Representations of relativistic particles of arbitrary spin in Poincaré, Lorentz, and Euclidean covariant formulations of relativistic quantum mechanics

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Background: Relativistic treatments of quantum mechanical systems are important for understanding hadronic structure and dynamics at sub-nucleon scales. Relativistic invariance of a quantum system means that there is an underlying unitary representation of the Poincaré group. This is equivalent to the requirement that the quantum observables (probabilities, expectation values and ensemble averages) for equivalent measurements performed in different inertial reference frames are identical. Many different representations are used in practice, including Poincaré covariant forms of dynamics, representations based on Lorentz covariant wave functions, Euclidean covariant representations and representations generated by Lorentz covariant fields.

Purpose: Wave functions for relativistic states are typically matrix elements of interacting relativistic states in a basis of free relativistic states. These wave functions are needed to evaluate matrix elements of currents and current commutators that probe the structure of hadronic systems. The purpose of this work is to illustrate the relation between the different representations of states in relativistic quantum mechanics.

Method: The starting point is a description of a particle of mass $m$ and spin $j$ using irreducible representations of the Poincaré group. Since any unitary representation of the Poincaré group can be decomposed into a direct integral of irreducible representations, these are the basic building blocks of any relativistically invariant quantum theory. These representations are used to construct equivalent Lorentz covariant representations, Euclidean covariant representations, and mass $m$ spin $j$ fields.

Results: Equivalent descriptions for positive mass representations of arbitrary spin are presented in each of these frameworks.

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I. INTRODUCTION

Relativistic quantum mechanical models are important for modeling hadronic structure. Experiments using electromagnetic and weak probes are designed to investigate the structure of hadronic targets. The relevant theoretical quantities are matrix elements of current operators between initial and final hadronic states in different inertial frames. The finer the resolution of the probe, the larger the momentum difference between the initial and final hadronic states. First principles calculations of the initial and final hadronic wave functions with quantifiable errors are challenging, especially when they are needed in different inertial frames. Relativistic models of the hadronic states provide a consistent treatment of initial and final states in different inertial reference frames.

There are many different formulations of relativistic quantum mechanical models. In this work the relation between different quantum mechanical descriptions of relativistic particles is systematically developed. In order to take advantage of the relations discussed in this work it is necessary to first have a dynamical model. While it is beyond the scope of this paper to discuss dynamical models, typical relativistic wave functions are matrix elements between an interacting relativistic state and a non-interacting relativistic basis state. For example, in describing a nucleus as a system of constituent nucleons, the nuclear state is the solution of a dynamical equation expressed in a basis of free nucleon states. In this work the focus is on deriving the relation between different relativistic descriptions of these particle states. This applies to both the interacting relativistic states and the relativistic free-particle basis states.

In 1939 Wigner [1] showed that the relativistic invariance of a quantum system is equivalent to the requirement that there is a unitary ray representation of the Poincaré group on the Hilbert space of the quantum system. This is the mathematical formulation of the physical requirement that quantum observables (probabilities, expectation values and ensemble averages) for equivalent measurements performed in different inertial reference frames are identical. Physically this means that equivalent quantum measurements in isolated systems cannot be used to distinguish inertial

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frames. This quantum mechanical formulation of relativistic invariance focuses on the invariance of measurements, rather than the transformation properties of equations, which is used in classical formulation of relativistic invariance.

Relativistically invariant quantum systems are represented using Poincaré covariant methods, Lorentz covariant methods, Euclidean covariant methods, and Lorentz covariant fields. Each method provides a different representation of the same physical system. Each representation has different advantages. The purpose of these notes is to exhibit the relation between these different representations. While most of the content of this exposition can be found in references, it is difficult to find all of the relations in one place.

The starting point is the realization that any unitary representation of the Poincaré group can be decomposed into a direct sum/integral of irreducible representations. These are the basic building blocks of any relativistically invariant quantum theory. The construction of the direct integral is the dynamical problem, which is mathematically equivalent to the simultaneous diagonalization of the Casimir operators (mass and spin) of the Poincaré Lie algebra. This is the relativistic analog of diagonalizing a non-relativistic center of mass Hamiltonian. This will not be discussed in this work, but it is a non-trivial dynamical problem. Wave functions of these irreducible states are matrix elements of these states with free-particle relativistic basis states. The free particle states could be irreducible basis states or tensor products of irreducible basis states. The relevant observation of this work is that once these wave functions are found in one representation, the results of this work can be applied to determine the corresponding relativistic wave functions in different representations.

Because of this it is sufficient to understand the relation between the different representations of the irreducible representations. This work considers only positive-mass positive-energy representations of the Poincaré group. These are the relevant representations for hadronic states.

The next section summarizes the notation used in the rest of this paper and gives a brief description of the essential elements of the Poincaré group. Section three discusses the construction of positive mass, positive energy unitary irreducible representations of the Poincaré group for a particle of any (positive) mass and spin. Single-particle states are represented by simultaneous eigenstates of a complete set of commuting observables that are functions of the infinitesimal generators of the Poincaré group. These basis states span a one-particle subspace, and the structure of the unitary representation of the Poincaré group on that subspace is fixed by the choice of commuting observables and group theory. For a given choice of commuting observables, there is a largest subgroup of the Poincaré group where the transformations are independent of the mass. These subgroups are called kinematic subgroups. Dirac identified basis choices with the largest kinematic subgroups. He referred to them as defining “forms of dynamics”. Kinematic subgroups are useful because for transformations in this subgroup, dynamical Poincaré transformations on interacting states can be computed by applying the inverse kinematic transformation to the free particle basis states. This avoids the need to explicitly compute the dynamical transformations.

Section four gives an introduction to SL(2, C) which is related to the Lorentz group like SU(2) is related to SO(3). SL(2, C) plays a central role in the construction of Lorentz covariant descriptions of particles. Euclidean covariant descriptions of particles and Lorentz covariant fields. This section includes a complete description of all of the properties of SL(2, C) that are needed in relativistic quantum theories.

Section five discusses Lorentz covariant descriptions of particles. In these representations the SU(2) Wigner rotations are decomposed into products of SL(2, C) matrices. The momentum-dependent parts are absorbed into the definition of the wave functions. The result is a new wave function that transforms in a Lorentz covariant way. In this representation the Hilbert space inner product acquires a non-trivial kernel, which removes the momentum dependence that was absorbed in the wave functions. The resulting kernel is a free-particle Wightman function. In addition, the SU(2) identity, $R = (R^I)^{-1}$ for the SU(2) Wigner rotations leads to two inequivalent decompositions of the Wigner rotation into products of SL(2, C) matrices. The inequivalent representations are related by space reflection. The treatment of space reflection in these representations is discussed.

Section six exhibits Euclidean covariant Green functions that lead to all of the covariant representations constructed in section five. The interesting feature of this representation is that no analytic continuation is needed to show equivalence with the Lorentz covariant representation.

Section seven discusses the construction of free Lorentz covariant fields using the occupation number representation in the Lorentz covariant description of particles. In section eight the covariant fields are used to construct local covariant fields. Section nine contains a brief summary.

II. THE POINCARÉ GROUP

The Poincaré group is the group of space-time coordinate transformations that preserve the form of the source-free Maxwell’s equations. It is also the group that relates different inertial coordinate systems in special relativity.
In what follows the space and time coordinates of events are labeled by components of a four vector

\[ x^\mu = (ct, x^1, x^2, x^3). \]  

The convention for the Lorentz metric tensor is

\[ \eta^{\mu\nu} = \eta_{\mu\nu} := \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \]  

and repeated indices are assumed to be summed. This choice of metric is natural for developing the relation with Euclidean representations.

Poincaré group is the group of point transformations that preserve the proper time between events:

\[ \Delta \tau_{xy}^2 = (x^0 - y^0)^2 - |\mathbf{x} - \mathbf{y}|^2 = -\eta_{\mu\nu}(x^\mu - y^\mu)(x^\nu - y^\nu). \]  

The general form of a point transformation, \( x'^\mu = f^\mu(x) \), that preserves (3) is

\[ f^\mu(x) = x'^\mu = \Lambda^\mu_\nu x^\nu + a^\mu \]  

where \( a^\mu \) and \( \Lambda^\mu_\nu \) are constants and the Lorentz transformation \( \Lambda^\mu_\nu \) satisfies

\[ \eta^{\mu\nu} = \Lambda^\mu_\alpha \Lambda^\nu_\beta \eta^{\alpha\beta}. \]  

These relations can be derived by differentiating

\[ (f^\mu(x) - f^\mu(y))(f^\nu(x) - f^\nu(y))\eta_{\mu\nu} = (x^\mu - y^\mu)(x^\nu - y^\nu)\eta_{\mu\nu}. \]  

with respect to \( x \), setting \( x \) to 0, and then doing the same with \( y \). In matrix form equation (5) has the form

\[ \eta = \Lambda \eta \Lambda^t \]  

which indicates that \( \Lambda \) is a real orthogonal transformation with respect to the Lorentz metric. Equations (1) and (7) are relativistic generalizations of the fundamental theorem of rigid body motion which asserts that any motion that preserves the distance between points in a rigid-body is a composition of an orthogonal transformation and a translation.

Equation (7) implies that

\[ \det(\Lambda)^2 = 1 \]  

which follows from (8) that the Lorentz group has four topologically disconnected components distinguished by

\[ \det(\Lambda) = 1 \quad \Lambda^0_0 \geq 1 \]  

\[ \det(\Lambda) = 1 \quad \Lambda^0_0 \leq -1 \]  

\[ \det(\Lambda) = -1 \quad \Lambda^0_0 \geq 1 \]  

\[ \det(\Lambda) = -1 \quad \Lambda^0_0 \leq -1. \]  

The component with \( \det(\Lambda) = 1 \) and \( \Lambda^0_0 \geq 1 \) contains the identity and is a subgroup. These Lorentz transformations are called proper Lorentz transformations. This subgroup is the symmetry group of special relativity. The other three components involve space and/or time reflections, which are not symmetries of the weak interaction. In what follows all Lorentz transformations will be assumed to be proper transformations unless otherwise specified.

The requirement that quantum observables are independent of inertial coordinate system requires that equivalent states in different inertial coordinate systems are related by a unitary (ray) representation of the proper subgroup of the Poincaré group. The Poincaré group has ten infinitesimal generators that can be expressed as components
of operators that transform as a four vector and an anti-symmetric rank-2 tensor under the unitary representation, $U(\Lambda)$, of the Lorentz group:

$$P^\mu = (H, P)$$  \hspace{1cm} (13)

$$J^{\mu\nu} = \begin{pmatrix}
0 & -K^1 & -K^2 & -K^3 \\
K^1 & 0 & J^3 & -J^2 \\
K^2 & -J^3 & 0 & J^1 \\
K^3 & J^2 & -J^1 & 0
\end{pmatrix}$$  \hspace{1cm} (14)

$$U(\Lambda)P^\mu U^\dagger(\Lambda) = (\Lambda^{-1})^\mu_\nu P^\nu$$  \hspace{1cm} (15)

$$U(\Lambda)J^{\mu\nu}U^\dagger(\Lambda) = (\Lambda^{-1})^\mu_\alpha(\Lambda^{-1})^\nu_\beta J^{\alpha\beta}. $$  \hspace{1cm} (16)

The Pauli-Lubanski vector is the four-vector operator defined by

$$W^\mu = -\frac{1}{2} \epsilon^{\mu\alpha\beta\gamma} P^\alpha J^\beta J^\gamma. $$  \hspace{1cm} (17)

The Lie algebra has two independent polynomial invariants

$$M^2 = -P^\mu P_\mu \quad \text{and} \quad W^2 = W^\mu W_\mu = -M^2 j^2. $$  \hspace{1cm} (18)

When the spectrum of the mass operator, $\sigma(M) > 0$, is positive spin operators are defined by

$$(0, j^x) := -\frac{1}{M} B^{-1}_x (P/M)_{\mu \nu} W^\nu$$  \hspace{1cm} (19)

where $B^{-1}_x (P/M)_{\mu \nu}$ is a matrix of operators that transform $P^\mu$ to $(M, 0, 0, 0)$:

$$B^{-1}_x (P/M)_{\mu \nu} P^\mu = (M, 0). $$  \hspace{1cm} (20)

A standard choice is the canonical (rotationless) boost $B_c(P/M)$ defined by

$$B_c(V := P/M) = \begin{pmatrix}
V^0 \\
V \\
\delta_{ij} + \frac{V^i V^j}{V^2}
\end{pmatrix}. $$  \hspace{1cm} (21)

The subscript $x$ indicates that both $B_x(P/M)$ and $j^x$ are not unique since for any $P$-dependent rotation $R_{xy}(P/M)$

$$B_\rho(P/M)_{\mu \nu} := B_x(P/M)_{\mu \rho} R_{xy}(P/M)^\rho_{\nu}$$  \hspace{1cm} (22)

gives another matrix of operators with property (20); however for any choice $(x)$ the Poincaré commutation relations imply

$$j^2 = W^2/M^2$$  \hspace{1cm} (23)

$$[j^i, j^m] = i \sum_n \epsilon^{imn} j^n$$  \hspace{1cm} (24)

$$[j^i, P^\mu] = 0. $$  \hspace{1cm} (25)

It follows from (19) that the different spin operators are related by

$$(0, j^x)^\mu := B^{-1}_x (P/M)^\mu_\rho B_\rho(P/M)^\rho_{\nu}(0, j^y)^\nu. $$  \hspace{1cm} (26)

The rotation

$$R_{xy}(P/M) := B^{-1}_x (P/M) B_y(P/M)$$  \hspace{1cm} (27)
is called a generalized Melosh rotation \[16\]. The interpretation of \( j \) is that it is the spin that would be measured in the rest frame of a particle if it was Lorentz transformed to the rest frame with the Lorentz transformation \( B_x^{-1}(P/M) \). This provides a mechanism to compare spins in different inertial frames. Different kinds of spin arise because products of rotationless Lorentz boosts can generate rotations. This means that the spin measured in the rest frame depends on the Lorentz transformation to the rest frame. Note that in spite of the 4 indices in (26), the spin is not a 4-vector.

The spin can alternatively be expressed as
\[
j^i_x = \varepsilon_{ijk} B_x^{-1}(P/M)^j \mu B_x^{-1}(P/M)^k \nu J^{\mu \nu},
\]
which can be interpreted as the angular momentum in the particle’s rest frame, which again depends on the Lorentz transformation used to get to the rest frame.

Representations of the Poincaré group can be built up out of irreducible representations. The classification of the irreducible representations depends on the spectrum of invariant operators \( M^2 \) and \( W^2 \) and the sign of \( P^0 \). Wigner [1] classified six classes of irreducible representations by the spectral properties of \( P^2 \) and \( P^0 \):

I. \( P^2 < 0 \), \( P^0 > 0 \)
II. \( P^2 < 0 \), \( P^0 < 0 \)
III. \( P^2 > 0 \)
IV. \( P^2 = 0 \), \( P^0 > 0 \)
V. \( P^2 = 0 \), \( P^0 < 0 \)
VI. \( P^0 = 0 \).

The physically interesting representations for particles are the ones with \(-P^2 = M^2 > 0, \quad P^0 > 0 \) (I) and \( P^2 = 0, \quad P^0 > 0 \) (IV) which are associated with massive and massless particles respectively.

The irreducible representations are induced from a subgroup that leaves a standard vector invariant in each of these classes.

### III. POINCARÉ COVARIANT POSITIVE MASS UNITARY IRREDUCIBLE REPRESENTATIONS

For a particle of mass \( m > 0 \) the mass, spin, and three components of the linear momentum, and one component of \( j_x \) are a maximal set of commuting self-adjoint functions of the infinitesimal generators of the Poincaré group. The standard vector can be taken as \((m,0,0,0)\). The rotation group is called the little group for these representations because it leaves the standard vector invariant. The mass and spin \(^2\) eigenvalues are fixed and label an irreducible subspace. Basis vectors can be taken as simultaneous eigenstates of this maximal set of commuting operators
\[
\lvert (m,j) p, \mu \rangle.
\]

In what follows the normalization convention
\[
\langle (m,j) p', \mu' \lvert (m,j) p, \mu \rangle = \delta(p' - p)\delta_{\mu' \mu}
\]
is used. The eigenvalue spectrum of both \( p \) and \( j_x \cdot \hat{z} \) is fixed by \( j \) and group properties (\( p \) can be boosted to any real value, and the spin components satisfy SU(2) commutation relations \[24\]).

