APERIODICITY: THE ALMOST EXTENSION PROPERTY AND UNIQUENESS OF PSEUDO-EXPECTATIONS

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Abstract. We prove implications among the conditions in the title for general $C^*$-inclusions $A \subseteq B$, and we also relate this to several other properties in case $B$ is a crossed product for an action of a group, inverse semigroup or an étale groupoid on $A$. We show that an aperiodic $C^*$-inclusion has a unique pseudo-expectation. If, in addition, the unique pseudo-expectation is faithful, then $A$ supports $B$ in the sense of the Cuntz preorder. The almost extension property implies aperiodicity, and the converse holds if $B$ is separable. A crossed product inclusion has the almost extension property if and only if the dual groupoid of the action is topologically principal. Topologically free actions are always aperiodic. If $A$ is separable or of Type I, then topological freeness, aperiodicity and having a unique pseudo-expectation are equivalent to the condition that $A$ detects ideals in all $C^*$-algebras $C$ with $A \subseteq C \subseteq B$. If, in addition, $B$ is separable, then all these conditions are equivalent to the almost extension property.

1. Introduction

Many important $C^*$-algebras may be described using crossed products for group actions and their generalisations. This makes it important to describe the ideal structure of crossed products or to decide whether they are purely infinite. Very satisfactory criteria for this were developed around 1980 by Olesen and Pedersen [26,28], Kishimoto [16,17], and Rieffel [35]. These were recently extended in [20] from ordinary group actions to Fell bundles over groups. Partial, twisted actions of groups are a special case of this. Another important generalisation of this theory is to actions of inverse semigroups by Hilbert bimodules. Such actions and their crossed products model $C^*$-algebras associated to Fell bundles over étale groupoids. The main new difficulties in this more general setting come from the non-Hausdorffness of locally compact groupoids. These were overcome recently in [23] where we proposed a construction of an essential crossed product. The latter coincides with the reduced crossed products for actions of groups or Hausdorff étale groupoids.

The articles by Olesen–Pedersen already study a rather large number of closely related properties for group actions. Two properties have, however, only come into focus more recently. The first is the uniqueness of pseudo-expectations, which is used, for instance, in [31,33,36]. The second is the almost extension property for a $C^*$-inclusion $A \subseteq B$, which says that there is a dense set of pure states on $A$ that extend uniquely to $B$. This property is used, for instance, in [11,25]. In this article, we relate these two properties to aperiodicity, topological freeness and to the property that a $C^*$-subalgebra $A \subseteq B$ detects ideals in all intermediate $C^*$-algebras $A \subseteq C \subseteq B$. Our findings are summarised in the diagram in Figure 1.

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If $B$ is the reduced crossed product for a group action or an essential crossed product for an action of an inverse semigroup or an étale groupoid, and $B$ is separable, then all the properties above are equivalent by Theorem 7.2. Here the action may be by automorphisms or by Hilbert bimodules. The latter case corresponds to Fell bundles.

The almost extension property is introduced by Nagy and Reznikoff [25]. It is weaker than the well known and extensively studied extension property due to Anderson [3]. Using a criterion by Anderson for pure states to extend uniquely, we prove the following: if a $C^*$-inclusion $A \subseteq B$ has the almost extension property, then it is aperiodic (we generalised aperiodicity from actions to general $C^*$-inclusions in [20]), and the converse holds when $B$ is separable (see Theorem 5.5). When $B$ is a crossed product (full or reduced) of an action of a group, inverse semigroup or an étale groupoid, then we characterise pure states that extend uniquely in terms of the isotropy of a dual groupoid. As a consequence, we show that $A \subseteq B$ has the (almost) extension property if and only if the dual groupoid is (topologically) principal (see Theorem 5.11). This is a far reaching generalisation of the recent result of Zarikian [37, Theorem 2.4] proved for group actions by automorphisms on unital $C^*$-algebras.

We call an action of a group, inverse semigroup or étale groupoid topologically free if the dual groupoid of the action is topologically free in the sense of [23]. In general, this is a weaker condition than being effective or topologically principal. For group actions, it is (at least formally) weaker than the condition used by Archbold–Spielberg [4] (we refer to [23 Section 2.4] for a careful comparison of these properties). We prove that topologically free actions are always aperiodic (see Corollary 4.8). So far, this was only known for actions on separable $C^*$-algebras,

**Figure 1.** Implications among properties of $C^*$-inclusions. For the left column, let $A \subseteq B$ be a $C^*$-inclusion, let $E: B \to I(A)$ be a pseudo-expectation and let $\mathcal{N}$ be the largest two-sided ideal in $B$ with $\mathcal{N} \subseteq \ker E$. For the whole diagram, assume that $B$ is an exotic crossed product for an action of an inverse semigroup $S$ and let $E$ be the canonical pseudo-expectation.
and the proof for actions by Hilbert bimodules (Fell bundles) was rather indirect. Our proof is based on the technique of excising states from \([1]\).

Let \(I(A)\) be Hamana’s injective envelope \([13]\). A \(\textit{pseudo-expectation}\) for a \(C^*\)-inclusion \(A \subseteq B\) is defined as a completely positive contraction \(E: B \rightarrow I(A)\) that extends the identity map on \(A\) (see \([31]\)). Unlike genuine conditional expectations, pseudo-expectations always exist because \(I(A)\) is injective. Pitts and Zarikian studied extensively the case when there is a unique pseudo-expectation \([33,36]\). We prove that any aperiodic \(C^*\)-inclusion \(A \subseteq B\) has a unique pseudo-expectation \(E: B \rightarrow I(A)\). We also improve the results in \([23]\) about general aperiodic inclusions. Namely, for any aperiodic inclusion \(A \subseteq B\), we prove that for any \(b \in B\) with \(b \geq 0\) and \(E(b) \neq 0\), there is \(a \in A\) with \(a \preceq b\) in the Cuntz preorder on \(B\). In particular, if the unique pseudo-expectation is faithful, this says that \(A\) \(\textit{supports}\) \(B\). This condition plays a crucial role in the study of pure infiniteness (see \([19,20,23]\)).

We say that \(A\) \(\textit{detects ideals}\) in \(B\) if \(J \cap A = 0\) for an ideal \(J \subseteq B\) implies \(J = 0\). If an action of a discrete group \(G\) on a \(C^*\)-algebra \(A\) is topologically free, then \(A\) detects ideals in \(A \rtimes_r G\) (see \([4]\)). The converse to this usually fails, except for very special groups like \(\mathbb{Z}\). We show, however, that an action has to be topologically free if \(A\) is separable or of Type I and \(A\) detects ideals in \(C\) for all intermediate \(C^*\)-algebras \(A \subseteq C \subseteq B\). This remains true for actions of inverse semigroups or étale groupoids. Moreover, it suffices here to consider intermediate \(C^*\)-subalgebras that are crossed products associated to a subgroup, inverse subsemigroup, or a subgroupoid, depending on the kind of action in question (see Propositions \([6,1]\) and \([6,4]\)). If, in addition, the inverse semigroup that acts is countable, then the dual groupoid is topologically free if and only if it is topologically principal by \([23, \text{Theorem 6.13}]\). And we show that this is equivalent to the almost extension property. Thus all properties studied here are equivalent for actions of countable inverse semigroups on \(C^*\)-algebras that are separable or of Type I. These equivalences are collected in \(\text{Theorem 7.2}\).

Many of our results are already interesting for group actions by automorphisms. In this case, the essential crossed product is the same as the reduced crossed product, and many technical issues do not occur. We expect that some readers are not familiar with the more general actions of inverse semigroups and étale groupoids. We have written the article in such a way that these readers should be able to get along by just ignoring these generalisations. To make the article easier to digest for such readers, we introduce the key concepts such as aperiodicity and topological freeness first for group actions and then sketch only briefly how they must be adapted to treat actions of inverse semigroups and étale groupoids. Readers who need the more general theory should consult \([23]\) and the references there for a more thorough introduction of inverse semigroup actions by Hilbert bimodules, Fell bundles over étale groupoids and their crossed products.

The paper is organised as follows. We start with an introduction of aperiodic actions and aperiodic inclusions in \(\text{Section 2}\). In \(\text{Section 3}\) we prove that any aperiodic inclusion \(A \subseteq B\) has a unique pseudo-expectation and that all pseudo-expectations have the technical property of being \(\textit{supportive}\) introduced in \([23]\). This strengthens our previous results and gives natural simplicity and pure infiniteness criteria for general aperiodic \(C^*\)-inclusions. In \(\text{Section 4}\) we elaborate on topological freeness for actions of groups, inverse semigroups and étale groupoids. We show that topologically free actions are aperiodic. The key result concerns a single Hilbert
bimodule (see Theorem [4.7]). The proof uses the concept of a net of elements excising a state from [1]. This concept is also used in Section [5] where we discuss the almost extension property and, along the way, the extension property, for pure states. We prove that the almost extension property implies aperiodicity for arbitrary C*-inclusions, and the converse holds in the separable case. We show that a crossed product inclusion has the almost extension property if and only if the dual groupoid is topologically principal. In Section [6] we prove that an action of an inverse semigroup or an étale groupoid is topologically free if A detects ideals in certain intermediate C*-algebras A ⊆ C ⊆ B, where B is an essential crossed product and A contains an essential ideal that is separable or of Type I. In Section [7] we summarise the results of this paper presented in Figure 1. Theorems 7.2 and 7.3 show that various properties are equivalent for actions of inverse semigroups or étale, locally compact groupoids on C*-algebras that are “essentially” separable, simple, or of Type I. These equivalences allow to weaken the assumptions and strengthen the conclusions in a number of results in the literature. We use them in Section [8] to characterise Cartan inclusions and C*-diagonals.

2. APERIODIC ACTIONS AND APERIODIC INCLUSIONS

Aperiodicity is one of the key concepts in this article. In this section, we discuss the definition of aperiodic inclusions and how it translates to aperiodicity for actions of groups, inverse semigroups and étale groupoids. Here we understand actions in a broad sense as Fell bundles over such objects. This section contains no new results.

Let A be a C*-algebra. Let

\[ H(A) := \{ \text{non-zero, hereditary C*-subalgebras of } A \}, \]

\[ A^+ := \{ a \in A : a \geq 0, \|a\| = 1 \}. \]

**Definition 2.1** ([20, Definition 4.1]). Let X be a normed A-bimodule. We say that \( x \in X \) satisfies *Kishimoto’s condition* if, for any \( D \in H(A) \) and any \( \varepsilon > 0 \), there is \( a \in D^+ \) with \( \|axa\| < \varepsilon \). We call X *aperiodic* if Kishimoto’s condition holds for all \( x \in X \) (we renamed this last condition in [23, Definition 5.9]).

**Example 2.2.** An automorphism \( \alpha \in \text{Aut}(A) \) defines a Hilbert A-bimodule \( A_\alpha \) as follows: it is A as a vector space, with the bimodule structure

\[ a \cdot x \cdot b := axa(b) \]

for \( a, b \in A, x \in A_\alpha \), and with the left and right inner products

\[ \langle x \mid y \rangle := xy^*, \quad \langle x \mid y \rangle := \alpha^{-1}(x^*y) \]

for \( x, y \in A_\alpha \). As a Hilbert bimodule, \( A_\alpha \) is given the norm \( \|x\| := \|\langle x \mid x \rangle\|^{1/2} \). This is equal to the C*-norm on A.

Kishimoto’s condition for the Hilbert A-bimodule \( A_\alpha \) is exactly the condition introduced by Kishimoto in [16, Lemma 1.1]. By [17, Theorem 2.1], \( A_\alpha \) is aperiodic if and only if there is no \( \alpha \)-invariant ideal \( J \subseteq A \) for which the Borchers spectrum of \( \alpha|_J \) is equal to \( \{1\} \subseteq T \). An automorphism with this property is often called freely acting or *properly outer* (see, for instance, [15,96]). A related condition due to Elliott asks for \( \|\alpha|_J - \text{Ad}(u)\| = 2 \) for all \( \alpha \)-invariant ideals \( J \subseteq A \) and all unitary multipliers \( u \) of \( J \). Kishimoto’s condition implies Elliott’s, and the converse also holds if \( A \) is separable (see [28, Theorem 6.6]). An even weaker condition is *pure outerness*, which says only that \( \alpha|_J \neq \text{Ad}(u) \) for all \( \alpha \)-invariant ideals \( J \subseteq A \) and all unitary multipliers \( u \) of \( J \) (see [55]). If \( A \) is simple, an automorphism is purely outer if and only if it is outer, and then it is already properly outer by Kishimoto’s Theorem from [16]. Purely outer automorphisms of Type I C*-algebras are properly...
outer as well. A purely outer automorphism of a separable C*-algebra that is not properly outer is described in [20, Example 2.14].

**Definition 2.3** ([23, Definition 5.14]). A C*-inclusion \( A \subseteq B \) is aperiodic if the Banach \( A \)-bimodule \( B/\mathbb{C} \) is aperiodic, that is, if for every \( x \in B, D \in \mathbb{H}(A) \) and \( \varepsilon > 0 \), there are \( a \in D_1^\varepsilon \) and \( y \in A \) with \( \|axa - y\| < \varepsilon \).

**Proposition 2.4.** Let \( \alpha : G \to \text{Aut}(A) \) be a group action of a discrete group \( G \). Form Hilbert bimodules \( A_{\alpha y} \) for \( y \in G \) as in Example 2.13. Let \( B := A \rtimes G \) be the full crossed product. The canonical inclusion \( A \to B \) is aperiodic if and only if if the normed \( A \)-bimodules \( A_{\alpha y} \) are aperiodic for all \( \alpha \in G \setminus \{1\} \).

**Proof.** The main point is the following. Let \( X \) be a normed \( A \)-bimodule and let \( X_i \) for \( i \in I \) be subbimodules such that \( \sum X_i \) is dense in \( X \). Give \( X_i \) the norm from \( X \). Then \( X \) is aperiodic if and only if each \( X_i \) is aperiodic. This follows easily from [20, Lemma 4.2] (see also [23, Lemma 5.12]). For \( g \in G \), let \( u_g \in \mathcal{M}(B) \) be the corresponding unitary in the multiplier algebra of the crossed product. Then \( A \cdot u_g \subseteq B \) is an \( A \)-subbimodule that is isomorphic to \( A_{\alpha_y} \) as a bimodule because \( a \cdot (b \cdot u_g) \cdot c = ab\alpha_y(c)u_g \) for all \( a, b, c \in A \). We claim that this isomorphism remains isometric as a map to \( B/A \), that is, \( \|a \cdot u_g + b\| \geq \|a\| \) for all \( a, b \in A, g \in G \setminus \{1\} \). The proof of the claim uses the regular representation \( A \rtimes G \to A \rtimes G \subseteq \mathbb{B}(L^2(G, A)) \) and that \( a \cdot u_g \) and \( b \) are orthogonal in \( L^2(G, A) \). Since \( \sum_{g \in G \setminus \{1\}} A \cdot u_g \) is dense in \( B/A \), the statement about sums of bimodules shows that the inclusion \( A \subseteq B \) is aperiodic if and only if each \( A_{\alpha_y} \) for \( y \in G \setminus \{1\} \) is aperiodic.

The same argument works when we replace \( A \rtimes G \) by \( A \rtimes \varepsilon \) or any exotic crossed product, that is, a C*-algebra \( B \) with surjective *-homomorphisms

\[
A \rtimes G \to B \to A \rtimes \varepsilon \ G
\]

that compose to the canonical quotient map \( A \rtimes G \to A \rtimes \varepsilon \ G \). We could also allow twisted actions or partial actions of \( G \). We turn right away to the most general kind of group actions, namely, Fell bundles.

