On the definition of geometric Dirac operators

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For the definition of a spin\(^c\) structure and its associated Dirac operators there can be found two different approaches in the literature. One of them uses lifts of the orthonormal frame bundle to principal spin\(^c\) bundles (cf. [Gil], [GH], [Frie] or [LM]) and the other one irreducible representations of the complex Clifford bundle (cf. [BD] or [Kar1,2]). The first approach is an offspring of vector and tensor calculus in its modern form as shaped by E. Cartan and Ch. Ehresmann whereas the second approach is rooted in physics, in particular, relativistic quantum mechanics. Although the second approach is favored nowadays, in defining spin structures most authors still rely on the first method. In this expository note we give a definition of spin structure and the corresponding Spin-Dirac operator purely in the spirit of irreducible representations and prove its equivalence with the usual definition.

This seems to be well known to people working in noncommutative geometry. At least it is used and taken for granted e.g. in [Con1-3], [Ren], and [Var]. The purpose of our note is to make this method accessible to a wider audience in mathematics and in physics and to direct attention to the so far mostly ignored work of G. Karrer who introduced spin\(^c\)-structures in this way already in 1962 and published his results in 1963 [Kar1] and 1973 [Kar2] (unfortunately in German); usually, this approach is credited to A. Connes (cf. [BD]).

Spinors first appeared in the theory of representations of the orthogonal group, in fact of its Lie algebra, in 1913 [Car] and then again in 1927 in connection with the Dirac equation [Dir]. The Schrödinger equation of classical quantum mechanics

\[
\frac{1}{i} \frac{\partial}{\partial t} \psi + \Delta \psi = 0
\]

(without external electro-magnetic field and ripped off of any physical meaning by setting the usual constants \(\hbar, m\) and \(c\) equal to 1) is of first order in the time variable and invariant under Galilei transformations, i.e. time and spatial translations and spatial rotations. To get a relativistic analogue again of first order in \(t\) and invariant under Lorentz transformations, P.A.M. Dirac was looking for a square root of the d’Alembert operator \(\Box = \frac{\partial^2}{\partial t^2} - \sum_{j=1}^{3} \frac{\partial^2}{\partial x_j^2}\) which governs the Klein-Gordon equation. He found

\[
\Box = \partial^2 = \left( \sum_{j=0}^{3} A_j \frac{\partial}{\partial x_j} \right)^2,
\]

where

\[
A_0 = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix} \quad \text{and} \quad A_j = \begin{pmatrix} 0 & \sigma_j \\ -\sigma_j & 0 \end{pmatrix} \in M_2(M_2(\mathbb{C})), \quad j = 1, 2, 3,
\]
with the Pauli matrices
\[\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \text{and} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.\]

The physicists encountered however difficulties in combining Dirac’s first order equation \(\frac{\partial}{\partial t} \psi = 0\) for the relativistic electron with the needs of general relativity, since the spinors \(\psi\) did not transform like vectors or tensors, and so, in first instance, had no geometrical meaning. We quote the words of C.G. Darwin in 1928 [Dar]:

> The relativity theory is based on nothing but the idea of invariance and develops from it the conception of tensors as a matter of necessity; and it is rather disconcerting to find that apparently something has slipped through the net, so that physical quantities exist, which it would be, to say the least, very artificial and inconvenient to express as tensors.

The following years saw various attempts to find a non-local version of the Dirac operator, which had to act on spinors; cf. [vdW1] for a detailed historical survey of concept of spin in physics. But still in 1937 E. Cartan in his book “La Théorie des Spineurs” noted unsurmountable difficulties to apply techniques of classical tensor calculus to spinors. Only in the fifties with the invention of principal bundles and its connections the spinors found their appropriate place in Riemannian geometry. It became possible to define the covariant derivative of spinors and finally around 1960 to define the Dirac operator. This has been achieved by E. Kaehler for the Dirac operator \(d + \delta\) in 1961 [Kae] and by M.F. Atiyah and I.M. Singer for the Spin-Dirac operator in 1962 [AS]. The definition of spin structures consists of two parts, a local and a global one. The local part is purely algebraic and will be treated in the first two sections. Here we sketch the most important results concerning Clifford algebras and refer to e.g. [Che], [Krb] or [LM] for a more detailed account. The global part is of topological nature. It will be exposed in the third section. In the fourth section we discuss spin structures in the setting of principle bundles. We conclude with the definition and some elementary properties of some geometric Dirac operators.

1 Clifford algebras

Clifford algebras solve an algebraic existence problem. To see this recall that the field of complex numbers arises in two ways. In the first instance it is merely a vector space that helps parametrize the Euclidean plane \(\mathbb{R}^2\) but in the second it is an algebra extending the real number field in which square roots exist and which contains an image of the group of rotations. In particular, only by this property we comprehend the law of multiplication of two negative numbers: \((-1)(-1) = 1\), since \(-1 = i^2\) is the composition of two rotations by 90 degrees. As is well known it took R.W. Hamilton ten years to find out in 1843 that there is no analogue in 3-space. One has to step out of ordinary space to find an algebra which contains \(\mathbb{R}^3\) as well its rotations, viz. the skew field of quaternions. What is the appropriate generalization to arbitrary dimension? Starting from a real vector space \(E\), one has e.g. the exterior algebra \(\bigwedge E\) introduced by
H.G. Grassmann in 1844. It contains $E$ and its multiplication $\wedge$ is anti-commutative on basis vectors:

$$e_i \wedge e_j + e_j \wedge e_i = 0.$$  

But then basis vectors $e_i$ are nilpotent, $e_i \wedge e_i = 0$. What we really need is a new multiplication $\cdot$ such that the basic vectors satisfy $e_i \cdot e_i = -1$. How to come to terms with this has first been observed by W.K. Clifford in 1876:

The system of quaternions differs from this, first in that the squares of the units, instead of being zero, are made equal to $-1$; and secondly in that the ternary product $\iota_1 \iota_2 \iota_3$ is made equal to $-1$.

I shall now examine the consequence of making, in a system of $n$ alternate numbers $\iota_1, \iota_2, \ldots, \iota_n$, the first of the modifications just named; namely I shall suppose that the square of each of the units is $-1$.

After the advent of modern abstract algebra the construction of Clifford’s “geometric algebra” runs as follows. We choose an inner product $\langle \cdot, \cdot \rangle$ on $E$, i.e., we assume a Euclidean vector space $(E, \langle \cdot, \cdot \rangle)$, and with respect to this inner product we choose an orthonormal basis $(e_i)_{1 \leq i \leq n}$ that satisfies

$$e_i \cdot e_j + e_j \cdot e_i = -2\delta_{ij} = -2\langle e_i, e_j \rangle.$$

This obtains from assuming

$$v \cdot v + \langle v, v \rangle = 0$$

for any $v \in E$. Just like the exterior algebra the new algebra we are looking for can now be constructed as a quotient of the tensor algebra $\mathcal{T}(E)$. Here we have to consider the two-sides ideal $\mathcal{J}(E) \subset \mathcal{T}(E)$, which is generated by elements $v \otimes v + \langle v, v \rangle 1, v \in E$.

**Definition 1** The $\mathbb{R}$-algebra $\mathcal{O}(E) = \mathcal{T}(E)/\mathcal{J}(E)$ (corresponding to a given Euclidean structure) is called the Clifford algebra of $E$. In case of $E = \mathbb{R}^n$ with its standard Euclidean structure we write $\mathcal{O}_n = \mathcal{O}(\mathbb{R}^n)$.

The product in $\mathcal{O}(E)$ will be denoted by $\cdot$, i.e. for $u, v \in \mathcal{O}(E)$ with $u = \pi(\tilde{u}), v = \pi(\tilde{v})$, where $\pi : \mathcal{T}(E) \rightarrow \mathcal{O}(E)$ denotes the natural projection, let $u \cdot v = \tilde{u} \otimes \tilde{v} + \mathcal{J}(E)$. We also denote by $\iota_E : E \rightarrow \mathcal{O}(E)$ the restriction of $\pi$ to $E$.

Just like the tensor algebra and the exterior algebra the Clifford algebra solves a universal problem.

**Theorem 1** Given an associative unital $\mathbb{R}$-algebra $A$ (with unit 1) and a linear map $f : E \rightarrow A$ with $f(v) \cdot f(v) = -\langle v, v \rangle 1$ for all $v \in E$, there is a unique homomorphism of $\mathbb{R}$-algebras, $\tilde{f} : \mathcal{O}(E) \rightarrow A$, such that the following diagram commutes

\[
\begin{array}{ccc}
\mathcal{O}(E) & \xrightarrow{\tilde{f}} & A \\
E \xrightarrow{f} & & \\
\end{array}
\]

In particular, the algebra $\mathcal{O}(E)$ together with the map $\iota_E : E \rightarrow \mathcal{O}(E)$ satisfying $\iota_E(v)^2 = -\langle v, v \rangle 1$ is uniquely determined by this property up to isomorphism.
Proof: We have

$$\iota_E(v)^2 = \pi(v)^2 = v \otimes v + \mathcal{T}(E) = -\langle v,v \rangle 1 + \mathcal{T}(E) = -\langle v,v \rangle 1 \quad \text{for } v \in E$$

(here $1 = 1 + \mathcal{T}(E)$ is the unity of $\mathcal{O}(E)$). Since $\mathcal{T}(E)$ is generated by $E$ as an algebra and since $\pi$ is surjective, $\mathcal{O}(E)$ is generated by $\iota_E(E)$. Now given a linear map $f : E \to A$ with $f(v)^2 = -\langle v,v \rangle 1_A$, $v \in E$, we have an extension to a homomorphism of algebras, $\otimes f : \mathcal{T}(E) \to A$, given by

$$\otimes f(v \otimes v + \langle v,v \rangle 1) = f(v)^2 + \langle v,v \rangle 1_A = 0,$$

and hence factorizes to a homomorphism of algebras, $\hat{f} : \mathcal{O}(E) \to A$. Then for $v \in E$ we have

$$\hat{f} \circ \iota_E(v) = \hat{f} \circ \pi(v) = \otimes f(v) = f(v)$$

and $\hat{f}$ is uniquely determined since $\iota_E(E)$ generates $\mathcal{O}(E)$.

Clifford algebras have entered quite different branches of modern mathematics and physics in the 100 years since their introduction by W.K. Clifford in 1876 [Cli] and independently by R. Lipschitz in 1880 [Lip]; cf. also his letter from Hades written by his medium A. Weil [Wei]. Clifford’s main purpose was to generalize H.G. Grassmann’s exterior algebra and R.W. Hamilton’s quaternions, whereas Lipschitz was looking for a parametrization of orthogonal transformations of $\mathbb{R}^n$. That Clifford algebras indeed meet both purposes turned out in 1935, when R. Brauer and H. Weyl [BW] gave a very elegant representation of the spin group.

In 1954 C. Chevalley [Che] gave the concise construction presented above. It allows the inner product to be replaced by an arbitrary symmetric bilinear form $\sigma : E \times E \to \mathbb{K}$, or, more precisely, by the corresponding quadratic form $Q$, and $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ by any field. We preferably consider $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ depending on $E$ being a real or a complex vector space. In general, one obtains Clifford algebras $\mathcal{O}(E,Q)$, in particular, for $Q = 0$ the exterior algebra. On $E = \mathbb{R}^{r+s}$ one considers the quadratic forms

$$Q_{r,s}(x) = \sum_{i=1}^{r} x_i^2 - \sum_{i=r+1}^{r+s} x_i^2$$

yielding the Clifford algebras $\mathcal{O}_{r,s}$. We take a look at some special examples.

**Examples**

1. For $E = \mathbb{R}$ with inner product $\langle x,y \rangle = xy$ we have $\mathcal{O}(\mathbb{R}) = \mathcal{O}_1 = \mathbb{C}$. For if $\iota_\mathbb{R}(x) = ix$, $x \in \mathbb{R}$, the algebra $\mathbb{C}$ is generated by $\iota_\mathbb{R}(\mathbb{R})$ since $\iota_\mathbb{R}(x)^2 = -\langle x,x \rangle 1$. Given an algebra $A$ and $f : \mathbb{R} \to A$ as above with $f(x)^2 = -\langle x,x \rangle 1_A$, we get

$$f(x) = xf(1),$$

since $f$ is linear, and

$$f(1)^2 = -1_A.$$

Defining

$$\hat{f}(x + iy) = x1_A + yf(1), \quad x,y \in \mathbb{R},$$

we obtain a homomorphism of $\mathbb{R}$-algebras and

$$\hat{f} \circ \iota_\mathbb{R}(y) = \hat{f}(iy) = yf(1) = f(y), \quad y \in \mathbb{R}.$$
2. The Clifford algebra $\mathcal{C}\ell_2$ is isomorphic with the skew field of quaternions, $\mathbb{H}$, which is generated by $i_t\mathbb{R}^2(e_1)$ and $j = t_2\mathbb{R}^2(e_2)$, since $k = ij$ and $i^2 = j^2 = k^2 = -1$, if $\{e_1, e_2\}$ denotes the standard basis of $\mathbb{R}^2$.

Remarks 1. By the universal property any isometry $f : (E, \langle \cdot, \cdot \rangle) \to (E', \langle \cdot, \cdot \rangle')$ induces a homomorphism of algebras $\mathcal{A}(f) : \mathcal{A}(E) \to \mathcal{A}(E')$: One simply has to lift the map $\bar{f} = \iota_{E'} \circ f$ that satisfies

$$\bar{f}(v)^2 = \iota_{E'}(f(v))^2 = -\langle f(v), f(v) \rangle' 1 = -\langle v, v \rangle 1$$

as in the (commutative) diagram

\[
\begin{array}{ccc}
\mathcal{A}(E) & \xrightarrow{\mathcal{A}(f)} & \mathcal{A}(E') \\
\iota_E & & \iota_{E'} \\
E & \xrightarrow{f} & E'
\end{array}
\]

Given another isometry $g : (E', \langle \cdot, \cdot \rangle') \to (E'', \langle \cdot, \cdot \rangle'')$, one has

$$\mathcal{A}(g \circ f) = \mathcal{A}(g) \circ \mathcal{A}(f).$$

Therefore, $\mathcal{A} : O(E) \to \text{Aut} \mathcal{A}(E)$ defines a homomorphism of groups.

2. The involution $\alpha : E \to E$, $\alpha(v) = -v$, $v \in E$, extends to an involution of $\mathcal{A}(E)$ again denoted by $\alpha$. Using $\alpha$ one defines a $\mathbb{Z}_2$-grading

$$\mathcal{A}(E) = \mathcal{A}(E)^0 \oplus \mathcal{A}(E)^1$$

by $\alpha|_{\mathcal{A}(E)^j} = (-1)^j \text{id}$, $j = 0, 1$; since $\alpha$ is a homomorphism, we have

$$\mathcal{A}(E)^i \cdot \mathcal{A}(E)^j \subset \mathcal{A}(E)^{i+j \mod 2}$$

turning $\mathcal{A}(E)^0$ into a subalgebra.

**Proposition 1** Given $v, w \in E$ with $\langle v, w \rangle = 0$ one has

$$\iota_E(v) \cdot \iota_E(w) + \iota_E(w) \cdot \iota_E(v) = 0.$$ 

More generally, given $x = \prod_{\ell=1}^n \iota_E(v_\ell) \in \mathcal{A}(E)^i$ and $y = \prod_{k=1}^m \iota_E(w_k) \in \mathcal{A}(E)^j$ with $\langle v_\ell, w_k \rangle = 0$ for all $\ell$ and $k$, one has

$$x \cdot y = (-1)^{ij} y \cdot x.$$ 

**Proof:** We compute $\iota_E(v + w)^2$ in two ways. One the one hand

$$\iota_E(v + w)^2 = -(v + w, v + w) 1 = -(v, v) 1 - (w, w) 1.$$
and on the other hand
\[\iota_E(v + w)^2 = \iota_E(v)^2 + \iota_E(w)^2 + \iota_E(v) \cdot \iota_E(w) + \iota_E(w) \cdot \iota_E(v).\]
Equating both sides gives the first assertion. The second one follows by induction since
\(\iota_E(E) \subset \mathcal{A}(E)^1.\)

In order to prove the basic structure theorem for Clifford algebras we need the notion of graded tensor product of two graded algebras. Given two unital \(\mathbb{R}\)-algebras \(A\) and \(B\) with units \(1_A\) and \(1_B\), resp., the tensor product \(A \otimes B\) turns into an \(\mathbb{R}\)-algebra if we put
\[(a \otimes b)(a' \otimes b') = aa' \otimes bb'\]
for \(a, a' \in A, b, b' \in B\). If \(A\) and \(B\) are \(\mathbb{Z}_2\)-graded, i.e. \(A = A^0 \oplus A^1\) and \(B = B^0 \oplus B^1\)
a \(\mathbb{Z}_2\)-grading of \(A \otimes B\) is defined by
\[(A \otimes B)^0 = A^0 \otimes B^0 \oplus A^1 \otimes B^1\]
\[(A \otimes B)^1 = A^1 \otimes B^0 \oplus A^0 \otimes B^1,\]
where the product is now given by
\[(a \otimes b)(a' \otimes b') = (-1)^{ij}aa' \otimes bb'\]
for \(a' \in A^i, b \in B^j\). To distinguish the two tensor products, we denote the graded tensor product of \(A\) and \(B\) by \(A \hat{\otimes} B\).

**Theorem 2** Any orthogonal splitting \(E = E_1 \oplus E_2\) gives rise to a canonical isomorphism of algebras \(\mathcal{A}(E)\) and \(\mathcal{A}(E_1) \hat{\otimes} \mathcal{A}(E_2)\).

