CLIFFORD ALGEBRA AS QUANTUM LANGUAGE*

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Abstract

We suggest Clifford algebra as a useful simplifying language for present quantum dynamics. Clifford algebras arise from representations of the permutation groups as they arise from representations of the rotation groups. Aggregates using such representations for their permutations obey Clifford statistics. The vectors supporting the Clifford algebras of permutations and rotations are plexors and spinors respectively. Physical spinors may actually be plexors describing quantum ensembles, not simple individuals. We use Clifford statistics to define quantum fields on a quantum space-time, and to formulate a quantum dynamics-field-space-time unity that evades the compactification problem. The quantum bits of history regarded as a quantum computation seem to obey a Clifford statistics.

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1 Spinors and plexors

Pauli represented electron rotations in SO(3) with elements of the Clifford algebra of a 3, and Dirac represented Lorentz transformation in SO(1,3) with elements of the Clifford algebra of a 4. It is unlikely that they knew that Wiman (1898) and Schur (1911) had represented permutations in the permutation group $S_N$ with elements of the Clifford algebra of $(N-1)\mathbb{1}$. A vector space (or module) on which these Clifford algebras are faithfully represented as endomorphism algebra (or ring) are called spinors for the orthogonal groups and plexors for the permutation groups. Plexors and spinors are isomorphic mathematical objects of different physical meaning. Spinors arise in 1-body quantum physics, plexors in $N$-body quantum physics with $N > 3$.

In the first years of quantum theory, physicists overlooked spinors because they do not occur in the tensor product of vectors. We then proceeded to overlook plexors until Nayak & Wilczek (1996) for much the same reason.

1.1 Clifford representation of the permutation groups

We write the free Clifford algebra over a quadratic space $V$ as

$$ \text{Cliff}(V) = 2^V $$

and the spinor space $S$ of $V$ defined by the isomorphism $2^V \cong S \otimes S^\dagger$, as

$$ S = \sqrt{2^V} = \sqrt{2^V}. $$

We construct a reducible Clifford representation $R$ of the permutation group $S_N$ by associating the $a$-th individual of the $N$-ad being permuted with a first-degree Clifford unit $i_a \in 2^{N-1}$, for $a = 1, \ldots, N$, obeying Clifford relations

$$ i_a i_b + i_b i_a = 2\delta_{ab}. $$

For quantum applications we define an adjoint $\dagger$ on $2^V$ by

$$ i_a^\dagger = i_a. $$

Then $R$ represents each swap $(ab) \in S_N$ (for $a < b$) by the Clifford difference $R(ab) = i_b - i_a$:

$$ R : S_N \to 2^{N-1}, (ab) \mapsto i_a - i_b. $$

It is straightforward to see that $R$ is a (projective!) representation of $S_N$. The $i_a$ make up an orthonormal basis for the first-degree subspace $\mathcal{V} = \text{Cliff}_1(N,0) \subset \text{Cliff}(N,0)$. Let us call $R$ the orthonormal Clifford representation of $S_N$.

$R'$, a useful variant of $R$, replaces (6) by

$$ i_a i_b + i_b i_a = 2t_{ab}, $$

where

$$ t_{ab} = \frac{N+1}{N-1} \delta_{ab} - \frac{2}{N-1}. $$

The form $t_{ab}$ is the inner product between two unit vectors $i_a, i_b$ from the center of a regular $N-1$-simplex $\sigma^{N-1} \subset (N-1)\mathbb{1}$ to its $N$ vertices $i_1, \ldots, i_N$. The $i_a$ of (6) can be identified with the $N$ simplex vertices permuted. We therefore call $R'$ the simplicial representation of $S_N$. It is isomorphic to a representation given

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1 In what follows, all representations and their homomorphisms are projective, and may be double-valued, unless they are stated to be linear or vector representations, which are single-valued. 1, 2, 3, \ldots represent real quadratic spaces of dimension 1, 2, 3, \ldots and of signature specified in context.

2 Also called “the symmetric group,” no doubt because its elements are not symmetric.
less symmetrically by Schur (1911). The $R'$ image of $S_N$ is an irreducible group of Clifford-algebra elements $\tilde{S}(N)$ that is a (universal) covering group $\tilde{S}_N$ of the discrete group $S_N$.

The concept of rotation presupposes physical concepts of angle and length. The concept of permutation does not. It presupposes the more primal concept of identity. Therefore permutation groups can enter theoretical physics at finer levels of resolution and higher energies than rotation groups. Plexors may be more basic than spinors.

Representations of permutation groups, and hence plexors, enter quantum physics by two specially deep routes, statistics and dynamics.

2 Clifford statistics

A statistics describes how we compose individual quantum elements into an aggregate quantum system. Such a transition from the individual to the collective is sometimes called “second quantization,” merely because it introduces new operators, but is better called quantification. Quantification has little to do with quantization and is thousands of years older.

A quantification or statistics is often defined by giving a representation of each permutation group $S_N$. A Clifford statistics represents permutations doubly by Clifford algebra elements of degree 1 [Finkelstein & Galiautdinov (2000)]. A cliffordon is a quantum whose aggregates have Clifford statistics. A squad is a quantum assembly of cliffordons.

The operators representing operations on a squad of cliffordons, including the observables of the squad, form the Clifford algebra of the one-cliffordon mode space. Modes of the squad are plexors of its Clifford algebra, representable by elements of a minimal left ideal of the algebra.

