GENERIC ROOT COUNTS AND FLATNESS
IN TROPICAL GEOMETRY

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Abstract. We use tropical and non-archimedean geometry to study generic root counts of families of polynomial equations. These families are given as morphisms of schemes $X \to Y$ that factor through a closed embedding into a relative torus over a parameter space $Y$. We prove a generalization of Bernstein’s theorem for these morphisms, showing that the root count of a single well-behaved tropical fiber spreads to an open dense subset of $Y$. We use this to express the generic root count of a wide class of square systems in terms of the matroidal degree of an explicit variety. This in particular gives tropical formulas for the birational intersection indices and volumes of Newton-Okounkov bodies from $\cite{KK10}$, and the generic root counts of the steady-state equations of chemical reaction networks. An important role in these theorems is played by the notion of tropical flatness, which allows us to infer generic properties of $X \to Y$ from a single tropical fiber. We show that the tropical analogue of the generic flatness theorem holds, in the sense that $X \to Y$ is tropically flat over an open dense subset of the Berkovich analytification of $Y$.

1. Introduction

In this paper we study the problem of determining the generic number of solutions of parametrized polynomial systems using tropical and non-archimedean geometry. These systems arise naturally in many areas, describing for instance the 27 lines on a smooth cubic surface, the dynamics of the Wnt signaling pathway $\cite{GHRS16}$, the motion of the Gough-Steward platform $\cite{SW05}$, and the central configurations in Newton’s $N$-body problem $\cite{HM06}$. One of the leading techniques in solving these systems numerically is homotopy continuation, which is capable of computing a single root in average polynomial time $\cite{Lai17}$. If however all roots are required, then finding a good upper bound on the total number of roots poses a major problem.

The archetypal example of a generic root count of a family of polynomial systems is Bézout’s theorem, which states that $n$ polynomials $f_1, \ldots, f_n$ in $n$ variables of degrees $d_1, \ldots, d_n$ generically have $\prod_{i=1}^{n} d_i$ solutions. Here, the family consists of all $n$-tuples $(f_1, \ldots, f_n)$ of polynomials of degrees $(d_1, \ldots, d_n)$ and being generic means that the number of solutions is attained on a Zariski open dense subset of the family.
Another example is the Bernstein-Kushnirenko theorem \cite{Kou76, Ber76}, where the family consists of systems with fixed monomial supports and the generic root count is the mixed volume of their Newton polytopes. A third prominent example can be found in the works of Kaveh and Khovanskii \cite{KK10, KK12}, who consider family of systems with fixed polynomial supports. Their generic root count is the birational intersection index, and in special cases it is the volume of the associated Newton-Okounkov body.

The goal of this paper is to develop a new approach for determining generic root counts using tropical geometry. We establish the notion of tropical flatness and use it to extend the Bernstein-Kushnirenko theorem to families of properly intersecting schemes. Herein the mixed volume is naturally replaced by the tropical intersection product, and the initial system criterion \cite[Theorem B]{Ber76} is rephrased in terms of tropical intersections. This allows us to introduce a method for constructing good re-embeddings of square polynomial systems, and we show how it can be applied to derive a formula for the number of steady states of chemical reaction networks \cite{Dic16}, and the birational intersection indices and volumes of the aforementioned Newton-Okounkov bodies from \cite{KK10, KK12}.

1.1. Outline and main results of this paper. In Section 2 we fix our notation and assumptions for the rest of the paper. The polynomial families in this paper are given by morphisms of schemes $X \to Y$, where $Y$ is the parameter space and $X$ is the base space. The parameter space $Y = \text{Spec}(A)$ will be integral of finite type over a non-archimedean field $K$, and given a generically finite morphism $X \to Y$, we define its generic root count $\ell_{X/Y}$ to be the length of the fiber of $X \to Y$ over the generic point $\eta$ of $Y$, see Section 2.2.

From Section 3 onwards, $T_Y = \text{Spec}(A[x_1^\pm, \ldots, x_n^\pm])$ will denote a relative $n$-dimensional torus over $Y$, and we will assume that $X \to Y$ factors through a closed embedding $X \to T_Y$. This naturally gives rise to the notion of a fiberwise tropicalization $\text{trop}(X_P)$, which is the tropicalization of the fiber of $X$ over a point $P$ in the Berkovich analytification $Y^{an}$ of $Y$. As fiberwise tropicalizations need not vary continuously, we introduce the notion of tropical flatness. We say $X \to T_Y$ is tropically flat over $P \in Y^{an}$ if, locally and after a valued field extension, there exist polynomials $f_i$ such that:

1. (Non-degeneracy) The coefficients of $f_i$ do not vanish around $P$.
2. (Tropical basis) The $f_i$ form a tropical basis of $X$ around $P$.

We then prove the tropical analogue of the generic flatness theorem:

**Theorem 3.22.** Let $Y = \text{Spec}(A)$ be an integral scheme of finite type over a non-archimedean field $K$. Then the tropically flat locus of a closed embedding of $Y$-schemes $X \to T_Y$ contains an open dense subset of $Y^{an}$.

Theorem 3.22 in particular implies that if $K$ is non-trivially valued and algebraically closed, then $Y(K)$ contains a dense open subset over which $X \to T_Y$ is
tropically flat. To prove this theorem, we study tropical bases of $X$ over points in $Y^\text{an}$ that give valuations of the function field $K(Y)$. For small enough neighborhoods of these points, we have that tropical bases continue to give tropical bases. The density in Theorem 3.22 then follows from an approximation argument.

If $X \to T_Y$ is tropically flat over a point, then many properties of the corresponding fiber of $X \to Y$ spread to an open dense subset of the parameter space $Y$, thus becoming generic properties. In Section 4, we investigate one instance of this phenomenon, and show that the tropical intersection number over a tropically flat point is the generic root count:

**Theorem 4.5.** Let $Y = \text{Spec}(A)$ be an integral scheme that is of finite type over a non-archimedean field $K$ and let $p: X \to Y$ be an affine morphism of finite type. Suppose that the following hold:

1. The morphism $p$ factors through a closed embedding $X \to T_Y$ of $Y$-schemes, where $T_Y$ is an $n$-dimensional torus over $Y$.
2. The closed subscheme $X \to T_Y$ is an intersection of closed subschemes $X_i \to T_Y$, $i = 1, \ldots, r$, that are generically Cohen-Macaulay and pure of relative dimension $k_i$ over $Y$ with
   \[ \sum_{i=1}^r (n - k_i) = n. \]
3. There is a $P \in Y^\text{an}$ over which the $X_i \to T_Y$ are tropically flat and the intersection $\bigcap_{i=1}^k \trop(X_{i,P})$ is bounded.

Then $X \to Y$ is generically finite and the generic root count $\ell_{X/Y}$ is the tropical intersection number $\prod_{i=1}^r \trop(X_{i,P})$.

Theorem 4.5 can be regarded as a generalization of Bernstein’s theorem to families of properly intersecting schemes. Here the mixed volume in [Ber76, Theorem A] is replaced by a tropical intersection number and the initial system criterion of [Ber76, Theorem B] is phrased in terms of tropical intersections. The concept of tropical flatness is crucial in Theorem 4.5, as otherwise the tropical intersection numbers would be able to vary from fiber to fiber.

To facilitate the use of Theorem 4.5 we introduce the notions of torus-equivariance and algebraic independence. We say that an embedding $X \to T_Y$ is torus-equivariant if there is an action $T_Y \times_Y Y \to Y$ such that $\lambda \cdot X_P = X_{\lambda \cdot P}$ for every point $(\lambda, P)$ of $T_Y \times_Y Y$, see Section 4.2 for more details. We furthermore say that the families $X_i \to Y$ from Theorem 4.5 are algebraically independent if there exist $X_i' \to Y_i$ with $\prod_{i=1}^r Y_i = Y$ such that $X_i' \times_Y Y = X_i$. In Proposition 4.19, we show that torus-equivariant and algebraically independent families satisfy the conditions of Theorem 4.5. In a sense, if a family is torus-equivariant then we can move the tropical fibers to obtain a stable intersection. This concept furthermore allows us to eliminate intersections in loci where the fibers $X_{i,P}$ are not Cohen-Macaulay.
From Section 5 on, we focus on square systems. The moduli of these square systems are captured by a finite graph which we call the degeneracy graph of \( X \to Y \), see Section 5.1. In Section 5.2, we propose a re-embedding of \( X \to Y \) that splits \( X \) into a linear part \( \hat{X}_{\text{lin}} \to Y \) and a nonlinear part \( \hat{X}_{\text{nlin}} \to Y \). We say that \( X \to Y \) is tropically modifiable if \( \hat{X}_{\text{lin}} \) is torus-equivariant, and \( \hat{X}_{\text{lin}} \to Y \) and \( \hat{X}_{\text{nlin}} \to Y \) are algebraically independent and of complementary codimension. For tropically modifiable \( X \to Y \), we obtain the following combinatorial formulas for their generic root counts.

**Theorem 5.11.** Let \( X \to Y \) be tropically modifiable. Then there exists a dense open subset \( U \subset T_Y \) such that \( X \cap U \to Y \) is generically finite. The generic root count is given by

\[
\ell_{X \cap U/Y} = \text{trop}(\hat{X}_{\text{lin},P}) \cdot \text{trop}(\hat{X}_{\text{nlin},P}) \tag{1}
\]

for explicit schemes \( \hat{X}_{\text{lin}} \to Y \) and \( \hat{X}_{\text{nlin}} \to Y \) and \( P \) in an open dense subset of \( Y^{an} \).

The open subset \( U \) in Theorem 5.11 can moreover be chosen to be the maximal one with \( \ell_{X \cap U/Y} = \text{trop}(\hat{X}_{\text{lin},P}) \cdot \text{trop}(\hat{X}_{\text{nlin},P}) \). We call this open subset the maximal compatible subset of the modification. It arises from intrinsic obstructions similar to those in the works of Kaveh and Khovanskii [KK10; KK12]. We describe the maximal compatible subset \( U \) and provide sufficient criteria for \( X \cap U = X \).

As the right hand side of Equation (1) depends only on the matroid of \( \hat{X}_{\text{lin},P} \), we refer to it as a matroidal degree of \( \hat{X}_{\text{nlin},P} \). If \( \text{trop}(\hat{X}_{\text{nlin},P}) \) is a tropical complete intersection, then the generic root count \( \ell_{X \cap U/Y} \) is expressible as a weighted sum of mixed volumes using cotransversal matroid decompositions. More generally, we can use a result by Jensen and Yu [JY16, Corollary 5.2], based on a result by McMullen [McM89, Lemma 20], to write \( \hat{X}_{\text{nlin}} \) and \( \hat{X}_{\text{lin}} \) as sums of powers of tropical hypersurfaces. This yields a formula for \( \ell_{X \cap U/Y} \) in terms of sums of mixed volumes.

In Section 6, we consider two classes of linearly parametrized square polynomial systems: systems with vertical dependencies inspired by steady state equations of chemical reaction networks [Dic16], and systems with horizontal dependencies inspired by polynomials with fixed polynomial support [KK10; KK12]. We show that the method from Section 5 is applicable to these systems. This in particular gives a formula for the birational intersection indices \([L_1, \ldots, L_n]\) from [KK10] for \( L_i \) subspaces of the function field of a torus in terms of tropical intersection numbers. Using [KK12, Theorem 4.9], we then also obtain a formula for the volumes of Newton-Okounkov bodies in terms of tropical intersection numbers.

These special cases provide simplifications of the tropical intersection numbers from Section 5 in two ways. Namely, for systems with vertical dependencies, we find that the \( \text{trop}(\hat{X}_{\text{nlin},P}) \) are tropical complete intersections, whereas for systems with horizontal dependencies the \( \text{trop}(\hat{X}_{\text{lin},P}) \) are tropical complete intersections. We demonstrate our technique in two examples: the stationary equations of the Kuramoto model [CMMN19] and Duffing oscillators [BMMT22].
1.2. **Connections to existing literature.** A landmark result in the subject of counting roots of polynomial equations is Bernstein’s theorem [Ber76; Kou76], which establishes the mixed volume as an upper bound for the number of roots of a square polynomial system. An algorithmic milestone in this area was the introduction of the technique of polyhedral homotopy continuation [HS95]. More recently, Kaveh and Khovanskii introduced the notion of Newton-Okounkov bodies [KK12], which in particular gives the root count for specialized systems associated to a single linear subspace. Our work is similar in purpose, but different in approach. To find generic root counts, we use techniques from non-archimedean and tropical geometry. Our approach allows us to treat overdetermined systems that come with a suitable decomposition, as well as systems with more general dependencies between coefficients. In particular, in Section 6.1 we discuss systems which arise as steady state equations of chemical reaction networks, and in Section 6.2 we discuss systems as considered by [KK12]. We show that our techniques can be applied to both types of system to give generic root counts as tropical intersection products.

Our work builds on two fundamental papers [OP13] and [OR13] by Osserman, Payne and Rabinoff, which relate tropical intersection theory to algebraic intersection theory. Our contribution with respect to these papers is that we are able to identify a precise context in which these results can be propagated to open dense subsets of arbitrary integral parameter spaces. The key ingredient is tropical flatness, which gives very precise control over the local variation of fiberwise tropicalizations. We furthermore obtain a generic tropical flatness theorem, showing that the condition imposed here is satisfied for most points in the parameter space.

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2. **Preliminaries**

In this section, we briefly review some basic concepts and fix our notation, which is close to the one used in [Gub13]. We also define generic root counts, decompositions and universal families of polynomial systems.

2.1. **Berkovich spaces.** All schemes in this paper are Noetherian unless mentioned otherwise. The residue field of a point \( P \in X \) is denoted by \( k(P) \). For a commutative unitary ring \( A \), we denote a polynomial ring in \( n \) variables over \( A \) by \( A[x_1, \ldots, x_n] \), and a Laurent polynomial ring in \( n \) variables over \( A \) by \( A[x_1^\pm, \ldots, x_n^\pm] \). Let \( K \) be a non-archimedean field with absolute value \(| \cdot | \). We allow the absolute value to be
trivial. The valuation associated to $(K, |·|)$ is $x \mapsto -\log|x|$, where $\log(\cdot)$ is the natural logarithm.

Let $X$ be a scheme that is locally of finite type over a non-archimedean field $K$. The Berkovich analytification $X^\text{an}$ of $X$ is the set of pairs $(P, k(P) \to \mathbb{R}_{\geq 0})$, where $P \in X$ and $k(P) \to \mathbb{R}_{\geq 0}$ is an absolute value extending the absolute value on $K$. Equivalently, we can view points of $X^\text{an}$ as equivalence classes of $L$-valued points of $X$, where $L$ is a valued field extension of $K$. If $p: X \to Y$ is a morphism of schemes, then we denote its analytification by $p^\text{an}: X^\text{an} \to Y^\text{an}$. There is a natural forgetful map $\pi: X^\text{an} \to X$ that sends $Q = (P, k(P) \to \mathbb{R}_{\geq 0})$ to $P$. We say that $Q$ is rational if the induced map $K \to k(\pi(Q))$ is an isomorphism. If $X = \text{Spec}(B)$, then $X^\text{an}$ coincides with the usual definition of multiplicative seminorms $B \to \mathbb{R}_{\geq 0}$ that restrict to the given norm on $K$. If $f \in B$ and $P \in X^\text{an}$, then we write $f(P)$ for the image of $f$ under $A \to k(\pi(P))$. Similarly, we write $|f(P)| \in \mathbb{R}_{\geq 0}$ for the image of $f$ under $B \to k(\pi(P)) \to \mathbb{R}_{\geq 0}$. The topology on $X^\text{an}$ is generated by
\[ \mathcal{B}(f, r_1, r_2) = \{ P : r_1 < |f(P)| < r_2 \} \]
for $r_1 \geq 0$ and $r_2 > 0$. If $X$ is given the Zariski topology and $X^\text{an}$ the topology above, then $\pi: X^\text{an} \to X$ is continuous. For more background information on Berkovich spaces and tropicalizations, we recommend [BPR16], [Gub13] and [Tem15].

