THE $h^*$-POLYNOMIALS OF LOCALLY ANTI-BLOCKING LATTICE POLYTOPES AND THEIR $\gamma$-POSITIVITY

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ABSTRACT. A lattice polytope $\mathcal{P} \subset \mathbb{R}^d$ is called a locally anti-blocking polytope if for any closed orthant $\mathbb{R}^d_\varepsilon$ in $\mathbb{R}^d$, $\mathcal{P} \cap \mathbb{R}^d_\varepsilon$ is unimodularly equivalent to an anti-blocking polytope by reflections of coordinate hyperplanes. In the present paper, we give a formula for the $h^*$-polynomials of locally anti-blocking lattice polytopes. In particular, we discuss the $\gamma$-positivity of the $h^*$-polynomials of locally anti-blocking reflexive polytopes.

INTRODUCTION

A lattice polytope is a convex polytope all of whose vertices have integer coordinates. A lattice polytope $\mathcal{P} \subset \mathbb{R}_{\geq 0}^d$ of dimension $d$ is called anti-blocking if for any $y = (y_1, \ldots, y_d) \in \mathcal{P}$ and $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$ with $0 \leq x_i \leq y_i$ for all $i$, it holds that $x \in \mathcal{P}$. Anti-blocking polytopes were introduced and studied by Fulkerson [11, 12] in the context of combinatorial optimization. See, e.g., [35]. For $\varepsilon \in \{-1, 1\}^d$ and $x \in \mathbb{R}^d$, set $\varepsilon x := (\varepsilon_1x_1, \ldots, \varepsilon_dx_d) \in \mathbb{R}^d$. Given an anti-blocking lattice polytope $\mathcal{P} \subset \mathbb{R}_{\geq 0}^d$ of dimension $d$, we define

$$\mathcal{P}^\pm := \{\varepsilon x \in \mathbb{R}^d : \varepsilon \in \{-1, 1\}^d, x \in \mathcal{P}\}.$$ 

Since $\mathcal{P}$ is an anti-blocking lattice polytope, $\mathcal{P}^\pm$ is convex (and a lattice polytope). Moreover, for any $\varepsilon \in \{-1, 1\}^d$ and $x \in \mathcal{P}^\pm$, we have $\varepsilon x \in \mathcal{P}^\pm$. The polytope $\mathcal{P}^\pm$ is called an unconditional lattice polytope ([23]). In general, $\mathcal{P}^\pm$ is symmetric with respect to all coordinate hyperplanes. In particular, the origin $0$ of $\mathbb{R}^d$ is in the interior $\text{int}(\mathcal{P}^\pm)$. Given $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_d) \in \{-1, 1\}^d$, let $\mathbb{R}^d_\varepsilon$ denote the closed orthant \{(x_1, \ldots, x_d) \in \mathbb{R}^d : x_i\varepsilon_i \geq 0 \text{ for all } 1 \leq i \leq d\}. A lattice polytope $\mathcal{P} \subset \mathbb{R}^d$ of dimension $d$ is called locally anti-blocking ([23]) if, for each $\varepsilon \in \{-1, 1\}^d$, there exists an anti-blocking lattice polytope $\mathcal{P}_\varepsilon \subset \mathbb{R}_{\geq 0}^d$ of dimension $d$ such that $\mathcal{P} \cap \mathbb{R}^d_\varepsilon = \mathcal{P}^\pm \cap \mathbb{R}^d_\varepsilon$. Unconditional polytopes are locally anti-blocking.

In the present paper, we investigate the $h^*$-polynomials of locally anti-blocking lattice polytopes. First, we give a formula for the $h^*$-polynomials of locally anti-blocking lattice polytopes in terms of that of unconditional lattice polytopes. In fact,

**Theorem 0.1.** Let $\mathcal{P} \subset \mathbb{R}^d$ be a locally anti-blocking lattice polytope of dimension $d$ and for each $\varepsilon \in \{-1, 1\}^d$, let $\mathcal{P}_\varepsilon$ be an anti-blocking lattice polytope of dimension $d$ such

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that \( \mathcal{P} \cap \mathbb{R}^d = \mathcal{P}^\pm \cap \mathbb{R}_{\varepsilon}^d \). Then the \( h^* \)-polynomial of \( \mathcal{P} \) satisfies
\[
h^*(\mathcal{P}, x) = \frac{1}{2^d} \sum_{\varepsilon \in \{-1, 1\}^d} h^*(\mathcal{P}_{\varepsilon}^\pm, x).
\]
In particular, \( h^*(\mathcal{P}, x) \) is \( \gamma \)-positive if \( h^*(\mathcal{P}_{\varepsilon}^\pm, x) \) is \( \gamma \)-positive for all \( \varepsilon \in \{-1, 1\}^d \).

Second, we discuss the \( \gamma \)-positivity of the \( h^* \)-polynomials of locally anti-blocking reflexive polytopes. A lattice polytope is called reflexive if the dual polytope is also a lattice polytope. Many authors have studied reflexive polytopes from viewpoints of combinatorics, commutative algebra and algebraic geometry. In [15], Hibi characterized reflexive polytopes in terms of their \( h \)-polynomials. To be more precise, a lattice polytope of dimension \( d \) is (unimodularly equivalent to) a reflexive polytope if and only if the \( h^* \)-polynomial is a palindromic polynomial of degree \( d \). On the other hand, in [23], locally anti-blocking reflexive polytopes were characterized. In fact, a locally anti-blocking lattice polytope \( \mathcal{P} \subset \mathbb{R}^d \) of dimension \( d \) is reflexive if and only if for each \( \varepsilon \in \{-1, 1\}^d \), there exists a perfect graph \( G_\varepsilon \) on \( [d] := \{1, \ldots, d\} \) such that \( \mathcal{P} \cap \mathbb{R}_\varepsilon^d = \mathcal{P}_{G_\varepsilon}^\pm \cap \mathbb{R}_{\varepsilon}^d \), where \( \mathcal{P}_{G_\varepsilon} \) is the stable set polytope of \( G_\varepsilon \). Moreover, every locally anti-blocking reflexive polytope possesses a regular unimodular triangulation. This fact and the result of Bruns–Römer [5] imply that its \( h^* \)-polynomial is unimodal.

In the present paper, we discuss whether the \( h^* \)-polynomial of a locally anti-blocking reflexive polytope has a stronger property, which is called \( \gamma \)-positivity. In [30], a class of lattice polytopes \( \mathcal{B}_G \) arising from finite simple graphs \( G \) on \( [d] \), which are called symmetric edge polytopes of type \( B \), was given. Symmetric edge polytopes of type \( B \) are unconditional, and they are reflexive if and only if the underlying graphs are bipartite. Moreover, when they are reflexive, the \( h^* \)-polynomials are always \( \gamma \)-positive. On the other hand, in [31], another family of lattice polytopes \( \mathcal{C}_P^{\varepsilon} \) arising from finite partially ordered sets \( P \) on \( [d] \), which are called enriched chain polytopes, was given. Enriched chain polytopes are unconditional and reflexive, and their \( h^* \)-polynomials are always \( \gamma \)-positive. Combining these facts and Theorem 0.1, we know that, for a locally anti-blocking reflexive polytope \( \mathcal{P} \), if every \( \mathcal{P} \cap \mathbb{R}_{\varepsilon}^d \) is the intersection of \( \mathbb{R}_{\varepsilon}^d \) and either an enriched chain polytope or a symmetric edge reflexive polytope of type \( B \), then the \( h^* \)-polynomial of \( \mathcal{P} \) is \( \gamma \)-positive (Corollary 3.2). By using this result, we show that the \( h^* \)-polynomials of several classes of reflexive polytopes are \( \gamma \)-positive.

In Section 4, we will discuss the \( \gamma \)-positivity of the \( h^* \)-polynomials of symmetric edge polytopes of type \( A \), which are reflexive polytopes arising from finite simple graphs. In [21], it was shown that the \( h^* \)-polynomials of the symmetric edge polytopes of type \( A \) of complete bipartite graphs are \( \gamma \)-positive. We will show that for a large class of finite simple graphs, which includes complete bipartite graphs, the \( h^* \)-polynomials of the symmetric edge polytopes of type \( A \) are \( \gamma \)-positive (Subsection 4.1). Moreover, by giving explicit \( h^* \)-polynomials of del Pezzo polytopes and pseudo-del Pezzo polytopes, we will show that the \( h^* \)-polynomial of every pseudo-symmetric simplicial reflexive polytope is \( \gamma \)-positive (Theorem 4.8).

