Schur Multipliers and Second Quandle Homology

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Abstract

We define a map from second quandle homology to the Schur multiplier and examine its properties. Furthermore, we express the second homology of Alexander quandles in terms of exterior algebras. Additionally, we present a self-contained proof of its structure and provide some computational examples.

Keywords: quandle; group homology; rack and quandle homology; Schur multiplier; exterior algebra; central extensions; semi-Hopfian group

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1 Introduction

A quandle \cite{Joy,Mat} is a set with a binary operation whose definition was motivated by knot theory. A primordial example of a quandle is a group with the binary operation given by conjugation. Rack (co)homology was introduced in \cite{FRS1,FRS2}. It was adapted into a quandle (co)homology theory to define topological invariants for knots of codimension two \cite{CJKLS}.

Since, there are adjoint functors between the category of quandles and the category of groups \cite{Joy}, it is natural to expect relations between the homology theories of quandles and groups. An example of such a relation is the analogy between central group extensions and quandle extensions \cite{Bro,BT,CENS}. Since, there are well-developed methods to compute group homology \cite{Bro,BT,Kar}, to gain more information, maps are constructed from the homology of quandles to that of groups \cite{CEGS,EG,Kab,Nos1}. However, these correspondences are often not isomorphisms. On the other hand, it was observed that the second homology of a Takasaki quandle of odd order is isomorphic to the Schur multiplier \cite{Mil,NP} of the underlying abelian group, that is, the exterior square of the group \cite{Mil,NP}.

In this paper, we demonstrate a new relation between the second quandle homology and the second group homology. Section 2 reviews quandles and quandle homology. In Section 3, we construct a homomorphism $T^*$ from the second quandle cohomology to the relative group cohomology by using diagram chasing techniques (Definition 3.1). In Theorem 3.3, we discuss the bijection after localization and conclude Section 3 by giving an algorithm to get an explicit presentation of the homomorphism $T^*$. In Section 4.1, we analyze the homomorphism when $X$ is an Alexander quandle and $(1 - t)$ is invertible. In this case, we see a close relation between the second homology of $X$ and the exterior square $X \wedge X$. We conclude Section 4 by briefly discussing the homomorphism $T^*$ for non-Alexander quandles.

\footnote{The subject was introduced by Schur in his work on the projective representations of finite groups \cite{Sch}. According to \cite{Kar}, the original definition of Schur multiplier $M(G)$ is the second cohomology with coefficients in $\mathbb{C}^\times$, the multiplicative group of invertible complex numbers. For a finite group $G$, we have $H_2(G, \mathbb{Z}) = H^2(G, \mathbb{C}^\times)$. The book by Beyl and Tappa \cite{BT} gives the definition of the Schur multiplicator, commonly known as the Schur multiplier, $M(G)$ via the Schur-Hopf formula and in this case, $M(G)$ is isomorphic to $H_2(G, \mathbb{Z})$.}
In Section 4.3, we give a self-contained algebraic proof of Theorem 4.1. Finally, in the last section we discuss Alexander quandles which are connected but not quasigroups.

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2 Preliminaries

In this section, we review basic quandle theory and the homology of groups and quandles.

2.1 Quandles and their properties

A quandle \([\text{Joy}, \text{Mat}]\) is a set, \(X\), with a binary operation \(\ast : X \times X \rightarrow X\) such that:

(I) (idempotency) the identity \(a \ast a = a\) holds for any \(a \in X\),

(II) (invertibility) the map \((\bullet \ast b) : X \rightarrow X\) defined by \(x \mapsto x \ast b\) is bijective for any \(b \in X\) and its inverse is denoted by \(\bullet \ast^{-1} b\),

(III) (right self distributivity) the identity \((a \ast b) \ast c = (a \ast c) \ast (b \ast c)\) holds for any \(a, b, c \in X\).

Consider an Abelian group \(X\) with an automorphism \(t : X \rightarrow X\). Then, \(X\) is a quandle with the operation \(x \ast y := tx + (1 - t)y\), and is called an Alexander quandle. Notice that \(X\) is also a \(\mathbb{Z}[t^{\pm 1}]\)-module. As a special case, if \(t = -1\), the quandle is called a Takasaki quandle [NP]. In addition, every group \(G\) has a quandle structure with operation \(g \ast h := h^{-1}gh\) and is called a conjugation quandle.

The group generated by the bijective maps \(\bullet \ast x\) is called the inner automorphism group of \(X\), and is denoted by \(\text{Inn}(X)\). Observe that \(\text{Inn}(X)\) acts on \(X\) from the right. Let \(\text{Stab}(x)\) be the stabilizer subgroup of \(\text{Inn}(X)\), for an element \(x \in X\). If the action is transitive, \(X\) is said to be connected. Furthermore, we define the associated group of \(X\), \(\text{As}(X)\), by the presentation:

\[
\langle e_x \ (x \in X) \mid e_xe_y = e_ye_{x+y} \ (x, y \in X) \rangle.
\]
In general, it is hard to concretely determine \( \text{As}(X) \); see [Cla] for the case of Alexander quandles. Analogous to \( \text{Inn}(X) \), \( \text{As}(X) \) acts on \( X \) from the right as follows, \( e_y(x) = x \ast y \). We denote this action by \( X \circlearrowleft \text{As}(X) \). Additionally, let \( \text{Stab}(x_i) \subset \text{As}(X) \) be the stabilizer subgroup and \( \iota_i \) denote the inclusion \( \text{Stab}(x_i) \subset \text{As}(X) \).

