A study of $\rho - \omega$ mixing in resonance chiral theory

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The strong and electromagnetic corrections to $\rho - \omega$ mixing are calculated using a SU(2) version of resonance chiral theory up to next-to-leading orders in $1/N_C$ expansion, respectively. Up to our accuracy, the effect of the momentum dependence of $\rho - \omega$ mixing is incorporated due to the inclusion of loop contributions. We analyze the impact of $\rho - \omega$ mixing on the pion form factor factor by performing numerical fit to the data extracted from $e^+e^- \rightarrow \pi^+\pi^-$ and $\tau \rightarrow \nu_\tau 2\pi$, while the decay width of $\omega \rightarrow \pi^+\pi^-$ is taken into account as a constraint. It is found that the momentum dependence is significant in a good description of the experimental data. In addition, based on the fitted values of the involved parameters, we analyze the decay width of $\omega \rightarrow \pi^+\pi^-$, which turns out to be highly dominated by the $\rho - \omega$ mixing effect.

I. INTRODUCTION

The study of $\rho - \omega$ mixing is a very interesting subject in hadron physics both theoretically and experimentally. The inclusion of $\rho - \omega$ mixing effect is crucial for a good description of the pion vector form factor in $e^+e^- \rightarrow \pi^+\pi^-$ process, which quantifies the hadronic vacuum polarization contribution to the anomalous magnetic moment of the muon. On the experimental side, several experimental collaborations, such as KLOE [1, 2] and BESIII [3], have recently launched measurements of the $e^+e^- \rightarrow \pi^+\pi^-$ with high statistics and high precision.

The $\rho - \omega$ mixing amplitude was assumed to be a constant or momentum-independent in the early stage of previous studies [4, 5]. The authors of Ref. [6] suspect the validity of the constant assumption and, based on a quark loop mechanism of $\rho - \omega$ mixing, they found that the mixing amplitude is significantly momentum-dependent. Since then, the use of various loop mechanisms for $\rho - \omega$ mixing is triggered in different models such as extended Nambu-Jona-Lasinio (NJL) model [7], the global color model [8], the hidden local symmetry model [9–11], and the chiral constituent quark model [12, 13].

In this work, we aim at studying $\rho - \omega$ mixing in a model independent way by invoking Resonance Chiral Theory (RχT) [14]. It provides a reliable tool to study physics in the intermediate energy region [15–20]. The tree-level calculation of $\rho - \omega$ mixing in the framework of RχT has been given in Refs. [21, 22], however, the tree-level mixing amplitude turns out to be momentum-independent. In order to implement the momentum dependence, here we will calculate the one-loop contributions as shown in Fig. 1. The $\rho - \omega$ mixing can be induced either by strong isospin-violating or by electromagnetic effects. The former is proportional to the mass difference between the $u, d$ quarks, i.e., $\Delta_{ud} = m_u - m_d$ and the latter is accompanied by the fine structure constant $\alpha$. In the present study, only the mixing effects linear in $\Delta_{ud} \alpha$ or $\alpha$ are under our consideration. Apart from the overall factors $\Delta_{ud} \alpha$ or $\alpha$, the large-$N_C$ counting rule proposed in Ref. [23] is imposed to truncate our perturbative calculation. Specifically, our calculations are truncated at next-to-leading order in the $1/N_C$ expansion for the strong and electromagnetic contributions. The ultraviolet (UV) divergence from the loops is cancelled by introducing counterterms with sufficient derivatives and the involved couplings are assumed to be beyond the leading order in $1/N_C$ expansion as claimed in Ref. [24].

We assess the impact of momentum-dependent $\rho - \omega$ mixing amplitude on the pion vector form factor by fitting to the experimental data extracted from the $e^+e^- \rightarrow \pi^+\pi^-$ process and $\tau \rightarrow \nu_\tau 2\pi$ decay in the energy region of 650–850 MeV. Besides, the decay width of $\omega \rightarrow \pi^+\pi^-$ is implemented as a constraint in the fit. It is known that, provided isospin invariance holds, the isovector part of the pion form factor in the $e^+e^- \rightarrow \pi^+\pi^-$ process and $\tau \rightarrow \nu_\tau 2\pi$ decay is related to the one in $\tau$ decays theoretically, via the conserved vector current assumption [25, 26]. Different effects of isospin breaking have been studied to describe the $e^+e^- \rightarrow \pi^+\pi^-$ annihilation data and $\tau$ decays data simultaneously [26–34], such as the short distance and long distance corrections in the $\tau$ partial decay width to two pions, charged and neutral $\rho$ mass and width difference, and $\rho - \omega$ mixing. In our study we will take into account all the above isospin breaking effects. Our fit result shows that the $\rho - \omega$ mixing amplitude is significantly momentum-dependent and its imaginary part is much smaller than real part. Based on the fitted values of the parameters, we also analyze the decay width of $\omega \rightarrow \pi^+\pi^-$ by including the effect of

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the $\rho - \omega$ mixing. It is found that the decay width is dominated by the $\rho - \omega$ mixing effect while the contribution from the direct coupling of $\omega \to \pi^+ \pi^-$ is negligible.

This paper is organized as follows. In Sec. II, we introduce the description of $\rho - \omega$ mixing. In Sec. III, we present the theoretical framework and elaborate on the calculation of the tree-level and loop contribution of $\rho - \omega$ mixing. In Sec. IV, the fit result is shown and the related phenomenology is discussed. A summary is given in Sec. V.

