ON COMPLEMENTED NON-ABELIAN CHIEF FACTORS OF A LIE ALGEBRA

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Abstract. The number of Frattini chief factors or of chief factors which are complemented by a maximal subalgebra of a finite-dimensional Lie algebra \( L \) is the same in every chief series for \( L \), by [13, Theorem 2.3]. However, this is not the case for the number of chief factors which are simply complemented in \( L \). In this paper we determine the possible variation in that number.

1. Preliminary Results

Throughout \( L \) will be a finite-dimensional Lie algebra with product \([,]\) over a field. We say that \( A \) is an \( L \)-algebra if it is a Lie algebra (with product denoted by juxtaposition) and there is a homomorphism \( \theta : L \to \text{Der } A \). Then \( A \) is also an \( L \)-module with action \( \cdot \) given by \( x.a = \theta(x)(a) \) and we have \( x(a_1a_2) = (x.a_1)a_2 - (x.a_2)a_1 \). If \( A \) is an ideal of \( L \) we will consider it as an \( L \)-algebra in the natural way.

Given such an \( L \)-algebra \( A \), we define the corresponding semi-direct sum \( A \rtimes L \) as the set of ordered pairs, where the multiplication is given by
\[
(a_1, x_1)(a_2, x_2) = (x_1a_2 - x_2a_1 + a_1a_2, [x_1, x_2])
\]
for all \( a_1, a_2 \in A \) and for all \( x_1, x_2 \in L \).

Let \( A \) and \( B \) two \( L \)-algebras. An (algebra) isomorphism \( \theta : A \to B \) is said to be an \( L \)-isomorphism if it is also an \( L \)-module isomorphism. Note that this is stronger than the definition used in [13], where \( \theta \) is only required to be an \( L \)-module isomorphism. However, the results proved there apply equally to this stronger version. When such a \( \theta \) exists we write \( A \cong_L B \). We say that \( A, B \) are \( L \)-equivalent, written \( A \sim_L B \) if there is an isomorphism \( \Phi : A \rtimes L \to B \rtimes L \) such that the following diagram commutes:
\[
\begin{array}{ccc}
0 & \to & A & \to & A \ltimes L & \to & L & \to & 0 \\
\downarrow \phi & & \downarrow \Phi & & \parallel & & & & \downarrow & \\
0 & \to & B & \to & B \ltimes L & \to & L & \to & 0
\end{array}
\]

In this case we say that the extensions \( A \to A \ltimes L \to L \) and \( B \to B \ltimes L \to L \) are equivalent. It is clear that \( L \)-equivalence is an equivalence relation.

If \( \phi : A \to B \) is an \( L \)-isomorphism, then putting \( \Phi((a, x)) = (\phi(a), x) \) defines an isomorphism \( \Phi : A \ltimes L \to B \ltimes L \) making the above diagram commutative. It follows that \( L \)-isomorphic \( L \)-algebras are \( L \)-equivalent. However, the converse is false. For example, if \( L = A \oplus B \), where \( A \) and \( B \) are isomorphic simple Lie algebras, then \( A \) and \( B \) are \( L \)-equivalent, but they are not \( L \)-isomorphic, as \( C_L(A) = B \) and \( C_L(B) = A \).

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If $B$ is an $L$-algebra we define a 1-cocycle of $L$ with values in $B$ to be a map $\beta \in Z^1(L, B)$ such that

$$\beta([x, y]) = x.\beta(y) - y.\beta(x) + \beta(x)\beta(y).$$

Then the map $\theta : L \to \text{Der } B$ given by $\theta(x) = \theta_x$ where $\theta_x(b) = \beta(x)b + x.b$ for all $x \in L$ and $b \in B$ is a homomorphism, and so we can define another $L$-module structure on $B$ by

$$x \odot b = \beta(x)b + x.b.$$ 

We denote the $L$-algebra with this $L$-module structure by $B_\beta$.

The following proposition gives us a useful criterion for two $L$-algebras to be equivalent.

**Proposition 1.1.** Let $A$ and $B$ be two $L$-algebras. They are $L$-equivalent if and only if there is a 1-cocycle $\beta \in Z^1(L, B)$ and an $L$-isomorphism $\phi$ from $A$ to $B_\beta$ (that is, $\phi(x.a) = x \odot \phi(a)$ for all $x \in L$, $a \in A$).

**Proof.** Suppose first that there is a 1-cocycle $\beta \in Z^1(L, B)$ and an $L$-isomorphism $\phi : A \to B_\beta$. Then, the map $\Phi : A \times L \to B \times L$ given by

$$\Phi((a, x)) = (\phi(a) + \beta(x), x)$$

shows that $A$ and $B$ are $L$-equivalent.

Conversely, suppose that they are $L$-equivalent under the isomorphism $\Phi : A \cong L \to B \times L$. Define $\beta : L \to B$ by $\beta(x) = \pi_1(\Phi((0, x)) - (0, x))$ where $\pi_1 : B \times L \to B : (b, x) \mapsto b$ is the projection map onto $B$. Then it is straightforward to check that $\beta \in Z^1(L, B)$ and that $\phi$ is an $L$-isomorphism from $A$ to $B_\beta$. \qed

If $A$ and $B$ are abelian and $L$-equivalent, they have the same dimension, and so are $L$-isomorphic. However, as we have seen, for nonabelian $L$-algebras, $L$-equivalence is strictly weaker than $L$-isomorphism.

Recall that the factor algebra $A/B$ is called a chief factor of $L$ if $B$ is an ideal of $L$ and $A/B$ is a minimal ideal of $L/B$. The Frattini ideal of $L$, denoted by $\phi(L)$, is the largest ideal of $L$ contained in the intersection of all of the maximal subalgebras of $L$. A chief factor $A/B$ is called Frattini if $A/B \subseteq \phi(L/B)$.

