Nonlinear generalized fractional differential equations with generalized fractional integral conditions

Samiha Belmor a,b, Chokkalingam Ravichandran b and Fahd Jarad c

aDepartment of Mathematics, University Mustapha Ben Boulaid, Batna, Algeria; bPG & Research Department of Mathematics, Kongunadu Arts and Science College (Autonomous), Coimbatore, India; cDepartment of Mathematics, Cankaya University, Ankara, Turkey

ABSTRACT
This research work is dedicated to an investigation of the existence and uniqueness of a class of nonlinear $\psi$-Caputo fractional differential equation on a finite interval $[0, T]$, equipped with nonlinear $\psi$-Riemann–Liouville fractional integral boundary conditions of different orders $0 < \alpha, \beta < 1$, we deal with a recently introduced $\psi$-Caputo fractional derivative of order $1 < q < 2$. The formulated problem will be transformed into an integral equation with the help of Green function. A full analysis of existence and uniqueness of solutions is proved using fixed point theorems: Leray–Schauder nonlinear alternative, Krasnoselskii and Schauder’s fixed point theorems, Banach’s and Boyd–Wong’s contraction principles. We show that this class generalizes several other existing classes of fractional-order differential equations, and therefore the freedom of choice of the standard fractional operator. As an application, we provide an example to demonstrate the validity of our results.

1. Introduction
The study of differentiation and integration to a fractional order has caught importance and popularity among researchers compared to classical differentiation and integration. Fractional operators used to illustrate better the reality of real-world phenomena with the hereditary property. For instance, various applications and comprehensive strategy of the fractional calculus are addressed in the works of Baleanu et al. [1,2], Abd-Elhameed et al. [3,4], Jarad et al. [5], Hafez et al. [6], and Youssri et al. [7]. A good review of different fractional operators can be found in [8]. It has been proved that differential equation with fractional-order process more accurately than integer-order differential equations do, and fractional arrangers provide excellent performance of the description of hereditary attributes than integer-order arrangers. Applications can be found in complex viscoelastic media, electrical spectroscopy, porous media, cosmology, environmental science, medicine (the modelling of infectious diseases), signal and image processing, materials, and many others. For detail, see analogous discussions to the topics in a comprehensive review by Sun et al. [9].

Fractional-order boundary value problems of nonlinear fractional differential equations have been extensively investigated by many authors. By applying various techniques of nonlinear analysis; many researchers have studied the existence of solutions of fractional-order differential equations supplemented by integral boundary conditions involve either classical, Riemann–Liouville, Hadamard, Erdélyi–Kober, or Katugampola type. For instance, in [10,11] Ahmad et al. applied the classical fixed point theory to nonlinear fractional differential equations with nonlocal generalized fractional integral boundary conditions. The author showed that the considered problems have a unique solution and unify some available results. In [12] Sun et al. investigated the existence of nonlinear fractional-order boundary value problems with nonlocal Erdélyi–Kober and generalized Riemann–Liouville type integral boundary conditions using Mawhin continuation theorem. We refer the reader to the survey by Agarwal et al. [13] which particularly had a chronological listing on major works in the investigation of the existence and uniqueness of differential equations and inclusions of fractional-orders with various boundary conditions.

In [14], Samko et al. presented fractional integrals and derivatives with unlike kernels. These fractional operators are now known as $\psi$-fractional operators, and it has been shown that these operators unify a wide class of fractional differentiations and integrations such as the aforementioned ones. As an application managing with the theory of $\psi$-fractional differentiation, we refer to [15], the reader will find other classifications of $\psi$-fractional differentiation with various applications in the work [16–18].

In [19], Almeida investigate the existence and uniqueness of solution for the following boundary value
problem of fractional differential equations
\[ ^cD^\alpha_{0^+;\psi} f(t, y(t)), \quad t \in [a, b], \quad 2 < \alpha < 3, \]
(1)
\[ y(a) = y_a, \quad y'(a) = y_{a1}, \quad y(b) = g^\alpha_{0^+;\psi} y(\delta), \]
(2)
where \(^cD^\alpha_{0^+;\psi}\) is the \(\psi\)-Caputo fractional derivative introduced in [20], \(g^\alpha_{0^+;\psi}\) is the \(\psi\)-Riemann–Liouville fractional integral presented in [14]. The existence results are obtained via the aid of some classical fixed point theorems.

We deal in this paper with a recently established \(\psi\)-Caputo fractional derivative within the framework of absolutely continuous functions proposed by Jarad and Abdeljawad [21]. We focus on the existence and uniqueness of solutions for a nonlinear fractional differential equation involving the \(\psi\)-Caputo fractional derivative, supplemented with \(\psi\)-Riemann–Liouville fractional integral boundary conditions of different orders
\[ ^cD^q_{0^+;\psi} y(t) = f(t, y(t)), \quad t \in I = [0, T], \quad 1 < q \leq 2, \]
(3)
\[ y(0) - \delta_\psi y(0) = \frac{1}{\Gamma(\alpha)} \int_0^\sigma (\psi(\sigma) - \psi(s))^{\alpha - 1} \psi'(s) g(s, y(s)) \, ds = I_{0^+;\psi}^0 g(\sigma, y(\sigma)), \]
(4)
\[ y(T) + \delta_\psi y(T) = \frac{1}{\Gamma(\beta)} \int_0^\eta (\psi(\eta) - \psi(s))^{\beta - 1} \psi'(s) h(s, y(s)) \, ds = I_{0^+;\psi}^\eta h(\eta, y(\eta)), \]
(5)
where \(0 < \alpha, \beta \leq 1, \psi' > 0\) on \([0, T]\), \(0 < \alpha, \eta < T\), \(f, g, h\) three real continuous functions defined on \([0, T] \times \mathbb{R}\), and \(\delta_\psi = (1/\psi'(t)) \, dt/d\psi\).

