τ-rigid modules for algebras with radical square zero *†

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Abstract

In this paper, we show that for an algebra Λ with radical square zero and an indecomposable Λ-module M such that Λ is Gorenstein of finite type or τM is τ-rigid, M is τ-rigid if and only if the first two projective terms of a minimal projective resolution of M have no non-zero direct summands in common. We also determined all τ-tilting modules for Nakayama algebras with radical square zero. Moreover, by giving a construction theorem we show that a basic connected radical square zero algebra admitting a unique τ-tilting module is local.

1 Introduction

In October of 2012, Adachi, Iyama and Reiten introduced the notion of τ-tilting modules which is a generalization of the classical tilting modules [APR, BB, HR]. τ-tilting modules which admit very similar properties to the classical tilting modules are very close to silting objects in [AiI] and the cluster tilting objects in 2-Calabi-Yau triangulated categories [IY]. So it is interesting to find τ-tilting modules for a given algebra. It is showed in [AIR] that all τ-tilting modules can be written as finite copies of direct sums of τ-rigid modules which was firstly introduced in [AuS]. To find the τ-tilting modules for given algebras, what we need to do is just to find the (indecomposable) τ-rigid modules for them.

Notice that Adachi, Iyama and Reiten showed that every τ-rigid module M has no common non-zero direct summands in the first and second projective terms of its minimal projective resolution. It is interesting to consider whether the τ-rigid modules can be determined by the non-existence of common direct summands in the first and second projective terms of their minimal projective resolutions. A positive answer to this question would make us be able to judge τ-rigid modules straightly. Unfortunately, it is far from being true. So we have to ask: (1) When can τ-rigid modules be determined by the non-existence of common direct summands in their minimal projective resolution?

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In addition, what we also want to know is to determine the structures of algebras from the properties of their \(\tau\)-rigid modules. It is well-known that a local algebra admits a unique \(\tau\)-tilting module, that is, all indecomposable \(\tau\)-rigid modules are projective. So it is natural to ask: (2) Is an algebra \(\Lambda\) local if it admits a unique \(\tau\)-tilting module? We should remark that a similar question for the classical tilting modules is not true in general since every non-local self-injective algebra admits a unique classical tilting module.

On the other hand, algebras with radical square zero have been studied by Auslander, Reiten and Smal\(\phi\) in [AuRS], which play an important role in classifying Nakayama algebras and stable equivalence. For the recent development of this class of algebras, we refer to [C] and [RX]. We should note that this kind of algebras make us be able to give more examples for algebras with best properties of \(\tau\)-rigid modules and non-trivial CM-free algebras (all finitely generated indecomposable Gorenstein projective modules are projective).

In this paper, we try to answer the two questions above over algebras with radical square zero. The paper is organized as follows:

In Section 2, we will recall some preliminaries on algebras with radical square zero. In Section 3, we give an answer to the first question above and prove the following:

**Theorem 1** Let \(\Lambda\) be a basic and connected Nakayama algebra with \(r^2 = 0\) which is not self-injective local and let \(n\) be the number of non-isomorphic simple modules. Then

1. Every indecomposable module \(M\) is \(\tau\)-rigid.
2. Every \(\tau\)-tilting module \(T\) is of the form \(S_1 \oplus S_2 \oplus \cdots \oplus S_t \oplus (\Lambda/P_0(\tau(S_1 \oplus \cdots \oplus S_t)))\), where \(S_j\) is simple for \(1 \leq j \leq t\), \(t\) is an integer such that \(0 \leq t \leq \text{int}(n/2)\) and \(\text{int}(m)\) denotes the largest integer less than or equal to \(m\) for any real number \(m\).

**Theorem 2** Let \(\Lambda\) be a basic and connected algebra with \(r^2 = 0\).

1. If \(\Lambda\) is self-injective local, then every indecomposable \(\tau\)-rigid module is projective.
2. If \(\Lambda\) is self-injective but not local, then every indecomposable module is \(\tau\)-rigid.
3. Let \(M\) be an indecomposable \(\Lambda\)-module. If \(\Lambda\) is representation finite of finite global dimension or \(\tau M\) is \(\tau\)-rigid, then \(M\) is \(\tau\)-rigid if and only if there is no non-zero direct summand of \(P_0(M)\) and \(P_1(M)\) in common, where \(P_0(M)\) and \(P_1(M)\) are the first and second projective terms of a minimal projective resolution of \(M\), respectively.

In Section 4, we will give a construction theorem to get indecomposable \(\tau\)-rigid modules from simple modules. This is very different from the mutation theorem in [AIR]. As a result, we can give an answer to the second question and prove the following:

**Theorem 3** Let \(\Lambda\) be a basic and connected algebra with \(r^2 = 0\). If \(\Lambda\) admits a unique
\(\tau\)-tilting module, then it is local.

In Section 5, we will give examples to show our results.

Throughout this paper, all algebras are basic connected non-semi-simple Artin algebras over a commutative Artin ring \(R\). \(D = \text{Hom}_R(-, I^0(R/r))\) is the ordinary dual, where \(r\) is the Jacobson radical of \(R\) and \(I^0(R/r)\) is the injective envelope of \(R/r\). All modules are finitely generated left \(\Lambda\)-modules if not claimed.

2 Properties for algebras with radical square zero

In this section we will recall some properties for algebras with radical square zero. Denote by \(r\) the Jacobson radical of an algebra \(\Lambda\). \(\Lambda\) is called radical square zero if \(r^2 = 0\). Let \(\Gamma\) be another algebra. We say that \(\Lambda\) is stable equivalent to \(\Gamma\) if there is an equivalence functor \(F: \text{mod}\Lambda \to \text{mod}\Gamma\), where \(\text{mod}\Lambda\) and \(\text{mod}\Gamma\) denote the associate module categories modulo the projective modules, respectively.

Now we can recall the following result for algebras with radical square zero from [AuRS, X, Theorem 2.4, Lemma 2.1].

Lemma 2.1 Let \(\Lambda\) be an algebra with \(r^2 = 0\). Denote by \(\Gamma\) the triangular matrix algebra

\[
\begin{pmatrix}
\Lambda/r & 0 \\
r & \Lambda/r
\end{pmatrix}
\]

(1)

and denote by \(F: \text{mod}\Lambda \to \text{mod}\Gamma\) the functor via \(F(M) = (M/rM, rM, f)\) and \(F(g) = (g_1, g_2)\) for any \(M, N, L \in \text{mod}\Lambda\) and \(g : N \to L\), where \(f : r \otimes_{\Lambda/r} M/rM \to rM\) is an epimorphism, \(g_1 : N/rN \to L/rL\) and \(g_2 : rN \to rL\) are induced by \(g\). Then

(1) \(F\) is an equivalence and hence \(\Lambda\) is stable equivalent to \(\Gamma\).

