CANCELLATION THEOREM FOR GROTHENDIECK-WITT-CORRESPONDENCES AND WITT-CORRESPONDENCES.

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Abstract. The cancellation theorem for Grothendieck-Witt–correspondences and Witt-correspondences between smooth varieties over an infinite perfect field $k$, char $k \neq 2$, is proved.

The result implies the the canonical functor $\Sigma^{\infty}_{Z_m} : \text{DM}^{GW}_{\text{eff}}(k) \rightarrow \text{DM}^{GW}(k)$ is fully faithful, and for any $X \in \text{Sm}_k$, and for any homotopy invariant sheaf with GW-transfers $\mathcal{F}$,

$$\text{Hom}_{\text{DM}^{GW}}(\text{M}^{GW}(X), \Sigma^{\infty}_{Z_m} \mathcal{F}[i]) \simeq \text{H}^{\prime}_{Nis}(X, \mathcal{F}),$$

where $\text{DM}^{GW}(k)$ is the category of GW-motives obtained by $\mathbb{G}_m$-stabilisation from the category of effective Grothendieck-Witt-motives $\text{DM}^{GW,eff}_{\text{eff}}(k)$ constructed in [7]. Similarly in the case of Witt-motives $\text{DM}^{W}(k)$.

1. Introduction.

1.1. The categories $\text{DM}^{GW}(k)$ and $\text{DM}^{W}(k)$. This article is devoted to the cancellation theorem in the categories of effective Grothendieck-Witt-motives and Witt-motives. This is the final article of the series of works ([7], [8]) devoted to the construction of the categories of GW-motives $\text{DM}^{GW}(k)$ and Witt-motives $\text{DM}^{W}(k)$ over an infinite perfect field $k$, char $k \neq 2$. The construction follows the Voevodsky-Suslin method originally used for the construction of the category of the Voevodsky motives $\text{DM}^{eff}_{Nis}(k)$, see [23], [24], [21], [19].

The construction of $\text{DM}^{GW}(k)$ starts with some additive category of correspondences between smooth varieties $GW^\otimes$, called the GW-correspondences defined in [8]. Namely, and for a pair of smooth affine varieties $X$ and $Y$ the group $GW^\otimes(X, Y)$ is defied as Grothendieck-Witt-group of quadratic space $(P, q)$ with $P \in k[Y \times X]$ mod such that $P$ is finitely generated projective over $k[X]$ and $q : P \simeq Hom_{k[X]}(P, k[X])$ being $k[Y \times X]$-linear isomorphism.

The cancellation theorem proved in the article yields the main property of motives of smooth varieties in $\text{DM}^{GW}(k)$ and $\text{DM}^{W}(k)$, i.e. isomorphism

$$(1.1) \quad \text{Hom}_{\text{DM}^{GW}(k)}(\text{M}^\times(X), \Sigma^{\infty}_{Z_m} \mathcal{F}[i]) = \text{H}^{\prime}_{Nis}(X, \mathcal{F}), * \in \{GW, W\},$$

for a homotopy invariant presheaf with GW-transfers (Witt-transfers) $\mathcal{F}$ and smooth variety $X$, where $\text{M}^{GW}(X)$ is the complex of Nisnevich sheaves $GW^\otimes_{Nis}(\Delta^\times \times - , X)$, and similarly for the case of Witt motives.

The category $\text{DM}^{GW}(k)$ gives a version of generalised motivic cohomology theory given by $X \mapsto \text{Hom}_{\text{DM}^{GW}}(\text{M}^{GW}(X), Z_{GW}(i)[j])$, where $Z_{GW}(i) = \text{M}^{GW}(\mathbb{G}_m^i)[i]$. As written in [3] example ..] this theory is equivalent to the generalised motivic cohomology defined by C’almes and Fasel in [5], so the result [] by Garkusha implies that $\text{DM}^{GW}(k)[1/p] \simeq \text{DM}(k)[1/p]$, $p = \text{char } k$. Consequently the Garkusha’s result [12] implies equivalence $\text{DM}^{GW}(k)_Q \simeq \mathcal{S}H(k)_Q$. Thus the computation of

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the GW-motives of smooth varieties presented here gives a fibrant replacement functor in $\mathcal{SH}(k)_Q$.

Let’s note also that the canonical adjunction $\mathcal{SH}(k) \cong \text{DM}^{\text{GW}}(k)$ is not equivalent.

It is expected that the category $\text{DM}^W(k)$ is equivalent to the category of Witt-motives $\text{DM}_W(k)$ constructed by Ananyevsky, Levine, Panin in [2] using the category of modules over the Witt-ring sheaf. As proven in [2] the category $\text{DM}_W(k)$ satisfies the Morel conjecture about Witt-motives, i.e. $\text{DM}_W(k)_Q \cong \mathcal{SH}^-(k)_Q$. In [2] such category was constructed. Usefulness of the reconstruction by the Voevodsky-Suslin method as above is that it gives an explicit fibrant replacements for the motives of smooth varieties that can be useful for computations.

The functor $\mathcal{SH}(k) \to \text{DM}^W(k)$ in some sense is the algebraic version of the realisation. To justify the last statement let’s say that it is known that $H^0(X_\mathbb{R}) = \text{Spec}_{\mathbb{R}} W(X)$, $H^1_{\text{sing}}(X_\mathbb{R}) = H^1_{\text{zar}}(X, \mathbb{C})$, where $X_\mathbb{R}$ is the realisation of an algebraic variety $X$, and $\mathbb{C}$ is sheaf of fundamental ideals in the Witt ring and $u > $ dim\text{Ker}(X). The the case zero cohomologies is the result by L. Mahe and J. Houdebine, [17,15], the general case is the result by J. Jacobson [16].

1.2. The cancellation theorem in $\text{DM}^{\text{GW}}(k)$ and $\text{DM}^W_{\text{eff}}(k)$. We give the formulation for the case of GW motives, the case of the W motive is similar. The main result of the article is

**Theorem 1.2 (Theorem 7.8).** For an infinite perfect field $k$, $\text{char} k \neq 2$, the canonical functor

$$
\Sigma_{\text{eff}}^{\infty} : \text{DM}_{\text{eff}}^{\text{GW}}(k) \to \text{DM}^{\text{GW}}(k), \quad A^* \mapsto (A^*, A^* \otimes G^1_m, \ldots, A^* \otimes G^1_m, \ldots)
$$

is a fully faithful embedding.

Combining the result of the article with the results of [7] we get the sequence of adjunctions

$$
\begin{align*}
L_{\text{GW}} &\dashv F_{\text{GW}}, \quad L_{A^1} \dashv R_{A^1,\text{GW}} \Sigma_{\text{eff}}^{\infty} \dashv \Omega^{\infty}_{\text{GW}}, \\
\mathbb{Z}(X) &\mapsto \mathbb{Z}_{\text{GW,Nis}}(X) \mapsto \mathbb{M}_{\text{eff,nb}}^{\text{GW}}(k) \mapsto \text{DM}^{\text{GW}}(k),
\end{align*}
$$

where $\mathbb{Z}_{\text{GW,Nis}}(X) = \text{GW}Cor_{\text{nis}}(-, X)$ is the Nisnevich sheafification, and $F_{\text{GW}}$ is forgetful. $L_{A^1}$ is the localisation with respect to $A^1$-equivalences. The adjoint functor $R_{A^1}$ is equivalent to the full embedding of the subcategory of motivic complexes, which are complexes with homotopy invariant cohomology sheaves. Under the last identification $L_{A^1}$ takes a complex $A^*$ to $\mathcal{H}om_{\text{DM}(\text{Sh}_{\text{et}(\text{GWCor}))}}(\Delta^*, A^*)$. Finally, the third adjunction is infinite-suspension and infinite-loop functors. The second adjunction in the sequence is a reflection, $R_{A^1}(L_{A^1}(A^*) \simeq A^*$. Theorem 1.2 implies that the third adjunction is a coreflection, $A \simeq \Omega^{\infty}_{\text{GW}}(\Sigma^{\infty}_{\text{GW}}(A))$.

Similar to in the original case of Cor-correspondences and other known cases Theorem 1.2 is a consequence of the following cancellation theorem.

**Theorem 1.4 (Corollary 7.5).** For any $A^*, B^* \in \text{DM}^{\text{GW}}(k)$, there is isomorphism on hom-groups

$$
\text{Hom}_{\text{DM}^{\text{GW}}(k)}(A^*, B^*) \simeq \text{Hom}_{\text{DM}^{\text{GW}}(k)}(A^*(1), B^*(1)),
$$

where $A^*(1) = A^* \otimes G^1_m$, and similarly for $\text{DM}^W_{\text{eff}}(k)$.

Equivalently this states that the adjunction $- \otimes G^1_m : \text{DM}^{\text{GW}}(k) \cong \text{DM}^{\text{GW}}(k) ; \text{Hom}(G^1_m, -)$ is a coreflection. The main case is the case of $A^* = M^{\text{GW}}(X), B^* = M^{\text{GW}}(Y)$, which is equivalent to the isomorphism

$$
\text{GW}Cor_{\text{nis}}(X \times \Delta^*, Y) \simeq \text{GW}Cor_{\text{nis}}(X \times G^1_m, \Delta^*, Y \times G^1_m)
$$

because of the equality $[14]$ for effective GW-motives

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1 We consider unbounded derived categories that allows to define the infinite loop functor $\Omega^{\infty}_{\text{GW}}$, but further in the text we’ll deal with bounded above derived categories, since the result after stabilisation by $G^1_m$ in both cases is the same, the GW-motives of smooth varieties and homotopy invariant presheaves with GW-transfers in formula [14] are bounded above.
1.3. Comparing with other cancellation theorems. Let's give short list of other known cancellation theorems proved for another categories of correspondences:

1) Cancellation for Cor-correspondences proved by V. Voevodsky in [22].
2) Cancellation for $K_0$-correspondences proved by Suslin in [20].
3) Cancellation for framed-motives proved by A. Ananievsky, G. Garkusha, I. Panin in [11].
4) Cancellation for Milnor-Witt correspondences in recent work [11] by J. Fasel, P. Östvaer.

In this subsection we give a brief overview of ingredients and steps of the proof of our cancellation theorem, and explain what is the similar and what differs proofs of other cancellation theorems. Following to the original scheme in [22], the main two ingredients for theorem [14] are

1) the construction of a ‘partially defined’ maps $GW Cor(X \times G_m, Y \times G_m)$ in $GW Cor(V, Y)$, which are inverse up to a canonical $A^1$-homotopy to the homomorphism $id_{G_m} \circ \mathbb{E}^{-} : GW^{\oplus}(X, Y) \to GW^{\oplus}(X \times G_m, Y \times G_m)$, (and similarly for $W Cor$);
2) the computation of the $A^1$-homotopy class $[T]$ of the permutation morphism $T$ on $G_m$ in the category of correspondences.

In the case of $Cor$ and $K_0$, $[T] = [-id_{G_m}]$ like as in $S\mathcal{H}^+(k)$. In the case of $GW^{\oplus}$, $[T] = [(-1)]$ (Prop. 2.20), like as for framed correspondences and $S\mathcal{H}(k)$. For $W Cor$, $[T] = [id_{G_m}]$ like as in $S\mathcal{H}^+(k)$.

The construction of the homomorphisms $\rho$ strongly depends on the definition of considered correspondences. Let’s note that the construction of the homotopy for $T$ of course depends in this too, but $\rho$ can not be obtained directly even if we have a functor between the categories of correspondences. Informally, $\rho$ can be thought about as $\cup$-product with the class of the diagonal $\Delta$ in $G_m \times G_m$, or it can be considered by analogy with the trace for linear operators.

1) For $Z \in Cor(X \times G_m, Y \times G_m)$, $\rho$ is defined by the intersection with a generic representor in the Chow-class of the diagonal $\Delta_{G_m}$, which is transversal with respect to $Z$. More precisely, $\rho_{n}(Z) = (Z \cap \Delta_{G_m} - (Z \cap \Delta_{G_m}(-f_n)))$, where $f_n = t^n - 1, f_n = t^n - n \in k[G_m \times G_m] = k(t, u)$. Then $\rho_{n}$ are defined on an exhausting filtration on the group $Cor(X \times G_m, Y \times G_m)$.

2) For $\Phi = [P] \in K_0(X \times G_m, Y \times G_m), P \in Coh(Y \times X)$, $\rho_{n}([P]) = [P/f^n] - [P/f^n - P] \in K_0(X, Y)$.

3) If $\Phi = [(\omega), g] \in Fr_n(X \times G_m, Y \times G_m)$, then $\rho([(\omega)]) = [(\omega), g] \in Fr_{n+1}(X, Y)$, where $\gamma$ is composition

4) In the case when correspondences are defined by a ring cohomology theory, $\rho$ is defined by cohomological multiplication with $pr^*(\Delta_{s}(1))$, where $X \times G_m \times Y \times G_m \xrightarrow{pr} G_m \times G_m \xrightarrow{\Delta} G_m$. Such construction is used for Milnor-Witt-correspondences. As well this can be used for the categories given by $(X, Y) \mapsto GW^{\oplus}_{fin}(Y \times X, \omega_{G_m \times Y})$ or $W^{\oplus}_{fin}(Y \times X, \omega_{G_m \times Y})$ (where $GW^{\oplus}$ denotes hermitian K-theory and subscript $fin$ means that we consider cohomology groups on $Y \times X$ with finite supports over $X$).

Since $GW^{\oplus}_{fin}$ and $W Cor_{fin}$ are defined by quadratic spaces, the closest case in a technical sense is $K_0$-corr. If $\Phi = [(P, q)]$, then $\rho(P, q) = (P^+, q^+) - [(P^-, q^-)]$, where $P^+ = P//f_n - P, P^- = P/f_n - P$. So the question is to equip sheaves $P/fP$ with quadratic forms In the case of affine schemes $P$ is $k[Y \times X]$-module finitely generated projective over $k[X)$, and $q: P \simeq Hom_{k[X]}(P, k[X \times G_m])$ is $k[X \times G_m \times Y \times G_m]$-linear isomorphism. So we should construct $k[Y \times X]$-linear isomorphism $q^1: P/fP \simeq Hom_{k[X]}(P/fP, k[X])$ for such $P$ that $P/fP$ is finitely generated projective over $k[X]$, where $f = f_n$ or $f_n$.

Firstly we choose some additional data, namely, $N \in k[A^1_X] = k[X][t]$ and $q \in k[X \times G_m \times Y \times G_m]$ such that the vanish locus $Z(N) \in X \times G_m$ is finite over $X$, and contains the image of $\text{Supp } P/fP$. CANCELLATION FOR GW-CORRESPONDENCES AND WITT-CORRESPONDENCES 3
under the projection on $X \times \mathbb{G}_m$, and such that for any $a \in P$, we have $N \cdot a = fg \cdot a$. Actually, we can put $N = \text{det} P(f)$. Then the construction is in four steps:

1. Put $(P', q') = (P, g) \otimes_{k[X]} k[X \times \mathbb{G}_m] / (N)$.
2. Define a linear homomorphism $t^N : k[Z(N)] \to k[X]$ as the junior term of the Euler trace $k[X][t] \to k[X][N]$, where $k[X][N]$ is identified with the subalgebra $k[X][t]$ generated by $N$.
3. Consider a quadratic space $(P', q'\circ t)$ over $k[X]$, where $q'(a, b) = t^N (q'(a, b))$.
4. Now we consider the quadratic form $q''(a, b) = q'^{(g \cdot a, b)}$. Then $q''$ is non-degenerate, and its kernel is exactly $FP'NP \subseteq FP\setminus P'$, which defines a quadratic form $q^\rho$ on $P'/FP$ which is the result of the construction. I.e. the resulting quadratic space is $(P/F, q^\rho)$.

By a formula this means $\rho_f (P, q) = \text{red} (g \cdot ((P, q) \circ \langle N \rangle))$ (see Def. 5.6), where $\langle N \rangle = (k[Z(N)], q^N) \in GW[\mathbb{C}^\infty(X, X \times \mathbb{G}_m)]$. $q^N (a, b) = t^N (ab), g \cdot (\cdot)$ denotes operation of multiplication of quadratic form by the function $g$, and $\text{red}(\cdot)$ denotes the reduction of degenerate quadratic from to the factor-space.

Since $\rho$ is well defined only if $P/F^{+/-}P$ is fin.gen. projective over $k[X]$ precisely we define the set of homomorphisms of presheaves $GW\text{Cor}(\mathbb{G}_m, Y \times \mathbb{G}_m) \preceq_{\text{GW, L}} R_\alpha^{GW,R} (id_{\mathbb{G}_m}) \cong \text{H}(\mathbb{G}_m, -)$ where $R_\alpha$ is a filtering systems $\lim_{\alpha \in \mathcal{A}} R_\alpha = GW\text{Cor}(\mathbb{G}_m, Y \times \mathbb{G}_m)$, see Lm 5.3 for the case of ‘left inverse’. In distinct to the cases of other correspondences $\alpha$ aren’t injections and $\mathcal{A}$ is not $\mathbb{Z}$, since it relates to the choice of the triple $(f, N, g)$ instead of a one function $f$. In sections 4, 5, 6 we show that this construction is correct and satisfies enough functoriality.

The construction above gives the homomorphism $\rho$ for presheaves. But to prove cancellation in the form of isomorphism (1.1) following the mentioned scheme we need to construct $\rho$ for sheaves $GW_{\text{eff}}(-, -)$. In distinct to the cases of Cor and framed correspondences the presheaves $GW_{\text{Cor}}(-, -)$ aren’t sheaves. That is why we prove firstly the cancellation theorem for presheaves, i.e. the quasi-isomorphism $GW\text{Cor}(X \times \Delta^*, Y) \simeq GW\text{Cor}(X \times \mathbb{G}_m^1 \times \Delta^*, Y \times \mathbb{G}_m^1)$. This yields that the adjunction $- \otimes_{\mathbb{G}_m} D^R_{\mathbb{G}_m} (\text{Pre}(GW\text{Cor})) \Rightarrow D^R_{\mathbb{G}_m} (\text{Pre}(GW\text{Cor})) : \text{Hom}((\mathbb{G}_m, -))$ is a coreflection, where $D^R_{\mathbb{G}_m} (\text{Pre}(GW\text{Cor}))$ is the localisation of $D^R_{\mathbb{G}_m} (\text{Pre}(GW\text{Cor}))$ with respect to $\text{H}^1$-equivalences. Next we show that this pair of functor actually is a coreflection after the Nisnevich localisation, which is the claim according the paragraph before (1.1). The presheave $K_0 (-, Y)$ are not sheaves too, and we think that our reasoning actually is equivalent, to the reasoning in [20]. But the approach used here is more functorial and it allows us to avoid references to internal arguments inside the proofs of some Voevodsky’s lemmas about presheaves with transfers (used in [20]), and it allows to avoid some technical details in this part of the text.

1.4. The text review: In the section 2 we recall the definition, and prove some properties of $QC\text{or}, GW^\oplus, WC\text{or}$, and the construction that produce a quadratic space from a regular function on a relative affine line. In section 3 we recall some facts about $DM_{GW}^W(k)$ and $DM_W^W(k)$.

In sections 4, 5, and 6 we prove some technical results on endomorphisms of locally free coherent sheaves of finite rank, and construct the homomorphisms $\rho^{GW,W,R}_\alpha, \rho^{GW,R}_\alpha, \rho^{W,R}_\alpha, \rho^{W,S}_\alpha$.

Finally, in section 7 we prove Cancellation Theorem 1.12, Theorem 1.12 and isomorphism (1.1).

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1.6. Notation and conventions: By default we assume schemes being of finite type and separated over a field $k$, char $k \neq 2$; $Sm_k$ is the category of smooth schemes over $k$; $Coh(X) = Coh_X$ denotes the category of coherent sheaves on a scheme $X$. Denote by $\Gamma_f$ a graph of a regular map $f : X \to Y$ and $f^{-1}(Z) = Z \times_Y X$. $k[X] = \Gamma(X, O(X))$, and for any $f \in k[X]$, $Z(f) = f^{-1}(0)$, and $Z_{\text{red}}(f)$ is its reduced subscheme. For $P \in Coh(X)$ we denote by $Supp P$ the closed subscheme in $X$ defined by the sheaf of ideals $\mathcal{I}(U) = \text{Ann} P \big|_U \subseteq k[U]$ and by $Supp_{\text{red}} P$ its reduced subscheme. Finally we denote by $F_{\text{nis}}$ the Nisnevich sheafification of a presheave $F$ on $Sm_k$. 
2. Quadratic correspondences

2.1. Categories $Q\text{Cor}$, $GW^\oplus$, $W\text{Cor}$. In this subsection we summarise definitions and used properties of $GW$-correspondences and Witt-correspondences, see [3], [7] for more details and proofs.

**Definition 2.1.** For a morphism of schemes $p: Y \to X$ let $\text{Coh}_{fin}(p)$ (or $\text{Coh}_{fin}(Y \to X)$) denotes the full subcategory of the category of coherent sheaves on $Y$ spanned by sheaves $F$ such that $\text{Supp} F$ is finite over $X$; and let $P(p)$ (or $P^X_Y$) denotes the full subcategory of $\text{Coh}_{fin}(Y)$ spanned by sheaves $F$ such that $p_*(F)$ is a locally free sheaf on $X$. For two schemes $X$ and $Y$ over a base scheme $S$ we denote $\text{Coh}_{fin}^S(X, Y) = \text{Coh}_{fin}(X \times_S Y \to X)$, $\mathcal{P}^S(X, Y) = \mathcal{P}(X \times_S Y \to X)$.

**Remark 2.2.** For affine schemes $X, Y$, $\mathcal{P}(Y \to X)$ is equivalent to the full subcategory in the $k[Y] - \text{Mod}$ spanned by finitely generated and projective modules over $k[X]$.