Our results here are new and generalize some known results in the literature for specific choices of the parameters involved. For instance, selecting \(\psi(t) = t, \alpha = \beta = 1, \eta \rightarrow T^-,\) and \(\sigma \rightarrow T^-\) in the problem (3)–(5), the boundary conditions take the form
\[ y(0) - y'(0) = \int_0^T g(s, y(s)) \, ds, \quad y(T) + y'(T) = \int_0^T h(s, y(s)) \, ds, \]
(6)
and the resulting problem meets the one studied in [22], reduces the one considered in [23] under a weakly sequentially continuity assumption imposed on \(f, g\) and \(h\). Studies in [24] use the technique of measures of noncompactness and the Mönch’s fixed point theorem. In case we choose \(\delta_\psi y(0) = \delta_\psi y(T) = 0\) the problem (3)–(5) generalize the one considered in [13]. An integer-order version of (3)–(5) was considered in [25] with \(q = 2, \psi(t) = t, \alpha = \beta = 1, \eta \rightarrow 1^-,\) and \(\sigma \rightarrow 1^-\), where the existence of solutions and extremal solutions are established. In summary, the present paper covers some interesting situations.

The paper is organized as follows. In Section 2, we present the main concepts of the generalized fractional calculus and give some valuable preliminary results. In Section 3, we prove the existence and uniqueness of solution to the problem (3)–(5) by using the standard fixed-point theorem. In Section 4, we present an illustrative example. At last, we conclude our results.

2. Preliminaries and mathematical background

We review a few definitions, notations and results of \(\psi\)-fractional integrals and derivatives which will be used throughout this paper.

By \(C([0, T], \mathbb{R})\) we denote the Banach space of all continuous functions from \([0, T]\) endowed with the norm defined by
\[ \|u\| = \sup_{[0, T]} |u(x)|. \]
Let \(AC([0, T], \mathbb{R})\) denotes the space of all absolutely continuous real valued function on \([0, T]\). Moreover, we define the space \(AC^\alpha_{\psi}([0, T], \mathbb{R})\) by
\[ AC^\alpha_{\psi}([0, T], \mathbb{R}) = \left\{ f : [0, T] \rightarrow \mathbb{R}; (\psi^{\alpha - 1} f)(t) \in AC([0, T], \mathbb{R}), \delta_\psi = \frac{1}{\psi'(t)} \right\}, \]
which is endowed with the norm defined by
\[ \|f\|_{AC^\alpha_{\psi}} = \sup_{k=0}^{n-1} \|\delta_\psi^k f(t)\|_{\infty}, \]
where \(\psi \in C^n([0, T], \psi'(t) > 0\) on \([0, T]\), and
\[ \delta_\psi = \delta_\psi \delta_\psi \ldots \delta_\psi. \]

Definition 2.1 ([14]): Let \(f : [0, T] \rightarrow \mathbb{R}\) be an integrable function. The \(\psi\)-Riemann–Liouville integral of order \(\alpha > 0\) of \(f\) for \(0 < t < T < +\infty\) is defined by
\[ I_{0^+;\psi}^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (\psi(t) - \psi(s))^{\alpha - 1} \psi'(s) f(s) \, ds, \]
(7)
Definition 2.2 ([21]): For \(0 < t < T < +\infty\), the \(\psi\)-Riemann–Liouville fractional derivative of order \(\alpha > 0\) of a function \(f \in AC^\alpha_{\psi}([0, T])\) is defined by
\[ ^cD^\alpha_{0^+;\psi} f(t) = \frac{\psi^{-\alpha}_{0^+;\psi}(\delta_\psi^\alpha f)(t)}{\Gamma(\alpha)} + \sum_{k=0}^{n-1} \frac{(\delta_\psi^k f)(0)}{\Gamma(k - \alpha + 1)} (\psi(t) - \psi(0))^{k-\alpha}, \]
\[ \psi' (s) \delta^\alpha_0 (s) \, ds \sum_{k=0}^{n-1} \frac{(\delta^k \psi) (0)}{\Gamma(k+1)} \times (\psi (t) - \psi (0))^{k-\alpha}, \]

(8)

if the integral exists, where \( n = [\alpha] + 1 \), \( \Gamma (x) \) is the Euler’s gamma function defined by \( \Gamma (x) = \int_0^\infty t^{x-1} e^{-t} \, dt. \)

**Definition 2.3 ([20,21]):** For \( f \in AC^\alpha_0 ([0, T]) \), the \( \psi \)-Caputo fractional derivative of order \( \alpha > 0 \), is defined by

\[ cD^\alpha_{0+\psi} f(t) = \frac{1}{\Gamma (n-\alpha)} \int_0^t (\psi (t) - \psi (s))^{-n+\alpha} \times \psi' (s) (\delta^\alpha_0 (s)) \, ds, \quad n = [\alpha] + 1. \]

Thus, if \( \alpha = n \in \mathbb{N} \) we have

\[ cD^\alpha_{0+\psi} f(t) = (\delta^\alpha_0 (f)) (t). \]

**Lemma 2.4 ([21]):** Given a function \( f \in AC^\alpha_0 ([0, T]) \) and \( \alpha \in \mathbb{R}^+ \), then

\[ \rho_{0+\psi} cD^\alpha_{0+\psi} f(t) = f(t) - \sum_{k=0}^{n-1} \frac{(\delta^k \psi) (0) f(t)}{k!} (\psi (t) - \psi (0))^k, \]

(9)

in particular, for \( 0 < \alpha < 1 \), we have

\[ \rho_{0+\psi} cD^{\alpha}_{0+\psi} f(t) = f(t) - f(0). \]

In the following auxiliary lemma, we solve the linear version of the problem (3)–(5).

**Lemma 2.5:** Let \( 1 < q \leq 2 \) and \( \psi, \phi_1, \phi_2 : [0, T] \to \mathbb{R} \) be continuous. A function \( y \in AC^1_\psi ([0, T], \mathbb{R}) \) is a solution of the fractional integral equation

\[ y(t) = L(t) + \int_0^T G_\psi (t, s) \psi (s) \, ds, \]

(10)

where

\[ L(t) = \frac{\psi (T) - \psi (t) + 1}{\psi (T) - \psi (0) + 2} \rho_{0+\psi} \phi_1 (\sigma) + \frac{\psi (t) - \psi (0) + 1}{\psi (T) - \psi (0) + 2} \rho_{0+\psi} \phi_2 (\eta), \]

here, \( \psi \) is an increasing function on \([0, T]\) such that \( \psi \in C^2 ([0, T], \mathbb{R}) \) and \( G_\psi (t, s) \) is a Green’s function given by

\[ G_\psi (t, s) = \frac{(\psi (t) - \psi (s))^{-q-1} - \psi (t) - \psi (0) + 1}{(\psi (T) - \psi (0) + 2) \Gamma (q)} \]

\[ \times \begin{cases} \psi (t) - \psi (s) \psi (s) - 1 & \text{if } 0 \leq s \leq t, \\
\psi (t) - \psi (s) \psi (s) - 1 & \text{if } t < s \leq T, \end{cases} \]

