I. INTRODUCTION

Model classification is an important subject in practice and has been studied extensively in statistics. For example, if we know that obtained data are drawn according to a particular good class of statistical models, well-established estimation methods for this class can be applied to make estimates on the statistical model. There are, of course, bad statistical models in the sense that it is extremely hard to make any statistical estimates even numerically. In classical statistics, there exists a variety of different parametric models studied in details. As a concrete example of a good statistical model, if we know that data is described by the standard linear response model with an equal variance, we can immediately apply the best linear unbiased estimator, which can be computed analytically. In reality, experimental data are affected by many unknown factors and considerable amounts of efforts have been devoted to study the general non-linear response models in statistics, see for example Refs. [1–5].

Information geometry offers a completely different motivation to the model classification problem based on the geometrical properties of parametric models [6]. The most famous model is the exponential family, or the log-linear model, defined as an auto-parallel sub-manifold with respect to the exponential connection. What is remarkable regarding the exponential family is that achievability of the Cramér-Rao (CR) bound for the finite sample is given as if and only if the parametric model is the exponential family and the parameter to be estimated is an m-affine parameter of the model.

The non-commutative extension of classical statistics to a quantum system was initiated in 1960s by Helstrom [7] and has been one of the fundamental problems in quantum information theory until today. The point-estimation problem about quantum states is the fundamental problem in theory and is also important for practical applications. In particular, recent advances in quantum metrology, quantum sensing, and quantum imaging, i.e., high precision measurement methods utilizing quantum resources, has triggered many activities in the field, see reviews on these subjects [8–13]. Despite these efforts in past, there exist many open problems regarding multi-parameter estimation problems. One such fundamental problem is an explicit expression for the optimal estimation strategy that sets the bound for the estimation error. In classical statistics, this estimation error is bounded by the well-known CR bound, and an optimal estimator is the maximum likelihood estimator that asymptotically achieves this bound. Importantly, the CR bound in statistics is analytically calculated by the classical Fisher information matrix of the statistical model. A quantum version of this result is still missing mainly due to a nontrivial optimization for the measurement degrees of freedom, and also partially due to the fact that there exist many quantum versions of the Fisher information in the quantum system. In particular, quantum CR bounds, which are defined by quantum Fisher information matrices, cannot be achieved even asymptotically in general. A unified understanding on this fundamental estimation error bound is given by the Holevo bound [14]. Unlike as in the classical case, this bound is expressed as a non-linear optimization problem.

Model classification for the quantum parametric models is also an important problem, but it seems that this problem is less attracted so far by the quantum information community. The first attempt of model classification for the quantum estimation theory was due to Holevo [14]. He introduced a particular class of quantum statistical models, called a D-invariant model, and showed that the right logarithmic derivative (RLD) CR bound can be achieved by the D-invariant model. Another non-trivial extension of model classification was studied by Nagaoka [15]. He defined the quantum exponential family, and showed that the symmetric logarithmic derivative (SLD) CR bound can be achieved uniformly by the quantum exponential family.

In this paper, we make an attempt at classifying quantum parametric models based on the ultimate precision...
bound, i.e. the Holevo bound [14]. One of the advantages of this approach is that we can immediately write down the achievable precision bound if a given model belongs to classes of models studied in this paper. The current paper is based on the results presented in Ref. [16], where we analyzed the structure of the Holevo bound in detail for a qubit system. We derived an explicit formula for any qubit model together with characterization of special classes of the qubit models. We also classified the D-invariant model for the general qudit model together with non-trivial characterization of this model. In this paper, we continue to explore possible classification of quantum parametric models into several classes in which the Holevo bound can be expressed in closed formulas. We are also motivated by analyzing the structure of the tangent space and several quantum metrics on the quantum-state space.

In this paper, we consider four different classes: The first class is the classical model where a quantum statistical model is reduced to a parametric model in classical statistics. This is because quantum statistical models defined by a set of positive semi-definite matrices with a unit trace contain classical statistical models as a special case. More precisely, when a given family of quantum states is simultaneously diagonalizable for all parameter values, the problem at hand can be reduced to the one in classical statistics. Though, this definition is trivial, it is important to characterize such the classical statistical model as properties of the tangent space. This is because the local property plays an important role in the quantum estimation theory in general. For the classical model, it is easy to see that the Holevo bound is simply reduced to the form of the classical Fisher information computed by the eigenvalues of the given quantum state.

The second class is known as the quasi-classical model defined by the condition imposing all symmetric logarithmic derivative (SLD) operators commute with each other. When this condition is satisfied, it is clear that we can construct an optimal measurement by diagonalizing all SLD operators simultaneously. This then achieves the SLD CR bound for any finite sample size. As indicated by the name, this class is still quantum and is different from the classical model in general.

The third class is known as the D-invariant model introduced by Holevo [14]. It was shown in Ref. [16] that the Holevo bound is equivalent to the right logarithmic derivative (RLD) CR bound if and only if the model is D-invariant. In this paper, we put further step into characterizing the D-invariant model.

The fourth class is when the Holevo bound coincides with the symmetric logarithmic derivative (SLD) CR bound. We call this class of models as the asymptotically classical in the sense that the model is asymptotically equivalent to a classical gaussian model in the local asymptotic normality (LAN) theory [17–22]. We note that the asymptotically classical model was introduced and analyzed in Refs. [16, 23]. In Ref. [24], the authors independently investigated the same problem and they called this condition as the compatibility condition. In this paper we give a more detailed analysis on their compatibility condition and derive several equivalent characterization of the asymptotically classical model.

The aim of this paper is not just to classify quantum models into classes mentioned above, but to derive several equivalent conditions characterizing each class for the parametric family of quantum states. The results are given by theorems in Sec. IV. We further examine relations among these classes. In Fig. 1, we summarize the relations among four different classes of quantum statistical models. Figure 2 in Sec. IV also represents a schematic diagram for one of the main results of this paper.

The content of this paper is summarized as follows. Sec. II provides preliminaries for notations and mathematical tools used in this paper. In Sec. II C, a few lemmas are proven to be useful for classifying quantum statistical models. In Sec. III, we list the definitions of four different classes of statistical models. Our main results are given in the next section. Sec. IV A gives the main theorems of this paper. In Sec. IV B, we discuss the meaning of the classical model in detail. Proofs for the theorems are given in Sec. IV C. Several examples are discussed in Sec. V to illustrate our findings. The last section, Sec. V, concludes the paper with a few remarks and open problems.

