On quantum cryptography with bipartite bound entangled states

Paweł Horodecki* and Remigiusz Augusiak

Faculty of Applied Physics and Mathematics, Gdańsk University of Technology,
Narutowicza 11/12, PL 80-952 Gdańsk, Poland

Abstract

Recently the explicit applicability of bound entanglement in quantum cryptography has been shown. In this paper some of recent results respecting this topic are reviewed. In particular relevant notions and definitions are reminded. The new construction of bound entangled states containing secure correlations is presented. It provides low dimensional $6 \otimes 6$ bound entangled states with nonzero distillable key.

1 Introduction

The explicit application of quantum entanglement in quantum information theory was the cryptographic protocol by Ekert [1]. The essential point of the protocol (cf. further modification [2]) was the entanglement monogamy principle (see [3]) which says that if the two particles are maximally entangled with each other then they are completely unentangled with any other (third) party. Hence, results of any correlation measurements on both particles must be completely safe from cryptographic point of view, as they are uncorrelated form results of any other measurement performed on the rest of the world. This point was further exploited in a nice application [4] of entanglement distillation [5] (cf. [6]). The idea of Ref. [4] is called quantum privacy amplification (QPA). Given stationary source of pure states $|\Psi_{ABE}\rangle$ describing Alice, Bob (which are cooperating) and Eve (eavesdropper) quantum correlations protocol QPA is focused on distilling maximally entangled states

$$|\Psi^{(d)}_+\rangle = \frac{1}{\sqrt{d}} \sum_{i=0}^{d-1} |i\rangle \otimes |i\rangle$$

(1.1)

from quantum entangled states

$$\varrho_{AB} = \text{Tr}_E |\Psi_{ABE}\rangle \langle \Psi_{ABE}|.$$ (1.2)

The distillation protocol uses local operations and classical communication (LOCC) which in presence of Eve are usually called local operations and public communication (LOPC). Once Alice and Bob distill maximally correlated states $|\Psi^{(d)}_+\rangle$ by entanglement monogamy they share log $d$ bits of classical secure bits. This can be done by performing local measurements on the state (1.1) in standard bases $\{|i\rangle_A\}_{i=0}^{d-1}$ and $\{|i\rangle_B\}_{i=0}^{d-1}$.

However, since 1998 it has been known that there is quantum entanglement called bound entanglement that can not be distilled to a pure form [7]. For a long time there

*email: pawel@mif.pg.gda.pl

---

arXiv:0712.3999v1 [quant-ph] 24 Dec 2007
was a common belief that distillation of secure key from quantum state is possible only when QPA is. In other words, that bound entanglement is useless for quantum cryptography. In fact the results of extensive analysis of two qubit case [8] naturally suggested equivalence of entanglement distillation protocols and secure key distillation.

Surprisingly it is not true, as it has been shown in papers [9, 10]. Before we shall recall main observations of the latter, let us point out the key ingredient of their reasoning. Namely, why QPA might be not necessary for distilling secure key? In fact, if Alice and Bob share maximally entangled state (1.1) they in a sense have much stronger security than they need. In fact they will get secure correlations if they measure the state in any pair of bases of the form: \(|e_i^{(A)}⟩ = U|i⟩_A \) and \(|e_i^{(B)}⟩ = U^*|i⟩_B \), where \(U \) stands for arbitrary unitary operation. It is crucial to understand that in quantum cryptography it would be enough to have a single basis, measurement in which could give secure correlations. To be more specific, the security requirement is to get (via local measurement of Alice and Bob in one basis) the state that is product with Eve’s degrees of freedom.

This leads us to the notion of ccq state [12], i.e., tripartite state of Alice, Bob and Eve that is, after local von Neumann measurements, classical on Alice and Bob parts and quantum on Eve part. With this notion one can summarize the idea of [9, 10] as follows: Alice and Bob should proceed to distill such a state \(\gamma_{AB'A'B'} \) that, after considering its purification \(|Ψ_{AB'A'B'E}⟩ \) and performing a local measurements in a standard product basis on its \(AB \) part the resulting ccq state \(\tilde{Ψ}_{ABE} \) is (i) product with respect to the division \(AB|E \) (ii) contains maximal classical correlations between Alice and Bob.

In this work we shall briefly describe main mathematical elements of the construction [9, 10], recall the idea of one-way distillation provided in [11] and provide a new construction of bound entangled states with secure quantum key.

2 Main notions of general secure key distillation scheme

Here we shall remind and discuss main ideas and notions introduced in Refs. [9, 11].

