Simulating Stochastic Inertial Manifolds
by a Backward-Forward Approach *

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Abstract

A numerical approach for the approximation of inertial manifolds of stochastic evolutionary equations with multiplicative noise is presented and illustrated. After splitting the stochastic evolutionary equations into a backward and a forward part, a numerical scheme is devised for solving this backward-forward stochastic system, and an ensemble of graphs representing the inertial manifold is consequently obtained. This numerical approach is tested in two illustrative examples: one is for a system of stochastic ordinary differential equations and the other is for a stochastic partial differential equation.

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1 Introduction

The concept of inertial manifolds for deterministic partial differential equations was introduced in 1980s [8, 16]. These manifolds are finite dimensional invariant manifolds that attract every trajectory at an exponential rate. They play an important role in the study of the long-time behavior of solutions, since through them the dynamics of a large system can be described by a finite dimensional system. To be more specific, the dimension of the state space is reduced by projecting the system onto the inertial manifold once the existence of the manifold has been proved. For certain dissipative nonlinear systems, the inertial manifold can be shown to exist and it exponentially attracts solution orbits [16]. In the case where the existence of the inertial manifold is unknown, approximate inertial manifolds are introduced [18, 31, 35, 29]. Several authors considered the construction of approximate inertial manifolds [35] as well as numerical simulations using these manifolds [17, 21, 24].

Inertial manifolds have also been considered for stochastic systems [6, 7, 9, 13, 14]. In one such study, Da Prato and Debussche [9] introduced the concept of a stochastic inertial manifold for an abstract stochastic evolutionary equation in a Hilbert space $H$ (with scalar product $\langle \cdot, \cdot \rangle$ and the induced distance $d(\cdot, \cdot)$)

$$
\begin{align*}
\{ & \quad du + (Au + f(u))dt = g(u)dW_t, \\
& \quad u(0) = u_0,
\}
\end{align*}
$$

where $A$ is a linear operator, $f$ and $g$ are two nonlinear functions, and $W_t$ is a Wiener process taking values in another Hilbert space $U$.

Although a theoretical framework for stochastic inertial manifolds has been set up [6, 13, 14], it is desirable to efficiently approximate stochastic inertial manifolds and perform simulations on them. Recently, Roberts introduced a normal form transformation of stochastic differential systems when the dynamics contains both slow modes and quickly decaying modes [33], in which algebraic techniques were used. This is intended for reduction of finite dimensional systems and it might be used to test the numerical techniques that are proposed for the infinite dimensional case and also to inspire future development.

In this paper, we introduce a numerical scheme for simulating the stochastic inertial manifold of a stochastic evolutionary system with multiplicative noise, which includes stochastic differential equations (SDEs) and stochastic partial differential equations (SPDEs). By projecting to a countable basis,
an SPDE could be converted to an infinite dimensional system of SDEs. The main idea is to solve a coupled backward-forward system of SDEs where the backward part is finite dimensional, but the forward part is either infinite or high dimensional. The forward Picard type iteration scheme \cite{1, 5, 12} is performed on the backward part, and an Euler discretization scheme is applied to the forward part. The graph for the stochastic inertial manifold is consequently obtained. Two examples, one for a system of SDEs and one for an SPDE, are presented to illustrate our backward-forward approach.

The rest of this paper is organized as follows. Section 2 formulates the problem and briefly reviews the analytical results \cite{9}. Section 3 introduces the numerical scheme and performs the error analysis. An approximation procedure is discussed in Section 4. Finally two numerical examples are presented in Section 5.

2 Problem formulation

We consider a stochastic evolutionary system in a Hilbert space $H$ of the form

$$
\frac{du}{dt} + Au = f(u)dt + g(u)dW_t, \quad (2)
$$

where the following conditions hold:

1. **Linear part.**
   $A$ is a self-adjoint operator in $H$ with eigenvalues
   \[ \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k \leq \cdots \leq \lambda_j \to +\infty. \]

2. **Nonlinear part.**
   $f : D(A^\alpha) \mapsto H$ and $g : D(A^\alpha) \mapsto L_2^0(D(A^\beta))$, for some $\alpha, \beta \in [0,1)$, are globally Lipschitz and bounded, with Lipschitz constants $L_f$ and $L_g$ respectively, i.e., for any $u, v \in D(A^\alpha),
   \begin{align*}
   | f(u) - f(v) |_H &\leq L_f | u - v |_{D(A^\alpha)},
   | g(u) - g(v) |_H &\leq L_g | u - v |_{D(A^\alpha)}. \]
   \end{align*}

   When $f$ and $g$ are only locally Lipschitz, but the corresponding deterministic system has a bounded absorbing set in an appropriate state space, it is possible to cut-off $f$ and $g$ to zero outside a ball containing the absorbing set. In this case the modified (“cut-off”) system has globally Lipschitz drift and noise intensity.

3. **Noise part.**
   Although the Wiener process $W_t$ may take values in a Hilbert space $U$, to be specific, here we just consider $W_t$ to be a two-sided one dimensional Wiener process, defined in a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and
adapted to a filtration $\mathcal{F}_t, t \in \mathbb{R}$; note that $\mathcal{F}_t$ is a two-parameter filtration [2] or a one-parameter filtration starting at time $-\infty$ instead of at time 0 [11]. More precisely, for each $t \in \mathbb{R}$,

$$\mathcal{F}^t_{-\infty} := \bigvee_{s \leq t} \mathcal{F}^s_t,$$

$$\mathcal{F}^t_s := \sigma(W(u), s \leq u \leq t),$$

which is the information generated by the Wiener process $W$ on the interval $(-\infty, t]$. We denote $\mathcal{F}^t_{-\infty}$ as $\mathcal{F}_t$ for simplicity here and henceforth.

The two-sided Wiener process $W_t$ is defined in terms of two independent Wiener processes $\hat{W}_t$ and $\tilde{W}_t$ ($t \geq 0$), as follows,

$$W_t = \begin{cases} \hat{W}_t, & \text{if } t \geq 0, \\ \tilde{W}_{-t}, & \text{if } t < 0. \end{cases}$$

The adaptedness means $W_t$ is measurable with respect to $\mathcal{F}_t$ for each $t$.

**Remark 1.** The two-parameter filtration defined in (4) requiring $\mathcal{F}^t_s \subset \mathcal{F}^{t'}_{s'}$, for $s' \leq s \leq t \leq t'$ is consistent with the well-known filtration for positive time. Indeed, $\{\mathcal{F}_t\}_{t \geq 0}$ is $\{\mathcal{F}^t_0\}_{t \geq 0}$ in the two-parameter setting. The only difference between one-parameter and two-parameter filtrations in the above setting is their starting time. Since the filtration specifies how the information is revealed in time, the property that a filtration is increasing corresponds to the fact the information is not forgotten. However, the generalization of filtration from one-parameter to two-parameter, while maintaining the property that a filtration is increasing, results in technical difficulties in constructing invariant manifolds since a backward SDE is encountered. Overcoming this difficulty will be discussed in detail in Remark 4.

As discussed in [2] [13] [14] [15], it is appropriate and convenient to consider the canonical sample space, by identifying sample paths of the Wiener process $W_t$ with continuous curves (passing through the origin at $t = 0$ since $W_0 = 0$). Namely, a sample path is now a point in the space $C(\mathbb{R}, \mathbb{R})$ of continuous functions: $W_t(\omega) = \omega(t)$. Therefore the sample space is taken to be

$$\Omega = \{\omega \in C(\mathbb{R}; \mathbb{R}) : \omega(0) = 0\}$$

and $\mathbb{P}$ is taken to be the Wiener measure. This is analogous to the situation of dice-tossing, where we take six face values, 1, 2, 3, 4, 5, and 6, as samples in the canonical sample space $\Omega = \{1, 2, 3, 4, 5, 6\}$. When we “toss” a Wiener process $W_t$, we see continuous (but nowhere differentiable) curves as “face values” or samples.
The Wiener shift $\theta_t$ is defined as a mapping in the canonical sample space $\Omega$, for each fixed $t \in \mathbb{R}$,

$$\theta_t : \Omega \rightarrow \Omega \quad \omega \mapsto \bar{\omega}$$

such that

$$\theta_t \omega (s) = \bar{\omega}(s) \triangleq \omega(t + s) - \omega(t), \quad s, t \in \mathbb{R}. \quad (5)$$

**Remark 2.** The Wiener shift defined in (5) is a measure preserving transformation, i.e.

$$\mathbb{P}(A) = \mathbb{P}(\theta_t^{-1} A), \forall A \in \mathcal{F},$$

where $\mathbb{P}$ is the Wiener measure and $\mathbb{P}(A) = \mathbb{P}(\theta_t A)$ is implied.

By a simple calculation, we see that $\theta_0 = \text{Id}$ (the identity mapping in $\Omega$) and $\theta_{s+t} = \theta_s \circ \theta_t$. Hence the Wiener shift is a deterministic dynamical system (or a flow) in $\Omega$. The above equation (5) means that

$$W_s(\theta_t \omega) = W_{t+s}(\omega) - W_t(\omega) \approx dW_t(\omega). \quad (6)$$

Thus $\theta_t$ is closely related to the noise in the stochastic system (2) and is often called the *driving flow*. The solution mapping satisfies the property [9]

$$S(t, s; \omega)x = S(t - s, 0; \theta_s \omega)x, \quad x \in H, \quad t \geq s, \quad \mathbb{P} \text{ a.s.} \quad (7)$$

**Definition 1.** A stochastic inertial manifold for (2) is a random family of manifolds $\{M(\omega)\}_{\omega \in \Omega}$ which is measurable with respect to $\mathcal{F}_0$ and satisfies the following three properties:

1. Each realization of $M(\omega)$ is a deterministic manifold: $M(\omega)$ is a Lipschitz (or smooth) manifold for $\mathbb{P}$–almost all $\omega \in \Omega$.