(11)

if and only if \( y \) is a solution of the following \( \psi \)-fractional BVP

\[ \rho_{0+\psi} cD^q_{0+\psi} y(t) = \psi (t), \quad t \in I := [0, T], \]

(12)

\[ y(0) - \delta_0 \psi y(0) = \rho_{0+\psi} \phi_1 (0), \]

(13)

\[ y(T) + \delta_0 \psi y(T) = \rho_{0+\psi} \phi_2 (0). \]

(14)

**Proof:** Applying the \( \psi \)-Riemann–Liouville operator \( \rho_{0+\psi} \) on both sides of Equation (12) and using Lemma 2.4, we obtain

\[ y(t) = c_1 + c_2 (\psi (t) - \psi (0)) + \rho_{0+\psi} \psi (t), \]

(15)

where \( c_1, c_2 \) are arbitrary constants. Taking the \( \delta_0 \psi \)-derivative (15) we get

\[ (\delta_0 \psi) y(t) = c_2 + \rho_{0+\psi}^{-1} \psi (t). \]

(16)

From (13) and (14), we get

\[ c_1 - c_2 = \rho_{0+\psi} \phi_1 (0), \]

(17)

and

\[ c_1 + c_2 (\psi (T) - \psi (0) + 1) + \rho_{0+\psi} \psi (T) + \rho_{0+\psi}^{-1} \psi (T) = \rho_{0+\psi} \phi_2 (0). \]

(18)

Equation (15) and (18) give

\[ c_2 = \frac{1}{\psi (T) - \psi (0) + 2} \rho_{0+\psi} \phi_2 (0). \]

(19)
and 
\[ c_1 = \frac{\psi(T) - \psi(0) + 1}{\psi(T) - \psi(0) + 2 R_{0^+}^2 \phi_1(\sigma)} + \frac{1}{\psi(T) - \psi(0) + 2 R_{0^+}^2 \phi_2(\eta)} - \frac{1}{\psi(T) - \psi(0) + 2 R_{0^+}^2} \left( R_{0^+}^2 \psi(T) + R_{0^+}^2 \psi(T) \right) \]. 

From (15), (19), (20), and using the fact that \( G \), we deal with the existence and uniqueness of solutions for the fractional-order boundary value problem (3)–(5) using certain fixed point theorems.

In this section, we deal with the existence and uniqueness of solutions for the fractional-order boundary value problem (3)–(5) using certain fixed point theorems.

Remark 3.1: From the expression of \( G\), it is clear that \( G\) is continuous on \( I \times I\), and hence is bounded. Thus, we let

\[ G = \sup \left\{ \int_0^T \left| G(t,s) \right|, t \in I \right\}. \]

In the following, for computational convenience, we set the following notations

\[ \Omega_1 = \frac{(\psi(T) + 1)(\psi(\alpha))}{(\psi(T) - \psi(0) + 2)(\alpha + 1)}, \quad \Omega_2 = \frac{(\psi(T) + 1)(\psi(\eta))}{(\psi(T) - \psi(0) + 2)(\beta + 1)}. \]

Based on Banach’s fixed point theorem [26], we state the uniqueness of solutions of the problem (3)–(5).

Theorem 3.2: Assume that the following hypotheses hold:

(H1) There exists a non-negative constant \( k > 0 \) such that

\[ |g(t,u) - g(t,v)| \leq k|u - v|, \]

for all \( t \in I \). for all \( u, v \in \mathbb{R} \).

(H2) There exists a non-negative constant \( k^* > 0 \) such that

\[ |g(t,u) - g(t,v)| \leq k^*|u - v|, \]

for all \( t \in I \).

(H3) There exists a non-negative constant \( k^{**} > 0 \) such that

\[ |h(t,u) - h(t,v)| \leq k^{**}|u - v|, \]

for all \( t \in I \).

Then the problem (3)–(5) has a unique solution on \( I \), provided that

\[ \left( \Omega_1 k^* + \Omega_2 k^{**} + TG\right) \leq 0, \]

where \( \Omega_1 \) and \( \Omega_2 \) are given by (23) and (24) respectively.

Proof: We reformulate the problem (3)–(5) as a fixed point problem by considering the operator

\[ N : C([0, T], \mathbb{R}) \rightarrow C([0, T], \mathbb{R}), \]

defined by the formula

\[ (Ny)(t) = L(t) + \int_0^T G(t,s)f(s,y(s)) \, ds, \]

where

\[ L(t) = \frac{(\psi(T) + 1)}{(\psi(T) - \psi(0) + 2)(\alpha)} \int_0^\sigma (\psi(\sigma) - \psi(s))^{\alpha - 1} \]

\[ \times \psi'(s)g(s,y(s)) \, ds + \frac{(\psi(T) + 1)}{(\psi(T) - \psi(0) + 2)(\beta)} \int_0^\eta (\psi(\eta) - \psi(s))^{\beta - 1} \]

\[ \times \psi'(s)h(s,y(s)) \, ds, \]

and the Green function \( G(t,s) \) is given by (11). It is well known that the fixed points of the operator \( N \) are solutions of the problem (3)–(5). By using the Banach contraction mapping principle, we shall show that \( N \) has a fixed point.

