Electromagnetic imaging of multiple-scattering small objects: Non-iterative analytical approach

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Abstract. Multiple signal classification (MUSIC) imaging method and the least squares method are applied to solve the electromagnetic inverse scattering problem of determining the locations and polarization tensors of a collection of small objects embedded in a known background medium. Based on the analysis of induced electric and magnetic dipoles, the proposed MUSIC method is able to deal with some special scenarios, due to the shapes and materials of objects, to which the standard MUSIC doesn’t apply. After the locations of objects are obtained, the nonlinear inverse problem of determining the polarization tensors of objects accounting for multiple scattering between objects is solved by a non-iterative analytical approach based on the least squares method.

1. Introduction

The electromagnetic inverse scattering problem of determining the locations and polarization tensors of a collection of small objects embedded in a known background medium is considered in this paper. The information of the objects is retrieved from the multistatic response (MSR) matrix generated by an array of transceivers [1–6].

Multiple signal classification (MUSIC) imaging method is widely used for locating point-like targets. MUSIC imaging was first developed in acoustic imaging, where scalar field is involved. The test function used to generate MUSIC pseudo-spectrum is the Green’s function of the background medium associated with a monopole source [3, 6–8]. Recently, Ammari et al. generalized MUSIC algorithm to electromagnetic imaging of small three-dimensional targets [2]. In case of electromagnetic scattering, the induced sources inside small objects are electric dipole and/or magnetic dipole, not monopole any more. The test function used to generate the MUSIC pseudo-spectrum is Green’s function of the background medium associated with an electric or magnetic dipole source with an arbitrary orientation [2]. When multiple scattering is taken into account, Ammari et al. proposed an approximate model, where two equivalent ellipsoids are constructed [2]. More recently, a MUSIC algorithm was proposed to locate closely spaced small anisotropic spheres, whose multiple scattering effect was described by the Foldy-Lax equation [9].

The aforementioned MUSIC algorithms [2, 9] assume three independent components of electric/magnetic dipoles are induced inside each scatterers. However, they cannot deal with degenerate scatterers in which only one or two components of electric or magnetic dipole are induced inside some small scatterers due to special shape or composing material of scatterers. For example, for an anisotropic small sphere, when part of components in the principal axes of
the permittivity tensor are equal to the permittivity of the background medium, the number of independent electric dipole components are less than three; although three components of electric dipole are theoretically induced inside a needle made of perfect electric conductor (PEC), the component along the major axis is much more dominant than the other two and in presence of noise it performs as if there was only one induced electric dipole. Both of the aforementioned MUSIC algorithms [2, 9] use as the test function of MUSIC pseudo-spectrum the Green’s function of the background medium associated with an electric or magnetic dipole source with an arbitrary orientation. In degenerate cases, standard MUSIC algorithms [2, 9] do not work because the arbitrarily chosen direction of test dipole is not necessarily located in the space spanned by actually induced independent dipole components.

In this paper, we propose a MUSIC algorithm for electromagnetic imaging of small objects that is able to deal with degenerate cases and accounts for multiple scattering effect. The proposed MUSIC algorithm determines not only the positions of small objects, but also the orientations of the independent dipole components in degenerate cases. The second is to propose a non-iterative analytical approach to retrieve polarization tensors of multiple-scattering objects after the locations of the objects are obtained. The proposed least squares method further develops the algorithm proposed in [9] so that it applies to degenerate cases.

2. Forward scattering problem

There are $M$ three-dimensional objects illuminated by time-harmonic electromagnetic waves radiated by an array of $N$ antenna units. The transceiver antenna units are located at $r_1, r_2, \ldots, r_N$, each of which consists of 3 small dipole antennas oriented in the $x$, $y$ and $z$ direction with length $d_x, d_y, d_z$ and driving current $I_x, I_y, I_z$, respectively, $i = 1, 2, \ldots, N$. The size of each of the $M$ spherical and ellipsoidal objects is much smaller than the wavelength so that Rayleigh scattering is observed. The centers of the scatterers are located at $r_1, r_2, \ldots, r_M$. The scatterers may be made of dielectric and permeable materials, or PEC, and may be isotropic or anisotropic. The shape and composing material of each small scatterer determine its polarization tensor $\tilde{\xi}_j$ ($\tilde{\zeta}_j$) [10, 11], which relates the induced electric (magnetic) current dipole $\Pi(r_j)$ ($Kl(r_j)$) inside the object to the total incident electric field $E_{t}^{\text{inc}}(r_j)$ (magnetic field $H_{t}^{\text{inc}}(r_j)$) by

$$\Pi(r_j) = \tilde{\xi}_j \cdot E_{t}^{\text{inc}}(r_j),$$  

$$Kl(r_j) = \tilde{\zeta}_j \cdot H_{t}^{\text{inc}}(r_j),$$

$j = 1, 2, \ldots, M$. The value of $\tilde{\xi}_j$ ($\tilde{\zeta}_j$) for isotropic and anisotropic spheres and ellipsoids can be found in Ref. [10].

