FINITE DIMENSIONAL HECKE ALGEBRAS

SUSUMU ARIKI

ABSTRACT. This article surveys development on finite dimensional Hecke algebras in the last decade. In the first part, we explain results on canonical basic sets by Geck and Jacon and propose a categorification framework which is suitable for our example of Hecke algebras. In the second part, we review basics of Kashiwara crystal and explain the Fock space theory of cyclotomic Hecke algebras and its applications. In the third part, we explain Rouquier’s theory of quasihereditary covers of cyclotomic Hecke algebras. We add detailed explanation of the proofs here. The third part is based on my intensive course given at Nagoya university in January 2007.

1. INTRODUCTION

In this article, we will explain current views on the modular representation theory of finite dimensional Hecke algebras. In the last decade, I followed the approach by Dipper and James, and it has become clear that the language from solvable lattice models, which uses terminology like Fock spaces, crystal bases, etc., is one of the most natural. On the other hand, Geck followed Lusztig’s approach and applied his methods to the modular representation theory of Hecke algebras. When we consider Hecke algebras of type $A$ and $B$, or more generally, cyclotomic Hecke algebras of classical type, the two approaches interact, and the study of various labelling sets of irreducible modules has stimulated an interest on cellular algebra structures on the Hecke algebras. In type $A$, we have theory of Specht and dual Specht modules. In type $B$, based on their earlier work [8] and [15], Bonnafé, Geck, Iancu and Lam [9] have conjectured in a very precise manner how Kazhdan-Lusztig cells should give various cellular algebra structures, and Geck, Iancu and Pallikaros [16] showed that the known cellular structure given by Dipper, James and Murphy is one of them. This suggests us a categorification framework for integrable highest weight $U_v(sl_n)$-modules with two specializations at $v = 0$ and $v = 1$.

In type $A$, we have $q$-Schur algebras, which has been an object of intensive study in the last several decades. By Leclerc-Thibon [34] and Varagnolo-Vasserot [40], the algebras also fit well in the categorification picture. Note that $q$-Schur algebras are cellular algebras of quasihereditary type. When the base field is $C$, Rouquier has showed that the category $O$ for the rational Cherednik algebra associated with the symmetric group is the $q$-Schur algebra [39], and quasihereditary structures of $O$ explains in some sense why we have the Specht and the dual Specht module theory. The result depends on his earlier work, one with Ginzburg, Guay and Opdam [19], the other with Brouté and Malle [10]. Observe that every piece that appears in the above story has its cyclotomic analogue. Hence, it is natural to expect that

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cyclotomic analogue of the story would be true. This is our current motivation of research, and even in type $B$, we have not reached a complete understanding, yet.

The paper is organized as follows. In the first part, we introduce the Hecke algebra and briefly explain results on the canonical basic sets by Geck and Jacon and the categorification framework in which the results sit in. To know more about the canonical basic sets and Hecke algebras, I recommend reading the survey [13].

In the second part, we explain the Fock space theory, mostly developed by the author and my collaborators, after explaining Kashiwara crystal. Its applications to Hecke algebras include the modular branching rule, the representation type, etc.

In the third part, we explain Rouquier’s theory of quasihereditary covers in terms of the category $\mathcal{O}$ for the rational Cherednik algebra. I reorganized the contents of [19] and [39] and explain the shortest way to reach the Rouquier’s result. The reader who have read [19] and [39] seriously would find that the proofs explained here are very reader friendly. I hope that this part provides a good preparation for reading [19] and [39].

The third part is based on my intensive course given at Nagoya university in January 2007. At the time, Shoji requested some written material of the lectures, and Kuwabara asked me if it could be in English. The third part partially answers their requests. I thank Shoji for inviting me to give the course.

2. HECKE ALGEBRAS WITH UNEQUAL PARAMETERS

2.1. The algebra. Following [33], we introduce the Hecke algebra, our main object of study.

**Definition 2.1.** We say that $(W, S, L)$ is a weighted Coxeter group if

(i) $(W, S)$ is a Coxeter group, $w \mapsto \ell(w)$, the length function.
(ii) $L: W \to \mathbb{Z}$ is such that $L(ww') = L(w) + L(w')$ if $\ell(ww') = \ell(w) + \ell(w')$.

**Remark 2.2.** Recall that $(st)^{m_{st}}/2 = (ts)^{m_{st}}/2$ if $m_{st}$ is even, $(st)^{(m_{st} - 1)/2} = (ts)^{(m_{st} - 1)/2}$ if $m_{st}$ is odd, is part of the defining relations of $W$. Giving $L$ is the same as giving a set of values \( \{L(s) \mid s \in S\} \) with the property that $L(s) = L(t)$ whenever $m_{st}$ is odd.

We say that $(W, S)$ is of finite type if $W$ is a finite group. $(W, S)$ is isomorphic to product of irreducible Coxeter groups of finite type, and the irreducible Coxeter groups are classified. By the above remark, if $(W, S, L)$ is a weighted Coxeter group such that $(W, S)$ is irreducible of finite type then $L$ takes at most 2 different values on $S$.

**Definition 2.3.** Let $\mathcal{A} = \mathbb{Z}[v, v^{-1}]$. Given weighted Coxeter group $(W, S, L)$, we define the associated Hecke algebra $\mathcal{H}(W, S, L)$, by generators $T_s$, for $s \in S$, and relations $(T_s - v^{L(s)})(T_s + v^{-L(s)}) = 0$ and

$$(T_sT_t)^{m_{st}}/2 = (T_tT_s)^{m_{st}}/2$$

if $m_{st}$ is even,

$$(T_sT_t)^{(m_{st} - 1)/2}T_s = (T_tT_s)^{(m_{st} - 1)/2}T_t$$

if $m_{st}$ is odd.

**Remark 2.4.** We may define multiparameter Hecke algebras when some of $m_{st}$ are even and not equal to 2, but to handle the modular representation theory of Hecke algebras of finite type, the above definition suffices. However, we also note that the definition is for general weighted Coxeter groups, and affine cases are very interesting examples which we do not cover in this article.
that well-known that Definition 2.5. 

Example 2.7. Note that (Kazhdan-Lusztig) (of second type). We write Remark 2.8. Let $A$ the $-vector spaces $G_Q = \sum_{i=0}^{\infty} (A)$. Define canonical basis The basis is called the Kazhdan-Lusztig basis (of second type). We write

\[
C_w = \sum_{y \in W} p_{y,w} T_y.
\]

Remark 2.8. Let $K = \mathbb{Q}(v)$, $A_0 = \{ c(v) \in K \mid c(v) \text{ is regular at } v = 0 \}$ and $A_\infty = \{ c(v) \in K \mid c(v) \text{ is regular at } v = \infty \}$. A $K$-vector space $V$ is balanced if there exist $\mathbb{Q}[v, v^{-1}]$-lattice $L$ of $V$, $A_0$-lattice $L_0$ and $A_\infty$-lattice $L_\infty$ of $V$ such that $E = L \cap L_0 \cap L_\infty$ satisfies the following three properties.

1. Any $\mathbb{Q}$-basis of $E$ is a free $\mathbb{Q}[v, v^{-1}]$-basis of $L$.
2. Any $\mathbb{Q}$-basis of $E$ is a free $A_0$-basis of $L_0$.
3. Any $\mathbb{Q}$-basis of $E$ is a free $A_\infty$-basis of $L_\infty$.

It is easy to see that if $V$ is balanced then we have a canonical isomorphism of $\mathbb{Q}$-vector spaces $G : L_\infty/v^{-1}L_\infty \simeq E$. The Kazhdan-Lusztig theorem says that $\mathcal{H}(W, S, L) \otimes K$ is balanced, and the basis 

\[
\{ C_w := G(T_w + v^{-1}L_\infty) \mid w \in W \}
\]

is not only $\mathbb{Q}[v, v^{-1}]$-basis but $A$-basis of $\mathcal{H}(W, S, L)$.

Definition 2.9. We write $C_xC_y = \sum_{z \in W} h_{x,y,z} C_z$, where $h_{x,y,z} \in A$. Define $a(z)$, for $z \in W$, by

\[
a(z) = \min \{ i \in \mathbb{Z}_{\geq 0} \cup \{ \infty \} \mid v^i h_{x,y,z} \in \mathbb{Z}[v], \text{ for all } x, y \in W \}.
\]

This is Lusztig’s $a$-function. In this subsection, we explain basics of the Kazhdan-Lusztig basis and the $a$-function following Lusztig and Geck.

Kazhdan and Lusztig proved Theorem 2.9 by inductively defining $C_w$. Hence, they also showed the following theorem at the same time.
Theorem 2.10. (1) Suppose that $L(s) = 0$. Then $C_s C_w = C_{sw}$.
(2) Suppose that $L(s) > 0$. Then

$$C_s C_w = \begin{cases} C_{sw} + \sum_{z: z < z < w} \mu^s_{z, w} C_z & (sw > w) \\ \tau^{-1}(s) + v^{-L(s)} C_w & (sw < w) \end{cases}$$

where $\mu^s_{z, w} \in A$ are bar invariant elements inductively defined by

$$\sum_{y < z < w, s < z} p_{y, z} \mu^s_{z, w} - v^{-L(s)} p_{y, w} \in v^{-1} \mathbb{Z}[v^{-1}],$$

for $y, w \in W$ such that $sy < y < w < sw$.

Theorem 2.10. (3) Suppose that $L(s) < 0$. Then

$$C_s C_w = \begin{cases} C_{sw} + \sum_{z: z < z < w} \mu^s_{z, w} C_z & (sw > w) \\ -(v^{-L(s)} + v^{-L(s)}) C_w & (sw < w) \end{cases}$$

where $\mu^s_{z, w} \in A$ are bar invariant elements inductively defined by

$$\sum_{y < z < w, s < z} p_{y, z} \mu^s_{z, w} + v^{-L(s)} p_{y, w} \in v^{-1} \mathbb{Z}[v^{-1}],$$

for $y, w \in W$ such that $sy < y < w < sw$.

$\mathcal{H}(W, S, L)$ has an $A$-linear antiautomorphism $\tau$ defined by $\tau(T_s) = T_{s^{-1}}$, for $s \in S$. Then, $\tau(T_w) = T_{w^{-1}}$, and $\tau(C_w) = C_{w^{-1}}$, thus $h_{x, y, z} = h_{y^{-1}, x^{-1}, z^{-1}}$ follows. In particular, we have $a(z) = a(z^{-1})$, for all $z \in W$. The next proposition is from [33, 8.4, 13.7, 13.8].

Proposition 2.11. Suppose that $L(s) > 0$, for all $s \in S$.

(1) $\oplus_{w: s < w} A C_w$ and $\oplus_{w: s < w} A C_w$ are left and right ideal of $\mathcal{H}(W, S, L)$, and $a(1) = 0$ follows.

(2) If $1 \neq z \in W$ then $a(z) \geq \min\{L(s) \mid s \in S\} > 0$.

(3) Suppose that $W$ is finite. Then $a(w) \leq L(w_0)$, for all $w \in W$, and the equality holds if and only if $w = w_0$.

Example 2.12. Let $(W, S)$ be of type $B_2$, and set $a = L(s_1), b = L(s_2)$. We consider the case $a > b > 0$. Write $T_1 = T_{s_1}$ and $T_2 = T_{s_2}$. Then

$$C_{s_1} = T_1 + v^{-a}, \quad C_{s_2} = T_2 + v^{-b}.$$ 

Further, nonexistence of $z$ with $s_1 z < z < s_2$ implies that

$$C_{s_1 s_2} = C_{s_1} C_{s_2} = T_1 T_2 + v^{-b} T_1 + v^{-a} T_2 + v^{-a-b}.$$ 

Similarly, we have

$$C_{s_2 s_1} = C_{s_2} C_{s_1} = T_2 T_1 + v^{-b} T_1 + v^{-a} T_2 + v^{-a-b}.$$ 

Explicit computation shows that

$$C_{s_1} C_{s_2 s_1} = T_1 T_2 T_1 + v^{-a} (T_1 T_2 + T_2 T_1) + v^{-2a} T_2 + (v^{-a-b} + v^{-a-b}) C_{s_1}.$$ 

Since $a - b > 0$ we subtract $(v^{-a-b} + v^{-a-b}) C_{s_1}$ to obtain

$$C_{s_1 s_2 s_1} = T_1 T_2 T_1 + v^{-a} (T_1 T_2 + T_2 T_1) + v^{-2a} T_2 + (v^{-a-b} + v^{-a-b}) C_{s_1},$$ 

and $C_{s_1} C_{s_2 s_1} = C_{s_1 s_2 s_1} + (v^{-a-b} + v^{-a-b}) C_{s_1}$. Similarly, we have

$$C_{s_2} C_{s_1 s_2} = T_2 T_1 T_2 + v^{-b} (T_1 T_2 + T_2 T_1) + v^{-2b} T_1 + (v^{-a-b} + v^{-a-b}) C_{s_2}.$$
and deduce $C_s C_{s_1 s_2} = C_{s_2 s_1 s_2}$.

Now, consider the longest element $w_0 = s_1 s_2 s_1 s_2 = s_2 s_1 s_2 s_1$. We have

$$C_{w_0} = T_1 T_2 T_1 + v - b T_1 T_2 T_1 + v - a T_2 T_1 T_2 + v - a - b (T_1 T_2 + T_2 T_1) + v - a - 2b T_1 + v - 2a - b T_2 + v - 2a - 2b.$$  

Then we get $C_{w_0}$ by explicit computation again. Applying $\tau$ we obtain $C_{w_0} = C_{w_0}$.

To summarize, $C_s C_{w_0} = (v^L(s) + v^{-L(s)}) C_{w_0}$, for $s \in S$, and we have

$$C_{s_1} C_{s_2} = C_{s_2} C_{s_1}, C_{s_2} C_{s_2} = C_{s_2}, C_{s_2} C_{s_2} = C_{s_2}.$$ 

and

$$C_{s_2} C_{s_2} = C_{w_0}, C_{s_1} C_{s_2} = C_{w_0} + (v^a - b + v^{-a} + b) C_{s_1 s_2}.$$ 

We have also obtained $C_{s_2} = C_{w_0} + (v^a - b + v^{-a} + b) C_{s_1 s_2}$. Using these formulas, we may further obtain the following.

$$C_{s_1 s_2} C_{s_2} = (v^b + v^{-b}) C_{s_1 s_2} + (v^b + v^{-b}) (v^a - b + v^{-a} + b) C_{s_1}$$

$$C_{s_1 s_2} = (v^b + v^{-b}) C_{w_0}$$

$$C_{s_1 s_2} C_{s_2} = (v^b + v^{-b}) C_{w_0} + (v^b + v^{-b}) (v^a - b + v^{-a} + b) C_{s_1 s_2}$$

$$C_{s_1 s_2} = C_{w_0} + (v^a - b + v^{-a} + b) C_{s_1 s_2}$$

$$C_{s_2 s_1} = C_{w_0} + (v^a - b + v^{-a} + b) C_{s_1 s_2}$$

$$C_{s_2 s_1} = C_{w_0} + (v^a + v^{-a}) C_{s_1}$$

$$C_{s_2 s_1} = C_{w_0} + (v^a + v^{-a}) C_{s_1 s_2}$$

$$C_{s_2 s_1} = (v^b + v^{-b}) C_{w_0} + (v^a - b + v^{-a} + b) C_{s_2 s_1 s_2}$$

$$C_{s_2 s_1} = (v^b + v^{-b}) C_{w_0} + (v^a + v^{-a}) C_{s_1 s_2}$$

$$C_{s_2 s_1} = (v^b + v^{-b}) C_{w_0} + (v^a + v^{-a}) C_{s_1 s_2}$$

Let $\Gamma_1 = \{1\}$, $\Gamma_2 = \{s_1, s_2 s_1\}$, $\Gamma_3 = \{s_2\}$, $\Gamma_4 = \{s_1 s_2, s_2 s_1 s_2\}$, $\Gamma_5 = \{s_1 s_2 s_1\}$, $\Gamma_6 = \{w_0\}$. We define $I_{\leq 1} = \mathcal{H}(W, S, L)$ and

$$I_{\leq 2} = \sum_{w \in \Gamma_2 \cup \Gamma_3 \cup \Gamma_4} AC_w,$$

$$I_{\leq 3} = \sum_{w \in \Gamma_3 \cup \Gamma_4 \cup \Gamma_5} AC_w,$$

$$I_{\leq 4} = \sum_{w \in \Gamma_4 \cup \Gamma_5 \cup \Gamma_6} AC_w,$$

$$I_{\leq 5} = \sum_{w \in \Gamma_5 \cup \Gamma_6} AC_w,$$

$$I_{\leq 6} = \sum_{w \in \Gamma_6} AC_w.$$  

These are left ideals. Further,

$$\mathcal{H}(W, S, L) = I_{\leq 1} \supseteq I_{\leq 2} \supseteq I_{\leq 3} \supseteq I_{\leq 4} \supseteq I_{\leq 5} \supseteq I_{\leq 6}$$

is a filtration by two-sided ideals. The sets $\Gamma_1, \Gamma_2 \cup \Gamma_4, \Gamma_3, \Gamma_5, \Gamma_6$ are called two-sided cells, and we may confirm that the $a$-function is constant on each two-sided cell. Namely, $a(z) = 0$ if $z \in \Gamma_1$, $a(z) = a$ if $z \in \Gamma_2 \cup \Gamma_4$, $a(z) = b$ if $z \in \Gamma_3$, $a(z) = 2a - b$ if $z \in \Gamma_5$, $a(z) = 2a + 2b$ if $z \in \Gamma_6$. This is a general phenomenon.

In the rest of the paper, we assume that

$W$ is finite and $L(s) > 0$, for all $s \in S$. 

**Definition 2.13.** Let $R$ be a commutative domain, $A \rightarrow R$ a ring homomorphism. Then we define $\mathcal{H}_R = \mathcal{H}(W, S, L) \otimes R$. We denote the image of $v \in A$ by $q^{1/2} \in R$.

It is known that $\mathcal{H}_R$ may be defined by the same defining relations as $\mathcal{H}(W, S, L)$.

$\mathcal{H}_R$ has a trace map $\text{tr} : \mathcal{H} \rightarrow A$ defined by $\text{tr}(T_w) = 0$ if $w \neq 1$, $\text{tr}(T_1) = 1$. For the symmetric bilinear form defined by $\langle h, h' \rangle = \text{tr}(hh')$, we have $\langle T_w^{-1}, T_w \rangle = \delta_{yw}$.

Hence, if $R$ is a field then $\mathcal{H}_R$ is a symmetric algebra.

Let $K = \mathbb{Q}(v)$ as before. Then $\mathcal{H}_K$ is split semisimple. The simple $\mathcal{H}_K$-modules are in bijection with $\text{Irr}(W)$, and we denote by $\{V^E \mid E \in \text{Irr}(W)\}$ the complete set of simple $\mathcal{H}_K$-modules. Then

$$\sum_{w \in W} \text{Tr}(T_w^{-1}, V^E)T_w$$

is a central element of $\mathcal{H}_K$, which acts on $V^E$ by the scalar

$$c_E = \frac{1}{\dim E} \sum_{w \in W} \text{Tr}(T_w^{-1}, V^E) \text{Tr}(T_w, V^E).$$

Observe that $A$ is integrally closed in $K$. The following is proved in [7.3.8].

**Proposition 2.14.** $\text{Tr}(T_w, V^E) \in A$, for $w \in W$ and $E \in \text{Irr}(W)$.

**Definition 2.15.** The $a$-invariant $a_E$, for $E \in \text{Irr}(W)$, is defined by

$$a_E = \min \{i \in \mathbb{Z}_{\geq 0} \mid v^i \text{Tr}(T_w, V^E) \in \mathbb{Z}[v], \text{ for all } w \in W.\}$$

**Proposition 2.16** (Lusztig). We may write $c_E = f_E v^{-2a_E} + \text{(higher terms)}$, for some $f_E \in \mathbb{Z}_{>0}$.

$c_E$ are called the Schur elements and we have

$$\text{tr} = \sum_{E \in \text{Irr}(W)} \frac{1}{c_E} \text{Tr}(-, V^E).$$

This result suggests that we may define $a_E$ for more general $A$-algebras that has a trace map. In fact, Jacon developed a theory of $a$-invariants for the cyclotomic Hecke algebra of type $(d, 1, n)$, which is also called the AK-algebra.

**Definition 2.17.** A field $R$ is $L$-good if $f_E$ is invertible in $R$, for all $E \in \text{Irr}(W)$.

The irreducible characters of generic Hecke algebras are explicitly known. Hence we may compute $f_E$ by substituting the parameters with $v^{L(s)}$ and expand $c_E$ into the Laurent series in $v$. When $L$ is the length function then we have ordinary notion of good primes. The following result of Geck shows that most primes are $L$-good.

**Lemma 2.18.** Suppose that $L(s) > 0$, for all $s \in S$. If the characteristic of $R$ is a good prime then $R$ is $L$-good.

In particular, if the characteristic of $R$ is different from 2, 3, 5 then $R$ is $L$-good. The following example, which is called the asymptotic case, was studied by [13, 14] and [16].

**Example 2.19.** Let $W$ be the Weyl group of type $B_n$. The Coxeter generators are denoted by $s_0, \ldots, s_{n-1}$ such that $s_1, \ldots, s_{n-1}$ generate the symmetric group of degree $n$. Write $L(s_0) = b, L(s_1) = \cdots = L(s_{n-1}) = a$ and suppose that $b > (n-1)a > 0$. Then $f_E = 1$, for all $E \in \text{Irr}(W)$, and all fields are $L$-good. On the other hand, 2 is a bad prime.
2.2. Cellularity.

