Quantum phase transitions and bifurcations: reduced fidelity as a phase transition indicator for quantum lattice many-body systems

Jin-Hua Liu, Qian-Qian Shi, Jian-Hui Zhao and Huan-Qiang Zhou
Centre for Modern Physics and Department of Physics, Chongqing University, Chongqing 400044, People’s Republic of China
E-mail: huanqiang.zhou@gmail.com

Received 23 July 2011, in final form 18 October 2011
Published 17 November 2011
Online at stacks.iop.org/JPhysA/44/495302

Abstract
We establish an intriguing connection between quantum phase transitions and bifurcations in the reduced fidelity between two different reduced density matrices for quantum lattice many-body systems with symmetry-breaking order. Our finding is based on the observation that in the conventional Landau–Ginzburg–Wilson paradigm a quantum system undergoing a phase transition is characterized in terms of spontaneous symmetry breaking that is captured by a local-order parameter, which in turn results in an essential change of the reduced density matrix in the symmetry-broken phase. Two quantum systems on an infinite lattice in one spatial dimension, i.e. a quantum Ising model in a transverse magnetic field and a quantum spin-1/2 XY model in an external magnetic field, are considered in the context of the tensor network algorithm based on the matrix product state representation.

PACS numbers: 03.67.-a, 03.65.Ud, 03.67.Hk
(Some figures may appear in colour only in the online journal)

Introduction
Recently, we have witnessed a growing interest in the study of quantum many-body systems in the context of the fidelity approach to quantum phase transitions (QPTs) [1, 2] with symmetry-breaking/topological orders [3–12]. In [4–7], it has been argued that the ground-state fidelity per site may be used to detect QPTs. Since the argument is solely based on the basic postulate of quantum mechanics on quantum measurements, it is expected that this approach is applicable to quantum lattice systems in any spatial dimensions, regardless of what type of internal order is present in quantum many-body states. In fact, it has been confirmed that the ground-state fidelity per site is able to describe QPTs arising from a spontaneous symmetry breaking (SSB) [4–8], the Kosterlitz–Thouless transition [9] and topological QPTs in the Kitaev model [10].
Actually, the ground-state fidelity per site, as a fundamental quantity, is well defined even in the thermodynamic limit, in contrast to the global ground-state fidelity itself. Remarkably, it may be computed in terms of the newly developed tensor network (TN) algorithms, such as the matrix product states (MPS) [13–15] in one spatial dimension, and the tensor product states [16], or equivalently, the projected entangled-pair states (PEPS) [17] in two and higher spatial dimensions.

In the conventional Landau–Ginzburg–Wilson paradigm, an SSB, which occurs when a system possesses a certain symmetry whereas the ground-state wavefunctions do not preserve it [18, 19], is quantified in terms of a local-order parameter. An intriguing feature of local-order parameters for quantum systems with symmetry-breaking order is bifurcations that they exhibit at critical points. Remarkably, such a bifurcation also manifests itself in the ground-state fidelity per site [8]. The advantage of the latter over local-order parameters lies in the fact that the ground-state fidelity per site is universal in the sense that it is not model dependent in contrast to model-dependent order parameters in characterizing QPTs in quantum lattice many-body systems.

The investigation that has been carried out so far mainly focuses on the ground-state wavefunctions for quantum lattice systems. Thus, the fidelity is equivalent to the overlap between two different ground states, which are pure states. However, as first discussed in [6] (see also [12]), the fidelity defined for two different mixed states is also useful to locate phase transition points. The mixed states are described by the reduced density matrices arising from tracing out the degrees of freedom in the environment surrounding a sub-system. Indeed, for a sub-system of a composite quantum system, a reduced density matrix is a basic notion that is indispensable for the analysis of a composite quantum system. In fact, the presence of a local-order parameter makes a difference in the reduced density matrices in the symmetric and symmetry-broken phases. From this, one may anticipate that the (reduced) fidelity between two different reduced density matrices for quantum lattice many-body systems with symmetry-breaking order should capture bifurcations arising from an SSB.

