Stationary Performance Analysis of Grishechkin Processor-Sharing Queues: An Integral Equation Approach

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Abstract

We compute the stationary performance metrics of a single server $M^X/G/1$ queue under a class of generalized processor-sharing scheduling policies that are proposed by Grishechkin. This class of processor-sharing policies allow service capacities to be allocated to jobs based on the amount of service they attained. In [11], Grishechkin derives an integral equation that is satisfied by the Laplace transform of the stationary performance metrics under these policies. Our main focus in this paper is to derive the solution to this integral equation. This is achieved through a series of transforms that convert the integral equation into a more tractable form, then solve it. Then we derive approximations to the density functions of the performance metrics through inverting the Laplace transforms. In the last part of the paper, we apply our results to some well-known scheduling policies that are either a special case of the Grishechkin processor-sharing policies or can be treated as the limits of a sequence Grishechkin processor-sharing policies. These examples include the egalitarian processor-sharing policy, the discriminatory processor-sharing policy, shortest residual processing time first rule, foreground-background policy and the time function scheduling policy.

Keywords: Queueing system; Laplace transform; Grishechkin processor-sharing

1 Introduction

In this paper, we consider a $M^X/G/1$ queue, that is, a single server queue with Poisson batch arrivals and general service distribution. We assume that the service distribution could follow any probability distribution with finite support. Therefore, the queuing model considered in the paper is fairly general and flexible. Our goal is to obtain an analytic characterization of its stationary performance metrics, such as the stationary system size under a class of general processor-sharing scheduling policies.
There are many reasons for studying a queueing model with a processor-sharing scheduling policy, which has been proved to be a powerful and flexible model in performance analysis. First, it has been demonstrated through extensive research from different aspects that a processor-sharing queue is an accurate mathematical model for the performance of a real life system that operates under scheduling policies such as round-robin or time-sharing, and these policies are commonly adapted in job scheduling for computer or communications networks, see, e.g. [18]. Secondly, in some recent study of fairness of the scheduling policies, processor-sharing policy often serves as a reference or a base on which other more complicated policies are developed and analyzed, see, e.g. [24, 9]. Fairness is a critical issue for many new applications, especially those in web services. These facts all contribute to the importance of the performance analysis of processor-sharing policies.

One important feature of processor-sharing is its flexibility. The basic principle of processor-sharing is to allocate resource capacity proportional to jobs that need to be processed. However, there can be many different ways to decide the proportion for each job, besides the equal proportion that corresponding to the egalitarian processor-sharing policy. Different regimes for determining the proportion can produce very different performances, which have to be considered very carefully when a scheduler is designed for any types of systems. A very natural scheme is to let the proportions be determined by the service already attained by the jobs, because this is usually a quantity that can be obtained with low overhead charged to many systems.

The success of the deployment of these policies and ideas in the design and performance analysis of the systems depends on efficient evaluation of a wide class of processor-sharing polices. Meanwhile, the theoretical development for understanding a processor-sharing queue still has plenty of room for improvements. It starts in the case of systems with memory-less assumptions on distributions (Poisson arrival or exponential service time), see, for example, Yashkov [25] for a survey on results of this nature, as well as other developments prior to 1987. For a general GI/GI/1 queue, Jean-Marie and Robert [16] study the transient behavior to obtain the rate of growth of the number of jobs in the system; Baccelli and Towsley [6] investigate stochastic ordering properties of the response time; Grishechkin [10, 11] establishes a strong approximation limit for the number of jobs in the system under heavy traffic; More recently, Gromoll et al. [12, 13] establish fluid and diffusion limits through measure-valued processes.

In his effort to obtain the heavy traffic limit of the steady state distribution of the performance metrics, Grishechkin establishes a connection between the state process of a $M^X/G/1$ queue under a very general class of processor-sharing policies and a Crump-Mode-Jagers branching process, then derives an integral equation for a wide class of performance metrics from the dynamics of the branching process. Grishechkin names this class of policies generalized processor-sharing policies. To avoid any confusion with the generalized processor sharing policy for multiclass networks, we will call this class of policies Grishechkin processor-sharing policy hereafter. Under a Grishechkin processor-sharing policy, each job in the system receives a portion of the processing capacity, and the proportion is determined by a random function whose variables are the amount of service each job has already received up to the moment under consideration. By choosing different random functions, Grishechkin processor-sharing
can cover a wide range of known scheduling policies. For example, the egalitarian processor-sharing policy is a special case of Grishechkin processor-sharing when the random function just assigns the same weight to each job in the system, and the policy of shortest residual processing time first rule can be treated as the limit of a sequence of Grishechkin processor-sharing policies with different parameters.

Due to the complicated nature of this integral equation, only the simplest case of egalitarian processor-sharing with unit demand has been solved in [11]. In this paper, our main contribution is to discover that several transforms can be applied to the integral equation so that it can be reduced to a more tractable form. To be more precise, we reduce the integral equation to a special form of combined first and second kind, and then we observe some special structures of the classical solutions to the first and second kinds of integral equation with the kernel appeared in our integral equation. These observations enable us to solve the transformed integral equation to obtain the Laplace transform of the stationary performance metrics. In the current paper, we will focus on one performance metric, the system size, that is, the number of jobs in the system. Similar approaches can be applied to other metrics that are also studied in [10, 11].

Then we devote our efforts to inverting the Laplace transform we obtained through the integral equation. Because in the course of solving the integral equation, we need to make use of the Laplace transform solution to the classical integral equation of the first kind, the final Laplace transform is essentially a two-dimensional Laplace transform. We adapt sophisticated Laplace transform inverting method to our problem, and obtain approximations to the stationary distribution of the performance metrics with accuracy guarantees.

