Continuous vacua in bilinear soliton equations

J. Hietarinta* and A. Ramani
CPT, Ecole Polytechnique
91128 Palaiseau, France
and
B. Grammaticos
LPN, Université Paris VII, Tour 24-14, 5ème étage
2 Place Jussieu, 75251 Paris, France

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Abstract

We discuss the freedom in the background field (vacuum) on top of which the
solitons are built. If the Hirota bilinear form of a soliton equation is given by
\[ A(D_x)G - F = 0, \quad B(D_x)(F \cdot F - G \cdot G) = 0 \]
where both \( A \) and \( B \) are even polynomials in their variables, then there can be a
continuum of vacua, parametrized by a vacuum angle \( \phi \). The ramifications of this
freedom on the construction of one- and two-soliton solutions are discussed. We
find, e.g., that once the angle \( \phi \) is fixed and we choose \( u = \arctan \frac{G}{F} \)
as the physical quantity, then there are four different solitons (or kinks) connecting
the vacuum angles \( \pm \phi, \pm \phi \pm \frac{\pi}{2} \) (defined modulo \( \pi \)).
The most interesting result is the existence of a “ghost” soliton; it goes over to the
vacuum in isolation, but interacts with “normal” solitons by giving them a finite
phase shift.

1 Introduction

The existence of multisoliton solutions has been considered as a very strong
indication of complete integrability of nonlinear evolution equations. The main
tool for this was created by Hirota, who has proposed his bilinear formalism based on the
observation that, expressed in the right variables, soliton solutions are just
polynomials in exponentials [1]. Indeed, when one starts from the solitary
wave solution

\[ u = \frac{k^2/2}{\cosh((kx - pt)/2)^2}, \quad k^3 - p = 0, \]

of the KdV equation

\[ u_t - 6uu_x + u_{xxx} = 0, \]

*permanent address: Department of Physics, University of Turku, 20500 Turku, Finland
and introduce the new dependent variable $F$ by $u = 2\partial_x^2 \log F$ one obtains simply $F = 1 + e^{kx - pt}$.

The KdV equation itself can be cast in the bilinear form using the new variable $F$. From the potential form of (2), which is $v_t - v_x^2 + v_{xxx} = 0$, where $v_x = u$, we find:

$$FF_t + FF_{xxx} - 4F_x F_{xx} + 3F_{xx}^2 = 0,$$

(3)

and using Hirota’s $D$ operators defined through:

$$D_x^a D_t^b \ldots F \cdot G = (\partial_x - \partial'_x)^a (\partial_t - \partial'_t)^b \ldots F(x,t,\ldots)G(x',t',\ldots)|_{x'=x,t'=t,\ldots},$$

(4)

we can write (3) as

$$(D_x^4 + D_x D_t) F \cdot F = 0.$$

(5)

Expressed in this new dependent variable $F$ the multisoliton solutions of the KdV equation can be systematically constructed (with $\eta = kx - pt$):

0): zero soliton (vacuum): $F = 1$

1): one soliton : $F = 1 + e^\eta$

2): two solitons : $F = 1 + e^{\eta_1} + e^{\eta_2} + A_{12} e^{\eta_1+\eta_2}$ with $A_{12} = \frac{k_1 - k_2}{k_1 + k_2}$

3): three solitons : $F = 1 + e^{\eta_1} + e^{\eta_2} + e^{\eta_3} + A_{12} e^{\eta_1+\eta_2} + A_{13} e^{\eta_1+\eta_3} + A_{23} e^{\eta_2+\eta_3} + A_{12} A_{13} A_{23} e^{\eta_1+\eta_2+\eta_3}$

etc.

We remark that the three-solitons solution does not contain any new free parameters, i.e. once $A_{ij}$ is fixed at the two-soliton level the parameters of the three- (and higher) soliton solutions are fixed. Thus, for the existence of these higher order solutions it is necessary that certain compatibility conditions be satisfied.

Returning to (3) we emphasize that there exists a solution corresponding to a ‘vacuum’, i.e. the absence of solitons. One can see that $F = \text{constant} \neq 0$ is a solution which (due to bilinearity) can be normalized to $F = 1$. Thus, the vacuum in the case of KdV is uniquely defined.