A **Fell bundle** \( A \) over a discrete group \( G \) is a family of Banach spaces \((A_g)_{g \in G}\) with bilinear, associative multiplication maps \( A_g \times A_h \to A_{gh} \) and conjugate-linear, antilinear actions \( \alpha_g : A \to A \) satisfying natural properties that turn the unit fibre \( A := A_1 \) into a C*-algebra, each \( A_g \) into a Hilbert \( A \)-bimodule, and the direct sum \( \bigoplus_{g \in G} A_g \) into a *-algebra. The full and the reduced section C*-algebras, \( C^*_f(G, A) \) and \( C^*_{\text{red}}(G, A) \), are defined by the \( \bigoplus_{g \in G} A_g \) as completions of \( \bigoplus_{g \in G} A_g \). A Fell bundle over \( G \) is called aperiodic if \( A_g \) for \( g \in G \setminus \{1\} \) is aperiodic [24, Definition 4.1].

The argument in the proof of Proposition 2.4 shows that the Fell bundle is aperiodic if and only if the inclusion of \( A \) into \( C^*_f(G, A) \) is aperiodic. Here we may replace \( C^*_f(G, A) \) by any C*-algebra with surjective *-homomorphisms \( C^*_f(G, A) \to B \to C^*_f(G, A) \) that compose to the canonical *-homomorphism \( C^*_f(G, A) \to C^*_f(G, A) \).

**Example 2.5.** For any kind of generalisation of a group \( G \), the “crossed product” \( B \) should be \( G \)-graded, that is, it should come with closed linear subspaces \( B_g \subseteq B \) for \( g \in G \) that satisfy \( B_g \cdot B_h \subseteq B_{gh} \) and \( B_{g^{-1}}^* = B_{g^{-1}} \) for \( g, h \in G \) and that \( \sum B_g \) is dense in \( B \). Then \( \{ (B_g)_{g \in G} \) with the multiplication and involution from \( B \) is a Fell bundle over \( G \). And the maps \( B_g \to B \) form a Fell bundle representation. So they induce a surjective *-homomorphism \( C^*_f(G, (B_g)_{g \in G}) \to B \). A \( G \)-grading is called topological if there is also a surjective *-homomorphism \( B \to C^*_f(G, A) \) as above so that the composite *-homomorphism \( C^*_f(G, A) \to C^*_f(G, A) \) is the canonical quotient map (see, for instance, [10]). Crossed products for twisted (partial) actions of \( G \) are \( G \)-graded by construction. Thus twisted (partial) actions define Fell bundles. The
full and reduced crossed product C*-algebras are naturally isomorphic to the full and reduced section C*-algebras of the corresponding Fell bundle.

Now let S be an inverse semigroup with unit 1 ∈ S. An action of the inverse semigroup S on a C*-algebra A by Hilbert bimodules consists of Hilbert A-bimodules E_t for t ∈ S and unitary multiplication maps μ_{t,u} : E_t ⊗_A E_u → E_{tu} for t, u ∈ S, such that μ_{t,u} is associative, E_1 = A, and μ_{t,1} and μ_{1,t} are the canonical maps for all t ∈ S (see [7, Definition 4.7]). Such an action is equivalent to a saturated Fell bundle over S and so it has a full and a reduced section C*-algebra (see [6, 9, 23]). We think of these as generalisations of full and reduced crossed products for group actions and denote them by A × S and A ×_ess S, respectively. In addition, we shall also use the essential crossed product A ×_ess S defined in [23], which in general is a quotient of A ×_ess S. We will discuss its definition when it becomes relevant.

An important difference between crossed products for group and inverse semigroup actions is that the images of E_t in A × S for t ∈ S are no longer linearly independent. The intersection of E_t with A in A × S is equal to the following ideal in A:

\[ I_{1,t} := \sum_{v ⪯ t}1 \cdot s(E_v) \]  

Here s(E_v) is the closed ideal generated by the inner products \( \langle x | y \rangle \) for \( x, y ∈ E_v \), and “\( ⪯ \)” is the standard partial order on S (\( v ⪯ t, 1 \) means that \( v \) is an idempotent and \( v = tv \)). Let

\[ I_{1,t}^t := \{ x ∈ A : x · I_{1,t} = 0 \} \]

be the annihilator of the ideal \( I_{1,t} \) in A. If S = G is a group, then for each t ∈ G\{1\} the sum in (2.1) is empty and hence \( I_{1,t} = 0 \) and \( I_{1,t}^t = A \). Thus \( A ×_1 G = A ×_ess G \).

The following proposition generalises Proposition 2.4 to inverse semigroup actions:

**Proposition 2.6** ([23] Proposition 6.3, Definition 6.1). Let \( \mathcal{E} \) be an action of an inverse semigroup on a C*-algebra A. Let B be a C*-algebra with surjective *-homomorphisms

\[ A × S → B → A ×_ess S \]

that compose to the quotient map \( A × S → A ×_ess S \). The inclusion \( A \subseteq B \) is aperiodic if and only if the A-bimodules \( \mathcal{E}_t · I_{1,t}^t \) for t ∈ S are aperiodic. In this case, we call the action \( \mathcal{E} \) aperiodic.

We call a C*-algebra B as in Proposition 2.6 an exotic crossed product for the action of S on A (see [23] Section 4.2).

Next we briefly discuss C*-algebras associated to étale groupoids. A topological groupoid G is étale if its range and source maps r and s are local homeomorphisms. An open set \( U ⊆ G \) is a bisection of G if s|_U and r|_U are injective. The bisections in G form a unital inverse semigroup Bis(G), where the space of units is the unit bisection.

**Example 2.7.** Let G be an étale groupoid with locally compact Hausdorff object space X. The full groupoid C*-algebra \( C^*(G) \) contains \( C_0(X) \) as a C*-subalgebra. By construction, the linear span of \( C_0 \)-functions on bisections in G is dense in \( C^*(G) \). For each bisection \( U ⊆ G \), the corresponding subspace \( C_0(U) ⊆ C^*(G) \) is a Hilbert \( C_0(X) \)-bimodule. The bimodule structure is \( (f_1 · f_2 : f_3)(g) := f_1(r(g))f_2(g)f_3(s(g)) \) for all \( f_1, f_3 ∈ C_0(X), f_2 ∈ C_0(U) \), \( g ∈ G \). The Banach spaces \( \mathcal{E}_U := C_0(U) \) form an action of Bis(G) on \( C_0(X) \) by Hilbert bimodules, whose full section C*-algebra is naturally isomorphic to \( C^*(G) \). Given a bisection \( U ⊆ G \), the submodule \( \mathcal{E}_U · I_{1,U}^t \) in Proposition 2.6 is \( C_0(U \setminus \overline{X}) \). Therefore, Proposition 2.6 says that the inclusion \( C_0(X) ⊆ C^*(G) \) is aperiodic if and only if the \( C_0(X) \)-bimodules \( C_0(U \setminus \overline{X}) \) with the supremum norm are aperiodic for all bisections U of G. It is easy to see that this
happens if and only if there is no non-empty open subset $U \subseteq G \setminus X$ with $r|_U = s|_U$, if and only if the set of $g \in G$ with $r(g) \neq s(g)$ is dense in $G \setminus X$. The same argument works if we replace the full groupoid $C^*$-algebra by the reduced one or by the essential one defined in [23].

We name the condition that characterises aperiodicity in the above example:

**Definition 2.8** ([23, Definition 2.20]). An étale groupoid $G$ with unit space $X \subseteq G$ is topologically free if there is no non-empty open subset $U \subseteq G \setminus X$ with $r|_U = s|_U$.

**Remark 2.9.** Topological freeness is weaker than similar popular conditions like being effective or topologically principal (see [23, Section 2.4]).

Let $G$ be an étale groupoid with locally compact and Hausdorff unit space $X$. A Fell bundle over the groupoid $G$ is an upper-semicontinuous bundle $A = (A_x)_{x \in G}$ of Banach spaces equipped with a continuous involution $^*: A \to A$ and a continuous partially defined multiplication $*:\{(a,b) \in A \times A: a \in A_{r(a)}, b \in A_{r(b)}, (\gamma_1, \gamma_2) \in G^{(2)}\} \to A$ that satisfy a number of natural properties (see [23] for details). Then $A := C_0(X, A|_X)$ is a $C_0(X)$-$C^*$-algebra and the space of $C_0$-sections $C_0(U, A|_U)$ for a bisection $U \in \text{Bis}(G)$ becomes a Hilbert $A$-bimodule. These Hilbert bimodules form a Fell bundle over the inverse semigroup $\text{Bis}(G)$. This Fell bundle is saturated when $A$ is saturated. In general, we change the construction by letting $S$ be the family of all Hilbert subbimodules of $C_0(U, A|_U)$ for all $U \in \text{Bis}(G)$, cf. [23, Lemma 7.3]. This defines a saturated Fell bundle by construction, which is the same as an action by Hilbert bimodules on $A$. And [23, Proposition 7.6 and 7.9 and Definition 7.12] give natural isomorphisms

\begin{equation}
C^*(G, A) \cong A \rtimes S, \quad C^*_e(G, A) \cong A \ltimes \text{ess } S, \quad C^*_\text{ess}(G, A) \cong A \ltimes \text{ess } S
\end{equation}

between the corresponding full, reduced and essential $C^*$-algebras.

**Definition 2.10.** We will call a Fell bundle $A = (A_x)_{x \in G}$ over an étale groupoid $G$ an action of the groupoid $G$ on $C_0(X, A|_X)$. An exotic crossed product for $A$ is a $C^*$-algebra $B$ with surjective $^*$-homomorphisms $C^*(G, A) \twoheadrightarrow B \twoheadrightarrow C^*_\text{ess}(G, A)$ that compose to the quotient map $C^*(G, A) \twoheadrightarrow C^*_\text{ess}(G, A)$.

**Definition 2.11.** A Fell bundle $A = (A_x)_{x \in G}$ over an étale groupoid $G$ is aperiodic if the corresponding inverse semigroup action described above is aperiodic.

Proposition 2.6 implies that the inclusion of $C_0(X, A|_X)$ into any exotic crossed product for a Fell bundle $A$ over $G$ is aperiodic if and only if the action $A$ is aperiodic. All results about actions of étale groupoids below are proven by reducing the statement to inverse semigroup actions as above.

### 3. Aperiodic inclusions and pseudo-expectations

In this section we improve [23, Theorem 5.28], the main theorem about general aperiodic inclusions in [23], by proving some results about the inclusion into the injective hull of a $C^*$-algebra. Namely, any aperiodic inclusion admits a unique pseudo-expectation, and all pseudo-expectations satisfy a technical condition that is needed in [23, Theorem 5.28]. Thus the conclusions of [23, Theorem 5.28] hold for all aperiodic inclusions together with their unique pseudo-expectation.

**Lemma 3.1.** Any normed $A$-bimodule $X$ contains a largest aperiodic $A$-subbimodule. For any $C^*$-inclusion $A \subseteq B$ the $C^*$-algebra $B$ contains a largest two-sided ideal which is aperiodic as an $A$-bimodule.

**Proof.** This follows from [20, Lemma 4.2] or [23, Lemma 5.12] because the closed linear span of a family of aperiodic subbimodules is again aperiodic. \qed
The largest aperiodic ideal \( N \subseteq B \) for a \( C^* \)-inclusion \( A \subseteq B \) plays an important role. By [23, Theorem 5.17], if the inclusion \( A \subseteq B \) is aperiodic, then \( N \) is the unique ideal for which \( A \cap N = 0 \) and the induced inclusion \( A \to B/N \) detects ideals in the following sense:

**Definition 3.2.** Let \( A \) be a \( C^* \)-subalgebra of a \( C^* \)-algebra \( B \). We say that \( A \) detects ideals in \( B \) if \( J \cap A = 0 \) implies \( J = 0 \) for every ideal \( J \) in \( B \). (Some authors then say that the \( C^* \)-inclusion \( A \subseteq B \) is essential (see [33]) or that it has the ideal intersection property.)

Assume for a moment that our aperiodic inclusion \( A \subseteq B \) carries a conditional expectation \( E : B \to A \). Then \( N \) is equal to the largest two-sided ideal contained in the kernel of \( E \) (see [23, Theorem 5.28]). This applies, in particular, to all crossed products for group actions. In this case, the quotient \( (A \rtimes G)/N \) is the reduced crossed product \( A \rtimes G \) because \( E \) induces a faithful conditional expectation \( A \rtimes G \to A \). A conditional expectation also exists for the inclusion \( C_0(X, A)(\chi) \subseteq C^*_\text{p}(G, A) \) if \( G \) is a Hausdorff, ātale, locally compact groupoid. Once again, \( C^*_\text{p}(G, A)/N \) is the reduced crossed product \( C^*_\text{p}(G, A) \). Thus \( A \subseteq A \rtimes G \) and \( C_0(X, A)(\chi) \subseteq C^*_\text{p}(G, A) \) detect ideals if the underlying action is aperiodic. The situation is different, however, for general inverse semigroup actions or Fell bundles over non-Hausdorff groupoids.

They do not admit a genuine conditional expectation. The way out is to consider “generalised” expectations, which take values in a larger \( C^* \)-algebra \( \hat{A} \supseteq A \):

**Definition 3.3.** A generalised expectation for a \( C^* \)-inclusion \( A \subseteq B \) consists of another \( C^* \)-inclusion \( A \subseteq \hat{A} \) and a completely positive, contractive map \( E : B \to \hat{A} \) that restricts to the identity map on \( A \).

Any generalised expectation is an \( A \)-bimodule map by [23, Lemma 3.2].

The identity map on \( B \) is a generalised expectation for any \( C^* \)-inclusion, and it cannot tell us anything interesting. Therefore, an extra condition on a generalised expectation is needed. The main theorem for general \( C^* \)-inclusions in [23] requires a generalised expectation which is “supportive” (see Definition 3.10 below). And it asserts that \( A \) supports \( B/N \) in the following sense:

**Definition 3.4.** Let \( B^+ \) be the set of positive elements in \( B \). We equip \( B^+ \setminus \{0\} \) with the Cuntz preorder \( \preceq \) introduced in [8]; for \( a, b \in B^+ \setminus \{0\} \), we write \( a \preceq b \) and say that \( a \) supports \( b \) (in \( B \)) if, for every \( \varepsilon > 0 \), there is \( x \in B \) with \( \|a - x^*bx\| < \varepsilon \).

We say that \( A \) supports \( B \) if for every \( b \in B^+ \setminus \{0\} \) there is \( a \in A^+ \setminus \{0\} \) with \( a \preceq_B b \).

**Definition 3.5.** A pseudo-expectation for a \( C^* \)-inclusion \( A \subseteq B \) is a generalised expectation \( E : B \to I(A) \) taking values in Hamana’s injective hull (see [13]).

The injectivity of \( I(A) \) implies that any \( C^* \)-inclusion has at least one pseudo-expectation. Having a unique pseudo-expectation is an important structural property, which has been advocated, in particular, by Pitts (see [31, 33, 36]).

Now we explain how we are going to improve the main theorem about general aperiodic inclusions in [23]. We show, first, that any aperiodic inclusion has a unique pseudo-expectation \( E \); secondly, that \( E \) is supportive; and, thirdly, that the largest aperiodic bimodule is \( \ker E \). When we put this information into [23, Theorem 5.28], then we get the following theorem:

**Theorem 3.6.** Let \( A \subseteq B \) be an aperiodic \( C^* \)-inclusion. Then there is exactly one pseudo-expectation \( E : B \to I(A) \) and \( \ker E \) is the largest aperiodic \( A \)-subbimodule in \( B \). Let \( N \) be the largest two-sided ideal contained in \( \ker E \). Then

1. for every \( b \in B^+ \) with \( b \notin N \), there is \( a \in A^+ \setminus \{0\} \) with \( a \preceq_B b \); in particular, \( A \) supports \( B/N \);
(2) if \( J \subseteq B \) is an ideal with \( J \cap A = 0 \), then \( J \subseteq N \); in particular, \( A \) detects ideals in \( B/N \), and \( B/N \) is the unique quotient of \( B \) with this property.

