Higher spin algebras as higher symmetries

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Abstract

The exhaustive study of the rigid symmetries of arbitrary free field theories is motivated, along several lines, as a preliminary step in the completion of the higher-spin interaction problem in full generality. Some results for the simplest example (a scalar field) are reviewed and commented along these lines.

Expanded version of the lectures presented at the “5th international school and workshop on QFT & Hamiltonian systems” (Calimanesti, May 2006).

1 Higher-spin interaction problem

Whereas covariant gauge theories describing arbitrary free massless fields on constant-curvature spacetimes of dimension $n$ are firmly established by means of the unitary representation theory of their isometry groups, it is still open to question whether non-trivial consistent self-couplings and/or cross-couplings among those fields may exist for $n \geq 2$, such that the deformed gauge algebra is non-Abelian. The goal of the present paper is to advocate that a lot of information on the interactions can be extracted from the symmetries of the free field theory.

The conventional local free field theories corresponding to unitary irreducible representations of the helicity group $SO(n-2)$, that are spanned by completely symmetric tensors, have been constructed a while ago (for some introductory reviews, see [1]). In order to have Lorentz invariance manifest and second order local field equations with minimal field content, the theory is expressed in terms of completely symmetric double-traceless tensor gauge fields $h_{\mu_1 \ldots \mu_s}$ of rank $s > 0$, the gauge transformation of which reads

$$\delta_\xi h_{\mu_1 \mu_2 \ldots \mu_s} = \nabla_{\mu_1} \xi_{\mu_2 \ldots \mu_s} + \text{cyclic},$$

(1)

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where \( \nabla \) is the covariant derivative with respect to the background Levi-Civita connection and “cyclic” stands for the sum of terms necessary to have symmetry of the right-hand-side under permutations of the indices.

The gauge parameter \( \xi \) is a completely symmetric traceless tensor field of rank \( s - 1 \). In this relativistic field theory, the “spin” is equal to the rank \( s \). For spin \( s = 1 \) the gauge field \( h_\mu \) represents the photon with \( U(1) \) gauge symmetry while for spin \( s = 2 \) the gauge field \( h_{\mu \nu} \) represents the graviton with linearized diffeomorphism invariance. The gauge algebra of field independent gauge transformations such as (1) is of course Abelian.

Non-Abelian gauge theories for “lower spin” \( s \leq 2 \) are well known and essentially correspond to Yang-Mills \((s = 1)\) and Einstein \((s = 2)\) theories for which the underlying geometries (principal bundles and Riemannian manifolds) were familiar to mathematicians before the construction of the physical theory. In contrast, the situation is rather different for “higher spin” \( s > 2 \) for which the underlying geometry (if any!) remains obscure. Due to this lack of information, it is natural to look for inspiration in the perturbative “reconstruction” of Einstein gravity as the non-Abelian gauge theory of a spin-two particle propagating on a constant-curvature spacetime (see e.g. [2] for a comprehensive review).

Let us denote by \( S^{(0)}[h_{\mu_1 \ldots \mu_s}] \) the Poincaré-invariant, local, second-order, quadratic, ghost-free, gauge-invariant action of a spin-\( s \)-symmetric tensor gauge field. In order to perform a perturbative analysis via the Noether method [3], the non-Abelian interaction problem for a collection of higher (and possibly lower) spin gauge fields is formulated as a deformation problem.

**Higher-spin interaction problem:** List all Poincaré-invariant local deformations

\[
S[h] = S^{(0)}[h] + \varepsilon S^{(1)}[h] + O(\varepsilon^2)
\]

of a positive sum, with at least one \( s > 2 \),

\[
S^{(0)}[h] = \sum_s S^{(0)}[h_{\mu_1 \ldots \mu_s}]
\]

of quadratic actions such that the deformed local gauge symmetries

\[
\delta^{(0)}_{\xi} h = \delta^{(1)}_{\xi} h + \varepsilon \delta^{(2)}_{\xi} h + O(\varepsilon^2)
\]

are already non-Abelian at first order, in the deformation parameters \( \varepsilon \) and do not arise from local redefinitions

\[
h \rightarrow h + \varepsilon \phi(h) + O(\varepsilon^2), \quad \xi \rightarrow \xi + \varepsilon \zeta(h, \xi) + O(\varepsilon^2)
\]

of the gauge fields and parameters.

