A FREE BOUNDARY PROBLEM FOR A CLASS OF PARABOLIC TYPE CHEMOTAXIS MODEL

HUA CHEN, WENBIN LV AND SHAOHUA WU

School of Mathematics and Statistics, Wuhan University
Computational Science Hubei Key Laboratory, Wuhan University
Wuhan, 430072, China

(Communicated by Tong Yang)

Abstract. In this paper, we study a free boundary problem for a class of parabolic type chemotaxis model in high dimensional symmetry domain $\Omega$. By using the contraction mapping principle and operator semigroup approach, we establish the existence of the solution for such kind of chemotaxis system in the domain $\Omega$ with free boundary condition.

1. Introduction. Understanding of the partially oriented movement of cells in response to chemical signals, chemotaxis, is of great significance in various contexts. This importance partly stems from the fact that when cells combined with the ability to produce the respective signal substance themselves, chemotaxis mechanisms are among the most primitive forms of intercellular communication. Typical examples include aggregation processes such as slime mold formation in Dictyostelium Discoideum discovered by K. B. Raper [17]. Then many mathematicians have made efforts to develop various models and investigated the problems from mathematical point of view. For a broad overview over various types of chemotaxis processes, we refer the reader to the survey [1, 3, 4, 11, 12, 23, 25, 26] and the references therein.

As we all know, in a standard setting for many partial differential equations, we usually assume that the process being described occurs in a fixed domain of the space. But in the real world, the following phenomenon may happen. At the initial state, a kind of amoeba occupied some areas. When foods become rare, they begin to secrete chemical substances on their own. Since the biological time scale is much slower than the chemical one, the chemical substances are full filled with whole domains and create a chemical gradient attracting the cells. In turn, the areas of amoeba may change according to the chemical gradient from time to time. In other words, a part of whose boundary is unknown in advance and that portion of the boundary is called a free boundary. In addition to the standard boundary conditions that are needed in order to solve the PDEs, an additional condition must be imposed at the free boundary. One then seeks to determine both the free boundary and the solution of the differential equations. The theory of free boundaries has seen great progress in the last thirty years; for the state of the field we refer to [9].

2010 Mathematics Subject Classification. Primary: 35A01; Secondary: 47D03, 35K57, 35R35.
Key words and phrases. Chemotaxis, free boundary problem, existence of solution.
This work is supported by the Fundamental Research Funds for the Central Universities (Grant No. 2014201020202) and National Natural Science Foundation of China (Grant No. 11131005).
In this paper, we consider the following high dimensional free boundary problem of a chemotaxis model. Such kind of models can be found in [2, 5, 24], and we can give more explanations in the appendix below.

\[
\begin{align*}
&u_t = \nabla(\nabla u - u\nabla v), \quad \text{in } \Omega \times (0, T), \\
&u = 0, \quad \text{in } \Omega \times (0, T) \setminus \Omega_t \times (0, T), \\
&-\nabla u \cdot \frac{\nabla \Phi}{|\nabla \Phi|} = k(x,t)u, \quad \text{on } \partial \Omega_t \times (0, T), \\
&\frac{u}{\partial t} = \nabla u \cdot \nabla \Phi - u \nabla v \cdot \nabla \Phi, \quad \text{on } \partial \Omega_t \times (0, T), \\
&v_t = \Delta v + u - v, \quad \text{in } \Omega \times (0, T), \\
&\frac{\partial v}{\partial n} = 0, \quad \text{on } \partial \Omega \times (0, T), \\
&v(x,0) = v_0(x), \quad \text{in } \Omega,
\end{align*}
\]

(1.1)

where we assume

- \( \Omega \subset \mathbb{R}^n \) is a bounded open set with smooth boundary \( \partial \Omega \) and \( n \) is unit outer normal vector of \( \partial \Omega \). Besides, \( \Omega \) is assumed as a symmetry domain, i.e. if \( x \in \Omega \) then \( -x \in \Omega \) as well.
- \( k(x,t) = k(|x|, t) \) is radial symmetric and satisfying the Lipschitz condition on \( |x| \), namely there exists a constant \( L > 0 \), such that
  \[ |k(|x_1|, t) - k(|x_2|, t)| \leq L |x_1| - |x_2|, \quad x_1, x_2 \in \Omega_t. \]  
  (1.2)

Also, \( k(x,t) \) is bounded on \( t \in [0, +\infty) \). In other words, there exists a constant \( c > 0 \) which depends on \( x \), such that
  \[ |k(x,t)| \leq c, \quad t \in [0, +\infty); \]  
  (1.3)

- \( u = u(x,t) \) is an unknown function of \( (x,t) \in \Omega_t \times (0, T) \) and it stands for the density of cellular slime molds. In other words, the density \( u(x,t) \) occupying the domain \( \Omega_t \), an open subset of \( \Omega \), in time \( t \) and \( u(x,t) = 0 \) in the outside of \( \Omega_t \);
- \( v = v(x,t) \) is an unknown function of \( (x,t) \in \Omega \times (0, T) \) and it stands for the concentration of chemical substances secreted by the slime molds;
- \( \Gamma_t : \Phi(x,t) = 0 \) is an unknown free boundary.

For general smooth domain \( \Omega \), the system (1.1) is based on the well-known chemotaxis model with fixed boundary

\[
\begin{align*}
&u_t = \nabla(\nabla u - u\nabla v), \quad \text{in } \Omega \times (0, T), \\
v_t = \Delta v + u - v, \quad \text{in } \Omega \times (0, T), \\
\frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = 0, \quad \text{on } \partial \Omega \times (0, T), \\
u(x,0) = u_0(x), \quad \text{in } \Omega, \\
v(x,0) = v_0(x), \quad \text{in } \Omega,
\end{align*}
\]

(1.4)

which was introduced by E.F. Keller and L.A. Segel [14]. The problem (1.4) is intensively studied by many authors (see for instance [6, 7, 8, 15, 18, 21, 22]). The initial functions \( u_0 \in C^0(\bar{\Omega}) \) and \( v_0 \in C^1(\bar{\Omega}) \) are assumed to be nonnegative. Within this framework, classical results state that

- if \( n = 1 \) then all solutions of (1.4) are global in time and bounded (see [16]);
- if \( n = 2 \) then
– in the case $\int_\Omega u_0(x)dx < 4\pi$, the solution will be global and bounded (see [10, 20]), whereas
– for any $m > 4\pi$ satisfying $m \in \{4k\pi \mid k \in \mathbb{N}\}$ there exist initial data $(u_0, v_0)$ with $m = \int_\Omega u_0(x)dx$ such that the corresponding solution of (1.4) blows up either in finite or infinite time, provided $\Omega$ is simply connected (see [13, 19]);

• if $n \geq 3$
  – given any $q > \frac{n}{2}$ and $p > n$ one can find a bound for $u_0$ in $L^q(\Omega)$ and for $\nabla v_0$ in $L^p(\Omega)$ guaranteeing that $(u, v)$ is global in time and bounded;
  – on the other hand, if $\Omega$ is a ball then for arbitrarily small mass $m > 0$ there exist $u_0$ and $v_0$ having $\int_\Omega u_0(x)dx = m$ such that $(u, v)$ blows up either in finite or infinite time (see [21]).

In one dimensional case, if $k$ is a positive constant H. Chen and S.H. Wu [2, 5, 24] studied the similar free boundary value problem (1.1) and established the existence and uniqueness of the solution for the system (1.1). However, to the best of our knowledge, high dimensional case for the free boundary value problem (1.1) will be more important. In view of the biological relevance of the particular case $n = 3$, we find it worthwhile to clarify these questions. In the present paper, we consider the system (1.1) on a high dimensional symmetry domain $\Omega$. In addition, the condition that $k$ is a positive constant in [2, 5, 24] seems too strict, it is also worthwhile to consider the system with non-constant coefficient $k$.

This paper is arranged as follows. In section 2, we rewrite the model and present the main result of the paper. In section 3, we use the operator semigroup approach to establish some estimates which are essential in the proof of the main result. In section 4, we shall give the proof of the main result.

