Generalised golden ratios over integer alphabets

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Abstract

It is a well known result that for $\beta \in (1, \frac{1+\sqrt{5}}{2})$ and $x \in (0, \frac{1}{\beta-1})$ there exists uncountably many $(\epsilon_i)_{i=1}^\infty \in \{0, 1\}^\mathbb{N}$ such that $x = \sum_{i=1}^\infty \epsilon_i \beta^{-i}$. When $\beta \in (\frac{1+\sqrt{5}}{2}, 2]$ there exists $x \in (0, \frac{1}{\beta-1})$ for which there exists a unique $(\epsilon_i)_{i=1}^\infty \in \{0, 1\}^\mathbb{N}$ such that $x = \sum_{i=1}^\infty \epsilon_i \beta^{-i}$. In this paper we consider the more general case when our sequences are elements of $\{0, \ldots, m\}^\mathbb{N}$. We show that an analogue of the golden ratio exists and give an explicit formula for it.

1 Introduction

Let $m \in \mathbb{N}$, $\beta \in (1, m+1]$ and $I_{\beta,m} = [0, \frac{m}{\beta-1}]$. Each $x \in I_{\beta,m}$ has an expansion of the form

$$x = \sum_{i=1}^\infty \frac{\epsilon_i}{\beta^i}$$

for some $(\epsilon_i)_{i=1}^\infty \in \{0, \ldots, m\}^\mathbb{N}$. We call such a sequence a $\beta$-expansion for $x$. For $x \in I_{\beta,m}$ we denote the set of $\beta$-expansions for $x$ by $\Sigma_{\beta,m}(x)$, i.e.,

$$\Sigma_{\beta,m}(x) = \left\{ (\epsilon_i)_{i=1}^\infty \in \{0, \ldots, m\}^\mathbb{N} : \sum_{i=1}^\infty \frac{\epsilon_i}{\beta^i} = x \right\}.$$ 

In [6] the authors consider the case when $m = 1$, they show that for $\beta \in (1, \frac{1+\sqrt{5}}{2})$ the set $\Sigma_{\beta,1}(x)$ is uncountable for every $x \in (0, \frac{1}{\beta-1})$. The endpoints of $[0, \frac{1}{\beta-1}]$ trivially have a unique $\beta$-expansion. In [5] it is shown that for $\beta \in (\frac{1+\sqrt{5}}{2}, 2]$ there exists $x \in (0, \frac{1}{\beta-1})$ with a unique $\beta$-expansion.

For $m \in \mathbb{N}$ we define $G(m) \in \mathbb{R}$ to be a generalised golden ratio for $m$ if for $\beta \in (1, G(m))$ the set $\Sigma_{\beta,m}(x)$ is uncountable for every $x \in (0, \frac{m}{\beta-1})$, and for $\beta \in (G(m), m+1]$ there exists $x \in (0, \frac{m}{\beta-1})$ for which $|\Sigma_{\beta,m}(x)| = 1$.

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In [11] the authors consider a similar setup. They consider the case where $\beta$-expansions are elements of $\{a_1, a_2, a_3\}^N$, for some $a_1, a_2, a_3 \in \mathbb{R}$. They show that for each ternary alphabet there exists a constant $G \in \mathbb{R}$ such that, there exists nontrivial unique $\beta$-expansions if and only if $\beta > G$. Moreover they give an explicit formula for $G$.

Our main result is the following.

**Theorem 1.1.** For each $m \in \mathbb{N}$ a generalised golden ratio exists and is equal to:

$$G(m) = \begin{cases} k + 1 & \text{if } m = 2k \\ \frac{k+1+\sqrt{k^2+6k+5}}{2} & \text{if } m = 2k+1. \end{cases}$$

**Remark 1.2.** $G(m)$ is a Pisot number for all $m \in \mathbb{N}$. Recall a Pisot number is a real algebraic integer greater than 1 whose Galois conjugates are of modulus strictly less than 1.

In section 6 we include a table of values for $G(m)$. We prove Theorem 1.1 in section 3. In section 4 we consider the set of points with unique $\beta$-expansion for $\beta \in (G(m), m+1]$, and in section 5 we study the growth rate and dimension theory of the set of $\beta$-expansions for $\beta \in (1, G(m))$.

## 2 Preliminaries

Before proving Theorem 1.1 we require the following preliminary results and theory. Let $m \in \mathbb{N}$ be fixed and $\beta \in (1, m+1]$. For $i \in \{0, \ldots, m\}$ we fix $T_{\beta,i}(x) = \beta x - i$. The proof of the following lemma is trivial and therefore omitted.

**Lemma 2.1.** The map $T_{\beta,i}$ satisfies the following:

- $T_{\beta,i}$ has a unique fixed point equal to $\frac{i}{\beta-1}$.
- $T_{\beta,i}(x) > x$ for all $x > \frac{i}{\beta-1}$,
- $T_{\beta,i}(x) < x$ for all $x < \frac{i}{\beta-1}$,
- $|T_{\beta,i}(x) - T_{\beta,i}(\frac{i}{\beta-1})| = \beta|x - \frac{i}{\beta-1}|$, for all $x \in \mathbb{R}$, that is $T_{\beta,i}$ scales the distance between the fixed point $\frac{i}{\beta-1}$ and an arbitrary point by a factor $\beta$.

Understanding where in $I_{\beta,m}$ these fixed points are will be important in our later analysis.

We let

$$\Omega_{\beta,m}(x) = \left\{ (a_i)_{i=1}^{\infty} \in \{T_{\beta,0} \ldots T_{\beta,m}\}^\mathbb{N} : (a_n \circ a_{n-1} \circ \ldots \circ a_1)(x) \in I_{\beta,m} \text{ for all } n \in \mathbb{N} \right\}.$$ 

Similarly we define

$$\Omega_{\beta,m,n}(x) = \left\{ (a_i)_{i=1}^{n} \in \{T_{\beta,0} \ldots T_{\beta,m}\}^n : (a_n \circ a_{n-1} \circ \ldots \circ a_1)(x) \in I_{\beta,m} \right\}.$$
Typically we will denote an element of \( \Omega_{\beta,m,n}(x) \) or any finite sequence of maps by \( a \). When we want to emphasise the length of \( a \) we will use the notation \( a^{(n)} \). We also adopt the notation \( a^{(n)}(x) \) to mean \((a_n \circ a_{n-1} \circ \ldots \circ a_1)(x)\).

**Remark 2.2.** It is important to note that if for some finite sequence of maps \( a, a(x) \notin I_{\beta,m} \) then we cannot concatenate \( a \) by any finite sequence of maps \( b \), such that \( b(a(x)) \in I_{\beta,m} \).

**Remark 2.3.** Let \( \beta \in (1, m+1] \), for any \( x \in I_{\beta,m} \) there always exists \( i \in \{0, \ldots, m\} \) such that \( T_{\beta,i}(x) \in I_{\beta,m} \). For \( \beta > m + 1 \) such an \( i \) does not always exist.

**Lemma 2.4.** \( |\Sigma_{\beta,m}(x)| = |\Omega_{\beta,m}(x)| \).

**Proof.** It is a simple exercise to show that

\[
\Sigma_{\beta,m}(x) = \{(e_i)_{i \in \mathbb{N}} \in \{0, \ldots, m\}^\mathbb{N} : x - \sum_{i=1}^{n} \frac{e_i}{\beta^i} \in \left[0, \frac{m}{\beta^n(\beta-1)}\right] \text{ for all } n \in \mathbb{N}\}.
\]

Following [8] we observe that

\[
\Sigma_{\beta,m}(x) = \{(e_i)_{i \in \mathbb{N}} \in \{0, \ldots, m\}^\mathbb{N} : x - \sum_{i=1}^{n} \frac{e_i}{\beta^i} \in \left[0, \frac{m}{\beta^n(\beta-1)}\right] \text{ for all } n \in \mathbb{N}\}
\]

\[
= \{(e_i)_{i \in \mathbb{N}} \in \{0, \ldots, m\}^\mathbb{N} : \beta^{-n}x - \sum_{i=1}^{n} e_i\beta^{n-i} \in I_{\beta,m} \text{ for all } n \in \mathbb{N}\}
\]

\[
= \{(e_i)_{i \in \mathbb{N}} \in \{0, \ldots, m\}^\mathbb{N} : (T_{\beta,e_1} \circ \ldots \circ T_{\beta,e_n})(x) \in I_{\beta,m} \text{ for all } n \in \mathbb{N}\}.
\]

Our result follows immediately. \( \square \)

By Lemma 2.4 we can rephrase the definition of a generalised golden ratio in terms of the set \( \Omega_{\beta,m}(x) \). This equivalent definition will be more suitable for our purposes. The set \( \Omega_{\beta,m,n}(x) \) will be useful when we study the growth rate and dimension theory of the set of \( \beta \)-expansions.

For a point \( x \in I_{\beta,m} \) we can take \( i \) to be the first digit in a \( \beta \)-expansion for \( x \) if and only if \( \beta x - i \in I_{\beta,m} \). This is equivalent to

\[
x \in \left[\frac{i}{\beta}, \frac{i \beta + m - i}{\beta(\beta-1)}\right],
\]

as such we refer to the interval \( \left[\frac{i}{\beta}, \frac{i \beta + m - i}{\beta(\beta-1)}\right] \) as the \( i \)-th digit interval. Generally speaking we can take \( i \) to be the \( j \)-th digit in a \( \beta \)-expansion for \( x \) if and only if there exists \( a \in \Omega_{\beta,m,j-1}(x) \) such that, \( a(x) \in \left[\frac{j}{\beta}, \frac{j \beta + m - j}{\beta(\beta-1)}\right] \). When \( x \) or an image of \( x \) is contained in the intersection of two digit intervals we have a choice of digit in our \( \beta \)-expansion for \( x \). Generally speaking any two digit intervals may intersect for \( \beta \) sufficiently small, however for our purposes we need only consider the case when the \( i \)-th digit interval intersects the adjacent \((i-1)\)-th or \((i+1)\)-th digit intervals, for some \( i \in \{0, \ldots, m\} \). Any intersection of this type is of the form

\[
\left[\frac{i}{\beta}, \frac{(i-1) \beta + m - (i-1)}{\beta(\beta-1)}\right],
\]
for some \( i \in \{1, \ldots, m\} \). In what follows we refer to the interval \( \left[ \frac{1}{\beta}, \frac{(i-1)\beta + m - (i-1)}{\beta(\beta-1)} \right] \) as the \( i \)-th choice interval. Both \( T_{\beta,i-1} \) and \( T_{\beta,i} \) map the \( i \)-th choice interval into \( I_{\beta,m} \). These intervals always exist and are nontrivial for \( \beta \in (1, m + 1) \).

