Moments, Sums of Squares, and Tropicalization

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Dedicated to Bernd Sturmfels on the occasion of his 60th birthday.

Abstract. We use tropicalization to study the duals to cones of nonnegative polynomials and sums of squares on a semialgebraic set $S$. The truncated cones of moments of measures supported on the set $S$ is dual to nonnegative polynomials on $S$, while “pseudo-moments” are dual to sums of squares approximations to nonnegative polynomials. We provide explicit combinatorial descriptions of tropicalizations of the moment and pseudo-moment cones, and demonstrate their usefulness in distinguishing between nonnegative polynomials and sums of squares. We give examples that show new limitations of sums of squares approximations of nonnegative polynomials. When the semialgebraic set is defined by binomial inequalities, its moment and pseudo-moment cones are closed under Hadamard product. In this case, their tropicalizations are polyhedral cones that encode all binomial inequalities on the moment and pseudo-moment cones.

1. Introduction

Understanding nonnegativity of polynomials in terms of sums of squares has been a central challenge in real algebraic geometry dating back to the work of Hilbert. The dual side of this problem is important in analysis and known as the moment problem. We now take a moment to introduce it.

For a semialgebraic set $S \subseteq \mathbb{R}^n$ and a finite subset $A \subseteq \mathbb{N}^n$, we consider the convex cone $M_A(S)$ of $A$-moments of measures supported on $S$. Despite extensive work this cone can be explicitly described in very few situations even when $S = \mathbb{R}^n$ and $A$ corresponds to all moments of degree at most $2d$ [CF96, CF91, dDS18, Sch17]. An important tool for understanding $M_A(S)$ comes from Positivstellensätze in real algebraic geometry: theorems on representing, via sums of squares, the dual cone of polynomials with support in $A$ which are nonnegative on $S$ [Sch91, Put93]. We denote the cone of linear functionals dual to the cone of “obviously nonnegative” polynomials generated by sums of squares by $\Sigma(S)_A^\vee$ and call such functionals “pseudo-moments”. Tropicalization of the cones of moments and pseudo-moments gives us “combinatorial shadows” of these sets. Our explicit descriptions of these shadows lead to interesting combinatorial questions, some of which have been considered in the context of SONC polynomials [Rez89, IdW16, KNT21].

Another way of understanding our results is through binomial inequalities in moments and pseudo-moments of measures supported on $S$. When the semialgebraic set $S$ is closed under Hadamard multiplication, the tropicalization $\text{trop } M_A(S)$ of the moment cone is a rational polyhedral cone. Its dual cone $(\text{trop } M_A(S))^\vee$ encodes all of the binomial inequalities in $A$-moments. Similarly, binomial moment inequalities that can proved via sums of squares correspond to another rational polyhedral cone, which may depend on a degree bound for the sums of squares construction. While polynomial inequalities valid on $M_A(S)$ are difficult to characterize, we can explicitly describe all binomial inequalities in moments and pseudo-moments by finding the extreme rays of the corresponding rational polyhedral cones. The use of tropicalizations to analyze the power of sums of squares method was first introduced in [BRST20] for analyzing graph density inequalities, and further developed in [BR21]. We take inspiration from some of their results and techniques, for instance the use of the Hadamard property to ensure that the tropicalization is a convex cone. However, to the best of our knowledge, this is the first instance where tropicalization is used to study the...
relationship between the moment and pseudo-moment cones. We believe that this just scratches the surface of applications of tropicalization in semi-algebraic geometry.

We start with a pair of examples which illustrate our setup and results.

**Example 1.1** (Motzkin Configuration on the Nonnegative Orthant). Let \( S = \mathbb{R}_{\geq 0}^2 \) be the nonnegative orthant and let \( A \subset \mathbb{N}^2 \) be the Motzkin configuration: \( A = \{(0,0), (1,2), (2,1), (1,1)\} \), which gives us the exponents of moments we are recording:

\[
m_{00} = \int_S 1 \, d\mu, \quad m_{12} = \int_S xy^2 \, d\mu, \quad m_{21} = \int_S x^2y \, d\mu, \quad m_{11} = \int_S xy \, d\mu.
\]

There is only one binomial inequality satisfied by \( A \)-moments of measures supported on \( S \):

\[
m_{00} m_{12} m_{21} \geq m_{311}^3. \tag{1}
\]

If we regard moments as functions on \( A \), then we see that moments are nonnegative log-convex functions on \( A \), and in fact inequalities coming from log-convexity are the only possibly binomial inequalities in \( A \)-moments for measures supported on the nonnegative orthant \( \mathbb{R}_{\geq 0}^n \) (see Theorem 4.2).

We now consider \( A \)-pseudo-moments of measures supported on \( \mathbb{R}_{\geq 0}^n \). Pseudo-moments are defined as linear functionals that are nonnegative on "obviously" nonnegative polynomials coming from sums of squares (see Section 5). We show in Example 5.2 that \( A \)-pseudo-moments of measures supported on \( \mathbb{R}_{\geq 0}^n \) satisfy log-midpoint-convexity inequalities:

\[
m_{\alpha} m_{\beta} \geq m_{\frac{\alpha+\beta}{2}}^2,
\]

with \( \alpha, \beta, \frac{\alpha+\beta}{2} \in A \). Moreover these inequalities generate all possible binomial inequalities valid on \( A \)-pseudo-moments. Since the Motzkin configuration contains no midpoints, we see that there are no binomial inequalities valid on \( A \)-pseudo-moments.

**Remark 1.2.** The combinatorial notions of convex and midpoint-convex functions on \( A \) are quite similar to what has been developed for analyzing certain sparse globally nonnegative polynomials and sums of squares arising from the arithmetic mean-geometric mean inequality. Such polynomials were originally called AGI-forms by Reznick in [Rez89] and were later called Sum of Nonnegative Circuit Polynomials (SONC) in [IdW16]. The only difference is that for analyzing global nonnegativity, it makes a difference whether points in \( A \) have all even coordinates or not, and for instance midpoint convexity has to hold only between even points in \( A \). As we will see in Theorem 4.12 and Example 5.1, this is precisely what happens for us as well when analyzing measures supported on all of \( \mathbb{R}^n \).

**Example 1.3** (Motzkin Configuration on the Square.). Let \( S = [0,1]^2 \subset \mathbb{R}^2 \) be the unit square given by inequalities \( 0 \leq x \leq 1, 0 \leq y \leq 1 \). Let \( A \subset \mathbb{N}^2 \) again be the Motzkin configuration. In addition to the log-convexity inequality (1), the following binomial moment inequalities are naturally valid on the unit square, since all variables lie between 0 and 1:

\[
m_{00} \geq m_{11}, \quad m_{11} \geq m_{12}, \quad m_{11} \geq m_{21}.
\]

Any binomial inequality in \( A \)-moments of measures supported on \( S \) can be obtained from the above inequalities and (1) via exponentiation and multiplication (see Example 4.7).

As we increase the degree \( d \), sums of squares provide increasingly better approximations to polynomials supported on \( A \) that are nonnegative on \( S \), and thus can, in principle, be used to provide increasingly sharper binomial inequalities for pseudo-moments (see Section 5 for more details). If we regard pseudo-moments as functions on \( A \) then increasing the degree allows us to use moments...
that lie outside of $A$. For instance we can show that $m_{00}m_{12} \geq 2m_{11}$ by combining the inequality $m_{12} \geq m_{22}$ with the log-midpoint-convexity inequality \( (2) \): $m_{00}m_{22} \geq m_{11}^2$.

We show that, in this case, the binomial $A$-pseudo-moment inequalities stabilize, and only the following binomial inequalities can be learned via sums of squares (regardless of the degree $d$):

$$m_{11} \geq m_{12}, \quad m_{11} \geq m_{21}, \quad m_{00}m_{12} \geq m_{11}^2, \quad m_{00}m_{21} \geq m_{11}^2.$$  

Therefore, for any degree $d$, sums of squares cannot prove the moment inequality $m_{00}m_{12}m_{21} \geq m_{11}^3$, and moreover, sums of squares remain quantifiably far away from certifying this inequality. \hfill \( \diamond \)

Remark 1.4. Since the unit square is compact, it follows from Schmüdgen’s Positivstellensatz \( \text{[Sch91]} \) that any polynomial $f$ strictly positive on the unit square has a sum of squares certificate. Therefore, as the degree increases, sums of squares provide an increasingly better approximation to all nonnegative polynomials supported on $A$. However, as we have seen, tropicalizations stabilize, and higher degree sums of squares do not have larger tropicalizations. This is due to the fact that trop$(S)$ only depends on the neighborhood of zero and the “neighborhood of infinity” contained in $S$. We give a simple example of this phenomenon below: Let $S$ be the planar triangle with vertices $(0,0)$, $(1,0)$ and $(1,1)$ and let $S_x$ be the quadrilateral with vertices $(0,0)$, $(1,0)$, $(1,1)$ and $(0,\varepsilon)$. Then we have $S_x \to S$ as $\varepsilon \to 0$, however trop$(S_x)$ is the nonpositive orthant for all $\varepsilon > 0$, and trop$(S)$ is the subset of the nonpositive orthant given by the inequality $x \geq y$. \hfill \( \diamond \)

Remark 1.5. The unit square is special in that all nonnegative polynomials have a sum of squares certificate. Example \( \text{[1.3]} \) also shows that even though every nonnegative polynomial is a sum of squares, there does not exist a degree bound for the certificate even for just the $A$-supported polynomials \( \text{[Mar08, 9.4.6 Example (1)]} \). \hfill \( \diamond \)

1.1. Main Results in Detail: Our main results are about the tropicalizations of moment cones and pseudo-moment cones for semi-algebraic sets with the Hadamard property. We say that a subset $S$ of $\mathbb{R}_{\geq 0}^n$ has the Hadamard property if $S$ is closed under coordinatewise (Hadamard) multiplication. Concretely, we focus on nonnegative orthants, hypercubes, and toric cubes to discuss our general results. Throughout, we fix a finite set $A \subset \mathbb{N}^n$ of exponents and consider the $A$-moments: $m_a = \int_S x^a$ for $a \in A$ (also known as truncated moment sequences).

We think of elements of the tropicalization of the moment cone (resp. pseudo-moment cone) as functions $h : A \to \mathbb{R}$ and describe the tropicalization mainly in terms of discrete convexity properties of these functions. For the moment cone, we have a general description of the tropicalization of $M_A(S)$ for any subset of the nonnegative orthant with the Hadamard property:

**Theorem** (Theorem \( \text{[4.2]} \)). Let $S \subset \mathbb{R}_{\geq 0}^n$ be a semialgebraic set with the Hadamard property such that the intersection of $S$ with the positive orthant is dense in $S$. The tropicalization of the $A$-moment cone $M_A(S)$ is the rational polyhedral cone of functions $h : A \to \mathbb{R}$ satisfying

$$\sum_{i=1}^r \lambda_i h(a_i) \geq h(b) \quad \text{for all } a_1, \ldots, a_r, b \in A, \lambda_i \geq 0, \sum_{i=1}^r \lambda_i = 1, \text{ with } \sum_{i=1}^r \lambda_i a_i - b \in \text{trop}(S)^\vee.$$  

In particular, all of the functions $h : A \to \mathbb{R}$ in $M_A(S)$ satisfy the following:

1. (Convexity) $\sum_{i=1}^r \lambda_i h(a_i) \geq h(b)$ for all $a_i, b \in A$, $\lambda_i \geq 0$, $\sum_{i=1}^r \lambda_i = 1$, $\sum_{i=1}^r \lambda_i a_i = b$;
2. (Nonincreasing) $h(a) \geq h(b)$ whenever $a - b \in \text{trop}(S)^\vee$.

The first type of inequality is the naive form of discrete convexity that arises in this context. The second type of inequality is where the set $S$ enters: The tropicalization of $S$ is a rational polyhedral
cone and \( \text{trop}(S)^\vee \) is its dual cone, which defines a partial order of \( \mathbb{R}^d \) – and the second inequality says that the functions in the tropicalization are order preserving in this sense. In the case \( S = \mathbb{R}_{\geq 0}^n \), we have \( \text{trop}(S)^\vee = \{0\} \), so the tropicalizations of the \( A \)-moment cones do not include inequalities of type \( (2) \). For \( S = [0, 1]^n \), we get \( \text{trop}(S)^\vee = \mathbb{R}_{\geq 0}^n \) and the inequalities of type \( (2) \) say that the functions \( h \in \text{trop}(M_A(S)) \) are non-increasing in the coordinate directions. However, the condition in Theorem 4.2 is stronger than the combination of conditions \( (1) \) and \( (2) \), see Example 4.3.

We can also think of this result as a general description of all binomial inequalities valid on the moment (by exponentiation). With the analogous result for pseudo-moment cones, we will see that these inequalities suffice in distinguishing moments from pseudo-moments in many important cases. Moreover, there is a rich combinatorial interplay between the geometry of the moment configuration \( A \) and the geometric and algebraic description of \( S \).

We study the convex cone of convex functions in the sense of the above theorem in Section 3 from the point of view of discrete and tropical geometry. In particular, we show in Theorem 3.14 that the convex cone of functions \( h: A \to \mathbb{R} \) with \( \sum \lambda_i h(a_i) \geq h(b) \) for all \( a_i, b \in A \) satisfying \( \sum \lambda_i a_i - b \in \text{trop}(S)^\vee \) is the tropical conical hull of \( A(\text{trop}(S)^\vee) = \{Au: u \in \text{trop}(S)^\vee\} \), where we think of \( A \) as a matrix.

We now move on to pseudo-moment cones, which are the dual cones to truncated preorderings or quadratic modules. We describe in detail how we truncate (in a total degree version) at the beginning of Section 5. For pseudo-moment cones, we focus on the case that the semialgebraic set \( S \) has an inequality description in terms of pure binomial inequalities.

**Theorem** (Theorem 5.7). Let \( S \subset \mathbb{R}_{\geq 0}^n \) be a semi-algebraic set defined by pure binomial inequalities \( g_i = x^{a_i} - x^{b_i} \geq 0 \) such that \( S \subset \overline{S} \cap \mathbb{R}_{\geq 0}^n \). Assume that the exponent vectors \( a_i - b_i \) of the binomials defining \( S \) generate the semigroup \( N = \text{trop}(S)^\vee \cap \mathbb{Z}^n \). For all sufficiently large \( d \) the tropicalization of \( QM_d(g_i)^\vee \) is the rational polyhedral cone \( F(S)_d \) given by the following inequalities:

1. **(Midpoint convexity:)** \( h(a_1) + h(a_2) \geq 2h(b) \) for all \( a_1, a_2, b \) such that \( |a_1| \leq d, |b| \leq d \) and \( a_1 + a_2 = 2b \);

2. **(Nonincreasing:)** \( h(a) \geq h(b) \) whenever \( a - b \in \text{trop}(S)^\vee \).

The inequalities in \( A \)-pseudo-moments provable by sums of squares of degree at most \( d \) are dual to the coordinate projection of \( F(S)_d \) onto the coordinates indexed by \( A \).

In the case of pseudo-moments, we need the additional assumption on the inequality description of \( S \) that the exponent vectors of the inequalities generate the semigroup of lattice points in the convex cone \( \text{trop}(S)^\vee \) to give the same inequalities of type \( (2) \) as in the case of moment cones. This is an assumption that, from a purely theoretical point of view, can be made without loss of generality by adding valid and redundant inequalities, if necessary. Without this assumption, we only get some inequalities of type \( (2) \), namely those corresponding to the lattice points in \( \text{trop}(S)^\vee \) that also lie in the semigroup generated by the exponent vectors.

Section 3 contains a comparison of the polyhedral convex cones given by convexity (the tropicalization of the moment cone) and given by mid-point convexity (the tropicalization of the pseudo-moment cone).

Our most intriguing observation is that tropicalizations of pseudomoment cones stabilize as the degree bound \( d \) grows. This means that for sufficiently large \( d \) the tropicalizations of pseudomoment cones remain the same, even though pseudomoments themselves provide a convergent approximation to the moment cone. This phenomenon was already observed in Example 1.3. We provide an explicit description of when stabilization occurs for the hypercube \( [0, 1]^n \) in Theorem 5.11. More examples of stabilization and a general theorem (in particular for semi-algebraic sets defined by pure binomial inequalities) are given in Section 5.4.
The rest of the paper is organized as follows. In Section 2, we introduce the necessary background in tropical geometry and build up the theory of convex sets with the Hadamard property, including moment cones and toric spectrahedra. In Section 3 we introduce cones of convex and mid-point convex functions on a lattice set $A \subset \mathbb{Z}^n$ and discuss their facet-defining inequalities. Section 4 is dedicated to understanding the tropicalization of the moment cones $M_A(S)$. The tropicalizations of the corresponding pseudo-moment cones are discussed in Section 5. In Section 6 we discuss open question and further research directions.

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2. Tropicalization, Hadamard property, and toric spectrahedra

2.1. Tropicalization. The tropicalization of a subset $S \subset (\mathbb{R}^*)^n$ is defined as the logarithmic limit set as in [Ale13]:

$$trop(S) = \lim_{t \to \infty} \{(\log_t |x_1|, \ldots, \log_t |x_n|) : x \in S\}.$$ 

It is often convenient to characterize it as an image under valuation as we will now explain. Let $\mathcal{R}$ be a real closed field with a compatible convex non-trivial non-archimedean valuation $\text{val} : \mathcal{R}^* \to \mathcal{R}$, such as the field of real Puiseux series with the order map, as in [JSY22]. The value group $\Gamma$ (the image of val) is dense in $\mathbb{R}$ since the valuation is nontrivial and the field is algebraically closed. For a point $x \in \mathcal{R}^n$, the tropicalization map is the negation of coordinate-wise valuation

$$trop(x) = (\text{val}(x_1), \ldots, \text{val}(x_n)).$$

For any subset $T \subset \mathcal{R}^n$, we define its tropicalization to be

$$trop(T) := \left\{(-\text{val}(x_1), \ldots, -\text{val}(x_n)) : x \in T \cap (\mathbb{R}^*)^n \right\} \subset \mathbb{R}^n$$

where the closure is taken in Euclidean topology of $\mathbb{R}^n$. By the proof of Lemma 6.4 in [JSY22], we have

$$trop(T) = trop(\overline{T})$$

for any subset $T \subset \mathcal{R}^n$ where the closure is taken under the non-archimedean norm. If $T$ is semialgebraic, $trop(T)$ is a closed polyhedral subset of $\mathbb{R}^n$ [AGS20].

Let us now return to a semialgebraic subset $S \subset \mathbb{R}^n$. Let $S_{\mathcal{R}}$ be the semialgebraic subset of $\mathcal{R}^n$ defined by the same semialgebraic expression (which, to be precise, means a first-order formula in the language of ordered rings) defining $S$, where $\mathcal{R}$ is a real closed field extension of $\mathbb{R}$ with a non-trivial valuation. We then define its tropicalization to be

$$trop(S) := trop(S_{\mathcal{R}}).$$

The tropicalization does not depend on the choice of the extension $\mathcal{R}$ or the choice of semialgebraic expression defining $S$, and it coincides with the logarithmic limit set $\text{Ale13}$ [JSY22].

We now discuss tropicalizing polynomial inequalities. Consider the tropical semiring $\mathcal{R}$ where the tropical addition $\oplus$ is taking maximum and the tropical multiplication $\odot$ is usual addition. For a polynomial $f \in \mathcal{R}[x_1, \ldots, x_n]$, let $trop(f)$ be the tropical polynomial obtained by replacing addition and multiplication with $\oplus$ and $\odot$ respectively and replacing the coefficients with the negative of their evaluations. For instance, for $f = (7\varepsilon^{-1} + 2)x_1^2 - (\pi \varepsilon^1 + 5\varepsilon^2)x_1x_2 + 1 \in \mathbb{R}[\varepsilon][x_1, x_2]$, we get $trop(f) = (1) \odot x_1 \odot x_1 \oplus (-1) \odot x_1 \odot x_2 \odot 0$, which is the same as $\text{max}\{1 + 2x_1, -1 + x_1 + x_2, 0\}$. 


Let us define the following notations:

\[
\{\text{trop}(f) \geq 0\} = \{x \in \mathbb{R}^n : \text{maximum in } \text{trop}(f) \text{ is attained for a positive term}\}
\]

\[
\{\text{trop}(f) > 0\} = \{x \in \mathbb{R}^n : \text{maximum in } \text{trop}(f) \text{ is attained only at positive term(s)}\}.
\]

In the above example, the terms \(1 + 2x_1\) and \(0\) are positive (as the tropicalizations of terms with positive coefficients) and \(-1 + x_1 + x_2\) is negative so that \(\text{trop}(f) \geq 0\) is described by the condition that \(1 + 2x_1 \geq -1 + x_1 + x_2\) or \(0 \geq -1 + x_1 + x_2\). By the definition of tropicalization,

\[
f(x) \geq 0 \implies \text{trop}(x) \in \{\text{trop}(f) \geq 0\}.
\]

Taking contrapositive and changing signs, we get

\[
\text{trop}(x) \in \{\text{trop}(f) > 0\} \implies f(x) > 0.
\]

For any semialgebraic set \(S \subset \mathbb{R}^n_{\geq 0}\), the Fundamental Theorem \[JSY22\] Theorem 6.9 implies that

\[
\text{trop}(S) = \bigcap_{f \geq 0 \text{ on } S} \{\text{trop}(f) \geq 0\}
\]

where the intersection can be taken to be finite. Conversely, any tropical polynomial inequality valid on \(\text{trop}(S)\) arises as the tropicalization of a polynomial inequality valid on \(S\) \[JSY22\] Lemma 6.8.

For semialgebraic sets \(S_1\) and \(S_2\) in \((\mathbb{R}^*)^n\) or \((\mathbb{R}^n)^n\), we have \(\text{trop}(S_1 \cup S_2) = \text{trop}(S_1) \cup \text{trop}(S_2)\) by definition. For the intersections we have the following which follows from \[JSY22\] Proposition 6.12 (see also \[AGS20\] Lemma 2.3).

**Lemma 2.1.** Let \(S_1\) and \(S_2\) be semialgebraic subsets of \((\mathbb{R}^*)^n\). Then

\[
\text{int}(\text{trop}(S_1) \cap \text{trop}(S_2)) \subset \text{trop}(S_1 \cap S_2) \subset \text{trop}(S_1) \cap \text{trop}(S_2).
\]

**Corollary 2.2** (see \[AGS20\] Corollary 4.8). Let \(S \subset (\mathbb{R}^n)^n\) be a semialgebraic set defined by inequalities \(f_1 \geq 0, \ldots, f_r \geq 0\). If the tropicalizations \(\text{trop}(f_i)\) cut out a set \(T = \{x \in \mathbb{R}^n : \text{trop}(f_1)(x) \geq 0, \ldots, \text{trop}(f_r)(x) \geq 0\}\) with regular support (meaning that it is equal to the closure of its interior), then \(\text{trop}(S) = T\).