In Section 5, we will discuss the \( \gamma \)-positivity of \( h^* \)-polynomials of twinned chain polytopes \( \mathcal{C}_{P,Q} \subset \mathbb{R}^d \), which are reflexive polytopes arising from two finite partially ordered sets \( P \) and \( Q \) on \( [d] \). In [39], it was shown that twinned chain polytopes \( \mathcal{C}_{P,Q} \) are locally
anti-blocking and each \( C_{P,Q} \cap \mathbb{R}^d_\varepsilon \) is the intersection of \( \mathbb{R}^d_\varepsilon \) and an enriched chain polytopes. Hence the \( h^* \)-polynomials of \( C_{P,Q} \) are \( \gamma \)-positive. We will give a formula for the \( h^* \)-polynomials of twinned chain polytopes in terms of the left peak polynomials of finite partially ordered sets (Theorem 5.3). Moreover, we will define enriched \((P,Q)\)-partitions of \( P \) and \( Q \), and show that the Ehrhart polynomial of the twined chain polytope \( C_{P,Q} \) of \( P \) and \( Q \) coincides with a counting polynomial of enriched \((P,Q)\)-partitions (Theorem 5.8).

This paper is organized as follows: In Section 1, we will review the theory of Ehrhart polynomials, \( h^* \)-polynomials, and reflexive polytopes. In Section 2, we will introduce several classes of anti-blocking polytopes and unconditional polytopes. In Section 3, we will investigate the \( h^* \)-polynomials of locally anti-blocking lattice polytopes. In particular, we will prove Theorem 0.1. We will discuss symmetric edge polytope of type A in Section 4, and twinned chain polytopes in Section 5.

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1. Ehrhart theory and Reflexive polytopes

In this section, we review the theory of Ehrhart polynomials, \( h^* \)-polynomials, and reflexive polytopes. Let \( \mathcal{P} \subset \mathbb{R}^d \) be a lattice polytope of dimension \( d \). Given a positive integer \( m \), we define

\[
L_{\mathcal{P}}(m) = |m \mathcal{P} \cap \mathbb{Z}^d|.
\]

Ehrhart [10] proved that \( L_{\mathcal{P}}(m) \) is a polynomial in \( m \) of degree \( d \) with the constant term 1. We say that \( L_{\mathcal{P}}(m) \) is the Ehrhart polynomial of \( \mathcal{P} \). The generating function of the lattice point enumerator, i.e., the formal power series

\[
\text{Ehr}_{\mathcal{P}}(x) = 1 + \sum_{k=1}^{\infty} L_{\mathcal{P}}(k) x^k
\]

is called the Ehrhart series of \( \mathcal{P} \). It is well known that it can be expressed as a rational function of the form

\[
\text{Ehr}_{\mathcal{P}}(x) = \frac{h^*(\mathcal{P}, x)}{(1 - x)^{d + 1}}.
\]

Then \( h^*(\mathcal{P}, x) \) is a polynomial in \( x \) of degree at most \( d \) with nonnegative integer coefficients (3.6) and it is called the \( h^* \)-polynomial (or the \( \delta \)-polynomial) of \( \mathcal{P} \). Moreover, one has \( \text{Vol}(\mathcal{P}) = h^*(\mathcal{P}, 1) \), where \( \text{Vol}(\mathcal{P}) \) is the normalized volume of \( \mathcal{P} \).

A lattice polytope \( \mathcal{P} \subset \mathbb{R}^d \) of dimension \( d \) is called reflexive if the origin of \( \mathbb{R}^d \) is a unique lattice point belonging to the interior of \( \mathcal{P} \) and its dual polytope

\[
\mathcal{P}^\vee := \{ y \in \mathbb{R}^d : \langle x, y \rangle \leq 1 \text{ for all } x \in \mathcal{P} \}
\]

is also a lattice polytope, where \( \langle x, y \rangle \) is the usual inner product of \( \mathbb{R}^d \). It is known that reflexive polytopes correspond to Gorenstein toric Fano varieties, and they are related to mirror symmetry (see, e.g., [3, 7]). In each dimension there exist only finitely many reflexive polytopes up to unimodular equivalence (25) and all of them are known up to dimension 4 (24). In [15], Hibi characterized reflexive polytopes in terms of their \( h^* \)-polynomials. We recall that a polynomial \( f \in \mathbb{R}[x] \) of degree \( d \) is said to be palindromic if
$f(x) = x^d f(x^{-1})$. Note that if a lattice polytope of dimension $d$ has interior lattice points, then the degree of its $h^*$-polynomial is equal to $d$.

**Proposition 1.1** ([15]). Let $\mathcal{P} \subset \mathbb{R}^d$ be a lattice polytope of dimension $d$ with $0 \in \text{int}(\mathcal{P})$. Then $\mathcal{P}$ is reflexive if and only if $h^*(\mathcal{P}, x)$ is a palindromic polynomial of degree $d$.

Next, we review properties of polynomials. Let $f = \sum_{i=0}^d a_i x^i$ be a polynomial with real coefficients and $a_d \neq 0$. We now focus on the following properties.

(RR) We say that $f$ is real-rooted if all its roots are real.

(LC) We say that $f$ is log-concave if $a_i^2 \geq a_{i-1} a_{i+1}$ for all $i$.

(UN) We say that $f$ is unimodal if $a_0 \leq a_1 \leq \cdots \leq a_k \geq \cdots \geq a_d$ for some $k$.

If all its coefficients are nonnegative, then these properties satisfy the implications

$$\text{(RR)} \Rightarrow \text{(LC)} \Rightarrow \text{(UN)}.$$  

On the other hand, the polynomial $f$ is $\gamma$-positive if $f$ is palindromic and there are $\gamma_0, \gamma_1, \ldots, \gamma_{[d/2]} \geq 0$ such that $f(x) = \sum_{i=0}^d \gamma_i x^i(1 + x)^{d-2i}$. The polynomial $\sum_{i=0}^d \gamma_i x^i$ is called the $\gamma$-polynomial of $f$. We can see that a $\gamma$-positive polynomial is real-rooted if and only if its $\gamma$-polynomial is real-rooted. If $f$ is a palindromic and real-rooted, then it is $\gamma$-positive. Moreover, if $f$ is $\gamma$-positive, then it is unimodal. See, e.g., [2, 34] for details.

For a given lattice polytope, a fundamental problem within the field of Ehrhart theory is to determine if its $h^*$-polynomial is unimodal. One famous instance is given by reflexive polytopes that possess a regular unimodular triangulation.

**Proposition 1.2** ([5]). Let $\mathcal{P} \subset \mathbb{R}^d$ be a reflexive polytope of dimension $d$. If $P$ possesses a regular unimodular triangulation, then $h^*(\mathcal{P}, x)$ is unimodal.

It is known that if a reflexive polytope possesses a flag regular unimodular triangulation all of whose maximal simplices contain the origin, then the $h^*$-polynomial coincides with the $h$-polynomial of a flag triangulation of a sphere ([5]). For the $h$-polynomial of a flag triangulation of a sphere, Gal ([13]) conjectured the following:

**Conjecture 1.3** (Gal Conjecture). The $h$-polynomial of any flag triangulation of a sphere is $\gamma$-positive.

### 2. Classes of anti-blocking polytopes and unconditional polytopes

In this section, we introduce several classes of anti-blocking polytopes and unconditional polytopes. Throughout this section, we associate each subset $F \subset [d]$ with a $(0,1)$-vector $e_F = \sum_{i \in F} e_i \in \mathbb{R}^d$, where each $e_i$ is $i$th unit coordinate vector in $\mathbb{R}^d$.

#### 2.1. $(0,1)$-polytopes arising from simplicial complexes

Let $\Delta$ be a simplicial complex on the vertex set $[d]$. Then $\Delta$ is a collection of subsets of $[d]$ with $\{i\} \in \Delta$ for all $i \in [d]$ such that if $F \in \Delta$ and $F' \subset F$, then $F' \in \Delta$. In particular $\emptyset \in \Delta$ and $e_0 = 0$. Let $\mathcal{P}_\Delta$ denote the convex hull of $\{e_F \in \mathbb{R}^d : F \in \Delta\}$. The following is an important observation.

**Proposition 2.1.** Let $\mathcal{P} \subset \mathbb{R}_{\geq 0}^d$ be a $(0,1)$-polytope of dimension $d$. Then $\mathcal{P}$ is anti-blocking if and only if there exists a simplicial complex $\Delta$ on $[d]$ such that $\mathcal{P} = \mathcal{P}_\Delta$. 
2.2. **Stable set polytopes.** Let $G$ be a finite simple graph on the vertex set $[d]$ and $E(G)$ the set of edges of $G$. (A finite graph $G$ is called simple if $G$ possesses no loop and no multiple edge.) A subset $W \subset [d]$ is called stable if, for all $i$ and $j$ belonging to $W$ with $i \neq j$, one has $\{i, j\} \notin E(G)$. We remark that a stable set is often called an independent set. Let $S(G)$ denote the set of stable sets of $G$. One has $\emptyset \in S(G)$ and $\{i\} \in S(G)$ for each $i \in [d]$. The stable set polytope $\mathcal{Q}_G = \operatorname{conv}(\{y_E \in \mathbb{R}^d : E \in S(G)\})$. Then one has $\dim \mathcal{Q}_G = d$. Since we can regard $S(G)$ as a simplicial complex on $[d]$, $\mathcal{Q}_G$ is an anti-blocking polytope.

