THE KISSING PROBLEM IN THREE DIMENSIONS

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Abstract

The kissing number $k(3)$ is the maximal number of equal size nonoverlapping spheres in three dimensions that can touch another sphere of the same size. This number was the subject of a famous discussion between Isaac Newton and David Gregory in 1694. The first proof that $k(3) = 12$ was given by Schütte and van der Waerden only in 1953. In this paper we present a new solution of the Newton-Gregory problem that uses our extension of the Delsarte method. This proof relies on basic calculus and simple spherical geometry.

Keywords: Kissing numbers, thirteen spheres problem, Newton-Gregory problem, Legendre polynomials, Delsarte’s method

1 Introduction

The kissing number $k(d)$ is the highest number of equal nonoverlapping spheres in $\mathbb{R}^d$ that can touch another sphere of the same size. In three dimensions the kissing number problem is asking how many white billiard balls can kiss (touch) a black ball.

The most symmetrical configuration, 12 billiard balls around another, is achieved if the 12 balls are placed at positions corresponding to the vertices of a regular icosahedron concentric with the central ball. However, these 12 outer balls do not kiss each other and may all be moved freely. So perhaps if you moved all of them to one side a 13th ball would possibly fit in?

This problem was the subject of a famous discussion between Isaac Newton and David Gregory in 1694 (May 4, 1694; see an interesting article [21] for details of this discussion). Most reports say that Newton believed the answer was 12 balls, while Gregory thought that 13 might be possible. However, Casselman [5] found some puzzling features in this story.

This problem is often called the thirteen spheres problem. Hoppe [9] thought he had solved the problem in 1874. But there was a mistake - an analysis of this mistake was published by Hales in 1994 [8] (see also [20]). Finally this problem

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was solved by Schütte and van der Waerden in 1953 [19]. A subsequent two-page sketch of an elegant proof was given by Leech [11] in 1956. Most people agree that Leech’s proof is correct, but there are gaps in his exposition, many involving sophisticated spherical trigonometry. (Leech’s proof was presented in the first edition of the well-known book by Aigner and Ziegler [1], the authors removed this chapter from the second edition because a complete proof would have had to include so much spherical trigonometry.) The thirteen spheres problem continues to be of interest, and new proofs have been published in the last few years by Hsiang [10], Maehara [13], Böröczky [3] and Austreicher [2].

The main progress in the kissing number problem in high dimensions was at the end of the 1970s. Levenshtein [12], and independently, Odlyzko and Sloane [16] (= [6, Chap.13]) using Delsarte’s method in 1979 proved that \( k(8) = 240 \) and \( k(24) = 196560 \). This proof is surprisingly short, clean, and technically easier than all proofs in three dimensions. However, \( d = 8, 24 \) are the only dimensions in which this method gives a precise result. For other cases (for instance, \( d = 3, 4 \)) the upper bounds exceed the lower.

We found an extension of the Delsarte method in 2003 [14](see details in [15]) that allowed us to prove the bound \( k(4) < 25 \), i.e. \( k(4) = 24 \). This extension also yields a proof \( k(3) < 13 \).

The first version of these proofs was relatively short, but used a numerical solution of some nonconvex constrained optimization problems. Later [15] these calculations were reduced to calculations of roots of polynomials in one variable.

In this paper we present a new proof of the Newton-Gregory problem. This proof needs just basic calculus and simple spherical geometry.

2 \( k(3) = 12 \)

Let us recall the definition of Legendre polynomials \( P_k(t) \) by the recurrence formula:

\[
P_0 = 1, \quad P_1 = t, \quad P_2 = \frac{3}{2} t^2 - \frac{1}{2}, \ldots, \quad P_k = \frac{2k - 1}{k} t P_{k-1} - \frac{k - 1}{k} P_{k-2}
\]

or equivalently

\[
P_k(t) = \frac{1}{2^k k!} \frac{d^k}{dt^k} (t^2 - 1)^k \quad (\text{Rodrigues’ formula}).
\]

Lemma 1. Let \( X = \{x_1, x_2, \ldots, x_n\} \) be any finite subset of the unit sphere \( S^2 \) in \( \mathbb{R}^3 \). By \( \phi_{i,j} = \text{dist}(x_i, x_j) \) we denote the spherical (angular) distance between \( x_i \) and \( x_j \). Then

\[
\sum_{i=1}^{n} \sum_{j=1}^{n} P_k(\cos(\phi_{i,j})) \geq 0.
\]
This lemma easily follows from Schoenberg’s theorem \[18\] for Gegenbauer (ultraspherical) polynomials $G_k^{(d)}$. (Note that $P_k = G_k^{(d)}$. ) For completeness we give a proof of Lemma 1 in the Appendix.