When multiple scattering between scatterers are taken into account, the total incident field $E_{t}^{\text{inc}}(r_j)$ ($H_{t}^{\text{inc}}(r_j)$) upon the $j$th scatterer includes both the incident field directly from antennas $E_{0}^{\text{inc}}(r_j)$ ($H_{0}^{\text{inc}}(r_j)$) and the scattered fields from other scatterers. The total incident fields are governed by the Foldy-Lax equation,

$$E_{t}^{\text{inc}}(r_j) = E_{0}^{\text{inc}}(r_j) + \sum_{m \neq j} \left\{ \frac{\omega \mu_0 \tilde{G}(r_j, r_m) \cdot \tilde{\zeta}_m \cdot E_{t}^{\text{inc}}(r_m) - \nabla g(r_j, r_m) \times \left[ \tilde{\zeta}_m \cdot H_{t}^{\text{inc}}(r_m) \right] }{2} \right\},$$

$$H_{t}^{\text{inc}}(r_j) = H_{0}^{\text{inc}}(r_j) + \sum_{m \neq j} \left\{ \frac{i \omega \epsilon_0 \tilde{G}(r_j, r_m) \cdot \tilde{\xi}_m \cdot H_{t}^{\text{inc}}(r_m) + \nabla g(r_j, r_m) \times \left[ \tilde{\xi}_m \cdot E_{t}^{\text{inc}}(r_m) \right] }{2} \right\},$$

where $\epsilon_0$ and $\mu_0$ are the permittivity and permeability of the homogeneous background medium, respectively, $\tilde{G}(r, r') = (\hat{I}_x + \frac{\nabla}{\omega}) g(r, r')$ is the dyadic Green’s function of the background medium. By defining a permittivity operator $[12] \chi(r, r')$, so that $\tilde{\chi}(r, r') \cdot A = \nabla g(r, r') \times A$ for an arbitrary vector $A$, we write Eqs. (3) and (4) in a matrix form,

$$\tilde{\psi}_{t}^{\text{inc}} = \tilde{\psi}_{0}^{\text{inc}} + \tilde{\Phi} \cdot \tilde{A} \cdot \tilde{\psi}_{t}^{\text{inc}},$$

(5)
where both $\bar{\psi}_t^{in}$ and $\bar{\psi}_0^{in}$ are 2$M$ dimensional vectors,

$$\bar{\psi}_t^{in} = \left[ E_t^{in}(r_1)^T, E_t^{in}(r_2)^T, \ldots, E_t^{in}(r_M)^T, \eta_0 H_t^{in}(r_1)^T, \ldots, \eta_0 H_t^{in}(r_M)^T \right]^T,$$

$$\bar{\psi}_0^{in} = \left[ E_0^{in}(r_1)^T, E_0^{in}(r_2)^T, \ldots, E_0^{in}(r_M)^T, \eta_0 H_0^{in}(r_1)^T, \ldots, \eta_0 H_0^{in}(r_M)^T \right]^T,$$

where the superscript $T$ denotes the transpose, $\Lambda$ is a diagonal matrix,

$$\bar{\Lambda} = \text{diag} \left[ \bar{P}_1, \bar{P}_2, \ldots, \bar{P}_M, \bar{P}_{M+1}, \bar{P}_{M+2}, \ldots, \bar{P}_{2M} \right],$$

where $\bar{P}_m = \eta_0 \tilde{\xi}_m$ for $m \leq M$ and $\bar{P}_m = (1/\eta_0) \tilde{\xi}_{m-M}$ for $m > M$, $\eta_0$ is the impedance of the background medium, and $\bar{\Phi}$ is a 6$M$-by-6$M$ matrix,

$$\bar{\Phi} = \left[ \begin{array}{cc} \bar{\alpha}, & -\bar{\beta} \\ \bar{\beta}, & \bar{\alpha} \end{array} \right],$$

where both $\bar{\alpha}$ and $\bar{\beta}$ consist of $M$-by-$M$ sub-matrices whose formulas in the $j$th row and $j'$th column $(j, j' = 1, 2, \ldots, M)$ are given by

$$\bar{\alpha}(j, j') = \begin{cases} ik_0 \bar{G}_0(r_j, r_{j'}) & j \neq j' \\ 0 & j = j' \end{cases},$$

and

$$\bar{\beta}(j, j') = \begin{cases} \tilde{\chi}(r_j, r_{j'}) & j \neq j' \\ 0 & j = j' \end{cases}.$$  \hspace{1cm} (10, 11)

When the $M$ small scatterers are illuminated by the $N$ units of dipole antennas, the incident fields at the positions of scatterers, i.e., $\bar{\psi}_0^{in}$ in Eq. (5) are given by

$$\bar{\psi}_0^{in} = \bar{T} \cdot \bar{D} \cdot \bar{I},$$

where $\bar{D} = \text{diag} \left[ d_{1x}, d_{1y}, d_{1z}, d_{2x}, d_{2y}, d_{2z}, \ldots, d_{Nz} \right]$, $\bar{I} = \eta_0 \left[ I_{1x}, I_{1y}, I_{1z}, I_{2x}, I_{2y}, I_{2z}, \ldots, I_{Nz} \right]^T$, and $\bar{T}$ is given by

$$\bar{T} = \left[ \bar{G}, \bar{X} \right]^T,$$

where both $\bar{G}$ and $\bar{X}$ consist of $N$-by-$M$ sub-matrices whose formulas in the $i$th row and $j$th column $(i = 1, 2, \ldots, N, j = 1, 2, \ldots, M)$ are given by

$$\bar{G}(i, j) = ik_0 \bar{G}_0(r_i', r_j),$$

and

$$\bar{X}(i, j) = \tilde{\chi}(r_i', r_j).$$  \hspace{1cm} (12)