(3) \( B \) is simple if and only if \( N = 0 \) and \( A \subseteq B1B \) for any non-zero ideal \( I \) in \( A \);

(4) if \( B \) is simple, then \( B \) is purely infinite if and only if all elements of \( A^+ \backslash \{0\} \) are infinite in \( B \).

**Remark 3.7.** The essential crossed products in [23] are defined using a generalised expectation into the local multiplier algebra, \( E : B \rightarrow \mathcal{M}_{loc}(A) \). There is a canonical embedding \( \iota : \mathcal{M}_{loc}(A) \rightarrow \mathcal{I}(A) \) by [12 Theorem 1]. Thus generalised expectations into \( \mathcal{M}_{loc}(A) \) become pseudo-expectations as well, and this change of view point affects neither the kernel ker \( E \) nor the largest two-sided ideal contained in ker \( E \).

As a result, if \( B = A \times S \) for an inverse semigroup action, then \( B/N \) is the essential crossed product \( A \rtimes_{\text{ess}} S \) defined in [23]. If \( B = C^*(G,A) \) is a section C*-algebra for a Fell bundle over an étale groupoid \( G \), then \( B/N \) is the essential section C*-algebra \( C^*_{\text{ess}}(G,A) \) as in [23]. Thus Theorem 3.6 contains criteria for C*-algebras of the form \( A \rtimes_{\text{ess}} S \) and \( C^*_{\text{ess}}(G,A) \) to be simple or purely infinite. In fact, these criteria are already proven in [23], using the generalised expectation \( E : B \rightarrow \mathcal{M}_{loc}(A) \).

The proof of Theorem 3.6 will occupy the rest of this section. The following concept is crucial for the proof:

**Definition 3.8.** A C*-inclusion \( A \subseteq \hat{A} \) is called anti-aperiodic if there are no non-zero aperiodic \( A \)-subbimodules of \( \hat{A} \).

The second and more difficult part of the proof will show that the inclusion \( A \hookrightarrow \mathcal{I}(A) \) is anti-aperiodic. The first part consists of several rather easy results about generalised expectations \( B \rightarrow \hat{A} \) when the inclusion \( A \subseteq \hat{A} \) is anti-aperiodic.

**Proposition 3.9.** Let \( B \supseteq A \subseteq \hat{A} \) be C*-inclusions such that \( A \subseteq B \) is aperiodic and \( A \subseteq \hat{A} \) is anti-aperiodic. Then there is at most one generalised expectation \( B \rightarrow \hat{A} \).

**Proof.** Let \( E_1, E_2 : B \rightarrow \hat{A} \) be two generalised expectations. The map \( E_1 - E_2 : B \rightarrow \hat{A} \) is an \( A \)-bimodule map that vanishes on \( A \). Thus it descends to a bounded \( A \)-bimodule map \( B/A \rightarrow \hat{A} \). Since \( B/A \) is an aperiodic \( A \)-bimodule, the range of the map \( E_1 - E_2 \) is an aperiodic \( A \)-subbimodule of \( \hat{A} \) by [23 Lemma 5.12]. Since \( A \subseteq \hat{A} \) is anti-aperiodic, the range must be 0. So \( E_1 = E_2 \).

**Definition 3.10** ([23 Definition 5.19]). A generalised expectation \( E : B \rightarrow \hat{A} \) is called supportive if no non-zero element of \( E(B^+) \) satisfies Kishimoto’s condition, that is, if, for any \( b \in B^+ \) with \( E(b) \neq 0 \), there are \( \delta > 0 \) and \( D \in \mathcal{H}(A) \) such that \( \|xE(b)x\| \geq \delta \) for all \( x \in D_1^+ \).

**Proposition 3.11.** Let \( B \supseteq A \subseteq \hat{A} \) be C*-inclusions. If \( A \subseteq \hat{A} \) is anti-aperiodic, then Kishimoto’s condition fails for any non-zero positive element in \( \hat{A} \), and so any generalised expectation \( E : B \rightarrow \hat{A} \) is supportive.

**Proof.** Let \( 0 \neq b \in A^+ \). Let \( A^1 := A \otimes C : 1 \) be the unital C*-algebra generated by \( A \) and extend the \( A \)-bimodule structure on \( A \) to an \( A^1 \)-bimodule structure. By assumption, the \( A \)-bimodule \( A^1b^{1/2}A^1 \) is not aperiodic. The set of elements that satisfy Kishimoto’s condition is a closed vector subspace by [20 Lemma 4.2]. The unital C*-algebra \( A^1 \) is spanned by unitaries. Hence there must be unitaries \( u, v \in A^1 \) such that Kishimoto’s condition fails for \( ub^{1/2}v \). That is, there are \( \delta > 0 \) and \( D \in \mathcal{H}(A) \) such that \( \|xub^{1/2}vx\| \geq \delta \) for all \( x \in D_1^+ \). Let \( y \in (u*Du)^+ \). Then \( y = u*xu \) for some \( x \in D_1^+ \) and hence

\[ \|yby\| = \|xub^{1/2}ux\| \geq \|xub^{1/2}vx^{1/2}v^{1/2}b^{1/2}u*b^12u^*x\| = \|xub^{1/2}vx\|^2 \geq \delta^2. \]
So Kishimoto’s condition fails for $b$. □

**Lemma 3.12.** Let $B \supseteq A \subseteq \hat{A}$ be $C^*$-inclusions such that $A \subseteq B$ is aperiodic and $A \subseteq \hat{A}$ is anti-aperiodic. Let $E : B \to \hat{A}$ be a generalised expectation. Then $\ker E \subseteq B$ is the largest aperiodic $A$-bimodule in $B$.

**Proof.** We first show that $\ker E$ is aperiodic. Since $E|_A = \text{Id}_A$, the quotient map from $E$ to $B/A$ is injective. We claim that it is a topological isomorphism. Then $\ker E$ inherits aperiodicity from $B/A$. Let $b \in \ker E$ and let $a \in A$. Then $a = E(a) = E(a - b)$. Since $E$ is contractive, this implies $\|a\| \leq \|a - b\|$. Then $\|b - a\| \geq \|b\| - \|a\| \geq \|b\| - \|b - a\|$. So $\|b - a\| \geq \|b\|/2$. This means that $\|b\|_{B/A} \geq \|b\|_{\ker E}/2$.

Now let $X \subseteq B$ be any aperiodic $A$-bimodule. Then $E(X) \subseteq \hat{A}$ with the norm from $\hat{A}$ is aperiodic by [23, Lemma 5.12]. Since $A \subseteq \hat{A}$ is anti-aperiodic, it follows that $E(X) = 0$. So $X \subseteq \ker E$. □

Now we begin to prove that the inclusion $A \subseteq I(A)$ is anti-aperiodic.

**Lemma 3.13.** Let $A \subseteq \hat{A}$ be a $C^*$-inclusion and let $X \subseteq \hat{A}$ be an aperiodic, closed $A$-subbimodule. The restriction of the quotient map $A \to \hat{A}/X$ is isometric.

**Proof.** If $X \subseteq \hat{A}$ is aperiodic, then so is $X^*$ because $\|ab^*a\| = \|a^*ba\|$. We get a stronger statement if we replace $X$ by the closed linear span of $X$ and $X^*$, and the latter is aperiodic as well by Lemma 3.1. Therefore, we may assume without loss of generality that $X = X^*$.

Let $a \in A$, $b \in X$ and $\varepsilon > 0$. There is $D \in \mathbb{H}(A)$ such that $\|xa^*ax\| \geq \|a^*a\| - \varepsilon/2$ for all $x \in D_+^+$ (see [20, Lemma 2.9]). Since $a^*b + b^*a \in X$, there is $x \in D_+^+$ such that $\|xa^*b + b^*a\|_X < \varepsilon/2$. Then

$$\|a + b\|^2 = \|a^*a + a^*b + b^*a + b^*b\| \geq \|xa^*a + a^*b + b^*a + b^*b\|_X \geq \|xa^*b + b^*a\|_X > \|xa^*b\| - \varepsilon/2 \geq \|a^*a\| - \varepsilon = \|a\|^2 - \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, this implies $\|a + b\| \geq \|a\|$. That is, the quotient norm on $\hat{A}/X$ restricts to the usual norm on $A$. □

**Lemma 3.14.** Let $A$ be a $C^*$-algebra and let $n \geq 1$. Let $X$ be an aperiodic normed $A$-bimodule. Then $M_n(X)$ is an aperiodic $M_n(A)$-bimodule.

**Proof.** Let $x = (x_{i,j})_{1 \leq i,j \leq n} \in M_n(X)$, $D \in \mathbb{H}(M_n(A))$ and $\varepsilon > 0$. We are going to check Kishimoto’s condition for this data. Equip $A^n$ with the standard right Hilbert $A$-module structure. Then $\mathbb{K}(A^n) \cong M_n(A)$, and $A^n$ is an $M_n(A)$-A-equivalence bimodule. Then $D \cdot A^n \subseteq A^n$ is a right $A$-submodule and $D \cong \mathbb{K}(D \cdot A^n)$. So $D \cdot A^n \neq 0$ and there is $\eta = (\eta_k)_{1 \leq k \leq n} \in D \cdot A^n \subseteq A^n$ with $\|\eta\| = 1$. Then [20, Lemma 2.9] applied to $[\eta] := \left< \eta, \eta \right>_A/\|\eta\| \in A$ gives $D_0 \in \mathbb{H}([\eta][\eta]^*)$ with $\|\eta b\| \geq (1 - \varepsilon)\|b\|$ for all $b \in D_0$. If $b \in D_0$, then

$$(3.1) \quad \|\eta b\|^2 = \|\eta b\| \cdot \|\eta b\|_A = \|b^*\| \|\eta\|^2 \|b\|^2 = \|\eta b\|^2 \geq (1 - \varepsilon)^2 \|b\|^2.$$

Put $w_{i,j} := \eta_i^* \cdot x_{i,j} \cdot \eta_j \in X$. Since $X$ is an aperiodic $A$-bimodule, so is $X^n$ by [20, Lemma 4.2]. Then there is $b \in (D_0)_+^+$ with $\|bw_{i,j}b\| < \varepsilon/(1 - \varepsilon)^2/n$ for all $1 \leq i, j \leq n$. Let $b_0 := b/\|b\|_A \in D_0^+$, so that $\|\eta b_0\| = 1$. Using (3.1), we get $\|\eta b_0 w_{i,j} b_0\| < \varepsilon/n$. The rank-one operator $\eta b_0 \otimes \eta b_0$ belongs to $\mathbb{K}(D \cdot A^n)^+_A$ because $\|\eta b_0 \otimes \eta b_0\| = \|\eta b_0\|^2 = 1$ and $\eta b_0 \in D \cdot A^n$. The isomorphism $\mathbb{K}(D \cdot A^n) \cong D$ maps it to an element $a \in D_+^+$. We claim that $\|a x a\| < \varepsilon$. For this computation, write
That is, the injective hull of the restriction of the quotient map is dense or, equivalently, is the unique pseudo-expectation. Thus \( x \in M_n(X) \) satisfies Kishimoto’s condition. 

**Lemma 3.15.** Let \( A \subseteq \hat{A} \) be a C*-inclusion and \( X \subseteq \hat{A} \) an aperiodic \( A \)-subbimodule. The restriction of the quotient map \( A \to \hat{A}/X \) is completely isometric.

**Proof.** Let \( n \geq 1 \). By definition, \( M_n(\hat{A}/X) \cong M_n(\hat{A})/M_n(X) \) with the quotient semi-norm. By Lemma 3.14, \( M_n(X) \subseteq M_n(\hat{A}) \) is aperiodic. Then Lemma 3.13 shows that the map \( M_n(\hat{A}) \to M_n(A)/M_n(X) \) is isometric.

**Proposition 3.16.** For any C*-algebra, the inclusion \( A \subseteq I(A) \) is anti-aperiodic. That is, the injective hull \( I(A) \) contains no non-zero aperiodic \( A \)-subbimodule.

**Proof.** Let \( \hat{A} = I(A) \) and let \( X \) be an aperiodic \( A \)-subbimodule of \( I(A) \). Lemma 3.15 says that the map \( A \to I(A)/X \) is completely isometric. Since \( I(A) \) is injective, the inclusion \( A \to I(A) \) extends to a completely contractive map \( h : I(A)/X \to I(A) \). Hence the composite map \( I(A) \to I(A)/X \to I(A) \) is completely contractive and it restricts to the identity map on \( A \). The rigidity of the injective envelope in [20, Corollary 15.7] implies that any such map is equal to the identity map on \( I(A) \). This can only happen if \( X = \varnothing \).

**Proof of Theorem 3.16.** A pseudo-expectation \( E : B \to I(A) \) exists because \( I(A) \) is injective. By Proposition 3.16 the inclusion \( A \subseteq I(A) \) is anti-aperiodic. Then Proposition 3.9 shows that \( E \) is the unique pseudo-expectation \( B \to I(A) \). Lemma 3.12 shows that ker \( E \) is the largest aperiodic bimodule in \( A \). By Proposition 3.11 \( E \) is supportive. Then the remaining assertions follow from [23, Theorem 5.28].

**Remark 3.17.** Is the inclusion \( A \subseteq I(A) \) aperiodic? While we do not have a proof for this, there is some positive evidence. First, if \( A \) is commutative, then \( I(A) = M_{\text{loc}}(A) \) (see [12, Theorem 1]); and the inclusion \( A \subseteq M_{\text{loc}}(A) \) is shown to be aperiodic in [23]. Secondly, the inclusion \( A \subseteq I(A) \) has a unique pseudo-expectation: it must be the identity map by the rigidity of \( I(A) \).

4. **Topological freeness implies aperiodicity**

In this section, we show that topologically free actions are aperiodic. This applies to actions of groups, actions of inverse semigroups by Hilbert bimodules, or Fell bundles over étale locally compact groupoids. The proof reduces to a statement about Hilbert bimodules. For Hilbert bimodules over separable C*-algebras, this is already shown in [20], where the proof is based on a statement about automorphisms shown by Olesen–Pedersen in [25]. Here we give a direct proof, which applies to arbitrary C*-algebras.

Let \( A \) be a C*-algebra and let \( \hat{A} \) be its spectrum. So \( \hat{A} \) is the set of unitary equivalence classes of irreducible representations of \( A \), equipped with the topology where the open subsets are \( \tilde{J} \subseteq \hat{A} \) for all ideals \( J \in A \). Any automorphism \( \alpha : A \to A \) induces a homeomorphism \( \hat{\alpha} : \hat{A} \to \hat{A} \) by \( \hat{\alpha}[\tilde{g}] = \tilde{g} \circ \alpha \) for \( [\tilde{g}] \in \hat{A} \). The automorphism \( \alpha \) is called topologically non-trivial if the set of \( [\tilde{g}] \in \hat{A} \) with \( \hat{\alpha}[\tilde{g}] \neq [\tilde{g}] \) is dense or, equivalently, \( \{[\tilde{g}] \in \hat{A} : [\tilde{g} \circ \alpha] = [\tilde{g}] \} \) has empty interior in \( \hat{A} \). More
explicitly, for every non-zero ideal $I$ in $A$, there is an irreducible representation $\rho$ of $I$ such that $\rho \circ \alpha$ and $\rho$ are not unitarily equivalent.

**Definition 4.1.** We call a group action $\alpha: G \to \text{Aut}(A)$ topologically free if, for each $g \in G \backslash \{1\}$, the automorphism $\alpha_g$ is topologically non-trivial.

**Remark 4.2.** The condition above differs slightly from the one in [4], which is used in a number of papers (including [20]) and that requires that the union $\bigcup_{k=1}^{n} \{ [g] \in \hat{A} : [g \circ \alpha_{g_k}] = [g] \} \text{ for } g_1, \ldots, g_n \in G \backslash \{1\}$ has empty interior in $\hat{A}$. The two conditions are equivalent when $A$ contains an essential ideal which is separable or of Type I (see [20, Proposition 9.7]). In general, the results in this article show that our (formally) weaker condition implies even stronger results than those in [4].

