A CONTINUOUS MODEL FOR TURBULENT ENERGY CASCADE

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ABSTRACT. In this paper we introduce a new PDE model in frequency space for the inertial energy cascade that reproduces the classical scaling laws of Kolmogorov’s theory of turbulence. Our point of view is based upon studying the energy flux through a continuous range of scales rather than the discrete set of dyadic scales. The resulting model is a variant of Burgers equation on the half line with a boundary condition which represents a constant energy input at integral scales. The viscous dissipation is modeled via a damping term. We show existence of a unique stationary solution, both in the viscous and inviscid cases, which replicates the classical dissipation anomaly in the limit of vanishing viscosity.

A survey of recent developments in the deterministic approach to the laws of turbulence, and in particular, to Onsager’s conjecture is given.

1. MOTIVATION FOR THE MODEL

1.1. Onsager and Kolmogorov. The Euler equations for the motion of an incompressible, inviscid fluid are

\begin{align}
\frac{\partial u}{\partial t} + (u \cdot \nabla) u &= -\nabla p \\
\nabla \cdot u &= 0,
\end{align}

where \(u(x, t)\) is a divergence free velocity vector and \(p(x, t)\) is the internal pressure. We consider the system in 3 spatial dimensions and we assume that the domain is either periodic or the entire space \(\mathbb{R}^3\). To obtain the energy equation we multiply (1.1) by \(u\) and integrate, using (1.2) to give

\[ \frac{1}{2} \frac{d}{dt} \int u^2 dx = -\int (u \cdot \nabla) u \cdot u dx. \]

Key words and phrases. Onsager’s conjecture, Kolmogorov turbulence, intermittency, Burgers equation.

The work of A. Cheskidov is partially supported by NSF grant DMS-0807827, S. Friedlander is partially supported by the NSF grant DMS-0849397, R. Shvydkoy acknowledges the support of NSF grant DMS-0907812.
We define the total energy flux Π by

\[ \Pi = \int (u \cdot \nabla)u \cdot u dx. \]  

For smooth solutions we can integrate by parts and use (1.2) to conclude that \( \Pi = 0 \) and hence energy conservation holds, i.e.

\[ \int |u(x, t)|^2 dx = \int |u(x, 0)|^2 dx \quad \text{for } t \geq 0. \]

However, in the context of turbulent flows in the limit of vanishing viscosity, it is appropriate to consider the Euler equations in the sense of distributions and impose only minimal assumptions on the regularity of the velocity field \( u \). In the absence of sufficient smoothness we cannot integrate by parts in (1.4) or even make sense of (1.4) and ensure that \( \Pi = 0 \). Conservation of energy might then be violated. Hence it is of interest to ask what are the minimal regularity assumptions on the velocity that ensures that (1.5) holds.

Observing that the integrand in (1.4) is cubic in \( u \) and contains one spatial derivative suggests that if \( u \) has Holder continuity \( h > 1/3 \), integration by parts is justified and \( \Pi = 0 \). In fact this was the conjecture made many years ago by Onsager in his seminal paper on statistical fluid dynamics [17]. More precisely, he conjectured that (a) every weak solution to the Euler equation with smoothness \( h > 1/3 \) conserves energy and (b) there exists a weak solution with \( h \leq 1/3 \) which does not conserve energy. Such putative energy dissipation due to the irregularity of the flow is called anomalous or turbulent dissipation. A detailed historical account of Onsager’s theory is given by Eyink and Sreenivasan [10].

All physical fluids are viscous, if only very weakly so. Turbulent fluids are believed to be described by the Navier-Stokes equations

\[ \frac{\partial u}{\partial t} + (u \cdot \nabla)u = -\nabla P + \nu \Delta u + f \]  
\[ \nabla \cdot u = 0, \]

where \( \nu \), which could be very small, is the coefficient of viscosity and \( f \) is an external force which supplies energy into the system. The “classical” Kolmogorov theory of turbulence predicts that energy dissipative solutions to the Euler equation may arise in the limit of vanishing viscosity for “generic” viscous flows that are governed by (1.6) - (1.7). In homogeneous, isotropic turbulence the mean kinetic energy per unit mass is defined by \( \langle E \rangle = \frac{1}{2} \langle |u|^2 \rangle \) while the energy density spectrum is defined by \( E(\kappa) = \frac{1}{2} \frac{d}{d\kappa} \langle |u_\kappa|^2 \rangle \). Here \( u_\kappa \) denotes the filtered velocity field containing all frequencies below a wave number \( \kappa \). Hence
\[ E = \int_0^\infty E(\kappa) d\kappa. \]

