Abstract

The purpose of this paper is to ensure the conditions of Gärtner-Ellis Theorem for evaluations of the empirical measure. We show that up-to-date conditions for ensuring the convergence to a quasi-stationary distribution can be applied efficiently. By this mean, we are able to prove Large Deviation results even with a conditioning that the process is not extinct at the end of the evaluation. The domain on which these Large Deviation results apply is implicitly given by the range of penalization for which one can prove the above-mentioned results of quasi-stationarity. We propose a way to relate the range of controlled deviations to the range of admissible penalization. Central Limit Theorems are deduced from these results of Large Deviations. As an application, we consider the empirical measure of position and jumps of a continuous-time process on an unbounded domain of \( \mathbb{R}^d \). This model is inspired by the adaptation of a population to a changing environment. Jumps makes it possible for the process to face a deterministic dynamics leading to high extinction areas.

0 Introduction

Large deviations theory on empirical measures started by Cramér in the 30s [9] (translated in [24]) and has been extensively generalized. The theory has initially developed to quantify these fluctuations for sums of independent variables, leading to the core Sanov theorem [15]. Proving a large deviations principle for correlated processes is much more intricate. A milestone in the theory is the series of papers by Donsker and Varadhan [10]-[14] and the dual approach followed by Gärtner and Ellis [18], [16]. The strategy of the former works is to build explicitly lower and upper large deviations bounds from the Tchebychev inequality and the Girsanov theorem [25]. On the other hand, the Gärtner–Ellis theorem relies on the existence and regularity of a free energy functional. These results have been strengthened into the exact value of the dominant term, what is called the strong large deviations [6]. The Central Limit Theorem is deduced and part of the proof.

These central ideas have guided the approach we follow while looking at the empirical measure of a general Markov process with possibly extinction and Feynman-Kac functional. It appears that in our setting, the free energy functional can be interpreted as the growth rate associated to a parametrized family of Feynman-Kac functionals. Each of these functionals produces a penalization of the original process for which we can study a conditional dynamics. This provides insight on the dynamics of the process leading to a specific large deviation of the empirical measure in a far future. In our result, to each admissible penalization \( \theta \) is associated a given growth rate \( \Lambda(\theta) \), a given typical deviation \( \psi(\theta) \), but also a quasi-stationary distribution \( \alpha^{(\theta)} \)
and a survival capacity $h^{(θ)}$. The two former quantities can actually be simply obtained from the two latter ones. Under our conditions, we deduce that both $α^{(θ)}$ and $h^{(θ)}$ are derivable in $θ$ (with respect respectively to the total variation and to the infinite norm). The fact that $ψ$ is derivable is ensured by these means. The expressions of these derivatives are calculated in terms of $α^{(θ)}$ and $h^{(θ)}$, and the semigroup $(P^{(θ)}_t)$. The strict convexity of the growth rate function $Λ$ exploits the convergence results of quasi-ergodicity, which is apparently not classical.

Looking at [6] or [20], it appears that strong large deviation results require an even more precise result, usually deduced from the characteristic functions. Yet, we could also deduce results of Central limit theorem that makes it possible to prove that the decay is actually quicker than the exponential rate.

Surprisingly to us, the use of results of convergence to quasi-stationary distribution appear rarely mentioned in the literature, and mainly in recent years. It is however present in the initial approach by Donsker and Varadhan. To our knowledge, such approach is only explicitly described in [23], in a few derived papers as in [21], and in [17]. As in our approach, the latter extend by this way the alternative approach for the convergence to QSDs that they have just obtained. Especially in the case of continuous-time and continuous-space processes, spectral methods are most commonly exploited and lead to the study of Dirichlet forms, as notably in [23] or [21]. In the case of discrete-time processes, perturbation theories have also been been introduced to look at such large deviations [19]. Strong conditions on the analyticity of the Fourier transform of the generator seem to be assumed and no conditioning on survival is considered there. Besides, no practical condition is given to extend the range of controlled deviations beyond a local estimate. The justifications of the derivability are however closely related.

The current paper exploits up-to-date techniques that we highlight in [26] and [27] in the sequel of [3] for the convergence to a unique QSD for general strong Markov processes in continuous-time and continuous-space. The adaptation of the criteria to discrete-time or discrete-space Markov processes can easily been derived.

The outline of the paper is as follow : Section 1 contains our more general results. After presenting the notions of exponential quasi-ergodicity on which we rely in Section 1.1 and 1.2, the implications for Large Deviations are stated in Section 1.3. Section 1.4 describes sufficient conditions for the exponential quasi-ergodicity. Section 1.5 contains additional results of quasi-ergodicity that we can deduce and are required for the application to Large Deviations. An illustration of these techniques is provided in Section 2, for which we prove that the range of controlled deviations is unbounded. The proofs of the results given in Section 2 are completed in Section 4.

1 Quasi-ergodicity

1.1 Notations

First we need some specification on the state space $X$, besides the fact that it is Polish:

By considering a strong Markov Process under Feynman-Kac penalization, we give ourselves $(Ω; (F_t)_{t≥0}; P^0; (X_t)_{t≥0}; (N_t)_{t≥0}; τ_θ)$, where $P^0$ is a probability measure on $Ω$, while $(F_t)$ is a filtration such that $X_t$ is a strong Markov process for $P^0$ until its extinction time $τ_θ$, while $(N_t)_{t≥0}$ a $F_t$-adapted additive functional (satisfying $N_{t+s} = N_t + N_s ◦ ϑ_t$, with $ϑ$ the shift-operator $ϑ_t(w)_s := w_{t+s}$).
We then define the biased measure with time-horizon of survival \( t \), for \( x \in X \) and \( A_t \in F_t \) by:

\[
P_t^x(A_t) := \mathbb{E}_x^0 \left[ \exp[N_t] ; A_t, t < \tau_0 \right],
\]

and extend it to any \( A \in \Omega \) with the Markov property of \((X_t, N_t)\) by:

\[
P_t^x(A) := \mathbb{E}_x^0 \left[ \exp[N_t] ; \mathbb{E}_x^0(\mathcal{A} \mid F_t), t < \tau_0 \right].
\]

We also consider the renormalized version \( Q_t^x \) of \( P_t^x \) (assuming it is well-defined). It is defined for \( A \in \Omega \) and initial condition \( \mu \in \mathcal{M}_1(X) \) by:

\[
Q_t^x(\mu)(A) := P_t^x(\mu)(A) / \langle \mu \mid \bar{h}_t \rangle
\]

where \( \bar{h}_t(x) := P_t^x(\Omega) \).

We also define the associated semi-group:

\[
\mu P_t(dx) := P_t^x(X_t \in dx)
\]

and the renormalized version:

\[
\mu A_t(dx) := \mu P_t(dx) / \langle \mu \mid \bar{h}_t \rangle = P_t^x(X_t \in dx) / P_t^x(\Omega).
\]

### 1.2 Notion of exponential quasi-ergodicity

As an extension of the properties given in \([27]\), we say that the Feynman-Kac strong Markov Process given by \((\Omega; (F_t)_{t \geq 0}; P_0; (X_t)_{t \geq 0}; (N_t)_{t \geq 0}; \tau_\partial)\) is exponentially quasi-ergodic if all of the following properties hold:

1. **Existence of the quasi-stationary distribution (QSD):**
   - There exists a unique QSD \( \alpha \).
   - \( \alpha \) is associated to some growth rate \( \lambda \):
     \[
     \forall t \geq 0, \ P_\alpha(t < \tau_0) = e^{\lambda t} \quad \text{so that as a QSD, } \alpha P_t = e^{\lambda t} \alpha \quad (1)
     \]
   - cf e.g. Theorem 2.2 in \([8]\). This motivates the definition:
     \[
     h_t(x) := e^{-\lambda t} \bar{h}_t(x) = \delta_x P_t(X) / \alpha P_t(X).
     \]

2. **Existence and convergence of the survival capacity:**
   - We have exponential convergence of \((h_t)_{t \geq 0}\) to \( h \) in the uniform norm (with the same rate \( \zeta \) as in inequality (2)). More precisely, there exists \( C > 0 \) such that:
     \[
     \forall t > 0, \ |h_t - h|_\infty \leq C e^{-\zeta t}. \quad (2)
     \]
   - where the function \( h \), which describes the 'survival capacity' of the initial condition, is positive, bounded on \( X \) and vanishes on \( \partial \). It also belongs to the domain of the infinitesimal generator \( \mathcal{L} \), associated with the semi-group \((P_t)_{t \geq 0}\) on \((B(X \cup \{\partial\}); \|\cdot\|_\infty)\), and satisfies:
     \[
     \mathcal{L} h = \lambda h, \quad \text{so } \forall t \geq 0, \ P_t h = e^{\lambda t} h. \quad (3)
     \]

3. **Convergence to the quasi-stationary distribution (QSD):**
   - We have exponential convergence to \( \alpha \) of the MCNE's. More precisely, there exists \( C, \zeta > 0 \) such that:
     \[
     \forall t > 0, \forall \mu \in \mathcal{M}_1(X), \ \|\mu A_t(dx) - \alpha(dx)\|_{TV} \leq \frac{C}{\langle \mu \mid h \rangle} e^{-\zeta t}. \quad (4)
     \]
1.3 Theorems

1.3.1 Large Deviations

Considering bounded measurable functions $G : X \mapsto \mathbb{R}$ and $F : X \times X \mapsto \mathbb{R}$, the latter vanishing on the diagonal, we denote:

$$(S_t \mid G) := \frac{1}{t} \int_0^t G(X_s) \, ds, \quad \langle J_t \mid F \rangle := \frac{1}{t} \sum_{s \leq t} F(X_{s-}, X_s), \quad \Psi_t := \langle S_t \mid G \rangle + \langle J_t \mid F \rangle \quad (1)$$

Our aim is to describe the large deviations of $\Psi_t$ conditionally on the survival of the process $X$ up to time $t$. Without loss of generality (given the possible choices for $G$ and $F$), these results naturally extend to any $\phi_1(x) := \frac{1}{\gamma}(\phi_1 J_t \mid F)$, with $\phi_1, \phi_2 > 0$.

Our interest will be on the $\theta \in \mathbb{R}$ such that the Feynman-Kac strong Markov Processes given by $(\Omega; (F_t)_{t \geq 0}; \mathbb{P}^\theta; (\theta t \Psi_t)_{t \geq 0}; \tau_{\theta})$, is exponentially quasi-ergodic. The set of such $\theta$ is denoted $\Theta_e$. For any $\theta \in \Theta_e$, the associated extinction rate, QSD and survival capacities are then denoted respectively $\lambda^{(\theta)}$, $\alpha^{(\theta)}$ and $h^{(\theta)}$.

We also consider the $\theta \in \mathbb{R}$ such that there exists $\delta \theta > 0$ such that $(\theta - \delta \theta, \theta + \delta \theta) \subset \Theta_e$ and that $\langle \alpha^{(\theta)} \mid h^{(\theta)} \rangle$ and $\langle \alpha^{(\theta)} \mid h^{(\theta)} \rangle$ are uniformly lower-bounded for $\theta \in (\theta - \delta \theta, \theta + \delta \theta)$. We finally require that the convergence parameters $C(\theta), \zeta(\theta)$ involved in the quasi-ergodicity estimates (2) and (3) can be chosen uniformly for $\theta \in (\theta - \delta \theta, \theta + \delta \theta)$. The set of $\theta$ satisfying both conditions is denoted $\Theta_e$.

We also assume that the jumps of $X$ are given by a Poisson Point Process with the rate $h(x, y) \nu(dy)$ for a jump from position $x$ to $y$, with $\nu$ some reference measure on $X$. The jump rate at position $x$ is the given by: $h_j(x) := \int_X h(x, y) \nu(dy)$. We assume in the following that $h_j(x)$ is bounded. (definition of the PPP for the first jump, from position $x$ to $y$ with law $h(x, y)/h_j(x) \nu(dy)$).

The two following quantities then play a crucial role in such Large Deviation estimates.

$$\psi_F(\theta) := \int_X \alpha^{(\theta)}(dx) \int_X \nu(dy) h(x, y) h^{(\theta)}(y) \exp[\theta F(x, y)] F(x, y),$$

$$\psi_G(\theta) := \int_X \alpha^{(\theta)}(dx) h^{(\theta)}(x) G(x) = \langle \beta^{(\theta)} \mid G \rangle, \quad \psi_{F,G}(\theta) := \psi_F(\theta) + \psi_G(\theta)$$

Theorem 1.1. Assume that $h_j$ is bounded, $0 \in \Theta_e$ and $\psi_{F,G}(\Theta_e)$ is not a singleton. Recall that $G$ and $F$ are bounded, the latter vanishing on the diagonal.

Then, $\psi_{F,G}$ is strictly increasing on $\Theta_e$. For any $\gamma \in \psi_{F,G}(\Theta_e \cap \mathbb{R}_+)$, any $\mu \in \mathcal{M}_2(X)$ :

$$\lim_{t \to \infty} \frac{-1}{t} \log \mathbb{P}_\mu[\Psi_t \leq \gamma \mid t < \tau_\theta] = \xi \circ \psi_{F,G}(\gamma).$$

Similarly, for any $\gamma \in \psi_{F,G}(\Theta_e \cap \mathbb{R}_-)$, any $\mu \in \mathcal{M}_2(X)$ :

$$\lim_{t \to \infty} \frac{-1}{t} \log \mathbb{P}_\mu[\Psi_t \geq \gamma \mid t < \tau_\theta] = \xi \circ \psi_{F,G}(\gamma).$$

1.3.2 The range of controlled deviations

Reversely, the existence of such penalization intensity $\theta$ under which $\Psi_t \sim \gamma$ can be deduced from an estimate that even more exceptional trajectories do not decay faster in time than exponential. We can thus deduce that strong values of $\theta$ shall be able to amplify trajectories for which $\Psi_t \leq \gamma$ so as to make them typical. This approach is illustrated by the following lemma.
Lemma 1.3.1. Assume that for some \( \theta_0 \in \Theta_e \), \( \gamma < \psi_{F,G}(\theta_0) \) and \( \mu \in \mathcal{M}_1(X) \), there exists \( C, \chi > 0 \) such that for any \( t \) sufficiently large:

\[
\mathbb{Q}_{\mu}^{(\theta_0),t}(\Psi_t \leq \gamma) \geq C \exp(-\chi t).
\]

Then, for any \( \epsilon > 0 \) and any \( \theta < \theta_0 - \chi/\epsilon \):

\[
\lim_{t \to \infty} \mathbb{Q}_{\mu}^{(\theta_0),t}(\Psi_t \leq \gamma + \epsilon) = 1.
\]

Symmetrically, assume that for some \( \theta_0 \in \Theta_e \), \( \gamma > \psi_{F,G}(\theta_0) \) and \( \mu \in \mathcal{M}_1(X) \), there exists \( C, \chi > 0 \) such that for any \( t \) sufficiently large:

\[
\mathbb{Q}_{\mu}^{(\theta_0),t}(\Psi_t \geq \gamma) \geq C \exp(-\chi t).
\]

Then, for any \( \epsilon > 0 \), any \( \theta > \theta_0 + \chi/\epsilon \):

\[
\lim_{t \to \infty} \mathbb{Q}_{\mu}^{(\theta_0),t}(\Psi_t \geq \gamma + \epsilon) = 1.
\]

One may obtain such estimate of the decay by looking at a restricted regeneration estimate like in the following lemma. Recall that we already know: \( \mathbb{P}_\mu(t < \tau_0) \sim \langle \mu \mid h \rangle \exp(\ell t) \).

