Recursively minimally-deformed oscillators

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Abstract

A recursive deformation of the boson commutation relation is introduced. Each step consists of a minimal deformation of a commutator \([a, a^\dagger] = f_k(\cdots; \hat{n})\) into \([a, a^\dagger]_{q_{k+1}} = f_k(\cdots; \hat{n})\), where \(\cdots\) stands for the set of deformation parameters that \(f_k\) depends on, followed by a transformation into the commutator \([a, a^\dagger] = f_{k+1}(\cdots, q_{k+1}; \hat{n})\) to which the deformed commutator is equivalent within the Fock space. Starting from the harmonic oscillator commutation relation \([a, a^\dagger] = 1\) we obtain the Arik-Coon and the Macfarlane-Biedenharn oscillators at the first and second steps, respectively, followed by a sequence of multiparameter generalizations. Several other types of deformed commutation relations related to the treatment of integrable models and to parastatistics are also obtained. The “generic” form consists of a linear combination of exponentials of the number operator, and the various recursive families can be classified according to the number of free linear parameters involved, that depends on the form of the initial commutator.

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1 Introduction

The study of deformed oscillators has already yielded a plethora of formal results and applications, but the attempts to introduce some order in the rich and varied choice of deformed commutation (quommutation) relations studied by different authors has so far achieved limited success. Of particular interest in this respect are the treatments due to Jannussis et al. [1, 2], Daskaloyannis [3, 4], McDermott and Solomon [5] and Meljanac et al. [6].

The following is a partial list of deformations that have been studied:

1. The Arik-Coon oscillator [7]

\[ [a, a^\dagger]_q \equiv aa^\dagger - qa^\dagger a = 1 \]

2. The Macfarlane-Biedenharn oscillator [8, 9]

\[ [a, a^\dagger]_q = q^{-\hat{n}} \]

that has independently been proposed by Sun and Fu [10].

3. The Chakrabarti-Jagannathan oscillator [11]

\[ [a, a^\dagger]_p = q^{-\hat{n}} \]

4. The Calogero-Vasiliev oscillator [12]

\[ [a, a^\dagger] = 1 + 2\nu(-1)^\hat{n}, \]

which for \( 2\nu = p - 1 \) is the Chaturvedi-Srinivasan parabose oscillator of order \( p \) [13].

5. The Brzeziński-Egusquiza-Macfarlane oscillator [14]

\[ [a, a^\dagger] = q^{-\hat{n}}(1 + 2\nu(-1)^\hat{n}) \]
6. Macfarlane’s $q$-deformed Calogero-Vasiliev oscillator \cite{15}

\[ aa^\dagger - q^{\pm(1+2\nu K)}a^\dagger a = [[1 + 2\nu K]]q^{\mp(\hat{n} + \nu - \nu K)}, \]

where $K = (-1)^n$ and $[[x]] = \frac{q^x - q^{-x}}{q - q^{-1}}$.

In the present contribution we introduce a notion of minimal deformation, that, along with the well known flexibility exhibited by the presentation of the quommutation relations within the Fock space, enables a recursive deformation procedure to be formulated, generating the various types of deformed oscillators listed above, thus yielding a certain classification principle. Moreover, the procedure proposed yields a multiparameter generalization of the quommutation relations and suggests that the “generic” structure involves sums of exponentials of the number operator.

2 Equivalence of quommutators and commutators

Let $a$ and $a^\dagger$ be two mutually conjugate operators and let $\hat{n}$ satisfy the commutation relations $[a, \hat{n}] = a$ and $[\hat{n}, a^\dagger] = a^\dagger$. It follows that $\hat{n}$ commutes with $a^\dagger a$ and with $aa^\dagger$. Furthermore, let

\[ a\alpha(\hat{n})a^\dagger - a^\dagger \beta(\hat{n})a = \gamma(\hat{n}) \]  

(1)

where $\alpha(\ell)$, $\beta(\ell)$ and $\gamma(\ell)$ are given functions such that $\alpha(\ell)$ does not vanish for integral and non-negative $\ell$. This quommutation relation contains the form studied by McDermott and Solomon \cite{5}, in which $\alpha(\hat{n}) = \gamma(\hat{n}) = 1$. It is a symmetrized version of that studied by Meljanac et al. \cite{6} (corresponding to $\alpha(\hat{n}) = 1$), to which it is easily shown to be equivalent. The transformations introduced below take place within a Fock space representation that is assumed to exist, possessing a non-degenerate ground state that satisfies $a|0 > = 0$. At least within this representation it is rather likely that the form introduced by McDermott and Solomon \cite{5} is sufficiently general. The non-degeneracy requirement of the ground state has recently been relaxed by several authors \cite{16-18} who introduced a doubly-degenerate ground
state that was found useful in the context of discussing intermediate statistics. We shall not pursue this extension. From the assumptions specified above it follows that

\[ a^\dagger |k\rangle = \sqrt{F(k + 1)}|k + 1\rangle \] (2)

and

\[ a|k + 1\rangle = \sqrt{F(k + 1)}|k\rangle \] (3)

where

\[ F(k) = \sum_{i=0}^{k-1} \frac{\gamma(i)\beta(i)\beta(i+1)\cdots\beta(k-1)}{\alpha(i+1)\alpha(i+2)\cdots\alpha(k)\beta(k-1)}. \] (4)

