Synthesis of multi-wavelength temporal phase-shifting algorithms optimized for high signal-to-noise ratio and high detuning robustness using the frequency transfer function

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Abstract: Synthesis of single-wavelength temporal phase-shifting algorithms (PSA) for interferometry is well-known and firmly based on the frequency transfer function (FTF) paradigm. Here we extend the single-wavelength FTF-theory to dual and multi-wavelength PSA-synthesis when several simultaneous laser-colors are present. The FTF-based synthesis for dual-wavelength PSA (DW-PSA) is optimized for high signal-to-noise ratio and minimum number of temporal phase-shifted interferograms. The DW-PSA synthesis herein presented may be used for interferometric contouring of discontinuous industrial objects. Also DW-PSA may be useful for DW shop-testing of deep free-form aspheres. As shown here, using the FTF-based synthesis one may easily find explicit DW-PSA formulae optimized for high signal-to-noise and high detuning robustness. To this date, no general synthesis and analysis for temporal DW-PSAs has been given; only had-hoc DW-PSAs formulas have been reported. Consequently, no explicit formulae for their spectra, their signal-to-noise, their detuning and harmonic robustness has been given. Here for the first time a fully general procedure for designing DW-PSAs (or triple-wavelengths PSAs) with desire spectrum, signal-to-noise ratio and detuning robustness is given. We finally generalize DW-PSA to higher number of wavelength temporal PSAs.

OCIS codes: (120.0120) Instrumentation, measurement, and metrology; (120.6650) Surface measurements, figure; (100.2650) Fringe analysis.

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1. Introduction

As far as we know, the first researcher to use dual-wavelength (DW) interferometry was Wyant in 1971 [1]. Wyant used two fixed laser-wavelengths \( \lambda_1 \) and \( \lambda_2 \) to test an optical surface with an equivalent wavelength of \( \lambda_{eq} = \lambda_1 \lambda_2 / | \lambda_1 - \lambda_2 | \) [2]. Thus typically \( \lambda_{eq} \) is much larger than either \( \lambda_1 \) or \( \lambda_2 \) (\( \lambda_{eq} \gg \{\lambda_1, \lambda_2\} \)). Double-wavelength (DW) interferometry was improved by Polhemus [3] and Cheng [4,5] using digital temporal phase-shifting.

On the other hand, Onodera et al. [6] used spatial-carrier, double-wavelength digital-holography (DH) and Fourier interferometry for phase-demodulation. This in turn was followed by a large number of multi-wavelength digital-holographic (DH) Fourier phase-demodulation methods in such diverse applications as interferometric contouring [7], phase-imaging [8], chromatic aberration compensation in microscopy [9]; single hologram DW microscopy [10]; comb multi-wavelength laser for extended range optical metrology [11], and a two-steps digital-holography for image quality improvement [12].

More recently temporal dual-wavelength phase-shifting algorithms (DW-PSAs) have been reworked by Abdelsalam et al. [14]. Even though Abdelsalam et al. give working PSA formulas they do not estimate their spectra, their signal-to-noise ratio, or their detuning and harmonics robustness. Kumar [15] and Baranda [16] also provided valid temporal PSA formulas but also failed to characterize their PSAs in terms of signal-to-noise, detuning and harmonic rejection. Another different approach was followed by Kulkarni and Rastogi [16] in which they have demodulated the two interesting phases by fitting a low-order polynomial to each phase. Their approach [17] worked well for the example provided but we think their method could easily cross-talk between fitted polynomials for complicated modulating phases [17]. Yet another approach by Zhang et al. was published [18-19]. Zhang used a simultaneous two-steps [18], and principal component interferometry [19] to solve the dual-wavelength phase-shifting measurement. Zhang et al. used 32 randomly phase-shifted interferograms [19]. Even though Zhang [19] could demodulate the two phases, they used 32 phase-shifted temporal interferograms. All these works on temporal DW-PSA [2-5,14-19] have given just specific DW-PSAs without explicit formulae for their spectra, signal-to-noise, detuning and harmonic robustness.

In contrast to previous ad-hoc temporal DW-PSA formulas, here we give a general theory for synthesizing DW-PSAs formalizing their spectrum, their signal-to-noise, and their detuning-harmonic robustness. At the risk of being repetitive, we emphasize that we are not
just giving particular DW-PSAs formulas as previously done [2-5. 14-19]. Here we are giving a general FTF-theory for synthesizing DW-PSA giving with explicit formulae for the most important characteristics of any PSA: spectra, signal-to-noise, detuning and harmonic robustness.

2. Spatial-carrier phase-demodulation for Dual-wavelength (DW) interferometry

Dual-wavelength digital-holography (DW-DH) is well understood and widely used [6-10]. As shown in Fig. 1, in DW-DH the two lasers beams are tilted to introduce spatial-carrier fringes [7]. In Fig. 1 both lasers beams are tilted in the x direction, but in general, for a better use of the Fourier space, one may tilt them independently along the x and y directions [11-14].