**Definition 2.20.** Let $R$ be a commutative domain, $A$ an $R$-algebra. $A$ is called **cellular** if there exist a finite poset $\Lambda$, a collection of finite sets $\{M(\lambda) \mid \lambda \in \Lambda\}$ and an $R$-linear antiautomorphism of $A$ which is denoted by $a \mapsto a^*$, for $a \in A$, such that

1. $A$ has a free $R$-basis $\cup_{\lambda \in \Lambda} \{C^\lambda_{ST} \mid S, T \in M(\lambda)\}$.
2. $(C^\lambda_{ST})^* = C^\lambda_{TS}$.
3. $A^{\lambda \mu} = \sum_{\mu \geq \lambda} \sum_{S,T \in M(\mu)} RC^\mu_{ST}$ is a two-sided ideal of $A$, for all $\lambda \in \Lambda$.
4. For each $h \in A$, $S, T \in M(\lambda)$, there exist $r(h, S, T) \in R$ such that we have
   \[ hC^\lambda_{SU} = \sum_{T \in M(\lambda)} r(h, S, T)C^\lambda_{TU} \mod A^{\lambda \mu}, \]
   for all $U \in M(\lambda)$.

König and Changchang Xi [31] showed that an $R$-algebra is cellular if and only if it is obtained from a particular construction which is called the iterated inflation of finitely many copies of $R$.

**Definition 2.21.** Assume that an $R$-algebra $A$ is cellular. Define an $A$-module $C^\lambda = \oplus_{S \in M(\lambda)} RC^\lambda_S$ by

\[ hC^\lambda_S = \sum_{T \in M(\lambda)} r(h, S, T)C^\lambda_T, \quad \text{for } h \in A. \]

$C^\lambda$ is called a **cell module**. $C^\lambda$ is equipped with a bilinear from defined by

\[ C^\lambda_U C^\lambda_V = \langle C^\lambda_S, C^\lambda_T \rangle C^\lambda_{UV} \mod A^{\lambda \mu}. \]

The radical $\text{Rad}_{\langle \; , \; \rangle} C^\lambda$ of the bilinear form is an $A$-submodule. Define $D^\lambda$ by

\[ D^\lambda = C^\lambda / \text{Rad}_{\langle \; , \; \rangle} C^\lambda. \]

The following are basic results on cellular algebras. See [22, 32] and [31].

**Theorem 2.22** (Graham-Lehrer). Let $R$ be a field, $A$ a cellular $R$-algebra.

(i) Nonzero $D^\lambda$'s form a complete set of simple $A$-modules. Further, if $D^\lambda \neq 0$ then it is absolutely irreducible and the Jacobson radical $C^\lambda$ coincides with $\text{Rad}_{\langle \; , \; \rangle} C^\lambda$. In particular, $\{C^\lambda \mid \lambda \in \Lambda\}$ is a complete set of simple $A$-modules if $A$ is semisimple.

(ii) If $D^\lambda \neq 0$, for all $\lambda \in \Lambda$, then $A$ is quasihereditary. In particular, $A$ has finite global dimension in this case.

(iii) Let $\Lambda^0 = \{\lambda \in \Lambda \mid D^\lambda \neq 0\}$, $D = ([C^\lambda : D^\mu])_{(\lambda,\mu)\in\Lambda^0\times\Lambda^0}$ the decomposition matrix. Then $D$ is unitriangular: $d_{\lambda \mu} \neq 0$ only if $\lambda \geq \mu$ and $d_{\mu \mu} = 1$.

(iv) The Cartan matrix of $A$ is of the form $C = tDD$.

**Theorem 2.23** (König-Xi). Let $R$ be a field, $A$ a cellular $R$-algebra.

(i) If $A$ is self-injective then it is weakly symmetric, that is, the head and the socle of any indecomposable projective $A$-module are isomorphic.

(ii) Assume that the characteristic of $R$ is odd. If another $R$-algebra $B$ is Morita equivalent to $A$ then $B$ is cellular.

Recall that the trivial extension $T(A) = A \oplus \text{Hom}_R(A, R)$ is a symmetric algebra whose bilinear form is given by $\langle a \oplus f, b \oplus g \rangle = f(b) + g(a)$, for $a, b \in A$ and $f, g \in \text{Hom}_R(A, R)$. 
**Theorem 2.24** (Xi-Xiang). Let $R$ be a field, $A$ a cellular $R$-algebra. Then we may define an antiautomorphism of $T(A)$ by $(a \oplus f)^* = a^* \oplus f^*$, where $f^*(b) = f(b^*)$, for $b \in A$, such that $T(A)$ is a cellular $R$-algebra. In particular, any cellular $R$-algebra is a quotient of a symmetric cellular $R$-algebra.

Now we return to Hecke algebras and state a result by Geck [14]. In fact, it is proved under more general assumption that part of the Lusztig conjectures namely (P2)-(P8) and (P15') [13, 5.2] hold. These are conjectures are about the structure constants $h_{x,y,z}$, and do not involve the base ring $R$. It is known that the conjectures hold when $L$ is the length function. In this case, $\mathcal{H}(W,S,L)^{\geq a} = \sum_{w \in W, \ell(w) \geq a} \mathcal{A}C_w$ is a two-sided ideal of $\mathcal{H}(W,S,L)$, for all $a \in \mathbb{Z}$. Then each successive quotient is a direct sum of Lusztig’s two-sided cells. By refining the ideal filtration, Geck has proved the following. Recall that $\mathcal{H}_R = \mathcal{H}(W,S,L) \otimes R$.

**Theorem 2.25** (Geck). Let $(W,S,L = \ell)$ be a weighted Coxeter group of finite type whose $L$ is the length function. Suppose that $R$ is $L$-good. Then $\mathcal{H}_R$ is cellular.

This opens a way to consider the possibilities to find analogues of many results that appeared in Specht module theory. For example, studying Young modules in this setting is important.

### 2.3. Canonical Basic Set

The first task in studying the modular representation theory of $\mathcal{H}_R$ is to determine the set of simple $\mathcal{H}_R$-modules. This very first task has already proven to be very interesting and deep.

Let $R$ be a field, $A \rightarrow R$ an algebra homomorphism. Its kernel is a prime ideal $\mathfrak{p}$ of $A$ and we may consider modular reduction between $\mathcal{H}_A$ and $\mathcal{H}_R$. For a simple $\mathcal{H}_R$-module, we denote by $[E : S]$ the multiplicity of $S$ in the modular reduction of $V_E$.

**Definition 2.26.** For a simple $\mathcal{H}_R$-module $S$, the $a$-invariant of $S$ is defined by

$$a_S = \min \{a_E \mid E \in \text{Irr}(W) \text{ such that } [E : S] \neq 0.\}$$

Geck and Rouquier used the $a$-invariant to label simple $\mathcal{H}_R$-modules.

**Definition 2.27.** We say that a subset $B$ of $\text{Irr}(W)$ is a canonical basic set if

(i) There is a bijection $B \simeq \text{Irr}(\mathcal{H}_R)$, which we denote $E \mapsto S_E$, such that $[E : S_E] = 1$ and $a_{S_E} = a_E$.

(ii) If a simple $\mathcal{H}_R$-module $S$ satisfies $[E : S] \neq 0$, for some $E \in \text{Irr}(W)$, then either $E \in B$ and $S \simeq S_E$ or $a_S < a_E$.

If a canonical basic set exists, then $E \notin B$ implies that $a_S < a_E$, for all $S$ with $[E : S] \neq 0$, and we have

$$B = \{E \in \text{Irr}(W) \mid a_E = a_S \text{ and } [E : S] \neq 0, \text{ for some } S \in \text{Irr}(\mathcal{H}_R).\}.$$  

As the right hand side is independent of the choice of $B$, we have the uniqueness of the canonical basic set. Note however that it may not exist. Geck showed that under the assumption that the Lusztig conjectures hold, the canonical basic set exists if $R$ is $L$-good. In particular, it implies the following result, which was proved in the early stage of their research.

**Theorem 2.28** (Geck-Rouquier). Suppose that $L$ is the length function and that $R$ is $L$-good. Then the canonical basic set exists.
In fact, it is a corollary of Theorem 2.25 and the canonical basic set is nothing but the index set of simple $\mathcal{H}_R$-modules given by the cellular algebra structure. In type $B_n$ we have a result for arbitrary $L$ by Geck and Jacon [17].

**Theorem 2.29** (Geck-Jacon). Let $\mathcal{H}_R = \mathcal{H}(W, S, L) \otimes R$ be of type $B_n$. Assume that the characteristic of $R$ is $0$. Then the canonical basic set exists.

The existence is still conjectural in positive characteristics unless $L$ is (a positive multiple of) the length function.

2.4. A categorification of integrable $U_v(\hat{\mathfrak{sl}}_e)$-modules. The canonical basic set is determined by Geck and Jacon for all types. Let us focus on Hecke algebras of type $B_n$. As we will explain in the next section for more general cyclotomic Hecke algebras, the author and Mathas used Kashiwara crystal for labelling simple $\mathcal{H}_R$-modules. The set of bipartitions that appeared in this labelling, which we call Kleshchev bipartitions, is a realization of the highest weight crystal $B(\Lambda)$ whose highest weight $\Lambda$ is determined by the parameters of $\mathcal{H}_R$. On the other hand, there was a different realization of the crystal $B(\Lambda)$ by Jimbo-Misra-Miwa-Okado in a purely solvable lattice model context, and Foda, Leclerc, Okado, Thibon and Welsh proposed another labelling of simple $\mathcal{H}_R$-modules that uses it. Then, Jacon found that the canonical basic set in type $B_n$ is precisely the set of Jimbo-Misra-Miwa-Okado bipartitions. Hence, the set of JMMO bipartitions gives module theoretic realization of the labeling of simple $\mathcal{H}_R$-modules proposed by Foda et al.

In the labelling by Kleshchev bipartitions, we used Specht module theory which was developed by Dipper, James and Murphy; they gave a cellular algebra structure on $\mathcal{H}_R = \mathcal{H}(W(B_n), S, L) \otimes R$, for any $L$. We showed that the labelling of simple $\mathcal{H}_R$-modules by Kleshchev bipartitions is nothing but the labelling induced by the cellular structure. Difference of Kleshchev and JMMO bipartitions is caused by the difference of the orders given on the nodes of bipartitions, but we seek for explanations in the representation theory of Hecke algebras why two (or more) different labelling sets appear naturally. The key seems to be various choices of the logarithm of the parameters of $\mathcal{H}_R$. It is now conjectured [9] that the choice would give a cellular algebra structure on $\mathcal{H}_R$ which is given by Kazhdan-Lusztig cells, and a parametrizing set of simple $\mathcal{H}_R$-modules, which we call Uglov bipartitions, as the one induced by the cellular algebra structure. There are two supporting evidences. Geck, Iancu and Pallikaros [16] showed that our labelling by Kleshchev bipartitions may be considered as a special case of this scheme, and Geck and Jacon [17] showed that the set of Uglov bipartitions is the canonical basic set, for any $L$, when the characteristic of $R$ is $0$.

This search for various cellular algebra structures may be viewed as a search for a categorification of integrable highest weight $U_v(\hat{\mathfrak{sl}}_e)$-modules $V_v(\Lambda)$ with two specializations at $v = 0$ and $v = 1$. Here, by specialization at $v = 0$ we mean the crystal $B(\Lambda)$, and specialization at $v = 1$ we mean the integrable highest weight $\hat{\mathfrak{sl}}_e$-module $V(\Lambda)$. Let $\mathcal{F}(\Lambda)$ be the higher level Fock space with highest weight $\Lambda$. It is the tensor product of $\mathcal{F}(\Lambda_m)$ which will be introduced in 3.2 below and the basis is given by multipartitions. Then we have the following diagram.

$$
B(\Lambda) \xleftarrow{v=0} V_v(\Lambda) \xrightarrow{v=1} V(\Lambda) \subseteq \mathcal{F}(\Lambda).
$$
We fix an embedding of $V_v(\Lambda)$ into one of various JMMO deformed Fock spaces $\mathcal{F}_v(\Lambda)_\mu$ in the middle, and realize $B(\Lambda)$ on the set of Uglov multipartitions on the left, and we categorify them. Then our categorification is to replace each weight space $V_v(\Lambda)_\mu$ of $V_v(\Lambda)$ with module category of a cellular algebra $A_\mu$, whose poset is the set of multipartitions which belong to $\mathcal{F}(\Lambda)_\mu$ such that

(i) the set $B(\Lambda)_\mu$ of Uglov multipartitions on the left coincides the parametrizing set of $\text{Irr}(A_\mu)$ induced by the cellular algebra structure on $A_\mu$,

(ii) $V(\Lambda)_\mu$ on the right coincides $\text{Hom}_Z(K_0(A_\mu\text{-modules}), \mathbb{C})$,

(iii) the embedding $V(\Lambda)_\mu \subseteq \mathcal{F}(\Lambda)_\mu$ coincides the dual of the decomposition map,

(iv) the Chevalley generators $e_i, f_i$ lift to functors among the module categories. Our candidates for $A_\mu$ are block algebras of cyclotomic Hecke algebras.

We expect to categorify the higher level Fock space $\mathcal{F}(\Lambda)$ itself by Rouquier’s theory of quasihereditary covers which uses rational Cherednik algebras associated with a root datum. Let $\Lambda$ to (iv) and

(iii) the embedding $V(\Lambda)_\mu \subseteq \mathcal{F}(\Lambda)_\mu$ coincides the dual of the decomposition map,

(iv) the Chevalley generators $e_i, f_i$ lift to functors among the module categories.

The last part is Yvonne’s conjecture. We said that the logarithm of the parameters of the Hecke algebra seems to control the cellular structure. Here, the logarithm appears as the parameters of the rational Cherednik algebra, and what we expect in (v) is that the quasihereditary algebra structure should induce the cellular algebra structure on the Hecke algebra.

3. The category of crystals

3.1. Kashiwara crystal. Let us recall the definition.

**Definition 3.1.** Let $A = (a_{ij})_{i,j \in I}$ be a generalized Cartan matrix,

$$(A, \Pi = \{\alpha_i\}_{i \in I}, \Pi' = \{h_i\}_{i \in I}, P, P' = \text{Hom}_Z(P, \mathbb{Z}))$$

a root datum. Let $\mathfrak{g} = \mathfrak{g}(A)$ be the Kac-Moody algebra associated with $A$. A set $B$ is a $\mathfrak{g}$-crystal if it is equipped with maps $\text{wt} : B \to P, \tilde{e}_i, \tilde{f}_i : B \to B \cup \{0\}$, $\epsilon_i, \varphi_i : B \to \mathbb{Z} \cup \{-\infty\}$ such that

1. $\varphi_i(b) = \epsilon_i(b) + \langle h_i, \text{wt}(b) \rangle$.

2. If $\bar{e}_i b \in B$ then

$$\text{wt}(\bar{e}_i b) = \text{wt}(b) + \alpha_i, \quad \epsilon_i(\bar{e}_i b) = \epsilon_i(b) - 1, \quad \varphi_i(\bar{e}_i b) = \varphi_i(b) + 1.$$  

3. If $\bar{f}_i b \in B$ then

$$\text{wt}(\bar{f}_i b) = \text{wt}(b) - \alpha_i, \quad \epsilon_i(\bar{f}_i b) = \epsilon_i(b) + 1, \quad \varphi_i(\bar{f}_i b) = \varphi_i(b) - 1.$$  

4. Let $b, b' \in B$. Then $\bar{f}_i b = b'$ if and only if $\bar{e}_i b' = b$.

1. They are not tensor product of $\mathcal{F}_v(\Lambda_m)$. 
We call this graph the crystal graph.

Definition 3.3. Let $\mathbf{g} = sl_2$, $\alpha$ the positive root, $\omega = \alpha/2$ the fundamental weight. Let $B(n\omega) = \{u_0, u_1, \ldots, u_n\}$ and define

$$\text{wt}(u_k) = n\omega - k\alpha, \quad \epsilon(u_k) = k, \quad \varphi(u_k) = n - k$$

and

$$\tilde{e}u_k = \begin{cases} u_{k-1} & (k > 0) \\ 0 & (k = 0) \end{cases}, \quad \tilde{f}u_k = \begin{cases} u_{k+1} & (k < n) \\ 0 & (k = n) \end{cases}.$$ 

Next, let $B(\infty) = \{u_k | k \in \mathbb{Z}_{\geq 0}\}$ and define

$$\text{wt}(u_k) = -k\alpha, \quad \epsilon(u_k) = k, \quad \varphi(u_k) = -k$$

and

$$\tilde{e}u_k = \begin{cases} u_{k-1} & (k > 0) \\ 0 & (k = 0) \end{cases}, \quad \tilde{f}u_k = u_{k+1}.$$

Then, $B(n\omega)$ and $B(\infty)$ are $\mathbf{g}$-crystals.

Example 3.2. (1) Let $g = sl_2$, $\alpha$ the positive root, $\omega = \alpha/2$ the fundamental weight. Let $B(n\omega) = \{u_0, u_1, \ldots, u_n\}$ and define

$$\text{wt}(u_k) = n\omega - k\alpha, \quad \epsilon(u_k) = k, \quad \varphi(u_k) = n - k$$

and

$$\tilde{e}u_k = \begin{cases} u_{k-1} & (k > 0) \\ 0 & (k = 0) \end{cases}, \quad \tilde{f}u_k = \begin{cases} u_{k+1} & (k < n) \\ 0 & (k = n) \end{cases}.$$ 

Next, let $B(\infty) = \{u_k | k \in \mathbb{Z}_{\geq 0}\}$ and define

$$\text{wt}(u_k) = -k\alpha, \quad \epsilon(u_k) = k, \quad \varphi(u_k) = -k$$

and

$$\tilde{e}u_k = \begin{cases} u_{k-1} & (k > 0) \\ 0 & (k = 0) \end{cases}, \quad \tilde{f}u_k = u_{k+1}.$$

Then, $B(n\omega)$ and $B(\infty)$ are $\mathbf{g}$-crystals.

(2) Let $B_i = \{b_i(a) | a \in \mathbb{Z}\}$. Define, for $a \in \mathbb{Z}$, $\text{wt}(b_i(a)) = a\alpha_i$,

$$\epsilon_j(b_i(a)) = \begin{cases} -a & (j = i) \\ -\infty & (j \neq i) \end{cases}, \quad \varphi_j(b_i(a)) = \begin{cases} a & (j = i) \\ -\infty & (j \neq i) \end{cases}$$

and

$$\tilde{e}_j(b_i(a)) = \begin{cases} b_i(a+1) & (j = i) \\ 0 & (j \neq i) \end{cases}, \quad \tilde{f}_j(b_i(a)) = \begin{cases} b_i(a-1) & (j = i) \\ 0 & (j \neq i) \end{cases}.$$ 

Then $B_i$ is a $\mathbf{g}$-crystal.

(3) Let $\Lambda \in P$ and $T_\Lambda = \{t_\Lambda\}$. Define

$$\text{wt}(t_\Lambda) = \Lambda, \quad \epsilon_i(t_\Lambda) = \varphi_i(t_\Lambda) = -\infty, \quad \tilde{e}_i t_\Lambda = \tilde{f}_i t_\Lambda = 0.$$ 

Then $T_\Lambda$ is a $\mathbf{g}$-crystal.

Definition 3.3. Let $B_1, B_2$ be $\mathbf{g}$-crystals. A crystal morphism is a map

$$f : B_1 \sqcup \{0\} \rightarrow B_2 \sqcup \{0\}$$

such that

(i) $f(0) = 0$.

(ii) Suppose that $b \in B_1$ and $f(b) \in B_2$. Then

$$\text{wt}(f(b)) = \text{wt}(b), \quad \epsilon_i(f(b)) = \epsilon_i(b), \quad \varphi_i(f(b)) = \varphi_i(b).$$

(iii) Suppose that $b, b' \in B_1$ and $f(b), f(b') \in B_2$. If $\tilde{f}_i b = b'$ then $\tilde{f}_i f(b) = f(b')$.

(iv) Suppose that $b, b' \in B_1$ and $f(b) = 0, f(b') \neq 0$. If $b = \tilde{e}_i b'$ (resp. $b = \tilde{f}_i b'$) then $\tilde{e}_i f(b') = 0$ (resp. $\tilde{f}_i f(b') = 0$).

If $f$ is injective, we say that $f$ is an embedding. If $f$ is bijective then we say that $f$ is an isomorphism. For example, the identity map is an isomorphism.
Remark 3.4. The definition of crystal morphism in [24], [26] and [28] are all different. In [25], which follows [27], (iv) is dropped. In [28], $f$ is assumed to map $B_1$ to $B_2$. Let us consider $B(2\omega) = \{u_0, u_1, u_2\}$ in Example 3.2(1). The map $f : B(2\omega) \cup \{0\} \to B(2\omega) \cup \{0\}$ defined by $f(0) = 0$, $f(u_i) = u_i$ for $i = 0, 1$, and $f(u_2) = 0$ satisfies (i), (ii) and (iii) but not (iv). As $B(2\omega)$ corresponds to the irreducible highest weight module $V_\omega(2\omega)$, we would like that the identity map is the only crystal endomorphism of $B(2\omega)$.

Note that a crystal morphism $f$ may not commute with $\tilde{e}_i$ and $\tilde{f}_i$. If $f$ commutes with them, we say that $f$ is a strict crystal morphism. The strictness further requires

(v) Suppose that $\tilde{e}_ib = 0$ (resp. $\tilde{f}_ib = 0$). Then $\tilde{e}_if(b) = 0$ (resp. $\tilde{f}_if(b) = 0$).