However, the vacuum solution is not always unique. In the following we will show that for certain multicomponent bilinear equations there is a continuum of vacua which cannot be scaled away. In this case the solitons turn out to interpolate between different vacua, and there is one instance where the soliton is a “ghost”: It is not visible alone, but only in interaction with others. This multiplicity of vacua puts also extra constraints on the existence of soliton solutions and thus on the integrability of the equation.

2 Continuous vacua for two-component bilinear equations

Bilinear equations can be classified according to the number of dependent functions one has to introduce in order to write them under Hirota’s form. Following the analysis in [2],
we concentrate on two generic classes of two-component bilinear equations which write:

\[
\begin{align*}
A(D\vec{x})G \cdot F &= 0, \\
B(D\vec{x})G \cdot F &= 0,
\end{align*}
\]

where \(A\) and \(B\) are respectively even and odd polynomials in the \(D\) operator (the modified-KdV family) and

\[
\begin{align*}
A(D\vec{x})(F \cdot F - G \cdot G) &= 0, \\
B(D\vec{x})F \cdot G &= 0,
\end{align*}
\]

where \(A\) and \(B\) are both even (the sine-Gordon family).

The even-odd case is ‘standard’ in that only usual vacua exist. We can write \(F = s, G = c\), but due to bilinearity we can scale this to \(F = 1, G = 1\), or \(F = 1, G = 0\), or \(F = 0, G = 1\) so that no continuous parameters remain.

Let us now focus on the even-even case (7), and look for the zero-soliton solution. Two types of standard vacua can readily be found. If \(A(0) = 0, B(0) \neq 0\) then \(G = 0, F = 1\) and \(F = 0, G = 1\) are the only possible vacua. Also if \(A(0) \neq 0, B(0) = 0\) we must take \(F = G = 1\) or \(F = -G = 1\). Note that if neither \(A(0)\) nor \(B(0)\) vanish there are no simple zero-soliton solution at all.

A novel possibility exists if both \(A(0)\) and \(B(0)\) vanish in (7). Then if \(F\) and \(G\) are both constants, one gets a vacuum solution, whatever their ratio. We can use a symmetric parametrisation of the continuous vacuum solution through

\[
F = \cos \phi, \quad G = \sin \phi,
\]

with \(\phi\) a free parameter.

The construction of a one-soliton solution on top of one of the standard vacua is straightforward and has been discussed e.g. in [2]. The question we will now address is whether one can find one-soliton solutions on top of the new vacuum (8). Let us start with the ansatz:

\[
F = c + Ce^\eta, \quad G = s + Se^\eta,
\]

where \(c = \cos \phi, s = \sin \phi, \eta = kx - pt + \text{const}\). Substituting into (9) we obtain the following conditions:

\[
B(\vec{p})(sC + cS) = 0, \quad A(\vec{p})(cC - sS) = 0,
\]

where \(\vec{p} = (k, p)\). Three cases can be distinguished, corresponding to three different types of solitons:

\(\alpha\) \(A(\vec{p}) = 0 \text{ and } B(\vec{p}) \neq 0\) then \(C = sM, S = cM\) for some constant \(M\). This constant can be absorbed into the constant in \(\eta\), so that

\[
F = c(1 + e^\eta), \quad G = s(1 - e^\eta).
\]

\(\beta\) \(A(\vec{p}) \neq 0 \text{ and } B(\vec{p}) = 0\) then

\[
F = c + se^\eta, G = s + ce^\eta.
\]
A typical situation where a $\gamma$-type soliton exists is when the Hirota polynomials $A$ and $B$ have a common factor $U$: $A = UV$, $B = UW$ and thus the dispersion manifold common to both $A$ and $B$ contains $U(\vec{p}) = 0$. Moreover, in many cases, (in particular for known integrable even-even systems), $U$ is just linear i.e. $U(\vec{p}) = \vec{λ} \cdot \vec{p}$. This will have further implications on the 2-solitons solution as we will see in the next section.

3 Constraints from the existence of two-soliton solutions

In addition to the continuous parametrization of the vacuum there is a further discrete ambiguity. We can observe this already from (11,12) when we look at the physical quantity $u = \arctan G/F$: It has different (constant) values at $\eta \to \pm \infty$, which suggest different vacua there. This will be discussed further in Sec. 5, but before doing that let us see what the formal requirement for the existence of two-soliton solution yields.