2. Invariance:

$$S(t, 0; \omega)M(\omega) = M(\theta_t \omega), \quad t \in \mathbb{R}^+, \quad \mathbb{P} \text{ a.s.}$$

3. Exponential attraction: for any $x \in H$

$$\lim_{t \to \infty} d(S(t, 0; \omega)x, M(\theta_t \omega)) = 0, \quad \text{exponentially in } L^2(\Omega),$$

where $u(t, s; \omega)$ is the unique solution for (2) defined for $t \in [s, +\infty)$ such that

$$u(s, s; \omega) = u_s$$

and $S(t, s; \omega)$ is the solution mapping

$$u(t, s; \omega) \triangleq S(t, s; \omega)u_s(\omega).$$
Remark 3. 1. \( M(\omega) \) is a “random variable” mapping all the samples (continuous functions) in the canonical sample space \( \Omega \) into the space of deterministic Lipschitz (or smooth) manifolds \([1]\), and thus is measurable with respect to \( F_0 \). Figure 1 explains the meaning of random family of manifolds \( \{M(\omega)\}_{\omega \in \Omega} \) heuristically.

2. The stochastic invariant property can be considered as a natural generalization of invariance property in the deterministic setting. In the deterministic case, \( S(t,0)M = M, \quad t \in \mathbb{R}^+ \), which can be seen as \( S(t,0)M(\omega) = M(\omega) \) since there is one and only one sample in such setting. In the stochastic setting, the invariance is in the sense of probability. More precisely, under the system evolution or solution mapping \( S \), if we start from somewhere on the manifold \( M(\omega) \), then after time \( t \), we will stand on manifold \( M(\theta_t \omega) \) whose likelihood of occurrence is the same as \( M(\omega) \). This is implied by the measure-preserving property of Wiener shift \( \theta_t \), i.e. \( P(\{\omega\}) = P(\{\theta_t \omega\}) \), see Remark 2.

From now on, we do not distinguish “\( \omega \)” and “\( \theta_t \omega \)” when we say “a fixed sample”.

Recall that the construction of an inertial manifold in the deterministic case amounts to finding a graph above an eigenspace of the linear operator.

![Figure 1: Sketch of a stochastic inertial manifold: Three random samples \((\omega, \hat{\omega}, \tilde{\omega})\) on the left, and the corresponding three realizations \((M(\omega), M(\hat{\omega}), M(\tilde{\omega}))\) of the manifold on the right.](image-url)
A. By analogy, we take a projection $P$ to a finite dimensional eigenspace $H^+$, and look for a function $\Phi$ from $H^+ = PH$ to $H^- = (I - P)H$ whose graph is invariant under the evolution of the stochastic system \[2\]. The analytical foundation of our numerical method is the combination of two well known methods for constructing deterministic inertial manifolds. One is the Lyapunov–Perron method, and the other one is the graph transform method \[8\]. Roughly speaking, the Lyapunov–Perron method looks for solutions of the original equation whose components on $(I - P)H$ are bounded for negative time. The graph transform method is to let an inertial manifold (typically, the flat manifold $PH$) evolve under the system evolution and to verify that the image of $PH$ at time $t$ is a graph which will converge to an invariant manifold as $t \to \infty$.

As introduced by Da Prato and Debussche \[9\], we reformulate \[2\] into a backward part and a forward part for time $t \in [-T, 0]$: \[u := X + Y,\]

\[
\begin{align*}
  dX + AX dt &= Pf(X + Y)dt + Pg(X + Y)dW_t, \\
  X(0) &= X_0, \\
  dY + AY dt &= Qf(X + Y)dt + Qg(X + Y)dW_t, \\
  Y(-T) &= 0,
\end{align*}
\]

where $P$ is a projection from $H$ to the eigenspace spanned by the first $k$ eigenvalues $\lambda_1, \ldots, \lambda_k$ of $A$, and $Q := I - P$. Here $k$ is determined by the eigenvalues and Lipschitz constants for the existence of stochastic inertial manifolds \[6, 9\]. Hence $X$ is of finite dimension $k$ and $Y$ is of infinite dimension.

The problem \[8\] involves a backward stochastic evolutionary equation.

**Remark 4.** At first glance, we might want to solve the equations \[8\] whose unknown is the pair $(X, Y)$, in the interval $[-T, 0]$. However, this type of problem does not have solutions in general. For the existence and uniqueness of solution of SDEs, in addition to the usual requirement as in the case of ODEs, it is also necessary for the solution to be adapted to the filtration generated by the noise. Since the filtration is a collection of fields, $\mathbb{F} := \{\ldots, \mathcal{F}_{-T}, \mathcal{F}_{-t}, \mathcal{F}_0, \mathcal{F}_t, \mathcal{F}_T, \ldots\}$, $\mathcal{F}_t \subset \mathcal{F}_{t+1}$, if we use the usual backward integration method, i.e., finding the solution at time $t$ by making use of the solution at time $t + 1$, then this $t$–solution is $\mathcal{F}_{t+1}$–measurable but not necessarily $\mathcal{F}_t$–measurable, which violates the definition of solution for SDEs.

In order to overcome this difficulty, the terminal value problem of SDE is reformulated in such a way as to allow a solution that is $\{\mathcal{F}_t\}_{t \leq 0}$–adapted.

By Proposition 3.1 introduced by Da Prato and Debussche \[9\], for every $X_0$ that is $\mathcal{F}_0$–measurable and square integrable, there exists a unique triple $(X, Y, M_t)$ such that
1. $X : [-T, 0] \mapsto PH$ is mean-square continuous and adapted,

2. $Y : [-T, 0] \mapsto QD(A^\alpha)$ is mean-square continuous and adapted,

3. $M_t$ is a square integrable martingale with values in $PH$,

and $(X, Y, M_t)$ solves the following combined backward-forward stochastic system, for time $t \in [-T, 0]$

$$
\begin{cases}
  dX + AX \, dt = Pf(X + Y) \, dt + dM_t, \\
  X(0) = X_0, \\
  dY + AY \, dt = Qf(X + Y) \, dt + Qg(X + Y) \, dW_t, \\
  Y(-T) = 0.
\end{cases}
$$

(9)

where

$$
M_t = X_t - X_0 - \int_t^0 AX_s ds + \int_t^0 Pf(X_s + Y_s) ds.
$$

Note that the solution of this system (9) sits on the interval $[-T, 0]$, the first $k$ components of $u$, i.e. $X$, travel backward from 0 to $-T$, and the remaining infinitely many components of $u$, i.e. $Y$, travel forward from $-T$ to 0.

Figure 2 is a schematic illustration of this backward-forward method.

For each fixed $T > 0$, define a mapping $\Phi_T$ via

$$
Y(0, \omega) := \Phi_T(X_0(\omega), \omega), \quad \omega \in \Omega.
$$

(10)
Da Prato and Debussche [9] showed that the limit of \((\Phi_T)_{T>0}\) as \(T \to \infty\) in \(L^2(\Omega)\) is the function \(\Phi\) whose graph above \(PH\) is the inertial manifold
\[
\mathcal{M}(\omega) = \{u = X + Y : Y = \Phi(X, \omega), X \in PH\}.
\]
In the next section we discuss how to numerically simulate (8) and then in Section 4 we approximate the mapping \(\Phi\) and thus obtain an approximate inertial manifold \(\mathcal{M}\).

3 A numerical scheme

In this section, we devise a numerical scheme to compute the solution of the backward-forward stochastic system (8). Our scheme is inspired by [4]. Some related references are [5, 12].

3.1 Main idea

Literally speaking, the backward-forward numerical iteration scheme for (9) works as follows: We start the first iteration by setting \(Y\) to be zero (flat manifold) for the entire time interval \([-T, 0]\), and \(X\) to be zero on \([-T, 0)\) together with the terminal \(X\) value to be any \(\mathcal{F}_0\)-measurable random variable. We obtain the \(X\) trajectory backward in time, by using future information of \((X, Y)\) at the previous iteration, and then we generate the \(Y\) trajectory future in time using past in time information of \((X, Y)\) at the previous iteration. The iteration is stopped when the distance of two consecutive \((X, Y)\) trajectories is less than a preset tolerance; we then use the terminal \(Y\) value at that iteration to approximate one point on the manifold \(\Phi\). We will illustrate the method in more detail in Section 4.

Before getting to the backward-forward approach, we need the following preparatory work.