The voltage induced by the scattered field at each component of the antenna array is obtained from Eqs. (5) and (12)

$$\bar{V} = \bar{D} \cdot \bar{E}^s = \bar{D} \cdot \bar{R} \cdot \bar{\Lambda} \cdot (\bar{I}_{6M} - \bar{\Phi} \cdot \bar{\Lambda})^{-1} \cdot \bar{T} \cdot \bar{D} \cdot \bar{I}$$

\hspace{1cm} (16)
where $\bar{V} = [V_{1x}, V_{1y}, V_{1z}, V_{2x}, V_{2y}, V_{2z}, \ldots, V_{Nz}]^T$, $\bar{I}_{6M}$ is a $6M$-dimensional identity matrix, $\bar{E} = [E_x(r'_1)^T, E_y(r'_2)^T, \ldots, E_N(r'_N)^T]^T$, and $\bar{R} = \begin{bmatrix} \bar{G}, -\bar{X} \end{bmatrix}$. Thus, the multi-static response (MSR) matrix that relates the induced voltages to the driving currents is

$$\bar{K} = \bar{D} \cdot \bar{R} \cdot \bar{\Lambda} \cdot (\bar{I}_{6M} - \bar{\Phi} \cdot \bar{\Lambda})^{-1} \cdot \bar{T} \cdot \bar{D}. \quad (17)$$

In a non-degenerate case, three independent electric dipole components and three independent magnetic dipole components are induced in each of the scatterers, ending up being $6M$ independent dipole sources. Thus, the rank of the MSR matrix is equal to $6M$ in case of $6M < 3N$. Note that the condition $6M < 3N$ is assumed throughout the paper. In a degenerate case, the rank of the MSR matrix is less than $6M$.

Firstly we consider the case where both electric and magnetic dipoles are induced in each scatterer, i.e., none of $\bar{P}_j$ ($j = 1, 2, \ldots, 2M$) in Eq. (8) is null. If only one or two independent electric dipole components are induced in the $j$th scatterer, its polarization tensor $\bar{P}_j$ can be effectively expressed as

$$\bar{P}_j = \bar{S}_j \cdot \bar{P}_j' \cdot \bar{S}_j^T, \quad (18)$$

where $\bar{S}_j$ is composed of unit vectors describing the directions of induced independent electric dipole components, and $\bar{P}_j'$ expresses the degenerate polarization tensor that is a square matrix whose size is equal to the number of induced independent electric dipole components. For a non-degenerate case, the matrix $\bar{S}_j$ can be chosen as an identity matrix of size three $\bar{I}_3$. A similar approach can be applied to matrix $\bar{P}_j'$ ($j > M$) by analyzing induced independent magnetic dipole components. Thus, the matrix $\bar{\Lambda}$ in Eq. (8) can be effectively expressed as

$$\bar{\Lambda} = \bar{S} \cdot \bar{\Lambda}' \cdot \bar{S}^T \quad (19)$$

where $\bar{S}$ and $\bar{\Lambda}'$ are generated by diagonally connecting $\bar{S}_j$ and $\bar{P}_j'$ ($j = 1, 2, \ldots, 2M$), respectively. It is highlighted that the square matrix $\bar{\Lambda}$ in Eq. (8) is rank deficient whereas the square matrix $\bar{\Lambda}'$ in Eq. (19) is rank full.

Secondly, we consider the case where either electric or magnetic dipole is not induced inside some scatterers, i.e., $\bar{P}_j$ in Eq. (8) is null for those scatterers. Define matrix $\bar{L}_j$ ($j = 1, 2, \ldots, M$) that is a zero matrix of size 3 by 3 if no electric dipole is induced in the $j$th scatterer, and identity matrix $\bar{I}_3$ otherwise. The matrix $\bar{L}_j$ for $j = M+1, M+2, \ldots, 2M$ is defined similarly. A matrix $\bar{L}$ is formed by diagonally connecting $\bar{L}_j$ ($j = 1, 2, \ldots, 2M$) in sequence followed by removing columns whose elements are all zero. Then the original rank deficient matrix in Eq. (8) can be effectively expressed as

$$\bar{\Lambda} = \bar{L} \cdot \bar{S} \cdot \bar{\Lambda}' \cdot \bar{S}^T \cdot \bar{L}^T, \quad (20)$$

where the square matrix $\bar{\Lambda}'$ is rank full.

