QCD in the axial gauge

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ABSTRACT

We review a recent attempt to deal with non-perturbative features of QCD by analytical means, using a manifestly gauge invariant, canonical approach.

INTRODUCTION

Suppose you would like to solve the elementary quantum mechanical problem of a particle in a central potential, but under the condition that only $s$-waves are “physical states”,

\[
H = \frac{p^2}{2m} + V(r) , \quad \vec{p} = \frac{1}{i} \frac{\partial}{\partial r} , \quad [H, \vec{L}] = 0 ,
\]

\[\vec{L}|_{\text{phys}} = 0 .\]

This is a typical example of a constrained system, formulated in terms of redundant variables. Here, the constraint can easily be resolved by transforming to polar coordinates,

\[
\left(-\frac{1}{2m} \frac{\partial^2}{\partial r^2} + V(r) \right) u(r) = Eu(r) , \quad u(r) = r\psi(r) .
\]

If we denote $r$ again by $x$, the result is suggestive of the “axial gauge” $y = z = 0$. (Note however that the condition $x > 0$ and the boundary condition $u(0) = 0$ for the radial wave function are remnants of the transition to curvilinear coordinates.) Rotational symmetry is guaranteed, irrespective of any further approximations to dynamics, since we are using the scalar variable $r$.

In the case of gauge theories, we face a similar situation since we start out with redundant variables. Can we find the analogue of polar coordinates in QED or QCD, of course with respect to local gauge symmetry rather than rotational symmetry? Let us

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consider QED first (Lenz et al., 1994a). The canonical formulation is most straightforward in the Weyl gauge \((A_0 = 0)\) since \(A_0\) has no conjugate momentum. All 3 spatial components of \(\vec{A}\) are then quantized. The Gauss law as constraint on the physical states accounts for the fact that only 2/3 of the variables (the transverse photons) are physical. The Hamiltonian in the Weyl gauge reads

\[
H = \int d^3x \left[ \psi^\dagger \left( -i\vec{\alpha} \vec{D} + \beta m \right) \psi + \frac{1}{2} \left( \vec{E}^2 + \vec{B}^2 \right) \right],
\]

with

\[
\vec{D} = \vec{\nabla} - ie\vec{A}, \quad \vec{E}(\vec{x}) = -\frac{1}{i} \frac{\delta}{\delta \vec{A}(\vec{x})}, \quad \vec{B} = \vec{\nabla} \times \vec{A}.
\]

\(H\) is invariant under local, time independent gauge transformations

\[
\vec{A} \to \vec{A} + \vec{\nabla} \beta, \quad \psi \to e^{ie\beta} \psi,
\]
generated by the Gauss law operator

\[
G(\vec{x}) = -\vec{\nabla} \vec{E} + e\rho, \quad [G(\vec{x}), H] = 0.
\]

Physical states are defined through the constraint

\[
G(\vec{x})|\text{phys}\rangle = 0.
\]

Here, unlike in the above toy model, the choice of gauge invariant variables is far from unique, as could have been guessed from the proliferation of “gauge choices” in the literature. To see how one can resolve the constraint, write down the Gauss law in the Schrödinger representation,

\[
\left( \vec{\nabla} - i \frac{\delta}{\delta \vec{A}} + e\rho \right) \Psi[\vec{A}, \psi] = 0.
\]

This (functional) first-order differential equation can be solved by a “plane wave” ansatz,

\[
\Psi[\vec{A}, \psi] = \exp \left\{ -i \int d^3x \vec{\Delta} \frac{1}{\Delta} \vec{\nabla} e\rho \right\} \Phi[\vec{A}_{\text{tr}}, \psi] := U^\dagger \Phi[\vec{A}_{\text{tr}}, \psi].
\]

Since \(U\) as defined in Eq. (10) is a unitary operator, the Schrödinger equation can be recast into the form

\[
H \Psi = E \Psi \to UHU^\dagger \Phi = E \Phi.
\]

