Bare canonicity of representable cylindric and polyadic algebras

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Abstract

We show that for finite \( n \geq 3 \), every first-order axiomatisation of the varieties of representable \( n \)-dimensional cylindric algebras, diagonal-free cylindric algebras, polyadic algebras, and polyadic equality algebras contains an infinite number of non-canonical formulas. We also show that the class of structures for each of these varieties is non-elementary. The proofs employ algebras derived from random graphs.

Keywords: canonical variety, canonical axiomatisation, algebras of relations, cylindric algebras, diagonal-free algebras, random graphs

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1. Introduction

The notion of the \textit{canonical extension} of a boolean algebra with operators (or ‘BAO’) was introduced by Jónsson and Tarski in a classical paper [15], generalising a construction of Stone [22]. It is an algebra whose domain is the power set of the set of ultrafilters of the original BAO, and its operations are induced from those of the BAO in a natural way. Canonical extensions are nowadays a key tool in algebraic logic, with a multitude of uses and generalisations.

A class of BAOs is said to be \textit{canonical} if it is closed under taking canonical extensions. In this paper we are concerned with the classes of representable \( n \)-dimensional cylindric algebras, diagonal-free cylindric algebras, polyadic algebras, and polyadic equality algebras, for finite \( n \geq 3 \). These four classes are varieties. They are non-finitely axiomatisable, and many further ‘negative’ results on axiomatisations are known (e.g., [1, 21]). However, the classes are canonical. Now [15] already established that positive equations are preserved by canonical extensions, and more generally, Sahlqvist equations are also preserved (see, e.g., [2]). This may suggest that the four classes might be axiomatisable by positive or Sahlqvist equations.

It turned out that the representable cylindric algebras are not Sahlqvist axiomatisable [23, footnote 1]. In this paper, we extend this result to a wider class of axioms and to all four classes.

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A first-order sentence is said to be \textit{canonical} if the class of its BAO models is canonical. Although some syntactic classes of canonical sentences (such as Sahlqvist equations) are known, canonicity is a semantic property that cannot be easily defined syntactically. For example, there is no algorithm to decide whether an equation is canonical [16, Theorem 9.6.1]. The goal of this paper is to show that there is no canonical axiomatisation of any of the four classes listed above. In fact, we will show that \textit{any first-order axiomatisation of any of them contains infinitely many non-canonical sentences}. We say that a canonical class of BAOs with this property is \textit{barely canonical}. Although the class is canonical, its canonicity emerges only ‘in the limit’ and does not reside in any finite number of axioms for it, however they are phrased.

There are a few related results in the literature. The class of representable relation algebras, proved to be canonical by Monk (reported in [18]), was shown in [13] to be barely canonical. Our proof in the current paper is similar but somewhat simpler: the use of finite combinatorics (finite Ramsey theorem, etc) in [13] is replaced by the use of first-order compactness. Bare canonicity of the ‘McKinsey–Lemmon’ modal logic was shown in [8].

We sketch the rough outline of the proof. Our aim is to convey the idea quickly, and the description will not be completely accurate in detail. Our construction uses polyadic equality-type algebras built from graphs. They are polyadic expansions of cylindric-type algebras constructed from graphs in [12], where it was shown (roughly) that such an algebra is representable if and only if its base graph has infinite chromatic number. (This was used in [12] to prove that the class of structures for the variety of representable \(n\)-dimensional cylindric algebras (finite \(n \geq 3\)) is non-elementary, a result generalised to diagonal-free, polyadic, and polyadic equality algebras in Theorem 9.7 below.) Here, we will cast this work in a wider setting by defining an elementary class \(K\) of three-sorted structures comprising a polyadic equality-type algebra \(A\), a graph \(G\), and a boolean algebra \(B\) of subsets of \(G\), with various relations tying them together quite closely. We will show that representability of \(A\) is equivalent to \(G\) having infinite chromatic number in the sense of \(B\). Both these properties can be defined by first-order theories, which therefore have the same models modulo the theory defining \(K\). It follows by compactness that if the class of representable algebras had a first-order axiomatisation using only canonical sentences, there would be a function \(f : \omega \rightarrow \omega\) such that whenever an algebra \(A\) has chromatic number at least \(f(k)\) (in the sense of some three-sorted structure), its canonical extension has chromatic number at least \(k\). We then borrow from [13] an inverse system of finite (random) graphs of chromatic number \(m\) whose inverse limit has chromatic number \(k\), for any chosen \(2 \leq k \leq m < \omega\). Using some results of Goldblatt [5] connecting canonical extensions with inverse limits, this yields an algebra of chromatic number \(m\) whose canonical extension has chromatic number \(k\). Since \(k, m\) are arbitrary, no function \(f\) as above can exist. A slight extension of the argument, using a little more compactness, shows that any first-order axiomatisation of the representable algebras has infinitely many non-canonical sentences.

\textit{Layout of paper.} In Section 2 we recall some basic notions of algebras of relations, representability, duality and canonicity. We define polyadic equality-type algebras over graphs in Section 3, and abstract generalisations of them in Section 4, where we also ascertain some of their elementary properties. This is continued in Section 5, where we study their ultrafilters. In Section 6 we introduce approximations to representations by means of systems of ultrafilters called ‘ultrafilter networks’, and lower-dimensional approximations of them called ‘patch systems’. This will allow us to prove in Section 7 that (roughly) an abstract algebra is representable if and only if its associated graph has infinite chromatic number. In Section 8 we introduce some needed material on direct and inverse systems. Assuming an axiomatisation with only finitely many
non-canonical formulas, we use direct and inverse systems in Section 9 to build an algebra that satisfies an arbitrary number of axioms, while its canonical extension satisfies only a bounded number, and thus obtain a contradiction. Section 10 lists some open problems.

Notation. We use the following notational conventions. We usually identify (notationally) a structure, algebra, or graph with its domain. For signatures \( L \subseteq L' \) and an \( L' \)-structure \( M \), we write \( M' 
\) for the \( L \)-reduct of \( M \).

Throughout the paper, the dimension \( n \) is a fixed finite positive integer and \( n \) is at least 3. It will often be implicit that cylindric algebras etc. are \( n \)-dimensional and that \( i, j, k, m, \) etc., denote indices \( < n \). We identify a non-negative integer \( m \) with the set \( \{0, 1, \ldots, m - 1\} \). If \( V \) is a set, we write \([V]^m\) for the set of subsets of size \( m \) of \( V \). We write \( \omega \) for the first infinite ordinal number. \( \wp(S) \) denotes the power set of a set \( S \).

For a function \( f : X \to Y \) we write \( \text{dom} f \) for its domain, \( \text{im} f \) for its image, and \( f[X'] \) for \( \{f(x') \mid x' \in X'\} \) when \( X' \subseteq X \). We use similar notation for \( m \)-ary functions, for \( m < \omega \) — e.g., in Definition 2.7. For functions \( f, g \), we write \( f \circ g \) for their composition: \( f \circ g(x) = f(g(x)) \). We omit brackets in function applications when we believe it improves readability. By \( \alpha U \), where \( \alpha \) is an ordinal, we denote the set of functions from \( \alpha \) to \( U \), so an \( \alpha \)-ary relation on \( U \) is a subset of \( \alpha U \). To keep the syntax similar to the finite case, we write \( x_i \) for \( x(i) \) if \( x \in \alpha U \) and \( i < \alpha \). Similarly, we write \( p_i \) for \( p(i) \) where \( p \in \prod_{\alpha \in \omega} U_i \) and \( i < \alpha \).

2. Algebras of relations

In this paper, we will consider four types of algebra: cylindric-type algebras, diagonal-free cylindric-type algebras, polyadic-type algebras, and polyadic equality-type algebras, all of dimension \( n \). They differ in their signatures and notion of representation. Here, we define them formally and recall some aspects of duality and canonicity for them.

2.1. Signatures and algebras

Definition 2.1. We let

1. \( L_{BA} = \{+, -, 0, 1\} \) denote the signature of boolean algebras,
2. \( L_{CA_n} = L_{BA} \cup \{c_i, d_{ij} \mid i, j < n\} \) denote the signature of \( n \)-dimensional cylindric algebras,
3. \( L_{Di_n} = L_{BA} \cup \{c_i \mid i < n\} \), denote the signature of \( n \)-dimensional diagonal-free cylindric algebras,
4. \( L_{PA_n} = L_{BA} \cup \{c_i, s_r \mid i, j < n, \sigma : n \to n\} \) denote the signature of \( n \)-dimensional polyadic algebras,
5. \( L_{PEA_n} = L_{BA} \cup \{c_i, d_{ij}, s_r \mid i, j < n, \sigma : n \to n\} \) denote the signature of \( n \)-dimensional polyadic equality-type algebras.

Here, the \( c_i \) (‘cylindrifications’) and \( s_r \) (‘substitutions’) are unary function symbols and the \( d_{ij} \) (‘diagonals’) are constants. By a cylindric-type algebra, we mean simply an algebra of signature \( L_{CA_2} \). Diagonal-free cylindric-type algebras, polyadic-type algebras, and polyadic equality-type algebras are defined analogously for the other signatures.

Our concern in this paper is with representable algebras of these four kinds, but we briefly note that abstract algebras have been defined as well: namely, cylindric algebras, diagonal-free cylindric algebras, polyadic algebras, and polyadic equality algebras. They are algebras of the above types that satisfy in each case a finite set of equations that can be found in [10, 11]. In
particular, cylindric algebras are defined in [10, Definition 1.1.1]. We will not use the formal definition so we do not recall it here, but the proofs of some later lemmas will be easier for readers familiar with basic computations in cylindric algebras. The material in [10, §1] is easily enough for what we need. Readers not so familiar can easily verify our claims directly in the specific algebras we are working with.

2.2. Representations

Natural examples of each kind of algebra arise from algebras of \( n \)-ary relations on a set.

**Definition 2.2.** A polyadic equality set algebra is a polyadic equality-type algebra of the form

\[
(\wp(V), \emptyset, V, \cup, V \setminus \cdot, C_i^U, D_i^U, S_{\sigma}^U | i, j < n, \sigma : n \to n),
\]

where \( U \) is a non-empty set, \( V = {}^nU \), and

1. \( D_i^U = \{ x \in V | x_i = x_j \} \) for \( i, j < n \),
2. \( C_i^U X = \{ x \in V | \exists y \in X \forall j < n(\sigma \neq i \to y_j = x_j) \} \) for \( i < n \) and \( X \subseteq V \),
3. \( S_{\sigma}^U X = \{ x \in V | x \circ \sigma \in X \} \), for \( \sigma : n \to n \) and \( X \subseteq V \).

A polyadic set algebra (cylindric set algebra) is the reduct of a polyadic equality set algebra to the signature \( L_{PA}^n \) (respectively, \( L_{CA}^n \)). Since \( L_{Df}^n \) has no operations connecting two different dimensions, a diagonal-free cylindric set algebra is defined rather differently, as an \( L_{Df}^n \)-algebra of the form

\[
(\wp(V), \emptyset, V, \cup, \setminus, C_i^V | i < n),
\]

where \( U_0, \ldots, U_{n-1} \neq \emptyset \), \( V = \prod_{i<n} U_i \), and \( C_i^V X = \{ x \in V | \exists y \in X \forall j \in n \setminus \{ i \}(y_j = x_j) \} \) for \( i < n \) and \( X \subseteq V \).

The only difference between the polyadic and diagonal-free cylindrifications is what the ‘unit’ \( V \) is. The polyadic operator \( C_i^U \) is the same as the diagonal-free operator \( C_i^V \). In the polyadic case we can simplify the notation to \( C_i^V \), but in the diagonal-free case we have no choice but to write \( C_i^V \).

**Definition 2.3.** An \( L_{PEA}^n \)-algebra is said to be representable if it is isomorphic to a subalgebra of a product of polyadic equality set algebras. The isomorphism is then called a representation. The class of all representable polyadic equality algebras of dimension \( n \) is called \( R_{PEA}^n \).

Exactly analogous definitions are made for \( L_{Df}^n \), \( L_{CA}^n \), and \( L_{PA}^n \), using the appropriate set algebras in each case. The classes of representable algebras for these are, respectively, \( R_{Df}^n \), \( R_{CA}^n \), and \( R_{PA}^n \).

It is known that \( R_{PEA}^n \), \( R_{PA}^n \), \( R_{CA}^n \), and \( R_{Df}^n \) are varieties (elementary classes defined by equations): see, e.g., [11, 3.1.108, 5.1.43]. For \( n \geq 3 \), they are not finitely axiomatisable [19, 14], and indeed we will see this later in Corollary 9.5.

2.3. Atom structures

We now recall a little duality theory, leading to canonicity, the topic of the paper. For more details, see, e.g., [15, 2] and [10, §2.7].
Definition 2.4. Let $L \supseteq L_{BA}$ be a functional signature (i.e., one with only function symbols and constants). We write $L_*$ for the relational signature consisting of a $(k+1)$-ary relation symbol $R_f$ for each $k$-ary function symbol $f \in L \setminus L_{BA}$. By an $(L)$-atom structure, we will simply mean an $L_*$-structure. We will sometimes refer to the elements of an atom structure as atoms.

Given an $L$-atom structure $S = (S, R_f | R_f \in L_*)$, we write $S^*$ for its complex algebra: $S^* = (\varphi(S), f | f \in L)$, where each $f \in L_{BA}$ is interpreted in the natural way as a boolean operation on $\varphi(S)$, and $f(x_1, \ldots, x_k) = \{ s \in S | S \models R_f(x_1, \ldots, x_k, s) \text{ for some } x_1 \in X_1, \ldots, x_k \in X_k \}$, for each $k$-ary $f \in L \setminus L_{BA}$ and $X_1, \ldots, X_k \subseteq S$. We sometimes identify each $s \in S$ with the atom $\{ s \}$ of $S^*$.

For the particular signature $L_{PEA_n}$, we will be defining atom structures in which the $R_f$ are equivalence relations and the $R_f^+$ are functions. So we adopt a slightly different definition of atom structure that is a little easier to specify in practice.

Definition 2.5. A polyadic equality atom structure is a structure

$$S = (S, D_{ij}, \equiv_i, =^\sim | i, j < n, \sigma : n \to n),$$

where $D_{ij} \subseteq S, \equiv_i$ is an equivalence relation on $S$, and $=^\sim : S \to S$ is a function. We regard $S$ as a standard $L_{PEA_n}$-atom structure in the sense of Definition 2.4 by interpreting $R_{d_{ij}}$ as $D_{ij}$, $R_{c_i}$ as $\equiv_i$, and letting $R_{c_i}(s, t)$ iff $t^\sim = s$.

2.4. Canonicity

One source of atom structures is from boolean algebra with operators (BAOs). These originated in [15], where they were called ‘normal BAOs’, and they are now familiar: see, e.g., [2]. Let $L$ be a functional signature containing $L_{BA}$. A class $K$ of $L$-BAO is an $L$-structure whose $L_{BA}$-reduct is a boolean algebra and in which each $f \in L \setminus L_{BA}$ defines a function that is normal (its value is zero whenever any argument is zero) and additive in each argument.

For example, if $S$ is an $L$-atom structure then $S^*$ is an $L$-BAO (note that the constants are vacuously normal and additive). Any algebra in RDF_n, RCA_n, RPA_n, and RPEA_n is easily checked to be a BAO for its signature.

Definition 2.7. Let $B$ be an $L$-BAO. We define the ultrafilter structure $B_\sigma$ to be the $L$-atom structure which has the set of ultrafilters of (the boolean reduct of) $B$ as domain and, for any $k$-ary function symbol $f \in L \setminus L_{BA}$ and $\mu_0, \ldots, \mu_{k-1}, v \in B_\sigma$,

$$B_\sigma \models R_f(\mu_0, \ldots, \mu_{k-1}, v) \iff f[\mu_0, \ldots, \mu_{k-1}] \subseteq v.$$  

The canonical extension of $B$, denoted by $B^\sigma$, is the $L$-BAO $(B_\sigma)^\sigma$. A class $K$ of $L$-BAOs is said to be canonical if $B \in K$ implies $B^\sigma \in K$. A first-order $L$-sentence $\theta$ is said to be canonical if $B \models \theta$ implies $B^\sigma \models \theta$ for every $L$-BAO $B$. Canonical extensions were introduced in [15], where it was shown that there is a canonical embedding of $B$ into $B^\sigma$ given by $b \mapsto \{ v \in B_\sigma | b \in v \}$, so justifying the use of ‘extension’. Canonical extensions of cylindric algebras are studied in [10, §2.7]. Canonical varieties in general have been intensely studied, for example by Goldblatt [7], and it is not hard to derive the following well known result. The proof we give follows [7]: [7, Theorem 4.6] proves by a stronger version of the same method that RCA_n and ICRS_n are canonical varieties for every ordinal $\alpha$. Canonicity of RCA_n is proved in a different way in [10, p.459].
Proposition 2.8. $\text{RD}F_n$, $\text{RCA}_n$, $\text{RPA}_n$, and $\text{RPEA}_n$ are canonical varieties.

