Abstract

Entanglement measures based on a logarithmic functional form naturally emerge in any attempt to quantify the degree of entanglement in the state of a multipartite quantum system. These measures can be regarded as generalizations of the classical Shannon-Wiener information of a probability distribution into the quantum regime. In the present work we introduce a previously unknown approach to the Shannon-Wiener information which provides an intuitive interpretation for its functional form as well as putting all entanglement measures with a similar structure into a new context: By formalizing the process of information gaining in a set-theoretical language we arrive at a mathematical structure which we call ”tree structures” over a given set. On each tree structure, a tree function can be defined, reflecting the degree of splitting and branching in the given tree. We show in detail that the minimization of the tree function, on, possibly constrained, sets of tree structures renders the functional form of the Shannon-Wiener information. This finding demonstrates that entropy-like information measures may themselves be understood as the result of a minimization process on a more general underlying mathematical structure, thus providing an entirely new interpretational framework to entropy-like measures of information and entanglement. We suggest three natural axioms for defining tree structures, which turn out to be related to the axioms describing neighbourhood topologies on a topological space. The same minimization that renders the functional form of the Shannon-Wiener information from the tree function then assigns a preferred topology to the underlying set, hinting at a deep relation between entropy-like measures and neighbourhood topologies.

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1 Introduction

In the past two decades, two well-established disciplines within Mathematics and Physics, namely Information Theory and Quantum Physics, have merged into a prospective new field on their own. This process was instigated by advances in Atomic and Molecular Physics which opened up the possibility of controlling the behaviour of matter by means of laser light down to very small length scales – from the nano regime, involving mesoscopic systems, even further down to the control of single atoms and molecules by means of appropriate laser radiation. What is more, even the possibility of reducing the irradiating source down to nanoscales has sprung up, and controlled single-photon sources have been experimentally proved to be possible. Together with these advances, the possibility of utilizing nanostructures, or even single molecules, as the fundamental building blocks for future quantum computers, has arisen. The study of information processing in such an environment, its limitations as well as its capabilities to exceed classical computational processes, then has been coined according to the two main pillars contributing to the new discipline – Quantum Information Theory.

However, it turns out that the scope of Quantum Information Theory is much wider than the range of its possible applications within Quantum Computation might suggest. Indeed, it has been conclusively demonstrated that Quantum Information Theory provides the main conceptual as well as computational foundation to tackle some of the basic unsolved problems within Quantum Mechanics which have haunted physicists for decades – the problem of Quantum Nonlocality [1, 2, 3], its presence signalled by the violation of appropriate Bell-Inequalities [4, 5, 6], and the nature of Entanglement between distinct quantum systems. These questions have a strong overlap with the mystery of how a ’classical world’ can emerge from a universe governed by Quantum Mechanics in which all superpositions of states are allowed, but still only a tiny subset of these, namely those which we perceive as ’classical’, can be ordinarily observed: A possible answer has been provided by the notion of Decoherence [7, 8, 9, 10, 11] — the decay of quantum correlations between systems which are subjected to the inevitable quantum noise from an environment which is ultimately understood to be the universe as a whole. Here again, the notion of entanglement emerges as a central quantity.

One of the basic problems in Quantum Information Theory (QIT) is then to find measures for the quantification of the degree of nonclassical correlations, or entanglement, between two physical systems. So far, entanglement has been shown to be quantifiable in two regimes, called ”finite” and ”asymptotic” [12, 13, 14]: The first one attempts to quantify the amount of entanglement within a single copy of a quantum state; the second one deals with tensor products of a large number of identical copies of a given state. Many of the entanglement measures proposed so far have a close relationship to the classical Shannon-Wiener information [15, 16, 17, 18] of a probability distribution, which is formally identical (up to a sign) to the thermodynamical entropy of the same distribution. For example, the so-called ”uniqueness theorem” [12, 19, 20] states that, under appropriate conditions, all entanglement measures coincide on pure bipartite states and are equal to the von Neumann entropy [21] – the quantum analogue of the classical Shannon-Wiener information – of the corresponding reduced density operators. It is this recurring fact which strongly hints at the relevance of entropy-like measures of information, such as the Shannon-Wiener information, both in the classical and in the quantum context.

In this work we present a new and rather unexpected approach to the concept of Shannon-Wiener information: We show that this quantity can be understood as the result of minimization of a so-called tree function on a mathematical structure, called tree structure, which we define and investigate in this work. We show in detail that the Shannon-Wiener information, as known up to date, may be obtained as the minimal value of the tree function, when the condition of minimization is imposed. This puts the notion of Shannon-Wiener information, and in turn, all other measures of information and entanglement which are based upon it, into a whole new context, which is presented in this text. Although tree-like objects are known in Information Theory [22], Complexity Theory and Discrete Mathematics, the framework presented here is a new and original cast of this theme, and differs from previously proposed concepts to such a degree that a complete and self-contained account of the new structure is justified; this account is given in the present text.

We now give a brief qualitative overview over the new structure and its main properties:
What are tree structures: A tree structure $B(X)$ over a given set $X$ is a subset of the power set $\mathcal{P}X$ of $X$ which is obtained by a continuous splitting of its nodes $b \in B(X)$ into smaller and ever smaller subsets; this splitting is described in terms of partitions of sets. Tree structures can be defined over sets of arbitrary cardinality, countable or non-countable. For infinite sets, the tree structures over $X$ are fractal-like objects [23]. There are three natural axioms governing tree structures, which are independent of whether the set $X$ is finite or infinite. These axioms give rise to preferred topologies on the underlying set $X$.

See sections 2, 4

How do tree structures arise: Tree structures arise in modelling processes of information gaining; they are designed to capture the operational aspect of this process. In such a model we assign a natural number to the outcome of an interaction between a unit that seeks to find a distinct but unknown element $x_0$ of a set $X$, and a unit that possesses this information, but renders only information about ”neighbourhoods” of the distinct element, as these neighbourhoods zoom more and more into $x_0$. These ”neighbourhoods” can be given a topological meaning.

See sections 3, 27

What are the typical structural elements: The main structural elements are the ”nodes” in the tree; these are subsets of the underlying set $X$ to which two characteristic numbers, the total number of elements, and the ”degree of splitting” in the next-level partition, are assigned. Strings of such nodes can be picturized by ”paths” in the tree structure. To every path in a finite tree structure, a natural number, called the ”amount” of the path, can be assigned, which represents the maximal number of Yes-No-questions that are necessary to single out the element $x$ for the given path in the given tree. Of central importance is the sum over the amounts of all complete paths in the tree; this sum will be called the ”amount function”. In this way, every tree structure over a given set $X$ can be assigned a unique value of the amount function. The amount functions are related to the ”tree function” which is a sum over cardinal numbers and degrees of splitting at every node in the tree. On certain subsets of trees, amount functions and tree functions coincide.

See sections 9, 10.3, 11, 13

The natural question concerning tree structures: Assigning a value of the tree function to every tree over $X$, we can ask on which trees the tree function takes its minimum. This question can be generalized, as constraints on the admissible trees can be imposed. The admissible trees then may be chosen so as to preserve a prescribed initial partition of $X$, which reflects a choice of ”weights” ($w_i$) for the path amounts in such a tree. This is analogous to choosing a probability distribution ($p_i$) for the paths in the admissible trees.

See sections 14, 18, 19, 21, 23, 26

The first main result concerning tree structures: If there are no constraints, then the minimal value of the tree function is close to $n \cdot \lg(n)$, where $\lg(n)$ is an integer approximation to the logarithm of $n$ with respect to the basis 2. Thus, the mean value of the amounts of $n$ paths in a complete tree over $X$ comes close to $\lg(n)$, which is the information gained in finding a distinct element among $n$ ”equally weighted” elements; or the entropy of $n$ distinct states, depending on the context. One of the central results of this work is that the functional form $\lg(n)$ of the entropy so defined is itself the result of a process of minimization, i.e. there is a more general functional form underlying, namely the tree function. These results can be generalized to the constrained case; here the terminal elements $b_i \in B(X)$ are endowed with weights $w_i$, so that the value of the tree function contains expressions like $n \cdot \lg(n) - \sum w_i \cdot \lg(w_i)$. Here we recognize the Shannon-Wiener information, or entropy,

$$- \sum \frac{w_i}{n} \cdot \log_2 \left( \frac{w_i}{n} \right) \simeq \frac{1}{n} \cdot \left[ n \cdot \lg(n) - \sum w_i \cdot \lg(w_i) \right]$$

of the probability distribution $\frac{w_i}{n}$, depending on the context. Again, we have the striking result that the functional form of the entropy is itself the result of a process of optimization of a more general expression,
namely the tree function, and the entropy, as usually known, is only the minimal value of this more general function.

See sections 20, 21, 23, 24, 26.

The second main result concerning tree structures: Every tree structure over a set $X$ defines a neighbourhood topology \textsuperscript{24} on $X$. As we vary the tree structures, so vary the topologies on $X$. A tree function on a given set of, possibly constrained, tree structures will single out preferred neighbourhood topologies, namely those for which the tree function becomes minimal. This defines something like an action principle for neighbourhood topologies on the set $X$, where the value of the action = tree function on the minimal trees is an entropy-like quantity.

See section 27.

— The plan of this report is as follows: In section 2 we recall the definition of partitions of sets. In section 3 we outline how a tree structure encodes the operational aspect of the problem of information gaining, which yields the concept of entropy/information. Three natural axioms describing tree structures are presented in section 4. In section 5 we recall some elementary facts on ordered sets; in section 6 we show how the set of all partitions of a given set $X$ is a partially ordered set. The concept of subtrees is discussed in section 7 while ideas concerning the sum, union, extension and completion of trees are introduced in section 8. After this preparation we define paths in a tree structure in section 9. In section 10 we show how a tree over $X$ selects a distinct subset of partitions of the underlying set $X$; here we introduce the important concepts of minimal and maximal partitions of the underlying set $X$ in the tree $B$, and the number $m(b)$ characterising the degree of splitting of a node in the tree. Then we come to the central notions in our theory: In section 11 we introduce amount functions on sets of tree structures. The technical section 12 contains a splitting theorem for amount functions. In section 13 we define the tree function on the set of all tree structures over $X$. The problem of minimizing trees is first taken up in section 14. We then introduce the concept of divisions in section 15 and explain its relation to partitions in section 16. In sections 17 and 18 we introduce optimal divisions of sets, and the concept of optimal trees based on optimal divisions. Section 19 defines minimal classes based on prescribed divisions. Section 20 introduces the integer approximation of the logarithm with respect to the base 2, together with some of its properties. The value of the optimal amount of an optimal tree is derived in 21. Section 22 introduces the notion of preoptimized trees, which is a tool of central import to the proof of the minimality of optimal trees. The latter result is approached in a series of propositions given in section 23. Section 24 reflects the same statements from the point of view of the mean path amount in a tree over $X$. In section 25 we introduce an important notion of structural similarity, encaptured in an appropriate definition of isomorphism between trees. In section 26 we outline how to find constrained minimal trees on which the functional form of the tree function contains expressions like $- \sum w_i \log (w_i)$. In the last section 27 we show how tree structures define neighbourhood topologies on $X$, and how the tree function selects distinct topologies according to a minimal principle.

This report is based on the preprint \textsuperscript{26}.

Notation convention: For the difference of two sets, which is commonly denoted as

$$A \setminus B = \{ x \in A | x \notin B \}$$

we shall use the notation $A - B \equiv A \setminus B$ instead.

2 Partitions

Let $X$ be a non-empty set. A partition $z$ of $X$ is a system of mutually disjoint non-empty subsets $\mu \subset X$ whose union is $X$, i.e.

$$(P1) \bigcup_{\mu \in z} \mu = X ,$$
(P2) \( \mu \neq \mu' \Rightarrow \mu \cap \mu' = \emptyset \).

The power set \( \mathcal{P}X \) of \( X \) is the set of all subsets of \( X \), including the empty set. That is to say, \( \mathcal{P}X \) contains the elements
\[
\emptyset , \{ x \} \text{ for } x \in X , \{ x, y \} \text{ for } x \neq y \in X , \ldots , X .
\]
We see that every partition is a subset \( z \subset \mathcal{P}X \) of the power set of \( X \).

The partition \( z_0 = \{ X \} \) will be called the trivial partition. The partition \( z \) is said to be complete if every element \( \mu \) of \( z \) contains precisely one element of \( X \), i.e. \( \# \mu = 1 \) for all \( \mu \in z \), or
\[
z = \left\{ \{ x \} \in \mathcal{P}X \mid x \in X \right\} .
\]

The set of all partitions \( z \) of \( X \) will be denoted by \( \mathcal{Z}(X) \). The set of all nontrivial partitions will be denoted by \( \mathcal{Z}^*(X) \), i.e. \( \mathcal{Z}^*(X) = \{ z \in \mathcal{Z}(X) \mid z \neq \{ X \} \} \).

3 Movitation for tree structures

We want to show how tree structures arise in the course of modelling processes of information gaining. We now describe such a model: Let \( X \) be a non-empty finite set, \( 0 < n \equiv \#X < \infty \). Let \( x_0 \in X \) be arbitrary. We want to find a numerical measure for the information that is gained when \( x_0 \) has been identified as a distinct object amongst \( n \) objects. Consider the interaction of two (information processing) units, the first one (storage unit) of which has stored the knowledge about \( x_0 \), and the second unit tries to identify \( x_0 \) amongst all \( n \) elements of \( X \). The only knowledge permitted to the second unit (search unit) is that all \( n \) choices are equally likely. The search unit starts by suggesting a partition of \( X \) to the storage unit; if the number of elements in the partition is \( m_1 \), then the search unit has to pose at most \((m_1 - 1)\) yes-no questions to the storage unit in order to identify the element of the partition that contains \( x_0 \). Next the search unit suggests a partition of the subset that contains \( x_0 \), and so on. This gives the following scheme: On level 1, we have a partition \( z(X) \in \mathcal{Z}(X) \) with \( \#z(X) = m_1 \) elements, i.e.
\[
z(X) = \{ X_1, \ldots, X_{m_1} \} .
\]

On level 2 of the emerging tree we partition all the subsets \( X_i \) in \( \{ X \} \): We decompose \( X_1 \) into \( m_2(1) \) non-empty subsets, \( X_2 \) into \( m_2(2) \) non-empty subsets, \ldots, \( X_{m_1} \) into \( m_2(m_1) \) subsets; here the subscripts 1, 2 in \( m_1, m_2 \) refer to the levels 1 and 2, respectively. Hence for \( i_1 = 1, \ldots, m_1 \) we have partitions \( z(X_{i_1}) \in \mathcal{Z}(X_{i_1}) \) with cardinality \( \#z(X_{i_1}) \equiv m_2(i_1) \),
\[
z(X_{i_1}) = \{ X_{i_1,1}, \ldots, X_{i_1,m_2(i_1)} \} .
\]

Now we continue along these lines: \( X_{1,1} \) is decomposed into \( m_3(1,1) \) subsets; \( X_{1,m_2(1)} \) is decomposed into \( m_3(1,m_2(1)) \) subsets; \ldots, \( X_{m_1,m_2(m_1)} \) is decomposed into \( m_3(m_1,m_2(m_1)) \) subsets, i.e. for \( i_1 = 1, \ldots, m_1, i_2 = 1, \ldots, m_2(i_1) \) we introduce a partition \( z(X_{i_1,i_2}) \in \mathcal{Z}(X_{i_1,i_2}) \) with cardinal number \( \#z(X_{i_1,i_2}) = m_3(i_1,i_2) \) such that
\[
z(X_{i_1,i_2}) = \{ X_{i_1,i_2,1}, \ldots, X_{i_1,i_2,m_3(i_1,i_2)} \} ,
\]

etc. Any of the subsets \( X_{i_1,i_2} \ldots \) emerging in this process is an element of the power set \( \mathcal{P}X \) of \( X \). The totality of all these subsets is a certain subset of the power set of \( X \) which we shall term a tree structure or simply a tree \( \mathcal{B}(X) \) over \( X \). Hence,
\[
\mathcal{B}(X) = \left\{ X, \right. \\
X_1, \ldots, X_{m_1}, \\
X_{1,1}, \ldots, X_{1,m_2(1)}, \ldots, X_{m_1,1}, X_{m_1,m_2(m_1)} \ldots \left\} .
\]
We see that the elements of a given tree structure \( \mathcal{B}(X) \) can obviously be labelled by series of the form

\[
\emptyset, \\
(1), \ldots, (m_1), \\
(1, 1), (1, m_2(1)), \ldots, (m_1, 1), (m_1, m_2(m_1)), \ldots.
\]

(8)

If the set \( X \) is infinite, there can be series \( 8 \) which extend forever. On the other hand, if \( X \) is finite, then each of these series is finite and can be denoted in the form \( (i_1, \ldots, i_n) \). In this case the cardinalities

\[
n(i_1, \ldots, i_n) \equiv \#X_{i_1, \ldots, i_n}
\]

(9)

are natural numbers.

Let \( b \) be a terminal element in the tree, with series \( (i_1, \ldots, i_n) \); this series may be called complete if \( n(i_1, \ldots, i_n) = 1 \), otherwise it will be called incomplete. Hence, in a tree over a finite set \( X \) with \( n = \#X \) we can have at most \( n \) distinct, complete series.

A tree \( \mathcal{B}(X) \) may be called complete if all series associated with terminal nodes are complete.

For a given set \( X \) let \( \mathcal{M}(X) \) denote the set of all tree structures over \( X \). The set of all complete tree structures over \( X \) will be denoted by \( \mathcal{C}(X) \). Clearly, \( \mathcal{C}(X) \subseteq \mathcal{M}(X) \) for \( \#X \geq 2 \).

— We have seen how tree structures emerge naturally in processes modelling information gaining. The basic properties of tree structures, as they present themselves from the above analysis, will be compiled in the next section.

### 4 Axiomatic definition of tree structures

We now suggest three natural axioms defining a tree structure as a set of subsets of \( X \), as motivated in eq. 7:

A **tree structure** \( \mathcal{B}(X) \) over \( X \) is a system of non-empty subsets \( b \subset X \) of \( X \) (hence a subset of the power set \( \mathcal{P}X \) of \( X \)) such that the following axioms hold:

(A1) \( X \subset \mathcal{B}(X) \).

(A2) If \( b, b' \in \mathcal{B}(X) \), then \( b \subset b' \) or \( b' \subsetneq b \) or \( b \cap b' = \emptyset \) [This is ”exclusive or”].

(A3) For all \( b, b' \in \mathcal{B}(X) \) there exists \( \overline{b} \in \mathcal{B}(X) \) such that \( b, b' \subset \overline{b} \).

Elements \( b \in \mathcal{B}(X) \) will be called the nodes in the tree \( \mathcal{B}(X) \). An element \( b \in \mathcal{B}(X) \) will be called primitive, if \( b \) contains only one element, i.e., \( b = \{y\} \) for some \( y \in X \). The tree structure \( \mathcal{B}(X) \) will be called complete if it contains all primitive elements, i.e., \( \{y\} \in \mathcal{B}(X) \) for all \( y \in X \). This definition is clearly consistent with the notion of completeness as given in the previous section 3.