3. Inverse scattering problem

3.1. MUSIC algorithm for estimating the positions of scatterers

3.1.1. Non-degenerate case In a non-degenerate case, six independent dipole components (three electric and three magnetic components) are induced in each scatterer, and the vector $\bar{V}$ representing the induced voltages in the antenna array is in the space $S_0$ spanned by the background Green’s function vectors associated with the $x$, $y$, and $z$ components of electric and magnetic dipole sources evaluated at the position of each scatterer, i.e., $\bar{V} \in S_0 = \text{span} \{ \bar{G}_x(r_j), \bar{G}_y(r_j), \bar{G}_z(r_j), \bar{X}_x(r_j), \bar{X}_y(r_j), \bar{X}_z(r_j); j = 1, 2, \ldots, M \}$, where $\bar{G}_x(r_j)$, $\bar{G}_y(r_j)$ and $\bar{G}_z(r_j)$ are the $[3(j-1)+1]^{th}$, $[3(j-1)+2]^{th}$, and $[3(j-1)+3]^{th}$ column of matrix
where \( \bar{G} \), respectively, and \( \bar{X}_x(\mathbf{r}_j), \bar{X}_y(\mathbf{r}_j) \) and \( \bar{X}_z(\mathbf{r}_j) \) the \( [3(j - 1) + 1]^{th} \), \( [3(j - 1) + 2]^{th} \), and \( [3(j - 1) + 3]^{th} \) column of matrix \( \bar{X} \), respectively. The Green’s vectors \( \bar{G}_i(\mathbf{r}) \) and \( \bar{X}_i(\mathbf{r}) \) \((l = x, y, z)\) evaluated at an arbitrary position \( \mathbf{r} \) can be defined similarly. On the other hand, the MSR matrix \( \bar{K} \) maps \( C^{3N} \), the vector space of complex \( 3N \)-tuples, to its range \( S_r \subseteq C^{3N} \). From the singular value decomposition analysis \( [13] \), the MSR matrix could be represented as \( \bar{K} \cdot \bar{v}_p = \sigma_p \bar{u}_p \) and \( \bar{K}^* \cdot \bar{u}_p = \sigma_p \bar{v}_p, p = 1, 2, \ldots, 3N \), where the superscript * denotes the Hermitian. The vector space \( C^{3N} \) can be decomposed into the direct sum of the range \( S_r = \text{span}\{ \bar{u}_p, \sigma_p > 0 \} \) and the orthogonal complement space \( S_n = \text{span}\{ \bar{u}_p, \sigma_p = 0 \} \) that is referred to as the noise space. It is easy to find that the two subspaces \( S_r \) and \( S_n \) are identical \( [1, 3, 9] \). Due to the orthogonality between the range \( S_r \) and the noise space \( S_n \), we have \( |\bar{u}_p^* \bar{G}_i(\mathbf{r}_m)| = 0 \) and \( |\bar{u}_p^* \bar{X}_i(\mathbf{r}_m)| = 0 \), for \( \sigma_p = 0, m = 1, 2, \ldots, M \) and \( l = x, y, z \). Define the MUSIC pseudo-spectrum as
\[
\Phi(\mathbf{r}) = \log_{10} \left( \frac{\sum_{p=0}^{p_0} |\bar{u}_p^* \bar{f}(\mathbf{r})|^2}{p_0} \right)^{-1/2},
\] (21)
where \( p_0 \) is the number of vanishing singular values and \( \bar{f}(\mathbf{r}) = \bar{f}(\mathbf{r})/||\bar{f}(\mathbf{r})|| \) is a normalized test function the test function \( \bar{f}(\mathbf{r}) \) which can be any linear combination of \( \bar{G}_x(\mathbf{r}), \bar{G}_y(\mathbf{r}), \bar{G}_z(\mathbf{r}), \bar{X}_x(\mathbf{r}), \bar{X}_y(\mathbf{r}), \) and \( \bar{X}_z(\mathbf{r}) \). The pseudo-spectrum becomes infinite at the position of each scatterer.

3.1.2. Degenerate case In the degenerate case, only one or two independent electric (or magnetic) dipole components are induced inside some of the scatterers. If only one independent electric dipole component is induced inside one of the scatterers, i.e., induced dipoles are all parallel to each other, in order to produce an infinite peak at the position of this scatterer, test function \( \bar{f}(\mathbf{r}) \) should represent the background Green’s function for an electric dipole source aligned with the induced dipole. If two independent electric dipole components are induced inside one of the scatterers, i.e., induced dipoles are all located in a plane, \( \bar{f}(\mathbf{r}) \) should represent the background Green’s function for an electric dipole source oriented in the aforementioned plane. Therefore, when the test function \( \bar{f}(\mathbf{r}) \) is a proper linear combination of the background Green’s functions associated with the \( x, y, \) and \( z \) components of the electric dipole source,
\[
\bar{f}(\mathbf{r}) = [G_x(\mathbf{r}), G_y(\mathbf{r}), G_z(\mathbf{r})] \cdot \hat{\alpha},
\] (22)
where \( \hat{\alpha} = [\alpha_x, \alpha_y, \alpha_z]^T \) is a unit vector representing the direction of the test electric dipole, so that the test dipole source is aligned with the induced dipoles if only one independent electric dipole component is induced or is in the plane of the induced dipoles if two independent electric dipole components are induced, the pseudo-spectrum will produce an infinite peak at the location of the scatterer. However, it is nontrivial to choose the correct value of \( \hat{\alpha} \) since for an arbitrary test point \( \mathbf{r} \) in the domain of interest, there may be zero, one, two, and three independent electric dipole components induced. If there is no electric dipole induced at a position, it is a waste of time to find the “correct” \( \hat{\alpha} \) since there is no \( \hat{\alpha} \) at all that produces an infinite peak at the test position. Moreover, since the discrete small scatterers occupy only a small fraction of the domain of interest, it is desirable for algorithm to make a quick decision on whether there is an electric dipole induced at the test position.