A statistics is abelian (or central, or scalar, respectively) as its representation of the permutation group is. The Bose and Fermi statistics are abelian, and the Maxwell, Clifford and braid statistics are non-abelian. Read & Moore (1992) suggested non-abelian statistics for quasi-particles of the fractional quantum Hall effect. Nayak & Wilczek (1996) recognized this as a Clifford statistics in the first application of the Schur (1911) theory to physics. While anyonic and braid statistics are confined to two dimensions, Clifford statistics works in any dimensionality.

2.1 Spinors describe aggregates

When a spinor of $SO_N$ describes one quantum individual with $N$ possible modes, a plexor of $S_N$ describes a complex of at least $N$ isomorphic individuals. Thus our belated encounter with Schur (1911) disabuses us of a long-held notion that spinors represent simpler entities than vectors, a notion that has blocked important research directions. Spinors represent aggregates.

This deserves to be said three times, differently:

2.2 Spinors describe aggregates

That spinors describe aggregates in quantum theory was already implicit in Cartan’s theory of spinors. Cartan starts from a complex vector space $\mathcal{V}$ with a symmetric quadratic form $g$. The complex space $\mathcal{V}$ can be decomposed (in many ways) into two maximal null subspaces $\mathcal{V}_\pm$ as $\mathcal{V} = \mathcal{V}_+ \oplus \mathcal{V}_-$. (A null space is one composed entirely of null vectors.) A Cartan spinor — relative to such a decomposition! — is an element of the Grassmann algebra $\bigwedge \mathcal{V}_+$. The spinor space $\mathcal{S}$ then has dimension

$$\text{Dim } \mathcal{S} = 2^{\left\lceil \frac{D-1}{2} \right\rceil}$$

Schur (1911) would call this a “representation group” of $S_N$, but today this seems apt to be confused with a group representation.
where $\text{Dim } V = D = 2\text{Dim } V_\pm$ and $\lceil N \rceil$ is the greatest integer $n \leq N$. Quantum theory interprets $V_\pm$ as mode spaces for a fermion and the Grassmann algebra $\sqrt{V_+}$ as the algebra of an aggregate of such fermions. Thus Cartan spinors describe fermionic quantum aggregates, not elementary individuals.

### 2.3 Spinors describe aggregates

The fact that a vector $v$ can be expressed through a spinor $\psi$ bilinearly as $v^\mu = \psi^\dagger_\sigma v^\sigma \psi$ is sometimes cited to indicate that the vector is less elementary. This is a categorical error. The relation $v^\mu = \psi^\dagger_\sigma v^\sigma \psi$ indicates that a spinor, like an aggregate of vectors, carries enough information to define a vector. But a vector cannot define a spinor. Therefore the vectorial object is a subobject of the spinorial object, not conversely.

Some have called Dirac’s spinor space a “square root of space-time,” because $v^\mu$ is quadratic in $\psi^\alpha$. More accurately, a spinor space $\mathcal{S}$ is the square root of the Clifford algebra $\mathcal{C} = 2^X \cong \mathcal{S} \otimes \mathcal{S}^\dagger$ over space-time; or half thereof, after reduction:

$$\mathcal{S} = \sqrt{2^X}. \quad (9)$$

$\text{Dim } \mathcal{S}$ is never less than $\text{Dim } X$, and is exponentially greater for high dimension.

Since plexors describe aggregates, plexor theory can be a quantum substitute for set theory. We apply it to quantum dynamics next.

### 3 Clifford dynamics

A dynamics too can be defined by a permutation. First we give a fixed set $\mathcal{V}$ of elementary processes that the system under study can undergo. Then we give a permutation

$$D : \mathcal{V} \to \mathcal{V}. \quad (10)$$

For any elementary process $s \in \mathcal{V}$, we interpret $Ds$ as the immediate successor of $s$ in the dynamical development defined by $D$. We write the group of such permutations of $\mathcal{V}$ as $S(\mathcal{V})$.

It is customary to avoid messy boundary questions by imagining experiments that have gone on forever and will continue forever. We shall use periodic boundary conditions. Now the experimental space-time region ultimately closes on itself, outside the interesting part of the experiment. This is no more fantastic than the infinite domain and it permits us to work in a finite-dimensional algebra.

A permutation $D$ partitions its domain $\mathcal{V}$ into a congruence of orbits. These are the threads that tie the dynamical elements together. They do not intersect. The intersecting geodesics and the light-cones of space-time must arise when we project orbits from 8 dimensions down to the 4 dimensions of space-time. They arise from and describe a quantum entanglement that occurs in the dynamical development.

Then a possible quantum concept of a dynamics mode $D$ is an operator $D$ in the algebra of a representation of $S(\mathcal{V})$; for example in the algebra of the covering group $\tilde{S}(\mathcal{V})$. Since $S_N$ is not simple in general, and distinguishes a basis in $SO_N$, we do not found our theory on this concept. Instead we imbed $S_N$ in the simple group $SO_N$ as the axis-permuting elements, and represent $SO_N$. We still interpret the operator $D$ as defining a dynamical succession.

Everything we know about a system is in the record of our dynamical operations on the system. A good language for quantum dynamics is then a language of great expressive power.