2. Main result.

2.1. Rewrite the model. In this subsection, we rewrite the model with the form of radial symmetry. We assume that the environment and solution are radially symmetric. Without loss of generality, we assume that $\Omega = B_1(0)$ which represents a unite ball centered in origin and that $u$ and $v$ are radially symmetric with respect to $x = 0$. The free boundary can be written as $r = |x| = h(t)$.

Let $(\tilde{u}, \tilde{v})$ denote the corresponding radial solution in $B_1(0) \times (0, T)$. In order to avoid confusion, we may write

$$\tilde{u}(r, t) = u(x, t), \quad \tilde{v}(r, t) = v(x, t),$$

with $r = |x| = (x_1^2 + \cdots + x_n^2)^{\frac{1}{2}} \in (0, 1)$ and

$$\frac{\partial r}{\partial x_i} = \frac{\partial}{\partial x_i} \left((x_1^2 + \cdots + x_n^2)^{\frac{1}{2}}\right) = \frac{x_i}{r}, \quad i = 1, 2, \cdots, n.$$

A simple calculation shows that

$$u_{x_i}(x, t) = \tilde{u}_{r}(r, t) \frac{x_i}{r}, \quad v_{x_i}(x, t) = \tilde{v}_{r}(r, t) \frac{x_i}{r},$$

$$u_{x_i, r}(x, t) = \tilde{u}_{rr}(r, t) \frac{x_i^2}{r^2} + \tilde{u}_{r}(r, t) \left(\frac{1}{r} - \frac{x_i^2}{r^3}\right),$$

and

$$v_{x_i, r}(x, t) = \tilde{v}_{rr}(r, t) \frac{x_i^2}{r^2} + \tilde{v}_{r}(r, t) \left(\frac{1}{r} - \frac{x_i^2}{r^3}\right).$$

• Reformulation of the boundary.
If $\Gamma_t : \Phi(r, t) = 0 \iff r - h(t) = 0$, then the condition of the free boundary convert into
\[
\tilde{k}(h(t), t)\bar{u}(h(t), t) + \overline{u_r}(h(t), t) = 0, \quad \text{for } \tilde{k}(r, t) = k(x, t),
\]
and
\[
-\tilde{u}(h(t), t)h_t(t) = \tilde{u}_{rr}(h(t), t) - \tilde{u}(h(t), t)\overline{v}_r(h(t), t).
\]
Actually, substituting (2.1) and
\[
\Phi_{x_i}(x, t) = \frac{\partial \Phi}{\partial r} \cdot \frac{\partial r}{\partial x_i} = \frac{x_i}{r}, \quad i = 1, 2, \ldots, n,
\]
into the third and forth equations of (1.1), we can easily get
\[
\tilde{k}(r, t)\bar{u}(r, t) = -\nabla u \cdot \frac{\nabla \Phi}{|\nabla \Phi|} = -\sum_{i=1}^{n} \bar{u}_{x_i}(r, t) \frac{x_i^2}{r^2} = -\bar{u}_r(r, t),
\]
and
\[
-\bar{u}(r, t)h_t(t) = u \frac{\partial \Phi}{\partial t} = \nabla u \cdot \nabla \Phi - u \nabla v \cdot \nabla \Phi = \bar{u}_r(r, t) - \bar{u}(r, t)\overline{v}_r(r, t), \quad (2.4)
\]
on $\Gamma_t$.
If $u > 0$ on $\Gamma_t$, then (2.4) is equivalent to
\[
h_t(t) = \tilde{k}(h(t), t) + \overline{v}_r(h(t), t).
\]
• Reformulation of the equation.
Substituting (2.1), (2.2) and (2.3) into the first and sixth equations of (1.1), we can easily obtain
\[
\tilde{u}_t(r, t) = u_t(x, t) = \Delta u - \nabla u \nabla v - u \Delta v
\]
\[
= \tilde{u}_{rr}(r, t) + \frac{n-1}{r} \tilde{u}_r(r, t) - \tilde{u}_r(r, t)\overline{v}_r(r, t)
\]
\[
- \tilde{u}(r, t) \left( \overline{v}_{rr}(r, t) + \frac{n-1}{r} \overline{v}_r(r, t) \right)
\]
\[
= r^{1-n} \left( r^{n-1} \tilde{u}_r(r, t) \right)' - r^{1-n} \left( r^{n-1} \tilde{u}(r, t)\overline{v}_r(r, t) \right)' ,
\]
or \( r^{n-1} \tilde{u}_t(r, t) = \left( r^{n-1} \tilde{u}_r(r, t) - r^{n-1} \tilde{u}(r, t)\overline{v}_r(r, t) \right)' , \quad (2.5) \)
and
\[
\tilde{v}_t(r, t) = v_t(x, t) = \Delta v - v + u
\]
\[
= \tilde{v}_{rr}(r, t) + \frac{n-1}{r} \tilde{v}_r(r, t) - \tilde{v}(r, t) + \overline{u}(r, t)
\]
\[
= r^{1-n} \left( r^{n-1} \overline{v}_r(r, t) \right)' - \tilde{v}(r, t) + \overline{u}(r, t),
\]
or \( r^{n-1} \tilde{v}_t(r, t) = \left( r^{n-1} \overline{v}_r(r, t) \right)' - r^{n-1} \tilde{v}(r, t) + r^{n-1} \overline{u}(r, t) . \quad (2.6) \)
Therefore, the model we are concerned here becomes
\[
\begin{aligned}
\bar{u}_t(r,t) &= \bar{u}_{rr}(r,t) + \frac{n-1}{r} \bar{u}_r(r,t) \\
-\bar{u}_r(r,t)\bar{v}_r(r,t) &= -\bar{u}(r,t) \left( \bar{v}_{rr}(r,t) + \frac{n-1}{r} \bar{v}_r(r,t) \right), \quad 0 < r < h(t), \quad 0 < t < T, \\
\bar{u}(r,t) &= 0, \quad h(t) < r < 1, \quad 0 < t < T, \\
\bar{u}_r(0,t) &= 0, \quad 0 < t < T, \\
\bar{u}_r(r,t) + \bar{k}(r,t)\bar{u}(r,t) &= 0, \quad \text{on} \ r = h(t), \quad 0 < t < T, \\
\bar{h}_t(t) &= \bar{k}(r,t) + \bar{u}_r(r,t), \quad \text{on} \ r = h(t), \quad 0 < t < T, \\
\bar{v}(r,0) &= \bar{v}_0(r), \quad 0 < r < b, \\
\bar{v}_t(r,t) &= \bar{v}_{rr}(r,t) + \frac{n-1}{r} \bar{v}_r(r,t) \\
-\bar{v}(r,t) + \bar{u}(r,t) &= 0, \quad 0 < r < 1, \quad 0 < t < T, \\
\bar{v}_r(0,t) &= 0, \quad 0 < t < T, \\
\bar{v}_r(1,t) &= 0, \quad 0 < t < T, \\
\bar{v}(r,0) &= \bar{v}_0(r), \quad 0 < r < 1,
\end{aligned}
\]
which corresponds to the equation with normal coordinate
\[
\begin{aligned}
\begin{array}{ll}
u_t = \nabla(\nabla u - u\nabla v), & \text{in} \quad B_{h(t)}(0) \times (0,T), \\
u = 0, & \text{in} \quad (B_1(0) \setminus B_{h(t)}(0)) \times (0,T), \\
\frac{\partial u}{\partial n}(x,t) + k(x,t)u(x,t) = 0, & \text{on} \quad \partial B_{h(t)}(0) \times (0,T), \\
h_t(t) = k(x,t) + \frac{\partial v}{\partial n}(x,t), & \text{on} \quad \partial B_{h(t)}(0) \times (0,T), \\
u(x,0) = u_0(x), & \text{in} \quad B_0(0), \\
v_t = \Delta v + u - v, & \text{in} \quad B_1(0) \times (0,T), \\
\frac{\partial v}{\partial n}(x,t) = 0, & \text{on} \quad \partial B_1(0) \times (0,T), \\
v(x,0) = v_0(x), & \text{in} \quad B_1(0).
\end{array}
\end{aligned}
\]