Proof. Let $\mathcal{K}_{\text{Fr}}$ be the class of all $(L_{\text{DF}})_n$-structures of the form $\{\prod_{i \in \mathcal{N}} U_i, R_i \mid i < n\}$, where $U_0, \ldots, U_{n-1} \neq \emptyset$ and $R_i((u_0, \ldots, u_{n-1}),(v_0, \ldots, v_{n-1}))$ iff $u_j = v_j$ for each $j \in n \setminus \{i\}$. Let $K_{\text{PEA}}$ be the class of $(L_{\text{PEA}})_n$-structures of the form $\{U,R,R_i \mid i,j < n$, $\sigma : n \to n\}$, where $U \neq \emptyset$, $R_i$ is defined in the same way as above, $R_i^j((u_0, \ldots, u_{n-1}))$ iff $u_j = u_j$, and $R_{j_i}((u_0, \ldots, u_{n-1}),(v_0, \ldots, v_{n-1}))$ iff $u_i = v_{\sigma(i)}$ for each $i < n$. Let $K_{\text{PA}}, K_{\text{CA}}$ be the class of reducts of structures in $K_{\text{PEA}}$ to the signatures $(L_{\text{PA}})_n$ and $(L_{\text{CA}})_n$, respectively.

We now assume familiarity with the notation of [7]. By Theorem 4.5 of [7], if $K$ is a class of atom structures with $\text{PuK} \subseteq \text{HSUD}_K$, then $S \subseteq \text{SU}_K$ is a canonical variety. By Theorem 2.2(2.5) of [7], $\text{PCm} = \text{CMUD}$ and $\text{SU} = \text{UD}_S$, so $S \subseteq \text{SU}_K = \text{SCmSU}_K$. Now let $K \subseteq \{\mathcal{K}_{\text{Fr}}, K_{\text{PEA}}, K_{\text{PA}}, K_{\text{CA}}\}$. Then $K$ is closed under ultraproducts, and under inner substructures (since no structure of the above forms has any proper inner substructures), so $\text{PuK} \subseteq K = \text{SK}$. Consequently, $\text{SCmK}$ — the closure of $\{S^+ \mid S \in K\}$ under subalgebras of products — is a canonical variety. But it follows from the definitions that $\text{SCmK}_{\text{PEA}} = \text{RPEA}_n$, and similar results hold for the other three classes. □

Notwithstanding this proposition, we will show that any first-order axiomatisation of any of these four varieties requires infinitely many non-canonical sentences.

3. Algebras from graphs

Here we will describe how to obtain polyadic equality type algebras from graphs. In this paper, graphs are undirected and loop-free. Recall that a set of nodes of a graph is independent if there is no edge between any two nodes in the set.

3.1. Atom structures from graphs

The first step is given by the following definitions (adapted from [12, Definition 3.5]), which construct a polyadic equality atom structure from a graph.

Notation. We let $E_q(n)$ denote the set of equivalence relations on $n$. If $\sim \in E_q(n)$ and $i < n$, we will write $\sim_i$ for the restriction of $\sim$ to $n \setminus \{i\}$.

Definition 3.1. Let $\Gamma = (V,E)$ be a graph. We let $\Gamma \times n$ denote the graph

$$(V \times n, \{(x,i),(y,j) \in V \times n \mid E(x,y) \lor i \neq j\})$$

consisting of $n$ copies of $\Gamma$ with all possible additional edges between copies.

Each graph $\Gamma$ will give rise to an atom structure whose `atoms’ will essentially be the pairs $(\bar{K},\bar{\sim})$, where $\sim \in E_q(n)$ and $\bar{K} : [n/\sim]^{\#-1} \to \Gamma \times n$ is a map such that if $\sim$ is equality on $n$ then the image of $\bar{K}$ is not an independent set in $\Gamma \times n$. Of course, $\bar{K} = \emptyset$ if $[n/\sim]^{\#-1} = \emptyset$. For notational simplicity, we will actually represent $\bar{K}$ by a possibly partial map $K : n \to \Gamma \times n$ in the following way. For each $j < n$ let $J = [i/\sim : i \in n \setminus \{j\}]$. If $|J| = n - 1$ then $K(j) = \bar{K}(J)$. Otherwise, $K(j)$ is undefined. This leads us to the following formal definition.

Definition 3.2. Fix a graph $\Gamma$. Let $S(\Gamma)$ be the set of all pairs $(\bar{K},\bar{\sim})$, where $K : n \to \Gamma \times n$ is a partial map and $\sim$ an equivalence relation on $n$ that satisfies the following:

1. If $|n/\sim| = n$, then $\text{dom}(K) = n$ and $\text{im}(K)$ is not independent.
2. If \(|n/\sim| = n - 1\), so that there is a unique \(\sim\)-class \([i, j]\) of size 2 with \(i < j < n\), say, then \(\text{dom}(K) = [i, j]\) and \(K(i) = K(j)\).

3. Otherwise, i.e. if \(|n/\sim| < n - 1\), \(K\) is nowhere defined.

For \((K, \sim), (K', \sim') \in S(\Gamma)\) and \(i, j < n\), we will write \(K(i) = K'(j)\) if either \(K(i)\) and \(K'(j)\) are both undefined, or they are both defined and are equal. According to this, if \(i \sim j\) then \(K(i) = K(j)\).

**Definition 3.3.** Let \(i < n\). A relation \(\sim \in \text{Eq}(n)\) is said to be \(i\)-distinguishing if \(\sim^{-1}(j \sim k)\) for all distinct \(j, k \in n \setminus \{i\}\). It is equivalent to saying that \(\sim_i\) is equality on \(n \setminus \{i\}\). A pair \((K, \sim) \in S(\Gamma)\) is said to be \(i\)-distinguishing if \(\sim\) is \(i\)-distinguishing.

**Remark.** If \((K, \sim) \in S(\Gamma)\), then \(K\) is defined on \(i < n\) if and only if \(\sim\) is \(i\)-distinguishing.

**Definition 3.4.** Let \(\Gamma\) be a graph. The polyadic equality atom structure

\[
\text{At}(\Gamma) = (S(\Gamma), D_{ij}, \equiv, \sim, \sigma : n \rightarrow n)
\]

is defined as follows:

1. \(D_{ij} = \{(K, \sim) \in S(\Gamma) \mid i \sim j\} \subseteq S(\Gamma)\), for \(i, j < n\).
2. \(\equiv\) is the equivalence relation on \(S(\Gamma)\) given by: \((K, \sim) \equiv (K', \sim')\) if and only if \(K(i) = K'(i)\) and \(\sim_i = \sim'_i\) for \(i < n\).
3. For each \(\sigma : n \rightarrow n\), the map \(\sim^{-\sigma} : S(\Gamma) \rightarrow S(\Gamma)\) is given by: \((K, \sim)^{\sigma} = (K^\sigma, \sim^\sigma)\), where
   - \(\sim^\sigma \in \text{Eq}(n)\) is defined by \(i \sim^\sigma j\) if \(\sigma(i) \sim \sigma(j)\) for \(i, j < n\),
   - \(K^\sigma(i)\) (for \(i < n\)) is defined if \(\sim^\sigma\) is \(i\)-distinguishing, and in that case, \(K^\sigma(i) = K(j)\), where \(j < n\) is the unique element satisfying \(j \notin \sigma[n \setminus \{i\}\).

We leave it to the reader to check that \((K^\sigma, \sim^\sigma)\) is well defined and in \(S(\Gamma)\), and that \(K^\sigma\) is determined by \(K\) and \(\sigma\) even though we cannot in general recover \(\sim^\sigma\) from them. Note that if \(\sigma\) is one-one then \(K^\sigma = K \circ \sigma\).

**Definition 3.5.** We write \(\mathcal{A}(\Gamma)\) for the \(n\)-dimensional polyadic equality type algebra \(\text{At}(\Gamma)^+\). Explicitly,

\[
\mathcal{A}(\Gamma) = (\wp(S(\Gamma)), \cup, \setminus, \emptyset, S(\Gamma), d_{ij}, c_i, s_\sigma \mid i, j < n, \sigma : n \rightarrow n),
\]

where \(d_{ij} = D_{ij}\) as above, and for \(X \subseteq S(\Gamma)\),

1. \(c_iX = \{(K, \sim) \in S(\Gamma) \mid \exists (K', \sim') \in X((K', \sim') \equiv_i (K, \sim))\}\),
2. \(s_\sigma X = \{(K, \sim) \in S(\Gamma) \mid (K, \sim)^\sigma \in X\}\).

An algebra of the form \(\mathcal{A}(\Gamma)\) will be called an algebra from a graph.

\(\mathcal{A}(\Gamma)\) is the expansion to the signature of polyadic equality algebras of a cylindric-type algebra, also written \(\mathcal{A}(\Gamma)\), that was defined in [12]. So some results proved for it also apply to the \(\mathcal{A}(\Gamma)\) defined above. Here is one (another is in Proposition 4.10 below):

**Proposition 3.6.** Let \(\Gamma\) be a graph. Then the cylindric reduct of \(\mathcal{A}(\Gamma)\) is an \(n\)-dimensional cylindric algebra.

**Proof.** This is proved in [17, Claim 3.4 and displayed line (4)]. \(\square\)
We will need to pick out certain elements of \( A(\Gamma) \), so that all the elements beneath are \( i \)-distinguishing and thus have \( K(i) \) defined on them.

**Definition 3.7.** Let \( A \) be a cylindric-type or polyadic equality-type algebra. For \( i < n \), define

\[
F_i = \prod \{-d_{jk} \mid j < k < n, \ j, k \neq i\}.
\]

We generally take \( F_i \) to be an element of the algebra under consideration (here, \( A \)).

**Remark.** Clearly, for an algebra from a graph \( A(\Gamma) \), \( F_i \) is just the sum of all the \( i \)-distinguishing atoms. For \((K, \sim) \in S(\Gamma)\), \( K(i) \) is defined if \((K, \sim) \in F_i\).

### 4. Algebra-graph systems

Proposition 3.6 establishes a relation between graphs and cylindric algebras. However, we need to study this relationship in a more abstract setting.

#### 4.1. Definitions

**Definition 4.1.** We denote by \( L_{AGS} \) the signature with three sorts \((A, G, B)\) and the following symbols:

1. \( A \)-sorted copies of the function symbols 0, 1, +, −, \( d_{ij}, c_i, s_{\sigma} \) of \( L_{PEA_n} \) for each \( i, j < n \) and \( \sigma: n \rightarrow n \) (with the obvious arities that make \( A \) into a polyadic equality-type algebra);
2. \( B \)-sorted copies of the function symbols 0, 1, +, − of \( L_{BA} \);
3. a binary (graph edge) relation symbol \( E \) on \( G \);
4. a binary relation symbol \( H \) on \( G \);
5. a binary relation symbol \( \in \) between the elements of \( G \) and \( B \);
6. a unary function symbol \( R_i: A \rightarrow B \) for each \( i < n \);
7. a unary function symbol \( S_i: B \rightarrow A \) for each \( i < n \).

We extend the definition of \( F_i \) (Definition 3.7) to \( L_{AGS} \)-structures in the obvious way. Sometimes we will regard \( F_i \) as an \( L_{AGS} \)-term.

**Definition 4.2.** For a graph \( \Gamma \), let \( M(\Gamma) \) be the 3-sorted \( L_{AGS} \)-structure

\[
(\mathcal{A}(\Gamma), \ G \times n, \ \varphi(\Gamma \times n))
\]

with operations defined as follows:

- The \( A \)-sorted and \( B \)-sorted symbols are interpreted on \( \mathcal{A}(\Gamma), \ \varphi(\Gamma \times n) \) in the natural way.
- \( E \) is interpreted as the edge relation on \( \Gamma \times n \).
- We have \( H(x, y) \) if and only if there is \( \ell < n \) such that \( x, y \in \Gamma \times \{\ell\} \).
- The relation \( \in \) denotes membership of elements of \( \Gamma \times n \) in the sets that are elements of \( \varphi(\Gamma \times n) \).
- Finally, we have
  
  \[
  R_i(a) = \{K(i) \mid (K, \sim) \in F_i \cdot a\} \quad \text{for} \ a \in \mathcal{A}(\Gamma),
  \]
  
  \[
  S_i(B) = \{(K, \sim) \in F_i \mid K(i) \in B\} \quad \text{for} \ B \in \varphi(\Gamma \times n).
  \]
A structure of the form $M(\Gamma)$ will be called a \textit{structure from a graph}.

We now define a theory that helps us talk about the subclass of all the $\text{L}_{\text{AGS}}$-structures similar to the ones derived from graphs.

\textbf{Definition 4.3.} A (first-order) $\text{L}_{\text{AGS}}$-formula is said to be $\mathcal{A}$-universal if it is of the form 

\[(\forall x_1, \ldots, x_m : \mathcal{A}) \varphi,\]

where $\varphi$ is an $\text{L}_{\text{AGS}}$-formula with no quantifiers over the $\mathcal{A}$-sort. We define $\mathcal{U}$ to be the set of $\mathcal{A}$-universal sentences that are true in all $\text{L}_{\text{AGS}}$-structures $M(\Gamma)$ for graphs $\Gamma$. An $\text{L}_{\text{AGS}}$-structure $M$ that is a model of $\mathcal{U}$ is called an \textit{algebra-graph system}.

This definition ensures that a good number of first-order statements that hold for algebras and structures from graphs, also hold in any algebra-graph system. It will allow us to prove many statements for algebra-graph systems, by just showing they are expressible by $\mathcal{A}$-universal sentences and hold in $M(\Gamma)$ for every graph $\Gamma$.

\subsection*{4.2. Basic properties of algebra-graph systems}

\textbf{Lemma 4.4.} In any algebra-graph system $M = (\mathcal{A}, G, B)$, the cylindric reduct of $\mathcal{A}$ is a cylindric algebra, and $B$ is a boolean algebra isomorphic to a subalgebra of $\mathcal{P}(G)$.

\textbf{Proof.} We know from Proposition 3.6 that an arbitrary algebra from a graph will satisfy all the axioms for cylindric algebras. These axioms are equations and can be recast in the obvious way as $\mathcal{A}$-universal $\text{L}_{\text{AGS}}$-sentences. Since these sentences hold in every $M(\Gamma)$, they hold in $M$. A similar argument shows that $B$ is a boolean algebra. Since the $\mathcal{A}$-universal sentences

- $\forall B, B' : B(p \in B \leftrightarrow p \in B') \implies B = B'$,
- $\forall p : G(p \in B \land \neg(p \in B))$,
- $\forall B, B' : B \forall p : G(p \in B + B' \leftrightarrow p \in B \lor p \in B')$,
- $\forall B : B \forall p : G(p \in -B \leftrightarrow \neg(p \in B))$

are in $\mathcal{U}$ and so are true in $M$, the function $B \mapsto \{p \in G \mid M \models p \in B\}$ is a boolean embedding from $B$ into $\mathcal{P}(G)$.

So in any algebra-graph system $(\mathcal{A}, G, B)$, Lemma 4.4 allows us to regard $B$ as a boolean algebra of subsets of $G$, and the $\text{L}_{\text{AGS}}$-relation symbol ‘$\in$’ as denoting genuine set membership.

Recall that $F_i = \prod_{j < k \leq n} d_{jk}$ from Definition 3.7.

\textbf{Lemma 4.5.} Let $M = (\mathcal{A}, G, B)$ be an algebra-graph system and let $i, j < n$. Then $F_i \cdot d_{ij} \leq F_j$ holds in $\mathcal{A}$.

\textbf{Proof.} It is enough to prove the lemma for algebras from graphs, as it is clearly a set of $\mathcal{A}$-universal first-order sentences. But this is easy and was done in [12, Lemma 4.2]. It is also easily seen to hold in cylindric algebras, of which $\mathcal{A}$ is one (by Lemma 4.4).

Now we examine the functions $R_i, S_i$.

\textbf{Lemma 4.6.} Let $M = (\mathcal{A}, G, B)$ be an algebra-graph system and let $i, j < n$ be distinct. Then:

(i) If $a, b \in \mathcal{A}$ and $a \leq b$ then $R_i(a) \leq R_i(b)$. 

(ii) If \( b \in \mathcal{A} \) and \( b \leq d_{ij} \) then \( R_i(b) = R_j(b) \).

(iii) The map \( f : \mathcal{A} \to \mathcal{B} \) given by \( f(a) = R_i(a \cdot d_{ij}) \) is a boolean homomorphism satisfying \( f(F_i \cdot d_{ij}) = 1 \).

(iv) If \( \mathcal{A} \) is the relativisation of the boolean reduct of \( \mathcal{A} \) to \( F_i \), then \( S_j : \mathcal{B} \to \mathcal{A} \) is a boolean homomorphism.

(v) If \( \mathcal{B} \) then \( c_i S_j(B) = S_i(B) \).

(vi) If \( a \in \mathcal{A} \) and \( a \leq F_i \), then \( S_i(R_i(a)) \geq a \).

(vii) If \( \mathcal{B} \), then \( R_i(S_j(B)) = B \). (Hence, \( S_j \) is injective and \( R_i \) surjective.)

**Proof.** Since the lemma can be easily expressed by \( \mathcal{A} \)-universal \( L_{AGS} \)-sentences, it is again sufficient to show that it is true for any structure \( M \) of the form \( M(\Gamma) \) from Definition 4.2. Recall from that definition that

\[
R_i(a) = \{ K(i) | (K, \sim) \in F_i \cdot a \} \quad \text{ for } a \in \mathcal{A} = \mathcal{A}(\Gamma),
\]

\[
S_j(B) = \{ (K, \sim) \in F_j | (K(i) \in B) \} \quad \text{ for } B \in \mathcal{B} = \wp(\Gamma \times n).
\]

First observe that:

(‡) For each \( p \in \Gamma \times n \), there is a unique atom \( (K_p, \sim) \in F_i \cdot d_{ij} \) with \( K_p(i) = p \). (Of course, \( K_p \) and \( \sim \) depend also on \( i, j \), which are fixed throughout the lemma.)