Let \(\{e_1, e_2, \ldots\}\) be an orthonormal basis for \(H\). In numerical analysis, we approximate the Hilbert space (could be infinite dimensional) by a \((k + l)\)-dimensional subspace, and project (2) into this subspace, i.e.
\[
du^{k+l} + A_{k+l}u^{k+l}dt = f_{k+l}(u^{k+l})dt + g_{k+l}(u^{k+l})dW.
\]
(11)
In theory, \(l\) could be infinite or a big natural number. The above projection is usually done by the Galerkin method. We denote the numerical approximations of \(X, Y\) by \(x, y\), respectively. Then \(PH = \mathbb{R}^k\) and \(QH = \mathbb{R}^l\),
\[
u^{k+l} = x + y, \quad x \in \mathbb{R}^k \text{ and } y \in \mathbb{R}^l,
\]
and
\[
\begin{cases}
  dx + Sx dt = \sum_{i=1}^k \langle e_i, f(X + Y) \rangle dt + dM_t, \\
  x(0) = x_0, \\
  dy + Uy dt = \sum_{i=k+1}^{k+l} \langle e_i, f(X + Y) \rangle dt + \sum_{i=k+1}^{k+l} \langle e_i, g(X + Y) \rangle dW_t, \\
  y(-T) = 0,
\end{cases}
\]
(12)
where $Sx := \sum_{i=1}^{k} (e_i, A_{k+i}u^{k+i})$, $Uy := \sum_{i=k+1}^{k+l} (e_i, A_{k+i}u^{k+i})$.

For each fixed $T > 0$, define a mapping $\Phi_T$ via

$$y(0, \omega) := \Phi_T(x_0(\omega), \omega), \quad \omega \in \Omega. \quad (13)$$

The limit of $(\Phi_T)_{T>0}$ as $T \to \infty$ in $L^2(\Omega)$ is the function $\Phi$, and its graph above $\mathbb{R}^k$ is the (approximate) inertial manifold $\mathcal{M}$

$$\mathcal{M}(\omega) = \{u = x + y : y = \Phi(x, \omega), \ x \in \mathbb{R}^k\}.$$ 

Therefore, our goal of constructing the stochastic inertial manifold $\mathcal{M}$ is to numerically solve (12) in order to obtain $\Phi_T$ for a large $T > 0$ and for a number of sample $\omega$’s.

For convenience, we denote

$$\sum_{i=1}^{k} (e_i, f(X + Y)) := f_1(x, y),$$

$$\sum_{i=k+1}^{k+l} (e_i, f(X + Y)) := f_2(x, y),$$

$$\sum_{i=k+1}^{k+l} (e_i, g(X + Y)) := g_2(x, y)$$

for the rest of this paper. In the following, we also denote $x(t)$ by $x_t$ and $y(t)$ by $y_t$.

In principle, the solution of (12) can be obtained as the limit of a Picard type iteration $(x^{(n)}, y^{(n)})$ as introduced by Da Prato and Debussche [9]. To be more precise, $(x_t^{(0)}, y_t^{(0)}) \equiv (0, 0)$, and $(x_t^{(n)}, y_t^{(n)})$ is the solution of the following iteration scheme

$$\begin{cases}
  x_t^{(n)} = \mathbb{E}\left(x_0 + \int_0^t Sx_s^{(n-1)}ds - \int_0^t f_1(x_s^{(n-1)}, y_s^{(n-1)})ds \right| \mathcal{F}_t), \\
  y_t^{(n)} = e^{-U(t+T)}y(-T) + \int_{-T}^t e^{-U(t+T-s)}f_2(x_s^{(n-1)}, y_s^{(n-1)})ds + \int_{-T}^t e^{-U(t+T-s)}g_2(x_s^{(n-1)}, y_s^{(n-1)})dW_s,
\end{cases} \quad (14)$$

where $t \in [-T, 0]$. The conditional expectation given $\mathcal{F}_t$ is introduced to guarantee that the solution is adapted, i.e., measurable with respect to the filtration $\mathcal{F}_t$, as is done in the theory of backward SDEs [31, 52].

Our goal is to find $\Phi_T$ via looking for $y(0, \omega)$ for given $x_0$ since $y(0) := \Phi_T(x_0, \omega)$. We now introduce a time discretization of the above iteration. Note that for the backward part $x$, the conditional expectation is still involved; for the forward part $y$, we use the Euler-Maruyama scheme in the discretization. In the numerical computation, every simulation corresponds to one sample, however, we cannot specify which sample we have actually chosen, but we do know that it is a validated sample in the sample space $\Omega$. 

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Suppose $h := T/N$, $t_i = -T + ih$, $i = 0, 1, \ldots, N$, taking $W_t$ to be a one-dimensional Wiener process, and denoting $\Delta W_i := W_{i+1} - W_i$, then a time discretization of (14) is

\[
\begin{aligned}
&x_{i}^{(n)} = E \left\{ x_0 + \sum_{j=i}^{N-1} \left[ Sx_{t_j}^{(n-1)} - f_1(x_{t_j}^{(n-1)}, y_{t_j}^{(n-1)}) \right] h \right\} \mathcal{F}_{t_i}, \\
y_{i}^{(n)} = e^{-U(t_i+T)} y(-T) + \sum_{j=0}^{i-1} e^{-U(t_i-t_j)} f_2(x_{t_j}^{(n-1)}, y_{t_j}^{(n-1)}) h + \sum_{j=0}^{i-1} e^{-U(t_i-t_j)} g_2(x_{t_j}^{(n-1)}, y_{t_j}^{(n-1)}) \Delta W_j.
\end{aligned}
\] (15)

In Section 4 we will explain how to implement this scheme. Note that we could also use the available $(x_{t_j}^{(n)}, y_{t_j}^{(n)})$, instead of $(x_{t_j}^{(n-1)}, y_{t_j}^{(n-1)})$, at the current iteration (for $j = 1, \ldots, i-1$) to calculate $y_{t_i}^{(n)}$ in the above scheme (15).

We now show the convergence of the above time discretized Picard iteration scheme (15).

### 3.2 Convergence of the numerical scheme

We now prove the convergence of the numerical scheme (15) devised in the last subsection. Namely, we prove the convergence of $(x_{t_i}^{(n)}, y_{t_i}^{(n)})$ to $(x_t, y_t)$ in a certain sense (see Theorem 1 below). To this end, we estimate $x_{t_i}^{(n)} - x_t(\infty), y_{t_i}^{(n)} - y_t(\infty)$ and $x_t(\infty) - x_t, y_t(\infty) - y_t$ in Lemma 1 and Lemma 2 respectively.

Define

\[
\begin{aligned}
x_{t_i}^{(\infty)} &= x_0, \\
y_{t_i}^{(\infty)} &= 0, \\
x_{t_i}^{(\infty)} &= E \left\{ \left( x_0 + \sum_{j=i}^{N-1} \left[ Sx_{t_j}^{(\infty)} - f_1(x_{t_j}^{(\infty)}, y_{t_j}^{(\infty)}) \right] h \right) \right\} \mathcal{F}_{t_i}, \\
y_{t_i}^{(\infty)} &= e^{-U h} y_{t_i-1}^{(\infty)} + \int_{t_i-1}^{t_i} e^{U(s-t_i-1)} f_2(x_{t_i-1}^{(\infty)}, y_{t_i-1}^{(\infty)}) ds + \int_{t_i-1}^{t_i} e^{U(s-t_i-1)} g_2(x_{t_i-1}^{(\infty)}, y_{t_i-1}^{(\infty)}) dW_s.
\end{aligned}
\]

Recall

\[
\begin{aligned}
x_{i}^{(0)} &= 0, \quad 0 \leq i \leq N-1, \quad x_{t_N} = x_0, \\
y_{i}^{(0)} &= 0, \quad 0 \leq j \leq N, \\
x_{i}^{(n+1)} &= E \left\{ \left( x_0 + \sum_{j=i}^{N-1} \left[ Sx_{t_j}^{(n)} - f_1(x_{t_j}^{(n)}, y_{t_j}^{(n)}) \right] h \right) \right\} \mathcal{F}_{t_i}, \\
y_{i}^{(n+1)} &= e^{-U h} y_{i-1}^{(n+1)} + \int_{t_{i-1}}^{t_i} e^{U(s-t_{i-1})} f_2(x_{t_{i-1}}^{(n)}, y_{t_{i-1}}^{(n)}) ds + \int_{t_{i-1}}^{t_i} e^{U(s-t_{i-1})} g_2(x_{t_{i-1}}^{(n)}, y_{t_{i-1}}^{(n)}) dW_s.
\end{aligned}
\]

**Lemma 1** (Iteration error). *Let the Lipschitz condition (3) be satisfied.*

Assume that

$$
|f_1(0,0)| + |f_2(0,0)| + |g_2(0,0)| + |x_0| \leq K,
$$
where \( C_1 \) depends on \( L_f, L_g, U, T \) and \( C_2 \) depends on \( L_f, L_g, K, U, T \).

