QUANTUM CLIFFORD ALGEBRAS

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Abstract. Quantum multiparameter deformation of real Clifford algebras is proposed. The corresponding irreducible representations are found.

Key words: Clifford algebras, quantum deformations

1. Introduction

As is well known there exists a direct relation between the exterior and the Grassmann algebras. Let us consider a unital algebra $C(x)$ over $C$ freely generated by the elements $x\Phi_1, \ldots, x\Phi_N$ and a two-sided ideal $J_0 \subset C(x)$ generated by $x\Phi_i x\Phi_j - x\Phi_j x\Phi_i$, with $i, j = 1, \ldots, N$. Now, the quotient algebra $M_0 = C(x)/J_0$ is freely generated by $x\Phi_1, \ldots, x\Phi_N$ subject to the commutativity relation

$$x\Phi_i x\Phi_j = x\Phi_j x\Phi_i. \tag{1}$$

Introducing a new set of generators $dx\Phi_k$, $k = 1, \ldots, N$, where $d$ is the exterior differential operator, we can extend this algebra first to an $A_0$-module and then to the exterior algebra $\Omega$, with the standard product given by

$$x\Phi_i dx\Phi_j = dx\Phi_j x\Phi_i, \tag{2}$$
$$dx\Phi_i dx\Phi_j = -dx\Phi_j dx\Phi_i. \tag{3}$$

The generators $dx\Phi_i$ define a finite dimensional subalgebra of $\Omega$ with the multiplication given by the exterior product $[3]$. This algebra is a differential realisation of the abstract Grassmann algebra generated by the set $\hat{\gamma}\Phi_i$, $i = 1, \ldots, N$ subject to the relations

$$\hat{\gamma}\Phi_i \hat{\gamma}\Phi_j + \hat{\gamma}\Phi_j \hat{\gamma}\Phi_i = 0. \tag{4}$$

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Clifford algebra is defined as the central extension of the algebra (5)
\[ \gamma \Phi_i \gamma \Phi_j + \gamma \Phi_j \gamma \Phi_i = 2g_{ij}I, \]
with \( g_{ij} = g_{ji} \).

Now, it is easy enough to apply this procedure to the Manin’s hyperplane. In this case we choose the ideal \( J \) leading to the following reordering rules
\[ x \Phi_i x \Phi_k = q_{ik} x \Phi_k x \Phi_i, \]
with \( q_{ik} \in \mathbb{C} - \{0\}, q_{ik} = q_{ji} - 1, q_{kk} = 1 \). Consequently, we obtain (6)
\[ [Bd \Phi_i \ dx \Phi_k = - q_{ik} \ dx \Phi_k \ dx \Phi_i ] \]
as basic rules for the two-form sector of the corresponding twisted exterior algebra. Similarly as in the standard case, identifying \( d \Phi_i \) with \( \hat{\Gamma} \Phi_i \), we obtain
\[ \hat{\Gamma} \Phi_i \hat{\Gamma} \Phi_k + q_{ik} \hat{\Gamma} \Phi_k \hat{\Gamma} \Phi_i = 0, \]
as a generalisation of the Grassmann algebra multiplication rules.

In the next sections of the paper we construct central extensions of the algebra (8).

2. Clifford Algebra \( C\Phi p, q \) in the Witt Basis

The canonical basis \( \{ \gamma \Phi \mu \} \) for the Clifford algebra \( C\Phi p, q \) is defined as follows
\[ \gamma \Phi \mu \gamma \Phi \nu + \gamma \Phi \nu \gamma \Phi \mu = 2g_{\mu \nu}I, \]
where
\[ g = \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix} \]
p \( \geq q \), \( \mu, \nu = 1, \ldots, (p + q) \).

