Rings in which all elements are sums or differences of nilpotents and potents

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Abstract. We examine those rings in which the elements are sums or differences of nilpotents and potents (also including in some special cases tripotents). Such decompositions of matrices over certain rings and fields are also studied. These results of ours somewhat support recent achievements presented in a publication due to Abyzov-Tapkin (Siber. Math. J., 2021).

In particular, letting $n$ be an arbitrary natural number, the class of weakly $n$-torsion clean rings is defined and studied. As direct applications of the facts presented above, some characterization theorems describing the structure of these rings up to an isomorphism and their matrix analogs, are established. The obtained results of ours somewhat supply recent achievements presented in a publication due to Danchev-Matczuk (Contemp. Math., 2019).

1. Introduction

Everywhere in the text of the present paper, all rings are assumed to be associative with unity which are not necessarily commutative unless explicitly specified something else. Our terminology and notations are in the most part standard being in agreement with those from [25]. Specifically, for such a ring $R$, $U(R)$ denotes the group of units, $\text{Id}(R)$ the set of idempotents and $J(R)$ the Jacobson radical of $R$, respectively. Besides, the finite field with $m$ elements will be denoted by $\mathbb{F}_m$, and $M_k(R)$ will stand for the $k \times k$ matrix ring over $R; \ k \in \mathbb{N}$. For an element $u$ of a group $G$, $o(u)$ will denote the order of $u$. The symbol $\text{LCM}(n_1, \ldots, n_k)$ will be reserved for the least common multiple of $n_1, \ldots, n_k$.

As usual, an element $d$ of a ring $R$ is termed nilpotent, provided $d^j = 0$ for some integer $j \geq 2$. Moreover, we will say a nil ideal $I$ of $R$ is nil of index $k$ if, for any $r \in I$, we have $r^k = 0$ and $k$ is the minimal natural number with this property. Likewise, we will say that $I$ is nil of bounded index if it is nil of index $k$, for some fixed $k$. Reciprocally, an element $t$ of $R$ is said to be potent if there is a natural number $i > 1$ with the property $t^i = t$. When $i = 2$ the element is called idempotent, whereas when $i = 3$ the element is called tripotent.

On the other vein, let us recall that a ring $R$ is said to be clean if, for every $r \in R$, there are $u \in U(R)$ and $e \in \text{Id}(R)$ with $r = e + u$. If, in addition, the commutativity condition $ue = eu$ is satisfied, the clean ring $R$ is called strongly clean. These rings were introduced by Nicholson in [26] and [27]. Both clean and strongly clean rings as well as their various specializations or generalizations are intensively studied since then (see, for instance, [7], [11], [16], [18], [22] and references within).

Our motivating tool is the following: A decomposition $r = e + u$ of an element $r$ in a ring $R$ will be called $n$-torsion clean decomposition of $r$ if $e \in \text{Id}(R)$ and $u \in U(R)$ is $n$-torsion, i.e., $u^n = 1$. We shall say that such a decomposition of $r$ is strongly $n$-torsion clean if, additionally, $e$ and $u$ commute. In the presence of these notations, we will say that the element $r$ is weakly $n$-torsion clean decomposed if $r = u + e$ or $r = u - e$. If, in addition, $ue = eu$, the element $r$ will be said to have a weakly $n$-torsion clean decomposition with the strong property.

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And so, the purpose of this article is to explore systematically the following new class of rings as well as a few matrix presentations into the sum or the difference of nilpotents and (some special) potent elements (e.g., the tripotent elements).

**Definition 1.1.** A ring $R$ is said to be *weakly $n$-torsion clean (with the strong property)* if there is $n \in \mathbb{N}$ such that every element of $R$ has a weakly $n$-torsion clean decomposition (with the strong property) and $n$ is the minimal possible natural in these two equalities.

Simple calculations show that the next two examples are somewhat surprising: (1) $\mathbb{Z}_3$ is simultaneously weakly 1-torsion clean and 2-torsion clean; (2) $\mathbb{Z}_5$ is simultaneously weakly 2-torsion clean and 4-torsion clean. Some more detailed information about these properties of rings will be given in the sequel.

In the case when the ring $R$ is commutative, we introduce the following additional notions: Let $g(x) = x^n - \sum_{i=0}^{n-1} a_i x^i \in R[x]$ be an unitary polynomial of degree $n \geq 1$. The companion matrix of $g(x) := g$ is the $n \times n$ matrix of the kind:

$$C(g) = \begin{pmatrix}
0 & 0 & \cdots & 0 & a_0 \\
1 & 0 & \cdots & 0 & a_1 \\
0 & 1 & \cdots & 0 & a_2 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 1 & a_{n-1}
\end{pmatrix}.$$

The trace of $g(x)$ is then defined as the trace of $C(g)$, namely $\text{tr}(g) = \text{tr}(C(g)) = a_{n-1}$. Moreover, the spectrum of the square matrix $A$ is denoted by $\text{spec}(A)$ and the block-diagonal matrix with quadratic blocks $A_1, \ldots, A_k$ on diagonals is designed as $A_1 \oplus \ldots \oplus A_k$.

In the present paper we mainly concentrate on the case of weakly $n$-torsion clean rings with the strong properties as well as on the ($n$-torsion clean) matrix presentations over certain fields. Resuming briefly, our further work is organized as follows: In the next section we state and prove some preliminary technicalities as well as we show the construction of some examples which motivated our further work. So, inspired by these statements, we continue with the main section in which we characterize the structure of those rings whose elements can be represented as a sum or a difference of tripotents and nilpotents. In fact, we establish in a series of assertions some structural characterizations of such rings (see, e.g., Theorems 3.1 and Theorem 3.4). Our approach includes some intensive number-theoretic computations as well as some instruments from the area of finite fields (see, for example, Theorems 2.6 and 2.10). Furthermore, as a direct application of these already established results, we try to describe comprehensively the introduced above weakly $n$-torsion clean rings in various ways (see, for instance, Theorems 4.5 and 4.6 as well as Theorems 4.12, 4.13, 4.17 and 4.20). We finish off our current studies with certain commentaries and a few still unsolved problems which immediately arise and which could be of some continuing interest.

### 2. Preliminary Technicalities

We first need some technical claims, which will be freely used in proving our chief statements by beginning with the following well-known assertion:

**Theorem 2.1 ([8, Theorem 2.3]).** Defining integers $B_l$ and $C_l$ as follows

$$B_l = (l - 3)2^{l-1} + 2, \quad l \geq 2,$$

$$C_l = (l + 1)2^{l-1} - 1, \quad l \geq 2,$$

then any prime $p$ such that

$$p^2 > \frac{1}{2} \left( B_l + (B_l^2 + 4C_l)^{1/2} \right),$$

has at least $l$ consecutive quadratic residues and at least $l$ consecutive quadratic non-residues in any complete set of residues (mod $p$).

The next technical claim is pivotal for our further considerations.
Lemma 2.2. Let $p$ be a prime integer, $\mathbb{F}_p$ the finite field of cardinality $p$ and $n \in \mathbb{N}$. Then the following two conditions are equivalent:

1. Every element of $\mathbb{F}_p$ is presentable as a sum of an $n$-potent and a tripotent;
2. If $p \in \{3, 5, 7\}$, then $\frac{p-1}{2} \mid n-1$; otherwise, $(p-1) \mid n-1$.

Proof. (1) $\Rightarrow$ (2). Given any element in $\mathbb{F}_p$ expressed as the sum of an $n$-potent and a tripotent. First, one readily sees that the set of $n$-potent elements coincides with the set of $m$-potent elements for some natural number $m$ having the property $(m-1) \mid (p-1)$. In this case, the relation $(m-1) \mid (n-1)$ is fulfilled, too.

If $m-1 = 1$, i.e., $m = 2$, then each element of $\mathbb{F}_p$ represents as a sum of an idempotent and a tripotent, that is, $\mathbb{F}_p = \{-1, 0, 1, 2\}$. Consequently, either $p = 2$ or $p = 3$. Thus $(p-1) \mid n-1$ when $p = 2$, and $(p-1) \mid n-1$ when $p = 3$.

Let us now $m-1 > 1$ and $A := \{a \in \mathbb{F}_p \mid a^m = 1\}$. Since $\mathbb{F}_p$ contains only the tripotents $\{0, \pm 1\}$, one deduces that $\mathbb{F}_p = A \cup (A-1) \cup (A+1) \cup \{-1\}$ (because $0 \in (A-1)$), yielding that $3(m-1) \geq p-1$. In this case the equality $3(m-1) = q-1$ is impossible. In fact, as for otherwise, we will have that the union $\mathbb{F}_p = A \cup (A-1) \cup (A+1) \cup \{-1\}$ has to be disjoint. Furthermore, if $m-1$ is even, then $-1 \in A$. However, if $m-1$ is odd, then $p = 3(m-1) + 1$ should be even, but $p$ is a prime and, by assumption, $m-1 > 1$, which is the desired contradiction.

So, one concludes that $(m-1) \mid (p-1)$ and $3(m-1) < p-1$. Therefore, either $m = p$, or $p$ is odd and $m = \frac{p+1}{2}$. We shall show that the last equality is possible only when $p \in \{3, 5, 7\}$.

To that goal, assume $m = \frac{p+1}{2}$. Hence every invertible $m$-potent is a quadratic residue modulo $p$. That is why, the representation of all elements of $\mathbb{F}_p$ as a sum of an $m$-potent and a tripotent is equivalent to the fact that, modulo $p$, do there not exist three consecutive quadratic residues. Now, the application of Theorem 2.1 gives the estimation of such primes $p$, namely if $l = 3$, then $B_3 = 2$, $C_3 = 15$ and $p^2 \leq \frac{1}{2} \left(2 + (2^2 + 4 \cdot 15)^\frac{1}{2}\right) = 5$, which implies that $p \leq 25$.

However, a direct inspection shows that:

(a) in the field $\mathbb{F}_3$ all elements are sums of two idempotents;
(b) in the field $\mathbb{F}_5$ all elements are sums of two tripotents;
(c) in the field $\mathbb{F}_7$ all elements are sums of a 4-potent and a tripotent;
(d) in the field $\mathbb{F}_{11}$ the elements $\{6, 7, 8\}$ are quadratic residues;
(e) in the field $\mathbb{F}_{13}$ the elements $\{6, 7, 8\}$ are quadratic residues;
(f) in the field $\mathbb{F}_{17}$ the elements $\{5, 6, 7\}$ are quadratic residues;
(g) in the field $\mathbb{F}_{19}$ the elements $\{12, 13, 14\}$ are quadratic residues;
(h) in the field $\mathbb{F}_{23}$ the elements $\{19, 20, 21\}$ are quadratic residues.

These calculations ensure our claim finally.

(2) $\Rightarrow$ (1). It is pretty clear that any element in the field $\mathbb{F}_p$ is a $p$-potent, whence it is trivially expressible as a sum of a $p$-potent and the zero tripotent. In this vein, in the field $\mathbb{F}_3$ each element is the sum of a tripotent and an idempotent, in the field $\mathbb{F}_5$ each element is the sum of two tripotents, and in the field $\mathbb{F}_7$ each element is the sum of a tripotent and a 4-potent.  

Theorem 2.3 (14 Theorem 1). The finite field $\mathbb{F}_q$ contains three consecutive primitive elements for all odd $q > 169$. Indeed, the only fields $\mathbb{F}_q$ (with $q$ odd) that do not contain three consecutive primitive elements are those for which $q = 3, 5, 7, 9, 13, 25, 29, 61, 81, 121, 169$.

With this in hand, we can now improve on Lemma 2.2 as follows:

Lemma 2.4. Suppose $q > 1$ is an integer, $\mathbb{F}_q$ is a finite field and $n \in \mathbb{N}$. Then the following two points are equivalent:

1. Every element of $\mathbb{F}_q$ can be presented as a sum of an $n$-potent and a tripotent;
2. If $q \in \{3, 5, 7, 9\}$, then $\frac{q-1}{2} \mid n-1$; otherwise, $(q-1) \mid n-1$.

Proof. (1) $\Rightarrow$ (2). Given $\mathbb{F}_q$ is a field of characteristic $p$. The case where $q = p$ is already settled in Lemma 2.2. Besides, in the case of even characteristic $p$ our claim follows directly from 3 Lemma 4. Assume now that $q > p$, $p$ is odd and any element in $\mathbb{F}_q$ can be expressed as a sum of an $n$-potent and an
idempotent. As already noted above, the set of all \( n \)-potents coincides with the set of \( m \)-potents for some integer \( m \) possessing the property \( (m - 1) \mid (q - 1) \). In that case, the relation \( (m - 1) \mid (n - 1) \) is valid as well.

