Spin Josephson effects of spin–orbit-coupled Bose–Einstein condensates in a non-Hermitian double well

Jia Tang\textsuperscript{1,3}, Zhou Hu\textsuperscript{1,3}, Zhao-Yun Zeng\textsuperscript{2}, Jinpeng Xiao\textsuperscript{2\dagger}, Lei Li\textsuperscript{2}, Yajiang Chen\textsuperscript{1}, Ai-Xi Chen\textsuperscript{1} and Xiaobing Luo\textsuperscript{1,2,⋆}

\textsuperscript{1} Department of Physics, Zhejiang Sci-Tech University, Hangzhou 310018, People’s Republic of China
\textsuperscript{2} School of Mathematics and Physics, Jinggangshan University, Ji’an 343009, People’s Republic of China

E-mail: xiaobingluo2013@aliyun.com

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Abstract
In this paper, we investigate the spin and tunneling dynamics of a spin–orbit-coupled noninteracting Bose–Einstein condensate in a periodically driven non-Hermitian double-well potential. Under high-frequency driving, we obtain the effective time-averaged Hamiltonian by using the standard time-averaging method, and analytically calculate the Floquet quasienergies, revealing that the parity-time (PT)-breaking phase transition appears even for arbitrarily small non-Hermitian parameters when the spin–orbit coupling strength takes half-integer value, irrespective of the values of other parameters used. When the system is PT-symmetric, we find numerically and analytically that in the broken PT-symmetric regions, there will exist the net spin current together with a vanishing atomic current, if we drop the contribution of the exponential growth of the norm to the current behaviors. When the system is non-PT-symmetric, though the quasienergies are partial complex, a stable net spin current can be generated by controlling the periodic driving field, which is accompanied by a spatial localization of the condensate in the well with gain. The results deepen the understanding of non-Hermitian physics and could be useful for engineering a variety of devices for spintronics.

Keywords: SO-coupling, current, non-Hermitian, PT-symmetric

(Some figures may appear in colour only in the online journal)

1. Introduction
Over the years an intense research effort has been made to investigate non-Hermitian systems both theoretically and experimentally \cite{1,2}. In conventional quantum mechanics, the Hermiticity requirement of a Hamiltonian guarantees the reality of the energy spectrum and conserved total probability. However, it is ubiquitous in nature that the quantum probability (the norm of the state) is effectively not conserved due to the exchange of energy, particles, and information between the environment and the quantum subsystem of our interest \cite{3}. An important development in the physics of non-Hermitian systems was the discovery, by Carl Bender and co-workers, that a broad class of non-Hermitian Hamiltonians can exhibit purely real spectra as long as the system possesses a combined parity and time-reversal symmetry, that is, parity-time (PT) symmetry \cite{4,5}. A distinctive characteristic in PT-symmetric systems is the spontaneous-symmetry-breaking phase transition, where the
spectral changes from all real (exact phase) to complex (broken phase) when the gain–loss coefficient exceeds a critical threshold. The exploitation of $PT$-symmetric systems with static (i.e. time-independent) potentials has been prolific. Recently, manipulation of the spontaneous $PT$-symmetry breaking (and non-Hermitian physics) by making use of periodic driving schemes has also attracted much attention [6–22]. For example, as shown in references [10, 20], the spontaneously-symmetry-breaking phase transition emerges even for arbitrarily weak gain–loss coefficient by adjusting the parameters of periodic driving field.

On another front, there have been remarkable progresses and research activities in the study of the quantum dynamics of Bose–Einstein condensates (BECs) in a double-well potential, due to the fundamental significance and numerous potential applications. The prototypical system of an atomic BEC in a double-well potential represents a bosonic Josephson junction (BJJ) [23, 24], i.e. a bosonic analogue of the well-known superconducting Josephson junction [25], and the coherent dynamical behaviors such as the Josephson oscillations and macroscopic quantum self-trapping have been observed experimentally [26, 27]. Early theoretical efforts have already been carried out in generalizing the Josephson junctions with scalar condensates to mixtures [28–31], or spinors [32–37], and a variety of fundamental tunneling phenomena have been uncovered. On the other hand, the controlled removal of atoms from a BEC was realized by using the experimental technique based on the electron microscopy [38, 39], which promotes the Boson–Josephson junction manipulated by a local dissipation as a governable open quantum system for implementing the switching between a self-trapping state and the macroscopic quantum tunneling regime [40]. In addition, experimental realization of a $PT$-symmetric two-well system of ultracold atoms is made possible by embedding it within additional time-dependent wells which act as particle reservoirs, as identified in the early proposal [41, 42].

Spin–orbit coupling (SOC), the interaction between the particle dynamics and its spin, has already been extensively studied in diverse branches of physics, which contributes to the electronic fine structure of atoms and condensed matter phenomena and applications like topological insulator [43], spin Hall effect [44], and spintronic [45]. In cold atom systems, SOC can be generated experimentally by coupling two hyperfine states of atoms via a pair of counterpropagating Raman lasers [46]. Such Raman-induced SOC opens new possibilities for investigating the Josephson effects (JEs) in two-component cold atom systems, for which two components can be explained as two hyperfine (pseudospin) atomic states. Recently, the role played by SOC on the tunneling dynamics of BECs in double-well potentials were addressed in several works [47–54]. In reference [47], Zhang and co-worker discovered that a net atomic spin current termed as spin JEs can be induced by the spin-dependent tunneling between two wells. Subsequently, reference [48] concentrated on the effect of atom–atom interactions and provided a classic study of self-trapping dynamics of the spin polarization and population imbalances of each bosonic pseudospin species. A parallel work analytically treated the quantum behavior of spin–orbit coupled BECs from the viewpoint of a two-mode Bose–Hubbard-like Hamiltonian [50]. The dynamical suppression of tunneling of spin–orbit-coupled noninteracting BEC in a double-well potential under periodic driving were reported in reference [52]. We also notice that two very similar four-level systems were studied in references [55, 56], which discussed the spin dynamics of a single spin–orbit-coupled electron in a double quantum dot. In view of the aforementioned achievements both in non-Hermitian physics and spin–orbit-coupled BJJ, it is natural to ask the following two important questions: can the $PT$-symmetry-breaking phase transition be induced for arbitrarily weak gain–loss coefficient if given certain suitably chosen values of SOC strength? Can the net atomic spin current still occur in the non-Hermitian two-well system of cold atoms with SOC?

