Gauge and Poincaré Invariant Regularization
and Hopf Symmetries

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Abstract

We consider the regularization of a gauge quantum field theory following a modification of
the Pochinski proof based on the introduction of a cutoff function. We work with a Poincaré
invariant deformation of the ordinary point-wise product of fields introduced by Ardalan,
Arfaei, Ghasemkhani and Sadooghi, and show that it yields, through a limiting procedure
of the cutoff functions, to a regularized theory, preserving all symmetries at every stage.
The new gauge symmetry yields a new Hopf algebra with deformed costructures, which is
inequivalent to the standard one.
1 Introduction

One of the most fascinating aspects of field theory is the renormalization programme and the role of the renormalization group. The regularization of field theory always requires the introduction of a scale, which may be taken as a momentum scale. In [1] Polchinski introduced a momentum cutoff function to find an exact renormalization group equation for a scalar theory. The procedure is of not easy applicability to gauge theories. This was performed later and a review of these methods with extended references can be found in [2]. The extension to nonabelian theories has remained problematic for long. Some results have been obtained [3], paying various prices like the necessity of an infinite number of colours, or the loss of Poincaré invariance (see for example [4]). In [5] it is introduced a formulation for which, although at any given scale the theory is not gauge invariant, the limiting theory is invariant.

In our case, the symmetry is maintained at all scales. The enhancement is due to the presence of an extra hidden Hopf symmetry and the use of a deformed product. For a different approach on the preservation of local and global symmetries see [6] and refs. therein.

Let us consider the following action (written in momentum space) for quantum electrodynamics for which a regularization has been implemented by a cutoff on the momenta:

\[ S_{\text{QED}} = \int_{\Lambda} \frac{d^4p}{(2\pi)^4} \left\{ \bar{\psi}(-p)(\gamma_\mu p^\mu + m)\psi(p) + e\bar{\psi}(-p) \int_{\Lambda} \frac{d^4q}{(2\pi)^4} \Theta_\Lambda(p-q) \gamma^\mu \tilde{A}_\mu(p-q)\psi(q) + \frac{1}{4} \left( \tilde{F}^{\mu\nu}(-p)\tilde{F}_{\mu\nu}(p) + \frac{1}{2\xi} p^\mu \tilde{A}^\mu(-p) p^\nu \tilde{A}_\nu(p) \right) \right\}. \]

Here by \( \int_{\Lambda} \frac{d^4p}{(2\pi)^4} \) we mean \( \int_{\Lambda}^0 p^3 dp \) (we are in the Euclidean case), and \( \Theta \) is the characteristic function of the four-sphere of radius \( \Lambda \)

\[ \Theta_\Lambda(p) = \begin{cases} 1 & p^2 < \Lambda \\ 0 & p^2 \geq \Lambda \end{cases} \]

(1.2)

The slight modification with respect to usual regularizations will be clear in the next sections. The theory described by (1.1) is free of UV divergences. The problem with cutoff regularization is that it destroys gauge invariance. In this paper we will show that such a regularized action has a hidden symmetry, which is a deformation of the usual U(1) symmetry of QED. The new symmetry is actually a Hopf symmetry with deformed costructures, which results to be inequivalent to the standard one. The tools we will be using have been developed for noncommutative quantum field theory (for a review see [7]), although we will only deal with commutative field theory.

Our starting point is the article [8], where it was shown that a suitable commutative deformation of the product among fields, inspired by noncommutative quantum field theory gives a new action, deformed by the presence of “cutoff functions”, which preserves gauge invariance, although in a deformed version. As long as the cutoff functions are analytic the new action is equivalent to the ordinary one, being related by a field redefinition. For the action not to be just a redefinition a non-analytic cut off is requested, which however destroys the associativity of the new product. We show in this paper that the desired result, a regularized gauge invariant action, can be obtained with a limiting procedure. Our strategy will be the following. We start...
by imposing the usual desirable space-time symmetries - translation and Lorentz invariance - and we find the form of the deformed product compatible with them. It is the commutative, non-local product introduced in [8] on the basis of [9]. The new action endowed with such a product is the deformed action considered in [8]. We then make the cutoff function converge towards a function which vanishes for momenta larger than a cutoff, and hence cannot be analytic. In this limit we recover the action [11].

A deformation of the product implies in general a deformation of the gauge group and, more precisely, it determines a deformation of the whole enveloping algebra (Hopf algebra). This is indeed what happens in our case. The Hopf algebra is effectively deformed and it is shown to be inequivalent to the classical one for any non-trivial choice of the cutoff function. This means that, although analytic cutoff functions may be reabsorbed with a field redefinition from the point of view of the algebra of fields, the Hopf algebra symmetry cannot be mapped into the undeformed one.