The mean energy dissipation rate per unit mass is defined by

\[ \varepsilon^\nu = \langle \nu | \nabla u^\nu |^2 \rangle, \]

where \( u^\nu \) is a solution to (1.6) - (1.7). Kolmogorov [15] predicted that the energy cascade mechanism in fully developed 3-dimensional turbulence produces a striking phenomenon, namely the persistence of non-vanishing energy dissipation in the limit of vanishing viscosity, i.e.

\[ \lim_{\nu \to 0} \varepsilon^\nu \to \varepsilon > 0 \]

where \( \varepsilon \) is the anomalous dissipation rate for the inviscid Euler system. The positivity of the limit in (1.9) is referred to as the dissipation anomaly. Let us now assume that \( f = f_{<\kappa_f} \) (i.e. \( f \) has finite Fourier support) and that solutions to (1.6) tend to a statistically stationary state with uniformly bounded mean energy. We multiply (1.6) by \( u_{<\kappa}^\nu \) and obtain

\[ \Pi^\nu(\kappa) = -\nu \langle |\nabla u_{<\kappa}^\nu |^2 \rangle + \langle f \cdot u_{<\kappa}^\nu \rangle. \]

If \( \kappa > \kappa_f \) we have \( \langle f \cdot u_{<\kappa}^\nu \rangle = \langle f \cdot u^\nu \rangle = \varepsilon^\nu > 0 \). On the other hand, by Bernstein’s inequality, \( \nu \langle |\nabla u_{<\kappa}^\nu |^2 \rangle \leq \nu \kappa^2 \langle |u^\nu |^2 \rangle \). Since the energy is uniformly bounded by assumption, we obtain from (1.10) that

\[ \lim_{\nu \to 0} \Pi^\nu(\kappa) = \lim_{\nu \to 0} \varepsilon^\nu = \varepsilon. \]

Thus in the limit of vanishing viscosity the average solution of the forced Euler equation inherits the anomalous dissipation rate \( \varepsilon \).

As Frisch [12] describes, a self-similarity hypothesis on the velocity increments in small (spatial) scales implies that the energy spectrum as a function of wave number \( \kappa \) has the power law

\[ E(\kappa) \sim \varepsilon^{2/3} \kappa^{-5/3} \]

in the “inertial” range \( \kappa \in [\kappa_f, \kappa_d] \). Here \( \kappa_d \) is the Kolmogorov dissipation wave number given by \( \kappa_d = (\varepsilon/\nu^3)^{1/4} \) and \( \kappa_f = \max \{|\kappa| : \kappa \in \text{supp} \hat{f} \} \). This power law is known as the K41 turbulence model. Although the 5/3 power law is consistent with much physical data, there are also experiments which indicate turbulent regimes with alternative power laws. In fact, Kolmogorov’s 1941 theory requires that the local velocity fluctuations are uniformly distributed over space. However, in reality dynamical stretching of the vortex filaments in 3-dimensional flows leaves some regions of the fluid domain with moderate turbulent activity and other regions with intense activity. This so called spatial intermittency should reasonably be accounted for in the description of
the scaling laws. The expressions for $E(\kappa)$ and $\kappa_d$ that incorporate the dimension $D$ of the effective dissipation region are

$$E(\kappa) \sim \varepsilon^{2/3} \kappa^{-(8-D)/3}$$

and

$$\kappa_d \sim (\varepsilon/\nu^3)^{1/(1+D)}$$

where $D \in [0, 3]$. Thus the classical $K41$ model corresponds to $D = 3$, i.e. uniform distribution over 3 dimensional space, while $D = 0$ corresponds to a fully intermittent model where energy cascades through scales and dissipates only on points.

1.2. Onsager’s Conjecture and Besov Spaces. In the past few years there have been a number of articles that address part (a) of Onsager’s conjecture. These include articles by Constantin et al [4], Eyink [5], Duchon and Robert [7]. It was shown that appropriate function spaces to examine the Euler equations in the context of Onsager’s conjecture are Besov spaces. In such spaces the notion of energy balance when the velocity is “a little smoother” than Holder $h > 1/3$ can be made precise. These are the natural spaces to work with in terms of a description of the energy flux phrased by a Littlewood-Paley decomposition which provides detailed information concerning the cascade of energy. Recently Cheskidov et al [2] obtained the largest Besov space where conservation of energy is ensured for the Euler equation. We note that to date there are no examples of Euler flows that possess some smoothness and confirm the second part of Onsager’s conjecture, although there are examples of “very weak” Euler solutions that violate the energy balance condition [5, 18, 19].