(2) \(F(M)\) is indecomposable if and only if \(M\) is indecomposable.

(3) \(F(M)\) is projective if and only if \(M\) is projective.

Recall that a morphism \(h : E \to M\) is called right minimal if for any \(l : E \to E\) \(h = hl\) implies that \(l\) is an isomorphism. \(h\) is right almost split if \(h\) is not a split epimorphism and for any \(m : N \to E\) which is not a split epimorphism there exists a \(t : N \to E\) such that \(m = ht\). Dually, one can define left minimal morphisms and left almost split sequences. An exact sequence \(0 \to A \xrightarrow{g} B \xrightarrow{h} C \to 0\) is called almost split if \(g\) is left almost split and \(h\) is right almost split. Now we are ready to recall the following properties of almost split sequences for algebras with \(r^2 = 0\) from [AuRS, V, Proposition 3.5, X, Proposition 2.5].

Lemma 2.2 Let \(\Lambda\) be an algebra with \(r^2 = 0\) and let \(0 \to A \xrightarrow{g} B \xrightarrow{h} C \to 0\) be an almost split sequence. Then
(1) $B$ is projective if and only if $A$ is non-injective simple. If $A$ is simple non-injective, then $h : B \to C$ is a projective cover.

(2) $B$ is injective if and only if $C$ is non-projective simple. If $C$ is simple non-projective, then $g : A \to B$ is an injective envelope.

**Lemma 2.3** Let $\Lambda$, $F$ and $\Gamma$ be as in Lemma 2.1 and let $0 \to A \overset{g}{\to} B \overset{h}{\to} C \to 0$ (* be an exact sequence such that $A$ and $C$ are indecomposable and $A$ is not simple. Then

(1) The sequence (*) is almost split in $\text{mod} \Lambda$ if and only if $0 \to F(A) \overset{F(g)}{\to} F(B) \overset{F(h)}{\to} F(C) \to 0$ is almost split in $\text{mod} \Gamma$.

(2) If (*) is almost split, then $F(A) = F(\tau_\Lambda C) = \tau_\Gamma F(C)$.

**Proof.** (2) follows from (1). \qed

The following result which gives a connection between morphisms in $\text{mod} \Lambda$ and $\text{mod} \Gamma$ is very important to the proof of the main results.

**Lemma 2.4** Let $\Lambda$, $F$ and $\Gamma$ be as in Lemma 2.1. Then

(1) For any $M, N \in \text{mod} \Lambda$ we have the following exact sequence of Abelian groups:

$$0 \to \text{Hom}_\Lambda(M, rN) \to \text{Hom}_\Lambda(M, N) \to \text{Hom}_\Gamma(F(M), F(N)) \to 0$$

(2) $\text{Hom}_\Lambda(M, N) \simeq \text{Hom}_\Gamma(F(M), F(N))$ if both $M$ and $N$ have no projective direct summands.

**Proof.** (1) follows from [AuRS, X, Lemma 2.1] and (2) follows from [AuRS, X, Lemma 2.3] and (1). \qed

In order to show the main result on Nakayama algebra with $r^2 = 0$, we need the following:

**Lemma 2.5** Let $\Lambda$ be a Nakayama algebra with $r^2 = 0$. Then every indecomposable module $M \in \text{mod} \Lambda$ is either simple or projective.

**Proof.** By using $r^2 = 0$ and [AsSS, V, Theorem 4.1]. \qed

### 3 $\tau$-rigid modules and minimal projective resolution

In this section, we will determine the $\tau$-rigid modules in terms of minimal projective resolution and try to answer the first question (see Theorem 3.4, Theorem 3.12 and Theorem 3.15). Firstly, we recall the notions of $\tau$-tilting modules and $\tau$-rigid modules in [AIR] and [AuS], respectively.

**Definition 3.1** For an algebra $\Lambda$, a $\Lambda$-module $M$ is called $\tau$-rigid if $\text{Hom}(M, \tau M) = 0$, where $\tau$ denotes the Auslander-Reiten translation. A module $N$ is $\tau$-tilting if it is $\tau$-rigid
and $|N| = |\Lambda|$, where $|N|$ denotes the number of non-isomorphic direct summands of $N$. Any \( \tau \)-rigid module is a direct summand of a \( \tau \)-tilting module. We also note that if \( \Lambda \) is hereditary then \( \tau \)-tilting modules and \( \tau \)-rigid modules coincide with tilting modules and rigid modules, respectively.

For any indecomposable $M$ in \( \text{mod} \Lambda \), if $M$ is projective, then it is \( \tau \)-rigid. So we can assume that $M$ is not projective. Denote by \( \cdots \rightarrow P_t(M) \rightarrow \cdots \rightarrow P_1(M) \rightarrow P_0(M) \rightarrow M \rightarrow 0 \) be a minimal projective resolution of $M$, where $t$ is a non-negative integer. And denote by $\Omega^iM$ the $i$-th syzygy of $M$ for any $i \geq 0$. Considering the almost split sequence $0 \rightarrow \tau M \rightarrow E \rightarrow M \rightarrow 0$, we have the following:

**Proposition 3.2** Let $\Lambda$ be an algebra with $r^2 = 0$. Then $\Lambda$ is self-injective local if and only if there is an almost split sequence $0 \rightarrow S \rightarrow P \rightarrow S \rightarrow 0$, where $S$ is a simple $\Lambda$-module.

**Proof.** $\Rightarrow$ Since $\Lambda$ is a basic connected local algebra, one can get the following exact sequence: $0 \rightarrow r \rightarrow \Lambda \rightarrow S \rightarrow 0$, where $S$ is simple and $r$ is the radical of $\Lambda$. Notice that $r^2 = 0$ then $r$ is semi-simple. Because $\Lambda$ is self-injective, we get $r$ is simple by [HuZ, Lemma 2.6], and hence $r \simeq S$. Then the sequence is almost split by [AsSS, IV, Proposition 3.11].