For, we may define \( \approx \in Eq(n) \) to be the (unique) \( i \)-distinguishing relation with \( i \sim j \) and define \( K_p \) by

\[
K_p(i) = K_p(j) = p, \quad K_p(k) \text{ undefined if } k \neq i, j.
\]

Then \( (K_p, \sim) \) is certainly a valid element of \( A \mathcal{A}(\Gamma) \) contained in \( F_i \) and \( d_{ij} \) and with \( K_p(i) = p \), and it is clearly the only such atom.

We now prove the lemma. Parts (i)–(ii) are easy and left to the reader.

(iii) It is clear that \( f(0) = 0 \) and \( f(a + b) = f(a) + f(b) \) for all \( a, b \in \mathcal{A} \). Let \( p \in \Gamma \times n \) be arbitrary. For any \( a \in \mathcal{A} \), by (‡) we have \( p \in f(a) \) iff \( (K_p, \sim) \in a \). Hence, \( p \not\in f(a) \) iff \( (K_p, \sim) \not\in a \), iff \( f(-a) \). This shows that \( f(-a) = -f(a) \). Hence also, \( f(1) = 1 \). So \( f \) is a boolean homomorphism. If \( p \in \Gamma \times n \) then \( (K_p, \sim) \in F_i \cdot d_{ij} \) and so \( p \not\in f(F_i \cdot d_{ij}) \). As \( p \) was arbitrary, \( f(F_i \cdot d_{ij}) = 1 \).

(iv) We require that \( S_j(0) = 0 \) and that \( S_j(B + B') = S_j(B) + S_j(B') \) and \( S_j(-B) = F_i - S_j(B) \) for each \( B, B' \in \mathcal{B} \). Recalling the definition of \( S_j(B) \), these requirements are easily checked.

(v) Let \( (K, \sim) \equiv (K', \sim') \in S_j(B) \), so that \( (K', \sim') \in F_i \) and \( K'(i) \in B \). Then \( \sim = \sim' \), so \( (K, \sim) \in F_i \); well as, and \( K(i) = K'(i) \in B \). Consequently, \( (K, \sim) \in S_j(B) \). Hence, \( c_i S_j(B) \subseteq S_j(B) \), and the converse is trivial.

(vi) Let \( (K, \sim) \in a \leq F_i \) be arbitrary. Then \( (K, \sim) \in F_i \), so \( K(i) \) is defined, \( K(i) \in R_i(a) \), and hence

\[
(K, \sim) \in \{ (K', \sim') \in F_i | K'(i) \in R_i(a) \} = S_j(R_i(a)).
\]

This shows that \( a \leq S_j(R_i(a)) \).

(vii) Let \( B \in \mathcal{B} \). First note that

\[
R_i(S_j(B)) = \{ (K(i) | (K, \sim) \in S_j(B) \} = \{ (K(i) | (K, \sim) \in F_i, K(i) \in B \} \subseteq B.
\]

For the converse, let \( p \in B \) be given. By (‡) above, \( (K_p, \sim) \in S_j(B) \), and \( p = K_p(i) \in R_i(S_j(B)) \).

This shows that \( B \subseteq R_i(S_j(B)) \). \( \square \)
Next, we examine the substitution operators.

**Definition 4.7.** For \( \sim \in E\Gamma(n) \) let \( d_\sim = \prod_{i,j \in \varepsilon} d_{ij} \cdot \prod_{i,j \in \varepsilon} \sim d_{ij} \).

**Lemma 4.8.** Let \( M = (\mathcal{A}, G, B) \) be an algebra-graph system, \( a \in \mathcal{A}, i, j < n, \) and \( \sigma, \tau : n \rightarrow n. \)

(i) The map \( s_\sim : \mathcal{A} \rightarrow \mathcal{A} \) is a boolean homomorphism.
(ii) \( s_{\sim \sigma} = s_\sigma \circ s_\sim. \)
(iii) \( s_\sim d_{ij} = d_{\sim(i)\sim(j)}. \)
(iv) If \( \sim \in E\Gamma(n), \) then \( s_\sim d_\sim \trianglelefteq d_\sim. \)
(v) If \( \sigma[n \setminus \{i\}] \neq n \setminus \{j\}, \) then \( R_\sim(s_\sigma a) \leq R_\sim(a). \)
(vi) If \( i \notin \im \sigma \) then \( c_i s_\sigma a = s_\sigma c_i a, \) and if \( \sigma \) is one-one then \( c_{\sim(i)} s_\sigma a = s_\sigma c_i a. \)

**Proof.** Again, it is enough to show that the lemma is true for an arbitrary structure \( M(\Gamma) \) from a graph \( \Gamma, \) as all statements are expressible by \( \mathcal{A} \)-universal first-order sentences.

(i) By the definitions, \( s_\sim \emptyset = \emptyset, s_\sim 1 = 1, \) and for any \( a, b \in \mathcal{A}(\Gamma), s_\sim(a + b) = [k \in S(\Gamma) \mid k^\sigma \in a + b] = [k \mid k^\sigma \in a] \cup [k \mid k^\sigma \in b] = s_\sim a + s_\sim b, \) and \( s_\sim(-a) = [k \in S(\Gamma) \mid k^\sigma \in -a] = S(\Gamma) \setminus [k \mid k^\sigma \in a] = -s_\sim a. \)

(ii) Let \( (K, \sim) \in S(\Gamma) \) be arbitrary. We claim that \( ((K, \sim)^\sigma)^\Gamma = (K, \sim)^{\sigma \Gamma}: \) that is,

\[
((K^\sigma)^\Gamma, (\sim^\sigma)^\Gamma) = (K^{\sigma \Gamma}, (\sim^{\sigma \Gamma})^\Gamma).
\]

For \( i, j < n, \) plainly \( i \sim (\sim)^\Gamma j \iff \tau(i) \sim \tau(j) \iff \sigma(\tau(i)) \sim \sigma(\tau(j)) \iff i \sim (\sim)^\Gamma j, \) so \( (\sim^\sigma)^\Gamma = \sim^{\sigma \Gamma}. \)

Let \( i \sim j. \) Then \( (K^\sigma)^\Gamma(i) \) is defined iff \( (\sim^\sigma)^\Gamma \) is \( i \)-distinguishing, iff \( \sim^{\sigma \Gamma} \) is \( i \)-distinguishing. \( K^{\sigma \Gamma}(i) \) is defined. In that case, \( (K^\sigma)^\Gamma(i) = K^\sigma(j) \) where \( j \notin \tau(n \setminus \{i\}). \) Then \( \sim^\sigma \) is \( j \)-distinguishing and \( K^{\sigma \Gamma}(j) = K(k) \) where \( k \notin \sigma[n \setminus \{j\}]. \) But now, \( k \notin \sigma \circ \tau(n \setminus \{i\}), \) so \( K^{\sigma \Gamma}(i) = K(k) \) as well.

This proves the claim. Consequently, \( s_\sim s_\sigma a = [k \in S(\Gamma) \mid k^\sigma \in s_\sigma a] = [k \mid (k^\sim)^\sigma \in a] = [k \mid k^\sim \sigma^{\Gamma} \in a] = s_\sim s_\sigma a. \)

(iii) We have \( s_\sim d_{ij} = [(K, \sim) \mid (K^\sigma, \sim^\sigma) \in d_{ij}] = [(K, \sim) \mid \sigma(i) \sim \sigma(j)] = d_{\sigma(i)\sigma(j)}. \)

(iv) By (i) and (iii),

\[
s_{\sim} d_\sim = s_{\sim} \left( \prod_{i,j \in \varepsilon, i^\sim = j} d_{ij} \cdot \prod_{i,j \in \varepsilon, i^\sim \neq j} -d_{ij} \right) = \prod_{i,j \in \varepsilon, i^\sim = j} s_{\sim} d_{ij} \cdot \prod_{i,j \in \varepsilon, i^\sim \neq j} -s_{\sim} d_{ij} = \prod_{i,j \in \varepsilon, \sigma(i) \sim \sigma(j)} d_{\sigma(i)\sigma(j)} \cdot \prod_{i,j \in \varepsilon, \sigma(i) \sim \sigma(j)} -d_{\sigma(i)\sigma(j)}.
\]

The last expression comprises some of the conjuncts (all of them, if \( \sigma \) is onto) \( d_\sim \) as required (with equality if \( \sigma \) is onto).

(v) Let \( p \in R_\sim(s_\sigma a), \) so that \( p = K(j) \) for some \( (K, \sim) \in s_\sigma a : F_i. \) Hence, \( (K^\sigma, \sim^\sigma) \in a \), and \( \sim \) is \( j \)-distinguishing. As \( \sigma[n \setminus \{i\}] = n \setminus \{j\}, \) it follows that \( \sim^\sigma \) is \( i \)-distinguishing. So \( (K^\sigma, \sim^\sigma) \in a : F_i, \) and \( K^\sigma(i) \) is defined and is plainly \( K(j), \) i.e. \( p. \) Hence \( p \in R_\sim(a) \) as required.

(vi) Let \( i \notin \im \sigma \) and \( (K, \sim) \equiv (K', \sim) \in s_\sigma a, \) so that \( (K'^\sigma, \sim^\sigma) \in a. \) We show that \( (K'^\sigma, \sim^\sigma) = (K', \sim') \). As \( \sim \equiv \sim' \) and \( i \notin \im \sigma, \) we have \( \sim' \equiv \sim' \). Take \( j < n \) such that \( \sim' \) is \( j \)-distinguishing. Then plainly \( \sigma[n \setminus \{j\}], \) so \( K'^\sigma(j) = K(i) = K'(i) = K'^\sigma(i). \) It follows that \( (K'^\sigma, \sim^\sigma) = (K', \sim') \in a, \) so \( (K, \sim) \in s_\sigma a. \) This proves that \( c_i s_\sigma a \leq s_\sigma c_i a. \) The converse is immediate by Lemma 4.4.

Now suppose that \( \sigma : n \rightarrow n \) is one-one. Then plainly, for any atoms \( (K, \sim), (K', \sim') \), we have \( (K'^\sigma, \sim^\sigma) = (K \circ \sigma, \sim') \), and \( (K, \sim) \equiv_{\sim(i)} (K', \sim') \) iff \( (K \circ \sigma, \sim') \equiv_j (K' \circ \sigma, \sim'). \)
Let \((K, \sim)\) be arbitrary. Then \((K, \sim) \in c_{\sigma(i)} \mathcal{E}_{\mathcal{A}} a\) iff there is \((K', \sim')\) with \((K, \sim) \equiv_{\sigma(i)} (K', \sim')\) and \((K' \circ \sigma, \sim'') \in a\), iff there is \((K', \sim')\) with \((K \circ \sigma, \sim'') \equiv_{i}(K' \circ \sigma, \sim'')\) and \((K' \circ \sigma, \sim'') \in a\), iff there is \((K', \sim')\) with \((K \circ \sigma, \sim'') \equiv_{i}(K', \sim') \in a\), iff \((K \circ \sigma, \sim'') \in c_{\sigma}, \text{iff } (K, \sim) \in s_{\sigma} c_{\sigma} a\) as required.

4.3. Simple algebras

Recall that a cylindric algebra \(\mathcal{A}\) is simple if \(|\mathcal{A}| > 1\) and for any algebra \(\mathcal{A}'\) with cylindric signature, any homomorphism \(\varphi : \mathcal{A} \to \mathcal{A}'\) is either trivial or injective. We will see that the cylindric reduct of the algebra part of an algebra-graph system is simple, so that if it is representable, it has a representation that is just an embedding into a single cylindric set algebra.

**Definition 4.9.** Let \(C\) be a class of BAOs of the same signature \(L\). An \(L\)-term \(d(x)\) satisfying

\[
d(a) = \begin{cases} 1 & \text{if } a > 0, \\ 0 & \text{if } a = 0. \end{cases}
\]

for each \(a \in \mathcal{A} \in C\), is called a discriminator term for \(C\).

**Proposition 4.10.** The term \(c_{1} \ldots c_{n-1}c_{n-1} \ldots c_{1} x\) is a discriminator term for the class of algebras from graphs \([\mathcal{A}(\Gamma) | \Gamma \text{ a graph}]\).

**Proof.** See line (5) in the proof of [12, Lemma 5.1].

We deduce the following in a standard way.

**Corollary 4.11.** In every algebra-graph system \((\mathcal{A}, \mathcal{G}, \mathcal{B})\), the cylindric-type reduct \(\mathcal{A} \upharpoonright L_{\mathcal{A}_{n}}\) of \(\mathcal{A}\) is simple, as is each of its subalgebras.

**Proof.** Let \(\mathcal{A}'\) be an algebra with cylindric signature, \(\mathcal{A}' \subseteq \mathcal{A} \upharpoonright L_{\mathcal{A}_{n}},\) and \(\varphi : \mathcal{A}' \to \mathcal{A}'\) a homomorphism. Since the statement that \(d(x) = c_{1} \ldots c_{n-1}c_{n-1} \ldots c_{1} x\) is a discriminator term is \(\mathcal{A}\)-universal, it follows from Proposition 4.10 that \(d(x)\) is a discriminator term for \(\mathcal{A}'\). Suppose \(\varphi\) is not injective, i.e. there are distinct \(a, b \in \mathcal{A}'\) such that \(\varphi a = \varphi b\). Then \((a - b) + (b - a) \neq 0\) and therefore

\[
\varphi(1) = \varphi d((a - b) + (b - a)) = d((\varphi a - \varphi b) + (\varphi b - \varphi a))
\]

\[
= d((\varphi a - \varphi a) + (\varphi a - \varphi a)) = \varphi d((a - a) + (a - a)) = \varphi d(0) = \varphi(0).
\]

So for any \(a \in \mathcal{A}'\), \(\varphi(a) = \varphi(a + 0) = \varphi(a) + \varphi(0) = \varphi(a) + \varphi(1) = \varphi(a + 1) = \varphi(1)\). Thus \(\varphi\) is trivial if it is not injective.

**Lemma 4.12.** Let \(\mathcal{A} \in \mathcal{RCA}_{n}\) be a representable cylindric algebra. If \(\mathcal{A}\) is simple, then it has a representation that is an embedding into a single cylindric set algebra.

**Proof.** There is a representation \(h : \mathcal{A} \to \prod_{k \in K} S_{k}\), where \(K\) is an index set and for each \(k \in K\), \(S_{k}\) is a non-empty base set and

\[
S_{k} = (\varphi^{0}(S_{k}), \cup, \setminus, 0, 0_{S_{k}}, D_{ij}^{S_{k}}, C_{i}^{S_{k}} | i, j < n)
\]

Because \(h\) is injective and \(|\mathcal{A}| > 1\), the index set \(K\) is non-empty. So choose \(\ell \in K\) and let \(\pi\) be the projection of \(\prod_{k \in K} S_{k}\) onto \(S_{\ell}\). Then \(\pi \circ h\) is certainly a homomorphism and because

\[
\pi \circ h(1) = a S_{\ell} 0 = \pi \circ h(0),
\]

it is non-trivial. But because \(\mathcal{A}\) is simple, \(\pi \circ h : \mathcal{A} \to S_{\ell}\) is injective and thus a representation that is an embedding into a single cylindric set algebra.
5. Ultrafilters

We now examine ultrafilters in algebra-graph systems.

5.1. Ultrafilter structures from algebra-graph systems

If \( M = (\mathcal{A}, G, \mathcal{B}) \) is an algebra-graph system then \( \mathcal{A} \) is an \( L_{\text{PEA}} - \text{BAO} \), since this statement is expressible by an \( \mathcal{A} \)-universal sentence true in every structure from a graph. So its ultrafilter structure \( \mathcal{A}_e \) (see Definition 2.7) is defined; it satisfies

\[
\begin{align*}
R_{d_i}(v) & \iff d_i \in v, \\
R_{c_i}(\mu, v) & \iff c_i[\mu] = \{ c_i a \mid a \in \mu \} \subseteq v, \\
R_{s_i}(\mu, v) & \iff s_i[\mu] = \{ s_i a \mid a \in \mu \} \subseteq v.
\end{align*}
\]

We view \( \mathcal{A}_e \) as a polyadic equality atom structure (Definition 2.5) by defining

\[
\mu \equiv_i v \iff R_{c_i}(\mu, v), \quad \forall \nu \in c_\nu \sigma.
\]

Lemma 5.1, the comment following it, and Lemma 5.3(v) show that this gives a well-defined polyadic equality atom structure which, when regarded as an \( L_{\text{PEA}} \)-atom structure as in Definition 2.5, yields the ultrafilter structure \( \mathcal{A}_e \) as above.

**Lemma 5.1.** Let \( M = (\mathcal{A}, G, \mathcal{B}) \) be an algebra-graph system and \( \sigma, \tau : n \to n \). Then for any ultrafilter \( v \) of \( \mathcal{A} \), the set \( v^\sigma \) is also an ultrafilter of \( \mathcal{A} \), and \( v^{\sigma \tau} = (v^\sigma)^\tau \).

**Proof.** By Lemma 4.8, \( s_\sigma : \mathcal{A} \to \mathcal{A} \) is a boolean homomorphism. It is well known and easily seen that for boolean algebras \( \mathcal{B}_1, \mathcal{B}_2 \), the preimage of an ultrafilter of \( \mathcal{B}_2 \) under a boolean homomorphism \( f : \mathcal{B}_1 \to \mathcal{B}_2 \) is an ultrafilter of \( \mathcal{B}_1 \). So \( v^\sigma \) is an ultrafilter of \( \mathcal{A} \). By Lemma 4.8(ii),

\[
\forall \nu \in c_\nu \sigma, \quad v^\sigma = \{ a \in \mathcal{A} \mid s_\nu(s_{\tau \nu}a) \in v \} = \{ a \in \mathcal{A} \mid s_{\tau \nu}a \in v^{\sigma \tau} \} = (v^\sigma)^\tau.
\]

It follows that \( R_{s_\nu}(\mu, v) \) if \( \mu \subseteq v^\sigma \), iff \( v^\sigma = \mu \) since both are ultrafilters.

