Existence of Solutions to the Second Boundary-Value Problem for the \( p \)-Laplacian on Riemannian Manifolds

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Abstract—We obtain necessary and sufficient conditions for the existence of solutions to the boundary-value problem

\[ \Delta_p u = f \quad \text{on} \quad M, \quad |\nabla u|^{p-2} \frac{\partial u}{\partial \nu} \bigg|_{\partial M} = h, \]

where \( p > 1 \) is a real number, \( M \) is a connected oriented complete Riemannian manifold with boundary, and \( \nu \) is the outer normal vector to \( \partial M \).

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1. INTRODUCTION AND PROBLEM STATEMENT

Let \( M \) be a connected oriented complete Riemannian manifold with boundary. We will consider the problem

\[ \Delta_p u = f \quad \text{on} \quad M, \quad |\nabla u|^{p-2} \frac{\partial u}{\partial \nu} \bigg|_{\partial M} = h, \quad (1.1) \]

where \( \Delta_p u = \nabla_i (g^{ij} |\nabla u|^{p-2} \nabla_j u), \quad p > 1 \), is the \( p \)-Laplace operator, \( \nu \) is the outer normal vector to \( \partial M \), and \( f \) and \( h \) with \( \text{supp} \ h \subset \partial M \) are distributions from \( D'(M) \).

As usual, by \( g_{ij} \) we denote the metric tensor consistent with the Riemannian connection, and by \( g^{ij} \), the tensor dual to the metric tensor, i.e., \( g_{ij} g^{jk} = \delta_i^k \). We have \( |\nabla u| = (g^{ij} \nabla_i u \nabla_j u)^{1/2} \). Following [1], by \( W^1_{p,\text{loc}}(\omega) \), where \( \omega \) is an open subset of \( M \), we mean the space of measurable functions belonging to \( W^1_p(\omega' \cap \omega) \) for any open set \( \omega' \subset M \) with compact closure. The space \( L_{p,\text{loc}}(\omega) \) is defined similarly.

By a solution of problem (1.1) we mean a function \( u \in W^1_{p,\text{loc}}(M) \) such that

\[ -\int_M g^{ij} |\nabla u|^{p-2} \nabla_j u \nabla_i \varphi \, dV = (f - h, \varphi) \]

for all \( \varphi \in C_0^\infty(M) \), where \( dV \) is the volume element of the manifold \( M \).

As a condition at infinity, we require that the solutions of (1.1) satisfy the relation

\[ \int_M |\nabla u|^p \, dV < \infty. \quad (1.2) \]

For brevity, we denote

\[ F = f - h. \quad (1.3) \]

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In the particular case where $f \in L_{1,\text{loc}}(M)$ and $h \in L_{1,\text{loc}}(\partial M)$, we will obviously have

$$(F, \varphi) = \int_M f \varphi dV - \int_{\partial M} h \varphi ds$$

for all $\varphi \in C_0^\infty(M)$, where $dV$ is the volume element of $M$ and $ds$ is the volume element of $\partial M$.

**Definition 1.** By the capacity of a compact set $K \subset \omega$ with respect to an open set $\omega \subset M$ we mean the quantity

$$\text{cap}_p(K, \omega) = \inf_{\varphi} \int_{\omega} |\nabla \varphi|^p dx,$$

where the infimum is taken over all functions $\varphi \in C_0^\infty(\omega)$, identically equal to 1 in a neighborhood of $K$. In the case $\omega = M$, we write $\text{cap}_p(K)$ instead of $\text{cap}_p(K, M)$. For an arbitrary closed set $H \subset M$, we put

$$\text{cap}_p(H) = \sup_K \text{cap}_p(K),$$

where the supremum is taken over all compact sets $K \subset H$. The capacity of an empty set is assumed to be zero.

**Definition 2.** A manifold $M$ is said to be $p$-hyperbolic if its capacity is positive, i.e., $\text{cap}_p(M) > 0$. Otherwise, $M$ is said to be $p$-parabolic.

If $M$ is a compact manifold, then it is obviously $p$-parabolic. It can also be shown that $\mathbb{R}^n$ is a $p$-parabolic manifold for $p \geq n$ and $p$-hyperbolic for $p < n$.

By $L_p^1(\omega)$, where $\omega$ is an open subset of $M$, we will denote the space of distributions $u \in \mathcal{D}'(\omega)$ for which $\nabla u \in L_p(\omega)$. The seminorm on $L_p^1(\omega)$ is defined by the equality

$$\|u\|_{L_p^1(\omega)} = \left( \int_\omega |\nabla u|^p dV \right)^{1/p}.$$  

It is known [2] that $L_p^1(\omega) \subset L_p(K)$ for any compact set $K \subset \omega$. It can also be shown that $L_p^1(\omega)/\langle 1 \rangle$ is a uniformly convex and, therefore, reflexive Banach space. By $\tilde{L}_p^1(\omega)$ we denote the closure of $C_0^\infty(\omega)$ in $L_p^1(\omega)$. By $\tilde{L}_p^1(\omega)^*$ we mean the space dual to $\tilde{L}_p^1(\omega)$, or, in other words, the space of linear continuous functionals on $\tilde{L}_p^1(\omega)$. The norm of the functional $l \in \tilde{L}_p^1(\omega)^*$ is defined by the equality

$$\|l\|_{\tilde{L}_p^1(\omega)^*} = \sup_{\varphi \in C_0^\infty(\omega), \|\varphi\|_{L_p^1(\omega)} = 1} |(l, \varphi)|.$$

**Proposition 1.** *For problem (1.1), (1.2) to be solvable, it is necessary and sufficient that the functional $F$ defined by formula (1.3) be continuous on the space $\tilde{L}_p^1(M)$.*

**Proof.** If $u$ is a solution of (1.1), (1.2), then

$$-\int_M g^{ij} |\nabla u|^{p-2} \nabla_j u \nabla_i \varphi dV = (F, \varphi)$$

for all $\varphi \in C_0^\infty(M)$, whence, by Hőlder’s inequality, we obtain

$$|(F, \varphi)| \leq \|u\|_{L_p^1(M)}^{p-1} \|\varphi\|_{L_p^1(M)}.$$

Extending $F$ to the entire space $\tilde{L}_p^1(M)$ by continuity, we complete the proof of necessity.