Example 3.5. Let $B$ be a $\mathfrak{g}$-crystal. Define a new crystal $(B, wt^\sigma, \tilde{e}_i^\sigma, \tilde{f}_i^\sigma, \tilde{\varphi}_i, \varphi_i)$ by

$$
wt^\sigma(b) = wt(\sigma^{-1}(b)), \quad \tilde{e}_i^\sigma(b) = \epsilon_i(\sigma^{-1}(b)), \quad \tilde{\varphi}_i(b) = \varphi_i(\sigma^{-1}(b)),
$$

$$
\tilde{f}_i^\sigma(b) = \sigma \tilde{e}_i \sigma^{-1}(b), \quad \tilde{f}_i^\sigma(b) = \sigma \tilde{f}_i^{-1}(b),
$$

where $\sigma : B \to B$ is a permutation. Then $f : B \cup \{0\} \to B \cup \{0\}$ defined by $f(0) = 0$ and $f(b) = \sigma(b)$ ($b \in B$) is an isomorphism, which is strict. Hence, if $B$ is given two crystal structures which are isomorphic, it does not mean that $\tilde{e}_i$ and $\tilde{f}_i$ of the two crystal structures coincide.

Example 3.6. Let $\mathfrak{g} = sl_2$ and $(B(\infty), wt, \tilde{e}, \tilde{f}, \epsilon, \varphi)$ as above. Define a new crystal

$$
B(\infty) \otimes T_{n\omega} = (B(\infty), wt + n\omega, \tilde{e}, \tilde{f}, \epsilon, \varphi + n).
$$

(1) The map $f : B(n\omega) \cup \{0\} \to B(\infty) \otimes T_{n\omega} \cup \{0\}$ defined by $f(0) = 0$ and $f(u_k) = u_k$, for $0 \leq k \leq n$, is a crystal morphism. However, $\tilde{f}u_n = 0$ in $B(n\omega)$ and $\tilde{f}u_n = u_{n+1} \neq 0$ in $B(\infty) \otimes T_{n\omega}$. Thus the morphism is not strict.

(2) The map $f : B(\infty) \otimes T_{n\omega} \cup \{0\} \to B(n\omega) \cup \{0\}$ defined by

$$
f(u_k) = \begin{cases} u_k & (k \leq n) \\ 0 & (k > n) \end{cases}
$$

is a strict crystal morphism.

Crystals and morphisms among them form a category, which are called the category of $\mathfrak{g}$-crystals. We have the notion of tensor product in the category.

Definition 3.7. Let $B_1, B_2$ be $\mathfrak{g}$-crystals. The tensor product $B_1 \otimes B_2$ is the set $B_1 \times B_2$ equipped with the crystal structure defined by

(1) $wt(b_1 \otimes b_2) = wt(b_1) + wt(b_2)$.

(2) $\tilde{e}_i(b_1 \otimes b_2) = \tilde{e}_ib_1 \otimes b_2$ if $\varphi_i(b_1) \geq \epsilon_i(b_2)$, $b_1 \otimes \tilde{e}_ib_2$ otherwise.

(3) $\tilde{f}_i(b_1 \otimes b_2) = \tilde{f}_ib_1 \otimes b_2$ if $\varphi_i(b_1) \geq \epsilon_i(b_2)$, $b_1 \otimes \tilde{f}_ib_2$ otherwise.

(4) $\epsilon_i(b_1 \otimes b_2) = \max\{\epsilon_i(b_1), \epsilon_i(b_2) - \langle h_i, wt(b_1) \rangle\}$.

(5) $\varphi_i(b_1 \otimes b_2) = \max\{\varphi_i(b_1) + \langle h_i, wt(b_2) \rangle, \varphi_i(b_2)\}$.

Example 3.8. Let $B$ be a $\mathfrak{g}$-crystal, $\Lambda \in P$. Then $B \otimes T_\Lambda$ is a crystal with the same $\tilde{e}_i, \tilde{f}_i, \epsilon_i$ as $B$ but wt and $\varphi_i$ are shifted by $\Lambda$ and $\Lambda(h_i)$. 

Recall that a monoidal category $\mathcal{C}$ is a category with a bifunctor $\mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$, an object $I$ called the unit object, such that natural isomorphisms

\[
\alpha_{B,B,B} : (B \otimes B_2) \otimes B_3 \simeq B_1 \otimes (B_2 \otimes B_3) \\
\lambda_B : I \otimes B \simeq B, \quad \rho_B : B \otimes I \simeq B
\]

satisfy axioms for $B_1 \otimes B_2 \otimes B_3 \otimes B_4$, $B_1 \otimes I \otimes B_2 \simeq B_1 \otimes B_2$ (the pentagon axiom and the triangle axiom) and $\lambda_I = \rho_I : I \otimes I \simeq I$.

For a crystal morphism $f : B_1 \rightarrow B_2$, we have crystal morphisms $B \otimes B_1 \rightarrow B \otimes B_2$ and $B_1 \otimes B \rightarrow B_2 \otimes B$ given by $b \otimes b' \mapsto b \otimes f(b')$ and $b \otimes b' \mapsto f(b) \otimes b'$, and the tensor product defines a bifunctor. The identity map gives a natural isomorphism $(B_1 \otimes B_2) \otimes B_3 \simeq B_1 \otimes (B_2 \otimes B_3)$. We have natural isomorphisms $T_0 \otimes B \simeq B$ and $B \otimes T_0 \simeq B$ given by the identity maps $t_0 \otimes b \mapsto b$ and $b \otimes t_0 \mapsto b$, and it gives the same map on $T_0 \otimes T_0$.

**Lemma 3.9.** The category of $\mathfrak{g}$-crystals is a monoidal category whose unit object is $T_0$.

**Remark 3.10.** Let $\mathcal{C}$ be a monoidal category with unit object $I$, $B$, $B'$ two objects of $\mathcal{C}$. Recall that $B'$ is the left dual of $B$ and $B$ is the right dual of $B'$ if there exist $\epsilon : I \rightarrow B \otimes B'$, $\eta : B' \otimes B \rightarrow I$ such that the composition

\[
B \simeq I \otimes B \xrightarrow{\epsilon \otimes \text{id}_B} (B \otimes B') \otimes B \simeq B \otimes (B' \otimes B) \xrightarrow{\text{id}_B \otimes \eta} B \otimes I \simeq B
\]

is equal to $\text{id}_B$ and the composition

\[
B' \simeq B' \otimes I \xrightarrow{\text{id}_{B'} \otimes \epsilon} B' \otimes (B \otimes B') \simeq (B' \otimes B) \otimes B' \xrightarrow{\eta \otimes \text{id}_{B'}} I \otimes B' \simeq B'
\]

is equal to $\text{id}_{B'}$. $\mathcal{C}$ is called rigid if every object has the left and the right duals. The category of $\mathfrak{g}$-crystals is not a rigid monoidal category. To see this, let $B(0)$ be the crystal $\{b_0\}$ with wt($b_0$) = 0. $\epsilon_i(b_0) = \varphi_i(b_0) = 0$, $\epsilon_i(b_0) = \tilde{f}_ib_0 = 0$. For any $B$, we have that $\varphi_i(b \otimes b_0) \neq -\infty$, which implies that there does not exist nonzero crystal morphism $B \otimes B(0) \sqcup \{0\} \rightarrow T_0 \sqcup \{0\}$ nor $T_0 \sqcup \{0\} \rightarrow B(0) \otimes B \sqcup \{0\}$. Hence, $B(0)$ does not have the dual.

**Remark 3.11.** The category of $\mathfrak{g}$-crystals is not a braided monoidal category. For example, $\mathfrak{sl}_2$-crystals $B(0) \otimes T_{n\omega}$ and $T_{n\omega} \otimes B(0)$ are not isomorphic if $n \neq 0$. Below we introduce crystals which come from integrable $U_q(\mathfrak{g})$-modules. For such crystals we have isomorphisms $B_1 \otimes B_2 \simeq B_2 \otimes B_1$, but we have to choose them functorial and they must satisfy the commutativity of moving $B_1$ step by step to the right

\[
B_1 \otimes B_2 \otimes B_3 \rightarrow B_2 \otimes B_1 \otimes B_3 \rightarrow B_2 \otimes B_3 \otimes B_1
\]

with swapping $B_1$ and $B_2 \otimes B_3$ at once. For $\mathfrak{g} = \mathfrak{sl}_2$ this is not satisfied.

**Remark 3.12.** In the case when $\mathfrak{g}$ is of affine type, we may consider $\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}]$-crystal, which is obtained by replacing $P$ with $P_{cl}$, which is $P$ modulo the null root, in the definition of $\mathfrak{g}$-crystal. Then we have other examples of $B_1 \otimes B_2 \simeq B_2 \otimes B_1$ given by combinatorial $R$-matrices for the affinizations of finite $\mathfrak{g}'$-crystals.

**Definition 3.13.** A crystal $B$ is seminormal if

\[
\epsilon_i(b) = \max\{n \in \mathbb{Z}_{\geq 0} \mid \tilde{e}_i^n b = 0\} \quad \text{and} \quad \varphi_i(b) = \max\{n \in \mathbb{Z}_{\geq 0} \mid \tilde{f}_i^n b \neq 0\}
\]

hold, for all $b \in B$. 
**Remark 3.14.** Let $U_v(t_{ij})$ be the subalgebra of $U_v(g)$ generated by $e_i, e_j, f_i, f_j$ and $v^h$, for $h \in P^\vee$. Let $\Lambda \in P$ be such that $\langle h_i, \Lambda \rangle \geq 0$ and $\langle h_j, \Lambda \rangle \geq 0$. Then, as we will explain below, we have the $t_{ij}$-crystal $B_{ij}(\Lambda)$ which is the crystal of the integrable highest weight $U_v(t_{ij})$-module with highest weight $\Lambda$.

Let $B$ be a $g$-crystal and consider it as a $t_{ij}$-crystal. If it is isomorphic to direct sum of $B_{ij}(\Lambda)$'s, for all $i, j \in I$ such that $t_{ij}$ is of finite type, we say that $B$ is normal. The following is proved in [26, 5.2].

**Lemma 3.15.** Let $B_1, B_2$ be seminormal (resp. normal) crystals. Then

1. $B_1 \otimes B_2$ is a seminormal (resp. normal) crystal.
2. Any crystal morphism $f : B_1 \to B_2$ is a strict crystal morphism.

**Corollary 3.16.** The category of seminormal (resp. normal) $g$-crystals is a monoidal category whose unit object is $B(0)$.

Recall that $K = Q(v)$. $A_0 = \{c(v) \in K \mid c(v)$ is regular at $v = 0\}$. Assume that the Cartan matrix $A$ is symmetrizable. Then we have the $K$-algebra $U_v(g)$, the quantized enveloping algebra associated with the root datum $(A, \Pi, \Pi', P, P^\vee)$.

Let $O_{\text{int}}$ be the full category of the BGG category consisting of integrable modules. Namely, $O_{\text{int}}$ consists of those $M \in U_v(g)$-Mod that satisfies

(i) $M$ admits a weight decomposition $M = \oplus_{\lambda \in P} M_{\lambda}$ such that $\dim K M_{\lambda} < \infty$.

(ii) There exists a finite set $U \subseteq P$ such that if $M_{\lambda} \neq 0$ then

$$\lambda \in U - \sum_{i \in I} \mathbb{Z}_{\geq 0} \alpha_i.$$  

(iii) The Chevalley generators $e_i$ and $f_i$ act locally nilpotently on $M$.

Let $M \in O_{\text{int}}$. Then we may define $\tilde{e}_i, \tilde{f}_i : M \to M$ by $\tilde{e}_i f_i^{(n)} u = f_i^{(n-1)} u$ and $\tilde{f}_i e_i^{(n)} u = f_i^{(n+1)} u$, for $u \in \text{Ker} e_i$. Here, $f_i^{(n)}$ is the $n$th divided power.

**Definition 3.17.** Let $M \in O_{\text{int}}$. An $A_0$-submodule $L = \oplus_{\lambda \in P} L_{\lambda}$ is called a crystal lattice of $M$ if $L_{\lambda} \subseteq M_{\lambda}$ and $L_{\lambda} \otimes K = M_{\lambda}$, for all $\lambda \in P$, $\tilde{e}_i L \subseteq L$ and $\tilde{f}_i L \subseteq L$, for all $i \in I$.

**Definition 3.18.** Let $M \in O_{\text{int}}$. A crystal basis of $M$ is a pair $(L, B = \cup_{\lambda \in P} B_{\lambda})$ such that

(i) $L = \oplus_{\lambda \in P} L_{\lambda}$ is a crystal lattice of $M$,

(ii) $B_{\lambda}$ is a $Q$-basis of $L_{\lambda}/uL_{\lambda}$, for all $\lambda \in P$.

(iii) $\tilde{e}_i B \subseteq B \cup \{0\}$ and $\tilde{f}_i B \subseteq B \cup \{0\}$, for all $i \in I$.

(iv) Let $b, b' \in B$. Then $\tilde{f}_i b = b'$ if and only if $\tilde{e}_i b' = b$.

If $(L, B)$ is a crystal basis of $M \in O_{\text{int}}$, then $B$ is a normal $g$-crystal. There are seminormal crystals which are not of this form. For normal crystals, no such example is known. The following theorem was proved by the famous grand loop argument.

**Theorem 3.19 (Kashiwara).** Let $M \in O_{\text{int}}$. Then there exists a unique crystal basis up to automorphism of $M$.

Let $\Lambda$ be a dominant integral weight. Then the irreducible highest weight $U_v(g)$-module $V_v(\Lambda)$ belongs to $O_{\text{int}}$. The crystal basis of $V_v(\Lambda)$ is denoted by $(L(\Lambda), B(\Lambda))$. The highest vector $v_\Lambda \in V_v(\Lambda)$ defines the highest weight element $u_\Lambda \in B(\Lambda)$. 
Remark 3.20. As \( O_{\text{int}} \) is a semisimple category, every object is a direct sum of \( V_v(\Lambda)'s \), which corresponds to the direct sum of \( B(\Lambda)'s \) in the category of crystals.

\[
\text{Hom}(B(\Lambda), B(\Lambda')) = \begin{cases} 
0 & (\Lambda \neq \Lambda') \\
\{0, \text{id}_{B(\Lambda)}\} & (\Lambda = \Lambda')
\end{cases}
\]

Hence \( \text{Hom}_{O_{\text{int}}}(V_v(\Lambda), V_v(\Lambda')) \) is a “linearization” of \( \text{Hom}(B(\Lambda), B(\Lambda')) \), and \( O_{\text{int}} \) is well controlled by the category of crystals. However, it is no more true when we compare their monoidal structures.

Recall that \( O_{\text{int}} \) is a braided monoidal category. (It is not rigid in general as long as we adopt the usual definition of the dual for general Hopf algebras: the dual \( V_v(\Lambda) = \text{Hom}_K(V_v, \mathbf{C}) \) is the lowest weight module, which does not belong to \( O_{\text{int}} \) unless \( g \) is of finite type.) Hence we have a natural isomorphism \( V_v(\Lambda) \otimes V_v(\Lambda') \simeq V_v(\Lambda') \otimes V_v(\Lambda) \) and this implies that \( B(\Lambda) \otimes B(\Lambda') \) and \( B(\Lambda') \otimes B(\Lambda) \) are isomorphic. However, as is mentioned above, the subcategory of these crystals is not braided. When \( g \) is of finite type, Henriques and Kamnitzer [24] gave the notion of commutator for the full category of crystals which consists of direct sums of \( B(\Lambda)'s \), and showed that it gives a coboundary monoidal structure.

The crystal \( B(\Lambda) \) produces a remarkable basis of \( V_v(\Lambda) \).

Theorem 3.21 (Kashiwara). Let \( L_0 = L(\Lambda) \) such that \( (L_0)_\Lambda = A_0 v_\Lambda \). Define the bar operation on \( V_v(\Lambda) \) by \( \overline{u} = v_\Lambda \) and \( \overline{f_i u} = f_i \overline{u} \), for \( i \in I \) and \( u \in V_v(\Lambda) \), and denote the Kostant-Lusztig form of \( U_v(g) \) by \( U_A(g) \). Set

\[
L_\infty = \overline{L(\Lambda)} \quad \text{and} \quad L = U_A(g)\overline{v_\Lambda}.
\]

Then \( (L, L_0, L_\infty) \) is a balanced triple. In particular, we have the canonical basis \( \{G(b) \mid b \in B(\Lambda)\} \) of \( V_v(\Lambda) \).

Let us consider \( U_v^-(g) \). Then it may be viewed as a module over the Kashiwara algebra (the algebra of deformed bosons), and we may define its crystal basis by the similar recipe. The crystal so obtained is the crystal \( B(\infty) \) and we also have the canonical basis \( \{G(b) \mid b \in B(\infty)\} \) of \( U_v^-(g) \). We have \( G(b) v_\Lambda = G(b') \), for a unique \( b' \in B(\Lambda) \), or \( G(b) v_\Lambda = 0 \). This defines a strict crystal epimorphism \( B(\infty) \otimes T_\Lambda \rightarrow B(\Lambda) \).

Theorem 3.22 (Kashiwara). There is an embedding \( B(\Lambda) \rightarrow B(\infty) \otimes T_\Lambda \), for each dominant integral weight \( \Lambda \), such that

1. The morphisms \( B(\Lambda) \otimes T^-_\Lambda \rightarrow B(\infty) \) form an inductive system and \( B(\infty) = \lim_{\Lambda \rightarrow \infty} B(\Lambda) \otimes T^-_\Lambda \).
2. The embedding is the section of \( B(\infty) \otimes T_\Lambda \rightarrow B(\Lambda) \).

We record two theorems which are useful to identify a crystal with \( B(\Lambda) \). The first is by Joseph [26, 6.4.21] and the second is by Kashiwara and Saito [30].

Theorem 3.23 (Joseph). Suppose that we are given a seminormal crystal \( D(\Lambda) \), for each dominant integral weight \( \Lambda \), such that

\[\text{Hom}(B(\Lambda), D(\Lambda)) = \begin{cases} 0 & (\Lambda \neq \Lambda') \\
\{0, \text{id}_{B(\Lambda)}\} & (\Lambda = \Lambda')
\end{cases}\]

That we adopt the definition of crystal morphism in [26] is important here.

As is well-known, Lusztig constructed the basis by geometrizing Ringel’s work when the generalized Cartan matrix is symmetric.
(i) There exists an element \(d_\Lambda \in D(\Lambda)\) of weight \(\Lambda\) and all the other elements of \(D(\Lambda)\) are of the form \(\bar{f}_{i_1} \cdots \bar{f}_{i_N} d_\Lambda\), for some \(i_1, \ldots, i_N \in I\).
(ii) The subcrystal of \(D(\Lambda) \otimes D(\Lambda')\) that is generated by \(d_\Lambda \otimes d_{\Lambda'}\) is isomorphic to \(D(\Lambda + \Lambda')\).

Then \(D(\Lambda) \simeq B(\Lambda)\), for all \(\Lambda\).

**Theorem 3.24** (Kashiwara-Saito). Let \(B\) be a \(\mathfrak{g}\)-crystal, \(b_0 \in B\) an element of weight 0. Suppose that

(i) \(\text{wt}(B) \subseteq \sum_{i \in I} \mathbb{Z}_{\leq 0} \alpha_i\).
(ii) \(b_0\) is the unique element of \(B\) of weight 0 and \(\epsilon_i(b_0) = 0\), for all \(i \in I\).
(iii) \(\epsilon_i(b)\) is finite, for all \(i \in I\) and \(b \in B\).
(iv) There exists a strict embedding \(\Psi_i : B \to B \otimes B_i\), for all \(i \in I\).
(v) \(\Psi_i(B) \subseteq \{b \otimes b_i(a) \mid b \in B, a \in \mathbb{Z}_{\leq 0}\}\).
(vi) If \(b \neq b_0\) then there exists \(i \in I\) such that \(\Psi_i(b) \in \{b \otimes b_i(a) \mid b \in B, a \in \mathbb{Z}_{\leq 0}\}\).

Then \(B \simeq B(\infty)\). If there also exists a seminormal crystal \(D\), a dominant integral weight \(\Lambda\) and an element \(d_\Lambda \in D\) of weight \(\Lambda\) such that

(v) \(d_\Lambda\) is the unique element of \(D\) of weight \(\Lambda\).
(vi) There is a strict epimorphism \(\Phi : B \otimes T_\Lambda \to D\) such that \(\Phi(b_0 \otimes t_\Lambda) = d_\Lambda\).
(vii) \(\Phi\) maps \(\{b \otimes t_\Lambda \in B \otimes T_\Lambda \mid \Phi(b \otimes t_\Lambda) \neq 0\}\) to \(D\) bijectively.

Then \(D \simeq B(\Lambda)\) and the section of \(\Phi\) given by the bijective map in (vii) gets identified with the embedding \(B(\Lambda) \to B(\infty) \otimes T_\Lambda\).

Kashiwara constructed a strict embedding \(\Psi_i : B(\infty) \to B(\infty) \otimes B_i\) that satisfies \(\Psi_i(b_0) = b_0 \otimes b_i(0)\), for any \(i \in I\), and showed that such an embedding is unique. Hence, \(\Psi_i\) in the above theorem is identified with this embedding.

