

A Mean Field Game Approach to Scheduling in Cellular Systems

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Abstract—We study auction-theoretic scheduling in cellular networks as a means to enable such value declarations. Our analysis is based on using the idea of mean field equilibrium (MFE). Here, agents model their opponents through a distribution over their action spaces and play the best response. The system is at an MFE if this action is itself a sample drawn from the assumed distribution. In our setting, the agents are smart phone apps that generate service requests, experience waiting costs, and bid for service from base stations. We show that if we conduct a second-price auction at each base station, there exists an MFE that would schedule the app with the (weighted) longest queue at each time. The result suggests that auctions can attain the same desirable results as queue-length-based scheduling with full information. We present results on the asymptotic convergence of a system with a finite number of agents to the mean field case, and conclude with simulation results illustrating the ease of computation of the MFE.

I. INTRODUCTION

There has recently been a rapid increase in the use of smart hand-held devices for Internet access. These devices are supported by cellular data networks, which carry the packets generated by apps running on these smart devices. These apps can be modeled as queues that arrive when the user starts the app, and depart when the user terminates that app. Packets generated by an app are buffered in a queue corresponding to that app. Queueing delays impact the quality of user experience (QoE) based on the app being used. Users move around cells that each has a base station, and scheduling a particular user provides service to the queue that represents his/her currently running app. User interest could shift from app to app, regardless of whether or not there are buffered packets. Hence, a queue might terminate and be replaced by a new one even if there are jobs waiting for service.

An important problem in cellular data networks is that for scheduling, i.e., determining which queues receive service at each time instant. Most work on scheduling has focused on the case of a finite number of infinitely long lived flows, with the objective being to maximize the total throughput. A seminal piece of work under this paradigm introduced the so-called max-weight algorithm [1]. Here, the drift of a quadratic Lyapunov function is minimized by maximizing the queue-length weighted sum of acceptable schedules. Later work (e.g., [2], [3], [4]) has used a similar approach in a variety of network scenarios. If queues arrive and depart, then a natural scheduling policy in the single server case is a Longest-Queue-First (LQF) scheme, in which each server serves the longest of the queues requesting service from it. LQF has many attractive properties, such as minimizing the expected value of the longest queue in the system.

The above approach assumes that the queue length values are given to the scheduler, and that it is aware of the function that maps the queue-length to cost on QoE of queuing. However, while the downlink queue lengths would naturally be available at a cellular base station, the only way to obtain uplink queue information is to ask the users themselves. However, a larger value of queue length results in a higher probability of being scheduled under all the above policies, implying an incentive to lie.

An appealing idea is to use a pricing scheme to inform scheduling decisions for cellular data access. These prices could be in the form of tokens issued by the cellular service provider that are used as currency in the service market. An example of a pricing approach is presented in [5], which describes an experimental trial of a system in which day-ahead prices are announced to users, who then decide on whether or not to use their 3G service based on the price at that time. However, these prices have to be determined through trial and error. Can we determine prices by using an auction?

Our key objective is to design an incentive compatible scheduling scheme that behaves in a (weighted) LQF-like fashion. We consider a system in which each app bids for service from the base station that it is currently connected to. The auction is conducted in a second-price fashion, with the winner being the app that bids highest, and the charge being the value of the second highest bid. It is well known that such an auction promotes truth-telling [6]. Would the scheduling decisions resulting from such auctions resemble that of LQF? Would conducting such an auction repeatedly over time with queues arriving and departing result in some form of equilibrium?

Mean Field Games

We investigate the existence of an equilibrium using the theory of Mean Field Games (MFG) [7], [8], [9], [10], [11], [12], [13]. In MFG, the players assume that each opponent would play an action drawn independently from a fixed distribution over its action space. The player then chooses an action that is the best response against actions drawn in this fashion. We say that the system is at Mean Field Equilibrium (MFE) if this best response action is itself a sample drawn from the assumed distribution.

The MFG framework offers a structure to approximate so-called Perfect Bayesian Equilibrium (PBE) in dynamic games.
PBE requires each player to keep track of their beliefs on the future plays of every other opponent in the system, and play the best response to that belief. This makes the computation of PBE intractable when the number of players is large. The MFG framework simplifies computation of the best response, and often turns out to be asymptotically optimal.

Work on MFGs has mostly focused on showing the existence, accuracy and stability of MFE \cite{1, 2, 3, 4, 5}. In the space of queueing systems, some recent work considers the game of sampling a number of queues and joining one \cite{6}. However, ours is a scheduling problem in queueing system interacting with an auction system, which we believe is unique.

In the space of applications, Iyer et al. \cite{7} study advertisers competing via a second price auction for spots on a webpage. The bid must lie in a finite real interval, and the winner can place an ad on the webpage. With time, the advertisers learn about the value of winning (probability of making a sale). Li et al. \cite{8} consider the problem of mechanism design for truthful state revelation (number of packets at each station) in a wireless D2D streaming system. Their main result is to generalize the Groves mechanism to the mean field regime. Both use some version of fixed point theorem to show existence of the MFE.

Neither of the above consider the use of auctions for service scheduling in queueing systems, which is the basis of our problem. In our setup, the state is the queue with arrivals and departures, and we allow bids to lie in the full positive real line. These considerations result in significant technical work to show existence and characterize the MFE. In an earlier version of this work \cite{9}, we presented a preliminary version of our work without proofs. The current work highlights the methodological contributions.

Overview of Paper

We introduce the Mean Field Game (MFG) in Section II. Here, a selected agent (app) has a belief $\rho$ about the bid distribution of the other agents, and assumes that their bids will be drawn independently from this distribution. The state of the agent is its current queue length, and it faces a per-time step cost that is a function of its queue length, which models the impact on QoE of queueing delay. The agent must place a bid based on the belief and its current state and belief about other agents.

We consider the problem of determining the cost minimizing bid function and the corresponding value function as a Markov Decision Process in Section III. We show that the Bellman operator corresponding to the MDP is a contraction mapping with a unique minimum, implying that value iteration would converge to the best response bid. Further, we show that the best response bid is monotone increasing in queue length. We call the bid distribution across agents that results from playing the best responses as $\gamma$.

We next prove the existence of the MFE in Sections IV- V by verifying the conditions of the Schauder Fixed Point Theorem. We need to show that the mapping between the assumed bid distribution $\rho$ to the resultant bid distribution $\gamma$ is continuous, and that the space in which $\gamma$ lies is contained in a compact subset of the space from which $\rho$ is drawn. In order to do this, we first show that the mapping between $\rho$ and best response bid function is continuous, and then show that the map between $\rho$ and the state (queue length) distribution is continuous. Putting these together yields the required continuity conditions. We then verify the conditions of the Arzelà-Ascoli Theorem for showing compactness.

We show in Section VI that the MFE in our case is an asymptotically accurate approximation of a PBE. The result follows from the fact that any finite subset of agents is unlikely to have interacted with any of the others as the number of agents becomes large. Finally, we present simulation results in Section VII showing that MFE computation is straightforward.

Discussion: In the case of a single cost function (homogeneous QoE for all apps) the best response bid function is monotone increasing in the queue length regardless of $\rho$. This implies that the service regime corresponding to MFE is identical to LQF (or a weighted version if we have different QoE classes for apps). Further, our simulations suggest that if the base stations were to compute the empirical bid distribution and return it to the users, the eventual bid distribution would be the MFE. Thus, the desirable properties of LQF are a natural result of auction-based scheduling, while the queue length distribution would be that generated by LQF.

II. MEAN FIELD GAME MODEL

We consider a system consisting of $N$ cells and $NM$ agents (apps), which are randomly permuted in these cells at each time instant, with each cell having exactly $M$ agents\footnote{Note that our results are essentially unchanged if $M$ is an independent random variable with finite support.}. The model is consistent on the likely evolution of the 5G cellular system, wherein we expect a large number of small, dense cells and much user mobility across different cells. The mobility model is identical to the basic framework used in work on mobile wireless networks \cite{10}. Each cell contains a base station, which conducts a second price auction to choose which agent to serve. Each agent must choose its bid in response to its state and its belief over the bids of its competitors.

Figure \cite{11} illustrates the MFG approximation, which is accurate in the limit as $N$ becomes large. An MFG is described from the perspective of a single candidate agent $i$, which assumes that the actions of all its competitors are drawn independently from some distribution. The asymptotic accuracy of the independence assumption follows from a standard argument on the propagation of chaos whose details are provided in Section VII. In Figure \cite{11} the auction (shown as blue/dark tiles) and the queue dynamics (shown as beige/light tiles). Since we focus on a single agent, we do not need to specify its identity explicitly, unless we wish to compare its actions with those of other agents. Hence, we will drop the index $i$ where possible for ease of notation.