$\Leftarrow$ By Lemma 2.2, one gets that $P$ is projective and injective. It is enough to prove that $\Lambda$ has a unique simple module $S$ up to isomorphism. On the contrary, Suppose that there is another simple $S' \neq S$. We claim that $\text{Hom}_\Lambda(P_0(S'), P_0(S)) = \text{Hom}_\Lambda(P_0(S), P_0(S')) = 0$, where $P_0(M)$ is the projective cover of $M$.

(1) $\text{Hom}_\Lambda(P_0(S'), P_0(S)) = 0$.

Suppose that there is an $f \in \text{Hom}_\Lambda(P_0(S'), P_0(S))$ such that $f \neq 0$, then $f$ is not epic since $P_0(S)$ is projective and $S \neq S'$. Denote by $\text{Im} f$ the image of $f$, then $\text{Im} f \subseteq rP_0(S)$. Notice that $0 \rightarrow S \rightarrow P \rightarrow S \rightarrow 0$ is almost split, then $P_0(S) \simeq P$ and $rP_0(S) \simeq S$ by Lemma 2.2, and hence $\text{Im} f = S$, then $f : P_0(S') \rightarrow S$ is epic, and hence $P_0(S') \simeq P_0(S)$. One gets a contradiction since $S \neq S'$.

(2) $\text{Hom}_\Lambda(P_0(S), P_0(S')) = 0$.

Suppose that there is a $g \in \text{Hom}_\Lambda(P_0(S), P_0(S'))$ such that $g \neq 0$, then $g$ is not epic and $\text{Img} \subseteq rP_0(S')$ by a similar argument in (1). Notice that $r^2 = 0$, then $rP_0(S')$ is semi-simple. So we get $\text{Img} = S$, and hence $j : S \hookrightarrow P_0(S')$. Then we have the following commutative diagram:

\[
\begin{array}{ccc}
P_0(S') & \xrightarrow{h} & 0 \\
| & | & | \\
\downarrow{j} & \downarrow{i} & \downarrow{\text{Id}} \\
0 \rightarrow S \rightarrow P \rightarrow S \rightarrow 0
\end{array}
\]
By Lemma 2.2 we get that $P_0(S) \simeq P$, $P$ is injective and $i : S \to P$ is an injective envelope. Then $i$ is an essential monomorphism implies that $h : P \to P_0(S')$ is monic, and hence $P \simeq P_0(S')$, a contradiction.

Since $\Lambda$ is connected, one gets the assertion by the claim and [AsSS, II, Lemma 1.6]. □

Now we can give a class of $\tau$-rigid modules over algebras with $r^2 = 0$ which are not local-self-injective.

**Proposition 3.3** Let $\Lambda$ be an algebra with $r^2 = 0$ which is not self-injective local and let $M$ be an indecomposable $\Lambda$-module. If $M$ satisfies (1) $\tau M$ is simple projective, or (2) both $\tau M$ and $M$ are simple, then $M$ is $\tau$-rigid.

**Proof.** Suppose that there is a non-zero $f \in \text{Hom}_\Lambda(M, \tau M)$. For both cases, we have $f$ is epic, and hence $M \simeq \tau M$. For the first case, we get that $M$ is projective, a contradiction. For the second one, by Lemma 2.2 there is an almost split sequence $0 \to S \to P \to S \to 0$ with $S$ a simple module. By Proposition 3.2 $\Lambda$ is self-injective local, a contradiction. □

**Remark** For any algebra $\Sigma$ and an indecomposable $\Sigma$-module $N$ with $\tau N$ simple projective, one can show that $N$ is $\tau$-rigid by formulating the proof of Proposition 3.3 (1).

Now we are in a position to state the $\tau$-tilting and $\tau$-rigid modules for Nakayama algebras with $r^2 = 0$.

**Theorem 3.4** Let $\Lambda$ be a Nakayama algebra with $r^2 = 0$ which is not self-injective local and let $n$ be the number of non-isomorphic simple modules. Then

(1) Every indecomposable module $M$ is $\tau$-rigid.

(2) Every $\tau$-tilting module $T$ is of the form $S_1 \oplus S_2 \oplus \cdots \oplus S_t \oplus (\Lambda/P_0(\tau(S_1 \oplus \cdots \oplus S_t)))$, where $S_j$ is simple for $1 \leq j \leq t$, $t$ is an integer such that $0 \leq t \leq \text{int}(n/2)$ and $\text{int}(m)$ denotes the largest integer less than or equal to $m$ for any real number $m$.

**Proof.** (1) By Lemma 2.5, $M$ is simple or projective for any indecomposable $M \in \text{mod} \, \Lambda$. If $M$ is projective, there is nothing to prove. If $M$ is simple non-projective, then $\tau M$ is simple, by Proposition 3.3, $M$ is $\tau$-rigid.

(2) For any $\tau$-tilting module $T$, we claim that there is at least one indecomposable $P$ as a direct summand of $T$.

On the contrary, suppose that $T = S_1 \oplus S_2 \oplus \cdots \oplus S_n$ with all $S_j$ simple non-projective for $1 \leq j \leq n$. Without loss of generality, we can assume that $\tau S_1 = S_2$ by Lemma 2.5 or (1). Then $0 \neq \text{Hom}_\Lambda(S_2, S_2) \subseteq \text{Hom}_\Lambda(T, \tau T) = 0$, a contradiction.

Next we will show if $S$ is a direct summand of $T$, then $P_0(S)$ is a direct summand of $T$.  

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If $S$ is projective, then the assertion holds true. We can assume that $S$ is not projective. Then we get the following almost split sequence: $0 \to S' \to P \to S \to 0$ with $S'$ simple by Lemma 2.2 and Lemma 2.5. Again by Lemma 2.2, $P$ is indecomposable projective and $P \cong P_0(S)$. Because $T$ is $\tau$-tilting and $S$ is direct summand of $T$, it is not difficult to show that $S'$ is not a direct summand of $T$. Similarly, if $S$ is not injective, then a simple module $S^*$ with $\tau S^* \cong S$ is not a direct summand of $T$. Since $\tau T$ is semi-simple, then $\text{Hom}_\Lambda(P_0(S), \tau T) = 0$, that is, $P_0(S)$ is in $\text{Fac}(T)$ by [AIR, Theorem 2.10], where $\text{Fac}(T)$ denotes the category consisting of factor modules of finite copies of direct sums of $T$. So $P_0(S)$ is a direct summand of $T$. Notice that $|T| = |\Lambda| = n$, so the number of simple direct summands of $T$ has to be at most $\text{int}(n/2)$.

