On the chromatic number of almost $s$-stable Kneser graphs

Peng-An Chen
Department of Applied Mathematics
National Taitung University
Taitung, Taiwan

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Abstract

In 2011, Meunier conjectured that for positive integers $n, k, r, s$ with $k \geq 2$, $r \geq 2$, and $n \geq \max\{r, s\}k$, the chromatic number of $s$-stable $r$-uniform Kneser hypergraphs is equal to $\lceil \frac{n-\max\{r, s\}(k-1)}{r-1} \rceil$. It is a strengthened version of the conjecture proposed by Ziegler (2002), and Alon, Drewnowski and Luczak (2009). The problem about the chromatic number of almost $s$-stable $r$-uniform Kneser hypergraphs has also been introduced by Meunier (2011).

For the $r = 2$ case of the Meunier conjecture, Jonsson (2012) provided a purely combinatorial proof to confirm the conjecture for $s \geq 4$ and $n$ sufficiently large, and by Chen (2015) for even $s$ and any $n$. The case $s = 3$ is completely open, even the chromatic number of the usual almost $s$-stable Kneser graphs.

In this paper, we obtain a topological lower bound for the chromatic number of almost $s$-stable $r$-uniform Kneser hypergraphs via a different approach. For the case $r = 2$, we conclude that the chromatic number of almost $s$-stable Kneser graphs is equal to $n - s(k - 1)$ for all $s \geq 2$. Set $t = n - s(k - 1)$. We show that any proper coloring of an almost $s$-stable Kneser graph must contain a completely multicolored complete bipartite subgraph $K_{\lceil \frac{t}{2} \rceil \lceil \frac{t}{2} \rceil}$. It follows that the local chromatic number of almost $s$-stable Kneser graphs is at least $\lceil \frac{t}{2} \rceil + 1$. It is a strengthened result of Simonyi and Tardos (2007), and Meunier’s (2014) lower bound for almost $s$-stable Kneser graphs.
1 Introduction

Let \([n]\) denote the set \(\{1, 2, \ldots, n\}\). For every nonempty subset \(S\) of \([n]\), \(\max(S)\) denotes the maximal element of \(S\). In particular, we define \(\max(\emptyset) = 0\). A mapping \(c : V \rightarrow [m]\) is a proper coloring of a graph \(G = (V, E)\) with \(m\) colors if none of the edges \(e \in E\) is monochromatic under \(c\). The chromatic number \(\chi(G)\) of a graph \(G\) is the smallest number \(m\) such that a proper coloring \(c : V \rightarrow [m]\) exists. For positive integers \(n, k\) and \(s\), a \(k\)-subset \(S \subseteq [n]\) is \(s\)-stable (resp. almost \(s\)-stable) if \(|S| = k\) and any two of its elements are at least "at distance \(s\) apart" on the \(n\)-path, that is, if \(s \leq |i - j| \leq n - s\) (resp. \(|i - j| \geq s\)) for distinct \(i, j \in S\). Hereafter, the symbols \(\binom{n}{k}\), \(\binom{n}{k}_{s\text{-stab}}\), and \(\binom{n}{k}_{s\text{-stab}}\) stand for the collection of all \(k\)-subsets of \([n]\), the collection of all \(s\)-stable \(k\)-subsets of \([n]\), and the collection of all almost \(s\)-stable \(k\)-subsets of \([n]\), respectively. Choosing \(k = 1\), we obtain that \(\binom{n}{1} = \binom{n}{1}_{s\text{-stab}} = \binom{n}{1}_{s\text{-stab}}\). One can easily check that \(\binom{n}{k}_{s\text{-stab}} \subseteq \binom{n}{k}_{s\text{-stab}} \subseteq \binom{n}{k}\).

The Kneser graph, denoted as \(KG(n, k)\), is defined for positive integers \(n \geq 2k\) as the graph having \(\binom{n}{k}\) as vertex set. Two vertices are defined to be adjacent in \(KG(n, k)\) if they are disjoint. Choosing \(k = 1\), we obtain the complete graph \(K_n\). The \(s\)-stable Kneser graph, denoted as \(KG^2(n, k)_{s\text{-stab}}\), is defined for positive integers \(n \geq sk\) as the graph having \(\binom{n}{k}_{s\text{-stab}}\) as vertex set. Two vertices are defined to be adjacent in \(KG^2(n, k)_{s\text{-stab}}\) if they are disjoint. Choosing \(s = 1\), we obtain the Kneser graph \(KG(n, k)\), whereas \(s = 2\) yields the stable Kneser graph or Schrijver graph, denoted as \(SG(n, k)\). The almost \(s\)-stable Kneser graph, denoted as \(KG^2(n, k)_{s\text{-stab}}\), is defined for positive integers \(n \geq sk\) as the graph having \(\binom{n}{k}_{s\text{-stab}}\) as vertex set. Two vertices are defined to be adjacent in \(KG^2(n, k)_{s\text{-stab}}\) if they are disjoint.

Kneser [18] conjectured that the chromatic number \(\chi(KG(n, k))\) of the Kneser graph \(KG(n, k)\) is equal to \(n - 2k + 2\). Kneser’s conjecture [18] was proved by Lovász [22] using the Borsuk-Ulam theorem; all subsequent proofs, extensions and generalizations also relied on Algebraic Topology results, namely the Borsuk-Ulam theorem and its extensions. Matoušek [23] provided the first combinatorial proof of Kneser’s conjecture [18]. Schrijver [28] found a fascinating family of subgraphs \(SG(n, k)\) of \(KG(n, k)\) that are vertex-critical with respect to the chromatic number. Ziegler [33, 34] provided a combinatorial proof of Schrijver’s theorem [28]. Meunier [24] provided another simple combinatorial proof of Schrijver’s theorem [28].
A hypergraph $\mathcal{H}$ is a pair $\mathcal{H} = (V(\mathcal{H}), E(\mathcal{H}))$, where $V(\mathcal{H})$ is a finite set and $E(\mathcal{H})$ a family of subsets of $V(\mathcal{H})$. The set $V(\mathcal{H})$ is called the vertex set and the set $E(\mathcal{H})$ is called the edge set. Let $r$ be any positive integer with $r \geq 2$. A hypergraph is said to be $r$-uniform if all its edges $S \in \mathcal{H}$ have the same cardinality $r$. A proper coloring of a hypergraph $\mathcal{H}$ with $t$ colors is a function $c : V \rightarrow [t]$ so that no edge $S \in \mathcal{H}$ is monochromatic, that is, every edge contains two elements $i, j \in S$ with $c(i) \neq c(j)$. Equivalently, no $c^{-1}(i)$ contains a set $S \in \mathcal{H}$. The chromatic number $\chi(\mathcal{H})$ of a hypergraph $\mathcal{H}$ is the smallest number $t$ such that there exists a $t$-coloring for $\mathcal{H}$. Throughout this paper, we suppose that $V(\mathcal{H}) = [n]$ for some positive integer $n$.

