Index Pairings for $\mathbb{R}^n$-Actions and Rieffel Deformations

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Abstract

With an action $\alpha$ of $\mathbb{R}^n$ on a $C^*$-algebra $A$ and a skew-symmetric $n \times n$ matrix $\Theta$ one can consider the Rieffel deformation $A_\Theta$ of $A$, which is a $C^*$-algebra generated by the $\alpha$-smooth elements of $A$ with a new multiplication. The purpose of this paper is to get explicit formulas for $K$-theoretical quantities defined by elements of $A_\Theta$. We assume there is a densely defined trace on $A$, invariant under the action. When $n$ is odd, for example, we give a formula for the index of operators of the form $P \pi_\Theta(u) P$, where $\pi_\Theta(u)$ is the operator of left Rieffel multiplication by an invertible element $u \in A + 1$, and $P$ is projection onto the nonnegative eigenspace of a Dirac operator constructed from the action $\alpha$. We first obtain the $K$-theoretical index as a Kasparov product and then apply to it a trace on the crossed product $A \rtimes_\alpha \mathbb{R}^n$ to obtain a real-valued index. The same quantity has the meaning of a pairing between a semifinite Fredholm module and a $K_1$-class of $A$, and as a pairing between cohomology and homology coming from the dynamical system $(A, \alpha, \mathbb{R}^n)$. The results are new also for the undeformed case $\Theta = 0$. The construction relies on two approaches to Rieffel deformations in addition to Rieffel’s original one: “Kasprzak deformation” and “warped convolution”. We end by outlining potential applications in mathematical physics.

1 Introduction

The pairing between $K$-theory and $K$-homology is central in Connes’ noncommutative geometry [Co4]. In this paper we will be concerned with $*$-subalgebras of a $C^*$-algebra $A$ with an action $\alpha$ of $\mathbb{R}^n$ as automorphisms of $A$ and with the original multiplication on $A$ replaced by a so-called “Rieffel product”. Such a product is
defined via the action \( \alpha \) together with a skew-symmetric \( n \times n \)-matrix \( \Theta \) and results in a new \(*\)-algebra \( A_\Theta \), the Rieffel deformation of \( A \), with this new product [Rie]. Our aim is to obtain \( K \)-theoretical quantities from operators acting as multiplication operators where the multiplication is the Rieffel product.

Connes also defined a, less well-known, Chern character for \( C^* \)-dynamical system [Co2]. He obtained the index of certain 0-order pseudodifferential operators associated with an extension of an iterated suspension of \( A \) by the crossed product \( A \rtimes_{\alpha} \mathbb{R}^n \).

On the other hand, we will show explicitly that the Thom isomorphism of Connes [Co] can in odd dimensions be directly realized as a generalized Toeplitz extension, an extension of \( A \) by \( A \rtimes_{\alpha} \mathbb{R}^n \) (for \( n = 1 \) this has been done before [Ji]). We cover the even case as well. Forming the analogous Toeplitz algebra for \( A_\Theta \) we can define an index for operators acting by left Rieffel multiplication with elements in \( A \). This index is an element of \( K_0(A \rtimes_{\alpha} \mathbb{R}^n) \). We assume the existence of a densely defined trace \( \tau \) on \( A \) which is invariant under the action. Then a scalar quantity is obtained by applying a semifinite trace \( \hat{\tau} \), which extends \( \tau \) to the von Neumann algebra \( \mathcal{N} := (A \rtimes_{\alpha} \mathbb{R}^n)' \), to the \( K \)-theory-valued index, using the Breuer-Fredholm theory surrounding \( \mathcal{N} \) and \( \hat{\tau} \).

One may ask how we ended up with crossed products and Toeplitz extensions when the original quest was to obtain an index pairing for operators acting by left Rieffel multiplication. The most straightforward approaches to do get complicated, hampered by the fact that \( \tau \) is not a trace for the Rieffel product. The solution turns out to be the transition described in Section 2.2 between product deformation and deformation of algebra elements into operators acting on the same Hilbert space. In that section we will clarify the relation between three different incarnations of the same deformation: (i) Rieffel deformation quantization by actions of \( \mathbb{R}^n \) [Rie], (ii) the Kasprzak deformation [Kas] in case of \( \mathbb{R}^n \), and (iii) the warped convolution introduced in [BLS]. The most important fact, proved in [Ne], is that the crossed products of \( A \) and \( A_\Theta \) are isomorphic. With this in hand the construction is carried out deceptively smoothly.

We obtain a formula for the Fredholm index of deformed operators using Toeplitz extensions and crossed products. Here we were inspired and helped by the magnificent paper [Le] which gives a formula for such a \( \hat{\tau} \)-index of a Fredholm Toeplitz operator \( T_u \) in terms of its invertible “symbol” \( u \in A \) as \( -(2\pi i)^{-1} \tau(u^{-1} \delta(u)) \), where \( \delta \) is the infinitesimal generator of \( \alpha \). This is the version of our formalism when \( \Theta \) is the zero matrix and \( n = 1 \) (more general Fredholm operators were considered in [Le] but we shall not need that). We can also use a trace which is not necessarily finite; this generalization was achieved for the \( n = 1 \) case already in [PR]. We mention that the result of [Le], [PR] has been of interest recently also in [CPS]. In fact, our initial idea was to generalize the proof in [Le], but it became evident that we could do better if we changed the strategy to that of a recent paper [CGPRS], which applies the machinery of the (nonunital semifinite) local index formula to prove the results from [PR]. The generalization to \( n \geq 1 \) is almost immediate once we have obtained an appropriate “Dirac operator” and representation of \( A \). We obtain the analogous formula for general deformation matrix \( \Theta \), also rather easily since we have identified the correct setting for Rieffel-deformed index theory (i.e. Toeplitz extensions).