To prove sufficiency, we take a sequence $\varphi_i \in C_0^\infty(M)$, $i = 1, 2, \ldots$, such that

$$\lim_{i \to \infty} J(\varphi_i) = \inf_{\varphi \in C_0^\infty(M)} J(\varphi),$$

where $J(\varphi) = \frac{1}{p} \int_M |\nabla \varphi|^p dV + (F, \varphi).$
The continuity of $F$ on $\tilde{L}^1_p(M)$ implies that the sequence $\{\varphi_i\}_{i=1}^{\infty}$ is bounded in the seminorm of the space $L^1_p(M)$; in particular,
\[
\lim_{i \to \infty} J(\varphi_i) = \inf_{\varphi \in C_0^\infty(M)} J(\varphi) > -\infty.
\]
In the sequence $\varphi_i + \langle 1 \rangle \in L^1_p(m)/\langle 1 \rangle$, $i = 1, 2, \ldots$, we choose a subsequence $\varphi_{i_j} + \langle 1 \rangle$, $j = 1, 2, \ldots$, weakly converging to some element $u + \langle 1 \rangle$ of the space $L^1_p(M)/\langle 1 \rangle$. Since $L^1_p(m)/\langle 1 \rangle$ is reflexive, such a subsequence exists. Let $R_m$ denote the convex hull of the set $\{\varphi_{i_j}\}_{j \geq m}$. By Mazur’s theorem, there exists a sequence $r_m \in R_m$, $m = 1, 2, \ldots$, such that
\[
\|u - r_m\|_{L^1_p(M)} \to 0 \quad \text{as} \quad m \to \infty.
\]
Because $r_m \in C_0^\infty(M)$, $m = 1, 2, \ldots$, it immediately follows that $u \in \tilde{L}^1_p(M)$. We also have
\[
J(r_m) \leq \sup_{j \geq m} J(\varphi_{i_j}), \quad m = 1, 2, \ldots,
\]
since $J$ is a convex functional. Thus, passing to the limit as $m \to \infty$, we obtain
\[
J(u) \leq \inf_{\varphi \in C_0^\infty(M)} J(\varphi).
\]
Since the inverse inequality is obvious, we have
\[
J(u) = \inf_{\varphi \in C_0^\infty(M)} J(\varphi),
\]
from which, by the variational principle, we obtain (1.4). In other words, $u$ is a solution of problem (1.1), (1.2).

Boundary-value problems in domains of various geometry and on smooth manifolds traditionally attract the attention of mathematicians [3]–[12]. In the case of bounded domains with irregular boundary, the solvability criterion of the Neumann problem for the $p$-Laplace operator was obtained in [12], where charges (differences of finite measures) were considered as boundary conditions and right-hand sides of the equation.

We obtain necessary and sufficient conditions for the solvability of the Neumann problem on Riemannian manifolds, which are a natural generalization of infinitely smooth domains in $\mathbb{R}^n$. These conditions are different for $p$-hyperbolic and $p$-parabolic manifolds. For example, in the simple case $M = \mathbb{R}^n \setminus B_1$, where $B_1$ is the unit ball in $\mathbb{R}^n$, $n \geq 2$, the exterior Neumann problem
\[
\Delta_p u = 0 \quad \text{on} \quad \mathbb{R}^n \setminus B_1, \quad |\nabla u|^{p-2} \frac{\partial u}{\partial \nu} = h, \quad \int_{\mathbb{R}^n \setminus B_1} |\nabla u|^p \, dx < \infty
\]
for $n > p$ has a solution for any function $h \in L_p/(p-1)(\partial B_1)$, while if $n \leq p$, then, for the existence of a solution, it is necessary and sufficient that
\[
\int_{\partial B_1} h \, dS = 0.
\]
In this sense, $p$-parabolic manifolds behave like bounded domains in $\mathbb{R}^n$ (see Corollaries 1 and 2).

2. THE CASE OF A FUNCTIONAL $F$ WITH COMPACT SUPPORT

**Theorem 1.** Let $M$ be a $p$-hyperbolic manifold, and let the functional $F$ defined by (1.3) have compact support. Then, for the existence of a solution of (1.1), (1.2), it is necessary and sufficient that $F$ be continuous on $\tilde{L}^1_p(\omega)$ for an open set $\omega$ such that $\text{supp} \, F \subset \omega$. 

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Theorem 2. Let $M$ be a $p$-parabolic manifold, and let the functional $F$ defined by (1.3) have compact support. Then, for the existence of a solution of (1.1), (1.2), it is necessary and sufficient that $F$ be continuous on $L_p^1(\omega)$ for an open set $\omega$ such that $\text{supp} \ F \subset \omega$ and, moreover,

$$
\lim_{s \to \infty} (F, \eta_s) = 0
$$

for a sequence $\eta_s \in C_0^\infty(M)$ such that

$$
\lim_{s \to \infty} \|\eta_s\|_{L_p^1(M)} = 0 \quad \text{and} \quad \eta_s|_K = 1, \quad s = 1, 2, \ldots,
$$

(2.2)

where $K$ is a compact set of positive measure.

Corollary 1. Let $M$ be a $p$-hyperbolic manifold with compact boundary, and let $h$ be a functional in $\mathcal{D}'(M)$ such that $\text{supp} \ h \subset \partial M$. Then, for the existence of a solution to the problem

$$
\Delta_p u = 0 \quad \text{on} \quad M, \quad |\nabla u|^{p-2} \frac{\partial u}{\partial \nu}|_{\partial M} = h
$$

(2.3)

satisfying condition (1.2), it is necessary and sufficient that $h$ be continuous on $L_p^1(\omega)$ for an open set $\omega$ such that $\partial M \subset \omega$.

Corollary 2. Let $M$ be a $p$-parabolic manifold with compact boundary, and let $h$ be a functional in $\mathcal{D}'(M)$ such that $\text{supp} \ h \subset \partial M$. Then, for the existence of a solution to problem (2.3), (1.2), it is necessary and sufficient that $h$ be continuous on $L_p^1(\omega)$ for an open set $\omega$ such that $\partial M \subset \omega$ and, moreover,

$$
(h, 1) = 0.
$$

(2.4)

Corollaries 1 and 2 immediately follow from Theorems 1 and 2. Assuming, without loss of generality, that $\overline{\omega}$ is a compact set, we only note that (2.4) implies (2.1) for any sequence $\eta_s \in C_0^\infty(M)$ satisfying (2.2), where $K = \overline{\omega}$. In the case of a $p$-parabolic manifold, such a sequence obviously exists. On the other hand, if $h \in L_p^1(\omega)^*$, then (2.1) will hold for any sequence $\eta_s \in C_0^\infty(M)$ satisfying (2.2), where $K = \overline{\omega}$. This, in turn, implies the validity of (2.4).

To prove Theorems 1 and 2, we need the following lemmas.

Lemma 1. Let $\text{cap}(K) = 0$ for a compact set $K$ of nonzero measure. Then $M$ is a $p$-parabolic manifold.