The conditions in Proposition 3.3 are not easy to be satisfied. In the following we will generalize it into a general framework. Denote by $\text{gl.dim} \Lambda$ the global dimension of $\Lambda$ and denote by $\text{pd}_\Lambda M$ the projective dimension of $M$. We have:

**Lemma 3.5** Let $\Lambda$ be an algebra with $r^2 = 0$. If $S$ is a simple module with $\text{pd}_\Lambda S = m < \infty$. Then $S$ is $\tau$-rigid. Moreover, if $\text{gl.dim} \Lambda = m < \infty$, then every simple module $S$ is $\tau$-rigid.

**Proof.** We only have to the first one since the last follows from the first. By [AIR, Proposition 1.2] a simple module $S$ is $\tau$-rigid if and only if it is rigid, that is, $\text{Ext}^1_\Lambda(S, S) = 0$. If $S$ is projective, there is nothing to prove. So we can assume that $m \geq 1$. Take the following part of a minimal projective resolution of $S$: $0 \to \Omega^1 S \to P_0(S) \to S \to 0$, where $\Omega^1 S$ denotes the first syzygy of $S$. One gets that $\text{pd}_\Lambda \Omega^1 S = m - 1$ since $\text{pd}_\Lambda S = m < \infty$. Since $r^2 = 0$, we have $\Omega^1 S \cong r P_0(S)$ is semi-simple and any direct summand of it is of projective dimension at most $m - 1$. So it is not difficult to show $\text{Ext}^1_\Lambda(S, S) \cong \text{Hom}_\Lambda(\Omega^1 S, S) = 0$. \hfill $\Box$

Denote by $\text{mod} \Lambda$ the associate modules category modulo injective modules and denote by $\text{Hom}_\Lambda(L, N)$ and $\text{Hom}_\text{mod} \Lambda(L, N)$ classes of morphisms from $L$ to $N$ in $\text{mod} \Lambda$ and $\text{mod} \Lambda$, respectively. Now we are in a position to state another main result on judging the $\tau$-rigid properties by simple modules.

**Theorem 3.6** Let $\Lambda$ be an algebra with $r^2 = 0$ and let $M$ be indecomposable with $\tau M$ simple. We have (1) $M$ is $\tau$-rigid if and only if $\tau M$ is $\tau$-rigid. (2) If $\text{pd}_\Lambda M < \infty$, then $M$ is $\tau$-rigid. Moreover, if $\text{gl.dim} \Lambda < \infty$, then $M$ is $\tau$-rigid.

**Proof.** Since (2) is a straight result of (1), Lemma 2.2 and Lemma 3.5, we only show (1).

$\leftarrow$ By the remark of Proposition 3.3, it is enough to show the case of $\tau M$ is not projective.

On the contrary, suppose that $M$ is not $\tau$-rigid, that is, $\text{Hom}(M, \tau M) \neq 0$. We get that
\( f \) is epic for any \( 0 \neq f \in \text{Hom}(M, \tau M) \) since \( \tau M \) is simple. By Lemma 2.2, one gets the following almost split sequence: \( 0 \to \tau M \xrightarrow{i} P_0(M) \to M \to 0 \). So \( P_0(\tau M) \) is a direct summand of \( P_0(M) \). Notice that \( i \) is left minimal, then by [AuRS, I, Theorem 2.4] \( \tau M \) can be embedded into \( P_0(\tau M) \), and hence a direct summand of \( rP_0(\tau M) \) since \( r^2 = 0 \). Then we have the following commutative diagram:

\[
\begin{array}{cccccc}
0 & \to & rP_0(\tau M) & \to & P_0(\tau M) & \xrightarrow{\pi} \tau M & \to 0 \\
& & \downarrow{\alpha} & & \downarrow{\beta} & & \\
0 & \to & \tau M & \to & E & \to & \tau M & \to 0
\end{array}
\]

with \( \alpha \) an epimorphism. By the snake lemma, \( \beta \) is also an epimorphism. Since \( \text{pd}_\Lambda M < \infty \), one can show that \( \text{pd}_\Lambda \tau M \) is of finite projective dimension by Lemma 2.2. Then by the assumption \( \tau M \) is \( \tau \)-rigid and hence the bottom row in the commutative diagram is split. So one gets an epimorphism \( P_0(\tau M) \to \tau M \oplus \tau M \), a contradiction.

\[ \Rightarrow \] Since \( M \) is \( \tau \)-rigid, one gets \( \text{Hom}_\Lambda(M, \tau M) = 0 \) which implies that \( \text{Hom}_\Lambda(M, \tau M) = 0 \). Notice that \( \tau : \text{mod}\Lambda \to \text{mod}\Lambda \) is an equivalence, one can get \( \text{Hom}_\Lambda(\tau M, \tau^2 M) = \text{Hom}_\Lambda(M, \tau M) = 0 \). By AR-formula one gets \( \text{Ext}_\Lambda^1(\tau M, \tau M) \simeq D\text{Hom}_\Lambda(\tau M, \tau^2 M) = 0 \). Then by [AIR, Proposition 1.2] \( \tau M \) is \( \tau \)-rigid since \( \tau M \) is simple. \( \square \)

To answer the first question, we need the following properties for \( \tau \)-rigid modules over hereditary algebras.

**Lemma 3.7** Let \( \Lambda \) be a hereditary algebra and let \( M \) be an indecomposable non-projective module. If \( \tau M \) is projective, then \( M \) is \( \tau \)-rigid.

**Proof.** Suppose that \( \text{Hom}_\Lambda(M, \tau M) \neq 0 \). Then there is a non-zero morphism \( f : M \to \tau M \). Since \( \Lambda \) is hereditary and \( \tau M \) is projective, one gets that \( \text{Im} f \) is projective and hence \( M \) is projective, a contradiction. \( \square \)

**Lemma 3.8** Let \( \Lambda \) be a hereditary algebra and let \( M \) be an indecomposable non-projective module. If \( \tau M \) is \( \tau \)-rigid, then \( M \) is \( \tau \)-rigid.

