Cohomology of abelian arrangements

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Abstract

An abelian arrangement is a finite set of codimension one abelian subvarieties (possibly translated) in a complex abelian variety. In this paper, we study the cohomology of the complement of an abelian arrangement, denoted by $M(\mathcal{A})$. For unimodular abelian arrangements, we provide a combinatorial presentation for a differential graded algebra whose cohomology is isomorphic to the rational cohomology of $M(\mathcal{A})$. Moreover, this DGA has a bi-grading that allows us to compute the mixed Hodge numbers.

1 Introduction

The goal of this paper is to study the cohomology of the complement of an abelian arrangement. Here, an abelian arrangement is a finite set of codimension one abelian subvarieties (possibly translated) in a complex abelian variety $X$. A special case of this is the configuration space of $n$ points on an elliptic curve $E$. This is the complement of an elliptic version of the braid arrangement in $E^n$, where for $1 \leq i < j \leq n$, we have an abelian subvariety $Y_{ij}$ given by the equation $e_i = e_j$. A more general special case is given by any $n \times \ell$ integer matrix, defining $\ell$ subvarieties in $E^n$. Here, each column of the matrix represents an algebraic map $\alpha : E^n \to E$, and we consider the subvariety $Y = \ker \alpha$ for each such $\alpha$.

Totaro [11] and Kriz [5] independently determined the cohomology of configuration spaces of smooth complex projective varieties. In particular, their work determines the cohomology in our special case of a configuration space on an elliptic curve. In this paper, we generalize Totaro’s method to compute the rational cohomology of the complement of any abelian arrangement $\mathcal{A}$ in a complex abelian variety $X$. We denote this complement by $M(\mathcal{A})$, and arrive at our results by studying the Leray spectral sequence of the inclusion $f : M(\mathcal{A}) \hookrightarrow X$. Specifically, we use Hodge theory to show degeneration of this spectral sequence at the $E_3$ term.
Our results are particularly nice in the case where $A$ is unimodular, which means that all intersections of subvarieties in $A$ are connected. In this case, we give a presentation of a differential graded algebra $A(A)^*$ in terms of the combinatorics of $A$ (the partially ordered set consisting of all intersections of subvarieties in $A$). The cohomology of $A(A)^*$ with respect to its differential is isomorphic as a graded algebra to the cohomology of $M(A)$, by Theorem 4.1. Moreover, $A(A)$ admits a second grading, and it is canonically isomorphic as a bi-graded algebra to $\text{gr} H^*(M(A); \mathbb{Q})$, the associated graded with respect to Deligne’s weight filtration. Thus it allows us to compute the mixed Hodge numbers of $M(A)$.

Remark 1.1. While the weight filtration on the cohomology of the complement of a linear or toroidal arrangement is trivial (by [7]), for an abelian arrangement it is always interesting. For example, consider a punctured elliptic curve $M(A) = E - \{p_1, \ldots, p_\ell\}$. Here, the first filtered piece of $H^1(M(A); \mathbb{Q})$ consists of the image of the restriction map from $H^1(E; \mathbb{Q})$, which is neither trivial nor surjective.

Levin and Varchenko [6] computed cohomology of elliptic arrangements with coefficients in a nontrivial rank one local system. Dupont [4] also studied the more general case of the complement to a union of smooth hypersurfaces which intersect like hyperplanes in a smooth projective variety. Dupont used a similar but alternative method to that presented in this paper to find the same model for cohomology as described in Section 3, but he does not give the combinatorial presentation in Section 4.

Acknowledgements. The author would like to thank her advisor Nick Proudfoot for his many helpful comments, suggestions, and guidance. The author would also like to thank Dev Sinha for his advice.

2 Preliminaries

We consider an arrangement $A = \{Y_1, \ldots, Y_\ell\}$ of smooth connected divisors in a smooth complex variety $X$, which intersect like hyperplanes. When we say that they intersect like hyperplanes, we mean that for every $p \in X$, there is a neighborhood $U \subseteq X$ of $p$, a neighborhood $V \subseteq T_pX$ of $0$, and a diffeomorphism $\varphi : U \to V$ that induces $Y_i \cap U \cong T_pY_i \cap V$ for all $Y_i \in A$.

Example 2.1. Let $A = \{Y_1, \ldots, Y_\ell\}$ be an abelian arrangement in an abelian variety $X$, that is, each $Y_i$ is, up to translation, a codimension-one abelian subvariety. Then the subvarieties in the arrangement intersect like hyperplanes.
A **component** of the arrangement \( \mathcal{A} \) is a connected component of an intersection \( Y_S := \bigcap_{Y \in S} Y \) for some subset \( S \subseteq \mathcal{A} \). Note that the intersections themselves need not be connected. We say that the arrangement is **unimodular** if the intersection \( Y_S \) is connected for all subsets \( S \subseteq \mathcal{A} \). The **rank** of a component is defined as its complex codimension in \( X \). For a subset \( S \subseteq \mathcal{A} \) with nonempty intersection \( Y_S \), consider a connected component \( F \) of \( Y_S \). If \( \text{rk}(F) = |S| \), we say that \( S \) is **independent**. Otherwise, \( \text{rk}(F) < |S| \) and we say that \( S \) is **dependent**. Also let \( M(\mathcal{A}) = X \setminus \bigcup_{Y \in \mathcal{A}} Y \) be the complement of the union of the divisors in \( \mathcal{A} \).