An element \( b \in \mathcal{B}(X) \) will be called refinable if \( b \) is not primitive; hence there exists \( b' \subsetneq b \). If none of the subsets \( b' \) which refine \( b \) lie in the given tree \( \mathcal{B}(X) \), we call the element \( b \) terminal in \( \mathcal{B}(X) \); in this case we shall also use the notation \( b = b_{\text{fin}} \). Thus, each node \( b \in \mathcal{B}(X) \) which is not terminal is refinable in the given tree. On the other hand, all primitive elements are trivially non-refinable, and hence must be terminal in \( \mathcal{B}(X) \).

Most of the definitions we will introduce in this work will be stated as general as possible, although our actual conclusions regarding the Shannon-Wiener information will be worked out on finite sets only.

### 5 Ordered sets

We recall some general definitions regarding ordered sets:

A non-empty set \( X \) is called ordered, if a relation ” \( \prec \) ” is defined on \( X \), satisfying:
(O1) For any two elements $a, b$ of $X$ either $a < b$ or $b < a$ or $a = b$ is true.

(O2) If $a < b$ and $b < c$ then $a < c$.

If the non-empty set $X$ contains a non-empty ordered subset $T$, then $X$ is said to be **partially ordered**. Hence every ordered set is partially ordered. To distinguish this case from a partial ordering we sometimes say that an ordered set $X$ is **totally ordered**.

If $X$ contains an element $x_0$ for which $x_0 < x$ for all $x \in X$ is true, we call $x_0$ the principal element in $X$ [or in the pair $(X, \prec)$, to be precise].

6 \ \ \mathcal{Z}(X) as a partially ordered set

On the set $\mathcal{Z}(X)$ of all partitions of $X$, a natural partial ordering " $\prec$ " can be introduced as follows: Let $z, z' \in \mathcal{Z}(X)$. The relation $z \prec z'$ is defined to be true if and only if every $b' \in z'$ is contained in some $b \in z$ according to $b' \subset b$, and there exists $b \in z$, $b' \in z'$ for which this inclusion is proper, $b' \not\subset b$. In this case we say that the partition $z'$ is a refinement of the partition $z$. If both $z$ and $z'$ are finite this implies in particular that $\#z < \#z'$.

Given two partitions $z, z'$, clearly none of the relations $z \prec z'$ or $z' \prec z$ or $z' = z$ need to be true; this is why the set $\mathcal{Z}(X)$ is only partially ordered.

If the partition $z$ of $X$ is kept fixed, we can think of the set of all partitions $\tilde{z}$ of $X$ for which $z$ is a refinement, $\tilde{z} \prec z$; they comprise the set
\[
\mathcal{Z}(X, z) \equiv \{ \tilde{z} \in \mathcal{Z}(X) \mid \tilde{z} \prec z \}.
\] (10)

7 \ \ \textbf{Subtrees}

Let $\mathcal{B}(X)$ be a given tree over $X$. Let $b \in \mathcal{B}(X)$. The set
\[
\mathcal{B}(b, X) \equiv \{ b' \in \mathcal{B}(X) \mid b' \subset b \}
\] (11)

will be called the subtree of $b$ with respect to $\mathcal{B}(X)$. By definition, $\mathcal{B}(b, X)$ is a tree structure over $b$, and hence an element of $\mathcal{M}(b)$.

If $b$ is non-refinable, and hence a terminal element in $\mathcal{B}(X)$, then $\mathcal{B}(b, X) = \{ b \}$ is trivial.

8 \ \ \textbf{Sum, union, extension, reduction, and completion of trees}

Let $\mathcal{B}(X)$ be a tree structure over $X$. Consider the elements $X_1, \ldots, X_{m_1}$ of level 1 in the partition $z(X)$ of $X$, as given in eq. (7) in section 3. For every $X_i$ we can think of the subtree $\mathcal{B}(X_i)$ over $X_i$ with respect to $\mathcal{B}(X)$. The relation of the subtrees $\mathcal{B}(X, X_i)$, $i = 1, \ldots, m_1$, to the "parent" tree will be described by saying that $\mathcal{B}(X)$ is the sum of the trees $\mathcal{B}(X, X_i)$. Now we see how to extend this definition to tree structures over sets which are not a priori subsets of a given set: Let $m \in \mathbb{N}$, let $X_1, \ldots, X_m \neq \emptyset$ be non-empty pairwise disjoint sets, i.e., $X_i \cap X_j = \emptyset$ for $i \neq j$. Let $\mathcal{B}(X_1), \ldots, \mathcal{B}(X_m)$ be tree structures over $X_1, \ldots, X_m$. Then the set $\sum_{i=1}^{m} \mathcal{B}(X_i)$, defined by
\[
\sum_{i=1}^{m} \mathcal{B}(X_i) \equiv \left[ \bigcup_{j=1}^{m} \mathcal{B}(X_j) \right] \cup \left\{ \bigcup_{j=1}^{m} X_j \right\}
\] (12)

will be called the sum of $\mathcal{B}(X_1), \ldots, \mathcal{B}(X_m)$. By construction this is a tree structure over the set $\bigcup_{j=1}^{m} X_j$ with subtrees $\mathcal{B}(X_1), \ldots, \mathcal{B}(X_m)$. 
Another construction is the \textit{union of trees}. This is defined as follows: Let $\mathcal{B}(X)$ be a tree structure, and let $b \in \mathcal{B}(X)$ be a terminal but non-primitive element. Then $\# b > 1$. Although $b$ is not further partitioned in the tree $\mathcal{B}(X)$, we can nevertheless consider tree structures over $b$ without reference to $\mathcal{B}(X)$. Let $\mathcal{B}(b)$ be such a tree over $b$. Then we can attach $\mathcal{B}(b)$ to $\mathcal{B}(X)$ by identifying $b \in \mathcal{B}(b)$ with $b \in \mathcal{B}(X)$; the resulting set is the union $\mathcal{B}(b) \cup \mathcal{B}(X)$, and is again a tree structure which will be called the \textit{union of the trees} $\mathcal{B}(b)$ and $\mathcal{B}(X)$.

A somewhat related, but more general, concept is the \textit{extension of trees}: Let $\mathcal{B}$ and $\mathcal{B}'$ be two tree structures over the same set $X$. We will say that $\mathcal{B}'$ is an \textit{extension} of $\mathcal{B}$ if $\mathcal{B}' \supseteq \mathcal{B}$. A special case of extension is the \textit{completion $\mathcal{B}_c$ of a tree $\mathcal{B}$}: This is defined to be a tree structure $\mathcal{B}_c$ over the same set $X$ as $\mathcal{B}$ that extends $\mathcal{B}$ and is complete, i.e., $\mathcal{B}_c$ contains all primitive nodes $\{x\}$ as $x$ runs through $X$. If $X$ is finite, every tree structure over $X$ admits such a completion; but clearly, there are many completions $\mathcal{B}_c$ for a given tree $\mathcal{B}$, which differ in the paths $q(\{x\})$ of the terminal nodes, see section 9 below.

Yet another construction is the \textit{reduction $\mathcal{B}'(X)$ of a tree $\mathcal{B}(X)$ by a subtree $\mathcal{B}(b,X)$}; this is just the operation inverse to the union of trees, as defined above: If $b$ is a given node in a given tree $\mathcal{B}(X)$, we can remove the subtree $\mathcal{B}(b,X)$ from $\mathcal{B}(X)$ by setting

\[ \mathcal{B}'(X) = \left[ \mathcal{B}(X) - \mathcal{B}(b) \right] \cup \{b\}. \]

The set $\mathcal{B}'(X)$ is a tree by construction and is obtained from $\mathcal{B}(X)$ by simply cutting off the branch containing all further partitions of $b$, but reattaching $b$ as a terminal element. This contains an important

**Splitting principle** Every tree $\mathcal{B}(X)$ can be expressed as the union of any of its subtrees $\mathcal{B}(b,X)$ with a cutoff tree $\mathcal{B}'(X)$,

\[ \mathcal{B}(X) = \mathcal{B}(b,X) \cup \mathcal{B}'(X), \]

where both trees on the right-hand side are subsets of the original tree and $\mathcal{B}'(X)$ is defined in eq. (13).

## 9 Paths in a tree structure

We now show that tree structures have a natural partial ordering. To this end we observe that there exist distinct subsets in a tree structure which can be totally ordered: Let $\mathcal{B}(X)$ be a given tree over $X$. Let $b \in \mathcal{B}(X)$. Then we call the set

\[ q(b) \equiv \{ \tilde{b} \in \mathcal{B}(X) \mid \tilde{b} \supset b \} \]

the \textit{path of $b$ in $\mathcal{B}(X)$}. $q(b)$ is certainly non-empty, since it always contains $X$ and $b$ itself. From the definition of $q(b)$ we see that a total ordering "$\prec"$ on $q(b)$ for all pairs of elements $(b',b'')$ of $q(b)$ can be defined by setting $b' \prec b''$ if and only if $b' \supset b''$. This makes the path $q(b)$ a totally ordered set, for all nodes $b \in \mathcal{B}(X)$. As a consequence, the tree $\mathcal{B}(X)$ is a partially ordered set. If $q(b)$ is finite, its cardinality is a natural number which we denote by $o(b)$,

\[ o(b) = \# q(b), \]

and which we shall call the \textit{length of the path $q(b)$ in the tree $\mathcal{B}(X)$}.

Given the natural ordering of the path $q(b)$ as defined above, we obviously have $b' \prec b$ for all $b' \in q(b)$, by construction of $q(b)$. Provided that $b \not\in X$, it follows that there always exists an element $b^- \in q(b)$ such that $b^- \prec b$ but $b' \prec b^-$ for all $b' \neq b^-$, $b$; this distinct element will be called the \textit{predecessor of $b$ in the tree $\mathcal{B}(X)$}. Thus all nodes $b \in \mathcal{B}(X)$ except for $X$ have a unique predecessor in $\mathcal{B}(X)$; $X$ itself has no predecessor in $\mathcal{B}(X)$. The predecessor is equal to the "smallest" node in $\mathcal{B}(X)$ which contains $b$ as a proper subset; obviously, its own path $q(b^-)$ coincides with the path $q(b)$ just up to $b$ itself,

\[ q(b) = q(b^-) \cup \{b\}, \quad b \not\in q(b^-). \]

**Notation conventions:** We introduce some notation conventions that will prove convenient in the sequel.

If $b \in \mathcal{B}(X)$ and $q(b)$ is the path of $b$ in $\mathcal{B}(X)$, then we denote

\[ \hat{q}(b) \equiv q(b) - \{b\}. \]
If $B(b', X)$ is a subtree of $B(X)$ and if $b \in B(b', X)$, then the path of $b$ in $B(b', X)$ will be denoted by

$$q_{B(b', X)}(b) \equiv \{ a \in B(b', X) \mid a \supseteq b \} .$$  

\section{Partitions compatible with a given tree}

Consider a given node $b$ in the tree $B(X)$. The subtree $B(b, X)$ defines a distinct set of partitions of $b$ in the following way: Each distinct partition $z$ is a collection $z = \{b_1', b_2', \ldots\}$ of mutually disjoint subsets $b_i' \subset b$ whose union is $b$ such that each $b_i'$ is also an element of $B(b, X)$. Thus, $z \subseteq B(b, X)$. Such a preferred partition of $b$ will be called \textit{compatible with the tree} $B(X)$. The set of all partitions of $b$ compatible with the tree $B(X)$ will be denoted by $\zeta(b)$,

$$\zeta(b) \equiv \{ z \in \mathcal{Z}(b) \mid z \subseteq B(b, X) \} .$$  

If it is necessary to point out that the compatibility is referred to the given tree $B(X)$ we shall also use the extended notation $\zeta(b, B)$. Similarly, we define $\zeta^*(b)$ to be the set of nontrivial partitions in $\zeta(b)$, i.e. $\zeta^*(b) = \zeta(b) - \{ \{b\}\}$.

\subsection{The maximal compatible partition $z_{\text{max}}(b)$}

The set $\zeta(X)$ contains several distinct partitions: Firstly, the trivial partition $\{X\}$; and secondly, the partition of $X$ which is constituted by all terminal nodes in $B(X)$. Similarly, if $b \in B(X)$ is arbitrary, then $\zeta(b)$ contains the trivial partition $\{b\}$ as well as the partition of $b$ which is constituted by all terminal elements in the subtree $B(b, X)$. The latter will be denoted by $z_{\text{max}}(b)$, and will be called the \textit{maximal partition of $b$ in the tree} $B(X)$. Its elements are those terminal nodes $b_{\text{fin}}$ of $B(X)$ which are also subsets of $b$. Equivalently we can say that the elements of the maximal partition $z_{\text{max}}(b)$ of $b$ are those terminal elements $b_{\text{fin}}$ of $B(X)$ whose paths $q(b_{\text{fin}})$ contain $b$,

$$z_{\text{max}}(b) = \{ b_{\text{fin}} \in B(X) \mid b \in q(b_{\text{fin}}) \} .$$  

Since the maximal partition is defined in terms of terminal elements it exhibits maximality in the following sense: $z_{\text{max}}(b)$ refines any other partition $z$ of $b$ which is compatible with $B(X)$,

$$z \preceq z_{\text{max}}(b) \quad \text{for all } z \in \zeta(b, B) .$$  

As a consequence, the number of elements in $z_{\text{max}}(b)$ is greater than or equal to the number of elements in any other $z \in \zeta(b, B)$,

$$\#z \leq \#z_{\text{max}}(b) \quad \text{for all } z \in \zeta(b, B) .$$  

If $b$ is terminal, then the maximal partition is the trivial partition, $z_{\text{max}}(b) = \{b\}$, as follows from eq. (21). In this case $\#z_{\text{max}}(b) = 1$.

If $b$ is not a terminal node then $z_{\text{max}}(b) \in \zeta^*(b)$. If $b = X$, then the union of all paths $q(b_{\text{fin}})$ with $b_{\text{fin}} \in z_{\text{max}}(X)$ renders the whole tree structure $B(X)$,

$$\bigcup_{b_{\text{fin}} \in z_{\text{max}}(X)} q(b_{\text{fin}}) = B(X) .$$  

The maximal partition is related to the concept of reduced trees, section 8, and the concept of the set of partitions $\zeta(b)$ compatible with $B(X)$, in the following way:

\textbf{Theorem 10.2.} Let $B(X)$ be a given tree over $X$. The set $\zeta(X)$ of partitions of $X$ compatible with $B(X)$ is comprised of the maximal partitions $z_{\text{max}}(X)$ of $X$ with respect to $B'(X)$, where $B'(X)$ ranges through all reduced trees 13 associated with $B(X)$.

\textbf{Proof:}

Let $z \in \zeta(X)$, then all elements $b_1, b_2, \ldots$ of $z$ are elements of $B(X)$. Consider the union of paths

$$\bigcup_z q(b_i) \equiv B'(X) ,$$  

(25)
where \(q(b_i)\) are the paths of \(b_i\) in \(\mathcal{B}(X)\). By construction, the right-hand side \(\mathcal{B}'(X)\) is a tree structure, and is also a reduction of the original tree \(\mathcal{B}(X)\), with maximal partition \(z_{\text{max}}(X, \mathcal{B}') = z\). Conversely, let \(\mathcal{B}'(X)\) be a reduction of \(\mathcal{B}(X)\) with maximal partition \(z'_{\text{max}}(X)\); then all elements \(b' \in z'_{\text{max}}(X)\) lie in \(\mathcal{B}(X)\) by definition of a reduced tree; hence \(z'_{\text{max}}(X)\) is compatible with \(\mathcal{B}(X)\). \(\blacksquare\)

10.3 The minimal compatible partition \(z_{\text{min}}(b)\)

Let \(b\) be a given non-terminal node in the tree. The partition of \(b\) that is obtained by stepping to the next level in the tree will be called the minimal partition \(z_{\text{min}}(b)\) of \(b\) in \(\mathcal{B}(X)\). The elements \(b'\) in the minimal partition are uniquely characterised by the feature that they all have the node \(b\) as their predecessor, and there are no further nodes \(b'\) which have this predecessor. We can therefore write

\[
z_{\text{min}}(b) = \{ b' \in \mathcal{B}(X) \mid (b')^- = b \} .
\] (26)

\(z_{\text{min}}(b)\) has another minimal property which can be alternatively used to define it as a set: The minimal partition \(z_{\text{min}}(b)\) of \(b\) is uniquely characterised by the fact that it is refined by any non-trivial partition of \(b\) compatible with \(\mathcal{B}(X)\)

\[
z_{\text{min}}(b) \preceq z \quad \text{for all} \quad z \in \zeta^*(b, \mathcal{B}) ,
\] (27)

and as a consequence contains the least number of elements,

\[
\#z_{\text{min}}(b) \leq \#z \quad \text{for all} \quad z \in \zeta^*(b, \mathcal{B}) .
\] (28)

This follows immediately from the definition (26) of \(z_{\text{min}}\).

The above definition (26) or, alternatively, eq. (27), is meaningful only when \(b\) is not a terminal element of the given tree \(\mathcal{B}(X)\). If \(b\) is terminal we define the minimal partition to be the trivial partition, \(z_{\text{min}}(b) = \{b\}\). In this case \(\#z_{\text{min}}(b) = 1\).

Let \(b\) be any non-terminal element of \(\mathcal{B}(X)\). Given the minimal partition \(z_{\text{min}}(b)\) of \(b\) in \(\mathcal{B}(X)\), we can split the subtree \(\mathcal{B}(b, X)\) accordingly into a sum of subtrees,

\[
\mathcal{B}(b, X) = \sum_{b' \in z_{\text{min}}(b)} \mathcal{B}(b', b) .
\] (29)

We shall make use of this fact frequently.

The number of elements in the minimal partition \(z_{\text{min}}(b, \mathcal{B})\) of \(b\) in the given tree \(\mathcal{B}(X)\) will be denoted as

\[
m(b) \equiv \#z_{\text{min}}(b, \mathcal{B}) .
\] (30a)

Similarly, the number of elements of \(b\) regarded as a set will be denoted by \(n(b)\),

\[
n(b) \equiv \#b .
\] (30b)

These quantities pertain to the nodes \(b \in \mathcal{B}(X)\) in a specific way and will play a crucial role in what follows. We must have

\[
n(b) = \sum_{a \in z_{\text{min}}(b)} n(a) .
\] (31)

For every \(b \in \mathcal{B}(X)\) the following inequality holds:

\[
1 \leq m(b) \leq n(b) .
\] (32)

Furthermore, if \(b \neq X\), then \(b\) has a unique predecessor \(b^-\), whose minimal partition \(z_{\text{min}}(b^-)\) has \(m(b^-)\) elements, one of which is just \(b\). Each of the nodes in \(z_{\text{min}}(b^-)\) contains at least one element, and there are \(m(b^-) - 1\) nodes apart from \(b\); hence

\[
n(b^-) \geq m(b^-) - 1 + n(b) ,
\] (33)

or

\[
m(b^-) - 1 \leq n(b^-) - n(b) .
\] (34)
10.4 Trees reduced by a partition

By means of the concept of a partition compatible with a given tree we can introduce a generalization of the idea of reduced trees as given in eq. (33) in section 8.