![Fig. 1 Schematics for DW-DH with a single tilted reference mirror [6]. The orange-light corresponds to the spatial superposition of the red and green lasers.](image)

The DW-DH obtained at the CCD camera in Fig. 1 may be modeled by,

\[
I(x, y, t) = a(x, y) + b_1(x, y) \cos[\varphi_1(x, y) + u_1x] + b_2(x, y) \cos[\varphi_2(x, y) + u_2x].
\]  

(1)

Here \(u_1x = x(2\pi / \lambda_1)\tan(\theta)\) and \(u_2x = x(2\pi / \lambda_2)\tan(\theta)\) are the spatial-carriers of the DW-DH. The reference mirror angle along the x axis is \(\theta\). The searched phases are \(\varphi_1(x, y) = (2\pi / \lambda_1)W_1(x, y)\) and \(\varphi_2(x, y) = (2\pi / \lambda_2)W_2(x, y)\); being \(W_1(x, y)\) and \(W_2(x, y)\) the measuring wavefronts. Figure 2 shows a Fourier spectrum of Eq. (1).

![Fig. 2. The hexagons represent the spatial filters, which demodulate the phases \(\varphi_1\) and \(\varphi_2\).](image)

The two hexagons in Fig. 2 are the quadrature filters that passband the desired analytic signals. After filtering, the inverse Fourier transform find the demodulated phases [1]. The advantage of DW-DH is that only one digital-hologram is needed to obtain \(\{\varphi_1,\varphi_2\}\); however its drawback is that just a fraction of the Fourier space \((u, v) \in [-\pi, \pi] \times [-\pi, \pi]\) is used (Fig 2). This limitation makes DW-DH not suitable for measuring discontinuous industrial objects [7]. In contrast, in DW-PSAs the full Fourier spectrum \((u, v) \in [-\pi, \pi] \times [-\pi, \pi]\) may be used.
3. Dual-wavelength (DW) temporal-carrier phase-shifting interferometry

The temporal phase-shifting double-wave interferogram may be modeled as,

\[ I(x, y, t) = a(x, y) + b_1(x, y) \cos \left( \frac{2\pi}{\lambda_1} t \right) + b_2(x, y) \cos \left( \frac{2\pi}{\lambda_2} t \right) \]

Where \( t \in (-\infty, \infty) \), and \( \varphi_1(x, y) = (2\pi / \lambda_1)W_1(x, y) \), \( \varphi_2(x, y) = (2\pi / \lambda_2)W_2(x, y) \) are the measuring phases. The parameter \( d \) is the PZT-step. The fringes background is \( a(x, y) \) and the lasers power must be about the same to obtain high fringe contrast: \( b_1(x, y) \approx b_2(x, y) \).

Figure 3 shows one possible set-up for a DW temporal phase-shifting interferometer.

![Fig. 3. A schematic example of a DW temporal-carrier interferometer [2-5] for surface measured with equivalent wavelength \( \lambda_{eq} \); the piezoelectric transducer is PZT.](image)

The motivation of using 2-wavelengths \( \lambda_1 \) and \( \lambda_2 \) (in spatial or temporal interferometry) is that interferometric measurements can be made with an equivalent wavelength \( \lambda_{eq} \) [2-19],

\[ \lambda_{eq} = \frac{\lambda_1 \lambda_2}{|\lambda_1 - \lambda_2|} \; ; \; \lambda_{eq} \gg (\lambda_1 \text{ or } \lambda_2) . \]  

(3)

With large \( \lambda_{eq} \) one may measure deeper surface discontinuities or topographies than using either \( \lambda_1 \) or \( \lambda_2 \) alone [2-19]. For a given PZT-step \( d \), the two angular-frequencies (in radians per interferogram) are given by,

\[ \omega_1 = \frac{2\pi}{\lambda_1} d \; , \; \text{and} \; \omega_2 = \frac{2\pi}{\lambda_2} d . \]  

(4)

Using this equation one may rewrite Eq. (2) as,

\[ I(x, y, t) = a(x, y) + b_1(x, y) \cos [\varphi_1(x, y) + \omega_1 t] + b_2(x, y) \cos [\varphi_2(x, y) + \omega_2 t] , \]

(5)

Here we have 5 unknowns: namely, \( \{a, b_1, b_2, \varphi_1, \varphi_2\} \). Therefore we need at least 5 phase-shifted interferograms to obtain a solution for \( \varphi_1(x, y) \) and \( \varphi_2(x, y) \). These are given by:

\[
\begin{align*}
I_0(x, y) &= a + b_1 \cos[\varphi_1] + b_2 \cos[\varphi_2], \\
I_1(x, y) &= a + b_1 \cos[\varphi_1 + \omega_1] + b_2 \cos[\varphi_2 + \omega_2], \\
I_2(x, y) &= a + b_1 \cos[\varphi_1 + 2\omega_1] + b_2 \cos[\varphi_2 + 2\omega_2], \\
I_3(x, y) &= a + b_1 \cos[\varphi_1 + 3\omega_1] + b_2 \cos[\varphi_2 + 3\omega_2], \\
I_4(x, y) &= a + b_1 \cos[\varphi_1 + 4\omega_1] + b_2 \cos[\varphi_2 + 4\omega_2].
\end{align*}
\]

(6)

For clarity, most \( (x, y) \) coordinates were omitted.
4. Fourier-spectrum for temporal DW-PSAs

The Fourier transform of the temporal interferogram (with $t \in (-\infty, \infty)$) in Eq. (5) is:

$$I(\omega) = a \delta(\omega) + \frac{h}{2} \left[ e^{i\phi_1} \delta(\omega - \omega_1) + e^{i\phi_2} \delta(\omega + \omega_1) \right] + \frac{h}{2} \left[ e^{i\phi_1} \delta(\omega - \omega_2) + e^{i\phi_2} \delta(\omega + \omega_2) \right].$$

(7)

All $(x,y)$ were omitted. As mentioned, $\omega_1 = (2\pi / \lambda_1)d$ and $\omega_2 = (2\pi / \lambda_2)d$ are the two temporal-carrier frequencies in radians/interferogram; Fig. 4 shows this spectrum.