The procedure illustrated can be extended to nonabelian theories as well.

We also notice that a cutoff in momentum space is similar (but not exactly equivalent) to having a minimal length. Recently there has been interest in electrodynamics with minimal length, see for example [10] and references therein.

2 Momentum Cutoff via a Deformed Product

The renormalization group describes the scaling of a field theory as a dimensional parameter is varied. The parameter is responsible for the regularization of the theory which would otherwise have divergencies. There are several ways to implement the regularization (see for example [11, Chapt. 7.5] for a quick comparison of the most popular schemes). The one used in [1] is to consider a deformation of the free propagator by a cutoff function in the momenta.

In [8] Ardalan, Arfaei, Ghasemkhani and Sadooghi proposed to implement the cutoff with the use of a deformed commutative product, employing the experience gained in the study of noncommutative field theory. For simplicity let us examine for the moment a scalar theory, the generalization to gauge theories will be done later in Sect. 3. The products considered (which we denote by ⋆) are associative products among functions on spacetime with the property:

\[ \int dx f \star g = \int dx g \star f. \]  \hfill (2.1)

It is not actually crucial that the integral be done with the usual measure, but it is necessary that there exist an integral with the tracial property, i.e. invariant for cyclic permutation of the factors. Another important property is the existence of derivations, that is operators satisfying the Leibnitz rule. For our case it is sufficient to assume the Leibnitz rule for the usual derivations:

\[ \partial_\mu (f \star g) = f \star \partial_\mu g + (\partial_\mu f) \star g. \]  \hfill (2.2)

*To avoid confusion we term “noncommutative” the theories on noncommutative spacetimes, like the ones built using deformed products like the Grönevold-Moyal ones. By “nonabelian” we indicate the case in which the gauge group is nonabelian, like ordinary Yang-Mills. A theory can be nonabelian and noncommutative.
The most studied product of this kind is the Grönewold-Moyal product \cite{12,13}, which is noncommutative, and reproduces the commutation rules of quantum mechanics adapted to spacetime: 
\[ x^\mu \star x^\nu - x^\nu \star x^\mu = i\theta^{\mu\nu}. \]

### 2.1 Poincaré Invariant star products

Let us consider space-time symmetries and let us start imposing translation invariance. We will discuss Lorentz invariance as a further restriction. In \cite{9} translation invariant associative products were introduced and classified on the basis of a suitable cohomology. The problem was further considered in \cite{14, 8}, and employed in the study of gauge theories. Let us briefly review the subject.

For our purposes a generic star product is an associative product between functions on $\mathbb{R}^d$ which depends on one or more parameters. In the limit in which these parameters vanish the product becomes the usual pointwise product. We contemplate the possibility that the star product be commutative. This will indeed be the main object of interest. Let $(\mathcal{A}, \mu)$ be the algebra of real functions with multiplication law
\[ \mu : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}. \tag{2.3} \]

A translation invariant associative product may be expressed as
\[ \mu(f \otimes h)(x) = (f \star h)(x) = \frac{1}{(2\pi)^d} \int dp^d dq^d dk^d e^{ip \cdot x} \tilde{f}(q) \tilde{h}(k) K(p, q, k) \tag{2.4} \]
where $\tilde{f}(q)$ is the Fourier transform of $f$. $K$ can be a distribution and has to implement translation invariance and associativity. The product of $d$-vectors is understood with the Minkowski or Euclidean metric: $p \cdot x = p_i x^i$. The usual pointwise product is reproduced with the choice $K(p, q, k) = \delta^d(k - p + q)$. Defining the translation by a vector $a$ as $T_a f(x) = f(x + a)$, by translation invariance of the product we mean the property
\[ T_a (f \star h) = T_a (f \star h). \tag{2.5} \]

Performing the Fourier transform we have
\[ \tilde{T}_a \tilde{f}(q) = e^{iap} \tilde{f}(q) \]
so that translation invariance imposes on the product (2.4) the condition
\[ e^{iap} \int dp^d dq^d dk^d e^{ip \cdot x} \tilde{f}(q) \tilde{h}(k) K(p, q, k) = \]
\[ = \int dq^d dk^d e^{iaq} e^{iak} e^{ip \cdot x} \tilde{f}(q) \tilde{h}(k) K(p, q, k). \tag{2.7} \]

At the distributional level this means
\[ K(p, q, k) = e^{i(k-p+q) \cdot a} K(p, q, k) \]
and it is solved by
\[ K(p, q, k) = K(p, q) \delta(k - p + q) \]
with $K$ a generic function to be further constrained. In [9] it was actually represented in exponential form, but there is no particular reason for that, as pointed out in [14]. As we will see, a different representation unveils a precious freedom in the choice of product representatives in each equivalence class.