2.2. **Main result.** Now we introduce the following space notations, which will be used in the main result here. For \( t_0 > 0 \), we define
\[
X_u^t(t_0) =
\]
\[
\begin{aligned}
C \left( [0,t_0], H^{t,p}(B_{h(t)}(0)) \right. & \cap \left\{ \frac{\partial u}{\partial n}(x,t) + k(x,t)u(x,t) = 0, x \in \partial B_{h(t)}(0) \right\} \\
X_v^{t+\frac{2}{p}}(t_0) = C \left( [0,t_0], H^{1+\frac{2}{p},p}(B_1(0)) \right. & \cap \left\{ \frac{\partial v}{\partial n}(x,t) = 0, x \in \partial B_1(0) \right\} \\
Y_u(t_0) = C^{1} \left( [0,t_0], L^p(B_{h(t)}(0)) \right) & \cap \left( X_u^{t+\frac{2}{p}}(t_0) \cap Y_u(t_0) \right) \\
Y_v(t_0) = C^{1} \left( [0,t_0], L^p(B_1(0)) \right) & \cap \left( X_v^{t+\frac{2}{p}}(t_0) \cap Y_v(t_0) \right).
\end{aligned}
\]

Our main result is:

**Theorem 2.1.** Assume \( k(x,t) \) is radial symmetric on \( x \) and satisfying the conditions (1.2) and (1.3). If \( u_0(x) \in H^{2,p}(B_0(0)), \quad u_0 > 0, \) and \( v_0(x) \in H^{2,p}(B_1(0)) \), are radial symmetric on \( x \), where \( 0 < b < 1 \) and \( b \) is a constant. Then there exist \( t_0 > 0 \) small enough, a radial symmetric pair \( (u(x,t), v(x,t)) \in (X_u^{t}(t_0) \cap Y_u(t_0)) \times (X_v^{t+\frac{2}{p}}(t_0) \cap Y_v(t_0)) \),
and a curve $\Gamma_t : |x| = h(t)$, which are the solutions of (2.8) for each $1 < \ell < 2$, $\frac{n}{p} < 1 - \frac{\ell}{2}$.

3. Some crucial estimates. In this section, we establish some crucial estimates, which will play the key roles in proving the local existence of radial symmetric solution of system (2.8) in high dimensional case.

3.1. A basic property of the solution.

Lemma 3.1. If $u_0(x) > 0$, $\int_0^{h(0)} r^{n-1} \tilde{u}_0(r)dr = M$, $\int_0^1 r^{n-1} \tilde{v}_0(r)dr = N$ and $(u, v)$ is the radial symmetrical solution of the system (1.1), then we have $u > 0$, $\int_0^{h(t)} r^{n-1} \tilde{u}(r,t)dr = M$ and $\int_0^1 r^{n-1} \tilde{v}(r,t)dr = e^{-t}(N - M) + M$.

Proof. Since $\tilde{u}_0(r) > 0$, by standard maximal principle of the parabolic equation, it follows that $\tilde{u} > 0$. Integrating the equation of (2.6) over $(0, h(t))$, we have

$$\int_0^{h(t)} r^{n-1} \tilde{u}_t(r,t)dr = \int_0^{h(t)} (r^{n-1} \tilde{u}_r(r,t) - r^{n-1} \tilde{u}(r,t)\tilde{v}_r(r,t))dr$$

$$= h^{n-1}(t) [\tilde{u}_r(h(t),t) - \tilde{u}(h(t),t)\tilde{v}_r(h(t),t)]$$

$$= - h^{n-1}(t)\tilde{u}(h(t),t)h_t(t),$$

where the fifth equation of (2.7) is used. Thus one has

$$\frac{d}{dt} \int_0^{h(t)} r^{n-1} \tilde{u}(r,t)dr = \int_0^{h(t)} r^{n-1} \tilde{u}_t(r,t)dr + h_t(t)h^{n-1}(t)\tilde{u}(h(t),t) = 0,$$

which implies that

$$\int_0^{h(t)} r^{n-1} \tilde{u}(r,t)dr = \int_0^{h(0)} r^{n-1} \tilde{u}(r,0)dr = M$$

as required.

Integrating the equation of (2.6) over $(0, 1)$, we have

$$\frac{d}{dt} \int_0^1 r^{n-1} \tilde{v}(r,t)dr$$

$$= \int_0^1 (r^{n-1} \tilde{v}_r(r,t))dr - \int_0^1 r^{n-1} \tilde{v}(r,t)dr + \int_0^{h(t)} r^{n-1} \tilde{u}(r,t)dr$$

$$= - \int_0^1 r^{n-1} \tilde{v}(r,t)dr + \int_0^{h(0)} r^{n-1} \tilde{u}(r,0)dr,$$

where the eighth and ninth equations of (2.7) is used. Through simple calculation, we can get

$$\int_0^1 r^{n-1} \tilde{v}(r,t)dr = e^{-t} \int_0^1 r^{n-1} \tilde{v}(r,0)dr + (1 - e^{-t}) \int_0^{h(0)} r^{n-1} \tilde{u}(r,0)dr$$

$$= e^{-t}(N - M) + M.$$

The proof of the lemma 3.1 is completed. \qed
3.2. Some basic properties of the system. Firstly, we define

\[ B = \left\{ h \in C[0, t_0] \mid h(0) = b, \left| \frac{h(t_1) - h(t_2)}{t_1 - t_2} \right| \leq M_0, t_1, t_2 \in (0, t_0), t_1 \neq t_2 \right\}, \]

where \( M_0 < \frac{1}{2} \left( \min(h, 1-h) \right) \) is a constant.

In this section, we shall establish some estimates which are important in the proof of the main result. For any fixed \( h(t) \in B \), we consider the following problems

\[
\begin{align*}
\begin{cases}
    u_t = \nabla(\nabla u - u \nabla v), & \text{in } B_{h(t)}(0) \times (0, t_0), \\
    v_t = \Delta v \leq v, & \text{in } B_{h(t)}(0) \times (0, t_0), \\
    \partial B_{h(t)}(0), & \text{on } \partial B_{h(t)}(0) \times (0, t_0), \\
    u(x, 0) = u_0(x), & \text{in } B_0(0),
\end{cases}
\end{align*}
\]

and

\[
\begin{align*}
\begin{cases}
    v_t = \Delta v + u - v, & \text{in } B_0(0) \times (0, t_0), \\
    \partial B_0(0), & \text{on } \partial B_0(0) \times (0, t_0), \\
    v(x, 0) = v_0(x), & \text{in } B_0(0).
\end{cases}
\end{align*}
\]

Lemma 3.2. If \( h(t) \in B \), \( u_0(x) \in H^2 \cap \{ \frac{\partial u}{\partial n}(x, 0) + k(x, 0)u(x, 0) = 0, x \in \partial B_0(0) \} \). Then for \( t_0 > 0 \) small enough and \( v \in X_{\ell}^2(t_0) \cap Y_{\ell}(t_0) \), the system (3.1) admits a unique solution \( u \in X_{\ell}^2(t_0) \cap Y_{\ell}(t_0) \) and for each \( 1 < \ell < 2, \frac{n}{p} < 1 - \frac{\ell}{2} \), we have

\[ \sup_{0 \leq t \leq t_0} \| u \|_{H^{\ell,p}(B_{h(t)}(0))} \leq C\| u(\cdot, 0) \|_{H^{2,p}(B_0(0))} + C_0^{\frac{1}{2} - \frac{\ell}{2}} \sup_{0 \leq t \leq t_0} \| u \|_{H^{\ell,p}(B_{h(t)}(0))} \cdot \sup_{0 \leq t \leq t_0} \| v \|_{H^{1+\frac{\ell}{2},p}(B_0(0))}, \]

where \( C \) depends on \( M_0 \) but is independent of \( t_0 \) and \( h(t) \in B \).