Lemma 1.3.2. Assume that there exists \( D_r, t_a, c_r, t_r > 0 \) such that:

\[
\mathbb{P}_\mu(X_{t_a} \in D_r \mid t_a < \tau_0) > 0,
\forall x \in D_r, \mathbb{P}_x(X_{t_a} \in D_r \mid \Psi_{t_a} \leq \gamma, t_a < \tau_0) \geq c_r.
\]

Then, there exists \( C > 0 \), \( \chi := -\log(c_r)/t_r \) such that, for any \( \epsilon \), for any \( t \) sufficiently large:

\[
\mathbb{P}_\mu(\Psi_t \leq \gamma + \epsilon ; t < \tau_0) \geq C \exp(-\chi t).
\]

Note however that we have a priori no guarantee that the process is exponentially quasi-ergodic under these penalization intensities \( \theta \). This property is however required to ensure that the asymptotic decay of probability is independent of the initial condition. It is also very useful to obtain regularity properties like in the Proposition 1.3.3.

1.3.3 Application for Central Limit Theorems

Theorem 1.2. For any \( \theta \in \Theta_e \), and any family \( \theta_t \) such that \( (\theta_t - \theta) = O(t^{-1/2}) \), the Central Limit Theorem holds for the Feynman-Kac penalizations associated with \( \theta_t \) at time \( t \). It means that for any \( \mu \in \mathcal{M}_1(X) \) and for any \( f \) bounded continuous from \( \mathbb{R} \) to \( \mathbb{R} \), the following convergence holds:

\[
\langle Q_{\mu}^{(\theta_t),t} \mid f \rangle = \frac{\mathbb{E}_\mu \left[ f(\sqrt{t}[\Psi_t - \psi_{F,G}(\theta_t)]) \exp(t \theta_t \Psi_t) \mid t < \tau_0 \right]}{\mathbb{E}_\mu \left[ \exp(t \theta_t \Psi_t) \mid t < \tau_0 \right]} \to \mathbb{E} \left[ f \left( \sqrt{\partial_{\theta} \psi_{F,G}(\theta)} \times N \right) \right],
\]

where \( N \) follows the standard normal distribution.

1.3.4 Gärtner-Ellis conditions: regularity and strict monotonicity

They involve the following formal derivatives \( \partial_{\theta}^F \mathcal{L}^{(g)} \), defined for any \( \mu \in \mathcal{M}_1(X) \) and \( g \) bounded measurable by:

\[
\langle \mu \mid \partial_{\theta}^F \mathcal{L}^{(g)} \mid g \rangle := \int_X \mu(dx) \int_X h(x,y) \nu(dy) \exp[\theta F(x,y)] F(x,y) g(y),
\]

\[
\langle \mu \mid \partial_{\theta}^G \mathcal{L}^{(g)} \mid g \rangle := \int_X \mu(dx) G(x) g(x), \quad \partial_{\theta} \mathcal{L}^{(g)} := \partial_{\theta}^G \mathcal{L}^{(g)} + \partial_{\theta}^F \mathcal{L}^{(g)}.
\]
Proposition 1.3.3. Assume that $F$ and $h_f$ are bounded. Then, $\psi_F, \psi_G$ and $\xi$ are derivable on $\Theta_e$, with the following derivatives:

$$
\partial_\theta \psi_G(\theta) = -\int_0^\infty d\tau \left[ \langle \alpha(\theta) | \partial_\theta \mathcal{L}^{(\theta)}(\cdot, e^{-t\lambda(\theta)} P_t^{(\theta)} \partial_\theta \mathcal{L}^{(\theta)} | h^{(\theta)} \rangle \right] + 2 \langle \alpha(\theta) | \partial_\theta \mathcal{L}^{(\theta)}(\cdot, e^{-t\lambda(\theta)} P_t^{(\theta)} \partial_\theta \mathcal{L}^{(\theta)} | h^{(\theta)} \rangle \\
- 2 \langle \alpha(\theta) | \partial_\theta \mathcal{L}^{(\theta)} | h^{(\theta)} \rangle \times \langle \alpha(\theta) | \partial_\theta \mathcal{L}^{(\theta)} | h^{(\theta)} \rangle \right],
$$

$$
\partial_\theta \psi_F(\theta) = \int_X \alpha(\theta)(dx) \int_X \nu(dy) h(x, y) h^{(\theta)}(y) \exp(\theta F(x, y)) |F(x, y)|^2 \\
- \int_0^\infty d\tau \left[ \langle \alpha(\theta) | \partial_\theta \mathcal{L}^{(\theta)}(\cdot, e^{-t\lambda(\theta)} P_t^{(\theta)} \partial_\theta \mathcal{L}^{(\theta)} | h^{(\theta)} \rangle \right] + 2 \langle \alpha(\theta) | \partial_\theta \mathcal{L}^{(\theta)}(\cdot, e^{-t\lambda(\theta)} P_t^{(\theta)} \partial_\theta \mathcal{L}^{(\theta)} | h^{(\theta)} \rangle \\
- 2 \langle \alpha(\theta) | \partial_\theta \mathcal{L}^{(\theta)} | h^{(\theta)} \rangle \times \langle \alpha(\theta) | \partial_\theta \mathcal{L}^{(\theta)} | h^{(\theta)} \rangle \right],
$$

$$
\partial_\theta \xi(\theta) = -\theta \partial_\theta \psi_{F,G}(\theta).
$$

More precisely, $\lambda^{(\theta)}$, $\alpha^{(\theta)}$ and $h^{(\theta)}$ are derivable as functions of $\theta \in \Theta_e$ with the following formulas for the derivatives:

$$
\partial_\theta \lambda^{(\theta)} = \langle \alpha^{(\theta)} | \partial_\theta \mathcal{L}^{(\theta)} | h^{(\theta)} \rangle = \psi_{F,G}(\theta),
$$

$$
\partial_\theta \alpha^{(\theta)}(dx) = \int_0^\infty d\tau \left[ -\langle \alpha^{(\theta)} | \partial_\theta \mathcal{L}^{(\theta)}(\cdot, e^{-t\lambda(\theta)} P_t^{(\theta)} \partial_\theta \mathcal{L}^{(\theta)} | h^{(\theta)} \rangle \right] + \langle \alpha^{(\theta)} | \partial_\theta \mathcal{L}^{(\theta)} | h^{(\theta)} \rangle \alpha^{(\theta)}(dx),
$$

$$
\partial_\theta h^{(\theta)}(x) = \int_0^\infty d\tau \left[ -\langle \delta_x | e^{-t\lambda^{(\theta)} P_t^{(\theta)}} \partial_\theta \mathcal{L}^{(\theta)} | h^{(\theta)} \rangle - \langle \alpha^{(\theta)} | \partial_\theta \mathcal{L}^{(\theta)} | h^{(\theta)} \rangle h^{(\theta)}(x) \right] + 2 \langle \alpha^{(\theta)} | \partial_\theta \mathcal{L}^{(\theta)} | h^{(\theta)} \rangle \cdot h^{(\theta)}(x).
$$

From the Gartner-Ellis theorem, see for instance p45 in [15], Theorem 1.4 is deduced from the property that $\psi_{F,G}$ is derivable on $\Theta_e$ and the following proposition that implies the strict convexity of $\lambda$ (where we also exploit the fact that $\psi_{F,G}(\theta) = \partial_\theta \lambda^{(\theta)}$).

Proposition 1.3.4. The function $\psi_{F,G}$ is strictly increasing on $\Theta_e$.

**Remark:** As stated in Propositions 1.5.2 and 1.5.3, $\psi_G(\theta)$ and $\psi_F(\theta)$ are the values for which for any $\mu$ and $\epsilon > 0$:

$$
\lim_{t \to \infty} \mathbb{Q}_\mu^{(\theta), t} \left[ \| S_t | G \| - \psi_G(\theta) \right] > \epsilon = 0.
$$

$$
\lim_{t \to \infty} \mathbb{Q}_\mu^{(\theta), t} \left[ \| J_t | F \| - \psi_F(\theta) \right] > \epsilon = 0.
$$

A priori, we do not require that $\Theta_e$ is a connected set or that $\theta$ is in the connected component of 0 in $\Theta_e$. We cannot think however on examples for which $\Theta_e$ would not be an interval. We see in Theorem 1.3.3 that $\psi_{F,G}$ is an increasing function. If $\Theta_e$ happens to be an interval (as we expect), the same will be for $\psi_{F,G}(\Theta_e)$.

1.4 Some criterion to find elements of $\Theta_e$

To justify the quasi-ergodicity of the process biased by $N^{(\theta)} := (\theta t \left[ \langle S_t | G \rangle + \langle J_t | F \rangle \right])_{t \geq 0}$, we propose to exploit the criteria given in [5] or [27]. These criteria have been obtained in the case of sub-Markovian semi-groups, where only an extinction event is considered. Provided that $F$ is non-negative and $\theta$ is non-positive, such Feynman-Kac penalization can be translated into the conditioning of the process by an extended extinction time, as shown in the next paragraphs.
Note also that \( Q_{\mu, t}^{(\theta)} \) is identical when \( G \) is replaced by \( G + \|G\|_{\infty} \) (due to the normalizing factor). We assume without loss of generality that \( G \) is also non-negative.

Let us define, with \( T \) an exponential variable of rate 1 independent of \( F_{\infty} \):

\[
\tau^{(\theta)}_{\partial} := \tau_{\partial} \wedge \inf \left\{ t \geq 0 : \theta \left[ \int_{0}^{t} G(X_s) \, ds + \sum_{s \leq t} F(X_{s-}, X_s) \right] > T \right\}.
\]

Then, for any initial condition \( \mu \) and positive measurable function \( f \), since \( F \) is positive:

\[
\mathbb{E}_{\mu} \left[ f(X_t) ; t < \tau^{(\theta)}_{\partial} \right] = \mathbb{E}_{\mu} \left[ f(X_t) P_{\mu} \left( T > \theta \left[ \int_{0}^{t} G(X_s) \, ds + \sum_{s \leq t} F(X_{s-}, X_s) \right] \right) ; t < \tau_{\partial} \right].
\]

In particular, considering also the case \( f \equiv 1 \):

\[
\mathbb{E}_{\mu} \left[ f(X_t) \mid t < \tau^{(\theta)}_{\partial} \right] = \langle \mu A^{N^{(\theta)}} \mid f \rangle.
\]

Of course, this seriously limits the range of application of Theorem 1.1. A paper in preparation aims at extending these results to semi-groups that are not necessarily sub-Markovian.

But for the moment, \( \theta \in \Theta_e \) if we can ensure one of the following set of assumptions, either (U) or the more general (A).

### 1.4.1 (U) : the uniform case

In \cite{3}, Champagnat and Villemonais provide necessary and sufficient conditions for having exponential quasi-ergodicity with a convergence to \( \alpha \) that is uniform over the initial condition. The interested reader may find the relation between the exponential rate of decay and the parameters of these assumptions. These two conditions, that we denote by (U), are the following:

- **(U1) : "Mixing property"
  There exists a probability measure \( \zeta \in \mathcal{M}_1(\mathcal{X}) \) and \( c, t > 0 \) such that:
  \[
  \forall x \in \mathcal{X}, \quad P_x [X_t \in dx ; t < \tau_{\partial}] \geq c \, \zeta(dx).
  \]

- **(U2) : "Asymptotic comparison of survival"
  \[
  \limsup_{t \to \infty} \sup_{x \in \mathcal{X}} \frac{P_x(t < \tau_{\partial})}{P\zeta(t < \tau_{\partial})} < \infty.
  \]

### 1.4.2 (A) : delay depending on the initial condition

The following conditions are taken from our previous work. One can find in \cite{26} and \cite{27} the proof of the exponential quasi-ergodicity and several applications to see how to derive these conditions.

