Tree morphisms, transducers, and integer sequences

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Abstract

The notion of transducer integer sequences is considered through a series of examples. By definition, transducer integer sequences are integer sequences produced, under a suitable interpretation, by finite automata encoding tree morphisms (length and prefix preserving transformations of words). Transducer integer sequences are related to the notion of self-similar groups and semigroups, as well as to the notion of automatic sequences.

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1 Introduction

It is known from the work of Allouche, Bétréma, and Shallit (see [1, 2]) that a square free sequence on 6 letters can be obtained by encoding the optimal solution to the standard Hanoi Towers Problem on 3 pegs by an automaton. Roughly speaking, given an input word which is the binary representation if the number $i$, the automaton ends in one of the 6 states. These states represent the six possible moves between the three pegs and if the automaton ends in state $q_{xy}$, this means that the one needs to move the top disk from peg $x$ to peg $y$ in step $i$ of the optimal solution. The obtained sequence over the 6-letter alphabet \{ $q_{xy}$ | $0 \leq x, y \leq 2$, $x \neq y$ \} is an example of an automatic sequence.

We choose to work with a slightly different type of automata, which under a suitable interpretation, produce integer sequences in the output. The difference with the above model, again roughly speaking, is that not only the final state matters, but the output depends on every transition step taken during the computation and both the input and the output words are interpreted as encodings of integers. The integer sequences that can be obtained this way are called transducer integer sequences. We provide some examples that illustrate the notion of a transducer integer sequence. All provided examples are related to the Hanoi Towers Problem on 3 pegs.

In recent years, a very fruitful line of research in group theory has led to the notion of a self-similar group [20] (also known as automata groups [10] or state closed groups [22]). Many challenging problems have been solved by using finite automata to encode groups of tree automorphisms with interesting properties, leading to solutions to outstanding problems. To name just a few, such examples are the first Grigorchuk group [9], solving the problem of Milnor on existence of groups of intermediate growth and the Day-von Neumann problem on existence of amenable but not elementary amenable groups, Basilica group [15, 7], providing an example of amenable but not subexponentially amenable group, Wilson groups [24], solving the problem of Gromov on existence of groups of non-uniform exponential growth, the realization of the lamplighter group $L_2$ by an automaton [17], leading to the solution of the Strong Atuyah Conjecture on

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L²-Betti numbers [15], and the recent solution to Hubbard’s Twisted Rabbit Problem in holomorphic dynamics [5]. The geometric language and insight coming from the interpretation of the action of the automata as tree automorphisms greatly simplifies the presentation and helps in the understanding of the underlying phenomena, such as self-similarity, contraction, branching, etc (see [10] [11] [3] [20] for definitions, examples, and details).

The language of finite automata has been proved extremely suitable in working with self-similarity phenomena. Indeed, in the case of automatic sequences, it is known from Cobham Theorem [8] that such sequences are precisely those that are obtained as images under codings of fixed points of uniform endomorphisms (limits of iterations of endomorphisms). The contracting self-similar groups have been related by Nekrashevych to finite partial self-coverings of orbispaces [20].

In the current article we use automata in the sense of transducers. As such, they generate self-similar groups (or semigroups) of tree automorphisms (or endomorphisms). In the same time, the output words are interpreted as encodings of integers, thus bringing the topic closer to the topic of automatic sequences. Thus, it is not surprising that the concrete examples of transducer integer sequences that are exhibited here all gave high level of self-similarity and can be defined as limits of certain iterations of sequences.

2 Tree morphisms and finite transducers

For \(k \geq 2\) denote \(X_k = \{0, 1, \ldots, k-1\}\). The free monoid \(X_k^*\) has the structure of a \(k\)-ary rooted tree \(X_k^*\) in which the empty word \(\emptyset\) is the root, the words of length \(n\) constitute the level \(n\) and each vertex \(v\) has \(k\) children, namely \(vx\), for \(x\) a letter in \(X_k^*\) (see Figure 1 for the ternary tree). The tree structure imposes order on \(X_k^*\), which is the well known prefix order. Namely, we say that \(u \leq v\) if \(u\) is a vertex on the unique geodesic from \(v\) to \(\emptyset\) in \(X_k^*\), which is equivalent to saying that \(u\) is a prefix of \(v\). A map \(\mu : X_{k_1}^* \to X_{k_2}^*\) is a tree morphism if it preserves the word length and the prefix relation, i.e.

\[|\mu(u)| = |u| \quad \text{and} \quad \mu(u) \leq \mu(\mu u),\]

for all words \(u\) and \(w\) over \(X_{k_1}\). In case \(k_1 = k_2\), morphisms are called endomorphisms and bijective endomorphisms are called automorphisms.

Every tree morphism \(\mu : X_{k_1}^* \to X_{k_2}^*\) can be decomposed as

\[\mu = \pi_\mu(\mu_0, \ldots, \mu_{k_1-1})\]

where \(\pi_\mu : X_{k_1} \to X_{k_2}\) is a map called the root transformation of \(\mu\) and \(\mu_x : X_{k_1}^* \to X_{k_2}^*\), \(x\) in \(X_{k_1}\), are tree morphisms called the sections of \(\mu\). The root permutation and the sections of \(\mu\) are uniquely determined by the recursive relation

\[\mu(xw) = \pi_\mu(x)\mu_x(w),\]
which holds for every letter $x$ and word $w$ over $X_k$. Thus the sections describe the action of $\mu$ on the $k_1$ subtrees hanging below the root in $X^*_k$, and the root transformation $\tau_{q}$ describes the action of $\mu$ at the root.

The tree morphisms act on the left and the composition is performed from right to left, yielding the formula

$$
\mu \nu = \pi_{\nu_{q_{\nu}}(x_{\nu})} \pi_{\nu_{q_{\mu}}(x_{\mu})} = \pi_{\nu_{q_{\nu_{\pi}}(x_{\nu})}} = \pi_{\nu_{q_{\nu_{\tau}}(x_{\nu})}} = \pi_{\nu_{q_{\tau_{q_{\nu_{\pi}}(x_{\nu})}}}}
$$