We recall the definition of a weak solution of the Euler equation. A vector field $u \in C_w([0, T] : L^2(\mathbb{R}^3))$ is a weak solution of the Euler equations with initial data $u_0 \in L^2(\mathbb{R}^3)$ if for every compactly supported test function $\psi \in C_0^\infty ([0, T] \times \mathbb{R}^3)$ with $\nabla_x \cdot \psi = 0$ and for every $0 \leq t \leq T$, we have

$$u(t)\psi(t) - u(0)\psi(0) - \int_0^t u \cdot \partial_s \psi ds = \int_0^t \int_{\mathbb{R}^3} u \cdot \nabla \psi \cdot u dx ds$$

and $\nabla_x \cdot u = 0$ in the sense of distributions.

We define the Littlewood-Paley energy flux $\Pi_j$ through a sphere in frequency space of radius $2^j$ as follows. For any divergence free vector field $u \in L^2(\mathbb{R}^3)$ we define

$$S_j u = u \ast \mathcal{F}^{-1}(\psi(\cdot 2^{-j}))$$
where $\psi(\xi)$ is a smooth nonnegative function supported in the ball of radius one centered at the origin and such that $\psi(\xi) = 1$ for $\xi \leq 1/2$ and $\mathcal{F}$ is the Fourier transform. We then define $\Pi_j$ as

\begin{equation}
\Pi_j = -\int_{\mathbb{R}^3} u \cdot \nabla S_j^2 u \cdot u \, dx.
\end{equation}

Using the test function $S_j^2 u$ in the weak formulation of the Euler equations we obtain

\begin{equation}
\frac{1}{2} \frac{d}{dt} \|S_j u\|_2^2 = -\Pi_j
\end{equation}

Cheskidov et al. [2] prove that the Littlewood-Paley energy flux of a divergence free vector field $u \in L^2$ satisfies the following estimate:

\begin{equation}
|\Pi_j| \lesssim \sum_{i=-1}^{\infty} 2^{-\frac{2}{3}j-i} 2^i \|u_i\|_3^3
\end{equation}

where $u_j$ is the $j$-th Littlewood-Paley piece of $u$ defined by

$$
  u_j = S_{j+1} u - S_j u.
$$

It follows from (1.19) that

\begin{equation}
\limsup_{j \to \infty} |\Pi_j| \lesssim C \limsup_{j \to \infty} (2^j \|u_j\|_3^3).
\end{equation}

Furthermore, an important feature of the bound (1.19) is that it is quasi-local in the sense of rapid decay when $|j-i|$ is large.

We define the Besov space $B^{1/3}_{3,c0}$ to be the space of all tempered distributions $u \in \mathbb{R}^3$ for which

\begin{equation}
\limsup_{j \to \infty} 2^{j/3} \|u_j\|_3 = 0.
\end{equation}

Hence, from (1.18) and (1.20) we obtain the following result.

Every weak solution $u$ to the Euler equation on a time interval $[0, T]$ which satisfies

\begin{equation}
\lim_{j \to \infty} \int_0^T 2^j \|u_j(t)\|_3^3 = 0
\end{equation}

conserves energy on the entire interval $[0, T]$. In particular, energy is conserved for every solution in the class $L^3[0, T]; B^{1/3}_{3,c0}) \cap C^w([0, T]; L^2)$.

In order to see more transparently the connection between (1.22) and the smoothness $1/3$ predicted by Onsager we rewrite (1.22) as follows:

\begin{equation}
\lim_{|y| \to 0} \frac{1}{|y|} \int_0^T \int_{\mathbb{R}^3} |u(x) - u(x-y)|^3 \, dx \, dt = 0.
\end{equation}
Hence the solution to the Euler equation needs to be a little better than 1/3 Holder continuous in the space-time average to ensure that energy is conserved. We call the Besov space $B^{1/3}_{3,\infty}$ Onsager critical. This is the space which contains distributions $u \in \mathbb{R}^3$ where $\lim_{j \to \infty} 2^{j/3} \|u_j\|_3$ is finite, but not necessarily zero. This is a critical space in which energy conservation for the Euler equation might be violated.

Applying the bound on the energy flux given by (1.19) to the Navier-Stokes equations gives a sufficient condition for the energy equality to hold, namely.

Let $u^\nu \in C_w([0,T]; L^2(\mathbb{R}^3)) \cap L^2([0,T]; H^1(\mathbb{R}^3))$ be a weak solution to (1.6)-(1.7) with

$$\lim_{j \to \infty} \int_0^T 2^j \|u^\nu_j(t)\|_3^3 \, dt = 0.$$  

Then $u^\nu$ satisfies the energy equality

$$\|u^\nu(t)\|_2^2 + 2\nu \int_0^t \|\nabla u^\nu(s)\|_2^2 \, ds = \|u^\nu(0)\|_2^2 + 2 \int_0^t f \cdot u^\nu(s) \, ds.$$  

In particular, (1.25) holds if $u \in L^3([0,T]; H^{5/6})$.