Consider the commutative diagram in (1). Let \( \psi : \text{As}(X) \to \text{Inn}(X) \) be the homomorphism which sends \( e_x \) to \( \cdot \ast x \). Then, we have the following group extensions:

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & \text{Ker}(\psi) & \longrightarrow & \text{As}(X) & \overset{\psi}{\longrightarrow} & \text{Inn}(X) & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \text{Ker}(\psi) & \longrightarrow & \text{Stab}(x_i) & \overset{\text{res}\psi}{\longrightarrow} & \text{Stab}(x_i) & \longrightarrow & 0
\end{array}
\]

(1) The horizontal sequences have been proven to be central extensions (see, for example, [Nos2, §2.3]).

### 2.2 Quandle Homology and Relative Group Homology

We now review the homology theory of quandles and groups. Let \( C^R_n(X) \) be the free \( \mathbb{Z} \)-module generated by \( X^n \), i.e., \( C^R_n(X) = \mathbb{Z}\langle X^n \rangle \). For \( n \leq 3 \), the differential \( \partial^R_n : C^R_n(X) \to C^R_{n-1}(X) \) is defined by

\[
\partial^R_1(x) := 0, \quad \partial^R_2(x, y) := (x) - (x \ast y),
\]

\[
\partial^R_3(x, y, z) := (x, z) - (x \ast y, z) - (x, y) + (x \ast z, y \ast z).
\]

Then, the second homology groups are given by

\[
H^R_2(X; \mathbb{Z}) := \frac{\text{Ker}(\partial^R_2)}{\text{Im}(\partial^R_3)}, \quad H^Q_2(X; \mathbb{Z}) := \frac{\text{Ker}(\partial^R_2)}{\text{Im}(\partial^R_3)}, \{ (a, a) \}_{a \in X}.
\]

The former is called (two term) rack homology [FRS1, FRS2], and the latter is called quandle homology [CJKLS]. Dually, for an abelian group \( A \), we can define quandle cohomology \( H^2_Q(X; A) \) (see [CJKLS] for details of general (co)homology).

Next, we review relative group (co)homology (see [BE, Bro]). Let \( A \) be a right \( \mathbb{Z}[G] \)-module. Let \( C^R_n(G; A) \) be \( A \otimes_{\mathbb{Z}[G]} \mathbb{Z}[G^n] \). Define the boundary map \( \partial_n(a \otimes (g_1, \ldots, g_n)) \in C^R_{n-1}(G; A) \) by the formula:

\[
a \otimes (g_2, \ldots, g_n) + \sum_{1 \leq i \leq n-1} (-1)^i a \otimes (g_1, \ldots, g_{i-1}, g_i g_{i+1}, g_{i+2}, \ldots, g_n) + (-1)^n (a \cdot g_n^{-1}) \otimes (g_1, \ldots, g_{n-1}).
\]
Since $\partial_{n-1} \circ \partial_n = 0$, we can define the homology $H_n^\varphi(G;A)$ in the usual way. As mentioned before, if $A$ is the trivial $\mathbb{Z}[G]$-module, the second homology is often referred to as the Schur multiplier (see [Bro, BT, Kar] for basic references).

Let $K \subset G$ be groups and set up the cochain groups as
\[ C_n^{gr}(G, K; A) := \text{Map}(G^n, A) \oplus \text{Map}((K)^{n-1}, A). \]
For $(h, k) \in C_n^{gr}(G, K; A)$, define $\partial^n(h, k) : G^{n+1} \times K^n \to A$ by the formula
\[ \partial^n(h, k)(a, b) = (h(\partial_{n+1}(a)), h(b) - k(\partial_n(b))), \]
where $(a, b) \in G^{n+1} \times K^n$. Then, we have a cochain complex $(C^{gr}_n(G, K; A), \partial^n)$, and we define the cohomology in the standard way. Since $C_n^{gr}(G, K; A)$ is defined as a mapping cone, we have the long exact sequence:
\[
\cdots \to H_{n-1}^{gr}(K; A) \xrightarrow{\delta_{n-1}} H_n^{gr}(G, K; A) \to H_n^{gr}(G; A) \to H_n^{gr}(K; A) \xrightarrow{\delta_n} H_{n+1}^{gr}(G; A) \to \cdots. \tag{4}
\]

Furthermore, for any central extension $0 \to K \to G \to \mathcal{G} \to 0$ and any trivial coefficient $A$, we recall the exact sequence
\[
0 \to H_1^{gr}(\mathcal{G}; A) \to H_1^{gr}(G; A) \xrightarrow{\delta^1} \text{Hom}(K, A) \to H_2^{gr}(\mathcal{G}; A) \to H_2^{gr}(G; A), \tag{5}
\]
which is called the 5-terms exact sequence (see, e.g., [Bro, Chapter II.5]).