Assume that Alice and Bob wish to communicate but without participation of an eavesdropper Eve. To this aim, as a source of quantum correlations, they use a quantum state \(\varrho_{AB'A'B'} \). Here, subsystem \(AA' (BB') \) belongs to Alice (Bob). Moreover, following [9], subsystem \(AB (A'B') \) is called the key part (shield part) of the given state. To make the considerations more formal, each subsystem of \(\varrho_{AB'A'B'} \) shall be represented by respective Hilbert space, i.e., Alice’s subsystems by \(H_A \) and \(H_{A'} \) and Bob’s by \(H_B \) and \(H_{B'} \). Hence, the key part of \(\varrho_{AB'A'B'} \) is defined on \(H = H_A ⊗ H_B \) and the shield part on \(H' = H_{A'} ⊗ H_{B'} \). Hereafter we shall be assuming that \(\text{dim} H_A = \text{dim} H_B = d, \text{dim} H_{A'} = \text{dim} H_{B'} = d_{A'} \). According to the results of [9, 10], let us remind the notion of ccq state. To this aim let us introduce a product basis defined on Hilbert space \(H_A ⊗ H_B = C^d ⊗ C^d , \) i.e.,

\[
\mathcal{B}_{AB} = \left\{ |e_i^{(A)}⟩ ⊗ |e_j^{(B)}⟩ \right\}_{i,j=0,...,d-1},
\]

where \(\{ |e_i^{(A)}⟩ \}_{i=0,...,d-1} \) and \(\{ |e_i^{(B)}⟩ \}_{i=0,...,d-1} \) are arbitrary bases spanning the Hilbert spaces, respectively, \(H_A \) and \(H_B \). Of course, these bases may be chosen to be standard and therefore \(\mathcal{B}_{AB} \) is the standard basis in \(C^d ⊗ C^d \). Then we have the following

Definition 1. (ccq state) We call the state \(\tilde{Ψ}_{ABE} \) a ccq state of initial state \(\varrho_{AB'A'B'} \) with respect to the basis \(\mathcal{B}_{AB} \) if \(\tilde{Ψ}_{ABE} \) is a result of measurement of
\( \varrho_{ABE} = \text{Tr}_{A'B'}|\Psi_{ABA'B'}\rangle \langle \Psi_{ABA'B'}| \) in the product basis \( \mathcal{B}_{AB} \). Here \( |\Psi_{ABA'B'}\rangle \) is a purification of \( \varrho_{ABA'B'} \).

As an illustrative example let us consider a density matrix \( \eta_{ABA'B'} \) acting on \( (\mathbb{C}^2)^{\otimes 4} \) and given by \( \eta_{ABA'B'} = p|0111\rangle \langle 0111| + (1-p)|1000\rangle \langle 1000| \). As one may easily verify its standard purification takes the form \( |\Psi_{ABA'B'}\rangle = \sqrt{p}|0111\rangle + \sqrt{1-p}|1000\rangle \). Therefore, the ccq state of \( \eta_{ABA'B'} \) with respect to standard basis 
\( \mathcal{B}_{AB}^{(st)} = \{ |i\rangle_A |j\rangle_B \}_{i,j=0}^1 \) is \( \bar{\eta}_{ABE} = p|1010\rangle \langle 1010| + (1-p)|0110\rangle \langle 0110| \).

Now, one can ask when a given state is said to be secure. As an answer one gives the following (see [9, 10])

**Definition 2.** (security) We call the state \( \varrho_{ABA'B'} \) **secure** with respect to the basis \( \mathcal{B}_{AB} \) if its ccq state is of the form
\[
\bar{\varrho}_{ABE} = \left[ \sum_{i,j=0}^{d-1} p_{ij} |e_i^{(A)}\rangle \langle e_j^{(B)}| |e_i^{(A)}\rangle \langle e_j^{(B)}| \right] \otimes \varrho_E. \tag{2.4}
\]

The security of such a state follows from the fact that Eve is completely uncorrelated from distribution represented by \( \mathcal{B}_{AB} \) system after Alice and Bob measurement. Note that if the distribution \( p_{ij} \) is homogenous, i.e., \( p_{ij} = 1/d; (i,j = 0, \ldots, d-1) \), then we say that \( \varrho_{ABA'B'} \) has a \( \mathcal{B}_{AB} \)-**key**.

