Closed formulas and determinantal expressions for higher-order Bernoulli and Euler polynomials in terms of Stirling numbers

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Abstract

In this paper, applying the Faà di Bruno formula and some properties of Bell polynomials, several closed formulas and determinantal expressions involving Stirling numbers of the second kind for higher-order Bernoulli and Euler polynomials are presented.

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1 Introduction

The classical Bernoulli polynomials $B_n(x)$ and Euler polynomials $E_n(x)$ are usually defined by means of the following generating functions:

$$
\frac{te^{xt}}{e^t-1} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!} \ (|t| < 2\pi) \quad \text{and} \quad \frac{2e^{xt}}{et+1} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!} \ (|t| < \pi).
$$

In particular, the rational numbers $B_n = B_n(0)$ and integers $E_n = 2^n E_n(1/2)$ are called classical Bernoulli numbers and Euler numbers, respectively.
As is well known, the classical Bernoulli and Euler polynomials play important roles in different areas of mathematics such as number theory, combinatorics, special functions and analysis.

Numerous generalizations of these polynomials and numbers are defined and many properties are studied in a variety of context. One of them can be traced back to Nörlund [7]: The higher-order Bernoulli polynomials $B_n^{(\alpha)}(x)$ and higher-order Euler polynomials $E_n^{(\alpha)}(x)$, each of degree $n$ in $x$ and in $\alpha$, are defined by means of the generating functions

\[
\left( \frac{t}{e^t - 1} \right)^\alpha e^{xt} = \sum_{n=0}^{\infty} B_n^{(\alpha)}(x) \frac{t^n}{n!},
\]

and

\[
\left( \frac{2}{e^t + 1} \right)^\alpha e^{xt} = \sum_{n=0}^{\infty} E_n^{(\alpha)}(x) \frac{t^n}{n!},
\]

respectively. For $\alpha = 1$, we have $B_n^{(1)}(x) = B_n(x)$ and $E_n^{(1)}(x) = E_n(x)$.

According to Wiki [2], "In mathematics, a closed-form expression is a mathematical expression that can be evaluated in a finite number of operations. It may contain constants, variables, certain 'well-known' operations (e.g., $+ - \times \div$), and functions (e.g., $n$th root, exponent, logarithm, trigonometric functions, and inverse hyperbolic functions), but usually no limit."

From this point of view, Wei and Qi [15] studied several closed form expressions for Euler polynomials in terms of determinant and the Stirling numbers of the second kind. Also, Qi and Chapman [8] established two closed forms for the Bernoulli polynomials and numbers involving the Stirling numbers of the second kind and in terms of a determinant of combinatorial numbers. Moreover, some special determinantal expressions and recursive formulas for Bernoulli polynomials and numbers can be found in [9].

In 2018, Hu and Kim [6] presented two closed formulas for Apostol-Bernoulli polynomials by aid of the properties of the Bell polynomials of the second kind $B_{n,k}(x_1, x_2, ..., x_{n-k+1})$ (see Lemma 2.1, below). Very recently, Dai and Pan [4] have obtained the closed forms for degenerate Bernoulli polynomials.

In this paper, we focus on higher-order Bernoulli and Euler polynomials in those respects, mentioned above. Firstly, we find some novel closed formulas for higher-order Bernoulli and Euler polynomials in terms of Stirling numbers of the second kind $S(n, k)$ via the Faà di Bruno formula for the
Bell polynomials of the second kind and the generating function methods. Secondly, we establish some determinantal expressions by applying a formula of higher order derivative for the ratio of two differentiable functions. Consequently, taking some special cases in our results provides the further formulas for classical Bernoulli and Euler polynomials, and numbers.

2 Some Lemmas

In order to prove our main results, we recall several lemmas below.