This is the equation corresponding to the radial Schrödinger equation \((3)\) in the above example, provided we can show that \(UHU^\dagger\) contains only the physical variables \(\vec{A}_{\text{tr}}, \psi\). Indeed, since the transformed Gauss operator

\[
UG(\vec{x})U^\dagger = \vec{\nabla} - i \frac{\delta}{\delta \vec{A}(\vec{x})}
\]

commutes with \(UHU^\dagger\), it is clear that the unitarily transformed Hamiltonian cannot contain the longitudinal part of \(\vec{A}\) any more. A simple calculation confirms this expectation, yielding

\[
UHU^\dagger = \int d^3x \left[ \psi^\dagger \left( -i\vec{\alpha} \vec{D}_{\text{tr}} + \beta m \right) \psi + \frac{1}{2} \left( \vec{E}_{\text{tr}}^2 + \vec{B}_{\text{tr}}^2 - e^2 \rho \frac{1}{\Delta} \rho \right) \right].
\]
We have thus rederived the standard Coulomb gauge Hamiltonian, including the familiar static Coulomb potential. The Gauss law constraint is resolved, and all redundant variables have been eliminated; in this quantum mechanical scheme of gauge fixing, $U$ is denoted as “unitary gauge fixing transformation” (Lenz et al., 1994a). In QED, the Coulomb gauge is clearly singled out on physics grounds, since it is only in this gauge that static sources decouple from the radiation field. However, many other gauges are conceivable; thus, for instance, the Gauss law can alternatively be resolved by the ansatz

$$
\Psi[\vec{A}, \psi] = \exp \left\{ -i \int d^3x A_3 \frac{1}{\partial_3} \left( -\vec{\nabla}_\perp \vec{E}_\perp + e\rho \right) \right\} \Phi[\vec{A}_\perp, \psi].
$$

(14)

This leads to the axial gauge ($A_3 = 0$) which, however, is less convenient from the point of view of atomic or molecular physics, since even static charges “radiate”. In QCD, it is much less obvious whether such a procedure can be carried through and what the preferred gauge choice is (Lenz et al., 1994b). The Weyl gauge Hamiltonian now reads

$$
H = \int d^3x \left[ \psi^\dagger \left( -i\vec{\alpha}\vec{D} + \beta m \right) \psi + \text{tr} \left( \vec{E}^2 + \vec{B}^2 \right) \right],
$$

(15)

with the chromo-electric and -magnetic fields ($a = 1, ..., N_c^2 - 1$)

$$
\vec{E}^a = -\frac{1}{i} \frac{\partial}{\partial A^a}, \quad \vec{B}^a = \vec{\nabla} \times \vec{A}^a + \frac{1}{2} g f^{abc} \vec{A}^b \times \vec{A}^c.
$$

(16)

The Gauss law is non-linear,

$$
(-\vec{D}\vec{E} + g\rho)|\text{phys}\rangle = 0, \quad (\vec{D}\vec{E})^a = \vec{\nabla}\vec{E}^a + g f^{abc} \vec{A}^b \vec{E}^c,
$$

(17)

hence it is much more difficult to resolve than the abelian one. If we want to implement the Coulomb gauge for instance, we have to solve an equation of the type ($\vec{E}_\ell = \vec{\nabla}\phi$)

$$
\vec{D}\vec{\nabla}\phi|\text{phys}\rangle = ...|\text{phys}\rangle
$$

(18)

for $\phi$. This is impossible (non-perturbatively) since one cannot identify analytically the zero-modes of the 2nd order partial differential operator $\vec{D}\vec{\nabla}$ (Gribov, 1978). By contrast, in order to implement the axial gauge, one only needs to solve a first order ordinary differential equation,

$$
D_3 E_3|\text{phys}\rangle = ...|\text{phys}\rangle.
$$

(19)

Since one has control over the zero modes of $D_3$, it becomes possible to derive a Hamiltonian formulated exclusively in terms of unconstrained, physical variables in the axial gauge (Lenz et al., 1994b). The whole procedure is quite involved, and I will make no attempt to go into any technical detail, nor even to show you the final Hamiltonian in full glory. Instead, in the following two sections, I will concentrate on one particularly instructive part of the Hamiltonian, trying to exhibit some basic physics for which the axial gauge is advantageous (Lenz et al., 1995).