Auction System: At each time step $k$, the agent of interest competes in a second price auction against $M-1$ other agents, whose bids are assumed to be independently drawn from a continuous, finite mean (cumulative) bid distribution $\rho$, with
Assumption 1. At each time $k$, the arrivals $\{A_k\}$ are i.i.d random variables distributed according to $\Phi$. We assume that $A_k \in [0, (A)]$, where $A$ is finite. Also, these random variables have a bounded density function, $\phi$ (i.e., $||\phi|| < c_\phi$, where $||.||$ is the sup norm).

Assumption 2. The regeneration values $\{R_k\}$ are i.i.d random variables distributed according to $\Psi$, and they have a bounded density $\psi$ (i.e., $||\psi|| < c_\psi$, where $||.||$ is the sup norm).

Assumption 3. The holding cost function $C : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is continuous, increasing and strictly convex. We also assume that $C$ is $O(q^m)$ for some integer $m$.

The polynomial form above is for technical reasons, but is not very restrictive since many convex functions can be approximated quite well.

Mean Field Equilibrium: The probability that the agent’s bid lies in the interval $[0, x]$ is equal to the probability that the agent’s queue length lies in some set whose best response is to bid between $[0, x]$. Thus, the probability of the bid lying in the interval $[0, x]$ is $\Pi_p(\theta^{-1}([0, x]))$, which we define as $\gamma(x)$. According to the assumed (cumulative) bid distribution, the probability of the same event is $\rho(x)$. If $\rho(x) = \gamma(x)$, it means that the assumed bid distribution is consistent with the best response bid distribution, and we have an MFE.

A. Agent’s decision problem

Let the candidate be agent $i$. Suppose that the belief over the bid of a random agent has cumulative distribution $\rho$. We assume that $\rho \in \mathcal{P}$ where,

$$\mathcal{P} = \{G|G \text{ is a continuous c.d.f, } \int (1 - G(x))dx < E\},$$

for some $E < \infty$, to be defined later. Besides, its current state, the information available with the agent about the market at any time prior to the auction only includes the following:

1) The bids it made in each previous auction from last regeneration.
2) The auctions that it won.
3) The payments made for the auctions won.

Let $H_{i,k}$ be the history vector containing the above information available to agent $i$ at time $k$. Suppose that the random variable representing the bid made by agent $i$ at time $k$ is denoted by $X_{i,k}$, with the random job size being denoted by $A$. Finally, the app can terminate at any time instant with probability $1 - \beta$. Based on these inputs, the agent needs to determine the value of its current state $V_\rho(q)$, and the best response bid to make $x = \theta_p(q)$. The assumption that only a single unit of service is provided at each base station is for simplicity of notation, and our results are unchanged if we are allowed to choose some $M < M$ agents as winners in each auction. The mechanism followed would be a $\bar{M} + 1^{th}$ price auction in that case.

Queueing System: The queueing dynamics are driven by the arrival process $\Phi$ and the probability of obtaining a unit of service being $p_\rho(x)$ as described above. When the user terminates an app, he/she immediately starts a fresh app, i.e., a new queue takes the place of a departing queue. The initial condition of this new queue is drawn from a regeneration distribution $\Psi$, whose support is $\mathbb{R}^+$. The invariant distribution associated with this queueing system (if it exists) is denoted $\Pi_p$.

We make the following assumptions.

Assumption 2. The regeneration values $\{R_k\}$ are i.i.d random variables distributed according to $\Psi$, and they have a bounded density $\psi$ (i.e., $||\psi|| < c_\psi$, where $||.||$ is the sup norm).

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1) The bids it made in each previous auction from last regeneration.
2) The auctions that it won.
3) The payments made for the auctions won.

Let $H_{i,k}$ be the history vector containing the above information available to agent $i$ at time $k$. Suppose that the random variable representing the bid made by agent $i$ at time $k$ is denoted by $X_{i,k}$, with the realized value being $x_{i,k}$. Also, let $\hat{X}_{i,k} = \max_{j \in M_{i,k}} X_{j,k}$, represent the maximum value of $M-1$ draws from the distribution $\rho$. Thus, $\hat{X}_{i,k}$ is the value of the highest opposing bid. The agent’s decision problem is to choose a bid function $\theta_i$, which maps its available information to a bid at each time $x_{i,k}$.

Since the time of regeneration $T_{i}^k$ is a geometric random variable, the expected cost of agent $i$ can be written as

$$V_{i,\rho}(H_{i,k}; \theta_i) = E \left[ \sum_{t=k}^{\infty} \beta^t [C(Q_{i,t}) + r_\rho(X_{i,t})] \right],$$

where the expectation is over future state evolutions. By replacing the belief with $\rho$, we have made an agent’s decision problem independent of other agents’ strategies. Hence, we represent the cost by $V_{i,\rho}(H_{i,k}; \theta_i)$. Also, $r_\rho(x) = E[\hat{X}_{i,k} 1(\hat{X}_{i,k} \leq x)]$ is the expected payment when $i$ bids $x$ under the assumption that the bids of other agents are distributed according to $\rho$. Hence, given $\rho$, the win probability in the auction is

$$p_\rho(x) = \mathbb{P}(\hat{X}_{i,k} \leq x) = \rho(x)^{M-1}.$$
The expected payment when bidding $x$ is
\[ r_\rho(x) = \mathbb{E}[X_{i,k} \mathbb{1}\{X_{i,k} \leq x\}] = xp_\rho(x) - \int_0^x p_\rho(u)du. \tag{3} \]

The state process $Q_{i,k}$ is Markov and has a transition kernel
\[ \mathbb{P}(Q_{i,k+1} \in B|Q_{i,k} = q, X_{i,k} = x) = \beta p_\rho(x)\mathbb{P}((q-1)^++A_k \in B) + \beta(1 - p_\rho(x))\mathbb{P}(q + A_k \in B) + (1 - \beta)\Psi(B), \tag{4} \]
where $B \subseteq \mathbb{R}^+$ is a Borel set and $x^+ \triangleq \max(x, 0)$. Recall that $A_k \sim \Phi$ is the arrival between $(k)^{th}$ and $(k+1)^{th}$ auction and $\Psi$ is density function of the regeneration process. In the above expression, the first term corresponds to the event that agent wins the auction at time $k$, while the second corresponds to the event that it does not. The last term captures the event that the agent regenerates after auction $k$. The agent’s decision problem can be modeled as an infinite horizon discounted cost MDP. It can be shown that there exists an optimal Markov deterministic policy for our MDP [16]. Then, from (1), the optimal value function of the agent is
\[ \hat{V}_{i,\rho}(q) = \inf_{\theta \in \Theta} \mathbb{E} \left[ \sum_{t=1}^{\infty} \beta^t [C(Q_{i,t}) + r_\rho(X_{i,t})] \middle| Q_{i,0} = q \right], \tag{5} \]
where $\Theta$ is the space of Markov deterministic policies.

Note that user index is redundant in the above expression as we are concerned with a single agent’s decision problem. In future notations, we will omit the user subscript $i$.

B. Invariant distribution

Given cumulative bid distribution $\rho$ and a Markov policy $\theta \in \Theta$, the transition kernel given by (4) can be re-written as,
\[ \mathbb{P}(Q_{k+1} \in B|Q_k = q) = \beta p_\rho(\theta(q))\mathbb{P}((q-1)^++A_k \in B) + \beta(1 - p_\rho(\theta(q)))\mathbb{P}(q + A_k \in B) + (1 - \beta)\Psi(B). \tag{6} \]

Then, we have an important result in the following lemma:

**Lemma 1.** The Markov chain described by the transition probabilities in (6) is positive Harris recurrent and has a unique invariant distribution.

**Proof:** From (6) we note that, $\mathbb{P}(Q_{k+1} \in B|Q_k = q) \geq (1 - \beta)\Psi(B)$, where $0 < \beta < 1$ and $\Psi$ is a probability measure. The result then follows from results in Chapter 12, Meyn and Tweedie [17].

We denote the unique invariant distribution by $\Pi_{\rho,\theta}$.

C. Mean field equilibrium

As described in the Introduction, the mean field equilibrium requires the consistency check that the bid distribution $\gamma$ induced by the invariant distribution $\Pi_{\rho,\theta_\rho}$ should be equal to the bid distribution conjectured by the agent, i.e., $\rho$. Thus, we have the following definition of MFE:

**Definition 1 (Mean field equilibrium).** Let $\rho$ be a bid distribution and $\theta_\rho$ be a stationary policy for an agent. Then, we say that $(\rho, \theta_\rho)$ constitutes a mean field equilibrium if
1. $\theta_\rho$ is an optimal policy of the decision problem in (5), given bid distribution $\rho$; and
2. $\rho(x) = \gamma(x) \triangleq \Pi_{\rho,\theta_\rho}^{-1}([0, x]), \forall x \in \mathbb{R}^+$, where $\Pi_{\rho} = \Pi_{\rho,\theta_\rho}$.

