Freeness of spherical Hecke modules of unramified $U(2, 1)$ in characteristic $p$

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Abstract

Let $F$ be a non-archimedean local field of odd residue characteristic $p$. Let $G$ be the unramified unitary group $U(2, 1)(E/F)$ in three variables, and $K$ be a maximal compact open subgroup of $G$. For an irreducible smooth representation $\sigma$ of $K$ over $\overline{F}_p$, we prove that the compactly induced representation $\text{ind}_K^G \sigma$ is free of infinite rank over the spherical Hecke algebra $\mathcal{H}(K, \sigma)$.

1 Introduction

In the last two decades, the area of $p$-modular representations of $p$-adic reductive groups has already had a vast development. In their pioneering work (\cite{BL94}, \cite{BL95}), Barthel–Livné gave a classification of irreducible smooth $\overline{F}_p$-representations of the group $GL_2$ over a local field of residue characteristic $p$, leaving the supersingular representations as a mystery. Almost ten years later, C. Breuil (\cite{Bre03}) classified the supersingular representations of $GL_2(\mathbb{Q}_p)$, which was one of the starting points of the mod-$p$ and $p$-adic local Langlands program, initiated and developed by Breuil and many other mathematicians (see \cite{Bre10} for an overview). Recently, a classification of irreducible admissible mod-$p$ representations of $p$-adic reductive groups has been obtained by Abe–Henniart–Herzig–Vignéras (\cite{AHHV17}), as a generalization of many previous works due to these authors. However, our knowledge of the mysterious supersingular representations is still very limited, and from the work of Breuil and Paškūnas (\cite{BP12}) it seems that a ‘classification’ of that, even for the group $GL_2(F)$ with $F \neq \mathbb{Q}_p$, is out of reach.

Smooth representations induced from maximal compact open subgroups, and their associated spherical Hecke algebras, are central objects in the study
of $p$-modular representation theory of $p$-adic reductive groups, see [BL94], [Her11a], [Abe13], [AHHV17]. As such an induced representation is naturally a left module over its spherical Hecke algebra, a general question is to ask what we can say about the nature of this module? Certainly, we might not expect much without any restrictions on the group under consideration. The only general result in this direction, as far as we know, is due to Große-Klönnne ([GK14]), see Remark 4.2 for a precise description of his result. In this note, we investigate this question for the unitary group in three variables.

We start to describe our result in detail. Let $E/F$ be an unramified quadratic extension of non-archimedean local fields with odd residue characteristic $p$. Let $G$ be the unitary group $U(2,1)(E/F)$ defined over $F$, and $K$ be a maximal compact open subgroup of $G$. For an irreducible smooth representation $\sigma$ of $K$ over $F$, the compactly induced representation $\text{ind}_K^G \sigma$ is naturally a left module over the spherical Hecke algebra $\mathcal{H}(K, \sigma) := \text{End}_G(\text{ind}_K^G \sigma)$.

Our main result is as follows:

**Theorem 1.1.** (Theorem 4.1)

The compactly induced representation $\text{ind}_K^G \sigma$ is a free module of infinite rank over $\mathcal{H}(K, \sigma)$.

Theorem 1.1 is an analogue to a theorem of Barthel–Livné ([BL94, Theorem 19]) for the group $GL_2(F)$, and we follow their approach in general. The underlying idea is naïve and depends on an analysis of the Bruhat–Tits tree of the group $G$. However, there is an essential difference when we prove a key ingredient (Lemma 4.4) in our case, where we find a more conceptual approach and reduce it to some simple computations on the tree of $G$.

Our freeness result has some natural applications, and we record some of them in this note.

The first one is that every non-trivial spherical universal module of $G$ is infinite dimensional (Corollary 4.6), which at least implies the existence of supersingular representations of $G$ containing a given irreducible smooth representation of $K$. The existence of supersingular representations, to our knowledge, was only proved very recently for most simple adjoint $p$-adic group ([Vig17]).

Next, following Große-Klönnne ([GK14, section 9]), we apply Theorem 1.1 to investigate integral structures in certain $p$-adic locally algebraic representations of $G$, and we formulate a conditional result for irreducible tamely ramified principal series (Theorem 5.1).
This note is organized as follows. In section 2, we set up the general notations and review some necessary background on the group $G$ and its Bruhat–Tits tree. In section 3, we study the Hecke operator $T$ in detail, and describe the image of certain invariant subspace of $\text{ind}^G_K\sigma$ under $T$. In section 4, we prove our main result. In section 5, we apply our main result to investigate $G$-invariant norms in certain local algebraic representations of $G$. In the final appendix 6, we provide a detail proof of the recursion relations in the spherical Hecke algebra of $G$.

2 Notations and Preliminaries

Let $F$ be a non-archimedean local field of odd residue characteristic $p$, with ring of integers $\mathfrak{o}_F$ and maximal ideal $\mathfrak{p}_F$, and let $k_F$ be its residue field of cardinality $q = p^f$. Fix a separable closure $F_s$ of $F$. Let $E$ be the unramified quadratic extension of $F$ in $F_s$. We use similar notations $\mathfrak{o}_E, \mathfrak{p}_E, k_E$ for analogous objects of $E$. Let $\varpi_E$ be a uniformizer of $E$, lying in $F$. Given a 3-dimensional vector space $V$ over $E$, we identify it with $E^3$ (the usual column space in three variables), by fixing a basis of $V$. Equip $V$ with the non-degenerate Hermitian form $h:

$$h : V \times V \to E, \ (v_1, v_2) \mapsto v_1^T \beta v_2, \ v_1, v_2 \in V.$$ 

Here, $-$ denotes the non-trivial Galois conjugation on $E/F$, inherited by $V$, and $\beta$ is the matrix

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

The unitary group $G$ is the subgroup of $GL(3, E)$ whose elements fix the Hermitian form $h:

$$G = \{ g \in GL(3, E) \mid h(gv_1, gv_2) = h(v_1, v_2), \text{for any } v_1, v_2 \in V \}.$$ 

Let $B = HN$ (resp, $B' = HN'$) be the subgroup of upper (resp, lower) triangular matrices of $G$, where $N$ (resp, $N'$) is the unipotent radical of $B$ (resp, $B'$) and $H$ is the diagonal subgroup of $G$. Denote an element of the following form in $N$ and $N'$ by $n(x, y)$ and $n'(x, y)$ respectively:
where \((x, y) \in E^2\) satisfies \(x\bar{x} + y + \bar{y} = 0\). Denote by \(N_k\) (resp, \(N'_k\)), for any \(k \in \mathbb{Z}\), the subgroup of \(N\) (resp, \(N'\)) consisting of \(n(x, y)\) (resp, \(n'(x, y)\)) with \(y \in p^k_E\). For \(x \in E^\times\), denote by \(h(x)\) an element in \(H\) of the following form:

\[
\begin{pmatrix}
  x & 0 & 0 \\
  0 & -\bar{x}^{-1} & 0 \\
  0 & 0 & \bar{x}^{-1}
\end{pmatrix}.
\]

We record the following useful identity in \(G\): for \(y \neq 0\),

\[
\beta n(x, y) = n(\bar{y}^{-1}x, y^{-1}) \cdot h(\bar{y}^{-1}) \cdot n'(\bar{y}^{-1}x, y^{-1}).
\]