Figure 5 shows two ideal quadrature filters $H_1(\omega)$ and $H_2(\omega)$ that could passband the desired analytic signals $\delta(\omega-\omega_1) \exp(i \varphi_1)$ and $\delta(\omega-\omega_2) \exp(i \varphi_2)$. Note how each filter is able to passband the desired signals from the same temporal interferograms.

5. Synthesis of DW-PSAs using 5-step temporal interferograms

The rectangular filters in Fig. 5 require a large number of temporal interferograms [1]. However, using the FTF we can synthesize 5-step bandpass filters by allocating 4 spectral-zeros at frequencies $\{-\omega_2,-\omega_1,0,\omega_1\}$ for $H_1(\omega)$, and at $\{-\omega_2,-\omega_1,0,\omega_2\}$ for $H_2(\omega)$ as,

$$H_1(\omega) = \left(1 - e^{i\omega} \right) \left[1 - e^{i(\omega + \omega_1)} \right] \left[1 - e^{i(\omega - \omega_1)} \right] \left[1 - e^{i(\omega + \omega_2)} \right] \left[1 - e^{i(\omega - \omega_2)} \right],$$

$$H_2(\omega) = \left(1 - e^{i\omega} \right) \left[1 - e^{i(\omega - \omega_1)} \right] \left[1 - e^{i(\omega + \omega_1)} \right] \left[1 - e^{i(\omega - \omega_2)} \right] \left[1 - e^{i(\omega + \omega_2)} \right].$$

(8)
From Eq. (8) one sees that by design, \( I(\omega)H_s(\omega) \) passband the signal \( \exp(i\varphi_1)\delta(\omega - \omega_1) \), while \( I(\omega)H_s(\omega) \) bandpass \( \exp(i\varphi_2)\delta(\omega - \omega_2) \). The impulse responses \( h(t) \) and \( h(t) \) of these two quadrature filters are then given by:

\[
h(t) = F^{-1}\{H_1(\omega)\} = \sum_{n=0}^{4} c_1(\omega_1, \omega_2) \delta(t-n),
\]
\[
h(t) = F^{-1}\{H_2(\omega)\} = \sum_{n=0}^{4} c_2(\omega_1, \omega_2) \delta(t-n).
\]  (9)

Here \( c_1(\omega_1, \omega_2) \) and \( c_2(\omega_1, \omega_2) \) are complex-valued coefficients that depend on the frequencies \( \{\omega_1, \omega_2\} \). Having \( h(t) \) and \( h(t) \), we obtain the two searched DW-PSAs as,

\[
\frac{1}{2}H_1(\omega) b(x,y)e^{i\varphi_1(x,y)} = \sum_{n=0}^{4} c_1(\omega_1, \omega_2) I_n(x,y),
\]
\[
\frac{1}{2}H_2(\omega) b(x,y)e^{i\varphi_2(x,y)} = \sum_{n=0}^{4} c_2(\omega_1, \omega_2) I_n(x,y).
\]  (10)

The explicit 5-step DW-PSA formula to estimate \( \varphi_1(x,y) \) is,

\[
A_1 \exp(i\varphi_1) = -e^{i\omega_1}I_0 + c_1(\omega_1, \omega_2) I_1 - c_2(\omega_1, \omega_2) I_2 + c_1(\omega_1, \omega_2) I_3 - e^{i(\omega_1-\omega_2)}I_4,
\]
\[
c_1(\omega_1, \omega_2) = 1 + e^{i\omega_1} + e^{i2\omega_2} + e^{i(\omega_1-\omega_2)} + e^{i(\omega_1-\omega_2)},
\]
\[
c_2(\omega_1, \omega_2) = 1 + e^{i\omega_1} + e^{i2\omega_2} + e^{i(\omega_1-\omega_2)} + e^{i(\omega_1-\omega_2)},
\]  (11)

Being \( A_1 = (1/2)H_1(\omega) b(x,y) \). Conversely the 5-step DW-PSA to estimate \( \varphi_2(x,y) \) is:

\[
A_2 \exp(i\varphi_2) = -e^{i\omega_1}I_0 + c_1(\omega_1, \omega_2) I_1 - c_2(\omega_1, \omega_2) I_2 + c_1(\omega_1, \omega_2) I_3 - e^{i(\omega_1-\omega_2)}I_4,
\]
\[
c_1(\omega_1, \omega_2) = 1 + e^{i\omega_1} + e^{i2\omega_2} + e^{i(\omega_1-\omega_2)} + e^{i(\omega_1-\omega_2)} + e^{i(\omega_1-\omega_2)}
\]
\[
c_2(\omega_1, \omega_2) = 1 + e^{i\omega_1} + e^{i2\omega_2} + e^{i(\omega_1-\omega_2)} + e^{i(\omega_1-\omega_2)} + e^{i(\omega_1-\omega_2)}.
\]  (12)

Being \( A_2 = (1/2)H_2(\omega) b(x,y) \). This is the basics for synthesizing DW-PSAs grounded on the FTF paradigm [1]. Previous papers on DW-PSAs [2-5,14-19] stop much shorter than this.