Let $B(X)$ be a given tree over the set $X$. Let $z \in \zeta(X, B)$ be a partition of $X$ compatible with the tree $B(X)$. Then we can construct a new tree $B(z)$ as follows: For each $b \in z$, we remove the subtree $B(b, X)$ from $B(X)$ but reattach $b$ as a terminal element; this is just the proper generalization of eq. (33). The set so obtained is again a tree by construction:

**Definition 10.5 (Tree reduced by a partition).** The tree structure $B(z)$ defined by

$$ B(z) \equiv \left[ B(X) - \bigcup_{b \in z} B(b, X) \right] \cup \left[ \bigcup_{b \in z} \{b\} \right], \quad (35) $$

is called the tree $B(X)$ reduced by the partition $z \in \zeta(X, B)$.

If $\tilde{b}$ is an element of the reduced tree $B(z)$, then the subtree of $\tilde{b}$ in the reduced tree will be denoted by $B(\tilde{b}, z)$.

11 Amount functions

From now on we explicitly assume that $X$ is a finite set. As a consequence, the quantities $m(b)$ and $n(b)$ are always finite natural numbers.

Let $b \in B(X)$ with $b \neq X$. Then $b^-$ exists, and the number of elements in $z_{\text{min}}(b^-)$ is $m(b^-)$. Now we think of $b$ as being distinct in the set of elements $b'$ comprising $z_{\text{min}}(b^-)$. Suppose we are presented the set $z_{\text{min}}(b^-) = \{b_1', \ldots, b_{m(b^-)}'\}$, as in the scenario laid out in section 8 and we are asked to find out which of the $b_i'$ is the distinct one. Presuming that no optimized search strategy is employed we have to expend at most $(m(b^-) - 1)$ questions in order to fulfill our task.

We can now extend this reasoning to the whole path $q(b)$: $b^-$ is distinct in the set of all $b''$ comprising the minimal decomposition $z_{\text{min}}(b^2^-)$, where $b^2^-$ denotes the predecessor of $b^-$ in $B(X)$. In order to determine $b^-$ amongst the $m(b^2^-)$ elements of $z_{\text{min}}(b^2^-)$ we have to expend at most $(m(b^2^-) - 1)$ questions. We can continue in this way up the whole path $q(b)$ until no predecessor $b^{k+1}^-$ exists any longer, in other words, $b^k^- = X$. The maximum number of questions to determine the distinct node $b \in B(X)$ we were seeking out is therefore the sum of all these contributions,

$$ e(b) \equiv \sum_{a \in q(b)} \left[ m(a) - 1 \right] = \left[ \sum_{a \in q(b)} m(a) \right] - o(b) + 1 \quad , \quad (36) $$

where the length of the path $o(b)$ was defined in eq. (10).

**Definition 11.1 (Amount of a node).** The quantity $e(b)$ in eq. (36) will be called the amount of $b$ in the tree $B(X)$.

When emphasizing the fact that the amount is dependent on the underlying tree we shall also use the notation $e_B(b)$.

**Remark:** In eq. (36), the element $b$ is excluded from summation, since $a \in q(b)$ only. If $b$ is terminal in $B(X)$ then $m(b) = 1$; in this case we can trivially extend the sums in (36) to range over the whole path $q(b)$, since the additional contribution $m(b) - 1$ is zero on account of $m(b) = 1$. It is then possible to write the path amount (36) as

$$ e(b) = \sum_{a \in q(b)} \left[ m(a) - 1 \right] \quad \text{for} \quad b = b_{\text{fin}} \in z_{\text{max}}(X) \quad . \quad (37) $$
We shall frequently make use of this convention.

Now let \( z \in \zeta(X) \) be an arbitrary partition of \( X \) compatible with the tree \( \mathcal{B}(X) \). Then every element \( b \in z \) has the uniquely defined path \( q(b) \subseteq \mathcal{B}(X) \). Hence it makes sense to sum up the amounts \( e_{B}(b) \) of each \( b \):

**Definition 11.2 (Total amount of partition).** The quantity \( G(z) \), defined by

\[
G(z) \equiv \sum_{b \in z} e_{\mathcal{B}(X)}(b) = \sum_{b \in z} \sum_{a \in q(b)} [m(a) - 1]
\]

is called the total amount of \( z \in \zeta(X) \) with respect to the tree \( \mathcal{B}(X) \). If \( z \) contains only one element we define the associated amount to be \( G(z) = 0 \).

When emphasizing the fact that the total amount is dependent on the underlying tree structure we shall also denote \( G(z) \equiv G_{\mathcal{B}}(z) \).

Now consider the maximal partition \( z_{\text{max}}(X) \) of \( X \) in \( \mathcal{B}(X) \): From

\[
G_{\mathcal{B}(X)} \equiv G(z_{\text{max}}(X)) = \sum_{b \in z_{\text{max}}(X)} e_{\mathcal{B}(X)}(b) = \sum_{a \in \mathcal{B}(X) - z_{\text{max}}(X)} [m(a) - 1]
\]

we see that in this case we sum over all but the terminable elements \( b \in \mathcal{B}(X) \); hence the total amount for the maximal partition of \( X \) in \( \mathcal{B}(X) \) is dependent on \( \mathcal{B}(X) \) only; this is reflected in our notation. The sum \( G_{\mathcal{B}(X)} \) therefore defines a map from the set of all tree structures over \( X \) into the natural numbers,

\[
G : \mathcal{M}(X) \rightarrow \mathbb{N}
\]

\[
\mathcal{B}(X) \rightarrow G_{\mathcal{B}(X)}
\]

**Definition 11.3 (Total amount of tree. Amount function).** The quantity \( G_{\mathcal{B}(X)} \) is called the total amount of the tree structure \( \mathcal{B}(X) \). The map \( G \), as defined in eq. \( \text{(40)} \), is called the amount function on \( \mathcal{M}(X) \).

**Remark 1:** If the tree \( \mathcal{B}(X) = \{X\} \) is trivial we again define the associated total amount to be zero, \( G_{\mathcal{B}(X)} = 0 \).

**Remark 2:** The total amount \( G(z) \) with respect to the partition \( z \in \zeta(X) \) compatible with the given tree \( \mathcal{B}(X) \), defined in eq. \( \text{(38)} \), is equal to the total amount \( G_{\mathcal{B}(z)} \) of the reduced tree \( \mathcal{B}(z) \), which is obtained by reducing \( \mathcal{B}(X) \) via \( z \) according to definition \( \text{(10.5)} \). We can use this fact to emphasize that the total amount of the partition \( G(z) \) is dependent on the underlying tree structure \( \mathcal{B}(X) \),

\[
G(z) \equiv G_{\mathcal{B}(z)}
\]

**Proposition 11.4 (Inequalities).** For every \( b \in \mathcal{B}(X) \) the following inequalities hold:

\[
o(b) - 1 \leq e(b) \leq n(X) - n(b) \leq n(X) - 1
\]

**Proof:**

Set \( o(b) = \kappa \) and \( q(b) = \{\beta_{1}, \ldots, \beta_{\kappa}\} \), with \( \beta_{1} = X \) and \( \beta_{\kappa} = b \). Then \( \hat{q}(b) = \{\beta_{1}, \ldots, \beta_{\kappa-1}\} \), and we must have

\[
e(b) = \sum_{j=1}^{\kappa-1} [m(\beta_{j}) - 1] = \sum_{j=2}^{\kappa} [m(\beta_{j-1}) - 1]
\]
For all $j \in \{1, \ldots, \kappa - 1\}$ we must have $m(\beta_j) \geq 2$. If this is inserted into eq. 13 we obtain the first inequality in 42. If eq. 34 is inserted into 43 we find

$$e(b) \leq \sum_{j=2}^\kappa \left[ n(\beta_{j-1}) - n(\beta_j) \right] = \sum_{j=1}^{\kappa-1} n(\beta_j) - \sum_{j=2}^\kappa n(\beta_j) = n(\beta_1) - n(\beta_\kappa) = n(X) - n(b).$$

(44)

This yields the second inequality in 42. The last inequality follows trivially from the fact that $n(b) \geq 1$. □

## 12 Induced Partitions

Let $B(X)$ be a given tree over $X$. Let $b \in B(X)$. For every $z \in \zeta(X)$ we can introduce the intersection

$$\sigma(z, b) \equiv z \cap B(b, X).$$

(45)

$\sigma(z, b)$ can be empty if all elements in $z$ are "coarser" than $b$, i.e., $b \subsetneq b'$ for precisely one $b' \in z$, and has zero intersection with the rest. If $\sigma(z, b)$ is non-empty, it is a partition of $b$ compatible with the subtree $B(b, X)$, as we show now:

**Theorem 12.1 (Induced partitions).**

(A) If $\sigma(z, b) \neq \emptyset$ then $\sigma(z, b) \in \zeta(b, B)$.

(B) Conversely, if $\tilde{z} \in \zeta(b, B)$, then there exists $z \in \zeta(X)$ such that $\sigma(z, b) = \tilde{z}$.

(C) **Definition:** If $\sigma(z, b)$ is non-empty it is called the partition of $b$ induced by $z$.

**Proof:**

Assume that $\sigma(z, b) \neq \emptyset$, then $\sigma(z, b) = \{b_1, b_2, \ldots\}$, where $b_i \in B(b, X)$. Now let $\mathcal{B} \in z - \sigma(z, b)$, then $\mathcal{B}$ cannot intersect $b$: For, $\mathcal{B}$ lies in $B(X)$ but not in the subtree $B(b, X)$ by assumption; hence if it intersects $b$ then it must contain $b$ properly, $\mathcal{B} \supset B$. But then it also contains all $b_i$ as subsets, which contradicts the fact that the $b_i$ are mutually disjoint. Thus, $\mathcal{B} \cap b = \emptyset$. It follows that the union of all $\mathcal{B} \in z - \sigma(z, b)$ has no intersection with $b$. As a consequence, the union of all $b_i$ must be equal to $b$, since $z$ is a partition of $X$. Furthermore, the $b_i$ are mutually disjoint, and lie in $B(b, X)$, from which it follows that $\{b_1, b_2, \ldots\} \in \zeta(b, X)$. This proves (A).

Let $\tilde{z} \in \zeta(b, B)$ be given. Now represent $B(X)$ as a union $B(X) = B'(X) \cup B(b, X)$ of trees as in section 8 where the reduced tree $B'(X)$ is given in eq. 13. Then the maximal partition $z_{\text{max}}(X, B')$ of $X$ in the reduced tree contains $b$ as an element. We now define a new partition by removing $b$ from $z_{\text{max}}(X, B')$ and replacing it by the set of elements in $\tilde{z}$,

$$z \equiv \left[ z_{\text{max}}(X, B') - \{b\} \right] \cup \tilde{z}.$$

(46)

The set $z$ so defined is obviously a partition of $X$ compatible with $B(X)$, hence $z \in \zeta(X, B)$, and, by construction, $z \cap B(b, X) = \tilde{z}$. This proves (B).

The concept of induced partitions is linked to the idea of refinements of partitions:

**Proposition 12.2 (Refinement of partitions).** Let $B(X)$ be a tree over $X$. Let $z, z' \in \zeta(X)$ with $z \prec z'$. Then

(A) $z - (z \cap z') \neq \emptyset$.

(B) $\sigma(z', b) \in \zeta^*(b)$ for all $b \in z - (z \cap z')$, whereas $\sigma(z', b) = \{b\}$ is the trivial partition for all $b \in z \cap z'$. 
for all \( b \) and \( z \).

Since \( z' \) is a refinement of \( z \), any element \( b' \) of \( z' \) is contained in some element \( b \) of \( z \) as a subset, \( b' \subset b \). \( z \cap z' \) contains all elements which are not partitioned under the refinement \( z \rightarrow z' \). This means that for all \( b \in z \cap z' \), \( z' \cap B(b, X) = \{b\} \), hence \( \sigma(z', b) \) is the trivial partition of \( b \). This proves the second statement in (B). On the other hand, if \( b \in z - (z \cap z') \), then \( b \) is undergoing a proper partition under the refinement \( z \rightarrow z' \). This implies that \( \sigma(z', b) \in \zeta^*(b) \), thus proving the first statement in (B). Since \( z' \) is refined there must exist at least one element of \( z \) that undergoes a proper partition, which says that \( z - (z \cap z') \) cannot be empty, hence (A). Finally,

\[
z' - (z' \cap z) = z' \cap \bigcup_{b \in z-(z \cap z')} B(b, X) = \bigcup_{b \in z-(z \cap z')} z' \cap B(b, X)
\]

for \( z' \geq (z \cap z') = b, X \in \zeta \).

(C)

\[
z' - (z' \cap z) = \bigcup_{b \in z-(z \cap z')} \sigma(z', b)
\]

Proof:

The splitting theorem describes the behaviour of total amount functions of reduced trees \( B(z) \) and \( B(z') \), where \( z' \) is a refinement of \( z \).

**Theorem 12.3 (Splitting theorem).** Let \( B(X) \) be a given tree over \( X \). Let \( z, z' \in \zeta(X) \) with \( z < z' \). Let \( B(z) \) and \( B(z') \) be the corresponding reduced trees. Then

\[
G(z') - G(z) = \sum_{b \in z-(z \cap z')} \left[ \#\sigma(z', b) - 1 \right] \cdot e_B(b) + \sum_{b \in z-(z \cap z')} G_B(b, z')
\]

Proof:

Using eq. (41) we have

\[
G(z') = \sum_{b' \in z'} e_{B(z')}(b') = \sum_{b' \in z' \cap z} e_{B(z')}(b') + \sum_{b' \in z'-(z' \cap z)} e_{B(z')}(b') = \\
= \sum_{b' \in z' \cap z} e_{B(z')}(b') + \sum_{b' \in z'-(z' \cap z)} \sum_{a' \in \hat{q}_{B(z')}(b')} \left[ m(a') - 1 \right] = \\
= \sum_{b' \in z' \cap z} e_{B(z')}(b') + ZS
\]

where

\[
ZS = \sum_{b' \in z'-(z \cap z')} \sum_{a' \in \hat{q}_{B(z')}(b')} \left[ m(a') - 1 \right]
\]

With the help of eq. (47) we can split the sums in \( ZS \) further:

\[
ZS = \sum_{b \in z-(z \cap z')} \sum_{b' \in \sigma(z', b)} \sum_{a' \in \hat{q}_{B(z')}(b')} \left[ m(a') - 1 \right]
\]

Since \( b \in z - (z \cap z') \) and \( b' \in \sigma(z', b) \), we have \( b \in \hat{q}_{B(z')}(b') \). But

\[
\{ a' \in \hat{q}_{B(z')}(b') \mid a' \subset b \} = \hat{q}_{B(b, z')}(b')
\]

and

\[
\{ a' \in \hat{q}_{B(z')}(b') \mid a' \supset b \} = \hat{q}_{B(z)}(b)
\]

for all \( b' \in \sigma(z', b) \) and \( b \in z - (z \cap z') \). Hence we can write

\[
\hat{q}_{B(z')}(b') = \{ a' \in \hat{q}_{B(z')}(b') \mid a' \supset b \} \cup \{ a' \in \hat{q}_{B(z')}(b') \mid a' \subset b \} = \\
= \hat{q}_{B(z)}(b) \cup \hat{q}_{B(b, z')}(b')
\]

Finally,

\[
\hat{q}_{B(z')}(b') = \{ a' \in \hat{q}_{B(z')}(b') \mid a' \supset b \} \cup \{ a' \in \hat{q}_{B(z')}(b') \mid a' \subset b \} = \\
= \hat{q}_{B(z)}(b) \cup \hat{q}_{B(b, z')}(b')
\]
This yields

$$ZS = \sum_{b \in z - (z \cap z')} \sum_{b' \in \sigma(z', b')} \left\{ \sum_{a' \in \mathcal{B}(z', b')} \left[ m(a') - 1 \right] + \right\} + \sum_{a' \in \mathcal{B}(z', b')(b')} \left[ m(a') - 1 \right] =$$

$$= \sum_{b \in z - (z \cap z') \ b' \in \sigma(z', b')} e_{\mathcal{B}(z)}(b) + \sum_{b \in z - (z \cap z') \ b' \in \sigma(z', b')} e_{\mathcal{B}(z', b')} =$$

$$= \sum_{b \in z - (z \cap z')} \#\sigma(z', b) \cdot e_{\mathcal{B}(z)}(b) + \sum_{b \in z - (z \cap z')} \sum_{b' \in \sigma(z', b')} e_{\mathcal{B}(z', b')} ,$$

where we have used eq. (36) for $e_{\mathcal{B}(z)}(b)$. We now insert $ZS$ into eq. (50) for $G(z')$:

$$G(z') = \sum_{b' \in z' \cap z} e_{\mathcal{B}(z')}(b') + \sum_{b \in z - (z \cap z')} \#\sigma(z', b) \cdot e_{\mathcal{B}(z)}(b) +$$

$$+ \sum_{b \in z - (z \cap z')} \sum_{b' \in \sigma(z', b')} e_{\mathcal{B}(z', b')} =$$

$$= \sum_{b \in z - (z \cap z')} e_{\mathcal{B}(z)}(b) + \sum_{b \in z - (z \cap z')} e_{\mathcal{B}(z)}(b) +$$

$$+ \sum_{b \in z - (z \cap z')} \left[ \#\sigma(z', b) - 1 \right] \cdot e_{\mathcal{B}(z)}(b) + \sum_{b \in z - (z \cap z')} G_{\mathcal{B}(z', b)}$$

where we have used the fact that, for elements $b \in z \cap z'$, the amount $e_{\mathcal{B}(z')}(b)$ of the path of $b$ in the tree $\mathcal{B}(z')$ is equal to the amount $e_{\mathcal{B}(z)}(b)$ of the path of $b$ in the tree $\mathcal{B}(z)$. Then the first two terms on the right-hand side of the last equation combine to give

$$\sum_{b \in z \cap z'} e_{\mathcal{B}(z')}(b) + \sum_{b \in z - (z \cap z')} e_{\mathcal{B}(z)}(b) = \sum_{b \in z} e_{\mathcal{B}(z)}(b) = G(z) .$$

If eq. (57) is inserted into eq. (55) we obtain eq. (49).

We see from eq. (49) that there are two contributions to the difference in the total amounts: The first one links the amounts $e_{\mathcal{B}(z)}(b)$ of the paths $q(b)$ of $b$ in $\mathcal{B}(z)$ to the "degree of splitting" $\#\sigma(z', b)$ of the set $b$ under the refinement $z \rightarrow z'$; the second one is the sum of all amounts $G_{\mathcal{B}(z', b)}$ of the subtrees $\mathcal{B}(b, z')$ of the larger tree $\mathcal{B}(z')$.