Let us start by combining solutions of the types $\alpha$ and $\beta$ above. First, for $\alpha + \alpha$ (which means that the dispersion relations are $A(\vec{p}_1) = A(\vec{p}_2) = 0$ for both of the individual solitons), the ansatz for $F$ and $G$ write:

\begin{align}
F &= c + ce^{h_1} + ce^{h_2} + Ke^{\eta_1 + \eta_2}, \\
G &= s - se^{h_1} - se^{h_2} + Le^{\eta_1 + \eta_2}.
\end{align}

and then we obtain from (11) $K = cM$, $L = sM$ with

\begin{equation}
M = -\frac{A(\vec{p}_1 - \vec{p}_2)}{A(\vec{p}_1 + \vec{p}_2)} = \frac{B(\vec{p}_1 - \vec{p}_2)}{B(\vec{p}_1 + \vec{p}_2)}.
\end{equation}

This implies in particular a condition for the existence of the two-solitons solutions: On the manifold $A(\vec{p}_1) = A(\vec{p}_2) = 0$ we must have:

\begin{equation}
A(\vec{p}_1 + \vec{p}_2)B(\vec{p}_1 - \vec{p}_2) + A(\vec{p}_1 - \vec{p}_2)B(\vec{p}_1 + \vec{p}_2) = 0.
\end{equation}

This condition reflects the fact that generically even-even bilinear equations do not have two-solitons solutions.

The $\beta + \beta$ solution leads to analogous results:

\begin{align}
F &= c + se^{h_1} + se^{h_2} - cMe^{\eta_1 + \eta_2}, \\
G &= s + ce^{h_1} + ce^{h_2} - sMe^{\eta_1 + \eta_2}.
\end{align}

with $M$ given by (14) and (15) as the existence condition, but now it should be understood on a different dispersion manifold, namely $B(\vec{p}_1) = B(\vec{p}_2) = 0$.

The case $\alpha + \beta$ is treated in a similar way. We start from

\begin{align}
F &= c + ce^{h_1} + ce^{h_2} + Ke^{\eta_1 + \eta_2}, \\
G &= s - se^{h_1} + ce^{h_2} + Le^{\eta_1 + \eta_2}.
\end{align}
on \( A(\vec{p}_1) = B(\vec{p}_2) = 0 \) (or \( B(\vec{p}_1) = A(\vec{p}_2) = 0 \)) and find \( K = -sM, L = cM \) with (14) and (15) as the compatibility condition (on a still different dispersion manifold). Thus for the standard type solitons \( \alpha \) and \( \beta \) the construction of two soliton solutions is straightforward and leads to the condition (15) on a suitable manifold.

More interesting are the cases where a \( \gamma \)-type soliton is involved. Let us combine one \( \alpha \)-type soliton with a \( \gamma \)-type one. The dispersion manifold in this case is \((A(\vec{p}_1) = 0) \cap (A(\vec{p}_2) = 0) \cap (B(\vec{p}_2) = 0) \) or \((A(\vec{p}_1) = 0) \cap (B(\vec{p}_1) = 0) \cap (A(\vec{p}_2) = 0) \). Let us choose the first case and write the solution as:

\[
F = c + ce^{\eta_1} + Ce^{\eta_2} + Ke^{\eta_1+\eta_2},
\]

\[
G = s - se^{\eta_1} + Se^{\eta_2} + Le^{\eta_1+\eta_2}.
\]

We readily find that \( K = CM, L = -SM \), with three possibilities for \( C, S \) and \( M \), namely:

1) type \( \gamma_1 \): \( C = c, S = s \), and

\[
M = \frac{A(\vec{p}_1 - \vec{p}_2)}{A(\vec{p}_1 + \vec{p}_2)}.
\]

2) type \( \gamma_2 \): \( C = s, S = -c \), and

\[
M = \frac{B(\vec{p}_1 - \vec{p}_2)}{B(\vec{p}_1 + \vec{p}_2)}.
\]

3) type \( \gamma_3 \): \( C \) and \( S \) are free and \( M \) given by (14) and therefore (15) must be satisfied again on the appropriate dispersion manifold.