**Proof.** If \( \tau M \) is projective, the assertion holds from Lemma 3.7. We only show the case \( \tau M \) is not projective. By [AsSS, IV, Corollary 2.15 (b)], one gets \( \text{Hom}_\Lambda(M, \tau M) \simeq \text{Hom}_\Lambda(\tau M, \tau^2 M) = 0 \). \( \square \)

Recall that an indecomposable module \( M \) over a hereditary algebra is preprojective if there is a non-negative integer \( j \) such that \( \tau^j M \) is a non-zero projective module. Then we have:

| 8 |
Proposition 3.9 [AuRS, VIII, Propositions 1.7, 1.13] Let $\Lambda$ be a hereditary algebra. Then (1) Every preprojective module $M$ is $\tau$-rigid. (2) If $\Lambda$ is of finite type, then every indecomposable module is $\tau$-rigid, and hence rigid.

Proof. (1) We can assume that $\tau^j M$ is projective for some non-negative integer $j$. By induction on $j$ and Lemma 3.8, one gets the assertion. Then by [AuRS, VIII, Proposition 1.13] and (1), one can show (2). □

Denote by $id_\Lambda M$ (resp. $id_\Lambda^o M$) the injective dimension of $M$ for an $M$ in $\text{mod } \Lambda$ (resp. $\text{mod } \Lambda^o$). Recall that an algebra $\Lambda$ is called Gorenstein if $id_\Lambda \Lambda = id_\Lambda^o \Lambda = n$ for some integer $n \geq 0$. We have the following:

Lemma 3.10 Let $\Lambda$ be a Gorenstein algebra with $r^2 = 0$. Then $\Lambda$ is either self-injective or of finite global dimension.

Proof. By [C], we can get that every algebra with $r^2 = 0$ is either self-injective or CM-free. Recall that an algebra is called CM-free if every finitely generated Gorenstein projective module is projective. We will show $\text{gl.dim } \Lambda = n$ if $id_\Lambda \Lambda = id_\Lambda^o \Lambda = n$ for some $n > 0$. By [AuR, Proposition 3.1] one can show that $\Omega^n M$ is Gorenstein projective and hence projective for any $M$ in $\text{mod } \Lambda$ since $\Lambda$ is CM-free. The assertion holds true. □

Notice that a self-injective algebra with $r^2 = 0$ is Nakayama by [AuRS, IV, Proposition 2.16]. Since the Nakayama case is completely classified in Proposition 3.2 and Theorem 3.4, we only have to find the $\tau$-rigid modules for algebras of finite global dimension with $r^2 = 0$.

Denote by $\text{Soc} M$ the socle of $M$. We have the following easy observation:

Lemma 3.11 Let $\Lambda$ be an algebra and let $M$ be an indecomposable and non-projective $\Lambda$-module. Then (1) $\text{Soc} \tau M \simeq \Omega^1 M / r \Omega^1 M$. (2) $\text{Soc} \tau M \simeq \Omega^1 M$ if $r^2 = 0$.

Proof. (1) Taking the following part of a minimal projective resolution of $M$: $P_1(M) \to P_0(M) \to M \to 0$ and then applying the functor $\text{Hom}_\Lambda (\cdot, \Lambda) = (-)^*$, one gets the following part of minimal projective resolution of $\text{Tr} M$: $P_0(M)^* \to P_1(M)^* \to \text{Tr} M \to 0$, where $\text{Tr}$ is the Auslander-Bridger transpose. Then applying the functor $\mathbb{D}$, one has a minimal injective resolution of $\tau M : 0 \to \tau M \to \mathbb{D}P_1(M)^* \to \mathbb{D}P_0(M)^*$. Then $\text{Soc} \tau M \simeq \text{Soc} \mathbb{D}P_1(M)^* \simeq \Omega^1 M / r \Omega^1 M$.

(2) Taking the following part of a minimal projective resolution of $M$: $0 \to \Omega^1 M \to$
$P_0(M) \to M \to 0$, one can get the following commutative diagram:

\[
\begin{array}{cccccc}
0 & \longrightarrow & \Omega^1 M & \longrightarrow & P_0(M) & \longrightarrow M & \longrightarrow 0, \\
\downarrow{g} & & \downarrow f & & \downarrow f & & \\
0 & \longrightarrow & rP_0(M) & \longrightarrow P_0(M) & \longrightarrow M/rM & \longrightarrow 0
\end{array}
\]

where $f$ is epic. Then by the snake lemma, one gets $g$ is a monomorphism. Since $r^2 = 0$, we have that $rP_0(M)$ is semi-simple and hence $\Omega^1 M$ is semi-simple. We are done. \qed

Now we are ready to determine all the $\tau$-rigid modules for a Gorenstein algebra of finite type with $r^2 = 0$. Combined with Proposition 3.2, Lemma 3.11 and Theorem 3.4, one can show:

**Theorem 3.12** Let $\Lambda$ be a Gorenstein algebra of finite type with $r^2 = 0$.

1. If $\Lambda$ is self-injective local, then every indecomposable $\tau$-rigid module is projective.
2. If $\Lambda$ is self-injective but not local, then every indecomposable module $M$ is $\tau$-rigid.
3. If $\Lambda$ is of finite global dimension, then an indecomposable module $M$ is $\tau$-rigid if and only if there is no non-zero direct summand of $P_0(M)$ and $P_1(M)$ in common.

**Proof.** (1) is clear and (2) is showed in Theorem 3.4.

(3) $\Rightarrow$ It is a straight result of [AIR, Proposition 2.5].

$\Leftarrow$ Without loss of generality, we can assume that $\tau M$ is not zero. If $\tau M$ is simple, then the assertion holds true by Theorem 3.6. Now we can assume that $\tau M$ is not simple. Let $\Gamma$ and $F$ be as in Lemma 2.1. By Lemma 2.1 and [AuRS, X, Proposition 1.1], we get that $\Gamma$ is hereditary of finite type. Then by Proposition 3.9(2), $F(M) \in \text{mod} \Gamma$ is $\tau$-rigid. So $\text{Hom}_\Gamma(F(M), F(\tau M)) \simeq \text{Hom}_\Gamma(F(M), \tau F(M)) = 0$ by Lemma 2.3. Then by Lemma 2.4 $M$ is $\tau$-rigid if and only if $\text{Hom}_\Lambda(M, r\tau M) = 0$. We show that $\text{Hom}_\Lambda(M, r\tau M) = 0$. Since $r^2 = 0$, we get that $r\tau M$ is semi-simple and hence a direct summand of $\text{Soc}\tau M \simeq \Omega^1 M$ by Lemma 3.11. Notice that there is no common direct summand of $P_0(M)$ and $P_1(M)$, one can show $\text{Hom}_\Lambda(M, \text{Soc}\tau M) = 0$ which implies that $\text{Hom}_\Lambda(M, r\tau M) = 0$. Then $M$ is $\tau$-rigid by Lemma 2.4. \qed

