ON THE REDUCEDNESS OF QUIVER SCHEMES

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Abstract. In this paper we prove that a quiver scheme in characteristic zero is reduced if the moment map is flat. We use the reducedness result to show that the equivariant integration formula computes the K-theoretic Nekrasov partition function of five dimensional \( N = 2 \) quiver gauge theories when the moment map is flat. We also give an explicit characterization of flatness of moment map for finite and affine type A Dynkin quivers with framings. As an application, we give a refinement of a theorem of Ivan Losev on the relation between quantized Nakajima quiver variety in type A and parabolic finite W-algebra in type A.

1. Introduction

Nakajima’s quiver varieties play important roles in representation theory, algebraic geometry, mathematical physics and other fields. For example they geometrically realize representations of Kac-Moody algebras [1], Yangians [2], quantum affine algebras [3], and they are the key objects in the study of Bethe/gauge correspondences [4]. They also show up as Higgs branches of three dimensional \( N = 4 \) quiver gauge theories [5].

Quiver varieties can be defined analytically as hyper-Kähler reductions of ADHM data of a quiver [6], and also algebraically as a GIT quotient [6]. The algebraic construction gives rise to a quiver scheme, whose underlying reduced scheme structure is the quiver variety. The formal definition is as follows.

Let \((Q, v)\) be a quiver, where \(Q\) is a finite directed graph with vertex set \(Q_0\) and arrow set \(Q_1\), and \(v \in \mathbb{N}^{Q_0}\) is called the dimension vector. Define the representation space of \((Q, v)\)

\[
R(Q, v) = \bigoplus_{a \in Q_1} \text{Hom}(\mathbb{C}^{v_t(a)}, \mathbb{C}^{v_h(a)}),
\]

where \(h(a)\) and \(t(a)\) are head and tail of an arrow \(a \in Q_1\). The reductive group \(G(v) = \prod_{i \in Q_0} \text{GL}(v_i)/\mathbb{C}^\times\) naturally acts on \(R(Q, v)\), where \(\mathbb{C}^\times\) embeds as the diagonal center and it acts trivially. There is a moment map \(\mu : T^* R(Q, v) \to g(v)^\ast\) associated to the \(\text{GL}(v)\)-action. Let \(Z\) be the subspace of \(\prod_{v_i \neq 0} \mathbb{C} \subset \mathbb{C}^{Q_0}\) that is annihilated by \(v\). Note that \(Z\) can be identified with \((g(v)/[g(v), g(v)])^\ast\), which is a subspace of \(g(v)^\ast\). Similarly let \(\Theta\) be the subspace of \(\prod_{v_i \neq 0} Q \subset \mathbb{C}^{Q_0}\) that is annihilated by \(v\). For \(\theta \in \Theta\), we denote the \(\theta\)-semistable (respectively \(\theta\)-stable) locus of \(\mu^{-1}(Z)\) by \(\mu^{-1}(Z)^{\theta-ss}\) (respectively \(\mu^{-1}(Z)^{\theta-s}\)). For the definition of \(\theta\)-(semi)stability, see [7].

Definition 1.1. Define the universal quiver scheme with stability condition \(\theta \in \Theta\) by the GIT quotient

\[
\mathcal{M}_Z^\theta(Q, v) := \mu^{-1}(Z)^{\theta-ss}/G(v).
\]

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And for \( \lambda \in Z \), define the quiver scheme with stability condition \( \theta \) by the GIT quotient
\[
(1.3) \quad \mathcal{M}_{\lambda}(Q, \nu) := \mu^{-1}(\lambda)^{\theta-ss}/G(\nu).
\]

Here we do not take reduced scheme structure.

In the study of using the cohomology of quiver schemes to geometrize representations of quantum algebras, it is fine to take the underlying reduced scheme structure, since this does not make a difference to the cohomology. However in other circumstances the potentially non-reduced scheme structure does raise an issue.

One example comes from the computation of K-theoretic Nekrasov partition function of 5d \( N = 2 \) quiver gauge theories, which is the equivariant K-theory class of \( \Gamma(M_{\lambda}^\theta(Q, \nu), \mathcal{O}_{M_{\lambda}^\theta(Q, \nu)}) \) for a generic stability parameter \( \theta \) (see Definition 2.9). Physicists proposed a way to compute it using the equivariant integration \([8, 9, 10]\), which effectively computes the equivariant K-theory class of the differential graded algebra
\[
(1.4) \quad \left( \mathbb{C}[T^*R(Q, \nu)] \otimes_{\mathbb{C}[\mathfrak{g}(\nu)^*]} \mathbb{C} \right)^{G(\nu)},
\]

via the Koszul resolution of \( \mathbb{C} \) as a \( \mathbb{C}[\mathfrak{g}(\nu)^*] \)-module. There are two potential issues in this proposal. The first is that \((1.4)\) might have non-zero homological degree components, which maps to zero in \( \Gamma(M_{\lambda}^\theta(Q, \nu), \mathcal{O}_{M_{\lambda}^\theta(Q, \nu)}) \) via the natural projection \( M_{\lambda}^\theta(Q, \nu) \to M_0(Q, \nu) \). One may resolve this issue by requiring that the moment map \( \mu : T^*R(Q, \nu) \to \mathfrak{g}(\nu)^* \) is flat, then \((1.4)\) is concentrated in homological degree zero, and this is in fact the case for many examples computed in \([10]\). Another issue is that the \( H^0 \) part of \((1.4)\) is \( \mathbb{C}[M_0(Q, \nu)] \), which might not be isomorphic to \( \Gamma(M_{\lambda}^\theta(Q, \nu), \mathcal{O}_{M_{\lambda}^\theta(Q, \nu)}) \): \( M_0^\theta(Q, \nu) \) and \( M_0(Q, \nu) \) can have different dimensions, and even if they have the same dimension, they are not isomorphic if there are nilpotent elements in \( \mathbb{C}[M_0(Q, \nu)] \). This is the place where reducedness becomes important. Nevertheless, we will see that if the moment map is flat, then both of the potential issues are resolved, namely the differential graded algebra \((1.4)\) is quasi-isomorphic to \( \Gamma(M_{\lambda}^\theta(Q, \nu), \mathcal{O}_{M_{\lambda}^\theta(Q, \nu)}) \) (see Corollary 2.16), in other words the equivariant integration formula holds under the assumption on the flatness of the moment map.

It seems to be unknown whether \( M_{\lambda}^\theta(Q, \nu) \) is in general reduced or not, though in some cases the reducedness have been shown. For example Gan and Ginzburg \([11]\) showed that when \( Q \) is a framed Jordan quiver, \( M_{\lambda}^\theta(Q, \nu) \) is reduced for all \( (\theta, \lambda) \in \Theta \times Z \); Crawley-Boevey \([12]\) showed that if \( \nu \in \Sigma_\lambda \) then \( M_\lambda(Q, \nu) \) is reduced, later Bellamy and Schedler \([13]\) extended Crawley-Boevey’s result to that if \( \nu \in \Sigma_{\lambda, \theta}, \lambda \in \mathbb{R}_Q^0 \cap Z \), then \( M_{\lambda}^\theta(Q, \nu) \) is reduced.

The main goal of this paper is to simultaneously generalize previous results on reducedness of quiver schemes of Gan and Ginzburg \([11]\), Crawley-Boevey \([12]\), Bellamy and Schedler \([13]\). The formal statement is as follows.

**Theorem A.** If the moment map \( \mu \) is flat along \( \mu^{-1}(\lambda)^{\theta-ss} \), then the scheme \( M_{\lambda}^\theta(Q, \nu) \) is reduced.

When \( Q \) is a framed Jordan quiver then \( \mu \) is flat (cf. Proposition 1.2). When \( \nu \in \Sigma_{\lambda, \theta} \) then \( \mu \) is flat along \( \mu^{-1}(\lambda)^{\theta-ss} \) [13 Proposition 3.28]. Therefore Theorem A indeed covers
previous results on reducedness of quiver schemes in [11,12,13]. A special case of the above theorem is the following.

**Theorem B.** If the moment map $\mu$ is flat, then $\forall(\theta, \lambda) \in \Theta \times Z$, the scheme $M^\theta_\lambda(Q,v)$ is reduced.

In fact Theorem A can be deduced from Theorem B, see Proposition 2.7.

**Remark 1.2.** If $\mu$ is flat along $\mu^{-1}(\lambda)^{ss}$ (assume it is non-empty), then the support of the dimension vector $v$ must be connected. In fact, if $\text{Supp}(v) = \text{Supp}(v^{(1)}) \sqcup \text{Supp}(v^{(2)})$ such that $v = v^{(1)} + v^{(2)}$, then the image of $\mu$ lies in $g(v^{(1)})^* \oplus g(v^{(2)})^*$, which is a proper linear subspace of $g(v)^*$, but the flatness of $\mu$ along $\mu^{-1}(\lambda)^{ss}$ implies that the image of $\mu$ is Zariski-dense in $g(v)^*$, we get a contradiction. However, this is just a feature of our definition of the group action that it does not take the possibility of multiple connected components of $\text{Supp}(v)$ into account, a more careful definition could separate the components and discuss flatness locally on each component. This is a straightforward generalization of the discussion in this paper, and in order to simplify notation we will not invoke this generalization throughout this paper.

**Remark 1.3.** One might wonder if Theorem B generalizes to other Hamiltonian reductions, namely if $M$ is a complex symplectic representation of complex algebraic group $G$ with a flat moment map $\mu : M \rightarrow g^*$, then is $M \sslash 0 G = \mu^{-1}(0)/G$ reduced or not? The answer is that $M \sslash 0 G$ is not reduced in general, for example the algebraic Uhlenbeck partial compactification of moduli space of framed SO(3)-instantons on $S^4$ with instanton number 4 is isomorphic to a Hamiltonian reduction with flat moment map, but it is not reduced, see [14,15] for more detail.

The paper is organized as follows.

In Section 2 we present preliminaries on quiver schemes that will be useful in the proof of Theorem A and we also derive some corollaries of Theorem A. Specifically we recall a criterion on the flatness of moment map in 2.1 and recall a theorem on étale transversal slice in 2.3; then reduces the proof of Theorem A to that of Theorem B. In 2.8 and 2.11 we discuss generic stability parameters and deduce Lemma 2.15 which is a key to induction step in the proof of Theorem B, then prove Corollary 2.16 which is useful in applications (for example the computation of K-theoretic Nekrasov partition functions). In 2.17 we apply the Theorem B to construct the $(\pm 1)$-reflection isomorphism for all $(\theta, \lambda) \in \Theta \times Z$, assuming that the moment map is flat.

In Section 3 we present the proof of Theorem B. The idea is to use the induction on $|v| = \sum_{i \in Q_0} v_i$, combined with Lemma 2.15 and a result of Crawley-Boevey recalled in Lemma 3.6. Then it boils down to some simple cases which are discussed in 3.1, 3.2 and the end of Section 3.

In Section 4 we discuss the application of Theorem B to the study of quantization of quiver schemes. In 4.1 we focus on affine type A Dynkin quivers and give an explicit description of the set of $(v,d)$ such that the moment map for the framed quiver $(Q,v,d)$ is flat (Proposition 4.2). In 4.4 we recall the definition of quantum Hamiltonian reduction and also its sheafified version, and show that in the quiver scheme case, if the moment map is flat and $v$ is indivisible then the quantized quiver scheme is isomorphic to certain sub-algebra of global sections of
quantized structure sheaf on a resolution (Proposition 4.9). Finally in 4.10 we present a refinement of a theorem of Ivan Losev on the relation between quantized Nakajima quiver schemes in type A and parabolic finite W-algebras in type A (Proposition 4.11).

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2. Preliminaries on Quiver Schemes

2.1. Criterion of flatness of the moment map. We recall a theorem of Crawley-Boevey here.

Lemma 2.2 ([12, Theorem 1.1]). The moment map \( \mu \) is flat if and only if

\[
\mathbf{p}(\mathbf{v}) \geq \sum_{t=1}^{r} \mathbf{p}(\mathbf{v}(t)),
\]

for all decomposition \( \mathbf{v} = \mathbf{v}^{(1)} + \cdots + \mathbf{v}^{(r)} \) into non-zero elements in \( \mathbb{N}Q_0 \). Here \( \mathbf{p} \) is the function

\[
\mathbf{p}(\mathbf{v}) = 1 - \frac{1}{2} \mathbf{v} \cdot C_Q \mathbf{v},
\]

where \( C_Q \) is the Cartan matrix of \( Q \).

It is worth mentioning that [10] gives an explicit description of the set of \( \mathbf{v} \) such that the moment map is flat in terms of \((-1)\)-reflections (see loc. cit. for the definition). We will come back to this point at the end of this section.

2.3. Transverse slice. One important tool of our proof of Theorem A is the following result essentially due to Bellamy and Schedler [13] of which the \( \theta = 0 \) case goes back to Crawley-Boevey [17]:

Lemma 2.4. Take \( \lambda \in \mathbb{Z} \), then for every point \( x \in \mu^{-1}(\lambda)^{\theta-ss} \) such that its orbit \( G(\mathbf{v}) \cdot x \) is closed in \( \mu^{-1}(\lambda)^{\theta-ss} \), \( G(\mathbf{v}) \cdot x \) has an \( \acute{\text{e}}\text{tale} \) transverse slice in \( x \in \mu^{-1}(\lambda)^{\theta-ss} \) such that it is \( \acute{\text{e}}\text{tale} \) locally isomorphic to an open neighborhood \( U \) of \( 0 \in \hat{\mu}^{-1}(0) \), where \( \hat{\mu} \) is the moment map for another quiver \((\hat{Q}, \hat{\mathbf{v}})\) such that \( G(\hat{\mathbf{v}}) \cong G(\mathbf{v})_x \) (stabilizer of \( x \)). In addition, this \( \acute{\text{e}}\text{tale} \) transverse slice gives rise to an isomorphism:

\[
\mathcal{M}_\lambda^\theta(\hat{Q}, \hat{\mathbf{v}})_{\hat{x}} \cong \mathcal{M}_0(\hat{Q}, \hat{\mathbf{v}})_{\hat{x}}.
\]

Here \((-)^\wedge\) means formal completion at \( x \). Moreover if \( \mu \) is flat at \( x \), then \( \hat{\mu} \) is flat.

