As is well known, the thermal history of the Universe during the radiation era depends on the effective degrees of freedom $g_{\ast \text{eff}}(T)$ or $g_\ast(T)$ contributed by the particles in thermal equilibrium at temperature $T$. The function $g_\ast(T)$ was first computed over the eV to TeV range in 1981\cite{1} and later updated in 1988\cite{2} and 1990\cite{3}, and has often been quoted without updating since then (e.g.\cite{4,5}). Over time our knowledge of the properties of the particles determining this function have undergone various modifications, particularly at high temperatures due to improved values for the top quark mass, improved limits for the Higgs boson, and improved detail on the hadron spectrum \cite{3}. This calls for an updating of $g_\ast(T)$.

Recall that the light nuclei are synthesized at temperatures $T \ll m_e$ when all particles have decoupled and $g_\ast$ receives contributions only from the three relativistic neutrino species and the photon. Each neutrino species then contributes $7/4$ degrees of freedom, corrected for the evolution of the neutrino temperature through the period of $e^\pm$ annihilation, and the photon contributes $g_\gamma = 2$. The function $g_\ast(T)$ then has the low-energy limit

$$g_\ast = g_\gamma + 3 \frac{7}{4} \left(\frac{4}{\pi^4}\right)^{4/3} \approx 3.363 . \quad (1)$$

At temperatures above this low-energy limit but in the ‘confined’ phase below the quark-hadron phase transition $T_c$, the hadrons contribute by the pions, $g_\pi = 3$, and by the low-energy tails of the heavier mesons and baryons. At all temperatures the three charged leptons contribute by $g_\ell = 3 \times \frac{2}{3}$, and the vector bosons by $g_W = g_Z = 6$ and $g_{\gamma\nu} = 3$. In the ‘deconfined’ phase above $T_c$ the hadron contributions are replaced by the six quarks, each of which contribute $g_q = 12$, and by the gluons, $g_g = 16$, and the Higgs boson, $g_H = 1$. We take their masses from the Review of Particle Physics \cite{6} which, for the $u$, $d$ and $s$ quarks implies current-quark masses.

We update the curves of the effective degrees of freedom for energy density $g_\ast(T)$ and for the entropy density $g_\ast S(T)$ during the era of radiation domination in the Universe. We find that a plain count of effective degrees of freedom sets an upper limit to the temperature of the quark-hadron transition at $T_c < 235$ MeV for the energy density and $T_c < 245$ MeV for the entropy density.

The total energy density of all species in equilibrium at photon temperature $T$ can be written \cite{6}

$$\rho = \frac{\pi^2}{30} g_\ast T^4 . \quad (2)$$

Summing over all species $i$ with mass $m_i$, the effective degrees of freedom $g_\ast$ for the energy density is given \cite{6} by

$$g_\ast = \sum_i \left(\frac{T_i}{T}\right)^4 g_i \frac{15}{\pi^4} \int_{x_i}^\infty \frac{\left(u^2 - x_i^2\right)^{1/2} u^2}{e^u \pm 1} du . \quad (3)$$

The entropy density is defined by

$$s = \frac{2\pi^2}{45} g_\ast S T^3 , \quad (4)$$

where $g_{\ast S}$ is the effective degrees of freedom . The contribution from all species $i$ to the entropy density is then \cite{6}

$$g_{\ast S} = \sum_i \left(\frac{T_i}{T}\right)^3 \frac{15g_i}{4\pi^4} \left(3 \int_{x_i}^\infty \frac{\sqrt{u^2 - x_i^2} u^2}{e^u \pm 1} du + \int_{x_i}^\infty \frac{\sqrt{u^2 - x_i^2}^3}{e^u \pm 1} du \right) . \quad (5)$$

These integrals can be evaluated numerically once one knows which particle species should be included in the sums, and what value to take for the transition temperature $T_c$.

To answer the first question, consider the thermalization of the $\rho(750)$ meson (in units of $\hbar = c = k = 1$).
The $\rho(750)$ number density at temperature $T = 0.2$ GeV is

$$N_\rho = n_{\text{spin}} (2\pi m_\rho T)^3/2 e^{-m_\rho/T} \approx 0.065 \text{ GeV}^3.$$  

The cms velocity in elastic $\pi + \rho \to \pi + \rho$ collisions $v_{\text{cms}} \approx 0.15$. Taking the cross-section to be about 30 mb or $\sigma \approx 7.7 \times 10^7$ GeV$^{-2}$ we find the reaction rate

$$\Gamma = \langle N_\rho v_{\text{cms}} \sigma \rangle \approx 7.5 \times 10^5 \text{ GeV}.$$  

This should be compared to the decay rate of the rho meson, $\Gamma_\rho = 0.15$ GeV. Thus there is plenty of time for rho mesons in the tail of their thermal distribution to thermalize in collisions against pions.