**CONFINEMENT**

Let us come back to the axial gauge in QED for a moment and point out one complication which we have ignored so far. We work in a finite box with periodic boundary
conditions (i.e., on a “torus”). In that case $A_3 = 0$ is not a legitimate gauge, simply because the 2-dimensional field $a_3(\vec{x}_\perp) = \int_0^L dz A_3(\vec{x}_\perp, z)$ is gauge invariant. Physically, it describes transverse photons polarized in the 3-direction and propagating in the (1,2) plane. In QED, this can easily be cured: Use the gauge $\partial_3 A_3 = 0$ (rather than $A_3 = 0$) by retaining $a_3(\vec{x}_\perp)$ and eliminate some other variables instead (Lenz et al., 1994a). In QCD, the corresponding gauge invariant quantities formed exclusively out of $A_3$ are the eigenvalues of the spatial Wilson loop winding around the torus (“Wilson line”),

$$P e^{ig \int_0^L dz A_3} = V e^{i\alpha_3 L} V^\dagger$$

($a_3$: diagonal matrix). Here, the residual 2-dimensional variables cause much more trouble than in QED. In fact, most of the work needed to resolve the non-abelian Gauss law has to do with these $(N_c - 1)$ lower dimensional, color neutral fields, and all the conspicuous non-perturbative features displayed by the gauge fixed Hamiltonian are somehow related to them (Lenz et al., 1994b). Among these features, most noteworthy is a Jacobian, reflecting the transition from Lie algebra ($A_3$) to group elements (spatial Wilson lines) in the process of gauge fixing.

Is this a purely technical matter, needed to properly define the theory in the infra-red, or is there some real physics associated with the $a_3$? By construction, it is clear that the 2-dimensional variables $a_3$ are those whose dynamics has been maximally simplified by the choice of the axial gauge; as such, they may teach us something about the dynamics of a whole class of variables for which they are representative (but which are not as simply described in our “coordinate frame”). In order to exhibit their physics content, let us do a very drastic approximation — truncate the axial gauge Hamiltonian by keeping only the terms in $a_3$ and the corresponding conjugate momenta (Lenz et al., 1995). Surprisingly, even this primitive version of the Hamiltonian still exhibits very interesting differences between QED and QCD, to which we now turn.

Formally, the truncated Hamiltonian for both QED and SU(2) Yang Mills theory reduces to the following 2-dimensional expression,

$$h = \int d^2x \left[ \frac{1}{2L} e_3^\dagger e_3 + \frac{L}{2} (\nabla_\perp a_3)^2 \right].$$

(21)

Here, the electric field energy is found to be

$$e_3^\dagger e_3 = -\frac{1}{J} \frac{\delta}{\delta a_3} J \frac{\delta}{\delta a_3}$$

(22)

with the SU(2) Haar measure

$$J = \sin^2 \left( \frac{g L a_3}{2} \right),$$

(23)

whereas $J = 1$ in the case of QED. This form of the kinetic energy requires some ultra-violet regularization, for which we choose a transverse lattice (lattice spacing $\ell$, fundamental lattice vectors $\vec{\delta}$). (For QED, this would not be necessary, but we do it anyway for ease of comparison.) Next, a “radial” wavefunctional is introduced as $\tilde{\Psi} = \sqrt{J} \Psi$, and the kinetic energy is reduced to standard form like in the above quantum mechanical example, with a corresponding boundary condition ($\tilde{\Psi} = 0$ whenever $J =$
In terms of the rescaled variable \( \varphi = g a_3 L/2 \), the “radial” Hamiltonian then becomes

\[
h = h_e + h_m = -\frac{g^2 L}{8\ell^2} \sum_{\vec{r}} \frac{\partial^2}{\partial \varphi_{\vec{r}}^2} + \frac{2}{g^2 L} \sum_{\vec{r},\delta} \left( \varphi_{\vec{r}+\delta} - \varphi_{\vec{r}} \right)^2 .
\] (24)