The rest of the paper will be organized as follows. In Sec. 2 we will give a detailed description of the Grishechkin processor-sharing policies, as well as the integral equation for the stationary system size. In Sec. 3 we list some basic facts on the Crump-Mode-Jagers branching process and describe Grishechkin’s results for connecting the queueing process and the branching process. In Sec. 4 we will derive the solution for the integral equation. In Sec. 5 we will summarize our results with a final expression of the stationary system size, and derive numerical procedures for several special cases. In Sec. 6 we will conclude the paper with a summary of our results and contributions, as well as a description of some ongoing research.

2 Models and Preliminaries

Let us describe the queueing system in detail, and the Grishechkin processor-sharing scheduling policies in particular. Jobs arrive at the queueing system following a compound Poisson process with rate $\Lambda$ and batch size distribution $B$ with finite support. For each job, the amount of service required is independent and follows an identical general distribution, let us denote it as $\ell$, and there exists $M > 0$ such that $P[\ell > M] = 0$. We assume that the arrival and service processes are independent, the inter-arrival and the batch distributions for different jobs are also independent. The server serves at the rate of one, and can serve multiple jobs simultaneously. At time $T = 0$, we assume that there are $N$ jobs in the system, and they attain no service before time $T = 0$. 


Each job \( n, n = 1, 2, \ldots \), is associated with a pair of independent stochastic processes, \((A_n(\cdot), D_n(\cdot))\), indexed by the amount of service it received. These pairs are i.i.d. for all jobs and a generic member \((A(\cdot), D(\cdot))\) satisfies,

- \( A(\cdot) \) and \( D(\cdot) \) have absolutely continuous sample path almost surely;
- The random variable \( \ell = \min\{T > 0, A(T) = 0\} \) is almost surely finite;
- The integral \( \int_0^\ell (A(y))^{-1} \, dy \) converges almost surely.

Intuitively, the process \( A(\cdot) \) reflects the weight of the job, and \( D(\cdot) \) the quantity of interest. Define \( Q(t) \) as the number of jobs in the system at any time \( t \), and \( V_i(t) \) as the amount of service attained by each job \( i, i = 1, 2, \ldots, Q(t) \), up to time \( t \). Then the policy will allocate service capacity among jobs in the system such that the following relationship for the service rates of job \( i \) must satisfy,

\[
\frac{dV_i(t)}{dt} = \frac{A_i(V_i)}{\sum_{j=1}^{Q(t)} A_j(V_j)}.
\]

That is, the sharing of the service capacity is determined by the distribution of \( A \) and the amount of service attained. Define

\[
Q^D(T) = \sum_{i: T_i \leq T} D_i(V_i(T)),
\]

where \( T_i \) denotes the arrival time of job \( i \). Then the main performance metric of interest is the stationary distribution of the above random variable. Depending upon the functional form of \( D(\cdot) \), \( Q^D(T) \) corresponds to different types of performance metrics at time \( T \),

- \( D(y) = 1\{y \in [0, \ell]\} \), \( Q^D(T) \) represents the system size;
- \( D(y) = 1\{y \in [0, \min\{a, \ell\}]\} \), \( Q^D(T) \) represents number of jobs with attained service less than \( a \);

where \( 1A \) denotes the indicator function of set \( A \), and \( Q^D \) denotes its stationary distribution. While the methods developed in this paper certainly are not restricted to, for the ease of exposition, we will only present the case when \( D(y) = 1\{y \in [0, \ell]\} \), i.e. the case in which \( Q^D(T) \) represents the system size, or equivalently, the number of jobs that are in the system, so, unless otherwise noted, we will omit the superscript \( D \).

3 Crump-Mode-Jagers branching process and the derivation of the main integral equation

It is shown in [11], see Theorem 2.1 in [11], that system size process will have the same probability law as that of the population of a Crump-Mode-Jagers branching process. In this section, for the completeness of the arguments, we will introduce some basic facts
about the Crump-Mode-Jagers branching process and sketch the main arguments in [10] for the derivation of the integral equations that the stationary performance metric will satisfy.

The following theorem of Grishechkin provides a description of the stationary distribution of the system size.

**Theorem 1.** The function $\phi(t; u, v)$ satisfies the following integral equation,

$$\phi(t; u, v) = E \left[ \exp \left\{ h(t; u, v) + \Lambda \int_0^t f(\phi(t - y; u, v) - 1)C(y)dy \right\} \right]. \quad (2)$$

where

$$h(t; u, v) := u\chi(t) + vC(t).$$

Our performance analysis is reduced to solving this integral equation.

### 4 Solving the Integral Equation

It is well-known that although very common in various fields of applied mathematics and engineering, an integral equation is rather difficult to solve, unless it is in the standard first or second kind integral equation forms, see, e.g. [23]. Our integral equation (2) does not appear to fall into those categories. In the following, we will take two steps in obtaining its solution. First, a sequence of transformations are applied to (2) aimed at simplifying it to a more tractable form. As a result, we found that (2) can be reduced to the form of a combined first and second kinds of integral equation. Then, in the second step, we derive the solution of the combined first and second kinds of integral equation solution making use of the special function form of the solution. There is no systematic approach to solve a combined first and second kinds of integral equation solution, to the best of our knowledge, our solution is among very few cases where an explicit solution has been derived.