We now recall some known results in quandle homology. Let $X$ be a quandle. Due to the action $X \rhd \text{Inn}(X)$, we have the orbit decomposition $\widetilde{X} = \sqcup_{i \in I} X_i$, where $I$ is the set of the orbits. For $i \in I$, we choose $x_i \in X_i$. Consider the induced map $(\iota_i)_* : H_1(\text{Stab}(x_i); \mathbb{Z}) \to H_1(\text{As}(X); \mathbb{Z})$ from the inclusion $\iota_i$ and the homomorphism $\epsilon_X : \text{As}(X) \to \mathbb{Z}$ which sends $e_x$ to 1. Notice that, since $e_{x_i} \in \widetilde{\text{Stab}(x_i)}$, the induced map
\[ \iota_* : H_1^{gr}(\widetilde{\text{Stab}(x_i)}) \longrightarrow \text{Im}((\iota_i)_*) = \mathbb{Z} \]
splits. Thus, we can fix the projection
\[ \mathcal{P}_i : H_1^{gr}(\widetilde{\text{Stab}(x_i)}; \mathbb{Z}) \longrightarrow \text{Ker}((\iota_i)_*) \subset H_1^{gr}(\widetilde{\text{Stab}(x_i)}; \mathbb{Z}). \tag{6} \]
Then, Eisermann [Eis] proved the following result about second rack and quandle homologies:

**Theorem 2.1 ([Eis])**. There are isomorphisms:
\[
H_2^R(X) \cong \bigoplus_{i \in I} H_1(\widetilde{\text{Stab}(x_i)}; \mathbb{Z}) \quad \text{and} \quad H_2^Q(X; \mathbb{Z}) \cong \bigoplus_{i \in I} \text{Ker}((\iota_i)_*). \tag{7}
\]
Moreover, the map $H_2^R(X; \mathbb{Z}) \to H_2^Q(X; \mathbb{Z})$ from the projection in (6) is equal to the direct sum of maps described in equation (4), that is, $\bigoplus_{i \in I} \mathcal{P}_i$. 

5
Let us define the type of $X$ by

$$\text{Type}(X) = \min \{ n \mid (\cdots (x \mathbin{\ast} y) \cdots) \mathbin{\ast} y = x \text{ for any } x, y \in X \} \in \mathbb{N} \cup \{\infty\}.$$

Regarding the second group homologies of $\text{As}(X)$, the fourth author showed the following:

**Theorem 2.2** ([Nos1]). Let $X$ be a connected quandle. Then, $H^2_{\text{gr}}(\text{As}(X); \mathbb{Z})$ is annihilated by $\text{Type}(X)$. In particular, by the 5-terms exact sequence [5], there is an isomorphism $\text{Ker}(\psi)_{(\ell)} \cong H^2_{\text{gr}}(\text{Inn}(X); \mathbb{Z})_{(\ell)} \oplus \mathbb{Z}_{(\ell)}$ for any prime $\ell$ with $(\ell, \text{Type}(X)) = 1$.

In addition, regarding the rack second cohomology, we recall the following result by Etingof and Graña:

**Theorem 2.3** ([EG]). For any quandle $X$ and any abelian group $A$, there is an isomorphism $H^2_{\text{gr}}(X; A) \cong H^1_{\text{gr}}(\text{As}(X); \text{Map}(X, A))$, where $\text{Map}(X, A)$ is regarded as a right $\mathbb{Z}[\text{As}(X)]$-module.

Although this isomorphism is elegant, it does not give much insight on how to compute the rack cohomology. In fact, there are only a limited numbers of examples obtained from Theorem 2.3.

### 3 Relation of quandle homology to Schur multipliers

In this section, we will define a map that gives a connection between the second quandle cohomology and Schur multipliers. Let $X$ be a quandle, and $A$ a trivial coefficient module. Recall the orbit decomposition $X = \sqcup_{i \in I} X_i$, and choose $x_i \in X_i$, as in the previous section.

For $i \in I$, consider the following commutative diagram:
Here, the horizontal sequences arise from the 5-terms exact sequences, and the left vertical sequence is obtained from the long exact sequence (4).

**Definition 3.1.** Using the above diagram, we define the homomorphism
\[
\mathcal{T}_i : H^1_{\text{gr}}(\tilde{\text{Stab}}(x_i); A) \longrightarrow \frac{H^2_{\text{gr}}(\text{Inn}(X), \text{Stab}(x_i); A)}{H^1_{\text{gr}}(\text{Stab}(x_i); A)}
\]
by setting
\[
\mathcal{T}_i(\phi) := (p^*)^{-1}(\delta^*_i \circ \text{res} \psi^* \circ \mathcal{P}_i^*(\phi)).
\]
Here, \(\mathcal{P}_i^*\) is a dual of (6). By diagram chasing, we can easily verify that \(\mathcal{T}_i\) is well-defined.

**Remark 3.2.** By definition the kernel \(\text{Ker}(\mathcal{T}_i)\) contains the image \(\iota^*_i(H^1_{\text{gr}}(\text{As}(X); A))\). By Theorem 2.1, the domain of the sum \(\bigoplus_{i \in I} \mathcal{T}_i\) can be replaced by \(H^2_Q(X; A)\). Namely,
\[
\bigoplus_{i \in I} \mathcal{T}_i : H^2_Q(X; A) \longrightarrow \bigoplus_{i \in I} \frac{H^2_{\text{gr}}(\text{Inn}(X), \text{Stab}(x_i); A)}{H^1_{\text{gr}}(\text{Stab}(x_i); A)}.
\]

In the connected case, we have the following results pertaining to the homomorphism \(\mathcal{T}_i = \mathcal{T}\).