In this paper, we propose an algorithm to determine not only the number of independent dipole components induced at a test position but also the direction of induced dipole component and the plane of induced dipoles when one and two independent dipole components are induced, respectively. If the test function \( \bar{f}(\mathbf{r}) \) is properly chosen, the pseudo-spectrum will produce an
infinite peak at a position where electric dipole is induced. It is equivalent to say that the following condition is satisfied

\[ \bar{\sigma}_p^* G_x(r), G_y(r), G_z(r) \cdot \hat{\alpha} = 0, \quad \sigma_p = 0 \]  

(23)

if unit vector \( \hat{\alpha} \) is properly chosen. Writing Eq. (23) in the matrix form, we have

\[ \tilde{W}(r) \cdot \hat{\alpha} = 0, \]  

(24)

where matrix \( \tilde{W}(r) \) is of size \( p_0 \)-by-3, where \( p_0 \) is the number of vanishing singular values and it is assumed \( p_0 \) is no less than 3 in this paper (In fact, this can be realized by utilizing sufficiently large number of antennas). For a test position \( r \), if the rank of \( \tilde{W}(r) \) is equal to 3, the solution to Eq. (24) is a null vector, which is contradict with the fact that \( \hat{\alpha} \) is a unit vector. In this case, there is no electric dipole induced at the test position. If the rank of \( \tilde{W}(r) \) is equal to 0, any unit vector \( \hat{\alpha} \) is a solution to Eq. (24), which indicates three independent electric dipole components are induced at the position. If the rank of \( \tilde{W}(r) \) is equal to 1, the solution to Eq. (24) forms a subspace spanned by two linearly independent unit vectors \( \hat{\alpha}_1 \) and \( \hat{\alpha}_2 \), which is identical to the plane of the induced electric dipoles. If the rank of \( \tilde{W}(r) \) is equal to 2, there is only one independent unit vector \( \hat{\alpha} \) satisfying Eq. (24), which is identical to the direction of the induced electric dipoles. The solution for the unit vector \( \hat{\alpha} \) can be obtained by QR-decomposition [13] of the matrix \( \tilde{W}(r) \) when the rank of \( \tilde{W}(r) \) is equal to 1 or 2.

A similar approach can be applied to determine the number of independent magnetic dipole components induced at a test position and the direction or the plane of induced dipoles for degenerated cases.

3.2. Non-iterative analytical algorithm for retrieving polarization tensors

The polarization tensors of scatterers can be retrieved after the positions of scatterers are located by the MUSIC method. The acoustic counterpart of this nonlinear inverse problem was dealt with by an iterative numerical approach, but the algorithm did not always yield a convergent result [3]. Recently two different non-iterative analytical algorithms were proposed [14, 15], which were shown to be superior to the iterative numerical approach. More recently, we applied the non-iterative analytical algorithm to the aforementioned electromagnetic inverse scattering problem [9], where only non-degenerate case is considered. In this paper, we generalize our algorithm to solve degenerate problems as well.

In general, the polarization tensor can be effectively expressed as that shown in Eq. (20), the substitution of which into Eq. (17) yields the MSR matrix \( \tilde{K} \) in terms of \( \tilde{\Lambda} \)

\[ \tilde{K} = \tilde{D} \cdot \tilde{R} \cdot \tilde{L} \cdot \tilde{S} \cdot \tilde{\Lambda}^T \cdot \tilde{L}^T \cdot (\tilde{I}_{6M} - \tilde{\Phi} \cdot \tilde{L} \cdot \tilde{S} \cdot \tilde{\Lambda}^T \cdot \tilde{L}^T)^{-1} \cdot \tilde{T} \cdot \tilde{D}. \]  

(25)

First, let

\[ \tilde{J} = \tilde{\Lambda}^T \cdot \tilde{L}^T \cdot (\tilde{I}_{6M} - \tilde{\Phi} \cdot \tilde{L} \cdot \tilde{S} \cdot \tilde{\Lambda}^T \cdot \tilde{L}^T)^{-1} \cdot \tilde{T} \cdot \tilde{D}, \]  

(26)

and treat it as an unknown in Eq. (25), the least squares solution of which is given by

\[ \tilde{J} = (\tilde{D} \cdot \tilde{R} \cdot \tilde{L} \cdot \tilde{S})^{\dagger} \cdot \tilde{K}, \]  

(27)

where \( ^\dagger \) denotes pseudoinverse of a matrix [13]. After \( \tilde{J} \) is obtained, we rewrite Eq. (26) as

\[ \tilde{J} = \tilde{\Lambda}^T \cdot \tilde{S}^T \cdot \tilde{L}^T \cdot (\tilde{T} \cdot \tilde{D} + \tilde{\Phi} \cdot \tilde{L} \cdot \tilde{S} \cdot \tilde{J}), \]  

(28)
where what is inside the parenthesis represents the total incident field onto scatterers. Since the matrix $\bar{A}'$ consists of diagonal blocks, Eq. (28) can be written as

$$
\bar{J}_i = \bar{A}'_i \cdot \left[ \bar{S}^T \cdot \bar{L}^T \cdot (\bar{T} \cdot \bar{D} + \bar{\Phi} \cdot \bar{L} \cdot \bar{S} \cdot \bar{J}) \right] \quad i = 1, 2, \ldots, I_{\text{tot}},
$$

(29)

where $\bar{A}'_i$ is the $i$th block matrix in the diagonal of $\bar{A}'$, $\bar{J}_i$ denotes the corresponding $i$th block row of $\bar{J}$, and $I_{\text{tot}}$ is the sum of the number of scatterers within which at least one electric dipole is induced and the number of scatterers within which at least one magnetic dipole is induced.