**Example 2.2.** We describe the motivation behind the terminology using our special case of an elliptic arrangement, where we have \( X = E^n \) and an \( n \times \ell \) integer matrix. As described in the introduction, each column corresponds to a map \( \alpha_i : E^n \to E \). Taking \( Y_i = \ker \alpha_i \) for \( i = 1, \ldots, \ell \) defines an abelian arrangement in \( X \).

In this case, an intersection \( Y_S \) is the kernel of an \( n \times |S| \) submatrix, taking the corresponding columns \( \alpha_i \) for \( Y_i \in S \). The codimension of \( Y_S \) is the rank of the corresponding matrix. In this way, the dependencies of the hyperplanes in \( \mathcal{A} \) correspond to the dependencies of the corresponding \( \alpha_i \)'s in \( \mathbb{Z}^n \).

Further suppose that \( \mathcal{A} \) is a unimodular arrangement and that the rank of the \( n \times \ell \) matrix is equal to \( n \). Then all \( n \times n \) submatrices will have determinant \( \pm 1 \) or 0. Otherwise, an intersection of subvarieties (that is, the kernel of the corresponding submatrix) would be disconnected. This agrees with the usual notion of a unimodular matrix.

Let \( F \) be a component of the arrangement \( \mathcal{A} \). For any point \( p \in F \), define an arrangement \( \mathcal{A}_F^{(p)} \) in the tangent space \( T_pX \) consisting of hyperplanes \( Y_F^{(p)} := T_pY \) for all \( Y \supseteq F \). If \( X \) has complex dimension \( n \), then \( \mathcal{A}_F^{(p)} \) is a central hyperplane arrangement in \( T_pX \cong \mathbb{C}^n \), and we denote its complement by \( M(\mathcal{A}_F^{(p)}) = T_pX \setminus \bigcup_{Y \supseteq F} Y_F^{(p)} \). This arrangement may be referred to as the localization of \( \mathcal{A} \) at \( F \), with respect to the point \( p \in F \).

**Remark 2.3.** We say that a point \( p \in F \) is a generic point of \( F \) if \( p \) is not contained in any smaller component of \( \mathcal{A} \). By our assumption that the divisors intersect like hyperplanes, for a generic point \( p \in F \), there is a neighborhood \( U \subseteq X \) of \( p \) such that \( U \cap M(\mathcal{A}) \cong M(\mathcal{A}_F^{(p)}) \).

**Remark 2.4.** Also by our assumption that the divisors intersect like hyperplanes, the intersection lattice of the arrangement \( \mathcal{A}_F^{(p)} \) does not depend
on the choice of $p \in F$. Since the cohomology of $M(A_F^{(p)})$ only depends on the combinatorics of $A_F^{(p)}$ (by [9]), we may write $H^*(M(A_F); \mathbb{Q})$ to mean the cohomology of $M(A_F^{(p)})$ for some (any) $p \in F$.

If $A$ is an abelian arrangement, then even more can be said. Not only does the cohomology not depend on the choice of $p \in F$, but for any two points $p$ and $q$ of $F$, we have a canonical homeomorphism (via translation) $M(A_F^{(p)}) \cong M(A_F^{(q)})$.

3 Rational Cohomology

Let $A = \{Y_1, \ldots, Y_t\}$ be a set of smooth connected divisors that intersect like hyperplanes in a smooth complex variety $X$, and denote the complement of their union in $X$ by $M(A)$. The inclusion $f : M(A) \hookrightarrow X$ gives a Leray spectral sequence of the form

$$H^p(X; R^q f_* \mathbb{Q}) \Rightarrow H^{p+q}(M(A); \mathbb{Q}).$$

Recall that $R^q f_* \mathbb{Q}$ is the sheafification of the presheaf on $X$ taking an open set $U$ to $H^q(U \cap M(A); \mathbb{Q})$. To make use of this spectral sequence, we need to examine the sheaves $R^q f_* \mathbb{Q}$.