This well-posed mathematical problem is expected to possess non-trivial solutions including higher-spin fields, as strongly indicated by Vasiliev’s works (for some reviews, see [4] and references therein) and deserves to be investigated further along systematic lines.
2 The Noether method

The assumption that the deformations are formal power series in some deformation parameters $\varepsilon$ enables to investigate the problem order by order. The crucial observation of any perturbation theory is that the first order deformations are constrained by the symmetries of the undeformed system. In the present case, the Noether method scrutinizes the gauge symmetry of the action, $\delta_\xi S = 0$. At zeroth order, the latter equation is satisfied by hypothesis. At first order, it reads

$$\delta^{(0)}_\xi S + \delta^{(1)}_\xi S = 0 .$$

This equation may be used to constrain the possible deformations by reinterpreting them as familiar objects of the undeformed gauge theory.

By definition, an observable of a gauge theory is a functional which is gauge-invariant on-shell, while a reducibility parameter of a gauge theory is a gauge parameter such that the corresponding gauge variation vanishes off-shell.

First-order deformations in terms of the undeformed theory:
- First-order deformations of the action are observables of the undeformed theory.
- First-order deformations of the gauge symmetries evaluated at reducibility parameters of the undeformed gauge theory define symmetries of the undeformed theory.

**Proof:** In (2) the infinitesimal variation $\delta^{(1)}_\xi S$ of the undeformed action is proportional to the undeformed Euler–Lagrange equations. This proves the first part of the theorem. Reducibility parameters $\xi$ of the undeformed gauge theory verify $\delta^{(0)}_\xi h = 0$ by definition. Inserting this fact into (2) with $\xi = \tilde{\xi}$ gives $\delta^{(1)}_\xi S = 0$, which is precisely the translation of the second part of the theorem.

In the mathematical literature, a (conformal) Killing tensor of a pseudo-Riemannian manifold is a symmetric tensor field $\xi$ such that its symmetrized covariant derivative with respect to the Levi–Civita connection, $\nabla_{\mu_1} \xi_{\mu_2 \ldots \mu_s} + \text{cyclic}$, vanishes (modulo a term proportional to the metric for conformal Killing tensors). Therefore, any reducibility parameter $\xi$ of the spin-$s$ symmetric gauge field theory on the constant-curvature spacetime $\mathcal{M}$ is identified with a Killing tensor of rank $s - 1$ of the manifold $\mathcal{M}$. The space of Killing tensors on any constant-curvature spacetime is known to be finite-dimensional [5], thus the linear gauge symmetries (1) are irreducible.

These results suggest two strategies for addressing the higher-spin interaction problem. The most ambitious one is the computation of all local observables of the free gauge theory associated to deformations of the gauge algebra. This result would provide the exhaustive list of algebra-deforming first order vertices, but this computation is technically demanding and seems out of reach in the completely general case. Nevertheless, the BRST reformulation of the problem [6] allowed the complete classification of non-Abelian deformations in various particular cases (see e.g.
the review [7] and references therein). Actually, a more humble strategy is the computation of all rigid symmetries of the free irreducible gauge theory. It is of interest because the knowledge of these rigid symmetries would strongly constrain the candidates for gauge symmetry deformations. Indeed, the constant tensors appearing in the rigid symmetries could be compared with the complete list [5] of constant-curvature space-time Killing tensors.

3 Free theory symmetries

Bosonic fields are usually described in terms of their components living in some subspace $V$ of the space $\otimes(\mathbb{R}^n)$ of tensors on $\mathbb{R}^n$ (e.g. $V = \otimes(\mathbb{R}^n)$ for symmetric tensor fields). The background metric of the constant-curvature spacetime induces some non-degenerate bilinear form on $V$. This defines a non-degenerate sesquilinear form $\langle \ | \rangle$ on the space $L^2(\mathbb{R}^n) \otimes V$ of square-integrable fields taking values in the countable space $V$ (the components). Let $\dagger$ stands for the adjoint with respect to the sesquilinear form $\langle \ | \rangle$.

Any quadratic action for bosonic fields $\psi$ can be expressed as a quadratic form

$$^{(0)}S[\psi] = \frac{1}{2} \langle \psi \ | \ K \ | \ \psi \rangle,$$

where the kinetic operator $K$ is self-adjoint, $K^\dagger = K$. Because the sesquilinear form $\langle \ | \rangle$ is non-degenerate, the Euler-Lagrange equation extremizing the quadratic action is the linear equation

$$\frac{\delta^{(0)}S}{\delta\langle\psi\rangle} = K|\psi\rangle = 0. \quad (4)$$

Moreover, the quadratic form $\langle \psi \ | \ K \ | \ \psi \rangle$ is degenerate if and only if the kinetic operator $K$ is degenerate. This happens if and only if there exists a linear operator $P$ (on $L^2(\mathbb{R}^n) \otimes V$) such that $KP = 0$. Infinitesimal gauge symmetries then read

$$^{(0)}\delta_{\chi}\langle\psi\rangle = P|\chi\rangle,$$

with gauge parameters $\chi$. The Noether identity is $P^\dagger K = (KP)^\dagger = 0$.