The aim of this paper is to answer the above concerns and questions by investigating the non-Hermitian system of a spin–orbit-coupled atom (or noninteracting BEC) in a periodically driven double-well potential. Fortunately, the answers to both of the questions above are definite ‘yes’. In such a system, we have the following main observations: (a) managing effective SOC alone can achieve $PT$-symmetry-breaking transition for arbitrarily small non-Hermitian parameters, which has the same effect as the use of periodic driving schemes; (b) despite existence of non-Hermiticity, a net spin current (i.e. the atomic current is zero while the spin current is nonzero), which is termed as spin JEs, can still exist, indicating that there are spin exchanges but no net-particle tunneling between the two wells of the potential.

2. Model

In the pioneering experiment by NIST group [46], the synthetic SOC was successfully realized by coupling two hyperfine states of atoms via a pair of Raman lasers. Relying on this experimental setup, we consider a single ultracold atom (or noninteracting BEC) with two hyperfine pseudospin states $|\uparrow\rangle$ and $|\downarrow\rangle$ in a periodically driven open double-well potential with synthetic SOC. Assuming that the pseudospin bosonic atom occupies the lowest state of each well, the quantum dynamics of a spin–orbit-coupled atom (or noninteracting BEC) confined in a periodically driven open double well can be rather generally described by the non-Hermitian Hamiltonian [53, 57]

$$
\hat{H} = \sum_\sigma [(\varepsilon + i\beta_\sigma) \hat{n}_\sigma - (\varepsilon + i\beta_\sigma) \hat{n}_\sigma] + \frac{\Omega}{2} \sum_{j,\sigma} \hat{c}^\dagger_{j\sigma} \hat{c}_{j'\sigma'} - \nu \left( \hat{c}^\dagger \hat{T} \hat{c} + H.c. \right).
$$

Here \( \hat{c}_j^\dagger = (\hat{c}_{j\uparrow}^\dagger, \hat{c}_{j\downarrow}^\dagger) \) (superscript T stands for the transpose), \( \hat{c}_{j\sigma} \) (\( \hat{c}_{j\sigma}' \)) describes the creation (annihilation) of a pseudospin \( \sigma = \uparrow, \downarrow \) in the \( j \)th \( (j = l, r) \) well, and \( H.c. \) denotes the Hermitian conjugate of the preceding term. \( \hat{n}_{j\sigma} = \hat{c}_{j\sigma}^\dagger \hat{c}_{j\sigma} \) represents the number operator for
effective Hamiltonian can be obtained by time averaging of systems. According to the well-established method, a static has been routinely employed in periodically driven quantum the high-frequency limit, the effective time-averaged Hamiltonian states can be obtained by solving the eigenvalue equation Schrödinger equation

In the experiments, the periodic driving, we incorporate gain and loss mechanisms by out-of-phase periodic modulation of depths of two wells, the non-Hermitian coefficient \( \alpha \) controls the periodic driving. In our model system is shown in figure 1. left well experiences gain while the right well loss. A schematic view of our model system is shown in figure 1. Through this paper, we have set \( h = 1 \) and let the system parameters \( \alpha, \omega, \beta, \nu, \Omega \) be in units of the reference frequency \( \omega_0 = 0.1 E_r = 2.25 \text{kHz} \) with \( E_r = \hbar^2 k_f^2 / (2m) \) being the single-photon recoil energy, and time be normalized in units of \( \omega_0^{-1} \). In the experiments, the system parameters can be adjusted in a wide range: \( \nu, \Omega, \beta, \nu, \omega_0 \) and \( \alpha \sim \omega \in [0, 100] / (\omega_0) \).

Since the Hamiltonian (1) is periodic in time with period \( \tau = 2\pi / \omega \), the Floquet theorem tells us that there exists a complete set of solutions to the time-dependent Schrödinger equation \( i\hbar \frac{\partial \psi(t)}{\partial t} = \tilde{H}(t) \) of the form \( \psi_n(t) = e^{-i\varepsilon_n t} \phi_n(t) \), where \( \phi_n(t) \) are Floquet states and \( \varepsilon_n \) are quasienergies [61–63]. The Floquet states can be obtained by solving the eigenvalue equation

\[
\left( \tilde{H} - i \frac{\partial}{\partial t} \right) \psi_n(t) = \varepsilon_n \phi_n(t).
\]

In general, it is hard to obtain the exact Floquet solutions of equation (2), but the quantum dynamics can be investigated analytically in high-frequency region where \( \omega \) is far greater than all other frequencies of the physical system. In the high-frequency limit, the effective time-averaged Hamiltonian can be derived by using a time-averaging method, which has been routinely employed in periodically driven quantum systems. According to the well-established method, a static effective Hamiltonian can be obtained by time averaging of the periodic-driving terms, i.e.

\[
\hat{H}_{\text{eff}} = \frac{\omega}{2\pi} \int_0^{\pi} dt \hat{S} \left[ -\nu \left( \hat{c}_l e^{-i\gamma t} + \hat{c}_r e^{i\gamma t} + \hat{H}\text{c.c.} \right) \right] \hat{S}^{-1}
\]

\[
+ \frac{\omega}{2\pi} \int_0^{\pi} dt \hat{S} \left[ \sum_{\sigma} \left( i\hbar \hat{n}_\sigma - i\beta \hat{n}_\sigma \right) + \frac{\Omega}{2} \sum_{j, \sigma} \hat{c}_j^\dagger \hat{c}_{j' \sigma} \right] \hat{S}^{-1},
\]

where \( \hat{S} = e^{-iA(t)\sum_n \left( \hat{n}_n - \hat{n}_n^\dagger \right)} \) and \( A(t) = \int_0^t [\alpha \cos(\omega t)] dt = \frac{\alpha}{2} \sin(\omega t) \). Implementing the integral in equation (3), we get the effective Hamiltonian

\[
\hat{H}_{\text{eff}} = \sum_{\sigma} \left( i\beta \hat{n}_\sigma - i\beta \hat{n}_\sigma \right)
\]

\[
+ \frac{\Omega}{2} \sum_{j, \sigma} \hat{c}_j^\dagger \hat{c}_{j' \sigma} - \left( \hat{c}_l^\dagger J c_{r \uparrow} + \hat{c}_r^\dagger J c_{l \uparrow} \right),
\]

where \( J = \nu e^{-i\gamma t} \Omega \left( \frac{\nu^2}{\omega^2} \right) \) with \( \Omega \left( \frac{\nu^2}{\omega^2} \right) \) being the zeroth-order Bessel function of variable \( \frac{\nu^2}{\omega^2} \).