5.2. Projections of ultrafilters

**Definition 5.2.** Let \( M = (\mathcal{A}, G, \mathcal{B}) \) be an algebra-graph system, let \( \mu \) be an ultrafilter of \( \mathcal{A} \), and let \( i \leq n \). We write \( \mu(i) \) for the set \( R_{i}[\mu] = \{ R_i(a) \mid a \in \mu \} \subseteq \mathcal{B} \) — the ‘\( i \)th projection of \( \mu \)’. We say that \( \mu \) is \( i \)-distinguishing if it contains \( F_i \).

Clearly, \( \mu \) is \( i \)-distinguishing iff it does not contain any of the \( d_{i_\beta} \) for distinct \( j, k \in n \setminus \{ i \} \). In this case, \( \mu(i) \) turns out to be an ultrafilter of \( \mathcal{B} \). The following lemma establishes this and other facts about projections of ultrafilters.

**Lemma 5.3.** Let \( M = (\mathcal{A}, G, \mathcal{B}) \) be an algebra-graph system, let \( i \leq n \), and let \( \mu, \nu \) be ultrafilters of \( \mathcal{A} \).

(i) The projection \( \mu(i) \) is an ultrafilter of \( \mathcal{B} \) if \( \mu \) is \( i \)-distinguishing, and \( \mathcal{B} \) (that is, the improper filter on \( \mathcal{B} \)), otherwise.

(ii) If \( j < n \) and \( d_{i_j} \in \mu \), then \( \mu(i) = \mu(j) \).

(iii) If \( i \neq j < n \) and \( \beta \) is an ultrafilter of \( \mathcal{B} \), then \( \alpha = \{ a \in \mathcal{A} \mid R_i(a \cdot d_{i_j}) \in \beta \} \) is the unique ultrafilter of \( \mathcal{A} \) with \( F_i, d_{i_j} \in \alpha \) and \( \alpha(i) = \beta \).
(iv) $\mu \equiv_{i} v$ iff $(a) d_{jk} \in \mu$ iff $d_{jk} \in v$ for all $j, k \in n \setminus \{i\}$, and $(b) \mu(i) = v(i)$.

(v) $\equiv_{i}$ is an equivalence relation on $\mathcal{A}_{i}$.

(vi) If $\sigma : n \to n$ and $\sigma[n \setminus \{i\}] = n \setminus \{j\}$, then $\mu^{\sigma}(i) = \mu(j)$.

Proof. (i) If $F_{i} \in \mu$, then $\mu(i) = \{B \in \mathcal{B} \mid S_{i}(B) \in \mu\}$. For, if $S_{i}(B) \in \mu$ then by Lemma 4.6(vii), $B = R_{i}(S_{i}(B)) \in \mu(i)$. Conversely, if $B \in \mu(i)$ then $B = R_{i}(a)$ for some $a \in \mu$ with $a \leq F_{i}$ (since $F_{i} \in \mu$). By Lemma 4.6(vii), $S_{i}(B) = S_{i}(R_{i}(a)) \geq a$ so $S_{i}(B) \in \mu$. Let $\mathcal{A}_{i}$ be the relationalisation of the boolean reduct of $\mathcal{A}$ to $F_{i}$. By lemma 4.6(iv), $S_{i} : \mathcal{B} \to \mathcal{A}_{i}$ is a boolean homomorphism. Now $F_{i} \in \mu$, so $\mu \cap \mathcal{A}_{i}$ is an ultrafilter of $\mathcal{A}_{i}$. So its preimage under $S_{i}$, namely $\mu(i)$, is an ultrafilter of $\mathcal{B}$.

If $-F_{i} \in \mu$, then for any $B \in \mathcal{B}$ we have $-F_{i} + S_{i}(B) \in \mu$. The statement that $R_{i}(-F_{i} + a) = R_{i}(a)$ for all $a$ is $\mathcal{A}$-universal and true in every structure from a graph, so it holds for $M$. By Lemma 4.6(vii), $R_{i}(-F_{i} + S_{i}(B)) = R_{i}(S_{i}(B)) = B$. So $B \in \mu(i)$. Since $\bar{B}$ was arbitrary, $\mu(i) = \mathcal{B}$.

(ii) This is obvious if $i = j$, so suppose $i \neq j$. Assume $d_{ij} \in \mu$. Let $R_{i}(a)$ be an element of $\mu(i)$ for some $a \in \mu$. Define $b = a \cdot d_{ij} \in \mu$. It follows from Lemma 4.6(ii),(iii) that $R_{i}(a) \geq R_{i}(b) = R_{i}(b) \in \mu(j)$. By (i), $\mu(j)$ is always a filter, so $R_{i}(a) \in \mu(j)$. Thus $\mu(i) \subseteq \mu(j)$. The converse inclusion holds by symmetry, so $\mu(i) = \mu(j)$.

(iii) By Lemma 4.6(iii), the map $a \mapsto R_{i}(a \cdot d_{ij})$ is a boolean homomorphism from $\mathcal{A}$ to $\mathcal{B}$. As $a$ is the preimage of $\beta$ under this map, it is an ultrafilter of $\mathcal{A}$. The lemma also shows that $R_{i}(F_{i} \cdot d_{ij}) = 1$, so $F_{i} \cdot d_{ij} \in a$. Plainly, $\alpha(i) \subseteq \beta$, so as $\beta$ is an ultrafilter of $\mathcal{B}$, by (i) we have $\alpha(i) = \beta$.

Let $\alpha'$ be any ultrafilter of $\mathcal{A}$ with $F_{i}, d_{ij} \in \alpha'$ and $\alpha'(i) = \beta$. If $a \in \alpha'$, then $a \cdot d_{ij} \in \alpha'$, so $R_{i}(a \cdot d_{ij}) \in \beta$. Hence, $a \in \{a \in \mathcal{A} \mid R_{i}(a \cdot d_{ij}) \in \beta\} = \alpha$. So $\alpha' \subseteq \alpha$, and since both sides are ultrafilters of $\mathcal{A}$, they are equal.

(iv) ($\implies$) Assume $\mu \equiv_{i} v$. For each $j, k \neq i$, we have $d_{jk} \in \mu \Rightarrow d_{jk} = c_{i} \cdot d_{jk} \in v$ (these equations are easily established by rewriting them as $\mathcal{A}$-universal sentences true in every structure from a graph, or using basic properties of cylindric algebras: see, e.g., [10, 1.3.3, 1.2.12]). As $\mu$ and $v$ are ultrafilters, this proves (a). Hence also, $F_{i} \in \mu \iff F_{i} \in v$.

We prove (b). If $-F_{i} \in \mu$, part (i) gives $\mu(i) = \mathcal{B} = v(i)$, proving (b). Assume then that $F_{i} \in \mu$. Then $\mu(i)$ and $v(i)$ are ultrafilters by part (i), so it is enough to show $\mu(i) \subseteq v(i)$. Let $B \in \mu(i)$ be arbitrary. Take $a \in \mu$ such that $B = R_{i}(a)$. By assumption, $c_{i}a \in v$. Note that the following holds for all structures from graphs:

$$\forall a : \mathcal{A}(R_{i}(a) = R_{i}(c_{i}a))$$

So $B = R_{i}(a) = R_{i}(c_{i}a) \in v(i)$ as required.

( $\iff$ ) For the converse, assume the hypotheses and let

$$D = \prod_{j \in \mathcal{A}(a), d_{jk} \in \mu} d_{jk} \cdot \prod_{j \in \mathcal{A}(a), d_{jk} \in \mu} -d_{jk},$$

so $D \in \mu \cap v$ by (a). Now the following statement holds in structures from graphs:

$$\forall a, b : \mathcal{A}(0 < a \leq D \land R_{i}(b) \leq R_{i}(a) \Rightarrow b \cdot D \leq c_{i}a).$$

For let $(K, \sim) \in b \cdot D$. If $K(i)$ is defined, then $K(i) \in R_{i}(b) \leq R_{i}(a)$, so we may pick $(K', \sim') \in a$ with $K'(i) = K(i)$. If $K(i)$ is undefined, let $(K', \sim') \in a$ be arbitrary (we use $a > 0$ here). Since
(K, ∼), (K′, ∼′) ∈ D, we have ∼ = ∼′, hence in the second case K′(i) is also undefined and K′(i) = K(i). So (K, ∼) ≡ (K′, ∼′), yielding (K, ∼) ∈ cDa.

Since the statement is \( \mathcal{A} \)-universal, it holds for \( M \). So if \( a ∈ µ \), then \( a · D ∈ µ \) and \( R_c(a · D) ∈ µ(i) = ν(i) \), so \( R_c(a · D) = R_c(b) \) for some \( b ∈ ν \). By the above, \( b · D ≤ c(a · D) ≤ c_1a \), and as \( b · D ∈ ν \), we have \( c_1a ∈ ν \) as well. So \( µ ≡ ν \) by definition.

(v) Immediate from (iv).

(vi) Let \( B ∈ µ^ω(i) \). Then \( B = R_c(a) \) for some \( a ∈ µ^ω \), so \( s_µa ∈ µ \) and \( R_c(s_µa) ∈ µ(j) \). By Lemma 4.8(v), which applies since \( σ[n \setminus {i}] = n \setminus {j} \), we have \( R_c(s_µa) ≤ R_c(a) = B \). As \( µ(j) \) is a filter, \( B ∈ µ(j) \) as well. As \( B \) was arbitrary, \( µ^ω(i) ≤ µ(j) \).

So by part (i), it only remains to show that if \( µ^ω(i) \) is an ultrafilter of \( B \) then so is \( µ(j) \). But as \( σ[n \setminus {i}] = n \setminus {j} \), the definition of \( F_i \) and Lemma 4.8(iii) yield

\[
\sum_{k,j \in \mathbb{V}(i), k \neq l} d_{ik} = \sum_{k,j \in \mathbb{V}(i), k \neq l} d_{ik} = F_j.
\]

So \( F_j ∈ µ^ω \) iff \( s_µF_i ∈ µ \), iff \( F_j ∈ µ \). By part (i), \( µ^ω(i) \) is an ultrafilter of \( B \) iff \( µ(j) \) is.

6. Networks and patch systems

In this section we introduce approximations to representations, called ultrafilter networks. They will be part of the game to construct representations. We will approximate the networks themselves by lower-dimensional objects that we call patch systems.

6.1. Ultrafilter networks

**Definition 6.1.** Let \( X \) be a set, \( i < n \), and \( v ∈ {}^n X \).

1. For \( w ∈ {}^n X \), we say \( v ≡ w \) if \( v_j = w_j \) for all \( j < n, j \neq i \).
2. If \( v_j ≠ v_k \) for all distinct \( j, k ∈ n \setminus {i} \), then \( v \) is called \( i \)-distinguishable.

**Definition 6.2.** Let \( M = (\mathcal{A}, G, \mathcal{B}) \) be an algebra-graph system. A cylindric ultrafilter network over \( \mathcal{A} \) is a pair \( N = (N_1, N_2) \), where \( N_1 \) is a set and \( N_2 : {}^n N_1 → \mathcal{A} \) is a map that satisfies the following for any \( v, w ∈ {}^n N_1 \):

1. For \( i, j < n \), we have \( d_{ij} ∈ N_2(v) \) if and only if \( v_i = v_j \).
2. If \( i < n \) and \( v ≡ w \), then \( N_2(v) ≡ N_2(w) \).

\( N \) is said to be a polyadic ultrafilter network if in addition:

3. For each \( σ : n → n \), we have \( N_2(σ ◦ v) = N_2(v)^ω \).

If \( N = (N_1, N_2) \) and \( M = (M_1, M_2) \) are ultrafilter networks, we write \( N ⊆ M \) to denote \( N_1 ⊆ M_1 \) and \( M_2 \upharpoonright {}^n N_1 = N_2 \). For a chain \( N^1 ⊆ N^2 ⊆ N^3 \) of ultrafilter networks \( N^k = (N_1^k, N_2^k) \), we write \( \bigcup_{k ∈ \omega} N_k \) for the ultrafilter network \( (\bigcup_{k ∈ \omega} N_1^k, \bigcup_{k ∈ \omega} N_2^k) \) (here we view the maps \( N_2^k \) formally as sets of ordered pairs). We will often write \( N \) for both \( N_1 \) and \( N_2 \).
6.2. Patch systems

Patch systems provide a way to assign ultrafilters on a graph to \((n-1)\)-sized subsets, or 'patches', of a set of nodes.

**Definition 6.3.** Let \(M = (\mathcal{A}, \mathcal{G}, \mathcal{B})\) be an algebra-graph system. A patch system for \(\mathcal{B}\) is a pair \(\mathcal{P} = (P_1, P_2)\), where \(P_1\) is a set and \(P_2 : [P_1]^{n-1} \to \mathcal{B}\) assigns an ultrafilter of \(\mathcal{B}\) to each subset of \(P_1\) of size \(n-1\). (If \(|P_1| < n-1\), then \(P_2 = \emptyset\).) A set \(V = \{v_0, \ldots, v_{n-1}\} \in [P_1]^n\) is said to be \(\mathcal{P}\)-coherent if the following is satisfied: For any \(B_i \in P_2(V \setminus \{v_i\})\) \((i < n)\), there are \(p_i \in \mathcal{G}\) with \(p_i \in B_i\) for each \(i < n\), such that \([p_0, \ldots, p_{n-1}]\) is not an independent subset of \(\mathcal{G}\). The patch system \(\mathcal{P}\) is said to be coherent if every set \(V \subseteq P_1\) of size \(n\) is \(\mathcal{P}\)-coherent.

**Lemma 6.4.** Let \(M = (\mathcal{A}, \mathcal{G}, \mathcal{B})\) be an algebra-graph system and \(\mathcal{P} = (P_1, P_2)\) a patch system for \(\mathcal{B}\). Let \(V = \{v_0, \ldots, v_{n-1}\} \in [P_1]^n\) and for each \(i < n\), let \(V_i = V \setminus \{v_i\}\). Then \(V\) is \(\mathcal{P}\)-coherent if and only if there exists an ultrafilter \(\mu\) of \(\mathcal{A}\) that is \(i\)-distinguishing and with \(\mu(i) = P_2(V_i)\) for each \(i < n\).

**Proof.** \((\implies)\) Assume \(V\) is \(\mathcal{P}\)-coherent. Define

\[
\mu_0 = \{S_i(B) \mid i < n, B \in P_2(V_i)\} \subseteq \mathcal{A}.
\]

We claim that \(\mu_0\) has the finite intersection property. By lemma 4.6(iv) and because \(P_2(V_i)\) is an ultrafilter of \(\mathcal{B}\), for each \(i < n\) the set \(\{S_i(B) \mid B \in P_2(V_i)\}\) is closed under finite intersection. So it is sufficient to consider arbitrary \(B_i \in P_2(V_i)\) and prove that \(S_0(B_0) \cdot S_1(B_1) \cdots S_{n-1}(B_{n-1}) \neq 0\). By the \(\mathcal{P}\)-coherence of \(V\), we can find \(p_i \in B_i\) for each \(i < n\) such that \([p_0, \ldots, p_{n-1}]\) is not an independent set. Now the following \(\mathcal{A}\)-universal sentence holds in structures \(M(\Gamma)\), because there is an atom \((K, \sim)\) that is \(i\)-distinguishing and such that \(K(i) = p_i\), for each \(i < n\):

\[
\forall B_0, \ldots, B_{n-1} : \mathcal{B} \left[ \exists p_0, \ldots, p_{n-1} : \mathcal{G} \left( \bigwedge_{i < n} p_i \in B_i \wedge \bigvee_{i < j < n} E(p_i, p_j) \right) \Rightarrow \bigwedge_{i < n} S_i(B_i) > 0 \right].
\]

We showed that the left hand side of the implication is satisfied, so the right hand side gives us that \(\mu_0\) has the finite intersection property, as claimed.

By the boolean prime ideal theorem, \(\mu_0\) extends to an ultrafilter \(\mu\) of \(\mathcal{A}\). Since plainly \(F_i = S_i(1^n) \in \mu\), we have that \(\mu\) is \(i\)-distinguishing for all \(i < n\). Moreover, if \(B \in P_2(V_i)\), then \(S_i(B) \in \mu_0 \subseteq \mu\), so by Lemma 4.6(vii), \(B = R_i(S_i(B)) \in \mu(i)\). Therefore \(P_2(V_i) = \mu(i)\) by Lemma 5.3(i), since both sides are ultrafilters of \(\mathcal{B}\).

\((\iff)\) Assume \(\mu\) is an ultrafilter of \(\mathcal{A}\) that is \(i\)-distinguishing for all \(i < n\) and with \(\mu(i) = P_2(V_i)\) for each \(i < n\). Choose arbitrary \(B_i \in P_2(V_i)\) for each \(i < n\). For each \(i < n\), we can choose \(b_i \in \mu\) such that \(R_i(b_i) = B_i\). Let \(b = \prod_{i < n} (b_i \cdot F_i) \in \mu\). Now the following \(\mathcal{A}\)-universal sentence holds by definition in structures from graphs, because we can take \((K, \sim) \in x\), and then \(im K\) is not independent and \(K(i) \in R_i(x)\) for each \(i:\)

\[
\forall x : \mathcal{A} \left[ 0 < x \leq \prod_{i < n} F_i \Rightarrow \exists p_0, \ldots, p_{n-1} : \mathcal{G} \left( \bigwedge_{i < n} p_i \in R_i(x) \wedge \bigvee_{i < j < n} E(p_i, p_j) \right) \right].
\]

So we can choose \(p_0, \ldots, p_{n-1}\) with \(p_i \in R_i(b) \subseteq R_i(b_i) = B_i\) and such that \([p_0, \ldots, p_{n-1}]\) is not independent. We conclude that \(V\) is \(\mathcal{P}\)-coherent. \(\square\)
6.3. Patch systems from cylindric networks

Here we show how to construct a coherent patch system from a cylindric ultrafilter network. We will need the following lemma to show that it is well defined. We adopt the standard notation that if \( i, j < n \) then \([i/j] : n \to n\) denotes the function given by \([i/j](i) = j\) and \([i/j](k) = k\) for \( k \neq i \).