Strong restrictions on $K$ come from the associativity requirement which reads
\[
\int dk^d K(p, k, q)K(k, r, s) = \int dk^d K(p, r, k)K(k, s, q).
\] (2.10)
This is nothing but the usual cocycle condition in the Hochschild cohomology. See [9] for details. Eq. (2.10) may be rephrased in terms of a condition for $K$ [14]
\[
K(p, q)K(q, r) = K(p, r)K(p - r, q - r).
\] (2.11)
Under rather mild assumptions of analicity the most general solution is provided by [14]
\[
K(p, q) = H^{-1}(p)H(q)H(p - q)e^{i\alpha(p,q)}
\] (2.12)
with $\alpha(p,q)$ a two cocycle in the appropriate cohomology [9] and $H(q)$ an arbitrary even real function. It can be shown that $\alpha(p,q)$ is constrained by the associativity condition and the cyclicity of the product [2.11] to be of the form
\[
\alpha(p,q) = \theta^{\mu\nu}p_\mu q_\nu + \partial\beta(p, q) = \theta^{\mu\nu}p_\mu q_\nu + \beta(q) - \beta(p) + \beta(p - q)
\] (2.13)
where we have emphasized the fact that it is defined up to a coboundary term, $\partial\beta(p, q)$, whose explicit form has been calculated. The function $\beta$ is a real odd function.

The matrix $\theta^{\mu\nu}$ is an antisymmetric and constant, responsible for the noncommutativity of the product. If $\theta = 0$ the product is commutative. Commutative products are associated to coboundaries.

The request of Lorentz invariance further constrains the form of the product. Indeed only the function $H$ survives, the term in $\theta$ being manifestly not invariant and the function $\beta$ being an odd function of the modulus of momenta. Hence, the requirement of full Poincaré invariance forces upon us the commutativity of the product $\dagger$. We stress however that this does not mean that the product is the usual pointwise one.

### 2.2 Regularized product

We will concentrate from now on the commutative, Poincaré invariant case, $\theta = 0, \beta = 0$. We will refer to field theories with the new product as *commutative deformed field theories*, not to be confused with noncommutative ones. The algebra of functions with the deformed product will be indicated with $(A_\star, \mu_\star)$ while the algebra of functions with point-wise product will be denoted by $(A_0, \mu_0)$.

The new star product, $\mu_\star$, acquires the form
\[
\mu_\star(f \otimes h)(x) = (f \star h)(x) = \int \frac{d^4p}{(2\pi)^4} \frac{1}{H(p)} e^{ipx} \int \frac{d^4q}{(2\pi)^4} [H(p - q)f(p - q)][H(q)\tilde{h}(q)]
\] (2.14)
\[\dagger\text{From now on, with an abuse of notation, we will indicate by } p, q, \ldots \text{ both the four-vector or its modulus. It will be clear from the context which of the two we are meaning.}\]
which is commutative and Poincaré invariant but non-local. $H = 1$ corresponds to the ordinary point-wise product $\mu_0$. It is possible to show that the deformed product enjoys the property

$$\int d^4x (f \ast h)(x) = \int \frac{d^4p}{(2\pi)^4} H^2(p) \tilde{f}(-p) \tilde{h}(p)$$

(2.15)

where we have used $H(0) = 1$. This is a consequence of the request that the algebra of functions be unital with ordinary unity

$$f \ast 1 = 1 \ast f = f.$$  

(2.16)

The identity (2.15) will be useful in computing the deformed QED action in momentum space and will play a crucial role in the limiting process that we will need to properly define the cutoff action (1.1).

It was argued in [8] that the freedom acquired with the product deformation through the function $H(p)$, may be exploited to regularize quantum field theories. In that proposal the function $H$ plays the role of a multiplicative cut off function in momentum space. Let us analyze better this. The presence of $H^{-1}$ in (2.14) implies the fact that the cutoff function cannot vanish, except possibly on a measure zero set if one considers it as a distribution. In particular it is impossible to consider a function which vanishes identically outside a sphere of finite radius. This is ultimately a consequence of the nonlocal nature of the convolution product of the Fourier transform. The idea suggested in [8] to consider only functions whose Fourier transform vanishes outside a sphere of radius $\Lambda' \gg \Lambda$ does not really work since the convolution product is nonlocal, and even if the starting functions have this property their product will not have it. It is easy to see that, since $\Theta_{\Lambda}(p)$ is not translationally invariant there is no way to obtain an associative product.