In this paper, we attempt to address this problem. We shall investigate two quantum models on an infinite lattice in one spatial dimension, i.e. a quantum Ising model in a transverse magnetic field and a spin-1/2 XYX model in an external magnetic field. Both systems possess a discrete symmetry group $Z_2$, but the ground-states break the symmetry. The SSB is reflected as a bifurcation [8] in the ground-state-reduced fidelity for both systems. Our results demonstrate that one may identify a phase transition point as a bifurcation point in the ground-state-reduced fidelity between two different reduced density matrices for quantum lattice many-body systems with symmetry-breaking order. Therefore, it offers an efficient means to distinguish symmetry-breaking order from topological order, both of which may be detected in terms of the ground-state fidelity per site [4–10].

The models

The first model we consider in this paper is the quantum Ising model in a transverse magnetic field in an infinite-size lattice in one spatial dimension. The Hamiltonian takes the form

$$H = - \sum_{i=-\infty}^{\infty} (S_i^x S_{i+1}^x + \lambda S_i^z),$$

1 Although our examples are restricted to quantum systems on an infinite lattice in one spatial dimension, the argument is valid for quantum systems in two and higher dimensions. In the latter case, the TN algorithm based on PEPS [26] is necessary.

2
where $S^{i}_{a}$ ($a = x, y, z$) are the spin-1/2 Pauli operators at site $i$ and $\lambda$ is the transverse magnetic field. The model is invariant under the symmetry operation: $S^{i}_{x} \rightarrow -S^{i}_{x}$ and $S^{i}_{z} \rightarrow S^{i}_{z}$ for all sites, which yields the $Z_2$ symmetry. As is well known, the system undergoes a second-order QPT at the critical field $\lambda_c = 1$ [20].

The second model is the spin-1/2 XYX model in an external magnetic field. The Hamiltonian can be written as

$$H = \sum_{i=-\infty}^{\infty} \left( S^{i}_{x} S^{i+1}_{x} + \Delta y S^{i}_{y} S^{i+1}_{y} + S^{i}_{z} S^{i+1}_{z} + h S^{i}_{z} \right), \quad (2)$$

where $S^{i}_{a}$ are the Pauli spin operators at site $i$, $\Delta y$ is a parameter describing the anisotropy in the internal space and $h$ is the external magnetic field. This model also possesses a $Z_2$ symmetry, with the symmetry operation: $S^{i}_{x} \rightarrow -S^{i}_{x}$, $S^{i}_{y} \rightarrow -S^{i}_{y}$ and $S^{i}_{z} \rightarrow S^{i}_{z}$ for all sites. Below we shall choose $\Delta y = 0.25$. In this case, the critical magnetic field is $h_c \sim 3.210(6)$ [21].

### The reduced density matrix

We are now in a position to clarify the difference of the reduced density matrices in the symmetric and symmetry-broken phases for both the quantum Ising model in a transverse magnetic field and the quantum spin-1/2 XYX model in an external magnetic field. The analysis will be carried out for both the one-site- and two-site-reduced density matrices, respectively. For the quantum Ising model in a transverse magnetic field, the one-site-reduced density matrix in the $Z_2$ symmetric phase takes the form

$$\rho_{\text{sing}} = \frac{1}{2} I + 2 \langle S_z \rangle S_z, \quad (3)$$

where $\langle S_z \rangle$ is the expectation value of $S_z$ in the ground state in the $Z_2$ symmetric phase, whereas the two-site-reduced density matrix in the $Z_2$ symmetric phase is

$$\rho_{\text{sing}} = \frac{1}{4} I + 4 \alpha_1 S_x \otimes S_x + 4 \alpha_2 S_z \otimes S_z + \alpha_3 I \otimes S_z + \alpha_4 S_z \otimes I. \quad (4)$$

Here, $\alpha_1 = \langle S_x \otimes S_x \rangle, \alpha_2 = \langle S_z \otimes S_z \rangle, \alpha_3 = \langle I \otimes S_z \rangle$ and $\alpha_4 = \langle S_z \otimes I \rangle$, with $I$ being the identity matrix.