**Lemma 6.5.** Let \( M = (\mathcal{A}, \mathcal{G}, \mathcal{B})\) be an algebra-graph system and \( N = (N_1, N_2)\) a cylindric ultrafilter network over \( \mathcal{A} \). Let \( i, j < n \) and \( v, w \in \mathcal{N}_1\). Then:

(i) \( N_2(v) \) is \( i\)-distinguishing if and only if \( v \) is \( i\)-distinguishing.

(ii) \( v \) is \( i\)-distinguishing iff \( v \circ [i/j] \) is \( j\)-distinguishing.

(iii) \( N_2(v)(i) = N_2(v \circ [i/j])(j)\).

(iv) If \([v_k \mid i \neq k < n] = [w_k \mid j \neq k < n]\) then \( N_2(v)(i) = N_2(w)(j)\).

**Proof.** (i) We have that \( N_2(v) \ni F_i \) if and only if it does not contain \( d_k \) for \( j < k < n \) and \( j, k \neq i \). But this is true if and only if \( v \) is \( i\)-distinguishing by the definition of cylindric ultrafilter networks.

(ii) Observe that \([v_k \mid i \neq k < n]\) is \( i\)-distinguishing if and only if \( v \) is \( i\)-distinguishing, and by part (i) and Lemma 5.3(i), \( N_2(v)(i) = \mathcal{B} = N_2(w)(j)\) as required. So assume that \( v \) is \( i\)-distinguishing, and hence that \( w \) is \( j\)-distinguishing. We may suppose without loss of generality that \( i = j = 0 \) (by (ii,iii)), we can just replace \( v \) by \( v \circ [i/0] \) and \( w \) by \( w \circ [j/0] \).

The proof is by induction on the highest number \( v, w \) disagree on: \( d(v, w) = \max\{k < n \mid v_k \neq w_k\} \). If they agree on everything or \( d(v, w) = 0 \), then \( v \equiv_0 w \), so \( N_2(v) \equiv_0 N_2(w) \) and Lemma 5.3(iv) gives us \( N_2(v)(0) = N_2(w)(0)\).

Assume now that \( d(v, w) = k > 0 \) and the claim holds if \( d(v, w) < k \). Since \([v\ell \mid 0 \neq \ell < n] = [w\ell \mid 0 \neq \ell < n]\), \( w_j = v_j \) for some \( 0 < j < n \). We have \( j \neq k \) by definition of \( k \). If \( j > k \), then \( w_j = v_j = w_k \), contradicting that \( w \) is \( 0\)-distinguishing. So \( 0 < j < k \). Now ‘swap’ the \( k \) and \( j \) entries of \( v \) — that is, define \( v' = v \circ [0/k] \circ [k/j] \circ [j/0] \).

By (iii), \( N_2(v)(0) = N_2(v')(0) \). By (ii), \( v' \) is also \( 0\)-distinguishing, and clearly \([v'\ell \mid 0 \neq \ell < n] = [w\ell \mid 0 \neq \ell < n]\). Also \( v'_j = v_j = w_k \) and \( v'_j = w_j \) for all \( \ell > k \), so \( d(v', w) < k \). So, using the induction hypothesis, we get \( N_2(v)(0) = N_2(v')(0) = N_2(w)(0) \).

The last part in the above lemma says that the \( i\)th projection is independent from the \( i\)th coordinate and the order of the elements in the vector. This allows us to define the following:

**Definition 6.6.** Let \( M = (\mathcal{A}, \mathcal{G}, \mathcal{B})\) be an algebra-graph system and \( N = (N_1, N_2)\) a cylindric ultrafilter network over \( \mathcal{A} \). We define \( \partial \mathcal{N} \) to be the patch system \( (N_1, P_2) \), where:

\[
P_2 : [N_1]^{n-1} \to \mathcal{B}_+, \quad \{v_0, \ldots, v_{i-1}, v_{i+1}, \ldots, v_{n-1}\} \mapsto N_2(v)(i),
\]
for each $i < n$ and $i$-distinguishing $v \in \#N_1$.

**Proposition 6.7.** Let $M = (\mathcal{A}, G, \mathcal{B})$ be an algebra-graph system and $N = (N_1, N_2)$ a cylindric ultrafilter network over $\mathcal{A}$. Then $\partial N$ is a well defined and coherent patch system for $\mathcal{B}$.

**Proof.** Let $\partial N = (N_1, P_2)$ as above. By Lemma 6.5(iv), $P_2([v_0, \ldots, v_{i-1}, v_{i+1}, \ldots, v_{n-1}]) = N_2(v)(i)$ is independent of the choice of $v_i$. By (i) of the lemma, $N_2(v)$ is $i$-distinguishing, so by Lemma 5.3(i), $N_2(v)(i) \in B_n$. So $\partial N$ is well defined. Let $V = \{v_0, \ldots, v_{n-1}\} \in [N_1]^{v_0}$. By Lemma 6.5(i), $\mu$ is $i$-distinguishing, and by definition of $\partial N$, $P_2(V_i) = \mu(i)$, for every $i < n$. By Lemma 6.4, $V$ is $\partial N$-coherent. As $V$ was arbitrary, $\partial N$ is coherent.

6.4. Polyadic networks from patch systems

A patch system contains a lot of the information in an ultrafilter network. Here we show that given a coherent patch system $P = (P_1, P_2)$, we can always find ultrafilters to assign to $n$-tuples of $P_1$ respecting $P_2$, and under fairly minimal conditions, they form a polyadic ultrafilter network.

**Lemma 6.8.** Let $M = (\mathcal{A}, G, \mathcal{B})$ be an algebra-graph system and $P = (P_1, P_2)$ a coherent patch system for $\mathcal{B}$. Let $v \in \#P_1$. Then there is an ultrafilter $\mu$ of $\mathcal{A}$ such that

1. For $i, j < n$, we have $d_{ij} \in \mu$ if and only if $v_i = v_j$.
2. $\mu(i) = P_2([v_j \mid j \in n \setminus \{i\}])$ for each $i < n$ such that $v$ is $i$-distinguishing.

**Proof.** There are three cases.

(a) If $|im(v)| = n$, then by Lemma 6.4 there is an ultrafilter $\mu$ of $\mathcal{A}$ that is $i$-distinguishing and with $\mu(i) = P_2([v_j \mid i \neq j < n])$, for all $i < n$.

(b) If $|im(v)| = n - 1$, there are unique $i < j < n$ such that $v_i = v_j$, and $v$ is $k$-distinguishing iff $k \in \{i, j\}$. By Lemma 5.3(iii),

$$\mu = \{a \in \mathcal{A} \mid R_i(a \cdot d_{ij}) \in P_2(im(v))\}$$

is an ultrafilter of $\mathcal{A}$ with $F_i, d_{ij} \in \mu$ (and hence $F_j \in \mu$ by Lemma 4.5), so for each $k, l < n$ we have $d_{il} \in \mu$ iff $v_k = v_l$. Also, $\mu(i) = P_2(im v_i)$. By Lemma 5.3(ii), $\mu(j) = P_2(im v_j)$ as well.

(c) If $|im(v)| < n - 1$, define $D = \prod_{i < j \in n \setminus \{i\}} d_{ij} \cdot \prod_{i < j \in n \setminus v_i} -d_{ij}$. In an algebra from a graph, $D$ would just be $\{0, \sim\}$ where $i \sim j$ if and only if $v_i = v_j$. So $D$ is an atom of $\mathcal{A}$, since this statement is $\mathcal{A}$-universal. We define $\mu$ to be the principal ultrafilter of $\mathcal{A}$ generated by $D$. Condition 2 holds vacuously as $v$ is never $i$-distinguishing.

**Lemma 6.9.** Let $M = (\mathcal{A}, G, \mathcal{B})$ be an algebra-graph system and $P = (N_1, P)$ a coherent patch system for $\mathcal{B}$. Suppose $N_2 : \#N_1 \to \mathcal{A}_+$ is a function satisfying the following, for any $v \in \#N_1$:

1. For $i, j < n$, we have $d_{ij} \in N_2(v)$ if and only if $v_i = v_j$.
2. $N_2(v)(i) = P([v_j \mid j \in n \setminus \{i\}])$ for each $i < n$ such that $v$ is $i$-distinguishing.
3. If $\sigma : n \to n$ and $v \circ \sigma : n \to N_1$ is one-one, then $N_2(v \circ \sigma) = N_2(v)^\sigma$.

Then $(N_1, N_2)$ is a polyadic ultrafilter network.
Proof. We check the conditions from Definition 6.2 defining ultrafilter networks. The first condition, that $d_{ij} \in N_2(v)$ if and only if $v_i = v_j$ (for $v \in \mathcal{N}_1$ and $i, j < n$), is given to us. It follows that $N_2(v)$ is $i$-distinguishing iff $v$ is $i$-distinguishing.

For the second condition, take $i < n$ and $v, w \in \mathcal{N}_1$ with $v \equiv w$. We require $N_2(v) \equiv_i N_2(w)$. By assumption (2) of the lemma, if $v, w$ are $i$-distinguishing we have

$$N_2(v)(i) = P_2((v_j \mid i \neq j < n)) = P_2((w_j \mid i \neq j < n)) = N_2(w)(i),$$

and if they are not, then by Lemma 5.3(i) we have $N_2(v)(i) = \mathcal{B} = N_2(w)(i)$. So by Lemma 5.3(iv), $N_2(v) \equiv_i N_2(w)$.

Lastly we check the third condition for ultrafilter networks. Let $\sigma : n \to n$, let $w = v \circ \sigma$, and let $\sim \in Eq(n)$ be given by $i \sim j$ iff $v_i = v_j$. Observe that $i \sim \sigma$ iff $w_i = w_j$. We check that

$$N_2(w) = N_2(v)^\sigma.$$

There are three cases. If $|n/\sim|=n=1$, then $v \circ \sigma$ is one-one and the result is given.

Suppose that $|n/\sim|=n-1$. Let $[i, j]$ be the unique $\sim$-class of size 2. By condition 1 of the lemma, $F_i, d_{ij} \in N_2(w)$. Also, if $k, l < n$ then $d_{kl} \in N_2(w)$ iff $k \sim \sigma \sim l$, iff $v_{\sigma(k)} = v_{\sigma(l)}$, iff $s_{\sigma}d_{kl} = d_{\sigma(k)\sigma(l)} \in N_2(v)$ by Lemma 4.8(iii), iff $d_{kl} \in N_2(v)^\sigma$. Therefore, $F_i, d_{ij} \in N_2(v)^\sigma$ as well.

So by the uniqueness part of Lemma 5.3(iii), it remains only to show that $N_2(w)(i) = (N_2(v)^\sigma)(i)$.

Now if $k, l \in n \setminus [i]$ and $\sigma(k) = \sigma(l)$, then certainly $k \sim \sigma \sim l$, so $k = l$ by assumption on $\sim\sigma$. Hence, $\sigma$ is one-one on $n \setminus [i]$, so $\sigma[n \setminus [i]] = n \setminus [l]$ for some $l < n$. We now obtain

$$N_2(w)(i) = P_2((w_k \mid k \in n \setminus [i])) \quad \text{by condition 2, since } w \text{ is } i\text{-distinguishing}$$

$$= P_2((v_{\sigma(k)} \mid k \in n \setminus [i])) \quad \text{by definition of } w$$

$$= P_2((v_k \mid k \in n \setminus [i])) \quad \text{since } \sigma[n \setminus [i]] = n \setminus [l]$$

$$= N_2(v)(l) \quad \text{by condition 2, since } v \text{ is } l\text{-distinguishing}$$

$$= (N_2(v)^\sigma)(i) \quad \text{by Lemma 5.3(vi)}.$$

Finally suppose that $|n/\sim| < n-1$. Then $d_{\sim}$ is an atom of $\mathcal{A}$ — this is true in algebras from graphs, because we have $d_{\sim} = \{0, \sim\}$, so it holds for $M$ since the statement is $\mathcal{A}$-universal.

So $N_2(w)$ is the principal ultrafilter generated by $d_{\sim}$.

Let $a \in N_2(v)^\sigma$ be arbitrary, so that $s_{\sigma}a \in N_2(v)$. By the first part, $d_{\sim} \in N_2(v)$, so $s_{\sigma}a \cdot d_{\sim} > 0$. By Lemma 4.8(i) and (iv), $s_{\sigma}(a \cdot d_{\sim}) = s_{\sigma}a \cdot s_{\sigma}d_{\sim} \geq s_{\sigma}a \cdot d_{\sim} > 0$, so $a \cdot d_{\sim} > 0$ as well. As $d_{\sim}$ is an atom, we obtain $a \geq d_{\sim}$ and $a \in N_2(w)$. This shows that $N_2(v)^\sigma \subseteq N_2(w)$, and equality follows since by Lemma 5.1 both sides are ultrafilters of $\mathcal{A}$.

\[\square\]

7. Chromatic number and representability

Here we show that the chromatic number of a graph $\Gamma$ and the representability of $\mathcal{A}(\Gamma)$ and its reducts are tied together.

Recall that the chromatic number $\chi(\Gamma)$ of a graph $\Gamma$ is the size of the smallest partition of the set of nodes of $\Gamma$ into finitely many independent sets, or $\infty$ if no such partition exists. Although the chromatic number is in general not first-order definable, we can define an analogue for algebra-graph systems with the following formula.
Definition 7.1. For each $k < \omega$, we define the following $L_{AGS}$-sentence:
\[
\theta_k = \forall B_0, \ldots, B_{k-1}: \mathcal{B}\left( \sum_{i \leq k} B_i = 1 \rightarrow \exists p, q : \mathcal{G}(p(q) \land \bigvee_{i \leq k} (p \in B_i \land q \in B_i)) \right),
\]
and $\Theta = \{ \theta_k | k < \omega \}$.

Then $M = (A, G, \mathcal{B}) \models \theta_k$ iff the chromatic number of $\mathcal{G}$ is larger than $k$ ‘as far as $\mathcal{B}$ can tell’. The true chromatic number of $\mathcal{G}$ may in principle be smaller, but $\mathcal{B}$ contains no independent sets witnessing this. However, $\mathcal{B}$’s estimate is correct when $\mathcal{B} = \varphi(\mathcal{G})$, as in structures of the form $M(\Gamma)$, and in a number of other circumstances too.

Definition 7.2. If $M = (A, G, \mathcal{B})$ is an algebra-graph system, we will say an element $B \in \mathcal{B}$ is an independent set, if there are no $p, q \in B$ such that $E(p, q)$.

7.1. Representable implies infinite chromatic number

This direction can be proved without further help, apart from some of the machinery from the preceding section and Ramsey’s theorem. It generalises [12, Proposition 5.4].

Theorem 7.3. Let $M = (A, G, \mathcal{B})$ be an algebra-graph system in which $\mathcal{B}$ is infinite. If the diagonal-free reduct of $A$ is representable, then $M \models \Theta$.

Proof. Suppose for a contradiction that the reduct of $A$ to the signature $L_{D_{\text{D}}}$ of diagonal-free cylindric algebras is representable but $M \not\models \theta_k$ for some $k < \omega$.

Recall (e.g., from [10, §1.6]) that for $a \in A$, $\Delta a = \{ i < n \mid c_i a \neq a \}$. Define $D = \{ a \in A \mid \Delta a \neq n \}$, and let $A'$ be the closure of $D$ under the boolean operations. We first claim that $A'$ is a subalgebra of $A$. By Lemma 4.4, the cylindric reduct of $A$ is a cylindric algebra. By basic cylindric algebra, or by $A$-universal sentences, $\Delta 0 = \Delta 1 = \Delta d_i = \emptyset$ for $i < n$; also, $\Delta d_{ij} = \{ i, j \}$ for distinct $i, j < n$, so since $n \geq 3$, $\Delta d_{ij} \neq n$; finally, if $a \in A'$ and $i < n$ then $i \notin \Delta c_i a$. So all these elements are in $D$ and hence in $A'$. Obviously, $A'$ is closed under $+$ and $\cdot$. By Lemma 4.8(vi), $D$ is closed under each $s_i$, so by Lemma 4.8(i), so is $A'$. This proves the claim.

Now let $N = (A', G, \mathcal{B})$. We claim next that $N$ is a substructure of $M$. Inspecting the function symbols of $L_{AGS}$, it suffices to show that $S_i(B)$ is $A'$ for every $B \in \mathcal{B}$ and $i < n$. But by Lemma 4.6(v), $i \notin \Delta S_i(B)$, so $S_i(B) \in D \subseteq A'$. This proves the claim.