II. PRELIMINARIES

A quantum system $\mathcal{H}$ is a $d$-dimensional Hilbert space on the complex number. Denote by $\mathcal{L}(\mathcal{H})$ a set of (bounded) linear operators from $\mathcal{H}$ to itself, and by $\mathcal{L}_b(\mathcal{H})$ a set of linear and hermite operators from $\mathcal{H}$ to itself. A quantum state is a positive semi-definite operator.
on $\mathcal{H}$ with unit trace. Let us denote a set of all quantum states on $\mathcal{H}$ by $\mathcal{S}(\mathcal{H})$ and all full-ranked quantum states by $\mathcal{S}_\infty(\mathcal{H})$. A quantum statistical model is defined by a parametric family of quantum states

$$\mathcal{M} := \{ \rho_\theta \in \mathcal{S}(\mathcal{H}) \mid \theta = (\theta^1, \ldots, \theta^n) \in \Theta \},$$

where $\Theta$ is an open subset of $\mathbb{R}^n$. As in classical statistics, we impose several regularity conditions, such as one-to-one smooth mapping: $\theta \mapsto \rho_\theta$, differentiability, linearity independence of partial derivatives $\partial \rho_\theta / \partial \theta^i$ with respect to the coordinates $(\theta^j)$, non-degeneracy for the eigenvalues, and so on. In the following discussions, we assume all these regularity conditions to avoid non-regular behaviors of the statistical model. In particular, we mainly consider a quantum statistical model of full-rank states unless stated explicitly.

A. Tangent space and quantum Fisher information

We define two quantum versions of the logarithmic derivative, the quantum score functions, as follows. For a given quantum state $\rho_\theta$ and any operators $X, Y \in \mathcal{L}(\mathcal{H})$, define the symmetric logarithmic derivative (SLD) and right logarithmic derivative (RLD) inner product by

$$\langle X, Y \rangle^S_{\rho_\theta} := \frac{1}{2} \text{tr} (\rho_\theta (Y X^\dagger + X^\dagger Y)),
$$

$$\langle X, Y \rangle^R_{\rho_\theta} := \text{tr} (\rho_\theta Y X^\dagger),$$

respectively, where $X^\dagger$ denotes the hermite conjugate of $X$. The $i$th SLD and RLD operators, $L_i$ and $\tilde{L}_i$, are formally defined by the solutions to the operator equations:

$$\partial_i \rho_\theta = \frac{1}{2} (\rho_\theta L_{\theta,i} + L_{\theta,i} \rho_\theta),
$$

$$\partial_i \rho_\theta = \rho_\theta \tilde{L}_{\theta,i},$$

for $i = 1, 2, \ldots, n$, where $\partial_i := \partial / \partial \theta^i$ denotes the partial derivative with respect to $\theta^i$. It is not difficult to see that the SLD operators are hermite, whereas RLD operators are not in general.

The SLD and RLD Fisher information matrices (quantum Fisher metric) are defined by

$$G_\theta := [g_{\theta,i,j}] \text{ with } g_{\theta,i,j} := \langle L_{\theta,i}, L_{\theta,j} \rangle^S_{\rho_\theta};$$

$$\tilde{G}_\theta := [\tilde{g}_{\theta,i,j}] \text{ with } \tilde{g}_{\theta,i,j} := \langle \tilde{L}_{\theta,i}, \tilde{L}_{\theta,j} \rangle^R_{\rho_\theta},$$

respectively. It is known that the SLD Fisher information is the smallest and the real part of RLD is the largest operator monotone metrics on the quantum state space $\mathcal{S}_\infty(\mathcal{H})$.

The SLD tangent space is define by the linear span of SLD operators:

$$T_\theta(\mathcal{M}) := \text{span}_\mathbb{R} \{ L_{\theta,i} \} \subset \mathcal{L}_h(\mathcal{H}),$$

and the RLD tangent space is defined by the linear span of RLD operators with complex coefficients:

$$\tilde{T}_\theta(\mathcal{M}) := \text{span}_\mathbb{C} \{ \tilde{L}_{\theta,i} \} \subset \mathcal{L}(\mathcal{H}).$$

Let $G_\theta^{-1} = [g_{\theta,i,j}^{-1}]$ be the inverse of the SLD Fisher information and $\tilde{G}_\theta^{-1} = [\tilde{g}_{\theta,i,j}^{-1}]$ be the inverse for the RLD case. It is convenient to introduce the following linear combinations of the logarithmic derivative operators

$$L_{\theta,j} := \sum_{j=1}^n g_{\theta,j}^{ij} L_{\theta,j}, \quad \tilde{L}_{\theta,j} := \sum_{j=1}^n \tilde{g}_{\theta,j}^{ij} \tilde{L}_{\theta,j}.$$

By definitions, $\{ L_{\theta,j} \}$ forms a dual basis for the inner product space $\langle \cdot, \cdot \rangle^S_{\rho_\theta}$; $\langle L_{\theta,j}, L_{\theta,j} \rangle^S_{\rho_\theta} = \delta_{ij}$, and we shall call it the SLD dual operator. The same statement holds for the RLD case.

Noting that the SLD and RLD operators are a sort of exponential representation of the tangent vector $\partial_i$, we can show the next lemma.

Lemma II.1 For $\forall X \in \mathcal{L}(\mathcal{H})$, and $\forall f \in C^\infty(\mathbb{R})$, the following holds.

$$\langle f(L_{\theta,i}), X \rangle^S_{\rho_\theta} = \langle f(\tilde{L}_{\theta,i}), X \rangle^R_{\rho_\theta},$$

Proof: We note that the definitions of logarithmic derivative operators gives

$$\langle L_{\theta,i}, X \rangle^S_{\rho_\theta} = \langle \tilde{L}_{\theta,i}, X \rangle^R_{\rho_\theta} = \text{tr} (\partial_i \rho_\theta X),$$

and repeated applications of this relation proves

$$\langle (L_{\theta,i})^k, X \rangle^S_{\rho_\theta} = \langle (\tilde{L}_{\theta,i})^k, X \rangle^R_{\rho_\theta},$$

for any integer power $k$. It is then easy to prove Eq. (7).

B. Commutation operator

For a given quantum statistical model (1), we introduce a super-operator $\mathcal{D}$ from $\mathcal{L}(\mathcal{H})$ to itself, whose action on $X \in \mathcal{L}(\mathcal{H})$ is defined by the operator equation:

$$[\rho_\theta, X] := \rho_\theta X - X \rho_\theta = i \rho_\theta \mathcal{D}_\rho(X) + i \mathcal{D}_\rho(X) \rho_\theta.$$

The operator $\mathcal{D}_\rho$, called a commutation operator, was introduced by Holevo, and the detail can be found in his book [14]. By definition, we can check that the operator $\mathcal{D}_\rho$ is linear. Denoting the identity operator $I$, the following relationship holds

$$L_{\theta,i} = (I + i \mathcal{D}_\rho)(\tilde{L}_{\theta,i}),$$

which can be proven by the direct calculation.