An irreducible unitary representation of the Poincaré group in this basis can be constructed by considering the action of elementary Poincaré transformations on the rest, \( (p = 0) \), eigenstates. On these states rotations can only affect the spin variables since they leave the rest four-momentum (standard vector) unchanged. The total spin constrains the structure of the transformation - it must be a \( 2j + 1 \) dimensional irreducible unitary representation of \( SU(2) \):
\[
U(R,0)\lvert (m,j) 0, \mu \rangle = \lvert (m,j) 0, \nu \rangle D^j_{\nu \mu}[R]
\]
where (see the Appendix)
\[
D^j_{\nu \mu}[R] = \langle j, \nu \lvert U(R,0)\lvert j, \mu \rangle =
\]
are the \(2j+1\) dimensional unitary representations of \(SU(2)\) in the \(|j,\mu\rangle\) basis where

\[
R = e^{i\theta \sigma} = \sigma_0 \cos(\theta/2) + i\theta \cdot \sigma \sin(\theta/2) = \begin{pmatrix} R_{++} & R_{+-} \\ R_{-+} & R_{--} \end{pmatrix}.
\]

(33)

Here \(\sigma_0\) is the \(2 \times 2\) identity and \(\sigma\) are the Pauli spin matrices. The Wigner function \(D[R]\) is a degree \(2j\) polynomial in the components of \(R\). It follows from (32) that \(D_j^{\nu_\mu}[R^*] = (D_{\nu\mu}[R])^*\) and \(D_j^{\nu_\mu}[R^t] = D_{\nu\mu}[R]\).

Space-time translations of the rest state introduce a phase

\[
U(I,a)|(m,j)0,\mu\rangle := e^{-ia^\mu m}(m,j)0,\mu\rangle,
\]

(34)

while Lorentz boosts are unitary operators that change the rest vector to \(p^\mu = (\sqrt{m^2 + p^2}, p)\). A different type of spin is associated with each type of Lorentz boost. The \(x\)-spin is the spin that is unchanged when the basis vector is transformed to a rest vector with the inverse boost \(B_x^{-1}(p/m)\). The following definition is consistent with the requirement that the \(x\)-spin is unchanged when transformed to the rest frame with the inverse boost \(B_x^{-1}(p/m)\):

\[
U(B_x(p/m),0)|(m,j)0,\mu\rangle := |(m,j)p,\mu\rangle \sqrt{\frac{\omega_m(p)}{m}}.
\]

(35)

where \(\omega_m(p) := \sqrt{m^2 + p^2}\) is the energy of the particle. The Jacobian is chosen to make the boost unitary for states with the normalization \((30)\). This can be seen by considering the Lorentz invariant measure

\[
\int d^4p \delta(p^2 + m^2)\theta(p^0) = \int \frac{dp}{2\omega_m(p)} = \int \frac{dp'}{2\omega_m(p')}
\]

where \(p' = \Lambda p\). It follows that

\[
I = \int |p|dp|p| = \int |p'||dp'|(p') = \int |p| \frac{dp'}{2\omega_m(p')} dp'\langle p| = \int |p| \frac{2\omega_m(p)}{2\omega_m(p')} \omega_m(p')d\omega_m(p')
\]

(37)

which leads to the identification

\[
|p'(p)\rangle = |p\rangle \sqrt{\frac{\omega_m(p)}{\omega_m(p')}}
\]

(38)

A general unitary representation of the Poincaré group on any basis state can be expressed as a product of these elementary transformations on rest states using the group representation property:

\[
U(\Lambda, a)|(m,j)p,\nu\rangle = U(I, a)U(\Lambda, 0)|(m,j)p,\nu\rangle =
\]

\[
U(I, a)U(\Lambda, 0)U(B_x(p/m),0)|(m,j)0,\nu\rangle \sqrt{\frac{m}{\omega_m(p)}} =
\]

\[
U(B_x(\Lambda p/m),0)U(B_x^{-1}(\Lambda p/m),0)U(I, a)U(\Lambda, 0)U(B_x(p/m),0)|(m,j)0,\nu\rangle \sqrt{\frac{m}{\omega_m(p)}} =
\]

\[
U(B_x(\Lambda p/m),0)U(I, B_x^{-1}(\Lambda p/m)a)U(B_x^{-1}(\Lambda p/m),0)U(\Lambda, 0)U(B_x(p/m),0)|(m,j)0,\nu\rangle \sqrt{\frac{m}{\omega_m(p)}} =
\]

\[
e^{i\Lambda p a}|(m,j)\Lambda p,\mu\rangle D_{\nu\mu}^{\Lambda p}[B_x^{-1}(\Lambda p/m)\Lambda B_x(p/m)] \sqrt{\frac{\omega_m(\Lambda p)}{\omega_m(p)}}.
\]

(39)
The rotation
\[ R_{wx}(\Lambda, p) := B_{x}^{-1}(\Lambda p/m) \Lambda B_{x}(p/m) \]  
(40)
is called a spin \( x \) Wigner rotation. The final result is the mass \( m \) spin \( j \) irreducible unitary representation of the Poincaré group in the momentum-spin-x basis:

\[ U(\Lambda, a) [(m, j)p, \mu] = e^{i\Lambda p \cdot a} [(m, j)\Lambda p, \nu] D_{\nu \mu}^{ij}[R_{wx}(\Lambda, p)] \sqrt{\frac{\omega_{m}(\Lambda p)}{\omega_{m}(p)}}. \]  
(41)

Since \( U(\Lambda, a) \) is defined as a product of unitary transformations, it is unitary.

The momentum labels can be replaced by any three functions, \( f(p) = f(p, m) \), of the four momentum \( p^{\mu} \) and the spins can be replaced by any type of spin. These replacements correspond to choosing a basis using a different set of commuting observables. Each replacement is just a unitary change of basis. The general form of the change of basis transformation is

\[ |(m, j)f, \mu\rangle_{y} = \]

\[ |(m, j)p(f, m), \nu\rangle_{x} D_{\nu \mu}^{ij}[R_{xy}(p/m)] \sqrt{\frac{\partial p(f, m)}{\partial f}}. \]  
(42)

Combining this with (39) gives the resulting unitary representation of the Poincaré group in the transformed basis

\[ U(\Lambda, a) [(m, j)f, \mu\rangle_{y} = \]

\[ e^{i\Lambda(f) \cdot a} [(m, j)f(\Lambda p), \nu\rangle_{y} D_{\nu \mu}^{ij}[B_{y}^{-1}(\Lambda p(f)/m) \Lambda B_{y}(p(f)/m)] \sqrt{\frac{\partial f(\Lambda p)}{\partial f(p)}}. \]  
(43)

There are four cases that are commonly used. They are distinguished by having some simplifying properties:

\[ f = p, \quad B_{x}(p/m) = B_{x}(p/m) \quad \frac{\partial f(\Lambda p)}{\partial f(p)} = \frac{\omega_{m}(\Lambda p)}{\omega_{m}(p)} \]  
(44)

\[ f = v = p/m, \quad B_{x}(p/m) = B_{x}(p/m) \quad \frac{\partial f(\Lambda p)}{\partial f(p)} = \frac{\omega_{1}(\Lambda v)}{\omega_{1}(v)} \]  
(45)

\[ f = \hat{p} := (p^{+}, p_{\perp}), \quad B_{x}(p/m) = B_{f}(p/m) \quad \frac{\partial f(\Lambda p)}{\partial f(p)} = \frac{(\Lambda p)^{+}}{p^{+}} \quad p^{+} := p^{0} + p^{3}; \quad p_{\perp} = (p^{1}, p^{2}) \]  
(46)

\[ f = p, \quad B_{x}(p/m) = B_{h}(p/m) \quad \frac{\partial f(\Lambda p)}{\partial f(p)} = \frac{\omega_{m}(\Lambda p)}{\omega_{m}(p)} \]  
(47)

these choices are associated with an instant, point, front-form, or Jacob-Wick helicity dynamics. The boost \( B_{x}(p/m) \) is a rotationless boost, \( B_{f}(p/m) \) is a light-front preserving boost, and \( B_{h}(p/m) \) is a helicity boost. These choices lead to different spin observables. The different types of boosts will be defined later. The first three cases are distinguished by the choice of a kinematic subgroup. The kinematic subgroup is the subgroup of the Poincaré group where \( \Lambda p(f) \cdot a, \quad B_{y}^{-1}(\Lambda p(f)/m) \Lambda B_{y}(p(f)/m) \) and \( \frac{\partial f(\Lambda p)}{\partial f(p)} \) are all independent of \( m \). Since the transformations that relate these representations involve the mass, they will generally have different kinematic subgroups. The choices (44) are the largest kinematic subgroups. Kinematic subgroups are useful in dynamical theories because transformations, \( (\Lambda, a) \), in the kinematic subgroup can be computed exactly without having to diagonalize the mass and spin operators using

\[ \langle \phi_{0} | U_{f}(\Lambda, a) | \phi_{1} \rangle = \langle \phi_{1} | U_{0}^{\dagger}(\Lambda, a) | \phi_{0} \rangle. \]  
(48)
Explicit forms of the unitary irreducible representations of the Poincaré group in each of these bases are given below

\[
U(\Lambda, a)(m, j) p, \nu) = e^{i\Lambda^\mu a_{\mu}}(m, j) A_{\mu} D_{\mu\nu}^j [R_{wc}(\Lambda, p)] \sqrt{\frac{\omega_m(\Lambda p)}{\omega_m(p)}}
\]

(instant form)

\[
U(\Lambda, a)(m, j) \hat{p}, \nu) = e^{i\Lambda^\mu a_{\mu}}(m, j) \hat{A}_{\mu} D_{\mu\nu}^j [R_{wf}(\Lambda, p)] \sqrt{\frac{\omega_m(\Lambda p)}{\omega_m(p)}}
\]

(front form)

\[
U(\Lambda, a)(m, j) v, \nu) = e^{i\Lambda^\mu a_{\mu}}(m, j) v D_{\mu\nu}^j [R_{wc}(\Lambda, v)] \sqrt{\frac{\omega_1(\Lambda v)}{\omega_1(v)}}
\]

(point form)

\[
U(\Lambda, a)(m, j) p, \nu) = e^{i\Lambda^\mu a_{\mu}}(m, j) p D_{\mu\nu}^j [R_{wc}(\Lambda, p)] \sqrt{\frac{\omega_m(\Lambda p)}{\omega_m(p)}}
\]

(Jacob-Wick form). These bases are called instant-form, front-form, point-form, and Jacob Wick helicity bases.

In the instant-form case the kinematic subgroup is the six-parameter three-dimensional Euclidean group. In the point-form case, the kinematic subgroup is the six parameter Lorentz group, and in the light-front case the kinematic subgroup is the seven parameter subgroup that leaves the plane \(x^+ = x^0 + x^3 = 0\) invariant.

The light-front boosts have the distinguishing feature that they form a subgroup - so light-front Winger rotations of light-front boosts are the identity. The light-front representation has the largest kinematic subgroup. It is a natural representation for deep inelastic scattering.

The canonical boost has the distinguishing property that the Wigner rotation of a rotation is the rotation. This property is unique to the canonical boost and is useful for adding angular momenta. Both the point-form and instant-form representations use canonical boosts to define the spins.

The helicity boost has the property that the Wigner rotation of any Lorentz transformation is a phase. The helicity spin is related to the canonical spin by \([17] j_h \cdot z = j_c \cdot \hat{p} \). These are the most commonly used Poincaré covariant representations of single-particle states. They are equivalent representations of a free mass \(m\) spin \(j\) particle. They are related by the unitary transformations \([12]\). These unitary equivalences also apply to dynamical theories after the mass and spin are diagonalized.

These representations are the closest representations of single particle states to non-relativistic representations, but they are not the only representations used to describe relativistic particles. In addition to these there are representations that are manifestly Lorentz covariant and representations that are also Euclidean covariant. In order to understand the relation of these representations to the Poincaré covariant representations constructed in this section it is useful to introduce the group \(SL(2, \mathbb{C})\), of complex \(2 \times 2\) matrices with unit determinant, which is the covering group of the Lorentz group. The relation between \(SL(2, \mathbb{C})\) and the Lorentz group is analogous to the relation between \(SU(2)\) and the rotation group \(SO(3)\). It will be developed in the next section.

IV. \(SL(2, \mathbb{C})\)

In order to motivate the connection of \(SL(2, \mathbb{C})\) with the Lorentz group it is useful to represent space-time coordinates by \(2 \times 2\) Hermitian matrices

\[
X = x^\mu \sigma_\mu = \begin{pmatrix}
x^0 + x^3 & x^1 - i x^2 \\
x^1 + i x^2 & x^0 - x^3
\end{pmatrix}
= \begin{pmatrix}
x^+ \\
x^0 - x^3
\end{pmatrix}
\]

(53)
The inverse is
\[ x^\mu = \frac{1}{2} \text{Tr}(\sigma_\mu X) = \frac{1}{2} \text{Tr}(X \sigma_\mu) \] (54)
which follows from properties of the Pauli matrices
\[ \sigma_i \sigma_j = \delta_{ij} \sigma_0 + i \epsilon_{ijk} \sigma_k \] (55)
\[ \text{Tr}(\sigma_i) = 0 \quad \text{Tr}(\sigma_0) = 2 \quad \text{Tr}(AB) = \text{Tr}(BA). \] (56)
The determinant of $X$ is the square of the proper time:
\[ \det(X) = (x^0)^2 - \mathbf{x} \cdot \mathbf{x} = -\eta_{\mu\nu} x^\mu x^\nu = \tau^2. \] (57)
Taking complex conjugates of (54) gives
\[ x^{\mu*} = \frac{1}{2} \text{Tr}(\sigma^{\mu*} X^*) = \frac{1}{2} \text{Tr}((\sigma^{\mu*} X^*)^\dagger) = \frac{1}{2} \text{Tr}(X^\dagger \sigma_\mu) = \frac{1}{2} \text{Tr}(X^\dagger \sigma_\mu) = \frac{1}{2} \text{Tr}(\sigma_\mu X^\dagger). \] (58)
This will be equal to $x^\mu$ if and only if $X = X^\dagger$.
It follows that any linear transformation that preserves both the Hermiticity and the determinant of $X$ must be a real Lorentz transformation.

A general linear transformation of the matrix $X$ has the form
\[ X' = AXB. \] (59)
Hermiticity of $X'$ requires
\[ AXB = B^\dagger X A^\dagger \] (60)
or
\[ A^{-1} B^\dagger X = X B A^{-1\dagger} \] (61)
for any Hermitian $X$. If $X$ is set to the identity this becomes
\[ C := B A^{-1\dagger} = A^{-1} B^\dagger = C^\dagger. \] (62)
Using (62) in (61) gives
\[ CX = XC. \] (63)
This means that for any Hermitian $X$
\[ [X, C] = 0. \] (64)
Since this must be true for $X = \sigma_\mu$ and any complex matrix can be expressed as $M = m^\mu \sigma_\mu$, it follows that $C$ commutes with every complex $2 \times 2$ matrix, so it must be proportional to the identity, $C = cI$, with a real constant $c$ (by Hermiticity). This leads to the relation
\[ B = c A^\dagger. \] (65)
The condition on the determinant requires
\[ c^2 |\text{det}(A)|^2 = 1. \] (66)
The magnitude of $c$ can be absorbed into the matrices by redefining $A \to A' = \frac{1}{\sqrt{|c|}} A$. Then $c = \pm 1$ which gives
\[ B = \pm A^\dagger. \] (67)
The (-1) changes the sign of all components of $X$ so it corresponds to a space-time reflection, which is not in the proper subgroup of the Lorentz group (the component connected to the identity). It follows that
\[ X' = AXA^\dagger \quad \text{det}(A) = 1. \] (68)
The determinant could be allowed to have a phase, but the $\dagger$ will cause the phases to cancel, so there is no loss of generality in choosing the determinant to be 1.
It follows that any $SL(2, \mathbb{C})$ matrix $A$ defines a real proper Lorentz transformation by
\[ \Lambda^\mu_\nu = \frac{1}{2} \text{Tr}(\sigma_\mu A \sigma_\nu A^\dagger). \] (69)
**General form of $A$**

A general invertible complex $2 \times 2$ matrix can always be expressed in exponential form

$$ A = e^M = e^{m^\mu \sigma_\mu}. $$

(70)

The requirement that

$$ 1 = \det(A) = e^{m^\mu \text{Tr}(\sigma_\mu)} = e^{2m^0} $$

holds for $m^0 = n\pi i$. This gives

$$ A = \pm e^{z \sigma} $$

(72)

where $z$ is a complex vector. The minus sign can be absorbed in $z$ since

$$ -I = e^{i\pi \sigma \cdot \hat{a}} $$

(73)

for any unit vector $\hat{a}$, so a general $A \in SL(2, \mathbb{C})$ has the form

$$ A = e^{z \sigma}. $$

(74)

Note that both $A$ and $-A$ have determinant 1 and lead to the same Lorentz transformation since the (-) signs cancel in

$$ X' = AXA^\dagger. $$

(75)

This is the same behavior exhibited by $SU(2)$.