A very important ingredient of construction discussed here is a special class of controlled unitary operations (see [9, 10]) that we recall by the following

**Definition 3.** (twisting) Let \( U_{ij}^{(A'B')} \) be certain unitary operations acting on subsystem \( A'B' \). For a given basis \( \mathcal{B}_{AB} \) we call the operation
\[
U = \sum_{i,j=0}^{d-1} |e_i^{(A)}\rangle \langle e_j^{(B)}| |e_i^{(A)}\rangle \langle e_j^{(B)}| \otimes U_{ij}^{(A'B')} \tag{2.5}
\]

\( \mathcal{B}_{AB} \)-twisting or shortly, twisting.

The importance of such a class of operations follows from the fact that applied to a given state \( \varrho_{ABA'B'} \), \( U \) do not change its ccq state. It means that if we take \( \varrho_{ABA'B'} \) and \( \varrho_{ABA'B'} = U \varrho_{ABA'B'} U^\dagger \), their ccq states are exactly the same, i.e., \( \bar{\varrho}_{ABE} = \tilde{\varrho}_{ABE} \).

Now we define the central notion of the generalised approach provided in [9, 10]. This is the notion of private state that has log \( d \) bits of secure key encoded in its \( AB \) part (of \( d \) \( d \) type).

**Definition 4.** (pdit) Let \( \varrho_{ABA'B'} \) is a density operator acting on the Hilbert space \( \mathcal{H} \otimes \mathcal{H}' \) and \( \varrho_{ABA'B'} \) is a density matrix acting on \( \mathcal{H}' \). Let \( U_i^{(A'B')} \) (\( i = 0, \ldots, d-1 \)) be certain unitary operations acting on \( A'B' \) system. Then we call the state \( \varrho_{ABA'B'} \) **private state** or pdit with respect to the basis \( \mathcal{B}_{AB} \) if it is of the form
\[
\varrho_{ABA'B'} = \frac{1}{d} \sum_{i,j=0}^{d-1} |e_i^{(A)}\rangle \langle e_j^{(B)}| |e_i^{(A)}\rangle \langle e_j^{(B)}| \otimes U_i^{(A'B')} \varrho_{ABA'B'} U_j^{(A'B')\dagger}. \tag{2.6}
\]

Hereafter, as usual, we shall denote the private dits by \( \gamma_{ABA'B'}^{(d)} \). In the case when the dimension of the key part is \( d = 2 \) on each side, we have to do with **private bit** or pbit.

Now it is important to pose the question: **What is the essential feature that allows private bit to be truly private?** Namely, a detailed analysis shows [9, 10] that ccq state
of private bit $\gamma^{(d)}_{ABA'B'}$ is the same as the ccq state of the following state (called \textbf{basic pdit})

$$\mathcal{P}^{(d)}_+ \otimes \sigma_{ABA'B'},$$

(2.7)

where $\sigma_{ABA'B'} = \text{Tr}_{AB} \gamma^{(d)}_{ABA'B'}$ and $\mathcal{P}^{(d)}_+$ is a projector onto $|\Psi^{(d)}_+\rangle$. This is because it is always possible for a given pbit to find such a twisting under which $\gamma^{(2)}_{ABA'B'}$ transforms it to $|\Psi^{(d)}_+\rangle$ and conversely. Thus, after performing measurement in the basis $P_{ABA'}$, the physical system $ABE$ is in the same state irrespective of whether before Alice and Bob state was (2.7) or just a pdit $\gamma^{(d)}_{ABA'B'}$. Hence the security with respect to the measurement in that particular basis is the same as if Alice and Bob really shared maximal entangled state $\mathcal{P}^{(d)}_+$. This is the key observation for understanding the essence of private dit.

We conclude the preliminary section recalling the definition of distillable key \cite{9, 10} and related theorem.

\textbf{Definition 5. (distillable key)} Let $\sigma_{AB}$ be a density matrix acting on $\mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B}$ and let $\mathcal{P}_n$ be a sequence of LOCC operations such that $\mathcal{P}_n(\sigma^{\otimes n}_{AB}) = \Sigma_n$, where $\Sigma_n$ is defined on $\mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B} \otimes \mathcal{H}_A \otimes \mathcal{H}_B$. The set $\mathcal{P} = \{ \mathcal{P}_n \}_{n=1}^{\infty}$ is said to be a \textbf{pdit distillation protocol} of a given state $\sigma_{AB}$ if the following relation

$$\lim_{n \to \infty} \left\| \Sigma_n - \gamma^{(d_n)}_{ABA'B'} \right\|_{\text{Tr}} = 0.$$  

(2.8)