A symmetry of the quadratic action [3] is an invertible linear pseudodifferential operator $U$ preserving the quadratic form $\langle \ | K \ | \rangle$. In other words,

$$U^\dagger KU = K.$$

The group of off-shell symmetries is the group of symmetries of the quadratic action endowed with the composition $\circ$ as product. A symmetry generator of the quadratic action [3] is a linear differential operator $T$ which is self-adjoint with respect to the quadratic form $\langle \ | K \ | \rangle$. More concretely,

$$KT = T^\dagger K.$$

Any symmetry generator $T$ defines a symmetry $U = e^{iT}$ of the quadratic action [3]. If $T = T^\dagger$ then the linear operator $T$ is a symmetry generator
of the quadratic action if and only it commutes with $K$. The \emph{real Lie algebra of off-shell symmetries} is the algebra of symmetry generators of the quadratic action endowed with $i$ times the commutator as Lie bracket, \{ , \} := i [ , ].

A \emph{symmetry of the linear equation} (4) is a linear differential operator $T$ obeying

$$KT = SK,$$

for some linear operator $S$. Such a symmetry $T$ preserves the space $\text{Ker}K$ of solutions to the equations of motion. Any symmetry generator $T$ of the action (3) is always a symmetry of the equation of motion (4) with $S = T^\dagger$ in (5). A symmetry $T$ is \emph{trivial on-shell} if $T = RK$ for some linear operator $R$. Such an on-shell-trivial symmetry is always a symmetry of the field equation (4), since it obeys (5) with $S = KR$. The algebra of on-shell-trivial symmetries obviously forms a left ideal in the algebra of linear differential operators endowed with the composition $\circ$ as multiplication. Furthermore, it is also a right ideal in the algebra of symmetries of the linear equation (4). The \emph{complex associative algebra of on-shell symmetries} is the associative algebra of symmetries of the linear equation quotiented by the two-sided ideal of on-shell-trivial symmetries. The \emph{complex Lie algebra of on-shell symmetries} is the algebra of on-shell symmetries endowed with the commutator as Lie bracket.

Notice that when $K$ is non-degenerate, a linear operator $T = RK$ is a symmetry generator of the quadratic action (3) if and only if $R$ is self-adjoint. Moreover, the Lie subalgebra of such on-shell-trivial symmetry generators is an ideal in the Lie algebra of off-shell symmetries.

\section{Higher-spin algebras}

Let $\mathfrak{g}$ be the Lie algebra corresponding to the finite-dimensional \emph{(conformal)} isometry group $G$ of the constant-curvature spacetime of dimension $n > 2$. For $n = 2$, the spacetime may be arbitrary and the conformal algebra is of course infinite-dimensional. If the free field theory is relativistic, then $\mathfrak{g}$ is linearly realized on the space $L^2(\mathbb{R}^n) \otimes V$ (respectively, $\text{Ker}K$) of off-shell (resp. on-shell) fields. This induces a linear realization of the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ over $\mathbb{C}$. The real form of this realization corresponding to the self-adjoint operators, endowed with $i$ times the commutator as Lie bracket, is nowadays referred to as \emph{(conformal) on/off-shell higher-spin algebra of the constant-curvature spacetime} (see e.g. \cite{8} for an elementary introduction to such algebraic structures). The name comes from the fact that its generators are in “higher-spin” representations of the Lorentz group, and the algebra is said to be “on” or “off” shell whether the algebra is realized on the space of solutions of the Euler-Lagrange equations or not.

The isometry algebra $\mathfrak{g}$ of a constant-curvature spacetime is a module of the Lorentz subalgebra $\mathfrak{o}(n - 1, 1) \subset \mathfrak{g}$ for the adjoint representation. This module decomposes as the sum of two irreducible $\mathfrak{o}(n-1,1)$-modules: the “translations” are in the vector module $\cong \mathbb{R}^n$ while the boosts and rotations are in the antisymmetric module $\cong \wedge^2(\mathbb{R}^n)$. These representations are labelled by one-column Young diagrams of, respectively, one and two
cells. The number of columns is associated with the spin. The fact that the generators of $\mathcal{U}(\mathfrak{g})$ are in higher-spin representations is summarized in the following result.