- **(A0) : "Exhaustion of \( \mathcal{X} \)"
  There exists a sequence \( (\mathcal{D}_\ell)_{\ell \geq 1} \) of closed subsets of \( \mathcal{X} \) such that:
  \[
  \forall \ell \geq 1, \quad \mathcal{D}_\ell \subset \mathcal{D}_{\ell+1} \quad \text{and} \quad \bigcup_{\ell \geq 1} \mathcal{D}_\ell = \mathcal{X}.
  \]

Let \( \mathcal{D} \) be the set of compact subsets of \( \mathcal{X} \) that are included in at least one \( \mathcal{D}_\ell \). We recall also that for any set \( \mathcal{D} \), we defined its exit and its hitting times as:

\[
T_{\mathcal{D}} := \inf \{ t \geq 0 ; X_t \notin \mathcal{D} \}, \quad \tau_{\mathcal{D}} := \inf \{ t \geq 0 ; X_t \in \mathcal{D} \}.
\]
(A1) : "Mixing property"
There exists a probability measure $\zeta \in \mathcal{M}_1(\mathcal{X})$ such that, for any $\ell \geq 1$, there exists $L > \ell$ and $c, t > 0$ such that:
\[
\forall x \in \mathcal{D}_E, \quad \mathbb{P}_x[X_t \in dx; \ t < \tau_0 \wedge T_{\mathcal{D}_E}] \geq c \zeta(dx).
\]

(A2) : "Escape from the Transitory domain"
For a given $\rho > 0$ and $\mathcal{D}_E \in \mathcal{D}$:
\[
e_T := \sup_{\{x \in \mathcal{X}\}} \mathbb{E}_x(\exp[\rho(\tau_0 \wedge \tau_{\mathcal{D}_E})]) < \infty.
\]
The exponential moment in the previous condition is required to be larger than the following "survival estimate":
\[
\rho_S := \sup \left\{ \rho \geq 0 \left| \sup_{t \geq 1} \liminf_{\tau_0 \wedge \tau_{\mathcal{D}_E}} \mathbb{P}_\zeta(t < \tau_\theta \wedge T_{\mathcal{D}_E}) = 0 \right\} \vee 0.
\]

(A3) : "Asymptotic comparison of survival"
For a given $\mathcal{D}_E \in \mathcal{D}$ and $\zeta \in \mathcal{M}_1(\mathcal{X})$:
\[
\lim_{t \to \infty} \sup_{x \in \mathcal{D}_E} \mathbb{P}_x(t < \tau_\theta) \mathbb{P}_\zeta(t < \tau_\theta) < \infty
\]

This last assumption may be difficult to ensure and it may be useful to rather rely on the following one:

(A3F) : "Absorption with failures"
Given $\zeta \in \mathcal{M}_1(\mathcal{X})$, $\rho > 0$ and $\mathcal{D}_E \in \mathcal{D}$, for any $\epsilon \in (0, 1)$, we can find $t, c > 0$ such that for any $x \in \mathcal{D}_E$ there exists a stopping time $U_A$ such that:
\[
\{\tau_\theta \wedge t \leq U_A\} = \{U_A = \infty\} \quad \text{and} \quad \mathbb{P}_x(U_A = \infty, t < \tau_\theta) \leq \epsilon \exp(-\rho t),
\]
while for some stopping time $V$, possibly depending on $x$:
\[
\mathbb{P}_x(X(U_A) \in dx'; \ U_A < \tau_\theta) \leq c \mathbb{P}_\zeta(X(V) \in dx'; \ V < \tau_\theta).
\]
We further require that there exists a stopping time $U^*_A$ extending $U_A$ in the following sense:
- $U^*_A := U_A$ on the event $\{\tau_\theta \wedge U_A < \tau_{\mathcal{D}_E}\}$, where $\tau_{\mathcal{D}_E} := \inf\{s \geq t : X_s \in \mathcal{D}_E\}$.
- On the event $\{\tau_{\mathcal{D}_E} \leq \tau_\theta \wedge U_A\}$ (which by construction equals $\{\tau_{\mathcal{D}_E} \leq \tau_\theta\} \cap \{U_A = \infty\}$) and conditionally on $\mathcal{F}_{\tau_{\mathcal{D}_E}}$, the law of $U^*_A$ coincides with the one of $\bar{U}_{Abs}$ for a realization $\bar{X}$ of the Markov process $(X_t, t \geq 0)$ with initial condition $\bar{X}_0 := X(\tau_{\mathcal{D}_E})$ and independent of $X$ conditionally on $X(\tau_{\mathcal{D}_E})$. (that is a strong Markov property at time $\tau_{\mathcal{D}_E}$ for the law of $\bar{X}$).

As presented in Lemma 3.0.1 of [26] and Remark 2.4.1 of [27], the value of $\rho_S$ and the pairs $(\rho, \mathcal{D}_E)$ for which Assumption (A3F) holds do not depend on the specific choice of $\zeta$ as long as it satisfies Assumption (A1).

1.4.3 The general sets of Assumptions
We say that assumption (A) holds, whenever:
*Assumptions (A0), (A1C) hold as well as (A2) for some $\zeta \in \mathcal{M}_1(\mathcal{X})$. Moreover, there exist $\rho > \rho_S$ and $\mathcal{D}_E \in \mathcal{D}$ such that assumptions (A4) and either (A3) or (A3F) hold.*

By Theorems 2.1, 2.2, and 2.3 in [27], Assumption (A) with the extinction time $\tau_0^{(\theta)}$ is sufficient to ensure $\theta \in \Theta_c$. 

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1.4.4 How do the assumptions change with $\theta$?

Since the jump rate and $F$ are bounded, assumption $(A1_G)$ for $\tau_0$ implies its extension $(A1_G^{(\theta)})$ for $\tau_0^{(\theta)}$ with any $\theta < 0$. Similarly, we expect that assumption $(A1)$ for $\tau_0$ generalizes simply into its extension $(A1^{(\theta)})$. In this view, the dependency of $m$ and $t$ in $n$ shall keep the same, while the efficiency $c$ should be smaller as $|\theta|$ gets large. Likewise, assumption $(A3_F)$ shall generalize, up to a lower value of $c$. Since the extinction is larger with $\tau_0^{(\theta)}$ than with $\tau_0$, the event $\{U_A = \infty, t < \tau_0^{(\theta)}\}$ is still exceptional for the rate $\rho$. The generalization of $(A2)$ is also immediate. The main issue is however that the value $\rho_s^{(\theta)}$ might largely drop as $\theta$ gets large. Therefore, the reference value for $\rho^{(\theta)}$ in $(A3_F^{(\theta)})$ and $(A2^{(\theta)})$ has to be adjusted, which might become difficult. After all, the uniqueness might stop to hold after a specific threshold.

In conclusion, if we can prove $(A)$ for $\tau_0$ while imposing any positive value for $\rho$, the proof should imply quite easily (still depending on the arguments involved) its generalized version with $\tau_0^{(\theta)}$. We can find an example of this generalization in our first illustration.

It can however be false when one refers to $(A)$ with only the requirement that $\rho > \rho_s$, in part because the process may become degenerate. As we shall see in our second illustration, too strong a penalization may lead the associated $Q$-process to stay confined in a subspace of $X$ (for instance of lower dimension). The rate of decay might then depend strongly on the initial condition.

1.4.5 Alternative assumptions?

Extended results have concurrently been obtained for cases where the QSD is not the limit for any initial condition and the survival capacity is a priori unbounded [5], [1]. Under their conditions, there is still a Yaglom limit, i.e. a QSD that attracts the distributions starting from any Dirac mass. The current approach may be adapted to this case, yet require to specify some control on the tails of distributions.

1.5 Main steps towards the proofs of Theorems 1.3.3 and Proposition 1.3.4

Lemma 1.5.1. Assume (4) and (2). Then, there exists $\zeta, C > 0$ such that for any signed measure $\mu$ with $||\mu||_{TV} = 1$ (in particular for any probability measure):

$$\left\| \mu(e^{-t \lambda} P_t) - \langle \mu \mid h \rangle \alpha \right\|_{TV} \leq C \exp[-\zeta t].$$

As a corollary, for any signed measure $\mu$ with finite total variation and such that $\langle \mu \mid h \rangle = 0$, we have:

$$\left\| \mu(e^{-t \lambda} P_t) \right\|_{TV} \leq C \exp[-\zeta t] \|\mu\|_{TV}.$$

In addition, for any bounded measurable function $\psi$ such that $\langle \alpha \mid \psi \rangle = 0$, we have:

$$\|e^{-t \lambda} P_t \psi\|_\infty \leq C \exp[-t \zeta] \|\psi\|_\infty.$$

This Lemma is Lemma 4.2.2 in [27], and we refer for its proof to the latter article.

Proposition 1.5.2. Assume the same condition of quasi-ergodicity as in Theorem 1.4. Then, for any bounded measurable function $\hat{G} : X \mapsto \mathbb{R}$:

$$\lim_{t \to \infty} \mathbb{E}_\mu^{(\theta),t} \left[ \frac{1}{T} \int_0^T \hat{G}(X_s) ds - \langle \alpha^{(\theta)} \mid h^{(\theta)} \times \hat{G} \rangle \right] > \epsilon = 0.$$
Proposition 1.5.3. Assume the same condition of quasi-ergodicity as in Theorem 1.1. Then, for any bounded measurable function \( \hat{F} : X \times X \to \mathbb{R} \) vanishing on the diagonal:

\[
\lim_{t \to \infty} Q^{(\theta)}_t \left[ \sum_{s \leq t} \hat{F}(X_{s-}, X_s) - \int_{X \times X} J^{(\theta)}(dx, dy) \hat{F}(x, y) \right] > \epsilon = 0.
\]

with the notation

\[
J^{(\theta)}(dx, dy) := \alpha^{(\theta)}(dx) h(x, y) \nu(dy) \exp[F(x, y)] h^{(\theta)}(y).
\]

The integral involved in the previous probability shall be written in short as \( \langle J^{(\theta)} \mid \hat{F} \rangle \).

As a first step towards Theorem 1.1, we prove the following intermediate proposition.

Proposition 1.5.4. Assume that \( F \) and \( h, J \) are bounded. Then, \( \lambda^{(\theta)}, \alpha^{(\theta)}, h^{(\theta)}, \psi_G \) and \( \psi_F \) are Lipschitz-continuous on \( \Theta_v \).

Remark: To give some insight on proposition 1.5.2, it is noticeable that the distribution \( h^{(\theta)}(x) \alpha^{(\theta)}(dx) \) is actually the stationary distribution of what is called the Q-process. This process can be seen as the initial process conditioned upon the event that the extinction epoch \( \tau^{(\theta)}_\partial \) happens in a far future. The following results regarding this process are obtained in [3] (in the context of uniform convergence) and simply extended to the contexts of [26] and [27].

1. There exists a family \( (Q_x)_{x \in X} \) of probability measures on \( \Omega \) defined by:

\[
\lim_{t \to \infty} Q^{(\theta)}_t(\Lambda_s) = Q_x(\Lambda_s),
\]

for all \( \mathcal{F}_t \)-measurable set \( \Lambda_s \). The process \( (\Omega; (\mathcal{F}_t)_{t \geq 0}; (X_t)_{t \geq 0}; (Q_x)_{x \in X}) \) is an \( X \)-valued homogeneous strong Markov process.

2. The transition kernel of the Markov process \( X \) under \( (Q_x)_{x \in X} \) is given by:

\[
g(x; t; dy) = e^{-\lambda t} h(y) \delta_x P_t(dy) / h(x).
\]

The Q-process can thus be described as the \( h \)-transform of the initial process \( P \), by analogy to the original definition of \( h \)-transform where \( h \) is required to be harmonic (corresponding to \( \lambda = 0 \)).

3. The measure \( \beta(dx) := h(x) \alpha(dx) \) is the unique invariant probability measure under \( Q \). Moreover, for any measure \( \mu \in M_1(X) \) such that \( (\mu \mid h) < \infty \):

\[
\forall t > 0, \quad \|Q^{(\theta)}_t[X_t \in dx] - \beta(dx)\|_{1/h} \leq C \inf_{u \geq 0} \|\mu - u \beta\|_{1/h} e^{-\zeta t},
\]

where \( Q^{(\theta)}_t(dw) := \int_X \mu(dx) Q_x(dw), \quad \|\nu\|_{1/h} := \left\|\frac{\nu(dx)}{h(x)}\right\|_{TV} \).

These results imply that conditionally on the event \( \{T < \tau^{(\theta)}_\partial\} \), and given \( t = pT \) for \( p \in (0, 1) \), the law of \( X_t \) is approximated by \( \beta(dx) \) as \( T \to \infty \).
2 Illustrative example: Mutations compensating a drift leading to extinction

2.0.1 Description of the process:

We consider as an example the process $X$ given in [24] where jumps are to compensate a drift leading to the deadly regions for the process. For simplicity, the drift is assumed to be constant, which leads us to the following system:

$$X_t = x - vt e_1 + \int_{[0,t] \times \mathbb{R}^d \times \mathbb{R}^+} w 1_{\{w \leq h(x_{t-}, w)\}} M(ds, dw, du) \quad \text{as long as } t < \tau_d, \quad (9)$$

$M$ is a Poisson point process (PPP) over $\mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}_+$, with intensity $\pi(ds, dw, du) = ds \, dw \, du$, while $h(x, w)$ describes the rate of invasion of a new mutation of size $w$ in a population with type $x$, and the change in type is given by the constant speed $v > 0$. Moreover, the extinction occurs at time $\tau_\theta$ with the rate $\rho_c(x), x \in \mathbb{R}^d$, i.e.:

$$\tau_\theta := \inf\{t \geq 0 : \int_0^t \int_{\mathbb{R}^d} \rho_c(x, w) M^E(ds, du) \geq 1\},$$

where $M^E$ is again a PPP, with intensity $ds \, dw$, independent of $M$.

- $(E_1)$ $\rho_c$ is locally bounded and $\lim_{\|x\| \to \infty} \rho_c(x) = +\infty$. Also, explosion implies extinction:

$$\tau_\theta \leq \sup_{\{\ell \geq 1\}} T_{D_\ell}.$$  

$(E_2)$ The jump-rate is assumed to be globally bounded:

$$\forall x \in \mathbb{R}^d, \quad 0 < h_f(x) := \int_{\mathbb{R}^d} h(x, w) dw \leq \rho_f'.$$

$h$ is measurable and for any compact $K \subset \mathbb{R}^d$, there exists:

$(E_3)$ a lower-bound $h_\lambda > 0$ on jumps able to compensate the drift, with some $0 < \delta S < S$:

$$\forall x \in K, \forall w \in B(S e_1, \delta S), \quad h(x, w) \geq h_\lambda.$$

$(E_4)$ a tightness estimate for the jumps:

for any $\epsilon > 0$, there exists $n_\epsilon$ such that:

$$\forall x \in K, \int_{\mathbb{R}^d} h(x, w)/h_f(x) 1_{\{\|w\| \geq n_\epsilon\}} dw \leq \epsilon.$$

$(E_5)$ an upper-bound $h_\nu$ on the density for each jump:

$$\forall x \in K, \forall w \in \mathbb{R}^d, \quad \|h(x, w)\| \leq h_\nu h_f(x).$$

$(E_6)$ $G : \mathbb{R}^d \to \mathbb{R}_+$ is positive measurable bounded. $F : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}_+$ is positive measurable bounded and vanishes on the diagonal, with $\theta < 0$.

**Theorem 2.1.** Consider the process $X$ given by equation (9) under $(E)$. Then, with $D_\ell := \bar{B}(0, \ell), \ell \geq 1$, $(A^{(0)})$ holds for any $\theta > 0$, that is $\theta \in \Theta_\cdot$ Moreover, there are lower-bound on $\zeta = \zeta^{(0)}$ and upper-bound on $C(\mu) = C^{(0)}(\mu)$ uniformly on the set of Feynman-Kac penalization $N^{(\theta)}_t := \theta \big[ \int_0^t G(X_s) ds + \sum_{s \leq t} F(X_{s-}, X_s) \big]$ for $\theta$ in any compact set of $\mathbb{R}$.

