Abstract

The aim of this paper is to give identities which are generalizations of the formulas given by Koornwinder [J. Math. Phys. 30, (1989)] and Hamdi-Zeng [J. Math. Phys. 51, (2010)]. Our proofs are much simpler than and different from the previous investigations.

1 Introduction

Let $W$ be the Weyl algebra generated by $p$ and $q$ with the relation $[p, q] := pq - qp = 1$. In this paper, we prove the following theorems.

Theorem 1.1. We put $T := pq + qp$. We obtain

\begin{equation}
2^n \sum_{k=0}^{m} \binom{m}{k} p^k q^n p^{m-k} = 2^n \sum_{k=0}^{n} \binom{n}{k} q^k p^m q^{n-k}
\end{equation}

\begin{equation}
= \begin{cases} 
2^{n!} i^{-n} P_n^{(1+\frac{m-n}{2})} \left( \frac{i(T+m-n)}{2}, \frac{\pi}{2} \right) p^{m-n} & (m \geq n), \\
2^{n!} i^{-m} q^{-n} P_n^{(1+\frac{n-m}{2})} \left( \frac{i(T+n-m)}{2}, \frac{\pi}{2} \right) & (n \geq m).
\end{cases}
\end{equation}

In particular, we have

\begin{equation}
\sum_{k=0}^{n} \binom{n}{k} p^k q^n p^{n-k} = \sum_{k=0}^{n} \binom{n}{k} q^k p^n q^{n-k} = n! i^{-n} P_n^{(\frac{1}{2})} \left( \frac{iT + \pi}{2}, \frac{\pi}{2} \right).
\end{equation}

Here $P_n^{(\alpha)}(x; \phi)$ is the Meixner-Pollaczek polynomial given by the hypergeometric series

\begin{equation}
P_n^{(\alpha)}(x; \phi) := \frac{(2\alpha)_n}{n!} e^{i n \phi} \binom{\mathbf{2F}1 \left[ -n, \alpha + i x, 1 - e^{-2i\phi} \right]}. 
\end{equation}

Theorem 1.2. Let $T_{m,n}$ be the sum of all possible terms containing $m$ factors of $p$ and $n$ factors of $q$. We have

\begin{equation}
T_{m,n} = \begin{cases} 
\frac{m!}{2^m} \binom{m+n}{m} i^{-n} P_n^{(1+\frac{m-n}{2})} \left( \frac{i(T+m-n)}{2}, \frac{\pi}{2} \right) p^{m-n} & (m \geq n), \\
\frac{2^n}{m!} \binom{m+n}{m} i^{-m} q^{-n} P_n^{(1+\frac{n-m}{2})} \left( \frac{i(T+n-m)}{2}, \frac{\pi}{2} \right) & (n \geq m).
\end{cases}
\end{equation}
In particular, we have ([2], [3], [4])

\[
T_n := T_{n,n} = \frac{n!}{2^n} \binom{2n}{n} i^{-n} P_n^{(\frac{1}{2})} \left( \frac{i T}{2}, \pi \right).
\]

The formula (1.5) for \(T_n\) was first observed by Bender, Mead and Pinsky ([2]), and proved by Koorwinder ([6]). The idea of the proof in [6] is to consider the irreducible unitary representations of the Heisenberg group and some analysis for special functions. Moreover, a combinatorial proof was given by Hamdi and Zeng ([5]). They used the rook placement interpretation of the normal ordering of the Weyl algebra and gave also a proof of (1.2), which was first observed by [3]. Our results extend these to general \(m\) and \(n\).

The proofs given in this paper are much simpler than the investigations ([6], [3]). Actually, we only use some basic properties of the Weyl algebra and a certain transformation formula of the hypergeometric function. Our proofs clarify the reason why (1.2) and (1.5) are equal up to constant, which is not explained in [5].