Let
\[
f(t) = \frac{2431}{80} t^9 - \frac{1287}{20} t^7 + \frac{18333}{400} t^5 + \frac{343}{40} t^4 - \frac{83}{10} t^3 - \frac{213}{100} t^2 + \frac{t}{10} - \frac{1}{200}.
\]

**Remark.** This polynomial of degree 9 is satisfying the assumptions of the extended Delsarte’s method \[14, 15\]. An algorithm for calculating suitable polynomials is presented in the Appendix of \[15\].

**Lemma 2.** Suppose $X = \{x_1, x_2, \ldots, x_n\} \subset S^2$. Then
\[
S(X) := \sum_{i=1}^{n} \sum_{j=1}^{n} f(\cos(\phi_{i,j})) \geq n^2.
\]

**Proof.** The expansion of $f$ in terms of $P_k$ is
\[
f = \sum_{k=0}^{9} c_k P_k = P_0 + \frac{8}{5} P_1 + \frac{87}{25} P_2 + \frac{33}{20} P_3 + \frac{49}{25} P_4 + \frac{1}{10} P_5 + \frac{8}{25} P_9.
\]

We have $c_0 = 1$, $c_k \geq 0$, $k = 1, 2, \ldots, 9$. Using Lemma 1 we get
\[
S(X) = \sum_{k=0}^{9} c_k \sum_{i=1}^{n} \sum_{j=1}^{n} P_k(\cos(\phi_{i,j})) \geq \sum_{i=1}^{n} \sum_{j=1}^{n} c_0 P_0 = n^2.
\]

If $n$ unit spheres kiss the unit sphere in $\mathbb{R}^3$, then the set of kissing points is an arrangement on the central sphere such that the (Euclidean) distance between any two points is at least 1. So the kissing number problem can be stated in another way: How many points can be placed on the surface of $S^2$ so that the angular separation between any two points is at least $60^\circ$?

**Lemma 3.** Suppose $X = \{x_1, x_2, \ldots, x_n\}$ is a subset of $S^2$ such that the angular separation $\phi_{i,j}$ between any two distinct points $x_i, x_j$ is at least $60^\circ$. Then
\[
S(X) = \sum_{i=1}^{n} \sum_{j=1}^{n} f(\cos(\phi_{i,j})) < 13n.
\]

We give a proof of Lemma 3 in the next section.

**Theorem.** $k(3) = 12$.

**Proof.** Suppose $X$ is a kissing arrangement on $S^2$ with $n = k(3)$. Then $X$ satisfies the assumptions in Lemmas 2 and 3. Therefore, $n^2 \leq S(X) < 13n$. From this $n < 13$ follows, i.e. $n \leq 12$. From the other side we have $k(3) \geq 12$, showing that $n = k(3) = 12$. \[\Box\]
3 Proof of Lemma 3.

We need one fact from spherical trigonometry, namely the law of cosines:

\[
\cos \phi = \cos \theta_1 \cos \theta_2 + \sin \theta_1 \sin \theta_2 \cos \varphi,
\]

for a spherical triangle \(ABC\) with sides of angular lengths \(\theta_1, \theta_2, \phi\) and \(\angle BAC = \varphi\) (Fig. 1). For \(\varphi = 90^\circ\), this reduces to the spherical Pythagorean theorem: \(\cos \phi = \cos \theta_1 \cos \theta_2\).

**Proof.** 1. The polynomial \(f(t)\) satisfies the following properties (see Fig.2):

(i) \(f(t)\) is a monotone decreasing function on the interval \([-1, -t_0]\);
(ii) \(f(t) < 0\) for \(t \in (-t_0, 1/2]\);
where \(f(-t_0) = 0, t_0 \approx 0.5907\).

These properties hold because \(f(t)\) has only one root \(-t_0\) on \([-1, 1/2]\), and there are no zeros of the derivative \(f'(t)\) (eighth degree polynomial) on \([-1, -t_0]\).

Let \(S_i(X) := \sum_{j=1}^{n} f(\cos(\phi_{i,j}))\), then \(S(X) = \sum_{i=1}^{n} S_i(X)\). From this it follows

that if \(S_i(X) < 13\) for \(i = 1, 2, \ldots, n\), then \(S(X) < 13n\).