The functors $k[Y]-\text{mod} \to k[Y]-\text{mod}^{\oplus}: M \to Hom_{k[X]}(M, k[X])$ for finite morphisms $Y \to X$ of affine schemes, defines in a canonical way a functor $D_X: \text{Coh}_{fin}(Y_X) \to \text{Coh}_{fin}(Y_X)^{\oplus}$ for any morphism of schemes $Y \to X$. This gives an exact category with duality $(\mathcal{P}(Y \to X), D_X)$. For any schemes $X, Y$ and $Z$ over a base scheme $S$ the tensor product over $Y$ induce a functor of categories with duality

$$
\cap: (\mathcal{P}^S(Y, Z), D_Y) \times (\mathcal{P}^S(X, Y), D_X) \to (\mathcal{P}^S(X, Z), D_X),
$$

where $(\mathcal{P}^S(Y, Z), D_Y) \times (\mathcal{P}^S(X, Y), D_X) = (\mathcal{P}^S(X, Y) \times \mathcal{P}^S(Y, Z), D_X \times D_Y)$. This functor is natural in $X, Y$ and $Z$ and satisfies the associativity axiom.

**Definition 2.4.** Let $(\mathcal{C}, D)$ be an exact category with duality, then

1) a pair $(P, q)$, where $P \in \mathcal{C}$ and $q: P \to D(P)$ is symmetric morphism, is called a quadratic pre-space, and let’s denote by $\text{preQ}(\mathcal{C}, D)$ the set of isomorphism classes of quadratic pre-spaces;

2) a quadratic space is a pair $(P, q)$ whenever $q: P \to D(P)$ is isomorphism, and we denote by $Q(\mathcal{C}, D)$ the set of isomorphism classes of quadratic spaces;

3) $GW^\oplus(\mathcal{C}, D)$ denotes the Grothendieck-Witt-group of $Q(\mathcal{C}, D)$ in respect to the direct sums, 

4) $W(\mathcal{C}, D)$ denotes the Witt-group of $(\mathcal{C}, D)$, i.e. factor-group of $GW(\mathcal{C}, D)$ by classes of metabolic spaces (see [4]).

Some times we write $Q(\mathcal{P}(X, Y))$ for $Q(\mathcal{P}(X, Y), D_X)$ since we always consider the duality $D_X$.

**Definition 2.5.** Let $k$ be a field. The category $Q\text{Cor}_k$ is the category such that objects of $Q\text{Cor}_k$ are smooth varieties and morphisms $Q\text{Cor}(X, Y) = Q(\mathcal{P}(X, Y), D_X)$. The composition is induced by the functor $\mathcal{P}^S$, and the identity morphism $Id_X$ is defined by the class of quadratic space $(\mathcal{O}(\Delta), 1)$ where $\Delta: \Delta \to X \times X$. To shortify notations we write sometimes $Q(X, Y)$ for $Q\text{Cor}(X, Y)$. The categories $GW^\oplus_k$ and $W\text{Cor}_k$ are the additive categories with the same objects and such that $GW^\oplus_k(X, Y) = GW^\oplus(\mathcal{P}^X_Y, D_X)$, $W\text{Cor}_k(X, Y) = W(\mathcal{P}^X_Y, D_X)$, the composition is induced by the one of $Q\text{Cor}$. A presheave with $GW$-transfers is an additive presheave $F: GW\text{Cor}_k \to Ab$ similarly for presheaves with Witt-transfers.

**Notation 2.6.** For $X, Y, Z \in Sm_k$ and a pair of quadratic spaces $(P_1, q_1) \in Q\text{Cor}(X, Y)$, $(P_2, q_2) \in Q\text{Cor}(Y, Z)$ we call $(P_2, q_2) \circ (P_1, q_1)$ the composition of the quadratic spaces and denote it also sometimes by $(P_2, q_2) \circ (P_1, q_1)$ or by $(P_2 \otimes_Y P_1, q_2 \otimes_Y q_1)$.

**Definition 2.7.** We write $\Phi_1 \xrightarrow{k^1} \Phi_2$ for $\Phi_1, \Phi_2 \in GW^\oplus_k(X, Y)$ whenever there is $\Theta \in GW^\oplus_k(X \times A^1, Y)$ such that $\Theta \circ i_0 = \Phi_1, \Theta \circ i_1 = \Phi_2$, where $i_0, i_1: X \to X \times A^1$ denotes zero and unit sections. In such a situation we call $\Theta$ by $A^1$-homotopy joining $\Phi_1$ and $\Phi_2$. Denote by $GW^\oplus_{k}$ the factor-category of $GW^\oplus_k$ such that morphisms are classes up to homotopy equivalence.

**Theorem 2.8** (see [6], theorem 1, and [7]). Suppose is a homotopy invariant presheave with $GW$-transfers (Witt-transfers). Then (a) for a local essentially smooth $U$ over a base filed $k$, the restriction homomorphism $F(U) \to F(\eta)$ is injective, where $\eta \in U$ is generic point.
Theorem 2.9 (see [7]). For $F$ as above and for any open subschemes $V_1 \subset V_2 \subset \mathbb{A}_k^1$, $k = k(X)$, $X \in Sm_k$, the restriction homomorphism $F(V_2) \to F(V_1)$ is injective.

Theorem 2.10 (see [7], theorem 3.1 and corollary 4.13; [10], theorem 3; and [5]). For a presheave with GW-transfers (Witt-transfers) $F$ over a field $k$, the Nisnevich sheaf $F_{nis}$ and Nisnevich cohomology presheaves $h_{nis}(F_{nis})$ are equipped with the structure of presheaves with GW-transfers (Witt-transfers) in a canonical way.

If $F$ is homotopy invariant, then $H^*_{nis}(U) = 0$ for any open $U \subset \mathbb{A}_k^1$, $k = k(X)$, $X \in Sm_k$. If in addition $k$ be infinite, perfect, char $k \neq 2$, then the associated Nisnevich sheaf $F_{nis}$ and Nisnevich cohomology presheaves $h_{nis}(F_{nis})$ are homotopy invariant.

For any pair of quadratic spaces $(P_1, q_1) \in Q(P^1_{X_1})$, $(P_2, q_2) \in Q(P^1_{X_2})$, the isomorphism

$$q_1 \otimes_k q_2: P_1 \otimes_k P_2 \cong D_{X_1}(P_1) \otimes_k D_{X_2}(P_2) \cong D_{X_1 \times X_2}(P_1 \otimes_k P_2)$$

defines a quadratic space $(P_1 \otimes_k P_2, q_1 \otimes_k q_2) \in Q(P^1_{X_1 \times X_2})$ and if at least one of spaces $(P_1, q_1)$, $(P_2, q_2)$ is metabolic then $(P_1 \otimes_k P_2, q_1 \otimes_k q_2)$ is metabolic. So we have the following:

Lemma 2.11. The tensor product over a base field $k$ induce a functors $- \otimes_k -: QCor_k \times QCor_k \to QCor_k$, and $- \boxtimes -: GW^\otimes_k \times GW^\otimes_k \to GW^\otimes_k$ (and similarly for WCOr), such that $X \boxtimes Y = X \times Y$.

Remark 2.12. We use symbol $\boxtimes$ here, to distinct products on the categories of correspondences from the tensor product on the category of presheaves with transfers for which we use the symbol $\otimes$.

Remark 2.13. Since $GW^\otimes_k$ is additive category, all mentioned operations, functors and presheaves with GW-transfers can be passed to the Karoubi envelope $Kar(GWCor_k)$. Similarly for WCOr$_k$.

2.2. Construction of quadratic correspondences by a function on an oriented curve.

Lemma 2.14. Let $X \in Sm_k$ and $f \in k[X \times \mathbb{G}_m]$ such that $Z(f)$ is finite over $X$ and $Z(f) \neq \emptyset$. Then there is a unique regular map $\overline{f}: \mathbb{P}^1 \to X \times \mathbb{G}_m$. Moreover for such a map $\overline{f}$ the morphism $(\overline{f}, pr_X): \mathbb{P}^1 \times X \to \mathbb{P}^1 \times X$ is finite and flat and the scheme theoretical preimage $\overline{f}^{-1}(0)$ is equal to the vanish locus $Z(f)$.

Proof. If the map $\overline{f}$ exists, then $\Gamma_{\overline{f}} = \Gamma_f \subset \mathbb{P}^1 \times X \times \mathbb{P}^1$. Hence if $\overline{f}$ exists then it is unique.

Let $l_0$ be the minimal integers such that there is $r \in \Gamma(\mathbb{P}^1 \times X, O(l_0 + l_0))$: $f = r/(l_0 \cdot l_0)$. We are going to prove that $Z(f) = Z(r) \subset \mathbb{P}^1 \times X$. Since $Z(f)$ is finite over $X$, it is closed in $\mathbb{P}^1_X$. On the other hand $Z(f) = Z(r) \cap X \times \mathbb{G}_m$, so it is open in $Z(r)$. Then $Z(f)$ is disjoint component of $Z(r)$, and let $Z(r) = Z(f) \sqcup Z_0 \sqcup Z_0$, where $Z_0 \subset \mathbb{P}^1 \times X$ and $Z_0 \subset \mathbb{P}^1 \times X$. If $Z_0 \neq \emptyset$, then $\dim Z_0 \geq \dim (X \times \mathbb{G}_m) - 1 = \dim (\mathbb{P}^1 \times X)$; hence $Z_0 = \mathbb{P}^1 \times X$ and $r|_{\mathbb{P}^1 \times X} = 0$. This contradicts to the minimality of the choice of $l_0$. So $Z_0 = \emptyset$, and similarly $Z_0 = \emptyset$. Then $Z(f) = Z(r) \cap X \times \mathbb{G}_m = Z(r)$.

Define $\overline{f} = [r : l_0 \cdot l_0]: \mathbb{P}^1 \times X \to \mathbb{P}^1$. Then from the above we have $Z(f) = \overline{f}^{-1}(0)$. Since $\dim Z(f) \geq \dim \mathbb{P}^1_X - 1 = \dim X$ and since $Z(f)$ is finite over $X$, it follows that projection $Z(f) \to X$ is surjective. Hence the fibre of $Z(f)$ over each point $x \in X$ is not empty and it is not equal to $\mathbb{P}^1_x$. Hence the fibre of the map $(\overline{f}, pr_X): \mathbb{P}^1 \times X \to \mathbb{P}^1 \times X$ over each point $x \in \mathbb{P}^1 \times X$ is not empty and it is not equal to $\mathbb{P}^1_x$, and so $(\overline{f}, pr_X)$ is quasi-finite. Thus the morphism $(\overline{f}, pr_X)$ is quasi-finite and projective, and whence it is finite. Finally, since it is finite morphism of equidimensional smooth varieties, it is flat.

Definition 2.15. For any $U \in Sm_k$ and open subscheme $V \subset \mathbb{A}^1 \times U$, we denote by $FP(U, V)$ the set of regular maps $\overline{f}: \mathbb{P}^1 \times U \to \mathbb{P}^1$ such that $\overline{f}^{-1}(0)$ is finite over $U$ and $\overline{f}^{-1}(0) \subset V$. Let $FP(U, V, Z) \subset FP(U, V)$, $f \in FR(U, V, Z)$ iff $\overline{f}^{-1}(0) = Z \sqcup Z'$.

Proposition 2.16. There are
1) a natural (along base changes) map $FP(U, V) \to Q(P(U, V))$ such that any function $\mathcal{F} \in FP(U, V)$ goes to some quadratic space $(\mathcal{F}) = (\mathcal{O}(f^{-1}(0)), u)$.

2) a natural map $FP(U, V, Z) \to Q(P(U, Z))$ such that any function $\mathcal{F} \in FP(U, V, Z)$ goes to some quadratic space $(\mathcal{F}, Z) = (\mathcal{O}(Z), u)$, where $u \in k[Z]^\times$.

Proof. 1) Let $F = (\mathcal{F}, pr_U)$ for $\mathcal{F} \in FP(U)$. Let’s denote the source of the morphism $F$ as $Y$ and the target by $X$. Then $X$ and $Y$ are isomorphic to two copies of the relative projective lines over $X$ and morphism $F$ is finite, The Grothendieck duality for a finite morphism of smooth projective varieties leads to the natural isomorphism $\omega_X \simeq \text{Hom}_Y(f_*(\mathcal{O}(X)), \mathcal{O}(Y))$, and functoriality of this isomorphism in respect to endomorphisms of $\mathcal{O}(X)$ implies that it defines isomorphism $\omega_X \simeq D_\mathcal{F}(\mathcal{O}(X))$ that can be considered as symmetric quadratic space $(\mathcal{O}(X), \mathcal{I}) \in Q(P(f), D_\mathcal{F}(\omega_\mathcal{Y} \otimes \omega_\mathcal{X}^\times))$, where $\mathcal{L} = f^*(\omega_\mathcal{Y}) \otimes \omega_\mathcal{X}^\times$, and $D_\mathcal{F}(\mathcal{F}) = D_\mathcal{Y}(\mathcal{F}) \otimes \mathcal{L}$. Then put $(\mathcal{O}(Z), q_j) = i^*(\mathcal{O}(X), \mathcal{I}) \in Q(P(f), D_\mathcal{X}(f^*(\omega_\mathcal{Y} \otimes \omega_\mathcal{X}^\times)))$, where $j: Z \hookrightarrow \mathbb{A}_U^1, i: 0_U \hookrightarrow \mathbb{P}_U^1$. Now since the standard trivialisation of $\omega(\mathbb{A}_U^1)$ given by differential of coordinate gives an identification $i^*(\omega_\mathcal{Y}) \simeq \mathcal{O}(U)$ and $j^*(\omega_\mathcal{X}) \simeq \mathcal{O}(Z)$, we get required quadratic space $(\mathcal{O}(Z), q_j) \in Q(P(f), D_\mathcal{X})$.

2) The claim follows from that if $\mathcal{F}^{-1}(0) = Z \sqcup Z'$ then any quadratic space $(\mathcal{O}(\mathcal{F}^{-1}(0)), q)$ splits in a canonical way into the sum $(\mathcal{O}(Z), u) \oplus (\mathcal{O}(Z'), u')$. □

Definition 2.17. Let $U \in Sm_k, V \subset \mathbb{A}^1$ be open subscheme and $f: V \times U \to \mathbb{A}^1$ be regular function. Then we denote by $(f, V \times U) \in Q(P(U, V \times U))$ the image under the map from Proposition 2.16 of $\mathcal{F}$ given by differential of coordinate gives an identification $i^*(\omega_\mathcal{Y}) \simeq \mathcal{O}(U)$ and $j^*(\omega_\mathcal{X}) \simeq \mathcal{O}(Z)$, we get required quadratic space $(\mathcal{O}(Z), q_j) \in Q(P(f), D_\mathcal{X})$.

Lemma 2.18. Assume char $k \neq 2$, then for any finite scheme $Z$ over a variety $X$ and quadratic space $(\mathcal{O}(Z \times \mathbb{A}^1), q) \in Q(Coh_{fin}(Z \times \mathbb{A}^1 \to X \times \mathbb{A}^1))$, there is an isomorphism of quadratic spaces $(\mathcal{O}(Z), q_0) \simeq (\mathcal{O}(Z), q_1)$, where $q_0 = i_0^*(q), q_1 = i_1^*(q)$ and $i_0, i_1: X \to X \times \mathbb{A}^1$ denote zero and unit sections.

Lemma 2.19. For any regular function $q$ on a scheme $Z$ (over the base field $k$, char $k \neq 2$) such that $q|_{Z_{red}} = 1$ there is a square root $t \in \mathcal{O}(Z)$: $t^2 = q \in \mathcal{O}(Z)$.

Proof of Lemma 2.18. Any quadratic form on $\mathcal{O}(Z \times \mathbb{A}^1)$ is defined by invertible regular function $q$ on $Z \times \mathbb{A}^1$. In the case of reduced $Z$ any such function is constant along $\mathbb{A}^1$, and hence $q_0 = i_0^*(q) = i_1^*(q) = q_1$, that implies the claim.

In general case we have that restriction $q|_{Z_{red} \times \mathbb{A}^1}$ is constant (along $\mathbb{A}^1$). Hence $i_0^*(q)$ and $i_1^*(q)$ are equal on $Z_{red}$. To prove the claim let’s note that any function on $Z$ that is equal to 1 on $Z_{red}$ has a square root and applying this to fraction $i_0^*(q)/i_1^*(q)$ we get isomorphism of quadratic spaces $i_0^*(\mathcal{O}(Z \times \mathbb{A}^1), q)$ and $i_0^*(\mathcal{O}(Z \times \mathbb{A}^1))$. □

2.3. Locally splitting homomorphisms.

Lemma 2.20. For any endomorphism of coherent sheaves $e \in End_{Coh_{fin}(X, Y)}(P)$ for any schemes $X, Y$ and $P \in Coh_{fin}(X, Y)$, there is a unique commutative diagram

$$
\begin{array}{cccccc}
\text{Ker } D(e) & \longrightarrow & D_X(P) & \longrightarrow & \text{Im } D(e) & \longrightarrow & D_X(P) & \longrightarrow & \text{Coker } D(e) \\
\downarrow & & \downarrow w & & \downarrow & & \downarrow & & \\
D(\text{Coker } e) & \longrightarrow & D_X(P) & \longrightarrow & D_X(\text{Im } e) & \longrightarrow & D_X(P) & \longrightarrow & D(\text{Ker } e)
\end{array}
$$

Proof. The vertical arrows in the diagram are defined by universal properties of kernels and cokernels. □
Definition 2.22. A morphism of coherent sheaves \( e: P_1 \to P_2 \) on a scheme \( X \) is called locally splitting whenever both short exact sequences in the diagram

\[
\begin{array}{c}
\text{kere} & \xrightarrow{e} & P_1 & \xrightarrow{\text{Im}e} & P_2 & \xrightarrow{\text{Coker}e} \text{kerc}
\end{array}
\]
splits locally on \( X \). A morphism \( e: P_1 \to P_2 \in \text{Con}_{\text{fin}}(X,Y) \) for a varieties \( X,Y \) is called locally splitting whenever its direct image along the projection \( X \times Y \to X \) is locally splitting on \( X \).

Lemma 2.23. 1) For any varieties \( X,Y \) the subcategory of locally splitting morphisms is an abelian subcategory of \( \text{Coh}_{\text{fin}}(X,Y) \), and the functor \( D_X \) is exact on the subcategory of locally splitting morphisms in \( \text{Coh}_{\text{fin}}(X,Y) \).

2) For any locally splitting morphism the vertical arrows in the commutative diagram (2.21) are isomorphisms.

3) Any short exact sequence in \( \mathcal{P}(X,Y) \) is locally splitting.

Proof. 1) Since for any locally splitting homomorphism \( e \), the homomorphisms \( \text{kere} \) and \( \text{coker}e \) are locally splitting the first statement follows.

The second claim follows from that base change of the duality \( D_X \) along the open immersion \( U \hookrightarrow X \) (or along the embedding of local subscheme \( X_x \hookrightarrow X \)) is equal to \( D_U (D_{X_x}) \), and for any scheme \( U \) the functor \( D_U \) sends splitting sequences to splitting ones.

2) The claim follows from the previous point, since by definition exact functor preserves kernels, images, and cokernels.

3) The last statement follows form that short exact sequence of locally free coherent sheaves is locally splitting.

\[ \Box \]

2.4. Homotopy for permutation on \( G^\Delta^1_m \times G^\Delta^1_m \).

Lemma 2.24. For a homotopy invariant presheave with GW-transfers \( F \) the homomorphism \( F(G^\Delta^1_m \times G^\Delta^1_m) \to F(G^\Delta^1_m \times G^\Delta^1_m, \Delta_{G^\Delta^1_m}) \) is injective.

Proof. The prove is based on the same method on the prove of the injectivity on the relative affine line. Indeed, the method proves the injectivity for a pair of open subsets \( U \subset V \) in the relative affine line \( A^1_X \) such that \( U = A^1 - (T \cap D) \), \( V = A^1 - T \), where \( T \) is quasi-finite and \( D \) is finite over \( X \). \( \Box \)

Definition 2.25. Let \( pr^G_m: G_m \to pt \), and let \( 1: pt \to G_m \) denotes the unit section. Let \( G^\Delta^1_m = \text{Coker}(G_m \to pt) \in \text{Kar}(GW\text{Cor}_k) \) and let \( 1_{G^\Delta^1_m} = 1 \circ pr^G_m \). Finally, let \( pr^\Delta^1_m: GW^\otimes(G_m, G_m) \to GW^\otimes(G^\Delta^1_m, G^\Delta^1_m) \) denote the canonical projection to a direct summand, and let \( e = (id_{G_m} - 1_{G_m}) \circ - \circ (id_{G^\Delta^1_m} - 1_{G^\Delta^1_m}) : GW^\otimes(G_m, G_m) \to GW^\otimes(G^\Delta^1_m, G^\Delta^1_m) \) be the corresponding idempotent. The same notation we use for \( W\text{Cor} \).