Note that the game theoretic definition of the MFE considers the existence of an invariant distribution at a fixed time as the number of agents becomes asymptotically large. In keeping with extending the ideas of a Bayesian Nash Equilibrium to the system with a large number of agents, the definition does not require the occupancy distribution to converge to the invariant distribution from an arbitrary initial condition as time becomes large [12], [13], [9].

A main result of this work is showing the existence of an MFE.

### III. Properties of Optimal Bid Function

The decision problem given by (3) is an infinite horizon, discounted Markov decision problem. The optimality equation or Bellman equation corresponding to the decision problem is
\[ \hat{V}_\rho(q) = C(q) + \beta\mathbb{E}_{A}(\hat{V}_\rho(q + A)) + \inf_{x \in \mathbb{R}^+} \left[ r_\rho(x) - p_\rho(x)\beta\mathbb{E}_A \left( \hat{V}_\rho(q + A) - \hat{V}_\rho((q - 1)^++A) \right) \right], \tag{7} \]
where $A \sim \Phi$, and we use the notation $\max(0, z) = z^+$. Note that the decision problem above is independent of the regeneration distribution $\Psi$, since the game simply ends at any time with probability $1 - \beta$ from the agent’s perspective. However, from a system perspective, the Markov chain describing the state transition is correctly represented by (6).

Define the set of functions
\[ \mathcal{V} = \left\{ f : \mathbb{R}^+ \rightarrow \mathbb{R}^+ : \sup_{q \in \mathbb{R}^+} \frac{f(q)}{w(q)} < \infty \right\}, \tag{8} \]
where $w(q) = \max\{C(q), 1\}$. Note that $\mathcal{V}$ is a Banach space with $w$-norm,
\[ \|f\|_w = \sup_{q \in \mathbb{R}^+} \frac{f(q)}{w(q)} < \infty. \]

Also, define the operator $T_\rho$ as
\[ (T_\rho f)(q) = C(q) + \beta\mathbb{E}_A f(q + A) + \inf_{x \in \mathbb{R}^+} \left[ r_\rho(x) - p_\rho(x)\beta(\mathbb{E}_A(f(q + A) - f((q - 1)^++A))) \right], \tag{9} \]
where $f \in \mathcal{V}$. It is straightforward to show that the infimum in the above operator occurs at
\[ \beta \Delta f(q)^+; \tag{10} \]
where $\Delta f(q) = E_A(f(q + A) - f((q - 1)^+ + A))$. Then, substituting from (2), (3) and (10), (9) can be rewritten as

$$\langle T_\rho f \rangle (q) = C(q) + \beta E_A f(q + A) - \int_0^{\beta A f(q_n)} p_\rho(u) du.$$  \hspace{1cm} (11)

The following lemma characterizes the optimal solution.

**Lemma 2.** Given a cumulative bid distribution $\rho$, 

1. There exists a $j \in \mathbb{N}$ such that $T_\rho^j : \mathcal{V} \to \mathcal{V}$ is a contraction mapping. Hence, there exists a unique $f^* \in \mathcal{V}$ such that $T_\rho f^* = f^*$, and for any $f \in \mathcal{V}$, $T_\rho^n f \to f^*$ as $n \to \infty$.
2. The fixed point $f^*$ of operator $T_\rho$ is the unique solution to the optimality equation (7), i.e., $f^* = \hat{V}_\rho$.
3. Let, $\hat{\theta}_\rho(q) = \beta \Delta \hat{V}_\rho(q)^+$. Then, $\hat{\theta}_\rho$ is an optimal policy.

**Proof:** The proof is similar to Theorem 8.3.6 in [19]. An exception to be noted here is that the action space in our case is not a compact set which violates Assumption 8.3.1(a) in [19]. However, this assumption can be overridden if the statement of Lemma 8.3.8(a) in [19] holds true. This applies to our case since, as derived in [10], a minimizer exists for the infimum operator in (9) for every $q$. Further, Theorem 8.3.6 is specified with $j = 1$ (a one-step contraction). Hence, we replace Assumption 8.3.2(b) with an appropriate condition to obtain a $j$-step contraction. Please refer to Appendix A for details.

**Corollary 3.** An optimal policy of the agent’s decision problem (5) is given by

$$\hat{\theta}_\rho(q) = \beta E_A \left[ \hat{V}_\rho(q + A) - \hat{V}_\rho((q - 1)^+ + A) \right].$$

We now establish that $\hat{V}_\rho$ and $\hat{\theta}_\rho$ are continuous and increasing functions.

**Lemma 4.** Given a cumulative bid distribution function $\rho$

1. $\hat{V}_\rho$ is a continuous increasing function.
2. $\hat{\theta}_\rho$ is a continuous strictly increasing function.

**Proof:** Let $f \in \mathcal{V}$. Suppose $f$ is a continuous monotone increasing function. We first prove that $T_\rho f$ is also a continuous monotone increasing function. Since $T_\rho^n f \to \hat{V}_\rho$ according to Statement 2 of Lemma [2] we conclude that $\hat{V}_\rho$ also has the same property.

Let $q > q'$. Then,

$$T_\rho f(q) - T_\rho f(q') = C(q) - C(q') + \beta E_A (f(q + A) - f(q' + A)) + \inf_x [p_\rho(x) - \beta p_\rho \rho(x) E_A (f(q + A) - f((q - 1)^+ + A))]$$

$$- \inf_x [p_\rho(x) - \beta p_\rho \rho(x) E_A (f(q' + A) - f((q - 1)^+ + A))] \geq \beta E_A (f(q + A) - f(q' + A)) + \beta \min_x [E_A (f(q + A) - f(q' + A))] \geq 0,$$

where (a) follows from the assumption that $C(.)$ is an increasing function, and (b) follows from the assumption that $f(.)$ is an increasing function.

To prove that $T_\rho f$ is continuous consider a sequence $\{q_n\}$ such that $q_n \to q$. Since $f$ is a continuous function, $f(q_n + A) \to f(q + A)$ in $\mathcal{V}$.

Further, Theorem 8.3.6 holds true. This applies to our case since, as derived in [10], a minimizer exists for the infimum operator in (9) for every $q$. Further, Theorem 8.3.6 is specified with $j = 1$ (a one-step contraction). Hence, we replace Assumption 8.3.2(b) with an appropriate condition to obtain a $j$-step contraction. Please refer to Appendix A for details.

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To prove that $T_\rho f$ is continuous consider a sequence $\{q_n\}$ such that $q_n \to q$. Since $f$ is a continuous function, $f(q_n + A) \to f(q + A)$. Then, by using dominated convergence theorem, we have $E_A f(q_n + A) \to E_A f(q + A)$ and $E_A f((q_n - 1)^+ + A) \to E_A f((q - 1)^+ + A)$. Also, $\Delta f(q_n) \geq 0$ as $f$ is an increasing function. Then, from (11), we get that

$$T_\rho f(q_n) = C(q_n) + \beta E_A f(q_n + A) - \int_0^{\beta A f(q_n)} p_\rho(u) du$$

$$= C(q) + \beta E_A f(q + A) - \int_0^{\beta A f(q)} p_\rho(u) du = T_\rho f(q).$$

Hence, $T_\rho f$ is a continuous function. This yields Statement 1 in the lemma.

Now, to prove the second part, assume that $\Delta f$ is an increasing function. First, we show that $\Delta T_\rho f$ is an increasing function. Let $q > q'$. From (11), for any $a < A$ we write

$$T_\rho f(q) - T_\rho f(q') = C(q) - C(q') + \beta E_A f(q + A) - \beta E_A f(q' + A) + \beta \min_x [E_A (f(q + A) - f(q' + A))]$$

$$\geq \beta E_A (f(q + A) - f(q' + A)) + \beta \min_x [E_A (f(q + A) - f(q' + A))].$$

It can be easily verified that $E_A f(q + a + A) - E_A f(q + A) - E_A f(q + a - 1) + a + A) \geq 0$ if $f$ is increasing (due to Statement 1 of this lemma). From the assumption that $\Delta f$ is increasing, the last two terms in the above expression are also non-negative. Now, taking expectation on both sides, we obtain $\Delta T_\rho f(q) - \Delta T_\rho f(q') \geq \Delta C(q) - \Delta C(q') \geq 0$. Therefore, from Statements 2 and 3 of Lemma [2] we have

$$\hat{\theta}_\rho(q) - \hat{\theta}_\rho(q') = \beta \Delta \hat{V}_\rho(q) - \beta \Delta \hat{V}_\rho(q') \geq \Delta C(q) - \Delta C(q') \geq 0.$$

Here, the last inequality holds since $C$ is a strictly convex increasing function.

**IV. Existence of MFE**

The main result showing the existence of MFE is as follows.