(1)

Up to conjugacy, the group \(G\) has two maximal compact open subgroups \(K_0\) and \(K_1\) ([Hij63],[Tit79]), which are given by:

\[
K_0 = \left( \begin{array}{ccc} o_E & o_E & o_E \\ o_E & o_E & o_E \\ o_E & o_E & o_E \end{array} \right) \cap G, \quad K_1 = \left( \begin{array}{ccc} o_E & o_E & p_E^{-1} \\ p_E & o_E & o_E \\ p_E & p_E & o_E \end{array} \right) \cap G.
\]

The maximal normal pro-\(p\) subgroups of \(K_0\) and \(K_1\) are respectively:

\[
K_0^1 = 1 + \mathfrak{w}_EM_3(o_E) \cap G, \quad K_1^1 = \left( \begin{array}{ccc} 1 + p_E & o_E & o_E \\ p_E & 1 + p_E & o_E \\ p_E^2 & p_E & 1 + p_E \end{array} \right) \cap G.
\]

Let \(\alpha\) be the following diagonal matrix in \(G\):

\[
\begin{pmatrix}
  \mathfrak{w}_E^{-1} & 0 & 0 \\
  0 & 1 & 0 \\
  0 & 0 & \mathfrak{w}_E
\end{pmatrix},
\]

and put \(\beta' = \beta \alpha^{-1}\). Note that \(\beta \in K_0\) and \(\beta' \in K_1\).

Let \(K\) be one of the two maximal compact open subgroups of \(G\) above, and \(K^1\) be the maximal normal pro-\(p\) subgroup of \(K\). We identify the finite group \(\Gamma_K = K/K^1\) with the \(k_F\)-points of an algebraic group defined over \(k_F\), denoted also by \(\Gamma_K\): when \(K = K_0\), \(\Gamma_K\) is \(U(2,1)(k_E/k_F)\), and when \(K = K_1\), \(\Gamma_K\) is \(U(1,1) \times U(1)(k_E/k_F)\). Let \(\mathbb{B}\) (resp, \(\mathbb{B}'\)) be the upper (resp, lower) triangular subgroup of \(\Gamma_K\), and \(\mathbb{U}\) (resp, \(\mathbb{U}'\)) be its unipotent radical. The Iwahori subgroup \(I_K\) (resp, \(I'_K\)) and pro-\(p\) Iwahori subgroup \(I_{1,K}\) (resp, \(I'_{1,K}\)) in \(K\) are the preimages of \(\mathbb{B}\) (resp, \(\mathbb{B}'\)) and \(\mathbb{U}\) (resp, \(\mathbb{U}'\)) in \(K\). We have the following Bruhat decomposition for \(K\):

\[
\text{4}
\]
\[ K = I \cup I\beta K, \]

where \( \beta_K \) denotes the unique element in \( K \cap \{ \beta, \beta' \} \), \( I \) is either \( I_K \) or \( I'_K \).

We end this part by recalling some facts on the Bruhat–Tits tree \( \triangle \) of \( G \). Denote by \( X_0 \) the set of vertices of \( \triangle \), which consists of all \( \mathfrak{o}_E \)-lattices \( \mathcal{L} \) in \( E^3 \), such that

\[ \varpi_E \mathcal{L} \subseteq \mathcal{L}^* \subseteq \mathcal{L}, \]

where \( \mathcal{L}^* \) is the dual lattice of \( \mathcal{L} \) under the Hermitian form \( h \), i.e., \( \mathcal{L}^* = \{ v \in E^3 \mid h(v, \mathcal{L}) \in p_E \} \).

Let \( v, v' \) be two vertices in \( X_0 \) represented by \( \mathcal{L} \) and \( \mathcal{L}' \). The vertices \( v \) and \( v' \) are adjacent, if:

\[ \mathcal{L}' \subset \mathcal{L} \text{ or } \mathcal{L} \subset \mathcal{L}'. \]

When \( v \) and \( v' \) are adjacent, we have the edge \( (v, v') \) on the tree.

Let \( \{e_{-1}, e_0, e_1\} \) be the standard basis of \( E^3 \). We consider the following two lattices in \( E^3 \):

\[ \mathcal{L}_0 = \mathfrak{o}_E e_{-1} \oplus \mathfrak{o}_E e_0 \oplus \mathfrak{o}_E e_1, \quad \mathcal{L}_1 = \mathfrak{o}_E e_{-1} \oplus \mathfrak{o}_E e_0 \oplus p_E e_1. \]

Denote respectively by \( v_0, v_1 \) the vertices represented by \( \mathcal{L}_0 \) and \( \mathcal{L}_1 \), which are then adjacent. The group \( G \) acts on \( X_0 \) in a natural way with two orbits, i.e.,

\[ X_0 = \{ G \cdot v_0 \} \cup \{ G \cdot v_1 \}. \]

For \( i = 0, 1 \), the stabilizer of \( v_i \) in \( G \) is exactly the maximal open compact subgroup \( K_i \), and the stabilizer of the edge \( (v_0, v_1) \) is the intersection \( K_0 \cap K_1 \).

For a vertex \( v \in X_0 \), the number of vertices adjacent to \( v \) is equal to \( q^{c_v} + 1 \), where \( c_v \) is either 3 or 1, depending on \( v \in \{ G \cdot v_0 \} \) or \( \{ G \cdot v_1 \} \).

For a maximal compact open subgroup \( K \), we will write \( c_K \) for \( c_v \), if \( v \) is the unique vertex on the tree stabilized by \( K \).

Unless otherwise stated, all the representations of \( G \) and its subgroups considered in this note are smooth over \( \mathbb{F}_p \).
3 The spherical Hecke operator $T$

3.1 The spherical Hecke algebra $\mathcal{H}(K, \sigma)$

Let $K$ be a maximal compact open subgroup of $G$, and $(\sigma, W)$ be an irreducible smooth representation of $K$. As $K^1$ is pro-$p$, $\sigma$ factors through the finite group $\Gamma_K = K/K^1$, i.e., $\sigma$ is the inflation of an irreducible representation of $\Gamma_K$.

It is well-known that $\sigma_{I_1,K}$ and $\sigma'_{I_1,K}$ are both one-dimensional, and that the natural composition map $\sigma_{I_1,K} \rightarrow \sigma \rightarrow \sigma_{I_1,K}'$ is non-zero, i.e., an isomorphism of vector spaces ([CE04, Theorem 6.12]). Denote by $j_\sigma$ the inverse of the composition map just mentioned. For $v \in \sigma_{I_1,K}$, we have $j_\sigma(\bar{v}) = v$, where $\bar{v}$ is the image of $v$ in $\sigma_{I_1,K}'$. When viewed as a map in $\text{Hom}_{\mathfrak{F}_p}(\sigma, \sigma_{I_1,K})$, the $j_\sigma$ factors through $\sigma_{I_1,K}'$, i.e., it vanishes on $\sigma(\sigma'_{I_1,K})$.

**Remark 3.1.** There is a unique constant $\lambda_{\beta K, \sigma} \in \mathfrak{F}_p$, such that $\beta K \cdot v - \lambda_{\beta K, \sigma} v \in \sigma(\sigma'_{I_1,K})$, for $v \in \sigma_{I_1,K}$. The value of $\lambda_{\beta K, \sigma}$ is known: it is zero unless $\sigma$ is a character ([HV12, Proposition 3.16]), due to the fact that $\beta K \notin I_K \cdot I'_K$.