Lemma 2.26. Let \( T \) denote the transposition on \( G^\Delta^1_m \times G^\Delta^1_m \), then

\[
[T] = \langle [-1] \cdot id_{G^\Delta^1_m \times G^\Delta^1_m} \rangle \in GW^\otimes_k(G^\Delta^1_m \times G^\Delta^1_m, G^\Delta^1_m \times G^\Delta^1_m).
\]

Proof. Let \( i_Q: Q = Z(t^2 - at + m) \subseteq A^1 \times G_m \), where we use \( k[A^1 \times G_m] = k[a,m,-m] \). Define

\[
\Pi = (t^2 - at + m, Q) \in GW^\otimes(A^1 \times G_m, Q), \quad C: G_m \times G_m \to A^1 \times G_m: C(x,y) = (x,y).
\]

Then \( \Pi \circ C = \langle (t-x)(t-y), Q \times G_m \times G_m \rangle \subseteq GW\text{Cor}(G_m \times G_m, Q) \). The map \( G_m \times G_m \to A^1 \times A^1 \times G_m: (x,y) \mapsto (x+y,x+y) \) induce isomorphism \( G_m \times G_m \cong Q \). Hence

\[
(2.27) \quad (\Pi \circ i)^{(-1)} = (x-y)(t-x,Z(t-x)) + (y-x)(t-y,Z(t-y)) =
\]

\[
\langle x-y \rangle \cdot (id_{G_m \times G_m} + (-1) \circ T) \in GW\text{Cor}(G_m \times G_m - \Delta_{G_m}, Q),
\]

where \( i: G_m \times G_m - \Delta_{G_m} \hookrightarrow G_m \times G_m \).
On the other hand
\[ \Pi \circ sect \circ \mu \circ i = (1 - xy) \circ [(x, y) \mapsto (1, xy)] + (xy - 1) \circ [(x, y) \mapsto (xy, y)], \]
where \( \mu : \mathbb{G}_m \times \mathbb{G}_m \to \mathbb{G}_m \) : \((x, y) \mapsto xy\) and \( sect : \mathbb{G}_m \to \mathbb{A}^1 \times \mathbb{G}_m \) : \((a, m) \mapsto (m + 1, m)\); and whence \( e \circ \Pi \circ sect \circ \mu \circ i \in 0 \in GWCor_k(\mathbb{G}_m \times \mathbb{G}_m, \mathbb{G}_m^\times \times \mathbb{G}_m^\times). \)

Now using homotopy \( h : \mathbb{A}^1 \times \mathbb{G}_m \to \mathbb{A}^1 \times \mathbb{G}_m \) : \((a, m) \mapsto (a(1 - \lambda) + (1 + m)\lambda, m)\), and since \( h \circ i_0 = id_{\mathbb{A}^1 \times \mathbb{G}_m}, h \circ i_1 = sect \circ \mu \), we get
\[ pr^\lambda(id_{\mathbb{G}_m} + (-1) \circ T) = ((x - y)^{-1} \circ \Pi \circ C \circ i) = \]
\[ = ((x - y)^{-1}) \circ \Pi \circ sect \circ \mu \circ C \circ i = 0 \in GWCor_k(\mathbb{G}_m \times \mathbb{G}_m - \Delta_{\mathbb{G}_m}, \mathbb{G}_m^\times \times \mathbb{G}_m^\times). \]

Now the claim follows form that homomorphism \( GWCor_k(\mathbb{G}_m^\times \times \mathbb{G}_m^\times, \mathbb{G}_m^\times \times \mathbb{G}_m^\times) \to GWCor_k(\mathbb{G}_m^\times \times \mathbb{G}_m^\times, \mathbb{G}_m^\times \times \mathbb{G}_m^\times - \Delta_{\mathbb{G}_m^\times}, \mathbb{G}_m^\times \times \mathbb{G}_m^\times) \) is injective. (see lemma 2.24). \( \square \)

**Remark 2.28.** In the previous proof in the equation we use base change along closed embeddings for the duality isomorphism for finite morphisms \( f : X \to X \) is equal to the isomorphism defined by canonical isomorphisms \( \mathcal{O}(X) \simeq f^*(\mathcal{O}(X)) \) and \( \omega(X) \simeq f^*(\omega(X)) \).

Let’s note also that for this computation it is enough to use flat base changes, and actually base changes along isomorphisms. To do this we need to apply base change for the square defined by two copies of morphism \( Q \to \mathbb{A}^1 \times \mathbb{G}_m \), identity isomorphism on \( \mathbb{A}^1 \times \mathbb{G}_m \) and isomorphism \( T \) on \( Q \) (using isomorphism \( Q \simeq \mathbb{G}_m \times \mathbb{G}_m \)). Then we get that \((t - x_1)(t - x_2)z(t - x_1) = (\lambda) \) and \(((t - x_1)(t - x_2))z(t - x_1) = (\lambda) \) for some invertible function \( \lambda \in k[\mathbb{G}_m^\times]^\times \). So the required equality \([id_{\mathbb{G}_m^\times + \Delta}] = [(-1)id_{\mathbb{G}_m^\times + \Delta}] \in GWCor_k(\mathbb{G}_m^\times - \Delta, \mathbb{G}_m^\times - \Delta) \) follows after the multiplication of quadratic forms in (2.27) and in other equalities form the proof above by \( \lambda^{-1} \).

**Corollary 2.29.** 1) \([T] = [id_{\mathbb{G}_m^\times}] \) in \( WCOr_k \), 2) \([C_3] = [id_{\mathbb{G}_m^\times}] \) in \( GWCor_k \), where \( C_3 \) denotes the 3-cycle permutation.

**Proof.** 1) The claim follows form that \([1]) + [(-1)] = 0 \in WCOr_k(pt, pt), \) since it is class of a metabolic space. 2) The claim follows from that a 3-cycle is a composition of two transpositions and \([(-1)]^2 = id_{pt} \in GWCor_k(pt, pt). \) \( \square \)

3. Categories of GW-motives and Witt-motives

Starting form this section we assume that the base field \( k \) is infinite, perfect, and \( char k \neq 2 \). Here we recall the results of [2] on the categories of effective GW-motives and Witt-motives used in the article. We consider the case of GW-motives, and the case of Witt-motives is similar.

Let \( Pre \) and \( Sh_{Nis} \) denote categories of presheaves and Nisnevich sheaves on \( Sm_k \), \( Pre(GWCor) \) (or \( Pre_{GW} \)) the category of presheaves with GW-transfers, \( Sh_{Nis}(GWCor) \) (or \( Sh_{GW} \)) the category of Nisnevich sheaves with GW-transfers. It is proven in [2] that \( Sh_{Nis}(GWCor) \) is abelian.

**Definition 3.1.** The category of effective GW-motives \( DM_{eff}^{GW}(k) \) is the full subcategory in \( \mathcal{D}^\bullet((Sh_{Nis}(GWCor)) \) spanned by the complexes with homotopy invariant cohomology sheaves. The functor \( M_{GW} : Sm_k \to DM_{eff}^{GW}(k) \), which sends \( X \) to \( Sm_k \) to its effective GW-motive, is defined as \( M_{GW}^{eff}(X) = Hom_{\mathcal{D}^\bullet(Pre_{GW})}(\Delta^\bullet, GWCor_{nis}(-, X)) = GWCor_{nis}(- \times \Delta^\bullet, X) \), where \( \Delta^\bullet \) denotes affine simplexes.

**Theorem 3.2.** Suppose \( k \) is an infinite perfect filed, \( char k \neq 2 \).

There is an adjunction \( L^{pre}_{GW} : \mathcal{D}^\bullet((Pre_{GW}) \leftarrow DM_{GW}^{GW}(k) : R^{pre}_{GW} \) such that \( R^{pre}_{GW} \) is equivalent to the full embedding functor of the subcategory spanned by complexes with homotopy invariant cohomology presheaves, \( L^{pre}_{GW} \) is equal to the localisation with respect to morphisms of the form \( Z_{GW}(X) \times \mathbb{Z}_{GW}(K) \to X \), for all \( X \in Sm_k \), and \( L^{pre}_{GW} \circ R^{pre}_{GW}(Z_{GW}(X)) = GW^{\otimes}(\Delta^\bullet \times -, X) \).
Moreover the functors $L^\text{pre}_{\mkern 1mu \mathcal{A}^1}$ and $R^\text{pre}_{\mkern 1mu \mathcal{A}^1}$ are exact with respect to the Nisnevich quasi isomorphisms. The localisation of $D^\text{eff}_{\mathcal{A}^1}(k)$ with respect to Nisnevich quasi isomorphisms is equivalent to $\text{DM}^\text{eff}(k)$. The adjunction $L^\text{pre}_{\mkern 1mu \mathcal{A}^1} \dashv R^\text{pre}_{\mkern 1mu \mathcal{A}^1}$ induces the adjunction $L_{\mkern 1mu \mathcal{A}^1} : D^-(Sh^\text{GW}_{\mkern 1mu k}) \rightleftarrows \text{DM}^\text{eff}(k) : R_{\mkern 1mu \mathcal{A}^1}$ such that $L_{\mkern 1mu \mathcal{A}^1} \circ R_{\mkern 1mu \mathcal{A}^1}(\mathcal{C}_\text{GW,nis}(X)) = M^\text{GW}(X)$.

**Corollary 3.3.** For an infinite perfect field $k$, char $k \neq 2$, $X \in \text{Sm}_k$ and a motivic complex $\Lambda^* \in \text{DM}^\text{eff}(k)$ there is a natural isomorphism $\text{Hom}_{\text{DM}^\text{eff}(k)}(\Lambda^*, \Sigma^\infty_{\text{DM}^\text{eff}}(\Lambda^*[i]) \cong H^i_{\text{Eff}}(X, \Lambda^*)$.

The following lemma is used in section 7 to deduce the sheaf cancellation theorem form the presheaf form one.

**Lemma 3.4.** Let $U$ is local essential smooth $k$-scheme, and $V \subset \mathbb{A}^1_k$ is open over an infinite perfect base filed $k$, char $k \neq 2$. Let $\mathcal{F} \in \text{Pre}^\text{GW}$ (or $\text{Pre}^W$) be homotopy invariant. Then $\mathcal{F}_{\text{nis}}(U \times V) \cong \mathcal{F}(U \times V), H^i_{\text{nis}}(U \times V, \mathcal{F}_{\text{nis}}) \cong 0$, for $i > 0$.

**Proof.** Consider the case of $GW^\text{op}$. Let $\eta \in U$ denote the generic point. By Th. 2.10 $U \mapsto H^i_{\text{nis}}(U \times V, \mathcal{F}_{\text{nis}})$ are homotopy invariant presheaves with $GW^\text{op}$-transfers. Hence Th. 2.9 yields the injection $H^i_{\text{nis}}(U \times V, \mathcal{F}_{\text{nis}}) \rightarrow H^i_{\text{nis}}(\eta \times V, \mathcal{F}_{\text{nis}})$ for all $i$, and by Th. 2.10 $\mathcal{F}_{\text{nis}}(\eta \times V) \cong \mathcal{F}(\eta \times V)$, $H^i_{\text{nis}}(\eta \times V, \mathcal{F}_{\text{nis}}) = 0$, for $i > 0$.

Thus since the injection $\mathcal{F}(U \times V) \hookrightarrow \mathcal{F}(\eta \times V)$ factors as $\mathcal{F}(U \times V) \rightarrow \mathcal{F}_{\text{nis}}(U \times V) \rightarrow \mathcal{F}_{\text{nis}}(\eta \times V) \cong \mathcal{F}(\eta \times V)$, it follows that $\nu : \mathcal{F}(U \times V) \rightarrow \mathcal{F}_{\text{nis}}(U \times V)$ is injective. On other side, applying Th. 2.9 to the hom. inv. presheaves $\text{Coker}(\nu), \text{Coker}(\nu)(- \times V) \in \text{Pre}^\text{GW}$, $\text{Coker}(\nu)(U \times V) = \text{Coker}(\nu)(\eta \times V), \text{Coker}(\nu)(\eta') = 0$, where $\eta'$ is generic point of $\eta \times V$. Thus $\nu$ is an isomorphism. □

As shown in [7] there is a tensor structure on the category $\text{DM}^\text{eff}(k)$ such that $M^\text{GW}(\Lambda \otimes M^\text{GW}(X) \cong M^\text{GW}(X \times Y)$. (Note that this tensor structure doesn’t relates to the tensor structure $\otimes$ on the category of correspondences defined in Lemma 2.11.) Let $G^\text{eff}_m = (G^\text{eff}_m)^{\otimes k} \in \text{DM}^\text{eff}(k)$, where $G^\text{eff}_{m^k} = \text{Cone}(pt \to G_m)$, and for any $\Lambda^* \in \text{DM}^\text{eff}(k)$, we denote $\Lambda^*(n) = G^\text{eff}_{m^k} \otimes \Lambda^*$.

**Definition 3.5.** Since $\text{DM}^\text{eff}(k)$ is the full subcategory in $D^-(Sh^\text{GW}_k)$, it is dg-category and it is equipped with the injective model structure. Define the category of (non-effective) motives $\text{DM}^\text{eff}(k)$ as the category of $G^\text{eff}_m$-spectra with respect to $\text{DM}^\text{eff}(k)$. So objects of $\text{DM}^\text{eff}(k)$ are sequences $E = (E^*, s_0, E^*_1, s_1, \ldots, E^*_n, s_n, \ldots)$, $E^* \in \text{DM}^\text{eff}(k)$, $s_i \in \text{Hom}_{\text{DM}^\text{eff}}(E^*_i(1), E^*_{i+1})$ and for such two sequences $E = (E_i)$ and $F = (F_i)$ the homomorphism group is defined as $\text{Hom}_{\text{DM}^\text{eff}}(E, F) = \lim_{\text{colim}} \text{Hom}_{\text{DM}^\text{eff}}(E_i, F_i)$.

A functor $\Sigma^\infty_{\text{DM}^\text{eff}} : \text{DM}^\text{eff}(k) \rightarrow \text{DM}^\text{GW}(k)$ takes a motivic complex $E^*$ to the spectrum $(E^*, E^*(1), \ldots E^*(n), \ldots)$ with $s_i$ being identity morphisms $id_{E^*(i)}$.

4. NORM OF THE MULTIPLICATION ENDOMORPHISM $m_F^P$.

**Definition 4.1.** Let $p : V \rightarrow S$ a morphism of schemes, $P \in \mathcal{P}(V \rightarrow S)$, and $f \in k[V]$. Then the multiplication by $f$ defined the endomorphism $m^P_f(p) \in \text{End}(p_*(P))$, and denote $\text{det}(p) = \prod_{r} m^P_f(p) \in \text{End}(p_*(P)) = k[S]$, where $r = \text{rank}_{\mathcal{O}(S)} p_*(P)$.

**Lemma 4.2.** For any morphism of schemes $p : V \rightarrow S$, $P \in \mathcal{P}(V \rightarrow S)$, $P \neq 0$ and $f \in k[V]$, we have $Z_{\text{red}}(Np(f)) = \mu_0(\text{Supp}(P) / fP)$, where $P / fP = \text{Coker}(m^P_f)$.

**Proof.** One can easily check that for any point $z \in S$, $N_p(f) = \text{det}(p_z)$, where $p_z$ is a fibre of $P$ over $z$. So $z \not\in Z_{\text{red}}(Np(f))$ ⇔ the fibre $(m^P_f)_z$ is invertible ⇔ $P / fP = 0$. □

**Definition 4.3.** For any $p : V \rightarrow S$, a homomorphism $h : P \rightarrow Q \in \text{Coh}(V)$ is called universally injective over $S$ if $VF \in \text{Coh}(S)$ the homomorphism $h \otimes p^*(F)$ is injective.
Lemma 4.4. Suppose $V \xrightarrow{\alpha} U \xrightarrow{\beta} S$ are morphisms of schemes, $\alpha$ is affine, and $h: P \to Q \in \text{Coh}(V)$; then $h$ is universally injective over $S$, iff $\alpha_*(h)$ is universally injective on $S$.

2) Suppose $p: V \to S$ is affine and flat, and $h: P \to Q \in \text{Coh}(V)$; then the following conditions are equivalent: (1) $h$ is universally injective over $S$, (2) $p_!(\text{Coker}(h)) = \text{Coker}(p_!(h)) \in \text{Coh}(S)$ is flat, (3) the fibre $h_x = i_x^*(h)$ is injective for any $x \in S$, where $i_x: x \to S$ is the injection.

Proof. 1) The claim follows form that the direct image functor with respect to an affine morphism is exact. 2) (1) $\iff$ (2): By the first point of the lemma (1) is equivalent that $p_!(h)$ is universally injective. Now we use the reasoning used by Suslin in the case of $K_0$-corr. Since $p$ is flat, and $P$ and $Q$ are flat objects in $\text{Coh}(V)$, we see that $p_!(P)$ and $p_!(Q)$ are flat on $S$ too. Since $p_!(h)$ is universally injective, computing $\text{Tor}_i(p_!(\text{Coker}(h)))$ using the flat resolvent given by $0 \to p_!(P) \to p_!(Q) \to p_!(\text{Coker}(h))$, we see that $\text{Tor}_i(p_!(\text{Coker}(h, G)) = 0 \forall G \in \text{Coh}(S), i > 0$. The equivalence (2) $\iff$ (3) follows form that $F \in \text{Coh}(S)$ is flat iff $\text{Tor}_i(F, k(x)) = 0$ for all points $x \in S$.

Lemma 4.5. Let $X$, $Y$ be schemes over the base field $k$, let $P \in \mathcal{P}(\mathbb{G}_m \times X, \mathbb{G}_m \times Y)$, $f \in \mathbb{k}[\mathbb{G}_m \times X \times \mathbb{G}_m \times Y]$, and $pr_X: X \times \mathbb{G}_m \times Y \times \mathbb{G}_m \to X$ denote the canonical projection; then $pr_X(P/fP)$ is coherent on $X$ if and only if $\text{Supp} P/fP$ is finite over $X$.

Proof. The claim follows from the general fact that direct image of a coherent $\mathcal{F}$ sheaf along a affine morphism $p: T \to S$ is coherent iff $\text{Supp} \mathcal{F}$ is finite over $S$.

Lemma 4.6. Let $P \in \mathcal{P}(\mathbb{G}_m \times X, \mathbb{G}_m \times Y)$, $f \in \mathbb{k}[\mathbb{G}_m \times X \times \mathbb{G}_m \times Y]$, $\mathcal{N} = \text{det}_P(f) \in \mathbb{k}[X \times \mathbb{G}_m]$.

(1) Then $m_{\mathcal{N}, P} = m_f \circ m_{\mathcal{N}, P}$ for some $g \in \mathbb{k}[\mathbb{G}_m \times X \times \mathbb{G}_m \times Y]$.

(2) $\forall \nu \in \text{Coh}(m_f)$, $\nu^P$ (1) is equivalent that $h$ are flat objects in $\mathcal{N}$.

(3) Suppose $P/\mathcal{N}$ is finite over $X$, then $\nu^P$ splits locally (see Def. [2.23]), and $fP', gP', P/\mathcal{N}$, $P/fP'$ are locally free of a finite rank.

Proof. 1) Consider $p(\lambda) = \text{det}_P(\lambda - m_f)$, $p(\lambda) = \lambda^{\text{rank} P} + a_1 \lambda^{\text{rank} P-1} + \cdots + a_{\text{rank} P}$, $a_i \in \mathbb{k}[\mathbb{G}_m \times X]$, where $m_f = m_f$. Since $\text{det}_P(f) = 0$, it follows that $p(\lambda) = (\lambda - f)^g$ for some $g(\lambda) \in \mathbb{k}[\mathbb{G}_m \times X \times \mathbb{G}_m \times Y]$. Hence $\mathcal{N} = p(0) = fg$, where $g = g(0)$.

2) Suppose $m_f$ is injective. Since $\mathcal{N} = fg$, $m_f$, and $m_{\mathcal{N}}$ are injective. Since $\text{Supp} P$ is finite over $\mathbb{G}_m$, for any $g \in \mathbb{k}[X \times \mathbb{G}_m \times Y \times \mathbb{G}_m]$, $m_f$ is injective iff $pr_{X \times \mathbb{G}_m}(m_h)$ is injective. By the definition of $\mathcal{P}(X \times \mathbb{G}_m, Y \times \mathbb{G}_m)$ the sheaf $pr_{X \times \mathbb{G}_m}(m_h)$ is coherent and locally free. Hence $pr_{X \times \mathbb{G}_m}(m_h)$ is injective iff the homomorphism $\nu^p(pr_{X \times \mathbb{G}_m}(m_h))$ induced over generic point $\nu \to X \times \mathbb{G}_m$ is injective. Then $\nu^p(pr_{X \times \mathbb{G}_m}(m_h))$ is injective iff $\text{det}(\nu^p(pr_{X \times \mathbb{G}_m}(m_h))) \in k(\nu)$ is invertible, since $\text{dim}_{k(\nu)}(pr_{X \times \mathbb{G}_m}(m_h)) < \infty$. Now since $\text{det}(\nu^p(pr_{X \times \mathbb{G}_m}(m_h))) \in k(\nu)$, we see that $m_{\mathcal{N}}$ is injective.
\[ \dim Z_\infty = \dim (X \times \mathbb{G}_m) - 1 = \dim (\infty \times X), \text{ and } Z_\infty = \infty \times X. \]  
So this contradicts the minimality of \( l_\infty \). Thus \( Z_\infty = \emptyset \), and similarly \( Z_0 = \emptyset \).

So we get \( \tilde{Z}(r) = \tilde{Z}(r) \cap X \times \mathbb{G}_m = \tilde{Z}(N) \). Hence \( Z(N) \) is projective over \( X \). Since as was shown \( Z(N)_{red} \) is finite over \( X \), it follows that \( Z(N) \) is quasi-finite over \( X \). Whence \( Z(N) \) is finite over \( X \).

3') Let \( x \in X \). By the same reasoning as in (2') applied to the fibre \( x^* (P) \) over a point \( x \in X \) we see that if \( (m_f^\nu)_x \) is injective, then \( (m_f^\nu)_x \) and \( (m_f^\nu)_x \) are injective. And moreover, the \( (m_f^\nu)_x \) is injective iff \( m_f^\nu(P_x) \) is invertible, where \( \nu \) is generic point of \( X \times \mathbb{G}_m \). If \( \text{Supp}(P/fP) \) is finite over \( X \), then \( \mu^\nu(P/fP) = 0 \), so \( \text{coker}(m_f^\nu(P_x)) = 0 \), and hence \( m_f^\nu(P_x) \) is invertible, since \( \nu^\nu(P_x) \) is finite dimensional vector space over \( k(\nu) \).

5. \( \cup \)-PRODUCT OF QUADRATIC SPACES WITH A FUNCTION.

We construct some operation \( \rho \) which takes \( (P,q) \in Q(X \times \mathbb{G}_m, Y \times \mathbb{G}_m) \) and a suitable \( f \in k[\mathbb{G}_m \times \mathbb{G}_m] \) to some \( (P/fP,q') \in Q(X,Y) \) (see Def. 5.9). Informally it can be considered as a \( \cup \)-product of a quadratic space with a regular function. The function \( f \) is applicable to \( (P,q) \) only if \( Z(f) \) is finite over \( Y \times \mathbb{G}_m \) and transversal to \( \text{Supp} P \), but it isn’t enough. Moreover, by the construction \( \rho \) depends on a triple of functions \( (f,N,\tilde{g}) \), where \( N \in k[X \times \mathbb{G}_m], \tilde{g} \in k[X \times \mathbb{G}_m \times Y \times \mathbb{G}_m] \) such that \( N = f g, Z(N) \) is finite over \( X \) (see Def. 5.5). A set of \( \tilde{T} \)-triples that we can use in our construction with a given quadratic space \( (P,q) \) we call as \( (P,q) \)-applicable \( \tilde{T} \)-triples.

So to apply our construction we should firstly prove the existence of an applicable triple (Lemma 5.11) and a partly independence form the choice of \( N \) and \( g \). Also we show that the operation is well defined on Grothendieck-Witt and Witt groups.

In this and the next section we assume that \( char k \neq 2 \), where \( k \) is the base field.