Locally anti-blocking reflexive polytopes are characterized by stable set polytopes. A **clique** of $G$ is a subset $W \subset [d]$ which is a stable set of the complement graph $\overline{G}$ of $G$. The **chromatic number** of $G$ is the smallest integer $t \geq 1$ for which there exist stable sets $W_1, \ldots, W_t$ of $G$ with $[d] = W_1 \cup \cdots \cup W_t$. A finite simple graph $G$ is said to be **perfect** if, for any induced subgraph $H$ of $G$ including $G$ itself, the chromatic number of $H$ is equal to the maximal cardinality of cliques of $H$. See, e.g., [2] for details on graph theoretical terminologies.

**Proposition 2.2** ([23]). Let $\mathcal{P} \subset \mathbb{R}^d$ be a locally anti-blocking lattice polytope of dimension $d$. Then $\mathcal{P} \subset \mathbb{R}^d$ is reflexive if and only if, for each $\varepsilon \in \{-1, 1\}^d$, there exists a perfect graph $G_{\varepsilon}$ on $[d]$ such that $\mathcal{P} \cap \mathbb{R}_{\varepsilon}^d = \mathcal{Q}_{G_{\varepsilon}}^\perp \cap \mathbb{R}_{\varepsilon}^d$.

2.3. **Chain polytopes and enriched chain polytopes.** Let $(P, <_P)$ be a partially ordered set (poset, for short) on $[d]$. A subset $A$ of $[d]$ is called an antichain of $P$ if all $i$ and $j$ belonging to $A$ with $i \neq j$ are incomparable in $P$. In particular, the empty set $\emptyset$ and each 1-element subset $\{i\}$ are antichains of $P$. Let $\mathcal{A}(P)$ denote the set of antichains of $P$. In [37], Stanley introduced the chain polytope $\mathcal{C}_P$ of $P$ defined by

$$\mathcal{C}_P := \operatorname{conv}(\{e_A \in \mathbb{R}^d : A \in \mathcal{A}(P)\})$$

It is known that chain polytopes are stable set polytopes. Indeed, let $G_P$ be the finite simple graph on $[d]$ such that $\{i, j\} \in E(G_P)$ if and only if $i <_P j$ or $j <_P i$. We call $G_P$ the comparability graph of $P$. It then follows that $\mathcal{A}(P) = S(G_P)$. Hence the chain polytope $\mathcal{C}_P$ is the stable set polytope $\mathcal{Q}_{G_P}$. Therefore, chain polytopes are anti-blocking polytopes. We remark that any comparability graph is perfect.

On the other hand, the **enriched chain polytope** $\mathcal{C}_{P(\varepsilon)}$ of $P$ is the unconditional lattice polytope defined by

$$\mathcal{C}_{P(\varepsilon)} := \mathcal{C}_P^\perp.$$ 

In [31], it was shown that the Ehrhart polynomial of $\mathcal{C}_{P(\varepsilon)}$ coincides with a counting polynomial of left enriched $P$-partitions. We assume that $P$ is naturally labeled. A map $f : P \to \mathbb{Z} \setminus \{0\}$ is called an **enriched $P$-partition** ([38]) if, for all $x, y \in P$ with $x <_P y$, $f$ satisfies

(i) $|f(x)| \leq |f(y)|$;

(ii) $|f(x)| = |f(y)| \Rightarrow f(y) > 0$.

A map $f : P \to \mathbb{Z}$ is called a **left enriched $P$-partition** ([33]) if, for all $x, y \in P$ with $x <_P y$, $f$ satisfies
(i) \(|f(x)| \leq |f(y)|\);
(ii) \(|f(x)| = |f(y)| \Rightarrow f(y) \geq 0\).

We denote \(\Omega^{(\ell)}_P(m)\) the number of left enriched \(P\)-partitions \(f : P \rightarrow \mathbb{Z}\) with \(|f(x)| \leq m\) for any \(x \in P\), which is called the left enriched order polynomial of \(P\).

**Proposition 2.3** (31). Let \(P\) be a naturally labeled finite poset on \([d]\). Then one has

\[
L^{(\ell)}_P(x) = \Omega^{(\ell)}_P(m).
\]

Given a linear extension \(\pi = (\pi_1, \ldots, \pi_d)\) of a finite poset \(P\) on \([d]\), a left peak of \(\pi\) is an index \(1 \leq i \leq d - 1\) such that \(\pi_{i-1} < \pi_i > \pi_{i+1}\), where we set \(\pi_0 = 0\). Let \(pk^{(\ell)}(\pi)\) denote the number of left peaks of \(\pi\). Then the left peak polynomial \(W^{(\ell)}_P(x)\) of \(P\) is defined by

\[
W^{(\ell)}_P(x) = \sum_{\pi \in \mathcal{L}(P)} x^{\pi^{(\ell)}(\pi)},
\]

where \(\mathcal{L}(P)\) is the set of linear extensions of \(P\).

**Proposition 2.4** (31). Let \(P\) be a naturally labeled finite poset on \([d]\). Then the \(h^*\)-polynomial of \(\mathcal{C}^{(e)}_P\) is

\[
h^*(\mathcal{C}^{(e)}_P, x) = (x + 1)^d W^{(\ell)}_P\left(\frac{4x}{(x+1)^2}\right).
\]

In particular, \(h^*(\mathcal{C}^{(e)}_P, x)\) is \(\gamma\)-positive.

Note that if \(Q\) is a finite poset which is obtained from \(P\) by reordering the label, then \(\mathcal{C}^{(e)}_P\) and \(\mathcal{C}^{(e)}_Q\) are unimodularly equivalent. Hence the \(h^*\)-polynomials of enriched chain polytopes are always \(\gamma\)-positive.

2.4. **Symmetric edge polytopes of type B.** Let \(G\) be a finite simple graph on \([d]\). We set

\[
B_G := \text{conv}\{0, e_1, \ldots, e_d\} \cup \{e_i + e_j : \{i, j\} \in E(G)\}.
\]

Then \(B_G = \mathcal{P}_\Delta\) where \(\Delta\) is a simplicial complex on \([d]\) obtained by regarding \(G\) as a 1-dimensional simplicial complex. The symmetric edge polytope of type B of \(G\) is the unconditional lattice polytope defined by

\[
\mathcal{B}_G := B_G^\perp.
\]

**Proposition 2.5** (30). Let \(G\) be a finite simple graph on \([d]\). Then \(\mathcal{B}_G\) is reflexive if and only if \(G\) is bipartite.

A hypergraph is a pair \(\mathcal{H} = (V, E)\), where \(E = \{e_1, \ldots, e_n\}\) is a finite multiset of non-empty subsets of \(V = \{v_1, \ldots, v_m\}\). Elements of \(V\) are called vertices and the elements of \(E\) are the hyperedges. Then we can associate \(\mathcal{H}\) to a bipartite graph \(\text{Bip}\mathcal{H}\) with a bipartition \(V \cup E\) such that \(\{v_i, e_j\}\) is an edge of \(\text{Bip}\mathcal{H}\) if \(v_i \in e_j\). Assume that \(\text{Bip}\mathcal{H}\) is connected. A hypertree in \(\mathcal{H}\) is a function \(f : E \rightarrow \{0, 1, \ldots\}\) such that there exists a spanning tree \(\Gamma\) of \(\text{Bip}\mathcal{H}\) whose vertices have degree \(f(e) + 1\) at each \(e \in E\). Then we say that \(\Gamma\) induces \(f\). Let \(B_\mathcal{H}\) denote the set of all hypertrees in \(\mathcal{H}\). A hyperedge \(e_j \in E\) is said to be *internally active* with respect to the hypertree \(f\) if it is not possible to decrease
In particular, \( h^* \) denotes the projection map. (Here \( \pi \) is the generating function \( I_{\mathcal{H}}(x) = \sum_{f \in B_{\mathcal{H}}} x^{\deg f} \). It is known [22, Proposition 6.1] that \( \deg I_{\mathcal{H}}(x) \leq \min\{|V|, |E|\} - 1 \). If \( G = \text{Bip}\mathcal{H} \), then we set \( I_G(x) = I_{\mathcal{H}}(x) \).

Assume that \( G \) is a bipartite graph with a bipartition \( V_1 \cup V_2 = [d] \). Then let \( \tilde{G} \) be a connected bipartite graph on \([d+2]\) whose edge set is

\[
E(\tilde{G}) = E(G) \cup \{(i,d+1) : i \in V_1\} \cup \{(j,d+2) : j \in V_2 \cup \{d+1\}\}.
\]

**Proposition 2.6** (30). Let \( G \) be a bipartite graph on \([d]\). Then \( h^* \)-polynomial of the reflexive polytope \( B_G \)
is

\[
h^*(\mathcal{B}_G, x) = (x+1)^d \tilde{I}_G \left( \frac{4x}{(x+1)^2} \right).
\]

In particular, \( h^*(\mathcal{B}_G, x) \) is \( \gamma \)-positive.