Proof. The statement of \( \acute{\text{e}}\text{tale} \) transverse slice and local isomorphism are essentially [13] Theorem 3.8], and the argument in loc. cit. works at the full scheme-theoretic level, i.e. not just for underlying reduced scheme structure. And we also emphasize that their argument works for general \( \lambda \in \mathbb{Z} \), not just for \( \lambda \in \mathbb{R}Q_0 \cap \mathbb{Z} \). We only need to show that if \( \mu \) is flat at \( x \), then \( \hat{\mu} \) is flat.
Recall some of the ingredients of the construction of étale transverse slice. Since the stabilizer $G(\nu)_x$ is reductive by Matsushima’s theorem [18], there is $G(\nu)_x$-stable complement $L$ of $g(\nu)_x$ in $g(\nu)$. By [17, Lemma 4.1], the $G(\nu)_x$-submodule $g(\nu) \cdot x \subset T^* R(Q, \nu)$ is isotropic, and by [17 Corollary 2.3], there exists a coisotropic $G(\nu)_x$-module complement $C$ to $g(\nu) \cdot x$ in $T^* R(Q, \nu)$. Let $W = (g(\nu) \cdot x)^\perp \cap C$. The composition of $\mu : T^* R(Q, \nu) \to g(\nu)^*$ with the restriction map $g(\nu)^* \to g(\nu)_x^*$ is denoted by $\mu_x$, and the restriction of $\mu_x$ to $W$ is denoted by $\hat{\mu}$. Then there is an identification $W \cong T^* R(\hat{Q}, \hat{\nu})$, with the moment map being $\hat{\mu}$.

Note that the natural map $\eta : G(\nu) \times G(\nu)_x C \to T^* R(Q, \nu)$ is étale at $(1, 0) \in G(\nu) \times G(\nu)_x C$, therefore the composition $\mu \circ \eta$ is flat at $(1, 0)$. Since $\mu|_C$ agrees with the composition $\mu \circ \eta \circ \iota$, where $\iota : C = G(\nu)_x \times G(\nu)_x C \hookrightarrow G(\nu) \times G(\nu)_x C$ is the natural embedding, we see that $\mu|_C$ is flat at $0 \in C$, thus $\mu_x|_C$ is flat at $0 \in C$. According to [17, Lemma 4.7] we have $C = C^+ \oplus W$ so there is a projection map $p : C \to W$, and it is easy to see that $\mu \circ p = \mu_x|_C$, thus $\hat{\mu}$ is flat at $0 \in W$. Then it follows that $\hat{\mu}$ is flat by the $C^\times$-equivariance of $\hat{\mu}$, here the $C^\times$ acts on the doubling of $\hat{Q}$ by scaling all arrows with weight one.

**Remark 2.5.** $(\hat{Q}, \hat{\nu})$ is determined as following. Suppose that $x$ decomposes as

$$x = x^{\oplus k_1} \oplus \cdots \oplus x^{\oplus k_r}$$

where $x_t$ are non-isomorphic $\theta$-stable representations, and let $\nu^{(t)}$ be the dimension vector of $x_t$. Then the vertex set of $\hat{Q}$ is $\{1, \cdots, r\}$ with dimension vector $\hat{\nu}_t = k_t$, $t \in \hat{Q}_0$.

The adjacency matrix of $\hat{Q}$ is determined from its Euler form $(-, -)_{\hat{Q}}$, i.e. the sum of adjacency matrix and its transpose, which is

$$\langle \hat{e}_t, \hat{e}_u \rangle_{\hat{Q}} = 2\delta_{tu} - \nu^{(t)} \cdot C_Q \nu^{(u)}.$$

Here $\hat{e}_t$ is the dimension vector on $\hat{Q}$ such that it is $1$ at vertex $t$ and zero elsewhere. As a corollary, we have

$$\hat{p} \left( \sum_t k_t \hat{e}_t \right) = p \left( \sum_t k_t \nu^{(t)} \right).$$

**Lemma 2.6.** For a quiver $(Q, \nu)$, the reduced scheme $M^\theta_\lambda(Q, \nu)_{\text{red}}$ is normal for all $(\theta, \lambda)$. Moreover, following statements are equivalent:

1. The moment map $\mu$ is flat along $\mu^{-1}(\lambda)^{\theta-ss}$.
2. For every decomposition $\nu = \sum_{t=1}^r k_t \nu_t$ such that there exists mutually non-isomorphic $\theta$-stable representations $x_t$ of dimension $\nu_t$ and $\lambda \cdot \nu_t = 0$, then

$$p(\nu) \geq \sum_{t=1}^r k_t p(\nu_t).$$

3. $\dim M^\theta_\lambda(Q, \nu) = 2p(\nu)$.

**Proof.** In the view of Lemma [24], for every point $x \in M^\theta_\lambda(Q, \nu)$, the formal completion $M^\lambda_\nu(Q, \nu)_{\hat{x}}$ is isomorphic to $M^\theta_\lambda(\hat{Q}, \hat{\nu})_{\hat{x}}$ of some quiver $(\hat{Q}, \hat{\nu})$, then the normality of $M^\theta_\lambda(Q, \nu)_{\text{red}}$ follows from [17 Theorem 1.1].
\[ (1) \Rightarrow (3): \text{For every point } x \in \mu^{-1}(\lambda)^{\theta-ss} \text{ with closed } G(v)\text{-orbit, we attach the quiver } (\hat{Q}, \hat{v}) \text{ to it, and by Lemma } \ref{lemma:isomorphism} \mu \text{ is flat, so we have} \]
\[ \dim M_\lambda^\theta(Q, v)_x^\lambda = \dim M_0(\hat{Q}, \hat{v})_0^\lambda = 2\hat{p}(\hat{v}) = 2p(v). \]

Here we use the equation (2.5).

\[ (3) \Rightarrow (2): \text{Suppose there is a decomposition } v = \sum_{t=1}^r k_t v_t \text{ such that there exists mutually non-isomorphic } \theta\text{-stable representations } x_t \text{ of dimension } v_t \text{ and } \lambda \cdot v_t = 0, \text{ then let } x = x_1^{\oplus k_1} \oplus \cdots \oplus x_r^{\oplus k_r} \text{ be a point in } \mu^{-1}(\lambda)^{\theta-ss} \text{ with closed } G(v)\text{-orbit, and we attach the quiver } (\hat{Q}, \hat{v}) \text{ to it}, \text{ then we have } \dim M_0(\hat{Q}, \hat{v})_0^\lambda = 2\hat{p}(\hat{v}), \text{ therefore } \dim M_0(\hat{Q}, \hat{v}) = 2\hat{p}(\hat{v}). \text{ By } \cite{II} \text{ Theorem 1.3} \text{ we see that } \hat{\mu} \text{ is flat, thus by } \cite{II} \text{ Theorem 1.1} \text{ we have} \]
\[ p(v) = \hat{p}(\hat{v}) \geq \sum_{t=1}^r k_t \hat{p}(\hat{e}_t) = \sum_{t=1}^r k_t p(v_t). \]

\[ (2) \Rightarrow (1): \text{Generalizing } \cite{II} \text{ Corollary 6.4} \text{ to } \theta\text{-stable representations, of which the proof follows verbatim as } loc. \ cit., \text{ we have} \]
\[ \dim \xi^{-1}(M_\lambda^\theta(Q, v)_\tau) \leq v \cdot v - 1 + p(v) + \sum_{t=1}^r p(v_t), \]

Here \( \tau = (k_1, v_1; \cdots; k_r, v_r) \) is a representation type such that \( v_1 \cdot \theta = v_t \cdot \lambda = 0 \), and \( M_\lambda^\theta(Q, v)_\tau \) \( \text{is the locus of representation type } \tau, \text{ and } \xi: \mu^{-1}(\lambda)^{\theta-ss} \rightarrow M_\lambda^\theta(Q, v) \text{ is the quotient map. Therefore we have} \]
\[ \dim \mu^{-1}(\lambda)^{\theta-ss} \leq v \cdot v - 1 + 2p(v). \]

On the other hand, \( \mu^{-1}(\lambda) \) is a fiber of morphism between two smooth schemes of relative dimension \( v \cdot v - 1 + 2p(v) \), every irreducible component of \( \mu^{-1}(\lambda) \) has dimension \( \geq v \cdot v - 1 + 2p(v) \). Since \( \mu^{-1}(\lambda)^{\theta-ss} \) is open in \( \mu^{-1}(\lambda) \), this forces \( \dim \mu^{-1}(\lambda)^{\theta-ss} = v \cdot v - 1 + 2p(v) \), and therefore \( \mu \) is flat along \( \mu^{-1}(\lambda)^{\theta-ss} \) by Miracle flatness theorem. \( \square \)

An immediate consequence of Lemma \ref{lemma:isomorphism} is the following.

**Proposition 2.7.** Theorem \cite{II} implies Theorem \ref{thm:main}.

**Proof.** Assume that Theorem \cite{II} holds. By Lemma \ref{lemma:isomorphism} there is an isomorphism \( M_\lambda^\theta(Q, v)_x^\lambda \cong M_0(\hat{Q}, \hat{v})_0^\lambda \) of schemes, and \( \hat{\mu} \) is flat for the quiver \( (\hat{Q}, \hat{v}) \) so \( M_0(\hat{Q}, \hat{v}) \) is reduced, and thus \( M_\lambda^\theta(Q, v) \) is reduced at \( x \). Let \( x \) runs through the set of points in \( \mu^{-1}(\lambda)^{\theta-ss} \) with closed \( G(v)\text{-orbits}, \text{ this set maps surjectively to } M_\lambda^\theta(Q, v) \text{ by geometric invariant theory, hence } M_\lambda^\theta(Q, v) \text{ is reduced.} \( \square \)

2.8. Generic parameters.

**Definition 2.9.** We show that \( (\theta, \lambda) \in \Theta \times Z \) is generic if

\[ (2.6) \quad (\theta, \lambda) \in (\Theta \times Z) \setminus \bigcup_{v' \in \mathbb{Z}^Q_0} v' \perp v, \quad 0 < v' < v. \]

Here the partial order on \( \mathbb{Z}^Q_0 \) is such that \( v^{(1)} \leq v^{(2)} \) if \( v^{(2)} - v^{(1)} \in \mathbb{N}^Q_0 \). The set of generic \((\theta, \lambda)\) is denoted by \((\Theta \times Z)_0\), and we use the notation \( \Theta_0 \) (resp. \( Z_0 \)) to denote the
intersection of \((\Theta \times Z)_0\) with \(\Theta \times \{0\}\) (resp. \(\{0\} \times Z\)). If \(\theta \in \Theta_0\) (resp. \(\lambda \in Z_0\)) we show that it is generic.

**Remark 2.10.** It is well-known that \(M^{\theta}(Q, v)\) is smooth if \((\theta, \lambda)\) is generic and \(v\) is indivisible, i.e. \(\exists v' \in \mathbb{N}^{Q_0}, k \in \mathbb{Z}_{>1}\) such that \(v = kv'\). This can be proved as following. If \((\theta, \lambda)\) satisfies \((2.6)\), then every \(\theta\)-semistable representation in \(\mu^{-1}(\lambda)\) is \(\theta\)-stable, i.e.

\[
\mu^{-1}(\lambda)_{\theta-ss} = \mu^{-1}(\lambda)_{\theta-st},
\]

because the dimension vector of a proper sub-representation, denoted by \(v'\), must be orthogonal to \(\lambda\), therefore \(\theta \cdot \lambda \neq 0\), thus the representation is \(\theta\)-stable, and henceforth the stability implies that \(\mu\) is smooth along \(\mu^{-1}(\lambda)\) and the action of \(G(v)\) is free. This proves the smoothness result.

### 2.11. Relation between different stability conditions.

Denote the quiver scheme with zero stability by \(M_Z(Q, v)\) and \(M_\lambda(Q, v)\). Then there is a projective morphism

\[
p^\theta : M^{\theta}_Z(Q, v) \longrightarrow M_Z(Q, v).
\]

**Lemma 2.12.** \(p^\theta|_{Z_0} : M^{\theta}_Z(Q, v)|_{Z_0} \longrightarrow M_Z(Q, v)|_{Z_0}\) is isomorphism.

**Proof.** If \(\lambda \in Z_0\), then every representation in \(\mu^{-1}(\lambda)\) is automatically \(\theta\)-semistable.

**Lemma 2.13.** If the moment map \(\mu\) is flat, and \((\theta, \lambda)\) is generic, and moreover assume that either \(\lambda \in Z_0\) or \(\lambda \in \mathbb{R}^{Q_0}\), then \(M^{\theta}_\lambda(Q, v)\) is reduced and irreducible.