Finally note that $A(z) = e^{z \sigma}$ maps the complex plane into $SL(2, \mathbb{C})$, so any path in $SL(2, \mathbb{C})$ is parameterized by a path in the complex plane that can be contracted to the identity, which implies that $SL(2, \mathbb{C})$ is simply connected.

**Polar decomposition - generalized Melosh rotations and canonical boosts**

$SL(2, \mathbb{C})$ matrices $A$ have polar decompositions

$$ A = (AA^\dagger)^{1/2}(A^\dagger A)^{-1/2} = A(A^\dagger A)^{-1/2}(A^\dagger A)^{1/2} $$

(76)

where $(AA^\dagger)^{1/2}$ and $(A^\dagger A)^{1/2}$ are positive Hermitian matrices and $(AA^\dagger)^{-1/2}A$ and $A(A^\dagger A)^{-1/2}$ are $SU(2)$ matrices. Define

$$ P_l := (AA^\dagger)^{1/2} \quad U_r := (AA^\dagger)^{-1/2}A $$

$$ P_r := (A^\dagger A)^{1/2} \quad U_l := A(A^\dagger A)^{-1/2}. $$

(77)

(78)

Equation (76) implies that a general $SL(2, \mathbb{C})$ matrix $A$ has decompositions of the form

$$ A = P_l U_r = U_l P_r. $$

(79)

The positive Hermitian $SL(2, \mathbb{C})$ matrices have the form

$$ P = e^{\rho \sigma / 2} = \cosh(\rho/2)\sigma_0 + \hat{\rho} \cdot \sigma \sinh(\rho/2) $$

(80)

while the unitary $SL(2, \mathbb{C})$ ones have the form

$$ U = e^{i\theta \sigma / 2} = \cos(\theta/2)\sigma_0 + i\hat{\theta} \cdot \sigma \sin(\theta/2). $$

(81)

The factor of $1/2$ is a convention motivated by the $4 \times 4$ matrix representations of the Lorentz group.

The Lorentz transformation $\Lambda^\mu_\nu$ is related to the $SL(2, \mathbb{C})$ matrix $A$ by

$$ \Lambda^\mu_\nu = \frac{1}{2} \text{Tr}(\sigma_\mu A \sigma_\nu A^\dagger). $$

(82)
It can be computed for both real and imaginary \( z \). In the positive case it is a rotationless or canonical boost. In the unitary case it is a rotation.

\( SL(2, C) \) representatives of canonical boosts are given by:

\[
A = e^{i \rho \sigma}.
\] (83)

This \( A \) has the property that it transforms \( (m, 0) \) to

\[
p'^{\mu} \sigma_{\mu} = A m \sigma_0 A^\dagger \quad \text{where} \quad p'^{\mu} = (\sqrt{m^2 + p^2}, p) = \frac{1}{2} \text{Tr}(\sigma^\mu A m \sigma_0 A^\dagger),
\] (84)

which represents a Lorentz boost with rapidity \( \rho \) defined by

\[
\hat{\rho} = \hat{p} = \hat{v}
\] (85)

and

\[
\sinh(\rho) = \frac{|\hat{p}|}{m} = |v|
\] (86)

\[
\cosh(\rho) = \frac{\hat{p}^0}{m} = v^0
\] (87)

\[
\sinh(\frac{\rho}{2}) = \sqrt{\frac{\hat{p}^0 - m}{2m}} = \sqrt{\frac{v^0 - 1}{2}}
\] (88)

\[
\cosh(\frac{\rho}{2}) = \sqrt{\frac{\hat{p}^0 + m}{2m}} = \sqrt{\frac{v^0 + 1}{2}}
\] (89)

with

\[
A = B_c(v) := B_c(p/m) = \cosh(\rho/2)\sigma_0 + \sinh(\rho/2)\hat{v} \cdot \sigma =
\]

\[
\sqrt{\frac{v^0 + 1}{2}} \sigma_0 + \sqrt{\frac{v^0 - 1}{2}} \hat{v} \cdot \sigma =
\]

\[
\frac{1}{\sqrt{2(v^0 + 1)}} \left((v^0 + 1)\sigma_0 + \hat{v} \cdot \sigma\right) =
\]

\[
\frac{1}{\sqrt{2m(\hat{p}^0 + m)}} \left((\hat{p}^0 + m)\sigma_0 + \hat{p} \cdot \sigma\right)
\] (90)

\[
B_c^\dagger(v) = B_c(v).
\] (91)

The inverse of a canonical boost can be computed by reversing the sign of \( p \) or \( v \) or \( \hat{p} \)

\[
B_c^{-1}(v) = \sigma_2 B_c^\dagger(v) \sigma_2 = \cosh(\omega/2)\sigma_0 - \sinh(\omega)\hat{v} \cdot \sigma =
\]

\[
\sqrt{\frac{v^0 + 1}{2}} \sigma_0 - \sqrt{\frac{v^0 - 1}{2}} \hat{v} \cdot \sigma =
\]

\[
\frac{1}{\sqrt{2(v^0 + 1)}} \left((v^0 + 1)\sigma_0 - \hat{v} \cdot \sigma\right) =
\]
\[ \frac{1}{\sqrt{2m(p^0 + m)}} \left( (p^0 + m)\sigma_0 - p \cdot \sigma \right) . \] (92)

This is NOT true for a general boost. Note that in all of the above expressions for the boosts, \( \epsilon^0 \text{ or } p^0 \) represent "on-shell" quantities.

Finally an important observation in what follows is

\[ B_{\epsilon}\left( p/m \right)^2 = e^{\epsilon \cdot \sigma} = \cosh(\rho)\sigma_0 + \hat{p} \cdot \sigma \sinh(\rho) = \frac{1}{m} p^\mu \sigma_\mu \] (93)

where \( p^0 = \sqrt{m^2 + \vec{p}^2} \). This is a square of the Hermitian matrix, \( e^{\epsilon \cdot \sigma}/2 \), so it is a positive Hermitian matrix.

**Inequivalence of conjugate representation:** \( A \neq SA^*S^{-1} \)

\( SL(2, \mathbb{C}) \) matrices have some important properties. Both \( SL(2, \mathbb{C}) \) and the complex conjugate representation are representations, but they are inequivalent. This means that there is NO single similarity transformation \( S \) that relates the two representations

\[ A^* = SAS^{-1} \] (94)

for all \( A \). To show this note that if (94) holds it follows that for \( A = e^{\frac{z}{2} \sigma^\epsilon} \) that

\[ z \cdot S\sigma S^{-1} = z^* \cdot \sigma^* \] (95)

for all complex \( z \). This can be rewritten

\[ z \cdot S\sigma S^{-1} = -z^* \cdot \sigma_2 \sigma \sigma_2. \] (96)

For the special case that \( z = iy \) is pure imaginary this becomes

\[ y \cdot S\sigma S^{-1} = y \cdot \sigma_2 \sigma \sigma_2. \] (97)

This is because \( \sigma_2 \) is imaginary and anti-commutes with \( \sigma_1 \) and \( \sigma_3 \). Thus for imaginary \( z, S = \sigma_2 C \) where \( C \) is a matrix that commutes with \( \sigma \). The only matrix commuting with all of the Pauli matrices is a constant multiplied by the identity. It follows that \( S = \sigma_2 C \) and \( S^{-1} = C^{-1} \sigma_2 \). The constant factor can be taken as 1 since it does not change the overall similarity transformation. For real \( z \) this requires

\[ \sigma_2 \sigma \sigma_2 = \sigma \] (98)

which is NOT true for \( \sigma_1 \) and \( \sigma_3 \). This shows that in general there is NO \( S \) satisfying

\[ A^* = SAS^{-1} \] (99)

for all \( A \in SL(2, \mathbb{C}) \), however it was demonstrated that

\[ R^* = \sigma_2 R \sigma_2 \] (100)

for all \( A = R \in SU(2) \).

Equation (100) is special case of the general property of \( SL(2, \mathbb{C}) \) matrices

\[ \sigma_2 A \sigma_2 = (A^\dagger)^{-1} \quad \sigma_2 A^* \sigma_2 = (A^\dagger)^{-1}. \] (101)

Equations (99) and (100) mean that while \( SU(2) \) representations are equivalent to the complex conjugate representations, this relation is not true for \( SL(2, \mathbb{C}) \) representations. This fact has implications for structure of Lorentz covariant descriptions of free particles and the treatment of space reflections in these representations.

**Complex Lorentz transformations**
If both $A, B \in SL(2, \mathbb{C})$ then for

$$Y := AXB^t$$

(102)

it still follows that

$$\det Y = \det X \quad \text{but} \quad Y^\dagger \neq Y.$$  

(103)

This means that the pair $(A, B)$ represents a transformation that preserves the proper time, $-x^2 = -y^2$ with $y^{\mu*} \neq y^\mu$ i.e. it is a complex Lorentz transformation.

If $\sigma_0$ is replaced by $i\sigma_0$ and $\sigma_{\epsilon \mu}$ is defined by

$$\sigma_{\epsilon \mu} := (i\sigma_0, \sigma)$$

(104)

then

$$\det(x^\mu_{\epsilon} \sigma_{\epsilon \mu}) = -(x^0_{\epsilon})^2 - x \cdot x$$

(105)

which is ($-$) the square of the Euclidean length of $x^\mu_{\epsilon}$. The Euclidean four vector $x^\mu_{\epsilon}$ can also be represented by a $2 \times 2$ matrix:

$$X_{\epsilon} = x^\mu_{\epsilon} \sigma_{\epsilon \mu}$$

(106)

which can be inverted using

$$x^\mu_{\epsilon} = \frac{1}{2} \text{Tr}(\sigma^\dagger_{\epsilon \mu} X_{\epsilon}).$$

(107)

It follows from (105) that

$$X'_{\epsilon} = AX_{\epsilon}B^t \quad \det(A) = \det(B) = 1$$

(108)

also preserves the Euclidean distance. This means that

$$O^{\mu \nu}(A, B) = \frac{1}{2} \text{Tr}(\sigma^\dagger_{\epsilon \mu} A \sigma_{\epsilon \nu} B^t).$$

(109)

is a complex four-dimensional orthogonal transformation. The result of these observations is that $SL(2, \mathbb{C}) \times SL(2, \mathbb{C})$ represents both complex Lorentz and complex orthogonal transformations. The transpose is included in (108) so the group multiplication property has the form

$$(A', B')(A, B) = (A'A, B'B)$$

(110)

where each factor represents matrix multiplication.

If both $A$ and $B$ are $SU(2)$ matrices, then $(A, B)$ defines a real four-dimensional orthogonal transformation. To show reality when $A$ and $B$ are $SU(2)$ matrices note that the transformed coordinates are

$$y^\mu_{\epsilon} = \frac{1}{2} \text{Tr}(\sigma^\dagger_{\epsilon \mu} AX_{\epsilon}B^t).$$

(111)

Taking complex conjugates (for $A, B \in SU(2)$)

$$y^{\mu*}_{\epsilon} = \frac{1}{2} \text{Tr}(\sigma^\dagger_{\epsilon \mu} A^* X^\epsilon_B B^t).$$

(112)

For $SU(2)$ matrices (100) gives

$$A^* = \sigma_2 A \sigma_2 \quad B^{*t} = \sigma_2 B^t \sigma_2.$$  

(113)

Using (113) in (112) gives

$$y^{\mu*}_{\epsilon} = \frac{1}{2} \text{Tr}(\sigma^\dagger_{\epsilon \mu} \sigma_2 A \sigma_2 X^\epsilon_B \sigma_2 B^t \sigma_2).$$

(114)
For real \( x^\mu \)
\[
\sigma_2 X^*_e \sigma_2 = -X_e
\]
so (114) becomes
\[
y^\mu = \frac{1}{2} \text{Tr}(-\sigma^\dagger_{e\mu} \sigma_2 AX_e B^t \sigma_2) = \frac{1}{2} \text{Tr}(-\sigma_2 \sigma^\dagger_{e\mu} AX_e B^t) = \frac{1}{2} \text{Tr}(\sigma^\dagger_{e\mu} AX_e B^t) = y^\mu.
\]
This shows that pairs of \( SU(2) \) matrices represent real four-dimensional orthogonal transformations. These considerations are relevant for Euclidean representations of relativistic particles.

**Rotations and canonical boosts**

\( SU(2) \) rotations have the form
\[
R = e^{i \hat{\theta} \cdot \sigma / 2} = \cos(\theta/2) \sigma_0 + i \hat{\sigma} \sin(\theta/2)
\]
corresponding to a rotation about the \( \hat{\theta} \) axis by \( \theta \).

The canonical boosts have the important property that the Wigner rotation of a rotation is the rotation. This is shown below. The following notation is used: \( R \) represents an \( SU(2) \) rotation and \( \mathbf{R} \) represents the corresponding \( SO(3) \) rotation:
\[
R e^{\hat{\rho} \cdot \sigma / 2} \mathbf{R}^t = e^{\hat{\rho} \cdot \sigma / 2} R \mathbf{R}^t = e^{\hat{\rho} \cdot (R \sigma)} = e^{\hat{\rho} \cdot (\mathbf{R} \rho)}.
\]
This can be written as
\[
RB_c(p/m)R^t = B_c(Rp/m)
\]
or
\[
R = B_c^{-1}(Rp/m)RB_c(p/m) = R_{wc}(R,p/m).
\]
This property is unique to canonical boosts. The important property is that the Wigner rotation of a rotation is the rotation, independent of \( p \). This means that if a rotation is applied to a many-particle system, where each particle has a different momentum, all of the particles’ spins will Wigner rotate the same way - independent of their momenta. This allows them to be coupled with ordinary Clebsch-Gordan coefficients. Adding angular momenta is most easily preformed by transforming all of the spins to canonical spins.

**Melosh Rotations**

In order to add spins it is necessary to first convert them to canonical spins so they can be added. After adding the spins they can be converted back to their original spin representation. The matrices that transform the spins are generalized Melosh rotations (the original Melosh transformation relates light-front spins to canonical spins).

If a general boost is right multiplied by the inverse of a canonical boost the result is a \( SU(2) \) rotation, since it maps zero momentum to zero momentum
\[
R_{cx}(p/m) = B_c^{-1}(p/m)B_x(p/m).
\]
This can be expressed in the form
\[
B_x(p/m) = B_c(p/m)R_{cx}(p/m)
\]
where \( R_{\text{ex}}(p/m) \) is the \( SU(2) \) (rotation) from the polar decomposition \([17]\) of \( B_x(p/m) \). This is called a generalized Melosh rotation. For \( A = B_x(p) \) the generalized Melosh rotation is given by

\[
R_{\text{ex}} := (AA^\dagger)^{-1/2} A = (B_x(p/m)B_x(p/m)^\dagger)^{-1/2} B_x(p/m).
\]

while the associated canonical boost is

\[
B_c(p/m) = (AA^\dagger)^{1/2}.
\]

An important observation is that

\[
B_x(p/m)B_x^{\dagger}(p/m) = B_c(p/m)R_{\text{ex}}(p/m)R_{\text{ex}}^\dagger(p/m)B_c(p/m) = B_c^2(p/m) = \frac{\not p}{m} \theta \not p \frac{\not p}{m}
\]

independent of \( x \). This is a consequence of the polar decomposition of the \( SL(2, \mathbb{C}) \) matrices. It will be used to show that Dirac’s forms of dynamics are irrelevant in Lorentz and Euclidean covariant representations of relativistic quantum mechanics.

The generalized Melosh rotations are used to change the type of spins \((y \to x)\):

\[
\langle (m,j)|p,\mu\rangle_x = U(B_x(p/m))\langle (m,j)|0,\mu\rangle_x \sqrt{\frac{m}{\omega_m(p)}} = U(B_y(p/m))U(B_y^{-1}(p/m)B_x(p/m))\langle (m,j)|0,\mu\rangle_x \sqrt{\frac{m}{\omega_m(p)}} = U(B_y(p/m))\langle (m,j)|0,\nu\rangle_x D_{\nu\mu}^{ij}[B_y^{-1}(p/m)B_x(p/m)] \sqrt{\frac{m}{\omega_m(p)}} = \langle (m,j)|p,\nu\rangle_y D_{\nu\mu}^{ij}[B_y^{-1}(p/m)B_x(p/m)]
\]

\( SL(2, \mathbb{C}) \) representations of light-front boosts:

The light-front is the hyper-plane defined by points satisfying \( x^+ = x^0 + x^3 = 0 \). The kinematic subgroup of the light front is the subgroup of Poincaré group that preserves \( x^+ = 0 \).