II. GENERIC DESCRIPTION OF $\rho - \omega$ MIXING

In the isospin basis $|I, I_3\rangle$, we define $|\rho_1\rangle \equiv |1, 0\rangle$ and $|\omega_1\rangle \equiv |0, 0\rangle$ for convenience. The mixing between the isospin states of $|\rho_1\rangle$ and $|\omega_1\rangle$ can be implemented by considering the self-energy matrix

$$
\Pi_{\mu\nu} = T_{\mu\nu} \left( \begin{array}{cc} \Pi_{\rho\rho}(s) & \Pi_{\rho\omega}(s) \\ \Pi_{\omega\rho}(s) & \Pi_{\omega\omega}(s) \end{array} \right),
$$

with $T_{\mu\nu} \equiv g_{\mu\nu} - \frac{g_{\mu\nu}p^2}{p^2} 1$ and $s \equiv p^2$. The off-diagonal matrix element $\Pi_{\rho\omega}(s)$ is non-zero, e.g., due to isospin-broken effect, and it therefore carries the information of $\rho - \omega$ mixing. Subsequently, the dressed propagator has the form [35]

$$
D_{\mu\nu} = g_{\mu\nu} \left( \begin{array}{c} 1/s_\rho \\ \frac{\Pi_{\omega\rho}(s)}{s_\rho s_\omega} \\ 1/s_\omega \end{array} \right) \equiv g_{\mu\nu} D^I(s),
$$

where the abbreviations $s_\rho$ and $s_\omega$ are defined by

$$
s_\rho \equiv s - \Pi_{\rho\rho}(s) - m_\rho^2,
$$

$$
s_\omega \equiv s - \Pi_{\omega\omega}(s) - m_\omega^2.
$$

In above the vector-current conservation has been used to eliminate the longitudinal part proportional to $p_\mu$. Furthermore, we have also neglected terms of $\Pi_{\omega\omega}^2(s)$, since they correspond to contributions at two-loop order and are beyond our accuracy. $m_\rho$ and $m_\omega$ are bare masses of the $\rho$ and $\omega$ mesons, respectively.

The $\rho - \omega$ mixing, i.e., mixing between the physical states of $\rho^0$ and $\omega$, is obtainable by introducing the following relation

$$
\begin{pmatrix} \rho^0 \\ \omega \end{pmatrix} = C \begin{pmatrix} \rho_1 \\ \omega_1 \end{pmatrix}, \quad C = \begin{pmatrix} 1 \\ -\epsilon_1 \\ \epsilon_2 \\ 1 \end{pmatrix}
$$

with $\epsilon$ being the mixing parameter. The matrix of dressed propagators corresponding to physical states is diagonal. Moreover, it can be connected to the matrix $D^I(s)$ in Eq. (2) through

$$
\begin{pmatrix} 1/s_\rho & 0 \\ 0 & 1/s_\omega \end{pmatrix} = C \begin{pmatrix} 1/s_\rho & \Pi_{\rho\omega}/s_\omega \\ \Pi_{\rho\omega}/s_\omega & 1/s_\omega \end{pmatrix} C^{-1}.
$$

Solving the above equation and neglecting higher-order terms of $\mathcal{O}(\epsilon^2)$ and $\epsilon \Pi_{\rho\omega}$, one obtains:

$$
\epsilon_1 = \frac{\Pi_{\rho\omega}(M_\omega^2)}{s_\rho - s_\omega}, \quad \epsilon_2 = \frac{\Pi_{\rho\omega}(M_\omega^2)}{s_\rho - s_\omega}.
$$

The two mixing parameters should be just different with each other slightly, see Ref. [35] for more details.

III. CALCULATIONS IN RESONANCE CHIRAL THEORY

In this section we will calculate the mixing amplitude $\Pi_{\rho\omega}(s)$ using RχT so as to study its momentum dependence. The information of $\rho - \omega$ mixing is encoded in the off-diagonal element of the self-energy matrix, which can be decomposed as

$$
\Pi_{\rho\omega}(s) = \Delta_{ud} S_{\rho\omega}(s) + 4\pi \alpha E_{\rho\omega}(s),
$$

where $\Delta_{ud} = m_u - m_d$ is the mass difference between $u, d$ quarks, and $\alpha$ denotes the fine-structure constant. In above, $S_{\rho\omega}(s)$ and $E_{\rho\omega}(s)$ stand for the structure functions of strong and electromagnetic interactions, respectively.

In the present work, the diagrams in Fig. 1 are needed for a calculation in RχT up to NLO in $1/N_C$ expansion. As will be seen below, the LO contributions of $S_{\rho\omega}(s)$ and $E_{\rho\omega}(s)$ are different: the former starts at $\mathcal{O}(N_C^0)$ while the latter does at $\mathcal{O}(N_C^1)$. Therefore, their corresponding NLO contributions are of $\mathcal{O}(N_C^{-1})$ and $\mathcal{O}(N_C^0)$, respectively. In what follows, all the diagrams in Fig. 1 will be calculated by using effective Lagrangians constructed in the framework of RχT.

A. Resonance chiral theory and Tree-level amplitudes

In RχT, the vector resonances are described in terms of antisymmetry tensor fields with the normalization

$$
\langle 0|V_{\mu\nu}|V, p\rangle = iM_V^{-1}\{p_\mu\epsilon_\nu(p) - p_\nu\epsilon_\mu(p)\},
$$

with $\epsilon_\mu$ being the polarization vector. The kinetic Lagrangian of vector resonances takes the form [14]

$$
\mathcal{L}_{kin}(V) = -\frac{1}{2} \nabla^\lambda V_{\lambda\mu} \nabla_\nu V^{\nu\mu} - \frac{M_V^2}{2} V_{\mu\nu} V^{\mu\nu},
$$

where $M_V$ is the mass of the vector resonances in the chiral limit. Here the vector mesons are collected in a $2 \times 2$ matrix

$$
V_{\mu\nu} = \left( \begin{array}{cc} \rho^0 + \sqrt{2}\rho^- & \frac{1}{\sqrt{2}}\rho^0 + \frac{1}{\sqrt{2}}\omega^- \\ \frac{1}{\sqrt{2}}\rho^0 + \frac{1}{\sqrt{2}}\omega^- & \rho^+ + \sqrt{2}\omega^\prime \end{array} \right)_{\mu\nu}.
$$
Besides, the covariant derivative and chiral connection are defined by
\[ \nabla_{\mu} V = \partial_{\mu} V + [\Gamma_{\mu}, V], \]
\[ \Gamma_{\mu} = \frac{1}{2}(u^+(\partial_{\mu} - i\gamma_{\mu})u + u(\partial_{\mu} - il_{\mu})u^+). \] (11)