If there is a subalgebra, $M$ such that $L = A + M$ and $B \subseteq A \cap M$, we say that $A/B$ is a supplemented chief factor of $L$, and that $M$ is a supplement of $A/B$ in $L$. Also, if $A/B$ is a non-Frattini chief factor of $L$, then $A/B$ is supplemented by a maximal subalgebra $M$ of $L$.

If $A/B$ is a chief factor of $L$ supplemented by a subalgebra $M$ of $L$, and $A \cap M = B$ then we say that $A/B$ is complemented chief factor of $L$, and $M$ is a complement of $A/B$ in $L$. When $L$ is solvable, it is easy to see that a chief factor is Frattini if and only if it is not complemented.

The centralizer of an $L$-algebra $A$ in $L$ is $C_L(A) = \{x \in L \mid x.a = 0 \text{ for all } a \in A\}$. We will denote an algebra direct sum by ‘⊕’, whereas ‘‘ will denote a direct sum of the underlying vector space only. Then the following proposition gives a criterion for a nonabelian chief factor to be complemented.

**Proposition 1.2.** Let $A_1/B_1$ be a nonabelian chief factor of $L$. Then, $A_1/B_1$ is complemented in $L$ if and only if there exists an $L$-algebra $B$ such that $A_1/B_1 \cong_L B$, and $A_1 \cong C_L(B)$. 

Proof. \((\Rightarrow)\) Suppose that \(A_1/B_1\) is complemented in \(L\) and \(M\) is a complement of \(A_1/B_1\) in \(L\). Since \(L = M + A_1\), for each \(x \in L\) we can write \(x = m_x + a_x\) for some \(m_x \in M\) and \(a_x \in A_1\). We consider the \(L\)-algebra \(B\) whose underlying algebra is \(A_1/B_1\) with the module operation:

\[
\wedge : L \times B \to B \\
(x, b) \mapsto [m_x, a_b] + B_1
\]

where \(b = a_b + B_1\) \((a_b \in A_1)\). Define the 1-cocycle \(\beta \in Z^1(L, B)\) as:

\[
\beta : L \to B \\
x \mapsto \beta(x) = a_x + B_1 \quad (a_x \in A_1)
\]

It is immediate that both are well defined mappings and that \(\beta\) is a 1-cocycle. Let \(\phi : A_1/B_1 \to B\) be given by, \(\phi(a_1 + B_1) = a_1 + B_1\) for all \(a_1 + B_1 \in A_1/B_1\). Then we can define another module structure on \(B\) using \(\beta\) and, for all \(x \in L\) and for all \(a_b + B_1 \in A_1/B_1\), we have

\[
\phi([x, a_b + B_1]) = \phi([x, a_b] + B_1) = [x, a_b] + B_1 = [a_x + B_1, a_b + B_1] + [m_x, a_b + B_1] = [\beta(x), a_b + B_1] + x \wedge (a_b + B_1) = x \circ (a_b + B_1).
\]

Hence \(\phi\) is an \(L\)-isomorphism and \(A_1/B_1 \cong_L B_\beta\). Then, using Proposition 1.1, we have that \(A_1/B_1 \cong_L B\). Also

\[
C_L(B) = \{ x \in L \mid x \wedge B = 0_B \}
\]

\[
= \{ m_x + a_x \in L \mid [m_x, a_b] + B_1 = B_1 \text{ for all } b \in B \}
\]

\[
= C_M(A_1/B_1) \oplus A_1
\]

whence \(A_1 \subseteq C_L(B)\).

\((\Leftarrow)\) Assume now that \(B\) is an \(L\)-algebra, \(A_1/B_1 \cong_L B\) and \(A_1 \subseteq C_L(B)\). We need to show that \(A_1/B_1\) is complemented in \(L\). Since \(A_1/B_1 \cong_L B\) we have an \(L\)-isomorphism \(\phi : B \to (A_1/B_1)_\alpha\) where \(\alpha \in Z^1(L, A_1/B_1)\), by Proposition 1.1, and \(A_1 \subseteq C_L(B)\). If \(b \in B\), then

\[
\phi(b) = \phi(b + a_1, b) = \phi(b) + \alpha(b) + \phi(b) = \phi(b) + [\alpha(a_1), \phi(b)] + [a_1 + B_1, \phi(b)]
\]

so \([\alpha(a_1) + a_1 + B_1, \phi(b)] = B_1\) for all \(b \in B\); that is,

\[
\alpha(a_1) + a_1 + B_1 \in C_L(A_1/B_1) \cap A_1/B_1 = B_1,
\]

since \(A_1/B_1\) is a nonabelian chief factor of \(L\). Hence \(\alpha(a_1) = -a_1 + B_1\).

Put \(M = Ker(\alpha)\). Let \(x \in L\) and \(\alpha(x) = a_1 + B_1\). Then

\[
\alpha(x + a_1) = \alpha(x) + \alpha(a_1) = (a_1 + B_1) + (-a_1 + B_1) = B_1
\]

so \(x + a_1 \in M\). Hence \(L = M + A_1\). If \(m \in M \cap A_1\) we have \(B_1 = \alpha(m) = -m + B_1\), so \(M \cap A_1 = B_1\) and \(M\) is a complement of \(A_1/B_1\) in \(L\). \(\square\)

Recall that,

(i) the socle of \(L\), \(Soc(L)\) is the sum of all of the minimal non-zero ideals of \(L\); and

(ii) if \(U\) is a subalgebra of \(L\), the core of \(U\), \(U_L\), is the largest ideal of \(L\) contained in \(U\). We say that \(U\) is core-free in \(L\) if \(U_L = 0\).

We shall call \(L\) primitive if it has a core-free maximal subalgebra. Then we have the following characterisation of primitive Lie algebras.
Theorem 1.3. ([13] Theorem 1.1)

(i) A Lie algebra $L$ is primitive if and only if there exists a subalgebra $M$ of $L$ such that $L = M + A$ for all minimal ideals $A$ of $L$.