In \( SL(2, \mathbb{C}) \) the Lorentz transformations in this subgroup are represented by lower triangular matrices. \( SL(2, \mathbb{C}) \) representatives of light-front boosts are given by:

\[
B_f(v) := \begin{pmatrix} \sqrt{v^+} & 0 \\ v_+ / \sqrt{v^+} & \sqrt{v^+} \end{pmatrix} = \begin{pmatrix} \alpha & 0 \\ \beta / \alpha & 1 / \alpha \end{pmatrix} \tag{128}
\]

\[
B_f^{-1}(v) := \begin{pmatrix} 1 / \sqrt{v^+} & 0 \\ -v_+ / \sqrt{v^+} & \sqrt{v^+} \end{pmatrix} = \begin{pmatrix} 1 / \alpha & 0 \\ -\beta / \alpha & \alpha \end{pmatrix} \tag{129}
\]

\[
B_f^{\dagger}(v) := \begin{pmatrix} \sqrt{v^+} & 0 \\ 0 & 1 / \sqrt{v^+} \end{pmatrix} = \begin{pmatrix} \alpha & \beta^* / \alpha \\ 0 & 1 / \alpha \end{pmatrix} \tag{130}
\]

\[
\hat{B}_f(v) := \begin{pmatrix} 1 / \sqrt{v^+} & 0 \\ -v_+ / \sqrt{v^+} & \sqrt{v^+} \end{pmatrix} = \begin{pmatrix} 1 / \alpha & \beta^* / \alpha \\ 0 & \alpha \end{pmatrix} \tag{131}
\]

where \( \alpha := \sqrt{v^+} = \sqrt{p^+ / m} \) and \( \beta := v_+ := (p_1 + ip_2) / m \). In \( \hat{A} \) and in what follows the notation \( \hat{A} := (A^\dagger)^{-1} \) is used.

These lower triangular matrices with real quantities on the diagonal form a group. This is the subgroup of light-front boosts. The light-front boost subgroup can be expressed in terms of the light-front components of the four momentum and mass as:

\[
B_f(p) := \frac{1}{\sqrt{mp^+}} \begin{pmatrix} p^+ & 0 \\ p_\perp & m \end{pmatrix} \tag{132}
\]

\[
B_f^{-1}(p) := \frac{1}{\sqrt{mp^+}} \begin{pmatrix} m & 0 \\ -p_\perp & p^+ \end{pmatrix} \tag{133}
\]
These boosts are used to define light front spins. These boost form a subgroup, since lower triangular $SL(2, \mathbb{C})$ matrices with real values on the diagonal form a subgroup. This implies that light-front boosts do not change the light front spin.

**$SL(2, \mathbb{C})$ representations of Helicity boosts:**

Helicity boosts are defined by

$$B_h(p/m) := B_c(p/m) R(\hat{z} \rightarrow \hat{p}) = R(\hat{z} \rightarrow \hat{p}) B_c(p_z/m).$$

where the rotation

$$R(\hat{z} \rightarrow \hat{p}) = \sqrt{\frac{1 + \hat{z} \cdot \hat{p}}{2}} \sigma_0 + \sqrt{\frac{1 - \hat{z} \cdot \hat{p}}{2}} (\hat{z} \times \hat{p}) \cdot \sigma_{\hat{z} \times \hat{p}}.$$ (138)

The associated helicity-spin Wigner rotation is

$$R_{wh}(\Lambda, p) = R^{-1}(\hat{z} \rightarrow \hat{\Lambda} \hat{p}) B_c^{-1}(\Lambda p/m) \Lambda B_c(p/m) R(\hat{z} \rightarrow \hat{p})$$ (139)

which is always a rotation about the $z$ axis. Because of this property the Wigner D-function of the Jacob-Wick helicity Wigner rotation is always a phase.

The helicity spin and canonical spin are related by

$$j_h \cdot \hat{z} = j_c \cdot \hat{p}$$ (140)

so the $z$ component of the helicity spin is the canonical spin projected in the direction of the momentum. This projection is the better known definition of the Jacob-Wick helicity.

**Lorentz Spinors:**

The transformation property of a four vector represented by a $2 \times 2$ Hermitian matrix can be expressed in tensor form as

$$X^{a\dot{a}} \rightarrow X'^{a\dot{a}} := A^{ab} A^{*\dot{a}\dot{b}} X'^{b\dot{b}}$$ (141)

where repeated matrix indices are assumed to be summed over two values. This looks like a rank two tensor with one index transforming under $SL(2, \mathbb{C})$ and one under the inequivalent complex conjugate representation.

This motivates the definition of Lorentz spinors. These are two-component vectors that transform under either of these representations.

The two-component spinors are characterized by their transformation properties

$$\xi^a \rightarrow \xi'^a = A^{ab} \xi^b \quad \xi^{\dot{a}} \rightarrow \xi'^{\dot{a}} = A^{*\dot{a}\dot{b}} \xi^{\dot{b}}$$ (142)

where a sum over repeated spinor indices is assumed. These transformation properties define two different types of two spinors that transform under the regular and complex conjugate representations of $SL(2, \mathbb{C})$. The upper undotted or dotted indices identify the transformation properties. These are referred to as right- and left-handed spinors respectively. The reason for this designation will be discussed later.
It is possible to construct Lorentz invariant quadratic forms with either of these types of spinors. This follows from the general property of $SL(2,C)$ matrices \([101]\):

\[
\sigma_2 A \sigma_2 = (A^{-1})^t.
\] (143)

This leads to the definition of the metric spinor

\[
\varepsilon_{ab} = -\varepsilon^{ab} = i(\sigma_2)_{ab} \quad \varepsilon_{\dot{a}\dot{b}} = -\varepsilon^{\dot{a}\dot{b}} = i(\sigma_2)_{\dot{a}\dot{b}}
\] (144)

and lower indexed spinors

\[
\xi_a := \varepsilon_{ab} s^b \quad \xi_{\dot{a}} := \varepsilon_{\dot{a}\dot{b}} s^\dot{b}.
\] (145)

The transformation properties of the lower index spinors are

\[
\xi_a \rightarrow \xi_a' = \varepsilon_{ab} A^{bc} \varepsilon_{\dot{d}\dot{e}} \xi_{\dot{e}} = (A^t)^{-1ab} \xi_b
\] (146)

and

\[
\xi_{\dot{a}} \rightarrow \xi_{\dot{a}}' = \varepsilon_{\dot{a}\dot{b}} A^{\dot{b}\dot{c}} \varepsilon^{\dot{d}\dot{e}} \xi_{\dot{e}} = (A^\dagger)^{-1\dot{a}\dot{b}} \xi_{\dot{b}}.
\] (147)

The metric spinor, $\varepsilon_{ab}$ could also be taken to be $(\sigma_2)_{ab}$. It has the advantage that there are no sign changes on raising and lowering indices, but the disadvantage is that it is not real. Equations (146)-(147) show that the lower undotted and dotted indices have different transformation properties than the corresponding upper indices.

The metric spinor can be used to construct Lorentz invariant scalars by contracting upper and lower indexed spinors of the same type (dotted or undotted)

\[
\chi_a^b \xi^a = (A^t)^{-1ab} \chi_b A^{ac} \xi^c = \chi_b (A)^{-1ba} A^{ac} \xi^c = \chi_a \xi^a
\] (148)

and

\[
\chi_{\dot{a}} \xi^\dot{a} = (A^\dagger)^{-1\dot{a}\dot{b}} \chi_{\dot{b}} A^{\star \dot{a} \dot{c}} \xi^\dot{c} = \chi_{\dot{b}} (A)^{\star -1\dot{b} \dot{a}} A^{\star \dot{a} \dot{c}} \xi^\dot{c} = \chi_{\dot{a}} \xi^\dot{a}.
\] (149)

It follows from the anti-symmetry of $\varepsilon_{ab}$ that

\[
\xi^a \xi_a = \varepsilon_{ab} \xi^a \xi^b = 0 \quad \xi^\dot{a} \xi_{\dot{a}} = \varepsilon_{\dot{a}\dot{b}} \xi^\dot{a} \xi^\dot{b} = 0.
\] (150)

The tensor product of a 2-spinor with its complex conjugate,

\[
X^{ab} := \xi^a \xi^b,
\] (151)

defines a real four vector; since it is Hermitian and the determinant vanishes this defines a light-like four vector. It follows from \([158]\) and \([149]\) that

\[
\xi^a \chi_a \quad \xi^\dot{a} \chi_{\dot{a}}
\] (152)

are both invariant quadratic forms under $SL(2,C)$. These forms are neither positive nor sesquilinear. Thus they cannot be used to construct a positive invariant scalar product. However in terms of the spinor indices it is useful to define the following 4-momentum dependent $2 \times 2$ Hermitian matrices that transform like products of right and left handed spinors

\[
P^{\dot{a}\dot{a}} := (p^\mu \sigma_\mu)^{\dot{a}\dot{a}}
\] (153)

\[
P_{a\dot{a}} := p^\mu (\sigma_2 \sigma_\mu \sigma_2)_{a\dot{a}}
\] (154)

\[
P^{\dot{a}a} = (p^\mu \sigma_\mu)^{\dot{a}a}
\] (155)

\[
P_{a\dot{a}} = (p^\mu \sigma_\mu \sigma_2)_{a\dot{a}}
\] (156)
The matrices \([157, 158, 159, 160]\) are all positive definite (see \([61]\)) if \(p\) is a time-like positive energy four vector. They satisfy the following covariance properties
\[
A^{ab} P^{bc} A^{cd} := (\Lambda p)^{\mu} \sigma_{\mu i}^{ad}
\]
(157)
\[
(A^\dagger)^{-1}_{ab} P_{bc}(A^\dagger)^{-1}_{cd} := (\Lambda p)^{\mu} (\sigma_2 \sigma_1 \sigma_2)_{\mu i}^{ad}
\]
(158)

\[
A^{\ast \dot{a} \dot{b}} P^{\dot{b}c} A^{tcd} = (\Lambda p)^{\mu} \sigma_{\mu \dot{i}}^{\ast \dot{a} \dot{d}}
\]
(159)
\[
(A^\dagger)^{-1}_{\dot{a} \dot{b}} P_{\dot{b}c} A^{-1}_{cd} = (\Lambda p)^{\mu} (\sigma_2 \sigma_1 \sigma_2)_{\mu \dot{i}}^{\ast \dot{a} \dot{d}}
\]
(160)

Because they are positive they can be used as kernels of the invariant positive sesquilinear forms:
\[
\xi_\alpha \xi_\dot{\alpha} P^{\alpha \dot{\alpha}} = \xi_\alpha \xi_\dot{\alpha} P^{\alpha \dot{\alpha}} \geq 0
\]
(161)
\[
\xi_\alpha \xi_\dot{\alpha} P_{\alpha \dot{\alpha}} = \xi_\alpha \xi_\dot{\alpha} P_{\alpha \dot{\alpha}} \geq 0.
\]
(162)

The following identity is important in what follows,
\[
(p^\mu \sigma_2 \sigma_1 \sigma_2)^{\alpha \dot{a}} = (pP)^{\mu} \sigma_{\mu \alpha \dot{a}}
\]
(163)

where \(P\) represents a space reflection.

The matrices \(P^{\alpha \dot{a}}/m, P^{\dot{a} \alpha}/m, P^{\alpha \dot{a}}/m, P_{\alpha \dot{a}}/m\) are all \(SL(2, \mathbb{C})\) matrices. The \(SL(2, \mathbb{C})\) spins can be added like \(SU(2)\) spins with \(SU(2)\) Clebsch-Gordan coefficients. This is because the \(SU(2)\) identities
\[
\sum \langle j, \mu | j_1, \mu_1, j_2, \mu_2 \rangle D^{j_1}_{\mu \mu_1}[R] D^{j_2}_{\mu \mu_2}[R] \langle j, \mu' | j_1, \mu_1', j_2, \mu_2' \rangle - D^{j}_{\mu \mu'}[R] = 0,
\]
(164)
\[
\sum \langle j, \mu | j_1, \mu_1, j_2, \mu_2 \rangle D^{j}_{\mu \mu'}[R] \langle j, \mu' | j_1, \mu_1', j_2, \mu_2' \rangle - D^{j}_{\mu \mu'}[R] D^{j}_{\mu \mu'}[R] = 0
\]
(165)

also hold when \(R\) is replaced by a \(SL(2, \mathbb{C})\) matrix \(A\). This follows because both sides of these equations are finite degree polynomials in the four components of \(R\) which are entire analytic functions of real angles. This means that the left side of these equations are entire functions of three complex angles that vanish when all three angles are real. It follows by analytic continuation that they vanish for complex angles. Thus they hold when \(R \rightarrow A\) for \(A \in SL(2, \mathbb{C})\). This means that there are higher spin versions of the positive kernels \([157, 160]\). In the next section the same method will be used to show that \(D^{j}_{\mu \nu}[A]\) is a \(2j + 1\) dimensional representation of \(SL(2, \mathbb{C})\).

These relations can be used use to construct \(2j + 1\) dimensional representations of \(SL(2, \mathbb{C})\) that transform under
\[
D^{j}_{\mu \nu}[A], D^{j}_{\mu \nu}[A^\dagger], D^{j}_{\mu \nu}[(A^\dagger)^{-1}], \text{ or } D^{j}_{\mu \nu}[(A^\dagger)^{-1}].
\]
(166)

from the corresponding 2-component \(j = 1/2\) spinors. In these expression the notation using the upper and lower dotted and undotted indices is not used.

V. LORENTZ COVARIANT REPRESENTATIONS

The unitary representation of the Poincaré group for a particle of mass \(m\) and spin \(j\) has the form \([61]\)
\[
U(\Lambda, a)(m, j)|p, \nu\rangle = e^{i\Lambda p \cdot a}(m, j)\Lambda p, \mu) D^{j}_{\mu \nu}[R_{\nu w}(\Lambda)] \frac{\omega_m(\Lambda p)}{\omega_m(p)}.
\]
(167)

or one of the related forms \([49, 52]\).

In what follows the notation for \(SL(2, \mathbb{C})\) matrices
\[
\tilde{\Lambda} := (A^\dagger)^{-1} = \sigma_2 A^* \sigma_2
\]
(168)
is used. The spin-$x$ Wigner rotation can be written in either of two equivalent ways

\[ D^j_{\nu\mu}[B_x^{-1}(\Lambda p/m)\Lambda B_x(p/m)] = D^j_{\nu\mu}[\tilde{B}_x^{-1}(\Lambda p/m)\tilde{\Lambda}\tilde{B}_x(p/m)], \]

(169)

where $\Lambda$ and $\Lambda$ are related by (82). This is because $\tilde{R} := (R^\dagger)^{-1} = R$ for $R \in SU(2)$.

The Wigner function can be written in either of two equivalent ways

\[ D^j_{\nu\mu}[e^{z_\theta}], \]

(170)

is a finite degree polynomial of entire analytic functions of the three components of $\theta$. It satisfies the group representation property (the matrix indices are suppressed):

\[ D^j[R_2]D^j[R_1] - D^j[R_2R_1] = 0 \]

(171)

for $R_1, R_2 \in SU(2)$. Since the left side is an entire function of all 6 angle variables, $(\theta_1, \theta_2)$, that is 0 for all real variables, by analytic continuation the group representation property holds for complex angles $i\theta \rightarrow \theta = \rho + i\theta$. It follows that $D^j[A]$ is a also 2$j + 1$ dimensional representation of $SL(2, \mathbb{C})$.

This means that the Wigner rotation can be factored. There are two possible factorizations that arise because while $\tilde{R} = R$, this is not true for the $SL(2, \mathbb{C})$ transformations that are used to define the Wigner rotation. This is due to the inequivalence of the two conjugate representations of $SL(2, \mathbb{C})$. This leads to the following two factorizations of the Wigner rotation

\[ D^j_{\nu\mu}[B_x^{-1}(\Lambda p/m)\Lambda B_x(p/m)] = \left(D^j[B_x^{-1}(\Lambda p/m)]D^j[A]D^j[B_x(p/m)]\right)_{\nu\mu} \]

(172)

and

\[ D^j_{\nu\mu}[B_x^{-1}(\Lambda p/m)\Lambda B_x(p/m)] = \left(D^j[B_x^{-1}(\Lambda p/m)]D^j[\tilde{\Lambda}]D^j[\tilde{B}_x(p/m)]\right)_{\nu\mu}. \]

(173)

Using these factorizations and the group representation property, can be equivalently written as

\[ U(\Lambda, a) |(m, j)p, \nu\rangle = \sqrt{\omega_m(p)} e^{i\Lambda p - a} |(m, j)p, \nu\rangle \]

(174)

or

\[ U(\Lambda, a) |(m, j)p, \nu\rangle = \sqrt{\omega_m(p)} e^{i\Lambda p - a} |(m, j)p, \nu\rangle \]

(175)

This leads to the definition of two types of Lorentz covariant states

\[ |(m, j)p, \mu\rangle_{\text{cov}} := \]

(176)

and

\[ |(m, j)p, \nu\rangle_{\text{cov}} := \]

(177)

These are called right and left handed Lorentz covariant states.

