Strong geodesic convex functions of order m

Akhlad Iqbal* and Izhar Ahmad**

*Department of Mathematics, Aligarh Muslim University, Aligarh-202002, India
**Department of Mathematics and Statistics, King Fahd University of Petroleum and Minerals, Dhahran 31261, Saudi Arabia

emails: akhlad6star@gmail.com; drizhar@kfupm.edu.sa

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Abstract. Strong geodesic convex function and strong monotone vector field of order m on Riemannian manifolds have been established. A characterization of strong geodesic convex function of order m for the continuously differentiable functions has been discussed. The relation between the solution of a new variational inequality problem and the strict minimizers of order m for a multiobjective programming problem has also been established.

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1 Introduction

The theory of convex functions has significant applications in optimization and variational inequality problems. Many scientists and researchers have explored it broadly in finite as well as infinite dimensional linear spaces, for the details see [1-7]. Because of its vast applications many generalizations have been proposed. Rapcsak [8] and Udriste [9] presented an innovative generalization, in which the line segment is replaced by the geodesic and Euclidean space is replaced by the Riemannian manifolds.

Motivated by the fact that the model of convex function unveils all its applications and outcomes only when it is developed on Riemannian manifolds, we define strong geodesic convex function and strongly monotone vector field of order $m$ on Riemannian manifolds.

Some variational inequality problems on Euclidean spaces can not be solved using classical technique but can be solved on manifolds. Nemeth [3] introduced the concept of variational inequality problem (VIP) on Hadamard manifolds while Li et. al. [10] discussed its existence and uniqueness on Riemannian manifolds.

Motivated by the above mentioned research work, we introduce variational inequality problem and establish a relation between its solution and the strict minimizers of order $m$ for a multiobjective programming problem.

2 Preliminaries

Let $(M, g)$ be a complete $n$-dimensional Riemannian manifold with Riemannian connection $\nabla$. Let $T_xM$ be the tangent space at $x \in M$ and $\langle ., . \rangle_x$ denotes the scalar product on $T_xM$ with the associated norm denoted by $\|\cdot\|_x$. A vector field $V$ on $M$ is a map from $M$ to $TM$ which associates with each pint $x \in M$ a vector $V(x) \in T_xM$. Let $\gamma_{xy} : [0, 1] \to M$ be a geodesic joining the points $x, y \in M$ such that $\gamma_{xy}(0) = x$ and $\gamma_{xy}(1) = y$. For the basic definitions and concepts of Riemannian geometry one can see ([11],
Rapcsak [8] defined geodesic convexity as follows.

**Definition 2.1.** [8] A set $A \subseteq M$ is called geodesic convex if a geodesic joining any two points $x, y \in A$ belongs to $A$.

**Definition 2.2.** [8] A real valued function $f : A \to R$ is called geodesic convex if

$$f(\gamma_{xy}(t)) \leq (1 - t)f(x) + tf(y)$$

for every $x, y \in A$ and $t \in [0, 1]$.

Udriste [9] has defined totally convex set as follows.

**Definition 2.3.** [9] A set $A \subseteq M$ is called totally convex if $A$ contains every geodesic $\gamma_{xy}$ of $A$ whose end points $x$ and $y$ are in $A$.

### 3 Strong geodesic convex functions of order $m$

Lin et al. [13] extended the concept of convexity to strong convexity of order $m$ on $R^n$ as follows.

**Definition 3.1.** [13] Let $X$ be a convex subset of $R^n$. A function $f : X \to R$ is said to be strongly convex of order $m$ if there exists a constant $c > 0$ such that

$$f(tx + (1 - t)y) \leq tf(x) + (1 - t)f(y) - ct(1 - t) \parallel x - y \parallel^m$$

for any $x, y \in X$ and $t \in [0, 1]$.

Motivated by Lin et al. [13], we introduce the concept of strong geodesic convex function of order $m$.

**Definition 3.2.** Suppose $A \subseteq M$ is a geodesic convex set of $M$. A function $f : M \to R$ is said to be strongly geodesic convex of order $m > 0$ on $A$ if
there exists a constant $c > 0$ such that

$$f(\gamma_{xy}(t)) \leq (1 - t)f(x) + tf(y) - ct(1 - t) \| \dot{\gamma}_{xy}(t) \|^m$$

for every $x, y \in A$ and $t \in [0, 1]$

**Remark.** Let $c = 0$. Then the above definition becomes the definition of geodesic convex defined by Rapcsak [8].

**Theorem 3.1.** Suppose $A \subseteq M$ is a geodesic convex set and $f : M \to R$ be continuously differentiable on $A$. Then, $f$ is strongly geodesic convex of order $m$ on $A$ if and only if there exists a constant $c > 0$, such that,

$$f(y) \geq f(x) + \dot{\gamma}_{xy}(f)(x) + c \| \dot{\gamma}_{xy}(t) \|^m, \quad \forall \ x, y \in A$$  \hspace{1cm} (1)

