( c \) which has to be zero since \( \min_L \phi_0 = \min_L \tilde{\phi}_0 = 0 \). This proves uniqueness.

**Remarks and questions.** There are many interesting questions arising from this context. First of all the above calculations show that the mean curvature form of a Lagrangian submanifold is closed if the ambient space is a Ricci-flat Calabi-Yau manifold. Therefore it makes sense to define the mean curvature class \([H]\) as the cohomology class of the mean curvature form. If \( L \) is a minimal Lagrangian submanifold then clearly \([H] = 0 \). The question is, if any Lagrangian submanifold with trivial mean curvature class can be deformed into a minimal Lagrangian submanifold of the same topological type. If \( L \) is compact, orientable and if the first Betti number of \( L \) vanishes then \( H \) is exact. Are minimal Lagrangian submanifolds automatically special Lagrangian? Under what conditions can one deform a Lagrangian submanifold into a special Lagrangian? Is this possible if and only if the mean curvature class of the initial surface vanishes? We also note that the mean curvature vector has to vanish at least in one point if the Lagrangian surface cannot admit an everywhere nonzero, smooth tangent vector field because the complex structure would map this vector field to a smooth and nowhere vanishing tangent vector field. If the mean curvature form is exact and \( L \) is closed then the mean curvature has to vanish at least in two points, since then \( H = d\phi \) and the smooth function \( \phi \) has at least one maximum and one minimum on \( L \).

Another remarkable fact is that the pair (Lagrangian, special Lagrangian) seems to be analogous to the pair (U(n) holonomy, SU(n) holonomy). Since it has turned out in the past that the mean curvature flow and the Ricci flow are always giving analogous results, one might ask under what conditions does the Ricci flow or perhaps a different flow preserve the holonomy of a metric. There are already some interesting results with respect to that question [1].

If one wants to deform a special Lagrangian manifold through special Lagrangian manifolds then the corresponding 1-forms have to be closed and coclosed, i.e. harmonic. This is sufficient and necessary and has been proven in [6] (we only proved that this is necessary). Unfortunately there is no canonical harmonic 1-form known which could be used to deform such a surface. The most obvious approach would be to define a coordinate system on the moduli space of special Lagrangian manifolds by requiring that the cohomology class does not change, thus giving something like geodesic normal coordinates (see [8]). However there are still great difficulties in particular the existence of such a flow has not been proven. Assuming that a special Lagrangian manifold moves under a flow generated by a harmonic 1-form one immediately gets a good insight in the structure of the evolution equations by studying the corresponding evolution equations for the mean curvature flow on Lagrangian submanifolds. This is very analogous to the situation of a hypersurface evolving in an ambient space. In that case the equations induced by the mean curvature flow and those coming from different curvature functions are very similar (see [7]) and the equations for a Lagrangian surface flowing along its mean curvature are also analogous to those coming from a deformation of special Lagrangian surfaces by harmonic 1-forms.
harmonic 1-forms as we have already seen. Thus the study of the mean curvature flow on the moduli space of Lagrangian submanifolds in a Ricci-flat Calabi-Yau manifold would help much in understanding what happens to special Lagrangian surfaces.

Any smooth submanifold has a mean curvature vector and therefore it would be possible to investigate the mean curvature flow in arbitrary codimensions. But so far this has only been done for hypersurfaces (e.g. [5]). The reason is that a hypersurface has a unique (up to the sign) unit normal vector. Given a basis of the tangent space one can therefore identify this basis with a basis of the normal space which in that case has dimension 1. For a Lagrangian submanifold this can be achieved by using the complex structure $J$ and this makes it possible to compare the different second fundamental forms on $L$ which is a severe problem for a general submanifold.

There might be a greater number of canonical deformations. Given any smooth family of functions $f(t)$ on $L$ one gets closed 1-forms $\theta_t$, namely $\theta_t = df_t$. Obviously $df_t$ is closed independent of the metric on $L$ and therefore independent of the immersion of $L$ into $M$. The corresponding flow (if it exists) would preserve the Lagrangian structure since always $\frac{d}{dt} \omega = d\theta_t$. On the other hand there is a great number of canonical, smooth functions on $L$. These functions are the eigenfunctions of the Laplace operator, i.e. $\Delta f_i + \lambda_i f = 0$. It would be interesting to study the behaviour of $L$ under these flows, i.e. with $\Delta_t f(t) + \lambda_i(t) f(t) = 0$.

Acknowledgements. During the preparation of this paper the author has been a Post-Doc fellow of the Alexander von Humboldt Foundation and would like to thank S.T. Yau for his hospitality and his interest in this paper. The author also thanks E. Zaslow for useful discussions.

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