On the other hand, as long as the cut-off function is analytic the new product belongs to the same cohomology class as the point-wise product. In that case it was shown in [15] that the ultraviolet behaviour of the deformed quantum field theory remains unmodified with respect to ordinary quantum field theory. In other terms, a non vanishing cut-off function provides essentially a field redefinition, basically the multiplication of the Fourier transforms of the fields by the function $H$, which cannot improve the ultraviolet regime. Indeed we have, for each invertible $H$, an isomorphism between the deformed and undeformed algebra of fields,

$$\varphi : (A_\ast, \mu_\ast) \rightarrow (A_0, \mu_0)$$

(2.17)

which, in momentum space, reads

$$\tilde{\varphi}(f)(p) = H(p) \tilde{f}(p)$$

(2.18)

$$\tilde{\varphi}(f \ast g)(p) = \tilde{\varphi}(f) \bullet \tilde{\varphi}(g)$$

(2.19)

with $\bullet$ the undeformed convolution.

Despite these warnings we will show in Sect. 4.2 that the regularized theory with the sharp cutoff can be properly defined as the limit $H(p) \rightarrow \Theta_{\Lambda}(p)$ of well defined theories with analytic cutoff.

\[\text{However we will show in Sect. 4.3 that this equivalence does not extend to the algebra of symmetries.}\]
As noted already in [8] the theory with a sharp cutoff is akin to putting the theory on a lattice, although in that case the momentum is periodic, which is not true in this case. A closer analogy is with quantum field theories on fuzzy spaces (see [16] and refs therein) where the regularization is achieved without destroying the symmetry. The realization of fuzzy noncompact spaces may be found in [17] with an application to the regularization of scalar field theory. A related case is the implementation of the cutoff via the eigenvalues of a generalized Dirac operator. This is the basis of the spectral action [18] and of the finite mode regularization (see for example [19, 20, 21]).

3 Deformed gauge symmetry

It is further proved in the article [8] that the deformation procedure can be implemented for gauge theories and that the gauge symmetry is preserved by the deformation adopted, although in a modified form. What remained to be understood is the nature of the symmetry.

Let \( f, h \in (A_0, \mu_0) \). The pointwise product may be defined in terms of the convolution product in momentum space

\[
(f \cdot h)(p) = \frac{d^4q}{(2\pi)^4} \tilde{f}(p - q)\tilde{h}(q) \quad (3.1)
\]

so that

\[
\mu_0(f \otimes h)(x) = (f \cdot h)(x) = \int \frac{d^4p}{(2\pi)^4} e^{ipx} (\tilde{f} \cdot \tilde{h})(p).
\]

Analogously we introduce the deformed convolution product \( \star \)

\[
(f \star h)(p) = \frac{1}{H(p)} \int \frac{d^4q}{(2\pi)^4} H(p - q)\tilde{f}(p - q)\tilde{H}(q)h(q) \quad (3.3)
\]

so to have

\[
\mu_\star(f \otimes h)(x) = (f \star h)(x) = \int \frac{d^4p}{(2\pi)^4} e^{ipx} (\tilde{f} \star \tilde{h})(p).
\]

Ordinary gauge theories with gauge group \( \hat{\mathbb{G}} \) are modified replacing the point-wise product with the non-local product (3.4). The resulting field theories are invariant under the deformed gauge transformations

\[
\phi(x) \rightarrow g_\star(x) \triangleright \star \phi(x) = \exp_\star (i\alpha^\prime(x)T) \triangleright \star \phi(x).
\]

\( \triangleright \star \) indicates generically the action of the group and later of the algebra, which implies (for the nonabelian case) both a matrix multiplication and a \( \star \) product. In the U(1) case the notation is redundant, it being \( (\alpha \triangleright \star \phi)(x) = \alpha(x) \ast \phi(x) \), but the more general notation will be useful later on. Here \( T_i \) are the Lie algebra generators, \( T_i \in \mathfrak{g} \) and the gauge group elements \( g_\star(x) \) are defined as star exponentials

\[
g_\star(x) = \exp_\star (i\alpha^\prime(x)T) = 1 + i\alpha^\prime(x)T_i - \frac{1}{2}(\alpha^i \ast \alpha^j)(x)T_iT_j + \ldots
\]

\( ^5 \)In our notation \( \hat{G} \) is the Lie group, with Lie algebra \( \mathfrak{g} \). The hatted objects \( \hat{G}, \hat{\mathfrak{g}} \) indicate respectively the group and the algebra of gauge transformations, i.e. functions from spacetime in the group or algebra.
At the infinitesimal level we have then

\[ \phi(x) \longrightarrow \phi(x) + i(\alpha \triangleright \phi)(x) \]  

(3.7)

with

\[ (\alpha \triangleright \phi)(x) = i \left( \alpha^j(x) \ast (T_j \triangleright \phi) \right)(x) \]  

(3.8)

and \( T_j \) is in the appropriate representation to the field \( \phi \). The construction is valid for fields with non zero spin as well. In [8] quantum electrodynamics is explicitly considered.