Hence, \(aa^\dagger = F(\hat{n} + 1), a^\dagger a = F(\hat{n})\) and the commutator \([a, a^\dagger]_Q \equiv aa^\dagger - Qa^\dagger a\) is

\[ [a, a^\dagger]_Q = \frac{\gamma(\hat{n})}{\alpha(\hat{n} + 1)} + \sum_{i=0}^{\hat{n}-1} \frac{\gamma(i)\beta(i)\beta(i+1)\cdots\beta(\hat{n}-1)}{\alpha(i+1)\alpha(i+2)\cdots\alpha(\hat{n})} \left( \frac{1}{\alpha(\hat{n} + 1)} - \frac{Q}{\beta(\hat{n} - 1)} \right) \] (5)

where \(Q\) is arbitrary, but will usually be chosen to be equal to unity, and where the appearance of the number operator within the upper summation limit (as well as within the summand) has a well defined meaning when applied to any Fock state.

Consider the following example:

Let

\[ aa^\dagger - a^\dagger q^{\hat{n}+1}a = 1, \]

i.e., \(\alpha(\hat{n}) = 1, \beta(\hat{n}) = q^{\hat{n}+1}, \gamma(\hat{n}) = 1\). This is equivalent to

\[ [a, a^\dagger] = 1 + (q^\hat{n} - 1) \sum_{j=0}^{\hat{n}-1} q^j \frac{j(2\hat{n}+j-1)}{2}. \]

The commutation relation obtained for \(q = -1\), i.e., \(aa^\dagger - a^\dagger(-1)^{\hat{n}+1}a = 1\), can be transformed with the aid of the identity

\[ \sum_{j=0}^{2k} (-1)^{(j-1)j} = 1 \]

into the equivalent form \([a, a^\dagger] = (-1)^{\hat{n}}\), whose significance was discussed by Quesne and Vansteenkiste [19].
As a further example we consider the $q$-deformed Calogero-Vasiliev oscillator, proposed by Macfarlane [15]. This oscillator can be transformed into

$$\left[ a, a^\dagger \right]_Q = \frac{1}{2(q-q^{-1})} \left\{ q^n (q - Q)(q^{2
u} + 1) + q^{-\bar{n}} (Q - q^{-1})(q^{-2
u} + 1) + (-q)^{\bar{n}} (Q + q)(q^{2\nu} - 1) + (-q)^{-\bar{n}} (Q + q^{-1})(1 - q^{-2\nu}) \right\}.$$  \hspace{1cm} (6)

This can be done either by starting from the commutator quoted in the introduction and applying the procedure illustrated above, or, more simply, using the expressions for $aa^\dagger$ and for $a^\dagger a$ presented by Macfarlane [15]. In either case, the expression obtained is written separately for $\hat{n}$ even and for $\hat{n}$ odd, and the two expressions are combined with coefficients of the form $\frac{1}{2} + (-1)^{\hat{n}}$ and $\frac{1}{2} (1 - (-1)^{\hat{n}})$, respectively. Some further minor rearrangement yields eq. 6, that consists of a linear combination of four exponentials in $\hat{n}$ (three, if $Q$ is chosen to be equal to $q$, $q^{-1}$, $q$, or $q^{-1}$).

Another “exotic” commutator is [20, 21]

$$aa^\dagger - q^\hat{n} + 1 + \frac{1}{q(q^n + 1)} a^\dagger a = 1$$

i.e., $\alpha(\hat{n}) = 1$, $\beta(\hat{n}) = \frac{q^{\hat{n}+1} + 1}{q(q^n + 1)}$, $\gamma(\hat{n}) = 1$. In this case

$$[a, a^\dagger]_Q = 1 + \frac{q^{\hat{n}+1}(q - Q) + 1 - qQ}{q^2 - 1} (1 - q^{-\hat{n}}),$$

which, for $q = Q$ reduces to the Macfarlane-Biedenharn oscillator $[a, a^\dagger]_q = q^{-\hat{n}}$.