**Definition 5.1.** Let \( X \) be a scheme, \( P \in \text{coh}(X) \), \( q \) a symmetric quadratic form \( q \colon P \rightarrow D_X(P) \), and \( e \in \text{End}(P) \), \( q \circ e = D(e) \circ q \in \text{Hom}(P,D_X(P)) \); then denote by \( e \cdot (P,q) \) the pair \( (P,q') \), where \( q' = q \circ e \in \text{Hom}(P,D_X(P)) \), which symmetric quadratic form, since \( e \) is self-adjoint.

**Definition 5.2.** For any \( (P,q) \in \text{pre}Q(\text{coh}_{fin}(X,Y)) \), we denote by \( \text{red}(P,q) \) (or \( (P,q)_{red} \)) the pair \( (\text{Im}(q,q_{red}) \in \text{pre}Q(\text{Coh}_{fin}(X,Y)), \text{where} q_{red} : \text{Im} (3q) \rightarrow D(3q) \) is a unique homomorphism such that \( q = D(\text{Im} q) \circ q_{red} \circ \text{im} q \) defined by universal property of the kernels and cokernels.

**Lemma 5.3.** Let \( (P,q) \in Q(P(X,Y)) \) and \( e : P \rightarrow P \) be a locally splitting (see Def. 2.22) self-adjoint endomorphism; then \( \text{red}(e \cdot (P,q)) \in Q(P(X,Y)) \).

**Proof.** By definition \( (e \cdot q)_{red} \) is equal to composition \( \text{Im}e \rightarrow \text{Im}DX(e) \xrightarrow{w} D_X(\text{Im}e) \), where \( w \) is defined by universal property of the image. So there is the commutative diagram

\[
\begin{array}{cccccc}
\text{Ker} e^\nu & \rightarrow & P & \rightarrow & \text{Im} e^\nu & \rightarrow & P \xrightarrow{q} \text{Coker} e \\
\downarrow \quad & & \downarrow & & \downarrow \quad & & \downarrow \\
\text{Ker} DX(e)^\nu & \rightarrow & DX(P) & \xrightarrow{\text{Im}DX(e)} & \text{Im}DX(e) & \xrightarrow{\text{Coker} DX(e)} \\
\downarrow \quad & & & & \downarrow \quad & & \downarrow \\
DX(\text{Coker} e)^\nu & \rightarrow & DX(P) & \xrightarrow{DX(\text{Im}e)^\nu} & DX(P) & \rightarrow & DX(\text{Ker} e).
\end{array}
\]
Since \((P, q) \in Q(P(X, Y))\), we see that \(q\) is isomorphism and hence \(u\) is isomorphism. Since \(e\) is locally splitting, if follows from Lm 2.22 that \(w\) is isomorphism. The claim follows. \(\square\)

**Remark 5.4.** For any \(P\) as above \(\text{Supp } P/fP\) is finite over \(X\) iff \(pr_{X, i}(P/fP) \in \text{coh}(X)\).

**Definition 5.5.** For a pair of varieties \(X, Y\) we call by an \(T\)-tripe a triple \((f, N, g), f, g \in k[X \times G_m \times Y \times G_m], N \in k[X \times G_m]\) such that \(Z(N)\) is finite over \(X\).

An \(T\)-tripe \((f, N, g),\) is called applicable with respect to \((P, q)\) or just a \((P, q)\)-triple iff \(\text{Supp } P/fP\) is finite over \(X\) and \(m_{N,P} = m_{f,P} \circ m_{g,P} \in \text{End}(P)\).

A \((P, q)\)-triple \((f, N, g)\) is called normal \((P, q)\)-triple, whenever there are regular functions \(N_{ad}\) on \(X \times G_m\) and \(g_{\text{min}}\) on \(X \times G_m \times Y \times G_m\), such that \(N = N_P(f)N_{ad}\) and \(g = g_{\text{min}}N_{ad}\).

**Definition 5.6 (The map \(\rho\)).** For an \((P, q)\)-triple \((f, N, g)\) (see Definition 5.5) put
\[
\rho_{(f, N, g)}(P, q) = \text{red}(m_{f,P} \cdot ((P, q) \circ (N, X \times G_m))) \in \text{preQ}(P(X, Y))
\]
(see Def. 2.17 5.1 5.2)

**Lemma 5.7.** For a \((P, q)\)-triple \((f, N, g),\) if \((P, q) \in Q(P(X \times G_m, Y \times G_m))\) then \(\rho_{(f, N, g)}(P, q) \in Q(P(X, Y))\)

**Proof.** The claim follows from the point (3") of Lm 1.6 and Lm 5.3 \(\square\)

**Lemma 5.8.** Let \((P_1, q_1) \in \text{preQ}(P(X, Y)),\) and \((P_2, q_2) \in Q(P(Y, Z),\) then
1) for any locally splitting \(e \in \text{End}(P)\) (see Def. 2.22), and any \((P_2, q_2) \in Q(P(Y, Z),\) one has
\[
\text{red}(e \cdot ((P_2, q_2) \circ (P_1, q_1))) = (P_2, q_2) \circ \text{red}(e \cdot (P_1, q_1)) \in Q(P(X, Z))
\]
2) for a commuting self-adjoint endomorphisms \(e_1, e_2 \in \text{End}(P),\) one has
\[
(e_2 \circ e_1) \cdot (P, q) = e_2 \cdot (e_1 \cdot (P, q)) \quad \text{and} \quad \text{red}(e_2 \circ e_1) \cdot (P, q) = \text{red}(e_2 \cdot e_1 \cdot (P, q)),
\]
where \(e_2\) at the right side denotes the restriction \(e_2|_{\text{End}(P)}\), which is well defined since \(e_1 e_2 = e_2 e_1\).

**Proof.** The point (1) is equivalent to that \(\text{Ker } q_1 \otimes P_2 \subset (\text{Ker } q_1 \otimes Y, q_2)\) (see Lm 2.4 for \(q_1 \otimes Y, q_2)\). Straightforward verification shows that \(\text{Ker } q_1 \otimes P_2 \subset (\text{Ker } q_1 \otimes Y, q_2)\). On other hand by Lm. 5.3 \(\text{red}(e \cdot (P_1, q_1)) \in Q(P(X, Y))\) and then by the discussion before Def. 2.20 \((P_2, q_2) \circ \text{red}(e \cdot (P_1, q_1)) \in Q(P(X, Y))\); hence \(\text{Ker } q_1 \otimes P_2 = (\text{Ker } q_1 \otimes Y, q_2)\). Point (2) easily follows from the definitions. \(\square\)

**Lemma 5.9.** Let \((P, q) \in Q(P(X \times G_m, Y \times G_m))\) and \((f, N, g)\) be \((P, q)\)-triple, then
1) \(N = N_{\text{ad}}N_{\text{ad}}\) and \(g = g_{\text{ad}}N_{\text{ad}}\), we have \(\rho_{(f, N, g)}(P, q) = \text{red}(g_{\text{ad}} \cdot (N, X \times G_m) \circ (P, q))\).
2) \(id_{G_m} \otimes \rho_{(f, N, g)}(P, q) \simeq \rho_{(f, N, g)}(id_{G_m} \otimes P)\).
3) For any metabolic \((P, q) \in Q(P(X \times G_m, Y \times G_m))\) and \((P, q)\)-triple \((f, N, g), \rho_{(f, N, g)}(P, q)\) is metabolic.

**Proof.** 1) Using Lm 5.8 and Def 5.6 we see \(\rho_{(f, N, g)}(P, q) = \text{red}(m_{g,P} \cdot (N, X \times G_m) \circ (P, q)) = \text{red}(m_{g_{\text{ad}},P} \cdot m_{N_{\text{ad}},P} \cdot (N, X \times G_m) \circ (P, q)) = \text{red}(g_{\text{ad}} \cdot (N, X \times G_m) \circ (P, q))\).

2) The claim follows, since operations used in the def. 5.6 commute with \(id_{G_m} \otimes \).

3) We can assume that \(X\) is affine. By Def 5.6 and Lm 4.6 (3") \(f g = N\), and \(\rho_{(f, N, g)}(P, q) = \text{red}(P', q') = (P'', q''),\) and \(f g = N\), where \((P', q') = (P, q) \circ (N, X \times G_m)\) and \((P'', q'') = P'/fP' = P/fP'.\) Moreover \(L'\) is subalgebraic in \((P, q)\), then \(L' = P'/NP\) is subalgebraic subspace of \((P', q')\), and \(L'' = L'/fL\) is subalgebraic in \((P'', q'').\) Since diagram 2.21 is self-dual, if follows that \(D(P''/L'')\) is subalgebraic in \((D(P''), q''^{-1})\), and so \(L''\) subalgebraic subspaces in \((P'', q'').\) \(\square\)

**Lemma 5.10.** Let \((P, q) \in Q(P(X \times G_m, Y \times G_m))\), and \((f, N', g)\) and \((f, N, g')\) are \((P, q)\)-triples.
1) Then if \(N' = N\), then \(\rho_{(f, N', g)}(P, q) \simeq \rho_{(f, N, g')}(P, q)\).
2) The following conditions holds:
Lemma 5.12. 1) Suppose $Z$ subscheme of $P/fP$. By Lemma 4.2 it is easy to show that $m_g.p = m_{g'},p$. Since both $T'$-triples are applicable for $(P,q)$, we can assume that $m_{g,p} \circ m_{f,p} = m_{g',p} \circ m_{f,p}$, i.e., $\frac{f}{\text{Supp}} = \frac{\tilde{g}}{\text{Supp} P}$. Thus if we show that multiplication by $f$ on $\mathcal{O}(\text{Supp} P)$ is injective, then the claim follows.

Since $Z(N)$ is finite over $X$, it follows that $\text{Supp} P$ is injective on $\mathcal{O}(X \times \mathbb{G}_m)$. Denote by $i$: $\text{Supp} P \rightarrow X \times \mathbb{G}_m \times Y \times \mathbb{G}_m$ the canonical injection, and by $\text{pr}: X \times \mathbb{G}_m \times Y \times \mathbb{G}_m \rightarrow X \times \mathbb{G}_m$ the projection to the first two multiplicands. Let $P' = i*(P)$, then by Definition 4.2 the direct image $P'' = \text{pr}_* (i_! (P'))$ is locally free coherent sheaf of finite rank on $X \times \mathbb{G}_m$. Hence $\text{Supp} P''$ is injective.

The morphism $\text{pr} \circ i$ is finite and consequently it is affine, hence direct image functor $\text{pr}_* \circ i_*$ is exact an faithful. So $\text{Supp} P''$ is injective. Thus $m_{f,P}$ is injective. 

Proof. By Lemma 5.12 since $m^{(P')}_{g,p} = m^{(P)}_{g,N} = m^{(P')}_{g,N_\text{ad}} \circ m^{(P)}_{N_\text{ad},N-min} = m^{(P)}_{N_\text{ad}} \circ m^{(P)}_{g,N_\text{ad}} \circ m^{(P)}_{f}$, we can assume that $g = g_{\text{min} N_\text{ad}}$, and by the same reason $g' = g_{\text{min} N'_\text{ad}}$. Then by Lemma 5.12

$$\rho_{(f,N,g)}(P,q) = \text{red}(g_{\text{min} \cdot N_\text{ad}} \cdot (\langle N, X \times \mathbb{G}_m \rangle) \circ (P,q)),$$

$$\rho_{(f,N,g')}(P,q) = \text{red}(g'_{\text{min} \cdot N'_\text{ad}} \cdot (\langle N', X \times \mathbb{G}_m \rangle) \circ (P,q)).$$

So to prove the claim it is enough to prove that $\text{red}(N_\text{ad} \cdot (\langle N, X \times \mathbb{G}_m \rangle) \simeq \text{red}(N'_\text{ad} \cdot (\langle N', X \times \mathbb{G}_m \rangle))$.

By assumption $N = a_1 t_1 + \cdots + a_c t_c \in k[X]|(t) = k[X \times \mathbb{G}_m]$ and $N' = a_1 t_1 + \cdots + a_c t_c \in k[X]|(t) = k[X \times \mathbb{G}_m]$, for some integers $d \geq c$ and regular functions $a_1, a_c \in k[X]$. Since $Z(N)$ is finite over $X$, it follows that $a_1$ and $a_c$ are invertible. Consider affine homotopy between these quadratic correspondences defined by the $T'$-triple $(f', \tilde{N}, \tilde{g})$: $\tilde{N}_\text{ad} = N_\text{ad}(1 - \lambda) + N'_\text{ad} \lambda$, $\tilde{N} = \tilde{N}_\text{ad}(N_{\text{min}}$, $\tilde{g} = N_\text{ad} g_{\text{min}}$, which is the quadratic space $\text{red}(N_\text{ad} \cdot (\langle N, X \times \mathbb{G}_m \rangle))$. Then $\text{Supp} \mathcal{O}(Z(\tilde{N})) \simeq \mathcal{O}(X \times \mathbb{G}_m \times \mathbb{A}^1)/N_{\text{min}} = \mathcal{O}(Z(N_{\text{min}}) \times \mathbb{A}^1)$, and $\text{red}(N_\text{ad} \cdot (\langle N, X \times \mathbb{G}_m \times \mathbb{A}^1 \rangle) = \mathcal{O}(Z(N_{\text{min}}) \times \mathbb{A}^1), u)$, for some invertible regular function $u$ on $Z(N_\text{ad} \cdot (\langle N, X \times \mathbb{G}_m \rangle) = i_0^*(\mathcal{O}(Z(N_{\text{min}}) \times \mathbb{A}^1), u)$, where $i_0, i_1$ denotes zero and unit sections of $X \times \mathbb{A}^1$ as usual. The claim follow since the quadratic spaces above are isomorphic by Lemma 5.13.

Lemma 5.11. For any $(P,q) \in Q(P \times \mathbb{G}_m, Y \times \mathbb{G}_m)$ and $f \in k[X \times \mathbb{G}_m \times Y \times \mathbb{G}_m]$ such that $\text{Supp} P/fP$ is finite over $X$, there is an $(f,P)$-triple of the form $(f, N_\text{ad} (f), g)$.

Proof. By Lemma 1.2 $Z(\text{red}(N_\text{ad}(f))) = p(\text{Supp} \text{red} P/fP)$, where $p$: $\text{Supp} P \rightarrow X \times \mathbb{G}_m$. Hence $Z(\text{red}(N_\text{ad}(f))$ is finite over $X$. Whence closure of $Z(\text{red}(N_\text{ad}(f))$ in $P_{\mathbb{G}_m}^1$ is equal to $Z(N_\text{ad}(f))$. Then $Z(N_\text{ad}(f))$ is projective over $X$ and hence it is finite. By Lemma 4.2 (1) there is $g \in k[X \times \mathbb{G}_m \times Y \times \mathbb{G}_m]$, such that $m_{N_\text{ad}(f)},p = m_{f},p \circ m_{g},p$. Thus $(f, N_\text{ad}(f), g)$ is $(P,q)$-triple.

Lemma 5.12. 1) Suppose $n, m$ are positive integers such that $m > n$, $X \in S_m k$, $g', q' \in k[X \times \mathbb{A}^1] = k[X]|(t)$ are regular functions that are monic polynomials of degree $m - n$ and such that $Z,g \times \mathbb{G}_m$ and $Z(g'|X \times \mathbb{G}_m)$ are finite over $X$, and suppose $\tau \in k[X \times \mathbb{A}^1]$ is a function such that $Z(\tau)$ is a subscheme of $X \times \mathbb{G}_m$ that is finite over $X$; then

$$p_{X \times \mathbb{A}^1} \circ [\text{red}(g \cdot (\langle \tau, g' \times \mathbb{A}^1 \rangle)] = p_{X \times \mathbb{A}^1} \circ [\text{red}(g' \cdot (\langle \tau, g \times \mathbb{G}_m \times X \rangle))] \in GW \text{Cor}(X, Y),$$

where $p_{X} : X \times \mathbb{A}^1 \rightarrow X$. 

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2) Suppose \( n, m \in \mathbb{Z}, m > n > 0 \); then there are an invertible \( \beta_{n,m} \in k^* \) and some \( h_{n,m} \in GWCor(k^1, pt) \) such that for any \( X \) and \( g = t^{m-n} + a_{m-n-1}t^{m-n-1} + \cdots + a_1t + 1 \in k[k^1 \times X] = k[X][t] \), we have
\[
(id_X \otimes h) \circ i_0 = pr_X \circ (\text{red}(g \cdot (g(t^n - 1), X \times \mathbb{G}_m)) - [\text{red}(g \cdot (g(t^n - t), X \times \mathbb{G}_m))]),
\]
\[
(id_X \otimes h) \circ i_1 = (\beta_{n,m}) \otimes id_X \in GWCor(k, X),
\]
where \( i_0, i_1 : X \to X \times k^1 \) denotes the zero and unit sections, and \( pr^X : X \to pt \), and \( pr^X \times \mathbb{G}_m : X \times \mathbb{G}_m \to X \) denote the canonical projections.

Proof. 1) Consider a quadratic space \( \tilde{Q} = \text{red}(g \cdot (\tilde{g}, X \times k^1)) \in QCor(X, X \times k^1) \), where \( \tilde{g} = g(1 - \lambda) + g' \lambda \in k[X \times k^1 \times k^1] \subset k[X \times \mathbb{G}_m \times k^1] \). Since \( O(Z(\tilde{g}, \tau)) \to \mathcal{O}(Z(\tau)) \), then \( \tilde{Q} = (\mathcal{O}(Z(\tau) \times k^1), \tilde{g}) \) and Lemma 2.18 yields the claim.

2) Firstly consider the case of \( X = pt \) and \( g = (t^{m-n} + 1) \). Define
\[
\tilde{h}_{n,m} = pr^{k^1} \circ [\text{red}((t^{m-n} + 1) \cdot ((t^{m-n} + 1)(t^n - 1) + (t^n - t)\lambda]), A^1)]) \in GWCor(pt, pt).
\]
Then since \( Z((t^{m-n} + 1)(t^n - 1) \subset \mathbb{G}_m \), we get \( \tilde{h}_{n,m} \circ i_0 = pr^{k^1} \circ [\text{red}((t^{m-n} + 1) \cdot ((t^{m-n} + 1) (t^n - 1), A^1)] = pr^{\mathbb{G}_m} \circ [((t^{m-n} + 1)(t^n - 1), \mathbb{G}_m)] \).

On other side, \( Z((t^{m-n} + 1)(t^n - t) = Z((t^{m-n} + 1)(t^n - 1)) \cap \mathbb{G}_m \). On other side, \( Z((t^{m-n} + 1)(t^n - t) = ((t^{m-n} + 1)(t^n - 1)) \cap \mathbb{G}_m \), and the module (sheaf) of the quadratic space \( \text{red}((t^{m-n} + 1)(t^n - t)) \) is isomorphic to \( k[t]/(t^n - 1) \cap k[t]/t \). Then \( (t^{m-n} + 1)(t^n - t) = (t^{m-n} + 1)(t^n - t), [0]) \) is rank one quadratic spaces over \( k \).

Hence
\[
\tilde{h}_{n,m} \circ i_1 = pr^{k^1} \circ [\text{red}((t^{m-n} + 1) \cdot ((t^{m-n} + 1)(t^n - t), A^1)] =
pr^{\mathbb{G}_m} \circ [\text{red}((t^{m-n} + 1) \cdot ((t^{m-n} + 1)(t^n - t), \mathbb{G}_m)] \oplus \langle \beta_{n,m} \rangle,
\]
for some invertible \( \beta_{n,m} \in k^* \). Thus we can put
\[
h_{n,m} = \tilde{h}_{n,m} - pr^{\mathbb{G}_m} \circ [\text{red}((t^{m-n} + 1) \cdot ((t^{m-n} + 1)(t^n - t), \mathbb{G}_m)] \circ pr^{\mathbb{G}_m \times k^1},
\]
where \( pr^{\mathbb{G}_m \times k^1} : \mathbb{G}_m \times k^1 \to \mathbb{G}_m \).

Now consider any \( X \in Sm_k \) and a function \( g \) as in lemma. Form the first point it follows that
\[
[\text{red}(g \cdot (g(t^n - 1), X \times \mathbb{G}_m))] - [\text{red}(g \cdot (g(t^n - t), X \times \mathbb{G}_m))] = [\text{red}((t^{m-n} + 1) \cdot ((t^{m-n} + 1)(t^n - 1), X \times \mathbb{G}_m))],
\]
and the claim follows from that constructions from Definition 5.1 Definition 5.2 and Proposition 2.19 respects base change.

\[\square\]

6. An inverse homomorphism for \(- \otimes id_{\mathbb{G}_m^1}\)

6.1. Filtering system of functions. In this subsection we define two systems of functions \( f_n^+/- \) on \( \mathbb{G}_m \times \mathbb{G}_m \) indexed by positive integers, define some special \( \tau \)-triples relating to the functions \( f_n^+/- \), and prove properties of such \( \tau \)-triples needed in the construction of the homomorphisms that are left and right inverse for the homomorphism \(- \otimes id_{\mathbb{G}_m^1}\).

Definition 6.1. Using identification \( k[\mathbb{G}_m \times \mathbb{G}_m] = k[u](u) \) let's define two regular function \( f_n^+ = t^n - 1 \), \( f_n^- = t^n - u \). Denote by the same symbols inverse images of these functions on \( X \times \mathbb{G}_m \times Y \times \mathbb{G}_m \). A pair of \( \tau \)-triples (Def. 5.3) \( \tau = (f_n^+, N^+, g^+), (f_n^-, N^-, g^-) \) is called an \( \tau \)-bi-triple of degree \((n, m)\) or by \((n, m)\)-bi-triple whenever \( N^+, N^- \in k[X][t] \subset [X \times \mathbb{G}_m] \) and \( N^+ = t^m + a_{m-1}t^{m-1} + \cdots + a_1t - 1, N^- = t^m + a_{m-1}t^{m-1} + \cdots + a_2t^2 - t \).
For any quadratic space \( Q \in QCor(X \times \mathbb{G}_m, Y \times \mathbb{G}_m) \), \((n, m)\)-bi-triple \( \tau \) is called normal and applicable for \( Q \) iff such are both of \( \Gamma \)-triples in \( \tau \) (see Def. \ref{def:3.5}), and we denote \( \rho^*_\tau(Q) = \rho(f^*_\tau, N^\tau, +, g^\tau)(Q) \), \( \rho^*_\tau(Q) = \rho(f^*_\tau, N^\tau, -, g^\tau)(Q) \).