As $M \models \mathcal{U}$ and all sentences in $\mathcal{U}$ are $A$-universal, it follows that $N \models \mathcal{U}$. So $N$ is also an algebra-graph system in which $\mathcal{B}$ is infinite. By Lemma 4.4 and Corollary 4.11, the cylindric reduct $A' \upharpoonright L_{Cn}$ is a simple cylindric algebra. It is generated by $D$, and its diagonal-free reduct is representable (since the diagonal-free reduct of $A$ is). It follows from a theorem of Johnson [14, Theorem 1.8(ii)] that $A' \upharpoonright L_{Cn}$ is representable as a cylindric algebra. So by Lemma 4.12, there is a cylindric representation $h$ that embeds $A' \upharpoonright L_{Cn}$ into a single cylindric set algebra $S = (\varphi(\mathcal{S} \cup \emptyset, \cup, \cup, \emptyset), S_i, D^n, C^n) : i < n$ with base set $S$.

Let $N$ be the ultrafilter network with nodes $S$ and $N(s) = \{ a \in A' \mid s \in h(a) \} \in A'$, for $s \in S$. This is easily seen to be a well-defined cylindric ultrafilter network over $A'$. Furthermore, by Proposition 6.7 we can make it into a well-defined and coherent patch system $\partial N$.

Now $M \not\models \theta_k$ means that the following is true in $M$ and therefore $N$:
\[
\exists B_0, \ldots, B_{k-1} : \mathcal{B}\left( \sum_{i \leq k} B_i = 1 \land \forall p, q : \mathcal{G}(p \land q \rightarrow \neg E(p, q)) \right).
\]
So $G$ is the union of $k$ independent sets from $B$: say, $B_0, \ldots, B_{k-1}$.

Since $B$ is infinite, by Lemma 4.6(vii) $A$ is also infinite. As $h$ is injective, $S$ is infinite and therefore $S$ as well. So we can choose infinitely many pairwise distinct elements $s_0, s_1, \ldots$ from $S$. Now define a map $f: [\omega]^{n-1} \to k$ by letting $f([i_1, \ldots, i_{n-1}])$ be the least $j < k$ such that $B_j \in \partial N([s_{i_1}, \ldots, s_{i_{n-1}}])$. By Ramsey’s theorem [20], we can choose the elements so that $f$ has constant value $c$, say. Now consider $[s_0, \ldots, s_{n-1}]$. Since $f$ is constant, $B_c \in \partial N([s_j | i \neq j < n])$ for all $i < n$. Because $\partial N$ is coherent, we can choose $p_0, \ldots, p_{n-1} \in B_c$ so that $\{p_0, \ldots, p_{n-1}\}$ is not an independent set. But this is impossible since $B_c$ is independent.

\[\square\]

### 7.2. Infinite chromatic number implies representable

For the other direction, we define a game that allows us to build a polyadic representation for $\mathcal{A}$ if $M = (\mathcal{A}, G, B) \models \Theta$ (i.e., $G$ has infinite chromatic number in the sense of $B$).

**Definition 7.4.** Let $M = (\mathcal{A}, G, B)$ be an algebra-graph system. A game $G(\mathcal{A})$ is an infinite sequence of polyadic ultrafilter networks

\[N_0 \subseteq N_1 \subseteq \ldots\]

built by the following rules. There are two players, named $\forall$ and $\exists$. The game begins with the (unique) one-point network $N_0$. There are $\omega$ rounds. In round $t < \omega$, the current network (at the start of the round) is $N_t$, and player $\forall$ chooses an $n$-tuple $v \in {}^n N_t$, a number $i < n$ and an element $a \in \mathcal{A}$ such that $c_i a \in N_t(v)$. The other player $\exists$ then has to respond with a polyadic ultrafilter network $N_{t+1} \supseteq N_t$ such that there is $w \in {}^n N_{t+1}$ with $w \equiv v$ and $a \in N_{t+1}(w)$. She wins the game if she can play a network that satisfies these constraints in each round.

**Lemma 7.5.** Let $M = (\mathcal{A}, G, B)$ be an algebra-graph system. If $\exists$ has a winning strategy in the game $G(\mathcal{A})$, then $\mathcal{A}$ is a representable polyadic equality algebra.

**Proof.** By the downward Löwenheim–Skolem–Tarski theorem (see e.g. [3]), there is a countable elementary subalgebra $\mathcal{A}_0$ of $\mathcal{A}$. Let $\mathcal{N}_0 \subseteq N_1 \subseteq \cdots$ be a play of the game $G(\mathcal{A})$ in which $\forall$ plays every possible move in $\mathcal{A}_0$ and $\exists$ uses her winning strategy in $G(\mathcal{A})$ to respond. Define $\mathcal{N} = \bigcup_{t \in \omega} N_t$. This is certainly a polyadic ultrafilter network over $\mathcal{A}$, as all the $N_t$ are polyadic ultrafilter networks. Now define:

\[h: \mathcal{A}_0 \to (\wp(\mathcal{N}), \cup, \emptyset, {}^n \mathcal{N}, D^N, C^N, S^N | i, j < n, \sigma: n \to n)\]

\[a \mapsto \{ v \in {}^n \mathcal{N} | a \in N_t(v) \}.
\]

It can be checked that $h$ is a homomorphism. Recall from Corollary 4.11 that $\mathcal{A}_0$ is simple. So, since $h(1) = {}^n \mathcal{N} \neq \emptyset = h(0)$, the map $h$ is injective. This shows that $\mathcal{A}_0$ is representable, and because (by Proposition 2.8) $\text{RPEA}_n$ is a variety, $\mathcal{A}$ is representable as well.

\[\square\]

**Remark.** The converse of the lemma also holds, but is not needed here.

**Lemma 7.6.** In any algebra-graph system $M = (\mathcal{A}, G, B)$, $H$ defines an equivalence relation on $G$ with $n$ classes, each of which is in $B$.

**Proof.** The statement that $H$ defines an equivalence relation with $n$ classes is an $\mathcal{A}$-universal statement true in every structure $M(\Gamma)$, and hence it is true in $M$. Each equivalence class is in $B$ since the following $\mathcal{A}$-universal sentence is true in every $M(\Gamma)$, and hence in $M$:

\[\forall x: G \exists B: B \forall y: G(y \in B \leftrightarrow H(x, y)).\]

\[\square\]
Lemma 7.7. Let \( M = (\mathcal{A'}, \mathcal{G}, \mathcal{B}) \) be an algebra-graph system such that \( M \models \Theta \). Let \( X \) be an equivalence class of \( H \). Then there is an ultrafilter \( \nu \) of \( \mathcal{B} \) that contains \( X \) but contains no independent sets.

Proof. Let \( v_0 = \{ B \in \mathcal{B} \mid X\setminus B \text{ is independent} \} \). Then \( v_0 \) contains \( X \) (clearly), and has the finite intersection property: Suppose for a contradiction that for \( B_0, \ldots, B_{k-1} \in v_0 \) we have \( B_0 \cdot B_1 \cdots B_{k-1} = 0 \). Then
\[
X = X - (B_0 \cdot B_1 \cdots B_{k-1}) = (X - B_0) + (X - B_1) + \cdots + (X - B_{k-1}).
\]

So \( X \) is the union of \( k \) independent sets in \( \mathcal{B} \). Now in any structure \( M(\Gamma) \), if an \( H \)-class is the union of \( k \) independent sets in \( \mathcal{B} \), then copies of these sets for every \( H \)-class lie in \( \mathcal{B} \), so that \( \Gamma \) is the union of \( nk \) independent sets in \( \mathcal{B} \) — that is, \( M(\Gamma) \models \neg \theta_{ab} \). This implication is \( \mathcal{A} \)-universal, so it holds in \( M \). Hence, \( M \not\models \theta_{ab} \), a contradiction to \( M \models \Theta \). Thus \( v_0 \) has the finite intersection property and, by the boolean prime ideal theorem, it can be extended to an ultrafilter \( \nu \), which contains \( X \) but no independent set (because it contains the complement).

Remark. The converse of Lemma 7.7 also holds, but is not needed here.

The following ‘converse’ of Theorem 7.3 generalises [12, Proposition 5.2].

Theorem 7.8. Let \( M = (\mathcal{A}, \mathcal{G}, \mathcal{B}) \) be an algebra-graph system. If \( M \models \Theta \), then \( \mathcal{A} \) is representable as a polyadic equality algebra.

Proof. By Lemma 7.5 it is sufficient to show that player \( \exists \) has a winning strategy in the game \( G(\mathcal{A}) \). Suppose we are in round \( i \) and the current polyadic ultrafilter network is \( N_i \). According to the rules, player \( \forall \) chooses \( a \in \mathcal{A}, i < n \) and \( v \in \mathcal{N}_i \) with \( c_i a \in N_i(v) \). The other player \( \exists \) now has to respond with a network \( N_{i+1} \supseteq N_i \) that contains some tuple \( w \in \mathcal{N}_{i+1} \) such that \( v \equiv_i w \) and \( a \in N_{i+1}(w) \). If there is already such a \( w \) in \( N_i \) then she can just respond with the unchanged network \( N_i \). So we assume in the following that there is no such \( w \).

Step 1. Let \( N_{i+1} = N_i \cup \{z\} \), where \( z \notin N_i \) is a new node. Let the tuple \( w \) be defined by \( w \equiv_i v \) and \( w_j = z \). We will first try to find an ultrafilter of \( \mathcal{A} \) for \( w \). To help \( \exists \) win the game, the ultrafilter should contain \( a \). We achieve this by showing that the following set has the finite intersection property:
\[
\mu_0 = \{ a \cup \{ -d_{ij} \mid i \neq j < n \} \cup \{ c_i b \mid b \in N_i(v) \} \}.
\]

Let \( D = \prod_{j \neq i} -d_{ij} \). We claim that \( c_i(a \cdot D) \notin N_i(v) \). Assume for contradiction that \( c_i(a \cdot D) \notin N_i(v) \). Clearly, \( D + \sum_{j \neq i} d_{ij} = 1 \). Therefore, \( c_i a = c_i(a \cdot D) + \sum_{j \neq i} c_i(a \cdot d_{ij}) \in N_i(v) \). So there is \( j \neq i \) such that \( c_i(a \cdot d_{ij}) \in N_i(v) \). Let \( v' = v \circ [i/j] \). Then \( v \equiv v', \) so by definition of ultrafilter networks, \( N_i(v) \equiv N_i(v') \). So \( c_i(a \cdot d_{ij}) = c_i(c_i(a \cdot d_{ij})) \in N_i(v') \) as well. But \( v'_j = v'_j \), and therefore by definition of ultrafilter networks, \( d_{ij} \in N_i(v') \). Thus \( d_{ij} \cdot c_i(a \cdot d_{ij}) \in N_i(v') \). In algebras from graphs (and in cylindric algebras generally) we certainly have
\[
\forall a : \mathcal{A}, d_{ij} \cdot c_i(a \cdot d_{ij}) \leq a
\]

This is \( \mathcal{A} \)-universal, and hence \( a \in N_i(v') \). But this contradicts our assumption that no suitable tuple \( w \) exists in \( \mathcal{N}_i \). So we must have \( c_i(a \cdot D) \in N_i(v) \) as claimed.

Now, if \( \mu_0 \) failed the finite intersection property, there would be \( b_0, \ldots, b_{m-1} \in N_i(v) \) such that \( a \cdot D \cdot c_i b_0 \cdots c_i b_{m-1} = 0 \). Then by cylindric algebra, \( 0 = c_i(a \cdot D \cdot c_i b_0 \cdots c_i b_{m-1}) = c_i(a \cdot D) \cdot c_i b_0 \cdots c_i b_{m-1} \in N_i(v) \), a contradiction. Thus \( \mu_0 \) has the finite intersection property.
By the boolean prime ideal theorem, player $∃$ can choose an ultrafilter $μ$ of $𝒜$ that contains $μ_0$. By construction, $N_i(v) ≡ μ$. Moreover,
$$d_{jk} ∈ μ \iff w_j = w_k \quad (⋆)$$
for all $j, k < n$, because for $j ≠ i$ we have $w_i ≠ w_j$ and $-d_{ij} ∈ μ$, and for $j, k ≠ i$,
$$w_j = w_k \Rightarrow v_j = v_k \Rightarrow d_{jk} ∈ N_i(v) \Rightarrow d_{jk} = c_i d_{jk} ∈ μ,
\quad w_j ≠ w_k \Rightarrow v_j ≠ v_k \Rightarrow -d_{jk} ∈ N_i(v) \Rightarrow -d_{jk} = c_i -d_{jk} ∈ μ.$$

**Step 2.** $∃$ also needs to define ultrafilters for all the remaining new tuples containing $z$. She can do this with the help of the patch system $P = (N_{r+1}, P_2)$, defined as follows.

- For each set of ‘old’ nodes $V ∈ [N_i]^{n-1}$, we define $P_2(V) = ∂N_i(V)$.
- For each $j < n$, define $W_j = \{w_k \mid j ≠ k < n\}$. For each $W_j$ of size $n - 1$, she has to define $P_2(W_j)$.

For the case $j = i$, if $|W_i| = n - 1$ then because $W_i ⊆ N_i$, she already defined $P_2(W_i) = N_i(v)(i) = μ(i)$ (by Lemma 5.3(iv)).

Now consider the $j ≠ i$ with $|W_j| = n - 1$. Then $z ∈ W_j ∉ N_i$. We showed in $(⋆)$ that $μ$ is $j$-distinguishing if $w$ is, so $μ(j)$ is an ultrafilter of $B$ in that case. So we define $P_2(W_j) = μ(j)$.

Note that this is well defined, because if there is $k ≠ j$, such that $W_k = W_j$, then $w_j = w_k$, and thus by $(⋆) d_{jk} ∈ μ$ and by Lemma 5.3(ii), $μ(j) = μ(k)$.

- For the remaining $W ∈ [N_{r+1}]^{n-1}$ that contain $z$, but that are not contained in $im(w)$, we use a single ultrafilter constructed as follows. Recall from Lemma 7.6 that $H$ is an equivalence relation on $G$ with exactly $n$ equivalence classes, say $G_1, \ldots, G_n$, which are contained in $B$.

In structures from graphs we have:
$$∀x, y : G(¬H(x, y) → E(x, y)). \quad (†)$$

As $(†)$ is $𝒜$-universal, it is true for $H$ on $G$. Now each of the $μ(j)$ for $j ≠ i$, if an ultrafilter of $B$, contains exactly one of the $G_j$. There are at most $n - 1$ such $j$, so there must be at least one $G_j$ that is not contained in any $μ(j)$ that is an ultrafilter. We are given that $M ⊨ Θ$, so by Lemma 7.7 there is an ultrafilter $v$ of $B$ containing $G_j$ and no independent sets. We define $P_2(W) = v$ for all the remaining $W ∈ [N_{r+1}]^{n-1}$.

We check that $P$ is a coherent patch system. Let $U = \{u_0, \ldots, u_{n-1}\} ∈ [N_{r+1}]^n$ and write $U_j$ for $U \setminus \{u_j\}$ for each $j < n$. We need to check that $U$ is $P$-coherent:

- If $z ∉ U$, then $U ⊆ N_i$ and $U$ is $P$-coherent because $N_i$ is a polyadic, hence cylindric network, so by Proposition 6.7, $∂N_i$ is coherent.
- If $U = im(w)$, then $U$ is $P$-coherent by Lemma 6.4.
- In the case where $z ∈ U$ and $|U \cap im(w)| = n - 1$, we can find $j, k < n$ such that $z ∈ U_j = U \cap im(w)$ and $z ∈ U_k ∉ im(w)$. Then, by the above, $G_i ∈ v = P_2(U_k)$. Moreover, by the choice of $ℓ$, there is $m ≠ ℓ$ such that $G_m ∈ P_2(U_j)$.

Take any $X_j ∈ P_2(U_j)$ for each $r < n$. Choose $p_r ∈ X_r$, for each $r < n$, with $p_j ∈ X_j \cdot G_m$ and $p_ℓ ∈ X_ℓ \cdot G_ℓ$. Since $ℓ ≠ m$ and therefore $H(p_j, p_ℓ)$ does not hold, we have $E(p_j, p_ℓ)$ by $(†)$. Thus $\{p_0, \ldots, p_{n-1}\}$ is not independent.
• In the remaining cases, $z \in U$ and $|U \cap \text{im}(w)| < n - 1$. Then there are distinct $j, k < n$ such that $z \in U_j, U_k \not\subseteq \text{im}(w)$. So by the above, we have $P_2(U_j) = P_2(U_k) = v$.

Take any $X_r \in P_2(U_r)$ for each $r < n$. Then $X_j, X_k \in v$, and thus $X_j \cdot X_k \in v$ and is therefore not independent. So there are $p_j, p_k \in X_j \cdot X_k$ such that $E(p_j, p_k)$. For the other $s \neq j, k$ just choose any $p_s \in X_s$. Then \{p_0, \ldots, p_{n-1}\} is not independent.

This shows that $\mathcal{P}$ is coherent.

We are nearly ready to define $\mathcal{N}_{i+1}$. First, define an equivalence relation $\simeq$ on the set of one-one tuples in $^{n}\mathcal{N}_{i+1} \setminus ^{n}\mathcal{N}_i$, by: $u \equiv u'$ iff there is a permutation $\sigma$ of $n$ such that $u \circ \sigma = u'$. Choose a representative $u_\varepsilon$ of each $\simeq$-class $\varepsilon$, ensuring that if $w$ is one-one then it is chosen as a representative. We now define an ultrafilter $\mathcal{N}_{i+1}(u)$ of $\mathcal{A}$ for each $u \in ^{n}\mathcal{N}_{i+1}$ as follows.

U1. If $u \in ^{n}\mathcal{N}_i$ we set $\mathcal{N}_{i+1}(u) = \mathcal{N}_i(u)$.

U2. Define $\mathcal{N}_{i+1}(w) = \mu$.