**Remark 2.1** Let $X = \text{Spec}(B)$ be of finite type over a non-archimedean field $K$ and let $P \in X^\text{an}$ with field of definition $L = k(\pi(P))$. The point $P$ induces a canonical point $P_L \in X^\text{an}_L$ by the following procedure. We represent $P$ as a ring homomorphism $B = K[x_1, \ldots, x_n]/(g_1, \ldots, g_m) \to L$. Then $P_L$ is given by the natural ring homomorphism $B_L = L[x_1, \ldots, x_n]/(g_1, \ldots, g_m) \to L$. If $K \to M$ is a valued field extension different from $K \to L$, then $P$ in general does not induce a canonical point of $X^\text{an}_M$. In this case we simply choose a point $P_M$ lying over $P$. This exists by [Gub13, Lemma 2.3] applied to $X^\text{an}_M \to X^\text{an}_M \to X^\text{an}$. Here $M$ is the completion of $M$ with respect to the given valuation. If $P$ is rational, then there is only one point $P_M$ lying over $P$ for any valued field extension $K \to M$.

**2.2. Generic properties.** In this section, we define generic root counts, generically Cohen-Macaulay morphisms and generically pure morphisms. Determining generic root counts will be main focus of the paper; the other two are important conditions required to find generic root counts.

Let $p: X \to Y$ be locally of finite type with $Y$ integral. If $p$ is generically finite, then there are finitely many points lying over the generic point $\eta$ of $Y$. By [Stacks22, Lemma 02NG], we then have that the generic fiber $X_\eta$ is a finite $k(\eta)$-scheme, where $k(\eta)$ is the residue field of $\eta$. This is associated to a finite $k(\eta)$-algebra, whose $k(\eta)$-dimension we call the length or degree of $X_\eta$. 
Definition 2.2 The **generic root count** of \( p: X \to Y \) is the length of the \( k(\eta) \)-scheme \( X_\eta \). We denote it by \( \ell_{X/Y} \). If \( X \to Y \) is not generically finite, then we set \( \ell_{X/Y} = \infty \).

Example 2.3 Suppose \( Y = \text{Spec}(A) \) is integral and \( p: X \to Y \) is affine with \( X = \text{Spec}(B) \). Then

\[
\ell_{X/Y} = \dim_{K(A)}(B \otimes_A K(A)),
\]

where \( K(A) \) is the function field of \( A \).

Remark 2.4 By our assumptions, \( X \) is Noetherian and thus \( p: X \to Y \) is quasi-separated and quasi-compact. If \( p \) is generically finite, then there is an open affine \( U \subset Y \) such that \( p^{-1}(U) \to U \) is finite \([\text{Stacks22, Lemma 02NW}]\). Using Grothendieck’s generic freeness theorem, we may shrink \( U \) and assume that \( p_U: p^{-1}(U) \to U \) is finite and free. The rank of this morphism is then exactly \( \ell_{X/Y} \). For every \( y \in U \), we then have that the length of the fiber of \( p_U \) over \( y \) is \( \ell_{X/Y} \).

We say that a morphism of schemes \( p: X \to Y \) is generically Cohen-Macaulay if there is an open dense \( U \subset Y \) such that \( p^{-1}(U) \to U \) is Cohen-Macaulay. We can rephrase this in terms of generic fibers as follows.

Lemma 2.5 Let \( Y \) be integral and let \( p: X \to Y \) a morphism of finite type. Suppose that the generic fiber \( X_\eta \) is Cohen-Macaulay. Then there is an open dense subset \( U \subset Y \) such that the irreducible components of the fibers of \( f \) over \( U \) are all of constant dimension \( k \).

Proof. We can assume by generic flatness that \( p \) is flat. Consider the open subset \( W \) from \([\text{Stacks22, Lemma 045U}]\). Its complement \( Z = X \setminus W \) is closed and the fiber of \( Z \to Y \) over the generic point is empty by assumption. This gives an open set \( U_\eta \) containing \( \eta \) for which \( p^{-1}(U_\eta) \cap Z = \emptyset \) by \([\text{Stacks22, Lemma 02NE}]\). The induced morphism \( p^{-1}(U_\eta) \to U_\eta \) is automatically Cohen-Macaulay. □

Definition 2.6 Let \( p: X \to Y \) be a morphism of finite type. We say that \( p \) is **generically pure** if there exists an open dense subset \( U \subset Y \) such that the irreducible components of the fibers of \( f \) over \( U \) are all of constant dimension \( k \).

Lemma 2.7 Let \( p: X \to Y \) be a morphism of finite type. Suppose that the fiber over some \( y \in Y \) is pure of dimension \( k \). Then there is an open dense subset \( V \subset \{y\} = Z \) such that the base change \( p_Z: p^{-1}(Z) \to Z \) has pure fibers of dimension \( k \) over \( V \).

Proof. We follow the proof of \([\text{Stacks22, Lemma 055A}]\). We first reduce to \( Y = Z \) integral and find a surjective finite étale map \( \pi: Y' \to V \) for an open dense \( V \subset Y \) such that the generic fiber \( X'_y \) consists of geometrically irreducible components \( X'_{i,y} \) (there is only one point lying over \( y \) by construction, so we denote it by \( y \) again). These are necessarily pure of dimension \( k \) again, see \([\text{Stacks22, Lemma 04KX}]\) and \([\text{Stacks22, Section 07NB}]\) for instance. Let \( X'_{i,y} \) be the closure of \( X'_{i,y} \) in \( X' \). The proof
of \cite[Lemma 055A]{Stacks22} then shows that we can find an open \( V' \subset Y' \) such that the fibers of the \( X'_i \) are geometrically irreducible and \( X' = \bigcup_i X'_i \) over \( V' \). We can furthermore assume by shrinking \( V' \) that all the fibers of the \( X'_i \) are of dimension \( k \).

We now have a morphism \( \pi: V' \to \pi(V') \) such that the base change \( p_\pi \) of \( p \) is relatively pure of dimension \( k \). Consider the proof of \cite[Lemma 0556]{Stacks22}, giving a bijection between the geometrically irreducible components of \( p \) over \( \pi(V) \) and \( p' \) over \( V \). Under this bijection the Krull dimensions of the irreducible components are not changed, since they are obtained by base change over a field extension. We thus obtain the desired result. \( \square \)

**Corollary 2.8** If the generic fiber \( X_\eta \) is pure of dimension \( k \), then \( f \) is generically pure.

**Proof.** This follows from Lemma 2.7. \( \square \)

We note here that generically Cohen-Macaulay morphisms are almost generically pure. Namely, if \( p: X \to Y \) is Cohen-Macaulay, then there exist open and closed subschemes \( X_k \) with \( \bigsqcup_k X_k = X \) such that the \( X_k \) are pure of relative dimension \( k \) over \( Y \). In our theorems, we assume that there is only one closed and open subscheme \( X = X_k \), so that \( p \) is pure of relative dimension \( k \).

### 2.3. Closed embeddings into relative tori

In this section, we introduce notation for morphisms \( X \to Y \) that factor through closed embeddings into tori. These provide a natural framework for fiberwise tropicalizations, as we will see in Section 3. We furthermore define decompositions of these closed embeddings.

**Notation 2.9** For the remainder of the article, we write \( A \) and \( B \) for algebras of finite type over a non-archimedean field \( K \), and \( p: X \to Y \) for a morphism of finite type between their spectra \( X = \text{Spec}(B) \) and \( Y = \text{Spec}(A) \). We assume that \( Y \) is integral, so that the generic root count \( \ell_{X/Y} \) is defined. We write \( T_{Y,n} = \text{Spec}(A[x_1^+, \ldots, x_n^+]) \) for the relative standard \( n \)-dimensional torus over \( Y \). If \( n \) is clear from context, then we denote it by \( T_Y \). We assume that \( p \) factors through a closed embedding of \( Y \)-schemes \( \phi: X \to T_Y \), and we denote the ideal of the embedding by \( I \subseteq A[x_1^+, \ldots, x_n^+] \). For any Zariski open \( V \) in \( Y \), we write \( A_V \) for the corresponding coordinate ring and \( I_V \) for the corresponding ideal in \( A_V[x_1^+, \ldots, x_n^+] \).

Let \( Q \in Y_{\text{an}} \). We write \( I_Q \) for the image of \( I \otimes_A k(\pi(Q)) \) in \( k(\pi(Q))[x_1^+, \ldots, x_n^+] \). The corresponding scheme gives the fiber of \( V(I) \) over \( \pi(Q) \). For \( f \in A[x_1^+, \ldots, x_n^+] \) and \( G \subseteq A[x_1^+, \ldots, x_n^+] \), we similarly write \( f_Q \) and \( G_Q \) for the images of \( f \) and \( G \) under \( A[x_1^+, \ldots, x_n^+] \to k(\pi(Q))[x_1^+, \ldots, x_n^+] \).

**Definition 2.10** A decomposition of the closed embedding \( \phi \) is a finite set of ideals \( I_1, \ldots, I_k \subseteq A[x_1^+, \ldots, x_n^+] \) such that \( I = \sum_{i=1}^k I_i \). Writing \( X_i = \text{Spec}(A[x_1^+, \ldots, x_n^+]/I_i) \)
for the corresponding closed subschemes, we then have an isomorphism
\[ X \simeq \prod_{T_Y} X_i, \]
where the right-hand side is the fiber product over the embeddings \( \phi_i: X_i \to T_Y \).

We say a decomposition is square if \( k = n \) and \( I_i = (f_i) \) for \( f_i \in A[x_1^\pm, \ldots, x_n^\pm] \). We will assume that the \( f_i \) are chosen so that \( V(f_i) \) is relatively and generically pure of codimension one. We also refer to these as square polynomial systems. The mixed volume of the decomposition is the normalized mixed volume of the Newton polytopes of the \( f_1, \ldots, f_n \). We denote it by \( \text{MV}(f_1, \ldots, f_n) \).

**Remark 2.11** For more general decompositions, we can view the tropical intersection number of a general fiber as the analogue of the mixed volume, see Proposition 4.19.

### 2.4. Universal families

We now define universal families of polynomial systems with a fixed monomial support. Square systems are studied in more detail in Sections 5 and 6.

**Definition 2.12** Let \( n, k \in \mathbb{N}, \) and \([k] = \{1, \ldots, k\} \). A fixed monomial support is a finite subset \( S \subset \mathbb{Z}^n \times [k] \) such that \( S \cap (\mathbb{Z}^n \times \{i\}) \neq \emptyset \) for every \( i \in S \). We write \( S_i \subset \mathbb{Z}^n \) for the projection of \( S \cap (\mathbb{Z}^n \times \{i\}) \) onto \( \mathbb{Z}^n \), and we will generally assume that \(|S_i| > 1\). The parameter ring associated to a fixed monomial support \( S \) is the ring \( A = K[c_{i,\alpha} | (\alpha, i) \in S] \) generated by variables \( c_{i,\alpha} \) with \((\alpha, i) \in S\).

Consider the (Laurent) polynomials
\[ f_i = \sum_{\alpha \in S_i} c_{i,\alpha} x^\alpha \in A[x_1^\pm, \ldots, x_n^\pm].\]
These are the universal polynomials with fixed monomial support \( S \). We set \( X = \text{Spec}(A[x_1^\pm, \ldots, x_n^\pm]/(f_1, \ldots, f_k)) \) and \( Y = \text{Spec}(A) \). If \( k = n \), we call \( X \to Y \) the universal family of square polynomial systems over \( K \) with monomial support \( S \).

We will see in Section 4 that the universal family \( \text{Spec}(B) \to \text{Spec}(A) \) of square polynomial systems with a fixed monomial support \( S \) is generically finite and that its generic root count is the mixed volume of the Newton polytopes of the \( f_i \). Similar statements hold for the universal families of square systems with fixed polynomial supports, see Section 5.2.

### 3. Fiberwise tropicalizations and generic flatness

In this section, we study fiberwise tropicalizations for morphisms \( p: X \to Y \) that factor through a closed embedding into a torus. We introduce the notion of tropical flatness, as these fibers need not vary continuously. For families that are tropically flat over a point, we show that the fiberwise tropicalizations are locally constant after
a non-archimedean base change. We then show that closed embeddings $X \to T_V$ are generically tropically flat, see Section 3.2.

3.1. Fiberwise tropicalizations and flatness. Let $T_{K,n}$ be an $n$-dimensional torus over a non-archimedean field. Recall that there is a natural tropicalization map $T_{K,n}^\an \to \mathbb{R}^n$, see [BPR16] and [Gub13]. For a scheme $Y = \text{Spec}(A)$ of finite type over $K$, we similarly define a relative tropicalization map

$$\text{trop}: T_Y^\an \to \mathbb{R}^n$$

as follows. For any $P \in T_Y^\an$ with ring homomorphism $\psi: A[x_1^\pm, \ldots, x_n^\pm] \to k(\pi(P))$, we set

$$\text{trop}(P) = (-\log|\psi(x_1)|, \ldots, -\log|\psi(x_n)|).$$

We use this to define fiberwise tropicalizations of closed subschemes of $T_Y$.

**Definition 3.1** Let $P \in Y^\an$. We define the fiberwise tropicalization of $\phi: X \to T_Y$ over $P$ to be

$$\text{trop}(X_P) := \text{trop}(\phi^\an((p^\an)^{-1}(P))).$$

Given a decomposition $\phi_i: X_i \to T_Y$ and corresponding $p_i: X_i \to Y$, the fiberwise tropical prevariety over $P$ is $\bigcap_i \text{trop}(X_{i,P})$.

**Remark 3.2** If we represent a point $P \in Y^\an$ as a morphism $A \to k(\pi(P))$ together with a valuation on $k(P) = k(\pi(P))$, then the fiberwise tropicalization of the torus over $P$ can be identified with the tropicalization of the Berkovich spectrum of $A[x_1^\pm, \ldots, x_n^\pm] \otimes_A k(P) = k(P)[x_1^\pm, \ldots, x_n^\pm]$. The fiberwise tropicalization of a closed subscheme $X \to T_Y$ is then the tropicalization of the corresponding ideal in $k(P)[x_1^\pm, \ldots, x_n^\pm]$. In particular, we find that the fiberwise tropicalization of a closed subscheme has the usual structure of a tropical variety in $\mathbb{R}^n$.

General tropical fibers can behave quite erratically, so we impose the following continuity conditions.

**Definition 3.3** A $K$-rational local tropical basis for $X \to T_Y$ around $P$ is an open neighborhood $U$ of $P$, a Zariski open $V \subset Y$ with $\pi(U) \subset V$, and generators $f_1, \ldots, f_k$ of $I_V$ satisfying:

1. (Non-degeneracy) $c_{i,a}(Q) \neq 0$ for all $Q \in U$, where $f_i = \sum c_{i,a}x^a$ and $c_{i,a} \in A_V$.
2. (Tropical basis) $f_1, Q, \ldots, f_k, Q$ form a tropical basis of $I_Q$ for every $Q \in U$, i.e.,

$$\text{trop}(X_Q) = \bigcap_{i=1}^k \text{trop}(V(f_i)_Q).$$

A local tropical basis at $P$ is a valued field extension $K \to L$, a point $P_L$ lying over $P$, and an $L$-rational local tropical basis at $P_L$ for $X_L \to T_{Y,L}$.

