EXISTENCE AND A GENERAL DECAY RESULTS FOR A VISCOELASTIC PLATE EQUATION WITH A LOGARITHMIC NONLINEARITY

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Abstract. In this paper, we consider a viscoelastic plate equation with a logarithmic nonlinearity. Using the Galerkin method and the multiplier method, we establish the existence of solutions and prove an explicit and general decay rate result. This result extends and improves many results in the literature such as Gorka [19], Hiramatsu et al. [27] and Han and Wang [26].

1. Introduction. In this paper, we deal with the existence and decay of solutions of the following plate problem:

\[
\begin{align*}
\begin{cases}
\frac{\partial^2 u}{\partial t^2} + \Delta^2 u + u - \int_0^t g(t - s)\Delta^2 u(s)ds = ku \ln |u|, & \text{in } \Omega \times (0, \infty), \\
\frac{\partial u}{\partial \nu} = 0, & \text{in } \partial \Omega \times (0, \infty), \\
u(x, 0) = u_0(x), & \text{in } \Omega, \\
u_t(x, 0) = u_1(x), & \text{in } \Omega,
\end{cases}
\end{align*}
\]

where \( \Omega \) is a bounded domain of \( \mathbb{R}^2 \) with a smooth boundary \( \partial \Omega \), \( \nu \) is the unit outer normal to \( \partial \Omega \) and \( k \) is a small positive real number. The kernel \( g \) satisfies some conditions to be specified later.

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1.1. Plate problems. Concerning the study of plates, Lagnese [31] studied a viscoelastic plate equation and showed that the energy decays to zero as time goes to infinity by introducing a dissipative mechanism on the boundary of the system. Rivera et al. [43] proved that the first and second order energy, associated with the solutions of the viscoelastic plate equation, decay exponentially provided that the kernel of the memory also decays exponentially. Komornik [29] investigated the energy decay of a plate model under weak growth assumptions on the feedback function. Messaoudi [36] studied the following problem:

\[
\begin{align*}
    u_{tt} + \Delta^2 u + |u_t|^{m-2}u_t &= |u|^{p-2}u, & \text{in } Q_T = \Omega \times (0, T), \\
    u &= \frac{\partial u}{\partial \nu} = 0, & \text{on } \Gamma_T = \partial \Omega \times [0, T), \\
    u(x, 0) &= u_0(x), \ u_t(x, 0) = u_1(x), & \text{in } \Omega,
\end{align*}
\]  

(1.2)

established an existence result and showed that the solution continues to exist globally if \( m \geq p \), and blows up in finite time if \( m < p \) and the initial energy is negative. This result was later improved by Chen and Zhou [13].

For boundary damping, Santos and Junior [44] studied the stability of the following system:

\[
\begin{align*}
    u_{tt} + \Delta^2 u &= 0, & \text{in } \Omega \times (0, \infty), \\
    u &= \frac{\partial u}{\partial \nu} = 0, & \text{on } \Gamma_0 \times (0, \infty), \\
    -u + \int_0^1 g_1(t - s)\beta_1 u(s)ds &= 0, & \text{on } \Gamma_1 \times (0, \infty), \\
    \frac{\partial u}{\partial \nu} + \int_0^1 g_2(t - s)\beta_2 u(s)ds &= 0, & \text{on } \Gamma_2 \times (0, \infty), \\
    u(0, x) &= u_0(x), \ u_t(0, x) = u_1(x), & \text{in } \Omega,
\end{align*}
\]

(1.3)

where \( \beta_1 = \Delta u + (1 - \mu) B_1 u \) and \( \beta_2 u = \frac{\partial \Delta u}{\partial \mu} + (1 - \mu) \frac{\partial B_2 u}{\partial \eta} \)

with

\[
B_1 u = 2\nu_1 \nu_2 u_{xy} - \nu_1^2 u_{yy} - \nu_2^2 u_{xx} \quad \text{and} \quad B_2 u = (\nu_1 - \nu_2) u_{xy} + \nu_1 \nu_2 (u_{yy} - u_{xx}).
\]

For more results in this direction, see [3, 22, 26, 30, 32].

1.2. Viscoelastic problems. Since the pioneer works of Dafermos [15, 16] in 1970, where the general decay was discussed, problems related to viscoelasticity have attracted a great deal of attention and many results of existence and long-time behavior have been established. The importance of the viscoelastic properties of materials has been realized because of the rapid developments in rubber and plastic industry. Many advances in the studies of constitutive relations, failure theories and life prediction of viscoelastic materials and structures were reported and reviewed in the last two decades [14]. Hrusa [28] considered a one-dimensional nonlinear viscoelastic equation of the form

\[
u_{tt} - cu_{xx} + \int_0^t m(t - s)(\psi(u_x(s)))_x ds = f(x, t)
\]

and proved several global existence results for large data and an exponential decay result for strong solutions when \( m(s) = e^{-s} \) and \( \psi \) satisfies certain conditions. In [17], Dassios and Zafiropoulos considered a viscoelastic problem in \( \mathbb{R}^3 \) and proved a polynomial decay result for exponentially decaying kernels. After that, Rivera [42] considered equations for linear isotropic homogeneous viscoelastic solids of integral type which occupy bounded domains or the whole space \( \mathbb{R}^n \). In the bounded-domain case and for exponentially decaying memory kernels and regular solutions,
he showed that the sum of the first and the second energy decays exponentially. Whereas, the decay is polynomial when the body occupies the whole space $\mathbb{R}^n$, even if the relaxation function is of an exponential decay.

For quasilinear problems, Cavalcanti et al. [7] studied, in a bounded domain, the following equation:

$$|u_t|^a u_{tt} - \Delta u - \Delta u_{tt} + \int_0^t g(t - \tau) \Delta u(\tau) d\tau - \gamma \Delta u_t = 0, \quad \text{in } \Omega \times (0, \infty), \quad (1.4)$$

for $\rho > 0$. A global existence result for $\gamma \geq 0$, as well as an exponential decay result for $\gamma > 0$, have been established. This latter result was then extended to a situation, where $\gamma = 0$, by Messaoudi and Tatar [39, 40], and exponential and polynomial decay results have been established in the absence, as well as in the presence, of a source term. In all the above mentioned works, the rates of decay in relaxation functions were either of exponential or polynomial type. In [8], Cavalcanti et al. considered

$$u_{tt} - \Delta u + \int_0^t g(t - s) \Delta u(s) ds + a(x) u_t + |u|^{p-1} u = 0, \quad \text{in } \Omega \times (0, \infty),$$

where $a : \Omega \to \mathbb{R}^+$ is a function which may vanish on a part of the domain $\Omega$ but satisfies $a(x) \geq a_0$ on $\omega \subset \Omega$ and $g$ satisfies, for two positive constants $\xi_1$ and $\xi_2$,

$$-\xi_1 g(t) \leq g'(t) \leq -\xi_2 g(t), \quad \forall t \in \mathbb{R}^+$$

and established an exponential decay result under some restrictions on $\omega$. Berrimi and Messaoudi [4] established the result of [8], under weaker conditions on both $a$ and $g$, to a problem where a source term is competing with the damping term. Cavalcanti and Oquendo [10] considered the following problem:

$$u_{tt} - k_0 \Delta u + \int_0^t \text{div}[a(x)g(t - s)\Delta u(s)] ds + b(x) h(u_t) + f(u) = 0, \quad \text{in } \Omega \times (0, \infty) \quad (1.5)$$

and established, for $a(x) + b(x) \geq \rho > 0$, an exponential stability result for $g$ decaying exponentially and $h$ linear, and a polynomial stability result for $g$ decaying polynomially and $h$ nonlinear. Li et al. [34] treated (1.5) with $b(x) = 0$ and $f(u) = -|u|^{\gamma} u$, $\gamma > 0$. They showed the global existence and uniqueness of global solution of problem (1.5) and established uniform decay rate of the energy under suitable conditions on the initial data and the relaxation function $g$. For more general decay results, Messaoudi [37, 38] considered

$$u_{tt} - \Delta u + \int_0^t g(t - \tau) \Delta u(\tau) d\tau = b|u|^{q-2} u, \quad \text{in } \Omega \times (0, \infty) \quad (1.6)$$

for $q \geq 2$, $b \in \{0, 1\}$ and $g$ satisfying (A1) and (A2) below with $p = 1$, and established a more general decay result, from which the usual exponential and polynomial decay rates are only special cases. Guesmia et al. [24] studied the well-posedness and stability of the following coupled two wave equations:

$$\begin{cases}
\left\{\begin{array}{l}
u_{tt} - \Delta u + \int_0^t g_1(t - \tau) \Delta u(\tau) d\tau + |u_t|^{m-1} u_t = f_1(u, v), \quad \text{in } \Omega \times (0, \infty), \\
u_{tt} - \Delta v + \int_0^t g_2(t - \tau) \Delta v(\tau) d\tau + |v_t|^{r-1} v_t = f_2(u, v), \quad \text{in } \Omega \times (0, \infty),
\end{array}\right.
\end{cases} \quad (1.7)$$

where $m, r \geq 1$, $f_1$ and $f_2$ are given functions satisfying some hypotheses, and $g_1$ and $g_2$ are like $g$ in (A1) and (A2) below (with $p = 1$, $\xi_1$ and $\xi_2$ instead of $\xi$). They established the same stability estimate as in [37, 38] with $\xi = \min\{\xi_1, \xi_2\}$. Very recently, Messaoudi and Al-Khulaifi [41] considered (1.4) with $\gamma = 0$, where
the relaxation function satisfies (2.2) below, and established a more general decay result. For the case of memories acting on the boundary of domain, we refer the readers to [9, 23] and the references therein.