### 3 Recursive deformation of the harmonic oscillator

In the following we will be interested in what appears to be a somewhat more restricted framework. Starting from $[a, a^\dagger] = f_0(\hat{n})$ let us assume that at the $k$th step of a recursive procedure to be fully explicated below we have obtained the commutation relation

$$[a, a^\dagger] = f_k(\hat{n}).$$

We define the next minimal deformation to be

$$[a, a^\dagger]_{q_k+1} = f_k(\hat{n}).$$
This minimally-deformed relation implies that, in the Fock-space representation,

\[ a^\dagger |\ell> = \sqrt{F_{k+1}(\ell+1)}|\ell+1> \]

and

\[ a|\ell+1> = \sqrt{F_{k+1}(\ell+1)}|\ell> \]

where

\[ F_{k+1}(\ell) = \sum_{i=0}^{\ell-1} q_i f_k(\ell - i). \]  \hspace{1cm} (7)

It follows that

\[ [a, a^\dagger] = f_{k+1}(\hat{n}) \]

where

\[ f_{k+1}(\hat{n}) \equiv F_{k+1}(\hat{n} + 1) - F_{k+1}(\hat{n}). \]

This recurrence relation can also be written in the form

\[ f_{k+1}(\hat{n}) = \sum_{i=0}^{\hat{n}} q_{k+1}^{\hat{n}-i} \left( f_k(i) - f_k(i-1) \right) \]

provided that we define \( f_k(-1) \equiv 0 \). From the recurrence relation it follows that if \( \lim_{q_1 \to 1, q_2 \to 1, \ldots, q_k \to 1} f_k(\ell) = 1 \) for \( \ell = 0, 1, \ldots, \), then \( \lim_{q_1 \to 1, q_2 \to 1, \ldots, q_{k+1} \to 1} f_{k+1}(\ell) = 1 \). In other words, for all \( k \), if \( f_k(\hat{n}) \) is a deformation of unity, so is \( f_{k+1}(\hat{n}) \).

It will be convenient to define

\[ \Phi_k(\ell) = \sum_{0 \leq i_1, i_2, \ldots, i_k \atop (i_1 + i_2 + \cdots + i_k = \ell - k + 1)} q_1^{i_1} q_2^{i_2} \cdots q_k^{i_k}, \]

which is easily shown to satisfy the limiting property

\[ \lim_{q_1 \to 1, q_2 \to 1, \ldots, q_k \to 1} \Phi_k(\ell) = \binom{\ell}{k-1}. \]

Note that

\[ \Phi_1(\ell) = q_1^\ell \]

\[ \Phi_2(\ell) = \frac{q_1^\ell - q_2^\ell}{q_1 - q_2} = \frac{q_1^\ell}{q_1 - q_2} + \frac{q_2^\ell}{q_2 - q_1} \]

6
\[ \Phi_3(\ell) = \frac{q_1^\ell}{(q_1 - q_2)(q_1 - q_3)} + \frac{q_2^\ell}{(q_2 - q_1)(q_2 - q_3)} + \frac{q_3^\ell}{(q_3 - q_1)(q_3 - q_2)} \]

or, in general,

\[ \Phi_k(\ell) = \sum_{i=1}^{k} \frac{q_i^\ell}{\prod_{m=1}^{k} (q_i - q_m)} \]

the prime indicating that \( m \neq i \).

Let us now take

\[ f_0(\ell) = \begin{cases} 1 & \text{for } \ell \geq 0 \\ 0 & \text{for } \ell < 0 \end{cases} \]

i.e., start from the conventional harmonic oscillator commutation relation, \([a, a^\dagger] = 1\). With this initial value it can be shown that for \( k \geq 1 \)

\[ f_k(\hat{n}) = \sum_{j=0}^{k-1} (-1)^{k-1-j} \binom{k-1}{j} \Phi_k(\hat{n} + j) = \sum_{i=1}^{k} \omega_{k,i} q_i^\hat{n} \tag{8} \]

where \( \omega_{k,i} = \prod_{m=1}^{k} (\frac{q_i-1}{q_i-q_m}) \). Applying the residue theorem to the function

\[ f(z) = \frac{(z-1)^\ell}{\prod_{m=1}^{k} (z-q_m)} \]

we obtain

\[ \sum_{i=1}^{k} \frac{(q_i-1)^\ell}{\prod_{m=1}^{k} (q_i-q_m)} = \begin{cases} 1 & \ell = k-1 \\ 0 & 0 \leq \ell < k-1 \end{cases} \tag{9} \]

The case \( \ell = k-1 \) yields

\[ \sum_{i=1}^{k} \omega_{k,i} = 1, \]

which clarifies the significance of eq. 8, suggesting that the coefficients \( \omega_{k,i}, i = 1, 2, \ldots, k \), are the weights in an appropriate average. Substituting eq. 8 in eq. 4 we obtain, for \( k \geq 1 \),

\[ F_{k+1}(\ell) = \sum_{i=1}^{k} \frac{(q_i-1)^{k-1}}{\prod_{m=1}^{k+1} (q_i-q_m)} (q_i^\ell - q_{i+k}^\ell) = \sum_{i=1}^{k+1} \frac{(q_i-1)^{k-1}}{\prod_{m=1}^{k+1} (q_i-q_m)} q_i^\ell, \]

where use was made of the identity \( \sum_{i=1}^{k+1} \frac{(q_i-1)^{k-1}}{\prod_{m=1}^{k+1} (q_i-q_m)} = 0 \) that corresponds to \( \ell = k-2 \) in eq. 3. Using the latter identity once more we obtain the equivalent form