The Goldstone bosons originating from the spontaneous breaking of the \( SU(2)_L \times SU(2)_R \) chiral symmetry are nonlinearly parametrized as
\[ u = \exp\{i \frac{\Phi}{\sqrt{2} F}\}, \quad \Phi = \left( \frac{1}{\sqrt{2}} \pi^0 - \frac{\pi^+}{\sqrt{2}} \right). \] (12)
with \( F \) being the pion decay constant.

In the isospin limit, the standard Lagrangian describing the interactions between \( V_{\mu\nu} \) and Goldstone bosons or electromagnetic fields are given by
\[ \mathcal{L}_{2}(V) = \frac{F_{V}}{2\sqrt{2}} V_{\mu\nu} f_{+}^{\mu\nu} + iG_{V} \langle V_{\mu\nu} u^\mu u^\nu \rangle, \] (13)
with the relevant building blocks defined by
\[ f_{\pm}^{\mu\nu} = u F_{L}^{\mu\nu} u^+ \pm u^+ F_{R}^{\mu\nu} u, \]
\[ u_{\mu} = i[u^+(\partial_{\mu} - ir_{\mu})u - u(\partial_{\mu} - il_{\mu})u^+]. \] (14)
Here \( F_{L,R}^{\mu\nu} \) are field strength tensors composed of the left and right external sources \( l_{\mu} \) and \( r_{\mu} \), and \( F_{V}, G_{V} \) are real couplings.

The LO isospin-breaking effect is introduced by the Lagrangian
\[ \mathcal{L}^{\rho\omega}_{2} = v_{8}[V_{\rho\omega} V^{\mu\nu} \chi_{+}], \] (15)
with \( \chi_{+} = u^+ \chi u^+ + u \chi^+ u \) and \( \chi = 2B_{0}(s + ip) \). Here \( v_{8} \) is an unknown coupling constant. However, it can be determined by considering the mass relations of the vector mesons at \( O(p^3) \) in terms of the quark counting rule [21], which leads to: \( v_{8} = 1/8. \)

With the above preparations, one is now able to calculate the tree amplitudes. The tree-level strong contribution, corresponding to diagram (a) in Fig. 1, turns out to be
\[ S^{(a)}_{\rho\omega} = 2M_{\rho}, \] (16)
which is counted as \( O(N_{C}^0) \), since \( M_{\rho} \sim O(N_{C}^0) \). The tree-level electromagnetic contribution is from diagram (b) in Fig. 1 and the amplitude can be obtained by using the Lagrangian in Eq. (13):
\[ F^{(b)}_{\rho\omega} = \frac{F_{\rho} F_{\omega}}{3}, \] (17)
with the \( N_{C} \)-counting order being \( O(N_{C}^1) \), due to \( F_{\rho} \sim F_{\omega} \sim \sqrt{N_{C}} \). As mentioned in the beginning of this section, the leading-order strong and electromagnetic contributions indeed start at different orders in \( 1/N_{C} \) expansion.

**B. Loop contributions**

The relevant loop diagrams contributing up to our accuracy are shown in the second and third lines of Fig. 1. Diagrams (c) and (d) contribute to the strong correction at \( O(N_{C}^{-1}) \), which are next-to-leading order compared to diagram (a). Likewise, with respect to diagram (b), diagrams (e)-(h) lead to electromagnetic corrections at next-to-leading order, i.e. \( O(N_{C}^{0}) \). In our calculation below, the necessary isospin-breaking vertices are constructed based on the basic chiral building blocks taken from \( \chi PT [36] \) and \( R\chi T [14] \).

1. **Diagram (c): \( \pi\pi \) loop**

The vertex of \( \rho_I \to \pi^+\pi^- \) can be read from the Lagrangian in Eq. (13). For the isospin-violating vertex of
\[ \omega_I \rightarrow \pi^+\pi^- \text{, we construct the following Lagrangian} \]

\[ \mathcal{L}_{\omega_I \rightarrow \pi^+\pi^-} = a_1 i (V_{\mu\nu} \{ \chi_+, u^{\mu} u^{\nu} \}) + a_2 i (V_{\mu\nu} u^\mu \chi_+ u^{\nu}) \]

\[ = (a_1 - \frac{1}{2} a_2) \frac{8 \sqrt{2} B_0}{F^2} \Delta_{\omega I} \omega_\alpha \pi^{\pi - \alpha} \pi^{\pi - \beta} . \tag{18} \]

For convenience, we define the combination \( a \equiv a_1 - \frac{1}{2} a_2 \). The \( \pi \pi \)-loop contribution can be obtained by calculating the integral

\[ i \Pi_{\rho \omega} \epsilon_{\rho \mu} \epsilon_{\omega} \]  

\[ = - \frac{16 \sqrt{2} G V B_0 a (m_u - m_d) p^2}{F^4} \epsilon_{\rho \mu} \epsilon_{\omega} \nu} \right) \frac{(2\pi)^n}{(p - k)^2 - m_\pi^2} \tag{19} \]

where \( p \) and \( k \) denote the momenta of the exchanged vector mesons and either of the exchanged pions, respectively. After integrating, the structure function can be extracted, which reads

\[ S^{(c)}_{\rho \omega} = \frac{\sqrt{2} G V B_0 a}{12 F^4 \pi^2} \left\{ (1 - \frac{6 m_\pi^2}{p^2}) (\lambda_\pi - \ln \frac{m_\pi^2}{\mu^2}) \right. \]

\[ + \frac{5}{3} \frac{m_\pi^2}{p^2} + \sigma^3 \ln \left( \frac{\sigma + 1}{\sigma - 1} \right) \right\} \tag{20} \]

where \( \sigma \equiv \sqrt{1 - 4 m_\pi^2 / p^2} \) and \( \lambda_\pi \equiv \frac{1}{2} - \gamma_E + 1 + \ln 4 \pi \) with \( \epsilon = 2 - \frac{4}{3} \) and \( \gamma_E \) being the Euler constant.