**Theorem 3.3.** Let \(X\) be a connected quandle, and fix \(x_0 \in X\). Let \(\ell\) be a prime number which is coprime to \(\text{Type}(X)\). Suppose that the localization \(H^1_{\text{gr}}(\text{Stab}(x_0); A)(\ell)\) is zero. Then, the localized map \(\mathcal{T}(\ell)\) gives an isomorphism
\[
\mathcal{T}(\ell) : H^2_Q(X; A)(\ell) \sim \rightarrow H^2_{\text{gr}}(\text{Inn}(X), \text{Stab}(x_0); A)(\ell).
\]

**Proof.** From Theorem 2.2 it follows that the localized map \((\delta^*_i)(\ell)\) is surjective and the kernel is equal to \(\iota^*_i(H^1_{\text{gr}}(\text{As}(X); A)(\ell))\). Since \(H^1_{\text{gr}}(\text{Stab}(x_0); A)(\ell) = 0\), the localized \(\mathcal{T}(\ell)\) is surjective, and the kernel is also equal to \(\iota^*_i(H^1_{\text{gr}}(\text{As}(X); A)(\ell))\). Hence, bijectivity is established.

A dual reconsideration of the above proof results in the following theorem.

**Theorem 3.4.** Let \(X\) be a connected quandle. Using the same notation as in Theorem 3.3, we get the homomorphism:
\[
\mathcal{T}_\ell : \text{Ker}\left( H^2_{\text{gr}}(\text{Inn}(X), \text{Stab}(x_0); \mathbb{Z}) \rightarrow H^1_{\text{gr}}(\text{Stab}(x_0); \mathbb{Z}) \right) \longrightarrow H^2_Q(X; \mathbb{Z}),
\]
such that the map localized at \(\ell\) yields the following isomorphism:
\[
\mathcal{T}_\ell : H^2_{\text{gr}}(\text{Inn}(X), \text{Stab}(x_0); \mathbb{Z})(\ell) \sim \rightarrow H^2_Q(X; \mathbb{Z})(\ell),
\]
provided that \(\ell\) is a prime number, \(\ell\) and \(\text{Type}(X)\) are relatively prime, and \(H^1_{\text{gr}}(\text{Stab}(x_0); \mathbb{Z})(\ell)\) vanishes.
We now emphasize the advantages of the homomorphism $\mathcal{T}$. While it is true that Theorem 2.1 gives a way to compute second quandle (co)homology, in general, it is hard to definitively determine $\text{As}(X)$ and $\text{Ker}(\psi)$. In contrast, according to Theorems 3.3 and 3.4 we do not need any information of $\text{As}(X)$ and $\text{Ker}(\psi)$. What we do need is $\text{Inn}(X)$ and its second group homology. Furthermore, the map $T_i$ can be concretely described as follows.

For this, by the definitions, we shall only examine $\delta_1^*$ and the isomorphisms in (7) in detail. For a general central extension $0 \to K \to G \to \mathcal{G} \to 0$, we give a concrete description of the delta map $\delta_1^* : H^1_{gr}(K; A) \to H^2_{gr}(G; A)$. For $\phi \in H^1_{gr}(K, A)$, choose a representative 1-cocycle $F : K \to A$ with $[F] = \phi$, and a section $s : \mathcal{G} \to G$. We define a map

$$\delta(F) : \mathcal{G}^2 \longrightarrow A \text{ by } \delta(F)(g, h) := F(s(g)s(h)s(gh)^{-1})$$

for any $g, h \in \mathcal{G}$. Then, $\delta_1^*(\phi)$ is equal to $[\delta(F)]$ (see [Ron] for a proof).

We now explain the isomorphism in (7). For $i \in I$ and any element $[g] \in \widetilde{\text{Stab}}(x_i)$, we choose a representative $g = e_{x_1}^{\epsilon_1} \cdots e_{x_n}^{\epsilon_n}$ for some $x_1, \ldots, x_n \in X$ and $\epsilon_1, \ldots, \epsilon_n \in \{\pm 1\}$. Now, consider the correspondence

$$\widetilde{\text{Stab}}(x_i) \ni e_{x_1}^{\epsilon_1} \cdots e_{x_n}^{\epsilon_n} \longmapsto \sum_{j=1}^n \epsilon_j \left((x_i \ast_1 x_1) \ast_2 \cdots \ast_{j-1} x_{j-1}\right) \ast_{j-1} x_j, x_j \in \mathbb{Z}^R_2(X).$$

Since this correspondence is additive, we have an induced map $H^1_{gr}(\widetilde{\text{Stab}}(x_i); \mathbb{Z}) \to H^2_{gr}(X)$. This map is exactly equal to the isomorphism in (7) (see also [Nos2 §5.2] for the proof).

When $X$ is of finite order, there are methods to determine the finite group $\text{Inn}(X)$ (see for example, [EMR] and [Nos2, Appendix B]) and it is not so hard to choose a section $\text{Inn}(X) \to \text{As}(X)$. In summary, if we find an explicit representative of a quandle 2-cocycle $\phi : X^2 \to A$, we can compute $\mathcal{T}(\phi)$.