Lemma 2.1 ([3, p. 134 and 139]) The Bell polynomials of the second kind, or say, partial Bell polynomials, denoted by $B_{n,k} (x_1, x_2, ..., x_{n-k+1})$ for $n \geq k \geq 0$, defined by

$$B_{n,k} (x_1, x_2, ..., x_{n-k+1}) = \sum_{\sum_{i=1}^{\infty} l_i = n, \sum_{i=1}^{n} i_l = k} \prod_{i=1}^{l-1} \frac{n!}{l_i!} \prod_{i=1}^{n} \left( \frac{x_i}{i!} \right)^{l_i}.$$

The Faà di Bruno formula can be described in terms of the Bell polynomials of the second kind $B_{n,k} (x_1, x_2, ..., x_{n-k+1})$ by

$$\frac{d^n}{dt^n} f \circ h (t) = \sum_{k=0}^{n} f^{(k)} (h (t)) B_{n,k} (h' (t), h'' (t), ..., h^{(n-k+1)} (t)).$$  \hspace{1cm} (2.1)

Lemma 2.2 ([3, p. 135]) For $n \geq k \geq 0$, we have

$$B_{n,k} (abx_1, ab^2 x_2, ..., ab^{n-k+1} x_{n-k+1}) = a^k b^n B_{n,k} (x_1, x_2, ..., x_{n-k+1}),$$  \hspace{1cm} (2.2)

where $a$ and $b$ are any complex number.

Lemma 2.3 ([3, p. 135]) For $n \geq k \geq 0$, we have

$$B_{n,k} (1, 1, ..., 1) = S(n, k),$$  \hspace{1cm} (2.3)

where $S(n, k)$ is the Stirling numbers of the second kind, defined by [3, p. 206]

$$\frac{(e^t - 1)^k}{k!} = \sum_{n=k}^{\infty} S(n, k) \frac{t^n}{n!},$$
Lemma 2.4 ([5]) For \( n \geq k \geq 1 \), we have

\[ B_{n,k} \left( \frac{1}{2} , \frac{1}{3} , \ldots , \frac{1}{n-k+2} \right) = \frac{n!}{(n+k)!} \sum_{i=0}^{k} (-1)^{k-i} \binom{n+k}{k-i} S(n+i,i). \]

(2.4)

Lemma 2.5 ([1, p. 40, Entry 5]) For two differentiable functions \( p(x) \) and \( q(x) \neq 0 \), we have for \( k \geq 0 \)

\[ \frac{d^k}{dx^k} \left[ \frac{p(x)}{q(x)} \right] = \frac{(-1)^k}{q^{k+1}} \begin{vmatrix} p & q & 0 & \ldots & 0 & 0 \\ p' & q' & q & \ldots & 0 & 0 \\ p'' & q'' & \binom{2}{1}q' & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ p^{(k-2)} & q^{(k-2)} & \binom{k-2}{1}q^{(k-3)} & \ldots & q & 0 \\ p^{(k-1)} & q^{(k-1)} & \binom{k-1}{1}q^{(k-2)} & \ldots & \binom{k-1}{k-2}q' & q \\ p^{(k)} & q^{(k)} & \binom{k}{1}q^{(k-1)} & \ldots & \binom{k}{k-2}q'' & (k-1)q' \end{vmatrix} \]

(2.5)

In other words, the formula (2.5) can be represented as

\[ \frac{d^k}{dx^k} \left[ \frac{p(x)}{q(x)} \right] = \frac{(-1)^k}{q^{k+1}} \left| W_{(k+1) \times (k+1)} (x) \right|, \]

where \( \left| W_{(k+1) \times (k+1)} (x) \right| \) denotes the determinant of the matrix

\[ W_{(k+1) \times (k+1)} (x) = \begin{bmatrix} U_{(k+1) \times 1} (x) & V_{(k+1) \times k} (x) \end{bmatrix}. \]

Here \( U_{(k+1) \times 1} (x) \) has the elements \( u_{l,1} (x) = p^{(l-1)} (x) \) for \( 1 \leq l \leq k+1 \) and \( V_{(k+1) \times k} (x) \) has the entries of the form

\[ v_{i,j} (x) = \begin{cases} \binom{i-1}{j-1} q^{(i-j)} (x), & \text{if } i-j \geq 0; \\ 0, & \text{if } i-j < 0, \end{cases} \]

for \( 1 \leq i \leq k+1 \) and \( 1 \leq j \leq k \).
3 Closed formulas

In this section, we give closed formulas for higher-order Bernoulli and Euler polynomials.

