Collective oscillation modes of a superfluid Bose–Fermi mixture

Wen Wen 1,2, Ying Wang 1 and Jianyong Wang 1
1 Department of Mathematics and Physics, Hohai University, Changzhou 213022, People’s Republic of China
2 College of Science, Hohai University, Nanjing 210098, People’s Republic of China
3 School of Science, Jiangsu University of Science and Technology, Zhenjiang 212003, People’s Republic of China
E-mail: wenwen1666ma@163.com

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Abstract
In this work, we present a theoretical study for the collective oscillation modes, i.e. quadrupole, radial and axial mode, of a mixture of Bose and Fermi superfluids in the crossover from a Bardeen-Cooper-Schrieffer (BCS) superfluid to a molecular Bose–Einstein condensate (BEC) in harmonic trapping potentials with cylindrical symmetry of experimental interest. To this end, we start from the coupled superfluid hydrodynamic equations for the dynamics of Bose–Fermi superfluid mixtures and use the scaling theory that has been developed for a coupled system. The collective oscillation modes of Bose–Fermi superfluid mixtures are found to crucially depend on the overlap integrals of the spatial derivations of density profiles of the Bose and Fermi superfluids at equilibrium. We not only present the explicit expressions for the overlap density integrals, as well as the frequencies of the collective modes provided that the effective Bose–Fermi coupling is weak, but also test the valid regimes of the analytical approximations by numerical calculations in realistic experimental conditions. In the presence of a repulsive Bose–Fermi interaction, we find that the frequencies of the three collective modes of the Bose and Fermi superfluids are all upshifted, and the change speeds of the frequency shifts in the BCS–BEC crossover can characterize the different groundstate phases of the Bose–Fermi superfluid mixtures for different trap geometries.

1. Introduction
The recent experimental realization of a mixture of Bose and Fermi superfluids in ultracold atoms has generated much interest in the properties of this new state of matter [1–6], and led to a surge of theoretical activity [7–18]. Collective excitations that characterize a system’s response to small perturbations constitute one of the main sources of information for understanding the physics of many-body systems [19, 20]. In the last two decades, collective modes have been extensively investigated to understand the properties of atomic gases in various systems [21–38], including bosons [39, 40], fermions [41, 42], and multi-components, such as spin–orbit-coupled bosons [43] and fermions [44], Bose–Bose mixtures [45] and degenerate Bose–Fermi mixtures [46, 47].

The Bose–Fermi superfluid mixture [1–6], which is different from other coupled systems obtained previously [43–47], can provide a unique setting for studying and understanding the properties of interacting quantum systems belonging to different quantum statistics. The fermion–fermion interactions can be widely tuned with a magnetic field Feshbach resonance. For the strongly-repulsive interaction, it is a mixture of two Bose–Einstein condensate (BEC), in which one made of atoms and the other of molecules. For the weakly-attractive interaction it is a mixture of a Bose superfluid with a Bardeen-Cooper-Schrieffer (BCS)-type superfluid. Such strong interactions are difficult to generate with bosonic atoms. However, three body losses and recombination processes are significantly lower with fermions due to the Pauli principle. The lifetime of a mixture of Bose and Fermi superfluids has been demonstrated experimentally to be in the order of a few seconds [1]. Such stability allows us to observe this mixture oscillating back and forth in harmonic traps over numerous periods without visible damping, and study how the Bose–Fermi interaction affects the dipole modes of Bose and Fermi superfluids. The long-lived center–of–mass oscillations have been realized in the Bose–Fermi...
superfluid mixtures of $^7$Li–$^6$Li [1, 2], $^{41}$K–$^6$Li [6], and $^{174}$Yb–$^6$Li [5], and the Bose–Fermi interaction gives rise to a rich behavior. In the presence of a repulsive Bose–Fermi interaction, the frequencies of the dipole oscillations of the Bose and Fermi superfluids are both downshifted in the weakly confined direction [1, 5, 6], and the frequency shifts increase monotonically from the BCS side to the BEC side [48, 49]. In contrast, the frequency in the tight confinement for the Bose superfluid is upshifted, whereas the frequency for the Fermi superfluid is still downshifted [6]. The frequency shifts show non-monotonic and resonantlike behaviors in both directions around the BCS side [6], which may be originated from the effects of fermionic pairs breaking [50].

It is naturally to ask how Bose–Fermi interaction affects the collective modes of Bose and Fermi superfluids in the BCS–BEC crossover, which is of great interest recently. However, a theoretical study for the collective oscillation modes of Bose–Fermi superfluid mixtures in a realistic experimental situation is highly nontrivial. In this work, to study quadrupole, radial and axial modes in cylindrically symmetric traps, we start from the coupled superfluid hydrodynamic equations describing the dynamics of the Bose–Fermi superfluid mixtures. The scaling method for coupled systems is then applied, and the eigenvalue equations for the coupled collective modes are obtained, which are crucially sensitive to the overlap integrals of the spatial derivations of the Bose and Fermi densities at groundstate. To present the explicit expressions for these integrals, we use a perturbative approximation for the coupled Bose and Fermi density profiles and in the overlap region the Fermi density is replaced by the value in the trap center. The analytical results for the frequencies of collective oscillation modes are calculated in realistic experimental parameters [6], and the valid regimes of the analytical approximations are confirmed by the numerical calculations. In the presence of a repulsive Bose–Fermi interaction, we find that the frequencies for the collective modes of the Bose and Fermi superfluids are all upshifted, and the frequency shifts for the Fermi superfluid are smaller than the bosons, due to a larger number of the particles. For a fixed repulsive Bose–Fermi interaction, the frequency shifts increase from the BCS side to the BEC side, and the behaviors of the frequency shifts for the Fermi and Bose superfluids are different in the BEC regime. The different changes of the frequency shifts of the collective modes can be used to characterize different ground configurations of the mixtures [51–54] in different trap geometries. This is because that the frequency shifts not only originate from the Bose–Fermi interaction, but also are sensitive to the spatial distributions of the equilibrium Bose and Fermi densities. The Bose superfluid is localized in a small region in the trapping center within the Fermi superfluid, which produces a depletion of the Fermi density due to the repulsive Bose–Fermi interaction. As the interaction energy of the Fermi superfluid decreases from the BCS side to the BEC side, such depletion becomes more pronounced and its boundary are steeper. Thus the overlap integrals of the spatial deviations of these two densities increase, which result in a stronger coupling and larger frequency shifts. In the BEC regime the depletion of the Fermi density in the center is completely, which is accompanied by significant increases of the collective mode frequencies of the Bose–Fermi superfluid mixtures. In recent experiments on the degenerate Bose–Fermi mixture of $^{41}$K–$^6$Li [46], a rapid frequency upshift of the breathing mode of the bosons is observed, which is attributed by the emergent interface when the mixture undergoes phase separation [55] by increasing the repulsive interspecies interaction. We hope that our theoretical results can provide a reference for future experiments on collective oscillation modes of Bose–Fermi superfluid mixtures in the BCS–BEC crossover.