We can solve the eigenvalue equation with the time-independent effective Hamiltonian (4),

\[
\hat{H}_{\text{eff}} |\varphi_n^\prime \rangle = E_n |\varphi_n^\prime \rangle,
\]

where \( |\varphi_n^\prime \rangle \) and \( E_n \) are eigenvectors and eigenvalues, respectively. Since the effective Hamiltonian is not self-adjoint, we should be aware that the eigenvectors corresponding to distinct eigenvalues are not orthogonal, i.e. \( \langle \varphi_m^\prime | \varphi_n^\prime \rangle \neq 0 \) for \( m \neq n \). In the non-Hermitian system, both the right vectors \( |\varphi_n^\prime \rangle \) [defined by equation (5)] and the left eigenvectors \( |\varphi_n^\prime \rangle \) [defined by \( \hat{H}_{\text{eff}} |\varphi_n^\prime \rangle = E_n |\varphi_n^\prime \rangle \) ] together form a complete set of biorthogonal basis, and the orthogonality is given by \( \langle \varphi_m^\prime | \varphi_n^\prime \rangle = \delta_{m,n} \).

Note that the unitary transformation operator \( \hat{S} \) has the same period \( \tau = 2\pi / \omega \) as the Hamiltonian (1). Thus, through the inverse transformation, we can construct the approximate expressions of Floquet solutions to the original Hamiltonian (1) as follows

\[
|\psi_n(t)\rangle = |\varphi_n(t)\rangle e^{-i\varepsilon_n t} = \hat{S} |\varphi_n^\prime \rangle e^{-iE_n t},
\]

where \( |\varphi_n(t)\rangle = \hat{S} |\varphi_n^\prime \rangle \) inherits the period of the driving force, satisfying \( |\varphi_n(t + \tau)\rangle = |\varphi_n(t)\rangle \). This implies that \( |\varphi_n(t)\rangle = \hat{S} |\varphi_n^\prime \rangle \) are the so-called Floquet states and the eigenvalues \( E_n \) in equation (5) are the corresponding analytical quasienergies.

Taking the \( \sigma \) Wannier state \( |j, \sigma\rangle = c_{j \sigma}^\dagger |0\rangle \) localized in the \( j \)th \(( j = l, r)\), we as well as \( \alpha \), we expand the quantum state of system (1) as \( |\Psi(t)\rangle = \sum_{j, \sigma} \epsilon_{j \sigma}(t) |j, \sigma\rangle \), where \( \epsilon_{j \sigma}(t) \) indicates the probability amplitude of finding a pseudospin-\( \sigma \) atom to be localized in the \( j \)th well. In the Wannier representation, the Hamiltonian operators can be represented in form of \( 4 \times 4 \) matrix. By diagonalizing the effective Hamiltonian (4), we get the eigenvalues (approximate quasienergies) as

\[
E_{1,2} = \frac{1}{2} \left( i\beta_l - i\beta_r \mp \sqrt{m-w} \right),
\]

\[
E_{3,4} = \frac{1}{2} \left( i\beta_l - i\beta_r \mp \sqrt{m+w} \right).
\]
where

\[ m = 4 |\nu J_0 (2\alpha/\omega)|^2 + \Omega^2 - (\beta_1 + \beta_1)^2, \]

\[ w = 2\Omega \sqrt{4 |\nu J_0 (2\alpha/\omega)|^2 \cos^2(\gamma \pi) - (\beta_1 + \beta_1)^2}. \]  

(8)

The analytical Floquet states and quasienergies provide basic concepts and tools for treatment of the periodically driven system (1), from which all available time-dependent information about the system can be deduced. At any time, the quantum state can be expanded in the basis of the Floquet eigenstates, namely,

\[ |\Psi (t)\rangle = \sum_{n=1}^4 a_n |\varphi_n (t)\rangle e^{-i\epsilon_n t} = \sum_{n=1}^4 a_n |\varphi_n' (t)\rangle e^{-i\epsilon' n t}, \]  

(9)

where \( a_n \) are components of the quantum state, which are time-independent and determined by the initial state, i.e. \( |\Psi (t = 0)\rangle = \sum_{n=1}^4 a_n |\varphi_n \rangle \). Given the initial state \( |\Psi (t = 0)\rangle = (c_{1\uparrow} (0), c_{1\downarrow} (0), c_{2\uparrow} (0), c_{2\downarrow} (0))^T \), we can determine the components \( (a_1, a_2, a_3, a_4)^T = U(c_{1\uparrow} (0), c_{1\downarrow} (0), c_{2\uparrow} (0), c_{2\downarrow} (0))^T \), where \( U \) is transformation matrix between the Wannier-state space and the eigenvector space. Note that the operator \( U \) is necessarily not unitary, and each column of \( U^{-1} \) and \( U^\dagger \) is given by a right (\( |\varphi_n'\rangle \)) and left (\( \langle \varphi_n' | \)) eigenvector, respectively.

3. Currents in the \( \mathcal{PT} \)-symmetric systems

First, we consider the situation of balanced gain and loss, where the loss (gain) coefficients of two wells take the same values, \( \beta_1 = \beta_1 = \beta \). In such situation, Hamiltonian (1) is \( \mathcal{PT} \) symmetric because of \( \mathcal{PT}H = H \mathcal{PT} \), where the parity operator \( \mathcal{P} \) corresponds to the exchange of the two wells numbered by \( l \) and \( r \), and the time reversal operator is defined as \( T: t \rightarrow -t + l_0 \) (\( l_0 \) is an appropriate time point), \( i \rightarrow -i \).

In this work, our focus is placed on the current behaviors of non-Hermitian system under action of SOC and periodic driving. To this end, we introduce the population imbalance between the two wells

\[ P_a = \langle \Psi (t) | z | \Psi (t) \rangle = \langle \Psi (t) | \hat{z} \otimes \hat{l} | \Psi (t) \rangle, \]  

(10)

and magnetization

\[ P_s = \langle \Psi (t) | A | \Psi (t) \rangle = \langle \Psi (t) | \hat{z} \otimes \hat{\sigma}_z | \Psi (t) \rangle, \]  

(11)

where \( \hat{z} \) denotes the position operator, \( \hat{z} = \sum_{j=1,L} \langle j | \hat{z} | j \rangle \langle j | \), and \( \hat{l} \) is the corresponding identity operator. By convention, we set the center location of the double-well potential as the origin of coordinates (such that \( \langle \hat{l} | \hat{z} | \hat{l} \rangle = -\langle \hat{r} | \hat{z} | \hat{r} \rangle \)), and drop the additive physically-irrelevant constant \( \langle \hat{l} | \hat{z} | \hat{l} \rangle \), which leads the position operator to be simplified as \( \hat{z} = \hat{l} - |\hat{r}\rangle \langle \hat{r}| \). In the Wannier representation, \( \hat{c} = \hat{z} \otimes \hat{\sigma}_z = \text{diag} (1, 1, -1, -1), L = \hat{z} \otimes \hat{\sigma}_z = \text{diag} (1, -1, -1, 1) \), such that we have [50],

\[ P_a = |c_{1\uparrow}|^2 + |c_{1\downarrow}|^2 - |c_{2\uparrow}|^2 - |c_{2\downarrow}|^2, \]

\[ P_s = |c_{1\uparrow}|^2 - |c_{1\downarrow}|^2 - |c_{2\uparrow}|^2 + |c_{2\downarrow}|^2. \]  