Recall that we have an anti-automorphism of \(U^-\mathfrak{g}\) defined by \(f_i^* = f_i\). It induces the star crystal structure on \(B(\infty)\) defined by

\[
\text{wt}^*(b) = \text{wt}(b^*), \quad \epsilon_i^*(b) = \epsilon_i^*(b^*), \quad \varphi_i^*(b) = \varphi_i(b^*),
\]
\[
\bar{e}_i^* b = (\bar{e}_i b^*)^*, \quad \bar{f}_i^* b = (\bar{f}_i b^*)^*.
\]

The next proposition is from [27, 8.1.8.2].

**Proposition 3.25** (Kashiwara). Let \(\Psi_i\) and \(\Lambda\) be as above.

1. The image of the strict embedding \(\Psi_i\) is given by
\[
\{b \otimes b_i(a) \mid b \in B(\infty), \epsilon_i^*(b) = 0, a \leq 0\}\.
\]
2. The image of the embedding \(B(\Lambda) \to B(\infty) \otimes T_\Lambda\) is given by
\[
\{b \otimes t_\Lambda \mid \epsilon_i^*(b) \leq h_i, \Lambda, \text{for any } i \in I\}.
\]

3.2. **Realizations of crystals.** Kashiwara crystal has many realizations. Each realization has its own advantage and in the case when we may transfer a result in one realization to a result in the other realization, it would lead to a very nontrivial consequence. This is exactly the case when we apply the theory of crystals to the modular representation theory of Hecke algebras. We have obtained classification of simple modules, decomposition matrices, representation type of the whole algebra, the modular branching rule, so far. This is the aim of the next subsection, and as a preparation for this, we explain various realizations here.
(1) Realization by crystal bases: This is already explained. When \( g \) is of type \( A^{(1)}_1 \), it is closely related to soliton theory and solvable lattice models.

Recall that the study of the Kadomtsev-Petviashvili equations by Sato school lead to the understanding of the Fock space as a \( \mathfrak{gl}_\infty \)-module, and then, by reduction to the Korteweg-de Vries equation etc, we obtain a \( g \)-module.

Let \( \lambda = (\lambda_1, \lambda_2, \ldots) \) be a partition. Then, we assign its beta numbers, in which we have an ambiguity. However, this ambiguity precisely amounts to the choice of the coloring of the nodes of \( \lambda \), or equivalently, the choice of the highest weight for the vacuum. Let us fix \( m \in \mathbb{Z}/e\mathbb{Z} = I \). For a node \( x \in \lambda \) which lies in the \( a \)-th row and the \( b \)-th column, we color \( x \) with its residue \( r(x) \) defined by \( m - a + b \). Then \( \mathcal{F} = \bigoplus \mathbb{Q} \lambda \), the space of these colored partitions, becomes a \( g \)-module via

\[
\begin{align*}
e_i \lambda &= \sum_{\mu : r(\lambda/\mu) = i} \mu, \\
f_i \lambda &= \sum_{\mu : r(\mu/\lambda) = i} \mu
\end{align*}
\]

and definitions for the Cartan part. We denote the module by \( \mathcal{F}(\Lambda_m) \). The Fock space is \( \bigoplus_{m \in \mathbb{Z}/e\mathbb{Z}} \mathcal{F}(\Lambda_m) \), although we call \( \mathcal{F}(\Lambda_m) \)'s also Fock spaces.

Later, they studied the XXZ model with periodic boundary conditions. The XXZ model is one of the important models for spin chains. Then, they found \( U_v(\mathfrak{sl}_2) \)-symmetry in the model, and they introduced the deformed Fock space \( \mathcal{F}_v = \bigoplus \mathcal{F}(\lambda) \), which becomes a \( U_v(\mathfrak{g}) \)-module after a choice of the coloring of partitions. We denote the module by \( \mathcal{F}_v(\Lambda_m) \). The vacuum has the weight \( \Lambda_m \), and the \( U_v(\mathfrak{g}) \)-submodule generated by the vacuum is isomorphic to \( V_v(\Lambda_m) \). The observation was that the space of states of half infinite spin chains looks like \( \bigoplus_{m \in \mathbb{Z}/e\mathbb{Z}} \mathcal{F}_v(\Lambda_m) \).

Misra and Miwa showed that \( \bigoplus \mathcal{A}_{\lambda} \lambda \lambda \) is a crystal lattice of \( \mathcal{F}_v(\Lambda_m) \), and that the set of partitions is its crystal. Thus, its connected component that contains the empty partition is isomorphic to \( B(\Lambda_m) \). Note that \( B(\Lambda_m) \) is realized as a subcrystal of the crystal of partitions.

(2) Realization by Young diagram/Young tableaux and Young walls: This gave new treatment of classical objects in algebraic combinatorics. For example, the set of \( e \)-restricted/\( e \)-regular colored partitions is a realization of \( B(\Lambda_m) \). This is the crystal which appeared in (1) as the connected component which contains the empty partition.

(3) Path realization: This came from the same effort to understand the spin model. Let \( g \) be of affine type. Following standard notation, we have a special node \( 0 \in I \) and the Cartan subalgebra has the basis \( \{ h_i \mid i \in I \} \cup \{ d \} \).

The canonical central element \( c = \sum h_i \) is defined by the requirement that \( \gcd \{ c_i \mid i \in \mathbb{Z}_{>0} \mid i \in I \} = 1 \). Let \( \Lambda \) be a dominant integral weight such that \( \langle d, \Lambda \rangle = 0 \). Then the ground state \( p_\Lambda = \cdots \otimes b_1 \otimes b_0 \), which corresponds to the highest weight element, and the other excited states \( \cdots \otimes p_1 \otimes p_0 \) where \( p_k = b_k \), for sufficiently large \( k \), form the crystal \( B(\Lambda) \), which explained the appearance of the crystal in the XXZ model. In the XXZ model, we have \( b_k = + \) or \( b_k = - \), for all \( k \). In the crystal language, \( \{ +, - \} \) is a perfect crystal.
Definition 3.26. Let \( l \) be a positive integer, which is called a level. Let \( B \) be a finite \( \mathfrak{g}' \)-crystal. We say that \( B \) is a perfect crystal of level \( l \) if it satisfies the following.

(i) There exists a finite dimensional \( U_\psi(\mathfrak{g}') \)-module with crystal basis \((\mathcal{L}, B)\), for some crystal lattice \( \mathcal{L} \).

(ii) \( B \otimes B \) is connected.

(iii) There exists \( \lambda_0 \in P_d \) such that \( \text{wt}(B) \subseteq \lambda_0 - \sum_{i \neq 0} \mathbb{Z}_{\geq 0} \alpha_i \).

(iv) There is the unique element of weight \( \lambda_0 \) in \( B \).

(v) For any dominant integral weight \( \lambda = \sum_{i \in I} \lambda_i \Lambda_i \in P_d = \bigoplus_{i \in I} \mathbb{Z} \Lambda_i \) with \( \langle c, \lambda \rangle = l \), there exist unique vectors \( b^\lambda \in B \) and \( b_\lambda \in B \) such that \( \epsilon_i(b^\lambda) = \lambda_i \) and \( \varphi_i(b^\lambda) = \lambda_i \), for all \( i \in I \).

(vi) \( B_\lambda \) be a finite \( B \)-crystal. We say that \( B_\lambda \) is more useful for us. Let \( \mathcal{B}(\Lambda) \). Kashiwara introduced different treatment of the path model, which is more useful for us. Let \( \mathcal{B}(\Lambda) \) be crystals. A map \( \psi : B \to B' \) is called a crystal morphism of amplitude \( h \) if

\[
\text{wt}(\psi(b)) = h \text{wt}(b), \quad \epsilon_i(\psi(b)) = h \epsilon_i(b), \quad \varphi_i(\psi(b)) = h \varphi_i(b),
\]

\[
\psi(\bar{e}_i b) = \bar{e}_i^h \psi(b), \quad \psi(\bar{f}_i b) = \bar{f}_i^h \psi(b),
\]

for all \( b \in B \). Kashiwara showed that there is a unique crystal isomorphism of amplitude \( h \) for \( B(\Lambda) \to B(h\Lambda) \subseteq B(\Lambda)^{\otimes h} \), and that if \( h \) is sufficiently divisible then the image stabilizes in the sense that there exist \( b_1, \ldots, b_s \) and \( 0 = a_0 < \cdots < a_s = 1 \) which are independent of \( h \) such that

\[
b \mapsto b_1^{\otimes a_1} \otimes b_2^{\otimes (a_2 - a_1)} \otimes \cdots \otimes b_s^{\otimes (1 - a_{s-1})},
\]

for sufficiently divisible \( h \). Set \( \nu_j = \text{wt}(b_j) \), for \( 1 \leq j \leq s \). Then the map

\[
b \mapsto (\nu_1, \ldots, \nu_s; a_0, \ldots, a_s)
\]

gives the Littelmann path model realization. When \( \mathfrak{g} \) is of type \( \mathfrak{A}_{s-1}^{(1)} \) and \( B(\Lambda) = B(\Lambda_m) \), which is realized on partitions,
then $b_j$ are e-cores. Thus, a variant of the Littelmann path model is that $b \in B(\Lambda_m)$ is expressed by $(\nu_1, \ldots, \nu_s; a_0, \ldots, a_s)$, where $\nu_1 \geq \cdots \geq \nu_s$ are e-cores.

(5) Polyhedral realization: This is the embedding $B(\infty) \to \cdots \to B_{i_2} \otimes B_{i_1}$ given by the strict morphisms $\Psi_i : B(\infty) \to B(\infty) \otimes B_i$, for $i \in I$. This induces the embedding

$$B(\Lambda) \to \cdots \to B_{i_2} \otimes B_{i_1} \otimes T_{\Lambda}.$$ 

There are other realizations in terms of irreducible components of Nakajima quiver varieties, Mirkovic-Vilonen cycles/polytopes, Nakajima monomials, etc.

3.3. Kashiwara crystals and Hecke algebras. Let us return to our theme, Hecke algebras. We mainly focus on type $B$ or its generalization to type $(d, 1, n)$. Broué, Malle and Rouquier introduced cyclotomic Hecke algebras. Let $W$ be a complex reflection group, $\mathcal{A}$ the arrangement consisting of reflection hyperplanes in the defining $W$-module $V$. Then they introduced the Knizhnik-Zamolodchikov equation on the complement $V \setminus \mathcal{A}$, and the cyclotomic Hecke algebra is defined as a quotient of the group algebra of $\pi_1((V \setminus \mathcal{A})/W)$. When $W = G(d, 1, n)$, we have the AK-algebra over $\mathbb{C}$, which is obtained from the definition of $\mathcal{H}(W(B_n), S, L)$ by replacing the quadratic relation for $T_0$ with $(T_0 - q^{r_1}) \cdots (T_0 - q^{r_d}) = 0$. Hecke algebras of type $B$ are special cases of these algebras. Assume that $q$ is a primitive $e$th root of unity and $e \geq 2$. Then we consider $g$ of type $A_{e-1}^{(1)}$ and the dominant integral weight $\Lambda = \sum_{j=1}^{d} \Lambda_{\gamma_j}$. We have $V \otimes \Lambda_{\gamma_j} \subseteq \mathcal{F}_v(\Lambda_{\gamma_j})$, for $1 \leq j \leq d$. By using the coproduct

$$\Delta(e_i) = 1 \otimes e_i + e_i \otimes v^{-h_i}, \quad \Delta(f_i) = f_i \otimes 1 + v^{h_i} \otimes f_i,$$

$$\Delta(v^h) = v^h \otimes v^h, \text{ for } h \in P^\vee,$$

we define

$$\mathcal{F}_v(\Lambda) = \mathcal{F}_v(\Lambda_{\gamma_d}) \otimes \cdots \otimes \mathcal{F}_v(\Lambda_{\gamma_1}).$$

Denote $\lambda^{(d)} \otimes \cdots \otimes \lambda^{(1)}$ by $\Lambda = (\lambda^{(1)}, \ldots, \lambda^{(d)})$. The set of colored multipartitions becomes the $g$-crystal associated with $\mathcal{F}_v(\Lambda)$. The $U_v(g)$-submodule generated by the vacuum of $\mathcal{F}_v(\Lambda)$ is isomorphic to $V \otimes \Lambda$ and it defines the connected component of the crystal of the colored multipartitions. We say that $\Lambda$ is Kleshchev if it belongs to the component. Hence, we have a realization of $B(\Lambda)$ by Kleshchev multipartitions, and we identify them hereafter. Now, we have the canonical basis $\{G(\mu) \mid \mu : Kleshchev\}$ of $V \otimes \Lambda \subseteq \mathcal{F}_v(\Lambda)$. We may expand $G(\mu)$ in $\mathcal{F}_v(\Lambda)$ and write

$$G(\mu) = \sum_{\Lambda} d_{\Lambda, \mu}(v) \Lambda.$$ 

It is known that $d_{\Lambda, \mu}(v) \in v\mathbb{Z}_{\geq 0}[v]$ if $\Lambda \neq \mu$, and $d_{\Lambda, \lambda}(v) = 1$. We specialize at $v = 1$ and write $G(\mu) = \sum_{\Lambda} d_{\Lambda, \mu}(1) \Lambda$ by abuse of notation. This is an equality in the nondeformed Fock space $\mathcal{F}(\Lambda)$. We denote the $g$-submodule of $\mathcal{F}(\Lambda)$ generated by the vacuum by $V(\Lambda)$.

Denote by $\mathcal{H}^\Lambda_{\mathfrak{A}}(q)$ the cyclotomic Hecke algebra whose parameters are specified above. Dipper, James and Mathas showed that $\mathcal{H}^\Lambda_{\mathfrak{A}}(q)$ is a cellular $R$-algebra and the poset is the set of multipartitions. The cell modules are called Specht modules and we denote them by $S_{\Lambda}$. $D_{\Lambda}$ is the module obtained by factoring out the radical
of the bilinear form on $S^\Lambda$. An old theorem of mine then says the following. In (1), we see that the cellular algebra structure fits well in the crystal picture.

**Theorem 3.27 (A).** Let $\mathfrak{g}$, $\Lambda$, $V(\Lambda) \subseteq F(\Lambda)$ and $\mathcal{H}_n^\Lambda(q)$ be as above.

1. $D^\Lambda \neq 0$ if and only if $\mu$ is Kleshchev. Hence, the union for $n \geq 0$ of the set of (isomorphism classes of) simple $\mathcal{H}_n^\Lambda(q)$-modules has a structure of $\mathfrak{g}$-crystal, which is isomorphic to $B(\Lambda)$.

2. Let $K_n(\Lambda) = K(\mathcal{H}_n^\Lambda(q)\text{-mod}) \otimes \mathbb{Q}$, or equivalently, $K^{\text{split}}(\mathcal{H}_n^\Lambda(q)\text{-proj}) \otimes \mathbb{Q}$. Then $K(\Lambda) = \oplus_{n \geq 0} K_n(\Lambda)$ becomes a $\mathfrak{g}$-module, which is isomorphic to the $\mathfrak{g}$-module $V(\Lambda)$.

3. Identify $K(\Lambda)$ with $V(\Lambda) \subseteq F(\Lambda)$ in the unique way by the requirement that $D^\Lambda = P^\Lambda$ corresponds to $\emptyset$. Let $P^\Lambda$ be the projective cover of $D^\Lambda$. Then we have

$$[P^\Lambda] = \sum_{\lambda} d_{\lambda, \mu} D^\lambda,$$

where $d_{\lambda, \mu} = [S^\lambda : D^\mu]$ are decomposition numbers.

4. Suppose that the characteristic of $R$ is 0. Then $[P^\Lambda] = G(\mu)$. In particular, we have $d_{\lambda, \mu} = d_{\lambda, \mu} (1)$.

The proof uses results of Lusztig and Ginzburg on affine Hecke algebras and Lusztig’s construction of $U_v(\mathfrak{g})$, which is the generic composition algebra of the Ringel-Hall algebra of the cyclic quiver. In [2] and [4] we explained the materials which were used in the proof of the theorem. Note that the Hall polynomials of the cyclic quiver were given by Jin Yun Guo. A generalization of the above theorem to affine Hecke algebras of type $B$ is attempted by Enomoto and Kashiwara. This involves a new type of crystals, called symmetric crystals.

The theorem also suggests how to label block algebras. Recall that the block algebras of Hecke algebras of type $A$ are labelled by $e$-cores. Recall also that if $\mathfrak{g}$ is of twisted type or simply-laced non-twisted type and $\Lambda$ has level 1, then the weight poset of $V(\Lambda)$ has the form $\sqcup_{e \in W(\Lambda) (\nu - \mathbb{Z}_{\geq 0} \delta)}$. Since the $W$-orbit through the empty partition is the set of $e$-cores, the set of partitions $\mu$ with $\text{wt}(\mu) \in e - \mathbb{Z}_{\geq 0} \delta$ has the unique $e$-core of weight $\nu$. Thus, two partitions $\lambda$ and $\lambda'$ have the same $e$-core if and only if $\text{wt}(\lambda)$ and $\text{wt}(\lambda')$ belong to the same $e - \mathbb{Z}_{\geq 0} \delta$. However, $\text{ht}(\Lambda - \lambda) = \text{ht}(\Lambda - \lambda') = n$ implies that this is equivalent to $\text{wt}(\lambda) = \text{wt}(\lambda')$. In conclusion, the block algebras of $\mathcal{H}_n^\Lambda(q)$ are parametrized by the weight set of $V(\Lambda_m)$. This picture is proved to be true for general $\Lambda$ by Lyle and Mathas [33].

**Theorem 3.28 (Lyle-Mathas).** The block decomposition of $\mathcal{H}_n^\Lambda(q)$ is the same as the weight space decomposition $\bigoplus_{\mu : \text{ht}(\Lambda - \mu) = n} V(\Lambda)_{\mu}$. In particular, the block algebras are labelled by

$$\{ \mu \in P \mid \text{ht}(\Lambda - \mu) = n, V(\Lambda)_{\mu} \neq 0 \}.$$

Denote by $B^\lambda_\mu(q)$ the block algebra labelled by $\mu$. Then the set of simple $B^\lambda_\mu(q)$-modules is $\{ D^\lambda \mid \lambda \in B(\Lambda)_{\mu} \}$ and if the characteristic of $R$ is 0, we can compute the decomposition matrix of $B^\lambda_\mu(q)$.

We explain four more applications of the theory of crystals and the canonical bases. The first is the modular branching rule. Let $D^\lambda$ be a simple $\mathcal{H}_n^\Lambda(q)$-module. Its restriction to $\mathcal{H}_{n-1}^\Lambda(q)$ is not semisimple in general. The modular branching rule is an explicit formula to describe which simple $\mathcal{H}_{n-1}^\Lambda(q)$-modules appear in the socle. This is a suitable generalization of the ordinary branching problem.
In [23] Grojnowski and Vazirani proved that $\text{Soc}(D^\lambda |_{\mathcal{H}_n^\Lambda(q)})$ is multiplicity free. They also showed that the set of simple $\mathcal{H}_n^\Lambda(q)$-modules becomes a crystal, which is isomorphic to $B(\Lambda)$ again. Note however that crystal isomorphisms do not respect the labelling of simple $\mathcal{H}_n^\Lambda(q)$-modules. In the next theorem from [5], we proved that the isomorphism in question is the identity map.

**Theorem 3.29** (A). The modular branching rule is given by

$$\text{Soc} D^\mu |_{\mathcal{H}_{n-1}^\Lambda(q)} \simeq \bigoplus_{i \in I = \mathbb{Z}/e\mathbb{Z}} D^{\tilde{e}_i \mu}.$$  

Thus, the modular branching rule has a very crystal theoretic description.

The second is about representation type [3]. For Hecke algebras of type $A$, it was settled by Erdmann and Nakano by different methods. We have obtained the result for any $L$, but instead of preparing further notations, we state the result only in the case when $L$ is the length function. Blockwise determination is in progress.

**Theorem 3.30** (A). Let $W$ be a finite Weyl group without exceptional components, $P_W(x) = \sum_{w \in W} x^{\ell(w)}$ the Poincaré polynomial of $W$. We suppose that $R$ is an algebraically closed field. Then $H_R = H(W, S, L = \ell) \otimes R$ is

(i) semisimple if $P_W(q) \neq 0$.

(ii) of finite type but not semisimple if

$$\max\{k \in \mathbb{Z}_{\geq 0} \mid (x - q)^k \text{ divides } P_W(x)\} = 1.$$

(iii) of tame type but not of finite type if $q = -1 \neq 1$ and

$$\max\{k \in \mathbb{Z}_{\geq 0} \mid (x - q)^k \text{ divides } P_W(x)\} = 2.$$

(iii) of wild type otherwise.

The third is about an old conjecture of Dipper, James and Murphy. When they started the study of $\mathcal{H}_n = \mathcal{H}(W(B_n), S, L) \otimes R$ motivated by classification of simple modules of finite groups of Lie type in the non-defining characteristic case, Kashiwara crystal was not available. The Specht module theory they constructed is the special case of the cellular structure we explained above, and they conjectured when $D^\lambda$ was nonzero. The idea resembles the highest weight theory. Define the Jucys-Murphy elements $t_1, \ldots, t_n$ by $t_1 = T_0$ and $t_{i+1} = T_i t_i T_i$, for $1 \leq i \leq n - 1$. They generate a commutative subalgebra $T_n$, which plays the role of the Cartan subalgebra.

(i) One dimensional representations of $T_n$ are called weights.

(ii) For an $\mathcal{H}_n$-module, the generalized simultaneous eigenspace decomposition of the module is called the weight decomposition.