In general, for algebras mentioned in Theorem 3.12 (3) we don’t know whether there is a common direct summand in $P_0(M)$ and $P_1(M)$ for an indecomposable $M$ (see Example 5.3). However, we get the following:

**Proposition 3.13** Let $\Lambda$ be an algebra of finite global dimension with $r^2 = 0$ and let $M$ be an indecomposable module. If $M/rM$ is simple, then $P_0(M)$ and $P_1(M)$ have no non-zero direct summands in common.
Proof. Denote by \( S = M/rM \). It is enough to show that \( \text{Hom}_\Lambda(\Omega^1 M, S) = 0 \). On the contrary, suppose that \( \text{Hom}_\Lambda(\Omega^1 M, S) \neq 0 \). Then \( S \) is a direct summand of \( \Omega^1 M \) since \( r^2 = 0 \) and \( S \) is simple. Moreover, we have the following commutative diagram with exact rows:

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & \Omega^1 M & \longrightarrow & P_0(M) & \longrightarrow & M & \longrightarrow & 0 \\
& & \downarrow f & & \downarrow & & \downarrow g & & \\
0 & \longrightarrow & \Omega^1 S & \longrightarrow & P_0(S) & \longrightarrow & S & \longrightarrow & 0 \\
\end{array}
\]

Since \( g \) is epic, one can show that \( f : \Omega^1 M \to \Omega^1 S \) is a monomorphism, and hence \( \Omega^1 M \) is a direct summand of \( \Omega^1 S \) because \( r^2 = 0 \). So \( S \) is a direct summand of \( \Omega^1 S \), that is \( \text{pd}_\Lambda S \leq \text{pd}_\Lambda S - 1 < \infty \), a contradiction. \(\square\)

Denote by \( l(M) \) the length of \( M \). As a result of Proposition 3.13, we can get:

**Corollary 3.14** Let \( \Lambda \) be an algebra of finite global dimension with \( r^2 = 0 \) and \( M \) be an indecomposable module. If \( l(M) \leq 2 \), then \( P_0(M) \) and \( P_1(M) \) have no non-zero direct summands in common.

**Proof.** By Proposition 3.13, it suffice to prove that \( M/rM \) is simple. If \( l(M) = 1 \) then \( M \) is simple, the assertion holds true. If \( l(M) = 2 \) then \( rM = \text{Soc}M \) is simple, and hence \( M/rM \) is simple. \(\square\)

At the end of this section we will give a method to find more \( \tau \)-rigid modules for any algebra with \( r^2 = 0 \). As we know all the projective \( \Lambda \)-modules are \( \tau \)-rigid. In case that the algebra \( \Lambda \) is not self-injective, there must be some indecomposable \( M \) such that \( \tau M \) is projective. It is interesting to know whether \( M \) is \( \tau \)-rigid. A more general question is: Whether is \( M \) \( \tau \)-rigid if \( \tau M \) is \( \tau \)-rigid? To answer this question, we have

**Theorem 3.15** Let \( \Lambda \) be an algebra with \( r^2 = 0 \) and let \( M \) be an indecomposable \( \Lambda \)-module such that \( \tau M \) is \( \tau \)-rigid. Then \( M \) is \( \tau \)-rigid if and only if there is no non-zero direct summand of \( P_0(M) \) and \( P_1(M) \) in common.

**Proof.** \( \Rightarrow \) By [AIR, Proposition 2.5].

\( \Leftarrow \) If \( \tau M \) is simple, then by Theorem 3.6 (1) \( M \) is \( \tau \)-rigid if and only if \( \tau M \) is \( \tau \)-rigid. Then the assertion holds by the assumption \( \tau M \) is \( \tau \)-rigid.

Now we can assume that \( \tau M \) is not simple. Let \( \Gamma \) and \( F \) be as in Lemma 2.1. We claim that \( \text{Hom}_\Gamma(F(M), F(\tau M)) = 0 \).

Since \( \tau M \) is \( \tau \)-rigid, we get \( \text{Hom}_\Lambda(\tau M, \tau^2 M) = 0 \) and hence \( \text{Hom}_\Lambda(\tau M, \tau^2 M) = 0 \).
Notice that $\tau : \text{mod}\ A \rightarrow \text{mod}\ A$ is an equivalence, we get that $\text{Hom}_A(M, \tau M) = 0$. If $\tau M$ is not projective, then we get $\text{Hom}_\Gamma(F(M), F(\tau M)) = 0$ by Lemma 2.4. If $\tau M$ is projective, then $\text{Hom}_\Gamma(F(M), F(\tau M)) = 0$ since $F(\tau M)$ is projective and $\Gamma$ is hereditary. Otherwise, one can get a non-zero $g : F(M) \rightarrow F(\tau M)$. So $\text{Im} g$ is projective, and hence $F(M)$ is projective, that is, $M$ is projective by Lemma 2.1, a contradiction.

By Lemma 2.4, we only have to show $\text{Hom}_A(M, r\tau M) = 0$. One can get the assertion by a similar argument in Theorem 3.12.

For a non-Nakayama algebra $A$ with $r^2 = 0$, by Theorem 3.15 one can find the $\tau$-rigid modules one by one from the projective vertices of the $AR$-quiver of $A$ since here $A$ is not self-injective. For the Nakayama case with $r^2 = 0$, we refer to Theorem 3.4 or Theorem 3.12. On the other hand, it is interesting to study the structure of algebras in terms of indecomposable $\tau$-rigid modules. Compared with Theorem 3.10 and Lemma 2.1, we end this section with an open question which is closed to algebras of finite type.

**Question** Let $A$ be an algebra with radical square zero. If all the indecomposable modules are $\tau$-rigid, then $A$ is of finite type.

### 4 $\tau$-rigid modules and local algebras

In this section we firstly introduce a theorem to get a class of indecomposable $\tau$-rigid modules from simple modules (here we don’t need $A$ to be radical square zero). This method is very different from the mutation theorem in [AIR]. As a result, we give a partial answer to the second question.

**Theorem 4.1** Let $A$ be an algebra and let $S$ be a simple $A$-module such that the first syzygy $\Omega^1S$ is non-zero semi-simple.