For any hypergraph $\mathcal{H} = (V(\mathcal{H}), E(\mathcal{H}))$ and positive integer $r \geq 2$, the general Kneser hypergraph $KG^r(\mathcal{H})$ of $\mathcal{H}$ is an $r$-uniform hypergraph has $E(\mathcal{H})$ as its vertex set and the edge set consisting of all $r$-tuples of pairwise disjoint edges of $\mathcal{H}$. The Kneser hypergraph $KG^r([n])$ is an $r$-uniform hypergraph which has $([n])$ as vertex set and whose edges are formed by the $r$-tuples of disjoint $k$-element subsets of $[n]$. Choosing $r = 2$, we obtain the ordinary Kneser graph $KG(n, k)$. Hereafter, for positive integers $n, k, r$ with $r \geq 2$ and $n \geq rk$, the hypergraph $KG^r([n])$ is denoted by $KG^r(n, k)$. Erdős [10] conjectured that, for $n \geq rk$,

$$\chi(KG^r(n, k)) = \left\lceil \frac{n - r(k - 1)}{r - 1} \right\rceil.$$

The conjecture settled by Alon, Frankl and Lovász [5]. Above-mentioned results are generalized in many ways. One of the most promising generalizations is the one found by Dol’nikov [9] and extended by Krč [19, 20]. Finding a lower bound for chromatic number of Kneser hypergraphs has been studied in the literature, see [9, 19, 20, 26, 29, 30, 33, 34].

The $r$-colorability defect of $\mathcal{H}$, denoted by $cd^r(\mathcal{H})$, is the minimum number of vertices that should be removed from $\mathcal{H}$ so that the induced subhypergraphs on the remaining vertices has the chromatic number at most $r$. Dol’nikov [9] (for $r = 2$) and Krč [19, 20] proved that

**Theorem 1** (Dol’nikov-Krč theorem). Let $\mathcal{H}$ be a hypergraph and assume that $\emptyset$ is not an edge of $\mathcal{H}$. Then

$$\chi(KG^r(\mathcal{H})) \geq \left\lceil \frac{cd^r(\mathcal{H})}{r - 1} \right\rceil$$

for any integer $r \geq 2$. 
These results were also generalized by Ziegler \cite{33,34}. Note that if we set \( \mathcal{H} = \left( [n], \binom{[n]}{k} \right) \), then this result implies \( \chi(KG^r(n, k)) = \left\lceil \frac{n-r(k-1)}{r-1} \right\rceil \) bringing in Alon, Frankl and Lovász’s result [5].

The \( s \)-stable \( r \)-uniform Kneser hypergraph \( KG^r(n, k)_{s\text{-stable}} \) is an \( r \)-uniform hypergraph which has \( \binom{n}{k}_{s\text{-stable}} \) as vertex set and whose edges are formed by the \( r \)-tuples of disjoint \( s \)-stable \( k \)-element subsets of \( [n] \). Note that an \( s \)-stable 2-uniform Kneser hypergraph is simply an \( s \)-stable Kneser graph. Hereafter, for positive integers \( n, k, r, s \) with \( k \geq 2, r \geq 2 \) and \( n \geq \max(\{r, s\})k \), the hypergraph \( KG^r(n, k)_{s\text{-stable}} \) is denoted by \( KG^r(n, k)_{s\text{-stable}} \).

The \textit{almost} \( s \)-stable \( r \)-uniform Kneser hypergraph \( KG^r(n, k)_{\sim s\text{-stab}} \) is an \( r \)-uniform hypergraph which has \( \binom{n}{k}_{\sim s\text{-stab}} \) as vertex set and whose edges are formed by the \( r \)-tuples of disjoint almost \( s \)-stable \( k \)-element subsets of \( [n] \). Note that an almost \( s \)-stable 2-uniform Kneser hypergraph is simply an almost \( s \)-stable Kneser graph. Hereafter, for positive integers \( n, k, r, s \) with \( k \geq 2, r \geq 2 \) and \( n \geq \max(\{r, s\})k \), the hypergraph \( KG^r(n, k)_{\sim s\text{-stab}} \) is denoted by \( KG^r(n, k)_{\sim s\text{-stab}} \).

Ziegler \cite{33,34} gave a combinatorial proof of the Alon-Frankl-Lovász theorem [5]. He was inspired by a combinatorial proof of the Lovász theorem found by Matoušek [23]. At the end of his paper, Ziegler made the supposition that \( \chi(KG^r(n, k)_{r\text{-stab}}) = \chi(KG^r(n, k)) \) for any \( n \geq rk \). Alon, Drewnowski and Luczak make this supposition an explicit conjecture in [4].

**Conjecture 1.** Let \( n, k, \) and \( r \) be positive integers such that \( k \geq 2, r \geq 2, \) and \( n \geq rk \). Then \( \chi(KG^r(n, k)_{r\text{-stab}}) = \left\lceil \frac{n-r(k-1)}{r-1} \right\rceil = \chi(KG^r(n, k)) \).

Alon, Drewnowski, and Luczak [4] also confirmed Conjecture 1 for \( r \) is a power of 2. Meunier [24] proposed the following conjecture:

**Conjecture 2.** Let \( n, k, r, \) and \( s \) be positive integers such that \( k \geq 2, r \geq 2, \) and \( n \geq \max(\{r, s\})k \). Then \( \chi(KG^r(n, k)_{s\text{-stab}}) = \left\lceil n-\max(\{r, s\})(k-1) \right\rceil \).

The case \( s = r \) is the Alon-Drewnowski-Luczak-Ziegler conjecture \cite{1,33,34}. Conjecture 2 is a strengthened version of Conjecture 1.

Ziegler \cite{33,34} pointed out that for positive integers \( n, k, r, s \) with \( r \geq 2, k \geq 2, \) and \( n \geq sk \), \( cd^r\left( \binom{n}{k}_{s\text{-stab}} \right) = \max(\{n-rs(k-1), 0\}) \). Since \( KG^r(n, k)_{s\text{-stab}} \) is an induced subhypergraph of \( KG^r(n, k)_{\sim s\text{-stab}} \), \( cd^r\left( \binom{n}{k}_{\sim s\text{-stab}} \right) \geq cd^r\left( \binom{n}{k}_{s\text{-stab}} \right) \). We can easily obtain the following \( r \)-colorability defect result of \( \binom{n}{k}_{s\text{-stab}} \) with the same proof work of Ziegler \cite{33,34}.
Lemma 1. Let $n, k, r, s$ be positive integers such that $r \geq 2$, $k \geq 2$, and $n \geq sr$. Then $\cd^r \binom{n}{k} \tilde{s}_{\text{stab}} = \cd^r \binom{n}{k} s_{\text{stab}} = \max \{n - rs(k-1), 0\}$.

Combining the Dol’nikov-Kříž Theorem and Lemma 1, we have that

$$\chi(KGr(n, k) s_{\text{stab}}) \geq \left\lceil \frac{n - rs(k-1)}{r-1} \right\rceil$$

and

$$\chi(KGr(n, k) \tilde{s}_{\text{stab}}) \geq \left\lceil \frac{n - rs(k-1)}{r-1} \right\rceil$$

for any $n \geq rs(k-1) + \max \{r, s\}$. We would like to find the substantial improvement of the Dol’nikov-Kříž lower bounds for $\chi(KGr(n, k) s_{\text{stab}})$ and $\chi(KGr(n, k) \tilde{s}_{\text{stab}})$.

First, we prove the following lemma, the proof follows a very similar scheme as the proofs of Ziegler [33, 34] and Meunier [24].

Lemma 2. Let $n, k, r, s$ be positive integers such that $k \geq 2$, $r \geq 2$, and $n \geq \max \{r, s\}k$. Then $\chi(KGr(n, k) s_{\text{stab}}) \leq \chi(KGr(n, k) \tilde{s}_{\text{stab}}) \leq \left\lceil \frac{n - \max \{r, s\}(k-1)}{r-1} \right\rceil$.

Proof. Since $KGr(n, k) \tilde{s}_{\text{stab}}$ is an induced subhypergraph of $KGr(n, k)$. If $r > s$, then $\chi(KGr(n, k) s_{\text{stab}}) \leq \chi(KGr(n, k) \tilde{s}_{\text{stab}}) \leq \chi(KGr(n, k)) = \left\lceil \frac{n - r(k-1)}{r-1} \right\rceil = \left\lceil \frac{n - \max \{r, s\}(k-1)}{r-1} \right\rceil$.