Finally, the least squares solution of $\bar{A}'$ is obtained as

$$
\bar{A}'_i = (\bar{S}^T \cdot \bar{L}^T \cdot (\bar{T} \cdot \bar{D} + \bar{\Phi} \cdot \bar{L} \cdot \bar{S} \cdot \bar{J}))^\dagger \quad i = 1, 2, \ldots, I_{\text{tot}}.
$$

(30)

Note that Eq. (30) yields the exact solution when there is no noise in the measurement of MSR matrix $\bar{K}$ and the positions of scatterers are exactly estimated by MUSIC pseudo-spectrum.

4. Numerical examples

The proposed inversion method is tested through two numerical examples. In both numerical examples, the transceiver antenna units are distributed in a 9-by-9 grid pattern with increment $2.5\lambda$ in a plane $z = 1.5\lambda$. The center of the grid is directly above the origin and the length of each antenna is equal to $\lambda/100$, i.e., $d_{x} = d_{y} = d_{z} = \lambda/100$, $i = 1, 2, \ldots, N$. We first calculate the noise-free MSR matrix $\bar{K}$ using the forward model Eq. (25), and then add white Gaussian noise $\kappa$. The noisy matrix $\bar{K} + \kappa$ is treated as the measured MSR matrix. The noise level is quantified by the signal-to-noise ratio (SNR) in dB defined as $20 \log_{10} \frac{||\bar{K}||}{||\kappa||}$, where $|| \cdot ||$ denotes the Frobenius norm of a matrix.

4.1. Inverse scattering problem for two anisotropic spheres

Two anisotropic spheres are located at $\mathbf{r}_1 = (0.05\lambda, 0.05\lambda, 0)$ and $\mathbf{r}_2 = (-0.05\lambda, -0.05\lambda, 0)$, respectively. Their permittivities and permeabilities are $\bar{\epsilon}_1 = \text{diag}[5\varepsilon_0, \varepsilon_0, \varepsilon_0]$, $\bar{\mu}_1 = 4\varepsilon_0$ and $\bar{\mu}_1 = \mu_0$, $\bar{\mu}_2 = \text{diag}[3\mu_0, 3\mu_0, \mu_0]$, respectively. The radii of both spheres are $\lambda/30$. The electric and magnetic polarization tensors for anisotropic spheres are given by [10]

$$
\bar{\xi} = -i4\pi k_0^3 \frac{1}{\eta_0} \cdot \text{diag} \left[ \frac{\epsilon^{(1)} - \epsilon_0}{\epsilon^{(1)} + 2\epsilon_0}, \frac{\epsilon^{(2)} - \epsilon_0}{\epsilon^{(2)} + 2\epsilon_0}, \frac{\epsilon^{(3)} - \epsilon_0}{\epsilon^{(3)} + 2\epsilon_0} \right],
$$

(31)

$$
\bar{\zeta} = -i4\pi k_0^3 \eta_0 \cdot \text{diag} \left[ \frac{\mu^{(1)} - \mu_0}{\mu^{(1)} + 2\mu_0}, \frac{\mu^{(2)} - \mu_0}{\mu^{(2)} + 2\mu_0}, \frac{\mu^{(3)} - \mu_0}{\mu^{(3)} + 2\mu_0} \right],
$$

(32)

respectively, where $\epsilon^{(1)}$, $\epsilon^{(2)}$, and $\epsilon^{(3)}$ ($\mu^{(1)}$, $\mu^{(2)}$, and $\mu^{(3)}$) are diagonal values of permittivity (permeability) tensor. Thus only one independent electric dipole along the first principal axis can be induced in the first sphere, and no magnetic dipole is induced in it. For the second sphere, three independent electric dipole components are induced, and two independent magnetic dipole components are induced in the plane containing the first and second principal axes. In this numerical example, the principal axes of the spheres are rotated, and their Euler angles [16] ($\alpha$, $\beta$, $\gamma$) are ($-\pi/4$, 0, 0) and (0, $\pi/6$, 0), respectively.