**Proposition 2.5.** Suppose for any \( x \in (0, \frac{m}{\beta-1}) \) there always exists a finite sequence of maps that map \( x \) into the interior of a choice interval, then \( \Omega_{\beta,m}(x) \) is uncountable.

The proof of this proposition is essentially contained in the proof of Theorem 1 in [17].

**Proof.** Let \( x \in (0, \frac{m}{\beta-1}) \). Suppose there exists \( n \in \mathbb{N} \) and \( a \in \Omega_{\beta,m,n}(x) \) such that \( a(x) \in (\frac{1}{\beta}, \frac{(i-1)\beta + m - (i-1)}{\beta(\beta-1)}) \), for some \( i \in \{1, \ldots, m\} \). As \( a(x) \) is an element of the interior of a choice interval both \( T_{\beta,i-1}(a(x)) \in (0, \frac{m}{\beta-1}) \) and \( T_{\beta,i}(a(x)) \in (0, \frac{m}{\beta-1}) \). As such our hypothesis applies to both \( T_{\beta,i-1}(a(x)) \) and \( T_{\beta,i}(a(x)) \), and we can assert that there exists a finite sequence of maps that map these two distinct images of \( x \) into the interior of another choice interval. Repeating this procedure arbitrarily many times it is clear that \( \Omega_{\beta,m}(x) \) is uncountable.

By Proposition 2.5 to prove Theorem 1.1 it suffices to show that for \( \beta \in (1, G(m)) \) every \( x \in (0, \frac{m}{\beta-1}) \) can be mapped into the interior of a choice interval, and for \( \beta \in (G(m), m + 1] \) there exists \( x \in (0, \frac{m}{\beta-1}) \) that never maps into a choice interval.

We define the switch region to be the interval

\[
\left[ \frac{1}{\beta}, \frac{(m-1)\beta + 1}{\beta(\beta-1)} \right].
\]

The significance of this interval is that if a point \( x \) has a choice of digit in the \( j \)-th entry of a \( \beta \)-expansion, then there exists \( a \in \Omega_{\beta,m,j-1}(x) \) such that \( a(x) \in \left( \frac{1}{\beta}, \frac{(m-1)\beta + 1}{\beta(\beta-1)} \right] \). The following lemmas are useful in understanding the dynamics of the maps \( T_{\beta,i} \) around the switch region, understanding these dynamics will be important in our proof of Theorem 1.1.

**Lemma 2.6.** For \( \beta \in (1, \frac{m + \sqrt{m^2 + 1}}{2}) \) and \( x \in (0, \frac{m}{\beta-1}) \) there exists a finite sequence of maps that map \( x \) into the interior of our switch region.

**Proof.** If \( x \) is contained within the interior of the switch region we are done, let us suppose otherwise. By the monotonicity of the maps \( T_{\beta,0} \) and \( T_{\beta,m} \) it suffices to show that

\[ T_{\beta,0} \left( \frac{1}{\beta} \right) < \frac{(m-1)\beta + 1}{\beta(\beta-1)} \quad \text{and} \quad T_{\beta,m} \left( \frac{(m-1)\beta + 1}{\beta(\beta-1)} \right) > \frac{1}{\beta}. \]

Both of these inequalities are equivalent to \( \beta^2 - m\beta - 1 < 0 \), applying the quadratic formula we can conclude our result.

**Remark 2.7.** When \( m = 1 \) the switch region is a choice interval. An application of Lemma 2.4, Proposition 2.5 and Lemma 2.6 yields the result stated in [8], i.e., for \( \beta \in (1, \frac{1 + \sqrt{5}}{2}) \) and \( x \in (0, \frac{1}{\beta-1}) \) the set \( \Sigma_{\beta,1}(x) \) is uncountable.
Lemma 2.8. For $\beta \in (1, \frac{m+2}{2})$ every $x$ in the interior of the switch region is contained in the interior of a choice interval.

Proof. It suffices to show that for each $i \in \{1, 2, \ldots, m - 1\}$ the $(i - 1)$-th and $(i + 1)$-th digit intervals intersect in a nontrivial interval. This is equivalent to

$$\frac{i + 1}{\beta} < \frac{(i - 1)\beta + m - (i - 1)}{\beta(\beta - 1)},$$

a simple manipulation yields that this is equivalent to $\beta < \frac{m+2}{2}$. □

We refer the reader to Figure 1 for a diagram depicting the case where $\beta < \frac{m+2}{2}$. For $i \in \{1, 2, \ldots, m - 1\}$ and $\beta \geq \frac{m+2}{2}$ the interval

$$\left[\frac{(i - 1)\beta + m - (i - 1)}{\beta(\beta - 1)}, \frac{i + 1}{\beta}\right]$$

is well defined. We refer to this interval as the $i$-th fixed digit interval. The significance of this interval is that if a point $x$ is contained in the interior of the $i$-th fixed digit interval only $T_{\beta, i}$ maps $x$ into $I_{\beta, m}$. Similarly we define the 0-th fixed digit interval to be $[0, \frac{1}{\beta}]$ and the $m$-th fixed digit interval to be $[\frac{(m-1)\beta+1}{\beta(\beta-1)}, \frac{m}{\beta-1}]$. Understanding how the different $T_{\beta, i}$'s behave on these intervals will be important when it comes to constructing generalised golden ratios in the case where $m$ is odd.
3 Proof of Theorem 1.1

We are now in a position to prove Theorem 1.1; for ease of exposition we reduce our analysis to two cases, when \( m \) is even and when \( m \) is odd.

3.1 Case where \( m \) is even

In what follows we assume \( m = 2k \) for some \( k \in \mathbb{N} \).

**Proposition 3.1.** For \( \beta \in (1, k + 1) \) every \( x \in (0, \frac{m}{\beta - 1}) \) has uncountably many \( \beta \)-expansions.

**Proof.** By Lemma 2.4 and Proposition 2.5 it suffices to show that every \( x \in (0, \frac{m}{\beta - 1}) \) can be mapped into the interior of a choice interval. It is a simple exercise to show that \( \frac{m+2}{2} < \frac{m^2+\sqrt{m^2+4}}{2} \) for all \( m \in \mathbb{N} \), as such for \( \beta \in (1, k + 1) \) we can apply Lemma 2.6 therefore there exists a sequence of maps that map \( x \) into the interior of the switch region. By Lemma 2.8 every point in the interior of our switch region is contained in the interior of a choice interval.

**Proposition 3.2.** For \( \beta \in (k + 1, m + 1] \) there exists \( x \in (0, \frac{m}{\beta - 1}) \) with a unique \( \beta \)-expansion.

**Proof.** It suffices to show that there exists \( x \in (0, \frac{m}{\beta - 1}) \) that never maps into a choice interval.

We consider the point \( \frac{k}{\beta - 1} \), we will show that this point has a unique \( \beta \)-expansion. This point is contained in the \( k \)-th digit interval and is the fixed point under the map \( T_{\beta,k} \). To show that it has a unique \( \beta \)-expansion it suffices to show that it is not contained within the \( (k - 1) \)-th or \( (k + 1) \)-th digit intervals, this is equivalent to

\[
\frac{(k - 1)\beta + m - (k - 1)}{\beta(\beta - 1)} < \frac{k}{\beta - 1} < \frac{k + 1}{\beta}.
\]

Both of these inequalities are equivalent to \( \beta > k + 1 \).

Figure 2 describes the construction of our point with unique \( \beta \)-expansion for \( \beta \in (k + 1, m + 1] \). By Proposition 3.1 and Proposition 3.2 we can conclude Theorem 1.1 in the case where \( m \) is even.

3.2 Case where \( m \) is odd

The analysis of the case where \( m \) is odd is somewhat more intricate. In what follows we assume \( m = 2k + 1 \) for some \( k \in \mathbb{N} \). Before finishing our proof of Theorem 1.1 we require the following technical results.

**Lemma 3.3.** For \( \beta \in (1, k + 2) \) the fixed point of \( T_{\beta,i} \) is contained in the interior of the choice interval \( \left[ \frac{i}{\beta}, \frac{(i-1)\beta + m - (i-1)}{\beta(\beta - 1)} \right] \) for \( i \in \{1, \ldots, k\} \), and in the interior of the choice interval \( \left[ \frac{i+1}{\beta}, \frac{i\beta + m - i}{\beta(\beta - 1)} \right] \) for \( i \in \{k + 1, \ldots, m - 1\} \).
Figure 2: A point with unique $\beta$-expansion for $\beta \in (k + 1, m + 1]$.

**Proof.** Let $i \in \{1, \ldots, k\}$. To show that the fixed point $\frac{i}{\beta - 1}$ is contained in the interior of the interval $[\frac{i}{\beta}, \frac{(i-1)\beta + m - (i-1)}{\beta(\beta - 1)}]$ it suffices to show that

$$\frac{i}{\beta - 1} < \frac{(i-1)\beta + m - (i-1)}{\beta(\beta - 1)}.$$ 

This is equivalent to $\beta < m + 1 - i$, which for $\beta \in (1, k + 2)$ is true for all $i \in \{1, \ldots, k\}$. The case where $i \in \{k + 1, \ldots, m - 1\}$ is proved similarly.

**Corollary 3.4.** For $\beta \in \left[\frac{2k+3}{2}, k + 2\right]$ the map $T_{\beta,i}$ satisfies $T_{\beta,i}(x) = \beta(x - \frac{i}{\beta - 1})$ for all $x$ contained in the $i$-th fixed digit interval for $i \in \{1, \ldots, k\}$, and $\frac{i}{\beta - 1} - T_{\beta,i}(x) = \beta(\frac{i}{\beta - 1} - x)$ for all $x$ contained in the $i$-th fixed digit interval for $i \in \{k + 1, \ldots, m - 1\}$.