**Lemma 6.5.** Suppose \( (P, g) \in QCor(X \times \mathbb{G}_m, Y \times \mathbb{G}_m) \) and \( \tau_1, \tau_2 \) are two normal \( Q \)-applicable \((n, m)\)-bi-triples; then \( \rho^{-1}_{\tau_1}(P, g) \simeq \rho^{-1}_{\tau_2}(P, g) \).

**Proof.** The claim follows form definitions, Lemma \ref{lem:5.3} and Lemma \ref{lem:5.10}. □

**Definition 6.3.** Define ordered set \( \mathfrak{A} = \{ (r, n, m) \in \mathbb{Z}^3 \mid n < rn \} \) with order \( (r_2, n_2, m_2) > (r_1, n_1, m_1) \) if \( r_2 > r_1, n_2 > n_1 \) and \( m_2 - n_2 > m_1 - n_1 \).

**Lemma 6.4.** Ordered set \( \mathfrak{A} \) is filtering, i.e. for any \( \alpha_1, \alpha_2 \in \mathfrak{A} \) there is \( \alpha_3 > \alpha_1, \alpha_2 \).

**Proof.** For two triples \((r_1, n_1, m_1)\) and \((r_2, n_2, m_2)\), the triple \((r_3, n_3, m_3)\) \((r_1, n_1, m_1)\), \((r_2, n_2, m_2)\) where \( r_3 = \max(r_1, r_2), n_3 = \max(n_1, n_2), m_3 = \max(m_2 - n_2) \). □

**Definition 6.5.** For any \( Q = (P, q) \in QCor(X \times \mathbb{G}_m, Y \times \mathbb{G}_m) \) we put \( -M_P = \deg N_P(m_P, u) \) (under the identification \( k[G_m \times G_m] = k(t, u) \)).

**Lemma 6.6.** For any \( X, Y \in Sm_k \) and \((P, q) \in Q(\mathbb{P}(X \times \mathbb{G}_m, Y \times \mathbb{G}_m)) \) there is an integer \( N_P \) such that for all \( N > N_P \):

1. \( \text{Supp } P/f^+_nP \) is finite over \( X \), \( N_P f_n^+P = (t^n - 1)^r \in k[X](t) \subset k[X \times \mathbb{G}_m] \).
2. \( \text{Supp } P/f^+_nP \) is finite over \( X \), \( N_P f_n^+P = t^{-M_P} t^{n+\alpha_m-1}t^{m-1}\alpha_0q \in k[X][t] \subset k[X \times \mathbb{G}_m] \),

where \( n = rn + M \) and \( r = \text{rank}_{k[X \times \mathbb{G}_m]} P \), and \( a_0 \) is some invertible regular function on \( X \).

**Proof.** By Definition \ref{def:2.1} for any \( P \in \mathbb{P}(X \times \mathbb{G}_m, Y \times \mathbb{G}_m) \), \( p_*(P) \) is locally free coherent sheave on \( X \times \mathbb{G}_m \) of finite rank \( r \). Hence for any \( f \in k[X \times \mathbb{G}_m] \), \( \text{Supp } P/fP \) is finite over \( Z(f) \) that is finite over \( X \), and \( N_P f = g \). In particular \( \text{Supp } P/f^+_nP \) is finite over \( Z(t^n - 1) \subset X \times \mathbb{G}_m \), which is finite over \( X \), and \( N_P f_n^+P = (t^n - 1)^r \). This proves the first point.

The proof of the second point is contained in the proof of proposition 4.1 in \cite{20}. Let’s briefly repeat it. If \( N_P (x - u) = x^n + b_{n-1}x^{n-1} + \ldots + b_1x + b_0 \) be characteristic polyomin of operator \( m_P(u) \) on \( p_*(P) \), then \( b_i \in k[X \times \mathbb{G}_m] = k[X](t) \) and \( b_0 = N_P(u) \) is invertible, since \( m_P(u) \) is invertible operator. Let \( b_i = c_{i, d_i} t^d_i + c_{i, d_i-1} t^{d_i-1} + \ldots + c_{i, d_i+t} t^{d_i+1} + c_{i, d_i+t} t^{d_i} \) and then \( b_0 = t^c_0 c_{0, 0}, \ c_{0, 0} \in k[X]^* \). For \( x = t^n \) we get \( N_P(t^n - u) = t^{rn} + b_{n-1} t^{r(n-1)} + \ldots + b_1 t^r + b_0 \) and for enough big \( n \), namely for \( n > N_P = \text{max}(-c_i + d_i) \), the function \( t^{-c_0} N_P(t^n - u) \) is monic polynomial in \( t \) with coefficients in \( k[X] \) and its zero term is equal to \( c_0 c_0 \) and so it is invertible. Now let’s note that by definition \( c_0 = M_P \). So we get that for \( n > N_P \), \( N_P f_n^+P = t^{-M_P} t^{n+\alpha_m-1}t^{m-1}\alpha_0q \), and hence \( Z(N_P f_n^+P) \) is finite over \( X \) and hence \( \text{Supp } P \) is finite over \( X \).

**Definition 6.7.** For any quadratic space \( (Q, q) \in QCor(X \times \mathbb{G}_m, Y \times \mathbb{G}_m) \) put \( \alpha_Q = (\text{rank}_{k[X \times \mathbb{G}_m]} P, N_P, \max(r N_P, r N_P, M_P + M_P)) \in \mathfrak{A} \).

**Lemma 6.8.** For any \( Q = (P, q) \in QCor(X \times \mathbb{G}_m, Y \times \mathbb{G}_m) \) for all \( \alpha = (r, n, m) \) there is a normal \( Q \)-applicable \((n, m)\)-bi-triple (see Def. \ref{def:6.3} and Def \ref{def:6.11}).

**Proof.** Lemmas \ref{lem:6.3} and \ref{lem:6.11} yields that for \( n > N \) there is a pair of applicable \( \Gamma \)-triples \((f^+_n, N_P(f_n^+), g^+), (f^-_n, N_P(f_n^-), g^-) \) that is \((n, m - M)\)-bi-triple. Denote \( \delta_n = m - rn + M \), then for any \( N_{ad} = t^m + \cdots + 1 \in k[X][t] \subset k[X \times \mathbb{G}_m] \) pair \((f^+_n, N_P(f_n^+), N_{ad}, g^+ N_{ad}), (f^-_n, N_P(f_n^+), N_{ad}, g^- N_{ad}) \) is applicable \((n, m)\)-bi-triple.

**Lemma 6.9.** For some quadratic space \( Q = (P, q) \in QCor(X \times \mathbb{G}_m, Y \times \mathbb{G}_m) \), for any morphism of schemes \( v : Y \to Y' \), we have \( \alpha_Q' = \alpha_Q \), where \( Q' = v \circ Q \).

**Proof.** Let \( Q' = (P', q') \) the claim follows form that direct images of sheaves \( P \) and \( P' \) on \( X \times \mathbb{G}_m \times \mathbb{G}_m \) are isomorphic. □
6.2. Sets with action of two commuting idempotents. In this subsection we fix notations relating to the compositions of quadratic spaces $QCor(X \times \mathbb{G}_m, Y \times \mathbb{G}_m)$ with maps $1_X \times \mathbb{G}_m \colon X \times \mathbb{G}_m \to X \overset{1} \to X \times \mathbb{G}_m$ and $1_Y \times \mathbb{G}_m \colon Y \times \mathbb{G}_m \to Y \overset{1} \to Y \times \mathbb{G}_m$.

**Definition 6.10.** Let $P$ be a semi-group and $Q$ be a $P$-set (i.e. set with action of $P$), and let $F \subset Q$ be any subset. Then denote by $CCl_P(F)$ the maximal subset of $F$ closed under action of $P$, and denote by $St_P(Q)$ the subset of $P$-stable elements, i.e.
\[
CCl_P(F) = \{ s \in F \colon P \cdot s \in F \}, \quad St_P(Q) = \{ s \in Q \colon g \cdot s = s, \forall g \in P \}.
\]

If $Q(-, -) \to Sm_k \times Sm_k \to Set$ is a bi-functor with $P$-action (i.e. functor $Sm_k \times Sm_k \to P - Set$), and $F \subset Q$ is a $P$-set, we get $P$-Set bi-functor inclusions $CCl_P(F) \subset Q$, $St_P(Q) \subset Q$.

**Definition 6.11.** Let $I^d$ denote semi-group with four elements $\{1, p_1, p_2, p_{1,2}\}$ such that $p_1^2 = p_1$, $p_2^2 = p_2$, $p_1 p_2 = p_{1,2}$. Then if $I^d$-Set $A$ is abelian group then it is exactly abelian group with two idempotents $p_1, p_2 \in End(A)$. Let’s denote
\[
dot^A = (id_A - p_1)(id_A - p_2) \in End(A),
\]
and then $St_{I^d}(A) = \text{Im} (\text{dot}^A)$, and we get homomorphisms $p : A \to St_{I^d}(A)$: $i, p \circ i = id_{St_{I^d}(A)}$.

We define the action of $I^d$ on the bi-functor $(X, Y) \mapsto Q(P(X \times \mathbb{G}_m, Y \times \mathbb{G}_m))$ as follows:
\[
p_1 \cdot (P, q) = 1_X \times \mathbb{G}_m \circ (P, q), \quad p_2 \cdot (P, q) = (P, q) \circ 1_Y \times \mathbb{G}_m,
\]
and $p_{1,2} \cdot (P, q) = 1_X \times \mathbb{G}_m \circ (P, q) \circ 1_Y \times \mathbb{G}_m$.

6.3. Domains of left and right inverse homomorphisms. In this subsection we define a filtering system of additive bi-functors $L^*_\alpha, N^*_\alpha, R^*_\alpha : Sm_k \times Sm_k^\text{op} \to Ab, \ast \in \{GW, W\}$ with homomorphisms $L^GW \to GWCor(X \times \mathbb{G}_m, Y \times \mathbb{G}_m^\Delta)$, $N^GW \to GWCor(X, Y)$, $R^GW \to GWCor(X \times \mathbb{G}_m, Y \times \mathbb{G}_m^\Delta)$ (and similarly for $WCor$). Bi-functors $R^*_\alpha$ and $L^*_\alpha$ play a role of domains of partly defined homomorphisms $\rho^{\ast, L}_\alpha : L^*_\alpha \to N^*_\alpha$ and $\rho^{\ast, R}_\alpha : R^*_\alpha \to GWCor(X, Y)$, that are left and right inverse homomorphisms to the homomorphism $- \otimes id_{\mathbb{G}_m}$, and $N^*_\alpha$ plays the role of codomain for $\rho^{\ast, R}_\alpha$.

**Definition 6.12.** Let $Q \subset QCor(X, Y)$ be any subset for smooth varieties $X, Y \in Sm_k$, then let’s denote by $Met(Q) \subset Q$ the subset of the metabolic spaces; by $\Delta^1(Q, \emptyset)$ the set of triples $Q_1, Q_2, Q_3 \in Q : Q_3 = Q_2 \otimes Q_1$; and denote $W(Q) = \text{WCor}(Met(Q) \to GWQ)$.

Note that any action of $Q$ on a set, that commutes with direct sums and sends metabolic subspaces to metabolic, induce the group action on $GWQ$.

6.3.1. For any $\alpha = (r, n, m) \in \mathfrak{A}$ let’s define bi-functors
\[
\tilde{L}^Q(-, -), L^Q : Sm_k^\text{op} \times Sm_k \to Set, \quad \tilde{L}^GW, L^GW, \tilde{L}^W, L^W : Sm_k^\text{op} \times Sm_k \to Ab.
\]
\[
\tilde{L}^Q(X, Y) = \{ Q \in Q(P(X \times \mathbb{G}_m, Y \times \mathbb{G}_m))(\alpha > \alpha_Q) \}, \quad L^Q = CCl_{I^d}(\tilde{L}^Q), \quad L^GW = GW(L^Q),
\]
\[
\tilde{L}^W = W(L^Q), \quad L^GW = St_{I^d}(\tilde{L}^GW), \quad L^W = St_{I^d}(L^W),
\]
(see Def. 6.5 and Lemma 6.9 for the second row; see Def. 6.10 Def. 6.11 and Def. 6.12 for the third row). In other words $L^GW = \text{Im}(\text{dot}^I)$, where $\text{dot}^I \in End(L^GW)$ is the idempotent defined by the composition $(id_X \times \mathbb{G}_m - 1_X \times \mathbb{G}_m) \circ - (id_Y \times \mathbb{G}_m - 1_Y \times \mathbb{G}_m)$. (see Def. 6.11). Set in addition $L^\Delta = QCor(X \times \mathbb{G}_m, Y \times \mathbb{G}_m), L^\Delta = GWCor(X \times \mathbb{G}_m, Y \times \mathbb{G}_m)$, $L^\Delta = WCor(X \times \mathbb{G}_m, Y \times \mathbb{G}_m)$.

**Definition 6.14.** For any $\alpha = (r, n, m) \in \mathfrak{A}$, define the bi-functor $N^\Delta : Sm_k^\text{op} \times Sm_k \to Set$ as the preimage of $L^\Delta$ under $- \otimes id_{\mathbb{G}_m}$, i.e.
\[
N^\Delta(X, Y) = (- \otimes id_{\mathbb{G}_m})^{-1}(L^\Delta(X, Y)) = \{(P, q) \in Q(P(X, Y))(P, q) \otimes id_{\mathbb{G}_m} \in L^\Delta\};
\]
and define bi-functors $\tilde{N}_\alpha^{GW}, N_\alpha^{GW}, \tilde{N}_\alpha^{W}, N_\alpha^{W}: Sm_{k}^{op} \times Sm_{k} \to Ab: N_\alpha^Q = CCl_{T, \alpha}(\tilde{N}_\alpha^Q)), \tilde{N}_\alpha^{GW} = GW(N_\alpha^Q), N_\alpha^{GW} = St_{T, \alpha}(\tilde{N}_\alpha^{W}), N_\alpha^{W} = St_{T, \alpha}(\tilde{N}_\alpha^{GW})$ (see Def. 6.10 Def. 6.11 and Def. 6.12). Set in addition $N_\infty^Q = QCor(X, Y), N_\infty^{GW} = GWCor(X, Y), N_\infty^W = WCor(X, Y)$.

**Lemma 6.15.** There are maps

$$h_{per}^{+/-} : QCor(X \times G_{m} \times G_{m}, Y \times G_{m} \times G_{m}) \to QCor(X \times G_{m} \times G_{m} \times A^{1}, Y \times G_{m} \times G_{m})$$

such that $h^{+/-}$ are natural in $X$ and $Y$ and preserve direct sums and metabelian spaces, and maps (not necessarily natural)

$$h_{per,b}^{+/-}, h_{per,d}^{+/-} : QCor(X \times G_{m} \times G_{m}, Y \times G_{m} \times G_{m}) \to QCor(X \times G_{m} \times G_{m}, Y \times G_{m} \times G_{m}), * \in \{0, 1\}$$

such that for any $Q \in QCor(X \times G_{m} \times G_{m}, Y \times G_{m} \times G_{m})$ the spaces $h_{per,b}^{+/-}(Q)$ split into the sum of spaces that supports are sent to the unit under the projection to at least one of multiplicands $G_{m}$, and such that

$$(6.16) i_{0}(h_{per}(Q)) \oplus h_{per,b}^{+/-}(Q) \oplus h_{per,d}^{+/-}(Q) \cong Q \oplus i_{0}(h_{per}(Q)) \oplus h_{per,b}^{+/-}(Q) \oplus h_{per,d}^{+/-}(Q)$$

$$i_{1}(h_{per}(Q)) \oplus h_{per,b}^{+/-}(Q) \oplus h_{per,d}^{+/-}(Q) \cong T_{Y} \circ Q \circ T_{X} \oplus i_{1}(h_{per}(Q)) \oplus h_{per,b}^{+/-}(Q) \oplus h_{per,d}^{+/-}(Q),$$

where $T_{X} = id_{X} \boxtimes T, T_{Y} = id_{Y} \boxtimes T$ are transpositions on $X \times G_{m} \times G_{m}$ and $Y \times G_{m} \times G_{m}$, and $i_{0}, i_{1}$ denote zero and unit sections.

**Proof.** Lemma 2.20 implies that there are

$$\Delta^{+/-} \in QCor(G_{m} \times G_{m} \times A^{1}, G_{m} \times G_{m}), h_{0}, h_{1} \in QCor(G_{m} \times G_{m}, G_{m} \times G_{m}):$$

$$(6.17) i_{0}(h^{+}) \oplus h^{0} = (k[\Delta], 1) \oplus h^{0}, i_{1}(h^{-}) \oplus h^{1} = (k[\Delta], 1) \oplus h^{1},$$

and such that spaces $h_{0}$ and $h_{1}$ splits into the sum of spaces that supports are contained in $pr_{i}^{-1}(1)$, where $pr_{i}: G_{m} \times G_{m} \times G_{m} \to G_{m}$ for $1 \leq i \leq 4$ denotes projections to multiplicands. The operation of composition at the left with the class $[\Delta^{+} - [\Delta^{-}]$ defines a homotopy on $GWCor(X \times G_{m}^{+}, X \times G_{m}^{-})$ that permutes multiplicands in the domain. The operation of composition at the right with $[\Delta^{+} - [\Delta^{-}]$ defines a homotopy that permutes multiplicands in the codomain. Now composing these two homotopies we get the claim.

Indeed, let $i_{\Delta} : U \times A^{1} \to U \times A^{1} \times A^{1}$ denote the diagonal embedding for $U \in Sm_{k}$ and let

$$h_{1} \ast h_{2}(Q) = i_{\Delta}((h_{2} \circ (Q \circ h_{1}) \boxtimes A^{1})) \in QCor(U \times A^{1} \times A^{1}, U),$$

for any two quadratic spaces $h_{1} \in QCor(U \times A^{1}, U), h_{2} \in QCor(V \times A^{1}, V)$ and $Q \in QCor(U, V)$. Then we can define the required maps $h_{per}^{+/-}$ and $h_{per}$ as

$$h_{per}^{+/-} : Q \mapsto (id_{X} \boxtimes h^{+}) \ast (Y \boxtimes h^{+})(Q) \oplus (X \boxtimes h^{-})(Q) \ast (Y \boxtimes h^{-})(Q),$$

$$h_{per}^{-/-} : Q \mapsto (X \boxtimes h^{+})(Q) \ast (Y \boxtimes h^{-})(Q) \oplus (X \boxtimes h^{-})(Q) \ast (Y \boxtimes h^{+})(Q).$$

The equations (6.17) implies that

$$i_{0}(h_{per}(Q)) \oplus [h_{per}(Q)] = [Q] \in GWCor(G_{m} \times G_{m}, G_{m} \times G_{m}),$$

$$i_{1}(h_{per}(Q)) \oplus [h_{per}(Q)] = [T_{Y} \circ Q \circ T_{X}] \in GWCor(G_{m} \times G_{m}, G_{m} \times G_{m}),$$

and this implies existence of the maps $h_{per,b}^{+/-}, h_{per,d}^{+/-}, h_{per,b}^{0,+}, h_{per,d}^{0,+}, h_{per,b}^{1,+}, h_{per,d}^{1,+}$.

**Definition 6.18.** For any any $\alpha = (r, n, m) \in \mathbb{R}$ let’s define the bi-functor $R_{\alpha}^{Q} : Sm_{k}^{op} \times Sm_{k} \to Set$

$$R_{\alpha}^{Q}(X, Y) = \{Q = (P, q) \in L_{\alpha}^{Q}(X, Y) \cap N_{\alpha}^{Q}(X \times G_{m}, Y \times G_{m})|$$

$$T_{Y} \circ (Q \boxtimes id_{G_{m}})T_{X}, h_{per,b}^{+/-}(Q), h_{per,d}^{+/-}(Q) \in L_{\alpha}^{Q}(X \times G_{m}, Y \times G_{m}), * \in \{0, 1\},$$

$$h_{per}^{+/-}(Q) \in L_{\alpha}^{Q}(X \times G_{m} \times A^{1}, Y \times G_{m}), *, * \in \{+, -\},$$

$$\square$$
where \( h^*_{\text{per}, b}(Q), h^*_{\text{per}, b}(Q) \) denote maps from Lemma 6.15 and define bi-functors
\[
R^G_{\alpha}, R^G_{\alpha}, R^W_{\alpha}, R^W_{\alpha}: Sm^G_{\alpha} \times Sm_{\alpha} \to Ab
\]
\[
R^G_{\alpha} = CC\mathcal{I}_d((\tilde{R}^G_{\alpha})), N^G_{\alpha} = GW(R^G_{\alpha}), N^W = W(R^W_{\alpha}),
\]
\[
R^W = St\mathcal{I}_d((\tilde{R}^W_{\alpha})).
\]
(See Def. 6.10, Def. 6.11, and Def. 6.12). Set in addition \( R^G_{\infty} = QCor(X \times G^1_m, Y \times G^1_m), R^G_{\infty} = GWCor(X \times G^1_m, Y \times G^1_m), R^W_{\infty} = WCOr(X \times G^1_m, Y \times G^1_m) \).

**Lemma 6.19.** 1) For any \( Y \in Sm_k \), functors \( L^*_{\alpha}(-, Y), N^*_{\alpha}(-, Y), R^*_{\alpha}(-, Y), * \in \{Q, GW, W\} \) are additive, i.e. sends disjoint unions to products in corresponding category.

2) For any \( X \in Sm_k \) and \( \alpha \in \mathfrak{A} \), the functors \( L^*_{\alpha}(X, -), N^*_{\alpha}(X, -), R^*_{\alpha}(X, -), * \in \{GW, W\} \) are additive, i.e. \( L^*_{\alpha}, R^*_{\alpha} \) send disjoint unions in \( Sm_k \) to the direct sums in corresponding additive category.

**Proof.** 1) Firstly note that \( \tilde{L}^G_{\alpha}(X_1, \Pi X_2, Y) = L^G_{\alpha}(X_1, Y) \times L^G_{\alpha}(X_2, Y) \), since required conditions on rank can be checked on disjoint components, and pair of \( \Pi \)-triples on disjoint union is normal applicable \( (n, m) \)-bi-triple if it is so at each disjoint component. The claim for other functors defined in Def. 6.13 and Def. 6.18 follows from that additivity preserves under used in definitions operations and from that direct sums of quadratic spaces and property of metabolic spaces are compatible with disjoint union of schemes.