**Proof.** If \(\lambda \in Z_0\) then we can set \(\theta\) equals to zero since \(p^\theta : M^{\theta}_\lambda(Q, v) \rightarrow M_\lambda(Q, v)\) is isomorphism. Therefore we are in the setups of Bellamy and Schedler [13, 1.1]. We claim that \(v \in \Sigma_{\lambda, \theta}\) (see [13 Section 2] for notation), this means that for all decomposition \(v = v^{(1)} + \cdots + v^{(c)}\) into non-zero elements in \(\mathbb{N}^{Q_0}\) such that \(\forall t \in \{1, \cdots, r\}, v^{(t)} \cdot \lambda = v^{(t)} \cdot \theta = 0\), the inequality

\[
p(v) > \sum_{t=1}^{r} p(v^{(t)})
\]

holds. Since \((\theta, \lambda)\) is generic by assumption, we only need to show the inequality when \(v\) is divisible and \(v^{(t)} = k_t w\) for \(k_t \in \mathbb{N}_{>0}\) and \(w \in \mathbb{N}^{Q_0}\). By Lemma 2.2 we already have

\[
p(v) \geq \sum_{t=1}^{r} p(v^{(t)}),
\]

and it remains to show that the inequality must be strict, i.e. equality never holds. In effect, we expand two sides the above inequality as functions of \(w\):

\[
1 - \frac{1}{2} \left( \sum_{t=1}^{r} k_t \right) \geq \frac{1}{2} \left( \sum_{t=1}^{r} k_t^2 \right) \geq \frac{1}{2} \left( \sum_{t=1}^{r} k_t \right)^2,
\]

where \(w^{\mathbb{C}} \in \mathbb{C}^{Q_0}\) is the complexification of \(w\).
and conclude that \( w \cdot C_Q w < 0 \). Since \( w \cdot C_Q w \) is an even number, so \( w \cdot C_Q w \leq -2 \). Hence we have

\[
p(v) - \sum_{t=1}^{r} p(v(t)) \geq 1 - r + \left( \sum_{t=1}^{r} k_t \right)^2 - \left( \sum_{t=1}^{r} k_t^2 \right) = 1 - r + 2 \sum_{1 \leq t < u \leq r} k_t k_u \geq 1 - r + 2 \left( \frac{r}{2} \right) = (r - 1)^2 > 0,
\]

and our claim is justified. Then [13, Corollary 3.22] shows that the stable locus of \( \mathcal{M}^\theta_Z(Q, v) \), denoted by \( \mathcal{M}^\theta_Z(Q, v)^s \), is dense. And by [13, Theorem 3.27]

\[
\dim \xi^{-1}(\mathcal{M}^\theta_Z(Q, v) \setminus \mathcal{M}^\theta_Z(Q, v)^s) < \dim \mu^{-1}(\lambda)^{\theta−ss}.
\]

Here \( \xi : \mu^{-1}(\lambda)^{\theta−ss} \rightarrow \mathcal{M}^\theta_Z(Q, v) \) is the quotient map. Since \( \mu^{-1}(\lambda)^{\theta−ss} \) is Cohen-Macaulay hence equidimensional, this implies that the set of \( \theta \)-stable points in \( \mu^{-1}(\lambda)^{\theta−ss} \) is dense. Since \( \mu \) is smooth at \( \theta \)-stable points, we conclude that \( \mu^{-1}(\lambda)^{\theta−ss} \) is generically smooth. Combined with the Cohen-Macaulay property, we see that \( \mu^{-1}(\lambda)^{\theta−ss} \) is reduced, thus \( \mathcal{M}^\theta_Z(Q, v) \) is reduced.

It is easy to see that \( \forall \theta \in \Theta \) there exists \( \theta' \) generic and in the Euclidean neighborhood of \( \theta \) and such that

\[
\mu^{-1}(Z)^{\theta'−ss} \subset \mu^{-1}(Z)^{\theta−ss}.
\]

Then there exists projective morphism

\[
(2.9) \quad p_\theta^{\theta'} : \mathcal{M}^\theta_Z(Q, v) \longrightarrow \mathcal{M}^\theta_Z(Q, v).
\]

and it is compatible with (2.7) since \( p^{\theta'} = p^\theta \circ p_\theta^{\theta'} \). Note that \( p_\theta^{\theta'} |_{\mathcal{Z}_\theta} \) is isomorphism, therefore it is a birational morphism.

**Lemma 2.14.** If the moment map \( \mu \) is flat along \( \mu^{-1}(\lambda)^{\theta−ss} \), and moreover assume that \( \dim \mathcal{M}^\theta_Z(Q, v) = \dim \mathcal{M}^\theta_Z(Q, v) \), then \( \mu \) is flat along \( \mu^{-1}(\lambda)^{\theta−ss} \).

**Proof.** Apply the equivalence (1) \( \Leftrightarrow \) (3) in Lemma 2.6.

**Lemma 2.15.** If the moment map \( \mu \) is flat, and moreover assume that \( \mathcal{M}^\theta_Z(Q, v) \) is normal, then \( \mathcal{M}^\theta_Z(Q, v) \) is reduced if \( \mathcal{M}^\theta_Z(Q, v) \) is reduced, and if this is the case then the morphism \( p_\theta^{\theta'} : \mathcal{M}^\theta_Z(Q, v) \rightarrow \mathcal{M}^\theta_Z(Q, v) \) is birational.

**Proof.** Note that the normality of \( \mathcal{M}^\theta_Z(Q, v) \) implies that \( p_\theta^{\theta'} \) has connected fibers since it is an isomorphism over the open subset \( \mathcal{Z}_\theta \). In particular, the restriction of \( p_\theta^{\theta'} \) to any fiber over \( \lambda \in \mathcal{Z}_\theta \) gives rise to a birational morphism \( \mathcal{M}^\theta_Z(Q, v)_{\text{red}} \rightarrow \mathcal{M}^\theta_Z(Q, v)_{\text{red}} \) since both sides have the same dimension and \( p_\theta^{\theta'} \) has connected fibers. \( \mathcal{M}^\theta_Z(Q, v) \) is reduced by Lemma 2.13 and it remains to show that \( \mathcal{M}^\theta_Z(Q, v) \) is reduced.

We claim that \( R p_\theta^{\theta'} \mathcal{O}_{\mathcal{M}^\theta_Z(Q, v)} \) is concentrated in degree zero and \( p_\theta^{\theta'} \mathcal{O}_{\mathcal{M}^\theta_Z(Q, v)} \) is flat over \( Z \). In effect, since both \( \mathcal{M}^\theta_Z(Q, v) \) and \( \mathcal{M}^\theta_Z(Q, v) \) are flat over \( Z \) and \( p_\theta^{\theta'} \) is proper, \( R p_\theta^{\theta'} \mathcal{O}_{\mathcal{M}^\theta_Z(Q, v)} \)
is a relative perfect complex, and its formation commutes with base change \( Z' \to Z \). In particular, we base change to \( i : 0 \to Z \), and get
\[
Li^* Rp_{\theta*} \mathcal{O}_{\mathcal{M}^\theta_Z(Q,v)} \cong Rp_{\theta*} (\mathcal{M}^\theta_0(Q,v), \mathcal{O}_{\mathcal{M}^\theta_0(Q,v)}).
\]
The RHS is concentrated in degree zero, because both \( \mathcal{M}^\theta_0(Q,v)_{\text{red}} \) and \( \mathcal{M}^\theta_0(Q,v) \) (which is reduced by Lemma 2.13) have symplectic singularities [13, Theorem 1.2], in particular they have rational singularities, moreover the induced morphism between them is birational, so we conclude that \( R^k p_{\theta*} (\mathcal{M}^\theta_Z(Q,v), \mathcal{O}_{\mathcal{M}^\theta_Z(Q,v)}) \) for \( k > 0 \) (using the fact that any desingularization of \( \mathcal{M}^\theta_0(Q,v) \) is automatically a desingularization of \( \mathcal{M}^\theta_0(Q,v)_{\text{red}} \). Hence \( Rp_{\theta*} \mathcal{O}_{\mathcal{M}^\theta_Z(Q,v)} \) is concentrated in degree zero in an open neighborhood of \( \mathcal{M}^\theta_0(Q,v) \). Next we observe that there is a \( \mathbb{C}^x \) action on the quiver which scales every arrow with weight one, this action commutes with \( \text{GL}(v) \)-action and descends to actions on \( \mathcal{M}^\theta_Z(Q,v) \) and \( \mathcal{M}^\theta_0(Q,v) \) and \( p_{\theta*} \) is equivariant. Since \( Z \) has positive weights under this \( \mathbb{C}^x \)-action, \( \mathcal{M}^\theta_Z(Q,v) \) contracts to \( \mathcal{M}^\theta_0(Q,v) \) under this \( \mathbb{C}^x \)-action, henceforth \( Rp_{\theta*} \mathcal{O}_{\mathcal{M}^\theta_Z(Q,v)} \) is concentrated in degree zero on the whole \( \mathcal{M}^\theta_Z(Q,v) \). \( p_{\theta*} \mathcal{O}_{\mathcal{M}^\theta_Z(Q,v)} \) is flat over \( Z \) because it’s relatively perfect over \( Z \).

The punchline of above discussions is that we have a homomorphism between sheaves of rings
\[
\mathcal{O}_{\mathcal{M}^\theta_Z(Q,v)} \longrightarrow p_{\theta*} \mathcal{O}_{\mathcal{M}^\theta_Z(Q,v)},
\]
and it is an isomorphism on the open locus \( Z_0 \). By the assumption that \( \mathcal{M}^\theta_Z(Q,v) \) is normal, the homomorphism (2.10) is isomorphism. Since \( p_{\theta*} \mathcal{O}_{\mathcal{M}^\theta_Z(Q,v)} \) has base change property, specialization to arbitrary \( \lambda \in Z \) gives us isomorphism
\[
\mathcal{O}_{\mathcal{M}^\theta_\lambda(Q,v)} \cong p_{\theta*} \mathcal{O}_{\mathcal{M}^\theta_\lambda(Q,v)}.
\]
In particular, \( \mathcal{M}^\theta_\lambda(Q,v) \) is reduced if \( \mathcal{M}^\theta_\lambda(Q,v) \) is reduced.

An immediate consequence of Lemma 2.15 and Theorem 13 is the following:

**Corollary 2.16.** If the moment map \( \mu \) is flat, then \( \forall (\theta, \lambda) \in \Theta \times Z \), the projective morphisms \( p^\theta_{\lambda*} : \mathcal{M}^\theta_\lambda(Q,v) \to \mathcal{M}^\theta_\lambda(Q,v) \) are birational. In particular, we have a quasi-isomorphism of differential graded algebra
\[
(\mathbb{C}[T^*R(Q,v)] \otimes_{\mathbb{C}[\theta(v)]} \mathcal{O})_{G(v)} \cong \Gamma(\mathcal{M}^\theta_\lambda(Q,v), \mathcal{O}_{\mathcal{M}^\theta_\lambda(Q,v)}),
\]
where \( \mathbb{C} \) in the left hand side is the stalk of \( \mathcal{O}_{\theta(v)} \) at zero.

According to the Introduction, the quasi-isomorphism (2.11) shows that the equivariant integration formula in the physics literature [3, 9, 10] indeed computes the equivariant K-theory class of \( \Gamma(\mathcal{M}^\theta_\lambda(Q,v), \mathcal{O}_{\mathcal{M}^\theta_\lambda(Q,v)}) \), i.e. the K-theoretic Nekrasov partition function.

2.17. \((\pm 1)\)-Reflection isomorphisms. Consider the reflections \( s_i \) at loop-free vertex \( e_i \), which acts on the quiver data \((v, \lambda, \theta)\) as
\[
s_i v = v - (v, e_i)_Q e_i, \quad (s_i \lambda) j = \lambda_j - (e_i, e_j)_Q \lambda_i, \quad (s_i \theta)_j = \theta_j - (e_i, e_j)_Q \theta_i.
\]
The Lusztig-Maffei-Nakajima reflection isomorphism \cite[Theorem 26]{19} shows that if either $\lambda_i$ or $\theta_i$ is non-zero, then there is an isomorphism of schemes $\Phi_{s_i} : M^{\theta}_{\lambda}(Q, v) \cong M^{s_i, \theta}_{s_i, \lambda}(Q, s_i v)$. It is implicit in \cite{19} that the construction there actually holds for the full scheme structure, not just for reduced scheme structure. The key ingredient in the construction is an auxiliary scheme $Z_{\theta, \lambda}^{s_i}$ with projections

$$\mu^{-1}_v(\lambda)^{\theta-ss} \xleftarrow{p_i} Z^\theta_{i, \lambda} \xrightarrow{p'_i} \mu^{-1}_{s_i v}(s_i \lambda)^{s_i \theta-ss},$$

such that $p_i$ is $GL((s_i v)_i)$-torsor, and $p'_i$ is $GL(v_i)$-torsor. Moreover, there is an action of $G_i(v) = \prod_{j \neq i} GL(v_j) \times GL(v_i) \times GL((s_i v)_i)$ on $Z^\theta_{i, \lambda}$ such that $GL((s_i v)_i)$ acts through the torsor $p_i$ and $GL(v_i)$ acts through the torsor $p'_i$, and furthermore $p_i$ and $p'_i$ are $G_i(v)$-equivariant. Passing to the categorical quotient by $G_i(v)$, there are two isomorphisms:

$$M^{\theta}_{\lambda}(Q, v) \xleftarrow{\bar{p}_i} Z^\theta_{i, \lambda}/G_i(v) \xrightarrow{\bar{p}'_i} M^{s_i, \theta}_{s_i, \lambda}(Q, s_i v),$$

and we set $\Phi_{s_i} = \bar{p}'_i \circ \bar{p}_i^{-1}$. The condition that either $\lambda_i$ or $\theta_i$ is non-zero is crucial in the construction of $Z^\theta_{i, \lambda}$. In the following we will eliminate this condition, but only for flat $\mu_v$ and $(\pm 1)$-reflections.