U3. If $u \in ^{n}\mathcal{N}_{i+1} \setminus (^{n}\mathcal{N}_i \cup \{w\})$ is the representative of its $\simeq$-class or is not one-one, we use Lemma 6.8 to choose any ultrafilter $\mathcal{N}_{i+1}(u)$ of $\mathcal{A}$ satisfying the properties of that lemma. (These properties are exactly L1–L2 below.)

U4. Each remaining tuple $u$ is one-one but is not the representative of its $\simeq$-class $\varepsilon$. There is a unique $\sigma : n \rightarrow n$ such that $u = u_\varepsilon \circ \sigma$, and we set $\mathcal{N}_{i+1}(u) = (\mathcal{N}_{i+1}(u_\varepsilon))^{\sigma}$.

We check that $\mathcal{N}_{i+1}$ is a polyadic ultrafilter network. It is sufficient to check that each $u \in ^{n}\mathcal{N}_{i+1}$ satisfies the conditions of Lemma 6.9, namely:

L1. For $j, k < n$, we have $d_{jk} \in \mathcal{N}_{i+1}(u)$ if and only if $u_j = u_k$.

L2. $\mathcal{N}_{i+1}(u)(j) = P_2([u_k | k \in n \setminus \{j\})$ for each $j < n$ such that $u$ is $j$-distinguishing.

L3. If $\sigma : n \rightarrow n$ and $u \circ \sigma : n \rightarrow \mathcal{N}_{i+1}$ is one-one, then $\mathcal{N}_{i+1}(u \circ \sigma) = (\mathcal{N}_{i+1}(u))^{\sigma}$.

If $u \in ^{n}\mathcal{N}_i$ this is immediate because $\mathcal{N}_i$ is a polyadic ultrafilter network and by definition of $\mathcal{P}$. If $u = w$, L1 holds by choice of $\mu$, L2 by definition of $\mathcal{P}$ and because $\mu \equiv \mathcal{N}(v)$, and L3 by U4 above, since $w$ is the representative of its $\simeq$-class. If $u$ is not one-one then L1 and L2 hold by choice of $\mathcal{N}_{i+1}(u)$ in U3, and L3 holds vacuously. All that remains is the case where $u \not\equiv ^{n}\mathcal{N}_i \cup \{w\}$ is one-one. Let $e$ be the $\simeq$-class of $u$, and let $u = u_\varepsilon \circ \tau$ for some (unique) $\tau : n \rightarrow n$. Trivially if $u = u_\varepsilon$, and by U4 otherwise, $\mathcal{N}_{i+1}(u) = (\mathcal{N}_{i+1}(u_\varepsilon))^{\tau}$. Below, $j, k$ range over $n$.

1. For L1, $d_{jk} \in \mathcal{N}_{i+1}(u) = (\mathcal{N}_{i+1}(u_\varepsilon))^{\tau}$ iff $s_{d_{jk}} = d_{\tau(j)\tau(k)} \in \mathcal{N}_{i+1}(u_\varepsilon)$ by Lemma 4.8(iii), iff $(u_{\tau(j)} \cdot u_{\tau(k)}) = (u_{\tau(\tau(j))} \cdot u_{\tau(\tau(k))})$ by choice of $\mathcal{N}_{i+1}(u_\varepsilon)$, iff $u_j = u_k$ as required.

2. We check L2. Suppose that $u$ is $j$-distinguishing. Plainly, $\tau$ is one-one, so $\tau[n \setminus \{j\}] = n \setminus \{\tau(j)\}$. Consequently,

\[ \mathcal{N}_{i+1}(u)(j) = \mathcal{N}_{i+1}(u_\varepsilon)(j) \] by definition of $\mathcal{N}_{i+1}(u)$ in U4
\[ \mathcal{N}_{i+1}(u_\varepsilon)(\tau(j)) \] by Lemma 5.3(ii)
\[ P_2([u_{\tau(\tau(j))} | k \in n \setminus \{\tau(j)\}] \] by choice of $\mathcal{N}_{i+1}(u_\varepsilon)$
\[ P_2([u_{\tau(k)} | k \in n \setminus \{\tau(j)\}] \] as $n \setminus \{\tau(j)\} = \tau[n \setminus \{j\}]
\[ P_2([u_k | k \in n \setminus \{j\}] \] as $u = u_\varepsilon \circ \tau$.

3. For L3, suppose that $\sigma : n \rightarrow n$ and $u \cdot \sigma$ is one-one. We check that $\mathcal{N}_{i+1}(u \cdot \sigma) = (\mathcal{N}_{i+1}(u))^{\sigma}$. Plainly, $u \cdot \sigma = u_\varepsilon \circ \tau \circ \sigma$ and $\tau \circ \sigma$ is one-one. Using the definitions and Lemma 5.1,

\[ \mathcal{N}_{i+1}(u \circ \sigma) = \mathcal{N}_{i+1}(u_\varepsilon)^{\circ \sigma} = (\mathcal{N}_{i+1}(u_\varepsilon))^\sigma = \mathcal{N}_{i+1}(u)^\sigma, \]
as required.
8. Direct and inverse systems and duality

Here we examine morphisms between graphs, atoms structures, and algebras. Our algebras are constructed from atom structures based on graphs, so we will need to transform graph p-morphisms into p-morphisms of atom structures, and then, using duality, to embeddings of algebras. We will also consider direct and inverse systems, and their limits.

**Definition 8.1.** Let $\Gamma, \Delta$ be graphs. A map $f : \Gamma \to \Delta$ is said to be a graph p-morphism if for each $x \in \Gamma$, $f$ maps the set of neighbours of $x$ in $\Gamma$ surjectively onto the set of neighbours of $f(x)$ in $\Delta$.

**Definition 8.2.** Let $L \supseteq L_{BA}$ be a functional signature and let $S = (S, R_f \mid f \in L \setminus L_{BA})$ and $S' = (S', R'_f \mid f \in L \setminus L_{BA})$ be $L$-atom structures. Let $g : S \to S'$ be a function. We say that $g : S \to S'$ is a p-morphism of atom structures if for each $k$-ary $f \in L \setminus L_{BA}$, we have:

- **Forth:** $g$ is an $L_{BA}$-homomorphism: for every $x_1, \ldots, x_k, y \in S$, if $R_f(x_1, \ldots, x_k, y)$ then $R'_f(g(x_1), \ldots, g(x_k), g(y))$.

- **Back:** if $y \in S, x_1', \ldots, x_k' \in S'$, and $R'_f(x_1', \ldots, x_k', g(y))$, then there are $x_1, \ldots, x_k \in S$ such that $R_f(x_1, \ldots, x_k, y)$ and $g(x_i) = x_i'$ for $i = 1, \ldots, k$.

Our first lemma is straightforward.

**Lemma 8.3.** Let $\Gamma, \Delta$ be graphs and $f : \Gamma \to \Delta$ a surjective graph p-morphism. Let $f^\times : \Gamma \times n \to \Delta \times n$ be given by $f^\times(p, i) = (f(p), i)$ for $(p, i) \in \Gamma \times n$. Define

$$\hat{f} : \text{At}(\Gamma) \to \text{At}(\Delta), \quad (K, \sim) \mapsto (f^\times \circ K, \sim).$$

Then $\hat{f}$ is a surjective p-morphism of atom structures.

**Proof.** Plainly, $f^\times : \Gamma \times n \to \Delta \times n$ is a surjective graph p-morphism. We need to check the following:

(i) if $(K, \sim) \in \text{At}(\Gamma)$, then $\hat{f}(K, \sim) \in \text{At}(\Delta)$;

(ii) surjectivity of $\hat{f}$;

(iii) the forth property of the cylindrification relations, i.e. if we have $i < n$ and $(K^1, \sim^1) \equiv_i (K^2, \sim^2)$ then $\hat{f}(K^1, \sim^1) \equiv_i \hat{f}(K^2, \sim^2)$;

(iv) the back property of the cylindrification relations, i.e. if we have $i < n$ and $(J^2, \sim^2) \equiv_i \hat{f}(K^1, \sim^1)$, then there is $(K^2, \sim^2) \in \text{At}(\Gamma)$ such that $\hat{f}(K^2, \sim^2) = (J^2, \sim^2)$ and $(K^2, \sim^2) \equiv_i (K^1, \sim^1)$;

(v) substitutions are preserved, i.e. $(K, \sim) \in D_{ij} \iff \hat{f}(K, \sim) \in D_{ij}$;

(vi) substitutions are preserved: $\hat{f}((K, \sim)^\sigma) = (\hat{f}(K, \sim))^\sigma$. 

---

So by Lemma 6.9, $N_{i+1}$ is a polyadic ultrafilter network. We also have $N_{i+1} \supseteq N_i$, so $v \equiv_i v$, and $a \in \mu = N_{i+1}(w)$. The network $N_{i+1}$ is $\mathcal{F}$’s response to $\mathcal{V}$’s move in round $i$. So she is able to respond to any move made by $\mathcal{V}$ — she has a winning strategy. \qed
For (i), suppose \((K, \sim) \in \text{At}(\Gamma)\) and \(|n/\sim| = n\). Clearly the domain of \(K\) is preserved by \(\widehat{f}\). Moreover, since \(\text{im } K\) is not independent and \(f^\circ\) is a graph p-morphism, \(\text{im } K'\) is not independent either. The other cases follow directly from the definition of \(\widehat{f}\).

To show (ii) let \((K', \sim) \in \text{At}(\Delta)\). If \(K'\) is not defined anywhere, we let \(K\) be undefined everywhere as well. If there are \(i < j < n\) such that \(i \sim j\) and \(K'(i) = K'(j)\) is defined, then as \(f^\circ\) is surjective, there is \(p \in \Gamma \times n\) such that \(f^\circ(p) = K'(i)\). Define \(K(i) = K(j) = p\) and let \(K\) be undefined for the remaining values in that case. Finally, if \(K'\) is defined on all values \(i < n\), then \(\text{im } K'\) is not independent, so there are \(i < j < n\) such that there is an edge from \(K'(i)\) to \(K'(j)\). Since \(f^\circ\) is surjective, there is \(p_i \in \Gamma \times n\) such that there is an edge between \(p_i\) and \(p_j\) and \(f^\circ(p_i) = K'(i)\). For the remaining \(s \neq i, j\), using surjectivity we take any vertices \(p_s \in \Gamma \times n\) such that \(f^\circ(p_s) = K'(s)\). Now define \(K(s) = p_s\) for each \(s < n\). By construction, \((K, \sim) \in \text{At}(\Gamma)\) in all three cases, and \(\widehat{f}(K, \sim) = (K', \sim)\).

For (iii) we have for \((K^1, \sim^1), (K^2, \sim^2) \in \text{At}(\Gamma)\) and \(i < n\) that

\[
\begin{align*}
(K^1, \sim^1) &\equiv (K^2, \sim^2) \\
\implies K^1(i) &= K^2(i) \text{ and } \sim^1_\downarrow = \sim^2_\downarrow \\
\implies f^\circ(K^1(i)) &= f^\circ(K^2(i)) \text{ and } \sim^1_\downarrow = \sim^2_\downarrow \\
\implies \widehat{f}(K^1, \sim^1) &\equiv \widehat{f}(K^2, \sim^2).
\end{align*}
\]

For (iv), suppose that \((K^1, \sim^1) \in \text{At}(\Gamma), (J^1, \sim^1) \in \text{At}(\Delta)\), \(i < n\), and \(\widehat{f}(K^1, \sim^1) \equiv_i (J^1, \sim^1)\). Then

\[
f^\circ(K^1(i)) = J^1(i) \text{ and } \sim^1_\downarrow = \sim^2_\downarrow.
\]

Now take \((K^2, \sim^2)\) such that \(K^2(i) = K^1(i)\) (which may be undefined), and if \(j \neq i\), we choose \(K^2(j)\) from the \(f^\circ\)-pre-image of \(J^1(j)\) if \(J^1\) is defined for \(j\), and otherwise we leave \(K^2(j)\) undefined. It is not hard to do this in such a way that if \(j \sim \downarrow^2 k\) then \(K^2(j) = K^2(k)\), and if \(K^2\) is total then \(\text{im } K^2\) is not independent (here we use that \(\text{im } J^1\) is not independent and \(f^\circ\) is a graph p-morphism). Then \((K^2, \sim^2) \in \text{At}(\Gamma), \widehat{f}(K^2, \sim^2) = (J^1, \sim^1)\), and \((K^1, \sim^1) \equiv_i (K^2, \sim^2)\).

To see that diagonals are preserved (v), note that \((K, \sim) \in D_i \iff i \sim j \iff \widehat{f}(K, \sim) \in D_j\). For (vi), we have

\[
\begin{align*}
\widehat{f}(K, \sim)^\circ &\equiv (f^\circ \circ K^\circ, \sim^\circ), \\
(\widehat{f}(K, \sim))^\circ &\equiv ((f^\circ \circ K)^\circ, \sim^\circ).
\end{align*}
\]

Recall that in general, \(K^\circ(i)\) is defined iff \(\sim^\circ\) is \(i\)-distinguishing and is then \(K(j), \text{ where } j \notin \sigma[n \setminus \{i\}]\). So \(f^\circ \circ K^\circ(i)\) is defined iff \((f^\circ \circ K)^\circ(i)\) is defined, and in that case,

\[
f^\circ \circ K^\circ(i) = f^\circ(K(j)) = (f^\circ \circ K)(j) = (f^\circ \circ K)^\circ(i).
\]

So indeed, \(\widehat{f}((K, \sim)^\circ) = (\widehat{f}(K, \sim))^\circ\).

Next, we move from p-morphisms of atom structures to algebra embeddings.

**Lemma 8.4.** Let \(g : \text{At}(\Gamma) \to \text{At}(\Delta)\) be a surjective p-morphism. Then the map

\[
g^\circ : \mathcal{A}(\Delta) \to \mathcal{A}(\Gamma), \quad Y \mapsto \{x \in \text{At}(\Gamma) \mid g(x) \in Y\}
\]

is an injection that preserves \(\equiv, \sim, \downarrow, \circ\), and the \(\widehat{f}\) map of (ii) above. Also, for any \(i, j \in \text{At}(\Gamma)\), if \(i \equiv j\) then \(g^\circ(i) \equiv g^\circ(j)\), and if \(i \sim j\) then \(g^\circ(i) \sim g^\circ(j)\).

Prove this lemma.
is an algebra embedding. If \( f : \mathcal{A}(\Delta) \to \mathcal{A}(\Delta) \) is an embedding, then the map

\[
f_\ast : \mathcal{A}(\Gamma)_\ast \to \mathcal{A}(\Delta)_\ast, \quad \mu \mapsto \{a \in \mathcal{A}(\Delta) \mid f(a) \in \mu\}
\]

is a surjective p-morphism.

**Proof.** This is standard duality: see, e.g., [2, theorem 5.47].

We now scale up these results to direct and inverse systems of embeddings and p-morphisms, and their limits. We will need to consider only very special cases, so our definitions are highly restricted. More general definitions can be found in, e.g., [5, 11.5, 11.1]. We assume familiarity with direct products of arbitrary model-theoretic structures: see, for example, [3, Exercise 4.1.12].

**Definition 8.5.** Let \( L \supseteq L_{BA} \) be a functional signature.

1. A direct system of \( L \)-algebras and embeddings is a family \( \mathfrak{A} = (\mathcal{A}_k, h^k_i \mid k \leq l < \omega) \), where for each \( k \leq l \) \( \leq m < \omega \) we have: \( \mathcal{A}_k \) is an \( L \)-algebra, \( h^k_i : \mathcal{A}_k \to \mathcal{A}_l \) is an algebra embedding, \( h^k_l \) is the identity map on \( \mathcal{A}_k \), and \( h^m_l \circ h^k_l = h^m_k \).

   Its direct limit \( \lim \mathfrak{A} \) is the \( L \)-algebra defined as follows. Let \( D \) be the disjoint union of the domains of the algebras \( \mathcal{A}_k (k < \omega) \), and define a relation \( \sim \) on \( D \) by \( a \sim b \) iff for some \( k, l, m < \omega \) with \( k, l \leq m \) we have \( a \in \mathcal{A}_k \), \( b \in \mathcal{A}_l \), and \( h^m_k(a) = h^m_l(b) \). This is an equivalence relation. The domain of \( \lim \mathfrak{A} \) is defined to be the set \( D/\sim \) of \( \sim \) equivalence classes. Its algebra structure is defined as follows. Let \( f \in L \) have arity \( r \), and let \( a_1/\sim, \ldots, a_r/\sim \in D/\sim \). Suppose \( a_i \in \mathcal{A}_k \) for \( i = 1, \ldots, r \). Let \( m = \max(k_1, \ldots, k_r) \), and let \( b_i = h^m_k(a_i) \in \mathcal{A}_m \) for each \( i = 1, \ldots, r \). Then we define \( f(a_1/\sim, \ldots, a_r/\sim) = (f^\mathfrak{A}(b_1, \ldots, b_r))/\sim \). This can be checked to be well defined.

2. An inverse system of finite graphs and surjective graph p-morphisms is a family of the form \( \mathfrak{G} = (\Gamma, f^\mathfrak{G}_k \mid k \leq l < \omega) \), where for each \( k \leq l \leq m < \omega \) we have: \( \Gamma_k \) is a graph, \( f^\mathfrak{G}_k : \Gamma_k \to \Gamma_k \) is a surjective graph p-morphism, \( f^\mathfrak{G}_k \) is the identity map on \( \Gamma_k \), and \( f^\mathfrak{G}_l \circ f^\mathfrak{G}_k = f^\mathfrak{G}_k \). Its inverse limit \( \lim \mathfrak{G} \) is the subgraph of \( \prod_{k \leq l} \Gamma_k \) with domain \( \{p \in \prod_{k \leq l} \Gamma_k \mid f^\mathfrak{G}_k(p_k) = p_k \) for each \( k \leq l < \omega \).