**Proof.** We start with \(f : E \rightarrow \mathcal{A}(E_1) \hat{\otimes} \mathcal{A}(E_2)\) defined by
\[f(v_1 + v_2) = \iota_{E_1}(v_1) \otimes 1 + 1 \otimes \iota_{E_2}(v_2), v_k \in E_k.\]
Since \(\iota_{E_k}(v_k) \in \mathcal{A}(E_k)^1, 1 \in \mathcal{A}(E_k)^0,\) and \(v_1 \perp v_2,\) we get
\[f(v_1 + v_2)^2 = \iota_{E_1}(v_1)^2 \otimes 1 + 1 \otimes \iota_{E_2}(v_2)^2\]
\[= (-\langle v_1, v_1 \rangle - \langle v_2, v_2 \rangle)1 \otimes 1\]
\[= -\langle v_1 + v_2, v_1 + v_2 \rangle 1 \otimes 1\]
hence a unique homomorphism \(\tilde{f} : \mathcal{A}(E) \rightarrow \mathcal{A}(E_1) \hat{\otimes} \mathcal{A}(E_2)\) by Theorem 1. Likewise the isometries \(i_1 : E_1 \rightarrow E\) and \(i_2 : E_2 \rightarrow E\) induce homomorphisms \(\mathcal{A}(i_k), k = 1, 2,\)
and for \(x \in \mathcal{A}(E_1)^1, y \in \mathcal{A}(E_2)^1\) one has
\[\mathcal{A}(i_1)(x) \cdot \mathcal{A}(i_2)(y) = (-1)^{ij}\mathcal{A}(i_2)(y) \cdot \mathcal{A}(i_1)(x)\]
by the Proposition. Hence \(\tilde{g} : \mathcal{A}(E_1) \hat{\otimes} \mathcal{A}(E_2) \rightarrow \mathcal{A}(E)\) defined by
\[\tilde{g}(x \otimes y) = \mathcal{A}(i_1)(x) \cdot \mathcal{A}(i_2)(y), \quad x \in \mathcal{A}(E_1), y \in \mathcal{A}(E_2),\]
is a homomorphism; and a straightforward computation on generators shows that \(\tilde{f}\)
and \(\tilde{g}\) are mutual inverses. \(\Box\)
Remark An analogous result holds in case of a direct composition $E = E_1 \oplus E_2$ into $\mathbb{K}$-vector spaces with respect to a quadratic form $Q = Q_1 \oplus Q_2$.

Corollary Given an orthonormal basis $(e_i)_{1 \leq i \leq n}$ of $(E, \langle \cdot, \cdot \rangle)$ one obtains a basis

$$\{\iota_E(e_{k_1}) \cdots \iota_E(e_{k_r}) \mid 1 \leq k_1 < \cdots < k_r \leq n, \ r \geq 0\}$$

of $\mathcal{A}(E)$. In particular, $\dim \mathcal{A}(E) = 2^n$ and multiplication in $\mathcal{A}(E)$ is determined by the relations

$$\iota_E(e_k) \cdot \iota_E(e_k) = -1, \ \iota_E(e_k) \cdot \iota_E(e_\ell) + \iota_E(e_\ell) \cdot \iota_E(e_k) = 0 \quad \text{for} \ k \neq \ell.$$

Moreover one has $\mathcal{A}(E)^i = \text{span} \{\iota_E(e_{k_1}) \cdots \iota_E(e_{k_r}) \mid r = i \text{ mod } 2\}$.

Proof: We decompose $E$ orthogonally into

$$E = \bigoplus_{k=1}^n \mathbb{R}e_k$$

and apply Theorem 2 repeatedly using Example 1:

$$\mathcal{A}(E) \cong (\mathbb{R} \oplus \mathbb{R} \iota_E(e_1)) \hat{\otimes} \cdots \hat{\otimes} (\mathbb{R} \oplus \mathbb{R} \iota_E(e_n)).$$

It is clear that the multiplication is determined by the given relations. From

$$\alpha(\iota_E(e_{k_1}) \cdots \iota_E(e_{k_r})) = (-1)^r \iota_E(e_{k_1}) \cdots \iota_E(e_{k_r})$$

we obtain the final assertion. \hfill \square

From the Corollary we see that $\iota_E : E \to \mathcal{A}(E)$ is injective. Therefore, we can identify $E$ with its image $\iota_E(E)$ and multiply $v, w \in E$ within $\mathcal{A}(E)$, i.e., we write $v \cdot w$ instead of $\iota_E(v) \cdot \iota_E(w)$. We also extend the inner product of $E$ to the inner product of $\mathcal{A}(E)$ that renders the basis of the Corollary an orthonormal basis. Also note that an isomorphism $\mathcal{A}_{n-1} \cong \mathcal{A}_n^q$ is induced by $e_k \mapsto e_k \cdot e_n, \ k = 1, \ldots, n - 1$, given an orthonormal basis $\{e_1, \ldots, e_n\}$ of $\mathbb{R}^n$.

Since $\bigwedge E$ and $\mathcal{A}(E)$ have the same dimensions they are isomorphic as $\mathbb{R}$-vector spaces although not as $\mathbb{R}$-algebras. A canonical homomorphism $\phi : \bigwedge E \to \mathcal{A}(E)$ is given by

$$\phi(v_1 \wedge \cdots \wedge v_k) = \frac{1}{K} \sum_{\sigma \in S_k} (\text{sgn } \sigma) \ v_{\sigma(1)} \cdots v_{\sigma(k)}.$$

It is one-to-one since

$$\phi(e_{j_1} \wedge \cdots \wedge e_{j_k}) = e_{j_1} \cdots e_{j_k}$$

and actually an isometry if $\bigwedge E$ is equipped with the appropriate inner product. The inverse isomorphism $\sigma : \mathcal{A}(E) \to \bigwedge E$ is given by

$$\sigma(x) = c(x)1 \in \bigwedge E, \ c \in \mathcal{A}(E),$$

where $1 \in \mathbb{R} = \bigwedge^0 E$ and where $c : \mathcal{A}(E) \to \text{End}(\bigwedge E)$ denotes the unique extension of the linear map $c : E \to \text{End}(\bigwedge E)$ defined by

$$c(v)\omega = v \wedge \omega - v \wedge \omega, \quad \omega \in \bigwedge E, \ v \in E.$$
We already mentioned the generalized Clifford algebras $\mathcal{C}_{r,s}$. It is easily shown that they are generated by multiplying the standard basis elements $e_1, \ldots, e_{r+s}$ of $\mathbb{R}^{r+s}$ while respecting

$$e_k \cdot e_\ell + e_\ell \cdot e_k = \begin{cases} -2, & k = \ell \leq r \\ 2, & k = \ell > r \quad (\ast) \\ 0, & \text{else.} \end{cases}$$

If $n = r + s$ is even we put $\varepsilon = e_1 \cdot \ldots \cdot e_n$. Using $(\ast)$ we get

$$\varepsilon^2 = (-1)^{(n-1)+(n-2)+\cdots+1} e_1^2 e_2^2 \cdots e_n^2 = (-1)^{\frac{n(n-1)}{2}} (-1)^r 1$$

and we call $\mathcal{A}_{r,s}$ positive or negative if $\varepsilon^2 = +1$ or $-1$, respectively. Since the index of a quadratic form does not depend on the chosen basis we can speak of a positive or negative Clifford algebra $\mathcal{C}_{r,s}(E,Q)$ in case of any vector space of even dimension and any non-degenerate quadratic form.

**Theorem 3** Let $E = E_1 \oplus E_2$ and $Q = Q_1 \oplus Q_2$ with $\dim E_1$ even. Then

$$\mathcal{A}(E,Q) \cong \mathcal{A}(E_1,Q_1) \otimes \mathcal{A}(E_2,\pm Q_2),$$

the sign depending on $\mathcal{A}(E_1,Q_1)$ being positive or negative, respectively.

**Proof**: For $\varepsilon = e_1 \cdot \ldots \cdot e_n \in \mathcal{A}(E_1,Q_1)$, $n = \dim E_1$, one has

$$\varepsilon e_i = (-1)^{n-1} e_i \varepsilon = -e_i \varepsilon,$$

i.e., $\varepsilon v = -v \varepsilon$ for any $v \in E_1 \subset \mathcal{A}(E_1)$. We define

$$\varphi : E = E_1 \oplus E_2 \rightarrow \mathcal{A}(E_1,Q_1) \otimes \mathcal{A}(E_2,\pm Q_2)$$

by

$$\varphi(v_1, v_2) = v_1 \otimes 1 + \varepsilon \otimes v_2, \quad v_i \in E_i,$$

and obtain

$$\varphi(v_1, v_2)^2 = v_1^2 \otimes 1 + \varepsilon^2 \otimes v_2^2 + v_1 \varepsilon \otimes v_2 + \varepsilon v_1 \otimes v_2$$

$$= v_1^2 \otimes 1 \pm 1 \otimes v_2^2$$

$$= -(Q_1(v_1) + Q_2(v_2)) 1 \otimes 1$$

since $v_2^2 = -(\pm Q_2(v_2)) = \mp Q_2(v_2)$. Using Theorem 1 (more precisely the corresponding result for an arbitrary quadratic form) we obtain a homomorphism

$$\tilde{\varphi} : \mathcal{A}(E,Q) \rightarrow \mathcal{A}(E_1,Q_1) \otimes \mathcal{A}(E_2,\pm Q_2).$$

Since dimensions match we are reduced to verify that $\tilde{\varphi}$ is surjective. To this end it suffices to show that $v_1 \otimes 1$ and $1 \otimes v_2$ belong to the image of $\tilde{\varphi}$. But now we have

$$v_1 \otimes 1 = \tilde{\varphi}(\iota_E(v_1)) \quad \text{and} \quad 1 \otimes v_2 = \pm \tilde{\varphi}(\iota_E(v_2) \cdot \varepsilon),$$

which concludes the proof.

\[\square\]
Proposition 2  If dim $E$ is even and $\mathcal{A}(E,Q)$ positive, then
\[ \mathcal{A}(E,Q) \cong \mathcal{A}(E,-Q). \]

Proof: Employing the canonical maps $\iota_\pm : E \to \mathcal{A}(E,\pm Q)$ we put
\[ \varepsilon_\pm = \iota_\pm(e_1) \cdots \iota_\pm(e_n). \]
Now $f : E \ni v \mapsto \varepsilon_+ \cdot \iota_+(v) \in \mathcal{A}(E,Q)$ satisfies
\[ f(v)^2 = -\varepsilon_+^2 \cdot \iota_+(v)^2 = -( -Q(v)), \]
hence induces a homomorphism $\tilde{f} : \mathcal{A}(E,-Q) \to \mathcal{A}(E,Q)$. Moreover
\[ \tilde{f}(\varepsilon_- \cdot \iota_-(v)) = (f(e_1) \cdots f(e_n)) \cdot f(v) = (-1)^{(n-1)/2} \varepsilon_+^{n+2} \cdot \iota_+(v) = \pm \iota_+(v) \]
whereby $\tilde{f}$ is surjective, hence bijective. Q.E.D.

We have already determined $\mathcal{A}_{1,0} = \mathbb{C}$ and $\mathcal{A}_{2,0} = \mathbb{H}$. It is not difficult to see that
\[
\begin{align*}
\mathcal{A}_{0,1} & \cong \mathbb{R} \oplus \mathbb{R} = \mathbb{R}(1,1) + \mathbb{R}(1,-1), \\
\mathcal{A}_{0,2} & \cong M_2(\mathbb{R}) \quad \text{with} \quad e_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad e_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \\
\mathcal{A}_{1,1} & \cong M_2(\mathbb{R}) \quad \text{with} \quad e_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{and} \quad e_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.
\end{align*}
\]
Combining Theorem 3 with the last Proposition we obtain the following complete classification of Clifford algebras $\mathcal{A}_{r,s}$.

Theorem 4  The Clifford algebras $\mathcal{A}_{r+n,s+n}$ and $M_{2^n}(\mathcal{A}_{r,s})$ are isomorphic, in particular
\[ \mathcal{A}_{n,n} \cong M_{2^n}(\mathbb{R}), \quad \mathcal{A}_{r,s} \cong \begin{cases} M_{2^n}(\mathcal{A}_{r-s,0}), & r > s \\
M_{2^n}(\mathcal{A}_{0,s-r}), & r < s \end{cases} \]
and
\[ \mathcal{A}_{r+s+8} \cong \mathcal{A}_{r,s+8} \cong M_{16}(\mathcal{A}_{r,s}). \]

Proof: Since $\mathcal{A}_{1,1}$ is positive we get
\[ \mathcal{A}_{r+1,s+1} \cong \mathcal{A}_{r,s} \otimes \mathcal{A}_{1,1} \cong \mathcal{A}_{r,s} \otimes M_2(\mathbb{R}) \]
and repeatedly by Theorem 3
\[ \mathcal{A}_{r+n,s+n} \cong \mathcal{A}_{r,s} \otimes M_2(\mathbb{R}) \otimes \cdots \otimes M_2(\mathbb{R}) \cong \mathcal{A}_{r,s} \otimes M_{2^n}(\mathbb{R}) \cong M_{2^n}(\mathcal{A}_{r,s}). \]
Since $\mathcal{A}_{4,0}$ is positive again by Theorem 3 and by the Proposition we have
\[ \mathcal{A}_{8,0} \cong \mathcal{A}_{4,0} \otimes \mathcal{A}_{4,0} \cong \mathcal{A}_{4,0} \otimes \mathcal{A}_{0,4} \cong \mathcal{A}_{4,4} \cong M_{16}(\mathbb{R}) \]
and
\[ \mathcal{A}_{p+8,q} \cong \mathcal{A}_{p,q} \otimes \mathcal{A}_{8,0} \cong \mathcal{A}_{p,q} \otimes M_{16}(\mathbb{R}), \]
because $\mathcal{A}_{8,0}$ is positive, too.

We end up with the following table displaying the special Clifford algebras $\mathcal{A}_n = \mathcal{A}_{n,0}$

| $n$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|-----|---|---|---|---|---|---|---|---|---|
| $\mathcal{A}_n$ | $\mathbb{R}$ | $\mathbb{C}$ | $\mathbb{H}$ | $\mathbb{H} \oplus \mathbb{H}$ | $M_2(\mathbb{H})$ | $M_4(\mathbb{C})$ | $M_8(\mathbb{R})$ | $M_8(\mathbb{R}) \oplus M_8(\mathbb{R})$ | $M_{16}(\mathbb{R})$ |

2 Representations of Clifford algebras

We also need representations of abstract Clifford algebras. Recall that a representation $\rho : \mathcal{A}_n \to \text{End}(E)$ on a real (or complex) finite dimensional vector space $E$ is irreducible if for any decomposition $E = E_1 \oplus E_2$ into subspaces invariant under $\rho$ one has $E_1 = E$ or $E_2 = E$. In the reducible case one has $\rho = \rho_1 \oplus \rho_2$ with $\rho_j = \rho|_{E_j}$. Give any non-trivial representation $\rho$, one can find an inner product $\langle \cdot, \cdot \rangle$ on $E$ such that $\rho(x)$ acts orthogonal (or unitarily) on $E$ for all $x \in \mathbb{R}^n \subset \mathcal{A}_n$ with $|x| = 1$. One merely has to average a given inner product $\langle \cdot, \cdot \rangle'$ over the finite (multiplicative) group $G_n$ generated by $e_1, \ldots, e_n \in \mathcal{A}_n$, i.e. one puts

$$\langle v, w \rangle = \sum_{x \in G_n} \langle \rho(x)v, \rho(x)w \rangle', \quad v, w \in E.$$ 

Since $\rho(x)^2 = -|x|^2 I_E$, this amounts to

$$\langle \rho(x)v, w \rangle = -\langle v, \rho(x)w \rangle, \quad v, w \in E, \quad x \in \mathbb{R}^n,$$

i.e. $\rho(x)^* = -\rho(x)$. If this holds we call $\rho$ a skew-adjoint representation.

Now any representation $\rho$ can easily be decomposed into a direct sum of irreducible ones: Choosing $v \in E, v \neq 0$, one considers $E_v = \{ \rho(x)v \mid x \in \mathcal{A}_n \}$ which is invariant under $\rho$. Since $E^\perp_v$ is also invariant, successively splitting off invariant subspaces (in case also of $E_v$) one ends up with $E = \bigoplus_{j=1}^m E_j$ and $\rho = \bigoplus_{j=1}^m \rho_j$ where $\rho_j$ is irreducible. Two representations $\rho_j : \mathcal{A}_n \to \text{End}(E_j)$ are called equivalent if they are implemented by an isomorphism $T : E_1 \to E_2$, i.e.

$$T \rho_1(x) = \rho_2(x)T, \quad \text{for all} \quad x \in \mathcal{A}_n.$$ 

In our case we have $\mathcal{A}_n$ of the form $M_m(\mathbb{K})$ if $n \neq 3$ and 7 which being a simple algebra does not contain any non-trivial two-sided ideal. To see this consider elementary matrices $e_{ij}$ with entries 1 at $i, j$ and 0 elsewhere. Now given a two-sided ideal $V \subset M_m(\mathbb{K})$ and $x = \sum_{1 \leq i, j \leq m} x_{ij} e_{ij} \in V \setminus \{0\}$ there is an $x_{ij} \neq 0$ and therefore $e_{ij} = e_{ij}^* = e_{ij}^2 = e_{ij} x_{ij} e_{ij} \in V$. Since $e_{ij} e_{kl} = \delta_{kj} e_{il}$ all of the $e_{ij}$ belong to $V$, i.e. $V = M_m(\mathbb{K})$. In particular, we consider the left regular representation

$$\rho_L : M_m(\mathbb{K}) \to \text{End}(M_m(\mathbb{K})) \quad \rho_L(x)y = xy, \quad x, y \in M_m(\mathbb{K}).$$

It decomposed as $\rho_L = \bigoplus_{j=1}^m \rho_j$ with irreducible representations $\rho_j(x)e_{jj} = x e_{jj}$ on the left ideals $V_j = M_m(\mathbb{K}) e_{jj}$. If $\rho$ is an arbitrary faithful (i.e. injective) irreducible
representation it has to be equivalent to one of the \( \rho_j \) and hence to each of them. To prove this note that there is a \( v \in E \) and an \( x \in V_1 \) with \( \rho(x)v \neq 0 \). Now define \( T : V_1 \to E \) by \( T(y) = \rho(y)v, \ y \in V_1 \), and observe that

\[
T \rho_1(z)y = T(zy) = \rho(zy)v = \rho(z)\rho(y)v = \rho(z)Ty, \quad y \in V_1,
\]

hence by Schur’s Lemma \( T \) has to be an isomorphism since both representations are irreducible: \( \ker T \subset V_1 \) and \( \text{im } T \subset E \) are subspaces invariant under \( \rho_1 \) and \( \rho \), respectively, hence \( \text{im } T = E \) and \( \ker T = \{0\} \), since \( T \neq 0 \). Combining this with Theorem 4 and the table above we obtain:

**Theorem 5** For \( n \neq 3 \) and \( 7 \mod (8) \) the Clifford algebra \( \mathcal{C}_n \) has up to equivalence exactly one irreducible representation, viz. on \( \mathbb{R}^n \), where

\[
a_n = \begin{cases} 
1, & n = 0, \\
2, & n = 1, \\
4, & n = 2, 3, \\
8, & n = 4, 5, 6, 7,
\end{cases}
\]

and \( a_n + 8k = 2^{4k} a_n \). In cases \( n \equiv 3 \) or \( 7 \mod (8) \) there are exactly two non-equivalent irreducible representations on \( \mathbb{R}^n \).

