UNIFORM CONVERGENCE FOR COMPLEX \([0,1]\)-MARTINGALES

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Positive \(T\)-martingales were developed as a general framework that extends the positive measure-valued martingales and are meant to model intermittent turbulence. We extend their scope by allowing the martingale to take complex values. We focus on martingales constructed on the interval \(T = [0, 1]\) and replace random measures by random functions. We specify a large class of such martingales for which we provide a general sufficient condition for almost sure uniform convergence to a nontrivial limit. Such a limit yields new examples of naturally generated multifractal processes that may be of use in multifractal signals modeling.

1. Introduction.

1.1. Foreword about multifractal functions. Multifractal analysis is a natural framework to describe the heterogeneity that is reflected in the distribution at small scales of the Hölder singularities of a given locally bounded function or signal \(F : I \mapsto \mathbb{C}\) where \(I\) is an interval. The Hölder singularity of \(F\) can be defined, at every point \(t\), by

\[
h_F(t) = \liminf_{r \to 0^+} \frac{\log \text{Osc}_F([t - r, t + r])}{\log(r)}
\]

or

\[
h_F(t) = \liminf_{n \to \infty} \frac{\log_2 \text{Osc}_F(I_n(t))}{-n},
\]

where \(I_n(t)\) is the dyadic interval of length \(2^{-n}\) containing \(t\) and \(\text{Osc}_F(J) = \sup_{s, t \in J} |F(t) - F(s)|\). The multifractal analysis of \(F\) classifies points according to \(h_F\). It may compute the singularity spectrum of \(F\), that is, the Hausdorff dimension of the sets \(h_F^{-1}([h])\) for \(h \geq 0\) or, more roughly, measure the asymptotic number of dyadic intervals of generation \(n\) needed to cover the sets \(h_F^{-1}([h])\) by estimating the large deviation spectrum

\[
L_F(h) = \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{\log_2 \# \{J \in G_n, 2^{-n(h+\varepsilon)} \leq \text{Osc}_F(J) \leq 2^{-n(h-\varepsilon)}\}}{n},
\]
where $\mathcal{G}_n$ is the set of dyadic intervals of generation $n$. One says that $F$ is monofractal if there exists a unique $h \geq 0$ such that $E_F(h) \neq \emptyset$ or $L_F(h) \neq -\infty$. Otherwise, $F$ is multifractal (see [13, 25] for more details).

1.2. Motivations and methods to build multifractal processes. The main motivation for constructing and studying multifractal functions or stochastic processes comes from the need to model empirical signals for which the estimation of $L_F$ and related quantities reveals striking scaling invariance properties. These signals concern physical or social intermittent phenomena like energy dissipation in turbulence [11, 19, 20], spatial rainfall [12], human heart rate [29], internet traffic [26] and stock exchange prices [22]. Models of these phenomena are the statistically self-similar measures constructed in [2, 5, 17, 20]. These objects are special examples of limit of “$T$-martingales,” which consist in a class of random measures developed in [14, 15] after the seminal work [19] about Gaussian multiplicative chaos (see also [10, 30]). When $T = [0, 1]$, these martingales and their limit are also used to build models of nonmonotonic scaling invariant signals as follows: By performing a multifractal time change in Fractional Brownian motions or stable Lévy processes [2, 22, 25], by integrating a positive $[0, 1]$-martingale with respect to the Brownian motion or using such a martingale to specify the covariance of some Gaussian processes to get new types of multifractal random walks [2, 18], or by considering random wavelet series whose coefficients are built from a multifractal measure [1, 8].

1.3. A natural alternative construction. This paper considers the natural alternative to these constructions which allows the multiplicative processes involved in $[0, 1]$-martingales to take complex values.

Let us now recall what are $[0, 1]$-martingales. Let $(\Omega, \mathcal{B}, \mathbb{P})$ be a probability space, endow the interval $[0, 1]$ with the Borel $\sigma$-algebra $\mathcal{B}([0, 1])$ and the product space $[0, 1] \times \Omega$ with the product $\sigma$-algebra $\mathcal{B}([0, 1]) \otimes \mathcal{B}$. Let $(\mathcal{B}_n)_{n \geq 1}$ be a nondecreasing sequence of $\sigma$-algebras in $\mathcal{B}$. Also let $(Q_n)_{n \geq 1}$ be a sequence of complex-valued measurable functions defined on $[0, 1] \times \Omega$ such that for each $t \in [0, 1]$, $\{Q_n(t, \cdot), \mathcal{B}_n\}_{n \geq 1}$ is a martingale of expectation 1. Such a sequence of functions is called a $[0, 1]$-martingale. Given a Radon measure $\lambda$ on $[0, 1]$, for every $n \geq 1$ we can define the random complex measure $\mu_n$ whose density with respect to $\lambda$ is equal to $Q_n$.