For these states equations (173) and (175) have the form

\[ U(\Lambda)(m, j)p, \mu\rangle_{\text{cov}} = |(m, j)\Lambda p, \nu\rangle_{\text{cov}} D^j_{\nu\mu}[A] \]

(178)
\[ U(\Lambda) |(m, j)p, \mu\rangle_{\text{cov}} = |(m, j)p, \nu\rangle_{\text{cov}} D_{\nu \mu}^j[A]. \]  

(179)

This appears to violate the condition that there are no finite dimensional unitary representations of the Lorentz group. The reason that it does not is because the Hilbert space inner product in this representation has a non-trivial momentum-dependent kernel. To see this it is instructive to write out the inner product of two vectors in these representations. Starting with the Poincaré covariant representation

\[
\langle \psi | \phi \rangle = \int \langle \psi | (m, j)p, \mu \rangle_x dp_x \langle (m, j)p, \mu | \phi \rangle =
\]

\[
\int \langle \psi | (m, j)p, \nu \rangle_{\text{cov}} D_{\nu \mu}^j[B_x(p/m)B_x^\dagger(p/m)] \frac{dp}{\omega_m(p)} \langle (m, j)p, \mu | \phi \rangle =
\]

\[
\int \langle \psi | (m, j)p, \nu \rangle_{\text{cov}} D_{\nu \mu}^j[B_c(p/m)B_c(p/m)] 2\delta(p^2 + m^2) d^4p \delta(p^0)_{\text{cov}} \langle (m, j)p, \mu | \phi \rangle =
\]

\[
\int \langle \psi | (m, j)p, \nu \rangle_{\text{cov}} D_{\nu \mu}^j[p \cdot \sigma / m] 2\delta(p^2 + m^2) d^4p \delta(p^0)_{\text{cov}} \langle (m, j)p, \mu | \phi \rangle. \]  

(180)

Similarly for the left handed covariant representation

\[
\langle \psi | \phi \rangle =
\]

\[
\int \langle \psi | (m, j)p, \nu \rangle_{\text{cov}} D_{\nu \mu}^j[p \cdot \sigma^* / m] 2\delta(p^2 + m^2) d^4p \delta(p^0)_{\text{cov}} \langle (m, j)p, \mu | \phi \rangle. \]  

(181)

Here (127) was used to replace the \( x \)-boosts by canonical boosts. The Wigner functions in (180) have the form (suppressing the spin indices)

\[ D^j[p \cdot \sigma / m] = D^j[B^2_c] = D^j[B^\dagger_c] D^j[B_c] > 0 \]  

(182)

and in (183)

\[ D^j[p \cdot \sigma^* \sigma_2 / m] = D^j[B^{-2}_c] = D^j[B^{-\dagger}_c] D^j[B^{-1}_c] > 0 \]  

(183)

so they are positive kernels (note that these kernels are Hermitian since \( D^j_{\mu \nu}[A^\dagger] = D^j_{\nu \mu}[A] = (D^j_{\nu \mu}[A^\star])^* \) follows from (32).

The covariant kernels

\[ D_{\nu \mu}^j[p \cdot \sigma / m] 2\delta(p^2 + m^2) d^4p \delta(p^0) \]  

(184)

and

\[ D_{\nu \mu}^j[p \cdot \sigma^* \sigma_2 / m] 2\delta(p^2 + m^2) d^4p \delta(p^0) \]  

(185)

are spin \( j \)-Wightman functions for right and left handed free spin-\( j \) particles. They are \( 2j + 1 \) dimensional representations of the positive forms (157) and (160).

Because

\[ p \cdot \sigma^* \sigma_2 / m = (Pp) \cdot \sigma / m, \]

(186)

where \( P \) changes the sign of the spatial components of \( p \), the right and left handed representations are related by space reflection. All of these transformations are invertible so starting from any one of them it is possible to return to any standard Poincaré covariant description. As long as space reflection is not needed, these are all equivalent descriptions of a mass \( m \) spin \( j \) particle.

To understand the role of space reflections note that taking the complex conjugate of

\[ X' = AXA^\dagger \]  

(187)
implies
\[ X^* = A^* X A'. \] (188)

It follows that \( X \) and \( X^* \) transform under inequivalent representations of \( SL(2, \mathbb{C}) \). The operation \( X \to X^* \) changes the sign of \( y \) which is equivalent to a space reflection followed by a rotation by \( \pi \) about the \( y \) axis. This shows that space reflection maps right handed to left handed representations of the Hilbert space.

In the \( 2 \times 2 \) matrix representation space reflection is represented by
\[ X \to X' = \sigma_2 X^* \sigma_2. \] (189)

This operation changes \( A \) to \( \tilde{A} := \sigma_2 A^* \sigma_2 = (A^\dagger)^{-1} \). The problem with space reflections in Lorentz covariant representations is that the kernel of the Hilbert space representation changes, to the kernel for an inequivalent representation, so space reflection cannot be represented in the Hilbert space with the original Lorentz covariant kernel because it will not transform correctly with respect to Lorentz transformations.

The way to remedy this is to use a direct sum, where both kernels appear on the diagonal. Then space reflection can be realized on the direct sum space by changing the sign of \( p \) and interchanging the components of the direct sum.

In this case the representation of the Lorentz group is the chiral representation
\[ S[A] = \left( \begin{array}{cc} D^j[A] & 0 \\ 0 & D^j[\tilde{A}] \end{array} \right) \] (190)

and the kernel of the Hilbert space inner product is
\[ \delta(p^2 + m^2)\theta(p^0) \left( \begin{array}{cc} D^j[p \cdot \sigma/m] & 0 \\ 0 & D^j[p \cdot \sigma_2 \sigma^* \sigma_2/m] \end{array} \right) = \delta(p^2 + m^2)\theta(p^0) \left( \begin{array}{cc} D^j[p \cdot \sigma/m] & 0 \\ 0 & D^j[(Pp) \cdot \sigma/m] \end{array} \right). \] (191)

The operation of space reflection on wave functions in the doubled space becomes
\[ P \left( \begin{array}{c} \text{cov}\langle(m, j) | \mu, \phi_1 \rangle \\ \text{cov}\langle(m, j) | \mu, \phi_2 \rangle \end{array} \right) = \left( \begin{array}{c} \text{cov}\langle(m, j) - \mu, \phi_1 \rangle \\ \text{cov}\langle(m, j) - \mu, \phi_2 \rangle \end{array} \right). \] (192)

The kernels appearing in (191) arise naturally because they come from the \( SU(2) \) equivalence of \( R \) and \( \tilde{R} \), however the spin kernel \( D^j[p \cdot \sigma/m] \) could be replaced by \( D^j[p \cdot \sigma_2 \sigma^* \sigma_2/m] \) and \( D^j[p \cdot \sigma_2 \sigma^* \sigma_2/m] \) could be replaced by \( D^j[p \cdot \sigma^* /m] \) which involve different equivalent representations of the right and left handed spinor degrees of freedom.

An important observation is that the choice of kinematic variables replacing \( p \) and the choice of boost in the spin representation that characterize the Poincaré covariant forms of the dynamics has disappeared in the Lorentz covariant representations. The spins transform under a \( 2j + 1 \) dimensional representation of \( SL(2, \mathbb{C}) \). This means that there are “no forms of dynamics” in Lorentz covariant representations.

Another observation is that in the Lorentz covariant representations the Hilbert space kernels (184) and (185) have a mass dependence, which for free particles defines the dynamics. In a dynamical Lorentz covariant model the kernel of the Hilbert space inner product carries the dynamical content of the theory.

**\( SL(2, \mathbb{C}) \times SL(2, \mathbb{C}) \) spinors**

In order to understand the role played by the spinor degrees of freedom in Euclidean representations of relativistic quantum mechanics it is useful to define \( SL(2, \mathbb{C}) \times SL(2, \mathbb{C}) \) spinors.

Let \( Z := z^\mu \sigma_\mu \) denote a complex 4 vector represented as a \( 2 \times 2 \) matrix. Complex Lorentz transformations are given by
\[ Z \to Z' = AZB^t \] (193)

where both \( A \) and \( B \) are \( SL(2, \mathbb{C}) \) matrices.

In this representation complex space reflection, which transforms \((z^0, z^1, z^2, z^3) \) to \((z^0, -z^1, -z^2, -z^3) \) can be expressed in matrix form as
\[ Z \to Z' = PZ = \sigma_2 Z^t \sigma_2. \] (194)

The transformation properties of \( Z \) imply the transformation properties of \( Z' := PZ \):
\[ PZ \to PZ' = \sigma_2 (BZ^t A^\dagger) \sigma_2 = (B^{-1})^t PZA^{-1}. \] (195)
This means the under space reflection the complex spinor transformation properties are replaced by

\[ A \rightarrow (B^t)^{-1}, \quad B \rightarrow (A^t)^{-1}. \] (196)

This suggests defining right and left handed \( SL(2, \mathbb{C}) \times SL(2, \mathbb{C}) \) spinors by their transformation properties

\[ \xi^a \rightarrow A^{ab} \xi^b \] (197)

\[ \chi^{\dot{a}} \rightarrow B^{\dot{a}\dot{b}} \chi_\dot{b} \] (198)

\[ \xi_a \rightarrow ((A^t)^{-1})_{ab} \xi_b \] (199)

\[ \chi_{\dot{a}} \rightarrow ((B^t)^{-1})_{\dot{a}\dot{b}} \chi_{\dot{b}} \] (200)

These definitions recover the \( SL(2, \mathbb{C}) \) transformation properties of right and left handed spinors when \( B = A^* \). When \( (A, B) \in SU(2) \times SU(2) \) these relations define the transformation properties of right and left handed Euclidean spinors.

The definitions (197, 200) are consistent with the the upper and lower index spinors being related by \( \epsilon_{ab} \) and \( \epsilon^{ab} \):

\[ \xi_a = \epsilon_{ab} \xi^b \quad \chi_{\dot{a}} = \epsilon^{\dot{a}\dot{b}} \chi_\dot{b} \] (201)

and the contraction of an upper and lower index spinor of the same type (un-dotted or dotted) being invariant under \( SL(2, \mathbb{C}) \times SL(2, \mathbb{C}) \).

The Wigner functions \( D^j_{\mu\nu}[R] \) are finite degree polynomials in the four components of \( R \in SU(2) \) and the components of the \( SU(2) \) matrices are entire analytic functions of angles. This means that the relations

\[ D^j_{\mu\alpha}[R_2]D^i_{\alpha\nu}[R_1] - D^j_{\mu\nu}[R_2R_1] = 0 \] (202)

\[ \langle j, \mu|j_1, \mu_1, j_2, \mu_2 \rangle D^j_{\mu\nu_1}[R]\langle R|D^j_{\mu_2\nu_2}[R]\langle j, \nu|j_1, \nu_1, j_2, \nu_2 \rangle - D^j_{\mu\nu}[R] = 0 \] (203)

\[ \langle j, \mu|j_1, \mu_1, j_2, \mu_2 \rangle D^j_{\mu\nu}[R]\langle R|D^j_{\mu_2\nu_2}[R]\langle j, \nu|j_1, \nu_1, j_2, \nu_2 \rangle - D^j_{\mu_1\nu_1}[R]\langle R|D^j_{\mu_2\nu_2}[R] = 0 \] (204)

are entire functions of angles that vanish for all real values of the angles. Since these functions are entire, they hold for all complex angles by analytic continuation. This means that all three relations hold for \( R \in SU(2) \rightarrow A \in SL(2, \mathbb{C}) \).

This means that both the group representation property and addition of “spins” extend unchanged to \( SL(2, \mathbb{C}) \).

VI. EUCLIDEAN COVARIANT REPRESENTATIONS OF RELATIVISTIC QUANTUM MECHANICS

In the same way that Poincaré covariant representations were used to construct equivalent Lorentz covariant representations of any spin, the Lorentz covariant representations can be used to construct equivalent Euclidean covariant representations.

Euclidean formulations of relativistic quantum mechanics are used in path-integral representations, lattice calculations and with Schwinger-Dyson equations.

While the transformation from a Euclidean covariant formalism to a Lorentz covariant formalism normally requires an analytic continuation, a fully relativistic form of quantum mechanics can be formulated without explicit analytic continuation. It requires that the Euclidean analogs of the kernel of the inner product satisfies a condition called reflection positivity \[ \mathcal{R} \in [0, 1]. \] For irreducible representations this condition can be satisfied for any spin.

The Euclidean representation of relativistic quantum mechanics has a Hilbert space inner product that is defined by a kernel that is a Euclidean covariant distribution left multiplied by a Euclidean time reflection. Both the initial and final states have to vanish for negative Euclidean times. The requirement that the resulting quadratic form is non-negative is called reflection positivity.

In order to make contact with the Lorentz covariant representations discussed above consider vectors represented by Euclidean covariant spinor valued functions \( \langle \tau, x, \mu|\psi \rangle \) of four Euclidean space-time variables with support for positive Euclidean time. The transformation properties of the spinor degrees of freedom will be discussed in the next section.
In the Euclidean representation of relativistic quantum mechanics of a particle of mass \( m \) and spin \( j \) the quantum mechanical inner product is defined by

\[
\langle \phi | \psi \rangle := \frac{1}{\pi} \int \sum (\psi - \tau_x, x, \mu) e^{i p \cdot (x - y)} \frac{d^4 x}{p^2 + m^2} D^{ij}_{\mu \nu}[p \cdot \sigma_e / m] \langle \tau_y, y, \nu | \psi \rangle d^4 x d^4 y d^4 p =
\]

\[
\frac{1}{\pi} \int \sum (\phi | \tau_x, x, \mu) e^{-i p \cdot (x - y) + i p \cdot (x - y)} \frac{d^4 x}{p^0 - i \omega_m(p)} (p^0 + i \omega_m(p)) D^{ij}_{\mu \nu}[p \cdot \sigma_e / m] \langle \tau_y, y, \nu | \psi \rangle d^4 x d^4 y d^4 p
\]

(205)

where \( \omega_m(p) = \sqrt{p^2 + m^2} \) is the energy of a particle of mass \( m \) and momentum \( p \), and all of the integration variables are Euclidean. The \( - \) sign on \( \tau_x \) in the first term represents the Euclidean time reflection discussed above. In the second term the substitution \( \tau_x \rightarrow - \tau_x \) was made. This, along with the Euclidean time support condition of the wave functions, ensures that \( \tau_x + \tau_y \) in the exponent of the second term is positive.

To evaluate the \( p^0 \) integral, the \( p^0 \)’s appearing in \( D^{ij}_{\mu \nu}[p \cdot \sigma_e / m] \) can be replaced by \( -i \frac{\partial}{\partial \tau_y} \) acting on the initial wave function. The \( p^0 \) integral can then be evaluated by the residue theorem. The \( \tau_y \) derivatives can then moved back to \( D^{ij}_{\mu \nu}[p \cdot \sigma_e / m] \) by a finite number of integrations by parts, since it is a polynomial in the components of \( p \). This gives

\[
\int \langle \phi | \tau_x, x, \mu \rangle e^{-i \omega_m(p) \tau_x + i p \cdot x} d^4 x \frac{d^4 x}{\omega_m(p)} D^{ij}_{\mu \nu}[p m \cdot \sigma / m] e^{-i \omega_m(p) \tau_y - i p \cdot y} \langle \tau_y, y, \nu | \psi \rangle d^4 y
\]

(206)

where \( p_m = (\omega_m(p), p) \) and

\[
D^{ij}_{\mu \nu}[p \cdot \sigma_e / m] = D^{ij}_{\mu \nu}[p m \cdot \sigma / m].
\]

The resulting kernel

\[
\frac{d^4 x}{\omega_m(p)} D^{ij}_{\mu \nu}[p m \cdot \sigma / m]
\]

(207)

is exactly the Lorentz covariant measure appearing in (180). It follows that the Euclidean covariant distribution

\[
\frac{D^{ij}_{\mu \nu}[p \cdot \sigma_e / m]}{p^2 + m^2}
\]

(208)

is reflection positive because \( D^{ij}_{\mu \nu}[p \cdot \sigma_e / m] \) becomes a positive definite matrix after \( p^0 \) is set equal to \( -i \omega_m(p) \).