Let us compare the abelian and non-abelian cases:

i) QED:
Since the Jacobian is trivial, the Hamiltonian is quadratic and can be diagonalised by a standard lattice Fourier transform \( (\varphi_{\vec{r}} = \frac{1}{N} \sum_{\vec{k}} e^{2\pi i \vec{r} \cdot \vec{k}} / N \phi_{\vec{k}} \), with \( N = L/\ell \) the number of lattice sites in each transverse direction.) As a result of the magnetic coupling, collective excitations appear, with the dispersion relation

\[
\omega_{\vec{k}}^2 = \frac{4}{\ell^2} \sum_{\delta} \sin^2 \left( \frac{\pi \delta \vec{k}}{N} \right) \rightarrow \left( \frac{2\pi \vec{k}}{L} \right)^2 \quad (|\vec{k}| \ll N) .
\] (25)

In the limit \( L \to \infty \), they become just ordinary, massless photons. The ground state wave functional is Gaussian,

\[
\Psi \sim \prod_{\vec{k}} \exp \left( -\frac{4\ell^2}{g^2 L} \omega_{\vec{k}} \phi_{\vec{k}}^\dagger \phi_{\vec{k}} \right) ,
\] (26)

and the virial theorem ensures that fluctuations of electric and magnetic fields are equal,

\[
\langle \vec{E}^2 - \vec{B}^2 \rangle = 0 .
\] (27)

Hence the axial gauge is not a bad choice at all for pure QED: We have reduced the problem of solving a 3-dimensional vector field theory to that of a 2-dimensional scalar field theory, by an appropriate choice of coordinates. Although the Coulomb gauge is preferred for static charges, the axial gauge is well suited for studying the elementary gauge field excitations – a transverse field is naturally described in cartesian coordinates, the 3-axis pointing into the direction of the polarization vector.

ii) QCD:
Since \( \Psi = 0 \) whenever \( J = 0 \), we can restrict the variables \( \varphi_{\vec{r}} \) to the interval \([0, \pi]\). For large \( L \), \( h_e \) dominates over \( h_m \), and we are left with the simple quantum mechanics of infinite square well potentials at each site, totally decoupled from each other. The ground state is now uncorrelated in coordinate space,

\[
\tilde{\Psi}_0 \sim \prod_{\vec{r}} \sqrt{\frac{L}{\pi}} \sin \varphi_{\vec{r}} .
\] (28)

(Note that this corresponds to the “full” wavefunctional \( \Psi_0 = \text{const.}, E_0 = 0. \) Since \( \Psi_0 \) is an eigenstate of the electric field operator with vanishing eigenvalue, we find trivially an exact “dual Meissner effect”

\[
\langle \vec{E}^2 \rangle = 0 .
\] (29)

The vacuum has a magnetic condensate reminiscent of the QCD vacuum,

\[
\langle \vec{E}^2 - \vec{B}^2 \rangle < 0 .
\] (30)
Most importantly, there are no such excitations as “plane wave gluons”; the gap to the first excited state is

\[ \Delta E = \frac{3g^2L}{8\ell^2} \to \infty \quad (L \to \infty), \]  

so that these gluonic degrees of freedom get frozen in the thermodynamic limit. The magnetic contribution to the ground state energy can be evaluated perturbatively; one finds

\[ \langle 0| h_m|0 \rangle = \frac{4L}{g^2\ell^2} \left( \frac{\pi^2}{6} - 1 \right). \]

In contrast to the QED case, the vacuum has a precise value of the electric field (namely zero) and therefore the largest possible fluctuations in \( \vec{A} \) (“stochastic” vacuum). The excitation gap (31) can be understood by comparing it to lattice gauge theory: If one retains only \( A_3 \) and insists on Gauss’s law, the only gauge invariant excitations are electric flux lines in the 3-direction around the torus. Their energy in the strong coupling limit of Hamiltonian lattice gauge theory (Kogut and Susskind, 1975) is

\[ \Delta E = \frac{g^2}{2\ell} j(j+1) \left( \frac{L}{\ell} \right) = \frac{3g^2L}{8\ell^2} \quad (j = 1/2). \]

We thus recover the strong coupling string tension. Nevertheless, our approach is quite different from the standard lattice where one gets the same type of flux quantization and string tension also in QED, in the strong coupling limit (Rothe, 1992). Here, we obtain a qualitatively different behaviour in the two cases, the difference being exclusively due to the Jacobian.