First, the assumption of the absolute continuity of the $(A(\cdot), D(\cdot))$ enables us to write the integration equation as,

$$\phi(t; u, v) = \int_{R^3 \times [0,t]} \exp[h_1(t; u, v; z_1, z_2, z_3, z_4)] \times$$

$$\exp \left\{ \int_0^t [f(\phi(t - y; u, v)) - 1]h_2(y, z_1, z_2, z_3, z_4)dy \right\} g(z_1, z_2, z_3, z_4)dz_1dz_2dz_3dz_4.$$

where $g(z_1, z_2, z_3, z_4)$ is the joint density function of $(A(\cdot), D(\cdot), \ell)$ at time $z_4$, and

$$h_1(t; u, v; z_1, z_2, z_3, z_4) = [u\chi(z_1, z_2, z_3)(t) + vC(z_1, z_2, z_3)(t)]1\{t = z_4\}, \quad (3)$$

and

$$h_2(t, z_1, z_2, z_3, z_4) = C(z_1, z_2, z_3)(t)1\{t = z_4\}. \quad (4)$$

5
In the rest of the section, when transforms are applied to the integral equations, all the actions on \(z_1, z_2, z_3\) and \(z_4\), in fact, will be identical, meanwhile our results are not restricted by any specific function form \(g(\cdot)\) takes. Therefore, for the ease of exposition, we will just use one variable \(z\) and a univariate function \(g(z)\), with finite support \([0, M]\), to represent them, and restore their original form at the end of the derivation. Therefore, the integral equation under consideration will now bear the following form,

\[
\phi(t; u, v) = \int_{\mathbb{R}} \exp[h_1(t; u, v; z)] \times 
\exp \left\{ \int_0^t [f(\phi(t - y; u, v)) - 1]h_2(y, z)dy \right\} g(z)dz.
\]

(5)

with \(h_1(t; u, v; z)\) and \(h_2(t, z)\) in place for functions defined in (3) and (4).

Define an operator \(T : C^1(\mathbb{R}_+^4) \to C^1(\mathbb{R}_+^3)\), where \(C^1(\Omega)\) denotes the family of functions on domain \(\Omega\) with continuous derivatives, as follows,

\[
T(\psi(t; u, v; z)) = \int_{\mathbb{R}} \exp[h_1(t; u, v; z)]\psi(t; u, v; z)g(z)dz.
\]

Let us denote, \(\psi(t; u, v; w)\) as the solution of the following integral equation,

\[
\psi(t; u, v; w) = \exp \left[ \int_0^t -h_2(y, w)dy \right] 
\exp \left\{ \int_0^t h_2(y, w)f(T(\psi(t - y; u, v; z))dy \right\}.
\]

(6)

Then, it can be verified through direct calculation that,

**Lemma 2.**

\(\phi(t; u, v) = T(\psi(t, u, v, z))\)

is a solution to the original integral equation (5).

Taking logarithms, the integral equation (5) can be written in the following equivalent form,

\[
\log(\psi(t; u, v; w)) + \int_0^t h_2(y, w)dy = \int_0^t h_2(y, w)f(T(\psi(t - y; u, v; z))dy.
\]

Thus, we can further reduce the problem through the following lemma.
Lemma 3. Suppose that $\Psi(t, u, v, w_1, w_2, \ldots, w_n)$ is the solution to

$$
\log(\Psi(t, u, v, w_1, w_2, \ldots, w_n)) + \int_0^t \sum_{i=1}^n h_2(y, w_i) dy
= \int_0^t \sum_{i=1}^n \left[ \int \exp \left( \sum_{k=1}^f h_1(t; u, v, z_k) \right) \right.
\prod_{k=\ell+1}^n \nu(z_k) \prod_{k=1}^n g(z_k) \Psi(t, u, v, z_1, z_2, \ldots, z_n) dz_1 dz_2 \ldots dz_n] dy,
$$

where $\nu$ is any probability measure that has support on $[M, \infty)$, then,

$$
\phi(t; v, u) = T \left( (\Psi(t, u, v, z, \ldots, z)^\frac{1}{n}) \right)
$$
is the solution to the original integral equation (5).

Proof. First, let us consider the special case of $f(x) = x^n$ for some $n$, we can rewrite the equation as,

$$
\log(\psi(t, u, v, w)) + \int_0^t h_2(y, w) dy
= \int_0^t h_2(y, w) \left[ \int \exp \left( \sum_{k=1}^n h_1(t; u, v, z_k) \right) \right.
\prod_{k=1}^n g(z_k) \prod_{i=1}^n \psi(t - y, u, v, z_i) dz_1 dz_2 \ldots dz_n] dy.
$$

For each $i = 1, 2 \ldots, n$, define,

$$
\log(\psi(t, u, v, w_i)) + \int_0^t h_2(y, w_i) dy
= \int_0^t h_2(y, w_i) \left[ \int \exp \left( \sum_{k=1}^n h_1(t; u, v, z_k) \right) \right.
\prod_{k=1}^n g(z_k) \prod_{i=1}^n \psi(t - y, u, v, z_i) dz_1 dz_2 \ldots dz_n] dy.
$$

Next, define

$$
\Psi(t, u, v, w_1, w_2, \ldots, w_n) = \prod_{i=1}^n \psi(t, u, v, w_i).
$$
Therefore, $\Psi(t, u, v, w_1, w_2, \ldots, w_n)$ satisfies,

$$
\log(\Psi(t, u, v, w_1, w_2, \ldots, w_n)) + \int_0^t \sum_{i=1}^n h_2(y, w_i) dy = \int_0^t \sum_{i=1}^n h_2(y, w_i) \int_{\mathbb{R}^n} \exp \left[ \sum_{k=1}^n h_1(t; u, v; z_k) \right] \times \prod_{k=1}^n g(z_k) \Psi(t - y, u, v, z_1, z_2, \ldots, z_n) dz_1 dz_2 \ldots dz_n dy.
$$