(ii) Let $L$ be a primitive Lie algebra. Assume that $U$ is a core-free maximal subalgebra of $L$ and that $A$ is a non-trivial ideal of $L$. Write $C = C_L(A)$. Then $C \cap U = 0$. Moreover, either $C = 0$ or $C$ is a minimal ideal of $L$.

(iii) If $L$ is a primitive Lie algebra and $U$ is a core-free maximal subalgebra of $L$, then exactly one of the following statements holds:

(a) $Soc(L) = A$ is a self-centralising abelian minimal ideal of $L$ which is complemented by $U$; that is, $L = U + A$.

(b) $Soc(L) = A$ is a non-abelian minimal ideal of $L$ which is supplemented by $U$; that is $L = U + A$. In this case $C_L(A) = 0$.

(c) $Soc(L) = A \oplus B$, where $A$ and $B$ are the two unique minimal ideals of $L$ and both are complemented by $U$; that is, $L = A + U = B + U$. In this case $A = C_L(B), B = C_L(A)$, and $A, B$ and $(A + B) \cap U$ are nonabelian isomorphic algebras.

We say that $L$ is

- primitive of type 1 if it has a unique minimal ideal that is abelian;
- primitive of type 2 if it has a unique minimal ideal that is non-abelian; and
- primitive of type 3 if it has precisely two distinct minimal ideals each of which is non-abelian.

Let $A/B$ and $D/E$ be chief factors of $L$. We say that they are $L$-connected, if either they are $L$-isomorphic or there exists an epimorphic image of $L$ which is primitive of type 3 and whose minimal ideals are $L$-isomorphic to the given factors. The property of being $L$-connected is an equivalence relation on the set of chief factors. The set of chief factors of $L$ is denoted as:

$$CF(L) = \{A/B \mid A, B \text{ are ideals of } L, A/B \text{ is a chief factor of } L\}.$$  

Let

$$I_L(A) = \{x \in L \mid ad x \mid_A = ad a \text{ for some } a \in A\},$$

where $A$ is an $L$-algebra (and $ad x \mid_A$ refers to the module action of $x$ on $A$).

Lemma 1.4.  

(i) Let $A, B$ be ideals of a Lie algebra $L$ with $B \subseteq A$. Then $I_L(A/B) = A + C_L(A/B)$.

(ii) Let $A$ be an $L$-algebra with $C_L(A) \subseteq I_L(A)$. Then $I_L(A)/C_L(A)$ is isomorphic to a subalgebra of $A/Z(A)$.

(iii) $A$ is an abelian $L$-algebra if and only if $I_L(A) = C_L(A)$.

Proof.  

(i) We have

- $x \in I_L(A/B) \iff \exists a' \in A \text{ such that } [x, a] + B = [a', a] + B \ \forall a \in A$
- $\iff \exists a' \in A \text{ such that } [x - a', a] + B = B \ \forall a \in A$
- $\iff \exists a' \in A \text{ such that } [x - a', a] \in B \ \forall a \in A$
- $\iff \exists a' \in A \text{ such that } x - a' \in C_L(A/B)$
- $\iff x \in A + C_L(A/B)$

(ii) For $x \in I_L(A)$ let $a_x \in A$ be such that $x.a = a_x.a$ for all $a \in A$. Define $\theta : I_L(A) \to A/Z(A)$ by $\theta(x) = a_x + Z(A)$. Then it is straightforward to check that $\theta$ is well-defined and is a homomorphism. Moreover, $\ker(\theta) = C_L(A)$, whence the result.
(iii) This is straightforward.

Let $A$, $B$ be two $L$-algebras. If $A$ and $B$ are $L$-equivalent, then it is clear from Proposition 1.1 that $I_L(A) = I_L(B)$.

**Proposition 1.5.** Let $L$ be a Lie algebra and let $F_1$, $F_2 \in CF(L)$. Then the following assertions are equivalent:

(i) $F_1 \sim_L F_2$;

(ii) $F_1$ and $F_2$ are $L$-connected;

(iii) either $F_1 \cong_L F_2$ or there exist $E_i \in CF(L)$ such that $F_i \cong_L E_i$ for $i = 1, 2$, and the $E_i$’s have a common complement in $L$, which is a maximal subalgebra of $L$; and

(iv) either $F_1 \cong_L F_2$ or there exist $E_i \in CF(L)$ such that $F_i \cong_L E_i$ for $i = 1, 2$, and the $E_i$’s have a common complement in $L$.

**Proof.** From [13] we know that two abelian chief factors are $L$-equivalent if and only if they are $L$-isomorphic, and if and only if they are $L$-connected. Moreover, a complement $U$ of an abelian chief factor $A/B$ is a maximal subalgebra and $L/U$ is primitive of type 1 with $\text{Soc}(L/U) = C/U$ and $C/U \cong_L A$, by [13, Remarks following Proposition 2.5]. So we may assume that the chief factors are nonabelian and not $L$-isomorphic. Let $F_1 = A/B$ and $F_2 = D/E$, where $A, B, C, D$ are ideals of $L$.