3. \( h^* \)-Polynomials of Locally Anti-blocking Lattice Polytopes

In the present section, we prove Theorem 0.1 that is, a formula for the \( h^* \)-polynomials of locally anti-blocking lattice polytopes in terms of that of unconditional lattice polytopes. Given a subset \( J = \{j_1, \ldots, j_r\} \) of \([d]\), let

\[
\pi_J : \mathbb{R}^d \to \mathbb{R}^r, \quad \pi_J(x_1, \ldots, x_d) = (x_{j_1}, \ldots, x_{j_r})
\]
denote the projection map. (Here \( \pi_0 \) is the zero map.)

**Proposition 3.1.** Let \( \mathcal{P} \subset \mathbb{R}^d_{\geq 0} \) be an anti-blocking lattice polytope. Then we have

\[
h^*(\mathcal{P}, x) = \sum_{j=0}^{d} 2^j (x - 1)^{d-j} \sum_{J \subset [d], |J| = j} h^*(\pi_J(\mathcal{P}), x).
\]

**Proof.** The proof is similar to the discussion in [30, Proof of Proposition 3.1]. The intersection of \( \mathcal{P} \cap \mathbb{R}^d_{\geq 0} \) and \( \mathcal{P} \cap \mathbb{R}^d_{\leq 0} \) is of dimension \( d-1 \) if and only if \( \epsilon - \epsilon' \in \{\pm 2e_1, \ldots, \pm 2e_d\} \). Moreover, if \( \epsilon - \epsilon' = 2e_k \), then

\[
(\mathcal{P} \cap \mathbb{R}^d_{\geq 0}) \cap (\mathcal{P} \cap \mathbb{R}^d_{\leq 0}) = \mathcal{P} \cap \mathbb{R}^d_{\geq 0} \cap \mathbb{R}^d_{\leq 0} \simeq \pi_{[d]\setminus\{k\}}(\mathcal{P}) \cap \mathbb{R}^d_{\geq 0} \cap \mathbb{R}^d_{\leq 0} \sim \pi_{[d]\setminus\{k\}}(\mathcal{P}).
\]

Hence the Ehrhart polynomial \( L_{\mathcal{P}}(m) \) satisfies the following:

\[
L_{\mathcal{P}}(m) = \sum_{j=0}^{d} 2^j (-1)^{d-j} \sum_{J \subset [d], |J| = j} L_{\pi_J(\mathcal{P})}(m).
\]

Thus the Ehrhart series satisfies

\[
\frac{h^*(\mathcal{P}, x)}{(1-x)^{d+1}} = \sum_{j=0}^{d} 2^j (-1)^{d-j} \sum_{J \subset [d], |J| = j} \frac{h^*(\pi_J(\mathcal{P}), x)}{(1-x)^{|J|+1}},
\]
as desired. \( \square \)

We now prove Theorem 0.1.
Proof of Theorem 3.1. Given \( J = \{ j_1, \ldots, j_r \} \subset [d] \) and \( \varepsilon \in \{ -1, 1 \}^r \), let

\[
\mathbb{R}^d_{J, \varepsilon} = \{ x = (x_1, \ldots, x_d) \in \mathbb{R}^d : \pi_J(x) \in \mathbb{R}^r_{\varepsilon} \text{ and } x_j = 0 \text{ for all } j \not\in J \}.
\]

It then follows that \( \mathcal{P} \cap \mathbb{R}^d_{J, \varepsilon} \) is equal to \( \pi_J(\mathcal{P}_{\varepsilon})^\pm \cap \mathbb{R}^r_{\varepsilon} \), where \( \pi_J(\varepsilon') = \varepsilon \). Note that, given \( J = \{ j_1, \ldots, j_r \} \subset [d] \) and \( \varepsilon \in \{ -1, 1 \}^r \), we have \(|\{ \varepsilon' \in \{ -1, 1 \}^d : \pi_J(\varepsilon') = \varepsilon \}| = 2^{d-r} \). Thus

\[
h^* (\mathcal{P}, x) = \sum_{j=0}^d (x-1)^{d-j} \sum_{J \subset [d], |J| = j} \sum_{\varepsilon \in \{ -1, 1 \}^J} h^* (\mathcal{P} \cap \mathbb{R}^d_{J, \varepsilon}, x)
\]

\[
= \sum_{j=0}^d (x-1)^{d-j} \sum_{\varepsilon \in \{ -1, 1 \}^d} \sum_{J \subset [d], |J| = j} 1 \cdot 2^{d-j} h^* (\pi_J(\mathcal{P}_{\varepsilon}), x)
\]

\[
= \frac{1}{2^d} \sum_{\varepsilon \in \{ -1, 1 \}^d} \sum_{j=0}^d 2^j (x-1)^{d-j} \sum_{J \subset [d], |J| = j} h^* (\pi_J(\mathcal{P}_{\varepsilon}), x)
\]

\[
= \frac{1}{2^d} \sum_{\varepsilon \in \{ -1, 1 \}^d} h^* (\mathcal{P}_{\varepsilon}^\pm, x)
\]

by Proposition 3.1. \( \square \)

Combining Theorem 0.1 and Propositions 2.4 and 2.6 we have the following.

**Corollary 3.2.** Let \( \mathcal{P} \subset \mathbb{R}^d \) be a locally anti-blocking reflexive polytope. If every \( \mathcal{P} \cap \mathbb{R}^d_{\varepsilon} \) is the intersection of \( \mathbb{R}^d_{\varepsilon} \) and either an enriched chain polytope or a symmetric edge reflexive polytope of type B, then the \( h^* \)-polynomial of \( \mathcal{P} \) is \( \gamma \)-positive.

Finally, we conjecture the following:

**Conjecture 3.3.** The \( h^* \)-polynomial of any locally anti-blocking reflexive polytope is \( \gamma \)-positive.

Thanks to Theorem 0.1 and Proposition 2.2 in order to prove Conjecture 3.3 it is enough to study unconditional lattice polytopes \( \mathcal{D}_G^\pm \) where \( \mathcal{D}_G \) is the stable set polytope of a perfect graph \( G \).

4. **Symmetric edge polytopes of type A**

Let \( G \) be a finite simple graph on the vertex set \([d]\) and the edge set \( E(G) \). The symmetric edge polytope \( \mathcal{A}_G \subset \mathbb{R}^d \) of type A is the convex hull of the set

\[
A(G) = \{ \pm(e_i - e_j) \in \mathbb{R}^d : \{ i, j \} \in E(G) \}.
\]

The polytope \( \mathcal{A}_G \) is introduced in [26, 28] and called a “symmetric edge polytope of \( G \).”

**Example 4.1.** Let \( G \) be a tree on \([d]\). Then \( \mathcal{A}_G \) is unimodularly equivalent to a \((d - 1)\)-dimensional cross polytope. Hence we have \( h^*(\mathcal{A}_G, x) = (x+1)^{d-1} \).

It is known [26, Proposition 4.1] that the dimension of \( \mathcal{A}_G \) is \( d - 1 \) if and only if \( G \) is connected. Higashitani [20] proved that \( \mathcal{A}_G \) is simple if and only if \( \mathcal{A}_G \) is smooth Fano if and only if \( G \) contains no even cycles. It is known [26, 28] that \( \mathcal{A}_G \) is unimodularly
equivalent to a reflexive polytope having a regular unimodular triangulation. In particular, 
$h^\ast$-polynomial of $\mathcal{A}_G$ is palindromic and unimodal. For a complete bipartite graph $K_{\ell,m}$, it is known [21] that the $h^\ast$-polynomial of $\mathcal{A}_{K_{\ell,m}}$ is real-rooted and hence $\gamma$-positive.

4.1. Recursive formulas for $h^\ast$-polynomials. In this section, we give several recursive formulas of $h^\ast$-polynomials of $\mathcal{A}_G$ when $G$ belongs to certain classes of graphs. By the following fact, we may assume that $G$ is 2-connected if needed.

**Proposition 4.2.** Let $G$ be a graph and let $G_1,\ldots,G_s$ be 2-connected components of $G$. Then the $h^\ast$-polynomial of $\mathcal{A}_G$ satisfies

$$h^\ast(\mathcal{A}_G,x) = h^\ast(\mathcal{A}_{G_1},x) \cdots h^\ast(\mathcal{A}_{G_s},x).$$

**Proof.** Since $\mathcal{A}_G$ is the free sum of reflexive polytopes $\mathcal{A}_{G_1},\ldots,\mathcal{A}_{G_s}$, a desired conclusion follows from [4] Theorem 1.

The suspension $\hat{G}$ of a graph $G$ is the graph on the vertex set $[d+1]$ and the edge set

$$E(G) \cup \{\{i,d+1\} : i \in [d]\}.$$ 

We now study the $h^\ast$-polynomial of $\mathcal{A}_G$. Given a subset $S \subset [d]$,

$$E_S := \{e \in E(G) : |e \cap S| = 1\}$$

is called a cut of $G$. For example, we have $E_{\emptyset} = E_{[d]} = \emptyset$. In general, it follows that $E_S = E_{[d]\setminus S}$. We identify $E_S$ with the subgraph of $G$ on the vertex set $[d]$ and the edge set $E_S$. By definition, $E_S$ is a bipartite graph. Let $\text{Cut}(G)$ be the set of all cuts of $G$. Note that $|\text{Cut}(G)| = 2^{d-1}$. From Theorem [1.1] and Proposition [2.6] we have the following.