Define,

$$
h_4(t; u, v; z_1, z_2, \ldots, z_n) = \exp \left[ \sum_{k=1}^n h_1(t; u, v; z_k) \right] \prod_{k=1}^n g(z_k).
$$

Then the above expression can be written as,

$$
\log(\Psi(t, u, v, w_1, w_2, \ldots, w_n)) + \int_0^t \sum_{i=1}^n h_2(y, w_i) dy = \int_0^t \sum_{i=1}^n h_2(y, w_i) \int_{\mathbb{R}^n} h_4(t; u, v; z_1, z_2, \ldots, z_n) \Psi(t - y, u, v, z_1, z_2, \ldots, z_n) dz_1 dz_2 \ldots dz_n dy.
$$

In general, we have $f(x)$ in the following form $f(x) = \sum_{\ell=0}^n f_\ell x^\ell$. Notice that if we replace

$$
\exp \left[ \sum_{k=1}^n h_1(t; u, v; z_k) \right],
$$

in (8) by

$$
\exp \left[ \sum_{k=1}^\ell h_1(t; u, v; z_k) \right] \prod_{k=\ell+1}^n \nu(z_k),
$$

then the above logic applies to $x^\ell$, $\ell < n$, since $\nu$ has no mass in $[0, M]$. Overall, we can apply the same procedure for general function form $f(x)$, in which $h_4(t; u, v; z_1, z_2, \ldots, z_n)$ takes the following form,

$$
h_4(t; u, v; z_1, z_2, \ldots, z_n) = \sum_{\ell=0}^n f_\ell \exp \left[ \sum_{k=1}^\ell h_1(t; u, v; z_k) \right] \prod_{k=\ell+1}^n \nu(z_k) \prod_{k=1}^n g(z_k).
$$
Therefore, we can rewrite the equation as,

\[
\log(\psi(t, u, v, w)) + \int_0^t \sum_{i=1}^n h_2(y, w_i)dy = \int_0^t h_2(y, w) \left[ \int_{\mathbb{R}^n} \sum_{\ell=0}^n f_\ell \exp \left( \sum_{k=1}^n h_1(t; u, v; z_k) \right) \prod_{k=\ell+1}^n \nu(z_k) \prod_{k=1}^n g(z_k) \prod_{i=1}^n \psi(t - y, u, v, z_i)dz_1dz_2\ldots dz_n \right]dy,
\]

with the convention that \(\prod_{k=n+1}^n \nu(z_k) = 1\). The lemma can then be concluded in conjunction with Lemma 2. ✷

**Theorem 4.** The solution to the integral equation (7) bears the following form,

\[
\phi(t; u, v) = T \left( \exp[h_4(t, u, v, z, \ldots, z)] - n \int_0^t h_2(y, z) - \int_0^t R(t - y, z, \ldots, z)nh_2(y, z)\right) \]

where \(R(t, w_1, w_2, \ldots, w_n)\) is the inverse Laplace transform, with respect to \(p\), of

\[
\sum_{i=1}^n \hat{h}(p, w_i) \]

and \(\hat{h}(p, w_i)\) denotes the Laplace transform of \(h_2(t, w_i)\).

\[
\hat{h}_2(p, w_i) = \int_0^\infty h_2(t, w_i)e^{-pt}dt.
\]

**Proof** The equation (7) is a combined first and second kind integral equation. The techniques for solving this type of equations are of independent interest. For the one considered in the paper, the solution to its component of integral equation of the first kind has a special functional structure which enables us to solve the two components separately.

According to [22], the solution to the integral equation,

\[
\log(\Psi(t, u, v, w_1, w_2, \ldots, w_n)) = \int_{\mathbb{R}^n} h_4(t, u, v, z_1, z_2, \ldots, z_n) \Psi(t, u, v, z_1, z_2, \ldots, z_n)dz + \eta(t, u, v, w_1, w_2, \ldots, w_n)
\]

is in the following form,

\[
\log(\Psi(t, u, v, w_1, w_2, \ldots, w_n)) = h_4(t, u, v, w_1, w_2, \ldots, w_n) + \eta(t, u, v, w_1, w_2, \ldots, w_n).
\]
So it can be verified that if we can find a function
\[
\eta(t, u, v, w_1, w_2, \ldots, w_n)
\]
that satisfies,
\[
\eta(t, u, v, w_1, w_2, \ldots, w_n) + \sum_{i=1}^{n} \int_0^t h_2(y, w_i)
\]
\[
= \int_0^t \sum_{i=1}^{n} h_2(y, w_i) \eta(t - y, u, v, w_1, w_2, \ldots, w_n) dy,
\]
and plug it into (9), \(\Psi(t, u, v, w_1, w_2, \ldots, w_n)\) will be the solution to (7). Meanwhile, the integral equation (10) is a typical integral equation of the second kind, therefore, can be solved routinely by Laplace transform method. More specifically, we have, see e.g. [7],
\[
\eta(t, u, v, w_1, w_2, \ldots, w_n) = -\sum_{i=1}^{n} \int_0^t h_2(y, w_i)\]
\[
- \int_0^t R(t - y, w_1, w_2, \ldots, w_n) \sum_{i=1}^{n} h_2(y, w_i) dy
\]
In conjunction with lemmas 2 and 3, the result follows. □

5 Transform Expression of the Performance Measures and Inversion

In this section, we will first produce the final expression for the Laplace transform of our stationary performance metric, based upon results in the previous sections. Then, we will apply sophisticated Laplace inversion methods to several well-known scheduling policies that are either a special case of Grishechkin processor-sharing policy or can be treated as its limiting case. Especially, following the approach in [11], we demonstrate that some popular scheduling policies such as foreground-background, SRPT and time-function scheduling can be expressed as the limit of a sequence Grishechkin processor-sharing policies, then, by carefully selecting the parameters, we are able to obtain the expressions for the stationary performance metrics and their Laplace transform in fairly tractable forms.