Proof. By scalar coordinate transform, the system (3.1) is equivalent to the system

\[
\begin{align*}
\begin{cases}
    \bar{u}_t(r, t) = \bar{u}_{rr}(r, t) + \frac{n-1}{r} \bar{u}_r(r, t) - \bar{u}_r(r, t)\bar{v}_r(r, t), & 0 < r < h(t), 0 < t < t_0, \\
    \bar{u}(r, t) = 0, & h(t) < r < 1, 0 < t < t_0, \\
    \bar{u}_r(0, t) = 0, & 0 < t < t_0, \\
    \bar{u}_r(h(t), t) + \bar{k}(h(t), t)\bar{u}(h(t), t) = 0, & 0 < t < t_0, \\
    \bar{u}(r, 0) = \bar{u}_0(r), & 0 < r < b.
\end{cases}
\end{align*}
\]

Now we introduce a new transformation

\[
\begin{align*}
\begin{cases}
    \xi = \frac{r}{h(t)}, \\
    \tau = t,
\end{cases}
\end{align*}
\]

and set \( \bar{v}(\xi, \tau) = \bar{u}(h(t)\xi, t) \), \( \bar{v}(\xi, \tau) = \bar{v}(h(t)\xi, t) \) for \( 0 < \xi < 1, \) and \( \bar{v}(\xi, \tau) = 0 \) for \( \xi > 1 \) to straighten the free boundary \( \Gamma_t : r = h(t) \). A series of detailed calculation assert

\[
\begin{align*}
\begin{cases}
    \bar{v}_r = \bar{u}_t + \bar{u}_r\xi t', & \bar{v}_r = \bar{v}_t + \bar{v}_r\xi t', \\
    \bar{v}_{\xi} = \bar{v}_r h(\tau), & \bar{v}_{\xi} = \bar{v}_r h(\tau), \\
    \bar{v}_{\xi\xi} = \bar{u}_{rr} h^2(\tau), & \bar{v}_{\xi\xi} = \bar{v}_{rr} h^2(\tau).
\end{cases}
\end{align*}
\]
Thus $\Pi(\xi, \tau)$ satisfies
\[
\pi_{\tau} = \frac{1}{h^2(\tau)} \pi_{\xi\xi} + \frac{n-1}{h^2(\tau)} \pi_{\xi} - \frac{1}{h^2(\tau)} \pi_{\xi} \pi_{\xi} - \left( \frac{1}{h^2(\tau)} \pi_{\xi\xi} + \frac{n-1}{h^2(\tau)} \pi_{\xi} \right) \pi + \frac{\xi h'(\tau)}{h(\tau)} \pi_{\xi},
\]
0 < $\xi$ < 1, 0 < $\tau$ < $t_0$.

with initial and boundary data
\[
\begin{cases}
\pi_{\xi}(0, \tau) = 0, & 0 < \tau < t_0, \\
\pi_{\xi}(1, \tau) + \tilde{k}(h(\tau), \tau) h(\tau) \pi(1, \tau) = 0, & 0 < \tau < t_0, \\
\pi(\xi, 0) = \tilde{u}_0(h\xi), & 0 < \xi < 1.
\end{cases}
\]

This is a fixed boundary value problem. In order to solve this problem, we take a transform $|y| = \xi$, and let $u(y, \tau) = \pi(\xi, \tau)$ and $v = \pi(\xi, \tau)$, we have
\[
\frac{\partial u}{\partial n} = \nabla u \cdot n = \sum_{i=1}^{n} u_{y_i} \xi = \pi_{\xi},
\]
\[
\nabla u \nabla v = \sum_{i=1}^{n} u_{y_i} v_{y_i} = \sum_{i=1}^{n} \frac{y_i^2}{\xi^2} \pi_{\xi} \pi_{\xi} = \pi_{\xi} \pi_{\xi},
\]
and
\[
y_\nabla u = \sum_{i=1}^{n} y_i u_{y_i} = \sum_{i=1}^{n} y_i \frac{y_i}{\xi} \pi_{\xi} = \xi \pi_{\xi}.
\]

A simple calculation implies that
\[
u_{\tau} = \frac{1}{h^2(\tau)} \Delta u - \frac{1}{h^2(\tau)} \nabla u \nabla v - \frac{1}{h^2(\tau)} u \Delta v + \frac{h'(\tau)}{h(\tau)} y \nabla u, \quad y \in B_1(0), \quad 0 < \tau < t_0,
\]
with initial and boundary data
\[
\begin{cases}
\frac{\partial u}{\partial n}(y, \tau) + \tilde{k}(h(\tau), \tau) h(\tau) u(y, \tau) = 0, & y \in \partial B_1(0), \quad 0 < \tau < t_0, \\
u(y, 0) = u_0(\theta y), & y \in B_1(0).
\end{cases}
\]

Freezing the coefficient, the equation can be written as
\[
u_{\tau} = \frac{1}{h^2(0)} \Delta u - \frac{1}{h^2(\tau)} \nabla u \nabla v - \frac{1}{h^2(\tau)} u \Delta v + \frac{h'(\tau)}{h(\tau)} y \nabla u + \left( \frac{1}{h^2(\tau)} - \frac{1}{h^2(0)} \right) \Delta u, \quad y \in B_1(0), \quad 0 < \tau < t_0.
\]

(3.4)

For simplicity, we introduce some notations here. Let
\[L(0) = h^{-2}(0) \Delta.\]

We know $L(0)$ is a generator of holomorphic semigroup on $L^p(B_1(0))$ and
\[\Phi : D(L(0)) \to L^p(B_1(0)),\]
\[u \to \Phi(u) = -\frac{1}{h^2(\tau)} \nabla u \nabla v - \frac{1}{h^2(\tau)} u \Delta v + \frac{h'(\tau)}{h(\tau)} y \nabla u + \left( \frac{1}{h^2(\tau)} - \frac{1}{h^2(0)} \right) \Delta u,
\]
is Lipschitz, where $D(L(0)) = H_{\alpha}^2(B_1(0)) \cap \{ \frac{\partial u}{\partial n}(y, \tau) + \tilde{k}(h(\tau), \tau) h(\tau) u(y, \tau) = 0, y \in \partial B_1(0) \}$. So the problem (3.4) has a unique solution
\[u \in C([0, t_0], D(L(0))) \cap C^1([0, t_0], L^p(B_1(0)))) ,
\]
for each $t_0$. 
Let $t_1 > 0$ and $T_{t_1}(\tau)$ represent the operator semigroup on $L^p(\partial B_1(0))$ which is generated by
\[ L(t_1) = h^{-2}(t_1)\Delta, \]
where
\[ D(L(t_1)) = H^{t,p}(\partial B_1(0)) \cap \left\{ \frac{\partial u}{\partial n}(y, \tau) + \bar{k}(h(\tau), \tau)h(\tau)u(y, \tau) = 0, y \in \partial B_1(0) \right\}. \]

We know that $T_{t_1}(\tau)$ is a holomorphic semigroup on $L^p(\partial B_1(0))$ and
\[
\begin{align*}
    u(y, \tau) & = T_{t_1}(\tau)u(y, 0) - \int_0^\tau T_{t_1}(\tau - s) \frac{1}{h^2(s)} \nabla u \nablauds \\
    & \quad - \int_0^\tau T_{t_1}(\tau - s) \frac{1}{h^2(s)} u \Deltauds + \int_0^\tau T_{t_1}(\tau - s) \frac{h'(s)}{h(s)} y \nablauds \\
    & \quad + \int_0^\tau T_{t_1}(\tau - s) \left( \frac{1}{h^2(s)} - \frac{1}{h^2(t_1)} \right) \Deltauds. \tag{3.5}
\end{align*}
\]

There exists $t' \in [0, t_0]$ such that
\[
    \|u(\cdot, t')\|_{H^{t,p}} = \sup_{0 \leq \tau \leq t_0} \|u(\cdot, \tau)\|_{H^{t,p}},
\]

since $\|u(\cdot, \tau)\|_{H^{t,p}} \in C[0, t_0]$.