Weights are labelled by bitableaux. $\underline{\lambda}$ is $(Q, e)$-restricted if there exists a weight which corresponds to a bitableau of shape $\underline{\lambda}$ such that it appears in $S^\underline{\lambda}$ but does not appear in $S^\underline{\mu}$, for all $\underline{\mu} \prec \underline{\lambda}$. The following statement was the conjecture made by Dipper, James and Murphy. Jacon and I proved this conjecture in [6] by using one of the main results of [7].

**Theorem 3.31** (A-Jacon). $D^\lambda \neq 0$ if and only if $\underline{\lambda}$ is $(Q, e)$-restricted.

As we already know that $D^\lambda \neq 0$ if and only if $\underline{\lambda}$ is Kleshchev, what we actually proved is the assertion that $\underline{\lambda}$ is Kleshchev if and only if $\underline{\lambda}$ is $(Q, e)$-restricted.
The fourth is a remark on the Mullineux map for the symmetric group and the Hecke algebra of type A. The algebras have the involution $T_s \mapsto -T_s^{-1}$ and the involution induces a permutation of simple modules. The permutation is described by the transpose of partitions when the algebra is semisimple, but it was considered to be difficult to describe the permutation when the algebra is not semisimple. The Mullineux map was its conjectural description and it took long time before Kleshchev proved the conjecture. In [22], we have also proved that the Mullineux map is always given by transpose of partitions, if we work in the path model.

4. Quasihereditary covers of Hecke algebras

4.1. The rational Cherednik algebra. As our main object of study is $\mathcal{H}_n^A(q)$, we focus on the rational Cherednik algebra associated with $G(d,1,n)$. To introduce the algebra, we need many notations. Let

$$V = \mathbb{C}^n = \mathbb{C}e_1 \oplus \cdots \oplus \mathbb{C}e_n,$$

where $e_1, \ldots, e_n$ are standard basis vectors, that is, the $i^{th}$ entry of $e_k$ is $\delta_{ki}$. We write elements of $V$ by $y = \sum_{i=1}^n y_i e_i$, where $y_i \in \mathbb{C}$, for $1 \leq i \leq n$. Similarly, let

$$V^* = \text{Hom}_{\mathbb{C}}(V, \mathbb{C}) = \mathbb{C}e_1 \oplus \cdots \oplus \mathbb{C}e_n,$$

where $\langle x_i, e_j \rangle = \delta_{ij}$. We write elements of $V^*$ by $\xi = \sum_{i=1}^n \xi_i e_i$, where $\xi_i \in \mathbb{C}$, for $1 \leq i \leq n$. In the rest of the paper, we denote $G(d,1,n)$ by $W$.

**Definition 4.1.** Let $\zeta = \exp\left(\frac{2\pi \sqrt{-1}}{d}\right)$. We define $t_i \in W$, for $1 \leq i \leq n$, by

$$t_i e_k = \begin{cases} e_k & (k \neq i) \\ \zeta e_k & (k = i), \end{cases}$$

and $s_{ij;\alpha} \in W$, for $1 \leq i < j \leq n$ and $0 \leq \alpha < d$, by

$$s_{ij;\alpha} e_k = \begin{cases} e_k & (k \neq i,j) \\ \zeta^{-\alpha} e_j & (k = i) \\ \zeta^\alpha e_i & (k = j). \end{cases}$$

Then the set of complex reflections in $W$ is

$$S = \{t_i^a \mid 1 \leq i \leq n, 1 \leq a < d\} \cup \{s_{ij;\alpha} \mid 1 \leq i < j \leq n, 0 \leq \alpha < d\}.$$

Let $S_n$ be the symmetric group of degree $n$. $S_n$ acts on $V$ by $w e_k = e_{w(k)}$, and we may identify $S_n$ with $G(1,1,n) \subseteq G(d,1,n) = W$. Let $T = \{t_1, \ldots, t_n\}$ be the subgroup of $W$ generated by $t_1, \ldots, t_n$, which is isomorphic to $(\mathbb{Z}/d\mathbb{Z})^n$. $W$ is the semidirect product of $T$ (which is a normal subgroup of $W$) and $S_n$.

We denote the action of $w \in W$ on $V$ by $w : y \mapsto w(y)$, and the action of $w \in W$ on $V^*$ by $w : \xi \mapsto w(\xi)$. Note that

(i) if $t = t_1^{a_1} \cdots t_n^{a_n}$ then $t(x_k) = \zeta^{-a_k} x_k$,

(ii) if $w \in S_n$ then $w(x_k) = x_{w(k)}$.

For each $s \in S$, we have the reflection hyperplane $H_s = \{y \in V \mid s(y) = y\}$. Let $H_i = \text{Ker}(x_i)$ and $H_{ij;\alpha} = \text{Ker}(x_i - \zeta^\alpha x_j)$. If $s = t_i$ then $H_s = H_i$, and if $s = s_{ij;\alpha}$ then $H_s = H_{ij;\alpha}$. We denote the hyperplane arrangement $\{H_s \mid s \in S\}$ by $\mathcal{A}$.

**Definition 4.2.** (1) We define the set of roots $\Phi = \{\alpha_H \mid H \in \mathcal{A}\} \subseteq V^*$ by

$$\alpha_{H_i} = x_i \text{ and } \alpha_{H_{ij;\alpha}} = x_i - \zeta^\alpha x_j.$$
We define the set of coroots $\Phi^\vee = \{v_H \mid H \in \mathcal{A}\} \subseteq V$ by $v_{H_i} = e_i$ and $v_{H_{ij;\alpha}} = e_i - \zeta^{-\alpha}e_j$.

For each $H \in \mathcal{A}$, define $W_H = \{w \in W \mid w(y) = y, \text{for all } y \in H\}$ and $e_H = |W_H|$. If $H = H_i$ then $W_H = \langle t_i \rangle$ and $e_H = d$. If $H = H_{ij;\alpha}$ then $W_H = \langle s_{ij;\alpha} \rangle$ and $e_H = 2$.

As $V = H \oplus C_V H$ is a decomposition into a direct sum of $W_H$-modules, and $W_H$ acts trivially on $H$, $C_V H$ affords a faithful representation of $W_H$. Thus

$$W_H \simeq \{\exp(2\pi \sqrt{-1}a/e_H) \mid 0 \leq a < e_H\} \subseteq \text{GL}(C_V H),$$

and we may define a generator of the cyclic group $w_H \in W_H$ by $w_H \mapsto \exp(2\pi \sqrt{-1}/e_H)$.

In other words, we define $w_H = t_i$ if $H = H_i$ and $w_H = s_{ij;\alpha}$ if $H = H_{ij;\alpha}$.

**Definition 4.3.** $\epsilon_{h,k} = \frac{1}{e_H} \sum_{\alpha=0}^{e_H-1} \exp(2\pi \sqrt{-1}ak/e_H)w_H^\alpha$, for $0 \leq k < e_H$.

The $W_H$-module $C_{v_H,k}$ affords the representation $w_H \mapsto \exp(-2\pi \sqrt{-1}k/e_H)$.

**Definition 4.4.** Let $R$ be a commutative $\mathbb{C}$-algebra. We suppose that parameters $\kappa_1, \ldots, \kappa_{d-1}, h \in R$ are given. Define $\underline{\kappa} = (\kappa_i)_{i \in \mathbb{Z}/dz}$ and $\underline{h} = (h_i)_{i \in \mathbb{Z}/2z}$ by extending the kappa parameters by $\kappa_0 = \kappa_d = 0$ and by $h_1 = h$, $h_0 = h_2 = 0$. Then, for $0 \leq k < e_H$, we define

$$c_{h,k} = \begin{cases} \kappa_k & (H = H_i) \\ h_k & (H = H_{ij;\alpha}). \end{cases}$$

**Definition 4.5.** For $H \in \mathcal{A}$, we define

1. $\gamma_H = e_H \sum_{k=0}^{e_H-1} (c_{h,k+1} - c_{h,k}) \epsilon_{h,k} \in RW_H$.
2. $\alpha_H = e_H \sum_{k=0}^{e_H-1} c_{h,k} \epsilon_{h,k} \in RW_H$.

If $H = H_{ij;\alpha}$ then $\gamma_H = 2hs_{ij;\alpha}$ and $\alpha_H = h(1 - s_{ij;\alpha})$.

Now, we are ready to introduce the rational Cherednik algebra. Let $R$ be a commutative $\mathbb{C}$-algebra such that parameters $\kappa_1, \ldots, \kappa_{d-1}, h \in R$ are given. The $W$-action on $V$ and $V^*$ naturally defines $W$-action on $T(V \oplus V^*)\sharp W$. We have the smash product $T(V \oplus V^*)\sharp W$. We have the relations $wyw^{-1} = w(y)$ and $w\xi w^{-1} = w(\xi)$, for $y \in V$, $\xi \in V^*$ and $w \in W$.

**Definition 4.6.** The rational Cherednik algebra $H_R(\underline{\kappa}, \underline{h})$ (associated with $W$) is the $R$-algebra obtained from the $R$-algebra $T(V \oplus V^*)\sharp W \otimes_{\mathbb{C}} R$ by factoring out the two-sided ideal generated by

$$[y, \xi] = [\xi, y] - \sum_{H \in \mathcal{A}} \frac{\langle \xi, v_H \rangle}{\langle \alpha_H, v_H \rangle} \gamma_H, \ [y, y'], \ [\xi, \xi'],$$

where $y, y'$ run through $V$ and $\xi, \xi'$ run through $V^*$.

Let $D$ be the sheaf of algebraic differential operators on $V_{reg}$. Let us consider the trivial $W$-equivariant bundle $V_{reg} \times CW$ on $V_{reg}$ and let $\mathcal{O}(V_{reg}, CW)$ be the global sections. $W$ acts on the space by $(w \cdot f)(y) = w(f(w^{-1}(y)))$ as usual. $\mathcal{D}(V_{reg})$ also acts on the space and we have $w\partial_y w^{-1} = \partial_y$ and $w\xi w^{-1} = w(\xi)$, where $\xi$ is considered as a multiplication operator, and $\partial_y = \sum_{i=1}^{n} y_i \frac{\partial}{\partial x_i}$, for $y = \sum_{i=1}^{n} y_i e_i$. In particular, $\mathcal{O}(V_{reg}, CW)$ is a $\mathcal{D}(V_{reg})\sharp W$-module.
**Definition 4.7.** The operator

\[ T_y = \partial_y + \sum_{H \in A} \frac{\langle \alpha_H, y \rangle}{\alpha_H} a_H \in \mathcal{D}(V_{\text{reg}}) \mathbb{z} W \otimes_C R, \]

which acts on \( \mathcal{O}(V_{\text{reg}}, \mathbb{C} W) \otimes_C R \), is called the Dunkl operator.

The following lemma from \cite{12} is not difficult to prove.

**Lemma 4.8 (Etingof-Ginzburg).**

(1) We have an \( R \)-algebra monomorphism

\[ H_R(\mathbb{k}, h) \longrightarrow \mathcal{D}(V_{\text{reg}}) \mathbb{z} W \otimes_C R \]

given by \( \xi \mapsto \xi, \ y \mapsto T_y \) and \( w \mapsto w \).

(2) \( H_R(\mathbb{k}, h) = S(V^*) \otimes_C S(V) \otimes_C RW \) as an \( R \)-module. In particular, \( H_R(\mathbb{k}, h) \) is \( R \)-free of infinite rank with basis given by

\[ \{ x_1^{\alpha_1} \cdots x_n^{\alpha_n} e_1^{\beta_1} \cdots e_n^{\beta_n} w \mid \alpha_i \geq 0, \beta_i \geq 0, w \in W \}. \]

(3) If we localize the monomorphism in (1), we have

\[ \mathcal{O}(V_{\text{reg}}) \otimes_{\mathcal{O}(V)} H_R(\mathbb{k}, h) \simeq \mathcal{D}(V_{\text{reg}}) \mathbb{z} W \otimes_C R. \]

In particular, \( \mathcal{O}(V_{\text{reg}}) \otimes_{\mathcal{O}(V)} H_R(\mathbb{k}, h) \) is an \( R \)-algebra.

**Definition 4.9.** Let \( z = \sum_{H \in A} a_H \in RW \) and define

\[ \text{Eu} = -z + \sum_{i=1}^{n} \sum_{j} x_i e_i \in H_R(\mathbb{k}, h). \]

The element \( \text{Eu} \) is called the Euler element.

We have \( [\text{Eu}, y] = -y, \ [\text{Eu}, \xi] = \xi \) and \( [\text{Eu}, w] = 0 \). Hence \( H_R(\mathbb{k}, h) \) is a \( \mathbb{Z} \)-graded \( R \)-algebra.

**Definition 4.10.** Let \( \mathcal{O} \) be the full subcategory of \( H_R(\mathbb{k}, h) \)-\text{-mod} consisting of the objects that satisfy

(i) finitely generated as an \( H_R(\mathbb{k}, h) \)-module, i.e. \( \mathcal{O} \subseteq H_R(\mathbb{k}, h) \)-mod,

(ii) locally nilpotent as a \( S(V) \)-module.

Recall that an \( R \)-algebra \( A \) is filtered if there exists a family of \( R \)-submodules \( \{ F_p(A) \}_{p \in \mathbb{Z}_{\geq 0}} \) such that

(1) \( 1 \in F_0(A) \),

(2) \( F_p(A) \subseteq F_{p+1}(A) \) and \( \cup_{p \geq 0} F_p(A) = A \),

(3) \( F_p(A) F_q(A) \subseteq F_{p+q}(A) \).

An \( A \)-module \( M \) is filtered if there exists a family of \( R \)-submodules \( \{ F_p(M) \}_{p \in \mathbb{Z}} \) such that

(1) \( F_p(M) = 0 \), for sufficiently small \( p \),

(2) \( F_p(M) \subseteq F_{p+1}(M) \) and \( \cup_{p \in \mathbb{Z}} F_p(M) = M \),

(3) \( F_q(A) F_p(M) \subseteq F_{p+q}(M) \).

If \( N \) is an \( A \)-submodule of \( M \), then \( N \) is also filtered by \( F_p(N) = F_p(M) \cap N \). The following is well-known.

**Lemma 4.11.** Let \( A \) be a filtered algebra. Then an \( A \)-module \( M \) is finitely generated if and only if \( M \) is filtered such that \( \text{gr}(M) \) is finitely generated as a \( \text{gr}(A) \)-module.
The rational Cherednik algebra is a filtered algebra by the filtration
\[ F_p(H_R(\kappa, h)) = S(V^*) \otimes_{\mathbb{C}} S(V)_{p} \otimes R W. \]
As gr\((H_R(\kappa, h)) = S(V \oplus V^*)\) is an R-algebra, an \(H_R(\kappa, h)\)-submodule of a finitely generated \(H_R(\kappa, h)\)-module is again finitely generated. Hence \(\mathcal{O}\) is an Abelian category, and indecomposable objects in \(\mathcal{O}\) are indecomposable \(H_R(\kappa, h)\)-modules. In fact, \(\mathcal{O}\) is a Serre subcategory of \(H_R(\kappa, h)\)-Mod.

**Example 4.12.** Let \(E \in \text{Irr} W\) and let \(I = (e_1, \ldots, e_n)\) be the augmentation ideal of \(S(V)\). Then \((S(V)/I^{r+1}) \otimes_{\mathbb{C}} R\) is an \(S(V)\)-module. \(\Delta_r(E) = H_R(\kappa, h) \otimes S(V)_{r+1} \otimes R (S(V)/I^{r+1} \otimes_{\mathbb{C}} E) \otimes_{\mathbb{C}} R\).

Then \(\Delta_r(E) \in \mathcal{O}\). If \(r = 0\) we denote it \(\Delta(E)\) and call them standard modules.

The following values are important. They will determine the highest weight category structure of \(\mathcal{O}\).

**Definition 4.13.** Let \(E \in \text{Irr} W\). Then \(z\) acts on \(E \otimes_{\mathbb{C}} R\) by a scalar multiple. We denote the value by \(c_E \in R\).

**4.2. The existence of a progenerator.** As there are some confusions in \([19]\), we prove the existence theorem when the base ring \(R\) is a local ring. I do not know whether it holds for arbitrary Noetherian ring. One difficulty is that the rational Cherednik algebra is not module-finite over \(R\).

**Lemma 4.14.** Suppose that \(R\) is a local ring such that the residue field \(F\) contains \(\mathbb{C}\). For each \(a \in F\), define
\[ R(a) = \{ \alpha \in \bigcup_{E \in \text{Irr} W} (-c_E + \mathbb{Z}_{\geq 0}) \mid \bar{a} = a \}. \]
Let \(f_a(z) = \prod_{a \in R(a)}(z - \alpha)\) be a monic polynomial in \(R[z]\).
1. Let \(M \in \mathcal{O}\). Then \(M = \oplus_{a \in F} M_a\) where \(M_a = \{ m \in M \mid f_a(Eu)^N m = 0, \text{for sufficiently large } N \}\).
2. \(M \mapsto M_a\) is an exact functor from \(\mathcal{O}\) to \(RW\)-Mod.
3. \(M_a\) is a finitely generated \(R\)-module.

For the proof, consider the case \(M = \Delta_r(E)\) first.

**Definition 4.15.** For \(M \in \mathcal{O}\), define \(M^{\text{prim}} = \{ m \in M \midVm = 0 \}\).

Note that \(M^{\text{prim}} \neq 0\) whenever \(M \neq 0\), and \(M^{\text{prim}} \subseteq \sum_{E \in \text{Irr} W} M_{-\infty}\).

**Lemma 4.16.** Suppose that \(R\) is a Noetherian local ring such that the residue field \(F\) contains \(\mathbb{C}\). Then any \(M \in \mathcal{O}\) is a direct sum of finitely many indecomposable objects of \(\mathcal{O}\).

In fact, if we had a strictly increasing infinite sequence of submodules \(M_1 \subseteq M_1 \oplus M_2 \subseteq \cdots \subseteq M\) then we have a strictly increasing sequence \(M_1^{\text{prim}} \subseteq M_1^{\text{prim}} \oplus M_2^{\text{prim}} \subseteq \cdots \subseteq M^{\text{prim}}\), which is a contradiction since \(M^{\text{prim}}\) is a Noetherian \(R\)-module. Suppose that \(R\) is a complete regular local ring. Then the uniqueness of the decomposition follows from the existence of a progenerator which we will prove in this subsection. However, the case when \(R\) is a field is an easy case, and we record here the following basic results. The proof is by standard arguments.
Proposition 4.17 (Dunkl-Opdam). Let $R$ be a field of characteristic 0.

1. $\Delta(E)$ admits the eigenspace decomposition with respect to $E_u$. Further, the set of eigenvalues that appear in the eigenspace decomposition coincides with $-c_E + \mathbb{Z}_{\geq 0}$.

2. $\Delta(E)$ has a unique maximal $H_R(k, h)$-submodule and the eigenvalue $-c_E$ does not appear in its eigenspace decomposition with respect to $E_u$.

3. $\text{End}_O(\Delta(E)) = R$, for $E \in \text{Irr} W$.

4. Let $L(E) = \text{Top} \Delta(E)$, which is an irreducible $H_R(k, h)$-module. Then we have $[\Delta(E) : L(E)] = 1$ and if $[\text{Rad} \Delta(E) : L(E')] \neq 0$, for some $E' \in \text{Irr} W$, then $-c_{E'} \not\in -c_E + \mathbb{Z}_{\geq 0}$.

5. $\{L(E) \mid E \in \text{Irr} W\}$ is a complete set of irreducible objects of $O$.

6. Any $M \in O$ has a finite length as a $H_R(k, h)$-module.

7. $O$ is a Krull-Schmidt category.

Recall that an algebra $A$ is local if $A/\text{Rad} A$ is a division ring. (7) says that $\text{End}_O(M)$ is local, for any indecomposable object $M$ in $O$. In fact, we may apply the Fitting lemma to $M$ by (6), and (7) follows.

Definition 4.18. Let $R$ be a ring which contains $\mathbb{Z}$. We introduce a partial order on $R$ by $a \leq b$ if $a + \mathbb{Z}_{\geq 0} \subseteq b + \mathbb{Z}_{\geq 0}$. (Thus, $a \leq a - 1$.) The order induces a preorder on $\text{Irr} W$ by $E_1 \leq E_2$ if $-\overline{c_{E_1}} \leq -\overline{c_{E_2}}$.

Proposition 4.19. Let $R$ be a local ring such that the residue field $F$ contains $\mathbb{C}$. For $a \in F$, we define the full subcategory $O_{\leq a}$ of $O$ by

$$O_{\leq a} = \{M \in O \mid M_b = 0, \text{for } b \not\leq a\}.$$

Suppose that $\Delta(E) \in O_{\leq a}$ and that $-\overline{c_{E'}} \not\leq a$ when $-\overline{c_{E'}} > -\overline{c_E}$. Then $\Delta(E)$ is a projective object of $O_{\leq a}$.

Proof. Let $M \to N \to 0$ in $O_{\leq a}$ and take $0 \neq f \in \text{Hom}(\Delta(E), N)$. Fix $0 \neq v \in E$. Then, to show that $\text{Hom}(\Delta(E), M) \to \text{Hom}(\Delta(E), N)$ is surjective, it suffices to prove that there exists $m \in M$ such that (i) $m$ maps to $f(v)$, (ii) $RW_m \simeq E \otimes_C R$, (iii) $V m = 0$. By Lemma 4.14(2), we may choose $m \in M_{-\overline{c_E}}$ such that $m$ satisfies (i) and (ii). Suppose that $I' m \neq 0$ and $I'^{+1} m = 0$. Then

$$S^r(V) RW m \simeq I' RW m \subseteq M_{-\overline{c_{E'}} - r}.$$

Choose $E' \in \text{Irr} W$ such that $E'$ appears in $S^r(V) \otimes_C E$. Then we have $-\overline{c_{E'}} \leq a$ by $M \in O_{\leq a}$ and $-\overline{c_{E'}} = -\overline{c_E} - r \geq -\overline{c_E}$. Thus, $r = 0$ by the assumption and (iii) is also satisfied.