Given the solution of the integral equation, we can calculate the Laplace transform of the performance metric \(Q\).

**Theorem 5.** The Laplace transform of the stationary system size is given by
\[
E \exp(-uQ) = (1 - \rho) + \int_0^\infty f'(\kappa_1(t, u, 0))\kappa_2(t, u) dt,
\]
where

\[ \kappa_1(t, u, v) = \int_{\mathbb{R}} \exp[h_1(t; u, v; z)] \times \\
\left( \exp[h_4(t, u, v, z, \ldots, z) - n \int_0^t h_2(y, z) \right. \\
- \left. \int_0^t R(t - y, z, z, \ldots, z) n h_2(y, z) \frac{1}{n} \right) g(z) dz \]

\[ \kappa_2(t, u) = \int_{\mathbb{R}} \exp[h_1(t; u, 0; z)] \times \\
\left( \exp[h_4(t, u, 0, z, \ldots, z) - n \int_0^t h_2(y, z) \right. \\
- \left. \int_0^t R(t - y, z, z, \ldots, z) n h_2(y, z) \frac{1}{n} \right) \times \\
\frac{\partial}{\partial v} |_{v=0} h_1(t, u, v, z) g(z) dz + \\
\left( \exp[h_4(t, u, 0, z, \ldots, z) - n \int_0^t h_2(y, z) \right. \\
- \left. \int_0^t R(t - y, z, z, \ldots, z) n h_2(y, z) \frac{1}{n} \right) \times \\
\frac{\partial}{\partial v} |_{v=0} \kappa_3(t, u, v, z) g(z) dz, \]

and

\[ \kappa_3(t, u, v, z) = [h_4(t, u, 0, z, \ldots, z) - n \int_0^t h_2(y, z) \right. \\
- \left. \int_0^t R(t - y, z, z, \ldots, z) n h_2(y, z) \frac{1}{n}. \]

Apparently, this Laplace transform is in a very complicated functional and integral form, it is unrealistic to seek its inversion in closed-form. Especially, the function \( R(t, w_1, w_2, \ldots, w_n) \) is essentially an inverse Laplace transform, which makes our task essentially inverting a two-dimensional Laplace transform. This encourages us to use numerical procedures for inverting Laplace transform to approximate these functions. Extensive studies on a unified approach for inverting a Laplace transform are carried out in [2]. Among the methods discussed in [2], we select the Talbot method for its concise expression and high accuracy. In the following we summarize the main idea of this method, for details, see, e.g. [1] and [2].

For any function \( f \), the Laplace transform is defined by,

\[ \hat{f}(s) = \int_0^\infty e^{-st} f(t) dt. \]
Its inversion is given by the following Bromwich inversion integral,

\[ f(t) = \frac{1}{2\pi \sqrt{-1}} \int_C f(s)e^{st}ds, \quad t > 0, \quad (12) \]

where the contour \( C \) goes from \( c - \infty \sqrt{-1} \) to \( c + \infty \sqrt{-1} \) for \( c > 0 \). A unified Laplace inversion approach is to use rational functions to approximate the exponential function in the integrand. More specifically, use,

\[ e^z \approx \sum_{k=0}^{n} \frac{w_k}{\alpha_k - z}, \]

for some carefully selected complex numbers \( w_k \) and \( \alpha_k \). Then the Residue theorem will give a finite summation in terms of the evaluation of Laplace transform for the Bromwich integral (12), more specifically,

\[ f(t) = \frac{1}{t} \sum_{k=0}^{n} w_k \hat{f} \left( \frac{\alpha_k}{t} \right), \quad t > 0 \]

Different algorithms for Laplace inversion, such as the Gaver-Stehfest algorithm, Euler algorithm and Talbot algorithm, differ at the selection of the rational functions, i.e. \( w_k \) and \( \alpha_k \). Here, we will use the Talbot algorithm. Detailed analysis of the algorithm can be found in [2][1]. For any large integer \( I > 0 \), the Talbot method uses the following expression as an inversion of the Laplace transform \( \hat{f} \).

\[ f(t, I) = 2 \cdot \frac{1}{5t} \sum_{k=0}^{I-1} \text{Re} \left( \gamma_k \hat{f} \left( \frac{\delta_k}{t} \right) \right). \]

with

\[ \delta_0 = \frac{2m}{5}, \]
\[ \delta_k = \frac{2k\pi}{5} \left[ \cot \left( \frac{k\pi}{M} \right) + \sqrt{-1} \right], \quad k > 0 \]
\[ \gamma_0 = \frac{1}{2} e^{\delta_0}, \]
\[ \gamma_k = \left\{ 1 + \sqrt{-1} \left( \frac{k\pi}{I} \right)^2 \left[ + \cot \left( \frac{k\pi}{I} \right) \right] \right\} e^{\delta_k}, \quad k > 0. \]

5.1 Accuracy of the Laplace inversion

Here, we have a brief discussion on the accuracy of the Laplace inversion so that we will have a full picture on the approach we are taking.
Definition 6. For a large integer $M' > 0$, and $\alpha > 0$, we say that the inversion $f_q$ of the Laplace transform $f$ produces $\alpha M$ significant digits, if

$$\left| \frac{f(t) - f_q}{f(t)} \right| \approx 10^{-\alpha M'}$$

Then, it is known from [1], the output of the Talbot inversion produces $0.6I$ significant digits while requiring $I$ evaluations of the Laplace transform. In the case of two-dimensional inverting Laplace transform, such as the problem we are studying, it is demonstrated in [2] that we can apply the Talbot algorithm to both, and the overall algorithm still produces $0.6I$ significant digits while requiring $I^2$ evaluations of the Laplace transform.