Proof. First note that

\[
x_t^{(n+1)} = \mathbb{E}\{x_{t_{i+1}}^{(n+1)} + [Sx_t^{(n)} - f_1(x_t^{(n)}, y_t^{(n)})]h \mid \mathcal{F}_t\},
\]

\[
x_t^{(n)} = \mathbb{E}\{x_{t_{i+1}}^{(n)} + [Sx_t^{(n-1)} - f_1(x_t^{(n-1)}, y_t^{(n-1)})]h \mid \mathcal{F}_t\}.
\]

We now estimate

\[
\mathbb{E}[|x_t^{(n+1)} - x_t^{(n)}|^2] = \mathbb{E}\left( | \mathbb{E}\left\{ x_{t_{i+1}}^{(n+1)} - x_{t_{i+1}}^{(n)} + [Sx_t^{(n)} - Sx_t^{(n-1)} - (f_1(x_t^{(n)}, y_t^{(n)}) - f_1(x_t^{(n-1)}, y_t^{(n-1)})) h \mid \mathcal{F}_t\} \right\}|^2 \right)
\]

(Jensen's inequality)

\[
= \mathbb{E}\left( | x_{t_{i+1}}^{(n+1)} - x_{t_{i+1}}^{(n)} + [Sx_t^{(n)} - Sx_t^{(n-1)} - (f_1(x_t^{(n)}, y_t^{(n)}) - f_1(x_t^{(n-1)}, y_t^{(n-1)})) h |^2 \right)
\]

\[
= \mathbb{E}\left( | x_{t_{i+1}}^{(n+1)} - x_{t_{i+1}}^{(n)} |^2 + | [Sx_t^{(n)} - Sx_t^{(n-1)} - (f_1(x_t^{(n)}, y_t^{(n)}) - f_1(x_t^{(n-1)}, y_t^{(n-1)})) h |^2 \right.
\]

\[
+ 2 | x_{t_{i+1}}^{(n+1)} - x_{t_{i+1}}^{(n)} | \sqrt{h} \| [Sx_t^{(n)} - Sx_t^{(n-1)} - (f_1(x_t^{(n)}, y_t^{(n)}) - f_1(x_t^{(n-1)}, y_t^{(n-1)})) \sqrt{h} \right)
\]

\[
\leq \mathbb{E}\left\{ | x_{t_{i+1}}^{(n+1)} - x_{t_{i+1}}^{(n)} |^2 + | [Sx_t^{(n)} - Sx_t^{(n-1)} - (f_1(x_t^{(n)}, y_t^{(n)}) - f_1(x_t^{(n-1)}, y_t^{(n-1)})) h |^2 \right\}
\]

\[
+ 2 \sqrt{\mathbb{E}[|x_{t_{i+1}}^{(n+1)} - x_{t_{i+1}}^{(n)}|^2 h]} \sqrt{\mathbb{E}[|Sx_t^{(n)} - Sx_t^{(n-1)} - (f_1(x_t^{(n)}, y_t^{(n)}) - f_1(x_t^{(n-1)}, y_t^{(n-1))]|^2 h]}
\]

\[
\leq \mathbb{E}\left\{ | x_{t_{i+1}}^{(n+1)} - x_{t_{i+1}}^{(n)} |^2 + | [Sx_t^{(n)} - Sx_t^{(n-1)} - (f_1(x_t^{(n)}, y_t^{(n)}) - f_1(x_t^{(n-1)}, y_t^{(n-1)})) h |^2 \right\}
\]

\[
+ \Gamma h \mathbb{E}[|x_{t_{i+1}}^{(n+1)} - x_{t_{i+1}}^{(n)}|^2 + \Gamma^{-1} h \mathbb{E}[Sx_t^{(n)} - Sx_t^{(n-1)} - (f_1(x_t^{(n)}, y_t^{(n)}) - f_1(x_t^{(n-1)}, y_t^{(n-1)}))|^2]
\]

(Young's inequality \( ab \leq \frac{a^2}{2\Gamma} + \frac{b^2\Gamma}{2} \) where \( \Gamma > 0 \) will be choosen later)

\[
\leq (1 + \Gamma h) \left( \mathbb{E}\left[ | x_{t_{i+1}}^{(n+1)} - x_{t_{i+1}}^{(n)} |^2 \right] \right)
\]

\[
+ \frac{h + \Gamma^{-1}}{1 + \Gamma h} L_f^2 \mathbb{E}[|y_t^{(n)} - y_t^{(n-1)}|^2 h] + \frac{h + \Gamma^{-1}}{1 + \Gamma h} L_f^2 \mathbb{E}[|x_t^{(n)} - x_t^{(n-1)}|^2 h].
\]
Since $x_{t_i}^{(n+1)} = x_{t_i}^{(n)} = x_0$, by iterating the last inequality, we obtain

$$
\mathbb{E}[|x_{t_i}^{(n+1)} - x_{t_i}^{(n)}|^2] \leq L_f^2(h + \Gamma^{-1}) \left[ \sum_{j=i}^{N-1} \mathbb{E}[|y_{t_j}^{(n)} - y_{t_j}^{(n-1)}|^2] h + \sum_{j=i}^{N-1} \mathbb{E}[|x_{t_j}^{(n)} - x_{t_j}^{(n-1)}|^2] h \right].
$$

(16)

For the forward SDE part, since in the numerical scheme $y$ is only of finite dimension, we consider $y$ to be one-dimensional for simplicity, and estimate

$$
\mathbb{E}[|y_{t_i}^{(n+1)} - y_{t_i}^{(n)}|^2] = \mathbb{E}[e^{-Uh} \left( (y_{t_i}^{(n+1)} - y_{t_i}^{(n)}) + \int_{t_i}^{t_i+1} e^{U(s-t_i)}(f_2(x_{t_i}^{(n)}, y_{t_i}^{(n)}) - f_2(x_{t_i}^{(n-1)}, y_{t_i}^{(n-1)}))ds 
+ \int_{t_i}^{t_i+1} e^{U(s-t_i)}(g_2(x_{t_i}^{(n)}, y_{t_i}^{(n)}) - g_2(x_{t_i}^{(n-1)}, y_{t_i}^{(n-1)}))dW_s \right)]^2
$$

$$
\leq 3e^{-2Uh} \left( \mathbb{E}[|y_{t_i}^{(n+1)} - y_{t_i}^{(n)}|^2] + \mathbb{E}[\int_{t_i}^{t_i+1} e^{U(s-t_i)}(f_2(x_{t_i}^{(n)}, y_{t_i}^{(n)}) - f_2(x_{t_i}^{(n-1)}, y_{t_i}^{(n-1)}))ds]^2 
+ \mathbb{E}[\int_{t_i}^{t_i+1} e^{U(s-t_i)}(g_2(x_{t_i}^{(n)}, y_{t_i}^{(n)}) - g_2(x_{t_i}^{(n-1)}, y_{t_i}^{(n-1)}))dW_s]^2 \right)
$$

$$
\leq 3e^{-2Uh} \left( \mathbb{E}[|y_{t_i}^{(n+1)} - y_{t_i}^{(n)}|^2] + h^2 \mathbb{E}[L_f^2(|x_{t_i}^{(n)} - x_{t_i}^{(n-1)}| + |y_{t_i}^{(n)} - y_{t_i}^{(n-1)}|)]^2 
+ h \mathbb{E}[L_g^2(|x_{t_i}^{(n)} - x_{t_i}^{(n-1)}| + |y_{t_i}^{(n)} - y_{t_i}^{(n-1)}|)]^2 \right)
$$

$$
\leq 3e^{-2Uh} \left( \mathbb{E}[|y_{t_i}^{(n+1)} - y_{t_i}^{(n)}|^2] + (2h^2 L_f^2 + 2h L_g^2) \mathbb{E}[|x_{t_i}^{(n)} - x_{t_i}^{(n-1)}|]^2 
+ (2h^2 L_f^2 + 2h L_g^2) \mathbb{E}[|y_{t_i}^{(n)} - y_{t_i}^{(n-1)}|]^2 \right)
$$

$$
\leq 6e^{-2Uh}(h L_f^2 + L_g^2(\sum_{j=1}^{i-1} \mathbb{E}[|x_{t_j}^{(n)} - x_{t_j}^{(n-1)}|^2] h + \sum_{j=1}^{i-1} \mathbb{E}[|y_{t_j}^{(n)} - y_{t_j}^{(n-1)}|^2] h),
$$

(17)

where the last inequality is by iteration.

Combining (16) and (17), and letting $\tilde{M}_1 := \max\{L_f^2(h+\Gamma^{-1}), 6e^{-2Uh}(L_f^2+hL_g^2)\}$, we have:
Lemma 2 (Discretization error). Assume all the conditions as in Lemma 1 are satisfied, then

\[
\sup_{-T \leq t \leq 0} \left( \mathbb{E} \left| x_t - x_t^{(\infty)} \right|^2 + \mathbb{E} \left| y_t - y_t^{(\infty)} \right|^2 \right) < Ch.
\]
Proof. Note that

$$x_{t_i-1}^{(\infty)} = \mathbb{E}[x_{t_i}^{(\infty)} | \mathcal{F}_{t_i-1}] + [Sx_{t_i-1}^{(\infty)} - f_1(x_{t_i-1}^{(\infty)}, y_{t_i-1}^{(\infty)})](t_i - t_{i-1}).$$

For $t_{i-1} \leq t \leq t_i$, we have

$$x_{t_i-1}^{(\infty)} = \mathbb{E}[x_t^{(\infty)} | \mathcal{F}_{t_i-1}] + [Sx_{t_i-1}^{(\infty)} - f_1(x_{t_i-1}^{(\infty)}, y_{t_i-1}^{(\infty)})](t - t_{i-1}).$$