\[ F_k(\ell) = \sum_{i=1}^{k} \omega_{k,i} [\ell]_{q_i} \tag{10} \]
where $\lfloor \ell \rfloor_{q_i} = \frac{q_i^{\ell-1}}{q_i-1}$ is the Jackson $q_i$-(basic) integer. $F_k(\ell)$ is the weighted average of the Jackson $q$-deformations of the integer $\ell$, in the $k$ different bases $q_1, q_2, \ldots, q_k$. Thus, $F_1(\ell) = \frac{q_1^{\ell-1}}{q_1-1}$, $F_2(\ell) = \frac{q_1^{\ell}-q_2^{\ell}}{q_1-q_2} = \frac{q_1^{\ell-1}}{q_1-1} + \frac{q_2^{\ell-1}}{q_2-1}$, etc.

Using the Jackson $q$-derivative $qD_x g(x) \equiv g(qx) - g(x)$, we introduce the multi-parameter $q$-derivative

$$q_{q_1, q_2, \ldots, q_k} D_x \equiv \sum_{i=1}^k \omega_{q_i} q_i D_x,$$

which is a weighted average over the corresponding Jackson $q_i$-derivatives. In particular, the Macfarlane-Biedenharn $q$-derivative is a weighted average over Jackson $q_i$-derivatives with respect to $q$ and $q^{-1}$, i.e.,

$$q D_x = \omega_q qD_x + \omega_{q^{-1}} q^{-1} D_x$$

where $\omega_q = \frac{q^{-1}}{q-q^{-1}} = \frac{q^{1/2}}{q^{1/2}+q^{-1/2}}$ and $\omega_{q^{-1}} = \frac{q^{-1/2}}{q^{1/2}+q^{-1/2}}$. The multi-parameter $q$-derivative satisfies

$$q_{q_1, q_2, \ldots, q_k} D_x x^\ell = F_k(\ell) x^{\ell-1},$$

that enables the introduction of a corresponding $q$-exponential.

Thus, the minimal deformation of the conventional harmonic oscillator

$$[a, a^\dagger] = 1,$$

is the relation

$$[a, a^\dagger]_{q_1} = 1,$$

which is due to Arik and Coon [7]. It is easily found that $F_1(\ell) = \lfloor \ell \rfloor_{q_1} \equiv \frac{q_1^{\ell-1}}{q_1-1}$ and $f_1(\hat{n}) = q_1^{\hat{n}}$, i.e., the Arik-Coon oscillator is equivalent with

$$[a, a^\dagger] = q_1^{\hat{n}}.$$

This equivalence had been pointed out by Kumari et al. [22]. Eq. (11) suggests that the Arik-Coon oscillator gets more and more classical, with increasing $\hat{n}$, for $q_1 < 1$, and more and more quantal for $q_1 > 1$. In other, more picturesque words, we have an “energy dependent Planck’s constant”. This feature was discussed in refs. [23, 24], where it was referred to as the Tamm-Dancoff cut-off.
Continuing, we consider the minimal deformation of eq. (11), i.e., \([a, a^\dagger]_{q_2} = q_1 \hat{n}\). This is the Chakrabarti-Jagannathan \([11]\) two parameter oscillator, which for \(q_1 = q_2^{-1}\) reduces to the Macfarlane-Biedenharn \([8, 9]\) oscillator. The equivalent commutation relation is

\[
[a, a^\dagger] = f_2(\hat{n})
\]

where

\[
f_2(\hat{n}) = \Phi_2(\hat{n} + 1) - \Phi_2(\hat{n}).
\]

When \(q_1 = q_2^{-1}\) this expression reduces to the commutator \([a, a^\dagger] = \frac{q_1^{(n+\frac{1}{2})} + q_2^{-(n+\frac{1}{2})}}{q_1^{n} + q_2^{n}}, \) that (with a slight change of notation) was noted by Floreanini and Vinet \([25]\).

The equivalence between \([a, a^\dagger]_{q_2} = q_1 \hat{n}\) and \([a, a^\dagger] = f_2(\hat{n})\), and the fact that \(f_2(\hat{n})\) is symmetric in \(q_1\) and \(q_2\), implies the well known equivalence of \([a, a^\dagger]_{q_1} = q_2^{\hat{n}}\) and \([a, a^\dagger]_{q_2} = q_1^{\hat{n}}\). It is perhaps appropriate to emphasize that the latter equivalence, like the former, is only valid within the Fock space.

