SUBGROUP CONGRUENCES FOR GROUPS OF PRIME POWER ORDER

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Abstract. Given a $p$-group $G$ and a subgroup-closed class $\mathcal{X}$, we associate with each $\mathcal{X}$-subgroup $H$ certain quantities which count $\mathcal{X}$-subgroups containing $H$ subject to further properties. We show in Theorem I that each one of the said quantities is always $\equiv 1 \pmod{p}$ if and only if the same holds for the others. In Theorem II we supplement the above result by focusing on normal $\mathcal{X}$-subgroups and in Theorem III we obtain a sharpened version of a celebrated theorem of Burnside relative to the class of abelian groups of bounded exponent. Various other corollaries are also presented.

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1. Introduction and main results

The starting point of the work reported here was a theorem announced by Lior Yanovski. Yanovski wrote an e-mail to the “group-pub-forum” online discussion group enquiring if anyone was aware of an already existing proof for the following:

If $G$ is a finite $p$-group, then the number of maximal abelian subgroups of $G$ is $\equiv 1 \pmod{p}$.

Some discussion ensued, but ultimately no one claimed to have seen this result before. Marty Isaacs asked Yanovski to share his proof with the other pubbers and Yanovski obliged. (This result can now be found in [Yan21].) Later, Isaacs went on to generalise Yanovski’s theorem by replacing the condition “maximal abelian subgroups” with the stronger “maximal abelian subgroups of exponent $\leq p^k$, where $p^k > 2$”. (This generalisation is the main result in [IY22].)

The present paper is the culmination of our desire to look deeper both into Yanovski’s method of proof by Möbius inversion and to provide a unified framework for counting subgroups with specified properties within $p$-groups. In particular, we work with general classes of groups $\mathcal{X}$ which are subgroup-closed. Our first main result is the following.

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Theorem I. Let $G$ be a finite $p$-group and let $\mathcal{X}$ be a subgroup-closed class of finite groups. Then the following are equivalent:

(A) For each $\mathcal{X}$-subgroup $H$ of $G$, if $a_G(H) \neq 0$ then $a_G(H) \equiv 1 \pmod{p}$.

(B) For each $\mathcal{X}$-subgroup $H$ of $G$ we have that $b_G(H) \equiv 1 \pmod{p}$.

(C) For each $\mathcal{X}$-subgroup $H$ of $G$ and for all integers $k$ such that $|H| \leq p^k \leq \sup_G(H)$ we have that $c_{k,G}(H) \equiv 1 \pmod{p}$.

Here $a_G(H)$ is the number of $\mathcal{X}$-subgroups which contain $H$ as a maximal subgroup and $b_G(H)$ is the number of $\mathcal{X}$-subgroups which are maximal subject to containing $H$. In general, there may exist several subgroups which are maximal subject to containing $H$ and some of these certainly may have different orders. We write $\sup_G(H)$ to denote the smallest such order. Our notation $c_{k,G}(H)$ stands for the number of $\mathcal{X}$-subgroups which contain $H$ and lie at a given level, i.e. have fixed order $p^k$.

Our second main theorem acts as a supplement to the first and reads:

Theorem II. Let $G$ be a finite $p$-group and let $\mathcal{X}$ be a subgroup-closed class of finite groups. The group $G$ satisfies one (and thus all) of the conditions (A), (B), (C) in Theorem I if and only if $G$ satisfies the following two conditions:

1. Every maximal normal $\mathcal{X}$-subgroup of $G$ is also a maximal $\mathcal{X}$-subgroup of $G$.

2. For each normal $\mathcal{X}$-subgroup $H$ of $G$ and for all integers $k$ such that $|H| \leq p^k \leq \nsup_G(H)$ we have that $c_{k,G}(H) \equiv 1 \pmod{p}$.

Our notation $\nsup_G(H)$ is the appropriate analogue of $\sup_G(H)$ for normal subgroups. All notational conventions and details of our set-up will be reiterated and further explained in the next section.

In the last section of our paper we demonstrate how our main theorems may be used jointly with certain auxiliary results to recover the previously mentioned results of Yanovski and Isaacs, as well as a classical result due to Miller which guarantees the existence of abelian subgroups in a certain range. Moreover, we amplify a celebrated theorem due to Burnside. In particular, our last main result, which may reasonably be said to have some independent interest, asserts the following.