If the functions $Q_n$ take nonnegative values, then, with probability 1, the sequence of Radon measures $(\mu_n)_{n \geq 1}$ weakly converges to a measure $\mu$ ([14, 15]). This property is an almost straightforward consequence of the positive martingale convergence theorem and Riesz’s representation theorem. When the random functions $Q_n$ cease to be nonnegative, the martingales $Q_n(t)$ need not be bounded in $L^1$ norm; hence the total variations of the complex measures $\mu_n$ may diverge, and $(\mu_n)_{n \geq 1}$ need not converge almost surely weakly to an element of the dual
of $C([0, 1])$, the space of continuous complex-valued functions over $[0, 1]$. In this paper we rather consider the sequence of random continuous functions

$$F_n : t \in [0, 1] \mapsto \mu_n([0, t]) = \int_0^t Q_n(u) \, d\lambda(u).$$

Then the following questions arise naturally:

**Question 1.** Does there exist a general necessary and sufficient condition under which $(F_n)_{n \geq 1}$ converges almost surely uniformly to a limit which is non-trivial (i.e., different from 0) with positive probability?

**Question 2.** When the sequence $(F_n)_{n \geq 1}$ diverges, or converges to 0 in $C([0, 1])$, can a natural normalization of $F_n$ make it converge to a nontrivial multifractal limit $\tilde{F}$, at least in distribution?

**Question 3.** Consider the case of strong or weak convergence to a limit process $F$ or $\tilde{F}$ having scaling invariance properties. What is the multifractal nature of $F$ (or $\tilde{F}$), and does $F$ or $\tilde{F}$ possess the remarkable property to be naturally decomposed as a monofractal function in multifractal time, like for some other classes of multifractal functions [2, 21, 22, 27]?

We will introduce a subclass of complex $[0, 1]$-martingales, namely $\mathcal{M}$, such that for $(Q_n)_{n \geq 1} \in \mathcal{M}$, we have a general sufficient condition for the almost sure uniform convergence of $(F_n)_{n \geq 1}$ to a nontrivial limit, as well as a result of global Hölder regularity for the limit function (Theorem 2.1). Our result makes it possible to construct the complex extensions of some fundamental examples of statistically self-similar positive multiplicative cascades mentioned above (see Section 2.3 and an illustration in Figure 1).

Companion papers [3] and [4] provide further results and answers to the previous questions in the particular case of complex $b$-adic independent cascades (it is worth noting that these objects also play a role in the study of directed polymers in a random medium [9]).

Section 2 introduces the class $\mathcal{M}$, states Theorem 2.1 and provides fundamental examples in $\mathcal{M}$. Section 3 provides the proof of Theorem 2.1. We end this section with some definitions.

**1.4. Definitions.** Given an integer $b \geq 2$, we denote by $\mathcal{A}$ the alphabet \{0, \ldots, $b-1$\} and define $\mathcal{A}^* = \bigcup_{n \geq 0} \mathcal{A}^n$ (by convention $\mathcal{A}^0$ is the set reduced to the empty word denoted $\emptyset$). For every $n \geq 0$, the length of an element of $\mathcal{A}^n$ is by definition equal to $n$, and we denote it by $|w|$. For $w \in \mathcal{A}^*$, we define $t_w = \sum_{i=1}^{\lfloor |w| \rfloor} w_i b^{-i}$ and $I_w = [t_w, t_w + b^{-|w|})$. For $n \geq 1$ we define $T_n = \{t_w : w \in \mathcal{A}^n \} \cup \{1\}$ and then $T_* = \bigcup_{n \geq 1} T_n$. 


Fig. 1. A complex valued canonical dyadic cascade $F_n$ for $n = 9, 11, 15, 16, 17, 18$.

For any $t \in [0, 1]$ and $n \geq 1$, we denote by $t|n$ the unique word in $A^n$ such that $t \in I_{t|n}$. We also denote by $t|0$ the empty word.

If $f \in C([0, 1])$ we denote by $\|f\|_\infty$ the norm $\sup_{t \in [0,1]} |f(t)|$.

We denote by $(\Omega, \mathcal{B}, \mathbb{P})$ the probability space on which the random variables considered in this paper are defined.

2. A class of complex [0, 1]-martingales.

2.1. Definition. Consider a sequence of measurable complex functions

$$P_n: ([0, 1] \times \Omega, \mathcal{B}([0, 1]) \otimes \mathcal{B}) \mapsto (\mathbb{C}, \mathcal{B}(\mathbb{C})), \quad n \geq 1.$$ 

For $n \geq 1$ and $I$, a subinterval of [0, 1], let $\mathcal{F}^I_n$ be the $\sigma$-field generated in $\mathcal{B}$ by the family of random variables $\{P_m(t, \cdot)\}_{t \in I, 1 \leq m \leq n}$. Also let $\overline{\mathcal{F}}^I_n$ be the $\sigma$-field generated in $\mathcal{B}$ by the family of random variables $\{P_m(t, \cdot)\}_{t \in I, m > n}$. The $\sigma$-fields $\mathcal{F}^{[0,1]}_n$ and $\overline{\mathcal{F}}^{[0,1]}_n$ are simply denoted by $\mathcal{F}_n$ and $\overline{\mathcal{F}}_n$.

(P1) For all $t \in [0, 1]$, $P_n(t, \cdot)$ is integrable, and $\mathbb{E}(P_n(t, \cdot)) = 1$.

(P2) For every $n \geq 1$, $\mathcal{F}_n$ and $\overline{\mathcal{F}}_n$ are independent.