Let \( x, y \in C([0, T], \mathbb{R}) \). Then, for each \( t \in [0, T] \), one can obtain

\[ \left| (Nx)(t) - (Ny)(t) \right| \]

\[ \leq \frac{(\psi(T) + 1)}{(\psi(T) - \psi(0) + 2)(\alpha)} \int_0^\sigma |(\psi(\sigma) - \psi(s))^{\alpha - 1} \]

\[ \times \psi'(s) |g(s,x(s)) - g(s,y(s))| \, ds + \frac{(\psi(T) + 1)}{(\psi(T) - \psi(0) + 2)(\beta)} \int_0^\eta |(\psi(\eta) - \psi(s))^{\beta - 1} \]

\[ \times \psi'(s) |h(s,x(s)) - h(s,y(s))| \, ds + \int_0^T |G(t,s)||f(s,x(s)) - f(s,y(s))| \, ds. \]
Assume that the following hypothesis holds

\[ \frac{(\psi(T) + 1)(\psi(\sigma))^\alpha}{(\psi(T) - \psi(0) + 2)\Gamma(\alpha + 1)} k^*\|x - y\|_{\infty} \]
\[ + \frac{(\psi(T) + 1)(\psi(\eta))^\beta}{(\psi(T) - \psi(0) + 2)\Gamma(\beta + 1)} \]
\[ + TG_\psi Y_k\|x - y\|_{\infty} \]
\[ \leq (\Omega_1 k^* + \Omega_2 k^{**} + TG_\psi Y_k)\|x - y\|_{\infty}. \]

Taking the supremum, we obtain

\[ \|N(x) - N(y)\|_{\infty} \leq (\Omega_1 k^* + \Omega_2 k^{**} + TG_\psi Y_k)\|x - y\|_{\infty}. \]

Hence, by (25), \( N \) is a contraction. By the Banach fixed point theorem \( N \) has a fixed point which is a solution of the problem (3)–(5). The proof is now complete. \( \blacksquare \)

Now, our next uniqueness result for the problem (3)–(5) relies on Boyd–Wong Contraction Principle [27].

**Theorem 3.3:** Let \( X \) be a complete metric space and suppose \( T : X \to X \) satisfies

\[ d(Tx, Ty) \leq \Psi(d(x, y)), \]

for each \( x, y \in X \),

where \( \Psi : [0, +\infty) \to [0, +\infty) \) is upper semi-continuous from the right (i.e., \( r \downarrow f \geq 0 \Rightarrow \limsup_{n \to +\infty} \Psi(r) \leq \Psi(f(r)) \)) and satisfies \( 0 \leq \Psi(t) < t \) for all \( t > 0 \). Then \( T \) has a unique fixed point \( \bar{x} \) and \( T_n(x) \) converges to \( \bar{x} \) for each \( x \in X \).

**Theorem 3.4:** Assume that the following hypothesis hold

\( \text{(H4)} \) there exist \( \chi_1, \chi_2 \) and \( \chi_3 : \mathbb{R}^+ \to \mathbb{R}^+ \) upper semi-continuous from the right and non-decreasing functions such that

\[ |f(t, u) - f(t, v)| \leq \chi_1(|u - v|), \]
\[ |g(t, u) - g(t, v)| \leq \chi_2(|u - v|), \]
\[ |h(t, u) - h(t, v)| \leq \chi_3(|u - v|), \]

for all \( t \in [0, T] \) a.e. \( u \) and \( v \in \mathbb{R} \), and

\[ \Psi(t) := (\Omega_1\chi_2(t) + \Omega_2\chi_3(t) + TG_\psi \chi_1(t)) < t, \]

for all \( t > 0 \),

where \( \Omega_1 \) and \( \Omega_2 \) are given by (23) and (24) respectively.

Then the problem (3)–(5) has a unique solution.

**Proof:** We define the operator \( N \) as in (26). For any \( x, y \in C([0, T], \mathbb{R}) \), and \( t \in [0, T] \), by using (H4), we get

\[ \|N(x)(t) - N(y)(t)\| \]
\[ \leq \frac{(\psi(T) + 1)(\psi(\sigma))^\alpha}{(\psi(T) - \psi(0) + 2)\Gamma(\alpha + 1)} \int_0^t |(\psi(\sigma) - \psi(s))^{\alpha-1} \]
\[ \cdot \psi'(s)|g(s, x(s)) - g(s, y(s))| ds \]
\[ + \frac{(\psi(T) + 1)(\psi(\eta))^\beta}{(\psi(T) - \psi(0) + 2)\Gamma(\beta + 1)} \]
\[ + \int_0^T |G_0(t, s)|f(s, x(s)) - f(s, y(s))| ds \]
\[ \leq \frac{(\psi(T) + 1)(\psi(\sigma))^\alpha}{(\psi(T) - \psi(0) + 2)\Gamma(\alpha + 1)} \chi_2(\|x - y\|_{\infty}) \]
\[ + \frac{(\psi(T) + 1)(\psi(\eta))^\beta}{(\psi(T) - \psi(0) + 2)\Gamma(\beta + 1)} \chi_3(\|x - y\|_{\infty}) \]
\[ + TG_\psi \chi_1(\|x - y\|_{\infty}) \]
\[ \leq \psi(\|x - y\|_{\infty}). \]

By taking the supremum, we obtain,

\[ \|N(x)(t) - N(y)(t)\|_{\infty} \leq \psi(\|x - y\|_{\infty}). \]

Since \( \psi(t) < t \) for all \( t > 0 \), then Boyd–Wong’s contraction principle can be applied and \( N \) has a unique fixed point which is the unique solution of the problem (3)–(5). \( \blacksquare \)

**Remark 3.5:** Theorem 3.4 is a generalization of Theorem 3.2. Indeed for

\[ \chi_1(t) = kt, \quad \chi_2(t) = k^*t, \quad \text{and} \quad \chi_3(t) = k^{**}t, \]

condition (H4) becomes (H1–H3), and (31) is satisfied if and only if (25) holds.

Our next existence result for the problem (3)–(5) is based on Leray–Schauder nonlinear alternative [28].

**Theorem 3.6:** Assume that the following hypotheses hold

\( \text{(H5)} \) \( f : I \times \mathbb{R} \to \mathbb{R} \) is a continuous function.

\( \text{(H6)} \) There exists \( \phi_f \in L^1(I, \mathbb{R}^+) \), and a continuous and non-decreasing function \( \chi_1 : \mathbb{R}^+ \to \mathbb{R}^+ \) such that

\[ |f(t, u)| \leq \phi_f(t)\chi_1(|u|), \]

for all \( t \in I \) a.e. \( u \in \mathbb{R} \).