If, in particular, the semialgebraic set \(S\) is defined by binomial inequalities and their tropicalization, which are usual linear inequalities, cut out a full-dimensional set, then that polyhedral cone coincides with \(\text{trop}(S)\).

### 2.2. Tropical Convexity.

Now we recall some basics of tropical convexity. The **tropical conical hull** of a set \(S \subset \mathbb{R}^n\) is the set of all tropical linear combinations of points in \(S\) \[DS04\],

\[
\text{tcone}(S) = \{a_1 \odot s_1 \oplus (a_2 \odot s_2) \oplus \cdots \oplus (a_r \odot s_r) : r \in \mathbb{N}, s_1, \ldots, s_r \in S, a_1, \ldots, a_r \in \mathbb{R}\}.
\]

Here \(a \odot s = a \odot (s_1, \ldots, s_n)\) is the vector \((s_1 + a, s_2 + a, \ldots, s_n + a)\). A subset \(S \subset \mathbb{R}^n\) is called **tropically convex** if it equals its tropical conical hull. One of the fundamental theorems in the study of the tropicalization of convex sets is that for semialgebraic subsets in the positive orthant, the operations of tropicalization and conical hull commute.

**Proposition 2.3** (e.g. Lemma 8 \[AGS19\]). For any semialgebraic subset \(S \subset \mathbb{R}^n_{\geq 0}\), the tropicalization of the conical hull of \(S\) equals the tropical conical hull of \(\text{trop}(S)\). That is,

\[
\text{trop}(\text{cone}(S)) = \text{tcone}(\text{trop}(S)).
\]
The proof relies on the definition of \( \text{trop}(S) \) as the image of the set \( S_R \) under coordinate-wise valuation, as described in Section [2.1] and the observation that no cancellation in valuation can occur for elements in \( R_{>0} \). That is, for any \( s, t \in R_{>0} \), \( -\text{val}(s + t) = -\text{val}(s) \oplus -\text{val}(t) \). For any points \( x, y \in S_R \subset R^n_{>0} \) and scalars \( \lambda, \mu \in R_{>0} \) it follows that \( -\text{val}(\lambda x + \mu y) = (-\text{val}(\lambda) \oplus -\text{val}(x)) \oplus (-\text{val}(\mu) \oplus -\text{val}(y)) \). It follows that for any set \( S_R \subset R^n_{>0} \), the tropical conical hull of \( \text{trop}(S_R) \) coincides with the tropicalization of its conical hull. For any semialgebraic convex cone \( S \), \( \text{trop}(S_R) \) is thus already tropically convex. In particular, the tropicalization of a semialgebraic convex cone in \( R^n_{>0} \) is tropically convex.

The following lemma follows from [DS04, Section 3] and is more explicitly stated in [HLS19] and [LS19].

**Lemma 2.4.** For any set \( Y \subset R^n \), its tropical conical hull is

\[
\text{tcone}(Y) = \bigcap_{i=1}^{n} (Y + V_i)
\]

where \( V_i = \{ -x \in R^n : x_1 \oplus \cdots \oplus x_n = x_i \} \). In particular, the tropical conical hull of a convex polyhedron is again a convex polyhedron.

We apply the above lemma to the case where \( Y \) is a convex cone.

**Proposition 2.5.** Let \( Y \subset R^n \) be a convex cone. Then \( \text{tcone}(Y) \) is a convex cone and its dual cone has the form

\[
\text{tcone}(Y)^{\vee} = \bigcap_{i=1}^{n} (Y^{\vee} \cap U_i \cap H),
\]

where \( \sum \) is Minkowski addition, \( U_i \) is the orthant of \( R^n \) where the \( i \)-th coordinate is nonpositive and the rest are nonnegative, and \( H \) is the hyperplane in \( R^n \) perpendicular to the vector \( 1 \) of all 1s. In particular, any extreme ray of the dual cone \( Y^{\vee} \) has the form \( \sum_{i=1}^{n} \alpha_i e_i \), where \( e_i \) are the standard basis vectors, \( \sum_{i=1}^{n} \alpha_i = 0 \), and exactly one of the coefficients \( \alpha_i \) is negative.

**Proof.** Let \( H \subset R^n \) be the hyperplane of all vectors whose coordinates sum to 0, and let \( V_i = \{ -x \in R^n : x_1 \oplus \cdots \oplus x_n = x_i \} \). Then \( V_i = U_i + \text{span}(1) \), where \( U_i \) is the orthant in \( R^n \) defined by \( x_i \leq 0 \) and \( x_j \geq 0 \) for \( j \neq i \) and \( 1 \) is the vector of all 1s. By convex duality, it follows that \( V_i^{\vee} = U_i^{\vee} \cap H \). Moreover, \( U_i^{\vee} = U_i \) because \( U_i \) is an orthant. Since we know from Lemma 2.4 that \( \text{tcone}(Y) = \bigcap_{i=1}^{n} (Y + V_i) \), we have \( \text{tcone}(Y)^{\vee} = \bigcap_{i=1}^{n} (Y^{\vee} \cap U_i \cap H) \). In particular, any extreme ray of \( Y^{\vee} \) lies in \( Y^{\vee} \cap V_i \), and the proposition follows. \( \square \)

### 2.3. Hadamard Property

We say that a subset \( S \) of \( R^n \) has the **Hadamard property** if it is closed under Hadamard (coordinate-wise) multiplication

\[
R^n \times R^n \to R^n, \quad (x, y) \mapsto x \circ y := (x_1 y_1, \ldots, x_n y_n).
\]

**Proposition 2.6.** Let \( A \subset Z^n_{>0} \) be a finite set of nonnegative lattice points and let \( \varphi_A : R^n \to R^{|A|} \), \( x \mapsto (x^a : a \in A) \), be the corresponding monomial map. If \( S \subset R^n \) has the Hadamard property then so do

1. the image \( \varphi_A(S) \) of \( S \) under \( \varphi_A \),
2. the convex hull \( \text{conv}(S) \) and the conical hull \( \text{cone}(S) \), and
3. the closed moment cone \( M_A(S) \).

The moment cone is by definition the cone of all moment sequences \( (m_\alpha : \alpha \in A) \), that is \( m_\alpha \) is the integral \( \int_S x^\alpha d\mu \) of \( x^\alpha \) over the set \( S \) with respect to some measure \( \mu \). In general, this cone need not be closed. We denote its closure by \( M_A(S) \) and call it the closed moment cone. In the case
that the set $A$ is finite, the closed moment cone is also the closed conical hull of $\varphi_A(S)$. See, for example, [Sch17, Theorems 1.24 and 1.26].

Proof of Proposition 2.6. For (1), note that for $x, y \in S$, $\varphi_A(x) \circ \varphi_A(y) = \varphi_A(x \circ y)$ belongs to $\varphi_A(S)$.

For (2), let $x_1, \ldots, x_\ell, y_1, \ldots, y_m \in S$ and $\alpha_1, \ldots, \alpha_\ell, \mu_1, \ldots, \mu_m \in \mathbb{R}_{\geq 0}$. Then

$$\left(\sum_{i=1}^{\ell} \lambda_i x_i\right) \circ \left(\sum_{j=1}^{m} \mu_j y_j\right) = \sum_{i=1}^{\ell} \sum_{j=1}^{m} \lambda_i \mu_j (x_i \circ y_j).$$

This shows that the conical hull $cone(S)$ also has the Hadamard property. Moreover, if $\sum_i \lambda_i = 1$ and $\sum_j \mu_j = 1$, then $\sum_i^{\ell} \sum_j^{m} \lambda_i \mu_j = 1$. The expression above then shows that the convex hull of $S$ also has the Hadamard property.

For (3), as explained in the paragraph above the proof, the cone $M_A(S)$ is equal to $cone(\varphi_A(S))$. Then by parts (1) and (2), $M_A(S)$ has the Hadamard property. \hfill \Box

Example 2.7. A famous example of a set with Hadamard property is the cone of positive semidefinite $n \times n$ matrices $S^n_+$, which is closed under Hadamard products by the Schur product theorem. We can recover the Schur product theorem from Proposition 2.6 as follows. The cone $S^n_+$ is the convex hull of rank one matrices [Bar02, Chapter II.12] and the set $\{xx^*: x \in \mathbb{R}^n\}$ of rank 1 matrices is the image of $\mathbb{R}^n$ under the monomial map $\varphi_A$ where $A \subset \mathbb{Z}_{\geq 0}^n$ is the set of all vectors in $\mathbb{Z}_{\geq 0}^n$ where the sum of coordinates is two.

For any semialgebraic set $S$, its tropicalization is a rational polyhedral fan. If additionally, $S$ has the Hadamard property, then its tropicalization is closed under addition and therefore a rational convex polyhedral cone. By the above observation, the same is then also true for the tropicalization $\text{trop}(M_A(S))$ of the $A$-moment cone on $S$.

One motivation for studying the tropicalization of a set $S$ is to study the set of pure binomial inequalities that are valid on $S$. This is especially true when $S$ has the Hadamard property as the following two statements show.

Proposition 2.8. For any set $S \subset \mathbb{R}^n_{\geq 0}$, $\text{trop}(S)$ is contained in $cone(\log(S))$, with equality if $S$ has the Hadamard property.

The proposition still holds if the word cone is taken to mean just multiplying by positive constants, rather than conical hull.

Proof. Fix any base greater than one for log. The tropicalization of $S$ can be written as the pointwise limit as $t \to \infty$ of the set $\frac{1}{t} \log(S)$. For each $t > 0$, $\frac{1}{t} \log(S)$ is contained in the cone over $\log(S)$ and so the limit as $t \to \infty$ is contained in its closure, $cone(\log(S))$. Equality in the case that $S$ has the Hadamard property follows from [BRST20] Lemma 2.2. \hfill \Box

Proposition 2.9. Let $S \subset \mathbb{R}^n_{\geq 0}$ and $\alpha = \alpha_+ - \alpha_-$ where $\alpha_+, \alpha_- \in \mathbb{Z}_{\geq 0}^n$. The linear inequality $\sum_{i=1}^{n} \alpha_i x_i \geq 0$ holds on $cone(\log(S))$ if and only if the binomial inequality $x^{\alpha_+} \geq x^{\alpha_-}$ holds on $S$.

Proof. First, note that a linear inequality holds on $cone(\log(S))$ if and only if it holds on $\log(S)$. Consider a point $x \in S$. The inequality $x^{\alpha_+} \geq x^{\alpha_-}$ is equivalent to the Laurent inequality $x^\alpha \geq 1$ since $S \subset \mathbb{R}^n_{\geq 0}$. Because log is monotonic, $x^\alpha \geq 1$ if and only if $\sum_{i=1}^{n} \alpha_i \log(x_i) \geq 0$. \hfill \Box
Remark 2.10. Together, these propositions show that if $S \subseteq \mathbb{R}_{\geq 0}^n$ has the Hadamard property, then trop$(S)$ is a convex cone whose dual cone consists of the set of pure binomial inequalities that are valid on $S$. We only need to consider pure binomials in the presence of Hadamard property because they are the strongest possible binomial inequalities. An inequality of the form $x_1^{a_1} \cdots x_n^{a_n} \leq cx_1^{a_1} \cdots x_n^{a_n}$ holds on $S$ if and only if $\sum_{i=1}^n a_i \log(x_i) \leq \log(c)$ does. Since trop$(S) = \text{cone}((\log(S))$ is a convex cone, for any affine-linear inequality $\sum_{i=1}^n \alpha_i y_i \geq a_0$ that holds on trop$(S)$, the (stronger) linear inequality $\sum_{i=1}^n \alpha_i y_i \geq 0$ also holds.

We are particularly interested in semi-algebraic subsets of the nonnegative orthant given by pure binomial inequalities. These always have the Hadamard property.

Lemma 2.11. Let $S \subseteq \mathbb{R}^n$ be a semi-algebraic set given by the inequalities $x_i \geq 0$ $(i = 1, 2, \ldots, n)$ and pure binomial inequalities $g_j = x_{a_j} - x_{b_j} \geq 0$ $(j = 1, 2, \ldots, \nu)$. Then $S$ has the Hadamard property.

Proof. Let $x$ and $y$ be in $S$ and write $x \circ y$ for the Hadamard product $(x_i y_i; i = 1, 2, \ldots, n)$ of the two points. Then every entry of this vector is nonnegative so we need to only plug it into the pure binomials. For this, we compute

$$(x \circ y)^a - (x \circ y)^b = (x^a - x^b) \cdot y^a + (y^a - y^b) \cdot x^b$$

which shows that every binomial inequality $g_j$ defining $S$ also holds for the point $x \circ y$.

2.4. Tropicalizations of Toric Spectrahedra. In this section, we consider toric spectrahedra and some generalizations relevant for pseudo-moment cones in the pure binomial case (see Section 5.1).

A toric spectrahedron is a semialgebraic subset of $\mathbb{R}_{\geq 0}^n$ defined by the positive-semidefiniteness of a matrix with monomial (rather than linear) entries in a set of variables. Formally, let $A(x)$ be a symmetric matrix with entries that are monomials in variables $x = (x_1, \ldots, x_n)$ and let $S$ be the subset of $\mathbb{R}_{\geq 0}^n$ given by the values of $x$ which make matrix $A(x)$ positive semidefinite. By the Schur product theorem, every toric spectrahedron has the Hadamard property.

In the special case when each entry of $A(x)$ is one of the variables, so a monomial of degree 1, the toric spectrahedron $S$ is a spectrahedron in the usual sense as well. An important example is the Hankel spectrahedron, the dual convex cone to the cone of sums of squares.

As shown in [BRST20], the tropicalization of a toric spectrahedron $S = \{ x \in \mathbb{R}_{\geq 0}^n : A(x) \geq 0 \}$ is given by the tropicalization of the $2 \times 2$ principal minors of $A(x)$.

Formally, let $S'$ be the subset of $\mathbb{R}_{\geq 0}^n$ given by the values of $x$ which make all $2 \times 2$ principal minors of $A(x)$ nonnegative. Clearly $S \subseteq S'$, and since the $2 \times 2$ principal minors of $A(x)$ are pure binomials we see that trop$(S')$ is given by tropicalization of $2 \times 2$ principal minors of $A(x)$. In fact, trop$(S)$ fills up this entire trop$(S')$.

Proposition 2.12 (Theorem 4.4 [BRST20]). Suppose that trop$(S')$ has non-empty interior in $\mathbb{R}^n$ and no $2 \times 2$ principal minor of $A$ is identically zero. Then trop$(S) = \text{trop}(S')$ and both are given by tropicalization of $2 \times 2$ principal minors of $A$.

Remark 2.13. Note that when the toric spectrahedron $S$ is not contained in the nonnegative orthant, we can replace it with its image, $|S|$, under coordinate-wise absolute value. By the Schur product theorem, $S$ has Hadamard property, and therefore so does $|S|$. If $S'$ denotes the set of $x \in \mathbb{R}^n$ for which the principal $2 \times 2$ minors of $A(x)$ are positive semidefinite, then trop$(|S|) = \text{trop}(|S'|)$ and we can apply Proposition 2.12 to obtain all pure binomial inequalities in absolute values that are valid on $S$. \qed
We now extend this proposition to deal with differences of monomial matrices.

**Proposition 2.14.** Let $A(x)$ and $B(x)$ be symmetric matrices whose entries are pure monomials in the variables $x = (x_1, \ldots, x_n)$. Let $S$ be the subset of $\mathbb{R}^n_{\geq 0}$ consisting of points $x$ such that $A \succeq B \succeq 0$, or equivalently, $A - B \succeq 0$ and $B \succeq 0$. Let $S'$ be the subset of $\mathbb{R}^n_{> 0}$ given by the values of $x$ which make all $2 \times 2$ principal minors of $A$ and $B$ nonnegative, and the diagonal entries of $A - B$ nonnegative.

1. The set $S$ has the Hadamard property.
2. If $\text{trop}(S')$ has non-empty interior in $\mathbb{R}^n$ and no $2 \times 2$ principal minor of $A$ or $B$ is identically zero, then $\text{trop}(S) = \text{trop}(S')$ and both are given by tropicalization of $2 \times 2$ minors of $A$ and $B$ and diagonal entries of $A - B$.

**Proof.** To show claim (1), suppose that $x, y \in \mathbb{R}^n$ such that $A(x) \succeq B(x) \succeq 0$ and $A(y) \succeq B(y) \succeq 0$. Then $B(x \circ y) = B(x) \circ B(y) \succeq 0$ by the Schur product theorem. Also by the Schur product theorem

$$A(x \circ y) - B(x \circ y) = A(x) \circ (A(y) - B(y)) + (A(x) - B(x)) \circ B(y) \succeq 0,$$

and therefore $A(x \circ y) \succeq B(x \circ y)$ showing that $S$ has Hadamard property.

We now show claim (2). Since the $2 \times 2$ minors of $A$ and $B$ as well as the diagonal entries of $A - B$ are pure binomials, the set $S'$ has the Hadamard property, see Lemma 2.11. So Propositions 2.8 and 2.9 imply that $\text{trop}(S')$ is given by tropicalization of $2 \times 2$ minors of $A$ and $B$ as well as the diagonal entries of $A - B$. So we only need to show $\text{trop}(S) = \text{trop}(S')$. The inclusion $\text{trop}(S) \subseteq \text{trop}(S')$ is immediate because $S \subseteq S'$. To show the other inclusion $\text{trop}(S') \subseteq \text{trop}(S)$, it suffices to show that the interior of $\text{trop}(S')$ is contained in $\text{trop}(S)$, because $\text{trop}(S')$ is full-dimensional and $\text{trop}(S)$ is closed. This is a consequence of the following Lemma 2.15 below, with $r = 1$, $A_0(x) = A(x)$, $A_1(x) = B(x)$ and $c_1 = -1$.

Consider the polynomial matrix

$$A(x) = A_0(x) + \sum_{k=1}^r c_k A_k(x)$$

where $c_1, \ldots, c_r \in \mathbb{R}$ and each entry of $A_k(x)$ is a monomial in $x$. Specifically, suppose that the $(i,j)$th entry of $A_k(x)$ is $x^\beta$ where $\beta = \alpha_{ij}^k$. Let $S$ denote the set of points $x \in \mathbb{R}^n_{\geq 0}$ for which $A(x) \succeq 0$.

**Lemma 2.15.** Let $y \in \mathbb{R}^n$ be a vector strictly satisfying the tropical inequalities given by the positivity of the $2 \times 2$ minors of $A_k$ and the diagonal entries of $A_0 - A_k$. That is, $y \cdot \alpha_{ii}^0 > y \cdot \alpha_{ii}^k$ for all $i$ and all $k \neq 0$ and $y \cdot \alpha_{ii}^k + y \cdot \alpha_{jj}^k > 2y \cdot \alpha_{ij}^k$ for all $i \neq j$ and all $k$. Then for all sufficiently large $t \in \mathbb{R}$, $A(t^y)$ is positive definite and $y$ belongs to trop($S$). Here, $t^y$ denotes the vector $(t^{y_1}, \ldots, t^{y_n})$.

**Proof.** Showing that $A(t^y)$ is positive definite for all sufficiently large $t$ implies that $y$ belongs to trop($S$). We do this by expanding the principal minors of $A(t^y)$. Because the entries of $A_k(x)$ are monomial in $x$, the entries of $A_k(t^y)$ are powers of $t$. Specifically, the $(i,j)$th entry of $A_k(t^y)$ is $t^b$ where $b = y \cdot \alpha_{ij}^k$.

Consider the Laplace expansion of the determinant of $A(t^y)$, expanded as an exponential polynomial in $t$. We claim that $\sum_i y \cdot \alpha_{ii}^0$ will be the leading exponent of $t$ that will be obtained uniquely by the product of the diagonal of $A_0$. 

To see this, note that for all $k$
\[
2y \cdot \alpha_{ij}^k < y \cdot \alpha_{ii}^k + y \cdot \alpha_{jj}^k \leq y \cdot \alpha_{ii}^0 + y \cdot \alpha_{jj}^0
\]
where the second inequality is an equality only when $k = 0$. Formally, consider the exponent of $t$ in the Laplace expansion of the determinant of $A(t^y)$ given by the permutation $\pi$ of $[d]$ and a choice $\sigma : [d] \to \{0, 1, \ldots, r\}$ of one of the $r + 1$ terms in the $(i, \pi(i))$th entry of $A(t^y)$. The resulting exponent of $t$ is
\[
\sum_i y \cdot \alpha_{ii}^{\sigma(i)} \leq \frac{1}{2} \sum_i y \cdot \alpha_{ii}^{\sigma(i)} + y \cdot \alpha_{\pi(i)\pi(i)}^{\sigma(i)} \leq \sum_i y \cdot \alpha_{ii}^0.
\]
Moreover the equality is possible for the left and right hand sides only when $\pi$ is the identity permutation and $\sigma(i) = 0$ for all $i$.

This shows that the Laplace expansion of the determinant of $A(t^y)$ has the form $t^b$, where $b = \sum_i y \cdot \alpha_{ii}^0$, plus terms of strictly lower degree in $t$. For large enough $t \in \mathbb{R}$, the determinant of $A(t^y)$ will therefore be positive. The same argument can be made for all the principal minors of $A(t^y)$, showing that $A(t^y)$ is positive definite for large enough $t$. \qed

3. Convexity vs midpoint convexity

Tropicalizations of the moment cone and the dual cone to sums of squares deal with two different notions of convexity for real functions on lattice points. In this section we study combinatorial aspects of these notions and make the connection to moment and pseudo-moment cones in Sections 4 and 5.

Definition 3.1. Let $A \subset \mathbb{Z}^n$ be finite. We say that a function $h : A \to \mathbb{R}$ is convex if $\sum \lambda_i h(a_i) \geq h(b)$ whenever $\sum \lambda_i a_i = b$ with $a_i, b \in A$, $\lambda_i \geq 0$, and $\sum \lambda_i = 1$. We say that $f$ is midpoint convex if $h(a_1) + h(a_2) \geq h\left(\frac{a_1 + a_2}{2}\right)$ whenever $a_1, a_2, \frac{a_1 + a_2}{2} \in A$.

Remark 3.2. Every convex function is midpoint convex.

Any function $h : A \to \mathbb{R}$ induces a marked regular subdivision $\mathcal{S}_h$ of the point configuration $A$, by projecting back to $\mathbb{R}^A$ the lower faces of the “lifted polytope” $P_h = \text{conv}\{(a, h(a)) \in \mathbb{R}^A \times \mathbb{R} : a \in A\}$. A point $a \in A$ is marked in $\mathcal{S}_h$ if the corresponding point $(a, h(a))$ lies in a lower face of $P_h$. From this point of view, convex functions on $A$ are precisely the functions that induce a regular subdivision with all points marked. See [DLRS10] for precise definitions. Convex functions $h$ on $A$ are also precisely those that arise as the restriction of a maximum (tropical sum) of affine linear functions on $\mathbb{R}^n$.