By the Martingale Representation Theorem [23],

$$x_t^{(\infty)} = \mathbb{E}[x_t^{(\infty)} | \mathcal{F}_{t_i-1}] + \int_{t_i-1}^t Z_s^{(\infty)} dW_s,$$

we have

$$x_t^{(\infty)} = x_{t_i-1}^{(\infty)} - (t - t_{i-1})[Sx_{t_i-1}^{(\infty)} - f_1(x_{t_i-1}^{(\infty)}, y_{t_i-1}^{(\infty)})] + \int_{t_i-1}^t Z_s^{(\infty)} dW_s,$$

which yields

$$dx_t^{(\infty)} = [-Sx_{t_i-1}^{(\infty)} + f_1(x_{t_i-1}^{(\infty)}, y_{t_i-1}^{(\infty)})]dt + Z_t^{(\infty)} dW_t.$$ 

By Itô’s formula, we have

$$\mathbb{E}[x_t - x_t^{(\infty)}]^2 = \mathbb{E}[x_{t_i} - x_t^{(\infty)}]^2$$

$$- \mathbb{E} \int_{t_i}^t 2(x_s - x_t^{(\infty)})[-S(x_s - x_{t_i-1}^{(\infty)}) + f_1(x_s, y_s) - f_1(x_{t_i-1}^{(\infty)}, y_{t_i-1}^{(\infty)})]ds$$

$$- \mathbb{E} \int_{t_i}^t (Z_s - Z_s^{(\infty)})^2 ds.$$ 

Let

$$\delta x_t := x_t - x_t^{(\infty)}, \quad \delta Z_t := Z_t - Z_t^{(\infty)},$$

and $\delta f_t := (Sx_t - f_1(x_t, y_t)) - (Sx_{t_i-1}^{(\infty)} - f_1(x_{t_i-1}^{(\infty)}, y_{t_i-1}^{(\infty)}))$,

then define

$$A_t := \mathbb{E} | \delta x_t |^2 + \int_t^{t_i} \mathbb{E} | \delta Z_s |^2 ds - \mathbb{E} | \delta x_{t_i} |^2$$

$$\int_t^{t_i} \mathbb{E}[\delta Z_s \delta f_s] ds$$

$$\leq \mathbb{E}[C \int_t^{t_i} | x_s - x_{t_i-1}^{(\infty)} |] ds$$

$$\leq \int_t^{t_i} (\alpha \mathbb{E} | x_s - x_{t_i-1}^{(\infty)} |^2 + | y_s - y_{t_i-1}^{(\infty)} |^2) ds.$$
Since
\[ \mathbb{E} \left| x_s - x_{t_i-1} \right|^2 = \mathbb{E} \left| \int_{t_i-1}^s (-Sx_r + f_1(x_r, y_r)) dr + \int_{t_i-1}^s Z_r dW_r \right|^2 \]
\[ \leq 2 \left[ \mathbb{E} \left| \int_{t_i-1}^s (-Sx_r + f_1(x_r, y_r)) dr \right|^2 + \mathbb{E} \left| \int_{t_i-1}^s Z_r dW_r \right|^2 \right] \]
\[ \leq 2 [(s - t_{i-1}) \mathbb{E} \int_{t_i-1}^s (-Sx_r + f_1(x_r, y_r))^2 dr + \mathbb{E} \int_{t_i-1}^s Z_r^2 dr] \]
\[ \leq 2 [(s - t_{i-1}) \int_{t_i-1}^s [\mathbb{E}(-Sx_r + f_1(x_r, y_r) - f_1(0,0))^2 + \mathbb{E}(f_1(0,0))^2] dr \]
\[ + \mathbb{E} \int_{t_i-1}^s Z_r^2 dr \]
\[ \leq 2 [(s - t_{i-1}) \int_{t_i-1}^s (Cx_r^2 + 2L_f(x_r^2 + y_r^2) + C) dr + \mathbb{E} \int_{t_i-1}^s Z_r^2 dr] \]
\[ \leq Ch, \]
then we have
\[ \mathbb{E} \left| x_s - x_{t_i-1}^{(\infty)} \right|^2 \leq 2 [\mathbb{E} \left| x_s - x_{t_i-1} \right|^2 + \mathbb{E} \left| \delta x_{t_i-1} \right|^2] \]
\[ \leq C(h + \mathbb{E} \left| \delta x_{t_i-1} \right|^2) \]
for \( t_{i-1} \leq t \leq s \). On the other hand,
\[ \mathbb{E} \left| y_s - y_{t_{i-1}} \right|^2 \leq C + C \mathbb{E} \left| \int_{t_{i-1}}^s e^{U(r+T)} f_2(x_r, y_r) dr + \int_{t_{i-1}}^s e^{U(r+T)} g_2(x_r, y_r) dW_r \right|^2 \]
\[ \leq C + C [(s - t_{i-1}) \mathbb{E} \int_{t_{i-1}}^s e^{2U(r+T)} f_2^2(x_r, y_r) dr + \mathbb{E} \int_{t_{i-1}}^s e^{2U(r+T)} g_2^2(x_r, y_r) dr] \]
\[ \leq Ch, \]
and note also that \( y^{(\infty)} \) agrees with \( y \) at each grid point, i.e. \( y_{t_{i-1}}^{(\infty)} = y_{t_{i-1}} \), thus
\[ \mathbb{E} \left| y_s - y_{t_{i-1}}^{(\infty)} \right|^2 \leq 2 [\mathbb{E} \left| y_s - y_{t_{i-1}} \right|^2 + \mathbb{E} \left| \delta y_{t_{i-1}} \right|^2] \]
\[ \leq C (h + \mathbb{E} \left| \delta y_{t_{i-1}} \right|^2) \]
\[ = Ch. \]

By the definition of \( A_t \),
\[ A_t \leq \int_t^{t_1} \alpha \mathbb{E} \left| \delta x_s \right|^2 ds + \frac{C}{\alpha} \int_t^{t_1} [h + \mathbb{E} \left| \delta x_{t_i-1} \right|^2] ds \]
\[ \leq \int_t^{t_1} \alpha \mathbb{E} \left| \delta x_s \right|^2 ds + \frac{C}{\alpha} [h^2 + h \mathbb{E} \left| \delta x_{t_i-1} \right|^2] \]
For the forward part

\[ \mathbb{E} | \delta x_t |^2 \leq \mathbb{E} | \delta x_t |^2 + \int_t^{t_i} \mathbb{E} | \delta Z_s |^2 \, ds \]

\[ \leq \int_t^{t_i} \alpha \mathbb{E} | \delta x_s |^2 \, ds + \left( \mathbb{E} | \delta x_{t_i} |^2 + \frac{C}{\alpha} [h^2 + h\mathbb{E} | \delta x_{t_{i-1}} |^2] \right). \]

(18)

Let \( B_i := \mathbb{E} | \delta x_{t_i} |^2 + \frac{C}{\alpha} [h^2 + h\mathbb{E} | \delta x_{t_{i-1}} |^2] \), then by Gronwall’s inequality, \( \mathbb{E} | \delta x_t |^2 \leq B_i e^{\int_{t}^{t_i} \alpha ds} \leq B_i e^{\alpha h} \). Plugging it in the second inequality of (18), we have

\[ \mathbb{E} | \delta x_t |^2 + \int_t^{t_i} \mathbb{E} | \delta Z_s |^2 \, ds \leq B_i (1 + \alpha e^{\alpha h}). \]

(19)

By taking \( t = t_{i-1} \), we have

\[ \mathbb{E} | \delta x_{t_{i-1}} |^2 + \int_{t_{i-1}}^{t_i} \mathbb{E} | \delta Z_s |^2 \, ds \leq (1 + Ch)(\mathbb{E} | \delta x_{t_i} |^2 + \frac{C}{\alpha} [h^2 + h\mathbb{E} | \delta x_{t_{i-1}} |^2]), \]

and if we choose \( \alpha \gg C \),

\[ \mathbb{E} | \delta x_{t_{i-1}} |^2 + \int_{t_{i-1}}^{t_i} \mathbb{E} | \delta Z_s |^2 \, ds \leq (1 + Ch)(\mathbb{E} | \delta x_{t_i} |^2 + h^2). \]

(20)

Iterating the last inequality and recall that \( \mathbb{E} | \delta x_{t_N} |^2 = 0 \), we have for sufficiently small \( h \),

\[ \mathbb{E} | \delta x_{t_{i-1}} |^2 \leq (1 + Ch)^N (\mathbb{E} | \delta x_{t_N} |^2 + h) \leq Ch, \]

which yields \( B_i \leq Ch \), and by (19), we get for all \( t \in [t_{i-1}, t_i] \)

\[ \mathbb{E} | \delta x_t |^2 \leq Ch, \]

and the right hand side of the inequality does not depend on \( t \). Therefore,

\[ \sup_{-T \leq t \leq 0} \mathbb{E} | \delta x_t |^2 = \sup_{-T \leq t \leq 0} \mathbb{E} | x_t^{(\infty)} - x_t |^2 \leq Ch. \]

(21)

For the forward part \( y \), we have

\[ \mathbb{E} | y_t^{(\infty)} - y_t |^2 = \mathbb{E} \left( \int_t^{t_i} e^{-U(t-s)} [f_2(x_t^{(\infty)}, y_t^{(\infty)}) - f_2(x_s, y_s)] ds \right) \]

\[ + \int_{t_{i-1}}^{t} e^{-U(t-s)} [g_2(x_t^{(\infty)}, y_t^{(\infty)}) - g_2(x_s, y_s)] dW_s \right)^2 \]

\[ \leq 2h \mathbb{E} \int_{t_{i-1}}^{t} e^{-2U(t-s)} [f_2(x_t^{(\infty)}, y_t^{(\infty)}) - f_2(x_s, y_s)]^2 ds \]

\[ + 2 \mathbb{E} \int_{t_{i-1}}^{t} e^{-2U(t-s)} [g_2(x_t^{(\infty)}, y_t^{(\infty)}) - g_2(x_s, y_s)]^2 ds \]

\[ \leq [C(h + \mathbb{E} | \delta x_{t_{i-1}} |^2) + C(h + \mathbb{E} | \delta x_{t_{i-1}} |^2)] (2h^2C + 2hC) \]

\[ \leq Ch^2, \]

(22)
where the last inequality is due to (20). Consider (21) and (22), we have
\[ \sup_{-T \leq t \leq 0} \mathbb{E} | x_t^{(\infty)} - x_t |^2 + \mathbb{E} | y_t^{(\infty)} - y_t |^2 \leq Ch. \]

This proves the lemma.

