A Linear Upper Bound on the Weisfeiler-Leman Dimension of Graphs of Bounded Genus

Martin Grohe  
RWTH Aachen University, Aachen, Germany  
grohe@informatik.rwth-aachen.de

Sandra Kiefer  
RWTH Aachen University, Aachen, Germany  
kiefer@informatik.rwth-aachen.de

Abstract
The Weisfeiler-Leman (WL) dimension of a graph is a measure for the inherent descriptive complexity of the graph. While originally derived from a combinatorial graph isomorphism test called the Weisfeiler-Leman algorithm, the WL dimension can also be characterised in terms of the number of variables that is required to describe the graph up to isomorphism in first-order logic with counting quantifiers.

It is known that the WL dimension is upper-bounded for all graphs that exclude some fixed graph as a minor [17]. However, the bounds that can be derived from this general result are astronomic. Only recently, it was proved that the WL dimension of planar graphs is at most 3 [25].

In this paper, we prove that the WL dimension of graphs embeddable in a surface of Euler genus \( g \) is at most \( 4g + 3 \). For the WL dimension of graphs embeddable in an orientable surface of Euler genus \( g \), our approach yields an upper bound of \( 2g + 3 \).

1 Introduction
The Weisfeiler-Leman (WL) algorithm is a simple combinatorial graph isomorphism test. The 1-dimensional version of the algorithm, also known as colour refinement and naive vertex classification, is known since at least the mid 1960s, and it is widely used as a subroutine in almost all practical graph isomorphism tools (see, for instance, [9, 24, 33, 34]), but also in machine learning (see, for instance, [1, 21, 28, 36, 40]). The 2-dimensional version can be traced back to an article by Weisfeiler and Leman that appeared 50 years ago [41]. It is closely related to the algebraic theory of coherent configurations. The generalisation to higher dimensions is due to Babai (see [6, 8]), and again it plays an important role as a subroutine in graph isomorphism algorithms, albeit more on the theoretical side. Notably, the \((\log n)\)-dimensional version is used as a subroutine in Babai’s quasipolynomial graph isomorphism test [8].

The connection between the WL algorithm and logic was made by Immerman and Lander [23] and Cai, Fürer, and Immerman [8]. They showed that two graphs are distinguished by the \( k \)-dimensional WL algorithm if and only if they can be distinguished in the logic \( C^{k+1} \), the \((k + 1)\)-variable fragment of first-order logic which uses counting quantifiers of the form \( \exists \geq p x \). The connection between the WL algorithm and logical definability is at the core of some of the most interesting developments in descriptive complexity theory (see, for example, [17, 22, 38]). Only recently, it was noted that the WL algorithm (and thus the finite variable counting logic) has further surprising characterisations. In a breakthrough paper, Atserias and Maneva [4] showed that the dimension \( k \) of the WL algorithm required to distinguish two graphs corresponds to the level of the Sherali-Adams relaxation of the natural integer linear program for graph isomorphism testing (also see [20, 32]). This spawned a lot of work relating the WL algorithm to semidefinite programming [5, 37] and algebraic (Gröbner basis) approaches [7, 13] to graph isomorphism testing. These results can also be phrased in terms of propositional proof complexity. The latest facet of the theory is a characterisation in
terms of homomorphism counts from graphs of tree width \(k\) \[10\]. Various aspects of the WL algorithm and its relation to logic have been studied in detail in recent years (see, for instance, \[2\], \[3\], \[12\], \[26\], \[27\], \[30\]).

Cai, Fürer, and Immerman \[8\] proved that for every \(k\) there are non-isomorphic 3-regular graphs \(G_k, H_k\) of size \(O(k)\) that cannot be distinguished by the \(k\)-dimensional WL algorithm. Thus, as an isomorphism test, the \(k\)-dimensional WL algorithm is incomplete. But, in view of the wide variety of seemingly unrelated combinatorial, logical, and algebraic characterisations of the algorithm, we are convinced that the structural information the algorithm is able to detect is of fundamental importance.

The basic parameter of the algorithm is the dimension, corresponding to the number of variables in logical and the degree of polynomials in algebraic characterisations. It yields a structural invariant called the WL dimension of a graph \(G\) \[17\], defined to be the least \(k\) such that the \(k\)-dimensional WL algorithm distinguishes \(G\) from every graph \(H\) that is not isomorphic to \(G\) (we say that \(k\)-WL identifies \(G\)), or equivalently, the least \(k\) such that \(G\) can be characterised up to isomorphism (or identified) in the logic \(C_{k+1}\). It is also convenient to define the WL dimension of a class \(C\) of graphs to be the maximum of the WL dimensions of all graphs in \(C\) if this maximum exists, or \(\infty\) otherwise. We see the WL dimension as a measure for the inherent combinatorial or descriptive complexity of a graph or class of graphs. We are mostly interested in the relation between the WL dimension and other graph invariants.

Work in descriptive complexity shows that the WL dimension is bounded for many natural graph classes, among them trees \[23\], graphs of bounded tree width \[15\], planar graphs \[14\], graphs of bounded genus \[15, 16\], all graph classes that exclude some fixed graph as a minor \[17\], interval graphs \[29, 31\], and graphs of bounded rank width \[19\]. However, most of these results do not give explicit bounds on the WL dimension, and the bounds that can be derived from the proofs are usually bad. Only recently, the second author of this paper, jointly with Ponomarenko and Schweitzer, established an almost tight bound for planar graphs \[41\]: the WL dimension of planar graphs is at most \(3\), and there are planar graphs of WL dimension \(2\).

In this paper we establish bounds for graphs that can be embedded into an arbitrary surface, for example, a torus or a projective plane. By the classification theorem for surfaces (see \[35\], Theorem 3.1.3), up to homeomorphism (that is, topological equivalence) all surfaces fall into only two countably infinite families, the family \((S_k)_{k \geq 0}\) of orientable surfaces and the family \((N_\ell)_{\ell \geq 1}\) of non-orientable surfaces. For example, the sphere \(S_0\), the torus \(S_1\), and the double torus \(S_2\) are the first three orientable surfaces, and the projective plane \(N_1\) and the Klein bottle \(N_2\) are the first two non-orientable surfaces. The Euler genus \(eg(S)\) of a surface \(S\) is \(2k\) if \(S\) is homeomorphic to the orientable surface \(S_k\), and \(\ell\) if \(S\) is homeomorphic to the non-orientable surface \(N_\ell\). We define the Euler genus of a graph \(G\) to be the least \(g\) such that \(G\) is embeddable (that is, can be drawn without edge crossings) in a surface of Euler genus \(g\) (see Figure 1 for an example).

**Theorem 1.1.** The WL dimension of a graph of Euler genus \(g\) is at most \(4g + 3\).

For graphs embeddable in orientable surfaces, we can improve the bound further.

**Corollary 1.2.** The WL dimension of a graph embeddable in an orientable surface of Euler genus \(g\) is at most \(2g + 3\).

As mentioned above, it was first proved in \[15\] that the WL dimension of graphs of bounded genus is bounded. A more detailed proof of the same result can be found in the journal paper \[16\]. Neither of the two papers gives an explicit bound on the WL dimension.
The proof of [16] only yields a quadratic bound (in terms of the genus). It seems that the proof of [15] gives a linear bound, albeit with a large constant factor of at least $80$ (not all details are worked out there, so it is difficult to determine the exact bound). The proof in both of these papers is based on the fact that sufficiently large graphs of minimum degree at least 3 embedded in a surface will have a facial cycle of length at most 6. The proof we give here is completely different. It is based on the straightforward idea of removing a non-contractible cycle to reduce the genus and then applying induction. The problem with this idea is that we cannot define non-contractible cycles, only families of such cycles that may intersect in complicated patterns. Understanding these leads to significant technical complications, but in the end enables us to obtain a much better bound than the simpler proofs of [15] [16]. Our proof is based on a simplified version of a construction from [17, Chapter 15], applied there to graphs “almost embeddable” in a surface.

Outline of the Paper

In Section 2 we introduce the conventions as well as some topological notions and facts that we use throughout the paper. In Section 3 we introduce the WL dimension and relate it to logic. In Section 5, we introduce the graph-theoretic machinery that we need in the proof of our main theorem. The proof is outlined in Section 6. The detailed proof is long and complicated, and we defer it to a technical appendix.

2 Preliminaries

We introduce the definitions and conventions regarding notation in this paper, which mostly follow [17] Chapters 9 and 15.

2.1 Graphs

All graphs in this paper are finite, simple, and undirected. For a graph $G$, we denote by $V(G)$ and $E(G)$ its set of vertices and edges, respectively. We denote an edge between vertices $v$ and $w$ by $uv$. Depending on the context, we sometimes view the edge set $E(G)$ as a subset of $\binom{V(G)}{2}$ and sometimes as an irreflexive symmetric binary relation on $V(G)$; this should cause no confusion. The order of a graph $G$ is $|G| := |V(G)|$, and we let $|\!|G\!| := |E(G)|$.

For a set $V \subseteq V(G)$, we set $N^G(V) := \{w \mid w \in V(G) \setminus V, \exists v \in V : vw \in E(G)\}$. Here, and in similar notations, we omit the superscript $G$ if $G$ is clear from the context.

For two graphs $G$ and $H$, we denote by $G \cup H$ the graph with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$. A graph $H$ is a subgraph of $G$ (we write $H \subseteq G$) if $V(H) \subseteq V(G)$
and $E(H) \subseteq E(G)$. In this case we let $N(H) := N(V(H))$. We denote by $G[V] := (V, E(G) \cap \{uv \mid u, v \in V\})$ the subgraph of $G$ induced by $V$. For a set $X$ (not necessarily a subset of $V(G)$) we let $G \setminus X := G[V(G) \setminus X]$, and for a graph $H$, we let $G \setminus H := G \setminus V(H)$.

For $k \geq 1$, the graph $G$ is $k$-connected if $|G| > k$ and for every $V \subseteq V(G)$ with $|V| < k$, the graph $G \setminus V$ is connected. A $k$-separator of $G$ is a set $S \subseteq V(G)$ of size $|S| = k$ such that there are vertices $u, v \in V(G) \setminus S$ which belong to the same connected component of $G$, but to different connected components of $G \setminus S$.

Let $H \subseteq G$. For a connected component $A$ of $G \setminus H$, the vertices in $N(A) \subseteq V(H)$ are vertices of attachment of $A$. An $H$-bridge is a subgraph $B \subseteq (V(G), E(G) \setminus E(H))$ such that either $E = \{(u, v), \{uv\}\}$ for some edge $uv \in E(G) \setminus E(H)$ or $B$ is the union of a connected component $A$ of $G \setminus H$ together with all its vertices of attachment and all edges with at least one endvertex in $A$. The vertices of attachment of an $H$-bridge $B$ are the vertices in $V(B) \cap V(H)$. We denote the set of vertices of attachment of $B$ by $at(B)$.

An arc-coloured graph $(G, \chi)$ is a graph $G$ with a function $\chi : \{(u, v) \mid u \in V(G)\} \cup \{(u, v) \mid \{u, v\} \in E(G)\} \rightarrow C$, where $C$ is some set of colours. In an arc-coloured graph we interpret $\chi(u, v)$ as the vertex colour of $u$ and for $uv \in E(G)$ we interpret $\chi(u, v)$ as the colour of the arc from $u$ to $v$. In particular it may be the case that $\chi(u, v) \neq \chi(v, u)$, that is, the two orientations of an (undirected) edge $uv$ may receive different colours. A vertex-coloured graph is the special case of an arc-coloured graph where all arcs receive the same colour, say, $1$, that is, $\chi(u, v) = 1$ for all $u \neq v$. Whenever we refer to coloured graphs in this paper, we mean arc-coloured graphs. To simplify the notation, we usually do not mention the colouring explicitly and just denote an arc-coloured graph by $G$, implicitly assuming that the colouring is $\chi$.

For a (possibly coloured) graph $G$ and a sequence of vertices $v_1, \ldots, v_\ell$, we write $G_{v_1, \ldots, v_\ell}$ to denote the graph resulting from individualising every vertex $v_i$, i.e., by assigning every $v_i$ for $i \in [\ell]$ a unique colour. When comparing two graphs with individualised vertices $G_{v_1, \ldots, v_\ell}$ and $H_{v'_1, \ldots, v'_\ell}$ we assume that for $i \in [\ell]$, the two vertices $v_i$ and $v'_i$ have the same colours.

We write $G \cong H$ to indicate that the graphs $G$ and $H$ are isomorphic via a colour-preserving isomorphism. An automorphism of $G$ is an isomorphism from $G$ onto itself. The set of automorphisms of $G$ equipped with concatenation forms a group, also denoted by $\text{Aut}(G)$. For a vertex $v \in V(G)$, the orbit of $v$ is the set $\{\pi(v) \mid \pi \in \text{Aut}(G)\}$. A set $V \subseteq V(G)$ is called a block of $\text{Aut}(G)$ if for every $\pi \in \text{Aut}(G)$ it holds that $V \cap \pi(V) \in \emptyset, V$, i.e., if every automorphism of $G$ maps $V$ onto itself or onto a set that is disjoint to $V$.

For a set $W \subseteq V(G)$, let $G/W$ be the graph obtained from $G$ by identifying all vertices in $W$ and eliminating loops and parallel edges. We usually denote the vertex of $G/W$ representing the set $W$ by $w$. Formally, $G/W$ is the graph with vertex set $V(G/W) := V(G \setminus W) \cup \{w\}$ and edge set $E(G/W) := E(G \setminus W) \cup \{vw \mid v \in V(G \setminus W), \exists w' \in W : vw' \in E(G)\}$. Moreover, if $G$ has the colouring $\chi$ with range $C$, then $G/W$ has the colouring $\chi'$ where $\chi'(w, w) = 0$ and $\chi'(u, v) = \chi(u, v)$ and $\chi'(u, w) := \{\chi(u, w') \mid w' \in W\}$ and $\chi'(w, v) := \{\chi'(w', v) \mid w' \in W\}$ for all $u, v \in V(G \setminus W)$ and all $w \in W$. (We use $\{\ldots\}$ as notation for multisets.) For a subgraph $A \subseteq G$, we let $G/A := G/V(A)$, with the convention of denoting the vertex of $G/A$ representing $V(A)$ by $a$.

## 2.2 Topology

In this section we review basic notions of surface topology and graph embeddings. In our presentation and notation, we follow [11] Chapter 9. Many more details can be found there, in [35], and in [11] Appendix B.

We denote topological spaces like surfaces, curves, and embedded graphs by bold-face
In this paper, we only consider surfaces without boundary. We define the non-contractible simple closed curve in the class of all graphs of Euler genus at most $g$ embedded into the 2-sphere $S^2$. Then $\text{bd}_X(Y)$ is the boundary of $Y$ in $X$ to be the set of all points $x \in X$ such that every neighbourhood of $x$ has a nonempty intersection with both $Y$ and $X \setminus Y$. The interior of $Y$ is $\text{int}_X(Y) := Y \setminus \text{bd}_X(Y)$, and the closure of $Y$ is $\text{cl}_X(Y) := Y \cup \text{bd}_X(Y)$. We omit the subscript $X$ if the space, usually a surface, is clear from the context.

A surface is an arcwise connected 2-manifold (intuitively, a space that looks like a disk in a small neighbourhood of every point). Recall from the introduction that up to homeomorphism there are only two families $(S_g)_{g \geq 0}$ and $(N_g)_{g \geq 1}$ of surfaces. $S_0$ is the 2-sphere, and for $g \geq 1$, $S_g$ is the surface obtained from the 2-sphere by adding $g$ handles, and $N_g$ is the surface obtained from the 2-sphere by adding $g$ crosscaps. Intuitively, adding a handle to a surface means punching two holes into the surface and gluing a cylinder to these holes. Adding a crosscap means punching a hole into the surface and gluing a Möbius strip to this hole. The Euler genus $\text{eg}(S)$ of a surface $S$ is 2$g$ if $S$ is homeomorphic to $S_g$ and $g$ if $S$ is homeomorphic to $N_g$.

Let $g$ be a simple closed curve in a surface $S$. Then $g$ is contractible if it is the boundary of a closed disk in $S$, otherwise $g$ is non-contractible. If $g$ is non-contractible, we can obtain one or two surfaces of strictly smaller Euler genus by the following construction: we cut the surface along $g$; what remains is a surface with one or two holes in it. Then we glue a disk onto these hole(s) and obtain one or two simpler surfaces. For a more detailed description of this construction, see [11, Appendix B].

Formally, an embedded graph in a surface $S$ is a pair $G = (V(G), E(G))$ where $V(G) \subseteq S$ is a finite set and $E(G)$ is a set of simple curves in $S$ such that for all $e \in E(G)$, both endpoints and no internal point of $e$ are in $V(G)$ and any two distinct $e, e' \in E(G)$ have at most one endpoint and no internal points in common. $G$ denotes the point set $V(G) \cup \bigcup_{e \in E(G)} e \subseteq S$. Sometimes, we also regard $G$ as a topological (sub)space of $(S)$. The underlying graph of an embedded graph $G$ is the graph with vertex set $V(G)$ and edge set $\{e \cap V(G) \mid e \in E(G)\}$. We usually blur the distinction between an embedded graph $G$ and its underlying "abstract" graph. The faces of $G$ are the arcwise connected components of the space $S \setminus G$. It is easy to see that for every face $f$ of $G$ there is a subgraph $B \subseteq G$ such that the (topological) boundary $\text{bd}(f)$ of $f$ in $S$ is precisely $B$. We call $B$ a facial subgraph of $G$.

We say that an (abstract) graph $G$ is embeddable into a surface $S$ if it is isomorphic to (the underlying graph of) a graph embedded in $S$. The Euler genus $\text{eg}(G)$ of a graph $G$ is the least $g$ such that $G$ is embeddable into a surface of Euler genus $g$. It is useful to also define the orientable genus $\text{og}(G)$ of a graph $G$ to be the smallest $g$ such that $G$ is embeddable into $S_g$ and the non-orientable genus $\text{ng}(G)$ of $G$ to be the smallest $g$ such that $G$ is embeddable into $N_g$. Then $\text{eg}(G) = \min\{2\text{og}(G), \text{ng}(G)\}$.

The graphs of Euler genus 0 are precisely the planar graphs because a graph can be embedded into the 2-sphere $S_0$ if and only if it can be embedded into the plane $R^2$. The class of all graphs of Euler genus at most $g$ is denoted by $\mathcal{E}_g$.

A non-contractible cycle in a graph $G$ embedded in $S$ is a cycle $C \subseteq G$ such that $C$ is a non-contractible simple closed curve in $S$.

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1 In this paper, we only consider surfaces without boundary.
**Fact 2.1.** Let $S$ be a surface, and let $D, D' \subseteq S$ be closed disks such that $bd(D') \cap D$ is a simple curve. Then $D \cup D'$ is a closed disk.

**Fact 2.2** (see Fact 9.1.14, [17]). Let $G$ be a graph embedded in a surface $S$. Then either $G$ contains a non-contractible cycle or there is a closed disk $D \subseteq S$ such that $G \subseteq D$.

In the latter case, if $G$ is 2-connected, the disk $D$ can be chosen in such a way that there is a cycle $C \subseteq G$ such that $bd(D) = C$.

Let $S$ be a surface and let $G$ be a graph embedded in $S$. A set $X \subseteq S$ is $G$-normal if $X \cap G \subseteq V(G)$. The representativeness $p(G)$ of $G$ is the maximum $r \in \mathbb{N}$ such that every $G$-normal non-contractible simple closed curve $g$ in $S$ intersects $G$ in at least $r$ vertices. $G$ is polyhedrally embedded in $S$ if $G$ is 3-connected and $p(G) \geq 3$. Note that, particularly, every 3-connected plane graph is polyhedrally embedded in $S_0$. Polyhedrally embedded graphs have several useful properties (see [17] Fact 9.1.17). In particular, all facial subgraphs of a polyhedrally embedded graph are chordless and non-separating cycles [39]. Conversely, for every graph embedded in a surface, all contractible, chordless, and non-separating cycles are facial subgraphs (see [17] Lemma 9.1.15]). (Here a cycle $C \subseteq G$ is chordless if it is an induced subgraph of $G$, and it is non-separating if $G \setminus V(C)$ is connected.) This is a generalisation of the well-known theorem that the facial subgraphs of a 3-connected plane graph are precisely the chordless and non-separating cycles. It implies Whitney’s Theorem [12] that all plane embeddings of 3-connected planar graphs have the same facial cycles and that, up to homeomorphism, a 3-connected planar graph has a unique embedding into the sphere $S_0$.

### 3 Finite Variable Logic with Counting

Here we give a detailed introduction into the logic $\mathcal{C}$, the extension of $\text{FO}$ by counting quantifiers and its finite variable fragments, and we prove several technical lemmas.

We interpret the logic $\mathcal{C}$ over graphs, possibly coloured. In a logical context, we view a graph $G$ as a relational structure whose vocabulary consists of a single binary relation $E$. We view a coloured graph $(G, \chi)$ as a relational structure whose vocabulary contains, in addition to the binary relation symbol $E$, a binary relation symbol $R_c$ for every colour $c$ in the range of $\chi$. This relation symbol is interpreted by the set of all pairs $(u, v)$ such that $\chi(u, v) = c$.

An occurrence of a variable $x$ is free in a formula $\varphi$ if it is outside of all subformulas $\exists^p \varphi$. We often write $\varphi(x_1, \ldots, x_\ell)$ to indicate that the free variables of $\varphi$ are among $x_1, \ldots, x_\ell$. (Not all of these variables are required to appear in $\varphi$.) Then we also denote by $\varphi(y_1, \ldots, y_\ell)$ the result of substituting variables $y_1, \ldots, y_\ell$ for the free occurrences of $x_1, \ldots, x_\ell$.

For a graph $G$ and vertices $u_1, \ldots, u_\ell \in V(G)$, we write $G \models \varphi(u_1, \ldots, u_\ell)$ to denote that $G$ satisfies $\varphi$ if for all $i$ the variable $x_i$ is interpreted by $u_i$. Moreover, we write $\varphi[G, u_1, \ldots, u_\ell] = \varphi[x_1, \ldots, x_\ell]$ to denote the set of all $(\ell - i)$-tuples $(u_{i+1}, \ldots, u_\ell)$ such that $G \models \varphi(u_1, \ldots, u_\ell)$.

For a logic $L$ and two graphs $G$ and $H$, we say $L$ distinguishes $G$ and $H$ if there is a formula $\varphi \in L$ such that $G \models \varphi$ and $H \not\models \varphi$. Similarly $L$ identifies $G$ if for every graph $H \neq G$, it holds that $L$ distinguishes $G$ and $H$.

Atomic formulae in the language of (arc-coloured) graphs are of the form $x_1 = x_2$, $E(x_1, x_2)$, or $R_c(x_1, x_2)$, where $x_1, x_2$ are variables. $C$-formulae are constructed from the atomic formulae using negation $\neg \varphi$, disjunction $(\varphi \lor \psi)$, and counting quantifiers $\exists^p \varphi$ where $p \in \mathbb{N}_{\geq 1}$ and $x$ is a variable, and $\varphi, \psi$ are formulae. As abbreviations, we also use
We denote the subsets $C$ that

Note that for $G$ is consistent of all formulae with at most $k$ variables. If $p > k$, then $\exists^p x \varphi(x)$ is equivalent to $\exists x_1 \ldots \exists x_p \left( \bigwedge_{i \neq j} x_i \neq x_j \land \bigwedge_i \varphi(x_i) \right)$. However, we are mainly interested in the fragments $C^k$ of $C$ consisting of all formulae with at most $k$ variables. Observe that $C$ is only a syntactical extension of FO with not more expressive power, because $\exists^p x \varphi(x)$ is equivalent to $\exists x_1 \ldots \exists x_p \left( \bigwedge_{i \neq j} x_i \neq x_j \land \bigwedge_i \varphi(x_i) \right)$. However, we are mainly interested in the fragments $C^k$ of $C$ consisting of all formulae with at most $k$ variables. If $p > k$, then $\exists^p x \varphi(x)$ cannot be expressed in the $k$-variable fragment of FO, thus $C^k$ is strictly more expressive than the $k$-variable fragment of FO. The logics $C^k$ have played an important role in finite model theory since the 1980s.

We say a formula $\varphi \in C$ has width $k$ if every subformula of $\varphi$ has at most $k$ free variables. We denote the $C$-formulae of width $k$ by $C^k_w$.

Example 3.1. The following formula in $C^3$ has width 3:

$$\exists x_1(E(x, x_1) \land \exists x_2(E(x_1, x_2) \land \exists x_3(E(x_2, x_3) \land \exists x_4(E(x_3, x_4) \land \exists x_5 E(x_5, y))))$$

It is equivalent to the $C^4$-formula

$$\exists z(E(x, z) \land \exists x(E(z, x) \land \exists z(E(x, z) \land \exists x(E(z, x) \land \exists z E(z, y))))$$

We will use the following well-known characterisation of $C^k$.

Lemma 3.2. Every $C$-formula of width $k$ is equivalent to a $C^k$-formula.

We omit the straightforward proof. We note that to translate a $C$-formula of width $k$ into a $C^k$-formula, we only have to rename bound variables. Also note that every $C^k$-formula has width $k$.