1.3. Problems with logarithmic nonlinearity. The logarithmic nonlinearity is of much interest in physics, since it appears naturally in inflation cosmology and supersymmetric field theories, quantum mechanics and nuclear physics [1, 18]. This type of problems has applications in many branches of physics such as nuclear physics, optics and geophysics [2, 5, 19]. Birula and Mycielski [5, 6] studied the following problem:

\[
\begin{cases}
  u_{tt} - u_{xx} + u - \varepsilon u \ln |u|^2 = 0, & \text{in } [a,b] \times (0,T), \\
  u(a,t) = u(b,t) = 0, & \text{in } (0,T), \\
  u(x,0) = u_0(x), & u_t(x,0) = u_1(x), & \text{in } [a,b],
\end{cases}
\]  
(1.8)

which is a relativistic version of logarithmic quantum mechanics and can also be obtained by taking the limit \( p \to 1 \) for the \( p \)-adic string equation [20, 45]. In [11], Cazenave and Haraux considered

\[ u_{tt} - \Delta u = u \ln |u|^k, \text{ in } \mathbb{R}^3 \]  
(1.9)

and established the existence and uniqueness of the solution for the Cauchy problem. Gorka [19] used some compactness arguments and obtained the global existence of weak solutions for all \((u_0, u_1) \in H^1_0 \times L^2\) to the initial-boundary value problem (1.9) in the one-dimensional case. Bartkowski and Gorka [2] proved the existence of classical solutions and investigated the weak solutions for the corresponding one-dimensional Cauchy problem for equation (1.9). Hiramatsu et al. [27] introduced the following equation:

\[ u_{tt} - \Delta u + u + u_t + |u|^2 u = u \ln |u| \]  
(1.10)

to study the dynamics of Q-ball in theoretical physics and presented a numerical study. However, there was no theoretical analysis for the problem. In [25], Han proved the global existence of weak solutions, for all \((u_0, u_1) \in H^1_0(\Omega) \times L^2(\Omega)\), to the initial boundary value problem (1.10) in \( \Omega \subset \mathbb{R}^3 \).

In this paper, we are concerned with the well-posedness and stability of the plate problem (1.1) with a kernel \( g \) having an arbitrary growth at infinity (condition (2.2) below). The obtained stability results improve and generalize many results in the literature.

This paper is organized as follows. In section 2, we present some notations and material needed for our work. In section 3, we establish the local existence of the solutions of the problem. The global existence and the decay results are presented in section 4 and section 5, respectively.

2. Preliminaries. In this section, we present some notations and material needed in the proof of our results. We use the standard Lebesgue space \( L^2(\Omega) \) and Sobolev space \( H^2_0(\Omega) \) with their usual scalar products and norms. Throughout this paper, \( c \) is used to denote a generic positive constant.

We consider the following hypotheses:

(A1) \( g : \mathbb{R}^+ \to \mathbb{R}^+ \) is a \( C^1 \)- nonincreasing function satisfying

\[ g(0) > 0 \text{ and } 1 - \int_0^\infty g(s) ds = \ell > 0. \]  
(2.1)
(A2) There exist a nonincreasing differentiable function $\xi : \mathbb{R}^+ \to \mathbb{R}^+$, with $\xi(0) > 0$, and a constant $1 \leq p < \frac{3}{2}$ such that

$$g'(t) \leq -\xi(t)g^p(t), \forall t \in \mathbb{R}^+. \tag{2.2}$$

(A3) The constant $k$ in (1.1) satisfies $0 < k < k_0$, where $k_0$ is the positive real number satisfying

$$\sqrt{\frac{2\pi \ell}{k_0 c_p}} = e^{-\frac{3}{2} - \frac{1}{p}} \tag{2.3}$$

and $c_p$ is the smallest positive number satisfying

$$\|\nabla u\|_2^2 \leq c_p \|\Delta u\|_2^2, \forall u \in H_0^2(\Omega),$$

where $\|\cdot\|_2 = \|\cdot\|_{L^2(\Omega)}$.

Remark 2.1. The function $f(s) = \sqrt{\frac{2\pi \ell}{c_p s}} - e^{-\frac{3}{2} - \frac{1}{p}}$ is a continuous and decreasing function on $(0, \infty)$, with

$$\lim_{s \to 0^+} f(s) = \infty \text{ and } \lim_{x \to \infty} f(x) = -e^{-\frac{3}{2}}.$$

Then, there exists a unique $k_0 > 0$ such that $f(k_0) = 0$. Moreover,

$$e^{-\frac{3}{2} - \frac{1}{p}} < \sqrt{\frac{2\pi \ell}{c_p s}}, \forall s \in (0, k_0). \tag{2.4}$$

The energy functional associated with problem (1.1) is

$$E(t) = \frac{1}{2} \left( \|u_t\|_2^2 + \left(1 - \int_0^t g(s)ds\right) \|\Delta u\|_2^2 + \frac{k + 2}{2} \|u\|_2^2 \right)$$

$$- \frac{1}{2} \int_{\Omega} u^2 \ln |u|^tdx + \frac{1}{2} (go\Delta u)(t), \tag{2.5}$$

where

$$(go\Delta u)(t) = \int_0^t g(t-s)\|\Delta u(s) - \Delta u(t)\|_2^2 ds.$$  

Direct differentiation of (2.5), using (1.1), leads to

$$E'(t) = \frac{1}{2} (g' o\Delta u)(t) - \frac{1}{2} g(t)\|\Delta u\|_2^2 \leq \frac{1}{2} (g' o\Delta u)(t) \leq 0. \tag{2.6}$$

Lemma 2.2 ([21, 12], Logarithmic Sobolev inequality). Let $u$ be any function in $H_0^1(\Omega)$ and $a$ be any positive real number. Then

$$\int_{\Omega} u^2 \ln |u|dx \leq \frac{1}{2} \|u\|_2^2 \ln \|u\|_2^2 + \frac{a^2}{2\pi} \|\nabla u\|_2^2 - (1 + \ln a)\|u\|_2^2. \tag{2.7}$$

Corollary 2.3. Let $u$ be any function in $H_0^3(\Omega)$ and $a$ be any positive real number. Then

$$\int_{\Omega} u^2 \ln |u|dx \leq \frac{1}{2} \|u\|_2^2 \ln \|u\|_2^2 + \frac{c_p a^2}{2\pi} \|\Delta u\|_2^2 - (1 + \ln a)\|u\|_2^2. \tag{2.8}$$

Lemma 2.4 ([11], Logarithmic Gronwall inequality). Let $c > 0$, $\gamma \in L^1(0, T; \mathbb{R}^+)$ and assume that the function $w : [0, T] \to [1, \infty)$ satisfies

$$w(t) \leq c \left(1 + \int_0^t \gamma(s) w(s) \ln w(s) ds\right), \quad 0 \leq t \leq T. \tag{2.9}$$
Then
\[ w(t) \leq c \exp \left( c \int_0^t \gamma(s) ds \right), \quad 0 \leq t \leq T. \] (2.10)

3. Local existence. In this section, we state and prove the local existence result for problem (1.1).

Definition 3.1. Let \( T > 0 \). A function
\[ u \in C([0, T], H^2_0(\Omega)) \cap C^1([0, T], L^2(\Omega)) \cap C^2([0, T], H^{-2}(\Omega)) \]
is called a weak solution of (1.1) on \([0, T]\) if, for any \( w \in H^2_0(\Omega) \) and \( t \in [0, T] \),
\[
\begin{cases}
\int_\Omega u_t(x, t) w(x) dx + \int_\Omega \Delta u(x, t) \Delta w(x) dx + \int_\Omega u(x, t) w(x) dx \\
- \int_\Omega \Delta w(x) \int_0^t g(t-s) \Delta u(s) ds dx = \int_\Omega u(x, t) w(x) \ln |u(x, t)|^k dx,
\end{cases}
\]
(3.1)