**Proof:** Noting that \( \mathbb{C} \) is irreducibly represented in \( M_2(\mathbb{R}) \) by \( a + ib \mapsto \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \) and \( \mathbb{H} \) in \( M_2(\mathbb{C}) \subset M_4(\mathbb{R}) \) by \( z + wj \mapsto \begin{pmatrix} z & w \\ -\bar{w} & \bar{z} \end{pmatrix} \) the first assertion follows from the table.

For \( n = 3 \) or \( 7 \) one has \( \mathcal{O}_n \cong M_n(\mathbb{R}) \oplus M_n(\mathbb{R}) \) with \( \mathbb{K} = \mathbb{R} \) or \( \mathbb{H} \), respectively, and two irreducible representations on \( \mathbb{R}^n \cong \mathbb{R}^{2n} \) are given by \( \rho_1(x,y) = \rho(x) \) and \( \rho_2(x,y) = \rho(y) \). They are not equivalent since \( \rho_1(I_n,-I_n) = I_n \) and \( \rho_2(I_n,-I_n) = -I_n \).

Writing \( n = (2\ell + 1)16^\alpha2^\beta \) with \( \beta = 0, 1, 2, \) or \( 3 \) and \( \rho(n) = 8\alpha + 2^\beta \) the highest power of \( 2 \) dividing \( n \) being just \( a_{\rho(n)-1} \) we obtain:

**Corollary** The Clifford algebra \( \mathcal{O}_{\rho(n)-1} \) has a non-trivial representation on \( \mathbb{R}^n \). In particular, there are matrices \( A_1, \ldots, A_{\rho(n)-1} \in O(n) \) with \( A_i^2 = -I_n \) and \( A_iA_j = -A_jA_i, \ i \neq j, \ i, j = 1, \ldots, \rho(n) - 1 \).

**Proof:** Let \( n = p \cdot a_{\rho(n)-1}, \ p \) odd, and \( \delta : \mathcal{O}_{\rho(n)-1} \to \text{End}(\mathbb{R}^{a_{\rho(n)-1}}) \) the previous representation. Then

\[
\bar{\delta} = \bigoplus_{k=1}^p \delta : \mathcal{O}_{\rho(n)-1} \to \text{End} \left( \bigoplus_{k=1}^p \mathbb{R}^{a_{\rho(n)-1}} \right)
\]

is the one we are looking for, since, as seen before, we can choose an inner product that renders \( A_i \) orthogonal with respect to a suitable orthonormal basis.

The matrices \( A_j \) and the numbers \( a_{\rho(n)-1} \) which are guaranteed by the Corollary are often called Hurwitz-Radon matrices and Radon numbers, respectively, after A. Hurwitz [Hur] and J. Radon [Rad] who around 1920 independently constructed such matrices in order to factorize quadratic forms. They also solved the linear vector
field problem: There are exactly \( a_{\rho(2n)-1} \) linear vector fields, given by \( X_j(x) = A_j x \), \( x \in S^{2n-1} \subset \mathbb{R}^{2n} \), that are linearly independent at each point; cf. [Eck].

We also consider complex Clifford algebras \( \mathcal{O}_n^C = \mathcal{O}_n \otimes \mathbb{C} \) and their irreducible representations on complex vector spaces. Complexifying immediately entails

\[
\mathcal{O}_n^C \cong \begin{cases} 
M_{2k}(\mathbb{C}), & \text{if } n = 2k, \\
M_{2k}(\mathbb{C}) \oplus M_{2k}(\mathbb{C}), & \text{if } n = 2k + 1.
\end{cases}
\]

This also shows that (up to equivalence) \( \mathcal{O}_n^C \) has exactly one irreducible representation if \( n = 2k \) and exactly two if \( n = 2k + 1 \). The isomorphism with \( M_{2k}(\mathbb{C}) \) can be made explicit using the Pauli matrices \( \sigma_i \). The basis elements \( e_j, 1 \leq j \leq 2k \), are represented (up to a choice of sign) by the following skew-hermitian unitary matrices:

\[
A_{2\ell-1} = iA'_{2\ell-1} = \sigma_3 \otimes \cdots \sigma_3 \otimes \sigma_1 \otimes I_2 \otimes \cdots \otimes I_2, \quad 1 \leq \ell \leq k,
\]

\[
A_{2\ell} = iA'_{2\ell} = \sigma_3 \otimes \cdots \sigma_3 \otimes \sigma_2 \otimes I_2 \otimes \cdots \otimes I_2, \quad 1 \leq \ell \leq k.
\]

This is a simple consequence of the construction in Theorem 3. These matrices allow to classify complex Clifford algebras and their irreducible representations directly. If \( n = 2k \), i.e., \( \dim M_{2k}(\mathbb{C}) = 2^{2k} = \dim \mathcal{O}_n \) one only has to show that the representation \( \rho(e_j) = A_j \in M_{2k}(\mathbb{C}) \), \( j = 1, \ldots, 2k = n \), is faithful, i.e. that the matrices \( A_I = A_{i_1} \cdots A_{i_t}, 1 \leq i_1 < \cdots < i_t \leq 2k \), are linearly independent. To this end one uses the trace which defines an inner product on \( M_{2k}(\mathbb{C}) \) by \( \langle A, B \rangle = \text{tr}(A^*B) \).

Now for \( \ell \) even one has

\[
\text{tr}(A_I) = \text{tr}(A_{i_1}A_{i_2} \cdots A_{i_{\ell-1}}) = (-1)^{\ell-1} \text{tr}(A_I),
\]

hence \( \text{tr}(A_I) = 0 \), and for \( \ell < 2k \) odd and \( i_{\ell+1} \notin I \) one has

\[
\text{tr}(A_I) = - \text{tr}(A_I A_{i_{\ell+1}} A_{i_{\ell+1}}) = - \text{tr}(A_{i_{\ell+1}} A_I A_{i_{\ell+1}}) = (-1)^{\ell} \text{tr}(A_I),
\]

hence again \( \text{tr}(A_I) = 0 \). Given a linear combination \( \sum a_I A_I = 0 \) this implies

\[
0 = \text{tr} \left( \sum a_I A_I A_J \right) = \pm a_J 2^k.
\]

Note that the argument does not use the special shape of the matrices \( A_j \).

If \( n = 2k + 1 \) there is another matrix

\[
A_{2k+1} = -iA'_{2k+1} = -i\sigma_3 \otimes \cdots \otimes \sigma_3.
\]

However, the extended representation \( \rho : \mathcal{O}_{2k+1}^c \rightarrow M_{2k}(\mathbb{C}) \) by \( \rho(e_{2k+1}) = A_{2k+1} \) is no longer faithful, since

\[
\omega = i^{[(n+1)/2]} e_1 \cdots e_n = i^{k+1} e_1 \cdots e_{2k+1}
\]

is represented by \( \rho(\omega) = I_{2k} \).

A non-equivalent representation \( \rho' \) will be defined by \( \rho'(e_j) = -A_j, 1 \leq j \leq 2k + 1 \), since \( \rho'(\omega) = -I_{2k} \). To obtain a faithful (reducible) representation one takes the direct sum \( \rho \oplus \rho' : \mathcal{O}_{2k+1}^C \rightarrow M_{2k}(\mathbb{C}) \oplus M_{2k}(\mathbb{C}) \subset M_{2k+1}(\mathbb{C}) \).
Definition 2  

If \( \rho : C^n \rightarrow \text{End}(E) \) is an irreducible faithful representation, then the vector space \( E \cong C^{2^k} \) is called a space of spinors; usually, it will be denoted by \( S_0 \).

Remarks 1. Different realizations of \( S_0 \) will be given in the following examples.

2. In \( C_{2k} \), the element \( \omega = i^k e_1 \cdots e_{2k} \) satisfies \( \omega^2 = 1 \) and \( \omega \cdot e_j = -e_j \omega, j = 1, \ldots, 2k \), hence defines a \( Z_2 \)-grading on \( S_0 \), i.e. \( S_0 = S_0^0 \oplus S_0^1 \) where \( S_0^j = \frac{1}{2}(1 + (-1)^j \omega) S_0 \), \( j = 0, 1 \), are the so-called spaces of half-spinors.

The uniqueness of irreducible representations by complex \( 2^k \times 2^k \)-matrices that contain and generalize Pauli’s matrices \([\text{Pau}]\) has first been proved by P. Jordan and E. Wigner \([\text{JW}]\) using group theoretical arguments (in connection with the quantum theory of many electron systems in 1927). The shortest proof without any theory of real Clifford algebras can be found in H. Weyl’s “Group Theory and Quantum Mechanics” of 1931. He explicitly gives the matrices \( A_j \) and expresses by them all of the elementary matrices that generate the simple algebra \( M_{2k}(C) \); cf. also [BW] and [Wey]. We give his construction in the following example.

Examples 3. The reducible representation \( \rho_L : C^n \rightarrow \text{End}(C^n) \) be decomposed into a sum of irreducible ones if \( n = 2k \). In the first case one needs a minimal left ideal \( V \) to act on. Starting from an orthonormal basis \( \{ e_1, \ldots, e_{2k} \} \) of \( C^n \) one can construct \( V \) as follows: Put

\[
f_{\ell} = \frac{1}{\sqrt{2}}(e_{2\ell - 1} + ie_{2\ell}) \quad \text{and} \quad g_{\ell} = \frac{1}{\sqrt{2}}(e_{2\ell - 1} - ie_{2\ell})
\]

as well as

\[
p_{\ell}^+ = \frac{1}{2}(1 \pm ie_{2\ell - 1}e_{2\ell}) \quad \text{for} \quad 1 \leq \ell \leq k,
\]

hence \( p_{\ell}^+ = \frac{1}{2}f_{\ell}g_{\ell} \) and \( p_{\ell}^- = -\frac{1}{2}g_{\ell}f_{\ell} \). The idempotents \( p_{\ell}^\pm \) mutually commute, and for any \( n \)-tuple \( \varepsilon = (\varepsilon_1, \ldots, \varepsilon_k) \) with \( \varepsilon_j = \pm \) they define a projection \( p^\varepsilon = p_{\varepsilon_1}^+ \cdots p_{\varepsilon_k}^+ \).

Then \( V^\varepsilon = C_{2k} p^\varepsilon \) is a minimal left ideal and \( C_{2k} \cong \text{End}(V^\varepsilon) \).

Note that each projection \( p^\varepsilon \) is associated with an elementary matrix \( e_{jj} \) in \( M_{2k}(C) \), e.g. \( e_{11} \) with \( p = p^\varepsilon \) where \( \varepsilon = (1, \ldots, 1) \). Thus \( V \) is isomorphic with the vector space of matrices that have non-trivial entries only in its \( j^{th} \) column. Therefore, one has

\[
1 = \sum_{\varepsilon} p^\varepsilon.
\]

With regards to this example B.L. van der Waerden writes in 1966 [vdW2]:

If you want to determine the structure of an algebra or of a group defined by generating elements and relations and to find a representation of the algebra or group by linear transformations or by permutations, construct the regular representation.

4. To decompose the reducible representation \( c : C^n \rightarrow \text{End}(\bigwedge C^n) \) which in fact is equivalent to the previous one one starts with the orthogonal decomposition \( C^n = W \oplus \overline{W} \), where \( W \) or \( \overline{W} \) denote the subspaces spanned by \( g_{\ell} \) or \( f_{\ell} \), respectively. From the relations

\[
f_{j}f_{\ell} + f_{\ell}f_{j} = 0,
\]

\[
g_{j}g_{\ell} + g_{\ell}g_{j} = 0,
\]

\[
f_{j}g_{\ell} + g_{\ell}f_{j} = -2\delta_{j\ell}.
\]

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and since \( f_j g_I p = 0 \) if \( j \not\in I \) and \( f_j g_I p = (-1)^f 2 g_I p \) if \( I = I' \cup \{ j = i \} \) one obtains that the subspace
\[
V = \mathcal{A}_{2k}^c p = \bigwedge W p
\]
is a left ideal and isomorphic with \( \bigwedge W \) as a vector space. Modifying \( c \) on \( \bigwedge W \subset \bigwedge \mathbb{C}^{2k} \) by taking
\[
\hat{c}(w) = \sqrt{2} \left( \varepsilon(v) - i(\bar{v}) \right) \in \text{End} \left( \bigwedge W \right)
\]
for \( w = v + \bar{v} \in W \otimes \bar{W} \) one obtains the appropriate irreducible representation. Indeed, from the previous relations one easily verifies for
\[
w = \sum_{j=1}^k (x_j e_{2j-1} + y_j e_{2j}) = \frac{1}{\sqrt{2}} \sum_{j=1}^k (\bar{z}_j f_j + z_j g_j)
\]
with \( z_j = x_j + iy_j \) the relation \( \hat{c}(w)^2 = - \sum_{j=1}^k |z_j|^2 I = -|w|^2 I \).

The main problem with the space of spinors is that there is no canonical way to decompose a given representation, even a natural one as in the previous examples, into irreducible ones. Therefore, the spin structure to be defined in the next section and whose construction rests on a proper choice of irreducible representations will in its last analysis always be superficial.

We have shown that up to equivalence any representation \( \rho : \mathcal{A}_{2k} \rightarrow \text{End} (E) \) can be written as \( \rho_0 \otimes I : \mathcal{A}_{2k} \rightarrow \text{End} (S_0 \otimes W) \) with \( E \cong S_0 \otimes W \) and where
\[
\rho(v)(e \otimes w) = \rho_0(v)e \otimes w, \quad e \otimes w \in S_0 \otimes W.
\]

Now given \( S_0 \), at least, \( W \) is canonically defined. To see this we have to digress and recall some general results about tensor products.

Given two real (or complex) vector spaces \( E \) and \( F \), which moreover are right respectively left modules for some real (or complex) algebra \( A \) the tensor product \( E \otimes_A F \) is defined as the quotient space of \( E \otimes F \) by the subspace generated by \( va \otimes w - v \otimes aw \), \( v \in E, w \in F, a \in A \). It is the unique vector space with the following universal property. If \( H \) is a vector space and \( f : E \times F \rightarrow H \) is a bilinear \( A \)-balanced map, i.e. \( f(va,w) = f(v,aw) \) for \( v \in E, w \in F, a \in A \), then there is a unique linear map \( f_A \) such that the following diagram commutes:

\[
\begin{array}{ccc}
E \otimes_A F & \xrightarrow{f_A} & H \\
\downarrow{\gamma} & & \\
E \times F & \xrightarrow{f} & H
\end{array}
\]

Let \( B \) denote another algebra and let \( G \) be a left-\( B \)-module. Then the following results hold:
(a) If \( F \) is also a right-\( B \)-module (hence an \( (A,B) \)-bimodule), then \( E \otimes_A F \) is a right-\( B \)-module, \( F \otimes_B G \) an left-\( A \)-module, and
\[
(E \otimes_A F) \otimes_B G \cong E \otimes_A (F \otimes_B G)
\]
is a natural isomorphism.

(b) If $E$ is a $(B, A)$-bimodule, then $\text{Hom}_B(E, G)$, the space of $B$-module homomorphisms consisting of linear maps $f \in \text{Hom}(E, G)$, which satisfy $f(bv) = bf(v)$ for $v \in E$ and $b \in B$, is an left-$A$-module by $af(v) = f(va)$, $a \in A$, $v \in E$. One has the natural isomorphism

$$\text{Hom}_A((F, \text{Hom}_B(E, G)) \cong \text{Hom}_B(E \otimes_A F, G),$$

induced by $f \mapsto \tilde{f}$ with $\tilde{f}(u \otimes v) = f(v)(u)$ for $f \in \text{Hom}_A(F, \text{Hom}_B(E, G))$, $u \in E$, and $v \in F$.

(c) Moreover, by $(f \otimes v)(w) = f(w) \otimes v$ for $f \in \text{Hom}_B(G, E)$, $v \in F$, and $w \in G$, one obtains a natural homomorphism

$$\text{Hom}_B(G, E) \otimes_A F \cong \text{Hom}_B(G, E \otimes_A F),$$

which is one-to-one and onto if $F$ is a finitely generated projective module, i.e. a direct summand of $A^n$ for some $n \in \mathbb{N}$.