Example 3.3. For every $k \geq 0$ we define a $C^3_w$-formula $\text{dist}_{\leq k}$ such that for every graph $G$ and all vertices $u, u' \in V(G)$ it holds that $G \models \text{dist}_{\leq k}(u, u')$ if and only if $u$ and $u'$ have distance at most $k$ in $G$. We let

$$\text{dist}_{\leq k}(x, x') := \begin{cases} x = x' & \text{if } k = 0 \\ \exists y_k \left( E(x, y_k) \land \text{dist}_{\leq k-1}(y_k, x') \right) & \text{otherwise.} \end{cases}$$

Note that for $k \geq 1$, the $C^3_w$-formula $\text{dist}_{\leq k}(x, x') := \text{dist}_{\leq k}(x, x') \land \neg \text{dist}_{\leq k-1}(x, x')$ states that $x$ and $x'$ have distance exactly $k$. Moreover, in every graph of order at most $n$ the $C^3_w$-sentence $\text{conn}_n := \forall x \forall x' \text{dist}_{\leq n-1}(x, x')$ states that the graph is connected.

The following lemma bounds the number of variables needed for avoiding definable subsets.
Lemma 3.4. Let \( \varphi(x_1, \ldots, x_k, y) \in C_w^\ell \). Then there is a formula \( \text{comp}_\varphi(x_1, \ldots, x_k, y, y') \in C_w^{\max\{k+3, \ell\}} \) such that for all graphs \( G \) of order \( |G| \leq n \) and all \( u_1, \ldots, u_k, v, v' \in V(G) \),

\[
G \models \text{comp}_\varphi(u_1, \ldots, u_k, v, v') \iff v \text{ and } v' \text{ belong to the same connected component of } G \setminus \varphi[G, u_1, \ldots, u_k, y].
\]

Proof. Without loss of generality, we assume that in all formulae of the form \( \exists z \varphi \) that we consider, the variable \( z \) occurs free in \( \chi \). We let \( \psi(x_1, \ldots, x_k, y, y') \) be the formula obtained from the formula dist\(_{\leq n-1}\)(y, y') of Example 3.3 by replacing each subformula \( \exists z \varphi \) by \( \exists z (\neg \varphi(x_1, \ldots, x_k, z) \land \chi) \). Then, letting \( U := \varphi[G, u_1, \ldots, u_k, y] \), for all \( v, v' \in V(G) \setminus U \) we have \( G \models \psi[u_1, \ldots, u_k, v, v'] \) if and only if \( v \) and \( v' \) belong to the same connected component of \( G \setminus U \). Note that \( \psi(x_1, \ldots, x_k, y) \) has at most two free variables besides \( z \).

Now \( \text{comp}_\varphi(x_1, \ldots, x_k, y, y') := \neg \varphi(x_1, \ldots, x_k, y) \land \neg \varphi(x_1, \ldots, x_k, y') \land \psi(x_1, \ldots, x_k, y, y') \).

Lemma 3.5. Let \( n \geq 1 \), \( \ell \geq 3 \), and \( 1 \leq k \leq \ell \). Then for every \( C_w^\ell \)-formula \( \psi(x_1, \ldots, x_k) \) there is a \( C_w^k \)-formula \( \hat{\psi}(x_1, \ldots, x_k) \) such that for every graph \( G \) of order \( |G| \leq n \), every connected component \( A \) of \( G \), and all \( u_1, \ldots, u_k \in V(A) \), it holds that

\[
G \models \hat{\psi}(u_1, \ldots, u_k) \iff A \models \psi(u_1, \ldots, u_k).
\]

Proof. We construct \( \hat{\psi} \) by induction on \( \psi \). If \( \psi \) is atomic, then we simply let \( \hat{\psi} := \psi \). If \( \psi = \neg \varphi \) we let \( \hat{\psi} := \neg \hat{\varphi} \), and if \( \psi = \varphi \lor \varphi_2 \) we let \( \hat{\psi} := \hat{\varphi}_1 \lor \hat{\varphi}_2 \). The only interesting case is that \( \psi(x_1, \ldots, x_k) = \exists \varphi \varphi(x_1, \ldots, x_k, y) \). Note that the variable \( y \) may be among \( x_1, \ldots, x_k \). If this is the case, \( \varphi(x_1, \ldots, x_k, y) \) is the same formula as \( \varphi(x_1, \ldots, x_k) \). Without loss of generality we may assume that there is a \( j \leq k \) such that \( y \neq x_j \). This is obvious if \( k \geq 2 \). If \( k = 1 \), we can rename the bound variable \( y \) and choose \( j = 1 \). We let \( \hat{\psi}(x_1, \ldots, x_k) = \exists \varphi (\text{dist}_{\leq n-1}(x, y) \land \hat{\psi}(x_1, \ldots, x_k, y)) \), where dist\(_{\leq n-1}\) is the \( C_w^{\ell-1} \)-formula defined in Example 3.3.

Recall that the notation \( \psi(x_1, \ldots, x_k) \) merely says that the free variables of the formula \( \psi \) are among \( x_1, \ldots, x_k \); not all of these variables actually have to appear. Thus we can also apply the lemma to sentences \( \psi \) and obtain the following corollary.

Corollary 3.6. Let \( n \geq 1 \) and \( \ell \geq 3 \). Then for every \( C_w^\ell \)-sentence \( \varphi \) there is a \( C_w^k \)-formula \( \hat{\varphi}(x) \) such that for every graph \( G \) of order \( |G| \leq n \) and every \( u \in V(G) \) we have \( G \models \hat{\varphi}(u) \) if and only if \( A \models \varphi \) for the connected component \( A \) of \( u \) in \( G \).

Corollary 3.7. Let \( \ell \geq 3 \), and let \( G \) be a graph such that every connected component of \( G \) is identified by a \( C_w^\ell \)-sentence. Then \( G \) is identified by a \( C_w^\ell \)-sentence.

Observe that the corollary fails for \( \ell = 2 \). An example is the graph \( G \) that is the disjoint union of two triangles.

Lemma 3.8. Let \( k \geq 0 \), \( n \geq 1 \), \( \ell \geq 3 \) and let \( \psi \in C_w^\ell \) and \( \varphi(x_1, \ldots, x_k, y) \in C_w^{\max\{k+3, \ell\}} \). Then there is a formula \( \psi(x_1, \ldots, x_k, y) \in C_w^{\max\{k+3, \ell\}} \) such that for all graphs \( G \) of order \( |G| \leq n \) and all \( u_1, \ldots, u_k, v \in V(G) \) the following holds. Let \( U := \varphi[G, u_1, \ldots, u_k, y] \), and let \( A_v \) be the connected component of \( v \) in \( G \setminus U \) (assuming \( v \notin U \)). Then

\[
G \models \hat{\psi}(u_1, \ldots, u_k, v) \iff v \notin U \text{ and } A_v \models \psi.
\]
Proof. Again, without loss of generality, we assume that in all formulae of the form $\exists^p z \chi$ that we consider, the variable $z$ occurs free in $\chi$. We apply Corollary 3.6 to $\psi$ and obtain a $C_w^\ell$-formula $\hat{\psi}(y)$ such that for every graph $H$ of order at most $n$ and every $v \in V(H)$ we have $H \models \hat{\psi}(v)$ if and only if $A_v \models \psi$, where $A_v$ is the connected component of $v$ in $H$. In particular, this holds for the graph $H := G \setminus U$.

Without loss of generality we may assume that the variables $x_1, \ldots, x_k$ do not appear in $\hat{\psi}(y)$. We let $\hat{\psi}(x_1, \ldots, x_k, y)$ be the $C_w^\max\{k+\ell,m\}$-formula obtained from $\hat{\psi}(y)$ by replacing each subformula $\exists^p z \chi$ with $\exists^p z (\neg \varphi(x_1, \ldots, x_k, z) \land \chi)$. Then for all $v \in V(H)$ we have $G \models \hat{\psi}(v) \iff H \models \psi(v)$. We let

$$\tilde{\psi}(x_1, \ldots, x_k, y) := \neg \varphi(x_1, \ldots, x_k, y) \land \hat{\psi}(x_1, \ldots, x_k, y).$$

We need one more technical lemma which will be applied in one case of the proof of our main theorem in Section 6.1. The reason we put it here is that we do not want to interrupt the flow of the main argument later. The reader may safely skip the lemma on first reading the paper and get back to it later.

For the purposes of the lemma, we need a way to prevent some free variables from counting towards the width of a formula. We shall use the symbol $\circ$ as a special placeholder that can be substituted for the free occurrences of variables with the effect that this placeholder does not count as a variable for the width. For example, for $\psi(x, z) := \exists y (E(x, y) \land E(y, z)) \in C_3^w$, we have $\psi(o, z) \in C_2^w$ and $\psi(o, o) \in C_1^w$. Recall that for a graph $G$ and a subgraph $A \subseteq G$, by $G/A$ we denote the graph obtained from $G$ by identifying all vertices of $A$ and that $a$ is the vertex of $G/A$ corresponding to $A$.

Lemma 3.9. Let $0 \leq \ell < m < k$ and $\xi(x_1, \ldots, x_\ell, y), \psi(x_1, \ldots, x_m, z) \in C_k^w$ such that $\psi(o, \ldots, o, x_{\ell+1}, \ldots, x_m, o) \in C_w^{k-\ell}$. Then there is a formula $\varphi(x_1, \ldots, x_m) \in C_k^w$ such that the following holds.

Let $G$ be a graph and let $A \subseteq G$. Suppose $u_1, \ldots, u_m \in V(G) \setminus V(A)$ such that $V(A) = \xi[\{u_1, \ldots, u_\ell, y\}]$. Then

$$G \models \varphi(u_1, \ldots, u_m) \iff G/A \models \psi(u_1, \ldots, u_m, a).$$

Proof. Without loss of generality, we assume that every bound variable in $\psi$ does not occur free in $\varphi$ or $\xi$ and is not bound by a second quantifier in $\psi$.

We let $\varphi := \psi^*$, where we define the transformation $*$ inductively to eliminate the variable $z$ as follows.

For atoms $\alpha$ that do not mention $z$, we let $\alpha^* := \alpha$. Atoms with $z$ are treated as follows, where $x$ denotes a variable distinct from $z$. For equality atoms, we define $(z = z)^* := \text{true}$ and $(x = z)^* := \text{false}$. For atoms with predicate symbol $E$, we let $E(z, z)^* := \text{false}$, and $E(x, z)^* := \exists z (\xi(x_1, \ldots, x_\ell, z) \land E(x, z))$, and $E(z, x)^*$ analogous to $E(x, z)$. For atoms with predicate symbol $R_C$, where the colour $C$ is a multiset with $r$ distinct elements $c_1, \ldots, c_r$ of multiplicities $p_1, \ldots, p_r$, we define $R_C(z, z)^* := \text{false}$, and $R_C(x, z)^* := \forall_{j=1}^r \exists^p z (\xi(x_1, \ldots, x_\ell, z) \land R_{c_j}(x, z))$, and $R_C(z, x)^*$ analogous to $R_C(x, z)$.

Inductively, we define $(\neg \chi)^* := \neg \chi^*$ and $(\chi_1 \lor \chi_2)^* := (\chi_1^* \lor \chi_2^*)$. For the case $\exists^p \chi(x_1, \ldots, x_n, x, z)$ for $p \geq 2$, we define

$$\exists^p \chi^* := \left( \chi(x_1, \ldots, x_n, z, z)^* \land \exists^{p-1} x (\neg \xi(x_1, \ldots, x_\ell, x) \land \chi(x_1, \ldots, x_n, x, z)^*) \right) \lor \exists^{p-2} x (\neg \xi(x_1, \ldots, x_\ell, x) \land \chi(x_1, \ldots, x_n, x, z)^*).$$

Note that the formula $\chi(x_1, \ldots, x_n, z, z)^*$ is obtained by first substituting $z$ for $x$ in $\chi$ and then applying $*$ to the resulting formula to eliminate $z$. The case $p = 1$ is dealt with analogously.
To prove the correctness of the construction, we need to show that the free variables of $\psi^*$ are among $\{x_1, \ldots, x_m\}$ and $\psi^* \in \mathcal{C}_w^k$, and that $\psi^*$ has the correct meaning.

First, observe that a straightforward induction obtains that for every formula $\chi$, 

$$\text{free}(\chi^*) \subseteq (\text{free}(\chi) \setminus \{z\}) \cup \{x_1, \ldots, x_k\},$$ 

(2) 

where free$(\chi)$ denotes the free variables of $\chi$. Thus, free$(\psi^*) \subseteq \{x_1, \ldots, x_m\}$.

Second, observe that the condition $\psi(o, \ldots, o, x_{t+1}, \ldots, x_m, o) \in \mathcal{C}_w^{k-\ell}$ expresses that no subformula of $\psi$ (including $\psi$ itself) has more than $k - \ell$ free variables that are not contained in the set $\{x_1, \ldots, x_\ell, z\}$. So we can assume that all subformulae of $\psi$ satisfy this condition.

Now we are ready to prove $\psi^* \in \mathcal{C}_w^k$ by induction on $\psi$. For the base steps, note that $E(x, z)^*, E(z, x)^*, R_C(x, z)^*, R_C(z, x) \in \mathcal{C}_w^{\ell+2}$ and $\ell + 2 \leq k$; the other base cases are trivial.

For the inductive step, the case $\neg \chi$ is trivial. For the case $\chi_1 \lor \chi_2$ we exploit observations [2] and that $\chi$ has at most $k - \ell$ free variables not in $\{x_1, \ldots, x_\ell, z\}$. The case $\exists^2 p x \chi(x_1, \ldots, x_n, x, z)$ follows immediately by induction, since we have $\chi(x_1, \ldots, x_n, z, z)^* \in \mathcal{C}_w^k$ and $\chi(x_1, \ldots, x_n, x, z)^* \in \mathcal{C}_w^k$.

Finally, we show the following statement for every formula $\chi(x_1, \ldots, x_n, z)$, where $n \geq \ell$ and every bound variable in $\chi$ does not occur bound in $\chi$ and $\xi$ and is not bound by a second quantifier in $\chi$; for every graph $G$, every subgraph $A \subseteq G$, and all $u_1, \ldots, u_n \in V(G) \setminus V(A)$ such that $V(A) = \xi[G, u_1, \ldots, u, y]$ we have

$$G \models \chi^*(u_1, \ldots, u_n) \iff G/A \models \chi(u_1, \ldots, u, a).$$

(3) 

The proof is by induction on $\chi$. This statement in particular applies to $\psi(x_1, \ldots, x_\ell, z)$ and thus completes the proof of the lemma.

The base step for atomic formulae follows from the fact that $u_i \neq a$ for every $1 \leq i \leq n$ and the definition of $G/A$ and its colouring.

In the inductive step, the negation and disjunction cases are trivial. Now consider the case $\exists^2 p x \chi(x_1, \ldots, x_n, x, z)$. Recall the definition in [1]. To understand the following argument, it is important to know exactly which variables occur free in $(\exists^2 p x \chi(x_1, \ldots, x_n, x, z))^*$ and its constituent formulae. The formula $\chi^1 := \chi(x_1, \ldots, x_n, x, z)^*$ has free variables among $x_1, \ldots, x_n, x$; we write $\chi^1(x_1, \ldots, x_n, x)$ to make this explicit. The formula $\chi^2 := \chi(x_1, \ldots, x_n, z, z)^*$ has free variables among $x_1, \ldots, x_n$; we write $\chi^2(x_1, \ldots, x_n)$. The formula $(\exists^2 p x \chi)^*$ has free variables among $x_1, \ldots, x_n$; we write $(\exists^2 p x \chi)^*(x_1, \ldots, x_n)$.

Let $G$ be a graph, $A \subseteq G$, and all $u_1, \ldots, u_n \in V(G) \setminus V(A)$ such that $V(A) = \xi[G, u_1, \ldots, u, y]$.

By the induction hypothesis, for all $u \in V(G) \setminus V(A)$ we have

$$G \models \chi^1(u_1, \ldots, u_n, u) \iff G/A \models \chi(u_1, \ldots, u, u, a)$$

(4) 

and

$$G \models \chi^2(u_1, \ldots, u_n) \iff G/A \models \chi(u_1, \ldots, u, a).$$

(5) 

To prove the forward direction of (5), suppose that $G \models (\exists^2 p x \chi)^*(u_1, \ldots, u_n)$.

Case 1: $G \models \chi^2(u_1, \ldots, u_n)$ and there are pairwise distinct $u^1, \ldots, u^{p-1} \in V(G)$ such that for all $j$, $G \not\models \xi(u_1, \ldots, u_j, w^j)$ and $G \models \chi^1(u_1, \ldots, u_n, w^j)$.

Then it holds that $u^j = u$ by the assumption that $V(A) = \xi[G, u_1, \ldots, u, y]$. Furthermore, $G/A \models \chi(u_1, \ldots, u_n, u_a)$ by (4) and $G/A \models \chi(u_1, \ldots, u_n, w^j, a)$ by (4). Thus $a, u^1, \ldots, u^{p-1}$ witness that $G/A \models (\exists^2 p x \chi)(u_1, \ldots, u_n, x, a)$.

Case 2: There are pairwise distinct $u^1, \ldots, u^p \in V(G)$ such that for all $j$, $G \not\models \xi(u_1, \ldots, u_j, w^j)$ and $G \models \chi^2(u_1, \ldots, u_n, w)$. 


Then it holds that \( u^i \neq a \) by the assumption that \( V(A) = \xi[G, u_1, \ldots, u_k, y] \). Furthermore, \( G/A \models \chi(u_1, \ldots, u_n, v^i, a) \) by (4). Thus \( u^1, \ldots, u^p \) witness that \( G/A \models \exists x \forall y \chi(u_1, \ldots, u_n, x, a) \).

The backward direction of (3) is proved by reverting the same argument.

\[ \Box \]

4 The WL Dimension

We start by reviewing the \(k\)-dimensional WL algorithm (for short: \(k\)-WL) for \( k \geq 1 \).

The atomic type \( \text{atp}(G, \bar{u}) \) of a \( k \)-tuple \( \bar{u} = (u_1, \ldots, u_k) \) of vertices of a (possibly coloured) graph \( G \) is the set of all atomic facts satisfied by these vertices. The exact encoding is not important for us, the relevant property is that tuples \( \bar{u} = (u_1, \ldots, u_k) \) and \( \bar{v} = (v_1, \ldots, v_k) \) of vertices of graphs \( G, H \), respectively, have the same atomic type if and only if the mapping \( u_i \mapsto v_i \) is an isomorphism from the induced subgraph \( G[{u_1, \ldots, u_k}] \) to the induced subgraph \( H[{v_1, \ldots, v_k}] \).

Now \(k\)-WL is the algorithm that, given a graph \( G \), computes the following sequence of "colourings" \( C_i^k \) of \( V(G)^k \) for \( i \geq 0 \) until it returns \( C_i^k \) for the smallest \( i \) such that for all \( \bar{u}, \bar{v} \) it holds that \( C_i^k(\bar{u}) = C_i^k(\bar{v}) \iff C_{i+1}^k(\bar{u}) = C_{i+1}^k(\bar{v}) \). The initial colouring \( C_0^k \) assigns to each tuple its atomic type: \( C_0^k(\bar{u}) = \text{atp}(G, \bar{u}) \). In the \((i + 1)\)-st refinement round, the colouring \( C_{i+1}^k \) is defined by \( C_{i+1}^k(\bar{u}) := (C_i^k(\bar{u}), M_i(\bar{u})) \), where, for \( \bar{u} = (u_1, \ldots, u_k) \), \( M_i(\bar{u}) \) is the multiset

\[
\{(\text{atp}(G, (u_1, \ldots, u_k, v)), C_i^k(u_1, \ldots, u_{k-1}, v), C_i^k(u_1, \ldots, u_{k-2}, v, u_k), \ldots, C_i^k(v, u_2, \ldots, u_k)) \mid v \in V \}
\]

We say that \( k\)-WL distinguishes two graphs \( G, H \) if there is some colour \( c \) in the range of \( C_\infty^k \) such that the number of tuples \( \bar{u} \in V(G)^k \) with \( C_\infty^k(\bar{u}) = c \) is different from the number of tuples \( \bar{v} \in V(H)^k \) with \( C_\infty^k(\bar{v}) = c \). We say that \( k\)-WL identifies \( G \) if it distinguishes \( G \) from all graphs \( H \) not isomorphic to \( G \). The WL dimension of \( G \) is the least \( k \) such that \( k\)-WL identifies \( G \).

\[ \Delta \]

Definition 4.1 (see Definition 12, [25]). Let \( \mathcal{H} \) be a set of graphs. We say that the \( k\)-dimensional WL algorithm determines orbits in \( \mathcal{H} \) if for all coloured graphs \( (G, \lambda) \) and all coloured graphs \( (G', \lambda') \) (with colourings \( \lambda \) and \( \lambda' \) and all vertices \( s \in V(G) \) and \( s' \in V(G') \) the following holds: there exists an isomorphism from \( (G, \lambda) \) to \( (G', \lambda') \) mapping \( s \) to \( s' \) if and only if \( C_\infty^k(s) = C_\infty^k(s') \).

The following proposition is a useful correspondence between identification and determination of orbits in a graph.

\[ \Delta \]

Proposition 4.2. Let \( k \geq 1 \) be a natural number and let \( G \) be a coloured graph. Suppose \( k\)-WL identifies all vertex-coloured versions of \( G \). Then \((k + 1)\)-WL determines orbits on \( G \).

Proof. Let \( \chi^k \) denote the stable colouring computed by \( k\)-WL. Let \( G \) be a graph. Suppose there are a graph \( H \) and vertices \( v \in V(G), v' \in V(H) \) such that \( \chi^{k+1}(v) = \chi^{k+1}(v') \) holds. Then we can individualise \( v \) in \( G \) and \( v' \) in \( H \) and apply \( k\)-WL to these coloured graphs \( G_v \) and \( H_{v'} \). Since \( \chi^{k+1}(v) = \chi^{k+1}(v') \), we have that \( \{\chi^{k+1}(v, w_1, \ldots, w_k) \mid \{w_1, \ldots, w_k \} \in V^k(G)\} = \{\chi^{k+1}(v', w_1', \ldots, w_k') \mid \{w_1', \ldots, w_k' \} \in V^k(H)\} \). Thus, the graphs \( G_v \) and \( H_{v'} \) obtain isomorphic colourings under \( k\)-WL. By assumption, this implies \( G_v \cong H_{v'} \), which is equivalent to the existence of an isomorphism from \( G \) to \( H \) mapping \( v \) to \( v' \).

\[ \Box \]
For the following lemma, we assume that the reader is familiar with graph minors. For those who are not, we remark that for every $g \geq 0$ the class $\mathcal{E}_g$ of all graphs of Euler genus at most $g$ is closed under taking minors. We will only apply the lemma to these classes.

For a class $\mathcal{C}$ of (uncoloured) graphs, we let $\mathcal{C}^{*}$ be the class of all coloured graphs with underlying graph in $\mathcal{C}$.

Lemma 4.3 ([25]). Let $\mathcal{C}$ be a graph class that is closed under taking minors. Suppose $k$-WL identifies all 3-connected graphs in $\mathcal{C}^{*}$. Then $(k+1)$-WL identifies all graphs in $\mathcal{C}^{*}$.

Proof. By [25, Theorem 13], $(k+1)$-WL identifies all graphs in $\mathcal{C}^{*}$ if $(k+1)$-WL determines orbits on all 3-connected graphs in $\mathcal{C}^{*}$. Thus, the statement follows from Proposition 4.2. ▶

In this paper, we reason about the WL dimension in terms of logic, using the following correspondence.

Theorem 4.4 ([19, 23]). Let $k \geq 1$. Let $G$ and $H$ be graphs, possibly coloured, and $\bar{u} := (u_1, \ldots, u_k) \in V(G)^k$ and $\bar{v} := (v_1, \ldots, v_k) \in V(H)^k$. Then the following are equivalent:
1. $C^k_{\infty} (\bar{u}) = C^k_{\infty} (\bar{v})$;
2. $G \models \varphi(u_1, \ldots, u_k) \iff H \models \varphi(v_1, \ldots, v_k)$ for all $C^{k+1}$-formulae $\varphi(x_1, \ldots, x_k)$.

Recall that we say a graph $G$ is identified by the logic $C^k$ if there is a sentence $\text{iso}_G \in C^k$ such that for all graphs $H$ we have $H \models \text{iso}_G$ if and only if $H$ is isomorphic to $G$.

Corollary 4.5. A graph has WL dimension $k$ if and only if it is identified by $C^{k+1}$.

The WL dimension of a planar graph is at most 3 [25]. Using the previous corollary, we can re-phrase this as follows.

Theorem 4.6 (see [25]). For every colored planar graph $G$ there is a $C^4$-sentence $\text{iso}_G$ that identifies $G$.

In the following sections, we use these formulae characterising certain parts of a decomposition of $G$ in order to obtain a bound on the number of variables we need to identify the entire graph.

5 Shortest Path Systems, Patches and Necklaces

Here we introduce the graph-theoretic machinery necessary to prove our main theorem. Essentially, the definitions and results of this section are from [17, Chapter 15]. In fact, things are simpler here because [17, Chapter 15] deals with graphs almost embedded in a surface, whereas we only need to consider surface graphs. Sometimes, we need to change the definitions in order to improve the resulting bounds on the WL dimension later. Notably, our definition of necklaces is different from the one in [17]. This also requires an adaptation of the proof that reducing necklaces exist.

Definition 5.1. Let $G$ be a graph and $u, u' \in V(G)$. A shortest path system (spss) from $u$ to $u'$ is a family $Q$ of shortest paths in $G$ from $u$ to $u'$ such that every shortest path from $u$ to $u'$ in the subgraph $\bigcup_{Q \in \mathcal{Q}} Q$ is contained in $Q$.

We let $V(Q) := \bigcup_{Q \in \mathcal{Q}} V(Q)$ and $E(Q) := \bigcup_{Q \in \mathcal{Q}} E(Q)$ and $G(Q) := (V(Q), E(Q)) = \bigcup_{Q \in \mathcal{Q}} Q$. We call $Q$ trivial if $|V(Q)| \leq 2$, that is, if $G(Q)$ consists of a single vertex or a single edge.