We have seen that the simple degrees of freedom \( a_3 \) of the axial gauge are useful in order to study the existence or non-existence of plane wave gauge bosons. We find strong indications that QCD does not admit chromoelectric fields with constant polarisation vector over large distances. So far, we cannot say anything about the scale involved – what is large? This would clearly require taking into account the other gluonic degrees of freedom as well and going through some renormalization procedure. Nevertheless, we can get some information about the relevant scale by indirect methods, using lattice results as input. This is important in order to further assess the possible usefulness of the axial gauge.

**DECONFINEMENT**

It is generally believed that QCD undergoes a deconfining phase transition to a quark gluon plasma at a temperature \( T_c \approx 150 – 200 \) MeV (Müller, 1985). Can the axial gauge Hamiltonian formulation of QCD contribute anything useful to this issue? At first sight, the prospects look rather somber: Finite temperature field theory means compactification of the time direction, so that the Weyl gauge \( A_0 = 0 \) is no longer allowed (for the same reason that the axial gauge \( A_3 = 0 \) is not allowed on the torus). Moreover, there is evidence from lattice gauge calculations that the spatial Wilson loops show area law behaviour even above \( T_c \) (see e.g. Karsch, 1994). This seems to indicate that the “simple” variables \( a_3 \) are not particularly sensitive to the phase transition, and that consequently we would have no advantage over other approaches by using the axial gauge.
Fortunately, nature provides us with a very elegant way out of these problems. As
a matter of fact, one can study thermodynamic properties like the deconfining phase
transition by working strictly at $T = 0$, but in a spatial box contracted in one direction
(say, the 3-direction) to

$$L_3 = \beta = 1/T .$$

This result is intimately connected to Lorentz invariance (or, rather, Euclidean $O(4)$
invariance) and would not hold in non-relativistic field theories. Since it is rather
unfamiliar, let me explain it with a simple example: Consider the partition function

$$Z = \int D[\phi]e^{-\int d^4x L_E(x)}$$

for a generic field theory in a finite box at finite temperature in 2 cases:

I) $L_1, L_2, L_3 \gg \beta$

II) $\beta, L_1, L_2 \gg L_3$.

Case (I) corresponds to a hot system in a large, isotropic box, case (II) to a cold system
in a box contracted along the 3-direction. Covariance of the Euclidean action relates
these 2 situations which differ by the interchange of 2 coordinates, $x_3$ and the Euclidean
time $\tau$. Using the standard relations

$$E = -\frac{\partial}{\partial \beta} \ln Z , \quad p = \frac{1}{\beta} \frac{\partial}{\partial V} \ln Z ,$$

one finds for instance

$$\left( \frac{E}{V} \right)_I = -(p)_II ,$$

and vice versa. These symmetry relations have surprising and powerful consequences.
By way of example, consider the Casimir effect for a massless scalar field with periodic
boundary conditions (Toms, 1980). If the system is enclosed between two plates at
distance $L_3 = d$, there is an attractive force corresponding to a negative pressure

$$p = -\frac{\pi^2}{30} \frac{1}{d^4} .$$
On the other hand, the energy density for an ultra-relativistic ideal gas of scalar particles at finite temperature is given by the Stefan Boltzmann law

$$\frac{E}{V} = \frac{\pi^2}{30} T^4. \quad (40)$$

These two laws of seemingly unrelated parts of physics are indeed mapped onto each other by the substitution $d \leftrightarrow 1/T$. This observation – which has been exploited several times in the literature (cf. e.g. Toms, 1980; Koch et al., 1992) – opens a new perspective for studying thermodynamic properties of field theories in a technically and conceptually simpler way. All one has to do is to study the ground state in a different geometry.