Theorem III. Suppose that $G$ is a group of order $p^n$ and exponent $p^e$. Let $A \leq G$ be maximal among the normal abelian subgroups of $G$ having exponent $p^s$, where $s \geq 2$. If $|A| = p^r$ then

$$n \leq \frac{1}{2}(r - s + 1)(r - s + 2e - 2) + r,$$

and thus

$$r \geq -(e - s + 1/2) + \sqrt{(e - s + 1/2)^2 + 2n}.$$
**Theorem I.** We are ready to prove Theorem I. The first step of the proof \((A) \rightarrow \text{well. We may therefore assume that} \) is not a maximal \(\mathfrak{X}\)-subgroup and \(H < K\).

Given a prime \(p\), a finite \(p\)-group \(G\) and an \(\mathfrak{X}\)-subgroup \(H\) of \(G\), we define the following quantities

- \(a_G(H) := \{|H < K \leq G : K \in \mathfrak{X} \text{ and } |K : H| = p\}|\) so that \(a_G(H)\) is the number of minimal overgroups of \(H\) which are \(\mathfrak{X}\)-groups;
- \(b_G(H) := \{|H \leq K \leq G : K \text{ is a maximal } \mathfrak{X}\text{-subgroup}\}|\) so that \(b_G(H)\) is the number of maximal \(\mathfrak{X}\)-subgroups of \(G\) which contain \(H\);
- \(c_{k,G}(H) := \{|H \leq K \leq G : K \in \mathfrak{X} \text{ and } |K| = p^k\}|\) so that \(c_{k,G}(H)\) is the number of \(\mathfrak{X}\)-subgroups of \(G\) which contain \(H\) and have order \(p^k\);
- \(\sup_G(H) := \min\{|K : H \leq K \text{ and } K \text{ is a maximal } \mathfrak{X}\text{-subgroup}\}\); in other words, \(\sup_G(H)\) is the “lowest level” at which a maximal \(\mathfrak{X}\)-subgroup of \(G\) containing \(H\) occurs.
- \(\nsup_G(H) := \min\{|K : H \leq K \text{ and } K \text{ is a maximal normal } \mathfrak{X}\text{-subgroup}\}\); in other words, \(\nsup_G(H)\) is the “lowest level” at which a maximal normal \(\mathfrak{X}\)-subgroup of \(G\) containing \(H\) occurs.

Before we begin with the proof of Theorem I we record some preliminary observations.

**Remark 1.** For \(G\) and \(H\) as above we have \(a_G(H) = a_N(H)\), where \(N = N_G(H)\). This is easily seen as every \(\mathfrak{X}\)-subgroup of \(G\) directly above \(H\) (if there are any) normalises \(H\).

**Remark 2.** It is clear that in \((A)\) we need to distinguish between the cases \(a_G(H) = 0\) and \(a_G(H) \neq 0\) since if \(H\) is already a maximal \(\mathfrak{X}\)-subgroup of \(G\) then we have that \(a_G(H) = 0\).

**Remark 3.** If \(H\) is an \(\mathfrak{X}\)-subgroup of \(G\) and has order \(|H| = p^m\) then \(a_G(H)\) is simply \(c_{m+1,G}(H)\) (which, in turn, will be 0 in case \(H\) is a maximal \(\mathfrak{X}\)-subgroup of \(G\)).

**Remark 4.** Assuming that \(M\) is a maximal \(\mathfrak{X}\)-subgroup of \(G\) which contains \(H\), it is clear that \(\sup_G(H) \leq |M| = \sup_M(H)\).

We are ready to prove Theorem I. The first step of the proof \(((A) \rightarrow (B))\) resembles Yanovski’s method of proof using Möbius inversion that can be found in [Yan21].