The minimal deformation of eq. (12) yields \([a, a^\dagger]_{q_3} = f_2(\hat{n})\), which can be written in the equivalent commutator form

\[
[a, a^\dagger] = f_3(\hat{n})
\]

where

\[
f_3(\hat{n}) = \Phi_3(\hat{n} + 2) - 2\Phi_3(\hat{n} + 1) + \Phi_3(\hat{n}).
\]

Continuing the recursion we note that since \(f_k(\hat{n})\) is a symmetric polynomial in \(q_1, q_2, \cdots, q_k\) (cf. eq. (8)), it follows that the \(k\) relations

\[
[a, a^\dagger]_{q_i} = f_{k-1}(q_1, q_2, \cdots, q_{i-1}, q_{i+1}, \cdots, q_k; \hat{n}) \quad i = 1, 2, \cdots, k
\]

are all satisfied simultaneously with \([a, a^\dagger] = f_k(q_1, q_2, \cdots, q_k; \hat{n})\). Here, the dependence on the parameters is shown explicitly.

The present multiparameter deformation refers to a single coordinate, unlike the multiparameter quantum groups associated with the \(n\)-dimensional quantum space commutation relations [26-28].
At the $k$th step of the recursion we obtain a commutator which is equal to a linear combination of $k$ exponentials of the number operator, with coefficients that are fixed by the construction formulated. We shall now consider a more general starting point, involving a commutator that is equal to some polynomial in the number operator. It will be found that once the number of recursions exceeds the degree of the polynomial, the resulting commutator is again equal to a sum of exponentials, but now with a greater flexibility in the choice of the coefficients.

Taking
\[ f_0(\ell) = \begin{cases} 
\alpha_{0,1}\ell + \alpha_{0,0} & \text{for } \ell \geq 0 \\
0 & \text{for } \ell < 0
\end{cases} , \]
i.e., $[a, a^\dagger] = \alpha_{0,1}\hat{n} + \alpha_{0,0}$, we obtain
\[ f_1(\hat{n}) = \alpha_{1,0} + \alpha_{1,1}\hat{n} \]
where $\alpha_{1,0} = \frac{\alpha_{0,1}}{1-q_1}$ and $\alpha_{1,1} = \alpha_{0,0} - \frac{\alpha_{0,1}}{1-q_1}$. Thus,
\[ [a, a^\dagger] = 1 + 2\nu\hat{n} \]
is the first recursion of the relation $[a, a^\dagger] = f_0(\hat{n})$ with
\[ f_0(\ell) = \begin{cases} 
(1 - q_1)\ell + (1 + 2\nu) & \text{for } \ell \geq 0 \\
0 & \text{for } \ell < 0
\end{cases} . \]
In particular, the Calogero-Vasiliev oscillator corresponds to eq. (14) with $q_1 = -1$.

The second recursion yields
\[ [a, a^\dagger] = f_2(\hat{n}) = \alpha_{2,1}\hat{n} + \alpha_{2,2}\hat{n}^2 \]
where $\alpha_{2,1} = \alpha_{1,1}q_2^{-1}$ and $\alpha_{2,2} = \alpha_{1,0} + \alpha_{1,1}q_2^{-1}$. For $q_1 = q^{-1}$, $q_2 = -q^{-1}$, $q_3 = q$, $\alpha_{2,1} = 1$, $\alpha_{2,2} = 2\nu$, the minimal deformation of eq. (15) is the Brzeziński-Egusquiza-Macfarlane oscillator [14].

The third recursion yields
\[ f_3(\hat{n}) = \alpha_{3,1}\hat{n} + \alpha_{3,2}\hat{n}^2 + \alpha_{3,3}\hat{n}^3 \]
where

\[ \alpha_{3,1} = \frac{\alpha_{2,1}}{q_1 - q_3} (q_1 - 1) \]
\[ \alpha_{3,2} = \frac{\alpha_{2,2}}{q_2 - q_3} (q_2 - 1) \]
\[ \alpha_{3,3} = \frac{(q_3 - q_2)\alpha_{2,1} + (q_3 - q_1)\alpha_{2,2}}{(q_3 - q_1)(q_3 - q_2)} (q_3 - 1) , \]

etc.

Starting from a quadratic expression in the number operator

\[ f_0(\ell) = \begin{cases} \alpha_{0,2} \ell^2 + \alpha_{0,1} \ell + \alpha_{0,0} & \text{for } \ell \geq 0 \\ 0 & \text{for } \ell < 0 \end{cases} \]

we obtain

\[ f_1(\hat{n}) = \alpha_{1,1} \hat{n} + \alpha_{1,2} \hat{n} + \alpha_{1,3} \]

where

\[ \alpha_{1,1} = \frac{q_1 (\alpha_{0,2} + \alpha_{0,1}) + (\alpha_{0,2} - \alpha_{0,1})}{(q_1 - 1)^2} + \alpha_{0,0} \]
\[ \alpha_{1,2} = \frac{2\alpha_{0,2}}{1 - q_1} \]
\[ \alpha_{1,3} = \frac{q_1 (\alpha_{0,2} + \alpha_{0,1}) + (\alpha_{0,2} - \alpha_{0,1})}{(q_1 - 1)^2} \]