Then problem (1.1) has a weak solution
\[ u \in C([0, T], H^2_0(\Omega)) \cap C^1([0, T], L^2(\Omega)) \cap C^2([0, T], H^{-2}(\Omega)). \] (3.2)

Proof. To establish the existence of a solution to problem (1.1), we use the Faedo-Galerkin method. Let \( \{w_j\}_{j=1}^{\infty} \) be an orthogonal basis of the “separable” space \( H^2_0(\Omega) \) which is orthonormal in \( L^2(\Omega) \). Let \( V_m = \text{span} \{w_1, w_2, \ldots, w_m\} \) and let the projections of the initial data on the finite dimensional subspace \( V_m \) be given by
\[ u_0^m(x) = \sum_{j=1}^m a_j w_j(x) \quad \text{and} \quad u_1^m(x) = \sum_{j=1}^m b_j w_j(x), \]
where
\[ u_0^m \to u_0 \in H^2_0(\Omega) \quad \text{and} \quad u_1^m \to u \in L^2(\Omega), \quad \text{as} \ m \to \infty. \] (3.3)

We search for an approximate solution
\[ u^m(x, t) = \sum_{j=1}^m h_j^m(t) w_j(x) \]
of the approximate problem in \( V_m \):
\[
\begin{cases}
\int_\Omega \left( u_{tt}^m w + \Delta u^m \Delta w + u^m w - \int_0^t g(t-s) \Delta u^m(s) ds \Delta w \right) dx \\
= \int_\Omega w u^m \ln |u^m|^k dx, \quad \forall w \in V_m, \\
u^m(0) = u_0^m = \sum_{j=1}^m (u_0, w_j) w_j, \\
u_{tt}^m(0) = u_1^m = \sum_{j=1}^m (u_1, w_j) w_j.
\end{cases}
\]
(3.4)

This leads to a system of ODEs for unknown functions \( h_j^m(t) \). Based on standard existence theory for ODE, one can obtain functions
\[ h_j : [0, t_m] \to \mathbb{R}, \quad j = 1, 2, \ldots, m, \]
which satisfy (3.4) in a maximal interval \([0, t_m], t_m \in (0, T]\). Next, we show that \( t_m = T \) and that the local solution is uniformly bounded independent of \( m \) and \( t \).

For this purpose, let us replace \( w \) by \( u_t^m \) in (3.4) and integrate by parts to obtain
\[
\frac{d}{dt} E^m(t) \leq \frac{1}{2} (g' o \Delta u^m)(t) \leq 0, \]
(3.5)
where
\[
E^m(t) = \frac{1}{2} \left( \|u^m_t\|_2^2 + \left( 1 - \int_0^t g(s) \right) \|\Delta u^m\|_2^2 + \frac{k + 2}{2} \|u^m\|_2^2 \right) - \frac{1}{2} \int_\Omega |u^m|^2 \ln |u^m|^k dx + \frac{1}{2} (go \Delta u^m)(t).
\] (3.6)

From (3.5), we have
\[
E^m(t) \leq E^m(0).
\]
The last inequality together with (2.1) and the Logarithmic Sobolev inequality (2.8) lead to
\[
\begin{align*}
\|u^m_t\|_2^2 + (go \Delta u^m)(t) + \left( \ell - \frac{ka^2c_p}{2\pi} \right) \|\Delta u^m\|_2^2 + \left[ \frac{k + 2}{2} + k (1 + \ln a) \right] \|u^m\|_2^2 \\
&\leq C + \frac{k}{2} \|u^m\|_2^2 \ln \|u^m\|_2^2,
\end{align*}
\] (3.7)

where \( C = 2E^m(0) \). Choosing
\[
e^{-\frac{3}{2} - \frac{1}{4}} < a < \sqrt{\frac{2\pi \ell}{kc_p}}
\] (3.8)

will make
\[
\ell - \frac{ka^2c_p}{2\pi} > 0 \quad \text{and} \quad \frac{k + 2}{2} + k (1 + \ln a) > 0.
\]

This selection is possible thanks to (A3). So, we get
\[
(go \Delta u^m)(t) + \|u^m_t\|_2^2 + \|\Delta u^m\|_2^2 + \|u^m\|_2^2 \leq c \left( 1 + \|u^m\|_2^2 \ln \|u^m\|_2^2 \right).
\] (3.9)

Let us note that
\[
u^m(., t) = u^m(., 0) + \int_0^t \partial u^m/\partial s (., s) ds.
\]

Then, using Cauchy-Schwarz’ inequality, we get
\[
\|u^m(t)\|_2^2 \leq 2\|u^m(0)\|_2^2 + 2 \left\| \int_0^t \partial u^m/\partial s (., s) ds \right\|_2^2
\] (3.10)

\[
\leq 2\|u^m(0)\|_2^2 + 2T \int_0^t \|u^m_t(s)\|_2^2 ds,
\]

hence, inequality (3.9) gives
\[
\|u^m\|_2^2 \leq 2\|u^m(0)\|_2^2 + 2cT \left( 1 + \int_0^t \|u^m\|_2^2 \ln \|u^m\|_2^2 ds \right).
\] (3.11)

If we put \( C_1 = \max \{2cT, 2\|u^m(0)\|_2^2 \} \), (3.11) leads to
\[
\|u^m\|_2^2 \leq 2C_1 \left( 1 + \int_0^t \|u^m\|_2^2 \ln \|u^m\|_2^2 ds \right).
\]

Without loss of generality, we take \( C_1 \geq 1 \), which gives
\[
\|u^m\|_2^2 \leq 2C_1 \left( 1 + \int_0^t \left( C_1 + \|u^m\|_2^2 \right) \ln \left( C_1 + \|u^m\|_2^2 \right) ds \right).
\]

Applying the Logarithmic Gronwall inequality to the last inequality, we obtain the following estimate:
\[
\|u^m\|_2^2 \leq 2C_1 e^{2C_1 T} := C_2.
\]
Hence, from inequality (3.9) it follows that:
\[
(g_{e} \Delta u^{m})(t) + \| u_{t}^{m} \|^{2}_{2} + \| \Delta u^{m} \|^{2}_{2} + \| u^{m} \|^{2}_{2} \leq c (1 + C_{2} \ln C_{2}) := C_{3},
\]
where \( C_{3} \) is a positive constant independent of \( m \) and \( t \). This implies
\[
\sup_{t \in (0, T_{m})} \| u_{t}^{m} \|^{2}_{2} + \sup_{t \in (0, T_{m})} \| \Delta u^{m} \|^{2}_{2} + \sup_{t \in (0, T_{m})} \| u^{m} \|^{2}_{2} \leq 3C_{3}. \tag{3.12}
\]
So, the approximate solution is uniformly bounded independent of \( m \) and \( t \). Therefore, we can extend \( t_{m} \) to \( T \). Moreover, we obtain, from (3.12),
\[
\begin{cases}
 u^{m} \text{ is uniformly bounded in } L^{\infty}(0, T; H_{0}^{2}(\Omega)), \\
 u_{t}^{m} \text{ is uniformly bounded in } L^{\infty}(0, T; L^{2}(\Omega)),
\end{cases} \tag{3.13}
\]
which implies that there exists a subsequence of \((u^{m})\) (still denoted by \((u^{m})\)), such that
\[
\begin{cases}
 u^{m} \rightharpoonup u \text{ weakly * in } L^{\infty}(0, T; H_{0}^{2}(\Omega)), \\
 u_{t}^{m} \rightharpoonup u_{t} \text{ weakly * in } L^{\infty}(0, T; L^{2}(\Omega)), \\
 u^{m} \rightarrow u \text{ weakly in } L^{2}(0, T; H_{0}^{2}(\Omega)), \\
 u_{t}^{m} \rightarrow u_{t} \text{ weakly in } L^{2}(0, T; L^{2}(\Omega)).
\end{cases} \tag{3.14}
\]
Making use of Aubin-Lions’ theorem, we find, up to a subsequence, that
\[
u^{m} \rightarrow u \text{ strongly in } L^{2}(0, T; L^{2}(\Omega))
\]
and
\[
u^{m} \rightarrow u \text{ a.e. in } \Omega \times (0, T).
\]
Since the map \( s \rightarrow s \ln |s|^{k} \) is continuous on \( \mathbb{R} \), we have the convergence
\[
u^{m} \ln |u^{m}|^{k} \rightarrow u \ln |u|^{k} \text{ a.e. in } \Omega \times (0, T).
\]
Using the embedding of \( H_{0}^{2}(\Omega) \) in \( L^{\infty}(\Omega) \) (since \( \Omega \subset \mathbb{R}^{2} \)), it is clear that \( u^{m} \ln |u^{m}|^{k} \)
is bounded in \( L^{\infty}(\Omega \times (0, T)) \). Next, taking into account the Lebesgue bounded convergence theorem (\( \Omega \) is bounded), we get
\[
u^{m} \ln |u^{m}|^{k} \rightarrow u \ln |u|^{k} \text{ strongly in } L^{2}(0, T; L^{2}(\Omega)). \tag{3.15}
\]
Now, we integrate (3.4) over \((0, t)\) to obtain, for every \( w \in V_{m},\)
\[
\begin{align*}
\int_{\Omega} u_{t}^{m} w \, dx &= \int_{\Omega} u_{1}^{m} w \, dx + \int_{0}^{t} \int_{\Omega} \Delta u^{m}(s) \Delta w \, dx \, ds + \int_{0}^{t} \int_{\Omega} w^{m}(s) w \, dx \, ds \\
&\quad - \int_{0}^{t} \int_{\Omega} g(\tau - s) \Delta u^{m}(s) \, ds \Delta w \, dx \, d\tau \quad = \int_{\Omega} \int_{0}^{t} w u^{m}(s) \ln |u^{m}(s)|^{k} \, dx \, ds.
\end{align*} \tag{3.16}
\]
Convergences (3.3), (3.14) and (3.15) are sufficient to pass to the limit in (3.16), as \( m \rightarrow +\infty \), and get, for any \( w \in V_{m} \) and \( m \geq 1, \)
\[
\begin{align*}
\int_{\Omega} u_{t} w \, dx &= \int_{\Omega} u_{1} w \, dx - \int_{0}^{t} \int_{\Omega} \Delta u(s) \Delta w \, dx \, ds - \int_{0}^{t} \int_{\Omega} u(s) w \, dx \, ds \\
&\quad + \int_{0}^{T} \int_{\Omega} \Delta w(x) \int_{0}^{t} g(t - s) \Delta u(s) \, ds \, dx \, dt + \int_{0}^{t} \int_{\Omega} u(s) w \ln |u(s)|^{k} \, ds \, dx,
\end{align*} \tag{3.17}
\]
which implies that (3.17) is valid for any \( w \in H_{0}^{2}(\Omega) \). Using the fact that the terms in the right-hand side of (3.17) are absolutely continuous since they are functions
of \( t \) defined by integrals over \((0, t)\), hence it is differentiable for a.e. \( t \in \mathbb{R}^+ \). Thus, differentiating (3.17), we obtain, for a.e. \( t \in (0, T) \) and any \( w \in H_0^2(\Omega) \),