The height $\text{ht}^Q(v)$ of $v \in V(Q)$ is the distance from $u$ to $v$. The vertices in $\bigcap_{Q \in \mathcal{Q}} V(Q)$ are the articulation vertices of $Q$. An articulation vertex $v$ is proper if $v \neq u$ and $v \neq u'$. We denote the set of all articulation vertices of $Q$ by $\text{art}(Q)$.
For all \( u, u' \in V(G) \) such that there is a path from \( u \) to \( u' \) in \( G \), the canonical \( \text{sps} \) from \( u \) to \( u' \) in \( G \) is the set \( Q^G(u, u') \) of all shortest paths from \( u \) to \( u' \) in \( G \).

For a path \( Q \) and vertices \( u, v \in V(Q) \), we denote by \( uQv \) the segment of \( Q \) from \( u \) to \( v \). With every \( \text{sps} \) \( Q \) from \( u \) to \( u' \) we can associate a partial order \( \leq_Q \) on \( V(Q) \) by letting \( v \leq_Q w \) if \( v \) appears before \( w \) on some path \( Q \in \mathcal{Q} \). For \( v \leq_Q w \), we define the segment \( Q[v, w] \) to be the set of segments \( vQw \) from \( v \) to \( w \) of all paths \( Q \in \mathcal{Q} \) that contain both \( v \) and \( w \). Observe that \( Q[v, w] \) is an \( \text{sps} \) from \( v \) to \( w \).

\[\blacktriangleleft\text{Lemma 5.2 (17, Lemma 15.2.3).} \text{ Let } Q \text{ be an \( \text{sps} \). Then } Q \text{ is non-trivial and has no proper articulation vertices if and only if the graph } G(Q) \text{ is 2-connected.}\]

\[\blacktriangleleft\text{Lemma 5.3 (17, Lemma 15.2.4).} \text{ Let } Q \text{ be a non-trivial \( \text{sps} \) that has no proper articulation vertices. Then there are internally disjoint paths } Q, Q' \in \mathcal{Q}.\]

While shortest paths systems are defined with respect to abstract graphs, the following notions are defined with respect to embedded graphs. For the rest of the section, we make the following assumption.

\[\blacktriangleleft\text{Assumption 5.4.} \text{ } G \text{ is a graph polyhedrally embedded in a surface } S \text{ of Euler genus } g \geq 1.\]

\[\blacktriangleleft\text{Definition 5.5.} \text{ A patch in } G \text{ is an \( \text{sps} \) } Q \text{ in } G \text{ such that:}\]

(i) \( Q \) has no proper articulation vertices.

(ii) There is a closed disk \( D \subseteq S \) such that \( G(Q) \subseteq D \).

Fact 2.2 and Lemma 5.2 imply that if \( Q \) is a non-trivial patch then there is a unique disk \( D(Q) \) such that \( G(Q) \subseteq D \), and \( bd(D(Q)) = C(Q) \) for a cycle \( C(Q) \subseteq G(Q) \). Furthermore, \( C(Q) \) is a non-trivial patch then there is a unique disk \( D(Q) \) such that \( G(Q) \subseteq D \) and \( bd(D(Q)) = C(Q) \) for a cycle \( C(Q) \subseteq G(Q) \).

\[\blacktriangleleft\text{Definition 5.6.} \text{ A subgraph } H \subseteq G \text{ is simplifying if every connected component of } G \setminus H \text{ belongs to } \mathcal{E}_g.\]

A patch \( Q \) is simplifying if the graph \( G(Q) \) is simplifying.

\[\blacktriangleleft\text{Lemma 5.7 (17, Corollary 15.3.5).} \text{ For a non-simplifying subgraph } H \subseteq G, \text{ there is at most one connected component } A^* \text{ of } G \setminus H \text{ with } A^* \notin \mathcal{E}_{g-1}, \text{ and all other connected components are planar.}\]

It turns out that non-simplifying patches form the basic building blocks of our theory. Let \( Q \) be a non-trivial non-simplifying path. Let \( A^* \) be the unique connected of \( G \setminus V(Q) \) that is not planar (the existence and uniqueness of \( A^* \) follow from Lemma 5.7). Let \( G/A^* \) be the graph obtained from \( G \) by contracting the subgraph \( A^* \) to a single vertex \( a^* \). By [17, Corollary 15.4.5], \( G/A^* \) is a 3-connected planar graph. Figure 2 displays a schematic view of a patch \( Q \) with some attached (planar) connected components as well as the non-planar component \( A^* \), the disk \( D(Q) \), and the boundary cycle \( C(Q) \).

We define the internal graph of a non-trivial patch \( Q \) to be the graph \( I := I(Q) \) with vertex set \( V(I) := V(G) \cap D(Q) \) and edge set \( E(I) := \{ e \in E(G) \mid e \subseteq D(Q) \} \). Note that \( C(Q) \subseteq I \). Formally, the definitions of the graphs \( C(Q) \) and \( I(Q) \) do not only depend on the abstract graph \( G \) and the \( \text{sps} Q \), but on the embedding of \( G \) in \( S \). However, it can be proved that actually the graphs are invariant under embeddings.

\[\blacktriangleleft\text{Lemma 5.8 (17).} \text{ Let } Q \text{ be a non-simplifying patch in } G. \text{ Let } G' \text{ be a graph embedded in a surface } S' \text{ of Euler genus } g \text{ such that } G \text{ and } G' \text{ are isomorphic (as abstract graphs), and let } f \text{ be an isomorphism from } G \text{ to } G'. \text{ Then } Q' := f(Q') \text{ is a non-simplifying patch in } G', \text{ and it holds that } f(C(Q)) = C(Q') \text{ and } f(I(Q)) = I(Q').\]
This follows from [17, Lemma 15.4.10]. Intuitively, the reason this holds is that the 3-connected planar graph $G/A^*$ has a unique embedding (see Section 2.3).

**Corollary 5.9.** Let $u, u' \in V(G)$ and $Q := Q^G(u, u')$ such that $Q$ is a non-trivial non-simplifying patch. Let $f$ be an automorphism of $G$ such that $f(u) = u$ and $f(u') = u'$. Then $f(C(Q)) = C(Q)$ and $f(I(Q)) = I(Q)$.

We remark that the analogue of Corollary 5.9 for simplifying patches does not hold. (Figure 4 in Section 6.2 shows an example.) The analysis of simplifying patches is much more involved, and we defer it to Section 6.2.

The final objects we define in this section are *necklaces*.

**Definition 5.10.** A necklace in $G$ is a tuple $B := (u^0, Q^0, u^1, Q^1, u^2, Q^2)$, where $u^0, u^1, u^2 \in V(G)$ and $Q^i := Q^G(u^i, u^{i+1})$ (indices taken modulo 3) is the canonical sps from $u^i$ to $u^{i+1}$, such that the following conditions are satisfied:

1. $u^0, u^1, u^2$ are pairwise distinct.
2. $V(Q^i) \cap V(Q^{i+1}) = \{u^{i+1}\}$ (indices modulo 3).
3. There is a disk $D_i \subseteq S$ such that $G(Q^i) \subseteq D_i$.

For a necklace $B := (u^0, Q^0, u^1, Q^1, u^2, Q^2)$ we write $V(B)$ for the set $\bigcup_{i=0}^{2} \bigcup_{Q \in Q_i} V(Q)$ and $E(B)$ for $\bigcup_{i=0}^{2} \bigcup_{Q \in Q_i} E(Q)$, and we let $G(B) := (V(B), E(B))$. Moreover, we define the set of articulation vertices of $B$ to be $\text{art}(B) := \{u_0, u_1, u_2\} \cup \bigcup_{i=0}^{2} \text{art}(Q^i)$.

**Definition 5.11.** A necklace $B := (u^0, Q^0, u^1, Q^1, u^2, Q^2)$ is reducing if there are paths $Q^i \in Q^i$ such that $B := Q^1 \cup Q^2 \cup Q^3$ is a non-contractible cycle.

Figure 3 shows a reducing necklace on a torus with articulation vertices $u^0$, $u^1$, $u^2$.

**Lemma 5.12 (Necklace Lemma).** $G$ has a reducing necklace.

Essentially, this is [17, Lemma 15.5.8], with the necklaces corresponding to the belts there. But since apart from a renaming, we have also slightly changed the content of the definition of a necklace/belt, the proof also needs to be adapted. For the proof of the Necklace Lemma, we need one well-known fact and more complicated lemma from [17].

**Lemma 5.13.** Let $S$ be a surface, and let $g_1, g_2, g_3 \subseteq S$ be simple curves with the same endpoints and mutually disjoint interiors. Then $g_1 \cup g_2, g_2 \cup g_3$, and $g_1 \cup g_3$ are simple closed curves, and if $g_1 \cup g_2$ and $g_2 \cup g_3$ are contractible, then $g_1 \cup g_3$ is contractible as well.
Figure 3 A reducing necklace on a torus section.

For a proof, see [35, Proposition 4.3.1].

Lemma 5.14 ([17], Lemma 15.5.9). Let \( Q \) be an sps in \( G \) such that there is no disk \( D \subseteq S \) with \( G(Q) \subseteq D \), but for every proper segment \( Q' \) of \( Q \) there is a disk \( D' \subseteq S \) with \( G(Q') \subseteq D' \). Then there are internally disjoint paths \( Q, Q' \in Q \) such that \( Q \cup Q' \) is a non-contractible simple closed curve in \( S \).

With these tools at hand, we can now prove the existence of a reducing necklace in \( G \).

Proof of Lemma 5.12. By Fact 2.2, there is a cycle \( C \subseteq G \) such that \( C \) is a non-contractible simple closed curve in \( S \). We choose such a cycle \( C \) of minimum length. We let \( u_0, u_1, u_2 \in V(C) \) such that

\[
\left\lfloor \|C\|/3 \right\rfloor \leq \text{dist}_C(u_i, u_{i+1}) \leq \left\lceil \|C\|/3 \right\rceil, \tag{6}
\]

We let \( Q_i := Q^G(u_i, u_{i+1}) \) and \( B = (u_0, Q^0, u_1, Q^1, u_2, Q^2) \). Here and throughout the proof, indices \( i, j \) are taken from \( \mathbb{Z}_3 \) with addition modulo 3.

It follows from (6) that the \( u_i \) are mutually distinct. Thus \( B \) satisfies Condition 1 of Definition 5.10.

Let \( Q_i \) be the segment of \( C \) from \( u_i \) to \( u_{i+1} \) that does not contain \( u_i+2 \). Then \( C = Q^0 \cup Q^1 \cup Q^2 \).

Claim 1. Let \( P \subseteq G \) be a shortest path with distinct endvertices \( u, u' \in V(C) \) and no internal vertices in \( C \). Let \( Q, Q' \) be the two segments of \( C \) from \( u \) to \( u' \). Then \( P \cup Q \) or \( P \cup Q' \) is a non-contractible cycle. Furthermore, if \( P \cup Q \) is a non-contractible cycle, then \( \|Q'\| = \|P\| \), and if \( P \cup Q' \) is a non-contractible cycle, then \( \|Q\| = \|P\| \).

Proof. Clearly, since \( P \) has no internal vertices in \( C \), both \( P \cup Q \) and \( P \cup Q' \) are cycles. By Lemma 5.13 we know that \( P \cup Q \) or \( P \cup Q' \) is non-contractible. Say, \( P \cup Q \) is. Since \( C \) is a shortest non-contractible cycle, we have \( \|P \cup Q\| \geq \|C\| = \|Q \cup Q'\| \). Thus \( \|Q'\| \leq \|P\| \), and since \( P \) is a shortest path, equality holds.

Claim 2. Let \( Q \in Q^i \). Then \( V(Q) \cap V(C) \subseteq V(Q^i) \).

Proof. By symmetry, it suffices to prove the claim for \( i = 0 \). Suppose for contradiction that there is a path \( Q \in Q^0 \) with \( V(Q) \cap V(C) \not\subseteq V(Q^0) \). Fix \( Q \) to be such a path with the maximum number of edges in \( E(C) \).
Note that \( Q \neq Q^0 \). Thus, \( Q \not\subseteq C \), because the only paths in \( C \) from \( u^0 \) to \( u^1 \) are \( Q^0 \) and \( Q^1 \cup Q^2 \). However, \( Q \neq Q^0 \) and \( \|Q\| \leq \|Q^0\| < \|Q^1\| + \|Q^2\| = \|Q^1 \cup Q^2\| \), which implies \( Q \neq Q^1 \cup Q^2 \).

Recall that for a path \( Q \) and vertices \( v, w \in V(Q) \), we denote by \( vQw \) the segment of \( Q \) from \( v \) to \( w \). Throughout this proof, for a second path \( Q' \) with \( v, w \in V(Q') \), we denote by \( uQvQ'w \) the walk from \( u \) to \( w \) obtained by following \( Q \) from \( u \) to \( v \) and then following \( Q' \) from \( v \) to \( w \). (We also use this notation style to compose multiple segments of paths.)

Let \( u \neq u' \) be vertices in \( V(C) \) and let \( P = uQu'u' \) be a segment of \( Q \) with endpoints \( u, u' \). Then \( P \) is a shortest path from \( u \) to \( u' \). Let \( R, R' \) be the two segments of \( C \) with endpoints \( u, u' \). Then by Lemma 5.13 one of \( R \cup P \) and \( R' \cup P \) must be a non-contractible cycle, say \( R' \cup P \). Then \( \|P\| = \|R\| \).

**Case 1:** \( Q \) has an empty intersection with the interior of \( R \).

Then \( u^0Qu'u' \) is a path from \( u^0 \) to \( u^1 \) that has the same length as \( Q \), but more edges in \( E(C) \). This contradicts the maximality of \( Q \).

**Case 2:** The segment \( u^0Qu \) contains an internal vertex that lies in \( R \).

Let \( v \) be the first vertex of \( Q \) in \( R \). Then \( v \) appears on \( Q \) before \( u \). Let \( w \) be the last vertex of \( Q \) in \( R \) (possibly, \( w = u' \)). Then \( u^0QvRu'u' \) is a path from \( u^0 \) to \( u' \) that is shorter than \( Q \), which contradicts \( Q \) being a shortest path.

**Case 3:** The segment \( u^0Qu' \) contains an internal vertex that lies in \( R \).

Let \( w \) be the last vertex of \( Q \) in \( R \). Then \( w \) appears on \( Q \) after \( u' \). Let \( v \) be the first vertex of \( Q \) in \( R \) (possibly, \( v = u \)). Then \( u^0QvRu'u' \) is a path from \( u^0 \) to \( u' \) that is shorter than \( Q \), which again contradicts \( Q \) being a shortest path.

Thus, the segment \( P \) does not exist, which implies the claim.

Claim 3. \( Q^i \in Q^i \).

**Proof.** Again, by symmetry it suffices to prove the claim for \( i = 0 \). Let \( Q \in Q^0 \) with a maximum number of edges in \( E(C) \). Arguing with similar techniques as in the proof of Claim 2 we can show that \( Q = Q^0 \).

Claim 4. Let \( Q \in Q^i \) and \( Q' \in Q^{i+1} \). Then \( V(Q) \cap V(Q') = \{u^{i+1}\} \).

**Proof.** As usual, we assume \( i = 0 \).

Suppose that \( v_0 = u^0, v_1, v_2, \ldots, v_m = u^1 \) are the vertices in \( Q \cap Q^0 \) in the order in which they appear on \( Q^0 \). Then the vertices appear on \( Q \) in the same order, because by Claim 3 both \( Q^0 \) and \( Q \) are shortest paths. For \( i = 0, \ldots, m - 1 \), let \( P_i \) be the segment of \( Q \) from \( v_i \) to \( v_{i+1} \). Note that no internal vertex of \( P_i \) is in \( V(C) \). Thus either \( P_i = v_iQ^{0}v_{i+1} \) is a single edge or \( P_i \cup v_iQ^{0}v_{i+1} \) is a cycle. Since this cycle is shorter than \( C \), it must be contractible.

Let \( C_0 := C \), and for \( 1 \leq i \leq m - 1 \), let \( C_i \) be the cycle obtained from \( C_{i-1} \) by replacing the segment \( v_iQ^{0}v_{i+1} \) with \( P_i \). It follows from Claim 4 applied to the cycle \( C_{i-1} \) and the path \( P_i \) that each \( C_i \) is non-contractible. In particular,

\[
C' := C_{m-1} = u^0Qu^1v^2Q^2u^0
\]

is a non-contractible cycle of the same length as \( C \).

Thus, \( C' \) is also a shortest non-contractible cycle through \( u^0, u^1, u^2 \) and \( \text{dist}^C(u^i, u^j) = \text{dist}^C(u^i, u^j) \). This means that we can apply all previous claims to \( C' \) instead of \( C \). In particular, it follows from Claim 2 applied to \( C' \) and \( Q' \) that \( V(Q') \cap V(Q) = \{u^1\} \).
Claim 5. \(\text{Let } Q, Q' \subseteq G \) be paths from \(u^i\) to \(u^{i+1}\) such that \(\|Q\|, \|Q'\| \leq \|Q^i\|\). Then there is a disk \(D \subseteq S\) such that \(Q \cup Q' \subseteq \text{int}(D)\).

Proof. We have \(\|Q \cup Q'\| \leq \|Q\| + \|Q'\| \leq 2\|Q^i\| < \|C\|\). Thus the graph \(Q \cup Q'\) does not contain a non-contractible cycle, and by Fact 2.2 there is a closed disk \(D' \subseteq S\) such that \(Q, Q' \subseteq D'\). We can slightly increase \(D'\) to get a disk \(D\) such that \(Q, Q' \subseteq \text{int}(D)\). \(\square\)

Claim 6. There is a disk \(D \subseteq S\) such that \(G(Q^i) \subseteq D\).

Proof. Suppose for contradiction that there is no such disk. Let \(Q\) be a segment of \(Q^i\) such that there is no disk \(D \subseteq S\) with \(G(Q) \subseteq D\), but for every proper segment \(Q'\) of \(Q\) there is a disk \(D' \subseteq S\) with \(G(Q') \subseteq D'\). Then by Lemma 5.14, there are paths \(Q, Q' \in Q\) such that \(Q \cup Q'\) is a non-contractible simple closed curve in \(S\). This contradicts Claim 5. \(\square\)

We have already noted that \(B\) satisfies Condition 1 of Definition 5.10. It follows from Claim 4 that it satisfies Condition 2 and Claim 6 implies that it satisfies Condition 3 as well. Thus \(B\) is a necklace. Claim 5 implies that this necklace is reducing. \(\blacksquare\)

6 Upper Bound on the WL Dimension

Finally, in this section we give the proof of our main theorem (Theorem 1.1). By the correspondence between \(k\)-WL and the logic \(C^{k+1}\) as stated in Corollary 4.5, we need to prove that every graph of Euler genus at most \(g\) can be identified by a \(C^{g+1}\)-sentence. The proof is by induction on \(g\). The base step \(g = 0\) is Theorem 4.6.

For the inductive step, we make the following assumption.

Assumption 6.1. Assume \(g \geq 1\) and there is a natural number \(s \geq 4\) such that every graph in \(\mathcal{E}_{g-1}\) is identified by a \(C^s_{w}\)-sentence.

Our goal is to prove the following lemma (under Assumption 6.1). The lemma implies Theorem 1.1 by induction.

Lemma 6.2 (Inductive Step). For every coloured graph \(G\) in \(\mathcal{E}_g\) there is a sentence \(\text{iso}_G \in C^{s+4}_{w}\) that identifies \(G\).

The proof will proceed in a sequence of lemmas. Eventually, it will diverge into two main cases, to be dealt with in Subsections 6.1 and 6.2.

We first show that we can assume without loss of generality that \(\rho(G) \geq 3\).

Lemma 6.3. Let \(G\) be a coloured graph that has an embedding of representativity at most 2 into a surface of Euler genus at most \(g\). Then there is a sentence \(\text{iso}_G \in C^{s+2}_{w}\) that identifies \(G\).

Proof. Suppose that \(G\) is embedded in a surface \(S\) of Euler genus \(g\) with representativity \(\rho(G) \leq 2\). Let \(g\) be a \(G\)-normal non-contractible simple closed curve in \(S\) such that \(U := g \cap V(G)\) contains at most two vertices. We only consider the case that \(U = \{u_1, u_2\}\) for some \(u_1, u_2 \in V(G)\) (possibly equal), the case \(U = \emptyset\) follows similarly. Let \(H_1, \ldots, H_m\) be the connected components of \(G \setminus U\). Every \(H_i\) can be embedded into a simpler surface obtained from \(S\) by cutting along \(g\) and gluing (a) disk(s) on the hole(s). This means that \(\text{eg}(H_i) \leq g - 1\). We colour the vertices of \(H_i\) so as to encode the adjacencies to \(u_1\) and \(u_2\). By Assumption 6.1, there is a \(C^s_{w}\)-sentence \(\psi_i\) that identifies the coloured version of \(H_i\). Thus by Corollary 3.7 there is a \(C^s_{w}\)-sentence \(\psi\) that identifies the disjoint union of the coloured \(H_i\), that is, the coloured version of \(G \setminus \{u_1, u_2\}\). Now we can identify \(G\) by a sentence saying that there exist vertices \(x_1, x_2\) such that deleting these vertices leaves a graph satisfying \(\psi\) and having the correct adjacencies to \(x_1, x_2\). This requires \(s + 2\) variables. \(\blacksquare\)
So we can restrict our attention to graphs that only have embeddings of representativity at least 3. Furthermore, by Lemma 4.3 we can restrict our attention to 3-connected graphs (at the cost of 1 more variable). Recall that a polyhedral embedding is an embedding of representativity at least 3 of a 3-connected graph. Thus to prove Lemma 6.2 and thereby complete the proof of Theorem 1.1 it remains to prove the following lemma.

**Lemma 6.4.** Let $G$ be a coloured graph polyhedrally embedded in a surface $S$ of Euler genus $g$. Then there is a sentence $\text{iso}_G \in C^n_w$ that identifies $G$.

For the rest of the section, we fix a positive integer $k$. The intended meaning of $n$ is that it is the order of the target graph $G$. At this point we have fixed three numerical parameters: the Euler genus $g$, the number $s$ of variables required to identify graphs of smaller Euler genus, and the order $n$.

To prepare for the proof of Lemma 6.4, we define a number of useful concepts in $C^n_w$ for sufficiently small $k$.

We start the proof with a simple lemma that follows immediately from Assumption 6.1.

**Lemma 6.5.** Let $h < g$. Then there is a sentence $\text{genus}_h \in C^n_w$ such that for every graph $G$ of order $|G| \leq n$, the following holds:

$$G \models \text{genus}_h \iff G \in \mathcal{E}_h.$$  

**Proof.** Since there are only finitely many graphs of order at most $n$, we can let $\text{genus}_h$ be a disjunction over the $\text{iso}_H \in C^n_w$ for all $H \in \mathcal{E}_h$ with $|H| \leq n$. \hfill $\blacksquare$

In the following lemmas, we study the definability of shortest path systems, patches, and necklaces. Our strategy will then be to remove either a (definable) reducing necklace or a (definable) simplifying patch from the graph, then apply the induction hypothesis (Assumption 6.1) to the resulting simpler graph, and finally lift the identifying sentence to the original graph.

**Lemma 6.6.** There are formulae $\text{csps-vert}(x, x', y) \in C^n_w$, $\text{csps-edge}(x, x', y_1, y_2) \in C^n_w$, $\text{csps-art}(x, x', y) \in C^n_w$, and, for $i \geq 0$, formulae $\text{csps-height}_i(x, x', y) \in C^n_w$ and $\text{csps-art}_i(x, x', y) \in C^n_w$ such that for all connected graphs $G$ of order $|G| \leq n$ and all vertices $u, u' \in V(G)$,

$$\text{csps-vert}(G, u, u', y) = V(Q^G(u, u')),$$

$$\text{csps-edge}(G, u, u', y_1, y_2) = E(Q^G(u, u')),$$

$$\text{csps-art}(G, u, u', y) = \text{art}(Q^G(u, u')),$$

$$\text{csps-height}_i[G, u, u', y] = \{v \in V(Q^G(u, u')) \mid \text{ht}_{Q^G(u, u')}(v) = i\},$$

$$\text{csps-art}_i[G, u, u', y] = \{v\}, \text{ where } v \text{ is the } i\text{-th vertex when sorting art}(Q^G(u, u')) \text{ by height}.$$  

Recall that $Q^G(u, u')$ is the canonical sps from $u$ to $u'$, that is, the set of all shortest paths from $u$ to $u'$.

**Proof.** We let $\text{csps-vert}(x, x', y) := \bigvee_{k=0}^n \left( \text{dist}_{=k}(x, x') \land \bigvee_{i=0}^k \text{dist}_{=i}(x, y) \land \text{dist}_{=k-i}(y, x') \right)$, where $\text{dist}_{=k}(x, x')$ is the $C^n_w$-formula defined in Example 3.3. Note that $\text{csps-vert}(x, x', y) \in C^n_w$. 

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Since a vertex \( v \) lies on a shortest path from \( u \) to \( u' \) if and only if taking the shortest path from \( u \) to \( v \) and then to \( u' \) yields no detour, the formula \( \text{csps-vert} \) defines the desired set of vertices.

An edge is contained in \( E(G(u, u')) \) if and only if it connects an sps-vertex of a certain height \( h \) with an sps-vertex of height \( h + 1 \). Thus, it is easy to see that the formula for \( \text{csps-edge} \) can be constructed to have width 4.