Even if the theory of this paper would stop right here, this paper would contain a substantial improvement against current ad-hoc art in DW-PSA [2-5,14-19].

6. Signal-to-noise power-ratios for \( H_1(\omega) \) and \( H_2(\omega) \)

Here we review the signal-to-noise power-ratio formulas for PSA quadrature filters [1]. The signal-to-noise power-ratios for \( H_1(\omega) \) and \( H_2(\omega) \) are given by [1]:

\[
\text{SNR}_1 = \frac{\|H_1(\omega)\|^2}{\frac{1}{2\pi} \int_{-\infty}^{\infty} |H_1(\omega)|^2 \, d\omega}, \quad \text{SNR}_2 = \frac{\|H_2(\omega)\|^2}{\frac{1}{2\pi} \int_{-\infty}^{\infty} |H_2(\omega)|^2 \, d\omega}.
\]  (13)
These 2 SNR-formulas give the power of the demodulated signals $|H_1(\omega)|^2$ and $|H_2(\omega)|^2$ divided by their total noise-power $(1/2\pi)|H_1(\omega)|^2 d\omega$ and $(1/2\pi)|H_2(\omega)|^2 d\omega$.

7. Non-optimized DW-PSA design for wavelengths $\lambda_1 = 632.8\,\text{nm}$ and $\lambda_2 = 532.0\,\text{nm}$

Let us assume that we use a typical temporal frequency of $\omega = 2\pi/5$ radians per sample for the algorithm $H_i(\omega) e^{i\phi(x,y)}$. Having made this choice for $\omega$, the frequency $\omega_2$ is set to

$$d = \omega_1\left(\frac{\lambda_1}{2\pi}\right) = \omega_2\left(\frac{\lambda_2}{2\pi}\right) \Rightarrow \omega_1 = \omega_2 = \omega_1\left(\frac{\lambda_1}{\lambda_2}\right) \quad \therefore \omega_2 = 1.49 \text{ radians}.$$  

and the required PZT-step is $d = 126.6\,\text{nm}$. The DW-FTFs for these two frequencies are:

$$H_1(\omega) = (1 - e^{-i\omega}) \left[1 - e^{i(\alpha_1 + 1.49)}\right] \left[1 - e^{i(\alpha_1 - 1.49)}\right] \left[1 - e^{i(\alpha_1 + 1.26)}\right],$$  

$$H_2(\omega) = (1 - e^{-i\omega}) \left[1 - e^{i(\alpha_2 + 1.26)}\right] \left[1 - e^{i(\alpha_2 - 1.26)}\right] \left[1 - e^{i(\alpha_2 + 1.49)}\right].$$  

Figure 6 shows the magnitude plot of these two quadrature filters $H_1(\omega)$ and $H_2(\omega)$.

![Fig. 6 Spectral plots for the two DW-PSA. The crossed Dirac deltas are the filter-out signals. These two FTFs can demodulate $\phi_1$ and $\phi_2$ but with poor signal-to-noise performance.](image)

The signal-to-noise [1] for the searched signals $H_1(\omega)\exp(i\phi_1)$ and $H_2(\omega)\exp(i\phi_2)$ are,

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |H_1(\omega)|^2 d\omega = 0.94; \quad \frac{1}{2\pi} \int_{-\pi}^{\pi} |H_2(\omega)|^2 d\omega = 1.04; \quad \omega_1 = 1.26; \omega_2 = 1.49.$$  

For comparison, a 5-step least-squares PSA has a signal-to-noise power-ratio of 5 [1]. Thus $\omega_1 = 2\pi/5$ and $\omega_2 = 1.49$ were a bad choice; we can estimate $\phi_1(x,y)$ and $\phi_2(x,y)$ from the DW-PSAs in Eq. (11)-(12) using these temporal frequencies, but they are going to be noisy. Previous DW-PSAs efforts [2-5,14-19] only provide numeric-specific DW-PSAs formulas to obtain $\phi_1(x,y)$ and $\phi_2(x,y)$. However, they were silent about their Fourier spectra; their signal-to-noise; their harmonics rejection and their detuning robustness. All this useful and practical formulae are given here for the first time in terms of the FTF for designing DW-PSAs. Moreover, in contrast to previous art in DW-PSAs, Eq. (11) and Eq. (12) give infinite DW-PSA formulas for continuous pairs of temporal frequencies $(\omega_1,\omega_2) \in [-\pi,\pi] \times [-\pi,\pi]$.

8. Optimized joint signal-to-noise ratio synthesis for DW-PSAs

To find a better selection for $\omega_1 = (2\pi/\lambda_1)d$ and $\omega_2 = (2\pi/\lambda_2)d$, we construct a joint signal-to-noise ratio as,
\[ G_{S/N}(d) = \left( \frac{H_1(\omega_1)^2}{\frac{1}{2\pi} \int_\omega H_1(\omega)^2 d\omega} \right) \left( \frac{H_2(\omega_2)^2}{\frac{1}{2\pi} \int_\omega H_2(\omega)^2 d\omega} \right) ; \quad d \in [0, \lambda_{eq}]. \] (17)

The function \( G_{S/N}(d) \) is complicated and has many local maxima but, fortunately, it is one-dimensional. Thus we plot \( G_{S/N}(d) \) and look for a good maximum, and take the PZT-step \( d \). This PZT-step \( d \) is used to find the two specific DW-PSA (Eqs. (11)-(12)) which solves the dual-wavelength interferometric problem.