**Lemma 3.4** Let $f_1, \ldots, f_k$ be a $K$-rational local tropical basis at $P$ and let $K \to L$ be a valued field extension. Then $f_1, \ldots, f_k$ are an $L$-rational local tropical basis at any $P_L$ mapping to $P$. 
Proof. The morphism $X^\text{an}_L \to X^\text{an}$ is continuous, so the neighborhood $U$ of $P$ gives an open neighborhood $U_L$ of $P_L$. The $f_i$ then still form a tropical basis on $U_L$ since tropicalizations are invariant under field extensions by [MS15, Theorem 3.2.4]. □

Definition 3.5 We say a closed embedding $X \to T_Y$ is tropically flat over $P \in Y^\text{an}$ if it admits a local tropical basis at $P$. The tropically flat locus is the set of all $P \in Y^\text{an}$ such that $X \to T_Y$ is tropically flat at $P$.

Example 3.6 If $X = V(f)$ for some $f \in A[x_1^\pm, \ldots, x_n^\pm]$, then the fiberwise tropicalizations of $f = \sum c_\alpha x^\alpha$, $c_\alpha \in A$, near any rational $P$ outside $\bigcup_\alpha V(c_\alpha)$ are locally constant, as we will see in Lemma 3.9. This in particular implies that $X \to T_Y$ is tropically flat outside $\bigcup_\alpha V(c_\alpha)$.

In general it is difficult to show that a given $X \to T_Y$ is not tropically flat at a point $P$ if $P$ is outside the non-degenerate locus. We refer the reader to Example 4.7 for an example of an embedding that is not tropically flat over a non-degenerate $P \in Y^\text{an}$.

We will see in Section 4 that if $X \to T_Y$ is tropically flat at $P$, then many properties of the fiberwise tropicalization $\text{trop}(X_P)$ become generic properties of $p: X \to Y$. To prove this, we will need an important fact about tropically flat embeddings, namely that they are locally constant after a non-archimedean base change.

Lemma 3.7 Let $g \in A$ and $P \in Y^\text{an}$ with $r = |g(P)| \neq 0$. There is a non-empty open neighborhood $U$ of $P_L$ in $Y^\text{an}_L$ and an element $c \in L$ with $|c| = r$ such that
\[ |g(Q) - c(Q)| < r \]
for all $Q$ in $U$.

Proof. Take any element $c \in L$ such that $g(P) = c$ and consider the open ball $U_L = B(g-c,r) = \{Q \in Y^\text{an}_L : |(g-c)(Q)| < r\}$. For any $P_L \in Y^\text{an}_L$ mapping to $P$, we then have $P_L \in U_L$ since $g(P_L) = c$. The inequality in the lemma is then satisfied by definition. Note that such a point $P_L$ exists by Remark 2.1. □

Remark 3.8 By applying the non-archimedean triangle inequality to $g$ as in Lemma 3.7, we find that $g$ is locally constant after a base change to $L$. Without the base change, this is false: tropicalizations will generally not be locally constant. Take for instance a complete non-trivially valued algebraically closed field $K$ and the affine line $Y = \text{Spec}(K[x])$ over $K$. The function $f = x$ is then not locally constant on $Y^\text{an}$. Indeed, $-\log|x|$ is linear with absolute slope 1 on the line segment emanating from $x = 0$. It will however be locally constant after a base change.

Lemma 3.9 Let $P \in Y^\text{an}$ be a rational point and let $f = \sum c_\alpha x^\alpha$ be a non-constant polynomial in $A[x_1^\pm, \ldots, x_n^\pm]$. If $c_\alpha(P) \neq 0$ for all $c_\alpha$, then the fiberwise tropicalizations of $V(f)$ are constant on a non-empty open neighborhood of $P$. 

Proof. We apply Lemma 3.7 to every $c_{\alpha}$ to find a set of open neighborhoods $U_{\alpha}$ of $P$. By the non-archimedean triangle inequality, we have that $|c_{\alpha}(Q)|$ is constant on $U_{\alpha}$. We thus see that the tropicalizations of $f$ are constant over the neighborhood $U = \bigcap U_{\alpha}$ of $P$. □

3.2. Generic tropical flatness theorem. In this section, we prove that a closed embedding $X \to T_Y$ is generically tropically flat if $Y$ is integral. The main idea is to use Gröbner bases over valued fields to show that a tropical basis over a generic valuation $P \in Y^{an}$ gives a local tropical basis for an open subset of $Y^{an}$. By an approximation argument, we then show that the union of these open subsets is dense in $Y^{an}$.

Assumption 3.10 Throughout this section, except in Theorem 3.22, we assume that $K$ is algebraically closed, non-trivially valued, and complete.

We start with two important results that allow us to approximate the behavior of points in $Y^{an}$ using rational points.

Lemma 3.11 Let $K$ be as in Assumption 3.10 and let $\Gamma'$ denote the image of the absolute value $|\cdot|: K \to \mathbb{R}_{\geq 0}$. Let $c_1, \ldots, c_m \in A$ and $P \in Y^{an}$ be such that $|c_i(P)| = r_i \in \Gamma'$. Then there is a rational point $Q$ of $Y$ such that $|c_i(Q)| = r_i$. If $V$ is a strictly $K$-affinoid subdomain of $Y^{an}$ and $P \in V$, then the same holds for a $K$-rational point of $V$.

Proof. We write $\Gamma \subset \mathbb{R}$ for the value group of $K$, so that $\Gamma = \{-\log(r) : r \in \Gamma' \setminus \{0\}\}$. We extend the $c_i$ to a generating set of $A$. This gives a closed embedding

$$\phi: Y \to \mathbb{A}^m.$$ 

Let $v \in (\Gamma \cup \{\infty\})^m$ be the tropicalization of $\phi^{an}(P)$. This lies in the tropicalization of $\phi(Y)$, so by [Gub13, Proposition 3.8] we can find a $K$-rational point of $\phi(Y)$ that tropicalizes to $v$. This also gives a $K$-rational point $Q$ of $Y$ and it has the desired properties by construction.

If $V$ is a strictly $K$-affinoid subdomain of $Y^{an}$, then [Ber90, Proposition 2.1.15] shows that the $K$-rational points are dense in $V$. The embedding above gives $V$ as a closed subset of $\mathbb{A}^{m,an}$. The arguments in [Gub13, Proposition 3.8] can now be repeated to obtain the desired statement. □

Corollary 3.12 Let $K$ be as in Assumption 3.10 and let $P$, $c_i$, $r_i$, $Q$ be as in Lemma 3.11. Suppose $r_i \neq 0$. Then there is a non-empty open neighborhood $U$ of $Q$ with $|c_i(Q')| = r_i$ for all $Q' \in U$. The same holds for a strictly $K$-affinoid subdomain of $Y^{an}$.

Proof. This follows from Lemma 3.7. □
Definition 3.13 A (generic) valuation on $Y$ is a point $P \in Y^{an}$ such that $\pi(P)$ is the generic point of $Y$.

For points $Q \in Y^{an}$ close to a valuation $P$ in the sense of Corollary 3.12, we will show that the fiberwise tropicalizations trop($V(I)_P$) and trop($V(I)_Q$) coincide. In particular, this implies that a tropical basis of trop($V(I)_P$) gives a local tropical basis around $Q$. We first recall several concepts from [MS15, Section 2.4].

Definition 3.14 Let $Q \in Y^{an}$ and $w \in \mathbb{R}^n$. A finite set $G_Q \subseteq I_Q$ is called a Gröbner basis with respect to $w$, if $G_Q$ generates $I_Q$ and \{in_w(g) : g \in G_Q\}$ generates in_w(I_Q). We define the Gröbner polyhedron of $I_Q$ around $w$ to be $C_w(I_Q) = \text{cl}(\{w' \in \mathbb{R}^n : \text{in}_w(I_Q) = \text{in}_w(I_Q)\})$, where cl(·) denotes the closure with respect to the Euclidean topology. We denote the Gröbner complex of $I_Q$ by $\Sigma(I_Q) = \{C_w(I_Q) : w \in \mathbb{R}^n\}$. If $I_Q$ is homogeneous, then $\Sigma(I_Q)$ is a finite polyhedral complex and trop($V(I_Q)$) is the support of a subcomplex of $\Sigma(I_Q)$.

Lemma 3.15 Let $K$ be as in Assumption 3.10 and let $I$ be a homogeneous ideal. Let $P \in Y^{an}$ be a valuation, and $w \in \mathbb{R}^n$ a weight vector. Then there is a Zariski-open $V \subseteq Y$, a $\mathcal{G} \subseteq AV[x_1, \ldots, x_n]$, a rational point $Q \in V^{an}$, and an open neighborhood $U \subseteq V^{an}$ of $Q$ such that

1. $\mathcal{G}_P$ is a Gröbner basis of $I_P$ with respect to $w$,
2. $\mathcal{G}_{Q'}$ is a Gröbner basis of $I_{Q'}$ with respect to $w$ for all $Q' \in U$,
3. the monomial supports of $\mathcal{G}_{Q'}$ and $\mathcal{G}_P$ are equal for all $Q' \in U$,
4. the valuations of the coefficients of $\mathcal{G}_{Q'}$ and $\mathcal{G}_P$ coincide for all $Q' \in U$.

Proof. Fix a monomial ordering $>$ and let $\mathcal{G}_P$ be a Gröbner basis of $I_P$ with respect to $w$ as computed by [CM19, Algorithm 2.9]. In particular, $\mathcal{G}_P$ will satisfy a Buchberger-type criterion: if NF and $S$ denote the normal form and $S$-polynomial from [CM19], then NF_{\mathcal{G}_P}(S(g_i, g_j)) = 0 for every pair of polynomials in $\mathcal{G}_P$. This implies that there exist $h_i$ such that

$$S(g_i, g_j) = \sum h_k g_k,$$

where $h_k g_k \geq S(g_i, g_j)$ in the sense of [CM19]. The polynomials $g_i$ and $h_i$ are defined over a Zariski-open subset of $Y$. Using Corollary 3.12, we find that there is a rational point $Q$ and an open neighborhood $U$ of $Q$ such that $h_i g_i \geq S(g_i, g_j)$ still holds for the order defined by $>, Q'$ and $w$ for $Q' \in U$. This implies that the normal form of $S(g_i, g_j)$ with respect to this order is zero. We then conclude using Buchberger’s criterion that $\mathcal{G}_{Q'}$ is a Gröbner basis with respect to $w$.

Lemma 3.16 Let $K$ be as in Assumption 3.10 and let $I$ be a homogeneous ideal. Let $P \in Y^{an}$ be a valuation. Then there is an open neighborhood $U$ of a rational point $Q$ such that for all $Q' \in U$, we have that

$$\Sigma(I_P) = \Sigma(I_{Q'}).$$
Proof. We will show that $C_w(I_P) = C_w(I_Q)$ for all $w \in \mathbb{R}^n$. Note that, by the proof of [MS15, Proposition 2.5.2], $C_w(I_P)$ is uniquely determined by the monomial support of a Gröbner basis with respect to $w$ and the valuation of its coefficients. For every $\sigma \in \Sigma(I_P)$, fix a $w_\sigma \in \text{Relint}(\sigma)$, and let $G_\sigma$ be a Gröbner basis of $I_P$ with respect to $w_\sigma$. By Lemma 3.15 there is an open neighborhood $U$ of a rational point $Q \in Y^{\text{an}}$ such that every $G_\sigma$ remains a Gröbner basis of $I_Q$ with respect to $w_\sigma$ and the valuations of the coefficients of with respect to $Q'$ and $P$ coincide. Hence $\sigma = C_{w_\sigma}(I_Q)$, and thus $\Sigma(I_P) = \Sigma(I_{Q'})$. \hfill \Box

For the upcoming proofs, we will assume that $P$ admits an affinoid neighborhood $V$ such that $P$ reduces to a generic point of the canonical reduction $\tilde{V}$. We will see in Lemma 3.21 that these points form a dense subset of $Y^{\text{an}}$.

**Lemma 3.17** Let $K$ be as in Assumption 3.10. Let $P \in Y^{\text{an}}$ be a valuation that admits a strictly $K$-affinoid neighborhood $V$ in $Y^{\text{an}}$ such that $P$ reduces to a generic point of the canonical reduction $\tilde{V}$ of $V$ and let $w \in \text{trop}(V(I_P))$. Then there is an open neighborhood $U$ of a rational point $Q$ such that $w \in \text{trop}(V(I_Q))$.

*Proof.* Since tropicalizations commute with homogenizations, we may assume that $I$ is homogeneous. Consider the initial degeneration of $I_P$ at $w$. This scheme over the residue field of $k(P)$ is cut out by the initial forms of a Gröbner basis $G_P$ of $I_P$ with respect to $w$. We assume without loss of generality that $w = 0$ and that the elements of $G_P$ are defined over $A$, so that $G \subset A[x_1, \ldots, x_n]$. Consider a strictly $K$-affinoid neighborhood $V$ of $P$ with canonical reduction $\tilde{V}$ as in the statement of the lemma. By assumption, the intersection of the initial degeneration at $w = 0$ over $P$ with the torus is non-empty. We interpret this in terms of models over $R$ as follows. By scaling the generators of $G_P$, we obtain a closed subscheme $X \to T_{R_P}$ over the valuation ring $R_P$ of $P$ such that the toric initial degeneration of $I_P$ at $0$ is the special fiber of $X$, see [Gub13, Definition 5.1]. Consider the coefficients $c_i$ of the $G_P$ and the polynomials used in Lemma 3.15. Note that for every polynomial in $G_P$, we have that the coefficients have absolute value $\leq 1$ with respect to $P$ by the scaling process mentioned above. We denote their absolute values at $P$ by $r_i$. We then define the Laurent domain

$$W = \{Q \in V : |c_i(Q)| = r_i\},$$

which contains $P$ by construction. We write $\mathcal{B}$ for the affinoid algebra corresponding to $W$ and $\mathcal{B}^0$ for the ring of power-bounded elements in $\mathcal{B}$. Note that the polynomials in $G$ are defined over $\mathcal{B}^0$, so $X$ can be defined over $\text{Spec}(\mathcal{B}^0)$. For every $Q \in W$, we obtain a canonical $R_Q$-valued point of $\text{Spec}(\mathcal{B}^0)$. By construction, since $G$ forms a Gröbner basis over $W$, the fiber of $X$ over the closed point of $\text{Spec}(R_Q)$ is the toric initial degeneration of $I_Q$ at $w$. Let $\tilde{P}$ denote the point of $\text{Spec}(\mathcal{B}^0) \times_R k$ induced by $P$. Since the fiber over $\tilde{P}$ of $X$ is non-empty, there is a Zariski-open neighborhood of $\text{Spec}(\mathcal{B}^0) \times_R k$ over which the fibers of $X$ are non-empty. We consider a basic open
$D(\hat{h})$ contained in this Zariski-open neighborhood and we lift $\hat{h}$ to $W$ to obtain $h$. By restricting to an open neighborhood of $Y$, we can assume that $h \in A$. Note that if $Q$ reduces to a point in $D(\hat{h})$, then the intersection of the initial degeneration of $I$ over $Q$ with the torus will be non-empty. We now apply Corollary \ref{cor:local} to $W$ and $h$ to find a rational $Q$ and an open neighborhood over which $w \in \text{trop}(V(I)_Q)$. □

**Lemma 3.18** Let $K$ be as in Assumption \ref{assump:valu}. Let $P \in Y^\text{an}$ be a valuation that admits a strictly $K$-affinoid neighborhood $V$ such that $P$ reduces to a generic point of the canonical reduction $\tilde{V}$ of $V$. Then there is an open neighborhood $U$ of a rational point $Q \in V$ such that

$$\text{trop}(V(I)_P) = \text{trop}(V(I)_Q).$$

**Proof.** As in the proof of the previous lemma, we may assume that $I$ is homogeneous. We will use Corollary \ref{cor:local} to create the desired rational point $Q$ and open neighborhood $U$. To that end, we have to give a finite set of polynomials $c_i$ in $A$ or a localization thereof. In the previous lemmas, we also created neighborhoods using Corollary \ref{cor:local}. To obtain the desired open neighborhood in this lemma, we will expand our list of polynomials $c_i$ using polynomials from the previous lemmas.