Now we are ready to state the convergence theorem. For \( t \in [t_{i-1}, t_i] \), define \( x_t^{(n)} = x_{t_{i-1}}^{(n)} \) and \( y_t^{(n)} = y_{t_{i-1}}^{(n)} \). We can also define \( x_t^{(n)} \) and \( y_t^{(n)} \) as the linear interpolation among \( x_{t_{i-1}}^{(n)} \)'s and among \( y_{t_{i-1}}^{(n)} \)'s, respectively, and the following result also holds.

**Theorem 1** (Convergence). Assume that all the conditions in Lemma 1 are satisfied. Then
\[ \sup_{-T \leq t \leq 0} \left( \mathbb{E} | x_t - x_t^{(n)} |^2 + \mathbb{E} | y_t - y_t^{(n)} |^2 \right) \leq C (h + (\frac{1}{2} + Ch)^n), \]

where \( C > 0 \) will denote a generic constant depending only on the data \( L_f, L_g, K, U, T \). This implies the convergence of the numerical scheme (15), as \( h \to 0 \) and \( n \to \infty \).

**Proof.** Note that
\[
\begin{align*}
\mathbb{E} | x_t - x_t^{(n)} |^2 + \mathbb{E} | y_t - y_t^{(n)} |^2 \\
\leq \mathbb{E} | x_t - x_{t_{i-1}}^{(n)} |^2 + \mathbb{E} | x_{t_{i-1}}^{(\infty)} - x_{t_{i-1}} |^2 + \mathbb{E} | x_{t_{i-1}}^{(\infty)} - x_t^{(n)} |^2 \\
+ \mathbb{E} | y_t - y_{t_{i-1}}^{(n)} |^2 + \mathbb{E} | y_{t_{i-1}}^{(\infty)} - y_{t_{i-1}} |^2 + \mathbb{E} | y_{t_{i-1}}^{(\infty)} - y_t^{(n)} |^2 \\
\leq C (h + (\frac{1}{2} + Ch)^n)
\end{align*}
\]
by regularity of the true solution as well as Lemma 1 and Lemma 2. Taking supremum on both sides of the above inequality, we have
\[
\sup_{-T \leq t \leq 0} \left( \mathbb{E} | x_t - x_t^{(n)} |^2 + \mathbb{E} | y_t - y_t^{(n)} |^2 \right) \leq C (h + (\frac{1}{2} + Ch)^n).
\]

4  Approximation of the stochastic inertial manifold

Now we approximate the graph \( \Phi \) for the inertial manifold \( \mathcal{M} \). Recall that \( \Phi \) is approximated via
\[ y(0, \omega) := \Phi_T(x_0, \omega), \ \omega \in \Omega. \]
When $T$ is sufficiently big, $\Phi_T \approx \Phi$. So we need to evaluate $y(0, \omega)$ (which is $y(0)$ in Section 3). To this end, we need to compute, step by step, $x_{t_i}^{(n)}, y_{t_i}^{(n)}$ in (15). For the backward part $x_{t_i}^{(n)}$, the conditional expectations $E[\cdot \mid \mathcal{F}_{t_i}] \in L^2(\mathcal{F}_{t_i})$ in (15) will be approximated by their orthogonal projections $P_i$ on finite dimensional subspaces $\Lambda_i$ of $L^2(\mathcal{F}_{t_i})$, where $L^2(\mathcal{F}_{t_i})$ contains all the functions in $L^2(\Omega)$ that are $\mathcal{F}_{t_i}$-adapted.

Indeed, instead of computing $x_{t_i}^{(n)}$ as follows

$$x_{t_i}^{(n)} = E\left\{ \left[ x_0 - \sum_{j=i}^{N-1} \left( Sx_{t_j}^{(n-1)} + f_1(x_{t_j}^{(n-1)}, y_{t_j}^{(n-1)}) \Delta_j \right) \right] \right\},$$

we will compute its orthogonal projection $\hat{x}_{t_i}^{(n)}$:

$$\hat{x}_{t_i}^{(n)} = P_i E\left\{ \left[ x_0 - \sum_{j=i}^{N-1} \left( S\hat{x}_{t_j}^{(n-1)} + f_1(\hat{x}_{t_j}^{(n-1)}, y_{t_j}^{(n-1)}) \Delta_j \right) \right] \right\}. $$

Denoting a basis of the projection space $\Lambda_i$ by $\{\eta^i_1, \ldots, \eta^i_{D(i)}\}$, we then have

$$\hat{x}_{t_i}^{(n)} = \sum_{d=1}^{D(i)} \alpha_d^{(n)} \eta^i_d.$$

Set $\eta^i := (\eta^i_1, \ldots, \eta^i_{D(i)})'$ and $\alpha_i^{(n)} := (\alpha_{i,1}^{(n)}, \ldots, \alpha_{i,D(i)}^{(n)})'$, with prime denoting matrix transpose, the calculation for the backward part $x_{t_i}^{n}$ boils down to two aspects: one is the basis $\eta^i$, the other is the coefficient $\alpha_i^{(n)}$.

**Remark 5** (Basis). As in [27], a complete orthonormal basis of $L^2(\Omega, \mathcal{F}_t, \mathbb{P})$ is given by the Wick polynomials $\{T^t_a(\xi); a \in \mathcal{J}\}$,

$$T^t_a(\xi) := \Pi_{i=1}^\infty H_{a_i}(\xi_i(t)),$$

where $H_{a_i}(\cdot)$ are Hermite polynomials. Here

$$\mathcal{J} := \{a = (a_i, i \geq 1) \mid a_i \in \{0, 1, 2, \ldots\}, |a| = \sum_{i=1}^\infty a_i < \infty\},$$

$$\xi_i(t) := \int_0^t m_i(s)dW_s,$$

and $m_i(s), i = 1, 2, \ldots$, are a set of complete orthonormal basis in the Hilbert space $L^2([0, t])$. One choice of the basis elements are the Hermite polynomials of Brownian motion $W_t$. In fact,

$$a^1 = (1, 0, 0, \cdots), \quad T^t_a(\xi(t)) = H_1(\xi_1(t))H_0(\xi_2(t))H_0(\xi_3(t)) \cdots = H_1(\xi_1(t)),$$

$$a^2 = (2, 0, 0, \cdots), \quad T^t_a(\xi(t)) = H_2(\xi_1(t))H_0(\xi_2(t))H_0(\xi_3(t)) \cdots = H_2(\xi_1(t)),$$

$$\vdots$$

$$a^n = (n, 0, 0, \cdots), \quad T^t_a(\xi(t)) = H_2(\xi_1(t))H_0(\xi_2(t))H_0(\xi_3(t)) \cdots = H_2(\xi_1(t)).$$
Since $\xi_1(t) = \int_0^t m_1(s) dW_s = \int_0^t 1 dW_s = W_t$, the basis elements $\eta_d^i$'s are $H_d(W_t)$ which are adopted in our numerical implementation.

In principle, the coefficients $\alpha_i^{(n)}$ are calculated as follows:

$$\alpha_i^{(n)} = \beta_i^{-1} \mathbb{E} \left[ \eta^i \left( x_0 - \sum_{j=1}^{N-1} (S_h^{i(n-1)} + f_1(\hat{x}_j^{i(n-1)}, y_i^{i(n-1)})) \Delta_j \right) \right],$$

(25)

where $\beta_i := (\mathbb{E}[\eta_{i}^j \eta_{i}^q]_{p,q=1,...,D(i)})$ are the inner-product matrices associated with the basis.