The measure for the left handed (space reflected) representation is obtained by replacing

\[
\sigma_e \rightarrow \sigma_2 \sigma_e \sigma_2
\]

(209)

which changes the sign of the space components of \( \sigma_{e \mu} \). In this case

\[
\int \langle \phi | \tau_x, x, \mu \rangle e^{i p \cdot (x - y)} \frac{d^4 x}{p^2 + m^2} D^{ij}_{\mu \nu}[p \cdot \sigma_2 \sigma_e \sigma_2 / m] \langle \tau_y, y, \nu | \psi \rangle d^4 x d^4 y d^4 p =
\]

\[
\int \langle \phi | \tau_x, x, \mu \rangle e^{-i \omega_m(p) \tau_x + i p \cdot x} d^4 x \frac{d^4 x}{\omega_m(p)} D^{ij}_{\mu \nu}[p m \cdot \sigma_2 \sigma_e \sigma_2 / m] e^{-i \omega_m(p) \tau_y - i p \cdot y} \langle \tau_y, y, \nu | \psi \rangle d^4 y =
\]

\[
\int \langle \phi | \tau_x, x, \mu \rangle e^{-i \omega_m(p) \tau_x + i p \cdot x} d^4 x \frac{d^4 x}{\omega_m(p)} D^{ij}_{\mu \nu}[P p m \cdot \sigma / m] e^{-i \omega_m(p) \tau_y - i p \cdot y} \langle \tau_y, y, \nu | \psi \rangle d^4 y
\]

(210)

where

\[
\frac{d^4 x}{\omega_m(p)} D^{ij}_{\mu \nu}[P p m \cdot \sigma / m]
\]

(211)

which is the Lorentz covariant kernel (181) for left handed spinors. The positivity of the matrix \( D^{ij}_{\mu \nu}[P p m \cdot \sigma / m] \) implies that the Euclidean covariant distribution

\[
\frac{D^{ij}_{\mu \nu}[p \cdot \sigma_2 \sigma_e \sigma_2 / m]}{p^2 + m^2} = \frac{D^{ij}_{\mu \nu}[P p \cdot \sigma_e / m]}{p^2 + m^2}
\]

(212)
is also reflection positive. By defining

$$\langle p, \nu | \chi \rangle := \int e^{-\omega_m(p) \tau_0 - i p \cdot y} \langle \tau_0, y, \nu | \psi \rangle d^4 y$$ \hfill (214)

the norms can be expressed in the form

$$\langle \psi | \psi \rangle = \int \langle \chi | p, \mu \rangle \frac{dp}{\omega_m(p)} D_{\mu\nu}[p_m \cdot \sigma/m] \langle p, \nu | \chi \rangle$$ \hfill (215)

and

$$\langle \psi | \psi \rangle = \int \langle \chi | p, \mu \rangle \frac{dp}{\omega_m(p)} D_{\mu\nu}[P p_m \cdot \sigma/m] \langle p, \nu | \chi \rangle, \hfill (216)$$

for the right and left handed representations respectively. These expressions have the same form as the Lorentz covariant inner products with respect to the functions, $$\langle p, \nu | \chi \rangle$$, up to a multiplicative constant.

As in the Lorentz covariant case, in the Euclidean case the Euclidean covariant kernels are different for the right- and left-handed representations:

$$\frac{1}{p^2 + m^2} D_{\mu\nu}^i [p \cdot \sigma_e/m] \hfill (217)$$

$$\frac{1}{p^2 + m^2} D_{\mu\nu}^j [p \cdot \sigma_2 \sigma^\epsilon_2/m]. \hfill (218)$$

The $$SU(2) \times SU(2)$$ covariance property of the kernel (217) is

$$D^i[A] \frac{D^j[p \cdot \sigma_e/m]}{p^2 + m^2} D[B^i] = \frac{D^j[p \cdot A \sigma_e B^i/m]}{p^2 + m^2} = \frac{D^j[p \cdot O^i(A, B)\sigma_e/m]}{p^2 + m^2} =$$

$$\frac{D^j[O(A, B)p \cdot \sigma_e/m]}{p^2 + m^2} \frac{D^j[O(A, B)p \cdot \sigma_e/m]}{(O(A, B)p)^2 + m^2} \hfill (219)$$

Here $$p^2 = (O(A, B)p)^2$$ was used. The corresponding covariance property for the space reflected kernel, (218), can be obtained by taking the transpose of (219) and left and right multiplying by $$D_{\mu\nu}^i [\sigma_2] = (i)^{2\nu} \delta_{\mu-\nu}$$ which gives

$$D^i[\sigma_2 B \sigma_2] \frac{D^j[p \cdot \sigma_2 \sigma^\epsilon_2/m]}{p^2 + m^2} D[\sigma_2 A^i \sigma_2] = \frac{D^j[p \cdot \sigma_2 \sigma^\epsilon_2 \sigma_2/m]}{p^2 + m^2} D^i[A^i] =$$

$$\frac{D^j[O(A, B)p \cdot \sigma_2 \sigma^\epsilon_2 \sigma_2/m]}{(O(A, B)p)^2 + m^2} \hfill (220)$$

These results are abbreviated by

$$D[A] K_r(p) D[B^i] = K_r(O(A, B)p) \hfill (221)$$

$$D[B^i] K_i(p) D[A^i] = K_i(O(A, B)p) \hfill (222)$$

where $$K_r(p)$$ and $$K_i(p)$$ are the right and left handed reflection positive kernels (209) and (213).

While most treatments of Euclidean formulations of relativistic quantum theories involve an analytic continuation in time, the construction above shows how the right and left handed Lorentz covariant irreducible representations (178,179) are recovered in the Euclidean formulation without any analytic continuation. This reason for this is that reflection positivity, the spectral condition ($$m > 0$$), and the assumption that the Euclidean kernel is a tempered distribution ensures the existence of the analytic continuation, however for the purpose of formulating relativistic quantum mechanics the analytic continuation is not needed.

**Relativistic invariance in the Euclidean case**
Relativistic invariance in the Euclidean case is a consequence of the identities relating the Euclidean covariant inner product to the Lorentz covariant inner product and the Poincaré covariant inner product.

The relativistic transformation properties in the Euclidean representation can be understood from the observation that the complex orthogonal and complex Lorentz transformations have the same covering group, $SL(2, \mathbb{C}) \times SL(2, \mathbb{C})$. This means that the group of real Euclidean transformations can be identified with a subgroup of the complex Lorentz group. The real Euclidean group is a 10 parameter group. Each generator can be thought of generating a one-parameter subgroup of complex Poincaré transformations. This leads to a relation between the generators of real Euclidean transformations and real Poincaré transformations that can be realized in the Euclidean framework.

This relationship implies that the Poincaré Lie algebra is related to the Euclidean Lie algebra by multiplying the generators of real Euclidean transformations involving the Euclidean time by factors of $i$. The Euclidean generators involving the Euclidean time are the generator of Euclidean time translations and the generators of rotations in space-Euclidean time planes. The resulting Poincaré generators for time translation and canonical boosts are related to the generators of Euclidean time translation and rotations in Euclidean space-time planes by

$$H_m = iH_\epsilon \quad \text{and} \quad K \cdot \hat{n} = -iJ_{\hat{n},\tau}.$$ (223)

Both $H_m$ and $K$ become Hermitian operators with respect to the physical Hilbert space inner product $\langle \cdot, \cdot \rangle$ that includes the Euclidean time reflection. On the physical Hilbert space real Euclidean-time translations are represented by a contractive Hermitian semi-group [18] and the real rotations in space-Euclidean time planes are represented by local symmetric semi-groups [19] [20] [21]. The generators of these transformations are self-adjoint and are exactly the Poincaré generators discussed above.

The $2 \times 2$ matrix representation of ordinary rotations in both the Euclidean and Lorentz case can be represented by

$$X \rightarrow X' = AXB^t \quad X \rightarrow X'_e = AX_eB^t$$ (224)

where $(A, B) = (A, A^*)$ for $A \in SU(2)$.

Euclidean rotations in space-Euclidean-time planes can be represented by

$$X_e \rightarrow X'_e = AX_eB^t$$ (225)

where $(A, B) = (A, A^t)$ for $A \in SU(2)$, while rotationless Lorentz boosts can be represented by a transformation of the same form,

$$X \rightarrow X' = AX_eB^t$$ (226)

where $(A, B) = (A, A^t)$ and $A = A^\dagger$.

For a given $SU(2) \times SU(2)$ transformations $(A, B)$ there are four types of Euclidean spinor wave functions that are identified by their spinor transformation properties

$$\psi^\mu(j, p) \rightarrow \psi'^\mu(j, p) = \psi^\nu(j, O(A, B)p)D^\nu_{\mu}(A)$$ (227)

$$\psi_\mu(j, p) \rightarrow \psi'_\mu(j, p) = \psi_\nu(j, O(A, B)p)D^\nu_{\mu}(A^*)$$ (228)

$$\psi^\mu(j, p) \rightarrow \psi'^\mu(j, p) = \psi^\nu(j, O(A, B)p)D^\nu_{\mu}(B)$$ (229)

$$\psi_\mu(j, p) \rightarrow \psi'_\mu(j, p) = \psi_\nu(j, O(A, B)p)D^\nu_{\mu}(B^*)$$ (230)

In equations (227) (230) the bra-ket notation is not used in order to differentiate the different types of spinor wave functions. The first two are right-handed wave functions; the last two are left handed.

Representations of the Lorentz generators on each of these spinor wave functions are obtained by first constructing the finite transformations in (227) (230) using

$$(A(\lambda), B(\lambda))_r = (e^{i\frac{\lambda}{2}\hat{n} \cdot \sigma}, (e^{i\frac{\lambda}{2}\hat{n} \cdot \sigma})^\dagger)$$ (231)

for rotations about the $\hat{n}$ axis and

$$(A(\lambda), B(\lambda))_b = (e^{i\frac{\lambda}{2}\hat{n} \cdot \sigma}, (e^{i\frac{\lambda}{2}\hat{n} \cdot \sigma})^\dagger)$$ (232)

The relativistic transformation properties in the Euclidean representation can be understood from the observation that the complex orthogonal and complex Lorentz transformations have the same covering group, $SL(2, \mathbb{C}) \times SL(2, \mathbb{C})$. This means that the group of real Euclidean transformations can be identified with a subgroup of the complex Lorentz group. The real Euclidean group is a 10 parameter group. Each generator can be thought of generating a one-parameter subgroup of complex Poincaré transformations. This leads to a relation between the generators of real Euclidean transformations and real Poincaré transformations that can be realized in the Euclidean framework.

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$$\psi^\mu(j, p) \rightarrow \psi'^\mu(j, p) = \psi^\nu(j, O(A, B)p)D^\nu_{\mu}(B)$$ (229)

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$$(A(\lambda), B(\lambda))_r = (e^{i\frac{\lambda}{2}\hat{n} \cdot \sigma}, (e^{i\frac{\lambda}{2}\hat{n} \cdot \sigma})^\dagger)$$ (231)

for rotations about the $\hat{n}$ axis and

$$(A(\lambda), B(\lambda))_b = (e^{i\frac{\lambda}{2}\hat{n} \cdot \sigma}, (e^{i\frac{\lambda}{2}\hat{n} \cdot \sigma})^\dagger)$$ (232)
for rotations in the $\mathbf{n} - \tau$ plane.

Representations for the generator of ordinary rotations about the $\mathbf{n}$ axis are obtained by using $(A, B) = (A(\lambda), B(\lambda))$, differentiating with respect to $\lambda$, setting $\lambda$ to 0 and multiplying the result by $-i$.

Representations for the generator of rotationless boosts in the $\mathbf{n}$ direction are obtained by using $(A, B) = (A(\lambda), B(\lambda))$, differentiating each of (227-230) with respect to $\lambda$, setting $\lambda$ to 0 and multiplying the result by $-1$.

The Hamiltonian and linear momentum operators in the Euclidean representation are obtained by Fourier transforming each of (227-230) followed by

$$P = -i \nabla, \quad H = \frac{\partial}{\partial \tau}. \quad (233)$$

The resulting operators satisfy the Poincaré commutation relations and are Hermitian when they are used in the inner product (205). As in the Lorentz covariant case, the dynamics enters through the Euclidean kernel, which has all of the dynamics (mass dependence).

### VII. LORENTZ COVARIANT FIELDS

Covariant fields are useful for treating systems of many identical particles. In many-body quantum mechanics fields are associated with the occupation number representation. They are constructed from a single-particle basis $\{|n\rangle\}$, and operators, $a_n^\dagger$, that add and, $a_n$, that remove a particle in the $n$-th single-particle state. In this section the same methods are used to develop Lorentz covariant fields for systems of non-interacting particles of any spin. Locality of the fields is not assumed. Local fields will be discussed in the next section.

Given a single-particle basis, fields are operators defined by

$$\Psi(x) := \sum_n \langle x|n\rangle a_n \quad \Psi^\dagger(x) := \sum_n a_n^\dagger \langle n|x\rangle. \quad (234)$$

The field is independent of the choice of single-particle basis. In a plane-wave basis equations (234) become

$$\Psi(x) := \int d\mathbf{p} \langle x|\mathbf{p}\rangle a(\mathbf{p}) \quad \Psi^\dagger(x) := \int d\mathbf{p} a^\dagger(\mathbf{p}) \langle \mathbf{p}|x\rangle. \quad (235)$$

The time dependence is determined by solving the Heisenberg equations of motion

$$\frac{d\Psi(x,t)}{dt} = i[H, \Psi(x,t)]. \quad (236)$$

If $H$ is the free Hamiltonian the solution of the Heisenberg equations is

$$\Psi(x,t) := \int d\mathbf{p} \langle x|\mathbf{p}\rangle e^{-iE(\mathbf{p})t} a(\mathbf{p}) \quad \Psi^\dagger(x,t) := \int d\mathbf{p} a^\dagger(\mathbf{p}) e^{iE(\mathbf{p})t} \langle \mathbf{p}|x\rangle \quad (237)$$

where $E(\mathbf{p})$ is the energy of a particle with momentum $\mathbf{p}$.

The vector $|0\rangle$, represents the no particle state. It is defined by the conditions

$$a_n|0\rangle = 0 \quad \forall n \quad \langle 0|0\rangle = 1. \quad (238)$$

The creation and annihilation operators satisfy the commutation (anti-commutation) relations

$$[a_n, a_m^\dagger]|_\pm = \delta_{mn} \quad \text{or} \quad [a(\mathbf{p}), a_m^\dagger(\mathbf{p}')]|_\pm = \delta(\mathbf{p} - \mathbf{p}') \quad (239)$$

depending on whether the particles are Bosons or Fermions.

Free Lorentz covariant fields that transform under a finite-dimensional representation of $SL(2, \mathbb{C})$ can be constructed using the same method. In this case the plane wave states $\langle x|\mathbf{p}\rangle$ are replaced by Lorentz covariant plane wave states, and measure is replaced by the Lorentz invariant measure.

Because the Lorentz covariant states can transform under right or left-handed representations of $SL(2, \mathbb{C})$, the corresponding covariant fields will also have a handedness.
In this section right and left handed spin-$j$ fields are constructed with the following Poincaré covariance properties:

$$U(\Lambda, a) \Psi_{r\mu}(x) U^\dagger(\Lambda, a) = D_{\mu\nu}^j[A^{-1}] \Psi_{r\nu}(\Lambda x + a)$$  (240)

$$U(\Lambda, a) \Psi_{\mu}(x) U^\dagger(\Lambda, a) = D_{\mu\nu}^j[\tilde{A}^{-1}] \Psi_{\nu}(\Lambda x + a)$$  (241)

where $A$ and $\tilde{A}$ are related by (69).

The starting point is to define creation and annihilation operators that transform like single-particle irreducible states. These create or destroy particles with a momentum $p$ and a magnetic quantum number associated with the $x$-type of spin, as discussed in section 3.

The creation operators are assumed to have the following transformation properties

$$U(\Lambda, a)a^\dagger_x(p, \mu)(x) U^\dagger(\Lambda, a) = e^{-i\Lambda^\mu a^\dagger_x(D_{\mu\nu}^j[B^{-1}_x(p/m)]A B_x(p/m)]}\sqrt{\frac{\omega_m(\Lambda p)}{\omega_m(p)}} =$$

$$e^{-i\Lambda^\mu a^\dagger_x(D_{\mu\nu}^j[B^{-1}_x(p/m)]A B_x(p/m)]}\sqrt{\frac{\omega_m(\Lambda p)}{\omega_m(p)}}.$$  (242)

The two expressions above are identical because

$$B^{-1}_x(\Lambda p/m)A B_x(p/m) = R_{wz}(\Lambda, p/m) = (R^\dagger_{wz})^{-1}(\Lambda, p/m) = \tilde{B}^{-1}_x(\Lambda p/m)A \tilde{B}_x(p/m).$$  (243)

The transformation properties of the creation operator (242) is the same as the transformation properties a particle (244), except the sign of the phase is reversed because the time dependence of the operator is given by the Heisenberg equations of motion.