**Theorem 3.1** The higher-order Bernoulli polynomials $B^{(\alpha)}_n(x)$ can be expressed as

$$B^{(\alpha)}_n(x) = \sum_{k=0}^{n} \binom{n}{k} \sum_{i=0}^{k} (-\alpha)^i \frac{k!}{(k+i)!} \sum_{j=0}^{i} (-1)^{i-j} \binom{k+i}{i-j} S(k+j,j) x^{n-k},$$

where $S(n,k)$ is the Stirling numbers of the second kind and $\langle x \rangle_n$ denotes the falling factorial, defined for $x \in \mathbb{R}$ by

$$\langle x \rangle_n = \prod_{k=0}^{n-1} (x-k) = \begin{cases} x(x-1)...(x-n+1), & \text{if } n \geq 1; \\ 1, & \text{if } n = 0. \end{cases}$$

In particular $x = 0$, the higher-order Bernoulli numbers $B^{(\alpha)}_n$ possess the following form

$$B^{(\alpha)}_n = \sum_{i=0}^{n} (-\alpha)^i \frac{n!}{(n+i)!} \sum_{j=0}^{i} (-1)^{i-j} \binom{n+i}{i-j} S(n+j,j). \quad (3.1)$$

**Proof.** Let us begin by writing $\left(\frac{e^t - 1}{t}\right)^{\alpha} = \left(\int_1^e s^{t-1} ds\right)^{\alpha}$. From (2.1) and (2.4), we have

$$\frac{d^k}{dt^k} \left(\frac{e^t - 1}{t}\right)^{-\alpha} = \sum_{i=0}^{k} (-\alpha)^i \frac{k!}{(k+i)!} \sum_{j=0}^{i} (-1)^{i-j} \binom{k+i}{i-j} S(k+j,j) x^{n-k},$$

as $t \to 0$.
Also \((e^x)^{(k)} = x^k e^x \to x^k\), as \(t \to 0\). So, using the Leibnitz’s formula for the \(n\)th derivative of the product of two functions yields that

\[
\lim_{t \to 0} \frac{d^n}{dt^n} \left[ \left( \frac{e^t - 1}{t} \right)^{-\alpha} e^t \right] = \sum_{k=0}^{n} \binom{n}{k} \sum_{i=0}^{k} (-\alpha)_i \frac{k!}{(k+i)!} \sum_{j=0}^{i} (-1)^{i-j} \binom{k+i}{i-j} S(k+j,j) x^{n-k},
\]

which is equal to \(B_n^{(\alpha)}(x)\) from the generating function (1.1). For \(x = 0\), we immediately get the identity (3.1). ■

**Remark 3.2** For \(\alpha = 1\), noting the fact \((-1)_i = (-1)^i i!\), the equation (3.1) becomes

\[
B_n = \sum_{i=0}^{n} i! \binom{n}{i} \sum_{j=0}^{i} (-1)^j \binom{n+i}{i-j} S(n+j,j)
\]

\[
= \sum_{j=0}^{n} (-1)^j \frac{S(n+j,j)}{(n+j)!} \sum_{i=j}^{n} \binom{i}{j}
\]

\[
= \sum_{j=0}^{n} (-1)^j \frac{(n+1)_j}{(n+j)_j} S(n+j,j),
\]

which is [8, Eq. 1.3].

**Theorem 3.3** The higher-order Euler polynomials \(E_n^{(\alpha)}(x)\) can be represented as

\[
E_n^{(\alpha)}(x) = \sum_{k=0}^{n} \binom{n}{k} \sum_{i=0}^{k} (-\alpha)_i \frac{S(k,i)}{2^i} x^{n-k}.
\]

In particular, the higher-order Euler numbers \(E_n^{(\alpha)}\) have the following form

\[
E_n^{(\alpha)} = \sum_{k=0}^{n} \binom{n}{k} \sum_{i=0}^{k} (-\alpha)_i S(k,i) 2^{k-i}.
\]

**Proof.** By (2.1), (2.2) and (2.3), we have

\[
\frac{d^k}{dt^k} \left( \frac{e^t + 1}{2} \right)^{-\alpha} = \sum_{i=0}^{k} (-\alpha)_i \left( \frac{e^t + 1}{2} \right)^{-\alpha-i} B_{k,i} \left( \frac{e^t}{2}, \frac{e^t}{2}, ..., \frac{e^t}{2} \right)
\]