For any $x \in X$, we can take the unique smallest component $F_x$ of $A$ containing $x$. Note that $x$ is a generic point of $F_x$, and so for a small neighborhood $U$ around $x$, we have $U \cap M(A) \cong M(A^{(x)}_{F_x})$. This means that the stalk of our sheaf $R^q f_* \mathbb{Q}$ at $x$ is given by

$$H^q(U \cap M(A); \mathbb{Q}) \cong H^q(M(A^{(x)}_{F_x}); \mathbb{Q}).$$

Note that the rank-$q$ components (flats) of $A^{(x)}_{F_x}$ correspond exactly to the rank-$q$ components of $A$ that contain $F_x$. For such an $F$, we can consider the usual localization of $A^{(x)}_{F_x}$ (a central hyperplane arrangement) at the component (flat) corresponding to $F$ in $A^{(x)}_{F_x}$, denoted by $(A^{(x)}_{F_x})_F$. This is the same arrangement as $A^{(F)}_F$. Then Brieskorn’s Lemma [1, p. 27] implies that

$$H^q(M(A^{(x)}_{F_x}); \mathbb{Q}) \cong \bigoplus_{F \supseteq F_x, \text{rk}(F)=q} H^q(M((A^{(x)}_{F_x})_F); \mathbb{Q}) \cong \bigoplus_{F \supseteq F_x, \text{rk}(F)=q} H^q(M(A_F); \mathbb{Q}).$$

Since $x$ was a generic point of $F_x$, the rank-$q$ components containing $F_x$ are exactly the rank-$q$ components containing $x$. Thus, the stalk at $x \in X$ can
be decomposed as

$$(R^q f_* Q)_x \cong \bigoplus_{F \in x, \rk(F) = q} H^q(M(A_F); \mathbb{Q}).$$

Now, it is clear that the sheaf $R^q f_* Q$ is supported on the union of the rank-$q$ components of $A$. We will define a sheaf $\epsilon_F$, for each component $F$, that is supported on $F$. Then we will show that $R^q f_* Q$ is isomorphic to the direct sum of these constant sheaves $\epsilon_F$, taken over all rank-$q$ components. This will help us prove the following lemma:

**Lemma 3.1.** Let $A = \{Y_1, \ldots, Y_\ell\}$ be a set of smooth connected divisors that intersect like hyperplanes in a smooth complex variety $X$. Then

$$H^p(X; R^q f_* Q) \cong \bigoplus_{\rk(F) = q} H^p(F; \mathbb{Q}) \otimes H^q(M(A_F); \mathbb{Q}).$$

**Proof.** Let $F$ be a component of rank $q$, and consider the divisors in $A$ that contain $F$. These form an arrangement in $X$, which we denote by $A|_F$. Also denote its complement in $X$ by $M(A|_F)$. The inclusion $g^F : M(A|_F) \hookrightarrow X$ defines a sheaf $\epsilon_F := R^q g^F_* \mathbb{Q}$ on $X$. First, observe that the support of $\epsilon_F$ is equal to $F$. For any $x \in F$, there is a small neighborhood $U$ of $x$ in $X$ such that $U \cap M(A|_F) \cong M(A_F)$. This means that the stalk at any point $x \in F$ is $(\epsilon_F)_x \cong H^q(M(A_F); \mathbb{Q})$.

Moreover, the sheaf $\epsilon_F$ is constant on $F$. This is because, as we have previously discussed, $H^*(M(A_{\{x\}}); \mathbb{Q})$, and hence the stalks of $\epsilon_F$, can be canonically identified for any two points in $F$.

Let $\epsilon := \bigoplus_{\rk(F) = q} \epsilon_F$, a sheaf on $X$. The stalk at $x \in X$ is

$$\epsilon_x = \bigoplus_{\rk(F) = q} (\epsilon_F)_x \cong \bigoplus_{\rk(F) = q, x \in F} H^q(M(A_F); \mathbb{Q}).$$

For every open $U \subset X$, there is an inclusion $U \cap M(A) \hookrightarrow U \cap M(A|_F)$, which then induces a map $H^q(U \cap M(A|_F); \mathbb{Q}) \to H^q(U \cap M(A); \mathbb{Q})$. This gives a (presheaf) map $\epsilon \to R^q f_* Q$. It is also an isomorphism on stalks, hence a sheaf isomorphism $\epsilon \cong R^q f_* Q$.

Returning to the $E_2$ term of our Leray spectral sequence for the inclusion.
f : M(A) \hookrightarrow X, we now have that

\[ H^p(X; R^q f_* \mathbb{Q}) \cong H^p(X; \epsilon) \]

\[ \cong \bigoplus_{\text{rk}(F) = q} H^p(F; \epsilon_F) \]

\[ \cong \bigoplus_{\text{rk}(F) = q} H^p(F; \mathbb{Q}) \otimes H^q(M(A_F); \mathbb{Q}). \]

If we further take X to be a projective variety, then the $E_2$ term of the spectral sequence is all that is needed to calculate the cohomology of $M(A)$.

**Lemma 3.2.** Let $A = \{Y_1, \ldots, Y_n\}$ be a set of smooth connected divisors that intersect like hyperplanes in a smooth complex projective variety $X$, and denote its complement by $M(A)$. Then all differentials $d_j$ in the Leray spectral sequence for the inclusion $f : M(A) \hookrightarrow X$ are trivial for $j > 2$.