**Theorem 4.3.** Let $G$ be a finite graph on $[d]$. Then $\mathcal{A}_G$ is unimodularly equivalent to a locally anti-blocking reflexive polytope whose $h^\ast$-polynomial is

$$h^\ast(\mathcal{A}_G,x) = \frac{1}{2^{d-1}} \sum_{H \in \text{Cut}(G)} h^\ast(\mathcal{B}_H,x) = (x+1)^d f_G \left( \frac{4x}{(x+1)^2} \right),$$

where

$$f_G(x) = \frac{1}{2^{d-1}} \sum_{H \in \text{Cut}(G)} l_H(x).$$

In particular, $h^\ast(\mathcal{A}_G,x)$ is $\gamma$-positive. Moreover, $h^\ast(\mathcal{A}_G,x)$ is real-rooted if and only if $f_G(x)$ is real-rooted.

**Proof.** Let $\mathcal{P} \subset \mathbb{R}^d$ be the convex hull of

$$\{\pm e_1,\ldots,\pm e_d\} \cup \{\pm (e_i - e_j) : \{i,j\} \in E(G)\}.$$

Then $\mathcal{A}_G$ is lattice isomorphic to $\mathcal{P}$. Given $\varepsilon = (\varepsilon_1,\ldots,\varepsilon_d) \in \{-1,1\}^d$, let $S_{\varepsilon} = \{i \in [d] : \varepsilon_i = 1\}$. Then $\mathcal{P} \cap \mathbb{R}^d_\varepsilon$ is the convex hull of

$$\{0\} \cup \{\varepsilon_i e_i : i \in [d]\} \cup \{e_i - e_j : \{i,j\} \in E_{S_{\varepsilon}}, i \in S_{\varepsilon}\}.$$

Hence $\mathcal{P} \cap \mathbb{R}^d_\varepsilon = \mathcal{B}_{E_{S_{\varepsilon}}} \cap \mathbb{R}^d_\varepsilon$. Thus $\mathcal{P}$ is a locally anti-blocking polytope and

$$h^\ast(\mathcal{A}_G,x) = \frac{1}{2^{d-1}} \sum_{H \in \text{Cut}(G)} h^\ast(\mathcal{B}_H,x).$$
by Theorem \[0.1\]

Let \(G\) be a graph and let \(e = \{i, j\}\) be an edge of \(G\). Then the graph \(G/e\) obtained by the procedure

(i) Delete \(e\) and identify the vertices \(i\) and \(j\);

(ii) Delete the multiple edges that may be created while (i)
is called the graph obtained from \(G\) by contracting the edge \(e\). Next, we will show that, for any bipartite graph \(G\) and \(e \in E(G), h^*(\mathcal{A}_G, x)\) is \(\gamma\)-positive if and only if so is \(h^*(\mathcal{A}_{G/e}, x)\). In order to show this fact, we need the theory of Gröbner bases of toric ideals. Given a graph \(G\) on the vertex set \([d]\) and the edge set \(E(G) = \{e_1, \ldots, e_n\}\), let

\[\mathcal{R} = K[t_1, t_1^{-1}, \ldots, t_d, t_d^{-1}, s]\]

be the Laurent polynomial ring over a field \(K\) and let

\[\mathcal{G} = K[x_1, \ldots, x_n, y_1, \ldots, y_n, z]\]

be the polynomial ring over \(K\). We define the ring homomorphism \(\pi : \mathcal{G} \to \mathcal{R}\) by setting \(\pi(z) = s, \pi(x_k) = t_j^{-1}s\) and \(\pi(y_k) = t_j^{-1}s\) if \(e_k = \{i, j\} \in E(G)\) and \(i < j\). The toric ideal \(I_{\mathcal{G}_G}\) of \(\mathcal{A}_G\) is the kernel of \(\pi\). (See, e.g., [14] for details on toric ideals and Gröbner bases.) We now define the notation given in [21]. For any oriented edge \(e_i\), let \(p_i\) denote the corresponding variable, i.e. \(p_i = x_i\) or \(p_i = y_i\) depending on the orientation and let \(\{p_i, q_i\} = \{x_i, y_i\}\). Let \(\mathcal{G}(G)\) be the set of all binomials \(f\) satisfying one of the following:

(1) \[f = \prod_{e_i \in I} p_i - \prod_{e_i \in C \setminus I} q_i,\]

where \(C\) is an even cycle in \(G\) of length \(2k\) with a fixed orientation, and \(I\) is a \(k\)-subset of \(C\) such that \(e_\ell \notin I\) for \(\ell = \min\{i : e_i \in C\}\);

(2) \[f = \prod_{e_i \in I} p_i - z \prod_{e_j \in C \setminus I} q_i,\]

where \(C\) is an odd cycle in \(G\) of length \(2k + 1\) and \(I\) is a \((k + 1)\)-subset of \(C\);

(3) \[f = x_i y_i - z^2,\]

where \(1 \leq i \leq n\). Then \(\mathcal{G}(G)\) is a Gröbner basis of \(I_{\mathcal{G}_G}\) with respect to a reverse lexicographic order \(<\) induced by the ordering \(z < x_1 < y_1 < \cdots < x_n < y_n\) ([21] Proposition 3.8)). Here the initial monomial of each binomial is the first monomial. Using this Gröbner basis, we have the following.

**Proposition 4.4.** Let \(G\) be a bipartite graph on \([d]\) and let \(e \in E(G)\). Then we have

\[h^*(\mathcal{A}_G, x) = (x + 1)h^*(\mathcal{A}_{G/e}, x)\]

**Proof.** Let \(E(G) = \{e_1, \ldots, e_n\}\) with \(e = e_1 = \{i, j\}\). Since \(G\) is a bipartite graph, the Gröbner basis \(\mathcal{G}(G)\) above consists of the binomials of the form (1) and (3).

Since \(G\) has no triangles, the procedure (ii) does not occur when we contract \(e\) of \(G\). Hence \(E(G/e) = \{e'_2, \ldots, e'_n\}\) where \(e'_k\) is obtained from \(e_k\) by identifying \(i\) with \(j\). Let \(G'\)
be a graph obtained by adding an edge \( e' = \{d + 1, d + 2\} \) to the graph \( G/e \). Then \( \mathcal{G}(G') \) consists of all binomials \( f \) satisfying one of the following:

(4) \[
    f = \prod_{e_i \in I} p_i - \prod_{e_i \in C \setminus I} q_i,
\]

where \( C \) is an even cycle in \( G \) of length \( 2k \) with a fixed orientation and \( e_1 \not\in C \), and \( I \) is a \( k \)-subset of \( C \) such that \( e_\ell \notin I \) for \( \ell = \min\{i : e_i \in C\} \);

(5) \[
    f = \prod_{e_i \in I} p_i - z \prod_{e_i \in C \setminus I} q_i,
\]

where \( C \cup \{e_1\} \) is an even cycle in \( G \) of length \( 2k+2 \) and \( I \) is a \( (k+1) \)-subset of \( C \);

(6) \[
    f = x_1 y_1 - z^2,
\]

where \( 1 \leq i \leq n \). Hence \( \{\text{in}_{\leq}(f) : f \in \mathcal{G}(G)\} = \{\text{in}_{\leq}(f) : f \in \mathcal{G}(G')\} \). By a similar argument as in the proof of [19, Theorem 3.1], it follows that

\[
    h^\ast(\mathcal{A}_G, x) = h^\ast(\mathcal{A}_G, x) = h^\ast(\mathcal{A}_{\{e_1\}}, x) h^\ast(\mathcal{A}_{G/e}, x) = (x+1)h^\ast(\mathcal{A}_{G/e}, x),
\]

as desired. \( \square \)

From Theorem 4.3, Propositions 4.2 and 4.4, we have the following immediately.

**Corollary 4.5.** Let \( G \) be a bipartite graph on \([d]\). Then we have the following:

(a) The \( h^\ast \)-polynomial \( h^\ast(\mathcal{A}_G, x) = (x+1)h^\ast(\mathcal{A}_{G/e}, x) \) is \( \gamma \)-positive.

(b) If \( G \) is obtained by gluing bipartite graphs \( G_1 \) and \( G_2 \) along with an edge \( e \), then

\[
    h^\ast(\mathcal{A}_G, x) = (x+1)h^\ast(\mathcal{A}_{G/e}, x) = (x+1)h^\ast(\mathcal{A}_{G_1/e}, x) h^\ast(\mathcal{A}_{G_2/e}, x) = h^\ast(\mathcal{A}_{G_1}, x) h^\ast(\mathcal{A}_{G_2}, x) / (x+1).
\]

**Remark.** Corollary 4.5(b) was recently generalized in [8, Theorem 4.17].

### 4.2. Pseudo-symmetric simplicial reflexive polytopes.

A lattice polytope \( \mathcal{P} \subset \mathbb{R}^d \) is called pseudo-symmetric if there exists a facet \( \mathcal{F} \) of \( \mathcal{P} \) such that \( -\mathcal{F} \) is also a facet of \( \mathcal{P} \). Nill [27] proved that any pseudo-symmetric simplicial reflexive polytope \( \mathcal{P} \) is a free sum of \( \mathcal{P}_1, \ldots, \mathcal{P}_s \), where each \( \mathcal{P}_i \) is one of the following:

- cross polytope;
- del Pezzo polytope \( V_{2m} = \text{conv}(\pm e_1, \ldots, \pm e_{2m}, \pm (e_1 + \cdots + e_{2m})) \);
- pseudo-del Pezzo polytope \( \widetilde{V}_{2m} = \text{conv}(\pm e_1, \ldots, \pm e_{2m}, -e_1 - \cdots - e_{2m}) \).