(12)

Figure 2. Time-evolution curves of the population imbalance and magnetization (panels (a), (c) and (e)), and the atomic and spin currents (panels (b), (d) and (f)), obtained from the \( \mathcal{PT} \)-symmetric system (1) with the initial state prepared as \( |\Psi (0)\rangle = (c_{1\uparrow}, c_{1\downarrow}, c_{2\downarrow}, c_{2\uparrow})^T = (|\nu\rangle, 0, 0, |\omega\rangle)^T \). The parameters are \( \nu = 2, \Omega = 1, \beta = 0.2, \omega = 20 \), with (a) and (b) \( \alpha = 40, \gamma = 0 \); (c) and (d) \( \alpha = 40, \gamma = 0.5 \); (c) and (f) \( \alpha = 24, \gamma = 0 \). In panel (d), we put an inset with blue border for clear illustration of the exponential growth of the atomic current.

The corresponding atomic density current \( (I_a) \) and the spin current \( (I_s) \) are given by [47, 50]

\[ I_a = \frac{d |\langle \Psi (t) | z | \Psi (t) \rangle|^2}{dt}, \quad I_s = \frac{d |\langle \Psi (t) | A | \Psi (t) \rangle|^2}{dt}. \]  

(13)

According to the Schrödinger equation and the definition of currents, the atomic current and the spin current can be calculated as

\[ I_a = 2\beta_1 (|c_{1\uparrow}|^2 - |c_{1\downarrow}|^2) + 2\beta_2 (|c_{1\downarrow}|^2 - |c_{2\downarrow}|^2) + e^{i\pi \gamma} (c_{1\uparrow}^\dagger c_{1\downarrow}^\dagger - e^{i\pi \gamma} c_{1\downarrow}^\dagger c_{1\uparrow}^\dagger - e^{-i\pi \gamma} c_{1\downarrow}^\dagger c_{1\uparrow}^\dagger), \]

\[ I_s = 2\beta_1 (|c_{1\uparrow}|^2 - |c_{1\downarrow}|^2) + 2\beta_2 (|c_{1\downarrow}|^2 - |c_{2\downarrow}|^2) - e^{i\pi \gamma} (c_{1\uparrow}^\dagger c_{1\downarrow}^\dagger - e^{i\pi \gamma} c_{1\downarrow}^\dagger c_{1\uparrow}^\dagger + e^{-i\pi \gamma} c_{1\downarrow}^\dagger c_{1\uparrow}^\dagger - e^{i\pi \gamma} c_{1\downarrow}^\dagger c_{1\uparrow}^\dagger), \]

\[ - \Omega (c_{1\uparrow}^\dagger c_{1\downarrow}^\dagger c_{1\uparrow}^\dagger c_{1\downarrow}^\dagger - c_{1\downarrow}^\dagger c_{1\uparrow}^\dagger c_{1\uparrow}^\dagger c_{1\downarrow}^\dagger). \]  

(14)

In figure 2, we exhibit the time evolutions of population imbalance, magnetization, spin current and atomic current by numerically solving the time-dependent Schrödinger equation with Hamiltonian (1) for fixed parameters \( \nu = 2, \Omega = 1, \beta = 0.2, \omega = 20 \). The initial state is taken as \( |\Psi (0)\rangle = (c_{1\uparrow}, c_{1\downarrow}, c_{2\uparrow}, c_{2\downarrow})^T = (|\nu\rangle, 0, 0, |\omega\rangle)^T \). The blue solid line represents population imbalance (atomic current), and the red dashed line denotes magnetization (spin current), respectively. In figures 2(a) and (b), under the condition of \( \alpha = 40 \) and
\( \gamma = 0 \), we reveal that both the population imbalance (magnetization) and the atomic (spin) current exhibit periodic and stable oscillations, which implies that the system is in the unbroken \( PT \) phase. When the driving amplitude is fixed and only the effective SOC strength is changed to \( \gamma = 0.5 \), as shown in figures 2(c) and (d), we observe that the population imbalance and atomic current increase exponentially without bound, while the magnetization and spin current oscillate up and down around zero with oscillation amplitude increasing continuously, indicating that the system enters into the broken \( PT \) phase. In figure 2(d), we put an inset with blue border to clearly show the smooth exponential growth of the atomic current. Likewise, the transition from periodic (bounded) oscillation (unbroken \( PT \) phase) to secular (unbounded) growth (broken \( PT \) phase) can be also achieved by tuning the driving parameters, as illustrated in figures 2(e) and (f), where we only change the driving amplitude to \( \alpha = 24 \) and keep all the other parameters unchanged as compared to figures 2(a) and (b). The numerical results illustrate that apart from periodic driving schemes, the management of SOC strength provides a flexible alternative to control the \( PT \) phase transition.

Like in the undriven system, the Floquet \( PT \)-symmetric system (1) will be said to be in the unbroken \( PT \) phase whenever the quasienergies are all real, whereas it is said to be in the broken \( PT \) phase if complex conjugate quasienergies arise. The analytical expressions of quasienergies (7) allow us to determine the accurate boundaries between the unbroken \( PT \) and broken \( PT \) phases. From equation (7), under balanced gain and loss, the quasienergies become

\[
\begin{align*}
E_1 &= -E_2 = -\frac{1}{2} \rho_1, \quad E_3 = -E_4 = -\frac{1}{2} \rho_2, \\
\rho_1 &= \sqrt{m-w}, \quad \rho_2 = \sqrt{m+w}, \\
m &= 4\nu^2 J_0^2(2\alpha/\omega) + \Omega^2 - 4\beta^2, \\
w &= 4\Omega\sqrt{\nu J_0(2\alpha/\omega) \cos(\gamma \pi)} - \beta^2.
\end{align*}
\]

(15)

If the following two parameter relationships

\[
\begin{align*}
|\nu J_0(2\alpha/\omega)|^2 \cos^2(\gamma \pi) &\geq \beta^2, \\
4|\nu J_0(2\alpha/\omega)|^2 + \Omega^2 - 4\beta^2 &\geq 4\Omega\sqrt{\nu J_0(2\alpha/\omega)|^2 \cos^2(\gamma \pi) - \beta^2},
\end{align*}
\]