Assume that $r \leq s$. Let $t = \left\lfloor \frac{n - s(k-1)}{r-1} \right\rfloor$. Define the map

$$c : \binom{[n]}{k} \tilde{s}_{\text{stab}} \longrightarrow [t]$$

as follows:

Assume $S = \{i_1, i_2, \ldots, i_k\} \in \binom{[n]}{k} \tilde{s}_{\text{stab}}$ with $1 \leq i_1 < i_2 < \cdots < i_k \leq n$. Since $S$ is almost $s$-stable, $i_{k-j} \leq n - sj$ for $j = 0, \ldots, k-1$. Then we define its color

$$c(S) := \left\lfloor \frac{i_1}{r-1} \right\rfloor = \left\lfloor \frac{\min(S)}{r-1} \right\rfloor.$$

Thus we obtain a value $c(S)$ in $[t]$. One can easily check that $c$ is a proper coloring of $KGr(n, k) \tilde{s}_{\text{stab}}$. The proof is complete. \qed
It is obvious that the $s$-stable $r$-uniform Kneser hypergraph $\overline{KG^r(n, k)_{s-stab}}$ is an induced subhypergraph of the almost $s$-stable $r$-uniform Kneser hypergraph $\overline{KG^r(n, k)_{s-stab}}$. Combining Lemma 2, Conjecture 1, and Conjecture 2 we propose two conjectures with related to the almost $s$-stable $r$-uniform Kneser hypergraph $\overline{KG^r(n, k)_{s-stab}}$ as follows.

**Conjecture 3.** Let $n, k, r$ be positive integers such that $k \geq 2$, $r \geq 2$, and $n \geq rk$. Then $\chi(\overline{KG^r(n, k)_{s-stab}}) = \lceil \frac{n-r(k-1)}{r-1} \rceil = \chi(\overline{KG^r(n, k)})$.

**Conjecture 4.** Let $n, k, r, s$ be positive integers such that $k \geq 2$, $r \geq 2$, and $n \geq \max\{r, s\}k$. Then $\chi(\overline{KG^r(n, k)_{s-stab}}) = \lceil \frac{n-\max\{r, s\}(k-1)}{r-1} \rceil$.

## 2 Main results

In this section, we first discuss some proper coloring results about the chromatic number of $\overline{KG^r(n, k)_{s-stab}}$ for $r \leq s$. In addition, we show that Conjecture 3 is equivalent to Conjecture 4. Next, we obtain a topological lower bound of $\chi(\overline{KG^p(n, k)_{s-stab}})$ for prime $p$. Finally, we completely confirm the chromatic number of almost $s$-stable Kneser graphs is equal to $n - s(k - 1)$ for all $s \geq 2$. Let $t = n - s(k - 1)$. We also show that for any proper coloring of $\overline{KG^n(n, k)_{s-stab}}$ with $\{1, 2, \ldots, m\}$ colors ($m$ arbitrary) must contain a completely multicolored complete bipartite subgraph $\overline{K_{\lceil \frac{t}{2} \rceil \lfloor \frac{t}{2} \rfloor}}$. Moreover, we obtain that the local chromatic number of the usual almost $s$-stable Kneser graphs is at least $\lceil \frac{t}{2} \rceil + 1$.

### 2.1 $\chi(\overline{KG^r(n, k)_{s-stab}})$ for $r \leq s$

In this subsection, we would like to investigate $\chi(\overline{KG^r(n, k)_{s-stab}})$ for $r \leq s$. First, we provide the following lemma.

**Lemma 3.** Let $n, k, r, s$ be positive integers such that $k \geq 2$, $r \geq 2$, $s \geq r_1 \geq 2$, and $n \geq sk$. Assume that $\chi(\overline{KG^{r_1}(n, k)_{s-stab}}) = \lceil \frac{n-s(k-1)}{r_1-1} \rceil$. Then for any $r \leq r_1$, we have $\chi(\overline{KG^r(n, k)_{s-stab}}) = \lceil \frac{n-s(k-1)}{r-1} \rceil$.

**Proof.** Let $r$ be a positive integer such that $r_1 > r \geq 2$. It suffices to show that $\chi(\overline{KG^r(n, k)_{s-stab}}) = \lceil \frac{n-s(k-1)}{r-1} \rceil$. Assume that $\overline{KG^r(n, k)_{s-stab}}$ is
properly colored with \( t \) colors. Set \( \overline{n} = n + (r_1 - r)t \geq n \geq sk \). There exists a proper \( t \)-coloring \( c : \binom{[n]}{k}_{s-stab} \rightarrow [t] \) of the almost \( s \)-stable \( r \)-uniform Kneser hypergraph \( KG^r(n, k)_{s-stab} \). Now we construct a proper \( t \)-coloring \( f \) of the almost \( s \)-stable \( r_1 \)-uniform Kneser hypergraph \( KG^s(\overline{n}, k)_{s-stab} \). Let the map

\[
f : \binom{[\overline{n}]}{k}_{s-stab} \rightarrow [t]
\]

be defined as follows:

Assume \( S = \{i_1, i_2, \ldots, i_k\} \in \binom{[\overline{n}]}{k}_{s-stab} \) with \( 1 \leq i_1 < i_2 < \cdots < i_k \leq \overline{n} \). Since \( S \) is almost \( s \)-stable, \( i_{k-j} \leq \overline{n} - sj \) for \( j = 0, \ldots, k - 1 \).

**Case I.** \( S \subseteq [n] \), that is \( \max(S) \leq n \). We know that if \( S \in \binom{[\overline{n}]}{k}_{s-stab} \) and \( S \subseteq [n] \subseteq [\overline{n}] \), then \( S \in \binom{[n]}{k}_{s-stab} \). So we set

\[
f(S) = c(S).
\]

Clearly, \( f \) is well-defined. Hence we obtain a value \( f(S) \) in \([t]\).

**Case II.** \( S \setminus [n] \neq \emptyset \), that is \( \max(S) > n \). Then set

\[
f(S) = \left\lfloor \frac{\max(S) - n}{r_1 - r} \right\rfloor.
\]

One can easily check that \( f(S) \leq \left\lfloor \frac{\overline{n} - n}{r_1 - r} \right\rfloor = t \). Thus, we obtain a value \( f(S) \) in \([t]\).

We claim that \( f \) is a proper \( t \)-coloring of \( KG^{r_1}(\overline{n}, k)_{s-stab} \). Suppose to contrary that there are \( r_1 \) pairwise disjoint \( k \)-subsets \( T_1, T_2, \ldots, T_{r_1} \) in \( \binom{[\overline{n}]}{k}_{s-stab} \) and \( m \in [t] \) such that \( f(T_1) = f(T_2) = \cdots = f(T_{r_1}) = m \). Without lose of generality let \( \max(T_1) < \max(T_2) < \cdots < \max(T_{r_1}) \). We have \( \max(T_{r_1}) > \cdots > \max(T_r) > n \), otherwise \( T_1, T_2, \ldots, T_r \) in \( \binom{[n]}{k}_{s-stab} \) and \( c(T_1) = c(T_2) = \cdots = c(T_r) = m \) by the definition of \( f \). It is impossible since \( c \) is a proper coloring of \( KG^r(n, k)_{s-stab} \). Let \( q \in [r_1] \) such that

\[
\max(T_i) \leq n \text{ for } 1 \leq i \leq q,
\]

and

\[
\max(T_i) > n \text{ for } q + 1 \leq i \leq r_1.
\]

Hence \( q < r \). We know that \( T_1, T_2, \ldots, T_q \) in \( \binom{[n]}{k}_{s-stab} \). By definiton of \( f \), we have \( f(T_i) = c(T_i) = m \) for \( 1 \leq i \leq q \), and \( f(T_i) = \left\lfloor \frac{\max(T_i) - n}{r_1 - r} \right\rfloor = m \) for \( q + 1 \leq i \leq r_1 \).
Since \( q < r \), and then \( r_1 - q > r_1 - r \). It means that there are more than \( r_1 - r \) distinct positive integers \( \max(T_{q+1}), \max(T_{q+2}), \ldots, \max(T_{r_1}) \) such that \( \left\lceil \frac{\max(T_{q+1}) - n}{r_1 - r} \right\rceil = \left\lceil \frac{\max(T_{q+2}) - n}{r_1 - r} \right\rceil = \ldots = \left\lceil \frac{\max(T_{r_1}) - n}{r_1 - r} \right\rceil = m \). It is impossible. So we are done.