2. Diagram (d): \( \pi \)-tadpole loop

According to the Lorentz, \( P \) and \( C \) invariances, the Lagrangian corresponding to the interaction of \( \omega_I \rho \pi \pi \rangle \)

\[ \mathcal{L}_{\omega_I \rho \pi \pi} = b_1 (V_{\mu\nu} V^{\mu\nu} (u^{\alpha} u^{\alpha} \chi_+ + \chi_+ u^{\alpha} u^{\alpha} \chi_+)) \]

\[ + b_2 (V_{\mu\nu} V^{\mu\nu} u^{\alpha} \chi_+ u^{\alpha} \chi_+) + b_3 (V_{\mu\nu} \chi_+ V^{\mu\nu} u^{\alpha} u^{\alpha} \chi_+) \]

\[ + b_4 (V_{\mu\nu} V^{\mu\nu} (u^{\alpha} u^{\alpha} + u^{\alpha} \chi_+ u^{\alpha} \chi_+)) \]

\[ + b_5 (V_{\mu\nu} V^{\mu\nu} u^{\alpha} u^{\alpha} \chi_+ + V^{\alpha\beta} V_{\mu\nu} u^{\alpha} u^{\alpha} \chi_+ + V^{\alpha\beta} V_{\mu\nu} \chi_+ u^{\alpha} u^{\alpha} \chi_+ + V^{\alpha\beta} V_{\mu\nu} \chi_+ u^{\alpha} u^{\alpha} \chi_+ \]

\[ + b_6 (V_{\mu\nu} V^{\mu\nu} u^{\alpha} u^{\alpha} \chi_+ + V^{\alpha\beta} V_{\mu\nu} u^{\alpha} u^{\alpha} \chi_+ + V^{\alpha\beta} V_{\mu\nu} \chi_+ u^{\alpha} u^{\alpha} \chi_+ + V^{\alpha\beta} V_{\mu\nu} \chi_+ u^{\alpha} u^{\alpha} \chi_+ \]

\[ + b_7 (V_{\mu\nu} V^{\mu\nu} u^{\alpha} u^{\alpha} \chi_+ + V^{\alpha\beta} V_{\mu\nu} u^{\alpha} u^{\alpha} \chi_+ + V^{\alpha\beta} V_{\mu\nu} \chi_+ u^{\alpha} u^{\alpha} \chi_+ + V^{\alpha\beta} V_{\mu\nu} \chi_+ u^{\alpha} u^{\alpha} \chi_+ \]

\[ + g_8 (V_{\mu\nu} V^{\mu\nu} \chi_+ + V_{\mu\nu} V^{\mu\nu} \chi_+ + V_{\mu\nu} V^{\mu\nu} \chi_+) \tag{21} \]

Note that the \( v_8 (V_{\mu\nu} V^{\mu\nu} \chi_+) \) term, which contributes to the contact interaction of \( \rho - \omega \) mixing, also yields \( \omega_I \rho\pi \pi \) vertex. Though in Eq. (21) there are many terms with a large number of free couplings, the final result only depends on combinations of these couplings. For simplicity, the following two combinations are necessary, i.e.,

\[ \begin{aligned} h_1 & \equiv 6 b_1 - b_2 + 3 b_3 + b_4 - 2 g_1 - g_2, \\ h_2 & \equiv 4 b_5 - b_6 + 3 b_7 + 4 b_8 - b_9 + 3 b_{10} + 2 b_{11} + 2 b_{12} \\ & + 2 b_{13} + 2 b_{14} - 2 g_3 - 2 g_4 - 2 g_5. \end{aligned} \tag{22} \]

Furthermore, one can neglect the mass difference between the charged and neutral pions in the internal lines of loops, since the resultant difference is of higher orders beyond our consideration. As a result, the expanded form of Lagrangian (21) can be reduced simply to

\[ \mathcal{L}_{\omega_I \rho \pi \pi} = \frac{4 B_0}{F^2} h_1 (m_u - m_d) \rho_{\mu \nu} \omega^{\mu \nu} \pi_\alpha \pi^\alpha \]

\[ - \frac{2 B_0}{F^2} v_8 (m_u - m_d) \rho_{\mu \nu} \omega^{\mu \nu} \pi_\alpha \pi^\alpha \]

\[ + \frac{4 B_0}{F^2} h_2 (m_u - m_d) \rho_{\mu \nu} \omega^{\mu \nu} \pi_\alpha \pi^\alpha. \tag{23} \]

With the above Lagrangian, the \( \pi \)-tadpole contribution to the \( \rho - \omega \) mixing can be derived:

\[ i \Pi_{\rho \omega} \epsilon_{\rho \mu} \epsilon_{\omega} \]

\[ = \frac{4 i B_0}{F^2} h_1 (m_u - m_d) \epsilon_{\rho \mu} \epsilon_{\omega} \nu} \right) \frac{(2\pi)^n}{k^2 - m_\pi^2} - \frac{1}{F^2} v_8 (m_u - m_d) \]

\[ \times \epsilon_{\rho \mu} \epsilon_{\omega} \nu} \right) \frac{(2\pi)^n}{k^2 - m_\pi^2}. \tag{24} \]