A vertex \( v \) is an articulation vertex of \( G(u, u') \) if every shortest path from \( u \) to \( u' \) contains \( v \):

\[
\text{csps-art}(x, x', y) \coloneqq \text{csps-vert}(x, x', y) \land \\
\forall z \left( \text{csps-vert}(x, x', z) \rightarrow (\text{csps-vert}(x, y, z) \lor \text{csps-vert}(x', y, z)) \right).
\]

This formula has width 4.

Similarly, the height of \( v \) in \( G(u, u') \) is \( i \) if and only if \( v \) is contained in the sps and \( G \models \text{dist}_{i-1}(u, v) \). Thus, we can construct \( \text{csps-height}_i(x, x', y) \) with width 3.

By employing \( \text{csps-art} \) and \( \text{csps-height}_i \), we can also construct \( \text{csps-art}_i \), with width 4. ▪

The lemma shows how to define canonical shortest paths systems. We would also like to define patches and necklaces, but they depend on the embedding and since the embedding may not be unique, in general the property of an sps being a patch is not definable in a logic which only has access to the abstract graph and not the embedding. We therefore define “pseudo-patches” and “pseudo-necklaces” purely in terms of the abstract graph; in some situations they may serve as substitutes for the real object.

Definition 6.7. Let \( G \) be a graph.

1. A pseudo-patch in \( G \) is an sps that has no articulation vertices.
2. A pseudo-necklace in \( G \) is a tuple \( \mathcal{B} \coloneqq (u^0, Q^0, u^1, Q^1, u^2, Q^2) \), where \( u^0, u^1, u^2 \in V(G) \) and \( Q^i = Q^G(u^i, u^{i+1}) \) (indices taken modulo 3) is the canonical sps from \( u^i \) to \( u^{i+1} \), such that \( u^0, u^1, u^2 \) are pairwise distinct and \( V(Q^0) \cap V(Q^{i+1}) = \{u^{i+1}\} \) (indices modulo 3).

All the definitions for general sps apply to pseudo-patches, and we can generalise all definitions that do not refer to the embedding (for example, \( V(\mathcal{B}) \), \( E(\mathcal{B}) \), articulation vertices, et cetera) from necklaces to pseudo-necklaces. Observe that every patch is a pseudo-patch and every necklace is a pseudo-necklace.

Corollary 6.8. There are \( C^4_w \)-formulae

\[
\begin{align*}
\text{nl-vert}(x^0, x^1, x^2, y), & \quad \text{nl-edge}(x^0, x^1, x^2, y), \\
\text{nl-art}(x^0, x^1, x^2), & \quad \text{nl-art}_i(x^0, x^1, x^2, y),
\end{align*}
\]

such that for all connected graphs \( G \) of order \( |G| \leq n \) and all \( u^0, u^1, u^2 \in V(G) \) the following holds. If \( \mathcal{B} \coloneqq (u^0, Q^0, u^1, Q^1, u^2, Q^2) \) is a pseudo-necklace in \( G \), then

\[
\begin{align*}
\text{nl-vert}[G, u^0, u^1, u^2, y] & = V(\mathcal{B}); \\
\text{nl-edge}[G, u^0, u^1, u^2, y_1, y_2] & = E(\mathcal{B}); \\
\text{nl-art}[G, u^0, u^1, u^2, y] & = \text{art}(\mathcal{B}); \\
\text{nl-art}_i[G, u^0, u^1, u^2, y] & = \{v\}, \text{ where } v \text{ is the } i\text{-th vertex in the linear order of } \text{art}(\mathcal{B}) \text{ that orders the articulation vertices of } Q^0, Q^1, \text{ and } Q^2 \text{ by increasing height and puts the articulation vertices of } Q^0 \text{ before those of } Q^1 \text{ and the latter ones before those of } Q^2. 
\end{align*}
\]
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Proof. For a vertex \( v \in V(G) \), we have that \( v \in V(B) \) if and only if \( v \in Q^G(u^i, u^{i+1}) \) for some \( i \in \{0, 1, 2\} \) (indices taken modulo 3). Similarly, an edge is a necklace edge if and only if for some \( i \), it connects a vertex \( v \in V(Q^G(u^i, u^{i+1})) \) of a certain height \( h \) with a vertex of height \( h+1 \) in \( V(Q^G(u^i, u^{i+1})) \). Thus, containment in \( V(B) \) and in \( E(B) \) is definable in \( C^w_4 \).

A vertex \( v \) is an articulation vertex of \( B \) if \( v \) equals \( u_i \) or \( v \) is an articulation vertex of \( Q^G(u^i, u^{i+1}) \) for some \( i \in \{0, 1, 2\} \). Thus, we can construct \( \text{nl-art} \) to have width 4.

We can construct \( \text{nl-art} \), in a straightforward manner by employing the subformulae \( \text{nl-art} \) and \( \text{csps-height} \). ▶

From Lemmas \[3.4\] and \[6.6\], we obtain the following corollary.

**Corollary 6.9.** There is a formula \( \text{csps-comp}(x, x', y, y') \in C^w_5 \) such that for all connected graphs \( G \) of order \( |G| \leq n \) and all \( u, u', v, v' \in V(G) \),

\[
G \models \text{csps-comp}(u, u', v, v') \iff v \text{ and } v' \text{ belong to the same connected component of } G \setminus V(Q^G(u, u')).
\]

From Lemma \[3.8\] applied to the \( C^w_\omega \)-sentence \( \psi := \text{genus}_h \) of Lemma \[6.5\] and the \( C^w_\omega \)-formula \( \varphi(x, x', y) := \text{csps-vert}(x, x', y) \) of Lemma \[6.6\], we obtain the following corollary.

**Corollary 6.10.** Let \( h < g \). Then there is a formula \( \text{csps-comp-genus}_h(x, x', y) \in C^w_{g+2} \) such that for all connected graphs \( G \) of order \(|G| \leq n \) and all \( u, u' \in V(G) \) the following holds. Let \( Q := Q^G(u, u') \), and let \( A \) be the connected component of \( v \) in \( G \setminus V(Q) \) (assuming \( v \notin V(Q) \)). Then

\[
G \models \text{csps-comp-genus}_h(u, u', v) \iff v \notin V(Q) \text{ and } \text{eg}(A) \leq h.
\]

**Corollary 6.11.** There is a formula \( \text{csps-simplifying}(x, x') \in C^w_{g+2} \) such that for all connected graphs \( G \in \mathcal{E}_g \) of order \(|G| \leq n \) and all \( u, u' \in V(G) \),

\[
G \models \text{csps-simplifying}(u, u') \iff Q^G(u, u') \text{ is simplifying}.
\]

The formulae we have defined so far make no reference to an embedding of the input graph. However, if we want to talk about patches and necklaces, we need to take the embedding into account. For the rest of the section, we fix a specific embedded graph \( G \).

**Assumption 6.12.** \( G \) is a coloured graph of order \(|G| = n \) that is polyhedrally embedded in a surface \( S \) of Euler genus \( g \).

It is our goal to construct a \( C^w_{g+3} \)-sentence that identifies \( G \).

Intuitively, the following lemma says that even though the logical formulae only have access to the abstract graph and the disk of a patch and the internal graph depend on the embedding, we can still define the internal graph. This is non-trivial and somewhat surprising.

**Lemma 6.13.** There are formulae \( \text{int-vert}(x, x', y), \text{int-edge}(x, x', y_1, y_2), \text{bd-vert}(x, x', y), \)\( \text{bd-edge}(x, x', y_1, y_2) \) in \( C^w_\omega \) such that for all vertices \( u, u' \in V(G) \) for which \( Q := Q^G(u, u') \) is a non-trivial non-simplifying patch, the following holds:

\[
\begin{align*}
\text{int-vert}[G, u, u', y] &= V(I(Q)), \\
\text{int-edge}[G, u, u', y_1, y_2] &= E(I(Q)), \\
\text{bd-vert}[G, u, u', y] &= V(C(Q)), \\
\text{bd-edge}[G, u, u', y_1, y_2] &= E(C(Q)).
\end{align*}
\]
We let \( \psi \) be a vertex in \( V(G) \) such that \( Q := Q^G(u, u') \) is a non-trivial non-simplifying patch. Let \( D := D(Q) \), \( C := C(Q) \), and \( I := I(Q) \) (see Section 3).

By Lemma 5.7, the graph \( G \setminus V(Q) \) has a unique non-planar connected component \( A^* \). We let

\[
\text{planar-comp}(x, x', y) := \text{csps-comp-genus}_3(x, x', y) \quad \text{(see Corollary 6.10)}
\]

and

\[
\astar(x, x', y) := \neg \text{csps-vert}(x, x', y) \land \neg \text{planar-comp}(x, x', y).
\]

Note that both \( \text{planar-comp}(x, x', y) \) and \( \astar(x, x', y) \) are \( C^w_{5+2} \)-formulae. In fact, since we can identify planar graphs in the logic \( C^w \), we can construct these formulae as \( C^w \)-formulae.

For \( v \in V(G) \setminus V(Q) \), we have \( G \models \text{planar-comp}(u, u', v) \) if and only if the connected component of \( v \) in \( G \setminus V(Q) \) is planar, and \( G \models \astar(u, u', v) \) if and only if the connected component of \( v \) in \( G \setminus V(Q) \) is \( A^* \).

Let \( v_1 \) be a vertex in \( V(Q) \) that is adjacent to \( A^* \) and among all such vertices has minimal height in the sps, and let \( h := \text{ht}^Q(v_1) \). Since \( A^* \) is embedded outside of the disk \( D \), the vertex \( v_1 \) must be on the boundary cycle \( C \) of \( D \). There is at most one other vertex of height \( h \) on this cycle. Thus, even though \( v_1 \) is not unique, there are at most two choices. If there is a second vertex of height \( h \) adjacent to \( A^* \), let us call it \( v'_1 \).

Let

\[
\varphi_1(x, x', y_1) := \text{csps-height}_h(x, x', y_1) \land \exists z^*(E(z^*, y_1) \land \astar(x, x', z^*)) \land \neg \exists y_1 \left( \bigvee_{i=0}^{h-1} \text{csps-height}_i(x, x', y'_1) \land \exists z^*(E(z^*, y'_1) \land \astar(x, x', z^*)) \right).
\]

Then \( v_1 \) and possibly \( v'_1 \) are the only vertices in \( \varphi_1[G, u, u', y_1] \). Note that \( \varphi_1 \in C^w_6 \).

Recall that \( G/A^* \) denotes the graph obtained from \( G \) by contracting the connected subgraph \( A^* \) to a single vertex, which we call \( a^* \), and that the graph \( G/A^* \) is a 3-connected planar graph. By Whitney’s Theorem, the facial subgraph of a 3-connected plane graph is precisely the chordless non-separating cycles. In particular, they are independent of the embedding. Furthermore, every edge is contained in exactly two of these facial cycles. Let us consider the edge \( v_1 a^* \) in the graph \( G/A^* \). Let \( F \) and \( F' \) be the two facial cycles that contain this edge. Both \( F \) and \( F' \) contain exactly one neighbour of \( a^* \) distinct from \( v_1 \). Let \( v_2 \) and \( v_2' \) be these neighbours.

By Lemma 22, if we have a 3-connected planar graph \( H \) and three vertices \( w_1, w_2, w_3 \) on a common facial cycle, then after individualising these three vertices, the 1-dimensional WL algorithm computes a discrete colouring. By Theorem 4.4, this implies that for every vertex \( w \in V(H) \) there is a formula \( \psi_{H,w}(z_1, z_2, z_3, z) \in C^w_6 \) such that \( \psi_{H,w}[H, w_1, w_2, w_3, z] = \{ w \} \). We apply Lemma 22 to the graph \( G/A^* \) and the three vertices \( a^*, v_1, v_2 \) and obtain, for every vertex \( w \in V(G/A^*) = (V(G) \setminus A^*) \cup \{ a^* \} \), a formula \( \psi_w(z^*, y_1, y_2, z) \in C^w_6 \) such that \( \psi_w[G/A^*, a^*, v_1, v_2, z] = \{ w \} \).

Let \( w \in V(G) \setminus A^* \). Recall that \( \astar(x, x', y) \in C^w_6 \) and \( \psi_w(z^*, y_1, y_2, y) \in C^w_5 \). By Lemma 3.9 (applied to \( k = 6 \), \( \ell = 2 \), \( m = 5 \) and the formulæ \( \xi(x, x', z^*) := \astar(x, x', z^*) \) and \( \psi(y_1, y_2, y, z^*) := \psi_w(z^*, y_1, y_2, y) \)), there is a formula \( \tilde{\psi}_w(x, x', y_1, y_2, y) \in C^w_6 \) such that \( \tilde{\psi}_w[G, u, u', v_1, v_2, y] = \{ w \} \).

Since \( A^* \cap D = \emptyset \), we have \( V/I = V(G) \cap D \subseteq V(G \setminus A^*) \). We let

\[
\delta(x, x', y_1, y_2, z) := \bigvee_{w \in V(I)} \tilde{\psi}_w(x, x', y_1, y_2, z).
\]
Then \( \delta(G, u, u', v_1, v_2, z) = V(I) \). Thus \( \delta(x, x', y_1, y_2, z) \) is almost the formula \( \text{int-vert} \) we want, except that it has two additional parameters \( v_1, v_2 \) which we have to get rid of.

We will apply \cite{25} Corollary 26, which says that the 3-dimensional WL algorithm determines orbits in coloured 3-connected graphs. This implies that within a given graph, the 3-dimensional WL algorithm distinguishes two vertices if and only if they belong to different orbits of the automorphism group of the graph. It follows that for every 3-connected planar graph \( H \) and for every orbit \( O \) of the automorphism group of \( H \) there is a formula \( \xi_{H,O}(y_2) \in C_w^d \) such that \( \xi_{H,O}[H, y_2] = O \).

To eliminate the parameter \( v_2 \), we apply the corollary to the graph \( G/A^* \), but only after individualising the vertices \( a^* \) and \( v_1 \). (That is, we modify the colouring such that each of the two vertices has its own colour and is thus fixed by all automorphisms.) Let \( O_2 \) be the orbit of \( v_2 \) in this coloured graph. By the definition of \( v_2 \), either \( O_2 = \{v_2, v'_2\} \) or \( O_2 = \{v_2\} \). Since the graph \( G/A^* \) is 3-connected, by eliminating the colour relations for \( a^* \) and \( v_1 \) at the cost of new free variables \( z^* \) and \( y_1 \), we obtain a new formula \( \psi_2(z^*, y_1, y_2) \in C_w^d \) such that \( \psi_2(G/A^*, a^*, v_1, y_2) = O_2 \). Since \( \text{astar}(x, x', y) \in C_w^d \) and \( \xi_{H,O}(y) \in C_w^d \), by Lemma 3.9 (with \( k = 6, \ell = 2, m = 4 \) and the formulas \( \xi(x, x', z^*) := \text{astar}(x, x', z^*) \) and \( \psi(y_1, y_2, z^*) := \psi_2(z^*, y_1, y_2) \)), there is a formula \( \psi(x, x', y_1, y_2) \in C_w^d \) such that \( \psi_2(G, u, u', v_1, y_2) = O_2 \). We let

\[
\delta'(x, x', y_1, y_2) := \exists y_2 (\tilde{\psi}(x, x', y_1, y_2) \land \delta(x, x', y_1, y_2, z)).
\]

If \( O_2 = \{v_2\} \) then clearly \( \delta'[G, u, u', v_1, z] = \delta[G, u, u', v_1, v_2, z] = V(I) \). So suppose that \( O_2 = \{v_2, v'_2\} \), and let \( f \) be an automorphism of \( G \) with \( f(u) = u, f(u') = u', f(v_1) = v_1, \) and \( f(v_2) = v'_2 \). By Corollary 5.9 we have \( f(V(I)) = V(I) \) and thus

\[
\delta[G, u, u', v_1, v'_2, z] = \delta[f(G), f(u), f(u'), f(v_1), f(v'_2), z]
\]

\[
= f(\delta[G, u, u', v_1, v_2, z])
\]

\[
= f(V(I)) = V(I).
\]

It follows that

\[
\delta'[G, u, u', v_1, v'_2, z] \cup \delta[G, u, u', v_1, v_2, z] = V(I).
\]

So we have eliminated the parameter \( v_2 \). To eliminate \( v_1 \), we use essentially the same argument. Let \( O_1 \) be the orbit of \( v_1 \) in the graph \( G/A^* \) with the vertices \( a^*, u, u' \) individualised. Then either \( O_1 = \{v_1, v'_1\} \) for some \( v'_1 \neq v_1 \) or \( O_1 = \{v_1\} \).

By \cite{25} Corollary 26, there is a formula \( \xi_{G/A^*, O_1}(y_1) \in C_w^d \) such that

\[
\xi_{G/A^*, O_1}([G/A^*]_{a^*, u, u', y_1}) = O_1.
\]

Then by eliminating the colour relations for \( a^*, u, u' \) at the cost of new free variables \( z^*, x, x' \), we obtain a formula \( \psi_1(z^*, x, x', y_1) \in C_w^d \) such that \( \psi_1(G/A^*, a^*, u, u', y_1) = O_1 \). Since \( \text{astar}(x, x', y) \in C_w^d \) and \( \xi_{G/A^*, O}(y_1) \in C_w^d \), by Lemma 3.9 (with \( k = 7, \ell = 2, m = 3 \) and the formulas \( \xi(x, x', z^*) := \text{astar}(x, x', z^*) \) and \( \psi(x, x', y_1, z^*) := \psi_1(z^*, x, x', y_1) \)), there is a formula \( \psi_1(x, x', y_1) \in C_w^d \) such that \( \psi_1([G, u, u', y_1] = O_1 \).

We let

\[
\text{int-vert}(x, x', z) := \exists y_1 (\tilde{\psi}_1(x, x', y_1) \land \delta'(x, x', y_1, z)).
\]

Now a similar argument as above shows that \( \text{int-vert}[G, u, u', z] = V(I) \). Moreover, since \( \delta', \psi_1 \in C_w^d \), we have \( \text{int-vert} \in C_w^d \).

The formulae \( \text{int-edge}(x, x', y_1, y_2), \text{bd-vert}(x, x', y), \text{bd-edge}(x, x', y_1, y_2) \) can be defined similarly. ▷
Now we branch into two cases, depending on whether $G$ contains a simplifying patch or not.

### 6.1 Case 1: Absence of simplifying patches

Throughout this subsection, in addition to Assumption 6.12, we assume the following.

**Assumption 6.14.** $G$ does not contain any simplifying patches.

By Lemma 5.12, $G$ contains a reducing necklace $B$. We are going to define a subgraph $\text{Cut}(B)$ of $G$ that is obtained from $G$ by “cutting through the beads”. Since the necklace is reducing, the Euler genus of every connected component of $\text{Cut}(B)$ is at most $g - 1$ and we can identify it with a $C^s$-sentence. We colour $\text{Cut}(B)$ in such a way that we can reconstruct $G$ and identify it using only 3 more variables.

For a necklace $B := (u^0, Q^1, u^1, Q^2, u^2, Q^3)$ in $G$, let $u^i = u^i_1, u^i_2, \ldots, u^i_{n_i} = u^{i+1}$ be the articulation vertices of $Q^i$, ordered by height, and for $j \in \{0, \ldots, n_i - 1\}$ let $Q^i_j := [u^i_j, u^i_{j+1}]$ be the segment of $Q^i$ between $u^i_j$ and $u^i_{j+1}$. If the patch $Q^i_j$ is trivial, we denote its unique edge by $e^i_j$. If $Q^i_j$ is non-trivial, we let $D^i_j := D(Q^i_j)$.

The region of $B$ is the point set

$$R(B) := \bigcup_{i=0}^{2} \left( \bigcup_{0 \leq j \leq n_i - 1}^{Q \text{ non-trivial}} D^i_j \cup \bigcup_{0 \leq j \leq n_i - 1}^{Q \text{ trivial}} e^i_j \right).$$

Recall that the internal graph of a non-simplifying patch $Q$ is the graph $I := I(Q)$ with vertex set $V(I) := V(G) \cap D(Q)$ and edge set $E(I) := \{e \in E(G) \mid e \subseteq D(Q)\}$. We associate three subgraphs of $G$ with $B$:

**Definition 6.15.** The inside of $B$ is $I(B) := \bigcup_{i=0}^{2} \bigcup_{j=0}^{n_i - 1} I(Q^i_j)$.

The outside of $B$ is the graph $O(B)$ defined by

$$V(O(B)) := V(G) \setminus \text{int}(R),$$

$$E(O(B)) := E(G) \setminus \{e \in E(G) \mid e \cap \text{int}(R) \neq \emptyset\}.$$ 

The cut graph of $B$ is $\text{Cut}(B) := O(B) \setminus \text{art}(B)$.

**Lemma 6.16.** Suppose $B$ is a reducing necklace in $G$. Then every connected component of $\text{Cut}(B)$ is in $\mathcal{E}_{g-1}$.

**Proof.** This proof is a slight adaptation of the proof of [17 Lemma 15.5.6].

Let $R := R(B)$. For all $i, j$ such that $Q^i_j$ is non-trivial, we let $D^i_j := D(Q^i_j)$.

Let $B$ be a non-contractible cycle in $B$, whose existence is guaranteed by Definition 5.11.

Then $B \subseteq R$ is a simple closed curve, and for all $i, j$ such that $Q^i_j$ is non-trivial, the intersection $B_j^i := D^i_j \cap B$ is a simple curve in the disk $D^i_j$ with endpoints $u^i_j$ and $u^i_{j+1}$.

By slightly perturbing $B$, we obtain a homotopic simple closed curve $b \subseteq R$ such that for all $i, j$ with non-trivial $Q^i_j$ we have $b \cap bd(D^i_j) = \{u^i_j, u^i_{j+1}\}$. This new curve $b$ is still non-contractible, and it intersects $bd(R)$ only in the articulation vertices $u^i_j$ of $B$ and in the edges $e^i_j$ of the trivial patches $Q^i_j$.

This implies that for $H := \text{Cut}(B)$ we have $b \cap H = \emptyset$. Thus $H \subseteq G \setminus b$, and since $b$ is non-contractible, this implies that every connected component of $H$ is embeddable in a surface of Euler genus at most $g - 1$ obtained from $S$ by cutting along $g$ and gluing a disk on each hole.
Our next goal is to show that the cut graph is definable in $C_{w}^{w + 3}$. We start with the definability of patches.

From Lemma 6.13, we obtain that $C_{w}$ distinguishes the internal graph of a reducing necklace from the remainder of the graph.

**Corollary 6.17.** There are $C_{w}$-formulae

\[
\text{nl-int-vert}(x^0, x^1, x^2, y), \quad \text{nl-int-edge}(x^0, x^1, x^2, y_1, y_2).
\]

such that for $u^0, u^1, u^2 \in V(G)$ the following holds.

If $B := (u^0, Q^0, u^1, Q^1, u^2, Q^2)$ is a necklace in $G$, then

\[
\text{nl-int-vert}(G, u^0, u^1, u^2, y) = V(I(B));
\]

\[
\text{nl-int-edge}(G, u^0, u^1, u^2, y_1, y_2) = E(I(B)).
\]

**Proof.** Remember that we suppose Assumption 6.14. Thus, we can simply define

\[
\text{nl-int-vert}(x^0, x^1, x^2, y) := \bigvee_{i=0}^{2} \text{int-vert}(x^i, x^{i+1}, y).
\]

Similarly, we obtain the formula $\text{nl-int-edge}$ with the desired width.

In the following we show that $C_{w}$ distinguishes vertices in the outside and the cut graph of $B$ from the rest of the graph.

**Lemma 6.18.** There are $C_{w}$-formulae

\[
\text{nl-out-vert}(x^0, x^1, x^2, y), \quad \text{nl-out-edge}(x^0, x^1, x^2, y),
\]

\[
\text{nl-cut-vert}(x^0, x^1, x^2, y), \quad \text{nl-cut-edge}(x^0, x^1, x^2, y_1, y_2)
\]

such that for all $u^0, u^1, u^2 \in V(G)$ the following holds: if $B := (u^0, Q^0, u^1, Q^1, u^2, Q^2)$ is a necklace in $G$, then

\[
\text{nl-out-vert}(G, u^0, u^1, u^2, y) = V(O(B));
\]

\[
\text{nl-out-edge}(G, u^0, u^1, u^2, y_1, y_2) = E(O(B));
\]

\[
\text{nl-cut-vert}(G, u^0, u^1, u^2, y) = V(\text{Cut}(B));
\]

\[
\text{nl-cut-edge}(G, u^0, u^1, u^2, y_1, y_2) = E(\text{Cut}(B)).
\]

**Proof.** Let $R := R(B)$ and recall that $u^i = u^i_0, u^i_1, \ldots, u^i_n_i = u^{i+1}$ denote the articulation vertices of $B$, ordered by height, and that for $j \in \{0, \ldots, n_i - 1 \}$, we denote the segment of $Q^i$ between $u^i_j$ and $u^i_{j+1}$ by $Q^i_j := Q^i[u^i_j, u^i_{j+1}]$. Since by Assumption 6.14 all subpatches are non-simplifying, it holds that

\[
V(G) \cap \text{int}(R) = \bigcup_{i=0}^{2} \bigcup_{0 \leq j \leq n_i - 1} V(I(Q^i_j) \setminus C(Q^i_j)).
\]

Therefore,

\[
V(O(B)) = V(G) \setminus \bigcup_{i=0}^{2} \bigcup_{0 \leq j \leq n_i - 1} V(I(Q^i_j) \setminus C(Q^i_j)),
\]

\[
E(O(B)) = E(G) \setminus \bigcup_{i=0}^{2} \bigcup_{0 \leq j \leq n_i - 1} E(I(Q^i_j) \setminus C(Q^i_j)).
\]
Thus, we can just let
\[
nl-out-\text{-vert}(x^0, x^1, x^2, y) := \bigvee_{i=1}^{n} \exists z \exists z' (nl-\text{art}_i(x^0, x^1, x^2, z) \land \text{nl-}\text{-art}_{i+1}(x^0, x^1, x^2, z') \\
\land \neg \text{nl-}\text{-edge}(x^0, x^1, x^2, z, z') \land \text{bd-}\text{-vert}(z, z', y)) \\
\lor \text{nl-}\text{-art}(x^0, x^1, x^2, y) \lor \neg \text{nl-}\text{-int-}\text{-vert}(x^0, x^1, x^2, y),
\]
where the big disjunction expresses that the given vertex lies on the boundary of some disk of a non-trivial patch.