The dual action of $G$ on $\hat{A}$ has a transformation groupoid $\hat{A} \rtimes G$. This is an étale topological groupoid, where the unit space is $\hat{A}$, morphisms from $[\pi]$ to $[\rho]$ correspond to the elements $g \in G$ such that $g \cdot [\pi] = [\rho]$, and the topology on the arrow space $G \times \hat{A}$ is the product topology. The group action $\alpha$ on $A$ is topologically free if and only if the groupoid $\hat{A} \rtimes G$ is topologically free as in Definition 2.8. Similar dual groupoids are defined for actions of inverse semigroups by Hilbert bimodules and for Fell bundles over étale groupoids. We will use this to define topologically free actions of inverse semigroups and étale groupoids.

To generalise the dual action to Fell bundles, we must explain how a Hilbert $A$-bimodule $\mathcal{E}$ induces a partial homeomorphism of $\hat{A}$. Let $s(\mathcal{E})$ and $r(\mathcal{E})$ be the closed ideals in $A$ that are generated by the right and the left inner products. In symbols, $s(\mathcal{E}) = \overline{\text{span}}(\mathcal{E} | \mathcal{E})$ and $r(\mathcal{E}) = \overline{\text{span}}(\mathcal{E} | \mathcal{E})$. Let $g: A \to B(H)$ be an irreducible representation. The tensor product $\mathcal{E} \otimes_g H$ is non-zero if and only if $[g]$ belongs to $s(\mathcal{E})$. Then the left multiplication action of $A$ on $\mathcal{E} \otimes_g H$ is an irreducible representation that belongs to $r(\mathcal{E})$. The unitary equivalence class of the representation $\mathcal{E} \otimes_g H$ depends only on the class of $g$, and the map $[g] \mapsto [\mathcal{E} \otimes_g H]$ is a homeomorphism $\hat{\mathcal{E}}: s(\mathcal{E}) \to r(\mathcal{E})$.

**Definition 4.3** ([20, Definition 2.13]). A Hilbert $A$-bimodule $\mathcal{E}$ over a $C^*$-algebra $A$ is topologically non-trivial if for each ideal $J \triangleleft A$ there is $[g] \in \hat{J}$ with $\hat{\mathcal{E}}[g] \neq [g]$. 

**Definition 4.4** ([23, Definitions 2.14, 2.20]). Let $\mathcal{E} = (\mathcal{E}_t, (\mu_{t,u}), t \in S)$ be an action of an inverse semigroup on a $C^*$-algebra $A$ by Hilbert bimodules. Then each $\mathcal{E}_t$ for $t \in S$ defines a partial homeomorphism $\hat{\mathcal{E}}_t$ of $\hat{A}$. These partial homeomorphisms form an action of the inverse semigroup $S$ on $\hat{A}$. This action has an étale transformation groupoid, which is the dual groupoid of the action $\mathcal{E}$. The action $\mathcal{E}$ is called topologically free if this dual groupoid is topologically free.

By Proposition 2.3, the action $\mathcal{E}$ is aperiodic if and only if the Hilbert bimodules $\mathcal{E}_t \cdot I_{1,t}^+$ are aperiodic for all $t \in S$. There is a similar criterion for topological freeness:

**Lemma 4.5.** An inverse semigroup action $\mathcal{E} = (\mathcal{E}_t)_{t \in S}$ is topologically free if and only if the Hilbert $A$-bimodules $\mathcal{E}_t \cdot I_{1,t}^+$ for $t \in S$ are topologically non-trivial.

**Proof.** This lemma is a part of [23, Theorem 6.13]. We recall the relevant part of the proof. The dual groupoid $\hat{A} \rtimes S$ is covered by bisections associated to $t \in S$. The action of the bisection for $t \in S$ is the dual action of the Hilbert bimodule $\mathcal{E}_t$. Removing the closure of the unit bisection replaces the partial homeomorphism associated to $\mathcal{E}_t$ by the partial homeomorphism associated to $\mathcal{E}_t \cdot I_{1,t}^+$. Thus $\hat{A} \rtimes S$ is topologically free if and only if each $\mathcal{E}_t \cdot I_{1,t}^+$ is topologically non-trivial. □

**Example 4.6.** Let $\mathcal{A}$ be a Fell bundle over an étale groupoid $G$ with locally compact Hausdorff object space $X$. Then $G$ acts naturally on the spectrum $\hat{A}$ of the
C*-algebra $A := C_0(X, \mathcal{A})$. Every irreducible representation of $A$ factors through the evaluation map $A \to A_x$ for some $x \in X$. This defines a continuous map $\psi: \hat{A} \to X$. This is the anchor map of a $G$-action on $\hat{A}$. Namely, $\gamma \in G$ acts by the partial homeomorphisms $\psi_{\gamma}: \hat{A}_{s(\gamma)} \to \hat{A}_{s(\gamma)}$ induced by the Hilbert $A_{\eta(\gamma)}, A_{s(\gamma)}$-bimodule $A_{\gamma}$ (see, for instance, [14, Section 2]). The corresponding transformation groupoid is $\hat{A} \times G := \{(\gamma, [\pi]) \in G \times \hat{A} : s(\gamma) = \psi([\pi])\}$. The elements $(\eta, [\rho])$ and $(\gamma, [\pi])$ are composable if and only if $[\rho] = \psi_{\gamma}([\pi])$, and then their composite is $(\eta \gamma, [\pi])$. The inverse is given by $(\gamma, [\pi]) \mapsto (\gamma^{-1}, \psi_{\gamma}([\pi]))$. For the inverse semigroup action on $A$ used in the isomorphisms (2.2), there is also a natural isomorphism between the transformation groupoids

$\hat{A} \times G \cong \hat{A} \times S$

(see the discussion before Remark 7.4)). Therefore, we will call $\hat{A} \times G$ the dual groupoid for the groupoid action $A$, and we will say that the groupoid action $A$ is topologically free if this dual groupoid is topologically free.

The following theorem is the main result in this section:

**Theorem 4.7.** Let $A$ be a C*-algebra and let $\mathcal{E}$ be a Hilbert $A$-bimodule. If $\mathcal{E}$ is topologically non-trivial, then $\mathcal{E}$ is aperiodic.

**Corollary 4.8.** Any topologically free action of an inverse semigroup or an étale groupoid on a C*-algebra is aperiodic.

**Proof.** We have explained above why an inverse semigroup action $(\mathcal{E}_t)_{t \in S}$ is topologically free or aperiodic if and only if the Hilbert bimodules $\mathcal{E}_t, I^*_t$, are topologically non-trivial or aperiodic for all $t \in S$, respectively. Hence for such actions the assertion follows from Theorem 4.7. Actions of étale groupoids may be rewritten through inverse semigroup actions, and this preserves the properties of aperiodicity and topological freeness (see Definition 2.11 and Example 4.6). Thus the statement for inverse semigroup actions implies the statements for actions of groups and étale groupoids.

**Remark 4.9.** By [20, Theorem 8.1], an aperiodic Hilbert $A$-bimodule is topologically non-trivial provided $A$ contains an essential ideal which is separable or of Type I. In the separable case, this also follows from Theorem 5.5 below.

The authors do not know an action that is aperiodic but not topologically free. We speculate, however, that the counterexamples to Naimark’s problem by Akemann and Weaver may give such examples. Assuming a certain axiom in set theory, Akemann and Weaver [2] build a non-separable C*-algebra $A$ such that $\hat{A}$ has only one point, although $\hat{A}$ is not isomorphic to a C*-algebra of compact operators on any Hilbert space. Then $\hat{A}$ is simple. By Kishimoto’s Theorem, an automorphism of $\hat{A}$ is aperiodic if and only if it is outer. No automorphism of $\hat{A}$ is topologically free because $\hat{A}$ has only one point. We do not know whether $\hat{A}$ admits outer automorphisms. If one exists, then it would give a Fell bundle over $\mathbb{Z}$ or $\mathbb{Z}/p$ for a prime number $p$ that is aperiodic and not topologically free.

The proof of Theorem 4.7 will occupy the rest of this section. We shall use the concept of a net excising a state from [1]. We only need pure states, and then an excising net may be arranged to have additional useful properties. To simplify notation, we will add these extra properties to the definition. We begin with some preparation. For a pure state $f$, let $\varphi: A \to \mathbb{B}(H)$ be the GNS representation for $f$ and let $\xi \in H$ be the cyclic vector with $\langle \xi, |f(a)\xi\rangle = f(a)$ for all $a \in A$. Identify $\varphi$ with a direct summand in the universal representation $\psi: A \to \mathbb{B}(K)$, and identify the bidual $A^{\ast\ast}$ of $A$ with the bicommutant of $A$ in $\mathbb{B}(K)$. Since $\varphi$ is irreducible, the
orthogonal projection onto $C \cdot \xi$ is a minimal projection in the bicommutant of $\varrho$ and hence in $A''$. We denote this minimal projection in $A''$ by $P_f$.

**Definition 4.10.** Let $A$ be a C*-algebra and $f: A \to C$ a pure state. A decreasing net $(a_n)_{n \in N}$ in $A^+_f$ strongly excises $f$ if $f(a_n) = 1$ for $n \in N$, $$\lim \|a_n x a_n - f(x) a_n^2\| = 0,$$ for $x \in A$, and $(a_n)_{n \in N}$ converges towards $P_f$ in the strong topology in the bicommutant $A'' \subseteq B(K)$.

**Proposition 4.11.** Let $f$ be a pure state on a C*-algebra $A$ and let $D \in \mathbb{H}(A)$ be such that $f|_D$ is also a state. Then there is a net $(a_n)_{n \in N}$ in $D$ that strongly excises $f$ as a state on $A$.

**Proof.** Since $f|_D$ is a state, there is $d \in D^+_f$ with $f(d) = 1$. The construction in [1, Proposition 2.2] applied to $d$ gives a decreasing net $(a_n)_{n \in N}$ in $D$ with $$\lim \|a_n x a_n - f(x) a_n^2\| = 0$$ for all $x \in A$ and $f(a_n) = 1$ for all $n \in N$. It is observed in [2] that this net converges towards $P_f$ in the strong topology. Indeed, the net $(a_n)_{n \in N}$ converges strongly because it is a decreasing net of positive elements. The strong limit cannot be 0 because $f(a_n) = 1$ for all $n \in N$. Then the end of the proof of [1, Proposition 2.3] shows that the strong limit of $(a_n)_{n \in N}$ must be $P_f$. □

**Lemma 4.12.** Let $A$ be a C*-algebra, $E$ a Hilbert $A$-bimodule, and $f: A \to C$ a pure state. Let $(a_n)_{n \in N}$ be a net that strongly excises $f$. Let $\varrho: A \to \mathbb{B}(\mathcal{H})$ be the GNS representation of $f$. Assume that $E \otimes_A \varrho$ is not unitarily equivalent to $\varrho$. Then $$\lim \|a_n x a_n\| = 0 \quad \text{for all} \quad x \in \mathcal{E}.$$ 

**Proof.** Let $\xi \in H$ be the vector with $\langle \xi \mid \varrho(a) \xi \rangle = f(a)$ for all $a \in A$. For any $x \in H$, the vector $x \otimes \xi \in E \otimes_A H$ defines a positive linear functional $$g: A \to C, \quad a \mapsto \langle x \otimes \xi \mid a \cdot x \otimes \xi \rangle = \langle \xi \mid \varrho(\langle x \mid a \cdot x \rangle) \xi \rangle = f(\langle x \mid a \cdot x \rangle).$$ By assumption, the left multiplication representation of $A$ on $E \otimes_A \varrho$ is not unitarily equivalent to $\varrho$. Then the extension of this representation to $A''$ maps the projection $P_f$ to 0. The strong convergence $\lim a_n = P_f$ in the universal representation implies $\varrho(a_n) = 0$. Equivalently, $\lim f(\langle x \mid a_n \cdot x \rangle_A) = 0$. This equation generalises the claim in the middle of the proof of [2, Lemma 1]. From this point on, we closely follow the proof of [2, Lemma 1]. Let $\varepsilon > 0$. There is $n_0 \in N$ with $f(\langle x \mid a_n \cdot x \rangle_A) < \varepsilon/2$. Let $y := \langle x \mid a_{n_0} \cdot x \rangle_A \in A$. There is $n_1 \in N$ with $\|a_n(y - f(y))a_n\| < \varepsilon/2$ for $n \geq n_1$ because $(a_n)_{n \in N}$ excises $f$. If $n \geq n_0, n_1$, then $a_n \approx a_{n_0}$. We estimate $$\|a_n x a_n\|^2 \leq \|a_n^{1/2} x a_n\|^2 = \|\langle a_n^{1/2} x a_n \mid a_n^{1/2} x a_n \rangle\| = \|a_n \cdot \langle x \mid a_n x A \cdot a_n\| \leq \|a_n \cdot \langle x \mid a_n x A \cdot a_n\| \leq \|a_n(y - f(y))a_n\| + f(y)\|a_n^2\| < \varepsilon.$$

This finishes the proof. □

**Proof of Theorem 4.17.** Let $x \in E$ and $D \in \mathbb{H}(A)$. We check Kishimoto’s condition for this data. Let $ADA \subseteq A$ be the two-sided ideal generated by $D$. If $s(E) \cap ADA = 0$, then $E \cdot D = 0$. Then $\|x \cdot a\| = 0$ for any $a \in D^+_f$. This implies Kishimoto’s condition for $x$ and $D$. So we may assume $s(E) \cap ADA \neq 0$. Since $E$ is topologically non-trivial, there is an irreducible representation $\varrho: A \to \mathbb{B}(H)$ which is non-zero on $s(E) \cap ADA$ and such that the (irreducible) representation of $A$ on $E \otimes_A H$ by left multiplication is not unitarily equivalent to $\varrho$. Since $\varrho(ADA)H = H$, we get $\varrho(D)H \neq 0$. Let $\xi \in \varrho(D)H$ be a unit vector. Then $f: A \to C$, $a \mapsto \langle \xi \mid \varrho(a) \xi \rangle$, is a pure state that restricts to a state on $D$ and whose GNS representation is equivalent to $\varrho$. Hence Proposition 4.11 gives a net $(a_n)_{n \in N}$ in $D$ that strongly excises $f$. Then Lemma 4.12 shows that $\lim \|a_n x a_n\| = 0$. Thus $x$ satisfies Kishimoto’s condition. □
5. Aperiodicity and the almost extension property

In this section we will relate aperiodicity and topological freeness to the almost extension property introduced in [25]. The latter is a weakening of the extension property introduced in [3], which we also discuss.

**Definition 5.1 ([3, 25]).** Let \( A \subseteq B \) be a C*-inclusion. Let \( P_1(A \uparrow B) \) be the set of all pure states on \( A \) that extend uniquely to a state on \( B \). The C*-inclusion has the **almost extension property** if \( P_1(A \uparrow B) \) is weak-\( \ast \)-dense in the set \( P(B) \) of all pure states on \( A \). It has the **extension property** if \( P_1(A \uparrow B) = P(A) \).

Whether a pure state extends uniquely depends only on its GNS representation:

**Lemma 5.2.** Let \( f_1 \) and \( f_2 \) be two pure states on a C*-algebra \( A \). If their GNS representations are unitarily equivalent and \( f_1 \in P_1(A \uparrow B) \), then \( f_2 \in P_1(A \uparrow B) \).

**Proof.** By [30, Proposition 3.13.4], \( f_1 \) and \( f_2 \) have equivalent GNS representations if and only if there is a unitary \( u \in A + \mathbb{C} \cdot 1 \) with \( f_2(a) = f_1(uau^*) \) for all \( a \in A \). Clearly, the subset \( P_1(A \uparrow B) \) is invariant under conjugation by unitaries in \( A + \mathbb{C} \cdot 1 \).