As argued in the proof of Lemma 2.2 one infers that either \( m - 1 = q - 1 \), i.e., \( m = q \), or \( q \) is odd and \( m - 1 = \frac{q - 1}{2} \). We shall demonstrate now that the last case where \( m - 1 = \frac{q - 1}{2} \) is possible only when \( q = 9 \).

To that purpose, according to Theorem 2.3 provided \( q > 169 \) in the field \( \mathbb{F}_q \) there are three consecutive primitive elements (which are, obviously, square free). Consequently, provided \( q > 169 \) is odd, in the field \( \mathbb{F}_q \) there will exist elements which cannot be presented as a sum of a tripotent and a \( \left( \frac{q + 1}{2} \right) \)-potent. In the remaining case of fields, it could be derived by a direct check that only the fields \( \mathbb{F}_9, \mathbb{F}_{25}, \mathbb{F}_{81}, \mathbb{F}_{121} \) and \( \mathbb{F}_{169} \) do not contain three consecutive elements. In what follows, we shall interpret the field \( \mathbb{F}_q \) as the factor-ring \( \mathbb{F}_p[x]/(g(x)) \). Under this treating, the image of the element \( x \) will be assumed to be \( \xi \). So, we obtain that:

1. \( \mathbb{F}_{25} \cong \mathbb{F}_5[x]/(x^2 + x + 2) \). No one of the elements \( \xi^7 = 2\xi, \xi^{21} = 2\xi + 1, \xi^{23} = 2\xi + 2 \) is a 13-potent;
2. \( \mathbb{F}_{81} \cong \mathbb{F}_9[x]/(x^4 + x + 2) \). No one of the elements \( \xi^{25} = \xi^3 + \xi, \xi^{23} = \xi^3 + \xi + 1, \xi^7 = \xi^3 + \xi + 2 \) is a 41-potent;
3. \( \mathbb{F}_{121} \cong \mathbb{F}_{11}[x]/(x^2 + x + 7) \). No one of the elements \( \xi^{117} = 3\xi + 1, \xi^{27} = 3\xi + 2, \xi^{119} = 3\xi + 3 \) is a 61-potent;
4. \( \mathbb{F}_{169} \cong \mathbb{F}_{13}[x]/(x^2 + x + 2) \). No one of the elements \( \xi^9 = 9\xi + 7, \xi^{31} = 9\xi + 8, \xi^{11} = 9\xi + 9 \) is a 85-potent.

Therefore, in any of the fourth fields considered above, there is an element which is unrepresentable as a sum of a tripotent and a \( \left( \frac{q + 1}{2} \right) \)-potent. On the other hand, a direct verification illustrates that in the field \( \mathbb{F}_9 \cong \mathbb{F}_3[x]/(x^2 + x + 2) \) the 5-potent elements are only these: 0, 1, 2, 2\( \xi \) and \( \xi + 2 \). Thus, it is quite evident that each element in \( \mathbb{F}_9 \) can be written as a sum of a tripotent and a 5-potent, as required.

\[ (2) \Rightarrow (1). \] It follows immediately from Lemma 2.2.

Recall that a non-zero ring is said to be an integral ring or, in other words, an integral domain, provided that it is commutative and also does not possess non-trivial zero divisors (i.e., the product of any two non-zero elements is again non-zero).

We now continue with results concerning the full matrix ring. Concretely, the following surprising decomposable result is true:

**Lemma 2.5.** Let \( q > 1 \) be an integer and let \( R \) be an integral ring. If, for some \( n \in \mathbb{N} \), each matrix in \( M_n(R) \) is representable as a sum of a tripotent and a \( q \)-potent, then \( R \) is a finite field and

1. If \( |R| \in \{3, 5, 7, 9\} \), then \( \frac{|R| - 1}{2} \mid q - 1 \);
2. If \( |R| \not\in \{3, 5, 7, 9\} \), then \( (|R| - 1) \mid q - 1 \).

**Proof.** Suppose that for some \( n \in \mathbb{N} \) every element from \( M_n(R) \) can be written as a sum of a tripotent and a \( q \)-potent. Choosing \( a \in R \), there exist \( A, B \in M_n(R) \) with the properties \( aI_n = A + B \), \( A^0 = A \) and \( B^q = B \). Denote by \( F \) the field of fractions of \( R \). It is not too difficult to see that, for some invertible matrix \( C \in M_n(F) \), the matrix \( CAC^{-1} \) is upper triangular with elements on the main diagonal equal to either 0 or \( \pm 1 \) only. It now easily follows from the equality \( aI_n = CAC^{-1} + CBC^{-1} \) that either \( a^q = a \) or \( (a \pm 1)^q = (a \pm 1) \). In fact, the matrices \( aI_n \) and \( CAC^{-1} \) are upper triangular, hence the matrix \( CBC^{-1} \) is also upper triangular. Since \( (CAC^{-1})^3 = CAC^{-1} \) and \( (CBC^{-1})^q = CBC^{-1} \), similar equalities hold for the diagonal elements. In particular, the diagonal elements of the matrix \( CAC^{-1} \) are 0, \( \pm 1 \). Thus, comparing the diagonal elements for an element \( a \), one of the elements \( a, a + 1, a - 1 \) is a \( q \)-potent, as claimed. This means that \( R \) is a finite ring, and hence a finite field. Therefore, any element from \( R \) is the sum of a tripotent and a \( q \)-potent. We, finally, can employ Lemma 2.4 to get our claim, as expected.

As an alternative approach for possible new proof of Lemma 2.4 and for some further extensions of the obtained above number-theoretic results, we proceed thus. First of all, we reformulate in an equivalent manner the aforementioned lemma as follows:

**Theorem 2.6.** Let \( q \) be a prime power and \( n \) an integer such that \( 1 < n \leq q \) and \( (n - 1) \mid (q - 1) \). Then every element of \( \mathbb{F}_q \) is a sum of an \( n \)-potent and a tripotent if, and only if, either \( n = q \) or \( q \in \{3, 5, 7, 9\} \) and \( n = \frac{q - 1}{2} \).

We also need the following technicality.
Lemma 2.7 ([8], Theorem 2.3 with \( \ell = 3 \)). Suppose the prime \( p \) exceeds 25. Then there are three consecutive integers \( \gamma - 1, \gamma, \gamma + 1 \) that are each quadratic non-residues modulo \( p \).

The other result concerns primitive elements of a general finite field \( \mathbb{F}_q \), where \( q \) is a prime power. A primitive element is a generator of the (cyclic) multiplicative group. In its statement \( q \) is necessarily a power of an odd prime \( p \) (because, if \( p = 2 \), then \( \gamma - 1 = \gamma + 1 \)).

Theorem 2.8 ([14], Theorem 1). Let \( q \) be an odd prime power. Then the finite field \( \mathbb{F}_q \) contains three consecutive primitive roots \( \gamma - 1, \gamma, \gamma + 1 \) whenever \( q > 169 \). Indeed, the only fields that do not contain three consecutive primitive elements are those for which \( q \in S = \{3, 5, 7, 9, 13, 25, 29, 61, 81, 121, 169\} \).

Of course, if \( q \) is odd then a primitive element \( \gamma \) is a non-square and so is not a 3-potent.

Now, the proof of Theorem 2.8 is rather intricate both theoretically and computationally. Yet, all that is needed in the proof of Theorem 2.8 is the existence of three consecutive non-square elements. To do this, we replace Lemma 2.7 and Theorem 2.8 by a single result which is the principal aim of this section.

Theorem 2.9. Let \( q = p^r \), where \( p \) is an odd prime, and \( N_q \) be the number of triples \( \gamma - 1, \gamma, \gamma + 1 \) of consecutive non-square elements of \( \mathbb{F}_q \). Then

\[
N_q \geq \frac{1}{8}(q - 2\sqrt{q} - 3),
\]

with equality if, and only if, \( p \equiv 3 \mod(4) \) and \( n = 2m \), where \( m \) is odd.

Hence \( N_q \) is positive whenever \( q > 9 \).

In Theorem 2.8 if \( p > 3 \), a run of four consecutive non-squares, \( \gamma - 1, \gamma, \gamma + 1, \gamma + 2 \), say, contributes 2 to the quantity \( N_q \). The theorem is derived from a more general discussion giving the exact value of \( N_q \) in all cases.

Finally, we shall describe a theorem which is the proper equivalent in the theory of potents of Theorem 2.1. Revised proof of Theorem 2.6. We review the proof of Lemma 2.4 alluded to above in the wake of the newly established Theorem 2.8.

For example, if 2042024 = 1429², then \( c = 1673821 \) so that \( c > 0.8195q \) which means \( C \) is a large subset of \( \mathbb{F}_q \). This makes it apparently more likely that all members of \( \mathbb{F}_q \) could be sums of potents and tripotents. Nevertheless, Theorem 2.8 yields the following slightly curious assertion.

Theorem 2.10. Let \( q > 2 \) be a prime power. Then every element of \( \mathbb{F}_q \) is a sum of a potent (i.e., a member of \( C \)) and a tripotent if, and only if, \( q \in S \) as defined in Theorem 2.8.

2.1 Revised proof of Theorem 2.6

First, we deal with the case when \( q \) is a power of 2. In this situation, a tripotent is actually an idempotent and so \( C_3 = C_2 = \{0, 1\} \). Assume \( 1 < n - 1|q - 1 \). Write \( C_n + 1 = \{a + 1 : a \in C_n\} \). Then the elements 0, 1 are both in \( C_n \) and \( C_n + 1 \) and the condition that every member of \( \mathbb{F}_q \) is a sum of an \( n \)-potent and a tripotent is equivalent to

\[
C_n \cup (C_n + 1) = \mathbb{F}_q.
\]

If \( q = 2 \), necessarily \( n = 2 \) and there is nothing to prove. We can therefore suppose \( q \geq 4 \) and \( n - 1 \leq \frac{q - 1}{2} \).

Consequently, \( C_n \cup (C_n + 1) \) has cardinality at most \( 2n - 2 \leq \frac{2q - 2}{3} < q \), so that (2.2) cannot hold.
Now suppose \( q \) is odd. Then the set of tripotents \( C_3 = \{0, 1, -1\} \). Define \( C_n \) as \( \{a + 1 : a \in C_n\} \), respectively. Thus, for any \( n \) with \( n - 1 | q - 1 \), we have \( 0, 1 \in C_n, -1, 0 \in (C_n - 1) \) and \( 1 \in (C_n + 1) \) and the condition that every member of \( F_q \) is the sum of an \( n \)-potent and a tripotent is equivalent to

\[
D := C_n \cup (C_n - 1) \cup (C_n + 1) = F_q.
\]

Now, if \( d \) is the cardinality of \( D \), then \( d \leq 3n - 2 \). Hence, if \( n - 1 \leq \frac{q - 1}{2} \), i.e., \( n \leq \frac{q + 1}{2} \), then \( d \leq 3n < q \) and \( (2.3) \) cannot hold.

Furthermore, we can suppose \( n - 1 = \frac{q - 1}{2} \), and \( C_n \) is the set of squares (including 0) in \( F_q \). Now, by \( (2.3) \), it is not true that every element in \( F_q \) is the sum of an \( n \)-potent and a tripotent if and only if there exists a non-square \( \gamma \in F_q \) such that both \( \gamma + 1 \) and \( \gamma - 1 \) are also non-squares, i.e., \( \gamma \leq 9 \) utilizing Theorem 2.8. This establishes Theorem 2.6, as desired.

### 2.2. Three consecutive non-squares in \( F_q \)

We proceed to the argument which will verify Theorem 2.9 as pursued. Suppose throughout \( q \) is an odd prime and let \( \lambda \) be the quadratic character on \( F_q \). Thus

\[
(2.4) \quad \sum_{\alpha} \lambda(\alpha) = \sum_{\alpha \neq 0} \lambda(\alpha) = 0,
\]

where \( \sum_{\alpha} \) stands for \( \sum_{\alpha \in F_q} \) and \( \sum_{\alpha \neq 0} \) means that \( \alpha = 0 \) is excluded from the sum. We need further evaluations of character sums. The first can be found in [4 Theorem 2.1.2].

**Lemma 2.11.** Suppose \( q \) is an odd prime power. Let \( f(x) = x^2 + bx + c \in F_q[x] \). Assume \( b^2 - 4c \neq 0 \). Then

\[
\sum_{\alpha \in F_q} \lambda(f(\alpha)) = -1.
\]

The next result concerns the Jacobsthal sum \( J(a) = \sum_a \lambda(x^2 + a), a \in F_q \). Recall that

\[
\lambda(-1) = \begin{cases} 1, & \text{if } q \equiv 1 \pmod{4}, \\ -1, & \text{if } q \equiv 3 \pmod{4}. \end{cases}
\]

Hence, by replacing \( \alpha \) by \( -\alpha \) in the expression for \( J(\alpha) \), we see that if \( q \equiv 3 \pmod{4} \), then \( J(\alpha) = 0 \). On the other hand, when \( q \equiv 1 \pmod{4} \), we use the following evaluation from [24 Theorem 2].