**Remark 3.2.** There are unique integers $n_K$ and $m_K$ such that $N \cap I_{1,K} = N_{n_K}$ and $N' \cap I_{1,K} = N'_{m_K}$.

Let $\text{ind}_K^G \sigma$ be the compactly induced smooth representation, i.e., the representation of $G$ with underlying space $S(G, \sigma)$

$$S(G, \sigma) = \{ f : G \rightarrow W \mid f(kg) = \sigma(k) \cdot f(g), \text{ for any } k \in K \text{ and } g \in G, \text{ locally constant with compact support} \}$$

and $G$ acting by right translation. In this note, we will sometimes call $\text{ind}_K^G \sigma$ a maximal compact induction.

As usual ([BL94, section 2.3]), denote by $[g, v]$ the function in $S(G, \sigma)$, supported on $Kg^{-1}$ and having value $v \in W$ at $g^{-1}$. An element $g' \in G$ acts on the function $[g, v]$ by $g' \cdot [g, v] = [g'g, v]$, and we have $[gk, v] = [g, \sigma(k)v]$ for $k \in K$.

The spherical Hecke algebra $\mathcal{H}(K, \sigma)$ is defined as $\text{End}_G(\text{ind}_K^G \sigma)$, and by [BL94, Proposition 5] it is isomorphic to the convolution algebra $\mathcal{H}_K(\sigma)$ of all compactly support and locally constant functions $\varphi$ from $G$ to $\text{End}_{\mathfrak{F}_p}(\mathfrak{F}_p)$, satisfying $\varphi(kgk') = \sigma(k)\varphi(g)\sigma(k')$ for any $g \in G$ and $k, k' \in K$. Let $\varphi$ be the function in $\mathcal{H}_K(\sigma)$, supported on $K\alpha K$, and satisfying $\varphi(\alpha) = j_\sigma$. Denote
by \( T \) the Hecke operator in \( \mathcal{H}(K, \sigma) \), which corresponds to the function \( \varphi \), via the isomorphism between \( \mathcal{H}_K(\sigma) \) and \( \mathcal{H}(K, \sigma) \).

When \( K \) is hyperspecial, the following proposition is a special case of a theorem of Herzig ([Her11b]).

**Proposition 3.3.** The algebra \( \mathcal{H}(K, \sigma) \) is isomorphic to \( F_p[T] \).

**Proof.** Here, we give a straightforward proof by explicit computations, and the recursion relations in the algebra will be used later.

It suffices to consider the algebra \( \mathcal{H}_K(\sigma) \). Recall the Cartan decomposition of \( G \):

\[
G = \bigcup_{n \geq 0} K\alpha^n K.
\]

Let \( \varphi \) be a function in \( \mathcal{H}_K(\sigma) \), supported on the double coset \( K\alpha^n K \). Then, for any \( k_1, k_2 \in K \), satisfying \( k_1 \alpha^n = \alpha^n k_2 \), we are given \( \sigma(k_1)\varphi(\alpha^n) = \varphi(\alpha^n)\sigma(k_2) \).

When \( n = 0 \), \( \varphi(Id) \) commutes with all \( \sigma(k) \). As \( \sigma \) is irreducible, by Schur’s Lemma \( \varphi(Id) \) is a scalar.

For \( n > 0 \), let \( k_1 \in N'_{2n-1+m_K} \). As \( k_1 \in K^1 \), \( \sigma(k_1) = 1 \). Now \( k_2 = \alpha^{-n}k_1\alpha^n \in N'_{m_K-1} \), and we have \( \varphi(\alpha^n) = \varphi(\alpha^n) \cdot \sigma(k_2) \). We see \( \varphi(\alpha^n) \) factors through \( \sigma_{l_1,K} \). Similarly, for \( k_1 \in N_{n_K} \), \( k_2 = \alpha^{-n}k_1\alpha^n \in K^1 \), and we get \( \sigma(k_1)\varphi(\alpha^n) = \varphi(\alpha^n) \), that is to say \( \text{Im}(\varphi(\alpha^n)) \in \sigma^{l_1,K} \). In other words, \( \varphi(\alpha^n) \) only differs from \( j_\sigma \) by a scalar.

For \( n \geq 0 \), let \( \varphi_n \) be the function in \( \mathcal{H}_K(\sigma) \), supported on \( K\alpha^n K \), determined by its value on \( \alpha^n \): \( \varphi_0(Id) = Id, \varphi_n(\alpha^n) = j_\sigma, n > 0 \).

**Proposition 3.4.** \( \{\varphi_n\}_{n \geq 0} \) consists of a basis of \( \mathcal{H}_K(\sigma) \), and they satisfy the following convolution relations: for \( n \geq 1, l \geq 0 \),

\[
\varphi_1 \ast \varphi_n(\alpha^l) = \begin{cases} 
0, & l \neq n, n+1; \\
\ c \cdot j_\sigma, & l = n; \\
\ j_\sigma, & l = n + 1,
\end{cases}
\]

where \( c \) is some constant depending on \( \sigma \).

**Proof.** The convolution formulae in the proposition give that \( \varphi_1 \ast \varphi_n = c \cdot \varphi_n + \varphi_{n+1} \), which will matter to us later. In particular, it follows that the algebra \( \mathcal{H}_K(\sigma) \) is commutative. We leave the proof to the appendix 6.

Denote by \( T_n \) the operator in \( \mathcal{H}(K, \sigma) \) which corresponds to \( \varphi_n \). We then have similar composition of relations among \( \{T_n\}_{n \geq 0} \), namely
\[ T_1 \cdot T_n = c \cdot T_n + T_{n+1}, \]

and the assertion in the proposition follows. \(\square\)

### 3.2 The formula \(T[Id, v_0]\)

Let \(v\) be a vector in \(V\), and by [BL94, (8)] we have

\[ T[Id, v] = \sum_{g \in G/K} [g, \varphi(g^{-1}) \cdot v]. \tag{3} \]

As \(\varphi\) is supported on the double coset \(K\alpha K = K\alpha^{-1}K\), we decompose \(K\alpha^{-1}K\) into right cosets of \(K\):

\[ K\alpha^{-1}K = \bigcup_{k \in K/(K \cap \alpha^{-1}K\alpha)} k\alpha^{-1}K, \]

and we need to identify \(K/(K \cap \alpha^{-1}K\alpha)\) with some simpler set. Note firstly that \(I'_K\) contains \(K \cap \alpha^{-1}K\alpha\), and that

\[ I'_K/(K \cap \alpha^{-1}K\alpha) \cong N_{nK+1}/N_{nK+2}, \]

where, as mentioned in Remark 3.2, \(n_K\) is the unique integer such that \(N \cap I_K = N_{nK}\).