(P3) There exist two integers $b \geq 2$ and $N \geq 1$ such that for every $n \geq 1$ and every family $\mathcal{G}$ of $b$-adic subintervals of [0, 1] of generation $n$ such that $d(I, J) \geq Nb^{-n}$ for every $I \neq J \in \mathcal{G}$, the $\sigma$-algebra’s $\mathcal{F}^I_n$, $I \in \mathcal{G}$, are mutually independent, where $d(I, J) = \inf\{|t - s| : s \in I, t \in J\}$. 


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Under the properties (P1) and (P2), for each \( t \in (0, 1) \) the sequence

\[
Q_n(t, \cdot) = \prod_{k=1}^{n} P_k(t, \cdot)
\]

is a martingale of expectation 1 with respect to the filtration \( \mathcal{F}_n \) for each \( t \in (0, 1) \).

We denote by \( \mathcal{M} \) the class of martingales \((Q_n)_{n \geq 1}\) obtained as above and which satisfy properties (P1)–(P3).

We denote by \( \mathcal{M}' \) the subclass of \( \mathcal{M} \) of those \((Q_n)_{n \geq 1}\) which, in addition to (P1)–(P3), satisfy the statistical self-similarity property:

(P4) Let \( b \) be as in (P3). For every closed \( b \)-adic subinterval \( I \) of \([0, 1]\), let \( n(I) \) and \( S_I \), respectively, stand for the generation of \( I \) and the canonical affine map from \([0, 1]\) onto \( I \). The processes \( (P_n(I) + n \circ S_I)_{n \geq 1} \) and \( (P_n)_{n \geq 1} \) have the same distributions.

Let \( \lambda \) be a Radon measure on \([0, 1]\). If \((Q_n)_{n \geq 1} \in \mathcal{M} \), for \( n \geq 1 \), we define

\[
F_n(t) = \int_0^t Q_n(u) \, d\lambda(u).
\]

2.2. Convergence theorem for \((F_n)_{n \geq 1}\). Theorem 2.1 provides a sufficient condition for the almost sure uniform convergence of \( F_n \), as \( n \) tends to \( \infty \), to a limit \( F \) such that \( \mathbb{P}(F \neq 0) > 0 \). This condition is the extension of the condition introduced in Part II of [6] to show that when \((Q_n)_{n \geq 1}\) is nonnegative, the sequence of measures \( F'_n \) converge almost surely weakly to a random measure \( \mu \) such that \( \mathbb{P}(\mu \neq 0) > 0 \). When \((Q_n)_{n \geq 1}\) is not nonnegative, the uniform convergence of \( F_n \) is a more delicate issue.

For \( p \in \mathbb{R}_+ \) and \( n \geq 1 \) we define

\[
S(n, p) = \sum_{w \in \mathcal{A}^n} \lambda(I_w)^{p-1} \int_{I_w} \mathbb{E}(\{|Q_n(t)|^p\}) \, d\lambda(t)
\]

and

\[
\varphi(p) = \lim_{n \to \infty} \left( -\frac{1}{n} \log_b S(n, p) \right).
\]

We notice that \( \varphi \) is a concave function of \( p \), \( \varphi(0) \leq 0 \) by construction, and that due to our assumption that \( \mathbb{E}(Q_n(t)) = 1 \), we also have \( \varphi(1) \leq 0 \).

Theorem 2.1.

(1) Suppose that \( \varphi(p) > 0 \) for some \( p \in (0, 1) \), and that there exists a function \( \psi : \mathbb{N}_+ \to \mathbb{R}_+ \) such that \( \psi(n) = o(n) \) and \( \mathbb{E}(\sup_{t \in I_w} |Q_n(t)|^p) \leq \exp(\psi(n)) \times \mathbb{E}(|Q_n(t)|^p) \) for all \( n \geq 1 \), \( w \in \mathcal{A}^n \) and \( t \in I_w \). Then, with probability 1, \( F_n \) converge uniformly to 0 as \( n \to \infty \).

(2) Let \( p \in (1, 2] \). Suppose that \( \varphi(p) > 0 \). The functions \( F_n \) converge uniformly, almost surely and in \( L^1 \) norm, to a limit \( F \), as \( n \to \infty \). The function \( F \) is \( \gamma \)-Hölder continuous for all \( \gamma \in (0, \max_{q \in (1, p]} \varphi(q)/q) \). Moreover, \( \mathbb{E}(\|F\|_\infty^\gamma) < \infty \).
Remark 2.1. (1) The proof of Theorem 2.1(1) will show that this result does not require (P1), (P2) or (P3). The existence of the function $\psi$ corresponds to a kind of bounded distortion principle.

(2) Under the assumptions of Theorem 2.1(2), let $\beta = \min\{p \in [1, 2] : \varphi(p) = 0\}$. The nonnegative sequence $(Q_n^{(\beta)})_{n \geq 1} = (|Q_n|^{\beta})_{n \geq 1}$ is an element of $\mathcal{M}$, and by construction, the corresponding function $\varphi$ is positive near $1^+$. Consequently, the sequence $F_n^{(\beta)}$ defined by $\int_0^\cdot Q_n^{(\beta)}(u) \, du$ converges uniformly to a nondecreasing function $F^{(\beta)}$. Inspired by the results obtained in [4], it is natural to ask under which additional assumptions it is possible to write $F = B_{1/\beta} \circ F^{(\beta)}$ where $B_{1/\beta}$ is a monofractal function of exponent $1/\beta$.