Lemma 2. Let $M$ be a $p$-hyperbolic manifold. Then, for any compact set $K$, the space $L_p^1(M)$ is continuously embedded in $L_p(K)$. In other words,

$$
\|\varphi\|_{L_p(K)} \leq C \|\varphi\|_{L_p^1(M)}
$$

(2.5)

for all $\varphi \in C_0^\infty(M)$, where the constant $C > 0$ is independent of $\varphi$.

In the case $p = 2$, Lemmas 1 and 2 were proved in [5, Chap. 3, Sec. 1]. For $p > 1$, the proof is similar. For the convenience of the reader, we present this proof in full.

Proof of Lemma 1. Let $H$ be a compact set containing $K$, and let $\omega$ be a domain with compact closure such that $H \subset \omega$. Let $M$ be the union of a family of Lipschitz domains $\omega_i$ with compact closure such that $\overline{\omega} \subset \omega_i \subset \omega_{i+1}$, $i = 1, 2, \ldots$. Let $u_i$ denote a solution of the boundary-value problem

$$
\Delta_p u_i = 0 \quad \text{on} \quad \omega_i \setminus K, \quad u_i|_K = 1, \quad u_i|_{\partial \omega_i} = 0,
$$

$$
|\nabla u_i|^{p-2} \frac{\partial u_i}{\partial \nu}|_{\partial \omega_i \cap \partial M} = 0, \quad i = 1, 2, \ldots.
$$

According to the maximum principle, $0 \leq u_i(x) \leq u_{i+1}(x) \leq 1$ for all $x \in \omega_i$, $i = 1, 2, \ldots$. At the same time,

$$
\|u_i\|_{L_p^1(\omega_i)} = \text{cap}_p(K, \omega_i) \to 0 \quad \text{as} \quad i \to \infty.
$$
Thus, \( \{u_i\}_{i=1}^{\infty} \) is a fundamental sequence in \( W^1_p(\omega) \). Since \( \text{mes} \ K > 0 \), we obviously have \( u_i \to 1 \) in \( W^1_p(\omega) \) as \( i \to \infty \). It is known [11] that the sequence \( \{u_i\}_{i=1}^{\infty} \) has bounded Hölder norm in a neighborhood of of the set \( \partial \omega \); therefore, it has a subsequence converging to 1 uniformly on \( \partial \omega \).

According to the maximum principle, the function \( u_i \) does not exceed 1 on the set \( \omega \) and is not less than the infimum of this function on \( \partial \omega \). Thus, \( u_i \to 1 \) uniformly on \( \omega \) as \( i \to \infty \). Further, take the function \( \eta \in C^\infty(\mathbb{R}) \), equal to zero on the interval \((-\infty, 1/4]\) and 1 on \([3/4, \infty)\). It is easy to see that
\[
\|\eta \circ u_i\|_{L^p_\omega(\omega)} \leq \|\eta\|_{C(\mathbb{R})} \|u_i\|_{L^p_\omega(\omega)} \to 0 \quad \text{as} \quad i \to \infty,
\]
this allows us to assert that
\[
cap_p(H) \leq \lim_{i \to \infty} \|\eta \circ u_i\|_{L^p_\omega(\omega)} = 0. \quad \square
\]

**Proof of Lemma 2.** Take a Lipschitz domain \( \omega \) with compact closure such that \( K \subset \omega \). Assume on the contrary that there exists a sequence \( \psi_i \in C^\infty_0(M) \), satisfying the conditions
\[
\lim_{i \to \infty} \|\psi_i\|_{L^p_\omega(M)} = 0 \quad \text{and} \quad \|\psi_i\|_{L^p(\omega)} = \text{mes} \; \omega > 0, \quad i = 1, 2, \ldots \tag{2.6}
\]
Since \( W^1_p(\omega) \) is completely continuously embedded in \( L^p(\omega) \), there is a subsequence of the sequence \( \{\psi_i\}_{i=1}^{\infty} \) that converges in \( L^p(\omega) \). To avoid inconvenient indices, for this subsequence we will use the same notation \( \{\psi_i\}_{i=1}^{\infty} \). In view of (2.6), we see that \( \psi_i \to 1 \) in \( W^1_p(\omega) \) as \( i \to \infty \); therefore, a certain subsequence of this sequence, again denoted by \( \{\psi_i\}_{i=1}^{\infty} \), converges to 1 almost everywhere on \( \omega \).

By Egorov’s theorem, there exists a set \( E \subset \omega \) of positive measure such that \( \{\psi_i\}_{i=1}^{\infty} \) tends to 1 uniformly on \( E \). Because the \( \psi_i \) are continuous functions, the sequence \( \{\psi_i\}_{i=1}^{\infty} \) will also tend to 1 uniformly on the compact set \( \overline{E} \). Thus, taking the same function \( \eta \) as in the proof of Lemma 1, we will have
\[
cap_p(E) \leq \lim_{i \to \infty} \|\eta \circ \psi_i\|_{L^p_\omega(M)} = 0;
\]
hence, in view of Lemma 1, it follows that \( M \) is a \( p \)-parabolic manifold. The obtained contradiction proves that
\[
\|\varphi\|_{L^p(\omega)} \leq C\|\varphi\|_{L^p_\omega(M)}
\]
for all \( \varphi \in C^\infty_0(M) \), where the constant \( C > 0 \) is independent of \( \varphi \); this immediately implies (2.5). \( \square \)

**Proof of Theorem 1.** Necessity is obvious, because the continuity of the functional \( F \) on \( \tilde{L}^1_p(M) \) implies the continuity of \( F \) on \( \tilde{L}^1_p(\omega) \) for any open subset \( \omega \) of the manifold \( M \).