Hence, \( t \geq \left\lceil \frac{n - s(k - 1)}{r_1 - 1} \right\rceil \). It follows that \( t \geq n + (r_1 - r)t - s(k - 1) \iff (r_1 - 1)t \geq n + (r_1 - r)t - s(k - 1) \iff (r_1 - 1) \geq n - s(k - 1) \). Hence, we have

\[
t \geq \left\lceil \frac{n - s(k - 1)}{r_1 - 1} \right\rceil,
\]

that is, \( \chi(KG^r(n, k)_{s\text{-stab}}) \geq \left\lceil \frac{n - s(k - 1)}{r_1 - 1} \right\rceil \). From Lemma 2 we conclude that

\[
\chi(KG^r(n, k)_{s\text{-stab}}) = \left\lceil \frac{n - s(k - 1)}{r_1 - 1} \right\rceil.
\]

Conjecture 2 is confirmed by Jonsson [17] for \( r = 2^p \) and \( s = 2^q \) with positive integers \( p \leq q \), and by Chen [8] for \( r \) is a power of 2 and \( s \) is a multiple of \( r \). Combining Chen’s work [8], Lemma 2 and Lemma 3, we obtain the following immediate consequence about Conjecture 4:

**Corollary 1.** Let \( n, k, q, r \) and \( \alpha \) be positive integers such that \( k \geq 2 \) and \( r \geq 2 \). Assume that \( 2^q \geq r \). Then \( \chi(KG^r(n, k)_{\alpha 2^q\text{-stab}}) = \left\lceil \frac{n - \alpha 2^q(k - 1)}{r_1 - 1} \right\rceil \) for any \( n \geq \alpha 2^q k \).

Moreover, we obtain an immediate consequence of Lemma 3.

**Theorem 2.** Conjecture 3 is equivalent to Conjecture 4.

So it remains an interesting issue to verify the logical equivalence between Conjecture 1 and Conjecture 2.

### 2.2 Topological lower bound of \( \chi(KG^p(n, k)_{s\text{-stab}}) \) for prime \( p \)

This subsection is devoted to find the topological lower bounds of the chromatic number of \( KG^p(n, k)_{s\text{-stab}} \) and the chromatic number of \( KG^p(n, k)_{s\text{-stab}} \) for prime \( p \). First, we introduce the \( Z_p \)-Tucker lemma.
Throughout this paper, for any positive integer \( p \), we assume that \( Z_p = \{ \omega, \omega^2, \ldots, \omega^p \} \) is the cyclic and multiplicative group of the \( p \)th roots of unity. We emphasize that 0 is not considered as an element of \( Z_p \). We write \((Z_p \cup \{0\})^n\) for the set of all signed subsets of \([n]\). We define \(|X|\) to be the quantity \(|\{i \in [n]: x_i \neq 0\}|\).

Any element \( X = (x_1, x_2, \ldots, x_m) \in (Z_p \cup \{0\})^n \) can alternatively and without further mention be denoted by a \( p \)-tuple \( X = (X_1, X_2, \ldots, X_p) \) where \( X_j := \{ i \in [n] : x_i = \omega^j \} \). Note that the \( X_j \) are then necessarily disjoint. For two elements \( X, Y \in (Z_p \cup \{0\})^n \), we denote by \( X \subseteq Y \) the fact that for all \( j \in [p] \) we have \( X_j \subseteq Y_j \). When \( X \subseteq Y \), note that the sequence of non-zero terms in \((x_1, x_2, \ldots, x_n)\) is a subsequence of \((y_1, y_2, \ldots, y_n)\). For any \( X \in (Z_p \cup \{0\})^n \setminus \{(0, \ldots, 0)\} \), we write \( \text{max}(X) \) for the maximal element of \( X_1 \cup X_2 \cup \ldots \cup X_p \), that is, \( \text{max}(X) = \text{max}(X_1 \cup X_2 \cup \ldots \cup X_p) \).

Tucker’s combinatorial lemma [32] and Fan’s combinatorial lemma [15] are two powerful tools in combinatorial topology. The problem of finding a lower bound for the chromatic number of general Kneser hypergraphs via Tucker’s lemma by Ziegler [33, 34] and Fan’s combinatorial lemma [15] has been extensively studied in the literature, see [11, 12, 13, 14, 21, 23, 24, 25, 27, 29, 30, 31, 33, 34].

The following lemma was proposed by Meunier [24]. It is a variant of the \( Z_p \)-Tucker lemma by Ziegler [33, 34].

**Lemma 4 (\( Z_p \)-Tucker lemma).** Let \( p \) be a prime, \( n, m \geq 1 \), \( \alpha \leq m \), and let

\[
\lambda : (Z_p \cup \{0\})^n \setminus \{(0, \ldots, 0)\} \rightarrow Z_p \times [m]
\]

be a \( Z_p \)-equivariant map satisfying the following properties:

(i) for all \( X^{(1)} \subseteq X^{(2)} \in (Z_p \cup \{0\})^n \setminus \{(0, \ldots, 0)\} \), if \( \lambda_2(X^{(1)}) = \lambda_2(X^{(2)}) \leq \alpha \), then \( \lambda_1(X^{(1)}) = \lambda_1(X^{(2)}) \), and

(ii) for all \( X^{(1)} \subseteq X^{(2)} \subseteq \ldots \subseteq X^{(p)} \in (Z_p \cup \{0\})^n \setminus \{(0, \ldots, 0)\} \), if \( \lambda_2(X^{(1)}) = \lambda_2(X^{(2)}) = \ldots = \lambda_2(X^{(p)}) \geq \alpha + 1 \), then \( \lambda_1(X^{(i)}) \) are not pairwise distinct for \( i = 1, \ldots, p \).

Then \( \alpha + (m - \alpha)(p - 1) \geq n \).

Let \( H = (V(H), E(H)) \) be a hypergraph and \( r \) be an integer, where \( r \geq 2 \). The general Kneser hypergraph \( KG^r(H) \) is a hypergraph with the vertex set \( E(H) \) and the edge set

\[ E(KG^r(H)) = \{ e_1, \ldots, e_r : e_i \in E(H) \text{ and } e_i \cap e_j = \emptyset \text{ for each } i \neq j \in [r] \}. \]
In 2011, Meunier [24] investigated the chromatic number of almost 2-stable $r$-uniform Kneser hypergraphs. Dol’nikov-Kříž Theorem 1 was improved from the work of Alishahi and Hajiabolhassan [1] and their interesting notion of alternation number for general Kneser hypergraphs. Alishahi and Hajiabolhassan [1] generalized the proof techniques of Meunier [24]. Frick [12] investigated $\chi(KG_r(n, k)_{s-stab})$ for the case $r > s$. Recently, he shows that Conjecture 2 is true for $r > 6s - 6$ a prime power. Frick [13] makes significant progress via the topological Tverberg theorem. The proof techniques of Frick do not apply to the case $r \leq s$.