**Theorem 2.18.** If the moment map $\mu$ is flat, and $i \in Q_0$ is a loop-free vertex such that $(v, e_i)_Q = \pm 1$, then $\forall (\theta, \lambda) \in \Theta \times Z$, there is an isomorphism $\Phi_{s_i}^{\theta, \lambda} : M^\theta_{\lambda}(Q, v) \cong M^{s_i, \theta}_{s_i, \lambda}(Q, s_i v)$ such that the diagram

$$\begin{array}{ccc}
M^\theta_{\lambda}(Q, v) & \xrightarrow{\Phi_{s_i}^{\theta, \lambda}} & M^{s_i, \theta}_{s_i, \lambda}(Q, s_i v) \\
\downarrow & & \downarrow \\
M_{\lambda}(Q, v) & \xrightarrow{\Phi_{s_i}^{\theta, \lambda}} & M_{s_i, \lambda}(Q, s_i v)
\end{array}$$

commutes. Moreover if either $\lambda_i$ or $\theta_i$ is non-zero then $\Phi_{s_i}^{\theta, \lambda}$ agrees with the Lusztig-Maffei-Nakajima reflection isomorphism.

**Proof.** To begin with, we note that the moment map for $(Q, s_i v)$ is flat by Proposition \cite[Proposition 7.1]{16}. If either $\lambda_i$ or $\theta_i$ is non-zero then we define $\Phi_{s_i}^{\theta, \lambda}$ to be the Lusztig-Maffei-Nakajima reflection isomorphism. Note that in this case if $\lambda_i \neq 0$, the construction of $Z^\theta_{i, \lambda}$ fits into a commutative diagram

$$\begin{array}{ccc}
\mu^{-1}_v(\lambda)^{\theta-ss} & \xleftarrow{p_i} & Z^\theta_{i, \lambda} \xrightarrow{p'_i} \mu^{-1}_{s_i v}(s_i \lambda)^{s_i \theta-ss} \\
\downarrow & & \downarrow \\
\mu^{-1}_v(\lambda) & \xleftarrow{p_i} & Z^0_{i, \lambda} \xrightarrow{p'_i} \mu^{-1}_{s_i v}(s_i \lambda)
\end{array}$$

Thus the diagram
\[
\begin{array}{c}
M_\lambda^0(Q, v) \xrightarrow{\Phi_{s_i}^{\theta, \lambda}} M_{s_i, \lambda}^{s_i, \theta}(Q, s_i v) \\
\downarrow \\
M_\lambda(Q, v) \xrightarrow{\Phi_{s_i}^{0, \lambda}} M_{s_i, \lambda}^{s_i, \theta}(Q, s_i v)
\end{array}
\]

commutes.

If \( \lambda_i = \theta_i = 0 \), we define \( \Phi_{s_i}^{\theta, \lambda} \) as following. First we find a generic \( \theta' \) in the Euclidean neighborhood of \( \theta \) such that \( \mu^{-1}(\lambda)^{\theta'-ss} \subset \mu^{-1}(\lambda)^{\theta-s\theta} \), then by \( 2.16 \) the morphism \( p_0^{\theta'}: M_\lambda^{\theta'}(Q, v) \to M_\lambda^0(Q, v) \) is projective and birational. Since \( \theta' \) is generic, in particular \( \theta' \neq 0 \), we have Lusztig-Maffei-Nakajima reflection isomorphism \( \Phi_{s_i}^{\theta', \lambda}: M_\lambda^{\theta'}(Q, v) \cong M_{s_i, \lambda}^{s_i, \theta'}(Q, s_i v) \).

Taking affinization of the domain and codomain of \( \Phi_{s_i}^{\theta', \lambda} \), we get an isomorphism \( \Phi_{s_i}^{0, \lambda}: M_\lambda(Q, v) \cong M_{s_i, \lambda}(Q, s_i v) \).

After scaling by an integer, we assume that \( \theta \in \mathbb{Z}Q_0 \). By the construction of GIT quotient, there is an ample line bundle \( \mathcal{L}(\theta) \) on \( M_\lambda^0(Q, v) \) such that

\[
(2.13) \quad M_\lambda^0(Q, v) = \text{Proj} \bigoplus_{n \geq 0} \Gamma(M_\lambda^0(Q, v), \mathcal{L}(\theta)^{\otimes n}),
\]

as a scheme over \( \text{Spec} \Gamma(M_\lambda^0(Q, v), \mathcal{O}_{M_\lambda^0(Q, v)}) = M_\lambda(Q, v) \). Similarly there is an ample line bundle \( \mathcal{L}(s_i \theta) \) on \( M_{s_i, \lambda}^{s_i, \theta}(Q, s_i v) \) with similar property. We claim that

\[
(2.14) \quad p_0^{\theta'}(\mathcal{L}(\theta)) \cong (\Phi_{s_i}^{\theta', \lambda})^* p_{s_i, \theta}^* (\mathcal{L}(s_i \theta))
\]

In effect, the pullback of \( \mathcal{L}(\theta) \) to \( \mu^{-1}(\lambda)^{\theta-s\theta} \) is the \( G(\nu) \)-equivariant line bundle \( \prod_{j \in Q_0} \det(V_j^{s_i \theta-s\theta}) \), where \( V_j \) is the vector bundle on \( T^* R(Q, v) \) with fibers being the vector space at \( j \)th vertex.

Similar fact holds for the pullback of \( \mathcal{L}(s_i \theta) \) to \( \mu^{-1}(s_i \lambda)^{s_i \theta-s\theta} \). By the construction of \( Z_i^{\theta', \lambda} \) and the fact that \( \theta_i = (s_i \theta)_i = 0 \), we have

\[
p_{s_i}^* \xi^*(\mathcal{L}(\theta)) = \prod_{j \neq i} \det(V_j^{s_i \theta-s\theta}) = p_{s_i}^* \xi^{s_i} (\mathcal{L}(s_i \theta)).
\]

Here \( \xi: \mu^{-1}(\lambda)^{\theta-s\theta} \to M_\lambda^0(Q, v), \xi': \mu^{-1}(s_i \lambda)^{s_i \theta-s\theta} \to M_{s_i, \lambda}^{s_i, \theta}(Q, s_i v) \) are quotient maps. Therefore the claim follows. Since \( p_0^{\theta'} \) is a proper birational morphism between normal schemes, we have

\[
(2.15) \quad M_\lambda^0(Q, v) = \text{Proj} \bigoplus_{n \geq 0} \Gamma(M_\lambda^{\theta'}(Q, v), p_0^{\theta'} \mathcal{L}(\theta)^{\otimes n}).
\]

Similar fact holds for \( M_{s_i, \lambda}^{s_i, \theta}(Q, s_i v) \). Combining \( 2.15 \) with \( 2.14 \), we get an isomorphism \( \Phi_{s_i}^{\theta, \lambda}: M_\lambda^0(Q, v) \cong M_{s_i, \lambda}^{s_i, \theta}(Q, s_i v) \). Obviously the diagram

\[
\begin{array}{c}
M_\lambda^0(Q, v) \xrightarrow{\Phi_{s_i}^{\theta, \lambda}} M_{s_i, \lambda}^{s_i, \theta}(Q, s_i v) \\
\downarrow \\
M_\lambda(Q, v) \xrightarrow{\Phi_{s_i}^{0, \lambda}} M_{s_i, \lambda}(Q, s_i v)
\end{array}
\]
commutes.

We claim that $\Phi_{s_i}^{\theta, \lambda}$ does not depend on the choice of $\theta'$. If $\lambda_i \neq 0$ then $\Phi_{s_i}^{\theta, \lambda}$ is just Lusztig-Maffei-Nakajima reflection isomorphism, so we assume that $\lambda_i = 0$. In view of the above commutative diagram, it suffices to show that $\Phi_{s_i}^{\theta, \lambda}$ does not depend on the choice of $\theta'$. Consider the schemes

\[ S_{n,m} = \{ X_1, Y_1, X_2, Y_2 \mid X_1Y_1 = 0, X_2Y_2 = 0, Y_1X_1 = Y_2X_2 \} \]

\[ A_{n,m} = \{ X_1, Y_1 \mid X_1Y_1 = 0 \} \subset \text{Mat}(n \times (n + m)) \times \text{Mat}((n + m) \times n) \times \text{Mat}(m \times (n + m)) \times \text{Mat}((n + m) \times n) \]

\[ B_{n,m} = \{ X_2, Y_2 \mid X_2Y_2 = 0 \} \subset \text{Mat}(m \times (n + m)) \times \text{Mat}((n + m) \times n) \]

together with obvious projections:

\[ A_{n,m} \leftarrow S_{n,m} \rightarrow B_{n,m}. \]

The projections are equivariant under the natural $GL(n) \times GL(m) \times GL(n + m)$ actions on $S_{n,m}, A_{n,m}, B_{n,m}$. Define the quiver $Q^{(i)}$ as the sub-quiver of $Q$ deleting the vertex $i$ and all arrows $a$ such that $h(a) = i$ or $t(a) = i$. Let $v = v_i, m = (s_i v)_i$, then it is easy to see that there are closed emdeddings

\[ \mu^{-1}_v(\lambda) \subset T^* R(Q^{(i)}, v^{(i)}) \times A_{n,m} \subset T^* R(Q, v), \]

\[ \mu^{-1}_{s_i v}(s_i \lambda) \subset T^* R(Q^{(i)}, v^{(i)}) \times B_{n,m} \subset T^* R(Q, s_i v). \]

Consider the closed subscheme $Z_i^{\lambda} = \mu^{-1}_v(\mu^{-1}_v(\lambda)) \subset T^* R(Q^{(i)}, v^{(i)}) \times S_{n,m}$, it is easy to see that $Z_i^{\lambda}$ is the same as $\mu^{-1}_{s_i v}(s_i \lambda))$, then the natural projections

\[ \mu^{-1}_v(\lambda) \leftarrow Z_i^{\lambda} \rightarrow \mu^{-1}_{s_i v}(s_i \lambda) \]

are $G_i(v)$-equivariant. By the construction of [19, Definition 27], we see that $Z_i^{\theta', \lambda}$ is an open subscheme of $Z_i^{\lambda}$. Therefore we have a sequence of maps of rings:

\[ \mathbb{C}[M_\lambda(Q, v)] \rightarrow \mathbb{C}[Z_i^{\lambda}]^{G_i(v)} \rightarrow \Gamma(Z_i^{\theta', \lambda}, \mathcal{O}_{Z_i^{\theta', \lambda}})^{G_i(v)} = \Gamma(M_i^{\theta'}(Q, v), \mathcal{O}_{M_i^{\theta'}(Q, v)}). \]

Since the composition $\mathbb{C}[M_\lambda(Q, v)] \rightarrow \Gamma(M_i^{\theta'}(Q, v), \mathcal{O}_{M_i^{\theta'}(Q, v)})$ is isomorphism by Corollary 2.16 we see that the first map $\mathbb{C}[M_\lambda(Q, v)] \rightarrow \mathbb{C}[Z_i^{\lambda}]^{G_i(v)}$ is injective. On the other hand, the invariant theory shows that $G_i(v)$-invariant subalgebra of $\mathbb{C}[Z_i^{\lambda}]$ is generated by closed paths with arrows either in the doubled quiver $\overrightarrow{Q}$ or $X_2$ or $Y_2$, the equation $X_2Y_2 = 0, Y_1X_2 = Y_2X_2$ imply that $\mathbb{C}[Z_i^{\lambda}]^{G_i(v)}$ is generated by $\mathbb{C}[\mu^{-1}_v(\lambda)]^{G_i(v)}$, so the map $\mathbb{C}[M_\lambda(Q, v)] \rightarrow \mathbb{C}[Z_i^{\lambda}]^{G_i(v)}$ is surjective. We conclude that the natural morphism $\overrightarrow{p}_i : Z_i^{\lambda} / G_i(v) \rightarrow M_\lambda(Q, v)$ is isomorphism, and similarly the natural morphism $\overrightarrow{p}_i : Z_i^{\lambda} / G_i(v) \rightarrow M_{s_i \lambda}(Q, v)$ is isomorphism. From the above argument we have $\Phi_{s_i}^{\theta, \lambda} = \overrightarrow{p}_i \circ \overrightarrow{p}_i^{-1}$, and the latter does not depend on the choice of $\theta'$, therefore $\Phi_{s_i}^{\theta, \lambda}$ does not depend on the choice of $\theta'$. \qed
3. Proof of Theorem B

In this section we complete the proof of Theorem B thus proving Theorem A by Proposition 2.7. Since passing from \( Q \) to the support of dimension vector \( v \) does not affect the flatness, we will assume that \( v_i > 0 \) for all \( i \in Q_0 \) in this section.

3.1. Case of \(|Q_0| = 1\). In this case the vertex set is just one element \( Q_0 = \{1\} \), and \( \Theta = Z = \{0\} \). Suppose that there is only one arrow, then computation using Lemma 2.2 shows that the only value of \( v_1 \) such that the moment map \( \mu \) is flat is \( v_1 = 1 \), and in this case Theorem B trivially holds. Suppose that there are more than one arrows, then an easy computation shows that

\[
p(v) > \sum_{t=1}^{r} p(v^{(t)}),
\]

for all decomposition \( v = v^{(1)} + \cdots + v^{(r)} \) into non-zero elements in \( \mathbb{N}^{Q_0} \) and \( r > 1 \). In this case \( v \in \Sigma_0 \) (see comments after \[12\] Theorem 1.2 for notation), and in this case Crawley-Boevey shows that \( \mu^{-1}(0) \) is reduced and irreducible \[12\] Theorem 1.2, thus \( M_0(Q, v) \) is reduced hence normal, by \[17\] Theorem 1.1. This finishes the proof of the case when \(|Q_0| = 1\).