The properties useful in our discussion are given in the next lemma.
Lemma II.2 For \( \forall X, Y \in \mathcal{L}(\mathcal{H}) \), the following relations hold.
\[
\langle \mathcal{D}_{\rho_0}(X), Y \rangle^S_{\rho_0} = -\langle X, \mathcal{D}_{\rho_0}(Y) \rangle^S_{\rho_0},
\]
\[
\langle \mathcal{D}_{\rho_0}(X), Y \rangle^R_{\rho_0} = -\langle X, \mathcal{D}_{\rho_0}(Y) \rangle^R_{\rho_0}.
\]

Proof: The first relationship can be proven directly as
\[
2\langle \mathcal{D}_{\rho_0}(X), Y \rangle^S_{\rho_0} = \text{tr} \left( \rho_0 (\mathcal{D}_{\rho_0}(X)Y + Y \mathcal{D}_{\rho_0}(X)) \right)
= \text{tr} \left( \langle -i \rangle [\rho_0, X]Y \right) = -\text{tr} \left( \langle -i \rangle [\rho_0, Y]X \right) = -2\langle X, \mathcal{D}_{\rho_0}(Y) \rangle^S_{\rho_0}.
\]

Eq. (13) can be proven similarly. □

C. Basic lemmas

In this subsection, we list several lemmas that will be used in our discussion. We define two hermite matrices, \( Z_\theta, \tilde{Z}_\theta \) in terms of SLD and RLD dual operators as follows.
\[
Z_\theta := [z^i_j], \quad \tilde{Z}_\theta := [z^i_j],
\]
with
\[
z_{\rho,\alpha}^i := \langle L_0^i, L_0^j \rangle^S_{\rho_\alpha}, \quad z_{\rho,\alpha}^i := \langle L_0^i, L_0^j \rangle^R_{\rho_\alpha}.
\]

By definition, they are complex matrices in general. Hermiteness can be checked directly by
\[
(z_{\rho,\alpha}^i)^* = \text{tr} \left( \rho_\alpha L_0^i L_0^j \right) = z_{\alpha}^j,
\]
where * denotes its complex conjugate, and the matrix \( \tilde{Z}_\theta \) can be checked similarly.

Together with the SLD and RLD Fisher information matrices, we list four matrices for comparison:
\[
G_\theta^{-1} = [g_0^i], \quad \tilde{G}_\theta^{-1} = [\tilde{g}_0^i],
\]
\[
G_\theta^{-1} = [g_0^i], \quad \tilde{G}_\theta^{-1} = [\tilde{g}_0^i],
\]
with
\[
g_{\rho,\alpha}^i := \langle L_0^i, \tilde{L}_0^j \rangle^S_{\rho_\alpha}, \quad \tilde{g}_{\rho,\alpha}^i := \langle L_0^i, \tilde{L}_0^j \rangle^R_{\rho_\alpha}.
\]

By definition, \( \text{Re}(Z_\theta^{-1}) = G_\theta \) and \( \text{Re}(Z_\theta) = G_\theta^{-1} \) hold, where \( \text{Re} X := (X + X^*)/2 \) denotes the real part of \( X \in \mathcal{L}(\mathcal{H}) \) with \( X^* \) the complex conjugate of \( X \).

First, it is straightforward to see that the operator \( L_0^i - \tilde{L}_0^i \) has the following property.

Lemma II.3 \( L_0^i - \tilde{L}_0^i \) is orthogonal to the SLD tangent space \( T_\theta(M) \) with respect to \( \langle \cdot, \cdot \rangle^S_{\rho_\alpha} \), and is orthogonal to the RLD tangent space \( \tilde{T}_\theta(M) \) with respect to \( \langle \cdot, \cdot \rangle^R_{\rho_\alpha} \).

Proof: Direct calculation shows
\[
\langle L_{\theta,j}, L_0^i - \tilde{L}_0^i \rangle^S_{\rho_\alpha} = \langle L_{\theta,j}, L_0^i \rangle^S_{\rho_\alpha} - \langle L_{\theta,j}, \tilde{L}_0^i \rangle^S_{\rho_\alpha} = \langle L_{\theta,j}, L_0^i \rangle^S_{\rho_\alpha} - \langle L_{\theta,j}, \tilde{L}_0^i \rangle^R_{\rho_\alpha} = 0,
\]
where Lemma I.1 with \( f(x) = x \) is used to get the second line.

Orthogonality to the RLD tangent space with respect to the RLD inner product can be proven similarly. □

The following matrix inequalities between \( G_\theta, \tilde{G}_\theta, Z_\theta = [\langle L_0^i, L_0^j \rangle^S_{\rho_\alpha}], \) and \( \tilde{Z}_\theta = [\langle L_0^i, L_0^j \rangle^R_{\rho_\alpha}] \) are fundamental.

Lemma II.4 Two matrix inequalities
\[
Z_\theta \geq \tilde{G}_\theta^{-1}, \quad \tilde{Z}_\theta \geq G_\theta^{-1},
\]
hold where the equality conditions are same and is given by \( \forall i, L_0^i - \tilde{L}_0^i = 0 \).

Proof: Let \( m_0^i := L_0^i - \tilde{L}_0^i \) and define an \( n \times n \) hermite matrix,
\[
\tilde{M}_\theta := \langle m_0^i, m_0^j \rangle^R_{\rho_\alpha}.
\]
The matrix \( \tilde{M}_\theta \) is then positive semi-definite. Using Lemma II.3, we can also express matrix elements of \( \tilde{M}_\theta \) as
\[
\langle m_0^i, m_0^j \rangle^R_{\rho_\alpha} = \langle m_0^i, L_0^j \rangle^R_{\rho_\alpha} - \langle m_0^i, \tilde{L}_0^j \rangle^R_{\rho_\alpha} = \langle L_0^i, L_0^j \rangle^R_{\rho_\alpha} - \langle L_0^i, \tilde{L}_0^j \rangle^R_{\rho_\alpha} = \langle g_{\rho,\alpha}^i, g_{\rho,\alpha}^j \rangle.
\]

The second equality can be proven in the same way by starting with \( \tilde{M}_\theta := \langle \tilde{m}_0^i, \tilde{m}_0^j \rangle^S_{\rho_\alpha} \).

Next, define \( m_{\theta,i} := L_{\theta,i} - \tilde{L}_{\theta,i} \) and consider another hermite matrix \( \tilde{M}_\theta := \langle \tilde{m}_{\theta,i}, \tilde{m}_{\theta,i} \rangle^R_{\rho_\alpha} \). Following exactly the same logic as in Lemma II.4, we can prove the next lemma.