**Proof of Theorem I.** \((A) \rightarrow (B)\) We induce on the index \(|G : H|\). In case \(H = G\) we have (trivially) \(b_G(H) = 1\) and thus \(b_G(H) \equiv 1 \pmod{p}\). Observe that the exact same conclusion is reached if \(H\) is a maximal \(\mathfrak{X}\)-subgroup of \(G\), since \(b_G(H) = 1\) in that case as well. We may therefore assume that \(H\) is not a maximal \(\mathfrak{X}\)-subgroup. Denote by \(\mathcal{M}\) the set of maximal \(\mathfrak{X}\)-subgroups of \(G\) so that

\[
b_G(H) = \left|\{H \leq K \leq G : K \in \mathcal{M}\}\right|.
\]

Now let \(\mathcal{I}(G)\) be the set of subgroups of \(G\) and \(\delta_G : \mathcal{I}(G) \rightarrow \{0,1\}\) be the indicator function of the maximal \(\mathfrak{X}\)-subgroups of \(G\), i.e.

\[
\delta_G(K) = \begin{cases} 1, & \text{if } K \in \mathcal{M} \\ 0, & \text{otherwise.} \end{cases}
\]
Then \( b_G(H) = \sum_{H \leq K} \delta_G(K) \). Hence, by Möbius inversion, we get

\[
\delta_G(H) = \sum_{H \leq K} \mu(H, K) b_G(K). \tag{2.1}
\]

But \( \mu(H, H) = 1 \), while

\[
\mu(H, K) = \left\{ \begin{array}{ll}
(-1)^r p^{(r)}, & \text{if } H \leq K \text{ and } K/H \cong (C_p)^r \\
0, & \text{otherwise}
\end{array} \right.
\]

(cf. [KT84, Prop. 2.4]). Therefore equation (2.1) becomes

\[
\delta_G(H) = b_G(H) + \sum_{H < K} \mu(H, K) b_G(K) \equiv b_G(H) - \sum_{|K:H|=p} b_G(K) \pmod{p}; \tag{2.2}
\]

the congruence above holds true due to the fact that \( \mu(H, K) \equiv 0 \pmod{p} \) in case \( |K:H| \geq p^2 \). Equation (2.2) can now be rewritten as

\[
\delta_G(H) \equiv b_G(H) - \sum_{|K:H|=p} b_G(K) \equiv b_G(H) - \sum_{K \in \mathcal{X}} b_G(K) \pmod{p},
\]

where the second congruence follows from the fact that \( b_G(K) \) vanishes if \( K \not\in \mathcal{X} \). We conclude that

\[
b_G(H) \equiv \sum_{|K:H|=p} b_G(K) \pmod{p}
\]

since \( H \not\in \mathcal{M} \) and thus \( \delta_G(H) = 0 \). Notice that the number of terms in the sum above equals \( a_G(H) \) and that for each \( K \in \mathcal{X} \) such that \( |K:H| = p \) we have \( b_G(K) \equiv 1 \pmod{p} \) by the induction hypothesis applied to \( K \). Therefore

\[
b_G(H) \equiv a_G(H) \equiv 1 \pmod{p},
\]

where the last congruence holds by assumption.

(B) \( \rightarrow \) (C) If \( |H| = p^m \) the claim holds trivially for \( c_{m,G}(H) \), so we fix a \( k \) with \( p^m < p^k \leq \sup_G(H) \). We let \( \mathcal{M} = \{ M_1, \ldots, M_s \} \) be the set of maximal \( \mathcal{X} \)-subgroups of \( G \) which contain \( H \) and \( \mathcal{K} = \{ K_1, \ldots, K_t \} \) be the set of \( \mathcal{X} \)-subgroups of \( G \) which contain \( H \) and have order \( p^k \). We are assuming that \( s = b_G(H) \equiv 1 \pmod{p} \) and our goal is to prove that \( t = c_{k,G}(H) \equiv 1 \pmod{p} \). Note that every subgroup in \( \mathcal{M} \) contains some subgroup in \( \mathcal{K} \) and every subgroup in \( \mathcal{K} \) is contained in some subgroup in \( \mathcal{M} \).

For each \( i \) such that \( 1 \leq i \leq t \), let \( b_i \) be the number of subgroups in \( \mathcal{M} \) which contain \( K_i \). Observe that \( b_i = b_G(K_i) \) and thus \( b_i \equiv 1 \pmod{p} \) by assumption. Similarly, for each \( j \in \{ 1, \ldots, s \} \) let \( c_j \) be the number of members of \( \mathcal{K} \) which are contained in \( M_j \). Calculating the size of the set of ordered pairs \( (i, j) \) such that \( K_i \leq M_j \) in two ways yields

\[
\sum_{i=1}^{t} b_i = \sum_{j=1}^{s} c_j. \tag{2.3}
\]