**Corollary 12.4.** For the special case $z = z_{\text{min}}(X)$ and $z' = z_{\text{max}}(X)$ we have

$$G_{\mathcal{B}(X)} = \left[ m(X) - 1 \right] \cdot \sum_{b \in z_{\text{max}}(X)} \#z_{\text{max}}(b) + \sum_{b \in z_{\text{min}}(X)} G_{\mathcal{B}(b, X)} .$$

If the tree $\mathcal{B}(X)$ is complete then

$$G_{\mathcal{B}(X)} = \left[ m(X) - 1 \right] \cdot n(X) + \sum_{b \in z_{\text{min}}(X)} G_{\mathcal{B}(b, X)} .$$

**Proof:**

For $z = z_{\text{min}}(X)$, $z' = z_{\text{max}}(X)$ we have $G(z) = m(X) \cdot [m(X) - 1]$, as follows from eq. (35), and $G(z') = G_{\mathcal{B}(X)}$; since $\mathcal{B}(z_{\text{max}}(X)) = \mathcal{B}(X)$ is the total tree $\mathcal{B}(X)$. Furthermore, for $b \in z - (z \cap z')$ we have $\sigma(z', b) = z_{\text{max}}(b) \in$
\( \zeta(b) \), and \( G_{B(b,z')} = G_{B(b,X)} \). Then eq. (49) gives
\[
G_{B(X)} = m(X) \cdot \left[ m(X) - 1 \right] + \
+ \sum_{b \in z - (z \cap z')} \left[ \# z_{\text{max}}(b) - 1 \right] \cdot e_{B(X)}(b) + \sum_{b \in z - (z \cap z')} G_{B(b,X)} = \
= \left[ m(X) - 1 \right] \cdot \left\{ m(X) + \sum_{b \in z_{\text{min}}(X)} \left[ \# z_{\text{max}}(b) - 1 \right] \right\} + \
+ \sum_{b \in z_{\text{min}}(X)} G_{B(b,X)} . \tag{60}
\]

However, in all sums we can extend the range of \( b \) to take values in \( z \cap z' \) as well; for such an element \( b \), \( \# z_{\text{max}}(b) = 1 \), and \( G_{B(b,X)} = 0 \). Thus, \( b \) can be allowed to run over the whole set \( z = z_{\text{min}}(X) \),
\[
G_{B(X)} = \left[ m(X) - 1 \right] \cdot \left\{ m(X) + \sum_{b \in z_{\text{min}}(X)} \left[ \# z_{\text{max}}(b) - 1 \right] \right\} + \
+ \sum_{b \in z_{\text{min}}(X)} G_{B(b,X)} . \tag{61}
\]

The first and the third contribution in curly brackets cancel each other; thus, we arrive at eq. (58).

If the tree is complete then the maximal partition \( z_{\text{max}}(X) \) is complete, i.e.,
\[
z_{\text{max}}(X) = \{ \{ x_1 \}, \{ x_2 \}, \ldots, \} , \tag{62}
\]
where \( x_i \) are the elements of \( X \). In this case, each of the maximal partitions \( z_{\text{max}}(b) \) for \( b \in z_{\text{min}}(X) \) is complete, so that
\[
\# z_{\text{max}}(b) = \# b \text{ for all } b \in z_{\text{min}}(X) . \tag{63}
\]

Consequently,
\[
\sum_{b \in z_{\text{min}}(X)} \# z_{\text{max}}(b) = \# X = n(X) , \tag{64}
\]
from which eq. (59) follows.

\section{13 The tree function \( E_{B(X)} \)}

The next theorem will be the first main statement about the properties of amount functions, in that it expresses the amount of a tree as a function of the pairs of numbers \( (n(b), m(b)) \) at every node \( b \in B(X) \). To formulate this we need to define a new quantity:

\begin{definition}[Tree function]
Let \( B(X) \) be a tree structure over \( X \). The tree function \( E_{B(X)} \) of the tree \( B(X) \) is defined to be
\[
E_{B(X)} \equiv \sum_{b \in B(X)} n(b) \cdot \left[ m(b) - 1 \right] , \tag{65}
\]
where the sum runs over all nodes in the tree.
\end{definition}

\begin{theorem}[Tree function and total amount]
Let \( B(X) \) be a tree structure over \( X \).
\end{theorem}

\begin{enumerate}[(A)]
\item For a general tree,
\[
E_{B(X)} = \sum_{b \in z_{\text{max}}(X)} n(b) \cdot e_{B(X)}(b) . \tag{66}
\]
\end{enumerate}
If eqs. (71, 73) are inserted into the sum on the RHS of eq. (70) we obtain

\[ E_{B(X)} = G_{B(X)} \quad . \]  

These results say that, for a complete tree, the tree function coincides with the total amount in the tree, whereas if the tree is incomplete, then the tree function renders a weighted sum of the path amounts \( e_{B(X)}(b) \), the weights being equal to the cardinality \( n(b) \) of the terminal elements \( b \in z_{\text{max}}(X) \) in the incomplete tree.

**Proof:**

We first prove (A) by induction with respect to \( n \equiv \#X \): For \( n = 1 \), both left-hand side (LHS) and right-hand side (RHS) are zero and hence agree.

For \( n = 2 \), there are only two possible trees: Either, \( B(X) = \{X\} \) is the trivial tree, in which case \( z_{\text{max}}(X) = \{X\}, e_{B(X)}(X) = 0 \), and \( m(X) = 1 \), so that again, LHS and RHS agree to give zero. Or, \( B(X) = \{X, \{x_1\}, \{x_2\}\} \). In this case, the LHS is equal to

\[ E_{B(X)} = 2 \cdot (2 - 1) + 1 \cdot (1 - 1) + 1 \cdot (1 - 1) = 2 \quad . \]  

On the RHS, \( z_{\text{max}}(X) = \{\{x_1\}, \{x_2\}\} \), \( e_{B(X)}(b) = 1 \) and \( n(b) = 1 \) for \( b \in z_{\text{max}}(X) \), so that the RHS also yields 2.

We now perform the induction: We assume that eq. (66) holds for all possible sets \( X \) with \( \#X \in \{1, \ldots, n - 1\} \). We shall prove that eq. (66) is valid for sets \( X \) with \( \#X = n \) as well. If \( B(X) = \{X\} \) is trivial then eq. (66) holds trivially as before. Thus we can assume that the tree is nontrivial, which implies that the set \( X \) is properly split in the tree, hence \( \#z_{\text{min}}(X) > 1 \). As a consequence, the cardinality of each \( a \in z_{\text{min}}(X) \) must be smaller than that of \( X \),

\[ \#a < \#X = n \quad \text{for all } a \in z_{\text{min}}(X) \quad . \]  

Now we decompose \( X \) into a sum of subtrees \( B(a, X) \), where \( a \in z_{\text{min}}(X) \), as in eq. (12). Then the LHS of eq. (66) can be split into

\[ E_{B(X)} = n(X) \cdot \left[ m(X) - 1 \right] + \sum_{a \in z_{\text{min}}(X)} \sum_{b \in B(a, X)} n(b) \cdot \left[ m(b) - 1 \right] \quad . \]  

For each of the sums \( \sum_{b \in B(a, X)} \) on the RHS of (67), the induction assumption applies,

\[ \sum_{b \in B(a, X)} n(b) \cdot \left[ m(b) - 1 \right] = E_{B(a, X)} = \sum_{b \in z_{\text{max}}(a)} n(b) \cdot e_{B(a, X)}(b) \quad , \]  

where the maximal partition \( z_{\text{max}}(a) \) of \( a \) refers to the subtree \( B(a, X) \), but is clearly the same as with respect to the full tree \( B(X) \). Now assume that \( b \in z_{\text{max}}(a) \) for some \( a \in z_{\text{min}}(X) \), and consider the path amount of \( b \) in the full path \( q(b, B) \),

\[ e_{B(X)}(b) = \sum_{b' \in q(b, B)} \left[ m(b') - 1 \right] = \sum_{b' \in q(b, B)} \left[ m(b') - 1 \right] + m(X) - 1 = e_{B(a, X)}(b) + m(X) - 1 \quad . \]  

As a consequence,

\[ \sum_{b \in z_{\text{max}}(a)} n(b) \cdot e_{B(a, X)}(b) = \sum_{b \in z_{\text{max}}(a)} n(b) \cdot e_{B(X)}(b) - \left[ m(X) - 1 \right] \cdot \#a \quad . \]  

If eqs. (71, 73) are inserted into the sum on the RHS of eq. (70) we obtain

\[ \sum_{a \in z_{\text{min}}(X)} \sum_{b \in B(a, X)} n(b) \cdot \left[ m(b) - 1 \right] = - \left[ m(X) - 1 \right] \cdot \sum_{a \in z_{\text{min}}(X)} \#a + \sum_{a \in z_{\text{min}}(X)} \sum_{b \in z_{\text{max}}(a)} n(b) \cdot e_{B(X)}(b) \quad . \]  

(B) If \( B(X) \) is complete then

\[ E_{B(X)} = G_{B(X)} \quad . \]
However,
\[
\sum_{a \in z_{\text{min}}(X)} \#a = \#X = n(X) \quad ,
\]
and
\[
\sum_{a \in z_{\text{min}}(X)} \sum_{b \in z_{\text{max}}(a)} n(b) \cdot e_{B(X)}(b) = \sum_{b \in z_{\text{max}}(X)} n(b) \cdot e_{B(X)}(b) \quad .
\]

If eqs. (75, 76) are inserted into eq. (74), we obtain a contribution \(-[m(X) - 1]n(X)\) which cancels the same term in eq. (70), so that \(E_{B(X)}\) on the LHS of eq. (77) is equal to (76), which is what we have claimed in eq. (66). This finishes the proof of (A).

Now we prove (B). If \(B(X)\) is complete, then \(n(b) = 1\) for all \(b \in z_{\text{max}}(X)\). As a consequence, eq. (66) becomes
\[
E_{B(X)} = \sum_{b \in z_{\text{max}}(X)} e_{B(X)}(b) \quad ,
\]
but, according to eq. (59), the sum on the RHS of (77) is just \(G_{B(X)}\) by definition of \(G\). This proves (B).

14 Minimal classes

We now come to discuss the problem of minimizing the tree function \(E_{B(X)}\) on certain sets of tree structures. We will need a couple of new notions which we introduce in the sequel:

Consider the set \(\mathcal{M}(X)\) of all tree structures \(B(X)\) over \(X\). To every \(B \in \mathcal{M}(X)\) we can uniquely assign the minimal partition \(z_{\text{min}}(X)\) induced by \(B\) on \(X\); this assignment will be denoted by \(z_{\text{min}} : \mathcal{M}(X) \to \mathcal{Z}(X)\), \(B \mapsto z_{\text{min}}(X, B)\). Given \(z \in \mathcal{Z}(X)\), the inverse image \(z_{\text{min}}^{-1}(z)\) is the set of all tree structures \(B\) over \(X\) with the same minimal partition \(z_{\text{min}}(X)\) of \(X\).

Let \(n = \#X\). For \(1 \leq m \leq n\), let \(\mathcal{M}(X, m)\) denote the set of all tree structures over \(X\) whose minimal partition \(z_{\text{min}}(X)\) contains \(m\) elements. Since all \(\mathcal{M}(X, m)\) are disjoint, this defines a partition of \(\mathcal{M}(X)\),
\[
\mathcal{M}(X) = \bigcup_{1 \leq m \leq n} \mathcal{M}(X, m) \quad .
\]

We recall that the tree function \(E : \mathcal{M}(X) \to \mathbb{N}\) sends every tree over \(X\) to the sum over all \(n(b) \cdot [m(b) - 1]\), as \(b\) ranges through all nodes in the tree. We are interested in the minima of this map, as \(E\) is restricted to certain subsets of \(\mathcal{M}(X)\). We observe that it makes no sense to ask for the global minimum of \(E\) on \(\mathcal{M}(X)\), as the answer is trivial: In this case the minimum clearly is taken on the trivial tree \(B = \{X\}\), since \(E_{\{X\}} = G_{\{X\}} = 0\). Meaningful results are obtained, however, if we first focus on the subset of all complete trees \(\mathcal{C}(X) \subset \mathcal{M}(X)\); this inclusion is proper for \(\#X \geq 2\). We write \(\mathcal{C}(X, m)\) for the set of all complete trees with \(m\) elements in the minimal partition of \(X\). On the complete trees, the tree function \(E\) coincides with the total amount \(G\), as follows from statement (B) in theorem 13.2. Now we define
\[
\min(X) \equiv \min_{B \in \mathcal{C}(X)} E_{B} \quad ,
\]
and
\[
\min(X, m) \equiv \min_{B \in \mathcal{C}(X, m)} E_{B} \quad .
\]
In fact, \(\min(X)\) is a function of \(n = \#X\) only, and \(\min(X, m)\) is a function of \(n\) and \(m\) only,
\[
\min(n) \equiv \min(X) \quad , \quad \min(n, m) \equiv \min(X, m) \quad .
\]
These minima exist, since all tree functions take their values in the non-negative natural numbers. Thus it makes sense to speak of the set of all complete trees
\[
\text{Min}(X) \equiv E^{-1}\{ \min(X) \} \cap \mathcal{C}(X) \quad ,
\]
We introduce the trivial division \( n \div 2 \) division of \( m \) in many different ways, and for values of \( m \) ranging from 1 to \( n \). For a given \( m \), the numbers \( n_i \) can range between 1 and \( n \), and the \( n_i \) need not be mutually different. A decomposition of \( n \) in this form will be called a division of \( n \) into \( m \) terms. We can regard it as an \( m \)-tupel \( u = (n_1, \ldots, n_m) \) with positive integer components, \( n_i > 0 \), such that \( \sum n_i = m \). The set of all divisions of \( n \) into \( m \) terms will be denoted by \( U(n, m) \). If \( n \) is fixed and \( m \) varies from 1 to \( n \), the collection of all \( U(n, m) \) defines a partition of the set of all divisions \( U(n) \) of \( n \\

\[ U(n) \equiv \bigcup_{1 \leq m \leq n} U(n, m) \]  

We introduce the trivial division \( u_0 \equiv (n) \), and denote the set of all nontrivial divisions of \( n \) by \( U^*(n) \equiv U(n) - \{u_0\} \).

\( U(n, m) \) is a proper subset of

\[ H(n, m) \equiv \left\{ h \in \mathbb{R}^m \mid \sum_{i=1}^m h_i = n \right\} \subset \mathbb{R}^m \]

which is a hyperplane in \( \mathbb{R}^m \) whose least Euclidean distance to the origin is \( \frac{n}{m} \). The element of \( H(n, m) \) associated with the least distance will be denoted by \( \tilde{h} \); it has components \( \tilde{h} = \left( \frac{n}{m}, \ldots, \frac{n}{m} \right) \). Usually, \( n/m \) is not integer, so that \( \tilde{h} \neq U(n, m) \). However, there are always elements \( \tilde{n} \) of \( U(n, m) \) that come closest to \( \tilde{h} \). The minimal distance between these elements \( \tilde{n} \) and \( \tilde{h} \) ranges between 0 and \( \frac{\sqrt{m}}{2} \). If \( \tilde{h} \) coincides with a point in \( U(n, m) \), then \( \tilde{n} = \tilde{h} \) is uniquely defined. The bigger the distance between \( \tilde{h} \) and lattice points, the more elements \( \tilde{n} \) there are: If \( \tilde{h} \) lies at the center of a cube formed by elements of \( U(n, m) \), then there are \( 2^m \) candidates for \( \tilde{n} \), their distance from \( \tilde{h} \) being \( \frac{\sqrt{m}}{2} \) precisely. In this case each of the components \( \tilde{h}_i \) lies exactly between two integer values, \( \frac{n}{m} \pm \frac{1}{2} \in \mathbb{Z} \); thus, \( m \) must be even in this case. Whenever there is more than one \( \tilde{n} \), i.e. more than one element of \( U(n, m) \) with the same minimal distance to \( \tilde{h} \), they must be related by permutation of components.

There is another way to describe a division \( n = n_1 + \cdots + n_m \); this is in terms of occupation numbers \( t_k \) for all natural numbers \( k \) between 1 and \( n \) (and, in turn, even beyond), which express how often \( k \) appears as one of the terms \( n_i \) in a given decomposition of \( n \). Obviously, the description of a division of \( n \) into \( m \) terms is determined by the set of occupation numbers \( (t_1, t_2, \ldots) \) uniquely up to permutation of the terms \( n_i \) in the sum. Here comes the detailed definition:

Let \( n \in \mathbb{N} \), let \( 1 \leq m \leq n \). The \( m \)-tupel \( t \equiv (t_1, t_2, \ldots) \in \mathbb{N}_0 \times \mathbb{N}_0 \times \cdots \) will be called occupation numbers of the division of \( n \) into \( m \) terms, if it satisfies

\[ \sum_{k=1}^n t_k = m \quad \text{(86a)} \]

\[ \sum_{k=1}^n k \cdot t_k = n \quad \text{(86b)} \]

The first sum says that the number of terms in the division of \( n \) is \( m \); the second sum is just the decomposition of \( n \). Clearly, for \( k > n \) all occupation numbers \( t_k \) must vanish. For this reason we will now focus on the finite sequences \( t = (t_1, t_2, \ldots, t_n) \) of occupation numbers rather than the infinite ones, so that \( t \) ranges in \( \mathbb{N}_0^n \).

The trivial division as expressed by occupation numbers is \( t_0 \equiv (0, \ldots, 0, 1) \), i.e., \( t_n = 1 \), and all other components vanishing. The set of all occupation numbers of divisions of \( n \) into \( m \) terms will be denoted by
The relation between divisions $u$ and their associated occupation numbers $t$ is as follows: Every division $u = (n_1, \ldots, n_m)$ defines a unique $n$-tupel of occupation numbers $\kappa(u) \equiv (t_1, \ldots, t_n)$ by

$$
\kappa(u)_{a} = t_a = \sum_{i=1}^{m} \delta_{a,n_i} \quad \text{.}
$$

(87)

It follows readily that this indeed satisfies (86). Furthermore, every $n$-tupel $t$ of occupation numbers defines a unique naturally ordered division $u$ of $n$ by $m$ terms, $u_1 \leq u_2 \leq \cdots \leq u_m$. Now the inverse image $\kappa^{-1}(t)$ of an occupation number tupel $t$ is just the set of all divisions $u'$ that are related to the naturally-ordered division $u$ by permutation of components. Thus, every such inverse image has a naturally-ordered representative. We conclude that there is a $1$–$1$ relation between naturally-ordered divisions of $n$ and occupation numbers.

### 16 Partitions and divisions

Let $n = \#X$, let $z$ be an arbitrary partition of $X$, not necessarily related to a tree structure over $X$. Assume that the partition $z$ contains $m$ elements, $m = \#z$, where $z = \{b_1, \ldots, b_m\}$. $z$ defines a division $u(z)$ of $n$ into $m$ terms by $u = (\#b_1, \ldots, \#b_m)$. This defines the $u$-map $u : \mathcal{Z}(X) \to \mathcal{U}(n)$, $z \mapsto u(z)$. The associated occupation number will be written as $t(z)$ and has components

$$
t(z)_a = \sum_{b \in z} \delta_{a,\#b}
$$

(88)

for $a = 1, \ldots, n$. $t(z)_a$ will be called the $a$-th occupation number of the partition $z$. This defines the $t$-map $t : \mathcal{Z}(X) \to T(n)$, $z \mapsto t(z)$; it sends every partition of $X$ to the associated $n$-tupel of occupation numbers. The $u$-, $t$-maps are obviously surjective, since for every division of $n$ into $m$ terms one can construct an associated partition of $X$.

From the surjectivity of $u$ and $t$ and the fact that the map $\zeta_{\text{min}}$ sends $\mathcal{M}(X)$ onto the set of all partitions $\mathcal{Z}(X)$ we find $U(n) = (u \circ \zeta_{\text{min}})(\mathcal{M}(X))$ and $T(n) = (t \circ \zeta_{\text{min}})(\mathcal{M}(X))$, and furthermore, $U(n, m) = (u \circ \zeta_{\text{min}})(\mathcal{M}(X, m))$ and $T(n, m) = (t \circ \zeta_{\text{min}})(\mathcal{M}(X, m))$.