Lemma II.5 Two matrix inequalities
\[
G_\theta + \tilde{G}_\theta \tilde{Z}_\theta G_\theta \geq 2\tilde{G}_\theta, \quad G_\theta + G_\theta \tilde{Z}_\theta G_\theta \geq 2G_\theta,
\]
hold where the equality conditions are same and is given by \( \forall i, L_{\theta,i} - \tilde{L}_{\theta,i} = 0 \).
Finally, the commutation operator and logarithmic operators satisfy the following relations [28]. Importantly, the right hand side of three equations are expressed as a difference between two hermite matrices defined in Eqs. (17).

Lemma II.6

\[ \langle \hat{L}_{\theta}^i, iD_{\rho_{\theta}}(L_{\theta}^j) \rangle_{\rho_{\theta}}^S = z_{\theta}^{ij} - g_{\theta}^{ij} = i \Im z_{\theta}^{ij}, \]  
\[ \langle \hat{L}_{\theta}^i, iD_{\rho_{\theta}}(L_{\theta}^j) \rangle_{\rho_{\theta}}^S = \tilde{g}_{\theta}^{ij} - g_{\theta}^{ij}, \]  
\[ \langle \hat{L}_{\theta}^i, iD_{\rho_{\theta}}(L_{\theta}^j) \rangle_{\rho_{\theta}}^S = \tilde{z}_{\theta}^{ij} - z_{\theta}^{ij}, \]

hold for all parameter values \( \theta \in \Theta \).

Proof: Using definitions of the SLD and RLD inner product, and the commutation operator, we have

\[ \langle X, Y \rangle_{\rho_{\theta}}^R - \langle X, Y \rangle_{\rho_{\theta}}^S = \frac{1}{2} \tr (\rho_{\theta} [Y, X^\dagger]) \]
\[ = \frac{1}{2} \tr (\rho_{\theta} X Y) - \frac{1}{2} \tr (\rho_{\theta} Y X^\dagger) \]
\[ = \frac{1}{2} \tr (\rho_{\theta} [Y, X^\dagger]) \]
\[ = \frac{1}{2} \tr (\rho_{\theta} D_{\rho_{\theta}}(Y) X^\dagger + X^\dagger D_{\rho_{\theta}}(Y)) \]
\[ = \frac{1}{2} \langle X, iD_{\rho_{\theta}}(Y) \rangle_{\rho_{\theta}}^S, \]

for all \( X, Y \in \mathcal{L}(\mathcal{H}) \). Setting \( X = L_{\theta}^i \), \( Y = L_{\theta}^j \), we prove Eq. (21). Similarly, \( X = \hat{L}_{\theta}^i \), \( Y = \hat{L}_{\theta}^j \) gives Eq. (23).

Next, we observe

\[ g_{\theta}^{ij} = \langle \hat{L}_{\theta}^i, L_{\theta}^j \rangle_{\rho_{\theta}}^S, \quad \tilde{g}_{\theta}^{ij} = \langle \hat{L}_{\theta}^i, \hat{L}_{\theta}^j \rangle_{\rho_{\theta}}^R, \]

which can be directly checked. These relations then prove Eq. (22). \( \square \)

III. MODEL CLASS IN QUANTUM PARAMETRIC MODELS

In this section, we consider four different classes for quantum parametric models. The first class is a purely classical. The second class is so called a commutative model. The third and fourth ones are nontrivial, the D-invariant and asymptotically classical models.

A. Classical model

At each point \( \theta \in \Theta \), the quantum state \( \rho_{\theta} \) can be diagonalized with a unitary \( U_{\theta} \) as \( \rho_{\theta} = U_{\theta} \Lambda_{\theta} U_{\theta}^{-1} \), where a diagonal matrix,

\[ \Lambda_{\theta} = \begin{pmatrix} p_{\theta}(1) & 0 & \cdots & 0 \\ 0 & p_{\theta}(2) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & p_{\theta}(d) \end{pmatrix} \]

lists the eigenvalues of the state \( \rho_{\theta} \). By definition, \( \forall i, p_{\theta}(i) > 0 \) and \( \sum_{i=1}^d p_{\theta}(i) = 1 \). In other words, \( \Lambda_{\theta} \) can be regarded as an element of \( \mathcal{P}(d) := \) the set of all (positive) probability distributions on the set \( \{1, 2, \ldots, d\} \). When the unitary \( U_{\theta} \) is independent of \( \theta \) for all point in \( \Theta \), it is clear that any statistical problem is reduced to the classical one. With this identification, we have the following definition.

Definition III.1 (Classical statistical model) For a given parametric quantum statistical model (1), the model is said classical if the family of quantum states \( \rho_{\theta} \) can be diagonalized with a \( \theta \)-independent unitary \( U \) as

\[ \rho_{\theta} = U \Lambda_{\theta} U^{-1}, \]

for all parameter values \( \theta \in \Theta \).

In the following, we denote the set of all classical models on \( \mathcal{H} \) by \( \mathcal{M}_C \).

B. Quasi classical model

The second class of quantum statistical models has been known in the literature. It is called a quasi classical or commutative model.

Definition III.2 (Quasi classical model) A parametric quantum statistical model (1) is said quasi classical, if all SLD operators commute with each other at all point \( \theta \). That is,

\[ [L_{\theta,i}, L_{\theta,j}] = 0, \quad \forall i, j, \]

hold for all parameter values \( \theta \in \Theta \).

Clearly, if the model is classical, then it is also quasi classical. However, the converse statement does not hold in general. A simple counter example is discussed in Sec. V.B. It is also easy to see that any one-parameter model is automatically quasi-classical.

An important property of quasi classical models is that we can diagonalize all SLD operators simultaneously. It is then possible to perform a measurement that saturates the SLD CR bound defined in Eq. (30) explicitly. Let us denote the set of all quasi classical models on \( \mathcal{H} \) by \( \mathcal{M}_{QC} \).