\[
\int_\Omega u_{tt}(x,t)w(x)dx + \int_\Omega \Delta u(x,t)\Delta w(x)dx + \int_\Omega u(x,t)w(x)dx + \int \Delta w(x)\int_0^t g(t-s)\Delta u(s)dsdx = \int w(x)u(x,t)\ln|u(x,t)|^k dx.
\]  

(3.18)

To handle the initial conditions, we note that \( u^m \to u \) weakly in \( L^2(0, T; H_0^2(\Omega)) \) and \( u_t^m \to u_t \) weakly in \( L^2(0, T; L^2(\Omega)) \).

Thus, using Lion’s Lemma [33], we obtain

\[
u^m \to u \quad \text{in} \quad C([0, T], L^2(\Omega)).
\]  

(3.19)

Therefore, \( u^m(x, 0) \) makes sense and

\[
u^m(x, 0) \to u(x, 0) \quad \text{in} \quad L^2(\Omega).
\]

Also, we have

\[
u^m(x, 0) = u_0^m(x) \to u_0(x) \quad \text{in} \quad H_0^2(\Omega).
\]

Hence

\[
u^m(x, 0) = u_0(x).
\]

Now, multiply (3.4) by \( \phi \in C_0^\infty(0, T) \) and integrate over \((0, T)\), we obtain, for any \( w \in V_m \),

\[
- \int_0^T \int_\Omega u_t^m(t)w\phi'(t)dxdt = - \int_0^T \int_\Omega \Delta u^m(t)\Delta w\phi(t)dxdt
\]

\[
- \int_0^T \int_\Omega u^m(t)w\phi(t)dxdt + \int_0^T \int_\Omega \int_0^t g(t-s)\Delta u^m(s)ds\Delta w\phi(t)dxdt
\]

\[
+ \int_0^T \int_\Omega u^m(t)w\phi(t)\ln|w|^k dxdt.
\]

(3.21)

As \( m \to \infty \), we have, for any \( w \in H_0^2(\Omega) \) and any \( \phi \in C_0^\infty((0, T)) \),

\[
- \int_0^T \int_\Omega u_t(t)w\phi'(t)dxdt = - \int_0^T \int_\Omega \Delta u(t)\Delta w\phi(t)dxdt - \int_0^T \int_\Omega w\phi(t)dxdt
\]

\[
+ \int_0^T \int_\Omega \int_0^t g(t-s)\Delta u(s)ds\Delta w\phi(t)dxdt + \int_0^T \int_\Omega \phi(t)u\ln|u|^k dxdt.
\]

(3.22)

This means (see [35]),

\[
u_{tt} \in L^2([0, T], H^{-2}(\Omega)).
\]

Recalling that \( u_t \in L^2((0, T), L^2(\Omega)) \), we obtain

\[
u_t \in C([0, T), H^{-2}(\Omega)).
\]

So, \( u_t^m(x, 0) \) makes sense and

\[
u_t^m(x, 0) \to u_t(x, 0) \quad \text{in} \quad H^{-2}(\Omega).
\]

But

\[
u_t^m(x, 0) = u_t^m(x) \to u_1(x) \quad \text{in} \quad L^2(\Omega).
\]

Hence

\[
u_t(x, 0) = u_1(x).
\]
4. Global existence. In this section, we state and prove a global existence result under smallness conditions on the initial data \((u_0, u_1)\). For this purpose, we introduce the following functionals:

\[
J(t) = \frac{1}{2} \left( 1 - \int_0^t g(s) \, ds \right) \| \Delta u \|^2 \Omega + \| u \|^2 \Omega + (g \Delta u)(t) - \int_\Omega u^2 \ln |u|^k \, dx \tag{4.1}
\]

and

\[
I(t) = \left( 1 - \int_0^t g(s) \, ds \right) \| \Delta u \|^2 \Omega + \| u \|^2 \Omega + (g \Delta u)(t) - 3 \int_\Omega u^2 \ln |u|^k \, dx. \tag{4.2}
\]

Lemma 4.1. The following inequalities hold:

\[- kd_0 \sqrt{\Omega} c_*^2 \| \Delta u \|^2 \Omega \leq \int_\Omega u^2 \ln |u|^k \, dx \leq k c_*^3 \| \Delta u \|^3 \Omega, \quad \forall u \in H^3_0(\Omega), \tag{4.3}\]

where \(d_0 = \sup_{0 < s < 1} \sqrt{s} \ln s, |\Omega| \) is the Lebesgue measure of \(\Omega\) and \(c_* \) is the smallest embedding constant

\[
\left( \int_\Omega |u|^3 \, dx \right)^{\frac{1}{3}} \leq c_* \| \Delta u \|_2, \quad \forall u \in H^3_0(\Omega) \tag{4.4}
\]

\((c_* \) exists thanks to the embedding of \(H^3_0(\Omega) \) in \(L^\infty(\Omega)\)).

Proof. Let

\[\Omega_1 = \{ x \in \Omega : |u(x)| > 1 \} \text{ and } \Omega_2 = \{ x \in \Omega : |u(x)| \leq 1 \}.\]

So, using (4.4), we have

\[
\int_\Omega u^2 \ln |u|^k \, dx = \int_{\Omega_2} u^2 \ln |u|^k \, dx + \int_{\Omega_1} u^2 \ln |u|^k \, dx \\
\leq k \int_{\Omega_1} u^2 \ln |u| \, dx \leq k \int_{\Omega_1} |u|^3 \, dx \leq k \int_\Omega |u|^3 \, dx \leq kc_*^3 \| \Delta u \|^3 \Omega,
\]

this gives the right inequality in (4.3).

On the other hand, using Hölder’s inequality and (4.4), we find

\[
- \int_\Omega u^2 \ln |u|^k \, dx = - \int_{\Omega_2} u^2 \ln |u|^k \, dx - \int_{\Omega_1} u^2 \ln |u|^k \, dx \\
\leq -k \int_{\Omega_2} u^2 \ln |u| \, dx = k \int_{\Omega_2} u^2 \ln |u| \, dx \\
\leq kd_0 \int_{\Omega_1} |u|^2 \, dx \leq kd_0 \sqrt{\Omega} \left( \int_{\Omega} |u|^3 \, dx \right)^{\frac{1}{3}} \leq kd_0 \sqrt{\Omega} c_*^2 \| \Delta u \|^2 \Omega,
\]

which implies the left inequality in (4.3).