9. Example of optimized DW-PSA synthesis for \( \lambda_1 = 632.8\text{nm} \) and \( \lambda_2 = 532\text{nm} \)

The graph for the joint signal-to-noise ratio \( G_{S/N}(d) \) with \( \omega_1 = (2\pi / \lambda_1)d \), \( \omega_2 = (2\pi / \lambda_2)d \) and \( d \in [0, \lambda_{eq}] \) is shown next.

![Graph of \( G_{S/N}(d) \)](image)

**Fig. 7.** Graph of \( G_{S/N}(d) \). We kept the third local maximum at \( d = 0.225\lambda_{eq} = 751\text{nm} \), for which \( G_{S/N}(d) = 23.5 \). Each DW-PSA thus have a signal-to-noise of \( \sqrt{23.5} \approx 4.84 \).

The first good-enough maximum is \( G_{S/N}(0.225\lambda_{eq}) \approx 23.5 \) (in blue), being \( d = 0.225\lambda_{eq} \) or \( d = 751\text{nm} \). Note that most of this graph is less than 20; i.e. \( G_{S/N}(d) < 20 \). This means that taking a PZT-step within \( d \in [0, \lambda_{eq}] \) at random, the probability of landing in a very low signal-to-noise point is very high.

![Spectral plots for \( H_1(\omega) \) and \( H_2(\omega) \)](image)

**Fig. 8.** Spectral plots for \( H_1(\omega) \) and \( H_2(\omega) \) for the S/N-optimized DW-PSA. Note that \( \omega_1 = W[(2\pi / \lambda_1)d] = 1.2 \text{ rad} \) and \( \omega_2 = W[(2\pi / \lambda_2)d] = 2.6 \text{ rad} \); with \( W(x) = \arg[\exp(ix)] \).

Therefore in this section we have shown that even though the correct phases \( \phi_1(x, y) \) and \( \phi_2(x, y) \) can be found using Eq. (11) and Eq. (12), without plotting \( G_{S/N}(d) \) these DW-PSAs designs will have a low signal-to-noise power-ratio with high probability.
9. Example for DW-PSA phase-demodulation for $\lambda_1 = 632.8\text{nm}$ and $\lambda_2 = 532.0\text{nm}$

Figure 9 shows five computer-simulated interferograms to test the DW-PSAs found in previous section. The PZT-step is $d = 751\text{nm}$, giving a good signal-to-noise ratio. As mentioned, for large PZT-steps, the angular frequencies $(\omega_1, \omega_2)$ are wrapped and given by,

$$\omega_1 = \arg\left[\exp\left(\frac{i d 2\pi}{\lambda_1}\right)\right] = 2.6, \quad \omega_2 = \arg\left[\exp\left(\frac{i d 2\pi}{\lambda_2}\right)\right] = 1.2 \quad (18)$$

Using these angular frequencies in Eq. (11), the specific formula to estimate $\phi_1(x,y)$ is,

$$A_1(\omega_1) e^{i\phi_1} = -e^{2\omega_1 d} I_0 + (0.78 + 0.62i)I_1 - (0.5 - i)I_2 - (1 + 0.19i)I_3 - e^{-1\omega_1 d} I_4 \quad (19)$$

Also, from Eq. (12), the specific 5-step DW-PSA to estimate the signal $\phi_2(x,y)$ is,

$$A_2(\omega_2) e^{i\phi_2} = -e^{1\omega_2 d} I_0 + (0.8 + 0.6i)I_1 - (0.92 - 0.1i)I_2 + (0.65 - 0.77i)I_3 - e^{-1\omega_2 d} I_4. \quad (20)$$

![Interferograms](image)

Fig. 9. The upper row shows 5 simulated interferograms without noise. The lower panel shows the same interferograms corrupted with phase-noise uniformly distributed in $[0,\pi]$. The noisy fringes were low-pass filtered by a 3x3 averaging window.

Figure 10 shows the demodulated signals $\phi_1(x,y)$ and $\phi_2(x,y)$.

![Demodulated signals](image)

Fig. 10. The demodulated phases $\phi_1(x,y)$ and $\phi_2(x,y)$ corresponding to the noiseless (panel (a)) and noisy (panel (b)) 5-steps interferograms in Fig. 9. Please note that there is no cross-tackking between the two demodulated phases $\phi_1(x,y)$ and $\phi_2(x,y)$. 
Figure 10(a) shows the noiseless demodulated phases, while Fig. 10(b) shows the demodulated phases degraded with a phase noise uniformly distributed within $[0, \pi]$. Note that absolutely no cross-talking between the demodulated phases $\phi_1$ and $\phi_2$ appears.