The above lemma allows to replace the weak-\( \ast \)-density in \( P(A) \) in Definition 5.1 by a number of other conditions:

**Proposition 5.3.** The following are equivalent:

1. \( P_1(A \uparrow B) \) is weak-\( \ast \)-dense in \( P(A) \);
2. for every non-zero \( a \in A^\times \) there is \( f \in P_1(A \uparrow B) \) with \( f(a) \neq 0 \);
3. for each ideal \( I \not\subseteq A \) with \( I \neq 0 \), there is \( f \in P_1(A \uparrow B) \) with \( f|_I \neq 0 \);
4. the direct sum of the GNS representations for all \( f \in P_1(A \uparrow B) \) is faithful;
5. the image of \( P_1(A \uparrow B) \) is dense in \( \hat{A} \).

**Proof.** For every non-zero \( a \in A^\times \), there is \( f \in P(A) \) with \( f(a) \neq 0 \). If \( P_1(A \uparrow B) \) is dense in \( P(A) \), then \( f \) is the weak-\( \ast \)-limit of a net \( \{f_n\} \) in \( P_1(A \uparrow B) \). Then \( f_n(a) \neq 0 \) for some \( n \). So [1] implies [2]. The implications [2] \( \Rightarrow \) [3] \( \Rightarrow \) [4] are obvious.

For \( f \in P(A) \), let \( \varrho_f \) be its GNS representation. Condition [5] implicitly uses the map \( q : P(A) \to \hat{A}, f \mapsto [\varrho_f] \). Open subsets of \( \hat{A} \) are of the form \( f \) for ideals \( I \subseteq A \). So the density in [5] means that for each \( I \in \mathcal{I}(A) \) with \( I \neq 0 \), there is \( f \in P_1(A \uparrow B) \) with \( [\varrho_f] \neq 0 \). This follows from [4]. Moreover, the map \( q \) is continuous, open and surjective (see, for instance, [30, Theorem 4.3.3]). By Lemma 5.2, \( P_1(A \uparrow B) \) is the preimage of its image in \( \hat{A} \). Therefore, [1] and [5] are equivalent.

The following criterion by Anderson for a pure state to extend uniquely is similar to Kishimoto’s condition:

**Theorem 5.4 ([3, Theorem 3.2]).** Let \( A \subseteq B \) be a C*-inclusion. A pure state \( f : A \to \mathbb{C} \) extends uniquely to \( B \) if and only if for each \( x \in B \) and \( \varepsilon > 0 \), there is \( a \in A^+_1 \) with \( \|axa\|_{B/A} < \varepsilon \) and \( f(a) = 1 \).

By the definition of the quotient norm, \( \|axa\|_{B/A} < \varepsilon \) means that there is \( y \in A \) with \( \|axa - y\|_B < \varepsilon \).

Both Kishimoto’s condition and Anderson’s criterion for the almost extension property ask that for a given \( x \in B \) and \( \varepsilon > 0 \) there should be \( a \in A^+_1 \) with \( \|axa\|_{B/A} < \varepsilon \). In addition, Kishimoto’s condition asks that \( a \) may be taken from a specific non-zero hereditary subalgebra, whereas the almost extension property asks that \( f(a) = 1 \) for a given state \( f \), belonging to a weak-\( \ast \)-dense set of states.

The following theorem is the first main result of this section:
**Theorem 5.5.** C*-inclusions with the almost extension property are aperiodic. If B is separable, then A ⊆ B is an aperiodic inclusion if and only if A ⊆ B has the almost extension property.

**Proof.** Assume first that the inclusion A ⊆ B has the almost extension property. Let x ∈ B, ε > 0, and D ∈ ℋ(A). We are going to check Kishimoto’s condition for this data. Let ADA denote the two-sided ideal generated by D. Since we assumed the almost extension property, there is a pure state in P₁(A ↑ B) whose GNS representation g: A → ℋ(ℋ) belongs to AADA. Equivalently, g|D ≠ 0. Choose a unit vector ξ ∈ g(D)ℋ and let f be its vector state. It belongs to P₁(A ↑ B) by Lemma 5.2. The restriction f|D is a state whose associated cyclic representation is the restriction of g to D acting on ℋ(D)ℋ with cyclic vector ξ. This representation is the image of g|ADA under the Rieffel correspondence for the Morita–Rieffel equivalence between ADA and D. Thus it is again irreducible. The state f|D extends uniquely to A by [20, Lemma 2.9]. Since f extends uniquely to B, the state f|D belongs to P₁(D ↑ B). Anderson’s criterion in Theorem 5.4 gives a ∈ D₁⁺ with ||xx₁||₁ < ε and (f(a) = 1). This implies Kishimoto’s condition for x.

Conversely, let A ⊆ B be an aperiodic inclusion and let B be separable. We follow the proof of [23, Proposition 6.5] to show that the inclusion has the almost extension property. Let (xₙ)ₙ∈ℕ be a dense sequence in the unit ball of B. Let D ∈ ℋ(A). We recursively construct a sequence (eₙ)ₙ∈ℕ ∈ D₁⁺ such that

\[ eₙeₙ₊₁ = eₙ₊₁ \quad \text{and} \quad \|eₙxₙeₙ\|₁/B₁ < 2⁻ⁿ \]

for all n ∈ ℕ. To this end, we simultaneously construct auxiliary elements dₙ, yₙ ∈ D₁⁺ with eₙyₙ = yₙ and eₙyₙ ∈ C*(dₙ). Pick any d₀ ∈ D₁⁺. The functional calculus for d₀ gives elements e₀, y₀ ∈ C*(d₀) ⊆ D₁⁺ with e₀y₀ = y₀ as in the proof of [20, Lemma 2.9]. The estimate ||e₀x₀e₀||₁/B₁ < 2⁻⁰ is trivial. Assume eₙ, yₙ have been constructed as above. Let Dₙ₊₁ := \{z ∈ D : z eₙ = eₙz = z\}. This is a non-zero hereditary C*-subalgebra because it contains yₙ ≠ 0. Since B/A is aperiodic, there is dₙ₊₁ ∈ (Dₙ₊₁)₁⁺ with ||dₙ₊₁xₙ₊₁dₙ₊₁||₁/B₁ < 2⁻ⁿ⁻¹. As above, we use [20, Lemma 2.9] to choose elements eₙ₊₁, yₙ₊₁ ∈ C*(dₙ₊₁)₁⁺ with eₙ₊₁yₙ₊₁ = yₙ₊₁. Given ε > 0, we may choose ε > 0, we may choose eₙ₊₁ = f(dₙ₊₁) · dₙ₊₁ with ||f||₁ < 1 + ε. Thus we may choose eₙ₊₁ to also satisfy ||eₙ₊₁xₙ₊₁eₙ₊₁||₁/B₁ < 2⁻ⁿ⁻¹. This completes the recursion step.

Let K₀ := \{f ∈ A⁺ : ||f||₁ ≤ 1, f ≠ 0, f(eₙ) = 1\}. Any element of K₀ is a state of A, and the definition implies that K₀ is closed in the weak*-topology and contained in the unit ball. Therefore, K₀ is compact. If a convex combination of two states f, g belongs to K₀, then f, g ∈ K₀ because f(eₙ) ≤ 1 and g(eₙ) ≤ 1 for all states. Thus K₀ is a compact facet of the set of states. If f ∈ K₀, then eₙ fixes the cyclic vector in the GNS representation of f. This implies f(y) = f(eₙy) = f(yeₙ) for all y ∈ A. Then f(eₙ₋₁) = f(eₙ₋₁eₙ) = f(eₙ) = 1. This shows that K₀ ⊆ K₋₁. Then the intersection ∩ K_n of the decreasing chain of compact facets Kₙ of the state space of A must be a non-empty facet. Thus it contains a pure state ϕ. By construction, ϕ(eₙ) = 1 for all n ∈ ℕ.

Let x ∈ B and ε > 0. We claim that there is n ∈ ℕ with ||eₙxₙeₙ||₁/B₁ < ε. Rescaling x, we may assume without loss of generality that ||x|| ≤ 1. Since \{xₙ\} is dense in the unit ball of B, there is a subsequence (n(k))ₖ∈ℕ with limₖ xₙ(k) = x and limₖ(xₙ(k)) = ∞. There is k₀ ∈ ℕ with ||xₙ(k)|| < ε/2 for k ≥ k₀. There is k ≥ k₀ with 2⁻ⁿ(k) < ε/2. Then

\[ ||eₙ(k)xₙ(k)||₁ ≤ ||eₙ(k)(x - xₙ(k))eₙ(k)||₁ + ||eₙ(k)xₙ(k)eₙ(k)||₁ < ε/2 + 2⁻ⁿ(k) < ε. \]

By Anderson’s criterion in Theorem 5.4, the claim that we have just shown implies that ϕ extends uniquely to a state on B. Since ||ϕ||₁ = 1 by construction and D
was an arbitrary hereditary C*-subalgebra, this shows that the inclusion \( A \subseteq B \) has the almost extension property (see Proposition 5.3).

Remark 5.6. The separability assumption in the second part of Theorem 5.5 is needed, see Example 5.15 below.

Next we study when pure states extend uniquely to crossed products. We first work in the generality of inverse semigroup actions by Hilbert bimodules. Then we specialise to Fell bundles over groups and étale groupoids. The following proofs are inspired by the proof of [2] Theorem 2.1).

**Proposition 5.7.** Let \( \mathcal{E} \) be an action of a unital inverse semigroup \( S \) on a C*-algebra \( A \) by Hilbert bimodules. Let \( A \subseteq B \) be a C*-inclusion with a surjective *-homomorphism \( A \rtimes S \to B \) which restricts to the identity on \( A \). Let \( f \) be a pure state on \( A \) and let \( \varrho: A \to \mathcal{B}(\mathcal{H}) \) be its GNS representation. If \( [\varrho] \) has trivial isotropy in the dual groupoid of the action \( \mathcal{E} \), then \( f \in P_1(A \uparrow B) \).

**Proof.** Let \( N_{[\varrho]} \) be the directed set of open neighbourhoods of \([\varrho]\). Each element of \( N_{[\varrho]} \) has the form \( \tilde{J} \) for an ideal \( J \) in \( A \) with \( \varrho|_J \neq 0 \); equivalently, \( f|_J \) is a pure state on \( J \). Proposition 4.11 gives a net \((a_n)_{n \in N_J} \) in \( J \) that strongly excises \( f \).

We combine all these nets, indexing them by the directed set of open neighbourhoods of \([\varrho]\). The result is a net \((a_n)_{n \in N} \) in \( A \) that strongly excises \( f \) and such that for each \( n \in N \) there is \( n_0 \in N \) with \( a_n \in J \) for all \( n \geq n_0 \). We claim that \( \|a_nxa_n\|_{B/A} = 0 \) for all \( x \in B \).

The subset of elements \( x \in B \) with \( \|a_nxa_n\|_{B/A} = 0 \) is a norm closed vector subspace. Since the images of \( \mathcal{E}_t \) in \( B \) for \( t \in S \) are linearly dense, it suffices to check the claim for \( t \in S \) and \( x \in \mathcal{E}_t \). If \( t \) is such that \( \mathcal{E}_t \otimes \varrho \) is not unitarily equivalent to \( \varrho \), then the claim follows from Lemma 4.12. So assume \( \mathcal{E}_t \otimes \varrho \cong \varrho \). Since we assumed \([\varrho]\) to have trivial isotropy in the dual groupoid, it follows that \([\varrho]\) \( \in \tilde{I}_{1,t} \).

(For a group action, this only happens for \( t = 1 \).) Then there is \( n_0 \in N \) with \( a_n \in \tilde{I}_{1,t} \) for all \( n \geq n_0 \). Thus \( a_nxa_n \in \mathcal{E}_t \otimes \tilde{I}_{1,t} \). This is identified in \( A \rtimes S \) and hence in \( B \) with \( \tilde{I}_{1,t} \subseteq A \). So \( \|a_nxa_n\|_{B/A} = 0 \) for all \( n \geq n_0 \). This proves the claim. Then \( f \) extends uniquely to a state on \( A \rtimes S \) by Anderson’s criterion in Theorem 5.4.

**Proposition 5.8.** Let \( \mathcal{E} \) be an action of a unital inverse semigroup \( S \) on a C*-algebra \( A \) by Hilbert bimodules. Let \( A \subseteq B \) be a C*-inclusion with a surjective *-homomorphism \( B \to A \rtimes S \) that restricts to the canonical inclusion \( A \to A \rtimes S \).

Let \( f \) be a pure state on \( A \) and let \( \varrho: A \to \mathcal{B}(\mathcal{H}) \) be its GNS representation. If \([\varrho]\) \( \in A \) has non-trivial isotropy in the dual groupoid of the action \( \mathcal{E} \), then \( f \notin P_1(A \uparrow B) \).

**Proof.** By assumption, there is \( t \in S \) with \( \tilde{E}_t(a) = [\varrho] \) and \([\varrho] \notin \tilde{I}_{1,t} \) for the ideal \( I_{1,t} \) defined in (2.1); if \( S \) is a group, this simply means \( t \neq 1 \). Then \( \mathcal{E}_t \otimes \varrho \mathcal{H} = \{0\} \), so that there is \( x \in \mathcal{E}_t \) with \( x \otimes \xi \neq 0 \). The left multiplication representation of \( A \) on \( \mathcal{E}_t \otimes \varrho \mathcal{H} \) is unitarily equivalent to \( \varrho \). Since \( \varrho \) is irreducible, Kadison’s Transitivity Theorem allows us to choose an element \( a \in A \) so that the unitary intertwiner between these representations maps \( a \cdot (x \otimes \xi) \) to the canonical cyclic vector \( \xi \). We could have picked \( a \cdot x \) instead of \( x \) from the beginning, and we assume this to simplify notation. Then the unitary intertwiner maps \( x \otimes \xi \) to \( \xi \). So both vectors define the same vector state on \( A \). That is,

\[
(5.1) \quad f(a) = \langle \xi | \varrho(a)\xi \rangle = \langle x \otimes \xi | a \cdot x \otimes \xi \rangle = \langle \xi | \varrho(\langle x \mid a \cdot x \rangle)\xi \rangle = f(\langle x \mid a \cdot x \rangle)
\]

for all \( a \in A \). Let \( \tilde{x} \in B \) be a pre-image for \( x \) under the surjective map \( B \to A \rtimes S \).

We claim that \( \|a\tilde{x}a\|_{B/A} \geq 1 \) holds for all \( a \in A_\mathcal{E}_t \) with \( f(a) = 1 \).