Since $\psi_{F,G}$ is continuous and increasing, we know that $\psi_{F,G}(\Theta_\cdot \cap \mathbb{R}_+)$ is of the form $(\psi(-\infty), \psi(0))$. General results on the value of $\psi_{F,G}(-\infty)$ seem quite difficult to obtain, since it highly depends on the interplay between $F$ and the set of jump distributions that could compensate the drift. Refering to Lemma 1.3.1 and 1.3.2 and adapting the proof of (A1), we can propose the following condition to ensure that $\psi_{F,G}(-\infty) = 0$. 

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Proposition 2.0.1. Assume (E) and that for some $\epsilon > 0$, there exists an open set $D_\epsilon$ such that:

$$\sup \{ F(x, x + S_1 e_1 + z) \mid x \in D_\epsilon, z \in B(0, \delta S) \} \leq \epsilon.$$ 

Then for any $|\theta|$ sufficiently large ($\theta < 0$), with $G \equiv 0$:

$$\lim_{t \to \infty} \mathbb{Q}^{(\theta)}_\mu(t, (J_t \mid F) \leq 2\epsilon) = 1.$$ 

Given previous Theorem 2.1, this implies that for any $\gamma \in (2\epsilon, \psi_{F,G}(0))$, $\psi_{F,G}(\gamma)$ is well-defined and

$$-\frac{1}{t} \log \mathbb{P} [\langle J_t \mid F \rangle \leq \gamma \mid t < \tau_0] \xrightarrow{t \to \infty} \xi \circ \psi_{F,G}(\gamma).$$

Remarks: • We can obtain any value for $\gamma \in (0, \psi_{F,G}(0))$ provided that the above-mentioned condition holds for any $\epsilon > 0$. The value of $S$ can be adjusted provided that Assumption (A) still holds true. This condition is far from being necessary.

• We do not need to assume any comparison between $v$ and $h$. The extinction rate of the QSD can thus be very large, notably for large values of $v$.

2.0.2 Proof of Theorem 2.1

(A0) is clear. By (E), the rate of jumps from any compact set is bounded, so that (A1,G) holds. Since the extinction rate outside of $D_\epsilon$ tends to infinity while $\ell \to \infty$, for any $\rho > 0$, we can find some $D_\epsilon$ for which assumption (A2) is clear (cf. Subsection 3.1.2 in [26]). We defer to the end of this subsection the proof of:

Lemma 2.0.2. Under (E1,A2), for any $\ell \geq 1$, with $L := \ell + 1$, there exists $c, t > 0$ such that:

$$\forall x \in D_\ell, \quad \mathbb{P} [X_t \mid t < \tau_0 \wedge T_{DL}] \geq c \cdot 1_{D_\ell}(dx).$$

In particular, Assumption (A1) holds with $\zeta$ uniform over $D_\ell$.

Remark: 1) The main idea is that we are able to compensate the drift with the jumps and introduce a bit of fluctuations at each step, and thus we are able to ensure a dispersion to any part of the space (while keeping a bound on the norm).

2) Besides (A1), we need a lower-bounded density on any $D_\epsilon$ in order to prove assumption (A3) (with a reaching time and efficiency of course dependent on this set).

3) In a following paper [25], we extend this proof to a coupled process like in our first application.

Proof of Assumption (A3) Consider a compact set $D_E = \tilde{B}(0, n_\epsilon)$ and a test initial condition $x \in D_E$. We also assume that $\epsilon, \tilde{\rho}_S > 0$ are given. W.l.o.g. we assume that $\forall y \notin D_E, \rho_\epsilon(y) \geq \rho_E > \tilde{\rho}_S$. We first define $t_\xi$ by the relation:

$$\exp[\tilde{\rho}_S \times (2 n_\epsilon / v) - (\rho_E - \tilde{\rho}_S) \times (t_\xi - 2 n_\epsilon / v)] = \epsilon / 2.$$ 

The l.h.s. is decreasing and converges to 0 when $t_\xi \to \infty$, so that $t_\xi$ is well-defined. Let $T_{jp}$ be the first jump time of $X$. On the event $\{t_\xi < T_{jp}\}$, we set $U_A = \infty$. The choice of $t_\xi$ is done to ensure that the probability associated to the failure is indeed exceptional enough (with threshold $\epsilon / 2$ and time-penalty $\tilde{\rho}_S$). Any jump occurring before $t_\xi$ thus occur from a position $X(T_{jp}) \in \tilde{B}(0, n_\epsilon + v t_\xi) := K$. By (E), we can then define $n_w$ such that:

$$\forall x \in K, \quad \int_{\mathbb{R}^d} h(x, w) 1_{\{\|w\| \geq n_w\}} dw \leq \epsilon / 2 \times \exp[\tilde{\rho}_S t_\xi].$$
A jump size larger than $n_w$ is then the other criterion of failure.

On the event $\{T_{jp} \leq t\} \cap \{T_{jp} < \tau_0 \cap \|W\| \leq n_w\}$, where $W$ is the size of the first jump (at time $T_{jp}$), we set $U_A := T_{jp} \leq \tau_z$. Otherwise $U_A := \infty$.

In particular, $\{\tau_0 \land t \leq U_A\} = \{U_A = \infty\}$ is clear.

We prove next that the failures are indeed exceptional enough :

$$\mathbb{P}_x(U_A = \infty, \tau_z < \tau_0) \leq \mathbb{P}_x(T_{jp} \leq \tau_0 \land \|W\| > n_w) + \mathbb{P}_x(T_{jp} < \tau_0 \land \|W\| > n_w).$$

By the definition of $n_w$, we deal with the second term :

$$\mathbb{P}_x(T_{jp} < \tau_0 \land \|W\| > n_w) \leq \mathbb{P}_x(\|W\| > n_w | T_{jp} < \tau_0 \land \|W\| < n_w) \leq \epsilon/2 \times \exp[\tilde{\rho}_S \tau_z].$$

On the event $\{\tau_z \leq T_{jp} \land \tau_0\} : \forall x \leq \tau_z, X_t = x - vt e_1$. Thus, $X$ is outside of $\mathcal{D}_E$ in the time-interval $[2 n_c/v, \tau_z]$, with an extinction rate at least $\rho_E$. By the definition of $\tau_z$ :

$$\mathbb{P}_x(\tau_z \leq T_{jp} \land \tau_0 \leq \exp[-\rho_E (\tau_z - 2 n_c/v)] \leq \epsilon/2 \times \exp[\tilde{\rho}_S \tau_z].$$

This concludes : $$\mathbb{P}_x(U_A = \infty, \tau_z < \tau_0) \leq \epsilon \times \exp[\tilde{\rho}_S \tau_z].$$

On the other hand, since the density of jumps and the jump rate are bounded on $K$ :

$$\mathbb{P}_x(X(U_A) \in dx' | T_{jp} < \tau_z \land \tau_0, \|W\| \leq n_w) \leq h_v \mathbf{1}_{\{x' \in B(0, n_c + \ell z + n_w)\}} dx'.$$

We know also from Lemma 2.0.2 that there exists $t_M, c_M > 0$ such that :

$$\mathbb{P}_x(X(t_M) \in dx ; t < \tau_0) \geq c_M \mathbf{1}_{\{x' \in B(0, n_c + \ell z + n_w)\}} dx'.$$

With $U_A = t_M$ and $c_{obs} = h_v / c_M$ and inequality (13), this concludes the proof of (A3P), thus of Theorem 2.1 (given Lemma 2.0.2).

Proof of Lemma 2.0.2 : We wish to decompose the proof into elementary steps. The first idea is that we cut $\mathcal{D}_E$ into very small pieces, both for the initial value (to ensure the uniformity), and the final marginal (to ensure a density), around some reference values, resp. $x_i$ and $x_f$. We also cut the path from the neighborhood of the initial value $x_i$ to the neighborhood of the final value $x_f$ into elementary steps with only one jump in each. We see here that imposing one jump of size in $B(S e_1, \delta S)$ in a time-interval $t_0 := S/v$ gives to the marginal at time $t_0$ a bit of dispersion in a neighborhood of $x_f$.

Let us formalize the above explanation. Changing the scale of $X$, we may assume w.l.o.g. that $S = 1$. We also consider a characteristic length of dispersion by $r := \delta S/4$. Given some $\ell \geq 1$, $x_i \in \mathcal{D}_n$, $L := \ell + 2$ and $c > 0$, we define :

$$\mathcal{R}^{(L)}(c) := \{(t, x_f) \in \mathbb{R}^+ \times \mathbb{R}^d | \forall x_0 \in B(x_i, r), \mathbb{P}_{x_0}[X_t \in dx ; t < T_{D_L}] \geq c \mathbf{1}_{B(x_f, r)}(x) dx\}. $$

The proof then relies on the following three elementary lemmas :

Lemma 2.0.3. Given any $(\ell, L)$ with $\ell \geq 1$, $L = \ell + 2$, $x_i \in \mathcal{D}_n$, $t, c, c' > 0$ and $x \in B(x_i, r)$ :

$$(t, x) \in \mathcal{R}^{(L)}(c) \Rightarrow \{(t + s, y) \mid (s, y) \in \mathcal{R}^{(L)}(c')\} \subset \mathcal{R}^{(L)}(c \times c').$$
Lemma 2.0.4. Given any $L = \ell + 2$, $x_i \in D_\ell$, there exists $c_0, t_0, \delta t > 0$, such that:

$$[t_0, t_0 + \delta t] \times \{x_i\} \subset R^{(L)}(c_0).$$

Lemma 2.0.5. For any $x \in B(0, 2m - 2)$, there exists $t', c' > 0$ such that:

$$(t, x) \in R^{(m)}(c) \quad (\text{for } t, c > 0) \Rightarrow \{t + t'\} \times \{B(x, r) \subset R^{(m)}(c \times c').$$

Lemma 2.0.3 is just an application of the Markov property. The proofs of Lemmas 2.0.4 and 2.0.5 are postponed to Appendix A. Next we show how to deduce Lemma 2.0.2 from them.

Proof of Lemma 2.0.2 Step 1: from the vicinity of any $x_i \in D_\ell$ to the vicinity of any $x_f \in D_\ell$.

Let $K > \|x_f - x_i\|/r$. For $0 \leq k \leq K$, let $x_k := x_i + k(x_f - x_i)/K$. Exploiting Lemma 2.0.4, we choose $(t_0, c_0)$ such that $(t_0, x_i) \in R^{(m)}(c_0)$. By Lemma 2.0.3 and since $x_{k+1} \in B(x_k, r)$ for each $k$, and by induction for any $k \leq K$, there exists of $t_k, c_k > 0$ such that: $(t_k, x_k) \in R^{(m)}(c_k)$. In particular with $k = K$, we get some $t_f, c_f > 0$ such that $(t_f, x_f) \in R^{(m)}(c_f)$.

Step 2: a uniform time and efficiency.

By compactness, there exists $(x^\ell)_\ell \leq L$ such that: $D_\ell \subset \bigcup_{\ell \leq L}B(x^\ell, r)$. Let $t_f$ be the larger time of (as in step 1) needed to reach the vicinity of any $x_f \in \{x^\ell\}$ from any $x_i \in \{x^\ell\}$. To adjust the arrival time, we make the process stay some time around $x_i$. Let $t_a := t_0 \times [1 + t_0/\delta t]$. Then, for any $t \geq t_a$, we can find some $c > 0$ for which $(t, x_i) \in R^{(m)}(c)$. We let the reader deduce this corollary from Lemmas 2.0.3 and 2.0.2. Combining this corollary with Lemma 2.0.3, ensures, with $t_M := t_f + t_a$, a global lower-bound $c_M > 0$ on the efficiency:

$$\forall \ell, \ell' \leq L, \forall x_0 \in B(x^\ell, r), \quad P_{x_0}[X(t_M) \in dx \ ; \ t_M < T_m] \geq cM \chi_{B(x^\ell, r)}(x) dx.$$ 

Since $D_\ell \subset \bigcup_{\ell \leq L}B(x^\ell, r)$, this completes the proof of Lemma 2.0.2.

3 Proof of the results of Section 1

The proofs of this Section are ordered as follows: Lemmas 1.3.1, 1.3.2, Propositions 1.3.3, 1.3.4, and finally Theorem 1.3.3. Proposition 1.3.3 and Theorem 1.3.3 imply locally the conditions of Gärtner-Ellis Theorem. The last proofs are the one of Theorem 1.2 then of Theorem 1.1.