Similarly, we obtain the formula nl-out-edge of width 7.

To define that a vertex is contained in the cut graph, we just need to guarantee that it is contained in \(O(B)\) and that is not an articulation vertex of the necklace. Similarly, for an edge contained in \(O(B)\), to appear in \(\text{Cut}(B)\), its incident vertices must not be articulation vertices of \(B\). We obtain the desired \(C^0\)-formulae nl-cut-vert and nl-cut-edge.

We have collected all ingredients to show the statement from Lemma \ref{lem:iso} in case \(G\) contains no simplifying patches.

**Proof of Lemma \ref{lem:iso}** Case 1. We show that the statement holds if \(g \geq 1\) and \(G\) does not contain any simplifying patches.

Recall that by Assumption \ref{asmp:genus}, for every coloured graph \(H \in \mathcal{E}_{g-1}\), we assume the existence of a formula \(\text{iso}_H \in C^0_g\) such that for all graphs \(G'\) it holds that
\[
G' \models \text{iso}_H \iff G' \cong H.
\]

Let \(G\) be a coloured graph that does not contain any simplifying patches and is polyhedrally embedded in a surface \(S\) of genus \(g \geq 1\). Let \(\hat{G}\) be a second coloured graph such that there is no formula in \(C^{g+3}\) which distinguishes \(G\) and \(\hat{G}\). We show that \(G \cong \hat{G}\).

We may assume \(|\hat{G}| = |G|\), otherwise we can distinguish \(G\) and \(\hat{G}\) via the formula \(\exists v[G(v) = v]\).

\cite{grohe2018} Theorem 5 implies that if for some \(k \geq 3\), the logic \(C^k\) distinguishes all non-isomorphic pairs of coloured 2-connected graphs, then it distinguishes all pairs of non-isomorphic graphs in \(C\). Thus, if \(C^{g+3}\) distinguishes (the 2-connected graph) \(G\) from every non-isomorphic 2-connected coloured graph, then the same logic distinguishes \(G\) from every arbitrary non-isomorphic coloured graph and thus, it identifies \(G\). Hence, we can assume \(\hat{G}\) to be 2-connected.

Moreover, if \(\hat{G}\) is not 3-connected, then it has a separator of size 2 whereas \(G\) does not. Since for \(k \geq 3\), the \(k\)-dimensional WL algorithm distinguishes 2-separators from other pairs of vertices (see \cite{grohe2018} Corollary 14), by Corollary \ref{cor:3-connected} there is a formula in \(C^0_g\) which distinguishes \(G\) and \(\hat{G}\).

Hence, without loss of generality we may assume that \(\hat{G}\) is 3-connected.

By Lemma \ref{lem:reduce}, there is a reducing necklace \(B := (u^0, Q^0, u^1, Q^1, u^2, Q^2)\) in \(G\), which we fix for the rest of the proof. For a pseudo-necklace \(\hat{B} := (\hat{u}^0, \hat{Q}^0, \hat{u}^1, \hat{Q}^1, \hat{u}^2, \hat{Q}^2)\) in \(\hat{G}\), we say \(B\) and \(\hat{B}\) are isomorphic, and write \(B \cong \hat{B}\), if there is an isomorphism from \(G(B)\) to \(\hat{G}(\hat{B})\) mapping \(u^i\) to \(\hat{u}^i\) for \(i \in \{0, 1, 2\}\).

**Claim 1.** There is a formula \(\text{nl-iso}(x^0, x^1, x^2) \in C^0_g\) (not depending on \(\hat{G}\)) such that \(\hat{G} \models \text{nl-iso}(\hat{u}^i, \hat{u}^1, \hat{u}^2)\) if and only if \(\hat{B} := (\hat{u}^0, \hat{Q}^0, \hat{u}^1, \hat{Q}^1, \hat{u}^2, \hat{Q}^2)\) is a pseudo-necklace with \(B \cong \hat{B}\).

**Proof.** \(\hat{B}\) is a pseudo-necklace isomorphic to \(B\) if and only if for all \(i \in \{0, 1, 2\}\), the following two conditions hold for \(\hat{Q}^i := Q(\hat{u}^i, \hat{u}^{i+1})\).
1. \( \hat{G}(\hat{Q}) \cong G(Q) \) via an isomorphism mapping \( \hat{u}^i \) to \( u^i \) and \( \hat{u}^{-1} \) to \( u^{-1} \).

2. \( V(\hat{Q}) \cap V(\hat{Q}^{-1}) = \{ \hat{u}^i \} \).

Condition 2 is easy to express in \( C^6_\omega \). To treat Condition 1, let \( \text{sps-iso}' \in C^6_\omega \) be the sentence from Theorem 4.6 which identifies the planar coloured graph \( Q_{u_i,u_{i+1}} \) such that \( \hat{G} = \text{sps-iso}'(\hat{u}^i, \hat{u}^{-1}) \) if and only if \( \hat{G}(\hat{Q}) \cong G(Q) \) via an isomorphism that maps \( \hat{u}^i \) to \( u^i \) and \( \hat{u}^{-1} \) to \( u^{-1} \). We transform \( \text{sps-iso}' \) into a formula \( \text{sps-iso}(x^i, x^{i+1}) \) such that \( \hat{G}(\hat{Q}) \cong \text{sps-iso}(\hat{u}^i, \hat{u}^{-1}) \) if and only if \( \hat{G}(\hat{Q}) \cong G(Q) \) via an isomorphism that maps \( \hat{u}^i \) to \( u^i \) and \( \hat{u}^{-1} \) to \( u^{-1} \). To this end, we first replace in \( \text{sps-iso}' \) every \( R^i(z, z) \) with the formula \( z = x^i \) and every \( R^{i+1}(z, z) \) with \( z = x^{i+1} \). To relativise \( \text{sps-iso}' \) to the \( i \)-th shortest path system, we also replace subformulae of the form \( \exists y(\text{sps-vert}(x^i, x^{i+1}, y) \land \varphi) \) and \( E(y_1, y_2) \) with \( \text{sps-edge}(x^i, x^{i+1}, y_1, y_2) \).

By Lemma 6.6, the resulting formula \( \text{sps-iso}(x^i, x^{i+1}) \) is in \( C^6_\omega \).

Now we can define the desired \( C^6_\omega \)-formula:

\[
\text{nl-iso}(x^0, x^1, x^2) \defeq \bigwedge_{i=0}^2 \text{sps-iso}(x^i, x^{i+1}) \land \forall y \bigwedge_{i=0}^2 \left( (\text{sps-vert}(x^{i-1}, x^i, y) \land \text{csps-vert}(x^i, x^{i+1}, y)) \rightarrow y = x^i \right),
\]

where we take indices modulo 3.

For the remainder of this proof, let \( \hat{B} := (\hat{u}^0, \hat{Q}^0, \hat{u}^1, \hat{Q}^1, \hat{u}^2, \hat{Q}^2) \) be a pseudo-necklace in \( \hat{G} \) such that \( B \cong \hat{B} \). If no such pseudo-necklace exists, we can distinguish \( \hat{G} \) and \( \hat{G} \) in \( C^6_\omega \) using Claim 1. Let \( u^i = u^i_1, u^i_2, \ldots, u^i_n \) be the articulation vertices of \( Q^0 \) ordered by increasing height in \( B \). Since \( \hat{B} \cong B \), there is a bijection from \( \text{art}(B) \) to \( \text{art}(\hat{B}) \) mapping each articulation vertex to one of equal height in \( \hat{B} \). Thus, for simplicity, we use the same name for the two articulation vertices in \( B \) and \( \hat{B} \) of equal height. In the following, let \( Q_{i,j} := Q^0(u^i_j, u^i_{j+1}) \) and \( \hat{Q}_{i,j} := \hat{Q}^0(u^i_j, u^i_{j+1}) \). Note that the \( \hat{Q}_{i,j} \) are pseudo-patches. By our assumption that \( \hat{B} \cong B \), the pseudo-patch \( \hat{Q}_{i,j} \) is trivial if and only if the patch \( Q_{i,j} \) is.

Let \( I := I(B) \). Let \( \hat{I} \) be the graph with vertex set \( V(\hat{I}) := \text{nl-int-vert}[\hat{G}, \hat{u}^0, \hat{u}^1, \hat{u}^2, y] \) and edge set \( E(\hat{I}) := \text{nl-int-edge}[\hat{G}, \hat{u}^0, \hat{u}^1, \hat{u}^2, y_1, y_2] \). Since \( \hat{B} \) need not be a proper necklace (it might not comply with the third item in Definition 5.10), the graph \( \hat{I} \) is not necessarily the inside of a necklace. However, for simplicity, we also use the letter \( I \) to refer to such a “pseudo-inside” just as we also use \( B \) for all pseudo-necklaces. For simplicity, we call \( I \) and \( \hat{I} \) isomorphic, and we write \( I \cong \hat{I} \) if \( I_{u^0,u^1,u^2} \cong \hat{I}_{\hat{u}^0,\hat{u}^1,\hat{u}^2} \), i.e., if there is an isomorphism from \( I \) to \( \hat{I} \) mapping \( u^i \) to \( \hat{u}^i \) for every \( i \in \{0,1,2\} \). Note that every such isomorphism induces an isomorphism from \( B \) to \( \hat{B} \). We also define a “pseudo-inside” for all the pseudo-patches \( \hat{Q}_{i,j} \): we let \( \hat{I}(\hat{Q}_{i,j}) \) be the graph with vertex set \( \text{int-vert}[\hat{G}, \hat{u}^i_j, \hat{u}^i_{j+1}, y] \) and edge set \( \text{int-edge}[\hat{G}, \hat{u}^i_j, \hat{u}^i_{j+1}, y] \).

Claim 2. There is a formula inside-iso\((x^0, x^1, x^2) \in C^6_\omega \) (not depending on \( \hat{G} \)) such that \( \hat{G} \models \text{inside-iso}(\hat{u}^0, \hat{u}^1, \hat{u}^2) \) if and only if \( \hat{I} \cong I \).

Proof. We have that \( \hat{I} \cong I \) if and only if \( \hat{G} \) satisfies the following conditions for all \( i \in \{0,1,2\} \).

1a) \( I(\hat{Q}_{i,j})u^i_{j+1} = I(Q_{i,j})u^i_{j+1} \) for all \( j \in [n_i - 1] \).

1b) If \( i' = i + 1 \mod 3 \) and \( j = n_i - 1 \) and \( j' = 1 \), it holds that \( I(\hat{Q}_{i,j}) \cap I(\hat{Q}_{i',j'}) = \{ \hat{u}^{i+1} \} \).

1c) If \( j \in [n_i - 1] \) and \( j' = j + 1 \), it holds that \( I(\hat{Q}_{i,j}) \cap I(\hat{Q}_{i,j'}) = \{ u^{i+1}_j \} \).

1d) If \( i' = i + 1 \mod 3 \) and \( j \neq n_i - 1 \) and \( j' \neq 1 \), it holds that \( I(\hat{Q}_{i,j}) \cap I(\hat{Q}_{i',j'}) = \emptyset \). If \( |j' - j| \geq 2 \), it holds that \( I(\hat{Q}_{i,j}) \cap I(\hat{Q}_{i',j'}) = \emptyset \).
The “only if” follows from the localisation of \( I \) and the \( C \)-definability of \( I(Q_{i,j}) \). We now show the “if”-part. Consider isomorphisms \( \pi_{ij} \) witnessing Item (10). We define an isomorphism \( \pi \) to \( I \) by letting \( \pi(v) = \pi_{ij}(v) \) where \( i \) and \( j \) are such that \( I(Q_{i,j}) \) contains \( v \). Note that by Items (10), (13) and (14), if \( I(Q_{i,j}) \cap I(Q_{i',j'}) \neq \emptyset \) for \( Q_{i,j} \neq Q_{i',j'} \), then there is a unique vertex \( v \in I(Q_{i,j}) \cap I(Q_{i',j'}) \). In that case, Item (10) guarantees that the two possible images \( \pi(v) \) coincide. Thus, \( \pi \) is well-defined and it certainly is an isomorphism.

We still need to translate Items (11a)–(11d) into \( C \)-formulae. Since the subgraph \( I(Q_{i,j}) \) of \( G \) is embedded in the disk \( D(Q_{i,j}) \), it is planar. Hence, by Theorem 4.6, there is a sentence \( \text{disk-iso} \) which identifies \( I(Q_{i,j})u_j, u_{j+1} \). Let \( R_j \) and \( R_{j+1} \) be the relations that occur in \( \text{disk-iso} \) for the colours of \( u_j \) and \( u_{j+1} \), respectively.

To relativise \( \text{disk-iso} \) to \( I(Q_{i,j}) \), we replace every \( \exists \exists \varphi \) with \( \exists \exists \varphi \cap \text{int-vert}(x, y) \land \varphi \) and every \( E(x, y) \) with \( \text{int-edge}(x, y) \). Furthermore, we replace every \( R_j(z, z) \) with the formula \( \exists x \exists y (x = x_j \land y = y_{j+1} \land z) \). By Lemma 6.13, the resulting formula \( \text{disk-iso}(x_j, x_{j+1}) \) is in \( C_w \).

Again using Lemma 6.13, it is tedious but straightforward to construct a \( C_w \)-formula \( \text{disk-chain}(x^0, x^1, x^2) \) which checks if Items (11a), (11b) and (11d) hold. Now we can just let

\[
\begin{align*}
\text{inside-iso}(x^0, x^1, x^2) := \ & \text{nl-iso}(x^0, x^1, x^2) \land \text{disk-chain}(x^0, x^1, x^2) \land \\
\ & \bigwedge_{n=0}^{n-1} \bigwedge_{j=1}^{2} \exists \exists \exists' (\text{csps-art}_j(x^i, x^{i+1}, z) \land \\
\ & \text{csps-art}_{j+1}(x^i, x^{i+1}, z') \land \text{disk-iso}(z, z')).
\end{align*}
\]

In the following, we assume without loss of generality that \( I \equiv I \).

Since \( C(Q_{i,j}) \) is a cycle, the two sets of vertices of the segments on \( Q_{i,j} \) between \( u_j \) and \( u_{j+1} \) form blocks of the automorphism group of \( C(Q_{i,j})u_j, u_{j+1} \) and thus by Corollary 5.9 also of the automorphism group of \( G_{u_j, u_{j+1}} \). To be more precise, every automorphism of \( G \) that fixes \( u_j \) and \( u_{j+1} \) either leaves each of the two segments invariant or “swaps sides”, i.e., maps the two segments onto each other while preserving heights. Moreover, there is such an automorphism swapping sides in \( I(Q_{i,j}) \) if and only if there is a vertex \( v \in Q_{i,j} \) with \( v \notin \{u_j, u_{j+1}\} \) whose orbit of the automorphism group of \( G_{u_j, u_{j+1}} \) has size greater than 1. (In that case, it has size 2.)

Recall the definition of the cut graph \( \text{Cut}(B) \) from Definition 6.15. Also recall Lemma 6.18 where we introduced \( C_w \)-formulae \( \text{nl-cut-vert}(x^0, x^1, x^2, y) \), \( \text{nl-cut-edge}(x^0, x^1, x^2, y_1, y_2) \) defining the vertex set and edge set of the cut graph. We define a pseudo-cut graph \( \text{Cut}(B) \) of \( B \) by letting

\[
\begin{align*}
V(\text{Cut}(B)) := \ & \text{nl-cut-vert}(\hat{G}, u^0, u_1, \hat{u}_2, y), \\
E(\text{Cut}(B)) := \ & \text{nl-cut-edge}(\hat{G}, u^0, u_1, \hat{u}_2, y_1, y_2).
\end{align*}
\]

Let \( C(Q_{i,j}) \) be the graph with \( V(C(Q_{i,j})) = \text{bd-vert}(\hat{G}, u_j, u_{j+1}, y) \) and \( E(C(Q_{i,j})) = \text{bd-edge}(\hat{G}, u_j, u_{j+1}, y_1, y_2) \), where \( \text{bd-vert}(x, x', y) \), \( \text{bd-edge}(x, x', y_1, y_2) \) are the \( C_{w} \)-formulae from Lemma 6.13. Furthermore, let \( I^*(\hat{Q}_{i,j}) \) be the graph resulting from \( I(Q_{i,j})u_j, u_{j+1} \) by assigning all vertices in \( V(C(Q_{i,j})) \) a common distinct colour and proceeding similarly with \( E(C(Q_{i,j})) \). Define the graph \( I^*(\hat{Q}_{i,j}) \) similarly.

Let \( \text{Cut}^*(B) \) be the (coloured) graph resulting from \( \text{Cut}(B) \) by adding colours corresponding to the following unary and binary relations:
(2a) for each set \( J \subseteq |\text{art}(B)| \) a relation \( R_J \) with \( v \in R_J \) if and only if \( J = \{ i \mid vw \in E(G) \) for a \( w \in \text{nl-art}_i(G, u^0, u^1, u^2, w, y) \} \), where \( \text{nl-art}_i(x^0, x^1, x^2, y) \) is the \( C^1 \)-formula introduced in Corollary 6.8

(2b) for each \( i \in \{0,1,2\} \) and each \( j \in \{ n_i - 1 \} \) a relation \( R^1_{i,j} \) with \( v \in R^1_{i,j} \) if and only if \( v \in \text{bd-vert}[G, u^j, u^j_{i+1}, y] \).

(2c) for each \( i \in \{0,1,2\} \) and each \( j \in \{ n_i - 1 \} \) a relation \( R^2_{i,j} \) with \( e \in R^2_{i,j} \) if and only if \( e \in \text{bd-edge}[G, u^j, u^j_{i+1}, y_1, y_2] \).

(2d) for every orbit \( O \) of the automorphism group of \( I^*(Q_{i,j}) \) a relation \( R_O \) with \( v \in R_O \) if and only if \( v \in O \).

We show that all of the relations introduced in Items (2a)–(2d) can be defined in \( C^3_\omega \) by providing formulae that express containment in the relations. We omit the correctness proofs since they are straightforward.

(3a) For \( R \coloneqq R_J \), let

\[
\varphi_R(x^0, x^1, x^2, y) := \text{nl-cut-vert}(x^0, x^1, x^2, y) \land \bigwedge_{i \in J} \exists z (\text{nl-art}_i(x^0, x^1, x^2, z) \land E(y, z)) \\
\land \bigwedge_{i \in |\text{art}(B)| \setminus J} \neg \exists z (\text{nl-art}_i(x^0, x^1, x^2, z) \land E(y, z)).
\]

(3b) For \( R \coloneqq R^1_{i,j} \), let

\[
\varphi_R(x^0, x^1, x^2, y) := \text{nl-cut-vert}(x^0, x^1, x^2, y) \land \exists x^i_j \exists x^j_{i+1} (\text{csps-art}_j(x^i, x^{i+1}, x_j^i) \\
\land \text{csps-art}_{j+1}(x^i, x^{i+1}, x^i_{j+1}) \land \text{bd-vert}(x^j_i, x^j_{i+1}, y)).
\]

(3c) For \( R \coloneqq R^2_{i,j} \), let

\[
\varphi_R(x^0, x^1, x^2, y, z, y_2) := \text{nl-cut-edge}(x^0, x^1, x^2, y_1, y_2) \land \\
\exists x^i_j \exists x^j_{i+1} (\text{csps-art}_j(x^i, x^{i+1}, x^i_j) \land \text{csps-art}_{j+1}(x^i, x^{i+1}, x^i_{j+1}) \\
\land \text{bd-edge}(x^j_i, x^j_{i+1}, y_1, y_2)).
\]

(3d) Let \( R \coloneqq R_O \). By Proposition 4.2 and the correspondence from Corollary 4.5, since \( C^1 \) identifies \( I^*(Q_{i,j}) \), the logic \( C^3 \) determines orbits on \( I^*(Q_{i,j}) \). Thus, there is a \( C^3_\omega \)-formula \( \varphi'_R(x) \) such that for any graph \( H \) it holds that \( H \models \varphi'_R(v) \) if and only if there is an isomorphism \( \pi^* \) from \( I^*(Q_{i,j}) \) to \( H \) such that \( \pi^*(v) = v \) for some \( v \in O \).

We relativise \( \varphi'_R(x) \) to \( I(Q_{i,j}) \) by replacing every occurrence of the form \( \exists \psi(x_1, y_1) \) with

\[
\exists \psi(x_1) \exists \psi'(\text{csps-art}_j(x^0, x^i, z) \land \text{csps-art}_{j+1}(x^i, x^{i+1}, z', \pi^* \text{int-vert}(z, z', x) \land \psi))
\]

and proceeding similarly for the edges.

Let \( R^1_O \) and \( R^2_O \) be the colour relations corresponding to \( V(C(Q_{i,j})) \) and \( E(C(Q_{i,j})) \) in \( I^*(Q_{i,j}) \), respectively. We replace \( R^1_O(z, z) \) with

\[
\exists y \exists y' (\text{csps-art}_j(x^0, x^{i+1}, y) \land \text{csps-art}_{j+1}(x^i, x^{i+1}, y') \land \text{bd-vert}(y, y', z))
\]

and proceed similarly with \( R^2_O(z, z') \). Recall that \( R^1_O \) and \( R^2_{O+1} \) are the colour relations for \( u^j_i \) and \( u^j_{i+1} \), respectively. We replace \( R^1_O(z, z) \) with the formula \( \text{csps-art}_j(x^i, x^{i+1}, z) \) and do the analogous for \( R^2_{O+1}(z, z) \). By Corollary 6.8 and Lemma 6.13 the resulting formula \( \varphi_R(x^0, x^1, x^2, x) \) is in \( C^3_\omega \).

Our assumption \( \bar{B} \cong B \) implies that \( \text{art}(\bar{B}) = \text{art}(B) \). Define \( \text{Cut}^*(\bar{B}) \) as the graph resulting from \( \text{Cut}(\bar{B}) \) by interpreting each relation \( R \) from Items 2a, 2b, 2d as \( \varphi_R[\bar{G}, \bar{u}^0, \bar{u}^1, \bar{u}^2, y] \) and each \( R \) from Item 2c as \( \varphi_R[\bar{G}, \bar{u}^0, \bar{u}^1, \bar{u}^2, y_1, y_2] \).
Claim 3. \( \text{Cut}^*(\mathcal{B}) \equiv \text{Cut}^*(\hat{\mathcal{B}}) \) if and only if \( G_{u^0,u^1,u^2} \cong \hat{G}_{\tilde{u}^0,\tilde{u}^1,\tilde{u}^2} \).

Proof. We prove the backward direction first. Assume that \( G_{u^0,u^1,u^2} \cong \hat{G}_{\tilde{u}^0,\tilde{u}^1,\tilde{u}^2} \), and let \( \pi \) be an isomorphism from \( G \) to \( \hat{G} \) mapping \( u^i \) to \( \tilde{u}^i \). Since \( \mathcal{B} \cong \hat{\mathcal{B}} \), for each \( i,j \) the isomorphism \( \pi \) maps \( Q_{i,j} \) to \( \hat{Q}_{i,j} \). By Lemma 6.18 it holds that \( V(\text{Cut}(\mathcal{B})) = \text{nl-cut-vert} [G, u^0, u^1, u^2, y] \) and \( E(\text{Cut}(\mathcal{B})) = \text{nl-cut-edge} [G, u^0, u^1, u^2, y_1, y_2] \). Thus, \( \pi \) must map \( \text{Cut}(\mathcal{B}) \) to \( \text{Cut}(\hat{\mathcal{B}}) \). To see that \( \pi \) also induces an isomorphism between \( \text{Cut}^*(\mathcal{B}) \) and \( \text{Cut}^*(\hat{\mathcal{B}}) \), consider a vertex \( v \in V(\text{Cut}^*(\mathcal{B})) \) and suppose \( \pi(v) \) has a different colour (i.e. satisfies different colour relations) in \( \text{Cut}^*(\hat{\mathcal{B}}) \) than \( v \) in \( \text{Cut}^*(\mathcal{B}) \).

Let \( R \) be one of the unary relations from Items (40)–(44). Since \( \pi \) is an isomorphism which maps \( u^i \) to \( \tilde{u}^i \) for \( i \in \{0,1,2\} \), we have that

\[
v \in R(G) \iff v \in \varphi_R [G, u^0, u^1, u^2, y] \iff \pi(v) \in \varphi_R [\hat{G}, \tilde{u}^0, \tilde{u}^1, \tilde{u}^2, y] \iff \pi(v) \in R(\hat{G}).
\]

Similarly, we can show that every edge \( e \in E(\text{Cut}(\mathcal{B})) \) is mapped to an edge \( \pi(e) \in E(\text{Cut}(\hat{\mathcal{B}})) \) contained in the same colour relations. Thus, \( \pi \) induces an isomorphism between \( \text{Cut}^*(\mathcal{B}) \) and \( \text{Cut}^*(\hat{\mathcal{B}}) \), which concludes the backward direction of the proof.