The corresponding transformation properties for the annihilation operators can be obtained by taking the adjoint of (242):

$$U(\Lambda, a)a_x(p, \mu)(x) U^\dagger(\Lambda, a) = e^{i\Lambda^\mu a_x(D_{\mu\nu}^j[B_x(p/m)]A^\dagger \tilde{B}_x(\Lambda p/m)]a_x(D_{\mu\nu}^j[\Lambda p, \nu])}\sqrt{\frac{\omega_m(\Lambda p)}{\omega_m(p)}} =$$

$$e^{i\Lambda^\mu a_x(D_{\mu\nu}^j[B_x(p/m)]A^\dagger \tilde{B}_x(\Lambda p/m)]a_x(D_{\mu\nu}^j[\Lambda p, \nu])}\sqrt{\frac{\omega_m(\Lambda p)}{\omega_m(p)}}.$$  (244)

Local fields are linear combinations of fields with creation and annihilation operators that have the same covariance properties. The normal convention is to have the $SL(2, \mathbb{C})$ representation matrices to the left of the creation and annihilation operators as in (240) and (241).

This can be realized in (242) by using the $SU(2)$ identity (100)

$$R = \sigma_2(R^\dagger)^{-1}\sigma_2$$  (245)

in the Wigner rotation

$$R := B^{-1}_x(\Lambda p/m)A B_x(p/m).$$  (246)

which gives

$$D_{\mu\nu}^j(R) = D_{\mu\nu}^j(\sigma_2(R^\dagger)^{-1}\sigma_2) = D_{\mu\nu}^j((-\sigma_2)R^{-1}(-\sigma_2)) = D_{\mu\nu}^j(\sigma_2 R^{-1}\sigma_2)$$  (247)

the corresponding property of the Wigner functions (note the reversal $\mu \leftrightarrow \nu$ of the spin indices). Using this identity in (242) gives

$$U(\Lambda, a)D_{\mu\nu}^j[\sigma_2](p, \nu)(x) U^\dagger(\Lambda, a) = e^{-i\Lambda^\mu a_x(D_{\mu\nu}^j[B^{-1}_x(p/m)]A^{-1} B_x(\Lambda p/m)]\sigma_2)}a^\dagger_x(D_{\mu\nu}^j[\Lambda p, \nu])\sqrt{\frac{\omega_m(\Lambda p)}{\omega_m(p)}} =$$
Introducing the $\sigma_2$ factor gives the creation fields the same covariance properties as the annihilation fields. These operators determine the Poincaré transformation properties of the covariant spinor fields. Different types of spinor fields are distinguished by their covariance properties. General covariant fields of a given spin are built up out of four types of elementary covariant fields, that are classified as right (r) or left (l) handed and creation (c) or annihilation (a) fields. The subscripts $rc, lc, ra, la$ are used to distinguish the four different types of fields:

$$U(\Lambda, a)D_{\mu}^j[\sigma_2]a^\dagger_{x}(p, \nu)(x)U^\dagger(\Lambda, a) = e^{-i\lambda p \cdot a}D_{\mu}^j[B_x^j(p/m)\Lambda^\dagger(\Lambda p/m)\sigma_2]a^\dagger_{x}(\Lambda p, \nu)\sqrt{\omega_m(\Lambda p)}/\omega_m(p).$$ 

(248)

These fields have the same covariance properties as the annihilation fields. The transformation properties of (249-252) follow directly from the transformation properties of the creation and annihilation operators (242) and (244) and in all cases the 4-momenta are on shell: 

$$\Psi_{rcp}(\lambda) := \int \frac{d^4p}{(2\pi)^{3/2}\sqrt{\omega_m(p)}}D_{\mu}^j[B_x^j(p/m)\sigma_2]a^\dagger_{x}(p, \nu)\sqrt{\omega_m(p)} =$$ 

(249)

$$\Psi_{rap}(\lambda) := \int \frac{e^{ipx}}{(2\pi)^{3/2}\sqrt{\omega_m(p)}}D_{\mu}^j[B_x^j(p/m)]a_{x}(p, \nu)\sqrt{\omega_m(p)} =$$ 

(250)

$$\Psi_{lc\mu}(\lambda) = \int \frac{e^{-ipx}}{\sqrt{\omega_m(p)}}D_{\mu}^j[\bar{B}_x^j(p/m)\sigma_2]a^\dagger_{x}(p, \nu)\sqrt{\omega_m(p)} =$$ 

(251)

$$\Psi_{ta\mu}(\lambda) = \int \frac{e^{ipx}}{\sqrt{\omega_m(p)}}D_{\mu}^j[\bar{B}_x^j(p/m)]a_{x}(p, \nu)\sqrt{\omega_m(p)} =$$ 

(252)

where in all cases the 4-momenta are on shell:

$$p \cdot x = -\omega_m(p^2)x^0 + p \cdot x.$$ 

(253)

Note that $\Psi^\dagger_{xcp}(\lambda)$ is not the adjoint of $\Psi_{xap}(\lambda)$. This is because of the factor $\sigma_2$ that was introduced to make both fields have the same covariance property.

The transformation properties of (249-252) follow directly from the transformation properties of the creation and annihilation operators (242) and (244):

$$U(\Lambda, b)\Psi^\dagger_{rcp}(\lambda)U^\dagger(\Lambda, b) = D_{\mu}^j[(A^{-1})]^{\dagger}\Psi^\dagger_{rcp}(\Lambda x + b)$$ 

(254)

$$U(\Lambda, b)\Psi_{rap}(\lambda)U^\dagger(\Lambda, b) = D_{\mu}^j[(A^{-1})]^{\dagger}\Psi_{rap}(\Lambda x + b)$$ 

(255)

$$U(\Lambda, b)\Psi^\dagger_{lc\mu}(\lambda)U^\dagger(\Lambda, b) = D_{\mu}^j[A^{\dagger}]^{\dagger}\Psi^\dagger_{lc\mu}(\Lambda x + b)$$ 

(256)

$$U(\Lambda, b)\Psi_{ta\mu}(\lambda)U^\dagger(\Lambda, b) = D_{\mu}^j[A^{\dagger}]^{\dagger}\Psi_{ta\mu}(\Lambda x + b).$$ 

(257)

These fields can be multiplied by any normalization constants.
These transformation properties can be used to construct invariant operator densities. Invariant products are constructed by taking the product of a field of one handedness with the adjoint of a field of the opposite handedness and summing over the spins. Lorentz invariant Hermitian operators are obtained by adding the Hermitian conjugate to each of the invariant pairs. The following sums of products of left and right-handed fields are Hermitian and transform like Lorentz scalars:

$$\sum_\mu \left( \Psi_{l\mu}(x) \Psi_{r\mu}^\dagger(x) + \Psi_{r\mu}(x) \Psi_{l\mu}^\dagger(x) \right)$$

(258)

$$\sum_\mu \left( \Psi_{l\mu}(x) \Psi_{r\mu}(x) + \Psi_{r\mu}^\dagger(x) \Psi_{l\mu}(x) \right)$$

(259)

$$\sum_\mu \left( \Psi_{r\mu}(x) \Psi_{l\mu}(x) + \Psi_{l\mu}^\dagger(x) \Psi_{r\mu}(x) \right)$$

(260)

$$\sum_\mu \left( \Psi_{r\mu}^\dagger(x) \Psi_{l\mu}(x) + \Psi_{l\mu}^\dagger(x) \Psi_{r\mu}(x) \right)$$

(261)

Note that for free field these expressions are normal ordered. It is possible to make more complicated Lorentz invariant products of field operators using $SU(2)$ Clebsch-Gordan coefficients and the group representation properties of the Wigner functions.

The commutators or anti-commutators of the elementary fields with their true adjoints are

$$[\Psi_{r\mu}(x), \Psi_{r\nu}^\dagger(y)] = \int \frac{dp}{(2\pi)^3 \omega_m(p)} e^{ip(x-y)} D_{\mu\nu}^{ij}[\sigma^t, p] = 2 \int \frac{dp}{(2\pi)^3} \delta(p^2 + m^2) \theta(p^0) e^{ip(x-y)} D_{\mu\nu}^{ij}[\sigma^t, p]$$

(262)

$$[\Psi_{r\mu}(x), \Psi_{r\nu}^\dagger(y)] = 2 \int \frac{dp}{(2\pi)^3} \delta(p^2 + m^2) \theta(p^0) e^{ip(x-y)} D_{\mu\nu}^{ij}[\sigma \cdot Pp]$$

(263)

$$[\Psi_{l\mu}(x), \Psi_{l\nu}^\dagger(y)] = 2 \int \frac{dp}{(2\pi)^3} \delta(p^2 + m^2) \theta(p^0) e^{ip(x-y)} D_{\mu\nu}^{ij}[\sigma \cdot Pp]$$

(264)

$$[\Psi_{l\mu}(x), \Psi_{l\nu}^\dagger(y)] = 2 \int \frac{dp}{(2\pi)^3} \delta(p^2 + m^2) \theta(p^0) e^{ip(x-y)} D_{\mu\nu}^{ij}[\sigma^t \cdot Pp]$$

(265)

which are the spin-$j$ Wightman functions that define the kernels of the Lorentz covariant inner products. Note that $\sigma \cdot p$, $\sigma^t \cdot p$, $\sigma \cdot Pp$ and $\sigma^t \cdot Pp$ are all positive Hermitian matrices for time-like $p$, so these kernels are all positive distributions.

These fields are analogous to the non-relativistic fields; they add or remove particles in the occupation number representation. While they are not local, they are the basic building blocks of local free fields.

The creation and annihilation operators can be extracted from the right or left-handed fields using plane wave solutions of the Klein-Gordon equation and spinor matrices

$$f_m(p, x) := \int \frac{1}{\sqrt{\omega_m(p)(2\pi)^{3/2}}} e^{ip \cdot x}$$

(266)

$$a_{\pm}^\dagger(p, \mu) =$$

$$\frac{i}{2} D_{\mu\nu}^{ij} [\sigma_2 B^{-1}(p/m)] \int \left( \frac{\partial f_m(p, x)}{\partial t} \Psi_{r\nu}^\dagger(x) - \frac{\partial \Psi_{r\nu}^\dagger(x)}{\partial t} f_m(x) \right) dx$$

(267)

$$a_{\pm}(p, \mu) =$$
\begin{align*}
- \frac{i}{2} D_{\mu\nu}[B_x^{-1}(p/m)] \int \left( \frac{\partial f_m^*(p, x)}{\partial t} \Psi_{rav}(x) - \frac{\partial \Psi_{lav}^*(x)}{\partial t} f_m^*(x) \right)dx \\
&= a_x^+(p, \mu) \tag{268}
\end{align*}

\begin{align*}
\frac{i}{2} D_{\mu\nu}[\sigma_2 B_x^+(p/m)] \int \left( \frac{\partial f_m(p, x)}{\partial t} \Psi_{lav}^+(x) - \frac{\partial \Psi_{rav}^+(x)}{\partial t} f_m(x) \right)dx \\
&= a_x^+(p, \mu) \tag{269}
\end{align*}

\begin{align*}
- \frac{i}{2} D_{\mu\nu}[B_x^+(p/m)] \int \left( \frac{\partial f_m^+(p, x)}{\partial t} \Psi_{lav}(x) - \frac{\partial \Psi_{rav}^+(x)}{\partial t} f_m^+(x) \right)dx \\
&= a_x^+(p, \mu) \tag{270}
\end{align*}

where the integrals are evaluated at a common time.

Fields that transform linearly under space reflection can be constructed by taking a direct sum of a right and left handed field

\begin{align*}
\Psi_{e\mu}^+(x) \rightarrow \int e^{-ip \cdot x} \frac{dp}{\sqrt{\omega_m(p)}} \left( D_{\mu\nu}^+[B_x(p/m)\sigma_2] \Psi_{e\mu}^+(x) \right) a_x^+(p, \nu) \tag{271}
\end{align*}

where the matrix is a $2(2j + 1) \times (2j + 1)$ matrix.

The Poincaré transformation properties of these fields are

\begin{align*}
U(\Lambda, b) \left( \begin{array}{c}
\Psi_{rav}(x) \\
\Psi_{lav}(x)
\end{array} \right) U^\dagger(\Lambda, b) = \left( \begin{array}{cc}
D_{\mu\nu}^+[A^{-1}] & 0 \\
0 & D_{\mu\nu}^+[A^+]\end{array} \right) \left( \begin{array}{c}
\Psi_{rav}(\Lambda x + b) \\
\Psi_{lav}(\Lambda x + b)
\end{array} \right) \tag{272}
\end{align*}

Space reflection changes the sign of the space component of $x$ and interchanges the right- and left-handed components

\begin{align*}
P \Psi_{e\mu}^+(x) P^{-1} = \Psi_{e\mu}^+(P x) = P \left( \begin{array}{c}
\Psi_{rav}^+(x) \\
\Psi_{lav}^+(x)
\end{array} \right) P^{-1} = \left( \begin{array}{c}
\Psi_{rav}^+(P x) \\
\Psi_{lav}^+(P x)
\end{array} \right) \tag{273}
\end{align*}

The annihilation fields have the same structure

\begin{align*}
\Psi_{a\mu}(x) \rightarrow \left( \begin{array}{c}
\Psi_{rav}(x) \\
\Psi_{lav}(x)
\end{array} \right). \tag{274}
\end{align*}

The Poincaré transformation properties of the annihilation fields are

\begin{align*}
U(\Lambda, b) \left( \begin{array}{c}
\Psi_{rav}(x) \\
\Psi_{lav}(x)
\end{array} \right) U^\dagger(\Lambda, b) = \left( \begin{array}{cc}
D_{\mu\nu}^+[A^{-1}] & 0 \\
0 & D_{\mu\nu}^+[A^+]\end{array} \right) \left( \begin{array}{c}
\Psi_{rav}(\Lambda x + b) \\
\Psi_{lav}(\Lambda x + b)
\end{array} \right) \tag{275}
\end{align*}

Space reflection changes with sign of the space component of $x$ and interchanges the right and left handed components

\begin{align*}
P \Psi_{a\mu}(x) P^{-1} = \Psi_{a\mu}(x) = P \left( \begin{array}{c}
\Psi_{rav}(x) \\
\Psi_{lav}(x)
\end{array} \right) P^{-1} = \left( \begin{array}{c}
\Psi_{lav}(P x) \\
\Psi_{rav}(P x)
\end{array} \right). \tag{276}
\end{align*}

By analogy with the Dirac equation it is to useful to define

\begin{align*}
\Gamma^0 := \left( \begin{array}{cc}
0 & I \\
I & 0
\end{array} \right) \quad \Gamma^5 := \left( \begin{array}{cc}
I & 0 \\
0 & -I
\end{array} \right) \tag{277}
\end{align*}

In this notation equations \text{273} and \text{274} can be written as

\begin{align*}
P \Psi_{e\mu}^+(x) P^{-1} = \Gamma^0 \Psi_{e\mu}^+(P x) \tag{278}
\end{align*}

and

\begin{align*}
P \Psi_{a\mu}(x) P^{-1} = \Gamma^0 \Psi_{a\mu}(P x). \tag{279}
\end{align*}
Normally the linear combinations involve a particle creation operator with an antiparticle annihilation operator.

\[ \alpha \text{covariantly for any constants of the Poincaré group.} \]

Local free fields are constructed from linear combinations of creation and annihilation fields:

\[
\text{for use in many-body relativistic quantum mechanics, they are not suitable for use in local relativistic quantum field theory. Local free fields are constructed from linear combinations of creation and annihilation fields:}
\]

\[
\text{The fields constructed in the previous section transform covariantly, but they are not local. While they are sufficient for use in many-body relativistic quantum mechanics, they are not suitable for use in local relativistic quantum field theory. Local free fields are constructed from linear combinations of creation and annihilation fields:}
\]

\[
\text{Normally the linear combinations involve a particle creation operator with an antiparticle annihilation operator.}
\]

\[
\text{The commutator or anti-commutator of the linear combinations} \quad \left[ \alpha \Psi_{r\mu}(x) + \beta \Psi^\dagger_{r\mu}(y), \Psi_{r\mu}(x) + \beta \Psi^\dagger_{r\mu}(y) \right]_\pm =
\]

\[
\int \frac{dp}{\omega_m(p)} \left( |\alpha|^2 e^{ip(x-y)} D_{\mu\sigma}[B_x(p/m)] D_{\nu\alpha}^*[B_x(p/m)] \pm |\beta|^2 e^{-ip(x-y)} D_{\mu\sigma}[B_x(p/m)] D_{\nu\alpha}^*[B_x(p/m)] \right) =
\]

\[
\int \frac{dp}{\omega_m(p)} \left( |\alpha|^2 e^{ip(x-y)} D_{\mu\nu}^\dagger[\sigma \cdot p] \pm |\beta|^2 e^{-ip(x-y)} D_{\mu\nu}^\dagger[\sigma \cdot p] \right). \quad (286)
\]

For \((x-y)^2 > 0\) the integral \(22\)

\[
\int \frac{dp}{\omega_m(p)} e^{-ip(x-y)} = -\frac{4\pi m}{\sqrt{(x-y)^2}} K_1(m\sqrt{(x-y)^2}) \quad (287)
\]
is an even function of $x - y$. It follows that for $(x - y)^2 > 0$ this becomes

$$
(|\alpha|^2 D^i_{\mu,\nu}(-\sigma \cdot i\partial_x) + |\beta|^2 D^i_{\mu,\nu}(\sigma \cdot i\partial_x)) \int \frac{dp}{\omega_m(p)} e^{ip(x-y)} = 
$$

$$
(|\alpha|^2 (-)^{2j} + |\beta|^2) D^i_{\mu,\nu}(\sigma \cdot i\partial_x) \int \frac{dp}{\omega_m(p)} e^{ip(x-y)}
$$

(288)

For this to vanish $|\alpha|^2 = |\beta|^2$ and $(-)^{2j} = \pm 1$ this means that anti-commutation relations are required for $j$ half integral, commutation relations for $j$ integer.