### 10. Detuning-robust DW-PSA synthesis for $\lambda_1 = 632.8\text{ nm}$ and $\lambda_2 = 458\text{ nm}$

Let us assume that our PZT is poorly calibrated. Thus instead of having well-tuned frequencies at $\{\omega_1, \omega_2\}$ we have detuned frequencies at $\{\omega_1 + \Delta, \omega_2 + \Delta\}$, being $\Delta$ the amount of detuning. As Fig. 11 shows, the estimated phase $\hat{\phi}_2(x, y)$ is now be given by,

$$A_2 e^{-i \phi} = H_2(\omega_1 + \Delta) e^{-i \phi} + H_2(\omega_2 + \Delta) e^{-i \phi} + H_2(\omega_1 - \Delta) e^{i \phi} + H_2(\omega_2 - \Delta) e^{i \phi}. \quad (21)$$

The estimated phase $\hat{\phi}_2(x, y)$ then have cross-talking from $\{e^{-i \phi}, e^{i \phi}, e^{-i \phi}, e^{i \phi}\}$; conversely $\hat{\phi}_1(x, y)$ will be distorted by cross-talking from $\{e^{-i \phi}, e^{i \phi}, e^{-i \phi}, e^{i \phi}\}$.

![Figure 11](image1)

Fig. 11. Here we show the effect of detuning ($\Delta$), greatly exaggerated for clarity. The amount of linear detuning is $\Delta$ (radians/sample). The well-tuned frequencies are $\{\omega_1 - \omega_1, \omega_1, \omega_2\}$, while the detuned frequencies are $\{\omega_1 - \Delta, \omega_2 - \Delta, \omega_1 + \Delta, \omega_2 + \Delta\}$.

To ensure good detuning robustness we need double-zeros at the rejected frequencies. Therefore, we transform Eq. (8) (5-steps) to detuning-robust DW-FTFs (8-steps) as,

$$H_1(\omega) = \left(1 - e^{i \alpha}\right)^2 \left[1 - e^{-i (\omega - \omega_2)}\right] \left[1 - e^{i (\omega - \omega_1)}\right] \left[1 - e^{i (\omega + \omega_1)}\right].$$

$$H_2(\omega) = \left(1 - e^{i \alpha}\right)^2 \left[1 - e^{-i (\omega - \omega_2)}\right] \left[1 - e^{i (\omega - \omega_1)}\right] \left[1 - e^{i (\omega + \omega_2)}\right]. \quad (22)$$

Next, we plot $G_{S/N}(d)$ and look for a high local signal-to-noise maximum; see Fig. 12.

![Figure 12](image2)

Fig. 12. Graph of the joint signal-to-noise power-ratio $G_{S/N}(d)$ of the two detuning-robust FTF-filers in Eq. (22). The second maximum has a PZT-displacement of $d=381\text{ nm}$.

We choose the second maximum (in blue) where $G_{S/N}(0.23 \lambda_{eq}) = 44$, and $d = 381\text{ nm}$. Each (8-steps) DW-PSA filter in Eq. (22) has a signal-to-noise ratio of about $\sqrt{44} = 6.6$. Figure 13
shows the two 8-step DW-PSA detuning-robust FTFs. The spectral second-order zeroes are flatter, so they are frequency detuning $\Delta$ tolerant.

Fig. 13. Spectra of detuning-robust DW-PSA tuned at $\omega_1=2.5\text{rad}$ and $\omega_2=1.05\text{rad}$. The second-order zeroes tolerate a fair amount of frequency detuning $\Delta$.

11. Harmonic rejection in DW-PSAs

Figure 14 shows the harmonic response for the FTFs in Eq. (8). The red-sticks are the fringe harmonics at $(n\omega_1)$, and the green ones are the fringe harmonics at $(n\omega_2)$, $|n| \geq 2$.

Fig. 14. Amplitudes of the distorting harmonics for $|H_1(n\omega_1)|$, in red; and $|H_2(n\omega_2)|$, in green. The ideal result would be to bandpass just the Dirac’s deltas at $\omega = \omega_1$ and $\omega = \omega_2$.

The power of the desired analytic signals $|H_1(\omega_1)\exp(\phi_1)|^2$ and $|H_2(\omega_2)\exp(\phi_2)|^2$ with respect to the distorting harmonics is given by,

$$\begin{align*}
HR_1 &= \frac{|H_1(\omega_1)|^2}{\sum_{|n|\geq2} \left(\frac{1}{|n|^2}\right)^{2} \left[|H_1(n\omega_1)|^2 + |H_2(n\omega_2)|^2\right]} = 11.83, \\
HR_2 &= \frac{|H_2(\omega_2)|^2}{\sum_{|n|\geq2} \left(\frac{1}{|n|^2}\right)^{2} \left[|H_1(n\omega_1)|^2 + |H_2(n\omega_2)|^2\right]} = 12.2
\end{align*}
$$

(23)

Here we assumed that the amplitude of the harmonics decreases as $(1/|n|^2)$, so their power decreases as $(1/|n|^2)^2$. With this assumption, $H_1(\omega_1)$ and $H_2(\omega_2)$ have about 10-times more power than the total power sum of their harmonics $\{H_1(n\omega_1), H_1(n\omega_2), H_2(n\omega_1), H_2(n\omega_2)\}$.
Figure 15 shows five saturated phase-shifted interferograms. These 5 temporal interferograms are then phase demodulated using DW-PSAs, Eqs (11)-(12).

![Fig. 15. Five DW phase-shifted temporal interferograms with amplitude saturation.](image)

Figure 16 shows the demodulated phases $\phi_1$ and $\phi_2$ of the interferograms in Fig. 15.