The deformed Lie multiplication reads

\[ [\alpha, \tilde{\alpha}]_\star(x) = (\alpha \star \tilde{\alpha})(x) - (\tilde{\alpha} \star \alpha)(x). \]  

(3.9)

**Remark** In noncommutative field theory the definition (3.5) is problematic because we have

\[
(\alpha \star \tilde{\alpha})(x) - (\tilde{\alpha} \star \alpha)(x) = \left( (\alpha^i \star \tilde{\alpha}^j)(x) + (\tilde{\alpha}^j \star \alpha^i)(x) \right) [T_i, T_j] + \left( (\tilde{\alpha}^j \star \alpha^i)(x) - (\alpha^i \star \tilde{\alpha}^j)(x) \right) \{T_i, T_j\}
\]

(3.10)

which only closes for the group \( U(N) \) in the adjoint and fundamental representations. This problem is solved for example in twisted gauge theories (see for example [22]) where the star product is induced by a twist operator and the gauge transformations are twist-deformed. In the present case the definition (3.5) is perfectly viable for any Lie group, because the product is commutative, therefore the term proportional to the anticommutator \( \{T_i, T_j\} \) vanishes.

### 3.1 The Hopf algebra structures of ordinary gauge theory

The action of the group on the fields, when products of fields are considered, involves not only the Lie-algebra structure, but also the full Hopf-algebra structure, although for the pointwise multiplication the latter structure is trivial. This is not the place to describe all of aspects of the theory of Hopf algebras (for an introduction see for example [23]). In the following we will just recall the essential definitions.

When acting on the product of fields with a gauge transformation we need to extend the action of the gauge group. This is obtained via the coproduct. Since we will mostly deal with infinitesimal gauge transformations, \( \alpha(x) \in \hat{g} \), we review here the coproduct for the infinitesimal gauge generators while the Hopf algebra structure of finite gauge transformations and its deformation are discussed in the Appendix A.

The coproduct is properly defined on the universal enveloping algebra of \( \hat{g} \),

\[ \Delta : U(\hat{g}) \otimes U(\hat{g}) \longrightarrow U(\hat{g}). \]  

(3.11)

Its explicit form may be obtained on asking that the action of the group be an automorphism of the algebra \( \mathcal{A} \), i.e. that it be compatible with the multiplication law in \( \mathcal{A} \). We have

\[ \alpha \triangleright \mu \circ (f \otimes h) = \mu \circ (\rho \otimes \rho)(\Delta(\alpha)) \circ (f \otimes h). \]  

(3.12)

From the request that the coproduct be compatible with the ordinary point-wise product we obtain,

\[ \Delta_0(\alpha)(f \otimes h) = (\alpha \otimes \text{id} + \text{id} \otimes \alpha)(f \otimes h) = \alpha \triangleright f \otimes h + f \otimes \alpha \triangleright h \]  

(3.13)

\[ \Delta_0(\text{id}_{U(\hat{g})})(f \otimes h) = (\text{id}_{U(\hat{g})} \otimes \text{id}_{U(\hat{g})})(f \otimes h). \]  

(3.14)
This endows $U(\hat{g})$ with a cocommutative Hopf algebra structure, provided we define an algebra homomorphism $\epsilon$ called the counit and an algebra antihomomorphism $S$, the antipode,

$$
\epsilon : U(\hat{g}) \to \mathbb{C} \quad \epsilon(\alpha) = 0 \quad \forall \alpha \in \hat{g} \quad (3.15)
$$

$$
S : U(\hat{g}) \to U(\hat{g}) \quad S(\alpha) = -\alpha \quad \forall \alpha \in \hat{g}. \quad (3.16)
$$

It is easily verified that $\Delta_0, S, \epsilon$, so defined are mutually compatible, that is they satisfy the conditions (A.7)-(A.9). From (3.13) we have

$$
\alpha \triangleright (f \ast h)(x) = \mu_0 \circ \Delta_0(\alpha)(f \otimes h)(x) = ((\alpha \triangleright f) \cdot h)(x) + (f \cdot (\alpha \triangleright h))(x). \quad (3.17)
$$

This is the usual Leibnitz rule, with

$$
(\alpha \triangleright f)(x) = (\alpha^i \cdot (T_i \triangleright f))(x). \quad (3.18)
$$