**Universal enveloping algebra of isometries:** The universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ of the isometry algebra $\mathfrak{g}$ of an $n$-dimensional constant-curvature spacetime is an infinite-dimensional module of the general linear Lie algebra $\mathfrak{gl}(n)$, decomposing as an infinite sum of finite-dimensional irreducible $\mathfrak{gl}(n)$-modules labelled by the set of all Young diagrams, with multiplicity one, the first column of which has length $\leq n$.

**Proof:** The Poincaré-Birkhoff-Witt theorem states that the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ is isomorphic to the symmetric algebra $\odot(\mathfrak{g})$ as a vector space. As a $\mathfrak{gl}(n)$-module, the vector space $\mathfrak{g}$ is isomorphic to the sum $\mathbb{R}^n \oplus \wedge^2(\mathbb{R}^n)$ of irreducible modules. This leads to the following isomorphism of modules:

$$\odot(\mathfrak{g}) \cong \left( \odot(\mathbb{R}^n) \right) \otimes \left( \odot(\wedge^2(\mathbb{R}^n)) \right).$$  \hspace{1cm} (6)

The idea is to evaluate the right-hand-side of (6) using the available technology on Kronecker products of irreducible representations [9]. The module $\odot(\mathbb{R}^n)$ decomposes as the infinite sum of irreducible modules labelled by all one-row Young diagrams with multiplicity one. A formula of Littlewood for symmetric plethysms implies that the module $\odot(\wedge^2(\mathbb{R}^n))$ decomposes as the infinite sum of irreducible modules, with multiplicity one, labelled by all Young diagrams with columns of even lengths. The Kronecker product in (6) decomposes as the infinite sum of all the Kronecker products between a one-row Young diagram and a Young diagram with columns of even lengths, each with multiplicity one. Using the Littlewood–Richardson rule, one may show that the result of this computation is the infinite sum of irreducible modules labelled with all possible Young diagrams, each with multiplicity one. The Young diagrams whose first column has length greater than $n$ lead to vanishing modules, hence they do not appear in the series.

The higher-spin algebras are important in relativistic field theories because they always appear as spacetime symmetry algebras in the free limit.

**Spacetime symmetries of relativistic free field theories:** If the Lie algebra of off/on-shell symmetries contains the (conformal) isometry algebra $\mathfrak{g}$ of some constant-curvature spacetime $\mathcal{M}$, then it also contains the (conformal) off/on-shell higher-spin algebra of $\mathcal{M}$.

**Proof:** The Poincaré-Birkhoff-Witt theorem states that one can realize the universal enveloping $\mathcal{U}(\mathfrak{g})$ as Weyl-ordered polynomials in the elements of the Lie algebra $\mathfrak{g}$. The above theorem is proved by observing that any Weyl-ordered polynomial in on-shell symmetries is itself an on-shell symmetry. As observed in [10], the same is true for symmetry generators.

As an important corollary, the theorem implies that any relativistic free field theory has an infinite number of rigid symmetries, and therefore
it possesses an infinite number of conserved currents via the Noether theorem, as it is well known. Notice that relativistic integrable models are precisely such that they possess an infinite set of commuting rigid symmetries corresponding to an infinite set of conserved charges in involution. The infinite-dimensional subalgebra of symmetries of the free field theory generated by the translations only is, of course, Abelian. Actually, the factorization property is deeply related to the preservation of this subalgebra of symmetries at the interacting level [11]. Thus the relationship between higher-spin algebras and integrable models appears to be very intimate (see also [12] and references therein). The strong form of the Maldacena conjecture (in the large $N$ limit) and the integrability properties recently enlightened in this context are further indications of such a relationship.

Symmetries may be characterized by their action on the spacetime coordinates. A smooth change of coordinates is generated by a first-order linear differential operator. Therefore, a higher-order linear differential operator does not generate coordinate transformations. For instance, an isometry generator is a first-order linear differential operator corresponding to a Killing vector field, but the spacetime higher-symmetries are powers of such isometry generators, hence they are higher-order linear differential operators. They do not generate coordinate transformations and this explains why spacetime higher-symmetries are usually not considered in textbooks.

Let us focus on the first non-trivial example of free field theory: the quadratic action of a complex scalar field on an $n$-dimensional spacetime $\mathcal{M}$. In such case, the space $V = \mathbb{C}$ and the kinetic operator $K$ can be taken to be a constant mass term plus the Laplacian on $\mathcal{M}$,

\[
\Box = \nabla_\mu (0) \nabla^\mu .
\]

A scalar field is said to be conformal if its kinetic operator is the conformal Laplacian

\[
\Box = \frac{n - 2}{4(n - 1)} R ,
\]

where $R$ denotes the scalar curvature. The quadratic action and the linear equation are symmetric under the full conformal algebra $\mathfrak{o}(n, 2)$ if and only if the scalar field is conformal and has conformal weight $1 - n/2$.