In the $Z_2$ symmetry-broken phase, the presence of the nonzero local-order parameter $\langle S_z \rangle$ implies that the one-site-reduced density matrix takes the form

$$\rho_{\text{sing}} = \frac{1}{2} I + 2 \langle S_z \rangle S_z + 2 \langle S_z \rangle S_z. \quad (5)$$

Such a violation of the symmetry is also reflected in the two-site-reduced density matrix:

$$\rho_{\text{sing}} = \frac{1}{4} I + 4 \alpha_5 S_x \otimes S_x + 4 \alpha_6 S_z \otimes S_z + \alpha_7 I \otimes S_z + \alpha_8 S_z \otimes I + 4 \alpha_9 S_x \otimes S_x + 4 \alpha_{10} S_z \otimes S_z + \alpha_{11} I \otimes S_z + \alpha_{12} S_z \otimes I,$$

with $\alpha_5 = \langle S_x \otimes S_x \rangle, \alpha_6 = \langle S_z \otimes S_z \rangle, \alpha_7 = \langle I \otimes S_z \rangle$ and $\alpha_8 = \langle S_z \otimes I \rangle$.

For the quantum spin-1/2 XYX model in an external magnetic field, the one-site-reduced density matrix in the $Z_2$ symmetric phase takes the form

$$\rho_{\text{XYX}} = \frac{1}{2} I + 2 \langle S_z \rangle S_z, \quad (7)$$

while the two-site-reduced density matrix is

$$\rho_{\text{XYX}} = \frac{1}{4} I + 4 \beta_1 S_x \otimes S_x + 4 \beta_2 S_z \otimes S_z + 4 \beta_3 S_z \otimes S_z + \beta_4 I \otimes S_z + \beta_5 S_z \otimes I,$$

with $\beta_1 = \langle S_x \otimes S_x \rangle, \beta_2 = \langle S_z \otimes S_z \rangle, \beta_3 = \langle S_z \otimes S_z \rangle, \beta_4 = \langle I \otimes S_z \rangle$ and $\beta_5 = \langle S_z \otimes I \rangle$.

In the symmetry-broken phase, the one-site-reduced density matrix becomes

$$\rho_{\text{XYX}} = \frac{1}{2} I + 2 \langle S_z \rangle S_z + 2 \langle S_z \rangle S_z, \quad (9)$$

while the two-site-reduced density matrix is

$$\rho_{\text{XYX}} = \frac{1}{4} I + 4 \beta_1 S_x \otimes S_x + 4 \beta_2 S_z \otimes S_z + 4 \beta_3 S_z \otimes S_z + \beta_4 I \otimes S_z + \beta_5 S_z \otimes I,$$

with $\beta_1 = \langle S_x \otimes S_x \rangle, \beta_2 = \langle S_z \otimes S_z \rangle, \beta_3 = \langle S_z \otimes S_z \rangle, \beta_4 = \langle I \otimes S_z \rangle$ and $\beta_5 = \langle S_z \otimes I \rangle$. In this case, the critical magnetic field is $h_c \sim 3.210(6)$ [21].
whereas the two-site-reduced density matrix is
\[ \rho_{XYX} = \frac{1}{4} I + 4\beta_1 S_x \otimes S_x + 4\beta_2 S_x \otimes S_y + 4\beta_3 S_y \otimes S_x + \beta_4 I \otimes S_z + \beta_5 S_z \otimes I \]
\[ + 4\beta_6 S_x \otimes S_z + 4\beta_7 S_y \otimes S_z + \beta_8 I \otimes S_z + \beta_9 S_z \otimes I, \]
with \( \beta_6 = \langle S_x \otimes S_z \rangle, \beta_7 = \langle S_y \otimes S_z \rangle, \beta_8 = \langle I \otimes S_z \rangle \) and \( \beta_9 = \langle S_z \otimes I \rangle \).