1.3. Littlewood-Paley Framework for Intermittency. Let $u^\nu$ be a Leray-Hopf weak solution to the Navier-Stokes equations (1.6) - (1.7). We denote by $\langle \cdot \rangle$ the long time average. We define the Littlewood-Paley energy spectrum of $u^\nu$ by

$$E_{LP}(\kappa) = \frac{\langle \|u^\nu_j\|_2^2 \rangle}{\kappa}$$

for frequencies $\kappa \in [2^j, 2^{j+1}]$ and we define the mean energy dissipation rate by

$$\varepsilon^\nu = \nu \langle \|\nabla u^\nu\|_2^2 \rangle.$$  

If a family of individual realizations $\{u^\nu\}_{\nu<\nu_0}$ verifies Kolmogorov’s hypothesis that $\varepsilon^\nu \to \varepsilon > 0$, then the locality of the flux which is exhibited in the bound (1.19) suggests the following

$$2^j \langle \|u^\nu_0\|_3^3 \rangle \sim \varepsilon$$

for all $j$ sufficiently large. Here $u^0 = \lim_{\nu \to 0} u^\nu$. In other words, the limiting solution $u^0$ to the Euler equation is “on average” in the Onsager critical space $B^{1/3}_{3,\infty}$.

Eyink [9] showed that $B^{1/3}_{3,\infty}$ is consistent with the multi-fractal intermittency models of Frisch and Parisi [11]. Within the Littlewood-Paley framework we can model the intermittency correction, (see (1.13) and (1.14)) by assuming the relationship between $\varepsilon$ and $\|u^\nu_0\|_3$ given in
and fixing the saturation level in Bernstein’s inequalities. More precisely, in 3 dimensions we have

\[ \|u_j\|_3 \lesssim 2^{j/2} \|u_j\|_2. \]

Assuming that the region of active turbulence is bounded, (say, on a torus) we also have

\[ \|u_j\|_2 \lesssim \|u_j\|_3. \]

Hence for \(2^j \in [\kappa_f, \kappa_d]\) we write

\[ \|u_j\|_3 \sim 2^c j \|u_j\|_2 \]

for some \(c\) in the interval \([0, 1/2]\). Then from (1.26), (1.28) and (1.31) we recover the energy spectrum law

\[ E_{LP}(\kappa) \sim \frac{\varepsilon^{2/3}}{\kappa^{5/3+2c}} \]

with \(2^j\) being identified with \(\kappa\).

The analogy between (1.13) and (1.31) requires that \(D = 3 - 6c\). So, the fully saturated Bernstein’s inequality (i.e., \(c = 1/2\)) corresponds to a uniform distribution of modes \(u_j\) in each dyadic shell and hence strong localization in space (i.e. \(D = 0\)). On the other extreme, \(c = 0\) corresponds to a uniform distribution of \(u\) in physical space space (i.e. \(D = 3\)) and localization in frequency space which corresponds to the classical \(K41\) model.

2. A Continuous Model for the Energy Flux

Although there is abundant empirical evidence for Kolmogorov’s hypothesis that \(\lim_{\nu \to 0} \varepsilon' \to \varepsilon > 0\), this has not been rigorously proved for the Navier-Stokes to Euler limit. It is therefore of interest to examine simpler models that retain some of the essential features of the fluid equations and yet are tractable enough to allow a proof of Kolmogorov’s hypothesis. We now propose a PDE model for the turbulent energy spectrum in frequency space. We choose the scaling to include the intermittency correction that we described in sections 1.2 and 1.3. To motivate the model we start with a fully local version of the flux given by the bound in the inequality (1.20), namely

\[ \Pi_j \sim 2^j \|u_j\|_3^3. \]

We further assume that \(c\) in (1.31) is independent of \(j\). We thus obtain

\[ \Pi_j \sim 2^{(3c+1)j} \|u_j\|_2^3. \]
We now make a further step from the discrete expression for flux (2.2) to a continuous one by looking at the energy density function \( a(\kappa, t) \) defined by

\[
\|u(t)\|^2_2 = \int_0^\infty a^2(\kappa, t)d\kappa.
\]

Assuming that \( a \) ”does not vary much” in each dyadic shell, or disregarding energy density variations in each dyadic shell, we obtain

\[
\|u_j\|^2_2 \sim \int_{2^j}^{2^{j+1}} a^2(\kappa, t)d\kappa \sim \kappa a^2(\kappa, t).
\]

for all \( \kappa \in [2^j, 2^{j+1}] \). Thus, (2.2) becomes

\[
(2.3) \quad \Pi(\kappa) = \kappa^{3c+\frac{5}{2}} a^3(\kappa, t).
\]