**Higher symmetries of the conformal scalar field:** For the quadratic action of a complex conformal scalar field on a constant-curvature spacetime $\mathcal{M}$ of dimension $n \geq 2$, the following spaces over $\mathbb{R}$ are isomorphic:

- The Lie algebra of off-shell symmetries quotiented by the ideal of on-shell-trivial symmetry generators,
- A real form of the associative algebra of on-shell symmetries.
- The conformal on-shell higher-spin algebra,
- The real algebra of Weyl-ordered polynomials in the conformal Killing vector fields quotiented by the ideal generated by the conformal Laplacian, endowed with $i$ times the commutator as Lie bracket. The symbols of these differential operators,

\[
T = (-i)^r \xi^{r+1\cdots r} (x) \nabla_{r+1} \cdots \nabla_r + \text{lower + on-shell-trivial},
\]
may be represented by real traceless symmetric tensor fields $\xi$ which are conformal Killing tensors.

Moreover, in $n = 2$ dimensions the theorem is valid for an arbitrary spacetime manifold.

**Proof:** The theorem can be extracted from the results of [13] on flat spacetime of dimension $n > 2$ by taking into account that any constant-curvature spacetime $M$ can be seen as a conic in the projective null cone of the ambient space $\mathbb{R}^{n,2}$. The two-dimensional case is addressed by using the left/right-moving coordinates.

Notice that the on-shell higher-spin algebra of a non-conformal scalar field on a constant-curvature spacetime is a proper subalgebra of the universal enveloping algebra of the isometry algebra $g$: it decomposes as the infinite sum of irreducible $\mathfrak{o}(n-1,1)$-modules labelled by all two-row Young diagrams with multiplicity one, as reviewed in [4, 7]. This algebra is in one-to-one correspondence with the space of reducibility parameters of the infinite tower of symmetric tensor gauge fields where each field appears once and only once for each given spin $s > 0$. Moreover, notice that the $\text{AdS}_{n+1}/\text{CFT}_n$ correspondence for $n > 2$ in the weak tension/coupling limit also makes use of the isomorphism between the on-shell higher-spin algebra of a non-conformal scalar field on $\text{AdS}_{n+1}$ and the on-shell symmetry algebra of a conformal scalar field on $\mathbb{R}^{n-1,1}$ (see [14] for the correspondence at the level of conserved currents). Remark also that the conformal on-shell higher-spin algebra of a two-dimensional spacetime for a massless scalar field is isomorphic to the direct sum of $u(1)$ and the two Lie algebras of differential operators for the left and right moving sectors respectively. Each of such algebras of differential operators is isomorphic to the algebra $W_\infty$ with zero central charge [15].

The deep connection between higher-spin algebras and integrable models is exhibited by the following example in $n = 2$ dimensions.

**Higher symmetries of the interacting scalar field:** A non-linear action of a real scalar field on the two-dimensional Minkowski spacetime, without derivative interaction term, of the form

$$S[\phi] = \frac{1}{2} \langle \phi | \Box | \phi \rangle + \int d^2 x V(\phi), \quad V(\phi) = \mathcal{O}(\phi^2),$$

is invariant under an infinite number of local infinitesimal rigid symmetry transformations, independent of the coordinate $x^\mu$, if and only if

$$V(\phi) = \pm \left( \frac{m}{\alpha} \right)^2 \left( \cos (\alpha \phi) - 1 \right), \quad m \in \mathbb{R},$$

the parameter $\alpha$ is either purely real or imaginary. In such case, the field $\phi$ either corresponds to a free massless scalar field ($m = 0$), a free massive scalar field ($m \neq 0 \ , \ \alpha = 0$) or sine-Gordon theory ($m \neq 0 \ , \ \alpha \neq 0$).

Moreover, via linearisation, there is a one-to-one correspondence between:

- The set of on-shell non-trivial, polynomial in the field derivatives, coordinate-independent, symmetry transformations of the sine-Gordon Lagrangian

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• The Lie algebra of coordinate-independent off-shell symmetries of a free real scalar field quotiented by the ideal of on-shell-trivial symmetry generators,

• A proper Abelian Lie subalgebra of the on-shell higher-spin algebra of the Minkowski plane,

• The space of harmonic odd polynomials in the momenta $P_\mu = -i\partial_\mu$.