**Proof.** From the definition of strongly geodesic convex, we have

$$f(\gamma_{xy}(t)) \leq (1 - t)f(x) + tf(y) - ct(1 - t) \| \dot{\gamma}_{xy}(t) \|^m, \forall \ x, y \in A, \ t \in (0, 1]$$

or

$$f(x) + \frac{f(\gamma_{xy}(t)) - f(x)}{t} \leq f(y) - c(1 - t) \| \dot{\gamma}_{xy}(t) \|^m$$

Taking limit $t \to 0$, we get

$$f(x) + \dot{\gamma}_{xy}(0)(f) \leq f(y) - c \| \dot{\gamma}_{xy}(t) \|^m$$

or

$$f(y) \geq f(x) + \dot{\gamma}_{xy}(f)(x) + c \| \dot{\gamma}_{xy}(t) \|^m$$  \hspace{1cm} (2)

Conversely, let the given condition holds true for some $c > 0$. Changing $y$ with $x$, we get

$$f(x) \geq f(y) + \dot{\gamma}_{yx}(f)(y) + c \| \dot{\gamma}_{yx}(t) \|^m$$  \hspace{1cm} (3)

where $\dot{\gamma}_{yx}(t) = \gamma_{xy}(1 - t)$, $t \in [0, 1]$ is a geodesic joining $y$ with $x$. After fixing $t$ we get the point $\gamma_{xy}(t)$. Let $\gamma_{xy}(u)$, $u \in [t, 1]$ be the restriction for the geodesic arc that joins $\gamma_{xy}(t)$ and $y$. 

Setting \( u = t + s(1 - t) \), \( s \in [0,1] \), we obtain the reparametrization

\[
\alpha(s) = \gamma_{xy}(u(s)) = \gamma_{xy}(t + s(1 - t)), \quad s \in [0,1],
\]

where \( \alpha(0) = \gamma_{xy}(t) \), \( \frac{d\alpha(0)}{ds} = (1 - t) \frac{d\gamma_{xy}(t)}{dt} \).

Similarly, the restriction \( \overline{\gamma}_{yx}(u) = \gamma_{xy}(1 - u), \ u \in [1 - t, 1] \) is a geodesic joining \( \gamma_{xy}(t) \) with \( x \). Setting \( u = (1 - t) + st, \ s \in [0,1] \), we find the reparametrization

\[
\beta(s) = \overline{\gamma}_{yx}(1 - t + st) = \gamma_{xy}(t - st), \quad s \in [0,1],
\]

where \( \beta(0) = \gamma_{xy}(t), \frac{d\beta(0)}{ds} = -t \frac{d\gamma_{xy}(t)}{dt} \).

On replacing \( x \) with \( \gamma_{xy}(t) \) in (2) and \( \dot{\gamma}_{xy}(0) \) by \( \frac{d\alpha(0)}{ds} \), we get

\[
f(y) \geq f(\gamma_{xy}(t)) + (1 - t) \frac{d\gamma_{xy}(f)}{dt}(\gamma_{xy}(t)) + c \| \dot{\gamma}_{xy}(t) \|^m
\]

(4)

Analogously, replacing \( y \) with \( \gamma_{xy}(t) \) and \( \overline{\gamma}_{yx}(0) \) by \( \frac{d\beta(0)}{ds} \) in (3), we get

\[
f(x) \geq f(\gamma_{xy}(t)) - t \frac{d\gamma_{xy}(f)}{dt}(\gamma_{xy}(t)) + c \| \overline{\gamma}_{xy}(t) \|^m
\]

(5)

Multiplying (4) by \( t \), (5) by \( (1 - t) \) and then adding, we get

\[
f(\gamma_{xy}(t)) \leq (1 - t)f(x) + tf(y) - ct \| \dot{\gamma}_{xy}(t) \|^m - c(1 - t) \| \dot{\gamma}_{xy}(1 - t) \|^m
\]

\[
\leq (1 - t)f(x) + tf(y) - c't(1 - t) \| \dot{\gamma}_{xy}(t) \|^m
\]

Which shows that \( f \) is strongly geodesic convex of order \( m \).

Nemeth [14] defined monotone vector fields on Riemannian manifolds as follows.