3.2. Case of \( Q_0 = \{1, 2\} \) and \( v_1 = 1 \). In this case \( v_2 = n > 0 \) and \( (\theta, \lambda) \in \mathbb{Q} \times \mathbb{C} \). The main result of this subsection is following:

**Proposition 3.3.** If \( Q_0 = \{1, 2\} \) and \( v_1 = 1 \), then \( \mu^{-1}(0) \) is reduced.

**Proof.** Note that edge loop at vertex 1 only contributes a vector space to \( \mu^{-1}(0) \), without loss of generality we can assume that there is no edge loop at vertex 1. Since the quiver will be doubled when considering moment map, without loss of generality we can assume that all arrows between vertices 1 and 2 are pointing from 1 to 2, and denote the number of such arrows by \( k \). We split the discussion into three situations.

1. \( v \in \Sigma_0 \) (see comments after \[12\] Theorem 1.2 for notation), this condition is equivalent to that the inequality in Lemma 2.2 is strict, i.e. the equality never holds. In this situation \[12\] Theorem 1.2 shows that \( \mu^{-1}(0) \) is reduced and irreducible. An easy computation shows that \( v \in \Sigma_0 \) exactly when there are more than two edge loops at vertex 2, or there is one edge loop at vertex 2 and \( k > 1 \), or there is no edge loop at vertex 2 and \( k \geq 2n \).

2. If there is no edge loop in \( Q \), then the flatness of \( \mu \) is equivalent to \( k \geq 2n - 1 \) (see Lemma 2.2). The case \( k \geq 2n \) has been discussed above. If \( k = n = 1 \), then straightforward computation finds that \( \mu^{-1}(0) \) is reduced. The remaining cases are \( k = 2n - 1 > 1 \).

3. If there is one edge loop and and \( k = 1 \), then by the main result of Gan-Ginzburg \[11\], \( \mu^{-1}(0) \) is reduced in this case.

The idea of proof in the remaining cases in (2) is to show that every irreducible component of \( \mu^{-1}(0) \) is generically reduced, then flatness of \( \mu \) implies the Cohen-Macaulay property of

\(^{1}\)Dimension vector such that \( v_i > 0 \) for all \( i \in Q_0 \) is called sincere, cf. \[12\].
\( \mu^{-1}(0) \), which in turn implies that \( \mu^{-1}(0) \) is reduced.

**Remaining part of situation (2).** We claim that \( \mu^{-1}(0) \) has exactly two irreducible components. In effect consider the morphism \( \pi : \mu^{-1}(0) \to R(Q,v) \) defined by forgetting the cotangent direction, then [12] Lemma 4.3 implies that \( \mu^{-1}(0) \) is union of constructible subsets of which the maximal dimensional ones are \( \pi^{-1}(v^{(1)}, \ldots, v^{(r)}) \), where \( v = v^{(1)} + \cdots + v^{(r)} \) is a decomposition of \( v \) into non-zero elements in \( \mathbb{N}^Q \) such that \( v^{(1)} = 1 \), and

\[
(3.1) \quad p(v) = p(v^{(1)}) + \cdots + p(v^{(r)}).
\]

Here \( I(v^{(1)}, \ldots, v^{(r)}) \) is the constructible subset of \( R(Q,v) \) consisting of the representations whose indecomposable summands have dimension \( v^{(1)}, \ldots, v^{(r)} \). Under the assumption that \( k = 2n - 1 > 1 \), the only decompositions such that the equation (3.1) hold are \( (v^{(1)}, v^{(2)}) = (v - e_2, e_2) \) and \( v^{(1)} = v \). Here \( e_2 \) is the dimension vector \( e_{2i} = \delta_{2i} \). Note that both \( I(v - e_2, e_2) \) and \( I(v) \) consist of a single \( GL(n) \times GL(k) \)-orbit (by Gauss elimination), therefore preimages of both \( \pi^{-1}(v - e_2, e_2) \) and \( \pi^{-1}(v) \) are irreducible by [12] Lemma 3.4, and

\[
\dim \mu^{-1}(0) \setminus (\pi^{-1}(v - e_2, e_2) \cup \pi^{-1}(v)) < \dim \mu^{-1}(0).
\]

\( \mu^{-1}(0) \) is Cohen-Macaulay, so it is equidimensional, thus

\[
\mu^{-1}(0) = \pi^{-1}(v - e_2, e_2) \cup \pi^{-1}(v).
\]

Notice that \( \pi^{-1}(v) \) contains \( \theta \)-stable representations for \( (\theta_1, \theta_2) = (-n, 1) \), thus \( \pi^{-1}(v) \) is generically smooth.

For \( \pi^{-1}(v - e_2, e_2) \), consider a simple representation \( x \) of \( (Q, v - e_2) \) (simple representation exists for dimension vector \( v - e_2 \) by our previous discussion in the situation (1)) and take its direct sum with a trivial representation and regarded as a representation of \( (Q, v) \), still denoted by \( x \), then \( x \) is semisimple and \( \pi(x) \in I(v - e_2, e_2) \). According to Lemma 2.4 there is an étale transverse slice of \( G(v) : x \) in \( \mu^{-1}(0) \), which is étale locally isomorphic to an open neighborhood \( U \) of \( 0 \in \hat{\mu}^{-1}(0) \), where \( \hat{\mu} \) is the classical moment map for the quiver \( (\hat{Q}, \hat{v}) \). Here \( \hat{Q} \) is the quiver with vertex set \( \hat{Q}_0 = \{1, 2\} \), and the number of arrows between vertices 1 and 2 is \( k + 1 - n \), which equals to \( n > 1 \), and \( \hat{v}_1 = \hat{v}_2 = 1 \). Note that \( \hat{v} \in \Sigma_0 \) and this reduces to the situation (1), thus \( \mu^{-1}(0) \) is reduced in an open neighborhood of \( x \). Finally \( x \) is in the irreducible component \( \pi^{-1}(v - e_2, e_2) \), so the irreducible component \( \pi^{-1}(v - e_2, e_2) \) is generically reduced.

\( \square \)

**Corollary 3.4.** If \( Q_0 = \{1, 2\} \) and \( v_1 = 1 \), then Theorem 3.3 holds.

**Proof.** If either \( \theta \) or \( \lambda \) is non-zero, then \( (\theta, \lambda) \) satisfies (2.6), note that \( v \) is obviously indivisible, thus \( M^\theta_0(Q, v) \) and \( M^\lambda_0(Q, v) \) are smooth by Remark 2.10. So we focus on the case \( \theta = \lambda = 0 \). \( \mu^{-1}(0) \) is reduced by Proposition 3.3 and since \( \mu^{-1}(Z) \) is Cohen-Macaulay and \( \mu^{-1}(Z \setminus 0) \) is smooth, we conclude that \( \mu^{-1}(Z) \) is normal, and henceforth \( M_Z(Q, v) \) is normal. By Lemma 2.15 \( M_0(Q, v) \) is reduced.

\( \square \)

3.5. **General cases.** The last ingredient we need from Crawley-Boevey is the following technical lemma:
Lemma 3.6 ([17] Corollary 7.2). Let $X$ be a scheme over $\mathbb{C}$ with a reductive group $G$-action, assume that

1. Categorical quotient $q : X \to X/G$ exists and $q$ is affine,
2. There is an open $U \subset X/G$ such that $U$ is normal, and its complement $Y$ has codimension $\geq 2$ in $X/G$,
3. $X$ has property $(S_2)$ and $q^{-1}(Y)$ has codimension $\geq 2$ in $X$.

Then $X/G$ is normal.

Proof of Theorem 1.3. We assume that $Q$ is connected, otherwise the moment map can not be flat (see Remark 1.2). Define a partial order $\prec$ on the set of $(Q, v)$ by

$$(Q', v') \prec (Q, v) \text{ if } |v'| < |v|, \text{ or } |v'| = |v| \text{ and } |Q'| \geq 3 > |Q|.$$ 

Here $|v| = \sum_{i \in Q_0} v_i$. One can verify that $\prec$ is indeed a partial order. Note that if $(Q'', v'') \prec (Q', v') \prec (Q, v)$, then $|v''| < |v|$, therefore every chain $(Q, v) \succ (Q', v') \succ \cdots$ terminates after finite steps. We prove the theorem by induction on this partial order. Note that $|Q_0| = 1$ case has been proven in the subsection 3.1, so we assume that $|Q_0| \geq 2$ and the theorem is true for all $(Q', v')$ with flat moment map such that $(Q', v') \prec (Q, v)$.

- If $\theta$ is generic and $v$ is indivisible, then Remark 2.10 shows that $M^\theta_\lambda(Q, v)$ is smooth.

- If $\theta$ is generic and $v$ is divisible, i.e. $v = n w$, $n \in \mathbb{N}_{>1}$, $w \in N^{Q_0}$, then for arbitrary $\lambda \in Z$ we take $x \in \mu^{-1}(\lambda)^{\theta-ss}$ such that $G(v) \cdot x$ is closed in $\mu^{-1}(\lambda)^{\theta-ss}$. Either $x$ is simple, or $x$ decomposes as

$$x = x_1^{\oplus k_1} \oplus \cdots \oplus x_r^{\oplus k_r},$$

where $x_t$ are $\theta$-stable with dimension vectors $s_t w$. If $x$ is simple then $\mu$ is smooth at $x$. If $x$ is decomposable, then Lemma 2.3 shows that there is an isomorphism

$$M^\theta_\lambda(Q, v)^\wedge_x \cong M_0(\hat{Q}, \hat{v})^\wedge_0$$

for a new quiver $(\hat{Q}, \hat{v})$. We claim that

$$|\hat{v}| < |v|.$$ 

In effect, the equation $v = \sum_t k_t s_t w$ implies that $|\hat{v}| \leq |v|$ and the equality holds if and only $|w| = 1$, but if this is the case then $v$ is supported at a single vertex thus $|Q_0| = 1$, this contradicts with the assumption that $|Q_0| \geq 2$. Therefore $(\hat{Q}, \hat{v}) \prec (Q, v)$ and by induction hypothesis the Theorem 1.3 is true for generic $\theta$.

- If $|Q_0| \geq 3$, then we take $\lambda \neq 0$ and take $x \in \mu^{-1}(\lambda)^{\theta-ss}$ such that $G(v) \cdot x$ is closed in $\mu^{-1}(\lambda)^{\theta-ss}$. Either $x$ is simple, or $x$ decomposes as

$$x = x_1^{\oplus k_1} \oplus \cdots \oplus x_r^{\oplus k_r},$$

where $x_t$ are $\theta$-stable with dimension vectors $v^{(i)}$. If $x$ is simple then $\mu$ is smooth at $x$. If $x$ is decomposable, then Lemma 2.3 shows that there is an isomorphism

$$M^\theta_\lambda(Q, v)^\wedge_x \cong M_0(\hat{Q}, \hat{v})^\wedge_0$$
and we conclude that $M_{\lambda} \subset W_{\theta}$ with our choice of preimage in $\hat{\mathbf{Q}}$ for a new quiver $(\hat{\mathbf{Q}}, \hat{\mathbf{v}})$. We claim that

$$|\hat{\mathbf{v}}| < |\mathbf{v}|.$$ 

In effect, the equation $v = \sum_{t} k_t v^{(t)}$ implies that $|\hat{\mathbf{v}}| \leq |\mathbf{v}|$ and the equality holds if and only if $v^{(t)} = 1$ for all $t$, and $|\mathbf{v}^{(t)}| = 1$ is equivalent to $v^{(t)} = e_i$ for some vertex $i_t \in Q_0$, therefore $|\hat{\mathbf{v}}| = |\mathbf{v}|$ only happens when $\lambda \cdot e_i = 0$ for all $i \in Q_0$, but this forces $\lambda = 0$, which contradicts with our choice of $\lambda$. Hence we have

$$(\hat{\mathbf{Q}}, \hat{\mathbf{v}}) \prec (\mathbf{Q}, \mathbf{v})$$

and we conclude that $M_{\lambda}^{\theta}(\mathbf{Q}, \mathbf{v})$ is normal, because for every point in $M_{\lambda}(\mathbf{Q}, \mathbf{v})$ there is a preimage in $\mu^{-1}(\lambda)^{\theta-ss}$ with closed $GL(\mathbf{v})$-orbit. By the flatness of $\mu$, $M_{\lambda}^{\theta}(\mathbf{Q}, \mathbf{v})|_{Z_{\lambda}}$ is normal. Again by the flatness of $\mu$, $M_{\lambda}^{\theta}(\mathbf{Q}, \mathbf{v})$ has codimension $|Q_0| - 1 \geq 2$ in $M_{\lambda}^{\theta}(\mathbf{Q}, \mathbf{v})$, and $\mu^{-1}(0)^{\theta-ss}$ has codimension $|Q_0| - 1 \geq 2$ in $\mu^{-1}(Z)^{\theta-ss}$. Since $\mu^{-1}(Z)^{\theta-ss}$ is Cohen-Macaulay, we can apply the Lemma 3.6 and conclude that $M_{\lambda}^{\theta}(\mathbf{Q}, \mathbf{v})$ is normal. Since Theorem 3 is true for $M_{\lambda}^{\theta}(\mathbf{Q}, \mathbf{v})$ with generic $\theta$, we can apply Lemma 2.15 thus Theorem B is true for all $M_{\lambda}^{\theta}(\mathbf{Q}, \mathbf{v})$.