We obviously have \(\phi_{i,i} = 0\), so \(f(\cos \phi_{i,i}) = f(1)\). Note that our assumption on \(X\) (\(\phi_{i,j} \geq 60^\circ, i \neq j\)) yields \(\cos \phi_{i,j} \leq 1/2\). Therefore, \(\cos \phi_{i,j}\) lies in the interval \([-1, 1/2]\]. By (ii) we have \(f(\cos \phi_{i,j}) \leq 0\) whenever \(\cos \phi_{i,j} \in [-t_0, 1/2]\).

Let \(J(i) := \{j : \cos \phi_{i,j} \in [-1, -t_0]\}\). We obtain

\[
S_i(X) \leq T_i(X) := f(1) + \sum_{j \in J(i)} f(\cos \phi_{i,j}).
\]  

(1)

Let \(\theta_0 = \arccos t_0 \approx 53.794^\circ\). Then \(j \in J(i)\) iff \(\phi_{i,j} > 180^\circ - \theta_0\), i.e. \(\theta_j < \theta_0\), where \(\theta_j = 180^\circ - \phi_{i,j}\). In other words all \(x_{i,j}, j \in J(i)\), lie inside the spherical cap of center \(e_0\) and radius \(\theta_0\), where \(e_0 = -x_i\) is the antipodal point to \(x_i\).
2. Let us consider on $S^2$ points $e_0, y_1, \ldots, y_m$ such that

$$\phi_{i,j} := \text{dist}(y_i, y_j) \geq 60^\circ, \forall \ i \neq j, \ \theta_i := \text{dist}(e_0, y_i) < \theta_0 \ for \ 1 \leq i \leq m. \ (2)$$

Denote by $\mu$ the highest value of $m$ such that the constraints in (2) allow a nonempty set of points $y_1, \ldots, y_m$.

Suppose that $0 \leq m \leq \mu$ and $Y = \{y_1, \ldots, y_m\}$ satisfies (2). Let

$$H(Y) = H(y_1, \ldots, y_m) := f(1) + f(-\cos \theta_1) + \ldots + f(-\cos \theta_m),$$

$$h_m := \sup_Y \{H(Y)\}, \ h_{\text{max}} := \max\{h_0, h_1, \ldots, h_\mu\}. $$

It is clear that $T_i(X) \leq h_m$, where $m = |J(i)|$. From (1) it follows that $S_i(X) \leq h_m$. Thus, if we prove that $h_{\text{max}} < 13$, then we prove Lemma 3.

3. Now we prove that $\mu \leq 4$.

Suppose $Y = \{y_1, \ldots, y_m\} \subset S^2$ satisfies (2). By symmetry we may assume that $e_0$ is the North pole and $y_i$ has polar coordinates $(\theta_i, \varphi_i)$. Then from the law of cosines we have:

$$\cos \phi_{i,j} = \cos \theta_i \cos \theta_j + \sin \theta_i \sin \theta_j \cos(\varphi_i - \varphi_j).$$

Note that $\theta_i > 0$ for $m \geq 2$. Conversely, $y_i = e_0, \ \theta_j = \phi_{i,j} \geq 60^\circ > \theta_0$, a contradiction. From (2) we have $\cos \phi_{i,j} \leq 1/2$, then

$$\cos(\varphi_i - \varphi_j) \leq \frac{1/2 - \cos \theta_i \cos \theta_j}{\sin \theta_i \sin \theta_j}. \ \ (3)$$

Let

$$Q(\alpha, \beta) := \frac{1/2 - \cos \alpha \cos \beta}{\sin \alpha \sin \beta},$$

then

$$Q'_\alpha(\alpha, \beta) = \frac{\partial Q(\alpha, \beta)}{\partial \alpha} = \frac{2 \cos \beta - \cos \alpha}{2 \sin^2 \alpha \sin \beta}.$$ 

From this it follows that if $0 < \alpha, \beta \leq \theta_0$, then $\cos \beta > 1/2$ (because $\theta_0 < 60^\circ$); so then $Q'_\alpha(\alpha, \beta) > 0$, and $Q(\alpha, \beta) \leq Q(\theta_0, \beta) = Q(\beta, \theta_0) \leq Q(\theta_0, \theta_0)$. Therefore,

$$\frac{1/2 - \cos \theta_i \cos \theta_j}{\sin \theta_i \sin \theta_j} \leq \frac{1/2 - \cos^2 \theta_0}{\sin^2 \theta_0} = \frac{1/2 - t_0^2}{1 - t_0^2}. $$