Lemma 4.2. Assume that (A1) – (A3). Let \((u_0, u_1) \in H^3_0(\Omega) \times L^2(\Omega) \) such that

\[
I(0) > 0 \text{ and } \sqrt{\frac{54}{k}} c_*^3 \left( \frac{E(0)}{\ell} \right)^{\frac{1}{3}} < \ell. \tag{4.5}
\]

Then

\[
I(t) > 0, \quad \forall t \in [0, T). \tag{4.6}
\]
Proof. From (4.2), we have
\[ \int_\Omega u^2 \ln |u| \, dx = \frac{1}{3} \left( 1 - \int_0^t g(s) \, ds \right) \|\Delta u\|_2^2 + \frac{1}{3} \|u\|_2^2 + \frac{1}{3} (g_0 \Delta u)(t) - \frac{1}{3} I(t). \] (4.7)

Substitute (4.7) in (4.1), we find
\[ J(t) = \frac{1}{3} \left( 1 - \int_0^t g(s) \, ds \right) \|\Delta u\|_2^2 + \|u\|_2^2 + (g_0 \Delta u)(t) \right] + \frac{k}{4} \|u\|_2^2 + \frac{1}{6} I(t). \] (4.8)

Since \( I(0) > 0 \) and \( I \) is continuous on \([0, T]\), there exists \( t_0 \in (0, T) \) such that \( I(t) > 0 \), for all \( t \in [0, t_0) \). Let us denote by \( t_0 \) the biggest real number in \((0, T]\) such that \( I > 0 \) on \([0, t_0)\). If \( t_0 = T \), then (4.6) is satisfied.

We assume by contradiction that \( t_0 \in (0, T) \). Thus \( I(t_0) = 0 \) and
\[ \|\Delta u(t)\|_2^2 \leq \frac{3}{t} J(t) \leq \frac{3}{t} E(t) \leq \frac{3}{t} E(0), \quad \forall t \in [0, t_0). \] (4.9)

If \( \|\Delta u(t_0)\|_2^2 = 0 \), then (4.3) and (4.4) give
\[ 0 = I(t_0) = (g_0 \Delta u)(t_0) = \int_0^{t_0} g(s) \|\Delta u(s)\|_2^2 \, ds. \] (4.10)

Consequently, if \( g > 0 \) on \([0, t_0)\), we get
\[ \|\Delta u(s)\|_2 = 0, \quad \forall s \in [0, t_0). \]

Then
\[ I(t) = 0, \quad \forall t \in [0, t_0), \]
which is not true since \( I > 0 \) on \([0, t_0)\). If \( g \) is not positive on \([0, t_0)\), then let \( t_1 \in [0, t_0) \) the smallest real number such that \( g(t_1) = 0 \). Because \( g(0) > 0 \) and \( g \) is positive, nonincreasing and continuous on \( \mathbb{R}^+ \) (condition (A1)), then \( t_1 > 0 \) and \( g = 0 \) on \([t_1, \infty)\). Therefore, from (4.10), we deduce that
\[ 0 = \int_0^{t_0} g(s) \|\Delta u(s)\|_2^2 \, ds = \int_{t_1}^{t_0} g(s) \|\Delta u(s)\|_2^2 \, ds, \]
then \( \|\Delta u(s)\|_2 = 0 \), for any \( s \in [0, t_1) \), which implies that \( I(t) = 0 \), for any \( t \in [0, t_1) \). As before, this is a contradiction with the fact that \( I > 0 \) on \([0, t_0)\). Then we conclude that \( \|\Delta u(t_0)\|_2^2 > 0 \). On the other hand, we have
\[ I(t_0) \geq \ell \|\Delta u(t_0)\|_2^2 - 3 \int_\Omega u(t_0) \ln |u(t_0)| \, dx. \]

By using (4.9) and Lemma 4.1, we have
\[ I(t_0) \geq \left[ \ell - 3 k E(0) \right] \|\Delta u(t_0)\|_2^2. \]

By recalling (4.5), we arrive at \( I(t_0) > 0 \), which contradicts the assumption \( I(t_0) = 0 \). Hence, \( t_0 = T \) and then
\[ I(t) > 0, \quad \forall t \in [0, T). \]

The global existence can be easily established by repeating the steps of the proof of Theorem 3.1 [40].
5. Stability. In this section, we state and prove our stability result. We start by establishing several lemmas needed for the proof of our main result.

Lemma 5.1. Assume that \( g \) satisfies (A1). Then, for \( u \in H^2_0(\Omega) \), we have
\[
\int_\Omega \left( \int_0^t g(t-s)(u(t)-u(s)) \, ds \right)^2 \, dx \leq c(g \Delta u)(t)
\]
and
\[
\int_\Omega \left( \int_0^t g'(t-s)(u(t)-u(s)) \, ds \right)^2 \, dx \leq -c(g' \Delta u)(t).
\]

Proof.
\[
\int_\Omega \left( \int_0^t g(t-s)(u(t)-u(s)) \, ds \right)^2 \, dx = \int_\Omega \left( \int_0^t \sqrt{g(t-s)} \sqrt{g(t-s)}(u(t)-u(s)) \, ds \right)^2 \, dx.
\]
By applying Cauchy-Schwarz’ and Poincaré’s inequalities, we can show that
\[
\int_\Omega \left( \int_0^t g(t-s)(u(t)-u(s)) \, ds \right)^2 \, dx \leq \int_\Omega \left( \int_0^t g(t-s)ds \right) \left( \int_0^t (g(t-s))ds \right) \left( \int_0^t (g(t-s))ds \right) \, dx \leq (1-t)c(g \Delta u)(t)
\]
\[
\leq c(g \Delta u)(t).
\]
Similarly, the second inequality in Lemma 5.1 can be proved. \( \square \)

Lemma 5.2. Assume that \( g \) satisfies (A1) and (A2). Then
\[
\int_0^\infty \xi(t)g^{1-\sigma}(t)dt < \infty, \quad \forall \sigma < 2-p.
\]

Proof. Using (A1) and (A2), we easily see that, for any \( \sigma < 2-p \),
\[
\xi(t)g^{1-\sigma}(t) = \xi(t)g^{1-\sigma}(t)g^p(t)g^{-p}(t) \leq -g'(t)g^{1-\sigma-p}(t).
\]
Integrate the last inequality over \((0, \infty)\), we obtain
\[
\int_0^\infty \xi(t)g^{1-\sigma}(t)dt \leq -\int_0^\infty g'(t)g^{1-\sigma-p}(t)dt = \left[ \frac{g^{2-p-\sigma}(t)}{2-p-\sigma} \right]_0^\infty < \infty.
\]
\( \square \)

Similar to Cavalcanti and Oquendo [10], we can easily have the following lemma:

Lemma 5.3. Assume that (A1) – (A3) and (4.5) hold and \( u \) is a solution of (1.1). Then, for any \( 0 < \sigma < 1 \), we have
\[
(g \Delta u)(t) \leq c \left( \int_0^\infty g^{1-\sigma}(t)dt \right) E(0)^{\frac{p-1}{p-\sigma}} (g^p \Delta u)^{\frac{\sigma}{p+\sigma}}(t).
\]
By taking \( \sigma = \frac{1}{2} \), we get
\[
(g \Delta u)(t) \leq c \left( \int_0^t g^{\frac{1}{2}}(s)ds \right)^{\frac{2p-2}{p}} (g^p \Delta u)^{\frac{1}{p+1}}(t) \quad (5.3)
\]
and, for any $\epsilon_0 \in (0, 1),$
\[
(g \Delta u)_{\frac{1}{2\epsilon_0}}(t) \leq c_{\frac{1}{\epsilon_0}} \left( \int_0^t g^\frac{1}{2}(s)ds \right)^{\frac{2p-2}{2p-4}} (g^0 \Delta u)^{\frac{1}{2p-4}}(t).
\] (5.4)

**Corollary 5.4.** Assume that $(A1) - (A3)$ and $(4.5)$ hold and $u$ is a solution of $(1.1).$ Then
\[
\xi(t)(g \Delta u)(t) \leq c(-E'(t))^\frac{1}{p-1}
\] and, for any $\epsilon_0 \in (0, 1),$
\[
\xi(t)(g \Delta u)_{\frac{1}{2\epsilon_0}}(t) \leq c_{\epsilon_0}(-E'(t))^\frac{1}{p-1}\epsilon_0.
\] (5.6)