### 3.2 The deformed Hopf algebra

When the point-wise multiplication is deformed, the Leibnitz rule which is encoded in the coproduct changes accordingly. On generalizing (3.17) we have

$$
\alpha \triangleright (f \star h)(x) = \mu_* \circ \Delta_*(\alpha)(f \otimes h)(x) \quad (3.19)
$$

with a deformed coproduct to be determined. From the expression of the deformed product (3.4) and the deformed gauge action (3.5) we read off the expression of the new coproduct

$$
\Delta_*(\alpha)(f \otimes g) = (\alpha_* \otimes \text{id} + \text{id} \otimes \alpha_*) (f \otimes g) = \alpha \triangleright_* f \otimes g + f \otimes \alpha \triangleright_* g \quad (3.20)
$$

and

$$
(\alpha \triangleright_* f)(x) = (\alpha^i \ast T_i \triangleright f)(x) \quad (3.21)
$$

where the action of the ungauged Lie algebra in the appropriate representation is not modified. It can be verified that our definition is consistent with the coassociativity condition (A.7). Moreover, the coproduct $\Delta_*$ is cocommutative according to the definition in (A.4). We then define the antipode and the counit

$$
(S_*(\alpha) \triangleright f)(x) = - (\alpha \triangleright_* f)(x) \quad (3.22)
$$

$$
\epsilon_*(\alpha) = \epsilon(\alpha) = 0. \quad (3.23)
$$

These definitions are consistent with Eqs. (A.8), (A.9).

To summarize, the deformed universal enveloping algebra $U_*(\hat{g})$ is a cocommutative Hopf algebra with deformed Lie multiplication, $[\cdot, \cdot]_*$, deformed coproduct $\Delta_*$, deformed antipode, $S_*$, and undeformed counit $\epsilon$.

### 4 Deformed field theories

#### 4.1 The deformed Hopf symmetry

It is not difficult to verify that the action of quantum electrodynamics equipped with the star product (2.14)

$$
S_H = \int d^4 x \left\{ -\bar{\psi} \ast (i \gamma^\mu \partial_\mu - m) \psi + e \bar{\psi} \ast \gamma^\mu A_\mu \ast \psi + \frac{1}{4} F_{\mu \nu} \ast F^{\mu \nu} + \frac{1}{2\xi} (\partial_\mu A^\mu)^2 \right\} \quad (4.1)
$$

is a deformation of the usual electromagnetic field action.

\[\text{Page 8}\]
is invariant under the deformed U(1) Hopf algebra, when the deformed coproduct (3.20) is used. On generalizing (3.5) the deformed gauge transformations for matter and gauge fields explicitly read

\[ \psi(x) \rightarrow \psi(x) + ie(\alpha \ast \psi)(x), \quad \tilde{\psi}(x) \rightarrow \tilde{\psi}(x) - ie(\alpha \ast \tilde{\psi})(x), \quad A_\mu(x) \rightarrow A_\mu(x) - \partial_\mu \alpha(x). \quad (4.2) \]

They become in momentum space

\[ \delta \tilde{\psi}(p) = ieH^{-1}(p) \int \frac{d^4q}{(2\pi)^4} H(p - q)\tilde{\alpha}(p - q)H(q)\tilde{\psi}(q) \quad (4.3) \]

\[ \delta \tilde{\bar{\psi}}(p) = -ieH^{-1}(p) \int \frac{d^4q}{(2\pi)^4} H(p - q)\tilde{\alpha}(p - q)H(q)\tilde{\bar{\psi}}(q) \quad (4.4) \]

\[ \delta \tilde{A}_\mu(p) = ip_\mu \tilde{\alpha}(p). \quad (4.5) \]

On using (2.15) the action is rewritten as

\[ S_H = \int \frac{d^4p}{(2\pi)^4} \left\{ H(p)\tilde{\psi}(p)(\gamma_\mu p^\mu + m)H(p)\tilde{\psi}(p) \right. \]

\[ +eH(p)\tilde{\bar{\psi}}(-p) \int \frac{d^4q}{(2\pi)^4} H(p - q)\gamma_\mu \tilde{A}_\mu(p - q)H(q)\tilde{\bar{\psi}}(q) \]

\[ +\frac{1}{4}H^2(p) \left( \tilde{F}_{\mu\nu}(-p)\tilde{F}^{\mu\nu}(p) + \frac{1}{2\xi}p_\mu \tilde{A}_\mu(-p) p_\nu \tilde{A}_\nu(p) \right) \}. \quad (4.6) \]

Let us notice that, although the modified product (2.14) contains the inverse function of \( H \), this has disappeared from the action thanks to the identity (2.15).