As we will explain in Sections 4 and 5, convex functions correspond to tropicalizations of $A$-moment cones and midpoint convex functions correspond to tropicalizations of pseudo-moment cones. We will need the following notions.

A subset $T = \{a_0, \ldots, a_k, b\} \subset A$ with $k + 2$ elements is called a $k$-dimensional punctured simplex if $a_0, \ldots, a_k$ are affinely independent and $b$ lies in the $k$-dimensional simplex $\Delta = \text{conv}(a_0, \ldots, a_k)$, possibly on its boundary. We say that $T$ is an almost-empty simplex if it is punctured and in addition $\Delta \cap A = T$ and $b$ lies in the relative interior of $\Delta$.

We say that a punctured simplex $T = \{a_0, \ldots, a_k, b\} \subset A$ certifies the non-convexity of a function $h : A \to \mathbb{R}$ if $\sum_{i=0}^k \lambda_i h(a_i) < h(b)$ for scalars $\lambda_i \geq 0$ such that $\sum_{i=0}^k \lambda_i a_i = b$ and $\sum_{i=0}^k \lambda_i = 1$.

The next Lemma allows us to identify defining inequalities of the convex functions on a finite lattice set $A$. 
Lemma 3.3. If \( h : A \to \mathbb{R} \) is not convex then there is an almost-empty simplex \( T \subset A \) that certifies its non-convexity.

Proof. We first prove that there is a punctured simplex \( T \subset A \) that certifies the non-convexity of \( h \). Since \( h \) is not convex, there exists \( b \in A \) such that \((b, h(b))\) lies strictly above a lower face \( F \) of the lifted polytope \( P_h = \text{conv}\{(a, h(a)) : a \in A\} \). The face \( F \) projects down to a face \( F \) in the marked regular subdivision \( S_h \) induced by \( h \). The point \( b \) is then in the convex hull of the marked points of \( S_h \) that lie in \( F \), and by Carathéodory’s Theorem, \( b \) is in the convex hull of an affinely independent subset \( a_0, \ldots, a_k \) of them. The punctured simplex \( \{a_0, \ldots, a_k, b\} \subset A \) then certifies the non-convexity of \( h \).

Now, take a punctured simplex \( T = \{a_0, \ldots, a_k, b\} \subset A \) that certifies the non-convexity of \( h \) such that \( \text{conv}(T) \cap A \) has as few elements as possible. Note that \( b \) must lie in the interior of the simplex \( \Delta := \text{conv}(a_0, \ldots, a_k) \), as otherwise the proper face of \( \Delta \) where \( b \) lies would still certify the non-convexity of \( h \) and contain fewer points of \( A \). After adding a suitable affine function to \( h \), we can assume that \( h(a_i) = 0 \) for all \( i \) and \( h(b) > 0 \).

Assume for contradiction that \( T \) is not an almost-empty simplex, so \( \Delta \cap A \) contains a point \( a \) not in \( T \). Suppose first that \( h(a) \geq h(b) \). For \( i = 0, \ldots, k \), consider the \( k \)-dimensional (closed) affine cone \( C_i \) with vertex \( b \) and spanned by the rays \( a_j - b \) with \( j \neq i \). These affine cones \( C_0, \ldots, C_k \) form a fan that covers the whole affine span of \( \Delta \), and so \( a \in C_l \) for some \( l \). This implies that \( a \in \text{conv}(a_0, \ldots, a_l, a_k, b) \). But then \( T^l := (T \setminus \{a_l\}) \cup \{a\} \) is a punctured simplex that certifies the non-convexity of \( h \) and \( |\text{conv}(T^l) \cap A| < |\text{conv}(T) \cap A| \), which is a contradiction. Now, suppose that \( h(a) \leq h(b) \). Let \( C_l' \) be the reflection of the cone \( C_l \) across its vertex \( b \). Again, the collection of cones \( C_0', \ldots, C_k' \) cover the affine span of \( \Delta \), so \( a \in C_{l'} \) for some \( l' \). This implies that \( b \in \text{conv}(a_0, \ldots, a_{l'}, \ldots, a_k, b) \). But then \( T' := (T \setminus \{a_{l'}\}) \cup \{a\} \) is a punctured simplex that certifies the non-convexity of \( h \) and \( |\text{conv}(T') \cap A| < |\text{conv}(T) \cap A| \), which is a contradiction. \( \square \)

We now prove important structural properties of the cones of convex and midpoint-convex functions on a finite set of lattice points.

Proposition 3.4. The set
\[ K_A := \{h \in \mathbb{R}^A : h \text{ is convex}\} \]
is a polyhedral cone in \( \mathbb{R}^A \). Its facets correspond to inequalities of the form \( \sum_{i=0}^{k} \lambda_i h(a_i) \geq h(b) \) where \( \{a_0, \ldots, a_k, b\} \subset A \) is an almost-empty simplex and \( \lambda_i > 0 \) are the unique scalars such that \( \sum \lambda_i a_i = b \) and \( \sum \lambda_i = 1 \).

Similarly, the set
\[ M_A := \{h \in \mathbb{R}^A : h \text{ is midpoint convex}\} \]
is a polyhedral cone in \( \mathbb{R}^A \).

We describe the facets of the cone \( M_A \) later in Proposition 3.11.

Proof. By Lemma 3.3, the set \( K_A \) is the collection of real functions \( h \) on \( A \) that satisfy the linear inequalities \( \sum_{i=0}^{k} \lambda_i h(a_i) \geq h(b) \) for any almost-empty simplex \( \{a_0, \ldots, a_k, b\} \subset A \) and the unique scalars \( \lambda_i \geq 0 \) such that \( \sum \lambda_i a_i = b \) and \( \sum \lambda_i = 1 \). In particular, this implies that \( K_A \) is a polyhedral cone. To see that all of these are indeed facet-defining inequalities, note that for any almost-empty simplex \( T = \{a_0, \ldots, a_k, b\} \subset A \) there exists an \( h \in \mathbb{R}^A \) that does not satisfy the inequality corresponding to \( T \) but does satisfy all other inequalities. An example of such a function is obtained by taking a convex function \( h \) on \( A \setminus \{b\} \) satisfying \( h(a_i) = 0 \) for \( i = 0, \ldots, k \) and \( h(a) \gg 1 \) for all other \( a \in A \setminus \{a_0, \ldots, a_k, b\} \), and setting \( 0 < h(b) \ll 1 \).

Similarly, the set \( M_A \) is defined by finitely many linear inequalities, so it is a polyhedral cone. \( \square \)
Convex functions are always a subset of midpoint convex functions, and we classify sets $A$ where this inclusion is an equality.

**Corollary 3.5.** The inclusion of polyhedral cones $\mathcal{K}_A \subseteq \mathcal{M}_A$ is an equality if and only if $A$ does not contain an almost-empty simplex of dimension at least 2 and every 1-dimensional almost-empty simplex $\{a_0, a_1, b\} \subset A$ satisfies $\frac{a_0 + a_1}{2} = b$.

**Proof.** By Proposition 3.4, if every almost-empty simplex in $A$ is one dimensional and has the form $\{a_0, a_1, \frac{a_0 + a_1}{2}\}$ then all the facets of $\mathcal{K}_A$ are also facets of $\mathcal{M}_A$, and so $\mathcal{K}_A = \mathcal{M}_A$. Conversely, if $A$ contains an almost-empty simplex $T$ of dimension at least 2 or a 1-dimensional almost-empty simplex $T = \{a_0, a_1, b\} \subset A$ with $b \neq \frac{(a_0 + a_1)}{2}$, then a function $h \in \mathbb{R}^A$ that satisfies all almost-empty simplex inequalities except for the one corresponding to $T$ is an example of a function in $\mathcal{M}_A$ but not in $\mathcal{K}_A$. \hfill $\square$

**Remark 3.6.** It is not clear how to simply describe the lineality spaces and the extreme rays of the cones $\mathcal{K}_A$ and $\mathcal{M}_A$. Since $\mathcal{K}_A$ is a union of secondary cones, it is the positive span of the convex functions inducing coarsest subdivisions of $A$. However enumerating them appears to be difficult in general. On the other hand, the cone $\mathcal{M}_A$ is not always a union of secondary cones. Both cones $\mathcal{K}_A, \mathcal{M}_A \subset \mathbb{R}^A$ contain the $(n+1)$-dimensional linear subspace of affine linear functions on $A$:

$$L_A = \{h \in \mathbb{R}^A : h(w) = c \cdot w + d, \text{ where } c \in \mathbb{R}^n, d \in \mathbb{R}\}.$$  

However, their lineality space may be larger. For example, if $A = \{0, 1\}^2 \subset \mathbb{R}^2$ the four vertices of a square, then every function on $A$ is convex and mid-point convex.

By definition, the lineality space of $\mathcal{K}_A$ consists of all the functions $h$ on $A$ such that both $h$ and $-h$ are convex, which means that for every $a \in A$ the lifted point $(a, h(a))$ is in both the upper hull and the lower hull of $P_h = \text{conv}\{ (a, h(a)) : a \in A \}$. In particular, if $A$ contains a point in the interior of its convex hull, then the lineality space of $\mathcal{K}_A$ is equal to $L_A$. More generally if $A$ contains a point in the relative interior of a face of $\text{conv}(A)$, then any $h$ in the lineality space of $\mathcal{K}_A$ must be an affine linear function on that face. At the other extreme, if every point of $A$ lies in a simplicial face of $\text{conv}(A)$, then every vertex can be lifted independently and after that the lifts of other points are determined uniquely by affinely interpolating, so the lineality space of $\mathcal{K}_A$ consists of piecewise linear functions induced by arbitrary functions on vertices of $A$. $\diamond$

We now consider restrictions of convex and midpoint convex functions from a lattice set $E$ to a subset $A$. Let $A \subset E \subset \mathbb{Z}^n$ be finite subsets. We consider the natural projection

$$\pi_A : \mathbb{R}^E \to \mathbb{R}^A.$$

**Proposition 3.7.** Suppose $A \subset E \subset \mathbb{Z}^n$ are finite subsets. We have

$$\pi_A(\mathcal{K}_E) = \mathcal{K}_A.$$

If, in addition, $A = \text{conv}(A) \cap \mathbb{Z}^n$, we also have

$$\pi_A(\mathcal{M}_E) = \mathcal{M}_A.$$

**Proof.** The fact that $\pi_A(\mathcal{K}_E) = \mathcal{K}_A$ expresses the fact that any convex function $h \in \mathcal{K}_A$ can be extended to a convex function $\hat{h} \in \mathcal{K}_E$, for instance by defining $\hat{h} : E \to \mathbb{R}$ to be the height of the lower convex hull of the lifted points $\{(a, h(a)) : a \in A\} \subset \mathbb{R}^n \times \mathbb{R}$. If $A = \text{conv}(A) \cap \mathbb{Z}^n$, we can also extend any midpoint-convex function $h \in \mathcal{M}_A$ to a midpoint-convex function $\hat{h} \in \mathcal{M}_E$. For instance, fix an ordering of the set $E \setminus A = \{e_1, \ldots, e_m\}$ such that $\text{conv}(A \cup \{e_1, \ldots, e_i\}) \cap E = A \cup \{e_1, \ldots, e_i\}$ for all $1 \leq i \leq m$. Then, recursively for $i = 1, \ldots, m$, define $\hat{h}(e_i) \in \mathbb{R}$ to be sufficiently large so that
\[ \hat{h}(e_i) \geq 2\hat{h}(e) - h(e') \] for any \( e, e' \in A \cup \{e_1, \ldots, e_{i-1}\} \). Since \( e_i \) cannot be the midpoint of any two points \( e, e' \in A \cup \{e_1, \ldots, e_{i-1}\} \), this defines a midpoint-convex function \( \hat{h} \in M_E \) extending \( h \). \( \square \)

Suppose \( A \subset \mathbb{Z}^n \), and take \( E = \text{conv}(A) \cap \mathbb{Z}^n \). We are interested in understanding when the containment \( K_A \subset \pi_A(M_E) \) of polyhedral cones is an equality.

Given any \( S \subset \mathbb{Z}^n \), its set of midpoints is
\[
\text{Mid}(S) = \{(s + t)/2 \in \mathbb{Z}^n : s, t \in S \text{ and } s \neq t\}.
\]
Suppose \( V = \{v_0, \ldots, v_k\} \subset \mathbb{Z}^n \) is a set of affinely independent lattice points, and let \( F = \text{conv}(V) \cap \mathbb{Z}^n \). A subset \( S \subset F \) is called \emph{V-mediated} if \( S \supseteq V \) and \( S \setminus V \subset \text{Mid}(S) \). As V-mediated subsets are closed under union, there is a maximal V-mediated subset of \( F \), which we denote by \( V^* \). It can be computed by starting with the set \( F \) and repeatedly iterating the map \( X \mapsto V \cup \text{Mid}(X) \), i.e., iteratively discarding points that are not in \( V \) or are not midpoints of other points in the set. It was shown in \cite{Rez89} that this procedure stabilizes to the maximal V-mediated set \( V^* \).

\textbf{Example 3.8.} If \( V_1 = \{(0,0), (1,2), (2,1)\} \) then the maximal \( V_1^* \) is equal to \( V_1 \). The point \((1,1)\) is in \( \text{conv}(V_1) \cap \mathbb{Z}^n \), but not in \( V_1^* \). On the other hand, if \( V_2 = \{(0,3), (1,0), (3,1)\} \) then the maximal \( V_2^* \) is equal to the set of six points in \( \text{conv}(V_2) \cap \mathbb{Z}^n \); see Figure 1.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{triangles.png}
\caption{Two lattice triangles. Filled in red are the points in their corresponding maximal mediated sets.}
\end{figure}

Our interest in V-mediated sets comes from the following fact.

\textbf{Lemma 3.9} (\cite{Rez89}). Suppose \( V = \{v_0, \ldots, v_k\} \subset \mathbb{Z}^n \) is a set of affinely independent points, and let \( F = \text{conv}(V) \cap \mathbb{Z}^n \). Assume \( w \in F \setminus V \), and let \( T = V \cup \{w\} \). Then \( \pi_T(M_F) \) is equal to the halfspace \( K_T \subset \mathbb{R}^n \) if and only if \( w \) belongs to the maximal V-mediated subset \( V^* \).

\textbf{Corollary 3.10.} Let \( A \subset \mathbb{Z}^n \) be finite, and \( E \supseteq \text{conv}(A) \cap \mathbb{Z}^n \). Then \( K_A = \pi_A(M_E) \) if and only if for every almost-empty simplex \( T = \{a_0, \ldots, a_k, b\} \subset A \), the point \( b \) belongs to the maximal V-mediated subset \( V^* \), where \( V = \{a_0, \ldots, a_k\} \).

\textbf{Proof.} Suppose \( K_A = \pi_A(M_E) \). Fix an almost-empty simplex \( T = \{a_0, \ldots, a_k, b\} \subset A \), and let \( F = \text{conv}(T) \cap \mathbb{Z}^n \). By Proposition 3.7, we have \( \pi_F(M_E) = M_F \). It follows that \( \pi_T(M_E) = \pi_T(\pi_A(M_E)) = \pi_T(K_A) = K_T \), so by Lemma 3.9 we have that \( b \) belongs to the maximal V-mediated subset \( V^* \), where \( V = \{a_0, \ldots, a_k\} \).
Conversely, assume that for every almost-empty simplex $T = \{a_0, \ldots, a_k, b\} \subset A$, the point $b$ belongs to the maximal $V$-mediated subset $V^*$, where $V = \{a_0, \ldots, a_k\}$. By Lemma 3.9 for every almost-empty simplex $T \subset A$, the set $\pi_T(M_F)$ is equal to $K_T$, where $F = \text{conv}(T) \cap \mathbb{Z}^n$. By Proposition 3.7 we then have $K_T = \pi_T(M_F) = \pi_T(\pi_F(M_E)) = \pi_F(\pi_A(M_E))$, and thus $\pi_A(M_E)$ is contained in the halfspace $K_T \times \mathbb{R}^A \subset \mathbb{R}^A$. By Proposition 3.4 the cone $K_A$ is equal to the intersection of these halfspaces, and thus $\pi_A(M_E) = K_A$. \hfill \Box

The proof of Lemma 3.3 given in [Rez89] generalizes directly to provide a description of the facets of the cone $M_A$ of midpoint-convex functions.

**Proposition 3.11.** The facets of the polyhedral cone $M_A$ correspond to inequalities of the form $\frac{h(a_1) + h(a_2)}{2} \geq h(b)$ with $a_1, a_2, b \in A$, $a_1 + a_2 = 2b$, and $b$ not belonging to any subset $S \subset \text{conv}(a_1, a_2) \cap A$ such that:

\[(*) \text{ for all } s \in S \text{ with } s \neq a_1, a_2 \text{ there exist distinct } s_1, s_2 \in S \text{ satisfying } s_1 + s_2 = 2s \text{ and } \{s_1, s_2\} \neq \{a_1, a_2\}.\]

We give a simple example before the proof of the proposition.

**Example 3.12.** For $A = \{0, 1, 2, 3, 4\}$, the inequality $\frac{1}{2}(h(0) + h(4)) \geq h(2)$ is not a facet of $M_A$, as it is implied by the other inequalities. The same is true for $A = \{0, 2, 3, 4, 6\}$ and the inequality $\frac{1}{2}(h(0) + h(6)) \geq h(3)$. This is reflected by the fact that, in both of these cases, the set $A$ itself satisfies condition (*) above. However, for $A = \{0, 3, 4, 5, 8\}$, the inequality $\frac{1}{2}(h(0) + h(8)) \geq h(4)$ is a facet of $M_A$.

**Proof of Proposition 3.11.** Suppose the inequality $h(a_1) + h(a_2) \geq 2h(b)$ with $a_1, a_2, b \in A$, $a_1 + a_2 = 2b$ does not describe a facet of $M_A$. This inequality must then be implied by other inequalities defining $M_A$, that is, there exist $s_{i,1}, s_{i,2}, s_i \in A$ and $\lambda_i > 0$ such that $s_{i,1} + s_{i,2} = 2s_i$ for all $i$ and

$$
\sum \lambda_i(e_{s_{i,1}} + e_{s_{i,2}} - 2e_{s_i}) = e_{a_1} + e_{a_2} - 2e_b \in \mathbb{R}^A,
$$

with $\{s_{i,1}, s_{i,2}\} \neq \{a_1, a_2\}$. Take $S$ to be the set containing all points $s_{i,1}, s_{i,2}, s_i \in A$. Note that $S \subset \text{conv}(a_1, a_2) \cap A$, as Equation 3.5 implies that $\text{conv}(S) = \text{conv}(a_1, a_2)$. Furthermore, for any $s \in S$ with $s \neq a_1, a_2$, since the coefficient of $e_s$ in Equation 3.5 is non-positive (equal to 0 or $-2$), $s$ must be equal to some $s_i$, and thus $s_{i,1}, s_{i,2} \in S$ satisfy $s_{i,1} + s_{i,2} = 2s_i$ and $\{s_{i,1}, s_{i,2}\} \neq \{a_1, a_2\}$. We thus see that $b$ belongs to a set $S$ that satisfies condition (*).

Suppose now that $a_1, a_2, b \in A$, $a_1 + a_2 = 2b$, and $b$ belongs to a subset $S \subset \text{conv}(a_1, a_2) \cap A$ satisfying condition (*). Fix an ordering $S \setminus \{a_1, a_2\} = \{s_1, \ldots, s_n\}$ where $s_1 = b$. For each $s_i$ choose distinct $s_{j_i}, s_{k_i} \in S$ satisfying $s_{j_i} + s_{k_i} = 2s_i$ and $\{s_{j_i}, s_{k_i}\} \neq \{a_1, a_2\}$. Consider the matrix $M \in \mathbb{R}^{n \times n}$ whose diagonal entries $m_{i,i}$ are all equal to $-2$, entries $m_{j_i,i}$ and $m_{k_i,i}$ are equal to 1 whenever $s_{j_i}$ and $s_{k_i}$ are not in $\{a_1, a_2\}$, and all other entries are equal to 0. Note that every column of $M$ has at most two entries equal to 1, and all principal submatrices of $M$ must have a column with at most one entry equal to 1; for instance, the column in the submatrix corresponding to the point $s_1$ that is closest to $a_1$. By [Rez89] Lemma 4.3 applied to the matrix $-M^T$, we see that the matrix $M$ is invertible and all the entries of $M^{-1}$ are non-negative. This implies that the linear system $M \cdot x = (-2, 0, \ldots, 0)^T$ has a solution $x \in \mathbb{R}^n$ with all entries being non-negative. This in turn means that we can write

$$
c_1 e_{a_1} + c_2 e_{a_2} - 2e_b = \sum x_i(e_{s_{j_i}} + e_{s_{k_i}} - 2e_{s_i}) \in \mathbb{R}^A,
$$

with all the $x_i$ being non-negative. Now, for any $u \in \mathbb{R}^n$, consider the vector $\hat{u} := \sum_{a \in A}(u \cdot a)e_a \in \mathbb{R}^A$. All the summands on the right hand side of Equation 3.6 are orthogonal to $\hat{u}$, and thus the
same must be true for the left hand side, i.e., $u \cdot (c_1a_1 + c_2a_2 - 2b) = 0$. As this must be the case for any $u \in \mathbb{R}^n$, in fact we must have $c_1a_1 + c_2a_2 - 2b = 0$, and so $c_1 = c_2 = 1$. Equation (8) thus shows that the inequality $h(a_1) + h(a_2) \geq 2h(b)$ is implied by other inequalities defining $M_A$, and so it cannot define a facet of $M_A$.

If $P \subset \mathbb{R}^m$ is a convex cone, its dual cone is

$$P^\vee := \{ x \in \mathbb{R}^m : x \cdot v \geq 0 \text{ for all } v \in P \}.$$  

The discussion of convex and midpoint convex functions above applies directly to tropicalizations of the cones of $A$-moments and $A$-pseudo moments of measures supported on the entire nonnegative orthant $\mathbb{R}_{\geq 0}^n$. If we restrict the support to a semialgebraic subset $S$ of $\mathbb{R}^n$ with the Hadamard property, this leads to a more general notion of convexity and midpoint convexity with respect to a cone. In these applications the cone $C$ below is equal to trop$(S)^\vee$, the dual cone to the tropicalization of $S$. When $S = \mathbb{R}^n$, we have trop$(S)^\vee = \{0\}$.