- If $|Q_0| = 2$, the only $(\theta, \lambda)$ that is not generic is $(0, 0)$, so we only need to show that $M_{\lambda}(\mathbf{Q}, \mathbf{v})$ is normal, according to Lemma 2.13 and 2.15. In the view of Lemma 3.6 we need to show that there exists a Zariski open subset $U \subset M_{\lambda}(\mathbf{Q}, \mathbf{v})$ such that $U$ is reduced and $\dim \xi^{-1}(M_{\lambda}(\mathbf{Q}, \mathbf{v}) \setminus U) < \dim \mu^{-1}(0)$ where $\xi : \mu^{-1}(0) \rightarrow M_{\lambda}(\mathbf{Q}, \mathbf{v})$ is the quotient map. Lemma 2.4 shows that for every $x \in M_{\lambda}(\mathbf{Q}, \mathbf{v})$, the formal completion $M_{\lambda}(\mathbf{Q}, \mathbf{v})^\wedge_x$ is isomorphic to $M_{\lambda}(\hat{\mathbf{Q}}, \hat{\mathbf{v}})^\wedge_0$ for some $(\hat{\mathbf{Q}}, \hat{\mathbf{v}})$. So it is enough to show that there exists a constructible subset $W \subset M_{\lambda}(\mathbf{Q}, \mathbf{v})$ such that $\forall x \in M_{\lambda}(\mathbf{Q}, \mathbf{v}) \setminus W$ the quiver $(\hat{\mathbf{Q}}, \hat{\mathbf{v}})$ associated to $x$ is $\prec (\mathbf{Q}, \mathbf{v})$, and $\dim \xi^{-1}(W) < \dim \mu^{-1}(0)$.

The prospective choice of $W$ is $\bigcup_{\tau} M_{\lambda}(\mathbf{Q}, \mathbf{v})_{\tau}$ where $\tau$ are the representation types

$$(3.2) \quad \tau = (k_1, v^{(1)}; \cdots; k_r, v^{(r)}),$$

such that $v^{(t)}$ are $e_1$ or $e_2$ for all $t$. Precisely, we are going to show that one of the following situations must happen:

(a) $p(v) > v_1p(e_1) + v_2p(e_2)$.

(b) For representation type $\tau = (k_1, v^{(1)}; \cdots; k_r, v^{(r)})$ as above, and assume moreover

$$p(v) = \sum_{t=1}^{r} p(v^{(t)}),$$

then $r$ must be greater than 2.

If (a) is true, then we take $W = \bigcup_{\tau} M_{\lambda}(\mathbf{Q}, \mathbf{v})_{\tau}$, where $\tau = (k_1, v^{(1)}; \cdots; k_r, v^{(r)})$, so $\dim \xi^{-1}(W) < \dim \mu^{-1}(0)$ according to [17] Corollary 6.4, and $\forall x \in M_{\lambda}(\mathbf{Q}, \mathbf{v}) \setminus W$ the associated $(\hat{\mathbf{Q}}, \hat{\mathbf{v}})$ to $x$ must have $|\hat{\mathbf{v}}| < |\mathbf{v}|$ (the argument is the same as the $|Q_0| \geq 3$ case). If (b) is true, then we take $W = \bigcup_{\tau} M_{\lambda}(\mathbf{Q}, \mathbf{v})_{\tau}$, where $\tau = (k_1, v^{(1)}; \cdots; k_r, v^{(r)})$ such that

$$p(v) > \sum_{t=1}^{r} p(v^{(t)}).$$
In this situation $\dim \xi^{-1}(W) < \dim \mu^{-1}(0)$ according to \cite[Corollary 6.4]{17}, and $\forall x \in M_0(Q, v) \setminus W$ the associated $(\hat{Q}, \hat{v})$ to $x$ must have $|\hat{v}| > |v|$ or $|\hat{Q}_0| \geq 3$. Both situations imply the normality of $M_2(Q, v)$ by Lemma \ref{3.6}.

To show that one of (a) or (b) must happen, we proceed case-by-case. Let $(v_1, v_2) = (N, K)$ and we can assume that $N \geq 2, K \geq 2$ because the case of $v_1 = 1$ (or symmetrically $v_2 = 1$) has been proven in Corollary \ref{3.3}. Since the quiver will be doubled when considering moment map, without loss of generality we can assume that all arrows between vertices 1 and 2 are pointing from 1 to 2, and the adjacency matrix is

\begin{equation}
\begin{bmatrix}
    a & b \\
    0 & c
\end{bmatrix},
\end{equation}

note that $b > 0$ (in order that $Q$ is connected), and we assume that $a \leq c$ (otherwise we just reverse the direction of arrows).

(1) $c \geq a \geq 1$. We claim that (a) is true in this case. In effect,

$$p(v) - Np(e_1) - Kp(e_2) = (a - 1)(N - 1)N + (c - 1)(K - 1)K + bNK + 1 - N - K 
\geq NK - N - K + 1 > 0.$$ 

(2) $a = 0, c > 1$. We claim that (a) is true in this case. Consider $w = (w_1, w_2) = (1, K)$, then it is easy to see that $w \in \Sigma_0$, thus $p(w) > p(e_1) + Kp(e_2)$, therefore

$$p(v) \geq (N - 1)p(e_1) + p(w) > Np(e_1) + Kp(e_2).$$ 

(3) $a = 0, b > 1$. We claim that (a) is true in this case. Consider $w = (w_1, w_2) = (1, 1)$, then it is easy to see that $w \in \Sigma_0$, thus $p(w) > p(e_1) + p(e_2)$, therefore

$$p(v) \geq (N - 1)p(e_1) + (K - 1)p(e_2) + p(w) > Np(e_1) + Kp(e_2).$$ 

(4) $(a, b, c) = (0, 1, 0)$. We claim that this case can not happen. In fact

$$p(v) = -N^2 - K^2 + NK + 1 = 1 - (N - K)^2 - NK \leq 1 - NK < 0,$$

by our assumption that $N \geq 2, K \geq 2$. This contradicts with the flatness assumption.

(5) $(a, b, c) = (0, 1, 1)$ and $K > N + 1$. We claim that (a) is true in this case. In fact

$$p(v) - Np(e_1) - Kp(e_2) = NK - N^2 + 1 - K = (N - 1)(K - N - 1) > 0.$$ 

(6) $(a, b, c) = (0, 1, 1)$ and $K < N + 1$. This case can not happen, because

$$p(v) - Np(e_1) - Kp(e_2) = (N - 1)(K - N - 1) < 0,$$

this contradicts with the flatness assumption.

(7) $(a, b, c) = (0, 1, 1)$ and $K = N + 1 \geq 4$. This case can not happen, because

$$p(v) - (N - 2)p(e_1) - p(2e_1 + Ke_2) = NK - N^2 + 1 - (2K - 3) = 3 - K < 0,$$

this contradicts with the flatness assumption.

(8) $(a, b, c) = (0, 1, 1)$ and $N = 2, K = 3$. We claim that (b) is true in this case. By a straightforward computation we see that the only representation type $\tau = (k_1, v^{(1)}; \cdots; k_r, v^{(r)})$ such that $p(v) = \sum_{i=1}^{r} p(v^{(i)})$ is

$$\tau = (2, e_1; 1, e_2; 1, e_2; 1, e_2),$$
therefore $r = 4$ and (b) holds.

All cases have been covered, and this finishes the proof of Theorem 2.2. \hfill \Box

4. TYPE A DYNKIN QUIVERS AND FINITE $W$-ALGEBRAS

4.1. Flatness of moment maps for affine type A Dynkin quivers with framings.

Let $(Q, v, d)$ be a framed quiver with framing vector $d$ (assume $d \neq 0$). Following Crawley-Boevey, we define the associated unframed quiver $(Q^d, v)$ as $Q_0^d = Q_0 \cup \{\infty\}$ (union of vertices of $Q$ with an extra vertex denoted by $\infty$), and arrows in $Q^d$ are those from $Q$ and for each vertex $i \in Q_0$ attach $d_i$-copies of arrows from $\infty$ to $i$, and set $v_i^d = v_i$ if $i \in Q_0$ and $v_\infty^d = 1$. From the construction we see that the group

\begin{equation}
G(v^d) = \prod_{i \in Q_0^d} \text{GL}(v_i^d)/C^\times \cong \prod_{i \in Q_0} \text{GL}(v_i) =: \text{GL}(v)
\end{equation}

acts on $R(Q^d, v^d)$. Note that the deformation space $Z$ can be identified with $\prod_{v_i \neq 0} \mathbb{C}$, the identification is by $\lambda \mapsto \lambda^d$ such that $\lambda_i^d = \lambda_i$ for $i \in Q_0$ and $\lambda_\infty^d = -\sum_{i \in Q_0} \lambda_i v_i$. Similarly the stability space $\Theta$ can be identified with $\prod_{v_i \neq 0} Q$ via $\theta \mapsto \theta^d$ such that $\theta_i^d = \theta_i$ for $i \in Q_0$ and $\theta_\infty^d = -\sum_{i \in Q_0} \theta_i v_i$. We define the Nakajima quiver schemes associated to $(Q, v, d)$ as:

\begin{equation}
M_Z^\theta(Q, v, d) := \mathcal{M}_Z^d(Q^d, v^d), \ M_\lambda^\theta(Q, v, d) := \mathcal{M}_\lambda^d(Q^d, v^d).
\end{equation}

**Proposition 4.2.** If $Q$ is an affine type A Dynkin quiver, let $v$ and $d$ be dimension vector and framing vector (assume $d \neq 0$), then the moment map $\mu$ is flat if and only if the following condition is satisfied:

\begin{equation}
eq \sum_{i \in I} e_i \cdot (d - C_Q v) \geq -1,
\end{equation}

for arbitrary subset $I \subset Q_0$ such that nodes in $I$ are connected by arrows in $Q$, and $e_i = \sum_{i \in I} e_i$.

**Proof.** By Lemma 2.2, the flatness of $\mu$ is equivalent to that for every decomposition $v^d = v' + v^{(1)} + \cdots + v^{(r)}$ with $v^{(1)}, \ldots, v^{(r)}$ being roots of $Q$, the inequality $p(v^d) \geq p(v') + p(v^{(1)}) + \cdots + p(v^{(r)})$ holds.

Let $u = v^d - v'$, then it is easy to see that $p(v^{(1)}) + \cdots + p(u^{(r)}) \leq \min_{i \in Q_0} u_i$, and the equality can be achieved. On the other hand $p(v^d) - p(v') = u \cdot (d - C_Q v) + \frac{1}{2}(u, u)_Q$, so the flatness of moment map is equivalent to that $\forall u \in \mathbb{N}^{Q_0}$ such that $0 \leq u \leq v$, the inequality

\begin{equation}
u \cdot (d - C_Q v) + \frac{1}{2}(u, u)_Q \geq \min_{i \in Q_0} u_i \tag{4.4}
\end{equation}

holds. Let us discuss two situations separately: (1) $\text{Supp}(v) \neq Q$; (2) $\text{Supp}(v) = Q$.

In the case (1), the right hand side of (4.4) is zero and $Q$ can be regarded as a type $A$ quiver by removing one node with zero dimension vector. Moreover (4.4) holds if and only if it holds on each connected component of $\text{Supp}(v)$, so let us assume that $Q' = \text{Supp}(v)$ is connected. Let $I$ be a subset of $Q_0'$ such that nodes in $I$ are connected by arrows in $Q'$, and set $u = e_I$, then (4.4) implies that $e_I \cdot (d - C_Q v) \geq -1$. On the other hand if $e_I \cdot (d - C_Q v) \geq -1$ holds for all $I \subset Q_0'$ such that nodes in $I$ are connected by arrows in
Q′, then Lemma 4.3 implies that \( \forall u \in \mathbb{N}^Q \) such that \( 0 \leq u \leq v \) we have a decomposition 
\[
u = \sum_{i=1}^{n} e_{I_{i}}
\]
such that \( (e_{I_{i}}, e_{I_{j}})_{Q} \geq 0 \) for any pair \( 1 \leq \alpha, \beta \leq n \), therefore
\[
u \cdot (d - C_{Q}v) + \frac{1}{2}(u, u)_{Q} \geq \sum_{i=1}^{n} \left( e_{I_{i}} \cdot (d - C_{Q}v) + \frac{1}{2}(e_{I_{i}}, e_{I_{i}})_{Q} \right) \geq 0
\]
This shows that (4.4) is equivalent to (4.3) being true for all \( I \subset Q_{0} \) such that \( I \) is a contained in a connected component of \( \text{Supp}(v) \). Observe that if \( v = 0 \) then \( e_{I} \cdot (d - C_{Q}v) \geq 0 \), it follows that if (4.3) is true for all \( I \subset Q_{0} \) such that \( I \) is a contained in a connected component of \( \text{Supp}(v) \), then it is true for all \( I \). This completes the proof in the case (1).

In the case (2), by taking \( u = e_{I} \) we see that (4.3) implies that (4.3) holds for all \( I \). Note that the case \( I = Q_{0} \) is automatically true, since we assume that \( d \neq 0 \). Conversely, assume that (4.3) holds for all \( I \), then let us write \( u = m \delta + \sum_{i=1}^{n} e_{I_{i}} \) (\( \delta \) is the imaginary root of \( Q \)) such that \( I_{\alpha} \neq Q_{0} \) for all \( 1 \leq \alpha \leq n \), and \( (e_{I_{i}}, e_{I_{j}})_{Q} \geq 0 \) for any pair \( 1 \leq \alpha, \beta \leq n \), the existence of such decomposition follows from Lemma 4.3. Then we have
\[
u \cdot (d - C_{Q}v) + \frac{1}{2}(u, u)_{Q} \geq m \delta \cdot d + \sum_{i=1}^{n} \left( e_{I_{i}} \cdot (d - C_{Q}v) + \frac{1}{2}(e_{I_{i}}, e_{I_{i}})_{Q} \right) \geq m = \min_{i \in Q_{0}} u_{i}.
\]
This completes the proof in the case (2).

**Lemma 4.3.** If \( Q \) is an affine type A Dynkin quiver, and \( u \in \mathbb{N}^{Q_{0}} \), then there exists subsets \( I_{1}, \ldots, I_{n} \) of \( Q_{0} \) such that nodes in \( I_{\alpha} \) are connected by arrows in \( Q \) for all \( 1 \leq \alpha \leq n \), and \( u = \sum_{i=1}^{n} e_{I_{i}} \), and \( (e_{I_{i}}, e_{I_{j}})_{Q} \geq 0 \) for any pair \( 1 \leq \alpha, \beta \leq n \).