These differential operators $T$ may be represented by real traceless symmetric constant tensors $\lambda$:

$$T = i\lambda^{\mu_1\ldots\mu_{2q+1}}\partial_{\mu_1}\ldots\partial_{\mu_{2q+1}} + \text{on-shell-trivial}.$$

**Proof:** The first part of the theorem is a straightforward consequence of the results of [16] in the case when $V(\phi)$ is at least quadratic in $\phi$ (by hypothesis). The second part is proven by selecting all coordinate-independent symmetries of a free real scalar field and comparing them with the conserved currents of [16]. In both cases, the Noether correspondence between non-trivial conserved currents and non-trivial symmetries (see e.g. [17] for a precise statement of this isomorphism) is performed via the Hamiltonian formulation of a two-dimensional scalar field where one of the light-cone coordinate plays the role of “time.”

5 A gauge principle for higher-spins?

The analogy with lower-spins suggests to guess the full non-Abelian gauge theory by making use of the “gauge principle.” Moreover, this point of view actually provides a concrete motivation for using the higher-spin algebras in the interaction problem.

The idea is to consider some “matter” system described by a quadratic action (3) with some algebra of rigid symmetries. The rigid symmetries $U$ of this system are by definition in the “fundamental” representation of the algebra of off-shell symmetries of the action (3). Connections are usually introduced in order to “gauge” these rigid symmetries by allowing $U$ to be a smooth function on $\mathbb{R}^n$ taking values in the group of off-shell symmetries of the action (3). In order to construct a covariant derivative $D = \partial + \Gamma$, one introduces a connection defined as a covariant vector field $\Gamma_\mu$, taking values in the Lie algebra of off-shell symmetries and transforming as

$$|\psi\rangle \rightarrow U |\psi\rangle, \quad \Gamma \rightarrow U D U^{-1}, \quad (8)$$

in such a way that

$$D |\psi\rangle \rightarrow U D |\psi\rangle.$$

The minimal coupling is the replacement of all partial derivatives $\partial$ in the kinetic operator $K(\partial)$ by covariant derivatives $D$ which should ensure that the quadratic action $\langle \psi | K(D) |\psi\rangle$ is preserved by gauge symmetries (8). The connection transforms in the “adjoint” representation of the rigid symmetries while the matter field transforms in the “fundamental.” (More precisely, the covariant derivative transforms in the adjoint while the matter field belongs to a module of the gauge algebra.)

The introduction of a connection requires the introduction of some new dynamical fields: the “gauge” sector. In Yang-Mills gauge theories, the
rigid symmetry is internal and the connection is itself made of spin-1 gauge fields. For spacetime symmetries, the relation between the connection and the gauge field is more complicated. For instance, in Einstein gravity the Levi-Civita connection is expressed in terms of the first derivative of the metric via the torsionlessness and metricity constraints. In general, the spin-s tensor field propagating on a constant-curvature spacetime is expected to be the perturbation of some background field

\[ g_{\mu_1 \ldots \mu_s} = g_{\mu_1 \ldots \mu_s}^{(0)} + \varepsilon h_{\mu_1 \ldots \mu_s}, \]

so that the deformed gauge symmetries would be of the form

\[ \delta_{\xi} g_{\mu_1 \mu_2 \ldots \mu_s} = \varepsilon (D\xi)_{\mu_1 \mu_2 \ldots \mu_s}, \quad (9) \]

where the covariant derivative \( D = \nabla + \mathcal{O}(\varepsilon) \) starts as the covariant derivative with respect to the Levi–Civita connection for the spacetime metric plus non-minimal corrections. Thus the background connection is identified with the Levi-Civita connection for the background metric, and the linearization of (9) reproduces (1). Furthermore, the reducibility parameters of (1) exactly correspond to the gauge symmetries (9) leaving the background geometry invariant. In the present case, this group of rigid symmetries contains the isometry group \( g \) of the constant-curvature spacetime. The classical theory of (in)homogeneous pseudo-orthogonal groups tells us that completely symmetric tensor fields which are invariant under \( g \) are constructed from products of the background metric:

\[ g_{(\mu_1 \mu_2 \ldots \mu_s}^{(0)} g_{\mu_2 \ldots \mu_{2m−1}}^{(0)}. \]

Thus, along these lines, only even-spin symmetric tensor fields can be perturbations of a non-vanishing higher-spin background in a constant-curvature spacetime. The first-order deformation of the gauge symmetries (1) following from (9) would be of the schematic form

\[ \delta_{\xi} h_{\mu_1 \mu_2 \ldots \mu_s} = (D\xi)^{(1)}_{\mu_1 \mu_2 \ldots \mu_s}, \quad (10) \]

where \( D^{(1)} \) stands for the linearized connection (including the linearized Levi-Civita connection) and the dot stands for the action on the gauge parameter \( \xi \). The transformations (10) evaluated on Killing tensors \( \xi \) of the background spacetime would be rigid symmetry transformations of the free gauge theory. This property highly constrains the possible expressions for the linearized connection.