**Bifurcations in the reduced fidelity between two reduced density matrices and QPTs**

The reduced fidelity measures the distance between two quantum mixed states. Specifically, for two reduced density matrices \( \rho_{\lambda} \) and \( \rho_{\lambda'} \), the reduced fidelity \( F(\rho_{\lambda}, \rho_{\lambda'}) \) is defined to be
\[ F(\rho_{\lambda}, \rho_{\lambda'}) = \text{tr} \left( \sqrt{\rho_{\lambda}^{1/2} \rho_{\lambda'} \rho_{\lambda}^{1/2}} \right). \]

Here, \( \rho_\lambda \) and \( \rho_{\lambda'} \) are the reduced density matrices corresponding to two different values, \( \lambda \) and \( \lambda' \), of the control parameter \( \lambda \). Note that the reduced fidelity \( F(\rho_{\lambda}, \rho_{\lambda'}) \) is a function of \( \lambda \) and \( \lambda' \), which satisfies the following properties: (i) normalization \( F(\rho_{\lambda}, \rho_{\lambda}) = 1 \); (ii) symmetry \( F(\rho_{\lambda'}, \rho_{\lambda}) = F(\rho_{\lambda}, \rho_{\lambda'}) \); (iii) range \( 0 \leq F(\rho_{\lambda}, \rho_{\lambda'}) \leq 1 \).

The connection between a bifurcation point in the ground-state partial fidelity between two reduced density matrices and a critical point for quantum lattice systems undergoing QPTs with symmetry-breaking order may be established in the following way. In the conventional Landau–Ginzburg–Wilson paradigm, a quantum system undergoing a QPT is characterized in terms of an SSB that is captured by a local-order parameter, which in turn results in an essential change of the reduced density matrix in the symmetry-broken phase, as seen above. More precisely, consider a quantum system on an infinite lattice in one spatial dimension, with \( Z_2 \) as a symmetry group\(^2\). Suppose the \( Z_2 \) symmetry is spontaneously broken when the control parameter \( \lambda \) crosses a critical point \( \lambda = \lambda_c \), then the system undergoes a QPT with a nonzero local-order parameter in the symmetry-broken phase. Such a local-order parameter is defined in a local area \( \Omega \) on a lattice, and it is *not* invariant under the action of the symmetry operation generating the symmetry group \( Z_2 \), thus yielding degenerate ground states in the symmetry-broken phase. Therefore, the reduced fidelity \( F(\rho_{\lambda}, \rho_{\lambda'}) \), with the reference state \( \rho_{\lambda'} \) in the symmetry-broken phase, yields *two* different values, which correspond to two distinct values of the local-order parameter arising from two degenerate ground states, if \( \rho_{\lambda} \) is in the symmetry-broken phase, and *one* value, if \( \rho_{\lambda} \) is in the symmetric phase. Indeed, this simply follows from the transformation of the reduced density matrices under local unitary operations associated with the global symmetry group \( Z_2 \). As such, a bifurcation point occurs in the ground-state-reduced fidelity between two reduced density matrices \( F(\rho_{\lambda}, \rho_{\lambda'}) \), with the critical point being the bifurcation point. However, if we choose the reference state \( \rho_{\lambda'} \) in the symmetric phase, then no such bifurcation occurs, due to the invariance of the reduced density matrix \( \rho_{\lambda'} \) under the action of the symmetry operation. In addition, the same argument applies to the reduced fidelity defined on any local area \( \Omega' \) with \( \Omega \) as a subset\(^3\).

The remaining problem now is to efficiently compute the reduced density matrix on a local area from a given ground-state wavefunction for a quantum many-body system. This may be achieved by exploiting the TN algorithms based on the MPS representation [13–15] or the multi-scale entanglement renormalization ansatz (MERA) [22] in one spatial dimension. Notably, there is no efficient way to compute the ground-state fidelity per site in the context of the algorithm based on the MERA representation [23], in contrast to the reduced fidelity. This external algorithm based on the MERA representation [23], in contrast to the reduced fidelity. This

\(^2\) The extension of our argument to any other group is straightforward. It also applies to quantum lattice systems in any spatial dimensions.