The rate of this protocol is defined as

$$\mathcal{R}(\mathcal{P}) = \limsup_{n,d_n \to \infty} \frac{\log d_n}{n}$$

(2.9)

and distillable key of $\sigma_{AB}$ as

$$K_D(\sigma_{AB}) = \sup_{\mathcal{P}} \mathcal{R}(\mathcal{P}).$$

(2.10)

There is a problem however, since the above definition has a complicated form. It is hard to see whether given state fulfills the above condition or not. Fortunately, one can simplify the task providing necessary and sufficient conditions for nonzero distillable key that are more operational then the definition itself. Here we provide a summary of the conditions proven in \cite{9, 10}, which are enough to analyse cryptographic usefulness of many quantum states:

\textbf{Theorem 1.} The following three conditions are equivalent:

(i) finite number of copies of state $\varrho_{ABA'B'}$ can be transformed with some LOCC protocol into the state $\sigma^{(d)}_{ABA'B'}$ arbitrarily close in the trace norm to certain pbit $\gamma^{(2)}_{ABA'B'}$

(ii) finite number of copies of state $\varrho_{ABA'B'}$ can be transformed with some LOCC protocol into the state $\sigma_{ABA'B'}$ with trace norm of the element $A^{(A'B')}_{00,11}$ arbitrarily close to $1/2$, where the element is defined by the representation:

$$\varrho_{ABA'B'} = \sum_{i,j,k,l=0}^{1} \ket{e_i^{(A)} e_j^{(B)} e_k^{(A')} e_l^{(B')}} \bra{e_i^{(A)} e_j^{(B)} e_k^{(A')} e_l^{(B')}} \otimes A^{(A'B')}_{ij,kl}.$$  

(2.11)

(iii) the state $\varrho$ has nonzero distillable key, i.e., one has $K_D(\varrho) > 0$.

Moreover any convergence of $\|A^{(A'B')}_{00,11}\|_{\text{Tr}}$ to $1/2$ from (ii) during a given protocol is equivalent to a convergence of the state to a certain pbit during that protocol.

Especially the condition (ii) serves as a useful criterion which allows to adjudge an applicability of a given state to quantum cryptography. In next section we shall illustrate its power.


3 New class of bound entangled states with secure quantum key

In this section we present the main result. We provide a construction of a state that is useful for quantum cryptography simultaneously being bound entangled. Note that the construction, however based on that presented in [11], is different in details from known so far [9, 10, 11] and sheds some light on the still unexplored domain of bound entanglement. Hereafter it is assumed that $d = 2$, i.e., the key part of $\mathcal{H}_A' \otimes \mathcal{H}_B'$ consists of qubits and that $\dim \mathcal{H}_A = \dim \mathcal{H}_B = D$, which allows us to write $\mathcal{H} = \mathbb{C}^D \otimes \mathbb{C}^D$.

We may also assume for simplicity that $B$ is standard basis in $\mathbb{C}^D \otimes \mathbb{C}^D$.

3.1 The construction

To obtain a better insight into the construction we begin our considerations from an illustrative example with the shield part of dimension $D = 3$ on each side. This with the assumption that $d = 2$ makes the considered state to be of dimension 6 in each side. Finally, we show that the construction is possible for arbitrary $D \geq 3$.

3.1.1 The $6 \otimes 6$ case

At the very beginning suppose that Alice and Bob possess the following state (cf. [11])

$$\varrho_{ABA'B'} = \frac{11}{40} \begin{bmatrix} |X_3| & 0 & 0 & X_3 \\ 0 & |X_3^{T_{B'}}|^{T_{B'}} & 0 & 0 \\ 0 & 0 & |X_3^{T_{B'}}|^{T_{B'}} & 0 \\ X_3 & 0 & 0 & |X_3| \end{bmatrix}$$

where the superscript $T_{B'}$ denotes the partial transposition with respect to the system $B'$ and $X_3$ is a symmetric $9 \times 9$ matrix of the form

$$X_3 = \frac{1}{11} \begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

which is defined on the Hilbert space $\mathcal{H}_A' \otimes \mathcal{H}_B' = \mathbb{C}^3 \otimes \mathbb{C}^3$. In order to show that the matrix $\varrho_{ABA'B'}$ is Hermitian and nonnegative, i.e., represents a quantum state, let us observe that $X$ may be decomposed as $X = (1/11)(\mathcal{P}^{(3)}_+ - 2\mathcal{P}^{(3)} + \mathcal{Q}^{(3)})$. Here $\mathcal{P}^{(3)}_+$ is a projector onto the maximally entangled state $|\Psi^+_3\rangle = (1/\sqrt{3})\sum_{i=0}^{2}|ii\rangle|ii\rangle$ belonging to $\mathbb{C}^3 \otimes \mathbb{C}^3$ and $\mathcal{P}^{(3)}$, and $\mathcal{Q}^{(3)}$ are projectors given by relations