\( \text{(H7)} \) There exists \( \phi_g \in L^1(I, \mathbb{R}^+) \), and a continuous and non-decreasing function \( \chi_2 : \mathbb{R}^+ \to \mathbb{R}^+ \) such that

\[ |g(t, u)| \leq \phi_g\chi_2(|u|), \]

for all \( t \in I \) a.e. \( u \in \mathbb{R} \).
There exists \( \phi_h \in L^1(I, \mathbb{R}^+) \), and a continuous and non-decreasing function \( \chi_3 : \mathbb{R}^+ \to \mathbb{R}^+ \) such that
\[
|h(t,u)| \leq \phi_h x_3(|u|), \quad \text{for all } t \in I, a.e. u \in \mathbb{R}.
\]

There exists a number \( C > 0 \) such that
\[
\frac{C}{a\chi_2(C) + b\chi_3(C) + cG^*_\psi \chi_1(C)} > 1, \quad (H9)
\]

where
\[
a = \frac{(\psi(T) + 1)}{(\psi(T) - \psi(0) + 2)} \int_0^T \psi(\eta) \phi_2(t) \mathrm{d}s,
\]
\[
b = \frac{(\psi(T) + 1)}{(\psi(T) - \psi(0) + 2)} \int_0^T \psi(t) \phi_3(t) \mathrm{d}s,
\]
\[
c = \int_0^T \psi(t) \mathrm{d}s.
\]

Then the problem (3)–(5) has at least one solution on \( I \).

**Proof:** Consider the operator \( N \) defined by (26), by applying Leray–Schauder nonlinear alternative we will prove that \( N \) has a fixed point. The proof will be given in several steps

**Step 1:** \( N \) is continuous. Let \( y_n \) be a sequence such that \( y_n \to y \) in \( C([0,T],\mathbb{R}) \). Then for each \( t \in [0,T] \)
\[
|N(y_n)(t) - N(y)(t)|
\]
\[
\leq \frac{(\psi(T) + 1)}{(\psi(T) - \psi(0) + 2)} \int_0^T |(\psi(\sigma) - \psi(s))^{a-1}| \psi(\sigma) g(s,y_n(s)) - g(s,y(s))| \mathrm{d}s
\]
\[
\leq \frac{(\psi(T) + 1)}{(\psi(T) - \psi(0) + 2)} \int_0^T (\psi(\sigma) - \psi(s))^{a-1} |h(s,y_n(s)) - h(s,y(s))| \mathrm{d}s
\]
\[
\leq \|N(y_n)-N(y)\| \to 0, \quad \text{as } n \to +\infty.
\]

**Step 2:** \( F \) maps bounded sets into bounded sets in \( C([0,T],\mathbb{R}) \). For any \( \nu > 0 \) there exists a positive constant \( k \) such that for each \( y \in B_\nu = \{y \in C([0,T],\mathbb{R}) \mid \|y\|_\infty \leq \nu \} \), we have \( |N(y)| \leq I \). By using (H6)-(H8) for each \( t \in [0,T] \), one can obtain
\[
|N(y)(t)|
\]
\[
\leq \frac{(\psi(T) + 1)}{(\psi(T) - \psi(0) + 2)} \int_0^T (\psi(\sigma) - \psi(s))^{a-1} \psi(s) g(s,y(s)) |\mathrm{d}s.
\]

Thus, for each \( t \in [0,T] \)
\[
\|N(y)(t)\| \leq \chi_2(\nu) \frac{(\psi(T) + 1)}{(\psi(T) - \psi(0) + 2)} \int_0^T \psi(\sigma) \phi_2(t) \mathrm{d}s + \int_0^T \psi(t) \phi_3(t) \mathrm{d}s
\]
\[
\leq \chi_1(\|y\|_\infty) \chi_2(\nu) \int_0^T \psi(t) \mathrm{d}s.
\]

**Step 3:** \( N \) maps bounded sets into equicontinuous sets of \( C([0,T],\mathbb{R}) \). Let \( t_1, t_2 \in [0,T] \). If \( t_1 < t_2 \) then \( \psi(t_1) < \psi(t_2) \), and let \( y \in B_\nu \), where \( B_\nu \) is a bounded set of \( C([0,T],\mathbb{R}) \) as given in step 2. Then we obtain
\[
|N(y)(t_2) - N(y)(t_1)|
\]
\[
\leq \frac{(\psi(T) + 1)}{(\psi(T) - \psi(0) + 2)} \int_0^T \psi(t) - \psi(t_1) \mathrm{d}s
\]
\[
\times \int_0^\sigma (\psi(\sigma) - \psi(s))^{a-1} \psi(s) g(s,y(s)) |\mathrm{d}s.
\]

Since \( f, g \) and \( h \) are continuous functions, we have
\[
\|N(y_n)(t) - N(y)(t)\| \to 0, \quad \text{as } n \to +\infty.
\]
which tend to 0, since \( \psi(t_2) - \psi(t_1) \rightarrow 0 \) when \( t_2 - t_1 \rightarrow 0 \) independently of \( y \). In view of the Ascoli–Arzelà theorem and the consequence of Steps 1 to 3, we conclude that \( N \) is completely continuous.

Step 4: There exists an open set \( \mathcal{U} \subset C([0, T], \mathbb{R}) \) with \( y \neq \theta N(y) \) for \( \theta \in (0, 1) \) and \( y \in \partial \mathcal{U} \). Let \( y \) be a solution of \( y - \theta N(y) = 0 \) for \( \theta \in (0, 1) \). Then for \( t \in [0, T] \) we get

\[
|y(t)| = |\theta(Ny)(t)|
\]

\[
\leq \left( \frac{1}{\psi(T) - \psi(0)} \right) \left( \frac{1}{\psi'(0) + 2} \right) \int_0^T \left| (\psi(s) - \psi(s))^{\alpha-1}\psi'(s)||g(s, y(s))| \right| \chi(s) ds + \left( \frac{1}{\psi(T) - \psi(0)} \right) \left( \frac{1}{\psi'(0) + 2} \right) \int_0^T \left| (\psi(s) - \psi(s))^{\beta-1}\psi'(s)||h(s, y(s))| \right| \chi(s) ds.
\]

Taking the supremum for \( t \in [0, T] \), yields

\[
\|y\|_\infty \leq \frac{1}{\alpha \chi_2(\|y\|_\infty) + \beta \chi_3(\|y\|_\infty) + \chi_4(\|y\|_\infty)} \leq 1.
\]

By the condition (H9), there exists a positive constant \( C \) such that \( \|y\| \neq C \). Next we define \( \mathcal{U} = \{ y \in C([0, T], \mathbb{R}), \|y\| < C \} \) and note that the operator \( N : \partial \mathcal{U} \rightarrow C([0, T], \mathbb{R}) \) is continuous and completely continuous. From the choice of \( \mathcal{U} \), there is no \( y \in \partial \mathcal{U} \) such that \( y - \theta Ny = 0 \) for some \( \theta \in (0, 1) \). Therefore by the non-linear alternative of Leray–Schauder type, we deduce that \( N \) has a fixed point \( y \in \mathcal{U} \) which is a solution of the problem (3)–(5).