### 4 The second homology of Alexander quandles

Based on the computation of $\mathcal{T}$ in the previous section, we will concretely describe $\mathcal{T}$ where $X$ is an Alexander quandle of type $\text{Type}(X)$. That is, $X$ is a $\mathbb{Z}[t^{\pm 1}]$-module, and the quandle operation is defined by $x \ast y = tx + (1-t)y$. Recall that $\text{Type}(X)$ is equal to $\min\{n \in \mathbb{N} \mid t^n = \text{id}_X\}$ if it exists or $\infty$ otherwise, and $X$ is connected if and only if $1-t : X \to X$ is surjective.

We now analyze $\text{As}(X)$, when $(1-t)$ is invertible. Consider the homomorphism $X \otimes_{\mathbb{Z}} X \to X \otimes_{\mathbb{Z}} X$ which takes $x \otimes y$ to $x \otimes y - y \otimes tx$, and let $Q_X$ be its cokernel. Recall that Clauwens [Cla] showed a group isomorphism

$$\text{As}(X) \longrightarrow \mathbb{Z} \times X \times Q_X,$$
where the group operation on the right hand side is defined by
\[
(n, a, \alpha) \cdot (m, b, \beta) = (n + m, t^m a + b, [a \otimes t^m b] + \alpha + \beta).
\] (13)

4.1 Second quandle homology from Schur multipliers

In Section 4.3, we give a self-contained proof of the following theorem.

**Theorem 4.1.** Let \( X \) be an Alexander quandle with \((1 - t)\) invertible. The correspondence \( \mathbb{Z}<X^2> \ni (x, y) \mapsto x \otimes (1 - t)y \in X \otimes X \) gives rise to the isomorphism
\[
H_Q^2(X; \mathbb{Z}) \cong \frac{X \otimes \mathbb{Z} X}{\{x \otimes y - y \otimes tx\}_{x, y \in X}}.
\] (14)

**Remark 4.2.** The existence of the isomorphism in (14) is implicit in [Cla, IV]. In fact, since we can easily check that \( \text{Ker}(\iota) \cong Q_X \) from the group operation in (13), then the isomorphism \( H_Q^2(X) \cong \text{Ker}(\iota) \) in Theorem 2.1 readily implies the isomorphism in (14). However, Theorem 2.1 uses topological properties of the rack space [FRS1, FRS2], where the rack space is a geometrical realization of the rack complex. In contrast, Section 4.3 will give an independent algebraic proof of Theorem 4.1 and a concrete description of the isomorphism.

To describe the homomorphism \( T \), we consider the following transformation:

**Corollary 4.3.** Let \( X \) be an Alexander quandle with \((1 - t)\) invertible. Then, there is an isomorphism
\[
H_Q^2(X; \mathbb{Z}) \cong \frac{X \wedge \mathbb{Z} X}{\{x \wedge y - ty \wedge tx\}_{x, y \in X}}.
\] (15)

**Proof.** We can easily check that the correspondence \( X \otimes X \to X \otimes X \) which sends \( x \otimes y \) to \( x \otimes (1 - t)y \), defines a homomorphism from the right hand side of (14) to that of (15). Moreover, the inverse mapping is obtained from the correspondence \( X \otimes X \to X \otimes X \) which sends \( x \otimes y \) to \( x \otimes (1 - t)^{-1} y \).

**Remark 4.4.** If \( t = -1 \), the isomorphism in (15) has been proved in [NP]. Therefore, Corollary 4.3 is a generalization of [NP].

Next, we explicitly describe the homomorphism \( T \). It is not hard to check that \( \text{Inn}(X) \cong (\mathbb{Z}/\text{Type}(X)) \rtimes X \) (see for example, [Nos2, Proposition B.18]). In order to describe \( \delta_t^* \) in detail, we choose to define a section \( s : \text{Inn}(X) \to \text{As}(X) \) by \([n], x \mapsto (n, x, 0)\). The universal quandle 2-cocycle is represented by (see Corollary 4.9)
\[
\phi : X \times X \to Q_X; \quad (x, y) \mapsto [x \otimes (1 - t)y].
\] (16)
Then, according to the discussion in [13], we can easily show that the 2-cocycle of the group Inn($X$) is given by the map:

$$\mathcal{T}(\phi) : \text{Inn}(X)^2 \to \frac{X \wedge_{Z} X}{\{x \wedge y - tx \wedge ty\}_{x,y \in X}}; \quad ((n, x), (m, y)) \mapsto x \wedge y. \quad (17)$$

Finally, we will compute the domain of $\mathcal{T}_x$, i.e., the left hand side of (11), and check Theorem 3.4 when $X$ is an Alexander quandle and $(1 - t)$ is invertible. Since $\text{Inn}(X) \cong (\mathbb{Z}/\text{Type}(X)) \ltimes X$ as above, Stab($x_0$) $\cong \mathbb{Z}/\text{Type}(X)\mathbb{Z}$. Therefore, $H_k(\text{Stab}(x_0); \mathbb{Z})$ is annihilated by Type($X$). Hence, the left hand side is isomorphic to $H^g_2(\text{Inn}(X); \mathbb{Z})$ after localization at $\ell$. Then, by using transfer (see for example, [Bro, Sect. III.8]), we have the following isomorphisms:

$$H^g_2(\text{Inn}(X); \mathbb{Z})(\ell) \cong \frac{H^g_2(X; \mathbb{Z})}{\{ta - a\}_{ta \in H^g_2(X; \mathbb{Z})}}(\ell) \cong \frac{X \wedge_{Z} X}{\{x \wedge y - tx \wedge ty\}_{x,y \in X}}(\ell).$$

Here, the second isomorphism is obtained from $H^g_2(X; \mathbb{Z}) \cong X \wedge_{Z} X$ and using the fact that the action of $\mathbb{Z}/\text{Type}(X)$ on $H^g_2(X; \mathbb{Z})$ is compatible with the diagonal action on $X \wedge_{Z} X$ (see [Bro, Chapter V.6]). In particular, after comparing (16) with (17), we see that the localized $\mathcal{T}(\ell)$ is an isomorphism as in Theorem 3.4.