**Proof.** To show that higher differentials are trivial, we consider the weight filtration on

\[ H^p(X; R^q f_* \mathbb{Q}) \cong \bigoplus_{\text{rk}(F) = q} H^p(F; \mathbb{Q}) \otimes H^q(M(A_F); \mathbb{Q}). \]

Note that since $F$ is a smooth complex projective variety, $H^p(F; \mathbb{Q})$ is pure of weight $p$. Since $M(A_F)$ is the complement of a rational hyperplane arrangement, $H^q(M(A_F); \mathbb{Q})$ is pure of weight $2q$ (by [10]). This implies that $H^p(X; R^q f_* \mathbb{Q})$ is pure of weight $p + 2q$.

Now, the differentials must be strictly compatible with the weight filtration. Since the $(p, q)$ position on the $E_2$ term will also have weight $p + 2q$, the differential $d_j$ will map something of weight $p + 2q$ to something of weight $(p + j) + 2(q - j + 1) = p + 2q - j + 2$. Being strictly compatible with weights implies that the only nontrivial differential must be when $j = 2$. \qed

Moreover, if we consider only the cohomological grading (by $p + q$) on the $E_2$ term, we have the following theorem.

**Theorem 3.3.** Let $A = \{Y_1, \ldots, Y_n\}$ be a set of smooth connected divisors that intersect like hyperplanes in a smooth complex projective variety $X$. The rational cohomology of $M(A)$ is isomorphic as a graded algebra to the cohomology of $E_2(A)$ with respect to its differential.
Proof. By Lemma 3.2, the Leray spectral sequence degenerates at the $E_3$ term. This implies that the associated graded of $H^*(M(A); \mathbb{Q})$ with respect to the Leray filtration is isomorphic to the cohomology of $E_2(A)$.

The groups $E_{3}^{p,q} = E_{\infty}^{p,q}$ that contribute to the $k$-th rational cohomology (when $p + q = k$) each have distinct weight (as described in the proof of Lemma 3.2), and so the Leray filtration is exactly the weight filtration. By the work of Deligne [3, p. 81], the associated graded of $H^*(M(A); \mathbb{Q})$ with respect to its weight filtration is isomorphic to $H^*(M(A); \mathbb{Q})$ as an algebra.

Remark 3.4. The $E_2$ term of the spectral sequence forms a differential bi-graded algebra, denoted by $E_2(A)$. The main result of this section was that

$$H^*(E_2(A)) \cong \text{gr} H^*(M(A); \mathbb{Q}),$$

where the right hand side is the associated graded with respect to the weight filtration. In particular, if we consider the bi-grading (and not just the cohomological grading) of $E_2(A)$, we can compute the mixed Hodge numbers of $M(A)$.

Remark 3.5. The same method could be used to study the cohomology of an affine hyperplane arrangement in $\mathbb{C}^n$ or of a toric arrangement in $(\mathbb{C}^\times)^n$. In these cases, Lemma 3.1 applies, but Lemma 3.2 and Theorem 3.3 do not.

1. Let $A = \{H_1, \ldots, H_\ell\}$ be an affine arrangement of hyperplanes in a complex affine space $X$ of dimension $n$, and denote $M(A) = X \setminus \cup_i H_i$. For the Leray Spectral Sequence of the inclusion $f : M(A) \hookrightarrow X$, the $E_2$-term decomposes into $E_2^{0,q} = \bigoplus_{r_k(F) = q} H^q(M(A_F); \mathbb{Q})$ and $E_2^{p,q} = 0$ for $p \neq 0$. This forces the differentials to all be trivial, and we see that $E_2(A)$ is the Orlik-Solomon algebra $H^*(M(A); \mathbb{Q})$.

2. Let $A = \{T_1, \ldots, T_\ell\}$ be an arrangement of codimension-one subtori in a complex torus $X = (\mathbb{C}^\times)^n$, and denote the complement by $M(A) = X \setminus \cup_i T_i$. The $E_2$-term of the Leray Spectral Sequence for the inclusion $f : M(A) \hookrightarrow T$ decomposes into components, so that

$$E_2^{p,q} = \bigoplus_{r_k(F) = q} H^p(F; \mathbb{Q}) \otimes H^q(M(A_F); \mathbb{Q}).$$

Here, $F$ is a complex torus and so $E_2^{p,q}$ is pure of weight $2(p+q)$. Since the differentials $d_j$ respect the weights, $d_j$ must be trivial for all $j$. Thus, $E_2(A) \cong \text{gr}_k H^*(M(A); \mathbb{Q})$, the associated graded with respect to the Leray filtration. This decomposition of the cohomology is the
decomposition given by De Concini and Procesi in [2, Remark 4.3]. However, \( E_2(\mathcal{A}) \) and \( H^*(M(\mathcal{A}); \mathbb{Q}) \) are not isomorphic as algebras in this case.