At this point, observe that \( c_j = c_{k,M_j}(H) \). Since \( M_j \) is an \( \mathcal{X} \)-group, for every integer \( k \) with \( |H| \leq p^k \leq \sup_G(H) \leq |M| \) the quantity \( c_{k,M_j}(H) \) is simply the number of subgroups of \( M_j \) which have order \( p^k \) and contain \( H \). We now apply the standard result (cf. [Ber08, Thm. 5.14]) that the number of subgroups of a \( p \)-group at each level containing a given subgroup is \( \equiv 1 \pmod{p} \) to conclude that \( c_j \equiv 1 \pmod{p} \).
We look at the following sets of $X$.

We continue with the proof of our second main theorem.

We recall that $b_i \equiv 1 \pmod{p}$ for each $i \leq t$ and also $s \equiv 1 \pmod{p}$. Therefore the right-hand-side of (2.3) is congruent to 1 (mod $p$) and thus

$$1 \equiv \sum_{i=1}^{t} b_i \equiv t \pmod{p}.$$ 

Since $k$ was arbitrary, we have established the claim.

(C) $\rightarrow$ (A) In case $H$ is a maximal $X$-subgroup of $G$, we have $a_G(H) = 0$ trivially and thus (A) holds. Therefore we may assume that $\sup_G(H) > |H|$. Let $|H| = p^m$. Then $\sup_G(H) \geq p^{m+1}$, so by assumption

$$a_G(H) = c_{m+1,G}(H) \equiv 1 \pmod{p}$$

and thus (A) is true in all cases. Our proof is complete. $\blacksquare$

We continue with the proof of our second main theorem.

**Proof of Theorem II.** Assume first that $G$ satisfies (A), (B), (C) in Theorem I.

For the proof of (1), assume that $H$ is a maximal normal $X$-subgroup $H$ of $G$. According to condition (A) either $a_G(H) = 0$ or $a_G(H) \equiv 1 \pmod{p}$. The latter is not possible however, since it is easy to see that if $a_G(H) \not= 0$ then $a_G(H)$ is congruent (mod $p$) to the number of normal $X$-subgroups directly above it. Thus if $H$ is a maximal normal $X$-subgroup of $G$ it must be the case that $a_G(H) = 0$ and thus $H$ is a maximal $X$-subgroup.

It remains to show (2). Let $H \trianglelefteq G$ with $|H| = p^m$. If $H$ is a maximal normal $X$-subgroup then $c_{m,G}(H) = 1$ and (2) holds trivially with $k = m$. So we may assume that $H$ is not a maximal normal $X$-subgroup. Hence $p^m < \sup_G(H)$. Clearly for $k = m$ we have $c_{m,G}(H) = 1$ and the claim holds. So we fix $k$ with $p^m < p^k \leq \sup_G(H)$.

We look at the following sets of $X$-subgroups of $G$ that contain $H$

1. $\mathcal{M} = \{M_1, \ldots, M_t\}$ consists of maximal normal $X$-subgroups of order $p^k$. So if $p^k < \sup_G(H)$ we have $t = 0$.
2. $\mathcal{K} = \{K_1, \ldots, K_n\}$ consists of non-maximal $X$-subgroups of order $p^k$.
3. $\mathcal{L} = \{L_1, \ldots, L_s\}$ consists of non-normal maximal $X$-subgroups of order $\leq p^k$. The first $s_1$ of them $\{L_1, \ldots, L_{s_1}\}$ are of order exactly $p^k$.
4. $\mathcal{T} = \{T_1, \ldots, T_w\}$ consists of maximal $X$-subgroups of order $> p^k$.

First observe that if $S$ is an $X$-subgroup of $G$ that contains $H$ then the same holds for any $G$-conjugate $S^g$ of $S$. In addition $S$ is maximal if and only if $S$ is maximal. Hence the number $s_1$ of non-normal maximal $X$-subgroups of order $p^k$ is a multiple of $p$ as it is a union of conjugacy classes of subgroups that are not normal. The same holds for $s$ the total number of non-normal maximal $X$-subgroups of order $\leq p^k$ that contain $H$.