(16)

are satisfied, we obtain four all-real quasienergies, then the system is in the unbroken \( PT \) phase. Otherwise, if either one of the parametric relations in equation (16) is not satisfied, at least two of the quasienergies will become complex, then the system is in the broken \( PT \) phase. The ‘=’ signs taken in inequalities of (16) give the boundary (phase transition point) between unbroken \( PT \)-symmetric and broken \( PT \)-symmetric regions. According to equation (15), we have drawn the phase diagram by numerically computing the values of \([\text{Im}(E_2)] + [\text{Im}(E_4)]\), as shown in figure 3. By virtue of the relations \( E_1 = -E_2, E_3 = -E_4 \), we know that if \( |\text{Im}(E_2)| + |\text{Im}(E_4)| = 0 \) holds, all quasienergies naturally have no imaginary part. Thus, the blue areas with \([\text{Im}(E_2)] + [\text{Im}(E_4)] = 0\) correspond to the unbroken \( PT \)-symmetric regions where the quasienergy spectrum is entirely real, and the areas with other colors correspond to the broken \( PT \)-symmetric regions. In figure 3(a), we set the parameter \( \beta = 0.2 \) and plot \([\text{Im}(E_2)] + [\text{Im}(E_4)]\) as a function of \( \gamma \) and \( 2\alpha/\omega \), where the red lines mark the boundary between unbroken \( PT \)-symmetric and broken \( PT \)-symmetric regions. From figure 3(a), it is clearly seen that when either the driving parameters \( 2\alpha/\omega \) take the zeros of Bessel function such as \( 2\alpha/\omega = 2.4, 5.52, \ldots \), or the effective SOC is half-integer such as \( \gamma = 0.5, 1.5, \ldots \), the system is always in broken \( PT \) phase. The \( PT \) phase transition can also be observed in the parameter space \((\gamma, \beta)\) with fixed \( 2\alpha/\omega = 4 \) and the parameter space \((2\alpha/\omega, \beta)\) with \( \gamma = 0 \), as shown in figures 3(b) and (c) respectively. From these two plots, we further observe that when the effective SOC takes half-integer value or the driving parameters \( 2\alpha/\omega \) take the zeros of Bessel function, the \( PT \)-symmetry-breaking occurs for arbitrarily small gain–loss coefficient. The salient features can be readily inferred from the analytical expressions of quasienergies (15), where we find that when \( J_0(2\alpha/\omega) = 0 \) or \( \cos(\gamma \pi) = 0 \), the term \( w = 4\Omega\sqrt{\nu J_0(2\alpha/\omega) \cos(\gamma \pi)} - \beta^2 \) becomes purely imaginary for arbitrarily small \( \beta \). This implies that managing effective SOC alone can allow for spontaneous \( PT \)-symmetry-breaking transition for arbitrary values of the gain and loss parameter, which has the same effect as the use of periodic driving schemes.

As a next step we make further investigations on the current behaviors of the non-Hermitian system in both unbroken and broken \( PT \) phases. In the non-Hermitian system, the norm of the vector state (the quantum probability) \( N = \langle \Psi | \Psi \rangle = \Sigma_{j,a} |c_{j,a}|^2 \) is not conserved with time evolution. To eliminate the contribution of the norm to the physical quantities \([9, 67–69]\), we define the normalized population imbalance and the magnetization as

\[
P_{an} = \frac{P_{an}}{N}, \quad P_{an} = \frac{P_{an}}{N},
\]

(17)

and normalized atomic current and spin current,

\[
I_{an} = \frac{dP_{an}}{dt}, \quad I_{an} = \frac{dP_{an}}{dt}.
\]

(18)

The above definition is more physically reasonable to reflect the population transfer and current behaviors. Especially, when the system is in the broken \( PT \) phase, the norm \( N \) of quantum states will exponentially amplify with time due to the appearance of complex quasienergies. Therefore, it is necessary to cancel the contribution from the growth of the norm to the atomic population exchange and spin exchange between the two wells of the potential. As reported in reference \([47]\), a net spin current (i.e. the spin current is nonzero while atomic current is zero) can be induced in spin–orbit-coupled BJJ for weak Raman coupling. In the Hermitian case, our numerical investigation reveals that such a net spin current (together with a vanishing atomic current) can be observable for arbitrary values of Raman coupling strength, when the system is initialized in state \( |\Psi(0)\rangle = (c_{\uparrow}, c_{\downarrow}, c_{\uparrow}, c_{\downarrow}) = \left( \frac{1}{\sqrt{2}}, 0, 0, \frac{1}{\sqrt{2}} \right) \).
equation (\( w \)) in equation (15), the parameter space (a) \((\gamma, 2\alpha/\omega)\) with \(\beta = 0.2\), (b) \((\gamma, \beta)\) with \(2\alpha/\omega = 4\), (c) \((2\alpha/\omega, \beta)\) with \(\gamma = 0\). The different map colors specify different values of \(|\text{Im}(E_2)| + |\text{Im}(E_4)|\). The blue areas with \(|\text{Im}(E_2)| + |\text{Im}(E_4)| = 0\) indicate the unbroken \(\mathcal{PT}\)-symmetric regions where the quasienergies are entirely real, and the red lines are the boundaries between the unbroken and broken \(\mathcal{PT}\)-symmetric regions, across which the quasienergies change from being all real to partial complex. Note that when \(|\text{Im}(E_2)| + |\text{Im}(E_4)| = 0\), all of quasienergies are real.

\[
E_{1,2} = \pm \frac{1}{2} \sqrt{m - i\omega'} = \pm \frac{1}{2} \left( m^2 + w'^2 \right)^{1/4} \left( \cos \frac{\phi'}{2} + i \sin \frac{\phi'}{2} \right),
\]

\[
E_{3,4} = \pm \frac{1}{2} \sqrt{m + i\omega'} = \pm \frac{1}{2} \left( m^2 + w'^2 \right)^{1/4} \left( -\cos \frac{\phi'}{2} + i \sin \frac{\phi'}{2} \right).
\]

where we have defined \(m - i\omega' = (m^2 + w'^2)^{1/2} e^{\phi'}\) and \(m + i\omega' = (m^2 + w'^2)^{1/2} e^{i(2\pi - \phi')}\). For sufficiently strong gain–loss coefficient \(\beta\), we have \(m < 0, w' > 0\) and \(\phi' \in (\pi, \frac{3\pi}{2})\), such that

\[
\text{Im} (E_2) = \text{Im} (E_4) = \frac{1}{2} \left( m^2 + w'^2 \right)^{1/4} \sin \frac{\phi'}{2} > 0,
\]

\[
\text{Im} (E_1) = \text{Im} (E_3) = -\frac{1}{2} \left( m^2 + w'^2 \right)^{1/4} \sin \frac{\phi'}{2} < 0.
\]

That is to say, the imaginary parts of quasienergies \(E_1\) and \(E_3\) in equation (15) are negative, and the imaginary parts of quasienergies \(E_2\) and \(E_4\) are positive.