**Remark 3.6.** Another interesting result for an abelian arrangement \( \mathcal{A} \) in \( X \) comes from considering the deletion and restriction arrangements, with respect to some fixed \( Y_0 \in \mathcal{A} \). Here, we mean the analogous notion to the theory of hyperplane arrangements, where the deletion of \( Y_0 \) is the arrangement \( \mathcal{A}' = \mathcal{A} \setminus \{Y_0\} \) in \( X \) and the restriction to \( Y_0 \) is the arrangement of nonempty \( \{Y \cap Y_0 \} \in \mathcal{A}' \). In the theory of hyperplane arrangements, the long exact sequence of the pair \( (M(\mathcal{A}'), M(\mathcal{A})) \) relates the cohomologies of \( M(\mathcal{A}) \), \( M(\mathcal{A}') \), and \( M(\mathcal{A}'') \). Moreover, this long exact sequence splits into short exact sequences relating these cohomologies. In the abelian arrangement case, we can get the same kind of long exact sequence. However, it does not split into short exact sequences. To study the (non-trivial) boundary map, we can derive this long exact sequence in another way, by taking the long exact sequence induced by a short exact sequence of complexes

\[
0 \to E_2(\mathcal{A}')^* \to E_2(\mathcal{A})^* \to E_2(\mathcal{A}'')^* - 1 \to 0.
\]

The boundary map is then seen to be

\[
\pi_* : H^{i-1}(M(\mathcal{A}''); \mathbb{Q}) \to H^{i+1}(M(\mathcal{A}'); \mathbb{Q})
\]

where \( \pi : M(\mathcal{A}'') \hookrightarrow M(\mathcal{A}') \) is the closed immersion.

**Remark 3.7.** Dupont [4] independently found the same differential graded algebra as described here. He considers the cohomology of the complement of a union \( Y = Y_1 \cup \cdots \cup Y_\ell \) of smooth hypersurfaces which intersect like hyperplanes in a smooth complex projective variety \( X \), and for simplicity he assumes that the arrangement is unimodular. Dupont’s method uses the Gysin spectral sequence, which degenerates at the \( E_2 \) term and has a differential graded algebra \( M^*(X,Y) \) as the \( E_1 \) term. Setting \( \mathcal{A} = \{Y_1, \ldots, Y_\ell\} \), the differential graded algebras \( E_2(\mathcal{A}) \) and \( M^*(X,Y) \) are isomorphic. Moreover, Dupont constructs a wonderful compactification of these arrangements, so that the space \( X \setminus Y \) can be realized as the complement of a normal crossings divisor \( \bar{Y} \) in a smooth projective variety \( \bar{X} \). Dupont also shows functoriality of \( M^* \) so that \( M^*(X,Y) \) is quasi-isomorphic to \( M^*(\bar{X},\bar{Y}) \).

By the work of Morgan [8], the differential graded algebra \( M^*(\bar{X},\bar{Y}) \) is a model for the space \( X \setminus Y = \bar{X} \setminus \bar{Y} \), in the sense of rational homotopy theory. Since our differential graded algebra \( E_2(\mathcal{A}) \) is isomorphic to \( M^*(X,Y) \)
and hence quasi-isomorphic to $M^*(\bar{X}, \bar{Y})$, $E_2(A)$ is a model for the space $M(A) = X \setminus Y$.

4 Unimodular Abelian Arrangements

To explicitly describe the $\mathbb{Q}$-algebra structure of the $E_2$ term of the spectral sequence, we assume further that $A$ is a unimodular abelian arrangement. Recall that we allow the $Y_i \in A$ to be a translation of an abelian subvariety of $X$; denote this subvariety by $\bar{Y}_i$. For each $Y_i \in A$, let $E_i = X/\bar{Y}_i$ so that $\bar{Y}_i$ is the kernel of the projection $\alpha_i : X \to E_i$. The $E_2$ term is a bi-graded algebra with a differential, which we denote by $E_2(A)$. The $(p, q)$-th graded term is isomorphic to

$$\bigoplus_{\text{rk}(F) = q} H^p(F; \mathbb{Q}) \otimes H^q(M(A_F); \mathbb{Q})$$

by Lemma 3.1.

The multiplication of $E_2(A)$ can be described as follows: Let $x \otimes x'$ be in $H^p(F_1; \mathbb{Q}) \otimes H^q(M(A_{F_1}); \mathbb{Q})$, $y \otimes y'$ be in $H^p(F_2; \mathbb{Q}) \otimes H^q(M(A_{F_2}); \mathbb{Q})$. If $F_1 \cap F_2 = \emptyset$, then $(x \otimes x') \cdot (y \otimes y') = 0$. Otherwise, let $F = F_1 \cap F_2$ (which by unimodularity is a component of $A$), $p = p_1 + p_2$, and $q = q_1 + q_2$. Also let, for $j = 1, 2$, $\gamma_j : F \hookrightarrow F_j$ and $\eta_j : M(A_F) \hookrightarrow M(A_{F_j})$ be the natural inclusions. Then

$$(x \otimes x') \cdot (y \otimes y') = (\gamma_1^*(x) \cup \gamma_2^*(y)) \otimes (\eta_1^*(x') \cup \eta_2^*(y')),$$

an element of $H^p(F; \mathbb{Q}) \otimes H^q(M(A_F); \mathbb{Q})$.