$$\mathcal{P}^{(3)} = \sum_{i=0}^{2}|ii\rangle\langle ii| - \mathcal{P}^{(3)}_+, \quad \mathcal{Q}^{(3)} = \mathcal{I} - \sum_{i=0}^{2}|ii\rangle\langle ii|.$$
where $I_9$ stands for an identity acting on the Hilbert space $\mathbb{C}^3 \otimes \mathbb{C}^3$. These above projectors are orthogonal and therefore one obtains

$$|X_3| = \frac{1}{11} \left[ P^{(3)}_+ + 2P^{(3)} + Q^{(3)} \right] = \frac{1}{11} \left[ \begin{array}{ccccccc} \frac{5}{9} & 0 & 0 & 0 & -\frac{1}{3} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{3} & 0 & 0 & 0 & \frac{5}{9} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ -\frac{1}{3} & 0 & 0 & 0 & -\frac{1}{3} & 0 & 0 \\ \frac{5}{9} & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right].$$

(3.15)

From the above equation one infers two facts, first that the trace norm\(^1\) of $X_3$ is $||X_3||_T = 1$ and the second that $|X_3|^{T_{B'}} \geq 0$. Moreover, the matrix $X_3$ partially transposed with respect to the subsystem $B'$ may be written in the form $X_3^{T_{B'}} = (1/11) [2S^{(3)} - (I_9 - Q^{(3)})]$, where $S^{(3)}$ is a projector given by

$$S^{(3)} = \frac{1}{2} \left[ I_9 + \mathcal{V}^{(3)} - 2 \sum_{i=0}^2 |ii\rangle \langle ii| \right]$$

with $\mathcal{V}^{(3)}$ being the swap operator defined by relation $\mathcal{V}^{(3)}|\varphi_1\rangle |\varphi_2\rangle = |\varphi_2\rangle |\varphi_1\rangle$ for $|\varphi_i\rangle \in \mathbb{C}^3$ ($i = 1, 2$). Note that $I_9 - Q^{(3)}$ and $S^{(3)}$ are orthogonal projectors and therefore $X_3^{T_{B'}} = (1/11) [2S^{(3)} + I_9 - Q^{(3)}]$. Since $|X_3^{T_{B'}}|^{T_{B'}} = (1/11) [2P^{(3)} + Q^{(3)}]$, one may easily conclude that $|X_3^{T_{B'}}|^{T_{B'}} \geq 0$. This with the aid of the fact that the matrix $X_3$ is symmetric and real, ensures that $\varrho_{ABA'B'}$ represents a quantum state.

Now we are in position to prove that $\varrho_{ABA'B'}$ satisfies PPT (positive partial transpose) criterion \cite{[13],[14]}. To this aim we show that transposition with respect to subsystem $BB'$ preserves the positivity. Indeed, the state $\varrho_{ABA'B'}$ transposed with respect to the $BB'$ subsystem, remains a positive operator. To see this fact explicitly, we write

$$\varrho^{T_{B'B'}}_{ABA'B'} = \frac{11}{40} \begin{pmatrix} |X_3|^{T_{B'}} & 0 & 0 & 0 \\ 0 & X_3^{T_{B'}} & X_3^{T_{B'}} & 0 \\ 0 & X_3^{T_{B'}} & X_3^{T_{B'}} & 0 \\ 0 & 0 & 0 & |X_3|^{T_{B'}} \end{pmatrix}.$$  

(3.17)

Positivity of the above operator stems from two facts. As previously mentioned $|X_3|^{T_{B'}} \geq 0$ and on the other hand the off-diagonal elements are blocked by $|X_3^{T_{B'}}|$.  

### 3.1.2 Construction of general $2D \otimes 2D$ case

Trying to generalize the above investigations we start from the symmetric matrix

$$X_D = \frac{1}{D^2 + 2D - 4} \left[ (D - 2)P^{(D)}_+ - 2P^{(D)} + Q^{(D)} \right],$$

(3.18)

where, as previously, $P^{(D)}_+$ represents a projector onto maximally entangled state $|\Psi^{(D)}_+\rangle$. Orthogonal projectors $P^{(D)}$ and $Q^{(D)}$ are given by

$$P^{(D)} = \sum_{i=0}^{D-1} |ii\rangle \langle ii| - P^{(D)}_+, \quad Q^{(D)} = I_{D^2} - \sum_{i=0}^{D-1} |ii\rangle \langle ii|. \quad (3.19)$$

\(^1\)For an arbitrary matrix $\Xi$ the trace norm is defined by relation $||\Xi||_T = \text{Tr} \sqrt{\Xi^{\dagger} \Xi}$. 