3.1 Proof of Lemmas 1.3.1 and 1.3.2

For the proof of Lemma 1.3.1:

$$Q^\mu_{\ell}(\Psi_t > \gamma + \epsilon) \leq \frac{\exp[t(\theta - \theta_0) \times (\gamma + \epsilon)] E_{\mu}(\exp[\theta_0 \Psi_t] \ ; \ \Psi_t > \gamma + \epsilon , \ t < \tau_0)}{\exp[t(\theta - \theta_0) \times \gamma] E_{\mu}(\exp[\theta_0 \Psi_t] \ ; \ \Psi_t < \gamma , \ t < \tau_0)}$$

$$\leq \frac{\exp[-t(\theta_0 - \theta) \times \epsilon]}{Q^\mu_{\ell}(\Psi_t \leq \gamma)} \xrightarrow{t \to \infty} 0,$$

as soon as $(\theta_0 - \theta) \times \epsilon > \chi$. Considering the complementary event ends the proof of the first statement in Lemma 1.3.1. The second statement is deduced by symmetry.
For the proof of Lemma 1.3.2 since \( \langle J_{t_a} \mid F \rangle \) is a.s. finite, there exists \( \gamma_0 > 0 \) and \( c_0 > 0 \) such that:

\[
\mathbb{P}_\mu(X_{t_a} \in \mathcal{D}_r ; \langle J_{t_a} \mid F \rangle \leq \gamma_0 , \ t_a < \tau_0) \geq c_0. \tag{4}
\]

Consider \( t_k := t_a + k t_r \) for any \( k \geq K \) sufficiently large to ensure:

\[
\frac{t_a \gamma_0}{t_a + k t_r} \leq \epsilon/2, \quad \frac{\gamma + \epsilon}{k - 1} \leq \epsilon/2.
\]

Then, for any \( t \geq t_{K+1} \), choosing \( k \geq K \) such that \( t_{k-1} < t \leq t_k \), we have \( \langle J_t \mid F \rangle \leq \gamma + \epsilon \) and \( t < \tau_0 \) a.s. on the event

\[
\mathcal{R}_k := \{ X(t_a) \in \mathcal{D}_r \} \cap \left\{ \sum_{s \leq t_a} F(X_{s-}, X_s) \leq t_a \gamma_0 \right\} \
\cap \bigcap\limits_{1 \leq \ell \leq k} \{ X(t_\ell) \in \mathcal{D}_r \} \cap \left\{ \sum_{t_{\ell-1} < s \leq t_\ell} F(X_{s-}, X_s) \leq t_\ell \gamma \right\}.
\]

Indeed, on this event, since \( F \) is positive:

\[
\langle J_t \mid F \rangle \leq \frac{t_k}{t} \langle J_{t_a} \mid F \rangle \\
\leq (1 + \frac{t_r}{t_k - t_r}) \times \frac{t_a \gamma_0 + k t_r \gamma}{t_a + k t_r} \\
\leq (1 + \frac{1}{k - 1}) \times (\gamma + \epsilon/2) \leq \gamma + \epsilon.
\]

On the other hand, inductively by the Markov property:

\[
\mathbb{P}_\mu(\mathcal{R}_k) \geq \mathbb{P}_\mu(X_{t_a} \in \mathcal{D}_r ; \langle J_{t_a} \mid F \rangle \leq \gamma_0 , \ t_a < \tau_0) \times \left( \inf_{x \in \mathcal{D}_r} \mathbb{P}_x[X_{t_r} \in \mathcal{D}_r ; \langle J_{t_r} \mid F \rangle \leq \gamma , \ t_r < \tau_0] \right)^k \\
\geq c_0 \exp[-k t_r \chi] \geq c_0 \exp[(t_a - t_r) \chi] \exp[-t \chi],
\]

where we used in the following inequalities that \( \chi = -\log(c_r)/t_r \) and that \( t \geq t_a - t_r + k t_r \). This ends the proof of Lemma 1.3.2. \( \square \)

### 3.2 Proof of Proposition 1.5.2

We exploit the criteria given in Theorem 2.5 in [7]. The principles behind this Theorem are generalized to prove Proposition 1.5.3 by relying on the computation of some conditional expectation and variance. Since the actual form of the Feynman-Kac penalization does not matter here (also \( \theta \) is fixed), we drop the superscripts \( (\theta) \) in the following notations.

It is here sufficient to prove that, for any \( 0 < p < q < 1 \), \( \mu \in \mathcal{M}_1(X) \) and \( f, g \) bounded measurable:

\[
\lim_{t \to \infty} \mathbb{E}_\mu[f(X_{pt}) g(X_{qt}) \mid t < \tau_0] = \langle \beta \mid f \rangle \times \langle \beta \mid g \rangle. \tag{15}
\]

So we compute:

\[
\mathbb{E}_\mu[f(X_{pt}) g(X_{qt}) \mid t < \tau_0] = \frac{\langle \mu \mid h_{pt} \rangle}{\langle \mu \mid h_t \rangle} \int_X \mu A_{pt}(dx) f(x) \mathbb{E}_x \left[ g(X_{(q-p)t}) \exp[\lambda t (1-p)] \mid (1-p) t < \tau_0 \right] \\
= \frac{\langle \mu \mid h_{pt} \rangle}{\langle \mu \mid h_t \rangle} \int_X \mu A_{pt}(dx) f(x) h_{(q-p)t}(x) \int_X \delta_x A_{(q-p)t}(dy) g(y) h_{(1-q)t}(y). \tag{16}
\]

15
The same upper-bounds apply of course to the limiting versions, so that
the main idea is to use Palm formula in combination with the Markov property of

\[ 3.3 \text{ Proof of Proposition 1.5.3} \]

By Theorem 2.5 in [7], it concludes the proof of Proposition 1.5.2.

\[ \alpha \text{ large ensures that} \]

Finally, with the fact that

\[ \langle \text{the expectation and variance of} \ \rangle \]

Likewise, since our convergence in inequality (4) is uniform w.r.t. the initial condition \( x \in D_t \) :

\[ \| h\|_{\infty} \sup_{x \in D_t} \delta_x A_{(q-p)}(x) g(y) h_{(1-q)}(y) \]

The same upper-bounds apply to course to the limiting versions, so that \( \ell \geq 1 \) chosen sufficiently large ensures that \( \alpha(\mathcal{X} \setminus D_t) \) is sufficiently small to neglect the error of considering only \( D_t \).

Finally, with the fact that \( (\mu | h_{\mu}) / (\mu | h_t) \) converges to 1, we have indeed proved equation (16). By Theorem 2.5 in [7], it concludes the proof of Proposition 1.5.3.

\[ \text{3.3 Proof of Proposition 1.5.3} \]

The main idea is to use Palm formula in combination with the Markov property of \( X \) to compute the expectation and variance of \( \langle J_t | F' \rangle \) averaged over \( \exp \left[ \theta \sum_{t \leq s} F(X_{s-}, X_r) \right] \times f(X_t) \times 1_{(s \leq \tau_a)} \).

**Lemma 3.3.1.** Under the condition of Proposition 1.3.3:

\[ Q^{(\theta),t}_x \left( \frac{1}{\tau} \left[ \sum_{s \leq t} \hat{F}(X_{s-}, X_s) \right] \right) \xrightarrow{t \to \infty} \langle J^{(\theta)} | F' \rangle. \]

**Lemma 3.3.2.** Under the condition of Proposition 1.3.3:

\[ t \times (Q^{(\theta),t}_x \left[ \left( \frac{1}{\tau} \left[ \sum_{s \leq t} \hat{F}(X_{s-}, X_s) \right] \right]^2 \right] - Q^{(\theta),t}_x \left( \frac{1}{\tau} \left[ \sum_{s \leq t} \hat{F}(X_{s-}, X_s) \right] \right)^2 ) \xrightarrow{t \to \infty} \langle J^{(\theta)} | \hat{F}^2 \rangle. \]

**3.3.1 Conclusion of Proposition 1.5.3**

From the Markov inequality and Lemmas 3.3.1 and 3.3.2, we obtain for \( t \) sufficiently large:

\[ Q^{(\theta),t}_x \left( \| Z - \langle J^{(\theta)} | \hat{F} \rangle \| \geq \epsilon \right) \xrightarrow{t \to \infty} 0. \]
3.3.2 Computation of the expectation

We first represent the jumps of $X$ through some Poisson Point Process (PPP): 

$$\frac{1}{t} \sum_{s \leq t} \hat{F}(X_{s-}, X_s) = \frac{1}{t} \int_{[0,t]} \int_{\mathbb{R}} M^l(ds, dw, du) \hat{F}(X_{s-}, w) 1_{(u \leq h(X_{s-}, w))},$$

where $M^l$ is the PPP governing the jumps of $X$ with intensity $ds \nu(dw) du$.

\[\exp[-t \lambda] \mathbb{E}_x \left[ \frac{1}{t} \sum_{s \leq t} \hat{F}(X_{s-}, X_s) \right] \times \exp \left[ \theta \sum_{r \leq t} F(X_{r-}, X_r) \right]; \ t < \tau_0 \]

\[\frac{1}{t} \mathbb{E}_x \left[ \int_{[0,t]} \int_{\mathbb{R}} M^l(ds, dw, du) \hat{F}(X_{s-}, w) 1_{(u \leq h(X_{s-}, w))} \times \exp \left[ \theta \sum_{r \leq t} F(X_{r-}, X_r) - t \lambda \right] \times 1_{(t < \tau_0)} \right] \] 

\[= \frac{1}{t} \mathbb{E}_x \left[ \int_{[0,t]} \int_{\mathbb{R}} ds \nu(dw) du \hat{F}(X_{s-}, w) \times 1_{(u \leq h(X_{s-}, w))} \times \exp \left[ \theta \sum_{r < s} F(X_{r-}, X_r) - s \lambda \right] \times 1_{(s < \tau_0)} \right] \times \exp \left[ -(t-s) \lambda \right] \hat{h}_{t-s}^{(t)}(w) \]

where the integrability condition required for using the Palm formula in the last equality is justified below (for the Fubini theorem).

Note that the above definitions imply: \[\exp[-(t-s) \lambda] \hat{h}_{t-s}^{(t)}(w) = h_{t-s}(w).\] So the above integral can be rewritten as:

\[\mathbb{E}_x \left[ \int_{[0,t]} \int_{\mathbb{R}} dp \nu(dw) \hat{F}(X_{pt-}, w) \times h(X_{pt-}, w) \times \exp \left[ \theta \sum_{r < pt} F(X_{r-}, X_r) - pt \lambda \right] \times 1_{(pt < \tau_0)} \right] \times \exp[F(X_{pt-}, w)] \times h_{(1-p)t}^{(t)}(w) \]

\[= \int_{[0,t]} dp \left\langle \left[ \exp[-pt \lambda] \right] \delta_x P_{pt} \ | \Psi_{(1-p)t} \right\rangle \]

where \( \Psi_t(y) := \int \nu(dw) h(y, w) \hat{F}(y, w) \times \exp[F(y, w)] \times h_{t}^{(t)}(w) \)

by Fubini Theorem (where the integrability is ensured by the facts that \( \hat{F}, F \) and the jump rate \( h : y \mapsto \int \nu(dz) h(y, z) \) are bounded). We obtain punctual convergence of the terms in the integral by Lemma [15.5.1].

\[\left\| \exp[-t' \lambda] \delta_x P_{t'} - h^{(t)}(x) \alpha^{(t)} \right\| \leq C \exp[-t' \zeta] \]

\[\left| \Psi_t(z) - \Psi(z) \right| \leq C \exp[-t' \zeta] \times \left\| \hat{F} \right\|_\infty \times \exp[\left\| F \right\|_\infty] \times \left\| h \right\|_\infty \]

where \( \Psi(z) := \hat{F}(z, w) \times h(z, w) \times \exp[F(z, w)] \times h^{(t)}(w) \).

By the dominated convergence theorem:

\[\exp[-t \lambda] \mathbb{E}_x \left[ \frac{1}{t} \left\{ \sum_{s \leq t} \hat{F}(X_{s-}, X_s) \right\} \times \exp \left[ \theta \sum_{r \leq t} F(X_{r-}, X_r) \right]; \ t < \tau_0 \]

\[\xrightarrow{t \to \infty} h^{(t)}(x) \alpha^{(t)} | \Psi = h^{(t)}(x) (\mathcal{F}^{(t)} | \hat{F}) \].

We shall conclude the computation of the expectation by noting that

\[\exp[-t \lambda] \times \mathbb{E}_x \left[ \exp \left[ \theta \sum_{r \leq t} F(X_{r-}, X_r) \right]; \ t < \tau_0 \right] = h^{(t)}(x) \xrightarrow{t \to \infty} h^{(t)}(x).\]
3.3.3 Computation of the variance

\[
\exp[-t \lambda^{(\theta)}] E_x \left[ \left( \frac{1}{t} \sum_{s \leq t} \tilde{F}(X_{s^-}, X_s) \right)^2 \right] \times \exp \left[ \theta \sum_{s \leq t} F(X_{s^-}, X_s) \right] ; \ t < \tau_0
\]

\[
= \frac{1}{t} E_x \left[ \int_{[0,t]} \int_{\mathbb{R}_+} M^J(ds, dw, du) \tilde{F}(X_{s^-}, w)^2 \times 1\{u \leq h(X_{s^-}, w)\} \times \exp \left[ \theta \sum_{s \leq t} F(X_{s^-}, X_s) - t \lambda^{(\theta)} \right] \times 1_{(t < \tau_0)} \right] + \frac{2}{t} E_x \left[ \int_{[0,t]} \int_{\mathbb{R}_+} M^J(ds, dw, du) \tilde{F}(X_{s^-}, w) \times 1\{u \leq h(X_{s^-}, w)\} \times \exp \left[ \theta \sum_{s \leq t} F(X_{s^-}, X_s) - t \lambda^{(\theta)} \right] \times 1_{(t < \tau_0)} \right] \times \int_{(s,t]} \int_{\mathbb{R}_+} M^J(ds', dw', du') \tilde{F}(X_{s'-}, w') \times 1\{u \leq h(X_{s'-}, w')\} \times \exp \left[ \theta \sum_{r \leq t} F(X_{r^-}, X_r) - t \lambda^{(\theta)} \right] \times 1_{(t < \tau_0)} \right].
\]

The previous calculation ensures (by replacing \( \tilde{F} \) by \( \tilde{F}^2 \)) that the first term in the right-hand side shall be equivalent to: \( h^{(\theta)}(x) \langle \mathcal{J}^{(\theta)} \rangle \tilde{F}^2 / t \) as \( t \to \infty \). Thus, it is negligible. Concerning the second term, we can apply the Palm formula twice, so as to obtain:

\[
\frac{2}{t} E_x \left[ \int_{[0,t]} ds \int_X \nu(dw) j(X_{s^-}, w) \tilde{F}(X_{s^-}, w) \int_{[s,t]} ds' \int_X \nu(dw') j(X_{s'-}, w') \tilde{F}(X_{s'-}, w') \times \exp \left[ \theta \sum_{r \leq t} F(X_{r^-}, X_r) - t \lambda^{(\theta)} \right] \times 1_{(t < \tau_0)} \right],
\]

where by \( \tilde{X} \), we mean the process driven by \( M^J + \delta_{(s,w,u)} + \delta_{(s',w',u')} \), where we can assume w.l.o.g. that \( u \leq h(X_{s^-}, w) \) and \( u' \leq h(X_{s'-}, w') \). In particular, \( \tilde{X} \) can be identified with \( X \) on the time-interval \([0,s] \), jumps at time \( s \) with an effect \( w \), then behaves as a copy \( \tilde{X} \) of the process \( X \) starting from \( w \) at time \( s \) until time \( s' \), where it jumps again with an effect \( w' \). In the time-interval \([s', t]\), it finally behaves as a copy \( \tilde{X} \) of the process \( X \) starting from \( w' \) at time \( s' \) until time \( t \). Thus, the preceding integrals can be decomposed, by Fubini theorem, with \( p = s / t \) and \( q = s' / t \) into:

\[
2 \int_{[0,1]} dp \int_X \nu(dw) \int_{[p,1]} dq \int_X \nu(dw') \int_X \delta_{s} P^{(\theta)}_{\mu t}(dy) h(y, w) \tilde{F}(y, w) \exp [F(y, w)] \times \int_X \delta_{w} P^{(\theta)}_{(q-p)t}(dz) h(z, w') \tilde{F}(z, w') \exp [F(z, w')] h^{(\theta)}(w'),
\]

where \( \mu P^{(\theta)}_t \) := \( \exp[-t \lambda^{(\theta)}] \mu P_t \).