Let us prove sufficiency. Let \( F \) be continuous on \( \tilde{L}^1_p(\omega) \) for an open set \( \omega \) such that \( \text{supp} \; F \subset \omega \). We will show that, in this case, \( F \) will also be continuous in \( \tilde{L}^1_p(M) \). Let us take a function \( \psi \in C^\infty_0(\omega) \), equal to 1 in a neighborhood of \( \text{supp} \; F \). We have
\[
|(F, \varphi)| \leq \|F\|_{\tilde{L}^1_p(\omega)} \|\varphi\|_{L^1_p(\omega)}
\]
for all \( \varphi \in C^\infty_0(M) \), whence, due to the fact that \( (F, \varphi) = (F, \psi \varphi) \) and
\[
\|\psi \varphi\|_{L^1_p(\omega)} \leq \|\psi\|_{C(\omega)} \|\varphi\|_{L^1_p(\omega)} + \|\nabla \psi\|_{C(\omega)} \|\varphi\|_{L^p(\text{supp} \; \psi)}
\]
we have
\[
|(F, \varphi)| \leq \|F\|_{\tilde{L}^1_p(\omega)} (\|\psi\|_{C(\omega)} \|\varphi\|_{L^1_p(\omega)} + \|\nabla \psi\|_{C(\omega)} \|\varphi\|_{L^p(\text{supp} \; \psi)})
\]
for all \( \varphi \in C^\infty_0(M) \). Thus, to complete the proof, it remains to note that, by Lemma 2,
\[
\|\varphi\|_{L^p(\text{supp} \; \psi)} \leq C\|\varphi\|_{L^1_p(M)}
\]
for all \( \varphi \in C^\infty_0(M) \), where the constant \( C > 0 \) is independent of \( \varphi \). \( \square \)
Proof of Theorem 2. As in the case of Theorem 1, it is only necessary to prove sufficiency, because necessity is obvious. Let \( \Omega \) be a Lipschitz domain with compact closure containing \( K \) and \( \text{supp} F \). Without loss of generality, we can assume that the norms \( \| \eta_s \|_{W^1_p(\Omega)} \) are bounded by a constant independent of \( s \). If the last condition does not hold, then we replace \( \eta_s \) by

\[
\tilde{\eta}_s(x) = \begin{cases} 
0, & \eta_s(x) \leq 0, \\
n_s(x), & 0 < \eta_s(x) < 1, \\
1, & 1 \leq \eta_s(x), 
\end{cases} \quad s = 1, 2, \ldots.
\]  
(2.7)

Since \( W^1_p(\Omega) \) is completely continuously embedded in \( L_p(\Omega) \), there is a subsequence of the sequence \( \{ \eta_s \}_{s=1}^\infty \) converging to \( L_p(\Omega) \). We denote this subsequence by \( \{ \eta_s \}_{s=1}^\infty \) as well. By virtue of (2.2), we have

\[
\| 1 - \eta_s \|_{W^1_p(\Omega)} \to 0 \quad \text{as} \quad s \to \infty.
\]  
(2.8)

Suppose that \( \varphi \in C_0^\infty(M) \). By Poincaré’s inequality,

\[
\int_\Omega |\varphi - \varphi_j|^p \, dV \leq C \int_\Omega |\nabla \varphi|^p \, dV,
\]  
where

\[
\alpha = \frac{1}{\text{mes} \Omega} \int_\Omega \varphi \, dV.
\]  
(2.10)

Here and further in the proof of Theorem 2, by \( C \) we mean all possible positive constants depending only on \( p \), \( \omega \), \( \Omega \), and the support of the functional \( F \). Let us take a function \( \psi \in C_0^\infty(\omega \cap \Omega) \) equal to 1 in a neighborhood of \( \text{supp} F \). We denote

\[
\varphi_j' = (\varphi - \alpha \eta_j)(1 - \psi), \quad \varphi_j'' = (\varphi - \alpha \eta_j)\psi, \quad j = 1, 2, \ldots.
\]  
(2.11)

Since \( \varphi = \varphi_j' + \varphi_j'' + \alpha \eta_j \), we can assert that

\[
|(F, \varphi)| \leq |(F, \varphi_j')| + |(F, \varphi_j'')| + |\alpha||(F, \eta_j)|, \quad j = 1, 2, \ldots.
\]  
(2.12)

Because \( (F, \varphi_j') = 0 \) and \( \varphi_j'' \in C_0^\infty(\omega) \), this obviously implies the estimate

\[
|(F, \varphi)| \leq \|F\|_{L^p_\omega(\omega)} \cdot \|\varphi_j''\|_{L^p_\omega(\omega)} + |\alpha||(F, \eta_j)|, \quad j = 1, 2, \ldots.
\]

Combining this with the inequality

\[
\|\varphi_j''\|_{L^p_\omega(\omega)} \leq \|\psi||C_\omega||\varphi - \alpha \eta_j||_{L^p_\omega(\omega)} + \|\nabla \psi||C_\omega||\varphi - \alpha \eta_j||_{L_p(\omega)},
\]

we obtain

\[
|(F, \varphi)| \leq C\|F\|_{L^p_\omega(\omega)} \cdot (\|\varphi - \alpha \eta_j\|_{L^p_\omega(\omega)} + \|\varphi - \alpha \eta_j\|_{L_p(\omega)}) + |\alpha||(F, \eta_j)|, \quad j = 1, 2, \ldots.
\]

In the last formula, passing to the limit as \( j \to \infty \) and taking into account the relations

\[
\|\varphi - \alpha \eta_j\|_{L^p(M)} \leq \|\varphi\|_{L^p(M)} + |\alpha|\|\eta_j\|_{L^p(M)} \to \|\varphi\|_{L^p(M)} \quad \text{as} \quad j \to \infty,
\]  
(2.13)

\[
\|\varphi - \alpha \eta_j\|_{L_p(\Omega)} = \|\varphi - \alpha + \alpha(1 - \eta_j)\|_{L_p(\Omega)} \leq \|\varphi - \alpha\|_{L_p(\Omega)} + |\alpha||1 - \eta_j||_{L_p(\Omega)} \leq C\|\varphi\|_{L^p(\Omega)} + |\alpha||1 - \eta_j||_{L_p(\Omega)} \to C\|\varphi\|_{L^p(\Omega)} \quad \text{as} \quad j \to \infty,
\]  
(2.14)

we will have

\[
|(f, \varphi)| \leq C\|F\|_{L^p_\omega(\omega)} \cdot \|\varphi\|_{L^p(M)}.
\]

The proof is complete.

\[\square\]

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3. THE CASE OF A FUNCTIONAL \( f \) OF GENERAL FORM

We will assume that the manifold \( M \) admits a locally finite cover

\[
M = \bigcup_{i=1}^{\infty} \Omega_i
\]

of multiplicity \( k < \infty \), where the open sets \( \Omega_i \) are Lipschitz domains with compact closure such that \( \Omega_i \cap \Omega_{i+1} \neq \emptyset, \ i = 1, 2, \ldots \). Further, let \( \gamma: M \to (0, \infty) \) be a measurable function bounded away from zero and infinity on every compact subset of the manifold \( M \), and let \( \psi_i \in C_0^\infty(\Omega_i) \) be a partition of unity on \( M \) such that

\[
|\nabla \psi_i(x)|^p \leq \gamma(x), \quad x \in \Omega_i, \quad i = 1, 2, \ldots
\]

We will need the following well-known statement.

**Lemma 3** (Poincaré’s inequality). Let \( \omega \) be a Lipschitz domain with compact closure. Then

\[
\int_{\omega} |u - \overline{u}| \, dV \leq C \int_{\omega} \gamma^{1-1/p} |\nabla u| \, dV
\]

for any function \( u \in W_1^1(\omega) \), where

\[
\overline{u} = \frac{\int_\omega u \, dV}{\int_\omega \gamma \, dV},
\]

and the constant \( C > 0 \) is independent of \( u \).