The GNS representation \( \varrho \) of \( f \) induces a representation of \( A \rtimes S \) and thus a representation \( \omega \) of \( B \). We are going to verify \( \|\omega(a\tilde{x}a - y)\| \geq 1 \) for all \( y \in A \).
and \( a \in A_1^+ \) with \( f(a) = 1 \); this will finish the proof of the theorem. To build the representation \( \omega \), we first extend the \( S \)-action on \( A \) to the bidual \( A'' \). (This step is not needed in for group actions because then \( E \) takes values in \( A \).) We use the canonical conditional expectation \( E : A'' \rightarrow A'' \) to form a Hilbert \( A'' \)-module \( \ell^2(S, A'') \). Then \( \omega \) is the left multiplication action on the tensor product 

\[
K := \ell^2(S, A'') \otimes_{A''} (H, \varrho).
\]

Since \( A'' \) is unital, the Hilbert space \( K \) contains a copy of \( H \) of the form \( 1 \otimes H \). Let \( a, y \in A \). Then \( axa - y \in A \otimes_{\text{alg}} S \) maps the unit vector \( 1 \otimes \xi \) to the vector \( axa \otimes \xi - y \otimes \xi \) in \( K \). We claim that the summands \( axa \otimes \xi \) and \( y \otimes \xi \) are orthogonal. Indeed, their inner product is defined to be

\[
\langle \xi \mid \varrho^o \circ E((axa)^* \cdot y) \xi \rangle = f \circ E((axa)^* \cdot y).
\]

This vanishes because the expectation \( E \) multiplies with the support projection \( [I, I] \) of the ideal \( I_1 \), which is killed by \( f \) because \( \varrho(\xi) \notin I_1 \). So

\[
\|\omega(axa - y)\| \geq \|axa - y(1 \otimes \xi)\| = f(\langle axa \mid axa \rangle)^{1/2}.
\]

Recall that we assume \( f(a) = 1 \). Then \( \|\varrho(a)\xi\| = 1 \) and \( \langle \varrho(a)\xi \mid \xi \rangle = 1 \), and this implies \( \varrho(a)\xi = \xi \). So \( f(ay) = f(a) = f(ay) \) for all \( y \in A \). Using this and \( [3, 1] \), we compute

\[
f(\langle axa \mid axa \rangle) = f(\langle axa \mid axa \rangle) = f(\langle axa \mid axa \rangle) = f(\langle axa \mid axa \rangle) = f(a^2) = 1.
\]

This finishes the proof of the claim. Then \( f \) has more than one extension to a state on \( B \) by Anderson’s criterion in Theorem \( 5.4 \). \( \square \)

**Definition 5.9.** A groupoid is called **principal** if all points in \( G^0 \) have trivial isotropy. A topological groupoid \( G \) is called **topologically principal** if the set of \( x \in G^0 \) with trivial isotropy group is dense in \( G^0 \).

**Remark 5.10.** If an étale groupoid \( G \) is topologically principal, then it is topologically free. The converse holds when \( G \) has a countable cover by bisections and the unit space \( X \) contains a dense Hausdorff Baire space (see \cite{23} Corollary 2.26)). The case when \( X \) is not Hausdorff is more subtle (see the comment before \cite{23} Proposition 2.24)). In particular, the proof of \cite{34} Proposition 3.6.\( \text{(ii)} \) has a gap.

**Theorem 5.11.** Let \( S \) be a unital inverse semigroup that acts on a \( C^* \)-algebra \( A \) by Hilbert bimodules. Let \( B \) be a \( C^* \)-algebra with surjective *-homomorphisms \( A \times S \rightarrow B \rightarrow A \times S \) that compose to the quotient map \( A \times S \rightarrow A \times S \).

1. A pure state \( f \in \mathcal{P}(A) \) belongs to \( P_l(A \uparrow B) \) if and only if the GNS representation of \( f \) has trivial isotropy in the dual groupoid \( \hat{A} \times S \).

2. The inclusion \( A \rightarrow B \) has the almost extension property if and only if the dual groupoid \( \hat{A} \times S \) is topologically principal.

3. The inclusion \( A \rightarrow B \) has the extension property if and only if the dual groupoid is principal.

**Proof.** Propositions \( 5.7 \) and \( 5.8 \) combined give \( [1] \). Statement \( [1] \) implies \( [3] \). It also implies \( [2] \) because \( P_l(A \uparrow B) \) is dense in \( P(A) \) if and only if its image is dense in \( \hat{A} \) by Proposition \( 5.3 \). \( \square \)

Theorem \( 5.11 \) generalises a result of Zarikian for group actions by automorphisms on unital \( C^* \)-algebras (see \cite{37} Theorem 2.4)).

**Corollary 5.12.** Let \( \mathcal{A} \) be a Fell bundle over an étale groupoid \( G \) with a locally compact Hausdorff unit space \( X \). Put \( A := C_0(X, \mathcal{A}|X) \) and let \( B \) be a \( C^* \)-algebra with surjective maps \( C^*(G, A) \rightarrow B \rightarrow C^*_l(G, A) \) that compose to the quotient map \( C^*(G, A) \rightarrow C^*_l(G, A) \).
Remark 5.13. Assume the dual groupoid $\hat{A} \times G$ of a Fell bundle over an étale, locally compact groupoid gives rise to an action of an inverse semigroup such that the full, reduced, and essential crossed products and the dual groupoids are the same (see [23, Section 7]). The inverse semigroup is generated by bisections of $G$. So we may choose it to be countable if $G$ is covered by countably many bisections. Now all claims follow from Theorem 5.11. □

Proof. A Fell bundle over an étale, locally compact groupoid gives rise to an action of an inverse semigroup such that the full, reduced, and essential crossed products and the dual groupoids are the same (see [23, Section 7]). The inverse semigroup is generated by bisections of $G$. So we may choose it to be countable if $G$ is covered by countably many bisections. Now all claims follow from Theorem 5.11. □

Remark 5.13. Assume the dual groupoid $\hat{A} \times G$ of a Fell bundle over an étale, locally compact groupoid $G$ to be principal. Then [23, Lemma 7.15 and Proposition 7.18] imply that $C^*_\text{ess}(G, A) = C^*_{\text{ess}}(G, A)$.

In Proposition 5.7, $B$ may be $A \times S$, $A \rtimes S$, or $A \rtimes_{\text{ess}} S$. In fact, it may be any $S$-graded $C^*$-algebra $B$ ([21, Definition 6.15]). The $S$-grading is a family of closed subspaces $(B_t)_{t \in S}$ with $B_t = B_{t+}$ and $B_t \cdot B_u = B_{t+u}$ for all $t, u \in S$ and $\sum B_t = B$. Then the Banach spaces $B_t$ with the multiplication and involution from $B$ define an action of $S$ on $A$ by Hilbert bimodules. The inclusion maps $B_t \hookrightarrow B$ induce a canonical surjective $^*$-homomorphism from the crossed product for this action to $B$.

In contrast, Proposition 5.8 may fail for essential crossed products. That is, a pure state $f$ on $A$ may extend uniquely to the essential crossed product $A \rtimes_{\text{ess}} S$ without extending uniquely to $A \rtimes S$. We do not know how to characterise which pure states extend uniquely to $A \rtimes_{\text{ess}} S$. Even the equivalence in Theorem 5.11(2) may fail, as the following counterexample shows:

Example 5.14. There is a non-Hausdorff groupoid $G$ such that $C^*_{\text{ess}}(G) = C_0(G^0)$ although $G$ is not topologically principal. The construction starts with the uncountable groupoid $\Gamma := \bigoplus_{\gamma \in [0,1]} \mathbb{Z}/2$. Equip the trivial group bundle $[0,1] \times \Gamma$ over $[0,1]$ with the equivalence relation defined by

$$\left( t, \sum_{s \in [0,1]} a_s \right) \sim \left( x, \sum_{s \in [0,1]} b_s \right) \iff x = t \text{ and } a_t = b_t.$$

Let $G$ be the quotient of $[0,1] \times \Gamma$ by $\sim$, equipped with the quotient topology. This is a group bundle over $[0,1]$ with fibres $\mathbb{Z}/2$ at all points. Let $q : [0,1] \times \Gamma \to G$ be the quotient map. The sets $q([0,1] \times \{ \gamma \})$ for $\gamma \in \Gamma$ are open bisections of $G$ that cover $G$. So $G$ is an étale, non-Hausdorff groupoid. It is not topologically principal because each isotropy group is isomorphic to $\mathbb{Z}/2$. The unit bisection $G^0$ is dense in $G$ because it intersects $q([0,1] \times \{ \gamma \})$ for each $\gamma \in \Gamma$. It follows that the essentially defined conditional expectation $C^*(G) \to M_{\text{loc}}(C[0,1])$ is a $^*$-homomorphism to $C[0,1]$. Thus the inclusion map $C[0,1] \hookrightarrow C^*_{\text{ess}}(G)$ becomes an isomorphism. As a result, any (pure) state on $C[0,1]$ extends uniquely to $C^*_\text{ess}(G)$, although $G$ is not topologically principal.

It is crucial that the groupoid in Example 5.14 is not covered by a countable family of bisections. Indeed, Theorem 7.2 below implies the equivalence in Theorem 5.11(2) for all exotic crossed products provided $S$ is countable and $A$ contains an essential ideal that is separable or of Type I. In fact, under these assumptions the almost extension property is equivalent to a number of conditions including topological freeness and aperiodicity.
Example 5.15. Let $G$ be the group of affine isometries of $\mathbb{R}$. It is generated by translations and one reflection and therefore isomorphic to $\mathbb{R} \times \mathbb{Z}/2$. Give $G$ the discrete topology. The transformation groupoid for the action of $G$ on $\mathbb{R}$ is topologically free because each element of $G\setminus\{1\}$ fixes at most one point in $\mathbb{R}$. It is not topologically principal because each point in $\mathbb{R}$ is fixed by some element of $G\setminus\{1\}$. Our results show that the induced action of $G$ on $C_0(\mathbb{R})$ is topologically free and aperiodic. There is, however, no pure state on $C_0(\mathbb{R})$ that extends uniquely to $C_0(\mathbb{R}) \times G$.

6. Detection of ideals in intermediate algebras

The work of Olsen–Pedersen [26–28] shows that for actions of the group $\mathbb{Z}$ or $\mathbb{Z}/p$ for a square-free number $p$ on a separable $C^*$-algebra $A$, aperiodicity and topological freeness are not only sufficient, but also necessary for $A$ to detect ideals in the crossed product (see also [20]). In this section, we use this to prove that a stronger condition is always sufficient for an action to be topologically free. Namely, we require $A$ to detect ideals in all intermediate $C^*$-algebras between $A$ and the essential crossed product. It is natural to strengthen ideal detection in this way because the uniqueness of pseudo-expectations and aperiodicity are hereditary for such intermediate inclusions.

Proposition 6.1. Let $E$ be an action of a unital inverse semigroup $S$ on a $C^*$-algebra $A$. Assume that $A$ contains an essential ideal that is separable or of Type I. If $A$ detects ideals in $C$ for any $A \subseteq C \subseteq A \rtimes_{\text{ess}} S$, then the action $E$ is topologically free. In fact, it suffices to assume that $A$ detects ideals in $A \rtimes_{\text{ess}} T$ for any inverse subsemigroup $T \subseteq S$ that is generated by the idempotents in $S$ together with a single $t \in S$.

Before we prove this, we explain why $A \rtimes_{\text{ess}} T$ is contained in $A \rtimes_{\text{ess}} S$:

Lemma 6.2. Let $T \subseteq S$ be an inverse subsemigroup. If $T$ contains all idempotents in $S$, then $A \rtimes_{\text{ess}} T \subseteq A \rtimes_{\text{ess}} S$.

Proof. The inclusion $T \subseteq S$ induces a canonical $^*$-homomorphism $j: A \rtimes T \to A \rtimes S$. The issue is to prove that it descends to an injective $^*$-homomorphism between the essential crossed products. If $T$ contains all idempotents in $S$, then it follows that the ideals $I_{T,t}$ for $t \in T$ defined in (2.1) are the same when computed in $S$ or $T$. The generalised expectations $E^T: A \rtimes T \to \mathcal{M}_{\text{loc}}(A)$ and $E^S: A \rtimes S \to \mathcal{M}_{\text{loc}}(A)$ are defined in [23] Proposition 4.3. Since the ideals $I_{T,t}$ are the same in both cases, it follows easily that $E^S \circ j = E^T$. The generalised expectations on $A \rtimes_{\text{ess}} S$ and $A \rtimes_{\text{ess}} T$ induced by $E^S$ and $E^T$ are faithful by [23] Theorem 4.11. That is, $j(x)$ becomes 0 in $A \rtimes_{\text{ess}} S$ if and only if $E^S(j(x^*x)) = 0$, if and only if $E^T(x^*x) = 0$, if and only if $x$ becomes 0 in $A \rtimes_{\text{ess}} T$. This means that $j$ descends to an injective map $A \rtimes_{\text{ess}} T \to A \rtimes_{\text{ess}} S$. □

Remark 6.3. If an intermediate $C^*$-algebra $A \subseteq C \subseteq A \rtimes_{\text{ess}} S$ is equal to $A \rtimes_{\text{ess}} T$ for some $T$ as in Lemma 6.2 then it is $S$-graded, that is, $C$ is the closed linear span of $C \cap E_t$ for $t \in S$.

Proof of Proposition 6.1. We proceed by contradiction and assume that the action $E$ is not topologically free. Then, by Lemma 4.5, there are $t \in S$ and a non-zero ideal $J \subseteq I_{T,t}$ such that the Hilbert $A$-bimodule $F := E_t \cdot J$ is topologically trivial. That is, $F \otimes_A g \cong g$ for all irreducible representations $g$ of $J$. The key step in the proof is based on [20] Theorem 9.12 which, in turn, is based on the work of Olsen–Pedersen in [26–28]. That theorem is about a section $C^*$-algebra $C$ for a Fell bundle over $\mathbb{Z}$ or $\mathbb{Z}/p$ for a square-free number $p$ whose unit fibre $A$ contains an essential ideal that
is separable or of Type I. Then the inclusion \( A \subseteq C \) is topologically free if and only if \( A \) detects ideals in \( C \). We are going to build a Fell bundle over \( \mathbb{Z} \) or \( \mathbb{Z}/p \) from \( \mathcal{F} \) in such a way that its section \( \mathcal{C}^{*} \)-algebra \( C \) is an intermediate \( \mathcal{C}^{*} \)-algebra between \( A \) and \( A \rtimes_{\text{ess}} S \).

Let \( \mathcal{F}_{0} := J_{k}, \mathcal{F}_{k} := \mathcal{F} \otimes_{\alpha} k \) and \( \mathcal{F}_{-k} := (\mathcal{F}^{*}) \otimes_{\alpha} k \) for \( k > 0 \). These are slices for the inclusion \( A \subseteq A \rtimes_{\text{ess}} S \), and they form a Fell bundle over \( \mathbb{Z} \) with the multiplication and inclusion in \( A \rtimes_{\text{ess}} S \). The maps \( \mathcal{F}_{k} \rightharpoonup A \rtimes_{\text{ess}} S \) form a Fell bundle representation. They induce a \( * \)-homomorphism \( \varphi_{0} : \mathcal{C}^{*}(\mathbb{Z}, (\mathcal{F}_{k})) \to A \rtimes_{\text{ess}} S \). Assume first that \( \varphi_{0} \) is injective. By assumption, the Hilbert bimodule \( \mathcal{F}_{1} \) is topologically trivial. Hence the inclusion \( J \rightharpoonup \mathcal{C}^{*}(\mathbb{Z}, (\mathcal{F}_{j})) \) cannot detect ideals by [20] Theorem 9.12. Next, we replace \( \mathcal{C}^{*}(\mathbb{Z}, (\mathcal{F}_{j})) \) by an intermediate \( \mathcal{C}^{*} \)-algebra of the form \( A \rtimes_{\text{ess}} T \). Namely, we let \( T \) be the inverse subsemigroup generated by \( t \) and the idempotent elements of \( S \). So all elements of \( T \) are of the form \( t^{k} \cdot \varepsilon \) for some \( k \in \mathbb{Z} \) and an idempotent element \( \varepsilon \) of \( S \). Now \( J \subseteq A \) is an invariant ideal for the action of \( T \) on \( A \) and \( J \cdot (A \rtimes_{\text{ess}} T) = (A \rtimes_{\text{ess}} T) \cdot J = \mathcal{C}^{*}(\mathbb{Z}, (\mathcal{F}_{j})) \) (see [21] Proposition 6.19). Therefore, since \( J \rightharpoonup \mathcal{C}^{*}(\mathbb{Z}, (\mathcal{F}_{j})) \) does not detect ideals, neither does \( A \subseteq A \rtimes_{\text{ess}} T \). This finishes the proof in the case where \( \varphi_{0} \) is injective.