Let $p$ be a prime number. With the help of $Z_p$-Tucker’s Lemma 4, we obtain the following topological lower bound of the chromatic number of the almost $s$-stable $p$-uniform Kneser hypergraph $KG_p(n, k)_{s-stab}$. Our method is different from those of [1], [12, 13], and [24].

**Theorem 3.** Let $p$ be a prime number and $n, k, s$ be positive integers such that $s \geq 2$ and $n \geq \left( p + s - 2 \right) \left( k - 1 \right) + \max \{ \{ p, s \} \}$. Then $\chi(KG_p(n, k)_{s-stab}) \geq \left\lceil \frac{n - \left( p + s - 2 \right) \left( k - 1 \right) + \# \{ p, s \} }{p-1} \right\rceil$.

**Proof.** Assume that $KG_p(n, k)_{s-stab}$ is properly colored with $t$ colors. For $S \in \binom{[n]}{k}_{s-stab}$, we denote by $c(S)$ its color. We know that if $A$ is a nonempty subset of $[n]$, then $A$ must contain an almost $s$-stable 1-subset of $[n]$. In our approach, we need to introduce three functions. It should be emphasized that we shall use two functions $I(-)$ and $C(-)$ several times during the proof. Let $X = \langle x_1, x_2, \ldots, x_p \rangle \in (Z_p \cup \{0\})^n \setminus \{0\}^n$. We can write alternatively $X = \langle X_1, X_2, \ldots, X_p \rangle$. Define $I(X) := \max \{ \{ q \in [k] : T \text{ is an almost } s \text{-stable } q \text{-subset of } [n] \text{ and } T \subseteq X_j \text{ for some } j \in [p] \} \}$. Clearly, $I(X)$ is well-defined. Then we define $C(X) := \{ j \in [p] : X_j \text{ contains an almost } s \text{-stable } I(X) \text{-subset of } [n] \}$, and $M(X) := \max \{ \{ \max (T) \in [n] : T \text{ is an almost } s \text{-stable } I(X) \text{-subset of } [n] \text{ and } T \subseteq X_j \text{ for some } j \in [p] \} \}.

By definition of almost $s$-stable set, we know that if $X_j$ contains an almost $s$-stable $I(X)$-subset $A$ for some $j \in [p]$, then $X_j$ must contain an almost $s$-stable $I(X)$-subset $B$ with $\max (B) = \max (X_j)$. Hence, we can derive that $M(X) := \max \{ \{ \max (X_j) \in [n] : j \in C(X) \} \}$. One can easily check that if $|C(X)| = p$, then $M(X) = \max (X)$.
Set \( \alpha = (s - 1)(k - 1) \). Now, define the function

\[
\lambda : \ (Z_p \cup \{0\})^n \setminus \{(0, \ldots, 0)\} \rightarrow Z_p \times [m]
\]

with \( m = \alpha + (k - 1) + t \). We choose a total ordering \( \preceq \) on the subsets of \([n]\). This ordering is only used to get a clean definition of \( \lambda \).

**Case I.** \( \mathcal{I}(X) \leq k - 1 \).

If \( |\mathcal{C}(X)| = p \), then we know that \( \mathcal{M}(X) = \max(X) \). Let \( j \) be the index of \( X_j \) with \( \max(X_j) = \max(X) \).

Define \( \lambda(X) \) as

\[
\left( \omega^j, (\mathcal{I}(X) - 1)(s - 1) + \left( \max(X) - (s - 1) \left\lfloor \frac{\max(X)}{s - 1} \right\rfloor \right) + 1 \right).
\]

Note that \( \max(X) - (s - 1) \left\lfloor \frac{\max(X)}{s - 1} \right\rfloor \) is the remainder of \( \max(X) \) divided by \( s - 1 \). One can easily check that \( \lambda_2(X) \in \{1, 2, \ldots, \alpha\} \).

If \( |\mathcal{C}(X)| < p \), let \( j \) be the index of \( X_j \) with \( \max(X_j) = \mathcal{M}(X) \) and then \( \lambda(X) \) is defined to be \( (\omega^j, \alpha + \mathcal{I}(X)) \). One can easily check that \( \lambda_2(X) \in \{\alpha + 1, \ldots, \alpha + (k - 1)\} \).

**Case II.** \( \mathcal{I}(X) = k \). By definition of \( \mathcal{I}(X) \), at least one of the \( X_j \)'s with \( j \in [p] \) contains an almost \( s \)-stable \( k \)-subsets of \([n]\). Choose \( j \in [p] \) such that there is \( S \subseteq X_j \) with \( S \in \binom{[n]}{k \text{-stab}} \). In case several \( S \) are possible, choose the maximal one according to the total ordering \( \preceq \). Let \( j \) be such that \( S \subseteq X_j \) and define

\[
\lambda(X) := (\omega^j, c(S) + s(k - 1)).
\]

One can easily check that \( \lambda_2(X) \in \{\alpha + k, \ldots, m\} \). Clearly, \( \lambda \) is an \( \mathbb{Z}_p \)-equivariant map from \( (Z_p \cup \{0\})^n \setminus \{(0, \ldots, 0)\} \) to \( \mathbb{Z}_p \times [m] \).

Let \( X^{(1)} \subseteq X^{(2)} \subseteq (Z_p \cup \{0\})^n \setminus \{(0, \ldots, 0)\} \). Obviously, \( \max(X^{(1)}) \leq \max(X^{(2)}) \) and \( \mathcal{I}(X^{(1)}) \leq \mathcal{I}(X^{(2)}) \). If \( \lambda_2(X^{(1)}) = \lambda_2(X^{(2)}) \leq \alpha \), then \( |\mathcal{C}(X^{(1)})| = |\mathcal{C}(X^{(2)})| = p \), \( \mathcal{I}(X^{(1)}) \leq \mathcal{I}(X^{(2)}) \leq k - 1 \), and \( (\mathcal{I}(X^{(1)}) - 1)(s - 1) + \max(X^{(1)}) - (s - 1) \left\lfloor \frac{\max(X^{(1)})}{s - 1} \right\rfloor + 1 = (\mathcal{I}(X^{(2)}) - 1)(s - 1) + \max(X^{(2)}) - (s - 1) \left\lfloor \frac{\max(X^{(2)})}{s - 1} \right\rfloor + 1. \) It implies that \( 0 \leq (\mathcal{I}(X^{(2)}) - \mathcal{I}(X^{(1)}))(s - 1) = \left( \max(X^{(1)}) - (s - 1) \left\lfloor \frac{\max(X^{(1)})}{s - 1} \right\rfloor \right) - \left( \max(X^{(2)}) - (s - 1) \left\lfloor \frac{\max(X^{(2)})}{s - 1} \right\rfloor \right) \leq s - 2 \). Hence, we have \( \mathcal{I}(X^{(1)}) = \mathcal{I}(X^{(2)}) \) and \( \max(X^{(1)}) - (s - 1) \left\lfloor \frac{\max(X^{(1)})}{s - 1} \right\rfloor = \max(X^{(2)}) - (s - 1) \left\lfloor \frac{\max(X^{(2)})}{s - 1} \right\rfloor . \) That is,