A quite efficient way of defining tree morphisms is by using finite transducers. A finite $k_1$ to $k_2$ transducer is a 5-tuple $A = (Q, X_{k_1}, X_{k_2}, \tau, \pi)$, where $Q$ is a finite set of states, $X_{k_1}$ and $X_{k_2}$ are the input and output alphabets, $\tau : Q \times X_{k_1} \rightarrow Q$ is a map called the transition map of $A$, and $\pi : Q \times X_{k_2} \rightarrow X_{k_2}$ is a map called the output map of $A$. Every state $q$ of the finite transducer $A$ defines a tree morphism, also denoted $q$ by setting $q_x = \tau(q, x)$, for $x \in X_{k_1}$, and $\pi_q : X_{k_1} \rightarrow X_{k_2}$ to be the restriction of $\pi$ defined by $\pi_q(x) = \pi(q, x)$. Thus, for each state $q$ of $A$ we have

$$
q(\emptyset) = \emptyset \quad \text{and} \quad q(x)q_x = \pi_q(x)q_x(w),
$$

for $x$ a letter in $X_k$ and $w$ a word over $X_k$. When started at state $q$, the transducer reads the first input letter $x$, produces the first letter of the output according to the transformation $\pi_q$ and changes its state to $q_x$. The state $q_x$ then handles the rest of the input and output. The states of a $k$-ary transducer (transducer in which $k_1 = k_2 = k$) define $k$-ary tree endomorphisms. An invertible $k$-ary transducer is a transducer in which $k_1 = k_2 = k$ and the transformation $\pi_q$ is a permutation of $X_k$, for each state $q$ in $Q$. The states of an invertible $k$-ary transducer define $k$-ary tree automorphisms. When $k_1 \leq k_2$ and, for each state $q$, the vertex transformation $\pi_q$ is injective then every state of the transducer $A$ is an embedding of the $k_1$-ary tree into the $k_2$-ary tree. We call such a transducer an injective transducer.

The boundary $\partial X^*_k$ of the $k$-ary tree $X^*_k$ consists of all infinite (to the right) words over $X_k$. The boundary has a structure of an ultrametric space homeomorphic to a Cantor set. The recursive definition (2) applies to both finite and infinite words $w$. The action of a state $q$ of a $k$-ary transducer on the boundary $\partial X^*_k$ is by continuous maps, while the action of an invertible $k$-ary transducer is by isometries. More on these aspects of actions on rooted trees can be found in [10].

There are two common ways to represent finite $k_1$ to $k_2$ transducers by labeled directed graphs such as the ones in Figure 2. The graph on the left represents an invertible ternary transducer, denoted $A_H$. The vertices are the states, each state $q$ is labeled by its corresponding transformation (in this case permutation) $\pi_q$, and the edges labeled by the letters from $X_3$ define the transition function $\tau$ (for every $q$ in $Q$ and $x$ in $X_3$ there exists an edge from $q$ to $q_x = \tau(q, x)$ labeled by $x$). The graph on the right represents a 3 to 2 transducer. The vertices are the states and for each pair $(q, x)$ in $Q \times X_3$ an edge labeled by $x \mid \pi_q(x)$ connects $q$ to $q_x$. One can easily switch back and forth between the two formats. We refer to the second form (the one in which the output is indicated on the edges) as the Moore diagram of the automaton.

Figure 2: An invertible ternary transducer $A_H$ and a 3 to 2 transducer $A_L$
For $0 \leq i < j \leq 2$, the ternary tree automorphisms $a_{ij}$ from the automaton $A_h$ are defined recursively by
\[
\begin{align*}
a_{ij}(\emptyset) &= \emptyset, \\
a_{ij}(iw) &= jw, \\
a_{ij}(jw) &= iw, \\
a_{ij}(xw) &= xa_{ij}(w), & x \notin \{i, j\},
\end{align*}
\]
for a word $w$ over $X_3$. In simple terms, the only effect the transformation $a_{ij}$ has on a word $w$ over $X_3$ is that it changes the very first appearance of either of the symbols $i$ or $j$ in $w$ to the other symbol, if such an appearance exists. To simplify the notation, we write $a = a_{01}$, $b = a_{02}$, and $c = a_{12}$.

The state labeled by $id$ does not change any input word and represents the identity automorphism of the ternary tree. It is clear that $a$, $b$ and $c$ are self-invertible transformations of $X^*_3$, i.e $a^2 = b^2 = c^2 = id$.

The $3$ to $2$ tree morphisms defined by the transducer $A_L$ are defined recursively by
\[
\begin{align*}
\alpha(\emptyset) &= \emptyset, & \alpha(0w) &= 0\alpha(w), & \alpha(1w) &= 1\alpha(w), & \alpha(2w) &= 1\beta(w), \\
\beta(\emptyset) &= \emptyset, & \beta(0w) &= 1\alpha(w), & \beta(1w) &= 1\beta(w), & \beta(2w) &= 0\beta(w).
\end{align*}
\]

**Definition 2.1.** The semigroup (group) of $k$-ary tree endomorphisms (automorphisms) generated by all the states of an (invertible) $k$-ary transducer $A$ is called the semigroup (group) of $A$ and is denoted by $S(A)$ ($G(A)$).

The group $G(A_H)$ is introduced in [14], where it is called Hanoi Towers group on 3 pegs and denoted $H^{(3)}$ (in fact, one Hanoi Towers group $H^{(k)}$ is introduced for each number of pegs $k \geq 3$). The name is derived from the fact that the group $H^{(3)}$ models the well known Hanoi Towers Problem on 3 pegs.

To recall, the Hanoi Towers Problem on 3 pegs and $n$ disks is the following. In a valid $n$ disk configuration, disks of different size, labeled by 1, 2, ..., $n$ according to their size, are placed on three pegs, labeled 0,1 and 2, in such a way that no disk is placed on top of a smaller disk. In a single move the top disk from one peg can be moved and placed on top of another peg, as long as the newly obtained configuration is still valid. Initially all $n$ disks are placed on peg 0 and the problem asks for an optimal algorithm that moves all disks to another peg.

Each valid configuration of $n$ disks can be encoded by a word of length $n$ over $X_3$. The word $x_1 \ldots x_n$ represents the unique valid configuration in which disk $i$ is placed on peg $x_i$. The ternary tree automorphism $a_{ij}$ then represents a move between peg $i$ and peg $j$ (in either direction). For example the move between peg 0 and peg 2 illustrated in Figure 3 is encoded as $a_{02}(10221) = 12221$.

![Figure 3: A move between peg 0 and peg 2](image)

The action of $H^{(3)}$ on the ternary tree is spherically transitive, meaning that it is transitive on the levels of the tree. This is equivalent to the statement that any valid configuration on $n$ disks can be obtained from any other valid configuration on $n$ disks by legal moves.

Consider the stabilizer of the vertex $0^n$ in $H^{(3)}$, denoted $P_n$. The group $H^{(3)}$ acts on the set $H^{(3)}/P_n$ of left cosets of $P_n$. The action is described by the corresponding Schreier graph $\Gamma_n = \Gamma_n(H^{(3)}, P_n, S)$ of
$P_n$ with respect to the generating set $S = \{a, b, c\}$. The vertices are the cosets of $P_n$ and there is an edge connecting $hP_n$ to $shP_n$ for every coset $hP_n$ and generator $s$ in $S$. Since $h' \in hP_n$ if and only if $h'(0^n) = h(0^n)$ the vertices of the Schreier graph $\Gamma_n$ can be encoded by the vertices of the $n$-h level of the ternary tree (the coset $hP_n$ or labeled by $h(0^n)$) and two vertices are connected if and only if one is the image of the other under $s$, for some generator $s$ in $S$. The Schreier graph $\Gamma_3$ corresponding to level 3 of the ternary tree is given in Figure 4. Since all generators have order 2, no directions are indicated on the edges.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{schreier_graph.pdf}
\caption{The Schreier graph of $H^{(3)}$ at level 3}
\end{figure}

The sequence of graphs $\{\Gamma_n\}$ converges to an infinite graph $\Gamma$ in the space of pointed graphs based at $0^n$ (see [16] for definitions of this space), which is the Schreier graph $\Gamma = \Gamma(H^{(3)}, P, S)$, where $P = \cap_{n=0}^{\infty} P_n$ is the stabilizer of the infinite ray $0^\infty = 000\ldots$ on the boundary of the ternary tree. One can think of the limiting graph both as the Schreier graphs of the action of $H^{(3)}$ on the orbit of the infinite ray $0^\infty$ in $\partial X_3^*$ or as the model of Hanoi Towers Problem representing all valid configurations that can be reached from the configuration in which (countably) infinitely many disks are placed on peg 0 (this configuration corresponds to the infinite word $0^\infty$).