In practice, $\beta_i$'s are computed by their simulation-based estimators, such as Monte Carlo least squares estimators used here \([4]\). To this end, we are assuming to have $r$ (sufficiently large) independent copies $(\lambda W_i, \lambda \eta_d^i)$, $\lambda = 1, \ldots, r$, of $(W_i, \eta_d^i)$. Here the index $\lambda$ denotes copies. Then

$$\beta_i = (\mathbb{E}[\eta_{i}^j \eta_{i}^q]_{p,q=1,...,D(i)}) = \left( \begin{array}{cccc} \mathbb{E} \eta_{1}^i \eta_{1}^j & \mathbb{E} \eta_{1}^i \eta_{2}^j & \cdots & \mathbb{E} \eta_{1}^i \eta_{D(i)}^j \\ \mathbb{E} \eta_{2}^i \eta_{1}^j & \mathbb{E} \eta_{2}^i \eta_{2}^j & \cdots & \mathbb{E} \eta_{2}^i \eta_{D(i)}^j \\ \vdots & \vdots & \ddots & \vdots \\ \mathbb{E} \eta_{D(i)}^i \eta_{1}^j & \mathbb{E} \eta_{D(i)}^i \eta_{2}^j & \cdots & \mathbb{E} \eta_{D(i)}^i \eta_{D(i)}^j \end{array} \right),$$

is replaced by its Monte Carlo simulation $\tilde{\beta}_i$

$$\tilde{\beta}_i = \frac{1}{r} \left( \begin{array}{cccc} \sum_{\lambda=1}^r \lambda \eta_{1}^i \lambda \eta_{1}^j & \sum_{\lambda=1}^r \lambda \eta_{1}^i \lambda \eta_{2}^j & \cdots & \sum_{\lambda=1}^r \lambda \eta_{1}^i \lambda \eta_{D(i)}^j \\ \sum_{\lambda=1}^r \lambda \eta_{2}^i \lambda \eta_{1}^j & \sum_{\lambda=1}^r \lambda \eta_{2}^i \lambda \eta_{2}^j & \cdots & \sum_{\lambda=1}^r \lambda \eta_{2}^i \lambda \eta_{D(i)}^j \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{\lambda=1}^r \lambda \eta_{D(i)}^i \lambda \eta_{1}^j & \sum_{\lambda=1}^r \lambda \eta_{D(i)}^i \lambda \eta_{2}^j & \cdots & \sum_{\lambda=1}^r \lambda \eta_{D(i)}^i \lambda \eta_{D(i)}^j \end{array} \right),$$

and is further rewritten as

$$\tilde{\beta}_i = (\bar{A}_i)' \bar{A}_i = \frac{1}{r} \left( \sum_{\lambda=1}^r \lambda \eta_{a}^i \lambda \eta_{b}^i \right), \ a, b = 1, \ldots, D(i),$$

where

$$\bar{A}_i = \frac{1}{\sqrt{r}} \left( \begin{array}{c} \lambda \eta_{d}^i \\ \lambda \eta_{d}^i \\ \vdots \\ \lambda \eta_{d}^i \end{array} \right) = \frac{1}{\sqrt{r}} \left( \begin{array}{cccc} 1 \eta_{1}^i & 1 \eta_{2}^i & \cdots & 1 \eta_{D(i)}^i \\ 2 \eta_{1}^i & 2 \eta_{2}^i & \cdots & 2 \eta_{D(i)}^i \\ \vdots & \vdots & \ddots & \vdots \\ r \eta_{1}^i & r \eta_{2}^i & \cdots & r \eta_{D(i)}^i \end{array} \right).$$

The pseudo-inverse denoted by $A^\dagger := (A'A)^{-1}A'$ is used in the computation of $\beta_i$. 

20
The calculation of the coefficients $\alpha_i^{(n)}$ is to obtain the backward part $x_i^{(n)}$. Note that the calculation of $\alpha_i^{(n)}$ also needs the forward part $y_i^{(n-1)}$, which is calculated through Euler-Maruyama scheme. The following formulae illustrate how to update $\alpha_i^{(n)}$.

\[
\alpha_i^{(0)} = (0, 0, \ldots, 0)', \quad (26)
\]

\[
(\lambda x_{t_i}^{(n-1)})_{\lambda=1, \ldots, r} = \sum_{i}^{D(i)} \lambda x_{t_i}^{(n-1)}, \ldots, \lambda x_{t_i}^{(n-1)}',
\]

\[
(\lambda y_{t_i}^{(0)})_{\lambda=1, \ldots, r} = 0,
\]

\[
\lambda f(t_j)^{(n-1)} \triangleq \left( f(t_j \lambda x_{t_j}^{(n-1)}, \lambda y_{t_j}^{(n-1)}) \right) = \left( S_{\lambda x_{t_j}^{(n-1)}}, f_1(\lambda x_{t_j}^{(n-1)}, \lambda y_{t_j}^{(n-1)}) \right)_{\lambda=1, \ldots, r}, \quad (28)
\]

\[
\alpha_i^{(n)} = \frac{1}{\sqrt{T}} (\hat{A}_i)^{\dagger} (x_0 - \sum_{j=1}^{N-1} \lambda f(t_j)^{(n-1)} \Delta_j),
\]

\[
= \frac{1}{\sqrt{T}} (\hat{A}_i)^{\dagger} \left[ \left( \begin{array}{c} 1 x_0 \\ 2 x_0 \\ \vdots \\ r x_0 \end{array} \right) - \left( \begin{array}{c} 1 \sum_{j=i}^{N-1} f(t_j \lambda x_{t_j}^{(n-1)}, \lambda y_{t_j}^{(n-1)}) \\ 2 \sum_{j=i}^{N-1} f(t_j \lambda x_{t_j}^{(n-1)}, \lambda y_{t_j}^{(n-1)}) \\ \vdots \\ r \sum_{j=i}^{N-1} f(t_j \lambda x_{t_j}^{(n-1)}, \lambda y_{t_j}^{(n-1)}) \end{array} \right) \right] \Delta_j, \quad (30)
\]

\[
(\lambda y_{t_i}^{(n)})_{\lambda=1, \ldots, r} \triangleq (1 \hat{y}_{t_i}^{(n)}, 2 \hat{y}_{t_i}^{(n)}, \ldots, r \hat{y}_{t_i}^{(n)})',
\]

\[
= e^{-Uh} \left[ \lambda y_{t_i-1}^{(n-1)} + f_2(\lambda x_{t_i-1}^{(n-1)}, \lambda y_{t_i-1}^{(n-1)}) \Delta_j + g_2(\lambda x_{t_i-1}^{(n-1)}, \lambda y_{t_i-1}^{(n-1)}) \hat{W}_j \right]_{\lambda=1, \ldots, r}, \quad (31)
\]

where $\lambda x_{t_i}^{(n-1)} \triangleq (1 \hat{x}_{t_i}^{(n-1)}, \ldots, r \hat{x}_{t_i}^{(n-1)})$ are independent copies of $\hat{x}_{t_i}^{(n-1)}$ corresponding to independent copies of basis functions $\lambda \eta_d^{(n)}$.

Upon convergence, i.e.

\[
\max \left\{ \mathbb{E} \left| x_{t_i}^{(n)} - x_{t_i}^{(n-1)} \right|^2 + \mathbb{E} \left| y_{t_i}^{(n)} - y_{t_i}^{(n-1)} \right|^2 < \text{tol} \right\}, \quad (32)
\]

we have $y_{t_N}^{(n)}$ as our approximation of $y_0$. 

**Remark 6.** Although we only need the final grid point value $y_{t_N}$ (recall that $t_N = 0$) as our approximation of the inertial manifold $y_0 := \Phi_T(x_0)$, the
intermediate points \((x_{t_i}^{(n)}, y_{t_i}^{(n)})\) are approximated by \((\hat{x}_{t_i}^{(n)}, \hat{y}_{t_i}^{(n)})\) as follows

\[
\hat{x}_{t_i}^{(n)}(\omega) = \sum_{d=1}^{D(i)} \alpha_{i,d}^{(n)} \eta_i^d(\omega),
\]

\[
\hat{y}_{t_i}^{(n)}(\omega) = e^{-Uh}[\hat{y}_{t_{i-1}}^{(n)}(\omega) + f_2(\hat{x}_{t_{i-1}}^{(n)}(\omega), \hat{y}_{t_{i-1}}^{(n)}(\omega))\Delta_j + g_2(\hat{x}_{t_{i-1}}^{(n)}(\omega), \hat{y}_{t_{i-1}}^{(n)}(\omega))\Delta W_j(\omega)].
\]

Therefore, this approach of approximating stochastic inertial manifold also provides a way of solving backward-forward stochastic differential equations. As in equation (33), \(x_{t_i}^{(n)}(\omega)\)’s are approximated by its orthogonal projection \(\hat{x}_{t_i}^{(n)}(\omega) := \sum_{d=1}^{D(i)} \alpha_{i,d}^{(n)} \eta_i^d(\omega)\), where the randomness comes from the basis functions \(\eta_i^d\) rather than the coefficients \(\alpha_{i,d}^{(n)}\). The fact that \(\alpha\)’s are deterministic can be seen from (25), since \(\alpha\)’s are expectations. In the numerical simulation of \(\alpha\)’s, we utilize copies of \((x, y)\)’s (different copies corresponding to different sample paths \(\omega\)) to calculate the expectations.

All the conditions required by Bender and Denk in [4] are satisfied in our case, so the error analysis results for the Monte Carlo simulation also apply here.

Figure 3 demonstrates the procedure of computing stochastic inertial manifold.

Specifically, the computation is achieved in the following way as shown in Figure 3. We begin with the flat manifold, i.e. let all the \(y\)-values be
zero. In principle, we could take any other acceptable initial manifold and let it flow forward, and at the limit it would also approach the desired manifold. We thus start our iteration by letting the initial guess \( y \) be zero at each \( t_i \), see (28). In order to avoid nestings of conditional expectations, we also let initial guess \( x \) values to be zero except at terminal time \( t_N \), which corresponds to letting all but the final set of coefficients \( \alpha \) to be 0, see (26).