6
From Eq. (3.15) we have

\[ |X_D| = \frac{1}{D^2 + 2D - 4} \left[ (D - 2)P_+^{(D)} + 2P_+^{(D)} + Q^{(D)} \right] \]  

(3.20)

and therefore \(|X_D|^{T_{B'}} \geq 0\). Subsequently, after elementary steps, we may obtain

\[ X_D^{T_{B'}} = 1/((D^2 + 2D - 4)[2S^{(D)} - (I_{D^2} - Q^{(D)})]] \]

with \(S^{(D)}\) being defined as

\[ S^{(D)} = \frac{1}{2} \left[ I_{D^2} + \mathcal{V}^{(D)} - 2 \sum_{i=0}^{D-1} |ii\rangle \langle ii| \right], \]  

(3.21)

where \(\mathcal{V}^{(D)}\) is the swap operator acting on \(\mathbb{C}^{D} \otimes \mathbb{C}^{D}\). Again, the projectors \(I_{D^2} - Q^{(D)}\) and \(S^{(D)}\) are orthogonal and therefore

\[ |X_D|^{T_{B'}} = \frac{1}{D^2 + 2D - 4} \left[ 2S^{(D)} + I_{D^2} - Q^{(D)} \right]. \]  

(3.22)

Finally, performing partial transposition with respect to subsystem \(B'\), we have

\[ |X_D|^{T_{B'}}^{T_{B'}} = 1/((D^2 + 2D - 4)[2P_0^{(D)} + Q^{(D)}]] \]  

and therefore \(|X_D|^{T_{B'}}^{T_{B'}} \geq 0\). Now we can introduce a class of mixed states

\[ \varrho_{ABA'B'}^{(D)} = \frac{1}{4} \frac{D^2 + 2D - 4}{2D^2 + D - 2} \left[ \begin{array}{cccc} |X_D| & 0 & 0 & X_D \\ 0 & |X_D|^{T_{B'}} & 0 & 0 \\ 0 & 0 & |X_D|^{T_{B'}} & 0 \\ X_D & 0 & 0 & |X_D| \end{array} \right]. \]  

(3.23)

Again, by virtue of the fact that \(|X_D|^{T_{B'}} \geq 0\) one may infer that partial transposition with respect to subsystem \(BB'\) preserves the positivity of \(\varrho_{ABA'B'}^{(D)}\). In other words, the following matrix

\[ \varrho_{ABA'B'}^{(D)T_{BB'}} = \frac{1}{4} \frac{D^2 + 2D - 4}{2D^2 + D - 2} \left[ \begin{array}{cccc} |X_D|^{T_{B'}} & 0 & 0 & 0 \\ 0 & |X_D|^{T_{B'}} & X_D^{T_{BB'}} & 0 \\ 0 & X_D^{T_{B'}} & |X_D| & 0 \\ 0 & 0 & 0 & |X_D|^{T_{B'}} \end{array} \right]. \]  

(3.24)

possess the nonnegative eigenvalues.

### 3.2 Proof of nonzero distillable key \(K_D\)

Since the state \(\varrho_{ABA'B'}^{(D)}\), given by Eq. (3.23) is PPT one might expect that it is separable - it satisfies necessary (PPT) condition for separability [13]. However this is not the case. This is in agreement with the fact [14] that PPT condition is sufficient for separability only for the cases \(M \otimes N, MN \leq 6\) while here we have \(MN = (2D)^2 \geq 36\).

Like in [9] we shall prove nonseparability of \(\varrho_{ABA'B'}^{(D)}\) in a very nonstandard way. We simply show that the state has nonzero \(K_D\). Such a state must be entangled since, due to seminal result of Ref. [15], any separable state has \(K_D = 0\). Because state is PPT and entangled it must be bound entangled [7].