If $t' > 0$, then we take $t_1 = t'$ in (3.5) to get
\[
\begin{align*}
    u(y, t') & = T_{t'}(t')u(y, 0) - \int_0^{t'} T_{t'}(t' - s) \frac{1}{h^2(s)} \nabla u \nablauds \\
    & \quad - \int_0^{t'} T_{t'}(t' - s) \frac{1}{h^2(s)} u \Deltauds + \int_0^{t'} T_{t'}(t' - s) \frac{h'(s)}{h(s)} y \nablauds \\
    & \quad + \int_0^{t'} T_{t'}(t' - s) \left( \frac{1}{h^2(s)} - \frac{1}{h^2(t')} \right) \Deltauds.
\end{align*}
\]

In particular, we have
\[
\begin{align*}
    u(y, t') & = T_{t'}(t')u(y, 0) - \int_0^{t'} T_{t'}(t' - s) \frac{1}{h^2(s)} \nabla u \nablauds \\
    & \quad - \int_0^{t'} T_{t'}(t' - s) \frac{1}{h^2(s)} u \Deltauds + \int_0^{t'} T_{t'}(t' - s) \frac{h'(s)}{h(s)} y \nablauds \\
    & \quad + \int_0^{t'} T_{t'}(t' - s) \left( \frac{1}{h^2(s)} - \frac{1}{h^2(t')} \right) \Deltauds.
\end{align*}
\]

So we can obtain
\[
\begin{align*}
    \|u(\cdot, t')\|_{H^{t,p}} & \leq \|T_{t'}(t')u(\cdot, 0)\|_{H^{t,p}} + \int_0^{t'} \left\| T_{t'}(t' - s) \frac{1}{h^2(s)} \nabla u \nablauds \right\|_{H^{t,p}} ds \\
    & \quad + \int_0^{t'} \left\| T_{t'}(t' - s) \frac{1}{h^2(s)} u \Deltauds \right\|_{H^{t,p}} ds \\
    & \quad + \int_0^{t'} \left\| T_{t'}(t' - s) \frac{h'(s)}{h(s)} y \nablauds \right\|_{H^{t,p}} ds \\
    & \quad + \int_0^{t'} \left\| T_{t'}(t' - s) \left( \frac{1}{h^2(s)} - \frac{1}{h^2(t')} \right) \Deltauds \right\|_{H^{t,p}} ds \tag{3.6}
\end{align*}
\]

\[
= (S_1) + (S_2) + (S_3) + (S_4) + (S_5).
\]
If \( h(t) \in B \), then
\[
0 < b - M_0t_0 \leq h(t) \leq b + M_0t_0 < 1, \quad 0 \leq t \leq t_0.
\] (3.7)
The operator semigroup feature of \( T_t(t) \) necessary for the proof of this lemma is well known. There is a constant \( C > 0 \) which is dependent on \( M_0 \) but independent of \( t \) such that
\[
\| T_t(t)f \|_{H^ {\lambda+\varepsilon, p}} \leq C t^{-\frac{\lambda}{2} - \frac{1}{2}(\frac{1}{q} - \frac{1}{p})} \| f \|_{H^ {\lambda, q}}, \quad f \in H^ {\lambda, q},
\]
(3.8)
where \( 1 \leq q \leq p \leq +\infty \).

Using the facts (3.7) and (3.8), the terms of the right-hand side of (3.6) are estimated from above by
\[
(S_1) \leq C \| u(\cdot, 0) \|_{H^{\ell, p}} \leq C \| u(\cdot, 0) \|_{H^{2, p}};
\]
\[
(S_2) \leq C \int_0^{t'} (t' - s)^{-\frac{1}{2}} \| \nabla u \|_{L^p} \| \Delta u \|_{L^p} \leq C t^{1-\frac{1}{q}} \sup_{0 \leq s \leq t'} \| \nabla u \|_{L^p} \| \Delta u \|_{L^p}
\]
\[
\leq C t^{1-\frac{1}{q}} \sup_{0 \leq s \leq t'} \| u \|_{H^{\ell, p}} \| \Delta u \|_{L^p} \leq C t^{1-\frac{1}{q}} \sup_{0 \leq s \leq t'} \| u \|_{H^{1+\frac{1}{2}, p}};
\]
\[
(S_3) \leq C \int_0^{t'} (t' - s)^{-\frac{1}{2}} \| u \|_{H^{\ell, p}} \| \Delta u \|_{L^p} ds \leq C t^{1-\frac{1}{q}} \sup_{0 \leq s \leq t'} \| u \|_{H^{\ell, p}};\]
and
\[
(S_5) \leq \int_0^{t'} \left\| T_{t'}(t' - s) \frac{[h(t') + h(s)] [h(t') - h(s)]}{[h(s)]^2 [h(t')]^2} \right\|_{H^{\ell, p}} ds
\]
\[
\leq C \int_0^{t'} \| T_{t'}(t' - s) \|_{H^{\ell, p}} ds \leq C \int_0^{t'} (t' - s)^{\frac{1}{2}} \| \Delta u \|_{H^{\ell, p}} ds \leq C t^{1-\frac{1}{q}} \sup_{0 \leq s \leq t'} \| u \|_{H^{\ell, p}} .
\]
Thus, for \( t' \) small enough, it holds that
\[
\sup_{0 \leq s \leq t'} \| u \|_{H^{\ell, p}} \leq C \| u(\cdot, 0) \|_{H^{2, p}} + C t^{1-\frac{1}{q}} \sup_{0 \leq s \leq t'} \| u \|_{H^{\ell, p}} \sup_{0 \leq s \leq t'} \| \Delta u \|_{L^p} \sup_{0 \leq s \leq t'} \| \Delta u \|_{H^{1+\frac{1}{2}, p}}.
\]
In case of \( t' = 0 \), then for each \( 0 \leq t_2 \leq t_0 \), we have
\[
\sup_{0 \leq s \leq t_2} \| u \|_{H^{\ell, p}} \leq C \| u(\cdot, 0) \|_{H^{2, p}} + C t^{1-\frac{1}{q}} \sup_{0 \leq s \leq t_2} \| u \|_{H^{\ell, p}} \sup_{0 \leq s \leq t_2} \| \Delta u \|_{H^{1+\frac{1}{2}, p}}.
\]
From the two estimates above, we can easily deduce the conclusion. \( \square \)
Lemma 3.3. If $h(t) \in B$, $v_0(x) \in H^{2,p}(B_1(0)) \cap \{ \frac{\partial v}{\partial n}(x,0) = 0, x \in \partial B_1(0) \}$ and $u \in X^2_u(t_0) \cap Y_u(t_0)$. Then for $t_0$ small enough, the system (3.2) admits a unique solution $v \in X^2_v(t_0) \cap Y_v(t_0)$, and

$$
\sup_{0 \leq t \leq t_0} \| v \|_{H^{1+\frac{\ell}{2},p}(B_1(0))} \leq C \| v(\cdot,0) \|_{H^{2,p}(B_1(0))} + C t_0^{\frac{1}{2} - \frac{\ell}{4}} \sup_{0 \leq t \leq t_0} \| u \|_{L^p(B_h(t_0))}, \tag{3.9}
$$

where $1 \leq \ell \leq 2$ and $\frac{n}{p} < 1 - \frac{\ell}{2}$.

Proof. It is obvious that the problem (3.2) has a unique solution $v \in X^2_v(t_0) \cap Y_v(t_0)$. Moreover, we have

$$
v(x,t) = T(t)v(x,0) - \int_0^t T(t-s)v(x,s)ds + \int_0^t T(t-s)u(x,s)ds,
$$

where $T(t) = e^{t\Delta}$ and $D(\Delta) = H^{2,p}(B_1(0)) \cap \{ \frac{\partial v}{\partial n}(x,0) = 0, x \in \partial B_1(0) \}$. So we can obtain

$$
\| v(\cdot,t) \|_{H^{1+\frac{\ell}{2},p}} \leq \| T(t)v(\cdot,0) \|_{H^{1+\frac{\ell}{2},p}} + \int_0^t \| T(t-s)v(\cdot) \|_{H^{1+\frac{\ell}{2},p}} ds
$$

$$
+ \int_0^t \| T(t-s)u(\cdot) \|_{H^{1+\frac{\ell}{2},p}} ds
$$

$$
= (T_1) + (T_2) + (T_3).
$$

The terms of the right-hand side are estimated from above by

$$(T_1) \leq C \| v(\cdot,0) \|_{H^{2,p}}$$

$$(T_2) \leq C \int_0^t (t-s)^{-\frac{1}{2} - \frac{\ell}{4}} \| v \|_{L^p} ds \leq C t_0^{\frac{1}{2} - \frac{\ell}{4}} \sup_{0 \leq s \leq t_0} \| v \|_{H^{1+\frac{\ell}{2},p}}$$

and

$$(T_3) \leq C \int_0^t (t-s)^{-\frac{1}{2} - \frac{\ell}{4}} \| u \|_{L^p} ds \leq C t_0^{\frac{1}{2} - \frac{\ell}{4}} \sup_{0 \leq s \leq t_0} \| u \|_{L^p}.$$