A similar argument shows the following.

Lemma 4.20. Suppose that $R$ is a local ring such that the residue field $F$ contains $\mathbb{C}$. Let $E, E' \in \text{Irr} W$ be such that $E \not\simeq E'$. Then we have $\text{Ext}_O^1(\Delta(E), \Delta(E')) = 0$.

In fact, if $0 \to \Delta(E') \to M \to \Delta(E) \to 0$ is given, take $0 \neq v \in \Delta(E)_{-\overline{c_E}}$. Then we may choose $m \in M_{-\overline{c_E}}$ such that (i) $m$ maps to $v$, (ii) $V m = 0$. Hence the exact sequence splits.

Lemma 4.21. Suppose that $R$ is a local ring whose residue field $F$ contains $\mathbb{C}$. Let $a \in F$ and let $\{\Delta(E_1), \ldots, \Delta(E_m)\}$ be all of the standard modules that belong to $O_{\leq a}$. If we have projective objects $P_i$ of $O_{\leq a}$ such that $P_i \to \Delta(E_i) \to 0$, for $1 \leq i \leq m$, then $P = \bigoplus_{i=1}^m P_i$ is a progenerator of $O_{\leq a}$.
In fact, for any $M \in \mathcal{O}^{\leq a}$, we have $\Delta_r(CW)^{\oplus N} \to M \to 0$, for some $r$ and $N$. Note that $\Delta_r(CW)^{\oplus N}$ has a finite $\Delta$-filtration. Suppose that $\Delta(E')$ with $-c_{E'} \not\subseteq a$ appears in the $\Delta$-filtration. As any subquotient of $M \in \mathcal{O}^{\leq a}$ belongs to $\mathcal{O}^{\leq a}$, the image of $\Delta(E')$ vanishes. Hence, we have $P^{\oplus N'} \to M \to 0$, for some $N'$.

**Theorem 4.22.** Suppose that $R$ is a Noetherian local ring whose residue field contains $\mathbb{C}$. Then $\mathcal{O}$ has a progenerator which has a finite $\Delta$-filtration.

**Proof.** For $\gamma \in F/\mathbb{Z}$, define $\mathcal{O}^\gamma = \{ M \in \mathcal{O} \mid M_a = 0, \text{ for } a \not\in \gamma. \}$. Then we have $\mathcal{O} = \bigoplus_{\gamma \in F/\mathbb{Z}} \mathcal{O}^\gamma$. We show the existence of the desired progenerator for each $\mathcal{O}^\gamma$.

For this purpose, we assert that $\mathcal{O}^\gamma$ has a progenerator which is a quotient of an object with a finite $\Delta$-filtration. Let $\gamma = a_0 + \mathbb{Z}$. If $k$ is sufficiently small then $\mathcal{O}^{\leq a_0-k} = \mathcal{O}^\gamma$, and if $k$ is sufficiently large then $\mathcal{O}^{\leq a_0+k} = \{ 0 \}$. Hence we prove the assertion by induction on $k$. Suppose that we have the desired progenerator $Q$ for $\mathcal{O}^{\leq a_j}$. By Proposition 4.19 and Lemma 4.21, it suffices to show the existence of a projective object $P$ of $\mathcal{O}^{\leq a_{j+1}}$ such that $P \to Q \to 0$ and that $P$ is a quotient of an object with a finite $\Delta$-filtration. Write $Q = \bigoplus_{i=1}^l H_R(k, h) m_i$ such that $m_i \in Q_{a_i}$. Fix $N$ so that

$$a_i - N - 1 \not\in \bigcup_{E' \in \text{Irr} W} (-c_{E'} + \mathbb{Z}_{\geq 0}),$$

for $1 \leq i \leq l$. This implies that $M_{a_i-N-1} = 0$, for any $M \in \mathcal{O}$, by Lemma 4.24.

Moreover, since $Q$ is a quotient of an object with a finite $\Delta$-filtration and $f_a(Eu)$ acts as $0$ on $\Delta(E')_{a_i}$, for all $E' \in \text{Irr} W$, there exists $e$ such that $f_a(Eu)^e m = 0$, for $m \in Q_{a_i}$.

**Claim 1:** Let $M \in \mathcal{O}^{\leq a_i}$. Then $f_a(Eu)^{e+1} m = 0$, for $m \in M_{a_i}$.

In fact, since $M_a$ is a finitely generated $R$-module, there is

$$\varphi : \bigoplus_{E' : -c_{E'} = a} \Delta(E')^{\oplus m_{E'}} \to M$$

such that $\text{Coker} \varphi \in \mathcal{O}^{\leq a_i}$. In particular, there exists $Q^{\oplus m_Q} \to \text{Coker} \varphi \to 0$. Hence if $m \in M_{a_i}$, then $f_a(Eu)^e m \in (\text{Im} \varphi)_{a_i}$ and $f_a(Eu)^{e+1} m = 0$ follows.

Note that $\Delta_N(CW)$ is the tensor product $\mathcal{O}(V) \otimes_{\mathbb{C}} CW \otimes_{\mathbb{C}} S(V)/I^{N+1} \otimes_{\mathbb{C}} R$. Hence we have $1 := 1 \otimes 1 \otimes 1 \otimes 1 \in \Delta_N(CW)$. Define

$$R_i = \Delta_N(CW)/H_R(k, h) f_a(Eu)^{e+1}$$

and define $R'_i$ to be the $H_R(k, h)$-submodule of $R_i$ generated by $\bigoplus_{b \in a} (R_i)_b$. Then we let $P_i = R_i/R'_i$. Note that $P_i \in \mathcal{O}^{\leq a_i}$. We denote the image of $1$ in $P_i$ by $1_i$.

**Claim 2:** For $M \in \mathcal{O}^{\leq a_i}$, we have a natural isomorphism $\text{Hom}_\mathcal{O}(P_i, M) \simeq M_{a_i}$.

The isomorphism is given by $\varphi \mapsto \varphi(1_i)$. As $P_i = H_R(k, h) 1_i$, the injectivity is clear. On the other hand, if $m \in M_{a_i}$, then $I^{N+1} m \subseteq M_{a_i-N-1} = 0$. Hence we have $S(V)/I^{N+1} \to M$ defined by $1 \mapsto m$, which induces $\Delta_N(CW) \to M$. Since $f_a(Eu)^{e+1} m = 0$ by Claim 1, we obtain $R_i \to M$. We conclude that there exists $\varphi : P_i \to M$ such that $\varphi(1_i) = m$.

By Lemma 4.14(2) and Claim 2, $P_i$ is a projective object of $\mathcal{O}^{\leq a_i}$. $P_i$ is isomorphic to $\Delta_N(CW)$, and $\Delta_N(CW)$ has a finite $\Delta$-filtration. Thus, $P = \bigoplus_{i=1}^l P_i$ has the required properties, and the assertion is proved.
Now we have a progenerator of $\mathcal{O}^\gamma$ which is a quotient of an object with a finite $\Delta$-filtration. That this progenerator has a finite $\Delta$-filtration comes from the following claim, which is not difficult to prove.

Claim 3: If $M_1 \oplus M_2$ has a finite $\Delta$-filtration, then so does $M_1$ and $M_2$. $\square$

**Proposition 4.23.** Suppose that $R$ is regular. That is, $R$ is Noetherian and $R_p$ is a regular local ring, for all $p \in \text{Spec } R$. Let $H$ be an $R$-algebra, $\mathcal{C}$ a full subcategory of $H$-$\text{mod}$. Suppose that if $M \in \mathcal{C}$ and $M \to N \to 0$ in $H$-$\text{mod}$ then $N \in \mathcal{C}$, and that there exists a projective object $P$ of $\mathcal{C}$ such that

(i) $P$ is a finitely presented $H$-module,
(ii) $P$ is a projective $R$-module,
(iii) $\text{End}_\mathcal{C}(P)$ is a finitely generated $R$-module.

Then $A = \text{End}_\mathcal{C}(P)^\text{op}$ is a projective $R$-module. If $Q \in \mathcal{C}$ satisfies

(a) $Q$ is a projective $R$-module,
(b) $\text{Hom}_\mathcal{C}(P, Q)$ is a finitely generated $R$-module,

then $\text{Hom}_\mathcal{C}(P, Q)$ is a projective $R$-module.

**Proof.** As $R$ is Noetherian and by (b), $\text{Hom}_\mathcal{C}(P, Q)$ is a finitely presented $R$-module. Hence

$$\text{Ext}_R^1(\text{Hom}_\mathcal{C}(P, Q), ?)_p = \text{Ext}_R^1(\text{Hom}_\mathcal{C}(P, Q)_p, ?_p).$$

We also have $\text{Hom}_\mathcal{C}(P, Q)_p \simeq \text{Hom}_{H_p}(P_p, Q_p)$ by (i). Hence, $\text{Hom}_\mathcal{C}(P, Q)$ is a projective $R$-module if and only if $\text{Hom}_{H_p}(P_p, Q_p)$ is a free $R_p$-module. Hence we may assume that $R$ is a regular local ring from the beginning. Let $d$ be the Krull dimension of $R$, $(z_1, \ldots, z_d)$ be a system of parameters. Let $\mathcal{F} = \text{Hom}_\mathcal{C}(P, -)$ and define $Q_i = \mathcal{F}(Q/(z_1, \ldots, z_i)Q)$. By (a), we have the exact sequence

$$0 \to Q/(z_1, \ldots, z_i)Q \to Q/(z_1, \ldots, z_i+1)Q \to Q/(z_1, \ldots, z_i+1)Q \to 0.$$ 

Thus, $0 \to Q_i \to Q_i/z_i+1 \to 0$. In particular, all the $Q_i$ are finitely generated $R$-modules since $Q_0$ is so by (b). Let $R_i = R/(z_1, \ldots, z_i)$. We claim that $Q_i$ is a flat $R_i$-module. In fact, this is obvious when $i = d$. Suppose that $i < d$ and that $Q_{i+1}$ is a flat $R_{i+1}$-module. Then, since

(i) $Q_i$ is a finitely generated module over the Noetherian local ring $R_i$,
(ii) $z_{i+1} \in R_i$ is a non-invertible regular element,

$Q_i$ is a flat $R_i$-module if and only if $0 \to Q_i \to Q_i/z_{i+1} \to 0$. And $Q_{i+1} \simeq Q_i/z_{i+1}Q_i$ is a flat $R_{i+1}$-module. Hence, $Q_0$ is a flat $R$-module, which is a free $R$-module by (b). $\square$

**Theorem 4.24.** Suppose that $R$ is a regular local ring whose residue field $F = R/m$ contains $\mathbb{C}$. We denote by $\mathcal{O}(R)$ the category $\mathcal{O}$ for $H_R(k, h)$, and we identify $\mathcal{O}(F)$, the category $\mathcal{O}$ for $H_F(k, h)$, with the full subcategory of $\mathcal{O}(R)$ consisting of $M$ with $mM = 0$. Then, there exists a module-finite $R$-algebra $A$ such that we have

1. $A$ is a projective $R$-algebra, i.e. $A$ is projective as an $R$-module,
2. $\mathcal{O}(R) \simeq A$-$\text{mod}$,
3. $\mathcal{O}(F)$ corresponds to $A \otimes_R F$-$\text{mod}$ under the equivalence in (2),
4. $\Delta(E)$ corresponds to an $A$-module which is a finitely generated projective $R$-module under the equivalence.
For the proof, we take the progenerator $P$ constructed in Theorem 4.22, and define $A = \text{End}_O(P)^{\text{op}}$. Then we check the assumptions in Proposition 4.23 (ii) and (a) are obvious. Recall that if $0 \to L \to M \to N \to 0$ such that $L$ is finitely generated and $M$ is finitely presented then $N$ is finitely presented. Hence, that $\Delta_N(CW)$ is a finitely presented $H_R(\kappa, h)$-module implies that (i) holds. (iii) and (b) follow from the fact that $P_a$ and $Q_a$ are finitely generated $R$-modules, for $a \in F$, since $R$ is Noetherian.

Corollary 4.25. Suppose that $R$ is a complete regular local ring. Then the category $O$ is a Krull-Schmidt category.

Suppose that $R$ is a Noetherian commutative ring and $A$ is a module-finite projective $R$-algebra. Then a finitely generated $A$-module $M$ is projective if and only if the following hold.

(i) $M$ is a projective $R$-module,
(ii) for each closed point $x \in \text{Spec } R$, $M \otimes_R k(x)$ is a projective $A \otimes_R k(x)$-module.

By Theorem 4.24 if $M$ is $\Delta$-filtered then $\text{Hom}_O(P, M) \otimes_R F \simeq \text{Hom}_O(P, M \otimes_R F)$. Hence, we have the following corollary.

Corollary 4.26. Let $R$ be a regular local ring whose residue field $F$ contains $\mathbb{C}$. Then a $\Delta$-filtered object $M \in O(R)$ is a projective object if and only if $M \otimes_R F$ is a projective object of $O(F)$.

4.3. Highest weight category. The aim of the remaining part is to introduce Rouquier’s theory of quasihereditary covers.

Definition 4.27. Let $R$ be a commutative ring, $H$ an $R$-algebra, $C$ an $R$-linear Abelian category which is a subcategory of $H$-Mod. Let $\Lambda$ be a finite preordered set. Then, we say that $(C, \Lambda)$ is a highest weight category in weak sense if there exist objects $\{\Delta(\lambda) \mid \lambda \in \Lambda\}$ such that the following are satisfied.

(i) $\Delta(\lambda)$ is a projective $R$-module.
(ii) If $\text{Hom}_C(\Delta(\lambda'), \Delta(\lambda'')) \neq 0$ then $\lambda' \leq \lambda''$.
(iii) If $N \in C$ is such that $\text{Hom}_C(\Delta(\lambda), N) = 0$, for all $\lambda$, then $N = 0$.
(iv) For each $\lambda$, there exists a projective object $P(\lambda)$ of $C$ such that there is $P(\lambda) \to \Delta(\lambda) \to 0$ and that $\text{Ker}(P(\lambda) \to \Delta(\lambda))$ has a finite $\Delta$-filtration in which only $\Delta(\lambda')$ with $\lambda' > \lambda$ appear.

A highest weight category in weak sense $(C, \Lambda)$ is split if it also satisfies

(v) $\text{End}_C(\Delta(\lambda)) = R$, for all $\lambda$.

$\Delta(\lambda)$ are called standard objects. This definition drops the requirement that a highest weight category should be Artin in some sense. Hence, we add the phrase “in weak sense”. Following [39], we define as follows.

Definition 4.28. A highest weight category in weak sense $(C, \Lambda)$ is a highest weight category if $C \simeq A$-mod, for some module-finite projective $R$-algebra $A$.

Let us recall the usual definition of a highest weight category over a field. Note that we only require that $\Lambda$ is preordered. But this is not essential.

Definition 4.29. Let $R$ be a field. A category $C$ is an Artin category over $R$ if

(i) $C$ is an Abelian $R$-linear category,
object. Thus $\text{PM}$ finite $\Delta$-filtration admits a $\Delta$-filtration $\text{P}$ cover, which we denote by $\lambda$ are already proved in Proposition 4.17. By Theorem 4.24, $\Delta(\lambda)$ is Hom-finite. The other conditions but the existence of projective objects $\lambda$ nonzero $\Delta(\lambda)$. Then repeated use of Nakayama’s lemma implies that if $A$ is a highest weight category over a regular local ring $R$ generated $\lambda R$ is a highest weight category, for $\lambda R$ is a field. If $A$ is module-finite over a field $C$ highest weight category $\{\lambda \in \Lambda \}$ with the property that $\text{Ext}_C^1(\Delta(\lambda), \Delta(\lambda')) = 0$. Let $\text{End}_C(\lambda) = \text{End}_C(\Delta(\lambda))$. Suppose that $\text{End}_C(\lambda)$ is a highest weight category in weak sense. If $\lambda' \neq \lambda''$ then $\text{Ext}_C^1(\Delta(\lambda), \Delta(\lambda'')) = 0$. Next suppose that $\text{End}_C(\lambda)$ is a highest weight category over a field. If $\lambda' \neq \lambda''$ and $\lambda' \neq \lambda''$ then $\text{Hom}_C(\Delta(\lambda), \Delta(\lambda')) = 0$. If $\text{End}_C(\lambda)$ is a highest weight category over a regular local ring $R$ whose residue field is $F$, then repeated use of Nakayama’s lemma implies that if $A \otimes_R F$-mod is a highest weight category whose standard objects are $\{\Delta(\lambda) \otimes_R F \mid \lambda \in \Lambda\}$ and all $\Delta(\lambda)$ are projective $R$-modules, then $\lambda' \neq \lambda''$ and $\lambda' \neq \lambda''$ imply $\text{Hom}_C(\Delta(\lambda'), \Delta(\lambda'')) = 0$.

Lemma 4.31. Suppose that $R$ is a field. Then the two definitions of split highest weight category coincide.

That the usual definition implies Rouquier’s definition is clear. Hence we prove the converse. If $A$ is module-finite over a field $R$, it is a finite dimensional $R$-algebra, so $A$-mod is automatically an Artin category over $R$. By the conditions (ii) and (iv) we have $\text{Hom}_C(\lambda, \Delta(\lambda)) = \text{End}_C(\Delta(\lambda))$. If $\text{Top}(\lambda)$ is not irreducible, then $\dim_R \text{End}_C(\lambda) \geq 2$. Thus, $L(\lambda) = \text{Top}(\lambda)$ is irreducible. Now (i) is clear. If $[\text{Rad}(\lambda) : L(\mu)] \neq 0$ then we have a nonzero $\varphi : P(\mu) \to \text{Rad}(\lambda)$. Consider the $\Delta$-filtration $P(\mu) = 0 \supseteq F_1 = \text{Ker}(P(\mu) \otimes \Delta(\lambda))$. If $\varphi(F_1) = 0$ then $\varphi$ induces a nonzero $\Delta(\lambda) \to \Delta(\lambda)$, which contradicts $\dim_R \text{End}_C(\Delta(\lambda)) = 1$. Hence, there exists $\nu > \mu$ such that $\varphi$ induces a nonzero homomorphism $\Delta(\nu) \to \Delta(\lambda)$. Thus, $\lambda \geq \nu > \mu$ and (ii) is proved.

Theorem 4.32 (Guay). If $R$ is a field then the category $O$ for $H_R(\mathfrak{g}, h)$ is a split highest weight category.

In fact, by the argument which uses eigenvalues of $E_{\gamma}$, $\text{Hom}_O(X, Y)$ is a finitely generated $R$-module, for $X, Y \in O$, when $R$ is a Noetherian local ring. Hence $O$ is $\text{Hom}$-finite. The other conditions but the existence of projective objects $P(E)$ are already proved in Proposition 4.17. By Theorem 4.24, $\Delta(E)$ has the projective cover, which we denote by $P(E)$. Then, $P(E)$ is a direct summand of a $\Delta$-filtered object. Thus $P(E)$ is $\Delta$-filtered. Lemma 4.21 implies that any $M \in O^\gamma$ with a finite $\Delta$-filtration admits a $\Delta$-filtration $M = F_0 \supseteq F_1 \supseteq \cdots$ with the property that $F_i / F_{i+1} = \Delta(E')$ and $F_{i+1} / F_{i+2} = \Delta(E'')$ then $E' \leq E''$. Thus, if the $\Delta$-filtration $\text{Ker}(P(E) \to \Delta(E)) = F_1 \supseteq F_2 \supseteq \cdots \supseteq F_k \supseteq \cdots$.  

has the form $F_i/F_{i+1} = \Delta(E^{(i)})$ with $E^{(i)} > E$, for $1 \leq i < k - 1$, and $F_{k-1}/F_k = \Delta(E')$ with $E' \not\simeq E$, then $E' \leq E$ and we may move $\Delta(E')$ to the top of the filtration. This implies that $\text{Top} P(E) \supseteq L(E) \oplus L(E')$, which is a contradiction.

**Definition 4.33.** Let $R$ be a commutative ring, $H$ an $R$-algebra, $\mathcal{C}$ an $R$-linear Abelian category which is a subcategory of $H$-$\text{Mod}$. An object $L$ of $\mathcal{C}$ is $R$-$\text{split}$ if the canonical $R$-module homomorphism

$$L \otimes_{\text{End}_A(L)} \text{Hom}_C(L, P) \rightarrow P$$

is a split monomorphism in $R$-$\text{Mod}$, for all projective objects $P$ of $\mathcal{C}$.

It is not named but the definition is in [39]. The next lemma is from [39, 4.10].

**Lemma 4.34.** Let $R$ be a Noetherian local ring whose residue field is $F = R/m$, $A$ a module-finite projective $R$-algebra, $\mathcal{C}(R) = A$-$\text{mod}$. We denote the full subcategory $\{ M \in \mathcal{C} \mid mM = 0 \}$ by $\mathcal{C}(F) = A \otimes_R F$-$\text{mod}$. Let $L \in \mathcal{C}(R)$. Then, $L$ is an $R$-$\text{split}$ projective object of $\mathcal{C}(R)$ if and only if

(i) $L$ is a projective $R$-module,

(ii) $L \otimes_R F$ is an $F$-$\text{split}$ projective object of $\mathcal{C}(F)$.