Graphs similar to $\Gamma_n$, modeling the Hanoi Towers problem are well known in the literature, but there is a subtle difference. Namely, the difference with the corresponding graphs in [19] modeling the Hanoi Towers Problem is that the edges in $\Gamma_3$ are labeled (by the corresponding tree automorphisms) and our graphs have loops at the corners (corresponding to situations in which all disks are on one peg and the generator corresponding to a move between the other two pegs does not change anything), which turn them into 3-regular graphs. Finite dimensional permutational representations of $H^{(3)}$ based on the action on the
levels of the ternary tree were used in [14] to calculate the (Markov) spectrum of the graphs $\Gamma_n$ as well as the limiting infinite graph $\Gamma$. Among interesting properties of $\mathcal{H}(3)$ we mention that it is an amenable (but not subexponentially amenable), regular branch group over its commutator, it is not just infinite and its closure in the pro-finite group of ternary tree automorphisms is finitely constrained. Moreover, $\mathcal{H}(3)$ is (up to conjugation) the iterated monodromy group of the finite rational map $z \mapsto z^2 - \frac{16}{27}z$, whose Julia set is the Sierpiński gasket. This explains the fact that the sequence of Schreier graphs $\{\Gamma_n\}$ approximates the Sierpiński gasket. For more information on properties of $\mathcal{H}(3)$ we refer the interested reader to [14, 13, 12, 11].

When $k_1 \geq k_2$ every $k_1$ to $k_2$ tree morphism can also be thought of as a $k_1$-ary tree morphism, since the $k_2$-ary tree canonically embeds in the $k_1$-ary tree in obvious way. We calculate the semigroup $S(A_L)$ by thinking of the transducer $A_L$ as being a ternary transducer.

**Proposition 2.1.** The self-similar semigroup $S_L = S(A_L)$ is given by the semigroup presentation

$$S_L = \langle \alpha, \beta \mid \alpha^2 = \alpha, \alpha \beta = \beta \rangle.$$ 

In other words, $S(A_L)$ is the free cyclic semigroup generated by $\beta$ extended by a left identity element $\alpha$.

**Proof.** Since $\alpha$ acts trivially on the binary words (words over $\{0,1\}$) and the image of every ternary word under the elements of $S_L$ is a binary word, we have $\alpha \sigma = \sigma$, for every element $\sigma$ of $S_L$.

Denote by $\pi_1$ the transformation $X_3 \to X_3$ given by $\pi_1(x) = 1$, for $x$ in $X_3$. Note that $\pi_3 \pi_\sigma = \pi_1$, for all elements $\sigma$ of $S_L$. Calculations using (1) yield

$$\beta = \pi_3(\alpha, \beta),$$

$$\beta^2 = \pi_3(\beta \alpha, \beta, \alpha \beta) = \pi_3(\beta \alpha, \beta^2, \beta),$$

$$\beta^3 = \pi_3(\beta \beta \alpha, \beta^2 \beta, \beta \beta) = \pi_3(\beta^3 \alpha, \beta^3, \beta^2)$$

$$\cdots$$

$$\beta^n = \pi_3(\beta^{n-1} \alpha, \beta^n, \beta^{n-1}), \quad \text{for } n \geq 2.$$ 

Since $\pi_3 \neq \pi_1$ the tree morphism $\beta$ is different than any tree morphism $\beta^n$, for $n \geq 2$. On the other hand if $\beta^{n_1} = \beta^{n_2}$, for some $n_1, n_2 \geq 2$ then their sections at coordinate 2 must be equal, which forces $\beta^{n_1-1} = \beta^{n_2-1}$. Finite descent then shows that all positive powers of $\beta$ are distinct.

Further,

$$\alpha = \pi_3(\alpha, \alpha, \beta),$$

$$\beta^m \alpha = \pi_3(\beta^{m-1} \alpha, \beta^m \alpha, \beta^{m+1}), \quad \text{for } m \geq 1.$$ 

The powers of $\beta$ in coordinate 2 imply that all the elements $\beta^m \alpha$ are distinct for distinct values of $m$.

Finally, assuming $\beta^m \alpha = \beta^n$, for some $m$ and $n$, forces $2 \leq n = m + 2$, by comparing the sections at coordinate 2. However, $\alpha \neq \beta^2$ since they have different root transformation. For $m \geq 1$ the equality $\beta^m \alpha = \beta^{m+2}$ implies $\beta^{m-1} \alpha = \beta^{m+1}$, by comparing the sections at coordinate 1. Finite descent then finishes the proof. \qed

### 3 Transducer integer sequences

We first recall the well established notion of automatic sequence. The definition that follows is one of the equivalent definitions that can be found in [2].

A $k$-ary finite automaton with final state output ($k \geq 2$) is a 6-tuple $A = (Q, X_k, Y, s, \tau, \pi)$, where $Q$ is a finite set, called set of states, $X_k = \{0, \ldots, k - 1\}$ is the input alphabet, $Y$ is a finite set called the output alphabet, $s$ is an element in $Q$ called the initial state, $\tau : Q \times X \to Q$ is a map called transition map and $\pi : Q \to Y$ is a map called final state output map. Such an automaton defines an infinite sequence $y_0, y_1, y_2, \ldots$ over the output alphabet $Y$, called the final state output sequence of $A$, as follows. For a
natural number $i \geq 0$ let $[i]_k = i_0 \ldots i_m$ be any base $k$ representation of $i$ with $i = \sum_{j=1}^{m} i_j k^j$ (thus the least significant digit is written first). The term $y_i$ in the final state output sequence is defined as the image $\pi(q)$ of the state $q$ the automaton reaches as it reads the input word $[i]_k$ starting from the initial state $s$ (this output must be independent of the chosen representation of $i$). Thus

$$y_i = \pi(s, [i]_k),$$

where $\tau : Q \times X^* \to Q$ is the recursive extension of $\tau$ on $Q \times X^*$ defined by $\tau(q, \emptyset) = q$ and $\tau(q, xw) = \tau(\tau(q, x), w)$, for $q$ a state in $Q$, $x$ a letter in $X_k$ and and $w$ a word over $X_k$.