2) To show additivity in the second argument consider the pair of homomorphisms
\[
F^\oplus(X, Y_1) \oplus F^\oplus(X, Y_2) \to F^\oplus(X, Y_1 \Pi Y_2):
((P_1, q_1), (P_2, q_2)) \mapsto [P_1, q_1] + [P_2, q_2],
\]
where \( F \in \{L^G_{\alpha}, N^G_{\alpha}, R^G_{\alpha}\} \) and in the definition of the second map we use that support of any \( (P, q) \in Q(P(X, Y_1 \Pi Y_2)) \) splits into disjoint union and hence quadratic space splits into direct sum \( [P, q]_{X \times Y_1} \oplus [P, q]_{X \times Y_2} \). Checking that composition are identity we get isomorphism \( (F^\oplus) (X, Y_1 \Pi Y_2) \cong (F^\oplus) (X, Y_1) \oplus (F^\oplus) (X, Y_2) \) and checking that action of \( \mathcal{I}d^2 \) is compatible with this decomposition we get decomposition for \( St\mathcal{I}_d((F^\oplus)) \).

**Definition 6.20.** Lemma 6.19 implies that the presheaves of abelian groups \( L^*_{\alpha}, N^*_{\alpha}, R^*_{\alpha}, * \in \{GW, W\} \) can be uniquely lifted to a bi-additive bi-functors on Karoubi closure of the additivisation of \( Sm_k \), and in particular these presheaves are correctly defined on \( X \times G^1_m \) for \( X \in Sm_k \). We shall denote such lifts of these functors by the same symbols.

Precisely \( L^*(X \times G^1_m, Y \times G^1_m) \) is defined as
\[
L^*(X \times G^1_m, Y \times G^1_m) = \text{Im} \left( [P, q] \mapsto [P, q] - [1_{X \times G^1_m}(P, q)] - [1_{Y \times G^1_m}(P, q)] \right)
\]

**Lemma 6.21.** For any varieties \( X, Y \) and a quadratic space \( (P, q) \in Q(P(X \times G^1_m, Y \times G^1_m)) \) there is an \( \in \mathfrak{A} \) such that for all \( \alpha > \alpha \), \( (P, q) \in L^G_{\alpha}(X, Y) \) and \( (P, q) \in R^G_{\alpha}(X, Y) \).

**Proof.** Indeed by Definition 6.13 for any \( Q = (P, q) \) and \( \alpha > \alpha Q, (P, q) \in \tilde{L}^G_{\alpha} \). Then since definitions of \( L^G_{\alpha} \) (and \( R^G_{\alpha} \)) requires that some finite set of quadratic spaces lays in \( \tilde{L}^G_{\alpha} \) (and \( L^G_{\alpha} \) respectively), and since \( \mathfrak{A} \) is filtering, the claim follows.

**Lemma 6.22.** For any \( Y \in Sm_k \) and any \( \alpha_1, \alpha_2 \in \mathfrak{A} \cup \{\infty\} \) such that \( \alpha_1 < \alpha_2 \), there is inclusions of presheaves \( L^G_{\alpha_1} \subseteq L^G_{\alpha_2}, N^G_{\alpha_1} \subseteq N^G_{\alpha_2}, R^G_{\alpha_1} \subseteq R^G_{\alpha_2} \), and this induce homomorphisms of presheaves \( \gamma^L_{\alpha_1, \alpha_2}: L^*_{\alpha_1} \to L^*_{\alpha_2}, \gamma^N_{\alpha_1, \alpha_2}: N^*_{\alpha_1} \to N^*_{\alpha_2} \) and \( \gamma^R_{\alpha_1, \alpha_2}: R^*_{\alpha_1} \to R^*_{\alpha_2} \) (for \( * \in \{GW, W\} \)). Let’s denote
\( j_{\Gamma}^{GW,L} = \gamma_{\Gamma}^{GW,L} : L_{\alpha}^{GW}(-,-) \to GW Cor(X \times \mathbb{G}_{m}^{\Lambda}, Y \times \mathbb{G}_{m}^{\Lambda}) \), and similarly for \( j_{\Gamma}^{GW,N} \), \( j_{\Gamma}^{GW,R} \), and \( j_{\Gamma}^{W,L} \), \( j_{\Gamma}^{W,N} \), \( j_{\Gamma}^{W,R} \).

Proof. By Definition 6.13 the same statement holds for the bi-functor \( \bar{L}^{\tilde{Q}} \), and the claim follows by induction in the definition of the bi-functors \( L^{Q} \), \( N^{Q} \), \( R^{Q} \).

Lemma 6.23. For each \( * \in \{ Q, GW, W \} \) there are isomorphisms

\[
\lim_{\alpha \in \mathbb{N}} L_{\alpha}^{*}(X,Y) \simeq L_{*}^{*}, \quad \lim_{\alpha \in \mathbb{N}} N_{\alpha}^{*}(X,Y) \simeq N_{*}^{*}, \quad \lim_{\alpha \in \mathbb{N}} R_{\alpha}^{*}(X,Y) \simeq R_{*}^{*}.
\]

Proof. Lemma 6.21 yields that

\[
\bigcup_{\alpha \in \mathbb{N}} L_{\alpha}^{Q}(X,Y) = L_{Q}^{Q}, \quad \bigcup_{\alpha \in \mathbb{N}} \Delta^{1}(L_{\alpha}^{Q}(X,Y), \oplus) = \Delta^{1}(L_{Q}^{Q}, \oplus), \quad \text{and hence}
\]

\[
\lim_{\alpha \in \mathbb{N}} (L_{\alpha}^{Q}(X,Y))^{\oplus} = \lim_{\alpha \in \mathbb{N}} \mathbb{Z}(L_{\alpha}^{Q}(X,Y))/\mathbb{Z}(\Delta^{1}(L_{\alpha}^{Q}(X,Y), \oplus)) = \mathbb{Z}(L_{Q}^{Q}(X,Y))/\mathbb{Z}(\Delta^{1}(L_{Q}^{Q}(X,Y), \oplus)) = (L_{Q}^{Q}(X,Y))^{\oplus} = GW Cor(X \times \mathbb{G}_{m}, Y \times \mathbb{G}_{m}).
\]

Then action of the idempotents \( dot_{\alpha} \) (see Def. 6.11) commutes with homomorphisms in the direct limit, we get

\[
\lim_{\alpha \in \mathbb{N}} L_{\alpha}^{GW}(X,Y) = \lim_{\alpha \in \mathbb{N}} Im(dot_{\alpha}^{L}) = Im(dot_{\alpha}^{L_{\infty}}) = GW Cor(X \times \mathbb{G}_{m}, Y \times \mathbb{G}_{m}).
\]

Finally, since

\[
\bigcup_{\alpha \in \mathbb{N}} Met(L_{\alpha}^{Q}) = Met(L_{Q}^{Q}) \quad \text{(by Lemma 6.21), we have}
\]

\[
\lim_{\alpha \in \mathbb{N}} L_{\alpha}^{W}(X,Y) = WC Or(X \times \mathbb{G}_{m}, Y \times \mathbb{G}_{m}).
\]

The proof of the equalities for \( N_{*}^{*} \) and \( R_{*}^{*} \) is similar.

\[
\square
\]

6.4. Left and right inverse homomorphisms. Here we prove that homomorphisms \(- \times \mathbb{G}_{m}^{\Lambda}\) on \( GW Cor \) has a left and right inverse homomorphisms defined on presheaves \( P_{\Gamma}^{GW} \) and \( L_{\alpha}^{GW} \), and similarly for \( WC Or \). We prove precisely that \( \rho_{\Gamma}^{GW,L} \) and \( \rho_{\alpha}^{GW,R} \) defined in previous subsection becomes such inverse homomorphisms after the twist by some invertible function.

Lemma 6.24. For any \( \alpha = (r, n, m) \in \mathfrak{A} \), and any choice of sign \( * \in \{ +, - \} \) there are unique natural transformations \( \tilde{\rho}_{\alpha,*}^{Q,*} \in \{ Q, GW, W \} \) such that diagram

\[
\begin{array}{ccc}
Q Cor(-, Y) & \xrightarrow{\tilde{\rho}_{\alpha}^{Q,*}} & GW Cor(-, Y) \\
\downarrow & & \downarrow \\
\bar{L}_{\alpha}^{Q}(-, Y) & \xrightarrow{\tilde{\rho}_{\alpha}^{GW,*}} & \bar{L}_{\alpha}^{GW}(-, Y)
\end{array}
\]

\[
\begin{array}{ccc}
Q Cor(-, Y) & \xrightarrow{\tilde{\rho}_{\alpha}^{Q,*}} & GW Cor(-, Y) \\
\downarrow & & \downarrow \\
\bar{L}_{\alpha}^{Q}(-, Y) & \xrightarrow{\tilde{\rho}_{\alpha}^{GW,*}} & \bar{L}_{\alpha}^{GW}(-, Y)
\end{array}
\]

\[
\begin{array}{ccc}
Q(\mathcal{P}(\cdot \times \mathbb{G}_{m}, Y \times \mathbb{G}_{m})) & \xrightarrow{\tilde{\rho}_{\alpha}^{Q,*}} & GW Cor(X \times \mathbb{G}_{m}, Y \times \mathbb{G}_{m}) \\
\downarrow & & \downarrow \\
W Cor(X \times \mathbb{G}_{m}, Y \times \mathbb{G}_{m})
\end{array}
\]

is commutative, and such that for any \( (P, q) \in F_{r,n,m}(X,Y) \) and applicable \((n, m)\)-bi-triple \( \tau \),

\[
\tilde{\rho}_{\alpha,*}^{Q,*}(P, q) = \tilde{\rho}_{\alpha}^{Q,*}(P, q) \boxtimes (\beta_{n,m}^{-1}),
\]

where \( \beta_{n,m} \) are defined in Lemma 5.12.

Proof. Lemma 6.23 implies that for any \( Q = (P, q) \in L_{Q}^{Q} \) there is normal applicable \((n, m)\)-bi-triple.

Let \( \tau \) be such a bi-triple, then we can put \( \tilde{\rho}_{\alpha,*}^{Q,*}(Q) = \tilde{\rho}_{\alpha}^{Q,*}(Q) \boxtimes (\beta_{n,m}^{-1}) \), and by Lemma 6.22 the result is independence from the choice of \( \tau \).

Then to show that \( \tilde{\rho}_{\alpha,*}^{Q,*} \) induce homomorphisms \( \tilde{\rho}_{\alpha,*}^{GW,L} \) and \( \tilde{\rho}_{\alpha,*}^{GW,R} \) is enough to check that it preserves direct sums and sends metabolic spaces to metabolic. The second one follows immediate from
Lemma 6.25. Define a natural transformations
\[ p_\alpha^* = p_{\alpha, +}^* - p_{\alpha, -}^* - p_{\alpha, +, -}^* (P, q) = p_{\alpha, +}^* - ((id_{Y \times G_m} - 1_{Y \times G_m}) \circ (P, q) \circ (id_{X \times G_m} - 1_{X \times G_m})) \]
for \( * \in \{GW, W\} \) and \( \alpha = (r, n, m) \in \mathcal{A} \). Then we get commutative diagram
\[
\begin{array}{ccc}
GW Cor(X \times G_m, Y \times G_m) & \xrightarrow{\rho_{\alpha, GW}^*} & GW Cor(X, Y) \\
\downarrow & & \downarrow \\
W Cor(X \times G_m, Y \times G_m) & \xrightarrow{\rho_{\alpha, W}^*} & W Cor(X, Y).
\end{array}
\]

Remark 6.26. Note that homomorphisms \( p_\alpha^* \) are not injective and we don’t state and use that these homomorphisms are natural in \( \alpha \).

Definition 6.27. Let’s denote by \( \theta_\alpha^* : N_{GW}^0(-, -) \to L_{GW}^0(-, -) \) the natural transformations of presheaves induced by \( - \otimes \text{id}_{G_m} : QCor(X, Y) \to QCor(X \times G_m, Y \times G_m) \) (in particular \( \theta_\alpha^* = - \otimes \text{id}_{G_m} : GW Cor(X, Y) \to GW Cor(X \times G_m, Y \times G_m) \)).

Lemma 6.28. For any \( \alpha = (r, n, m) \in \mathcal{A} \) and a quadratic space \( Q \in \tilde{L}_G^0(X \times G_m, Y \times G_m) \),
\[ \tilde{p}_{\alpha}^{GW, L}([Q]) = [Q \otimes pr_X] \circ [\text{red}(g^+, (\alpha^+ n - 1), X \times G_m)] - [Q \otimes pr_X \circ \text{red}(g^-, (\alpha^- n - t), X \times G_m)] \otimes (\beta_{n, m}^{-1}), \]
for some functions \( g^{+/-} \in k[X \times G_m] \simeq k[X](t) \) with the leading term \( t^{m-r} \), where \( \beta_{n, m} \) are defined as in Lemma 6.2.

Proof. By definition
\[ \tilde{p}_{\alpha}^{GW, L}([Q]) = [(Q \otimes \text{id}_{G_m}) \circ \text{red}(g^+, (\alpha^+ n - 1), X \times G_m)] - [(Q \otimes \text{id}_{G_m}) \circ \text{red}(g^-, (\alpha^- n - t), X \times G_m)] \]
where \((f_n^+, \alpha^+, g^+), (f_n^-, \alpha^-, g^-)\) is applicable \((n, m)\)-bi-triple for \((Q \otimes \text{id}_{G_m})\). Denote \((P, q) = Q \otimes \text{id}_{G_m}\), then \(\text{Supp} P = X \times Y \times \Delta_{G_m}\), and let’s denote \(g^+ = i_{\Delta}^*(g^+), g^- = i_{\Delta}^*(g^-)\), where \(i_\Delta : X \times G_m \to X \times G_m \times Y \times G_m\). Then, since \(f_n^+ = t^n - 1, f_n^- = t^n - t, m_{\alpha^+, P} = m_{\alpha^-, P} \circ m_{\alpha^-, P}, m_{\alpha^-, P} = m_{\alpha^+, P} \circ m_{\alpha^-, P})\) and then \((f_n^+, (t^n - 1)g^+, g^+), (f_n^-, (t^n - t)g^-, g^-)\) is applicable \((n, m)\)-bi-triple for \((P, q)\). The claim follows by Lemma 6.2.

Lemma 6.29. For any \( \alpha = (r, n, m) \in \mathcal{A} \) and a quadratic space \( Q = (P, q) \in N_{GW}^0(X, Y) \),
\[ \tilde{p}_{\alpha}^{GW, L}([iy \circ Q \circ pr_X]) = 0, \]
where \(pr_X : X \times G_m \to X\) and \(iy : Y \to Y \times G_m\) is unit section.

Proof. Since \(\text{Supp}(iy \circ Q \circ pr_X) \subset X \times G_m \times Y \times 1\), then \(f_n^+ \mid_{\text{Supp} Q} = t^n - 1\). Let \(N_{ad}^+ = t^{m-n} + \cdots + (-1)^{r-1} t^{1-n}, N_{ad}^- = t^{m-n} + \cdots + (-1)^{r-1} t\) be regular functions in \(k[G_m] = k[t]\), and \(N_{ad}^+ = N_{ad}(t^{m-1}), N_{ad}^- = N_{ad}(t^{m-1}), g^+ = N_{ad}(t^{m-1})^{-1}, g^- = N_{ad}(t^{m-1})^{-1}\). Then pair \((f_n^+, N_{ad}^+, g^+),\)
$(f^{-}_n, N^-, g^-)$ is $(n,m)$-bi-triple applicable to $iY \circ Q \circ pr_X$, then using Lemma 6.2 and the second point of Lemma 6.12
\[
\tilde{\rho}^{GW,L}_\alpha([iY \circ Q \circ pr_X]) = [\rho^+_\alpha(iY \circ Q \circ pr_X)] - [\rho^-_\alpha(iY \circ Q \circ pr_X)] = \\
[iY \circ Q \circ pr_X \circ red(g^+ \langle N^+, X \times G_m \rangle)] - [iY \circ Q \circ pr_X \circ red(g^- \langle N^-, X \times G_m \rangle)] = \\
[iY \circ Q] \circ [pr_X \circ red(g^+ \cdot (t^n - 1), X \times G_m)] - pr_X \circ red(g^- \cdot (t^n - 1), X \times G_m)] = 0
\]

\[\square\]

**Lemma 6.30** (the left inverse). **Diagrams**

\[
\begin{array}{ccc}
N^{GW}_\alpha(X, Y) & \xrightarrow{\theta^{GW}_\alpha} & L^{GW}_\alpha(X, Y) \\
\downarrow{j^N_\alpha} & & \downarrow{j^L_\alpha} \\
GWCor(X, Y) & \xrightarrow{\theta^{GW}_\alpha} & GWCor(X \times G^1_{m}, Y \times G^1_{m})
\end{array}
\]

are natural in $X$ and $Y$ and

1) the left-up triangles in these diagrams is commutative and it is natural in $\alpha$, i.e. it commutes with morphisms $N^{GW}_\alpha \rightarrow N^{GW}_{\alpha'}$ and $\theta^{GW}_\alpha \rightarrow \theta^{GW}_{\alpha'}$.

2) left-up triangle is commutative for each $\alpha$ up to canonical $A^1$-homotopy, i.e. there are homomorphisms of presheaves $h^{GW,L}_\alpha: N^{GW}_\alpha(-, Y) \rightarrow GWCor(- \times A^1, Y)$: such that for any class $a \in N^{GW}_\alpha(X, Y)$ and $\alpha \in A$ there are equalities

\[
i^*_0(h^{GW,L}_\alpha(a)) = \rho^{GW}_\alpha(a \boxtimes id_{G_{m}}), i^*_1(h^{GW,L}_\alpha(a)) = a,
\]

where $i^*_0$ and $i^*_1$ denote homomorphism induced by zero and unit sections of $A^1$, and see Def. 6.28 and Def. 6.27 for $\rho^{GW}_\alpha$ and $\theta^{GW}_\alpha$.

And the same holds for $WCor$. (Note: we don’t state that homomorphisms $\rho^{GW}_\alpha$ and $h^{GW,L}_\alpha$ are natural in $\alpha$.)

**Proof.** We consider the case of $GWCor$, the case of $WCor$ is absolutely similar.

The first point follows from definitions. To prove the second we use second point of Lemma 5.12 and find homotopy $h \in GWCor(A^1 \rightarrow pt)$, then define $h_X \in GWCor(X \times A^1, X): h_X = id_X \boxtimes h$, and

\[
h^{GW,L}_\alpha(a) = j^{GW,N}_\alpha(a) \circ h_X \boxtimes (\beta^{L,-1}_{n,m}) \in GWCor(X \times A^1, Y),
\]

for any $a \in N^{GW}_\alpha(X, Y)$.

Let’s check required conditions. Let $a \in N^{GW}_\alpha(X, Y)$, $a = [Q]$ for $Q \in N^Q_{\alpha'}$. And let’s denote $Q' = Q \boxtimes id_{G_{m}}$. Then $Q' \circ 1_{X \times G_{m}} \simeq 1_{Y \times G_{m}} \circ Q' \simeq 1_{Y \times G_{m}} \circ Q \circ pr_X$, where $i_Y: Y \rightarrow Y \times G_{m}$ is unit section and $pr_X: X \times G_{m} \rightarrow X$ is projection. Then

\[
\rho^{GW,L}_\alpha(\theta^{GW}_\alpha(a)) = \rho^{GW,L}_\alpha([Q']) = \tilde{\rho}^{GW,L}_\alpha([Q] - [iY \circ Q \circ pr_X]).
\]

Now by Lemma 6.28 and by the second point of Lemma 6.12

\[
\tilde{\rho}^{GW,L}_\alpha([Q]) = \\
[Q \circ pr_X \circ [red(g^+ \cdot (t^n - 1), X \times G_m)] - red(g^- \cdot (t^n - t), X \times G_m)]) \boxtimes (\beta^{L,-1}_{n,m}) = \\
[Q \circ h_X \circ i_0 \boxtimes (\beta^{L,-1}_{n,m})] = i^*_0(h^{GW,L}_\alpha([Q])),
\]

and by Lemma 6.29 $\rho^{GW,L}_\alpha([iY \circ Q \circ pr_X]) = 0$. And on other side (by Lemma 5.12)

\[
[Q] = [Q \circ h_X \circ i_1 \boxtimes (\beta^{L,-1}_{n,m})] = i^*_1(h^{GW,L}_\alpha([Q])),
\]

so the claim follows. \[\square\]
Lemma 6.32 (the right inverse). For any $Y \in \text{Sm}_k$ and $\alpha \in \mathfrak{A}$, there is homomorphism of presheaves

$$h^\alpha_{GW,r}: R^\alpha_{GW}(-, Y) \to GW Cor((- \times \mathbb{A}^1) \times \mathbb{A}^1, Y \times \mathbb{G}^1_m),$$

where $i_0^\alpha(h^\alpha_{GW,r}(-)) = \rho^\alpha_{GW,R}(-) \boxtimes id_{\mathbb{G}^1_m}$, $i_1^\alpha(h^\alpha_{GW,r}) = J^\alpha_{GW,R},$

where $i_0^\alpha$ and $i_1^\alpha$ denotes natural transformations induced by zero and unit sections.

And there is $h^W_{GW,r}$ with the same property in respect to $R^W_{GW}$ and $W Cor$.

Proof. We consider the case of $GW Cor$, the case of $W Cor$ is absolutely similar.