Thus, for $t_0$ small enough, it holds that

$$
\sup_{0 \leq s \leq t_0} \| v \|_{H^{1+\frac{\ell}{2},p}} \leq C \| v(\cdot,0) \|_{H^{2,p}} + C t_0^{\frac{1}{2} - \frac{\ell}{4}} \sup_{0 \leq s \leq t_0} \| u \|_{L^p}.
$$

The result of Lemma 3.3 is proved. \qed

Lemma 3.4. If $h(t) \in B$, $u(x,0) \in H^{2,p}(B_0(0)) \cap \{ \frac{\partial u}{\partial n}(x,0) + k(x,0)u(x,0) = 0, x \in \partial B_0(0) \}$ and $v(x,0) \in H^{2,p}(B_1(0)) \cap \{ \frac{\partial v}{\partial n}(x,0) = 0, x \in \partial B_1(0) \}$. Then for $t_0$ small enough, the system (3.1) and (3.2) admit a unique solution

$$
u \in X^2_u(t_0) \cap Y_u(t_0), \quad v \in X^2_v(t_0) \cap Y_v(t_0).
$$

Moreover, we have

$$
\sup_{0 \leq t \leq t_0} \| u \|_{H^{\ell,p}(B_{h(t)}(0))} \leq 2C \| u(\cdot,0) \|_{H^{2,p}(B_h(0))},
$$

where $1 \leq \ell < 2$ and $\frac{n}{p} < 1 - \frac{\ell}{2}$.
Proof. Firstly, we consider \( w \in X^p \cap Y_u \) and \( w(x,0) = u(x,0) \). Let \( v = v(w) \) denote the corresponding solution of the equation

\[
\begin{cases}
v_t = \Delta v + w - v, & \text{in } B_1(0) \times (0,t_0), \\
\frac{\partial v}{\partial v}(x,t) = 0, & \text{on } \partial B_1(0) \times (0,t_0), \\
v(x,0) = v_0(x), & \text{in } B_1(0).
\end{cases}
\] (3.10)

Then the difference \( u = \Delta v + w - v \) by (3.3) and (3.9). Thus, for the solution \( v \) of the equation (3.10), we define \( u = u(v(w)) \) to be the corresponding solution of

\[
\begin{cases}
u_t = \nabla(\Delta u - u\nabla v), & \text{in } B_{h(t)}(0) \times (0,t_0), \\
u = 0, & \text{in } (B_1(0) \setminus B_{h(t)}(0)) \times (0,t_0), \\
\frac{\partial u}{\partial \nu}(x,t) + k(x,t)u(x,t) = 0, & \text{on } \partial B_{h(t)}(0) \times (0,t_0), \\
u(x,0) = u_0(x), & \text{in } B_h(0).
\end{cases}
\] (3.11)

Secondly, for the solution \( v \) of the equation (3.10), we define \( u = u(v(w)) \) to be the corresponding solution of

\[
\begin{cases}
u_t = \nabla(\Delta u - u\nabla v), & \text{in } B_{h(t)}(0) \times (0,t_0), \\
u = 0, & \text{in } (B_1(0) \setminus B_{h(t)}(0)) \times (0,t_0), \\
\frac{\partial u}{\partial \nu}(x,t) + k(x,t)u(x,t) = 0, & \text{on } \partial B_{h(t)}(0) \times (0,t_0), \\
u(x,0) = u_0(x), & \text{in } B_h(0).
\end{cases}
\]

Actually we have introduced a mapping \( Fw = u(v(w)) \). Now we take \( \mathcal{M} = 2C\|u(\cdot,0)\|_{H^{2,p}(B_h(0))} \) and a ball \( B_{\mathcal{M}} = \{ w \in X^p \cap Y_u \mid w(x,0) = u(x,0), \sup_{0 \leq s \leq t_0} \|w(\cdot,t)\|_{H^{1,p}(B_{h(t)}(0))} \leq \mathcal{M} \} \), where the constant \( C \) is given by (3.3). Thus the local existence of the solution will be established via contraction mapping principle. In fact, Lemma 3.2 shows that \( F \) maps \( B_{\mathcal{M}} \) into itself. Actually, we have

\[
\sup_{0 \leq s \leq t_0} \|Fw\|_{H^{1,p}(B_{h(t)}(0))} \leq C\|u(\cdot,0)\|_{H^{2,p}(B_h(0))} + Ct_0^{1-\frac{4}{p}} \sup_{0 \leq s \leq t_0} \|Fw\|_{H^{1,p}(B_{h(t)}(0))} \cdot \sup_{0 \leq s \leq t_0} \|v\|_{H^{1+\frac{4}{p},p}(B_1(0))}
\]

\[
\leq C\|u(\cdot,0)\|_{H^{2,p}(B_h(0))} + Ct_0^{1-\frac{4}{p}} \left( C\|v(\cdot,0)\|_{H^{2,p}(B_1(0))} + Ct_0^{1-\frac{4}{p}} \sup_{0 \leq s \leq t_0} \|w\|_{L^p(B_{h(t)}(0))} \right) \cdot \sup_{0 \leq s \leq t_0} \|Fw\|_{H^{1,p}(B_{h(t)}(0))}
\]

by (3.3) and (3.9). Thus, for \( t_0 \) small enough, it holds that

\[
\sup_{0 \leq s \leq t_0} \|Fw\|_{H^{1,p}(B_{h(t)}(0))} \leq \mathcal{M},
\]

i.e. \( F \) maps \( B_{\mathcal{M}} \) into \( B_{\mathcal{M}} \).

Next, we can prove that for \( t_0 \) small enough, \( F \) is a contract mapping. For \( w_1, w_2 \in B_{\mathcal{M}} \), let \( u_1, u_2 \) denote the corresponding solution of system (3.11) respectively. Then the difference \( u_1 - u_2 \) satisfies

\[
\begin{cases}
(u_1 - u_2)_t = \Delta (u_1 - u_2) - \nabla(u_1 - u_2)\nabla v_1 \\
- \nabla u_2 \nabla (v_1 - v_2) - (u_1 - u_2)\Delta v_1 \\
u_1 - u_2 = 0, & \text{in } (B_1(0) \setminus B_{h(t)}(0)) \times (0,t_0), \\
\frac{\partial (u_1 - u_2)}{\partial \nu}(x,t) + k(x,t)(u_1 - u_2)(x,t) = 0, & \text{on } \partial B_{h(t)}(0) \times (0,t_0), \\
(u_1 - u_2)(x,0) = 0, & \text{in } B_h(0).
\end{cases}
\]

Take scalar coordinate transform and set \( \tau_t(\xi,\tau) = \bar{u}_t(h(\tau)\xi,\tau), \quad \bar{v}_t(\xi,\tau) = \bar{v}_i(h(\tau)\xi,\tau), \)
Combining (3.12) and (3.13), we can easily get for

\[ C_t \]

Similar to the proof of Lemma 3.2, one can obtain that for each 1 < \( t \) < 2 and \( t_0 > 0 \) small enough

\[ \sup_{0 \leq s \leq t_0} \| Fw_1 - Fw_2 \|_{H^s(B_h(t_0))} \leq Ct_0^{\frac{1}{2}} \sup_{0 \leq s \leq t_0} \| v_1 - v_2 \|_{H^{s+p}(B_{h}(t_0))}, \]  

(3.12)

where \( C \) depends on \( M_0 \) but is independent of \( t_0 \) and \( h(t) \in B \).