5.2 Egalitarian Processor-sharing Queues

Let us consider the simplest egalitarian processor-sharing queue $M^X/G/1/PS$. In this system, $C(t) = A(t) = 1(t \in [0, \ell])$. So $h_2(y, z) = 1\{z \in [0, y]\}$, and

$$\hat{h}_2(p, w) = \int_0^\infty 1\{w \in [0, t]\} e^{-pt} dt = \int_0^\infty e^{-pt} dt = e^{-pw}.$$ 

Applying the Talbot method for integer $I_1 > 0$

$$R_1(t, w_1, w_2, \ldots, w_n, I_1) = \frac{2}{5t} \sum_{k=0}^{I_1-1} \text{Re} \left( \gamma_k \sum_{i=1}^n \frac{\exp(-w_i \delta_k/t)}{1 + \sum_{i=1}^n \exp(-w_i \delta_k/t)} \right).$$

Then the density function is given as, for integer $I_2 > 0$

$$\theta_Q(s) = \frac{2}{\delta s} \sum_{k=0}^{I_2-1} \text{Re} L_k,$$ 

where

$$L_k = \gamma_k \left[ (1 - \rho) + \int_0^\infty f'((\kappa_1(t, \delta_k/t, 0))\kappa_2(t, \delta_k/t)) dt \right],$$

where

$$R_1(t, w_1, w_2, \ldots, w_n, I_1)$$

will replace

$$R(t, w_1, w_2, \ldots, w_n)$$

in the definition of $\kappa_i$, $i = 1, 2, 3$. If we let both $I_1$ and $I_2$ be the order of some large integer $I > 0$, then the above expression produces $0.6I$ significant digits.

In [11], for the case of egalitarian processor-sharing, an integral equation is developed for another performance metric, the sojourn time. More specifically, let $W(T)$ be the sojourn time for a tagged job with processing time $\ell$, then, Theorem 6.2 in [11] gives,
Theorem 7. If $\rho < 1$ then $T \to \infty$, then $W(T) \to W$, and the Laplace transform of the random variable $W$ is given by

$$E[\exp(-uW)] = K(u, \ell) \exp(-u\ell) \exp \left[ \Lambda \int_0^\ell (f(S(y, u)) - 1)dy \right],$$

where $f(\cdot) \in C^1$ is the Z-transform for the arrival batch size. Here, $S(t, u)$ satisfies the following equation,

$$S(t, u) = E \exp \left[ -u \min(t, \ell) - \Lambda \int_0^t (1 - f(S(t - y, u))) dy \right],$$

and

$$K(u, b) = \Lambda (1 - \rho) \left[ \Lambda^{-1} + \frac{\partial}{\partial v} \right] \left. \left( f(\phi_b(t, u, v)) \right) \right|_{v=0} \int_0^\infty f(\phi_b(t; u, v)) dt$$

with $\phi_b$ satisfies,

$$\phi_b(t; v, u) = E \exp \left[ -u \int_{t+b}^{t+b} 1(y \in [0, \ell]) dy - v 1\{t \in [0, \ell]\} - \Lambda \int_0^\ell 1 - f(\phi_b(t - y; v, u)) dy \right],$$

(16)

It should be clear that both $S(t, u)$ and $\phi(t, u, v)$ are in the same form of integral equation as (2), hence, similar techniques can be applied to them. More precisely, Theorem 4 can be applied to these two integral equations with functions $h_1(\cdot)$ and $h_2(\cdot)$ defined as the following. For $S(t, u)$

$$h_1(t; u, v; z_1, z_2, z_3, z_4) = [-u \min(t, \ell)(t)] 1\{t = z_4\},$$

(17)

and

$$h_2(t, z_1, z_2, z_3, z_4) = 1\{t = z_4\},$$

(18)

and for $\phi_b$,

$$h_1(t; u, v; z_1, z_2, z_3, z_4) = [-u \int_{t+b}^{t+b} 1\{y \in [0, \ell]\} dy - v 1\{t \in [0, \ell]\}] 1\{t = z_4\},$$

(19)

and

$$h_2(t, z_1, z_2, z_3, z_4) = C(z_1, z_2, z_3)(t) 1\{t = z_4\}.$$
5.3 Discriminatory Processor-sharing Queues with random class assignment

Discriminatory processor-sharing queue was first studied by Kleinrock [17]. Jobs are grouped in $C$ classes, indexed by $c = 1, 2, \ldots, C$, each class carries a fixed weight $\nu_c$. In a system with $N_c$ jobs for each class $c$ jobs, the amount of service each job $x$ receives is determined by

$$\frac{\nu_{c(x)}}{\sum_{c=1}^{C} N_c \nu_c},$$

where $c(x)$ denotes the job class $x$ belongs to. Of course, it is easy to see that when all the $\nu_c$ are equal, the policy is just the ordinary processor-sharing. For an updated survey on the analysis of discriminatory processor-sharing policy, see, e.g. [3].

Here, we consider a queue under a policy that is a variation of the discriminatory processor-sharing policy. We allow the class characterization being randomly determined at the time of arrival. More specifically, a job will belong to class $c, c = 1, 2, \ldots, C$ with probability $\nu_c \sum_{c=1}^{C} \nu_c$.