**Lemma 2.12** (Katre and Rajwade, 1987). Suppose \( q = p^r \equiv 1 \pmod{4} \), where \( p \) is an odd prime. If \( p \equiv 3 \pmod{4} \) (so that \( r \) is even), let \( s = (-1)^{r/2} \sqrt{q} \). If \( p \equiv 1 \pmod{4} \), define \( s \) uniquely by \( q = s^2 + t^2, p \mid s, s \equiv 1 \pmod{4} \). Then

\[
J(a) = \begin{cases} -2s, & \text{if } a \text{ is a fourth power in } F_q, \\ 2s, & \text{if } a \text{ is a square but not a fourth power in } F_q. \end{cases}
\]

Now, let \( N_q \) be the number of consecutive triples of non-squares \( \gamma - 1, \gamma, \gamma + 1 \in F_q \). Evidently, we have

\[
(2.5) \quad N_q = \frac{1}{8} \sum_{\alpha \neq 0, \pm 1} (1 - \lambda(\alpha))(1 - \lambda(\alpha - 1))(1 - \lambda(\alpha + 1)).
\]

Now, set

\[
S_1 = \sum_{\alpha \neq 0, \pm 1} \lambda(\alpha); \quad S_2 = \sum_{\alpha \neq 0, \pm 1} \lambda(\alpha - 1); \quad S_3 = \sum_{\alpha \neq 0, \pm 1} \lambda(\alpha + 1)
\]

and

\[
T_1 = \sum_{\alpha \neq 0, \pm 1} \lambda(\alpha - 1); \quad T_2 = \sum_{\alpha \neq 0, \pm 1} \lambda(\alpha + 1); \quad T_3 = \sum_{\alpha \neq 0, \pm 1} \lambda(\alpha^2 - 1).
\]

Then

\[
N_q = \frac{1}{8} \left( q - 3 - \sum_{i=1}^{3} S_i + 3 \sum_{i=1}^{3} T_i - J(-1) \right).
\]
From [2.4],
\[
S_1 = \sum_{\alpha} \lambda(\alpha) - 1 - \lambda(-1) = -1 - \lambda(-1); \quad S_2 = -\lambda(-2) - \lambda(-1); \quad S_3 = -1 - \lambda(2),
\]
whereas, from Lemma [2.11]
\[
T_1 = \sum_{\alpha \neq -1} \lambda(\alpha(\alpha - 1)) = -1 - \lambda(2); \quad T_2 = -1 - \lambda(2); \quad T_3 = -1 - \lambda(-1).
\]
Further, \(J(-1) = 0\) if \(q \equiv 3 \pmod{4}\). But, when \(q \equiv 1 \pmod{4}\), by Lemma [2.12] we have
\[
J(-1) = \begin{cases} 
-2s, & \text{if } q \equiv 1 \pmod{8}, \\
2s, & \text{if } q \equiv 5 \pmod{8}.
\end{cases}
\]
We also have the well-known facts that
\[
\lambda(2) = \begin{cases} 
1, & \text{if } q \equiv \pm 1 \pmod{8}, \\
-1, & \text{if } q \equiv \pm 3 \pmod{8}.
\end{cases}
\]
and
\[
\lambda(-2) = \begin{cases} 
1, & \text{if } q \equiv 1 \text{ or } 3 \pmod{8}, \\
-1, & \text{if } q \equiv 5 \text{ or } 7 \pmod{8}.
\end{cases}
\]
We now evaluate \(N_q\) from (2.3) and the various expressions for \(S_i, T_i, J(-1)\). We require to consider five cases.

**Case 1:** If \(q \equiv 7 \pmod{8}\), then \(N_q = \frac{q - 7}{8}\).

**Proof.** Here \(\lambda(-1) = -1, \lambda(2) = 1, \lambda(-2) = 1\). Thus \(S_1 = 0, S_2 = 2, S_3 = -2, T_1 = T_2 = T_3 = 0\) while \(J(-1) = 0\). Hence
\[
8N_q = (q - 3 + 0 - 4) = q - 7.
\]

Small examples of Case 1 include \(N_7 = 0, N_{23} = 2\).

**Case 2:** If \(q \equiv 3 \pmod{8}\), then \(N_q = \frac{q - 3}{8}\).

**Proof.** Now \(\lambda(-1) = -1, \lambda(2) = -1, \lambda(-2) = -1\). Thus \(S_1 = S_2 = S_3 = T_1 = T_2 = T_3 = 0\). Also, \(J(-1) = 0\).

Small examples of Case 2 include \(N_3 = 0, N_{11} = 1, N_{19} = 2\).

**Case 3:** If \(q \equiv 5 \pmod{8}\), then
\[
N_q = \frac{q - 2s - 3}{8},
\]
where \(q = s^2 + t^2, s \equiv 1 \pmod{4}\).

**Proof.** Here \(q = p^r\), where also \(p \equiv 5 \pmod{8}\) and \(r\) is odd. We have \(\lambda(-1) = 1, \lambda(2) = \lambda(-2) = -1\). Hence, \(S_1 = -2, S_2 = S_3 = 0, T_1 = T_2 = 0, T_3 = -2\).

Further, let \(\gamma\) be a primitive element in \(\mathbb{F}_q\). Then \(-1 = \gamma^{\frac{q - 1}{4}}\) is the square of \(\gamma^{\frac{q - 1}{4}}\) but not a fourth power, since \(\frac{q - 1}{4}\) is odd. Hence \(J(-1) = 2s\).

In Case 3, since \(|s| < \sqrt{q}\), then \(N_q > \frac{1}{8}(q - 2\sqrt{q} - 3)\). Small examples of Case 3 include \(N_5 = 0\) (since \(5 = 1^2 + 2^2\), \(N_{13} = 2\) (since \(13 = (-3)^2 + 2^2\)), \(N_{29} = 2\) (since \(29 = 5^2 + 2^2\)).

**Case 4:** If \(q = p^r \equiv 1 \pmod{8}\), where \(p \equiv 1 \pmod{4}\), then
\[
N_q = \frac{q + 2s - 3}{8},
\]
where \(q = s^2 + t^2, s \equiv 1 \pmod{4}\).
Proof. Here $\lambda(-1) = \lambda(2) = \lambda(-2) = 1$. Hence, $S_1 = S_2 = S_3 = T_1 = T_2 = T_3 = -2$. This time $\frac{2-1}{4}$ is even and so $-1$ is a fourth power and $J(-1) = -2s$. □

Small examples of Case 4 include $N_{17} = 2$ (since $17 = 1^2 + 2^2$), $N_{25} = 2$ (since $25 = (-3)^2 + 2^2$), $N_{169} = 22$ (since $169 = 5^2 + 12^2$), $N_{289} = 32$ (since $289 = (-15)^2 + 8^2$).

Case 5: If $q = p^r \equiv 1 \pmod{8}, \text{where } p \equiv 3 \pmod{4},$ then $q$ is a square and

$$N_q = \frac{1}{8} \left( q + (-1)^{r/2} \sqrt{q} - 3 \right).$$

Proof. As in Case 4, each $S_i$ and $T_j$ has the value $-2$. Again $(-1)$ is a fourth power in $F_q$ so that, by Lemma \[2.12\], $J(-1) = -2s = -2(-1)^{r/2} \sqrt{q}$. □

In Case 5, when $r = 2m$ with $m$ odd, then $N_q = \frac{1}{8}(g - 2\sqrt{q} - 3)$. As can be observed from the formulae in every other case $N_q > \frac{1}{8}(q - 2\sqrt{q} - 3)$. Thus, Theorem \[2.9\] follows as a corollary from Cases 1-5.

Small examples of Case 5 include $N_9 = 0, N_{49} = 4, N_{81} = 12$.

2.3. Proof of Theorem \[2.10\] Given a prime power $q$, the set of potents $C$ of $F_q$ is defined by \[2.7\]. (The field $F_2$ is excluded for trivial reasons.)

First, suppose $q$ is odd. Then $0, 1 \in C$ and $-1 \in C - 1$. Hence the property that every member of $F_q$ is the sum of a potent and a tripotent is equivalent to the assertion that

$$C \cup (C - 1) \cup (C + 1) = F_q.$$

Suppose that $q \notin S$ as displayed in Theorem \[2.8\]. Then there exists $\gamma \in F_q$ such that each of $\gamma - 1, \gamma, \gamma + 1$ is a primitive element. Hence, $\gamma$ is not in the left-side of \[2.6\] and, therefore, \[2.7\] does not hold.

On the other hand, if $q \in S$, then by Theorem \[2.8\] for any $\gamma \in F_q$, we have that $\gamma \in (at least one of $C, C - 1, C + 1 = 1$ and so the relation \[2.6\] holds.

Finally, suppose $q > 2$ is even. Then $0, 1 \in (both) C$ and $C + 1$ and the fact that every element $\gamma \in F_q$ is the sum of a potent and a tripotent (= idempotent) is equivalent to an assertion that $C \cup (C + 1) = F_q$. But this cannot hold since it was shown already in \[13\] Theorem 2.1 that $F_q$ necessarily contains consecutive primitive elements, $\gamma, \gamma + 1$.

3. Main Results and Examples

We now have all the ingredients needed to prove the following assertion.

Theorem 3.1. Let $q > 1$ be an odd integer, and $R$ an integral ring which is not isomorphic to $F_3, F_5$ or $F_9$. Then the following seven issues are equivalent:

1. For each (for some) $n \in N$, every matrix in $M_n(R)$ can be presented as a sum of an idempotent matrix and a q-potent matrix;
2. For each (for some) $n \in N$, every matrix in $M_n(R)$ can be presented as a sum of a nilpotent matrix and a q-potent matrix;
3. For each (for some) $n \in N$, every matrix in $M_n(R)$ can be presented as a sum of a tripotent matrix and a q-potent matrix;
4. For each (for some) $n \in N$, every matrix in $M_n(R)$ can be presented as a sum or a difference of a q-potent matrix and an idempotent matrix;
5. Every element in $R$ is the sum of a q-potent and a tripotent;
6. Every element in $R$ is the sum or a difference of a q-potent and an idempotent;
7. $R$ is a finite field such that $(|R| - 1) | q - 1$.

Proof. If the ring $R$ is not isomorphic to $F_3, F_5, F_7$, or $F_9$, then the equivalence (1)-(7) follows from \[11\] Theorem 14] in combination with Lemma \[2.3\]. However, if $R \cong F_7$, then in virtue of Lemma \[2.3\] any element from $F_7$ is the sum of a tripotent and a q-potent, it follows that $(|F_7| - 1) | q - 1$. Therefore, the equivalence of points (1)-(7) in the case where $R \cong F_7$ follows at once from \[11\] Theorem 14], as pursued. □

It was shown in \[11\] Theorem 19, Corollary 20] that in the matrix ring $M_{3k}(F_3)$ there exists a matrix which cannot be presented as a sum of an idempotent and a tripotent. This example can be extended to the following one.
PROPOSITION 3.2. For every natural number $k$, in the ring $\mathbb{M}_3(\mathbb{F}_3)$ there is a matrix that is not representable neither in the sum nor in the difference of a tripotent and an idempotent.

**Proof.** Put $A = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \in \mathbb{M}_3(\mathbb{F}_3)$. Then, a routine check shows that the matrix $B = A \oplus \ldots \oplus A$ satisfies the wanted condition. Actually, referring to [1] Theorem 19, any matrix with a minimal polynomial $m(x) = x^3 - x \pm 1$ cannot be presented as a sum of an idempotent and a tripotent. If we assume for a moment that $B = f - e$, where $f^3 = f$ and $e^2 = e$, then the matrix $(-B)$ is plainly checked to be a sum of an idempotent and a tripotent. But the minimal polynomial of $(-B)$ is $x^3 - x + 1$, which contradicts the aforementioned theorem. \[ \square \]

It is also worthwhile noticing that by virtue of [1] Theorem 14, for every $n \in \mathbb{N}$ there is a matrix from $\mathbb{M}_n(\mathbb{F}_3)$ which is unrepresentable as a sum of an idempotent and a tripotent.

We continue our work with the next statement of some interest and importance to the subject.