**Definition 3.13.** Suppose $A \subset \mathbb{Z}^n$ is finite and $C \subset \mathbb{R}^n$ is a polyhedral cone. Define

$$K_{A,C} := \{ h : A \to \mathbb{R} : \sum \lambda_i h(a_i) \geq h(b) \text{ whenever } b, a_i \in A, \lambda_i \geq 0, \sum \lambda_i = 1, \text{ and } \sum \lambda_i a_i - b \in C \}$$

and

$$M_{A,C} := \{ h : A \to \mathbb{R} : h(a_1) + h(a_2) \geq 2h(b) \text{ whenever } b, a_i \in A \text{ satisfy } a_1 + a_2 = 2b, \text{ and } h(a_1) \geq h(a_2) \text{ whenever } a_i \in A \text{ satisfy } a_1 - a_2 \in C \}.$$

We have $K_{A,C} \subseteq M_{A,C}$. Note that these cones generalize the cones of convex and midpoint-convex functions on $A$; indeed, we have $K_A = K_{A,\{0\}}$ and $M_A = M_{A,\{0\}}$.

**Theorem 3.14.** The cone $K_{A,C}$ equals the tropical conical hull of $A(C^\vee) = \{ Au : u \in C^\vee \}$. That is, a function $h : A \to \mathbb{R}$ belongs to $K_{A,C}$ if and only if there exist $u_1, \ldots, u_r \in C^\vee$ and $c_1, \ldots, c_r \in \mathbb{R}$ for which

$$h(a) = \max \{ \langle u_1, a \rangle + c_1, \ldots, \langle u_r, a \rangle + c_r \}. \quad (7)$$

**Proof.** Let $Y = A(C^\vee)$. This is a polyhedral cone consisting of all functions of the form $h : A \to \mathbb{R}$ of the form $h(a) = \langle u, a \rangle$ for some $u \in C^\vee$. By definition, the tropical conical hull of $Y$ consists of functions of the form $\hat{h}$. First, we show that tcone$(Y) \subseteq K_{A,C}$. Suppose $h \in \text{tcone}(Y)$ and consider the extension $\hat{h} : \mathbb{R}^n \to \mathbb{R}$ obtained as the maximum of affine linear functions $\ell_j(x) = \langle u_j, x \rangle + c_j$. Since $u_j \in C^\vee$, $\ell_j(x) \geq \ell_j(y)$ whenever $x - y \in C$. It follows that $\hat{h}$ is convex and $\hat{h}(x) \geq \hat{h}(y)$ whenever $x - y \in C$. In particular, if $\sum \lambda_i a_i - b \in C$, then

$$\sum \lambda_i h(a_i) = \sum \lambda_i \hat{h}(a_i) \geq \hat{h} \left( \sum \lambda_i a_i \right) \geq \hat{h}(b) = h(b).$$

Therefore tcone$(Y) \subseteq K_{A,C}$.

For the reverse inclusion we show that the elements of $K_{A,C}$ satisfy all the inequalities given by the dual cone tcone$(Y^\vee)$. The cone $Y$ is the image of $C^\vee$ under the linear map $A$. By general convexity, the cone $Y^\vee$ is the intersection of $C$ with the image of the dual map of $A$. Explicitly, for each $a \in A$ and $h \in \mathbb{R}^A$, define $\delta_a(h) = h(a)$. The dual map of $A$ maps a point $\sum_{a \in A} \lambda_a \delta_a$ to $\sum_{a \in A} \lambda_a a \in \mathbb{R}^n$.

Suppose $\lambda = \sum_{a \in A} \lambda_a \delta_a$ is an extreme ray of tcone$(Y^\vee)$. By Proposition 2.5, this extreme ray only has one negative entry, say $\lambda_b$. We can rescale to make this entry $-1$. Then $\sum_{a \neq b} \lambda_a = 1$. This extreme ray imposes the condition $\sum_{a \neq b} \lambda_a h(a) \geq h(b)$ for elements $h : A \to \mathbb{R}$ of tcone$(Y)$. Moreover, since $\lambda$ belongs to tcone$(Y)^\vee$, its image $\sum_{a \in A} \lambda_a a = \sum_{a \neq b} \lambda_a a - b$ must belong to $C$.

By construction, $\lambda$ belongs to $K^\vee_{A,C}$.
Proposition 3.16. The cone $\mathcal{K}_{A,C}$ consists of all functions $h : A \to \mathbb{R}$ that can be extended to a convex function $\hat{h} : \mathbb{R}^n \to \mathbb{R}$ satisfying $\hat{h}(x_1) \geq \hat{h}(x_2)$ whenever $x_1 - x_2 \in C$. In particular, for any finite subset $E \subset \mathbb{Z}^n$ with $A \subset E$ we have

$$\mathcal{K}_{A,C} = \pi_A(\mathcal{K}_{E,C}) \subseteq \pi_A(\mathcal{M}_{E,C}).$$

Proof. One can check that the function $\hat{h}(x) = \max\{\langle u_1,x \rangle + c_1, \ldots, \langle u_r,x \rangle + c_r\}$ given by Theorem 3.14 is such an extension. \hfill \Box

The description by inequalities of $\mathcal{K}_{A,C}$ implies that the dual cone $\mathcal{K}_{A,C}^\vee$ is the conical hull of the vectors of the form $\sum \lambda_i a_i - b \in \mathbb{R}^A$ with $b, a_i \in A$, $\lambda_i \geq 0$, $\sum \lambda_i = 1$, and $\sum \lambda_i a_i - b \in C$. Similarly, the dual cone $\mathcal{M}_{A,C}^\vee$ is the conical hull of the vectors $e_a + e_b - 2c \in \mathbb{R}^A$ with $b, a_i \in A$ and $a_1 + a_2 = 2b$ together with the vectors $e_a - e_b \in \mathbb{R}^A$ with $a_i \in A$ and $a_1 - a_2 \in C$.

Proposition 3.16. Suppose $F$ is a face of the polyhedral cone $C \subset \mathbb{R}^n$. Then $\mathcal{K}_{A,F}^\vee$ is a face of $\mathcal{K}_{A,C}^\vee$, and similarly, $\mathcal{M}_{A,F}^\vee$ is a face of $\mathcal{M}_{A,C}^\vee$.

Proof. Since $F$ is a face of $C$, there exists $u \in \mathbb{R}^n$ such that $u \cdot c = 0$ for all $c \in F$ and $u \cdot c > 0$ for all $c \in C \setminus F$. Consider the vector $\hat{u} := \sum_{a \in A}(u \cdot a) e_a \in \mathbb{R}^A$. If $x = \sum \lambda_i a_i - b \in \mathbb{R}^A$ is one of the generators of $\mathcal{K}_{A,C}^\vee$ described above, the dot product $\hat{u} \cdot x$ is equal to $\sum \lambda_i (u \cdot a_i) - (u \cdot b) = u \cdot (\sum \lambda_i a_i - b)$. This number is non-negative for all the generators of $\mathcal{K}_{A,C}^\vee$, and it is equal to zero precisely when $\sum \lambda_i a_i - b \in F$. It follows that $\mathcal{K}_{A,F}^\vee$ is the face of $\mathcal{K}_{A,C}^\vee$ minimizing the dot product with $\hat{u}$.

A similar argument shows that $\mathcal{M}_{A,F}^\vee$ is the face of $\mathcal{M}_{A,C}^\vee$ minimizing the dot product with $\hat{u}$. Indeed, if $x = e_a + e_b - 2c \in \mathbb{R}^A$ is one of the generators of $\mathcal{M}_{A,C}^\vee$ then $\hat{u} \cdot x = 0$, and if $x = e_a - e_b \in \mathbb{R}^A$ with $a_1 - a_2 \in C$ then $\hat{u} \cdot x = 0$ if and only if $a_1 - a_2 \in F$. \hfill \Box

Proposition 3.17. Let $A \subset E \subset \mathbb{Z}^n$ be finite subsets and $C \subset \mathbb{R}^n$ a pointed polyhedral cone. If $\mathcal{K}_{A,C} = \pi_A(\mathcal{M}_{E,C})$ then $\mathcal{K}_A = \pi_A(\mathcal{M}_E)$.

A combinatorial characterization of the sets of lattice points $A$ for which $\mathcal{K}_A = \pi_A(\mathcal{M}_E)$ with $E \supseteq \text{conv}(A) \cap \mathbb{Z}^n$ was given in Corollary 3.10.

Proof. Suppose $\mathcal{K}_{A,C} = \pi_A(\mathcal{M}_{E,C})$. Taking dual cones we obtain

$$\mathcal{K}_{A,C}^\vee = \mathcal{M}_{E,C}^\vee \cap \mathbb{R}^A,$$

where $\mathbb{R}^A \subset \mathbb{R}^E$ denotes the coordinate subspace where $x_i = 0$ for all $i \notin A$. Since $C \subset \mathbb{R}^n$ is a pointed polyhedral cone, there exists $u \in \mathbb{R}^n$ such that $u \cdot c > 0$ for all $c \in C \setminus \{0\}$. Let $\hat{u} := \sum_{a \in E}(u \cdot a) e_a \in \mathbb{R}^E$, and denote by $\pi(\hat{u})$ its projection to $\mathbb{R}^A$. As explained in the proof of Proposition 3.16, the face of $\mathcal{K}_{A,C}^\vee$ consisting of all vectors orthogonal to $\pi(\hat{u})$ is equal to $\mathcal{K}_{A}^\vee$, and the face of $\mathcal{M}_{E,C}^\vee$ consisting of all vectors orthogonal to $\hat{u}$ is equal to $\mathcal{M}_{E}^\vee$. Since a vector in $\mathbb{R}^A$ is orthogonal to $\pi(\hat{u})$ if and only if it is orthogonal to $\hat{u}$, it follows that $\mathcal{K}_{A}^\vee = \mathcal{M}_{E}^\vee \cap \mathbb{R}^A$. Taking dual cones, we get $\mathcal{K}_A = \pi_A(\mathcal{M}_E)$, as desired. \hfill \Box

Proposition 3.17 shows that if the containment $\mathcal{K}_A \subset \pi_A(\mathcal{M}_E)$ is strict then for any fixed pointed cone $C \subset \mathbb{R}^n$ we also have $\mathcal{K}_{A,C} \subset \pi_A(\mathcal{M}_{E,C})$. Using the connection to moments and pseudo-moments described in Sections 4 and 5, this proposition shows that even if we increase degree bounds for sums of squares, there will exist valid binomial inequalities in moments that are not satisfied by pseudo-moments.
The following example shows that it is possible to have a sequence of cones \( \{C_n\}_{n \geq 1} \) — coming from a sequence of semialgebraic subsets \( \{S_n\}_{n \geq 1} \) of \( \mathbb{R}^n_{\geq 0} \) — such that for any \( n \) we have \( \mathcal{K}_{A,C_n} \subseteq \pi_A(\mathcal{M}_{E_n,C_n}) \), even though we have an equality in the limit \( n \to \infty \). Interpreted in terms of tropicalizations of moment and pseudo-moment cones, the example shows that different degenerations of semi-algebraic sets lead to different behavior for moments and pseudo-moments. See Example [6.3] for a different sequence of cones \( \{C_n\} \) where the equality in the limit does not hold, even as \( S \) shrinks to 1-dimensional subset of \( \mathbb{R}^2 \).

**Example 3.18.** For \( n \geq 1 \), consider the cone \( C_n \subset \mathbb{R}^2 \) spanned by the vectors \( (1, -(n + 1)) \) and \( (-1, n) \). Let \( A = \{(0, 0), (1, 2), (2, 1), (1, 1)\} \subset \mathbb{Z}^2 \), and take \( E_n \subset \mathbb{Z}^2 \) to be the subset

\[
E_n = A \cup \{(0, 1), (0, 2), \ldots, (0, n + 3)\} \cup \{(1, 3), (1, 4), \ldots, (1, n + 2)\} \cup \{(1, 0), (2, 0)\}.
\]

The cone \( \mathcal{K}_A \) is described by the single inequality \( h(0,0) + h(1,2) + h(2,1) \geq 3h(1,1) \), while the cone \( \pi_A(\mathcal{M}_{E_n}) \) is equal to all of \( \mathbb{R}^A \). In view of Proposition 3.17 this implies that \( \mathcal{K}_{A,C_n} \subseteq \pi_A(\mathcal{M}_{E_n,C_n}) \).

Nonetheless, we will show that

\[
\lim_{n \to \infty} \mathcal{K}_{A,C_n} = \lim_{n \to \infty} \pi_A(\mathcal{M}_{E_n,C_n}).
\]

In particular, the inequality \( h(0, 0) + h(1, 2) + h(2, 1) \geq 3h(1, 1) \) holds on the limit \( \lim_{n \to \infty} \pi_A(\mathcal{M}_{E_n,C_n}) \), despite not holding on \( \mathcal{M}_{E_n,C_n} \) for any finite \( n \).

The cone \( \mathcal{K}_{A,C_n} \) has four facets given by the following inequalities

\[
\begin{array}{ll}
h(1, 2) \geq h(2, 1), & h(0, 0) + h(1, 2) + h(2, 1) \geq 3h(1, 1), \\
\frac{1}{n + 2} h(0, 0) + \frac{n + 1}{n + 2} h(1, 2) \geq h(1, 1), & \frac{n}{n + 1} h(1, 1) + \frac{1}{n + 1} h(2, 1) \geq h(1, 2).
\end{array}
\]

Modulo its lineality space \( h(0,0) = h(1,1) = h(1,2) = h(2,1) \), it is the polyhedral cone generated by the four rays with entries \( (h(0,0), h(1,1), h(1,2), h(2,1)) \) equal to

\[
(1, \frac{1}{n+2}, 0, 0), \quad (1, \frac{n-1}{2n+1}, 0, 0), \quad (1, \frac{n+1}{2n+1}, 0, 0), \quad \text{and} \quad (1, 0, 0, 0).
\]

In the limit \( n \to \infty \), this tends to the cone generated by the same lineality space and the rays \( (1, 0, 0, 0) \) and \( (1, \frac{1}{2}, \frac{1}{2}, 0) \), which can be described by the inequalities

\[
h(1, 1) = h(1, 2) \geq h(2, 1) \quad \text{and} \quad h(0, 0) + h(1, 2) + h(2, 1) \geq 3h(1, 1).
\]

The cones \( \mathcal{M}_{E_n,C_n} \) satisfy the inequalities \( h(1, 1) \geq h(2, 2) \geq h(2, 1) \), and thus do so the projections \( \pi_A(\mathcal{M}_{E_n,C_n}) \). To see that the limit \( \lim_{n \to \infty} \pi_A(\mathcal{M}_{E_n,C_n}) \) satisfies \( h(1, 1) \leq h(2, 1) \), note that the restriction of any function \( h \in \mathcal{M}_{E_n,C_n} \) to the set \( \{(0, 0), (0, 1), \ldots, (0, n + 3)\} \) is convex. It follows that the cone \( \mathcal{M}_{E_n,C_n} \) satisfies

\[
h(1, 1) \leq h(0, n + 1) \leq \frac{2}{n + 3} h(0, 0) + \frac{n + 1}{n + 3} h(0, n + 3) \leq \frac{2}{n + 3} h(0, 0) + \frac{n + 1}{n + 3} h(2, 1).
\]

This implies that \( \lim_{n \to \infty} \pi_A(\mathcal{M}_{E_n,C_n}) \) satisfies \( h(1, 1) \leq h(1, 2) \).

To see that the inequality \( h(0,0) + h(1,2) + h(2,1) \geq 3h(1,1) \) also holds in the limit \( n \to \infty \), note that \( \mathcal{M}_{E_n,C_n} \) satisfies \( 2(h(1,0) + h(1,2)) \geq 4h(1,1) \) and \( h(0,0) + h(2,0) \geq 2h(1,0) \). Adding these two inequalities we obtain

\[
h(0,0) + h(1,2) + h(2,0) - (h(1,1) - h(1,2)) \geq 3h(1,1).
\]

Moreover, \( \mathcal{M}_{E_n,C_n} \) also satisfies

\[
h(1,1) - h(1,2) \geq h(1,2) - h(1,3) \geq \cdots \geq h(1,n) - h(1,n+1),
\]

so substituting in Equation (8) we get

\[
h(0,0) + h(1,2) + h(2,0) - (h(1,n) - h(1,n+1)) \geq 3h(1,1).
\]
Since \( h(2,0) \leq h(1,n) \) is valid in \( \mathcal{M}_{E_n,C_n} \), we further obtain \( h(0,0) + h(1,2) + h(1,n+1) \geq 3h(1,1) \). Adding \( h(2,1) - h(2,1) \) on the left gives

\[
h(0,0) + h(1,2) + h(2,1) + (h(1,n+1) - h(2,1)) \geq 3h(1,1).
\]

(10)

Now, the decreasing inequalities of \( \mathcal{M}_{E_n,C_n} \) also imply \( h(1,n+1) - h(2,1) \leq h(1,n+1) - h(1,n+2) \). Using midpoint inequalities as in [3] above and \( h(2,1) \geq h(1,n+2) \), we in fact have \( h(1,n+1) - h(2,1) \leq h(1,n+1) - h(1,n+2) \leq h(1,i) - h(1,i+1) \) for all \( i \in \{0, \ldots, n\} \). Adding all these inequalities together, we get

\[
(n+1)(h(1,n+1) - h(2,1)) \leq h(1,0) - h(1,n+1).
\]

Since \((-1,0) = n(1, -(n+1)) + (n+1)(-1, n) \in C_n\), the right hand side satisfies \( h(1,0) - h(1,n+1) \leq h(0,0) - h(1,n+1) \leq h(0,0) - h(2,1) \), and thus we get

\[
(n+1)(h(1,n+1) - h(2,1)) \leq h(0,0) - h(2,1).
\]

Finally, substituting this last inequality into Equation (10) we conclude that \( \mathcal{M}_{E_n,C_n} \) satisfies

\[
h(0,0) + h(1,2) + h(2,0) + \frac{1}{n+1} (h(0,0) - h(2,1)) \geq 3h(1,1).
\]

From this we conclude that the desired inequality \( h(0,0) + h(1,2) + h(2,1) \geq 3h(1,1) \) holds in the limit \( \lim_{n \to \infty} \pi_A(\mathcal{M}_{E_n,C_n}) \), and thus the claimed equality holds.

\[\Diamond\]

4. TROPICALIZING THE MOMENT CONE

Let \( S \) be a semialgebraic subset of the nonnegative orthant \( \mathbb{R}^n_{\geq 0} \) with the property that \( S \subset S \cap \mathbb{R}^n_{\geq 0} \), that is to say that the points in the intersection of \( S \) and the positive orthant are dense in \( S \). In this case, the tropicalization of \( S \) is the tropicalization of \( S \cap \mathbb{R}^n_{\geq 0} \).

Let \( A \subset \mathbb{Z}^n_{\geq 0} \) be a finite subset. By a slight abuse of notation we will also denote by \( A \) the matrix whose rows are the integer vectors of the point configuration \( A \).

**Lemma 4.1.** If \( S \) is a subset of the nonnegative orthant \( \mathbb{R}^n_{\geq 0} \) with \( S \subset S \cap \mathbb{R}^n_{\geq 0} \), then the tropicalization of its moment cone \( M_A(S) \) is

\[
trop(M_A(S)) = \text{tcone } A(\text{trop}(S)).
\]

**Proof.** The moment cone \( M_A(S) \) is the closure in \( \mathbb{R}[x_1, \ldots, x_n]^A \) of the convex cone generated by the point evaluations at the points of \( S \). The point evaluations have coordinates \( (x^\alpha : \alpha \in A) \) in the monomial basis (for \( x \in S \)). So the tropicalization of the set point evaluations is equal to the image of \( \text{trop}(S) \) under the linear map \( A \). By Proposition 2.3, the tropicalization of the conical hull equals the tropical convex hull of the tropicalization, which proves the claim. \[\square\]

The above lemma allows us to give an elegant description of tropicalizations of subsets of \( S \subset \mathbb{R}^n_{\geq 0} \) with the Hadamard property.

**Theorem 4.2.** Let \( S \subset \mathbb{R}^n_{\geq 0} \) be a semi-algebraic subset with the Hadamard property and such that \( S \subset S \cap \mathbb{R}^n_{\geq 0} \). The tropicalization of the moment cone \( M_A(S) \) for a support set \( A \subset \mathbb{Z}^n_{\geq 0} \) is the set of all functions \( h : A \to \mathbb{R} \) satisfying the inequalities

\[
\sum_{i=1}^r \lambda_i h(a_i) \geq h(b) \quad \text{for all} \quad a_1, \ldots, a_r, b \in A, \quad \lambda_i \geq 0, \quad \sum_{i=1}^r \lambda_i = 1 \quad \text{with} \quad \sum_{i=1}^r \lambda_i a_i - b \in \text{trop}(S)^\vee. \quad (11)
\]

That is, \( \text{trop}(M_A(S)) = \mathcal{K}_{A,C} \) where \( C = \text{trop}(S)^\vee \).
Proof. By Lemma 4.1, \( \text{trop}(M_A(S)) = \text{tcone}(A \text{trop}(S)) \). Let \( C = \text{trop}(S)^\vee \). By Theorem 3.14, the cone \( K_{A,C} \) coincides with the tropical convex hull of \( A(C)^\vee = A \text{trop}(S) \). \( \square \)

The condition in Theorem 4.2 cannot, in general, be split into two conditions, namely the convexity constraint \( \sum \lambda_i h(a_i) \geq h(b) \) for all convex combinations with \( a_i, b \in A \) and the nonincreasing property \( h(a) \geq h(b) \) whenever \( a - b \in \text{trop}(S)^\vee \) as the following example shows.

**Example 4.3.** Consider \( A = \{(3,0),(0,3),(2,2)\} \) and \( S = [0,1]^2 \). No point in \( A \) is a convex combination of the others and for any \( a \neq b \in A \), the difference \( a - b \) does not lie in \( \text{trop}(S)^\vee = \mathbb{R}^2_{\geq 0} \). Therefore any function \( h : A \to \mathbb{R} \) satisfies (1) and (2). However, \( \frac{1}{2} (3,0) + \frac{1}{2} (0,3) - (2,2) = (-\frac{1}{2}, -\frac{1}{2}) \) belongs to \( \text{trop}(S)^\vee \) and so the functions \( h : A \to \mathbb{R} \) in \( M_A(S) \) satisfy \( \frac{1}{2} h(3,0) + \frac{1}{2} h(0,3) \geq h(2,2) \). \( \diamond \)

We now discuss implications of Theorem 4.2 for \( \mathbb{R}^n_{\geq 0} \), the unit cube and toric cubes.

### 4.1. Positive orthant

Let \( S = \mathbb{R}_{\geq 0} \). Then \( \text{trop}(S) = \mathbb{R}^n \) and so \( \text{trop}(S)^\vee = \{0\} \). Theorem 4.2 applied to this situation gives the following characterization of the tropicalization of the moment cone. It recovers the result by Develin in [Dev06] that the tropical convex hull of a linear space \( A(\text{trop}(S)) \) is the set of convex functions on \( A \).