Suppose (i) and (ii). Then, by our assumptions on $A$ and $R$, $L$ is a projective object of $\mathcal{C}(R)$. Now, both $A$ and $L$ are free $R$-modules of finite rank and $L$ is a direct summand of a free $A$-module of finite rank. Thus

$$\text{Hom}_A(L, A) \otimes_R F = \text{Hom}_{A \otimes_R F}(L \otimes_R F, A \otimes_R F),$$

$$\text{End}_A(L) \otimes_R F = \text{End}_{A \otimes_R F}(L \otimes_R F).$$

Since $L \otimes_R F$ is $F$-split, $L \otimes_{\text{End}_A(L)} \text{Hom}_A(L, A) \otimes_R F \rightarrow A \otimes_R F$ is a monomorphism. Thus if we write $0 \rightarrow L \otimes_{\text{End}_A(L)} \text{Hom}_A(L, A) \rightarrow A \rightarrow L' \rightarrow 0$ then $\text{Tor}_1^R(F, L') = 0$. Hence, $L'$ is flat by [38, Theorem 22.3], and $L$ is $R$-split as desired. The other implication is clear.

**Lemma 4.35.** Let $R$ be a regular local ring whose residue field is $F$, $A$ a module-finite projective $R$-algebra, $\mathcal{C}(R) = A$-$\text{mod}$ and $\mathcal{C}(F) = A \otimes_R F$-$\text{mod}$. Suppose that a collection of $R$-free objects $\{ \Delta(\lambda) \mid \lambda \in \Lambda \} \subseteq \mathcal{C}(R)$ is given. If $\mathcal{C}(F)$ is a split highest weight category whose standard objects are $\{ \Delta(\lambda) \otimes_R F \mid \lambda \in \Lambda \}$, and whose projective objects are $\Delta \otimes_R F$-filtered, then $\Delta(\lambda)$ is $R$-$\text{split}$, for all $\lambda$.

As we only apply the result to $\mathcal{O}$, we add the assumption that projective objects are $\Delta$-filtered, but this is in fact automatic.

Let $\lambda$ be a maximal element of $\Lambda$, and let $P$ be a projective object of $\mathcal{C}(F)$. Then, since $\mathcal{C}(F)$ is a split highest weight category, $\Delta(\lambda) \otimes_R F$ is a projective object of $\mathcal{C}(F)$, there is a subobject $P_0$ of $P$ such that $P_0 \simeq (\Delta(\lambda) \otimes_R F)^{\oplus m}$, for some $m$, and $\text{Hom}_{\mathcal{C}(F)}(\Delta(\lambda) \otimes_R F, P/P_0) = 0$. We also have $\text{End}_{\mathcal{C}(F)}(\Delta(\lambda) \otimes_R F) = F$. Thus, $\Delta(\lambda) \otimes_R F$ is an $F$-$\text{split}$ projective object of $\mathcal{C}(F)$ by [39, Lemma 4.5]. As $\Delta(\lambda)$ is $R$-free, it is an $R$-$\text{split}$ projective object of $\mathcal{C}(R)$ by Lemma 4.34. Define

$$J = \text{Im} \left( \Delta(\lambda) \otimes_R \text{Hom}_A(\Delta(\lambda), A) \rightarrow A \right).$$

$J$ is a two-sided ideal of $A$. As $\Delta(\lambda)$ is $R$-$\text{split}$ and projective, [39, Lemma 4.5] implies that $A/J$ is a module-finite projective $R$-algebra, $\text{Hom}_A(\Delta(\lambda), A/J) = 0$ and $J \simeq (\Delta(\lambda)^{\oplus m}$, for some $m$. Thus, we may prove Lemma 4.35 by induction on $|\Lambda|$. Note that we need here $\text{Hom}_C(\Delta(\lambda'), \Delta(\lambda'')) = 0$, if $\lambda' \neq \lambda''$ and $\lambda' \neq \lambda''$, to guarantee that if $\lambda' \neq \lambda$ then $\Delta(\lambda')$ is an $A/J$-module. See [39, Lemma 4.4].
Now we have Theorem 4.36 below. The conditions (i), (ii) and (v) to be a split highest weight category are easy to check. By Theorem 4.24, we assume that objects of $\mathcal{O}(R)$ are finitely generated $R$-modules. Thus, $N = 0$ if and only if $N \otimes_R F = 0$. Hence Theorem 4.32 implies that (iii) holds. To verify (iv) by induction, we show that if $\lambda$ is maximal and $Q$ is a finitely generated projective $A/J$-module such that some $\Delta(\mu)$ with $\mu < \lambda$ appears in its $\Delta$-filtration, then we may find a finitely generated projective $A$-module $P$ such that

$$0 \to \Delta(\lambda)^{\otimes m} \to P \to Q \to 0,$$

for some $m$.

This is proved in [39, Lemma 4.9]. Take a surjective map $f : R^{\otimes m} \to \text{Ext}_A^1(Q, \Delta(\lambda))$. Then $f \in \text{Hom}_R(R^{\otimes m}, \text{Ext}_A^1(Q, \Delta(\lambda))) \simeq \text{Ext}_A^1(Q, \Delta(\lambda)^{\otimes m})$ gives the desired short exact sequence. Note that if $Q = Q_1 \oplus Q_2$ then we consider the direct sum of the short exact sequences for $Q_1$ and $Q_2$, so we may prove the assertion that $P$ is a projective $A$-module only when $Q = (A/J)^{\otimes n}$, for some $n$.

**Theorem 4.36** (Rouquier). Let $R$ be a regular local ring whose residue field $F$ contains $\mathbb{C}$. Then the category $\mathcal{O}(R)$ is a split highest weight category.

4.4. The Knizhnik-Zamolodchikov functor. In this subsection we consider the regular local ring

$$R = \mathbb{C}[[k_1 - \kappa_1, \ldots, k_{d-1} - \kappa_{d-1}, h - h]],$$

where $k_1, \ldots, k_{d-1}, h$ are indeterminates and $\kappa_1, \ldots, \kappa_{d-1}, h \in \mathbb{C}$. Let $H_R(k, h)$ be the rational Cherednik algebra. Then $H_R(k, h) \otimes_R \mathbb{C} = H_R(k, h)$. We denote the category $\mathcal{O}$ for $H_R(k, h)$ by $\mathcal{O}(R)$, and the category $\mathcal{O}$ for $H_R(k, h)$ by $\mathcal{O}$ itself.

**Definition 4.37.** For $M \in \mathcal{O}(R)$, we denote the sheaf $(\mathcal{O}_{V_{\text{reg}}} \otimes_{\mathbb{C}} R) \otimes_{\mathcal{O}(V)} \mathcal{O} M$ on $V_{\text{reg}}$ by $\hat{M}$. Similarly, we define $\hat{M}$, for $M \in \mathcal{O}$.

Let $X$ be a connected smooth algebraic variety over $\mathbb{C}$, let $\mathcal{M}$ be a free $O_X$-module of finite rank with a flat connection $\nabla$. Then we say that $\mathcal{M}$ is regular if $p^*\mathcal{M} = D_{C \times X} \otimes_{p^{-1}D_X} p^{-1}\mathcal{M}$, where $D_{C \times X} = O_C \otimes_{p^{-1}O_X} p^{-1}D_X$, for any smooth curve $C$ and $p : C \to X$, has regular singularities.

Let $X = V_{\text{reg}}$. Then $D_X \mathcal{M} \otimes_{\mathcal{O}} R$ is a localization of $H_R(k, h)$ with respect to $\mathcal{O}(X)$. For each $M \in \mathcal{O}(R)$, we may consider $\hat{M}$ as a $\mathcal{D}_X \mathcal{M} \otimes_{\mathcal{O}} R$-module via

$$\nabla_{\partial_y} = y - \sum_{a \in A} \frac{\langle \alpha_H, y \rangle}{\alpha_H} a_H.$$

Similarly, $\hat{M}$ is a $\mathcal{D}_X \mathcal{M}$-module, for $M \in \mathcal{O}$.

**Example 4.38.** If $M = \Delta(E) = R[x_1, \ldots, x_n] \otimes_{\mathbb{C}} E$, then, for $p \in R[x_1, \ldots, x_n]$ and $v \in E$, we have

$$\nabla_{\partial_y} (p \otimes v) = (\partial_y p) \otimes v - \sum_{i=1}^n \sum_{k=0}^{d-1} k_y x_i \frac{p}{x_i} \otimes d e_{H_i, k} v$$

$$- \sum_{1 \leq i \leq j \leq n} \sum_{a=0}^{d-1} h(y_i - \zeta^a y_j) \frac{p}{x_i - \zeta^a x_j} \otimes (1 - s_{ij, a}) v.$$

It is easy to see that any $\Delta(E) \in \mathcal{O}$ gives a regular holonomic $\mathcal{D}_X \mathcal{M}$-module whose characteristic variety is $T_X X$. Hence, $\hat{M}$ is regular, for all $M \in \mathcal{O}$.
By Deligne’s Riemann-Hilbert correspondence, the category of free $O_X$-modules of finite rank with a flat connection is equivalent to the category of $C_X$-modules of finite rank. Hence, $M \in O$ defines a $C_X$-module $Hom_{\mathcal{D}_{X^{an}}}(O_{X^{an}}, O_{X^{an}} \otimes O_X M)$. Here, the $O_{X^{an}}$ module is nothing but the $\mathcal{D}_{X^{an}}$ module $\mathcal{D}_{X^{an}}/\sum_{i=1}^{n} D_{X^{an}} \partial_{\omega_{i}}$. The $C_X$-module $\mathcal{O}$ is nothing but the sheaf of horizontal sections of $O_{X^{an}} \otimes O_X M$. As our connection satisfies $w \nabla_{\omega_{i}} w^{-1} = \nabla_{\omega_{i}(w)}$ for $w \in W$, it is in the category of $W$-equivariant $C_X$-modules of finite rank. The latter is equivalent to the category of $C_X/W$-modules of finite rank, and, by taking the monodromy, it is equivalent to the category of finite dimensional $C\pi_1(X/W)$-modules. Let us denote it by $\mathbb{C}\pi_1(X/W)$-mod. Hence, we have obtained an exact functor

$$KZ: O \rightarrow \mathbb{C}\pi_1(X/W)$$

The following is proved in [19, Proposition 5.9].

**Lemma 4.39.** Suppose that $v_i = \exp(2\pi \sqrt{-1}(\kappa_i + \frac{1}{2}))$, $1 \leq i \leq d - 1$, are different from $1$ and pairwise distinct, $q = \exp(2\pi \sqrt{-1}h) \neq -1$. Let $M, N \in O$ and suppose that $N$ is $\Delta$-filtered. Then $Hom_{\mathcal{D}_{X^{an}}}(O_{X^{an}}, M)$ is a locally constant free $C_X \otimes \mathbb{C} R$-module of rank $\sum_{E \in \text{Irr}} W | M : \Delta(E) | \dim_{\mathbb{C}} E$.

To show this, set $t_i = k_i - \kappa_i$, for $1 \leq i < d$, and $t_d = h - h$, then, we write $p = \sum_{n} p_n t^n M \in M$, where $p_n \in O_X$, and solve $\nabla_{\omega_{i}} p = 0$ by recursively solving the system of equations for $p_n$.

Let $K = \mathbb{C}((k_1 - \kappa_1, \ldots, k_{d-1} - \kappa_{d-1}, h - h))$ and denote by $\mathcal{O}(K)$ the category $\mathcal{O}$ for $H_K(k, h)$. We have $KZ(K) : \mathcal{O}(K) \rightarrow K\pi_1(X/W)$-mod. Results in [10] imply the following.

**Theorem 4.41** (Broué-Malle-Rouquier). There is an explicit choice of generators $\sigma_0, \ldots, \sigma_{n-1}$ of $\pi_1(X/W)$ such that the defining relations are given by

$$(\sigma_0 \sigma_1)^2 = (\sigma_1 \sigma_0)^2, \quad \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} (i \neq 0), \quad \sigma_i \sigma_j = \sigma_j \sigma_i (j \geq i + 2).$$

Further, $KZ(K)(M)$, for $M \in O(K)$, factors through

$$(\sigma_0 - 1)(\sigma_0 - v_1) \cdots (\sigma_0 - v_{d-1}) = 0, \quad (\sigma_i - 1)(\sigma_i + q) = 0, \quad \text{if } i \neq 0,$$

where $v_i = \exp(2\pi \sqrt{-1}(k_i + \frac{1}{2}))$, for $1 \leq i \leq d - 1$, and $q = \exp(2\pi \sqrt{-1}h)$.

For the proof of the second part, consider the KZ functor over $\mathbb{C}[k, h]$, which we denote by $KZ(\mathbb{C}[k, h])$. It is an exact functor by [11, Theorem 2.23]. If $(k, h)$ is in certain open dense subset of generic enough parameters, it is proved in [10] that $KZ(\Delta(CW))$ factors through the Hecke algebra. Now, we have

$$E_{pq}^2 = \text{Tor}^p_{\mathbb{C}[k, h]}(R^{-q}KZ(\mathbb{C}[k, h])(M), \mathbb{C}) \Rightarrow R^{p+q}KZ(M \otimes_{\mathbb{C}[k, h]} \mathbb{C}),$$
if $M \in \mathcal{O}(\mathbb{C}[k,h])$ is a flat $\mathbb{C}[k,h]$-module, by the Künneth spectral sequence. $E^2_{p,q} = 0$ if $q \neq 0$. Let $M = \Delta(\mathbb{C}W) \otimes_{\mathbb{C}} \mathbb{C}[k,h]$. Then $KZ(\mathbb{C}[k,h])(M)$ is also a flat $\mathbb{C}[k,h]$-module by Lemma 4.40. Hence

$$KZ(\mathbb{C}[k,h])(\Delta(\mathbb{C}W) \otimes_{\mathbb{C}} \mathbb{C}[k,h]) \otimes_{\mathbb{C}[k,h]} \mathbb{C} \simeq KZ(\Delta(\mathbb{C}W)),$$

and the right hand side factors through the Hecke algebra for the generic enough $(\mathfrak{g},h)$'s. Thus, $KZ(K)(\Delta(\mathbb{C}W) \otimes_{\mathbb{C}} K)$ factors through the Hecke algebra. Now, $\mathcal{O}(K)$ is a semisimple category whose irreducible objects are standard objects, which are direct summands of $\Delta(\mathbb{C}W) \otimes_{\mathbb{C}} K$. Hence, the result follows.

**Definition 4.42.** We denote by $\mathcal{H}_n(\mathfrak{g},q)$ the quotient of $R\pi_1(X/W)$ by the two-sided ideal generated by $(\sigma_0 - 1)(\sigma_0 - v_1) \cdots (\sigma_0 - v_{d-1})$ and $(\sigma_i - 1)(\sigma_i + q)$, for $i \neq 0$. Note that if $v_i = q^{\gamma_i}$, for $1 \leq i \leq d - 1$, then $\mathcal{H}_n(\mathfrak{g},q) = \mathcal{H}_n(\mathfrak{g})$ with $\Lambda = \Lambda_0 + \sum_{i=1}^{d-1} \Lambda_{\gamma_i}$.

**Lemma 4.43.** Let $R$ be a Noetherian commutative ring. Let $A$ be a module-finite projective $R$-algebra, $B$ an $R$-algebra. Suppose that there exists an exact functor $\mathcal{F} : A\text{-mod} \to B\text{-mod}$ such that $\mathcal{F}(A)$ is a finitely generated projective $R$-module. Then, $\mathcal{F} \simeq \text{Hom}_A(P,-)$, where the finitely generated projective $A$-module $P$ is given by the $(A,B)$-bimodule $P = \text{Hom}_R(\mathcal{F}(A),A)$. Hence $\text{Hom}_A(P,A) = \mathcal{F}(A)$. Thus, $\mathcal{G} = \text{Hom}_R(\mathcal{F}(A),-)$ gives the right adjoint of $\mathcal{F}$. If $B = \text{End}_A(P)^{\text{op}}$ then $\mathcal{F}G(M) \simeq M$, for $M \in B\text{-mod}$.

The starting point of Rouquier’s theory of quasihereditary covers is the theorem below, and we are reduced to a purely algebraic setting.

**Theorem 4.44.** Let $R = \mathbb{C}[k_1 - \kappa_1, \ldots, k_{d-1} - \kappa_{d-1}, h - h]$ and suppose that $v_i = \exp(2\pi i (\kappa_i + \frac{1}{d}))$, $1 \leq i \leq d - 1$, are different from 1 and pairwise distinct, $q = \exp(2\pi i h) \neq -1$. Then there is a projective object $P_{KZ} \in \mathcal{O}(R)$ such that

1. $KZ(R) \simeq \text{Hom}_{\mathcal{O}(R)}(P_{KZ},-): \mathcal{O}(R) \to \mathcal{H}_n(\mathfrak{g},q)$-mod, and it induces $KZ \simeq \text{Hom}_{\mathcal{O}}(P_{KZ} \otimes_R \mathbb{C},-): \mathcal{O} \to \mathcal{H}_n(\mathfrak{g},q)$-mod.

2. $\mathcal{H}_n(\mathfrak{g},q) \simeq \text{End}_{\mathcal{O}(R)}(P_{KZ})^{\text{op}}$ and $\mathcal{H}_n(\mathfrak{g},q) \simeq \text{End}_{\mathcal{O}}(P_{KZ} \otimes_R \mathbb{C})^{\text{op}}$.

3. $\mathcal{F} = KZ(R)$ has a right adjoint functor $G : \mathcal{H}_n(\mathfrak{g},q)$-mod $\to \mathcal{O}(R)$ and $\mathcal{F}G \simeq \text{Id}$.

First, we consider the case that $R = \mathbb{C}[k,h]_p$ where $p = (k - \kappa, h - h)$. Then the $KZ$ functor is representable by a projective object $P_{KZ}$ of $\mathcal{O}(R)$ by Lemma 4.43. As $P_{KZ}$ is a right $\text{End}_{\mathcal{O}(R)}(P_{KZ})^{\text{op}}$-module, $KZ(P_{KZ}) = \text{End}_{\mathcal{O}(R)}(P_{KZ})$ is a $(\text{End}_{\mathcal{O}(R)}(P_{KZ})^{\text{op}}, \text{End}_{\mathcal{O}(R)}(P_{KZ})^{\text{op}})$-bimodule. Namely, $a \cdot \varphi \cdot b = ab\varphi$, for $\varphi \in KZ(P_{KZ})$ and $a, b \in \text{End}_{\mathcal{O}(R)}(P_{KZ})^{\text{op}}$. Using the right action, we may define a $R$-algebra homomorphism $\rho : \mathcal{H}_n(\mathfrak{g},q) \to \text{End}_{\mathcal{O}(R)}(P_{KZ})^{\text{op}}$ by $h \cdot \varphi = \varphi \cdot \rho(h)(= \rho(h)\varphi)$, for $h \in \mathcal{H}_n(\mathfrak{g},q)$ and $\varphi \in KZ(P_{KZ})$. In particular, the $\mathcal{H}_n(\mathfrak{g},q)$-module structure on $KZ(P_{KZ})$ is the pullback of the $\text{End}_{\mathcal{O}(R)}(P_{KZ})^{\text{op}}$-module structure via $\rho$.

Suppose that $(\mathfrak{g}, h)$ is generic enough, then $\mathcal{O}$ is a semisimple category and $\text{End}_{\mathcal{H}_n(\mathfrak{g},q)}(KZ(\Delta(E))) \simeq \text{End}_{\mathcal{O}}(\Delta(E)) = \mathbb{C}$ implies that $\{KZ(\Delta(E)) \mid E \in \text{Irr} W\}$ is a set of pairwise non-isomorphic simple $\mathcal{H}_n(\mathfrak{g},q)$-modules. As $\dim_{\mathbb{C}} KZ(\Delta(E)) = \dim E$, we have

$$P_{KZ} \otimes_R \mathbb{C} = \bigoplus_{E \in \text{Irr} W} \Delta(E)^{\oplus \dim E}$$
and $\text{End}_\mathcal{O}(P_{KZ} \otimes_R \mathbb{C}) \simeq \oplus_{E \in \text{Irr } W} \text{End}_\mathcal{O}(KZ(\Delta(E)))$. Therefore,

$$\rho \otimes_R \mathbb{C} : \mathcal{H}_n(\mathbf{v}, \mathbf{q}) \rightarrow \text{End}_\mathcal{O}(P_{KZ} \otimes_R \mathbb{C})^{\text{op}}$$

is surjective. Comparing the dimensions, we have

$$\mathcal{H}_n(\mathbf{v}, \mathbf{q}) \otimes_R \mathbb{C} \simeq \text{End}_\mathcal{O}(P_{KZ} \otimes_R \mathbb{C})^{\text{op}} \simeq \text{End}_\mathcal{O}(R)(P_{KZ})^{\text{op}} \otimes \mathcal{O}(R) \mathbb{C}$$

and

$$\mathcal{H}_n(\mathbf{v}, \mathbf{q}) \otimes_R \mathbb{C}(\mathbf{k}, \mathbf{h}) \simeq \text{End}_\mathcal{O}(P_{KZ} \otimes_R \mathbb{C}(\mathbf{k}, \mathbf{h}))^{\text{op}}.$$ Hence, we also have

$$\mathcal{H}_n(\mathbf{v}, \mathbf{q}) \otimes_R K \simeq \text{End}_\mathcal{O}(P_{KZ} \otimes_R K)^{\text{op}}.$$}

Now we return to $R = \mathbb{C}[[k_1 - \kappa_1, \ldots, k_d - \kappa_d, h - h]]$, for arbitrary $(\mathbf{k}, \mathbf{h})$. Then, we have a projective object $P_{KZ}$ of $\mathcal{O}(R)$ such that $P_{KZ} \otimes_R K$ also represents $KZ(K)$. Thus, $\rho : \mathcal{H}_n(\mathbf{v}, \mathbf{q}) \rightarrow \text{End}_\mathcal{O}(R)(P_{KZ})^{\text{op}}$ is injective. On the other hand, we have

$$P_{KZ} \otimes_R \mathbb{C} = \oplus_{E \in \text{Irr } W} P(E)^{\oplus \dim KZ(L(E))}.$$ As $\text{End}_\mathcal{O}(P_{KZ} \otimes_R \mathbb{C}) \simeq \text{End}_\mathcal{H}_n(\mathbf{v}, \mathbf{q})(KZ(P_{KZ} \otimes_R \mathbb{C}))$ by Lemma 4.39, $KZ(P(E))$, for $E$ such that $KZ(L(E)) \neq 0$, are indecomposable $\mathcal{H}_n(\mathbf{v}, \mathbf{q})$-modules and

$$\text{Top } KZ(P_{KZ} \otimes_R \mathbb{C}) = \sum_{E \in \text{Irr } W} KZ(L(E))^{\oplus \dim KZ(L(E))}.$$ Thus, $\text{Top } \text{End}_\mathcal{O}(P_{KZ} \otimes_R \mathbb{C})$ is isomorphic to the direct sum of $\text{End}_\mathcal{H}(KZ(L(E)))$ over $E$ such that $KZ(L(E)) \neq 0$, and it follows that the composition map

$$\mathcal{H}_n(\mathbf{v}, \mathbf{q}) \rightarrow \text{End}_\mathcal{O}(P_{KZ} \otimes_R \mathbb{C})^{\text{op}} \rightarrow \text{Top } \text{End}_\mathcal{O}(P_{KZ} \otimes_R \mathbb{C})^{\text{op}}$$

is surjective. Hence $\rho \otimes_R \mathbb{C}$ is surjective as well. This implies that $\rho$ is surjective, and (2) follows.