Eventually, the explicit expression of the strong structure function has the form of

\[ S^{(d)}_{\rho \omega} = \frac{- m_\pi^4 B_0}{8 \pi^2 F^2} \left\{ (-16 v_8 + 4 h_1 m_\pi^2 + h_2 m_\pi^2) \right. \]

\[ \times \left( \lambda_\pi - \ln \frac{m_\pi^2}{\mu^2} \right) + \frac{h_2}{2} \right\}. \tag{25} \]

3. Diagrams (e)-(h): \( \pi^0 \gamma \) loops

In the loop diagrams (e)-(h), there are two types of vertices. The coupling of vector meson (V) as well as vector external source (J) to pseudoscalar (P) is labeled by VJP vertex for short. The interaction of two vector mesons and one pseudoscalar is called VVP vertex. The operators of VJP type are given in Ref. [37]:

\[ \mathcal{L}_{\text{VJP}} = \frac{c_1}{M_V} \epsilon_{\mu \nu \rho \sigma} \langle V^{\mu \nu} \chi_+ \rangle \nabla_\alpha u^\sigma \]

\[ + \frac{c_2}{M_V} \epsilon_{\mu \nu \rho \sigma} \langle V^{\mu} \alpha \rangle \nabla_\alpha u^\sigma \]
and the ones of VVP type are
\[
\mathcal{L}_{VVP} = d_1 \epsilon_{\mu
u\rho\sigma} \langle \{V_{\mu\nu}, V_{\rho\sigma}\} \nabla_{\alpha} u^{\sigma} \rangle + id_2 \epsilon_{\mu
u\rho\sigma} \langle \{V_{\mu\nu}, V_{\rho\sigma}\} \chi_{-} \rangle + d_3 \epsilon_{\mu
u\rho\sigma} \langle \{\nabla_{\alpha} V_{\mu\nu}, V_{\rho\sigma}\} u^{\sigma} \rangle + d_4 \epsilon_{\mu
u\rho\sigma} \langle \{\nabla_{\alpha} V_{\mu\nu}, V_{\rho\sigma}\} u_{\alpha} \rangle .
\]

The involved couplings or their combinations can be estimated by matching the leading operator product expansion of (VVP) Green function to the result calculated within R\(\chi\)T. Such a procedure leads to high energy constraints on the resonance couplings as follows [37]:
\[
4c_3 + c_1 = 0,
\]
\[
c_1 - c_2 + c_5 = 0,
\]
\[
c_5 - c_6 = \frac{N_c}{64\pi^2} \frac{M_V}{\sqrt{2} F_V},
\]
\[
d_1 + 8d_2 = -\frac{N_c}{64\pi^2} \frac{M_V^2}{F_V^2} + \frac{F_V^2}{4 F_V^2},
\]
\[
d_3 = -\frac{N_c}{64\pi^2} \frac{M_V^2}{F_V} + \frac{F_V^2}{8 F_V^2}.
\]

The mass of vectors in the chiral limit, \(M_V\), can be estimated by the mass of \(p(770)\) meson [38].

The loops diagrams (e)-(h) can be calculated simultaneously if the effective vertices of \(\rho^* \rightarrow \pi^* \gamma^*\) and \(\omega^* \rightarrow \pi^* \gamma^*\) are used, where a \(*\) stands for an off-shell particle. The explicit expression for \(\rho^* \rightarrow \pi^* \gamma^*\) reads
\[
i\mathcal{V}_{\rho\omega}^{\mu\nu\pi^*\gamma^*} = i\epsilon_{\mu\nu\rho\gamma} k^\rho c_\rho \frac{4\sqrt{2} e B_0}{3 M_p M_V F} c_1 (p - k) \cdot k - c_2 p \cdot (p - k) - 4 c_3 m_\pi^2 - c_5 p \cdot k + c_6 p^2\]
\[
+ i\epsilon_{\mu\nu\rho\gamma} k^\rho c_\rho \frac{4 F_V e B_0}{M MF(M^2 - k^2)} \times [d_1 (p - k)^2 + 8d_2 m_\pi^2 + 2d_3 p \cdot k] ,
\]
where \(p\) and \(k\) denote the momentum of the vector meson and the photon, respectively. Analogically, for \(\omega^* \rightarrow \pi^* \gamma^*\), one has
\[
i\mathcal{V}_{\omega\rho}^{\mu\nu\pi^*\gamma^*} = i\epsilon_{\mu\nu\rho\gamma} k^\rho c_\rho \frac{4\sqrt{2} e B_0}{3 M_p M_V F} c_1 (p - k) \cdot k - c_2 p \cdot (p - k) - 4 c_3 m_\pi^2 - c_5 p \cdot k + c_6 p^2\]
\[
+ i\epsilon_{\mu\nu\rho\gamma} k^\rho c_\rho \frac{4 F_V e B_0}{M MF(M^2 - k^2)} \times [d_1 (p - k)^2 + 8d_2 m_\pi^2 + 2d_3 p \cdot k] ,
\]

It should be stressed that there are two terms in each effective vertex. One corresponds to the case that the virtual photon is coupled to the \(V P\) system directly, while the other to the case that it is interacted through an intermediate vector meson. Note also that, throughout this work we only account for the corrections proportional to \(\Delta_{ud}\) or \(4\pi\alpha\), which implies the calculation of electromagnetic contribution can be carried out in the isospin limit, i.e., \(m_u = m_d\).