Note that a del Pezzo polytope is unimodularly equivalent to \( \mathcal{A}_{2m+1} \) where \( C_{2m+1} \) is an odd cycle of length \( 2m+1 \) (see [20]). The \( h^\ast \)-polynomial of \( \mathcal{A}_d \) was essentially studied in the following papers (see also the OEIS sequence A204621):

- Conway–Sloane [6, p.2379] computed \( h^\ast(\mathcal{A}_d, x) \) for small \( d \) by using results of O’Keeffe [32] and gave a conjecture on the \( \gamma \)-polynomial of \( h^\ast(\mathcal{A}_d, x) \) (coincides with the \( \gamma \)-polynomial in Proposition 4.7 below).
- General formulas for the coefficients of \( h^\ast(\mathcal{A}_d, x) \) were given by Ohsugi–Shibata [29] and Wang–Yu [40].

In order to give the \( h^\ast \)-polynomial of \( \widetilde{V}_{2m} \), we need the following lemma.
Lemma 4.6. Let $G$ be a connected graph. Suppose that an edge $e = \{i, j\}$ of $G$ is not a bridge. Let $\mathcal{P}_e$ be the convex hull of $A(G) \setminus \{e_i - e_j\}$. Then we have

$$h^*(\mathcal{P}_e, x) = \frac{1}{2}(h^*(A_G, x) + h^*(A_{G \setminus e}, x)),$$

where $G \setminus e$ is the graph obtained by deleting $e$ from $G$.

Proof. Note that $A_{G \setminus e} \subseteq \mathcal{P}_e \subseteq A_G$. Since $G$ is connected and $e$ is not a bridge of $G$, the dimension of each of $A_G$ and $A_{G \setminus e}$ is $d - 1$. Let $\mathcal{P}'_e$ denote the convex hull of $A(G) \setminus \{-e_i + e_j\}$, which is unimodularly equivalent to $\mathcal{P}_e$. Then $A_G$ and $\mathcal{P}_e$ are decomposed into the following disjoint union:

$$A_G = A_{G \setminus e} \cup (\mathcal{P}_e \setminus A_{G \setminus e}) \cup (\mathcal{P}'_e \setminus A_{G \setminus e}),$$

$$\mathcal{P}_e = A_{G \setminus e} \cup (\mathcal{P}_e \setminus A_{G \setminus e}).$$

Since $\mathcal{P}_e \setminus A_{G \setminus e}$ is unimodularly equivalent to $\mathcal{P}'_e \setminus A_{G \setminus e}$, we have a desired conclusion. $\square$

The $h^*$-polynomials of $V_{2m}$ and $\tilde{V}_{2m}$ are as follows:

**Proposition 4.7.** Let $C_d$ denote a cycle of length $d \geq 3$ and let $1 \leq m \in \mathbb{Z}$. Then we have

$$h^*(\mathcal{A}_{C_d}, x) = \sum_{i=0}^{\lfloor \frac{d+1}{2} \rfloor} \binom{2i}{i} x^i (x+1)^{d-2i-1},$$

$$h^*(V_{2m}, x) = \sum_{i=0}^{m} \binom{2i}{i} x^i (x+1)^{2m-2i},$$

$$h^*(\tilde{V}_{2m}, x) = (x+1)^{2m} + \sum_{i=1}^{m} \binom{2i-1}{i-1} x^i (x+1)^{2m-2i}.$$

In particular, the $h^*$-polynomials of $\mathcal{A}_{C_d}$, $V_{2m}$ and $\tilde{V}_{2m}$ are $\gamma$-positive.

Proof. The proof for $C_d$ is induction on $d$. First, we have $h^*(\mathcal{A}_{C_3}, x) = x^2 + 4x + 1 = (x+1)^2 + \binom{2}{1} x$. If $d \geq 4$ is even, then

$$h^*(\mathcal{A}_{C_d}, x) = (x+1)h^*(\mathcal{A}_{C_{d-1}}, x) = \sum_{i=0}^{\frac{d+1}{2}} \binom{2i}{i} x^i (x+1)^{d-2i-1} = \sum_{i=0}^{\frac{d-1}{2}} \binom{2i}{i} x^i (x+1)^{d-2i-1}.$$

Moreover, if $d = 2m + 1$ ($2 \leq m \in \mathbb{Z}$), then the coefficient of $x^m$ in

$$\sum_{i=0}^{\frac{d-1}{2}} \binom{2i}{i} x^i (x+1)^{d-2i-1} = (x+1)h^*(\mathcal{A}_{C_{d-1}}, x) + \binom{2m}{m} x^m$$

is $\sum_{i=0}^{m} \binom{2i}{i} \binom{2m-2i}{m-i} = 4^m = 2^{d-1}$ and other coefficient is arising from $(x+1)h^*(\mathcal{A}_{C_{d-1}}, x)$. By a recursive formula in [29, Theorem 2.3], we have

$$h^*(\mathcal{A}_{C_d}, x) = \sum_{i=0}^{\frac{d-1}{2}} \binom{2i}{i} x^i (x+1)^{d-2i-1}.$$
Since $V_{2m}$ is unimodularly equivalent to $\mathcal{A}_{2m+1}$, we have $h^*(V_{2m}, x) = h^*(\mathcal{A}_{2m+1}, x)$. By Lemma 4.6, it follows that

$$h^*(\tilde{V}_{2m}, x) = \frac{1}{2} \left( h^*(\mathcal{A}_{2m+1}, x) + h^*(\mathcal{A}_{2m+1}, x) \right)$$

$$= \frac{1}{2} \left( m \sum_{i=0}^{m} \binom{2i}{i} x^i (x+1)^{2m-2i} + (x+1)^{2m} \right)$$

$$= (x+1)^{2m} + \sum_{i=1}^{m} \binom{2i-1}{i-1} x^i (x+1)^{2m-2i}.$$

Thus it turns out that any pseudo-symmetric simplicial reflexive polytope is a free sum of reflexive polytopes whose $h^*$-polynomial are $\gamma$-positive. By [4, Theorem 1], we have the following.

**Theorem 4.8.** The $h^*$-polynomial of any pseudo-symmetric simplicial reflexive polytope is $\gamma$-positive.

**Proof.** From results by Nill [27], any pseudo-symmetric simplicial reflexive polytope is a free sum of cross polytopes, del Pezzo polytopes and pseudo-del Pezzo polytopes. On the other hand, by [4, Theorem 1], the $h^*$-polynomial of a free sum of reflexive polytopes $P_1, \ldots, P_s$ is equal to the product of their $h^*$-polynomials of $P_1, \ldots, P_s$. Hence by Example 4.1 and Proposition 4.7, it follows that the $h^*$-polynomial of any pseudo symmetric simplicial reflexive polytope is $\gamma$-positive.

4.3. **Classes of graphs such that $h^*(\mathcal{A}_G, x)$ is $\gamma$-positive.** Using results in the present section, for example, $h^*(\mathcal{A}_G, x)$ is $\gamma$-positive if one of the following holds:

- $G = \hat{H}$ for some graph $H$ (e.g., $G$ is a complete graph, a wheel graph);
- $G = \hat{H}$ for some bipartite graph $H$ (e.g., $G$ is a complete bipartite graph);
- $G$ is a cycle;
- $G$ is an outerplanar bipartite graph.

Moreover, we can compute $h^*(\mathcal{A}_G, x)$ explicitly in some cases. We give examples of such calculations for known formulas (for complete graphs [1], and for complete bipartite graphs [21]).