Furthermore, the definition of $c_{k,G}(H)$ implies that $c_{k,G}(H) = u + t + s_1$. Hence the previous observation implies

$$c_{k,G}(H) = u + t + s_1 \equiv u + t \pmod{p} \quad (2.4)$$

Now the total number $b_G(H)$ of maximal $X$-subgroups that contain $H$ is $s + t + w$. In addition, $b_G(H)$ is congruent to 1 (mod $p$), by hypothesis. Hence

$$1 \equiv b_G(H) = s + t + w \equiv t + w \pmod{p} \quad (2.5)$$
where the last equivalence holds because \( s \equiv 0 \pmod{p} \).

Let \( b_i = |\{ T_j \in \mathcal{T} \mid K_i \subseteq T_j \}| \) and \( c_j = |\{ K_i \in \mathcal{K} \mid K_i \subseteq T_j \}| \) for all \( i = 1, \ldots, u \) and all \( j = 1, \ldots, w \). The fact that every maximal normal \( \mathfrak{X} \)-subgroup is a maximal subgroup implies that none of the \( T_j \) contains any of the \( M_i \). So \( b_i, c_j \neq 0 \) and in addition,

\[
\sum_{i=1}^{u} b_i = \sum_{j=1}^{w} c_j. \tag{2.6}
\]

Observe that \( b_i = b_{G}(K_i) \) and thus \( b_i \equiv 1 \pmod{p} \) by assumption, since \( G \) satisfies Condition B. Also, \( c_j = c_{k,T_j}(H) \). But \( T_j \) is an \( \mathfrak{X} \)-group, thus the number of \( \mathfrak{X} \)-subgroups of \( T_j \) of order \( p^k \) containing \( H \) is equal to the total number of subgroups of \( T_j \) of order \( p^k \) containing \( H \). The latter is \( \equiv 1 \pmod{p} \) according to Theorem 5.14 in \([\text{Ber}08, \text{Thm. 5.14}]\). Hence

\[
c_j = c_{k,T_j}(H) \equiv 1 \pmod{p}.
\]

We conclude that equation (2.6) implies

\[
u \equiv w \pmod{p}
\]

This along with equations (2.4) and (2.5) provide

\[
c_{k,G}(H) \equiv u + t \equiv w + t \equiv b_{G}(H) \equiv 1 \pmod{p}.
\]

For the other direction, assume that \( G \) satisfies (1) and (2). We will show, using induction on \( |G : H| \), that (A) holds. The base case \( G = H \) is trivially true. If \( N_{G}(H) < G \), then since \( a_{G}(H) = a_{N}(H) \), where \( N = N_{G}(H) \), the claim follows by the induction hypothesis applied to \( H \) with respect to \( N \). We may thus assume that \( H \) is normal in \( G \). Suppose \( |H| = p^k \). If \( \text{nsup}_{G}(H) > |H| \), then \( c_{k+1,G} \equiv 1 \pmod{p} \) whence \( a_{G}(H) \equiv 1 \pmod{p} \). There only remains to consider the case \( \text{nsup}_{G}(H) = |H| \), i.e. when \( H \) is a maximal normal \( \mathfrak{X} \)-subgroup of \( G \). But then the assumption that every maximal normal \( \mathfrak{X} \)-subgroup of \( G \) is a maximal \( \mathfrak{X} \)-subgroup of \( G \) forces \( H \) to be a maximal \( \mathfrak{X} \)-subgroup of \( G \) and thus \( a_{G}(H) = 0 \). This completes the induction and proves the claim.

\[\blacksquare\]

3. Abelian Subgroups

The existence of (maximal) normal abelian subgroups of fixed order in a finite \( p \)-group \( G \) has been studied by several authors, see for example \([\text{Gla}06], [\text{GM}10], [\text{Ber}98]\). In this section we wish to explore how our Theorems I and II can be applied to the class of abelian groups of given order and bounded exponent.

We begin with the following.

**Theorem** ([\text{Ber}98, Thm. 1]). Let \( A \prec B \prec G \), where \( A, B \) are abelian subgroups of a \( p \)-group \( G \), \( |B : A| = p \), \( \exp(B) \leq p^k \) and \( p^k > 2 \). Let \( \mathcal{A} \) be the set of all abelian subgroups \( T \) of \( G \) such that \( A \prec T \), \( |T : A| = p \) and \( \exp(T) \leq p^k \). Then \( |\mathcal{A}| = 1 \pmod{p} \).