(3) Suppose that $\varphi$ is not positive over $[0, 2]$. In the case where the martingale $(F_n(1))_{n \geq 1}$ is not bounded in $L^2$ norm, inspired again by what is done in [4], it is natural to look at the process $F_n/\sqrt{\mathbb{E}(F_n(1)^2)}$ and seek for conditions under which it converges in distribution, as $n \to \infty$.

2.3. Examples.

Homogeneous $b$-adic independent cascades. We consider the complex extension of the nonnegative $[0, 1]$-martingales introduced in [20]. Let $b$ be an integer $\geq 2$ and for every $k \geq 0$ let $W^{(k)} = (W_0^{(k)}, \ldots, W_{b-1}^{(k)})$ be a vector such that each of its components is complex, integrable and has an expectation equal to 1. Then, consider $\{W^{(|w|)}(w)\}_{w \in \Sigma^*}$, a family of independent vectors such that for each $k \geq 0$ and $w \in \Sigma_k$ the vector $W^{(k)}(w)$ is a copy of $W^{(k)}$.

An element of $\mathcal{M}$ is obtained as follows. For $t \in [0, 1)$ and $n \geq 1$ let $P_n(t) = W_{tn}^{(n-1)}(t|n-1)$ and then $Q_n(t) = \prod_{k=1}^n P_k(t)$. If $\lambda$ is the inhomogeneous Bernoulli measure associated with a sequence of probability vectors $(\lambda^{(k)} = \lambda_0^{(k)}, \ldots, \lambda_{b-1}^{(k)})_{k \geq 0}$, then

$$\varphi(p) = \liminf_{n \to \infty} \left( -\frac{1}{n} \sum_{k=0}^{n-1} \log_b \mathbb{E} \left( \sum_{i=0}^{b-1} (\lambda_i^{(k)} |W_i^{(k)}|)^p \right) \right).$$

If all the vectors $W^{(k)}$ have the same distribution as a vector $W$, then $(Q_n)_{n \geq 1}$ belongs to $\mathcal{M}'$. Canonical cascades correspond to $W$ whose components are i.i.d. and $\lambda$ equal to the Lebesgue measure. Then a necessary and sufficient condition for the almost sure uniform convergence of $F_n$ to a nontrivial limit is $\varphi'(1^-) > 0$ if $W \geq 0$ [16, 17] and $\varphi(p) > 0$ for some $p \in (1, 2]$ for the special “monofractal” examples considered in [7].

Compound Poisson cascades. We consider the complex extension of the nonnegative $[0, 1]$-martingales introduced in [5]. Let $\nu$ be a positive Radon measure over $(0, 1]$ and denote by $\Lambda$ the measure $\text{Leb} \otimes \nu$ where $\text{Leb}$ stands for the Lebesgue measure over $\mathbb{R}$. We consider a Poisson point process $S$ of intensity $\Lambda$. 
To each point $M$ of $S$, we associate a random variable $W_M$ picked in a collection of random variables that are independent, independent of $S$, and are identically distributed with an integrable complex random variable $W$. We fix $\beta > 0$, and for $n \geq 1$ and $t \in [0, 1]$ we define the truncated cone $$\Delta C_n(t) = \{(t', r) : b^{-n} < r \leq b^{1-n}, t - \beta r/2 \leq t' < t + \beta r/2\}.$$ We obtain an element of $\mathcal{M}$ as follows. For $t \in [0, 1)$ and $n \geq 1$ we define $P_n(t) = e^{-\Lambda(\Delta C_n(t))} \mathbb{E}(W) - 1 \prod_{M \in S \cap \Delta C_n(t)} W_M$, and then $Q_n(t) = \prod_{k=1}^n P_k(t)$. If $\lambda$ is the Lebesgue measure and $\tilde{\beta}$ stands for $\limsup_{n \to \infty} n^{-1} \log \Lambda(\bigcup_{k=1}^n \Delta C_n)$, $\varphi(p) = p - 1 + \tilde{\beta}(p(\mathbb{E}(\Im W) - 1) - (\mathbb{E}(|W|^p) - 1))$. If, moreover, there exists $\delta > 0$ such that $\nu(dr) = \delta dr/r^2$, that is, if $\Lambda$ possesses scaling invariance properties, we have $\tilde{\beta} = \beta \delta$, and $(Q_n)_{n \geq 1}$ belongs to $\mathcal{M}'$.

**Log-infinitely divisible cascades.** This example is an extension of compound Poisson cascades when the weights $W_M$ take the form $\exp(L_M)$, and, in particular, the $W_M$ do not vanish. We use the notations of the previous section and take $\beta = \delta = 1$. Let $\psi$ be a characteristic Lévy exponent $\psi$ defined on $\mathbb{R}^2$, that is,

\begin{equation}
\psi : \xi \in \mathbb{R}^2 \mapsto i \langle \xi | a \rangle - Q(\xi)/2 + \int_{\mathbb{R}^2} (1 - e^{i \langle \xi | x \rangle} + i \langle \xi | x \rangle \mathbf{1}_{|x| \leq 1}) \pi(dx),
\end{equation}

where $a \in \mathbb{R}^2$, $Q$ is a nonnegative quadratic form and $\pi$ is a Radon measure on $\mathbb{R}^2 \setminus \{0\}$ such that $\int (1 \wedge |x|^2) \pi(dx) < \infty$.