We first note that there are finitely many $C_w(I_P) \in \Sigma(I_P)$. Suppose that $C_w(I_P) \subseteq \text{trop}(V(I)_P)$. By Lemma \ref{lem:local}, we can find a set of polynomials $c_i$ such that for $Q'$ in the induced $U$, we have that $w \in \text{trop}(V(I)_Q)$. Similarly, if $C_w(I_P) \not\subseteq \text{trop}(V(I)_P)$, then there is an $f_{w,P} \in I_P$ whose initial with respect to any $w' \in \text{Relint}(C_w(I_P))$ is monomial. Using Corollary \ref{cor:local}, we can expand our list of polynomials $c_i$ so that $f_{w,Q}$ remains monomial for $Q$ in the induced $U$. We now expand our list of polynomials one more time using the polynomials from Lemma \ref{lem:local} to obtain the desired $U$. If $C_w(I_P) \not\subseteq \text{trop}(V(I)_P)$, then in$_{w}(f_{w,Q})$ is monomial. Hence, $w \notin \text{trop}(V(I)_Q)$. But then $C_w(I_P) = C_w(I_Q) \not\subseteq \text{trop}(V(I)_Q)$. If $C_w(I_P) \subseteq \text{trop}(V(I)_P)$, then by construction $w \in \text{trop}(V(I)_Q)$. We then have $C_w(I_P) = C_w(I_Q) \subset \text{trop}(V(I)_Q)$. We conclude that $\text{trop}(V(I)_P) = \text{trop}(V(I)_Q)$. □

**Proposition 3.19** Let $K$ be as in Assumption \ref{assump:valu}. Let $P \in Y^\text{an}$ be a valuation that admits a strictly $K$-affinoid neighborhood $V$ such that $P$ reduces to a generic point of the canonical reduction $\tilde{V}$ of $V$. Suppose that $f_1, \ldots, f_m$ give a tropical basis over $P$. Then there is a rational point $Q$ and an open neighborhood $U$ of $Q$ such that $f_1, \ldots, f_m$ form a tropical basis over $Q'$ for all $Q' \in U$.

**Proof.** We may assume as before that $I$ and the $f_1, \ldots, f_m$ are homogeneous. By Lemma \ref{lem:local}, the tropicalizations of the $f_i$ are locally constant near a rational point. The result then follows from Lemma \ref{lem:local}. □

**Corollary 3.20** Let $K$ be as in Assumption \ref{assump:valu}, let $P, V$ and $f_1, \ldots, f_m$ be as in Proposition \ref{prop:local} and let $h \in A$. Then the open neighborhood $U$ can be chosen to satisfy $|h(Q')| = |h(P)|$ for all $Q' \in U$. 
Assumption

Finally, by Corollary 3.23, we have \(|h(P)| = |h(Q)|\), so that \(Q \in \mathbb{B}(h, r_0, r_1)\). □

**Corollary 3.23** Let \(Y\) be an integral scheme of finite type over a non-archimedean field \(K\) as in Assumption 3.10. Then there is an open dense subset of \(Y(K) \subset Y^{an}\) over which \(X \to TY\) is tropically flat.
Proof. We consider the set of rational $Q$ from the proof of Theorem 3.22 over which we have a local tropical basis. By the arguments in the proof, this is a dense open subset. Note that the basis is moreover defined over $K$ so we in fact have a $K$-rational local tropical basis. □

4. TROPICAL INTERSECTIONS AND GENERIC ROOT COUNTS

We now use the material from the previous two sections to show how generic properties of morphisms of schemes can be detected using tropical geometry. We will see that many properties of a single tropical fiber over a tropically flat point propagate to an open dense subset of the parameter space. In Section 4.1, we prove Theorem 4.5, which expresses the generic root count as a tropical intersection product. This also gives a standalone proof of Bernstein’s theorem, see Corollary 4.11. In Section 4.2 we study torus-equivariant and algebraically independent systems, and we prove Proposition 4.19. Finally, we discuss extensions of the results given here to analytic families of polynomial equations.

4.1. Generic root counts as tropical intersection numbers. In this section, we show that generic root counts can be expressed in terms of tropical intersection numbers, provided that we have a tropically flat proper intersection. This extends Bernstein’s theorem to possibly overdetermined families of polynomial equations with non-trivial relations among the coefficients. An important tool in proving this is the following lemma.

Lemma 4.1 Let $X$ be a scheme that is locally of finite type over a complete non-archimedean field $K$ and let $U \subset X$ be a dense open set. Then $U^{an}$ is open and dense.

Proof. This follows from [Ber93, Proposition 2.6.4]. □

We now show that the generic dimension is reflected in the fiber over a tropically flat point.

Proposition 4.2 Let $Y$ be irreducible. Let $P \in Y^{an}$ be a point over which $X \to T_Y$ is tropically flat and suppose that the fiberwise tropicalization of $X$ over $P$ is of dimension $k$. Then there is an open dense $U \subset Y$ such that $X_U \to U$ has fibers of dimension $k$.

Proof. We first note that Krull dimensions are unaffected by base changes $K \to L$, so we can assume that $P$ is $K$-rational and that $K$ is complete. Let $k'$ be the dimension of the generic fiber of $X$. By [Stacks22, Lemma 05F7], there is a (non-empty) open subset $U$ over which the fiber dimension is $k'$. By Lemma 3.9, there is an open $V$ around $P \in Y^{an}$ such that the tropicalizations are all of dimension $k$. By Lemma 4.1, $V$ intersects $U^{an}$ so we conclude that $k' = k$ using the tropical structure theorem [MS15, Theorem 3.3.5]. □
Example 4.3 Let $X \to T_Y$ be a closed embedding with an empty generic fiber. If the fiber of $X$ over some $P \in Y^{\text{an}}$ is non-empty, then the embedding is not tropically flat over $P$ by Proposition 4.2. More generally, if the dimension of the fiberwise tropicalization $\text{trop}(X_P)$ is higher than the generic dimension, then $X \to T_Y$ is not tropically flat over $P$.

Lemma 4.4 Let $X_1, \ldots, X_k$ be closed subschemes of complementary codimension in a torus $T_K$ over a field $K$, i.e., $\sum_{i=1}^{k} \text{codim}(X_i) = \text{dim}(T_K)$, and suppose that the $X_i$ intersect properly in a zero-dimensional set. Suppose that $\bigcap_{i=1}^{k} X_i$ lies in the Cohen-Macaulay-locus of every $X_i$. Then the higher intersection multiplicities vanish.

Proof. We will show that the higher intersection multiplicities for the $k$-fold diagonal $\Delta^k \subset T^k_K$ and $\prod_{i=1}^{k} X_i$ vanish. The support of the intersection corresponds to $\bigcap_{i=1}^{k} X_i$ embedded in the product by the map $D_T : T \to T^k$. Note that $\prod_{i=1}^{k} X_i$ is Cohen-Macaulay at every $D_T(z)$ for $z \in \bigcap_{i=1}^{k} X_i$. Indeed, this follows from [Stacks22, Lemma 0C0W] and [Stacks22, Lemma 045T(1)]. The lemma now follows from [Stacks22, Lemma 0B02].

Theorem 4.5 Let $Y = \text{Spec}(A)$ be an integral scheme that is of finite type over a non-archimedean field $K$, let $p : X \to Y$ be an affine morphism of finite type with Berkovich analytification $p^{\text{an}} : X^{\text{an}} \to Y^{\text{an}}$. Suppose that the following hold:

(1) The morphism $p$ factors through a closed embedding $X \to T_Y$ of $Y$-schemes, where $T_Y$ is an $n$-dimensional torus over $Y$.

(2) The closed subscheme $X \to T_Y$ is an intersection of closed subschemes $X_i \to T_Y$, $i = 1, \ldots, k$, that are generically Cohen-Macaulay and pure of relative dimension $k_i$ over $Y$ with

$$\sum_{i=1}^{k} (n - k_i) = n.$$

(3) There is a $P \in Y^{\text{an}}$ over which the $X_i$ are tropically flat and the tropical prevariety $\bigcap_{i=1}^{k} \text{trop}(X_i, P)$ is bounded.

Then $X \to Y$ is generically finite and the generic root count $\ell_{X/Y}$ is the tropical intersection number $\prod_{i=1}^{k} \text{trop}(X_i, P)$.

Proof. It suffices to prove the theorem after a base change $K \to L$, where $L$ is a valued field extension of $K$. Indeed, the local rank is stable under base change since the rank of a free module is stable under base change and being generically Cohen-Macaulay is stable under field extensions by [Stacks22, Lemma 00RJ] or Lemma 2.5. We can thus suppose that $K$ is complete and that $P$ is $K$-rational.

Since the $W_i$ are tropically flat over $P$, we can find a set of generators that form a tropical basis on an open neighborhood $B_1$ of $P$. By Lemma 3.9, there is an
open neighborhood $B_2$ of $P$ over which the tropicalizations of these generators are constant. Let $U_1 \subset Y$ be a dense open subset over which $p$ is locally free and let $U_2$ be a dense open subset over which $p$ is Cohen-Macaulay. Since $U_1^{an} \cap U_2^{an}$ is dense in $Y^{an}$ by Lemma 4.1, it intersects $B_1 \cap B_2$ in a point $Q$. As the the tropical prevariety is bounded, the fibers $X_{i,Q}$ are pure and their codimensions add up to $n$, we can apply [OR13, Corollary 6.13] to find that the tropical intersection number is equal to the sum of the algebraic intersection numbers. By Lemma 4.4, this sum is equal to the sum of the algebraic lengths. But again using the fact that free modules are stable under base change, we find that this sum is the local rank provided by Grothendieck’s theorem. This concludes the proof. □

Remark 4.6 The tropical intersection number in Theorem 4.5 is an algebraic intersection number in a suitable toric variety. Indeed, let $X(\Delta)$ be a toric variety such that $\Delta$ is a compatible compactifying fan for the trop$(X_{i,P})$ as in [OR13, Section 3]. Then by [OR13, Proposition 3.12], we find that the closures of the $X_{i,P}$ in $X(\Delta)$ only intersect in the dense torus. In particular, the tropical intersection number in Theorem 4.5 is equal to the algebraic intersection number $\prod_{i=1}^k X_{i,P}$.

The following examples shows the necessity of tropical flatness in Theorem 4.5:

Example 4.7 Let $K = \mathbb{C}\{\{t\}\}$ be the field of Puiseux series over $\mathbb{C}$ and let $Y = \text{Spec}(A)$ for $A = K[a]$. Consider the two subschemes $X_1$ and $X_2$ of the three-dimensional torus $T_Y = \text{Spec}(A[x^\pm, y^\pm, z^\pm])$ over $Y$ given by the ideals

$I_1 = (ax^2 + y^3 + x + 2y + 3 + az, x^2 + y^2 + 3x - y + 5)$ and $I_2 = (z - at)$.

We write $P$ for the point in $Y^{an}$ corresponding to the ring homomorphism $A \to K$ sending $a$ to 1. The fiberwise tropicalization of $X_1 = V(I_1)$ over $P$ is a tropical quartic curve in $\mathbb{R}^3$ and trop$(X_{2,P})$ is an affine hyperplane. These intersect transversally in one point of multiplicity two. The generic root count is however 4, so $X_1$ cannot be tropically flat at $P$. To illustrate this change in root counts, consider the points $P_\lambda$ given by sending $a \mapsto 1 + t^\lambda$ for $\lambda \in \mathbb{Q}$ large. The corresponding fiberwise tropicalizations can be found in Figure 1. We have that trop$(X_{1,P})$ and trop$(X_{2,P})$ intersect in two double points, one of which diverges as $\lambda \to \infty$.

By combining Theorems 3.22 and 4.5, we obtain the following.

Corollary 4.8 Suppose that $Y$, $X$ and the $X_i$ are as in Theorem 4.5. Suppose that $\bigcap_{i=1}^k \text{trop}(X_{i,P})$ is bounded for $P$ in a non-empty open subset of $Y^{an}$. Then there is a non-empty open subset $U$ of $Y^{an}$ such that the generic root count $\ell_{X/Y}$ is $\prod_{i=1}^k \text{trop}(X_{i,Q})$ for $Q \in U$.

Corollary 4.9 Suppose that $\ell_{X/Y} \neq \prod_{i=1}^k \text{trop}(X_{i,Q})$ for $Q$ in an open dense subset of $Y^{an}$. Then there is a dense subset $W$ of $Y^{an}$ such that the tropical prevariety $\bigcap_{i=1}^k \text{trop}(X_{i,Q})$ is unbounded for $Q \in W$. 

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**Note:** The text above is a continuation of the mathematical content from the previous page, focusing on tropical geometry and algebraic varieties, including detailed proofs, remarks, examples, and corollaries. The content is structured to provide a coherent narrative on tropical intersection numbers and their algebraic counterparts, illustrating the necessity of tropical flatness in certain cases and presenting examples and corollaries to support the theoretical developments.
We now explore the ramifications of Theorem 4.5 for square decompositions.

Corollary 4.10 Suppose that the $X_i$ in Theorem 4.5 are hypersurfaces $V(f_i)$ given by polynomials $f_i = \sum_{\alpha} c_{i,\alpha} x^\alpha$ and that the tropical prevariety $\bigcap_{i=1}^n \text{trop}(V(f_i)_P)$ is bounded for some $P \in Y^{\text{an}} \setminus \bigcup_{i,\alpha} V(c_{i,\alpha})$. Then the generic root count of $p: X \to Y$ is the normalized mixed volume $\text{MV}(f_1,\ldots,f_n)$.

Proof. The $V(f_i)$ are tropically flat, generically Cohen-Macaulay and pure of relative dimension $n - 1$ over the given locus. We can thus use Theorem 4.5 to conclude that the generic root count is the tropical intersection number. By [MS15, Theorem 4.6.8], this is the mixed volume, so we obtain the statement of the corollary. \hfill \Box

Corollary 4.11 (Bernstein-Kushnirenko) Let $\phi: X \to T_Y$ be the closed embedding associated to the universal family of square polynomial systems with fixed monomial supports $M_i$. Then the generic root count $\ell_{X/Y}$ is $\text{MV}(f_1,\ldots,f_n)$.

Proof. One easily finds a point $P \in Y^{\text{an}}$ for which the tropicalizations intersect in finitely many points (using for instance Lemma 4.16). The statement then follows from Corollary 4.10. \hfill \Box

For square systems, Corollary 4.9 can be simplified to give the following.

Corollary 4.12 Suppose that a square system over $Y$ given by polynomials $f_i = \sum_{\alpha} c_{i,\alpha} x^\alpha$ has nonzero generic root count less than the mixed volume. Then the fiberwise tropical prevariety over every $P \in Y^{\text{an}} \setminus \bigcup_{i,\alpha} V(c_{i,\alpha})$ is unbounded.
4.2. Torus-equivariant systems and generic root counts. In this section, we introduce the notion of torus-equivariance, which gives a natural condition under which Theorem 4.5 holds. It will be used in Sections 5 and 6 to obtain generic root counts for certain classes of systems.

The relative torus is a group scheme with group law $T_Y \times T_Y \rightarrow T_Y$.