**Example 3.3.** The matrix $A = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \in \mathbb{M}_3(\mathbb{F}_3)$ is neither a sum nor a difference of a tripotent and an idempotent.

**Proof.** Put $m(x) = x^3 - x - 1$.

Assume that $A = f + e \varepsilon$ for some $\varepsilon^2 = e$, $f^3 = f$ and $\varepsilon \in \{-1, 1\}$. It is straightforward that $A$, $A-1$ and $A+1$ are not tripotents. Thus, $\varepsilon \neq 0, 1$. Therefore, there exists a unit $C \in \mathbb{M}_3(\mathbb{F}_3)$ such that $CeC^{-1} = I_k \oplus (0)_{n-k}$ for some $1 \leq k \leq 2$. Put $A' = CAC^{-1}$ and $f' = CfC^{-1} = \begin{pmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{pmatrix}$ with $F_{11} \in \mathbb{M}_k(\mathbb{F}_3)$ and $F_{22} \in \mathbb{M}_{n-k}(\mathbb{F}_3)$. We have

$$A' = f' + \varepsilon(I_k \oplus (0)_{n-k}) = \begin{pmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{pmatrix} + \varepsilon \begin{pmatrix} I_k & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} F_{11} + \varepsilon I_k & F_{12} \\ F_{21} & F_{22} + I_{n-k} \end{pmatrix}.$$

Put $(f')^3 = \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix}$. It is clear that $f_{ij} = \sum_{a<b} F_{ia}F_{ab}F_{bj}$. Since $m(A') = 0$, we deduce

$$\begin{pmatrix} F_{11} + (1 + \varepsilon)I_k & F_{12} \\ F_{21} & F_{22} + I_{n-k} \end{pmatrix} = 1 + A' = (A')^3 = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}$$

for some $g_{ij}$. Taking into account that $(f')^3 = (f')$ we obtain the following equalities:

$$F_{11} + (1 + \varepsilon)I_k = g_{11} = (F_{11} + \varepsilon I_k)^3 + (F_{11} + \varepsilon I_k)F_{12}F_{21} + F_{12}F_{21}(F_{11} + \varepsilon I_k) + F_{12}F_{22}F_{21} =$$

$$= \left( \sum_{a<b} F_{ia}F_{ab}F_{bi} \right) + 3F_{11} + \varepsilon (3F_{11}^2 + I_k + 2F_{12}F_{21}) = 4F_{11} + \varepsilon (3F_{11}^2 + I_k + 2F_{12}F_{21}),$$

$$F_{12} = g_{12} = (F_{11} + \varepsilon I_k)^2 F_{12} + (F_{11} + \varepsilon I_k)F_{12}F_{22} + F_{12}F_{21}F_{12} + F_{12}F_{22}F_{22}^2 =$$

$$= \left( \sum_{a<b} F_{ia}F_{ab}F_{bi} \right) + \varepsilon (2F_{11}F_{12} + \varepsilon F_{12} + F_{12}F_{22}) = F_{12} + \varepsilon ((2F_{11} + \varepsilon I_k)F_{12} + F_{12}F_{22}),$$

$$F_{21} = g_{21} = F_{21}(F_{11} + \varepsilon I_k)^2 + F_{21}F_{12}F_{21} + F_{22}F_{21}(F_{11} + \varepsilon I_k) + F_{22}^2 F_{21} =$$

$$= \left( \sum_{a<b} F_{2a}F_{ab}F_{b1} \right) + \varepsilon (F_{21}(2F_{11} + \varepsilon I_k) + F_{22}F_{21}) = F_{21} + \varepsilon (F_{21}(2F_{11} + \varepsilon I_k) + F_{22}F_{21}),$$

$$F_{22} + I_{n-k} = g_{22} = F_{21}(F_{11} + \varepsilon I_k)F_{12} + F_{21}F_{12}F_{22} + F_{22}F_{21}F_{12} + F_{22}^3 =$$

$$= \left( \sum_{a<b} F_{2a}F_{ab}F_{b2} \right) + \varepsilon F_{21}F_{12} = F_{22} + \varepsilon F_{21}F_{12}.$$
This gives us the system of equations of the form

\[
\begin{align*}
F_1^2 + \varepsilon F_{11} &= 2\varepsilon I_k + F_{12} F_{21} \\
(2F_{11} + \varepsilon I_k) F_{12} &= -F_{12} F_{22} \\
F_{21} (2F_{11} + \varepsilon I_k) &= -F_{22} F_{21} \\
F_{21} F_{12} &= \varepsilon I_{n-k}
\end{align*}
\]

It follows that

\[
F_{22}^2 = (-F_{22} F_{21}) (-F_{12} F_{22}) = F_{21} (2F_{11} + \varepsilon I_k)^2 F_{12} = F_{21} (- (F_{11}^2 + \varepsilon F_{11}) + I_k) F_{12} = \\
= F_{21}((1 - 2\varepsilon) I_k - F_{12} F_{21}) F_{12} = (1 - 2\varepsilon) F_{21} F_{12} - (F_{21} F_{12})^2 = -2\varepsilon I_{n-k}.
\]

However, one can verify that neither 2 nor 3 is a square in \(\mathbb{F}_5\). Thus, it must be that \(n - k\) is even and so \(k = 1\). Hence, in this case, the ranks of \(F_{21}\) and \(F_{12}\) do not exceed 1, but on the other hand we have that \(F_{21} F_{12} = I_2\), a contradiction. \(\square\)

We now continue with a new kind of presentations of matrices, namely in the sum of a potent and a nilpotent of fixed order. In fact, in \([19], [20]\) and \([21]\) were intensively studied some matrix decompositions over certain fields and rings into sums of potents and nilpotents of order at most two.

We shall say that \(F\) is a potent field if, for every non-zero \(x \in F\), there exists an integer \(m = m(x) \geq 3\) such that \(x^{m-1} = 1\) (that is, each non-trivial element in \(F\) is a torsion unit).

So, we have now accumulated all the information necessary to establish the following assertion which somewhat expands the main results from \([6]\) and \([20]\) and which is our omnibus for a more detailed further exploration in the topic. Also, the next result could be viewed as a common extension of the corresponding results from \([15]\) and \([5]\), respectively.

**Theorem 3.4.** Let \(F\) be a potent field and \(n \in \mathbb{N}\). Then each non-zero matrix from \(M_n(F)\) can be presented as either

(i) a sum of a potent matrix and a square-zero matrix;
(ii) a sum of a torsion unit and a nilpotent;
(iii) a sum of a torsion unit and an idempotent.

**Proof.** (i) In the case where \(F\) is a finite potent field, we employ \([20]\) Corollary 2.7, Corollary 3.2].

Reciprocally, when \(F\) is an infinite potent field, one concludes that its characteristic must be a prime, say \(p\), as for otherwise we will have that the field \(\mathbb{Q}\) of rational numbers satisfies the inclusion \(\mathbb{Q} \subset F\) and, for example, the number \(2 \in F\), but 2 is not potent, a contradiction. Moreover, since every \(x \in F\) satisfies \(x^{m-1} = x\), one follows that \(x\) has to be an algebraic element over \(\mathbb{Z}_p\). Let us now consider \(A = (a_{ij}) \in M_n(F)\). Then \(A\) can be seen as a matrix over the field \(K\) generated by \(\mathbb{Z}_p\) and all the elements \(a_{ij}\), and since each \(a_{ij}\) appearing in \(A\) is algebraic over \(\mathbb{Z}_p\), the field \(K\) must be finite. Then, as in the first step, we may apply the result for finite fields, thus completing the arguments. Thus, by what we have shown so far, we might claim that even each square matrix over a field of prime characteristic, which is not necessarily potent, is the sum of a potent and a nilpotent of order less than or equal to 2.

(ii) Invoking \([9]\), every non-zero matrix is the sum of a unit and a nilpotent. Moreover, since \(F\) is a potent field, each invertible matrix is indeed a torsion unit, because as already showed above in point (i) it can be embedded into a finite field, so the result follows now directly.

(iii) Referring to \([23]\), every non-zero matrix is the sum of a unit and an idempotent. Besides, as already observed in both points (i) and (ii), the invertible matrix is actually a torsion unit, as expected. \(\square\)

It is worthwhile to notice that the decomposition from (i) does not hold for matrices over fields of characteristic 0 – just take for example \(A = 2I\), so if we assume the contrary that it could be written as \(P + N\) with \(P\) potent and \(N^2 = 0\), then \(P = 2I - N\) will satisfy \((P - 2I)^2 = 0\) and, on the other side, it satisfies \(P^k - P = 0\) for some \(k \in \mathbb{N}\), because \(P\) is potent; but then the minimal polynomial of \(P\) must divide these two polynomials which is manifestly impossible.

It is also worth of noticing that the decomposing of a matrix over a potent field into a (torsion) unit matrix plus a zero-square matrix is, definitely, untrue in general. In fact, suppose \(n \geq 4\) and let \(A = e_{12} \in M_n(F)\), where \(F\) is a potent field. If we, nevertheless, assume that \(A = U + N\), where \(U\) is a unit and \(N^2 = 0\), then
$U = A - N$. But the rank of a unit is always maximal (in this case), the rank of $A$ is one, and the rank of $N$ is $\leq 2$, so we surely cannot recover the rank $n$ from a matrix of rank 1 minus a matrix of rank at most 2.

Similarly, one may consider the matrix $A = e_{21} \in M_3(F)$ for $F$ being any finite field, we can never decompose $A = U + N$ with $U$ a unit and $N^2 = 0$, because $U$ must have rank equal to three and $N$ has rank at most one.

Let us notice also that in [12] were considered more restricted versions of the presentation established in point (iii) above, namely as a sum of a torsion unit of a fixed order and an idempotent.

4. Applications to (Weakly, Strongly) $n$-Torsion Clean Rings

Here we apply the results from the previous section to the variations of $n$-torsion cleanness. We first and foremost start with the following two technicalities.

**Proposition 4.1.** If $R$ is a commutative ring with $4 = 0$, then $R$ is weakly $2^n$-torsion clean if, and only if, $R$ is $2^n$-torsion clean.

**Proof.** One direction being elementary, we concentrate on the other one. If $r = u + e$, we are done, so let us assume that $r = u - e$. Thus, one easily checks that $r = (u - 2e) + e$, where $(u - 2e)^2 = u^2$ which gives that $(u - 2e)^n = u^{2n}$ for all $n \in \mathbb{N}$, as required. □

**Proposition 4.2.** Let $F$ be a field not isomorphic to any of the fields $\mathbb{F}_3$, $\mathbb{F}_5$ or $\mathbb{F}_9$. Then $F$ is weakly $n$-torsion clean if, and only if, $F$ is finite and $n = |F| - 1$.

**Proof.** Let $F$ be a weakly $n$-torsion clean field. Since $F$ contains only the trivial idempotents 0 and 1, it is clear by Lemma [4.3] below that $F$ is finite. Moreover, every element of $F$ is the sum of a $(n+1)$-potent and a tripotent. The utilization of Lemma [2.4] implies that either $|F| - 1 \mid n$ or $|F| \in \{3, 5, 7, 9\}$. But it can easily be checked that each element of a finite field $F$ has a weakly $(|F| - 1)$-torsion clean decomposition. Thus, it is enough to consider only the cases of $\mathbb{F}_3$, $\mathbb{F}_5$, $\mathbb{F}_7$ and $\mathbb{F}_9$.

So, a direct calculation shows that:

1. $\mathbb{F}_3$ is weakly 1-torsion clean;
2. $\mathbb{F}_5$ is weakly 2-torsion clean;
3. $\mathbb{F}_7$ is not a weakly 3-torsion clean, because $6 \in \mathbb{F}_7$ can not be represented as the sum of elements from the sets $\{1, 2, 4\}$ and $\{-1, 0, 1\}$. Therefore, $\mathbb{F}_7$ is 6-torsion clean and the wanted equality holds.
4. Considering $\mathbb{F}_9$ and knowing that $\mathbb{F}_9 \cong \mathbb{F}_3[x]/(x^2 + x + 2)$, we write $\xi$ for the image of $x$. A direct inspection shows that the invertible 4-potents of $\mathbb{F}_9$ are 1, 2, 2$\xi + 1$ and $\xi + 2$, whence it is clear that $\mathbb{F}_9$ is weakly 4-torsion clean.

Conversely, by analogous manipulations as above, it is obvious that every element of a finite field $F$ has a weakly $(|F| - 1)$-torsion clean decomposition, as required. □

The following technical lemma is crucial for our further considerations.

**Lemma 4.3.** Suppose that $R$ is a ring and the element $a \in R$ possesses weakly $n$-torsion clean decomposition with the strong property. Then the equality $(a^n - 1)((a \pm 1)^n - 1) = 0$ holds.