Then let $\rho = (\rho_1, \rho_2)$ be an independently scattered infinitely divisible random $\mathbb{R}^2$-valued measure on $\mathbb{R} \times \mathbb{R}_+^*$ with $\Lambda$ as control measure and $\psi$ as Lévy exponent (see [24] for the definition). In particular, for every Borel set $B \in \mathbb{R} \times \mathbb{R}_+^*$ and $\xi \in \mathbb{R}^2$ we have

$$\mathbb{E}(e^{i \langle \xi | \rho(B) \rangle}) = \exp(\psi(\xi) \Lambda(B)).$$

and for every finite family $\{B_i\}$ of pairwise disjoint Borel subsets of $\mathbb{R} \times \mathbb{R}_+^*$ such that $\Lambda(B_i) < \infty$, the random variables $\rho(B_i)$ are independent.

Let $I_1$ be the interval of those $\xi_1 \in \mathbb{R}$ such that $\int_{|x| \geq 1} e^{\xi_1 x_1} \pi(dx) < \infty$. The function $\psi$ has a natural extension $\tilde{\psi}$ to $\mathcal{D} = \mathbb{R}^2 \cup (-i I_1 \times \mathbb{R})$ given by the same expression as in (2.4) if we extend $Q$ to an Hermitian form on $\mathbb{C}^2$. Then for every $\xi \in \mathcal{D}$ and every Borel subset of $\mathbb{R} \times \mathbb{R}_+^*$ we have $\mathbb{E}(e^{i \langle \xi | \rho(B) \rangle}) = \exp(\tilde{\psi}(\xi) \Lambda(B))$. 

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Now, we assume that \( \xi_0 = (-i, 1) \in D \), and without loss of generality we set
\[
\tilde{\psi} := \widetilde{\psi} - \tilde{\psi}(\xi_0).
\]
Then, with the same definition of cones as in the previous section, if \( n \geq 1 \) and \( t \in [0, 1] \), we define
\[
P_n(t) = \exp[\langle \xi_0 | \rho(\Delta C_n(t)) \rangle] = \exp[\rho_1(\Delta C_n(t)) + i\rho_2(\Delta C_n(t))]
\]
and \( Q_n(t) = \prod_{k=1}^n P_n(t) \). If we take \( \lambda \) equal to the Lebesgue measure, and if \( p \in \mathbb{R} \) is such that \( (-ip, 0) \in D \), then
\[
(2.5) \quad \phi(p) = p - 1 - \tilde{\beta}\tilde{\psi}(-ip, 0).
\]
In the positive case, this construction that has been proposed has an extension of compound Poisson cascades in [2]. If \( \nu(dr) = dr/r^2 \), then \( (Q_n)_{n \geq 1} \) belongs to \( \mathcal{M}' \). In [2], a modification of \( P_1(t) \) is introduced, which yields a nice exact statistical scaling invariance property for the increments of the limit measure. It can be easily checked that this property, which is different from the statistical self-similarity imposed by (P4), also holds for the complex extension.

3. Proof of Theorem 2.1.

PROOF OF THEOREM 2.1(1). For any \( w \in \mathcal{A}^* \) and \( n \geq 1 \), define
\[
(3.1) \quad \Delta F_n(I_w) = F_n(t_w + b^{-n}) - F_n(t_w) = \int_{I_w} Q_n(t) \, d\lambda(t).
\]
We have \( \mathbb{E}(\| F_n \|_\infty^p) \leq \mathbb{E}(\sum_{w \in \mathcal{A}^n} |\Delta F_n(I_w)|^p) \leq \mathbb{E}(\sum_{w \in \mathcal{A}^n} |\Delta F_n(I_w)|^p) \), where we have used the subadditivity of \( x \geq 0 \mapsto x^p \) \((p \in (0, 1])\). Thus
\[
\mathbb{E}(\| F_n \|_\infty^p) \leq \sum_{w \in \mathcal{A}^n} \mathbb{E} \left( \left| \int_{I_w} Q_n(t) \, d\lambda(t) \right|^p \right)
\]
\[
\leq \sum_{w \in \mathcal{A}^n} \lambda(I_w)^p \mathbb{E} \left( \sup_{t \in I_w} |Q_n(t)|^p \right)
\]
\[
\leq \sum_{w \in \mathcal{A}^n} \exp(\psi(n))\lambda(I_w)^{p-1} \int_{I_w} \mathbb{E}(|Q_n(t)|^p) \, d\lambda(t)
\]
\[
= \exp(\psi(n))S(n, p).
\]
Due to the property of \( \psi(n) \), we have \( \lim sup_{n \to \infty} \log_b(\mathbb{E}(\| F_n \|_\infty^p))/n \leq -\psi(p) < 0 \). This implies the result. □

PROOF OF THEOREM 2.1(2). The two following crucial statements, which take natural and classical forms, will be proved at the end of the section.
PROPOSITION 3.1. There exists a constant $C_p > 0$ such that

$$
(\forall n \geq 2) \quad \mathbb{E}\left(\max_{t \in T_n} |F_n(t) - F_{n-1}(t)|^p\right) \leq C_p S(n, p).
$$

Consequently, for every $b$-adic number $t \in T_*$, $F_n(t)$ converges almost surely and in $L^p$ norm as $n \to \infty$.