Going back to (1.10) we assume that \( \kappa_f = 1 \), and the energy in the sub-inertial range is negligible. So, with (2.3) at hand we can write the energy balance relation as follows

\[
\frac{1}{2} \left( \int_1^\kappa a^2(\ell, t)d\ell \right)_t = -\kappa^{3c+\frac{5}{2}} a^3(\kappa, t) - \nu \int_1^\kappa \ell^2 a^2(\ell, t)d\ell.
\]

Differentiating in \( \kappa \) and cancelling \( a \) on both sides we obtain the following PDE

\[
(2.4) \quad a_t = -\left( \frac{5}{2} + 3c \right) \kappa^{\frac{3}{2}+3c} a^2 - 3\kappa^{\frac{3}{2}+3c} a a_\kappa - \nu \kappa^2 a
\]

We supplement this equation with the boundary condition

\[
(2.5) \quad a(1, t) = \varepsilon^{1/3}.
\]

Here \( \varepsilon \) represents the energy input rate coming from an external force. We thus disregard any particular detail of energy production and simply model it with our boundary condition. The input rate \( \varepsilon \) will subsequently be shown equal to the energy dissipation rate, hence the notation. Equation (2.4) can be easily simplified by rescaling the energy density \( a \) to \( b = \kappa^{\frac{3}{2}+c} a \). Thus, (2.4) becomes the following equation

\[
(2.6) \quad b_t = -3\alpha\kappa^\alpha b b_\kappa - \nu \kappa^2 b, \quad \kappa \geq 0,
\]

where \( \alpha = \frac{5}{3} + 2c \). The appropriate range of \( \alpha \) is \( [\frac{5}{3}, \frac{8}{3}] \), which exactly corresponds to the classical range of the energy density power laws with the spatial intermittency correction.

Remark 2.1. Equation (2.6) is a variant of the much studied Burgers equation. The one dimensional Burgers equation can be viewed as the most basic nonlinear PDE that has the bilinear structure of the nonlinearity of the Euler and Navier-Stokes equations. It can be invoked as a model for one dimensional compressible fluids. However there is
no clear physical basis for using Burgers equation in physical space as a model for turbulence. On the other hand, as we have argued, the locality of the energy flux manifested using Littlewood-Paley theory, motivates (2.6) as a PDE model for the turbulent cascade in frequency space.

Remark 2.2. In the past few decades a number of “toy models” for turbulence have been studied to test Kolmogorov’s theory. In particular, the derivation of the classical Desnyanskiy-Novikov discrete model, [6], follows a similar path. The flux there is modeled by taking

$$\Pi_j = 2^d a_j a_{j+1},$$

where $a_j^2$ represents the total energy in the $j$-th dyadic shell, while $d$ is an intermittency parameter with the appropriate range of values. The model is thus an infinite system of ODEs given by

$$\frac{d}{dt} a_j + \nu 2^{2j} a_j - 2^{d(j-1)} a_{j-1}^2 + 2^d a_j a_{j+1} = 0, \quad j = 0, 1, 2, \ldots,$$

where $a_{-1} = 0$. This model, as well as its inviscid versions, has been extensively studied by Katz and Pavlovic [13], Cheskidov and Friedlander [3], Kiselev and Zlatos [14], and others. To our knowledge the PDE model we present is the first continuous model.

In Section 3 we examine the inviscid ($\nu = 0$) form of (2.4) with boundary condition (2.5). We prove that there is a unique fixed point which is a global attractor. Moreover every solution reaches it in finite time. The inviscid equation exhibits anomalous dissipation and the average energy spectrum has the power law $\varepsilon^{2/3} \kappa^{-\alpha}$.

In Section 4 we turn to the viscous model $\nu > 0$ (2.4), which is in essence a damped Burgers equation. Again the PDE has a unique fixed point and this converges to the inviscid fixed point as $\nu \to 0$. The viscous fixed point reproduces Kolmogorov’s energy density spectrum in the inertial range and it becomes zero identically after the dissipation wave number $\kappa_d$. We further consider the Leray regularization of equation (2.4). We show that all bounded solutions of the regularized equation converge pointwise and in the metric of $L^2$-space to a fixed point which in turn converges to the fixed point of equation (2.4) in the limit as the regularization parameter goes to zero. The average dissipation rate for the viscous system converges to the anomalous dissipation rate for the inviscid system giving an example of the dissipation anomaly predicted by Kolmogorov.
3. INVISCID CASE

In this section we study the inviscid version of the model (2.4)

\[
\begin{cases}
    a_t = -3\kappa^\alpha \frac{\partial}{\partial \kappa} (\kappa^{2\alpha} a), & \kappa > 1, \\
    a(1, t) = \varepsilon^{1/3}, \\
    a(\kappa, 0) = a_0(\kappa) \geq 0,
\end{cases}
\]