As syntax for the dynamics language, abstract $\dagger$ algebra is not sufficient. It deals only with how to combine actions in series and parallel, by multiplication and addition, and how to reverse their internal chronological order, by forming the adjoint. It omits space-time fine structure, which is supplied in standard quantum theory by classical constructions prior to the definition of the algebra. How to express space-time concepts within the algebra is part of the problem of marrying quantum theory with gravity theory. We use a high-dimensional Clifford algebra to express quantum space-time.
The statistics and the dynamics roads to the permutation group join when we postulate that ordinary dynamical processes are aggregates too, namely of elementary dynamical processes. In the standard field theory the general process is composed of operations that go on everywhere in the system all the time, described by the Hamiltonian or Lagrangian density. Space-time is how the parts of the dynamics are interconnected.

Any quantum-dynamical theory must give the statistics of its elementary quantum-dynamical processes. In standard physics they are tagged with space-time coordinates, so they are distinguishable and implicitly obey Maxwell-Boltzman statistics.

3.1 Plexic dynamics

To learn the structure of the dynamical process, we dissect it into its atomic constituents and reassemble it out of them. Evidently these elements of dynamics must still have the nature of dynamical processes themselves. The dynamics is built out of elementary dynamical actions $\chi$, represented by arrows joining states. The simplest classical concept of dynamics is a topological dynamics $D$. This is usually presented as a map $D : S \rightarrow S$ of a set of states $S$ into itself. Instead we deal only with dynamical actions and not with states. We define the dynamics as a mapping $D$ sending each arrow $\chi$ to its dynamical successor $D\chi$.

A theory that puts states of being prior to modes of action is called ontic, the reverse praxic. Praxism is an acute case of the pragmatism of Charles Peirce and William James and the operationalism of Einstein and Heisenberg. Here we dissect dynamical operations into micro-operations as James proposed to analyze experience into micro-experiences.

This praxic concept of classical dynamics introduces our two principals: a set $X$ of atomic actions $\chi \in X$, and a semigroup $S(X)$ of possible dynamical developments

$$D : X \rightarrow X. \tag{11}$$

Dynamical developments are 1-1 mappings or permutations of $X \rightarrow X$.

To formulate a quantum dynamics we “quantize” the structure [1]. That is, we replace classical variables described by sample spaces with corresponding quantum variables described by full matrix algebras. Our prime variable is not space-time, as Einstein proposed, but the dynamical law. Anandan (1999) has proposed (like Newton) that dynamical law is variable and Smolin (1992, 1997) that it evolves. We sharpen this assumption and take the dynamical law as the only independent variable, on which all others depend.

The individual elementary quantum process making up the dynamics we call the chronon $\chi$. Its mode space $X$ and algebra $\operatorname{End}(X)$ replace the set $X$ of atomic actions. Its operator algebra $A(X) := \operatorname{End}(X)$ replaces and synthesizes the commutative Boolean algebra of $X$ and the arrow semigroup of ordered pairs $X \times X$. The nearest classical analogue of a chronon is not a space-time point, which has no natural dynamical successor, but a tangent or cotangent vector $(x, v)$ or $(x, p)$, which does. These form an 8-dimensional manifold, not a 4-dimensional one.

To describe an aggregate of chronons we need a statistics for the chronon.

Neither Fermi, Bose, nor Maxwell statistics will do. A dynamics is a permutation. A Fermi aggregate, like a classical set, is invariant under any permutation of its elements. It cannot represent its dynamics by its permutations. Nor can a Bose aggregate.

And Maxwell statistics are reducible.

Evidently chronons, to be permuted effectively, must be distinguishable, like classical space-time points, which are implicitly supposed to have Maxwell statistics.

In nature the ambient dynamics has modes with spin 1/2.

The simplest statistics that supports 2-valued representations of $S_X$ is the Clifford statistics. The operator algebra of this aggregate is a Clifford algebra $C = \operatorname{Cliff}(V)$ generated by individual unit modes $i_a$ obeying [4] The difference of two units $C(ab) = i_a - i_b$ represents their swap $(ab)$. We identify the individual mode space $V$ with the first-degree subspace $C_1 \subset C$. Here there is no doubt that the spinor represents an aggregate, namely the aggregate that the permutations permute.
Schur (1911) and Nayak & Wilczek (1996) use complex coefficients throughout. They represent some swaps by sums $i_a + i_b$ and others by differences $i_a - i_b$, depending on an arbitrary choice of $N-1$ generating swaps. It is not possible to represent all swaps $(ab)$ by sums $i_a + i_b$. But their representation is isomorphic to the the simplex representation, of all swaps by differences.

Choosing Clifford statistics for chronons expresses the distinguishability of events and the existence of spin 1/2. The grade of a Clifford element gives the minimal number of swaps or chronons in its factorization, corresponding roughly to classical phase-space volume.

In the quantum theory of a variable dynamics $D$, we distinguish between the dynamics $D$ and some dynamics operator $D$ that describes $D$ maximally, just as we distinguish between a hydrogen atom $H$ and a mode vector $\psi$ maximally describing $H$. A Hamiltonian is a kind of dynamics operator of the continuum limit.