In the simulation, 30dB Gaussian noise is added to the MSR matrix. In order to test the presence of induced electric dipoles, for each test point, the test function is chosen as the background Green’s function associated with an electric dipole whose orientation depends on
the rank of the matrix $\tilde{W}(r)$ defined in Eq. (24). However, regularization has to be taken due to the presence of noise. In doing so, we define a modified $\tilde{W}(r)$, in which $\tilde{G}_l(r) = G_l(r)/||G_l(r)||$, $l = x, y, z$. The rank of $\tilde{W}(r)$ is defined to be the number of singular values of the modified $\tilde{W}(r)$ whose magnitudes are no less than a predefined small number $\delta$. In our numerical simulations, $\delta$ is chosen to be 0.005. If the rank of $\tilde{W}(r)$ is zero or three, the test dipole is chosen to be in the $\hat{x}$ direction (It is also possible to choose the $\hat{y}$ or $\hat{z}$ direction, but it doesn’t matter). If the rank of $\tilde{W}(r)$ is one or two, the test dipole direction should be determined by the QR-decomposition method described in Section 3.1. The MUSIC pseudo-spectrum is plotted in Fig. 1(a), where we see two peaks at $(0.0625\lambda, 0.05\lambda, 0)$ and $(-0.05\lambda, -0.05\lambda, 0)$. Although the MSR matrix is contaminated with noise, the location of the first sphere is estimated with a slight error and the second one is correctly located. At the position $(0.0625\lambda, 0.05\lambda, 0)$, Eq. (24) has one independent solution $(0.72642, -0.68185, -0.085928)$, very close to the actual direction $(\sqrt{2}/2, -\sqrt{2}/2, 0)$ of the induced dipoles in the first sphere. At the position $(-0.05\lambda, -0.05\lambda, 0)$, the rank of $\tilde{W}(r)$ is zero and three independent solution are chosen to be $\hat{x}$, $\hat{y}$, and $\hat{z}$. The pseudo-spectrum generated from the test function associated with a magnetic dipole is shown in Fig. 1(b). Only one peak at $(-0.05\lambda, -0.05\lambda, 0)$ is located, which is corresponding to the second sphere. It is reasonable that the first sphere is not detected since there is no magnetic dipole induced in it because its permeability is the same as that of the background. At the position $(-0.05\lambda, -0.05\lambda, 0)$, Eq. (24) has two independent solutions $(0.86699, 0.0084903, -0.49825)$ and $(0.0097924, -0.99995, 0)$, both of which are almost perpendicular to $(\sin(\pi/6), 0, \cos(\pi/6))$, the normal of the plane of induced magnetic dipoles. Thus, the two obtained directions are almost in the plane of induced dipoles.

![Figure 1](image1.png)

(a) The test function is generated from an electric dipole. Both spheres are detected.

(b) The test function is generated from a magnetic dipole. The sphere at $(0.05\lambda, 0.05\lambda, 0)$ is not detected, as expected, since its permeability is equal to that of the background.

Figure 1. Logarithmic normalized pseudo-spectrum for test positions in the $z = 0$ plane. The MSR matrix is contaminated with 30dB Gaussian noise.

After the positions and orientations of induced dipoles are obtained, we apply Eq. (30) to retrieve the polarization tensors of the scatterers. The accuracy of the estimation of the polarization tensors is quantified by using a normalized percent error, defined by

$$E = \frac{\sqrt{\sum_{i=1}^{I_{tot}} ||\tilde{\Lambda}_i - \tilde{\Lambda}_i||^2}}{\sqrt{\sum_{i=1}^{I_{tot}} ||\tilde{\Lambda}_i||^2}} \times 100\%,$$

(33)
where $\tilde{\Lambda}'$ is the retrieved polarization tensor. The normalized percent error is $E = 2.15\%$ for this numerical example.

4.2. Inverse scattering problem for one needle and one disk

In the second numerical example, the inverse scattering problem for one needle and one disk is considered. Both the needle and the disk are made of PEC, and their geometries are described as spheroids. The major and minor semi-axes of the needle are $a_1 = 1/30\lambda$ and $b_1 = 1/1200\lambda$, respectively, and the major and minor semi-axes of the disk are $a_2 = 1/60\lambda$ and $b_2 = 1/2400\lambda$, respectively. Note that the ratios of their major axes to minor axes are both equal to 40. The electric and magnetic polarization tensors for a needle made of PEC is given by [10]

$$\tilde{\xi} = -i\frac{k}{\eta_0} \cdot \text{diag} \left[ \frac{8\pi}{3} a_1 b_1^2, \frac{8\pi}{3} a_1 b_1^2, \frac{4\pi}{3} a_1^3 \ln(a_1/b_1) \right], \quad (34)$$

$$\tilde{\zeta} = -i k \eta_0 \cdot \text{diag} \left[ -\frac{8\pi}{3} a_1 b_1^2, -\frac{8\pi}{3} a_1 b_1^2, -\frac{4\pi}{3} a_1 b_1^2 \right], \quad (35)$$

Since $a_1/b_1 \gg 1$, the induced electric dipole along the needle direction is much more dominant than other electric dipoles and magnetic dipoles. The electric and magnetic polarization tensors for a disk made of PEC is given by [10]

$$\tilde{\xi} = -i\frac{k}{\eta_0} \cdot \text{diag} \left[ \frac{16}{3} a_2^3, \frac{16}{3} a_2^3, \frac{4\pi}{3} a_2^2 b_2 \right], \quad (36)$$

$$\tilde{\zeta} = -i k \eta_0 \cdot \text{diag} \left[ -\frac{4\pi}{3} a_2^2 b_2, -\frac{4\pi}{3} a_2^2 b_2, -\frac{8}{3} a_2^3 \right], \quad (37)$$

Since $a_2/b_2 \gg 1$, two components of induced electric dipoles, both of which are in the plane of the disk, and one component of induced magnetic dipoles that is aligned with the normal direction of the disk are much more dominant than other components of electric and magnetic dipoles.