(i) $\Rightarrow$ (ii): Put $X = C_L(A/B)$ and $Y = C_L(D/E)$. Since $F_1$ and $F_2$ are nonabelian we have that $X \neq Y$, by [13, Theorem 2.1] Also, since $F_1 \sim_L F_2$, we have that $I_L(A/B) = I_L(D/E) := I$. Then $I = A + X = D + Y$, by Lemma 1.4. Also,

$$\frac{X + Y}{X} \subseteq A + X \cong_L A \quad \text{and} \quad \frac{X + Y}{Y} \subseteq D + Y \cong_L D.$$ 

So $I = X + Y$, since $X \neq Y$ and

$$X/X \cap Y \cong_L I/Y \cong_L D/E \quad \text{and} \quad Y/Y \cap X \cong_L I/X \cong_L A/B.$$ 

It thus suffices to show that $L/X \cap Y$ is primitive of type 3. Without loss of generality we can assume that $X \cap Y = 0$. Then $C_L(X) = Y$ and $C_L(Y) = X$. Moreover, since $\sim_L$ is an equivalence relation, we have $Y \sim_L X$. Thus there is a 1-cocyle $\alpha \in Z^1(L, Y)$ and an $L$-isomorphism, $\phi : Y \to X_\alpha$, by Proposition 1.1. We also have that $U = \text{Ker}(\alpha)$ complements $X$ in $L$, as in the proof of Proposition 1.2. Now let $y \in Y$ and $u \in Y \cap U$. Then

$$[u, \phi(y)] = 0 \quad \text{since} \quad \phi(y) \in X = C_L(Y) \quad \text{and} \quad u \in Y,$$ 

$$[\alpha(u), \phi(y)] = 0 \quad \text{since} \quad \alpha(u) = 0.$$ 

But also,

$$\phi([u, y]) = u \circ \phi(y) = [\alpha(u), \phi(y)] + [u, \phi(y)] = 0$$

whence, $[u, y] = 0$, since $\phi$ is injective. It follows that $u \in C_L(Y) \cap Y = X \cap Y = 0$ and so $Y \cap U = 0$. Thus $U$ is a maximal subalgebra of $L$ with trivial core and $F_1$ and $F_2$ are $L$-connected.

(ii) $\Rightarrow$ (iii): This follows immediately from the definition.

(iii) $\Rightarrow$ (iv): This is trivial.
Lemma 1.6. Let $N \to L \to L/N$ be a short exact sequence of Lie algebras, where $N$ is an ideal of $L$ and the arrows are the canonical inclusion and projection. If $A$ is an $L$-algebra, we have the following exact sequences:

$$0 \to Z^1(L/N, \mathcal{A}^N) \overset{\inf}{\longrightarrow} Z^1(L, \mathcal{A}) \overset{\res}{\longrightarrow} Z^1(N, \mathcal{A})$$

$$0 \to H^1(L/N, \mathcal{A}^N) \overset{\inf}{\longrightarrow} H^1(L, \mathcal{A}) \overset{\res}{\longrightarrow} H^1(N, \mathcal{A})^{L/N}$$

where $\inf$ and $\res$ denote the corresponding inflation and restriction maps.
Theorem 1.9. Let $A$ be a nonabelian irreducible $L$-algebra and let $N$ be an ideal of $L$ with $N \subseteq C_L(A)$. Then the following are equivalent:

1. $N \subseteq E_L(A)$
2. $Z^1(L, A) = Z^1(L/N, A)$
3. $H^1(L, A) = H^1(L/N, A)$

Proof. This follows from the above lemma. Note that the inflation is bijective if and only if the restriction is null and that is equivalent to $N \subseteq \ker(\alpha)$ for all $\alpha \in Z^1(L, A)$. □

The analogue of the following result for groups was proved using cohomology theory. Here we give a more direct proof for the Lie algebra case.

Theorem 1.8. If $A$ is an abelian irreducible $L$-algebra, then $E_L(A) = D_L(A)$.

Proof. Put $I = I_L(A) = C_L(A)$, by Lemma [43]. Let $\alpha \in Z^1(L, A)$. First note that $\alpha|_I$ is an $L$-homomorphism from $I$ into $A$, since

$$\alpha([x, y]) = x.\alpha(y) - y.\alpha(x) + \alpha(x)\alpha(y) = 0 \text{ for all } x, y \in I,$$

and

$$\alpha([x, i]) = x.\alpha(i) - i.\alpha(x) + \alpha(x)\alpha(i) = x.\alpha(i) \text{ for all } x \in L, i \in I.$$ 

It follows that $\alpha(I)$ is an $L$-submodule of $A$, and so $\alpha(I) = 0$ or $A$, by the irreducibility of $A$. The former implies that $D_L(A) \subseteq I \subseteq \ker(\alpha)$. So suppose that $\alpha(I) = A$. Then $I/I \cap \ker(\alpha) \cong L$. Moreover,

$$\dim(I + \ker(\alpha)) = \dim I + \dim \ker(\alpha) - \dim I \cap \ker(\alpha) = \dim A + \dim \ker(\alpha) = \dim \ker(\alpha) = \dim L.$$ 

It follows that $L = I + \ker(\alpha)$, and $I/I \cap \ker(\alpha)$ is complemented by $\ker(\alpha)$ (which is a subalgebra of $L$) and so is non-Frattini. Hence $D_L(A) \subseteq I \cap \ker(\alpha)$.

Thus, in either case, $D_L \subseteq \ker(\alpha)$, and $D_L \subseteq E_L(A)$.

Finally suppose that there exists $x \in E_L(A)$ such that $x \notin D_L(A)$. Then there exists $R \in D_L(A)$ such that $x \notin R$ but $x \in \ker(\alpha)$ for all $\alpha \in Z^1(L, A)$. Since $I/R$ is non-Frattini, there is a maximal subalgebra $M$ of $L$ such that $L = I + M$ and $I \cap M = R$. Now there is a cocycle $\beta \in Z^1(L, B)$ and an $L$-isomorphism $\phi$ from $I/R$ onto $A_{\beta}$, by Proposition [44]. Moreover $A_{\beta} = A$, since $A$ is abelian. So define $\alpha : L \rightarrow A$ by $\alpha(m) = 0, \alpha(i) = \phi(i + R)$. Then it is straightforward to check that $\alpha \in Z^1(L, A)$ and that $M = \ker(\alpha)$. Furthermore, $x \in I \cap M = R$, contradiction. Hence $E_L(A) \subseteq D_L(A)$ and equality results. □

In the rest of this section we investigate the case where $A$ is nonabelian.