Standard quantum theory uses modes in a complex $\dagger$ space $V$ whose $\dagger$ defines a non-singular sesquilinear form $\psi \dagger \phi$. Gauge invariance requires that the gauge generators be antihermitian, and the gauge group structure requires that some of them be nilpotent. Only in an indefinite sesquilinear space can a nilpotent other than the trivial $0$ be antihermitian. Therefore $\dagger$ is usually indefinite, and the quantum theories that work in Hilbert spaces, with their definite $\ast$, are not sufficiently relativistic for physics.

### 3.2 Field theory under the microscope

Clifford statistics also resolves a question has beset quantum space-time physics from its inception. What is the algebra of quantum fields on a quantum space-time? When we first asked this question [Finkelstein (1969)] we imagined that the q bits of the space-time quantum computer all commuted, and had serious difficulties with this question. Now that they all anti-commute it answers itself.

First the problem. In classical physics, the field fiber $F$ of field values and the Minkowski manifold $M$ of space-time points are combined (at least locally) by exponentiation into a space

$$\Phi = F^M$$

of fields, each field $f \in F^M$ being a function $f : M \to F$.

Question: How do we define the exponential $F^M$ when the classical spaces $F$ and $M$ have been replaced by operator algebras describing quantum field and space-time entities?

Answer: Take the logarithms of the algebras. This reduces the computation of the exponential to the computation of products, an already solved problem.

We expand on this answer a bit.

One’s first guess for the algebra $F^M$ is apt to be the algebra of linear morphisms $M \to F$, but this reduces merely to the tensor product $M^\dagger \otimes F$ when $M$ and $F$ are algebras, and represents merely a pair of one $M$ quantum and one $F$ quantum. This is to be expected, since $M$ and $F$ describe individuals, not aggregates, and a mapping from one individual to another is merely one ordered pair.

For a better answer, one must express algebraically the fact that $M$ is a plenum, not a point. All the points of $M$ are actual, not mutually exclusive possibilities.

This is just the case for the Clifford statistics, which permutes entities that are all present at once. We designate a real free Clifford algebra $C$ over a quadratic space $X$ with endomorphisms algebra $A = \text{End}(X)$ by

$$C = 2^X = \sqrt{2}^A = \text{Cliff}(X).$$

This makes $A$ a logarithm of $C$ and tells us how to define any quantum exponential of $C$:

$$C^M := (\sqrt{2}^A)^M := (\sqrt{2})^A \otimes M = \text{Cliff}(X \otimes Y)$$

where $M = \text{End}(Y)$.

It is not hard to see that the observed field algebras do have logarithm algebras, using the Chevalley (1954) representation of spinors within their Clifford algebra.
Definitions: An octad is a squad of 8 cliffordon, with neutral quadratic form. An octad space is a real neutral quadratic space
\[ \mathbf{8} = 4 \oplus \overline{4} \] (15)
of 8 dimensions, where the overline indicates a reversal of metric, \( \dagger \rightarrow -\dagger \). An octon is a hypothetical quantum whose mode space is \( \mathbf{8} \). An octadic space is a real neutral quadratic space whose dimension is a multiple of 8: The general octadic space is
\[ \mathcal{O} = 8 \oplus \ldots \oplus 8 = N\mathbf{8} \] (16)
with \( N > 0 \) terms.

3.3 Examples
The tangent-cotangent space to Minkowskian space-time \( M \) is the octad space \( M = M \oplus M^{\dagger} \), where \( M := 1 \oplus 3 \) is the Minkowski tangent space. The irreducible spinor spaces of \( \mathbf{8} \) are again octad spaces (Chevalley triality).

The Clifford algebra of an octadic space, with its neutral quadratic form, is algebra-isomorphic to the Clifford algebra of a space of the same dimension with a definite quadratic form.

Octad lemma An octadic chronon algebra \( 2^{N\mathbf{8}} \) factors as a Maxwell-Boltzmann ensemble of \( N \) octads each with algebra \( 2^{\mathbf{8}} \):
\[ 2^{N\mathbf{8}} = 2^{\mathbf{8}} \otimes \ldots \otimes 2^{\mathbf{8}} \quad (N \text{ terms}) \] (17)

In the limit \( N \rightarrow \infty \), this Maxwell-Boltzmann ensemble includes a Bose-Einstein aggregate of octads.

It is easy to see that this Bose-Einstein aggregate admits condensation into an 8-dimensional symplectic manifold isomorphic to the tangent bundle to space-time. A field of operators on space-time is a similar condensation of a squad of octads as \( N \rightarrow \infty \), \( \mathfrak{h} \rightarrow 0 \).

This is a great simplification. The group of a bundle is never simple; the base couples to the fiber without reverse coupling. In Galilean relativity the base was time, while in field theory the base is space-time, but the illness is the same, and the cure too: relativization. In (14) the field and the space-time are unified in the simple space-time-field entity \( S \). When we first attempted to express field theory in terms of q bits or chronons [Finkelstein (1969)] we imagined an absolute split between field fiber and space-time base. Now the field-space-time split appears to be a factorization of a field-space-time unity \( S \). It is as relative as the factorization of space-time into space/time.

4 Quantification operators
Each of the usual statistics, Fermi, Bose and Maxwell, has an operator-valued form \( Q^\dagger \) and dual form \( Q \) that defines how infinitesimal actions on the individual can act on the aggregate.