This theorem of Berkovich says, essentially, that if \( p, k \) are such that \( p^k > 2 \), \( G \) is a finite \( p \)-group and \( H \) is a subgroup of \( G \), then either \( a_{G}(H) = 0 \) or \( a_{G}(H) \equiv 1 \pmod{p} \) relative to the class \( \mathfrak{X} \) of finite abelian \( p \)-groups of exponent \( \leq p^k \).

Since (A) in Theorem I holds for the class \( \mathfrak{X} \) of finite abelian \( p \)-groups of exponent \( \leq p^k \) for some fixed choice of \( k \) (subject only to the condition \( p^k > 2 \)), we see that (B) holds as well and thus we recover the main result in the recent paper of Isaacs and Yanovski.
Corollary I ([IY22, Thm. C]). Let $P$ be a $p$-group, and suppose that $e > 2$ is a power of $p$. Also, let $H \leq P$ be an abelian subgroup with exponent dividing $e$, and let $n$ be the number of subgroups $A$ of $P$ that contain $H$ and that are maximal with respect to the property that $A$ is abelian and has exponent dividing $e$. Then $n \equiv 1 \pmod{p}$.

Moreover, given a finite $p$-group $G$ of exponent $p^k$ we let $\mathfrak{X}$ be the class of finite abelian $p$-groups of exponent $\leq p^k$. Then (A) in Theorem I holds for the class $\mathfrak{X}$ and thus (B) holds. The maximal $\mathfrak{X}$-subgroups which contain a given $\mathfrak{X}$-subgroup $H$ of $G$ are then simply the maximal abelian subgroups of $G$ containing $H$. The cardinality of the first set is our quantity $b_G(H) \equiv 1 \pmod{p}$ and thus we recover Yanovski’s result mentioned in the introduction.

A celebrated theorem due to Burnside asserts that if $G$ is a $p$-group of order $p^n$ and $A$ is a maximal abelian normal subgroup of $G$, where $|A| = p^n$, then $n \leq \frac{s(s+1)}{2}$. Since this holds for all maximal abelian normal subgroups of $G$, we see that $\text{nsup}_G(1) \geq p^s$ with $s$ the least positive integer for which $n \leq \frac{s(s+1)}{2}$ holds relative to the class of finite abelian groups $\mathfrak{A} = \mathfrak{A}$.

Now every finite $p$-group satisfies Theorem I with $\mathfrak{X} = \mathfrak{A}$ and thus every finite $p$-group $G$ satisfies Theorem II. If $k \leq s$ therefore, then $k$ lies in the range for which (2) of Theorem II works. But

$$\frac{k(k-1)}{2} \leq \frac{s(s-1)}{2} < n$$

where the last inequality is a consequence of the minimality of $s$. We deduce that $(\frac{k}{2}) < n$ and thus we recover the following classic result due to Miller.

Corollary II ([Ber08, Thm. 13.12]). Suppose that $G$ is a group of order $p^n$, $n, k \in \mathbb{N}$ and $n > (\frac{k}{2})$. Then the number of abelian subgroups of order $p^k$ in $G$ is congruent to 1 (mod $p$).

The next theorem provides an analogue of Burnside’s theorem for maximal abelian normal subgroups of given exponent.

In order to prove the theorem, we need the following auxiliary result which is essentially Lemma 1 in [Laf80]. We follow the notation in [Laf80] and write

$$\Omega(G) = \begin{cases} 
\Omega_2(G), & \text{if } p = 2 \\
\Omega_1(G), & \text{if } p > 2
\end{cases}$$

for any $p$-group $G$. We also write $d(G)$ for the minimum number of generators of $G$.

Lemma I. Let $C$ be a $p$-group of exponent $\leq p^s$. Assume further that $\Omega_s(C) \leq \mathbb{Z}(C)$. Then $|C|/|\Omega_s(C)| \leq |\Omega_1(C)|^{-s}$ for every integer $s$ with $p^s > 2$.