The next step is clearly to prove that this term converges to:

\[
\int_X \nu(dw) \int_X \nu(dw') h^{(\theta)}(x) \int_X \alpha^{(\theta)}(dy) h(y, w) \tilde{F}(y, w) \exp [F(y, w)] h^{(\theta)}(w) \times \int_X \alpha^{(\theta)}(dz) h(z, w') \tilde{F}(z, w') \exp [F(z, w')] h^{(\theta)}(w') \times h^{(\theta)}(x) \langle \mathcal{J}^{(\theta)} \rangle \tilde{F}^2,
\]

which is clear at least when \( h, \tilde{F} \) and \( F \) are bounded. Then, for \( Z = \frac{1}{t} \sum_{s \leq t} \tilde{F}(X_{s^-}, X_s) \), we deduce in particular that the variance of \( Z \) under \( \mathbb{Q}^t_x \) is of order \( O(t) \), namely equivalent to \( \langle \mathcal{J}^{(\theta)} \rangle \tilde{F}^2 / t \).
3.4 Proof of Proposition 1.3.4

Assume $\theta_1 < \theta_2$ and let $\epsilon > 0$. Recall the notations $\Psi_t$ and $\psi_{F,G}^{(\theta)}$ from [1]. From Proposition 1.5.3 for $t$ sufficiently large:

$$Q_t^{(\theta_2)}(\Psi_t \geq \psi_{F,G}^{(\theta_1)} - \epsilon) \geq Q_t^{(\theta_1)}(\Psi_t \geq \psi_{F,G}^{(\theta_1)} - \epsilon) \geq 3/4$$

$$Q_t^{(\theta_2)}(\Psi_t \leq \psi_{F,G}^{(\theta_2)} + \epsilon) \geq 3/4.$$ 

The intersection of both events is necessarily non-empty, implying that: $\psi_{F,G}^{(\theta_1)} \leq \psi_{F,G}^{(\theta_2)} + 2\epsilon$. Letting $\epsilon$ go to 0 concludes the proof that: $\theta \mapsto \psi_{F,G}^{(\theta)}$ is non-decreasing.

To prove that it is actually increasing, we shall exploit the following lemma, whose proof is deferred to the next subsection. 

**Lemma 3.4.1.** Assume that $\theta_1 < \theta_2$ in $\Theta_e$ are such that $\psi_{F,G}^{(\theta_1)} < \psi_{F,G}^{(\theta_2)}$.

Then, for any $\theta \in (\theta_1, \theta_2) \cap \Theta_e$, $\psi_{F,G}^{(\theta)} \in (\psi_{F,G}^{(\theta_1)}, \psi_{F,G}^{(\theta_2)})$.

Again, we assume $\theta_1 < \theta_2$ in $\Theta_e$. Since $\psi_{F,G}(\Theta_e)$ is not a singleton, and $\psi_{F,G}$ is non-decreasing on $\Theta_e$, we can find $\theta_0 \leq \theta_1, \theta_3 \geq \theta_2$ such that $\psi_{F,G}(\theta_0) < \psi_{F,G}(\theta_3)$. By Lemma 3.4.1, $\psi_{F,G}(\theta_1) = \psi_{F,G}(\theta_3)$ would imply $\theta_1 = \theta_3$. Since this statement contradicts $\theta_1 < \theta_2 \leq \theta_3$, we deduce that $\psi_{F,G}(\theta_1) < \psi_{F,G}(\theta_3)$. Similarly, the case $\psi_{F,G}(\theta_1) = \psi_{F,G}(\theta_2)$ is excluded by Lemma 3.4.1. $\psi_{F,G}$ is thus necessarily increasing on $\Theta_e$.

3.4.1 Proof of Lemma 3.4.1

The proof of this Lemma relies on the criterion given by Lemma 1.3.1 with small deviations from the behavior associated to $\theta_1, \theta_2$. By including the dynamics according $\theta_1$ in a small proportion of time, we can prove that, in order to slightly increasing the typical value of $\Psi_t$ for large $t$ from the reference given by $\theta_1$, the ‘cost of probability’ is small (vanishing with the threshold of increase). Thus, even small increase in $\theta$ must induce a typical value for $\Psi_t$ that is larger than $\psi_{F,G}(\theta_1)$.

Let $\zeta := (\psi_{F,G}(\theta_2) - \psi_{F,G}(\theta_1))/2$ and $u \in (0, 1)$.

From the Markov property and the fact that $t \Psi_t$ is an additive functional:

$$\mathbb{E}_\mu(t^{t \theta_1} \Psi_t \mid \Psi_t \geq \psi_{F,G}(\theta_1) + u \zeta, t < \tau_0)$$

$$\geq \exp\left[-t (\theta_2 - \theta_1) u \zeta \right] \int_X \nu_t(dx) \mathbb{E}_x \left(\exp[u t \theta_2 \Psi_{ut}] \mid \Psi_{ut} - \psi_{F,G}(\theta_2) < u \zeta, ut < \tau_0\right)$$

(5)

where $t' = (1 - u) t$ and

$$\langle \nu_{t'} \mid g \rangle := \mathbb{E}_\mu[g(X_{t'}) \exp(t' \theta_1 \Psi_{t'}) \mid \Psi_{t'} > \psi_{F,G}(\theta_1) - u \zeta, t' < \tau_0].$$

In the above inequality, we used the following property:

$$(1 - u) \psi_{F,G}(\theta_1) - u \zeta + u \psi_{F,G}(\theta_2) - u \zeta \geq \psi_{F,G}(\theta_1) + u \zeta [2 - u - (1 - u)] = \psi_{F,G}(\theta_1) + u \zeta.$$

Next, we prove that:

$$\left\| \exp(-t' \lambda^{(\theta_1)}) \nu_{t'} - (\mu \mid h^{(\theta_1)})\lambda^{(\theta_1)} \right\|_{TV} \to_{t' \to \infty} 0.$$  (6)
So let $g$ be a bounded measurable function.

$$\exp(-t'\lambda(\theta_1)) \langle \nu_{t'} \mid g \rangle = \langle \mu \mid h_{t'}^{(\theta_1)}Q_{\mu}^{(\theta_1),t'} [g(X_t')] \mid \Psi_{t'} > \psi_{F,G}(\theta_1) - u \zeta \rangle.$$

By propositions 1.5.2 and 1.5.3:

$$Q_{\mu}^{(\theta_1),t'} [g(X_t')] \mid \psi_{t'} \leq \psi_{F,G}(\theta_1) - u \zeta \leq \|g\|_\infty Q_{\mu}^{(\theta_1),t'} [\psi_{t'} \leq \psi_{F,G}(\theta_1) - u \zeta] \xrightarrow{t'\to\infty} 0.$$

Applying (1) on the system given by $\theta_1 \in \Theta_\kappa$, we deduce:

$$|Q_{\mu}^{(\theta_1),t'} [g(X_t')] - \alpha(\theta_1) \mid \|g\|_\infty \|A_{\alpha(\theta_1)} - A_{\alpha(\theta_1)}\|_{TV} \xrightarrow{t'\to\infty} 0.$$

By noting that $(h_{t'}^{(\theta_1)})$ is bounded and exploiting (2), we finally conclude the proof of (5).

Similarly, we can prove that:

$$\exp(-ut\lambda(\theta_2)) E_{\alpha(\theta_2)} \langle \exp[ut \theta_2 \Psi_{ut}] \mid \|\Psi_{ut} - \psi_{F,G}(\theta_2)\| < u \zeta, ut < t_0 \rangle \xrightarrow{t \to \infty} \langle \alpha(\theta_1) \mid h_{\alpha(\theta_1)} \rangle.$$

(7)

Furthermore:

$$\sup_{x,t} \exp(-ut\lambda(\theta_2)) E_{\alpha(\theta_2)} \langle \exp[ut \theta_2 \Psi_{ut}] \mid \|\Psi_{ut} - \psi_{F,G}(\theta_2)\| < u \zeta, ut < t_0 \rangle \leq \sup_t \|h_{\alpha(\theta_1)}\|_\infty < \infty.$$

(8)

Combining (6), (7) and (8), we deduce:

$$\exp \left(-t\lambda(\theta_1) - ut\lambda(\theta_2) - \lambda(\theta_1) \right) \int_X \nu_t(dx) E_{\alpha}(\exp[ut \theta_2 \Psi_{ut}] \mid \|\Psi_{ut} - \psi_{F,G}(\theta_2)\| < u \zeta, ut < t_0 \rangle \xrightarrow{t \to \infty} \langle \mu \mid h_{\alpha(\theta_1)} \times \alpha(\theta_1) \mid h_{\alpha(\theta_2)} \rangle.$$

(9)

Since $h(\theta)$ is positive on $X$ for any $\theta \in \Theta_\kappa$, this quantity is clearly positive. We then conclude from (5) that for any $t$ sufficiently large:

$$Q_{\mu}^{(\theta_1),t} [\Psi_t \geq \psi_{F,G}(\theta_1) + u \zeta] \geq \frac{\langle \alpha(\theta_1) \mid h_{\alpha(\theta_2)} \rangle}{2} \exp(-tu\chi),$$

(10)

where $\chi := (\theta_2 - \theta_1) \zeta + (\lambda(\theta_2) - \lambda(\theta_1)) > 0$.

From Lemma 1.3.1 with $\epsilon := \sqrt{u}$, any $\theta \in \Theta_\kappa$ such that $\theta > \theta_1 + \sqrt{u} \chi$ shall satisfy:

$$\psi_{F,G}(\theta) \geq \psi_{F,G}(\theta_1) + \sqrt{u} \chi.$$ By choosing $u$ sufficiently small, we deduce that any $\theta \in (\theta_1, \theta_2) \cap \Theta_\kappa$ satisfies $\psi_{F,G}(\theta) > \psi_{F,G}(\theta_1)$.

The proof is done in the same way to prove that $\psi_{F,G}(\theta) < \psi_{F,G}(\theta_2)$.

\[\square\]

### 3.5 Proof of Proposition 1.5.4

**Proposition 3.5.1.** Assume that for some $\theta, \vartheta \in \mathbb{R}$, $\|\alpha(\theta_1)(L(\theta) - L(\theta_1))\|_{TV} + |\lambda(\theta) - \lambda(\theta_1)| \leq \epsilon$ for some $\epsilon > 0$. Then, there exists $C > 0$ only depending on $c$ such that $\|\alpha(\theta_1) - \alpha(\theta_1)\|_{TV} \leq C \epsilon$.
In particular, we can deduce that $\theta \mapsto \alpha^{(\theta)}$ is locally Lipschitz provided we have a linear control on $\|\alpha^{(\theta)}(\mathcal{L}^{(\theta)} - \mathcal{L}^{(\theta)})\|_{TV}$ and uniform bound on $C$ and $\langle \alpha^{(\theta)} | h^{(\theta)} \rangle$ for $\theta, \vartheta$ in some compact set.

Conversely, we have similar results for the survival capacities :

**Proposition 3.5.2.** Assume that for some $\theta, \vartheta \in \mathbb{R}$, $\|\alpha^{(\theta)}(\mathcal{L}^{(\theta)} - \mathcal{L}^{(\theta)})\|_{TV} + \|\mathcal{L}^{(\theta)} - \mathcal{L}^{(\theta)}\|_\infty + |\lambda^{(\theta)} - \lambda^{(\theta)}| \leq \epsilon$ for some $\epsilon > 0$. Then, there exists $C > 0$ only depending on $\theta$ such that :

$$\|h^{(\theta)} - h^{(\theta)}\|_\infty \leq C \epsilon.$$ 

In the current situation, the conditions required for the Lipschitz-continuity are deduced from the extended versions required for the derivatives :

**Proposition 3.5.3.** Under the assumptions of Theorem 1.1, for any $\theta \in \Theta_v$ and uniformly over $\vartheta \in (\theta - \delta \theta, \theta + \delta \theta)$, we have :

$$\|\mathcal{L}^{(\vartheta)} - \mathcal{L}^{(\vartheta)}\| \to 0, \quad \|\partial_\vartheta \mathcal{L}^{(\vartheta)}\| < \infty, \quad \|\mathcal{L}^{(\vartheta)} - \mathcal{L}^{(\vartheta)} - (\vartheta - \theta) \partial_\vartheta \mathcal{L}^{(\vartheta)}\| = o(\vartheta - \theta).$$

$$\lambda^{(\vartheta)} - \lambda^{(\vartheta)} \to 0, \text{ as } \vartheta \to \vartheta.$$ 

**Remark :** The whole set of bounded measurable functions may not be included in the domains of $\mathcal{L}^{(\vartheta)}$ and $\mathcal{L}^{(\vartheta)}$ (which can be shown to be the same). Yet, the action of $\mathcal{L}^{(\vartheta)} - \mathcal{L}^{(\theta)}$ on this domain coincide with the one of the following operator, defined (by some abuse of notation) for any bounded function $f$ by :

$$[(\mathcal{L}^{(\vartheta)} - \mathcal{L}^{(\vartheta)}).f](x) := (\vartheta - \theta) G(x) f(x) + \int \nu(dy) h(x, y) (e^{\theta F(x,y)} - e^{\theta F(x,y)}) f(y).$$

When combined with Propositions 3.5.2 and 3.5.3 these results imply the following estimates at use in the proof of Theorem 1.1:

**Corollary 3.5.4.** Under the assumptions of Theorem 1.1, for any $\theta \in \Theta_v$ and uniformly over $\vartheta \in (\theta - \delta \theta, \theta + \delta \theta)$, we have :

$$\|\alpha^{(\vartheta)}(\mathcal{L}^{(\vartheta)} - \mathcal{L}^{(\vartheta)} - (\vartheta - \theta) \partial_\vartheta \mathcal{L}^{(\vartheta)}\|_{TV} = o(\vartheta - \theta)$$

$$\|\alpha^{(\vartheta)} - \alpha^{(\vartheta)}(\partial_\vartheta \mathcal{L}^{(\vartheta)} - \partial_\vartheta \lambda^{(\vartheta)})\|_{TV} \to 0, \text{ as } \theta \to \vartheta,$$

$$\lambda^{(\vartheta)} - \lambda^{(\vartheta)} - (\vartheta - \theta) \partial_\vartheta \lambda^{(\vartheta)} = o(\vartheta - \theta)$$

$$\|\mathcal{L}^{(\vartheta)} - \mathcal{L}^{(\vartheta)} - (\vartheta - \theta) \partial_\vartheta \mathcal{L}^{(\vartheta)}\|_\infty = o(\vartheta - \theta)$$

$$\|\partial_\vartheta \mathcal{L}^{(\vartheta)} - \partial_\vartheta \mathcal{L}^{(\vartheta)}(\mathcal{L}^{(\vartheta)} - \mathcal{L}^{(\vartheta)})\|_\infty \to 0, \text{ as } \theta \to \vartheta.$$ 

Without the assumption of a bounded jump rate and bounded $G$ and $F$, such results are challenging, since they a priori require to deal with the tails of the distributions $\alpha^{(\theta)}$ (a Taylor expansion of : $\theta \mapsto \exp[\theta F(x,w)]$ shall be required to deal with the convergence with order $o(\vartheta - \theta)$).