Therefore, if \( H \) can be chosen to be a cutoff function, the theory is fully regularized. It is gauge invariant by construction (with respect to the deformed symmetry discussed above), but it is also possible to prove the relevant Ward identities, the derivation mirrors the usual one and is described in detail in [8, App. B].

### 4.2 Limit of Hopf Algebras and non-analytic cutoff

For the product (2.14) to be defined it is necessary that the function \( H(p) \) do not vanish anywhere. But in that case, as we have already argued, the deformation is not very interesting because the new product is isomorphic to the point-wise one. The action (2.15) however can be defined for arbitrary cutoff functions, including those which identically vanish for \( p^2 \) larger than some scale. In this case the theory cannot be obtained from the ordinary theory via a field redefinition. For example in the event of non-analytic cutoff there is an actual loss of information, all momenta above \( \Lambda \) do not contribute. This is the interesting case, but we cannot simply cut the momenta, since we would lose associativity of the product. We thus consider a sequence of analytic cutoff functions which converge to the sharp cutoff \( \Theta_\Lambda(p) \).

\[ H_\varepsilon(p) \rightarrow \Theta_\Lambda(p). \quad (4.7) \]

A possible choice is for example the following sequence of functions

\[ H_\varepsilon(p) = \frac{1}{2} - \frac{1}{2} \tanh \left( \frac{p^2 - \Lambda^2}{e\Lambda^2} \right). \quad (4.8) \]
They converge to $\Theta_\Lambda(p)$ in the limit $\epsilon \to 0$. At each stage of the limiting procedure the action \((4.6)\) preserves the symmetries, while converging to the cutoff-action \((1.1)\) introduced at the beginning of this article, in the limit \((4.7)\).

The theory with the sharp cutoff cannot be defined with a deformed product, nevertheless, being a limit of Hopf-gauge invariant theories, it enjoys their symmetries, and the proof of the Ward identities of \([8]\) still goes through.

The limits we are taking here are to be understood in the weak (nonuniform) sense. At any stage the theory satisfies the Ward identities, and it has the full Hopf invariance.

### 4.3 Inequivalence of $U(\hat{g})$ and $U_*(\hat{g})$

The deformed symmetry of the QED action \((4.6)\), the Hopf algebra $U_*(u(1))$ is commutative (because $U(1)$ is Abelian) and cocommutative, because the deformed coproduct satisfies \((A.4)\), exactly as $U(u(1))$. Therefore we can ask whether the two Hopf algebras be or not equivalent. A precise definition of equivalence is in terms of homomorphisms. Two Hopf algebras $(\mathcal{A}, m, \Delta, S)$ $(\mathcal{B}, m', \Delta', S')$ are equivalent if there exists a map

$$\varphi : \mathcal{A} \rightarrow \mathcal{B}$$

which is

1. an algebra homomorphism,

$$\varphi \circ m = m' \circ \varphi$$

2. a coalgebra homomorphism,

$$(\varphi \otimes \varphi) \circ \Delta = \Delta' \circ \varphi$$

3. a Hopf algebra homomorphism, that is

$$\varphi \circ S = S' \circ \varphi.$$ 

Moreover it has to be compatible with the action on the algebra of fields. That is

$$\varphi(\alpha \triangleright \phi)(p) = \varphi(\alpha) \triangleright \varphi(\phi)$$ \hspace{1cm} (4.9)

It is not difficult to see that in the case of analytical $H$ there exists a map between $U(g)$ and $U_*(g)$ which is an algebra homomorphism. Eq. \((4.9)\) actually imposes that it be the map already defined for the algebra of fields \((2.18)\). This is the field redefinition we have already alluded. It is however not a Hopf algebra morphism because it is not difficult to check that it does not satisfy properties 2 and 3 above.

We conclude that although the deformed algebra with nonlocal product is isomorphic to the standard one with point-wise product, the deformed symmetry of the action \((4.6)\) is a genuine new symmetry.
5 Conclusions

Techniques borrowed from field theories on noncommutative spaces have been used in the context of the regularization of ordinary field theory. The main tool has been a commutative, Poincaré invariant deformed product. The deformed theory has been seen to possess a deformed symmetry, not only in the Lie algebra structure, but at the full Hopf algebra level. This regularized electrodynamics is an instance of the simplest possible Hopf algebra invariance, commutative, cocommutative, associative and coassociative. This is the "hidden" symmetry alluded to in [8]. The resulting theory is fully gauge invariant, and the general technique makes no distinction between Abelian and nonabelian symmetries.

The regularization depends on a cutoff function which, even if necessarily non-vanishing for the definition of the product, may be taken to converge to a sharp cutoff. In this limit the regularization is not just a field redefinition. The effective limiting theory does indeed depend only on the sector of the theory below the cutoff. At each stage the theory possesses internal and spacetime symmetries.