It remains to study the case when \( \varphi_{0} \) is not injective. Let \( E : A \rtimes_{\text{ess}} S \to \mathcal{M}_{\text{loc}}(A) \) be the canonical essentially defined expectation. By definition, if \( u \in S \), then \( E \) is the identity map on \( \mathcal{E}_{u} \cap I_{u,1} \) and vanishes on \( \mathcal{E}_{u} \cap I_{u,1}^{\perp} \). By construction, \( \mathcal{F}_{k} \subseteq \mathcal{E}_{t} \) for \( k \geq 0 \) and \( \mathcal{F}_{1} \subseteq \mathcal{E}_{t} \cap I_{t,1} \). We claim that there must be \( k \geq 2 \) for which \( \mathcal{F}_{k} \subseteq \mathcal{E}_{t} \cdot I_{t,1}^{k} \). Otherwise, \( \mathcal{F}_{k} \subseteq \mathcal{E}_{t} \cdot I_{t,1}^{k} \) holds for all \( k \geq 1 \). Then \( E|_{\mathcal{F}_{1}} = 0 \) for all \( k > 0 \). Then \( E \circ \varphi_{0} \) is equal to the canonical conditional expectation \( \mathcal{C}^{*}(\mathbb{Z}, (\mathcal{F}_{j})) \to A \). The latter is faithful because \( \mathbb{Z} \) is amenable, so \( \varphi_{0} \) is injective. Hence there are \( k \geq 1 \) for which \( \mathcal{F}_{k} \) is not contained in \( \mathcal{E}_{t} \cdot I_{t,1}^{k} \). We pick the minimal such \( k \). So \( E|_{\mathcal{F}_{k}} = 0 \) for \( n = 1, \ldots, k-1 \) and \( \mathcal{F}_{k} \cap \mathcal{E}_{t} \cdot I_{t,1}^{k} \neq 0 \). This intersection is equal to \( \mathcal{E}_{t} \cdot K \) for some non-zero ideal \( K \subseteq I_{t,1} \cap J \). We replace \( J \) by \( K \). This improves matters in such a way that \( \mathcal{F}_{k} \subseteq \mathcal{E}_{t} \cdot I_{t,1}^{k} \). This is contained in \( A \subseteq A \rtimes_{\text{ess}} S \), and it is, in fact, equal to the chosen ideal \( K \). It follows that \( \mathcal{F}_{n} : \mathcal{F}_{k} = \mathcal{F}_{n} \) for all \( n \geq 0 \). Then \( \mathcal{F}_{n} = I_{t,1} \), \( n = 1, \ldots, k \) is a Fell bundle over \( \mathbb{Z}/k \) and the inclusions \( \mathcal{F}_{n} \rightharpoonup A \rtimes_{\text{ess}} S \) form a Fell bundle representation. This induces an injective \( * \)-homomorphism \( \mathcal{C}^{*}(\mathbb{Z}/k, (\mathcal{F}_{n})) \to A \rtimes_{\text{ess}} S \). If \( k \) is square-free, then [20] Theorem 9.12 shows that \( K \) does not detect ideals in \( \mathcal{C}^{*}(\mathbb{Z}/k, (\mathcal{F}_{n})) \). In general, we write \( k = p \cdot k_{1} \) with a prime number \( p \). An argument as above shows that \( K \) does not detect ideals in \( \mathcal{C}^{*}(\mathbb{Z}/p, (\mathcal{F}_{n})) \), \( n = 1, \ldots, p \). Let \( T \) be the inverse subsemigroup generated by \( t^{k_{1}} \) and all idempotent elements of \( S \). The same argument as in the case where \( \varphi_{0} \) is injective shows that \( A \) does not detect ideals in \( A \rtimes_{\text{ess}} T \).

Proposition 6.4. Let \( A \) be a Fell bundle over an étale groupoid \( G \) with a locally compact Hausdorff unit space \( X \). Assume that \( A := C_{0}(X, \mathcal{A}|_{X}) \) contains an essential ideal that is separable or of Type I. Assume that \( A \) detects ideals in \( C_{\text{ess}}(H, \mathcal{A}|_{H}) \) for any open subgroupoid \( X \subseteq H \subseteq G \) that is generated by a single open bisection \( U \subseteq G \). Then the dual groupoid of the Fell bundle \( A \) is topologically free.

Proof. We may rewrite the essential section \( \mathcal{C}^{*} \)-algebra \( B := C_{\text{ess}}(G, A) \) as the essential crossed product \( A \rtimes_{\text{ess}} S \) for an inverse semigroup action on \( A \) (see Definition 2.11). The dual groupoids for both actions are the same, so that topological freeness is preserved. Proposition 6.1 shows that the dual groupoid \( A \rtimes G \) is topologically free if \( A \) detects ideals in all intermediate \( \mathcal{C}^{*} \)-algebras \( A \subseteq C \subseteq B \). It remains to show that the special intermediate algebras of the form \( C_{\text{ess}}(H, \mathcal{A}|_{H}) \) for open subgroupoids \( X \subseteq H \subseteq G \) suffice to carry through the proof of Proposition 6.1.

For this, we first describe the inverse semigroup \( S \). Each bisection \( U \subseteq G \) defines a subspace \( \mathcal{A}_{U} \subseteq C_{\text{ess}}(G, A) \) of sections supported in \( U \). This subspace
belongs to the inverse semigroup $S(A, B)$ in [23 Proposition 2.12]. We let $S$ be the inverse subsemigroup of $S(A, B)$ generated by $A_U \cdot J$ for all ideals $J$ in $A$. By definition, $B$ is graded by the inverse semigroup $S$. This gives a surjective $^*$-homomorphism $\varphi: A \rtimes S \to B$. Composing the canonical generalised expectation $B = C_{\text{ess}}(G, A) \to \mathcal{M}_{\text{loc}}(A)$ with $\varphi$ gives the canonical generalised expectation $A \rtimes S \to \mathcal{M}_{\text{loc}}(A)$. Therefore, $\varphi$ induces an isomorphism $A \rtimes_{\text{ess}} S \cong B$. Now assume that the dual groupoid $\hat{A} \rtimes G$ is not topologically free. Then there is a non-zero open bisection $U \subseteq \hat{A} \times G \setminus \hat{A}$ with $r|_U = s|_U$. We may assume that the bisection $U$ “lives” on a bisection $V \subseteq G$ on $\hat{A}$. More precisely, the partial homeomorphism corresponding to $U$ is the homeomorphism dual to the Hilbert $A$-bimodule $A \rtimes J$ for an ideal $J = \bigcup_{k \in \mathbb{N}} V_k$ is a non-empty open subgroupoid of $G$ containing the units. As in the proof of Proposition 6.1, we may arrange that either $V^p \subseteq X$ for some square-free number $p$ or $V^k \cap X = \emptyset$ for all $k \in \mathbb{N}$. Then $A_k := A_{V^k}$ for $k = 1, \ldots, p$ or for $k \in \mathbb{Z}$ defines a Fell bundle over the group $\mathbb{Z}/p$ or over $\mathbb{Z}$. The section $C^*_\text{ess}$ of this Fell bundle is isomorphic to $C^*_{\text{ess}}(H, A|_H)$. Now we can argue as in the proof of Proposition 6.1 that $A$ cannot detect ideals in $C^*_\text{ess}(H, A|_H)$ because the dual groupoid for the action of $H$ on $A$ is not topologically free.

**Corollary 6.5.** Let $A$ be a Fell bundle over a discrete group $G$. Assume that $A := A_1$ contains an essential ideal that is separable or of Type I. If $A$ detects ideals in $C^*_\text{ess}(H, A|_H)$ for any cyclic subgroup $H \subseteq G$, then the dual groupoid of the Fell bundle $A$ is topologically free.

If $A$ is arbitrary, then it is unclear whether detection of ideals in intermediate $C^*$-algebras implies that the action is topologically free. Only a weaker statement follows. Namely, the action is purely outer in the following sense:

**Definition 6.6.** A Hilbert $A$-bimodule $\mathcal{H}$ over a $C^*$-algebra $A$ is purely outer [20] if there is no non-zero ideal $J \subseteq \mathbb{I}(A)$ with $\mathcal{H} \cdot J \cong J$ as a Hilbert $A$-bimodule. An action $\mathcal{E}$ of an inverse semigroup on a $C^*$-algebra $A$ is purely outer [23] if the Hilbert $A$-bimodules $\mathcal{E}_t \cdot I^1_{1,t}$ are purely outer for all $t \in S$.

**Proposition 6.7.** Let $\mathcal{E}$ be an action of a unital inverse semigroup $S$ on a $C^*$-algebra $A$. Assume that $A$ detects ideals in $A \rtimes_{\text{ess}} T$ for any inverse subsemigroup $T \subseteq S$ that is generated by the idempotents in $S$ together with a single $t \in S$. Then the action $\mathcal{E}$ is purely outer.

**Proof.** Assume that the action is not purely outer. Then there are $t \in S$ and a non-zero ideal $J \subseteq I^1_{1,t}$ such that there is an isomorphism of Hilbert $A$-bimodules $\varphi: \mathcal{E}_t \cdot J \xrightarrow{\sim} J$.

As in the proof of Proposition 6.1, we may shrink $J$ and replace $t$ by $t^{k_1}$ for some $k_1 \in \mathbb{N}$ to arrange that either $J \subseteq I^1_{1,t^k}$ for all $k \geqslant 1$ or $J \subseteq I^1_{1,t^k}$ for $k = 1, \ldots, p$ and $J \subseteq I^1_{1,t^p}$ for some square-free number $p$; we put $p = 0$ in the first case. Let $T \subseteq S$ be generated by the idempotent elements of $S$ and $t$. Then $A \rtimes_{\text{ess}} T \subseteq A \rtimes_{\text{ess}} S$. The ideal $J$ is $T$-invariant or, equivalently, $J \cdot (A \rtimes_{\text{ess}} T) = (A \rtimes_{\text{ess}} T) \cdot J$.

The latter is canonically isomorphic to the section $C^*$-algebra of a Fell bundle over $\mathbb{Z}/p$.

We may, without loss of generality, assume that $S = T$, $A = J$ and $\mathcal{E}_t = \mathcal{E}_t \cdot J$ for all $t \in S$ to simplify notation. After this change, $A \rtimes_{\text{ess}} S$ becomes the section $C^*$-algebra of a Fell bundle over $\mathbb{Z}/p$. Since we replaced $A$ by $J$, the isomorphism $\varphi$ becomes an isomorphism $\varphi: \mathcal{E}_t \xrightarrow{\sim} A$. This generates isomorphisms

$$\mathcal{E}_t^k \cong \mathcal{E}_t^k \subseteq A \otimes_{\text{ess}} k \cong A$$
for all \( k \in \mathbb{N} \). If \( p = 0 \), then these maps together form a Fell bundle representation of our Fell bundle into \( A \). They induce a *-homomorphism \( A \rtimes \text{ess} S = A \rtimes S \to A \). Since it is the identity map on \( A \), its kernel is an ideal \( J \) in \( A \rtimes \text{ess} S \) with \( J \cap A = 0 \). If \( p \neq 0 \), then \( \mathcal{E}_p = A \) inside \( A \rtimes \text{ess} S \). So \( \varphi \) above generates a Hilbert bimodule isomorphism \( A \overset{\sim}{\to} A \). This is the same as a central unitary multiplier of \( A \). Shrinking the ideal \( J \) further, we may arrange that the spectrum of this central multiplier is not the entire unit circle. We assume this for simplicity, and without loss of generality. Then it has a \( p \)th root in the central multiplier algebra of \( A \). Multiplying \( \varphi \) with the inverse of this \( p \)th root gives another isomorphism \( \mathcal{E}_t \cong A \), such that the resulting map \( A = \mathcal{E}_p \overset{\sim}{\to} A \) is the identity. So we get a Fell bundle representation once again. As above, this induces a *-homomorphism \( A \rtimes \text{ess} S \to A \) that is the identity map on \( A \), and its kernel is a non-zero ideal \( J \) in \( A \rtimes \text{ess} S \) with \( J \cap A = 0 \). \( \square \)

A version of Proposition 6.1 for Fell bundles over étale, locally compact groupoids is also true. The details are left to the reader.

7. Equivalence of various conditions for crossed products

We have proven several implications among properties of a C*-inclusion \( A \subseteq B \). We summarised them in the diagram in Figure 1 (in the introduction) and added a few implications that are known from previous work.

The left column in Figure 1 is valid for any C*-inclusion \( A \subseteq B \). Here we pick a pseudo-expectation \( E : B \to I(A) \) and let \( \mathcal{N} \) be the largest ideal contained in \( \ker E \). The whole diagram is valid if \( B \) is an exotic crossed product for an action on \( A \) of an inverse semigroup, or an étale groupoid with locally compact Hausdorff unit space. In both cases, the dual groupoid is defined and there is a natural pseudo-expectation \( E \) for \( A \subseteq B \), namely, the canonical generalised expectation \( B \to \mathcal{M}_{\text{loc}}(A) \) from [23] composed with the inclusion \( \mathcal{M}_{\text{loc}}(A) \to I(A) \) from [12, Theorem 1] (see also Remark 5.7). We use this pseudo-expectation to define \( \mathcal{N} \). Then \( B/\mathcal{N} \) in Figure 1 is the essential crossed product for the action, as introduced in [23].

Let \( G \) be an étale groupoid with a locally compact Hausdorff unit space \( X \). An action of \( G \) on the C*-algebra \( A \) means a possibly non-saturated Fell bundle \( \mathcal{A} \) over \( G \) with \( A \cong \text{C}_0(X,A|_X) \) (see Definition 2.11). We will call an action of an étale groupoid purely outer if the corresponding inverse semigroup action is purely outer. The claims in Figure 1 for such actions of \( G \) follow from the corresponding claims for inverse semigroup actions. Fell bundles over groups and, more generally, for Fell bundles over Hausdorff, étale, locally compact groupoids, the essential and reduced crossed products are equal. More generally, this happens for inverse semigroup actions if the canonical pseudo-expectation is a genuine expectation, that is, its values are contained in \( A \).

Some of the implications in Figure 1 only work if the pseudo-expectation \( E \) that is used to define \( \mathcal{N} \) has the following property:

**Definition 7.1** ([23, Definition 3.7]). A generalised expectation \( E : B \to \hat{A} \) is symmetric if \( E(b^*b) = 0 \) for some \( b \in B \) implies \( E(bb^*) = 0 \).

A generalised expectation \( E \) is symmetric if and only if the largest two-sided ideal \( \mathcal{N} \) contained in \( \ker E \) is equal to \( \{ b \in B : E(b^*b) = 0 \} \), if and only if the induced generalised expectation \( B/\mathcal{N} \to \hat{A} \) is faithful (see [23, Proposition 3.6 and Corollary 3.8]). The canonical pseudo-expectation on an inverse semigroup crossed product is always symmetric by [23, Theorem 4.11]. This remains true for actions of étale groupoids because these are treated by a reduction to inverse semigroup actions.

We have now explained the meaning of Figure 1. Next, we give references for the various implications that are asserted there.
Proposition 5.7 implies that $A \subseteq B$ has the extension property if $\hat{A} \cong S$ is principal and that $A \subseteq B$ has the almost extension property if $\hat{A} \cong S$ is topologically principal. Proposition 5.8 implies the converse implications provided $A \cong S \Rightarrow B \Rightarrow A \cong S$, that is, $B$ sits between the full and reduced crossed products. Here we also use Proposition 5.6 as in the proof of Theorem 5.11. Example 5.14 shows that both converse implications can fail if $B = A \cong S$.

It is clear that the extension property implies the almost extension property. By Theorem 5.5 any inclusion with the almost extension property is aperiodic, and the converse holds if $B$ is separable. It is clear that a principal groupoid is topologically principal. Topologically principal groupoids are topologically free by [23, Lemma 2.23]. The converse implication for countable $S$ and $A$ containing an essential ideal which is separable or has a Hausdorff spectrum follows from [23, Theorem 6.13 and Corollary 2.6]. Example 5.15 shows that the converse implication may fail for uncountable $S$.