We will also assume that

\[
\sup_{i \in \mathbb{N}} C_i \left( \frac{1}{\int_{\Omega_i} \gamma \, dV} + \frac{1}{\int_{\Omega_{i+1}} \gamma \, dV} \right) \sum_{j=1}^{i} \int_{\Omega_j} \gamma \, dV < \infty
\]

in the case of a \( p \)-hyperbolic manifold \( M \) and

\[
\sup_{i \in \mathbb{N}} C_i \left( \frac{1}{\int_{\Omega_i} \gamma \, dV} + \frac{1}{\int_{\Omega_{i+1}} \gamma \, dV} \right) \sum_{j=i+1}^{\infty} \int_{\Omega_j} \gamma \, dV < \infty
\]

in the case of a \( p \)-parabolic manifold \( M \), where \( \mathbb{N} \) is the set of natural numbers and \( c_i > 0 \) is the constant in inequality (3.3) for the domain \( \omega = \Omega_i \cup \Omega_{i+1} \).

**Theorem 3.** Let \( M \) be a \( p \)-hyperbolic manifold. Then, for the existence of a solution of (1.1), (1.2), it is necessary and sufficient that

\[
\sum_{i=1}^{\infty} \|F\|_{L^p_{\gamma_i}(\Omega_i)^*} < \infty,
\]

where \( F \) is defined by (1.3).

**Theorem 4.** Let \( M \) be a \( p \)-parabolic manifold. Then, for the existence of a solution of (1.1), (1.2), it is necessary and sufficient that inequality (3.6) hold and, moreover, for some sequence \( \eta_s \in C_0^\infty(M) \), that conditions (2.1) and (2.2) be fulfilled.

The proof of Theorems 3 and 4 is based on Lemmas 4–6.

**Lemma 4.** Let \( \omega_1 \) and \( \omega_2 \) be measurable subsets of a Lipschitz domain \( \omega \subseteq M \) such that

\[
\gamma_i = \int_{\omega_i} \gamma \, dV > 0, \quad i = 1, 2.
\]

Then

\[
\frac{1}{\gamma_1} \int_{\omega_1} |u| \, dV \leq C \left( \frac{1}{\gamma_1} + \frac{1}{\gamma_2} \right) \int_{\omega} \gamma^{1-1/p} |\nabla u| \, dV + \frac{1}{\gamma_2} \int_{\omega_2} |u| \, dV
\]

for any function \( u \in W_1^1(\omega) \), where \( C > 0 \) is the constant in Poincaré’s inequality (3.3).
Proof. Taking into account (3.3), we obtain
\[ \int_{\Omega} \gamma |u - \overline{u}| \, dV \leq C \int_{\Omega} \gamma^{1-1/p} |\nabla u| \, dV, \]
whence, by virtue of the inequality $|u - \overline{u}| \leq |u - \overline{u}|$, we have
\[ \int_{\Omega} \gamma |u| \, dV - \int_{\Omega} \gamma |\overline{u}| \, dV \leq C \int_{\Omega} \gamma^{1-1/p} |\nabla u| \, dV \]
or, in other words,
\[ \frac{1}{\gamma_1} \int_{\Omega} \gamma |u| \, dV \leq C \frac{1}{\gamma_1} \int_{\Omega} \gamma^{1-1/p} |\nabla u| \, dV + |\overline{u}|. \quad (3.7) \]
By (3.3), we will also have
\[ \int_{\Omega} \gamma |u - \overline{u}| \, dV \leq C \int_{\Omega} \gamma^{1-1/p} |\nabla u| \, dV, \]
whence, by virtue of the inequality $|\overline{u}| - |u| \leq |u - \overline{u}|$, we immediately obtain
\[ \int_{\Omega} \gamma |\overline{u}| \, dV - \int_{\Omega} \gamma |u| \, dV \leq C \int_{\Omega} \gamma^{1-1/p} |\nabla u| \, dV, \]
which is obviously equivalent to the relation
\[ |\overline{u}| \leq C \frac{1}{\gamma_2} \int_{\Omega} \gamma^{1-1/p} |\nabla u| \, dV + \frac{1}{\gamma_2} \int_{\Omega} \gamma |u| \, dV; \]
Combining it with formula (3.7), we complete the proof. \[ \square \]

Lemma 5. Let condition (3.4) hold for the cover (3.1). Then
\[ \int_{\Omega} \gamma |\varphi|^p \, dV \leq C \int_{\Omega} |\nabla \varphi|^p \, dV \quad (3.8) \]
for any function $\varphi \in C^\infty_0(M)$, where the constant $C > 0$ depends only on $p$, on the multiplicity of the cover (3.1), and on the left-hand side of (3.4).

Proof. Let us denote the constant in Poincaré’s inequality (3.3) for $\omega = \Omega_i \cup \Omega_{i+1}$ by $C_i > 0$. We set
\[ S_i = \sum_{j=1}^{i} \int_{\Omega_j} \gamma \, dV, \quad \gamma_i = \int_{\Omega_i} \gamma \, dV, \quad i = 1, 2, \ldots. \]
Let us also assume that $S_0 = 0$. We have
\[ \int_{\Omega} \gamma |\varphi|^p \, dV \leq \sum_{i=1}^{\infty} \int_{\Omega_i} \gamma |\varphi|^p \, dV = \sum_{i=1}^{\infty} \frac{S_i - S_{i-1}}{\gamma_i} \int_{\Omega_i} \gamma |\varphi|^p \, dV \]
\[ = \sum_{i=1}^{\infty} S_i \left( \frac{1}{\gamma_i} \int_{\Omega_i} \gamma |\varphi|^p \, dV - \frac{1}{\gamma_{i+1}} \int_{\Omega_{i+1}} \gamma |\varphi|^p \, dV \right), \]
whence, by virtue of the inequality
\[ \frac{1}{\gamma_i} \int_{\Omega_i} \gamma |\varphi|^p \, dV \leq C_i \left( \frac{1}{\gamma_i} + \frac{1}{\gamma_{i+1}} \right)^p \int_{\Omega_i \cup \Omega_{i+1}} \gamma^{1-1/p} |\varphi|^{p-1} |\nabla \varphi| \, dV + \frac{1}{\gamma_{i+1}} \int_{\Omega_{i+1}} \gamma |\varphi|^p \, dV, \]
which follows from Lemma 4, we arrive at the estimate
\[ \int_{\Omega} \gamma |\varphi|^p \, dV \leq \sum_{i=1}^{\infty} S_i C_i \left( \frac{1}{\gamma_i} + \frac{1}{\gamma_{i+1}} \right)^p \int_{\Omega_i \cup \Omega_{i+1}} \gamma^{1-1/p} |\varphi|^{p-1} |\nabla \varphi| \, dV. \quad (3.9) \]
At the same time, according to Jensen’s inequality,
\[
S_i C_i \left( \frac{1}{\gamma_i} + \frac{1}{\gamma_{i+1}} \right) p \int_{\Omega_i \cup \Omega_{i+1}} \gamma^{1-1/p} |\varphi|^{p-1} |\nabla \varphi| \, dV
\leq \varepsilon \int_{\Omega_i \cup \Omega_{i+1}} \gamma |\varphi|^p \, dV + A S_i^p C_i^p \left( \frac{1}{\gamma_i} + \frac{1}{\gamma_{i+1}} \right)^p \int_{\Omega_i \cup \Omega_{i+1}} |\nabla \varphi|^p \, dV
\]
for any \( \varepsilon > 0 \), where the constant \( A > 0 \) depends only on \( \varepsilon \) and \( p \); so (3.9) allows us to state that
\[
\int_{M} \gamma |\varphi|^p \, dV \leq 2k \varepsilon \int_{M} \gamma |\varphi|^p \, dV + B \int_{M} |\nabla \varphi|^p \, dV
\]
for any \( \varepsilon > 0 \), where \( k \) is the multiplicity of the cover (3.1) and \( B > 0 \) is a constant depending only on \( \varepsilon, p, k \) and on the left-hand side of (3.4). Thus, taking \( \varepsilon > 0 \) sufficiently small in the last inequality, we complete the proof. \( \square \)