Let $L$ be a field over $K$ and let $\lambda \in T_Y(L)$ be an $L$-valued point. We denote the image of $\lambda$ in $Y(L)$ by $P$. The fiber of $T_Y$ over $P$ gives a torus $T_P$ defined over $L$ and by multiplying by $\lambda$, we obtain an isomorphism $\lambda: T_P \rightarrow T_P$. Let $\phi: X \rightarrow T_Y$ be a closed immersion. The fiber of $\phi: X \rightarrow T_Y$ over $P$ gives a closed immersion $\phi_P: X_P \rightarrow T_P$. We denote the image of $X_P$ under $\lambda \circ \phi_P$ by $\lambda \cdot X_P$. We use the same notation for points $\lambda \in \mathcal{T}_{Y}$. If we have an action $T_Y \times Y \rightarrow Y$, then we denote $L$-valued points of $T_Y \times Y$ by $(\lambda, P)$. The image of such a point under the group action is denoted by $\lambda \cdot P$.

Definition 4.13 We say that $\phi: X \rightarrow T_Y$ is torus-equivariant if there exists an action $T_Y \times Y \rightarrow Y$ such that for every field $L$ and $(\lambda, P) \in (T_Y \times Y)(L)$, we have $X_{\lambda \cdot P} = \lambda \cdot X_P$. Here the tori over two $L$-valued points of $Y$ are canonically identified.

Example 4.14 Consider the parameter ring $A = K[a_0, a_1, a_2]$ with polynomial system defined by $f = a_0 x + a_1 xy + a_2 y$. We define an action of $T_Y = \text{Spec}(A[\pm x, \pm y])$ on $Y = \text{Spec}(A)$ by the Hopf-algebra maps

$$a_0 \mapsto x^{-1} a_0, \quad a_1 \mapsto x^{-1} y^{-1} a_1, \quad a_2 \mapsto y^{-1} a_2.$$ 

Here we used the identification $A[\pm x, \pm y] \otimes_A A = A[\pm x, \pm y]$. It is easy to see that $A \rightarrow A[\pm x, \pm y]/(f)$ is a torus-equivariant system with respect to this action. In a sense, if the monomials in a polynomial contain free variables, then these can be used to define a torus-equivariant system.

Remark 4.15 We will need the torus-equivariance to eliminate the non-Cohen-Macaulay loci in Proposition 4.19. For systems that are automatically Cohen-Macaulay everywhere (e.g., square systems), we can make do with the weaker assumption of the tropicalizations being torus-equivariant, i.e., if for every $P \in Y_{an}$ and every $\lambda \in T_{\mathbb{A}}^\mathbb{A}$ mapping to $P$, there is a $Q \in Y_{an}$ such that $\text{trop}(X_Q) = \text{trop}(\lambda \cdot X_P) = \text{trop}(\lambda) + \text{trop}(X_P)$.

To prove Proposition 4.19, we will need the following lemma from tropical geometry, which shows that there are sufficiently many translations that give the stable intersection of two balanced polyhedral complexes in $\mathbb{R}^n$. A similar result also holds for $k$ polyhedral complexes.
Lemma 4.16 Let $\Sigma_1$ and $\Sigma_2$ be two balanced polyhedral complexes in $\mathbb{R}^n$. There is a dense open subset $U \subset \mathbb{R}^n \times \mathbb{R}^n$ such that for all $(\lambda_1, \lambda_2) \in U$, we have that $\lambda_1 + \Sigma_1$ and $\lambda_2 + \Sigma_2$ intersect transversally. This set moreover contains $\{0\} \times U_1$ and $U_2 \times \{0\}$ for suitable dense open subsets $U_i$ of $\mathbb{R}^n$.

Proof. This follows from the proof of [MS15, Proposition 3.6.12]. □

Lemma 4.17 Let $X_1, \ldots, X_k$ be closed pure subschemes of an $n$-dimensional torus $T_n$ over a non-archimedean field $K$ with

$$\sum_{i=1}^k \text{codim}(X_i) = \dim(T_n) = n.$$ 

There is a non-empty open subset $U \subset T_{an}^{n(k-1)}$ such that for $t = (t_2, \ldots, t_k) \in U$, we have that $X_1, t_2 X_2, \ldots, t_k X_k$ meet properly in finitely many points. Furthermore, for $t_1 := 1$ and $t = (t_2, \ldots, t_k) \in U$ the intersection $\bigcap_{i=1}^k t_i X_i$ lies in the Cohen-Macaulay locus of each $t_i X_i$ and the tropicalizations of the $t_i X_i$ meet transversally and properly.

Proof. Let $Z_i$ be the non-Cohen-Macaulay locus of $X_i$, which is proper closed subsets of the $X_i$ by [Stacks22, Lemma 00RG]. Consider the stable intersection of the tropicalizations of $X_1, X_2, \ldots, X_{k-1}$ and $Z_k$. By [MS15, Theorem 3.6.10] and the assumption on the codimensions, this is empty. The other $k - 1$ combinations also give empty stable intersections. For each of these, we obtain a dense open subset in $\mathbb{R}^{n(k-1)}$ for which the tropicalizations do not intersect by Lemma 4.16. We intersect these dense open subsets to obtain a dense open subset $V$. We can moreover shrink $V$ so that the trop($t_i X_i$) intersect properly and transversally everywhere. We then immediately find that $U = \text{trop}^{-1}(V)$ has the desired properties. □

Definition 4.18 Let $X_i$ be a decomposition of a closed subscheme $X$ of $T_Y$. We say this decomposition is algebraically independent if there exist parameter rings $A_i$ over $K$ with spectra $Y_i$ and closed subschemes $X'_i \to T_{Y_i}$ such that

1. $\prod_{i=1}^k Y_i = Y$ and
2. $X'_i \times_{Y_i} Y = X_i$.

We will identify $X'_i$ with $X_i$.

Proposition 4.19 Suppose the $k$ subschemes $X_i \to Y$ in Theorem 4.5 are algebraically independent, generically pure of complementary dimension, and that at least $k - 1$ families are torus-equivariant. Then the conclusion of Theorem 4.5 holds for $P$ in an open dense subset of $Y^{an}$.

Proof. For every $X_i$, we take a set of generators $f_{i,j}$ of the corresponding ideal. We write $U_0$ for the open subset of $Y$ over which the coefficients of the monomials of the $f_{i,j}$ are non-zero. Let $U_1 \subset Y$ be the open subset provided by Grothendieck’s generic freeness theorem and let $U_2$ be the open dense subset over which the $W_i$ are
pure. Let $U = U_0 \cap U_1 \cap U_2$ and $Q \in U^{an}$. We can assume that $Q$ is rational. Since the family is algebraically independent, we can find points $Q_i \in Y_i^{an}$ that give rise to $Q$. We write $X_{i,Q}$ for the fibers of the families and $Z_{i,Q}$ for their non-CM-loci. By Lemma 4.17, there is an open subset $V$ of $(t_1, ..., t_k) \in T_{K,nk}^{an}$ such that the $t_iX_{i,Q}$ meet in the CM-locus of each. Moreover, their tropicalizations meet transversally in finitely many points. Consider the torus-action

$$T_{Y_i,n} \times_{Y_i} Y_i \to Y_i$$

for each $i$. Applying the projection maps $Y \to Y_i$ to the open dense subset $U$, we obtain open dense subsets $U_i \subset Y_i$. Consider the map $T_{K,n} \to T_{Y_i,n} \times_{Y_i} Y_i \to Y_i$ induced by the point $\pi(Q_i)$ and the group action on $Y_i$. The inverse image of $U_i$ under this map is open and non-empty, and thus dense. Its analytification thus intersects the projection $V_i$ of $V$. By doing this for all $i$, we obtain a new point $Q$ such that the $X_{i,Q}$ meet in the CM-locus of each. We then conclude using Lemma 4.4.

**Remark 4.20** To find a generic point as in Theorem 4.19, we can use a point $P$ over which the $X_i \to Y_i$ are tropically flat. Indeed, the given open neighborhood $V$ of $P$ will then intersect the open set $U^{an}$, so we can use the tropicalizations of the given fibers over $P$.

We can use this to prove an invariance of tropical intersection numbers in torus-equivariant algebraically independent systems.

**Corollary 4.21** Consider a decomposition $X_i$ of an embedding $X \to T_Y$ as in Proposition 4.19. Then for a generic pair of points $P, Q \in Y^{an}$, we have

$$\prod \text{trop}(X_i,P) = \prod \text{trop}(X_i,Q).$$

**Remark 4.22** We note here that the results in Section 4 can be generalized to analytic families of polynomial equations. For many applications in practice this is important, since the functions in the parameters naturally contain analytic functions. For instance, one can consider the system

$$f_1 = \sin(a_1 + a_2)x^2 + \sin(a_1a_2)y^2 + \cos(a_1)x + a_4y + a_5,$n$$

$$f_2 = \cos(b_1)x^2 + \cos(b_1 + b_2)y^2 + b_3x + b_4y + b_5$$

over the ring $A = \mathbb{C}[[a_i, b_i]]$, where $\mathbb{C}$ is trivially valued. This is a $\mathbb{C}$-affinoid domain, so the material in [Ber93, Section 2] is again applicable. Note that the coefficients of the monomials in these types of equations can satisfy algebraic relations that are not always apparent. The tropical material presented here is however directly applicable, and we can find a ring homomorphism $A \to \mathbb{C}[[t]]$ with corresponding point $P$ such that the fiberwise tropical prevariety over $P$ is finite. This then implies that the generic root count is 4. More generally, one can consider polynomial equations...
defined over affinoid $K$-algebras $A$, where $K$ is any (complete) non-archimedean field. The material in this paper directly extends to this more general scenario.

5. Generic root counts of square systems

In this section, we focus on general square systems. In Section 5.1 we introduce the degeneracy graph of a square system, which characterizes all its finite degenerations. In Section 5.2 we study a special class of square systems for which we can express the generic root count in terms of tropical intersection numbers. This includes the steady-state equations of chemical reaction networks and the birational intersection indices $[L_1, ..., L_n]$ studied by Kaveh and Khovanskii in [KK10][KK12]. These applications will be studied in more detail in Section 6.

5.1. The degeneracy graph of a square system. Let $p : X \to Y$ be a generically finite square polynomial system given by polynomials $f_1, \ldots, f_n$ in $A[x_1, \ldots, x_n]$. We start with a basic lemma on the quasi-finite locus $QF(p) \subset Y$ for the morphism $p : X \to Y$.

**Lemma 5.1** $QF(p)$ is open.

**Proof.** Since $p : X \to Y$ is square, it is flat at every point $x$ lying over $y \in QF(p)$ by [Stacks22, Lemma 00ST] and [Stacks22, Lemma 00SW]. The inverse image of $QF(p)$ in $X$ is thus in the open flat locus $U$. Consider the restriction $p_u$ of $p$ to $U$, which is a flat morphism. By [Stacks22, Lemma 02NM], the locus of relative dimension zero of $p_u$ is open in $U$. The image of this locus under $p_u$ is exactly $QF(p)$. Since flat morphisms are open, we conclude that $QF(p)$ is open. $\square$

For every $y \in QF(p)$ with prime ideal $p_y$, we have that the induced system

$$\text{Spec}(B/p_yB) \to \text{Spec}(A/p_y)$$

is generically finite. We denote the generic root count of this system by $\ell_{X/Y,y}$. If $y$ is the generic point $\eta$ of $Y$, this gives the generic root count of $p$. We now show that root counts decrease under specialization, so that $\ell_{X/Y,y} \leq \ell_{X/Y}$ for every $y \in QF(p)$.

**Lemma 5.2** Let $y \in QF(p)$ with prime ideal $p_y$ and induced system $p_y : \text{Spec}(B/p_yB) \to \text{Spec}(A/p_y)$. Then $\ell_{X/Y,y} \leq \ell_{X/Y}$.

**Proof.** Let $D(h)$ be an open dense subset of $\text{Spec}(A)$ over which the root count is $\ell_{X/Y}$. By [Gub13, Proposition 4.14], the reduction map $\pi : D(h)^{an} \cap Y^{an,0} \to Y$ is surjective. We take a point $Q$ of $Y$ over which the root count is $\ell_{X/Y,y}$. By the above, we can then find a valuation ring $R$ with residue field $k$ and a point $P \in Y(R)$ such that $P_\eta$ lies in $D(h)(K)$ and $P_\eta = Q$. Here $P_\eta$ is the composition of $P : \text{Spec}(R) \to Y$ and $\text{Spec}(K) \to \text{Spec}(R)$, and $P_\eta$ is the composition of $P$ and $\text{Spec}(k) \to \text{Spec}(R)$. We can assume that $R$ is Henselian.
We consider the base change \( R \to B \otimes_A R =: M \). This is quasi-finite by construction. It is moreover flat by the considerations in Lemma 5.1. Since \( R \) is Henselian, we can write

\[
M = M_{\text{fin}} \times M_{\text{nfin}},
\]

where \( M_{\text{fin}} \) is finite and \( M_{\text{nfin}} \otimes_R k = (0) \). Moreover, \( M_{\text{fin}} \) is flat over \( R \), so it is free. We have \( \ell_{X/Y} = \text{rank}(M_\eta) = \text{rank}(M_{\text{fin},\eta}) + \text{rank}(M_{\text{nfin},\eta}) \) and \( \text{rank}(M_s) = \text{rank}(M_{\text{fin},s}) \). But \( \ell_{X/Y,y} = \text{rank}(M_{\text{fin},s}) = \text{rank}(M_{\text{fin},\eta}) \), so we obtain the statement of the lemma.

\[\square\]

**Remark 5.3** In general, if a system does not admit a square decomposition, then root counts need not decrease under specialization. We give two examples of this phenomenon. Consider the scheme \( Y = \text{Spec}(\mathbb{C}[x,y]/(y^2 - x^3 - x^2)) \) with normalization \( X \to Y \). Its generic root count is one, however its root count over the maximal ideal \((x,y)\) is two. This can be easily seen using the presentation \( X = \text{Spec}(\mathbb{C}[x,y,z]/(y^2 - x^3 - x^2, xz - y, x^2 - x - 1)) \).

For our second example, consider the scheme \( X_t = V(f_1, \ldots, f_6) \) over \( Y = \text{Spec}(\mathbb{C}[t]) \) given by

\[
\begin{align*}
  f_1 &= (x-t)(z-t), & f_3 &= yz, & f_5 &= x - z, \\
  f_2 &= (x-t)w, & f_4 &= yw, & f_6 &= y - w.
\end{align*}
\]

The generic root count of \( X_t \to Y \) is 2, but the root count of the fiber \( X_0 \) over \( t = 0 \) is 3. We thus see that generic root counts can increase under specialization. Note that in both cases, we necessarily have that the families are not flat, since otherwise the proof in Lemma 5.2 would have gone through verbatim.

By Lemma 5.2 and [Stacks22, Lemma 07RY], there are reduced closed subschemes

\[
\emptyset = Z_{-1} \subset Z_0 \subset Z_1 \subset \cdots \subset Z_{\ell_{X/Y}} = \text{QF}(p)
\]

such that the locus of rank \( i \) is \( Z_i \setminus Z_{i-1} \). Every \( Z_i \) has finitely many generic points which we denote by \( y_{i,j} \). These in turn correspond to minimal prime ideals \( p_{y_{i,j}} \). By abuse of notation, we write \( y \subset z \) for points \( y, z \) in \( Y \) if \( p_y \subset p_z \), so that \( z \) is a specialization of \( y \).

**Definition 5.4** The degeneracy graph of \( p: \text{Spec}(B) \to \text{Spec}(A) \) is the graph with vertices \( y_{i,j} \) and edges \( (y_{i,j}, y_{i-1,k}) \) for \( y_{i,j} \subset y_{i-1,k} \). For every \( y \in \text{QF}(p) \), there exists a \( y_{i,j} \) such that \( \ell_{X/Y,y} = \ell_{X/Y,y_{i,j}} \) and \( y_{i,j} \subset y \). We will call such a point \( y \) a witness of \( y_{i,j} \).