For \( p + q = 2n \) there exists another standard basis, the so called Witt basis [3, 4]. As we will see later this basis is suitable for a non-commutative generalisation of the Clifford algebra. In the Witt basis metric tensor takes the form
\[ G = \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix}, \]
while the new generators are defined by
\[ \gamma \Phi a_N = \frac{1}{\sqrt{2}}(\gamma \Phi a + i \gamma \Phi a + n), \quad \gamma \Phi b_N = \frac{1}{\sqrt{2}}(\gamma \Phi b - \gamma \Phi b + n), \]
\[ \gamma \Phi a_P = \frac{1}{\sqrt{2}}(\gamma \Phi a - i \gamma \Phi a + n), \quad \gamma \Phi b_P = \frac{1}{\sqrt{2}}(\gamma \Phi b + \gamma \Phi b + n), \]
where \( a = 1, \ldots, (n-q) \), \( b = (n-q+1), \ldots, n \). Using Eq. (8) we obtain the following commutation relations
\[ \gamma \Phi a_N \gamma \Phi b_N = - \gamma \Phi b_N \gamma \Phi a_N, \]
\[ \gamma \Phi a_P \gamma \Phi b_P = - \gamma \Phi b_P \gamma \Phi a_P, \]
\[ \gamma \Phi a_N \gamma \Phi b_P = - \gamma \Phi b_P \gamma \Phi a_N + \delta \Phi a_b I, \]
\( \alpha, \beta = 1, \ldots, n \). In particular, we notice that \( \gamma \Phi_{\alpha N} \Phi_2 = \gamma \Phi_{\alpha P} \Phi_2 = 0 \).

We see that \( \gamma \Phi_{\alpha N} \) and \( \gamma \Phi_{\alpha P} \) span two totally isotropic \( n \)-dimensional subspaces in the generating sector of the Clifford algebra \( C\Phi_p, q, p + q = 2n \).

The hermitian conjugation can always be chosen as

\[
\gamma \Phi \mu \Phi^\dagger = g \Phi \mu \gamma \Phi \mu, \tag{17}
\]

so that

\[
\gamma \Phi_{\alpha N} \Phi^\dagger = \gamma \Phi_{\alpha P}. \tag{18}
\]

Therefore \( \gamma \Phi_{\alpha N} (\gamma \Phi_{\alpha P}) \) behave like fermionic annihilation (creation) operators.

For \( p + q = 2n + 1 \), it is necessary to add an extra generator \( \gamma \Phi 0, \gamma \Phi \Phi^\dagger = \gamma \Phi 0 \) satisfying

\[
\gamma \Phi \Phi 2 = I, \quad \{ \gamma \Phi 0, \gamma \Phi_{\alpha N} \} = \{ \gamma \Phi 0, \gamma \Phi_{\alpha P} \} = 0. \tag{19}
\]

For \( p + q = 2n \), real forms \( C\Phi_p, q \) can be reconstructed via

\[
A = \sum \Phi_n - q_{\alpha}(z_\alpha \gamma \Phi_{\alpha N} + \bar{z}_\alpha \gamma \Phi_{\alpha P}) + \sum \Phi_{n_{\beta} = n - q - 1}(a_{\beta} \gamma \Phi_{\beta N} + b_{\beta} \gamma \Phi_{\beta P}), \tag{20}
\]

where \( z_\alpha \in C, a_\beta, b_\beta \in R \), while for \( p + q = 2n + 1 \) we have

\[
A' = A + a_0 \gamma \Phi 0, \quad a_0 \in R. \tag{21}
\]

3. Quantum Deformation of Clifford Algebras \( C\Phi_p, q \)

According to the standard procedure we try to deform the Clifford algebras in the Witt basis via the following Ansatz

\[
\Gamma \Phi_{\alpha N} \Gamma \Phi_{\beta N} = -q \Phi N_{\alpha \beta} \Gamma \Phi_{\beta N} \Gamma \Phi_{\alpha N}, \quad \Gamma \Phi_{\alpha P} \Gamma \Phi_{\beta P} = -q \Phi P_{\alpha \beta} \Gamma \Phi_{\beta P} \Gamma \Phi_{\alpha N}, \tag{22}
\]

\[
\Delta \Phi_{\alpha N} = \xi \Phi_{\alpha N} \Gamma \Phi_{\alpha N}, \quad \Delta \Phi_{\beta P} = \xi \Phi_{\beta P} \Gamma \Phi_{\beta P} \Delta, \tag{23}
\]

where \( \Gamma \Phi_{\alpha N} \Phi^\dagger = \Gamma \Phi_{\alpha P}, \Delta \Phi^\dagger = \Delta \).