PROPOSITION 3.2. Let $\gamma \in (0, \max_{q \in (1, p]} \varphi(q)/q)$. With probability 1, there exists $\eta_\gamma > 0$ such that for any $t, s \in T_*$ such that $|t - s| < \eta_\gamma$ we have

$$
(3.3) \quad \sup_{n \geq 1} |F_n(t) - F_n(s)| \leq C_\gamma |t - s|^\gamma,
$$

where $C_\gamma$ is a constant depending on $\gamma$ only.

Since $F_n(0) = 0$ almost surely for all $n \geq 1$, it follows from Propositions 3.2 and Ascoli–Arzela’s theorem that, with probability 1, the sequence of continuous functions $(F_n)_{n \geq 1}$ is relatively compact, and all the limits of subsequences of $F_n$ are $\gamma$-Hölder continuous for all $0 < \gamma < \max_{q \in (1, p]} \varphi(q)/q$. Moreover, Proposition 3.1 tells us that, with probability 1, $F_n$ is convergent over the dense countable subset $T_*$ of $[0, 1]$. This yields the uniform convergence of $F_n$ and the Hölder regularity of the limit $F$.

We then prove that $\|F(t)\|_\infty \leq M$ for $n \geq 1$, let $M_n = \max_{t \in T_n} |F_n(t)|$. We have

$$
(3.4) \quad M_{n+1} \leq M_n + \max_{t \in T_n} |F_{n+1}(t) - F_n(t)| + b \cdot \max_{w \in \mathcal{A}^{n+1}} |\Delta F_{n+1}(w)|.
$$

Then Minkowski’s inequality yields

$$
\|M_{n+1}\|_p \leq \|M_n\|_p + \max_{t \in T_n} |F_{n+1}(t) - F_n(t)|_p + b \cdot \max_{w \in \mathcal{A}^{n+1}} |\Delta F_{n+1}(w)|_p.
$$

Also, due to Proposition 3.1 we have $\sum_{n \geq 1} \max_{t \in T_n} |F_{n+1}(t) - F_n(t)|_p < \infty$. Moreover,

$$
\left(\sum_{w \in \mathcal{A}^{n+1}} \mathbb{E}(|\Delta F_{n+1}(w)|^p)^{1/p}\right) \leq S(n + 1, p)^{1/p},
$$

so $\sum_{n \geq 1} \max_{w \in \mathcal{A}^{n+1}} |\Delta F_{n+1}(w)|_p < \infty$. This implies $\sup_{n \geq 1} \|M_n\|_p < \infty$, and since, with probability 1, $F_n$ converges uniformly to $F_\infty$, and $T_*$ is dense in $[0, 1]$, we get $\|F(t)\|_p \leq \liminf_{n \to \infty} \|M_n\|_p < \infty$. In particular, $F$ belongs to $L^1$ and for every $n \geq 1$, the conditional expectation of $F$ with respect to $\mathcal{F}_n$ is well defined, and it converges almost surely and in $L^1$ norm to $F$ (see Proposition V-2-6 in [23]). It remains to prove that $F_n = \mathbb{E}(F|\mathcal{F}_n)$ almost surely. For every $t \in T_*$, we have shown that the martingale $(F_n(t), \mathcal{F}_n)_{n \geq 1}$ is uniformly integrable, so $F_n(t) = \mathbb{E}(F(t)|\mathcal{F}_n)$ almost surely. Consequently, since $T_*$ is countable, with probability 1, the restriction of $\mathbb{E}(F|\mathcal{F}_n)$ coincides with the function $F_n$ over $T_*$. Moreover, these two random functions are continuous and $T_*$ is dense in $[0, 1]$, so, with probability 1, they are equal.
Proof of Proposition 3.1. Fix \( n \geq 2 \) and denote the elements of \( T_n \) by \( t_j \), \( 0 \leq j \leq b^n \), where \( 0 = t_0 < t_1 < \cdots < t_{b^n} = 1 \). Also define \( J_j = [t_j, t_{j+1}] \) for \( 0 \leq j < b^n \). We can write
\[
F_n(t_j) - F_{n-1}(t_j) = \sum_{k=0}^{j-1} \int_{J_k} U(t)V(t) \, d\lambda(t)
\]
with \( U(t) = Q_{n-1}(t) \) and \( V(t) = P_n(t) - 1 \). Then we divide the family \( \{J_j\}_{0 \leq j < b^n} \) into \( bN \) sub-families, namely the \( \{J_{bNk+i}\}_{k \geq 0, 0 \leq bNk+i < b^n} \), for \( 0 \leq i \leq bN - 1 \). Also we define
\[
M_n = \max_{0 \leq j < b^n} |F_n(t_j) - F_{n-1}(t_j)|
\]
and remark that
\[
M_n \leq bN \max_{0 \leq j < b^n} \sum_{0 \leq i \leq bNk+i \leq j} \left| \int_{J_{bNk+i}} U(t)V(t) \, d\lambda(t) \right|^p.
\]
By raising both sides of the previous inequality to the power \( p \) we can get
\[
M_n^p \leq (bN)^p \max_{0 \leq j < b^n} \left| \int_{J_{bNk+i}} U(t)V(t) \, d\lambda(t) \right|^p
\]
(3.5)
\[
\leq (bN)^p \sum_{i=0}^{bN-1} \max_{0 \leq j < b^n} \left| \sum_{k \geq 0} \int_{J_{bNk+i}} U(t)V(t) \, d\lambda(t) \right|^p.
\]
We are going to use the following lemma. It is proved for real valued random variables in [28], and its extension to the complex case is immediate.