Once this is understood, it is not difficult to identify other relations between at first sight unrelated physical observables. One particularly amusing example is the following: In (Lenz et al., 1991), the Schwinger model was studied as a function of a parameter which measures how far off the light-cone the quantization surface is. The fermion condensate was evaluated analytically as a function of this parameter. It was also pointed out that the same function can be interpreted as dependence of the condensate on the size of the box. Independently, the finite temperature Schwinger model was investigated in (Sachs and Wipf, 1992), and again the fermion condensate evaluated in closed form as function of $T$. The two formulae (Eq. (3.105) of (Lenz et al., 1991) and Eq. (5.10) of (Sachs and Wipf, 1992)) agree exactly if one identifies the corresponding variables,

$$\eta' = \frac{\pi}{2} (\beta m_\gamma)^2, \quad (41)$$

a fact which seems to have been overlooked so far. Similarly, it is tantalizing to reinterpret corresponding findings for large $N_c$ QCD in (Lenz et al., 1991) as evidence for a chiral symmetry restoring phase transition at finite temperature, in contradiction to the common lore (McLerran and Sen, 1985; Ming Li, 1986).

Let us now return to axial gauge QED and QCD. We start with QED and go one step beyond the truncation of $H$, taking into account perturbatively the coupling of $a_3$ to the charged fermions. The simplest way to do this is to compute the effective potential for the zero mode of $a_3$ by evaluating the energy density of the Dirac sea in a constant background potential $a_3$. In view of the application to finite temperature field theory, we have to require anti-periodic boundary conditions for the fermions,

$$\psi(\vec{x}_\perp, L) = -\psi(\vec{x}_\perp, 0). \quad (42)$$

We can gauge away a constant $a_3$, provided we change these boundary conditions into quasi-periodic ones,

$$\psi(\vec{x}_\perp, L) = -e^{i a_3 L} \psi(\vec{x}_\perp, 0). \quad (43)$$

This yields the following discretization for the 3-component of the fermion momenta,

$$p_3 = \frac{\pi}{L} (2n + 1) - ea_3. \quad (44)$$

The effective potential is then given by

$$U_{\text{eff}}(a_3) = -2U(c) \quad (45)$$
with $U(c)$ the (heat-kernel regularized) sum over single particle energies,

$$
U(c) = \lim_{\lambda \to 0} \frac{1}{L} \int \frac{d^2 p_\perp}{(2\pi)^2} \sum_{n=-\infty}^{\infty} E(\vec{p}_\perp, n - c) e^{-\lambda E(\vec{p}_\perp, n - c)}, \tag{46}
$$

The variable $c$ is defined as

$$
c = \frac{eLa_3}{2\pi} - \frac{1}{2}, \tag{47}
$$

and

$$
E(\vec{p}_\perp, \nu) = \sqrt{p_\perp^2 + (2\pi\nu/L)^2}. \tag{48}
$$

Performing the integral in (46) and using the generating function for Bernoulli polynomials, one finds

$$
U(c) = \lim_{\lambda \to 0} \frac{1}{2\pi L} \frac{\partial^2}{\partial \lambda^2} \frac{1}{\lambda} \sum_{n=-\infty}^{\infty} e^{-2\pi\lambda|n-c|/L}
\begin{split}
&= \frac{2\pi^2}{3L^4}B_4(c)
\end{split} \tag{49}
$$

(valid for $|c| < 1$, and to be continued periodically outside this interval). We have dropped a $c$-independent, infinite constant. The result for the effective potential is therefore

$$
U_{\text{eff}}(a_3) = -\frac{4\pi^2}{3L^4}B_4(c)
\begin{split}
&= -\frac{4\pi^2}{3L^4} \left( c^2(1-c)^2 - \frac{1}{30} \right)
\end{split} \tag{50}
$$