The distinct occupation number $t_{\text{min}}(X) \equiv (t \circ \zeta_{\text{min}})(\mathcal{B})$ will be called the minimal division of $n = \#X$ in $\mathcal{B}(X)$.

**Definition 16.1 (Integer quotient).** For $n \in \mathbb{N}_0$, $m \in \mathbb{N}$, let

$$
\left[ \frac{n}{m} \right] \equiv \{ n' \in \mathbb{N}_0 \mid n' \cdot m \leq n \}
$$

(89)

denote the integer quotient of $n$ by $m$.

### 17 Optimal division

Let $1 \leq m \leq n$. Let $\nu \equiv \left[ \frac{n}{m} \right]$ be the integer quotient of $n$ by $m$; then $n = \nu \cdot m + r$ with $0 \leq r < m$. We construct a division of $n$ into $m$ terms according to

$$
(\nu, \ldots, \nu, \nu + 1, \ldots, \nu + 1)
$$

(90)

with $(m-r)$ occurrences of $\nu$ and $r$ occurrences of $(\nu+1)$. The associated occupation number is denoted as $\bar{t} \equiv \bar{t}(n, m) = (t_1, \ldots, t_n)$, with $t_2 = m - r$, $t_{\nu + 1} = r$, and $t_{\nu + \lambda} = 0$ for $\lambda \notin \{\nu, \nu + 1\}$. Consider the inverse image $\kappa^{-1}(\bar{t})$ of $\bar{t}$ under $\kappa$; every representative of this set will be called optimal division of $n$ by $m$, and will be denoted by $\bar{n}$. Obviously, the optimal divisions come closest to the $m$-tupel $\hat{h} = (\frac{n}{m}, \ldots, \frac{n}{m}) \in H(n, m) \subset \mathbb{R}^m$, where $\hat{h}$ is the element in $H(n, m)$ with least Euclidean distance to the origin; thus, they coincide with the
objects $\vec{n}$ introduced in section 15. We observe that $\kappa^{-1}(i)$ is the set of all elements $\vec{n}$ of $U(n, m)$ for which the Euclidean norm

$$\|\vec{n} - \vec{h}\| \leq \frac{\sqrt{m}}{2}. \tag{91}$$

We now prove an important lemma about optimal divisions:

**Lemma 17.1 (Optimal divisions).** Let $\|u\| = \sqrt{\sum_{i=1}^{m} u_i^2}$ denote the Euclidean norm of an element $u \in \mathbb{R}^m$. Let $u = (n_1, \ldots, n_m)$ be an element of $U(n, m)$. Then there exists a finite sequence $u^0, u^1, \ldots, u^f$ of elements in $U(n, m)$ with $u^0 = u$, $u^f = \vec{n}$ for some $\vec{n} = \kappa^{-1}(i)$, such that

$$\|u^0\| > \|u^1\| > \cdots > \|u^f\|, \tag{92}$$

and the step $u^\alpha \to u^{\alpha+1}$ involves alteration of two components of $u^\alpha$ only.

**Proof:**
Denote $M \equiv \{1, \ldots, m\}$ for short. For $(i, j) \in \mathbb{Z}^2$, $i \neq j$, we define an operation $S_{ij}$ on elements $\vec{h} \in \mathbb{R}^m$ by

$$S_{ij}(h_1, \ldots, h_m) \equiv (h_1, \ldots, h_i + 1, \ldots, h_j - 1, \ldots, h_m), \tag{93}$$

i.e., all components except $h_i$ and $h_j$ remain the same. By construction, $S_{ij}$ preserves $H(n, m)$, for if $\vec{h} \in H(n, m)$ then so is $S_{ij}\vec{h}$.

— We prove the statement: Let $u \in U(n, m)$, let $\Delta \equiv u - \vec{h}$. If $\Delta_j - \Delta_i \leq 1$ for all $i, j \in \mathbb{Z}^2$, then

$$u = \vec{n} \in \kappa^{-1}(i). \tag{94}$$

**Proof of (94):** If $\Delta = 0$ then the statement is trivial; hence assume $\Delta \neq 0$. Let $\Delta_{\text{max}}$ denote the maximal element in $\{\Delta_1, \ldots, \Delta_m\}$. $\Delta_{\text{max}}$ is certainly $> 0$; for, $\sum \Delta_i = \sum u_i - \sum \vec{h}_i = 0$, and there must be nonzero components of $\Delta_i$. Our starting assumption says $\Delta_{\text{max}} - \Delta_i \leq 1$, but on the other hand, $\Delta_i \leq \Delta_{\text{max}}$, hence

$$0 \leq \Delta_{\text{max}} - \Delta_i \leq 1 \quad \text{for all } i. \tag{95}$$

However, $\Delta_i - \Delta_j = u_i - u_j \in \mathbb{Z}$, hence the same must be true for the quantities $\Delta_{\text{max}} - \Delta_i$. We conclude that $\Delta_{\text{max}} - \Delta_i \in \{0, 1\}$. We have altogether $m$ components $\Delta_i$, which can take values of either $\Delta_{\text{max}}$ or $\Delta_{\text{max}} - 1$. Suppose there are $(m - r)$ components $\Delta_i = \Delta_{\text{max}}$, and $r$ components of the form $\Delta_{\text{max}} - 1$, where $0 \leq r \leq m$. We cannot have $r = m$, for otherwise none of the $\Delta_i$ would take the maximal value $\Delta_{\text{max}}$; thus, $r < m$. The sum over all $\Delta_i$ must vanish, from which it follows that $m\Delta_{\text{max}} = r$. An easy computation now gives $\|\Delta\|^2 = \frac{r(r - m)}{m}$. If $m$ is fixed, the expression on the RHS is zero for $r = 0$ and becomes maximal for $r = \frac{m}{2}$, in which case it takes the value $\frac{m^2}{4}$. Hence $\|u - \vec{h}\| \leq \frac{\sqrt{m}}{2}$, which implies that $u = \vec{n}$ by eq. (91). This proves the statement (94).

— Now we prove our lemma: We describe step 1 in constructing the series (92): Let $\Delta^0 \equiv u^0 - \vec{h}$. If $u^0 = \vec{n}$, there is nothing to prove. If $u^0 \neq \vec{n}$, we conclude from statement (94) that there exists a pair $(i, j) \in \mathbb{Z}^2$ with $i \neq j$ such that $\Delta^0_i - \Delta^0_j > 1$; since the left-hand side must be integer we must have, in fact, that $\Delta^0_j - \Delta^0_i \geq 2$. Now we define the new element $u^1 \equiv S_{ij}u^0$ for this choice of $(i, j)$. Let $\Delta^1 \equiv u^1 - \vec{h} = S_{ij}\Delta^0$. Since $\vec{h}$ has least distance to the origin, it is perpendicular to the hyperplane $H(n, m)$, whereas $\Delta^0, \Delta^1$ lie in this plane. Hence, by Pythagoras,

$$\|u^0\|^2 = \|\vec{h}\|^2 + \|\Delta^0\|^2, \tag{96}$$

or $\|u^1\|^2 - \|u^0\|^2 = \|S_{ij}\Delta^0\|^2 - \|\Delta^0\|^2$. The last expression is just $2(\Delta^0_j - \Delta^0_i + 1)$, which must be $\leq -2$ owing to $\Delta^0_j - \Delta^0_i \geq 2$. Thus,

$$\|u^1\|^2 \leq \|u^0\|^2 - 2 < \|u^0\|^2, \tag{97}$$

and only two components of $u^0$, namely $u^0_i$ and $u^0_j$ have been altered. This finishes step 1. In step 2 we check whether $u^1 = \vec{n}$ for some $\vec{n}$; if yes, the process terminates; if no, it continues in the same manner. Since every step $\alpha$ involves a decrease of $\|\Delta^\alpha\|^2$ by at least $-2$, the process must terminate after a finite number of steps. □
18 Optimal trees

A tree structure \( B_o = B_o(X) \) over the set \( X \) is called optimal over \( X \), if \( B_o \) is complete, and

\[
t(z_{\text{min}}(b)) = i(n(b), 2)
\]

(98)

for all non-terminal elements \( b \in B_o \). This means that every node \( b \) not belonging to the maximal partition \( z_{\text{max}}(X) \) of \( X \) is partitioned into two halves which are as close to being equal as possible, when stepping to the next level in the tree; and every terminal node contains only one element. The set of all optimal trees over \( X \) forms (for \( |X| > 2 \)) a proper subset of \( C(X) \), which will be denoted by \( O(X) \).

19 Minimal classes in \( T(n, m) \)

Every minimal tree \( B \in \text{Min}(X) \) maps into a certain partition \( z \) under \( z_{\text{min}} \), and into a certain occupation number \( t \) under the \( t \)-map. We shall be interested in the image of \( \text{Min}(X) \) under this sequence of maps, which we will denote as

\[
T_{\text{min}}(n) \equiv (t \circ z_{\text{min}})(\text{Min}(X))
\]

(99)

and which we shall call the global minimal class in \( T(n) \). For \( 1 \leq m \leq n \), the set

\[
T_{\text{min}}(n, m) \equiv (t \circ z_{\text{min}})(\text{Min}(X, m))
\]

(100)

will be termed the minimal class in \( T(n, m) \).

We note that we now have several distinct classes of occupation numbers in \( T(n) \): We have the class containing all optimal divisions of \( n \) by \( m \), \{ \( \bar{t}(n, 1), \bar{t}(n, 2), \ldots, \bar{t}(n, n) \} \); and on the other hand, the classes \( T_{\text{min}}(n, m) \). The relation between these will be investigated in the following developments.

Now let \( t \in T(n) \) arbitrary. We can study its inverse image \((t \circ z_{\text{min}})^{-1}(t) \cap C(X) \) in \( C(X) \). To every tree in this set we can assign the associated tree function \( E \); thus it makes sense to ask on which trees \( B \in (t \circ z_{\text{min}})^{-1}(t) \cap C(X) \), for a given division \( t \), the tree function \( E \) assumes its minimum. This minimum will be denoted by \( \text{min}(t) \); hence

\[
\text{min}(t) \equiv \min_{B \in (t \circ z_{\text{min}})^{-1}(t) \cap C(X)} E_B
\]

(101)

The associated subset of trees in \((t \circ z_{\text{min}})^{-1}(t) \cap C(X)\) that actually take this minimum will be denoted as \( \text{Min}(t) \),

\[
\text{Min}(t) \equiv E^{-1}(\text{min}(t)) \cap (t \circ z_{\text{min}})^{-1}(t) \cap C(X)
\]

(102)

20 Bases and integer logarithm

Let \( L \in \mathbb{N} \) with \( L \geq 2 \). Then the set

\[
\mathbb{B}_L \equiv \{ L^k \mid k \in \mathbb{N}_0 \}
\]

(103)

will be called basis over \( L \). The set \( \mathbb{B}_2 \) we shall also call binary basis. If no confusion is likely, \( \mathbb{B}_2 \) will be simply denoted by \( \mathbb{B} \).

**Definition 20.1 (Integer logarithm).** Let \( n \in \mathbb{N} \) be a natural number. The integer logarithm \( \lg_L(n) \) of \( n \) with respect to \( L \) is defined as

\[
\lg_L(n) \equiv \max \left\{ k \in \mathbb{N}_0 \mid L^k \leq n \right\}
\]

(104)

If no confusion is likely, the integer logarithm of \( n \) with respect to 2 will simply be written \( \lg(n) \equiv \lg_2(n) \). Clearly, \( \lg_L \) is a monotonically increasing function on \( \mathbb{N} \).

**Proposition 20.2 (Properties of integer logarithm).** The integer logarithm satisfies the following inequalities:
1. Let \( n, n' \in \mathbb{N} \) with \( n' \geq n \). Then
\[
\lg L(n') \geq \lg L(n).
\] (105a)

2. Let \( n, n' \in \mathbb{N} \). Then
\[
\lg L(n) + \lg L(n') \leq \lg L(n \cdot n').
\] (105b)

3. Let \( p \in \mathbb{N}_0 \). Then
\[
p \cdot \lg L(n) \leq \lg L(n^p).
\] (105c)

**Proof:**
\( \lg L(n) \) is the maximum in the set of those integers \( k \) which satisfy \( L^k \leq n \). As a consequence, \( L^{\lg L(n)} \leq n' \). Hence \( \lg L(n) \) lies in the set of those integers \( k' \) which satisfy \( L^{k'} \leq n' \) and consequently must be less than or equal to its maximum. This proves (105a).

Let \( k = \lg L(n) \) and \( k' = \lg L(n') \). Then \( L^k \leq n \) and \( L^{k'} \leq n' \), from which it follows that \( L^{k+k'} \leq nn' \), hence \( k + k' \leq \lg L nn' \). The converse is not necessarily true. — Furthermore, from \( L^k \leq n \) it follows that \( L^{pk} \leq n^p \), hence \( pk \leq \lg L(n^p) \). ■

**Lemma 20.3 (Standard decomposition).** Every natural number \( n \in \mathbb{N}_{\geq 0} \) has a standard decomposition
\[
n = 2^{\lg(n)} + R, \quad R \leq \frac{n}{2},
\] (106a)
\[
R < 2^{\lg(n)}.
\] (106b)

**Proof:**
We show that \( R \) must indeed be limited to be \( < n/2 \). Suppose to the contrary, then \( 2R \geq n \) for some \( n \). It follows that \( 2n = 2^{\lg(n)+1} + 2R \geq 2^{\lg(n)+1} + n \) or \( n \geq 2^{\lg(n)+1} \), implying \( \lg(n) \geq \lg(n) + 1 \), which is a contradiction. Hence (106a) holds. — Similarly, if \( R \geq 2^{\lg(n)} \), then \( n \geq 2^{\lg(n)+1} \), leading to a contradiction as before; thus, (106b) is true. ■

**Lemma 20.4.** The equation
\[
\lg(2\nu + 1) = \lg(2\nu) = \lg(\nu) + 1
\] (107)
holds for all integers \( \nu \in \mathbb{N} \).

**Proof:**
We use the decomposition (106a) for \( 2\nu \),
\[
2\nu = 2^{\lg(2\nu)} + R, \quad 0 \leq R < 2^{\lg(2\nu)}, \nu.
\] (108)

Assume that the first equation in (107) is not true, but rather
\[
\lg(2\nu + 1) = \lg(2\nu) + 1
\] (109)
We then have a chain of inequalities
\[
2\nu + 1 \geq 2^{\lg(2\nu+1)} = 2 \cdot 2^{\lg(2\nu)} =
\]
\[
= 2 \cdot [2^{\lg(2\nu)} + R] = 2 \cdot [2\nu + R] .
\] (110)
It follows that
\[
R \geq \nu - \frac{1}{2},
\] (111)
but \( R, \nu \) are integers, and therefore we must have
\[
R \geq \nu,
\] (112)
which contradicts the second inequality on the RHS of (108).

We now prove the second equation in (107): Let \( k \equiv \log(\nu) \). Then \( 2^k \leq \nu \), but \( 2^{k+1} > \nu \). By multiplying these inequalities with a factor of 2 we find \( 2^{k+1} \leq 2\nu \), but \( 2^{k+2} > 2\nu \). It follows that \( k + 1 \) has the maximum property with respect to \( 2\nu \), as required in definition 20.1 and therefore \( \log(2\nu) = k + 1 \), which proves the second equation in (107).

\[ \square \]

## 21 Optimal amount

**Theorem 21.1 (Amount of optimal trees).** Let \( n = \#X \) and \( B_o \in O(X) \). Let \( \log(n) \) denote the integer logarithm with respect to 2. Then

\[
E_{B_o} = G_{B_o} = n \cdot \log(n) + 2 \cdot \left[ n - 2 \log(n) \right] .
\]

This value is the same for all \( B_o \in O(X) \) and depends only on \( n \).

**Definition 21.2.** The common value of the amount of the optimal trees will be denoted by

\[
E(n) = G(n) \equiv E_{B_o} = G_{B_o} .
\]

**Proof:**

By induction with respect to \( n \). The statement is clear for \( n = 1 \), since in this case, \( E_{B_o} = G_{B_o} = 0 \), and \( \log(1) = 0 \).

Now perform the induction: Assume that eq. (113) holds for all \( 1 \leq n' < n \). Let \( X \) be a set with \( \#X = n \), let \( B_o \in O(X) \). Then the minimal partition \( z_{\min}(X) = \{ b_1, b_2 \} \) of \( X \) has two elements \( b_1 \) and \( b_2 \) whose cardinalities are as close to \( n/2 \) as possible. Use formula (59), together with the fact that \( m(1) = 2 \),

\[
G_{B_o} = n + \sum_{b \in z_{\min}(X)} G_{O(b,X)} .
\]

We have \( \#b \leq n - 1 \) for all \( b \in z_{\min}(X) \), and hence

\[
G_{O(b,X)} = \#b \cdot \log(\#b) + 2 \cdot \left[ \#b - 2 \log(\#b) \right]
\]

by assumption. We now must distinguish whether \( n \) is even or odd:

**Case 1:** \( n = 2\nu + 1 \). We apply formula (115), using the fact that \( \#b_1 = \nu + 1 \) and \( \#b_2 = \nu \). This gives

\[
G_{B_o} = 3n + (\nu + 1) \cdot \log(\nu + 1) + \nu \cdot \log(\nu) - 2 \log(\nu+1) + 2 \log(\nu+1) .
\]

Two subcases must be considered: \( \log(\nu) = \log(\nu + 1) \), or \( \log(\nu) + 1 = \log(\nu + 1) \). Consider first the case \( \log(\nu) = \log(\nu + 1) \),

\[
G_{B_o} = 3n + \nu \cdot \log(\nu) - 2 \log(\nu+2) .
\]

The first two terms give \( 2n + n \cdot \log(n) \), upon using eq. (107) in lemma 20.3. Multiple applications of the same equation then produce eq. (113). Now consider subcase \( \log(\nu + 1) = \log(\nu) + 1 \): This is the case if and only if \( \nu = 2^K - 1 \), where \( K \in \mathbb{N}_{>0} \). Therefore, \( K = \log(\nu + 1) \). On using \( \log(n) = \log(\nu + 1) \) we find

\[
G_{B_o} = 2n + n \log(n) + \nu + 1 - 2^K - 2^{K+1} .
\]

But \( \nu + 1 - 2^K = 0 \), and hence we arrive at (113) again.

**Case 2:** \( n = 2\nu \). In this case, \( \#b_1 = \#b_2 = \nu \), and formula (115) gives

\[
G_{B_o} = n + 2 \nu \cdot [\log(\nu) + 1] + 2 \nu - 2 \cdot 2 \log(\nu) ,
\]

which gives again (113), on using \( \log(n) = \log(\nu + 1) \).
Lemma 21.3 (Monotonicity of optimal amount). The optimal amount is a monotonically increasing function of $n$. In particular,

$$G(n+1) - G(n) = \lg(n) + 2 \quad \text{for all } n \in \mathbb{N} \quad .$$

Proof:
The result follows directly from eq. (113) for both cases $\lg(n) = \lg(n+1)$ and $\lg(n) + 1 = \lg(n+1)$.