Automata with final state output can be represented by labeled directed graphs similar to the ones representing transducers. The only significant difference is that each state $q$ is labeled by the corresponding output letter $\pi(q)$ and the initial state is indicated by an incoming arrow. As an example, consider the ternary automaton $A_{0-2}$ in Figure 5.

![Figure 5: A ternary automaton with final state output $A_{0-2}$](image)

**Definition 3.1.** A $k$-ary automatic sequence is an infinite sequence that can be obtained as the final state output sequence of some $k$-ary finite automaton.

By Cobham Theorem [8] a sequence over a finite alphabet is a $k$-ary automatic sequence if and only if it is an image under a coding of a fixed point of a $k$-uniform endomorphism.

Given a free monoid $X^*$ over a finite alphabet $X$, an endomorphism $\alpha : X^* \to X^*$ can be uniquely defined by specifying the images of the letters in $X$ under $\alpha$. Let there exists a letter $x$ in $X$ such that $\alpha(x) = xw$, where $w$ is non-empty word, and let $\alpha(x) \neq \emptyset$, for all letters $x$ in $X$. Then, for all $n \geq 0$ the $n$-th iterate $\alpha^n(x)$ is a proper prefix of the $(n+1)$-st iterate $\alpha^{n+1}(x) = \alpha(\alpha^n(x))$ and the limit $\lim_{n \to \infty} \alpha^n(x)$ is a well defined infinite sequence over $X$. In the particular case when the length of all the words $\alpha(x)$, $x \in X$, is equal to $k$, the morphism $\alpha$ is called a $k$-uniform endomorphism.

As an example, let $X = \{1, -1\}$ and denote by $w_\alpha$ the infinite binary sequence

$$w_\alpha = \lim_{n \to \infty} \alpha^n(1) = 11-1 \ 11-1 \ 1-1-1 \ 11-1 \ 11-1 \ 1-1-1 \ 11-1 \ 1-1 \ 1-1-1 \ldots$$

obtained by iterations, starting from 1, of the endomorphism $\alpha : X^* \to X^*$ given by (compare to the sequence A080846)

$$1 \mapsto 11-1 \quad -1 \mapsto 1-1-1.$$

A finite or infinite word $w$ over an alphabet $X$ is cube free if it does not contain a subword of the form $u w u$, where $u$ is a nontrivial finite word over $X$.

**Proposition 3.1.** The infinite binary sequence $w_\alpha$ is cube-free.

**Proof.** By the criterion of Richomme and Wlazinski [21], an easy way to verify that $w_\alpha$ is cube free is to observe that $\alpha(11-1-11-11-111-111-111-111-11-1) \text{ is cube free.}$

We offer two additional descriptions of $w_\alpha$. 
Define a sequence of words $w_n$ of length $3^n$ by

$$w_0 = 1,$$
$$w_{n+1} = w_n w_n' w_n''_n,$$

where $w_n'$ is obtained from $w_n$ by changing the middle symbol in $w_n$ from 1 to -1.

**Proposition 3.2.** The limit $\lim_{n\to\infty} w_n$ is well defined and is equal to $w_a$.

For an integer $i \geq 0$, let $(i)_k = i_0 i_1 \ldots$ be the sequence of digits in base $k$ representation of $i$, where $i = \sum_{j=0}^{\infty} i_j k^j$ (the sequence ends in infinitely many 0’s).

Call a natural number $i$ a 2-before-0 number if the least significant digit in the ternary representation $(i)_3$ of $i$ that is different from 1 is 2. Otherwise the number is called a 0-before-2 number. Define an infinite binary sequence $x_0, x_1, x_2, \ldots$, by

$$x_i = \begin{cases} 1, & \text{if } i \text{ is a 0-before-2 number} \\ -1, & \text{if } i \text{ is a 2-before-0 number} \end{cases}.$$

**Proposition 3.3.** The infinite binary sequence $x_0, x_1, x_2, \ldots$, is equal to $w_a$.

**Proposition 3.4.** The infinite binary sequence $w_a$ is a ternary automatic sequence. It can be obtained as the final state output sequence of the automaton $A_{0-2}$.

**Proof.** The only time the automaton $A_{0-2}$ produces -1 in the output is if it reaches the state $a_2$, which only happens if $i$ is a 2-before-0 number. \qed

We define now the notion of transducer integer sequence.

**Definition 3.2.** A $k_1$ to $k_2$ transducer integer sequence is a sequence of integers $\{z_i\}_{i=0}^\infty$ such that there exists a $k_1$ to $k_2$ transducer $A$ and a state $q$ in $A$ such that, for every $i \geq 0$, the output word $q((i)_{k_1})$ is the base $k_2$ representation of $z_i$.

It is implicit in the above definition that the state $q$ of $A$ maps the confinality class of $0^\infty$ in $\partial X_{k_1}$ to the confinality class of $0^\infty$ in $\partial X_{k_2}$ (the confinality class of $0^\infty$ is just the set of infinite words ending in $0^\infty$).

We keep our attention only to this class since it is the one describing non-negative integers.

As an easy example, let $A_T$ be the ternary transducer in Figure 6. The state labeled by $\sigma_0$ just rewrites all digits to 0. Clearly

$$\sigma_1 (1^n 0 w) = \sigma_1 (1^n 2 w) = 0^n 10^\infty$$

for any word $w$ in the confinality class of $0^\infty$. Since $[0^n 1]_3 = 3^n$ the obtained integer sequence $\{a_n\}_{n=0}^\infty$ is (compare to sequence A038500)

$$1, 3, 1, 1, 9, 1, 1, 3, 1, 1, 3, 1, 1, 27, 1, 1, 3, 1, 1, 3, 1, 1, 9, 1, 1, 3, 1, \ldots.$$

By thinking of the powers of 3 as an (infinite) alphabet, this sequence can be thought of as the fixed point of the iterations starting from 1 of the 3-uniform endomorphism defined by

$$x \mapsto 1, 3x, 1.$$

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![Figure 6: A ternary transducer $A_T$](image)
This sequence can also be defined by blocks $a_{[n]}$ of length $3^n$ as

$$a_{[0]} = 1 \quad a_{[n+1]} = a_{[n]}a'_{[n]}a_{[n]}$$

where $a'_{[n]}$ is obtained from $a_{[n]}$ by multiplying the middle term by 3.

A more interesting example is provided by the automaton $A_L$.

Let $N_2$ be the set of all non-negative integers whose base 3 representation does not use the digit 2 (they are listed in sequence A005836). Define a sequence $\{\ell_n\}_{n=0}^\infty$, called $L$-sequence, by

$$\ell_n = \ell_n^+ + \ell_n^-$$

where $\ell_n^+$ and $\ell_n^-$ are the unique non-negative integers such that $\ell_n^+, \ell_n^-, \ell_n^+ + \ell_n^- \in N_2$ and $n = \ell_n^+ - \ell_n^-$ (sequence A060374).