For the forward direction, assume that \( \text{Cut}^*(\mathcal{B}) \equiv \text{Cut}^*(\hat{\mathcal{B}}) \) via an isomorphism \( \pi \). Then since \( |\text{art}(\mathcal{B})| = |\text{art}(\hat{\mathcal{B}})| \), by Items (40) and (44), the isomorphism \( \pi \) can be extended to an isomorphism \( \pi' \) from the graph with vertex set \( V(\text{Cut}(\mathcal{B})) \cup \text{art}(\mathcal{B}) \) and whose edge set is the extension of \( E(\text{Cut}(\mathcal{B})) \) by all edges between \( V(\text{Cut}(\mathcal{B})) \cup \text{art}(\mathcal{B}) \) and \( \text{art}(\mathcal{B}) \) to the corresponding subgraph of \( \hat{G} \), where \( \pi' \) maps every articulation vertex of \( \mathcal{B} \) to one of equal height in \( \hat{\mathcal{B}} \).

Furthermore, by Items (55) and (56), the mapping \( \pi \) induces an isomorphism from \( C(Q_{i,j}) \) to \( C(\hat{Q}_{i,j}) \) which fixes \( u^i_j \) and \( u^i_{j+1} \) (and thus preserves heights). Let \( L_{i,j} \) and \( R_{i,j} \) as well as \( \hat{L}_{i,j} \) and \( \hat{R}_{i,j} \) be the two segments of \( C(Q_{i,j}) \) and \( C(\hat{Q}_{i,j}) \) between \( u^i_j \) and \( u^i_{j+1} \), respectively. Then for every pair \( (i,j) \), the isomorphism \( \pi \) either maps \( L_{i,j} \) to \( \hat{L}_{i,j} \) and \( R_{i,j} \) to \( \hat{R}_{i,j} \), or \( L_{i,j} \) to \( \hat{R}_{i,j} \) and \( R_{i,j} \) to \( \hat{L}_{i,j} \). Without loss of generality, assume the first.

If for every pair \( (i,j) \), the coloured graphs \( I^*(Q_{i,j}) \) and \( I^*(\hat{Q}_{i,j}) \) are isomorphic via an isomorphism \( \pi'_{i,j} \) mapping \( L_{i,j} \) to \( \hat{L}_{i,j} \), then the collection of the \( \pi'_{i,j} \) clearly extends \( \pi \) to an isomorphism between \( \mathcal{G} \) and \( \hat{\mathcal{G}} \).

Thus, suppose there is a pair \( (i,j) \) such that every isomorphism from \( I^*(Q_{i,j}) \) to \( I^*(\hat{Q}_{i,j}) \) swaps sides. Let \( v \in C(Q_{i,j}) \) and let \( O \) be the orbit of \( v \) with respect to the automorphism group of \( I^*(Q_{i,j}) \). Let \( R := R_O \). Then, by Item (56), it holds that \( v \in \varphi_R [G, u^0, u^1, u^2, x] \) but \( \pi(v) \notin \varphi_R [\hat{G}, \tilde{u}^0, \tilde{u}^1, \tilde{u}^2, x] \). However, this is a contradiction since \( \pi \) must respect all relations.

Thus, to check whether \( G_{u^0,u^1,u^2} \cong \hat{G}_{\tilde{u}^0,\tilde{u}^1,\tilde{u}^2} \) it suffices to consider the (pseudo-)cut graphs of \( G \) and \( \hat{G} \).

Claim 4. There is a formula \( \text{cut-iso}_G(x^0,x^1,x^2) \in \mathcal{C}_G^{n+3} \) (not depending on \( \hat{G} \)) such that \( \hat{G} \models \text{cut-iso}(\tilde{u}^0,\tilde{u}^1,\tilde{u}^2) \) if and only if \( \text{Cut}^*(\hat{\mathcal{B}}) \equiv \text{Cut}^*(\mathcal{B}) \).

Proof. By Lemma 6.16 every connected component of \( \text{Cut}(\mathcal{B}) \) is in \( \mathcal{E}_{g-1} \). Therefore, by Corollary 3.7 and by the induction assumption there is a sentence \( \text{cut-iso}_G \in \mathcal{C}_G^4 \) which identifies \( \text{Cut}^*(\hat{\mathcal{B}}) \). By replacing every subformula \( \exists x \exists y \varphi \) with \( \exists x \exists y (\text{nl-cut-vert}(x^0,x^1,x^2,x) \land \varphi) \) and \( E(x,y) \) with \( \text{nl-cut-edge}(x^0,x^1,x^2,x,y) \), we relativise \( \text{cut-iso}_G \) to the (pseudo-)cut graph. To transform it into a formula working also on the uncoloured cut graph, for every
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relation \( R \) from Items (2a), (2b), (2d), we replace each occurrence \( R(z, z) \) with \( \varphi_R(x^0, x^1, x^2, z) \) and proceed analogously for every \( R \) from Item (2c).

The resulting formula \( \text{cut-iso}_G(x^0, x^1, x^2) \) is in \( C_w^{+++} \) and it holds that

\[
\tilde{G} | \text{cut-iso}_G(u^0, \tilde{w}^1, \tilde{w}^2) \iff \text{Cut}^*(\tilde{B}) \cong \text{Cut}^*(B).
\]

Now the formula

\[
\text{iso}_G \triangleq \exists x^0 \exists x^1 \exists x^2 (\text{inside-iso}(x^0, x^1, x^2) \land \text{cut-iso}_G(x^0, x^1, x^2)) \in C_w^{+++}
\]

identifies \( G \). An application of Lemma 3.2 concludes the proof.

\[\Box\]

6.2 Case 2: Presence of simplifying patches

In this section, we still assume that \( G \) is polyhedrally embedded in the surface \( S \) of Euler genus \( g \) and that \( n = |G| \) (Assumption 6.12), but we replace Assumption 6.14 with the following assumption.

\[\blacktriangleright\textbf{Assumption 6.19.} G \text{ contains a simplifying patch.}\]

This case sounds simpler than the first one: we only need to remove a simplifying patch from our graph. The remaining pieces have smaller Euler genus and thus can be identified in the logic \( C_w^+ \). Thus, all we need to do is colour the pieces in such a way that we can reconstruct the original graph. The problem with this line of reasoning is that simplifying patches have a much more complicated structure than non-simplifying patches. For example, we cannot define the internal graph of a simplifying patch in the same way as we did for non-simplifying patches in Lemma 6.13. A consequence is that there is no easy way to reconstruct the original graph from the graph obtained by removing a simplifying patch.

The first lemma handles trivial simplifying patches, so that afterwards we can focus on non-trivial ones.

\[\blacktriangleright\textbf{Lemma 6.20.} If \( G \) has a trivial simplifying patch, then there is a sentence \( \text{iso}_G \in C_w^{+++} \) that identifies \( G \).
\]

\[\textbf{Proof}.\] If \( G \) has a trivial simplifying patch consisting of an edge \( uu' \), then each connected component of \( G \setminus \{u, u'\} \) is in \( E_{g-1} \) and can be identified by a sentence in \( C_w^+ \). From these sentences, we can construct a sentence in \( C_w^{+++} \) identifying \( G \) (arguing as in the proof of Lemma 6.3).

\[\Box\]

From now on, we make the following assumption.

\[\blacktriangleright\textbf{Assumption 6.21.} G \text{ contains no trivial simplifying patch.}\]

Recall the definition of a segment \( Q[v, v'] \) of an sps \( Q \) from Section 5. A subpatch of a patch \( Q \) is a segment of \( Q \) which is a patch, i.e., which has no proper articulation vertices. A patch \( Q \) is a minimal simplifying patch if \( Q \) is simplifying and all proper subpatches of \( Q \) are non-simplifying. We are mostly interested in minimal simplifying patches.

The internal region of an sps \( Q \) is the set

\[
R(Q) \triangleq \bigcup_{e \in E(Q)} e \cup \bigcup_{Q' \text{ non-trivial non-simplifying subpatch of } Q} D(Q').
\]

Note that if \( Q \) is a non-trivial patch, then \( R(Q) \subseteq D(Q) \), because \( D(Q') \subseteq D(Q) \) for every subpatch \( Q' \) of \( Q \). The regional graph of a patch \( Q \) is the graph \( J \triangleq J(Q) \) with vertex set
V(J) := V(G) \cap R(Q) \text{ and } E(J) := \{ e \in E(G) \mid e \subseteq R(Q) \}. \text{ It follows from Lemma 5.8 that the graph } J \text{ only depends on the abstract graph } G \text{ and not on the embedding of } G. \text{ Observe that if } Q \text{ is a non-trivial and non-simplifying patch, then } R(Q) = D(Q) \text{ and } J(Q) = I(Q).

Our first lemma shows that the regional graph of a patch is definable in \( C^\text{max}\{7, s+2\} \).

**Lemma 6.22.** There are \( C^\text{max}\{7, s+2\} \)-formulae \( J\text{-vert}(x, x', y) \) and \( J\text{-edge}(x, x', y_1, y_2) \) such that for all \( u, u' \in V(G) \) the following holds. If the canonical \( sps \ Q := Q^G(u, u') \) from \( u \) to \( u' \) is a patch, then for all \( u, u' \in V(G) \),

\[
J\text{-vert}[G, u, u', y] = V(J(Q)),
\]

\[
J\text{-edge}[G, u, u', y_1, y_2] = E(J(Q)).
\]

**Proof.** We first define a \( C^\omega_0 \)-formula \( csps\text{-path} \) such that \( G \models csps\text{-path}(u, u', v, w) \) if and only if there is a path \( Q \in Q \) such that \( v, w \in V(Q) \) and \( h^Q(v) \leq h^Q(w) \). We set

\[
\text{csps-path}(x, x', z, z') := \text{csps-vert}(x, x', z) \land \text{csps-vert}(x, x', z') \land \bigwedge_{i=0}^{n-1} \left( \text{dist}_{\omega_{i}}(x, x') \land \bigvee_{j \leq i} \left( \text{dist}_{\omega_{j}}(x, z) \land \bigvee_{k \leq i-j} \left( \text{dist}_{\omega_{k}}(z, z') \land \text{dist}_{\omega_{i-j-k}}(z', x') \right) \right) \right).
\]

Now we can let

\[
J\text{-vert}(x, x', y) := \text{csps-vert}(x, x', y) \lor \exists z \exists z' \left( \text{csps-path}(x, x', z, z') \land
\neg\text{csps-simplifying}(z, z') \land \neg E(z, z') \land \text{int-vert}(z, z', y) \right).
\]

By Corollary 6.11 and Lemma 6.13 we have \( J\text{-vert} \in C^\text{max}\{7, s+2\} \).

Similarly, we can define the \( C^\text{max}\{7, s+2\} \)-formula

\[
J\text{-edge}(x, x', y_1, y_2) := \text{csps-edge}(x, x', y_1, y_2) \lor \exists z \exists z' \left( \text{csps-path}(x, x', z, z') \land
\neg\text{csps-simplifying}(z, z') \land J\text{-vert}(z, z', y_1) \land J\text{-vert}(z, z', y_2) \land E(y_1, y_2) \right).
\]

For the remainder of this section, we fix a minimal simplifying patch of \( G \).

**Assumption 6.23.** \( Q := Q^G(u, u') \) is a minimal (non-trivial) simplifying patch of \( G \). Furthermore, \( D := D(Q), R := R(Q), \text{ and } J := J(Q) \).

By Lemma 6.22 the logic \( C^\text{max}\{7, s+2\} \) distinguishes the regional graph \( J \) from the remainder of the graph. Furthermore, since \( Q \) is simplifying, every connected component of \( G \setminus J \) is contained in \( E_{g-1} \). We need to branch into two cases once more.

**Case 2.1: \( G \setminus J \) is connected**

The proof in this case is similar to Case 1, but simpler. The key fact is that in this case we have \( R = D \). by Lemma 15.4.22(1). As remarked in Section 5 after Definition 5.5, there are paths \( Q, Q' \in Q \) such that \( C := Q \cup Q' \) is a cycle and \( bd(D) = C \). It turns out that the cycle \( C \) only depends on \( u, u' \) and the abstract graph \( G \) (that is, we have analogues of Lemma 5.3 and Corollary 5.9). Indeed, our first step in this case will be to define the cycle \( C \) in the logic \( C^7_w \).
Let $A_1, \ldots, A_m$ be the connected components of the graph $A^* := G \setminus J$. Each $A_i$ is embedded in $S \setminus D$, because each component of $G \setminus V(Q)$ embedded in $D$ belongs to $J$. Hence $N(A^*) \subseteq V(C)$. This implies that the graph $G/A^*$ obtained from $G$ by identifying all vertices in $A^*$ is planar. Furthermore, by Corollary 15.2.7, the graph $G/A^*$ is 3-connected.

**Lemma 6.24.** There are $\mathcal{C}_w^4$-formulae bd-vert$(x, x', y)$, bd-edge$(x, x', y_1, y_2)$ such that

\[
\begin{align*}
\text{bd-vert}(G, u, u', y) &= V(C), \\
\text{bd-edge}(G, u, u', y_1, y_2) &= E(C).
\end{align*}
\]

**Proof.** The proof is completely analogous to the proof of Lemma 6.13 just redefining the formula $\text{astar}$:

\[
\text{astar}(x, x', y) := \neg J\text{-vert}(x, x', y).
\]

Note that in the proof of Lemma 6.13 we never use that $A^*$ is connected.

Now we fix one additional vertex: let $u''$ be the neighbour of $u$ on the path $Q$. Let $k := |C|/2 = |Q| - 1 = |Q'|-1$. Using $u''$, we can enumerate the vertices of the cycle $C$ in the cyclic order given by letting $u_0 := u$, $u_1 := u''$, then moving along $Q$ to $u_k := u'$, and from there moving backwards along $Q'$ to the neighbour $u_{2k-1}$ of $u$ on the path $Q'$.

**Lemma 6.25.** For $0 \leq i \leq 2k-1$ there is a $\mathcal{C}_w^4$-formula $\text{bd-vert}_i(x, x', x'', y)$ such that $\text{bd-vert}_0(G, u, u', u'') = \{u\}$.

**Proof.** We let $\text{bd-vert}_0(x, x', x'', y) := (y = x)$, $\text{bd-vert}_1(x, x', x'', y) := (y = x'')$, and

\[
\text{bd-vert}_i := \neg \text{bd-vert}_{i-2}(x, x', x'', y) \land \exists y' (\text{bd-vert}_{i-1}(x, x', x'', y) \land \text{bd-edge}(x, x', y', y))
\]

for $2 \leq i \leq 2k-1$.

We are ready to finish this subcase.

**Proof of Lemma 6.4**. Case 2.1. Let $\tilde{G}$ be an arbitrary graph. We shall prove that if there is no $\mathcal{C}_w^{4k+3}$-formula distinguishing $G$ and $\tilde{G}$ then the two graphs are isomorphic.

So assume that there is no $\mathcal{C}_w^{4k+3}$-formula distinguishing $G$ and $\tilde{G}$. Then there are vertices $\bar{u}, \bar{u}', \bar{u}'' \in V(\tilde{G})$ such that for all $\mathcal{C}_w^{4k+3}$-formulae $\varphi(x, x', x'')$ we have $G \models \varphi(u, u', u'') \iff \tilde{G} \models \varphi(\bar{u}, \bar{u}', \bar{u}'')$. We fix such vertices $\bar{u}, \bar{u}', \bar{u}''$. We shall prove that there is an isomorphism from $G$ to $\tilde{G}$ mapping $u$ to $\bar{u}$, $u'$ to $\bar{u}'$, and $u''$ to $\bar{u}''$.

Let $\tilde{J}$ be the subgraph of $\tilde{G}$ with vertex set $V(\tilde{J}) := J\text{-vert}(\tilde{G}, \bar{u}, \bar{u}', y)$ and $E(\tilde{J}) := J\text{-edge}(\tilde{G}, \bar{u}, \bar{u}', y_1, y_2)$. Similarly, let $\tilde{C}$ be the subgraph of $\tilde{G}$ with vertex set $V(\tilde{C}) := \text{bd-vert}(\tilde{G}, \bar{u}, \bar{u}', y)$ and $E(\tilde{C}) := \text{bd-edge}(\tilde{G}, \bar{u}, \bar{u}', y_1, y_2)$. Then $\tilde{C} \subseteq \tilde{J}$ and $\tilde{C}$ is a cycle. For every $i \in \{0, \ldots, 2k-1 \}$, let $\tilde{u}_i$ be the unique vertex such that $\tilde{G} \models \text{bd-vert}_i(\bar{u}, \bar{u}', \bar{u}'', \tilde{u}_i)$. Then $V(\tilde{C}) = \{\tilde{u}_0, \ldots, \tilde{u}_{2k-1} \}$, and the vertices $\tilde{u}_i$ appear on $\tilde{C}$ in cyclic order starting from $\tilde{u}_0 = \bar{u}$ and $\tilde{u}_1 = \bar{u}''$. Moreover, $\tilde{u}_k = \bar{u}'$.

Now we individualise the vertices $u_i$ in $J$ and $\bar{u}_i$ in $\tilde{J}$ using the same colour. More formally, for every $i$ we introduce a new colour relation $R_i$ and define $R_i(J) := \{u_i\}$ and $R_i(\tilde{J}) := \{\bar{u}_i\}$. We observe that the obtained coloured versions of $J$ and $\tilde{J}$ satisfy the same $\mathcal{C}_w^4$-sentences, because the vertices $u_i$ and $\bar{u}_i$ are defined in terms of $u, u', u''$ and $\bar{u}, \bar{u}', \bar{u}''$, respectively, and for all $\mathcal{C}_w^4$-formulae $\varphi(x, x', x'')$ we have $G \models \varphi(u, u', u'') \iff \tilde{G} \models \varphi(\bar{u}, \bar{u}', \bar{u}'')$. Since $J$ is planar and every planar graph is identified by a $\mathcal{C}_4^4$-sentence, it follows that the coloured
graphs are isomorphic. Hence there is an isomorphism \( f : V(J) \to V(\hat{J}) \) such that \( f(u_i) = \hat{u}_i \) for \( 0 \leq i \leq 2k - 1 \).

We shall extend \( f \) to an isomorphism from \( G \) to \( \hat{G} \). Let \( \hat{A}^* := \hat{G} \setminus \hat{J} \). Note that \( N^\hat{G}(\hat{A}^*) \subseteq V(\hat{G}) \), because \( G \) and thus also \( \hat{G} \) satisfies the \( C_w^7 \)-formula

\[
\forall y \forall z (\neg J\text{-vert}(x, x', y) \land J\text{-vert}(x, x'', z) \land E(y, z) \to bd\text{-vert}(x, x', z)).
\]

We colour the graphs \( A^* = G\setminus J \) and \( \hat{A}^* \) using new colour relations \( R_I \) for \( I \subseteq \{0, \ldots , 2k - 1\} \). We let \( R_I(G) \) be the set of all \( v \in V(A^*) \) such that \( N^G(v) = \{ u_i \mid i \in I \} \), and similarly we let \( R_I(G) \) be the set of all \( \hat{v} \in V(\hat{A}^*) \) such that \( N^\hat{G}(\hat{v}) = \{ \hat{u}_i \mid i \in I \} \). Observe that there is a \( C_w^7 \)-formula \( \chi_I(x, x', x'', y) \) such that \( \chi_I[G, u, u', u'', y] = R_I(G) \) and \( \chi_I[\hat{G}, \hat{u}, \hat{u}', \hat{u}'', y] = R_I(\hat{G}) \). Thus the coloured graphs \( A^* \) and \( \hat{A}^* \) satisfy the same \( C_w^6 \)-sentences, because for all \( C_w^{++3} \)-formulae \( \varphi(x, x', x'') \) we have \( G \models \varphi(u, u', u'') \iff \hat{G} \models \varphi(\hat{u}, \hat{u}', \hat{u}'') \).

As \( Q \) is simplifying, all connected components of \( A^* \) are in \( E_{q-1} \). Hence by Assumption \ref{assumption:1} and Corollary \ref{corollary:3.7}, there is a \( C_w^6 \)-sentence \( \hat{A}^* \) that identifies \( A^* \). As \( A^* \) and \( \hat{A}^* \) satisfy the same \( C_w^6 \)-sentences, there is an isomorphism \( g \) from \( A^* \) to \( \hat{A}^* \). The colour relations \( R_I \) guarantee that \( f \cup g \) is an isomorphism from \( G \) to \( \hat{G} \).

\[ \blacktriangleleft \]

Case 2.2: \( G \setminus J \) is disconnected

In this case, we need to analyse the structure of the graph \( J \) in more detail. Let \( \hat{H}_1, \ldots , \hat{H}_\ell \) be the connected components of \( J \setminus \{ u, u' \} \). By the assumption of this case, we have \( \ell \geq 2 \). For every \( i \in [\ell] \), let \( H_i := J[V(\hat{H}_i) \cup \{ u, u' \}] \), and let \( Q_i \) be the set of all paths \( Q \in \hat{Q} \) such that \( Q \subseteq H_i \). Then \( Q_i \) is a shortest path system from \( u \) to \( u' \). We call the \( Q_i \) the fibres of \( Q \). Note that \( H_i \subseteq D \) for all \( i \). Let \( R_i := R(Q_i) \). Then \( R = \bigcup_{i=1}^\ell R_i \) and \( R_i \cap R_j = \{ u, u' \} \) for \( i \neq j \). Let \( f_1, \ldots , f_\ell \) be the arcwise connected components of \( D \setminus R \). By \cite{17} Lemma 15.4.22(2), we have \( \ell' = \ell - 1 \) and there is a permutation \( \pi \in S_\ell \) such that \( bd(f_i) \subseteq R_{\pi(i)} \cup R_{\pi(i+1)} \).

Without loss of generality, we assume that \( \pi \) is the identity, that is, \( bd(f_i) \subseteq R_i \cup R_{i+1} \).

It is not hard to see (and shown in the proof of \cite{17} Lemma 15.4.22) that there are paths \( Q_i' \in Q_i \) and \( Q_{i+1}' \in Q_{i+1} \) such that \( C_i := Q_i' \cup Q_{i+1} \) is a cycle and \( bd(f_i) = C_i \). Let \( f_i := S \setminus D \). Then there are paths \( Q_1 \in Q_1 \) and \( Q_2 \in Q_2 \) such that \( C_i := Q_i' \cup Q_{i} \) is a cycle and \( C_i = bd(D) = bd(f_i) \). For every \( i \) we have \( bd(R_i) = Q_i \cup Q_i' \). But note that \( Q_i \cup Q_i' \) is not necessarily a cycle.

We use the notation introduced in this section so far (that is, \( D, R, f_i, J, H_i, \hat{H}_i, Q_i \)) throughout the remainder of this subsection. Moreover, we always use indices modulo \( \ell \). For example, \( H_{i+1} \) refers to \( H_1 \).

\[ \blacktriangleleft \]

Example 6.26. Consider the graph shown in Figure \ref{figure:4}. The graph can be embedded into a torus in such a way that the red, blue, and green paths form a simplifying patch \( \mathcal{Q} := \mathcal{Q}_G(u, u') \). The disk \( D(\mathcal{Q}) \) is shown in grey; the region \( R(\mathcal{Q}) \) within \( D(\mathcal{Q}) \) in a darker grey. The patch has three fibres \( Q_1, Q_2, Q_3 \) shown in red, blue, green, respectively. The boundary cycle of \( D(\mathcal{Q}) \) consists of the leftmost red path and the rightmost green path from \( u \) to \( u' \). The two areas in light grey are \( f_1 \) (between red and blue) and \( f_2 \) (between blue and green). The regional graph \( J \) consists of the red, blue, and green paths and all black edges and vertices.

Observe that the graph has a second, different embedding into the torus in which \( \mathcal{Q} \) is still a patch, but the boundary of its disk consists of a green and a blue path (and therefore our numbering of the fibres would be different; the red fibre would be in the middle).

Recall the definition of a bridge from Section 2.1. Let \( B_1, \ldots , B_m \) be a list of all \( J \)-bridges in \( G \). If \( |B_j| \geq 3 \), let \( A_j \) be the connected component of \( G \setminus R \) associated with \( B_j \). If \( B_j \) is
just a single edge, let $A_j$ be the empty graph. In this case, we call $B_j$ trivial. Note that for each $j \in [m]$ there is an $i := i(j) \in [\ell]$ such that $B_j$ is embedded in $f_i$, or more precisely, in $cl(f_i)$. We say that $B_j$ is attached to $H_i$ if it has a vertex of attachment in $V(H_i)$. We say that $B_j$ connects $H_i$ and $H_{i'}$ if it is attached to both $H_i$ and $H_{i'}$. If $B_j$ is attached to $H_i$ then it is embedded in $f_i$ or in $f_{i-1}$ (indices modulo $m$). Thus, if $B_j$ connects $H_i$ and $H_{i'}$, then either $i' = i + 1$ and $B_j$ is embedded in $f_i$, or $i' = i - 1$ and $B_j$ is embedded in $f_{i-1}$.

Example 6.27. Consider again the graph shown in Figure 4 with the simplifying patch $Q$ detailed in Example 6.26. The graph $J$ (consisting of all red, green, blue, and black vertices and edges) has six bridges, all shown in pink. Three of these bridges are trivial. Note that in this example, all six bridges are planar; the non-planarity of the entire graph is a result of combining the bridges.

Observe that there is at most one $i \in [\ell]$ such that there is no $J$-bridge connecting $H_i$ and $H_{i+1}$. To see this, towards a contradiction suppose that there are $i$ and $i'$ with $i < i'$ such that there is no bridge connecting $H_i$ and $H_{i+1}$ and no bridge connecting $H_{i'}$ and $H_{i'+1}$. Then $\{u, u'\}$ separates $\hat{H}_i$ from $\hat{H}_{i+1}$, which is impossible since $G$ is 3-connected. If there is no bridge connecting $H_i$ and $H_{i+1}$, then we call $Q_i$ and $Q_{i+1}$ dangling fibres.