and

\[ f_2(\hat{n}) = \alpha_{2,1} \hat{n} + \alpha_{2,2} \hat{n} + \alpha_{2,3} \]

with appropriately defined coefficients. The next recursion yields

\[ f_3(\hat{n}) = \alpha_{3,1} \hat{n} + \alpha_{3,2} \hat{n} + \alpha_{3,3} \hat{n} , \]

where the coefficients \( \alpha_{3,1}, \alpha_{3,2}, \) and \( \alpha_{3,3}, \) that can be expressed in terms of \( \alpha_{0,0}, \alpha_{0,1}, \) and \( \alpha_{0,2}, \) can be chosen to agree with the coefficients of the \( q \)-deformed Calogero-Vasiliev oscillator, eq. 6, provided that \( Q \) is chosen to have one of the four values for which eq. 6 reduces to a sum of three exponentials, say \( Q = q, \) and \( q_1, q_2 \) and \( q_3 \) are chosen to be \( q^{-1}, -q \) and \( -q^{-1}, \) respectively.
Thus, starting with \( f_0(\hat{n}) \) that is a polynomial of degree \( k \) in \( \hat{n} \) we obtain, upon performing the recursive minimal deformation procedure, polynomials of decreasing degrees in \( \hat{n} \) combined with linear combinations of exponentials in \( \hat{n} \). After \( k \) steps we obtain just a linear combination of exponentials, but the original \( k \)th degree polynomial allows a corresponding number of coefficients in the linear combination to be chosen at will.

4 Normal ordering relations and multi-parameter deformed Stirling numbers

To derive a normal-ordering formula for the pair of operators \( a \) and \( a^\dagger \) satisfying \([a, a^\dagger]_q = f(\hat{n})\) we first use the identity

\[
[AB, C]_{q_1 q_2} = A[B, C]_{q_2} + q_2[A, C]_{q_1} B
\]

to derive the relation

\[
[a^\ell, a^\dagger]_{q^\ell} = \{\ell(\hat{n})\} a^{\ell-1}
\]  

(16)

where \( \{\ell(\hat{n})\} \equiv \sum_{i=0}^{\ell-1} q^{\ell-1-i} f(\hat{n} + i) \). For \( f(\hat{n}) = 1 \) we obtain \( \{\ell(\hat{n})\}_1 = [\ell]_{q_1} = q_1^{\ell-1}/q_1-1 \), that for \( q_1 = 1 \) is equal to \( \ell \). For \( f(\hat{n}) = q_1^{\hat{n}} \) we have \( \{\ell(\hat{n})\}_2 = q_1^\hat{n}[[\ell]]_{q_1 q_2} \) where \( [[\ell]]_{q_1 q_2} = q_2^{q_1-1} - q_1 q_2^{q_1-1} \).

Now, taking \( q = q_{k+1} \) and \( f(\hat{n}) = f_k(\hat{n}) \) (eq. 8) we obtain, for \( k \geq 1 \),

\[
\{\ell(\hat{n})\}_{k+1} = \sum_{i=1}^{k} (q_i)_{\hat{n}} \frac{q_i^{\ell} - q^{\ell}_{k+1}}{q_i - q_{k+1}} \omega_{k,i}.
\]

Thus, \( \{\ell(\hat{n})\}_{k+1} \) is a multiparameter, operator-valued deformation of the integer \( \ell \).

Using eq. (16) we obtain that the coefficients in the normal ordering formula

\[
(a^\dagger a)^m = \sum_{\ell=1}^{m} (a^\dagger)^\ell C_{m,\ell}(\hat{n}) a^\ell
\]  

(17)

satisfy the initial condition \( C_{1,1}(\hat{n}) = 1 \) and the recurrence relation

\[
C_{m+1,\ell}(\hat{n}) = q^{\ell-1} C_{m,\ell-1}(\hat{n} + 1) + \{\ell(\hat{n})\}_k C_{m,\ell}(\hat{n}).
\]
In the appropriate limits this relation reduces to the Stirling, $q$-Stirling and operator-valued $q$-Stirling coefficients, cf. ref. [29].

The normally-ordered form of an expression of the type $(a^\dagger a)^m$ is not invariant with respect to the different equivalent commutation and quommutation relations that the corresponding pair of operators $a$ and $a^\dagger$ satisfies. Starting from the commutation relation $[a, a^\dagger] = f_k(\hat{n})$ we obtain

\[
[a^\ell, a^\dagger] = \{\ell(\hat{n})\}_k
\]

where $\{\ell(\hat{n})\}_k = \sum_{i=0}^{\ell-1} f_k(\hat{n} + i) = \sum_{i=1}^{k} \omega_{k,i}[\ell]_{q_i}$ and $[\ell]_{q_i} = \frac{q_i-1}{q_i-1}$. Hence, a normal ordering expansion of the form of eq. (17) is obtained, with the coefficient satisfying the recurrence relation

\[
C_{m+1,\ell}(\hat{n}) = C_{m,\ell-1}(\hat{n} + 1) + \{\ell(\hat{n})\}_k C_{m,\ell}(\hat{n}) .
\]