This paper is organized as follows. The coupled hydrodynamic equations are introduced in section 2.1, and within the scaling theory the coupled set of differential equations for the relevant scaling parameters are derived in section 2.2. Subsequently, the dispersion relations and frequencies of the collective modes of the Bose–Fermi superfluid mixtures in cylindrically symmetric traps are obtained in section 2.3. The analytical derivations for the overlap density integrals are presented in section 2.4. In section 3, the frequencies of the collective modes of the Bose–Fermi superfluid mixtures in a realistic experimental setting and their physical properties for different trap geometries are discussed analytically, and the valid parameter regimes of the analytical approximations are demonstrated by the numerical calculations. Section 4 is for conclusion.

2. Basic equations

2.1. Superfluid hydrodynamic model

We consider a mixture of bosonic atoms and two spin components of fermionic atoms which is prepared in superfluid state at a low enough temperature. The dynamic properties of the Bose–Fermi superfluid mixture can be described by coupled hydrodynamic equations for superfluid. For bosons the hydrodynamic equations are given by [19, 56]

$$\frac{\partial n_b}{\partial t} + \nabla \cdot (n_b v_b) = 0,$$

(1a)
where the trapping potential acting on bosons is cylindrically symmetric with the form $V_{ext}^b(r) = m_b [\omega_{b,1}^2 (x^2 + y^2) + \omega_{b,2}^2 z^2]/2$, where $\omega_{b,1}$ and $\omega_{b,2}$ denotes the trapping frequencies and $m_b$ is the mass of a bosonic atom. The boson–boson interaction strength is related to the $s$-wave scattering length $a_b$ by $g_b = 4\pi \hbar^2 a_b/m_b$.

The superfluid hydrodynamic equations for fermions in terms of the density $n_f(r, t)$ and velocity field $\mathbf{v}_f(r, t)$ are given by [20, 57], respectively

$$
\frac{\partial n_f}{\partial t} + \nabla \cdot (n_f \mathbf{v}_f) = 0, \quad (2a)
$$

$$
\frac{m_f}{\mu_f} \frac{\partial \mathbf{v}_f}{\partial t} + \nabla \left[ V_{ext}^f + \mu(n_f) + \frac{1}{2} m_f \mathbf{v}_f^2 + g_{bf} n_b \right] = 0, \quad (2b)
$$

where the trapping potential acting on fermions is $V_{ext}^f(r) = m_f [\omega_{f,1}^2 (x^2 + y^2) + \omega_{f,2}^2 z^2]/2$ with $m_f$ the mass of a fermionic atom, and the total number of fermions in superfluid state is normalized by $N_f = \int n_f(r, t) \, d\mathbf{r}$. In contrast to a simple expression of the interaction strength in the bosonic part, the two-spin fermionic interaction is characterized by the equation of state $\mu(n_f)$. In order to obtain analytical results in various superfluid regimes in a unified way, we take a polytropic approximation [58, 59]

$$
\mu(n_f) = \mu_0 \left( \frac{n_f}{n_0} \right)^\gamma, \quad \mu_0 = \varepsilon_f \left[ \frac{\sigma(\eta)}{5} \frac{\partial \sigma(\eta)}{\partial \eta} \right]^{1/\gamma}, \quad (3a)
$$

$$
\gamma \equiv \gamma(\eta) = \frac{n_f}{\mu} \frac{\partial \mu}{\partial n_f} = \frac{2}{5} \frac{\sigma(\eta)}{\sigma(\eta) - \frac{2}{5} \sigma'(\eta)} + \frac{\sigma''(\eta)}{15} \frac{\sigma(\eta) - \frac{2}{5} \sigma'(\eta)}{\sigma(\eta)}, \quad (3b)
$$

where the reference chemical potential $\mu_0$ is proportional to the Fermi energy $\varepsilon_f = (\hbar k_f)^2/(2m_f) = \hbar (3N_f \omega_{f,1}^2 \omega_{f,2}^2)^{1/3}$ defined in a cylindrically symmetric trap and reference atomic number density is given by the density of noninteracting Fermi gas at the trapping center $n_0 = (2m_f \varepsilon_f)^{3/2}/(3\pi^2 \hbar^3)$. In order to be close to experimental observations, $\sigma(\eta)$ is based on the explicit expressions of fitting functions from ENS experimental data [60]. The effective polytropic index $\gamma$ and reference chemical potential $\mu_0$ are determined by $\sigma(\eta)$ as a function of the dimensionless interaction $\eta = 1/k_f a_f$, which have been plotted in [48].

The coupling between these two types of superfluids, we have introduced the boson–fermion interaction $g_{bf} = 2\pi \hbar^2 a_{bf}/m_{bf}$ at the mean-field level with boson–fermion scattering length $a_{bf}$ and reduced mass $m_{bf} = m_b m_f/(m_b + m_f)$. It is worth noting that in the BEC limit where $1/k_f a_f \gg 1$ and $a_f$ is comparable to $a_{bf}$, the boson–fermion interaction should be replaced by the boson–dimer interaction [8, 9].

The coupled hydrodynamic equations actually work in the Thomas–Fermi (TF) regime, in which they are analytically simpler to handle in the absence of quantum pressure terms. The TF approximation is valid, provided that the interactomic interaction energy is large enough to make the kinetic energy pressure negligible, i.e. in the large particle limit and collective excitations are of sufficiently long wavelength. By including the proper quantum pressure terms [61, 62], the coupled hydrodynamic equations are equivalent to the coupled order-parameter equations [49, 63]. In addition, the superfluid hydrodynamic equations only describe the dynamics of superfluid components, ignoring single particle excitation, normal components and temperature effects.

### 2.2. Scaling theory for a coupled system

To account for collective oscillation modes in a coupled system, we resort to a scaling theory [64, 65]. The basic idea behind the scaling method is to take appropriate scaling ansatz and simplify time-dependent problems into solving differential equations for the scaling parameters. It is specially suited for 3D hydrodynamic equations, whereby numerical simulations are very expensive. Moreover, it is enable to derive analytical approximations that provide a deep physical insight into the problem. This technique was first proposed in the context of BEC [64, 65], then used in the power-law equation of state for superfluid Fermi gases in the BCS–BEC crossover [25, 27, 66]. The calculated collective mode frequencies are shown to be in quantitative agreement with experiments [25, 27, 42]. The extension of the scaling theory to a coupled system was developed in the cases of degenerate Bose–Fermi mixtures [35, 67].