### 4.2 Examples of computations of the second homology of Alexander quandles

In this subsection, we compute the second homology of certain Alexander quandles using Corollary 4.3. The homomorphism $T : X \wedge_{Z} X \to X \wedge_{Z} X$ which sends $x \wedge y$ to $tx \wedge ty$ plays a key role.

First, we compute the second homology of a connected quandle of order $p^2$. Although the result is first proven by [IV], we give a simpler proof. To describe this, we say that a connected Alexander quandle $Y$ is special, if $Y \cong (\mathbb{Z}/p)^2$ and the determinant of $t : Y \to Y$ is 1.

**Proposition 4.5 ([IV, Proposition 5.9]).** Let $X$ be a connected quandle of order $p^2$. If $X$ is a special Alexander quandle, then $H^Q_2(X) \cong \mathbb{Z}/p$. Otherwise, $H^Q_2(X)$ is zero.

**Proof.** It is shown in [ECI] that $X$ is isomorphic to an Alexander quandle. Notice that, $X$ is either $\mathbb{Z}/p^2\mathbb{Z}$ or $(\mathbb{Z}/p\mathbb{Z})^2$. If $X = \mathbb{Z}/p^2\mathbb{Z}$, then $X \wedge X = 0$; hence, $H^Q_2(X)$ is zero.

Thus, we may suppose $X \cong (\mathbb{Z}/p\mathbb{Z})^2$. Since $X \wedge X \cong \mathbb{Z}/p$, $H^Q_2(X)$ is zero or $\mathbb{Z}/p$. Therefore, it is enough to show that $H^Q_2(X) \cong \mathbb{Z}/p$ if and only if $X$ is special. Recall the homomorphism $T : X \wedge_{Z} X \to X \wedge_{Z} X$, and that $H^Q_2(X) \cong \text{Coker}(1 - T)$. We can easily get $T = \det(t) \cdot \text{id}_{X \wedge X}$ by considering the eigenvalues of $t$. Hence, $H^Q_2(X) \cong \text{Coker}(1 - T)$ is $\mathbb{Z}/p$, if and only if $\det(t) = 1$, that is, $X$ is special.

As another example, we will compute the second homology of the Alexander quandle constructed using the polynomial $\Phi_n = (t^n - 1)/(t - 1)$. 


Proposition 4.6. Let $X$ be an Alexander quandle of the form $X = \mathbb{F}_p[t]/(\Phi_n)$ over $\mathbb{F}_p$, where $\mathbb{F}_p$ denotes the field of order $p$. Assume that $n$ and $p$ are coprime. Then, there is an isomorphism

$$H^Q_2(X; \mathbb{Z}) \cong (\mathbb{Z}/p)^{\lfloor \frac{n-1}{2} \rfloor}. \quad (18)$$

Proof. By assumption, $X$ is connected and considered over a field $\mathbb{F}_p$. Thus, $H^Q_2(X)$ is a vector space over $\mathbb{F}_p$ by Theorem 4.1. Thus, it is enough to show that $\dim(\text{Coker}(1 - T)) = \lfloor \frac{n-1}{2} \rfloor$. Let $\bar{\mathbb{F}_p}$ be the algebraic closure of $\mathbb{F}_p$, and $X_{\bar{\mathbb{F}_p}}$ be $\mathbb{F}_p \otimes_{\mathbb{Z}/p} X$. Fix $\mu \in \bar{\mathbb{F}_p}$ as an $n$-th root of unity. Then, we have the extension

$$\text{id}_{\bar{\mathbb{F}_p}} \otimes T : X_{\bar{\mathbb{F}_p}} \wedge_{\bar{\mathbb{F}_p}} X_{\bar{\mathbb{F}_p}} \to X_{\bar{\mathbb{F}_p}} \wedge_{\bar{\mathbb{F}_p}} X_{\bar{\mathbb{F}_p}}$$

and $\dim(\text{Coker}(1 - T)) = \dim(\text{Coker}(1 - \text{id}_{\bar{\mathbb{F}_p}} \otimes T))$. When considering $t : X_{\bar{\mathbb{F}_p}} \to X_{\bar{\mathbb{F}_p}}$ as a linear map, we can choose the eigenvectors $v_1, \ldots, v_{n-1}$ with $tv_i = \mu^i v_i$. Since, $\dim(X_{\bar{\mathbb{F}_p}}) = n - 1$, the vectors $v_1, \ldots, v_{n-1}$ give a basis of $X_{\bar{\mathbb{F}_p}}$. Then, notice $\text{id}_{\bar{\mathbb{F}_p}} \otimes T(\sum_{s,t} a_{s,t} v_s \wedge v_t) = \sum_{s,t} a_{s,t} \mu^{s+t} v_s \wedge v_t$, where $a_{s,t} \in \bar{\mathbb{F}_p}$. In particular, $T$ can be represented as a diagonal matrix and $\text{id}_{\bar{\mathbb{F}_p}} \otimes T(v_j \wedge v_{n-j}) = v_j \wedge v_{n-j}$, which implies that $\dim(\text{Coker}(1 - \text{id}_{\bar{\mathbb{F}_p}} \otimes T)) = \lfloor \frac{n-1}{2} \rfloor$, as required.