Combining this inequality and (3), we get

$$\cos(\varphi_i - \varphi_j) \leq \frac{1/2 - t_0^2}{1 - t_0^2}$$

Note that arccos$((1/2 - t_0^2)/(1 - t_0^2)) \approx 76.582^\circ > 72^\circ$. This implies that $m \leq 4$ because no more than four points can lie in a circle with minimum angular separation between any two points greater than $72^\circ$.
4. Now we have to prove that \( h_{\text{max}} = \max \{ h_0, h_1, h_2, h_3, h_4 \} < 13. \)

We obviously have \( h_0 = f(1) = 10.11 < 13. \)

From (i) follows that \( f(-\cos \theta) \) is a monotone decreasing function in \( \theta \) on \( [0, \theta_0] \). Then for \( m = 1 \) : \( \bar{H}(y_1) = f(1) + f(-\cos \theta_1) \) attains its maximum at \( \theta_1 = 0. \) So then

\[
h_1 = f(1) + f(-1) = 12.88 < 13.
\]

5. Let us consider for \( m = 2, 3, 4 \) an arrangement \( \{ e_0, y_1, \ldots, y_m \} \) in \( S^2 \) that gives \( H(Y) = h_m. \) Here \( y_i \neq e_0 \) (see 3). Note that in this arrangement, points \( y_k \) cannot be shifted towards \( e_0 \) because in this case \( H(Y) \) increases.

For \( m = 2 \) this yields \( e_0 \in y_1y_2, \) and \( \text{dist}(y_1, y_2) = 60^\circ. \) If \( e_0 \notin y_1y_2, \) then the whole arc \( y_1y_2 \) can be shifted towards \( e_0. \) If \( \text{dist}(y_1, y_2) > 60^\circ \), then \( y_1 \) (and \( y_2 \)) can be shifted towards \( e_0. \)

For \( m = 3 \) we prove that \( \Delta_3 = y_1y_2y_3 \) is a spherical regular triangle with edge length \( 60^\circ. \) As above, \( e_0 \in \Delta_3, \) otherwise the whole triangle can be shifted towards \( e_0. \) Suppose \( \text{dist}(y_1, y_i) > 60^\circ, \ i = 2, 3, \) then \( \text{dist}(y_1, e_0) \) can be decreased.

From this follows that for any \( y_i \) at least one of the distances \( \{ \text{dist}(y_i, y_j) \} \) is equal to \( 60^\circ. \) Therefore, at least two sides of \( \Delta_3 \) (say \( y_1y_2 \) and \( y_1y_3 \)) have length \( 60^\circ. \) Also \( \text{dist}(y_2, y_3) = 60^\circ, \) conversely \( y_3 \) (or \( y_2 \), if \( e_0 \in y_1y_3 \)) can be rotated about \( y_1 \) by a small angle towards \( e_0 \) (Fig.3).

For \( m = 4 \) we first prove that \( \Delta_4 := \text{conv} \ Y \) (the spherical convex hull of \( Y \)) is a convex quadrilateral. Conversely, we may assume that \( y_4 \in y_1y_2y_3. \)

The great circle through \( y_4 \) that is orthogonal to the arc \( e_0y_4 \) divides \( S^2 \) into two hemispheres: \( H_1 \) and \( H_2. \) Suppose \( e_0 \in H_1, \) then at least one \( y_i \) (say \( y_3 \)) belongs to \( H_2 \) (Fig.4). So the angle \( \angle e_0y_4y_3 \) greater than \( 90^\circ, \) then (again from the law of cosines) \( \text{dist}(y_3, e_0) > \text{dist}(y_3, y_4). \) Thus,

\[
\theta_3 = \text{dist}(y_3, e_0) > \text{dist}(y_3, y_4) > 60^\circ > \theta_0 - \text{ a contradiction.}
\]

Arguing as for \( m = 3 \) it is easy to prove that for any vertex \( y_i \) there are at least two vertices \( y_j \) at the distance \( 60^\circ \) from \( y_i. \) Note that the diagonals of \( \Delta_4 \) cannot be both of lengths \( 60^\circ. \) Conversely, at least one side of \( \Delta_4 \) is of length less than \( 60^\circ. \) Thus, \( \Delta_4 \) is a spherical equilateral quadrangle (rhomb) with edge length \( 60^\circ. \)