![Fig. 16. The two demodulated phases from the 5 saturated fringe patterns in Fig. 15.](image)

12. Multi-wavelength $\{\lambda_1,\lambda_2,\lambda_3,...,\lambda_n\}$ temporal phase-shifting interferometry

Here DW-PSA is generalized to 3-wavelengths. A simplified schematic of an interferometer simultaneously illuminated with 3-wavelengths $\{\lambda_1,\lambda_2,\lambda_3\}$ is shown in Fig. 17.

![Fig. 17. Simplified schematics for a temporal 3-wavelength phase-shifting interferometer.](image)

The continuous-time phase-shifted interferogram is,

$$I(x, y, t) = a + b_1 \cos(\phi_1 + \omega t) + b_2 \cos(\phi_2 + \omega_2 t) + b_3 \cos(\phi_3 + \omega_3 t).$$

Now we have 7 unknowns $\{a, b_1, b_2, b_3, \phi_1, \phi_2, \phi_3\}$; being $\{\phi_1, \phi_2, \phi_3\}$ the searched phases. Thus we need at least 7 temporal phase-shifted interferograms. Figure 18 shows the Fourier spectrum of this 3-wavelengths interferogram.
Therefore we need to construct 3-FTFs having at least 6 first-order zeroes (7-steps) as,

\[
H_1(\omega) = (1 - e^{i \omega}) \left[ 1 - e^{i(\omega - \omega_1)} \right] \left[ 1 - e^{i(\omega - \omega_2)} \right] \left[ 1 - e^{i(\omega - \omega_3)} \right],
\]

\[
H_2(\omega) = (1 - e^{i \omega}) \left[ 1 - e^{i(\omega - \omega_1)} \right] \left[ 1 - e^{i(\omega - \omega_2)} \right] \left[ 1 - e^{i(\omega - \omega_3)} \right],
\]

\[
H_3(\omega) = (1 - e^{i \omega}) \left[ 1 - e^{i(\omega - \omega_1)} \right] \left[ 1 - e^{i(\omega - \omega_2)} \right] \left[ 1 - e^{i(\omega - \omega_3)} \right].
\]

(25)

The spectrum \( H_1(\omega) \) rejects the analytic signals at \( \{ -\omega_3, -\omega_2, -\omega_1, 0, \omega_2, \omega_3 \} \); \( H_2(\omega) \) rejects the signals at \( \{ -\omega_3, -\omega_2, -\omega_1, 0, \omega_2, \omega_3 \} \); and \( H_3(\omega) \) rejects the analytic signals at \( \{ -\omega_3, -\omega_2, -\omega_1, 0, \omega_2, \omega_3 \} \). Therefore \( I(\omega)H_1(\omega) \) passband \( \exp(i\varphi_1)\delta(\omega - \omega_1) \); \( I(\omega)H_2(\omega) \) passband \( \exp(i\varphi_2)\delta(\omega - \omega_2) \), and finally \( I(\omega)H_3(\omega) \) passband \( \exp(i\varphi_3)\delta(\omega - \omega_3) \).

The joint signal-to-noise ratio optimizing criterion now reads,

\[
G_{S/N}(d) = \frac{\left| H_1(\omega_1) \right|^2}{\frac{1}{2\pi} \int_{-\pi}^{\pi} \left| H_1(\omega) \right|^2 d\omega} \frac{\left| H_2(\omega_1) \right|^2}{\frac{1}{2\pi} \int_{-\pi}^{\pi} \left| H_2(\omega) \right|^2 d\omega} \frac{\left| H_3(\omega_1) \right|^2}{\frac{1}{2\pi} \int_{-\pi}^{\pi} \left| H_3(\omega) \right|^2 d\omega}.
\]

(26)

We then find a convenient local maximum for \( G_{S/N}(d) \), obtaining a fixed PZT-step \( d \), and three angular-frequencies \( (\omega_1, \omega_2, \omega_3) \in [-\pi, \pi] \times [-\pi, \pi] \times [-\pi, \pi] \) as,

\[
\omega_1 = W\left( \frac{2\pi}{\lambda_1} d \right), \quad \omega_2 = W\left( \frac{2\pi}{\lambda_2} d \right), \quad \omega_3 = W\left( \frac{2\pi}{\lambda_3} d \right); \quad W(x) = \arg[\exp(i x)].
\]

(27)

The three impulse responses \{ \( h_1(t), h_2(t), h_3(t) \) \} are then given by,

\[
h_1(t) = F^{-1} \{ H_1(\omega) \} = \sum_{n=-6}^{6} c_1(n, \omega_1, \omega_2, \omega_3) \delta(t-n),
\]

\[
h_2(t) = F^{-1} \{ H_2(\omega) \} = \sum_{n=-6}^{6} c_2(n, \omega_1, \omega_2, \omega_3) \delta(t-n),
\]

\[
h_3(t) = F^{-1} \{ H_3(\omega) \} = \sum_{n=-6}^{6} c_3(n, \omega_1, \omega_2, \omega_3) \delta(t-n),
\]

(28)

Here \( c_1(n, \omega_1, \omega_2, \omega_3) \), \( c_2(n, \omega_1, \omega_2, \omega_3) \), \( c_3(n, \omega_1, \omega_2, \omega_3) \) are the complex coefficients of the PSAs, which now depend on the three temporal-carrier frequencies \( \{ \omega_1, \omega_2, \omega_3 \} \).