Secondly, we note the coset decomposition of \(K\) with respect to \(I'_K\):

\[ K = I'_K \cup \bigcup_{u \in N_{nK}/N_{nK+2}} \beta_K u I'_K. \]

In all, we may identify \(K/(K \cap \alpha^{-1}K\alpha)\) with the following set

\[ (N_{nK+1}/N_{nK+2}) \cup \beta_K \cdot (N_{nK}/N_{nK+2}), \]

and hence with the following set:

\[ \beta_K \cdot (N_{nK+1}/N_{nK+2}) \cup (N_{nK}/N_{nK+2}). \]

Using the previous identification, the above equation (3) becomes:

\[ T[Id, v] = \sum_{u \in N_{nK+1}/N_{nK+2}} [\beta_K u \alpha^{-1}, j_\sigma(\beta_K) v] + \sum_{u \in N_{nK}/N_{nK+2}} [u \alpha^{-1}, j_\sigma(u^{-1}) v], \tag{4} \]

where we note that \(N_{nK+1} \subset K^1\).
Recall we have a Cartan–Iwahori decomposition:

\[ G = \bigcup_{n \in \mathbb{Z}} K\alpha^n I_{1,K}. \quad (5) \]

Based on (5), we may describe the \( I_{1,K} \)-invariant subspace of \( \text{ind}^G_K \sigma \). By Frobenius reciprocity and an argument like that of [BL94, Proposition 5], we have \( (\text{ind}^G_K \sigma)^{I_{1,K}} = \{ f \in S(G, \sigma) \mid f(\kappa g i) = \sigma(k)f(g), \text{ for } k \in K, g \in G, i \in I_{1,K} \} \). Let \( f \) be a function in \( (\text{ind}^G_K \sigma)^{I_{1,K}} \), supported in \( K\alpha^n I_{1,K} \), so \( f \) is determined by its value at \( \alpha^n \). For any \( k \in K \), \( i \in I_{1,K} \) such that \( k\alpha^n = \alpha^ni \), \( f(\alpha^n) \) should satisfy \( \sigma(k)f(\alpha^n) = f(\alpha^n) \).

For \( n \geq 0 \), and \( u \in N_{nK} \), we get \( \sigma(u)f(\alpha^n) = f(\alpha^n) \), that is to say \( f(\alpha^n) \) is fixed by \( N_{nK} \). Similarly, for a negative \( n \), and \( u' \in N \cap I_{1,K}' = N_{mK-1}' \), we have \( \sigma(u')f(\alpha^n) = f(\alpha^n) \), which implies that \( f(\alpha^n) \) is fixed by \( N \cap I_{1,K}' \). Note that \( \sigma^{I_{1,K}} = \sigma^{N_{nK}} \) and \( \sigma^{I_{1,K}'} = \sigma^{N_{mK-1}'} \), as \( I_{1,K} = N_{nK} \cdot K^1 \) and \( I_{1,K} = N_{mK-1}' \cdot K^1 \), where \( K^1 \) acts trivially on \( \sigma \).

Recall that the subspace of \( I_{1,K} \)-invariants in \( \sigma \) is one-dimensional, and from now on we fix a non-zero \( v_0 \in \sigma^{I_{1,K}} \) throughout this note.

Let \( f_n \) be the function in \( (\text{ind}^G_K \sigma)^{I_{1,K}} \), supported on \( K\alpha^{-n} I_{1,K} \), such that

\[ f_n(\alpha^{-n}) = \begin{cases} \beta_K \cdot v_0, & n > 0, \\ v_0 & n \leq 0. \end{cases} \]

By the remarks above, we have:

**Lemma 3.5.** The set of functions \( \{ f_n \mid n \in \mathbb{Z} \} \) consists of a basis of the \( I_{1,K} \)-invariants of the maximal compact induction \( \text{ind}^G_K \sigma \).

It is useful to rewrite the function \( f_n \) in terms of a canonical \( G \)-transition of \( f_0 = [Id, v_0] \).

\[
\begin{align*}
f_{-m} &= \sum_{u \in N_{nK}/N_{nK+2m}} [u\alpha^{-m}, v_0], & \text{for } m \geq 0; \\
f_n &= \sum_{u' \in N_{mK}/N_{mK+2n-1}} [u'\alpha^n, \beta_K \cdot v_0], & \text{for } n > 0.
\end{align*}
\]

We now record the following formula \( T([Id, v_0]), \) i.e., \( T \cdot f_0 \), and we will do such thing for all the other \( f_n \) in the next subsection.

**Proposition 3.6.** \( T \cdot f_0 = f_{-1} + \lambda_{\beta_K, \sigma} \cdot f_1. \)

**Proof.** In (4), we take \( v \) as \( v_0 \). Then the first sum in (4) becomes \( \lambda_{\beta_K, \sigma} f_1 \), and the second sum is just the function \( f_{-1} \), as the group \( N_{nK} \) fixes \( v_0 \). We are done. \( \square \)

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3.3 The formula $Tf_n$ for $n \neq 0$

The purpose of this part is to push Proposition 3.6 further.

For $n \geq 0$, denote by $R_n^+(\sigma)$ (resp, $R_n^-(\sigma)$) the subspace of functions in $\text{ind}_K^G \sigma$ which are supported in the coset $K\alpha^n I_K$ (resp, $K\alpha^{-(n+1)} I_K$). Both spaces are $I_K$-stable. In our former notations, we indeed have $K\alpha^n I_K = K\alpha^n N_{nK}$, and $K\alpha^{-(n+1)} I_K = K\alpha^{-(n+1)} N'_{mK}$, for $n \geq 0$. Then we may rewrite $R_n^+(\sigma)$ and $R_n^-(\sigma)$ as follows:

Lemma 3.7. For $n \geq 0$, $R_n^+(\sigma) = [N_{nK} \alpha^{-n}, \sigma]$;
For $n \geq 1$, $R_{n-1}^-(\sigma) = [N'_{mK} \alpha^n, \sigma]$.

Remark 3.8. Note that $f_{-n} \in R_n^+(\sigma)^{j_1,\kappa}$ for $n \geq 0$, and $f_n \in R_{n-1}^-(\sigma)^{j_1,\kappa}$ for $n \geq 1$. From Lemma 3.5 and its argument, both $R_n^+(\sigma)^{j_1,\kappa}$ and $R_{n-1}^-(\sigma)^{j_1,\kappa}$ are one dimensional, hence they are generated by $f_{-n}$ and $f_n$ respectively.

Naturally we are interested in how the above $I_K$-subspaces are changed under the Hecke operator $T$, and the following is the first observation.

Proposition 3.9. (1). $T(R_0^+(\sigma)) \subseteq R_1^+(\sigma) \oplus R_0^-(\sigma)$.
(2). $T(R_n^+(\sigma)) \subseteq R_{n+1}^+(\sigma) \oplus R_n^-(\sigma) \oplus R_{n+1}^-(\sigma)$, $n \geq 1$.
(3). $T(R_n^-(\sigma)) \subseteq R_{n+1}^-\sigma) \oplus R_n^-(\sigma) \oplus R_{n+1}^-(\sigma)$, $n \geq 0$.

Here, we have put $R_{-1}^-(\sigma) = R_0^-(\sigma)$.

Proof. This Proposition can be roughly seen from the tree of $G$, but we want to make the inclusions in the statement more precisely, using the formula (4).

For (1), by the formula (4), we only need to note $\beta_K u\alpha^{-1} \in N'_{mK} \alpha K$ for $u \in N_{nK+1}$.

For (2), as $n \geq 1$, we note firstly that $\alpha^{-n} u\alpha^{-1} \in N_{nK} \alpha^{-(n+1)}$ for $u \in N_{nK}$.