(3.1)

where \( \alpha \in [5/3, 8/3] \). Note that the energy equality

\[
\frac{d}{dt} \frac{1}{2} \int_1^\infty a(\kappa, t)^2 d\kappa = \varepsilon
\]

is satisfied on some interval \( t \in (0, T) \) provided the solution satisfies the following smoothness condition (cf. (1.22)):

\[
\lim_{\kappa \to \infty} \int_{0}^{T} \kappa^{3\alpha/2} a^3 d\tau = 0,
\]

(3.3)

which is an analog to (1.22). So, \( \varepsilon > 0 \) represents the energy input rate in this model. The unique fixed point of (3.1) is given by

\[
A_0(\kappa) = \varepsilon^{1/3} \kappa^{-\alpha/2}
\]

(3.4)

We note that the fixed point does not satisfy (3.3). Moreover, it does not satisfy the energy equality since

\[
\frac{d}{dt} \frac{1}{2} \int_1^\infty A_0(\kappa)^2 d\kappa = 0 \neq \varepsilon.
\]

(3.5)

The anomalous energy dissipation rate is the difference between the energy input rate and the time derivative of the total energy. Thus using (3.2) and (3.5) we observe that the anomalous dissipation rate for the fixed point is exactly the energy input rate \( \varepsilon \). We will show that this also holds for every other solution asymptotically in time. In order to do this we will prove that \( A_0 \) is a global attractor.

We use the following change of variables:

\[
\xi = \kappa^{-1/\gamma}, \quad w(\xi, t) = \varepsilon^{-1/3} \xi^{-\alpha/2} a(\xi^{-\gamma}, \frac{1}{3} \varepsilon^{-1/3} \gamma t),
\]

where \( \gamma = \frac{1}{\alpha - 1} \). Then (3.1) reduces to Burgers equation

\[
w_t = ww_\xi, \quad 0 < \xi < 1
\]

(3.6)

with \( w(1, t) = 1, w(\xi, 0) = w_0(\xi) \geq 0 \). We extend it to Burgers equation on the whole real line

\[
\begin{cases}
    w_t = ww_\xi, & \xi \in \mathbb{R}, \\
    w(\xi, 0) = \tilde{w}_0(\xi),
\end{cases}
\]

(3.7)
where

\[ \tilde{w}_0(\xi) = \begin{cases} 
0, & \xi < 0, \\
w_0(\xi), & 0 \leq \xi \leq 1, \\
1, & \xi > 1.
\end{cases} \]

The weak solution to (3.7) is expressed using the Lax-Oleinik formula. Let

\[ h(y) = \int_0^y \tilde{w}_0(\xi) \, d\xi. \]

For all \( t \) and for all but at most countably many \( \xi \in \mathbb{R} \), there exists a unique \( y_*(\xi, t) \), such that

\[ \min_y \{ f(y) \} = f(y_*). \]

where

\[ f(y) = \frac{(\xi - y)^2}{2t} - h(y). \]

Then

\[ (3.8) \]

\[ w(\xi, t) = \frac{y_*(\xi, t) - \xi}{t} \]

is the weak solution to (3.7). Given \( \xi \in [0, 1] \), let \( t > 2(1 - \xi) \). We will show that \( y_*(\xi, t) = \xi + t \) and consequently \( w(\xi, t) = 1 \). Indeed, let \( y = \xi + s \). First, consider the interval \( s \geq 1 - \xi \). Then

\[ f(y) = \frac{s^2}{2t} - \int_0^1 w_0(\xi) \, d\xi - \xi - s + 1 > f(\xi + t) \]

provided \( s \neq t \). Since \( t > 2(1 - \xi) \), it follows that

\[ f(\xi + t) < -\int_0^1 w_0(\xi) \, d\xi, \]

and hence on the interval \( s < 1 - \xi \) we also have

\[ f(y) \geq \frac{s^2}{2t} - \int_0^1 w_0(\xi) \, d\xi \geq -\int_0^1 w_0(\xi) \, d\xi > f(\xi + t). \]

Therefore, (3.8) implies that \( w(\xi, t) = 1 \) for \( \xi \in (0, 1) \), \( t \geq 2 \) and hence, returning to the original variables,

\[ (3.9) \]

\[ a(\kappa, t) = A^0(\kappa), \quad t \geq \frac{2}{\delta} \varepsilon^{-1/3} \gamma. \]

Hence the average energy spectrum for solutions to the inviscid model is about \( \varepsilon^{2/3} \kappa^{-\alpha} \).
4. Viscous case