**Definition 3.3.** Let \( M \) be a Riemannian manifold and \( V \) be a vector field on \( M \). \( V \) is called monotone on \( M \) if for every \( x, y \in M \)

\[
\langle V(y), \dot{\gamma}_{xy}(0) \rangle \leq \langle V(x), \dot{\gamma}_{xy}(1) \rangle
\]

where \( \dot{\gamma} \) denotes the tangent vector of \( \gamma \) with respect to the arc length.
We define strong monotone vector field of order $m$ and establish a relation with strong geodesic convex function of order $m$.

**Definition 3.4.** Suppose $A \subset M$ is a geodesic convex set. A vector field $V$ on $A$ is called strongly monotone of order $m$ if there exists a constant $\beta > 0$ such that

$$\langle V(y), \dot{\gamma}_{xy}(1) \rangle - \langle V(x), \dot{\gamma}_{xy}(0) \rangle \geq \beta \| \dot{\gamma}_{xy}(t) \|^m, \quad \forall x, y \in A. \quad (6)$$

**Theorem 3.2.** Suppose $A \subset M$ is a geodesic convex set and $f : M \to \mathbb{R}$ be continuously differentiable on $A$. Then, $f$ is strongly geodesic convex of order $m$ on $A$ iff $\dot{\gamma}_{xy}(f)$ is strongly monotone of order $m$ on $A$.

**Proof.** Let $f$ be strongly geodesic convex of order $m$ on $A$. By Theorem 3.1, there exists a constant $c > 0$ such that (1) holds. Then, for any $x, y \in A$, we have

$$f(y) - f(x) \geq \dot{\gamma}_{xy}(f)(x) + c \| \dot{\gamma}_{xy}(t) \|^m,$$

$$f(x) - f(y) \geq \dot{\gamma}_{yx}(f)(y) + c \| \dot{\gamma}_{xy}(t) \|^m,$$

On adding, we get

$$\dot{\gamma}_{xy}(f)(x) + \dot{\gamma}_{yx}(f)(y) + 2c \| \dot{\gamma}_{xy}(t) \|^m \leq 0$$

or

$$\dot{\gamma}_{yx}(f)(x) - \dot{\gamma}_{yx}(f)(y) \geq 2c \| \dot{\gamma}_{xy}(t) \|^m$$

Which shows that if part is true.

Conversely, let (6) holds true and $V = \nabla f$. Set $t_i = \frac{i}{m+1}$, $i = 0, 1, ..., m+1$. By the mean value theorem, $\exists \xi \in (t_i, t_{i+1})$, $0 \leq i \leq m$

$$f(\gamma_{xy}(0) + t_{i+1}(\gamma_{xy}(1) - \gamma_{xy}(0))) - f(\gamma_{xy}(0) + t_i(\gamma_{xy}(1) - \gamma_{xy}(0)))$$

$$= (t_{i+1} - t_i)(\gamma_{xy}(1) - \gamma_{xy}(0))^T \nabla f(x + \xi_i(\gamma_{xy}(1) - \gamma_{xy}(0)))$$

It follows from (6),
\[ f(y) - f(x) = \sum_{i=0}^{m}[f(\gamma_{xy}(0) + t_{i+1}(\gamma_{xy}(1) - \gamma_{xy}(0))) - f(\gamma_{xy}(0) + t_{i}(\gamma_{xy}(1) - \gamma_{xy}(0)))] \]

\[ = \sum_{i=0}^{m}(t_{i+1} - t_{i})(\gamma_{xy}(1) - \gamma_{xy}(0))^T[\nabla f(x + \xi_{i}(\gamma_{xy}(1) - \gamma_{xy}(0)) - \nabla f(x)] \]

\[ + (\gamma_{xy}(1) - \gamma_{xy}(0))^T\nabla f(x) \]

\[ \geq \beta \| \dot{\gamma}_{xy}(t) \| \sum_{i=0}^{m} \xi_{i}^{m-1}(t_{i+1} - t_{i}) + (\gamma_{xy}(1) - \gamma_{xy}(0))^T\nabla f(x) \]

Taking limit \( m \to \infty \), we get

\[ f(y) - f(x) \geq \frac{\beta}{m} \| \dot{\gamma}_{xy}(t) \| \sum_{i=0}^{m} \xi_{i}^{m-1} + (\gamma_{xy}(1) - \gamma_{xy}(0))^T\nabla f(x). \]

Using theorem 3.1, the result follows.