\[
\mathcal{I}(X^{(1)}) = \mathcal{I}(X^{(2)}) \text{ and } \max(X^{(1)}) \equiv \max(X^{(2)}) \pmod{s - 1}.
\]
Now we show that $\lambda_1(X^{(1)}) = \lambda_1(X^{(2)})$. Assume that $\lambda_1(X^{(1)}) = \omega^i \neq \omega^j = \lambda_1(X^{(2)})$ for $i \neq j \in [p]$. It means that $\max(X_i^{(1)}) = \max(X^{(1)}) < \max(X^{(2)}) = \max(X_j^{(2)})$. Since $|\mathcal{C}(X^{(1)})| = p$, there exist two disjoint almost $s$-stable $\mathcal{I}(X^{(1)})$-subsets $A$ and $B$ such that $A \subseteq X_j^{(1)}$, $B \subseteq X_i^{(1)}$, and $\max(A) < \max(B) = \max(X^{(2)}) = \max(X_j^{(2)})$. From (1), we obtain that $\max(X_j^{(2)}) - \max(A) = (\max(X^{(2)}) - \max(X^{(1)})) + (\max(B) - \max(A)) \geq (s - 1) + 1 = s$. Since $X^{(1)} \subseteq X^{(2)}$, $A \subseteq X_i^{(1)} \subseteq X_j^{(2)}$. It means that there is an almost $s$-stable ($\mathcal{I}(X^{(1)}) + 1$)-subset $F = A \cup \{\max(X_j^{(2)})\}$ such that $F \subseteq X_j^{(2)}$. It follows that $\mathcal{I}(X^{(2)}) \geq \mathcal{I}(X^{(1)}) + 1 > \mathcal{I}(X^{(1)})$. It contradicts to (1). So we are done.

Let $X^{(1)} \subseteq X^{(2)} \subseteq \cdots \subseteq X^{(p)} \in (Z_p \cup \{0\})^n \setminus \{(0, \ldots, 0)\}$. If $\alpha + 1 \leq \lambda_2(X^{(1)}) = \lambda_2(X^{(2)}) = \cdots = \lambda_2(X^{(p)}) \leq \alpha + (k - 1)$, then $\mathcal{I}(X^{(1)}) = \mathcal{I}(X^{(2)}) = \cdots = \mathcal{I}(X^{(p)}) \leq k - 1$. Moreover, we have that for each $i \in [p]$, $|\mathcal{C}(X^{(i)})| < p$ and, there is almost $s$-stable $\mathcal{I}(X^{(p)})$-subset $T_i$ and $j_i \in [p]$ such that we have $T_i \subseteq X_j^{(i)}$. Since $X^{(1)} \subseteq X^{(2)} \subseteq \cdots \subseteq X^{(p)}$, $T_i \subseteq X_j^{(i)} \subseteq X_j^{(p)}$ for each $i \in [p]$. If all $\lambda_1(X^{(i)})$ would be distinct, then it would mean that all $j_i$ would be distinct, which implies that $|\mathcal{C}(X^{(p)})| = p$. It is a contradiction.

If $\lambda_2(X^{(1)}) = \lambda_2(X^{(2)}) = \cdots = \lambda_2(X^{(p)}) \geq \alpha + k$, then for each $i \in [p]$, there is $S_i \in \binom{n}{k}_{\text{s-stab}}$ and $j_i \in [p]$ such that we have $S_i \subseteq X_j^{(i)}$ and $\lambda_2(X^{(i)}) = c(S_i) + \alpha + (k - 1)$. If all $\lambda_1(X^{(i)})$ would be distinct, then it would mean that all $j_i$ would be distinct, which implies that the $S_i$ would be disjoint but colored with the same color, which is impossible since $c$ is a proper coloring.

Therefore, we can thus apply the $Z_p$-Tucker Lemma 4 and conclude that $n \leq \alpha + (m - \alpha)(p - 1) = (s - 1)(k - 1) + [(k - 1) + t](p - 1)$, that is $t \geq \left\lceil \frac{n - (p + s - 2)(k - 1)}{p - 1} \right\rceil$.

\[\square\]

Combining Lemma 2 and Theorem 3 for the case $s = 2$, we obtain the following result proposed by Meunier [24] via a different approach. As an approach to Conjecture 1, Meunier [24] settled Conjecture 4 for $s = 2$. 

12
Corollary 2. Let \( p \) be a prime number and \( n, k \) be positive integers such that \( n \geq pk \). Then \( \chi(KG^p(n, k)_{\sim 2-stab}) = \left\lceil \frac{n-p(k-1)-1}{p-1} \right\rceil \).

By Theorem 3, we also obtain a topological lower bound of the \( s \)-stable \( p \)-uniform Kneser hypergraph \( KG^p(n, k)_{s-stab} \) as follows.

Theorem 4. Let \( p \) be a prime number and \( n, k, s \) be positive integers such that \( p \geq 2 \), \( s \geq 2 \), and \( n \geq (p + s - 2)(k - 1) + (s - 1) + \max(\{p, s\}) \). Then \( \chi(KG^p(n, k)_{s-stab}) \geq \left\lceil \frac{n-(p+s-2)(k-1)-(s-1)}{p-1} \right\rceil \).

Proof. Let \( n = n - s + 1 \geq (p + s - 2)(k - 1) + \max(\{p, s\}) \geq sk \). Assume that \( KG^p(n, k)_{s-stab} \) is properly colored with \( t \) colors. We know that if \( S \) is an almost \( s \)-stable \( k \)-subset of \([n - s + 1]\), then \( S \) is an \( s \)-stable \( k \)-subset of \([n]\). That is, \( \binom{[\overline{n}]}{k}_{s-stab} \subseteq \binom{[n]}{k}_{s-stab} \). It means that \( KG^p(\overline{n}, k)_{s-stab} \) is \( t \)-colorable, and then \( \chi(KG^p(\overline{n}, k)_{s-stab}) \leq t \). From Theorem 3, we have

\[
t \geq \left\lceil \frac{n-(p+s-2)(k-1)-(s-1)}{p-1} \right\rceil = \left\lceil \frac{n-(p+s-2)(k-1)-(s-1)}{p-1} \right\rceil.
\]

By Theorem 4 for the case \( s = 2 \), we obtain the following result proposed by Alishahi and Hajiabolhassan [1] via a different approach.

Corollary 3. Let \( p \) be a prime number and \( n, k \) be positive integers such that \( n \geq pk + 1 \). Then \( \chi(KG^p(n, k)_{2-stab}) \geq \left\lceil \frac{n-p(k-1)-1}{p-1} \right\rceil \).

Alishahi and Hajiabolhassan [1] confirmed the chromatic number of \( 2 \)-stable \( r \)-uniform Kneser hypergraph \( KG^r(n, k)_{2-stab} \) unless \( r \) is odd and \( n \equiv k \) \((\text{mod } r-1)\). Recently, Frick [13] completely confirm Conjecture 2 for \( s = 2 \).

Let \( p \) be a prime number. Our results specialize to a substantial improvement of the Dol’nikov-Kříž lower bounds for \( \chi(KG^p(n, k)_{s-stab}) \) and \( \chi(KG^p(n, k)_{s-stab}) \) as well.