Recall that we approximate each \( x_{t_i}^{(n)} \) value by its finite dimensional orthogonal projection as in (24), so indeed we want to calculate \( \alpha_{t_i}^{(n)} \). The terminal value of \( x \) is set to be a \( \mathcal{F}_0 \)-measurable random variable. At each iteration, in order to update \( \alpha_{t_{i-1}}^{(n)} \) (30), we use copies of \( \lambda x_{t_{i-1}}^{(n-1)} \), \( \cdots \), \( \lambda x_{t_N}^{(n-1)} \) as well as \( \lambda y_{t_{i-1}}^{(n-1)} \), \( \cdots \), \( \lambda y_{t_N}^{(n-1)} \) (\( \lambda = 1, \ldots, r \)) from previous iteration, and then generate copies of \( x_{t_{i-1}}^{(n)} \) by virtue of copies of basis functions \( \eta \)'s, see (27). Copies of \( y_{t_{i-1}}^{(n)} \) are reproduced by an Euler-Maruyama scheme as in (31), in which different copies correspond to the sample \( \omega \)'s that are already chosen in basis functions \( \eta \)'s. This procedure is repeated until \( (x, y) \) converges in the mean square sense (32). We finally acquire the terminal value of \( y \) at the stopped iteration as the approximation of \( y_0 \).

![Figure 4: Schematic diagram of the numerical iteration.](image-url)
5 Examples

In this section, we test our backward-forward numerical scheme in two examples. One is a system of SDEs, and the other one is an SPDE (which is converted to a system of SDEs).

Example 1: A system of stochastic ordinary differential equations

\[ dX_t = (-a X_t - X_t Y_t)dt, \quad (35) \]
\[ dY_t = (-Y_t (1 + 2Y_t)^+ + X_t^2)dt + \sigma Y_t \circ dW_t, \quad (36) \]

where \( W_t \) is a scalar Wiener process, \( a \) and \( \sigma \) are real parameters, \( \circ \) indicates the Stratonovich interpretation of the noise term and \( f^+ \approx \max(f, 0) \).

Figure 5 is the phase portrait for the deterministic counterpart of Example 1 (\( \sigma = 0 \)).

Take \( \sigma = 0.1 \) and \( a = 0.1 \). Figure 6 shows several sample solution paths of the SDE system \((35)-(36)\).

Roberts [33] introduced a normal form transform method for stochastic differential systems with both slow modes and quickly decaying modes. The
(approximate) formula for the slow manifold (a type of inertial manifold) of
Example 1 obtained via his method is

\[ y \approx x^2 + 0.1x^2 \int_{-\infty}^{t} e^{-(t-\tau)} dW_\tau(\omega) \]  \hspace{1cm} (37)

and

\[ x \approx x + 0.1x \int_{-\infty}^{t} e^{-(t-\tau)} dW_\tau(\omega). \]  \hspace{1cm} (38)

In Figure, we plot the stochastic inertial manifold according to \( y \approx x^2 + 0.1x^2 \int_{-\infty}^{0} e^{\tau} dW_\tau \) and compare it with the stochastic inertial manifold from our backward-forward method for one sample path \( \omega \). There is a remarkable agreement of the shapes of the stochastic slow manifold obtained by the two distinct methods. For four samples in Figure, Figure then shows the discrepancy between the stochastic inertial manifold realised on three realisations \( \omega, \tilde{\omega}, \hat{\omega} \) and that realised on \( \omega \) via our backward-forward approach, respectively, i.e. \( M(\hat{\omega}) - M(\omega) \), \( M(\tilde{\omega}) - M(\omega) \), and \( M(\hat{\omega}) - M(\omega) \) in red, blue and green color.

**Example 2: A stochastic partial differential equation**

\[ \text{http://www.maths.adelaide.edu.au/anthony.roberts/sdesm.html} \]
We examine how effective the stochastic inertial manifold approach is in assisting the simulation of the long-term dynamics of an SPDE. We consider a stochastic partial differential equation

$$du_t = \nu \partial_x u_t \, dt + (u_t - u_t^3) \, dt + u_t \, dW_t, \quad 0 < x < 1,$$

(39)

$$u_t(0) = u_t(1) = 0,$$

$$u_0(x) = \sqrt{2} \sin(\pi x),$$

where $$\nu = 0.01$$, $$W_t$$ is a scalar Wiener process, and Itô interpretation of the noise term is adopted.

Let us first consider the dimension of the inertial manifold. Note that the eigenvalues of the operator $$\nu \partial_x + I$$ are $$-\nu i^2 \pi^2 + 1$$ with the corresponding eigenmodes $$e_i(x) = \sqrt{2} \sin(i\pi x)$$, for $$i = 1, 2, \ldots$$. Thus the dimension of the deterministic unstable eigenspace is 3. In this example, $$H = L^2(0, 1)$$. Let $$P$$ be the orthogonal projection to the (deterministic) unstable eigenspace $$H^+$$, spanned by eigenmodes $$e_i, i = 1, 2, 3$$.

The existence of the stochastic inertial manifold requires the nonlinear terms to be the globally Lipschitz. Here, in Example 2, although the drift term $$f(u) := u - u^3$$ is only locally Lipschitz, we can prepare the equation by replacing $$f$$ by the cutoff function $$\tilde{f}$$ which is defined to be $$f$$ in a bounded neighborhood centered at the origin, and 0 otherwise. The details of this procedure were described by Da Prato and Debussche [9].

Figure 7: Stochastic inertial manifold $$M(\omega)$$ calculated by the backward-forward method and by the normal form transform method for Example 1. Online version: Magenta from the backward-forward method and black from the normal form transform method.
With the Galerkin projection $u = \sum_i u_i(t)e_i(x)$, the evolutionary equations we will be working on become

$$\begin{align*}
\bar{p}: \quad & du_1(t) = \nu \lambda_1 u_1(t)dt + \langle u - u^3, e_1 \rangle dt + u_1(t)dW_t, \\
& du_2(t) = \nu \lambda_2 u_2(t)dt + \langle u - u^3, e_2 \rangle dt + u_2(t)dW_t, \\
& du_3(t) = \nu \lambda_3 u_3(t)dt + \langle u - u^3, e_3 \rangle dt + u_3(t)dW_t, \\
\bar{q}: \quad & du_4(t) = \nu \lambda_4 u_4(t)dt + \langle u - u^3, e_4 \rangle dt + u_4(t)dW_t,
\end{align*}$$

where we only keep four Fourier modes $u_i(t)e_i$, $i = 1, 2, 3, 4$ for $u$; $u_5(t), u_6(t), \ldots$ are simply ignored, as numerical simulations indicate that they are negligible in this specific case.

It is difficult to visualize the stochastic inertial manifold directly. But we can plot a “point” $p$ in $H^+$ and the corresponding “point” $q = \Phi(p, \omega)e_4(x)$ on the inertial manifold, separately. The “point” $p$ may be represented as $p \triangleq \sum_{i=1}^3 u_i(0, \omega)e_i(x)$ with coordinates $(u_1(0, \omega), u_2(0, \omega), u_3(0, \omega))$. The corresponding “point” $q$ on the stochastic inertial manifold is then computed through our backward-forward approach.

We plot three different realizations of a point $p$ versus space variable $x$ in the left panel of Figure 10 and then plot the corresponding point $q$ versus space variable $x$ separately in the right panel.
Figure 9: Discrepancy between different realisations of the stochastic inertial manifold in Example 1 via the backward-forward method. Online version: Magenta (top left) for the stochastic manifold realised on sample $\omega$, while red (top right), blue (bottom left) and green (bottom right) for discrepancies between stochastic manifold realised on samples $\bar{\omega}$, $\tilde{\omega}$ and $\hat{\omega}$ and stochastic manifold realised on sample $\omega$.

Figure 10: Example 2 – A “point” $p = u_1(0, \omega)e_1(x) + u_2(0, \omega)e_2(x) + u_3(0, \omega)e_3(x)$, with $u_1$, $u_2$ and $u_3$ being uniformly distributed random numbers, in $H^+$ and the corresponding “point” $q$ in the stochastic inertial manifold: Three samples of $p$ (left panel) and the corresponding three samples of $q$ (right panel). Online version: Corresponding $p$ and $q$ samples have the same color (green, red or black).
6 Discussion and conclusion

In this paper, we have devised a backward-forward numerical approach for computing stochastic inertial manifolds for stochastic evolutionary equations, including higher dimensional stochastic ordinary differential equations and stochastic partial differential equations. This approach is based on the stochastic inertial manifold theory of Da Prato and Debussche [9], which requires a backward-forward approximation formulation. In fact, our approach also provides a stand-alone numerical scheme for solving backward-forward SDEs.

Unlike deterministic evolutionary equations, we need to guarantee the adaptedness of solution processes with respect to an appropriate filtration for stochastic evolutionary equations. This makes the computation in the backward part of our numerical approach cumbersome, as it involves simulating conditional expectations which further requires a random basis. It would be nice to have a numerical scheme that uses only forward simulations. Clearly, a numerical scheme involving only forward simulations would be desirable; such schemes have been devised for deterministic problems (e.g. [20, 36]) and we are working on adapting them for the stochastic case.

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