Our next existence result for the problem (3)–(5) based on Leray–Schauder’s Degree theory [29].

**Theorem 3.7:** Along with condition (H4), assume that the following hypotheses hold

(H10) there exist constants \( \kappa \geq 0 \) and \( M > 0 \) such that

\[
|f(t, u)| \leq \kappa |u| + M, \quad \text{for} \ t \in I, \ a.e \ u \in \mathbb{R},
\]

(H11) there exist constants \( \vartheta \geq 0 \) and \( M^* \) such that

\[
|g(t, u)| \leq \vartheta |u| + M^*, \quad \text{for} \ t \in I, \ a.e \ u \in \mathbb{R},
\]

(H12) there exist constants \( \upsilon \geq 0 \) and \( M^{**} \) such that

\[
|h(t, u)| \leq \upsilon |u| + M^{**}, \quad \text{for} \ t \in I, \ a.e \ u \in \mathbb{R},
\]

where \( \kappa, \vartheta, \upsilon \) satisfying the following condition

\[
\vartheta \Omega_1 + \upsilon \Omega_2 + \kappa T G^*_\psi < 1,
\]

\( \Omega_1 \) and \( \Omega_2 \) are given by (23) and (24), respectively. Then the problem (3)–(5) has at least one solution on \([0, T] \).

**Proof:** Based on the fixed point problem, we have

\[
y = Ny,
\]

if and only if \( y \) is solution of the problem (3)–(5) where \( N : C([0, T], \mathbb{R}) \rightarrow C([0, T], \mathbb{R}) \) defined by 26. we simply need to establish the existence of at least one solution \( y \in C([0, T]) \) for 34. In order to complete the proof we set a ball \( \mathcal{B}_R \subset C([0, T]) \) with a constant radius \( R > 0 \) as follows

\[
\mathcal{B}_R = \{ y \in C([0, T]) : \max |y(t)| < R \}.
\]

We need to demonstrate that, \( N : \mathcal{B}_R \rightarrow C([0, T]) \) satisfies

\[
y \neq \lambda Ny, \quad \text{for all} \ u \in \partial \mathcal{B}_R, \text{and all} \ \lambda \in [0, 1].
\]

Let us define

\[
H(\lambda, y) = \lambda Ny,
\]

where \( \lambda \in [0, 1] \) and \( y \in C([0, T]) \). From Theorem 3.6, we know that the operator \( N \) is continuous, and completely continuous. Then, by the Arzèla–Ascoli theorem, a continuous map \( h_\lambda \) defined by \( h_\lambda(y) = y - H(\lambda, y) = y - \lambda Ny \) is also completely continuous. The following Leray–Schauder degrees are well defined unless (35) holds, then by the homotopy in-variance of topological degree, it is concluded that

\[
\text{deg}(h_\lambda, \mathcal{B}_R, 0) = \text{deg}(I - \lambda N, \mathcal{B}_R, 0) = \text{deg}(h_1, \mathcal{B}_R, 0) = \text{deg}(I, \mathcal{B}_R, 0) = 1 \neq 0,
\]

for \( 0 \in \mathcal{B}_R \), and \( I \) indicates the unit operator. By the non-zero property of Leray–Schauder degree, we have \( h_1(y) = y - Ny = 0 \) for at least \( x \in \mathcal{B}_R \). In order to prove 35, let us assume that \( y = \lambda Ny \) for some \( \lambda \in [0, 1] \) and for all \( t \in [0, T] \), thus

\[
|y(t)| = |\lambda(Ny)(t)|
\]

\[
\leq \left( \frac{1}{\psi(T) - \psi(0)} \right) \left( \frac{1}{\psi'(0) + 2} \right) \int_0^T \left| (\psi(s) - \psi(s))^{\alpha-1}\psi'(s)||g(s, y(s))| \right| \chi(s) ds + \left( \frac{1}{\psi(T) - \psi(0)} \right) \left( \frac{1}{\psi'(0) + 2} \right) \int_0^T \left| (\psi(s) - \psi(s))^{\beta-1}\psi'(s)||h(s, y(s))| \right| \chi(s) ds.
\]
Proof: Let $A$ and $B$ be two mappings of a Banach space $X$, such that

(i) $A$ is a contraction, and
(ii) $B$ is completely continuous.

Then, either

(a) the operator equation $y = A(y) + B(y)$ has a solution, or
(b) the set $E = \{u \in X: u = \lambda A(u/\lambda) + \lambda B(u)\}$ is unbounded for $0 < \lambda < 1$.

Theorem 3.8: Let $A$ and $B$ be two mappings of a Banach space $X$, such that

\[
(\Omega_1 k^* + \Omega_2 k^{**}) < 1,
\]

and

\[
\|y\|_\infty (1 - \Omega_2 k^* - \Omega_2 k^{**}) > 1,
\]

(36)

\[
g^{*\alpha - 1}(\psi(\sigma)) + h^\beta(\psi(\eta))^\beta + cG^*_\psi(\|y\|_\infty) > 1,
\]

(37)

holds, where $\Omega_1$, and $\Omega_2$ defined by (23), and (24) respectively, $g^{*} = \sup_{t \in [0,1]}|g(t,0)|$, $h^* = \sup_{t \in [0,1]}|h(t,0)|$, the problem (3)–(5) has at least one solution on $I$.