(d) If, however, $E$ is an right-$A$-module, $G$ a left-$B$-module, and $F$ a $(B, A)$-bimodule, then $\text{Hom}_A(E, F)$ is a left-$B$-module by $(bf)(v) = bf(v)$, $b \in B$, $v \in E$, and one has a natural isomorphism

$$E \otimes_A \text{Hom}_B(F, G) \cong \text{Hom}_B(\text{Hom}_A(E, F), G),$$

induced by $u \otimes f \mapsto \tilde{h}$ with $\tilde{h}(g) = f \circ g(u)$ for $u \in E$, $f \in \text{Hom}_B(F, G)$, and $g \in \text{Hom}_A(E, F)$.

We only need these results in the special case $A = \mathbb{C}$ and leave its proofs to the reader; cf. [AF].

As a simple consequence of the last one we obtain that the module $W$ in the decomposition $E = S_0 \otimes W$ can be chosen as $W = \text{Hom}_{\mathcal{O}_m^C}(S_0, E)$: Since $\mathcal{O}_m^C = \text{End}(S_0)$, there are isomorphisms

$$S \otimes \text{Hom}_{\mathcal{O}_m^C}(S_0, E) \cong \text{Hom}_{\mathcal{O}_m^C}(\text{Hom}(S_0, S_0), E) \cong \text{Hom}_{\mathcal{O}_m^C}(\mathcal{O}_m^C, E) \cong E.$$

We conclude this section and the algebraic part of the paper with a classical result of representation that will be essential in the proof of the main theorems of the next section. It is a special case of the Theorem of Skolem-Noether.

**Lemma** Let $A_1$ and $A_2$ be two isomorphic simple subalgebras of $M_n(\mathbb{C})$, say both isomorphic to $M_k(\mathbb{C})$. Then each isomorphism $\Phi : A_1 \rightarrow A_2$ is an inner automorphism $\text{Ad}(U)$ of $M_n(\mathbb{C})$, i.e., there is a $U \in GL_n(\mathbb{C})$ with

$$\Phi(a) = \text{Ad}(U)a = UaU^{-1}, \quad a \in A_2.$$

In particular, each automorphism of $M_k(\mathbb{C})$ is inner and each derivation $D$ of $M_k(\mathbb{C})$ is an inner derivation, i.e. given by

$$Da = \text{ad}(v)a = [v, a] = va - av, \quad a \in M_k(\mathbb{C}),$$

for some $v \in M_k(\mathbb{C})$.

**Proof:** The simple algebra $M_k(\mathbb{C})$ is represented by $A_1$ and $A_2$ in $M_n(\mathbb{C})$, respectively. There are decompositions $C^n = \bigoplus_{j=1}^t E_j$ and $C^n = \bigoplus_{j=1}^r F_j$ which reduce $A_1$ and
A_2$, respectively. $A_1$ and $A_2$ being isomorphic, one has $\ell = r$, and since the restricted irreducible representations have to equivalent, one has
\[ E_j \cong \mathbb{C}^k \cong F_j \quad \text{for all } j. \]

Now $U$ is given as the direct sum of such isomorphisms. To prove the second assertion, one simply has to take $k = n$ and to choose $A_2$ as the image of $A_1 = M_k(\mathbb{C})$ under a given automorphism. For the last assertion take $n = 2k$, $A_1 = \left\{ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \mid a \in M_k(\mathbb{C}) \right\}$, and $A_2 = \left\{ \begin{pmatrix} a & D(a) \\ 0 & a \end{pmatrix} \mid a \in M_k(\mathbb{C}) \right\}$ for a given derivation $D$. Then there is a $U = \begin{pmatrix} u & v \\ w & z \end{pmatrix}$ with
\[ \begin{pmatrix} u & v \\ w & z \end{pmatrix} \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix} = \begin{pmatrix} x & D(x) \\ 0 & x \end{pmatrix} \begin{pmatrix} u & v \\ w & z \end{pmatrix} \]
for all $x \in M_k(\mathbb{C})$. This entails $ux = xw$ and $zx = xz$ for all $x \in M_k(\mathbb{C})$, hence, by Schur’s Lemma, $u$ and $z$ are multiples of the identity. If say $z \neq 0$, the further condition $xw + D(x)z = vz$ on $D$ leads to $D = \text{ad}(z^{-1}v)$.

**Remark** The maps $\nu : GL_k(\mathbb{C}) \to \text{Aut}(M_k(\mathbb{C}))$, $\tau(u) = \text{Ad}(u)$ and $\mu : M_k(\mathbb{C}) \to \text{Der}(M_k(\mathbb{C}))$, $\mu(v) = \text{ad}(v)$ (into the space of derivations) are both onto but in general not one-to-one, since $\ker \nu \cong \mathbb{C}^* = \mathbb{C} \setminus \{0\}$ and $\ker \mu \cong \mathbb{C}$. In the case of $\mu$ one can, however, consider its restriction $\mu_0$ to the subspace $M_k^0(\mathbb{C})$ of matrices with vanishing trace and obtains an isomorphism.

### 3 Spinor bundles and Dirac operators

We now want to globalize the results of the previous section, i.e. to perform the constructions on vector bundles over smooth manifold.

**Definition 3** Let $E$ be a Euclidean vector bundle of rank $k$ over $M$. The vector bundle $\mathcal{O}(E) = \prod_{p \in M} \mathcal{O}(E_p)$ will be called the Clifford bundle of $E$. If $M$ is endowed with a Riemannian structure one particularly has $\mathcal{O}M = \mathcal{O}(TM)$, the Clifford bundle of $M$.

Starting from a local orthonormal frame $(e_i)_{1 \leq i \leq k}$ of $E$ over $U$ one obtains a local trivialization
\[ E|_U \cong v_q = \sum_{i=1}^k a_i e_i(q) \mapsto \varphi(v_q) = (q,(a_i)_{1 \leq i \leq k}) \in U \times \mathbb{R}^k \]
and $\varphi|_{E_q} : E_q \to \{q\} \times \mathbb{R}^k = \mathbb{R}^k$ is isometric for any $q \in U$. Choosing an atlas $\mathcal{A} = \{(U_\alpha, \varphi_\alpha) \mid \alpha \in A\}$ in this way one obtains a cocycle of transition maps with $g_{\alpha\beta} : U_\alpha \cap U_\beta \to O(k)$. To trivialize $\mathcal{O}(E|_U)$ we choose
\[ \mathcal{A}(\varphi_\alpha)(x_q) = \mathcal{A}(\varphi_\alpha|_{E_q})(x_q) \in \{q\} \times \mathcal{A}_k, \quad x_q \in \mathcal{A}(E_q) \]
Hom with the topological or geometrical structure of an oriented Riemannian manifold. We are now going to define geometric differential operators that are closely connected to the setting of

\[ \mathcal{A} \]

In particular, we can extend the natural isomorphisms at the end of the previous section. Specifically, we have the natural isomorphism

\[ \phi : \mathcal{A}(F) \rightarrow \mathcal{A}(G) \]

for any \( F, G \) in \( \mathcal{A} \)-modules. This makes \( \mathcal{A}(E) \) a smooth vector bundle. In particular,

\[ \{ e_{i_1} \cdots e_{i_k} \mid 1 \leq i_1 < \cdots < i_k \leq r, \ k = 0, \ldots, n \} \]

provides an orthonormal frame of \( \mathcal{A}(E) \). The Clifford bundle \( \mathcal{A}(E) \) depends on the Euclidean structure and is itself a Euclidean vector bundle. On the other hand, the \( C^\infty \)-structure and the Riemannian structure of \( \mathcal{A}(E) \) do not depend on the choice of the frame.

Each fiber of \( \mathcal{A}(E) \) comes with an algebra structure and fiber-wise multiplication makes \( C^\infty(\mathcal{A}(E)) \), the space of smooth sections, into an algebra, too. Suitably modifying the definition of a vector bundle one obtains the notion of an algebra bundle \( (A, \pi, M) \).

Each fiber \( \pi^{-1}(p) \) is a finite dimensional topological algebra with respect to the topology induced by \( A \), and at each point \( p \in M \) there exists a chart \( \varphi : \pi^{-1}(U) \rightarrow U \times A \) with a fixed given algebra \( A_0 \), such that

\[ \varphi|_{\pi^{-1}(q)} : \pi^{-1}(q) \rightarrow \{ q \} \times A_0 \]

is an algebra isomorphism for any \( q \in U \). We only consider the special case of unital algebras \( A_0 \) and \( A_p \) with unit elements \( e_0 \) and \( e_p \), respectively. Then we have a global section \( e \) in \( A \).

Alternatively, we may assume a bundle morphism \( (\mu, \text{id}_M) \) with \( \mu : A \otimes A \rightarrow A \) and \( \mu(e_p \otimes a_p) = a_p = \mu(a_p \otimes e_p) \) for any \( a_p \in A_p \).

Moreover a vector bundle \( F \) will be called a (left-)\( A \)-bundle if there is a bundle morphism \( \tau : A \otimes F \rightarrow F \) with

\[ \tau(a_p \otimes \tau(b_p \otimes v_p)) = \tau(\mu(a_p \otimes b_p) \otimes v_p) \quad \text{for} \quad a_p, b_p \in A_p, \ v_p \in F_p. \]

In other words, \( F_p \) is a left-\( A_p \)-module for any \( p \in M \) and

\[ \sigma \cdot s(p) = \sigma(p) \cdot s(p) = \tau(\sigma(p) \otimes s(p)) \quad p \in M \]

defines a smooth section, \( \sigma \cdot s \in C^\infty(F) \), for \( \sigma \in C^\infty(A) \) and \( s \in C^\infty(F) \). An \( A \)-bundle morphism is a bundle morphism \( (f, \text{id}_M) \) between two \( A \)-modules \( F \) and \( G \), that is an \( A_p \)-linear map from \( F_p \) to \( G_p \) for any \( p \in M \). The space of \( A \)-bundle morphisms, \( \text{HOM}_A(F, G) \), is a \( C^\infty(A) \)-module and can be identified with \( C^\infty(\text{Hom}_A(F, G)) \). Here \( \text{Hom}_A(F, G) \) is the sub-bundle of \( \text{Hom}(F, G) \), whose fibers are \( \text{Hom}_A(F_p, G_p) \), \( p \in M \).

In particular, we can extend the natural isomorphisms at the end of the previous section to the setting of \( A \)-bundles.

We are now going to define geometric differential operators that are closely connected with the topological or geometrical structure of an oriented Riemannian manifold \( M \).
**Definition 4**  A smooth vector bundle $E$ over $M$ is called a spinor bundle over $M$ if it is a left-$\mathcal{O}M$-bundle.

If the module structure is given by the morphism $\tau : \mathcal{O}M \otimes E \to E$ we also consider the bundle morphism $c_E : TM \to \text{End}(E)$, $c_E(v_p)(e_p) = \tau(v \otimes e)$, $v \in T_p M$, $e_p \in E_p$, induced by $\tau$ and its extension $c_E : \mathcal{O}M \to \text{End}(E)$ to a morphism of algebra bundles. To emphasize the underlying Clifford multiplication we sometimes denote a spinor bundle by $(E,c_E)$.

**Examples**  5. The Clifford bundle $\mathcal{O}M$ itself is a spinor bundle if

$$c_{\mathcal{O}M} : \mathcal{O}M \to \text{End}(\mathcal{O}M)$$

is in each fiber given by the left regular representation $\rho_L$.

6. Likewise the Grassmann bundle $\wedge^* M$, the exterior bundle of the cotangent bundle $T^*M$, is turned into a spinor bundle using the isomorphism of $\mathcal{O}M$ with $\wedge^* M$. Here and in the following we use the “musical isomorphisms” $^\flat : \wedge^* M \to \wedge^* M$ and its inverse $^\sharp : \wedge^* M \to \wedge^* M$ that extend the pairing between tangent vectors and cotangent vectors provided by the Riemannian metric $g$ of $M$.

7. Given a spinor bundle $E$ and a smooth vector bundle $F$ we can turn $E \otimes F$ into a spinor bundle, $E$ twisted by $F$. Here $\mathcal{O}M$ operates on $E \otimes F$ by $v \cdot (e \otimes f) = (v \cdot e) \otimes f$ for $e \otimes f \in E \otimes F$.

Given a spinor bundle $E$ over $M$ via the isomorphism $^\sharp : T^*M \to TM \subset \mathcal{O}M$ the bundle morphism $\tau$ induces a linear map $T : C^\infty(T^*M \otimes E) \to C^\infty(E)$ given by

$$T(\omega \otimes s)(p) = \tau(\omega(p)^\sharp \otimes s(p)), \ p \in M.$$  

It is easy to see that $T$ is a differential operator of order zero. To define more sophisticated differential operators on $C^\infty(E)$ we need a (Koszul) connection $\nabla$ on $E$, i.e. a linear differential operator $\nabla : C^\infty(E) \to C^\infty(T^*M \otimes E)$ satisfying

$$\nabla(fs) = df \otimes s + f\nabla s, \ f \in C^\infty(M), \ s \in C^\infty(E). \ (*)$$

For a vector field $X \in C^\infty(TM)$ this gives rise to a covariant derivative $\nabla_X$ that satisfies

$$\nabla_X(fs) = X(f)s + f\nabla_X(s).$$

The dual connection $\nabla^*$ on $C^\infty(E^*)$ can be defined by its covariant derivatives

$$\nabla_Xs^*(s) = X(s^*(s)) - s^*(\nabla_Xs)$$

for $s^* \in C^\infty(E^*)$, $s \in C^\infty(E)$, and $X \in C^\infty(TM)$. We also note the following elementary constructions that can be performed with connections $\nabla^E$ and $\nabla^F$ for vector bundles $E$ and $F$, respectively. By

$$\nabla^E \oplus F(s \oplus t) = \nabla^E s \oplus \nabla^F t,$$

and by

$$\nabla^E \oplus F(s \otimes t) = (\nabla^E s) \otimes t + \Psi(s \otimes (\nabla^F t)), \ s \in C^\infty(E), \ t \in C^\infty(F)$$

one defines connections $\nabla^E \oplus F$ for $E \oplus F$ and $\nabla^E \otimes F$ for $E \otimes F$. Here $\Psi$ is induced by the isomorphism of vector bundles, $\psi : E \otimes T^*M \otimes F \to T^*M \otimes E \otimes F$. In particular, one obtains a connection $\nabla^{\text{End}(E)}$ on $\text{End}(E) \cong E^* \otimes E$. 

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Definition 5 Let $E$ be a spinor bundle over $M$, and $\nabla$ a connection for $E$. Then

$$D = T \circ \nabla : C^\infty(E) \to C^\infty(E)$$

defines a first order differential operator, the Dirac operator associated with $(E, \nabla)$.

Proposition 3 Given a local orthonormal frame $(E_i)_{1 \leq i \leq m}$ of $TM$ over $U$ one has

$$Ds = \sum_{k=1}^m E_k \cdot \nabla E_k s$$

for $s \in C^\infty(E|_U)$.

Proof: Since $X = \sum_{k=1}^m \langle E_k, X \rangle E_k$ for $X \in C^\infty(TM)$ one has

$$\nabla_X s = \sum_{k=1}^m \langle E_k, X \rangle \nabla E_k s,$$

hence

$$\nabla s = \sum_{k=1}^m E_k^\flat \otimes \nabla E_k s$$

and so the representation of $D$ as stated. \(\square\)

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and so the representation of $D$ as stated. \(\square\)

With respect to a local frame $(s_j)_{1 \leq j \leq r}$ of $E$ a connection is given by

$$\nabla \left( \sum_{j=1}^r f_j s_j \right) = \sum_{j=1}^r \left( df_j \otimes s_j + f_j \sum_{k=1}^r \omega_{jk} \otimes s_k \right),$$

where the local connection form $\omega = (\omega_{jk})_{1 \leq j, k \leq r}$ defined on say $U$ uniquely determines $\nabla$ on $U$ and vice versa.

Recall that the tangent bundle of a Riemannian manifold $M$ itself comes with a unique torsion-free Riemannian connection, the Levi-Civita connection which we denote by $\overline{\nabla}$. Here torsion-free means that

$$\overline{\nabla}_X Y - \overline{\nabla}_Y X = [X, Y]$$

and Riemannian that

$$\langle \overline{\nabla}_X Y, Z \rangle + \langle Y, \overline{\nabla}_X Z \rangle = X\langle Y, Z \rangle$$

for any vector fields $X$, $Y$ and $Z$. Moreover, the Levi-Civita connection $\overline{\nabla}$ extends to $T^*M$ and to the tensor bundle by the previously mention constructions and also to the exterior bundle $\wedge^* M$ and to the Clifford bundle if we assume the product formula

$$\overline{\nabla}_X (\omega_1 \wedge \omega_2) = (\overline{\nabla}_X \omega_1) \wedge \omega_2 + \omega_1 \wedge (\overline{\nabla}_X \omega_2)$$

for forms $\omega_1, \omega_2 \in \Omega(M)$ respectively

$$\overline{\nabla}_X (\sigma_1 \cdot \sigma_2) = (\overline{\nabla}_X \sigma_1) \cdot \sigma_2 + \sigma_1 \cdot (\overline{\nabla}_X \sigma_2)$$

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for sections $\sigma_1, \sigma_2 \in \mathcal{C}^\infty(\Omega M)$. Combined with the action of the Clifford bundle we obtain Dirac operators that are defined on any oriented Riemannian manifold. The Dirac operator on $\Omega(M)$ has been introduced by E. Kähler in 1961 [Kae] and so is sometimes called Dirac-Kähler. The extension $\nabla$ to $\Lambda^* M$ also satisfies

$$\nabla(\sigma \cdot \omega) = (\nabla \sigma) \cdot \omega + \sigma \cdot (\nabla \omega)$$

in the sense that

$$\nabla_X (\sigma \cdot \omega) = (\nabla_X \sigma) \cdot \omega + \sigma \cdot (\nabla_X \omega)$$

for $X \in C^\infty(TM)$, $\omega \in \Omega(M)$, and $\sigma \in C^\infty(\Omega M)$, hence in both cases $\nabla$ and Clifford multiplication are compatible. Also recall that Clifford multiplication by unit tangent vectors $X_p \in T_p M$ is orthogonal on the spinor bundles $\Omega M$ and $\Lambda^* M$ equipped with the Riemannian metric induced by $g$. This suggests the following definition.