To gain analytical insight into the current behaviors in the broken \(\mathcal{PT}\)-symmetric region, we expand the quantum state at the initial time in the basis of Floquet modes, i.e. \(|\Psi(0)\rangle = \sum_{n=1}^{N} \alpha_n |\varphi_n(0)\rangle = \sum_{n=1}^{4} \alpha_n |\varphi_n'\rangle\). At time \(t\), the wave function evolves according to equation (9). As time increases, the components \(\alpha_n\) with \(|\text{Im}(E_n)| > 0\) will exponentially grow, and that of negative \(|\text{Im}(E_n)|\) exponentially decays. Thus, the asymptotic solution of the time-evolved quantum state can be written as

\[
|\Psi (t \to \infty)\rangle = a_2 |\varphi_2(t)\rangle e^{-i\xi_2 t} + a_4 |\varphi_4(t)\rangle e^{-i\xi_4 t}
= e^{i\text{Im}(E_2)t} \left( a_2 \hat{S} |\varphi_2'\rangle e^{-i\text{Re}(E_2)t} + a_4 |\varphi_4'\rangle e^{-i\text{Re}(E_4)t} \right),
\]

where \(S = \text{diag} \left( e^{-i\xi_2 \text{sin}(\omega t)}, e^{-i\xi_2 \text{sin}(\omega t)}, e^{i\xi_4 \text{sin}(\omega t)}, e^{i\xi_4 \text{sin}(\omega t)} \right)\).

From equation (21), we have the asymptotic norm

\[
N (t \to \infty) = \langle \Psi (t \to \infty) | \Psi (t \to \infty) \rangle
= e^{2\text{Im}(E_2)t} \left( |a_2|^2 + |a_4|^2 \right) e^{i\text{Re}(E_2) t} + \text{H.c.},
\]

\[
+ e^{2\text{Im}(E_4)t} \left( a_2^* a_4 |\varphi_2'\rangle |\varphi_4'\rangle e^{-i\text{Re}(E_2) t} + \text{H.c.} \right),
\]
and the asymptotic population imbalance and magnetization

\[ P_s(t \to \infty) = \langle \Psi(t \to \infty) | \zeta | \Psi(t \to \infty) \rangle \]

\[ = e^{2\text{Im}(E_4)\tau} \left[ |a_2|^2 \langle \varphi_2^4 \rangle | \varphi_2^4 \rangle + |a_4|^2 \langle \varphi_4^4 \rangle | \varphi_4^4 \rangle \right] \]

\[ + e^{2\text{Im}(E_4)\tau} \left[ a_2^*a_4 \langle \varphi_2^4 \rangle | \varphi_4^4 \rangle e^{\text{Im}(E_4-E_4)\tau} + \text{H.c.} \right], \]

(23)

\[ P_s(t \to \infty) = \langle \Psi(t \to \infty) | \Lambda | \Psi(t \to \infty) \rangle \]

\[ = e^{2\text{Im}(E_4)\tau} \left[ |a_2|^2 \langle \varphi_2^4 \rangle | \varphi_2^4 \rangle + |a_4|^2 \langle \varphi_4^4 \rangle | \varphi_4^4 \rangle \right] \]

\[ + e^{2\text{Im}(E_4)\tau} \left[ a_2^*a_4 \langle \varphi_2^4 \rangle | \varphi_4^4 \rangle e^{\text{Im}(E_4-E_4)\tau} + \text{H.c.} \right]. \]

(24)

In principle, we can explicitly derive the asymptotic forms of all the physical quantities by inserting the eigenvectors \( |\varphi_n^\pm\rangle \) of the effective Hamiltonian (3) into equations (22)–(24). Nevertheless, the derivation process is rather tedious and lengthy for general effective SOC strength. For simplicity, we take \( \gamma = 0 \) as an example for illustration. When \( \gamma = 0 \), the quasien-
ergies takes the simple form

\[ E_{1,2} = \pm \left( \frac{\Omega}{2} + i \sqrt{\beta^2 - \left[ \nu J_0 \left( \frac{2\alpha}{\omega} \right) \right]^2} \right), \]

\[ E_{3,4} = \pm \left( \frac{\Omega}{2} + i \sqrt{\beta^2 - \left[ \nu J_0 \left( \frac{2\alpha}{\omega} \right) \right]^2} \right), \]

(25)

with the corresponding eigenstates \( |\varphi_1^\pm\rangle = (\zeta, \zeta, 1, 1)^T \), \( |\varphi_2^\pm\rangle = (-\zeta, \zeta, -1, 1)^T \), where \( \zeta = i \left( \frac{\beta + \sqrt{\beta^2 - \nu J_0 \left( \frac{2\alpha}{\omega} \right)}}{\nu J_0 \left( \frac{2\alpha}{\omega} \right)} \right) \).

Armed with these, we immediately have

\[ \mathcal{N}(t \to \infty) = 2 \left( |a_2|^2 + |a_4|^2 \right) \left( |\zeta|^2 + 1 \right) e^{2\text{Im}(E_1)\tau}, \]

(26)

\[ P_s(t \to \infty) = 2 \left( |a_2|^2 + |a_4|^2 \right) \left( |\zeta|^2 - 1 \right) e^{2\text{Im}(E_1)\tau}, \]

(27)

\[ P_s(t \to \infty) = 2 \left( 1 - |\zeta|^2 \right) \left( a_2^*a_4 e^{i\Omega t} + a_4^*a_2 e^{-i\Omega t} \right) e^{2\text{Im}(E_1)\tau}. \]

(28)

According to equation (17), we obtain

\[ P_{an}(t \to \infty) = \frac{P_s(t \to \infty)}{\mathcal{N}} = \frac{|\zeta|^2 - 1}{|\zeta|^2 + 1}. \]

(29)

\[ P_{an}(t \to \infty) = \frac{P_s(t \to \infty)}{\mathcal{N}} = \frac{1 - |\zeta|^2 \left( a_2^*a_4 e^{i\Omega t} + a_4^*a_2 e^{-i\Omega t} \right) \left( |\zeta|^2 + 1 \right)}{\left( |a_2|^2 + |a_4|^2 \right)}. \]

(30)

From the definition for normalized atomic current and spin current, we get

\[ I_{an} = \frac{dP_{an}}{dt}(t \to \infty) = 0. \]

(31)

Equations (31) and (32) mean that if we drop the contribution of the exponential growth of the norm to the current in the broken \( \mathcal{PT} \)-symmetric region, for the case of \( \gamma = 0 \), the spin current is nonzero while the atomic current is zero. Along the very same line of reasoning, for the broken \( \mathcal{PT} \)-symmetric phase, we can analytically demonstrate that in the specific case of \( \gamma = 0.5 \) (or half-integer values of \( \gamma \)), the situation is the same as in \( \gamma = 0 \), where the normalized atomic current is zero and normalized spin current is not zero. As we know, the normalized current is more appropriate to quantify the non-Hermitian physics than its unnormalized counterpart. In this sense, we can say that the net spin current (zero \( I_{an} \) and nonzero \( I_{an} \)) exists in the broken \( \mathcal{PT} \)-symmetric region, which we find is nevertheless not a general feature for arbitrary values of effective SOC strength. The above conclusions are independent of the preparation of initial state.