To prove the cryptographic use of \(\varrho_{ABA'B'}^{(D)}\), below we show that there exist a LOCC protocol that allows Alice and Bob to approach arbitrarily closely to some pbit \(\gamma_{ABA'B'}^{(2)}\). Note that, obviously, since LOCC operations preserves PPT property (see [7]), the resulting state is still bound entangled. Given \(k\) copies of the state \(\varrho_{ABA'B'}\) in the \(i\)-th step of the protocol Alice and Bob perform the following operations:

- They take the state \(\varrho_{ABA'B'}^{(i-1)}\) \((i = 1, \ldots, k - 1)\) and one of remaining \(k - i + 1\) copies of \(\varrho_{ABA'B'}\) (here \(\varrho_{ABA'B'} = \varrho_{ABA'B'}\)).
• They perform C-NOT operation treating qubits $A$ and $B$ of $\rho_{ABA'B'}$ as source qubits and that of $\rho^{(i-1)}_{ABA'B'}$ as target qubits.

• They perform a measurement of target qubits in computational basis and then compare their results. If both of them have the same results (00 or 11) then they keep the source state. Otherwise they get rid of it.

After performing all $k$ steps, with some probability they arrive at the following state

$$\rho^{(D,k)}_{ABA'B'} = \frac{1}{N_{D,k}} \begin{bmatrix}
|X_D|^{\otimes k} & 0 & 0 & X^{\otimes k} \\
0 & \left(|X_D^{T_B'}|^{T_B'}\right)^{\otimes k} & 0 & 0 \\
0 & 0 & \left(|X_{D'}^{T_B'}|^{T_B'}\right)^{\otimes k} & 0 \\
X^{\otimes k} & 0 & 0 & |X_D|^{\otimes k}
\end{bmatrix}, \quad (3.25)$$

where (for $D \geq 3$)

$$N_{D,k} = 2\text{Tr}|X_D|^{\otimes k} + 2\text{Tr}\left(|X_D^{T_B'}|^{T_B'}\right)^{\otimes k} = 2\left[1 + \left(\frac{D^2}{D^2 + 2D - 4}\right)^k\right] \to 2. \quad (3.26)$$

If we define, according to (2.11) the matrix, $A^{(A'B')}_{00,11}(D,k) = (1/N_{D,k})|X_D|^{\otimes k}$ we can see that $||A^{(A'B')}_{00,11}(D,k)||_{\text{Tr}} \to 1/2$ whenever $k \to \infty$, for $D \geq 3$. This means that repeating the whole procedure described above, one may find such a $k$ that the trace norm of the upper right block of $\rho^{(D,k)}_{ABA'B'}$ is close to 1/2 with arbitrary precision. According to the the Theorem 1 (see Section 2) this convergence guarantees that the original state $\rho^{(D)}_{ABA'B'}$ defined by the formula (3.23) satisfies $K_D > 0$. The Theorem 2 guarantees in particular, that (like it was in [9]) the above sequence of bound entangled states $\rho^{(D,k)}_{ABA'B'}$ approaches private bit.

4 Summary and discussion

We have summarized main elements of general scheme of distillation of secure key [9, 10]. The central notion of the scheme is the idea of private bit (with its natural generalization - private dit) which is the state that, in general, consists of two parts: the key part $AB$ and the shield part $A'B'$. The first contains a bit of secure key. The role of the second part is in a sense - to protect the key form Eve. A surprising fact, found in [9] is that PPT bound entangled states can approach private bit in trace norm. Since this convergence is a necessary and sufficient condition to distill secure key form the original state, this means that bound entangled states can serve as a source of distillable key [9, 10].

First bound entangled states with nonzero distillable key $K_D$ were provided in [9]. They required however very high dimensions. The small ($4 \otimes 4$) bound entangled states with nonzero distillable key were provided later in paper [11].

Here we have provided new class of small (of, among others, $6 \otimes 6$ type) bound entangled states with that property. We have proven this fact applying an easy criterion form [9] showing that a given sequence of quantum states approaches the sequence of private bits. The LOCC protocol applied to produce such a sequence was of two-way type. As noticed in [9] at some point the elements of the sequence got one-way distillable key which can be distilled with help of Devetak-Winter protocol. The original problem was that such states were of very high dimensions. Quite surprisingly the low-dimensional bound entangled states provided in [11] represent one-way distillable key. It was proven with help of the observation that any biased mixture of two private bits with second of them having the key part rotated locally
with $\sigma_x$ Pauli matrix contains nonzero distillable key. Namely one has the following theorem [11]:

**Theorem 2.** For two private bits $\gamma_1^{(2)}$, $\gamma_2^{(2)}$ one-way distillable key of the mixture of the form:

$$\rho = p_1 \gamma_1^{(2)} + p_2 \sigma_x^{(A)} \gamma_2^{(2)} \sigma_x^{(A)}$$

(4.27)

where $\sigma_x^{(A)} = [\sigma_x]_A \otimes I_{A'B'}$ satisfies $K_D^B(\rho) \geq 1 - h(p_1)$ with binary entropy $h(p_1)$. Here $K_D^B(\rho)$ stands for cryptographic key distillable from $\rho$ with help of forward classical communication. The natural question is whether one can modify the $6 \otimes 6$ bound entangled states provided in the present paper to get mixture of two private bits of the above form. Our first analysis has shown that most probably it is impossible to turn our example into a state of the form (4.27) while keeping bound entanglement property. In Ref. [11] the construction leading to bound entangled state of the form (4.27) was based on some properties of states that were used in locking entanglement measures effects. May be that was the reason why the construction was successful there. Still there is a natural question about other methods to construct low-dimensional bound entangled states with one-way distillable key.

Of course the most important open problem is whether any entangled bipartite state contains nonzero distillable key or not. In multipartite case it is not true - there are states that are entangled but no secure key between any of the parties can be distilled [16]. In bipartite case lack of such states is guaranteed (via entanglement distillation approach [11]) only for $2 \otimes 2$ and $2 \otimes 3$ cases, since, as shown in [17] [18], all the entangled states can be distilled to singlet form in those cases. For $d \otimes d$ with $d \geq 4$ it is known that at least some bound entangled (i.e. nondistillable to singlets) states have nonzero distillable key [11]. No example of $3 \otimes 3$ or $2 \otimes 4$ states with that property is still known.

**Acknowledgments**

The author thank Maciej Demianowicz and Michal Horodecki for fruitful discussions. The work is partially supported by Polish Ministry of Scientific Research and Information Technology grant under the (solicited) project no. PBZ-MIN-008/P03/2003 and by EC grants: RESQ, contract no. IST-2001-37559.

**References**

[1] A. Ekert, Phys. Rev. Lett. 67, 661 (1991).
[2] C. H. Bennett, G. Brassard, and N. D. Mermin, Phys. Rev. Lett. 68, 557 (1992).
[3] B. M. Terhal, IBM J. Res. Dev. 48, 71 (2004).
[4] D. Deutsch, A. Ekert, R. Jozsa, C. Macchiavello, S. Popescu, and A. Sanpera, Phys. Rev. Lett. 77, 2818 (1996).
[5] C. H. Bennett, G. Brassard, S. Popescu, B. Schumacher, J. Smolin, and W. K. Wootters, Phys. Rev. Lett. 76, 722 (1996).
[6] C. H. Bennett, D. P. Di Vincenzo, J. Smolin, and W. K. Wootters, Phys. Rev. A 54, 3814 (1997).
[7] M. Horodecki, P. Horodecki, and R. Horodecki, Phys. Rev. Lett. 80, 5239 (1998).
[8] N. Gisin and M. Wolf, Phys. Rev. Lett. 83, 4200 (1999); A. Acin, L. Massanes, and N. Gisin, Phys. Rev. Lett. 91, 167901 (2003).

---

2Binary entropy of the distribution $\{p_1, p_2\}$ is defined as $h(p_1) = -p_1 \log_2 p_1 - p_2 \log_2 p_2$
[9] K. Horodecki, M. Horodecki, P. Horodecki, and J. Oppenheim, Phys. Rev. Lett. 94, 160502 (2005).
[10] K. Horodecki, M. Horodecki, P. Horodecki, and J. Oppenheim, quant-ph/0506189.
[11] K. Horodecki, L. Pankowski, M. Horodecki, and P. Horodecki, quant-ph/0506203.
[12] I. Devetak and A. Winter, Proc. R. Soc. A 461, 207 (2005).
[13] A. Peres, Phys. Rev. Lett. 77, 1413 (1996).
[14] M. Horodecki, P. Horodecki, and R. Horodecki, Phys. Lett. A 223, 1 (1996).
[15] M. Curty, M. Lewenstein, and N. Lütkenhaus, Phys. Rev. Lett. 92, 217903 (2004).
[16] R. Augusiak and P. Horodecki, quant-ph/0405187; R. Augusiak and P. Horodecki, Phys. Rev. A 73, 012318 (2005).
[17] M. Horodecki, P. Horodecki, and R. Horodecki, Phys. Rev. Lett. 78, 574 (1997).
[18] M. Horodecki and P. Horodecki, Phys. Rev. A 59, 4026 (1999).