**Proof.** Let $i \in \{1, \ldots, k\}$, by Lemma 3.3 the $i$-th fixed digit interval is to the right of the fixed point of $T_{\beta,i}$, our result follows from Lemma 2.1. The case where $i \in \{k + 1, \ldots, m - 1\}$ is proved similarly.

**Lemma 3.5.** Suppose $\beta \in \left[\frac{2k+3}{2}, \frac{k+1+\sqrt{k^2+6k+5}}{2}\right]$ and $x$ is an element of the $i$-th fixed digit interval for some $i \in \{1, \ldots, m - 1\}$. For $i \in \{1, \ldots, k\}$

$$T_{\beta,i}(x) < \frac{k\beta + m - k}{\beta(\beta - 1)}$$

and for $i \in \{k + 1, \ldots, m - 1\}$

$$T_{\beta,i}(x) > \frac{k+1}{\beta}.$$
**Proof.** By the monotonicity of the maps $T_{\beta,i}$ it is sufficient to show that

$$T_{\beta,i}\left(\frac{i+1}{\beta}\right) < \frac{k\beta + m - k}{\beta(\beta - 1)}$$

for $i \in \{1, \ldots, k\}$, and

$$T_{\beta,i}\left(\frac{(i-1)\beta + m - (i-1)}{\beta(\beta - 1)}\right) > \frac{k+1}{\beta},$$

for $i \in \{k+1, \ldots, m-1\}$. Each of these inequalities are equivalent to $\beta^2 - (k+1)\beta - (k+1) < 0$. Our result follows by an application of the quadratic formula. \qed

**Proposition 3.6.** For $\beta \in (1, \frac{k+1 + \sqrt{k^2 + 6k + 5}}{2})$ every $x \in (0, \frac{m}{\beta-1})$ has uncountably many $\beta$-expansions.

**Proof.** The proof where $\beta \in (1, \frac{2k+3}{2})$ is analogous to that given in the even case. As such, in what follows we assume $\beta \in \left[\frac{2k+3}{2}, \frac{k+1 + \sqrt{k^2 + 6k + 5}}{2}\right)$. We remark that

$$\frac{k+1 + \sqrt{k^2 + 6k + 5}}{2} \leq \frac{m + \sqrt{m^2 + 4}}{2}$$

and

$$\frac{k+1 + \sqrt{k^2 + 6k + 5}}{2} < k+2,$$

for all $k \in \mathbb{N}$. We can therefore use Lemma 2.6 and Corollary 3.4. Let $x \in (0, \frac{m}{\beta-1})$, we will show that there exists a sequence of maps that map $x$ into the interior of a choice interval, by Lemma 2.4 and Proposition 2.5 our result follows. By Lemma 2.6 there exist a finite sequence of maps that map $x$ into the interior of the switch region. Suppose the image of $x$ is not contained in the interior of a choice interval, then it must be contained in the $i$-th fixed digit interval for some $i \in \{1, \ldots, m-1\}$. By repeatedly applying Corollary 3.4 and Lemma 3.5 the image of $x$ must eventually be mapped into the interior of a choice interval. \qed

We refer the reader to Figure 3 for a diagram illustrating the case where $m = 2k+1$ and $\beta \in \left(\frac{2k+3}{2}, \frac{k+1 + \sqrt{k^2 + 6k + 5}}{2}\right)$.

**Proposition 3.7.** For $\beta \in \left(\frac{k+1 + \sqrt{k^2 + 6k + 5}}{2}, m+1\right]$ there exists $x \in (0, \frac{m}{\beta-1})$ that has a unique $\beta$-expansion.

**Proof.** We will show that the points

$$\frac{k\beta + (k+1)}{\beta^2 - 1} \text{ and } \frac{(k+1)\beta + k}{\beta^2 - 1}$$

have a unique $\beta$-expansion. The significance of these points is that

$$T_{\beta,k}\left(\frac{k\beta + (k+1)}{\beta^2 - 1}\right) = \frac{(k+1)\beta + k}{\beta^2 - 1}$$
To show that these points have a unique $\beta$-expansion it suffices to show that
\[
\frac{(k - 1)\beta + m - (k - 1)}{\beta(\beta - 1)} < \frac{k\beta + (k + 1)}{\beta^2 - 1} < \frac{k + 1}{\beta},
\]
(2)

and
\[
\frac{k\beta + (m - k)}{\beta(\beta - 1)} < \frac{(k + 1)\beta + k}{\beta^2 - 1} < \frac{k + 2}{\beta}.
\]
(3)

The left hand side of (2) is equivalent to $0 < \beta^2 - k\beta - (k + 2)$ which is equivalent to
\[
\frac{k + \sqrt{k^2 + 4k + 8}}{2} < \beta,
\]
however
\[
\frac{k + \sqrt{k^2 + 4k + 8}}{2} < \frac{k + 1 + \sqrt{k^2 + 6k + 5}}{2}
\]
for all $k \in \mathbb{N}$, therefore the left hand side of (2) holds. The right hand side of (2) is equivalent to $0 < \beta^2 - (k + 1)\beta - (k + 1)$. So (2) holds by the quadratic formula.

The right hand side of (3) is equivalent to $0 < \beta^2 - k\beta - (k + 2)$ which we know to be true by the above. Similarly the left hand side of (3) is equivalent to $0 < \beta^2 - (k + 1)\beta - (k + 1)$,
which we also know to be true. It follows that both \( \frac{k\beta + (k+1)}{\beta^2 - 1} \) and \( \frac{(k+1)\beta + k}{\beta^2 - 1} \) are never mapped into a choice interval and have a unique \( \beta \)-expansion for \( \beta \in \left( \frac{k+1+\sqrt{k^2+6k+5}}{2}, m + 1 \right) \).

We refer the reader to Figure 4 for a diagram describing the points we constructed with unique \( \beta \)-expansion for \( \beta \in \left( \frac{k+1+\sqrt{k^2+6k+5}}{2}, m + 1 \right) \). By Proposition 3.6 and Proposition 3.7 we can conclude Theorem 1.1.

4 The set of points with unique \( \beta \)-expansion

In this section we study the set of points whose \( \beta \)-expansion is unique for \( \beta \in (\mathcal{G}(m), m + 1] \). Let

\[
U_{\beta,m} = \left\{ x \in I_{\beta,m} \mid |\Sigma_{\beta,m}(x)| = 1 \right\}
\]

and

\[
W_{\beta,m} = \left\{ x \in \left( \frac{m + 1 - \beta}{\beta - 1}, 1 \right) \mid |\Sigma_{\beta,m}(x)| = 1 \right\}.
\]

The significance of the set \( W_{\beta,m} \) is that if \( x \in U_{\beta,m} \), then it is a preimage of an element of \( W_{\beta,m} \). In [9] the authors study the case where \( m = 1 \), they show that the following theorems hold.

**Theorem 4.1.** The set \( U_{\beta,1} \) satisfies the following:

1. \( |U_{\beta,1}| = \aleph_0 \) for \( \beta \in (\frac{1+\sqrt{5}}{2}, \beta_c) \).
2. \(|U_{\beta,1}| = 2^{\aleph_0}\) for \(\beta = \beta_c\).

3. \(U_{\beta,1}\) is a set of positive Hausdorff dimension for \(\beta \in (\beta_c, 2]\).

**Theorem 4.2.** The set \(W_{\beta,1}\) satisfies the following:

1. \(|W_{\beta,1}| = 2\) for \(\beta \in \left(\frac{1+\sqrt{5}}{2}, \beta_f\right]\), where \(\beta_f\) is the root of the equation \(x^3 - 2x^2 + x - 1 = 0, \ \beta_f = 1.75487\ldots\)

2. \(|W_{\beta,1}| = \aleph_0\) for \(\beta \in (\beta_f, \beta_c)\)

3. \(|W_{\beta,1}| = 2^{\aleph_0}\) for \(\beta = \beta_c\)

4. \(W_{\beta,1}\) is a set of positive Hausdorff dimension for \(\beta \in (\beta_c, 2]\).

Here \(\beta_c \approx 1.78723\) is the Komornik-Loreti constant introduced in [12]. It is the smallest value of \(\beta\) for which \(1 \in U_{\beta,1}\). Moreover \(\beta_c\) is the unique solution of the equation

\[
\sum_{i=1}^{\infty} \frac{\lambda_i}{\beta^i} = 1,
\]

where \((\lambda_i)_{i=0}^{\infty}\) is the Thue-Morse sequence (see [3]), i.e. \(\lambda_0 = 0\) and if \(\lambda_i\) is already defined for some \(i \geq 0\) then \(\lambda_{2i} = \lambda_i\) and \(\lambda_{2i+1} = 1 - \lambda_i\). The sequence \((\lambda_i)_{i=0}^{\infty}\) begins

\[(\lambda_i)_{i=0}^{\infty} = 0110\ 1001\ 1001\ 0110\ 1001\ \ldots\ .\]

In [2] it was shown that \(\beta_c\) is transcendental. For \(m \geq 2\) we define the sequence \((\lambda_i(m))_{i=1}^{\infty} \in \{0, \ldots, m\}^\mathbb{N}\) as follows:

\[
\lambda_i(m) = \begin{cases} 
k + \lambda_i - \lambda_{i-1} & \text{if } m = 2k \\
k + \lambda_i & \text{if } m = 2k + 1.\end{cases}
\]

We define \(\beta_c(m)\) to be the unique solution of

\[
\sum_{i=1}^{\infty} \frac{\lambda_i(m)}{\beta^i} = 1.
\]

In [13] the authors proved that \(\beta_c(m)\) is transcendental and the smallest value of \(\beta\) for which \(1 \in U_{\beta,m}\). In section 6 we include a table of values for \(\beta_c(m)\). We begin our study of the sets \(U_{\beta,m}\) and \(W_{\beta,m}\) by showing that the following proposition holds.