**Corollary 4.4.** The tropicalization \( \text{trop}(M_A(\mathbb{R}^n_{\geq 0})) \) is the cone of convex functions on \( A \). \( \square \)

**Example 4.5.** Consider the Motzkin configuration in the plane shown in Figure 1a, i.e. \( A = \{(0,0),(1,2),(2,1),(1,1)\} \). By Corollary 4.4 we see that \( \text{trop}(M_A(\mathbb{R}^n_{\geq 0})) \) consists of functions \( h : A \to \mathbb{R} \) satisfying:

\[
    h(0,0) + h(1,2) + h(2,1) \geq 3h(1,1).
\]

\( \diamond \)

### 4.2. The Cube

Let \( C_n = [0,1]^n \) be the \( n \)-dimensional cube. Then \( \text{trop}(C_n) \) is the nonpositive orthant of \( \mathbb{R}^n \). Since this orthant is also self-dual as a convex cone, Theorem 4.2 implies the following description of the tropicalization of the moment cone on (hyper-)cubes.

**Corollary 4.6.** The following four sets are equal to each other up to natural identifications.

1. \( \text{trop}(M_A(C_n)) \), the tropicalization of the moment cone of the unit \( n \)-cube.
2. The tropical conical hull of the polyhedral cone spanned by columns of the matrix \( -A \).
3. The polyhedral cone in \( \mathbb{R}^A = \{h : A \to \mathbb{R}\} \) defined by inequalities of the form

\[
    \lambda_1 h(a_1) + \cdots + \lambda_r h(a_r) \geq h(b)
\]

where \( a_1, \ldots, a_r, b \) are in \( A \), \( \lambda_i \geq 0 \), \( \sum \lambda_i = 1 \), and \( \lambda_1 a_1 + \cdots + \lambda_r a_r \) is coordinate-wise at most \( b \). \( \square \)

**Example 4.7.** We revisit the Motzkin configuration in the plane of Example 4.5. The tropicalization \( \text{trop}(M_A(C_n)) \) is the tropical conical hull of the two dimensional convex cone with extreme rays \( (0,-1,-2, -1) \) and \( (0,-2,-1,-1) \). Using Proposition 2.5 we can compute that \( \text{trop}(M_A(C_n)) \) consists of functions \( h : A \to \mathbb{R} \) satisfying:

\[
    h(1,1) \geq h(1,2), \quad h(1,1) \geq h(2,1), \quad \text{and} \quad h(0,0) + h(1,2) + h(2,1) \geq 3h(1,1).
\]

The corresponding binomial moment inequalities for measures supported on \( [0,1]^2 \) are:

\[
    m_{(1,1)} \geq m_{(1,2)}, \quad m_{(1,1)} \geq m_{(2,1)}, \quad \text{and} \quad m_{(0,0)} m_{(1,2)} m_{(2,1)} \geq m_{(1,1)}^3.
\]

We only need to consider the pure binomials, by Remark 2.10. \( \diamond \)
The following example gives a better illustration of the difference between binomial moment inequalities for the orthant and the cube.

**Example 4.8.** Consider the following bivariate moments \( A = \{(4,0), (0,4), (3,2)\} \). Since these points are in convex position, the tropicalization \( \text{trop}M_A(\mathbb{R}_{\geq 0}^2) \) is all of \( \mathbb{R}^3 \), and there are no binomial inequalities satisfied by these moments for measures supported on \( \mathbb{R}_{\geq 0} \).

For the cube \( C = [0,1]^2 \) the tropicalization \( \text{trop}M_A(C_2) \) consists of functions \( h : A \to \mathbb{R} \) satisfying:

\[
\begin{align*}
    h(4,0) + h(0,4) &\geq 2h(3,2) \quad \text{and} \quad 3h(4,0) + h(0,4) \geq 3h(1,1).
\end{align*}
\]

These inequalities correspond to inequalities \( \frac{1}{2}(4,0) + \frac{1}{2}(0,4) \leq (3,2) \) and \( \frac{3}{4}(4,0) + \frac{1}{4}(0,4) \leq (3,2) \) coordinate-wise in \( \mathbb{R}^2 \). The corresponding binomial moment inequalities are:

\[
\begin{align*}
    m(4,0)m(0,4) &\geq m^2_{(3,2)} \quad \text{and} \quad m^3_{(4,0)m(0,4)} \geq m^3_{(1,1)}.
\end{align*}
\]

\( \diamond \)

### 4.3 Toric Cubes

Consider a subset \( S \subseteq [0,1]^n \) defined by binomial inequalities that is the closure of its positive points \( S \cap (0,1)^n \). Such a set was called a toric cube in [EHST13]. They also showed that every toric cube \( S \subseteq [0,1]^n \) can be parametrized by a monomial map \( \varphi : [0,1]^d \to [0,1]^n \) taking \( t = (t_1, \ldots, t_d) \) to \( (t^{b_1}, \ldots, t^{b_n}) \) for appropriate exponent vectors \( b_i \). Let \( Q \) be a \( d \times n \) matrix with columns \( b_1, \ldots, b_n \). We can think of \( Q \) as a map from \( \mathbb{N}^n \) to \( \mathbb{N}^d \).

Let \( A = \{a_1, \ldots, a_k\} \subset \mathbb{N}^n \) be a collection of lattice points and consider the moment cone \( M_A(S) \). The pullback via the monomial parametrization \( \varphi \) of \( S \) gives a new configuration \( A' \) whose points are given by \( Qa_i \) for \( 1 \leq i \leq k \). So the moment cone \( M_A(S) \) is the same as \( M_{A'}([0,1]^d) \).

**Example 4.9.** Let \( S \) be the subset of \( [0,1]^2 \) given by the inequalities \( y \geq x^2 \) and \( y \leq x^2 \). We can see that \( S \) is the image of \( [0,1]^2 \) under the map \( (t_1,t_2) \mapsto (t_1t_2, t_1^2t_2^3) \). Then \( Q \) is given by \( \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix} \).

Consider again the Motzkin configuration in the plane \( A = \{(0,0), (1,2), (2,1), (1,1)\} \). The new configuration \( A' = Q(A) \) is given by \( A' = \{(0,0), (5,7), (4,5), (3,4)\} \).

\( \diamond \)

We apply Theorem 4.2 to \( M_{Q(A)}([0,1]^d) \) to obtain a description of \( \text{trop}(M_A(S)) \). We use that \( \text{trop}(S) = Q^T \text{trop}([0,1]^d) \), which implies by duality, using \( \text{trop}([0,1]^d)^{\vee} = \mathbb{R}_{\geq 0}^n \), the description of the relevant cone appearing in Theorem 4.2 namely \( \text{trop}(S)^{\vee} = Q(\mathbb{R}_{\geq 0}^n) \).

**Corollary 4.10.** The following sets are equal.

1. \( \text{trop}(M_A(S)) \), the tropicalization of the moment cone
2. The tropical conical hull of the polyhedral cone spanned by columns of the matrix \( -A' \), where the points in \( A' = Q(A) \) are written as the rows.
3. The polyhedral cone of functions \( h : A \to \mathbb{R} \) defined by inequalities of the form \( \lambda_1 h(a_1) + \cdots + \lambda_r h(a_r) \geq h(b) \) where \( \lambda_i \geq 0, \lambda_1 + \cdots + \lambda_r = 1 \), and \( \lambda_1 Q(a_1) + \cdots + \lambda_r Q(a_r) \leq b \) coordinate-wise in \( \mathbb{R}^d \). \( \Box \)

**Example 4.11** (Example 4.9 continued). Using Proposition 2.5 we can compute that \( \text{trop} M_A(S) \) consists of functions \( h : A \to \mathbb{R} \) satisfying:

\[
\begin{align*}
    h(2,1) &\geq h(1,2), \\
    h(1,2) + 2h(1,1) &\geq 3h(2,1), \\
    h(0,0) + 3h(2,1) &\geq 4h(1,1), \\
    h(0,0) + h(1,2) + h(2,1) &\geq 3h(1,1).
\end{align*}
\]
The corresponding binomial moment inequalities for measures supported on $S$ are:

\[
m_{(2,1)} \geq m_{(1,2)}, \quad m_{(1,2)}m_{(1,1)}^2 \geq m_{(2,1)}^3, \quad m_{(0,0)}m_{(2,1)} \geq m_{(1,1)}^3, \quad m_{(0,0)}m_{(1,2)}m_{(2,1)} \geq m_{(1,1)}^3.
\]

Observe that since $S$ is a subset of $[0,1]^2$ the binomial inequalities for $[0,1]^2$ from Example 4.7 hold, but we also acquire additional inequalities.

4.4. Measures supported on all of $\mathbb{R}^n$. For measures supported on all of $\mathbb{R}^n$ we now find a nice description of the tropicalization of the moment cone, where even points in $A$ play a more prominent role.

The cone $\text{trop}(M_A(\mathbb{R}^n))$ naturally contains $\text{trop}(M_A(\mathbb{R}^n_{\geq 0}))$ because every measure with support in $\mathbb{R}^n_{\geq 0}$ is, by extension by 0, a measure on $\mathbb{R}^n$. So one way to formulate our question is to ask: which linear inequalities valid on $\text{trop}(M_A(\mathbb{R}^n_{\geq 0}))$, generated by convexity inequalities on almost-empty simplices, are also valid on $\text{trop}(M_A(\mathbb{R}^n))$?

We call an almost-empty lattice simplex even if all of its vertices are even lattice points (whereas the interior lattice point is not required to be even). We now state the main theorem of this subsection; it is very similar to Proposition 4.4 but we now only consider even almost-empty simplices.

**Theorem 4.12.** A function $h : A \rightarrow \mathbb{R}$ belongs to $\text{trop}(M_A(\mathbb{R}^n))$ if and only if $h$ is convex on all almost-empty even simplices in $A$.

We first establish helpful facts before proving this theorem.

**Lemma 4.13.** Let $A_+$ and $A_-$ be disjoint subsets of $A$. Suppose that a pure binomial inequality

\[
\prod_{\alpha \in A_+} |m_\alpha|^{a_\alpha} \geq \prod_{\beta \in A_-} |m_\beta|^{a_\beta}
\]

is valid on $M_A(\mathbb{R}^n)$ for some $a_\alpha, a_\beta \in \mathbb{Z}_{>0}$. Then all lattice points in $A_+$ must be even.

**Proof.** It suffices to show that for any $\gamma \in A$ with $\gamma \not\in 2\mathbb{N}$, there exists a measure on $\mathbb{R}^n$ with $m_\gamma = 0$ and all other moments non-zero.

We first deal with the univariate case $n = 1$. We need to show that for any odd $k \in A$ there is a measure on $\mathbb{R}$ whose $k$th moment is 0 and all others are non-zero. We proceed by induction on $d = \lceil \max(A)/2 \rceil$. If the $(d+1) \times (d+1)$ matrix $(m_{i+j})_{0 \leq i,j \leq d}$ is positive definite, then there is a measure $\mu$ on the real line with moments $(m_0, m_1, \ldots, m_{2d})$. For $d = 0$, there are no odd moments, and so there is nothing to show. Now let $m_k = 0$ and, by induction, suppose that we have a choice of $(m_0, m_1, \ldots, m_{2d-2}) \in \mathbb{R}^{2d-1}$ so that $(m_{i+j})_{0 \leq i,j \leq d-1}$ is positive definite. If $k \neq 2d-1$, then let $m_{2d-1}$ be any nonzero real number. Otherwise, if $k = 2d-1$, we set $m_{2d-1} = 0$. For sufficiently large $m_{2d} \in \mathbb{R}_+$, the resulting matrix $(m_{i+j})_{0 \leq i,j \leq d}$ will be positive definite, meaning that $(m_0, m_1, \ldots, m_{2d})$ is the vector of moments of some measure on $\mathbb{R}$, all of which are nonzero except for $m_k$. In fact, we can choose an atomic measure on $\mathbb{R}$, meaning that for some $t_1, \ldots, t_{d+1} \in \mathbb{R}$ and nonnegative weights $w_1, \ldots, w_{d+1} \in \mathbb{R}_{\geq 0}$, $m_j = \sum_{i=1}^{d+1} w_i t_i^j$ for all $j = 0, \ldots, 2d$.

Now we leverage the univariate case to prove the multivariate case. Suppose that for some $\gamma = (\gamma_1, \ldots, \gamma_n) \in A_+$, $\gamma_1$ is odd. Let $v = (v_1, \ldots, v_n)$ be an element of $\mathbb{N}^n$ such that $v_1$ is odd and $v_2, \ldots, v_n$ are even, and $\langle \alpha, v \rangle \neq \langle \beta, v \rangle$ for any pair of distinct elements $\alpha, \beta$ in $A$. As $\langle \alpha, v \rangle = \langle \beta, v \rangle$ defines a hyperplane in $\mathbb{R}^n$, there are many such choices.
Therefore if binomial inequalities are valid on $\mathbb{R}$, we can finish the proof of Theorem 4.12.

The image of $A$ under $\langle \cdot, v \rangle$ is a finite subset of $\mathbb{N}$. Moreover $\langle \gamma, v \rangle$ is odd. By the univariate case, there are some $t_1, \ldots, t_d \in \mathbb{R}$ and nonnegative weights $w_1, \ldots, w_d \in \mathbb{R}_{\geq 0}$, so that $\sum_{i=1}^d w_i t_i^j$ is zero for $j = \langle \gamma, v \rangle$ and nonzero for all $j = \langle \alpha, v \rangle$ where $\alpha \in A \setminus \{ \gamma \}$. We can extend this to the desired measure on $\mathbb{R}^n$ by considering an atomic measure with points $t_i^v = (t_i^{v_1}, \ldots, t_i^{v_n})$ and the same weights. Then for any $\alpha \in \mathbb{N}^n$, the corresponding moment is

$$m_\alpha = \sum_{i=1}^d w_i \prod_{j=1}^n (t_i^v)^{\alpha_j} = \sum_{i=1}^d w_i t_i^{(v, \alpha)}.$$ 

By construction, this is a measure on $\mathbb{R}^n$ with $m_\gamma = 0$ and $m_\alpha \neq 0$ for all other $\alpha$.

**Lemma 4.14.** A function $h : A \to \mathbb{R}$ belongs to $\text{trop}(M_A(\mathbb{R}^n))$ if and only if $h$ satisfies all inequalities $\sum_{a \in A} \lambda_a h(a) \geq 0$ valid on $\text{trop}(M_A(\mathbb{R}^n_{\geq 0}))$ where $\lambda_a > 0$ only for even points $a \in A$.

**Proof.** We think in terms of pure binomial inequalities in absolute values of moments of measures. Suppose that a pure binomial inequality in absolute values of moments is valid on $M_A(\mathbb{R}^n)$. Then by Lemma 4.13, we know that this inequality only has even moments on the “greater” side. Moreover, this inequality must be valid for all measures supported on $\mathbb{R}^n$, and therefore also for all measures supported on $\mathbb{R}^n_{\geq 0}$. This gives us one inclusion.

Given a measure $\mu$ on $\mathbb{R}^n$ define a measure $|\mu|$ on $\mathbb{R}^n_{\geq 0}$ via $|\mu|(B) = \sum_{g \in \{\pm 1\}^n} \mu(B^g)$ for $B \subseteq \mathbb{R}^n_{\geq 0}$, where $g$ ranges over all possible coordinate sign changes. We can also think of $|\mu|$ as a measure on all of $\mathbb{R}^n$ which gives zero weight to subsets outside of $\mathbb{R}^n_{\geq 0}$.

The $a$-th moment of $|\mu|$ is a sum of the absolute values of the $a$-th moment of the restriction of $\mu$ to each orthant. That is, $\int x^a \, d|\mu|$ equals $\sum_{g \in \{\pm 1\}^n} \int_{B^g} x^a \, d\mu$, where $B_g$ denotes the orthant in $\mathbb{R}^n$ with sign pattern $g$. If $a$ is even, then each term is already positive and so the $a$-th moment of $\mu$ and $|\mu|$ agree. For arbitrary $a \in \mathbb{N}^n$, the $a$-th moment of $\mu$ is upper bounded by that of $|\mu|$.

Therefore if $\mu$ fails a pure binomial inequality in absolute values of moments, where all of the terms on the “greater” side are even, then $|\mu|$ will fail this inequality as well. This shows no additional pure binomial inequalities are valid on $M_A(\mathbb{R}^n)$ besides those that have all even terms on the “greater” side and are valid on $M_A(\mathbb{R}^n_{\geq 0})$.

We can finish the proof of Theorem 4.12.

**Proof of Theorem 4.12.** Let $Y = \text{trop}(M_A(\mathbb{R}^n))$ and $Z = \text{trop}(M_A(\mathbb{R}^n_{\geq 0}))$. It follows from Lemma 4.14 that $Y^\vee = Z^\vee \cap T$ where $T$ the set of vectors in $\mathbb{R}^{|A|}$ where coordinates with non-even indices are non-positive. Since the cone $Y$ is tropically convex it follows from Proposition 2.5 that

$$Y^\vee = \sum_{\alpha \in A} (Y^\vee \cap U_\alpha),$$

where $\sum$ stands for Minkowski addition and $U_\alpha$ is the orthant of $\mathbb{R}^{|A|}$ where the $\alpha$-indexed coordinate is nonpositive and the rest are nonnegative. Using the above centered equation on the $Y^\vee$ inside the Minkowski sum we see that

$$Y^\vee = \sum_{\alpha \in A} (Z^\vee \cap U_\alpha \cap T).$$

We observe that $U_\alpha \cap T$ is the set of vectors in $\mathbb{R}^{|A|}$ where $\alpha$-indexed coordinate is nonpositive, all non-even indexed coordinates (except potentially $\alpha$) are zero, and all even-indexed coordinates (except potentially $\alpha$) are nonnegative. We see that $U_\alpha \cap T$ is a face of $U_\alpha$, and therefore extreme rays of $Z^\vee \cap U_\alpha \cap T$ are extreme rays of $Z^\vee \cap U_\alpha$. It follows from Lemma 3.3 that the extreme rays of $Y^\vee$ come from almost-empty even simplices as desired. 

Example 4.15. We consider the doubled Motzkin configuration $\mathcal{A} = (0,0), (2,4), (4,2), (2,2)$ in the plane. By Theorem 4.12 we see that $\trop M_\mathcal{A}(\mathbb{R}_+)$ consists of functions $h: \mathcal{A} \to \mathbb{R}$ satisfying:

$$h(0,0) + h(2,4) + h(4,2) \geq 3h(2,2).$$

\[\square\]

4.5. Recovering The Moment Cone from Tropicalization. Although tropicalization is a quite forgetful operation in general, sometimes we can recover the moment cone from its tropicalization. If our configuration $A$ is an almost empty simplex then the moment cone is defined by a single binomial inequality and thus completely determined by its tropicalization. See also [DHNdW20, Lemma 3.5]

Proposition 4.16. Let $A \subset \mathbb{Z}^n$ be almost-empty simplex with vertices $v_0, \ldots, v_n$ and interior lattice point $w$. Write $w = \sum_{i \in I} \lambda_i v_i$ with $\lambda_i > 0$ for $i \in I$ and $\sum \lambda_i = 1$. The moment cone $M_A(\mathbb{R}_+)$ equals the set of $(m_{v_0}, \ldots, m_{v_n}, m_w) \in \mathbb{R}_{+}^{n+2}$ for which $\prod_{i \in I} m_{v_i}^\lambda \geq m_w$.

Proof. The proof proceeds by convex duality and we first consider the cone of polynomials supported on $A$ that are nonnegative on $\mathbb{R}_{+}^n$. The polynomial

$$c_0 x^{v_0} + \cdots + c_n x^{v_n} \geq c_w x^w$$

for all $x \in \mathbb{R}_{+}^n$ if and only if $c_0, \ldots, c_n$ are nonnegative and $c_w = \prod_{i = 0}^n \left( \frac{v_i}{x} \right)^{\lambda_i}$. This can be understood via the arithmetic-geometric mean inequality (in short AM/GM), which states that for every $y \in \mathbb{R}_{+}^{n+1}$,

$$\lambda_0 y_0 + \cdots + \lambda_n y_n \geq y_0^{\lambda_0} \cdots y_n^{\lambda_n},$$

with equality if and only if all coordinates $y_i$ with $i \in I$ are equal. Letting $y_i = (c_i/\lambda_i)x^{v_i}$ for $i \in I$ and applying AM/GM gives that

$$c_0 x^{v_0} + \cdots + c_n x^{v_n} \geq \sum_{i \in I} c_i x^{v_i} \geq x^w \prod_{i \in I} \left( \frac{c_i}{\lambda_i} \right)^{\lambda_i}.$$

The second inequality is tight when all the $y_i$’s are equal, meaning $(c_i/\lambda_i)x^{v_i} = (c_j/\lambda_j)x^{v_j}$ for all $i, j$. We can find such a point $x \in \mathbb{R}_{+}^n$ by solving the system of $|I| - 1 \leq n$ affine-linear equations $\langle \log(x), v_i - v_j \rangle = \log(c_j/\lambda_j) - \log(c_i/\lambda_i)$. To see that the first inequality must also be tight even when $I \neq \{0,\ldots,n\}$, consider a vector $\alpha \in \mathbb{R}^n$ so that $\langle \alpha, v_i \rangle = a$ for all $i$ with $\lambda_i > 0$ and $\langle \alpha, v_j \rangle < a$ whenever $\lambda_j = 0$. Consider rescaling $x \in \mathbb{R}^n$ coordinate-wise by $t^a$. As $t \to \infty$, the limit of $t^{-a}(c_0(t^a \cdot x)^{v_0} + \cdots + c_n(t^a \cdot x)^{v_n})$ equals $\sum_{i \in I} c_i x^{v_i}$. On the other hand, since $\sum_i \lambda_i v_i = w$, we see that $\langle \alpha, x \rangle = a$, $t^{-a}(t^a \cdot x)^w$ equals $t^{(\alpha,w)-a}x^w = x^w$. So the right hand side is invariant under this rescaling. Since the inequality must hold for all $t$, we see that the coefficient of $x^w$ cannot be improved.

Using this characterization let us derive defining inequalities for the dual cone $M_A(\mathbb{R}_{+}^n)$ as follows. The dual cone $M_A(\mathbb{R}_{+}^n)$ is contained in $\mathbb{R}_{+}^{n+2}$, and by convex duality, is the set of points $(m_{v_0}, \ldots, m_{v_n}, m_w)$ for which $\sum_{i=0}^n c_i m_{v_i} \geq c_w m_w$ for all polynomials $\sum_{i=0}^n c_i x^{v_i} \geq c_w x^w$ nonnegative on $\mathbb{R}_{+}^n$. By coordinate scaling, we see that a point $(m_\alpha)_{\alpha \in A}$ belongs to $M_A(\mathbb{R}_{+}^n)$ if and only if $(m_\alpha x^\alpha)_{\alpha \in A}$ belongs to $M_A(\mathbb{R}_{+}^n)$ for every $x \in \mathbb{R}_{+}^n$. 