\(^3\) Moreover, one may establish the monotonicities of the reduced fidelity \( F(\rho_{\lambda}, \rho_{\lambda'}) \) with \( \lambda \) for a fixed \( \lambda' \), along a similar line to the ground-state fidelity per site [4], which may be used to identify stable fixed points.
characterizes an essential difference between the reduced fidelity approach and the fidelity per site approach from a practical perspective in the context of the MERA algorithms.

The results

In the context of the TN algorithm initiated by Vidal [24], the problem to find the system’s ground-state wavefunctions amounts to computing the imaginary time evolution for a given initial state $|\Psi(0)\rangle$: $|\Psi(\tau)\rangle = \exp(-H\tau)|\Psi(0)\rangle/\exp(-H\tau)|\Psi(0)\rangle$. An efficient way to achieve this task is to exploit the Suzuki–Trotter decomposition [25], which allows us to reduce the imaginary time evolution operation to a product of two-site evolution operators acting on sites $i$ and $i+1$: $U(i, i+1) = \exp(-\hbar^{(i,i+1)}\delta\tau)$, $\delta\tau \ll 1$. In addition, any wavefunction admits an MPS representation in a canonical form; attached to each site is a three-index tensor $U_{i,r}^a \equiv U_{i,r}^a$ and to each bond a diagonal singular value matrix $\lambda_A$ or $\lambda_B$, depending on the evenness and oddness of the $i$th site and the $i$th bond. Here, $s$ is a physical index, $s = 1, \ldots, d$, with $d$ being the dimension of the local Hilbert space, and $l$ and $r$ denote the bond indices, $l, r = 1, \ldots, \chi$, with $\chi$ being the truncation dimension. The action of a two-site gate $U(i, i+1)$ may be absorbed by performing a singular value decomposition, thus resulting in the update of the MPS representation. Repeating this procedure until the ground-state energy converges, one may generate the ground-state wavefunctions in the MPS representation. We emphasize that, in practice, we adjust the truncation dimension $\chi$ to identify a critical point from bifurcation points in the reduced fidelity $F(\rho_{\lambda,\rho_{\lambda}})$. In fact, it saves a lot of computational resources to perform simulations for relatively small values of $\chi$. Usually, a shift in the bifurcation points occurs due to the finiteness of $\chi$. Therefore, it is necessary to perform an extrapolation with respect to $\chi$ to locate the critical point.

In figure 1(a), we plot the ground-state-reduced fidelity $F(\rho_{\lambda,\rho_{\lambda}})$ between the one-site-reduced density matrices for the quantum Ising model in a transverse field with the field strength $\lambda$ as the control parameter. Here, we choose $\rho_{\lambda}$ ($\lambda' = 0.9$) as a reference state, which breaks the $Z_2$ symmetry. The one-site-reduced fidelity can distinguish two degenerate ground states with a bifurcation point as a pseudo-phase transition point $\lambda_{\chi} [8]$. When the parameter $\lambda$ is tuned beyond such a pseudo-transition point, two degenerate ground states vanish, implying that the system undergoes a phase transition. One observes that, with $\chi$ increasing, the pseudo-phase transition point $\lambda_{\chi}$ moves toward the exact value $1$. Performing an extrapolation of $\lambda_{\chi}$ with respect to $\chi$, we obtain $\lambda_c = 1.00233$. In figure 1(b), we show the two-site-reduced fidelity for the quantum Ising model in a transverse magnetic field. The same reference state is selected as in the case of the one-site-reduced fidelity. We observe that a bifurcation also occurs in the two-site partial fidelity. Indeed, it yields the same pseudo-phase transition points $\lambda_{\chi}$, and thus the same critical point $\lambda_c$.