The scaling ansatz for the time-dependent density profiles for the Bose and Fermi superfluids are chosen as follows [35, 67], respectively
\[ n_b(x, y, z, t) = \frac{1}{b_x(t) b_y(t) b_z(t)} n_b^0 \left( \frac{x}{b_x(t)}, \frac{y}{b_y(t)}, \frac{z}{b_z(t)} \right), \] (4a)

\[ n_f(x, y, z, t) = \frac{1}{a_x(t) a_y(t) a_z(t)} n_f^0 \left( \frac{x}{a_x(t)}, \frac{y}{a_y(t)}, \frac{z}{a_z(t)} \right), \] (4b)

where \( n_b^0 \) and \( n_f^0 \) are equilibrium density distributions for the Bose and Fermi superfluids, respectively. The scaling ansatz for the velocity fields can be obtained by inserting the scaling ansatz equations (4a) and (4b) into the equation of continuity equations (1a) and (2a), respectively

\[ \mathbf{v}_b(x, y, z, t) = \left( \frac{x}{b_x(t)}, \frac{y}{b_y(t)}, \frac{z}{b_z(t)} \right), \]

\[ \mathbf{v}_f(x, y, z, t) = \left( \frac{x}{a_x(t)}, \frac{y}{a_y(t)}, \frac{z}{a_z(t)} \right). \]

Substituting the scaling ansatz for the densities (4a), (4b) and the velocity fields (5a), (5b) into the equations (1b) and (2b), we arrive at the differential equations for the scaling parameters

\[ R_{ba} \frac{d^2 b_i}{dt^2} + \omega_{bi}^2 b_i R_{bi} + \frac{g_b}{m_b b_i} \int \frac{\partial n_b^0}{\partial r_i} R_{bi} dt + \frac{g_{bf}}{m_f} \int \frac{\partial n_f^0}{\partial r_i} R_{fi} dt = 0, \]

\[ R_{fi} \frac{d^2 a_i}{dt^2} + \omega_{fi}^2 a_i R_{fi} + \frac{g_f}{m_f a_i} \int \frac{\partial n_f^0}{\partial r_i} R_{fi} dt + \frac{g_{bf}}{m_b} \int \frac{\partial n_b^0}{\partial r_i} R_{bi} dt = 0, \]

where we have introduced the time-dependent coordinates \( R_b = [x/b_x(t), y/b_y(t), z/b_z(t)] \) and \( R_f = [x/a_x(t), y/a_y(t), z/a_z(t)] \). It is seen that we transfer the time-dependent problems for the coupled hydrodynamic equations into solving the ordinary differential equations for \( b_i(t) \) and \( a_i(t) \), with \( i = x, y, z \) respectively, and the disturbances of density profiles are expressed by these scaling parameters. In the equilibrium states equations (6a) and (6b) reduce to

\[ m_b \omega_{bi}^2 r_i + \frac{\partial n_b^0}{\partial r_i} + g_{bf} \frac{\partial n_f^0}{\partial r_i} = 0, \]

\[ m_f \omega_{fi}^2 r_i + \frac{\partial n_f^0}{\partial r_i} + g_{bf} \frac{\partial n_b^0}{\partial r_i} = 0. \]

In order to obtain scaling solutions for a coupled system, i.e. in the presence of the boson–fermion interaction \( g_{bf} \) a useful strategy is developed that is assuming the scaling form of the solution \( \alpha \) priori and fulfilling it on an average by integrating over the spatial coordinates [35, 67]. Combining the differential equations (6a), (6b) with the equilibrium states (7a), (7b) and carrying out the spatial integration, we obtain the following expressions

\[ \frac{d^2 b_i}{dt^2} + \omega_{bi}^2 b_i - \frac{\omega_{bi}^2}{b_i} \int \frac{\partial n_b^0}{\partial r_i} R_{bi} dt \left[ \frac{1}{b_i} \int \frac{\partial n_f^0}{\partial r_i} R_{bi} \right] = 0, \]

\[ \frac{d^2 a_i}{dt^2} + \omega_{fi}^2 a_i - \frac{\omega_{fi}^2}{a_i} \int \frac{\partial n_f^0}{\partial r_i} R_{fi} dt \left[ \frac{1}{a_i} \int \frac{\partial n_b^0}{\partial r_i} R_{fi} \right] = 0, \]

with \( b \equiv [b_x, b_y, b_z] \) and \( a \equiv [a_x, a_y, a_z] \). \( \langle R_{bi}^2 \rangle_b = (1/N_b) \int \delta R_b n_b^0 (R_b) R_{bi}^2 dR_b \) and \( \langle R_{fi}^2 \rangle_f = (1/N_f) \int \delta R_f n_f^0 (R_f) R_{fi}^2 dR_f \) correspond to the mean square radii of the Bose and Fermi superfluids in the \( i \) axis, respectively.

Due to the collective oscillations considered here are small around the equilibrium states, by expanding \( n_b^0 (b b_i) \) \( n_f^0 (a a_i) \) one can simplify equations (8a) and (8b) as

\[ \frac{d^2 b_i}{dt^2} + \omega_{bi}^2 b_i - \frac{\omega_{bi}^2}{b_i} \int \frac{\partial n_f^0}{\partial r_i} R_{bi} dt \left[ \frac{1}{b_i} \int \frac{\partial n_b^0}{\partial r_i} R_{bi} \right] = 0, \] (9a)
\[
\frac{d^2 a_i}{dt^2} + \omega_i^2 a_i - \frac{\omega_i^2}{a_i \prod_j a_j} a_j + \omega_i^2 \left( \frac{1}{a_i \prod_j a_j} - \frac{1}{a_i \prod_j a_j^2} \right) F_i + \sum_k \frac{\omega_k^2}{a_i \prod_j a_j} \left( \frac{b_k}{a_i} - 1 \right) F_k = 0. \tag{9b}
\]

The dimensionless parameters proportional to \( g_{gf} \) are given by
\[
B_{ik} = \frac{g_{gf}}{N_b m_b \omega_i^2 (R_i^2)_{b}} \int dr \frac{\partial n_i^0 (r)}{\partial r} \frac{\partial n_k^0 (r)}{\partial r}, \tag{10a}
\]
\[
F_i = \frac{g_{gf}}{N_i m_f \omega_i^2 (R_i^2)_{f}} \int dr \frac{\partial n_i^0 (r)}{\partial r} n_i^0 (r), \tag{10b}
\]
\[
F_k = \frac{g_{gf}}{N_f m_f \omega_k^2 (R_k^2)_{f}} \int dr \frac{\partial n_k^0 (r)}{\partial r} n_k^0 (r), \tag{10c}
\]
where we have replaced the variables \( R_b \) and \( R_f \) by \( r \) for simplicity in equations (10a)–(10c), because the each integration is relevant to either \( R_b \) or \( R_f \).