Proof. First note that $\Omega(C) \leq \Omega_s(C) \leq \mathbb{Z}(C)$ for every integer $s$ with $p^s > 2$. Hence Corollary 1 of Lemma 1 in [Laf74] implies that $\Omega_i(C) = \{x \in C \mid x^{p^i} = 1\}$ for every $i \geq 0$. Therefore $\Omega_{i+1}(C)/\Omega_i(C)$ is an elementary abelian group for all $i \geq 1$. So

$$|\Omega_{i+1}(C)/\Omega_i(C)| \leq |\Omega_{i+1}(C)/\Phi(\Omega_{i+1}(C))| \leq d(\Omega_{i+1}(C)).$$

In addition, for every $i \geq 1$, we have

$$\Omega(\Omega_i(C)) \leq \Omega(C) \leq \Omega_s(C) \leq \mathbb{Z}(C).$$
Applying Corollary 2 in [Laf74] on $\Omega_i(C)$ we get
\[ d(\Omega_i(C)) \leq d(\Omega_1(\Omega_i(C))) = d(\Omega_1(C)), \]
where the last equality follows from the fact that $\Omega_1(\Omega_i(C)) = \Omega_1(C)$.

Now notice that if $d(\Omega_1(C)) = t$ then $|\Omega_1(C)| = p^t$. Hence for every $i \geq s$ we get $|\Omega_{i+1}(C)/\Omega_i(C)| \leq p^t$. As $C = \Omega_s(C)$ we need to repeat the argument $(e-s)$ times to get $|C|/|\Omega_s(C)| \leq p^{(e-s)}$, and the result follows. \hfill \qed

We are now ready to prove our last main theorem.

**Proof of Theorem III.** We denote by $d = d(A)$ the rank of the subgroup $A$ and begin with the observation that $d \leq r - s + 1$. This is so because $A = B \times D$ for some subgroups $B, D$ of $A$, where $B$ is cyclic of order $p^s$. Thus
\[ d(A) = d(D) + 1 \leq r - s + 1, \]
the inequality being a consequence of $p^d(D) \leq |D|$.

Our strategy will be to write
\[ |G| = |G : C| |C : A| |A|, \]
where $C := C_G(A)$, and to obtain bounds for the indices involved. As regards the quantity $|G : C|$, the $N/C$ Theorem tells us that $|G : C| \leq |\text{Aut}(A)|_p$, the $p$-part of $|\text{Aut}(A)|$. But
\[ |\text{Aut}(A)|_p \leq p^{\frac{1}{2}d(2r-d-1)} \leq p^{\frac{1}{2}(r-s+1)(2r-(r-s+1)-1)} = p^{\frac{1}{2}(r-s+1)(r+s-2)}. \]

The first inequality follows from the fact that $|\text{Aut}(A)|$ divides $p^{d(r-d)} \cdot \text{GL}_d(p)$ (cf. [Hup67, Satz 3.19]). To see why the second inequality is true, let $f(x) = \frac{1}{2}x(2r - x - 1)$ and note that
\[ f(d) - f(r - s + 1) = \frac{1}{2}(2 + d - r - s)(r - s + 1 - d) \leq 0; \]
we have $2 + d - r - s \leq 0$ since $2 \leq s$ and $d \leq r$, while $r - s + 1 - d \geq 0$ by our initial observation.

Regarding the quantity $|C : A|$, we use Lemma I to bound it. By a theorem of Alperin [Alp64] $\Omega_s(C) = A$ and thus $|C : A| \leq |\Omega_1(C)|^{e-s}$ by Lemma I. But
\[ |\Omega_1(C)| = p^{d(\Omega_1(C))} \leq p^d(A) \leq p^{e-s+1}. \]

The equality is a consequence of the fact that $\Omega_1(C) \leq \Omega_s(C) = A \leq Z(C)$.

The first inequality follows from the containment $\Omega_1(C) \leq \Omega_s(C) = A$ and the fact that $A$ is abelian. Thus
\[ |C : A| \leq (p^{r-s+1})^{e-s} = p^{(e-s)(r-s+1)}. \]

Then
\[ |G| = |G : C| |C : A| |A| \leq p^{\frac{1}{2}(r-s+1)(r+s-2)} p^{(e-s)(r-s+1)} p^r \]
\[ = p^{\frac{1}{2}(r-s+1)(r+s+2e-2)+r}, \]
which is clearly equivalent to what we wanted to prove.