In particular, consider the case that $F_1 = Y_i$ and $F_2 = X$. For $1 \otimes g$ in $H^0(Y_i; \mathbb{Q}) \otimes H^1(M(A_{Y_i}); \mathbb{Q})$ and $x \otimes 1$ in $H^p(X; \mathbb{Q}) \otimes H^0(M(A_X); \mathbb{Q})$, we have

$$(1 \otimes g) \cdot (x \otimes 1) = \gamma_i^*(x) \otimes g \in H^p(Y_i) \otimes H^q(M(A_{Y_i}); \mathbb{Q}).$$

Since $Y_i$ is (a possible translation of) the kernel of some map $\alpha_i : X \to E_i$, the kernel of $\gamma_i^*$ contains the image of $\alpha_i^*$ in positive degree. This means that for $p > 0$, and any element $x \in H^p(X; \mathbb{Q})$ that is in the image of $\alpha_i^*$, $(1 \otimes g) \cdot (x \otimes 1) = 0$.

We further observe that the row $q = 0$ inherits an algebra structure from $H^*(X; \mathbb{Q})$, and the column $p = 0$ inherits an algebra structure from the Orlik-Solomon algebra. In particular, if $\cap_{Y \in A} Y \neq \emptyset$, then the column $p = 0$ inherits an algebra structure from $H^*(M(A_0); \mathbb{Q})$ where $A_0$ is the localization at the intersection of all hyperplanes in $A$. These algebras are
generated in degree one; moreover, they will generate the entire $E_2(A)$ algebra. This is because the map $\gamma^* : H^*(X; \mathbb{Q}) \to H^*(F; \mathbb{Q})$, where $F$ is a component and $\gamma : F \hookrightarrow X$ is the natural inclusion, is surjective.

Since the algebra is generated by $E_{2,0}^1$ and $E_{2,1}^0$, it suffices to describe the differential on $H^0(Y_i; \mathbb{Q}) \otimes H^1(M(A_{Y_i}); \mathbb{Q})$ for each $Y_i \in A$. This has a canonical generator, since the Orlik-Solomon algebra $H^*(M(A_{Y_i}); \mathbb{Q})$ has a canonical generator in degree one. The differential here is determined by the differential of the Leray spectral sequence for the inclusions $X \setminus Y_i \hookrightarrow X$, which takes the generator to $[Y_i] \in H^2(X; \mathbb{Q})$.

Now we will describe an algebra $A(A)$, determined by the arrangement $A$, and prove in Theorem 4.1 that this algebra is isomorphic to $E_2(A)$. Let $B(A) = H^*(X; \mathbb{Q})[g_1, \ldots, g_l]$, a graded-commutative, bigraded algebra over $\mathbb{Q}$, where $H^i(X; \mathbb{Q})$ has degree $(i, 0)$ and each $g_j$ has degree $(0, 1)$. Let $I(A)$ be the ideal in $B(A)$ generated by the following relations:

1. $g_1 \cdots g_k$ whenever $\cap_{i=1}^k Y_{y_i} = \emptyset$.

2. $\sum_{j=1}^k (-1)^{j-1} g_{1j} \cdots \hat{g}_j \cdots g_{k}$ whenever $Y_{1j}, \ldots, Y_{kj}$ are dependent.

3. $\alpha_i^*(x)g_i$, where $\alpha_i$ defines $Y_i$ and $x \in H^1(E_i; \mathbb{Q})$.

For notational purposes, denote $g_C = g_{i_1} \cdots g_{i_k}$ for $C = \{Y_{i_1}, \ldots, Y_{i_k}\}$ and $\partial g_C = \sum_{j=1}^k (-1)^{j-1} g_{i_1} \cdots \hat{g}_j \cdots g_{i_k}$.

Let $A(A) = B(A)/I(A)$. Since $I(A)$ is homogeneous with respect to the grading on $B(A)$, $A(A)$ is a bi-graded algebra over $\mathbb{Q}$. Moreover, there is a differential on $A(A)$ defined by $dg_i = [Y_i] \in H^2(X; \mathbb{Q})$ and $dx = 0$ for $x \in H^*(X; \mathbb{Q})$.

**Theorem 4.1.** Assume that $A = \{Y_1, \ldots, Y_l\}$ is a unimodular abelian arrangement. Then there is an isomorphism of bi-graded differential algebras

$$\varphi : A(A) \to E_2(A).$$

Before we prove this theorem, we’ll show an example in which this presentation can be used to compute the cohomology of $M(A)$. Moreover, if we consider the bi-grading on $A(A) \cong E_2(A)$, then we can compute the dimension of $gr_j H^i(M(A); \mathbb{Q})$, the associated graded with respect to the weight filtration. By Remark 3.3, the $(p, q)$-th graded piece of $H^i(A(A))$ will be isomorphic to $gr_{p+2q} H^{p+q}(M(A); \mathbb{Q})$. We encode the information about dimension in a two-variable polynomial $H(t, u)$, where the coefficient of $t^i u^j$ is the dimension of $gr_j H^i(M(A); \mathbb{Q})$. 