**Remark 5.5** Consider the universal family \( p: X \to Y \) of square polynomial systems with fixed monomial supports \( M_i \). By Corollary 4.11, the generic root count is the corresponding mixed volume. Suppose that we take a subset \( M'_i \subset M_i \) of every monomial support, giving a new universal family \( X' \to Y' \). The degeneracy graph of \( X' \to Y' \) is then naturally a minor of the degeneracy graph of \( X \to Y \).
More generally, the degeneracy graph associated to any irreducible closed subscheme $Y' \subset Y$ with generic point in the quasi-finite locus will give a minor of the degeneracy graph of $p: X \to Y$.

5.2. Modifications of a square system. In this section, we discuss a general strategy for expressing the generic root count of a square system $X \to Y$ as a tropical intersection number. This is done using appropriate re-embeddings, also referred to as tropical modifications [Kal15]. Similar to the works of Kaveh and Khovanskii [KK10; KK12], we may need to restrict to an open dense subset $U$ of the relative torus $T_Y$ and consider the generic root count of $X_U := X \cap U \to Y$ instead of $X \to Y$.

Definition 5.6 Let $f_1, \ldots, f_n \in A[x_1^\pm, \ldots, x_n^\pm]$. Choose $p_{i,j} \in A$ and $q_j \in K[x_1^\pm, \ldots, x_n^\pm]$ such that

$$f_i = \sum_{j=1}^m p_{i,j} \cdot q_j. \tag{2}$$

Note that some $p_{i,j}$ may be zero and the $q_j$ are not necessarily pairwise distinct.

We write $T_Y = \text{Spec}(A[x_1^\pm, \ldots, x_n^\pm])$ for the torus and $X = V(f_1, \ldots, f_n)$. For any open subset $U$ of $T_Y$, we write $X_U = X \cap U$, which we again consider as a $Y$-scheme. Let

$$\hat{C} := A[x_i^\pm, w_j^\pm | 1 \leq i \leq n, 1 \leq j \leq m],$$

and

$$\hat{f}_i := \sum_{j=1}^m p_{i,j} w_j \quad \text{for } 1 \leq i \leq n,$$

$$\hat{h}_j := w_j - q_j \quad \text{for } 1 \leq j \leq m.$$ 

Here the latter two are polynomials in $\hat{C}$. Set $\hat{B} := \hat{C}/(\hat{f}_i, \hat{h}_j | 1 \leq i \leq n, 1 \leq j \leq m)$ and $\hat{X} := \text{Spec}(\hat{B})$. The scheme $\hat{X}$ admits a natural embedding into the relative torus $\hat{T}_Y := \text{Spec}(\hat{C})$ over $Y = \text{Spec}(A)$, and the two ideals $\hat{I}_{\text{lin}} = (\hat{f}_i | 1 \leq i \leq n)$ and $\hat{I}_{\text{nl}} = (\hat{h}_j | 1 \leq j \leq m)$ give a decomposition of $\hat{X} \to \hat{T}_Y$. We write $\hat{X}_{\text{lin}} = V(\hat{I}_{\text{lin}})$ and $\hat{X}_{\text{nl}} = V(\hat{I}_{\text{nl}})$. Note that $\hat{X}_{\text{lin}}$ is linear, and that $\hat{X}_{\text{nl}}$ is a constant scheme over $Y$, which makes $\hat{X}_{\text{lin}}$ and $\hat{X}_{\text{nl}}$ algebraically independent. We refer to $\hat{X} \to Y$ as the modification of $X \to Y$ derived from the representation in Equation (2).

Note that the modification of $X \to Y$ depends on the choice of $p_{i,j}$ and $q_j$. If the modification satisfies the conditions of Proposition 4.19, then we immediately obtain a formula for the root count:

Corollary 5.7 Suppose that $\hat{X}_{\text{lin}}$ is torus-equivariant and of generic codimension $n$. Then $\hat{X} \to Y$ is generically finite and $\ell_{\hat{X}/Y} = \text{trop}(\hat{X}_{\text{lin}, P}) \cdot \text{trop}(\hat{X}_{\text{nl}, P})$ for $P$ in an open dense subset of $Y^{an}$. 
Proof. We only have to verify that \( \hat{X}_{\text{lin}} \) is generically pure and of the right codimension. This follows from the fact that \( \hat{X}_{\text{lin}} \) is isomorphic to an open subset of the \( n \)-dimensional torus over \( Y \). □

We now focus on three aspects of Corollary 5.7 which will serve as guides for the remainder of this section:

(a) The assumption that \( \hat{X}_{\text{lin}} \) is of generic codimension \( n \).

(b) The assumption that \( \hat{X}_{\text{lin}} \) is torus-equivariant.

(c) The fact that Corollary 5.7 gives a formula for \( \ell_{\hat{X}/Y} \) and not \( \ell_{X/Y} \).

In general, Assumptions (a) and (b) need not be satisfied for a representation of the \( f \) as in Equation (2). However, Lemma 5.8 shows that (a) is satisfied in all cases of interest and Lemma 5.9 shows that (b) is guaranteed by a set of natural algebraic conditions on the \( p_{i,j} \). For (c), we will see that \( \ell_{X_U/Y} = \ell_{\hat{X}/Y} \) for an open dense subset \( U \subset T_Y \). This open subset also plays an important role in [KK10; KK12].

**Lemma 5.8** Suppose that \( X_U \to Y \) is generically finite for some open dense \( U \). Then \( \hat{X}_{\text{lin}} \to Y \) is generically a complete intersection of generic codimension \( n \).

**Proof.** Suppose that \( \hat{X}_{\text{lin}} \to Y \) is not generically a complete intersection of generic codimension \( n \). Then there is a linear relation over the function field of \( Y \) among the \( \hat{f}_i \). But this implies that there is a linear relation among the \( f_i \), contradicting the fact that \( X_U \to Y \) is generically finite. □

**Lemma 5.9** Let \( Y = \text{Spec}(A) \), where \( A = K[a_1, ..., a_m] \). Suppose that the \( p_{i,j} \) from Definition 5.6 are linear and homogeneous. Suppose that there exist subrings \( A_j \subset A \) generated by linear polynomials in \( A \) with \( \otimes K A_j = A \) and \( p_{i,j} \in A_j \). Then \( \hat{X}_{\text{lin}} \) is torus-equivariant.

**Proof.** To define the action, we consider a set of independent linear generators \( z_{j,k} \) of \( A_j \). The torus action sends a vector \( (w_i) \) to \( (\lambda_i w_i) \). We then send the \( z_{j,k} \) to \( z_{j,k}/\lambda_k \). One easily checks that \( \hat{X}_{\text{lin}} \) is torus-equivariant with respect to this action. □

**Assumption 5.10** If \( \hat{X}_{\text{lin}} \) is torus-equivariant and of generic codimension \( n \), then we say that \( X \to Y \) is **tropically modifiable**. We will assume for the remainder of this section that \( X \to Y \) is tropically modifiable.

We now address point (c). As with the previous two, there is no guarantee that \( \ell_{\hat{X}/Y} = \ell_{X/Y} \). Instead, we will see that generic root count of the modification \( \hat{X} \) is the generic root count of \( X_U \) for an open dense \( U \) in \( T_Y \). In the following the we will show what \( U \) is, why \( U \) is necessary, and when \( X_U = X \).

**Theorem 5.11** Suppose that \( X \to Y \) is tropically modifiable as in Assumption 5.10 and let \( U = \cap_{i=1}^m D(q_i) \). Then \( X_U \to Y \) is generically finite and \( \ell_{X_U/Y} = \ell_{\hat{X}/Y} = \text{trop}(\hat{X}_{\text{lin},P}) \cdot \text{trop}(\hat{X}_{\text{lin},P}) \) for \( P \) in an open dense subset of \( Y^{\text{an}} \).
Proof. Note that the solutions of $\hat{X} \to Y$ correspond exactly to solutions of $X_U \to Y$ since the $q_j$ are nonzero. Moreover, the multiplicities are preserved by the linear nature of the modification, hence the generic root counts coincide. □

Example 5.12 Note that without passing to $X_U$, Theorem 5.11 is not true in general. For instance, consider the system
\[ f_1 = a_1(x - 1) + a_2(y - 1) \quad \text{and} \quad f_2 = a_3(x - 1) + a_4(y - 1) \]
over $Y = \text{Spec}(A)$ with $A = K[a_1, a_2, a_3, a_4]$. The unique solution for every non-zero choice of parameters of this system is $(1, 1)$. The tropical intersection number of the modification is however 0, which gives the number of solutions with $x \neq 1$ or $y \neq 1$.

We would now like to extend the open subset from Theorem 5.11 so that the tropical intersection number derived from $\hat{X}_{\text{lin}}$ and $\hat{X}_{\text{nilin}}$ still gives a valid generic root count. We start with a basic tool in extending this root count.

Lemma 5.13 Suppose that $U_i \supset U_0 = \bigcap_{j=1}^m D(q_j)$ are open subsets of $T_Y$ such that the $X_{U_i} \to Y$ are generically finite with $\ell_{X_{U_i}/Y} = \ell_{X_{U_0}/Y}$. Let $U = \bigcup U_i$. Then $X_U \to Y$ is generically finite with $\ell_{X_U/Y} = \ell_{X_{U_0}/Y}$.

Proof. Any point in the generic fiber of $X_U \to Y$ lies in the generic fiber of $X_{U_i} \to Y$ for some $i$. But these all lie in the generic fiber of $X_{U_0} \to Y$, which quickly gives the desired equality. □

It now follows that there is a largest open subset $U \supset \bigcap_{j=1}^m D(q_j)$ such that $X_U$ has generic root count $\ell_{\hat{X}/Y}$.

Definition 5.14 Let $U$ be the union of all open subsets $U_i \supset \bigcap_{j=1}^m D(q_j)$ such that $\ell_{X_{U_i}/Y} = \ell_{\hat{X}/Y}$. We call this the maximal compatible open subset for the modification.

Example 5.12 shows that this $U$ can be a strict subset of $T_Y$. In Proposition 5.16, we will give a criterion that allows us to extend $\bigcap_{j=1}^m D(q_j)$ to a larger open set $U$ with $\ell_{X_U/Y} = \ell_{\hat{X}/Y}$. We first give some preliminary definitions.

Definition 5.15 Write $Z = \text{Spec}(A[x_i^\pm, w_j \mid 1 \leq i \leq n, 1 \leq j \leq m])$. Any subset $J \subseteq [m]$ gives rise to a toric stratum $H_J = \bigcap_{j \in J} D(w_j) \cap \bigcap_{j \in J} V(w_j)$ of $Z$. Note that every $H_J$ is isomorphic to a standard torus over $Y$. Let $\hat{I}_{\text{lin}, a}$ and $\hat{I}_{\text{nilin}, a}$ be the ideals generated by the $\hat{f}_i$ and $\hat{h}_j$ in $A[x_i^\pm, w_j \mid 1 \leq i \leq n, 1 \leq j \leq m]$. We write $\hat{X}_{\text{lin}, a} = V(\hat{I}_{\text{lin}, a})$ and $\hat{X}_{\text{nilin}, a} = V(\hat{I}_{\text{nilin}, a})$ for the corresponding subschemes in $Z$.

Proposition 5.16 Let $J \subseteq [m]$ be a subset and suppose that
\[ \text{codim}(\hat{X}_{\text{lin}, a} \cap H_J, H_J) + \text{codim}(\hat{X}_{\text{nilin}, a} \cap H, H_J) > \dim(H_J) \] (3)
for every $J'$ with $J' \cap J = \emptyset$. Here the codimensions are taken over the generic point $\eta$ of $Y$. Then $\ell_{\hat{X}/Y} = \ell_{X_{U,J}/Y}$, where $U_J = \bigcap_{j \in J} D(q_j)$.

**Proof.** We first note that every $\hat{X}_{\text{lin},a} \cap H$ is torus-equivariant. For $H$ as in the statement of the proposition, we can use [MS15, Theorem 3.6.10] and Proposition 4.2 to conclude that $(\hat{X}_{\text{lin},a} \cap H) \cap (\hat{X}_{\text{nlinc},a} \cap H)$ is generically empty. Suppose that there is a point $P$ in $X_{U,\eta}$ with $q_j(P) = 0$ for some $q_j$. Then $q_j \notin T_1$, so we can find a stratum $H$ such that $P$ gives a point of $(\hat{X}_{\text{lin},a} \cap H) \cap (\hat{X}_{\text{nlinc},a} \cap H)$. But the fiber of this scheme over $\eta$ is empty, a contradiction. \[\square\]

**Remark 5.17** Note that codim$(\hat{X}_{\text{lin},a} \cap H,H)$ is relatively easy to compute. Computing codim$(\hat{X}_{\text{nlinc},a} \cap H,H)$ on the other hand generally involves calculating a Gröbner basis, which can be unfeasible. In certain key cases we can still easily give a a good lower bound for codim$(\hat{X}_{\text{nlinc},a} \cap H,H)$ however, so that we can again apply Proposition 5.16.

**Corollary 5.18** Let $J \subset \{ m \}$ be a subset such that for every $f_i$ there exists a $j \in J$ with $p_{i,j} \neq 0$. Then $\ell_{\hat{X}/Y} = \ell_{X_{U,J}/Y}$, where $U_J = \bigcap_{j \in J} D(q_j)$.

**Proof.** Indeed, the condition implies that the codimension of the linear space does not change. Using Krull’s Hauptidealsatz, one then easily finds that the inequality from Proposition 5.16 holds for every $H$ with $S' \cap S_1 = \emptyset$. \[\square\]

**Corollary 5.19** Suppose that for every $f_i$, there exists a $j$ with $p_{i,j} \neq 0$ and $q_j$ monomial. Then $\ell_{X/Y} = \ell_{\hat{X}/Y}$.

**Example 5.20** Consider the system $X = V(f_1, f_2)$ over $Y = \text{Spec}(\mathbb{C}[a_1, a_2, a_3, a_4])$ given by the polynomials

$$f_1 = a_1(x - 1) + a_2(y - 2), \quad \text{and} \quad f_2 = a_3(x - 1) + a_4(y - 4).$$

The maximal open subsets we obtain from Corollary 5.18 are $D(x - 1)$ and $D(y - 2) \cap D(y - 4)$. Moreover, their union $U$ is not the entire torus. Using Proposition 5.16 with $J = \emptyset$, we however easily see that the root count is also valid over $T_Y$. We thus see that the maximal compatible subset can be strictly larger than the ones obtained from Corollary 5.18.

We conclude this section with a few comments on the linear scheme $\hat{X}_{\text{lin}}$. In order to make trop$(\hat{X}_{\text{lin},P})$ generic enough, we can use the following lemma.

**Lemma 5.21** Consider the non-zero Plücker vectors $z_{i,j}$ of $\hat{X}_{\text{lin}}$ as elements of the parameter ring $A$. For any $P \in Y^{an}$ such that $z_{i,j}(P) \neq 0$ for all $z_{i,j}$, we have that $\ell_{\hat{X}/Y} = \text{trop}(\hat{X}_{\text{lin},P}) \cdot \text{trop}(\hat{X}_{\text{nlinc},P})$. 

Proof. By Lemma 3.7, there is an open neighborhood \( V \) of \( P \) such that \( |z_{i,j}(Q)| = |z_{i,j}(P)| \neq 0 \) for all \( Q \in V \). This intersects the open dense subset from Proposition 4.19, which gives the statement of the lemma. \( \square \)

Let \( \Sigma \) be a pure balanced polyhedral complex and let \( \Sigma_{\text{lin}} \) be a tropical linear space of complementary dimension. We can degenerate \( \Sigma_{\text{lin}} \) to its Bergman fan without changing the intersection number \( \Sigma \cdot \Sigma_{\text{lin}} \). This number therefore only depends on the matroid of the tropical linear space. We can thus define the following.