**Lemma 6.** Let condition (3.5) hold for the cover (3.1). Then, for any function \( \varphi \in C^\infty(M) \) that vanishes on the set \( \Omega_1 \), inequality (3.8) holds, where the constant \( C > 0 \) depends only on \( p \), on the multiplicity of the cover (3.1), and on the left-hand side of (3.5).

**Proof.** We set
\[
S_i = \sum_{j=i+1}^{\infty} \int_{\Omega_j} \gamma \, dV, \quad \gamma_i = \int_{\Omega_i} \gamma \, dV, \quad i = 1, 2, \ldots .
\]
Condition (3.5) means, in particular, that \( S_i < \infty \) for all natural numbers \( i \). As above, by \( C_i > 0 \) we denote the constant in the inequality (3.3) for the domain \( \omega = \Omega_i \cup \Omega_{i+1} \).

It can be seen that
\[
\int_{M} \gamma |\varphi|^p \, dV \leq \sum_{i=1}^{\infty} \int_{\Omega_{i+1}} \gamma |\varphi|^p \, dV = \sum_{i=1}^{\infty} \frac{S_i - S_{i+1}}{\gamma_{i+1}} \int_{\Omega_{i+1}} \gamma |\varphi|^p \, dV
\]
\[
= \sum_{i=1}^{\infty} S_i \left( \frac{1}{\gamma_{i+1}} \int_{\Omega_{i+1}} \gamma |\varphi|^p \, dV - \frac{1}{\gamma_i} \int_{\Omega_i} \gamma |\varphi|^p \, dV \right),
\]
from whence, in view of the inequality
\[
\frac{1}{\gamma_{i+1}} \int_{\Omega_{i+1}} \gamma |\varphi|^p \, dV \leq C_i \left( \frac{1}{\gamma_i} + \frac{1}{\gamma_{i+1}} \right) p \int_{\Omega_i \cup \Omega_{i+1}} \gamma^{1-1/p} |\varphi|^{p-1} |\nabla \varphi| \, dV + \frac{1}{\gamma_i} \int_{\Omega_i} \gamma |\varphi|^p \, dV,
\]
which follows from Lemma 4, we obtain (3.9). In conclusion, it remains to repeat the argument given in the proof of Lemma 5. \( \square \)

**Corollary 3.** If the cover (3.1) satisfies condition (3.4), then the manifold \( M \) is \( p \)-hyperbolic.

**Proof.** Indeed, let \( K \) be a compact set of nonzero measure. Applying Lemma 5, we obtain
\[
0 < \int_{K} \gamma \, dV \leq C \int_{M} |\nabla \varphi|^{p} \, dV
\]
for any function \( \varphi \in C^\infty(M) \) equal to 1 in a neighborhood of \( K \); here the constant \( C > 0 \) is independent of \( \varphi \). Thus, \( \text{cap}(K) > 0 \). \( \square \)

**Proof of Theorem 3.** We will follow an idea proposed in [11]. Let us assume that problem (1.1), (1.2) has a solution. In this case, \( F \) is a continuous functional on \( \tilde{L}^1_p(M) \). Let us show the validity of inequality (3.6). Take functions \( \varphi_i \in C^\infty_c(\Omega_i) \) such that
\[
\|\varphi_i\|_{\tilde{L}^1_p(\Omega_i)} = 1, \quad (F, \varphi_i) \geq \frac{1}{2} \|F\|_{\tilde{L}^1_p(\Omega_i)^*}, \quad i = 1, 2, \ldots .
\]
Indeed, let

$$k_0$$

It is easy to see that

Thus,

$$\text{whence, passing to the limit as } j \to \infty, \text{ we obtain}$$

On the other hand, obviously,

Thus, taking into account the relation

where almost all terms on the right-hand side are zero; hence

Setting

we will have

$$\sum_{i=1}^{j} \|F\|_{L^p(\Omega_i)^*}^p \leq k \sum_{i=1}^{j} \|F\|_{L^p(\Omega_i)^*}^p \int_{\Omega_i} |\nabla \psi_i|^p dV = k \sum_{i=1}^{j} \|F\|_{L^p(\Omega_i)^*}^p,$$

where \(k\) is the multiplicity of the cover (3.1), we obtain

$$\left( \sum_{i=1}^{j} \|F\|_{L^p(\Omega_i)^*}^p \right)^{1/p} \leq 2k \|F\|_{\hat{L}^1_p(M)^*}, \quad j = 1, 2, \ldots.$$  

Combining the last inequality with inequality (3.10), we conclude that

$$\left( \sum_{i=1}^{j} \|F\|_{L^p(\Omega_i)^*}^p \right)^{1-1/p} \leq 2k \|F\|_{\hat{L}^1_p(M)^*}, \quad j = 1, 2, \ldots,$$

whence, passing to the limit as \(j \to \infty\), we obtain (3.6).