Therefore, $A(t) = \nu_c \mathbf{1}\{w \in [0, t]\}$ and $C(t) = A(t) = \mathbf{1}\{t \in [0, \ell]\}$. So $h_2(y, z) = \mathbf{1}\{z \in [0, y]\}$, and

$$\hat{h}_2(p, w) = \int_0^\infty \nu_c \mathbf{1}\{w \in [0, t]\} e^{-pt} dt = \int_w^\infty \nu_c e^{-pt} dt = \nu_c e^{-pw}.$$

Following our approach for the egalitarian processor-sharing, apply the Talbot method for integer $I_1 > 0$, we obtain,

$$R_2(t, w_1, w_2, \ldots, w_n, I_1) = \frac{2}{5t} \sum_{k=0}^{I_1-1} \text{Re} \left( \gamma_k \sum_{i=1}^{n} \frac{\mu_c \exp(-w_i \delta_k/t)}{1 + \sum_{i=1}^{n} \mu_c \exp(-w_i \delta_k/t)} \right).$$

Plug the above

$$R_2(t, w_1, w_2, \ldots, w_n, I_1)$$

as function

$$R(t, w_1, w_2, \ldots, w_n)$$

in the definition of $\kappa_i, i = 1, 2, 3$ in equation [13], for some $I_1$, we obtain a guaranteed approximation for the density function of the performance depending on the selection of $I_1$ and $I_2$. 

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\[ Q_k = -\frac{1}{N} \sum_{i=1}^{n} \gamma \left( \frac{1}{N+1}, \frac{t(N+w)}{\sigma_k} \right) - \gamma \left( \frac{1}{N+1}, \frac{t(N)}{\sigma_k} \right) \sum_{i=1}^{n} \gamma \left( \frac{1}{N+1}, \frac{t(N+w)}{\sigma_k} \right) - \gamma \left( \frac{1}{N+1}, \frac{t(N)}{\sigma_k} \right), \tag{21} \]

### 5.4 Shortest Residual Processing Time First

In [11], it is shown that another popular queue scheduling policy, the shortest residual processing time (SRPT) rule, can be treated as the limit of a sequence of Grishechkin processor-sharing policies with carefully chosen parameters. In particular, for any positive integer \( N = 1, 2, \ldots \), let \( c_N \) be a sequence of functions satisfying

\[ c_N(T_1)/c_N(T_2) \to \infty, \quad \text{as} \quad N \to \infty \quad \text{for any fixed} \quad T_2 > T_1 > 0. \]

Then define,

\[ A_N(T) = c_n(T)1 \{ T \leq \ell \}. \]

Denote \( Q_N^\phi \) as the state process of the \( N \)-th system, and \( Q^\phi \) as the state process of a system that follows the SRPT rule, then Lemma 7.1 in [11] indicates that \( Q_N^\phi \) converges to \( Q^\phi \) in distribution, as \( N \to \infty \).

To facilitate the calculation, we select

\[ c_N(y) = (\ell - y)N, \]

it is easy to see that this sequence satisfies the assumptions. Now,

\[ A_N(T) = (\ell - T)^{N+1}1 \{ T \leq \ell \}. \]

\[ R_N(u) = -\frac{1}{N}(\ell - N - (\ell - u) - N \]

\[ C_N(u) = -\frac{1}{N}(\ell - N - Ny)^{-\frac{1}{N}-1}. \]

Hence,

\[ h_2(p, w) = -\frac{1}{N}p^{-\frac{1}{N}+1} \left[ \gamma \left( \frac{1}{N+1}, \frac{\ell^N - w}{p} \right) - \gamma \left( \frac{1}{N+1}, \frac{\ell^N}{p} \right) \right]. \]

where \( \gamma(s, x) \) denotes the lower incomplete Gamma function, which is defined as

\[ \gamma(s, x) = \int_0^x t^{s-1} e^{-t} dt. \]

For any \( M_1 > 0 \), we have,

\[ R^N_3(t, w_1, \ldots, w_n, M_1) = \frac{2}{5t} \sum_{k=0}^{M_1-1} \text{Re}(Q_k), \]

where \( Q_k \) is defined in (21).

Again, plug the above

\[ R^N_3(t, w_1, \ldots, w_n, I_1) \]

as function

\[ R(t, w_1, \ldots, w_n) \]

in the definition of \( \kappa_i, i = 1, 2, 3 \) in equation (13) for properly chosen \( I_2 \), the density function can be computed to desired accuracy.
5.5 The Foreground-Background Queue

Foreground-background policy is another policy that can be analyzed using the techniques discussed in this paper, because it is a policy that allocates the service capacity according to the service attained for each individual job. To be more specific, foreground-background policy gives priority to that job that has the least amount of service attained, and if there is more than one such jobs, then the server is shared equally among these jobs. It is known that under heavy-tailed distribution, there is a strong positive correlation between large attained service and large remaining service, therefore, under this circumstance, foreground-background policy is considered as a surrogate for SRPT when the total amount of service requirement for each job cannot be determined by the scheduler.

As pointed out in [11], similar logic to the above in the case of the shortest remaining service rule applies here. So, for any positive integer \( N = 1, 2, \ldots \), let \( c_N \) be a sequence of functions satisfying \( c_N(T_1)/c_N(T_2) \to \infty \), as \( N \to \infty \) for any fixed \( T_2 > T_1 > 0 \). Then define,

\[
A_N(T) = c_N(T)1\{T \leq \ell\}.
\]