It remains to check the following, which completes the argument of (2):

$$\alpha^{-n} \beta_K u\alpha^{-1} \in \begin{cases} \alpha^{1-n} K, & u \in N_{nK+2}; \\
N_{nK} \alpha^{-n} K, & u \in N_{nK+1} \setminus N_{nK+2}. \end{cases}$$

For (3), let $n$ be a non-negative integer. At first, we see

$$\alpha^{n+1} \beta_K u\alpha^{-1} \in N'_{mK} \alpha^{n+2} K,$$

for $u \in N_{nK+1}/N_{nK+2}$. Next, we check the following, which finishes the proof of (3):
\[ \alpha^{n+1} u \alpha^{-1} \in \begin{cases} 
 N'_m \alpha^n K, & u \in N_{nK+2}; 
 N'_m \alpha^{n+1} K, & u \in N_{nK+1} \setminus N_{nK+2}; 
 N'_m \alpha^{n+2} K, & u \in N_{nK} \setminus N_{nK+1}. 
 \end{cases} \]

Remark 3.10. The argument tells us more: for \( f \in R^+_n(\sigma) \), we can indeed detect the parts of \( T f \) which lie in \( R^+_n(\sigma) \) and \( R^+_{n+1}(\sigma) \).

Corollary 3.11. For \( n \in \mathbb{Z} \setminus \{0\} \), we have
\[ T \cdot f_n = c_n \cdot f_n + f_{n+\delta_n}, \]
for some constant \( c_n \), and \( \delta_n \) is given by
\[ \delta_n = \begin{cases} 
 1 & n > 0; 
 -1 & n < 0. 
 \end{cases} \]

Proof. Suppose \( n = -m \) is a negative integer, and we will prove that \( T f_{-m} = c_{-m} + f_{-m-1} \)
for some \( c \in \overline{\mathbb{F}}_p \).

By (2) of Proposition 3.9, \( T f_{-m} \in R^+_{m-1}(\sigma) \oplus R^+_m(\sigma) \oplus R^+_{m+1}(\sigma) \). As \( f_{-m} \)
is \( I_{1,K} \)-invariant, and \( T \) preserves \( I_{1,K} \)-invariants, Lemma 3.5 and Remark 3.8 imply that
\[ T f_{-m} = c_{m-1} f_{m+1} + c_m f_{-m} + c_{m+1} f_{-m-1}, \]
for some \( c_i \in \overline{\mathbb{F}}_p \).

We need to evaluate the function \( T f_{-m} \) at \( \alpha^{m-1} \) and \( \alpha^{m+1} \). Recall that \( f_{-m} = \sum_{u \in N_{nK}/N_{nK+2m}} u \alpha^{-m} \cdot f_0 \), and by Proposition 3.6 we get:
\[ T f_{-m} = \sum_{u \in N_{nK}/N_{nK+2m}} u \alpha^{-m}(f_{-1} + \lambda_{\sigma} \alpha f_1). \]

We need to estimate in which Cartan–Iwahori double cosets the elements \( \alpha^{m-1} u \alpha^{-m} \) and \( \alpha^{m+1} u \alpha^{-m} \) might belong, for \( u \in N_{nK}/N_{nK+2m} \).

We have firstly that:
\[ \alpha^{m-1} u \alpha^{-m} \in \begin{cases} 
 K \alpha^{-1} I_K, & u \in N_{2m-2+nK}; 
 K \alpha^{(u)} I_K, & u \in N_{nK} \setminus N_{2m-2+nK}. 
 \end{cases} \]
Here, in the second inclusion above, \( l(u) \) is some integer smaller than \(-1\), depending on \( u \). To see that, let \( c(u) \) be the largest integer such that \( u \in N_{c(u)} \), and the assumption \( u \in N_{n_K} \setminus N_{2m-2+n_K} \) means that \( d(u) := c(u) - (2m - 2) < n_K \). We now apply the equality (1):

\[
\alpha^{m-1}u\alpha^{-m} = \beta \cdot \beta (\alpha^{m-1}u\alpha^{1-m}) \cdot \alpha^{-1} = \beta \cdot u_1 \alpha^{d(u)}i_2 \cdot \alpha^{-1},
\]

where \( u_1 \in N_{-d(u)} \), \( i_2 \in I_K \cap B' \). The last expression of above identity gives us that

\[
\alpha^{m-1}u\alpha^{-m} \in K\alpha^{d(u)-1}I_K,
\]

and the assertion on \( l(u) \) follows.

Now we may determine the value of \( c_{m-1} = Tf_{-m}(\alpha^{m-1}) \). The list above immediately gives that \( f_{-1}(\alpha^{m-1}u\alpha^{-m}) = 0 \), for any \( u \in N_{n_K} / N_{n_K + 2m} \). As \( f_1 \) is supported on \( K\alpha^{-1}I_K \), the above list reduces us to look at the sum

\[
\sum_{u \in N_{2m-2+n_K} / N_{2m+n_K}} f_1(\alpha^{m-1}u\alpha^{-m}) = \sum_{u_1 \in N_{n_K} / N_{n_K + 2m}} u_1 \beta_K v_0,
\]

which is clearly zero by splitting it as a double sum, observing that \( N_{n_K + 1} \subseteq K^1 \). In all we have shown \( c_{m-1} = 0 \).

In a similar way, we have

\[
\alpha^{m+1}u\alpha^{-m} \in \begin{cases} K\alpha I_K, & u \in N_{2m+n_K}; \\ K\alpha^{l'(u)}I_K, & u \in N_{n_K} \setminus N_{2m+n_K}. \end{cases}
\]

Here, in the second inclusion above, \( l'(u) \) is some integer smaller than \(-1\), depending on \( u \), which is seen by applying (1) again.

Therefore, we have \( f_1(\alpha^{m+1}u\alpha^{-m}) = 0 \), for any \( u \in N_{n_K} / N_{n_K + 2m} \), and the following

\[
f_{-1}(\alpha^{m+1}u\alpha^{-m}) = \begin{cases} v_0, & u \in N_{2m+n_K}; \\ 0, & u \in N_{n_K} \setminus N_{2m+n_K}. \end{cases}
\]

In summary, we have proved \( c_{m+1} = 1 \).

The exact value of \( c_m \) will not be used in the paper, so we don’t record it here\(^1\).

The other half of the corollary can be dealt in a completely similar manner, and we are done here.

\(^1\)The exact value of \( c_m \) depends on the representation \( \sigma \): it is non-zero only if \( \sigma \) is a character (see [Xu14, 3.7]).
Remark 3.12. Among other things, what matters to us of the above corollary is that the coefficient of $f_{n+δ_n}$ is 1, especially it is non-zero.

4 Freeness of spherical Hecke modules

In this section, we prove the main result of this note, as an application of Proposition 3.6 and Corollary 3.11.

Throughout this section, let $K$ be a maximal compact open subgroup of $G$, and $σ$ be an irreducible smooth representation of $K$.

Theorem 4.1. The maximal compact induction $\text{ind}_K^Gσ$ is free of infinite rank over $\mathcal{H}(K, σ)$.

Before we continue, we give some remarks on related works in the literature.

Remark 4.2. In $[GK14]$, for a split group $G$ over $F$, Elmar Große-Klönne has proved the spherical universal module of $G = G(F)$ with trivial coefficients is free, under the condition that $F$ is $\mathbb{Q}_p$, and that $G$ has connected center, and that the Coxeter number of $G$ is $p$-small ($[GK14$, Corollary 6.1, Theorem 8.2$]$).