Corollary 4.8 shows that topologically free inverse semigroup actions are aperiodic. The inclusion $A \subseteq B$ is aperiodic if and only if the action is aperiodic (see [23, Proposition 6.3], which is copied here in Proposition 2.6). Theorem 3.6 shows that an aperiodic inclusion has a unique pseudo-expectation.

If $A \subseteq B$ is an aperiodic inclusion, then so is $A \subseteq C$ for any intermediate $C^*$-algebra $A \subseteq C \subseteq B/N$ because subbimodules of aperiodic bimodules remain aperiodic (see [23, Lemma 5.12]). Therefore, we may apply Theorem 3.6 to the inclusion $A \subseteq C$. If the pseudo-expectation on $B$ is symmetric, then the induced pseudo-expectation $B/N \to I(A)$ is faithful. Therefore, Theorem 3.6 implies that $A$ supports $C$ for any $A \subseteq C \subseteq B/N$. It is easy to see that $A$ detects ideals in $C$ if it supports $C$ (see [23, Lemma 5.27]). Assume that $A \subseteq B$ comes from an inverse semigroup (or étale groupoid) action and that $A$ contains an essential ideal that is separable or of Type I. Proposition 5.14 says that the dual groupoid is topologically free if $A$ detects ideals in all intermediate $C^*$-algebras $A \subseteq C \subseteq A \cong S$.

It is shown by Pitts and Zarikian that $A$ detects ideals in $C$ for all $A \subseteq C \subseteq B/N$ if and only if all pseudo-expectations $B/N \to I(A)$ are faithful (see [33, Theorem 3.5]). If $E$ is symmetric, then the pseudo-expectation $B/N \to I(A)$ induced by $E$ is faithful. If, in addition, $E$ is the only pseudo-expectation $B \to I(A)$, then it follows that $A$ detects ideals in $C$ for all $A \subseteq C \subseteq B/N$.

Restrict to actions of inverse semigroups once again. If $A$ detects ideals in all intermediate $C^*$-subalgebras $A \subseteq C \subseteq B/N$, then the action is purely outer by Proposition 6.7. By [23, Theorem 6.13], actions that are aperiodic or topologically free are purely outer, and the converse holds if $A$ contains an essential ideal that is separable or of Type I.

This explains all the implications in Figure 1.

A very important fact is that there are some full cycles of implications if $A$ contains an essential ideal that is separable or simple or of Type I. Therefore, many of the conditions in Figure 1 become equivalent under suitable assumptions. We state two very similar theorems of this type.

**Theorem 7.2.** Let $A$ be a $C^*$-algebra that contains an essential ideal that is separable or of Type I. Let $A \subseteq B$ be the inclusion into an exotic crossed product for an action of an inverse semigroup $S$ or an étale groupoid $G$ on $A$. Assume that the inverse semigroup that acts is countable or that the étale groupoid that acts is covered by countably many bisections. Let $B_{\text{ess}}$ be the corresponding essential crossed product, which is a quotient of $B$. The following are equivalent:

- the dual groupoid of the action is topologically principal;
- the dual groupoid of the action is topologically free;
- $A \subseteq B$ has the almost extension property;
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- $A \subseteq B$ is aperiodic;
- $A \subseteq B$ has a unique pseudo-expectation;
- $A$ supports all intermediate $C^*$-algebras $A \subseteq C \subseteq B_{\text{ess}}$;
- $A$ detects ideals in all intermediate $C^*$-algebras $A \subseteq C \subseteq B_{\text{ess}}$;
- $A$ detects every essential crossed product $A \rtimes_{\text{ess}} T$ for an inverse subsemigroup $T \subseteq S$ that contains all idempotent elements of $S$, or in every essential section $C^*$-algebra $C^*_{\text{ess}}(H,A)$ for an open subgroupoid $H \subseteq G$ that contains the space of units of $G$.

**Theorem 7.3.** Let $A$ be a $C^*$-algebra that contains an essential ideal that is simple or of Type I. Let $A \subseteq B$ be the inclusion into an exotic crossed product for an action of an inverse semigroup $S$ or an étale groupoid $G$ on $A$. Let $B_{\text{ess}}$ be the corresponding essential crossed product, which is a quotient of $B$. The following are equivalent:

- the action on $A$ is purely outer;
- $A \subseteq B$ is aperiodic;
- $A \subseteq B$ has a unique pseudo-expectation;
- $A$ supports all intermediate $C^*$-algebras $A \subseteq C \subseteq B_{\text{ess}}$;
- $A$ detects ideals in all intermediate $C^*$-algebras $A \subseteq C \subseteq B_{\text{ess}}$;
- $A$ detects ideals in every essential crossed product $A \rtimes_{\text{ess}} T$ for an inverse subsemigroup $T \subseteq S$ that contains all idempotent elements of $S$, or in every essential section $C^*$-algebra $C^*_{\text{ess}}(H,A)$ for an open subgroupoid $H \subseteq G$ that contains the space of units of $G$.

**Proof.** Both theorems follow mostly from the implications in Figure 1. Under our assumptions, $E$ is symmetric. Propositions 6.1 and 6.7 give topological freeness and pure outerness of an inverse semigroup action when $A$ detects ideals in intermediate $C^*$-algebras of the form $A \rtimes_{\text{ess}} T$. The étale groupoid version of Proposition 6.1 is Proposition 6.4, and the étale groupoid version of Proposition 6.7 is similar and left to the reader. □

**Remark 7.4.** A mistake has crept into the hypotheses of [20, Theorem 9.12]. Namely, if $A$ contains an essential ideal that is simple, then it is unclear whether aperiodicity implies topological freeness (see also Remark 4.9). We only know that conditions (9.12.1)–(9.12.6) and (9.12.11) in [20, Theorem 9.12] are equivalent. Some of these equivalences are shown in Theorem 7.3 in greater generality.

**Remark 7.5.** Zarikian proved in [36, Theorem 3.5] that the inclusion $A \subseteq A \rtimes G$ for an action of a discrete group $G$ is aperiodic if and only if it has a unique pseudo-expectation. One direction in this implication is proven in Theorem 3.6. We do not know whether, conversely, any inclusion with a unique pseudo-expectation is aperiodic. It seems likely that this is true for inclusions into crossed products for inverse semigroup actions. Under separability assumptions, this is contained in Theorem 7.2 (see also Proposition 8.2 below).

**Remark 7.6.** Kennedy and Schafhauser introduced in [15] a cohomological invariant for discrete amenable group actions by automorphisms whose vanishing implies that aperiodicity (proper outerness) and detection of ideals are equivalent. Using our results, we see that this invariant detects whether detection of ideals already implies detection of ideals in all intermediate $C^*$-subalgebras.

Many results in the $C^*$-algebra literature have been proven under one of the assumptions in Figure 1. The implications proven here often allow to strengthen the conclusions of such results or weaken assumptions. As an example, we discuss a classical result of Archbold and Spielberg [4]. It says that the inclusion $A \subseteq A \rtimes G$ for an action of a discrete group $G$ detects ideals if the dual groupoid satisfies a
condition that is between topological freeness and topological principality, called
*AS topologically free* in [29] (see also Remark 4.2). According to Figure 1 we now get the same conclusion whenever the action is topologically free or aperiodic. In fact, we get the stronger statement that $A$ supports the reduced crossed product, which may help to prove that the reduced crossed product is purely infinite (see Theorem 3.6).

In our recent paper [23], we did not yet know Theorem 4.7 and therefore proved results about detection of ideals both for aperiodic and AS topologically free actions. Now we see that there is no need for a separate treatment for (AS) topologically free actions. So [23, Theorem 6.14] is no longer needed.

Two implications in Figure 1 only work if $E: B \to I(A)$ is symmetric. We can still get similar statements without this assumption. These, however, depend explicitly on the pseudo-expectation $E$ and not only on the ideal $N$ defined by it:

**Lemma 7.7.** Let $A \subseteq C \subseteq B$ be an intermediate $C^*$-algebra. Let $N_{E|_C}$ be the largest two-sided ideal in $C$ contained in $\ker E$. If $A \subseteq B$ is aperiodic, then $A$ supports $C/N_{E|_C}$. If $A \subseteq B$ has a unique pseudo-expectation, then $A$ detects ideals in $C/N_{E|_C}$.

**Proof.** If $A \subseteq B$ is aperiodic or has a unique pseudo-expectation, then the same is true for the inclusion $A \subseteq C$. Theorem 3.6 applied to this inclusion shows that $A$ supports $C/N_{E|_C}$ if $A \subseteq C$ is aperiodic. And [33, Proposition 3.1] shows that $A$ detects ideals in $C/N_{E|_C}$ if $A \subseteq C$ has a unique pseudo-expectation. □

**Lemma 7.8.** If $E$ is not symmetric, then there is an intermediate $C^*$-algebra $C$ for which $N_{E|_C} \neq N_E \cap C$.

**Proof.** We are going to construct $C$ as the pre-image of an intermediate $C^*$-algebra $A \subseteq C \subseteq B/N$ such that the pseudo-expectation $C \to I(A)$ induced by $E$ is not almost faithful. The induced pseudo-expectation on $B/N$ is almost faithful, but not faithful. Then there is $x \in B/N$ with $x \neq 0$, but $E(x^*x) = 0$. Let $C$ be the $C^*$-subalgebra generated by $A$ and $x^*x$. It is easy to see that $E$ vanishes on the two-sided ideal in $C$ generated by $x^*x$ in $C$ (compare the proof of [33, Theorem 3.5]). Therefore, $N_{E|_C} \neq 0$. □

**Example 7.9.** Let $B = M_2(\mathbb{C})$ and let $A := \mathbb{C}: E_{11} \subseteq B$. It is well known that states on hereditary $C^*$-subalgebras extend uniquely (see [30, Proposition 3.1.6]). Since $A$ is a hereditary subalgebra in $B$, the inclusion $A \subseteq B$ has the extension property. This implies that it is aperiodic and that it has a unique pseudo-expectation. The latter is the obvious expectation $E: B \to A$, $(T_{ij})_{1 \leq i, j \leq 2} \mapsto T_{11}$. This expectation is almost faithful because $B$ is simple, but not faithful. So $N = 0$. It follows that $A$ supports $B$. Now let $C \subseteq B$ be the $C^*$-subalgebra of diagonal matrices. Then $E|_C$ is not faithful. And $A \subseteq C$ does not detect ideals, so it cannot support $C$ either.

8. Applications to Cartan subalgebras and Kumjian’s diagonals

We end this article with two applications of our results to Cartan subalgebras of some kind. Since this is the only place where we use regular inclusions, normalisers, and twists of groupoids, we do not define these concepts here. Our regular inclusions are non-degenerate by definition. First we apply our characterisation of the extension property to Kumjian’s $C^*$-diagonals. Kumjian noted that his $C^*$-diagonals have the extension property (see [18, Proposition 1.4 and Theorem 3.1]). The following proposition removes the separability assumption from this result and shows that the extension property characterises Kumjian’s diagonals.

**Proposition 8.1.** Let $C_0(X) = A \subseteq B$ be a regular commutative $C^*$-subalgebra with a faithful conditional expectation $E: B \to A$. The following are equivalent:

- $A$ supports $B$.
- $A$ detects ideals in $B$.
- $A$ has a unique pseudo-expectation.

□
The twisted groupoid $\{(H, \Sigma)\}$ in (3) is unique up to isomorphism.

Proof. By [18, Proposition 1.4], (1) implies (2). Assume (2). Then the inclusion $A \subseteq B$ is aperiodic by Theorem 5.5. Then [22, Corollary 7.6] gives an isomorphism $B \cong C_r^\ast(H, \Sigma)$ for a unique twisted groupoid $\{(H, \Sigma)\}$, where $H$ is a Hausdorff, étale, locally compact groupoid with the unit space $X$. Here $H$ is the dual groupoid of the inclusion. Since $C_0(X) \subseteq C_r^\ast(H, \Sigma)$ has the extension property, $H$ is also principal by Corollary 5.12. Thus (2) implies (3). It remains to show that (3) implies (1) (as proven already in [18, Lemma 2.12]). Since $H$ is principal, $H \setminus X$ is covered by bisections $U \subseteq H$ with the property that $s(U) \cap r(U) = \emptyset$. If $b \in C_r^\ast(H, \Sigma)$ is supported in such a bisection, then it is a normaliser for $C_0(X)$ with $b^2 = 0$. Such elements are linearly dense in $\ker E$.

Next we apply our theory to Exel’s noncommutative Cartan subalgebras, which we have recently characterised in [22]. A noncommutative Cartan inclusion is a regular $C^\ast$-inclusion $A \subseteq B$ with a faithful conditional expectation $E : B \to A$ and an extra property, for which many equivalent forms are given in [22, Theorem 4.2]. One of them says that $B \cong A \rtimes_r S$ for a closed and purely outer action $E$ of an inverse semigroup $S$ on $A$, with an isomorphism that restricts to the canonical embedding on $A$. It is already noted in [22, Theorem 6.3] that aperiodicity implies the equivalent conditions in [22, Theorem 4.2]. Here we observe that the unique pseudo-expectation property also does that. This characterises Renault’s (commutative) Cartan inclusions in a new way (see [34]). It also characterises Exel’s Cartan subalgebras $A \subseteq B$, provided $A$ has an essential ideal that is simple or of Type I.

**Proposition 8.2.** Let $A \subseteq B$ be a regular $C^\ast$-inclusion with a faithful conditional expectation $E : B \to A$. Consider the following conditions:

1. $A \subseteq B$ has the almost extension property;
2. $A \subseteq B$ is aperiodic;
3. $A \subseteq B$ has a unique pseudo-expectation;
4. $A \subseteq B$ is a noncommutative Cartan subalgebra in the sense of Exel [9].

Then (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4). If $A$ contains an essential ideal that is simple or of Type I, then (2) $\Rightarrow$ (4) are equivalent. If, in addition, $B$ is separable, then all the conditions (1) $\Rightarrow$ (4) are equivalent.

Proof. Recall that if (4) holds, then $B \cong A \rtimes_r S = A \rtimes_{\operatorname{ess}} S$ for a closed purely outer action of an inverse semigroup $S$. Hence all the implications except (3) $\Rightarrow$ (4) are included in Figure 1. Implication (3) $\Rightarrow$ (4) follows from [22, Theorem 4.2] and the following lemma, which checks condition (1) of [22, Theorem 4.2].

**Lemma 8.3.** Let $A \subseteq B$ be a $C^\ast$-inclusion with a unique pseudo-expectation. For every ideal $I$ in $A$, there is at most one conditional expectation for the inclusion $I \subseteq IBI$.

Proof. Let $I$ be an ideal in $A$ and let $E : IBI \to I$ be a conditional expectation. We are going to prove that $E$ extends to a pseudo-expectation $B \to I(A)$. Since the latter is unique, it follows that $E$ is also unique.

Let $I^\perp$ be the annihilator of $I$ in $A$. We extend $E$ to a conditional expectation for the inclusion $I + I^\perp \subseteq IBI + I^\perp$ by putting $E|_{I^\perp} := \Id_{I^\perp}$. This extends to
a strictly continuous conditional expectation $\hat{E} : \mathcal{M}(IBI + I^\perp) \to \mathcal{M}(I + I^\perp)$ by Lemma 4.6]. Since $IBI + I^\perp$ is an essential ideal in $IBI + A$ we may treat the latter as a subalgebra of $\mathcal{M}(IBI + I^\perp)$. In this way, we get a generalised conditional expectation $\hat{E} : \mathcal{M}(IBI + I^\perp) \to \mathcal{M}(I + I^\perp)$. Since $I + I^\perp$ is an essential ideal in $A$ we have $\mathcal{M}(I + I^\perp) \subseteq \mathcal{M}_{\text{loc}}(A) \subseteq I(A)$. Since $I(A)$ is injective, the map $\hat{E} : B \supseteq IBI + A \to \mathcal{M}(I + I^\perp) \subseteq I(A)$ extends to a pseudo-expectation $B \to I(A)$.

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