4.3 Proof of Theorem 4.1

A quandle $X$ is a quasigroup if there exists a unique $c$ such that $a * c = b$ for any $a, b \in X$. We denote by $a \circ b$ such an element $c$. Observe that if $X$ is a quasigroup, then $X$ is connected. In particular, an Alexander quandle $X$ is a quasigroup if and only if $(1 - t)$ is invertible. We need the following basic properties proven in [NP](see Lemma 2.1, Corollary 2.2).

Proposition 4.7 ([NP]). Let $X$ be a quasigroup quandle. Choose any element $d_0$ in $X$. Then,

(I) For any element $d_0$ in $X$, the quotient map $C^Q_2 \to C^Q_2/((d_0, b))$ is an isomorphism when restricted to $\text{Ker} \partial^R_2$. Here, $(d_0, b)$ is the subgroup of $C^Q_2$ generated by elements of the form $(d_0, b)$ for any $b \in X$.

(II) The induced map

$$H^Q_2(X) \to \frac{\mathbb{Z}[X \times X]}{\{(a, a), (d_0, a), \partial^R_3(a, b, c)\}_{a, b, c \in X}}. \quad (19)$$

is an isomorphism.

Furthermore, we will also need the following proposition:

Proposition 4.8. Let $X$ be a quasigroup quandle. Choose any element $d_0$ in $X$. In the quotient on the right hand side of (19), the relation $(c, d_0) = 0$ holds for any $c \in X$. 

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Proof. In the quotient, compute $\partial_R^3(d_0, a, d_0)$ as $(d_0, d_0) - (d_0 * a, d_0) - (d_0, a) + (d_0 * d_0, a * d_0) = -(d_0 * a, d_0)$. Since $X$ is quasigroup, $d_0 * a$ can be regarded as any element $c$. Therefore, we have $(c, d_0) = 0$.

Now, we give a self-contained proof of Theorem 4.1. Let $X$ be an Alexander quandle such that $(1 - t)$ is invertible, and $d_0 = 0$. In a quasigroup quandle we can write $[a, (1 - t)b]$ for $(a, b)$. Then, the boundary map in (2) has the form:

$$\partial_3(a, b, c) = (a, c) - (a * b, c) - (a, b) + (a * c, b * c) = [a, (1 - t)c] - [a, (1 - t)b] - [ta + (1 - t)b, (1 - t)c] + [ta + (1 - t)c, (1 - t)(tb + (1 - t)c)].$$  \hspace{1cm} (20)

Proof of Theorem 4.1. The following isomorphism follows from Propositions 4.7 and 4.8:

$$H^Q_2(X; \mathbb{Z}) \cong \frac{\mathbb{Z}(X \times X)}{\{[a, a(1 - t)], [a, 0], [0, a], (20)\}_{a, b, c \in X}}. \hspace{1cm} (21)$$

The correspondence $[a, b] \mapsto a \otimes b$ gives rise to an epimorphism

$$\frac{\mathbb{Z}(X \times X)}{\{[a, a(1 - t)], [a, 0], [0, a], (20)\}_{a, b, c \in X}} \longrightarrow \frac{X \otimes \mathbb{Z} X}{\{x \otimes y - y \otimes tx\}_{x, y \in X}}.$$

The goal is to show that this epimorphism is an isomorphism. For this, it is enough to show that $[,]$ is bilinear and $[a, b] = [b, ta]$.

First, we use (20) to show that the equality $[a, b] = [b, ta]$ holds. If we replace $a$ by $x$, $b$ by $(1 - t)^{-1}y$, and $c$ by $(1 - t)^{-1}z$, the relation (20) in the right hand side of (21) is equivalent to

$$[x, z] - [x, y] - [tx + y, z] + [tx + z, ty + (1 - t)z] = 0.$$  \hspace{1cm} (22)

For $z = 0$, $x = 0$, and $y = -tx$, we obtain, respectively,

$$[tx, ty] = [x, y],$$  \hspace{1cm} (23)

$$[y, z] = [z, ty + (1 - t)z],$$  \hspace{1cm} (24)

$$[x, -tx] = [x, z] + [tx + z, -t^2x + (1 - t)z].$$  \hspace{1cm} (25)

Note that by (24) the relation in (25) can be reduced to

$$[x, -tx] = [x, z] + [-x, tx + z].$$  \hspace{1cm} (26)

In addition, using (24), the relation in (22) is equivalent to

$$0 = [x, z] - [x, y] - [tx + y, z] + [y + (t - 1)x, tx + z].$$  \hspace{1cm} (27)

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After replacing \( z \) with \( y \) in (26) we get
\[
[x, -tx] = [x, y] + [-x, tx + y].
\] (28)

Then, using (26) and (28), we have
\[
0 = -[-x, tx + y] + [-x, tx + z] - [tx + y, z] + \[y + (t - 1)x, tx + z\].
\] (29)

In addition, if we replace \((tx + y)\) by \(Y\) and \((tx + z)\) by \(Z\), then (29) changes to
\[
0 = [-x, Z] - [-x, Y] + [Y, Z - tx] - [Y - x, Z].
\] (30)

Then, after the substitution \( Z = 0 \), the formula (30) gives the desired equality \([-x, Y] = Y, -tx\) for any \( x, Y \in X \).