2 Proof of Theorem 1.1

The operations \(L_A, R_A \in \text{End}_C(W)\) are respectively left and right multiplications, that is,

\[
L_A X := AX, \quad R_A X := XA, \quad (A, X \in W).
\]

We introduce some useful operators ([7]).

\[
\tilde{\text{ad}}(A) := L_A + R_A.
\]

We remark that \(L, R : W \to \text{End}_C(W)\) are linear, hence \(\tilde{\text{ad}}\) is also linear. In addition, since \(\tilde{\text{ad}}(A)^N 1 = 2^N A^N\), we obtain the following lemma immediately.

**Lemma 2.1.** Let \(t_1, \ldots, t_n\) be indeterminates. For any \(N \in \mathbb{Z}_{\geq 0}\), we obtain

\[
\left\{ \sum_{k=1}^{n} t_k \tilde{\text{ad}}(A_k) \right\}^N 1 = 2^N \left\{ \sum_{k=1}^{n} t_k A_k \right\}^N.
\]

In particular, we have

\[
(t_1 \tilde{\text{ad}}(p) + t_2 \tilde{\text{ad}}(q))^N 1 = 2^N (t_1 p + t_2 q)^N.
\]

**Remark 2.2.** When \(N = n\) in Lemma 2.1, comparing the coefficients of \(t_1 \cdots t_n\) on both sides of the (2.3), we obtain the following formula immediately.

\[
F(\tilde{\text{ad}}(A_{\sigma})).1 = 2^n F(A_{\sigma}).
\]

Here, \(A_n := (A_1, \ldots, A_n), \tilde{\text{ad}}(A_{\sigma}) := (\tilde{\text{ad}}(A_1), \ldots, \tilde{\text{ad}}(A_n))\) and

\[
F(A_n) := \sum_{\sigma \in \mathfrak{S}_n} A_{\sigma(1)} \cdots A_{\sigma(n)}, \quad F(\tilde{\text{ad}}(A_{\sigma})) := \sum_{\sigma \in \mathfrak{S}_n} \tilde{\text{ad}}(A_{\sigma(1)}) \cdots \tilde{\text{ad}}(A_{\sigma(n)}).
\]

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Lemma 2.3. The operators $\tilde{\text{ad}}(p)$ and $\tilde{\text{ad}}(q)$ are commutative.

Proof. Obviously $L_A$ and $R_B$ are commutative. Since $L$ is a homomorphism and $R$ is an anti-homomorphism, we have

$$[\text{ad}(p), \tilde{\text{ad}}(q)] = [L_p + R_p, L_q + R_q] = [L_p, L_q] + [R_p, R_q] = L_{pq - qp} - R_{pq - qp} = 0.$$ 

\hfill $\square$

Proposition 2.4.

(2.7) $\tilde{\text{ad}}(p)^m \tilde{\text{ad}}(q)^n \cdot 1 = 2^n \sum_{k=0}^{m} \binom{m}{k} p^k q^n m^{n-k}.$

Proof. Since $L_A$ and $R_B$ are commutative, $L$ is a homomorphism and $R$ is an anti-homomorphism, we obtain

$$\tilde{\text{ad}}(p)^m \tilde{\text{ad}}(q)^n \cdot 1 = (L_p + R_p)^m 2^n q^n = 2^n \sum_{k=0}^{m} \binom{m}{k} L_{pq}^k R_{pq}^{m-k} q^n = 2^n \sum_{k=0}^{m} \binom{m}{k} p^k q^n m^{n-k}.$$ 

On the other hand, since $\tilde{\text{ad}}(p)$ and $\tilde{\text{ad}}(q)$ are commutative, we have

$$\tilde{\text{ad}}(p)^m \tilde{\text{ad}}(q)^n = \tilde{\text{ad}}(q)^n \tilde{\text{ad}}(p)^m.$$ 

Hence, the second equality of (2.7) can be proved in the same way. \hfill $\square$

Remark 2.5. Wakayama\cite{7} has constructed the oscillator representation of the simple Lie algebra $\mathfrak{sl}_2$ by $\tilde{\text{ad}}$ and $\text{ad}$ in $\text{End}_C(W)$ and then, proves that $\tilde{\text{ad}}(p)^n \tilde{\text{ad}}(q)^n \cdot 1$ satisfies the difference equation of the Meixner-Pollaczek polynomials.