In each statistics the individual \( I \) has a mode vector space \( \mathcal{V}(I) \) and operator algebra \( \mathcal{A}(I) \). The aggregate has a mode vector space \( \mathcal{V}(S) \) and operator algebra \( \mathcal{A}(S) \). Let \( d\mathcal{A} \) be the infinitesimal Lie algebra of \( \mathcal{A} \) with Lie product \( [\alpha, \beta] := \alpha\beta - \beta\alpha \). The quantification operator \( Q^\dagger \) is a linear morphism
\[ Q^\dagger : \mathcal{V}(I) \rightarrow \mathcal{A}(S), \; \psi \mapsto Q^\dagger \psi, \] (18)
transforming a mode vector (or a ket) \( \psi \in \mathcal{V}(I) \) for the individual to an operator \( Q^\dagger \psi \in \mathcal{A}(S) \) for the aggregate. The operators \( \psi^\dagger Q \) generate \( \mathcal{A}(S) \) \( \dagger \)-algebraically. The mapping
\[ \langle \ldots \rangle : d\mathcal{A}(I) \rightarrow d\mathcal{A}(S), \; w \mapsto \bar{w} = Q^\dagger wQ \] (19)
is a Lie-algebra homomorphism.

We call the \( Q \) with these properties, when it exists, the quantification operator of the statistics. If \( w \) represents an additive quantity or infinitesimal transformation of the individual, we call \( Q^\dagger wQ \) the quantified \( w \).
for the quantified system. The quantification operators of Fermi, Bose and Maxwell statistics map individual mode vectors \( \psi \) into annihilation operators \( Q^\dagger \psi \), and individual quantities \( q \) into additive total quantities \( Q^\dagger qQ \). Clifford statistics also has a quantification operator \( Q^0 \), which maps mode vectors into swaps instead of creation operators.

The Clifford quantification operator \( Q^0 \) obeys the Clifford law

\[
(\forall v \in V) \quad (Q^0 v)^2 = ||v||
\]

(20)

We chose the sign in (3) so that the mapping (19) is a Lie algebra homomorphism, preserving the commutation relations of the individual within those of the aggregate, as for Fermi and Bose statistics. This is just the familiar fact that the commutators \( L_{ab} = [a,b] \) generate a representation of the orthogonal group.

We write the quantification operators for Clifford, Fermi, and Bose statistics as \( Q^0, Q^1, \) and \( Q^2 \). The numerical subscripts count the independent imaginaries in the coefficient field \( \mathbb{R}, \mathbb{C}, \mathbb{H} \) of the classical group of the statistics. We write \( Q_M \) for the Maxwell-Boltzmann quantification operator. We call the most important aggregates (Maxwell) sequences, (Bose) sibs, (Fermi) sets and (Clifford) squads for brevity. We call the Clifford composite a squad to remind us that it is an essentially quantum structure. There are classical sets and classical sibs, but no classical squads. Clifford statistics, like anyonic and other multivalued statistics, involves quantum superposition more deeply than the single-valued composites such as the sequence, sib, or set.

For example if \( L \) is a component of individual angular momentum, then for Fermi quantification \( Q = Q^1 \) and Bose quantification \( Q = Q^2, Q^\dagger LQ \) is the total angular momentum of the aggregate. These quantifications totalize the operator on which they act.

The Maxwell quantification operator \( Q_M \) does not totalize. The quantified operator \( Q_M^\dagger LQ_M \) represents the \( L \) of only the last individual in the sequence, not the total \( \omega \). Totals have somewhat more complicated expressions in Maxwell statistics.

Clifford quantification, like Fermi and Bose, totalizes.

Clifford statistics relates a quadratic space \( V \), its endomorphism algebra \( A \), its Clifford algebra \( C \), and its spinor space \( S \), by the commutative diagram

\[
\begin{array}{ccc}
V & \xrightarrow{\text{Endo}} & A \\
\text{Sq} & \downarrow \text{Cliff} & \text{Sq} \\
S & \xrightarrow{\text{Endo}} & C
\end{array}
\]

(21)

When we apply Clifford statistics to dynamics, \( V \) is the mode space \( X \) for a chronon. The composite system described by a spinor of \( S \) consists of all the chronons transpiring in the experimental space-time region. The algebra \( A \) consists of endomorphisms of \( V \). The Clifford algebra \( C \) consists of descriptions \( D \) of the global dynamics of the squad.

The real Clifford algebra \( C = \text{Cliff}(V, \mathbb{R}) \) of Clifford statistics is the endomorphism algebra of an underlying spinor module \( S \) over one of the five rings \( \mathbb{R}, \mathbb{C}, \mathbb{H}, 2\mathbb{R}, 2\mathbb{H} \), depending on the dimension and signature of \( V \) according to the spinorial chessboard [Budinich & Trautman (1988)].

Clifford statistics can readily simulate Bose and Fermi statistics with pseudo-bosons and pseudo-fermions. In (17), Clifford statistics exactly simulates an aggregate of octads obeying mutual Maxwell-Boltzmann statistics and internal Clifford statistics at the same time.