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Example 4.2. Let $X = E^2$ for an elliptic curve $E$, and let $\alpha_i : E^2 \to E$ be projection onto the $i$-th coordinate. Consider the arrangement $\mathcal{A} = \{Y_1, Y_2, Y_3\}$ in $E^2$ with $Y_1 = \ker \alpha_1$, $Y_2 = \ker \alpha_2$, and $Y_3 = \ker(\alpha_1 - \alpha_2)$. Pick generators $x$ and $y$ for $H^*(E; \mathbb{Q})$. Then $H^*(E^2; \mathbb{Q})$ is generated by $x_1 = \alpha_1^*(x)$ and $y_1 = \alpha_1^*(y)$ for $i = 1, 2$. This then implies that the algebra $B(\mathcal{A})$ is the exterior algebra with generators $\{x_1, y_1, x_2, y_2, g_1, g_2, g_3\}$.

The relations in $I(\mathcal{A})$ can be written as

1. no relations of the type $gs$ (since all intersections are nonempty)
2. $g_2g_3 - g_1g_3 + g_1g_2$ (since $\{Y_1, Y_2, Y_3\}$ is minimally dependent)
3. $x_1g_1, y_1g_1, x_2g_2, y_2g_2, (x_1 - x_2)g_3, (y_1 - y_2)g_3$.

The differential of $A(\mathcal{A}) = B(\mathcal{A})/I(\mathcal{A})$ is defined by $dx_i = 0, dy_i = 0, dg_1 = [Y_1] = x_1y_1, dg_2 = [Y_2] = x_2y_2$, and $dg_3 = [Y_3] = x_1y_1 - x_1y_2 - x_2y_1 + x_2y_2$.

Computing cohomology, the polynomial described above becomes

$$H(t, u) = 1 + 4tu + 3t^2u^2 + 2t^2u^3.$$  

Setting $u = 1$, we obtain the Poincaré polynomial $P(t) = 1 + 4t + 5t^2$.

Proof of Theorem 4.1. First, we show that there is a surjective homomorphism $\varphi$, by defining a map $\theta : B(\mathcal{A}) \to E_2(\mathcal{A})$ which induces $\varphi$ as follows: Let

$$\theta(g_i) := 1 \otimes e_i \in H^0(Y_i; \mathbb{Q}) \otimes H^1(M(\mathcal{A}_Y); \mathbb{Q})$$

and for $x \in H^i(X; \mathbb{Q})$, let

$$\theta(x) := x \otimes 1 \in H^i(X; \mathbb{Q}) \otimes H^0(M(\mathcal{A}_X); \mathbb{Q}).$$

We have already observed that $E_2(\mathcal{A})$ is generated by $E_2^{1,0}$ and $E_2^{0,1}$. Even more explicitly, the algebra is generated by $1 \otimes e_i$ and $x \otimes 1$, where $e_i$ is the canonical generator of $H^1(M(\mathcal{A}_Y); \mathbb{Q})$ and $x \in H^1(X; \mathbb{Q})$. Since $\theta$ maps to these generators, it must be surjective.

By our observations above, it is easy to see that $\theta(gs) = 0$ whenever $Y_S = \emptyset$. For relation (2), suppose $S$ is a dependent subset of $\mathcal{A}$. Then

$$\theta(\partial gs) = 1 \otimes (\partial e_S) \in H^0(Y_S; \mathbb{Q}) \otimes H^{rk(S)}(M(\mathcal{A}_Y); \mathbb{Q})$$

which is zero since $\partial e_S = 0$ in the Orlik-Solomon algebra $H^*(M(\mathcal{A}_Y); \mathbb{Q})$. Also, by our observations above, $\theta(\alpha_i^*(x)g_i)$ is equal to zero. Therefore, $\theta(I(\mathcal{A})) = 0$ and hence $\theta$ induces the desired surjection $\varphi : A(\mathcal{A}) \to E_2(\mathcal{A})$. 

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We can decompose $B(A)$ with respect to the components of the arrangement, $B(A) = \bigoplus_F B_F$, where $B_F$ is the $\mathbb{Q}$-vector space spanned by $xg_S$ for all standard tuples $S$ of hyperplanes in $A$ whose intersection is exactly $F$, and all $x \in H^*(X; \mathbb{Q})$. The ideal $I(A)$ is homogeneous with respect to this grading. Thus, $A(A)$ can be decomposed by $A(A) = \bigoplus_F A_F$, where $A_F = B_F/I_F$ with $I_F := I(A) \cap B_F$.

The $E_2$ term of the Leray spectral sequence can also be graded by the components. Here, we have $E_2(A) = \bigoplus_F E_2(F)$, where for each component $F$, $E_2(F) = H^*(F; \mathbb{Q}) \otimes H^{rk(F)}(M(A_F); \mathbb{Q})$.