C. Asymptotic bound: Holevo bound

In this subsection, we give a brief summary of the asymptotic theory on quantum state estimation [26]. As in classical statistics, we are given \( N \)-tensor product of identically and independently distributed (i.i.d.) quantum states \( \rho_{\theta}^\otimes N := \rho_{\theta} \otimes \rho_{\theta} \otimes \cdots \otimes \rho_{\theta} \) on \( \mathcal{H} \). We perform a measurement \( \hat{\Pi}^{(N)} \) on \( \rho_{\theta}^\otimes N \), which is described by a set of matrices under certain conditions, to infer an unknown parameter value \( \theta \). The estimation error of the measurement \( \hat{\Pi}^{(N)} \) is evaluated by the standard mean-square error (MSE) matrix \( V_{\theta}^{(N)}[\hat{\Pi}^{(N)}] \). In the asymptotic theory
of quantum state estimation, one minimizes the weighted trace of the MSE matrix under an additional condition as follows.

\[
C_\theta[W] := \inf_{\hat{\pi}(N)} \limsup_{N \to \infty} N \text{Tr} \left\{ WV_\theta(N)[\hat{\pi}(N)] \right\},
\]

(28)

where \( W > 0 \) is an arbitrary positive-definite weight matrix and a.u. stands for asymptotically unbiased. The first order estimation error bound (28) is usually referred to as the Cramér-Rao (CR) type bound in the literature.

There have been many mathematical works to obtained an alternative expression for the CR bound in terms of information quantities, such as the quantum Fisher information matrix. Unlike classical statistics, where the information quantities, such as the quantum Fisher information matrix, are defined by

\[
\mathcal{X}_\theta := \{ \hat{X} = (X^1, X^2, \ldots, X^n) | \forall i X^i \in \mathcal{L}_h(\mathcal{H}),
\]

\[
\forall i \text{Tr} (\rho_\theta X^i) = 0, \forall i, j \text{tr} \left( \frac{\partial \rho_\theta}{\partial \theta^i} X^j \right) = \delta^i_j \}.
\]

Introduce an \( n \times n \) hermite matrix \( H_\theta[\hat{X}] := [(X^1, X^2)_{\rho_\theta}]_R \), and we define the function \( h_\theta[\hat{X}|W] \) by

\[
h_\theta[\hat{X}|W] := \text{Tr} \left\{ W \text{Re} H_\theta[\hat{X}] \right\} + \text{Tr} \left\{ |W|^2 \text{Im} H_\theta[\hat{X}] W^\dagger \right\},
\]

where \( |X| = \sqrt{X^\dagger X} \) denotes the absolute value of a linear operator \( X \), and \( \text{Im} X := (X - X^*)/2i \) denotes the imaginary part of \( X \in \mathcal{L}(\mathcal{H}) \). The following theorem establishes that the Holevo bound is equal to the CR type bound.

**Theorem III.3** For a quantum statistical model satisfying the regularity conditions, \( C_\theta[W] = C^{H}_\theta[W] \) holds for all weight matrices.

Proofs based on different assumptions can be found in Refs. [17–22]. The Holevo bound is regarded as unification of previously known bounds [27], such as the SLD and RLD CR bounds:

\[
C^{S}_\theta[W] := \text{Tr} \left\{ W C^{-1}_\theta \right\},
\]

(30)

\[
C^{H}_\theta[W] := \text{Tr} \left\{ W \text{Re} \tilde{C}^{-1}_\theta \right\} + \text{Tr} \left\{ |W|^2 \text{Im} \tilde{C}^{-1}_\theta W^\dagger \right\}.
\]

(31)

The relation ship \( C^{H}_\theta[W] \geq \max\{C^{S}_\theta[W], C^{H}_\theta[W]\} \) holds for all \( W > 0 \) [14].

### D. D-invariant model

Holevo introduced an important class of quantum statistical models based on the commutation operator \( D_{\rho_0} \) [14].

**Definition III.4 (D-invariant model (Holevo))** A quantum statistical model (1) is called D-invariant at \( \theta \), if the Holevo bound is identical to the RLD CR bound for all weight matrices.

Mathematically, this condition is expressed as \( \forall X \in T_\theta(M), D_{\rho_0}(X) \in T_\theta(M) \) at \( \theta \). For our discussion, we will focus on the D-invariant model at all \( \theta \) (global D-invariance). For (globally) D-invariant models, the Holevo bound can be expressed analytically and coincides with the RLD CR bound [14], i.e., \( \forall W > 0, C^{H}_\theta[W] = C^{H}_\theta[W] \), and its achievability was discussed in the literature.

Based on the result of Ref. [16], we have another definition for the D-invariant model.

**Definition III.5 (D-invariant model 2)** A quantum statistical model (1) is called D-invariant at \( \theta \), if the Holevo bound is identical to the RLD CR bound for all positive weight matrices.

The equivalence between two definitions was proven [16].

**Theorem III.6** The Holevo bound is identical to the RLD CR bound for all weight matrices, if and only if the quantum statistical model is D-invariant in the sense of Definition III.4.

The set of all D-invariant models is denoted by \( \mathcal{M}_D \).

### E. Asymptotically classical model

The last class of quantum statistical models is when the Holevo bound coincides with the SLR CR bound.

**Definition III.7** A quantum statistical model (1) is called asymptotically classical, if the Holevo bound is identical to the SLD CR bound for all positive weight matrices.

Mathematically, this definition is expressed by the condition: \( \forall W > 0, C^{H}_\theta[W] = C^{H}_\theta[W] \). We shall denote the set of all asymptotically classical models by \( \mathcal{M}_\text{AC} \).

### IV. MODEL CLASSIFICATION AND CHARACTERIZATION

In this section, we give classification of quantum statistical models based on the notations and concept introduced in Sec. III. We first list the results on several equivalent characterization of each model class. Discussions on the results are presented followed by the proofs in Sec. IV C.
A. Results

1. Classical model

The following theorem characterizes the classical model.

**Theorem IV.1** For a given (regular) quantum statistical model (1), the following conditions are all equivalent.

1. The model is classical (Def. III.2).
2. \( \forall X \in T_\theta(M), [X, \rho_\theta] = 0 \).
3. \( \forall X \in \tilde{T}_\theta(M), [X, \rho_\theta] = 0 \).
4. \( G_\theta = \tilde{G}_\theta \).
5. \( \forall i, L_{\theta,i} = \tilde{L}_{\theta,i} \).
6. \( D_{\rho_\theta}(T_\theta(M)) = 0 \).
7. \( D_{\rho_\theta}(\tilde{T}_\theta(M)) = 0 \).
8. The model is D-invariant and asymptotically classical.

Here we remind that all statements are made for global aspect of the model, that is for all point \( \theta \in \Theta \).

2. D-invariant model

In Ref. [16], we derived several equivalent characterizations of the D-invariant model, which are summarized in the following theorem.