With the help of the effective vertices, the \(\pi\gamma\) loop contribution, i.e., the sum of the loops diagrams (e)-(h), can be expressed as:
\[
i\Pi_{\rho\omega} = \frac{1}{p^2} \int \frac{d^4 k}{(2\pi)^4} \frac{i}{(p - k)^2 - m_{\pi}^2 - \omega (p \cdot k)}
\]
\[
\times \left\{ \frac{32 e^2}{3 M_V^2 F_V^2} \left[ c_1 (p - k) \cdot k - c_2 (p - k) \cdot p - 4 c_3 m_\pi^2 - c_5 p \cdot k + c_6 p^2 \right]^2
\]
\[
- \frac{16 \sqrt{2} F_V e^2}{3 M_V^2 F_V^2} \left[ \frac{1}{M_{\rho}^2 - k^2} + \frac{1}{M_{\rho}^2 - k^2} \right]
\]
\[
\times \left[ d_1 (p - k)^2 + 8d_2 m_\pi^2 + 2d_3 p \cdot k \right]^2 \right\} .
\]

The further calculation is straightforward but the result of the extracted electromagnetic structure function \(E_{\omega\rho}^{\pi\gamma} = E_{\omega\rho}^{(c)} + E_{\omega\rho}^{(f)} + E_{\omega\rho}^{(d)} + E_{\omega\rho}^{(h)}\) is too lengthy to be shown here. It is worthy noting that in our numerical computation we will use the high energy constraints in Eq. (28) together with the fitted parameters given in Ref. [18], therefore, all the parameters involved in \(E_{\omega\rho}^{\pi\gamma}\) are known.

4. counterterms and renormalized amplitude

Up to now, the total contribution of \(\rho - \omega\) mixing can be expressed as
\[
\Pi_{\rho\omega} = \Delta_{ad} \left[ S_{\rho\omega}^{(a)} + S_{\rho\omega}^{(c)} + S_{\rho\omega}^{(d)} + 4\pi\alpha \left( E_{\omega\rho}^{(b)} + E_{\omega\rho}^{(f)} + E_{\omega\rho}^{(h)} \right) \right] ,
\]
which is still unrenormalized. The resonance chiral theory is unrenormalizable in the sense that the amplitude has to be renormalized order by order with increasing number of counterterms when the accuracy of the calculation is improved. In our case, the tree amplitudes, \(S_{\rho\omega}^{(a)}\) and \(E_{\rho\omega}^{(b)}\), can only absorb the ultraviolet divergence proportional to \(p^0\). In order to cancel the \(O(p^2)\), \(O(p^4)\) and \(O(p^6)\) stemming from the loop contribution \(S_{\rho\omega}^{(c)}\) and
$E_{\rho\omega}^\gamma$, additional counterterms are needed. For this purpose, we construct

$$
\mathcal{L}_{ct} = Y_A(V_{\mu\nu}V^{\mu\nu}\chi_+) - \frac{1}{2} Y_B(\nabla^\lambda V_{\lambda\mu}\nabla_\nu V^{\mu\nu}\chi_+ ) + \frac{Y_C}{2}(\nabla^2 V^{\mu\nu}\chi_+ , (\nabla_\nu, \nabla^\sigma) V_{\nu\sigma} )
+ \frac{Y_C}{4}(\{V_{\nu\sigma}, \nabla_\lambda\} V^{\mu\nu}\chi_+ , (\nabla_\sigma, \nabla^\alpha) V_{\nu\alpha} )
+ \frac{Y_C}{4}(\{\nabla_\nu, \nabla_\sigma\} V^{\nu\sigma}\chi_+ , (\nabla_\nu, \nabla_\sigma) V_{\mu\nu} )
+ \frac{Z_A F_v}{2\sqrt{2}}(V_{\mu\nu} f^\mu_+ ) + \frac{Z_B F_v}{2\sqrt{2}}(V_{\mu\nu} \nabla^2 f^{\mu_+} )
+ \frac{Z_C F_v}{2\sqrt{2}}(V_{\mu\nu} \nabla^4 f^{\mu_+} ) + \frac{Z_D F_v}{2\sqrt{2}}(V_{\mu\nu} \nabla^6 f^{\mu_+} ) \tag{32}
$$

We adopt the $\overline{\text{MS}} - 1$ subtraction scheme and absorb the divergent pieces proportional to $\lambda_\infty$ by the bare couplings in the counterterms. Consequently, the remanent finite pieces of counterterms can be written as:

$$
\Pi_{\rho\omega}^c = X_W^r p^6 + X_Z^r p^4 + X_R^r p^2 , \tag{33}
$$

with

$$
X_W^r = \frac{8\pi\alpha F_\rho F_\omega}{3} (Z_D^r + Z_B^r Z_C^r) ,
X_Z^r = \frac{4\pi\alpha F_\rho F_\omega}{3} (2Z_C^r + Z_B^2 - 16M_\rho(m_u - m_d)Y_C^r + Y_Z^r ) ,
X_R^r = \frac{8\pi\alpha F_\rho F_\omega}{3} Z_B^r - 4M_\rho(m_u - m_d)Y_B^r . \tag{34}
$$

In summary, the UV-renormalized mixing amplitude reads

$$
\Pi_{\rho\omega}^c (p^2) = 2M_\rho(m_u - m_d) + \frac{4\pi\alpha F_\rho F_\omega}{3} + \frac{\sqrt{2} G_\rho B_{\rho\omega}}{12 F_\rho^2} (m_u - m_d)p^4 \left( \frac{6m_\rho^2}{p^2} - 1 \right) \ln \frac{m_\rho^2}{\mu^2} + \frac{5}{3} - \frac{8m_\rho^2}{p^2} + \sigma^r \frac{(\sigma + 1)}{\sigma - 1} \left( \frac{m_\rho^2}{\mu^2} - \frac{h_2}{2} \right)
+ \left( -16v_8 + 4h_1 m_\rho^2 + h_2 m_\rho^2 \right) \ln \frac{m_\rho^2}{\mu^2} - \frac{h_2}{2} \right)
+ \frac{E_{\rho\omega}^\gamma (p^2)}{m_\rho^2} + X_W^r p^6 + X_Z^r p^4 + X_R^r p^2 , \tag{35}
$$

where a bar indicates the divergences are subtracted. As discussed in Ref. [35], there is an important constraint on the mixing amplitude, namely, it should vanish as $p^2 \to 0$. Thus the final expression of the renormalized mixing amplitude should be

$$
\Pi_{\rho\omega}^c (p^2) = \Pi_{\rho\omega}^c (p^2) - \Pi_{\rho\omega}^c (0) , \tag{36}
$$

where an additional finite shift is imposed so as to guarantee that the constraint $\Pi_{\rho\omega}^c (0) = 0$ is satisfied.