It suffices to show that, as $\mathbb{Q}$-vector spaces, $A_F \cong E_2(F)$. We do this by examining $A_F$. We have $B_F \cong \bigoplus_S H^*(X; \mathbb{Q}) \cdot g_S$, where the direct sum is taken over all standard tuples $S$ of hyperplanes in $A$ with $Y_S = F$. If we consider just the ideal $I_1$ generated by relations (1) and (2), then

$$B_F/(I_1 \cap B_F) \cong \bigoplus_S H^*(X; \mathbb{Q}) \cdot g_S,$$

where the sum is taken over all non-broken circuits $S$ with $Y_S = F$. This is because relations (1) and (2) just the Orlik-Solomon relation on the $g_i$'s associated to $F$.

Next, we claim that relation (3) implies that for all $Y_i \supseteq F$, all $S \subseteq A$ with $Y_S = F$, and all $x \in H^1(E_i; \mathbb{Q})$, we have $\alpha_i^*(x)g_S \in I$. This implies that relation (3) depends only on the component $F$, and not on the choice of subset $S$. This claim is clearly true when $Y_i \in S$. If $Y_i \notin S$, then take a maximal independent subset of $S$, denoted by $T$. Then $C := T \cup \{Y_i\}$ is a dependent set, and $Y_C = Y_T = F$. We may assume, for ease of notation, that our hyperplanes are ordered so that $g_S = g_{(S-T)}g_T$ and $g_C = g_i g_T$. Then we have $g_T - g_i \partial g_T = \partial g_C \in I$, since $C$ is dependent. This implies that

$$\alpha_i^*(x)g_S = \alpha_i^*(x) g_{(S-T)}g_T$$

$$= \alpha_i^*(x) g_{(S-T)}(g_T - g_i \partial g_T) + \alpha_i^*(x) g_{(S-T)}g_i \partial g_T$$

$$\in I.$$

Let $J_F$ be the ideal in $H^*(X; \mathbb{Q})$ generated by $\alpha_i^*(x)$ for all $Y_i \supseteq F$ and $x \in H^1(E_i; \mathbb{Q})$. Now, since $H^*(F; \mathbb{Q}) \cong H^*(X; \mathbb{Q})/J_F$, we must have that

$$A_F \cong \bigoplus H^*(F; \mathbb{Q}) \cdot g_S$$

where the sum is taken over all non-broken circuits $S$ with $Y_S = F$. This is then isomorphic to $H^*(F; \mathbb{Q}) \otimes H^q(M(A_F); \mathbb{Q}) \cong E_2(F)$, since the non-broken circuits form a basis for $H^q(M(A_F); \mathbb{Q})$. 

Remark 4.3. If $A$ is not unimodular, we can still define the bi-graded differential algebra $A(A)$ and the homomorphism $\varphi : A(A) \to E_2(A)$, but it will no longer be surjective. The problem is that if an intersection $Y_S$ of subvarieties has multiple components, the image of $\varphi$ will include the element $1 \in H^0(Y_S; \mathbb{Q})$, but it will not include the corresponding classes for the individual components.

References

[1] E. Brieskorn. Sur les groupes de tresses [d’aprè s V. I. Arnol’d]. In Séminaire Bourbaki, 24ème année (1971/1972), Exp. No. 401, pages 21–44. Lecture Notes in Math., Vol. 317. Springer, Berlin, 1973.

[2] C. De Concini and C. Procesi. On the geometry of toric arrangements. Transform. Groups, 10(3-4):387–422, 2005.

[3] P. Deligne. Poids dans la cohomologie des variétés algébriques. In Proceedings of the International Congress of Mathematicians (Vancouver, B. C., 1974), Vol. 1, pages 79–85. Canad. Math. Congress, Montreal, Que., 1975.

[4] C. Dupont. Hypersurface arrangements and a global Brieskorn-Orlik-Solomon theorem. ArXive-prints, Feb. 2013.

[5] I. Kriz. On the rational homotopy type of configuration spaces. Ann. of Math. (2), 139(2):227–237, 1994.

[6] A. Levin and A. Varchenko. Cohomology of the complement to an elliptic arrangement. In A. Bjorner, F. Cohen, C. Concini, C. Procesi, and M. Salvetti, editors, Configuration Spaces, CRM Series, pages 373–388. Scuola Normale Superiore, 2012.

[7] E. Looijenga. Cohomology of $\mathcal{M}_3$ and $\mathcal{M}_3^1$. In Mapping class groups and moduli spaces of Riemann surfaces (Göttingen, 1991/Seattle, WA, 1991), volume 150 of Contemp. Math., pages 205–228. Amer. Math. Soc., Providence, RI, 1993.

[8] J. W. Morgan. The algebraic topology of smooth algebraic varieties. Inst. Hautes Études Sci. Publ. Math., (48):137–204, 1978.

[9] P. Orlik and H. Terao. Arrangements of hyperplanes, volume 300 of Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. Springer-Verlag, Berlin, 1992.
[10] B. Z. Shapiro. The mixed Hodge structure of the complement to an arbitrary arrangement of affine complex hyperplanes is pure. *Proc. Amer. Math. Soc.*, 117(4):931–933, 1993.

[11] B. Totaro. Configuration spaces of algebraic varieties. *Topology*, 35(4):1057–1067, 1996.