**Proof.** Without loss of generality, let us assume that \( \text{Supp}(u) \) is connected, otherwise we can restrict to connected components. Let us take \( I_{1} \) to be the set of nodes in \( \text{Supp}(u) \), note that for any \( J \subset I_{1} \), we have \( (e_{J}, e_{I_{1}})_{Q} \geq 0 \). By induction on \( \sum_{i \in Q_{0}} u_{i} \), we have \( I_{2}, \ldots, I_{n} \) such that \( u - e_{I_{1}} = \sum_{i=2}^{n} e_{I_{i}} \) and \( (e_{I_{i}}, e_{I_{j}})_{Q} \geq 0 \) for any pair \( 2 \leq \alpha, \beta \leq n \). Now \( I_{\alpha} \subset I_{1} \) for all \( \alpha \), thus it suffices to take \( I_{1}, \ldots, I_{n} \).

**4.4. Quantization of quiver schemes.** Let \( \mathcal{A}_{h} \) be a flat \( \mathbb{C}[h] \) algebra such that \( A : = \mathcal{A}_{h}/\overline{\mathcal{A}}_{h} \) is commutative. Suppose that \( \mathbb{C}^{\times} \) acts on \( \mathcal{A}_{h} \) by automorphisms such that \( h \) has weight 2. Let \( g \) be a Lie algebra with an action on \( \mathcal{A}_{h} \), i.e. a Lie homomorphism \( g \rightarrow \text{Der}(\mathcal{A}_{h}) \). Suppose that \( \Phi_{h} : g \rightarrow \mathcal{A}_{h} \) is a \( \mathbb{C}[h] \)-linear map such that image of \( \Phi_{h} \) is in the weight 2 subspace of \( \mathcal{A}_{h} \). We furthermore assume that \( \Phi_{h} \) is a Lie algebra homomorphism, where the Lie bracket \( [-, -]_{h} \) on \( \mathcal{A}_{h} \) is \( [a, b]_{h} = h[a, b] \).

**Definition 4.5.** A Lie algebra homomorphism \( \Phi_{h} : g \rightarrow \mathcal{A}_{h} \) as above is called a quantum moment map if \( \forall a \in g, b \in \mathcal{A}_{h}, \)
\[
[\Phi_{h}(a), b]_{h} = a \cdot b.
\]
Here \( a \cdot b \) means the action of \( a \) on \( b \).

Let \( \chi \in (g/[g, g])^{*} \) and form the shift \( g_{\chi} = \{ a - h(\chi, a) \} \subset g \oplus \mathbb{C}h \), then it is easy to see that \( (\mathcal{A}_{h}(g_{\chi}))^{0} \) is a two-sided ideal of \( \mathcal{A}_{h}^{0} \). More generally, let \( U \) be a linear space and \( \varphi : U \rightarrow (g/[g, g])^{*} \) be a linear map, and form the shift \( g_{U} = \{ a - \varphi^{*}(a) \} \subset g \oplus U^{*} \), then \( (\mathcal{A}_{h}[U^{*}]\Phi_{h}(g_{U}))^{0} \) is a two-sided ideal of \( \mathcal{A}_{h}^{0}[U^{*}] \).
Definition 4.6. Define the quantum Hamiltonian reductions $A_h \sslash_h \mathfrak{g} = A_h^\theta / (A_h \Phi_h(\mathfrak{g}_h))_h$ and $A_h \sslash_U \mathfrak{g} = A_h^\theta [U^*] / (A_h [U^*] \Phi_h(\mathfrak{g}_U))_h$.

Note that we also have classical Hamiltonian reduction $A \sslash_U \mathfrak{g} = A^\theta [U^*] / (A [U^*] \Phi(\mathfrak{g}_U))_h$, and it is easy to see that there is a natural algebra homomorphism $(A_h \sslash_U \mathfrak{g}) / (h) \to A \sslash_U \mathfrak{g}$.

Lemma 4.7. Suppose that $\mathfrak{g}$ is reductive, and $\{e_1, \cdots, e_n\}$ is a basis of $\ker(\varphi^*) \subset \mathfrak{g}$ such that $\{\Phi(e_i)\}_{i=1}^n$ is a regular sequence in $A = A_h / (h)$, then there is an isomorphism

$$ (A_h \sslash_U \mathfrak{g}) / (h) \cong A \sslash_U \mathfrak{g}. $$

Proof. See [20, Lemma 3.3.1].

Apply the above construction to the quiver representation of $(Q, \mathfrak{v})$, and we set $A_h = W_h(T^* R(Q, \mathfrak{v}))$ (Weyl algebra of the symplectic vector space $T^* R(Q, \mathfrak{v})$) with $\mathbb{C}^\times$ acting on the vector space $T^*(R, \mathfrak{v})$ of weight $-1$, $G = G(\mathfrak{v})$, and $\Phi_h : \mathfrak{g}(\mathfrak{v}) \to W_h(T^* R(Q, \mathfrak{v}))$ is the (unique) degree 2 lift of the comoment map $\mu^* : \mathfrak{g}(\mathfrak{v}) \to \mathbb{C}[T^* R(Q, \mathfrak{v})]$. For a linear map $U \to Z = [\mathfrak{g}(\mathfrak{v})/[\mathfrak{g}(\mathfrak{v}), \mathfrak{g}(\mathfrak{v})]]^*$, call the quantum Hamiltonian reduction $W_h(T^* R(Q, \mathfrak{v})) \sslash_U \mathfrak{g}(\mathfrak{v})$ the quantized quiver scheme, denoted by $A_h \sslash_U (Q, \mathfrak{v})$. Since $\mathfrak{g}(\mathfrak{v})$ is reductive, the natural homomorphism $A_h \sslash_U (Q, \mathfrak{v}) / (h) \to \mathbb{C}[M_U(Q, \mathfrak{v})]$ is surjective, where $M_U(Q, \mathfrak{v}) = M_Z(Q, \mathfrak{v}) \times Z U$. A special case of Lemma 4.7 reads:

Lemma 4.8. If the moment map $\mu : T^* R(Q, \mathfrak{v}) \to \mathfrak{g}(\mathfrak{v})^*$ is flat, then the epimorphism $A_{h,U}(Q, \mathfrak{v}) / (h) \to \mathbb{C}[M_{U}(Q, \mathfrak{v})]$ is an isomorphism, and $A_{h,U}(Q, \mathfrak{v})$ is flat over $\mathbb{C}[U][h]$.

Proof. If $\mu$ is flat then any basis $\{e_1, \cdots, e_n\}$ of $\ker(\varphi(\mathfrak{v}) \to U^*)$ is mapped to a regular sequence in $\mathbb{C}[T^* R(Q, \mathfrak{v})]$, so by [4.7] $A_{h,U}(Q, \mathfrak{v}) / (h) \to \mathbb{C}[M_{U}(Q, \mathfrak{v})]$ is an isomorphism. Moreover the associated graded of $A_{h,U}(Q, \mathfrak{v})$ with respect to the $h$-filtration is $\mathbb{C}[M_U(Q, \mathfrak{v})][h]$, which is flat over $\mathbb{C}[U][h]$, therefore $A_{h,U}(Q, \mathfrak{v})$ is flat over $\mathbb{C}[U][h]$.

We can sheafify the construction of quantum Hamiltonian reduction, at the cost of taking $h$-completion. We will not repeat the definition here, see [20, 3.4] for detail. A particular case that we will focus on is the following: $G(\mathfrak{v})$ acts on $\tilde{W}_h(T^* R(Q, \mathfrak{v}))$ with quantum moment map $\Phi_h$, here $\tilde{W}_h(T^* R(Q, \mathfrak{v}))$ is the completion of the Weyl algebra of symplectic vector space $T^* R(Q, \mathfrak{v})$ in the $h$-adic topology, and it can be sheafified on $T^* R(Q, \mathfrak{v})$, assume that $\mathfrak{v}$ is indivisible and $\theta \in \Theta$ is a generic stability condition, then for every linear map $U \to Z = ([\mathfrak{g}(\mathfrak{v})/[\mathfrak{g}(\mathfrak{v}), \mathfrak{g}(\mathfrak{v})]]^*)$ we have a sheaf of associative flat $\mathbb{C}[h]$-algebras $\tilde{W}_h(T^* R(Q, \mathfrak{v})) \sslash_U G(\mathfrak{v})$ on the smooth scheme $\mathcal{M}^\theta_U(Q, \mathfrak{v})$ (see Remark 2.10), denoted by $\mathcal{O}_{h,\mathcal{M}^\theta_U(Q, \mathfrak{v})}$. Note that $\mathcal{O}_{h,\mathcal{M}^\theta_U(Q, \mathfrak{v})} / (h) \cong \mathcal{O}_{\mathcal{M}^\theta_U(Q, \mathfrak{v})}$, i.e. $\mathcal{O}_{h,\mathcal{M}^\theta_U(Q, \mathfrak{v})}$ is a quantization of $\mathcal{O}_{\mathcal{M}^\theta_U(Q, \mathfrak{v})}$.

Proposition 4.9. Suppose that the quiver $(Q, \mathfrak{v})$ is such that $\mathfrak{v}$ is indivisible and the moment map $\mu : T^* R(Q, \mathfrak{v}) \to \mathfrak{g}(\mathfrak{v})^*$ is flat, then the natural homomorphism

$$ W_h(T^* R(Q, \mathfrak{v}))^G(\mathfrak{v}) \to \Gamma(\mathcal{M}^\theta_U(Q, \mathfrak{v}), \mathcal{O}_{h,\mathcal{M}^\theta_U(Q, \mathfrak{v})}) $$

gives rise to an isomorphism between $A_{h,U}(Q, \mathfrak{v})$ and the sub-algebra of $\mathbb{C}^\times$-finite elements in $\Gamma(\mathcal{M}^\theta_U(Q, \mathfrak{v}), \mathcal{O}_{h,\mathcal{M}^\theta_U(Q, \mathfrak{v})})$. 


Proof. We need to check the conditions (i), (ii), and (iii) in [20 Lemma 4.2.4]. Condition (i) is checked by Lemma 4.8. Condition (ii) holds because $v$ is indivisible and $\theta$ is generic; condition (iii) follows from that $p^\theta : M_h^\lambda(Q, v) \to M_\lambda(Q, v)$ is birational and Poisson for all $\lambda \in Z$ (see Corollary 2.16), and the Poisson structure on $M_h^\lambda(Q, v)$ is symplectic. \qed

4.10. Finite $W$-algebras. Let $G$ be a reductive algebraic group, $\mathfrak{g}$ be the Lie algebra of $G$. Pick a nilpotent element $e \in \mathfrak{g}$ and choose an $\mathfrak{sl}_2$-triple $e, f, h$. Set $\chi \in \mathfrak{g}^*$ be $(e, -)$. Consider the grading $\mathfrak{g} = \bigoplus_i \mathfrak{g}(i)$ by the eigenvalues of $\text{ad}(h)$. It is easy to see that the skew-symmetric form $\langle \xi, \eta \rangle = \chi((\xi, \eta))$ is non-degenerate on $\mathfrak{g}(-1)$. Pick a Lagrangian $l \subset \mathfrak{g}(-1)$ and set $m := l \oplus \bigoplus_{i \leq -2} \mathfrak{g}(-i)$. Note that $\chi \in (m/[m, m])^*$. Since $m$ is nilpotent, it exponentiates to an algebraic subgroup $M \subset G$. The finite $W$-algebra $W_h$ is defined as the quantum Hamiltonian reduction $U_h(\mathfrak{g}) \sslash_{\chi} M$, where $U_h(\mathfrak{g}) = T(\mathfrak{g})/(xy - yx - h[x, y] \mid x, y \in \mathfrak{g})$.

Let $P \subset G$ be a parabolic subgroup and $P_0 = [P, P]$, and let $a = p/p_0$. Consider the $\mathbb{C}[a^*][h]$-algebra $A_{h}$ defined as $\mathbb{C}^\times$-finite elements in $\Gamma(G/P, D_h(G/P_0)^{P/F_h})$, where $D_h(G/P_0)$ is the sheaf of $h$-adic differential operators and the action of $P/P_0$ on $D_h(G/P_0)$ is induced from the right action on $G/P_0$, and the $\mathbb{C}^\times$-action on $D_h(G/P_0)$ is inherited from $\mathbb{C}^\times$-action on $T^*(G/P_0)$ by scaling cotangent fibers of weight $-2$. Note that the $\mathbb{C}[a^*]$-algebra structure comes from a modification of the moment map $\Phi_h : a \subset U_h(\mathfrak{g}) \to A_h$, which is defined as $a \mapsto \Phi_h(a) - (\rho, a)h$, where $\rho$ is the half sum of positive roots of $\mathfrak{g}$. The parabolic finite $W$-algebra $W_{h, a}$ is defined as the quantum Hamiltonian reduction $A_h \sslash_{\chi} M$.