**Proposition 3.5.** The $L$-sequence is a ternary transducer integer sequence. It is generated by the transducer $A_L$ with initial state $\alpha$.

*Proof.* When the current input digit of $n$ is 0, the corresponding digit in $\ell_n^-$ must be 0. Indeed if it were 1, then the corresponding digit in $\ell_n^+$ would be $0 + 1 = 1$, which would force the corresponding digit in $\ell_n$ to be 2. Thus the corresponding digit in $\ell_n^-$ is 0, and so are the digits in $\ell_n^+$ and $\ell_n$. This corresponds to the first column under $\alpha$ in the following table.

| $n$ | $\alpha$ | $\beta$ |
|-----|---------|---------|
| 0   | 0 1 2   | 0 1 2   |
| $\ell_n$ | 0 0 1 | 0 1 0 |
| $\ell_n^+$ | 0 1 0 | 1 0 0 |
| $\ell$ | 0 1 1* | 1 1* 0* |

Similarly, if the current input digit in $n$ is 1, the corresponding digit in $\ell_n^-$ must be 0, in $\ell_n^+$ must be 1 and in $\ell_n$ must be 1. If the current input digit in $n$ is 2 then the corresponding digit in $\ell_n^-$ must be 1. Indeed if it were 0 then the corresponding digit in $\ell_n^+$ would be $2 + 0 = 2$. Thus the corresponding digit in $\ell_n^-$ is 1, in $\ell_n^+$ is 0 and in $\ell_n$ is 1. However, in this case there is a carryover for the next digit (indicated by the * in the table. This is why a second state $\beta$ is introduced (this state corresponds to the situation in which there is a carryover in the addition $n + \ell_n^- = \ell_n^+$). The entries in the right half of the table (those corresponding to $\beta$) can be treated similarly, by taking into account the carryover.  

Let $\{p_n\}_{n=0}^\infty$ be the sequence defined by

$$p_0 = 0, \quad p_n = \sum_{i=0}^{n-1} w_i a_i, \quad \text{for } n \geq 1$$

where the sequence $\{w_n\}_{n=0}^\infty$ providing the signs is the cube free sequence generated by the automaton $A_{0-2}$ and $\{a_n\}_{n=0}^\infty$ is the transducer sequence generated by $A_T$.

**Proposition 3.6.** The sequence $\{p_n\}$ is equal to the $L$-sequence.

*Proof.* We have $p_0 = 0 = \ell_0$ and, for $n$ a positive integer and $w$ a word over $X_3$,

$$\alpha(0w + 1) = \alpha(1w) = 1\alpha(w) = 0\alpha(w) + 1 = \alpha(0w) + 1, \quad \alpha(1^n0w + 1) = \alpha(21^n-10w) = 11^n-11\alpha(w) = 1^n0\alpha(w) + 3^n = \alpha(1^n0w) + 3^n,$$

$$\alpha(1^n2w + 1) = \alpha(21^n-12w) = 11^n-10\beta(w) = 1^n1\beta(w) - 3^n = \alpha(1^n2w) - 3^n,$$

$$\alpha(2^n0w + 1) = \alpha(0^n1w) = 0^n1\alpha(w) = 10^n-11\alpha(w) - 1 = \alpha(2^n0w) - 1, \quad \alpha(2^n1w + 1) = \alpha(0^n2w) = 0^n1\beta(w) = 10^n-11\alpha(w) - 1 = \alpha(2^n1w) - 1.$$
In each case the change in the value of $\alpha(i)$ is exactly $w_ia_i$, i.e., for all $i$,

$$\ell_{i+1} = \alpha(i + 1) = \alpha(i) + w_ia_i = \ell_i + w_ia_i$$

and therefore the sequence of partial sums $\{p_n\}$ is exactly the $L$-sequence. \hfill $\square$

The sequence $\{\ell_n\}_{n=0}^\infty$ can also be described as a fixed point of an endomorphism over the alphabet consisting of the elements of $N_2$. The iterations start at 0 and the endomorphism is given by

$$0 \mapsto 0, 1 \quad x \mapsto 3x + 1, 3x, 3x + 1, \quad \text{for } x \geq 1.$$  

4 Relation to Hanoi Towers Problem

In this section we exhibit a connection between Hanoi Towers Problem, the automatic cube free sequence $\{w_n\}$ and the transducer sequence $\{a_n\}$.

Define a matrix $K_n$ of size $3^n \times n$ with entries in $X_3$ by

$$K_1 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \quad K_{n+1} = \begin{bmatrix} K_n \\ K_n R \\ K_n \end{bmatrix},$$

where the matrix $K_n^R$ is obtained from the matrix $K_n$ by flipping $K_n$ along the horizontal axis, and $0_n$, $1_n$ and $2_n$ are column vectors with $3^n$ entries equal to 0, 1 and 2, respectively. Denote the infinite limit matrix $\lim_{n \to \infty} K_n$ by $K$.

For example, the transpose of $K_3$ is given by

$$K_3^T = \begin{bmatrix} 012 & 210 & 012 & 210 & 012 & 210 & 012 \\ 000 & 111 & 222 & 222 & 111 & 000 & 000 \\ 000 & 000 & 000 & 000 & 111 & 111 & 111 \end{bmatrix}.$$  

The limiting matrix $K$ is well defined due to the fact that $K_n$ appears as the upper left corner in $K_{n+1}$. By definition, the indexing of the rows of $K$ starts with 0 while the indexing of the columns starts with 1.

A sequence $w_0, \ldots, w_{k-1}$ of words of length $n$ over $X_k$ is a $k$-ary Gray code of length $n$ if all words of length $n$ over $X_k$ appear exactly once in the sequence and any two consecutive words differ in exactly one position.

**Proposition 4.1.** The $3^n$ rows of the matrix $K_n$ represent a ternary Gray code of length $n$.

By interpreting the rows of $K$ as ternary representations of integers, we obtain the sequence

$$0, 1, 2, 5, 4, 3, 6, 7, 8, 17, 16, 15, 12, 13, 14, 11, 10, 9, \ldots,$$

which is not included in The On-Line Encyclopedia of Integer Sequences (as of December 2006).

We observe that the successive rows in $K$ are obtained from each other by applying the ternary tree automorphism $a$ at odd steps and $c$ at even steps (the automorphisms $a$ and $c$ are defined by $A_H$ - the automaton generating the Hanoi Towers group).