We say that two fibres $Q_i$ and $Q_{i'}$ are adjacent if $|i - i'| = 1$ or $\{i, i'\} = \{1, \ell\}$. Note that $Q_i$, $Q_{i'}$ are adjacent if $i \neq i'$ and either there is a $J$-bridge that connects $H_i$ and $H_{i'}$ or both $Q_i$ and $Q_{i'}$ are dangling fibres. This means that we can detect the cyclic adjacency structure on the fibres $Q_i$ just by looking at the bridges connecting them. It follows that the cyclic order of the fibres only depends on the abstract graph $G$ and not on its embedding.

Lemma 6.28. There are $\mathcal{C}_w^{max\{7,s+2\}}$-formulae $\text{same-H}(x, x', y_1, y_2)$, $\text{bconn-H}(x, x', y_1, y_2)$,
adj-H(x, x', y_1, y_2) such that for all w_1, w_2 \in V(G) we have:

\[ G \models \text{same-H}(u, u', w_1, w_2) \iff \text{there is an } i \in [\ell] \text{ such that } w_1, w_2 \in V(H_i), \]

\[ G \models \text{bconn-H}(u, u', w_1, w_2) \iff \text{there are distinct } i, i' \in [\ell] \text{ and } j \in [m] \text{ such that } w_1 \in V(\tilde{H}_i) \text{ and } w_2 \in V(\tilde{H}_{i'}) \text{ and } B_j \text{ connects } \tilde{Q}_i \text{ and } \tilde{Q}_{i'}, \]

\[ G \models \text{adj-H}(u, u', w_1, w_2) \iff \text{there is an } i \in [\ell] \text{ such that } w_1 \in V(\tilde{H}_i) \text{ and } w_2 \in V(\tilde{H}_{i-1}) \cup V(\tilde{H}_{i+1}). \]

**Proof.** By definition of \( H_i \), there is an \( i \in [\ell] \) such that \( w_1, w_2 \in V(H_i) \) if and only if \( w_1, w_2 \in V(J) \) and either \( \{w_1, w_2\} \cap \{u, u'\} \neq \emptyset \) or \( w_1 \) and \( w_2 \) belong to the same \( \tilde{H}_i \). Thus, we can let

\[
\text{same-H}(x, x', y_1, y_2) := \text{J-vert}(x, x', y_1) \land \text{J-vert}(x, x', y_2) \land (y_1 = x \lor y_1 = x' \lor y_2 = x \lor y_2 = x' \lor \varphi(x, x', y_1, y_2)),
\]

where \( \varphi(x, x', y_1, y_2) \) is a \( C^w_{\max(7, s+2)} \)-formula stating that \( y_1, y_2 \) belong to the same connected component of \( J \setminus \{u, u'\} \). Using \( \text{J-vert}(x, x', y) \) and \( \text{J-edge}(x, x', y_1, y_2) \) as building blocks, it is easy to define such a formula.

There is a \( J \)-bridge that connects fibres \( \tilde{Q}_i \) and \( \tilde{Q}_{i'} \) if and only if there is a path \( P \subseteq G \setminus \{u, u'\} \) from a vertex \( w_1 \in V(\tilde{H}_i) \) to a vertex \( w_2 \in V(\tilde{H}_{i'}) \) with all internal vertices in \( G \setminus J \). Let \( \psi(x, x', z_1, z_2) \) be a \( C^w_{\max(7, s+2)} \)-formula such that \( G \models \psi(u, u', w_1, w_2) \) if and only \( w_1, w_2 \in V(J) \setminus \{x, x'\} \) and there is a path from \( w_1 \) to \( w_2 \) with all internal vertices in \( V(G \setminus J) \). We can easily construct such a formula using \( \text{J-vert}(x, x', y) \) and \( \text{J-edge}(x, x', y_1, y_2) \) as building blocks. Now we let

\[
\text{bconn-H}(x, x', y_1, y_2) := \text{J-vert}(x, x', y_1) \land \text{J-vert}(x, x', y_2) \land \neg \text{same-H}(x, x', y_1, y_2) \land \exists z_1 \exists z_2 (\bigwedge_{i=1}^2 \text{same-H}(x, x', y_1, z_i) \land \psi(x, x', z_1, z_2)).
\]

Recall that two fibres are adjacent if and only if either there is a \( J \)-bridge that connects them or both fibres are dangling. To define dangling fibres, we use the following formula:

\[
\text{dangling}(x, x', y) := \forall y' \forall y'' (\text{bconn-H}(x, x', y, y') \land \text{bconn-H}(x, x', y, y'') \land \neg \text{same-H}(x, x', y', y'')).
\]

Then \( G \models \text{dangling}(u, u', v) \) if and only if \( v \) belongs to a dangling fibre.

We let

\[
\text{adj-H}(x, x', y_1, y_2) := \text{bconn-H}(x, x', y_1, y_2) \lor \left( \neg \text{same-H}(x, x', y_1, y_2) \land \bigwedge_{i=1}^2 \text{dangling}(x, x', y_1) \right).
\]

**Lemma 6.29.** There is a vertex \( u'' \in V(J) \) and for every \( i \in [\ell] \) a \( C^w_{\max(7, s+2)} \)-formula \( \text{H-vert}_i(x, x', y) \) such that

\[ \text{H-vert}_i[G, u, u', u'', y] = V(H_i). \]

Before we prove the lemma, let us remark that \( H_i \) is an induced subgraph of \( G \). Therefore there is no need for a formula \( \text{H-edge} \) defining \( E(H_i) \).
Proof of Lemma 6.29. It will be easier to define formulae $H\text{-}vert_i^\circ(x, x', x'', y)$ such that $H\text{-}vert_i^\circ[G, u, u', u''], w] := V(\tilde{H}_i)$. Then we let

$$H\text{-}vert_i(x, x', x'', y) := H\text{-}vert_i^\circ(x, x', x'', y) \forall y = x \lor y = x'.$$

We let

$$\text{same-}H^\circ(x, x', y_1, y_2) := \text{same-}H(x, x', y_1, y_2) \land \bigwedge_{i=1}^2 \left(y_i \neq x \land y_i \neq x'\right).$$

If $\ell = 2$, we choose an arbitrary $u'' \in V(\tilde{H}_1)$, and we let

$$H\text{-}vert_i^\circ(x, x', x'', y) := \text{same-}H^\circ(x, x', x'', y),$$

$$H\text{-}vert_i^\circ(x, x', x'', y) := J\text{-}vert(x, x', y) \land \neg \text{same-}H(x, x', x'', y).$$

In the following, we assume $\ell \geq 3$. If there are dangling fibres, we proceed as follows. Suppose $Q_{i-1}$ and $Q_i$ are dangling. We choose an arbitrary $u'' \in V(\tilde{H}_1)$. We let

$$H\text{-}vert_i^\circ(x, x', x'', y) := \text{same-}H^\circ(x, x', x'', y),$$

$$H\text{-}vert_i^\circ(x, x', x'', y) := J\text{-}vert(x, x', y) \land \text{bconn-}H(x, x', x'', y),$$

and for $2 \leq j \leq \ell - 1$

$$H\text{-}vert_i^{\circ}(x, x', x'', y) := J\text{-}vert(x, x', y) \land \neg H\text{-}vert_i^{\circ}(x, x', x'', y) \land$$

$$\exists y' \left( (H\text{-}vert_i^{\circ}(x, x', x'', y') \land \text{bconn-}H(x, x', y')) \right).$$

In the following, we assume that there are no dangling fibres. For $i \in [\ell]$, denote by $(i, i+1)$-bridge a $J$-bridge that connects $Q_i$ and $Q_{i+1}$. Since there is no dangling fibre, for all $i \in [\ell]$ there is at least one $(i, i+1)$-bridge.

Suppose that for some $i$ there is a vertex $v \in V(\tilde{H}_i)$ that is a vertex of attachment of an $(i, i+1)$-bridge, but not of an $(i-1, i)$-bridge. Then we let $u'' := v$. As before, $H\text{-}vert_i^\circ(x, x', x'', y) := \text{same-}H^\circ(x, x', x'', y)$. To define $H\text{-}vert_i^{\circ}(x, x', x'', y)$, we let $\psi(x, x', z_1, z_2)$ be a $\max\{7, \omega+2\}$-formula such that $G \models \psi(u, u', w_1, w_2)$ if and only $w_1, w_2 \in V(J) \setminus \{x, x'\}$ and there is a path from $w_1$ to $w_2$ with all internal vertices in $V(G \setminus J)$. Then we let

$$H\text{-}vert_i^{\circ}(x, x', x'', y) := \neg H\text{-}vert_i^\circ(x, x', x'', y) \land \exists y' \left( \psi(x, x', x'', y') \land \text{same-}H^\circ(x, x', y') \right).$$

For $2 \leq j \leq \ell - 1$, we define $H\text{-}vert_i^{\circ}(x, x', x'', y)$ as above:

$$H\text{-}vert_i^{\circ}(x, x', x'', y) := J\text{-}vert(x, x', y) \land \neg H\text{-}vert_i^{\circ}(x, x', x'', y) \land$$

$$\exists y' \left( (H\text{-}vert_i^{\circ}(x, x', x'', y') \land \text{bconn-}H(x, x', y')) \right).$$

In the following, we assume that for every $i \in [\ell]$ and every $v \in V(\tilde{H}_i)$, either $v$ is a vertex of attachment of both an $(i-1, i)$-bridge and an $(i, i+1)$-bridge (we call $v$ doubly-attached) or it is neither a vertex of attachment of an $(i-1, i)$-bridge nor of an $(i, i+1)$-bridge (we call $v$ unattached). Observe that if $v$ is doubly-attached then it is an articulation vertex of the sps $Q_i$.

Claim 1. Let $i \in [2, \ell - 1]$. Then no vertex $v \in V(\tilde{H}_i)$ is unattached.

Proof. Suppose towards a contradiction that $v \in V(\tilde{H}_i)$ is unattached. Suppose first that $v \in V(Q)$ for some path $Q \in Q_i$ from $u$ to $u'$. As we have no dangling fibres, there is at least
one doubly-attached vertex in $V(\hat{H}_1)$. Since all doubly-attached vertices are articulation vertices, every doubly-attached vertex in $V(\hat{H}_1)$ appears on the path $Q$. Let $u$ be the last doubly-attached vertex on $Q$ before $v$, or if no such vertex exists, let $u := u$. Let $u'$ be the first doubly-attached vertex on $Q$ after $v$, or if no such vertex exists, let $u' := u$. Then by our assumption we have $u \neq u$ or $u \neq u'$. Say, $w' \neq u'$. Now $\{w, w'\}$ separates $v$ from $u'$. This is an easy consequence of the fact that the graph that is the union of $H_{i−1}, H_i, H_{i+1}$, all $(i−1, i)$-bridges, all $(i, i+1)$-bridges, and all $J$-bridges that have all their vertices of attachment in $H_i$ is embedded in the disk $D$. However, this contradicts $G$ being 3-connected.

It remains to consider the case that $v \in V(J) \setminus V(Q_i)$. Then $v \in V(I(Q'))$ for some non-trivial non-simplifying subpatch $Q' = Q_i[v_1, v_2]$ of $Q_i$. Note that $Q'$ contains no articulation vertices of $Q_i$ except for (possibly) $v_1$ and $v_2$: otherwise it would not be a path. Thus every vertex $v' \in V(Q') \setminus \{v_1, v_2\}$ is also unattached. We pick such a $v'$. It is contained in a path $Q \in Q_i$. Therefore, we can apply the same argument as above to $v'$ instead of $v$ and again obtain a contradiction.

Let unattached$(x, x', y)$ be a $C^\max\{7, s+2\}$-formula such that $G \models \text{unattached}(u, u', v)$ if and only if $v \in V(J) \setminus \{u, u'\}$ and $v$ is unattached. It is straightforward to construct such a formula. Suppose next that there is an unattached vertex $v$. Then $v \in V(\hat{H}_1)$ or $v \in V(\hat{H}_2)$. Say, $v \in V(\hat{H}_1)$. Let $u''$ be an arbitrary vertex in $V(\hat{H}_1)$. We let

\[
\begin{align*}
\text{H-vert}_1^1(x, x', x'', y) &:= \text{same-H}^0(x, x', x'', y), \\
\text{H-vert}_1^2(x, x', x'', y) &:= \text{J-vert}(x, x', y) \land \neg\text{H-vert}_1^1(x, x', x'', y) \land \\
& \exists y' \left( \text{same-H}^0(x, x', y, y') \land \text{unattached}(x, x', y') \right)
\end{align*}
\]

and for $2 \leq i \leq \ell − 1$

\[
\begin{align*}
\text{H-vert}_i^2(x, x', x'', y) &:= \text{J-vert}(x, x', y) \land \neg\text{H-vert}_{i-1}^2(x, x', x'', y) \land \\
& \exists y' \left( \text{H-vert}_{i-1}^2(x, x', x'', y') \land \text{bconn-H}(x, x', y', y) \right).
\end{align*}
\]

In the following, we assume that there is no unattached vertex. This implies that every $Q_i$ consists of just one path $Q_i$. For every $i$, let $L_i$ be the graph that is the union of the paths $Q_i$ and $Q_i+1$ and all $(i, i+1)$-bridges. Observe that $L_1, \ldots, L_{\ell−1}$ are planar, because they are embedded in the disk $D$. In fact, all these $L_i$ have a planar embedding where $Q_i \cup Q_{i+1}$ is a facial cycle. Note that if $L_\ell$ also has such a planar embedding, then the graph $\bigcup_{i=1}^{\ell} L_i$ is planar.

Suppose $L_\ell$ does not have a planar embedding where $Q_\ell \cup Q_1$ is a facial cycle. Using the fact that planarity is expressible in $C^4$, we can construct a $C^\max\{7, s+2\}$-formula planar-$L(x, x', y, y')$ such that $G \models \text{planar-L}(u, u', v, v')$ if and only if for some $i \in [\ell]$ the following conditions are satisfied:

- either $v \in V(\hat{H}_i)$ and $v' \in V(\hat{H}_{i+1})$, or $v' \in V(\hat{H}_i)$ and $v \in V(\hat{H}_{i+1})$;
- $L_i$ has a planar embedding where $Q_i \cup Q_{i+1}$ is a facial cycle.

We choose $u'' \in V(\hat{H}_1)$, and we let

\[
\begin{align*}
\text{H-vert}_1^3(x, x', x'', y) &:= \text{same-H}^0(x, x', x'', y), \\
\text{H-vert}_1^4(x, x', x'', y) &:= \text{J-vert}(x, x', y) \land \\
& \exists y' \left( \text{bconn}(x, x', y, y') \land \neg\text{planar-L}(x, x', y, y') \land \text{same-H}^0(x, x', x'', y') \right)
\end{align*}
\]

and for $2 \leq i \leq \ell − 1$, as before,

\[
\begin{align*}
\text{H-vert}_i^3(x, x', x'', y) &:= \text{J-vert}(x, x', y) \land \neg\text{H-vert}_{i-1}^3(x, x', x'', y) \land \\
& \exists y' \left( \text{H-vert}_{i-1}^3(x, x', x'', y') \land \text{bconn-H}(x, x', y', y) \right).
\end{align*}
\]
In the following, we assume that $L_\ell$ has a planar embedding where $Q_{\ell} \cup Q_1$ is a facial cycle. Then the graph $L := \bigcup_{i=1}^{\ell} L_i$ is planar. However, $G$ is not planar. The graphs $G$ and $L$ differ only in the $J$-bridges that are attached to just a single fibre. Let us call a $J$-bridge whose vertices of attachment are in the fibre $Q_i$ (and thus on the path $Q_i$) an $i$-bridge. Due to the 3-connectivity of $G$, every $i$-bridge has at least 3 vertices of attachment in $V(Q_i)$. Furthermore, since all vertices of $Q_i$ are doubly-attached (i.e., they are vertices of attachment of both an $(i-1)$-bridge and an $(i,i+1)$-bridge), there is no way to attach an $i$-bridge for some $i \in \{2, \ldots, \ell - 1\}$ without destroying the embedding in the disk $D$. This is easy to see considering the fact that an $i$-bridge embedded in, say, $f_i$ has two vertices $v$, $v'$ of attachment of distance at least two in $Q_i$ and it thus “blocks” the vertex between $v$ and $v'$ in $V(Q_i)$ from being attached to a vertex in $Q_{i+1}$ (cf. Corollary 9.1.2, [17]). Hence there can only be $i$-bridges for $i = 1$ and $i = \ell$, and must be at least one such bridge, because otherwise $G = L$ would be planar. Say, there is an $\ell$-bridge. We can easily construct a $C_{\text{max}}^{(7,\ell+2)}$-formula $\text{self-bridge}(x,x',y)$ such that $G \models \text{self-bridge}(u',u'',y)$ if and only if $u'' \in V(H_i)$ for some $i$ such that there is an $i$-bridge. We choose $u'' \in V(H_i)$ and let

\[
\begin{align*}
H_{\text{-vert}}(x,x',x'',y) &:= \text{same-H}^0(x,x',x'',y), \\
H_{\text{-vert}}(x,x',x'',y) &:= \text{self-bridge}(x,x',y) \land \neg \text{same-H}(x,x',x'',y)
\end{align*}
\]

and for $2 \leq i \leq \ell - 1$, as before,

\[
\begin{align*}
H_{\text{-vert}}(x,x',x'',y) &:= \text{J-vert}(x,x',y) \land \neg H_{\text{-vert}}(x,x',x'',y) \land \\
&\exists y' \bigl( H_{\text{-vert}}(x,x',x'',y') \land \text{bconn-H}(x,x',y',y) \bigr).
\end{align*}
\]

This completes the proof.

In the following, we fix a vertex $u''$ that is chosen according to Lemma 6.29.

**Assumption 6.30.** $u'' \in V(J)$ is a fixed vertex such that $H_{\text{-vert}}(G,u,u',u'',y) = V(H_i)$ for every $i \in [\ell]$.

A $J$-bridge $B$ is an inner bridge if it has at least one vertex of attachment in $\bigcup_{i=1}^{\ell-1} V(H_i)$. Note that all inner bridges are embedded in the disk $D$. Let $K$ be the union of $J$ with all inner bridges. Then $K$ is a planar graph embedded in $D$. Using Lemma 3.3 for $\varphi := \text{J-vert}(x,x',y)$, we can construct $C_{\text{max}}^{(7,\ell+2)}$-formulae that define membership in $K$.

**Corollary 6.31.** There are $C_{\text{max}}^{(7,\ell+2)}$-formulae $K_{\text{-vert}}(x,x',x'',y)$, $K_{\text{-edge}}(x,x',x'',y_1,y_2)$ such that

\[
\begin{align*}
K_{\text{-vert}}(G,u,u',u'',y) &= V(K), \\
K_{\text{-edge}}(G,u,u',u'',y_1,y_2) &= E(K).
\end{align*}
\]

Finally, we are ready to complete the proof of Lemma 6.4.

**Proof of Lemma 6.4.** Case 2.2. Let us briefly recall our main assumptions for this case:

- $G$ is a graph of order $n = |G|$ polyhedrally embedded in a surface $S$ of Euler genus $g \geq 1$.
- $Q = Q^G(u,u')$ is a non-trivial simplifying patch in $G$ with $\ell \geq 2$ fibres.
- $u''$ is a vertex that allows us to identify the fibres of $Q$ via the formulae of Lemma 6.29.

We continue to use the notation introduced in this section, such as $D$, $J$, $Q_i$, $H_i$ and $H_i$, etcetera.

Moreover, we define

\[
h := \text{dist}(u,u').
\]
Let $\hat{G}$ be an arbitrary graph. We shall prove that if there is no $C_{w^{s+3}}$-formula distinguishing $G$ and $\hat{G}$, then the two graphs are isomorphic.

So assume that there is no $C_{w^{s+3}}$-formula distinguishing $G$ and $\hat{G}$. Then $|\hat{G}| = n$ and $\hat{G} \not\in E_{w^{-1}}$. Furthermore, there are vertices $\hat{u}, \hat{u}', \hat{u}'' \in V(\hat{G})$ such that for all $C_{w^{s+3}}$-formulae $\varphi(x, x', x'')$ we have $G \models \varphi(u, u', u'') \iff \hat{G} \models \varphi(\hat{u}, \hat{u}', \hat{u}'')$. We fix such vertices $\hat{u}, \hat{u}', \hat{u}''$. We shall prove that there is an isomorphism from $G$ to $\hat{G}$ mapping $u$ to $\hat{u}$, $u'$ to $\hat{u}'$, and $u''$ to $\hat{u}''$.

Let $\hat{Q} \coloneqq Q(\hat{G}(\hat{u}, \hat{u}'))$. We say $Q$ and $\hat{Q}$ are isomorphic, and write $Q \cong \hat{Q}$, if there is an isomorphism from $G(Q)$ to $\hat{G}(\hat{Q})$ mapping $u$ to $\hat{u}$ and $u'$ to $\hat{u}'$. Thus, in the following we always regard $u, u', u''$ and the corresponding $\hat{u}, \hat{u}', \hat{u}''$ as distinguished vertices that isomorphisms need to respect.

Just as in the proof of Claim 1 of Case 1, we have a formula $\text{sps-iso}(x, x') \in C_w^6$ (not depending on $\hat{G}$) such that $\hat{G} \models \text{sps-iso}(\hat{u}, \hat{u}')$ if and only if $\hat{Q}$ is a pseudo-patch with $\hat{Q} \cong Q$.

Hence $\hat{Q}$ is a non-trivial pseudo-patch in $\hat{G}$. Let $J$ be the graph with vertex set $V(J) := J\text{-vert}(\hat{G}, \hat{u}, \hat{u}', y)$ and edge set $E(J) := J\text{-edge}(\hat{G}, \hat{u}, \hat{u}', y_1, y_2)$. Note that $\hat{u}'' \in V(J)$, because $u'' \in V(J) = J\text{-vert}(G, u, u', y)$. Therefore, we call $J$ and $\hat{J}$ isomorphic, and write $J \cong \hat{J}$, if $J_{u,u',u''} \cong \hat{J}_{\hat{u},\hat{u}',\hat{u}''}$, that is, there is an isomorphism from $J$ to $\hat{J}$ that maps $u$ to $\hat{u}$, $u'$ to $\hat{u}'$, and $u''$ to $\hat{u}''$. Note that every such isomorphism induces an isomorphism from $G(Q)$ to $\hat{G}(\hat{Q})$.

Let $T \subseteq V(\hat{J})$ be the set of vertices of attachment of all $J$-bridges in $\hat{G}$, and, similarly, let $\hat{T}$ be the set of vertices of attachment of all $\hat{J}$-bridges in $\hat{G}$.

**Claim 1.** There is a formula $J\text{-iso}(x, x', x'') \in C_w^{\max\{7, s+2\}}$ such that $\hat{G} \models J\text{-iso}(\hat{u}, \hat{u}', \hat{u}'')$ if and only if $J \cong \hat{J}$ via an isomorphism that maps $T$ to $\hat{T}$.

**Proof.** Let $J^*$ be the graph resulting from $J_{u,u',u''}$ by assigning all vertices in $T$ a common distinct colour (however maintaining the individual colours for $u, u', u''$). Since $J$ is planar, the claim follows by relativising a sentence $J\text{-iso}' \in C_w^6$ which identifies $J^*$ to the subgraph whose vertex and edge set is defined by the formula $J\text{-vert}$ and $J\text{-edge}$, respectively, and by replacing the colour relations for $u, u'$, and $u''$ with equations of the form $z = x$, $z = x'$, $z = x''$ and the colour relation for $T$ with $\exists z'(E(z, z') \land \neg J\text{-edge}(x, x', z', z'))$. By Lemma 6.22 the formula has width $\max\{7, s + 2\}$.

In the following, we assume without loss of generality that $\hat{J} \cong J$ and we only consider isomorphisms which preserve the property of being a vertex of attachment.

We intend to equip certain supergraphs of $J$ and $\hat{J}$ with colour relations such that the coloured graphs are isomorphic if and only if $G$ and $\hat{G}$ are isomorphic via an isomorphism mapping $u$ to $\hat{u}$, $u'$ to $\hat{u}'$, and $u''$ to $\hat{u}''$. Then we show that the coloured supergraph of $J$ can be identified in $C_{w^{s+3}}$.

First recall that for every $J$-bridge its vertices of attachment lie on a shortest path from $u$ to $u'$. Thus, by Claim 1 we can assume the same for $\hat{G}$, since we can express the sps-containment of a vertex of attachment. Hence, each element in $V(J)$ and $V(\hat{J})$ which is a vertex of attachment of a bridge has a well-defined height in $Q$ and $\hat{Q}$, respectively.

For $i \in [\ell]$, let $H_i$ be the induced subgraph of $G$ with vertex set $H\text{-vert}(\hat{G}, \hat{u}, \hat{u}', \hat{u}'' y_i)$. Then $\hat{J} = \bigcup_{i=1}^\ell H_i$, because there is a $C_{w}^{\max\{7, s+2\}}$-formula which expresses that $J = \bigcup_{i=1}^\ell H_i$.