Recall that, if $A$ is an $L$-algebra, then $\alpha : L \rightarrow A$ is a 1-cocyle if and only if $\alpha^* : L \rightarrow A \ltimes L$ given by:

$$\alpha^*(x) = (\alpha(x), x)$$

is a homomorphism and $\alpha \mapsto \alpha^*(L)$ defines a bijection between $Z^1(L, A)$ and the set of all complements of $A$ in $A \ltimes L$. Then

$$\ker(\alpha) = \alpha^*(L) \cap L$$

We can give the following characterization:

Theorem 1.9. Let $A$ be a nonabelian irreducible $L$-algebra. Then;

$$E_L(A)A \ltimes L = \cap\{C_L(B) \mid B \sim_L A\}.$$
Proposition 1.11. Let $A$ be a nonabelian irreducible $L$-algebra such that $C_L(A) \subset I_L(A)$. Then,

$$D_L(A) \subseteq C_L(A) \iff I_L(A)/C_L(A) \cong_L A$$

Proof. Put $I := I_L(A)$, etc. If $I/C \cong_L A$, then it is not Frattini, since it is nonabelian. It follows from the definition of $D_L(A)$ that $D_L(A) \subseteq C_L(A)$.

Suppose now that $D_L(A) \subseteq C_L(A)$. Then $A$ is nonabelian, so $I \neq C_L(A)$. Moreover, $I/D$ is completely reducible (as in [13, Theorem 3.2]), so $I_L(A)/C_L(A) \cong_L A$. \qed

Corollary 1.12. Let $A$ be a nonabelian irreducible $L$-algebra such that

$$\{B \in CF(L) \mid B \sim_L A\} \neq \emptyset.$$ 

Then

$$D_L(A) = \cap\{C_L(B) \mid B \sim_L A, B \in CF(L)\}$$

Let $A$ be a nonabelian irreducible $L$-algebra. We set

$$J_L(A) = \cap\{C_L(B) \mid B \sim_L A, B \not\sim_L F, F \in CF(L)\}$$

if $\{B \mid B \sim_L A, B \not\sim_L F, F \in CF(L)\} \neq \emptyset$ and we put $J_L(A) = I_L(A)$, otherwise.

Proposition 1.13. Let $A$ be a nonabelian irreducible $L$-algebra. Then

$$I_L(A) = J_L(A) + D_L(A)$$

and

$$J_L(A) \cap D_L(A) = E_L(A)_{A \times L}$$
Proof. It is clear that $J_L(A) \cap D_L(A) = E_L(A)_{A \neq L}$. Let $B \sim_L A$ be such that $B \not\equiv_L F$ if $F \in CF(L)$, and put $S := C_L(B)$, $I := I_L(A)$, etc. Then there is a $1$-cocyle $\alpha \in Z^1(L, B)$ and an $L$-isomorphism $\phi : A \to B_\alpha$, by Proposition 1.1. Let $x \in S$ and $a \in A$. Then

$$\phi(xa) = x \circ \phi(a) = \alpha(x)\phi(a) + x.\phi(a) = \alpha(x)\phi(a),$$

so $xa = \phi^{-1}\alpha(x)a$. It follows that $x \in I$ and $S \subseteq I$. Now, $Z(B) = 0$, since $B$ is nonabelian and an irreducible $L$-algebra, so there is a monomorphism $\theta : I/S = I_L(B)/S \to B$, by Lemma 1.3. Moreover, $\theta$ cannot be surjective, since $I/S \in CF(L)$. Hence $I/S$ is isomorphic to a proper subalgebra of $B$.

Now $D + S$ is an ideal of $I$ and $I/D$ is completely reducible $L$-algebra, with its irreducible components $L$-equivalent to $A$ (as in [13, Theorem 3.2]), and thus to $B$. If $A_i/D$ is an irreducible component of $I/D$, then $A_i \subseteq S$, as in Proposition 1.2. It follows that $D + S = I$.

Suppose that $D + J \subseteq I$. Let $I/R$ be a chief factor of $L$ such that $D + J \subseteq R$. Then, $I/R \cong_L A$ because $D \subseteq R$. As $J \subseteq R$, there exists $B \sim_L A$ with $B \not\equiv_L F$ if $F \in CF(L)$, such that $I/C_L(B)$ has a factor isomorphic to $I/R$, contradicting the fact that $\dim(I/C_L(B)) < \dim A$. \hfill \Box

2. On Complemented Chief Factors

Let $L$ be a Lie algebra. We say that a chief factor of $L$ is a $c$-factor if it is complemented in $L$ by a subalgebra, and that it is an $m$-factor if it is complemented by a maximal subalgebra of $L$; otherwise we say that it is a $c'$-factor, respectively an $m'$-factor.

Observe that, an abelian chief factor is an $m$-factor (respectively an $m'$-factor) if and only if it is a $c$-factor (respectively, a Frattini chief factor).

Let $A/B$ and $C/D$ be chief factors of $L$. We write $A/B \triangleleft C/D$ if $A = B + C$ and $B \cap C = D$. If $A/B \triangleleft C/D$, $A/B$ is a Frattini chief factor and $C/D$ is supplemented by a maximal subalgebra of $L$, then we call this situation an $m$-crossing, and denote it by $[A/B \triangleleft C/D]$.

We say that two chief factors $A/B$ and $C/D$ of $L$ are $m$-related if one of the following holds.

1. There is a supplemented chief factor $R/S$ such that $A/B \triangleleft R/S \triangleleft C/D$.
2. There is an $m$-crossing $[U/V \triangleleft W/X]$ such that $A/B \triangleleft V/X$ and $W/X \triangleleft C/D$.
3. There is a Frattini chief factor $Y/Z$ such that $A/B \triangleleft Y/Z \triangleleft C/D$.
4. There is an $m$-crossing $[U/V \triangleleft W/X]$ such that $A/B \triangleleft U/V$ and $U/W \triangleleft C/D$.