6
\[
= \sum_{i=0}^{k} \langle -\alpha \rangle_i \left( \frac{e^t + 1}{2} \right)^{-\alpha - i} \left( \frac{e^t}{2} \right)^i B_{k,i} (1, 1, \ldots, 1)
\]

\[
\rightarrow \sum_{i=0}^{k} \langle -\alpha \rangle_i \frac{S(k,i)}{2^i}, \quad \text{as } t \to 0.
\]

So, from the Leibnitz’s rule again and the generating function for higher-order Euler polynomials, given by (1.2), we obtain that

\[
E_n^{(\alpha)}(x) = \lim_{t \to 0} \frac{d^n}{dt^n} \left[ \left( \frac{e^t + 1}{2} \right)^{-\alpha} e^{xt} \right]
\]

\[
= \sum_{k=0}^{n} \binom{n}{k} \sum_{i=0}^{k} \langle -\alpha \rangle_i \frac{S(k,i)}{2^i} x^{n-k}.
\]

Taking special case gives the closed form (3.2) immediately for higher-order Euler numbers. ■

**Remark 3.4** For \( \alpha = 1 \), the counterpart closed formula for Euler polynomials can be derived. Moreover, setting more special cases leads to similar formula for Euler numbers.

## 4 Determinantals expressions

This section is devoted to demonstrate some determinantal expressions for higher order Bernoulli and Euler polynomials.

**Theorem 4.1** The higher-order Bernoulli polynomials \( B_n^{(\alpha)}(x) \) can be represented in terms of the following determinant as

\[
B_n^{(\alpha)}(x) = (-1)^n \begin{vmatrix}
1 & \gamma_0 & 0 & \ldots & 0 & 0 \\
x & \gamma_1 & \gamma_0 & \ldots & 0 & 0 \\
x^2 & \gamma_2 & (\frac{2}{1})\gamma_1 & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
x^{n-2} & \gamma_{n-2} & (\frac{n-2}{1})\gamma_{n-3} & \ldots & \gamma_0 & 0 \\
x^{n-1} & \gamma_{n-1} & (\frac{n-1}{1})\gamma_{n-2} & \ldots & (\frac{n-2}{1})\gamma_1 & \gamma_0 \\
x^n & \gamma_n & (\frac{n}{1})\gamma_{n-1} & \ldots & (\frac{n-2}{2})\gamma_2 & (\frac{n}{n-1})\gamma_1
\end{vmatrix},
\]
where
\[ \gamma_n = \sum_{i=0}^{n} \langle \alpha \rangle_i \frac{n!}{(n+i)!} \sum_{j=0}^{i} (-1)^{i-j} \binom{n+i}{i-j} S(n+j,j). \]

Proof. We use Lemma 2.5 for \( p(t) = e^{xt} \) and \( q(t) = ((e^t - 1)/t)^\alpha \). Note that if we proceed similar manipulations to the proof of Theorem 3.1, then, we deduce that
\[
\lim_{t \to 0} \frac{d^n}{dt^n} q(t) = \sum_{i=0}^{n} \langle \alpha \rangle_i \frac{n!}{(n+i)!} \sum_{j=0}^{i} (-1)^{i-j} \binom{n+i}{i-j} S(n+j,j) := \gamma_n.
\]

So, we have
\[
\frac{d^n}{dt^n} \left[ \frac{e^{xt}}{((e^t - 1)/t)^\alpha} \right] = \frac{(-1)^n}{((e^t - 1)/t)^{(n+1)\alpha}}
\]
\[
\rightarrow (-1)^n
\]

as \( t \to 0 \). From the generating function, given by (1.1), we reach the desired result. \( \blacksquare \)