Using this lemma we can prove:

Theorem 21.4 (Total amount for unsymmetric divisions). Let $n \in \mathbb{N}_{>0}$. Consider two divisions of $n$ into two terms,

$$n_1 + n_2 = n'_1 + n'_2 \quad .$$

If

$$\left(n'_1\right)^2 + \left(n'_2\right)^2 \geq n_1^2 + n_2^2 \quad ,$$

then

$$G(n'_1) + G(n'_2) \geq G(n_1) + G(n_2) \quad .$$

Proof:

Eq. (122a) implies that some $\Delta$ exists such that $n'_1 = n_1 + \Delta$ and $n'_2 = n_2 - \Delta$. Without loss of generality we can assume that $\Delta \geq 0$. If $\Delta = 0$ then there is nothing to prove, hence we can assume $\Delta > 0$. Then (122c) is equivalent to

$$G(n_1 + \Delta) - G(n_1) \geq G(n_2) - G(n_2 - \Delta) \quad .$$

The LHS and RHS can be written as sums over differences $G(n_1 + \Delta) - G(n_1 + \Delta - 1)$, etc. Using eq. (121) we find that (123) is equivalent to

$$\sum_{j=0}^{\Delta-1} \lg(n_1 + j) \geq \sum_{j=0}^{\Delta-1} \lg(n_2 - \Delta + j) \quad .$$

Now a short computation shows that (122b) implies

$$n_1 \geq n_2 - \Delta \quad .$$

Now eq. (105a) in proposition 20.2 shows that $\lg(n_1 + j) \geq \lg(n_2 - \Delta + j)$ for each pair of terms in (124). This theorem is used in proving the important

Theorem 21.5 (Amount of non-optimal divisions). Let $u = (n_1, \ldots, n_m) \in U(n, m)$, let $\bar{n} = (\bar{n}_1, \ldots, \bar{n}_m)$ $\in \kappa^{-1}(\bar{f})$ be an optimal division of $n$ by $m$. Then

$$\sum_{i=1}^{m} G(n_i) \geq \sum_{i=1}^{m} G(\bar{n}_i) \quad .$$

Proof:

Clearly, the RHS of (126) is independent of the representative $\bar{n} \in \kappa^{-1}(\bar{f})$, as the representatives differ only by permutations of components. According to lemma 17.1 there exists a finite sequence $u^0, u^1, \ldots, u^f$ of elements in $U(n, m)$ with $u^0 = u$ and $u^f = \bar{n}$ for some $\bar{n} \in \kappa^{-1}(\bar{f})$, such that $\|u^0\| > \|u^1\| > \cdots > \|u^f\|$, and the step $u^\alpha \rightarrow u^{\alpha+1}$ involves alteration of two components of $u^\alpha$ only, namely $u^\alpha_{i+1} = u^\alpha_i + 1$ and $u^\alpha_{j+1} = u^\alpha_j - 1$. As a consequence,

$$\left(u^\alpha_i\right)^2 + \left(u^\alpha_j\right)^2 > \left(u^\alpha_{i+1}\right)^2 + \left(u^\alpha_{j+1}\right)^2 \quad .$$

(127)
whereas \( u_k^a = u_k^{a+1} \) for all \( k \not\in \{i,j\} \). Thus, using theorem \([127]\) with \([127]\) implies that

\[
\sum_{i=1}^{m} G(u_i^a) \geq \sum_{i=1}^{m} G(u_i^{a+1}) .
\] (128)

This inequality holds for every step \( \alpha \to \alpha + 1 \), and hence \([126]\) follows.

## 22 Preoptimized trees

The concept of preoptimization is required as a necessary intermediate step in order to solve the problem of finding the global minimal class \( \text{Min}(n) \). Let \( n = \# X \).

**Definition 22.1 (Preoptimized trees).** A tree \( B \in \mathcal{M}(X) \) is called preoptimized if every subtree \( B(b, X) \) of \( B \) based on elements \( b \in z_{\text{min}}(X) \) in the minimal partition of \( X \) in \( B \) is optimal.

Thus, the only "degrees of freedom" of varying a preoptimized tree are the different choices of minimal partitions \( z_{\text{min}}(X) \), where these choices can be effectively described by the set of all divisions \( U(n) \) of \( n \) into \( m \) terms, for \( m = 1, \ldots, n \). Every preoptimized tree is complete by definition. The subset of all preoptimized trees over \( X \) in \( \mathcal{M}(X) \) will be denoted by \( p\mathcal{O}(X) \subset \mathcal{C}(X) \). \( p\mathcal{O}(X) \) is comprised of the disjoint subsets \( p\mathcal{O}(X, m) \) of preoptimized trees with \( m \) elements in the minimal partition \( z_{\text{min}}(X) \); hence we have a partition of \( p\mathcal{O}(X) \) according to

\[
p\mathcal{O}^*(X) = \bigcup_{2 \leq m \leq n} p\mathcal{O}(X, m) = p\mathcal{O}(X) - \{ \{X\} \} .
\] (129)

The structure of the subsets \( \mathcal{O}(X), p\mathcal{O}(X), p\mathcal{O}^*(X), \ldots, \) etc., is independent of the nature of the underlying set \( X \) but depends only on the number \( n = \# X \) of elements in it. We can therefore write \( \mathcal{O}(X) = \mathcal{O}(n), p\mathcal{O}(X) = p\mathcal{O}(n), p\mathcal{O}^*(X) = p\mathcal{O}^*(n), \) etc., when appropriate.

On the subsets just described, the tree function \( E \) coincides with the total amount by theorem \([132]\) since all trees are complete. There, \( E = G \) takes the minima

\[
p\text{min}(X) \equiv \min_{B \in p\mathcal{O}^*(X)} E_B ,
\] (130a)

and

\[
p\text{min}(X, m) \equiv \min_{B \in p\mathcal{O}(X, m)} E_B .
\] (130b)

In \([130a]\) we have restricted the trees to the set \( p\mathcal{O}^*(X) \), since for the trivial preoptimized partition \( z_{\text{min}}(X) = \{X\} \), there is nothing to optimize since there are no subtrees; and the total amount \( G \) of this tree, as well as the corresponding tree function \( E \), is zero.

In accord with definitions \([130]\) we introduce the sets of all preoptimized trees for which \( G = E \) takes the corresponding minima:

\[
\text{Min}_p(X) \equiv \text{Min}_p(n) \equiv E^{-1}(p\text{min}(X)) \cap p\mathcal{O}^*(X) ,
\]

\[
\text{Min}_p(X, m) \equiv \text{Min}_p(n, m) \equiv E^{-1}(p\text{min}(X, m)) \cap p\mathcal{O}(X, m) .
\] (131)

Obviously, \((t \circ z_{\text{min}})(p\mathcal{O}^*(X)) = T^*(n)\), and \((t \circ z_{\text{min}})(p\mathcal{O}(X, m)) = T(n, m)\). Hence \( p\mathcal{O}(X, m) \) can be partitioned according to

\[
p\mathcal{O}(X, m) = \bigcup_{t \in T(n, m)} [ (t \circ z_{\text{min}})^{-1}(t) \cap p\mathcal{O}(X) ] .
\] (132)

In all of the discussions so far the nature of the set \( X \) was immaterial; the only thing that matters is the number \( n = \# X \) of elements in \( X \). Thus, we could replace in each of the quantities above the symbol \( X \) by \( n \).
Now let \( t \in T(n), \ m = \sum_{i=1}^{n} t_i \) and \( B \in (t \circ z_{\min})^{-1}(t) \cap p\mathcal{O}(X) \) such that \( m = \#z_{\min}(X) \) for the corresponding minimal partition of \( X \). From eq. \((59)\) in corollary \((2.2)\) we have
\[
E_B = G_B = n(m - 1) + \sum_{b \in z_{\min}(X)} G_{B(b,X)} \quad .
\]
(133)

But all subtrees \( B(b,X) \) are optimal, hence \( G_{B(b,X)} \) coincides with \( G(#b) \) according to eq. \((113)\) in theorem \((21.3)\) and the first term \( n(m - 1) \) is constant for fixed \( t \). Here, the common value of all optimal trees with \( #b \) elements in the underlying set is denoted as \( G(#b) \), according to definition \((21.2)\). Thus \( E_B \) is constant on \((t \circ z_{\min})^{-1}(t) \cap p\mathcal{O}(X) \) and hence descends to a map, again denoted by
\[
E : T(n) \to \mathbb{N} \quad , \quad E(t) \equiv E_B \quad ,
\]
(134)

for any choice of representative \( B \in (t \circ z_{\min})^{-1}(t) \cap p\mathcal{O}(X) \). Now \((133)\) can be expressed as
\[
E(t) = n(m - 1) + \sum_{k=1}^{n} t_k \cdot G(k) \quad ,
\]
(135)

for all \( t \in T(n,m) \). Furthermore, we write \( E(u) = E(t) \) for any division \( u \in \kappa^{-1}(t) \).

In section \((14)\) we have introduced the minimum \( \min(n,m) \) of the tree function on the subset of complete trees \( B \in \mathcal{C}(X) \) which has \( m \) elements in its minimal partition \( z_{\min}(X) \), where the base set is \( X \) with \( n = #X \). This subset corresponds to the set \( T(n,m) \) defined in section \((15)\) and hence \( \min(n,m) \) is the minimum of the descended map \( E \), formula \((133)\), in \( T(n,m) \). The relation of the quantity \( \min(n,m) \) to the preoptimized minimum \( p \min(n,m) \) introduced above is as follows:

**Proposition 22.2 (Preoptimized minima).** Let \( n = #X \) and \( 1 \leq m \leq n \), then
\[
\min(n,m) \leq \min(n,m) \quad .
\]
\[
(136)
\]

**Proof:**
The set of all preoptimized trees \( p\mathcal{O}(n,m) \) in general is a proper subset of \( \mathcal{M}(n,m) = (t \circ z_{\min})^{-1}(T(n,m)) \). Hence, the minimum \( p \min(n,m) \) of \( E \) taken on \( p\mathcal{O}(n,m) \) need not be the global minimum \( \min(n,m) \) on \( T(n,m) \).

In the next section we shall compare the values \( E(t) \) with \( E(\bar{t}) \) at the optimal division \( \bar{t} \in T(n,m) \).

### 23 Minimality of the optimal division

In eq. \((90)\) in section \((17)\) we have defined the optimal division \( \bar{t} \) of \( n \) by \( m \) terms. In this section we show that the preoptimized trees for which the minimal partition \( z_{\min}(X) \) is optimal, or equivalently, for which \( t \circ z_{\min}(X) = \bar{t} \), are actually the minimal ones in \( \mathcal{M}(n,m) \), i.e. they lie in \( \min(n,m) \). First we show that they are the minimal ones in the set of all preoptimized trees \( p\mathcal{O}(n,m) \):

**Theorem 23.1 (Minimality of optimal partition 1).** Let \( \bar{t} = \bar{t}(n,m) \) be the occupation number of the optimal division of \( n \) by \( m \). Then
\[
E(t) \geq E(\bar{t})
\]
(137)

for all \( t \in p\mathcal{O}(n,m) \).

As a consequence we have
\[
p \min(n,m) = E(\bar{t}) \quad ,
\]
(138)

and the inverse image of \( \bar{t} \) in \( p\mathcal{O}(X) \) must therefore lie in \( \min_p(n,m) \),
\[
(t \circ z_{\min})^{-1}(\bar{t}) \cap p\mathcal{O}(X) \subset \min_p(n,m) \quad .
\]
(139)
Now define a new division $\bar{\kappa}$ holds.

Remark: The inclusion in eq. (139) is proper in general. This means that there exist elements in $\text{Min}_p(n, m)$ whose associated minimal partition is not optimal. As an example, consider $n = 6$, $X = \{1, \ldots, 6\}$, with optimal amount

$$G(6) = 6 + G(3) + G(3) = 6 + 5 + 5 = 16.$$  

Now compare this with the complete tree $B = B_{po}(2) + B_{po}(4)$, which is a sum of the preoptimized trees $B_{po}(2)$ and $B_{po}(4)$, respectively. $B$ is non-optimal, since the minimal partition $z_{\text{min}}(X)$ is based on the non-optimal division $(2, 4)$ of 6. The fact that $B_{po}(4)$ is preoptimized implies that elements $b \in z_{\text{min}}(4)$ have cardinality $\#b = 2$, and hence the fact that the whole tree is complete implies that the subtrees $B(b, B_{po}(4))$ are optimal. Thus, $G_{B_{po}(4)} = G(4) = 8$, whereas $G_{B_{po}(2)} = G(2) = 2$, and thus the tree $B$ has a total amount of

$$G_B = 6 + 8 + 2 = 16,$$

which coincides with $G(6)$ in eq. (142) even though the tree $B$ is not optimal.

The next theorem explains how $p \min(n, m)$ changes for fixed $n$ as $m$ increases:

**Theorem 23.2 (Monotonicity of the preoptimized minimum).** Let $n \geq 2$ and $1 \leq m \leq n$. Then

$$p \min(n, m + 1) > p \min(n, m).$$  

Proof:

The case $m = 1$ yields $p \min(n, 1) = 0$, whereas $p \min(n, 2) = G(n) > 0$ whenever $n \geq 2$. Thus we certainly have $p \min(n, 1) < p \min(n, 2)$. Therefore assume now that $m \geq 2$. Let $n$ be optimally divided by $(m + 1)$ according to $n = \nu \cdot (m + 1) + r$, where $\nu = \left\lfloor \frac{n}{m+1} \right\rfloor$ and $r < m + 1$. The naturally ordered representative of $\kappa^{-1}(\bar{\kappa})$ is

$$\bar{\nu} = \left(\underbrace{\nu, \ldots, \nu}_{m+1-r}, \underbrace{\nu + 1, \ldots, \nu + 1}_{r}\right).$$

According to this decomposition we have from eq. (138) in theorem 23.1 and eq. (135) that

$$E(\bar{\nu}) = p \min(n, m + 1) = nm + (m + 1 - r) \cdot G(\nu) + r \cdot G(\nu + 1).$$

Now define a new division $u$ of $n$ into $m$ terms by

$$u \equiv (u_1, \ldots, u_m) \equiv (\bar{n}_2, \ldots, \bar{n}_m, \bar{n}_1 + \bar{n}_{m+1}).$$
The value of $E$ on any preoptimized tree whose minimal partition $z_{\min}(X)$ corresponds to $u$ can be computed using eq. (133),

$$E(u) = n(m-1) + (m-r) \cdot G(\nu) + (r-1) \cdot G(\nu+1) + G(2\nu+1)$$

$r > 0$ \hspace{1cm} (148a)

$$E(u) = n(m-1) + (m-1) \cdot G(\nu) + G(2\nu)$$

$r = 0$ \hspace{1cm} (148b)

Assume first that $r > 0$: In this case we have $u_m = \bar{n}_1 + \bar{n}_{m+1} = 2\nu + 1$. Use eqs. (138, 146, 148a) to compute

$$p_{\min}(n, m+1) - E(u) = n + G(\nu) + G(\nu+1) - G(2\nu+1)$$

(149)

However, the amount $G(2\nu+1)$ in the optimal tree $B_o(2\nu+1)$ over a set with $(2\nu+1)$ elements can be expressed using eq. (59) in corollary 12.4 as

$$G(2\nu+1) = (2\nu+1) + G(\nu) + G(\nu+1)$$

(150)

so that (149) yields

$$p_{\min}(n, m+1) - E(u) = \nu(m-1) + (r-1)$$

(151)

Since $r \geq 1$ and $m \geq 2$, the RHS is $> 0$. – For the case $r = 0$ we use eq. (148b) to obtain in the same way as above

$$p_{\min}(n, m+1) - E(u) = n + 2G(\nu) - G(2\nu)$$

(152)

However, $G(2\nu) = 2\nu + 2G(\nu)$, so that (152) becomes

$$p_{\min}(n, m+1) - E(u) = \nu(m-1)$$

(153)

which is again greater than zero. To finish our argument we use eqs. (137, 138) in theorem 23.1 which imply that

$$E(u) \geq p_{\min}(n, m)$$

(154)

Now eqs. (152 – 154) imply the result in eq. (144). \hfill \blacksquare

The chain of inequalities in eq. (144) points out that the minimum $p_{\min}(n, 2)$ with respect to a binary optimal division of the set $X$ is the lowest in the set of all minima $p_{\min}(n, m)$. It follows from eq. (129) that $p_{\min}(n, 2)$ is therefore the global minimum in $pO^*(n)$. Hence, on using the notation (130a),

Corollary 23.3 (Minimality of bidivisions). $p_{\min}(n, 2)$ is the global minimum of $E$ on $pO^*(n)$,

$$p_{\min}(n, 2) = p_{\min}(n)$$

(155)

The next theorem explains the role of the optimal trees in the present context:

Theorem 23.4 (Minimality of optimal division 2). Let $\#X = n \geq 2$. Then the optimal trees minimize the tree function on the set of all preoptimized trees with two elements in $z_{\min}(X)$, and hence on all preoptimized trees. In symbols,

$$O(X) \subset \text{Min}_p(X, 2) \subset \text{Min}_p(X)$$

(156a)

and

$$G(n) = p_{\min}(n, 2) = p_{\min}(n)$$

(156b)

Proof:

Let $B_o \in O(X)$, then in particular, $B_o$ is preoptimized, and furthermore, the minimal partition $z_{\min}(X)$ is optimal, i.e., $(t \circ z_{\min})(B_o) = \bar{i}(n, 2)$. Then (137) says that $E(\bar{i}) = G(n)$ is the minimum in $\text{Min}_p(X, 2)$. As a consequence we must have the first inclusion in (156a), and the first equality in (156b) must hold. The second inclusion in (156a) is a consequence of corollary 23.3. \hfill \blacksquare

Now we come to the main theorem of this work:
Theorem 23.5 (Optimal trees are globally minimal). Let \( \# X = n \geq 2 \). Then the optimal trees over \( X \) belong to the globally minimal trees over \( X \), i.e.,

\[
\mathcal{O}(X) \subset \text{Min}(X),
\]

and

\[
G(X) = \min(X) = \min(n).
\]

**Proof:**

Since all trees involved in the present discussion are complete, the tree function \( E \) always coincides with the total amount \( G \) of the tree, as follows from theorem 13.2. We prove (157) by induction with respect to \( n = \# X \).

**Induction step:** We assume that \( \mathcal{O}(n') \subset \text{Min}(n') \) (158) for all \( 2 \leq n' \leq n - 1 \). We prove (157) for \( n = \# X \) by showing that \( G_B \geq G(n) \) for every complete tree \( B \in \mathcal{C}(X) \) over \( X \). Let \( B \in \mathcal{C}(X) \), let \( z_{\text{min}}(X) = (b_1, \ldots, b_m) \) be the minimal partition of \( X \) in \( B \). Let \( u = (u_1, \ldots, u_m) = (\# b_1, \ldots, \# b_m) \). Now apply eq. (59) in corollary 12.4:

\[
G_B = n(m - 1) + \sum_{j=1}^{m} G_{B(b_j, X)}.
\]

The subtrees need not be optimal, hence assumption (158) implies that

\[
G_{B(b_j, X)} \geq G(u_j) \quad \text{for all } j = 1, \ldots, m.
\]

Thus, (159) implies that

\[
G_B \geq n(m - 1) + \sum_{j=1}^{m} G(u_j) \equiv G_{B'}.
\]

where the RHS of the last formula defines the total amount of the **preoptimized** tree

\[
B' \equiv \sum_{b \in z_{\text{min}}(X)} \mathcal{O}(b) \in \mathcal{P}(\text{O}(X), m).
\]

Now \( G_{B'} = E(t) \), where \( t \) is the occupation number of \( u \), \( t = \kappa(u) \); hence, by eqs. (137, 138) in theorem 23.1 we must have \( G_{B'} \geq E(\bar{t}) = p \min(n, m) \), where \( \bar{t} \) is now the optimal division of \( n \) by \( m \). From (141) in theorem 23.2 we know that \( p \min(n, m) \geq p \min(n, 2) \). Using (156b) in theorem 23.3 we have \( p \min(n, 2) = p \min(n) = G(n) \). Thus, putting all these inequalities together,

\[
G_B \geq G(n) = G(X).
\]

which proves the theorem.