Finally, we show the bilinearity of the bracket \([,]\). By the previous equality, it is enough to show that \([y', Z] + [x', Z] = [x' + y', Z]\) for any \( x', y', Z \in X \). Now, by applying the equality \([Y, Z - tx] = [t^{-1}Z - x, Y]\) to (30), we have
\[
[-x, Z] - [Y - x, Z] = [-x, Y] - [t^{-1}Z - x, Y].
\] (31)

Then, the substitution, \( Z \mapsto Y, Y \mapsto t^{-1}Z \) transforms (31) into
\[
[-x, Y] - [t^{-1}Z - x, Y] = [-x, t^{-1}Z] - [t^{-1}Y - x, t^{-1}Z].
\] (32)

Hence, combining (31) with (32) yields the equation
\[
[-x, Z] - [Y - x, Z] = [-x, t^{-1}Z] - [t^{-1}Y - x, t^{-1}Z].
\] (33)

In particular, when \( x = Y \), we have the equality \([-x, Z] = [-x, t^{-1}Z] - [(t^{-1} - 1)x, t^{-1}Z]\). Thus, (33) is reduced to
\[
[Y - x, Z] + [((t^{-1} - 1)x, t^{-1}Z] = [t^{-1}Y - x, t^{-1}Z] = [Y - tx, Z],
\]
where we use (23) for the last equality. Then, replacing \((Y - x)\) by \(y'\), and \((1 - t)x\) by \(x'\), we have \([y', Z] + [x', Z] = [x' + y', Z]\).

From the proof, we have the following corollary.

**Corollary 4.9.** The map \( X^2 \to X \otimes X/\{x \otimes y - y \otimes tx\} \), which sends \((x, y)\) to \(x \otimes (1 - t)y\), is a quandle 2-cocycle.

### 4.4 Non-Alexander quandles

Up until now, we mainly focused on Alexander quandles. In general, it is not easy to compute \( H^2_\mathbb{Q}(X) \) using Schur multipliers. In fact, we have verified that, for any connected non-Alexander quandle of order \( \leq 15 \), the map \( T \) in Definition 3.1 is zero. Consider the following example.
Example 4.10. Let $\mathfrak{S}_n$ be the permutation group on $n$ elements, and let $X$ be the subset \{${g^{-1}(12)g}$ $\mid$ $g \in \mathfrak{S}_n$\} $\subset \mathfrak{S}_n$. Then, $X$ is of order $n(n - 1)/2$, and the conjugacy operation makes $X$ into a connected quandle of type 2. Then, $\text{Inn}(X) \cong \mathfrak{S}_n$ and $\text{Stab}(x_0) = \mathfrak{S}_{n-1}$. In particular, $H_2^0(\text{Stab}(x_0); \mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$. Moreover, it is known that $H_2^0(X) \cong \mathbb{Z}/2\mathbb{Z}$ for $n > 3$ (See [Eis, Example 1.18]), but by diagram chasing in (8), the map $\mathcal{T}$ turns out to be zero.

Furthermore, there are few examples of connected quandles $X$ such that the orders of $H_2^0(X)$ and $\text{Type}(X)$ are relatively prime. Thus, in the future one may consider finding other families of quandles for which Theorem 3.4 is applicable.

5 Alexander quandles and semi-Hopfian groups

In this section, we give an example of an Alexander quandle which is connected but not a quasigroup.

Analogous to the Hopfian condition for groups, we define a semi-Hopfian Abelian group as follows:

Definition 5.1. An Abelian group $X$ is called semi-Hopfian if for every epimorphism $f : X \to X$, such that $(1 - f)$ is an isomorphism, $f$ is also an isomorphism.

It is clear that every Abelian Hopfian group is semi-Hopfian. In particular, every finitely generated Abelian group is semi-Hopfian. Examples of non semi-Hopfian groups are well known [Br]. For instance, consider the following construction:

Construction 5.2. Consider the countable direct sum of the group $\mathbb{Z}$ indexed by positive numbers:

$$X = \bigoplus_{i > 0} \mathbb{Z}^{(i)}$$

where $1_i$ is the identity of $\mathbb{Z}^{(i)}$.

Let $t : X \to X$ be an epimorphism defined on the basis by $f(1_i) = 1_{i-1}$ for $i > 1$ and $f(1_1) = 0$. Clearly $f$ is not a monomorphism. We have $(1 - f)(1_i) = t_i - t_{i-1}$ and observe that $(1 - t)$ is invertible since:

$$(1 - t)^{-1}(1_i) = 1_i + 1_{i-1} + \ldots + 1_1.$$

Construction 5.2 gives an example of an Alexander quandle which is connected but not a quasigroup. Namely, if we put $t = 1 - f$, then $(X, \ast)$, with $a \ast b = ta + (1 - t)b$, is a quandle which is not a quasigroup since $1 - t = f$ is not invertible. However, it is connected since $(1 - t)X = X$.

\footnote{A group $X$ is called a Hopfian group if every epimorphism from $X$ to $X$ is an isomorphism.}
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