**Theorem 5.** There exists an MFE $(\rho, \hat{\theta}_\rho)$ such that

$$\rho(x) = \gamma(x) \triangleq \Pi_{\rho} \left( \hat{\theta}_\rho^{-1}(0, x) \right), \forall x \in \mathbb{R}^+.$$
We first introduce some useful notation. Let \( \Theta = \{ \theta : \mathbb{R} \rightarrow \mathbb{R}, \sup_{q \in \mathbb{R}^+} \left| \frac{\theta(q)}{w(q)} \right| < \infty \} \). Note that \( \Theta \) is a normed space with \( w \)-norm. Also, let \( \Omega \) be the space of absolutely continuous probability measures on \( \mathbb{R}^+ \). We endow this probability space with the topology of weak convergence. Note that this is same as the topology of point-wise convergence of continuous cumulative distribution functions.

We define \( \theta^* : \mathcal{P} \rightarrow \Theta \) as \((\theta^*(\rho))(q) = \hat{\theta}_\rho(q)\), where \( \hat{\theta}_\rho(q) \) is the optimal bid given by Corollary 3. It can easily verified that \( \hat{\theta}_\rho \in \Theta \). Also, define the mapping \( \Pi^* \) that takes a bid distribution \( \rho \) to the invariant workload distribution \( \Pi_\rho(\cdot) \). Later, using Lemma 8 we will show that \( \Pi_\rho(\cdot) \in \Omega \). Therefore, \( \Pi^* : \mathcal{P} \rightarrow \Omega \). Finally, define \( \mathcal{F} \) as \((\mathcal{F}(\rho))(x) = \gamma(x) = \Pi_\rho(\hat{\theta}_\rho^{-1}((0,x]))\). Lemma 11 will show that \( \mathcal{F} \) maps \( \mathcal{P} \) into itself.

Now to prove the above theorem we need to show that \( \mathcal{F} \) has a fixed point, i.e \( \mathcal{F}(\rho) = \rho \).

**Theorem 6** (Schauder Fixed Point Theorem). Suppose \( \mathcal{F}(\mathcal{P}) \subset \mathcal{P} \). If \( \mathcal{F}(\cdot) \) is continuous, and \( \mathcal{F}(\mathcal{P}) \) is contained in a convex and compact subset of \( \mathcal{P} \), then \( \mathcal{F}(\cdot) \) has a fixed point.

In next section, we show that the mapping \( \mathcal{F} \) satisfies the conditions of the above theorem, and hence it has a fixed point. Note that \( \mathcal{P} \) is a convex set. Therefore, we just need to show that the other two conditions are satisfied.

V. MFE Existence: Proof

A. Continuity of the map \( \mathcal{F} \)

To prove the continuity of mapping \( \mathcal{F} \), we first show that \( \theta^* \) and \( \Pi^* \) are continuous mappings. To that end, we will show that for any sequence \( \rho_n \rightarrow \rho \) in \( w \)-norm and \( \Pi^*(\rho_n) \rightarrow \Pi^*(\rho) \) (where \( \rightarrow \) denotes weak convergence). Finally, we use the continuity of \( \theta^* \) and \( \Pi^* \) to prove that \( \mathcal{F}(\rho_n) \rightarrow \mathcal{F}(\rho) \).

**Step 1: Continuity of \( \theta^* \)**

**Theorem 7.** The map \( \theta^* \) is continuous.

**Proof:** Define the map \( V^* : \mathcal{P} \rightarrow \mathcal{V} \) that takes \( \rho \) to \( \hat{V}_\rho(\cdot) \). We begin by showing that \( \|\hat{\theta}_\rho - \hat{\theta}_{\rho_2}\| \leq K \|\hat{V}_{\rho_1} - \hat{V}_{\rho_2}\| \), which means that the continuity of the map \( V^* \) implies the continuity of the map \( \theta^* \).

Then we show two simple properties of the Bellman operator. The first is that for any \( \rho \in \mathcal{P} \) and \( f_1, f_2 \in \mathcal{V} \),

\[
\|T_\rho f_1 - T_\rho f_2\| \leq K \|f_1 - f_2\| \tag{12}
\]

for some large \( K \), independent of \( \rho \). This result is available in Lemma 18 in Appendix B-A.

Second, let \( T_{\rho_1} \) and \( T_{\rho_2} \) be the Bellman operators corresponding to \( \rho_1, \rho_2 \in \mathcal{P} \) and let \( f \in \mathcal{V} \). We show that

\[
\|T_{\rho_1} f - T_{\rho_2} f\| \leq 2(M - 1)K_1 \|f\|_w \|\rho_1 - \rho_2\|. \tag{13}
\]

This result is available in Lemma 19 in Appendix B-A.

We then have

\[
\|T_{\rho_1} \hat{V}_{\rho_1} - T_{\rho_2} \hat{V}_{\rho_2}\| \leq \|T_{\rho_1} \hat{V}_{\rho_1} - T_{\rho_1}^{j-1} T_{\rho_1} \hat{V}_{\rho_2}\| + \|T_{\rho_1}^{j-1} T_{\rho_1} \hat{V}_{\rho_2} - T_{\rho_2} \hat{V}_{\rho_2}\| + \cdots
\]

\[
\leq \hat{K}^{j-1} \|\hat{V}_{\rho_1} - \hat{V}_{\rho_2}\| + \|T_{\rho_1}^{j-1} \hat{V}_{\rho_2} - T_{\rho_2} \hat{V}_{\rho_2}\| \tag{14}
\]

Here, \( \hat{K} \) and \( \hat{K}^{j-1} \) follow from \( \hat{K} \) and \( \hat{K}^{j-1} \), respectively.

Finally, from \( \hat{K} \) and \( \hat{K}^{j-1} \), we get

\[
\|\hat{V}_{\rho_1} - \hat{V}_{\rho_2}\| \leq 2(m - 1)K \hat{K} \|\rho_1 - \rho_2\| \|\hat{V}_{\rho_2}\| \tag{18}
\]

Therefore, if \( \|\hat{V}_{\rho_1} - \hat{V}_{\rho_2}\| < \frac{1}{2} \), then

\[
\|\hat{V}_{\rho_1} - \hat{V}_{\rho_2}\| \leq 4(m - 1)K \hat{K} \|\rho_1 - \rho_2\|. \tag{19}
\]

Hence, the maps \( V^* \) and \( \theta^* \) are continuous.

**Step 2: Continuity of the map \( \Pi^* \)**

Let \( \Pi_{\rho,\hat{\theta}(\cdot)} \) be the invariant distribution generated by any \( \hat{\theta} \). Recall that \( \Pi^* \) takes \( \rho \) to probability measure \( \Pi_{\rho,\hat{\theta}(\cdot)} \). First, we show that \( \Pi_{\rho,\hat{\theta}(\cdot)} \in \Omega \), where \( \Omega \) is the space of absolutely continuous measures (with respect to Lebesgue measure) on \( \mathbb{R}^+ \).

**Lemma 8.** For any \( \rho \in \mathcal{P} \) and any \( \theta \in \Theta \), \( \Pi_{\rho,\theta}(\cdot) \) is absolutely continuous with respect to the Lebesgue measure on \( \mathbb{R}^+ \).

**Proof.** \( \Pi_{\rho,\theta}(\cdot) \) can be expressed as the invariant queue-length distribution of the dynamics

\[
q \rightarrow \begin{cases} 
Q' + A & \text{with probability } \beta \\
R & \text{with probability } (1 - \beta),
\end{cases}
\]

where \( A \sim \Phi \) and \( R \sim \Psi \), and \( Q' \) is a random variable with distribution generated by the conditional probabilities

\[
\mathbb{P}(Q' = q|q) = 1 - p_\rho(\hat{\theta}(q))
\]

\[
\mathbb{P}(Q' = (q - 1)^+|q) = p_\rho(\hat{\theta}(q))
\]
Let $\Pi'$ be the distribution of $Q'$. Then for any Borel set $B$, $\Pi$ can be expressed using the convolution of $\Pi'$ and $\Phi$:

$$
\Pi_{\rho,\theta}(B) = \beta \int_{-\infty}^{\infty} \Phi(B-y)d\Pi'(y) + (1-\beta)\Psi(B).
$$

If $B$ is a Lebesgue null-set, then so is $B-y \ \forall y$. So, $\Phi(B-y) = 0$ and $\Psi(B) = 0$ and therefore $\Pi_{\rho}(B) = 0$.

We now develop a useful characterization of $\Pi_{\rho,\theta}$. Let

$$
\gamma^{(k)}_{\rho,\theta}(B|q) = \mathbb{P}(Q_k \in B|\text{no regeneration}, Q_0 = q)
$$

be the distribution of queue length $Q_k$ at time $k$ induced by the transition probabilities conditioned on the event that $Q_0 = q$ and that there are no regenerations until time $k$. We can now express the invariant distribution $\Pi_{\rho,\theta}(\cdot)$ in terms of $\gamma^{(k)}_{\rho,\theta}(\cdot|q)$ as in the following lemma.