Thus, the Arik-Coon quommutation relation gives rise to the $q$-Stirling numbers as the coefficients in the normally-ordered expansion, but if the (equivalent) commutation relation $[a, a^\dagger] = q \hat{n}$ is used to effect the normal ordering, the coefficients are operator-valued. As an illustration consider $(a^\dagger a)^2$, which, in terms of the Arik-Coon quommutator is given by

\[
(a^\dagger a)^2 = q(a^\dagger)^2 a^2 + a^\dagger a ,
\]

whereas in terms of the equivalent commutator becomes

\[
(a^\dagger a)^2 = (a^\dagger)^2 a^2 + a^\dagger q \hat{n} a .
\]

These two normally-ordered expressions are related to one another via the identity $q \hat{n} = (q - 1)a^\dagger a + 1$.

5 The inverse problem

The following inverse problem may sometimes be of interest: Given a commutator of some form, can it be transformed into a quommutator that, in some sense, is of simpler form? To motivate this problem we recall that the normal ordering problem for the Arik-Coon oscillator

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\([a, a^\dagger]_q = 1\) yields the \(q\)-Stirling numbers as coefficients, whereas the equivalent commutator relation, \([a, a^\dagger] = q^\hat{n}\), yields a normal ordering expansion with a new type of \(\hat{n}\) dependent ("operator valued") \(q\)-Stirling numbers. Given the latter commutator, we may wish to obtain the equivalent quommutator that, in this case, yields a simpler normal-ordering formula.

Thus, given \([a, a^\dagger] = \phi(\hat{n})\), where \(\phi(0) = 1\), it can be shown straightforwardly that
\[
a a^\dagger - a^\dagger \beta(\hat{n}) a = 1,
\]
where
\[
\beta(\hat{n}) = \sum_{i=1}^{\hat{n}+1} \phi(i) \sum_{i=0}^{\hat{n}} \phi(i).
\]
As an example we take \(\phi(\hat{n}) = q^\hat{n}\) that yields \(\beta(\hat{n}) = q\), thus reproducing the Arik-Coon quommutator.

A somewhat different inverse problem involves the transformation of
\([a, a^\dagger] = \phi(\hat{n})\)
into the equivalent form
\([a, a^\dagger]_Q = \Phi(\hat{n})\),
choosing \(Q\) so as to make \(\Phi(\hat{n})\) as simple as possible, for a given \(\phi(\hat{n})\). Since in the above quommutation relation \(\alpha(\hat{n}) = \beta(\hat{n}) = 1\), we obtain
\[
\Phi(\hat{n}) = \phi(\hat{n}) + (1 - Q) \sum_{i=0}^{\hat{n}-1} \phi(i),
\]
(cf. eq. 5). Thus, \(\phi(\hat{n}) = q^\hat{n}\) yields
\[
\Phi(\hat{n}) = q^\hat{n} \left( \frac{q - Q}{q - 1} \right) + \frac{Q - 1}{q - 1}.
\]
The “best choice” is very clear in this case, i.e., \(Q = q\), yielding \(\Phi(\hat{n}) = 1\).

Taking
\[
\phi(\hat{n}) = \alpha q_1^\hat{n} + \beta q_2^\hat{n}
\]
we obtain
\[
\Phi(\hat{n}) = \alpha q_1^\hat{n} \left( \frac{q_1 - Q}{q_1 - 1} \right) + \beta q_2^\hat{n} \left( \frac{q_2 - Q}{q_2 - 1} \right) + \alpha \frac{Q - 1}{q_1 - 1} + \beta \frac{Q - 1}{q_2 - 1}
\]
In this case we have two equally good choices of $Q$, i.e., $Q = q_1$ and $Q = q_2$. The former yields

$$[a, a^\dagger]_{q_1} = \Phi(\hat{n}) = \beta q_2 q_1 - q_1 = \alpha + \beta \frac{q_1 - 1}{q_2 - 1}.$$  

For the special case $\phi(\hat{n}) = 1 + 2\nu p$ we obtain $
\sum_{i=0}^{\ell-1} \phi(i) = \ell + 2\nu \frac{p-1}{p}$, so, setting $Q = p$

$$[a, a^\dagger]_p = 1 + 2\nu + (1 - p)\hat{n}.$$  

Hence, $[a, a^\dagger]_p = 1 + 2\nu p$ is equivalent with $[a, a^\dagger]_p = 1 + 2\nu (1 - 1)\hat{n}$ is equivalent with $\{a, a^\dagger\} = 1 + 2\nu + 2\hat{n}$, cf. ref. [30]. The latter is related to the realization of $osp(n/2, R)$ in terms of parabosons, presented by Palev [31].