2.3. Collective oscillation modes

In order to clearly understand the collective oscillation modes obtained from the scaling method, let us first discuss the results without the boson–fermion interaction \( g_{gf} \). In this case all dimensionless parameters (10a)–(10c) disappear, and equations (9a) and (9b) decouple and describe the Bose and Fermi superfluids alone, respectively. In a cylindrical symmetry, the frequencies of the harmonic traps are \( \omega_{b,\perp} (\omega_{f,\perp}) = \omega_{b,\parallel} (\omega_{f,\parallel}) \) and \( \omega_{b,\parallel} (\omega_{f,\parallel}) = \lambda_b \omega_{b,\parallel} (\lambda_f \omega_{f,\parallel}) \), with the trap anisotropy \( \lambda_b = \omega_{b,\parallel} / \omega_{b,\perp} \) (\( \lambda_f = \omega_{f,\parallel} / \omega_{f,\perp} \)) for the Bose (Fermi) superfluid. The resulting eigenvalues of the scaling equations are three mode frequencies [23, 24]. By examining the signs of the eigenvectors, one can find that one is (radial) quadrupole mode illustrated in figure 1(a), which only supports in the radial plane with no excitation in axial direction, and the radii oscillating out of phase with each other. The frequency of the quadrupole mode [22, 23, 30] is, respectively
\[
\omega_{b,\perp}^2 = 2 \omega_{b,\perp}^2, \quad \omega_{b,\parallel}^2 = 2 \omega_{b,\parallel}^2, \quad \omega_{f,\perp}^2 = 2 \omega_{f,\perp}^2, \quad \omega_{f,\parallel}^2 = 2 \omega_{f,\parallel}^2 \tag{11}
\]
for Bose and Fermi superfluids. The other two mode frequencies for Bose and Fermi superfluids are given by [23, 25, 27]
\[
\frac{\omega_{b,\perp}^2}{\omega_{b,\parallel}^2} = 2 + \frac{3}{2} \lambda_b^2 \pm \frac{1}{2} \sqrt{9 \lambda_b^4 - 16}, \tag{12a}
\]
\[
\frac{\omega_{f,\perp}^2}{\omega_{f,\parallel}^2} = (\gamma + 1)^2 + \frac{\gamma + 2}{2} \lambda_f^2 \pm \sqrt{1 + \gamma + \frac{\gamma^2 + 2}{2} \lambda_f^2 - 2(\gamma + 3) \lambda_f^2}, \tag{12b}
\]
where + and − refer to the radial and axial mode, respectively. As shown in figures 1(b) and (c), respectively, the radial mode features an in-phase oscillation for all radii in the radial and axial directions, while the axial mode corresponds to two radii in radial direction oscillating in phase with each other, but out of phase with the radius in the axial direction. Differently from the quadrupole mode, the radial and axial modes are relevant to the anisotropy of the trap. In a highly elongated trap (\( \lambda_b, \lambda_f \ll 1 \)) [25, 27, 42], the frequency of the radial mode reduces to \( \omega_{b,\perp} = 2 \omega_{b,\perp} (\omega_{f,\perp} = 2 \omega_{f,\perp}^2) \), which coincides with the radial (transverse) breathing mode.
where the ratios of radial to axial excitation amplitude are given by
\( M_{zz} \) and \( M_{ff} \), and the axial mode is
\[ \omega_{f+} = \sqrt{3} \omega_{f2} (\omega_{r+} = \sqrt{3} \omega_{f2}) \], in a spherical trap (\( \lambda_{bf} = 1 \)), the frequency of the radial mode reduces to \( \omega_{f+} = \sqrt{3} \omega_{f2} (\gamma + 2) \), and the axial mode is
\[ \omega_{f+} = \sqrt{10/3} \omega_{f1} (\gamma + 2) \). In a spherical trap (\( \lambda_{bf} = 1 \)), the frequency of the radial mode is \( \omega_{r+} = \sqrt{3} \omega_{r2} (\gamma + 2) \), which is also referred as monopole mode [22], and the axial mode frequency is \( \omega_{f+} = \sqrt{2} \omega_{f2} (\gamma + 2 \omega_{f2}) \), recovering to the quadrupole mode equations (11).

In the presence of the Bose–Fermi interaction \( g_{bf} \), the collective modes of the Bose and Fermi superfluids are coupled each other and their frequencies are varied. For the quadrupole mode, the linearization of equations (9a), (9b) around the equilibrium states result in the eigenvalue function \( dP/dT = M_{P}P \), where the vector notation is \( \mathbf{P}^T \equiv (\delta b, \delta b, \delta a, \delta a) \) and matrix \( M_{q} \) is written in a cylindrical coordinate (\( \lambda, \gamma, z \) ) by
\[
M_{q} = \begin{pmatrix}
(3 - \frac{3}{4} B_{pp} \omega_{r1}^2)_{b+} & (1 - \frac{1}{4} B_{pp} \omega_{r1}^2)_{b+} & \frac{3}{4} B_{pp} \omega_{r1} \omega_{r2} \gamma & \frac{1}{4} B_{pp} \omega_{r2} \\
(1 - \frac{1}{4} B_{pp} \omega_{r1}^2)_{b+} & (3 - \frac{3}{4} B_{pp} \omega_{r1}^2)_{b+} & \frac{3}{4} B_{pp} \omega_{r1} \omega_{r2} \gamma & \frac{1}{4} B_{pp} \omega_{r2} \\
\frac{3}{4} F_{pp} \omega_{r1}^2 \gamma & \frac{3}{4} F_{pp} \omega_{r1}^2 \gamma & G_{r1} \omega_{r2} \gamma & H_{r1} \gamma \\
\frac{3}{4} F_{pp} \omega_{r1}^2 \gamma & \frac{3}{4} F_{pp} \omega_{r1}^2 \gamma & H_{a2} \omega_{r2} \gamma & G_{a2} \gamma
\end{pmatrix}
\]
(13)

with \( G = 2 + \gamma + (\gamma - 1)F_{r1} - 3F_{pp}/4 \) and \( H = \gamma + (\gamma - 1)F_{r1} - F_{pp}/4 \). By examining the dispersion relations (13), one can find that if the elements for the couplings of the amplitudes of the bosonic and fermionic excitations in the matrix are small which implies that the effective coupling is weak, the explicit expressions for the frequencies of the collective oscillation modes can be presented [35]. The high- (+) and low-lying (−) frequencies of the quadrupole mode are then explicitly expressed by
\[
\omega_{r+}^2 = \frac{1}{2} \left( 2 - \frac{1}{2} B_{pp} \right)_{b+} \omega_{r1}^2 + \left( 2 - \frac{1}{2} B_{pp} \right)_{b+} \omega_{r1}^2 + \left( 2 - \frac{1}{2} D_{pp} \right) \omega_{r1}^2
\]
(14)