In figure 2(a), the ground-state one-site-reduced fidelity $F(\rho_{h,\rho_{h}})$ for the quantum spin-1/2 XYX model in an external magnetic field $h$ is plotted, with the magnetic field strength $h$ as the control parameter. We choose $\rho_{h}$, with $h' = 3.05$, in the $Z_2$ symmetry-broken phase, as the reference state. The reduced fidelity $F(\rho_{h,\rho_{h}})$ distinguishes two degenerate ground states in the symmetry-broken phase. The bifurcation point $h_x$ resulted from the one-site-reduced fidelity is the pseudo-phase transition point, which is quite close to the known critical value $h_c \sim 3.210(6)$ [21], if the truncation dimension $\chi$ is large enough. Performing an extrapolation with respect to $\chi$ yields the critical point $h_c = 3.2049$. In figure 2(b), we plot the two-site-reduced fidelity for the quantum XYX model. Here, the same reference state as in the case of the one-site partial fidelity has been chosen. The two-site-reduced fidelity $F(\rho_{h,\rho_{h}})$ is also able to detect the $Z_2$ SSB. Indeed, it again yields the same bifurcation points $h_x$, and thus the same critical point $h_c$. 
Figure 1. (a) The ground-state one-site-reduced fidelity $F(\rho_\lambda, \rho_{\lambda'}')$ for the quantum Ising model in a transverse magnetic field. We choose $\rho_{\lambda'}'$ (with $\lambda' = 0.9$) as the reference state, which is in the $Z_2$ symmetry-broken phase. The pseudo-phase transition point $\lambda_{\chi}$ occurs as a bifurcation point. When we enlarge the truncation dimension $\chi$, the pseudo-phase transition point $\lambda_{\chi}$ is getting closer to the exact value. Inset: the critical point $\lambda_c$ is determined from an extrapolation of the pseudo-phase transition point $\lambda_{\chi}$ with respect to the truncation dimension $\chi$. Here, the fitting function is $\lambda_{\chi} = \lambda_c + a\chi^{-b}$, with $\lambda_c = 1.00233$, $a = 0.39373$ and $b = 2.43904$. (b) The ground-state two-site-reduced fidelity $F(\rho_\lambda, \rho_{\lambda'}')$, for the quantum Ising model in a transverse magnetic field. The same reference state $\rho_{\lambda'}'$ ($\lambda' = 0.9$) has been chosen as in the case of the one-site reduced fidelity. Both insets indicate the same pseudo-critical points for the ground-state one-site- and two-site-reduced density matrices for the system. This is expected due to the fact that they result from the same set of the ground states.

Summary

We have established an intriguing connection between QPTs and bifurcations in the reduced fidelity between two different reduced density matrices for quantum lattice many-body systems with symmetry-breaking order. Our work is based on the newly developed TN algorithms, which produce degenerate ground states in the symmetry-broken phase for quantum lattice
systems under QPTs arising from an SSB. Two quantum systems on an infinite lattice in one spatial dimension, i.e. the quantum Ising model in a transverse magnetic field and the quantum spin-1/2 XYX model in an external magnetic field, have been investigated in the context of the TN algorithm based on the MPS representation. Here, we stress that the reduced fidelity approach does not have any advantage over the fidelity per site approach in the context of the MPS representation. However, it does allow us to take advantage of the MERA algorithms, which are not efficient to compute the fidelity per site.
In conclusion, our work not only presents a practical way to locate a transition point, but also offers an efficient means to distinguish symmetry-breaking order from topological order, given that the ground-state fidelity per site is able to detect a phase transition irrespective of the internal order for a quantum many-body system. This follows from the fact that no bifurcation would, by definition, occur at a critical point for a quantum many-body system with topological order.

Acknowledgments

This work is supported in part by the National Natural Science Foundation of China (grant nos 10774197 and 10874252), the Natural Science Foundation of Chongqing (grant no CSTC, 2008BC2023), the Chongqing University Postgraduates’ Science and Innovation Fund, (project no 200911C1A0060322).