**Proof.** Multiply both sides of (5.3) by $\xi(t)$ and use (5.2) and (2.6) to obtain
\[
\xi(t)(g \Delta u)(t) \leq c\xi^\frac{2p-2}{2p-4}(t) \left( \int_0^t g^\frac{1}{2}(s)ds \right)^{\frac{2p-2}{2p-4}} \xi^\frac{1}{2p-4}(t)(g^0 \Delta u)^{\frac{1}{2p-4}}(t)
\] \[
\leq c\left( \int_0^t \xi(s)g^\frac{1}{2}(s)ds \right)^{\frac{2p-2}{2p-4}} (\xi g^0 \Delta u)^{\frac{1}{2p-4}}(t)
\] \[
\leq c\left( \int_0^\infty \xi(s)g^\frac{1}{2}(s)ds \right)^{\frac{2p-2}{2p-4}} (-g^0 \Delta u)^{\frac{1}{2p-4}}(t)
\]
\[
\leq c(-E'(t))^\frac{1}{p-1}.
\] (5.7)

For the proof of (5.6), using (5.5) and because $\xi$ is nonincreasing, we obtain
\[
\xi(t)(g \Delta u)_{\frac{1}{2\epsilon_0}}(t) = \xi^\frac{1}{\epsilon_0}(t) (\xi(t)(g \Delta u)(t))_{\frac{1}{\epsilon_0}} \leq c_{\epsilon_0}(-E'(t))^\frac{1}{p-1}\epsilon_0.
\]

\[\square\]

**Lemma 5.5.** Assume that $(A1) - (A3)$ and $(4.5)$ hold. Then the functional
\[
\psi(t) = \int_\Omega uu dx
\]
satisfies, along the solutions of $(1.1),$
\[
\psi'(t) \leq ||u_t||_2^2 - \frac{\ell}{2}||\Delta u||_2^2 - ||u||_2^2 + \int_\Omega u^2 \ln|u|^k dx + c(g \Delta u)(t).
\] (5.8)

**Proof.** By using Eq. $(1.1),$ we easily see that
\[
\psi'(t) = ||u_t||_2^2 - ||\Delta u||_2^2 - ||u||_2^2 + \int_\Omega \Delta u \int_t^\infty g(t-s)\Delta u(s)ds dx
\] \[
+ \int_\Omega u^2 \ln|u|^k dx.
\] (5.9)

We now use Lemma 5.1 and Young’s inequality, to obtain, for any $\mu > 0,$
\[
\int_\Omega \Delta u(t) \left( \int_0^t g(t-s)\Delta u(s)ds \right) dx
\] \[
\leq \left( 1 - \ell + \frac{\mu}{2} \right) ||\Delta u||_2^2 + \frac{1}{2\mu}(1 - \ell)(g \Delta u)(t).
\] (5.10)

By choosing $\mu = \ell$ and combining (5.9) and (5.10), we obtain (5.8). \[\square\]
Lemma 5.6. Assume that (A1) – (A3) and (4.5) hold. Then the functional
\[ \chi(t) = - \int_0^t \int_\Omega g(t-s)(u(t) - u(s))dsdx \]
satisfies, along the solutions of (1.1) and for any \( \epsilon_0 \in (0, 1) \) and \( \delta > 0 \),
\[ \chi'(t) \leq \delta \| \Delta u \|^2 + \frac{c}{\delta} (g \Delta u)(t) + \frac{c}{\delta} (-g' \Delta u)(t) + \left( \delta - \int_0^t g(s)ds \right) \| u_t \|^2 \]  
(5.11)

Proof. Direct computations, using (1.1), yield
\[ \chi'(t) = \int_\Omega \Delta u \int_0^t g(t-s)(\Delta u(t) - \Delta u(s))dsdx + \int_\Omega u \int_0^t g(t-s)(u(t) - u(s))dsdx 
+ \int_\Omega \int_0^t g(t-s)(\Delta u(t) - \Delta u(s))ds \int_0^t g(t-s)\Delta u(s)dsdx 
- \int_\Omega u \ln |u|^k \int_0^t g(t-s)(u(t) - u(s))dsdx 
- \int_\Omega u_t \int_0^t g'(t-s)(u(t) - u(s))dsdx 
- \left( \int_0^t g(s)ds \right) \int_\Omega u_t^2 dx. \]  
(5.12)

Similarly to (5.9), we estimate the right-hand side terms of (5.12). So, by using
Young’s inequality, the first term gives, for any \( \delta > 0 \),
\[ \int_\Omega \Delta u \int_0^t g(t-s)(\Delta u(t) - \Delta u(s))dsdx \leq \frac{\delta}{4} \| \Delta u \|^2 + \frac{c}{\delta} (g \Delta u)(t). \]  
(5.13)

Using Lemma 5.1, Young’s and Poincaré’s inequalities, the second and fifth terms lead to
\[ \int_\Omega u \int_0^t g(t-s)(u(t) - u(s))dsdx \leq \frac{\delta}{4} \| \Delta u \|^2 + \frac{c}{\delta} (g \Delta u)(t) \]  
(5.14)
and
\[ - \int_\Omega u_t \int_0^t g'(t-s)(u(t) - u(s))dsdx \leq \delta \| u_t \|^2 - \frac{c}{\delta} (g' \Delta u)(t). \]  
(5.15)

Similarly, the third term can be estimated as follows
\[ \int_\Omega \int_0^t g(t-s)(\Delta u(t) - \Delta u(s))ds \int_0^t g(t-s)\Delta u(s)dsdx \leq \frac{\delta}{4} \| \Delta u \|^2 + c \left( 1 + \frac{1}{\delta} \right) (g \Delta u)(t). \]  
(5.16)

Let \( \epsilon_0 \in (0, 1) \) and \( f(s) = s^{\epsilon_0} (| \ln s | - s) \). Notice that \( f \) is continuous on \((0, \infty)\) and
its limit at 0 is 0, and its limit at \( \infty \) is \( -\infty \). Then \( f \) has a maximum \( d_{\epsilon_0} \) on \([0, \infty)\),
so the following inequality holds:
\[ s | \ln s | \leq s^2 + d_{\epsilon_0} s^{1-\epsilon_0}, \forall s > 0. \]  
(5.17)
Applying this inequality to \( u \ln |u| \), using the embedding of \( H_0^2(\Omega) \) in \( L^\infty(\Omega) \) and performing the same calculations as before, we get, for any \( \delta_1 > 0 \),
\[
\int_\Omega u \ln |u|^k \int_0^t g(t-s)(u(t)-u(s))dsdx \\
\leq k \int_\Omega (u^2 + d_{co}|u|^{1-c_{\psi}}) \int_0^t g(t-s)(u(t)-u(s))dsdx \\
\leq c \int_\Omega |u| |t|^{1-c_{\psi}}(u(t)-u(s))ds dx \\
+ \delta_1 \int_\Omega u^2 dx + c_{co, \delta_1} \int_\Omega \left| \int_0^t g(t-s)(u(t)-u(s))ds \right|^{2-d_{\psi}} dx \\
\leq c\delta_1 ||\Delta u||_2^2 + c \delta_1 \int_\Omega \left| \int_0^t g(t-s)(u(t)-u(s))ds \right|^{1-d_{\psi}} dx \\
+ c_{co, \delta_1} \int_\Omega \left| \int_0^t g(t-s)(u(t)-u(s))ds \right|^{1-d_{\psi}} dx,
\]
then, putting \( \frac{2}{3} = \frac{d_{\psi}}{2} \) and using H"older’s inequality and Lemma 5.1, we find
\[
\int_\Omega u \ln |u|^k \int_0^t g(t-s)(u(t)-u(s))dsdx \leq \frac{\delta}{4} ||\Delta u||_2^2 + \frac{c}{\delta} (go\Delta u)(t) \\
+ c_{co, \delta}(go\Delta u)^{1/1-d_{\psi}}(t). \tag{5.18}
\]
The above inequalities imply (5.11). \( \Box \)

**Lemma 5.7.** Assume that (A1) – (A3) and (4.5) hold and let \( \epsilon_0 \in (0,1) \). Then, for \( k \) small enough, there exist two positive constants \( \epsilon_1 \) and \( \epsilon_2 \) such that the functional
\[
L(t) = E(t) + \epsilon_1 \psi(t) + \epsilon_2 \chi(t)
\]
satisfies
\[
L \sim E \tag{5.19}
\]
and, for any \( t_0 > 0 \), there exists a positive constant \( m \) such that
\[
L'(t) \leq -mE(t) + c(go\Delta u)(t) + c_{co}(go\Delta u)^{1/1-d_{\psi}}(t), \quad \forall t \geq t_0. \tag{5.20}
\]