For the radial and axial mode, the eigenvector corresponds to \( \mathbf{P}^T \equiv (\delta b, \delta b, \delta a, \delta a) \) and the matrix \( M \) is defined by
\[
M = \begin{pmatrix}
(4 - B_{pp}) \omega_{b1}^2 & (1 - B_{pp}) \omega_{b1}^2 & B_{pp} \omega_{b1} \omega_{b2} & B_{pp} \omega_{b1} \omega_{b2} \\
(2 - B_{pp}) \omega_{b1}^2 & (3 - B_{pp}) \omega_{b1}^2 & B_{pp} \omega_{b1} \omega_{b2} & B_{pp} \omega_{b1} \omega_{b2} \\
F_{pp} \omega_{b1} \omega_{b2} & F_{pp} \omega_{b1} \omega_{b2} & M_{b1} \omega_{b1} \omega_{b2} & M_{b1} \omega_{b1} \omega_{b2} \\
F_{pp} \omega_{b1} \omega_{b2} & F_{pp} \omega_{b1} \omega_{b2} & M_{b2} \omega_{b1} \omega_{b2} & M_{b2} \omega_{b1} \omega_{b2}
\end{pmatrix}
\]
(15)

with \( M_{pp} = 2 \gamma + 2 + 2(\gamma - 1)F_{r1} - F_{pp}, M_{b2} = \gamma + (\gamma - 1)F_{r1} - F_{pp}, \) and \( M_{b2} = \gamma + (\gamma - 1)F_{r1} - F_{pp} \). The analytical expressions in the case of a weak coupling for the high- and low-lying frequencies for the radial \( \omega_{r+}^2 \) and axial \( \omega_{a+}^2 \) modes are more complicated and given by, respectively
\[
\omega_{r+}^2 = \frac{1}{2(2\gamma - 1) + 2 \omega_{r1}^2} \left[ A_{r2} \omega_{r1}^2 (2\gamma - 1) + A_{r1} (\gamma + 1) \right]
\]
\[
\omega_{a+}^2 = \frac{1}{2(2\gamma - 1) + 2 \omega_{a1}^2} \left[ A_{a2} \omega_{a1}^2 (2\gamma - 1) + A_{a1} (\gamma + 1) \right]
\]
(16a)

\[
\omega_{r-}^2 = \frac{1}{2(2\gamma - 1) + 2 \omega_{r1}^2} \left[ A_{r2} \omega_{r1}^2 (2\gamma - 1) + A_{r1} (\gamma + 1) \right]
\]
\[
\omega_{a-}^2 = \frac{1}{2(2\gamma - 1) + 2 \omega_{a1}^2} \left[ A_{a2} \omega_{a1}^2 (2\gamma - 1) + A_{a1} (\gamma + 1) \right]
\]
(16b)

with
\[
A_{r1} = [M_{b1} (4 - B_{pp}) + (1 - B_{pp})] \omega_{b1}^2 + [M_{b1} (2 - B_{pp}) + (3 - B_{pp})] \omega_{b2}^2,
\]
\[
A_{r2} = (M_{b1} B_{pp} + B_{pp}) \omega_{b1}^2 + (M_{b1} B_{pp} + B_{pp}) \omega_{b2}^2,
\]
\[
A_{a1} = (M_{b1} B_{pp} + B_{pp}) \omega_{b1}^2 + (M_{b1} B_{pp} + B_{pp}) \omega_{b2}^2,
\]
\[
A_{a2} = (M_{b1} B_{pp} + B_{pp}) \omega_{b1}^2 + (M_{b1} B_{pp} + B_{pp}) \omega_{b2}^2,
\]
(17a)

\[
A_{r2} = (M_{b1} B_{pp} + B_{pp}) \omega_{b1}^2 + (M_{b1} B_{pp} + B_{pp}) \omega_{b2}^2,
\]
\[
A_{a1} = (M_{b1} B_{pp} + B_{pp}) \omega_{b1}^2 + (M_{b1} B_{pp} + B_{pp}) \omega_{b2}^2,
\]
\[
A_{a2} = (M_{b1} B_{pp} + B_{pp}) \omega_{b1}^2 + (M_{b1} B_{pp} + B_{pp}) \omega_{b2}^2,
\]
(17b)

where the ratios of radial to axial excitation amplitude are given by \( M_{r+}^2 = \omega_{r1}^2 / \omega_{a1}^2 \) and \( M_{r+} = \gamma \omega_{r1}^2 / \omega_{a1}^2 \), with \( \omega_{b+} \) and \( \omega_{f+} \) being the frequencies of the radial (+) and axial (−)
modes of the Bose and Fermi superfluids alone in equations (12a) and (12b). The expressions for the dimensionless parameters $B_{\alpha \beta}$, $F_{\alpha \beta}$, and $F_{\alpha}$ are shown in Appendix A, which are integrals in terms of the density profiles of the Bose and Fermi superfluids at equilibrium and responsible for the coupling of the Bose and Fermi components.

2.4. Analytical approximations for the integral terms
In order to obtain the dimensionless parameters, one should first obtain the spatial overlap integrals in terms of the equilibrium density profiles of the Bose and Fermi superfluids. Equations (1b) and (2b) at groundstates give the density profiles of the Bose and Fermi superfluids coupled each other, which are written in cylindrical coordinate as

$$n^{0}_b(r, z) = \frac{1}{g_b} \mu_b - \frac{1}{2} m_b (\omega_x^2 r^2 + \omega_y^2 z^2) - g_{bf} n^{0}_f(r, z),$$  
(18a)

$$n^{0}_f(r, z) = \frac{n_0}{\mu^0_f} \left[ \mu_f - \frac{1}{2} m_f (\omega_x^2 r^2 + \omega_y^2 z^2) - g_{bf} n^{0}_b(r, z) \right]^{1/\gamma},$$  
(18b)

where the bulk chemical potentials $\mu_b$ and $\mu_f$ are determined by the total numbers of bosons and fermions, respectively. Without the boson–fermion interaction $g_{bf}$, the explicit expressions for the density profile of the Bose superfluid are [19, 56]

$$n^{00}_b(r, z) = \frac{1}{g_b} \max \left[ \mu_b - V^{0*}_{\text{ext}}(r, z), 0 \right], \quad \mu_b = \left( \frac{15 N_b \hbar^2 a_b \sqrt{m_b} \omega_x^2}{4 \sqrt{2}} \right)^{2/5},$$  
(19)

and the Fermi density profiles along the BCS–BEC crossover [39] are

$$n^{00}_f(r, z) = \frac{n_0}{\mu^0_f} \max \left[ \mu_f - V^{0*}_{\text{ext}}(r, z), 0 \right]^{1/\gamma},$$  

$$\mu_f = \epsilon_f \left( \sigma(\eta) - \frac{\eta \sigma'(\eta)}{5} \right) \frac{1}{8 \Gamma(\frac{1}{\gamma} + 1)}.$$

The density profiles $n^{00}_b$ and $n^{00}_f$ in the absence of $g_{bf}$ can be regarded as the zero-order approximation for the density profiles (18a) and (18b). By a perturbative expansion, the density profiles $n^{01}_b$ and $n^{01}_f$ at the first-order approximation can be naturally obtained by replacing $n^{0}_b$ by the zero-order results $n^{00}_b$ in equation (18a) and $n^{0}_f$ by $n^{00}_f$ in equation (18b), correspondingly. Differently from the previous works by the numerical methods [35, 67], we apply analytical approximations for the integrals [48] that holds in the recent experimental situations, then give explicit expressions for the dimensionless parameters.