We now digitally grab 7 phase-shifted interferograms given by,

\[
I_n = a + b_1 \cos[\varphi_1 + n\omega_1] + b_2 \cos[\varphi_2 + n\omega_2] + b_3 \cos[\varphi_3 + n\omega_3]; \quad n = 0, \ldots, 6.
\]

(29)

Obtaining the three searched quadrature analytic signals as,
\[ A_n(x,y) \exp[i \varphi(x,y)] = \sum_{n=0}^{6} c_n(\omega_1, \omega_2, \omega_3) I_n(x,y), \]

where \[ A_n(x,y) = (1/2)H_n(\omega_n) \delta(x,y). \] By mathematical induction, one may see that a 4-wavelength \( \{ \lambda_1, \lambda_2, \lambda_3, \lambda_4 \} \) phase-shifting algorithm would need at least 9 phase-shifted interferograms, requiring FTFs having 8 first-order zeroes, etc, etc.

13. Conclusions

The problem that was solved here may be stated as follows: Having a laser interferometer simultaneously illuminated with fixed wavelengths \( \{ \lambda_1, \lambda_2, \ldots, \lambda_K \} \) and a single PZT phase-shifter, find \( K \) phase-shifting algorithms (PSAs) which phase-demodulate \( \{ \varphi_1, \varphi_2, \ldots, \varphi_K \} \) for each laser-color, with high signal-to-noise and no cross-taking among these phases.

This was solved as follows (for \( K=2 \), and \( K=3 \) in section 12),

a) To start, we synthesize two 5-step quadrature-filters (PSA-spectra, Eq. (8)) that bandpass \( \exp(i\varphi_1) \) and \( \exp(i\varphi_2) \) from 5 phase-shifted interferograms (Eq. (6)) as,

\[ H_t(\omega) = \left(1 - e^{-i\omega t}\right) \left[1 - e^{i(\alpha_1 - \omega t)}\right] \left[1 - e^{i(\alpha_2 - \omega t)}\right] \left[1 - e^{i(\alpha_3 - \omega t)}\right], \]

\[ H_s(\omega) = \left(1 - e^{-i\omega s}\right) \left[1 - e^{i(\alpha_1 - \omega s)}\right] \left[1 - e^{i(\alpha_2 - \omega s)}\right] \left[1 - e^{i(\alpha_3 - \omega s)}\right]. \]

b) We then jointly optimize \( \{ H_t(\omega), H_s(\omega) \} \) for high signal-to-noise ratio \( G_{15}(d) \) (Fig. 7) and obtained the PZT-step \( d \) at which that local maximum occurs.

c) Having an optimum PZT-step \( d \), we then calculate the tuning frequencies \( \omega_1 = (2\pi / \lambda_1)d \), \( \omega_2 = (2\pi / \lambda_2)d \), which substituted back into \( \{ H_t(\omega), H_s(\omega) \} \) give the specific DW-PSAs that phase-demodulate \( \varphi_1(x,y) \) and \( \varphi_2(x,y) \) (Eqs. (11)-(12)).

d) We then plot (Fig. 8) the S/N-optimized designs \( \{ H_t(\omega), H_s(\omega) \} \) to gauge their spectral behavior within \( \omega \in [-\pi, \pi] \) (Fig. 8).

e) We also plotted (Fig. 14) the S/N-optimized \( \{ H_t(\omega), H_s(\omega) \} \) designs for an extended frequency range of \( \omega \in [-30\pi, 30\pi] \), to gauge their harmonic-rejection.

f) With the S/N-optimized \( \{ H_t(\omega), H_s(\omega) \} \) designs we quantified the harmonic-rejection capacity for each DW-PSA (Eq. (23)).

g) For poor PZT-calibration we modified \( \{ H_t(\omega), H_s(\omega) \} \) by raising the first-order zeroes to second-order zeroes, i.e. \( (\omega - \alpha_1) \Rightarrow (\omega - \alpha_1)^2 \), \( (\omega - \alpha_2) \Rightarrow (\omega - \alpha_2)^2 \), etc.; making \( \{ H_1(\omega), H_2(\omega) \} \) robust to detuning at the rejected frequencies (Fig. 13).

h) We used the S/N-optimized designs to phase-demodulate 5 phase-shifted interferograms (Figs. 9-10) with high signal-to-noise and no phase cross-taking.

i) Finally we extended the DW-PSA theory to 3-wavelengths \( \{ \lambda_1, \lambda_2, \lambda_3 \} \); further \( K \)-wavelengths \( \{ \lambda_1, \lambda_2, \ldots, \lambda_K \} \) generalization of this theory is just a matter of mathematical induction.

Finally, two examples of DW-PSA demodulation with \( \{ \lambda_1 = 632.8nm, \lambda_2 = 532nm \} \) which illustrate the behavior of the synthesized PSAs were given. As far as we know, previous art on DW-PSAs [2-5,14-19] only provided ad-hoc multi-wavelength PSA designs. Thus, this is the
first time that a general theory for synthesizing and analyzing multi-wavelength temporal phase-shifting algorithms is presented, and from which one may derive quantifying formulas for: (a) the PSAs spectra for each wavelength, (b) the PSAs signal-to-noise robustness for each wavelength, (c) the PSAs detuning sensitivity, and (d) the PSAs harmonics rejection for each wavelength.

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