On the other hand, we have

\[
\begin{align*}
(v_1 - v_2)_{t} &= \Delta (v_1 - v_2) + (w_1 - w_2) - (v_1 - v_2), \quad \text{in} \quad B_1(0) \times (0, T), \\
\frac{\partial (v_1 - v_2)}{\partial n}(x, t) &= 0, \quad \text{on} \quad \partial B_1(0) \times (0, T), \\
(v_1 - v_2)(x, 0) &= 0, \quad \text{in} \quad B_1(0).
\end{align*}
\]

Similar to the proof of Lemma 3.3, one can obtain that for \( t_0 > 0 \) small enough

\[ \sup_{0 \leq s \leq t_0} \| v_1 - v_2 \|_{H^{s+p}(B_{h}(t_0))} \leq Ct_0^{\frac{1}{2}} \sup_{0 \leq s \leq t_0} \| w_1 - w_2 \|_{L^p(B_{h}(t_0))}. \]  

(3.13)

Combining (3.12) and (3.13), we can easily get for \( t_0 \) small enough, \( F \) is a contract mapping. This completes the proof of Lemma 3.4.

4. The proof of Theorem 2.1. For each \( h(t) \in B \), by Lemma 3.4 we know that there exists a pair

\[ u \in \mathbb{X}_u(t_0) \cap Y_u(t_0), \quad v \in \mathbb{X}_v^{1 + \frac{1}{2}}(t_0) \cap Y_v(t_0), \]

which is the solution of the system (3.1) and (3.2).

Now we will use Schauder theorem to prove the result of Theorem 2.1.

Set

\[ g(t) = b + \int_0^t k(x, s) ds + \int_0^t \frac{\partial v}{\partial n}(x, s) ds, \quad x \in \partial B_{h(t)}(0). \]

Then we have

\[
\left| \frac{g(t_1) - g(t_2)}{t_1 - t_2} \right| \leq \left| \frac{1}{t_1 - t_2} \int_{t_1}^{t_2} k(x, s) \right| ds + \left| \frac{1}{t_1 - t_2} \int_{t_1}^{t_2} \frac{\partial v}{\partial n}(x, s) \right| ds
\]

\[
\leq \sup_{0 \leq t \leq t_0} \| k(\cdot, t) \|_{L^\infty} + C \sup_{0 \leq t \leq t_0} \| \nabla v(\cdot, t) \|_{L^\infty} \text{ (4.1)}
\]

\[
\leq C + C \left( \| v(\cdot, 0) \|_{H^{2,p}(B_1(0))} + \| u(\cdot, 0) \|_{H^{2,p}(B_1(0))} \right).
\]
Let $M_1$ denote the constant at the right hand of (4.1). If $t_0$ is small enough, then
\[ \frac{1}{2} \min \{ b, 1 - b \} > M_1. \]

We choose $M_0 = M_1$ in $B$, then it is clear that $B \subset C[0, t_0]$ is a compact and convex set.

Define $G : h(t) \to g(t)$, therefore $G$ maps $B$ into itself. Next we will demonstrate that $G$ is continuous. Then the Schauder theorem yields that there exist a pair $(u, v)$ and a curve $\Gamma : r = h(t)$ which are the solution of (2.7).

For $h_1(t), h_2(t) \in B$, let $(u_1, v_1), (u_2, v_2)$ represent the corresponding solutions of (2.8) respectively and $(\tilde{u}_1, \tilde{v}_1), (\tilde{u}_2, \tilde{v}_2)$ represent the corresponding solutions of (2.7) respectively, then
\begin{align*}
G(h_1) - G(h_2) &= g_1(t) - g_2(t) \\
&= \int_0^t (k(x, s) - k(y, s)) \, ds + \int_0^t \left( \frac{\partial v_1}{\partial n}(x, s) - \frac{\partial v_2}{\partial n}(y, s) \right) \, ds \\
&= (I_1) + (I_2),
\end{align*}
where $x \in \partial B_{h_1(t)}(0)$, $y \in \partial B_{h_2(t)}(0)$.

The terms on the right-hand side are estimated from above by
\begin{align*}
(I_1) &\leq L \int_0^t |h_1(s) - h_2(s)| \, ds \leq Lt \sup_{0 \leq s \leq t_0} |h_1 - h_2|; \\
(I_2) &\leq \int_0^t \left| \frac{\partial v_1}{\partial n}(x, s) - \frac{\partial v_2}{\partial n}(y, s) \right| \, ds \\
&\quad + \int_0^t \left| \frac{\partial v_1}{\partial n}(y, s) - \frac{\partial v_2}{\partial n}(y, s) \right| \, ds \\
&= (I_{21}) + (I_{22}).
\end{align*}

On the following, we will focus on the term $(I_2)$. Thus, we have
\begin{align*}
(I_{21}) &\leq C t_0 \sup_{0 \leq s \leq t_0} \| \nabla v_1(\cdot, s) \|_{C^{\frac{1}{2}, \frac{1}{2}}} \sup_{0 \leq s \leq t_0} |h_1 - h_2|^{\frac{1}{2} - \frac{1}{2}} \leq C t_0 \sup_{0 \leq s \leq t_0} |h_1 - h_2|^{\frac{1}{2} - \frac{1}{2}};
\end{align*}

On the other hand, we have
\[ \begin{cases} (v_1 - v_2)_t = \Delta (v_1 - v_2) + (u_1 - u_2) - (v_1 - v_2), & \text{in } B_1(0) \times (0, t_0), \\ \frac{\partial (v_1 - v_2)}{\partial n}(x, t) = 0, & \text{on } \partial B_1(0) \times (0, t_0), \\ (v_1 - v_2)(x, 0) = 0, & \text{in } B_1(0). \end{cases} \]

By Lemma 3.3, it holds that
\[ \left| \frac{\partial v_1}{\partial n}(y, s) - \frac{\partial v_2}{\partial n}(y, s) \right| \leq C t_0^{\frac{1}{2} - \frac{1}{2}} \sup_{0 \leq \tau \leq t_0} \| u_1 - u_2 \|_{L^p(\nu_{\tau}(0))}. \]
Let $\bar{h}(t) = \max\{h_1(t), h_2(t)\}$ and $\underline{h}(t) = \min\{h_1(t), h_2(t)\}$, then we have from Lemma 3.4
\[
\|u_1 - u_2\|_{L^p(B_{\underline{\Xi}(t)}(0))}^p = \int_0^\infty |\tilde{u}_1 - \tilde{u}_2|^p r^{n-1} dr + \int_{B_{\underline{\Xi}(t)}(0) \setminus B_{\bar{\Xi}(t)}(0)} |u_1 - u_2|^p dx \\
\leq \int_0^\infty |\tilde{u}_1 - \tilde{u}_2|^p r^{n-1} dr + C|h_1 - h_2|.
\]
Taking
\[
\left\{\begin{array}{l}
\xi = \frac{r}{\bar{\Xi}(t)}; \\
\tau = t,
\end{array}\right.
\]
and set $\Xi_i(\xi, \tau) = \tilde{u}_i(h(t)\xi, t), \Xi_i(\xi, \tau) = \tilde{u}_i(h(t)\xi, t), i = 1, 2$. Thus $\Xi_1 - \Xi_2$ satisfies
\[
\left\{\begin{array}{l}
(\Xi_1 - \Xi_2)\xi = \frac{n}{2^\Xi(t)}(\Xi_1 - \Xi_2)\xi \\
+ \left(\frac{n-1}{2^\Xi(t)} + \frac{\Xi_k(t)}{2^\Xi(t)}\right)(\Xi_1 - \Xi_2)\xi \\
- \frac{1}{2^\Xi(t)}(v_1)\xi(\Xi_1 - \Xi_2)\xi \\
- \frac{1}{2^\Xi(t)}(v_1)\xi(\Xi_1 - \Xi_2)\xi \\
- \frac{n-1}{2^\Xi(t)}(v_1)\xi(\Xi_1 - \Xi_2) \\
- \frac{1}{2^\Xi(t)}(v_1)\xi(\Xi_1 - \Xi_2) \\
- \frac{1}{2^\Xi(t)}(v_1)\xi, \\
(\Xi_1 - \Xi_2)\xi(0, t) = 0, \\
(\Xi_1 - \Xi_2)\xi(0, t) = 0, \\
(\Xi_1 - \Xi_2)\xi(1, t) = \int_0^1 (\Xi_1 - \Xi_2)(t, \tau) \Xi_h(\tau)(\Xi_1 - \Xi_2)(1, \tau) d\tau \\
= F, \\
(\Xi_1 - \Xi_2)(\xi, 0) = 0, \\
0 < \xi < 1, \\
0 < \tau < t_0,
\end{array}\right.
\]
where
\[
F = \left\{\begin{array}{l}
-F_1, \\
F_2,
\end{array}\right.
\]
and
\[
F_1 = h_1(t) \left[\tilde{u}_{2r}(h_1(\tau), \tau) - \tilde{u}_{2r}(h_2(\tau), \tau)\right] \\
+ h_1(t) \left[1_k(h_1(\tau), \tau) + 1_k(h_2(\tau), \tau)\right] \\
+ h_2(t) \left[1_k(h_2(\tau), \tau) + 1_k(h_1(\tau), \tau)\right] \\
+ h_2(t) \left[1_k(h_2(\tau), \tau) + 1_k(h_1(\tau), \tau)\right].
\]
It is trivial that
\[
\sup_{0 \leq \tau \leq t_0} \int_0^1 (\Xi_1 - \Xi_2)^p \xi^{n-1} d\xi \to 0
\]
as $h_1 \to h_2$ on $C[0, t_0]$. 