**Definition 6** Let $E$ be complex vector bundle with a Hermitian metric $\langle \cdot, \cdot \rangle$, a connection $\nabla$ and a left $\Omega M_{\mathbb{C}}$-module structure $c_E$. We call the triple $(E, \nabla, \langle \cdot, \cdot \rangle)$ a Dirac triple and, for short, $E$ a Dirac bundle if the given data are compatible, i.e. if

1. $c_E$ is a skew-adjoint representation in each fiber,
2. $\nabla$ is a compatible connection, i.e.

$$\nabla(\sigma \cdot s) = (\nabla \sigma) \cdot s + \sigma \cdot \nabla s \quad \sigma \in C^\infty(\Omega M), \; s \in C^\infty(E),$$

3. $\nabla$ is a Riemannian connection, i.e.

$$\langle \nabla_X s_1, s_2 \rangle + \langle s_1, \nabla_X s_2 \rangle = X(\langle s_1, s_2 \rangle), \quad X \in C^\infty(TM), \; s_1, s_2 \in C^\infty(E).$$

**Remarks**

1. By definition of the Levi-Civita connection on $\Omega M$ to ensure (2) it suffices that

$$\nabla(X \cdot s) = (\nabla X) \cdot s + X \cdot \nabla s$$

for $X \in C^\infty(TM) \subset C^\infty(\Omega M)$ and $s \in C^\infty(E)$.

2. If $(E, \nabla^n_E)$ is a Dirac bundle and $F$ is a Riemannian vector bundle with Riemannian connection $\nabla^F$, then $(E \otimes F, \nabla^E \otimes \nabla^F)$ with Clifford multiplication as in Example 3 is again a Dirac bundle, since for $s_1 \in C^\infty(E)$, $s_2 \in C^\infty(F)$, and $\sigma \in C^\infty(\Omega M)$ one has

$$\nabla^E \otimes \nabla^F (\sigma \cdot (s_1 \otimes s_2)) = \nabla^E (\sigma \cdot s_1) \otimes s_2 + (\sigma \cdot s_1) \otimes \nabla^F s_2$$

$$= (\nabla \sigma) \cdot s_1 \otimes s_2 + (\sigma \cdot \nabla^E s_1) \otimes s_2 + (\sigma \cdot s_1) \otimes \nabla^F s_2$$

$$= (\nabla \sigma) \cdot (s_1 \otimes s_2) + (\nabla^E s_1 \otimes s_2) + (\sigma \cdot (s_1 \otimes \nabla^F s_2))$$

$$= (\nabla \sigma) \cdot (s_1 \otimes s_2) + \sigma \cdot (\nabla^E \otimes \nabla^F (s_1 \otimes s_2)).$$

Condition (1) also holds, since for $X \in C^\infty(TM)$

$$\langle X \cdot (s_1 \otimes t_1), s_2 \otimes t_2 \rangle = \langle (X \cdot s_1) \otimes t_1, s_2 \otimes t_2 \rangle$$

$$= \langle X \cdot s_1, s_2 \rangle \langle t_1, t_2 \rangle = -\langle s_1, X \cdot s_2 \rangle \langle t_1, t_2 \rangle$$

$$= -\langle s_1 \otimes t_1, (X \cdot s_2) \otimes t_2 \rangle$$

$$= -\langle s_1 \otimes t_1, X \cdot (s_2 \otimes t_2) \rangle.$$
In this way we obtain a Dirac operator with coefficients in the bundle $F$ or a Dirac operator by twisting the Dirac operator $D^E$ on $E$ with the connection $\nabla^F$. It will be denoted by $D^E \otimes \nabla^F$ or simply by $D^E \otimes I_F$.

It is well known that any complex vector bundle can be equipped with a Hermitian structure and with a Riemannian connection. Recall that one defines inner products and connections locally and in a second step uses partitions of unity to paste the local data to obtain global ones. So, in general, there is a lot of freedom to do this. In case of a complex spinor bundle one can ask whether these data can be chosen to satisfy (1) to (3). We shall prove that this can indeed be achieved. But before doing so we address the question of uniqueness, i.e. the impact that irreducibility has on the choice of these data.

**Proposition 4** Let $S$ be an irreducible complex spinor bundle with a Hermitian metric $\langle \cdot, \cdot \rangle$ and a connection $\nabla$ satisfying properties (1) to (3). Then the following results hold:

(a) Any Hermitian metric $\langle \cdot, \cdot \rangle'$ with property (1) is of the form

$$\langle \cdot, \cdot \rangle' = \lambda \langle \cdot, \cdot \rangle$$

for some positive real-valued function $\lambda \in C^\infty(M)$.

(b) Any connection $\nabla'$ with property (2) is of the form

$$\nabla' = \nabla + \omega$$

for some complex-valued one-form $\omega \in \Omega^1(M, \mathbb{C})$.

(c) If moreover $\nabla'$ is a Riemannian connection with respect to the given metric, the one-form $\omega$ is purely imaginary, i.e. $\nabla' = \nabla + i \eta$ for some real-valued one-form $\eta \in \Omega^1(M, \mathbb{R})$.

**Proof:** (a) For $p \in M$ let $T \in \text{End}(S_p)$ be a hermitian endomorphism, such that

$$\langle s_1, s_2 \rangle' = \langle Ts_1, s_2 \rangle$$

for all $s_1, s_2 \in S_p$. Then

$$\langle TX_p \cdot s_1, s_2 \rangle = \langle X_p \cdot s_1, s_2 \rangle' = -\langle s_1, X \cdot s_2 \rangle' = -\langle Ts_1, X_p \cdot s_2 \rangle = \langle X_p \cdot Ts_1, s_2 \rangle$$

for all $X_p \in T_p M$. Since the $X_p$ generate $\text{End}(S_p)$, $T$ commutes with each element of $\text{End}(S_p)$, hence by Schur’s Lemma $T = \lambda I$ with $\lambda \in \mathbb{C}$. Since $T$ is hermitian and positive, we have $\lambda \in \mathbb{R}$.

(b) Analogously we conclude that the section $\phi = \nabla'_X - \nabla_X$ into $\text{End}(S)$, which satisfies

$$\phi(\sigma \cdot s) = \sigma \cdot \phi(s)$$

for all $\sigma \in C^\infty(\Omega M_C)$ and $s \in C^\infty(S)$ because of the derivation property that $\phi = \omega(X)I$ with $\omega(X) \in \mathbb{C}$.

(c) This is immediate, since $\omega = \nabla' - \nabla$ has to be skew-hermitian, i.e. $\overline{\omega} = -\omega$. 

**Theorem 6** Let $E$ be a complex spinor bundle over the Riemannian manifold $M$ (of dimension $m = 2n$). Then there are a Hermitian structure and a Riemannian connection for $E$ compatible with Clifford multiplication which possess properties (1) and (2).

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Proof: It suffices to prove this locally. Using a partition of unity local metrics as well as local connections can be pasted to global ones ensuing properties (1) to (3). Let \((U, \varphi)\) be a chart of \(M\) at \(p \in M\) trivializing \(E|_U\). We shall show that on a possibly smaller \(U\) there are complex vector bundles \(S\) and \(W\) with \(E|_U = S \otimes W\) and the \(\mathcal{O}M\)-action irreducible on \(S\) and trivial on \(W\). By the previous remarks it suffices to consider only \(S\) and to equip \(W\) with an arbitrary Hermitian structure and an arbitrary Riemannian connection. Starting from a local orthonormal frame \(\{E_1, \ldots, E_m\}\) of \(TM|_U\) we obtain sections \(p^s \in C^\infty(\mathcal{O}M_C|_U)\) consisting of orthogonal projections. If \(s_1 \in C^\infty(E|_U)\) is a non-vanishing section one has \(p^s(q) \cdot s_1(q) \neq 0\) in the possibly smaller open set \(U\) for some \(\varepsilon\). Then

\[
f(q, (a_\sigma)_{\sigma \in G_m}) = \sum_{\sigma \in G_m} a_\sigma \sigma(q)p^s(q) \cdot s_1(q), \quad q \in U, \quad (a_\sigma)_{\sigma \in G_m} \in \mathbb{C}^{[G]},
\]
defines a vector bundle morphism \(f : U \times \mathbb{C}^{[G_m]} \to E|_U\) of constant rank \(\text{rk}_f(p, \cdot) = N\) hence \(F_1 = \text{Im} f\) is a subbundle of \(E\) whose fibers are irreducible \(\mathcal{O}_m\)-modules. We have \(E = F_1 \oplus F_1^\perp\) and proceeding likewise with a second non-vanishing section \(s_2 \in C^\infty(F_1^\perp|_U)\) etc. we eventually obtain that \(E|_U \cong S \otimes W\) as a \(\mathcal{O}U_\mathbb{C}\)-bundle where \(S = F_1\) and \(W = \varepsilon f_1\). Now the products of sections \(E_j\) in \(C^\infty(\mathcal{O}M|_U)\) generate a finite group \(G_m\). Given an arbitrary Hermitian structure \((\cdot, \cdot)'\) on \(E|_U\) we may define a new one by putting

\[
\langle v, w \rangle = \sum_{\sigma \in G_m} \langle \sigma(q) \cdot v, \sigma(q) \cdot w \rangle', \quad v, w \in E_q.
\]

Now given the irreducible spinor bundle \(S\) we have an isomorphism of algebra bundles \(\Phi : \mathcal{O}M_C|_U \to \text{End}(S)\). Extending the Levi-Civita connection \(\nabla\) to \(\mathcal{O}M|_U\) and then to \(\mathcal{O}M_C|_U\), by \(\Phi^{-1}\) we induce a connection \(\nabla S = \Phi \nabla \Phi^{-1}\) on \(\text{End}(S)\). We only have to show that \(\nabla S = \nabla^{\text{End}(S)}\), i.e. induced by a Riemannian connection \(\nabla S\) on \(S\). This one will automatically possess property (2), since

\[
\nabla S(\sigma \cdot s) = \nabla S(\Phi(\sigma)(s)) = \nabla(\Phi(\sigma))(s) + \Phi(\sigma)(\nabla S s) = \Phi(\nabla S \sigma)(s) + \Phi(\sigma)(\nabla S s) = \nabla S(\sigma \cdot s) + \sigma \cdot \nabla S s.
\]

Note that from the Remark concluding section 2 we have sub-bundles \(\text{End}_0(S)\) of fiber-wise endomorphisms with trace 0 and \(\text{Der}(S)\) of fiber-wise derivations of \(\text{End}(S)\), as well as a bundle isomorphism \(\mu_0 : \text{End}_0(S) \to \text{Der}(S)\).

If \(\nabla 0\) is an arbitrary Riemannian connection on \(S\) and \(\nabla 0\) the connection induced on \(\text{End}(S)\), then \(\eta = \nabla - \nabla 0\) is a section in \(T^* M \otimes \text{End} \left( \text{End}(S) \right)\) and from the derivation property even a section in \(T^* M \otimes \text{Der}(S)\). For \(\gamma = \mu_0^{-1}\eta\) we then have

\[
\nabla_X t - \nabla_{0X} t = \gamma(X) t - t \gamma(X), \quad X \in C^\infty(TM), \quad t \in C^\infty(\text{End}(S)).
\]

And putting

\[
\nabla_{0X} s = \nabla_{0X} s + \gamma(X)(s), \quad s \in C^\infty(S),
\]

we obtain, for the induced connection on \(\text{End}(S)\),

\[
\nabla_{0X} t = \nabla_{0X} t + \gamma(X) t - t \gamma(X),
\]

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hence $\nabla^0_0 = \nabla$. Although we started from a Riemannian connection $\nabla_0$ the construction does not guarantee that $\nabla^0_0$ is also a Riemannian connection. Now putting

$$\langle s_1, s_2 \rangle' = X(\langle s_1, s_2 \rangle) - \langle \nabla^S_{\partial \bar{X}} s_1, s_2 \rangle - \langle s_1, \nabla^S_{\partial \bar{X}} s_2 \rangle$$

we get a sesquilinear form on $S$, hence $\langle \cdot, \cdot \rangle' = \omega(X)\langle \cdot, \cdot \rangle$ by (a) of the Proposition. It is easily seen, that $\omega$ is a (real-valued) one-form. We can finally put

$$\nabla^S = \nabla^S_0 + \frac{1}{2} \omega$$

which by (b) of the Proposition satisfies property (2) and by a simple computation is seen to be a Riemannian connection with respect to $\langle \cdot, \cdot \rangle$.

For $E$ we have found, at least locally, a decomposition $E = S \otimes W$ with $\mathcal{A}M$ acting irreducibly on $S$. However, there are topological obstructions for a global such decomposition to hold. We come back to this point later on. However, if $S$ is given globally, $W$ is naturally determined by $W = \text{Hom}_{\mathcal{A}M}(S, E)$. It is easy to show that this is indeed a sub-bundle of the bundle of $\text{Hom}(S, E)$. This gives rise to the following definition.

**Definition 7** An oriented Riemannian manifold $M$ of dimension $m = 2n$ is said to be spin$^c$ if there is a complex spinor bundle $S$ over $M$ with $\mathcal{A}M \otimes \mathbb{C} \cong \text{End}(S)$.

If $M$ is spin$^c$ any spinor bundle $E$ can be written as $E = S \otimes W$ with some complex vector bundle $W$. In particular, for any further irreducible spinor bundle $S'$ there exists a complex line bundle $L$ with $S' = S \otimes L$, viz. $L = \text{Hom}_{\mathcal{A}M}(S, S')$. Given $S$ we can now make it a Dirac bundle by properly choosing a Hermitian structure and a Riemannian connection $\nabla$. However, this connection is only determined up to an additional purely imaginary one-form. Most desirable would be a unique connection on $S$ induced by the Levi-Civita connection of $M$. Then the connection on any further Dirac bundle $S' = S \otimes L$ could be chosen as the product connection only depending on the connection on the line bundle $L$. To ensure this we need a spin structure for $M$ given by an additional structure on $S$.

We start with the algebraic setting and consider the complex vector space $S_0$ of spinors bearing an operation of the real Clifford algebra $\mathcal{A}_m$. This is not irreducible but depending on the dimension $m = 2n = 8k + 2\ell$ one can find an irreducible real subspace of $S_0$. More precisely, there exist an antilinear map $\theta_0 : S_0 \to S_0$ with $\theta_0^2 = I_{S_0}$ for $\ell = 0$ or $3$ and $\theta_0^2 = -I_{S_0}$ for $\ell = 1$ or $2$, a so-called structural map. In the first case $S_0$ carries a real structure, in the second case a quaternionic structure. This is obvious if $\ell = 0$ or $3$, since then $\mathcal{A}_m = M_{2^\ell}(\mathbb{R})$ is acting irreducibly on $\mathbb{R}^{2n}$ and the complex Clifford algebra and the spinor space $S_0 = \mathbb{C}^{2^n}$ are obtained therefrom by complexification.

Here $\theta$ can be chosen the complex conjugation $c : \mathbb{C}^{2^n} \to \mathbb{C}^{2^n}$ taken component-wise. In the other two cases we consider the explicit representation of $S_0 = \mathbb{C}^{2^n}$, and by periodicity may restrict to $k = 0$. With $c$ as before and $\tau = i\sigma_2$ we now put $\theta_0 = \tau \circ c$ if $\ell = 1$ and $\theta_0 = (\tau \circ \sigma_3) \circ c$, if $\ell = 2$. Then $\theta_0$ is the required structural map, and in all cases it commutes with the representation of $\mathcal{A}_m$. The antilinear map $\theta_0 : S_0 \to S_0$ can also be seen as a linear map $\theta_0 : S_0 \to \bar{S}_0$, where by $\bar{S}_0$ we denote the complex vector space $S_0$ with scalar multiplication changed to $\lambda \cdot v = \bar{\lambda}v$, $\lambda \in \mathbb{C}$, $v \in S_0$. The
representation of $\mathcal{G}_m$ on $S_0$ induces a representation of $\mathcal{G}_m$ on $\bar{S}_0$, and extending both representations to $\mathcal{G}_m^\infty$ we obtain an element $\theta_0$ of Hom$_{\mathcal{G}_m^\infty}(S_0, \bar{S}_0)$ with $\theta_0^2 = \pm I_{S_0}$.

Since $c$ and $\tau$ both depend on a basis of $S_0$ it is in general not possible to extend this local construction to a global one on the spinor bundle $S$. So at first we will assume a global structural map and afterwards will establish sufficient conditions for it existence.

**Definition 8** Let $M$ be an oriented Riemannian manifold of dimension $m = 8n + 2\ell$. We say that $M$ carries a spin structure or that $M$ is spin, if $M$ is spin$^c$ and if the irreducible complex spinor bundle $S$ allows a structural map $\theta \in \mathcal{C}^\infty(\text{Hom}_{\mathcal{G}_M}(S, \bar{S}))$ with $\theta^2 = I_S$ or $\theta^2 = -I_S$ inducing respectively a real ($\ell = 0$ or 3) or quaternionic ($\ell = 1$ oder 2) structure on $S$, that is compatible with the complex conjugation of $\mathcal{G}_M = \mathcal{G} \otimes \mathbb{C}$.

**Remark** Equivalently, we may require the existence of a real spinor bundle on which the real Clifford bundle $\mathcal{G}$ acts irreducibly on each fiber. If $\ell = 3$ or 4 one can choose the fixed-point bundle of $\theta$, and conversely the complexified real spinor bundle will define a spin structure.