References

[1] Sachdev S 1999 Quantum Phase Transitions (Cambridge: Cambridge University Press)
[2] Wen X-G 2004 Quantum Field Theory of Many-Body Systems (Oxford: Oxford University Press)
[3] Zanardi P and Paunkovič N 2006 Phys. Rev. E 74 031123
[4] Zhou H-Q and Barjačkarevič J P 2008 J. Phys. A: Math. Theor. 41 412001
[5] Zhou H-Q, Zhao J-H and Li B 2008 J. Phys. A: Math. Theor. 41 492002
[6] Zhou H-Q 2007 arXiv:0704.2945
[7] Zhou H-Q, Orús R and Vidal G 2008 Phys. Rev. Lett. 100 080601
[8] Zhao J-H, Wang H-L, Li B and Zhou H-Q 2010 Phys. Rev. E 82 061127
[9] Wang H-L, Zhao J-H, Li B and Zhou H-Q 2011 J. Stat. Mech. L10001
[10] Zhao J-H and Zhou H-Q 2009 Phys. Rev. B 80 014405
[11] Zanardi P, Cozzini M and Giorda P 2007 J. Stat. Mech. L02002
   Oelkers N and Links J 2007 Phys. Rev. B 75 115119
   Buonsante P and Vezzani A 2007 Phys. Rev. Lett. 98 110601
   Cozzini M, Iomiciciou R and Zanardi P 2007 Phys. Rev. B 76 104420
   Campos Venuti L and Zanardi P 2007 Phys. Rev. Lett. 99 095701
   You W-L, Li Y-W and Gu S-J 2007 Phys. Rev. E 76 022101
   Gu S-J et al 2008 Phys. Rev. B 77 245109
   Tzeng Y C and Yang M F 2008 Phys. Rev. A 77 012311
   Yang M F 2007 Phys. Rev. B 76 180403
   Fjaerestad J O 2008 J. Stat. Mech. P07011
   Chen S, Wang L, Hao Y and Wang Y 2008 Phys. Rev. A 77 032111
   Invernizzi C and Paris M 2010 J. Mod. Opt. 57 198–206
   Khan A and Pieri P 2009 Phys. Rev. A 80 012303
   Santos G et al 2010 Phys. Rev. A 81 063621
[12] Paunkovič N, Sacramento P D, Nogueira P, Vieira V R and Dugaev V K 2008 Phys. Rev. A 77 052302
   Kwok H M, Ho C S and Gu S-J 2008 Phys. Rev. A 78 062302
   Ma J, Xu L, Xiong H and Wang X 2008 Phys. Rev. E 78 051126
   Eriksson E and Johannesson H 2009 Phys. Rev. A 79 060301
[13] Fannes M, Nachtergaele B and Werner R F 1992 Commun. Math. Phys. 144 443
   Fannes M, Nachtergaele B and Werner R F 1994 J. Funct. Anal. 120 511
   Ostlund S and Rommer S 1995 Phys. Rev. Lett. 75 3537
[14] Perez-García D et al 2007 Quantum Inform. Comput. 7 401 (arXiv:quant-ph/0608197)
[15] Verstraete F, Forras D and Cirac J I 2004 Phys. Rev. Lett. 93 227205
[16] Takasaki H, Hikihara T and Nishino T 1999 J. Phys. Soc. Japan 68 1537
   Nishino T et al 2000 Nucl. Phys. B 575 504
[17] Verstraete F and Cirac J I 2004 arXiv:cond-mat/0407066
[18] Anderson P W 1997 Basic Notions of Condensed Matter Physics (The Advanced Book Program) (Reading, MA: Addison-Wesley)
[19] Coleman S 1975 *An Introduction to Spontaneous Symmetry Breakdown and Gauge Fields, Laws of Hadronic Matter* ed A Zichichi (New York: Academic)
[20] Lieb E, Schultz T and Mattis D 1961 *Ann. Phys.* 60 407
Pleuty P 1970 *Ann. Phys.* 57 79
[21] Kurmann J et al 1982 *Physica A* 112 235
Pfeuty P 1970 *Ann. Phys.* 57 79
[22] Kurmann J et al 1982 *Physica A* 112 235
Dmitriev D V et al 2002 *J. Exp. Theor. Phys.* 95 538
Roscilde T et al 2004 *Phys. Rev. Lett.* 93 167203
[23] Vidal G and Vidal G 2009 *Phys. Rev. B* 79 144108
[24] Vidal G 2008 private communication
[25] Vidal G 2007 *Phys. Rev. Lett.* 98 070201
[26] Suzuki M 1990 *Phys. Lett. A* 146 319
[27] Jordan J et al 2008 *Phys. Rev. Lett.* 101 250602