**Example 4.9 ([1]).** By Theorem 4.3, we have

$$h^*(\mathcal{A}_{K_d}, x) = h^*(\mathcal{A}_{K_{d-1}}, x) = \frac{(x+1)^{d-1}}{2^{d-2}} \sum_{H \in \text{Cut}(K_{d-1})} I_H \left( \frac{4x}{(x+1)^2} \right).$$

If the edge set of $H \in \text{Cut}(K_{d-1})$ is $E_S$ with $S \subset [d-1]$, then $H$ is a complete bipartite graph $K_{|S|, d-1-|S|}$ and $I_H(x) = \sum_{i \geq 0} \binom{|S|}{i} \binom{d-1-|S|}{i} x^i$. (Here $K_{0, d-1}$ denotes an empty
Let $(V,E)$ be a graph. It then follows that
\[
h^*(\mathcal{A}_{K_d}, x) = \frac{1}{2^{d-1}} \sum_{k=0}^{d-1} \binom{d}{k} \sum_{i=0}^{\lfloor \frac{d-1}{2} \rfloor} 4^i \binom{k}{i} \binom{d-k-1}{i} x^i (x+1)^{d-1-2i}
\]
\[
= \frac{1}{2^{d-1}} \sum_{i=0}^{\lfloor \frac{d-1}{2} \rfloor} 4^i x^i (x+1)^{d-1-2i} \sum_{k=i}^{d-1} \binom{d-1}{k} \binom{d-k-1}{i}
\]
\[
= \frac{1}{2^{d-1}} \sum_{i=0}^{\lfloor \frac{d-1}{2} \rfloor} 4^i x^i (x+1)^{d-1-2i} \binom{2i}{i} \binom{d-1}{2i}
\]
\[
= \frac{1}{2^{d-1}} \sum_{i=0}^{\lfloor \frac{d-1}{2} \rfloor} \binom{d-1}{2i} x^i (x+1)^{d-1-2i}.
\]

**Example 4.10** ([21]). Let $G = K_{m,n}$. Then $\tilde{G} = K_{m+1,n+1}$ and
\[
h^*(\mathcal{A}_{K_{m+1,n+1}}, x) = (x+1)h^*(\mathcal{A}_{K_{m,n}}, x) = \frac{(x+1)^{m+n+1}}{2^{m+n-1}} \sum_{H \in \text{Cut}(K_{m,n})} I_H \left( \frac{4x}{(x+1)^2} \right).
\]

Let $V_1 \cup V_2$ be the partition of the vertex set of $K_{m,n}$, where $|V_1| = m$ and $|V_2| = n$. If the edge set of $H \in \text{Cut}(K_{m,n})$ is $E_S$ with $S \subset [m+n]$, then $H$ is the disjoint union of two complete bipartite graphs $K_{k,\ell}$ and $K_{m-k,n-\ell}$, and hence
\[
I_H(x) = \left( \sum_{i \geq 0} \binom{k}{i} \binom{\ell}{i} x^i \right) \left( \sum_{j \geq 0} \binom{m-k}{j} \binom{n-\ell}{j} x^j \right),
\]
where $k = |V_1 \cap S|$ and $\ell = n - |V_2 \cap S|$. It then follows that
\[
h^*(\mathcal{A}_{K_{m+1,n+1}}, x) = \frac{x+1}{2^{m+n}} \sum_{k=0}^{m} \sum_{\ell=0}^{n} \binom{m}{k} \binom{n}{\ell} \sum_{i=0}^{\min(k,\ell)} 4^i \binom{k}{i} \binom{\ell}{i} x^i (x+1)^{k+\ell-2i}
\]
\[
= \frac{1}{2^{m+n}} \sum_{i,j \geq 0} 4^{i+j} x^i (x+1)^{n+m-2(i+j)+1} \sum_{k=i}^{m-j} \binom{m}{k} \binom{m-k}{j} \sum_{\ell=i}^{n-j} \binom{n}{\ell} \binom{\ell}{i} \binom{n-\ell}{j}.
\]

Since
\[
\sum_{k=i}^{m-j} \binom{m}{k} \binom{m-k}{j} = \sum_{k=i}^{m-j} \binom{m}{i+j} \binom{m-(i+j)}{k-i} = 2^{m-(i+j)} \binom{m}{i+j} \binom{i+j}{i},
\]
we have
\[ h^* \left( \mathcal{A}_{K_{m+1,n+1}}(x) \right) = \sum_{i \geq 0} \sum_{j \geq 0} \binom{i+j}{i} \binom{m}{i} \binom{n}{j} x^{i+j} (x+1)^{m+n-2(i+j)+1} \]
\[ = \min(m,n) \sum_{\alpha=0}^{\min(m,n)} \binom{2\alpha}{\alpha} \binom{m}{\alpha} \binom{n}{\alpha} x^{\alpha} (x+1)^{m+n-2\alpha+1} \]
\[ = \min(m,n) \sum_{\alpha=0}^{\min(m,n)} \binom{2\alpha}{\alpha} \binom{m}{\alpha} \binom{n}{\alpha} x^{\alpha} (x+1)^{m+n-2\alpha+1}. \]

Finally, we conjecture the following:

**Conjecture 4.11.** The \( h^* \)-polynomial of any symmetric edge polytope of type A is \( \gamma \)-positive.

5. **Twinned Chain Polytopes**

In this section, we will apply Theorem 0.1 to twinned chain polytopes. For two lattice polytopes \( \mathcal{P}, \mathcal{Q} \subset \mathbb{R}^d \), we set \( \Gamma(\mathcal{P}, \mathcal{Q}) := \text{conv}(\mathcal{P} \cup (-\mathcal{Q})) \subset \mathbb{R}^d \).

Let \( P \) and \( Q \) be two finite posets on \([d]\). The *twinned chain polytope* of \( P \) and \( Q \) is the lattice polytope defined by
\[ C_{P,Q} := \Gamma(C_P, C_Q). \]

Then \( C_{P,Q} \) is reflexive. Moreover, \( C_{P,Q} \) has a flag, regular unimodular triangulation all of whose maximal simplices contain the origin ([16, Proposition 1.2]). Hence we obtain the following:

**Corollary 5.1.** Let \( P \) and \( Q \) be two finite posets on \([d]\). Then the \( h^* \)-polynomial of \( C_{P,Q} \) coincides with the \( h \)-polynomial of a flag triangulation of a sphere.

In [39, Proposition 2.2] it was shown that \( C_{P,Q} \) is locally anti-blocking. In general, for two finite posets \( (P, \prec_P) \) and \( (Q, \prec_Q) \) with \( P \cap Q = \emptyset \), the ordinal sum of \( P \) and \( Q \) is the poset \( (P \oplus Q, \prec_{P\oplus Q}) \) on \( P \oplus Q = P \cup Q \) such that \( i \prec_{P\oplus Q} j \) if and only if (a) \( i, j \in P \) and \( i \prec_P j \), or (b) \( i, j \in Q \) and \( i \prec_Q j \), or (c) \( i \in P \) and \( j \in Q \). Given a subset \( I \) of \([d]\), we define the induced subposet of \( P \) on \( I \) to be the finite poset \( (P_I, \prec_P) \) on \( I \) such that \( i \prec_P j \) if and only if \( i \prec_P j \).

**Proposition 5.2** ([39, Proposition 2.2]). Let \( P \) and \( Q \) be two finite posets on \([d]\). Then for each \( \epsilon \in \{-1, 1\}^d \), it follows that
\[ C_{P,Q} \cap \mathbb{R}^d = C_{P_\epsilon \oplus Q_\epsilon} \cap \mathbb{R}^d, \]
where \( I_\epsilon = \{ i \in [d] : \epsilon_i = 1 \} \).

From this result, Theorem 0.1 and Proposition 2.4 we obtain the following:
Theorem 5.3. Let $P$ and $Q$ be two finite posets on $[d]$. Then one has
\[
h^*(\varepsilon_{P,Q} x) = \frac{1}{2^d} \sum_{\varepsilon \in \{-1,1\}^d} h^*(\varepsilon_{R_e} x) = (x+1)^d f_{P,Q} \left( \frac{4x}{(x+1)^2} \right),
\]
where $I_e = \{i \in [d] : \varepsilon_i = 1\}$ and $R_e$ is a naturally labeled poset which is obtained from $P_e \oplus Q_{e}$ by reordering the label and
\[
f_{P,Q}(x) = \frac{1}{2^d} \sum_{\varepsilon \in \{-1,1\}^d} W_{R_e}(x).
\]
In particular, $h^*(\varepsilon_{P,Q} x)$ is $\gamma$-positive. Moreover, $h^*(\varepsilon_{P,Q} x)$ is real-rooted if and only if $f_{P,Q}(x)$ is real-rooted.