Lemma 3.1. Let \( p \in (1, 2] \). There exists a constant \( C_p > 0 \) such that for every \( n \geq 1 \) and every sequence \( \{V_j\}_{1 \leq j \leq n} \) of independent and centered complex random variables we have
\[
\mathbb{E}\left( \max_{1 \leq k \leq n} \left| \sum_{j=1}^{k} V_j \right|^p \right) \leq C_p \sum_{j=1}^{n} \mathbb{E}(|V_j|^p).
\]

Due to (P3), for each \( 0 \leq i \leq bN - 1 \), the restrictions of the function \( V(t) \) to the intervals \( J_{bNk+i} \), \( 0 \leq bNk + i < b^n \), are centered and independent. Also, due to (P2), the functions \( U(t) \) and \( V(t) \) are independent. Consequently, by taking the conditional expectation with respect to \( \mathcal{F}_{n-1} \) in (3.5) and using Lemma 3.1 we get for each \( 0 \leq i \leq bN - 1 \)
\[
\mathbb{E}\left( \max_{0 \leq j < b^n} \left| \sum_{k \geq 0} \int_{J_{bNk+i}} U(t)V(t) \, d\lambda(t) \right|^p \bigg| \mathcal{F}_{n-1} \right)
\]
\[
\leq C_p \sum_{k \geq 0} \mathbb{E}\left( \left| \int_{J_{bNk+i}} U(t)V(t) \, d\lambda(t) \right|^p \bigg| \mathcal{F}_{n-1} \right).
\]
This implies

\[(3.6) \quad \mathbb{E}(M_n^p | \mathcal{F}_{n-1}) \leq \tilde{C}_p \sum_{0 \leq j \leq b^n} \mathbb{E} \left( \left| \int_{I_j} U(t)V(t) \, d\lambda(t) \right|^p \right| \mathcal{F}_{n-1}) \]

with \(\tilde{C}_p = C_p(bN)^{p+1}\). Now, since \(p > 1\), the Jensen inequality yields

\[\left| \int_{I_j} U(t)V(t) \, d\lambda(t) \right|^p \leq \lambda(I_j)^{p-1} \int_{I_j} |U(t)V(t)|^p \, d\lambda(t).\]

Moreover, since \(\mathbb{E}(|P_n(t)|) \geq 1\) and \(p \geq 1\), we have

\[(3.7) \quad \mathbb{E}(|V(t)|^p) \leq 2 \left( 1 + \mathbb{E}(|P_n(t)|^p) \right) \leq 2^p \mathbb{E}(|P_n(t)|^p).\]

Thus, taking the expectation in (3.6) yields

\[\mathbb{E}\left( \max_{t \in T_n} |F_n(t) - F_{n-1}(t)|^p \right) \leq 2^p \tilde{C}_p S(n, p),\]

that is, (3.2). If \(\varphi(p) > 0\), by definition of \(\varphi\), \(S(n, p)\) converges exponentially fast to 0; hence the series \(\sum_{n \geq 1} S(n, p)^{1/p}\) converge and, due to (3.2) and the fact that \(T = \bigcup_{n \geq 0} T_n\), \(F_n(t)\) converges almost surely and in \(L^p\) norm as \(n \to \infty\) for all \(t \in T\). \(\square\)

**Proof of Proposition 3.2.** Recall (3.1). Let \(q \in (1, p]\) such that \(\varphi(q) > 0\). It follows from (P1) that \((\Delta F_n(I_w))_{n \geq 1}\) is a martingale, so Doob’s and then Jensen’s inequalities yield a constant \(C_q\) such that for \(n \geq 1\)

\[\mathbb{E}\left( \max_{1 \leq k \leq n} |\Delta F_k(I_w)|^q \right) \leq C_q \mathbb{E}(|\Delta F_n(I_w)|^q) \leq C_q \lambda(I_w)^{q-1} \int_{I_w} \mathbb{E}(|Q_n(t)|^q) \, d\lambda(t).\]

Consequently

\[(3.8) \quad \sum_{w \in A^n} \mathbb{E}\left( \max_{1 \leq k \leq n} |\Delta F_k(I_w)|^q \right) \leq C_q S(n, q).\]