Of course, any spin manifold is spin$^c$ but the converse does not hold in general. We address this question in the next section. Here we only prove the following general characterization.

**Theorem 7** Let $M$ be an oriented Riemannian manifold $M$ with spin$^c$ structure given by the irreducible complex spinor bundle $S$. Then $S$ defines a spin structure, i.e., allows a global structural map $\theta$ if and only if the vector bundle Hom$_{\mathcal{G}_M}(S, \bar{S})$ is trivial.

**Proof:** We already know that Hom$_{\mathcal{G}_M}(S, \bar{S})$ is a complex line bundle: Each fiber contains a $\mathcal{G}(T_p M)$-linear isomorphism $\theta_p : S_p \to \bar{S}_p$ and by irreducibility of $S_p$ and $\bar{S}_p$ and Schur’s Lemma any $\mathcal{G}(T_p M)$-linear map $\theta_p' : S_p \to S_p$ satisfies $\theta_p' \circ \theta_p = \lambda I_{S_p}$ for some $\lambda \in \mathbb{C}$. In case of a spin structure $\theta$ defines a non-vanishing section in Hom$_{\mathcal{G}_M}(S, \bar{S})$, hence Hom$_{\mathcal{G}_M}(S, \bar{S})$ is trivial. Conversely, if this bundle is trivial and if $\theta'$ is a non-vanishing section, then $\theta'^2 = \lambda I_S$ for some non-vanishing map $\lambda \in \mathcal{C}^\infty(M, \mathbb{C})$. But

$$\lambda(p) \hat{\theta}_p(v) = \hat{\theta}_p \circ \hat{\theta}_p \circ \hat{\theta}_p(v) = \hat{\theta}(\lambda(p) v) = \overline{\lambda(p)} \hat{\theta}(v),$$

i.e., $\lambda \in \mathcal{C}^\infty(M)$ is real-valued, and replacing $\hat{\theta}$ by $\theta = |\lambda|^{-1/2} \hat{\theta}$ we obtain a structural map.

Now given an irreducible complex spinor bundle $S$ and a structural map $\theta$) we can choose a Riemannian structure compatible with Clifford multiplication and such that $\theta$ is an isometry. Moreover, we can choose a Riemannian connection $\nabla^S$ with properties (1) and (2) uniquely determined up to a purely imaginary one-form. If we also require that $\nabla^S$ is compatible with $\theta$, i.e.

$$\nabla^S s = (\text{id}_{T^* M} \otimes \theta) \nabla^S (\theta \circ s), \quad s \in \mathcal{C}^\infty(S),$$

then such a connection $\nabla^S$ is uniquely determined:
Theorem 8 If $M$ is a spin manifold of dimension $m = 2n$ with corresponding spinor bundle $S$ and structural map $\theta$, then:

(a) On $S$ there exists a Riemannian structure compatible with Clifford multiplication and with $\theta$, i.e. $(\theta(s_1), \theta(s_2)) = \langle s_1, s_2 \rangle$ for $s_1, s_2 \in C^\infty(S)$.

(b) There is a unique Riemannian connection $\nabla^S$ with properties (1) and (2) and compatible with $\theta$.

Proof: (a) We consider $\theta$ as an antilinear map on $S$ and change a given Riemannian metric $(\cdot, \cdot)'$ with property (1) to

$\langle s_1, s_2 \rangle = \frac{1}{2}((s_1, s_2)' + \langle \theta(s_1), \theta(s_2) \rangle')$.

Then the new metric will also be compatible with Clifford multiplication. Moreover, one has

$\langle \theta(s_1), \theta(s_2) \rangle = \frac{1}{2}((\theta(s_1), \theta(s_2))' + \langle \theta^2(s_1), \theta^2(s_2) \rangle') = \langle s_1, s_2 \rangle$.

In particular, $\langle \theta(s_1), s_2 \rangle = \pm \langle \theta(s_2), s_1 \rangle$, hence $\langle \theta(s_1), s_1 \rangle = 0$ in the quaternionic case.

(b) It suffices to prove uniqueness. We choose a local orthonormal frame $s_j$ of $S$ (which is a local orthonormal frame of $\bar{S}$ simultaneously) and the corresponding local connection form $\omega$. In the real case $S$ is a complexified real spinor bundle, and we can choose the frame such that $\theta(s_j) = s_j$, $j = 1, \ldots, 2^n$. In the quaternionic case we can choose the frame such that $s_{2^{n-1} + j} = \theta(s_j)$, $j = 1, \ldots, 2^{n-1}$. By compatibility of $\nabla^S$ and $\theta$ in the real case we obtain

$\nabla^S \theta(s_j) = \sum_{i=1}^{2^n} \omega_{j,i}s_i = \sum_{i=1}^{2^n} \bar{\omega}_{j,i}s_i = \theta(\nabla^S s_j)$,

i.e. $\omega = \bar{\omega}$. In the quaternionic case we obtain

$\nabla^S \theta(s_j) = \sum_{i=1}^{2^n} \omega_{j+2^{n-1},i}s_i = \sum_{i=1}^{2^{n-1}} \bar{\omega}_{j,i}s_i + \sum_{i=1}^{2^{n-1}} \bar{\omega}_{j,i+2^{n-1}}s_i = \sum_{i=1}^{2^{n-1}} \bar{\omega}_{j,i}s_i = \theta(\nabla^S s_j)$,

for $j = 1, \ldots, 2^{n-1}$, i.e.

$\omega_{j+2^{n-1},i} = \begin{cases} \bar{\omega}_{j,i+2^{n-1}}, & i = 1, \ldots, 2^{n-1}, \\ \bar{\omega}_{j,i-2^{n-1}}, & i = 2^{n-1} + 1, \ldots, 2^n, \end{cases}$

and, in particular, $\omega_{jj} = \bar{\omega}_{j+2^{n-1},j+2^{n-1}}$ for $j = 1, \ldots, 2^{n-1}$.

Thus, in both cases addition of a purely imaginary one-form is prohibited.

Examples 8. Any oriented complex manifold (or, more generally, an almost-complex manifold) is spin: Since the complex cotangent bundle $T^*M_C$ splits orthogonally $T^*M_C = (T^cM)^* \oplus (T^cM)^* = \bigwedge^{1,0} M \oplus \bigwedge^{0,1} M$, we can choose $S = \bigwedge^* T^c M = \bigwedge^0 M$. 

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9. However, in general a complex manifold is not spin, e.g. it can be proved that $\mathbb{C}P^n$ is spin if and only if $n$ is odd.

10. Any oriented compact hyper surface $M \subset \mathbb{R}^{2n+1}$ (that is the boundary of a compact $2n + 1$-dimensional submanifold $N$ with boundary) is spin:

Using the matrices $A_j \in M(\mathbb{C}^2)$, $j = 1, \ldots, 2n + 1$ the Clifford multiplication $E_j \cdot v = A_j v$ for $v \in \mathbb{C}^2 = S_0$ and the standard orthonormal frame $E_1, \ldots, E_{2n+1}$ of $\mathbb{R}^{2n+1}$ makes $\mathbb{R}^{2n+1} \times \mathbb{C}^2$ a (trivial) complex spinor bundle over $\mathbb{R}^{2n+1}$. If we restrict to $M$ and consider $TM$ as a subbundle of $TN|_M$ the Clifford modules $\{p\} \times \mathbb{C}^2$, $p \in M$, are irreducible $\mathfrak{O}(T_pM_C)$ modules, since we can generate $\mathfrak{O}^2_{2n+1}$ by an orthonormal basis $E_1(p), \ldots, E_{2n}(p)$ of $T_pM$ and the exterior normal vector $E_{2n+1}(p) = X_N(p)$. Therefore, $H = M \times S_0$ defines a spinor bundle for $M$. Since, moreover, the $E_j$ as real linear combinations of the $E_j$ also commute with the structural map $\theta$ of $S_0$, we even have a spin structure. The grading operator on $H$ is defined by $\epsilon = -iX_N$, with the exterior normal vector field $X_N$ at $M$. If $M = S^{2n}$, we have $X_N(x) = \sum_{k=1}^{2n+1} x_k E_k$ and $H^{n(\text{mod } 2)}$, the bundles of half-spinors are non-trivial smooth vector bundles.

Special examples are oriented compact surfaces $T_3$ in $\mathbb{R}^3$ or spheres $S^{2n}$ in $\mathbb{R}^{2n+1}$. The former allow $2^{2g}$ different spin structures whereas there is only one spin structure on $S^{2n}$. To see this we need the following result.

**Theorem 9** Let $M$ be a connected oriented Riemannian manifold. If $M$ carries a spin structure, then all of the non-equivalent spin$^c$ structures are parametrized by $H^2(M, \mathbb{Z})$.

If moreover $M$ is spin, then all of the different spin structures are parametrized by $H^1(M, \mathbb{Z}_2) \cong \text{Hom}(\pi_1(M), \mathbb{Z}_2)$. In particular, $M$ allows at most one spin structure if $M$ is simply connected.

**Proof:** Starting from an irreducible complex spinor bundle $S$, any further irreducible complex spinor bundle on $M$ is of the form $S' = S \otimes L$ where $L = \text{Hom}_{\mathfrak{O}_M}(S, S')$. If $S'$ and $S'' = S(E) \otimes L'$ are isomorphic as spinor bundles, i.e. determine equivalent spin$^c$ structures, there is a $\Phi \in \text{Iso}_{\mathfrak{O}_M}(S', S'')$, and so $L \cong L'$. This shows that $H^2(M, \mathbb{Z})$ acts transitively on the set of different spin$^c$ structures. Now for $\text{Hom}_{\mathfrak{O}_M}(S', S')$ we obtain

$$\text{Hom}_{\mathfrak{O}_M}(S', S') \cong \text{Hom}_{\mathfrak{O}_M}(S \otimes C L, S \otimes C \tilde{L}) \cong \text{Hom}(L, \text{Hom}_{\mathfrak{O}_M}(S, S \otimes C \tilde{L}))$$

$$\cong L^* \otimes C \text{Hom}_{\mathfrak{O}_M}(S, S \otimes C \tilde{L}) \cong L^* \otimes C \text{Hom}_{\mathfrak{O}_M}(S, S \otimes C \tilde{L})$$

if $\text{Hom}_{\mathfrak{O}_M}(S, S)$ is trivial. Therefore, there is a structural map on $S'$ if and only if $L^* \otimes \tilde{L} \cong \tilde{L}^2$ is trivial. If $H^2(M, \mathbb{Z})$ has no 2-torsion, $\tilde{L}$ has to be trivial, too, and likewise $L$. In any case different spin structures are classified by isomorphism classes of real line bundles, i.e., by $H^1(M, \mathbb{Z}_2)$; cf. [Kar].

**Remarks 1**. We always started with the Clifford bundle of the tangent bundle. Only with literate changes we can start with a real Riemannian vector bundle $E$ of even rank. A spin$^c$ structure is then given by a complex spinor bundle $S(E)$ with $\mathfrak{O}^c(E)$ acting irreducibly on the fibers, and a spin structure by an additional structural map compatible with Clifford multiplication. If $E$ comes with a Riemannian connection $\nabla^E$ there is unique connection $\nabla^{S(E)}$ on $\mathfrak{O}(E)$ and in the spin case a unique Riemannian connection $\nabla^{S(E)}$ on $S(E)$ that satisfy properties (1) and (2) and

$$\nabla^{S(E)}(\sigma \cdot s) = \nabla^{\mathfrak{O}(E)}(\sigma) \cdot s + \sigma \cdot (\nabla^{S(E)}s),$$

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for $\sigma \in C^\infty(\mathcal{A}(E))$, $s \in C^\infty(S(E))$.

2. On an oriented Riemannian vector bundle $E$ of odd rank $m = 2n + 1$ (in particular, on an odd-dimensional Riemannian manifold) spin$^c$ or spin structures can be defined, too. Here a spin$^c$ structure is given by a complex spinor bundle $S(E)$, on which $\mathcal{A}^C(E)$ acts irreducibly, and where for each oriented orthonormal frame $e_1(p), \ldots, e_m(p)$ of $E_p$, the element $i^{n+1}c_1(p)\cdots e_m(p)$ acts as $I_{E_p}$.

4 Spin groups and principal bundles

There are topological obstructions for a spin$^c$ or a spin structure to exist on a manifold $M$. We know that if $M$ is spin$^c$ and $S$ an irreducible complex spinor bundle structure then $M$ is spin if and only if $L = \text{Hom}_{\mathcal{A}Mc}(S, S)$ is trivial. Now if $M$ is simply connected this can be decided by computing a topological invariant. It is well known (cf. [Sdr2]) that $L$ is trivial if and only if the first Chern class $c_1(L)$ vanishes. But this does not apply in general if $M$ is not simply connected. Then the obstructions are better expressed in terms of the so-called second Stiefel-Whitney class $w_2(TM)$, an element of $H^2(M, \mathbb{Z}_2)$ (cf. [Hae]). This is a cohomology class with coefficients in $\mathbb{Z}_2 = \{\pm 1\}$, and can be represented by lifts of cocycles of $SO(n)$-valued transition maps to the covering group $\text{Spin}(n)$.

At this point we have to digress and take a closer look at the covering group $\text{Spin}(n)$ of $SO(n)$. Here again Clifford algebras are the appropriate tool to generalize classical constructions. We first inspect how Clifford algebras help represent orthogonal transformations. It is well known that $S^3 \subset \mathbb{H}$ is the two-fold simply connected covering of the Lie group $SO(3)$. Identifying $\mathbb{R}^3$ with $\text{Im} \mathbb{H} = \{is + jt + ku \in \mathbb{H} \mid s, t, u \in \mathbb{R}\}$ an element $x \in S^3 = \{y \in \mathbb{H} \mid |y|^2 = \bar{y}y = 1\}$ acts on $\mathbb{R}^3$ by

$$\text{Ad}_x(v) = xvx^{-1} = xv\bar{x}, \quad v \in \mathbb{R}^3.$$ 

Note that $x$ and $-x$ define the same element of $SO(3)$. More generally one could use any $x \in \mathbb{H}^* = \mathbb{H} \setminus \{0\}$ since $\text{Ad}_x = \text{Ad}_{x/|x|}$.

To find the covering group of $SO(n)$ for $n \geq 4$ or of $SO(E)$ for a Euclidean vector space $E$ we start from the regular group $G\mathcal{A}(E)$ of invertible elements of the algebra $\mathcal{A}(E)$. For $x \in E \setminus \{0\} \subset G\mathcal{A}(E)$ and $v \in E \subset \mathcal{A}(E)$ we have $x \cdot v + v \cdot x = -2(x, v)1$, hence

$$- \text{Ad}_x(v) = v - 2\frac{(x, v)}{(x, x)}x.$$ 

From a geometric point of view this is the reflection at the hyperplane perpendicular to $x$. Using the involution $\alpha$ (that induces the grading $\mathcal{A}(E) = \mathcal{A}(E)^0 \oplus \mathcal{A}(E)^1$) we pass over to the “twisted” adjoint representation on $E$ given by

$$\widetilde{\text{Ad}}_x(v) = \alpha(x)vx^{-1}$$ 

which is is naturally defined on the Clifford group

$$\Gamma(E) = \{x \in G\mathcal{A}(E) \mid \alpha(x)vx^{-1} \in E \text{ for all } v \in E\}.$$ 

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Proposition 5 The twisted adjoint representation $\tilde{\Ad} : \Gamma(E) \to \text{Aut}(E)$ is a homomorphism of groups and induces an exact sequence

$$1 \to \mathbb{R}^* \to \Gamma(E) \xrightarrow{\tilde{\Ad}} O(E) \to 1.$$  

Any $x \in \Gamma(E)$ can be written as $x = v_1 \cdots v_k$, $v_i \in E$, $v_i \neq 0$, $i = 1, \ldots, k$.

**Proof:** Obviously, $\tilde{\Ad}$ is a homomorphism. Next we show that $x \in \mathbb{R}^* = \mathbb{R} \setminus \{0\} \subset \Gamma(E)$ if $\alpha(x)v = xv$ for all $v \in E$ or equivalently if this holds elements $v$ of an orthonormal basis $(e_i)_{1 \leq i \leq n}$ of $E$. To this end we write $x = x^0 + x^1 \in \mathcal{O}(E)^0 \oplus \mathcal{O}(E)^1$ with $x^0 = a_i^0 + e_i b_i^1$ and $x^1 = a_i^1 + e_i b_i^0$, where $a_i^0$ and $b_i^j$ are of degree $j \pmod{2}$ and both do not contain $e_i$. Then we get

$$\alpha(x)e_i = (x^0 - x^1)e_i = e_i(a_i^0 - a_i^1) + b_i^1 + b_i^0$$

and

$$e_i x = e_i(x^0 + x^1) = e_i(a_i^0 + a_i^1) - b_i^1 - b_i^0,$$

which entails $b_i^0 = b_i^1 = 0$, i.e. $x \in \mathbb{R}^*$.