**Proof.** For the proof of (5.19), we see that, using similar calculations as before,
\[
|L(t) - E(t)| = |\epsilon_1 \psi(t) + \epsilon_2 \chi(t)| \\
\leq c(\epsilon_1 + \epsilon_2) \left( ||u||_2^2 + ||\Delta u||_2^2 + (go\Delta u)(t) \right),
\]
therefore, from (4.6) and (4.8), we obtain
\[
|L(t) - E(t)| \leq c(\epsilon_1 + \epsilon_2) \left( \frac{1}{2} ||u||_2^2 + J(t) \right) = c(\epsilon_1 + \epsilon_2) E(t),
\]
then
\[
(1 - c(\epsilon_1 + \epsilon_2)) E(t) \leq L(t) \leq (1 + c(\epsilon_1 + \epsilon_2)) E(t).
\]
Hence, for \( \epsilon_1, \epsilon_2 > 0 \) satisfying
\[
1 - c(\epsilon_1 + \epsilon_2) > 0, \tag{5.21}
\]
the equivalence (5.19) holds.
Now, we prove inequality (5.20). Since $g$ is positive and $g(0) > 0$ then, for any $t_0 > 0$, we have
\[
\int_0^t g(s)ds \geq \int_0^{t_0} g(s)ds = g_0 > 0, \quad \forall t \geq t_0.
\]
By using (2.6), (5.8), (5.11) and the definition of $E(t)$, then, for $t \geq t_0$ and any $m > 0$, we have
\[
L'(t) \leq -mE(t) - \left( \varepsilon_2(g_0 - \delta) - \varepsilon_1 - \frac{m}{2} \right) \|u_t\|^2_2
- \left( \frac{\ell}{2} \varepsilon_1 - \varepsilon_2 \delta - \frac{m}{2} \right) \|\Delta u\|^2_2
- \left( \varepsilon_1 - \frac{(k + 2)m}{4} \right) \|u\|^2_2
\]
\[
+ \left( \varepsilon_1 + \varepsilon_2 \frac{c}{\delta} + \frac{m}{2} \right) (g_0 \Delta u)(t)
+ \left( \frac{1}{2} - \frac{c\varepsilon_2}{\delta} \right) (g'\Delta u)(t) + \varepsilon_2 c_{\varepsilon_0, \delta} (g_0 \Delta u)^{\frac{1}{\alpha+1}}(t). \tag{5.22}
\]
Using the Logarithmic Sobolev inequality, for $0 < m < 2\varepsilon_1$, we get
\[
L'(t) \leq -mE(t) - \left( \varepsilon_2(g_0 - \delta) - \varepsilon_1 - \frac{m}{2} \right) \|u_t\|^2_2
- \left( \frac{\ell}{2} \varepsilon_1 - \varepsilon_2 \delta - \frac{m}{2} \right) \|\Delta u\|^2_2
- \left( \varepsilon_1 - \frac{m(k + 2)}{4} \right) + k \left( \varepsilon_1 - \frac{m}{2} \right) (1 + \ln a) + k \left( \frac{m}{4} - \frac{\varepsilon_1}{2} \right) \|u\|^2_2
\]
\[
+ \left( \varepsilon_1 + \varepsilon_2 \frac{c}{\delta} + \frac{m}{2} \right) (g_0 \Delta u)(t)
+ \left( \frac{1}{2} - \frac{c\varepsilon_2}{\delta} \right) (g'\Delta u)(t) + \varepsilon_2 c_{\varepsilon_0, \delta} (g_0 \Delta u)^{\frac{1}{\alpha+1}}(t). \tag{5.23}
\]
At this point we choose $\delta$ so small that
\[
g_0 - \delta > \frac{1}{2}g_0 \quad \text{and} \quad \delta < \frac{\ell g_0}{16}.
\]
Whence $\delta$ is fixed, the choice of any two positive constants $\varepsilon_1$ and $\varepsilon_2$ satisfying
\[
\frac{g_0}{4} \varepsilon_2 < \varepsilon_1 < \frac{g_0}{2} \varepsilon_2 \tag{5.24}
\]
will make
\[
k_1 := \varepsilon_2(g_0 - \delta) - \varepsilon_1 > 0 \quad \text{and} \quad k_2 := \frac{\ell}{2} \varepsilon_1 - \varepsilon_2 \delta > 0.
\]
Then, we choose $\varepsilon_1$ and $\varepsilon_2$ so small so that (5.21) and (5.24) remain valid and, further,
\[
\frac{1}{2} - \frac{c\varepsilon_2}{\delta} > 0.
\]
Consequently, we get (5.19) and
\[ L'(t) \leq -mE(t) - \left( k_1 - \frac{m}{2} \right) \|u_t\|_2^2 \]
\[ - \left( k_2 - \frac{m}{2} - k \left( \varepsilon_1 - \frac{m}{2} \right) \frac{c_p a^2}{2\pi} \right) \|\Delta u\|_2^2 \]
\[ - \left( \varepsilon_1 - \frac{m(k + 2)}{4} + k \left( \varepsilon_1 - \frac{m}{2} \right) (1 + \ln a) + k \left( \frac{m}{4} - \frac{\varepsilon_1}{2} \right) \ln \|u\|_2^2 \right) \|u\|_2^2 \]
\[ + c(g_0\Delta u)(t) + c_{\alpha_0,\gamma}(g_0\Delta u)^{1+\frac{1}{\alpha}}(t). \]
(5.25)

Then, using (3.8) and selecting \( m \) and \( k \) so small that
\[ \alpha_1 = k_1 - \frac{m}{2} > 0, \quad \alpha_2 = k_2 - \frac{m}{2} - k \left( \varepsilon_1 - \frac{m}{2} \right) \frac{c_p a^2}{2\pi} > 0 \]
and
\[ \alpha_3 = \varepsilon_1 - \frac{m(k + 2)}{4} + k \left( \varepsilon_1 - \frac{m}{2} \right) (1 + \ln a) + k \left( \frac{m}{4} - \frac{\varepsilon_1}{2} \right) \ln \|u\|_2^2 > 0. \]
Therefore, we arrive at the desired result (5.20).

**Remark 5.8.** Using (2.1), (2.5), (4.1), (4.6) and (4.8), we have
\[ E(t) = J(t) + \frac{1}{2}\|u_t(t)\|_2^2 \geq J(t) \geq \frac{l}{6}\|\Delta u(t)\|_2^2, \]
then, using (2.6),
\[ \|\Delta u(t)\|_2^2 \leq \frac{6}{l} E(t) \leq \frac{6}{l} E(0). \]
(5.26)

So, from (2.6) and using Young’s inequality, we get
\[ |E'(t)| = \frac{1}{2} g(t) \|\Delta u(t)\|_2^2 - \frac{1}{2} (g' o \Delta u)(t) \]
\[ \leq \frac{1}{2} g(t) \|\Delta u(t)\|_2^2 - \int_0^t g'(t-s) \left( \|\Delta u(t)\|_2^2 + \|\Delta u(s)\|_2^2 \right) ds \]
\[ \leq \frac{6}{l} \left( \frac{1}{2} g(t) + 2g(0) - 2g(t) \right) E(0) \]
\[ \leq cE(0). \]
(5.27)

**Theorem 5.9.** Let \( (u_0, u_1) \in H_0^2(\Omega) \times L^2(\Omega), \quad \epsilon \in (0, 2p - 1) \) and \( t_0 > 0 \). Assume that (A1) – (A3) and (4.5) hold. Then, for \( k \) small enough, there exists a positive constant \( K \) such that the solution of (1.1) satisfies
\[ E(t) \leq K \left( 1 + \int_{t_0}^t \xi^{2p-1+\epsilon}(s) ds \right)^{\frac{-1}{2p-1+\epsilon}}, \quad \forall t \geq t_0. \]
(5.28)

Moreover, if there exist \( \epsilon_1 \in (0, 2p - 1) \) and \( t_0 > 0 \) such that
\[ \int_{t_0}^\infty \left( 1 + \int_{t_0}^t \xi^{2p-1+\epsilon_1}(s) ds \right)^{\frac{-1}{2p-1+\epsilon_1}} dt < \infty, \]
(5.29)