Similar results are obtained for left handed spinors. The only difference is that $D^j(\sigma \cdot p)$ is replaced by $D^j(\sigma \cdot Pp)$.

Thus right and left handed spin $j$ free local fields have the form

$$
\Psi_{rloc\mu}(x) = Z(\Psi_{r\mu}(x) \pm \Psi^\dagger_{r\mu}(x))
$$

(289)

$$
\Psi_{lloc\mu}(x) = Z(\Psi_{l\mu}(x) \pm \Psi^\dagger_{l\mu}(x))
$$

(290)

where $Z$ is a normalization constant. Locality does not fix the $\pm$ sign. Local fields where space reflection acts linearly can be constructed from these by taking the direct sum of a right and left handed local field:

$$
\Psi^\dagger_{loc\mu}(x) \rightarrow \begin{pmatrix} \Psi^\dagger_{rloc\mu}(x) \\ \Psi^\dagger_{lloc\mu}(x) \end{pmatrix}
$$

(291)

This structure will be used to construct a spin 1/2 field satisfying the Dirac equation. The structure of the gamma matrices follow from the $SL(2, \mathbb{C})$ transformation properties of the Pauli matrices and the $2 \times 2$ identity. The relevant representation of $SL(2, \mathbb{C})$ for a Dirac field is the direct sum of a right and left handed representation of $SL(2, \mathbb{C})$:

$$
S(A) = \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & (A^\dagger)^{-1} \end{pmatrix}
$$

(292)

The representation of the $\gamma$-matrices follow from the transformation properties of four vectors represented by $2 \times 2$ Hermitian matrices:

$$
X := x^\mu \sigma_\mu \quad X' = AXA^\dagger \quad A \in SL(2, \mathbb{C})
$$

(293)

This can be expressed in terms of the components of $x$ as

$$
\sigma_\mu A^\mu, x^\nu = A x^\nu.
$$

(294)

Equating the coefficients of $x^\nu$ gives

$$
A x^\nu = \sigma_\mu A^\mu
$$

(295)

Multiplying both sides of this equation by $\sigma_2$ and taking complex conjugates gives

$$
\tilde{A} \sigma_2 \sigma^\nu x^\mu = \sigma_2 \sigma^\nu \sigma_2 A^\nu
$$

(296)

Equations (295) and (296) can be combined into a single equation

$$
\begin{pmatrix} A & 0 \\ 0 & (A^\dagger)^{-1} \end{pmatrix} \begin{pmatrix} 0 & \sigma_\mu \\ \sigma_2 \sigma^\nu \sigma_2 & 0 \end{pmatrix} \begin{pmatrix} A^{-1} & 0 \\ 0 & A^\dagger \end{pmatrix} = \begin{pmatrix} 0 & \sigma_\nu \\ \sigma_2 \sigma^\nu \sigma_2 & 0 \end{pmatrix} \Lambda^\nu
$$

(297)

which shows that the matrices

$$
\gamma_\nu := \begin{pmatrix} 0 & -\sigma_\nu \\ -\sigma_2 \sigma^\nu \sigma_2 & 0 \end{pmatrix}
$$

(298)

transform like four vectors with respect to the similarity transformation

$$
S(A) \gamma_\mu S(A^{-1}) = \gamma_\nu \Lambda^\nu
$$

(299)
where the \(-\) sign is a convention. With this convention
\[
\gamma^0 = \begin{pmatrix} 0 & \sigma_0 \\ \sigma_0 & 0 \end{pmatrix} \quad \gamma^\mu = \begin{pmatrix} 0 & \sigma_2 / \sigma_0 \\ -\sigma_0 & 0 \end{pmatrix}
\]  
and
\[
\gamma^5 = \gamma_5 = i \gamma^0 \gamma^1 \gamma^2 \gamma^3 = \begin{pmatrix} \sigma_0 & 0 \\ 0 & -\sigma_0 \end{pmatrix}
\]  
(301)

In order to construct the Dirac field using matrix multiplication it is useful to define the \(4 \times 2\) matrix
\[
u_{c\mu} := \begin{pmatrix} \sigma_0 \\ \sigma_0 \end{pmatrix}.
\]  
(302)

The Dirac field is a linear combination of the form [133 134]:
\[
\Psi_\alpha(x) = \int \frac{d\mathbf{p}}{\sqrt{m(p)}} (e^{i p \cdot x} S(B_x(p/m)) c_d u_{d\mu} a_\mu(p, \mu) + e^{-i p \cdot x} (\gamma_5 S(B_x(p/m)) c_c u_{c\mu} \sigma_2 u_{\mu} b^\dagger_2(p, \mu))
\]  
(303)

The \(\gamma_5\) commutes with \(S(B_x(p/m))\), and anti-commutes with \(\gamma^\mu\). While it changes the sign on the lower two components, it is consistent with the freedom to choose the sign of \(\alpha\) and \(\beta\) in the locality constraint [280] and [291]. Multiplying (302) by the Dirac operator \((-i \gamma^\mu \partial / \partial x^\mu + m I)\) gives
\[
(-i \gamma^\mu \partial / \partial x^\mu + m I) \Psi(x) =
\]
\[
\int \frac{d\mathbf{p}}{\sqrt{m(p)}} (e^{i p \cdot x} (p \cdot \gamma + m) S(B_x(p/m)) u a_\mu(p) + e^{-i p \cdot x} (-p \cdot \gamma + m) \gamma_5 S(B_x(p/m)) u \sigma_2 b^\dagger_2(p)) =
\]
\[
\int \frac{d\mathbf{p}}{\sqrt{m(p)}} (e^{i p \cdot x} (p \cdot \gamma + m) S(B_x(p/m)) u a_\mu(p) + e^{-i p \cdot x} \gamma_5 (p \cdot \gamma + m) S(B_x(p/m)) u \sigma_2 b^\dagger_2(p))
\]  
(304)

This vanishes because
\[
(p \cdot \gamma + m) S(B_x(p/m)) = S(B_x(p/m)) S^{-1}(B_x(p/m)) (p \cdot \gamma + m) S(B_x(p/m)) = S(B_x(p/m)) (m \gamma^0 + m I)
\]  
(305)

which vanishes when applied to \(u\) or \(u \sigma_2\).

The quantities
\[
u_{x\mu}(p) := \sqrt{m} (S(B_x(p/m)) u_{c\mu}
\]  
(306)

\[
u_{x\mu}(p) := \sqrt{m} (\gamma_5 S(B_x(p/m)) u \sigma_2)_{c\mu}
\]  
(307)

are Dirac spinors. Note that both the spinors and creation and annihilation operators depend on the choice of boost, \(B_x(p/m)\), but the field itself is independent of this choice.

**IX. SUMMARY**

Relativistically invariant treatments of quantum mechanics are needed to understand physics on distance scales that are small compared to the Compton wavelength of the relevant particles. Of particular importance is the need to consistently calculate matrix elements of hadronic currents when the initial and final hadronic states are in different Lorentz frames.

Relativistic invariance in quantum mechanics means that measurements of quantum observables - probabilities, expectation values and ensemble averages cannot be used to distinguish inertial coordinate systems. This is equivalent to the requirement that equivalent operators and states in different inertial coordinate systems are related by a unitary ray representation of the Poincaré group on the Hilbert space of the quantum theory. Unitary representations of
the Poincaré group can always be decomposed into direct integrals of irreducible representations. This step is the relativistic analog of diagonalizing the Hamiltonian in non-relativistic quantum theory. The structure of the invariant mass \( m \) spin \( j \) irreducible subspaces are fixed by group theory. Different treatments of relativistic quantum theory use different ways of representing these elementary building blocks of the theory. Since each representation has its own advantages, it is important to know precisely how different representations are related.

In this work mass \( m \) spin \( j \) irreducible representations of the Poincaré group were constructed using a basis of simultaneous eigenstates of independent commuting functions of the Poincaré generators. The relevant Hilbert space was the space of square integrable functions of the eigenvalues of these operators. The eigenvalue spectrum of these commuting observables is fixed by properties of the Poincaré group. The transformation properties of the Poincaré generators led to an explicit unitary representation of the Poincaré group on this representation of the Hilbert space. Different choices of the commuting observables lead to different representations that are related by unitary transformations.

Factoring the Wigner rotations that appear in these irreducible representations into products of Lorentz \( SL(2, \mathbb{C}) \) transformations, and using group representation properties of \( SL(2, \mathbb{C}) \), led to equivalent Lorentz covariant representations, where the states transform under finite dimensional representations of \( SL(2, \mathbb{C}) \). In these representations the Hilbert space inner product has a non-trivial kernel, which was shown to be, up to normalization, the two-point Wightman function of a free quantum field theory.

These Lorentz covariant representations where then shown to be derivable from a representation of the Hilbert with a Euclidean covariant kernel and a Euclidean time reflection on the final states. In this representation of the Hilbert space inner product involves an integral over Euclidean variables; it does not require analytic continuation.

Finally covariant fields are constructed from the Lorenz covariant wave functions. These fields have the property that the vacuum expectation value of products of two fields recover the free field Wightman functions that appear in the kernel of the Lorentz covariant representations.

While the Lorentz covariant, Euclidean covariant, and field representations were constructed starting with irreducible representations of the Poincaré group, the process could easily be reversed by factoring the Wightman functions.

While these relations indicate how the many different representations that are used in applications are related, there is no discussion of how to treat the dynamics. This can be very different in each representation. The resulting wave functions can be Poincaré covariant, Lorentz covariant, or Euclidean covariant. This work is limited to understanding the relation between the different irreducible representations of the Hilbert space.

Another class of objects that are used in calculations of strongly interacting systems are quasipotential methods. Quasipotential wave functions were not explicitly discussed because they are not not quantum probability amplitudes, but they are matrix elements of products of covariant fields applied to the vacuum with Poincaré covariant states. Their transformation properties follow from the transformation properties of the fields and Poincaré covariant states.

**Appendix: Wigner D-functions**

The Wigner functions as functions of \( SU(2) \) matrix elements are used extensively in these notes. The following properties (1) \( D_{\mu \nu}^j[R] \) is a homogeneous polynomial of degree \( 2j \) in the components of \( R \in SU(2) \) (2) the coefficients of the polynomial are real (3) and \( D_{\mu \nu}^j[R^\dagger] = D_{\nu \mu}^j[R] \) were all used in this work. Most derivations in the literature are for the expression in terms of Euler angles, rather than in terms of the \( SU(2) \) matrix elements.

The most straightforward derivation of the formula (32) for the Wigner \( D \)-function in terms of the \( SU(2) \) matrix elements uses Schwinger’s formulation of the angular momentum algebra using creation and annihilation operators. The main elements of this formalism are pair of creation and annihilation operators. Angular momentum state are relabeled with

\[
n_{\pm} := j \pm m
\]

which are related to the standard angular momentum labels by

\[
j := \frac{1}{2}(n_+ + n_-) \quad m := \frac{1}{2}(n_+ - n_-)
\]

\[
|n_+, n_-\rangle := |j, m\rangle
\]

The creation and annihilation operators are defined by

\[
a_{\pm}^\dagger |n_\pm\rangle = \sqrt{n_\pm + 1}|n_\pm + 1\rangle
\]
\[ a_\pm |n_\pm \rangle = \sqrt{n_\pm} |n_\pm - 1 \rangle \]  

(A.5)

With these definitions

\[ J_\pm = a_\pm^\dagger a_\mp \quad J_z = \frac{1}{2} (a_+^\dagger a_+ - a_-^\dagger a_-) \]  

(A.6)

\[ \mathbf{J} = \frac{1}{2} \left( \begin{array}{c} a_+^\dagger a_- \\ a_-^\dagger a_+ \end{array} \right) \left( \begin{array}{c} a_- \\ a_+ \end{array} \right) \]  

(A.7)

This can be used to show

\[ e^{i\mathbf{J} \cdot \mathbf{\theta}} a_\mp e^{-i\mathbf{J} \cdot \mathbf{\theta}} = (a_+^\dagger R_{++} + a_-^\dagger R_{--}) \]  

(A.8)

where

\[ R = \cos\left(\frac{\theta}{2}\right) \mathbf{I} + i \hat{\mathbf{\theta}} \cdot \mathbf{\sigma} \sin\left(\frac{\theta}{2}\right) \]  

(A.9)

Normalized angular momentum eigenstates have the form

\[ |n_+, n_-\rangle = \frac{(a_+^\dagger)^{n_+} (a_-^\dagger)^{n_-}}{\sqrt{n_+}! \sqrt{n_-}!} |0, 0\rangle \]  

(A.10)

Combining these results gives

\[ D_{m'm}^{j}[R] = \langle n_+', n_-'|e^{i\mathbf{J} \cdot \mathbf{\theta}} |n_+, n_-\rangle = \]  

\[ \frac{1}{\sqrt{n_+! n_-! n_0! n_0!}} (0, 0) (a_+^\dagger)^{n_+} (a_-^\dagger)^{n_-} (a_+^\dagger R_{+} + a_-^\dagger R_{--})^{n_+} (a_+^\dagger R_{-} + a_-^\dagger R_{--})^{n_-} |0, 0\rangle \]  

(A.11)

(A.12)

(this vanishes unless \( j = j' \)). Expanding \((a_+^\dagger R_{+} + a_-^\dagger R_{--})^{n_+}\) and \((a_+^\dagger R_{-} + a_-^\dagger R_{--})^{n_-}\) using the binomial series and properties of the creation and annihilation operators gives the result...

\[ [1] \text{E. P. Wigner, Annals Math. 40, 149 (1939).} \]
\[ [2] \text{P. A. M. Dirac, Rev. Mod. Phys. 21, 392 (1949).} \]
\[ [3] \text{P. A. M. Dirac, Proc. Roy. Soc. Lond. A183, 284 (1945).} \]
\[ [4] \text{S. Weinberg, Phys. Rev. 134, B882 (1964).} \]
\[ [5] \text{S. Weinberg, Phys. Rev. 133, B1318 (1964).} \]
\[ [6] \text{S. Weinberg, Phys. Rev. 181, 1893 (1969).} \]
\[ [7] \text{S. Weinberg, The Quantum Theory of Fields, vol. I (Cambridge University Press, NY, 1995).} \]
\[ [8] \text{R. F. Streater and A. S. Wightman, PCT, Spin and Statistics, and All That (Princeton Landmarks in Physics, 1980).} \]
\[ [9] \text{K. Osterwalder and R. Schrader, Commun. Math. Phys. 42, 281 (1975).} \]
\[ [10] \text{K. Osterwalder and R. Schrader, Commun. Math. Phys. 31, 83 (1973).} \]
\[ [11] \text{M. Jacob and G. C. Wick, Annals of Physics 7, 404 (1959).} \]
\[ [12] \text{B. Bakamjian and L. H. Thomas, Phys. Rev. 92, 1300 (1953).} \]
\[ [13] \text{V. Bargmann, Annals Math. 59, 1 (1954).} \]
\[ [14] \text{E. P. Wigner, Rev. Mod. Phys. 29, 255 (1957).} \]
\[ [15] \text{A. S. Wightman, L’Invariance Dans La Mecanique Quantique Relativiste, vol. 7 (Hermann, Paris, 1960).} \]
\[ [16] \text{H. J. Melosh, Phys. Rev. D9, 1095 (1974).} \]
\[ [17] \text{W. N. Polyzou, W. Gockle, and H. Witala, Few Body Syst. 54, 1667 (2013).} \]
\[ [18] \text{J. Glimm and A. Jaffe, Quantum Physics - A functional Integral Point of View (Springer, 1981).} \]
\[ [19] \text{A. Klein and L. L., J. Functional Anal. 44, 121 (1981).} \]
\[ [20] \text{A. Klein and L. L., Comm. Math. Phys 87, 469 (1983).} \]
\[ [21] \text{J. Frohlich, K. Osterwalder, and E. Seiler, Annals Math. 118, 461 (1983).} \]
\[ [22] \text{N. N. Bogoliubov and D. V. Shirkov, Introduction to the theory of quantized fields (Wiley-Interscience, 1959).} \]
\[ [23] \text{J. Schwinger, On Angular Momentum (Dover, NY, 1955).} \]