Let us now consider the expansion of the minimally coupled action for the “matter” sector in power series of \( \varepsilon \):

\[ \frac{1}{2} \langle \psi \mid K(D) \mid \psi \rangle = \frac{1}{2} \langle \psi \mid K(\partial) \mid \psi \rangle + \varepsilon \langle \psi \mid J \rangle + \mathcal{O}(\varepsilon^2), \]

where \( J \) denotes a set of symmetric tensors which are bilinear in \( \psi \) and their derivatives. Assuming that the “matter” sector is strictly distinct from the “gauge” sector, the gauge invariance of the complete action at first order in \( \varepsilon \) requires the symmetric tensors \( J^{\mu_1 \mu_2 \ldots \mu_s} \) to be conserved.
up to terms proportional to the “matter” free field equations (and derivatives thereof) corresponding to first-order deformations

\[ \delta^{(1)} |\psi\rangle = U |\psi\rangle \]

of the gauge transformations of the “matter” sector, where \(U\) is a linear differential operator depending linearly on \(\xi\) and its derivatives. At zeroth order in \(\epsilon\), the “gauge” group does not act on the matter. Therefore, at leading order, the transformation law (8) reads as (11). Via the Noether correspondence, the space of all rigid symmetries of the “matter” quadratic action determines the space of all on-shell-conserved currents bilinear in the “matter” fields. The latter ones determine, at first order, the “fundamental” representation of the “gauge” group. The transformations (11) evaluated on Killing tensors \(\xi\) must define off-shell symmetries of the “matter” quadratic action. Their algebra algebra is non-Abelian, hence the “gauge” algebra is already non-Abelian at first order.

As a suggestive example, one may consider a “matter” sector containing only a single scalar field.

**Noether cubic couplings of a scalar field:** The minimally coupled action of a complex scalar field on flat spacetime, given by

\[ S[\phi] = \frac{1}{2} \langle \phi | \Box - m^2 | \phi \rangle - \epsilon \int d^n x \, h_{\mu_1 \mu_2 \ldots \mu_s} \phi^{\mu_1 \mu_2 \ldots \mu_s} + \mathcal{O}(\epsilon^2), \]

is invariant at first order in \(\epsilon\), for any symmetric tensor field \(\xi^{\mu_1 \mu_2 \ldots \mu_{s-1}}\), under infinitesimal symmetry transformations

\[ \delta^0_{\xi} h_{\mu_1 \mu_2 \ldots \mu_s} = \delta^0_{\xi} h_{\mu_1 \mu_2 \ldots \mu_s} + \mathcal{O}(\epsilon), \]

and

\[ \delta^0_{\xi} |\phi\rangle = \epsilon T |\phi\rangle + \mathcal{O}(\epsilon^2), \]

where the symbol of the differential operator \(T\) is represented by \(\xi\) and the lower order terms depend on derivatives of \(\xi\).

\[ T = (-i)^{s-1} \xi^{\mu_1 \ldots \mu_{s-1}} \partial_{\mu_1} \ldots \partial_{\mu_{s-1}} + \text{lower + on-shell-trivial}, \]

if and only if the on-shell-conserved current \(J\) is equivalent to a Noether current associated to the coordinate-independent off-shell symmetries of the free scalar field. This defines a one-to-one correspondence between equivalence classes of such symmetric Noether currents \(J\), bilinear in \(\phi\) and its derivatives, and equivalence classes of such deformations \(\delta^0_{\xi} \phi\) at first order.

**Proof:** The explicit equation expressing the gauge invariance of the minimal coupled action for any symmetric tensor field \(\xi(x)\) of rank \(s-1\) precisely states that the symmetric tensor \(J\) of rank \(s\) is conserved modulo terms proportional to field equation of the scalar field \(\phi\). The one-to-one correspondence, precisely explained in [17], between equivalence classes of on-shell conserved currents and equivalence classes of off-shell symmetry transformations shows explicitly that \(J\) is necessarily related to a coordinate-independent transformation of the form (12). In turn, these
transformations are obtained by evaluating the transformation \( \xi^\mu \), at lowest order in \( \varepsilon \) and on gauge parameters \( \xi \) equal to constant Killing tensors. The sufficiency is proven by making use of the symmetric conserved currents of \( [18] \). The second part of the theorem follows from the fact that trivial currents define trivial deformations and conversely, as it can be seen explicitly.