Similar conclusions can be reached in the \( \beta + \gamma \) case, mutatis mutandis.

Let us finally consider the interaction of two \( \gamma \)-type solitons. Here, the dispersion manifold is: \((A(\vec{p}_1) = 0) \cap (B(\vec{p}_1) = 0) \cap (A(\vec{p}_2) = 0) \cap (B(\vec{p}_2) = 0) \). The two-soliton solution is written

\[
F = c + C_1e^{\eta_1} + C_2e^{\eta_2} + Ke^{\eta_1+\eta_2},
\]

\[
G = s + S_1e^{\eta_1} + S_2e^{\eta_2} + Le^{\eta_1+\eta_2}.
\]

For notational convenience, we introduce the quantities \( \varphi \) and \( \psi \) through:

\[
K = s\varphi + c\psi, \quad L = c\varphi - s\psi,
\]

and readily find

\[
\varphi = -(C_1S_2 + C_2S_1) \frac{B(\vec{p}_1 - \vec{p}_2)}{B(\vec{p}_1 + \vec{p}_2)},
\]

\[
\psi = -(C_1C_2 - S_1S_2) \frac{A(\vec{p}_1 - \vec{p}_2)}{A(\vec{p}_1 + \vec{p}_2)}
\]

provide that \( A(\vec{p}_1 + \vec{p}_2) \neq 0 \) and \( B(\vec{p}_1 + \vec{p}_2) \neq 0 \).
However, if the two $\gamma$-type solitons are obtained for $A = UV$, $B = UW$ through $U(\vec{p}_1) = U(\vec{p}_2) = 0$, and if moreover $U$ is linear, then $U(\vec{p}_1 \pm \vec{p}_2) = 0$ as well and $A(\vec{p}_1 \pm \vec{p}_2) = B(\vec{p}_1 \pm \vec{p}_2) = 0$. In that case with $F$ and $G$ given by (19) one has $A(F \cdot F - G \cdot G) = BF \cdot G = 0$ for arbitrary $C_i$, $S_i$, $K$ and $L$, and $\varphi$ and $\psi$ are still totally free at this stage.

As we have seen above, the quantities $C_i$ and $S_i$ $(i = 1, 2)$ are in general fixed by the interaction of the $\gamma$-type solitons with $\alpha$- and $\beta$-type ones, except in the special case of $\gamma_3$ where (15) would also be satisfied on the appropriate dispersion manifold. Thus if $A(\vec{p}_1 + \vec{p}_2) \neq 0$ and $B(\vec{p}_1 + \vec{p}_2) \neq 0$ the only possible values of $C_i$, $S_i$ and consequently $\varphi$ and $\psi$ are:

a) $\gamma_1 + \gamma_1$: $C_1 = C_2 = c$, $S_1 = S_2 = s$ and

$$\varphi = -2sc \frac{B(\vec{p}_1 - \vec{p}_2)}{B(\vec{p}_1 + \vec{p}_2)}, \; \psi = (s^2 - c^2) \frac{A(\vec{p}_1 - \vec{p}_2)}{A(\vec{p}_1 + \vec{p}_2)}$$

b) $\gamma_1 + \gamma_2$: $C_1 = c$, $C_2 = s$, $S_1 = s$, $S_2 = -c$ and

$$\varphi = (c^2 - s^2) \frac{B(\vec{p}_1 - \vec{p}_2)}{B(\vec{p}_1 + \vec{p}_2)}, \; \psi = -2sc \frac{A(\vec{p}_1 - \vec{p}_2)}{A(\vec{p}_1 + \vec{p}_2)}$$

c) $\gamma_2 + \gamma_2$: $C_1 = C_2 = s$, $S_1 = S_2 = -c$ and

$$\varphi = 2sc \frac{B(\vec{p}_1 - \vec{p}_2)}{B(\vec{p}_1 + \vec{p}_2)}, \; \psi = (c^2 - s^2) \frac{A(\vec{p}_1 - \vec{p}_2)}{A(\vec{p}_1 + \vec{p}_2)}$$

In the exceptional case 3) above where $C$ and $S$ are free, the $\gamma$-type soliton would remain completely free even at this stage.