4.5. Faithful covers.

**Definition 4.45.** Let $R$ be a Noetherian commutative ring, $A$, $B$ module-finite $R$-algebras, $P$ a finitely generated projective $A$-module. Suppose that

(i) $B \simeq \text{End}_A(P)^{\text{op}},$

(ii) $A$-mod is a split highest weight category and projective $A$-modules are $\Delta$-filtered.

Then, $A$ (or $A$-mod) is a $n$-faithful quasihereditary cover of $B$ (or $B$-mod) if

$$F = \text{Hom}_A(P, -) : A$-mod \rightarrow B$-mod$$

satisfies the condition

$$\text{Ext}_A^i(M, N) \simeq \text{Ext}_{B}(F(M), F(N)) \ (0 \leq i \leq n),$$

for any $\Delta$-filtered $A$-modules $M, N$.

Let $S(\lambda) = F(\Delta(\lambda))$, for $\lambda \in \Lambda$. We call them Specht modules.

**Lemma 4.46.** Let $R$ be a Noetherian commutative ring, $A$, $B$ module-finite $R$-algebras, $P$ a finitely generated projective $A$-module, $U$ an additive full subcategory of $A$-mod. Suppose that $B \simeq \text{End}_A(P)^{\text{op}}$ and that $U$ contains $A$. Define functors $F = \text{Hom}_A(P, -)$ and $G = \text{Hom}_B(F(A), -)$. Then, we have the following.

(1) The following are equivalent.

(a) $\text{Hom}_A(M, N) \simeq \text{Hom}_B(F(M), F(N))$, for $M, N \in U$.

(b) $M \simeq G(F(M))$, for $M \in U$.

(2) Suppose that $\text{Hom}_A(M, N) \simeq \text{Hom}_B(F(M), F(N))$, for $M, N \in U$. Then, the following are equivalent.
We also consider the following conditions.

V which we write Ass(Ext^1_{\mathcal{A}}(\mathcal{F}(M), \mathcal{F}(N))) \ (i = 0, 1), for M, N \in \mathcal{U}.

Let R be a regular local ring whose residue field contains \mathbb{C}, A a module-finite projective R-algebra, P a finitely generated projective A-module. We denote

\[ \mathcal{F} = \text{Hom}_{R}(P, -) : \mathcal{C} \to B \text{-mod}, \]

\[ B = \text{End}_{R}(P)^{\text{op}} \text{ and } \mathcal{C} = A \text{-mod}, \mathcal{C}(p) = A \otimes_{R} (R/p) \text{-mod}, \text{ for } p \in \text{Spec } R. \]

We also denote the quotient field of R/p by Q(p).

We consider the following conditions, for p \in \text{Spec } R such that R/p is regular and A \otimes_{R} Q(p) is split semisimple.

(I) There exist a finite preordered set \Lambda and \{\Delta(\lambda) \in \mathcal{C} \mid \lambda \in \Lambda\} such that

(i) \Delta(\lambda) is R-free, for \lambda \in \Lambda.

(ii) \mathcal{C}(p) is a split highest weight category whose standard objects are

\[ \{\Delta(\lambda) \otimes_{R} (R/p) \in \mathcal{C} \mid \lambda \in \Lambda\}. \]

(iii) Projective objects of \mathcal{C}(p) are \Delta-filtered.

(II) B satisfies the following.

(i) \mathcal{B} \otimes_{R} (R/p) \cong \text{End}_{R}(P \otimes_{R} (R/p))^{\text{op}}.

(ii) \mathcal{F} \text{ restricts to } \mathcal{F}(p) = \text{Hom}_{R}(P \otimes_{R} (R/p), -) : \mathcal{C}(p) \to B \otimes_{R} (R/p) \text{-mod}.

(III) If M \in \mathcal{C}(p) is \Delta-filtered then \mathcal{F}(p)(M) is a finitely generated projective \mathcal{R}/p-module.

We also consider the following conditions.

(IV) \mathcal{F}(m) : A \otimes_{R} (R/m) \text{-mod} \to B \otimes_{R} (R/m) \text{-mod} is a 0-faithful cover, for the maximal ideal m \in \text{Spec } R.

(V) Let K be the quotient field of R. Then A \otimes_{R} K is a split semisimple K-algebra.

If R = \mathbb{C}[[k_1 - \kappa_1, \ldots, k_{d-1} - \kappa_{d-1}, h, h]] and suppose v_i = \exp(2\pi \sqrt{-1}(\kappa_i + \frac{q}{d})), 1 \leq i \leq d-1, are different from 1 and pairwise distinct, q = \exp(2\pi \sqrt{-1}h) \neq -1, then we already know that \mathcal{O}(R) \simeq A \text{-mod}, for some module-finite projective R-algebra A, such that A \otimes_{R} C is a 0-faithful cover of \mathcal{H}_{R}(\kappa, q) and the above conditions are satisfied. The following is a key result [39, Proposition 4.42].

**Proposition 4.47.** Let R be a regular local ring whose residue field contains \mathbb{C}, A a module-finite projective R-algebra, P a finitely generated projective A-module such that the conditions (I)-(V) are satisfied. Then A is a 1-faithful cover of B.

**Proof.** The proof is induction on dim R. If dim R = 0 then A is split semisimple and Ext^1_{\mathcal{B}}(\mathcal{F}(A), \mathcal{F}(M)) = 0, for M \in A-\text{mod}. Suppose dim R > 0. Let M \in A-\text{mod} be \Delta-filtered. We argue that Ext^1_{\mathcal{B}}(\mathcal{F}(A), \mathcal{F}(M)) \neq 0 leads to a contradiction. Define

\[ Z = \text{supp Ext}^1_{\mathcal{B}}(\mathcal{F}(A), \mathcal{F}(M)), \]

which we write V(p_1) \cup \cdots \cup V(p_r), where p_1, \ldots, p_r are minimal elements of Ass(Ext^1_{\mathcal{B}}(\mathcal{F}(A), \mathcal{F}(M))), the set of associated primes of Ext^1_{\mathcal{B}}(\mathcal{F}(A), \mathcal{F}(M)). As P and \mathcal{F}(A) are finitely presented,

\[ \text{Ext}^1_{\mathcal{B}}(\mathcal{F}(A), \mathcal{F}(M))_p = \text{Ext}^1_{\mathcal{B}}(\mathcal{F}(A_p), \mathcal{F}(M_p)). \]

Thus, we may assume r = 1 and Z = V(m), where m is the maximal ideal of R, without loss of generality. Take an open set

\[ D(\alpha) \subseteq \{p \in \text{Spec } R \mid \mathcal{C}(p) \text{ is semisimple.}\} \neq \emptyset. \]
We apply the functor and we have a morphism of exact sequences from proved that \(0\)-faithful cover of \(R\). Nakayama’s lemma implies that \(\text{Ann} \mathfrak{A} \subset R\). Consider \(m\) is an isomorphism by the induction hypothesis and Lemma 4.46 (1). Hence, \(E\) only need the exact sequence for since \(H\) page, and we obtain

\[
E_2^{p,q} = \bigoplus_{-i+j=q} \text{Tor}_R^p(R^{-q}G(F(M)), R/\pi) \implies R^{p+q}G(F(M \otimes_R (R/\pi)) \text{.}
\]

since \(H^{p+q}(C_\bullet \otimes_R (R/\pi)) \cong R^{p+q}G(F(M \otimes_R (R/\pi))\). The spectral sequence degenerates at the \(E_2\) page, and we obtain

\[
0 \to R^nG(F(M)) \otimes_R (R/\pi) \to R^nG(F(M \otimes_R (R/\pi)))
\]

\[
\to \text{Tor}_R^1(R^{n+1}G(F(M)), R/\pi) \to 0.
\]

We only need the exact sequence for \(n = 0\) below. Define

\[
\varphi : G(F(M) \otimes_R (R/\pi)) \to G(F(M \otimes_R (R/\pi))) \cong G(F(M \otimes_R (R/\pi)).
\]

Then, the composition map

\[
M \otimes_R (R/\pi) \to G(F(M) \otimes_R (R/\pi)) \xleftarrow{\varphi} G(F(M \otimes_R (R/\pi))
\]

is an isomorphism by the induction hypothesis and Lemma 4.46 (1). Hence, \(\varphi\) is surjective. Using the exact sequence above, we have

(a) \(M \otimes_R (R/\pi) \simeq G(F(M) \otimes_R (R/\pi)) \xleftarrow{\varphi} G(F(M \otimes_R (R/\pi))\),

(b) \(\text{Tor}_R^1(R^1G(F(M), R/\pi)) = 0\).

(b) implies that

\[
0 \to R^1G(F(M)) \xrightarrow{\varphi} R^1G(F(M)) \rightarrow R^1G(F(M) \otimes_R (R/\pi)) \to 0.
\]

However, \(m \in \text{Ass}(R^1G(F(M)))\) implies that there exists \(x \in R^1G(F(M))\) such that \(\text{Ann}_{\mathfrak{A}}(x) = m\), and \(\pi \in m\) implies that \(\pi x = 0\). This is absurd. We have proved that \(R^1G(F(M)) = 0\), for any \(\Delta\)-filtered \(M \in A\text{-mod}\).

Since \(F(M)\) is projective as an \(R\)-module, we have

\[
0 \rightarrow F(M) \xrightarrow{\varphi} F(M) \rightarrow F(M) \otimes_R (R/\pi) \to 0.
\]

We apply the functor \(G\) to the exact sequence and use (a). Then,

\[
0 \rightarrow G(F(M)) \xrightarrow{\varphi} G(F(M)) \rightarrow G(F(M) \otimes_R (R/\pi)) \to 0
\]

and we have a morphism of exact sequences from

\[
0 \rightarrow M \xrightarrow{\varphi} M \rightarrow M \otimes_R (R/\pi) \to 0.
\]

Let \(K = \text{Ker}(M \rightarrow G(F(M))\). Then, we may show that

\[
0 \rightarrow K \otimes_R (R/\pi) \rightarrow M \otimes_R (R/\pi) \simeq G(F(M) \otimes_R (R/\pi)).
\]

Thus, \(K \otimes_R (R/\pi) = 0\) and Nakayama’s lemma implies that \(K = 0\). Now, we consider \(M \subset G(F(M)\) and use (a) again. Then \((G(F(M)/M) \otimes_R (R/\pi) = 0\) and Nakayama’s lemma implies \(G(F(M)) \simeq M\). Lemma 4.46 (1) implies that \(A\) is a \(0\)-faithful cover of \(B\), and \(A\) is a \(1\)-faithful cover of \(B\) by Lemma 4.46 (2). \(\square\)
4.6. **Uniqueness of quasihereditary covers.** Now we are prepared to state main results in [39]. Let us start with definitions and consequences of Proposition 4.47.

**Definition 4.48.** Suppose that \( \mathcal{C} \) and \( \mathcal{C}' \) are highest weight categories over \( R \). Let \( \{ \Delta(\lambda) \mid \lambda \in \Lambda \} \) and \( \{ \Delta(\lambda') \mid \lambda' \in \Lambda' \} \) be standard objects of \( \mathcal{C} \) and \( \mathcal{C}' \) respectively. We say that \( \mathcal{C} \) and \( \mathcal{C}' \) are **equivalent highest weight categories** if there is an equivalence \( \mathcal{C} \simeq \mathcal{C}' \) and a bijection \( \Lambda \simeq \Lambda' \) such that if \( \lambda \) corresponds to \( \lambda' \) then \( \Delta(\lambda) \) goes to \( \Delta(\lambda') \otimes_R U \), for an invertible \( R \)-module \( U \).

**Definition 4.49.** Let \( R \) be a Noetherian commutative ring, \( B \) a module-finite \( R \)-algebra. Let \( \mathcal{F} : A \text{-mod} \to B \text{-mod} \) and \( \mathcal{F}' : A' \text{-mod} \to B \text{-mod} \) be quasihereditary covers. If there is an equivalence of highest weight categories \( \mathcal{K} : A \text{-mod} \simeq A' \text{-mod} \) such that \( \mathcal{F}' \mathcal{K} = \mathcal{F} \), we say that \( A \) and \( A' \) are **equivalent quasihereditary covers**.

**Theorem 4.50** ([39] Proposition 4.44, Corollary 4.45). Let \( R \) be a Noetherian commutative ring, \( B \) a module-finite \( R \)-algebra. Let \( \mathcal{F} : A \text{-mod} \to B \text{-mod} \) and \( \mathcal{F}' : A' \text{-mod} \to B \text{-mod} \) be \( \lambda \)-faithful quasihereditary covers. Suppose that the preorders on \( \text{Irr} B \) are compatible. Then \( A \) and \( A' \) are equivalent quasihereditary covers.

**Theorem 4.51** ([39] Theorem 4.48]). Let \( R \) be a Noetherian commutative domain, \( K \) its quotient field. Suppose that \( B \) is a module-finite projective \( R \)-algebra such that \( B \otimes_R K \) is split semisimple. Let \( \mathcal{F} : A \text{-mod} \to B \text{-mod} \) and \( \mathcal{F}' : A' \text{-mod} \to B \text{-mod} \) be \( \lambda \)-faithful quasihereditary covers. Suppose that the preorders on \( \text{Irr} B \) are compatible. Then \( A \) and \( A' \) are equivalent quasihereditary covers.

4.7. **The category \( \mathcal{O} \) as quasihereditary covers.** Rouquier’s motivation for developing the theory of quasihereditary covers which we have explained so far, is to prove the following. My motivation to write this survey is to explain and advertise his beautiful ideas to prove the theorem.

**Theorem 4.52** (Rouquier). Suppose that \( v_i = \exp(2\pi \sqrt{-1}(\kappa_i + \frac{1}{2})) \), \( 1 \leq i \leq d-1 \), are different from 1 and pairwise distinct, \( q = \exp(2\pi \sqrt{-1}h) \neq -1 \). Then the category \( \mathcal{O} \) for the rational Cherednik algebra \( H_C(k, h) \) is a quasihereditary cover of the Hecke algebra \( \mathcal{T}_q(v, q) \). If it is the Hecke algebra associated with the symmetric group, then \( \mathcal{O} \) and the module category of the \( q \)-Schur algebra over \( C \) are equivalent quasihereditary covers.

The idea of the proof of the second part is to lift the \( 0 \)-faithful cover over \( C \) to a \( 1 \)-faithful cover over \( R = C[[k_1 - \kappa_1, \ldots, k_{d-1} - \kappa_{d-1}, h - h]] \), by using Proposition 4.47 and apply the uniqueness result Theorem 4.51.

As we explained in 2.4, Theorem 4.52 is closely related to the Fock space theory in the second part, categorification of JMMO deformed Fock spaces, and the results of Geck and Jacon on the canonical basic sets in the first part, cellular structures of Hecke algebras. See [39] 6.5 for some conjectures in this field. I also recommend reading papers [20] and [21] by Iain Gordon.

**References**

[1] S. Ariki, On the decomposition numbers of the Hecke algebra of \( G(m, 1, n) \), *J. Math. Kyoto Univ.* 36 (1996), 789–808.

[2] S. Ariki, *Representations of quantum algebras and combinatorics of Young tableaux*. Univ. Lecture Series 26, Amer. Math. Soc., 2002. Errata in Appendix of [3].
[3] S. Ariki, Hecke algebras of classical type and their representation type, *Proc. London Math. Soc. (3)* 91 (2005), 355–413. Corrigendum in *Proc. London Math. Soc. (3)* 92 (2006), 342–344.

[4] S. Ariki, Modular representations of Hecke algebras of classical type, *Sugaku Expositions* 20 (2007), 15–41. Translated from Japanese original: *Sugaku* 56 (2004), 113–136.

[5] S. Ariki, Proof of the modular branching rule for cyclotomic Hecke algebras, *J. Algebra* 306 (2006), 290–300.

[6] S. Ariki and N. Jacon, Dipper-James-Murphy’s conjecture for Hecke algebras of type B, [arXiv:math/0703447](http://arxiv.org/abs/math/0703447).

[7] S. Ariki, V. Kreiman and S. Tsuchioka, On the tensor product of two basic representations of $U_q(sl_2)$, *Adv. Math.*, to appear.

[8] C. Bonnafé and L. Iancu, Left cells in type $B_n$ with unequal parameters, *Represent. Theory* 7 (2003), 587–609.

[9] C. Bonnafé, M. Geck, L. Iancu and T. Lam, On domino insertion and Kazhdan–Lusztig cells in type $B_n$, [arXiv:math/0609279](http://arxiv.org/abs/math/0609279).

[10] M. Geck, Modular representation theory of Hecke algebras, In: *Group representation theory* (EPFL, 2005; eds. M. Geck, D. Testerman and J. Thévenaz), 301–353, Presses Polytechniques et Universitaires Romandes, EPFL-Press, 2007.

[11] M. Geck and L. Iancu, Lusztig’s $a$-function in type $B_n$ in the asymptotic case, *Nagoya Math. J.* 182 (2006), 199–240.

[12] M. Geck and N. Jacon, Canonical basic sets in type $B_n$, *J. Algebra* 306 (2006), 104–127.

[13] M. Geck and G. Pfeiffer, *Characters of finite Coxeter Groups and Iwahori-Hecke algebras*. London. Math. Soc. Monographs, New Series 21, Oxford, 2000.

[14] I. Gordon, *Introduction to Quantum Groups and Crystal Bases*. Graduate Stud-ies in Math. 42, Amer. Math. Soc., 2002.

[15] A. Joseph, *Quantum Groups and Their Primitive Ideals*. Ergebnisse der Mathematik und ihrer Grenzgebiete 29, Springer-Verlag, 1995.

[16] M. Kashiwara, *On crystal bases*, *CMS. Conf. Proc.* 16, (1995), 155–197.

[17] M. Kashiwara, *Bases cristallines des groupes quantiques*. Cours Spécialisés 9, Soc. Math. France, 2002.

[18] A. Konig and C. C. Xi, *Cellular algebras: inflations and Morita equivalences*, *J. London Math. Soc. (2)* 60 (1999), 706–722.
[34] B. Leclerc and J. Y. Thibon, Canonical bases of $q$--deformed Fock spaces, *Int. Math. Res. Notices* **2** (1996), 447–456.

[35] S. Lyle and A. Mathas, Blocks of cyclotomic Hecke algebras, *Adv. Math.* **216** (2007), 854–878.

[36] A. Mathas, *Iwahori-Hecke Algebras and Schur Algebras of the Symmetric Group*. Univ. Lecture Series 15, Amer. Math. Soc., 1999.

[37] A. Mathas, The representation theory of the Ariki-Koike and cyclotomic $q$-Schur algebras, In: *Representation theory of algebraic groups and quantum groups*. Adv. Studies Pure Math., 261–320, 2004.

[38] H. Matsumura, *Commutative Ring Theory*. Cambridge studies in advanced mathematics 8, Cambridge University Press, 1989.

[39] R. Rouquier, $q$-Schur algebras and complex reflection groups I, [arXiv:math/0509252](http://arxiv.org/abs/math/0509252).

[40] M. Varagnolo and E. Vasserot, On the decomposition matrices of the quantized Schur algebra, *Duke Math. J.* **100** (1999), 267–297.

[41] C. C. Xi and Dajing Xiang, Cellular algebras and Cartan matrices, *Linear Algebra Appl.* **365** (2003), 369–388.

S.A.: Research Institute for Mathematical Sciences, Kyoto University, Kyoto 606-8502, Japan

E-mail address: ariki@kurims.kyoto-u.ac.jp