Since $O(E)$ is generated by reflections it is at least contained in the image of $\tilde{\Ad}$. It remains to show $\tilde{\Ad}(\Gamma(E)) \subseteq O(E)$, i.e. $|\tilde{\Ad}_x(v)| = |v|$ for $v \in E$. To prove this we consider the anti-automorphism of $\mathcal{O}(E)$ induced by

$$x = v_1 \cdots v_k \mapsto x^t = v_k \cdots v_1$$

and the anti-automorphism

$$\mathcal{O}(E) \ni x \mapsto \bar{x} = \alpha(x^t) = (\alpha(x))^t \in \mathcal{O}(E)$$

which allows to extend the quadratic form $\mathcal{E} \ni v \mapsto v \cdot \bar{v} = \langle v, v \rangle 1 = |v|^2 1 \in \mathcal{O}(E)$ to the so-called spinor norm

$$\mathcal{O}(E) \ni x \mapsto N(x) = x \cdot \bar{x} \in \mathcal{O}(E)$$

of $\mathcal{O}(E)$. Since the anti-automorphisms leave $\Gamma(E)$ invariant, we have $N(\Gamma(E)) \subseteq \Gamma(E)$. Actually $N(\Gamma(E)) \subseteq \mathbb{R}^*$, because

$$\tilde{\Ad}_{N(x)}(v) = \alpha(\alpha(x)v(x^t)x)^{-1} = x^t \alpha(x)vx^{-1} \alpha(x)^{-1} = (\alpha(x)^{-1})\alpha(x)vx^{-1}x^t = v.$$ 

Now $N|_{\Gamma(E)}$ is a homomorphism of groups, since

$$N(xy) = xy \bar{xy} = xy\alpha(y^t)\alpha(x^t) = xN(y)\alpha(x^t) = N(x)N(y),$$

as $N(\Gamma(E)) \subseteq \mathbb{R}^*$. In particular,

$$N(\alpha(x)x^{-1}) = N(\alpha(x))N(v)N(x)^{-1} = N(v)N(\alpha(x))N(x)^{-1} = N(v),$$

since $N(\alpha(x)) = \alpha(x)x^t = \alpha(N(x)) = N(x) \in \mathbb{R}^*$, and we conclude

$$|\tilde{\Ad}(v)|^2 = |\alpha(x)vx^{-1}|^2 = |v|^2,$$

i.e. $\tilde{\Ad}_x \in O(E)$.  \qed
**Definition 9** We put $\text{Pin}(E) = N^{-1}(1) \cap \Gamma(E)$ and define the spin group of the Euclidean vector space $E$ by $\text{Spin}(E) = \text{Pin}(E) \cap \mathcal{O}(E)^0$. In the case $E = \mathbb{R}^n$ with its standard inner product we write $\text{Spin}(n)$ instead of $\text{Spin}(\mathbb{R}^n)$.

**Remarks**

1. The group $\text{Spin}(E)$ is compact, in fact a Lie group as a closed subgroup of the group of invertibles of the algebra $\mathcal{O}(E)$.
2. Of course, $\text{Pin}(E)$ and $\text{Spin}(E)$ both depend on the Euclidean structure. More generally, one can also define $\text{Spin}(E,Q)$ for a real vector space $E$ and a non-degenerate quadratic form $Q$.
3. One has $\text{Pin}(E) = \{v_1 \cdots v_k \in \mathcal{O}(E) \mid v_i \in E, \langle v_i, v_i \rangle = 1, i = 1, \ldots, k\}$ and $\text{Spin}(E) = \{v_1 \cdots v_{2k} \in \mathcal{O}(E) \mid v_i \in E, \langle v_i, v_i \rangle = 1, i = 1, \ldots, 2k\}$.

**Corollary** The groups $\text{Pin}(E)$ and $\text{Spin}(E)$ fit into the following exact sequences

$$1 \to \mathbb{Z}_2 \to \text{Pin}(E) \to O(E) \to 1$$

$$1 \to \mathbb{Z}_2 \to \text{Spin}(E) \to SO(E) \to 1.$$  

In particular,  

$$1 \to \mathbb{Z}_2 \to \text{Spin}(n) \to SO(n) \to 1$$

is exact, i.e., $\text{Spin}(n)$ is a non-trivial two-sheeted covering of $SO(n)$. For $n \geq 3$ it is simply connected, i.e. the universal covering group of $SO(n)$.

**Proof:** Given $x \in \Gamma(E)$ and $\lambda = 1/\sqrt{N(x)}$ one has $\lambda x \in \text{Pin}(E)$ hence  

$$\widetilde{\text{Ad}}|_{\text{Pin}(E)} : \text{Pin}(E) \to O(E)$$

is onto and  

$$\ker \widetilde{\text{Ad}}|_{\text{Pin}(E)} = \{\lambda \in \mathbb{R}^* \mid N(\lambda) = \lambda^2 = 1\} \cong \mathbb{Z}_2.$$  

Any element of $SO(E)$ may be written as $\widetilde{\text{Ad}}_v \cdots \widetilde{\text{Ad}}_{v_{2k}}$ hence  

$$\rho = \widetilde{\text{Ad}}|_{\Gamma(E) \cap \mathcal{O}(E)^0} : \Gamma(E) \cap \mathcal{O}(E)^0 \to SO(E)$$

is onto with $\ker \rho = \mathbb{R}^*$. Now the restriction to $\text{Spin}(E)$ yields the analogous exact sequence. To prove the last assertion we only have to find a continuous path connecting $+1$ and $-1$ in $\text{Spin}(n)$. To this end we choose $e_1, e_2 \in \mathbb{R}^n$ with $e_1 \perp e_2$, $|e_i| = 1$, and  

$$c(t) = \exp(2\pi t e_1 \cdot e_2) = \cos 2\pi t + e_1 \cdot e_2 \sin 2\pi t$$

$$= (e_1 \cos \pi t + e_2 \sin \pi t) \cdot (-e_1 \cos \pi t + e_2 \sin \pi t),$$

for $0 \leq t \leq \frac{1}{2}$. Thus the covering is non-trivial and $\text{Spin}(n)$ is connected. For $n \geq 3$ it is also simply connected by the classical topological result $\pi_1(\text{SO}(n)) = \mathbb{Z}_2$, $n \geq 3$.  

If $E_\mathbb{C}$ is the complexification of $E$ with $\mathbb{C}$-linear extension $Q_\mathbb{C}$ of $Q$, then $\mathcal{O}(E_\mathbb{C}, Q_\mathbb{C})$ and $\mathcal{O}(E,Q) \otimes \mathbb{C}$ are isomorphic. We put $\alpha(x \otimes z) = \alpha(x) \otimes z$ and $(x \otimes z)^t = x^t \otimes z$ and with $\mathbb{C}$-linear extension $N$ as before we also define $\text{Pin}^\mathbb{C}(E)$ and the group $\text{Spin}^\mathbb{C}(E) \subset \mathcal{O}^0(E, Q) \otimes \mathbb{C}$. The latter is isomorphic with $\text{Spin}(E) \times S^1/\mathbb{Z}_2$ where $\mathbb{Z}_2 = \{(1,1),(-1,-1)\}$. If $E = \mathbb{R}^n$ we simply denote it by $\text{Spin}^\mathbb{C}(n)$. The group $\text{Spin}^\mathbb{C}(E)$ is also compact and fits into
the exact sequences

\[ 1 \to S^1 \to \text{Spin}^c(E) \xrightarrow{\rho_0} SO(E) \to 1, \]
\[ 1 \to \text{Spin}(E) \to \text{Spin}^c(E) \xrightarrow{\rho_1} S^1 \to 1, \]

where the left hand homomorphisms are canonical inclusions and the right hand ones are defined by \( \rho_0([(x,z)]) = \rho(x) \) and \( \rho_1([(x,z)]) = z^2 \), \((x,z) \in \text{Spin}(E) \times S^1\), respectively.

Usually, spin and spin\(^c\) structures are defined with the help of corresponding principal bundles; cf. [BH] and [Mil]. One starts with the orthonormal frame bundle \( P_{SO(m)} \) of the tangent bundle of an \( m \)-dimensional oriented Riemannian manifold \( M \) (or of an oriented Riemannian vector bundle of rank \( m \)). A spin structure for \( M \) consists of a principal bundle \( P_{\text{Spin}(m)} \) with structure group \( \text{Spin}(m) \) and a two-sheeted covering

\[ \xi : P_{\text{Spin}(m)} \to P_{SO(m)} \quad \text{with} \quad \xi(pg) = \xi(p)\rho_0(g), \quad p \in P_{\text{Spin}(m)}, \quad g \in \text{Spin}(m), \]

where \( \rho_0 : \text{Spin}(m) \to SO(m) \) is the standard covering. A spin\(^c\) structure is given by a principal bundle \( P_{\text{Spin}^c(m)} \) and a map

\[ \xi : P_{\text{Spin}^c(m)} \to P_{SO(m)} \quad \text{with} \quad \xi(pg) = \xi(p)\rho_0(g), \quad p \in P_{\text{Spin}^c(m)}, \quad g \in \text{Spin}^c(m) \]

where \( \rho_0 : \text{Spin}^c(m) \to SO(m) \) is again the standard map.

To show that this approach is equivalent with the one presented so far one has to go two ways. A spinor bundle can be obtained as an associated bundle: If \( F \) is a real or a complex vector space, which is also a \( \mathcal{O}_m \)-module or a \( \mathcal{O}_m^c \)-module with compatible inner product, representations \( \rho : \text{Spin}(m) \to SO(F) \) or \( \rho : \text{Spin}^c(m) \to U(F) \) will be induced by left-multiplication with elements of \( \text{Spin}(m) \subset \mathcal{O}_m^0 \) or \( \text{Spin}^c(m) \subset \mathcal{O}_m^0 \otimes \mathbb{C} \), respectively. Then \( S = P_{\text{Spin}(m)} \times_p F \) is a real or a complex spinor bundle, which moreover is irreducible if \( F = S_0 \) the space of spinors.

If on the other hand a spin structure is given by an irreducible complex spinor bundle \( S \) the corresponding principal bundles can be recovered as follows. First recall that \( P_{SO(m)} \) can be considered as the subset of \( \text{Hom}_c(M \times \mathbb{R}^m, TM) \) that consists of all orientation preserving isometries \( f_p : \mathbb{R}^m \to T_p M, \quad p \in M \). Then we define \( P_{\text{Spin}(m)} \) and \( P_{\text{Spin}^c(m)} \) to be proper subsets of \( \text{Hom}_c(M \times S_0, S) \). In the second case it consists of all isometries \( \phi_p : S_0 \to S_p \) that respect the decompositions \( S_0^0 \oplus S_0^1 \) and \( S_p^0 \oplus S_p^1 \) and satisfy \( \phi_p(v \cdot \phi_p^{-1}) \in T_v M \subset \mathcal{O}(T_v M_C) \) for all \( v \in \mathbb{R}^m \subset \mathcal{O}_m^c = \text{End}(S_0) \). In the first case we additionally require that these isometries respect the real or quaternionic structure. The map \( \xi : P_{\text{Spin}^c(m)} \to P_{SO(m)} \) is now defined by \( \xi(\Phi_p) = \text{Ad}(\Phi_p) \). Then one has

\[ \xi(\Phi_p g) = \text{Ad}(\Phi_p g) = \text{Ad}(\Phi_p) \circ \text{Ad}(g) = \xi(\Phi_p) \rho_0(g). \]

This action from the right is transitive, since for \( \Phi_p, \Phi'_p \in P_{\text{Spin}^c(m)} \) one has \( \Phi_p = \Phi_p' \circ (\Phi_p')^{-1} \circ \Phi_p \) and by definition \( x = (\Phi_p')^{-1} \circ \Phi_p \in \mathcal{O}_m^0 \otimes \mathbb{C} \) as well as \( N(x) = 1 \), hence \( x \in \text{Spin}^c(m) \). Here \( N(x) = 1 \) does hold, since \( x \) is unitary and since \( \bar{x} = x^* \) for \( x \in \mathcal{O}_m \otimes \mathbb{C} = \text{End}(S_0) \) as \( \alpha(v) = -v = v^* \) for \( v \in \mathbb{R}^m \). In the spin case \( \xi \) is defined likewise and obviously such an element \( x \) belongs to \( \text{Spin}(m) \).

Finally, we can come back to the topological obstructions that decide upon spin\(^c\) or spin structures. A spin\(^c\) structure can be supplied if and only if \( w_2(TM) \) is the mod 2-reduction of some integral cohomology class (or, what amounts to the same, if the
integral Stiefel-Whitney class $W_3(TM)$ vanishes). A spin structure exists if and only if $w_2(TM) = 1$. We refer to [Kar2], where the first assertion is proved explicitly and the second one implicitly — in the case of a spin structure in the notation of [Kar2] one has to replace $\mathbb{C}^*$ by $\mathbb{R}^*$, which makes $l_1$ automatically an isomorphism. These conditions can be checked combinatorially. We refer to [Gil] and [LM] for some specific computations. In particular, it is proved that $w_2(T\mathbb{C}P^n)$ and $w_2(T\mathbb{R}P^{2n+1})$ only vanish if $n$ is odd. Using deeper results of algebraic topology one can show that any compact oriented 3-manifold is spin (since according to E. Stiefel it is parallelizable) and that any compact oriented 4-manifold is spin$^c$ (according to a theorem of Whitney; cf. [HH]).

5 The geometric Dirac operators

Now we want to look more closely at some Dirac operators. First we consider the special case $M = \mathbb{R}^m$ with its standard metric and the global orthonormal frame $E_j = \frac{\partial}{\partial x_j}$, $j = 1, \ldots, m$, of $T\mathbb{R}^m$. If $V$ is an $n$-dimensional $\mathfrak{gl}_m$-module defined by an algebra homomorphism $\rho : \mathfrak{gl}_m \to \text{End}(V)$, say $\rho(e_j) = A_j$, $\rho(1) = I_V$, and $(v_i)_{1 \leq i \leq n}$ is a basis of $V$ a global frame on $E = \mathbb{R}^m \times V$ is given by $s_i(p) = (p, v_i)$, $p \in \mathbb{R}^m$, $i = 1, \ldots, n$. Let $\nabla$ denote a flat connection on $E$, i.e. $\omega \equiv 0$ with respect to the frame $s_i$, hence $\nabla f s_i = df \otimes s_i$ for $f \in C^\infty(\mathbb{R}^m)$. Then Clifford multiplication $E_j \cdot s_i$ is given by

$$(E_j \cdot s_i)(p) = (p, A_j(v_i)),$$

hence

$$D\left(\sum_{i=1}^n f_i s_i\right)(p) = \sum_{j=1}^m \sum_{i=1}^n E_j \cdot \nabla E_j (f_i s_i)(p) = \left(p, \sum_{j=1}^m \sum_{i=1}^n \frac{\partial}{\partial x_j} f_i(p) A_j(v_i)\right).$$

Let $A_j$ also denote the matrix with respect to the basis $(v_i)$. Then a (local) representation of $D$ is given by

$$Df = \sum_{j=1}^n A_j \frac{\partial}{\partial x_j} f,$$

where $f = (f_1, \ldots, f_n)^T$. In particular

$$D^2 f = \sum_{j,k=1}^m A_j A_k \frac{\partial}{\partial x_j} \frac{\partial}{\partial x_k} f = \Delta \otimes I_n f = -\sum_{j=1}^m \frac{\partial^2}{\partial x_j^2} f,$$

since

$$A_j A_k + A_k A_j = \rho(e_j \cdot e_k + e_k \cdot e_j) = \rho(-2\delta_{jk}) = -2\delta_{ik} I_V.$$ 

Thus $D$ is a square-root of $\Delta$. In cases $m = 1, 2$ we have the following classical operators.

**Examples** 11. If $m = 1$, i.e. $\mathfrak{gl}_1 = \mathbb{C}$, we choose $V = \mathbb{C} \cong \mathbb{R}^2$ with $\rho(e_1) = i$ and obtain $D = i \frac{\partial}{\partial x}$. 

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12. If \( m = 2 \), i.e. \( \mathcal{A}_2 = \mathbb{H} \), we choose \( V = \mathbb{H} \) with \( \rho(e_1) = i \), \( \rho(e_2) = j \) and get a grading \( \mathcal{A}_2 = \mathcal{A}_2^0 \oplus \mathcal{A}_2^1 \cong \mathbb{C} \oplus \mathbb{C} \) by

\[
\mathcal{A}_2^0 \ni u + ve_1 \mapsto u + iv \in \mathbb{C} \\
\mathcal{A}_2^1 \ni ve_2 \mapsto u + iv \in \mathbb{C}.
\]

Identifying \( E = \mathbb{R}^2 \times X \) and \( C \times (\mathbb{C} \oplus \mathbb{C}) \) the Dirac operator \( D = i \frac{\partial}{\partial x_1} + j \frac{\partial}{\partial x_2} \) becomes

\[
\frac{1}{2}D(f \oplus g) = -\frac{\partial}{\partial z} g \oplus \frac{\partial}{\partial \bar{z}} f, \quad f, g \in C^\infty(\mathbb{C}).
\]

If we write \( D = D^0 \oplus D^1 \) with \( D^j : C^\infty(\mathbb{R}^2 \times \mathcal{A}_2^j) \to C^\infty(\mathbb{R}^2 \times \mathcal{A}_2^{j+1} \mod 2) \) then \( \frac{1}{2}D^0 \) is just the Cauchy-Riemann operator \( \bar{\partial} = \frac{\partial}{\partial \bar{z}} \) which is studied in the theory of complex functions.

If \( M \) is a (compact) oriented Riemannian manifold there are several Dirac operators related to additional geometric structures. We cannot go into the analytic properties of these Dirac operators; cf. [Gil], [LM], or [Sdr1,2]. We only note that they are symmetric (elliptic) differential operators.