In our numerical computation, the scale $\mu$ will be set to $M_\rho$ and we use $(m_u - m_d) = -2.49$ MeV provided by particle data group (PDG) [39]. Furthermore, we can define

$$
f_4 = \frac{m_\rho^2 B_0}{8\pi^2 F^2} (m_u - m_d) \left( (-16v_8 + 4h_1 m_\rho^2 \right.
+ h_2 m_\rho^2) \ln \frac{m_\rho^2}{\mu^2} - \frac{h_2}{2} \right) , \tag{37}
$$

and in principle the unknown parameters in Eq. (35) are $a$, $f_4$, $X_W^r$, $X_Z^r$ and $X_R^r$.  

IV. THE EFFECT OF $\rho - \omega$ MIXING ON PION VECTOR FORM FACTOR

The mass and width of $\rho$ meson are conventionally determined by fitting to the data of $e^+e^- \to \pi^+\pi^-$ and $\tau \to \nu_\tau 2\pi$ [39], where various mechanisms are introduced to describe the $\rho - \omega$ mixing effect. To avoid intervening by their $\rho - \omega$ mixing mechanisms, we do not employ their extracted values for the mass and width, rather, we set the mass $M_\rho$, the relevant couplings $G_\rho$ and $F_\omega$ to be free parameters in our fit. As for the width, a energy-dependent form will be imposed, which is supposed to be dominated by the two $\pi$ decay channel [40]:

$$
\Gamma_\rho (s) = \frac{s M_\rho}{9\pi F_\rho^2} (1 - 4m_\rho^2/s)^{1/2} . \tag{38}
$$

For the narrow-width resonance $\omega$, we take $M_\omega = 782.65$ MeV and $\Gamma_\omega = 8.49$ MeV from PDG [39]. The physical coupling $F_\omega$ can be extracted from the decay width of $\omega \to e^+e^-$. Using the Lagrangian $\frac{F_v}{2\sqrt{2}}(V_{\mu\nu} f^{\mu_+} )$, one can derive the decay width

$$
\Gamma_\omega^{e+e^-} = \frac{4\alpha^2 F_\omega^2 (2m_\rho^2 + M_\rho^2) \sqrt{M_\omega^2 - 4m_\rho^2}}{27 M_\rho^3} , \tag{39}
$$

and get $F_\omega \approx 138$ MeV. With the decay widths given above, $s_\rho$ and $s_\omega$ in Eq. (3) now can be rewritten as

$$
s_\rho \approx s - M_\rho^2 + iM_\rho \Gamma_\rho (s) ,
\Gamma_\omega \approx s - M_\omega^2 + iM_\omega \Gamma_\omega . \tag{40}
$$

The experimental data considered in this work are the pion form factor $F_e(p^2)$ of the $e^+e^- \to \pi^+\pi^-$ process [1–3, 41–45] and $\tau \to \nu_\tau 2\pi$ decay [25, 46] in the energy region of 650–850 MeV, and the decay width of $\omega \to \pi^+\pi^-$ [39].

The Feynman amplitude for the process $\gamma^* \to \pi^+\pi^-$, proceeding via virtual intermediate hadrons, i.e., $\rho$, $\omega$ and their mixing, is described by [35]

$$
M_{\gamma^* \to \pi\pi} = M_{\gamma^* \to \rho_1} \frac{1}{s_\rho} M_{\rho_1 \to \pi\pi}
+ M_{\gamma^* \to \omega_1} \frac{1}{s_\omega} M_{\omega_1 \to \pi\pi}
+ M_{\gamma^* \to \omega_1} \frac{1}{s_\omega} M_{\omega_1 \to \pi\pi} .
$$
FIG. 2: Fit results for the pion form factor in the $e^+e^- \rightarrow \pi^+\pi^-$ process (left panel) and $\tau \rightarrow \nu_\tau 2\pi$ process (right panel). The data of $e^+e^-$ annihilation are taken from the OLYA and CMD [41], CMD2 [42, 43], DM1 [44], SND [45], KLOE [1, 2], BESIII [3] collaborations. The $\tau$ decay data are taken from the ALEPH [16] and CLEO [25] collaborations. The solid lines are our theoretical predictions.

\[
F_{\pi}^{ex}(p^2) = 1 - \frac{G_\rho F_\rho p^2}{F_s^2} - \frac{G_\rho F_\rho p^2}{3F_s^2} - \frac{4\sqrt{2}aB_\omega (m_u - m_d)p^2}{3F_s^2} \frac{1}{s_\omega},
\]

where $M_\rho$ and $G_\rho$ are the mass and weak coupling constant of the $\rho$ meson, respectively, $F_s$ is the decay constant of the $\rho$ meson, $a$ is the coupling constant of $\rho \omega \chi$ contact interaction, and $B_\omega$ is the coefficient of the $\rho \omega \chi$ contact interaction. $s_\omega$ is the helicity 1 component of the $\omega$ meson polarized in the $\rho \omega \chi$ contact interaction. The fit results are plotted in the Fig. 2. One can see that the experimental data of pion form factor, especially the kink around the mass of $\omega$ in the $e^+e^- \rightarrow \pi^+\pi^-$ process, is well described.

| Fit results |
|----------------|
| $M_\rho$ [MeV] | 775.3 ± 0.3 |
| $G_\rho$ [MeV] | 67.0 ± 3.0 |
| $F_s$ [MeV] | 152.9 ± 6.8 |
| $a$ [MeV$^{-1}$] | $(-1.28 \pm 0.7) \times 10^{-6}$ |
| $X_0$ [MeV$^{-1}$] | $(7.3 \pm 0.2) \times 10^{-11}$ |
| $X_1$ [MeV$^{-1}$] | $(-5.5 \pm 0.6) \times 10^{-11}$ |
| $X_2$ [MeV$^{-2}$] | $(-1.1 \pm 0.1) \times 10^{-4}$ |
| $f_\rho$ [MeV$^2$] | $(1.3 \pm 0.4) \times 10^4$ |

TABLE I: The fit results of the parameters.