(1) Suppose that $S$ is not a direct summand of $\Omega^1S$. Let $S_1$ be a simple submodule of $\Omega^1S$ and let $m$ be the maximal integer such that $S_1^m$ is a direct summand of $\Omega^1S$. Then there is an exact sequence $0 \rightarrow \Omega^1S/S_1^m \rightarrow P_0(S) \rightarrow M \rightarrow 0$ with $M$ indecomposable $\tau$-rigid.

(2) Assume that $S$ is a direct summand of $\Omega^1S$. Let $n$ be the maximal integer such that $S^n$ is a direct summand of $\Omega^1S$.

(a) If $\Omega^1S \simeq S^n$, then there is no non-projective indecomposable $\tau$-rigid module $N$ with the projective cover $P_0(N) \simeq P_0(S)$.

(b) If $\Omega^1S \not\simeq S^n$, then we can get the following exact sequence $0 \rightarrow \Omega^1S/S^n \rightarrow P_0(S) \rightarrow N \rightarrow 0$ such that $N$ is indecomposable non-projective $\tau$-rigid.

**Proof.** (1) By the assumption of (1), one can get that the simple module $S$ is non-projective
τ-rigid since \( \text{Ext}^1_{\Lambda}(S, S) \simeq \text{Hom}(\Omega^1 S, S) = 0 \).

If \( \Omega^1 S \simeq S_1^m \), there is nothing to prove.

Now we can assume that \( \Omega^1 S \not\simeq S_1^m \). Since \( \Omega^1 S \) is semi-simple, we get that a monomorphism \( \Omega^1 S/S_1^m \hookrightarrow \Omega^1 S \hookrightarrow P_0(S) \), and hence we have the desired exact sequence \( 0 \rightarrow \Omega^1 S/S_1^m \rightarrow P_0(S) \rightarrow M \rightarrow 0 \). It remains to prove that \( M \) is indecomposable \( \tau \)-rigid.

Since \( P_0(S) \) is indecomposable and projective, one can show that \( M \) is indecomposable and \( P_0(M) \simeq P_0(S) \) by the exact sequence above. In the following we show that \( M \) is \( \tau \)-rigid.

By [AIR, Proposition 1.2(a)], it is enough to show that \( \text{Ext}^1_{\Lambda}(M, N) = 0 \) for any \( N \in \text{Fac} M \), where \( \text{Fac} M \) is the full subcategory consisting of factor modules of finite copies of direct sums of \( M \).

By the construction of \( M \), we have the following commutative diagram with exact rows

\[
\begin{array}{ccccccccc}
0 & \rightarrow & \Omega^1 M & \rightarrow & P_0(M) & \overset{a}{\rightarrow} & M & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & \Omega^1 S & \rightarrow & P_0(S) & \rightarrow & S & \rightarrow & 0
\end{array}
\]

Here \( \Omega^1 M \simeq \Omega^1 S/S_1^m \) and by snake lemma one gets an exact sequence

\[
0 \rightarrow \Omega^1 M \rightarrow \Omega^1 S \rightarrow S_1^m \rightarrow 0 \quad (\ast 1)
\]

Since \( N \) is in \( \text{Fac} M \), then there is a minimal positive integer \( t \geq 1 \) such that \( g : M^t \rightarrow N \) is an epimorphism. By [AuRS, I, Theorem 2.2] it is not difficult to show that \( P_0(N) \simeq P_0(M)^t \simeq P_0(S)^t \). Hence we get an epimorphism \( h : N \rightarrow S^t \). Then we have the following commutative diagram with exact rows

\[
\begin{array}{ccccccccc}
0 & \rightarrow & \Omega^1 N & \rightarrow & P_0(N) & (\simeq P_0(S)^t) & \rightarrow & N & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow h & & \downarrow & & \\
0 & \rightarrow & \Omega^1 S^t & \rightarrow & P_0(S)^t & \rightarrow & S^t & \rightarrow & 0
\end{array}
\]

Notice that \( h \) is an epimorphism and \( \Omega^1 S \) is semi-simple, by snake lemma we have two exact sequences

\[
0 \rightarrow \Omega^1 N \rightarrow \Omega^1 S^t \rightarrow L \rightarrow 0 \quad (\ast 2)
\]

\[
0 \rightarrow L \rightarrow N \rightarrow S^t \rightarrow 0 \quad (\ast 3)
\]

On the other hand, we have the following commutative diagram
Since $g$ is an epimorphism, we get a monomorphism $l : \Omega^1 M^t \to \Omega^1 N$. Combining the exact sequence $(*)_1$ and $(*)_2$, we have the following commutative diagram:

\[
\begin{array}{ccccc}
0 & \longrightarrow & \Omega^1 M^t & \longrightarrow & P_0(M)^t(\simeq P_0(S)^t) \\
& & \downarrow \exists & & \downarrow g \\
0 & \longrightarrow & \Omega^1 N & \longrightarrow & P_0(N)(\simeq P_0(S)^t) \\
& & \downarrow \exists f & & \downarrow \\
& & 0 & & 0
\end{array}
\]

By snake lemma again, we get that $f$ is an epimorphism. Notice that $S_{1}^m$ is semi-simple, then $L$ is a direct summand of $S_{1}^m$. Applying the functor $\text{Hom}_\Lambda(M, -)$ to the exact sequence $(*)_3$, one can get that $\text{Ext}_\Lambda^1(M, S) \simeq \text{Hom}_\Lambda(\Omega^1 M, S) = 0$. Similarly, one can get $\text{Ext}_\Lambda^1(M, S_1) = 0$ and hence $\text{Ext}_\Lambda^1(M, L) = 0$. Then one gets $\text{Ext}_\Lambda^1(M, N) = 0$. We are done.

(2) We only prove (a) since the proof of (b) is very similar to the proof of (1). It is easy to show that $S$ is not $\tau$-rigid. Suppose that there is an indecomposable $\tau$-rigid module $N$ such that $P_0(S) \simeq P_0(N)$. Then $N \ncong S$ and we have the following commutative diagram

\[
\begin{array}{ccccc}
0 & \longrightarrow & \Omega^1 N & \longrightarrow & P_0(N) \\
& & \downarrow \exists & & \downarrow \\
0 & \longrightarrow & \Omega^1 S & \longrightarrow & P_0(S) \\
& & \downarrow \exists & & \downarrow \\
& & 0 & & 0
\end{array}
\]

By snake lemma, one get that $\Omega^1 N$ is a direct summand of $\Omega^1 S$, and hence has $S$ as one of its direct summand. That means $P_0(N)$ and $P_1(N)$ have a non-zero direct summand $P_0(S)$. But $N$ is $\tau$-rigid, by using [AIR, Proposition 2.5], one gets a contradiction. □

**Remark** One can easily show that algebras with radical square zero satisfy the condition of Theorem 4.1. For a non-local algebra $\Gamma$ with radical square zero, there is at least $2n - m$ indecomposable $\tau$-rigid modules, where $n$ and $m$ is the number of non-isomorphic simple modules and the number of non-isomorphic simple projective modules, respectively.