If \( M \) is of even dimension \( m = 2k \) we have a global section \( \omega \in C^\infty(\Omega^M) \) which is locally given by

\[
\omega = i^k E_1 \cdots E_m
\]

with respect to an oriented orthonormal frame \( (E_i)_{1 \leq i \leq m} \) of \( TM \). Obviously, one has \( \omega^2 = 1 \) and \( \omega \cdot X = -X \cdot \omega \) for \( X \in C^\infty(TM) \). Since \( \omega \) does not depend on the local frame we may assume \( \nabla_{E_i} E_j(p) = 0 \) at a fixed point \( p \in M \) and conclude that

\[
\nabla_{E_i} \omega(p) = i^j \sum_{j=1}^m E_1 \cdots \nabla_{E_i} E_j \cdots E_m(p) = 0,
\]

hence \( \nabla \omega = 0 \). Using \( \omega \) any Dirac bundle \( E \) on \( M \) will be graded by \( E^0 = (1 + \omega) \cdot E \) and \( E^1 = (1 - \omega) \cdot E \). For \( s \in C^\infty(\mathcal{E}) \) and \( X \in C^\infty(TM) \) we then obtain

\[
\nabla_X s = (-1)^j \nabla_X (\omega \cdot s) = (-1)^j (\nabla_X \omega) \cdot s + \omega \cdot \nabla_X s = (-1)^j \omega \cdot \nabla_X s,
\]

i.e. \( \nabla_X s = \frac{1}{2}(1 + (-1)^j \omega) \nabla_X s \in C^\infty(\mathcal{E}) \), and

\[
X \cdot s = (-1)^j X \cdot \omega \cdot s = (-1)^j \omega \cdot X \cdot s,
\]

hence \( X \cdot s = \frac{1}{2}(1 + (-1)^j \omega) \cdot X \cdot s \in C^\infty(\mathcal{E}) \). Since \( (s \cdot \omega, t) = (s, \omega \cdot t) \) for \( s, t \in C^\infty(\mathcal{E}) \), the decomposition \( E = E^0 \oplus E^1 \) is orthogonal. This gives rise to the following definition:

**Definition 10** Let \( E \) be a Dirac bundle on \( M \). An orthogonal decomposition \( E = E^0 \oplus E^1 \) is called admissible if

1. \( \mathcal{A}^j(M) \mathcal{E}^j \subset E^{i+j \mod 2} \),
2. \( \nabla_X s \in C^\infty(\mathcal{E}) \) for \( s \in C^\infty(\mathcal{E}) \), \( X \in C^\infty(TM) \).
Now given an admissible Dirac bundle \( E = E^0 \oplus E^1 \) the corresponding Dirac operator induces first order differential operators
\[
D^i : C_0^\infty(E^i) \to C_0^\infty(E^{i+1 \mod 2}), \quad i = 0, 1.
\]
If \( M \) is compact \( D \) and \( D^i \) are elliptic and extend to bounded operators on appropriate Sobolev space. These extensions are Fredholm operators, i.e. have an index
\[
\text{ind } D = \dim \ker D - \dim \text{Coker } D \in \mathbb{Z}.
\]
Of course, \( \text{ind } D = 0 \) but \( \text{ind } D^0 = \dim \ker D^0 - \dim \ker D^1 \) turns out to be an interesting geometric invariant. We already met the Kähler-Dirac operator \( D = \nabla^\omega \). Since the Dirac bundle \( (\bigwedge^* M = \bigwedge^\text{ev} M \oplus \bigwedge^\text{odd} M, \nabla) \) is admissible, we obtain \( D = D^0 \oplus D^1 \) and \( \text{ind } D^0 = \chi(M) \), the Euler characteristic of \( M \). There is a different admissible decomposition of \( \bigwedge^* M \) given by \( \omega \). If \( M \) is of dimension \( m = 4k \) the index of the corresponding Dirac operator is just the signature of \( M \); cf. [Gil], [LM], or [Sdr2]. We can now, finally, define the Spin-Dirac or Atiyah-Singer operator.

**Definition 11** Let \( M \) be a compact spin manifold of dimension \( m = 2n \) with spinor bundle \( S \), and \( \nabla^S \) the unique connection on \( S \) that is induced by the Levi-Civita connection. The Dirac operator associated with the Dirac bundle \( S \) is called the Spin-Dirac operator or Atiyah-Singer operator and will be denoted by \( D_{AS} \).

The Dirac bundle \( S = S^0 \oplus S^1 \) with decomposition induced by \( \omega \) is admissible, i.e. \( D_{AS} = D_{AS}^0 \oplus D_{AS}^1 \). The operator \( \mathcal{D} = D_{AS}^0 \) is also often called the Spin-Dirac operator. Its index \( \text{ind } \mathcal{D} = \hat{A}(M) \) is called the \( \hat{A} \)-genus of \( M \). It has topological significance which is expressed by the famous Atiyah-Singer index theorem:
\[
\text{ind } \mathcal{D} = \int_m \hat{A}(TM)
\]
where \( \hat{A}(TM) \in H^m(M, \mathbb{R}) \) is the cohomology class first introduced by F. Hirzebruch in 1954; cf. [Gil] for a detailed history of the subject matter.

We finally take a closer look on the Dirac-Laplace operator \( D^2 \) and on its relation to the curvature tensor \( \text{curv}(\nabla) \) which is defined by
\[
\text{curv}(\nabla)(X,Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}
\]
for vector fields \( X \) and \( Y \). Here the brackets denote the respective commutators. Note that the curvature tensor \( \text{curv}(\nabla)(X,Y) \) of a Riemannian connection is skew-adjoint,
\[
\langle \text{curv}(X,Y)s, t \rangle = -\langle s, \text{curv}(X,Y)t \rangle(R)
\]
and that
\[
\text{curv}(\nabla)(X,Y)(\sigma \cdot s) = \text{curv}(\nabla)(X,Y)(\sigma) \cdot s + \sigma \cdot \text{curv}(\nabla)(X,Y)s, (D)
\]
if \( \nabla \) satisfies property (2) of a Dirac triple. Recall that the second covariant derivative

\[ \nabla^2_{X,Y} : C^\infty (E) \to C^\infty (E), \]

is defined for \( X, Y \in C^\infty (TM) \) by

\[ \nabla^2_{X,Y} s = \nabla_X \nabla_Y s - \nabla_{\nabla_X Y} s, \quad s \in C^\infty (E), \]

and the curvature tensor of \( \nabla \) is given by

\[ \text{curv} (\nabla)(X,Y) = \nabla^2_{X,Y} - \nabla^2_{Y,X}. \]

Since \( \nabla^2_{X,Y} s(p) \) only depends on \( X_p \) and \( Y_p \), it makes \( \nabla^2 \) and \( \text{curv} (\nabla) \) tensors with values in \( E_p \). The Bochner-Laplace operator of the connection \( \nabla \) is defined as

\[ \nabla^* \nabla s = - \text{tr} (\nabla^2 \cdot s), \quad s \in C^\infty (E), \]

i.e. as

\[ \nabla^* \nabla s = - \sum_{j=1}^m \nabla^2_{E_j,E_j} s \]

when computed in some local orthonormal frame \( (E_j)_{1 \leq j \leq m} \). This definition does not depend on the chosen frame. Moreover, \( \nabla^* \nabla : C^\infty (E) \to C^\infty (E) \) is a second order (elliptic) differential operator.

Using the smooth section \( \mathcal{R} \in C^\infty (\text{Hom} (E,E)) \) given by

\[ \mathcal{R}(s) = \frac{1}{2} \sum_{j,k=1}^m E_j \cdot E_k \cdot \text{curv} (\nabla)(E_j,E_k)(s) \]

we obtain the following fundamental result.

**Theorem 10 (Bochner-Weitzenböck)** Let \( E \) be a Dirac bundle over \( M \) with associated Dirac operator \( D \). Then the Dirac-Laplace operator satisfies \( D^2 \)

\[ D^2 = \nabla^* \nabla + \mathcal{R}. \]

**Proof:** With the frame \( (E_j)_{1 \leq j \leq m} \) at \( p \) as above we have

\[
D^2 = \sum_{j,k=1}^m E_j \cdot \nabla E_j (E_k \cdot \nabla E_k) = \sum_{j,k=1}^m E_j \cdot E_k \cdot \nabla_{E_j} \nabla_{E_k} \\
= \sum_{j,k=1}^m E_j \cdot E_k \cdot \nabla^2_{E_j,E_k} \\
= - \sum_{j=1}^m \nabla^2_{E_j,E_j} + \sum_{1 \leq j < k \leq m} E_j \cdot E_k \cdot (\nabla^2_{E_j,E_k} - \nabla^2_{E_k,E_j}) \\
= \nabla^* \nabla + \mathcal{R}. 
\]
A simple but important application is the following vanishing theorem.

**Corollary** If \( M \) is compact and connected and \( \mathcal{R}(p) \) positive semi-definite for all \( p \in M \) and positive definite for at least one \( p \), then the differential equation \( D^2 s = 0 \) has only the trivial solution \( s = 0 \), i.e. there are no non-trivial harmonic sections.

**Proof.** For a fixed point \( p \in M \) one can choose \((E_j)_{1 \leq j \leq m}\) such that \( \nabla_{E_j} E_j(p) = 0 \), and given sections \( s, t \in C^{\infty}(E) \) there is a vector field \( X \) with

\[
\langle X, Y \rangle = \langle \nabla_Y s, t \rangle_E, \quad Y \in C^{\infty}(TM).
\]

These data help to prove that

\[
\langle \nabla^* \nabla s, t \rangle(p) = -\sum_{j=1}^{m} \langle \nabla_{E_j} \nabla_{E_j} s, t \rangle(p)
= -\sum_{j=1}^{m} (E_j \langle \nabla_{E_j} s, t \rangle - \langle \nabla_{E_j} s, \nabla_{E_j} t \rangle)(p)
= -\text{div} X(p) + \langle \nabla s, \nabla t \rangle(p).
\]

When integrated over \( M \), by Gauß’ theorem, the divergence term does not occur and we obtain

\[
0 \leq \int_{M} \langle \mathcal{R}(s), s \rangle = -\int_{M} \langle \nabla^* \nabla s, s \rangle = -\int_{M} \langle \nabla s, \nabla s \rangle \leq 0,
\]

if \( D^2 s = 0 \). Therefore, \( \nabla s = 0 \) and \( \|s\| \) is constant, since \( \nabla \) is Riemannian. Assuming \( s(p) \neq 0 \) and \( \mathcal{R}(p) \) positive definite gives \( \int_{M} \langle \mathcal{R}(s), s \rangle_E > 0 \), which cannot hold. \( \square \)

There are a lot of special cases of the Bochner-Weitzenböck formula. The Bochner-Weitzenböck formula for the Laplace operator can already be found in Weitzenböck’s monograph “Invarianztheorie” of 1923. It has been rediscovered and applied in 1946 by S. Bochner [Boc]. Here we only consider one special case and deduce a special vanishing result.

**Theorem 11** If \( M \) is a spin manifold with spinor bundle \( S \) and connection \( \nabla^S \), then the Spin-Dirac-Laplace operator \( D^2_{AS} \) and the Bochner-Laplace operator \( \nabla^S \ast \nabla^S \) are related by

\[
D^2_{AS} = \nabla^S \ast \nabla^S + \frac{1}{4} \tau.
\]

Here \( \tau \) denotes the scalar curvature of the Riemannian manifold \( M \).

**Proof.** We only have to prove that \( \mathcal{R} = \frac{1}{4} \tau \). It suffices to show that with respect to a local orthonormal frame \( \{E_1, \ldots, E_m\} \) of \( TM \) the curvature \( \text{curv} (\nabla^S) \) is given by

\[
\text{curv} (\nabla^S)(X, Y) = \frac{1}{4} \sum_{k,\ell=1}^{m} \langle R(X, Y) E_k, E_\ell \rangle E_k \cdot E_\ell, \quad X, Y \in C^{\infty}(TM |_U)(*)
\]
Now it is a straightforward computation to show that for fixed vector fields \( A \), \( I \) the right-hand side of (\ref{eq:as}) shares the same properties (R) and (D) as the left-hand-side and so does their difference \( \tau \). In particular, by (D) it commutes with the left-action of \( \mathcal{O} \) and so acts as multiplication by an element \( \gamma \) which by (R) is skew-adjoint, i.e. \( \gamma = i\eta \) with \( \eta \in C^\infty(M, \mathbb{R}) \). Actually, \( \eta \) has to vanish, since \( \tau \) also respects the real structure on \( S \), i.e. commutes with the structural map \( \theta \).

This Bochner-Weitzenböck formula for the Spin-Dirac operator is used by A. Lichnerowicz \cite{Lic} to prove the following vanishing theorem. The relation of \( D^2_{AS} \) and the scalar curvature had however already been noted by E. Schrödinger in 1932 \cite{Sch}.

**Corollary** (Lichnerowicz) Let \( M \) be a compact spin manifold with positive scalar curvature. Then there are no harmonic spinors on \( M \). If \( \dim M = 4k \), then \( \hat{A}(M) = 0 \).

**Proof:** The first assertion is immediate while the second one is a consequence of the Atiyah-Singer index theorem.

We also study the twisted Dirac operator \( D \otimes I_E \), where \( E \) is a Hermitian vector bundle over \( M \) with connection \( \nabla^E \), i.e. the Dirac operator of the Dirac bundle \((S \otimes E, \nabla^S \otimes E)\). Let \( \mathcal{R}^E : C^\infty(S \otimes E) \to C^\infty(S \otimes E) \) denote the zero order differential operator, which for sections \( \sigma \otimes s \) and the frame \( (E_i)_{1 \leq i \leq m} \) is defined by

\[
\mathcal{R}^E(\sigma \otimes s) = \frac{1}{2} \sum_{j,k=1}^{m} E_j \cdot E_k \cdot \sigma \otimes \text{curv}(\nabla^E)(E_j, E_k)s.
\]

**Theorem 12** Let \( M \) be spin and \( S \) and \( E \) as before. Then the Spin-Dirac operator \( D_{AS} \otimes I_E \) and the Bochner-Laplace operator \( \nabla^* \nabla \) of the tensor bundle \( S \otimes E \) are related by

\[
(D_{AS} \otimes I_E)^2 = \nabla^* \nabla + \frac{1}{4} \tau + \mathcal{R}^E.
\]

Here \( \tau \) is again the scalar curvature of \( M \).
Proof: For \( \sigma \in C^\infty(S) \) and \( s \in C^\infty(E) \) we have
\[
\nabla^{S \otimes E}(\sigma \otimes s) = (\nabla^S \sigma) \otimes s + \sigma(\nabla^E s).
\]
This entails
\[
\text{curv} (\nabla^{S \otimes E})(\sigma \otimes s) = \text{curv} (\nabla^S)(\sigma) \otimes s + \sigma \otimes \text{curv} (\nabla^E)(s)
\]
and
\[
\mathcal{R}(\sigma \otimes s) = \frac{1}{2} \sum_{j,k=1}^m E_j \cdot E_k \cdot \text{curv} (\nabla^{S \otimes E})(E_j, E_k)(\sigma \otimes s)
\]
\[
= \frac{1}{2} \sum_{j,k=1}^m E_j \cdot E_k \cdot (\text{curv} (\nabla^S)(E_j, E_k)(\sigma)) \otimes s
\]
\[
+ \frac{1}{2} \sum_{j,k=1}^m E_j \cdot E_k \cdot \sigma \otimes \text{curv} (\nabla^E)(E_j, E_k)(s)
\]
\[
= \frac{1}{4} \tau(\sigma \otimes s) + \mathcal{R}^E(\sigma \otimes s).
\]

Remark For the Spin\(^c\)-Dirac operator \( D_{S^c} \) there is a Bochner-Weitzenböck formula, too. If \( M \) is spin a spin\(^c\) structure is given by \( S^c = S \otimes L \) for some complex line bundle \( L \). Choosing the product connection on \( S^c \) with some Hermitian connection \( \nabla^L \) on \( L \) the square of the corresponding Spin\(^c\)-Dirac operator \( D_{S^c} \) satisfies
\[
D_{S^c}^2 = \nabla^* \nabla + \frac{1}{4} \tau + \mathcal{R}^L.
\]
Because of
\[
\mathcal{R}^L(\sigma \otimes s) = \frac{1}{2} \sum_{j,k=1}^m E_j \cdot E_k \cdot \sigma \otimes \text{curv} (\nabla^L)(E_j, E_k)(s)
\]
\[
= \frac{1}{2} \sum_{j,k=1}^m E_j \cdot E_k \cdot \sigma \otimes \Omega^L(E_j, E_k)(s)
\]
\[
= \frac{1}{2} \sum_{j,k=1}^m \Omega^L(E_j, E_k)E_j \cdot E_k \cdot \sigma \otimes s = (\Omega^L \cdot \sigma) \otimes s
\]
(with \( \Omega^L \) denoting the curvature form of \( \nabla^L \)) we obtain
\[
D_{S^c}^2 = \nabla^* \nabla + \frac{1}{4} \tau + \Omega^L.
\]
Since this computation is local, we can also apply it in the general non-spin case. Although \( S^c \) is a product \( S \otimes L \) only locally the line bundle \( L_{S^c} = \text{Hom}_{\mathcal{A}_{MC}}(S^c, S^c) \) = \( L \otimes L \) is nevertheless globally defined. Choosing a Hermitian connection, the corresponding curvature form \( \Omega \) satisfies \( \Omega = 2\Omega^L \). Thus we obtain the Weitzenböck formula
\[
D_{S^c}^2 = \nabla^* \nabla + \frac{1}{4} \tau + \frac{1}{2} \Omega.
\]
The index formula for $D^0_{S^c}$ can also be established by a local computation (cf. [Sdr2]). With $c = c_1(L_{S^c})$ one obtains

$$\text{ind } D^0_{S^c} = \int_M e^{c/2} \hat{A}(TM).$$

The non-vanishing of the $\hat{A}$-genus is the simplest obstruction for a Riemannian metric with positive scalar curvature. N. Hitchin [Hit] has introduced an invariant $\alpha(M)$, which can be defined for spin manifolds $M$ of any dimension and which coincides with $\hat{A}(M)$ if $m = 4k$. It again vanishes in case of positive scalar curvature. For simply connected manifolds $\alpha(M) = 0$ is even sufficient for such a metric to exist as S. Stolz [Sto] proved in 1989; cf. [RS] for a survey of the current state.

The Bochner-Weitzenböck formula for the Spin$^c$-Dirac operator on oriented compact 4-manifolds is the footing of the so-called Seiberg-Witten theory in which the theoretical physicists N. Seiberg and E. Witten initiated new differential topological invariants in 1994. These lead to new essential contributions for the classification of 4-manifolds [Mor].

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