Since $R^0_{GW}(X,Y) \subset N^Q_{\alpha}(X \times \mathbb{G}_m, Y \times \mathbb{G}_m)$ (see Def. 6.15), it follows that there is a natural transformation

$$(6.33) \quad h^l_{\alpha}: R^\alpha_{GW}(-, Y) \to GW Cor((- \times \mathbb{A}^1, Y): i_0^\alpha(h^l_{\alpha}(a)) = \rho^\alpha_{GW,l}(\theta^\alpha_{GW}(a)), i_1^\alpha(h^l_{\alpha}(a)) = a.$$  

Next define natural maps

$$h^Q_{\alpha,+/-,-per}: R^Q_{\alpha}(X,Y) \to L^Q_{\alpha}(X \times \mathbb{G}_m \times \mathbb{A}^1, Y \times \mathbb{G}_m); h^Q_{\alpha,+/-,-per}(Q) = h^l_{\alpha}(-)(\theta^Q_{\alpha}(Q)),$$

where $h^Q_{\alpha,+/-,-per}$ are maps from Lemma 6.15. Denote by $\tilde{h}_{\alpha, GW,+}$, $\tilde{h}_{\alpha, GW,-}$ the induced homomorphisms of Grothendieck-Witt groups, and set

$$\tilde{h}^\alpha_{GW,per} = \tilde{h}^\alpha_{GW,+} - \tilde{h}^\alpha_{GW,-}; \tilde{h}^\alpha_{GW}(X,Y) \to L^\alpha_{GW}(X \times \mathbb{G}_m \times \mathbb{A}^1, Y \times \mathbb{G}_m),$$

where $p$ denotes projection $L^\alpha_{GW}(X \times \mathbb{G}_m \times \mathbb{A}^1, Y \times \mathbb{G}_m) \to L^\alpha_{GW}(X \times \mathbb{A}^1, Y \times \mathbb{G}_m).$ Then equations (6.10) implies that for any $a \in R^\alpha_{GW}(X,Y),$

$$(6.34) \quad i_0^\alpha(h^\alpha_{GW,pre-r,per}(a)) = \theta^\alpha_{GW}(a), i_1^\alpha(h^\alpha_{GW,pre-r,per}(a)) = T_Y \circ \theta^\alpha_{GW}(a) \circ T_X,$$

since for any $Q \in Q Cor(X \times \mathbb{G}_m \times \mathbb{G}_m, Y \times \mathbb{G}_m \times \mathbb{G}_m)$, we have $p(p_{GW,L}([h^\alpha_{GW}(Q)]) = 0.$ (With ore details: for any $y \in \mathbb{G}_m \times \mathbb{G}_m \times \mathbb{G}_m$ we have $p_{GW,L}([Q]) = 0.$, and for any $Q \in Q Cor(X \times \mathbb{G}_m \times \mathbb{G}_m, Y \times \mathbb{G}_m \times \mathbb{G}_m)$ such that Supp $Q \subset X \times \mathbb{G}_m \times \mathbb{G}_m$ we have $p([Q]) = 0.$ (and similarly for $Y$ instead of $X$).

Now define required homotopy as composition of homotopies $h^l_{\alpha}$ and $\rho^\alpha_{GW} \circ h^\alpha_{GW,pre},$ i.e.

$$h^\alpha_{GW,r} = \rho^\alpha_{GW} \circ h^\alpha_{GW,pre} - h^l_{\alpha} + id_{L^\alpha_{GW}(-, Y)}.$$  

Then form (6.33) and (6.34) we get

$$i_0^\alpha(h^\alpha_{GW,r}(a)) = i_\alpha, i_1^\alpha(h^\alpha_{GW,r}(a)) = \theta^\alpha_{GW,R}(a)$$

where in the second equality we use that

$$\rho^\alpha_{GW}(-)(T_Y \circ (Q \boxtimes \mathbb{G}_m) \circ T_X) = \rho^\alpha_{GW}(-)((\mathbb{G}_m \boxtimes Q) = \mathbb{G}_m \boxtimes \rho^\alpha_{GW}(-)Q = \rho^\alpha_{GW}(-)Q \boxtimes \mathbb{G}_m,$$

for any $Q \in R^Q_{\alpha}(X,Y)$ (see Lemma ?? for the middle equality).

\[\square\]

7. Cancellation theorem.

Firstly we prove cancellation in the category $D^\alpha_{GW}(k) = D^\alpha_!(\text{Pre}(GW Cor))$ (see sect. 3).

Let’s recall that there is a canonical functor $\text{Sm}_k \to D^\alpha_{GW}(k)$ which we denote $C^*(V) = C^*(GW Cor(-, V)) = \mathcal{H}om_{\text{Pre}(GW Cor)}(\Delta^*, Z_{GW}(V)).$ In the section we consider the case of GW-correspondences. All statements and proofs in the section holds for $W Cor$ as well.

Notation 7.1. Denote by $h^\alpha_{GW}(\mathcal{F}) = h^\alpha(C^*(\mathcal{F})) = h^\alpha(\mathcal{H}om(\Delta^*, \mathcal{F}))$ the presheaves of cohomologies of $\mathcal{F} \in \text{Pre}(GW).$ And for any $s: \mathcal{F} \to \mathcal{G} \in \text{Pre}(GW),$ $h^{s\alpha}_{GW}(s): h^{s\alpha}_{GW}(\mathcal{F}) \to h^{s\alpha}_{GW}(\mathcal{G})$ is the induced homomorphism.
Lemma 7.2. For any $Y \in S_{\alpha k}$,
$$
\lim_{\alpha \in A} h_{\alpha}^{i}(L_{\alpha}^{GW}(−, Y)) \simeq \lim_{\alpha \in A} h_{\alpha}^{i}(R_{\alpha}^{GW}(−, Y)) \simeq h_{\alpha}^{i}(GW\circ(−, \times G_{m}, Y \times G_{m}))
$$

Proof. Lm. [6.23] yields that $\lim_{\alpha \in A} R_{\alpha}^{GW}(X, Y) = \lim_{\alpha \in A} L_{\alpha}^{GW}(X, Y) = Q(X \times G_{m}, Y \times G_{m})$. By Lm. [6.23] again for any $(P_{1}, q_{1}), (P_{2}, q_{2}) \in Q(P(X \times G_{m}, Y \times G_{m}))$ there is a triple of integers $(r, m, n)$ such that $(P_{1}, q_{1}), (P_{2}, q_{2}) \in R_{\alpha}^{GW}(X, Y) \subset L_{\alpha}^{GW}(X, Y)$. Hence $\lim_{\alpha \in A} R_{\alpha}^{GW}(X, Y) \simeq \lim_{\alpha \in A} L_{\alpha}^{GW}(X, Y) \simeq GW\circ(−, X \times G_{m}, Y \times G_{m})$. Since the direct limits of abelian groups along filtering systems are exact, the claim follows.

Lemma 7.3. For any $\alpha \in A$ and integer $i$
1) $h_{\alpha}^{i}(\rho^{GW,L}_{\alpha}(\theta^{GW}_{\alpha})) = h_{\alpha}^{i}(\rho^{GW,N}_{\alpha}(\theta^{GW}_{\alpha})); h_{\alpha}^{i}(N^{GW}_{\alpha}(−, Y)) = h_{\alpha}^{i}(GW\circ(−, Y))$.
2) $h_{\alpha}^{i}(\theta^{GW}_{\infty}) \circ h_{\alpha}^{i}(\rho^{GW,R}_{\alpha}); h_{\alpha}^{i}(R^{GW}_{\alpha}(−, Y)) = h_{\alpha}^{i}(GW\circ(−, X \times G_{m}^{1}, Y \times G_{m}^{1}))$.

(see Lemma 6.22). Therefore $\lim_{\alpha \in A} \theta^{GW}_{\alpha} \simeq GW\circ(−, X \times G_{m}^{1}, Y \times G_{m}^{1})$ and see def [6.22] for $\theta_{\alpha}, \alpha \in A$.

Proof. The cylinder triangulation and Lm. [6.30] Lm. [6.32] gives homotopies of the chain complexes
$$
h_{\alpha}: C^{\alpha}(GW\circ(−, Y)) \to C^{\alpha}((GW\circ(−, Y))[−1],
$$
$$
h_{\alpha}: C^{\alpha}(L^{GW}(−, Y)) \to C^{\alpha}(GW\circ(−, X \times G_{m}^{1}, Y \times G_{m}^{1}))[−1]$$. \]

such that $d_{\alpha}(a) = a - \rho^{GW,L}(a) \otimes id_{G_{m}^{1}}$, and $d_{\alpha}(a) = id(a) - \rho^{GW,R}(a) \otimes id_{G_{m}^{1}}$, for $\alpha \in GW\circ(−, X \times G_{m}^{1}, Y \times G_{m}^{1})$. The claim follows.

Theorem 7.4. For any $X, Y \in S_{\alpha k}$ the functor $- \otimes id_{C^{\alpha}}$ induces the natural quasi-isomorphism
$$
GW\circ(−, X \times G_{m}^{1}, Y \times G_{m}^{1}) \simeq GW\circ(−, X \times G_{m}^{1}, Y \times G_{m}^{1})
$$

Proof. First we shall prove that $\text{Coker} h^{i}_{\alpha}(\theta^{GW}_{\infty}) = 0$. Lm [7.3] (2) gives us the commutative diagram
$$
\begin{array}{ccc}
\text{Coker}(\theta^{GW}_{\infty}) & \to & \text{Coker}(\theta^{GW}_{\infty}) \\
\downarrow \quad & \quad & \downarrow \\
\text{Coker}(\theta^{GW}_{\infty}) & \to & \text{Coker}(\theta^{GW}_{\infty}) \\
\end{array}
$$

Since the category of factor objects of $R_{\infty}$ is an ordered set, for any $\alpha < \alpha' \in A, c_{\alpha} = c_{\alpha} \circ c_{\alpha'},$ where $c_{\alpha} : \text{Coker}(\alpha^{GW,R}) \to \text{Coker}(\alpha^{GW,R})$. Hence there is a surjection $\lim_{\alpha \in A} \text{Coker}(\alpha) \to \text{Coker}(\theta^{GW}_{\infty})$ In the same time by Lemma 7.2 $\lim_{\alpha \in A} \text{Coker}(\alpha) \to \text{Coker}(\theta^{GW}_{\infty}) = 0$. Thus $\text{Coker}(\theta^{GW}_{\infty}) = 0$.

Now we prove injectivity. For any integer triples $\alpha = (r, m, n), \alpha' = (r', m', n')$ homomorphisms $\theta^{GW}_{\alpha}$ and $\theta^{GW}_{\alpha'}$ commute with homomorphisms $\gamma_{\alpha, \alpha'}^{GW,N}$ and $\gamma_{\alpha, \alpha'}^{GW,L}$ (see Lemma 6.22). Therefore $\lim_{\alpha \in A} \theta^{GW}_{\alpha} \simeq GW\circ(−, Y) \simeq L^{GW}(−, Y)$. Since injective limits of abelian groups is exact it follows that $\text{Ker}(\theta^{GW}_{\infty}) = \text{Ker}(\lim_{\alpha \in A} \theta^{GW}_{\alpha}) = \lim_{\alpha \in A} \text{Ker}(\theta^{GW}_{\alpha}) = 0$. \]

Corollary 7.5. For all $A^{\bullet}, B^{\bullet} \in D^{GW}_{\alpha k}(k)$, there is a natural isomorphism
$$
\text{Hom}_{D^{GW}_{\alpha k}(k)}(A^{\bullet}, B^{\bullet}) \simeq \text{Hom}_{D^{GW}_{\alpha k}(k)}(A^{\bullet}(1), B^{\bullet}(1))
$$
Proof. Since $M^\text{GW}(X)(1) = M^\text{GW}(X) \otimes M^\text{GW}(G^\Delta_1) \simeq M^\text{GW}(X \times G^\Delta_1)$, and since the objects $M^\text{GW}(X)$ generate the category $\text{DM}_{\text{eff}}^\text{GW}(k)$, it is enough to prove isomorphism $\text{Hom}_{\text{DM}_{\text{eff}}^\text{GW}}(X, Y[i]) \to \text{Hom}_{\text{DM}_{\text{eff}}^\text{GW}}(X \times G^\Delta_1, Y \times G^\Delta_1[i])$ for all $X, Y \in \text{Sm}_k$ and $i \in \mathbb{Z}$.

The adjunction $\text{D}^{-}(\text{Pre}(\text{GW} \text{Cor})) \subseteq \text{D}^+_k[(\text{Pre}(\text{GW} \text{Cor})) (\text{Th }??)]$ yields $\text{Hom}_{\text{DM}_{\text{eff}}^\text{GW}}(X, Z^\text{GW}(X), Z^\text{GW}(Y)[i]) \simeq \text{Hom}_{\text{DM}_{\text{eff}}^\text{GW}}(Z^\text{GW}(X), C^*(Z^\text{GW}(Y))[i]) = H^i\text{GW}^\Delta(X \times \Delta^*, Y)$. So the claim follows from Th ?? □

Consider the functor $\text{Hom}_{\text{DM}_{\text{eff}}^\text{GW}}(G^\Delta_1, -) : \text{Pre}^\text{GW} \to \text{Pre}^\text{GW}$, which is exact on $\text{Pre}^\text{GW}$, and induces a functor $\text{Hom}_{\text{DM}_{\text{eff}}^\text{GW}}(G^\Delta_1, -)$ on $\text{D}^{-}(\text{Pre}^\text{GW})$. Now since the last functor commutes with the functor $C^* = \text{Hom}_{\text{DM}_{\text{eff}}^\text{GW}}(\Delta^*, -)$, it follows that $\text{Hom}_{\text{DM}_{\text{eff}}^\text{GW}}(G^\Delta_1, -)$ is exact in respect to $\Delta^*$-quasi-isomorphisms. Thus it induces a functor $\text{Hom}_{\text{DM}_{\text{eff}}^\text{GW}}(G^\Delta_1, -) \to \text{D}^+_k[G^\Delta_1]$

Since the functor $- \otimes G^\Delta_1$ on $\text{D}^{-}(\text{Pre}^\text{GW})$ preserves the class of morphisms $X \times \Delta^1 \to X$ it is $\Delta^1$-exact too. So the adjunction $- \otimes G^\Delta_1 \to \text{Hom}(G^\Delta_1, -)$ on $\text{D}^{-}(\text{Pre}^\text{GW})$ yields an adjunction (7.6)

\[-\otimes G^\Delta_1 : \text{D}^+_k(G^\Delta_1) \to \text{D}^+_k[G^\Delta_1] : \text{Hom}(G^\Delta_1, -)\]

The last adjunction is a coreflection by Corollary 7.5 i.e. $A^* \simeq \text{Hom}(G^\Delta_1, A^* \otimes G^\Delta_1)$. We’ll show that the same holds for derived functors in $\text{DM}_{\text{eff}}^\text{GW}(k)$ that is considered as localisation of $\text{D}^+_k[G^\Delta_1]$ in respect to Nisnevich-quasi-isomorphisms.

Proposition 7.7. The functors $- \otimes G^\Delta_1$ and $\text{Hom}(G^\Delta_1, -)$ on the category $\text{D}^+_k[G^\Delta_1]$ are exact in respect to Nisnevich quasi-isomorphisms.

Proof. The functor $- \otimes G^\Delta_1$ is exact in respect to Nisnevich quasi-isomorphisms, since it preserves Nisnevich squares (see ?? for detailed discussion).

To prove the second claim it is enough to show that for a locally trivial homotopy invariant presheave with GW-transfers $F$ the complex of presheaves $\text{Hom}(G_m, F)$ is Nisnevich acyclic. In the same time $\text{Hom}(G_m, F) = F(\Delta^* \otimes G_m \times -) \simeq F(G_m \times -)$, and Lm 3.4 yields the claim. □

Theorem 7.8. For an infinite perfect field $k$, char $k \neq 2$ the canonical functors $\text{DM}_{\text{eff}}^\text{GW}(k) \to \text{DM}_{\text{eff}}^\text{GW}(k)$ are fully faithful embeddings.

Proof. Since the functors $- \otimes \text{D}^+_k[G^\Delta_1]$ and $\text{Hom}_{\text{DM}_{\text{eff}}^\text{GW}}(G^\Delta_1, -)$ on the category $\text{D}^+_k[G^\Delta_1]$ are exact in respect to Nisnevich quasi-isomorphisms by Proposition 7.7. Hence the adjunction (7.6) yields the adjunction $- \otimes G^\Delta_1 : \text{DM}_{\text{eff}}^\text{GW}(k) \to \text{DM}_{\text{eff}}^\text{GW}(k) : \text{Hom}(G^\Delta_1, -)$, and this adjunction is a coreflection too, that is equal to the claim. □

Corollary 7.9. For an infinite perfect field $k$, char $k \neq 2$, $X \in \text{Sm}_k$ and a motivic complex $A^* \in \text{DM}_{\text{eff}}^\text{GW}(k)$ there is a natural isomorphism $\text{Hom}_{\text{DM}_{\text{eff}}^\text{GW}}(M^\text{GW}(X), \Sigma_{G^\Delta_1} A^*[i]) \simeq H^i_{\text{Nis}}(X, A^*)$.

Proof. The claim follows immediately from Corollary 3.3 and Th 7.8. □

8. Appendix: The functor from the frame-correspondences to GWCor.

Definition 8.1. Suppose $X$ and $Y$ are pair of schemes, then a frame-correspondence on the rank $n$ between $X$ and $Y$ is a set $(V, Z, \phi, g)$, where $Z$ is a closed subset in $\mathbb{A}^n_X$, $\nu V \to \mathbb{A}^n_X$ is an etale morphism such that $\nu^{-1}(Z) \simeq Z$ (in other words $(V, Z) \to (\mathbb{A}^n_X, Z)$ is a Nisnevich neighbourhood), $\phi : V \to \mathbb{A}^n$, and $g : V \to Y$ are regular maps.

Denote by $Fr_n(X, Y)$ the set of isomorphism classes of frame-correspondences on the rank $n$ between $X$ and $Y$ up to a shrinking of the Nisnevich neighbourhood $(V, Z) \to (\mathbb{A}^n_X, Z)$, and let $Fr_n^* (X, Y) = \prod_n Fr_n(X, Y)$; denote by $Fr_*$ the category with objects smooth schemes and morphisms $Fr_n(X, Y)$ and the composition as defined in [13]: denote by $\mathbb{Z} Fr_*$ an additive category with objects smooth schemes and morphisms $\mathbb{Z} Fr_n(X, Y)/(\nu ((V, Z, \phi, g) - [\nu (V, Z, \phi, g)]) - [\nu (V, Z, \phi, g)])$, and denote by $Fr_+^{eff}$ and $\mathbb{Z} Fr_+^{eff}$ the full subcategories spanned by affine schemes.
Definition 8.2. For any integer n and integers \(d_1, \ldots, d_n\) denote \(Fr_{d_1, \ldots, d_n}(X, Y)\) a subset in \(Fr_n(X, Y)\) consisting of frame-correspondences \((V, Z, \phi_1, \ldots, \phi_n, g)\) such that for any \(i\) there is a section \(s_i \in \Gamma(\mathbb{P}_X^n, \mathcal{O}(d_i))\): \(s_i|_{\mathbb{P}^n_X} = x_i^{d_i}\), \(\phi_i = \nu^*(s_i/x_i^d)\), where \((x_0, x_1, \ldots, x_n)\) are coordinates on \(\mathbb{P}^n_X\), \(Z(x_0) = \mathbb{P}^n_X \setminus \mathbb{A}^n_X\).

Definition 8.3. For any frame correspondence \(\Phi = (V, Z, \phi_1, \ldots, \Phi_n, g) \in Fr_n(X, Y)\) denote by \(S(\Phi)\) the subset in \(Fr_n(\Phi)\) consisting of elements \(\tilde{\Phi} = (V', Z, \phi'_1, \ldots, \phi'_n, g)\) such that \(\phi'_i = \phi_i\mid_{Spec(k[A^n]/J(Z)^2)} = 0\) for all \(i\).

Let’s write \(\Phi \sim \Phi'\) iff \(\Phi' \in S(\Phi)\). This defines equivalence relation on frame-correspondences.

Lemma 8.4. For any \(\Phi = (V, Z, \phi, g), \Phi' = (V', Z, \phi', g) \in Fr_n(X, Y)\) if \(\Phi \sim \Phi'\) then there is an affine homotopy connecting \(\Phi\) and \(\Phi'\), i.e. \(\Phi \sim \Phi'\).

Proof. The required homotopy is given by \((V \times A^n, V, Z, \alpha \phi + (1 - \alpha)\phi', g) \in Fr_n(X \times A^n, Y)\).

Lemma 8.5. Suppose \(X\) is affine. Then for any \(\Phi \in Fr_n(X, Y)\) there is integer \(d\), such that for any integers \(d_i > d, i = 1, \ldots, n\), there is \(\tilde{\Phi} \in Fr_{n,d_1,\ldots,d_n}(X, Y) \cap S(\Phi)\). Proof. The claim follows from that for an affine scheme \(X\) the sheave \(\mathcal{O}(1)\) on \(\mathbb{P}^n_X\) is ample.

Construction 8.6. For any \(\Phi \in Fr_{d_1, \ldots, d_n}(X, Y)\) we construct a quadratic space \(Q(\Phi) \in QCor(X, Y)\) in the following way:

Consider map \(f: A^n_X \to A^n_X\), and denote \(A \to B\) corresponding homomorphism of sheaves of algebras over \(X\). Since \(\phi_i\) is polynom with leading coefficient \(x_i^{d_i}\) then \(f\) is finite morphism of smooth varieties and \(B\) is finite flat over \(A\). The Grothendieck duality theorem gives us isomorphism

\[\text{Hom}_B(B, A) \simeq \omega_B \otimes \omega_A^{-1}\]

Next using trivialisation of the canonical classes \(\omega_B\) and \(\omega_A\) defined by coordinate functions on relative affine space we get isomorphism

\[\text{Hom}_A(B, A) \simeq B\]

Now base change along the embedding by zero section \(X \to \mathbb{A}^n_X\) gives us isomorphism

\[\text{Hom}_{\mathcal{O}(X)}(\mathcal{O}(Z), \mathcal{O}(X)) \simeq \mathcal{O}(Z)\]

Proposition 8.7. 1) for any \(\Phi_1, \Phi_2 \in Fr_{d_1, \ldots, d_n}(X, Y), \Phi_1 \in S(\Phi_2), Q(\Phi_1) \simeq Q(\Phi_2)\); 2) for any \(\Phi_1 \in Fr_{d_1, \ldots, d_n}(X, Y), \Phi_2 \in Fr_{d_1, \ldots, d_n}(X, Y), \Phi_2 \in S(\Phi_2), Q(\Phi_1) \simeq Q(\Phi_2)\).