Then we have the following result.

**Proposition 2.1.** Let $L$ be a Lie algebra over any field, let $H$ and $K$ be ideals of $L$ with $H \subseteq K$, and let

$$H = X_0 < X_1 < X_2 < \ldots < X_n = K$$

and

$$H = Y_0 < Y_1 < Y_2 < \ldots < Y_m = K$$

be two sections of chief series of $L$ between $H$ and $K$. Then $n = m$ and there exists a unique permutation $\pi$ in $S_n$ such that $X_i/X_{i-1}$ and $Y_{\pi(i)}/Y_{\pi(i)-1}$ are $m$-related. In particular,
(i) $X_i/X_{i-1} \cong L Y_{\pi(i)}/Y_{\pi(i)-1}$

(ii) $X_i/X_{i-1}$ and $Y_{\pi(i)}/Y_{\pi(i)-1}$ are simultaneously $m$-factors or $n$-factors.

(iii) If $X_i/X_{i-1}$ and $Y_{\pi(i)}/Y_{\pi(i)-1}$ are $m$-factors, then they have a maximal subalgebra of $L$ as a common complement.

Proof. This follows from [13, Theorems 2.9 and 2.7].

In particular, the number of $m$-factors in any chief series of $L$ are the same. But this is no longer true for $c$-factors, in spite of the equivalence between (3) and (4) in Proposition [15], as we shall see in a later example.

If $S$ is a subalgebra of $L$, the normizer of $S$ in $L$ is defined as

$$N_L(S) = \{ x \in L \mid [x,S] \subseteq S \}.$$  

Lemma 2.2. Assume that $B^*/B$ is a $c'$-factor and that $A^*/A$ is a $c$-factor of $L$, both of which are nonabelian and such that $B^*/B \subseteq A^*/A$. Let $I = I_L(A^*/A)$ and $C = C_L(A^*/A)$. Then

(i) $I/C \subseteq B^*/B$ and $I/C$ is a $c'$-factor;

(ii) there exists an ideal $X$ of $L$ with $X \subseteq I$ such that $X/N \subseteq A^*/A$, where $N = X \cap C$, $I/C \cap X/N$ and $X/N$ is a $c$-factor;

(iii) there exists a supplement $F$ of $I/C$ in $L$ such that $L/N$ is isomorphic to the natural semi-direct sum of $I/C$ by $F/C$; and

(iv) $L/C$ is a primitive Lie algebra of type 2 and $\text{soc}(L/C) = I/C$.

Proof. We have that $B^*/B \subseteq A^*/A$, so $A^* + B = B^*$ and $A^* \cap B = A$. Also, $[B,A^*] + A = A$ or $A^*$. But the latter implies that $A^* \subseteq A + B \cap A^* = A$, a contradiction, so $[B,A^*] \subseteq A$; that is, $B \subseteq C$. Hence $B^* + C = A^* + B + C = A^* + C = I$, by Lemma [13] and $B^* \cap C = (A^* + B) \cap C = B + A^* \cap C = B + A = B + A^* \cap B = B$. Thus $I/C \subseteq B^*/B$. Suppose that $I/C$ is a $c$-factor of $L$. Then there is a subalgebra $U$ of $L$ such that $L = I + U$ and $I \cap U = C$. But now $L = B^* + C + U = B^* + U$ and $B^* \cap U = B^* \cap I \cup U = B^* \cap C = B$, so $B^*/B$ is a $c'$-factor, a contradiction. Thus, $I/C$ is a $c'$-factor of $L$ and we have (i).

Let

$$A = A_0 < A_1 < \ldots < A_n = C$$

be part of a chief series of $L$ between $A$ and $C$. Then

$$A^* = A^* + A_0 < A^* + A_1 < \ldots < A^* + A_n = I$$

is part of a chief series of $L$ between $A^*$ and $I$. Suppose that $(A^* + A_i)/A_i$ is a $c$-factor for some $1 \leq i \leq n-1$. Then there is a subalgebra $U$ such that $L = A^* + A_i + U$ and $(A^* + A_i) \cap U = A_i$. Then $A_i \subseteq U$ so $L = A^* + U = A^* + A_{i-1} + U$ and $(A^* + A_{i-1}) \cap U = A^* \cap U + A_{i-1} = A_{i-1}$, since $A^* \cap U \subseteq A^* \cap A_i = A$. Thus $(A^* + A_{i-1})/A_{i-1}$ is a $c$-factor. It follows that $(A^* + A_k)/A_k$ is a $c$-factor and $(A^* + A_{k+1})A_{k+1}$ is a $c'$-factor for some $0 \leq k \leq n-1$, since $A^*/A$ is a $c$-factor and $I/C$ is a $c'$-factor. Put $N = A_k$, $X = A^* + A_k$, $Y = A_{k+1}$ and $M = A^* + A_{k+1}$.

Then it is straightforward to check that

$$I/C \subseteq M/Y \subseteq X/N \subseteq A^*/A$$

where $M/Y$ is a $c'$-factor and $X/N$ is a $c$-factor and we have (ii).

Without loss of generality we may assume that $N = 0$. Let $U$ be a complement of $X$ in $L$, so $L = X + U$ and $X \cap U = 0$, and consider, $K = U \cap C$. Then $[X,C] \subseteq X \cap C = N = 0$, since $I/C \subseteq X/N$, so $K$ is an ideal of $L$. We have
$U + (K + X) = L$ and $U \cap (K + X) = K + (U \cap X) = K$. It follows that $(K + X)/K$ is a $c$-factor of $L$. Also $K + X + C = X + C = A^* + A_k + C = A^* + C = I$ and $(K + X) \cap C = K + X \cap C = K$, so $I/C \triangleleft (K + X)/K$ and we may assume that $K = 0$.