$U_{\text{eff}}$ has extrema at $c = 0, \frac{1}{2}, 1$, the minimum corresponding to $c = \frac{1}{2}$ or $a_3 = \frac{2\pi}{eL}$. Expanding $U_{\text{eff}}$ to 2nd order around the minimum, we find

$$
U_{\text{eff}} \left( \frac{2\pi}{eL} + \delta a_3 \right) \simeq -\frac{7}{4} \left( \frac{\pi^2}{45L^4} \right) + \frac{1}{2} \left( \frac{e^2}{3L^2} \right) \delta a_3^2. \tag{51}
$$

From this expression, replacing $L$ by $\beta = 1/T$, we can read off the familiar free energy density of an ideal gas of massless fermions,

$$
f = -\frac{7}{4} \left( \frac{\pi^2T^4}{45} \right), \tag{52}
$$

as well as the “electric mass” of the photon (Kapusta, 1989),

$$
m_{el}^2 = \frac{1}{3}e^2T^2, \tag{53}
$$

in spite of the fact that we have only been dealing with ground state properties. This gives a first impression of the potential use of the axial gauge, if at the same time one reinterprets the contracted box in terms of finite temperature field theory.

Let us now turn to the corresponding computations for QCD. Here, if one interchanges 3- and time directions, the simple variables $a_3$ in the axial gauge become the eigenvalues of the thermal Wilson lines, the standard order parameters for the confinement-deconfinement transition. This is obviously a welcome feature which means that we
have chosen a useful set of variables and are in a good position to derive an effective theory for this order parameter in the spirit of Landau Ginzburg theory.

Ignoring quarks (which could be taken into account easily as well in the present approximation), we evaluate the zero-point energy of the “perpendicular” gluons $A_1, A_2$ in the presence of a constant background field $a_3$. We treat the $\vec{A}_\perp$ as free fields, except for the minimal coupling to $a_3$. Again, if $a_3$ is constant, its effect is equivalent to changing the boundary conditions in 3-direction from periodic to quasi-periodic ones. The expression for the vacuum energy density can be written down in complete analogy to the QED example. We consider SU(2) Yang-Mills theory and introduce the rescaled field variable

$$c = \frac{gL}{2\pi} a_3 \in [0, 1].$$

(54)

The vacuum energy density of $\vec{A}_\perp$ will serve as effective potential for $a_3$. It can be decomposed as

$$U_{\text{eff}} = 2U(c) + U(0),$$

(55)

where the two different terms come from the (color) charged and neutral components of $\vec{A}_\perp$, respectively. The function $U(c)$ is exactly the same as the one given in Eq. (46). This yields for $U_{\text{eff}}$, Eq. (53), the form

$$U_{\text{eff}} = \frac{4\pi^2}{3L^4} \left( c^2(1-c)^2 - \frac{1}{20} \right).$$

(56)

(here again, this function has to be continued periodically outside of the interval $0 \leq c \leq 1$). The value of $U_{\text{eff}}$ in the minima reproduces the free energy density of an ideal gluon gas, if we replace $L$ by $\beta = 1/T$,

$$U_{\text{eff}}|_{c=0} = -\frac{\pi^2}{15L^4} \rightarrow -\frac{\pi^2}{45} T^4 (N_c^2 - 1) \quad (N_c = 2).$$

(57)

Using the same substitution, we reproduce the “one-loop effective potential” for the ($\vec{x}$-independent) thermal Wilson line, which has been discussed extensively in the literature (first for SU(2) by Weiss, 1981). A look at the original references shows that our derivation is significantly simpler.

In the standard approach to finite temperature QCD, one infers the gluon electric mass from the 2nd derivative of $U_{\text{eff}}$ in a minimum,

$$m_{\text{el}}^2 = \frac{\partial^2 U}{\partial a_3^2}|_{c=0} = \frac{2g^2}{3L^2} \rightarrow \frac{1}{3} N_c g^2 T^2$$

(58)

From this point of view, it would seem that QED and QCD behave quite similarly indeed. However, so far, we have only discussed the effective potentials and ignored the difference in the kinetic energy of $a_3$, which was crucial to “confine” plane wave gluons. In our approach, we see no justification for disregarding these effects, which are overwhelming at low temperature.