**Proof.** Assuming first that $a = v + e$ is the desired weakly $n$-torsion clean decomposition of $a$ satisfying $ve = ev$, we derive as in [17] that the equation $(a^n - 1)((a - 1)^n - 1) = 0$ is valid.

So, assume now that $a = v - e$, where $v^n = 1$, $e^2 = e$ and $ve = ev$. Hence $ve = (a + 1)e$, so that $v^n = ((a + 1)e)^n = (a + 1)^ne$. But $a^n - 1 = (a^n - 1)e$, that is, $(a^n - 1)(1 - e) = 0$ whence $(a + 1)^n = e = a^n e - a^n + 1$. By plain manipulations, we deduce that $1 = (a + 1)^n - a^n e + a^n$ implying $a^n - 1 = -(a + 1)^n + a^n e$ implying $(a^n - 1)e = a^n e - (a + 1)^ne$ implying $a^n - 1 = (a^n - (a + 1)^n)e$. Consequently, $(a^n - 1)e = (a^n - (a + 1)^n)e$ and so $(a + 1)^n = e = (a + 1)^n - 1 = 0$. Finally, $(a^n - 1)((a + 1)^n - 1) = (a^n - 1)(a + 1)^n - 1 = 0$, as expected. □

The following assertion is useful, but its proof is a slight version of that from [17], so we omit the details leaving them to the interested reader.

**Lemma 4.4.** Let $n \in \mathbb{N}$ and let $R$ be a ring satisfying the identity $(x^n - 1)((x \pm 1)^n - 1) = 0$. Then the following two points hold:
(1) $R$ has finite non-zero characteristic; \\
(2) $J(R)$ is a nil-ideal.

Now we are in a position to establish the following theorem.

**Theorem 4.5.** Let $n \in \mathbb{N}$. Suppose $R$ is a weakly $n$-torsion clean ring having the strong property. Then the following items hold:

(1) $R$ is a PI-ring satisfying the polynomial identity $(x^n - 1)((x \pm 1)^n - 1) = 0;$

(2) $R$ has finite non-zero characteristic;

(3) $J(R)$ is a nil-ideal.

**Proof.** The claim follows at once by combination of Lemmas [13] and [14]. $\square$

Subsuming the assertions alluded to above along with the methods developed in [17], we now arrive at our central statement.

**Theorem 4.6.** For a ring $R$, the following two conditions are equivalent:

(1) There exists $n \in \mathbb{N}$ such that $R$ is a weakly $n$-torsion clean abelian ring;

(2) The ring $R$ is abelian weakly clean such that $U(R)$ is of finite exponent.

By combining the presented above ideas, in some close similarity to [17], we derive the following consequence.

**Corollary 4.7.** For a ring $R$, the following two points are equivalent:

(1) $R$ is weakly $n$-torsion clean with the strong property for some $n \in \mathbb{N}$;

(2) $R$ is weakly clean with the strong property and $U(R)$ is of finite exponent.

Furthermore, taking into account Lemma [2,3] or Proposition [4,2], one sees that all (weakly) $n$-torsion clean fields have to be finite. In this direction, [11] Theorem 14] gives a complete description of those finite fields whose matrices are a sum of an idempotent and a $q$-potent for some odd integer $q > 1$. In particular, if the field $\mathbb{F}_q$ is not isomorphic to $\mathbb{F}_3$, then each finite matrix over it is the sum of an idempotent and a $q$-potent. However, this is not true for fields of characteristic 2. To avoid this restriction on the number $q$ to be odd, we just will speak about the representations of matrices over $\mathbb{F}_q$ of an idempotent and a $(\mathrm{LCM}(q - 1, 2) + 1)$-potent.

The next technical key for our further investigations completely demonstrates what we said above.

**Lemma 4.8.** Let $q \geq 5$ be an integer which is power of a prime, and let $p = p(x) \in \mathbb{F}_q[x]$ be an unitary polynomial of degree $n \geq 1$. Put $d = \mathrm{LCM}(q - 1, 2) + 1$. Then the matrix $C(p) \in \mathbb{M}_n(\mathbb{F}_q)$ is the sum of an idempotent matrix and an invertible $d$-potent matrix.

**Proof.** Let us fix an arbitrary primitive element $\xi$ of the field $\mathbb{F}_q$ such that $\xi \neq 1 - \xi$. Under this assumption, if $q = 5$, then we can choose $\xi$ to be equal to the element 3. Since $q \geq 5$, there exists an element $k \in \mathbb{F}_q$, having the property $0 \notin k + \{-1, 0, 1, 2, -\xi, \xi - 1\}$. Since the matrix $C(p) - kI_n$ is obviously cyclic, for some invertible matrix $V \in \mathbb{M}_n(\mathbb{F}_q)$ and an unitary polynomial $p_1 \in \mathbb{F}_q[x]$ the next equality is fulfilled:

$$C(p) - kI_n = VC(p_1)V^{-1}.$$ 

Assume now that $n \geq 2$ and that $\mathrm{tr}(p_1) \neq 1 - k$. We will differ three basic cases about $\mathrm{tr}(p_1)$.

**Case 1:** Assume $\mathrm{tr}(p_1) = 1$. In accordance with [11] Lemma 3] there is a decomposition $C(p_1) = e + f$, where $e^2 = e$, $f^2 = f$ and $\mathrm{spec}(f) \subseteq \{-1, 0, 1\}$. Obviously, the $d$-potent $f + kI_n$ inverts satisfying the equality

$$C(p) - kI_n = V(e + f)V^{-1} + kI_n = VeV^{-1} + V(f + kI_n)V^{-1}.$$ 

**Case 2:** Assume $\mathrm{tr}(p_1) = 0$. According to [11] Lemma 2] there is a decomposition $C(p_1) = e + f$, where $f^q = f = f^d$, $e^2 = e$ and $\mathrm{spec}(f) \subseteq \{-1, 0, -\xi, \xi - 1\}$. In particular, if $m = 2$, then the decomposition of the matrix $C(p_1)$ has the form

$$
\begin{pmatrix}
0 & a_0 \\
1 & 0
\end{pmatrix} = 
\begin{pmatrix}
1 - \xi & \xi(1 - \xi) \\
1 & \xi
\end{pmatrix} + 
\begin{pmatrix}
\xi - 1 & a_0 - \xi(1 - \xi) \\
0 & -\xi
\end{pmatrix}.
$$

Then the $d$-potent $f + kI_n$ inverts and also satisfies the equality $C(p) = VeV^{-1} + V(f + kI_n)V^{-1}$.
Case 3: Assume $\text{tr}(p_1) \neq 0, 1$. Bearing in mind [11] Lemma 1, we can decompose $C(p_1) = e + f$, where $f^d = f$ and $\text{spec}(f) \subseteq \{-1, 0, \text{tr}(p_1)-1\}$. In view of the choice of the element $k$, the $d$-potent element $f + kI_n$ is seen to be invertible and one may write that $C(p) = VeV^{-1} + V(f + kI_n)V^{-1}$.

We next assume for a moment that $\text{tr}(p_1) = a_{n-1} = 1 - k$. It follows from the initial choice of the element $k$ that $1 - k \notin \{-1, 0, 1\}$. Since the matrix $-C(p_1)$ is cyclic, for some invertible matrix $W \in M_n(\mathbb{F}_q)$ and some unitary polynomial $p_2 \in \mathbb{F}_q[x]$ the following equality is true

$$-C(p_1) = WC(p_2)W^{-1}.$$  

Moreover,

$$\text{tr}(p_2) = \text{tr}(-C(p_1)) = k - 1 \neq 0, 1.$$ 

Therefore, utilizing [11] Lemma 1, we get that $C(p_2) = e + f$, where $\text{spec}(f) \subseteq \{-1, 0, k - 2\}$. Furthermore, one deduces that

$$C(p_1) + kI_n = -WC(p_2)W^{-1} + kI_n = W(-e - f)W^{-1} + kI_n = W(I_n - e)W^{-1} + W(-f + (k - 1)I_n)W^{-1}.$$ 

Also, the equality $(-f + (k - 1)I_n)^{q} = (-1)^q f + (k - 1)I_n$ holds. Consequently, the element $-f + (k - 1)I_n$ must be a $d$-potent. However, because of the inclusion $\text{spec}(-f + (k - 1)I_n) \subseteq \{k, k - 1, 1\}$, one infers that $V(-f + (k - 1)I_n)V^{-1}$ is an invertible $d$-potent.

Finally, one remains to treat the case when $n = 1$. For the element $a \in \mathbb{F}_q$, which differs from $-k$, the decomposition $a = e_a + f_a = 0 + a$ ensures the invertibility of the $q$-potent $(f_a + k)$. If, however, we have that $a = -k$, then we may write that $-k = e_{-k} + f_{-k} = 1 + (-k - 1)$, as required.  

Let us now indicate that Lemma [4.8] restricted our attention on fields containing at least five elements. On the other side, in [17] conjectured that in the ring $M_n(\mathbb{F}_2)$ each element is the sum of an idempotent and an invertible $(n + 1)$-potent. However, it follows from [11] and our Proposition [5.2] that the structure of the matrices considered over $\mathbb{F}_3$ are also not completely described. Nevertheless, we can offer in the sequel some description of matrices over the field $\mathbb{F}_4$ consisting of four elements.

The next technicality is pretty easy but useful, so we state it here with a short proof only for the sake of completeness of the exposition and for the reader’s convenience.

**Lemma 4.9.** Let $R$ be a commutative ring, $r$ and $s$ polynomials over $R$, and $A \in M_n(R)$, $B \in M_{n,p}(R)$, $C \in M_{p}(R)$ such that $r(A) = 0$ and $s(C) = 0$. Then the equality $(rs)(M) = 0$ holds for the upper triangular block-matrix

$$M = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix}$$

**Proof.** It straightforwardly follows from the equation $(rs)(M) = r(M)s(M)$, which we leave to be proved by the interested reader.  

The next claim is essential for our further studies.

**Lemma 4.10.** Let $p \in \mathbb{F}_4[x]$ be an unitary polynomial of degree $n \geq 3$ such that the number $n$ is odd. Then the matrix $C(p) \in M_n(\mathbb{F}_4)$ is the sum of an idempotent matrix and an invertible $7$-potent matrix.

**Proof.** We fix an arbitrary primitive element $\xi$ of the field $\mathbb{F}_4$. Likewise, let $n = 2k+1$ for some natural number $k$. We shall consider three major cases about $\text{tr}(p)$.

*Case 1:* Assume $\text{tr}(p) = 0$. Since the matrix $C(p) - \xi I_n$ is cyclic, for some invertible matrix $V \in M_n(\mathbb{F}_4)$ and some unitary polynomial $p_1 \in \mathbb{F}_4[x]$ the following equality is valid:

$$C(p) - \xi I_n = VC(p_1)V^{-1},$$

where $\text{tr}(p_1) = \xi$. We also define the idempotent $e = (1) \oplus A_1 \oplus A_2 \ldots \oplus A_k \in M_n(\mathbb{F}_4)$, where $A_1 = A_2 = \cdots = A_k = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$. In this case, the matrix $C(p_1) - e$ is an upper triangular block: precisely, we have that

$$C(p_1) - e = \begin{pmatrix} H & T \\ 0 & 1 + \xi \end{pmatrix},$$

where $H = B_1 \oplus B_2 \oplus \ldots \oplus B_k$ and $B_1 = B_2 = \cdots = B_k = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$.

Since the matrix $H$ annihilates by the polynomial $r(x) = x(x - 1)$, while the matrix $(1 + \xi)$ annihilates by the polynomial $s(x) = x - (1 + \xi)$, Lemma [4.11] allows us to conclude that the matrix $C(p) - e$ vanishes by
the product rs, so that it is a 4-potent. As in the proof of Lemma 4.8, we observe that \((C(p_1) - e) + \xi I_n\) is an invertible 4-potent, which turned out that the matrix \(C(p)\) is the sum of an idempotent and an invertible 7-potent.

Case 2: Assume \(\text{tr}(p) \in \{\xi, \xi + 1\}\). Since the matrix \(C(p) - \text{tr}(p)I_n\) is cyclic, for some invertible matrix \(V \in M_n(\mathbb{F}_4)\) and an unitary polynomial \(p_1 \in \mathbb{F}_4[x]\) the following equality is valid:

\[
C(p) - \text{tr}(p)I_n = VC(p_1)V^{-1},
\]

where \(\text{tr}(p_1) = 0\). As in the preceding Case 1, we define the idempotent \(e \in M_n(\mathbb{F}_4)\). Hence the matrix \(C(p_1) - e\) is an upper triangular block: specifically, we have that \(C(p_1) - e = \begin{pmatrix} H & T \\ 0 & 1 + \text{tr}(p) \end{pmatrix}\), where \(H = B_1 \oplus B_2 \oplus \ldots \oplus B_k\) and all matrices \(B_i\) are as in Case 1 alluded to above.