Observe that the map

$$u + x \mapsto (x + C, u + C)$$

where $u \in U$ and $x \in X$ defines an epimorphism between $L = X + U$ and the natural semidirect sum $I/C \rtimes ((U + C)/C)$. Furthermore, it is easy to check that the kernel of this map is $N$, so putting $F = U + C$ proves (iii).

We have

$$\frac{I}{C} = \frac{A^* + C}{C} \cong \frac{A^*}{A^* \cap C} = \frac{A^*}{A},$$

and $[U_L, I] \subseteq C$, whence $[U_L, A^*] \subseteq A$ and $U_L \subseteq C$. This establishes (iv). \hfill \Box

We can construct an example of the situation in Lemma 2.2 as follows.

**Example 2.3.** Let $L_0$ be a primitive Lie algebra of type 2 with $\text{Soc}(L_0) = X_0$, where $X_0$ is not complemented in $L_0$, and let $U_0$ be a supplement to $X_0$ in $L_0$. So, for example, we could take $L_0 = \text{sl}(2) \otimes \mathcal{O}_m + 1 \otimes \mathcal{D}$, $X_0 = \text{sl}(2) \otimes \mathcal{O}_m$, $U_0 = (F \sigma_0 + F \sigma_1) \otimes \mathcal{O}_m + 1 \otimes \mathcal{D}$ where $\mathcal{O}_m$ is the truncated polynomial algebra in $m$ indeterminates, $\mathcal{D}$ is a solvable subalgebra of $\text{Der}(\mathcal{O}_m)$, $\mathcal{O}_m$ has no $\mathcal{D}$-invariant ideals, and the ground field is algebraically closed of characteristic $p > 5$ (see [7, Theorem 6.4]).

Put $Y_0 = U_0 \cap X_0$. Then $X_0$ is a $U_0$-algebra and so we can form the semi-direct sum

$$L = X_0 \rtimes U_0 = \{(x, u) \mid x \in X_0, u \in U_0\}.$$

Put $X = \{(x, 0) \mid x \in X_0\}$, $U = \{(0, u) \mid u \in U_0\}$. Then $L = X + U$, $X \cap U = 0$ and $X$ is an ideal of $L$.

Now let $B = \{(0, y) \mid y \in Y_0\}$, $W = \{(y, 0) \mid y \in Y_0\}$. Putting $C = \{(y, -y) \mid y \in Y_0\}$ and $I = X + C$, we have that

1. $C$ is an ideal of $L$, $X \cap C = 0$, $I = X + B$ and $B = U \cap I$;
2. $X \cap (B + C) = W$, $B + W = B + C$, $W$ is an ideal of $C + U$ and $[W, C] = 0$;
3. $W \not\subseteq U \Rightarrow B \not\subseteq U \subseteq C$.

Consider the following chief series of $L$.

$$0 < C < I < \ldots < L \text{ and } 0 < X < I < \ldots < L.$$

We have the situation of Lemma 2.2 with $N = 0$. Then $I/C$ is a $c'$-factor and $X/0$ is a $c$-factor as in the lemma. Suppose that $C$ is complemented in $L$, so there is a maximal subalgebra $M$ of $L$ such that $L = C + M$. Then $[C, X] = 0$ so $X \cap M$ is an ideal of $L$. But $X$ is a minimal ideal of $L$, so $X \cap M = 0$ or $X \subseteq M$. The former implies that $\dim(X + M) = \dim X + \dim M > \dim Y_0 + \dim M = \dim C + \dim M = \dim L$, which is impossible. The latter implies that $L = I + M$ and $I \cap M = (C + X) \cap M = C \cap M + X = X$, so $M$ is a complement for $I/X$. If the chief factors between $I$ and $L$ are the same in each series then the second series has one more complemented chief factor than the first.

Then we have the following proposition.

**Proposition 2.4.** Suppose that, in the situation of Proposition 2.4, $X_1/X_{i-1}$ and $Y_{\pi(i)}/Y_{\pi(i)-1}$ are $m'$-factors. Then we have:
(a) both factors are $c'$-factors; or
(b) both factors are nonabelian $c'$-factors; or
(c) both factors are nonabelian, one of them is a $c$-factor, the other one is a $c'$-factor, and there exist ideals $I, C, X, N$ of $L$ of satisfying (i)-(iv) of Lemma 2.2.

Proof. Assume that $X_i/X_{i−1}$ is a $c'$-factor and that $Y_{\pi(i)}/Y_{\pi(i)−1}$ is a $c$-factor. As these two chief factors are $m$-related, one of the following situations arises.

1. There is a supplemented chief factor $R/S$ such that

$$X_i/X_{i−1} \not\lhd R/S \not\rhd Y_{\pi(i)}/Y_{\pi(i)−1}.$$ 

Since $X_i/X_{i−1}$ is a $c'$-factor, so is $R/S$. We are thus in the situation of Lemma 2.2 with $B^* = R, B = S, A^* = Y_{\pi(i)}, A^* = Y_{\pi(i)−1}$.

2. There is an $m$-crossing $[U/V \setminus W/X]$ such that

$$X_i/X_{i−1} \not\lhd V/X \text{ and } W/X \not\rhd Y_{\pi(i)}/Y_{\pi(i)−1}.$$ 

Then [14 Theorem 2.4] implies that $[U/W \setminus V/X]$ is also an $m$-crossing. Suppose that $W/X$ is supplemented by the maximal subalgebra $M$ of $L$, so $L = W + M$ and $X \subseteq W \cap M$. Then $L = W + M = U + M$. If $V \subseteq M$ then $V \subseteq U \cap M$ and $U/V$ is supplemented by $M$, contradicting the fact that $U/V$ is Frattini. Hence $V \not\subseteq M$. It follows that $L = V + M$. Moreover, $X \subseteq V \cap M \subseteq V$. As $V/X$ is a chief factor of $L$ we have $V \cap M = X$, and so $V/X$ is a $c$-factor. But then $X_i/X_{i−1}$ is a $c$-factor, which is a contradiction. Thus this case cannot occur.