The parameters \( q \Phi N_{\alpha \beta}, q \Phi P_{\alpha \beta}, q \Phi_{\alpha N}, \xi \Phi_{\alpha P} \) should satisfy a number of conditions following from the consistency with associativity and the hermitian conjugation rules. Solving these constraints, we obtain (for \( q \Phi N(P)_{\alpha \alpha} \neq -1 \))

\[
\xi \Phi_{\alpha N} = q \Phi - 1_{\alpha \alpha}, \quad \xi \Phi_{\alpha P} = q_{\alpha \alpha}, \quad q_{\alpha \beta} = q_{\beta \alpha}, \tag{25}
\]

\[
q \Phi N_{\alpha \beta} = q_{\alpha \beta}/q_{\alpha \beta}, \quad q \Phi P_{\alpha \beta} = q_{\alpha \beta}/q_{\alpha \beta}, \quad q_{\alpha \beta} q_{\beta \alpha} = q_{\alpha \alpha} q_{\beta \beta}. \tag{26}
\]

Now, demanding the extension be central, we are led to

\[
q_{\alpha \alpha} = 1, \tag{27}
\]

so that

\[
|q_{\alpha \beta}| = 1, \quad q \Phi N_{\alpha \beta} = q_{\beta \alpha}, \quad q \Phi P_{\alpha \beta} = q_{\beta \alpha}. \tag{28}
\]

and \( \Delta \) can be chosen as \(+I\) (notice that \( \gamma_N \gamma_P \) is normal, so positive definite).

\(^1\) The case \( q \Phi N(P)_{\alpha \alpha} = -1 \) leads to deformations of the symplectic or Crumeyrolle Clifford algebra.
Resulting is the following deformation of the Clifford algebra, obtained as a central extension of the $q$-deformed Grassmann algebra:

\[
\Gamma \Phi_\alpha_N \Gamma \Phi_\beta_N = -q_{\beta\alpha} \Gamma \Phi_\beta_N \Gamma \Phi_\alpha_N,
\]
\[
\Gamma \Phi_\alpha_P \Gamma \Phi_\beta_P = -q_{\beta\alpha} \Gamma \Phi_\beta_P \Gamma \Phi_\alpha_P,
\]
\[
\Gamma \Phi_\alpha_N \Gamma \Phi_\beta_P = -q_{\alpha\beta} \Gamma \Phi_\beta_P \Gamma \Phi_\alpha_N + 2 \delta_{\Phi\alpha\beta} I,
\]
with \( \Gamma \Phi_\alpha_N \Phi^\dagger = \Gamma \Phi_\alpha_P \) and \( \alpha, \beta = 1, \ldots, n \), where

\[
|q_{\alpha\beta}| = 1, \quad \overline{q}_{\alpha\beta} = q_{\beta\alpha}.
\]

The odd case \((p + q = 2n + 1)\) can be treated by extending the above algebra by

\[
\Gamma \Phi_0 \Phi^\dagger = \Gamma \Phi_0, \quad \Gamma \Phi_0 \Phi^2 = I
\]

together with the Ansatz

\[
\Gamma \Phi_0 \Gamma \Phi_{N(P)} = -q \Phi_{N(P)} \Gamma \Phi_0, \quad \Gamma \Phi_0 \Gamma \Phi_{N(P)} = -q \Phi_{P} \Gamma \Phi_0 \Gamma \Phi_N.
\]

If we additionally demand the existence of the classical limit, we will obtain

\[
q \Phi_{N(P)}(\alpha) = 1,
\]

i.e.

\[
\Gamma \Phi_0 \Gamma \Phi_{N(P)} = -\Gamma \Phi_{N(P)} \Gamma \Phi_0.
\]

4. Representations

Here we construct a Fock space of representations of the deformed Clifford algebra. We consider even and odd cases separately and we conclude this section with the example of the deformed Dirac matrices in four dimensions.