Now, the inequality
\[ 2n \leq r^2 + r(2e - 2s + 1) + (s - 1)(s - 2e + 2). \]
along with the fact that for every \( e \geq 2 \) we get \((s - 1)(s - 2e + 2) \leq 0 \) (since \( s \leq e \leq 2e - 2 \)) implies
\[
2n \leq r^2 + r(2e - 2s + 1).
\]
Solving the quadratic for \( r \) yields \( r \geq -(e - s + 1/2) + \sqrt{(e - s + 1/2)^2 + 2n} \). This completes the proof of the theorem.

In Theorem III we have not addressed the case \( s = 1 \) which corresponds to the class of elementary abelian subgroups of a \( p \)-group. But this case has already been handled by Laffey in [Laf80]. The main result of that paper is the following.

**Theorem** ([Laf80, Thm. 1]). Let \( G \) be a finite \( p \)-group, and suppose that \( |G| = p^n \), \( \exp(G) = p^r \) and \( p^r = \min \{|A|\} \), where the minimum is taken over all maximal elementary abelian normal subgroups \( A \) of \( G \). Then
\[
n \leq \begin{cases} 
\frac{1}{2}r(3r - 1) + er, & \text{if } p = 2, \\
\frac{1}{2}r(r - 1) + er, & \text{if } p > 2.
\end{cases}
\]

The quantity \( p^r \) in Laffey’s result is precisely our \( n_{\sup G}(1) \) relative to the class \( \mathcal{Y} \) of elementary abelian \( p \)-groups. Moreover, Laffey’s inequality implies that (relative to the class \( \mathcal{Y} \))
\[
n_{\sup G}(1) = r \geq \begin{cases} 
-y + \sqrt{y^2 + 2n/3}, & \text{if } p = 2, \\
-y + \sqrt{y^2 + 2n}, & \text{if } p > 2
\end{cases}
\]
where
\[
y = \begin{cases} 
\frac{1}{16}(2e - 1) & \text{if } p = 2, \\
\frac{1}{12}(2e - 1) & \text{if } p > 2.
\end{cases}
\]

Thus we can apply Theorem II (2) to the class \( \mathcal{Y} \) (of elementary abelian subgroups) when \( H = 1 \) to deduce the following.

**Corollary III.** Suppose that \( G \) is a group of order \( p^n \) and exponent \( p^e \), while \( y \) is the function of \( e \) defined above. Then for all \( 1 \leq k \) such that
\[
k \leq \begin{cases} 
-y + \sqrt{y^2 + 2n/3}, & \text{if } p = 2, \\
-y + \sqrt{y^2 + 2n} & \text{if } p > 2
\end{cases}
\]
the number of elementary abelian subgroups of order \( p^k \) is \( \equiv 1 \) (mod \( p \)). In particular, there exists at least one elementary abelian normal subgroup of order \( p^k \) for all such \( k \).

Additionally, solving the inequality
\[
y + \sqrt{y^2 + 2n} \geq m
\]
allows one to conclude that a finite \( p \)-group \( G \) of odd order \( p^n \) and exponent \( p^e \) is guaranteed to have an elementary abelian normal subgroup of order \( p^k \) for all \( k \leq m \) provided that
\[
e \leq \frac{n}{m} - \frac{m - 1}{2}
\]
holds. Compare the corollary above with Theorems 6 and 7 in [Ber98].

4. SOME PROBLEMS AND QUESTIONS

In this final section we outline some open problems and questions, hoping thereby to stimulate further research on this topic.

The only instances of classes $\mathcal{X}$ for which Theorem I holds universally (i.e. for all finite $p$-groups) are subclasses of $\mathcal{A}$, the class of finite abelian groups. One natural problem here is the following:

**Problem 1.** Find other classes $\mathcal{X}$ which are not subclasses of $\mathcal{A}$ such that all finite $p$-groups satisfy Theorem I with respect to $\mathcal{X}$ or prove that no such classes exist.

**Question 2.** What can we conclude if a group $G$ satisfies $\sup_G(1) = n\sup_G(1)$?

Finally, is a certain “converse” true? That is:

**Question 3.** Suppose that $\mathcal{X}$ enjoys the following property: for all $p$-groups $G$, if a subgroup is a maximal normal $\mathcal{X}$-subgroup of $G$ then it is also a maximal $\mathcal{X}$-subgroup of $G$. Is it then true that every $p$-group $G$ satisfies the conclusions of Theorem I with respect to $\mathcal{X}$?

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