The needle is located at the origin, with orientation $(\pi/3, \pi/4, 0)$, and the disk is located at $(0.05\lambda, 0.05\lambda, 0.05\lambda)$, with orientation $(0, \pi/6, 0)$. Thus the dominant electric dipole induced in the needle is in the direction $(\sin(\pi/4) \cos(\pi/3), \sin(\pi/4) \sin(\pi/3), \cos(\pi/4))$. The dominant magnetic dipole induced in the disk is in the normal direction of the disk, $(\sin(\pi/6), 0, \cos(\pi/6))$, and the dominant components of electric dipoles are in the plane of the disk.

Firstly, we use noise-free MSR matrix to produce MUSIC pseudo-spectra, where the test function is associated with the $\hat{x}$ oriented electric or magnetic dipole although the chosen direction is different from the directions of the aforementioned four dominant dipoles. The pseudo-spectra for test positions in the $z = 0$ and $z = 0.05\lambda$ planes are shown in Fig. 2. We see that both scatterers are detected. Similar results are produced by the test function associated with the $\hat{y}$ or $\hat{z}$ oriented electric and magnetic dipoles. The reason for this is that although the aforementioned four dipole components are dominant, the other eight dipole components also contribute to the scattered field so that the range of the MSR matrix is spanned by a total of 12 Green’s functions associated with the three electric and three magnetic dipoles at each scatterer. We plot thirty largest singular values of the MSR matrix in Fig. 3(a), where we see four dominant singular values and the next eight are much larger than the rest so that the range of the MSR matrix is spanned by the singular vectors corresponding to the twelve leading singular values.

However, the results are quite different when the MSR matrix is contaminated with noise. When $40\text{dB}$ Gaussian noise is added to the MSR matrix, its thirty largest singular values are plotted in Fig. 3(b). Comparing Fig. 3(b) with Fig. 3(a), we find that although the first four
(a) The test function is generated from an electric dipole oriented in the x axis. The position of the needle is correctly detected.

(b) The test function is generated from a magnetic dipole oriented in the x axis. The position of the needle is correctly detected.

(c) The test function is generated from an electric dipole oriented in the x axis. The position of the disk is correctly detected.

(d) The test function is generated from a magnetic dipole oriented in the x axis. The position of the disk is correctly detected.

**Figure 2.** Logarithmic normalized pseudo-spectrum for test positions in the $z = 0$ plane for (a) and (b), and in the $z = 0.05\lambda$ plane for (c) and (d). The MSR matrix is noise-free.

**Figure 3.** Thirty largest singular values of the MSR matrix
singular values are still dominant the next eight are contaminated with noise and are in the same level in magnitude as the rest singular values.

Next, for the noisy MSR matrix, we produce the MUSIC pseudo-spectrum using the background Green’s function associated with the test dipole whose direction is determined by the QR-decomposition method described in Section 3.1. Regularization method presented in the first numerical example is also used here.

(a) The test function is generated from an electric dipole. The position of the needle is correctly detected.

(b) The test function is generated from a magnetic dipole. The position of the needle is not correctly detected.

(c) The test function is generated from an electric dipole. The position of the disk is correctly detected.

(d) The test function is generated from a magnetic dipole. The position of the disk is correctly detected.

Figure 4. Logarithmic normalized pseudo-spectrum for test positions in the $z = 0$ plane for (a) and (b), and in the $z = 0.05\lambda$ plane for (c) and (d). The MSR matrix is contaminated with 40dB Gaussian noise.

The needle is correctly located by the test function associated with an electric dipole, and its position as shown in Fig. 4(a) is $(0, 0.0031\lambda, 0)$, very close to the actual position (the origin). The direction of the test electric dipole used at this position is $(0.35505, 0.60415, 0.7134)$, which is close to the actual direction. In comparison, the needle is not correctly located by the test function associated with a magnetic dipole since it is different from the four aforementioned dipoles. We observe from Fig. 4(b) that the value of peak in the pseudo-spectrum are much smaller, the peak spreads out instead of focusing at a point, and peak is not located at the actual position of the needle. The position of the disk is correctly predicted by the test function associated with both an electric dipole and a magnetic dipole. The predicted position in Fig. 4(c) and Fig. 4(d) are both $(0.05\lambda, 0.05\lambda, 0.05\lambda)$, exactly where the disk is located. Two orientations of the test electric dipole are $(0.86467, 0.032643, -0.50128)$ and $(0.037725, -0.99929, 0)$, both of which are almost perpendicular to the normal of the disk $(\sin(\pi/6), 0, \cos(\pi/6))$. The obtained direction of the test magnetic dipole is $(0.48973, 0.0049726, 0.87186)$, which is almost the same
as the normal of the disk. After the positions and orientations of induced dipoles are obtained, we apply Eq. (30) to retrieve the polarization tensors of the scatterers. The normalized percent error is $E = 1.43\%$ for this numerical example.

5. Conclusion
A MUSIC algorithm for electromagnetic imaging of small objects is proposed that is able to deal with both non-degenerate and degenerate cases and accounts for multiple scattering effect. The proposed MUSIC algorithm determines not only the positions of small objects, but also the orientations of independent dipole components induced in the scatterers. After the locations of the objects are obtained, the nonlinear inverse problem of determining the polarization tensors of the objects accounting for multiple scattering is solved by a non-iterative analytical approach based on the least squares method. Despite its nonlinearity, the inverse scattering problem of determining the locations and polarization tensors of small scatterers is tackled by the MUSIC imaging method and the retrieval method based on the least squares method, both of which do not iteratively evaluate associated forward scattering problem.

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