**Proposition 4.3.** Let \(m \geq 2\), then \(|U_{\beta,m}| \geq \aleph_0\) for \(\beta \in (\mathcal{G}(m), m + 1]\).

Combining Proposition 4.3 with the results presented in [14] the following analogue of Theorem 4.1 is immediate.
Theorem 4.4. For \( m \geq 2 \) the set \( U_{\beta,m} \) satisfies the following:

1. \( |U_{\beta,m}| = \aleph_0 \) for \( \beta \in (G(m), \beta_c(m)) \)
2. \( |U_{\beta,m}| = 2^{\aleph_0} \) for \( \beta = \beta_c(m) \)
3. \( U_{\beta,m} \) is a set of positive Hausdorff dimension for \( \beta \in (\beta_c(m), m+1] \).

Proof of Proposition 4.3. To begin with let us assume \( m = 2k \) for some \( k \in \mathbb{N} \), in this case \( G(m) = k+1 \).

It is a simple exercise to show that for \( \beta \in (k+1, m+1] \)

\[
T_{\beta,n}\left(\frac{k}{\beta-1}\right) = \frac{k}{\beta^n(\beta-1)} < \frac{1}{\beta}
\]

for all \( n \in \mathbb{N} \). By Proposition 3.2 we know that \( k \beta - 1 \) has a unique \( \beta \)-expansion. It follows from (4) that \( T_{\beta,n}\left(\frac{k}{\beta-1}\right) \) is never mapped into a choice interval and therefore has a unique \( \beta \)-expansion. As \( n \) was arbitrary we can conclude our result. The case where \( m = 2k+1 \) is proved similarly, in this case we can consider preimages of \( \frac{k \beta + (k+1)}{\beta^2-1} \).

We also show that the following analogue of Theorem 4.2 holds.

Theorem 4.5. If \( m = 2k \) the set \( W_{\beta,m} \) satisfies the following:

1. \( |W_{\beta,m}| = 1 \) for \( \beta \in (G(m), \beta_f(m)) \), where \( \beta_f(m) \) is the root of the equation

\[
x^2 - (k+1)x - k = 0, \quad \beta_f(m) = \frac{k+1 + \sqrt{k^2 + 6k + 1}}{2}
\]

2. \( |W_{\beta,m}| = \aleph_0 \) for \( \beta \in (\beta_f(m), \beta_c(m)) \)
3. \( |W_{\beta,m}| = 2^{\aleph_0} \) for \( \beta = \beta_c(m) \)
4. \( W_{\beta,m} \) is a set of positive Hausdorff dimension for \( \beta \in (\beta_c(m), m+1] \).

If \( m = 2k+1 \) the set \( W_{\beta,m} \) satisfies the following:

1. \( |W_{\beta,m}| = 2 \) for \( \beta \in (G(m), \beta_f(m)) \), where \( \beta_f(m) \) is the root of the equation

\[
x^3 - (k+2)x^2 + x - (k+1) = 0
\]

2. \( |W_{\beta,m}| = \aleph_0 \) for \( \beta \in (\beta_f(m), \beta_c(m)) \)
3. \( |W_{\beta,m}| = 2^{\aleph_0} \) for \( \beta = \beta_c(m) \)
4. \( W_{\beta,m} \) is a set of positive Hausdorff dimension for \( \beta \in (\beta_c(m), m+1] \).

Remark 4.6. \( \beta_f(m) \) is a Pisot number for all \( m \in \mathbb{N} \).

Using Theorem 4.4 to prove Theorem 4.5 it suffices to show that statement 1 holds in both the odd and even cases and \( |W_{\beta,m}| \geq \aleph_0 \) for \( \beta > \beta_f(m) \) in both the odd and even cases. In section 6 we include a table of values for \( \beta_f(m) \).
4.1 Proof of Theorem 4.5

The proof of Theorem 4.5 is more involved than Theorem 4.4 and as we will see requires more technical results. The following is taken from [14]. Firstly let us define the lexicographic order on \( \{0, \ldots, m\}^\infty \), we say that \((x_i)_{i=1}^\infty < (y_i)_{i=1}^\infty \) with respect to the lexicographic order if there exists \( n \in \mathbb{N} \) such that \( x_i = y_i \) for all \( i < n \) and \( x_n < y_n \) or if \( x_1 < y_1 \). For a sequence \((x_i)_{i=1}^\infty \in \{0, \ldots, m\}^\mathbb{N} \) we define \((x_i)_{i=1}^\infty = (m-x_i)_{i=1}^\infty \). We also adopt the notation \((\epsilon_1, \ldots, \epsilon_j)_{i=1}^\infty \) to denote the element of \( \{0, \ldots, m\}^\mathbb{N} \) obtained by the infinite concatenation of the finite sequence \((\epsilon_1, \ldots, \epsilon_j)\). Let the sequence \((d_i(m))_{i=1}^\infty \in \{0, \ldots, m\}^\mathbb{N} \) be defined as follows: let \( d_1(m) \) be the largest element of \( \{0, \ldots, m\} \) such that \( \frac{d_1(m)}{\beta} \leq 1 \), and if \( d_1(m) \) is defined for \( i < n \) then \( d_i(m) \) is defined to be the largest element of \( \{0, \ldots, m\} \) such that \( \sum_{i=1}^{n} \frac{d_i(m)}{\beta} = 1 \). The sequence \((d_i(m))_{i=1}^\infty \) is called the quasi-greedy expansion of 1 with respect to \( \beta \); it is trivially a \( \beta \)-expansion for 1 and the largest infinite \( \beta \)-expansion of 1 with respect to the lexicographic order not ending with \( (0)_{i=1}^\infty \). We let

\[
S_{\beta,m} = \left\{ (\epsilon_i)_{i=1}^\infty \in \{0, \ldots, m\}^\mathbb{N} : \sum_{i=1}^{\infty} \epsilon_i / \beta^i \in W_{\beta,m} \right\},
\]

it follows from the definition of \( W_{\beta,m} \) that \( |W_{\beta,m}| = |S_{\beta,m}| \) and to prove Theorem 4.5 it suffices to show that equivalent statements hold for \( S_{\beta,m} \). The following lemma which is essentially due to Parry [13] provides a useful characterisation of \( S_{\beta,m} \).

Lemma 4.7.

\[
S_{\beta,m} = \left\{ (\epsilon_i)_{i=1}^\infty \in \{0, \ldots, m\}^\mathbb{N} : (\epsilon_i, \epsilon_{i+1}, \ldots) < (d_1, d_2, \ldots) \text{ and } (d_1, d_2, \ldots) < (\epsilon_i, \epsilon_{i+1}, \ldots) \text{ for all } i \in \mathbb{N} \right\}
\]

Remark 4.8. If \( \beta < \beta' \) then the quasi-greedy expansion of 1 with respect to \( \beta \) is lexicographically strictly less than the quasi-greedy expansion of 1 with respect to \( \beta' \). As a corollary of this we have \( S_{\beta,m} \subseteq S_{\beta',m} \) for \( \beta < \beta' \).

Proposition 4.9. For \( \beta \in (\mathcal{G}(m), \beta_f(m)) \) \( |S_{\beta,m}| = 1 \) when \( m \) is even, \( |S_{\beta,m}| = 2 \) when \( m \) is odd and \( |S_{\beta,m}| \geq 8_0 \) for \( \beta \in (\beta_f(m), m+1) \).

By the remarks following Theorem 4.5 this will allow us to conclude our result.

Proof. We begin by considering the case where \( m = 2k \). When \( \beta = \beta_f(m) \) we have \((d_i(m))_{i=1}^\infty = (k+1, k-1)^\infty \) and by Lemma 4.7

\[
S_{\beta_f(m),m} = \left\{ (\epsilon_i)_{i=1}^\infty \in \{0, \ldots, m\}^\mathbb{N} : (\epsilon_i, \epsilon_{i+1}, \ldots) < (k+1, k-1)^\infty \right\}
\]

By our previous analysis we know that for \( \beta \in (\mathcal{G}(m), m+1) \) the point \( \frac{k}{\beta} \) has a unique \( \beta \)-expansion, the \( \beta \)-expansion of this point is the sequence \((k)_{i=1}^\infty \). By Remark 4.8 to prove \( |S_{\beta,m}| = \)
In this section we study the growth rate of the sequences 

4.2 The growth rate of

for \(1\) result we firstly require the following lemma. 

Clearly if \(\epsilon_i = k - 1\) then \(\epsilon_{i+1} = k + 1\). Therefore if \(\epsilon_i \neq k\) for some \(i\), then \((\epsilon_i, \epsilon_{i+1}, \ldots)\) must equal \((k - 1, k + 1)^\infty\) or \((k + 1, k - 1)^\infty\). By Lemma 4.7 this cannot happen and we can conclude that \(S_{\beta_f(m), m} = \{(k)^\infty\}\). For \(\beta \in (\beta_{f,m}, m + 1]\), we can construct a countable subset of \(S_{\beta,m}\); for example all sequences of the form \(S k - k + 1\). 

Theorem 4.10. The following theorem summarises the growth rate of each of these sequences. 

We now consider the case where \(m = 2k + 1\), when \(\beta = \beta_f(m)\) we have \((d_i(m))_{i=1}^\infty = (k + 1, k + 1, k, k)^\infty\) and

\[
\begin{align*}
S_{\beta_f(m), m} &= \{(\epsilon_i)_{i=1}^\infty \in \{0, \ldots, m\}^\mathbb{N} : (\epsilon_i, \epsilon_{i+1}, \ldots) < (k + 1, k + 1, k, k)^\infty \text{ and } \\
(k, k + 1, k + 1)^\infty &< (\epsilon_i, \epsilon_{i+1}, \ldots) \text{ for all } i \in \mathbb{N}\}.
\end{align*}
\]

By our earlier analysis we know that \(\{(k, k + 1)^\infty, (k + 1, k)^\infty\} \subset S_{\beta,m} \text{ for } \beta \in (G(m), m + 1]\). By Remark 4.8 to prove \(|S_{\beta,m}| = 2\) for \(\beta \in (G(m), \beta_f(m)]\) it suffices to show that \(S_{\beta_f(m), m} = \{(k, k + 1)^\infty, (k + 1, k)^\infty\}\). By an analogous argument to that given in [9] we can show that if \((\epsilon_i)_{i=1}^\infty \in S_{\beta_f(m), m}\) then \(\epsilon_i = k\) implies \(\epsilon_{i+1} = k + 1\), and \(\epsilon_i = k + 1\) implies \(\epsilon_{i+1} = k\). Clearly any element of \(S_{\beta_f(m), m}\) must begin with \(k\) or \(k + 1\) and we may therefore conclude that \(S_{\beta_f(m), m} = \{(k, k + 1)^\infty, (k + 1, k)^\infty\}\). To see that \(|W_{\beta,m}| \geq \aleph_0\) for \(\beta > \beta_f(m)\) we observe that \((k + 1, k)^j(k + 1, k + 1, k, k)^\infty \in S_{\beta,m}\) for all \(j \in \mathbb{N}\), for \(\beta > \beta_f(m)\). \(\square\)

4.2 The growth rate of \(G(m), \beta_f(m)\) and \(\beta_c(m)\)

In this section we study the growth rate of the sequences \((G(m))_{m=1}^\infty, (\beta_f(m))_{m=1}^\infty\) and \((\beta_c(m))_{m=1}^\infty\). The following theorem summarises the growth rate of each of these sequences.