**Lemma 9.** For any bid distribution $\rho \in \mathcal{P}$ and for any stationary policy $\theta \in \Theta$, the Markov chain described by the transition probabilities $\gamma^{(k)}_{\rho,\theta}(\cdot|q)$ has a unique invariant distribution $\Pi_{\rho,\theta}(\cdot)$. Also $\Pi_{\rho,\theta}$ and $\Psi_{\rho,\theta}$ are related as follows:

$$
\Pi_{\rho,\theta}(B) = \sum_{k \geq 0} (1-\beta)\beta^k \mathbb{P}(\gamma^{(k)}_{\rho,\theta}(B|Q=0)),
$$

where $\mathbb{P}(\gamma^{(k)}_{\rho,\theta}(B|Q=0)) = \int \gamma^{(k)}_{\rho,\theta}(B|q)d\Psi(q)$.

**Proof:** $\gamma^{(k)}_{\rho,\theta}(B|q)$ is the queue length distribution assuming no regeneration has happened yet, and the regeneration event occurs with probability $\beta$ independently of the rest of the system. It is then easy to find $\Pi_{\rho,\theta}(B)$ in terms of $\gamma^{(k)}_{\rho,\theta}(B|q)$ by simply using the properties of the conditional expectation, and the theorem follows. Note that in $\mathbb{P}(\gamma^{(k)}_{\rho,\theta}(B|Q=0))$, the random variable is the initial condition of the queue, as generated by $\Psi$. Full details are available in Appendix B-B.

We shall now prove the continuity of $\Pi^*$ in $\rho$.

**Theorem 10.** The mapping $\Pi^* : \mathcal{P} \rightarrow \Omega$ is continuous.

**Proof:** By Portmanteau theorem [20], we only need to show that for any sequence $\rho_n \rightarrow \rho$ in $\mathcal{P}$, and any open set $B$, $\liminf_{n \rightarrow \infty} \Pi_{\rho_n}(B) \geq \Pi_{\rho}(B)$. By Fatou’s lemma,

$$
\liminf_{n \rightarrow \infty} \Pi_{\rho_n}(B) = \liminf_{n \rightarrow \infty} \sum_{k = 0}^{\infty} (1-\beta)\beta^k \mathbb{P}(\gamma^{(k)}_{\rho_n}(B|Q=0)) \geq \sum_{k = 0}^{\infty} (1-\beta)\beta^k \mathbb{P}(\liminf_{n \rightarrow \infty} \gamma^{(k)}_{\rho_n}(B|Q=0))
$$

where $Q \sim \mathbb{P}_{\rho}$. Let $\gamma^{(k)}_{\rho_n} = \gamma^{(k)}_{\rho_n}(B|q)$. We prove in Lemma [20] (see Appendix B-B) that $\liminf_{n \rightarrow \infty} \gamma^{(k)}_{\rho_n}(B|q) \geq \gamma^{(k)}_{\rho}(B|q)$ for every $q \in \mathbb{R}^+$, and the proof follows.

**Step 3:** Continuity of the mapping $F$

Now, using the results from Step 1 and Step 2, we establish continuity of the mapping $F$. First, we show that $\mathcal{D}(\rho) \in \mathcal{P}$.

**Lemma 11.** For any $\rho \in \mathcal{P}$, let $\gamma(x) = (F(\rho))(x) = \Pi_{\rho}(\theta^{-1}_{\rho}([0,x]))$, $x \in \mathbb{R}^+$. Then, $\gamma \in \mathcal{P}$.

**Proof:** From the definition of $\Pi_{\rho}$, it is easy to see that $\gamma$ is a distribution function. Since $\theta_{\rho}$ is continuous and strictly increasing function as shown in Lemma [4], $\theta_{\rho}^{-1}(\{x\})$ is either empty or a singleton. Then, from Lemma [8] we get that $\Pi_{\rho}(\theta_{\rho}^{-1}(\{x\})) = 0$. Together, we get that $\gamma(x)$ has no jumps at any $x$ and hence it is continuous.

To complete the proof, we need to show that the expected bid under $\gamma(\cdot)$ is finite. In order to do this, we construct a new random process $\tilde{Q}_k$ that is identical to the original queue length dynamics $Q_k$, except that it never receives any service. We show that this process stochastically dominates the original, and use this property to bound the mean of the original process by a finite quantity independent of $\rho$. Full details are presented in Appendix B-C.

We now have the main theorem.

**Theorem 12.** The mapping $F : \mathcal{P} \rightarrow \mathcal{P}$ given by $(F(\rho))(x) = \Pi_{\rho}(\theta_{\rho}^{-1}([0,x]))$ is continuous.

**Proof:** Let $\rho_n \rightarrow \rho$ in uniform norm. From previous steps, we have $\theta_{\rho_n} \rightarrow \theta_{\rho}$ in $w$-norm and $\Pi_{\rho_n} \Rightarrow \Pi_{\rho}$. Then, using Theorem 5.5 of Billingsley [20], one can show that the push-forwards also converge:

$$
\Pi_{\rho_n}(\theta_{\rho_n}^{-1}(\cdot)) \Rightarrow \Pi_{\rho}(\theta_{\rho}^{-1}(\cdot)).
$$

Then, $F(\rho_n)$ converges point-wise to $F(\rho)$ as it is continuous at every $x$, i.e., $(F(\rho_n))(x) \rightarrow (F(\rho))(x)$ for all $x \in \mathbb{R}^+$.

It is easy to show that in the norm space $\mathcal{P}$, point-wise convergence implies convergence in uniform norm. This result is proved in Lemma [21] in Appendix B-C. This completes the proof.

**B. $F(\mathcal{P})$ contained in a compact subset of $\mathcal{P}$**

We show that the closure of the image of the mapping $F$, denoted by $\overline{F(\mathcal{P})}$, is compact in $\mathcal{P}$. As $\mathcal{P}$ is a normed space, sequential compactness of any subset of $\mathcal{P}$ implies that the subset is compact. Hence, we just need to show that $\overline{F(\mathcal{P})}$ is sequentially compact. Sequential compactness of a set $F(\mathcal{P})$ means the following: if $\{\rho_n\} \subset F(\mathcal{P})$ is a sequence, then there exists a subsequence $\{\rho_{n_j}\}$ and $\rho \in F(\mathcal{P})$ such that $\rho_{n_j} \rightarrow \rho$. We use Arzelá-Ascoli theorem and uniform tightness of the measures in $F(\mathcal{P})$ to show the sequential compactness. The version that we will use is stated below:

**Theorem 13 (Arzelá-Ascoli Theorem).** Let $X$ be a $\sigma$-compact metric space. Let $\mathcal{G}$ be a family of continuous real valued functions on $X$. Then the following two statements are equivalent:

1. For every sequence $\{g_n\} \subset \mathcal{G}$ there exists a subsequence $g_{n_j}$ which converges uniformly on every compact subset of $X$.

2. The family $\mathcal{G}$ is equicontinuous on every compact subset of $X$ and for any $x \in X$, there is a constant $C_x$ such that $|g(x)| < C_x$ for all $g \in \mathcal{G}$.

Suppose a family of functions $\mathcal{D} \subset \mathcal{P}$ satisfies the equivalent conditions of the Arzelá-Ascoli theorem and in addition satisfy the uniform tightness property, i.e., $\forall \epsilon > 0$ there exists an $x_{\epsilon}$ such that for all $f \in \mathcal{D}$, $f(x_{\epsilon}) \geq 1 - \epsilon$. Then, for any sequence $\{\rho_n\} \subset \mathcal{D}$, there exists a subsequence $\{\rho_{n_j}\}$ that
converges uniformly on every compact set to a continuous increasing function \( \rho \) on \( \mathbb{R}^+ \). As \( D \) is uniformly tight it can be shown that \( \rho_{i,j} \) converges uniformly to \( \rho \) and that \( \rho \in \mathcal{P} \). Therefore, \( D \) is sequentially compact in the topology of uniform norm.

In the following, we show that \( \mathcal{F}(\mathcal{P}) \) satisfies uniform tightness property and condition 2 in Arzelà-Ascoli theorem. First verifying the conditions of Arzelà-Ascoli theorem, note that the functions in consideration are uniformly bounded by 1. To prove equicontinuity, consider a \( \gamma = \mathcal{F}(\rho) \) and let \( x > y \).

\[
\gamma(x) - \gamma(y) = \Pi_{\rho}(\theta_{\rho}(q) < x) - \Pi_{\rho}(\theta_{\rho}(q) \leq y) = \Pi_{\rho}(y < \theta_{\rho}(q) \leq x)
\]

(23)

Lemma 14. For any interval \([a, b]\), \( \Pi_{\rho}([a, b]) < c \cdot (b - a) \), for some large enough \( c \).