3. An inverse system of \( L \)-atom structures and surjective p-morphisms is a family of the form \( \mathfrak{S} = (S_k, g^\mathfrak{S}_k \mid k \leq l < \omega) \), where for each \( k \leq l \leq m < \omega \) we have: \( S_k \) is an \( L \)-atom structure, \( g^\mathfrak{S}_k : S_k \to S_k \) is a surjective p-morphism, \( g^\mathfrak{S}_k \) is the identity map on \( S_k \), and \( g^\mathfrak{S}_l \circ g^\mathfrak{S}_k = g^\mathfrak{S}_k \). Its inverse limit \( \lim \mathfrak{S} \) is the sub-atom structure of \( \prod_{k \leq l} S_k \) with domain \( \{s \in \prod_{k \leq l} S_k \mid g^\mathfrak{S}_k(s_k) = s_k \) for each \( k \leq l < \omega \).

Our earlier work allows us to transform some kinds of system into others.

**Definition 8.6.** Let \( \mathfrak{G} = (\Gamma, \nu^\mathfrak{G}_k \mid k \leq l < \omega) \) be an inverse system of graphs and surjective p-morphisms. In the notation of Lemmas 8.3 and 8.4, define

\[
\text{At}(\mathfrak{G}) = (\text{At}(\Gamma_k), \nu^\mathfrak{G}_k \mid k \leq l < \omega),
\]

\[
\mathcal{A}(\mathfrak{G}) = (\mathcal{A}(\Gamma_k), \nu^\mathfrak{G}_k^+ \mid k \leq l < \omega),
\]

\[
\mathcal{A}(\mathfrak{G})_\ast = (\mathcal{A}(\Gamma_k)_\ast, (\nu^\mathfrak{G}_k)_\ast^+ \mid k \leq l < \omega).
\]
It is almost immediate from these lemmas that \( \text{At}(\mathfrak{G}) \) is an inverse system of atom structures and surjective p-morphisms, \( \mathcal{A}(\mathfrak{G}) \) is a direct system of BAOs and embeddings, and \( \mathcal{A}(\mathfrak{G})_\sim \) is again an inverse system of atom structures and surjective p-morphisms.

**Proposition 8.7.** Let \( \mathfrak{G} = (\Gamma_k, \nu^k_l \mid k \leq l < \omega) \) be an inverse system of finite graphs and surjective p-morphisms. Then:

(i) \( \lim \mathcal{A}(\mathfrak{G})_\sim \cong \lim \mathcal{A}(\mathfrak{G})_\neq \)

(ii) \( \mathcal{A}(\mathfrak{G})_\sim \cong \text{At}(\mathfrak{G}) \), where isomorphism of inverse systems is defined in the obvious way

(iii) \( \lim \text{At}(\mathfrak{G}) \cong \text{At}(\lim \mathfrak{G}) \)

(iv) \( \lim \mathcal{A}(\mathfrak{G})_\sim \cong \text{At}(\lim \mathfrak{G}) \)

**Proof.** Part (i) is a consequence of important results of Goldblatt [5, theorems 10.7, 11.2, 11.6]. Goldblatt proved these results for modal algebras, but they generalise easily to BAOs.

For part (ii), as each \( \text{At}(\Gamma_k) \) is finite, \( \mathcal{A}(\Gamma_k)_\sim \cong \text{At}(\Gamma_k) \) (see, e.g., [5, theorems 9.2, 10.7]), and this can be easily extended to show that \( \mathcal{A}(\mathfrak{G})_\sim \cong \text{At}(\mathfrak{G}) \).

For part (iii), write \( \Gamma = \lim \mathfrak{G} \). We define maps

\[
\begin{align*}
f : \text{At}(\Gamma) &\to \lim \text{At}(\mathfrak{G}) \\ g : \lim \text{At}(\mathfrak{G}) &\to \text{At}(\Gamma)
\end{align*}
\]

as follows. Let \( (K, \sim) \in \text{At}(\Gamma) \) be arbitrary. Thus, \( K : n \to \Gamma \times n \) is a partial map satisfying the conditions of Definition 3.2. For \( k \leq \omega \) define a partial map \( K_k : n \to \Gamma_k \times n \) with the same domain as \( K \), by \( K_k(i) = (p_k, j) \), where \( i \in \text{dom} K \) and \( K(i) = (p, j) \in (\prod_{k \leq \omega} \Gamma_k) \times n \). It can easily be checked that \( K_k \) also meets the conditions of Definition 3.2, so that \( (K_k, \sim) \in \text{At}(\Gamma_k) \). Define \( f((K, \sim)) = ((K_k, \sim) \mid k \leq \omega) \). Clearly, this value is in \( \lim \text{At}(\mathfrak{G}) \).

We now define \( g \). Let \( \sigma \in \lim \text{At}(\mathfrak{G}) \) be arbitrary. So \( \sigma \) has the form \( ((K_k, \sim) \mid k \leq \omega) \) where \( (K_k, \sim) \in \text{At}(\Gamma_k) \) and \( K_k = (\nu_k^l)^\sigma \circ K_l \) for each \( k \leq l \leq \omega \). The relation \( \sim \) and the set \( D = \text{dom} K_k \) do not depend on \( k \). We define a map \( K : D \to \Gamma \times n \) as follows.

For each \( i \in D \), there are \( p \in \Gamma \) and \( s \in n \) such that \( K_k(i) = (p_k, s) \) for each \( k \leq \omega \). Define \( K(i) = (p, s) \in \Gamma \times n \). It can be verified that \( (K, \sim) \in \text{At}(\Gamma) \). We will check the requirement that if \( D = n \) then \( \text{im} K \) is not an independent set in \( \Gamma \times n \). For each \( k \leq \omega \), since \( (K_k, \sim) \in \text{At}(\Gamma_k) \), \( \text{im} K_k \) is not independent and there are \( i_k < j_k < n \) such that \( (K_k(i_k), K_k(j_k)) \) is an edge of \( \Gamma_k \times n \). Choose \( i < j < n \) such that \( (i, j) \equiv (i_s, j_k) \) for infinitely many \( k < \omega \). Since for each \( k \leq l < \omega \), the map \( (\nu_k^l)^\sigma : \Gamma_l \times n \to \Gamma_k \times n \) preserves graph edges, it follows that \( (K_k(i), K_k(j)) \) is an edge of \( \Gamma_k \times n \) for every \( k < \omega \), and hence that \( (K(i), K(j)) \) is an edge in \( \Gamma \times n \), as required. The other requirements are easy to check. So indeed, \( (K, \sim) \in \text{At}(\Gamma) \).

We now define \( g(\sigma) = (K, \sim) \).

We leave it to the reader to check that \( f, g \) are mutual inverses and preserve \( R_{d_l}, \equiv_l \), and \(-^\sigma\).

For part (iv), by parts (i), (ii) and (iii) we have

\[
(\lim \mathcal{A}(\mathfrak{G}))_\sim \cong \lim(\mathcal{A}(\mathfrak{G})_\sim) \cong \lim \text{At}(\mathfrak{G}) \cong \text{At}(\lim \mathfrak{G}).
\]

\[\square\]

9. Applications

Here we prove our two main theorems (Theorems 9.4 and 9.7 below).
9.1. Canonical axiomatisations

Here, we use direct and inverse systems to build a certain algebra, and apply the results from the previous sections to show that it can be made to satisfy an arbitrary number of representability axioms, while its canonical extension only satisfies a bounded number. It will follow that any first-order axiomatisation of the representable cylindric algebras (and various other classes) has infinitely many non-canonical axioms.

Our argument is based on the following result. It is from [13, Lemma 4.1], but it can be proved in a rather simpler way by modifying the argument of [9, Theorem 4]. Both proofs use similar random graphs. Recall that $\chi(\Gamma)$ denotes the chromatic number of a graph $\Gamma$.

**Theorem 9.1.** Suppose that $2 \leq \ell \leq k < \omega$. Then there exists an inverse system of finite graphs

$$\Gamma_0 \overset{f_0}{\twoheadleftarrow} \Gamma_1 \overset{f_1}{\twoheadleftarrow} \cdots,$$

where the $f_i$ are surjective graph $p$-morphisms, such that $\chi(\Gamma_s) = k$ for every $s < \omega$, and $\chi(\lim\leftarrow \Gamma_s) = \ell$.

**Definition 9.2.** Let us define some $L_{AGS}$-theories.

1. Fix a universal axiomatisation $\Pi$ of $RPEA_n$ — such an axiomatisation exists because $RPEA_n$ is a variety (Proposition 2.8).
2. Also fix any first-order axiomatisation $\Delta$ of $RDf_n$.

We regard $\Pi$ and $\Delta$ as $\mathcal{A}$-sorted $L_{AGS}$-theories in the obvious way.

3. Let $\Phi$ be the following $L_{AGS}$-theory, expressing that $B$ is infinite:

$$\Phi = \{ \phi_m | m < \omega \} \text{ where } \phi_m = \exists B_0, \ldots, B_{m-1} : \mathcal{B}( \bigwedge_{i<j<m} B_i \neq B_j ).$$

Also recall from Definition 7.1 that $\Theta = \{ \theta_k | k < \omega \}$ expresses that $G$ has infinite chromatic number in the $\mathcal{B}$-sense. The theory $U$ defining algebra-graph systems was laid down in Definition 4.3.

**Definition 9.3.** For $L_{DL_n} \subseteq L \subseteq L_{PEA_n}$, we write $RL$ for the class of $L$-algebras having a representation respecting all the $L$-operations.

We can now prove the main result of the paper.

**Theorem 9.4.** Let $L$ be a signature satisfying $L_{DL_n} \subseteq L \subseteq L_{PEA_n}$. Then any first-order axiomatisation of $RL$ contains infinitely many non-canonical axioms.

**Proof.** Suppose for a contradiction that $T = T_C \cup T_{NC}$ is a first-order axiomatisation of $RL$, where every sentence in $T_C$ is canonical and $T_{NC}$ is finite. We regard $T$ equally as an $\mathcal{A}$-sorted $L_{AGS}$-theory in the natural way. Referring to Definition 9.2, we plainly have $\Pi \models T \models \Delta$. Also, by Theorem 7.3 we have $U \cup \Phi \cup \Delta \models \Theta$, and by Theorem 7.8 we have $U \cup \Theta \models U \cup \Theta \models \Pi$. Using this and first-order compactness, and bearing in mind that $\theta_l \models \theta_l$ whenever $l \leq k < \omega$, we see that:

1. there is $\ell < \omega$ such that $\mathcal{U} \cup \{ \theta_{\ell} \} \models T_{NC}$,
2. there is a finite $T_0 \subseteq T_C$ such that $\mathcal{U} \cup \Phi \cup T_0 \cup T_{NC} \models \theta_{\ell+1}$,
3. there is a finite $\Pi_0 \subseteq \Pi$ such that $\Pi_0 \models T_0$.
4. there is $k < \omega$ such that $k \geq \ell$ and $\mathcal{U} \cup \{\theta_k\} \models \Pi_0$.

Using Theorem 9.1, take finite graphs $\Gamma_0, \Gamma_1, \ldots$ such that $\chi(\Gamma_s) = k + 1$ for all $s < \omega$.

$$\Gamma_0 \xrightarrow{f_0^1} \Gamma_1 \xrightarrow{f_1^2} \cdots,$$

where the $f_j$ are surjective graph p-morphisms, and, writing $\Gamma = \lim \Gamma_s$, we have $\chi(\Gamma) = \ell + 1$.

Using Lemmas 8.3 and 8.4, we obtain embeddings:

$$A(\Gamma_0) \hookrightarrow A(\Gamma_1) \hookrightarrow \cdots.$$

Define $A = \lim \rightarrow A(\Gamma_s)$. Then, because $\chi(\Gamma_s) = k + 1$, we have $M(\Gamma_s) \models \mathcal{U} \cup \{\theta_k\}$, so $A(\Gamma_s) \models \Pi_0$ for each $s < \omega$. As the sentences in $\Pi$ are universal, they are preserved by direct limits, and we therefore have $A \models \Pi_0$ and hence $A \models T_0$. As all sentences in $T_0$ are canonical, $A^\sigma \models T_0$ as well. Moreover, from Proposition 8.7(iv) we get

$$A^\sigma = (\lim \rightarrow A(\Gamma_s))^\sigma = \text{Art}(\text{lim} \rightarrow \Gamma_s) = \text{Art}(\Gamma),$$

and thus $A^\sigma \equiv A(\Gamma)$ and $M(\Gamma) \equiv (A^\sigma, \Gamma, \varphi(\Gamma))$. We chose the graphs so that $\chi(\Gamma) = \ell + 1$. So $M(\Gamma) \models \mathcal{U} \cup \{\theta_{\ell}\}$ and hence $A^\sigma \models T_{\text{NC}}$. As $\Gamma$ is plainly infinite, $\varphi(\Gamma)$ is also infinite, and so $M(\Gamma) \models \mathcal{U} \cup \Phi \cup T_0 \cup T_{\text{NC}}$ and hence $M(\Gamma) \models \theta_{\ell+1}$. So $\chi(\Gamma) > \ell + 1$, a contradiction. \qed

**Corollary 9.5.** Any first-order axiomatisation (for example, any equational axiomatisation) of any of the following classes has infinitely many non-canonical sentences:

1. the class $\text{RDi}_n$ of representable $n$-dimensional diagonal-free cylindric algebras,
2. the class $\text{RCA}_n$ of representable $n$-dimensional cylindric algebras,
3. the class $\text{RPA}_n$ of representable $n$-dimensional polyadic algebras,
4. the class $\text{RPEA}_n$ of representable $n$-dimensional polyadic equality algebras.

Hence, none of the classes is finitely axiomatisable, nor does it have an axiomatisation where only finitely many axioms are not Sahlqvist equations.

**Proof.** Immediate from Theorem 9.4 and because Sahlqvist equations are canonical. \qed

9.2. Strongly representable atom structures

The main result of [12] (Theorem 6.1) showed that for each finite $n \geq 3$, the class $\text{Str \, RCA}_n$ of strongly representable $n$-dimensional cylindric algebra atom structures is non-elementary. We finish with a generalisation of this to other signatures. It has already been proved by Sayed Ahmed (draft of untitled monograph, 2010) using the same algebras.

**Definition 9.6.** Let $L_{\text{DL}} \subseteq L \subseteq L_{\text{PEA}}$. An $L$-atom structure $\mathcal{S}$ is said to be **strongly representable** if $\mathcal{S}^+ \in \text{RL}$ (see Definition 9.3 for $\text{RL}$).

**Theorem 9.7.** For any $L_{\text{DL}} \subseteq L \subseteq L_{\text{PEA}}$, the class of strongly representable $L$-atom structures is non-elementary. In another common notation, the class $\text{Str \, RL}$ of structures for $\text{RL}$ is non-elementary.
Proof. A celebrated result of Erdős [4] shows that for all $k < \omega$ there is a finite graph $G_k$ with chromatic number and girth (length of the shortest cycle) both at least $k$. Let $\Gamma_k$ be the disjoint union of the $G_\ell$ for $k \leq \ell < \omega$: this time, no edges are added between copies. Plainly, $\Gamma_k$ has infinite chromatic number, and its girth is at least $k$. By Theorem 7.8 applied to $M(\Gamma_k)$, $A(\Gamma_k) \restriction L \in RL$, so that $(A\Gamma_k) \restriction L_+$ is strongly representable.

Now let $\Gamma$ be a non-principal ultraproduct of the $\Gamma_k$. Then $\Gamma$ is infinite, and by Łoś’s theorem it has girth at least $k$ for all finite $k$, since this property is first-order definable. Hence, $\Gamma$ has no cycles, so its chromatic number is at most two. By Theorem 7.3, the diagonal-free reduct of $A(\Gamma)$ is not representable, and hence neither is its $L$-reduct. So $(A\Gamma) \restriction L_+$ is not strongly representable.

But it is easily seen that the operation $A(-)$ commutes with ultraproducts, and it follows that $(A\Gamma) \restriction L_+$ is isomorphic to an ultraproduct of the $(A\Gamma_k) \restriction L_+$. This shows that the class of strongly representable $L$-atom structures is not closed under ultraproducts and so cannot be elementary.

10. Conclusion

We have proved that every variety of representable algebras of relations whose signature lies between that of $\mathbb{RD}_n$ and $\mathbb{RPEA}_n$ (for finite $n \geq 3$) is barely canonical, in that (although canonical) it cannot be axiomatised by first-order sentences only finitely many of which are not themselves canonical. As far as we know, it is an open question whether various other varieties of algebras of relations are also barely canonical, including infinite-dimensional diagonal-free, cylindric, polyadic (equality) and quasi-polyadic (equality) algebras, classes of relativised set algebras such as $\mathbb{Crs}_n$, $\mathbb{D}_n$, $\mathbb{G}_n$ ($n \geq 3$), and various classes of neat reducts, such as $\mathbb{S}\mathbb{P}_{\mathbb{R}n}\mathbb{C}\mathbb{A}_m$ for $3 \leq n < m < \omega$, and $\mathbb{S}\mathbb{P}_{\mathbb{R}n}\mathbb{C}\mathbb{A}_m$ for $5 \leq n < \omega$. Some of these (such as $\mathbb{G}_\omega$) are not even known to be varieties. A wider question is to find a more general method for proving bare canonicity.

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