As the three- or more-soliton solutions exist only for integrable systems, we cannot continue the analysis for general $A$ and $B$ polynomials. The only known integrable cases for which $\gamma$-type solitons exist at all do factorize with a linear $U$, so only the study of the three-solitons solutions could determine the quantities $\varphi$ and $\psi$ which remain free at the two-soliton level. This staggered determination of the N-soliton solutions from the study of the $(N+1)$-one was first remarked in our work [3] on “static” solitons, i.e. precisely the case where the common factor $U$ is just $D_t$.

4 Is there a free $\gamma$-type soliton for integrable systems?

The fact that the parameters of a $\gamma$-type soliton may remain undetermined even at the level of two-soliton solutions raises questions about the nature of such a solution. That is, should we call this solution a ‘soliton’ or not. The point is that it is in general not true that any value of $C$ and $S$ in (9) define a soliton. For instance, let us assume that the common factor of $A$ and $B$ is just $U = D_t$. Then, since $U$ is a factor of both $A$ and $B$ it is easy to convince oneself that any time-independent $F$ and $G$ will satisfy the equations.
However, not any time-independent object can be called a time-independent soliton: there is the additional requirement that upon interaction with any moving object it should re-emerge unchanged, maybe up to a shift in position. The condition for the \( \gamma \)-type object defined above to satisfy this criterion is that it reduces to one of the cases \( \gamma_1 (C = c \text{ and } S = s) \) or \( \gamma_2 (C = s \text{ and } S = -c) \) defined above, unless condition (15) happens to be satisfied on the appropriate dispersion manifold, in which case the \( \gamma \)-type soliton is still free at this stage. Now we should note, that while (15) is satisfied on the \( \alpha + \alpha, \alpha + \beta \) and \( \beta + \beta \) manifolds for all the known integrable equations, it is not satisfied on the \( \alpha + \gamma \) or \( \beta + \gamma \) manifolds for those few known equations where \( \gamma \)-type solitons exist. This means that the \( \gamma \)-type solitons are in fact fixed at this stage for the integrable cases.

From the analysis of the conditions for the existence of the two-solitons solutions we obtained the condition (15) on various dispersion manifolds depending on the soliton solutions under consideration. This condition is a very strong one. To start with, it constitutes a first necessary condition for integrability. Thus, whenever the partial differential equation under consideration has a Hirota form (7) that possesses a continuous vacuum (i.e. \( A(0) = B(0) = 0 \)), condition (15) above may serve as a first check for the integrability of the equation.

To illustrate this, let us consider the even-even bilinear equation with Hirota polynomials

\[
A = D_x^3 D_t + D_y D_t + a D_x^2, \quad B = D_x(D_t + b D_x).
\]

(22)

If we write condition (15) on \( A(\vec{p}_1) = A(\vec{p}_2) = 0 \) we find the necessary condition \( a = b = 0 \). The same condition is obtained on the manifold \( A(\vec{p}_1) = 0, k_2 = 0 \) (from the first factor of \( B \)). On the other hand, if one wants to satisfy \( B(\vec{p}_2) = 0 \) through the second factor, i.e. \( p_2 + bk_2 = 0 \) then (15) is never satisfied, even for \( a = b = 0 \). In this last special case, and in that case only, however, (15) is not needed for integrability, as for \( p_2 = 0, a = b = 0 \) both \( A \) and \( B \) vanish and we have in fact a \( \gamma_1 \)- or \( \gamma_2 \)-type soliton. Finally \( a = b = 0 \) is a necessary and sufficient condition for the existence of two-soliton solutions of all possible types \((\alpha, \beta, \gamma_1, \gamma_2)\). One has thus obtained quickly the only integrable subcase of (22) at the two-soliton level. If one instead uses only the standard-type vacuum, one has to go to three soliton solutions (and in exceptional cases even to four soliton solutions) in order to restrict the values of \( a \) and \( b \).