Let $c_0, \ldots, c_n \in \mathbb{R}_{\geq 0}$ and $c_w = \prod_{i \in I} \left( \frac{c_i}{\lambda_i} \right)^{\lambda_i}$. By the arguments above, $\sum_i c_i m_{v_i} x^{v_i} \geq c_w m_w x^w$ for all $x \in \mathbb{R}_{\geq 0}^n$ if and only if

$$c_w m_w \leq \prod_{i \in I} \left( \frac{c_i m_{v_i}}{\lambda_i} \right)^{\lambda_i} = c_w \prod_{i \in I} m_{v_i}^{\lambda_i}.$$  

This shows that $M_A(\mathbb{R}_{\geq 0}^n)$ equals the set of $(m_{\alpha})_{\alpha \in A}$ in $\mathbb{R}_{\geq 0}^{n+2}$ satisfying $m_w \leq \prod_{i \in I} m_{v_i}^{\lambda_i}$. \hfill $\square$

5. Truncated pseudo-moment cones

The goal of this section is to understand the tropicalizations of the dual cones of cones of sums of squares (e.g. $\Sigma_{\chi}$) and more generally, dual cones to truncations of preorders and quadratic modules. We observe an interesting phenomenon of stabilization, which we now explain informally.

Let $S$ be a compact basic closed semialgebraic set defined by inequalities $g_i(x) \geq 0$, $i = 1, \ldots, k$. We can build a set of obviously nonnegative polynomials on $S$ by combining sums of squares and the known nonnegativity of polynomials $g_i$; this set is called the preorder generated by $\{g_i\}_{i=1}^k$. By Schmüdgen’s Positivstellensatz any polynomial $f$ strictly positive on $S$ has an obviously nonnegative representation $[\text{Sch91}]$. We note that this representation may (and often does) involve sums of squares of degree significantly larger than that of the positive polynomial $f$. As a consequence, we see that as we allow representations of larger and larger degree, obviously nonnegative polynomials fill up the interior of the cone of nonnegative polynomials. We call the dual cones to obviously nonnegative polynomials the cones of pseudo-moments. We see that the cones of pseudo-moments provide a convergent outer approximation to the cone of $A$-moments supported on $S$, as we allow degree bounds to grow.

To tropicalize the cones of obviously nonnegative polynomials we restrict ourselves to sets $S$ contained in the nonnegative orthant and defined by binomial inequalities. Tropicalization of $M_A(S)$ encodes all pure binomial inequalities in $A$-moments of measures supported on $S$. One would expect that, as the degree bounds grow, tropicalizations of pseudo-moment cones provide a convergent approximation to trop $M_A(S)$. However, as we will show below, often this does not happen. In fact tropicalizations of pseudo-moment cones stabilize and after a finite number of steps we do not learn any new binomial inequalities in moments, even as the degrees grow arbitrarily large. We discuss stabilization when $S$ is the unit cube in Section 5.2 and for more general sets $S$ in Section 5.4.

We ask in Question 6.1 whether this finite stabilization phenomenon occurs generally for sets defined by binomial inequalities.

We begin with the simple case of globally nonnegative polynomials, where one cannot pick up more obviously nonnegative polynomials by increasing the degree, since we have no generators $g_i$ to cancel out high-degree terms.

**Example 5.1.** Consider the cone

$$\Sigma_{n,2k} = \left\{ \sum_{i=1}^r f_i^2 : r \in \mathbb{N}, f_i \in \mathbb{R}[x_1, x_2, \ldots, x_n]_{\leq k} \right\}$$

in the real vector space $\mathbb{R}[x_1, x_2, \ldots, x_n]_{\leq k}$ with the monomial basis and tropicalize its dual cone $\Sigma_{n,2k}^\vee \subset \mathbb{R}[x_1, x_2, \ldots, x_n]^*$ in the dual basis. This cone $\Sigma_{n,2k}^\vee$ consists of all linear functionals $\ell \in \mathbb{R}[x_1, x_2, \ldots, x_n]^*$ such that the Hankel matrix $(\ell(x^{\alpha+\beta}))_{|\alpha|,|\beta| \leq k}$ is positive semidefinite, see for instance [BPT13, Section 4.6].

This Hankel matrix is a matrix filled with variables $y_{\delta}$, and so we can apply [BRST20, Theorem 4.4] to conclude that the tropicalization is given by the tropicalizations of the $2 \times 2$ minors of the Hankel
For the doubled Motzkin configuration detects the difference between the cones \( k = R \) the finite-dimensional subspace. In this case, the preorder has a nice property, namely that the intersection of words, is defined by taking absolute values. These tropicalizations are linear inequalities of the form

\[
y_{2\alpha} + y_{2\beta} \geq 2y_{\alpha + \beta},
\]

where \( y_\delta \) is the dual coordinate in \( \mathbb{R}[x_1, x_2, \ldots, x_n]^* \) corresponding to the monomial \( x_\delta^\beta \). In other words, \( \text{trop}(\Sigma_{n,2k}) \) consists of all functions \( h: k\Delta_{n-1} \to \mathbb{R} \) that satisfy even midpoint convexity, i.e. \( h(v) + h(w) \geq 2h(\frac{1}{2}(v + w)) \) for all \( v, w \in k\Delta_{n-1} \) with only even coordinates.

More generally, if we consider \( \Sigma_A = \{ \sigma = \sum_{r=1}^r f_i^2 : r \in \mathbb{N}, \text{supp}(\sigma) \subset A \} \) for a finite set \( A \subset \mathbb{Z}^n \), then its dual cone \( \Sigma_A^\vee \) is the projection of \( \Sigma_{n,2k}^\vee \) (where \( k \) is chosen such that \( A \subset k\Delta_{n-1} \)) onto \( \mathbb{R}^n \). Since tropicalization commutes with coordinate projections in this case, the same holds for the tropicalizations, i.e., the tropicalization of \( \Sigma_A^\vee \) is the projection of the tropicalization of \( \Sigma_{n,2k}^\vee \) onto \( A \). So a function \( h: A \to \mathbb{R} \) is contained in \( \text{trop}(\Sigma_A^\vee) \) if and only if it can be extended to a function \( \tilde{h}: \text{conv}(A) \cap \mathbb{Z}^n \to \mathbb{R} \) such that \( \tilde{h}(v) + \tilde{h}(w) \geq 2\tilde{h}(\frac{1}{2}(v + w)) \) for all \( v, w \in \text{conv}(A) \cap 2\mathbb{Z}^n \). We can further extend \( \tilde{h} \) to a function on \( k\Delta_{n-1} \) satisfying the same property, by assigning to points in \( k\Delta_{n-1} \setminus \text{conv}(A) \) values of a convex function which is sufficiently larger than the values of \( \tilde{h} \) on \( \text{conv}(A) \cap \mathbb{Z}^n \).

For the doubled Motzkin configuration \( A = \{(0,0), (2,4), (4,2), (2,2)\} \) in the plane from Example 4.15 there are no midpoints in \( A \), and therefore \( \text{trop}(\Sigma_A^\vee) \) is \( \mathbb{R}^4 \). This shows that tropicalization detects the difference between the cones \( \Sigma_A^\vee \) and \( M_A(\mathbb{R}^n) \).

More generally, we want to consider pseudo-moments on semi-algebraic subsets of the nonnegative orthant \( \mathbb{R}^n_{\geq 0} \). So rather than considering the cone sums of squares, we consider truncated quadratic modules or preorders. These are convex cones inside the infinite-dimensional vector space \( \mathbb{R}[x_1, x_2, \ldots, x_n] \) and we truncate them (that is to say look at finite-dimensional versions) in order to tropicalize them. There are various ways to do this and we want to illustrate the general procedure with the following basic case, the nonnegative orthant itself.

**Example 5.2.** The nonnegative orthant \( \mathbb{R}_{\geq 0} \) is defined by the inequalities \( x_i \geq 0 \) for \( i = 1, 2, \ldots, n \) so we consider the set

\[
\text{PO}(x_1, \ldots, x_n) = \left\{ \sum_{\alpha \in \{0,1\}^n} \sigma_\alpha x^\alpha : \sigma_\alpha \text{ sum of squares of polynomials for all } \alpha \in \{0,1\}^n \right\}
\]

of obviously nonnegative polynomials on \( \mathbb{R}_{\geq 0} \), called the preorder generated by \( x_1, x_2, \ldots, x_n \). To get a finite-dimensional version of this cone, we restrict it to the space \( \mathbb{R}[x_1, x_2, \ldots, x_n]_A \) of all polynomials with support in \( A \subset \mathbb{Z}^n \), where \( A \) is a finite set. For simplicity, let us also assume that \( A = \text{conv}(A) \cap \mathbb{Z}^n \).

In this case, the preorder has a nice property, namely that the intersection of \( \text{PO}(x_1, \ldots, x_n) \) with the finite-dimensional subspace \( \mathbb{R}[x_1, x_2, \ldots, x_n]_A \) consists of exactly those elements of the form \( f = \sum_{\alpha \in \{0,1\}^n} \sigma_\alpha x^\alpha \), where the sum of squares \( \sigma_\alpha \) satisfy the obviously sufficient property that \( \text{NewtonPolytope}(\sigma_\alpha) + \sum_{i=1}^n \alpha_i e_i \cap \mathbb{Z}^n \subset A \) (see for example [Net09] Theorem 5.2). This is also necessary here because each extreme term of \( \sigma_\alpha \) has a positive coefficient (since it is a sum of squares of real polynomials) and the extreme term of \( f \) in any direction \( v \) is attained at one of the extreme terms of \( \sigma_\alpha x^\alpha \).

Write \( \text{PO}_A \) for the intersection of \( \text{PO}(x_1, \ldots, x_n) \) and \( \mathbb{R}[x_1, \ldots, x_n]_A \). Setting \( x_i = z_i^2 \) for new variables \( z_i \), we can identify \( \text{PO}_A \) with the cone of sums of squares in \( \mathbb{R}[z_1, z_2, \ldots, z_n]_{A'} \), where \( A' \) consists of all points \( 2\alpha \) with \( \alpha \in A \). By the previous Example 5.1, we conclude that the
tropicalization of the dual cone \( PO_A \) is the set of functions \( h : A \to \mathbb{R} \) that are midpoint-convex, i.e. satisfy \( h(v) + h(w) \geq 2h(\tfrac{1}{2}(v + w)) \) for all \( v, w \in A \) such that \( \tfrac{1}{2}(v + w) \in A \).

In other words \( \text{trop}(PO_A) \) consists of secondary cones of \( A \) corresponding to subdivisions in which for every \( v, w, x \in A \) satisfying \( 2x = v + w \), if \( v \) and \( x \) are marked in the subdivision, then \( w \) is also marked.

The following result shows that tropicalization can detect, in some special situations at least, if nonnegative polynomials are sums of squares (in a dual sense).

**Proposition 5.3.** Let \( A \subset \mathbb{Z}^n_{\geq 0} \) be an almost empty simplex with vertices \( v_0, \ldots, v_n \) and interior point \( w \) with \( w = \lambda_0 v_0 + \lambda_1 v_1 + \ldots + \lambda_n v_n \) with \( \lambda_i \geq 0 \) and \( \sum_{i=0}^n \lambda_i = 1 \). The cone of \( A \)-moments \( M_A(\mathbb{R}^n_{\geq 0}) \) of measures supported on \( \mathbb{R}^n_{\geq 0} \) is equal to \( PO_A \) if and only if their tropicalizations agree, i.e. \( \text{trop}(M_A(\mathbb{R}^n_{\geq 0})) = \text{trop}(PO_A) \).

**Proof.** The “only if” direction is clear, and we need to show that equality of tropicalizations implies equality of the actual cones.

From Proposition 4.16, the moment cone \( M_A(\mathbb{R}^n_{\geq 0}) \) is defined by the inequality \( \prod_{i \in I} m_{v_i}^{\lambda_i} \geq m_w \) where \( I = \{ i : \lambda_i > 0 \} \). Its tropicalization \( \text{trop}(M_A(\mathbb{R}^n_{\geq 0})) \) is then given by the single linear inequality \( \sum_{i \in I} \lambda_i y_{v_i} \geq y_w \). If trop \( (M_A(\mathbb{R}^n_{\geq 0})) = \text{trop}(PO_A) \), then this linear inequality also holds on \( \text{trop}(PO_A) \). By Propositions 2.9 and 2.8 it follows that \( \prod_{i \in I} m_{v_i}^{\lambda_i} \geq m_w \) holds on \( PO_A \). Therefore \( PO_A \subseteq M_A(\mathbb{R}^n_{\geq 0}) \). The other containment is immediate. \( \square \)

A difference in tropicalizations directly implies that the original cones are different, as in the following example.

**Example 5.4.** We revisit the Motzkin configuration \( A \) in the plane of Example 4.5. In this case there are no midpoints in \( A \), and therefore \( \text{trop} PO_A \) (with the notation as in the previous example) is \( \mathbb{R}^4 \). However, the tropicalization of the moment cone is a half-space. The difference between \( \text{trop} PO_A \) and \( \text{trop} M_A(\mathbb{R}^2_{\geq 0}) \) of course implies that \( PO_A \neq M_A(\mathbb{R}^2_{\geq 0}) \) and dually, that there are nonnegative polynomials on \( \mathbb{R}^2_{\geq 0} \) that are not in \( PO_A \). \( \diamond \)

However, it can happen that the tropicalizations of truncated moment cones and truncated pseudo-moment cones are equal even though the original cones are different.

**Example 5.5.** Let \( A \subset \mathbb{Z}^n_{\geq 0} \) be the collection of nonnegative integer points with total sum of coordinates equal to 2. The cone \( P_A(\mathbb{R}^n_{\geq 0}) \) of quadratic forms in \( n \) variables that are nonnegative on the nonnegative orthant is known as the cone of copositive quadratic forms. The cone \( PO_A \) (with the notation of Example 5.2) is the cone of quadratic forms that can be written as a sum of a positive semidefinite quadratic form and a form with nonnegative coefficients. It is known that for \( n \geq 5 \) we have strict containment \( PO_A \subsetneq P_A(\mathbb{R}^n_{\geq 0}) \) (see [DDGH13]). Therefore we also have strict containment of the duals: \( M_A(\mathbb{R}^n_{\geq 0}) \subsetneq PO_A^\vee \). However, all convexity relations in \( A \) are midpoint relations, so we have \( \text{trop} PO_A^\vee = \text{trop} M_A(\mathbb{R}^n_{\geq 0}) \). \( \diamond \)

Truncation will often not have the nice properties that we see above on the nonnegative orthant, and we need to discuss truncation in general. Since our focus is on subsets of the nonnegative orthant, we restrict our attention to quadratic modules and preorders that are closed under multiplication by monomials. Let \( g_1, g_2, \ldots, g_r \) be fixed polynomials in \( \mathbb{R}[x_1, x_2, \ldots, x_n] \) (that we usually assume to be binomials or even pure binomials). We fix the following notation: Set \( g_0 = 1 \) and let \( \Sigma^2 \)
denote the cone of sums of squares in $\mathbb{R}[x_1, \ldots, x_n]$. We define the truncated quadratic module and preorder generated by $g_1, \ldots, g_r$ to be

$$\text{QM}_d(g_1, \ldots, g_r) = \left\{ \sum_{J \subseteq [n], i \in \{0\} \cup [r]} \sigma_{J,i} x^J g_i : \sigma_{J,i} \in \Sigma^2, \deg(\sigma_{J,i} x^J g_i) \leq d \text{ for all } J,i \right\},$$

and

$$\text{PO}_d(g_1, \ldots, g_r) = \left\{ \sum_{J \subseteq [n], J \subseteq [r]} \sigma_{J,I} x^J g_I : \sigma_{J,I} \in \Sigma^2, \deg(\sigma_{J,I} x^J g_I) \leq d \text{ for all } J,I \right\},$$

respectively, where $g_I^I$ denotes $\prod_{i \in I} g_i$ with the convention that $g_0^0 = 1$. Moreover, we slightly abuse notation and identify a subset $J$ with its indicator vector in $\{0,1\}^n$.

The dual cones are defined in terms of (pseudo-)moment and localizing matrices. Specifically, for a given degree $d \in \mathbb{N}$, linear function $\ell \in \mathbb{R}[x_1, x_2, \ldots, x_n]_d^*$, and polynomial $f \in \mathbb{R}[x_1, \ldots, x_n]$ with degree $\leq d - k$, we define the symmetric matrix $M_{k,\ell}(f) = (\ell(x^{\gamma + \delta}))_{|\gamma|,|\delta| \leq k}$, whose rows and columns are indexed by the monomials of degree at most $k$ that we order lexicographically. Then the dual cones of the truncated quadratic module and preorder above are given by positivity conditions on the localizing matrices:

$$\text{QM}_d(g_1, g_2, \ldots, g_r)^\vee = \left\{ \ell \in \mathbb{R}[x_1, x_2, \ldots, x_n]_d^* : M_{d(J,i),\ell}(x^J g_i) \geq 0 \text{ for all } J,i \right\},$$

$$\text{PO}_d(g_1, g_2, \ldots, g_r)^\vee = \left\{ \ell \in \mathbb{R}[x_1, x_2, \ldots, x_n]_d^* : M_{d(J,I),\ell}(x^J g_I) \geq 0 \text{ for all } J,I \right\}.$$

Here we take the degrees $d(J,i) = \lfloor \frac{d - \deg(x^J g_i)}{2} \rfloor$ and $d(J,I) = \lfloor \frac{d - \deg(x^J g_I)}{2} \rfloor$, respectively.

5.1. The pure binomial case. We work with semi-algebraic subsets $S \subset \mathbb{R}^n_{\geq 0}$ of the nonnegative orthant defined by pure binomial inequalities $x^a - x^b \geq 0$. For a pure binomial $f = x^a - x^b$, the matrix $M_{k,\ell}(f)$ equals $M_{k,\ell}(x^a) - M_{k,\ell}(x^b)$. This is a matrix of the form $A - B$ as in Proposition 2.14, where the entries of $A = M_{k,\ell}(x^a)$ and $B = M_{k,\ell}(x^b)$ are variables corresponding to the coordinates $(\ell(x^\gamma))_{|\gamma| < d}$ of the linear functional $\ell$. We use this to show that the cone $\text{QM}_d(g_1, g_2, \ldots, g_r)^\vee$ has the Hadamard property when the polynomials $g_i$ are pure binomials.

**Lemma 5.6.** Let $g_1, \ldots, g_r$ be pure binomials and consider the semi-algebraic set $S \subset \mathbb{R}^n$ defined by $x_i \geq 0$ and $g_j \geq 0$. If $S$ is full-dimensional, then $\text{QM}_d(g_1, g_2, \ldots, g_r)^\vee$ has non-empty interior and the Hadamard property.

**Proof.** If $S$ is full-dimensional, then there can be no polynomial that is both nonnegative and nonpositive on $S$. That is, the convex cone of polynomials in $\mathbb{R}[x_1, \ldots, x_n]_d$ that are nonnegative on $S$ is pointed. Since $\text{QM}_d(g_1, g_2, \ldots, g_r)$ is a subset of this cone, it is also pointed, implying that its dual cone is full dimensional.

To see that $\text{QM}_d(g_1, g_2, \ldots, g_r)^\vee$ has the Hadamard property, we first consider the case of a single binomial ($r = 1$), say $g = x^a - x^b$. The cone $\text{QM}_d(g)^\vee$ is defined by the positive semidefiniteness of matrices of the form $M_{d(J,0),\ell}(x^J) \succeq 0$ and $M_{d(J,1),\ell}(x^J) \succeq 0$.

Fix $J \subseteq [n]$ and $k = d(J,1)$. We can write $M_{k,\ell}(x^J g) = M_{k,\ell}(x^{J+a}) - M_{k,\ell}(x^{J+b})$. The two matrices on the right hand side are principal minors of the localizing matrices of squarefree monomials. To be precise write $J + a = K + 2\alpha'$ and $J + b = L + 2\beta'$ where $K, L \subseteq [n]$ and $\alpha', \beta' \in \mathbb{Z}^n_{\geq 0}$. Then $M_{k,\ell}(x^{J+a}) = M_{k,\ell}(x^{K+2\alpha'})$ is a principal minor of $M_{m,\ell}(x^K)$ for $m \geq k + |\alpha'|$. Similarly $M_{k,\ell}(x^{J+b}) = M_{k,\ell}(x^{L+2\beta'})$ is a principal minor of $M_{m,\ell}(x^L)$ for $m \geq k + |\beta'|$.

Consider the spectrahedron $S_J$ in $\mathbb{R}[x_1, \ldots, x_n]_d^*$ defined by the conditions $M_{d(K,0),\ell}(x^K) \succeq 0$, $M_{d(L,0),\ell}(x^L) \succeq 0$, and $M_{d(J,1),\ell}(x^J) \succeq 0$. These constraints have the form $A \succeq 0$, $B \succeq 0$, and $A_I - B_P \succeq 0$, where $A_I$, $B_P$ are principal minors of $A$ and $B$, respectively. By Proposition 2.14(1),
the spectrahedron $S_J$ has the Hadamard property. Since $QM_d(g)^\vee$ is the intersection of $S_J$ over all $J \subseteq [n]$, we see that $QM_d(g)^\vee$ has the Hadamard property as well.

Similarly, for $r > 1$, we can write $QM_d(g_1, \ldots, g_r)^\vee$ as the intersection of $QM_d(g_i)^\vee$ for $i = 1, \ldots, r$. Since each individual set has the Hadamard property, so does their intersection.

All together, we have that $QM_d(g_1, \ldots, g_r)^\vee$ is a full-dimensional set in $\mathbb{R}[x_1, \ldots, x_n]^d$ with the Hadamard property. 

We now determine tropicalization of truncated quadratic modules for the case where all polynomials $g_i$ are pure binomials.

**Theorem 5.7.** Let $S \subset \mathbb{R}^n_{\geq 0}$ be a semi-algebraic set defined by pure binomial inequalities $g_i = x^{a_i} - x^{b_i}$ such that $S \subset \mathbb{R}^n \cap \mathbb{Z}_{\geq 0}$.

Assume that the vectors $v_i = a_i - b_i$ generate the semigroup $N = trop(S)^\vee \cap \mathbb{Z}^n$. For all sufficiently large $d$ the tropicalization of $QM_d(g_i)^\vee$ is the rational polyhedral cone $F(S)_d$ given by the following inequalities:

1. $h(a_1) + h(a_2) \geq 2h(b)$ for all $a_1, a_2, b$ such that $|a_i| \leq d$, $|b| \leq d$ and $a_1 + a_2 = 2b$;
2. $h(a) \geq h(b)$ whenever $|a| \leq d$, $|b| \leq d$, and $a - b \in trop(S)^\vee$.