In the lower-spin case, one recovers the standard minimal coupling procedure. For \( s = 1 \), the minimal coupling stops at second order in \( \varepsilon \) since \( J^\mu \) is the \( U(1) \) current and \( h_\mu \) is the Abelian vector gauge field. For \( s = 2 \), the minimal coupling at first order is the usual coupling between a spin-two gauge field and the energy-momentum tensor \( J^{\mu \nu} \) leading to the coordinate transformations of the scalar field, generated by the vector fields \( T = -i\xi^\mu(x)\partial_\mu \). The commutators of such infinitesimal transformations close and define the Lie bracket of vector fields, so the underlying gauge symmetry algebra may already be guessed at first order for gravity: it is the Lie algebra of smooth vector fields, \( i.e. \) the Lie algebra for the group of diffeomorphisms. The minimally coupled action is obtained to all orders by introducing the Levi-Civita connection.

In the higher-spin case, it should be stressed that the trace conditions on the gauge field and parameter have not been stated in the former proposition because they may indeed be relaxed in order to simplify its formulation. (Nevertheless, these constraints may be included by consistently imposing weaker conservation laws on double-traceless currents.) Moreover, it is convenient to remove trace constraints for searching a Non-Abelian higher-spin gauge symmetry algebra. Actually, the trace constraints may be removed for free field theories in several ways (see \( [19] \) for some reviews, and \( [20] \) for the latest developments). The Lie algebra of gauge transformations \( [12] \) for the infinite tower of all gauge parameters \( 1 \leq s < \infty \) is a real form of the algebra of linear differential operators on \( \mathbb{R}^n \) endowed with \( i \) times the commutator as Lie bracket. Notice also that the unital associative algebra of linear differential operators on \( \mathbb{R}^n \) is isomorphic to the universal enveloping algebra of vector fields on \( \mathbb{R}^n \). (Strictly speaking, this is true only for polynomial vector fields and differential operators, more sophisticated mathematical statements may be required for smooth functions, but this point is only technical.) More concretely, the symbol of a differential operator of order \( r \) is represented by a symmetric tensor field of rank \( r \). In the light of these remarks, it is tempting to conjecture that, for higher-spin gauge theories, the algebra of Hermitian differential operators,

\[
T = \frac{1}{2} \left( \sum_r (-i)^r \xi^{\mu_1 \ldots \mu_r}(x) \partial_{\mu_1} \ldots \partial_{\mu_r} + \text{Hermitian conjugate} \right),
\]

generalizes the algebra of infinitesimal diffeomorphisms for gravity. Another argument in favour of this conjecture may be presented in the “gauge” sector by looking at the metric-like formulation of higher-spins arising from the frame-like formulation of Vasiliev, at first order in the coupling constant \( [21] \).
6 Conclusion

The conclusion is that there are two complementary but distinct ways of using rigid symmetries of the free theory in order to guess the proper gauge symmetry principle of higher-spin gauge theories.

On the one hand, the infinite set of rigid symmetries of the free (or, maybe, even integrable) “matter” sector, might be gauged by the introduction of a connection via a minimal coupling prescription. The idea of using a massive scalar field as free matter sector and an infinite tower of massless symmetric tensor fields as interacting gauge sector is in agreement with the isomorphism between the off-shell higher-spin algebra and the space of reducibility parameters. (If tensor fields are used as free “matter” sector, then the symmetry algebra could be larger. Following the lines of the Vasiliev construction in such case, the structure of the universal enveloping algebra points towards a larger infinite tower of gauge fields including mixed-symmetry tensors.)

On the other hand, in the free “gauge” sector, rigid symmetries linked to reducibility parameters may arise from the linearization of the gauge symmetries of some non-linear action. Thus the complete knowledge of the rigid symmetries of free higher-spin gauge theories would indicate what can be the linearized connection.

Acknowledgments

I. Bakas, G. Barnich, N. Boulanger, T. Damour and J. Remmel are thanked for very useful exchanges. The author is grateful to the organizers for their invitation to this enjoyable meeting and the opportunity to present his lecture. The Institut des Hautes Études Scientifiques de Bures-sur-Yvette is acknowledged for its hospitality.

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