As a example of the integral $I_{\rho \varphi}$, we derive it as following

$$I_{\rho \varphi} = \int \rho d\rho dz \frac{\partial n^{00}_b}{\partial \varphi} \rho \frac{\partial n^{00}_f}{\partial \varphi},$$  
(21a)

$$\rho \frac{\partial n^{00}_f}{\partial \varphi} = \frac{m_f}{g_{bf}} \left( \frac{\partial n^{00}_f}{\partial \varphi} \right)_{r=0} \frac{m_f \omega^2_{\perp}}{g_b} \left[ \frac{\partial \mu^{00}_f}{\partial \varphi} \right]_{r=0} - \left[ \frac{\partial \mu^{00}_f}{\partial \varphi} \right]_{r=0},$$  
(21b)

$$\left( \frac{g_{bf}}{g_b} - \frac{m_f \omega^2_{\perp}}{m_b \omega^2_{\perp}} \right) \int \rho d\rho z \rho^3 = \frac{8}{7 \pi} \mu_b N_b \left[ \frac{\partial n^{00}_b}{\partial \varphi} \right]_{r=0} \left[ \frac{\partial \mu^{00}_b}{\partial \varphi} \right]_{r=0} - \left[ \frac{\partial \mu^{00}_b}{\partial \varphi} \right]_{r=0} - 1,$$

$$\left( \frac{g_{bf}}{g_b} - \frac{m_f \omega^2_{\perp}}{m_b \omega^2_{\perp}} \right),$$  
(21c)

where $\left( \frac{\partial n^{00}_f}{\partial \varphi} \right)_{r=0}$ is given by (18a). The recent experimental conditions for the Bose–Fermi superfluid mixtures [1, 5, 6] mainly share two common characteristics. First, the numbers of bosons and fermions are both large at order of $10^4$–$10^6$ magnitude, and the number of bosons is one order smaller than fermions, thus the bosons can be regarded as a mesoscopic impurity immersed in Fermi superfluids. Second, the boson–fermion interaction $g_{bf}$ is small. Since $\mu_f(N_f)$ is much larger than $\mu_b(N_b)$ and $n_f$ is much smaller than $n_b$, due to the stronger interaction, the last terms on the right sides of equations (18a) and (18b) are relatively smaller than the first terms and the Bose and Fermi density profiles are weakly coupled.
By using a perturbative method, in the first step of the integration (21a), we have substituted the density profiles \( n_g^0 \) and \( n_f^0 \) by the first-order approximations \( n_g^{01} \) and \( n_f^{01} \), respectively. Since the Bose superfluid is weakly interacting and the atomic number is smaller than the fermionic counterpart, the spatial distribution of the Bose superfluid only overlaps with the Fermi superfluid in a small central region. In the second step (21b), we approximate the Fermi density by the central value and the region of integration is the volume \( V_B \) of the Bose superfluid. Based on the above analysis for the parameters and these approximations, the explicit expression for the integral \( I_{\mu\nu} \) is presented in (21c).

The scattering length for the USTC experimental setting is given by the analytical results denoted by the open circles are obtained by solving the eigenvalue matrix [21]. The explicit expressions for all involved integrals and the dimensionless parameters are presented in appendix B.

3. Results and discussions

In this section, we apply the theoretical results of the previous section to a realistic experimental situation as an example to discuss how the Bose–Fermi interaction affects the frequencies of the collective modes of the Bose and Fermi superfluids in the BCS–BEC crossover in different trap geometries. Our theoretical results can be also easily extended to other experimental settings. We choose the parameters of the experiment performed at University of Science and Technology of China (USTC) on the dipole oscillations of a Bose–Fermi superfluid mixture of \( ^{41}\text{K}–^{6}\text{Li} \) with a large mass-imbalance [6]. The experiment is performed in a cigar-shaped trap \( (\lambda_B = \omega_{Bz}/\omega_{B\perp} = 0.04, \lambda_f = \omega_{fz}/\omega_{f\perp} = 0.06) \), with the radial and axial frequencies of the harmonic trap for bosons (fermions) being \( \omega_{B\perp} = 2\pi \times 170.7 \text{ Hz} \) (\( \omega_{f\perp} = 2\pi \times 295.4 \text{ Hz} \)) and \( \omega_{Bz} = 2\pi \times 6.295 \text{ Hz } \) (\( \omega_{fz} = 2\pi \times 16.453 \text{ Hz} \)). In the following calculations, the frequency in the radial direction is fixed, and the axial frequency and the anisotropy \( \lambda_B = \omega_{Bz}/\omega_{B\perp} (\lambda_f = \omega_{fz}/\omega_{f\perp}) \) are varied to realize different trap geometries. For a spherical trap \( (\lambda_B = 1, \lambda_f = 1) \), the frequencies are given by \( \omega_{Bz} = 2\pi \times 170.7 \text{ Hz } \) and \( \omega_{fz} = 2\pi \times 295.4 \text{ Hz} \). For a disk-shaped trap \( (\lambda_B = 7, \lambda_f = 11) \), the axial frequencies are enlarged by a same factor \( \omega_{Bz} = 2\pi \times 6.295 \times 189 = 2\pi \times 1190 \text{ Hz } \) and \( \omega_{fz} = 2\pi \times 16.453 \times 189 = 2\pi \times 3109 \text{ Hz} \).

The total numbers of \( ^{41}\text{K} \) bosons and \( ^{6}\text{Li} \) fermions are given by \( N_B = 2.3 \times 10^5 \) and \( N_f = 1 \times 10^6 \), respectively. The scattering lengths of boson–boson \( a_B = 60.5a_0 \) and boson–fermion \( a_{BF} = 60.2a_0 \) (\( a_0 \) the Bohr radius) are fixed. The scattering length \( a_f \) of two-spin fermionic atoms is tunable across the BCS–BEC crossover through a Feshbach resonance.