Theorem 4.10. 1. \(G(2k) = k + 1\) for all \(k \in \mathbb{N}\).

2. \(\beta_f(2k) - (k + 2) = O(1/k)\).

3. \(\beta_c(2k) - (k + 2) \to 0\) as \(k \to \infty\).

4. \(G(2k + 1) - (k + 2) = O(1/k)\).

5. \(\beta_f(2k + 1) - (k + 2) \to 0\) as \(k \to \infty\).

6. \(\beta_c(2k + 1) - (k + 2) \to 0\) as \(k \to \infty\).

The proof of this theorem is somewhat trivial but we include it for completion. To prove this result we firstly require the following lemma.

Lemma 4.11. The sequence \(\beta_c(m)\) is asymptotic to \(\frac{m}{2}\), i.e., \(\lim_{m \to \infty} \frac{\beta_c(m)}{m/2} = 1\).
Proof. Suppose \( m = 2k \). It is a direct consequence of the definition of \( \lambda_i(m) \) and \( \beta_c(m) \) that the following inequalities hold

\[
\sum_{i=0}^{\infty} \frac{k-1}{\beta_c(m)^i} \leq \beta_c(m) \leq \sum_{i=0}^{\infty} \frac{k+1}{\beta_c(m)^i},
\]

which is equivalent to

\[
\frac{k-1}{1 - \frac{1}{\beta_c(m)}} \leq \beta_c(m) \leq \frac{k+1}{1 - \frac{1}{\beta_c(m)}}.
\]

Dividing through by \( m/2 \) and using the fact that \( \beta_c(m) \to \infty \) we can conclude our result. The case where \( m = 2k + 1 \) is proved similarly.

We are now in a position to prove Theorem 4.10.

Proof of Theorem 4.10. Statements 1, 2 and 4 are an immediate consequence of Theorem 1.1 and Theorem 4.5. It remains to show statements 3 and 6 hold; statement 4 will follow from the fact that \( G(2k+1) < \beta_f(2k+1) < \beta_c(2k+1) \). It is immediate from the definition of \( \lambda_i(m) \) that if \( m = 2k \) then

\[
\beta_{c,m} = k + 1 + \frac{k}{\beta_c(m)} + \sum_{i=2}^{\infty} \frac{\lambda_{i+1}(m)}{\beta^i}.
\]

Our result now follows from Lemma 4.11 and the fact that \( \sum_{i=2}^{\infty} \frac{\lambda_{i+1}(m)}{\beta_c(m)^i} \to 0 \) as \( m \to \infty \). The case where \( m = 2k + 1 \) is proved similarly.

5 The growth rate and dimension theory of \( \Sigma_{\beta,m}(x) \)

To describe the growth rate of \( \beta \)-expansions we consider the following. Let

\[
\mathcal{E}_{\beta,m,n}(x) = \{ (\epsilon_1, \ldots, \epsilon_n) \in \{0, \ldots, m\}^n \mid \exists (\epsilon_{n+1}, \epsilon_{n+2}, \ldots) \in \{0, \ldots, m\}^{\mathbb{N}} : \sum_{i=1}^{\infty} \frac{\epsilon_i}{\beta^i} = x \},
\]

we define an element of \( \mathcal{E}_{\beta,m,n}(x) \) to be a \( n \)-prefix for \( x \). Moreover, we let

\[
\mathcal{N}_{\beta,m,n}(x) = |\mathcal{E}_{\beta,m,n}(x)|
\]

and define the growth rate of \( \mathcal{N}_{\beta,m,n}(x) \) to be

\[
\lim_{n \to \infty} \frac{\log_{m+1} \mathcal{N}_{\beta,m,n}(x)}{n},
\]
when this limit exists. When this limit does not exist we can consider the lower and upper growth rates of $N_{\beta,m,n}(x)$, these are defined to be
\[
\liminf_{n \to \infty} \frac{\log_{m+1} N_{\beta,m,n}(x)}{n} \quad \text{and} \quad \limsup_{n \to \infty} \frac{\log_{m+1} N_{\beta,m,n}(x)}{n}
\]
respectively.

In this paper we also consider $\Sigma_{\beta,m}(x)$ from a dimension theory perspective. We endow $\{0, \ldots, m\}^\mathbb{N}$ with the metric $d(\cdot, \cdot)$ defined as follows:
\[
d(x, y) = \begin{cases} 
(m + 1)^{-n(x,y)} \quad &\text{if } x \neq y, \text{ where } n(x,y) = \inf\{i : x_i \neq y_i\} \\
0 \quad &\text{if } x = y.
\end{cases}
\]
We will consider the Hausdorff dimension of $\Sigma_{\beta,m}(x)$ with respect to this metric. It is a simple exercise to show that following inequalities hold:
\[
\dim_H(\Sigma_{\beta,m}(x)) \leq \liminf_{n \to \infty} \frac{\log_{m+1} N_{\beta,m,n}(x)}{n} \leq \limsup_{n \to \infty} \frac{\log_{m+1} N_{\beta,m,n}(x)}{n}. \tag{5}
\]
The case where $m = 1$ is studied in [4], [8] and [10]. In [4] and [8] the authors show that for $\beta \in (1, \frac{1+\sqrt{5}}{2})$ and $x \in (0, \frac{1}{\beta-1})$ we can bound the lower growth rate and Hausdorff dimension of $\Sigma_{\beta,1}(x)$ below by some strictly positive function depending only on $\beta$, in [10] the growth rate is studied from a measure theoretic perspective. Our main result is the following.

**Theorem 5.1.** For $\beta \in (1, \mathcal{G}(m))$ and $x \in (0, \frac{m}{\beta-1})$ the Hausdorff dimension of $\Sigma_{\beta,m}(x)$ can be bounded below by some strictly positive constant depending only on $\beta$.

By (5) a similar statement holds for both the lower and upper growth rates of $N_{\beta,m,n}(x)$. Replicating the proof of Lemma 2.4 it is a simple exercise to show that the following result holds.

**Proposition 5.2.** $N_{\beta,m,n}(x) = |\Omega_{\beta,m,n}(x)|$

By Proposition 5.2 we can identify elements of $\Omega_{\beta,m,n}(x)$ with elements of $\mathcal{E}_{\beta,m,n}(x)$, as such we also define an element of $\Omega_{\beta,m,n}(x)$ to be a $n$-prefix for $x$. To prove Theorem 5.1 we will use a method analogous to that given if [4]. We construct an interval $I_{\beta} \subset I_{\beta,m}$ such that, for each $x \in I_{\beta}$ we can generate multiple prefixes for $x$ of a fixed length depending on $\beta$ that map $x$ back into $I_{\beta}$. As we will see Theorem 5.1 will then follow by a counting argument. As was the case in our previous analysis we reduce the proof of Theorem 5.1 to two cases.

### 5.1 Case where $m$ is even

In what follows we assume $m = 2k$ for some $k \in \mathbb{N}$. To prove Theorem 5.1 we require the following technical lemma.

**Lemma 5.3.** For each $\beta \in (1, k+1)$ there exists $\epsilon_0(\beta) > 0$ such that, if $x \in \left[\frac{1}{\beta}, \frac{1}{\beta} + \epsilon_0(\beta)\right)$ then $T_{\beta,0}(x) \in \left[\frac{1}{\beta} + \epsilon_0(\beta), \frac{(m-1)\beta+1}{\beta(\beta-1)} - \epsilon_0(\beta)\right]$, and similarly if $x \in \left(\frac{(m-1)\beta+1}{\beta(\beta-1)} - \epsilon_0(\beta), \frac{(m+1)\beta+1}{\beta(\beta-1)}\right]$ then $T_{\beta,m}(x) \in \left[\frac{1}{\beta} + \epsilon_0(\beta), \frac{(m-1)\beta+1}{\beta(\beta-1)} - \epsilon_0(\beta)\right]$. 

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Figure 5: The interval $I_{\beta}$ in the case where $m = 2$ and \( \beta \in (1, 2) \).

**Proof.** This follows from Lemma 2.6 and a continuity argument.

For each \( i \in \{1, \ldots, m - 1\} \) we let \( \epsilon_{i}(\beta) = \frac{1}{2}\left(\frac{\beta+1}{\beta(\beta-1)} - \frac{i+1}{\beta}\right) \). If \( \beta \in (1, k + 1) \) then \( \epsilon_{i}(\beta) > 0 \) for each \( i \in \{1, \ldots, m - 1\} \). We define the interval $I_{\beta} = [L(\beta), R(\beta)]$ where $L(\beta)$ and $R(\beta)$ are defined as follows:

\[
L(\beta) = \min \left\{ T_{\beta,1} \left( \frac{1}{\beta} + \epsilon_{0}(\beta) \right), \min_{i \in \{1, \ldots, m-1\}} T_{\beta,i+1} \left( \frac{i+1}{\beta} + \epsilon_{i}(\beta) \right) \right\}
\]

and

\[
R(\beta) = \max \left\{ T_{\beta,m-1} \left( \frac{m-1}{\beta} + \frac{1}{\beta(\beta-1)} - \epsilon_{0}(\beta) \right), \max_{i \in \{1, \ldots, m-1\}} T_{\beta,i-1} \left( \frac{i+1}{\beta} + \epsilon_{i}(\beta) \right) \right\}
\]

We refer to Figure 5 for a diagram illustrating the interval $I_{\beta}$ in the case where $m = 2$ and $\beta \in (1, 2)$.