**Remark :** We could also exploit the tails of $h$, yet it might be quite flat at a level distinct from 0 (notably when descent from infinity occurs), so that it shall be less convenient.
3.5.1 Proof that Propositions 3.5.1, 3.5.2, and 3.5.3 imply Corollary 3.5.4

The estimates (13), (15), (17) and (18) are easily implied by these Propositions. Thanks to Propositions 3.5.1 and 3.5.3, we know that \( \alpha^{(\omega)} \) converges in total variation to \( \alpha^{(\theta)} \) at most linearly in \( |\vartheta - \theta| \).

We thus deduce that \( \langle \alpha^{(\omega)} | h^{(\theta)} \rangle \) tends to \( 1 = \langle \alpha^{(\theta)} | h^{(\theta)} \rangle \), with an error at most linear in \( |\vartheta - \theta| \).

This makes it possible to improve the estimate (13), by exploiting the following expression.

\[
\begin{align*}
\lambda^{(\omega)} - \lambda^{(\theta)} - \langle \vartheta - \theta, \partial_\vartheta \lambda^{(\theta)} \rangle &= \frac{\langle \alpha^{(\omega)} | \mathcal{L}^{(\omega)} - \mathcal{L}^{(\theta)} | h^{(\theta)} \rangle}{\langle \alpha^{(\omega)} | h^{(\theta)} \rangle} - \langle \vartheta - \theta, \partial_\vartheta \mathcal{L}^{(\theta)} | h^{(\theta)} \rangle \\
&= \langle \alpha^{(\theta)} | \mathcal{L}^{(\theta)} - \mathcal{L}^{(\omega)} | h^{(\theta)} \rangle - \langle \alpha^{(\omega)} | \mathcal{L}^{(\omega)} - \mathcal{L}^{(\theta)} | h^{(\theta)} \rangle.
\end{align*}
\]

This concludes the proof of Corollary 3.5.4. \( \square \)

**Remark:** In our specific situation, all the convergences listed in Proposition 3.5.3 and Corollary 3.5.4 actually occur at rate \( O(|\vartheta - \theta|^2) \) rather than \( O(|\vartheta - \theta|) \).

3.5.2 Proof of Proposition 3.5.3

Since \( (\vartheta - \delta \vartheta, \theta + \delta \vartheta) \) is bounded, we assume without loss of generality:

\[
\theta, \vartheta \in [-\vartheta_\circ, \vartheta_\circ]
\]

(19)

From the definitions given in (2) and (3):

\[
\left\| \partial_\vartheta^{\alpha} \mathcal{L}^{(\theta)} \right\| \leq \| h \|_\infty \| F \|_\infty \exp[\| \theta \|_\infty] < \infty, \quad \left\| \partial_\vartheta^{\alpha} \mathcal{L}^{(\omega)} \right\| \leq \| G \|_\infty < \infty,
\]

Now that \( \| \partial_\vartheta \mathcal{L}^{(\theta)} \| < \infty \) is clear, the estimate on \( \mathcal{L}^{(\theta)} - \mathcal{L}^{(\omega)} \) can be deduced for instance by the Taylor-Lagrange formula:

\[
\| \mathcal{L}^{(\theta)} - \mathcal{L}^{(\omega)} \| \leq (\vartheta - \theta) \sup_{\vartheta \in [-\vartheta_\circ, \vartheta_\circ]} \| \partial_\vartheta \mathcal{L}^{(\omega)} \| \to 0.
\]

(20)

We turn on the next estimate, recalling equation (13). Let \( f \) be a measurable bounded function. By definition, \( \partial_\vartheta^{\alpha} \mathcal{L}^{(\theta)} f = G \times f \) so that this linear part is easily handled.

\[
\begin{align*}
\int \nu(dy) \ h(x, y) \ (e^{\vartheta F(x,y)} - e^{\theta F(x,y)}) f(y) - \langle \delta \vartheta, \partial_\vartheta^{\alpha} \mathcal{L}^{(\omega)} | f \rangle &= \int \nu(dy) \ h(x, y) \ f(y) [e^{\vartheta F(x,y)} - e^{\theta F(x,y)} - \nu(x,y) F(x,y) e^{\theta F(x,y)}].
\end{align*}
\]

By the Taylor-Lagrange formula, we deduce:

\[
\| \mathcal{L}^{(\theta)} - \mathcal{L}^{(\omega)} \| \leq (\vartheta - \theta)^2 \left\| h \right\|_\infty \| F \|_\infty^2 \exp[\| \vartheta \|_\infty] F \|_\infty
\]

This concludes the proof of the norm estimates.

For the control of \( |\lambda^{(\omega)} - \lambda^{(\theta)}| \), we propose to use the following:

\[
\langle \alpha^{(\omega)} | \mathcal{L}^{(\omega)} - \lambda^{(\omega)} - \mathcal{L}^{(\theta)} + \lambda^{(\theta)} | h^{(\theta)} \rangle = \langle \alpha^{(\omega)} | \mathcal{L}^{(\omega)} - \mathcal{L}^{(\theta)} | h^{(\theta)} \rangle - \langle \alpha^{(\omega)} | \mathcal{L}^{(\omega)} - \lambda^{(\theta)} | h^{(\theta)} \rangle = 0
\]

which simplifies into the following formula:

\[
(\lambda^{(\omega)} - \lambda^{(\theta)}) \times \langle \alpha^{(\omega)} | h^{(\theta)} \rangle = \langle \alpha^{(\omega)} | \mathcal{L}^{(\omega)} - \mathcal{L}^{(\theta)} | h^{(\theta)} \rangle,
\]

(17)

The convergence is deduced from the previous norm estimate (20) with the assumed uniform lower-bound on \( \langle \alpha^{(\omega)} | h^{(\theta)} \rangle \) for \( \vartheta \) in \( (\theta - \delta \vartheta, \theta + \delta \vartheta) \). \( \square \)

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3.5.3 Proof of Proposition 3.5.1

We want here to control $\alpha^{(d)} - \alpha^{(k)}$, $h^{(d)} - h^{(k)}$ and $\lambda^{(d)} - \lambda^{(k)}$ when $d \in \mathbb{R}$ is close to $c$. Let:

$$\delta \mu := \alpha^{(d)}(\mathcal{L}^{(d)} - \lambda^{(k)}),$$

so that $||\delta \mu||_{TV} = \|\alpha^{(d)}(\mathcal{L}^{(d)} - \mathcal{L}^{(k)} - \lambda^{(k)} + \lambda^{(k)})\|_{TV} \leq \|\alpha^{(d)}(\mathcal{L}^{(d)} - \mathcal{L}^{(k)})\|_{TV} + |\lambda^{(k)} + \lambda^{(k)}| \leq \varepsilon$.

We prove in the following the equality between the two following quantities:

$$\delta^1 \alpha := \alpha^{(d)} - (\alpha^{(d)} | h^{(k)}) \alpha^{(k)}, \delta^2 \alpha(dx) := \int_0^\infty dt \langle \delta \mu | e^{-t \lambda^{(k)}} P_t^{(d)}(dx) \rangle.$$

Note that this measure is indeed well-defined because $\langle \delta \mu | h^{(k)} \rangle = 0$ so that Lemma 1.5.1 ensures an exponential tail. By noting that $\delta \mu = \delta^1 \alpha(\mathcal{L}^{(d)} - \lambda^{(k)})$, we deduce the following equalities for any bounded measurable function $g$:

$$\langle \delta^2 \alpha | g \rangle = \int_0^\infty d \langle \delta \mu | e^{-t \lambda^{(k)}} P_t^{(d)} | g \rangle = \int_0^\infty d \langle \delta^1 \alpha | \mathcal{L}^{(d)} - \lambda^{(k)} | e^{-t \lambda^{(k)}} P_t^{(d)} | g \rangle = \int_0^\infty d \langle \delta^1 \alpha | e^{-t \lambda^{(k)}} P_t^{(d)} | g \rangle = \langle \delta^1 \alpha | g \rangle, \text{ with again Lemma 1.5.1 and } \langle \delta^1 \alpha | h^{(k)} \rangle = 0.$$

This concludes indeed that $\delta^2 \alpha = \delta^1 \alpha = \alpha^{(d)} - (\alpha^{(d)} | h^{(k)}) \alpha^{(k)}$. Moreover, with the constants $C, \zeta = C^{(d)}, \zeta^{(k)}$ from Lemma 1.5.1 we deduce:

$$\|\delta^1 \alpha\|_{TV} \leq (C/\zeta) \|\delta \mu\|_{TV}.$$

Then, it implies in particular:

$$\langle \alpha^{(d)} | h^{(k)} \rangle = 1 - \delta^1 \alpha(X) = 1 + (C/\zeta) \|\delta \mu\|_{TV},$$

$$\|\alpha^{(d)} - \alpha^{(k)}\|_{TV} \leq \|\delta^1 \alpha\|_{TV} + |1 - \langle \alpha^{(d)} | h^{(k)} \rangle| = 2 (C/\zeta) \|\delta \mu\|_{TV} \leq 2 (C/\zeta) \varepsilon.$$

This concludes the proof of Proposition 3.5.1. \square

Remark: • The constant $2(C/\zeta)$ could be explicitly related to the parameters involved in the set of Assumption (AK), or (AKF) (cf. Subsection 1.4) provided the sets $D_E, D_s$ and the measure $\zeta$ are kept the same. Thus, it shall be possible to get an upper-bound uniform for $c$ in any compact set.

• The proof that $\|\alpha^{(d)}(\mathcal{L}^{(d)} - \mathcal{L}^{(k)})\|_{TV}$ is small is a priori highly dependent on $F$ and the model (notably on where $\alpha^{(d)}$ has its mass).

3.5.4 Proof of Proposition 3.5.2

The proof follows very similar arguments as the one of Proposition 3.5.1 where the role of $h$ and $\alpha$ are reversed, with the infinite norm replacing the total variation.
We then notably prove:

\[ h^{(\theta)} - \langle \alpha^{(\theta)} \mid h^{(\theta)} \rangle h^{(\theta)} = \int_0^\infty dt \, e^{-t\lambda^{(\theta)}} P_t^{(\theta)} (\mathcal{L}^{(\theta)} - \lambda^{(\theta)}) h^{(\theta)} \]

while \[ \| (\mathcal{L}^{(\theta)} - \lambda^{(\theta)}) h^{(\theta)} \|_\infty \leq \| (\mathcal{L}^{(\theta)} - \lambda^{(\theta)}) h^{(\theta)} \|_\infty + |\lambda^{(\theta)} - \lambda^{(\theta)}| \].

Note also that the previous result already ensured that \[ \langle \alpha^{(\theta)} \mid h^{(\theta)} \rangle \] is close to 1. \[ \square \]

### 3.6 Proof of Theorem 1.3.3

The quantity

\[ \partial \lambda^{(\theta)} = \langle \alpha^{(\theta)} \mid \partial \mathcal{L}^{(\theta)} \mid h^{(\theta)} \rangle = \psi_{F,G}(\theta) \]

indeed defines the derivative of \( \lambda^{(\theta)} \) along \( \theta \) in \( \Theta_e \) thanks to Corollary 3.5.4.

#### 3.6.1 Derivative of the QSD

We note from our definition of \( \partial \alpha^{(\theta)} \):

\[ \langle \partial \alpha^{(\theta)} \mid \mathcal{L}^{(\theta)} - \lambda^{(\theta)} \rangle = \int_0^\infty dt \, \partial_t \langle \delta \mu \mid e^{-t\lambda^{(\theta)}} P_t^{(\theta)} \rangle = \delta \mu \]

where \( \delta \mu := -\langle \alpha^{(\theta)} \mid \partial \mathcal{L}^{(\theta)} \rangle + \langle \alpha^{(\theta)} \mid \partial \mathcal{L}^{(\theta)} \mid h^{(\theta)} \rangle \alpha^{(\theta)} \)

is such that \( \langle \delta \mu \mid h^{(\theta)} \rangle = 0 \). Denote:

\[ \Delta \alpha := \alpha^{(\theta)} - \alpha^{(\theta)} - (\vartheta - \theta) \partial \alpha^{(\theta)}. \]

Then, with the previous result on the growth rate:

\[ \langle \Delta \alpha \mid \mathcal{L}^{(\theta)} - \lambda^{(\theta)} \rangle = \langle \alpha^{(\theta)} \mid \mathcal{L}^{(\theta)} - \lambda^{(\theta)} - (\vartheta - \theta) \partial \mathcal{L}^{(\theta)} \rangle + \langle \lambda^{(\theta)} - \lambda^{(\theta)} - (\vartheta - \theta) \partial \lambda^{(\theta)} \rangle \alpha^{(\theta)} \]

By Corollary 3.5.4 this term is \( o(\vartheta - \theta) \) in total variation. The inversion procedure then implies that:

\[ \Delta \alpha - \langle \Delta \alpha \mid h^{(\theta)} \rangle \alpha^{(\theta)} \]

is also \( o(\vartheta - \theta) \) in total variation. We deduce from \( \Delta \alpha(X) = 0 \) that \( \langle \Delta \alpha \mid h^{(\theta)} \rangle \) is \( o(\vartheta - \theta) \) itself. The multiple of \( \alpha^{(\theta)} \) in the integral of \( \partial \alpha^{(\theta)} \) is indeed chosen so as to ensure \( \partial \alpha^{(\theta)}(X) = 0 \). Thus \( \Delta \alpha \) itself is \( o(\vartheta - \theta) \) in bounded variation, concluding the proof. \[ \square \]