One comes to the same conclusion when one studies the reaction $\gamma + \rho \to \gamma + \rho$: the reaction rate is then $1.7 \times 10^4$ GeV for an approximate cross-section of 0.1 mb.

One also comes to the same conclusion when one turns to heavier meson resonances and to baryon resonances. We therefore include all known meson and baryon resonances up to 3 GeV mass in the sums in $g_*$ and $g_{\ast S}$ at temperatures below $T_c$. This is the procedure followed also by the earlier studies.

In spite of the strong exponential decrease in the number density of non-relativistic species with heavy masses $m_i > T_c$, such species will contribute significantly below $T_c$. For example the proton and neutron contribute 0.25 each to $g_*(T)$ at $T = 150$ MeV (versus an unattenuated 3.5) even though their mass of roughly 939 MeV is 6.3 times higher. The sum of unattenuated mesonic and baryonic degrees of freedom below 3 GeV mass is $g_{\text{mesons}} = 495$ and $g_{\text{baryons}} = 320$, respectively. As a result $g_*(T)$ rises quite rapidly for temperatures above $T = 100$ MeV.

Near the transition temperature $T_c$, the ideal gas approximation breaks down and the transition from the deconfined quark phase to the confined hadronic phase can only be studied with finite temperature lattice simulations. No reliable value for $T_c$ has yet been obtained, but it is believed to be near 170 MeV. Srednicki et al. proposed that the transition my be smooth and bracketed by the cases $T_c = 150$ MeV and 400 MeV. Their procedure to obtain smooth curves in the context of the ‘bag’ model has only been elucidated by Sarkar in a footnote on p. 1513 which refers to a private communication from K. A. Olive. But whether the change is smooth or discontinuous still remains open.

In Fig. 1 we plot $g_*(T)$ and $g_{\ast S}(T)$ as (almost) discontinuous functions of $\log T$ for the range 100 GeV > $T$ > 10 keV. For purposes of the figure we have somewhat arbitrarily chosen the discontinuity at $T_c = 200$ MeV (as in $\sigma$) and a smoothing naive transition function that mixes the quark and hadronic phases over a range of roughly 5 MeV on either side of $T_c$. Since it is quite laborious to calculate the curves in Fig. 1 and difficult to read them precisely there, we give here some numerical approximations over the range $T = 155 - 300$ MeV.

For the quark phase we have

$$g_*(T) = 2.388 \times 10^{-4} T^{2.287}, \quad \text{max error 5.3%},$$

$$g_{\ast S}(T) = 3.631 \times 10^{-4} T^{2.192}, \quad \text{max error 5.5%}.$$  

(8)

For the quark phase we have

$$g_*(T) = 24.604 \times T^{0.1713}, \quad \text{max error 0.2%},$$

$$g_{\ast S}(T) = 22.961 \times T^{0.1810}, \quad \text{max error 0.2%}.$$  

(9)

It is instructive to exhibit $g_*(T)$ in the quark phase and the hadronic phase separately, as we have done in Fig. 1. This shows a very well defined cross-over at 235 MeV. Taking this temperature to be $T_c$ one obtains a curve which is indistinguishable from the curves in Fig. 1 (because of the logarithmic scales). The cross-over temperature is defined quite precisely due to the steepness of the hadronic phase, but it does of course depend to some extent on the precision of the hadronic resonance energies, as well as on unknown particles and interactions above 1 TeV.

Our most interesting conclusion is that a plain count of effective degrees of freedom sets an upper limit to the temperature of the quark-hadron transition. From the $g_*(T)$ cross-over one has

$$T_c \lesssim 235 \text{ MeV},$$  

(10)

and from the $g_{\ast S}(T)$ cross-over

$$T_c \lesssim 245 \text{ MeV}.$$  

(11)

Higher $T_c$ values would give an unphysical spike and lower values would increase the discontinuous drop in $g_*(T)$ just below $T_c$.

Acknowledgments

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[1] K. A. Olive, D. N. Schramm G. Steigman, Nucl. Phys. B 180, 497 (1981)

[2] M. Srednicki, W. Watkins and K. A. Olive, Nucl. Phys. B
FIG. 1: Effective degrees of freedom for the energy density $g_\ast(T)$ and for the entropy density $g_{s\ast}(T)$. The quark-hadron transition is chosen to occur at 200 MeV.

FIG. 2: Effective degrees of freedom $g_\ast(T)$ due to the quark and hadronic phases, respectively. For the quark phase we have taken $T_c = 150$ MeV, for the hadronic phase $T_c = 300$ MeV. The cross-over occurs at 235 MeV.