We, consequently, deduce that

\[
(C(p_1) - e) + \text{tr}(p)I_n = \begin{pmatrix} H + \text{tr}(p)I_{n-1} & T \\ 0 & 1 + \text{tr}(p) \end{pmatrix}.
\]

But the matrix \(H + \text{tr}(p)I_{n-1}\) annihilates by the polynomial \(r(x) = (x - \text{tr}(p))(x - 1 - \text{tr}(p))\), and the matrix \((1 + \text{tr}(p))\) annihilates by the polynomial \(s(x) = x - 1 - \text{tr}(p)\), so that the application of Lemma 4.9 is a guarantor that the matrix \((C(p_1) - e) + \text{tr}(p)I_n\) vanishes by the product rs, and so by the polynomial \(x^7 - x = x(x^3 - 1)^2\) as well. That is why, \((C(p_1) - e) + \text{tr}(p)I_n\) is an invertible 7-potent, whence the matrix \(C(p)\) is the sum of an idempotent and an invertible 7-potent.

Case 3: Assume \(\text{tr}(p) = 1\). Again \(C(p) - (1 + \xi)I_n = VC(p_1)V^{-1}\) with \(\text{tr}(p_1) = \xi\). Similarly, one chooses \(e = (1) \oplus A_1 \oplus A_2 \oplus \ldots \oplus A_k\) such that \(A_1 = A_2 = \cdots = A_k = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}, A_k = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\). In that case, the matrix \((C(p_1) - e)\) is an upper triangular block: concretely, we have that \(C(p_1) - e = \begin{pmatrix} H & T \\ 0 & \xi \end{pmatrix}\), where

\[
H = B_1 \oplus B_2 \oplus \ldots \oplus B_k \quad \text{and} \quad B_1 = B_2 = \cdots = B_{k-1} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, B_k = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.
\]

Furthermore, one derives that

\[
(C(p_1) - e) + (1 + \xi)I_n = \begin{pmatrix} H + (1 + \xi)I_{n-1} & T \\ 0 & 1 \end{pmatrix}.
\]

Since \(H + (1 + \xi)I_{n-1}\) annihilates by the polynomial \(r(x) = (x - 1 - \xi)(x - \xi)^2\), and the matrix \((\xi)\) annihilates by the polynomial \(s(x) = x - 1\), one more applying Lemma 4.9 leads us to the fact that the matrix \((C(p_1) - e) + (1 + \xi)I_n\) vanishes the product rs, and thus by the polynomial \(x^7 - x = x(x^3 - 1)^2\). Now, the matrix \((C(p_1) - e) + (1 + \xi)I_n\) must be an invertible 7-potent, whence the matrix \(C(p)\) is the sum of an idempotent and an invertible 7-potent.

The next observation is crucial for our further developments.

**Lemma 4.11.** Let \(p \in \mathbb{F}_4[x]\) be an unitary polynomial of degree \(n \geq 2\) and let the number \(n\) be even. Then the matrix \(C(p) \in M_n(\mathbb{F}_4)\) is the sum of an idempotent matrix and an invertible 7-potent matrix.

**Proof.** Assume that \(n = 2k \geq 2\). Fix an arbitrary primitive element \(\xi\) of the field \(\mathbb{F}_4\). We further define the element \(d \in \mathbb{F}_4\) in the following manner: \(d = 1 + \xi\) if \(\text{tr}(p) = 1 + \xi\), or \(d = \xi\) otherwise. The choice of \(d\) guarantees that the elements \(d, 1 + d\) and \(1 + \text{tr}(p) + d\) are non-zero for each value of \(\text{tr}(p)\).

Since the matrix \(C(p) - dI_n\) is cyclic, for some invertible matrix \(V \in M_n(\mathbb{F}_4)\) and unitary polynomial \(p_1 \in \mathbb{F}_4[x]\) the following equality is true

\[
C(p) - dI_n = VC(p_1)V^{-1},
\]

where \(\text{tr}(p_1) = \text{tr}(p)\). Also, define the idempotent \(e = A_1 \oplus A_2 \oplus \ldots \oplus A_k \in M_n(\mathbb{F}_4)\), where \(A_1 = A_2 = \cdots = A_k = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}\). In that case, the matrix \((C(p_1) - e)\) is an upper triangular block: exactly, we have that

\[
C(p_1) - e = \begin{pmatrix} H & T \\ 0 & 1 + \text{tr}(p) \end{pmatrix},
\]

where \(H = (0) \oplus B_1 \oplus B_2 \oplus \ldots \oplus B_k\) and \(B_1 = B_2 = \cdots = B_k = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\).
We thus obtain that

\[(C(p_1) - e) + dI_n = \left( \begin{array}{cc} H + dI_{n-1} & T \\ 0 & 1 + \text{tr}(p) + d \end{array} \right)\].

Because the matrix \(H + dI_{n-1}\) is annihilated by the polynomial \(r(x) = (x - d)(x - 1 - d)\), whereas the matrix \((1 + \text{tr}(p) + d)\) annihilates by the polynomial \(s(x) = x - (1 + \text{tr}(p) + d)\), one may conclude with the aid of Lemma 4.10 that the matrix \((C(p_1) - e) + dI_n\) vanishes by the product \(rs\), and hence by the the polynomial \(x^7 - x = x(x^3 - 1)^2\). Therefore, \((C(p_1) - e) + dI_n\) has to be an invertible 7-potent, whence \(C(p)\) must be a sum of an idempotent and an invertible 7-potent. \(\square\)

We are now ready to establish the following chief affirmation:

**Theorem 4.12.** Let \(q \in \mathbb{N}\) be a prime power and let \(F = \mathbb{F}_q\) be a field with at least 4 elements. Set \(d = \text{LCM}(q - 1, 2) + 1\). Then, for any \(n \in \mathbb{N}\), every matrix from the ring \(M_n(F)\) can be written as a sum of an idempotent matrix and an invertible \(d\)-potent matrix.

**Proof.** Given \(A \in M_n(F_q)\) with \(q \geq 4\). One checks that the matrix \(A\) is similar to a matrix of the type \(A_1 \oplus A_2 \oplus \ldots \oplus A_k\), where \(A_i\) is a Frobenious block for each \(1 \leq i \leq k\). If \(q \geq 5\), then Lemma 4.10 tells us that the matrix \(A\) is the sum of an idempotent matrix and an invertible \(d\)-potent matrix. But if \(q = 4\), then \(d = 7\) and we may apply Lemmas 4.10 and 4.11. \(\square\)

As a valuable consequence, we derive:

**Corollary 4.13.** Suppose \(q > 1\) is an odd integer and \(R\) is an integral ring not isomorphic to any of the fields \(\mathbb{F}_2^\prime\) or \(\mathbb{F}_3\). Then the following three conditions are equivalent:

1. For every (for some) \(n \in \mathbb{N}\), each matrix in the matrix ring \(M_n(R)\) can be expressed as a sum of an idempotent matrix and an invertible \(q\)-potent matrix;
2. For every (for some) \(n \in \mathbb{N}\), each matrix in the the matrix ring \(M_n(R)\) can be expressed as a sum of an idempotent matrix and a \(q\)-potent matrix;
3. \(R\) is a finite field and \(|R| - 1| q - 1\).

In addition, if \(2 \in U(R)\) and \(R\) is not isomorphic to \(\mathbb{F}_3\), \(\mathbb{F}_5\) or \(\mathbb{F}_9\), then the conditions (1) – (3) are also equivalent to:

4. For every (for some) \(n \in \mathbb{N}\), each matrix in the matrix ring \(M_n(R)\) can be expressed as a sum of an involution and an invertible \(q\)-potent matrix.

**Proof.** The equivalence of points (2) and (3) follows immediately from \[11\, Theorem 14\]. Besides, the implication (1) \(\Rightarrow\) (2) is clear, and the implication (3) \(\Rightarrow\) (1) follows at once from Theorem 4.12.

(3) \(\Rightarrow\) (4). Let \(x \in M_n(R)\). Then, by implication (3) \(\Rightarrow\) (1), one writes that \((x + 1)/2 = e + u\), where \(u\) is an invertible \(q\)-potent matrix and \(e^2 = e\). Therefore, \(x = 2u + (2e - 1)\), where \(2u = (2u)^q\) is an invertible element, and \((2e - 1)^2 = 1\), as required.

(4) \(\Rightarrow\) (3). Follows directly from Theorem 3.1. \(\square\)

The following main statement illustrates what happens in the matrix ring in the considered situations. Specifically, we have:

**Theorem 4.14.** Suppose \(q > 1\) is an odd integer and \(R\) is a commutative ring which does not possess a homomorphic image isomorphic to \(\mathbb{F}_3\) and \((q - 1) \in U(R)\). The following items are equivalent:

1. Every matrix in \(M_n(R)\) is the sum of an idempotent matrix and an invertible \(q\)-potent matrix;
2. There exists a positive integer \(n\) such that each matrix in \(M_n(R)\) is the sum of an idempotent matrix and an invertible \(q\)-potent matrix;
3. Every matrix in \(M_n(R)\) is the sum of an idempotent matrix and a \(q\)-potent matrix;
4. There exists a positive integer \(n\) such that each matrix in \(M_n(R)\) is the sum of an idempotent matrix and a \(q\)-potent matrix;
5. The ring \(R\) satisfies the identity \(x^q = x\).

**Proof.** Implications (1) \(\Rightarrow\) (2), (2) \(\Rightarrow\) (4) and (3) \(\Rightarrow\) (4) are pretty obvious, so we omit the check of their validity.

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(4) $\Rightarrow$ (5). Take $n$ such that every matrix in $\mathbb{M}_n(R)$ is the sum of an idempotent matrix and a $q$-potent matrix. Consulting with Corollary [4.13] for every prime ideal $I$ of the ring $R$, the quotient ring $R/I$ is a field satisfying the identity $x^q = x$. So, $J(R) = Nil(R)$ is true and thus $R$ is semi-regular by [28] Lemma 16.6.

Consider now a maximal indecomposable factor $S = R/I$ of the ring $R$. By using [28] Remark 29.7(2)], Proposition 32.2] and Corollary [4.13 the factor-ring $S$ is a local ring, the quotient $S/J(S)$ is a field of characteristic $p$ in which the identity $x^q = x$ holds, and $J(S)$ is a nil-ideal. We next wish to prove that $J(S) = 0$. To achieve the claim, we assume on the contrary that $J(S) \neq 0$. However, if $pS \neq J(S)$, then $J(S/pS) \neq 0$ and so there exists nonzero $a \in J(S/pS)$ with identity $a^2 = 0$. By hypothesis, we write that

$$aI_n = E_1 + E_2, E_1^2 = E_1, E_2^2 = E_2,$$

for some $E_1, E_2 \in \mathbb{M}_n(S/pS)$. Since it is well known that every idempotent matrix is diagonalizable over a local commutative ring, it can be assumed without loss of generality that the matrices $E_1, E_2$ are of diagonal form. And since $a \neq 0$, it must be that $E_1 \neq 0$. Therefore, for some element $b$ on the main diagonal of the matrices $E_2$, the equalities $a = 1 + b, b^{q-1} = 1$ hold. Then

$$0 = a^p = (1 + b)^p = 1 + b^p,$$

and hence $b^p = -1$. Since by condition $(q - 1, p) = 1$, the equality $1 = t_1(q - 1) + t_2p$ holds for some integers $t_1$ and $t_2$. But since $t_2$ is an odd number, we then have $b = (b^{q-1})^{t_1} (b^p)^{t_2} = -1$ which yields $a = 0$, that is the desired contradiction.

If now $pS = J(S)$, then $J(S) \neq 0$ implies $pS \neq p^2S$. We put $a = p + p^2S \in S/p^2S$. By hypothesis, there exists $E_1, E_2 \in \mathbb{M}_n(S/p^2S)$ such that

$$aI_n = E_1 + E_2, E_1^2 = E_1, E_2^2 = E_2.$$

From $a^2 = 0$, we readily see that

$$aI_n - E_2 = E_1 = E_1^2 = -E_2a + qaI_nE_2^{-1} = -E_2 + qaI_nE_2^{-1}.$$

Then $aI_n = qaI_nE_2^{-1}$. It once again can be assumed without loss of generality that the matrices $E_1, E_2$ are of diagonal form. Therefore, for some element $b$ on the main diagonal of the matrices $E_2$, the equalities $a = ab^{q-1}$ hold. Since $b^q = b$, we obtain $(q - 1)ab = 0$ and since $(q - 1) \in U(S/p^2S)$, we arrive at $ab = 0$. Consequently, $a = ab^{q-1} = 0$, which is a new contradiction. Thus, finally, $J(S) = 0$, which substantiates our claim.