Since $T = pq + qp$ and $pq - qp = 1$, we have

(2.8) $pq = \frac{T + 1}{2}, \quad qp = \frac{T - 1}{2}.$

The proof of the following lemma is straightforward.

Lemma 2.6. (1) Let $f(T) \in C[T], l \in Z_{\geq 0}$. We have

(2.9) $p^l f(T) = f(T + 2l) p^l, \quad q^l f(T) = f(T - 2l) q^l.$

(2) For any $l \in Z_{\geq 0}$, we have

(2.10) $p^l q^l = \left(\frac{1 + T}{2}\right)_l, \quad q^l p^l = (-1)^l \left(\frac{1 - T}{2}\right)_l.$

Here, $(x)_l := x(x + 1) \cdots (x + l - 1), (x)_0 := 1$. 

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Proposition 2.7.

\[(2.11) \quad n!i^{-n}P_n^{(\alpha)}\left(\frac{ix}{2};\frac{\pi}{2}\right) = \sum_{k=0}^{n} \binom{n}{k}(-1)^k \left(\alpha - \frac{x}{2}\right)_k \left(\alpha + \frac{x}{2}\right)_{n-k}.
\]

Proof. It follows from the formula (2.3.14) in \[1\] that

\[(LHS) = (2\alpha)_{n}F_{1}\left[-n, \alpha - \frac{x}{2}; \frac{\pi}{2}\right] = (\alpha + \frac{x}{2})_{n}F_{1}\left[-n - \alpha - \frac{x}{2} + 1; -1\right] = (RHS).
\]

Remark 2.8. One may also prove this proposition using the generating function for Meixner-Pollaczek polynomials.

We now prove Theorem \[1.1\] as follows. If \(m \geq n\),

\[2^m \sum_{k=0}^{n} \binom{n}{k}q^kp^m_q^{n-k} = 2^m \sum_{k=0}^{n} \binom{n}{k}(-1)^k \left(1 - \frac{T}{2}\right)_k \left(1 + \frac{T}{2}\right)_{n-k} \left(\frac{1}{2}\right)_k \left(1 + \frac{T}{2} + m - n\right)_{n-k} p^{m-n}.
\]

The second equality follows from (2.10), the third from (2.9) and the fourth from (2.11). By Proposition 2.4, the case of \(n \geq m\) can be proved in the same way.

3 Proof of Theorem \[1.2\]

Comparing the coefficients of \(t^m_t^n\) on both sides in (2.4) for \(N = m + n\), one obtain the key Proposition.

Proposition 3.1. For any \(m, n \in \mathbb{N}\), we have

\[(3.1) \quad T_{m,n} = \frac{1}{2^{m+n}} \frac{(m+n)!}{m!n!} \hat{\alpha}(p)^m \hat{\alpha}(q)^n.
\]

Theorem \[1.2\] follows immediately from (3.1), (2.7) and (1.1).

Remark 3.2. (1) If \(m \geq n\), then we have the following result immediately by Theorem \[1.2\] and (2.10).

\[(3.2) \quad T_{m,n}q^{m-n} = \frac{n!}{2^n} \binom{m+n}{n}i^{-n} \left(\frac{1 + T}{2}\right)_m \tilde{P}_n^{(1+m-n)} \left(\frac{i(T + m - n)}{2}; \frac{\pi}{2}\right).
\]
The case of \( n \geq m \) is similar.

(2) If \( m \geq n \), then an explicit expression of the Poincare-Birkhoff-Witt theorem for \( T_{m,n} \) follows from (1.4), (1.3) and (2.10).

\[
T_{m,n} = \frac{1}{2^n (m-n)!} \binom{m+n}{n} \sum_{k \geq 0} \binom{n}{k} 2^k (1 + m - n)_k q^k p^{k+m-n}.
\]

The case of \( n \geq m \) is similar.

Recently, a generalization of Theorem 1.2 using the multivariate Meixner-Pollaczek polynomials in the framework of the Gelfand pair has been established in [4]. Another proof of [4] in our current approach would be desirable.

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