**Proof:** The proof follows easily from our characterization of \( \Pi_{\rho} \) in terms of \( \gamma_{\rho}(k) \).

The above lemma and equation (23) imply that \( \gamma(x) - \gamma(y) \leq c (\theta_{\rho}^{-1}(x) - \theta_{\rho}^{-1}(y)) \). To show equicontinuity, it is enough to show that \( \limsup_{y \uparrow x} \gamma(x) - \gamma(y) \leq K(x) \) for some \( K \) independent of \( \rho \), which will show now.

\[
\limsup_{y \uparrow x} \frac{\gamma(x) - \gamma(y)}{x - y} = \limsup_{y \uparrow x} \frac{\Pi_{\rho}(y < \theta_{\rho}(y) \leq x)}{x - y} = \limsup_{y \uparrow x} \frac{\theta_{\rho}^{-1}(x) - \theta_{\rho}^{-1}(y)}{x - y} \leq c \limsup_{y \uparrow x} \frac{\theta_{\rho}^{-1}(x) - \theta_{\rho}^{-1}(y)}{x - y} = c \limsup_{y \uparrow x} \frac{\theta_{\rho}^{-1}(x) - \theta_{\rho}^{-1}(y)}{\theta_{\rho}^{-1}(x) - \theta_{\rho}^{-1}(y)}
\]

Let \( x' = \theta_{\rho}^{-1}(x) \) and \( y' = \theta_{\rho}^{-1}(y) \). Now,

\[
\limsup_{y \uparrow x} \frac{\gamma(x) - \gamma(y)}{x - y} \leq c \limsup_{y' \uparrow x'} \frac{x' - y'}{\theta_{\rho}(x') - \theta_{\rho}(y')}
\]

\[
= c \limsup_{y' \uparrow x'} \frac{x' - y'}{\beta \Delta V(x') - \beta \Delta V(y')}
\]

\[
\leq c \frac{1}{H(x')}
\]

Where,

\[
0 < H(x') = \begin{cases} E_A[C'(x' + A) - C'(x - 1 + A)] & x' > 1 \\ E_A[C'(x' + A)] & x' \leq 1 \end{cases}
\]

and \( C'(x) = \frac{dC(x)}{dx} \).

Finally, we have the following lemma showing that \( \mathcal{F}(\mathcal{P}) \) is uniformly tight.

**Lemma 15.** \( \mathcal{F}(\mathcal{P}) \) is uniformly tight, i.e., for any \( \epsilon > 0 \) and any \( f \in \mathcal{F}(\mathcal{P}) \), there exists an \( x_\epsilon \in \mathbb{R} \) such that \( 1 - \epsilon \leq f(x_\epsilon) \leq 1 \).

**Proof:** From Lemma 11 we have \( \mathcal{F}(\mathcal{P}) \subseteq \mathcal{P} \). Hence, the expectation of the bid distributions in \( \mathcal{F}(\mathcal{P}) \) is bounded uniformly. An application of Markov inequality will give uniform tightness.

VI. APPROXIMATION RESULTS: PBE AND MFE

In this section we prove that the mean field policy is an \( \epsilon \)-Nash equilibrium. We have the following theorem:

**Theorem 16.** Let \((\rho, \hat{\theta}_\rho)\) constitute an MFE. Suppose at time 0 the queue length of the users is set independently across users according to \( \Pi_{\rho} \); and that their initial belief is also consistent. Also, suppose that all queues except queue 1 play the MFE policy \( \hat{\theta}_\rho \). Then, for any policy \( \theta^N \) of queue 1 that may be history dependent and any \( q \in \mathbb{R}^+ \), we have

\[
\limsup_{N \to \infty} V_{1, \mu_{1,0}}^N(q; \hat{\theta}_\rho) - V_{1, \mu_{1,0}}^N(q; \theta^N, (\hat{\theta}_\rho)^{-1}) \leq 0,
\]

where \( \mu_{1,0} = \Pi_{\rho} \) and the superscript \( N \) has been added to explicitly indicate the dependence on the number of cells.

The main idea behind the proof is a result called propagation of chaos, and it identifies conditions under which any finite subset of the state variables are independent from each other. We state this result now in our context. We only provide brief sketches of proofs in this section, since the methodology is much the same as [21] and space constraints do not allow us to present the full version of the proofs here.

**Lemma 17** (Propagation of chaos). For any fixed indices \( i_1, \ldots, i_k \), let \( \mathcal{L}(Q_{i_1}^N(t), \ldots, Q_{i_k}^N(t)) \) denote the probability law of the \( k \)-tuple of corresponding queues in the \( MN \)-queue system, at time \( t \). Suppose that \( \mathcal{L}(Q_{i_1}^N(0), \ldots, Q_{i_k}^N(0)) = \otimes^k \Pi_{\rho} \), where \((\rho, \hat{\theta}_\rho)\) is the solution to the MFE equation. Also, suppose that all queues are following mean field equilibrium strategy. Then for any \( T > 0 \), we have

\[
\mathcal{L}(Q_{i_1}^N(T), \ldots, Q_{i_k}^N(T)) \Rightarrow \otimes^k \Pi_{\rho},
\]

as \( N \to \infty \).

**Proof:** We shall only consider the case \( k = 2 \); the proof of the general case is similar. We can follow the proof of Theorems 4.1 and 5.1 in Graham and Meleard [21]. The proof is divided into two parts; the first part proves that for any two agents \( i \) and \( j \),

\[
\| \mathcal{L}(Q_{i_1}^N, Q_{j_1}^N) - \mathcal{L}(Q_{i_1}^N) \otimes \mathcal{L}(Q_{j_1}^N) \|_D \to 0,
\]

where the subscript \( D \) refers to the total variation norm. In the second, we show that \( \mathcal{L}(Q_{i_1}^N) \Rightarrow \Pi_{\rho} \). Both parts rely on studying interaction graphs, defined in [21], which characterize the amount of interactions that any finite subset of agents may have had in the past.

The proof of Theorem 16 is as follows. Suppose we start at time \( t = 0 \) with queue length of agent 1 being \( Q_1(0) \). We can choose a time \( T \) large enough so that the value added by auctions occurring after time \( T \) is less than \( \epsilon \), due to discounting. Thus, the difference in value contributed by these auctions, when using policy \( \theta^N \) and \( \hat{\theta}_\rho \) can be bounded by \( 2\epsilon \), and we can restrict attention to the first \( T \) auctions.

Using ideas similar to Lemma 17, we show that probability of the event \( E^N \) that other agents that interact with agent 1 at time \( t \) have never been influenced by agent 1 goes to 1.
as $N$ becomes large. Thus, the belief of distribution of queue lengths of other agents encountered converges to $\Pi_{\rho}$ according to Lemma \[17\] Then we can show that for any $\epsilon > 0$ and $N$,
\[
V_{1,\mu_{1}}(q; \theta_{\rho}, (\theta_{\rho})^{-1}) - V_{1,\mu_{1}}(q; \theta^{N}, (\theta_{\rho})^{-1}) \leq \epsilon,
\]
which yields the desired result.

VII. Simulation Results

We now turn to computing the MFE distribution. We simulate a large system with 100,000 users distributed among 10,000 cells with 10 users per cell. For simplicity of simulation, we truncate and discretize both state and bid spaces. The truncated state space is $S = \{0.01m, 0 \leq m \leq 2000\}$, while the bid space is $X = \{0.15m, 0 \leq m \leq 3000\}$. The job arrival and regeneration distributions are both chosen to be uniform over interval $[0, 1]$. The service rate of each base station is assumed to be 5 units per time slot. Finally, the holding cost function is chosen as $C(q) = q^2$.

Our simulation simply follows the choices made by each agent and calculating the empirical distribution that would result at each time step. Let $\rho(x) = \min\{0.001x, 1\}$, $x \in \mathcal{X}$ and $\Pi_{\rho} = \Psi$. For every positive integer $n$, do the following:

1) Compute the optimal value function, $\hat{V}_n$, which is the unique solution to the following equation,
\[
\hat{V}_n(q) = C(q) + \beta E_A[\hat{V}_n(q + A)] - \sum_{x \leq \beta \hat{V}_n(q), x \in \mathcal{X}} p_{\rho_n}(x),
\]
where $\Delta \hat{V}_n(q) = E_A[\hat{V}_n(q + A)] - E_A[\hat{V}_n((q - 1)^{+} + A)]$. We apply value iteration (Section 6.10 \[22\]) to compute an approximate solution to the above equation.

2) Compute the optimal bid function, $\hat{\rho}_n$ as
\[
\hat{\rho}_n(q) = \beta E_A[\hat{V}_n(q + A) - \hat{V}_n((q - 1)^{+} + A)].
\]

3) Next, all agents employ the optimal bid policy from Step 2 and update their states. Then we compute the candidate steady state distribution by evaluating the empirical distribution of state.