In order to test the approximations in section 2.4, we compare the analytical results with the numerical calculations in figure 2. We show the relevant integrals and the mean square radii as a function of the dimensionless parameter \( 1/(k_B T) \) for the USTC experimental setting (cigar-shaped case). The dashed lines are the analytical results defined in appendix B, and the discrete data are the results from solving the coupled equilibrium hydrodynamic equations (18a) and (18b) numerically through a self-consistent iterative procedure [48]. One can find that the analytical results agree well with the numerical calculations, but showing obvious discrepancy when \( 1/(k_B T) \) is larger than 1. In order to explain such difference, in figure 3 we plot the numerical results for the axial density profiles at \( r = 0 \) for the fermions (upper panels) and bosons (lower panels) in the BCS–BEC crossover. One can find that for a fixed Bose–Fermi interaction, as the interaction energy of the Fermi superfluid decreases from the BCS side (figure 3(a)) to the BEC regime (figure 3(e)), the depletion of the Fermi superfluid density in the center caused by the bosons becomes more and more pronounced, until it is completely in the BEC regime (see figures 3(d) and (e)). Thus in the BEC side, the analytical approximation of substituting the Fermi density distribution in the integral region by the value of the Fermi superfluid alone in the center is unreliable. It should be pointed that even though the Fermi density is zero in the center for the cases of figures 3(a) and (c), the integrals for the spatial overlaps of these two densities still remain. In addition, for the mean square radii of the Bose and Fermi superfluids shown in figures 2(g) and (h), compared with the analytical results (dashed lines) actually corresponding to the noninteracting Bose and Fermi superfluids, numerical results (discrete data) show that the repulsive Bose–Fermi interaction has a very slight affect on the Fermi superfluids due to a larger number of particles, while the radii of the Bose superfluids decrease obviously, suggesting that the bosonic cloud is compressed by the outer shell of fermions.

In figure 4, we present the frequencies of the quadrupole oscillation modes of the Bose–Fermi superfluid mixtures in the BCS–BEC crossover for different trap geometries. The numerical results denoted by the open circles are obtained by solving the eigenvalue matrix (13) directly, in which the relevant dimensionless...
The parameters are determined by the numerical calculations for the Bose and Fermi superfluid density profiles from the coupled equilibrium hydrodynamic equations \[48\]. By examining the signs of the eigenvectors, one can identify the corresponding quadrupole modes of the Fermi and Bose superfluids. In contrast, the analytical results are shown by the solid lines, which are calculated from the expressions \([14]\) combined with the analytical results for the dimensionless parameter and the mean square radii in appendix B. The positive (\(+\)) and negative (\(−\)) roots of the analytical expressions \([14]\), as the frequencies of the harmonic trap for fermions are larger than bosons, actually correspond to the frequencies of the Fermi and Bose counterparts, respectively. The frequencies of the quadrupole modes of the Bose and Fermi superfluid without interacting are also plotted by dashed lines for comparison.

It is clearly seen that in the presence of a repulsive Bose–Fermi interaction, the frequencies of the quadrupole modes of the Fermi and Bose superfluids are all upshifted. Both the analytical and numerical results show that
the upshifted values increase from the BCS side to the BEC regime, however, its behavior for the Fermi and Bose superfluids exhibits differently in the BEC regime. The frequency of the quadrupole mode of the Fermi superfluid interacting with bosons increases monotonically from the BCS side to the BEC regime, and due to a larger number, the induced frequency shifts and their changes around the unitary limit are smaller than the Bose counterpart. However, the numerical results for the Fermi superfluid show a very rapid increase for \( \frac{1}{k_f a_f} \), some of which are plotted in the corresponding insets of figure 4(a). Such significant rise can be understood by reexamining the equilibrium density profiles shown in figure (3). For a fixed repulsive Bose–Fermi interaction, as the interacting strength of the Fermi superfluid decreases from the BCS side to the BEC regime, the density of the Fermi superfluid overlapping with the bosons in the center of the trap decreases. Despite of the density reduction in the overlap region, the frequency shifts are actually sensitive to the spatial deviations of these two density distributions. From figures 3(a)–(e), the boundaries of the overlap regions become sharper and sharper, which result in more rapid increases of the overlap integrals and the effective Bose–Fermi coupling, and faster increases of the frequency shifts. However, the analytical results show an obvious slower increase, since the approximations of the analytical analysis underestimate these integrals (see figure 2).

Differently from the fermionic counterpart, the numerical results for the frequency shifts of the Bose superfluid show non-monotonic increases from the BCS side to BEC regime for the spherical (figure 4(b2)) and

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### Figure 4. The frequencies of the (radial) quadrupole oscillation modes of the Fermi (upper panels) and Bose (lower panels) superfluids with a repulsive Bose–Fermi interaction as a function of the dimensionless parameter \( \frac{1}{k_f a_f} \) for different trap geometries: (1) cigar-shaped trap \( (\lambda_b = 0.04, \lambda_f = 0.06) \), (2) spherical trap \( (\lambda_b = 1, \lambda_f = 1) \), and (3) disk-shaped trap \( (\lambda_b = 7, \lambda_f = 11) \). The solid lines represent the analytical results, while the numerical calculations are shown by the open circles, in which some larger values of the Fermi superfluids in the BEC regime are plotted in the corresponding insets. The dashed lines correspond to the Bose and Fermi superfluids alone.
the disk-shaped (figure 4(b3)) cases. Here to realize the traps from cigar-shaped to disk-shaped we increase the axial trapping frequencies, which actually also enhance the overlaps of the Bose and Fermi densities and the Bose–Fermi coupling effects. This is the reason why the frequency shifts in the disk-shaped case are largest and the discrepancy between the analytical and numerical results is most significant. We find that in the BEC regime $\frac{1}{(k_f a_f)} > 1.5$ where the depletion of the Fermi superfluid is completely repelled by the bosons, although the overlap integrals for spatial derivations of the densities and the effective Bose–Fermi coupling increases monotonically, the frequency shifts of the Bose superfluids in the spherical (figure 4(b2)) and disk-shaped (figure 4(b3)) traps increasingly reach to a peak, then decrease slightly. Therefore one can find that the different increase speeds of the frequency shifts can be used to discriminate different equilibrium configurations of Bose–Fermi superfluid mixtures in the BCS–BEC crossover.