2.3 Coloring results about \( KG^2(n, k)_{s-stab} \)

This subsection deals with some coloring results about the almost \( s \)-stable Kneser graphs \( KG^2(n, k)_{s-stab} \). As a special case \( r = 2 \) of Conjecture 2, Meunier [24] proposed the following conjecture:
Let $n, k, s$ be positive integers such that $k \geq 2$, $s \geq 2$ and $n \geq sk$. Then

$$\chi \left( KG^2(n, k)_{s, \text{stab}} \right) = n - s(k - 1).$$

Meunier [24] showed that $\chi \left( KG^2(sk + 1, k)_{s, \text{stab}} \right) = s + 1$. Jonsson [17] confirmed that $\chi \left( KG^r(n, k)_{s, \text{stab}} \right) = n - s(k - 1)$ for $s \geq 4$, provided $n$ is sufficiently large in terms of $s$ and $k$. For the case $r = 2$ and $s$ even, Chen [8] proved that Conjecture 2 is true. It follows that $\chi \left( KG^2(n, k)_{s, \text{stab}} \right) = n - s(k - 1)$ for all even $s$ and for $s \geq 4$ and $n$ sufficiently large. Besides the case $s = 2$, the case $s = 3$ is completely open.

The proof of Theorem 5 makes use of the $Z_p$-Tucker Lemma [11]. We provide a different approach to investigate the topological lower bound of the chromatic number of the almost $s$-stable $r$-uniform Kneser hypergraphs. We consider the case $r = 2$. Combining Lemma 2 and Theorem 3, we completely confirm the chromatic number of the usual almost $s$-stable Kneser graphs is equal to $n - s(k - 1)$ for all $s \geq 2$.

**Theorem 5.** Let $n, k, s$ be positive integers such that $k \geq 2$, $s \geq 2$ and $n \geq sk$. Then

$$\chi \left( KG^2(n, k)_{s, \text{stab}} \right) = n - s(k - 1).$$

Recall the definition of $(Z_p \cup \{0\})^n$ for $p = 2$. We write $\{+,-,0\}^n$ for the set of all signed subsets of $[n]$, the family of all pairs $(X^+, X^-)$ of disjoint subsets of $[n]$. Such subsets can alternatively be encoded by sign vectors $X \in \{+,-,0\}^n$, where $X_i = +$ denotes that $i \in X^+$, while $X_j = -$ means that $j \in X^-$. The positive part of $X$ is $X^+ := \{i \in [n] : X_i = +\}$, and analogously for the negative part $X^-$. For every signed subset $(X^+, X^-)$ in $\{+,-,0\}^n$, the signed subset $(X^-, X^+)$ can be encoded by sign vector $-X$, that is, $(-X)^+ = X^-$ and $(-X)^- = X^+$. For example, $-(0 + -) = (0 - +)$. We write $|X|$ for the number of non-zero signs in $X$, that is, $|X| = |X^+| + |X^-|$. Let $X \in \{+,-,0\}^n \setminus \{0\}^n$. We write $\max(X)$ for the maximal element of $X^+ \cup X^-$, that is, $\max(X) = \max(X^+ \cup X^-)$.

In the following, we shall switch freely between the different notations for signed sets. For sign vectors, we use the usual partial order from oriented matroid theory, which is defined componentwise with $0 \leq +$ and $0 \leq -$. Thus $X \leq Y$, that is $(X^+, X^-) \leq (Y^+, Y^-)$, holds if and only if $X^+ \subseteq Y^+$ and $X^- \subseteq Y^-$. For positive integers $m$ and $n$, and an antipodal subset $S \subseteq \{+,-,0\}^n \setminus \{0\}^n$, a labeling map $\lambda : S \rightarrow \{\pm 1, \pm 2, \ldots, \pm m\}$ is called antipodal if $\lambda(-X) = -\lambda(X)$ for all $X$. 

14
Let $n$ be a positive integer and let $sd([-1, +1]^n)$ be the barycentric subdivision of the $n$-cube $[-1, +1]^n$. We denote $\partial(sd([-1, +1]^n))$ to be the boundary of $sd([-1, +1]^n)$. In the rest of this paper, the vertex set of $\partial(sd([-1, +1]^n))$ can be identified with $\{+,-,0\}^n \setminus \{0\}^n$. Every set with $n$ vertices in $\{+,-,0\}^n \setminus \{0\}^n$ form an $(n-1)$-simplex if they can be arranged in a sequence $X_1 \leq X_2 \leq \cdots \leq X_n$ satisfying $|X_j| = j$ for $j = 1, 2, \ldots, n$.

Fan [15] proposed a combinatorial formula on the barycentric subdivision of the octahedral subdivision of $n$-sphere $S^n$. The following lemma corresponds to Fan’s combinatorial lemma [15] applied to $\partial(sd([-1, +1]^n))$.

**Lemma 5 (Octahedral Fan’s lemma).** Let $m, n$ be positive integers. Suppose $\lambda : \{+,-,0\}^n \setminus \{0\}^n \rightarrow \{\pm 1, \pm 2, \ldots, \pm m\}$ satisfies (i) $\lambda$ is antipodal and (ii) $X \leq Y$ implies $\lambda(X) \neq -\lambda(Y)$ for all $X, Y$. Then there are $n$ signed sets $X_1 \leq X_2 \leq \cdots \leq X_n$ in $\{+,-,0\}^n \setminus \{0\}^n$ such that $\{\lambda(X_1), \lambda(X_2), \ldots, \lambda(X_n)\} = \{+a_1, -a_2, \ldots, (-1)^{n-1}a_n\}$, where $1 \leq a_1 < a_2 < \cdots < a_n \leq m$. In particular, $m \geq n$.

We say that a graph is completely multicolored in a coloring if all its vertices receive different colors. The existence of large colorful bipartite subgraphs in a properly colored graph has been extensively studied in the literature, see [1, 2, 3, 6, 7, 21, 25, 29, 30, 31]. Simonyi and Tardos in 2007 [30] improved Dol’nikov’s theorem. The special case for Kneser graphs is due to Ky Fan [16].

**Theorem 6 (Simonyi-Tardos theorem).** Let $\mathcal{H}$ be a hypergraph and assume that $\emptyset$ is not an edge of $\mathcal{H}$. Let $r = cd^2(\mathcal{H})$. Then any proper coloring of $KG^2(\mathcal{H})$ with colors $1, \ldots, t$ (t arbitrary) must contain a completely multicolored complete bipartite graph $K_{\lceil \frac{r}{2} \rceil, \lfloor \frac{r}{2} \rfloor}$ such that the $r$ different colors occur alternating on the two parts of the bipartite graph with respect to their natural order.

In 2014, Meunier [25] found the first colorful type result for uniform hypergraphs to generalize Simonyi and Tardos’s work [30]. Follow the proof technique of Theorem [3] we propose the following strengthened result of the Simonyi-Tardos Theorem [6] about the case $\mathcal{H} = \binom{[n]}{k}_{s \text{-stab}}$ via the Octahedral Fan’s Lemma [5].

**Theorem 7.** Let $n$, $k$, $s$, and $t$ be positive integers such that $k \geq 2$, $s \geq 2$, $n \geq sk$, and $t = n - s(k-1)$. Then any proper coloring of $KG^2(n, k)_{s \text{-stab}}$ with
\{1,2,\ldots,m\} \text{ colors (m arbitrary) must contain a completely multicolored complete bipartite subgraph } K_{\left\lceil \frac{t}{2} \right\rceil \left\lfloor \frac{t}{2} \right\rfloor} \text{ such that the } t \text{ different colors occur alternating on the two sides of the bipartite graph with respect to their natural order.}

\textbf{Proof.} Assume that } KG(n,k)_{s,stab} \text{ is properly colored with } m \text{ colors. For } S \in \binom{[n]}{k}_{s,stab}, \text{ we denote by } c(S) \text{ its color. Let } X = (X^+,X^-) \in \{+, -, 0\}^n \setminus \{0\}^n. \text{ Set } 
\mathcal{I}(X) := \max(\{q \in [k] : T \text{ is an almost } s\text{-stable subset of size } q, \text{ and } T \subseteq X^+ \text{ or } T \subseteq X^-\}).