4) Finally, compute the empirical bid distribution, $\rho_{n+1}(x) = \Pi_{\rho}([\theta_{\rho}^{-1}([0, x])]$. If $||\rho_n - \rho_{n+1}|| < \epsilon$, then $\rho = \rho_{n+1}$ and exit. Otherwise, set $n = n + 1$ and go to Step 1.

If the algorithm converges, then its output distribution $\rho$ is an approximation of the MFE bid distribution.

We simulated the algorithm for three set of parameters: 1. $\beta = 0.9, M = 10$, 2. $\beta = 0.95, M = 10$ and 3. $\beta = 0.9, M = 15$. Also, we chose the accuracy parameter $\epsilon = 0.008$. Figure 2 shows that the algorithm converges in less than 20 iterations in all three cases. In each iteration, Step 1 (value iteration) is the most computationally intensive. It converges in 80 recursions, with each recursion having to update $|S|$ number of variables, and with each variable update requiring at most $|X|$ number of arithmetic operations. All together, the computational complexity of each iteration is in the order of $80 \times |S| \times |X|$ arithmetic operations.

The queue length distributions at MFE are shown in Figure 3. We observe that the distribution curves exhibit a rightward shift with increase in $\beta$ or $M$. Note that larger $\beta$ makes queues live longer without regeneration, while higher $M$ reduces each individual’s average service rate. Hence, the queues get longer on average. We show the optimal bid functions at MFE in Figure 4. As expected from our analysis, the bid functions are monotonically increasing in queue length.

VIII. Discussion and Concluding Remarks

Our algorithm for computing the MFE immediately suggests a simple implementation scheme. Suppose each mobile device has a network interface manager on which the human user sets up priorities for different apps. The interface manager also should be aware of the cost functions corresponding to the QoE of different apps. Now, suppose that the base stations were to calculate the empirical bid distribution at each time instant, and return it to the interface manager. The interface manager plays its best response to this bid distribution. Value iteration could be done either on each device or at a data center and provided as a look up table to the interface manager. The base stations combine all the bids to create a new empirical bid distribution. Such a proceeding is essentially identical to the algorithm that we employed above, and would converge in a similar fashion.

We had assumed a single cost function for the agent (app), but as long as there are a finite set of cost functions (corresponding to a finite number of app types) we can incorporate the cost function as part of the state of the agent, with the cost function being chosen according to some distribution at each regeneration (corresponding to choosing to start a new app with some probability). Such a modification causes no changes to any of our analysis. Then the mean field bid distribution accounts for the distribution of cost functions (app popularities), and the agent takes a decision based on this distribution as before. The only difference to the achieved equilibrium is that it now follows a weighted version of LQF, with the weights corresponding to the cost on QoE of the competing apps.

To summarize our work, we explored the question of whether it is possible to design an scheduling policy that allows for declaration of value by humans in the loop and effects on QoE, and which has the attractive properties of a (weighted) LQF service regime. We used a mean field framework to show that as the number of agents in the system becomes large, this objective can indeed be fulfilled using a second price auction at each server. Our design appears to lend itself well to implementation and this will be our future goal.

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APPENDIX A

PROOF OF LEMMA \[2\]

We may rewrite the the definition of $T_p$ in \[9\] as

$$ (T_p f)(q) = \inf_{x \in \mathbb{R}^+} S_f(q, x) $$

where $S_f(q, x) = C(q) + r_p(x) + \beta E_Q_1[f(Q_1)|q, x]$. Given the current state and action pair, $(q, x)$, the first two terms in $S_f(q, x)$ constitute the current cost, while the last term is the future expected cost, where $Q_1$ is one-step future state variable. Further, from \[9\], we have

$$ \mathbb{E}_{Q_1}[f(Q_1)|q, x] = (1 - p_\theta(x))E_{A}[f(q + A)] + p_\theta(x)E_{A}[f((q - 1)^+ + A)]. $$

The proof proceeds through a verification of the assumptions of Theorem 8.3.6 in \[19\]. An exception is that action space in our case is not a compact set which violates Assumption 8.3.1(a) in \[19\]. However, this assumption can be overridden if the statement of Lemma 8.3.8(a) in \[19\], equivalently Condition (3) below, holds true. Further, we desire to show the existence of a $j \in \mathbb{N}$ such that $T^j_p$ is a contraction mapping. Since Theorem 8.3.6 is derived for $j = 1$, we replace Assumption 8.3.2(b) with Condition (5) given below.

Now, we prove the following statements.  

1. $C(q) + r_p(x)$ is a continuous function in $x \in \mathbb{R}^+$.  
2. $\mathbb{E}_{Q_1}[f(Q_1)|q, x]$ is continuous in $x \in \mathbb{R}^+$ for every $f \in \mathcal{V}$.  
3. For any $f \in \mathcal{V}$, there exists a measurable function $\theta_f : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ such that $\theta_f(q)$ attains minimum in \[24\]. Further, $S_f(q, \theta_f(q))$ is a measurable function for any $f \in \mathcal{V}$.  
4. There exists a nonnegative constant $c_1$ such that $\sup_{x} |C(q) + r_p(x)| \leq c_1 w(q)$ where $w(q) = \max\{C(q), 1\}$.  
5. There exists $j \in \mathbb{N}$ and $c_2$ with $c_2 \leq 1$ such that $\beta^j \sup_{x} E_{Q_j}[w(Q_j)|q, \bar{x}] \leq c_2 w(q) + c_3$ where $Q_j$ is $j$-step future state variable and $\bar{x}$ is a $j$-length sequence of actions.  
6. The function $\mathbb{E}[w(Q_1)|q, x]$ is continuous in $x \in \mathbb{R}^+$.

Condition (1) and (2) are obvious from the continuity of $r_p(x)$ and $p_\theta(x)$. Further, as derived in \[10\], $\theta_f(q) = \Delta f(q)^+$ where $\Delta f(q) = E_A[f(q + A)] - f((q - 1)^+ + A)$. The measurability of functions $\theta_f(q)$ and $S_f(q, \theta_f(q))$ are evident from their definitions. Condition (4) holds true from the definition of $w(q)$ and from the fact that $r_p(x) \leq \lim_{y \rightarrow \infty} r_p(y) < (M - 1) \int_0^\infty (1 - \rho(x)) dx < (M - 1)E$ where the last inequality follows as $\rho \in \mathcal{P}$. Condition (5)
follows from the fact that,
\[
\beta^j E_Q [w(Q_j) | q, \vec{a}] \leq \beta^j \max \{ 1, C(q + jA) \} \\
\leq \beta^j (k_1 w(q) + k_2),
\]
where $\vec{a}$, as defined in Assumption 1, is the maximum arrival possible between any two adjacent auctions and $k_1 > 0$, $k_2$ are some constants independent of $j$. The above results follows from (25) and the definition of $w(q)$. Then, there exists a $j$ such that $\beta^j k_1 = c_2 < 1$ and hence (5) holds. Finally, the last condition follows from Condition (2) as $w(q) \in \mathcal{V}$.

Given that the above conditions are met, we can prove the first statement of Lemma 2. The proof is essentially identical to that of Theorem 8.3.6 in [19]. The second statement of the lemma can be obtained by comparing (9) and (7). The last part of the lemma follows from (10).

**APPENDIX B**

**PROOFS FROM SECTION V**

In this section, we present details of proofs that were omitted from Section V. We divide this section into parts based on the development of that section.

### A. Proofs Pertaining to Section V-A: Step 1

**Lemma 18.** Suppose $\rho_1, \rho_2 \in \mathcal{P}$. Then, $||\hat{\rho}_{p_1} - \hat{\rho}_{p_2}||_{\mathcal{W}} \leq K ||V_{p_1} - V_{p_2}||_{\mathcal{W}}$

**Proof:** For any $q \in \mathbb{R}^+$, by the definition of $\hat{\rho}$, we have,
\[
|\hat{\rho}_{p_1}(q) - \hat{\rho}_{p_2}(q)|
\leq \beta [E_A[V_{p_1}(q + A) - V_{p_2}((q - 1)^+ + A)]]
\leq \beta [E_A[V_{p_1}(q + A) - V_{p_2}((q - 1)^+ + A)]]
\leq \beta [V_{p_1} - V_{p_2} w(q) + w((q - 1)^+ + A)]
\leq K ||V_{p_1} - V_{p_2}||_{\mathcal{W}} w(q)
\]

**Lemma 19.** Let $\rho \in \mathcal{P}$ and $f_1, f_2 \in \mathcal{V}$. Then,
\[
\|T_{\rho} f_1 - T_{\rho} f_2\|_{\mathcal{W}} \leq K \|f_1 - f_2\|_{\mathcal{W}}
\]