Therefore,

\[
\hat{h}_2(p, w) = (N + 1)^{\frac{-N}{N + 1}} \int_0^w y^{\frac{-N}{N + 1}} e^{-yp} dy
\]

where \( \gamma(s, x) \) denotes the lower incomplete Gamma function, which is defined as \( \gamma(s, x) = \int_0^x t^{s-1}e^{-t}dt \). For any \( I_1 > 0 \), we have,

\[
R_N^4(t, w_1, w_2, \ldots, w_n, M_1) = \frac{2}{M_1} \sum_{k=0}^{I_1-1} \text{Re}(Q_k),
\]

where

\[
Q_k = \frac{1 + \sum_{i=1}^n \gamma_k(N + 1)^{\frac{-N}{N + 1}} \frac{\delta_k}{t} \left( \frac{1}{N + 1}, \frac{w_i t}{\delta_k} \right)}{\sum_{i=1}^n \gamma_k(N + 1)^{\frac{-N}{N + 1}} \frac{\delta_k}{t} \left( \frac{1}{N + 1}, \frac{w_i t}{\delta_k} \right)}.
\]
Plug the above
\[ R_N^N(t, w_1, w_2, \ldots, w_n, I_1) \]
as function
\[ R(t, w_1, w_2, \ldots, w_n) \]
in the definition of \( \kappa_i, i = 1, 2, 3 \) in equation (13), we obtain a guaranteed approximation for the density function of the performance metric.

### 5.6 Time Function Scheduling

Time function scheduling is another scheduling policy that was first studied by Kleinrock, see, e.g. [13]. Further studies can be found in [8] and [19]. Under this policy, jobs are grouped in \( C \) classes, and a weight \( \nu_c > 0 \) is assigned to each class \( c = 1, 2, \ldots, C \). The server serves only one job at a time. For any class \( c \) job in the system, a time function is calculated in terms of its cumulative waiting time and its class weight \( \nu_c \), in the literature, such as, [18], [8] and [19], this function is basically taking the form of a linear function of the waiting time with \( \nu_c \) as the slope. The scheduling policy is to assign the job that has the highest value of its time function to be served by the server.

Again as in Sec. 5.3, we assume that a job is randomly assigned to a class, and the probability distribution is denoted by \( \mu_c \).

Following similar derivations as those for the SRPT rule in [11], we can show that the above defined time function scheduling policy can also be treated as a limit of Grishechkin processor-sharing policies. For any positive integer \( N = 1, 2, \ldots \), let \( c_N \) be a sequence of functions satisfying \( c_N(T_1)/c_N(T_2) \to \infty \), as \( N \to \infty \) for any fixed \( T_2 > T_1 > 0 \). Then define,
\[ A_N(T) = c_N(T)\mu_c 1\{T \leq \ell\}. \]

Now, denote \( Q_N^\phi \) as the state process of the \( N \)-th system following a Grishechkin processor-sharing policy with \( A \) defined as above, and \( Q^\phi \) as the state process of a system that follows the time function rule. We can demonstrate that \( Q_N^\phi \) converge to \( Q^\phi \) in distribution, as \( N \to \infty \).

Now, let us select \( c_N(y) = y^{-N} \). Following the same calculation as in foreground-background queue, we have, Hence,
\[ h_2(p, w) = \nu_c(N + 1) \frac{N}{N+1} p - \frac{N}{N+1} \gamma \left( \frac{1}{N+1}, \frac{w}{p} \right). \]

Similarly, for any \( I_1 > 0 \), we have,
\[ R_N^N(t, w_1, w_2, \ldots, w_n, M_1) = \frac{2}{5t} \sum_{k=0}^{I_1-1} \text{Re}(Q_k), \]
where
\[ Q_k = \frac{1 + \sum_{i=1}^n \mu_c \gamma_k(N + 1) \frac{N}{N+1} \frac{1}{T} - \frac{1}{N+1} \gamma \left( \frac{1}{N+1}, \frac{w_i}{t} \right)}{\sum_{i=1}^n \mu_c \gamma_k(N + 1) \frac{N}{N+1} \frac{1}{T} - \frac{1}{N+1} \gamma \left( \frac{1}{N+1}, \frac{w_i}{t} \right)}. \]
The rest of the calculation can follow the same approach as for the foreground-background policy.

6 Conclusions

In this paper, we analyze the stationary performance of a queueing system under a general class of processor sharing scheduling policies. Our main contribution is obtaining a solution to a complicated integral equation that plays a critical role in queueing analysis. The methods we derived in solving the integral equation appear to be of independent interest to many other problems in mathematics and engineering. Meanwhile, we adopted a sophisticated numerical Laplace inversion scheme, so that the relative error of the numerical inversion can be easily controlled. These results have important implications in the development of numerical computational package softwares for performance analysis and optimal control.

As we have demonstrated in the paper, because they allow service capacities to be dynamically determined by the attained service for each job, this general class of scheduling policies, Grishechkin processor-sharing policies, are very powerful and flexible mathematical models. Their performance analysis leads to deeper understanding and analysis of many popular scheduling policies. It also provides a building block for potential development of adaptive scheduling policies that can be used for different performance requirements. A typical example is to eliminate the independence assumptions on stochastic processes $A_i$ for different $i$. More precisely, let $(A_1, A_2, \ldots, A_n)$ be dependent stochastic processes indexed by an infinite-dimensional sequence of service attained for each job, then,

$$\frac{dV_i(t)}{dt} = \frac{A_i(V_1, V_2, \ldots)}{\sum_{j=1}^{Q(t)} A_j(V_1, V_2, \ldots)}$$

We aim to use some more generalized branching process to characterize this type of scheduling policy. Once this can be achieved, policies such as SRPT, FB and time function scheduling can be directly incorporated instead of resorting to limit.

The integral equation studied in the paper has a very complicated form, however, it is not extremely eccentric. Quite often, applications in mathematics, engineering and economics also produce integral equations of a similar form. In fact, some stochastic control and optimal stopping problems appear to be very closely related to integral equations of this type, see, e.g. a recent result on evaluation of finite horizon Russian option in [21]. Therefore, our other line of research is to apply some of the techniques we developed here to integral equations arising in other areas.

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