**Proposition 5.4.** Let \( \beta \in (1, k + 1) \). There exists \( n(\beta) \in \mathbb{N} \) such that, for each \( x \in I_{\beta} \) there exists two elements \( a, b \in \Omega_{\beta,m,n(\beta)}(x) \) such that \( a(x) \in I_{\beta} \) and \( b(x) \in I_{\beta} \).
Proof. Let \( x \in \mathcal{I}_\beta \). Without loss of generality we may assume that \( \epsilon_0(\beta) \) is sufficiently small such that \( \mathcal{I}_\beta \) contains the switch region. By Lemma 2.6 there exists a sequence of maps \( a \) that map \( x \) into the interior of our switch region. By Lemma 5.3 we may assume that \( a(x) \in \left[ \frac{1}{\beta} + \epsilon_0(\beta), \frac{(m-1)\beta+1}{\beta(\beta-1)} - \epsilon_0(\beta) \right] \).

The distance between the endpoints of \( \mathcal{I}_\beta \) and the endpoints of \( I_{\beta,m} \) (the fixed points of the maps \( T_{\beta,0} \) and \( T_{\beta,m} \)) can be bounded below by some positive constant, by Lemma 2.1 \( T_{\beta,0} \) and \( T_{\beta,m} \) both scale the distance between their fixed points and a general point by a factor \( \beta \), therefore we can bound the length of our sequence \( a \) above by some constant \( n_s(\beta) \in \mathbb{N} \) that does not depend on \( x \). We will show that we can take \( n(\beta) = n_s(\beta) + 1 \).

We remark that

\[
\left[ \frac{1}{\beta} + \epsilon_0(\beta), \frac{(m-1)\beta+1}{\beta(\beta-1)} - \epsilon_0(\beta) \right] = \left[ \frac{1}{\beta} + \epsilon_0(\beta), \frac{2}{\beta} \right]
\]

\[
\cup \left[ \frac{(m-2)\beta+2}{\beta(\beta-1)}, \frac{(m-1)\beta+1}{\beta(\beta-1)} - \epsilon_0(\beta) \right]
\]

\[
m-2 \sum_{i=1}^{m-2} \left[ \frac{(i-1)\beta+m-(i-1)}{\beta(\beta-1)}, \frac{i+2}{\beta} \right]
\]

\[
m-1 \sum_{i=1}^{m-1} \left[ \frac{i+1}{\beta}, \frac{(i-1)\beta+m-(i-1)}{\beta(\beta-1)} \right]
\]

We now proceed via a case analysis.

- If \( a(x) \in \left[ \frac{1}{\beta} + \epsilon_0(\beta), \frac{2}{\beta} \right] \) then \( T_{\beta,0}(a(x)) \in \mathcal{I}_\beta \) and \( T_{\beta,1}(a(x)) \in \mathcal{I}_\beta \).

- If \( a(x) \in \left[ \frac{(m-2)\beta+2}{\beta(\beta-1)}, \frac{(m-1)\beta+1}{\beta(\beta-1)} - \epsilon_0(\beta) \right] \) then \( T_{\beta,m-1}(a(x)) \in \mathcal{I}_\beta \) and \( T_{\beta,m}(a(x)) \in \mathcal{I}_\beta \).

- If \( a(x) \in \left[ \frac{(i-1)\beta+m-(i-1)}{\beta(\beta-1)}, \frac{i+2}{\beta} \right] \) for some \( i \in \{1, \ldots, m-2\} \) then \( T_{\beta,i}(a(x)) \in \mathcal{I}_\beta \) and \( T_{\beta,i+1}(a(x)) \in \mathcal{I}_\beta \).

- We reduce the the case where \( a(x) \in \left[ \frac{i+1}{\beta}, \frac{(i-1)\beta+m-(i-1)}{\beta(\beta-1)} \right] \) for some \( i \in \{1, \ldots, m-1\} \) to two subcases. If \( a(x) \in \left[ \frac{i+1}{\beta}, \frac{i+1}{\beta} + \epsilon_i(\beta) \right] \) then by the monotonicity of our maps, both \( T_{\beta,i-1}(a(x)) \) \( \in \mathcal{I}_\beta \) and \( T_{\beta,i}(a(x)) \) \( \in \mathcal{I}_\beta \). Similarly, in the case where \( a(x) \in \left[ \frac{i+1}{\beta} + \epsilon_i(\beta), \frac{(i-1)\beta+m-(i-1)}{\beta(\beta-1)} \right] \) both \( T_{\beta,i}(a(x)) \) \( \in \mathcal{I}_\beta \) and \( T_{\beta,i+1}(a(x)) \) \( \in \mathcal{I}_\beta \).

We’ve shown that for any \( x \in \mathcal{I}_\beta \) there exists \( n(x) \leq n_s(\beta) + 1 \) such that two distinct elements of \( \Omega_{\beta,m,n(x)}(x) \) map \( x \) into \( \mathcal{I}_\beta \). If \( n(x) < n_s(\beta) + 1 \) then we can concatenate our two elements of \( \Omega_{\beta,m,n(x)}(x) \) by an arbitrary choice of maps of length \( n_s(\beta) + 1 - n(x) \) that map the image of \( x \) into \( \mathcal{I}_\beta \). This ensures that we can take our sequences of maps to be of length \( n_s(\beta) + 1 \).

For \( \beta \in (1, k+1) \) and \( x \in (0, \frac{m}{\beta-1}) \) we may assume that there exists a sequence of maps \( a \) that maps \( x \) into \( \mathcal{I}_\beta \). We denote the minimum number of maps required to do this by \( j(x) \). Replicating
arguments given in [4] we can use Proposition 5.4 to construct an algorithm by which we can generate two prefixes of length \(n(\beta)\) for \(a^{(j(x))}\). Repeatedly applying this algorithm to successive images of \(a^{(j(x))}\) we can generate a closed subset of \(\Sigma_{\beta,m}(x)\). We denote this set by \(\sigma_{\beta,m}(x)\) and the set of \(n\)-prefixes for \(x\) generated by this algorithm by \(\omega_{\beta,m,n}(x)\). Replicating the proofs given in [4] we can show that the following lemmas hold.

**Lemma 5.5.** Let \(x \in (0, \frac{m}{\beta-1})\). Assume \(n \geq j(x)\) then

\[
|\omega_{\beta,m,n}(x)| \geq 2^{\frac{n-j(x)}{n(\beta)}-1}.
\]

**Lemma 5.6.** Let \(x \in (0, \frac{m}{\beta-1})\). Assume \(l \geq j(x)\) and \(b \in \omega_{\beta,m,l}(x)\), then for \(n \geq l\)

\[
|\{a = (a_i)_{i=1}^n \in \omega_{\beta,m,n}(x) : a_i = b_i \text{ for } 1 \leq i \leq l\}| \leq 2^\frac{m-1}{n(\beta)+2}.
\]

With these lemmas we are now in a position to prove Theorem 5.1 in the case where \(m\) is even. The argument used is analogous to the one given in [4], which is based upon Example 2.7 of [7].

**Proof of Theorem 5.1 when \(m = 2k\).** By the monotonicity of Hausdorff dimension with respect to inclusion it suffices to show that \(\text{dim}_H(\sigma_{\beta,m}(x))\) can be bounded below by a strictly positive constant depending only on \(\beta\). It is a simple exercise to show that \(\sigma_{\beta,m}(x)\) is a compact set; by this result we may restrict to finite covers of \(\sigma_{\beta,m}(x)\). Let \(\{U_n\}_{n=1}^N\) be a finite cover of \(\sigma_{\beta,m}(x)\). Without loss of generality we may assume that all elements of our cover satisfy \(\text{Diam}(U_n) < (m+1)^{-j(x)}\). For each \(U_n\) there exists \(l(n) \in \mathbb{N}\) such that

\[
(m + 1)^{-l(n)+1} \leq \text{Diam}(U_n) < (m + 1)^{-l(n)}.
\]

It follows that there exists \(z^{(n)} \in \{0, \ldots, m\}^{l(n)}\) such that, \(y_i = z_i^{(n)}\) for \(1 \leq i \leq l(n)\), for all \(y \in U_n\). We may assume that \(z^{(n)} \in \omega_{\beta,m,l(n)}(x)\), if we supposed otherwise then \(\sigma_{\beta,m}(x) \cap U_n = \emptyset\) and we can remove \(U_n\) from our cover. We denote by \(C_n\) the set of sequences in \(\{0, \ldots, m\}^\mathbb{N}\) whose first \(l(n)\) entries agree with \(z^{(n)}\), i.e.

\[
C_n = \left\{(\epsilon_i)_{i=1}^\infty \in \{0, \ldots, m\}^\mathbb{N} : \epsilon_i = z_i^{(n)} \text{ for } 1 \leq i \leq l(n)\right\}.
\]

Clearly \(U_n \subset C_n\) and therefore the set \(\{C_n\}_{n=1}^N\) is a cover of \(\sigma_{\beta,m}(x)\).