**Proposition 4.2.** For $j \geq 0$, define $t_{2j} = (ca)^j$ and $t_{2j+1} = a(ca)^j$. Let $k_i$ denote the $i$-th row in the matrix $K$. Then

$$k_i = t_i(k_0).$$

**Proof.** The proof is by induction on $n$ in $K_n$. The crucial observation for the inductive step is that the last row in $K_n$ is $2^n0^n\infty$ and is obtained by applying $c$ in step $3^n - 1$. In the next step applying $a$ to $2^n0^n\infty$ produces $2^n1^n\infty$. Alternate applications of $c$ and $a$ ($3^n - 1$ total) do not affect the 1 in the position $n+1$, but backtrack the word in the first $n$ entries from $2^n$ back to $0^n$, thus producing $0^n1^n\infty$ at step $3^n - 1 + 1 + 3^n - 1 = 2 \cdot 3^n - 1$. The last taken step is $a$ so $c$ takes $0^n1^n\infty$ to $0^n2^n\infty$ and then alternate applications of $a$ and $c$ change the first $n$ entries again from $0^n$ to $2^n$ in $3^n - 1$ steps, eventually producing $2^n+10^n\infty$ in $2 \cdot 3^n - 1 + 1 + 3^n - 1 = 3^{n+1} - 1$ steps, alternating between $a$ and $c$. \hfill $\square$
It is clear that the rows of $K_{i}$ constitute the whole confinality class of $0^{\infty}$. Thus the subgroup $\langle a, c \rangle$ acts transitively on this class. Since the order of both $a$ and $c$ is 2 this means that $\langle a, c \rangle$ is the infinite dihedral group $D_{\infty}$. The transitivity of the action of $\langle a, c \rangle$ on the confinality class of $0^{\infty}$ is equivalent to the known fact that any valid $n$ disk configuration can be obtained from any other in a restricted version of Hanoi Towers Problem in which no disk can move between pegs 0 and 2 (in our terminology, applications of the automorphism $b$ are not allowed). Figure 7 shows the path taken by $(ca)^{13}$ from 000 to 222 in $\Gamma_{3}$.

Order all configurations (words in the confinality class of $0^{\infty}$) according to their position in the matrix $K$ (small configurations correspond to rows with small index). When $b$ is applied to any configuration $k_{i}$ the obtained configuration $b(k_{i})$ is either larger or smaller than $k_{i}$. Based on this alternative define an infinite sequence $\{d_{i}\}_{i=0}^{\infty}$ over $X = \{1,-1\}$ by

$$d_{i} = \begin{cases} 1, & \text{if } b(k_{i}) > k_{i} \\ -1, & \text{if } b(k_{i}) < k_{i} \end{cases}$$

Call this sequence the $b$-direction sequence. Further, define an integer sequence $\{b_{i}\}_{i=0}^{\infty}$ by $b_{i} = |i-j|$, where $j$ is the index of the configuration $k_{j} = b(k_{i})$. Call this sequence the $b$-change index sequence.

**Proposition 4.3.** The $b$-direction sequence is exactly the cube free automatic sequence $\{w_{n}\}$ generated by $A_{0-2}$ and the $b$-change index sequence is exactly the transducer integer sequence $\{a_{n}\}$ generated by $A_{T}$.

**Proof.** The proof is by induction on blocks of size $3^{n}$. Observe that in each matrix $K_{n}$ the configuration which is half way between $0^{n}$ and $2^{n}$ is $1^{n}$. The size $3^{n}$ blocks of the $b$-change sequence satisfy a relation of the form $b_{i+1} = b'_{i} b''_{i} b'''_{i}$, where $b'_{i}$, $b''_{i}$ and $b'''_{i}$ are obtained from $b_{i}$ by possible changes in the

![Figure 7: The ternary Gray code path generated by $a$ and $c$ in $H^{(3)}$ at level 3](image-url)
middle term, corresponding to the configuration $1^n0$, $1^n1$ and $1^n2$, respectively. The reason is that all other configurations contain 0 or 2 in a position before $n+1$ and therefore the changes made by the automorphism $b$ are already accounted for in the sequence $b_{[n]}$. Since $b(1^n0) = 1^n2$, $b(1^{n+1}0) = 1^{n+2}$, and $b(1^n2) = 1^n0$ and the distance between $1^n0$ and $1^n2 = b(1^n0)$ along the ca path is $(3^n-1)/2+1+(3^n-1)+1+(3^n-1)/2 = 3^{n+1}$, we see that the only change is that the middle term in $b'_{[n]}$ is multiplied by 3. Similarly, the $3^n$ size blocks of the $b$-change index sequence satisfy a relation of the form $b_{[n+1]} = b'_{[n]} b''_{[n]}$. However, the changes in $b(1^n0) = 1^n2$ and $b(1^{n+1}0) = 1^{n+2}$ are in the positive direction, while the change in $b(1^n2) = 1^n0$ is in the negative direction (we are just traveling along the same $b$ edge as in $b(1^n0) = 1^n2$ but in the opposite direction).

\section{Optimal configurations in Hanoi Towers Problem}

Define a matrix $M_n$ of size $2^n \times n$ with entries in $X_2$ by

\[ M_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad M_{n+1} = \begin{bmatrix} M_n & 0_n \\ M_R^n & 1_n \end{bmatrix}, \]

where the matrix $M_R^n$ is obtained from the matrix $M_n$ by flipping $M_n$ along the horizontal axis, and $0_n$ and $1_n$ are column vectors with $2^n$ entries equal to 0 and 1, respectively. The $2^n$ rows of the matrix $M_n$ represent a binary Gray code of length $n$. Denote the infinite limit matrix $\lim_{n \to \infty} M_n$ by $M$. We observe that the successive rows in $M$ are obtained from each other by applying the binary tree automorphism $f$ at odd steps and the automorphism $g$ at even steps, where $f$ and $g$ are given by the invertible transducer $A_D$ defined by $A_D$ and generated by $f$ and $g$ is the infinite dihedral group $D_{\infty}$.

\[ A_D : \quad 0 \xrightarrow{0} \text{id} \quad f(01) \xrightarrow{1} 0 \quad g(0) \xrightarrow{0} 0 \quad \lambda_0 \xrightarrow{0} 0 \quad \lambda_1(01) \xrightarrow{1} 1 \]

\[ A_{L_2} : \quad 0 \xrightarrow{0} \text{id} \quad f(01) \xrightarrow{1} 0 \quad g(0) \xrightarrow{0} 0 \quad \lambda_0 \xrightarrow{0} 0 \quad \lambda_1(01) \xrightarrow{1} 1 \]

Figure 8: Two binary invertible transducers: $A_D$ and $A_{L_2}$

\begin{proposition}
For $j \geq 0$, define $s_{2j} = (gf)^j$ and $s_{2j+1} = f(gf)^j$. Let $m_i$ denote the $i$-th row in the matrix $M$. Then

\[ m_i = s_i(m_0). \]

\end{proposition}

Consider the transducer in the right half of Figure 8. It is known \cite{17} (see also \cite{23} \cite{6}) that the group $G(A_{L_2})$ is the lamplighter group $L_2$ which is the wreath product of the cyclic group of order 2 (representing a switch) and the infinite cyclic group (representing moves between consecutive lamps). The realization of the lamplighter group $L_2$ by the automaton $A_{L_2}$ was used by Grigorchuk and Zuk \cite{17} to calculate the spectrum of the Markov operator on the Cayley graph of $L_2$, which then lead to the solution of Strong Atiyah Conjecture in \cite{15}.

\begin{proposition}
For $i = 2^n - 1$, the row $i$ word $m_i(n)$ in the matrix $M_n$ (the $i$-th Gray code word of length $n$) is equal to $\lambda_0([i]_2)^R$, where $R$ denotes word reversion and $[i]_2$ is the length $n$ representative of $i$ (including leading zeros, if necessary).