The deformation of products seems a promising tool, not only for renormalization issues, but also for the performance of actual calculations and may turn out to be particularly useful for nonabelian theories.

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A Appendix

We review in this appendix the Hopf algebra structure of finite (as opposed to infinitesimal) gauge transformations, and introduce their deformation.

Let $C\hat{G}$ be the group algebra of $\hat{G}$ over the complexes, with multiplication $m$. The coproduct is a homomorphism from $C\hat{G}$ to $C\hat{G} \otimes C\hat{G}$

$$\Delta : C\hat{G} \rightarrow C\hat{G} \otimes C\hat{G}, \forall g \in \hat{G}, f, h \in A, \quad g \triangleright (f \otimes h) = (\rho \otimes \rho)(\Delta(g)) \circ (f \otimes h)$$ (A.1)

with $\rho$ a representation of $C\hat{G}$ on the algebra of fields. $\triangleright$ indicates the action of the gauge group. The explicit form of the coproducts is obtained on asking that the action of the group be an automorphism of the algebra $A$, i.e. that it be compatible with the multiplication law in $A$,

$$g \triangleright \mu(f \otimes h)(x) = \mu(\Delta(g) \triangleright (f \otimes h)(x)).$$ (A.2)

The ordinary coproduct, compatible with the point-wise multiplication, is easily obtained by Eq. (A.2) to be, for group-like elements

$$\Delta_0(g) = g \otimes g.$$ (A.3)
The rule is compatible with the multiplication $m$, therefore it can be uniquely extended to the whole of $\mathbb{C}\hat{G}$. The coproduct $\Delta_0$ is cocommutative, i.e. it satisfies

$$\tau \circ \Delta_0 = \Delta_0$$  \hspace{1cm} (A.4)

with $\tau$ the permutation operator, $\tau(f \otimes h) = h \otimes f$.

The group algebra $(\mathbb{C}\hat{G}, m, \Delta)$ is a bialgebra. It is turned into a Hopf algebra if we also have an algebra homomorphism, $\epsilon$ called the counit

$$\epsilon : \mathbb{C}\hat{G} \rightarrow \mathbb{C} \epsilon(g) = 1 \forall g \in \hat{G}$$  \hspace{1cm} (A.5)

and an algebra antihomomorphism $S$, the antipode

$$S : \mathbb{C}\hat{G} \rightarrow \mathbb{C}\hat{G} \ S(g) = g^{-1} \forall g \in \hat{G}. \hspace{1cm} (A.6)$$

Both can be uniquely extended to the whole group algebra.

For a given Hopf algebra $H$ the coproduct, the counit and the antipode have to satisfy the relations below:

$$\begin{align*}
(\Delta \otimes \mathbb{I}) \circ \Delta &= (\mathbb{I} \otimes \Delta) \circ \Delta \hspace{1cm} (A.7) \\
(\mathbb{I} \otimes \epsilon) \circ \Delta &= \mathbb{I}(\epsilon \otimes \mathbb{I}) \circ \Delta \hspace{1cm} (A.8) \\
m \circ (\mathbb{I} \otimes S) \circ \Delta &= m \circ (S \otimes \mathbb{I}) \circ \Delta = 1_H \circ \epsilon. \hspace{1cm} (A.9)
\end{align*}$$

Eq. (A.7) is the coassociativity condition. These relations have to be verified when we introduce the Hopf algebra structures of the deformed symmetry.

To conclude this section we briefly describe the Hopf algebra structure of deformed gauge transformations.

In total analogy with the universal enveloping algebra, we can deform the group algebra, $\mathbb{C}\hat{G}$, together with its structures and co-structures. Group elements $g(x)$ are replaced by star exponentials as in Eq. (3.6). The group multiplication $m$ is thus replaced by $m_*$

$$m_* \circ (g_* \otimes \tilde{g}_*) = g_* \ast \tilde{g}_*.$$  \hspace{1cm} (A.10)

The deformed coproduct is obtained as previously, on requesting that it be an automorphism for the algebra of fields $(A_*, \mu_*)$. The antipode is obtained by consistency with the coproduct. They read respectively

$$\begin{align*}
\Delta_*(g)(f \otimes h) &= g_* \triangleright_* f \otimes h + f \otimes g_* \triangleright_* h \hspace{1cm} (A.11) \\
S_*(g) \triangleright_* f &= (g_*^{-1}) \triangleright_* f \hspace{1cm} (A.12)
\end{align*}$$

and the counit is undeformed. It can be verified that these definitions satisfy the consistency conditions $[A.7]$ and $[A.9]$.

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