24 Mean path amount and quadratic deviation

From section 21 formula (113), we immediately see that the mean path amount \( \frac{1}{n} \sum e_i \) will be close to \( \log(n) \). We can make this statement more precise:

**Definition 24.1 (Mean path amount).** The mean path amount \( \bar{e}_B \) in the tree \( B(X) \) is defined to be

\[
\bar{e}_B \equiv \min \{ \eta \in \mathbb{N} \mid \eta \cdot n \geq G_B \}.
\]
Thus
\[ G_B = \bar{e}_B \cdot n - r , \quad \text{with } 0 \leq r < n . \quad (165) \]

In particular:

**Proposition 24.2 (Mean path amount in optimal trees 1).** In an optimal tree \( B = B_o \),
\[ \bar{e}_{B_o} = \begin{cases} \log(n) , & n \in \mathbb{B} \\ \log(n) + 1 , & n \notin \mathbb{B} \end{cases} . \quad (166) \]

**Proof:**

If \( n \in \mathbb{B} \) then \( \bar{e}_o = \log(n) \) follows immediately from (113), since \( n = 2^{\log(n)} \) in this case.

Now assume that \( n \notin \mathbb{B} \). From the maximum property of \( \log(n) \) it follows that \( n < 2^{\log(n) + 1} \). This can be rearranged to give
\[ 2 \left[ n - 2^{\log(n)} \right] < n . \quad (167) \]

If we add \( n \log(n) \) on both sides of this inequality we obtain
\[ n \log(n) + 2 \left[ n - 2^{\log(n)} \right] < n \left[ \log(n) + 1 \right] . \quad (168) \]

However, the LHS is just \( E_{B_o} \) according to formula (113), and hence
\[ E_{B_o} < n \left[ \log(n) + 1 \right] . \quad (169) \]

On the other hand, the same formula (113) says that
\[ E_{B_o} \geq n \log(n) . \quad (170) \]

Since \( n \notin \mathbb{B} \) we have \( n = 2^{\log(n)} + r \), where \( 0 < r < 2^{\log(n)} \). In this case the inequality in formula (170) becomes proper, and thus \( \log(n) + 1 \) satisfies the minimum property in formula (164), definition 24.1. \( \blacksquare \)

We now come back to eq. (165), \( G_B = (\bar{e}_B \cdot n - r) \), where \( r < n \). Now let us define the \( n \)-tupel
\[ (\bar{e}_1, \ldots, \bar{e}_n) \equiv \left( \bar{e}_B, \ldots, \bar{e}_B, \bar{e}_B - 1, \ldots, \bar{e}_B - 1 \right) . \quad (171) \]

Thus, for any tree \( B \) over \( X \) (which need not be optimal) we have
\[ G_B = \sum_{i=1}^{n} e_i = \sum_{i=1}^{n} \bar{e}_i = (n - r) \cdot \bar{e}_B + r \cdot (\bar{e}_B - 1) , \quad (172) \]

where we have used eqs. (165, 171). Introducing the \( n \)-tupel of deviations
\[ \Delta e \equiv (\Delta e_1, \ldots, \Delta e_n) \equiv (e_1 - \bar{e}_1, \ldots, e_n - \bar{e}_n) , \quad (173) \]

and the total quadratic deviation in \( B(X) \) by
\[ \sigma_{tot}^2(B) \equiv \sum_{i=1}^{n} (\Delta e_i)^2 , \quad (174) \]

we find on using (172) that
\[ \sigma_{tot}^2 = \sum_{i=1}^{n} (e_i^2 - \bar{e}_i^2) + 2 \sum_{i=n-r+1}^{n} \Delta e_i . \quad (175) \]

We now present some statements about the mean path amount in optimal trees. In every tree \( B \), the elements \( b_1, \ldots, b_k \) in the maximal partition \( z_{\text{max}}(X) \) can be labelled so that the associated path amounts are monotonically decreasing, \( e_1 \geq e_2 \geq \cdots e_k \). In particular, if \( \bar{e}_{B_o} \) is the mean path amount in the optimal tree \( B_o \) as defined in eq. (164), then we have:
Theorem 24.3 (Mean path amount in optimal trees 2). Let \( G(n) \) be the amount of the optimal tree \( B_o \) with \( n = \#X \). Let \( r = n \cdot e_{B_o} - G(n) \). Then, if \( n \in \mathbb{E} \),

\[
e_i = \log(n) \tag{176a}
\]

for all \( i = 1, \ldots, n \), whereas for \( n \not\in \mathbb{E} \),

\[
e_i = \begin{cases} 
\log(n) + 1 & , \quad i = 1, \ldots, n-r \\
\log(n) & , \quad i = n-r+1, \ldots, n
\end{cases} \tag{176b}
\]

Hence, in any case,

\[
e_i = \bar{e}_i \quad , \quad i = 1, \ldots, n \quad , \tag{177}
\]

where the tupel \((\bar{e}_1, \ldots, \bar{e}_n)\) was defined in eq. (174).

Proof:

We first prove (176a) by induction with respect to \( \log(n) \): For \( n = 1, 2 \), corresponding to \( \log(n) = 0, 1 \), (176a) is trivially satisfied. Now choose \( k \equiv \log(n) > 1 \) and suppose that (176a) is true for \( k-1 \). From formula (36) we know that the amount \( e(b) \) of a path in any optimal tree is equal to \( \#q(b) - 1 \), since \( m(b) = 2 \) for all non-terminal elements, while \( m(b) = 1 \) for all terminal ones. Thus, the amount \( e(b) \) of any terminal element is 1 plus the amount of the same element in the subtree \( B(a, X) \), where \( a \in z_{\min}(X) \) and \( b \subset a \). By assumption, \( n = 2^k \), hence the minimal partition of \( X \) contains two elements \( a_1 \) and \( a_2 \) both of which must have the same cardinality \( \#a_1 = \#a_2 = 2^{k-1} \). Let \( b \) be any terminal element in the tree such that \( b \subset a_1 \), say. Then \( e_{B(a, X)}(b) = k-1 \) by assumption. In the full tree, the path length of the same element \( b \) is greater by just one, hence \( e_{B_o}(b) = k = \log(n) \), which confirms (176a).

Now assume \( n \not\in \mathbb{E} \). We first show: The path lengths \( e_i \) can mutually differ at most by \( \pm 1 \),

\[
|e_i - e_j| \in \{0, 1\} \quad . \tag{178}
\]

We prove this statement by induction with respect to \( n \): For \( n = 1 \) and \( n = 2 \) the path lengths in the optimal trees are 0 and 1, respectively, and hence (178) is satisfied. For \( n = 3 \) the path amounts in the optimal tree are \( e_1 = e_2 = 2 \) and \( e_3 = 1 \); again, (178) is satisfied. Now let \( n \geq 4 \) and assume that statement (178) is true for all \( 1 \leq n' \leq n - 1 \). Let \( b, b' \) be any two elements in the maximal partition \( z_{\max}(X) \) of \( X \). Let \( a, a' \) denote those elements in the minimal partition \( z_{\min}(X) \) for which \( b \subset a \) and \( b' \subset a' \) (this includes the possibility that \( a = a' \)). Then \( \#a, \#a' < n \), and the induction assumption applies to the path amounts in the optimal subtrees \( B(a, X) \) and \( B(a', X) \): Namely, since \( e(b) = e_{B(a, X)}(b) + 1 \) and \( e(b') = e_{B(a', X)}(b') + 1 \) we must have

\[
|e(b) - e(b')| = |e_{B(a, X)}(b) - e_{B(a, X)}(b')| \in \{0, 1\} \quad . \tag{179}
\]

This proves formula (178).

From formula (178) we infer that there exist integers \( \alpha \) and \( k \) such that

\[
\alpha(k + 1) + (n - \alpha)k = G(n) \quad , \tag{180}
\]

where \( 0 \leq \alpha < n \). If \( \alpha \) were zero we would have \( G(n) = nk \), and together with eq. (113) it would follow that

\[
n \left[ \log(n) + 2 - k \right] = 2^{\log(n)+1} \quad . \tag{181}
\]

By means of prime number factorisation of the factors on the left-hand side we conclude that \( n \) must take the form \( n = 2^K \) for some integer \( K \), thus implying \( n \not\in \mathbb{E} \), which contradicts the initial assumption. Hence we really have \( \alpha > 0 \). Now eq. (180) can be written in the form

\[
G(n) = n \cdot k + \alpha \quad , \quad 0 < \alpha < n \quad . \tag{182}
\]

If \( n \) and \( G(n) \) are given numbers we can consider (182) as an equation for the unknowns \( \alpha, k \). If the restriction \( 0 < \alpha < n \) is upheld, then the solution for the pair \((\alpha, k)\) is unique. Now consider formula (113) for \( G(n) \),

\[
G(n) = n \cdot \log(n) + 2R \quad , \quad R = n - 2^{\log(n)} \quad . \tag{113}
\]
From formula (106a) in lemma 20.3 we know that $2R < n$. Thus, the pair

$$\alpha = 2R, \quad k = \lg(n)$$  \hfill (183)

is the unique solution to the system (182). As a consequence, amongst the $c_i$ there must be $(n-2R)$ occurrences of $\lg(n)$ and $2R$ occurrences of $\lg(n)+1$,

$$e_i = \begin{cases}
\frac{\lg(n)+1}{\lg(n)}, & i = 1, \ldots, 2R \\
\frac{\lg(n)}{\lg(n)}, & i = 2R+1, \ldots, n
\end{cases}$$  \hfill (184)

It remains to show that $n-r = 2R$: To this end we write down formula (165) for the case at hand, i.e., an optimal tree with $n \not\in \mathcal{B}$, in which case (166) applies,

$$G(n) = \lceil \lg(n) + 1 \rceil \cdot n - r$$  \hfill (185)

Comparison of (185) with formula (113) gives $2R = n-r$, hence (176b) is proved. \hfill \blacksquare

## 25 Isomorphic trees

In this section we formulate a notion of structural similarity between trees $\mathcal{B}$ and $\mathcal{B}'$ which no longer need to be defined over the same set $X$. This will lead to an appropriate notion of isomorphism of trees.

Consider a tree $\mathcal{B}$ over a set $X$. The structure of the tree $\mathcal{B}$ is captured in the set of its nodes $b$, and the degree of splitting $m(b)$ associated with each node. The particular nature of the underlying set $X$, just as the particular value $n(b)$ of the cardinality of the nodes, is not a primary structure-determining element. To see this we can construct different trees from the given tree $\mathcal{B}$ which exhibit the same structure: To this end consider the maximal partition $z_{\max}(X, \mathcal{B}) = \{c_1, \ldots, c_K\}$ of $X$ in $\mathcal{B}$. The elements $c_i \in z_{\max}(X, \mathcal{B})$ are terminal in this tree and are never partitioned further; this means that their "internal structure" is immaterial, as far as the tree $\mathcal{B}$, and its internal structure, are concerned. Now consider any collection $\{c'_1, \ldots, c'_K\}$ of non-empty, mutually disjoint sets $c'_i$ with $i = 1, \ldots, K$, and let $X' \equiv \bigcup_i c'_i$. We can think of constructing a new tree $\mathcal{B}'$ by replacing every terminal element $c_i$ in the old tree $\mathcal{B}$ by the corresponding element $c'_i$. Then there is a 1–1 relation between nodes $b \in \mathcal{B}$ and $b' \in \mathcal{B}'$; moreover, the minimal partitions $z_{\min}(b, \mathcal{B}(X))$ and $z_{\min}(b, \mathcal{B}'(X'))$ are the same for all nodes $b$ and $b'$ which correspond to each other. In particular, the degree of splitting $m(b')$ is the same as $m(b)$ for such nodes. Obviously, both trees have the same cardinality, $\#\mathcal{B}' = \#\mathcal{B}$. This idea can be made precise in the following

**Definition 25.1 (Isomorphism of trees).** Let $\mathcal{B}$ and $\mathcal{B}'$ be trees over sets $X$ and $X'$ with the same cardinality, $\#\mathcal{B} = \#\mathcal{B}'$. $\mathcal{B}$ and $\mathcal{B}'$ are isomorphic if there exists a bijection $i : \mathcal{B} \rightarrow \mathcal{B}'$ such that

$$m \circ i(b) = m(b) \quad \text{for all } b \in \mathcal{B}.$$  \hfill (186)

For a given pair $\mathcal{B}$ and $\mathcal{B}'$ of trees there can exist more than one isomorphism.

**Proposition 25.2 (Paths in isomorphic trees).** Let $i : \mathcal{B} \rightarrow \mathcal{B}'$ be an isomorphism of trees. Then the path assignment $b \mapsto q(b)$ commutes with $i$,

$$q \circ i = i \circ q.$$  \hfill (187a)

As a consequence, isomorphic trees have the same path amounts,

$$e_{iB}(i(b)) = e_{B}(b) \quad \text{for all } b \in \mathcal{B}.$$  \hfill (187b)

**Proof:**

Since isomorphic trees have the same basic structure in the sense that they have the same number of nodes, and each node has the same degree of splitting, the paths in isomorphic trees have the property that

$$q(i(b)) = i(q(b)) \quad \text{for all } b \in \mathcal{B},$$  \hfill (188)
which gives \( (187a) \). Furthermore, \( m \circ i = i \circ m \) by definition of isomorphism. Then the statement \( (187b) \) follows immediately from eq. \( (186) \) in definition 11.1.

For a given tree \( B \) we can think of the category \([B]\) of isomorphic trees. Even if \( B \) is defined over a finite set \( X \), general elements of \([B]\) need no longer share this property; all that is required is that they have the same number of nodes and the same degree of splitting \( m(b) \) as the original tree \( B \) at each node \( b \). From proposition 25.2 we learn that all trees in the category \([B]\) have the same path lengths \( o(b) = \#q(b) \), and the same path amounts \( c(b) \). In general they differ in the total amount \( E_B \), however.

There exists a stronger form of isomorphism which can be defined for trees which are built over the same base set \( X \):

**Definition 25.3 (Equivalent trees).** Let \( B \) and \( B' \) be trees over the same base set \( X \). \( B \) and \( B' \) are said to be equivalent if they are isomorphic and share the same maximal partition of \( X \),

\[
z_{\text{max}}(X, B) = z_{\text{max}}(X, B') \quad .
\]

Let us assume that \( X \) is finite. Then there exists an integer \( K \) such that \( z_{\text{max}}(X, B) = \{c_1, \ldots, c_K\} \) and \( z_{\text{max}}(X, B') = \{c'_1, \ldots, c'_K\} \). Equivalence of \( B \) and \( B' \) then means that there exists a permutation \( \pi \) of \( K \) elements such that

\[
c'_j = c_{\pi(j)} \quad \text{for all } j = 1, \ldots, K \quad .
\]

But \( B \) and \( B' \) are isomorphic, hence \( c'_j = i(c_j) \), where \( i : B \to B' \) is an appropriate isomorphism. By eq. \( (187b) \), the path amounts are related by \( e_B(c'_j) = e_B(c_j) \equiv e_j \). Let \( w_j \equiv n(c_j) \); then it follows from eq. \( (66) \) in theorem 13.2 that the following statements are true:

**Theorem 25.4 (Tree function on equivalent trees).** Let \( B \) and \( B' \) be equivalent trees over the finite set \( X \). Then

\[
\begin{align*}
E_B &= \sum_{j=1}^{K} w_j \cdot e_j \quad , \\
E_{B'} &= \sum_{j=1}^{K} w_{\pi(j)} \cdot e_j \quad ,
\end{align*}
\]

where \( \pi \) is a permutation of \( K \) objects.

Each category \([B]\) contains preferred elements \( S \) which we can construct as follows: Let \( B \in [B] \), and consider the maximal partition \( z_{\text{max}}(X, B) = \{c_1, \ldots, c_K\} \) as before. Now define the set \( X_S \equiv \{1, \ldots, K\} \). \( S \) is now defined to be a tree over \( X_S \), isomorphic to \( B \), and is obtained by replacing every terminal element \( c_i \) in the maximal partition \( z_{\text{max}}(X, B) \) by the one-element set \( \{i\} \). More generally, every node \( b = \bigcup_{i_1, \ldots, i_k} c_{i_k} \) is replaced by the set \( \{i_1, \ldots, i_k\} \). By construction, the tree so obtained has the same number of nodes as \( B \) and has the same degree of splitting at each node. However, everything about the internal structure of the terminal elements \( c_i \) in the maximal partition of \( B \) has been stripped away, so that the tree now incorporates nothing more than the inherent structure which is shared by all trees in the category \([B]\). It is then befitting to call such a tree \( S \) a "skeleton" of \( B \), and hence a skeleton in the respective category. The defining criterion of a skeleton is the fact that all terminal elements are one-element sets, i.e., that \( S \) is complete:

**Definition 25.5 (Skeleton).** A tree \( S \in [B] \) which is complete is called a skeleton in the category \([B]\).

Thus, it is the skeletons which embody the inherent structure in the category \([B]\).

**Proposition 25.6 (Tree function on isomorphic trees).** Let \( B \) be a tree over \( X \). Let \( S \) be a skeleton in the category \([B]\). Then

\[
E_B = \sum_{c \in z_{\text{max}}(X, B)} n(c) \cdot e_S(i(c)) \quad ,
\]

where \( i : B \to S \) is the associated isomorphism.
Proof:
From eq. [25.2] in proposition 25.2 we know that \( e_B(c) = +S(i(c)) \) for all \( c \in \text{z}_{\text{max}}(X, B) \); if this is inserted into eq. [66] in theorem 13.2, (192) follows.

It follows that the tree function \( E \), when restricted to the category \([B]\), takes its minimum on the skeletons \( S \in [B] \), since for these, \( n(c) = 1 \) for all \( c \in \text{z}_{\text{max}}(X_S, S) \).

26 Restricted minimal problems

In the previous sections we have solved the problem of minimizing the tree function \( E \) on the set of all unconstrained complete trees over the set \( X \). By unconstrained we mean that no conditions on the possible trees \( B \) over \( X \) were imposed other than requiring that \( B \) must not be trivial. We now investigate how to extend the framework we have worked in so far in order to obtain tree functions that contain expressions like \( \sum p_i \log_2(p_i) \) in the functional form of their minimal value, when restricted to certain classes of tree structures over \( X \).

Amongst the countless ways to constrain the set of admissible trees we shall consider the following two cases only: For a given partition \( z \) of the base set \( X \) we first study the set of all trees preserving the partition \( z \); and then, the set of all trees containing \( z \).