We define the function

$$\lambda : \{+, -, 0\}^n \setminus \{0\}^n \longrightarrow \{\pm 1, \pm 2, \ldots, \pm (s(k-1) + m)\}$$

as follows.

\textbf{Case I.} \(\mathcal{I}(X) \leq k - 1\).

If there exist two disjoint almost \(s\)-stable \(\mathcal{I}(X)\)-subsets \(S,T\) such that \(S \subseteq X^+\) and \(T \subseteq X^-\), then we define \(\lambda(X)\) as

$$\pm \left( (\mathcal{I}(X) - 1)(s - 1) + \left( \max(X) - (s - 1) \left\lceil \frac{\max(X)}{s-1} \right\rceil \right) + 1 \right),$$

where the sign indicates which of \(\max(X^+)\) or \(\max(X^-)\) equals \(\max(X)\). Note that \(\max(X) - (s - 1) \left\lceil \frac{\max(X)}{s-1} \right\rceil \) is the remainder of \(\max(X)\) divided by \(s - 1\). Thus we obtain a value \(\lambda(X)\) in the set \(\{\pm 1, \pm 2, \ldots, \pm (s-1)(k-1)\}\).

Otherwise, define \(\lambda(X)\) as

$$\pm (\mathcal{I}(X) + (s - 1)(k - 1)),$$

where the sign indicates which of \(X^+\) or \(X^-\) can contain an almost \(s\)-stable of size \(\mathcal{I}(X)\). Thus we obtain a value \(\lambda(X)\) in the set \(\{\pm((s-1)(k-1) + 1), \ldots, \pm s(k - 1)\}\).

\textbf{Case II.} \(\mathcal{I}(X) = k\). By definition of \(I(X)\), at least one of \(X^+\) and \(X^-\) contains an almost \(s\)-stable \(k\)-subset. Among all almost \(s\)-stable \(k\)-subsets included in \(X^+\) and \(X^-\), select the one having the largest color. Call it \(S\). Then define

$$\lambda(X) = \pm (c(S) + s(k - 1)),$$

where the sign indicates which of \(X^-\) or \(X^+\) the subset \(S\) has been taken from. Thus we obtain a value \(\lambda(X)\) in the set \(\{\pm(s(k-1) + 1), \ldots, \pm(s(k-1) + m)\}\).
Clearly, $\lambda$ is antipodal. Follows the similar scheme as the proof of Theorem \ref{thm:main} one can easily check that $X \leq Y$ implies $\lambda(X) \neq -\lambda(Y)$ for all $X, Y$. Applying Octahedral Fan’s Lemma \ref{lem:ff} there are $n$ signed sets $X_1 \leq X_2 \leq \cdots \leq X_n$ in $\{+,-,0\}^n \setminus \{0\}^n$ such that $\{\lambda(X_1), \lambda(X_2), \ldots, \lambda(X_n)\} = \{+a_1, -a_2, \ldots, (-1)^{n-1}a_n\}$ where $1 \leq a_1 < a_2 < \cdots < a_n \leq m$. Since $1 \leq |X_1| < |X_2| < \cdots < |X_n| \leq n$, we have $|X_i| = i$ for $i = 1, 2, \ldots, n$. From the definition of $\lambda$, we know that $\max(\{\lambda(X_1), |\lambda(X_2)|, \ldots, |\lambda(X_{s(k-1)})|\}) < |\lambda(X_{s(k-1)+1})| < |\lambda(X_{s(k-1)+2})| < \cdots < |\lambda(X_n)|$. Therefore, we have $\lambda(X_i) = (-1)^{i-1}a_i$ for $i = s(k-1) + 1, s(k-1) + 2, \ldots, n$. Assume that $n$ is even. If $s(k-1)$ is even, then there exist $A_1, A_3, \ldots, A_{n-s(k-1)-1}$ in $^{[n]}\binom{k}{s}$ \textit{such that} $A_i \subseteq X^+_{s(k-1)+i} \subseteq X^+_n$ and $c(A_i) = a_{s(k-1)+i}$ for $i = 1, 3, \ldots, n-s(k-1)-1$. Also, there exist $A_2, A_4, \ldots, A_{n-s(k-1)}$ in $^{[n]}\binom{k}{s}$ \textit{such that} $A_i \subseteq X^-_{s(k-1)+i} \subseteq X^-_n$ and $c(A_i) = a_{s(k-1)+i}$ for $i = 2, 4, \ldots, n-s(k-1)$. Since $X^+_n \cap X^-_n = \emptyset$, the induced subgraph on vertices 

$$\{A_1, A_3, \ldots, A_{n-s(k-1)-1}\} \cup \{A_2, A_4, \ldots, A_{n-s(k-1)}\}$$

is a complete bipartite graph which is the desired subgraph. If $s(k-1)$ is odd, then there exist $A_1, A_3, \ldots, A_{n-s(k-1)}$ in $^{[n]}\binom{k}{s}$ \textit{such that} $A_i \subseteq X^-_{s(k-1)+i} \subseteq X^-_n$ and $c(A_i) = a_{s(k-1)+i}$ for $i = 1, 3, \ldots, n-s(k-1)$. Also, there exist $A_2, A_4, \ldots, A_{n-s(k-1)}$ in $^{[n]}\binom{k}{s}$ \textit{such that} $A_i \subseteq X^+_{s(k-1)+i} \subseteq X^+_n$ and $c(A_i) = a_{s(k-1)+i}$ for $i = 2, 4, \ldots, n-s(k-1)-1$. Since $X^+_n \cap X^-_n = \emptyset$, the induced subgraph on vertices 

$$\{A_1, A_3, \ldots, A_{n-s(k-1)}\} \cup \{A_2, A_4, \ldots, A_{n-s(k-1)-1}\}$$

is a complete bipartite graph which is the desired subgraph. Now, assume that $n$ is odd. We can prove the statement with the same kind of proof works when $n$ is even (omitted here) directly. This completes the proof. \hfill \square

In a graph $G = (V, E)$, the \textit{closed neighborhood} of a vertex $u$, denoted $N[u]$, is the set $\{u\} \cup \{v \in V \mid uv \in E\}$. The \textit{local chromatic number} of a graph $G = (V; E)$, denoted $\chi_l(G)$, is the maximum number of colors appearing in the closed neighborhood of a vertex minimized over all proper colorings:

$$\chi_l(G) = \min_c \max_v |c(N[v])|,$$

where the minimum is taken over all proper colorings $c$ of $G$. This number has been defined in 1986 by Erdős, Füredi, Hajnal, Komjáth, Rödl, and Seress.
[11]. Meunier [25], by using his colorful theorem, generalized the Simon Tardos lower bound [30] for the local chromatic number of Kneser graphs to the local chromatic number of Kneser hypergraphs. The local chromatic number of uniform Kneser hypergraphs have been extensively studied in the literature, see [1, 2, 3, 25, 29, 30]. We investigate the lower bound for the local chromatic number of almost $s$-stable Kneser graphs. The following result is an immediate consequence of Theorem 7.

**Theorem 8.** Let $n$, $k$, and $s$ be positive integers such that $k \geq 2$, $s \geq 2$, and $n \geq sk$. Then

$$\chi_l(KG^2(n, k)_{s\text{-stab}}) \geq \left\lceil \frac{n - s(k - 1)}{2} \right\rceil + 1.$$ 

Our lower bound improves Simonyi and Tardos [30], and Meunier’s [25] lower bounds for the local chromatic number of $KG^2(n, k)_{s\text{-stab}}$ as well.

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18
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