In this section we study the viscous model (2.4) (4.1)

\[
\begin{aligned}
  a_t &= -3a\kappa^\alpha \frac{\partial}{\partial \kappa} (\kappa^{\frac{\alpha}{2}} a) - \nu \kappa^2 a, \quad \kappa > 1, \\
  a(1, t) &= \varepsilon^{1/3}, \\
  a(\kappa, 0) &= a_0(\kappa), a_0 \in L^2, a_0 \geq 0,
\end{aligned}
\]

where \( \alpha \in [5/3, 8/3] \). There exists a unique fixed point to (4.1) given
by

\[
(4.2) \quad A^\nu(\kappa) = \begin{cases} \\
  \kappa^{-\alpha/2} \left[ \varepsilon^{1/3} + \frac{\nu}{3(3 - \alpha)} (1 - \kappa^{3-\alpha}) \right], & 1 \leq \kappa \leq \kappa_d, \\
  0, & \kappa > \kappa_d,
\end{cases}
\]

where \( \kappa_d \) is Kolmogorov’s dissipation wavenumber described in Section
1.1. For the model it is explicit and given by

\[
(4.3) \quad \kappa_d = \left[ 1 + \frac{3(3 - \alpha)\varepsilon^{1/3}}{\nu} \right]^{\frac{1}{1-\alpha}}.
\]

To see the parallel with the classical expressions for \( \kappa_d \) we note that
for \( \nu \) small one has

\[
\kappa_d \sim \left( \frac{\varepsilon}{\nu^3} \right)^{\frac{1}{4}}, \quad \text{for} \quad \alpha = \frac{5}{3},
\]

and

\[
\kappa_d \sim \frac{\varepsilon}{\nu^2}, \quad \text{for} \quad \alpha = \frac{8}{3}.
\]

In the limit of vanishing viscosity we immediately obtain from (4.2)
and (3.4) that

\[
(4.4) \quad A^\nu(\kappa) \to A^0(\kappa),
\]

uniformly on any finite interval \([1, \kappa_0] \) as \( \nu \to 0 \). With a little more
effort we can show that the convergence also takes place in \( L^2([1, \infty)) \).
Indeed,

\[
\int_1^{\kappa_d} |A^\nu(\kappa) - A^0(\kappa)|^2 d\kappa \leq \varepsilon^{2/3} \int_{\kappa_d}^{\kappa_0} \kappa^{-\alpha} d\kappa + \frac{\nu^2}{9(3 - \alpha)^2} \int_1^{\kappa_d} \kappa^{-\alpha} (1 - \kappa^{3-\alpha})^2 d\kappa.
\]

Since \( \kappa_d \to \infty \) we see that the first integral vanishes as \( \nu \to 0 \). The
second integral behaves like \( \nu^2 \) for \( \alpha < 7/3 \), like \( \nu^2 \log(\nu) \) for \( \alpha = 7/3 \),
and like \( \nu^{(\alpha-1)/(3-\alpha)} \) for \( 7/3 < \alpha \leq 8/3 \). So, within our range of \( \alpha \) the
second integral vanishes too as \( \nu \to 0 \).
In order to study the time dependent solutions to the viscous system (4.1) we utilize the same change of variables as in the previous section
\[ \xi = \kappa^{-1/\gamma}, \quad w(\xi, t) = \varepsilon^{-1/3} \xi^{-2/3} a(\xi^{-\gamma}, \frac{1}{3} \varepsilon^{-1/3} t), \]
where \( \gamma = \frac{1}{\alpha - 1} \). Then (4.1) reduces to the damped Burgers equation
\[ w_t = w w_{\xi} - \mu \xi^{-2\gamma} w, \quad 0 < \xi < 1 \]
with \( w(1, t) = 1, \ w(\xi, 0) = w_0(\xi) \geq 0, \ \int_0^1 \xi^2 w_0^2(\xi) d\xi < \infty \). Here \( \mu = \frac{1}{3} \nu \varepsilon^{-1/3} \gamma \). The equation has a unique fixed point
\[ W(\xi) = \begin{cases} 
1 + \frac{\mu}{2\gamma - 1} (1 - \xi^{1-2\gamma}), & \xi_d < \xi \leq 1, \\
0, & 0 \leq \xi \leq \xi_d,
\end{cases} \]
where
\[ \xi_d = \left[ 1 + \frac{2\gamma - 1}{\mu} \right]^{-\frac{1}{1-2\gamma}}. \]
Note that \( \xi_d \to 0 \) as \( \mu \to 0 \). Note that \( \kappa_d = \xi_d^{-\gamma} \).