Remark 4.3. For a unramified and adjoint-type $p$-adic group (e.g., $\text{PGL}_n(F)$), Xavier Lazarus conjectured in $[Laz99]$ that over an algebraic closed field of characteristic different from $p$, a maximal compact induction from the trivial character is flat over the corresponding spherical Hecke algebra, and it was proved by Bellaiche–Otwinowska in $[BO03]$ for $\text{PGL}_3$ and $σ = 1^2$.

We start to prove Theorem 4.1. At first, we recall some general setting.

For $n \geq 0$, denote by $B_{n,σ}$ the set of functions in $\text{ind}_K^Gσ$ which are supported in the ball $B_n$ of the tree of radius $2n$ around the vertex $v_K$ (the unique vertex on $Δ$ stabilized by $K$). Let $C_{n,σ}$ be the set of functions in $\text{ind}_K^Gσ$ which are supported in the circle $C_n$ of radius $2n$ around the vertex $v_K$. Both the set $B_{n,σ}$ and $C_{n,σ}$ are indeed $K$-stable spaces, and we may write them in term of our former notations as:

$^2$In $[GK14$, Remark after Corollary 6.1$]$, Große-Klönne has pointed out that freeness is indeed proved in $[BO03]$, even only flatness is claimed.
\[ C_0,\sigma = R_0^+(\sigma), C_n,\sigma = R_n^+ (\sigma) \oplus R_{n-1}^-(\sigma), \text{ for } n \geq 1. \]
\[ B_{n,\sigma} = \bigoplus_{k \leq n} C_{k,\sigma}, \text{ for } n \geq 0. \]

We prefer to define \( B_{n,\sigma} \) and \( C_{n,\sigma} \) in terms of the tree, as it will make some formulation of later argument easier.

**Lemma 4.4.** For \( n \geq 0 \), let \( f \in B_{n+1,\sigma} \). If \( Tf \in B_{n+1,\sigma} \), then \( f \in B_{n,\sigma} \).

**Proof.** Denote by \( M_{n+1,\sigma} \) the subspace of \( B_{n+1,\sigma} \) consisting of functions \( f \) such that \( T f \in B_{n+1,\sigma} \). The assertion in the lemma means that \( M_{n+1,\sigma} \subseteq B_{n,\sigma} \).

Assume there exists an \( f \in M_{n+1,\sigma} \setminus B_{n,\sigma} \). As \( B_{n+1,\sigma} = B_{n,\sigma} \oplus C_{n+1,\sigma} \), we may write \( f \) uniquely as \( f' + f'' \), for some \( f' \in B_{n,\sigma} \) and some \( f'' \in C_{n+1,\sigma} \). We see firstly that \( f'' = f - f' \in M_{n+1,\sigma} \), where we note that \( B_{n,\sigma} \subseteq M_{n+1,\sigma} \), by the fact \( B_{n,\sigma} \subseteq B_{n+1,\sigma} \) and \( TB_{n,\sigma} \subseteq B_{n+1,\sigma} \) (Proposition 3.9). As \( f \notin B_{n,\sigma} \), we must have \( f'' \neq 0 \). Thus, we have shown the space \( C_{n+1,\sigma} \cap M_{n+1,\sigma} \neq 0 \).

By definition, all the spaces \( B_{n,\sigma} \) and \( C_{n,\sigma} \) are \( I_K \)-stable, for \( n \geq 0 \). The space \( M_{n+1,\sigma} \) is then also \( I_K \)-stable by its definition, as \( T \) respects \( G \)-action. Therefore, we have a non-zero \( I_K \)-stable space \( C_{n+1,\sigma} \cap M_{n+1,\sigma} \). As \( I_{1,K} \) is pro-\( p \), there is some non-zero function \( f^* \in C_{n+1,\sigma} \cap M_{n+1,\sigma} \), which is fixed by \( I_{1,K} \) ([BL94, Lemma 1]).

By Remark 3.8, the \( I_{1,K} \)-invariant subspace of \( C_{n+1,\sigma} \) is two dimensional with the basis \( \{ f_{n+1}, f_{-(n+1)} \} \). By writing \( f^* \) as a linear combination of \( f_{n+1} \) and \( f_{-(n+1)} \), Corollary 3.11 implies that \( T f^* \) does not lie in \( B_{n+1,\sigma} \), a contradiction. \( \square \)

**Remark 4.5.** As we will see, Lemma 4.4 is the key used to prove Theorem 4.1. Our argument above is a bit formal up to the application of Corollary 3.11, and as mentioned in the introduction, is essentially different from that of Barthel–Livné ([BL94, Lemma 20, 21]), which depends crucially on the representation \( \sigma \) (of \( GL_2(k_F) \)) with an explicit basis. Our strategy is applicable to give a new proof of [BL94, Lemma 20], and certain details will appear elsewhere. Note also that [BL94, Lemma 21] is indeed used in the last step of the argument of [GK14, Corollary 6.1].

**Proof of Theorem 4.1.** We proceed to complete the argument of Theorem 4.1. Using Lemma 4.4, by induction we find a non-empty subset \( A_n \) of \( C_{n,\sigma} \), satisfying that \( \sqcup_{2k+2 \leq 2n} T^k A_k \) forms a basis of \( B_{n,\sigma} \).
For \( n = 0 \), take \( A_0 \) to be a basis of the space \( C_{0, \sigma} \). Assume the former statement is done for \( n \). Then we need to show the set \( \cup_{k \leq n} T^i A_k \) is linearly independent.

Assume the claim is false and we have a non-trivial linear combination of elements from \( \cup_{k \leq n} T^i A_k \). As \( \cup_{k \leq n} T^i A_k \) is the disjoint union of \( \cup_{k \leq n} T^i A_k \) and \( \cup_{k \leq n} T^i A_k \), we get a function \( f \), as a linear combination of elements from \( \cup_{k \leq n} T^i A_k \), which lies in the ball \( B_{n, \sigma} \), and satisfies that \( Tf \in B_{n, \sigma} \). Now Lemma 4.4 ensures that \( f \in B_{n-1, \sigma} \). This means that the projection of \( f \) to the circle \( C_n \) is zero. Note that the induction hypothesis for \( n \) already implies that the set of all projections of \( \cup_{k \leq n} T^i A_k \) to \( C_n \) is a basis for \( C_{n, \sigma} \), hence the former statement about \( f \) forces its vanishing. We are done for the claim in the last paragraph.

We then proceed to choose a subset \( A_{n+1} \) of the space \( C_{n+1, \sigma} \), which completes the set \( \cup_{k \leq n} T^i A_k \) to be a basis of \( B_{n+1, \sigma} \). This is certainly possible, and we only need to complete the set of projections of \( \cup_{k \leq n+1} T^i A_k \) to \( C_{n+1} \) to be a basis of \( C_{n+1, \sigma} \).

We note that the subsets \( A_n \) chosen above are non-empty for all \( n \geq 0 \): the cardinality of \( A_0 \) is equal to the dimension of \( \sigma \). For \( n \geq 1 \), the cardinality of \( A_n \) is equal to \( \dim C_{n, \sigma} - \dim C_{n-1, \sigma} \).