Furthermore, by virtue of the above reasoning, all Pierce stalks of $R$ are isomorphic to the finite fields $F_i$ with $i - 1 \mid q - 1$. Invoking [28] Corollary 11.10], the ring $R$ has identity $x^q = x$.

(5) $\Rightarrow$ (1). (5) $\Rightarrow$ (3). Let $n$ be an arbitrary natural number and take $A \in \mathbb{M}_n(R)$. Consider the subring $S$ of the ring $R$, generated by the elements of the matrix $A$. One straightforwardly verifies that the ring $S$ is finite. Hence, one decomposes $S \cong P_1 \times \ldots \times P_m$, for some finite fields $P_i$ with identities $x^q = x$ and for any $1 \leq i \leq m$. Now, with Corollary [4.13] at hand, every $P_i$ satisfy the conditions of points (1) and (3), whence so does $S$.

We observe that Theorem [4.14] requires the condition $q - 1 \in U(R)$, which is crucial for obtaining this result. As a matter of fact, let us take an odd prime $p$ and consider the ring $\mathbb{Z}/p^2\mathbb{Z}$. So, we come to the following assertion.

**Lemma 4.15.** Suppose that $p$ is an odd prime and $q \in \mathbb{N}$. Then the following two conditions are equivalent:

1. Every element of $\mathbb{Z}/p^2\mathbb{Z}$ is the sum of an idempotent and a $q$-potent.
2. $p(p-1) \mid q - 1$.

**Proof.** (1) $\Rightarrow$ (2). Suppose that every element of $\mathbb{Z}/p^2\mathbb{Z}$ is the sum of an idempotent and a $q$-potent. Since $U(\mathbb{Z}/p^2\mathbb{Z})$ is a cyclic group of order $p(p-1)$, there exists $q' \in \mathbb{N}$ such that $q' - 1 \mid p(p-1)$ and the set of $q$-potents of $\mathbb{Z}/p^2\mathbb{Z}$ coincides with the set of $q'$-potents. If $q' - 1 = p(p-1)$, then $p(p-1) \mid q - 1$, as required. Otherwise, the inequality $q' - 1 \leq \frac{p(p-1)}{2}$ holds, and cardinality of the set of elements that are the sum of an idempotent and a $q'$-potent is not greater than the number $2(1 + \frac{p(p-1)}{2}) < p^2$.

(2) $\Rightarrow$ (1). It is clear that every element of $\mathbb{Z}/p^2\mathbb{Z}$ is either a $(p(p-1)+1)$-potent or the sum of 1 and a $(p(p-1)+1)$-potent, as required. 

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The next example will materialize our observations from above.

**Example 4.16.** Suppose that \( p \) is an odd prime and \( q \in \mathbb{N} \). Then the following two conditions are equivalent:

1. Every element of \( M_2(\mathbb{Z}/p^2\mathbb{Z}) \) is the sum of an idempotent matrix and a \( q \)-potent matrix;
2. \( p(p-1) \mid q - 1 \).

**Proof.** (1) \( \Rightarrow \) (2). Take \( a \in \mathbb{Z}/p^2\mathbb{Z} \). Then there exist matrices \( E_1, E_2 \in M_2(\mathbb{Z}/p^2\mathbb{Z}) \), such that \( aI_2 = E_1 + E_2 \), \( E_1^2 = E_1 \), \( E_2^2 = E_2 \). Since it is well known that every idempotent matrix is diagonalizable over a local commutative ring, it can be assumed without loss of generality that the matrices \( E_1, E_2 \) are of diagonal form. We, therefore, can conclude with the aid of Lemma 4.15 that every element of \( \mathbb{Z}/p^2\mathbb{Z} \) is the sum of an idempotent and a \( q \)-potent, and \( p(p-1) \mid q - 1 \).

(2) \( \Rightarrow \) (1). Take \( A \in M_2(\mathbb{Z}/p^2\mathbb{Z}) \) and let \( \pi : M_2(\mathbb{Z}/p^2\mathbb{Z}) \to M_2(\mathbb{Z}/p\mathbb{Z}) \) denote the reduction map. The matrix \( \pi(A) \) is similar to its rational canonical form. Then the standard theory of determinants will imply that every invertible matrix in \( M_2(\mathbb{Z}/p\mathbb{Z}) \) can be lifted upon \( \pi \). Therefore, without loss of generality, we may assume that \( A \) is presentable in one of the two following forms: \( \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \) or \( \begin{pmatrix} 0 & a \\ 1 & b \end{pmatrix} \), where \( a, b \in \mathbb{Z}/p^2\mathbb{Z} \) and \( j \in J(M_2(\mathbb{Z}/p^2\mathbb{Z})) \). We shall now distinguish three basic cases as follows:

**Case 1:** Suppose that \( A = \begin{pmatrix} 0 & a \\ 1 & b \end{pmatrix} \).

If \( b \not\equiv 1 \mod p^2 \), then
\[
\begin{pmatrix} 0 & a \\ 1 & b \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} + \begin{pmatrix} 0 & a \\ 0 & b-1 \end{pmatrix}.
\]
Since \( b-1 \in U(\mathbb{Z}/p^2\mathbb{Z}) \), we have \((b-1)\cdot\text{gcd}(p-1, b-1) = 1 \) and \( b \) is annihilated by the polynomial \( x^{\text{gcd}(p-1, b-1)} - 1 \). In view of Lemma 4.9, the matrix \( \begin{pmatrix} 0 & a \\ b & 1 \end{pmatrix} \) is annihilated by \( x(x^{\text{gcd}(p-1, b-1)} - 1) \), i.e., it is a \((p(p-1) + 1)\)-potent.

If \( b \equiv 1 \mod p^2 \), then
\[
\begin{pmatrix} 0 & a \\ 1 & b \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} -1 & a \\ 0 & b \end{pmatrix}.
\]
Since \( x^{p(p-1)} - 1 = (x+1)g(x) \) for some polynomial \( g(x) \) over \( \mathbb{Z} \) and \( b+1 \) is invertible in \( \mathbb{Z}/p^2\mathbb{Z} \), we conclude that \( b \) is annihilated by \( g(x) \). In virtue of Lemma 4.9, the matrix \( \begin{pmatrix} -1 & a \\ 0 & 1 \end{pmatrix} \) is annihilated by \( (x+1)g(x) \), i.e., it is a \((p(p-1) + 1)\)-potent.

**Case 2:** Suppose that \( A = \begin{pmatrix} pk & a \\ 1 + pm & b \end{pmatrix} \). Take \( u = (1 + pm)^{-1} \). We have
\[
\begin{pmatrix} 1 & 0 \\ 0 & u \end{pmatrix} \begin{pmatrix} pk & a \\ 1 + pm & b \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & u^{-1} \end{pmatrix} = \begin{pmatrix} pk & au^{-1} \\ 1 & b \end{pmatrix}.
\]
Next,
\[
\begin{pmatrix} 1 & -pk \\ 0 & 1 \end{pmatrix} \begin{pmatrix} pk & au^{-1} \\ 1 & b \end{pmatrix} \begin{pmatrix} 1 & pk \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & au^{-1} - bpk \\ 1 & b + pk \end{pmatrix}.
\]
Thus Case 2 is reduced to Case 1.

**Case 3:** Suppose that \( A = \begin{pmatrix} a & pk \\ pm & b \end{pmatrix} \).

If \( a \) and \( b \) are both units, then
\[
\begin{pmatrix} a & pk \\ pm & b \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} a & pk \\ pm & b \end{pmatrix}.
\]
It is enough to show that \( \begin{pmatrix} a & pk \\ pm & b \end{pmatrix}^{p(p-1)} = I_2 \).

If \( a = b \), then
\[
\begin{pmatrix} a & pk \\ pm & a \end{pmatrix}^{p(p-1)} = \left(aI_2 + \begin{pmatrix} 0 & pk \\ pm & 0 \end{pmatrix}^{p(p-1)}\right)^{p(p-1)} = a^{p(p-1)}I_2 + \frac{p(p-1)}{2}a^{p(p-1)-1}\begin{pmatrix} 0 & pk \\ pm & 0 \end{pmatrix} = I_2,
\]
because $2 \mid (p - 1)$.

If $a \neq b$ and $a - b \in U(\mathbb{Z}/p^2\mathbb{Z})$, then simple induction shows that

$$
\begin{pmatrix}
a & pk \\
p^m & b
\end{pmatrix}^r =
\begin{pmatrix}
a^r & pk \sum_{i=0}^{r-1} a^i b^{r-i} \\
p^m \sum_{i=0}^{r-1} a^i b^{r-i} & b^r
\end{pmatrix}
=\begin{pmatrix}
a^r & pk\frac{a^r - a^r}{a-b} \\
p^m \frac{a^r - a^r}{a-b} & b^r
\end{pmatrix}
$$

Since $a$ and $b$ are units, we have $\begin{pmatrix}a & pk \\p^m & b\end{pmatrix}^{p(p-1)} = I_2$.

If $a - b \in p\mathbb{Z}/p^2\mathbb{Z}$ and $a - b \neq 0$ then

$$
0 = a^{p(p-1)} - b^{p(p-1)} = (a - b) \left( \sum_{i=0}^{p(p-1)-1} a^i b^{p(p-1)-1-i} \right).
$$

Thus $\sum_{i=0}^{p(p-1)-1} a^i b^{p(p-1)-1-i} \in p\mathbb{Z}/p^2\mathbb{Z}$ and

$$
\begin{pmatrix}
a & pk \\
p^m & b
\end{pmatrix}^{p(p-1)} =
\begin{pmatrix}
a^{p(p-1)} & pk \sum_{i=0}^{p(p-1)-1} a^i b^{p(p-1)-1-i} \\
p^m \sum_{i=0}^{p(p-1)-1} a^i b^{p(p-1)-1-i} & b^{p(p-1)}
\end{pmatrix} = I_2.
$$

If, however, $a$ and $b$ are both not units, then

$$
\begin{pmatrix}
a & pk \\
p^m & b
\end{pmatrix} = \begin{pmatrix}1 & 0 \\0 & 1\end{pmatrix} + \begin{pmatrix}a-1 & pk \\
p^m & b-1\end{pmatrix}
$$

and $\begin{pmatrix}a-1 & pk \\
p^m & b-1\end{pmatrix}$ is a $(p(p-1) + 1)$-potent, as we saw earlier.

Finally, if $b$ and $a - 1$ are units, then

$$
\begin{pmatrix}
a & pk \\
p^m & b
\end{pmatrix} = \begin{pmatrix}1 & pk \\
p^m & 0\end{pmatrix} + \begin{pmatrix}a-1 & 0 \\
0 & b\end{pmatrix}
$$

is the sum of idempotent and a $(p(p-1) + 1)$-potent. We thus got a similar decomposition if $a$ and $b - 1$ are units, as expected. \hfill \square

We are now in a position to prove the following:

**Theorem 4.17.** Suppose that $q > 1$ is an odd integer and $R$ is an integral ring not isomorphic to $\mathbb{F}_3$ having characteristic different to 2. Then the following two conditions are equivalent:

1. For every (for some) $n \in \mathbb{N}$, the ring $\mathcal{M}_n(R)$ is $(q - 1)$-torsion clean;
2. $R$ is a finite field and $|R| = q$.

**Proof.** It follows by combining Corollary 4.13 with Lemma 4.21. \hfill \square

With this at hand, we can extract the next two consequences.

**Corollary 4.18.** Suppose $k, n > 1$ are naturals. Then the ring $\mathcal{M}_n(\mathbb{F}_{2^k})$ is $d$-torsion clean, where $d \in \{2^k - 1, 2^{k+1} - 2\}$.

**Proof.** It follows from Lemma 4.22 that $2^k - 1 \mid d$. But Theorem 4.19 enables us that $d \leq 2^{k+1} - 2$, as asked for. \hfill \square

**Corollary 4.19.** Let $p$ be an odd prime, and $q = p^\alpha$ for some integer $\alpha \geq 0$. If $R$ is an integral ring of characteristic not equal to 2 and $|R| > 9$, then the following four conditions are equivalent:

1. For every (for some) $n \in \mathbb{N}$, each matrix in the matrix ring $\mathcal{M}_n(R)$ can be expressed as a sum of an idempotent matrix and an invertible $q$-potent matrix;
2. For every (for some) $n \in \mathbb{N}$, each matrix in the matrix ring $\mathcal{M}_n(R)$ can be expressed as a sum or a difference of an invertible $q$-potent matrix and an idempotent matrix;
(3) For every (for some) \( n \in \mathbb{N} \), each matrix in the matrix ring \( M_n(R) \) can be expressed as a sum of a tripotent matrix and a \( q \)-potent matrix;

(4) \( R \) is a finite field with \(|R| - 1 \mid q - 1 \).