Proof: We define the operators $A, B : C([0,T], \mathbb{R}) \rightarrow C([0,T], \mathbb{R})$ by

\[
(Ay)(t) = \int_0^T G_\psi(s,t)\|f(s,y(s))\|ds,
\]

(38)

\[
(By)(t) = \int_0^T G_\psi(t,s)f(s,y(s))\|ds,
\]

where $G_\psi$ is a Green’s function given by (11). The operator $A$ is a contraction map from the Banach space $C([0,T], \mathbb{R})$ into itself. Indeed, by using conditions (H1) and (H2), for $x, y \in C([0,T], \mathbb{R})$, we get

\[
|Ay(t) - Ax(t)| \leq \int_0^T |f(s,y(s)) - f(s,x(s))|ds,
\]

(39)

which, on taking supremum over $t \in [0,T]$, yields

\[
\|Ay(t) - Ax(t)\|_\infty \leq (\Omega_1 k^* + \Omega_2 k^{**})\|y - x\|_\infty.
\]

Therefore $A$ is a contraction as (36) holds. Clearly $B$ is completely continuous by using condition (H6). Thus, we just need to prove that $E := \{y \in C([0,T]) : y = \lambda A(y/\lambda) + \lambda B(y), \lambda \in (0,T)\}$ is bounded. Let $y \in E$; then for each $t \in [0,T]$, we are led to the homotopy equation

\[
y(t) = \lambda A(y/\lambda)(t) + \lambda B(y)(t).
\]

From (H2), (H3) and (H6) we have

\[
|y(t)| \leq \lambda(\psi(T) + 1)\psi(|\sigma|)
\]

\[
\times \left(\left|g(s, y(s)/\lambda) - g(t,0)\right| + |g(t,0)|\right)ds,
\]

(40)

\[
+ \lambda(\psi(T) + 1)\psi(|\eta|),
\]

(41)

\[
+ \lambda \int_0^T |G_\psi(s,t)||f(s,y(s))|ds,
\]

(42)

\[
\leq \psi(T) + 1|f(s,y(s))|ds,
\]

(43)

\[
+ g^* \sup_{0 < \sigma < T} |G_\psi(s,t)||f(s,y(s))|ds.
\]

(44)
By means of (37), it follows that there exist
fulness of the main results. Let us consider the following
In this section, we give an example to illustrate the use-
4. Example
In this section, we give an example to illustrate the use-
fulness of the main results. Let us consider the following
fractional boundary value problem
\begin{align*}
\mathcal{D}_t^q y(t) &= \frac{\sin\left(\frac{\pi}{2} t\right)}{5\pi} \log(y) \quad \text{for } t > 0, \\
(t, y) &\in [0, 1] \times [e, +\infty), \\
y(0) &= 1, \\
y(T) &= \frac{1}{3} y(T), \
\end{align*}
where \( 1 < q \leq 2, \quad \sigma = \frac{1}{2}, \quad \eta = \frac{1}{2}, \quad \text{and } T = 1. \)
Set
\begin{align*}
f(t, x) &= \frac{\sin\left(\frac{\pi}{2} t\right)}{5\pi} \log(x), \\
g(t, x) &= \frac{x}{1 + 5x^2}, \\
h(t, x) &= \frac{x}{1 + 3x^2},
\end{align*}
where \( 1 < q \leq 2, \quad \sigma = \frac{1}{2}, \quad \eta = \frac{1}{2}, \quad \text{and } T = 1. \)
Let us take \( \psi(t) = t^3 + 1. \) Clearly \( \psi \) is an increasing function on \([0, 1]\) and \( \psi(0) = 3t^2 \) is a continuous function on \([0, 1].\)
Let \( u, v \in [e, +\infty), \) and \( t \in [0, 1]. \) Then, we have
\begin{align*}
|f(t, u) - f(t, v)| &\leq \frac{\sin\left(\frac{\pi}{2} t\right)}{5\pi} \left| \log(u) - \log(v) \right| \\
&\leq \frac{\sin\left(\frac{\pi}{2} t\right)}{5\pi} \left| \frac{\log(u)}{1 + \log(u)} - \frac{\log(v)}{1 + \log(v)} \right| \\
&\leq \frac{\sin\left(\frac{\pi}{2} t\right)}{5\pi} \left| \frac{\log(u) - \log(v)}{1 + \log(u)(1 + \log(v))} \right| \\
&\leq \frac{\sin\left(\frac{\pi}{2} t\right)}{5\pi} \left| \frac{\log(u) - \log(v)}{1 + \log(u)(1 + \log(v))} \right| \\
&\leq \frac{\sin\left(\frac{\pi}{2} t\right)}{5\pi} \left| u - v \right| \leq \frac{1}{5\pi} |u - v|,
\end{align*}
Thus, condition (H1) holds with \( k = 1/5\pi. \) It is easy to show that conditions (H2) – (H3) are satisfied with \( k = \frac{1}{5\pi}, k^* = \frac{1}{5\pi}. \)
From (11) Green’s function \( G_\psi \) is given by
\begin{align*}
G_\psi(t, s) &= \begin{cases}
\frac{(t^3 - s^3)q - 1}{\Gamma(q)} - \frac{(t^3 + 1)(1 - s^3)\beta - 1}{(t^3 + 1)\Gamma(q)}, & 0 \leq s < t, \\
\frac{(t^3 + 1)(1 - s^3)\beta - 1}{t^2 + 1}, & t \leq s \leq 1.
\end{cases}
\end{align*}
From (42) we have
\begin{align*}
\int_0^1 G_\psi(t, s) ds &= \int_0^t G_\psi(t, s) ds + \int_t^1 G_\psi(t, s) ds \\
&= - \frac{3\pi}{\Gamma(q + 1)} - \frac{3\pi^2}{\Gamma(q + 1)} (1 - t^3)\beta \\
&\quad + \frac{3\pi^2}{\Gamma(q + 1)} (1 - t^3)\beta \\
&\quad + \frac{3\pi^2}{\Gamma(q + 1)} (1 - t^3)\beta \\
&\quad + \frac{3\pi^2}{\Gamma(q + 1)} (1 - t^3)\beta.
\end{align*}
A simple calculation yields
\begin{align*}
G_\psi &\leq \frac{3}{\Gamma(q + 1)} + \frac{3}{\Gamma(q)}.
\end{align*}
Now
\begin{align*}
\Omega_1 k^* + \Omega_2 k^* + TG_\psi k &= \frac{(\psi(t))^2}{3\pi^2 (\alpha + 1)} + \frac{(\psi(t))^2}{16\pi (\beta + 1)} \\
&\quad + \frac{3}{5\pi} \left( \frac{\Gamma(q + 1)}{\Gamma(q + 1)} \right) < 1,
\end{align*}
thus condition (25) is satisfied with \( T = 1, q \in (1, 2], \) and for each \( \alpha, \beta \in (0, 1). \) By Theorem 3.2, the problem (39)–(41) has a unique solution on \([0, 1].\)

5. Conclusion
The fractional calculus has been attracting many scien-
tists because of the good results obtained when the tra-
tional derivatives are replaced with fractional deriva-
tives. Hence, studying the qualitative properties such as the existence and uniqueness of solutions to differential
equations in the framework of fractional derivatives has gained importance. In this article, we discussed the existence and uniqueness of a certain class of boundary
value problem in the frame of \( \psi \)-Caputo fractional
derivatives and with boundary conditions expressed in terms of \( \psi \)-Riemann–Liouville fractional integrals. This class of boundary value problem is new and is the generalization of some systems discussed in the literature. In fact, the \( \psi \)-fractional operators contain within themselves some conventional fractional operators such as the Riemann–Liouville and Hadamard fractional operators and many others.