In summary, we have chosen a family of non-empty sets \( A_n \subset C_{n, \sigma} \) so that \( \cup_{n \geq 0} \cup_{k \leq n} T^i A_k \) is a basis of the maximal compact induction \( \text{ind}^G_K \sigma \). In particular, the infinite set \( \cup_{n \geq 0} A_n \) is a basis of \( \text{ind}^G_K \sigma \) over \( \mathcal{H}(K, \sigma) \).

The following interesting application is straightforward:

**Corollary 4.6.** For any non-constant polynomial \( P \), the \( G \)-representation \( \text{ind}^G_K \sigma / (P(T)) \) is infinite dimensional.

**Proof.** It suffices to consider \( P \) is a linear polynomial \( T - \lambda \). By writing \( \text{ind}^G_K \sigma \) as \( \bigoplus \mathbb{F}_p[T] \), we see

\[
\text{ind}^G_K \sigma / (T - \lambda) \cong \bigoplus \mathbb{F}_p[T] / (T - \lambda) \cong \bigoplus \mathbb{F}_p[T] / (T - \lambda) \cong \bigoplus \mathbb{F}_p,
\]

and the assertion follows.

**Remark 4.7.** For some \( P(T) \), all the irreducible quotients of \( \text{ind}^G_K \sigma / (P(T)) \) are supersingular representations. For such a \( P(T) \), the above Corollary implies that supersingular representations containing \( \sigma \) do exist.
5 Invariant norms in $p$-adic smooth principal series

In this part, we apply Theorem 4.1 to investigate the existence of $G$-invariant norms in certain locally algebraic representations of $G$ ([STP01, Appendix]). For the background and already known results on this problem, we refer the readers to the excellent survey article [Sor15]. Here, we follow along the lines in [GK14, section 9].

Let $L$ be a finite extension of $\mathbb{Q}_p$, and assume $E$ is contained in $L$. Let $\varepsilon : B \to L^\times$ be a smooth character, and consider the principal series $P(\varepsilon) = \text{Ind}_B^G \varepsilon$ of $G$:

$$\{ f : G \to L \text{ locally constant} \mid f(bg) = \varepsilon(b)f(g) \forall b \in B, g \in G \}$$

and the group $G$ acts by right translation. We are interested to know whether and when $P(\varepsilon)$ admits a $G$-invariant norm.

Assume $\sigma_L$ is the restriction to $K$ of an irreducible $\mathbb{Q}_p$-rational representation of $G$ on a finite dimensional $L$-vector space. As $K$ is open, the representation $\sigma_L$ is still irreducible. We assume further that $\sigma_L$ is contained in $P(\varepsilon)$, i.e., the following space

$$\text{Hom}_L(G)(\text{ind}_K^G \sigma_L, \text{Ind}_B^G \varepsilon) \cong \text{Hom}_L(K)(\sigma_L, \text{Ind}_B^G \varepsilon|_K) \cong (\text{Ind}_B^G \varepsilon)^K, \sigma_L$$

is non-zero. By our assumption, the Satake-Hecke algebra $\mathcal{H}_L(K, \sigma_L) := \text{End}_G(\text{ind}_K^G \sigma_L)$ is isomorphic to $\text{End}_G(\text{ind}_K^G 1)$ ([BS07, Lemma 2.1]), especially it is commutative. The Iwasawa decomposition $G = BK$ implies that the above space is one-dimensional, and the natural action of $\mathcal{H}_L(K, \sigma_L)$ on it gives us a character $\chi_{\sigma_L} : \mathcal{H}_L(K, \sigma_L) \to L$.

Then, we have an induced $G$-equivariant map:

$$M_{\chi_L}(\sigma_L) := \text{ind}_K^G \sigma_L \otimes_{\mathcal{H}_L(K, \sigma_L)} \chi_{\sigma_L} \to \text{Ind}_B^G \varepsilon$$ (6)

Take a $K$-stable $\sigma_L$-lattice $\sigma$ in $\sigma_L$, which is equivalent to finding a $K$-invariant norm $| \cdot |_{\sigma_L}$ on $\sigma_L$. This gives the maximal compact induction $\text{ind}_K^G \sigma_L$ a sup-norm, and then the $G$-representation $M_{\chi_L}(\sigma_L)$ inherits a canonical quotient seminorm. Note that such a seminorm may have non-zero kernel, and as a result it is not necessarily a norm ! However, if $M_{\chi_L}(\sigma_L)$ admits a $G$-invariant norm, the previous canonical seminorm on it must be a norm (see the explanation in [Sor15, page 7, 8]).
We assume the reduction $\sigma \otimes_{o_L} k_L$ of $\sigma$ is an irreducible representation of $\Gamma_K$ over the residue field $k_L$. We consider the restriction $\chi$ of $\chi_{\sigma_L}$ to the subalgebra $H_{o_L}(K, \sigma) := \text{End}_G(\text{ind}_G^K \sigma)$. Note that Theorem 4.1 implies that $\text{ind}_G^K \sigma$ is free over $H_{o_L}(K, \sigma)$. When $\chi$ takes values in $o_L$, the $o_L$-module:

$$M_{\chi}(\sigma) := \text{ind}_G^K \sigma \otimes_{H_{o_L}(K, \sigma)} \chi$$

is free (over $o_L$), which then is a $G$-stable free $o_L$-submodule of $M_{\chi}(\sigma) \otimes_{o_L} L \cong M_{\chi L}(\sigma_L)$. If the map (6) is an isomorphism, we in turn get a $G$-stable free $o_L$-lattice in the principal series $P(\varepsilon)$, as required. In summary:

**Theorem 5.1.** Assume $\sigma \otimes_{o_L} k_L$ is an irreducible representation of $\Gamma_K$ over the residue field $k_L$, and

(a). The map (6) is an isomorphism.

(b). The character $\chi_{\sigma_L} |_{H_{o_L}(K, \sigma)}$ takes values in $o_L$.

Then the principal series $P(\varepsilon)$ admits a $G$-invariant norm.

We end this section with some remarks on the conditions in last theorem:

**Remark 5.2.** 1) By [Key84, section 7], the principal series $P(\varepsilon)$ is irreducible if $\varepsilon \neq \varepsilon^*$, and in such a case the map (6) being an isomorphism is equivalent to require it to be injective.

2) We have a further look of the following space:

$$\text{Hom}_{L[K]}(\sigma_L, \text{Ind}_B^K \varepsilon |_K) \cong \text{Hom}_{L[K]}(\sigma_L, \text{Ind}_B^K \varepsilon)$$

$$\subseteq \text{Hom}_{o_L}(\sigma, (\text{Ind}_B^K \varepsilon)^{K^1})$$

$$\cong \text{Hom}_{o_L}(\Gamma_K)(\sigma, \text{Ind}_B^K \varepsilon_0),$$

where $\varepsilon_0 := \varepsilon |_{B \cap K}$. The first isomorphism is by the decomposition $G = BK$. The second inclusion is due to that the group $K^1$ acts trivially in the lattice $\sigma$. Note that the space $(\text{Ind}_B^K \varepsilon)^{K^1} \neq 0$ implies the character $\varepsilon$ is trivial on $B \cap K^1$, and the character $\varepsilon_0$ factors through $(B \cap K)/(B \cap K^1) \cong \mathbb{B}$. Hence, we may identify $(\text{Ind}_B^K \varepsilon)^{K^1}$ with $\text{Ind}_B^K \varepsilon_0$, which gives the last isomorphism. The last space in the display above is non-zero if and only if ([Enn63]):

(1). $\sigma$ contains the trivial character of $U$, i.e., $\sigma^U \neq 0$,

(2). The action of $\mathbb{B}$ on $\sigma^U$ contains the character $\varepsilon_0$.