The analytical (solid lines) and numerical (open circles) results for the frequencies of the radial modes of the Bose and Fermi superfluids as a function of the dimensionless parameter $\frac{1}{(k_f a_f)}$ are shown in figure 5. The results for noninteracting Bose and Fermi superfluids (dashed lines) are also plotted for comparison. In contrast to the quadrupole mode in the absence of the Bose–Fermi interaction, the radial mode frequency of the Fermi superfluid shows a non-monotonical variation as a function of the dimensionless parameter in figure 5(a), which are also relevant to the trap anisotropy. As discussed in section 2.3, the radial mode of a single superfluid in a highly elongated trap reduces to the radial (transverse) breathing mode [23, 24, 46], which is featured by the

![Figure 5](image)
radii in the transverse plane oscillating in phase with each other without axial excitation. In the insets of figures 5 (a1) and (b1), we compare the numerical results for the frequencies of the radial modes (open circles) of the Fermi and Bose superfluids in the cigar-shaped traps to those of the radial breathing modes (dots), respectively. The radial breathing modes are calculated by numerically solving the eigenvalue matrix (13) and distinguished by the signs of the eigenvectors. The frequencies of these two modes are shown to be in agreement, which implies that in the presence of the Bose–Fermi interaction such symmetry of the system is still preserved.

In figures 6 (a) and (b), we show the frequencies of the axial modes of the Fermi and Bose superfluids, respectively, as a function of the dimensionless parameter $1/(k_1a_f)$ for different trap geometries. For the cases in the cigar-shaped and disk-shaped traps separately in the insets of figures 6(a1) and (a3), and the squares of the frequencies for the Bose counterpart in the insets of figures 6(b1) and (b3), respectively. The reason for the failure of the analytical expressions for the axial modes in the highly-anisotropic traps is that from the explicit expressions (16a), (16b) and the eigenvalue matrix (15), one can find the elements $A_{k_2 k_1}^{\pm}$ characterizing the coupling of the Bose and Fermi superfluids, are not only determined by the dimensionless parameters as the quadrupole mode, but also rely on the ratio $\frac{M_{bf}}{\omega}$ of the radial excitation amplitude to axial excitation amplitude. We find that for the cases of the axial modes in the highly-

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**Figure 6.** The frequencies of the axial oscillation modes of the (upper panels) Fermi and (lower panels) Bose superfluids with a repulsive Bose–Fermi interaction as a function of the dimensionless parameter $1/(k_1a_f)$ for the different trap geometries. The numerical calculations are shown by the open circles, and some larger values of the Fermi superfluids in the BEC regime are plotted in the corresponding insets of (a2) and (a3). The analytical results are denoted by the solid lines, and the cases in the anisotropic traps (i.e. (a1) and (a3) for the Fermi superfluid and (b1) and (b3) for the Bose superfluid) are shown in the corresponding insets. The dashed lines correspond to the Bose and Fermi superfluids alone.
Appendix A. The dimensionless parameters

\[ B_{\alpha\beta} = \frac{g_{bf}}{N_0 m_b \omega_{2a}^2 \rho_{\alpha}^{2} \rho_{\beta}} \int dr \frac{\partial n_{b}^{0}}{\partial r_{\alpha}} \frac{\partial n_{b}^{0}}{\partial r_{\beta}} \]  

(1.1a)
\[ F_{N,\beta} = \frac{g_{\text{sf}}}{N_f m_f \omega_{f0}^2 (R_{N,\beta}^2)^2} \int \, \partial n_f^0 \partial n_0^0 \partial n_0^0 \partial n_0^0 \partial r_{\beta} \partial r_\gamma, \]
\[ F_0 = \frac{g_{\text{sf}}}{N_f m_f \omega_{f0}^2 (R_{N,0}^2)^2} \int \, \partial n_f^0 \partial n_0^0 \partial n_0^0 \partial n_0^0 \partial r_0 \partial n_0^0, \]

with the mean square radii in the \( \alpha = {\perp, z} \) directions given by \( \langle R_{N,0}^2 \rangle = (1/N_f) \int \, \partial r_0^2 \partial n_0^0 \) for the Bose superfluid and \( \langle R_{N,\beta}^2 \rangle = (1/N_f) \int \, \partial r_{\beta}^2 \partial n_0^0 \) for the Fermi superfluid.

**Appendix B. The analytical results for the integrals and dimensionless parameters**

The explicit expressions for the integrals \( I_{\alpha,\beta} \) and \( I_\alpha \) with \( \alpha, \beta = \rho(\perp), z \) are given by, respectively

\[ I_{\alpha,\beta} = \int \rho \partial r \partial \rho \partial z \partial n_0^0 \partial r_{\alpha} \partial r_{\beta} = \frac{i_{\alpha,\beta}}{2\pi} \rho \partial r \partial \rho \partial z, \]
\[ C_{\alpha,\beta} = \left( \frac{\partial n_0^0}{\partial r_{\alpha}} \right) \left( \frac{\partial n_0^0}{\partial r_{\beta}} \right) \left( m_f \omega_{f0}^2 \right)^2 = \left( \frac{g_{\text{sf}}}{g_0} \right) \left( \frac{m_f \omega_{f0}^2}{m_f \omega_{f0}^2} \right), \]
\[ I_\alpha = \int \rho \partial r \partial z \partial n_0^0 = \frac{n_0^0 (0)}{\pi}, \]
\[ C_\alpha = g_{\text{sf}} \left( \frac{\partial n_0^0}{\partial r_{\alpha}} \right) \left( m_f \omega_{f0}^2 \right)^2 = 1, \]

and the dimensionless parameters are given by

\[ B_{\alpha,\beta} = \frac{2\pi g_{\text{sf}}}{N_f m_f \omega_{f0}^2 (R_{N,\beta}^2)^2} I_{\alpha,\beta} = \frac{g_{\text{sf}}}{N_f m_f \omega_{f0}^2 (R_{N,\beta}^2)^2} I_{\alpha,\beta}, \]
\[ E_{\alpha,\beta} = \frac{2\pi g_{\text{sf}}}{N_f m_f \omega_{f0}^2 (R_{N,\beta}^2)^2} I_{\alpha,\beta} = \frac{g_{\text{sf}}}{N_f m_f \omega_{f0}^2 (R_{N,\beta}^2)^2} I_{\alpha,\beta} \]
\[ F_\alpha = \frac{2\pi g_{\text{sf}}}{N_f m_f \omega_{f0}^2 (R_{N,0}^2)^2} I_{\alpha,\beta} = \frac{g_{\text{sf}}}{N_f m_f \omega_{f0}^2 (R_{N,0}^2)^2} I_{\alpha,\beta}, \]

where \( \langle R_{N,\beta}^2 \rangle = (1/N_f) \int \, \partial r_{\beta}^2 \partial n_0^0 \) and \( \langle R_{N,0}^2 \rangle = (1/N_f) \int \, \partial r_0^2 \partial n_0^0 \) are the equilibrium TF radii for the Bose (Fermi) superfluid alone is given by \( R_{\text{eq}} = \sqrt{2\mu_f / (m_f \omega_{f0}^2)} \)

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