On the other hand, it is known that, from $h^*(\varepsilon_{P,Q} x)$, we obtain the $h^*$-polynomials of several non-locally anti-blocking lattice polytopes arising from the posets $P$ and $Q$. The order polytope $\mathcal{O}_\varepsilon$ (\[37\]) of $P$ is the $(0, 1)$-polytope defined by
\[
\mathcal{O}_\varepsilon := \{x \in [0, 1]^d : x_i < x_j \text{ if } i < j\}.
\]
Given two lattice polytopes $\mathcal{P}, \mathcal{Q} \subset \mathbb{R}^d$, we define
\[
\mathcal{P} \ast \mathcal{Q} := \text{conv}((\mathcal{P} \times \{0\}) \cup (\mathcal{Q} \times \{1\})) \subset \mathbb{R}^{d+1},
\]
which are called the Cayley sum of $\mathcal{P}$ and $\mathcal{Q}$, and define
\[
\Omega(\mathcal{P}, \mathcal{Q}) := \text{conv}((\mathcal{P} \times \{1\}) \cup (-\mathcal{Q} \times \{-1\})) \subset \mathbb{R}^{d+1}.
\]

Proposition 5.4 (\[16\] Theorem 1.1). Let $P$ and $Q$ be two finite posets on $[d]$. Then one has
\[
h^*(\varepsilon_{P,Q} x) = h^*(\Gamma(\mathcal{O}_P, \mathcal{O}_Q), x).
\]
Further, if $P$ and $Q$ has a common linear extension, then we obtain
\[
h^*(\varepsilon_{P,Q} x) = h^*(\Gamma(\mathcal{O}_P, \mathcal{O}_Q), x).
\]

Proposition 5.5 (\[18\] Theorem 1.4). Let $P$ and $Q$ be two finite posets on $[d]$. Then one has
\[
(1+x)h^*(\varepsilon_{P,Q} x) = h^*(\Omega(\mathcal{O}_P, \mathcal{O}_Q), x).
\]
Further, if $P$ and $Q$ has a common linear extension, then we obtain
\[
(1+x)h^*(\varepsilon_{P,Q} x) = h^*(\Omega(\mathcal{O}_P, \mathcal{O}_Q), x).
\]

Proposition 5.6 (\[17\] Theorem 4.1). Let $P$ and $Q$ be two finite posets on $[d]$. Then one has
\[
h^*(\varepsilon_{P,Q} x) = h^*(\mathcal{O}_P \ast \mathcal{O}_Q, x).
\]

From these propositions and Theorem 5.3, we obtain the following:

Corollary 5.7. Let $P$ and $Q$ be two finite posets on $[d]$. Then the $h^*$-polynomials of $\Gamma(\mathcal{O}_P, \mathcal{O}_Q)$, $\Omega(\mathcal{O}_P, \mathcal{O}_Q)$, $\mathcal{O}_P \ast \mathcal{O}_Q$ and $\Omega(\mathcal{O}_P, \mathcal{O}_Q)$ are $\gamma$-positive. Furthermore, if $P$ and $Q$ has a common linear extension, then the $h^*$-polynomials of $\Gamma(\mathcal{O}_P, \mathcal{O}_Q)$ and $\Omega(\mathcal{O}_P, \mathcal{O}_Q)$ are also $\gamma$-positive.
In the rest of section, we introduce enriched \((P, Q)\)-partitions and show that the Ehrhart polynomial of \(\mathcal{C}_{P, Q}\) coincides with a counting polynomial of enriched \((P, Q)\)-partitions. Assume that \(P\) and \(Q\) are naturally labeled. We say that a map \(f : [d] \to \mathbb{Z}\) is an enriched \((P, Q)\)-partition if, for all \(x, y \in [d]\), \(f\) satisfies

- \(x <_P y\), \(f(x) \geq 0\) and \(f(y) \geq 0\) \(\Rightarrow\) \(f(x) \leq f(y)\);
- \(x <_Q y\), \(f(x) \leq 0\) and \(f(y) \leq 0\) \(\Rightarrow\) \(f(x) \geq f(y)\).

For a map \(f : [d] \to \mathbb{Z}\), we set \(m(f) = \min\{\{0\} \cup \{f(x) : x \in [d]\}\}\) and \(M(f) = \max\{\{0\} \cup \{f(x) : x \in [d]\}\}\). For each \(0 < m \in \mathbb{Z}\), let \(\Omega^{(e)}_{P, Q}(m)\) denote the number of enriched \((P, Q)\)-partitions \(f : [d] \to \mathbb{Z}\) with \(M(f) - m(f) \leq m\).

**Theorem 5.8.** Let \(P\) and \(Q\) be two finite posets on \([d]\). Then one has

\[
L_{\mathcal{E}_{P,Q}}(m) = \Omega^{(e)}_{P, Q}(m).
\]

**Proof.** Denote \(F(m)\) the set of enriched \((P, Q)\)-partitions with \(M(f) - m(f) \leq m\). We show that there exists a bijection from \(m\mathcal{E}_{P,Q} \cap \mathbb{Z}^d\) to \(F(m)\).

Take \(f \in F(m)\) and set \(m(f) = a\) and \(M(f) = b\). We set

\[
I = \{i \in [d] : f(i) \geq 0\}.
\]

Let

\[
x_i = \begin{cases} 
   f(i) & \text{if } i \in I \text{ is minimal in } P_l, \\
   \min\{f(i) - f(j) : i \text{ covers } j \text{ in } P_l\} & \text{if } i \in I \text{ is not minimal in } P_l, \\
   -|f(i)| & \text{if } i \in \mathcal{T} \text{ is minimal in } Q_l, \\
   -\min\{|f(i)| - |f(j)| : i \text{ covers } j \text{ in } Q_l\} & \text{if } i \in \mathcal{T} \text{ is not minimal in } Q_l.
\end{cases}
\]

Assume that \(I = \{1, \ldots, k\}\) and \(\mathcal{T} = \{k + 1, \ldots, d\}\). Then we have \((x_1, \ldots, x_k) \in b\mathcal{E}_{P_l}\) and \((x_{k+1}, \ldots, x_d) \in a\mathcal{E}_{Q_l}\) by a result of Stanley [37, Theorem 3.2]. Hence one obtains \((x_1, \ldots, x_d) \in b\mathcal{E}_{P_l} \oplus a\mathcal{E}_{Q_l} \subset m\mathcal{E}_{P,Q}\), where \(b\mathcal{E}_{P_l} \oplus a\mathcal{E}_{Q_l}\) is the free sum of \(b\mathcal{E}_{P_l}\) and \(a\mathcal{E}_{Q_l}\).

Similarly, in general, it follows that \((x_1, \ldots, x_d) \in m\mathcal{E}_{P,Q}\). Therefore, the map \(\varphi : F(m) \to m\mathcal{E}_{P,Q} \cap \mathbb{Z}^d\) defined by \(\varphi(f) = (x_1, \ldots, x_d)\) for each \(f \in F(m)\) is well-defined.

Take \((x_1, \ldots, x_d) \in m\mathcal{E}_{P,Q} \cap \mathbb{Z}^d\). We set

\[
I = \{i \in [d] : x_i \geq 0\}.
\]

We define a map \(f : [d] \to \mathbb{Z}\) by

\[
f(i) = \begin{cases} 
   \max\{x_{j_1} + \cdots + x_{j_k} : j_1 <_{P_l} \cdots <_{P_l} j_k = i\} & \text{if } i \in I, \\
   -\max\{|x_{j_1}| + \cdots + |x_{j_k}| : j_1 <_{Q_l} \cdots <_{Q_l} j_k = i\} & \text{if } i \in \mathcal{T}.
\end{cases}
\]

Assume that \(I = \{1, \ldots, k\}\) and \(\mathcal{T} = \{k + 1, \ldots, d\}\). Then one has \((x_1, \ldots, x_d) \in m\mathcal{E}_{P_l} \oplus (\mathcal{E}_{Q_l}) \cap \mathbb{Z}^d\). Moreover, for some integers \(a\) and \(b\) with \(a \leq 0 \leq b\) and \(b - a \leq m\), it follows that \((x_1, \ldots, x_k) \in b\mathcal{E}_{P_l}\) and \((x_{k+1}, \ldots, x_d) \in a\mathcal{E}_{Q_l}\). We define \(f_1 : I \to \mathbb{Z}\) by \(f_1(i) = f(i)\), and \(f_2 : \mathcal{T} \to \mathbb{Z}\) by \(f_2(i) = -f(i)\). From [37, Proof of Theorem 3.2], it follows that \(0 \leq f_1(i) \leq b\) for any \(i \in I\) and \(f_1(x) \leq f_1(y)\) if \(x <_{P_l} y\), and \(0 \geq f_2(i) \geq a\) for
any \( i \in \mathcal{I} \) and \( f_2(x) \leq f_2(y) \) if \( x <_\mathcal{I} y \). Therefore, \( f : [d] \to \mathbb{Z} \) is an enriched \((P, Q)\)-partition with \( M(f) - m(f) \leq b - a \leq m \), namely, \( f \in F(m) \). Similarly, in general, it follows that \( f \in F(m) \). Thus, the map \( \psi : m\mathcal{E}_{P,Q} \cap \mathbb{Z}^d \to F(m) \) defined by \( \psi(x)(i) = f(i) \) for each \( x = (x_1, \ldots, x_d) \in m\mathcal{E}_{P,Q} \cap \mathbb{Z}^d \) is well-defined.

Finally, we show that \( \varphi \) is a bijection. However, this immediately follows by the above and the argument in [37] Proof of Theorem 3.2. \( \square \)

Since \( \mathcal{E}_{P,Q} \) is reflexive, we obtain the following:

\[ \textbf{Corollary 5.9.} \text{ Let } P \text{ and } Q \text{ be two finite naturally labeled posets on } [d]. \text{ Then } \Omega_{P,Q}^{(e)}(m) \text{ is a polynomial in } m \text{ of degree } d \text{ and one has} \]

\[ \Omega_{P,Q}^{(e)}(m) = (-1)^d \Omega_{P,Q}^{(e)}(-m - 1). \]

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