In figure 5, we show the time-evolution curves of the normalized physical quantities such as the population imbalance (magnetization) and the atomic (spin) current, on the basis of full numerical analysis of the system (1) with the same initial state as before. The system parameters are fixed as \( \nu = 2, \Omega = 1, \omega = 20, \alpha = 40 \), with (a) and (b) \( \beta = 0.8, \gamma = 0 \) and (c) and (d) \( \beta = 0.65, \gamma = 0.8 \). The main figures show the time evolutions of the normalized physical quantities, and the insets in each panel give the time evolutions of the unnormalized counterparts. As illustrated in the two insets of the left column, the unnormalized population imbalances (blue lines) for both parameter sets show unbounded growth as a signature of the broken \( \mathcal{PT} \) phase. The difference is that the unnormalized population imbalance shows smooth exponential behavior for the case of \( \gamma = 0 \), while for \( \gamma = 0.8 \), due to the general nature that the complex Floquet eigenstates are not necessarily orthogonal in non-Hermitian system, the unnormalized population imbalance shows oscillatory growth (exponential growth plus periodic oscillation), rather than smooth exponential growth. As shown in figures 5(a) and (b) with \( \beta = 0.8 \) and \( \gamma = 0 \), the normalized population imbalance oscillates during the initial short time interval, after which it is asymptotically unchanged with time evolution, and as the time evolves, the vanishing normalized atomic current will be a consequence (see the blue line in main figure of panel (b)). At the same time, the normalized magnetization and normalized spin current (see red dashed lines) show stable periodic oscillations after a transient decay. When \( \beta = 0.65 \) and \( \gamma = 0.8 \) are set as shown in figures 5(c) and (d), the normalized population imbalance rapidly tends to a small-amplitude periodic oscillation around a nonzero value, and as a result, the normalized atomic current asymptotically tends to a stable periodic oscillation around zero with very small oscillating amplitude. Additionally, the normalized magnetization and normalized spin current exhibit stable periodic oscillations with large amplitude as the increase in time. As the numerical simulations demonstrate, the net spin current (zero \( I_{an} \) and nonzero \( I_{an} \)) exists in the broken \( \mathcal{PT} \)-symmetric region, but it is truly not a general character for
4. Currents under unbalanced gain and loss

In this section, we turn to explore the current behaviors of the non-$PT$-symmetric system with unbalanced gain and loss ($\beta_l \neq \beta_r$). We assume $\beta_l < \beta_r$, which represents a dissipative system with the particle loss in the right well greater than the gain in the left well. Generally, in such a situation, all of $\text{Im}(E_n)$ are less than zero and the atomic probabilities will asymptotically decay to zero. However, if we tune the driving parameters to satisfy the following conditions:

$$4|\nu J_0(2\nu/\omega)|^2 + \Omega^2 = (\beta_l + \beta_r)^2, \beta_l - \beta_r = -\sqrt{\Omega x},$$  

where $x = \sqrt{(\beta_l + \beta_r)^2 - 4(\nu J_0(2\nu/\omega))^2 \cos^2(\gamma \pi)}$, from equation (7) the quasienergies are given by

$$E_1 = i(\beta_l - \beta_r) - \frac{1}{2}\sqrt{\Omega x}, E_2 = \frac{1}{2}\sqrt{\Omega x},$$  

$$E_3 = i(\beta_l - \beta_r) + \frac{1}{2}\sqrt{\Omega x}, E_4 = -\frac{1}{2}\sqrt{\Omega x}.$$  

That is to say, two of the quasienergies are real, and the imaginary parts of the other two are less than zero. This will result in that the populations and currents tend to be stable as time increases. According to equation (9), at $t \to \infty$, the asymptotic solution of the quantum state can be written as

$$|\Psi(t \to \infty)\rangle = a_2|\varphi_2(t)\rangle e^{-i\xi t} + a_4|\varphi_4(t)\rangle e^{i\xi t},$$  

where $\tilde{S} = \text{diag}(e^{-i\frac{\sqrt{2}}{\Omega} \sin(\omega t)}, e^{i\frac{\sqrt{2}}{\Omega} \sin(\omega t)}, e^{i\frac{\omega}{2} \sin(\omega t)}, e^{-i\frac{\omega}{2} \sin(\omega t)})$. In our analysis, we focus on the case $\gamma = 0.5$, but similar behaviors can be obtained for other choices of effective SOC as well. When $\gamma = 0.5$, from equation (34) we can simplify the two real quasienergies as $E_2 = \sqrt{\Omega (\beta_l + \beta_r)}$, $E_4 = -\sqrt{\Omega (\beta_l + \beta_r)}$, with the corresponding eigenstates $|\varphi_2\rangle = (\xi, \xi, 1, 1)^T$, $|\varphi_4\rangle = (-\xi^*, -\xi^*, -1, 1)^T$, where $\xi = \frac{\sqrt{2\beta_l - \beta_r - i\sqrt{\beta_l \beta_r - \Omega^2}}}{2\sqrt{\nu J_0(2\nu/\omega)}}$. Applying them to equation (35) yields

$$c_{\uparrow, \downarrow}(t \to \infty) = a_2 \xi e^{-\frac{\sqrt{\Omega (\beta_l + \beta_r)}}{2}} + a_4 \xi^* e^{\frac{\sqrt{\Omega (\beta_l + \beta_r)}}{2}} \times e^{i\frac{\sqrt{2}}{\Omega} \sin(\omega t)},$$  

$$c_{\downarrow, \uparrow}(t \to \infty) = a_2 \xi e^{-\frac{\sqrt{\Omega (\beta_l + \beta_r)}}{2}} + a_4 \xi^* e^{\frac{\sqrt{\Omega (\beta_l + \beta_r)}}{2}} \times e^{i\frac{\omega}{2} \sin(\omega t)}.$$  