We now colour vertices in $V(G \setminus J)$ by their “attachment pattern” in $J$. For every $v \in V(G \setminus J)$, we let

$$S(v) := \llbracket \{(i, j) \mid w \in N(v) \cap V(H_i), w \text{ has height } j \text{ in } Q \} \rrbracket.$$ (7)
That is, for each \( w \in N(v) \cap V(H_i) \) of height \( j \), the set \( S(v) \) contains one separate copy of \((i, j)\). Similarly, for \( \vhat \in V(G \setminus \hat{J}) \) we let
\[
\hat{S}(\vhat) := \{(i, j) \mid \vhat \in N(\vhat) \cap V(H_i), \vhat \text{ has height } j \text{ in } \hat{G}\}.
\]
Let \( A \) be a connected component of \( G \setminus J \). We view \( A \) as a coloured graph where (in addition to colours that may have already been present in \( G \)) each vertex \( v \) is coloured by the multiset \( S(v) \). Since \( Q \) is simplifying, \( eg(A) \leq g - 1 \), and thus there is a \( C_w^g \)-sentence \( \text{bridge-iso}_A \) that identifies \( A \). We shall transform it into a \( C_w^{g+3} \)-formula \( \text{bridge-iso}_A(x, x', x'', y) \) such that \( \hat{G} \models \text{bridge-iso}_A(\hat{u}, \hat{u}', \hat{u}'', \hat{v}) \) if and only if \( A \) is isomorphic to the connected component of \( \hat{v} \) in \( \hat{G} \setminus \hat{J} \) via an isomorphism \( \pi \) that preserves the attachment patterns, that is, \( S(v) = \hat{S}(\pi(v)) \) for all \( v \in V(A) \).

Note that if a bridge in \( G \) is attached to two vertices \( w \) and \( w' \) with the same label pair \((i, j)\), then it must hold that \( w = w' \). Thus, for any vertex \( v \) in \( G \), the multiset \( S(v) \) is actually a set, i.e., each tuple occurring in the multiset has multiplicity 1. However, in \( \hat{G} \) this might not be the case.

To relativise \( \text{bridge-iso}_A \) to the connected component of a vertex \( v \) in \( G \setminus J \), we use the formula \( \text{comp}_v \) from Lemma 3.4 for \( \varphi := J(\text{vert}(x, x', y)) \) and replace every \( \exists \psi \) with \( \exists \{\text{comp}_{J(\text{vert}(x, x', y, z)} \setminus \psi \}. \) Since \( J(\text{vert}(x, x', y)) \) has width \( \max\{7, s + 2\} \), Lemma 3.4 yields that \( \text{comp}_{J(\text{vert}(x, x', y, z)} \setminus \psi \} = C_w^{\max(7,s+2)} \).

To account for the colours, we define for each multiset \( S \) of label pairs \((i, j)\) a relation \( R_S \) with \( v \in R_S \) if and only if \( S(v) = S \). Note that all label pairs that can occur are contained in \([m] \times \{0, \ldots, h\}\). Let us denote the multiplicity of a pair \((i, j)\) in a multiset \( S \) by \( \text{mult}_S(i, j) \).

We let
\[
\text{att-pat}_S(x, x', x'', y) := \bigwedge_{(i, j) \in [m] \times \{0, \ldots, h\}} \exists^{\text{comp}}(\text{H-vert}(x, x', x'', y) \land \text{cpsp-height}(j, x', x'', z) \land E(y, z))
\]
Then \( G \models \text{att-pat}_S(u, u', u'', w) \iff S(v) = S \). Note that by Lemma 6.29 we have \( \text{att-pat}_S \in C_w^{\max(7,s+2)} \). We replace every \( R_S(z, z) \) in \( \text{bridge-iso}_A \) with \( \text{att-pat}_S(x, x', x'', y) \) and obtain the desired formula \( \text{bridge-iso}_A(x, x', x'', y) \), which has width \( s + 3 \).

In the following, we only consider \( J \)-bridges and \( \hat{J} \)-bridges. If the reference to \( J \) or \( \hat{J} \) is clear from the context, we often do not mention it explicitly and simply use the term “bridge”.

Recall that every \( J \)-bridge is either an \( i \)-bridge with all vertices of attachment in a single fibre \( H_i \) or an \((i, i+1)\)-bridge with vertices of attachment in two adjacent fibres \( H_i \) and \( H_{i+1} \) for some \( i \in [\ell] \).

Recall (from the paragraph preceding Corollary 6.31) that an inner bridge is a \( J \)-bridge which has at least one vertex of attachment in \( \bigcup_{i=2} V(H_i) \) and that the graph \( K \) is the union of \( J \) with all inner bridges. \( K \) is a planar graph embedded in \( D \). By Corollary 6.31 we have \( C_w^{g+3} \)-formulae \( K(\text{vert}(x, x', x'', y)) \) and \( K(\text{edge}(x, x', y, y_1, y_2)) \) such that \( V(K) = K(\text{vert}(G, u, u', u'', y)) \) and \( E(K) = K(\text{edge}(G, u, u', u'', y_1, y_2)) \). We let \( K \) be the subgraph of \( \hat{G} \) with vertex set \( V(\hat{K}) = V(K) \) and edge set \( E(\hat{K}) = E(K) \).

A bridge is critical if it is not an inner bridge (see Figure 5). Observe that a bridge is critical if it is either an \( \ell \)-bridge or an \( 1 \)-bridge or an \((\ell, 1)\)-bridge. Let \( B_{\text{crit}} \) denote the set of all critical \( J \)-bridges. Similarly, let \( B_{\text{crit}} \) be the set of all \( \hat{J} \)-bridges in \( \hat{G} \) whose vertices of attachment are contained in \( V(\hat{Q}) \cup V(\hat{Q}_1) \) (where \( \hat{Q} \) is the set of all paths \( \hat{Q} \in \hat{Q} \) such that \( \hat{Q} \subseteq \hat{H}_i \)) with each \( J \)-bridge \( B \in B_{\text{crit}} \) we associate a type \( \eta(B) \) as follows.

- If \( B \) consists of a single edge \( v \) then \( \Theta(B) = \{(i, j), (i', j')\} \), where \((i, j) = (0, 0)\) if \( v = u \), \((i, j) = (h, 0)\) if \( v = u' \), and otherwise \( v \in V(H_i) \) and \( j = \text{dist}(u, v) \), and similarly
Figure 5 A simplifying patch with an inner bridge (green), multiple critical bridges (blue, orange, red) and a super-critical bridge (blue). Note that without $A_1''$, the graph would have an opposite pair \{$A_1', A_1''\}$. 

\[(i', j') = (0, 0) \text{ if } v' = u, \quad (i', j') = (0, h) \text{ if } v' = u', \text{ and otherwise } v' \in V(\hat{H}_v) \text{ and } j' = \text{dist}(u, v').\]

If $|B| \geq 3$, let $A := B \setminus V(J)$ be the connected component of $G \setminus J$ associated with $B$. We view $A$ as a coloured graph (with colours representing the attachment patterns $S(v)$ as above) and choose a label $\theta_A$ for the isomorphism type of $A$ (in such a way that $\theta_A = \theta_{A'} \iff A \cong A'$). We let $\theta(B) := \{\theta_A\}$.

We can define the type $\hat{\theta}(B)$ of a $\hat{J}$-bridge $\hat{B} \in \hat{B}_{\text{crit}}$ similarly. Observe that there is a bijection $\beta: B_{\text{crit}} \to \hat{B}_{\text{crit}}$ such that $\hat{\theta}(B) = \hat{\theta}(\beta(B))$ for all $B \in B_{\text{crit}}$. To see this, note that we can use the formulae $\text{bridge-iso}_A$ to construct for every type $\theta$ a $C_{w+3}$-formula that encodes the number of bridges of type $\theta$.

Observe that if two critical bridges have the same type, then either both are $(\ell, 1)$-bridges or both are $\ell$-bridges.

Recall that $\text{at}(B)$ denotes the set of vertices of attachment of a bridge $B$. Let us call bridges $B, B'$ aligned if $\text{at}(B) = \text{at}(B')$. We show that being aligned can be defined in $C_{w}^{\max(7, s+2)}$. Let

\[
\text{at-vert}(x, x', y, z) := \text{J-vert}(x, x', z) \land \exists z'(E(z, z') \land \text{comp}_{\text{j-vert}}(x, x', z', y)).
\]

Then if $v \notin J$, it holds that $G \models \text{at-vert}(u, u', v, w)$ if and only if $w$ is a vertex of attachment of some $J$-bridge that contains $v$. By Lemmas 3.4 and 6.22 the formula at-vert has width $\max(7, s + 2)$. Now we can define

\[
\text{aligned}(x, x', y, y') := \neg\text{comp}_{\text{j-vert}}(x, x', y, y') \land \\
\quad \forall z(\text{at-vert}(x, x', y, z) \leftrightarrow \text{at-vert}(x, x', y', z)).
\]

Then if $v, v' \notin J$, it holds that $G \models \text{aligned}(u, u', v, v')$ if and only if $v$ and $v'$ are contained in distinct $J$-bridges $B$ and $B'$, respectively, and $B$ and $B'$ are aligned. Furthermore, aligned has width $\max(7, s + 2)$ since at-vert has width $\max(7, s + 2)$. The two formulae can easily be modified to also capture the case that $v$ or $v'$ itself is a vertex of attachment (and the case of trivial bridges, but we do not need this for our purposes).
Recall that for all critical bridges $B \in \mathcal{B}_{\text{crit}}$ we have $\text{at}(B) \subseteq V(Q_1) \cup V(Q_r)$. Let

$$Z := \text{art}(Q_1) \cup \text{art}(Q_r),$$

$$\hat{Z} := \text{art}(\hat{Q}_1) \cup \text{art}(\hat{Q}_r).$$

**Claim 2.** Let $B, B' \in \mathcal{B}_{\text{crit}}$ such that $\theta(B) = \theta(B')$.

1. If $B$ and $B'$ are not aligned, then

$$\text{at}(B) \cap \text{at}(B') = \text{at}(B) \cap Z = \text{at}(B') \cap Z,$$

and $B$ or $B'$ is embedded in $D$.

2. At most one of $B, B'$ is embedded in $D$.

**Proof.** Let $\theta := \theta(B) = \theta(B')$. Suppose that $\text{at}(B) = \{v_1, \ldots, v_r\}$ and $\text{at}(B') = \{v'_1, \ldots, v'_{r'}\}$. Since the type of a bridge contains information about the number of vertices of attachment, their fibres, and their height, we have $s = s'$, and without loss of generality we may assume that for every $i$ the vertices $v_i$ and $v'_i$ belong to the same fibre and have the same height in this fibre. Thus, $v_i$ is an articulation vertex of its fibre if and only if $v'_i$ is one (and in this case they are equal). Hence $\text{at}(B) \cap Z = \text{at}(B') \cap Z$ and therefore we have

$$\text{at}(B) \cap Z = \text{at}(B) \cap \text{at}(B') \cap Z \subseteq \text{at}(B) \cap \text{at}(B').$$

Furthermore, for every fibre $i$ and every height $j$ there are at most two vertices $v, v' \in V(H_i)$ of height $j$ that may be vertices of attachment of a bridge, one on the path $Q_i$ and one on the path $Q'_i$. It follows that every vertex in $\text{at}(B) \cap \text{at}(B')$ lies in $V(Q_i) \cap V(Q'_i)$. Hence, every path from $v$ to $v'$ in $Q_i$ passes through $v$ and therefore, $v$ is an articulation vertex of $Q_i$. This proves $\text{at}(B) \cap \text{at}(B') \subseteq \text{at}(B) \cap Z$ and hence equality.

This means that if $B$ and $B'$ are not aligned, one of them has some vertices of attachment in $V(Q_i) \setminus V(Q'_i)$ and the other has some vertices of attachment in $V(Q'_i) \setminus V(Q_i)$. Thus, one must be embedded in $f_{i-1}$ and one in $f_i$. At least one of these sets is a subset of $D$.

Since $G$ is 3-connected, we have $r \geq 3$, and this means we cannot embed both $B$ and $B'$ into $D$, because this would violate planarity (reasoning via a $K_{3,3}$-minor).

Note that for a connected component of $G \setminus J$ or $\hat{G} \setminus \text{setminus} \hat{J}$, there is only a bounded number of possible isomorphism types $\theta_A$. Thus, we can check whether two bridges have the same isomorphism type using the formulae $\text{bridge-iso}_A$, $\text{aligned}$, $\text{att-vert}$ and requiring the variable $z$ in the definition of $\text{csp-arg}$ (cf. Lemma 6.6) additionally to be in $V(H_1)$ and $V(H_r)$, respectively, we can show that all restrictions the claim imposes on $G$ are definable in $C_{3r+3}^+$. Thus, for all $\hat{B}, \hat{B}' \in \hat{B}_{\text{crit}}$, if $\hat{B}$ and $\hat{B}'$ are not aligned, then $\text{at}(\hat{B}) \cap \text{at}(\hat{B}') = \text{at}(\hat{B}) \cap \hat{Z} = \text{at}(\hat{B}') \cap \hat{Z}$.

There is an interesting special case of pairs of critical bridges that we need to deal with separately. Consider a type $\theta$ such that there are exactly two bridges $B, B' \in \mathcal{B}_{\text{crit}}$ of type $\theta$, and these two bridges are not aligned. Then by Claim 2 either both $B$ and $B'$ are $t$-bridges or both are 1-bridges. Moreover, the claim implies that exactly one of them is embedded in $f_1 \cup f_{r-1} \subseteq D$. Say, $B$ is embedded in $f_i \cup f_{r-1}$. We call $\{B, B'\}$ an opposite pair. That is, an opposite pair in $G$ is an unordered pair $\{B, B'\}$ of bridges $B, B' \in \mathcal{B}_{\text{crit}}$ such that $\theta(B) = \theta(B')$, there is no $B'' \in \mathcal{B}_{\text{crit}} \setminus \{B, B'\}$ such that $\theta(B'') = \theta(B)$, and $B, B'$ are not aligned.

**Claim 3.** Let $\{B_1, B'_1\}, \ldots, \{B_p, B'_p\}$ be a list of all opposite pairs of $G$. Then the graph

$$K^+ := K \cup \bigcup_{i=1}^p (B_i \cup B'_i)$$

is planar.

Proof. Recall that for every $\Theta(\overline{v})$ we let $\theta(\overline{v})$ be the unique $\Theta(\overline{v})$ such that $\theta(\overline{v}) = \theta(\overline{v'})$, there is no $\overline{v''} \in \hat{B}_{\text{crit}} \setminus \{\overline{B}, \overline{B'}\}$ such that $\theta(\overline{v''}) = \theta(\overline{B})$, and $\overline{B}, \overline{B'}$ are not aligned. Let $\hat{B}_1, \hat{B}_1', \ldots, \hat{B}_{\ell}, \hat{B}_{\ell}'$ be a list of all opposite pairs in $\hat{G}$. It is easy to see that $p = p'$. We let

$$\hat{K} = \hat{K} \cup \bigcup_{i=1}^{p} (\hat{B}_i \cup \hat{B}_i')}.$$ 

It is easy to construct $C_{w^{+3}}^{\text{K-plus-vert}}(x, x', x'', y)$ and $C_{w^{+3}}^{\text{K-plus-edge}}(x, x', x'', y_1, y_2)$ such that $V(K^+) = C_{w^{+3}}^{\text{K-plus-vert}}(G, u, u', u'', y_1, y_2)$, $E(K^+) = C_{w^{+3}}^{\text{K-plus-edge}}(G, u, u', u'', y_1, y_2)$, $V(\hat{K}^+) = C_{w^{+3}}^{\text{K-plus-vert}}(\hat{\overline{G}}, \hat{\overline{u}}, \hat{\overline{u}'}, \hat{\overline{u}'}, \hat{\overline{u}'}, y_1, y_2)$, and $E(\hat{K}^+) = C_{w^{+3}}^{\text{K-plus-edge}}(\hat{\overline{G}}, \hat{\overline{u}}, \hat{\overline{u}'}, \hat{\overline{u}'}, y_1, y_2)$.

Let us call a $J$-bridge $B$ super-critical if it is critical, but not contained in an opposite pair. Let $B_{\text{sc}}$ be the set of all super-critical $J$-bridges (see Figure 5). Similarly, we call a $J$-bridge $\tilde{B}$ super-critical if it is critical, but not contained in an opposite pair, and let $\tilde{B}_{\text{sc}}$ be the set of all super-critical $\tilde{J}$-bridges.

Observe that the bijection $\beta$ between $B_{\text{crit}}$ and $\hat{B}_{\text{crit}}$ defined above induces a bijection between $B_{\text{sc}}$ and $\hat{B}_{\text{sc}}$. Moreover, we have $G = K^+ \cup \bigcup_{B \in B_{\text{sc}}} B$, and this implies $\hat{G} = \hat{K}^+ \cup \bigcup_{\hat{B} \in \hat{B}_{\text{sc}}} \hat{B}$.

Next, we expand $K^+$ and $\hat{K}^+$ by new colours that encode the information about which vertices are bridges or are attached to which vertices. For every $v \in V(K^+)$, let

$$\Theta(v) := \{\theta(B) \mid B \in B_{\text{crit}}, v \in \text{at}(B)\}.$$ 

Moreover, we let

$$\Phi(v) := \begin{cases} x & \text{if } v = u, \\ x' & \text{if } v = u', \\ x'' & \text{if } v = u'', \\ i & \text{if } v \in V(\hat{H}) \setminus \{u''\} \text{ for some } i \in \ell, \\ \bot & \text{if } v \in V(\hat{K}) \setminus V(J). \end{cases}$$

We view $\Theta(v)$ and $\Phi(v)$ as additional colours of the vertices of $K^+$ and in the following view $K^+$ as a coloured graph where these new colours are incorporated. We note that for every colour $c \in \text{rg}(\Phi)$ we have a $C_{w^{+3}}^{\text{bridge-iso}}$-formula $\Phi_c(x, x', x'', y)$ such that $G \models \Phi_c(u, u', u'', v) \iff \Phi(v) = c$. Similarly, using the formulae $\text{bridge-iso}_A$ defined above, for every colour $c \in \text{rg}(\Theta)$ we can construct a $C_{w^{+3}}^{\text{bridge-iso}}$-formula $\Theta_c(x, x', x'', y)$ such that $G \models \Theta_c(u, u', u'', v) \iff \Theta(v) = c$.

We can use these formulae to transfer the colouring to the graph $\hat{K}^+$: for $\hat{v} \in V(\hat{K}^+)$, we let $\Phi(\hat{v})$ be the unique $c \in \text{rg}(\Phi)$ such that $\hat{G} \models \Phi_c(\hat{u}, \hat{u}', \hat{u}'', \hat{v})$. If there is more than one or no such $c$, then the graphs can be distinguished by a $C_{w^{+3}}^{\text{bridge-iso}}$-formula. Similarly, for $\hat{v} \in V(\hat{K}^+)$, we let $\Theta(\hat{v})$ be the unique $c \in \text{rg}(\Theta)$ such that $\hat{G} \models \Theta_c(\hat{u}, \hat{u}', \hat{u}'', \hat{v})$. In the following, we regard $K^+$ and $\hat{K}^+$ as coloured graphs with these colours, in addition to the colours inherited from $G$ and $\hat{G}$. 


Claim 4.

\[ K^+ \cong \hat{K}^+. \]

Proof. The graphs \( K^+ \) and \( \hat{K}^+ \) with all colours are definable in \( G \) by \( C^{\nu+3}_w \)-formulae using the three parameters \( u, u', u'' \). Moreover, \( K^+ \) is a planar graph, and thus there is a \( C^4_w \)-formula that identifies it. From this formula and the formulae defining membership in the subgraphs \( K^+ \) and \( \hat{K}^+ \) we can construct a \( C^{\nu+3}_w \)-formula that would distinguish \( G \) and \( \hat{G} \) if \( K^+ \) and \( \hat{K}^+ \) were non-isomorphic.

In the following, we let \( \pi \) be an isomorphism from \( K^+ \) to \( \hat{K}^+ \). It is our goal to extend \( \pi \) to an isomorphism from \( G \) to \( \hat{G} \). For this, we need to extend \( \pi \) to all super-critical bridges. We process the bridges by type. So let \( \theta \) be a type. Let \( B_\theta \) be the set of all \( B \in B_{\text{sc}} \) with \( \theta(B) = \theta \), and similarly, let \( \hat{B}_\theta \) be the set of all \( \hat{B} \in \hat{B}_{\text{sc}} \) with \( \hat{\theta}(B) = \theta \). Then the bijection \( \beta \) between \( B_{\text{sc}} \) and \( \hat{B}_{\text{sc}} \) defined above induces a bijection between \( B_\theta \) and \( \hat{B}_\theta \).

We shall construct an extension \( \pi_\theta \) of \( \pi \) that is an isomorphism from \( K^+ \cup \bigcup_{B \in B_\theta} B \) to \( \hat{K}^+ \cup \bigcup_{\hat{B} \in \hat{B}_\theta} \hat{B} \). We can easily combine all the \( \pi_\theta \) to one isomorphism from \( G \) to \( \hat{G} \), because they all coincide on \( K^+ \) and the intersection between any two bridges is in \( K^+ \) and \( \hat{K}^+ \), respectively.

Suppose first that all \( B, B' \in B_\theta \) are aligned. Then for all \( \hat{B}, \hat{B}' \in \hat{B}_\theta \) we have \( \pi(\hat{B}) = \pi(\hat{B}') \), since \( \pi \) is \( C^{\nu+3}_w \)-definable in \( G \). Note that \( \beta \) induces an isomorphism from \( \bigcup_{B \in B_\theta} B \) to \( \bigcup_{\hat{B} \in \hat{B}_\theta} \hat{B} \). We can easily extend this isomorphism to an isomorphism from \( K^+ \cup \bigcup_{B \in B_\theta} B \) to \( \hat{K}^+ \cup \bigcup_{\hat{B} \in \hat{B}_\theta} \hat{B} \) because the attachment pattern is encoded in the colouring of the bridges.

Suppose next that there are \( B_1, B_2 \in B_\theta \) that are not aligned. Then \( B_\theta \geq 3 \), because otherwise \( \{B_1, B_2\} \) would be an opposite pair. Say, \( B_\theta = \{B_1, B_2, \ldots, B_m\} \). Without loss of generality we assume that every \( B_i \) is a 1-bridge. The case that every \( B_i \) is an \( \ell \)-bridge or every \( B_i \) is an \( (\ell, 1) \)-bridge can be dealt with in the same way. By Item 1 of Claim 2 one of \( B_1 \) and \( B_2 \), say \( B_1 \), is embedded in \( f_1 \). But then by Item 2 of Claim 2 the bridges \( B_2, \ldots, B_p \) are not embedded in \( D \). By Item 1 again, \( B_2, \ldots, B_p \) are aligned. For every \( i \in [p] \), let \( X_i := \pi(B_i) \cap \hat{B}(Q_i) \) and \( Y_i := \pi(B_i) \setminus X_i \). Then by Item 1 we know that \( X_1 = X_2 = \cdots = X_i \) and \( Y_1 = Y_2 = \cdots = Y_i \) and \( V_1 \cap Y_i = \emptyset \) for \( i \geq 2 \). Now the key observation is that the vertices in \( Y_i \) have a different colour than the vertices in \( Y_2 \), because they are attached to a different number of bridges of type \( \theta \). The isomorphism \( \pi \) maps \( Y_1 \) to a set \( \hat{Y}_1 \) of vertices that are attached to exactly one bridge of type \( \theta \), and it maps \( Y_2 \) to a set \( \hat{Y}_2 \) of vertices that are attached to \( p - 1 \) bridges of type \( \theta \). Moreover, it maps \( X := X_1 \) to a set \( \hat{X} \) of vertices that are attached to \( p \) bridges of type \( \theta \). We can now extend the isomorphism \( \pi \) by mapping \( B_1 \) to the unique bridge of type \( \theta \) that is attached to the vertices in \( \hat{Y}_1 \) and by mapping \( B_2, \ldots, B_p \) to the \( p - 1 \) bridges of type \( \theta \) that are attached to \( \hat{Y}_2 \). \( \square \)

This completes the proof of Lemma [6.1] and thus also the proof of Theorem 1.1. We finally prove the bound \( 2g + 3 \) if the surface \( S \) that \( G \) is embedded into is orientable.

Proof of Corollary 1.2. The Euler genus of an orientable surface is always even. Suppose \( G \) is a graph embeddable in an orientable surface of Euler genus \( g \). Since the subgraphs obtained by cutting through the beads are also embeddable in orientable surfaces of smaller Euler genus, their Euler genus is at least 2 smaller than the Euler genus of \( G \). Therefore, inductively proceeding as described in the previous section, redefining \( s \) to be the number of variables needed for graphs embeddable in orientable surfaces of Euler genus at most \( g - 2 \), we can improve our bound from Theorem 1.1 to \( 2g + 3 \). \( \square \)
7 Concluding Remarks

The WL dimension is a measure for the combinatorial and descriptive complexity of a graph. In view of its numerous, seemingly unrelated characterisations in terms of logic, algebra, mathematical programming, and homomorphisms, we can arguably regard the WL dimension as a natural and robust graph invariant.

We have proved an upper bound of $4g + 3$ for the WL dimension of graphs of Euler genus $g$ and showed that if $G$ is known to be embeddable on an orientable surface of Euler genus $g$, the bound improves to $2g + 3$. The immediate question that remains is how tight our bound is.

We believe that by refining our arguments in some places it might be possible to reduce the bound from Theorem 1.1 to $3g + 3$ or even $2g + 3$; any further improvement seems to require substantial additional ideas. It is conceivable that the WL dimension of planar graphs is 2. If this is the case, the additive term in our bound would automatically drop to 2.

In terms of lower bounds, using the so-called CFI construction \[8\] it is easy to prove a linear lower bound of $\epsilon \cdot g$ for the WL dimension of graphs of Euler genus $g$, albeit with a rather small constant $\epsilon > 0$. To close the gap between upper and lower bound, it may be worthwhile to spend some effort on improving the lower bound.

Beyond graphs of bounded genus, we can try to determine the WL dimension of other graph classes and tie the WL dimension to other graph invariants. A natural target would be the class of all graphs that exclude the complete graph $K_\ell$ as a minor. We know that the WL dimension of this class is bounded \[17\]. But even an exponential bound of the WL dimension in terms of $\ell$ would be major progress.

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