Notice
\[
\int_0^\beta (\bar{u}_1 - \bar{u}_2)^p r^{n-1} dr = \int_0^1 (\bar{\pi}_1 - \bar{\pi}_2)^p |\bar{h}(t)|^n \xi^{n-1} d\xi \\
= |\bar{h}(t)|^n \int_0^1 (\bar{\pi}_1 - \bar{\pi}_2)^p \xi^{n-1} d\xi,
\]

hence we have
\[
(I_{22}) \leq C t_0^{\frac{3}{2} - \frac{\ell}{4}} |\bar{h}(t)|^{\frac{3}{2}} \left( \sup_{0 \leq \ell \leq t_0} \int_0^1 (\bar{\pi}_1 - \bar{\pi}_2)^p \xi^{n-1} d\xi \right)^{\frac{3}{2}} + C t_0^{\frac{3}{2} - \frac{\ell}{4}} \sup_{0 \leq \ell \leq t_0} |h_1 - h_2|^{\frac{3}{2}}.
\]

As \( \|h_1 - h_2\|_{C[0,t_0]} \) converges to zero, \((I_1)\) and \((I_2)\) converge to zero. From this we can get that \( \sup_{0 \leq \ell \leq t_0} |G(h_1) - G(h_2)| \) also converges to zero, which shows that the map \( G \) is continuous on \( C[0,t_0] \). Now the Schauder theorem yields that there exist a pair \((u,v)\) and a curve \( \Gamma(t) : r = h(t) \) which are the solution of (2.8).

5. Appendix. In this section, let us recall the construction of the problem. All of the material here can be found in [5].

Let \( \Omega \subset \mathbb{R}^n \) be a bounded open domain and \( \Omega_0 \subset \Omega \) be an open sub-domain. Assume a population density \( u(x,0) \) occupying the domain \( \Omega_0 \), and in the outside of \( \Omega_0 \) the population density \( u(x,0) \equiv 0 \) and the external signal \( v \) occupying \( \Omega \). For \( t > 0 \), \( u(x,t) \) spreads to domain \( \Omega_t \subset \Omega \). Let \( \partial \Omega_t \) denote the boundary of \( \Omega_t \) and \( \mathbf{n}_t \) denote the outer normal vector of \( \partial \Omega_t \), then \( \partial \Omega_t \times (0,T) \) is the free boundary.

The spatial diffusion of species is referred to the free boundary of \( \Omega_t \) which is occupied by the specie at the time \( t \geq 0 \). Observe the flux is increasing with respect to the density of the specie, so it would be reasonable to suppose that flux is proportional to the density. Thus we have following flux condition on \( \partial \Omega_t \),
\[
-j = -\nabla u \cdot \mathbf{n}_t + \chi u \nabla v \cdot \mathbf{n}_t,
\]
where \( k(x,t) \) is positive function, and \( \frac{\partial u(x,t)}{\partial t} > 0 \) is mass flow ratio.

On the other hand, noticing that the full flux on \( \partial \Omega_t \) is
\[
\mathbf{j} = -\nabla u \cdot \mathbf{n}_t + \chi u \nabla v \cdot \mathbf{n}_t,
\]
By conservation of population, one has
\[
u n_t = -\nabla u \cdot \mathbf{n}_t + \chi u \nabla v \cdot \mathbf{n}_t \quad \text{on} \quad \partial \Omega_t, \tag{5.1}
\]
where \( v_{n_t} \) is the normal diffusion velocity of \( \partial \Omega_t \). Assume \( \Gamma_t : \Phi(x,t) = 0 \), then
\[
v_{n_t} = \left( \frac{dx_1}{dt}, \frac{dx_2}{dt}, \cdots, \frac{dx_n}{dt} \right) \cdot \mathbf{n}_t = \left( \frac{dx_1}{dt}, \frac{dx_2}{dt}, \cdots, \frac{dx_n}{dt} \right) \cdot \frac{\nabla \Phi}{|\nabla \Phi|}, \tag{5.2}
\]
where \( x = (x_1, x_2, \cdots, x_n) \) and \( \nabla = \left( \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \cdots, \frac{\partial}{\partial x_n} \right) \).

Notice that
\[
\frac{\partial \Phi}{\partial t} + \frac{\partial \Phi}{\partial x_1} \cdot \frac{dx_1}{dt} + \frac{\partial \Phi}{\partial x_2} \cdot \frac{dx_2}{dt} + \cdots + \frac{\partial \Phi}{\partial x_n} \cdot \frac{dx_n}{dt} = 0, \tag{5.3}
\]
thus (5.2) and (5.3) give
\[
v_{n_t} = -\frac{1}{|\nabla \Phi|} \frac{\partial \Phi}{\partial t}. \tag{5.4}
\]
Combining (5.1) and (5.4) we obtain
\[
u \frac{\partial \Phi}{\partial t} = \nabla u \cdot \nabla \Phi - \chi u \nabla v \cdot \nabla \Phi \quad \text{on} \quad \partial \Omega_t.
\]
At last we obtain the conditions of the free boundary $\Gamma_t$
\[-\nabla u \cdot \frac{\nabla \Phi}{|\nabla \Phi|} = ku \quad \text{on} \quad \partial \Omega_t,\]
and
\[u \frac{\partial \Phi}{\partial t} = \nabla u \cdot \nabla \Phi - \chi u \nabla v \cdot \nabla \Phi \quad \text{on} \quad \partial \Omega_t.\]
Therefore the full free boundary problem reads
\[
\begin{cases}
  u_t = \nabla(\nabla u - \chi u \nabla v), & \text{in} \quad \Omega_t \times (0,T), \\
  u = 0, & \text{in} \quad \Omega \times (0,T) \setminus \Omega_t \times (0,T), \\
  -\nabla u \cdot \frac{\nabla \Phi}{|\nabla \Phi|} = ku, & \text{on} \quad \partial \Omega_t \times (0,T), \\
  u \frac{\partial \Phi}{\partial t} = \nabla u \cdot \nabla \Phi - \chi u \nabla v \cdot \nabla \Phi, & \text{on} \quad \partial \Omega_t \times (0,T), \\
  u(x,0) = u_0(x), & \text{in} \quad \Omega_0, \\
  v_t = \Delta v + u - v, & \text{in} \quad \Omega \times (0,T), \\
  \frac{\partial v}{\partial n} = 0, & \text{on} \quad \partial \Omega \times (0,T), \\
  v(x,0) = v_0(x), & \text{in} \quad \Omega,
\end{cases}
\]
where $\Gamma_t : \Phi(x,t) = 0$ is the free boundary.

**Acknowledgments.** The authors of this paper would like to thank the referee for the comments and helpful suggestions.

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Received February 2015; revised May 2015.

E-mail address: chenhua@whu.edu.cn
E-mail address: lvwenbin@whu.edu.cn
E-mail address: wush8@sina.com