26.1 Trees preserving a partition

A complete tree has a maximal partition of \( X \) which is complete, i.e., the elements of \( \text{z}_{\text{max}}(X) \) are comprised by the one-element subsets \( \{x\} \) for \( x \in X \). Trivially, every partition \( z \) of \( X \) preserves \( \text{z}_{\text{max}}(X) \) in the sense that \( \text{z}_{\text{max}} \) is a refinement of every partition \( z \) of \( X \). We now generalize this reasoning to the case where \( \text{z}_{\text{max}}(X) \) is no longer complete: We want to prescribe a partition \( z \) of \( X \) such that the relation \( z' \preceq z \) is true for all \( z' \in \zeta(X, B) \) compatible with \( B \). In particular, for the maximal partition of \( X \) in \( B \) we must have \( \text{z}_{\text{max}}(X) \preceq z \).

If such a relation is true we shall say that the tree \( B(X) \) preserves the partition \( z \). In general, the prescribed partition \( z \) that is preserved by the admissible trees \( B \) need not be an element of \( \zeta(X, B) \) itself; in this case it induces a non-trivial partition on at least one of the elements \( b \in \text{z}_{\text{max}}(X) \) which are terminal in \( B \), so that the resulting refinement \( z \) of \( \text{z}_{\text{max}}(X) \) is compatible with the resulting extension of \( B \). Alternatively, we can have \( z = \text{z}_{\text{max}}(X) \); in this case, \( z \) is the most refined partition compatible with the tree \( B \). These ideas lead to

Definition 26.2 (\( z \)-preserving, \( z \)-complete trees). Let \( B \in \mathcal{M}(X) \), let \( z \in \mathcal{Z}(X) \) be a partition of \( X \). \( B \) is called \( z \)-preserving if \( z' \preceq z \) for all \( z' \in \zeta(X, B) \). \( B \) is called \( z \)-complete if \( z_{\text{max}}(X, B) = z \).

It is clear that, without further conditions, it makes no sense to ask for the minimum of \( E \) on the set of all \( z \)-preserving trees over \( X \), as the answer is trivial: If \( z \) is given, the minimum is taken on the trivial tree \( B = \{X\} \), since the trivial tree preserves every partition. And even if this trivial solution is excluded, then the tree function \( E \) takes its minimum on any binary non-trivial partition of \( X \) which preserves \( z \); the associated value of the minimum can be inferred immediately from eq. (66) to be \( E = n \).

However, a meaningful minimal problem can be given on the smaller set of \( z \)-complete trees, which we shall denote by \( \mathcal{C}(z) \). The minimum of \( E \) on \( \mathcal{C}(z) \) will be denoted by \( \text{Min}_+(z) \). The subset of all trees in \( \mathcal{C}(z) \) on which \( E \) actually takes the minimum will be written as \( \text{Min}_+(z) \); it coincides with the set \( E^{-1}(\text{min}_+(z)) \cap \mathcal{C}(z) \).

In the present work we shall not attempt to solve this minimal problem; however, we provide a necessary condition which arises in the course of its study:

Proposition 26.3. Let \( z = \{c_1, \ldots, c_K\} \) be the common maximal partition in \( \mathcal{C}(z) \). Let \( B \in \text{Min}_+(z) \). Then,

\[
w_i < w_j \quad \Rightarrow \quad e_i \geq e_j ,
\]

where \( w_k \equiv n(c_k) \), and \( e_k \) are the path amounts of \( c_k \) in \( B \).
Proof:
Let $B \in \text{Min}_+(z)$ with path amounts $e_k$, $k = 1, \ldots, K$. From eq. (190) in theorem 16.2 we know that
\begin{equation}
E_B = \sum_{k=1}^{K} w_k \cdot e_k .
\end{equation}
Assume that there exist $i \neq j$ with $w_i < w_j$ such that $e_i < e_j$. We define a new tree $B'$ by the statements: (1) $B'$ is equivalent to $B$; and (2) the maximal partition $z_{\text{max}}(X, B') = \{c'_1, \ldots, c'_K\}$ of $X$ in $B'$ is such that
\begin{equation}
c'_k = \tau(i, j) c_k ,
\end{equation}
where $\tau(i, j)$ is the transposition of $i$ and $j$. The path amounts are the same by definition of equivalence,
\begin{equation}
e_{B'}(c_k) = e_B(c_k) = e_k ,
\end{equation}
hence the tree function on $B'$ takes the value
\begin{equation}
E_{B'} = \sum_{k \neq i, j} w_k \cdot e_k + w_j \cdot e_i + w_i \cdot e_j .
\end{equation}
It follows that
\begin{equation}
E_{B'} - E_B = (w_j - w_i)(e_i - e_j) < 0 ,
\end{equation}
and hence $E_{B'} < E_B$, which contradicts the minimal property of $B$. Thus, the initial assumption was wrong, and implication (193) must be true. $\blacksquare$

26.4 Trees containing a partition

Another construction is the set of trees containing the partition $z$: The idea is that we can constrain trees by requiring that all admissible trees contain the elements of a prescribed partition; this leads to the

Definition 26.5 (Trees containing a partition). Let $z \in Z(X)$. The tree $B$ is said to contain the partition $z$ if
\begin{align}
z \subset B(X) & \iff a \in B(X) \quad \text{for all } a \in z \\
\text{(199a)} & \quad \text{(199b)}
\end{align}
is true.

Without further conditions, the minimum of $E$ will always be taken on a tree for which the prescribed partition $z$ coincides with the maximal partition in this tree; for, any further splitting, beyond the nodes $a \in z$, can only increase the value of $E$. It follows that there are two possibilities for meaningful minimal problems: (1) We require that, for all admissible trees, the maximal partition $z_{\text{max}}(X, B)$ agrees with $z$; or, (2) we require that all admissible trees are complete. Clearly, case (1) agrees with the minimal problem on the set $C(z)$ of all $z$-complete trees as discussed in the last paragraph after definition 26.2; the minimum of $E$ is $\text{min}_+(z)$, and the set of all trees on which the minimum is taken is $\text{Min}_+(z)$. Case (2) defines another meaningful minimal problem which nevertheless can be traced back to case (1): Suppose that $B$ is an admissible tree with respect to case (2); then $B$ can be regarded as the completion of a reduced tree $B'$, where $B'$ is an element in the set $C(z)$ of $z$-complete trees. It is then clear that minimal trees with respect to case (2) are those for which the reduction $B'$ is minimal in $C(z)$, in other words, $B' \in \text{Min}_+(z)$, and for which the subtrees $B(b_i, X)$, $b_i \in z$, are optimal. The minimal value of the tree function $E$ in case (2) then will be a sum of $\text{min}_+(z)$ and another sum over expressions $w_i \lg(w_i) + 2[w_i - 2^{\lg(w_i)}]$, where $w_i = n(b_i)$, $b_i \in z$,
\begin{equation}
E_{\text{min}} = \text{min}_+(z) + \sum_{i=1}^{K} \left\{ w_i \cdot \lg(w_i) + 2 \cdot \left[w_i - 2^{\lg(w_i)}\right] \right\} .
\end{equation}
Here we have again assumed that $\#z \equiv K$. 
26.6 Trees with a prescribed minimal partition $z_{\text{max}}(X)$

Another minimal problem can be obtained by prescribing the minimal partition $z_{\text{min}}(X) = z$ of $X$ and demanding that all trees in this class be complete. We shall denote the associated class of trees by $C_{\text{c}}(z)$. The minimum of $E$ taken in $C_{\text{c}}(z)$ will be written as min$_{\text{c}}(z)$, while the subset of $C_{\text{c}}(z)$ on which this minimum is actually taken will be denoted by Min$_{\text{c}}(z)$; the latter coincides with the intersection $E^{-1}\{\text{min}_{\text{c}}(z)\} \cap C_{\text{c}}(z)$.

The solution to this minimal problem is readily found: Let us suppose that the minimal partition of $X$ is prescribed to be

\[
\begin{align*}
    z_{\text{min}}(X) &= z = \{c_1, \ldots, c_K\} , \\
    w_i &= n(c_i) , \quad i = 1, \ldots, K .
\end{align*}
\]

Since all admissible trees are complete, by eq. (67) in theorem 13.2 the tree function $E$ on $M_{\text{c}}(z)$ coincides with the total amount function $G$. Eq. (79) in corollary 12.4 then implies that

\[E_B = n(K - 1) + \sum_{i=1}^{K} G_{B(c_i, X)} ,\]

for all $B \in M_{\text{c}}(z)$. It follows that $E_B$ becomes minimal if all subtrees $B(c_i, X)$ become optimal; in other words, if

\[G_{B(c_i, X)} = G(w_i) \quad \text{for all } i = 1, \ldots, K .\]

But the values of the quantities $G(w_i)$ are given in eq. (113) of theorem 21.1

\[G(w_i) = w_i \cdot \lg(w_i) + 2 \cdot \left[ w_i - 2^{\lg(w_i)} \right] .\]

On inserting (203) into (202) we have proven:

**Theorem 26.7 (Minimal problem on $C_{\text{c}}(z)$).** The minimum of $E$ on $C_{\text{c}}(z)$ is equal to

\[
\begin{align*}
    \text{min}_{\text{c}}(z) &= n(K - 1) + \sum_{i=1}^{K} \left\{ w_i \cdot \lg(w_i) + 2 \cdot \left[ w_i - 2^{\lg(w_i)} \right] \right\} .
\end{align*}
\]

The set Min$_{\text{c}}(z)$ contains those trees which are sums of optimal trees over the quantities $w_i$,

\[B = \sum_{i=1}^{K} B_{\alpha(c_i)} \Rightarrow B \in \text{Min}_{\text{c}}(z) .\]

We can rewrite the result (205) in such a way that probability-like quantities $\frac{w_i}{n} \in \mathbb{R}$ appear:

\[
\frac{1}{n} \text{min}_{\text{c}}(z) - \lg(n) = (K - 1) +
\]

\[+ \sum_{i=1}^{K} \frac{w_i}{n} \left[ \lg(w_i) - \lg(n) \right] + \frac{1}{n} \sum_{i=1}^{K} 2 \cdot \left[ w_i - 2^{\lg(w_i)} \right] .\]

The quantity $\left[ \lg(w_i) - \lg(n) \right]$ is evidently an approximation to $\log_2(\frac{w_i}{n})$, so that the right-hand side of (207) contains an integer approximation to the Shannon-Wiener entropy with respect to the ”probabilities” $p_i = \frac{w_i}{n}$, where $i = 1, \ldots, K$.

27 Tree structures and neighbourhood topology

Finally, we want to put forward arguments to show how tree structures define a neighbourhood topology on the underlying set $X$. We now allow the set $X$ to have arbitrary cardinality; in particular, $X$ can be non-countable. We recall that the path $q(b)$ of a node $b \in B(X)$ was defined to be the set of all elements $b'$ in $B$ containing $b$ as a subset. We now extend this definition so as to speak of the path of any single element $x \in X$ in the base set: For every $x \in X$ there exists precisely one terminal element $b_x \in B$ such that $x \in b$; we can then decree that the path of the element $x$ in the tree $B$ be the path of the associated terminal element, and this assignment will be unique. Thus,
Definition 27.1 (Path of points in base set X). Let $B \in M(X)$ be a given tree over the base set $X$. Let $x \in X$, let $b_x$ be the uniquely determined terminal node in $B$ which contains $x$ as an element. Then the path of $x$ in $B$ is defined by

$$q(x) \equiv q(\{b_x\}) \quad .$$

(208)

If the degree of splitting $m(b)$ remains finite at every node $b \in B$, the path of $x$ will be a countable subset of the tree $B$.

Proposition 27.2 (The path of points). Let $B$ be a given tree over $X$. Then

$$b \in q(x) \iff x \in b \quad .$$

(209)

Proof:

Let $x \in X$ and assume that $b \in q(x)$. There exists a unique $b' \in z_{\text{max}}(X,B)$ such that $x \in b'$. Thus, $q(b') = q(x)$, and therefore $b \supset b' \ni x$, which proves the implication from left-to-right.

Conversely, let $b'$ be the unique terminal element in $z_{\text{max}}(X,B)$ such that $x \in b'$. Then $x \in b$ implies that $b \cap b' \neq \emptyset$. Now, axiom (A2) in section 4 implies that either $b \subseteq b'$ or $b' \subseteq b$. The first inclusion cannot be true since $b'$ is a terminal element; thus, $b' \subseteq b$, hence it follows that $b \in q(b') = q(x)$.

We now show that the given tree structure $B$ over $X$ defines a neighbourhood topology on $X$. We recall that a neighbourhood topology $N$ assigns a collection $N(x)$ of distinct subsets $N$ of $X$ to every point $x$ in $X$; $N$ is just the collection of all $N(x)$. The elements $N \in N(x)$, which are subsets of $X$, are called

\begin{itemize}
  \item \textit{neighbourhoods of $x$ in the topology $N$,} if they satisfy the axioms \cite{24}
\end{itemize}

(N1) If $N$ is a neighbourhood of $x$, then $x \in N$.

(N2) If $N$ is a subset of $X$ containing a neighbourhood of $x$, then $N$ is a neighbourhood of $x$.

(N3) The intersection of two neighbourhoods of $x$ is again a neighbourhood of $x$.

(N4) Any neighbourhood $N$ of $x$ contains a neighbourhood $M$ of $x$ such that $N$ is a neighbourhood of each point of $M$.

The pair $(X,N)$ is then called a \textit{topological space}. Furthermore, a \textit{base} for the neighbourhoods at $x$ is a set $\text{Basis}(x)$ of neighbourhoods of $x$ such that every neighbourhood $N$ of $x$ contains an element $b \in \text{Basis}(x)$. Now we define the path $q(x)$ to be a neighbourhood base for $x$, and a subset $N \subset X$ to be a neighbourhood of $x$ if and only if there exists a $b \in q(x)$ that is contained in $N$. The result is indeed a neighbourhood topology on $X$:

Theorem 27.3 (Trees and neighbourhood topology). Every tree structure $B(X)$ over $X$ defines a neighbourhood topology on $X$.

Proof:

If $N$ is a neighbourhood of $x$ in our sense then it contains an element $b \in q(x)$ and therefore contains $x$ as an element, even if the path $q(x)$, or the tree $B$, does not contain $\{x\}$ as an element; thus, (N1) is satisfied. (N2) is fulfilled automatically by our definition. Let $N$ and $N'$ be two neighbourhoods of $x$; then they both contain elements $b$ and $b'$ of the same path $q(x)$, and hence at least one of the relations $b \supset b'$ or $b \supset b'$ is satisfied. We can assume without loss of generality that the latter is the case; then the intersection of $N$ and $N'$ contains $b'$ and hence is a neighbourhood of $x$, thus (N3) is satisfied. Finally, let $N$ be a neighbourhood of $x$; then $N$ contains some $b \in q(x)$, which itself is a neighbourhood of $x$. Then, for every $y \in b$, $b$ lies in the path of $y$, as follows from proposition 27.2. Hence $b$ is a neighbourhood of $y$. Consequently, $N$ is a neighbourhood for each $y \in b$. This shows that (N4) is satisfied.
It is clear that the set of all possible tree structures over the given set $X$ may be constrained in many different ways, for example, by imposing the conditions discussed in section 26.1. On each constrained set of trees, the tree function $E$ will take a minimum, which is an entropy-like quantity, and will single out those trees on the constrained set on which the minimum is actually taken. The associated trees then define preferred topologies on the underlying set by means of the construction given above. We see that this looks distinctly like an action principle for topologies on the set $X$, the role of the action being played by the tree function, the degrees of freedom being expressed by the different trees over $X$, and the minimal value of the action=tree function $E$ being associated with an entropy-like quantity.

28 Summary

We have presented a comprehensive account of a new mathematical structure, called tree structure, which arises in the formalisation of the operational aspects of information gaining. It was shown that a given set of tree structures can be endowed with a tree function whose value is related to the maximal number of yes-no questions which are necessary to identify a given node in the tree. The question of minimality of the tree function on these sets of trees can be posed. It was shown that, on unconstrained trees, the minimal value of the tree function is related to the dyadic logarithm of the number of elements in the base set; whilst, on constrained sets of trees, the tree function takes minima whose functional form is similar to the Shannon-Wiener information, or entropy, of a probability distribution. We have presented three natural axioms governing tree structures. It was subsequently demonstrated that these axioms can be related to the axioms describing neighbourhood topologies on a given set. As a consequence, every tree structure defines a neighbourhood topology on a set. The minimisation of a tree function on a set of tree structures over a base set then opens up the possibility to obtain preferred neighbourhood topologies, namely those which are related to minimal trees over the given base set. This phenomenon has the distinct flavour of an action principle, distinguishing certain preferred neighbourhood topologies by means of a minimal principle.

Acknowledgements

Hanno Hammer acknowledges support from EPSRC grant No. GR/86300/01.

References

[1] G. Weihs, T. Jennewein, C. Simon, H. Weinfurter, and A. Zeilinger, Phys. Rev. Lett. 81, 5039 (1998).
[2] O. Nairz, B. Brezger, M. Arndt, and A. Zeilinger, Phys. Rev. Lett. 87, 160401 (2001).
[3] M. Arndt, O. Nairz, J. Vos-Andreae, C. Keller, G. van der Zouw, and A. Zeilinger, Nature 401, 680 (1999).
[4] J. S. Bell, Speakable and unspeakable in quantum mechanics. Cambridge University Press, Cambridge (1988).
[5] A. Peres, Found. Phys. 29, 589 (1999).
[6] A. Peres, Fortsch. Phys. 48, 531 (2000).
[7] W. H. Zurek, Phys. Rev. D 24, 1516 (1981).
[8] W. H. Zurek, Phys. Rev. D 26, 1862 (1982).
[9] W. H. Zurek, Prog. Theor. Phys. 89, 281 (1993).
[10] W. H. Zurek, Phys. Today 44, 36 (1991).
[11] E. Joos, H. Zeh, C. Kiefer, D. Giulini, J. Kupsch, and I.-O. Stamatescu, *Decoherence and the Appearance of a Classical World in Quantum Theory*. Springer, Berlin and Heidelberg (2003).

[12] M. Horodecki, P. Horodecki, and R. Horodecki, *Phys. Rev. Lett.* **84**, 2014 (2000). quant-ph/9908065.

[13] M. J. Donald, M. Horodecki, and O. Rudolph (2002). quant-ph/0105017.

[14] M. A. Nielsen, Ph.D. dissertation, The University of New Mexico (1998). quant-ph/0011036.

[15] C. E. Shannon, *Bell System Tech. J.* **27**, 379 (1948).

[16] N. Wiener, *Cybernetics or Control and Communication in the Animal and the Machine*. Wiley, New York (1948).

[17] C. E. Shannon, *Bell System Tech. J.* **28**, 656 (1949).

[18] C. E. Shannon and W. Weaver, *The Mathematical Theory of Communication*. University of Illinois Press, Urbana (1949).

[19] S. Popescu and D. Rohrlich, *Phys. Rev. A* **56**, R3319 (1997). quant-ph/9610044.

[20] G. Vidal, *J. Mod. Opt.* **47**, 355 (2000). quant-ph/9807077.

[21] J. von Neumann, *Gött. Nachr.*, 273 (1927).

[22] R. Baierlein, *Atoms and Information Theory*. Freeman and Company (1971).

[23] K. J. Falconer, *The Geometry of Fractal Sets*. Cambridge University Press, Cambridge (1985).

[24] R. Brown, *Topology*. Ellis Harwood Limited (1988).

[25] H. Heuser, *Lehrbuch der Analysis, Teil 2*. Teubner, Stuttgart (1988).

[26] H. Hammer, hep-th/9811118.