**Definition 5.22** Let \( W = V(h_1, \ldots, h_m) \) be a closed subscheme determined by linear forms \( h_i \in A[x_1^\pm, \ldots, x_n^\pm] \). The generic matroid of \( W \) is the matroid of \( W_\eta \), where \( \eta \) is the generic point of \( Y \). Let \( M \) be a matroid on \( [n] \) and let \( \Sigma_{\text{lin}} \) be a tropical linear space of dimension \( \rho(M) \) with matroid \( M \). Let \( X \subseteq \mathcal{T}_{K,n} \) be a subvariety such that \( \dim(X) + \rho(M) = n \). The matroidal degree of \( X \) is

\[
\deg_M X := \Sigma_{\text{lin}} \cdot \text{trop}(X).
\]

**Corollary 5.23** Let \( U \supset \bigcap_{i=1}^m D(q_i) \) be the maximal compatible subset. Then \( \ell_{X_U/Y} = \deg_M(\hat{X}_{\text{lin},P}) \), where \( M \) is the generic matroid of \( \hat{X}_{\text{lin}} \).

To compute the matroidal degree, it may be advantageous to replace \( \text{trop}(\hat{X}_{\text{lin}}) \) with a different tropical linear space. For example, it might be beneficial to consider a translate of its Bergman fan, as the intersection will generally involve fewer maximal cells.

### 6. Applications to systems with linear dependencies

In this section, we explore square systems with linear dependencies between the coefficients of their polynomials. We will focus on two special types of dependencies which we call vertical and horizontal dependencies. The first is inspired by the steady-state equations of chemical reaction networks [Dic16], and the second is inspired by equations with fixed polynomial supports [KK12]. For each of these, either \( \hat{X}_{\text{lin}} \) or \( \hat{X}_{\text{lin}} \) is a tropical complete intersection, so that the tropical intersection product from Section 5.2 simplifies.

**Assumption 6.1** Throughout this section, the parameter space \( Y \) will be the \( m \)-dimensional affine space \( \mathbb{A}^m = \text{Spec}(K[a_1, \ldots, a_m]) \) and the coefficients of the polynomials \( f_1, \ldots, f_n \in K[a_1, \ldots, a_m][x_1^\pm, \ldots, x_n^\pm] \) will be linear and homogeneous in \( a_1, \ldots, a_m \).

#### 6.1. Square systems with vertical parameter dependencies.

In this section, we consider a class of parametrized polynomial systems inspired by the steady-state equations of chemical reaction networks [Dic16].
Definition 6.2 Let \( x^{\alpha_1}, \ldots, x^{\alpha_m} \) be the monomials of \( f_1, \ldots, f_n \). We say \( f_1, \ldots, f_n \) have vertical parameter dependencies, if there is a decomposition \( A = \bigotimes_{j=1}^m A_j \) of the parameter ring \( A \), such that each \( f_i \) is of the form
\[
 f_i = \sum_{j=1}^m p_{i,j} \cdot x^{\alpha_j} \quad \text{with } p_{i,j} \in A_j.
\]
In other words, the Macaulay matrix \( (p_{i,j}) \in A^{n \times m} \) of \( f_1, \ldots, f_n \) has algebraic dependencies along its columns, but not rows.

Definition 6.3 Let \( f_1, \ldots, f_n \) have vertical parameter dependencies. Choosing \( q_j = x^{\alpha_j} \) in Definition 5.6 we obtain:
\[
 \hat{C} := A[x_i^+, w_j^+ \mid 1 \leq i \leq n, 1 \leq j \leq m],
\]
\[
 \hat{f}_i := \sum_{j=1}^m p_{i,j} w_j \quad \text{for } 1 \leq i \leq n,
\]
\[
 \hat{h}_j := w_j - x^{\alpha_j} \quad \text{for } 1 \leq j \leq m.
\]
By Lemma 5.9, the resulting \( \hat{X}_{\text{lin}} = V((\hat{f}_1, \ldots, \hat{f}_n)) \) is torus-equivariant. Note moreover that trop(\( \hat{X}_{\text{lin}, P} \)) is a fixed linear subspace independent of \( P \). We will refer to the resulting \( \hat{X} \to Y \) as the modification for vertical dependencies.

For vertical dependencies, one exceptionally nice property of trop(\( \hat{X}_{\text{lin}, P} \)) is that it is a tropical complete intersection, i.e., it arises as a stable intersection
\[
 \text{trop(} \hat{X}_{\text{lin}, P} \text{)} = \text{trop(} V(\hat{h}_1)_P \text{)} \cap_{\text{st}} \cdots \cap_{\text{st}} \text{trop(} V(\hat{h}_m)_P \text{)}.
\]
This means that the intersection product with trop(\( \hat{X}_{\text{lin}, P} \)) equals the intersection product with all the trop(\( \hat{h}_j)_P \)):

**Proposition 6.4** Let \( f_1, \ldots, f_n \) have vertical parameter dependencies. Then, for \( P \) as in Lemma 5.21, the generic root count of \( X \to Y \) is
\[
 \text{trop(} \hat{X}_{\text{lin}, P} \text{)} \cdot \prod_{j=1}^m \text{trop(} V(\hat{h}_j)_P \text{)}.
\]

**Proof.** Indeed, Corollary 5.19 shows that the maximal compatible subset is \( T_Y \), so that \( \ell_{X/Y} = \ell_{\hat{X}/Y} \).

Proposition 6.4 allows us to compute the generic root count via mixed volumes, eliminating the need to compute any set-theoretic intersections of tropical hypersurfaces:

**Remark 6.5** Let \( M \) denote the matroid of trop(\( \hat{X}_{\text{lin}, P} \)) in Proposition 6.4. If \( M \) is cotransversal, then its Bergman fan is a tropical complete intersection, so that \( B(M) = \text{trop(} V(\hat{\ell}_1)_P \text{)} \cap_{\text{st}} \cdots \cap_{\text{st}} \text{trop(} V(\hat{\ell}_n)_P \text{)} \) for linear polynomials \( \hat{\ell}_i \). Consequently, the generic root count coincides with the mixed volume MV(\( \hat{\ell}_1, \ldots, \hat{\ell}_n, \hat{h}_1, \ldots, \hat{h}_m \)).
If $M$ is not cotransversal, then it can be expressed as a signed linear combination of cotransversal matroids, see [Ham17]. Consequently, the generic root count coincides with a signed linear combination of mixed volumes.

We conclude this section with an example that showcases how our techniques can be applied to steady-state equations of reaction networks.

Example 6.6 Consider the following reaction network:

$$X_{i-1} + E \xrightarrow{a_{1,i}} Y_{1,i} \xrightarrow{k_{1,i}} X_i + E \quad X_i + F \xrightarrow{a_{2,i}} Y_{2,i} \xrightarrow{k_{2,i}} X_{i-1} + F \quad i = 1, ..., n.$$ 

It describes the standard model of distributive sequential phosphorylation on $n$ sites, see for example [FRW20, Section 2]. Under the assumption of mass-action kinematics, the corresponding evolution equations are given by

$$\dot{x}_i = f_i := -a_{1,i}x_i x_E - a_{2,i}x_i x_F + d_{1,i+1}y_{1,i+1}$$

$$+ d_{2,i}y_{2,i} + k_{1,i}y_{1,i} + k_{2,i+1}y_{2,i+1}, \quad i = 0, \ldots, n,$$

$$\dot{y}_{1,i} = f_{1,i} := a_{1,i}x_{i-1} x_E - (d_{1,i} + k_{1,i})y_{1,i}, \quad i = 1, \ldots, n,$$

$$\dot{y}_{2,i} = f_{2,i} := a_{2,i}x_i x_F - (d_{2,i} + k_{2,i})y_{2,i}, \quad i = 1, \ldots, n,$$

$$\dot{x}_E = f_E := \sum_{i=1}^{n} -a_{1,i}x_{i-1} x_E + (d_{1,i} + k_{1,i})y_{1,i},$$

$$\dot{x}_F = f_F := \sum_{i=1}^{n} -a_{2,i}x_i x_F + (d_{2,i} + k_{2,i})y_{2,i},$$

where $a_{2,0} = d_{2,0} = k_{1,0} = 0$ and $a_{1,n+1} = d_{1,n+1} = k_{2,n+1} = 0$. The solutions of the evolution equations describe a three-dimensional set of steady states. In order to obtain finitely many solutions, observe that the following quantities are conserved:

$$E_{tot} = g_{E} := x_E + \sum_{i=1}^{n} y_{1,i} - c_E,$$

$$F_{tot} = g_{F} := x_F + \sum_{i=1}^{n} y_{2,i} - c_F,$$

$$X_{tot} = g_{X} := \sum_{i=0}^{n} x_i + \sum_{i=1}^{n} (y_{1,i} + y_{2,i}) - c_F$$

To make the system square, we can omit a suitable subset of the evolution equations depending on the conservation equations. In the system above, we may omit $f_0, f_E, f_F$ and consider the system consisting of $f_i, f_{1,i}, f_{2,i}$ for $i = 1, \ldots, n$ and $g_{E}, g_{F}, g_{X}$.

If the conservation equations $g_E, g_F$ and $g_X$ were generic, in the sense that each of its monomials come with a unique parameter as in the evolution equations, then the
modifications in Definition 6.3 would be immediately applicable, so that the generic root count is given by the tropical intersection number in Proposition 6.4.

To address non-generic conservation equations, a slight adaptation of the modification and decomposition in Definition 6.3 suffices: Let $\hat{f}_i, \hat{f}_{1,i}, \hat{f}_{2,i}$ and $\hat{h}_j$ be defined as in Definition 6.3. This means that the $\hat{f}_i, \hat{f}_{1,i}, \hat{f}_{2,i}$ are linear and that there is an $\hat{h}_j$ per vertex in the reaction network. Set

$$\hat{X}_{\text{lin}} := V(f_i, f_{1,i}, f_{2,i} | i = 1, \ldots, n), \quad \hat{X}_{\text{con}} := V(g_E, g_F, g_X),$$

$$\hat{X}_{\text{nlin}} := V(h_j | j = 1, \ldots, m).$$

One can show that $\text{trop}(\hat{X}_{\text{nlin}}, P)$ and $\text{trop}(\hat{X}_{\text{con}}, P)$ intersect transversally, and that $\hat{X}_{\text{lin}}$ is equivariant and of complementary dimension. Hence the conditions in Proposition 4.19 are satisfied, so that

$$\ell_{X/Y} = \text{trop}(\hat{X}_{\text{lin}}, P) \cdot \text{trop}(\hat{X}_{\text{nlin}}, P) \cdot \text{trop}(\hat{X}_{\text{con}}, P)$$

for $P$ in an open dense subset of $Y^\text{an}$. We thus see that the generic root count remains expressible as a tropical intersection product between a tropical linear space $\text{trop}(\hat{X}_{\text{lin}}, P) \cap \text{trop}(\hat{X}_{\text{con}}, P)$ and a classical linear space $\text{trop}(\hat{X}_{\text{nlin}}, P)$. The above technique works for arbitrary chemical reaction networks, provided that the generic codimension of $\hat{X}_{\text{lin}}$ is as expected as in Lemma 5.8.

### 6.2. Square systems with horizontal parameter dependencies

In this section, we consider a class of parametrized square systems inspired by the work of Kaveh and Khovanskii [KK12].

**Definition 6.7** Let $x^{a_1}, \ldots, x^{a_s}$ be the monomials of $f_1, \ldots, f_n$. We say $f_1, \ldots, f_n$ have horizontal parameter dependencies, if there is a decomposition of the parameter ring $A = \bigotimes_K A_i$, such that each $f_i$ is of the form

$$f_i = \sum_{j=1}^s p_{i,j} \cdot x^{a_j} \quad \text{with } p_{i,j} \in A_i.$$  

In other words, the Macaulay matrix $(p_{i,j}) \in A^{n \times s}$ of $f_1, \ldots, f_n$ has algebraic dependencies along its rows, but not columns. Note that we can also write $f_i$ as

$$f_i = \sum_{j=1}^m a_{i,j} \cdot q_j \quad \text{with } a_{i,j} \in A_i,$$

where the non-zero $a_{i,j}$ are algebraically independent.

**Definition 6.8** Let $f_1, \ldots, f_n$ have horizontal parameter dependencies. Choosing the $q_j$ in Definition 5.6 to be the $g_{i,j}$ of Definition 6.7, we obtain:

$$\hat{C} := A[x_i^+, w_j^+ | 1 \leq i \leq n, 1 \leq j \leq m].$$
\[ \hat{f}_i := \sum_{j=1}^{m} a_{i,j} w_j \quad \text{for } 1 \leq i \leq n, \]
\[ \hat{h}_j := w_j - q_j \quad \text{for } 1 \leq i \leq n, 1 \leq j \leq m. \]

As before by Lemma 5.9, the resulting \( \hat{X}_{\text{lin}} = V(\langle \hat{f}_1, \ldots, \hat{f}_n \rangle) \) is torus-equivariant and \( \text{trop}(\hat{X}_{\text{lin},P}) \) is independent of \( P \). We will refer to the resulting \( \hat{X} \to Y \) as the modification for horizontal dependencies.

Recall that, for vertical dependencies, \( \text{trop}(\hat{X}_{\text{lin}}) \) can be a general tropical linear space while \( \text{trop}(\hat{X}_{\text{lin},P}) \) is a tropical complete intersection. In a reversal of roles, for horizontal dependencies, \( \text{trop}(\hat{X}_{\text{lin}}) \) is a tropical complete intersection while \( \text{trop}(\hat{X}_{\text{lin},P}) \) can be a general tropical variety:

**Proposition 6.9** Let \( f_1, \ldots, f_n \) have horizontal parameter dependencies and let \( U \) be the maximal compatible subset. Then, for \( P \) as in Lemma 5.21, the generic root count of \( X_U \to Y \) is

\[ \text{trop}(\hat{X}_{\text{lin},P}) \cdot \prod \text{trop}(V(\hat{f}_i)_P). \]

**Remark 6.10** (Comparison to the works of Kaveh and Khovanskii) Let \( X \) be an \( n \)-dimensional irreducible variety over the complex numbers, and let \( L_1, \ldots, L_n \subset \mathbb{C}(X) \) be linear subspaces of the function field. In [KK10], Kaveh and Khovanskii define the birational intersection index \( [L_1, \ldots, L_n] \), which records the generic number of solutions to the system given by \( h_1, \ldots, h_n \), where \( h_i \) are generic elements of \( L_i \), outside a suitable support of the \( L_i \). In [KK12], a suitable higher-rank valuation \( \mathbb{C}(X) \setminus \{0\} \to \mathbb{Z}^n \) is used to attach a convex body \( \Delta_L \), the Newton-Okounkov body, to any subspace \( L \). In [KK12, Theorem 4.9] it is then proved that

\[ [L, \ldots, L] = \frac{n! \cdot \deg(\Phi_L)}{\text{ind}(A_L)} \cdot \text{vol}(\Delta_L), \]

where \( \Phi_L \) is the Kodaira map and \( \text{ind}(A_L) \) is the index of a certain subgroup, see [KK10; KK12] for more details. If the Kodaira map is a birational isomorphism, then both \( \deg(\Phi_L) \) and \( \text{ind}(A_L) \) are equal to 1, so that the formula reduces to \( [L, \ldots, L] = n! \text{vol}(\Delta_L) \).

In comparison, our paper always assumes that the ambient variety \( X \) is a torus, though it may be over any field \( K \). Let \( q_{i,1}, \ldots, q_{i,k_i} \) be a basis of \( L_i \) and consider

\[ f_i = \sum_{j=1}^{k_i} a_{i,j} q_{i,j} \quad \text{for } i = 1, \ldots, n. \]

These give a square system \( X \to Y \) with horizontal parameter dependencies. The open subset used in [KK12, Definition 4.5 (1)] is a subset of the maximal compatible subset. Namely, if we write \( Z_i = \bigcap_{j: p_{i,j} \neq 0} V(q_j) \), then \( Z_L \) from [KK12, Section 4.2]
is \( \bigcup_{i=1}^{n} \mathbb{Z} \). Its complement is then easily seen to be a union of open subsets \( U \) as in Corollary 5.18.