**Acknowledgments**

The authors would like to thank the reviewers and the editor for their valuable comments.

**Disclosure statement**

No potential conflict of interest was reported by the authors.

**ORCID**

Samih Belmor
http://orcid.org/0000-0002-1659-4734

Chokkalingam Ravichandran
http://orcid.org/0000-0003-0214-1280

Fahd Jarad
http://orcid.org/0000-0002-3303-0623

**References**

[1] Baleanu D, Fernandez A. On fractional operators and their classifications. Mathematics. 2019;7(9):830.

[2] Baleanu D, Diethelm K, Scalas E, et al. Fractional calculus models and numerical methods. Boston (MA): World Scientific; 2012. (Series on complexity, nonlinearity and chaos).

[3] Abd-Elhameed WM, Youssri YH. Sixth-Kind Chebyshev spectral approach for solving fractional differential equations. Int J Nonlinear Sci Numer Simul. 2019;20(2):191–203.

[4] Abd-Elhameed WM, Youssri YH. Fifth-kind orthonormal Chebyshev polynomial solutions for fractional differential equations. Comput Appl Math. 2018;37(3):2897–2921.

[5] Jarad F, Harikrishnan S, Shah K, et al. Existence and stability results to a class of fractional random implicit differential equations involving generalized Hilfer fractional derivative. Discrete Cont Dyn-S. 2019;13:723–739.

[6] Hafez RM, Youssri YH. Legendre-collocation spectral solver for variable-order fractional functional differential equations. Comput Methods Differ Equ. 2019;8:99–110.

[7] Youssri YH, Hafez RM. Chebyshev collocation treatment of Volterra–Fredholm integral equation with error analysis. Arab J Math. 2019; DOI:10.1007/s40465-019-0243-y

[8] Teodoru GS, Machado JT, De Oliveira EC. A review of definitions of fractional derivatives and other operators. J Comput Phys. 2019;388:195–208.

[9] Sun H, Zhang Y, Baleanu D, et al. A new collection of real world applications of fractional calculus in science and engineering. Commun Nonlinear Sci Numer Simul. 2018;64:213–231.

[10] Ahmad B, Ntouyas SK, Zhou Y, et al. A study of fractional differential equations and inclusions with nonlocal Erdelyi–Kober type integral boundary conditions. Bull Math Soc. 2018;44(5):1315–1328.

[11] Ahmad B, Alghanmi M, Ntouyas SK, et al. Fractional differential equations involving generalized derivative with Stieltjes and fractional integral boundary conditions. Appl Math Lett. 2018;84:111–117.

[12] Sun Q, Meng S, Cui Y. Existence results for fractional order differential equation with nonlocal Erdelyi–Kober and generalized Riemann–Liouville type integral boundary conditions at resonance. Adv Differ Equ. 2018;2018:243.

[13] Agarwal RP, Benchohra M, Hamani S. A survey on existence results for boundary value problems of nonlinear fractional differential equations and inclusions. Acta Appl Math. 2010;109(3):973–1033.

[14] Samko SG, Kilbas AA, Marichev OL. Fractional integrals and derivatives, theory and applications. Yverdon: Gordon and Breach; 1993.

[15] Voyiadjis GZ, Sumelka W. Brain modelling in the framework of anisotropic hyperelasticity with time fractional damage evolution governed by the Caputo-Almeida fractional derivative. J Mech Behav Biomed Mat. 2019;89:209–216.

[16] Harikrishnan S, Shah K, Baleanu D, et al. Note on the solution of random differential equations via \( \psi \)-Hilfer fractional derivative. Adv Differ Equ. 2018;1:224.

[17] Ozarslan R, Ercan A, Bas E. \( \beta \)-type fractional Sturm–Liouville Coulomb operator and applied results. Math Meth Appl Sci. 2019;1–12. DOI:10.1002/mma.5769

[18] Panakhov E, Ercan A, Bas E, et al. Hilfer fractional spectral problem via bessel operator. TWMS J Pure Appl Math. 2019;10(2):199–211.

[19] Almeida R. Fractional differential equations with mixed boundary conditions. B Malays Math Sci Soc. 2019;42(4):1687–1697.

[20] Almeida R. A Caputo fractional derivative of a function with respect to another function. Commun Nonl Sci Numer Simul. 2017;44:460–481.

[21] Jarad F, Abdeljawad T. Generalized fractional derivatives and Laplace transform. Discrete Cont Dyn-S. DOI:10.3934/dcdss.2020039

[22] Benchohra M, Graef JR, Hamani S. Existence results for boundary value problems with nonlinear fractional differential equations. Appl Anal. 2008;87(7):851–863.

[23] Benchohra M, Graef JR, Mostefai FZ. Weak solutions for nonlinear fractional differential equations on reflexive Banach spaces. Electron J Qual Theo Differ Equ. 2010;54:1–10.

[24] Benchohra M, Cabada A, Seba D. An existence result for nonlinear fractional differential equations on Banach spaces. Bound Value Probl. 2009;2009:1–11. Art. ID 628916.

[25] Belarbi A, Benchohra M, Dhage BC. Existence theory for perturbed boundary value problems with integral boundary conditions. Georgian Math J. 2006;13:215–228.

[26] Granas A, Dugundji J. Fixed point theory. New York: Springer; 2003.

[27] Boyd DW, Wong JSW. On nonlinear contractions. Proc Am Math Soc. 1969;20:485–464.

[28] Smart DR. Fixed point theorems. London: Cambridge University Press; 1974.

[29] Leray J, Schauder J. Topologie et équations fonctionnelles. Ann Sci Ecole Norm S. 1934;3(1):45–78.

[30] Burton TA, Kirk C. A fixed point theorem of Krasnoselskii–Schaefer type. Math Nachr. 1998;189:23–31.