Let us now show that condition (3.6) implies the continuity of the functional \(F\) on the space \(\hat{L}^1_p(M)\). Indeed, let \(\varphi \in C_0^\infty(M)\). Since \(\text{supp } \varphi\) is compact and (3.1) is a locally finite cover, it follows that the support of the function \(\varphi\) can only intersect with finitely many the domains \(\Omega_i\). Thus,

$$\varphi = \sum_{i=1}^{\infty} \psi_i \varphi,$$

where almost all terms on the right-hand side are zero; hence

$$|(F, \varphi)| \leq \sum_{i=1}^{\infty} \left| (F, \psi_i \varphi) \right| \leq \sum_{i=1}^{\infty} \|F\|_{\hat{L}^1_p(\Omega_i)^*} \|\psi_i \varphi\|_{L^1_p(\Omega_i)}$$

$$\leq \left( \sum_{i=1}^{\infty} \|F\|_{L^p(\Omega_i)^*}^p \right)^{(p-1)/p} \left( \sum_{i=1}^{\infty} \|\psi_i \varphi\|_{L^1_p(\Omega_i)}^p \right)^{1/p}. \quad (3.11)$$

It is easy to see that

$$\|\psi_i \varphi\|_{L^1_p(\Omega_i)}^p = \int_{\Omega_i} |\nabla (\psi_i \varphi)|^p dV \leq 2^p \int_{\Omega_i} |\nabla \psi_i|^p |\varphi| dV + 2^p \int_{\Omega_i} \psi_i^p |\nabla \varphi|^p dV,$$

whence, in view of (3.2) and the fact that \(0 \leq \psi_i^p \leq \psi_i \leq 1\) on the set \(\Omega_i\), we have

$$\|\psi_i \varphi\|_{L^1_p(\Omega_i)}^p \leq 2^p \int_{\Omega_i} \gamma |\varphi| dV + 2^p \int_{\Omega_i} \psi_i |\nabla \varphi|^p dV, \quad i = 1, 2, \ldots.$$  

Thus,

$$\sum_{i=1}^{\infty} \|\psi_i \varphi\|_{L^1_p(\Omega_i)}^p \leq 2^p k \int_{\Omega_i} \gamma |\varphi| dV + 2^p \int_{\Omega_i} |\nabla \varphi|^p dV,$$
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where \( k \) is the multiplicity of the cover (3.1); hence from Lemma 5 we obtain the estimate

\[
\sum_{i=1}^{\infty} \|\psi_i \varphi\|_{L^p_k(\Omega)}^p \leq C \int_M |\nabla \varphi|^p \, dV,
\]

where the constant \( C > 0 \) depends only on \( p, k \), and the left-hand side of (3.4); therefore, relation (3.11) allows us to state that

\[
|(F, \varphi)| \leq C^{1/p} \left( \sum_{i=1}^{\infty} \|F\|_{L^p_k(\Omega)}^{p/(p-1)} \right)^{(p-1)/p} \left( \int_M |\nabla \varphi|^p \, dV \right)^{1/p}.
\]

(3.12)

The proof of Theorem 3 is complete. \( \square \)

Proof of Theorem 4. Necessity is proved in the same way as in the case of Theorem 3. Note only that, in view of the \( p \)-parabolicity of the manifold \( M \), there is a sequence \( \eta_s \in C_0^\infty(M) \) satisfying (2.2). This sequence will also satisfy (2.1), because the existence of a solution of problem (1.1), (1.2) implies the continuity of \( F \) on \( L^1_p(M) \).

We will now prove sufficiency. Suppose that \( \eta_s \in C_0^\infty(M) \) is a sequence satisfying conditions (2.1) and (2.2). Take a Lipschitz domain \( \Omega \) with compact closure such that \( K \subset \Omega \) and \( \bar{\Omega}_1 \subset \Omega \). Without loss of generality, we can assume that the norms \( \|\eta_s\|_{W^1_p(\Omega)} \) are bounded by some constant independent of \( s \). Otherwise, we will replace \( \eta_s \) by (2.7). Since \( W^1_p(\Omega) \) is completely continuously embedded in \( L^p(\Omega) \), there exists a subsequence of the sequence \( \eta_s \) which converges to \( L^p(\Omega) \). We will preserve the same notation \( \eta_s \) for this subsequence. Taking into account (2.2), we can assert that (2.8) holds.

Let \( C \) denote various positive constants depending only on \( p \), the cover (3.1), the partition of unity \( \{\psi_i\}_{i=1}^{\infty} \), the set \( \Omega \), and the left-hand side of (3.5). Let \( \varphi \in C_0^\infty(M) \), and also let \( \alpha \) be the real number defined by formula (2.10). In view of Poincaré’s inequality, the estimate (2.9) is valid. Also suppose that \( \varphi_j^\prime \) and \( \varphi_j^\prime\prime \) are defined by (2.11), where \( \psi \in C_0^\infty(\Omega) \) is a function equal to 1 on \( \Omega_1 \). For any positive integer \( j \), we have

\[
\varphi_j = \sum_{i=1}^{\infty} \psi_i \varphi_j^\prime,
\]

where almost all terms on the right-hand side are zero; therefore,

\[
|(F, \varphi_j)| \leq \sum_{i=1}^{\infty} |(F, \psi_i \varphi_j^\prime)| \leq \sum_{i=1}^{\infty} \|F\|_{L^p_k(\Omega)} \|\psi_i \varphi_j^\prime\|_{L^p_k(\Omega)}
\]

\[
\leq \left( \sum_{i=1}^{\infty} \|F\|_{L^p_k(\Omega)}^{p/(p-1)} \right)^{(p-1)/p} \left( \sum_{i=1}^{\infty} \|\psi_i \varphi_j^\prime\|_{L^p_k(\Omega)}^p \right)^{1/p}.
\]

Thus, replacing the function \( \varphi \) in the argument that has led to estimate (3.12) by \( \varphi_j^\prime \) and Lemma 5 by Lemma 6, we obtain

\[
|(F, \varphi_j)| \leq C \left( \sum_{i=1}^{\infty} \|F\|_{L^p_k(\Omega)}^{p/(p-1)} \right)^{(p-1)/p} \left( \int_M |\nabla \varphi_j^\prime|^p \, dV \right)^{1/p}.
\]

(3.13)

It is not difficult to verify that

\[
\left( \int_M |\nabla \varphi_j^\prime|^p \, dV \right)^{1/p} \leq 1 - \|\psi\|_{C(\Omega)} \|\varphi - \alpha \eta_j\|_{L^p_k(M)} + \|\nabla \psi\|_{C(\Omega)} \|\varphi - \alpha \eta_j\|_{L^p(\Omega)}
\]

and, moreover, (2.13) and (2.14) hold; therefore, (3.13) implies the estimate

\[
\limsup_{j \to \infty} |(F, \varphi_j)| \leq C \left( \sum_{i=1}^{\infty} \|F\|_{L^p_k(\Omega)}^{p/(p-1)} \right)^{(p-1)/p} \|\varphi\|_{L^p_k(M)}.
\]