**Proof:** Using the characterization of $T_{\rho}$ from eq. (11), we have that, for any $q \in \mathbb{R}^+$,
\[
|T_{\rho} f_1(q) - T_{\rho} f_2(q)|
\leq \beta |f_1(q) - f_2(q)| K_1 w(q) + \int \beta |\Delta f_1(q) - \Delta f_2(q)|
\leq \beta |f_1(q) - f_2(q)| K_1 w(q) + \beta |\Delta f_1(q) - \Delta f_2(q)|
\leq \beta(K_1 + K_2)|f_1(q) - f_2(q)| w(q)
\]

### B. Proofs Pertaining to Section V-A: Step 2

**Proof of Lemma 2** For brevity, denote $\Pi_{\rho, \theta} \cdot (\cdot)$ be $\Pi(\cdot)$ and $\Upsilon_{\rho, \theta}^{(k)} = \Upsilon_{\rho}^{(k)}$. Let $-\tau$ be the last time before 0 the chain regenerated. We have
\[
\Pi(B) = \sum_{k=0}^{\infty} \mathbb{P}(B, \tau = k)
\]
\[
= \sum_{k=0}^{\infty} \mathbb{P}(\tau = k) \mathbb{P}(B | \tau = k)
\]
Since the regeneration events are independent of the queue-length and occur geometrically with probability $(1 - \beta)$, $\mathbb{P}(\tau = k) = (1 - \beta)^k$. Hence,
\[
\Pi(B) = \sum_{k=0}^{\infty} (1 - \beta) \beta^k \mathbb{P}(Q_0 \in B | \tau = k)
\]
\[
= \sum_{k=0}^{\infty} (1 - \beta) \beta^k \mathbb{E}[\mathbb{1}_{Q_0 \in B} | \tau = k, Q_{-k} = Q] | \tau = k
\]
\[
= \sum_{k=0}^{\infty} (1 - \beta) \beta^k \mathbb{E}_{\mathcal{W}} \Upsilon_{\rho}^{(k)}(B|Q)
\]
\[
\text{since } Q_{-k} \sim \mathcal{W} \text{ given } \tau = k.
\]

**Lemma 20.** $\text{lim inf}_{n \to \infty} \Upsilon_{\rho}^{(k)}(B|q) \geq \Upsilon_{\rho}^{(k)}(B|q)$

**Proof:** The proof proceeds through mathematical induction on $k$. For $k = 0$, we have $\Upsilon_{\rho}^{(0)}(B|q) = \mathbb{1}_{q \in B}$ and hence the hypothesis holds true. Suppose that the hypothesis is true till $k = m - 1$. To prove the lemma, we just need to verify that the hypothesis holds for $k = m$. Let $\mathbb{P}_{q, \rho} \cdot (\cdot)$ be the one step transition kernel of the queue dynamics conditioned on the following facts: the initial state is $q$, the bids are generated according to the optimal policy given by Corollary 3 and no regeneration. Verify that $\mathbb{P}_{q, \rho} \cdot (\cdot) \implies \mathbb{P}_{q, \rho} \cdot (\cdot)$ by considering the integrals of a bounded continuous function. Then, by Skorokhod representation theorem, there exists $X_n$ and $X$ on common probability space such that $X_n \sim \mathbb{P}_{q, \rho}$, $X \sim \mathbb{P}_{q, \rho}$, and $X_n \to X$ a.s. We have,
\[
\text{lim inf } \Upsilon_{\rho}^{(m)}(B|q) = \text{lim inf } \mathbb{E}[\Upsilon_{\rho}^{(m-1)}(B|X)]
\]
\[
\geq \mathbb{E}[\text{lim inf } \Upsilon_{\rho}^{(m-1)}(B|X)]
\]
\[
\geq \mathbb{E}[\Upsilon_{\rho}^{(m-1)}(B|X)]
\]
\[
= \Upsilon_{\rho}^{(m)}(B|q)
\]
where eq. (33) follows from Fatou’s lemma, and eq. (34) follows from the induction hypothesis.

### C. Proofs Pertaining to Section V-A: Step 3

**Details of proof of Lemma 17** To complete the proof, we need to show that the expected bid under the cumulative distribution function $\hat{\rho}$ is bounded from above by a constant
that is independent of \( \hat{\rho} \). To that end, define a new Markov random process \( \hat{Q}_k \) with the probability transition matrix

\[
P(\hat{Q}_{k+1} \in B | \hat{Q}_k = q) = \beta \mathbb{1}_{(q+\hat{A} \in B)} + (1 - \beta) \Psi_R(B)
\]

where \( \hat{A} \) is the maximum possible arrival between any two consecutive auction instants. The process \( \hat{Q}_k \) has an invariant distribution which is given by,

\[
\hat{\Pi}(B) = \sum_{k=0}^{\infty} (1 - \beta)^k \mathbb{E}_{\Psi_R} \left( \mathbb{1}_{(q+k\hat{A} \in B)} \right).
\]

The proof of the above result is identical to that of Lemma 9. For any \( q \) given, the above probability measure \( \Psi \) stochastically bounds the probability measure in eq. (41). Therefore, it can be shown that \( \hat{\Pi} \) stochastically dominates \( \Pi_{\rho} \) for all \( \rho \in \mathcal{P} \), i.e., \( \Pi_{\rho} \ll \hat{\Pi} \).

Now, the expected value of the optimal bid function \( \hat{\rho}_\rho(q) \) under \( \Pi_{\rho} \) satisfies,

\[
\mathbb{E}_{\Pi_{\rho}}[\hat{\rho}_\rho(q)] \leq \mathbb{E}_{\hat{\Pi}}[\hat{\rho}(q + \hat{A})] 
\leq \sum_{k=0}^{\infty} (1 - \beta)^k \mathbb{E}_{\Psi_R} \left( \hat{V}_\rho(q + (k + 1)\hat{A}) \right)
\]

Above, the first inequality follows from stochastic dominance of \( \hat{\Pi} \) and the second inequality is due to the definition of optimal bid function.

From \( \mathbb{E}_{\Psi_R}(\hat{V}_\rho(q + (k + 1)\hat{A})) \leq \sum_{k=0}^{\infty} \beta^k C(q + k\hat{A}) \) independent of \( \rho \). Since \( C(q) \in O(q^m) \) for some \( m \), we have \( \hat{V}_\rho(q) \in O(q^m) \). Then, \( \mathbb{E}_{\Psi_R}(\hat{V}_\rho(q + (k + 1)\hat{A})) \in O(k^m) \) as the moments of \( \Psi_R \) are bounded. This directly gives that \( \mathbb{E}_{\Pi_{\rho}}[\hat{\rho}(q)] \) is bounded by the same constant that is independent of \( \rho \) and, hence independent of \( \hat{\rho} \).

Lemma 21. In \( \mathcal{P} \), pointwise convergence implies uniform convergence.

Proof: Let \( \rho_n, \rho \in \mathcal{P} \) and \( \rho_n \to \rho \) point-wise. Given \( \epsilon > 0 \), choose \( L \) large enough so that \( \rho(L) > 1 - \epsilon \). Since \( \rho \) is continuous function by definition, it is uniformly continuous on the compact set \([0, L]\). Therefore, we can construct a sequence \( 0 = x_1 < x_2 < \cdots < x_k = L \) such that and \( |\rho(x_{i+1}) - \rho(x_i)| \) and \( \rho_n(x_i) \) such that \( x_i < y < x_{i+1} \),

\[
|\rho(y) - \rho_n(y)| \leq |\rho(y) - \rho(x_i)| + |\rho(x_i) - \rho_n(x_i)| + |\rho_n(x_i) - \rho_n(y)|
\]

While if \( L < y \), then

\[
|\rho(y) - \rho_n(y)| \leq 5\epsilon
\]

Therefore, \( |\rho(y) - \rho_n(y)| < 5\epsilon \) for all \( n > J \) and hence \( \rho_n \) converges to \( \rho \) uniformly.

D. Proofs Pertaining to Section V-B

Proof of Lemma 14. We know that \( \Pi([a, b]) = \sum_{k=0}^{\rho(L) - 1} (1 - \beta)^k \mathbb{E}_{\Psi_R}(\Psi(\rho)) \).

\[
\mathbb{E}_{\Psi_R}(\Psi(\rho)) = \mathbb{E}(\mathbb{1}_{(Q_0 + A_k - D_k \in [a, b])}|Q_0) = \mathbb{E}(\mathbb{1}_{(Q_0 + A_k - D_k \in [a, b])}|D_k, Q_0)|Q_0)
\]

The above results hold since the random variable \( A_k \) is independent of \( Q_0 \) and \( D_k \) for any \( k \) and it has a bounded density function. Therefore, \( \mathbb{E}_{\Psi_R}(\Psi(\rho)) \leq c \cdot (b - a) \) for all \( k > 0 \). For \( k = 0 \), we know that \( \Psi_R \) has a bounded density which implies \( \Psi_R([a, b]) \leq c \cdot (b - a) \). These two results prove that there is a large enough \( c \) such that \( \Pi_{\rho}([a, b]) \leq c \cdot (b - a) \).