\end{proposition}

We can define a variation on the notion of transducer integer sequences as sequences that can be obtained from transducers by reading the input starting from the most significant digit (and interpreting the output as starting from the most significant digit). Call these sequences SF transducer sequences (for significant
first). Since the sequence of binary Gray code words can be obtained by feeding the binary representations of integers, most significant digit first, into \( A_{L_2} \) starting at \( \lambda_0 \), we see that the sequence A003188 of integers

\[
0, 1, 3, 2, 6, 7, 5, 4, \ldots
\]

represented by the binary Gray code words is a SF binary transducer sequence. On the other hand, this sequence is not a binary transducer sequence. This is clear since the transformation \((i)_2 \rightarrow m_i\) does not preserve prefixes. Namely \(0^\infty \rightarrow 0^\infty\), while \(010^\infty \rightarrow 110^\infty\).

We offer two 2 to 3 transducers each of which generates all the configurations on the geodesic lines between the regular configurations \(0^n, 1^n\) and \(2^n\) (depending on chosen initial state). The first one uses the order prescribed by the binary Gray code, while the other uses the natural order.

**Proposition 5.3.** The transducer \( \mathcal{O}_H \) in Figure 9 generates the optimal configurations in Hanoi Towers Problem. More precisely, for \( x, y \in \{0, 1, 2\} \), \( x \neq y \), starting at state \( t_{xy} \), and feeding the reversal \( m_i(n)^R \) of the length \( n \) row \( i \) binary Gray code word from \( M_n \) into \( \mathcal{O}_H \) produces the reverse of the length \( n \) ternary word representing the unique \( n \) disk configuration at distance \( i \) along the geodesic from \( x^n \) to \( y^n \) in \( \Gamma_n \).

\[
\mathcal{O}_H : \quad \begin{array}{cccc}
0/0 & 0/0 & 0/1 & 0/1 \\
1/0 & 1/1 & 0/2 & 0/2 \\
t_{01} & t_{10} & t_{12} & t_{20} \\
t_{02} & t_{10} & t_{12} & t_{21} \\
1/2 & 1/2 & & \\
\end{array}
\]

Figure 9: A ternary transducer generating/recognizing optimal configurations

**Proof.** For any permutation \( x, y, z \) of the three letters in \( X_3 \) the states of the transducer \( \mathcal{O}_H \) have (as tree morphisms) the decomposition

\[
t_{xy} = \pi_{xy}(t_{xz}, t_{yz}),
\]

where \( \pi_{xy} = \left( \begin{array}{c}
0 \\
x \\
y
\end{array} \right) \).

It is well known that the unique geodesic path of length \( 2^n - 1 \) from \( x^n \) to \( y^n \) connects \( x^n \) to \( z^{n-1}x \) in the first \( 2^{n-1} - 1 \) steps, then in the next step the largest disk is moved to get the configuration \( z^{n-1}y \) and then the last \( 2^{n-1} - 1 \) steps are used to connect \( z^{n-1}y \) to \( y^n \).

Since we want to use Gray code words to describe the configurations along the way, we observe that in the first part of the geodesic from \( x^n \) to \( y^n \) (corresponding to the last digit in the Gray code being 0) the last digit in the reached configurations is \( x \), while in the second part (corresponding to the last digit in the Gray code being 1) the last digit in the reached configuration is \( y \). This explains the root transformations in the above decomposition.

As for the sections, in the first part of the geodesic (last digit 0 in the Gray code) the configurations corresponding to the first \( n - 1 \) digits describe the path from \( x^{n-1} \) to \( z^{n-1} \), while in the second part (last digit 1 in the Gray code) the configurations corresponding to the first \( n - 1 \) digits describe the path from \( z^{n-1} \) to \( y^{n-1} \) in the natural order and from \( y^{n-1} \) to \( z^{n-1} \) in the Gray code word order (because of the flip in the second half of the Gray code). Thus the section at 0 is \( t_{xz} \) and the section at 1 is \( t_{yz} \).

It is apparent from the above proof that the following is also true.
Proposition 5.4. The transducer $O'_H$ in Figure 10 generates the optimal configurations in Hanoi Towers Problem. More precisely, for $x, y \in \{0, 1, 2\}$, $x \neq y$, starting at state $q_{xy}$, and feeding the reversal $[i]_2^R$ of the length $n$ binary representative of $i$ (including leading 0’s if needed) into $O'_H$ produces the reverse of the length $n$ ternary word representing the unique $n$ disk configuration at distance $i$ along the geodesic from $x^n$ to $y^n$ in $\Gamma_n$.

Proof. Observe that, for any permutation $x, y, z$ of the three letters in $X_3$ the states of the transducer $O'_H$ have (as tree morphisms) the decomposition $q_{xy} = \pi_{xy}(q_{xz}, q_{zy})$, where $\pi_{xy} = \begin{pmatrix} 0 & 1 \\ x & y \end{pmatrix}$. This is precisely the decomposition that corresponds to the natural order in the previous proof.

The automaton $O'_H$, started at $q_{01}$, generates the sequence A055661

$$0, 1, 7, 8, 17, 15, 12, 13, \ldots,$$

but only when all input words are adjusted by leading zeros to have odd length, and it gives the sequence

$$0, 2, 5, 4, 22, 21, 24, 26, \ldots,$$

which does not appear in The On-Line Encyclopedia of Integer Sequences (as of December 2006), when the input words are adjusted to have even length. In fact, the former sequence records the integers whose ternary representations give the configurations in the Hanoi Towers Problem on the geodesic line in the infinite Schreier graph $\Gamma_0^\infty$ determined by applying repeatedly the automorphisms $a$, $b$ and $c$ (in that order) and the latter records the integers whose ternary representations give the configurations on the geodesic line in $\Gamma_0^\infty$ determined by applying repeatedly the automorphisms $b$, $a$ and $c$ (in that order). There is nothing strange in this split, since it is known that the optimal solution transferring disks to peg 1 follows different paths depending on the parity of the number of disks.

By flipping the input and the output symbol in the automata $O_H$ and $O'_H$ we obtain two automata that can be used to recognize the configurations on the geodesic lines between $0^\infty$, $1^\infty$ and $2^\infty$ and encode them either by using the Gray code words or binary representations.
More generally, when $A = (Q, X_k, X_3, \tau, \pi)$ is injective transducer one can define a partial inverse transducer $A^{-1} = (Q^{-1}, X_3, X_k, \tau^{-1}, \pi^{-1})$ in which $Q = \{ q^{-1} \mid q \in Q \}$, and $\tau : Q^{-1} \times X_3 \to Q^{-1}$ and $\pi : Q^{-1} \times X_k \to X_k$, are partial maps, defined by $\tau^{-1}(q^{-1}, y) = p$ and $\pi^{-1}(p^{-1}, y) = x$ whenever $\tau(q, x) = p$ and $\pi^{-1}(p, x) = y$.