then, for any \( \epsilon \in (0, p) \) and \( t_0 > 0 \), there exists a positive constant \( \widetilde{K} \) such that the solution of (1.1) satisfies
\[ E(t) \leq \widetilde{K} \left( 1 + \int_{t_0}^t \xi^{p+\epsilon}(s) ds \right)^{\frac{-1}{p+1+\epsilon}}, \quad \forall t \geq t_0. \]
(5.30)
Proof. We multiply (5.20) by \(\xi(t)\) and use Corollary 5.4 and (5.27) to get, for any \(t \geq t_0\),
\[
\xi(t)L'(t) \leq -m\xi(t)E(t) + c(-E'(t))^{\frac{1}{p-1}} + c(-E'(t))^{\frac{1}{(p-1)(1+\epsilon_0)}}
\]
\[
\leq -m\xi(t)E(t) + c(-E'(t))^{\frac{1}{p-1}} + c(-E'(t))^{\frac{1}{(p-1)(1+\epsilon_0)}}
\]
\[
+ c(-E'(t))^{\frac{1}{(p-1)(1+\epsilon_0)}}
\]
\[
\leq -m\xi(t)E(t) + c(-E'(t))^{\frac{1}{(p-1)(1+\epsilon_0)}}
\]
\[
\forall t \geq t_0.
\]
Multiply the last inequality by \(\xi'(t)E(t)\), where \(\gamma = (2p-1)(1+\epsilon_0)-1\), and notice that \(\xi' \leq 0\) to obtain
\[
\xi^{\gamma+1}(t)E(t)\xi'(t)L(t) \leq -m\xi^{\gamma+1}(t)E^{\gamma+1}(t) + c(\xi(t))^\gamma (t)(-E'(t))^{\frac{1}{\gamma+1}}, \quad \forall t \geq t_0.
\]
Use of Young’s inequality, with \(q = \gamma + 1\) and \(q^* = \frac{\gamma+1}{\gamma}\), gives, for any \(\varepsilon' > 0\),
\[
\xi^{\gamma+1}(t)E(t)\xi'(t)L(t) \leq -m\xi^{\gamma+1}(t)E^{\gamma+1}(t) + c(\varepsilon'\xi^{\gamma+1}(t)E^{\gamma+1} - c\varepsilon' E'(t))
\]
\[
= -(m - \varepsilon'c)\xi^{\gamma+1}(t)E^{\gamma+1} - cE'(t), \quad \forall t \geq t_0.
\]
We then choose \(0 < \varepsilon' < \frac{m}{c}\) and recall that \(\xi' \leq 0\) and \(E' \leq 0\), to get, for \(c_1 = m - \varepsilon'c\),
\[
\left(\xi^{\gamma+1}E^\gamma L + cE\right)'(t) \leq \xi^{\gamma+1}(t)E(t)\xi'(t)L(t) \leq -c_1\xi^{\gamma+1}(t)E^{\gamma+1}(t) - cE'(t), \quad \forall t \geq t_0,
\]
which implies
\[
\left(\xi^{\gamma+1}E^\gamma L + cE\right)'(t) \leq -c_1\xi^{\gamma+1}(t)E^{\gamma+1}(t), \quad \forall t \geq t_0.
\]
Let \(F = \xi^{\gamma+1}E^\gamma L + cE\). Then \(F \sim E\) (thanks to (5.19)) and
\[
F'(t) \leq -c_1\xi^{\gamma+1}(t)E^{\gamma+1}(t) = -c_1\xi^{(2p-1)(1+\epsilon_0)}(t)E^{(2p-1)(1+\epsilon_0)}(t), \quad \forall t \geq t_0.
\]
Integrating over \((t_0, t)\) and using the fact that \(F \sim E\), we obtain (5.28) with \(\epsilon = (2p-1)\epsilon_0\).

To establish (5.30), we use the idea of Messaoudi and Al-Khulaifi [41]. Let
\[
\eta(t) = \int_0^t \|\Delta u(t) - \Delta u(t-s)\|^2 ds.
\]
Using (5.28), (5.29) and (5.26), we have
\[
\eta(t) \leq 2 \int_0^t \left(\|\Delta u(t)\|^2 + \|\Delta u(t-s)\|^2\right) ds
\]
\[
\leq \frac{12}{T} \int_0^t (E(t) + E(t-s)) ds
\]
\[
\leq \frac{24}{T} \int_0^t E(s) ds < \frac{24}{T} \int_0^\infty E(s) ds < \infty.
\]
This implies that
\[
\sup_{t > 0} \eta^{\frac{1}{\beta}}(t) < \infty.
\]
(5.32)
Assume that \(\eta(t) > 0\). Then, because \(\xi\) is nonincreasing, we find
\[
\xi(t)(g \circ \Delta u)(t) \leq \frac{\eta(t)}{\eta(t)} \int_0^t (\xi^\beta(s)g^\beta(s))^{\frac{1}{\beta}} \|\Delta u(t) - \Delta u(t-s)\|^2 ds.
\]
Applying Jensen’s inequality to get
\[
\xi(t)(g \circ \Delta u)(t) \leq \eta(t) \left( \frac{1}{\eta(t)} \int_0^t \xi^p(s) g^p(s) \|\Delta u(t) - \Delta u(t-s)\|_2^2 \, ds \right)^{\frac{1}{p}}.
\]

Therefore, using (A2) and (5.32) we obtain
\[
\xi(t)(g \circ \Delta u)(t) \leq \eta^{-\frac{1}{p}}(t) \left( \xi^{p-1}(0) \int_0^t \xi(s) g^p(s) \|\Delta u(t) - \Delta u(t-s)\|_2^2 \, ds \right)^{\frac{1}{p}}
\]
\[
\leq c(-g' \circ \Delta u)^\frac{1}{p}(t),
\]
and then, according to (2.6),
\[
\xi(t)(g \circ \Delta u)(t) \leq c(-E'(t))^\frac{1}{p}.
\]
(5.33)

So, since \( \xi \) is nonincreasing,
\[
\xi(t)(g \circ \Delta u) \leq (\xi^\alpha(t)(g \circ \Delta u)(t))^{\frac{1}{1+\alpha}}
\]
\[
\leq (\xi^\alpha(0)(g \circ \Delta u)(t))^{\frac{1}{1+\alpha}}
\]
\[
\leq c(\xi(t)(g \circ \Delta u)(t))^{\frac{1}{1+\alpha}}
\]
\[
\leq c(-E'(t))^{\frac{1}{p(1+\alpha)}}.
\]
(5.34)

If \( \eta(t) = 0 \), then \( s \to \Delta u(s) \) is a constant function on \([0, t]\). Therefore,
\[
(g \circ \Delta u)(t) = 0,
\]
and hence (5.33) and (5.34) hold.

Now, multiplying (5.20) by \( \xi(t) \) and using (5.27), (5.33) and (5.34) to find, for any \( t \geq t_0 \) (as for (5.31)),
\[
\xi(t)L'(t) \leq -m\xi(t)E(t) + c(-E'(t))^{\frac{1}{p}} + c(-E'(t))^{\frac{1}{p(1+\alpha)}}
\]
\[
\leq -m\xi(t)E(t) + c(-E'(t))^{\frac{1}{p(1+\alpha)}} (-E'(t))^{\frac{1}{p(1+\alpha)}} + c(-E'(t))^{\frac{1}{p(1+\alpha)}}
\]
\[
\leq -m\xi(t)E(t) + c(-E'(t))^{\frac{1}{p(1+\alpha)}}, \quad \forall t \geq t_0.
\]
(5.35)

Inequality (5.31) with \( 2p - 1 \) replaced by \( p \) is exactly (5.35). Then, the proof of (5.30) can be completed as for the one of (5.28) (by taking \( \gamma = p(1 + \epsilon_0) - 1 \) and \( \epsilon = p\epsilon_0 \)). This completes the proof of our main result.

**Remark 5.10.** We note here that \( 2p - 2 + \epsilon \) and \( p - 1 + \epsilon \) can be arbitrary close to \( 2p - 2 \) and \( p - 1 \), respectively, since \( \epsilon \) can be arbitrary close to zero. On the other hand, in the absence of the logarithmic “forcing” term \( (k = 0) \), the estimates (5.17) and (5.18) drop out and, consequently, (5.20) takes the form
\[
L'(t) \leq -mE(t) + c(go \Delta u)(t), \quad \forall t \geq t_0.
\]
(5.36)

In this case, we obtain the following result:

**Theorem 5.11.** Let \( (u_0, u_1) \in H^2_0(\Omega) \times L^2(\Omega) \) and \( t_0 > 0 \). Assume that (A1) – (A2) hold. Then, there exists a positive constant \( K \) such that the solution of (1.1) satisfies, for all \( t \geq t_0 \),
\[
E(t) \leq Ke^{-\lambda \int_{t_0}^t \xi(s) \, ds} \quad \text{if} \quad p = 1
\]
(5.37)
and
\[ E(t) \leq K \left( 1 + \int_{t_0}^t \xi^{2p-1}(s) ds \right)^{-\frac{1}{2p-2}} \quad \text{if} \quad 1 < p < \frac{3}{2}, \quad (5.38) \]

Moreover, if \( 1 < p < \frac{3}{2} \) and
\[ \int_0^\infty \left( 1 + \int_{t_0}^t \xi^{2p-1}(s) ds \right)^{-\frac{1}{2p-2}} dt < \infty, \quad (5.39) \]

then
\[ E(t) \leq K \left( 1 + \int_{t_0}^t \xi^p(s) ds \right)^{-\frac{1}{p-1}}, \quad \forall t \geq t_0. \quad (5.40) \]

**Remark 5.12.** This result \((k = 0)\) improves and generalizes many results in the literature such as Han and Wang [26].

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