By our assumption, the space $\sigma^U_L (= \sigma^U_{N \cap K})$, which contains $\sigma^U$, is just the weight space underlying the highest weight of $\sigma_L$ ([STP01, 1', Proposition 3.4]). Hence, (1) holds automatically, and the $o_L$-module $\sigma^U \neq 0$ is of rank one. Now (2) is to say the group $\mathbb{B}$ acts on $\sigma^U$ as the character $\varepsilon_0$. 

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3) To make the Theorem work, we have to assume that the reduction \( \sigma \otimes L_k \) of \( \sigma \) is irreducible as a representation of \( \Gamma_K \). However, even if the group \( K \) is hyperspecial, such a condition may fail for a general \( \sigma_L \) unless its highest weight lies in the closure of the lowest alcove.

6 Appendix: Proof of Proposition 3.4

Proof of Proposition 3.4. The argument is slightly modified from the author’s thesis ([Xu14]). By definition, for \( n \geq 1, l \geq 0 \),

\[
\varphi_1 * \varphi_n(\alpha^l) = \sum_{g \in G/K} \varphi_1(g)\varphi_n(g^{-1}\alpha^l).
\]

As the support of \( \varphi_1 \) is \( K\alpha K = \bigcup_{g \in K/(K \cap \alpha K\alpha^{-1})} gaK \), the previous sum becomes

\[
\sum_{g \in K/(K \cap \alpha K\alpha^{-1})} \varphi_1(g)\varphi_n(\alpha^{-1}g^{-1}\alpha^l) = \sum_{g \in K/I_K} \sum_{g_2 \in N'_{m_K}/N'_{m_K+1}} \varphi_1(g_1g_2\alpha)\varphi_n(\alpha^{-1}g_2^{-1}g_1^{-1}\alpha^l),
\]

where, we note that \( K \supseteq I_K \supseteq K \cap \alpha K\alpha^{-1} \), and that \( I_K/(K \cap \alpha K\alpha^{-1}) \cong N'_{m_K}/N'_{m_K+1} \).

To proceed, we use the Bruhat decomposition \( K = I_K \cup I_K\beta_K I_K \) to split the above sum further into two parts, say,

\[
\sum_1 = \sum_{g_2 \in N'_{m_K}/N'_{m_K+1}} \varphi_1(\beta_Kg_2\alpha)\varphi_n(\alpha^{-1}g_2^{-1}\beta_K\alpha^l)
\]

and

\[
\sum_2 = \sum_{g \in N'_{m_K-1}/N'_{m_K}} \sum_{g_2 \in N'_{m_K}/N'_{m_K+1}} \varphi_1(g_1g_2\alpha)\varphi_n(\alpha^{-1}g_2^{-1}g_1^{-1}\alpha^l).
\]

Firstly we claim \( \sum_1 \) is always 0. As \( \varphi_1 \in \mathcal{H}_K(\sigma) \), \( \sum_1 \) is simplified as

\[
\sum_1 = \sum_{g_2 \in N'_{m_K}/N'_{m_K+1}} \sigma(\beta_K)j_\alpha \varphi_n(\alpha^{-1}g_2^{-1}\beta_K\alpha^l).
\]

We note that \( \alpha^{-1}g_2^{-1}\beta_K\alpha^l \in K\alpha^{-(l+1)}K \), hence we only need to consider the case that \( l + 1 = n \). In this case, the sum \( \sum_1 \) is reduced to

\[
\sum_1 = \sum_{g_2} \sigma(\beta_K)j_\alpha \sigma(\beta_K)j_\sigma,
\]

which is equal to 0, as it sums over the same term \( q^{|\kappa|} \) times.

For the remaining \( \sum_2 \), we note the part \( \sum_2 \) for which \( g_1 \in N_{m_K-1} \setminus N'_{m_K} \) is equal to 0. A simple calculation using (1) gives

\[\text{Communicated to us by Florian Herzig.}\]
\[
\alpha^{-1}g_2^{-1}g_1^{-1}\alpha^l = g'\alpha^{-(l+1)}g'',
\]
for some \(g' \in N_{nK+2}\) and \(g'' \in K\). As a result, we have \(\sum'_2 = 0\), if \(l \neq n - 1\). When \(l = n - 1\), one can rewrite \(\sum'_2\) as
\[
\sum'_2 = \sum_{g_2} (\sum_{g_1} f'),
\]
in which \(f'\) is a function only related to \(g_1\). We get \(\sum'_2 = q^\sigma \cdot (\sum_{g_1} f') = 0\).

The other part of \(\sum_2\), denoted by \(\sum''_2\), depends on the values of \(l\) and \(n\):
\[
\sum_{g_2 \in N_{nK}/N_{nK+1}} \varphi_1(g_2\alpha)\varphi_n(\alpha^{-1}g_2^{-1}\alpha^l) = \begin{cases} j_\sigma, & l = n + 1, \\ c \cdot j_\sigma, & l = n, \\ 0, & \text{otherwise}, \end{cases}
\]
where \(c\) is a constant.

From the definition of \(\varphi_1\), the sum \(\sum''_2\) is reduced to
\[
\sum''_2 = \sum_{g_2} j_\sigma \varphi_n(\alpha^{-1}g_2^{-1}\alpha^l).
\]
If \(l = 0\), a term in \(\sum''_2\) is non-zero only if \(n = 1\); but in this case, the sum itself is clearly zero.

We assume \(l \geq 1\). For the term in the sum \(\sum''_2\) corresponding to \(g_2 \in N_{nK+1}\), it becomes \(j_\sigma \varphi_n(\alpha^{l-1})\), as \(\alpha^{-1}g_2^{-1}\alpha \in I_{l,K}\) and note that \(j_\sigma\) is trivial on \(\sigma(I_{l,K})\). Hence, such term is non-zero only if \(l = n + 1\), and in this case it is equal to \(j^2_\sigma = j_\sigma\). For the remaining terms in \(\sum''_2\), we write \(g_2^{-1}\) as \(g\) for short, and put \(g^* = \beta_Kg\beta_K\). By using (1) again we get
\[
\alpha^{-1}g^l = \alpha^{-1}\beta_Kg^*\beta_K\alpha^l \in K^1h\beta_K\alpha^lK^1
\]
for some diagonal matrix \(h \in K\) depending on \(g = g_2^{-1}\). Therefore, each term is non-zero only if \(l = n\), and in that case we get the sum \(\sum''_2\) as
\[
\sum_h j_\sigma \sigma(h\beta_K)j_\sigma.
\]
As we have just shown that the other parts of the whole sum contribute zero to its value at \(\alpha^n\), the above sum must be a multiple of \(j_\sigma\), say \(c \cdot j_\sigma\) for some constant \(c\). In all, we are done.

\[\Box\]

For the concrete form of \(f'\), one needs to distinguish \(l = 0\) or not.
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