Since there are only finitely many elements in our cover there exists \(J\) such that \((m + 1)^{-J} \leq \text{Diam}(U_n)\) for all \(n\). We consider the set \(\omega_{\beta,m,J}(x)\). Since \(\{C_n\}_{n=1}^N\) is a cover of \(\sigma_{\beta,m}(x)\) each \(a \in \omega_{\beta,m,J}(x)\) satisfies \(a_i = z_i^{(n)}\) for \(1 \leq i \leq l(n)\), for some \(n\). Therefore

\[
|\omega_{\beta,m,J}(x)| \leq \sum_{n=1}^N \left|\left\{a \in \omega_{\beta,m,J}(x) : a_i = z_i^{(n)} \text{ for } 1 \leq i \leq l(n)\right\}\right|.
\]
By counting elements of $\omega_{\beta,m,J}(x)$ and Lemmas 5.5 and 5.6 we observe the following:

$$2^{J_{\beta,m}(x)} - 1 \leq |\omega_{\beta,m,J}(x)|$$

$$\leq \sum_{n=1}^{N} \left| \{ a \in \omega_{\beta,m,J}(x) : a_i = z_i^{(n)} \text{ for } 1 \leq i \leq l(n) \} \right|$$

$$\leq \sum_{n=1}^{N} 2^{J_{\beta,m}(x)-2}$$

$$= 2^{J_{\beta,m}(x)-2} \sum_{n=1}^{N} 2^{-(l(n)+1)}$$

$$\leq 2^{J_{\beta,m}(x)-2} \sum_{n=1}^{N} \text{Diam}(U_n) \frac{\log m+1}{n(\beta)}.$$

Dividing through by $2^{J_{\beta,m}(x)-2}$ yields

$$\sum_{n=1}^{N} \text{Diam}(U_n) \frac{\log m+1}{n(\beta)} \geq 2^{J_{\beta,m}(x) - 3n(\beta)-1}$$

the right hand side is a positive constant greater than zero that does not depend on our choice of cover. It follows that $\dim_H(\sigma_{\beta,m}(x)) \geq \frac{\log m+1}{n(\beta)}$, our result follows. \hfill \Box

### 5.2 Case where $m$ is odd

In what follows we assume $m = 2k + 1$ for some $k \in \mathbb{N}$. For $\beta \in (1, \frac{2k+3}{2})$ the proof of Theorem 5.1 is analogous to the even case for $\beta \in (1, k+1)$. As such, in what follows we assume $\beta \in \left[\frac{2k+3}{2}, k+1+\sqrt{k^2+6k+3}\right)$. The significance of $\beta \in \left[\frac{2k+3}{2}, k+1+\sqrt{k^2+6k+3}\right)$ is that for $i \in \{1, \ldots, m-1\}$ the $i$-th fixed digit interval is well defined.

Before defining the interval $I_{\beta}$ we require the following. We let

$$\epsilon_i(\beta) = \begin{cases} \frac{1}{2} \left( \frac{(i-1)\beta + m - (i-1)}{\beta(\beta-1)} - \frac{1}{\beta-1} \right) & \text{if } i \in \{1, \ldots, k\} \\ \frac{1}{2} \left( \frac{1}{\beta-1} - \frac{i+1}{\beta} \right) & \text{if } i \in \{k+1, \ldots, m-1\} \end{cases}$$

By Lemma 5.5 $\epsilon_i(\beta) > 0$ for all $i \in \{1, \ldots, m-1\}$ for $\beta \in (1, k+2)$. Before proving an analogue of Proposition 5.4 we require the following technical lemmas. It is a simple exercise to show that the following analogue of Lemma 5.5 holds.

**Lemma 5.7.** For each $\beta \in \left[\frac{2k+3}{2}, k+1+\sqrt{k^2+6k+3}\right)$ there exists $\epsilon_0(\beta) > 0$ such that, if $x \in \left[\frac{1}{\beta}, \frac{1}{2}\right] + \epsilon_0(\beta)$ then $T_{\beta,0}(x) \in \left[\frac{1}{\beta}, \frac{1}{2}\right] + \epsilon_0(\beta)$, and similarly if $x \in \left(\frac{(m-1)\beta+1}{\beta(\beta-1)} - \epsilon_0(\beta), \frac{(m-1)\beta+1}{\beta(\beta-1)} \right]$ then $T_{\beta,n}(x) \in \left[\frac{1}{\beta}, \frac{1}{2}\right] + \epsilon_0(\beta)$, $\frac{(m-1)\beta+1}{\beta(\beta-1)} - \epsilon_0(\beta)$.
Lemma 5.8. Let $\beta \in \left[\frac{2k+3}{2}, \frac{k+1+\sqrt{k^2+6k+5}}{2}\right]$. For each $i \in \{1, \ldots, k-1\}$ there exists $\epsilon_i^*(\beta) > 0$ such that, if $x \in \left[\frac{(i-1)\beta+m-(i-1)}{\beta(\beta-1)} - \epsilon_i(\beta), \frac{i+1}{\beta} + \epsilon_i^*(\beta)\right]$ then $T_{\beta,i}(x) < \frac{k+2}{\beta} + \epsilon_{k+1}$. Similarly for $i \in \{k+2, \ldots, m-1\}$ there exists $\epsilon_i^*(\beta) > 0$ such that, if $x \in \left[\frac{(i-1)\beta+m-(i-1)}{\beta(\beta-1)} - \epsilon_i^*(\beta), \frac{i+1}{\beta} + \epsilon_i(\beta)\right]$ then $T_{\beta,i}(x) > \frac{(k-1)\beta+m-(k-1)}{\beta(\beta-1)} - \epsilon_k$.

Proof. By the analysis given in the proof of Lemma 3.5 for $i \in \{1, \ldots, k-1\}$ $T_{\beta,i}(\frac{i+1}{\beta}) < \frac{k^2+6k+5}{\beta(\beta-1)}$ for $\beta \in (1, \frac{k+1+\sqrt{k^2+6k+5}}{2})$. However, for $\beta \in \left[\frac{2k+3}{2}, \frac{k+1+\sqrt{k^2+6k+5}}{2}\right]$ $\frac{k^2+6k+5}{\beta(\beta-1)} \leq \frac{k+2}{\beta}$. The existence of $\epsilon_i^*(\beta)$ then follows by a continuity argument and the monotonicity of the maps $T_{\beta,i}$.

The case where $i \in \{k+2, \ldots, m-1\}$ is proved similarly. \hfill \square

We are now in a position to define the interval $I_\beta$. Let $I_\beta = [L(\beta), R(\beta)]$ where

$$L(\beta) = \min \left\{ T_{\beta,1}(\frac{1}{\beta} + \epsilon_0(\beta)), T_{\beta,k+1}\left(\frac{k\beta + k + 1}{\beta^2 - 1}\right), \min_{i \in \{2, \ldots, k\}} \left\{ T_{\beta,i}\left(\frac{i}{\beta} + \epsilon_{i-1}(\beta)\right)\right\}, \min_{i \in \{k+2, \ldots, m\}} \left\{ T_{\beta,i}\left(\frac{i}{\beta} + \epsilon_{i-1}(\beta)\right)\right\} \right\}$$

and

$$R(\beta) = \max \left\{ T_{\beta,k}\left(\frac{(k+1)\beta + k}{\beta^2 - 1}\right), T_{\beta,m-1}\left(\frac{(m-1)\beta + 1}{\beta(\beta-1)} - \epsilon_0(\beta)\right), \max_{i \in \{1, \ldots, k\}} \left\{ T_{\beta,i-1}\left(\frac{(i-1)\beta + m - (i-1)}{\beta(\beta-1)} - \epsilon_i(\beta)\right)\right\}, \max_{i \in \{k+2, \ldots, m-1\}} \left\{ T_{\beta,i-1}\left(\frac{(i-1)\beta + m - (i-1)}{\beta(\beta-1)} - \epsilon_i(\beta)\right)\right\} \right\}.$$

For ease of exposition in Figure 6 we give a diagram illustrating the interval $I_\beta$, in the case where $m = 3$ and $\beta \in \left[\frac{3}{2}, 1 + \sqrt{3}\right]$.

Proposition 5.9. Let $\beta \in \left[\frac{2k+3}{2}, \frac{k+1+\sqrt{k^2+6k+5}}{2}\right]$. There exists $n(\beta) \in \mathbb{N}$ such that, for each $x \in I_\beta$ there exists two elements $a, b \in \Omega_{\beta,m,n(\beta)}(x)$ such that $a(x) \in I_\beta$ and $b(x) \in I_\beta$.

Proof. Without loss of generality we may assume that $\epsilon_0(\beta)$ is sufficiently small such that $I_\beta$ contains the switch region. By Lemma 2.6 there exists a sequence of maps $a$ that map $x$ into the switch region. As the endpoints of $I_\beta$ are bounded away from the endpoints of $I_{\beta,m}$ we can bound the length of $a$ above by some $n_\epsilon(\beta) \in \mathbb{N}$. Moreover, by Lemma 5.7 we may assume that $a(x) \in \left[\frac{1}{\beta} + \epsilon_0(\beta), \frac{(m-1)\beta + 1}{\beta(\beta-1)} - \epsilon_0(\beta)\right]$. As in the even case it is useful to treat $\left[\frac{1}{\beta} + \epsilon_0(\beta), \frac{(m-1)\beta + 1}{\beta(\beta-1)} - \epsilon_0(\beta)\right]$ as the union of subintervals. We observe that

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Figure 6: The interval $I_\beta$ in the case where $m = 3$ and $\beta \in \left[\frac{5}{2}, 1 + \sqrt{3}\right)$.

\[
\left[\frac{1}{\beta} + \epsilon_0(\beta), \frac{(m - 1)\beta + 1}{\beta(\beta - 1)} - \epsilon_0(\beta)\right] = \left[\frac{1}{\beta} + \epsilon_0(\beta), \frac{m}{\beta(\beta - 1)} - \epsilon_1(\beta)\right]
\]

\[
\bigcup_{i=2}^{m-1} \left[\frac{i}{\beta} + \epsilon_{i-1}(\beta), \frac{(i - 1)\beta + m - (i - 1)}{\beta(\beta - 1)} - \epsilon_i(\beta)\right]
\]

\[
\bigcup_{i=k+2}^{k-1} \left[\frac{\beta(i - 1)\beta + m - (i - 1)}{\beta(\beta - 1)} - \epsilon_i(\beta), \frac{i + 1}{\beta} + \epsilon_i(\beta)\right]
\]

\[
\bigcup_{i=k+2}^{m-1} \left[\frac{(i - 1)\beta + m - (i - 1)}{\beta(\beta - 1)} - \epsilon_i(\beta), \frac{i + 1}{\beta} + \epsilon_i(\beta)\right]
\]
Without loss of generality we may assume that $\epsilon_0(\beta), \epsilon_i(\beta), \epsilon'_i(\beta)$ are all sufficiently small such that each of the above intervals in our union are well defined and nontrivial. We now proceed via a case analysis.