**Theorem IV.2** Given a quantum statistical model, the following conditions are equivalent.

1. \( M \) is D-invariant at \( \theta \).
2. \( \forall i, D_{\rho_\theta}(L^i_{\theta}) = \sum_j (\text{Im } Z_\theta)^j L_{\theta,j} \).
3. \( Z_\theta = \tilde{G}_\theta^{-1} \).
4. \( \forall i, L^i_{\theta} = \tilde{L}^i_{\theta} \).
5. \( \forall X^i \in \mathcal{L}_h(\mathcal{H}), X^i = L^i_{\theta} \perp T_\theta(M) \) with respect to \( \langle \cdot, \cdot \rangle_{\rho_\theta} \) implies \( X^i = \tilde{L}^i_{\theta} \perp \tilde{T}_\theta(M) \) with respect to \( \langle \cdot, \cdot \rangle_{\tilde{\rho}_\theta} \).

3. Asymptotically classical model

With this notion of the asymptotically classical model, we have the following result.

**Theorem IV.3** For a regular quantum statistical model, the following equivalences hold:

1. \( M \) is asymptotically classical.
2. \( \exists W_0 > 0, C^H_\theta[W_0] = C^S_\theta[W_0] \).

B. Discussion on Theorem IV.1

In this subsection, we discuss the meaning and its statistical consequences of Theorem IV.1.
We first note that two conditions 2 and 3 are nothing but condition (33). This is straightforward to understand if we regard $\partial \rho_\theta / \partial \theta^i$ as an m-representation of the tangent vector $\partial / \partial \theta^i$ and $L_{\theta,i}$ as an e-representation of it with respect to the SLD Fisher metric. The statement applies for the RLD case.

2. Quantum Fisher information

Condition 4 states that two quantum Fisher information matrices are identical. If this is the case, in fact, all possible monotone metric on $\mathcal{S}(\mathcal{H})$ are identical. In other words, they collapse to the single monotone metric. This is due to the facts that 1) the imaginary part of the RLD Fisher information vanishes, and 2) the SLD Fisher is the minimum and the real RLD Fisher is the maximum monotone metric (Petz’s theorem) [25].

We note that this result, equivalence between condition 1 and condition 4, was also stated in Ref. [30].

Next, we can contrast condition 5 to the condition for a D-invariant model in Lemma II.1: $L_{\theta} = \tilde{L}_{\theta}$ for all i. This latter condition is not equivalent to $G_{\theta} = \tilde{G}_{\theta}$ in general unless the imaginary part of the RLD Fisher information vanishes. Thus, condition 5 is a stronger condition than the condition for the D-invariant model as should be.

3. Tangent space

Condition 6 (or 7) means that the SLD tangent space is in the kernel of the commutation operator $D$. We split the SLD operator into two parts: a classical part and quantum part where the latter is defined by the change in a unitary direction. Since the $D$ operator maps the commutation relationship to the anti-commutation relationship as in Eq. (10), the quantum part of the SLD operator is expressed in terms of the commutation operator. With more analysis, we can show that the condition for the classical model is equivalent to vanishing of the quantum part of SLD operators. See also discussion given in Ch. 7 of the book [6].

4. Asymptotic bound

The last equivalent condition is a rather straightforward consequence once we combining all ingredients presented in the lemmas and other equivalent conditions for the classical model. However, the statistical implication of this condition is non-trivial in the sense that we only consider properties of asymptotically achievable bounds. One is the bound for the D-invariant model, and the other is the bound for the asymptotically classical model. Another implication of this equivalence is that there is no genuine quantum statistical model that is both D-invariant and asymptotically classical.

C. Proofs

1. Proof for Theorem IV.1

We give a proof for Theorem IV.1. As we stated before, all conditions below are about all parameter values $\theta$ unless otherwise stated.

Equivalence to 2 and 3:

First, we note that the definition of the classical model is equivalent to the commutativity of $\rho_\theta$ for all different values $\theta$, that is,

$$[\rho_\theta , \rho_{\theta'}] = 0 \text{ for all } \theta \neq \theta'. \quad (32)$$

By the standard matrix analysis, this is equivalent to:

$$\forall i, \left[ \partial / \partial \theta^i \rho_\theta , \rho_\theta \right] = 0. \quad (33)$$

From the definitions of the SLD and RLD operators, we can show that condition (33) is equivalent to $[L_{\theta,i} , \rho_\theta] = 0$ for all $i$. This is condition 2. Similarly, condition (33) can be converted to $[\tilde{L}_{\theta,i} , \rho_\theta] = 0$ for all $i$, which is condition 3. □

Equivalence to 4 and 5:

If the model is classical, the SLD operator $L_{\theta,i}$ commutes with the quantum state. Hence, operator equations (3) defining the SLD and RLD operators are identical. Since the SLD and RLD operator are uniquely defined, we obtain $L_{\theta,i} = \tilde{L}_{\theta,i}$ for all $i$. Next, assume condition 5, then matrices $\tilde{G}_{\theta}$ and $Z_{\theta}^{-1}$ are identical. Noting Re $Z_{\theta}^{-1} = G_{\theta}$, we get condition 4.

Last, suppose condition 4, $G_{\theta} = \tilde{G}_{\theta}$, then from Lemma II.3, this is possible if and only if Im $Z_{\theta} = 0$ and $L_{\theta} = \tilde{L}_{\theta}$ for all $i$. Since $g_{\theta,ij} = g_{\theta',ij}$, the latter condition leads to $L_{\theta,i} = \tilde{L}_{\theta,i}$ for all $i, j$. □

Equivalence to 6 and 7:

Condition 6 is to say that the SLD tangent space is in the kernel of the commutation operator. From definition of the commutation operator and the fact that $X \rho + \rho X = 0$ implies $X = 0$ if $\rho > 0$, we have

$$\ker D_{\rho} = \{ X \in \mathcal{L}(\mathcal{H}) \mid [X, \rho_\theta] = 0 \}. \quad (34)$$

This then immediately establishes equivalence between condition 1 and condition 6. A similar argument applies for condition 7. □

Equivalence to 8:

When the model is classical, conditions 4 and 5 give $L_{\theta} = \tilde{L}_{\theta}$ for all i (D-invariance). Combining it with $L_{\theta,i} = \tilde{L}_{\theta,i}$ leads to $Z_{\theta} = G_{\theta}^{-1}$. Hence, the definitions for D-invariant and asymptotically classical model are clearly satisfied, if the model is classical. Conversely, suppose that the model is D-invariant, $G_{\theta}^{-1} = Z_{\theta}$, and asymptotically classical, $Z_{\theta} = G_{\theta}^{-1}$. Then, it gives condition 4, $G_{\theta} = \tilde{G}_{\theta}$. □
2. Proof for Theorem IV.3

Proof: First: The third condition \( \text{Im} Z_\theta = 0 \) implies \( \forall W > 0, C^H_\theta[W] = C^S_\theta[W] \). This is because of \( C^H_\theta[W] \geq C^S_\theta[W], \forall W > 0 \) and the direct substitution gives \( h_0[\bar{L}_\theta[W] = C^S_\theta[W] + \text{Tr} \{ W^\dagger \text{Im} Z_\theta W^{\frac{1}{2}} \} = C^S_\theta[W] \).