6. Now we introduce the function \( F_1(\psi), \) \( \psi \in [60^\circ, 2\theta_0]; \)

\[
F_1(\psi) := \max_{\psi/2 \leq \theta \leq \theta_0} \{ \bar{F}_1(\theta, \psi) \}, \quad \bar{F}_1(\theta, \psi) = f(-\cos \theta) + f(-\cos(\psi - \theta)).
\]

So if \( \text{dist}(y_i, y_j) = \psi, \) then

\[
f(-\cos \theta_i) + f(-\cos \theta_j) \leq F_1(\psi). \tag{4}
\]

Therefore,

\[
H(y_1, y_2) \leq h_2 = f(1) + F_1(60^\circ) \approx 12.8749 < 13.
\]

\footnote{For given \( \psi, \) the value \( F_1(\psi) \) can be found as the maximum of the 9th degree polynomial \( \Omega(s) = \bar{F}_1(\theta, \psi), \ s = \cos(\theta - \psi/2), \) on the interval \( [\cos(\theta_0 - \psi/2), 1]. \)}
7. Let \( m = 4 \), \( d_1 = \text{dist}(y_1, y_3), d_2 = \text{dist}(y_2, y_4) \). Since \( \Delta_4 = y_1y_2y_3y_4 \) is a spherical rhomb, we have \( \cos(d_1/2) \cos(d_2/2) = 1/2 \) (Pythagorean theorem, the diagonals \( y_1y_3, y_2y_4 \) of \( \Delta_4 \) are orthogonal). So if

\[
\rho(s) := 2 \arccos \frac{1}{2 \cos(s/2)},
\]

then

\[
\rho(d_1) = d_2, \quad \rho(d_2) = d_1, \quad \rho(90^\circ) = 90^\circ, \quad \rho(s) = s.
\]

Suppose \( d_1 \leq d_2 \). The inequalities \( \theta_i \leq \theta_0 \) yield \( d_2 \leq 2\theta_0 \). Then

\[
\rho(2\theta_0) \leq d_1 \leq 90^\circ \leq d_2 \leq 2\theta_0.
\]

Now we consider two cases:

1) \( \rho(2\theta_0) \leq d_1 < 77^\circ \), and

2) \( 77^\circ \leq d_1 \leq 90^\circ \).

1) Clearly, \( F_1(\psi) \) is a monotone decreasing function in \( \psi \). Then (4) implies

\[
f(-\cos \theta_1) + f(-\cos \theta_3) \leq F_1(d_1) \leq F_1(\rho(2\theta_0)),
\]

\[
f(-\cos \theta_2) + f(-\cos \theta_4) \leq F_1(d_2) = F_1(\rho(d_1)) < F_1(\rho(77^\circ)),
\]

so then

\[
H(Y) < f(1) + F_1(\rho(2\theta_0)) + F_1(\rho(77^\circ)) \approx 12.9171 < 13.
\]

2) In this case we have

\[
H(Y) \leq f(1) + F_1(77^\circ) + F_1(90^\circ) \approx 12.9182 < 13.
\]

Thus, \( h_4 < 13 \).

8. Our last step is to show that \( h_3 < 13 \).

Since \( \Delta_3 \) is a regular triangle, \( H(Y) = f(1) + f(-\cos \theta_1) + f(-\cos \theta_2) + f(-\cos \theta_3) \) is a symmetric function in the \( \theta_i \), so it is sufficient to consider the case \( \theta_1 \leq \theta_2 \leq \theta_3 \leq \theta_0 \).

In this case \( R_0 \leq \theta_3 \leq \theta_0 \), where \( R_0 = \arccos \sqrt{2/3} \approx 35.2644^\circ \) is the (spherical) circumradius of \( \Delta_3 \).

\[\text{A more detailed analysis shows } h_3 \approx 12.8721, \ h_4 \approx 12.4849.\]
Let \( y_c \) be the center of \( \Delta_3 \). We have \( \gamma := \angle y_1 y_3 y_c = \angle y_2 y_3 y_c \). Using the law of cosines for the triangle \( y_1 y_3 y_c \), we get \( \gamma = \arccos \sqrt{2/3} \), i.e. \( \gamma = R_0 \).