Since there are discontinuous solutions to the damped Burgers equation (4.5), we now consider a Leray-type regularization of the equation. Such regularizations have been used to approximate weak solutions to the Burgers equations [11, 16]. Consider the following regularized equation:
\[ w_t = v_\delta w_{\xi} - \mu \xi^{-2\gamma} w, \quad v_\delta = w * \phi_\delta, \quad 0 < \xi < 1, \ t > 0, \]
with the boundary conditions \( w(1, t) = 1, \ w(0, t) = 0 \). Here \( \phi_\delta(\xi) = \frac{1}{\delta} \phi(\xi/\delta) \), where \( \phi(\xi) \) is smooth, nonnegative, and such that \( \int \phi(\xi) d\xi = 1, \ \text{supp} \phi = (-1, 0) \). First, note that there exists a unique fixed point \( w = W_\delta(\xi) \) to (4.5), which is smooth, nonnegative, monotonically increasing, and with \( W_\delta(0) = 0 \). Now consider characteristics \( \eta_\delta(t) \), which are solutions to
\[ \frac{d}{dt} \eta_\delta(t) = -v_\delta(\eta_\delta(t), t). \]
It is easy to see that characteristics do not intersect on \((0, 1)\). Indeed, take \( \varepsilon \in (0, 1) \) and consider two solutions \( \eta'_\delta(t), \eta''_\delta(t) \) to (4.9). As long as they belong to \((\varepsilon, 1)\) we have
\[ \frac{d}{dt} |\eta'_\delta(t) - \eta''_\delta(t)| \lesssim \delta^{-3/2} |\eta'_\delta(t) - \eta''_\delta(t)| \left( \int_\varepsilon^1 w_\delta^2(\xi, t) d\xi \right)^{1/2} \]
\[ \lesssim \delta^{-3/2} |\eta'_\delta(t) - \eta''_\delta(t)| \frac{1}{\varepsilon} \left( \int_\varepsilon^1 \xi^2 w_\delta^2(\xi, t) d\xi \right)^{1/2}. \]
Since \( \int_0^1 \xi^2 w_\delta^2(\xi, t) \, d\xi \) is non-increasing, the characteristics do not intersect.

Along characteristics we have
\[
\frac{d}{dt} w_\delta(\eta_\delta(t), t) = -\mu \eta_\delta^{-2\gamma} w_\delta(\eta_\delta(t), t).
\]
Moreover, along every characteristic curve that starts from the boundary \( \eta = 1 \) we have that \( w_\delta \) is equal to the fixed point \( W_\delta \), i.e.,
\[
w_\delta(\eta_\delta(t), t) = W_\delta(\eta_\delta(t)),
\]
provided \( \eta_\delta(t_0) = 1 \) for some \( t_0 \geq 0 \). Now consider the characteristic curve \( \eta_\delta^0(t) \) with \( \eta_\delta^0(0) = 1 \). Note that \( \eta_\delta^0(t) \) is decreasing, positive, \( \eta_\delta^0(t) \to 0 \) as \( t \to \infty \), and
\[
|a^\nu(t)| := \int_1^\infty a^\nu(\kappa, t)^2 \, d\kappa, \quad \|a^\nu(t)\| := \int_1^\infty \kappa^2 a^\nu(\kappa, t)^2 \, d\kappa.
\]

Due to the energy inequality we have
\[
\frac{1}{2T} |a^\nu(T)|^2 - \frac{1}{2T} |a^\nu(0)|^2 \leq -\nu \frac{1}{T} \int_0^T \|a^\nu(t)\|^2 \, dt + \varepsilon.
\]
Hence,
\[
\limsup_{T \to \infty} \frac{1}{T} \int_0^T \nu \|a^\nu(t)\|^2 \, dt \leq \varepsilon.
\]
On the other hand, note that the fixed point \( A^\nu \) satisfies the energy equality
\[
\nu \|A^\nu\|^2 = \varepsilon.
\]
Now for any $\eta > 0$, there exists $N$, such that
\[
\nu \int_1^N \kappa^2 A^\nu(\kappa)^2 \, d\kappa \geq \nu \|A^\nu\|^2 - \eta.
\]
Since $a^\nu(t) \to A^\nu$ in $L^2$, we have
\[
\liminf_{T \to \infty} \frac{1}{T} \int_0^T \int_1^N \nu \kappa^2 a^\nu(\kappa, t)^2 \, d\kappa \, dt \geq \nu \int_1^N \kappa^2 A^\nu(\kappa)^2 \, d\kappa \geq \nu \|A^\nu\|^2 - \eta.
\]
Therefore,
\[
\liminf_{T \to \infty} \frac{1}{T} \int_0^T \nu \|a^\nu(t)\|^2 \, dt \geq \nu \|A^\nu\|^2 = \epsilon.
\]
Then we obtain
\[
\epsilon := \lim_{\nu \to 0} \lim_{T \to \infty} \frac{1}{T} \int_0^T \nu \|a^\nu(t)\|^2 \, dt > 0.
\]
i.e., in the limit of vanishing viscosity the energy dissipation rate is positive and equal to the energy input rate or anomalous dissipation rate for the inviscid model.

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