**Proof.** By Lemma 5.6, the toric spectrahedron $QM_d(g_i)^\vee$ has non-empty interior and it is full-dimensional. By Proposition 2.14(2), its tropicalization is by two types of linear inequalities: Type 1 is the tropicalization of the $2 \times 2$ minors of the localizing matrices $M_{d(\ell, 0), \ell}(x^j)$ of monomials, which give the midpoint inequalities $h(a_1) + h(a_2) \geq 2h(b)$ for all $a_1, a_2, b \in \mathbb{Z}_{\geq 0}^n$ of total degree at most $d$. More precisely, there is a vector $J \in \{0, 1\}^n$ and vectors $a_1, a_2 \in \mathbb{Z}_{\geq 0}^n$ such that $a_i = 2a_1 + J$ because there sum is an even vector $2b$. With this notation, the $2 \times 2$ minor whose tropicalization gives the desired inequality is the minor of $M_{d(\ell), \ell}(x^j)$ indexed by the exponent vectors $a_1$ and $a_2$.

Type 2 comes from tropicalizing the diagonal entries of the localizing matrices $M_{d(\ell, 0), \ell}(x^j g_i)$, which give the inequalities $h(a) \geq h(b)$ for $a - b \in trop(S)^\vee \cap \mathbb{Z}_{\geq 0}^n$. More precisely, we can write $a - b$ as a nonnegative integer combination of the exponent vectors $v_i = a_i - b_i$ of the binomials defining $S$ because they generate the semigroup $trop(S)^\vee \cap \mathbb{Z}_{\geq 0}^n$ by assumption. Write $a - b = \sum_j \lambda_j v_j$. Without loss of generality, assume that $\lambda_1$ is not zero so that $a - b = v_1 + \sum_j \lambda_j' v_j$ for $\lambda_1' \in \mathbb{Z}_{\geq 0}^n$. Write the vector $\sum_j \lambda_j' v_j$ as $2a + J$ with $a \in \mathbb{Z}_{\geq 0}^n$ and $J \in \{0, 1\}^n$. The desired inequality is the tropicalization of the diagonal entry of $M_{d(\ell, 0), \ell}(x^j g_i)$ indexed by $a$ where $g_1 = x^{a_1} - x^{b_1}$ is the binomial corresponding to the exponent vector $v_1 = a_1 - b_1$. 

The assumption that the vectors $v_i = a_i - b_i$ coming from the defining binomials of $S$ generate the semigroup $trop(S)^\vee \cap \mathbb{Z}^n$ is not a strong restriction in the sense that we can add redundant binomial inequalities to a chosen inequality description of $S$ so that the assumption is satisfied.

**Proposition 5.8.** Let $S \subset \mathbb{R}^n_{\geq 0}$ be a semi-algebraic set defined by pure binomial inequalities $g_i = x^{a_i} - x^{b_i}$ with the property that $S \subset \mathbb{R}^n \cap \mathbb{Z}_{\geq 0}^n$. By adding valid binomial inequalities to the description of $S$ as a semi-algebraic set, we can assume that the exponent vectors $a_i - b_i$ of the binomials generate the semigroup $trop(S)^\vee \cap \mathbb{Z}^n$.

**Proof.** By Lemma 2.11, the set $S$ has the Hadamard property so that $trop(S)$ is a convex cone. Since the polynomials $g_i = x^{a_i} - x^{b_i}$ are pure binomials, the vector $v_i = a_i - b_i$ associated to the exponents is in $trop(S)^\vee$, which shows $\text{cone}(v_1, \ldots, v_r) \subset trop(S)^\vee$. For the reverse inclusion, suppose that $trop(S)^\vee$ is strictly larger. Dually, this means that $\text{cone}(v_1, \ldots, v_r)^\vee$ is strictly larger than $trop(S)$. Since the linear inequality $\langle v_i, y \rangle \geq 0$ exponentiates exactly to $x^{a_i} - x^{b_i} \geq 0$ for $x_i = \exp(y_i)$, this shows that $\text{cone}(v_1, \ldots, v_r)^\vee = trop(S)$, a contradiction (see Proposition 2.9).
This implies that every element of a generating set for the semigroup $\text{trop}(S)^\vee \cap \mathbb{Z}^n$ is a conic combination of $v_1, \ldots, v_r$. By exponentiating, such inequalities are redundant for $S$ and we can therefore add them all to the inequality description.

Now fix a finite set $A \subset \mathbb{Z}^n_{\geq 0}$. Write $\mathbb{R}[x_1, x_2, \ldots, x_n]_A$ for the vector space of polynomials whose support is contained in $A$. By intersecting convex cones of polynomials with $\mathbb{R}[x_1, x_2, \ldots, x_n]_A$, we see that, up to changing degree bounds, tropicalizations of dual cones to quadratic modules and preorders behave in the same way.

**Lemma 5.9.** Let $S \subset \mathbb{R}^n$ be a full-dimensional semi-algebraic set defined by the inequalities $x_i \geq 0$ for $i \geq 0$ and pure binomial inequalities $g_i = x^{a_i} - x^{b_i} \geq 0$ ($i = 1, 2, \ldots, r$) with $a_i, b_i \in \mathbb{Z}^n_{\geq 0}$. Let $A \subset \mathbb{Z}^n_{\geq 0}$ be fixed and for $d \geq \max\{|\gamma| : \gamma \in A\}$, denote by $\pi_A$ the projection $\pi_A : \mathbb{R}[x_1, x_2, \ldots, x_n]_d^* \to \mathbb{R}[x_1, x_2, \ldots, x_n]_d^*$ given by restriction. Consider the cones

$$Q_d = \text{QM}_d(g_1, \ldots, g_r) \quad \text{and} \quad P_d = \text{PO}_d(g_1, \ldots, g_r).$$

1. The tropicalization of $P_d^\vee$ is contained in the tropicalization of $Q_d^\vee$ after projection, i.e., $\pi_A(\text{trop}(Q_d^\vee)) \supseteq \pi_A(\text{trop}(P_d^\vee))$.
2. For sufficiently large $D \in \mathbb{N}$, the tropicalization of $Q_D^\vee$ is contained in the tropicalization of $P_d^\vee$ after projection, i.e., $\pi_A(\text{trop}(Q_D^\vee)) \subseteq \pi_A(\text{trop}(P_d^\vee))$.

**Proof.** Part (1) is immediate: Since $Q_d \subseteq P_d$ and convex duality reverses inclusion, we have $Q_d^\vee \supseteq P_d^\vee$. Taking tropicalization preserves these inclusions, which implies the claim.

For Part(2), let $D \geq n + 2d + 2e$ where $e = \deg(\prod_{i \in [r]} g_i)$. By Lemma 5.6, $\text{trop}(Q_D^\vee)$ is a full dimensional convex cone. Therefore it suffices to show that $\pi_A(y)$ is in $\pi_A(\text{trop}(P_d^\vee))$ for every interior point $y = \text{trop}(Q_D^\vee)$.

Consider a point $y = (y(\gamma))_{|\gamma| \leq D}$ in the interior of $\text{trop}(Q_D^\vee)$. Then the tropicalization of the diagonal and principal $2 \times 2$ minors of the localizing matrices $M_{k,\ell}(x^J g_i)$ are strictly positive at $y$ for $2k + \deg(x^J g_i) \leq D$. In particular,

$$y(a_i + J + 2\gamma) > y(b_i + J + 2\gamma)$$

and

$$y(J + 2\gamma) + y(J + 2\delta) > 2y(J + \gamma + \delta)$$

for all $J \subseteq [n]$, $i = 1, \ldots, r$, and $|\gamma|, |\delta| \leq k$. We can write any $\delta \in \mathbb{Z}^n_{\geq 0}$ in the form $\delta = J + 2\gamma$ for some $J \subseteq [n]$ and $\gamma \in \mathbb{Z}^n_{\geq 0}$, giving that $y(a_i + \delta) > y(b_i + \delta)$ for all $|\delta| \leq D - \deg(g_i)$, and so in particular all $|\delta| \leq D - e$.

Let $\ell \in \mathbb{R}[x_1, x_2, \ldots, x_n]_d^*$ be defined by $\ell(x^\gamma) = tu^{\langle \gamma \rangle}$, where we will choose $t \in \mathbb{R}_{>0}$ sufficiently large. Now consider the localizing matrix $M_{m,\ell}(x^J g^I)$ for some $I \subseteq [r]$ where $2m + \deg(x^J g^I) \leq d$.

By linearity, we can expand $g^I$ and write

$$M_{m,\ell}(x^J g^I) = M_{m,\ell}(x^{J+\sum_{i \in T} a_i}) + \sum_{\emptyset \neq T \subseteq I} (-1)^{|T|} M_{m,\ell}(x^{J+\sum_{i \in I \setminus T} a_i + \sum_{j \in T} b_j}).$$

We claim that this is a polynomial matrix of the form required for Lemma 2.15. The entries in each of the above matrices are variables of the form $\ell(x^\gamma)$ for $|\gamma| \leq m$. First, we check that $T = \emptyset$ maximizes the tropicalization of the diagonal entries in the above sum. For all $T \neq \emptyset$ and all $|\gamma| \leq m$, we have

$$y \left( J + 2\gamma + \sum_{i \in T} a_i \right) > y \left( J + 2\gamma + \sum_{i \in I \setminus T} a_i + \sum_{j \in T} b_j \right)$$

This follows from the inequalities $y(a_i + \delta) > y(b_i + \delta)$ for all $|\delta| \leq n + 2m + e \leq D - e$ and induction on $|T|$. 


Now fix some \( T \subseteq I \) and let \( \eta \) denote the integer vector \( J + \sum_{i \in I \setminus T} a_i + \sum_{j \in T} b_j \). If \( J' \subseteq [n] \) is the set of indices for which \( \eta \) is odd, then we can rewrite \( \eta = J' + 2\theta \) for some \( \theta \in \mathbb{Z}^n_{\geq 0} \). Consider the principal \( 2 \times 2 \) minor of \( M_{m,\ell}(x^\eta) \) given by \( \gamma \neq \delta \) with \( |\gamma|,|\delta| \leq m \). To check that the tropicalization of this minor is positive at \( y \), we need to show that

\[
2y(\eta + \gamma + \delta) < y(\eta + 2\gamma) + y(\eta + 2\delta).
\]

Indeed, by using the \( 2 \times 2 \) minor of \( M_{d,\ell}(x^{J'}) \) corresponding to \( (\theta + \gamma, \theta + \delta) \), we see that

\[
2y(\eta + \gamma + \delta) = 2y(J' + 2\theta + \gamma + \delta) < y(J' + 2(\theta + \gamma)) + y(J' + 2(\theta + \delta))
= y(\eta + 2\gamma) + y(\eta + 2\delta).
\]

By Lemma 2.15, the matrix \( M_{m,\ell}(x^J y^I) \) is positive definite for sufficiently large \( t \). We can take \( t \) large enough so that this holds for all \( J \subseteq [n] \) and \( I \subseteq [\ell] \), giving \( \ell \in P_d^r \), and \( y \in \text{trop}(P_d^\ell) \).

5.2. The cube. As an application, we consider the \( 0/1 \)-cube \( [0,1]^n \subset \mathbb{R}^n \) with its natural inequality description \( x_i \geq 0 \) and \( g_i = 1 - x_i \geq 0 \) for \( i = 1,2,\ldots,n \). We apply the results of the previous section to show not only that the tropicalizations of the dual cones of the truncated preorder and quadratic module agree in sufficiently high degree but we give a nice combinatorial description of when this happens.

**Lemma 5.10.** Let \( a_1,a_2,\ldots,a_n \in \mathbb{N} \) and let \( B = ([0,a_1] \times [0,a_2] \times \ldots \times [0,a_n]) \cap \mathbb{Z}^n \subset \mathbb{R}^n \) be a lattice box. Let \( h: \mathbb{B} \to \mathbb{R} \) be a midpoint convex function that is non-increasing in coordinate directions (i.e. \( h(\alpha + e_i) \leq h(\alpha) \) for all \( \alpha \in B \) such that \( \alpha + e_i \in B \)). Then the extension \( \hat{h}: \mathbb{Z}^n_{\geq 0} \to \mathbb{R} \) that assigns to every \( \alpha \in \mathbb{Z}^n_{\geq 0} \) the value \( h(\gamma) \), where \( \gamma \) is the lattice point in \( B \) that is closest to \( \alpha \), is midpoint convex and non-increasing in coordinate directions.

**Proof.** To prove this claim, define the map

\[
\varphi: \mathbb{Z}_+^n \to B, \quad \varphi(\alpha) = (\min\{a_1,\alpha_1\}, \min\{a_2,\alpha_2\}, \ldots, \min\{a_n,\alpha_n\}).
\]

Then the extension \( \hat{h} \) is the equal to \( h \circ \varphi \). It is straightforward to see that \( \hat{h}(\alpha + e_i) \leq \hat{h}(\alpha) \) because \( \varphi(\alpha + e_i) \) is either equal to \( \alpha \) or \( \alpha + e_i \).

So we only have to show that \( \hat{h}(\alpha) + \hat{h}(\beta) \geq 2\hat{h}(\delta) \) for all \( \alpha,\beta,\delta \in \mathbb{Z}_+^n \) with \( \alpha + \beta = 2\delta \).

We prove this by case distinction and analysis of \( \varphi \). For this, we construct two vectors \( v,w \in \mathbb{Z}_+^n \) such that \( \varphi(\alpha) + v, \varphi(\beta) + w \in B \) and \( \varphi(\alpha) + v + \varphi(\beta) + w = 2\varphi(\delta) \): If \( \alpha_i < a_i \) and \( \beta_i < a_i \), set \( v_i = 0 \) and \( w_i = 0 \). If \( \alpha_i \geq a_i \) and \( \beta_i \geq a_i \), again set \( v_i = 0 \) and \( w_i = 0 \). If \( \alpha_i < a_i \) and \( \beta_i \geq a_i \), and \( \delta_i \geq a_i \), set \( v_i = a_i - \alpha_i \) and \( w_i = 0 \). If \( \alpha_i < a_i \) and \( \beta_i \geq a_i \), and \( \delta_i < a_i \), set \( v_i = \beta_i - a_i \) and \( w_i = 0 \). Symmetrically, if \( \alpha_i \geq a_i \), \( \beta_i < a_i \), and \( \delta_i \geq a_i \), set \( v_i = 0 \) and \( w_i = a_i - \beta_i \). Lastly, if \( \alpha_i \geq a_i \), \( \beta_i < a_i \), and \( \delta_i \geq a_i \), set \( v_i = 0 \) and \( w_i = \alpha_i - a_i \).

With this, we are done, because

\[
\hat{h}(\alpha) + \hat{h}(\beta) = h(\varphi(\alpha)) + h(\varphi(\beta)) \geq h(\varphi(\alpha) + v) + h(\varphi(\beta) + w) \geq 2h(\varphi(\delta)) = 2\hat{h}(\delta),
\]

where we used the midpoint convexity of \( h \) since \( \varphi(\alpha) + v + \varphi(\beta) + w = 2\varphi(\delta) \). \( \square \)

Fix a finite set \( A \subset \mathbb{Z}_+^n \). The cubical hull of \( A \) is the smallest lattice box

\[
B = ([\ell_1,u_1] \times [\ell_2,u_2] \times \ldots \times [\ell_n,u_n]) \cap \mathbb{Z}^n
\]

(with \( \ell_i \leq u_i \in \mathbb{Z}_+^n \)) that contains the set \( A \). We now show stabilization of tropicalizations of pseudo-moment cones on the unit cube.
Theorem 5.11. Let \( Q_m \) be the truncated quadratic module \( QM_m(x_1, x_2, \ldots, x_n, g_1, g_2, \ldots, g_n) \) with \( g_i = 1 - x_i \). A function \( h: A \to \mathbb{R} \) belongs to \( \pi_\Delta(Q_m^n) \) for all \( m \in \mathbb{N} \) if and only if it is non-increasing in coordinate directions and it can be extended to a midpoint convex function on the cubical hull of \( A \).

Proof. Proposition 2.14 tells us that a function \( h: A \to \mathbb{R} \) is in \( \text{trop}(Q_m^n) \) if and only if it is midpoint convex and non-increasing in coordinate directions. Indeed, the localizing matrices \( M_{m,\ell}(g_i) \) are differences of the Hankel matrix and the localizing matrix of a variable, i.e. of \( M_{m,\ell}(1) \) and \( M_{m,\ell}(x_i) \), which play the roles of \( A \) and \( B \) in Proposition 2.14 (Formally, we apply Proposition 2.14 to the block sum of all of these matrices.) The \( 2 \times 2 \) minors of a localizing matrix of a monomial tropicalize to midpoint convexity conditions. The tropicalization of the diagonal entries of \( M_{m,\ell}(x^j g_i) \) give the condition that the function \( h \) is non-increasing when moving in a coordinate direction.

So if \( h \in \text{trop}(Q_m^n) \) for all \( m \in \mathbb{N} \), then it is mid-point convex and non-increasing in coordinate directions on \( m\Delta_{n-1} \), which contains the cubical hull of \( A \) for sufficiently large \( m \). This shows the first implication.

For the other direction, we explicitly describe in Lemma 5.10 an extension from the cubical hull to \( \mathbb{Z}^n_{\geq 0} \) that is midpoint-convex. The appropriate restriction of this extension is in \( \text{trop}(Q_m^n) \) for all \( m \in \mathbb{N} \). □

Example 5.12. The cubical hull of the Motzkin configuration \( A = \{(0, 0), (1, 1), (2, 1), (1, 2)\} \) is the square \( \{0, 1, 2\}^2 \). The inequalities defining the convex cone of mid-point convex functions \( h: \{0, 1, 2\}^2 \to \mathbb{R} \) with \( h(\alpha) = y_\alpha \) for \( \alpha \leq \beta \) (in the partial ordering on \( \mathbb{Z}^2 \) given by entry-wise comparing the vectors) and \( y_{20} + y_{2\beta} \geq 2y_{\alpha+\beta} \) for all \( \alpha, \beta \in \{0, 1\}^2 \). Given values of \( h \) on \( A \), note that \( \min\{y_{12}, y_{21}\} \geq y_{22} \) and \( y_{00} + y_{22} \geq 2y_{11} \). Eliminating \( y_{22} \) gives inequalities on the values of \( h \) on \( A \), namely \( y_{00} + \min\{y_{12}, y_{21}\} \geq 2y_{11} \). We claim that the set of functions \( h: A \to \mathbb{R} \) that can be extended to a mid-point convex function on the cube are given by

\[
y_{11} \geq y_{12}, \quad y_{11} \geq y_{21}, \quad y_{00} + y_{12} \geq 2y_{11}, \quad \text{and} \quad y_{00} + y_{21} \geq 2y_{11}.
\]

The inequality \( y_{00} \geq y_{11} \) is implied by the first and third inequalities. Given a function satisfying these inequalities, we can extend it to \( \{0, 1, 2\}^2 \) by defining the missing function values to be as large as possible given the constraints that \( y_{\alpha} \geq y_{\beta} \). That is, we define \( y_{01}, y_{02}, y_{10}, \) and \( y_{20} \) to be equal to \( y_{00} \) as well as \( y_{22} \) to be the minimum of \( y_{12} \) and \( y_{21} \). □

5.3. Toric Cubes. We now discuss application of the above results to toric cubes. The situation is more complicated than for 0/1 cube, and we are not able to prove that the stabilization phenomenon holds generally. In Section 5.4 we show stabilization in the case that tropicalization of the toric cube is contained in the strictly negative orthant (except for the origin).

Given a full-dimensional toric cube \( S \subseteq [0, 1]^n \) defined by \( x_i \geq 0, \ 1 - x_i \geq 0 \), and pure binomial inequalities \( g_j \geq 0 \) (\( j = 1, 2, \ldots, r \)) that is the image of \( [0, 1]^d \) with \( d \geq n \) under a monomial map \( \varphi \), the map \( \varphi \) induces a map \( Q \) between the lattices \( \mathbb{Z}^n \) and \( \mathbb{Z}^d \). Let \( L \) be the image of this map, which is a sublattice of \( \mathbb{Z}^d \). We consider \( A' = Q(A) \subset L \). We would like to apply our previous results to this point configuration in the lattice \( L \). The problem is that the cones do not get mapped to each other in general. To set this up, we write \( \varphi^*: \mathbb{R}[x_1, \ldots, x_n] \to \mathbb{R}[t_1, \ldots, t_d] \) for the induced map on coordinate rings that maps \( x_i \) to the \( i \)-th coordinate monomial in the map \( \varphi \). The issue with pulling back our results via the transpose of this map is simply that \( \varphi^*QM(g_1, \ldots, g_r) \) is usually strictly contained in the quadratic module of the cube \( [0, 1]^d \).

Example 5.13. Let \( S \) be the subset of \( [0, 1]^2 \) given by the inequalities \( x_2 \geq x_1^2 \) and \( x_2^2 \geq x_2 \). Then \( S \) is the image of \( [0, 1]^2 \) under the map \( \varphi: [0, 1]^2 \to S, \ (t_1, t_2) \mapsto (t_1 t_2, t_1^2 t_2^2) \). The corresponding
map $Q$ of lattices is described by the integer matrix whose columns correspond to the exponent vectors of the monomials in the coordinates of $\varphi$. In this example, we get

$$Q = \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}$$

This is even a unimodular lattice transformation. In particular, we have $L = \mathbb{Z}^2$ here. However, $\varphi^*(x_2 - x_1^2) = t_1^2 t_2^3 (1 - t_1)$ is not in the quadratic module generated by $t_1$, $t_2$, $1 - t_1$, and $1 - t_2$. To see this, suppose it had a representation

$$t_1^2 t_2^3 (1 - t_1) = \sigma_0 + \sigma_1 t_1 + \sigma_2 t_2 + \sigma_3 (1 - t_1) + \sigma_4 (1 - t_2)$$

with sums of squares of polynomials $\sigma_i \in \mathbb{R}[t_1, t_2]$. By setting $t_1 = 0$ in this identity, we get

$$0 = \sigma_0(0, t_2) + \sigma_2(0, t_2)t_2 + \sigma_3(0, t_2) + \sigma_4(0, t_2)(1 - t_2).$$

Since the right hand side is in a pointed quadratic module (of the interval $[0, 1] \subset \mathbb{R}$), every term must be 0 and therefore, every sum of squares must be identically 0 which means that it is divisible by $t_2^2$. By plugging back in, we see that every term on the right hand side in the original identity is divisible by $t_1$. Cancelling one $t_1$ from it and repeating the argument, we get a representation

$$t_2^3 (1 - t_1) = \sigma'_0 + \sigma'_1 t_1 + \sigma'_2 t_2 + \sigma'_3 (1 - t_1) + \sigma'_4 (1 - t_2).$$

Symmetrically, we can go through the same process to see that the identity is divisible by $t_2^3$ and end up with the case

$$t_2(1 - t_1) = s_0 + s_1 t_1 + s_2 t_2 + s_3 (1 - t_1) + s_4 (1 - t_2)$$

for sums of squares $s_i \in \mathbb{R}[t_1, t_2]$. To find a contradiction here, we proceed as before by setting $t_2 = 0$ to conclude that the polynomial $(1 - t_1)$ would have to be in the quadratic module generated by $t_2$, $t_1 t_2$, $(1 - t_1) t_2$, and $(1 - t_2) t_2$. This is impossible because every generator of this quadratic module vanishes for $t_2 = 0$ which means that $(1 - t_1)$ would have to be a sum of squares, contradiction.