**PROOF.** It follows from Lemma 2.5 and Corollary 4.13.

We thus have obtained the following characterization statement.

**Theorem 4.20.** Let \( p \) be an odd prime, and \( q = p^\alpha \) for some integer \( \alpha \geq 0 \). If \( R \) is an integral ring of characteristic not equal to 2 and \(|R| > 9\), then the following three conditions are equivalent:

(1) For every (for some) \( n \in \mathbb{N} \), the ring \( M_n(R) \) is \((q - 1)\)-torsion clean;

(2) For every (for some) \( n \in \mathbb{N} \), the ring \( M_n(R) \) is weakly \((q - 1)\)-torsion clean;

(3) \( R \) is a finite field and \(|R| = q\).

In addition, we have the following two results.

**Lemma 4.21.** Let \( q > 1 \) be an integer, \( F_q \) a finite field and \( n \in \mathbb{N} \). Then the following two items are equivalent:

(1) Every element of \( F_q \) admits an \( n \)-torsion clean presentation;

(2) \((q - 1) \mid n\).

**PROOF.** (1) \( \Rightarrow \) (2). In view of Lemma 2.4 it is necessary to consider only the fields \( F_3, F_5, F_7, F_9 \). Assume in a way of contradiction that each element of these fields is \( n \)-torsion clean, but \( n \) is not divisible by \((q - 1)\).

To get the desired contrary, we first observe the obvious fact that the set of \((n + 1)\)-potents in the field \( F_q \) coincides with the set of \((1 + \gcd(n, q - 1))\)-potents. But \( \frac{(q - 1)}{2} \mid n \) in virtue of Lemma 2.4, whence \( \gcd(n, q - 1) = \frac{q - 1}{2} \) and so in the field every element is the sum of an idempotent and of an invertible \( \left(\frac{q + 1}{2}\right)\)-potent. Since the number of \( \left(\frac{q + 1}{2}\right)\)-potents is exactly \( \frac{q - 1}{2} \), one follows that the number of \( n \)-torsion clean elements does not exceed \( q - 1 \), which is impossible, as wanted.

(2) \( \Rightarrow \) (1). It is straightforward.

**Lemma 4.22.** Let \( q > 1 \) be an integer and let \( R \) be an integral ring. If, for some \( n \in \mathbb{N} \), each matrix from the ring \( M_n(R) \) is \( n \)-torsion clean, then \( R \) is a finite field and \(|R| - 1 \mid n\).

**PROOF.** Owing to Lemma 2.3 the ring \( R \) is necessarily a finite field. Moreover, all elements of \( R \) admit an \( n \)-torsion clean presentation. But now Lemma 4.21 assures that \(|R| - 1 \mid n\), as promised.

It was also asked in [17] whether if the ring \( R \) strongly \( n \)-torsion clean, the equality \( n = \exp(U(R)) \) is true? We shall partially settle this query by using the following helpful assertion.

**Theorem 4.23** ([8 Theorem 6]). Let \( F \) be a finite field of characteristic \( p, n, q \in \mathbb{N} \), and \( q > 1 \) is odd. The following statements are equivalent:

(1) Every matrix \( A \in M_n(F) \) is the sum of a \( q \)-potent and an idempotent that commute;

(2) The number \( N = |F| - 1, |F|^2 - 1, ..., |F|^n - 1|p^l \) is a divisor of \( q - 1 \), where \( p^l \) is the least non-negative integer power of \( p \) that is greater or equal to \( m \).

Actually, in the proof of the implication (2) \( \Rightarrow \) (1) was obtained a stronger result like this: every matrix \( A \in M_n(F) \) is the sum of an invertible \( q \)-potent and an idempotent that commute. In particular, one can be seen that the number \(|F| - 1, |F|^2 - 1, ..., |F|^m - 1|p^l \) is equal exactly to \( \exp(U(M_n(F))) \).

We, thereby, can extract the following important consequence.

**Corollary 4.24.** Let \( F \) be a finite field of odd characteristic and \( n \in \mathbb{N} \). Then the ring \( M_n(F) \) is strongly \( \exp(U(M_n(F))) \)-torsion clean.

**5. Concluding Discussion and Unsettled Problems**

We begin this section with the next comments pertaining to the decomposing of matrices into certain elements.
Remark 5.1. We offer now the following statement:

Let $R$ be a finite commutative ring such that $J^s(R) = \{0\}$ for some $s \in \mathbb{N}$. Then every matrix in $M_n(R)$ for any $n \geq 1$ can be expressed as the sum of a potent matrix and a nilpotent matrix of order at most $s$.

Sketching the idea for the attack of the proof, as it is well-known, every finite commutative ring is a direct product of local rings, so we may assume $R$ to be local. Moreover, as $R/J(R)$ is a field, every square matrix over $R/J(R)$ has a rational canonical form. So, concentrating on one block of the rational canonical form, it is enough to show that a matrix $A + X$, where $A \in M_n(R)$ is in rational canonical form (for instance, for $n = 3$, we may consider $A = \begin{pmatrix} 0 & 0 & a \\ 1 & 0 & b \\ 0 & 1 & c \end{pmatrix}$ with $a, b, c \in R$) and $X \in M_n(J(R))$, has the desired decomposition. In fact, if the $(1, n)$th entry of $A$ is a unit, then it is obvious that $A$ is a unit, as well. But since $R$ is finite, one verifies that $A$ must be a torsion unit and thus potent. Furthermore, as $X^s = 0$, the sum $A + X$ is surely the wanted decomposition.

Now, suppose the $(1, n)$th entry of $A$ is in $J(R)$, we may assume that it is zero after merging it into $X$ if necessary. In this case, one plainly sees that

$$A + X = (A + E_{1n} + X) - E_{1n}. $$

Since $A + E_{1n}$ is again a unit, and $X \in M_n(J(R))$, one easily checks that the sum $A + E_{1n} + X$ will be still a unit. Also, as $(-E_{1n})^s = 0$, we are through as above.

Thus, in conclusion, every matrix over $R$ which does not lie in the Jacobson radical of the matrix ring has the decomposition $U + N$, where $U$ is a torsion unit and $N^s = 0$, as expected. However, when the given matrix lies in the Jacobson radical of the matrix ring, one easily observes that it is the unique sum of a zero potent and a nilpotent of index $\leq s$, as asserted.

However, the most difficult part in the present construction, which gives the incompleteness of the argumentation, is the following one: We do not know how to get a unit from the matrix $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ due to the second nilpotent $1 \times 1$ block.

As an immediate consequence, we eventually might derive the following assertion, which is actually the main result from [21] (see, also, [19]) proved there in the case where $k = 2$. We, however, will formulate the more general and conceptual version for an arbitrary natural number $k$ as follows:

Let $p$ be a prime and $k \geq 2$ an integer. Then each matrix in $M_n(\mathbb{Z}_{p^k})$ for any $n \geq 1$ is the sum of a potent and a nilpotent of order not exceeding $k$.

Thus, in closing, we pose the following three open questions of some importance for the further development of the topic.

Question 1. Let $n \in \mathbb{N}$ be arbitrary. Are weakly $n$-torsion clean rings (in particular, weakly $n$-torsion clean rings having the strong property) always clean?

The next query is closely related to Theorem 3.4 and the comments after its proof as well as to the subjects respectively considered in [15] and [5], which mainly treated periodic rings (see, for a more account, also [19], [20] and [21]).

Problem 1. Let $F$ be a (potent) field. For any $n \geq 1$ find those matrices from $M_n(F)$ which are a sum of a (torsion) unit and a square-zero nilpotent.

Question 2. Is it true that the matrix ring over any (finite) field is the sum of two potents?

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References

[1] A.N. Abyzov and D.T. Tapkin, Rings over which every matrices are sums of idempotent and $q$-potent matrices, Siberian Math. J. 62 (2021), 1–13.
[2] A.N. Abyzov, D.T. Tapkin, When is every matrix over a ring the sum of two tripotents?, Lin. Algebra & Appl. 630 (2021), 316–325.
[3] A.N. Abyzov and D.T. Tapkin, On rings with $x^n - x$ nilpotent, J. Algebra & Appl. 21 (2022).
[4] B.C. Berndt, R.J. Evans and K.S. Williams, Gauss and Jacobi Sums, Canadian Mathematical Society Series of Monographs and Advanced Texts, John Wiley & Sons, Inc., New York, 1998, xii+583.
[5] A.D. Bouzidi, A. Cherchema and A. Leroy, Exponents of skew polynomials over periodic rings, Commun. Algebra 49 (4) (2021), 1639–1655.
[6] S. Breaz, G. Călugăreanu, P. Danchev and T. Micu, Nil-clean matrix rings, Lin. Algebra & Appl. 439 (2013), 3115–3119.
[7] S. Breaz, P. Danchev and Y. Zhou, Rings in which every element is either a sum or a difference of a nilpotent and an idempotent, J. Algebra & Appl. 15 (2016).
[8] D.A. Buell and R.H. Hudson, On runs of consecutive quadratic residues and quadratic non residues, BIT Numerical Mathematics 24 (2) (1984), 243–247.
[9] G. Călugăreanu and T.-Y. Lam, Fine rings: A new class of simple rings, J. Algebra & Appl. 15 (9) (2016).
[10] V. Camillo, T.J. Dorsey and P.P. Nielsen, Dedekind-finite strongly clean rings, Commun. Algebra 42 (2014), 1619–1629.
[11] H. Chen, On uniquely clean rings, Commun. Algebra 39 (2011), 189–198.
[12] A. Cimpean and P. Danchev, $n$-torsion clean and almost $n$-torsion clean rings, Russian Mathematics 65 (1) (2021), 47–56.
[13] S.D. Cohen, Consecutive primitive roots in a finite field, Proc. Amer. Math. Soc. 93 (1985), 189–197.
[14] S.D. Cohen, T.O. e Silva and T.S. Trudgian, On consecutive primitive elements in a finite field, Bull. Lond. Math. Soc. 47 (3) (2015), 418–426.
[15] J. Cui and P.V. Danchev, Some new characterizations of periodic rings, J. Algebra & Appl. 19 (12) (2020).
[16] P.V. Danchev, Invo-clean unital rings, Commun. Korean Math. Soc. 32 (2017), 19–27.
[17] P.V. Danchev and J. Matczuk, $n$-torsion clean rings, Contemp. Math. 727 (2019), 71–82.
[18] P.V. Danchev and W.Wm. McGovern, Commutative weakly nil clean unital rings, J. Algebra 425 (2015), 410–422.
[19] P. Danchev, E. García and M.G. Lozano, On some special matrix decompositions over fields and finite commutative rings, Proceedings of the Fiftieth Spring Conference of the Union of Bulgarian Mathematicians 50 (2021), 95–101.
[20] P. Danchev, E. García and M.G. Lozano, Decompositions of matrices into diagonalizable and square-zero matrices, Lin. & Multilin. Algebra 70 (12) (2022).
[21] P. Danchev, E. García and M.G. Lozano, Decompositions of matrices into potent and square-zero matrices, Internat. J. Algebra & Computat. 32 (2) (2022), 251–263.
[22] A.J. Diesl, Nil clean rings, J. Algebra 383 (2013), 197–211.
[23] J. Han and W.K. Nicholson, Extensions of clean rings, Commun. Algebra 29 (6) (2001), 2589–2595.
[24] S.A. Katre and A.R. Rajwade, Resolution of the sign ambiguity in the determination of the cyclotomic numbers of order 4 and the corresponding Jacobsthal sum, Math. Scand. 60 (1987), 52–62.
[25] T.-Y. Lam, A First Course in Noncommutative Rings, Second Edition, Graduate Texts in Math., vol. 131, Springer-Verlag, Berlin-Heidelberg-New York, 2001.
[26] W.K. Nicholson, Lifting idempotents and exchange rings, Trans. Amer. Math. Soc. 229 (1977), 269–278.
[27] W.K. Nicholson, Strongly clean rings and Fitting’s lemma, Commun. Algebra 27 (1999), 3583–3592.
[28] A.A. Tuganbaev, Rings Close to Regular, Springer Netherlands (Kluwer), Dordrecht-Boston-London, 2002, 362 pp.

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