- If $a(x) \in \left[\frac{a}{\beta} + \epsilon_0(\beta), \frac{m}{\beta(\beta-1)} - \epsilon_1(\beta)\right]$ then $T_{\beta,0}(a(x)) \in I_{\beta}$ and $T_{\beta,1}(a(x)) \in I_{\beta}$.

- If $a(x) \in \left[\frac{m}{\beta} + \epsilon_{m-1}(\beta), \frac{(m-1)\beta+1}{\beta(\beta-1)} - \epsilon_0(\beta)\right]$ then $T_{\beta,m-1}(a(x)) \in I_{\beta}$ and $T_{\beta,m}(a(x)) \in I_{\beta}$.

- Suppose $a(x) \in \left[\frac{(k-1)\beta+m-(k-1)}{\beta(\beta-1)} - \epsilon_k(\beta), \frac{k+2}{\beta} + \epsilon_{k+1}(\beta)\right]$. If $a(x) \in \left[\frac{k\beta+k+1}{\beta^2-1}, \frac{(k+1)\beta+k}{\beta^2-1}\right]$ then $T_{\beta,k}(a(x)) \in I_{\beta}$ and $T_{\beta,k+1}(a(x)) \in I_{\beta}$. If $a(x) \in \left[\frac{(k-1)\beta+m-(k-1)}{\beta(\beta-1)} - \epsilon_k(\beta), \frac{k\beta+k+1}{\beta^2-1}\right]$ then we are bounded distance away from the fixed point of the map $T_{\beta,k}$, by Lemma 2.1 we know that $T_{\beta,k}$ scales the distance between $a(x)$ and the fixed point of $T_{\beta,k}$ by a factor $\beta$, therefore we can bound the number of maps required to map $a(x)$ into $\left[\frac{k\beta+k+1}{\beta^2-1}, \frac{(k+1)\beta+k}{\beta^2-1}\right]$. By a similar argument, if $a(x) \in \left[\frac{(k+1)\beta+k}{\beta^2-1}, \frac{k+2}{\beta} + \epsilon_{k+1}(\beta)\right]$ we can bound the number of maps required to map $a(x)$ into $\left[\frac{k\beta+k+1}{\beta^2-1}, \frac{(k+1)\beta+k}{\beta^2-1}\right]$. By the above we can assert that when $a(x) \in \left[\frac{(k-1)\beta+m-(k-1)}{\beta(\beta-1)} - \epsilon_k(\beta), \frac{k+2}{\beta} + \epsilon_{k+1}(\beta)\right]$ there exists two distinct sequences of maps whose length we can bound above by some $n_\beta(\beta) \in \mathbb{N}$ that map $a(x)$ into $I_{\beta}$.

- If $a(x) \in \left[\frac{i}{\beta} + \epsilon_{i-1}(\beta), \frac{(i-1)\beta+m-(i-1)}{\beta(\beta-1)} - \epsilon_i(\beta)\right]$ for some $i \in \{2, \ldots, k-1\}$ then $T_{\beta,i-1}(a(x)) \in I_{\beta}$ and $T_{\beta,i}(a(x)) \in I_{\beta}$.

- If $a(x) \in \left[\frac{i}{\beta} + \epsilon_i(\beta), \frac{(i-1)\beta+m-(i-1)}{\beta(\beta-1)} - \epsilon'_i(\beta)\right]$ for some $i \in \{k+2, \ldots, m - 1\}$ then $T_{\beta,i-1}(a(x)) \in I_{\beta}$ and $T_{\beta,i}(a(x)) \in I_{\beta}$.

- If $a(x) \in \left[\frac{(i-1)\beta+m-(i-1)}{\beta(\beta-1)} - \epsilon_i(\beta), \frac{i+1}{\beta} + \epsilon'_i(\beta)\right]$ for some $i \in \{1, \ldots, k-1\}$ then $a(x)$ is a bounded distance away from the fixed point of the map $T_{\beta,i}$, by Lemma 2.1 we know that $T_{\beta,i}$ scales the distance between $a(x)$ and its fixed point by a factor $\beta$, therefore we can bound the number of maps required to map $a(x)$ outside of the interval $\left[\frac{(i-1)\beta+m-(i-1)}{\beta(\beta-1)} - \epsilon_i(\beta), \frac{i+1}{\beta} + \epsilon'_i(\beta)\right]$ by some $n_i(\beta) \in \mathbb{N}$. If $a(x)$ has been mapped into an interval covered by one of the above cases we are done, if not it has to be mapped into another interval of the form $\left[\frac{(i-1)\beta+m-(i-1)}{\beta(\beta-1)} - \epsilon_j(\beta), \frac{i+1}{\beta} + \epsilon'_i(\beta)\right]$. By Corollary 3.4 and Lemma 5.8 we know that $i < j \leq k + 1$. Repeating the previous step as many times as is necessary we can ensure that within $\sum_{i=1}^{k-1} n_i(\beta)$ maps, $a(x)$ has to be mapped into an interval that was addressed in one of our previous cases.

- The case where $a(x) \in \left[\frac{(i-1)\beta+m-(i-1)}{\beta(\beta-1)} - \epsilon_i(\beta), \frac{i+1}{\beta} + \epsilon_i(\beta)\right]$ for some $i \in \{k+2, \ldots, m-1\}$ is analogous to the case where $a(x) \in \left[\frac{(i-1)\beta+m-(i-1)}{\beta(\beta-1)} - \epsilon_i(\beta), \frac{i+1}{\beta} + \epsilon'_i(\beta)\right]$ for some $i \in \{1, \ldots, k-1\}$.

We’ve shown that for any $x \in I_{\beta}$ there exists $n(x) \in \mathbb{N}$ such that, two distinct elements of $\Omega_{\beta,m,n(x)}(x)$ map $x$ into $I_{\beta}$, moreover $n(x) \leq n_s(\beta) + n_c(\beta) + \sum_{i=1}^{k-1} n_i(\beta)$. We take $n(\beta)$ to equal $n_s(\beta) + n_c(\beta) + \sum_{i=1}^{k-1} n_i(\beta)$. If $n(x) < n(\beta)$ then as in the even case we concatenate
our image of $x$ by an arbitrary sequence of maps of length $n(\beta) - n(x)$ that map $x$ into $\mathcal{I}_\beta$, this ensures our sequences of maps are of length $n(\beta)$.

Repeating the analysis given in the case where $m$ is even we can conclude Theorem 5.1 in the case where $m$ is odd.

6 Open questions and a table of values for $G(m)$, $\beta_f(m)$ and $\beta_c(m)$

We conclude with a few open questions and a table of values for $G(m)$, $\beta_f(m)$ and $\beta_c(m)$.

- In [1] the authors study the order in which periodic orbits appear in the set of uniqueness. When $m = 1$ they show that as $\beta \nearrow 2$ the order in which periodic orbits appear in the set of uniqueness is intimately related to the classical Sharkovskii ordering. It is natural to ask whether a similar result holds in our general case.

- In [18] it is shown that when $m = 1$ and $\beta = \frac{1 + \sqrt{5}}{2}$ the set of numbers: $x = \frac{(1 + \sqrt{5})n}{2} \mod 1$ for some $n \in \mathbb{N}$ have countably many $\beta$-expansions, while the other elements of $(0, \frac{1}{\beta - 1})$ have uncountably many $\beta$-expansions. Does an analogue of this statement hold in the case of general $m$?

- Let $p_1, \ldots, p_k$ be points in $\mathbb{R}^d$ such that the polyhedra $\Pi$ with these vertices is convex. Let \{\(f_i\)\}_{i=1}^{k}$ be the one parameter family of maps given by \(f_i(x) = \lambda x + (1 - \lambda)p_i\), where $\lambda \in (0, 1)$ is our parameter. As is well known there exists a unique $S_\lambda$ such that $S_\lambda = \bigcup_{i=1}^{k} f_i(S_\lambda)$. We say that $(\epsilon_i)_{i=1}^{\infty} \in \{1, \ldots, k\}^\mathbb{N}$ is an address for $x \in S_\lambda$ if $\lim_{n \rightarrow \infty}(f_{\epsilon_n} \circ \ldots \circ f_{\epsilon_1})(0) = x$. We ask whether an analogue of the golden ratio exists in this case, i.e, does there exist $\lambda^*$ such that for $\lambda \in (\lambda^*, 1)$ every $x \in S_\lambda \setminus \{p_1, \ldots, p_k\}$ has uncountably many addresses, but for $\lambda \in (0, \lambda^*)$ there exists $x \in S_\lambda \setminus \{p_1, \ldots, p_k\}$ with a unique address. In [16] the author shows that an analogue of the golden ratio exists in the case when $d = 2$ and $k = 3$.

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Table 1: Table of values for $G(m)$, $\beta_f(m)$ and $\beta_c(m)$

| $m$  | $G(m)$     | $\beta_f(m)$ | $\beta_c(m)$ |
|------|------------|---------------|---------------|
| 1    | $\frac{1+\sqrt{5}}{2} \approx 1.61803\ldots$ | 1.75488\ldots  | 1.78723\ldots |
| 2    | 2          | $1+\sqrt{2} = 2.41421\ldots$ | 2.47098\ldots  |
| 3    | $1+\sqrt{3} \approx 2.73205\ldots$ | 2.89329\ldots  | 2.90330\ldots |
| 4    | 3          | $\frac{3+\sqrt{17}}{2} = 3.56155\ldots$ | 3.66607\ldots  |
| 5    | $\frac{5+\sqrt{21}}{2} \approx 3.79129\ldots$ | 3.93947\ldots  | 3.94583\ldots |
| 6    | 4          | $2+\sqrt{2} = 4.64575\ldots$ | 4.75180\ldots  |
| 7    | $2+2\sqrt{2} \approx 4.82843\ldots$ | 4.96095\ldots  | 4.96496\ldots |
| 8    | 5          | $\frac{5+\sqrt{41}}{2} = 5.70156\ldots$ | 5.80171\ldots  |
| 9    | $\frac{5+\sqrt{45}}{2} \approx 5.85410\ldots$ | 5.97273\ldots  | 5.97537\ldots |
| 10   | 6          | $3+\sqrt{4} = 6.74166\ldots$ | 6.83469\ldots  |

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