Proof. 1) Consider the frame correspondence \(\Theta = (Z \times \mathbb{A}^1, \mathbb{A}^n - Z', \varphi, g) \in Fr_n(X \times \mathbb{A}^1, Y)\), where \(\varphi = \lambda \varphi_1 + (1 - \lambda) \varphi_2, Z(\varphi) = Z' \cap \mathbb{A}^1\). The support \(Z(\varphi)\) is a closed subscheme in \(\mathbb{A}^n_X\) finite over \(X\) and so we can apply construction 5.0 to \(\Theta\). Then \(Q(\Theta) = (k[Z \times \mathbb{A}^1], g) \in QCor(X, Y)\) for some invertible function \(g \in k[Z \times \mathbb{A}^1]^*\) and it follows from lemma 2.18 that \(Q(\Theta) \simeq Q(\Theta \circ i_1)\) and whence \(GW(\Phi_0) = GW(\Theta \circ i_0) = GW(\Theta \circ i_1) = GW(\Phi_1)\).

2) Let \(\Phi_1 = (Z, \phi, g), \phi = (\phi^k), k = 1, 2; \phi^1 = s_1^k/x_0^{d_1}, \phi^2 = s_1^k/x_0^{d_1}\) for otherwise \(k\) and \(i\). Consider the frame correspondence \(\Theta = (Z \times \mathbb{A}^1, \mathbb{A}^n - Z', \lambda \varphi_1 + (1 - \lambda) \varphi_2, g) \in Fr_n(X \times \mathbb{A}^1, Y)\), where \(\varphi = \lambda \varphi_1 + (1 - \lambda) \varphi_2, Z(\varphi) = Z' \cap \mathbb{A}^1\).

Let \(\Gamma\) be the graph of the regular map \((\varphi, \nu_r): \mathbb{A}^n \times X \times \mathbb{A}^1 \to \mathbb{A}^n \times X \times \mathbb{A}^1\). Then \(\Gamma = \mathbb{A}^n \times \mathbb{A}^n \times X \times \mathbb{A}^1\), \(\widetilde{Z} = Z(\varphi) \subset \mathbb{P}^n \times X \times \mathbb{A}^1, \widetilde{s} = (\tilde{s}_i), \tilde{s}_i = s_i - T_i x_0^{d_1} \in \Gamma(\mathbb{P}^n \times X \times \mathbb{A}^1, \mathcal{O}(d_1))\), \(s_i = \lambda s_1 + (1 - \lambda) s_2 \in \Gamma(\mathbb{P}^n \times X \times \mathbb{A}^1, \mathcal{O}(d_1))\), and \(T_i\) denote coordinate functions on the multiplicand \(\mathbb{A}^n\) and \(\lambda\) denotes the coordinate of the multipicand \(\mathbb{A}^1\).

By the same reason as in lemma 5.3, we can choose some sections \(s'_i \in \Gamma(\mathbb{P}^n \times X \times \mathbb{A}^1, \mathcal{O}(d_1))\), \(s'_i|_{Z(Z(\mathbb{A}^1))} = s_i|_{Z(Z(\mathbb{A}^1))}\), \(s'_i = x_0^{d_1}, i = 2, \ldots, n\). Let’s put \(s'_i = s'_{i} - T_i x_0^{d_1}, i = 2, \ldots, n\).
Then consider the closed subscheme \( \tilde{Z} = Z(\tilde{s}) \subset \mathbb{P}^n \times A^n \times X \times A^1 \times (A-1)^{n-1} \), \( \tilde{s} = (\tilde{s}_i) \), \( \tilde{s}_i = \alpha_i \tilde{s}_i (1 - \alpha_i) \tilde{s}_i' \in \Gamma(\mathbb{P}^n \times A^n \times X \times A^1 \times (A-1)^{n}, \mathcal{O}(d)) \), where \( \alpha_i \) denote coordinates on the last multiplicant \((A-1)^n\).

Then \( \tilde{Z} \) is a smooth scheme over \( X \) and the projection \( \tilde{Z} \to A^n \times X \times A^1 \times (A-1)^n \) is finite. Applying the Duality theorem we get the \( k[\tilde{Z}] \)-linear isomorphism

\[
\tilde{q} : \text{Hom}_{k[A^n \times X \times A^1 \times (A-1)^n]}(k[\tilde{Z}], k[A^n \times X \times A^1 \times (A-1)^n]) \simeq \omega_X \times A^1 \times A^n(\tilde{Z}) \otimes \omega(A^n)^{-1},
\]

which can be considered as a morphism of coherent sheaves on \( \tilde{Z} \) and let \( \tilde{q} \) be the restriction of \( \tilde{q}' \) to the affine part \( \tilde{Z} \cap A^n \times X \times A^1 \times (A-1)^n \). It is easy to see that \( \tilde{F} \) is a graph of the morphism \( A^n \times X \times A^1 \times (A-1)^n \to A^n \) defined by the function , hence the projection \( \tilde{F} \to A^n \times X \times A^1 \times (A-1)^n \) (here we skip the second multiplicant \( A^n \)) is isomorphism.

The fibre of \( \tilde{Z} \) over \( A^n \times X \times 0 \times 0 \) is equal to the \( X \)-smooth closed subscheme \( \tilde{Z}_1 = Z(s_1^1 + T_1 x_0^1) \subset \mathbb{P}^n \times A^n \times X \times 0 \times 0 \simeq \mathbb{P}^n \times A^n \times X \times 0 \times 0 \), and base change in the Duality theorem yields that the fibre of \( \tilde{q} \) is equal to the quadratic space \( \tilde{q}_1 : \text{Hom}_{k[k[A^n \times X]}(k[\tilde{Z}_1], k[A^n \times X]) \simeq \omega_X(\tilde{Z}_1) \otimes \omega(A^n)^{-1} \) using in the construction applying to \( \Phi_1 \). Hence the fibre of \( Q = (k[\tilde{Z}], \tilde{q}) \) over \( 0 \times X \times 0 \times 0 \) is equal to \( Q(\Phi_1) \).

On the other side the \( X \)-smooth scheme \( \tilde{Z}_2 = Z(s_1^1 + T_1 x_0^1, s_2^1 + T_1 x_0^2) \subset \mathbb{P}^n \times A^n \times X \times 1 \times 0 \simeq \mathbb{P}^n \times A^n \times X \times 0 \times 0 \) is the disjoint component of the fibre of \( \tilde{Z} \) over \( A^n \times X \times 1 \times 0 \) and by similarly to the above the fibre of \( Q \) over \( 0 \times X \times 1 \times 0 \) is equal to \( Q(\Phi_2) \).

Thus the fibre of \( Q \) over \( 0 \times X \times A^1 \times 0 \) defines the quadratic space \((k[Z \times A^1], q)\) that is homotopy joining \( Q(\Phi_1) \) and \( Q(\Phi_2) \). So the claim follows by lemma 2.18.

Let’s present the using duality theorem.

**Theorem 8.8.** Suppose \( f : Y \to X \) is finite Cohen-Macaulay morphism of smooth affine schemes over the base \( S \); then there is a \( k[Y]-\)linear isomorphism \( q : \text{Hom}_{k[Y]}(k[Y], k[X]) \simeq \omega(Y) \otimes \omega_S(X)^{-1} \) that is natural in respect to base changes, i.e. for a diagram with Cartesian squares

\[
\begin{array}{ccc}
Y' & \to & X' \\
\downarrow & & \downarrow \\
Y & \to & S
\end{array}
\]

we have

\[
q' = q \otimes_{k[S]} k[S']
\]

**Proof.** The claim is a particular case of the Duality theorem from [20] in combination with the Base Change theorem from [27] or [28]. Another link is the proposition 2.1 in [20].

Using lemma 8.5 and proposition 8.7 we see that the construction 8.6 define a map \( Fr_a(X,Y) \to QCor(X,Y) \) for affine smooth schemes \( X,Y \). The pseudo-functoriality and the base change in the Duality theorem above yields that \( Q(\Phi_1 \circ \Phi_2) = Q(\Phi_1) \circ Q(\Phi_2) \), so we get a functor \( Fr_a^{a/1} \to QCor \). Moreover this induce a functor \( ZFr_a^{a/1} \to GWCor \), since if the support of \( Z \) of a frame-corrrespondence \( \Phi \) splits into disjoint union \( Z = Z_1 \sqcup Z_2 \) then for any \( \Phi' = (Z, s, g) \in S(\Phi) \), \( \Phi'_i = (Z_i, s, g) \in S((Z_i, \phi, g)), i = 1, 2 \), and by construction 8.3 \( Q(\Phi') = Q(\Phi_1) \cup Q(\Phi_2) \).

**Theorem 8.10.** There is a functor \( Fr_a^{a/1} \to QCor \) that takes \( \Phi \) to \( Q(\Phi) \), and there is a functor \( ZFr_a^{a/1} \to GWCor \) that takes \( [\Phi] \) to \( [Q(\Phi)] \).
Corollary 8.11. There are functors $Fr_* \rightarrow QCor_{nis}$, $ZFr_* \rightarrow GWCor_{nis}$, where $QCor_{nis}(-, Y)$ and $GWCor_{nis}(-, Y)$ are Nisnevich sheafification of $QCor(-, Y)$ and $GWCor(-, Y)$.

Remark 8.12. We can shorten the proof of the second point of the lemma by applying the following variant of the duality theorem to the scheme $\tilde{Z}$ and open subscheme $\tilde{Z} \cap A^n \times A^n \times X \times A^1$.

The duality theorem: Let $f: X \rightarrow Y$ be finite Cohen-Macaulay morphism, and $i: U \hookrightarrow X$ a smooth open subscheme; then there is an $\mathcal{O}(Y)$-linear isomorphism $\tau_{f,U}: \text{Hom}_{\mathcal{O}(Y)}(f_*(\mathcal{O}(X)), \mathcal{O}(Y))|_U \simeq (f \circ i)_*\omega_S(U)$ that is natural in respect to base change and natural in respect shrinking of $U$. I.e. for a diagram with Cartesian squares

$$
\begin{array}{ccc}
U' & \rightarrow & Y' \\
\downarrow & & \downarrow \\
U & \rightarrow & Y \\
\end{array}
$$

we have

$$q' = q \otimes_{k[S]} k[S'],
$$

and if $U' \subset U$ then $\tau_{f,U'}|_{U'} = \tau_{f,U}$.

Indeed, it is enough to the case of the schemes $X$ that are full intersection of sections in relative projective plain, i.e. $X = Z(s) \subset \mathbb{P}^n_k$, $s = (s_1, \ldots, s_k)$, $s_i \in \Gamma(\mathbb{P}^n_k, \mathcal{O}(d_i))$. In this case the Duality theorem above can be proved by using the explicit formula for dualisable complex given by Koszul complex $\wedge_{i=1,\ldots,n}[\mathcal{O} \rightarrow \mathcal{O}(d_i)] \in K^0(\mathbb{P}^n_k)$. So the required base change property can be checked explicitly using base change for Koszul complex and the isomorphism with the canonical class follows from the splitting of the Koszul complex in this case.

Remark 8.14. It is possible to get the functor for all schemes using the Duality theorem in the following form: For any scheme of finite type $S$ over the base field $k$ there $s$ a pseudo-functor $f'$ from the category of morphisms of finite type of $S$-schemes to the triangulated (dg-categories) $f': \text{Sch}_{ft,ln} \rightarrow \text{Tr}': X \rightarrow D^+_{qc}(X)$ with isomorphism $f'(X) \simeq \omega_S(X)$ for smooth $X$, and duality isomorphism $\text{Hom}_{D^+_{qc}(X)}(f_*(F), f'(G)) \simeq \text{Hom}_{D^+_{qc}(Y)}(f_*(F), G)$ for $F \in D^+_{qc}(X)$, and projective $f$ is compatible with base changes along arbitrary morphisms $S' \rightarrow S$ and Cohen-Macaulay $\tilde{f}$ and along open immersions $Y' \rightarrow Y$ and arbitrary morphisms $f$.

Let’s briefly describe the construction:

Let $\Phi = (\iota: \mathcal{V} \rightarrow \mathbb{A}^n_X, Z, \phi, g)$ be a frame-correspondence. Consider the morphism $f: \mathcal{V} \rightarrow \mathbb{A}^n_X$ defined by $\phi$ and projection to $X$. Shrinking $\mathcal{V}$ we may assume that $f$ is quasi-finite and let $\mathcal{V} \rightarrow \mathcal{T} \rightarrow \mathbb{A}^n_X$ be factorisation of $f$ such that $i$ is an open immersion and $\mathcal{T}$ is finite, such factorisation exists by the Main Zarisky theorem. Now applying the duality theorem to the finite morphism $\mathcal{T}$ we get an $\mathcal{O}(\mathcal{T})$-linear isomorphism $\tilde{q}: \text{Hom}_{\mathcal{O}(\mathcal{T})}(\mathcal{T}_*(\mathcal{O}(\mathcal{T})), \mathcal{O} \ast \mathbb{A}^n_X) \simeq \omega^1_{\mathcal{T} / \mathbb{A}^n_X}$, where $\omega_{\mathcal{T} / \mathbb{A}^n_X}$ is dualizable complex. Then using the isomorphism $\omega_{\mathcal{T} / \mathbb{A}^n_X} \simeq \omega_{\mathcal{V} / \mathcal{X} \cap X} \otimes \omega^{-1}_{\mathcal{X} / \mathcal{V}}$ and trivialisation of $\omega_{\mathcal{V} / \mathcal{X}}$ induced by $\nu$ and trivialisation of $\omega_{\mathcal{X} / \mathcal{V}}$ we get an isomorphism

$$\text{Hom}_{\mathcal{O}(\mathcal{X})}(\mathcal{T}_*(\mathcal{O}(\mathcal{T})))|_\mathcal{V} \simeq \mathcal{O}(\mathcal{V}).$$

Finally using base change along $0_X \rightarrow \mathbb{A}^n_X$ we get the $\mathcal{O}(Z)$-linear isomorphism $\text{Hom}(p_*(\mathcal{O}(Z)), \mathcal{O}(X)) \simeq \mathcal{O}(Z)$.

To prove the functoriality let’s note that though $\mathcal{T}$ is not Cohen-Macaulay but it is Cohen-Macaulay over generic point of $\mathbb{A}^n_X$ and so combining the base change theorem along open immersions (for arbitrary projective morphisms) with base change for Cohen-Macaulay morphisms (along an arbitrary morphism of schemes) we deduce that $\tilde{q}$ satisfies compositions axiom. We leave details for further consideration.
9. Appendix: Spectral category and non-commutative category of \( GW \)-correspondences.

We see that the definition \( \mathbb{E} \) (with the composition functor) gives rise to the category of correspondences enriched over the exact (additive) categories with duality, and over dg-categories with duality.

**Definition 9.1.** Let \( S \) be the noetherian base scheme of a finite dimension. Denote by \( \text{Cat}^\text{dual}_{\text{Ch}(S)} \) the category of dg-categories with duality (and duality preserving functors).

Denote by \( \mathcal{P}^D(S) \) the category enriched over \( \text{Cat}^\text{dual}_{\text{Ch}(S)} \) with objects being finite type schemes over \( S \), and the morphism-category for \( X, Y \in \text{Sch}_S \) being \( (\mathcal{P}(X, Y), D_X) \) (see def. ?).

Let \( GW_S \) be the category enriched over the spectra \( SH \) with objects finite type schemes over \( S \) and morphism-spectra \( GW_S(X, Y) \) be the Hermitian K-theory spectra of \( (D(\mathcal{P}(X, Y)), D_X) \).

Consider the category \( \text{Cat}^\text{dual}_{\text{Ch}(S)} \) enriched over \( \text{Cat}^\text{dual}_{\text{Ch}(S)} \) with objects being dg-categories with duality \( \mathcal{X} = (\mathcal{P}_X, D_X) \) with smooth dg-category \( \mathcal{P}_X \), and for \( \mathcal{X}, \mathcal{Y} \in \text{Cat}^\text{dual}_{\text{Ch}(S)} \) the morphism-category \( \text{Cat}^\text{dual}_{\text{Ch}(S)}(\mathcal{X}, \mathcal{Y}) = \text{Funct}(\mathcal{X}, \mathcal{Y}) \) is a category of functors \( \text{Funct}(\mathcal{X}, \mathcal{Y}) \) equipped with the duality \( F \to D_Y \circ F \circ D_X \), where \( D_X \) and \( D_Y \) are the dualities on \( \mathcal{X}, \mathcal{Y} \).

By the same way as above we can define the category \( GW_{nc} \) enriched over spectra form the category \( \text{Cat}^\text{dual}_{\text{Ch}(S)} \).

The definition of the category \( GW \) enriched over spectra actually is parallel to the definition of spectral category in \( \mathbb{E} \) by Garkusha and Panin, where the case of usual K-theory is considered, though the intermediate step of the category enriched over the categories is not discussed explicitly. And following the technique of \( \mathbb{E} \) one can define the category of motives related to \( GW \).

In the same time we can consider the following definition is the encroachment of the category of non-commutative varieties (spaces) with the dualities on the dg-categories. (we follow the construction presented in \( \mathbb{E} \)).

**Definition 9.2.** Define the \((\infty, 1)\)-category \( \mathcal{D}^\text{dual}_{\text{idem}}(S) \) as the localisation \( \mathcal{D}^\text{dual}_{\text{idem}}(S) \) at \( N\text{is} \) of \( \text{Cat}^\text{dual}_{\text{Ch}(S)} \) with objects being finite type schemes over \( S \) and morphism-categories \( \mathcal{D}^\text{dual}_{\text{idem}}(S) \) spanned by objects that goes to the dg-categories of finite type under the mentioned forgetful functor.

Denote by \( \overline{\text{NcS}}^\text{dual}(S) \) the opposite category to \( \mathcal{D}^\text{dual}_{\text{idem}}(S) \).

Let’s note that the \((\infty, 1)\)-categories in the definition above can be equipped with symmetric monoidal structures, which we denote by the symbol \( \otimes \).

**Remark 9.3.** In the definition above we need to use several different universes saying about additive categories with duality, the categories of additive categories with duality and the category of correspondences enriched over the last one.

Now we can apply the technique of \( \mathbb{E} \) to get the categories of motives form the categories of correspondences.

**Definition 9.4.** Define

\[
DM^GW(S) = L_{\text{nis}}L_{A^1}\mathcal{P}(GW)[\mathbb{P}^1, \infty]^{-1}, \quad DM^nc_{GW}(S) = L_{\text{nc-nis}}L_{A^1, nc}\mathcal{P}(GW_{nc}),
\]

where \( \mathcal{P}(\mathcal{C}) \) denotes the category of functors on \( \mathcal{C} \) with values in \( \text{SH} \), \( L_{A^1}, L_{A^1, nc}, L_{\text{nis}}, L_{\text{nc-nis}} \) are the localisation with respect to the classes of morphisms \( w_{A^1}, w_{ncA^1}, w_{\text{nis}}, w_{ncNis}, w_{A^1} \) is the class of morphisms of the form \( X \times A^1 \to X, X \in Sm_S \), \( w_{ncA^1} \) is the class of morphisms of the form \( X \otimes A^1 \to X, X \in NcS(S) \), \( w_{ncNis} \) of the form \( U \coprod_{L'} X' \to X \) defined by Nisnevich squares...
is invertible in $DM$ objects. Similar to that fact that The correspondences given by $GW$ groups or an $w$

Proof. The functor are defined by the universal properties and due that fact that $(D(P[\mathbb{P}^1]), D_{\mathbb{P}^1}, 1)$ is invertible in $DM_{nc}^{GW}(k)$ (because of full exceptional set of linear bundles on $\mathbb{P}^1$).

The second claim is because the functor $Hom_{Cat_{CH}(S)}(X, pt) \simeq (P(X), DX)$ respects $w_{nc,k}$ and $w_{nc,NcS}$.

\textbf{conjecture 9.6.} The canonical functor $SH(k)_Q \rightarrow DM_{nc}^{GW}(k)_Q$ is fully faithful.

The conjecture is based on that as follows form the recent results by Garkusha and result by Bachmann and Fasel $DM^{GW}(k)_Q \simeq SH(k)_Q$. So the full embedding form the conjecture above probable is decomposed as

$$SH(k)_Q \simeq DM^{GW}(k)_Q \simeq DM_{nc}^{GW}(k)_Q \hookrightarrow DM_{nc}^{GW}(k)_Q.$$ 

Actually the second question should follow form the general equivalence relation between $(\infty, 1)$-categories and categories enriched over $H_*$. The question which looks being the most difficult (and doubtful) is the equivalence $DM^{GW}(k)_Q \simeq DM_{nc}^{GW}(k)_Q$.

Another point we’d like to discuss is the following. In the definition of $\overline{NcS}(S)$ we replaced the objects. Similar to that fact that The correspondences given by $GW$ groups or an $SL$-orientable cohomology theory can be defined in the category $Sm_S$ and in the same time on the category of smooth varieties equipped with a line bundle, which has effect for the twisting of the cohomology groups. The corresponding categories of motives are equivalent with the identification of a pair $(X, L)$ and a pair of varieties $(T_L, T_L \rightarrow X)$ (i.e. Cone$(T_L \rightarrow X \rightarrow T_L)$, where $T_L$ is the Tome space of the bundle $L$, and $X$ is the zero section. In the same time in the context of non-commutative varieties, it is known that smooth proper non-commutative spaces satisfy duality theorem and so there is a canonical duality which we can use in the definition of $\overline{NcS}$ restricted to a smooth proper non-commutative varieties. So it looks being natural to ask the question:

\textbf{conjecture 9.7.} To reconstruct the category $\overline{NcS}(S)$ or $DM_{nc}^{GW}(k)$ without replacement of the objects of $NcS$ (which are dg-categories with out any additional structures) and with replacement morphisms only.

Let’s give a version of the answer which is based on the equivalence of the information given by functor $f^*$ of the categories with dualities and the information given by the pair of adjunctions $f^* \vdash f^*$ and $f^! \vdash f_!$. Define $Cat_{CH}(S)$ as the category with objects being dg-categories and any morphism form $A$ to $B$ is a pair of dg-functors $F, G$ with natural equivalence $t: F \simeq G$. $NcS^{\sim}$ which is defined in a similar way to $NcS$ starting form $Cat_{CH}(S)$.

\textbf{conjecture 9.8.} The $(\infty, 1)$-category $DM_{nc}^{GW}(k)$ is equivalent to $L_{nc-nis}L_{k,nc}P(NcS^{\sim})$.

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