Remark 4.2 We mention that for the special case \( x = 0 \), the analog determinantal expression for higher-order Bernoulli numbers \( B_n^{(\alpha)} \) can be offered. Moreover, for \( \alpha = 1 \), and \( \alpha = 1 \) and \( x = 0 \), the similar representations can be obtained for classical Bernoulli polynomials and numbers, respectively.
**Theorem 4.3** The higher-order Euler polynomials $E_n^{(α)}(x)$ can be represented in terms of the following determinant as

$$E_n^{(α)}(x) = (-1)^n \begin{vmatrix} 1 & β_0 & 0 & \ldots & 0 & 0 \\ x & β_1 & β_0 & \ldots & 0 & 0 \\ x^2 & β_2 & \binom{2}{1}β_1 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ x^{n-2} & β_{n-2} & \binom{n-2}{1}β_{n-3} & \ldots & β_0 & 0 \\ x^{n-1} & β_{n-1} & \binom{n-1}{1}β_{n-2} & \ldots & \binom{n-1}{2}β_1 & β_0 \\ x^n & β_n & \binom{n}{1}β_{n-1} & \ldots & \binom{n}{2}β_2 & \binom{n}{n}β_1 \end{vmatrix},$$

where

$$β_n = \sum_{i=0}^{n} \langle α \rangle_i \frac{S(n, i)}{2^i}.$$

**Proof.** The proof can be verified by proceeding as in the proof of Theorem 4.1. So, we omit it. ■

**Remark 4.4** The special values of the Bell polynomials of the second kind $B_{n,k}$ are worthy in combinatorics and number theory. In this respect, $B_{n,k}$ has been applied in order to cope with some difficult problems and obtain significant results in many studies (see for example [10–14]).

**References**

[1] N. Bourbaki, *Functions of a Real Variable, Elementary Theory*, Translated from the 1976 French Original by Philip Spain. Elements of Mathematics (Berlin). Springer, Berlin, 2004.

[2] *Closed-form expression*. https://en.wikipedia.org/wiki/Closed-form_expression.

[3] L. Comtet, *Advanced Combinatorics: The Art of Finite and Infinite Expansions*, Revised and Enlarged Edition. D. Reidel Publishing Co., Dordrecht 1974.

[4] L. Dai and H. Pan, *Closed forms for degenerate Bernoulli polynomials*, Bull. Aust. Math. Soc. **101** (2020), no. 2, 207–217.
[5] B.-N. Guo and F. Qi, *An explicit formula for Bernoulli numbers in terms of Stirling numbers of the second kind*, J. Anal. Number Theory 3 (2015), no. 1, 27–30.

[6] S. Hu and M.-S. Kim, *Two closed forms for the Apostol–Bernoulli polynomials*, Ramanujan J. 46 (2018), no. 1, 103–117.

[7] N. E. Nörlund, *Vorlesungen über Differenzenrechnung*, Springer-Verlag, Berlin, 1924.

[8] F. Qi and R. J. Chapman, *Two closed forms for the Bernoulli polynomials*, J. Number Theory, 159 (2016), 89–100.

[9] F. Qi and B.-N. Guo, *Some Determinantal Expressions and Recurrence Relations of the Bernoulli Polynomials*, Mathematics, 4 (2016), no. 4, 1–11.

[10] F. Qi, *Derivatives of tangent function and tangent numbers*, Appl. Math. Comput. 268 (2015), 844–858.

[11] F. Qi and M.-M. Zheng, *Explicit expressions for a family of the Bell polynomials and applications*, Appl. Math. Comput. 258 (2015), 597–607.

[12] F. Qi and B.-N. Guo, *Explicit formulas for special values of the Bell polynomials of the second kind and for the Euler numbers and polynomials*, Mediterr. J. Math. 14 (2017), no. 3, Article 140, 14 pages.

[13] F. Qi, D. Lim, and B.-N. Guo, *Explicit formulas and identities for the Bell polynomials and a sequence of polynomials applied to differential equations*, Rev. R. Acad. Cienc. Exactas Fis. Nat. Ser. A Mat. RACSAM, 113 (2019), 1–9.

[14] F. Qi, *An Explicit Formula for the Bell Numbers in Terms of the Lah and Stirling Numbers*, Mediterr. J. Math. 13 (2016), 2795–2800.

[15] C.-F. Wei and F. Qi, *Several closed expressions for the Euler numbers*, J. Inequal. Appl. 2015, 219 (2015).