(3.14)
Since $\text{supp } \psi \subset \Omega$, it follows that the function $\varphi''_j$ can be represented as

$$\varphi''_j = \sum_{\Omega \ni \Omega_i \neq \emptyset} (\varphi - \alpha \eta_j) \psi_i.$$  

Note that there are finitely many domains $\Omega_i$ satisfying the condition $\Omega \cap \Omega_i \neq \emptyset$, because $\Omega$ is a compact set and the cover (3.1) is locally finite. Thus,

$$|(F, \varphi''_j)| \leq \sum_{\Omega \ni \Omega_i \neq \emptyset} (F, (\varphi - \alpha \eta_j) \psi_i) \leq \sum_{\Omega \ni \Omega_i \neq \emptyset} \|F\|_{L^1_p(\Omega_i)}^* \|\varphi - \alpha \eta_j\|_{L^1_p(\Omega_i)}.$$  

At the same time,

$$\|(\varphi - \alpha \eta_j) \psi_i\|_{L^1_p(\Omega_i)} \leq \|\psi_i\|_{C(\Omega)} \|\varphi - \alpha \eta_j\|_{L^1_p(\Omega)} + \|\nabla(\psi_i)\|_{C(\Omega)} \|\varphi - \alpha \eta_j\|_{L^p(\Omega)},$$

whence, in view of (2.13) and (2.14), we have

$$\limsup_{j \to \infty} \|(\varphi - \alpha \eta_j) \psi_i\|_{L^1_p(\Omega_i)} \leq C \|\varphi\|_{L^1_p(M)}$$

for all $i$ such that $\Omega \cap \Omega_i \neq \emptyset$. Thus, we can state that

$$\limsup_{j \to \infty} |(F, \varphi''_j)| \leq C \sum_{\Omega \ni \Omega_i \neq \emptyset} \|F\|_{L^1_p(\Omega_i)}^* \|\varphi\|_{L^1_p(M)}.$$  

By Hölder's inequality,

$$\sum_{\Omega \ni \Omega_i \neq \emptyset} \|F\|_{L^1_p(\Omega_i)}^* \leq N^{1/p} \left( \sum_{\Omega \ni \Omega_i \neq \emptyset} \|F\|_{L^1_p(\Omega_i)}^{p/(p-1)} \right)^{(p-1)/p},$$

where $N$ is the number of domains $\Omega_i$ satisfying the condition $\Omega \cap \Omega_i \neq \emptyset$; therefore, the following estimate holds:

$$\limsup_{j \to \infty} |(F, \varphi''_j)| \leq C \left( \sum_{\Omega \ni \Omega_i \neq \emptyset} \|F\|_{L^1_p(\Omega_i)}^{p/(p-1)} \right)^{(p-1)/p} \|\varphi\|_{L^1_p(M)}.$$  

Combining this estimate with (2.1), (2.12), and (3.14), we will have

$$|(F, \varphi)| \leq C \left( \sum_{i=1}^{\infty} \|F\|_{L^1_p(\Omega_i)}^{p/(p-1)} \right)^{(p-1)/p} \|\varphi\|_{L^1_p(M)}.$$  

The proof of Theorem 4 is complete. \hfill \square

**Example 1.** Let $M$ be a subset of $\mathbb{R}^n$ of the form $\{x = (x', x_n) : |x'| \leq x_n^\lambda, x_n \geq 0\}$ with boundary smoothed near zero, where $n \geq 2$ and $\lambda \geq 0$ is a real number. The manifold $M$ is $p$-hyperbolic if and only if

$$n > p \quad \text{and} \quad \lambda > \frac{p-1}{n-1}.$$  

Indeed, if at least one of the inequalities in (3.15) does not hold, then, setting

$$\varphi_{r,R}(x) = \varphi \left( \frac{\ln R/|x|}{\ln R/r} \right), \quad 0 < r < R,$$

where $\varphi \in C^\infty(\mathbb{R})$ is a function equal to zero in a neighborhood of the interval $(-\infty, 0]$ and 1 in a neighborhood of $[1, \infty)$, we immediately obtain

$$\text{cap}_p(\overline{B_r}) \leq \int_{B_r} |\nabla \varphi_{r,R}|^p \, dv \to 0 \quad \text{as} \quad R \to \infty$$

for all $r > 0$, where $B_r = \{x \in M : |x| < r\}$. Thus, $\text{cap}(M) = 0$.  

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If both inequalities in (3.15) hold, then, taking
\[ \Omega_1 = \{ x \in M : |x| < 4 \}, \quad \Omega_i = \{ x \in M : 2^{i-1} < |x| < 2^{i+1} \}, \quad i = 2, 3, \ldots, \]
and \( \gamma(x) = c(1 + |x|)^{-p} \), where \( c > 0 \) is a sufficiently large real number, we easily construct a partition of unity \( \psi_i \in C_0^\infty(\Omega_i) \) satisfying condition (3.2). Since (3.4) holds, Corollary 3 implies that \( M \) is a \( p \)-hyperbolic manifold.

**Example 2.** Let \( M \) be the manifold from Example 1. We assume that \( h \) is a measure on \( \partial M \) with density \( (1 + |x|^\sigma)^{-p} \). If \( M \) is a \( p \) hyperbolic manifold or, in other words, if inequalities (3.15) hold, then, in view of Theorem 3, for the existence of solutions to problem (2.3), (1.2), it is necessary and sufficient that
\[
\sigma < \begin{cases} 
\frac{\lambda n(p-1)}{p} - (1 - \lambda) \left( 2 - \frac{1}{p} \right), & \lambda < 1, \\
\frac{n(p-1)}{p}, & 1 \leq \lambda.
\end{cases}
\]
Indeed, using estimates based on embedding theorems, we can show that
\[
\| h \|_{L^p_\lambda(\Omega_i)^*} \lesssim \begin{cases} 
2^{i(\sigma + \lambda n(p-1)/p + (1-\lambda)(2-1/p))}, & \lambda < 1, \\
2^{i(\sigma + n(p-1)/p)}, & 1 \leq \lambda,
\end{cases}
\]
where \( \Omega_i, \ i = 1, 2, \ldots, \) is the cover constructed in Example 1.

Note that, in the case of a \( p \)-parabolic manifold \( M \), a solution to (2.3), (1.2) does not exist for any \( \sigma \), because condition (2.1) is not fulfilled.

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