In the following we will focus on the structure of algebras and the homological properties of algebras for which all $\tau$-rigid modules are projective. To prove the main result of this section, we need the following lemmas.
Lemma 4.2 Let \( \Lambda \) be an algebra such that all \( \tau \)-rigid modules are projective, then \( \Lambda \) has no simple projective module.

Proof. Suppose that there is a simple projective module \( S \). Then one can get an AR-sequence \( 0 \to S \to E \to M \to 0 \). By Proposition 3.3, \( M \) is \( \tau \)-rigid. But \( M \) is not projective since \( \tau M \simeq S \neq 0 \).

Lemma 4.3 Let \( \Lambda \) be an algebra such that all \( \tau \)-rigid modules are projective and let \( S \) be a simple \( \Lambda \)-module.

(1) Then there is a non-zero direct summand of \( P_0(S) \) and \( P_1(S) \) in common.

(2) If in addition \( \Lambda \) is radical square zero, then \( S \) is a direct summand of \( \Omega^1 S \).

Proof. (1) By Lemma 4.2, we get that there is no projective simple module. By the assumption, \( S \) is not \( \tau \)-rigid. By [AIR, Proposition 1.2 (a)], \( 0 \neq \text{Ext}^1_{\Lambda}(S,S) \simeq \text{Hom}_\Lambda(\Omega^1 S, S) \). Then one gets the assertion.

(2) is a straight result of (1).

Now we can state the main theorem of this section.

Theorem 4.4 Let \( \Lambda \) be an algebra with radical square zero. If \( \Lambda \) admits a unique \( \tau \)-tilting module, then \( \Lambda \) is local.

Proof. Firstly, we claim that for any simple module \( S \), \( \Omega^1 S \simeq S^t \) for some positive integer \( t \). By Lemma 4.3, we get that \( S \) is a direct summand of \( \Omega^1 S \). By Theorem 4.1 (2)(b), \( \Omega^1 S/S^t \) must be zero (otherwise, there will be an indecomposable non-projective \( \tau \)-rigid module). The assertion holds.

Next we will show that there is a unique simple module \( S \) in \( \text{mod} \Lambda \). Suppose there is another simple module \( S' \). Then by the claim above we get that there is a positive integer \( m \) such that \( \Omega^1 S' \simeq S^m \). So one can get \( \text{Hom}_\Lambda(P_0(S), P_0(S')) = \text{Hom}_\Lambda(P_0(S'), P_0(S)) = 0 \). Notice that \( \Lambda \) is basic and connected, this is a contradiction.

In Theorem 4.4, if \( \Lambda \) is a finite dimensional algebra over an algebraically closed field \( K \), one can get that the quiver of \( \Lambda \) is just one vertex with several cycles. Then one determines the structure of the algebras completely. After finishing Theorem 4.4, the author was told by Professor Iyama that he can prove that a basic connected algebra with a unique \( \tau \)-tilting module is a local algebra by mutation.

5 Examples

In this section we give examples to show our results. Let \( Q \) be a quiver. Denote by \( P(i) \), \( I(i) \) and \( S(i) \) the indecomposable projective module, indecomposable injective module and
the simple module according to the vertex $i \in Q$, respectively. The following example is a Nakayama algebra with $r^2 = 0$.

**Example 5.1** Let $\Lambda$ be given by the quiver:

```
  2  \downarrow a \\
  a    \downarrow a \\
  1 \leftarrow a \\
```

with relations $a^2 = 0$. By Theorem 3.4, every indecomposable module is $\tau$-rigid. The $\tau$-tilting modules are of the following forms:

1. 0-simple module. $P(1) \oplus P(2) \oplus P(3) \oplus P(4)$
2. 1-simple module. $P(1) \oplus S(1) \oplus P(3) \oplus P(4), P(1) \oplus P(2) \oplus S(2) \oplus P(4)$
3. $P(1) \oplus P(2) \oplus P(3) \oplus S(3), S(4) \oplus P(2) \oplus P(3) \oplus P(4)$

In the following we give an example to show that there does exist an algebra of finite global dimension with $r^2 = 0$ which is of finite type but not Nakayama.

**Example 5.2** Let $\Lambda$ is given by the quiver:

```
  1  \downarrow a \\
  a    \downarrow a \\
  2 \leftarrow a \\
```

with relations $a^2 = 0$. Then

1. $\Lambda$ is a representation finite algebra of global dimension 2 with $r^2 = 0$.
2. $\tau S(3) \simeq S(4)$ and $\tau I(3) \simeq S(3)$. So $S(3)$ and $I(3)$ are $\tau$-rigid by Theorem 3.6.
3. By Theorem 3.12 or Theorem 3.15, Corollary 3.14 and (2), every indecomposable $\Lambda$-module is $\tau$-rigid.

To show Theorem 3.12 and Theorem 3.15, in the following we will construct an algebra $\Lambda$ and an indecomposable $\Lambda$-module $M$ such that $\Lambda$ is of finite type and finite global dimension with $r^2 = 0$ and there is no non-zero direct summand of $P_0(M)$ and $P_1(M)$ in common.

**Example 5.3** Let $\Lambda$ is given by the quiver:

```
  2  \uparrow a \\
  a    \uparrow a \\
  3 \leftarrow a \\
```

With relations $a^2 = 0$. The following example is a Nakayama algebra with $r^2 = 0$.
with relations $a^2 = 0$. Then

(1) $\Lambda$ is a representation finite algebra of global dimension 2 with $r^2 = 0$.

(2) The injective module $I(3)$ has a minimal projective resolution:

$$P(2) \bigoplus P(3) \to P(2) \bigoplus P(4) \to I(3) \to 0.$$ By Theorem 3.12 (3), $I(3)$ is not $\tau$-rigid.

(3) $\tau^2 I(3) \simeq S(2)$, then by Theorem 3.6 (2) $\tau I(3)$ is $\tau$-rigid. So $\tau M$ is $\tau$-rigid can not imply that $M$ is $\tau$-rigid in general.

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