The most striking difference between the Clifford statistics and Fermi statistics cannot be read from their algebras. Orthogonal \( \psi \)'s anticommute in both statistics, and every fermionic algebra is algebra-isomorphic to a neutral Clifford algebra, but fermions are identical and cliffordon are not. An algebra does not define its interpretation. The exchange of two fermions is represented by factor exchange, one stipulates, and hence by \( -1 \), which is projectively equivalent to the identity. The exchange of cliffordon 1 and 2, however, is represented by \( i_1 - i_2 \). Cliffordon swaps, far from being trivial, scalar or central, may generate the entire aggregate action algebra.
5 Chronon dynamics

We hypothesize that the dynamics of a suitably isolated physical system is a squad of elementary dynamical processes or chronons, and that the ambient vacuum breaks its Clifford algebra \( \mathcal{C} \) down into many mutually commuting local octadic Clifford algebras as in (17).

We construct a simple finite-dimensional Clifford algebra \( \mathcal{C} = \bar{A} \) that approaches (or “contracts to”) the Minkowski manifold algebra, the associative algebra \( \mathcal{A} = \mathcal{A}(x^\mu, \partial_\mu) \) of coordinates \( x \) and derivations \( \partial \) of space-time differential geometry. \( \mathcal{A} \) may be regarded as a generalization of the Heisenberg algebra of \( x \) and \( p \) and a variant of the Bose-Einstein algebra. Expanding it into a Clifford algebra is mathematically akin to approximating bosonic fields with fermionic ones, the process of bosonization.

Since \( \mathcal{A} \) is infinite-dimensional, the dimensions of \( \mathcal{C} \) and its orthogonal group are huge, like the number of phase-space cells in the experiment, which is likely \( \gg 10^{20} \) for atomic experiments, and approach infinity in the contraction to the continuum.

Chronons and the basic Clifford variables \( i_a \) that represent them are pre-local in the extreme, since they all anticommute. Nevertheless they are the raw material of our universe, we propose. This only apparently clashes with quasilocality, due to (17).

5.1 Localization

We begin construction with an octadic chronon space \( \mathbb{8N} \) and its Clifford algebra \( \mathcal{C} \).

We decompose \( \mathcal{C} \) into \( N \) mutually commuting octad algebras of independent local variables \( \gamma_\mu(n) \) obeying local commutation relations

\[
\{ \gamma_\mu(n), \gamma_\nu(m) \} = t_{\nu\mu} \delta_{nm}
\]

(22)

We define the Lorentz algebra and group of such a Clifford algebra by the usual expression for the angular momentum of a spin aggregate,

\[
\bar{L}_{\nu\mu} := \frac{1}{2} \sum_{n=1}^{N} \sum_{\beta=0}^{1} \gamma_\nu\mu(n,\beta)
\]

(23)

We assume that \( p_\mu \) and \( x_\mu \) for the \( N \) squads are, like \( L_{\mu\nu} \), additively composed of terms from each squad, corresponding to how the displacement \( \Delta x = \int_C dx(\tau) \) along a curve \( C : x = x(\tau) \) is an integral over \( C \) of a contribution from each differential element \( dx(\tau) \):

\[
\Delta x_\mu = \sum_{n,\beta} \delta x_\mu(n,\beta), \quad \Delta p_\mu = \frac{1}{2N} \sum_{n,\beta} \delta p_\mu(n,\beta).
\]

(24)

Here we use \( \hbar = c = \mathbb{1} = 1 \).

Suppose each tetrad has contributions

\[
\delta x_\mu = 2^{-1/2} \gamma_\mu, \quad \delta p_\mu = 2^{-1/2} \gamma^\dagger \gamma_{\mu}, \quad \delta i = \gamma^\dagger.
\]

(25)

The unit of \( x \) is \( \mathbb{1} \) and the unit of \( p \) is \( \hbar/\mathbb{1} \), while \( i \) is dimensionless.

Then for each tetrad

\[
[\delta x_\nu, \delta p_\mu] = \delta i \delta_{\nu\mu}, \quad [\delta i, \delta x_\nu] = +2\delta p_\nu, \quad [\delta i, \delta p_\mu] = -2\delta x_\mu.
\]

(26)

The first of equations (26) makes \( \gamma^\dagger \) the expansion of Heisenberg’s \( i \), much as Hestenes (1966) proposed. Presumably this builds in a violation of parity.

The second and third tell us that the expanded \( i \) generates the symplectic symmetry between \( x \) and \( p \), as Segal (1951) proposed. On the chronon scale, this violates Heisenberg’s commutation relations seriously. In the standard quantum theory \( i \) is central.
We recover the Heisenberg commutation relations as a contraction of the Clifford relations by summation and a subsequent correlation:

\[
\hat{x}^\mu = \sum_{n,\beta} \delta x^\mu(n, \beta), \\
\hat{p}_\mu = \frac{1}{2N} \sum_{n,\beta} \delta p_\mu(n, \beta), \\
\hat{\gamma}_i = \sum_{n,\beta} \delta i(n, \beta)
\]  

(27)

Here and in what follows, the breve on a variable, like \(\hat{\gamma}_i\) above, is a semantic annotation rather than a syntactic one. It declares that the accented variable is an expansion of the standard unaccented one, and reduces to it upon contraction. It thus tells us how to measure the variable in the contracted domain of experience. If one set of physical concepts successfully covers both domains, as in quantum theory and relativity, for example, we drop such labels.

Feynman (1971) proposed that the space-time coordinate-difference operator is the sum of many mutually commuting tetrads of Dirac vectors:

\[
\Delta x^\mu = \text{Const} \sum_\gamma \gamma^\mu(n)
\]  

(28)

as a quantum form of the proper-time Heisenberg-Dirac equation \(dx^\mu/d\tau = \gamma^\mu(\tau)\). Feynman’s proposal returns in (27) as a result of the octad lemma. Each of his \(\gamma\)'s represents one chronon \(i^a\) in a large octadic squad.