**Definition 3.5.** Suppose \( A \subset M \) is a geodesic convex set. A vector field \( V \) on \( A \) is called strongly pseudomonotone of order \( m \) if

\[ \langle V(x), \dot{\gamma}_{xy}(0) \rangle + \beta \| \dot{\gamma}_{xy}(t) \|^m \geq 0 \Rightarrow \langle V(y), \dot{\gamma}_{xy}(1) \rangle \geq 0 \quad \forall \ x, y \in A. \quad (7) \]

**Proposition 3.1.** Every strongly monotone vector field of order \( m \) is strongly pseudomonotone of order \( m \).

**Proof.** Let \( V \) on \( A \) be strongly monotone of order \( m \), then

\[ \langle V(y), \dot{\gamma}_{xy}(1) \rangle - \langle V(x), \dot{\gamma}_{xy}(0) \rangle \geq \beta \| \dot{\gamma}_{xy}(t) \|^m \]

or

\[ \langle V(y), \dot{\gamma}_{xy}(0) \rangle + \beta \| \dot{\gamma}_{xy}(t) \|^m \leq \langle V(x), \dot{\gamma}_{xy}(1) \rangle \]

Let \( \langle V(y), \dot{\gamma}_{xy}(0) \rangle + \beta \| \dot{\gamma}_{xy}(t) \|^m \geq 0 \), then

\[ \langle V(x), \dot{\gamma}_{xy}(1) \rangle \geq 0. \]

Hence, \( V \) is strongly pseudomonotone of order \( m \).
4 Variational Inequality Problem

Let $M$ be a complete Riemannian manifold and $A \subseteq M$ be a non empty set of $M$. Let $\Gamma^A_{x,y}$ denotes the collection of all geodesics from $x$ to $y$ such that $\gamma_{xy} \in A$. Suppose that $T = (T_1, T_2, ..., T_k)$, where $T_i : A \to 2^{TM}$ be a set-valued vector field on $A$. The variational inequality problem is to find $x \in A$ and $v \in T_xM$ such that

$$\langle v, \dot{\gamma}_{xx}(0) \rangle \not< 0 \quad \text{for all } x \in A,$$

where $\langle v, \dot{\gamma}_{xx}(0) \rangle = (\langle v_1, \dot{\gamma}_{x}(0) \rangle, \langle v_2, \dot{\gamma}_{x}(0) \rangle, ..., \langle v_k, \dot{\gamma}_{x}(0) \rangle), v_i \in T_iM, i = 1, 2, ..., k$.

The multiobjective optimization problem (MOP) is to find a strict minimizer of order $m$ for

$$\text{minimize } f(x) = (f_1(x), f_2(x), ..., f_k(x)), \quad x \in A$$

**Theorem 4.1.** Let $f_i, i = 1, 2, ..., k$ be strongly convex of order $m$ on $A$. Then $\bar{x} \in A$ is the solution of VIP with $T_{\bar{x}}M = \nabla f_i(\bar{x}), i = 1, 2, ..., k$, iff $\bar{x}$ is a strict minimizer of order $m$ for the MOP.

**Proof.** Suppose $\bar{x}$ is the solution of VIP but is not a strict minimizer of order $m$ for MOP. Then for $c > 0$, there exists some $x^* \in A$, such that

$$f(x^*) < f(\bar{x}) + c \| \dot{\gamma}_{x^*}(0) \|^m$$

or

$$f_i(x^*) < f_i(\bar{x}) + c_i \| \dot{\gamma}_{x^*}(0) \|^m, \quad i = 1, 2, ..., k.$$

Since $f_i, i = 1, 2, ..., k$, are strongly geodesic convex of order $m$ on $A$, the above inequality implies

$$\langle v_i, \dot{\gamma}_{x^*}(0) \rangle < 0, \quad \forall \, v_i \in T_{x^*}M = \nabla f_i(\bar{x}), \quad i = 1, 2, ..., k,$$

that is

$$\langle v, \dot{\gamma}_{x^*}(0) \rangle < 0 \quad \forall \, v \in T_{x^*}M, \quad x^* \in A,$
which contradicts the assumption that $\overline{x}$ is the solution of the VIP.

Conversely, let $\overline{x}$ be a strict minimizer of order $m$ for (MOP) but is not a solution of (VIP). Therefore, there exists an $x^* \in A$ such that

$$\langle v_i, \dot{\gamma}(x^*)(0) \rangle < 0, \quad \forall \ v_i \in T_M = \nabla f_i(\overline{x}), \ i = 1, 2, ..., k.$$

Using the definition of strongly geodesic convexity of order $m$ for $f_i, i = 1, 2, ..., k$, we get

$$f_i(x^*) - f_i(\overline{x}) < c \| \dot{\gamma}(t) \|^m,$$

which contradicts the strict minimizer condition. Consequently, $\overline{x}$ is not a strict minimizer of order $m$ for (MOP) and hence $\overline{x}$ is a solution of (VIP).

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