In Fig. 3, contributions at different orders to the real and imaginary parts of the pion form factor $F_{\pi}^{ex}(s)$ are displayed. The leading-order contribution (mixing-effect irrelevant) includes the contact interaction and the $\rho$-mediated mechanism, namely the first two terms on the right side of Eq. (41). The next-to-leading-order contribution includes the $\rho-\omega$ mixing term and the direct...
the same order, but with opposite sign. Note that from Eq. (41) that the dominant contribution is from the imaginary part around the peak, and accounts for that the $e^+e^-$ data are higher than the $\tau$ data in that region as shown in Fig. 2. Similar behavior has also been observed in Ref. [11] where the $\rho - \omega$ mixing was treated in hidden local symmetry model.

In Fig. 4, we plot the real and imaginary parts of the mixing amplitude $\Pi_{\rho\omega}(s)$. It is found that the real part is dominant almost in all the region and its momentum-dependence is significant. Compared to the real part, the imaginary part is rather small. For the imaginary part, the contributions from $\pi\pi$ loop and $\pi\gamma$ loop are of the same order, but with opposite sign. Note that the $\pi$-tadpole is real and $s$-independent as can be seen from Eq. (25). The smallness of the imaginary part is consistent with the observation in Refs. [5, 48], though therein the effect of direct $\omega_\rho \to \pi^+\pi^-$ was not taken into account and even in [5] the isospin breaking is considered to be purely electromagnetic origin. We also note that larger imaginary part is obtained in [8, 13] by using global color model and a chiral constituent quark model, respectively. However, our finding is more reliable in the sense that it is based on a model-independent description of the $\rho - \omega$ mixing and, moreover, constraint from experimental data is imposed by means of fitting.

The values of $\Pi_{\rho\omega}$ at physical masses of $\rho$ or $\omega$ are interesting since they are related to the mixing parameters given in Eq. (6). To that end, we obtain: at $s = M_\rho^2$, $\Pi_{\rho\omega}(M_\rho^2) = (-2380 - 40.8i)\text{ MeV}^2$ and $\epsilon_2 = 0.21$; at $s = M_\omega^2$, $\Pi_{\rho\omega}(M_\omega^2) = (-2743.4 - 44.4i)\text{ MeV}^2$ and $\epsilon_1 = 0.24$. As expected, $\epsilon_1$ and $\epsilon_2$ come out to be almost the same. Note that, in the numerical calculation of $\epsilon_i$, we have neglected the small imaginary part of the mixing amplitude as well as the widths of the $\rho$ and $\omega$ resonances. This leads to a real number of $\epsilon_i$ and hence a probability interpretation can be assigned.

Using the central values of the fitted parameters in Table I, we calculate the decay width of $\omega \to \pi^+\pi^-$

$$\Gamma_{\omega \to \pi^+\pi^-} = \frac{1}{192\pi F^4} (M_\omega^2 - 4m_n^2)^{3/2} \sqrt{2} B_0(m_u - m_d) a + \frac{2G_F \Pi_{\rho\omega}(M_\omega^2)}{M_\rho^2 - M_\omega^2 - i(M_\omega \Gamma_\rho - M_\rho \Gamma_\rho)}.$$

FIG. 3: The real and imaginary parts of the fitted form factor $F_{\pi\pi}^e(s)$. The black solid and red dashed lines represent our best results of the real and imaginary parts, respectively. The blue dotted and cyan dash-dot-dotted lines correspond to the leading order and the second order contributions of the real parts, respectively. The magenta dash-dotted and green short dash-dash-dotted lines denote the leading order and second order contributions of the imaginary parts.
\[
\rho \to \omega \quad \text{is less than the second term due to the } \rho - \omega \text{ mixing by two orders. In other words, the direct } \omega I \pi \pi \text{ coupling only affects the decay width less than one percent. Within 1\sigma uncertainties, our theoretical value of the branching fraction is } B(\omega \to \pi^+ \pi^-) = (1.53 \pm 0.10) \times 10^{-2}, \text{ which agrees with the values given in PDG} [39] \text{ and by the recent dispersive analysis} [49].
\]

\section*{V. SUMMARY}

We have analyzed the \( \rho - \omega \) mixing within the framework of resonance chiral theory. Based on the effective Lagrangians constructed under the guidance of various symmetries, we calculate the \( \rho - \omega \) mixing amplitude up to next-to-leading order in large \( 1/N_C \) expansion. Importantly, the momentum-dependent effect is implemented due to the inclusion of loops in our calculation. The values of the resonance couplings are determined by fitting to the data of the pion vector form factor extracted from the \( e^+e^- \to \pi^+\pi^- \) process and \( \tau \to \nu_\tau 2\pi \) decay. The decay width of \( \omega \to \pi^+\pi^- \) is served an additional constraint in the fit as well. It is found that the imaginary part of the pion form factor \( F_2^{\pi\pi}(s) \) is enhanced largely around the \( \rho \) peak. The \( \rho - \omega \) mixing amplitude is dominated by its real part almost in all the region, which is significantly momentum-dependent. On the contrary, its imaginary part is relatively small. We also find that \( \rho - \omega \) mixing plays a major role in the decay width of \( \omega \to \pi^+\pi^- \), and its contribution is two orders of magnitude larger than that from the direct \( \omega I \pi \pi \) coupling.

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