5.2 Correlation

We now have far too many \(i\)'s: one for every tetrad. In the standard physics there is only one imaginary \(i\) and one Clifford vector \(\gamma^\mu\) for the whole system. We correlate all the \(\delta i\) so that they are effectively one \(i\), and all the tetrads \(\gamma^\mu(n)\) so that they are effectively one tetrad \(\gamma^\mu\). We suppose that the vacuum Bose-Einstein condensation establishes this correlation. The local departure of \(\hat{\gamma}_i\) from its global mean \(i\) is presumably a Higgs operator.

To be sure, the tetrads of the tetrad lemma obey Maxwell statistics, not Bose. But the mode space of Maxwell statistics is the direct sum of the spaces of all the tensorial statistics, including Fermi and Bose and parastatistics. Any Bose projection is a fortiori a Maxwell projection.

We require a projection operator \(P \in \mathcal{C}\) expressing this alleged Bose-Einstein correlation, projecting onto the symmetric subspace.

We have constructed this symmetrizer elsewhere [Finkelstein (2000)]. The symmetric subspace of \(\mathcal{S}\) is \(PS\), the \(P\)-image of \(\mathcal{S}\). The restrictions of the \(i(n)\) to \(PS\) are all equal to \(P\hat{i}P\), the restriction of \(i\) to \(\mathcal{S}\), with \(i\) given by (27).

It is then straightforward to see that this restriction of \(\hat{i}\) is a square root of the restriction of \(-1\) to \(P\):

\[
(P\hat{i}P)^2 = -P
\]  

(29)

It remains to be seen whether the variable \(P\hat{i}P\) is sufficiently close to being central in processes close to the vacuum. If so then it can pass for the physical \(i\) of quantum mechanics in such processes.

The Clifford elements

\[
L_{\mu\nu} = \sum_{\beta,\tau} [i_{\mu,\beta,\tau}, i_{\nu,\beta,\tau}]
\]  

(30)

are infinitesimal generators of the connected Lorentz group \(\mathcal{L}\) of this model. The operators \(x\) and \(p\) constructed by (27) are covariant under the connected Lorentz group \(\mathcal{L}\).
6 Relativistic dynamics operator

To recover Maxwell statistics in the classical space-time limit we were forced to order the octads with a “proper time” index $\tau$. This permits us to formulate an “octad-cycling” dynamics operator $D_0$ that advances $\tau$ by unity and so is a scalar invariant, unlike the Hamiltonian, which increases $t$ and is one component of a vector.

A covariant development in proper time is generated by a rest mass operator just as a coordinate-time development is generated by a Hamiltonian energy operator. A covariant proper-time dynamics with a second-order mass operator is used by Schwinger (1949) among others. The present theory is a kind of “quantum square-root” of such theories.

$D_0$ is then an ordered product of 8 disjoint cycles of length $N$ according to Clifford statistics. Defining $\Delta i_{\mu,\beta,\tau} := (i_{\mu,\beta,\tau+1} - i_{\mu,\beta,\tau})$, we write this basic dynamics operator as

$$D_0 = \prod_\beta \prod_\mu \prod_\tau \Delta i_{\mu,\beta,\tau}. \quad (31)$$

In the $\tau$-ordering, $\Delta i_{\mu,\beta,\tau}$ is to be treated as a whole with one index $\tau$, not divided between $\tau$ and $\tau+1$. Each cycle of (31) swaps the $\mu,\beta$ chronon of octad $\tau$ with the $\mu,\beta$ chronon of octad $\tau+1$ according to the orthonormal statistics.

Now two apparently separate conceptual streams, special relativity and combinatorics, merge almost unexpectedly:

**The octadic chronon dynamics $D_0$ is Lorentz invariant.**

The proof is straightforward. It rests on the familiar fact that the Dirac top gamma $\gamma^5$ is invariant under the connected Lorentz group. This confluence gives us renewed hope for a chronon dynamics.

$D_0$ has several easily constructed brothers that are also Lorentz-invariant and also shift tetrads or octads forward in $\tau$ by small steps.

$D_0$ is still unsatisfactory for many reasons. Above all, it does not define the Minkowskian metrical structure of space-time. In a simple theory, Poincaré invariance of the ambient dynamics operator $D$ is not enough. $D$ must also define the space-time metric.

Physically, we must know the momentum of a moving particle to predict its next position sharply. This means a coupling among the 8 components of space-time-energy-momentum. The dynamics $D_0$ is an uncoupled development. Lorentz-Invariant couplings are now under study.

7 Simplifying the standard model

The simplification of the space-time structure of the standard model that we have performed so far suggest that we can simplify its internal structure as well by dissolving the separation between field variables and space-time variables. This was the goal of our earlier Fermi quantizations of space-time, and it comes closer in the present Clifford quantization.

The standard theory needs internal variables to describe hypercharge, isospin, color, and family because it assumes that the immediate neighborhood of any event is exactly Minkowskian. *At any one point all gauge vectors can be transformed to zero by a gauge transformation.*

In the case of gravity this assumption of the standard model specializes to Einstein’s equivalence principle. We call this italicized proposition the generalized equivalence principle when we intend to include gravity among the gauge fields.