6 Non-Fock space representations of the deformed commutation relations

The equivalence between quommutators and corresponding commutators, presented in section 2, is a central ingredient of the recursive minimal deformation procedure introduced in section 3. It was noted in section 2 that the transformation proposed is being carried out within the Fock space representation. Deformed oscillator algebras are known to have additional, non-Fock space, representations [32, 33] that are characterized by the existence of a Casimir operator with non-trivial eigenvalues [34, 35]. We stress that there is no reason to expect these non-Fock space representations to be the same for different ways of writing the commutation relation that are equivalent within the Fock space. While we do not wish to delve in a detailed analysis of these non-Fock space representations for the different algebras discussed, the following general observations indicate some of the features to be expected.

The algebra $[a, a^\dagger] = f_k(\hat{n}) = F_k(\hat{n} + 1) - F_k(\hat{n})$ has a Casimir operator

$$C_k = F_k(\hat{n}) - a^\dagger a .$$

This can be shown by noting that $[C_k, a^\dagger] = (F_k(\hat{n}) - F_k(\hat{n} - 1))a^\dagger - a^\dagger [a, a^\dagger] = 0$. In the Fock-space representation a state $|0> \equiv F_k(0)\equiv 0$ is satisfied. Since, furthermore, $F_k(\hat{n})|0 >= F_k(0)|0 >$ and $F_k(0) \equiv 0$, it follows that within this
representation $C_k$ has eigenvalue 0. The non-Fock representations are characterised by non-vanishing eigenvalues of the Casimir operator.

The minimal deformation of the algebra just discussed, $[a, a^\dagger]_{q_{k+1}} = f_k(\hat{n})$, has a Casimir operator as well, i.e., $\tilde{C}_k = \mu_k(\hat{n}) - \nu_k(\hat{n}) a^\dagger a$, where $\mu_k(\hat{n})$ and $\nu_k(\hat{n})$ should be determined so as to satisfy the condition $[\tilde{C}_k, a^\dagger] = 0$. By adding a suitable constant one can set $\mu_k(0) = 0$, so that the Casimir operator vanishes on the Fock space representation. To determine $\mu_k(\hat{n})$ and $\nu_k(\hat{n})$ we note that

$$[\tilde{C}_k, a^\dagger] = (\mu_k(\hat{n}) - \mu_k(\hat{n} - 1) - \nu_k(\hat{n}) f_k(\hat{n} - 1)) a^\dagger + (\nu_k(\hat{n} - 1) - q_{k+1} \nu_k(\hat{n})) (a^\dagger)^2 a .$$

A sufficient condition for the vanishing of $[\tilde{C}_k, a^\dagger]$ is

$$\mu_k(\hat{n}) - \mu_k(\hat{n} - 1) = \nu_k(\hat{n}) f_k(\hat{n} - 1)$$

$$\nu_k(\hat{n}) = q_{k+1}^{-1} \nu_k(\hat{n} - 1)$$

From eq. [19] we obtain $\nu_k(\hat{n}) = q_k^{-\hat{n}}$, where the normalization $\nu_k(0) = 1$ (which is consistent with the choice $\mu_k(0) = 0$ made above) was chosen. Consequently, eq. [18] becomes a recurrence relation for $\mu_k(\hat{n})$, i.e., $\mu_k(\hat{n}) = \mu_k(\hat{n} - 1) + q_k^{-\hat{n}} f_k(\hat{n} - 1)$. This recurrence relation, along with the initial condition $\mu_k(0) = 0$, is satisfied by

$$\mu_k(\hat{n}) = \sum_{i=1}^{\hat{n}} q_{k+1}^{-\hat{n}} f_k(i - 1) .$$

For $f_k(\hat{n}) = \sum_{i=1}^{k} \omega_{k,i} q_i^{\hat{n}}$, cf. eq. 8, we obtain

$$\mu_k(\hat{n}) = q_k^{-\hat{n}} \sum_{j=1}^{k+1} \omega_{k+1,j} [\hat{n}] q_j = q_k^{-\hat{n}} F_{k+1}(\hat{n})$$

where the last equality follows from eq. [10]. It follows that

$$\tilde{C}_k = q_k^{-\hat{n}} (F_{k+1}(\hat{n}) - a^\dagger a) = q_k^{-\hat{n}} C_{k+1} .$$

The Casimir operators introduced can be used to investigate the non-Fock space representations of the various deformed oscillators presented, along the lines of refs. [32-35].
7 Conclusions

A recursive minimal-deformation of a commutator into a quommutator, followed by a transformation of the resulting quommutator into a new commutator, to which it is equivalent within the corresponding Fock space, has been introduced. The familiar deformed oscillators have been obtained at appropriate steps of this recursive construction, along with multiparameter generalizations that would be difficult to guess otherwise. This recursive scheme provides a classification of the existing deformed-oscillators. The multi-parameter generalizations may appeal to investigators who would like to use the deformed oscillator framework in order to fit molecular or nuclear vibrational spectra, and in similar contexts in which further flexibility would be useful. To what extent they offer hints of further fundamental developments remains to be seen.

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