Here \( \bar{L}_\theta = (L^1_\theta, L^2_\theta, \ldots, L^n_\theta) \in X_\theta \) is the collection of SLD dual operators. This means the set of SLD dual operators is the optimal achieving the lowest value in the definition of the Holevo bound (29).

By definition, the first condition obviously implies the second one: \( 3W_0 > 0, C^H_\theta[W_0] = C^S_\theta[W_0] \).

To show that the existence of a weight matrix \( W \) satisfying \( C^H_\theta[W] = C^S_\theta[W] \) implies the vanishing of the imaginary part of the matrix \( Z_\theta \), we prove the contraposition. That is, if \( \text{Im} Z_\theta \neq 0 \), then \( C^H_\theta[W] > C^S_\theta[W] \) holds for all weight matrices \( W \). Let us use the following substitution for optimizing the Holevo function:

\[
\bar{X} = (L^1_\theta, L^2_\theta, \ldots, L^n_\theta) + (K^1_\theta, K^2_\theta, \ldots, K^n_\theta),
\]

(34)

where \( K^i_\theta \) \( (i = 1, 2, \ldots, n) \) are tangent operators orthogonal to all SLD operators \( L_\theta \), with respect to the SLD inner product. With this, the Holevo function reads

\[
h_0[\bar{X}[W] = C^S_\theta[W] + \text{Tr} \{ W^\dagger \text{Im} (Z_\theta + K_\theta) W^{\frac{1}{2}} \},
\]

(35)

where \( n \times n \) matrix \( K_\theta = [(K^1, K^2, \ldots, K^n_\theta)] \) is hermite. We note that the last two terms:

\[
\text{Tr} \{ W^\dagger \text{Im} K_\theta \} + \text{Tr} \{ W^\dagger \text{Im} (Z_\theta + K_\theta) W^{\frac{1}{2}} \}
\]

is strictly positive since it vanishes if and only if \( \text{Re} K_\theta = 0 \) and \( \text{Im} (Z_\theta + K_\theta) = 0 \) hold. But these two conditions cannot be satisfied due to the assumption \( Z_\theta \neq 0 \) and the positivity of the matrix \( K_\theta \). Therefore, we show that if \( \text{Im} Z_\theta \neq 0 \), we have \( C^H_\theta[W] > C^S_\theta[W] \) for all \( W > 0 \). Finally, \( \text{Im} Z_\theta = 0 \iff \forall i, j, \text{tr} \{ (\rho_0[L_{\theta,i}, L_{\theta,j}]) = 0 \} \) can be shown by elementary algebra. Collecting these arguments proves Theorem IV.3. □

3. Proof for Corollary IV.4

Proof: Equivalence in condition 1:

Since \( G_\theta = \tilde{G}_\theta \iff G_\theta^{-1} = \tilde{G}_\theta^{-1} \), the first equivalence is immediate.

To prove the second equivalence to \( \tilde{G}_\theta^{-1} = \tilde{Z}_\theta \) in 1, let us assume first that a model is classical. Condition 7 of Theorem IV.1 gives

\[
D_{\rho_0}(\tilde{L}_\theta) = 0, \forall i.
\]

(36)

Then, Eq. (23) of Lemma II.6 yields \( \tilde{g}^{ij}_\theta - \tilde{z}^{ij}_\theta = 0 \) for all \( i, j \). Conversely, if \( \tilde{G}_\theta^{-1} = \tilde{Z}_\theta \) holds, we have the following equivalence from the first matrix inequality in Lemma II.5.

\[
\forall i, L_{\theta,i} = \tilde{L}_{\theta,i} \iff G_\theta + \tilde{G}_\theta \tilde{Z}_\theta \tilde{G}_\theta = 2\tilde{G}_\theta
\]

\[
\iff G_\theta + \tilde{G}_\theta G_\theta^{-1} \tilde{G}_\theta = 2\tilde{G}_\theta
\]

\[
\iff G_\theta = \tilde{G}_\theta.
\]

This proves the converse part.

The last equivalence to \( Z_\theta = \tilde{Z}_\theta \) in 1 is proven as follows. A classical model gives this condition is straightforward. Conversely, if this condition is satisfied, the second matrix inequality of Lemma II.4 is then expressed as

\[
Z_\theta \geq G_\theta^{-1}.
\]

(37)

Noting \( G_\theta^{-1} = \text{Re} Z_\theta \), this inequality concludes \( \text{Im} Z_\theta = 0 \). (Otherwise, the matrix inequality does not hold.) This then shows that the model is asymptotically classical, and we have \( G_\theta^{-1} = Z_\theta = \tilde{Z}_\theta \). The condition \( G_\theta^{-1} = \tilde{Z}_\theta \) holds if and only if the model is D-invariant from Lemma II.4. Therefore, the model is asymptotically classical and D-invariant, i.e., the classical model. □

Equivalence in condition 2:

The first equivalence is already proven in Theorem IV.3, whose proof is given in Ref. [16]. Here we note that both conditions can be proven immediately if we use Lemma II.4. □

Equivalence in condition 3:

This is proven in Theorem IV.3. □

V. EXAMPLES

A. Qubit models

When the dimension of the Hilbert space is two, i.e., a qubit system, we can explicitly work out classification of models. To analyze a given qubit model, it is convenient to use the Bloch vector representation of qubit states. Define a three dimensional real vector \( s_\theta = (s^i_\theta) \) for \( i = 1, 2, 3 \) by

\[
s^i_\theta := \text{tr} \{ \rho_0 \sigma_i \},
\]

(38)

where \( \sigma_i \) are the standard Pauli matrices. Since the mapping \( s_\theta \mapsto \rho_0 \) is bijective, a quantum statistical model for the qubit case can be defined as

\[
\mathcal{M} = \{ s_\theta \mid \theta \in \Theta \}.
\]

(39)

Based on the Bloch vector \( s_\theta \), we can derive closed formulas for the quantum score functions (SLD and RLD logarithmic derivative operators) and the quantum Fisher information matrices. (See, for example, Ref. [16].) In Ref. [16], we derived the following conditions for a given model 39 to be the D-invariant and asymptotically classical.

1. \( \mathcal{M} \) is D-invariant.

\[
\iff |s_\theta| \text{ is independent of } \theta.
\]
2. $\mathcal{M}$ is asymptotically classical.
   $\Leftrightarrow \partial_i s_0 \times \partial_j s_0 \ (\forall i \neq j)$ is orthogonal to $s_0$.