We have supposed that the field variables actually describe finite quasilocal defects of size $\mathcal{M}$ in the vacuum condensate. In the continuum limit $\mathcal{M} \to 0$, these vanish, and surrogate variables have to be invented. Variables that describe the condensate in maximal quantum detail, with $\mathcal{M} \neq 0$, should suffice to describe its defects.

This approach to simplification avoids the compactification problem that plagues theories of the Kaluza kind. It replaces mysteriously small extra dimensions by physically small neighborhoods. The same chronon
variables $i^a$ that combine to form the external $x$'s and $\partial$'s, also combine into the internal $\gamma$'s and $\tau$'s, the fields, and the Lagrangian, a contraction of the dynamics operator. In the condensate many of these degrees of freedom are frozen. A high-energy interaction thaws and excites some of them. We require that the same $i$'s give us the external modes when they sing in unison and the internal modes when they sing polyphonically.

8 Discussion

The revolutions of the past hundred years of physics have simplified certain algebras. Several non-simple algebras still wait to be simplified, notably the Heisenberg algebra of quantum theory, the algebra of the field bundle, and the algebra of dynamics. These couple to each other so that the next revolution must likely simplify them all at once. The result will be a non-local quantum theory.

We have given one obvious simplification of these algebras, postulating a dynamics operator $D$ that contracts to the standard Hamiltonian and Lagrangian in a suitable limit but consists of many elementary quantum actions, chronons. The key unifying element is the Clifford statistics for the chronon.

This leads us to a quantum correspondent for the standard gauge principle: Remote comparisons are effected by a succession of quantum swaps. It also suggests a host of relativistic candidates for the dynamics operator $D$.

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References

[1] Anandan, J. (1999) Are there dynamical laws? Foundations of Physics 29(11), 1647-1672
[2] Atiyah, M. F., R. Bott and A. Shapiro (1964). Clifford modules. Topology 3 Supplement 1, 3-38.
[3] Budinich, P. and A. Trautman (1988). The Spinorial Chessboard. Springer, Berlin.
[4] Chevalley, C. (1954) The Algebraic Theory of Spinors. Columbia Univ. Press, New York.
[5] Feynman, R. P. (1961) Private communication, Palo Alto. Feynman described an attempt he made as a graduate student and dropped because it was “too hard.” He was unable to locate a reference to it.
[6] Finkelstein, D. (1969) Space-time code. Physical Review 184, 1261-1271. Finkelstein, D. (1972) Space-time code III. Physical Review D5, 2922.
[7] Finkelstein, D. (1996). Quantum Relativity. Springer, New York.
[8] Finkelstein, D. and A. Galiautdinov (2000). Clifford statistics. Submitted for publication. Also hep-th/0005039.
[9] Finkelstein, D. and E. Rodriguez (1984). The quantum pentacle. International Journal of Theoretical Physics 23, 887-894.
[10] Finkelstein, D. and E. Rodriguez (1984). Relativity of Topology and Dynamical Law, International Journal of Theoretical Physics 23, 1065-1097. See p. 1074. We did not notice that the Clifford statistics of this paper was 2-valued until 1998, when we met it in the work of Schur and Wilczek.
[11] W. H. Gottschalk and G. A. Hedlund (1955). *Topological dynamics*. American Mathematical Society. Colloquium publications, v.36. Providence.

[12] Hestenes, D. (1966) Space-Time Algebra. Gordon and Breach Science.

[13] Inönü, E. and E. P. Wigner (1952). On the contraction of groups and their representations. *Proceedings of the National Academy of Sciences* 39 510-524.

[14] Nayak, C. and F. Wilczek, *Nuclear Physics* B479, 529 (1996)

[15] Nazarov, M. Young’s symmetrizers for projective representations of the symmetric group. *Advances in Mathematics* 127, 190-257 (1997).

[16] Read, N. and Moore, G. Fractional Quantum Hall Effect and Nonabelian Statistics. *Progress of Theoretical Physics (Kyoto) Supplement* 107 157 (1992)

[17] Schur, I. (1911). Über die Darstellung der symmetrischen und der alternierenden Gruppen durch gebrochene lineare substitutionen. *Journal für die reine und angewandte Mathematik* 139, 155-250. On the representation of the symmetric and alternating group by fractional linear substitutions. Translated by M.-F. Otto. *International Journal of Theoretical Physics*, this issue.

[18] Segal, I. E. (1951). A class of operator algebras which are determined by groups. *Duke Mathematical Journal* 18, 221.

[19] Smolin, Lee. (1992) Did the universe evolve? *Classical and Quantum Gravity* 9, 173-191. Oxford, 1997.

[20] Smolin, Lee. *The Life of the Cosmos*. Oxford, 1997.

[21] Wilczek, F. (1997). Some examples in the realization of symmetry. Talk given at Strings ’97, 18-21 June 1997, Amsterdam, The Netherlands. *Nuclear Physics* Proceedings Supplement. Also hep-th/9710135.

[22] Wilczek, F. (1998). Private communication.

[23] Wilczek, F. (1999) Projective Statistics and Spinors in Hilbert Space. Submitted for publication. Also hep-th/9806228.

[24] Wiman, A. (1898) *Mathematische Annalen* 47, 531; 52, 243.