By using Markov’s inequality as well as (3.8) and Proposition 3.1, we get

\[
\mathbb{P}\left( \max_{w \in A^n} \max_{0 \leq k \leq n} |\Delta F_k(I_w)| > b^{-n\gamma} \text{ or } \max_{t \in T_n} |F_n(t) - F_{n-1}(t)| > b^{-n\gamma} \right)
\leq \sum_{w \in A^n} \mathbb{P}\left( \max_{0 \leq k \leq n} |\Delta F_k(I_w)| > b^{-n\gamma} \right) + \mathbb{P}\left( \max_{t \in T_n} |F_n(t) - F_{n-1}(t)| > b^{-n\gamma} \right)
\leq \sum_{w \in A^n} b^{n\gamma q} \cdot \mathbb{E}\left( \max_{0 \leq k \leq n} |\Delta F_k(I_w)|^q \right) + b^{n\gamma q} \cdot \mathbb{E}\left( \max_{t \in T_n} |F_n(t) - F_{n-1}(t)|^q \right)
\leq C_q b^{n\gamma q} S(n, q),
\]
where $C_q$ is another constant depending only on $q$. Since $\gamma \in (0, \varphi(q)/q)$, by definition of $\varphi(q)$ the series $\sum_{n \geq 1} b^{n\gamma} q S(n, q)$ converges, and by the Borel–Cantelli lemma, with probability 1, there exists $n_1$ such that for all $n \geq n_1$,

\begin{align}
(3.9) \quad \max_{w \in \mathcal{A}^n} \max_{0 \leq k \leq n} |\Delta F_k(I_w)| \leq b^{-n\gamma} \quad \text{and} \quad \max_{i \in T_n} |F_n(t) - F_{n-1}(t)| \leq b^{-n\gamma}.
\end{align}

Now fix $n \geq n_1$. We are going to prove by induction that for all $M \geq n + 1$ and $t, s \in T_M$ such that $0 < t - s < b^{-n}$ we have

\begin{align}
(3.10) \quad \Delta_M(t, s) \leq 2b \sum_{m=n+1}^M b^{-m\gamma} \quad \text{where} \quad \Delta_M(t, s) = \max_{0 \leq k \leq M} |F_k(t) - F_k(s)|.
\end{align}

If $M = n + 1$, then there exist $i$ and $i'$ with $0 < i - i' < 2b$ such that $t = ib^{-(n+1)}$ and $s = i'b^{-(n+1)}$, so due to (3.9) applied at generation $n + 1$,

\begin{align}
\Delta_{n+1}(t, s) \leq (i - i')b^{-(n+1)\gamma} \leq 2b \cdot b^{-(n+1)\gamma}.
\end{align}

Now let $M \geq n + 1$ and suppose that (3.10) holds for all $n + 1 \leq m \leq M$. Let $t, s \in T_M$ such that $0 < t - s < b^{-n}$. If there is no element of $T_M$ between $s$ and $t$, then (3.9) yields $\Delta_{M+1}(t, s) \leq (b - 1)b^{-(M+1)\gamma}$. Otherwise, consider $\bar{t} = \max\{u \in T_M : u \leq t\}$ and $\bar{s} = \min\{u \in T_M : u \geq s\}$. We have

\begin{align}
&\bar{s} - s \leq (b - 1)b^{-(M+1)}, \quad \bar{t} - \bar{s} < b^{-n}.
\end{align}

Since $\bar{s}$ and $\bar{t}$ belong to $T_M \subseteq T_{M+1}$, we deduce from (3.9) that

\begin{align}
\begin{cases}
\max\{\Delta_{M+1}(t, \bar{t}), \Delta_{M+1}(\bar{s}, s)\} \leq (b - 1)b^{-(M+1)\gamma}, \\
\max\{|F_{M+1}(\bar{s}) - F_M(\bar{s})|, |F_{M+1}(\bar{t}) - F_M(\bar{t})|\} \leq b^{-(M+1)\gamma}.
\end{cases}
\end{align}

Also, due to (3.10) we have $\Delta_M(\bar{t}, \bar{s}) \leq 2b \sum_{m=n+1}^M b^{-m\gamma}$. Consequently,

\begin{align}
\Delta_{M+1}(t, s) &\leq \Delta_{M+1}(t, \bar{t}) + \Delta_{M+1}(\bar{s}, s) + \Delta_M(\bar{t}, \bar{s}) + |F_{M+1}(\bar{s}) - F_M(\bar{s})| + |F_{M+1}(\bar{t}) - F_M(\bar{t})| \\
&\leq 2(b - 1)b^{-(M+1)\gamma} + 2b \sum_{m=n+1}^M b^{-m\gamma} + 2b^{-(M+1)\gamma},
\end{align}

so (3.10) holds for $m = M + 1$. Let $C_\gamma = 2b/(1 - b^{-\gamma})$. Letting $M$ tend to infinity in (3.10) yields that $\max_{k \geq 1} |F_k(t) - F_k(s)| \leq C_\gamma b^{-(n+1)\gamma}$ for all $n \geq n_1$ and $t, s \in T_n$ such that $|t - s| \leq b^{-n}$. Now, for $t, s \in T_n$ with $|t - s| \leq b^{-n_1}$, there is a unique $n \geq n_1$ such that $b^{-(n+1)} \leq |t - s| < b^{-n}$ and $\max_{k \geq 1} |F_k(t) - F_k(s)| \leq C_\gamma b^{-(n+1)\gamma} \leq C_\gamma |t - s|\gamma$. The conclusion comes from the density of $T_n$ in $[0, 1]$ and the continuity of the $F_k$. □
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