4.1. The Even Case \((p + q = 2n)\)

We define the vacuum state

\[
\Gamma \Phi_\alpha_N |0\rangle = 0, \quad \langle 0 | 0 \rangle = 1.
\]

The basis of the Fock space can be defined via

\[
|\sigma_1, \ldots, \sigma_n\rangle = \left( \frac{\Gamma \Phi_{n,P}}{\sqrt{2}} \right) \Phi_{\sigma_n} \cdots \left( \frac{\Gamma \Phi_{1,P}}{\sqrt{2}} \right) \Phi_{\sigma_1} |0\rangle,
\]

where \( \sigma_\alpha = 0, 1 \).

Consequently

\[
\Gamma \Phi_\alpha_N |\sigma_1, \ldots, \sigma_n\rangle = \sqrt{2} \delta_{\sigma_\alpha} \prod \Phi_{n_\beta=1+n} (-q_{\alpha\beta}) \Phi_{\sigma_\beta} |0\rangle, \quad \Gamma \Phi_\alpha_P |\sigma_1, \ldots, \sigma_n\rangle = \sqrt{2} \delta_{\sigma_\alpha} \prod \Phi_{n_\beta=1+n} (-q_{\beta\alpha}) \Phi_{\sigma_\beta} |0\rangle, \quad \Gamma \Phi_\alpha_P |\sigma_1, \ldots, \sigma_n\rangle.
\]
4.2. The Odd Case \((p + q = 2n + 1)\)

When \(p + q = 2n + 1\), the representations of \(C\Phi_p, q\) can be easily derived from the representations of even \(C\Phi_p, q\) by the following procedure. First, we replace the even case vacuum \(|0\rangle\) by \(|0\pm\rangle\), defined by Eqs. (37) and

\[
\Gamma\Phi_0|0\pm\rangle = \pm|0\pm\rangle. \quad (41)
\]

The Fock space is generated from the vacuum by the actions of the raising operators \(\Gamma\Phi^\alpha_{\pm}\), precisely as in the even case. The action of \(\Gamma\Phi^\alpha_{\pm}\) on a standard state \(|\sigma_1, \ldots, \sigma_n\pm\rangle\) is given by Eqs. (39–40) and

\[
\Gamma\Phi_0|\sigma_1, \ldots, \sigma_n\pm\rangle = \pm(-\Phi\Sigma\sigma_\alpha|\sigma_1, \ldots, \sigma_n\pm\rangle. \quad (42)
\]

4.3. Example

Let us consider the simplest non-trivial example: \(n = 2, p + q = 4, q_{12} = \kappa, |\kappa| = 1\). Explicitly, we have the following algebra

\[
\Gamma\Phi_1N\Gamma\Phi_2N = -\kappa\Phi - 1\Gamma\Phi_2N\Gamma\Phi_1N, \quad \Gamma\Phi_1P\Gamma\Phi_2P = -\kappa\Phi - 1\Gamma\Phi_2P\Gamma\Phi_1P,
\]

\[
\Gamma\Phi_1N\Gamma\Phi_2P = -\kappa\Gamma\Phi_2P\Gamma\Phi_1N, \quad \Gamma\Phi_2N\Gamma\Phi_1P = -\kappa\Gamma\Phi_1P\Gamma\Phi_2N,
\]

\[
\{\Gamma\Phi_1N, \Gamma\Phi_1P\} = \{\Gamma\Phi_2N, \Gamma\Phi_2P\} = 2I.
\]

and

\[
(\Gamma\Phi_1N)\Phi_2 = (\Gamma\Phi_2N)\Phi_2 = (\Gamma\Phi_1P)\Phi_2 = (\Gamma\Phi_2P)\Phi_2 = 0.
\]

According to the Eqs. (39–40) we obtain the following deformation of the Dirac matrices

\[
\Gamma\Phi_1P = \sqrt{2} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \Gamma\Phi_1N = \sqrt{2} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\kappa \Phi - 1 \end{pmatrix},
\]

\[
\Gamma\Phi_2P = \sqrt{2} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad \Gamma\Phi_2N = \sqrt{2} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.
\]

5. Related Topics

Let us conclude with pointing out a number of interesting problems. One may investigate relations of the above twisted Clifford algebras to:

- multiparameter \(q\)-deformations of Spin and pseudo-orthogonal groups (corresponding to quantum groups);
- \(q\)-spinors;
- exotic statistics (anyons, etc.).

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