Denote the angle \( \angle e_0 y_3 y_c \) by \( u \). Then (see Fig.5)

\[
\begin{align*}
\cos \theta_1 &= \cos 60^\circ \cos \theta_3 + \sin 60^\circ \sin \theta_3 \cos (R_0 - u), \\
\cos \theta_2 &= \cos 60^\circ \cos \theta_3 + \sin 60^\circ \sin \theta_3 \cos (R_0 + u),
\end{align*}
\]

where \( 0 \leq u \leq u_0 := \arccos(\cot \theta_3 / \sqrt{3}) - R_0 \). Note that if \( u = u_0 \), then \( \theta_2 = \theta_3 \); \( u = 0 \) yields \( \theta_1 = \theta_2 \); and if \( 0 < u < u_0 \), then \( \theta_1 < \theta_2 < \theta_3 \).

For fixed \( \theta_3 = \psi \), \( H(y_1, y_2) \) is a polynomial of degree 9 in \( s = \cos u \). Denote by \( F_2(\psi) \) the maximum of this polynomial on the interval \([\cos u_0, 1]\).

Let

\[ \{\psi_1, \ldots, \psi_6\} = \{R_0, 38^\circ, 41^\circ, 44^\circ, 48^\circ, \theta_0\}. \]

It is clear that \( F_2(\psi) \) is a monotone increasing function in \( \psi \) on \([R_0, \theta_0]\). From the other side, \( f(\cos \psi) \) is a monotone decreasing function in \( \psi \). Therefore for \( \theta_3 \in [\psi_i, \psi_{i+1}] \) we have

\[ H(Y) = H(y_1, y_2) + f(\cos \theta_3) < w_i := F_2(\psi_{i+1}) + f(\cos \psi_i). \]

Since,

\[ \{w_1, \ldots, w_5\} \approx \{12.9425, 12.9648, 12.9508, 12.9606, 12.9519\}, \]

we get \( h_3 < \max\{w_i\} < 13 \).

Thus, \( h_m < 13 \) for all \( m \) as required.

\[ \square \]

**Appendix. Proof of Lemma 1.**

In this proof we are using Schoenberg’s original proof \[18\] which is based on the addition theorem for Gegenbauer polynomials.\[3\] The addition theorem for Legendre polynomials was discovered by Laplace and Legendre in 1782-1785:

\[
P_k(\cos \theta_1 \cos \theta_2 + \sin \theta_1 \sin \theta_2 \cos \varphi) = P_k(\cos \theta_1)P_k(\cos \theta_2) + 2 \sum_{m=1}^{k} \frac{(k-m)!}{(k+m)!} P_k^m(\cos \theta_1)P_k^m(\cos \theta_2) \cos m\varphi
\]

\[
= \sum_{m=0}^{k} c_{m,k} P_k^m(\cos \theta_1)P_k^m(\cos \theta_2) \cos m\varphi,
\]

where

\[
P_k^m(t) = (1 - t^2)^\frac{m}{2} \frac{d^m}{dt^m} P_k(t).
\]

(See details in \[14\] and \[17\].)

\[3\] Pfender and Ziegler \[17\] give a proof as a simple consequence of the addition theorem for spherical harmonics. This theorem is not so elementary. The addition theorem for Legendre polynomials can be proven by elementary algebraic calculations.
Proof. Let $X = \{x_1, \ldots, x_n\} \subset S^2$ and $x_i$ has spherical (polar) coordinates $(\theta_i, \varphi_i)$. Then from the law of cosines we have:

$$\cos \varphi_{i,j} = \cos \theta_i \cos \theta_j + \sin \theta_i \sin \theta_j \cos \varphi_{i,j}, \quad \varphi_{i,j} := \varphi_i - \varphi_j,$$

which yields

$$\sum_{i,j} P_k(\cos \varphi_{i,j}) = \sum_{i,j} \sum_{m=0}^k c_{m,k} P_k^m(\cos \theta_i) P_k^m(\cos \theta_j) \cos m\varphi_{i,j}$$

$$= \sum_m c_{m,k} \sum_{i,j} u_{m,i} u_{m,j} \cos m\varphi_{i,j}, \quad u_{m,i} = P_k^m(\cos \theta_i).$$

Let us prove that for any real $u_1, \ldots, u_n$

$$\sum_{i,j} u_i u_j \cos m\varphi_{i,j} \geq 0.$$

Pick $n$ vectors $v_1, \ldots, v_n$ in $\mathbb{R}^2$ with coordinates $v_i = (\cos m\varphi_i, \sin m\varphi_i)$. If $v = u_1 v_1 + \ldots + u_n v_n$, then

$$0 \leq ||v||^2 = \langle v, v \rangle = \sum_{i,j} u_i u_j \cos m\varphi_{i,j}.$$

This inequality and the inequalities $c_{m,k} > 0$ complete our proof. \qed

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