Let us try to understand at least qualitatively the effect of the Jacobian. Following (Polchinski, 1992), we first assume that the effective potential derived for a constant $a_3$ can be taken over for $\vec{a}_\perp$-dependent $a_3$ as well, with the same functional form. The
effective potential to be added to $h_e + h_m$, Eq. (24), is then (up to an irrelevant constant term)

$$u_{\text{eff}} = \frac{4\ell^2}{3\pi^2L^3} \sum_{\vec{r}} \varphi_{\vec{r}}^2 (\pi - \varphi_{\vec{r}})^2 . \quad (59)$$

Secondly, we model the effect of the Jacobian by approximating the infinite square well potential by a harmonic oscillator potential, chosen such as to reproduce the gap between ground state and first excited state, eq. (31),

$$u_{\text{conf}} = \frac{9g^2L}{32\ell^2} \sum_{\vec{r}} (\varphi_{\vec{r}} - \pi/2)^2 . \quad (60)$$

The resulting potential $u_{\text{tot}} = u_{\text{eff}} + u_{\text{conf}}$ now exhibits the correct $Z_2$ “center symmetry”, whereas without the confining potential it has a discrete translational symmetry both in QED and QCD. With a quartic potential and a quadratic one (possibly changing sign),

$$u_{\text{tot}} = \left( \frac{9g^2L}{32\ell^2} - \frac{2\ell^2}{3L^3} \right) \sum_{\vec{r}} (\varphi_{\vec{r}} - \pi/2)^2 + \frac{4\ell^2}{3\pi^2L^3} \sum_{\vec{r}} (\varphi_{\vec{r}} - \pi/2)^4 + \text{const.} , \quad (61)$$

we are now in the standard situation for a second order phase transition and can determine the gluon mass below and above the critical temperature. It is convenient to introduce 2 masses, the electric gluon mass of Eq. (58) and the “confining mass” of Eq. (31),

$$m_{\text{el}}^2 = \frac{2g^2}{3L^2} , \quad m_{\text{conf}}^2 = \left( \frac{3g^2L}{8\ell^2} \right)^2 . \quad (62)$$

The critical temperature (using $L = 1/T$) where the quadratic term in $u_{\text{tot}}$ vanishes is then determined by the condition $m_{\text{conf}}^2 - m_{\text{el}}^2/2 = 0$, or

$$T_c^4 = \frac{27g^2}{64\ell^4} . \quad (63)$$

The expressions for the gluon mass in the confined and deconfined phase are $m_{\text{eff}}^2 = m_{\text{conf}}^2 - m_{\text{el}}^2/2$ and $m_{\text{eff}}^2 = m_{\text{el}}^2 - 2m_{\text{conf}}^2$, respectively. These results are by no means realistic, but are reminiscent of discussions of the phase transition in the strong coupling limit (Gross, 1983). Nevertheless, it is interesting that the crudest conceivable approximation to the gauge fixed Hamiltonian has already the potential of describing both confinement and deconfinement of gluons. Much more work is needed in order to properly understand how the confining effects (related to the Jacobian) are overcome at high temperature and how the well established perturbative features of QCD can be recovered in the present framework.

We finish with the following simple observation, which is again related to the question of length scales. If we translate the critical temperature as determined from lattice gauge calculations (Karsch, 1994) into a critical length, we learn that the phase transition to the quark gluon plasma takes place if we contract space in 3-direction to

$$L_\text{c} = 1/T_c \simeq 1.0 - 1.3 \text{ fm} . \quad (64)$$

Thus, the variables $a_3$ should be qualitatively representative for a large class of waves indeed – all those whose polarization vector does not change over distances appreciably
larger than 1 fm. On the other hand, $L_c$ also sets the scale for the length at which quark and gluon degrees of freedom become essential – a rather large value from the point of view of nuclear physics.

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