Conversely, we could use (15) in order to derive the possible forms of bilinear PDE’s which would possess a two-solitons solution in the presence of a continuous vacuum. However, the general solution of the functional equation (15) under the constraints defining the dispersion manifold, seems to be a formidable task, (in particular because it may happen that for some special values of the parameters the existence of a solution is not obtained through (15), but instead because a \( \beta \)-soliton reduces to \( \gamma \)-type, as happened in the case of (22) above). Moreover, in [4] we have used general arguments based on singularity analysis and derived all the possible bilinear even-even PDE’s that could be candidates for integrable equations, and on the light of these results finding the general solution of (15) does not seem to be necessary. In conclusion, (22) with \( a = b = 0 \) is the only integrable pair with a continuous vacuum (and \( A \neq B \)).
5 The physical vacuum and solitons interpolating between them

Let us now return to the question of the vacuum angle. We noted already that even when \( \phi \) is fixed there are in fact several ‘physical’ vacua, and the soliton solutions connect them. First evidence of that is obtained when we recall that bilinear equations are invariant under a simultaneous change of phase. Thus (9) can also be written as

\[
\tilde{F} := e^{-\eta}F = e^{-\eta}c + C, \\
\tilde{G} := e^{-\eta}G = e^{-\eta}s + S,
\]

so that the soliton seems to be build on top of the vacuum \( \tilde{F} = C, \tilde{G} = S \).

Further information on the vacuum is obtained when we use the physical quantities. It is well known that \( F \) or \( G \) alone do not have physical meaning as they blow up when \( \eta \to \infty \). The typical physical variable is

\[
u = \arctan(G/F),
\]

so let us see how the vacua look like from the point of view of \( u \). The starting vacuum (8) yields \( u = \phi \), and that value is obtained also from (9) when \( \eta \to -\infty \). When \( \eta \to \infty \) we find that the limiting value is different for different solitons:

\[
\alpha: \phi \to -\phi, \\
\beta: \phi \to \frac{\pi}{2} - \phi, \\
\gamma_1: \phi \to \phi, \\
\gamma_2: \phi \to \phi - \frac{\pi}{2}.
\]

The solitons do therefore connect different values of the vacuum. Note that from the point of view of \( u \) the vacuum angle is defined only modulo \( \pi \).

When we look at the two-soliton solutions exhibited in Sec. 3 we observe that it is really this change in the angle that characterizes the soliton. For example (17) connects from vacuum angle \( \phi \to \phi - \frac{\pi}{2} \) and the intermediate vacuum angle is \( -\phi \) or \( \frac{\pi}{2} - \phi \), depending on which order the \( \alpha \) and \( \beta \) kinks appear (see Figure 1). That is, the values of the soliton’s leftside and rightside vacuum angles may change during the interaction, but the operation made on the vacuum angle will stay invariant and may be associated with the soliton. Figure 2 shows how the \( \alpha \)-soliton interpolates between different vacua.

The soliton \( \gamma_1 \) is quite curious, because in isolation it goes over to the vacuum. It is therefore a kind of “ghost” soliton, and is invisible when taken isolated. However, when it interacts with a normal soliton it manifests itself in an unambiguous way. Figure 3 shows the time evolution of an \( \alpha + \gamma_1 \) pair and the effect of the \( \gamma_1 \)-soliton is only in a phase shift in the evolution of the \( \alpha \) soliton.
6 Conclusions

To conclude, we remark that the (continuous) vacuum multiplicity is an interesting property of even-even bilinear equations which allows surprising phenomena to occur. One of most remarkable is the existence of hidden solitons which appear only in interaction with “normal” solitons. For these systems it turns out that already the existence of general two-soliton solutions can be used for investigating the integrability in a simple manner. Unfortunately, integrable systems having this property are quite rare and few examples are known to date.

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Figure captions

Figure 1:

The time evolution of the $\alpha + \beta$ soliton solution of (22) with $a = b = 0$ (the integrable case). The four different vacuum levels are clearly visible. (For this equation the $\beta$-soliton is $x$-independent.)

Figure 2:

This figure shows how the $\alpha$-soliton (11) interpolates different vacua. For figure a) we have assumed that $e^\eta$ in (11) is positive, for figure b) that it is negative.

Figure 3:

Contour lines for the time evolution of the $\alpha + \gamma_1$ soliton. For $t \to \pm \infty$ only the $\alpha$ soliton is visible, the $\gamma_1$ manifests itself only at the point where the $\alpha$ soliton goes over it.