In [KK12, Definition 4.5 (2)], it is furthermore required that the solutions are non-degenerate in the sense that the generic fiber of the morphism \( X \to Y \) is étale. This will generally not be the case if the characteristic of the base field \( K \) is positive.

For example, the parametrized polynomial \( f = a_1 + a_2x^p \) over the ring \( A = \mathbb{F}_p[a_1, a_2] \) has generic root count \( p \), but the number of generic solutions where the morphism is étale is zero. If the characteristic of the base field \( K \) is zero, then the proof of [KK10, Proposition 5.7] implies that \( X_U \to Y \) is generically étale for the maximal compatible subset \( U \), from which we obtain

\[
[L_1, \ldots, L_n] = \text{deg}_M(\hat{\mathcal{X}}_{\text{lin}, P}) = \ell_{X_U/Y}
\]

using Corollary 5.23 and our earlier observation on the open subset used in the definition of \([L_1, \ldots, L_n]\).

We end this section with two examples from literature, in which we highlight two different ideas to simplify the tropical intersection product in Proposition 6.9:

1. In Example 6.11, \( \hat{\mathcal{X}}_{\text{lin}} \) is quasi-linear in the sense that it is the preimage of a linear space under a finite toric morphism.
2. In Example 6.12, \( \hat{\mathcal{X}}_{\text{lin}} \) is a tropical complete intersection and we showcase how one can simplify the resulting mixed volume.

**Example 6.11** (Kuramoto model) Consider the following polynomials whose zero sets give the stationary equations of the Kuramoto model [CMMN19, Equation F3]:

\[
f_i := \sum_{j=1}^{N} a_{i,j} (x_i x_j^{-1} - x_j x_i^{-1}) - b_i \quad \text{for } i = 1, \ldots, N - 1 \text{ and } x_N := 1. \tag{4}
\]

Note that the parameters were renamed to \( a_{i,j} \) and \( b_i \), and some constants which do not change the generic root count were omitted. Moreover, [CMMN19] also studies specializations of (4) where some \( a_{i,j} \) are set to 0. In our set-up, this degenerates the matroid corresponding to \( \hat{\mathcal{X}}_{\text{lin}} \) and leaves \( \hat{\mathcal{X}}_{\text{lin}} \) unchanged.

The modification in Definition 6.8 yields

\[
\hat{C} := A[x_i, w_{i,j} | 1 \leq i, j < N, i \neq j],
\]

\[
\hat{f}_i := \sum_{j \neq i} a_{i,j} w_{i,j} - b_i \quad \text{for } 1 \leq i < N,
\]

\[
\hat{h}_{i,j} := w_{i,j} - (x_i x_j^{-1} - x_j x_i^{-1}) \quad \text{for } 1 \leq i, j < N, i \neq j.
\]

Note that the maximal compatible subset here is \( T_Y \). By Proposition 6.9, the generic root count equals \( \text{trop}(\hat{\mathcal{X}}_{\text{lin}, P}) \cdot \prod_{i=1}^{N-1} \text{trop}(V(\hat{f}_i)_P) \). We determine the polyhedral complex \( \text{trop}(\hat{\mathcal{X}}_{\text{lin}, P}) \) by pulling back the tropicalization of a suitable linear space under a toric map.
We first define an automorphism \( \kappa_1 \) of the torus by sending \( x_i \) to \( x_i \), and \( w_{i,j} \) to \( w_{i,j} := x_i x_j w_{i,j} \). We then define a Kummer map \( \kappa_2 \) by sending \( x_i \) to \( z_i := x_i^2 \) and \( w_{i,j} \) to \( w_{i,j}^\prime \). The composition \( \kappa = \kappa_2 \circ \kappa_1 \) defines a finite map of tori of degree \( 2^{N-1} \) and the space \( X_{\text{lin}} \) is the inverse image of the linear space \( \hat{X}_{\text{lin}}' = \bigcap V(w_{i,j}^\prime - (z_i - z_j)) \) under \( \kappa \). One checks that the tropical volume of the polynomials in System (5) is the same root count in terms of two explicit polyhedral complexes.

Example 6.12 Fix \( N > 0 \), and consider the following polynomials for \( i = 1, \ldots, N \):
\[
\hat{f}_i = a_{1,i} u_i (u_i^2 + v_i^2) + a_{2,i} u_i + a_{3,i} v_i + a_{4,i} + \sum_{j \neq i} c_{j,i} v_j,
\]
\[
\hat{g}_i = b_{1,i} v_i (u_i^2 + v_i^2) + b_{2,i} u_i + b_{3,i} v_i + b_{4,i} + \sum_{j \neq i} d_{j,i} u_j.
\]
Here, the \( a_{j,i}, b_{j,i}, c_{j,i}, d_{j,i} \) are the parameters and the \( u_i, v_i \) are the variables. This polynomial system describes the steady states of coupled Duffing oscillators. In [BMMT22], Breiding, Michałek, Monid and Telen use Newton-Okounkov bodies and Khovanskii bases to show that the generic root count of this system is \( 5^N \). We will show here how the same root count can be obtained using our results.

Applying our modifications in Definition 6.8 yields:
\[
\hat{f}_i = a_{1,i} w_{i,1} + a_{2,i} u_i + a_{3,i} v_i + a_{4,i} + \sum_{j \neq i} c_{j,i} v_j, \quad \hat{h}_{i,1} = w_{i,1} - u_i (u_i^2 + v_i^2),
\]
\[
\hat{g}_i = b_{1,i} w_{i,1} + b_{2,i} u_i + b_{3,i} v_i + b_{4,i} + \sum_{j \neq i} d_{j,i} u_j, \quad \hat{h}_{i,2} = w_{i,2} - v_i (u_i^2 + v_i^2),
\]
which we can reformulate to the following generating set
\[
\hat{f}_i = a_{1,i} w_{i,1} + a_{2,i} u_i + a_{3,i} v_i + a_{4,i} + \sum_{j \neq i} c_{j,i} v_j, \quad \hat{h}_i = w_{i,1} - u_i (u_i^2 + v_i^2),
\]
\[
\hat{g}_i = b_{1,i} w_{i,2} + b_{2,i} u_i + b_{3,i} v_i + b_{4,i} + \sum_{j \neq i} d_{j,i} u_j, \quad \hat{\rho}_i = v_i w_{i,1} - u_i w_{i,2}.
\]
As before, we have that the maximal compatible subset is \( T_Y \), so that \( \ell_{X/Y} = \ell_{\hat{X}/Y} \).

We will first show that the trop(\( V(\hat{h}_i)_P \)) and trop(\( V(\hat{\rho}_i)_P \)) intersect transversally, which combined with Proposition 6.9 implies that the generic root count is the mixed volume of the polynomials in System (5). We then use previous work by Bihan and Soprunov [BS19] to reduce this mixed volume to \( 5^N \).

To see that the aforementioned tropical hypersurfaces intersect transversally, observe that trop(\( V(\hat{h}_i)_P \)) consists of three maximal cells, while trop(\( V(\hat{\rho}_i)_P \)) consists
of only one. Letting $\sigma_i$ denote a maximal cell of $\text{trop}(V(\hat{h}_i)_{P})$ and $\tau_i$ the maximal cells of $\text{trop}(V(\hat{\rho}_i)_{P})$, we have

$$\sigma_i \subseteq \begin{cases} 
(e_{w_i,1} - 3e_{u_i})^\perp & \text{or} \\
(e_{w_i,1} - e_{u_i} - 2e_{v_i})^\perp & \text{or} \\
(2e_{u_i} - 2e_{v_i})^\perp,
\end{cases}$$

and

$$\tau_i = (e_{v_i} + e_{w_i,1} - e_{u_i} + e_{w_i,2})^\perp.$$ 

It is straightforward to check that, regardless of the choice of $\sigma_i$, the normal vectors of $\sigma_1, \tau_1, \ldots, \sigma_N, \tau_N$ specified above will always be linearly independent, which in turn implies that the cells intersect transversally. This can for example be done by constructing a matrix of normal vectors, where the rows are indexed by the maximal cells and the columns are indexed by the unit vectors in the following ordering:

$$\begin{array}{cccccccc}
& & e_{w_1,2} & \cdots & e_{w_N,2} & e_{w_1,1} & e_{u_1} & e_{v_1} & \cdots & e_{w_N,1} & e_{u_N} & e_{v_N} \\
\tau_1 & \vdots & & & & & & & & & & \\
\vdots & & & & & & & & & & & \\
\tau_N & & & & & & & & & & & \\
\sigma_1 & & & & & & & & & & & \\
\vdots & & & & & & & & & & & \\
\sigma_N & & & & & & & & & & & \\
\end{array}$$

Regardless of the choice of $\sigma_i$, the matrix of normal vectors will always be in row-echelon form, and hence of full rank.

To show that the mixed volume of the Newton polytopes of the polynomials in Equation (5) is $5^N$, recall [BS19, Proposition 3.2] which states that for polytopes $Q_1, \ldots, Q_n$ and $P_1 \subseteq Q_1$ in $\mathbb{R}^n$:

$$\text{MV}(P_1, Q_2, \ldots, Q_n) = \text{MV}(Q_1, Q_2, \ldots, Q_n)$$

$$\iff \forall u \in \mathbb{R}^n \text{ with } \text{MV}(Q_2^u, \ldots, Q_n^u) > 0 : P_1 \cap Q_1^u \neq \emptyset. \quad (6)$$

Here $P^u$ denotes the face of $P$ minimizing scalar product by $u$ and the polytopes in $\text{MV}(Q_2^u, \ldots, Q_n^u)$ are considered to be polytopes of $u^\perp \cong \mathbb{R}^{n-1}$. We will use this result to show that the Newton polytopes of $\hat{f}_i$ and $\hat{g}_i$ can be replaced by the Newton polytopes of

$$a_{1,i}w_{i,1} + a_{2,i}u_i + a_{3,i}v_i + a_{4,i} \quad \text{and} \quad b_{1,i}w_{i,2} + b_{2,i}u_i + b_{3,i}v_i + b_{4,i}$$

without changing the mixed volume. A quick computation then reveals the mixed volume to be $5^N$. 
Let \((u, v, w)\) be a vector whose minimum on the Newton polytope of \(\hat{f}_i\) is uniquely attained at a vertex corresponding to a monomial in \(\sum_{j \neq i} c_{i,j} v_j\). Here, \(u_i, v_i, w_{i,j}\) represent weights on the variables \(u_i, v_i, w_{i,j}\). Without loss of generality, we may assume that \(i = 1\) and that the monomial is \(v_2\), so that the assumption implies:

\[
v_2 < w_{1,1}, \quad v_2 < u_1, \quad v_2 < 0, \quad \text{and} \quad v_2 < v_j \quad \text{for} \quad j \neq 2.
\]

We will show that the mixed volume in Expression (6) is zero. This is done by assuming that none of the polytopes in it are vertices and proving that it either leads to a contradiction or to mixed volume 0.

In the following, we will use \((p)\) as a shorthand for a tropical equation. In other words, \((p)\) means that the Newton polytope of \(p\) minimizing \((u, v, w)\) is not a vertex, or equivalently that the minimum of \(\text{trop}(p)(u, v, w)\) is attained at least twice. We distinguish between three cases:

- \(u_2 < v_2\): From \((\hat{h}_2)\) and \((\hat{f}_2)\) we obtain \(w_{2,1} = 3a_2\) and \(w_{2,1} = u_2\), respectively. Together, they imply \(u_2 = 0\), which contradicts \(u_2 < v_2 < 0\).

- \(u_2 > v_2\): From \((\hat{p}_2)\) we obtain \(w_{2,2} = 3v_2 < v_2\), the last inequality simply following from \(v_2 < 0\). By \((\hat{g}_2)\) we then get that \(w_{2,2} = u_j\) for some \(j \neq 2\). We therefore have \(u_j < v_2\). Considering \((\hat{f}_j)\), we thus obtain \(w_{j,1} = u_j < v_j\). From \(u_j < v_j\) and \((\hat{h}_j)\) we get \(w_{j,1} = 3a_j\), which together with the previous \(w_{j,1} = u_j\) implies \(u_j = 0\). This contradicts \(u_j < v_2 < 0\).

- \(u_2 = v_2\): From \((\hat{p}_2)\) and \((\hat{h}_2)\) we obtain \(w_{2,1} = w_{2,2}\) and \(w_{2,1} \geq 3a_2 = u_2 + 2v_2\) respectively. We again distinguish between three cases:
  - \(u_2 = v_2 > w_{2,1} = w_{2,2}\): The minimum of \(\text{trop}(f_2)(u, v, w)\) is attained uniquely at \(w_{2,1}\), hence the mixed volume is 0.
  - \(u_2 = v_2 < w_{2,1} = w_{2,2}\): Suppose that \(v_2 < u_i\) for all \(i \neq 2\). Then the minimum of both tropical polynomials \(\text{trop}(f_2)\) and \(\text{trop}(g_2)\) evaluated at \((u, v, w)\) is attained at the monomials \(u_2\) and \(v_2\), hence the mixed volume is 0.
  - Suppose that \(v_2 \geq u_i\) for some \(i \neq 2\). Then \(u_i \leq v_2 < v_i\) and from \((\hat{h}_i)\) we obtain \(w_{i,1} = 3a_i < u_i\). This implies that the minimum of \(\text{trop}(f_i)\) evaluated at \((u, v, w)\) is uniquely attained at the monomial \(w_{i,1}\), contradicting \((\hat{f}_i)\).

- \(u_2 = v_2 = w_{2,1} = w_{2,2}\): From \((\hat{h}_2)\) we obtain \(w_{2,1} \geq 3a_2\). Combined with the assumptions \(u_2 = w_{2,1}\) and \(w_{2,1} = v_2 < 0\), this implies \(w_{2,1} > 3a_2\). Hence the minimum of \(\text{trop}(h_2)(u, v, w)\) is attained uniquely at \(u_2^2\) and \(w_2^2\). Our assumptions on \(\hat{f}_i\) imply that the minimum in \(\text{trop}(f_2)(u, v, w)\) is attained at \(w_{2,1}\), \(u_2\), and \(v_2\). We again distinguish between three cases:
  - If the minimum in \(\text{trop}(g_2)(u, v, w)\) is attained at \(v_2\) and \(u_j\) for \(j \neq 2\), then from our initial assumption, we obtain \(u_j < v_j\) and thus \(w_{j,1} = 3a_j < u_j\). But then the minimum in \(\text{trop}(f_j)(u, v, w)\) is uniquely attained at \(w_{j,1}\), contradicting \((\hat{f}_j)\).
  - If the minimum in \(\text{trop}(g_2)(u, v, w)\) is attained at \(u_j\) and \(u_k\) for \(j, k \neq 2\) and \(j \neq k\), then the minimum in \(\text{trop}(f_j)(u, v, w)\) is attained at \(w_{j,1}\) and \(u_j\). As before, this implies that \(w_{j,1} = 3a_j = 0\), which contradicts \((\hat{f}_j)\).
The remaining case is where the minimum in \( \text{trop}(\hat{g}_2)(u, v, w) \) is attained at \( w_{2,2}, u_2, \) and \( v_2 \). But then the mixed volume of the Newton polytopes of \( \text{in}(h_{1,1}) = u_2(u_2^2 + v_2^2), \text{in}(f_2), \text{in}(g_2) \) is zero, since they contain a common lineality space.

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