**Proposition 5.5.** The inverse transducer $O_{H}^{-1}$, recognizes the optimal configurations in Hanoi Towers Problem. More precisely, starting at the inverse state $q_{xy}^{-1}$, $x, y \in X_3$, $x \neq y$, and feeding ternary words of length $n$ into the inverse transducer $O_{H}^{-1}$, only reversals of ternary words representing configurations on the geodesic from $x^n$ to $y^n$ in $\Gamma_n$ are read entirely by the transducer and, for such configurations, the output of the corresponding binary Gray code word of length $n$ is produced in the output.

**Proposition 5.6.** The inverse transducer $O_{H'}^{-1}$, recognizes the optimal configurations in Hanoi Towers Problem. More precisely, starting at the inverse state $q_{xy}^{-1}$, $x, y \in X_3$, $x \neq y$, and feeding ternary words of length $n$ into the inverse transducer $O_{H'}^{-1}$, only reversals of ternary words representing the configurations on the geodesic from $x^n$ to $y^n$ in $\Gamma_n$ are read entirely by the transducer and, for such configurations, the output represents reversals of the binary representation of the distance to $x^n$.

For example, the configuration 10021 is not accepted starting from the state $q_{01}^{-1}$ (after it is fed into $O_{H'}^{-1}$ as 12001 it stops after reading the first 4 symbols in state $q_{20}^{-1}$ and it cannot read the last symbol). This simply means that this configuration is not on the geodesic between $0^5$ and $1^5$. On the other hand, 20021 is read completely and it produces the output 01101, which says that the configuration 20021 is on the geodesic between $0^5$ and $1^5$ and its distance to $0^5$ is $2 + 4 + 16 = 22$. If we read 20021 starting at state $q_{10}^{-1}$ in $O_{H'}^{-1}$ we obtain the output 10010, which confirms that the configuration 20021 is on the geodesic between $1^5$ and $0^5$ and that its distance to $1^5$ is $1 + 8 = 9$.

**References**

[1] J.-P. Allouche, J. Bétréma, and J. O. Shallit. Sur des points fixes de morphismes d’un monoïde libre. *RAIRO Inform. Théor. Appl.*, 23(3):235–249, 1989.

[2] Jean-Paul Allouche and Jeffrey Shallit. *Automatic sequences*. Cambridge University Press, Cambridge, 2003.

[3] Laurent Bartholdi, Rostislav Grigorchuk, and Volodymyr Nekrashevych. From fractal groups to fractal sets. In *Fractals in Graz 2001*, Trends Math., pages 25–118. Birkhäuser, Basel, 2003.

[4] Laurent Bartholdi, Rostislav I. Grigorchuk, and Zoran Šunič. Branch groups. In *Handbook of algebra, Vol. 3*, pages 989–1112. North-Holland, Amsterdam, 2003.

[5] Laurent Bartholdi and Volodymyr Nekrashevych. Thurston equivalence of topological polynomials. (to appear in Acta Mathematica), 2006.

[6] Laurent Bartholdi and Zoran Šunič. Some solvable automaton groups. In *Topological and asymptotic aspects of group theory*, volume 394 of *Contemp. Math.*, pages 11–29. Amer. Math. Soc., Providence, RI, 2006.

[7] Laurent Bartholdi and Bálint Virág. Amenability via random walks. *Duke Math. J.*, 130(1):39–56, 2005.

[8] Alan Cobham. Uniform tag sequences. *Math. Systems Theory*, 6:164–192, 1972.

[9] R. I. Grigorchuk. On Burnside’s problem on periodic groups. *Funktsional. Anal. i Prilozh.*, 14(1):53–54, 1980.

[10] R. I. Grigorchuk, V. V. Nekrashevich, and V. I. Sushchanskiı. Automata, dynamical systems, and groups. *Tr. Mat. Inst. Steklova*, 231(Din. Sist., Avtom. i Beskon. Gruppy):134–214, 2000.
[11] Rostislav Grigorchuk, Volodymyr Nekrashevych, and Zoran Šunić. Hanoi towers group on 3 pegs and its pro-finite closure. Oberwolfach Reports, 25:15–17, 2006.

[12] Rostislav Grigorchuk, Volodymyr Nekrashevych, and Zoran Šunić. Hanoi towers groups. Oberwolfach Reports, 19:11–14, 2006.

[13] Rostislav Grigorchuk and Zoran Šunić. Self-similarity and branching in group theory. to appear in Lecture Notes of London Mathematical Society 339, 2003.

[14] Rostislav Grigorchuk and Zoran Šunić. Asymptotic aspects of Schreier graphs and Hanoi Towers groups. C. R. Math. Acad. Sci. Paris, 342(8):545–550, 2006.

[15] Rostislav I. Grigorchuk, Peter Linnell, Thomas Schick, and Andrzej Żuk. On a question of Atiyah. C. R. Acad. Sci. Paris Sér. I Math., 331(9):663–668, 2000.

[16] Rostislav I. Grigorchuk and Andrzej Żuk. On the asymptotic spectrum of random walks on infinite families of graphs. In Random walks and discrete potential theory (Cortona, 1997), Sympos. Math., XXXIX, pages 188–204. Cambridge Univ. Press, Cambridge, 1999.

[17] Rostislav I. Grigorchuk and Andrzej Żuk. The lamplighter group as a group generated by a 2-state automaton, and its spectrum. Geom. Dedicata, 87(1-3):209–244, 2001.

[18] Rostislav I. Grigorchuk and Andrzej Żuk. On a torsion-free weakly branch group defined by a three state automaton. Internat. J. Algebra Comput., 12(1-2):223–246, 2002.

[19] Andreas M. Hinz. The Tower of Hanoi. Enseign. Math. (2), 35(3-4):289–321, 1989.

[20] Volodymyr Nekrashevych. Self-similar groups, volume 117 of Mathematical Surveys and Monographs. American Mathematical Society, Providence, RI, 2005.

[21] G. Richomme and F. Wlazinski. Some results on $k$-power-free morphisms. Theoret. Comput. Sci., 273(1-2):119–142, 2002.

[22] Said Sidki. Automorphisms of one-rooted trees: growth, circuit structure, and acyclicity. J. Math. Sci. (New York), 100(1):1925–1943, 2000.

[23] P. V. Silva and B. Steinberg. On a class of automata groups generalizing lamplighter groups. to appear in Internat. J. Algebra Comput.

[24] John S. Wilson. On exponential growth and uniformly exponential growth for groups. Invent. Math., 155(2):287–303, 2004.

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