#### 3.6.2 Derivative of the survival capacity

Likewise, we note from our definition of \( \partial h^{(\theta)} \):

\[ \langle \mathcal{L}^{(\theta)} - \lambda^{(\theta)} \rangle \partial h^{(\theta)} = -\partial \mathcal{L}^{(\theta)} h^{(\theta)} + \langle \alpha^{(\theta)} \mid \partial \mathcal{L}^{(\theta)} \rangle h^{(\theta)} \]

Denote:

\[ \Delta h := h^{(\theta)} - h^{(\theta)} - (\vartheta - \theta) \partial h^{(\theta)}. \]

Then, with the previous result on the growth rate:

\[ \langle \mathcal{L}^{(\theta)} - \lambda^{(\theta)} \rangle \Delta h = \langle \mathcal{L}^{(\theta)} - \lambda^{(\theta)} - (\vartheta - \theta) \partial \mathcal{L}^{(\theta)} \rangle h^{(\theta)} + \langle \mathcal{L}^{(\theta)} - \lambda^{(\theta)} - (\vartheta - \theta) \partial \lambda^{(\theta)} \rangle h^{(\theta)} \]

\[ - (\vartheta - \theta) \langle \partial \mathcal{L}^{(\theta)} - \partial \lambda^{(\theta)} \rangle \rangle (h^{(\theta)} - h^{(\theta)}). \]
By Corollary 3.5.3, this term is $o(\bar{\theta} - \theta)$ in infinite norm. The inversion procedure then implies that: $\Delta^2 h := \Delta h - \langle \alpha^{(s)} \mid \Delta h \rangle h^{(s)}$ is also $o(\bar{\theta} - \theta)$ in infinite norm. Next, we want to justify that $\langle \alpha^{(s)} \mid \Delta h \rangle$ is $o(\bar{\theta} - \theta)$. We first remark that our definitions are made to ensure:

$$\langle \alpha^{(s)} \mid \partial_\theta h^{(s)} \rangle = \int_0^\infty dt \langle \alpha^{(s)} \mid \partial_\theta \mathcal{L}^{(s)} \mid h^{(s)} - h^{(s)}(t) \rangle = -\langle \partial_\theta \alpha^{(s)} \mid h^{(s)} \rangle.$$  

This leads to compute:

$$\langle \alpha^{(s)} \mid \Delta^2 h \rangle = 1 - \langle \alpha^{(s)} \mid h^{(s)} \rangle - (\bar{\theta} - \theta) \langle \alpha^{(s)} \mid \partial_\theta h^{(s)} \rangle - \langle \alpha^{(s)} \mid \Delta h \rangle \times \langle \alpha^{(s)} \mid h^{(s)} \rangle$$

$$= \langle \alpha^{(s)} \mid h^{(s)} \rangle - (\bar{\theta} - \theta) \langle \alpha^{(s)} \mid \partial_\theta h^{(s)} \rangle + \langle \alpha^{(s)} \mid \partial_\theta h^{(s)} \rangle - \langle \alpha^{(s)} \mid \Delta^2 h \rangle.$$  

This ensures that $\langle \alpha^{(s)} \mid \Delta h \rangle = o(\bar{\theta} - \theta)$, concluding the proof that $\partial_\theta h^{(s)}$ is indeed the derivative of $c \mapsto h^{(s)}$.

\[\square\]

3.6.3 Derivative of the averaged jumps effects

$$\psi_{F,G}(\bar{\theta}) - \psi_{F,G}(\theta) = \langle \alpha^{(s)} - \alpha^{(s)} \mid \partial_\theta \mathcal{L}^{(s)} \rangle \|h^{(s)}\| + \langle \alpha^{(s)} \mid \partial_\theta [\mathcal{L}^{(s)}]_{\theta = 0} \mid h^{(s)} - h^{(s)}(0) \rangle$$

$$+ \langle \alpha^{(s)} \mid \partial_\theta \mathcal{L}^{(s)} \rangle \|h^{(s)}\| + \langle \alpha^{(s)} \mid \partial_\theta h^{(s)} \rangle + \langle \alpha^{(s)} \mid \partial_\theta^2 \mathcal{L}^{(s)} \rangle \|h^{(s)}\|$$

Then, we consider the error terms. For the first term:

$$\langle \alpha^{(s)} - \alpha^{(s)} \mid \partial_\theta \mathcal{L}^{(s)} \rangle \|h^{(s)}\| = \langle \bar{\theta} - \theta \rangle \langle \alpha^{(s)} \mid \partial_\theta \mathcal{L}^{(s)} \rangle \|h^{(s)}\| + \langle \bar{\theta} - \theta \rangle \langle \partial_\theta \alpha^{(s)} \mid \partial_\theta \mathcal{L}^{(s)} \rangle \|h^{(s)}\|$$

$$= \langle \bar{\theta} - \theta \rangle \langle \alpha^{(s)} \mid \partial_\theta \mathcal{L}^{(s)} \rangle \|h^{(s)}\| + \langle \bar{\theta} - \theta \rangle \langle \partial_\theta \alpha^{(s)} \mid \partial_\theta \mathcal{L}^{(s)} \rangle \|h^{(s)}\|.$$  

We are thus able to prove that the above error term is $O((\bar{\theta} - \theta)^2)$.  

By Corollary 3.5.3 and the previous result on the derivatives of the QSD and the survival capacities.
3.7 Proof of Theorem 1.2

The proof is inspired by the one of Lévy’s Theorem and consists in ensuring the tightness of the variables and characterize their limits through the Laplace transform.

Step 1: Convergence of the Laplace transform

Let \( \theta \in \Theta_0 \) and \( \mu \in M_1(X) \). By definition and Lemma 1.5.1, it implies that there exists \( C, \gamma > 0 \) such that for any \( \delta \in [\theta - \delta \theta, \theta + \delta \theta] \):

\[
\left| \exp[-\lambda^{(\theta)} t] E_\mu(\exp[-\theta t \Psi_t] ; t < \tau_0) - \langle \mu \mid h^{(\theta)} \rangle \right| \leq C e^{-\gamma t}.
\]

W.l.o.g., we also assume that \( \langle \mu \mid h^{(\theta)} \rangle \) is uniformly lower-bounded for \( \theta \in [\theta - \delta \theta, \theta + \delta \theta] \).

We define the family of measure \((\pi_t)_{t>0}\) by the fact that for any \( f \) bounded continuous (non-measurable non-negative):

\[
\langle \pi_t \mid f \rangle := \langle Q^{(\theta),t} \mid f \rangle = \frac{E_\mu \left[ f(\sqrt{\lambda} \Psi_t - \psi_{F,G}(\theta)) \exp(t \theta \Psi_t) ; t < \tau_0) \right]}{E_\mu \left[ \exp(t \theta \Psi_t) ; t < \tau_0 \right]}.
\]

Let \( \eta \in \mathbb{R} \).

\[
\int_\mathbb{R} \pi_t(dx) e^{\eta x} \leq \frac{E_\mu \left[ \exp(\eta \sqrt{\lambda} \Psi_t - \psi_{F,G}(\theta)) \exp(t \theta \Psi_t) ; t < \tau_0 \right]}{E_\mu \left[ \exp(t \theta \Psi_t) ; t < \tau_0 \right]} \leq \exp \left[ \lambda^{(\theta + \eta/\sqrt{\lambda}) t - \lambda^{(\theta) t} + \eta \psi_{F,G}(\theta) \sqrt{\lambda}} \right] \times \frac{\exp \left[ \lambda^{(\theta) t} - \lambda^{(\theta) t} + \eta \psi_{F,G}(\theta) \sqrt{\lambda}} \right]}{E_\mu \left[ \exp(t \theta \Psi_t) ; t < \tau_0 \right]}
\]

By applying (15) to \( \theta = t \eta/\sqrt{\lambda} \) for \( t \) sufficiently large and recalling that \( \langle \mu \mid h^{(\theta)} \rangle \) converges to \( \langle \mu \mid h^{(0)} \rangle \) as \( \theta \to \theta \), so as \( t \to \infty \), the ratio in the r.h.s. tends to 1. Now, exploiting Theorem 1.3.8 and the Taylor-Young formula:

\[
\lambda^{(\theta + \eta/\sqrt{\lambda})} = \lambda^{(\theta)} + \psi_{F,G}(\theta) (t \eta - \theta + \eta \sqrt{\lambda}) + \frac{1}{2} \partial_\theta \psi_{F,G}(\theta) (t \eta - \theta + \eta \sqrt{\lambda})^2 + o(1/t),
\]

\[
\psi_{F,G}(\theta) = \psi_{F,G}(\theta) + \partial_\theta \psi_{F,G}(\theta) (t \eta - \theta + \eta \sqrt{\lambda}) + o(1/\sqrt{\lambda})
\]

\[
\lambda^{(\theta + \eta/\sqrt{\lambda})} - \lambda^{(\theta) t} + \eta \psi_{F,G}(\theta) \sqrt{\lambda} = \frac{1}{2} \partial_\theta \psi_{F,G}(\theta) \left[ \sqrt{\lambda} (t \eta - \theta + \eta \sqrt{\lambda})^2 - t (t \eta - \theta)^2 - 2 \eta \sqrt{\lambda} (t \eta - \theta) \right] + o(1)
\]

\[
\longrightarrow \frac{1}{2} \partial_\theta \psi_{F,G}(\theta) \eta^2.
\]

Injecting these convergence results into (19) yields the expected convergence result:

\[
\int_\mathbb{R} \pi_t(dx) e^{\eta x} \longrightarrow \exp \frac{1}{2} \partial_\theta \psi_{F,G}(\theta) \eta^2,
\]

where we recognize the Laplace transform of the centered Gaussian distribution with variance \( \partial_\theta \psi_{F,G}(\theta) \).

Step 2: tightness and uniqueness

Let \( A > 0 \). By the Markov inequality and for \( t \) sufficiently large:

\[
\pi_t([A, \infty)) \leq \int_\mathbb{R} \pi_t(dx) e^{x} \leq C e^{-A},
\]

where we exploited (20). Reasoning similarly for \( \pi_t((-\infty, -A]) \), we conclude that the family \( \pi_t \) is tight. Again by (20), the Laplace transform of any limit point of \( \pi_t \) is necessarily the one of the
Gaussian distribution. This distribution is thus the unique limit point of the tight sequence \( \pi_t \), from which it is classical that \( \pi_t \) converges in distribution to the centered Gaussian distribution with variance \( \partial_\theta \psi_{F,G}(\theta) \). \( \square \)

**Remark:** As pointed out in [2], the conditions of Gärtner-Ellis Theorem for the Large Deviations are not sufficient for the proof of the Central Limit Theorem without additional hypothesis. One usually relies on the use of the Fourier transform, for which our argument of convergence is unclear. Yet, we have stronger results of convergence here, although not clearly sufficient for the strong large deviations (as in [20]).

### 3.8 Proof of Theorem 1.1

As previously mentioned, the logarithmic limit is deduced from the Gärtner-Ellis conditions (see e.g. [20] for a proof of the local version). Thanks to Propositions 1.3.4 and 1.3.3, these conditions are clearly satisfied.

We now exploit Theorem 1.2 to prove that the decay is quicker than the exponential rate. The proof is adapted from the one of Theorem 1 in [20], and weakened given our current knowledge. Let \( \mu \in \mathcal{M}_1(X) \) and \( \gamma \in \psi_{F,G}^{-1}(\Theta_v \cap \mathbb{R}_- \cap \mathbb{R}_- \cap \mathbb{R}_+) \) and \( \theta \in \Theta_v \cap \mathbb{R}_- \) the unique value for which \( \psi_{F,G}(\theta) = \gamma \).

The first part is to relate the probability of interest to the random variable \( V_t := \sqrt{\Psi_t - \gamma} \sigma_t \), with \( \sigma_t := \sqrt{\partial_\theta \psi_{F,G}(\theta)} \) under \( P_{\mu(t)} \), which is asymptotically gaussian.

\[
\mathbb{P}_{\mu} [ \Psi_t \leq \gamma ; t < \tau_\partial ] = \mathbb{E}_{\mu(t)}^{(\theta)} \left( \exp\left[ -\theta t (\Psi_t - \gamma) \right] ; \Psi_t \leq \gamma \right) \\
\times \exp\left[ -\theta t \gamma \right] \times \mathbb{E}_{\mu(t)}^{(\theta)} \left( \exp\left[ -\theta t \Psi_t \right] ; t < \tau_\partial \right) \\
= \mathbb{E}_{\mu(t)}^{(\theta)} \left( \exp\left[ -\theta \sigma_t \sqrt{t} V_t \right] ; V_t \leq 0 \right) \times (\mu_h^{(\theta)}) \exp[-\xi(\theta)t + \lambda^{(\theta)}t] \tag{21}
\]

Let \( A > 0 \). Thanks to Theorem 1.2, we deduce that for \( t \) sufficiently large:

\[
\mathbb{E}_{\mu(t)}^{(\theta)} \left( \exp[-\theta \sigma_t \sqrt{t} V_t] ; V_t \leq 0 \right) \leq \mathbb{E}_{\mu(t)}^{(\theta)} \left( \exp[A V_t] ; V_t \leq 0 \right) \\
\xrightarrow{t \to \infty} \mathbb{E}(\exp[AN] ; N \leq 0), \tag{22}
\]

where \( N \sim \mathcal{N}(0,1) \). By the dominated convergence theorem, this last expression converges to 0 as \( A \) tends to \( \infty \). It implies that the expectation converges to 0 as \( t \) tends to \( \infty \).

We also recall that \( (\mu | h^{(\theta)}_t) \) tends to \( \langle \mu | h^{(\theta)} \rangle \), \( (\mu | h^{(\theta)}_t) \) tends to \( \langle \mu | h^{(\theta)} \rangle > 0 \), where \( \langle \mu | h^{(\theta)}_t \rangle = \exp[-\lambda^{(\theta)}t] \mathbb{P}_\mu(t < \tau_\partial) \).

Injecting these bounds in (21), we can conclude that:

\[
\exp[-\xi(\theta)t] \mathbb{P}_\mu [ \Psi_t \leq \gamma ; t < \tau_\partial ] \xrightarrow{t \to \infty} 0 \quad \square
\]

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