According to the definition for population imbalance and magnetization, we obtain

$$P_a(t \to \infty) = \left(|c_{\uparrow\uparrow}|^2 + |c_{\downarrow\downarrow}|^2 - |c_{\uparrow\downarrow}|^2 - |c_{\downarrow\uparrow}|^2\right)_{t \to \infty} = 2(|\xi|^2 - 1)(|a_2|^2 + |a_4|^2).$$
To corroborate the above analytical results, the population imbalance and magnetization (panels (a), (c) and (e)), and the atomic and spin currents (panels (b), (d) and (f)) obtained from the non-$P^T$-symmetric system (1) under the condition (33). The parameters are set as (a) and (b) $\nu = 2$, $\Omega = 1$, $\alpha = 22.52$, $\omega = 40$, $\beta_1 = 1$, $\beta_2 = 2$, $\gamma = 0$; (c) and (d) $\nu = 2$, $\Omega = 1$, $\alpha = 7.16$, $\omega = 40$, $\beta_1 = 1$, $\beta_2 = 3$, $\gamma = 0.5$. In panels (a) and (d), the initial states are the same and taken as $|\Psi(0)\rangle = \left(\frac{1}{\sqrt{2}}, 0, 0, \frac{1}{\sqrt{2}}\right)^T$. (e) and (f) Parameters are the same as in (c) and (d), but with different initial state $|\Psi(0)\rangle = \left(\sqrt{0.4}, 0, 0, \sqrt{0.6}\right)^T$.

$P_c(t \rightarrow \infty) = \left| c_{1\uparrow} \right|^2 - \left| c_{4\uparrow} \right|^2 - \left| c_{1\downarrow} \right|^2 + \left| c_{4\downarrow} \right|^2 \bigg|_{t \rightarrow \infty}$

$= 4 \left| a_2 \right| \left| a_4 \right| \left( \left| \xi \right|^2 + 1 \right) \cos \theta_4 - \theta_2 + \sqrt{\Omega (\beta_1 + \beta_2)} \right|_{t \rightarrow \infty}$,

where $a_2 = |a_2|e^{i\theta_2}$, $a_4 = |a_4|e^{i\theta_4}$. Substituting equation (36) into equation (14), we get the atomic current and spin current as

$I_a(t \rightarrow \infty) = 0$,

$I_s(t \rightarrow \infty) = - 4 \sqrt{\Omega (\beta_1 + \beta_2)} \left| a_2 \right| \left| a_4 \right| \left( \left| \xi \right|^2 + 1 \right)$

$\times \sin \left[ \theta_4 - \theta_2 + \sqrt{\Omega (\beta_1 + \beta_2)} t \right]$. (38)

To corroborate the above analytical results, the population imbalance and magnetization (panels (a), (c) and (e)) and the atomic and spin currents (panels (b), (d) and (f)) are plotted versus time, by direct integration of the time-dependent Schrödinger equation with Hamiltonian (1). In figure 6, we consider two scenarios with $\gamma = 0$ and $\gamma = 0.5$, and take two sets of the parameter: (a) and (b) $\nu = 2$, $\Omega = 1$, $\alpha = 22.52$, $\omega = 40$, $\beta_1 = 1$, $\beta_2 = 2$, $\gamma = 0$ and (c) and (d) $\nu = 2$, $\Omega = 1$, $\alpha = 7.16$, $\omega = 40$, $\beta_1 = 1$, $\beta_2 = 3$, $\gamma = 0.5$, to match the condition (33). The initial states for both cases are taken as $|\Psi(0)\rangle = (c_{1\uparrow}, c_{1\downarrow}, c_{4\uparrow}, c_{4\downarrow})^T = \left(\frac{1}{\sqrt{2}}, 0, 0, \frac{1}{\sqrt{2}}\right)^T$. For both cases, we observe that after certain period of time, the population imbalances tend to steady nonzero positive values which signal the spatial localization of the condensate in the amplifying well, whereas magnetization exhibits periodic oscillation. As a consequence, the atomic currents tend to zero, while the spin currents are always nonzero, as illustrated in figures 6(b) and (d). In figures 6(e) and (f), we carry out the numerical studies of the populations and current behaviors with the same parameters as those in figures 6(c) and (d), but only with different initial state $|\Psi(0)\rangle = \left(\sqrt{0.4}, 0, 0, \sqrt{0.6}\right)^T$.

Apparentely, though the initial state is altered, the same physical effect (zero atomic current and nonzero spin current) can be attained, and the conclusion remains unaffected, except for the magnitudes of asymptotic population imbalance and oscillating amplitudes of magnetization (spin current). With the system parameters and initial states presented in figures 6(c) and (e), according to equation (37), we analytically calculate the asymptotic values of population imbalances, which read 0.63 and 0.56 as indicated by the horizontal lines, showing good agreement with the numerical results. Thus, we demonstrate, both analytically and numerically, that the net spin current and steady state with spatial localization can be accessible in the dissipative Floquet spin–orbit-coupled system by adjusting the driving parameters.

5. Conclusions

We have analytically and numerically explored the populations and current behaviors of a single spin–orbit-coupled bosonic atom held in an open double well under periodic driving. Under the high-frequency driving, we have deduced the effective Hamiltonian by using the time-averaging method, and obtained the analytical Floquet states and quasienergies of the considered system. We have explored the joint effects of SOC and periodic driving on Josephson tunneling and current behaviors in a non-Hermitian double-well system. Interestingly, it is found that if the values of SOC strength are taken of half-integer numbers, no matter what values the other parameters are taken, the $P^T$ phase transition can appear even for arbitrarily small gain–loss coefficients.

In addition, we have revealed that the net spin current (zero atomic current and nonzero spin current) effect can not exist in the unbroken $P^T$-symmetric region, whereas it can survive in the broken $P^T$-symmetric region, if we drop the contribution of the growth of the norm to the current behaviors. In other words, in the broken $P^T$-symmetric region, the normalized atomic current can be zero while the normalized spin current can be nonzero at the same time. Nevertheless, the existence of net spin current (zero normalized atomic current and nonzero normalized spin current) in the broken $P^T$-symmetric region is not a general feature for arbitrary values of SOC strength. When the dissipation is greater than the gain, we...
have found that the net spin current effect can be truly realized by tuning the parameters of the periodic driving field to match certain conditions. In such a non-$\mathcal{PT}$-symmetric system, the periodic driving enables the system to approach a steady state accompanied with atomic localization phenomenon, for which the atomic current is zero while the spin current is nonzero.

Data availability statement

The data that support the findings of this study are available upon reasonable request from the authors.

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Jia Tang and Zhou Hu contributed equally to this work.

ORCID iDs

Jinpeng Xiao https://orcid.org/0000-0003-0848-804X
Xiaobing Luo https://orcid.org/0000-0002-3724-4848

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