3. There is a Frattini chief factor $Y/Z$ such that

$$X_i/X_{i−1} \not\lhd Y/Z \not\rhd Y_{\pi(i)}/Y_{\pi(i)−1}.$$ 

Since $Y_{\pi(i)}/Y_{\pi(i)−1}$ is a $c$-factor, so is $Y/Z$. But $Y/Z$ is Frattini, so this is impossible and this case cannot occur.

4. There is an $m$-crossing $[U/V \setminus W/X]$ such that

$$X_i/X_{i−1} \not\lhd U/V \text{ and } U/W \not\rhd Y_{\pi(i)}/Y_{\pi(i)−1}.$$ 

Then [14 Theorem 2.4] implies that $[U/W \setminus V/X]$ is also an $m$-crossing, so $U/W$ is a $c'$-factor, contradicting the fact that $Y_{\pi(i)}/Y_{\pi(i)−1}$ is a $c$-factor. Hence this case cannot occur either.

Let $A$ be an irreducible $L$-algebra. We say that $A$ is of $cc'$-type in $L$ if there exist two chief series of $L$ in which case (c) of Proposition 2.4 holds with $A \sim_L X_i/X_{i−1}$. (Clearly this forces $A$ to be nonabelian.)

**Proposition 2.5.** Let $v$ be the number of equivalence classes of irreducible $L$-algebras of $cc'$-type. Then the number of complemented chief factors on two chief series of $L$ differs by at most $v$.

Proof. A consequence of Proposition 2.4 is that, on a chief series of $L$, for each nonabelian crown there is at most one $m'$-factor. If the crown corresponds to a factor of $cc'$-type, this shows that on each chief series there is at most one $c'$-factor corresponding to the crown. □
Theorem 2.6. Let $A$ be a nonabelian irreducible $L$-algebra. Then $A$ is of $cc'$-type in $L$ if and only if

$$E_L(A) \subset D_L(A) \subset I_L(A)$$

and $\text{Soc}(P)$ is a $c'$-factor of $P$, where $P$ is the corresponding primitive epimorphic image of $L$.

Proof. Put $E = E_L(A)$, $D = D_L(A)$ and $I = I_L(A)$ and suppose that $E \subset D \subset I$. Then, since $D_L(A) \neq \emptyset$, there is an ideal $R$ of $L$ such that $I/R \sim_L A$. Also, $J_L(A) \neq I$, since otherwise $D = E$, by Proposition 1.12, so there is an ideal $B$ of $L$ with $B \sim_L A$ and $B \not\sim_L F$ if $F \in \mathcal{C}F(L)$. Put $H = C_L(B)$. Then $H \subseteq I_L(B) = I$, by Proposition 1.1 and $H \neq I$, by Lemma 1.3 (iii), so $H \subset I$. Put $K = H \cap R$. Then $H/K \cong_L (H + R)/R$. Moreover, if $H + R = R$ then $H = R$, $H \subseteq I = I_L(B)$ and $D_L(B) = D \subseteq R = H$, so $I_L(B)/C_L(B) \cong_L B$, by Lemma 1.10 contradicting the fact that $B \not\cong_L F$ if $F \in \mathcal{C}F(L)$. It follows that $H + R = I$ and $H/K \cong_L I/R$, whence $A \sim_L H/K$. By Proposition 1.2, $H/K$ is a $c'$-factor of $L$ and $I/R$ is a $c'$-factor and so we have that $A$ is of $cc'$-type.

Conversely, if $A$ is of $cc'$-type, from the definition we obtain ideals $I$, $C$, $X$, and $N$ of $L$ and a subalgebra $U$ of $L$ such that $I/C \sim_L A$, $I/C$ and $X/N$ are $m'$-factors, $I/C$ is a $c'$-factor, $I/C \subseteq X/N$, and $U$ complements $X/N$ in $L$ (using the same notation as Proposition 2.2). Note that $I = C + X$ and $C \cap X = N$, so $I/C \cong_L X/N$, whence $C_L(X/N) = C_L(I/C) = C$, by Lemma 2.2 (iii). Now $[L, U \cap C] = [U + X, U \cap C] \subseteq U \cap C$ since $[X, C] \subseteq N \subseteq U \cap C$, so $U \cap C$ is an ideal of $L$. Also

$$\frac{X + U \cap C}{U \cap C} \cong \frac{X}{N}.$$ 

As in the proof of Proposition 2.2, we obtain an $L$-algebra $B, B \sim_L A$ such that

$$C_L(B) = X + C_L(X/N) = X + U \cap C.$$

Suppose now that there exists $F \in \mathcal{C}F(L)$, such that $F \cong_L B$. Then $I_L(F) = I_L(B) = I$ and $C_L(F) = C_L(B) = X + U \cap C$, and so $F \cong_L I/(X + U \cap C)$. It follows that $I/(X + U \cap C) \sim_L I/C$, which is a primitive Lie algebra of type 2, by Lemma 2.2 (iv). But

$$\frac{L}{X + U \cap C} \cong_L \frac{L}{U \cap C} \left/ \frac{X + U \cap C}{U \cap C} \right.,$$

so $L/U \cap C$ is primitive of type 3. It follows that $(X + U \cap C)/U \cap C$ is an $m$-factor, and hence so is $X/N$, by [1] Lemma 2.1], a contradiction.

Moreover we have that

$$D_L(A) \subseteq C \subseteq I, \quad J_L(A) \subseteq X \subseteq I,$$

which completes the proof. ☐

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