The equivalent condition for the D-invariant model immediately tells us that any unitary model on the qubit system is D-invariant. The converse statement is, of course, not true in general. For example, the following two-parameter model is D-invariant, but not unitary.

$$\mathcal{M} = \{ s_\theta = (\theta^1, \theta^2, \sqrt{s_0^2 - (\theta^1)^2 - (\theta^2)^2}) \mid \theta \in \Theta \}, \quad (40)$$

where $s_0 \in (0, 1)$ is a fixed constant and the parameter takes values within the region $\Theta \subset \mathbb{R}^2$ satisfying the positivity condition for the state.

Next, we can work out whether or not there exists a classical qubit model. It is straightforward to show that there cannot be any multi-parameter classical qubit model under the regularity condition, and thus only one-parameter classical model exists. The reason is simply because there can be a single parameter classical model embedded in a $2 \times 2$ matrix space. Any multi-parameter classical model becomes a non-regular model.

Finally, we ask if there can be a quasi-classical model in a qubit system. It turns out that there exists no such a quasi-classical qubit model. This is due to the fact that imposing commutativity between the SLD operators leads to a non-regular model.

To prove this statement, we note the commutation condition for the SLD operators is expressed in terms of the Bloch vectors as

$$[L_{\theta,i}, L_{\theta,j}] = 0 \Leftrightarrow \partial_i s_\theta \times \partial_j s_\theta = 0. \quad (41)$$

Consider a two-parameter qubit model. The condition $\partial_1 s_\theta \times \partial_2 s_\theta = 0$ is equivalent to linearly dependence of two vectors $\partial_1 s_\theta, \partial_2 s_\theta$. This then implies the existence of a function $c : \Theta \to \mathbb{R}$ such that $L_{\theta,1} = c(\theta)L_{\theta,2}$ holds. This contradicts with linearly independence of the tangent vectors. Note that, if this is the case, the dimension of the tangent space is one rather than two. The case of three parameter models can be checked similarly.

### B. Non-classical quasi-classical model

As we mentioned earlier, there exists a quantum statistical model that is quasi-classical (all SLD operators commute with each other) and non-classical. It is straightforward to observe that such cases arise if a model is non-regular. For example, quantum states are not full rank. Below, we give a simple regular statistical model in a three-dimensional quantum system ($d = 3$).

We consider the following two-parameter model:

$$\mathcal{M} := \{ \rho_\theta \mid \theta = (\theta^1, \theta^2) \in \Theta \}, \quad (42)$$

where a constant $c \in \mathbb{R} \ (c \neq 1)$ and smooth function $\lambda(\theta^j)$ are chosen arbitrary as long as the corresponding classical model for $\Lambda_{\theta^i}$

$$\mathcal{M}_1 := \{ p_{\theta^1, \theta^2} = (\lambda(\theta^1), c\lambda(\theta^1), 1 - (1 + c)\lambda(\theta^1)) \mid \theta^1 \in \Theta \},$$

satisfies $\mathcal{M}_1 \in \mathcal{P}(3)$. The SLD operators are calculated as

$$L_{\theta,1} = U_{\theta^2} \frac{\lambda}{\lambda} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -m(\theta^1) \end{pmatrix} U_{\theta^2}^{-1}, \quad (45)$$

$$L_{\theta,2} = U_{\theta^2} \frac{1 - c}{1 + c} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} U_{\theta^2}^{-1}, \quad (46)$$

where $\lambda = d\lambda(\theta^1)/d\theta^1$, $m(\theta^1) = 1 - (1 - (1 + c)\lambda(\theta^1))^{-1}$. To have a regular quantum model, we also impose $\lambda \neq 0$ for all $\theta^1$. It is clear that two SLD operators commute with each others for all $\theta$. The RLD operators are

$$\tilde{L}_{\theta,1} = L_{\theta,1}, \quad (47)$$

$$\tilde{L}_{\theta,2} = U_{\theta^2} \begin{pmatrix} 1 & 0 & -i(1 - c) \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} U_{\theta^2}^{-1}. \quad (48)$$

We can show that the SLD Fisher information matrix is diagonal and is given by

$$g_{\theta,11} = \frac{\lambda}{\lambda}(2 + m(\theta^1)^2), \quad (49)$$

$$g_{\theta,12} = g_{\theta,21} = 0, \quad (50)$$

$$g_{\theta,22} = \left( \frac{2}{1 + c} \right)^2 \lambda(\theta^1). \quad (51)$$

Whereas the RLD Fisher information matrix is

$$\tilde{g}_{\theta,11} = g_{\theta,11}, \quad (52)$$

$$\tilde{g}_{\theta,12} = \tilde{g}_{\theta,21} = 0, \quad (53)$$

$$\tilde{g}_{\theta,22} = \frac{(1 - c)^2(1 + c)}{c} \lambda(\theta^1). \quad (54)$$

It is easy to see that $\tilde{g}_{\theta,22} \geq g_{\theta,22}$ with equality if and only if $c = 1$, which is excluded. Therefore, $G_\theta \neq \tilde{G}_\theta$ holds and this model is not classical by Theorem IV.1.
VI. CONCLUDING REMARKS

We have derived classification and several equivalent characterizations of quantum parametric models based on the estimation error bound, the Holevo bound. Three classes are mainly discussed in this paper: the classical model, D-invariant model, and asymptotically classical model. We have also given relationships among these classes. In particular, the classical model can be viewed as the intersection of the D-invariant and asymptotically classical models. The classical model has several different interpretations based on the geometrical point of view. Although all conditions are related to another, they show different side-sights on the classical model as a sub-model of the general quantum statistical model. We have also analyzed the quasi-classical model, in which all SLD operators commute with each other, and have shown that it is still quantum model.

Before closing the paper, we list several open questions to be addressed. In this paper, we have focused on the global aspects of the quantum statistical models only. First extension is to analyze local properties of each class of the quantum model. This is important to understand their properties from the point of view of information geometry. In Ref. [16], we analyzed the local properties for the D-invariant and asymptotically classical models. Therefore, it is interesting to see where the local classical model is a useful concept or not. Second, we don’t know how much local properties determine the global property for a given model. An interesting question is then to ask whether we can characterize the class model globally by local conditions. Third, we have only used two different quantum Fisher information matrices, the SLD and RLD Fisher, together with their dual matrices $Z_\theta$ and $\tilde{Z}_\theta$. We expect that other families of quantum Fisher information should also give model classification and characterization. Last, there should other important classes for the quantum parametric models other than discussed in this paper. These are some of untouched questions in this paper, and should be examined in the subsequent publication.

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