We focus on the type A case. Explicitly, $G = SL_N$, and fix $r_1, \ldots, r_n \in \mathbb{Z}_{\geq 0}$ with $\sum_{i=1}^n r_i = N$, then $r_1, \ldots, r_n$ defines a parabolic subgroup $P \subset SL_N$ as the stabilizer of a partial flag $\mathcal{F} = (0 \subset F_1 \subset F_2 \subset \cdots \subset F_n = \mathbb{C}^N)$ with $\dim F_j = \sum_{i=1}^j r_i$. Also pick $d = (d_1, \ldots, d_{n-1}) \in \mathbb{Z}_{\geq 0}^{n-1}$ with $\sum_{i=1}^{n-1} id_i = N$ and let $e \in \mathfrak{sl}_N$ be a nilpotent element whose Jordan type is $(1^{d_1}, 2^{d_2}, \ldots, (n-1)^{d_{n-1}})$. Define $v = (v_1, \ldots, v_{n-1})$ by

$$v_{n-i} = r_n, \quad v_i = \sum_{j=i+1}^n r_j - \sum_{j=i+1}^n (j-i)d_j. \quad (4.8)$$

Below we assume that all $v_i$'s are non-negative. Let $Q$ be an $A_{n-1}$ quiver so we can identify $v$ as a dimension vector and $d$ as a framing vector. View $a = p/p_0$ as the space $\{\text{diag}(x_1, \ldots, x_1, \ldots, x_n, \ldots, x_n)\}$ matrices such that $x_i$ appears $r_i$ times with $\sum_{i=1}^n r_i x_i = 0$. Map $a$ to $\mathbb{C}^{Q_0}$ by sending $\text{diag}(x_1, \ldots, x_n)$ to $\sum_{i=1}^{n-1} (\sum_{j=1}^{n-i} r_j x_j) \epsilon_i$. The composition of this map with the natural projection $\mathbb{C}^{Q_0} \to Z^* = \mathfrak{g}(v)/[\mathfrak{g}(v), \mathfrak{g}(v)]$ defined by

$$\epsilon_i \mapsto \begin{cases} \frac{1}{v_i} \text{Id}_{v_i} & , v_i \neq 0 \\ 0 & , v_i = 0 \end{cases}$$

gives rise to a map $a \to Z^*$. Note that this is an isomorphism if all $r_i$'s and $v_i$'s are positive. Define the $\mathbb{C}[Z][h]$-algebra

$$W_{h, Z}^P := W_{h, a} \otimes_{\mathbb{C}[a^*]} \mathbb{C}[Z],$$

where $W_{h, a}$ is the parabolic finite $W$-algebra associated to the aforementioned parabolic subgroup $P$ and nilpotent element $e$. Apply the construction in the previous subsection and
we obtain a $\mathbb{C}[Z][\hbar]$-algebra

$$A_{h,Z}(Q, v, d) := A_{h,Z}(Q^d, v^d).$$

Losev showed in [20, Theorem 5.3.3] that if all $v_i$'s are positive and $r_i \geq r_{i+1}$ then $A_{h,Z}(Q, v, d)$ is $\mathbb{C}[Z][\hbar]$-linearly isomorphic to $W^P_{h,Z}$.

Henceforth there is a $\mathbb{C}^\times$-equivariant even quantization $\mathcal{O}_{h,\hat{S}(e, P)}$ on the deformed Slodowy variety $\hat{S}(e, P)$ (see Proposition 4.11).

Proposition 4.11. There exists a $\mathbb{C}[Z][\hbar]$-linear epimorphism $A_{h,Z}(Q, v, d) \to W^P_{h,Z}$ of graded associative algebras. Moreover it is an isomorphism if and only if $r_i - r_j \geq -1$ for all $1 \leq i < j \leq n$.

Proof. Recall that $W^P_{h,a}$ is the algebra of $\mathbb{C}^\times$-finite elements in $\Gamma(\hat{S}(e, P), \mathcal{O}_{h,\hat{S}(e, P)})$ for a $\mathbb{C}^\times$-equivariant even quantization $\mathcal{O}_{h,\hat{S}(e, P)}$ on the deformed Slodowy variety $\hat{S}(e, P)$ (see [20, Lemma 5.2.1(2)]). Note that the Slodowy variety $S(e, P)$ is $\mathbb{C}^\times$-equivariantly symplectomorphic to $M^\theta_0(Q, v, d)$ [21], where $\theta = (-1, \ldots, -1)$ is a generic stability condition. The aforementioned quantized structure sheaf of $M^\theta_0(Q, v, d)$, denoted by $\mathcal{O}_{h,M^\theta_0(Q,v,d)}$, is also an even quantization, moreover the quantum period maps $Z \to H^2_{DR}(M^\theta_0(Q, v, d)) \cong H^2_{DR}(S(e, P))$ and $\alpha^* \to H^2_{DR}(S(e, P))$ are intertwined by the map $Z \to \alpha^*$ [20, Lemma 4.6.5], henceforth there is a $\mathbb{C}^\times$-equivariant isomorphism between formal schemes $S(e, P) \times_{\alpha^*} \hat{Z} \cong \tilde{M}^\theta_0(Q, v, d)$ together with a $\mathbb{C}^\times$-equivariant isomorphism between sheaf of flat $\mathbb{C}[\hbar]$-algebras $\mathcal{O}_{h,\tilde{M}^\theta_0(Q,v,d)} \cong \mathcal{O}_{h,\tilde{M}^\theta_0(Q,v,d)}$, where $\tilde{Z}$ is the completion of $Z$ at 0 and $\tilde{M}^\theta_0(Q, v, d)$ is the completion of $M^\theta_0(Q, v, d)$ along $M^\theta_0(Q, v, d)$. Thus there exists a $\mathbb{C}[Z][\hbar]$-linear isomorphism between $W^P_{h,Z}$ and the algebra of $\mathbb{C}^\times$-finite elements in $\Gamma(\tilde{M}^\theta_0(Q, v, d), \mathcal{O}_{h,\tilde{M}^\theta_0(Q,v,d)})$.

By [21, Theorem 12], the projection $p^\theta: M^\theta_0(Q, v, d) \to M_0(Q, v, d)$ maps $M^\theta_0(Q, v, d)$ bijectively to its image in $M_0(Q, v, d)$ and its image is a normal variety, thus $R^i p^\theta_! \mathcal{O}_{M^\theta_0(Q,v,d)} = 0$ for all $i > 0$ and the natural map $\mathbb{C}[M_0(Q, v, d)] \to \Gamma(M^\theta_0(Q, v, d), \mathcal{O}_{h,M^\theta_0(Q,v,d)})$ is surjective. Since the $\mathbb{C}^\times$-action on $\tilde{M}^\theta_0(Q, v, d)$ is positive, we conclude that $R^i p^\theta_! \mathcal{O}_{M^\theta_0(Q,v,d)} = 0$ for all $i > 0$ and the natural map $\mathbb{C}[M_0(Q, v, d)] \to \Gamma(M^\theta_0(Q, v, d), \mathcal{O}_{h,M^\theta_0(Q,v,d)})$ is surjective.

Following verbatim argument as the proof of [20, Lemma 4.2.4], there is a $\mathbb{C}[Z][\hbar]$-linear map of graded associative algebras $A_{h,Z}(Q, v, d)^{\wedge h} \to \Gamma(\tilde{M}^\theta_0(Q, v, d), \mathcal{O}_{h,\tilde{M}^\theta_0(Q,v,d)})$ and it fits into a commutative diagram

$$
\begin{array}{ccc}
A_{h,Z}(Q, v, d)^{\wedge h} & \longrightarrow & \Gamma(\tilde{M}^\theta_0(Q, v, d), \mathcal{O}_{h,\tilde{M}^\theta_0(Q,v,d)}) \\
\downarrow & & \downarrow \\
\mathbb{C}[M_0(Q, v, d)] & \longrightarrow & \Gamma(M^\theta_0(Q, v, d), \mathcal{O}_{M^\theta_0(Q,v,d)})
\end{array}
$$

where $A_{h,Z}(Q, v, d)^{\wedge h}$ is the completion of $A_{h,Z}(Q, v, d)$ in the $h$-adic topology. The bottom horizontal arrow and vertical arrows are epimorphisms, thus the top horizontal arrow is also an epimorphism since both domain and codomain are flat over $\mathbb{C}[\hbar]$ and it is surjective modulo $\hbar$. Since $A_{h,Z}(Q, v, d)$ is the sub-algebra of $\mathbb{C}^\times$-finite elements in $A_{h,Z}(Q, v, d)^{\wedge h}$,

2If all $v_i$’s are positive and $r_i \geq r_{i+1}$, then all $r_i$’s are positive since $r_n = v_{n-1} > 0$, and it follows that the map $a \to Z^*$ is an isomorphism.
it follows that $A_{h,Z}(Q, v, d)$ maps surjectively to the sub-algebra of $\mathbb{C}^*$-finite elements in $\Gamma(M^\theta_Z(Q, v, d), O_{h,M^\theta_Z(Q,v,d)})$. Since $\mathbb{C}^*$-finite elements in $\Gamma(M^\theta_Z(Q, v, d), O_{h,M^\theta_Z(Q,v,d)})$ are the same as $\mathbb{C}^*$-finite elements in $\Gamma(M^\theta_Z(Q, v, d), O_{h,M^\theta_Z(Q,v,d)})$, the first assertion is proven.

Next, suppose that the epimorphism $A_{h,Z}(Q, v, d) \to \mathcal{W}^P_{h,Z}$ we constructed above is an isomorphism, then it is an isomorphism modulo $Z^*$ and $\hbar$. Therefore, the composition

$$A_{h,Z}(Q, v, d)/(Z^*, \hbar) \to \mathbb{C}[M_0(Q, v, d)] \to \Gamma(M^\theta_0(Q, v, d), O_{M_0^\theta(Q,v,d)})$$

is an isomorphism. This implies that $\mathbb{C}[M_0(Q, v, d)] \to \Gamma(M^\theta_0(Q, v, d), O_{M_0^\theta(Q,v,d)})$ is injective, in particular $\dim M_0(Q, v, d) = \dim M^\theta_0(Q, v, d)$, then by Lemma 2.6 the moment map for $(Q, v, d)$ is flat. By Proposition 4.2 the flatness of moment map is equivalent to $e_i \cdot (d - C_Q v) \geq -1$ for arbitrary subset $I \subset Q_0$ such that nodes in $I$ are connected by arrows in $Q$. For $1 \leq i < j \leq n$, let $e_{i,j} = \sum_{k=i}^{j-1} e_k$, then it is easy to see that $e_{i,j} \cdot (d - C_Q v) = r_i - r_j$, thus the flatness of moment map is equivalent to $r_i - r_j \geq -1$ for all $1 \leq i < j \leq n$.

Conversely, assume that $r_i - r_j \geq -1$ for all $1 \leq i < j \leq n$, then the moment map is flat, thus by Proposition 4.9 $A_{h,Z}(Q, v, d)$ maps isomorphically to $\mathbb{C}^*$-finite elements in $\Gamma(M^\theta_Z(Q, v, d), O_{h,M^\theta_Z(Q,v,d)})$, which is isomorphic to $\mathcal{W}^P_{h,Z}$ by our previous discussion. □

References

[1] Hiraku Nakajima. Quiver varieties and Kac-Moody algebras. *Duke Mathematical Journal*, 91(3):515–560, 1998.
[2] Michela Varagnolo. Quiver varieties and Yangians. *Letters in Mathematical Physics*, 53(4):273–283, 2000.
[3] Hiraku Nakajima. Quiver varieties and finite dimensional representations of quantum affine algebras. *Journal of the American Mathematical Society*, pages 145–238, 2001.
[4] Davesh Maulik and Andrei Okounkov. Quantum groups and quantum cohomology. *arXiv preprint arXiv:1211.1287*, 2012.
[5] Kenneth Intriligator and Nathan Seiberg. Mirror symmetry in three dimensional gauge theories. *Physics Letters B*, 387(3):513–519, 1996.
[6] Hiraku Nakajima. Instantons on ALE spaces, quiver varieties, and Kac-Moody algebras. *Duke Mathematical Journal*, 76(2):365–416, 1994.
[7] Alastair D. King. Moduli of representations of finite dimensional algebras. *The Quarterly Journal of Mathematics*, 45(4):515–530, 1994.
[8] Nikita Nekrasov and Sergey Shadchin. ABCD of instantons. *Communications in mathematical physics*, 252(1):359–391, 2004.
[9] Agostino Butti, Davide Forcella, Amihay Hanany, David Vegh, and Alberto Zaffaroni. Counting chiral operators in quiver gauge theories. *Journal of High Energy Physics*, 2007(11):092, 2007.
[10] Sergio Benvenuti, Amihay Hanany, and Noppadol Mekareeya. The Hilbert series of the one instanton moduli space. *Journal of High Energy Physics*, 2010(6):1–40, 2010.
[11] Wee Liang Gan and Victor Ginzburg. Almost-commuting variety, D-modules, and Cherednik algebras. *International Mathematics Research Papers*, 2006:26439, 2006.
[12] William Crawley-Boevey. Geometry of the moment map for representations of quivers. *Compositio Mathematica*, 126(3):257–293, 2001.
[13] Gwyn Bellamy and Travis Schedler. Symplectic resolutions of quiver varieties. *arXiv preprint arXiv:1602.00164*, 2016.
[14] Jaeyou Choy. Corrigendum and addendum to “Moduli spaces of framed symplectic and orthogonal bundles on $\mathbb{P}^2$ and the K-theoretic Nekrasov partition functions”[J. Geom. Phys. 106 (2016) 284–304]. *Journal of Geometry and Physics*, 110:343–347, 2016.
Jaeyoo Choy. Moduli spaces of framed symplectic and orthogonal bundles on $\mathbb{P}^2$ and the K-theoretic Nekrasov partition functions. Journal of Geometry and Physics, 106:284–304, 2016.

Xiuping Su. Flatness for the moment map for representations of quivers. Journal of Algebra, 298(1):105–119, 2006.

William Crawley-Boevey. Normality of Marsden-Weinstein reductions for representations of quivers. arXiv preprint math/0105247, 2001.

Andrea Maffei. A remark on quiver varieties and Weyl groups. Annali della Scuola Normale Superiore di Pisa-Classe di Scienze, 1(3):649–686, 2002.

Ivan Losev. Isomorphisms of quantizations via quantization of resolutions. Advances in Mathematics, 231(3-4):1216–1270, 2012.

Andrea Maffei. Quiver varieties of type A. Commentarii Mathematici Helvetici, 80(1):1–27, 2005.

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