We expect that this rather simple example (that could be fixed by considering the preorder of the square $[0, 1]^2$ instead of its quadratic module) has generalizations to higher dimensions (where preorders are not saturated anymore) that cannot be fixed in this simple way.

To understand toric cubes, we go back to Theorem 5.7. Let $\Delta_d \subset \mathbb{Z}^n$ denote the set of nonnegative integer vectors with sum of coordinates at most $d$.

**Proposition 5.14.** Let $Q \in \mathbb{Z}_{\geq 0}^{d \times n}$ be a matrix of exponent vectors in the columns and $C$ the corresponding toric cube. A function $h : A \to \mathbb{R}$ belongs to trop$C$ if and only if for every $d \geq 0$ the function $h$ can be extended to a function $\hat{h} : \Delta_d \to \mathbb{R}$ satisfying the following 3 conditions: midpoint convexity, non-increasing in coordinate directions, and non-increasing in directions $\beta - \alpha$ for extreme rays $\delta$ of $(\text{trop} C)^\vee$ written as $\delta = \alpha - \beta$ with nonnegative integer vectors $\alpha$ and $\beta$ (and appropriate scaling of $\delta$ so that it is integer).

**Proof.** By taking logs, the parametrization of the toric cube becomes a linear map from $\mathbb{R}^d \to \mathbb{R}^n$ whose matrix is $Q^T$. So the tropicalization trop$C$ of $C$ is (up to sign) the image of the nonnegative orthant under the map $Q^T$. Therefore the dual polyhedral cone $(\text{trop} C)^\vee$ is (by general duality) the same as the nonnegative orthant in $\mathbb{R}^d$ intersected with the image of the linear map $Q : \mathbb{R}^n \to \mathbb{R}^d$. The extreme rays of this cone $(\text{trop} C)^\vee$ correspond to the binomial inequalities defining $C$ (as a subset of $[0, 1]^n$, that is in addition to the inequalities $x_i \geq 0$ and $1 - x_i \geq 0$). Applying Theorem 5.7 to the moment matrices in this setup directly results in three types of conditions: midpoint convexity (from the $2 \times 2$ minors of $M(x^J)$; non-increasing in coordinate directions from the diagonals of $M(x^J(1 - x_i))$; and non-increasing in directions $\beta - \alpha$ for extreme rays $\delta$ of $T^\vee$.}
written as $\delta = \alpha - \beta$ with nonnegative integer vectors $\alpha$ and $\beta$ (and appropriate scaling of $\delta$ so that it is integer).

It seems much harder to show stabilization for toric cubes than it was for unit cubes (in Theorem 5.11). We need to show a bound the size of $\Delta_d$ in Theorem 5.7, such that extending to $\Delta_d$ allows us to automatically extend to $\Delta_{d+i}$ for all $i \geq 0$. In the next section we do show stabilization in the case that tropicalization of the toric cube is contained in the negative orthant (except for the origin).

5.4. Stabilization. Let $A$ be a finite subset of $\mathbb{Z}^n_{\geq 0}$ and let $C$ be a pointed rational polyhedral cone in $\mathbb{R}^n$. Define the cone $K$ to be the intersection of the finitely many translations of $-C$ by the points in $A$:

$$K = \cap_{a \in A} (a - C).$$

Recall that a function $h$ from some subset of $\mathbb{Z}^n_{\geq 0}$ to $\mathbb{R}$ is non-decreasing (with respect to $C$) if $a - b \in C$ implies that $h(a) \geq h(b)$.

**Proposition 5.15.** Let $B \subseteq \mathbb{Z}^n_{\geq 0}$ be any finite subset containing $A$. If a function $h : A \to \mathbb{R}$ can be extended to a mid-point convex, non-decreasing function $\hat{h} : B \to \mathbb{R}$ then it has such an extension with $\hat{h}(b) \geq \min_{a \in A} h(a)$ for all $b \in B$ and $\hat{h}(b) = \min_{a \in A} h(a)$ for all $b \in K$.

**Proof.** Suppose that $h$ has some extension that is mid-point convex and non-decreasing function with respect to $C$. The set of mid-point convex, non-decreasing extensions of $h$ is then non-empty. It is also closed under taking point-wise maximum. For each $b \in B$, we choose $\hat{h}_b$ to be an element of this cone as follows. If the value of $\hat{h}(b)$ is bounded from above over all mid-point convex and non-decreasing extensions $\hat{h}$ of $h$, then we choose $\hat{h}_b$ to be any extension achieving this maximum value. If not, then we choose can choose $\hat{h}_b$ to be an extension such that $\hat{h}_b(b) > \min_{a \in A} h(a)$.

Define $\hat{h} : B \to \mathbb{R}$ to be the point-wise maximum of $\hat{h}_b$ over all $b \in B$. This is a mid-point convex, non-decreasing extension of $h$. We claim that $\hat{h}(b) \geq \min_{a \in A} h(a)$ for all $b \in B$.

Let $\lambda$ denote the minimum value of $\hat{h}(b)$ over all $b \in B$ and let $B_\lambda$ denote the set of points achieving this minimum $B_\lambda = \{b \in B : \hat{h}(b) = \lambda\}$. The set $\text{conv}(B_\lambda) - C$ is convex. Let $b \in B_\lambda$ be an element that is an extreme point of this set. Some upper bound on the value of $\hat{h}(b)$ must be tight. If $b \not\in A$, the potential upper bounds have the form $\hat{h}(b) \leq \hat{h}(c)$ where $b - c \in -C$ or $\hat{h}(b) \leq \frac{1}{2}(\hat{h}(a) + \hat{h}(c))$ where $a + c = 2b$. The first inequality implies that $\hat{h}(c) = \lambda$ and thus $c \in B_\lambda$. Since $b = c + (b - c)$, this contradicts the extremality of $b$. Similarly, if the inequality $\hat{h}(b) \leq \frac{1}{2}(\hat{h}(a) + \hat{h}(c))$ is tight, then $\hat{h}(a) = \hat{h}(c) = \lambda$, showing that both $a$ and $c$ belong to $B_\lambda$, and contradicting the extremality of $b$. It follows that $b$ must belong to $A$ and that $\lambda = \min_{a \in A} h(a)$ This shows that for all $b \in B$, $\hat{h}(b) \geq \min_{a \in A} h(a)$.

By definition, for all points in $b \in K$, $\hat{h}(b) \leq \min_{a \in A} h(a)$, showing equality. \qed

**Proposition 5.16.** Suppose that the interior of $-C$ strictly contains the nonnegative orthant, so that the complement of any translate of $K$ in $\mathbb{Z}^n_{\geq 0}$ is finite and define

$$\hat{A} = \{2b - a : a, b \in \mathbb{Z}^n_{\geq 0} \setminus K\}.$$ 

The set $\hat{A}$ is finite and a function $h : A \to \mathbb{R}$ can be extended to a mid-point convex, non-decreasing (with respect to $C$) function $\hat{h} : \mathbb{Z}^n_{\geq 0} \to \mathbb{R}$ if and only if it can be extended to a mid-point convex, non-decreasing function $\hat{h} : \hat{A} \to \mathbb{R}$.

Note that taking $a = b$ shows that $\hat{A}$ contains $\mathbb{Z}^n_{\geq 0} \setminus K$. 

To see that it is non-decreasing, suppose that
\[ \hat{h} : \hat{A} \to \mathbb{R} \]
we can assume that \( \hat{h}(b) \geq \min_{a \in A} h(a) \) for all \( b \in \hat{A} \) and \( \hat{h}(b) = \min_{a \in A} h(a) \)
for all \( b \in K \cap \hat{A} \). We then define an extension \( \hat{h} : \mathbb{Z}_{\geq 0}^n \to \mathbb{R} \) by defining \( \hat{h}(b) = \min_{a \in A} h(a) \)
for all \( b \notin \hat{A} \) and claim that this extension is both mid-point convex, non-decreasing function.

To see that it is non-decreasing, suppose that \( b, c \in \mathbb{Z}_{\geq 0}^n \) with \( c - b \in C \). If \( b \in K \), and \( \hat{h}(b) = \min_{a \in A} h(a) \) is the minimum value taken by \( \hat{h} \) on all of \( \mathbb{Z}_{\geq 0}^n \). So in particular, \( \hat{h}(b) \leq \hat{h}(c) \).

If \( b \notin K \), then there is some \( a \in A \) for which \( b \notin a - C \). This implies that \( c \) also does not belong to \( a - C \), since \( b = c + (b - c) \) and \( b - c \in (-C) \). If neither \( b \) nor \( c \) belong to \( K \), then they both belong to \( \hat{A} \) and \( \hat{h}(b) \leq \hat{h}(c) \) by assumption.

To see that \( \hat{h} \) is mid-point convex, suppose that for some \( a, b, c \in \mathbb{Z}_{\geq 0}^n \), \( a + c = 2b \). Note that
\[ \hat{h}(b) \leq \hat{h}(a) + \hat{h}(c) \]
is immediately satisfied. If \( a \) and \( c \) belong to \( K \), then by convexity, so does \( b \). Otherwise, \( b \) and at least one of \( \{a, c\} \) belong to \( \mathbb{Z}_{\geq 0}^n \backslash K \).
Without loss of generality, suppose \( a, b \in \mathbb{Z}_{\geq 0}^n \backslash K \). Then, by construction, \( c = 2b - a \in \hat{A} \), giving that \( \hat{h}(b) \leq \hat{h}(a) + \hat{h}(c) \).

\[ \square \]

Example 5.17. Consider the exponents \( A = \{(0,0),(1,0),(0,1),(1,1)\} \) and the semialgebraic set
\[ S = \{(x_1,x_2) \in \mathbb{R}_{\geq 0}^2 : x^2 < y < x^{1/2}\} \]. Then trop\((S)\) is a cone with extreme rays \((-2,-1), (-1,-2)\) and \(-C = -\text{trop}(S)^\vee\) is a convex cone with extreme rays \((-1,2)\) and \((2,-1)\), which strictly contains the positive orthant. Following the construction above, we take \( K = \cap_{a \in A} (a - C) \) to be the intersection of all translations of \(-C\) by the points of \( A \). In this case, this is just the single translate, \((1,1) - C\). There are only finitely many points in \((\mathbb{Z}_{\geq 0}^2 \backslash K) \cup A\), namely \( \{(0,0),(1,0),(2,0),(0,1),(1,1),(0,2)\} \). We then take \( \hat{A} = \{2b - a : a, b \in (\mathbb{Z}_{\geq 0}^2 \backslash K) \cup A\} \cap \mathbb{Z}_{\geq 0}^2 \), which gives
\[ \hat{A} = \{(0,0),(1,0),(2,0),(3,0),(4,0),(0,1),(1,1),(2,1),(0,2),(1,2),(0,3),(0,4)\} \].

By Proposition 5.16, a function \( h : A \to \mathbb{R} \) can be extended to a function \( \hat{h} : \mathbb{Z}_{\geq 0}^n \to \mathbb{R} \) that is mid-point convex and non-decreasing function with respect to \( C = \text{trop}(S)^\vee\) if and only if it can be extended to a mid-point convex, non-decreasing function \( \hat{h} : \hat{A} \to \mathbb{R} \). A computation shows that this set of functions, with is cut out by the four inequalities
\[ h(1,0) \geq h(1,1), \quad h(0,0) + h(1,0) \geq 2h(0,1), \quad 2h(0,0) + h(1,1) \geq 3h(0,0), \]
\[ h(0,1) \geq h(1,1), \quad h(0,0) + h(0,1) \geq 2h(1,0), \quad 2h(0,0) + h(1,1) \geq 3h(0,1). \]
The last inequality, for example, is a convex combination of the inequalities \( \hat{h}(1,1) \geq \hat{h}(0,3) \), \( \hat{h}(0,1) + \hat{h}(0,3) \geq 2\hat{h}(0,2) \) and \( \hat{h}(0,0) + \hat{h}(0,2) \geq 2\hat{h}(0,1) \) on the set of non-decreasing, midpoint convex functions \( h : \hat{A} \to \mathbb{R} \). The extension to a function \( \hat{h} : \mathbb{Z}_{\geq 0}^2 \to \mathbb{R} \) is given by \( \hat{h}(\alpha) = h(1,1) \) for all \( \alpha \notin \hat{A} \).

\[ \diamond \]

Example 5.18. For a less symmetric example, consider the Motzkin configuration of moments, \( A = \{(0,0),(1,1),(1,2),(2,1)\} \). over the semialgebraic set \( S = \{(x,y) \in \mathbb{R}_{\geq 0}^2 : y^2 \geq x \geq y^3\} \). The cone \(-C = -\text{trop}(S)^\vee\) has extreme rays spanned by \((1,-2)\) and \((-1,3)\). In this case, the intersection \( K \) of all translates of \(-C\) by the points in \( A \) equals the translate \((2,1) - C\).

The set \((\mathbb{Z}_{\geq 0}^2 \backslash K) \cup A\) is finite and given by \( \{(0,j) : 0 \leq j \leq 6\} \cup \{(1,j) : 0 \leq j \leq 3\} \cup \{(2,0),(2,1)\} \).
To construct the set \( \hat{A} \), we take the set of all points completing a mid-point triple with two out of three points coming from this set, giving
\[ \hat{A} = \{(0,j) : 0 \leq j \leq 12\} \cup \{(i,j) : i \in \{1,2\}, 0 \leq j \leq 6\} \cup \{(i,j) : i \in \{2,4\}, 0 \leq j \leq 2\} \].
We project the 33-dimensional convex cone of functions $\hat{h}: |\hat{A}| \to \mathbb{R}$ that are midpoint convex and non-decreasing with respect to $C = \text{trop}(S)^\vee$ onto the function values in $A$. This gives a four-dimensional cone defined by the inequalities
\[
h(0, 0) + 3h(1, 2) \geq 4h(1, 1), \quad 4h(0, 0) + h(1, 2) + 5h(2, 1) \geq 10h(1, 1), \\
2h(1, 1) + h(2, 1) \geq 3h(1, 2), \quad \text{and } h(1, 2) \geq h(2, 1).
\]
For example, the second inequality is a convex combination of the inequalities $\hat{h}(1, 2) \geq \hat{h}(0, 5), \quad 4\hat{h}(0, 0) + \hat{h}(0, 5) \geq 5\hat{h}(0, 1)$ and $\hat{h}(0, 1) + \hat{h}(2, 1) \geq 2\hat{h}(1, 1)$ on the function values of $\hat{h}: \hat{A} \to \mathbb{R}$. This four-dimensional cone has a one-dimensional lineality space spanned by $(1, 1, 1, 1)$. In $\mathbb{R}^4/\mathbb{R}(1, 1, 1, 1)$ it is a pointed cone over a quadrilateral with extreme rays $3\hat{h}(1, 1) = 33 > 32 = h(0, 0) + h(1, 2) + h(2, 1)$.

\[\diamond\]

6. Further Directions

To conclude, we highlight some open questions.

One of the main questions left open in the theory developed above is whether for an arbitrary set $S \subseteq \mathbb{R}^n_{\geq 0}$ defined by binomial inequalities and a set $A \subseteq \mathbb{Z}^n$, the tropicalizations of the $A$-pseudo-moments of $S$ stabilize as the degree bounds increase. Here is a precise version of this question with the same truncation that we used in Section 5.

**Question 6.1.** Let $S \subseteq \mathbb{R}^n_{\geq 0}$ defined by pure binomial inequalities $g_i = x^{a_i} - x^{b_i}$ as in Theorem 5.7. For any arbitrary finite set $A \subseteq \mathbb{Z}^n_{\geq 0}$, does there exist $D_A \in \mathbb{Z}^n_{\geq 0}$ such that for all $d \geq D_A$,
\[
\pi_A(\text{trop}(Q_{D_A}^M)) = \pi_A(\text{trop}(Q_d^A))?
\]
Here $Q_d$ denotes $\text{QM}_d(g_1, \ldots, g_r)$. The question also makes sense, of course, for other ways of bounding the degrees of the sum-of-squares multipliers.

For $S = [0, 1]^n$, this is a consequence of Theorem 5.11. Other sufficient conditions are given Section 5.3. We note that by Theorem 5.7 this can be phrased as a question purely about polyhedral combinatorics. In the language of Section 2, Question 6.1 can be restated as follows:

**Question 6.2.** Let $C$ be a rational polyhedral cone. Given an arbitrary finite set $A \subseteq \mathbb{Z}^n_{\geq 0}$, does there exist a finite set $E \subseteq \mathbb{Z}^n_{\geq 0}$ containing $A$ so that for all $E' \supseteq E$,
\[
\pi_A(\mathcal{M}_{E,C}) = \pi_A(\mathcal{M}_{E',C})?
\]
In the questions above, the set $S$ is fixed and the degree bounds change. Another type of stabilization one might consider comes from changing the set $S$.

**Example 6.3.** Consider the Motzkin configuration $A = \{(0, 0), (1, 2), (2, 1), (1, 1)\}$ and the sets $S_n = \{(x, y) \in \mathbb{R}^2_+: y^{n/(n+1)} \leq x \leq y^{(n+1)/n}\}$. The set $S_n$ is symmetric in $x$, $y$ and approaches the line segment between $(0, 0)$ and $(1, 1)$ as $n \to \infty$. The cone $C_n = -\text{trop}(S_n)^\vee$ is spanned by the vectors $(-n, n + 1)$ and $(n + 1, -n)$.

For any $d \geq 4$ and $n \geq 2$, we find the following inequalities on functions $h: \Delta_d \to \mathbb{R}$ in $\mathcal{M}_{\Delta_d, C_n}$:
\[
\frac{3}{4}h(0, 0) + \frac{5}{4}h(2, 1) \geq 2h(1, 1), \quad \frac{7}{4}h(2, 1) + \frac{1}{4}h(0, 0) \geq 2h(1, 2), \quad h(2, 1) + h(1, 1) \geq 2h(1, 2)
\]
as well as their images under the reflection \( h(i, j) \mapsto h(j, i) \). The first comes from the midpoint inequality on the triple \((0, 1), (1, 1), (2, 1)\) and the inequalities \( h(0, 1) \leq \frac{1}{4}(3h(0, 0) + h(0, 4)) \) and \( h(0, 4) \leq h(2, 1) \). The second and third from from the midpoint inequality on the triple \((0, 3), (1, 2), (2, 1)\) along with the constraints \( h(0, 3) \leq \frac{1}{4}(h(0, 0) + 3h(0, 4)) \leq \frac{1}{4}(h(0, 0) + 3h(2, 1)) \)

and \( h(0, 3) \leq h(1, 1) \), respectively.

In \( \mathbb{R}^4/\mathbb{R}(1, 1, 1, 1) \), these six inequalities define a convex cone over a hexagon with extreme rays

\[
(h(0, 0), h(1, 1), h(1, 2), h(2, 1)) = (1, 0, 0, 0), (8, 3, 0, 0), (8, 3, 1, 0), (8, 3, 0, 1), (8, 2, 1, 0), (8, 2, 0, 1).
\]

One can check that all of these extend to a mid-point convex function \( \tilde{h} : \mathbb{Z}_2^3_0 \to \mathbb{R} \) that is non-decreasing with respect to the cone \( C_n \). Therefore, these points belong to tropicalization of the pseudo-moment cone for any degree bound \( d \geq 4 \). Here we see a different kind of stabilization. For every \( n \geq 2 \), the tropicalization of the \( A \)-pseudomoment cone is the same.

Note that the ray spanned by \((8, 3, 0, 0)\) does not satisfy the AM-GM inequality \( h(0, 0) + h(1, 2) + h(2, 1) \geq 3h(1, 1) \). Since the cones \( \pi_A(\mathcal{M}_{\Delta_4, C_n}) \) stabilize at \( n = 2 \), this inequality cannot be obtained in the limit as \( n \to \infty \), even though it does hold for the \( A \)-pseudomoments of the limit set \( S_{\infty} = \{ (t, t) : t \in [0, 1] \} \).

We contrast this behavior with that of Example 3.18 which again concerns the pseudomoments of the Motzkin configuration \( A = \{(0, 0), (1, 1), (1, 2), (2, 1)\} \) but with the semialgebraic sets \( S_n = \{(x, y) \in \mathbb{R}_2^2_0 : y^n \geq x \geq y^{n+1}\} \) and cones \( C_n = \text{trop}(S_n)^\vee \). While for any finite \( n \), the convexity inequality \( h(0, 0) + h(1, 2) + h(2, 1) \geq 3h(1, 1) \) for functions \( h : A \to \mathbb{R} \) is not implied by mid-point convexity and the non-decreasing condition with respect to \( C_n \), it can be obtained as a limit of such inequalities as \( n \to \infty \). It would be interesting to better understand the distinction between the behavior in these two examples.

**Question 6.4.** For what sequences of semialgebraic sets \((S_n)\) with limit \( S \) can we find all convexity inequalities \( \text{trop}(\mathcal{M}_A(\mathbb{R}^n_{\geq 0})) \) as the limits of inequalities in the tropical pseudomoment cone of \( S_n \)?

Tropicalizations of not necessarily semialgebraic sets with the Hadamard property were considered in [BRST20] and [BR21]. In particular it was shown in [BR21] Lemma 2.2] that the tropicalization of a subset of \( \mathbb{Z}^n_2 \) which has the Hadamard property and is closed under addition is a max-closed convex cone, i.e. it is a convex cone which is additionally closed under tropical addition. More generally, we ask whether the tropicalization of a convex cone is necessarily max-closed.

**Question 6.5.** Let \( C \subset \mathbb{R}^n_{\geq 0} \) be a convex cone. Is it true that \( \text{trop} C \) is max-closed?

Many of the tropicalized sets in this paper are both tropically convex and convex in the classical sense. Max-closed convex cones contained in \( \mathbb{R}^n_{\geq 0} \) were studied in [BR21] and [DMP22]. One way of describing a convex body is by its extreme points, whose convex hull recovers the set. For sets that are convex in both notions, one can ask about the minimal generating set from which one can recover the set. To this end, we define the double-hull of a set \( S \subset \mathbb{R}^n \) to be the tropical convex hull of the convex hull of \( S \).

**Question 6.6.** What are the double-hull extreme rays (that is the minimal generating set for the double-hull operation) of the convex cones \( \mathcal{K}_A \) and \( \mathcal{M}_A \), modulo their lineality spaces?

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