We therefore put many pieces together. We discuss the \( K \)- and \( KK \)-theoretical aspects and show that the mentioned relation between the crossed products of \( A \) and \( A_\Theta \) ensures that the process carries over to \( A_\Theta \) at every step. We will also see that this crossed product approach gives an example of the general construction in
[KNR] where a $K_0$-valued spectral flow was defined.

In more detail, we have a separable $C^*$-algebra $A$ and a lower semicontinuous $\alpha$-invariant faithful trace $\tau$ on $A$ with dense domain $\text{Dom}(\tau) \subset A$. There is a strongly continuous action $\alpha$ of $\mathbb{R}^n$ on $A$ by $*$-automorphism, where $n \in \mathbb{N}$. We assume that $A$ is a subalgebra of $B(H)$ where $H$ denotes a Hilbert space such that $\alpha$ is unitarily implemented. The crossed product acts on the Hilbert space $L^2(\mathbb{R}^n, H)$. For $K$-theoretical reasons, the Hilbert space we use is $H = \mathbb{C}^N \otimes L^2(\mathbb{R}^n, H)$ where $N = 2^{(n-1)/2}$ for odd $n$ and $N = 2^{n/2}$ for even $n$ (see Section 2.4).

We consider the Rieffel deformation $A_\Theta$ of $A$ by the action $\alpha$ of $\mathbb{R}^n$ on $A$ and a skew-symmetric $n \times n$ matrix $\Theta$, and we represent $A_\Theta$ on the same space $H$. Any element $a \in A_\Theta$ acts on $H$ as an operator $\pi^\Theta(a) := 1_N \otimes \pi^\Theta(a) \in B(H)$ where $\pi^\Theta(a)$ means Rieffel multiplication by $a$. We show in Section 2.2 that such an operator corresponds to an operator with deformed symbol $a^\Theta$ in $A$ and with undeformed action on $H$. We let $D_1, \ldots, D_n$ be generators of the unitary group on $H$ implementing $\alpha$ in $H$ and form the Dirac operator

$$D := \sum_{k=1}^n \gamma^k \otimes D_k$$

using generators $\gamma^k$ of the $n$-dimensional complex Clifford algebra $\mathbb{C}_n$ acting irreducibly on $\mathbb{C}^N$. For even $n$ there is a $\mathbb{Z}_2$-grading of $H$ which we write as $H = H_+ \oplus H_-$. We will find a $*$-subalgebra $\mathcal{C}$ of $A$ of elements which are both sufficiently “smooth” with respect to $\hat{D}$ and “integrable” with respect to $(\hat{D}, \hat{\tau})$. It is from elements of this algebra that our $K$-theoretical quantities can be explicitly calculated. Let $\delta$ denote the infinitesimal generator of $\alpha$. By a standard procedure [LN] we extend the trace $\tau$ to a trace on the algebra of adjointable operators on $S^2(\mathbb{R}^n)$, so that it is well-defined on elements like $a^\Theta$. Let $\times_\Theta$ denote the Rieffel product, i.e. the multiplication in the deformed algebra $A_\Theta$.

**Theorem 1.** Let $n$ odd. For each unitary $u \in 1 + \mathcal{C}$,

$$\text{Index}_\tau(T^\Theta_u) = \text{Sf}(\hat{D}, \pi^\Theta(u^*) \hat{D}, \pi^\Theta(u))$$

$$= \langle [u^\Theta], [\mathcal{C}, \hat{D}, H] \rangle$$

$$= -\frac{2^n}{(2\pi i)^n n!} \tau\left((u^* \times_\Theta \delta(u))^n\right)$$

$$= \langle \text{Ch}_\tau(u^\Theta), [\mathbb{R}^n] \rangle$$

where $\text{Sf}(\hat{D}, u^* \hat{D} u)$ denotes the spectral flow between $\hat{D}$ and $\pi^\Theta(u^*) \hat{D} \pi^\Theta(u)$, the $\hat{\tau}$-index $\text{Index}_\hat{\tau}(T_u)$ concerns the Toeplitz operator $T^\Theta_u := P \pi^\Theta(u) P$ defined via the positive spectral projection $P$ of $\hat{D}$, and the bracketed pairings will be explained as we proceed.

Let $n$ even. For each projection $e \in \mathcal{C}$,

$$\text{Index}_\tau(\pi^\Theta(e) F_+ \pi^\Theta(e)) = \langle [e^\Theta], [\mathcal{C}, \hat{D}, H] \rangle$$

$$= \frac{2^n}{(2\pi i)^n n!} \tau\left((e \times_\Theta \delta(e) \times_\Theta \delta(e))^n\right)$$

$$= \langle \text{Ch}_\tau(e^\Theta), [\mathbb{R}^n] \rangle,$$
where \( F_+ : \mathcal{H}_+ \to \mathcal{H}_- \) is the \((+,−)\) part of \( F_\Theta := \mathcal{D}(1 + \mathcal{D}^2)^{-1/2} \) under the splitting \( \mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_- \).

Taking \( \Theta \) to be the zero matrix, Proposition 7 gives a generalization of the \( n = 1 \) formulae in [Le], [PR], [CGPRS]. If furthermore \( A = C(\mathbb{S}^1), \mathcal{C} = C^\infty(\mathbb{S}^1) \), with \( \mathcal{D} = \sqrt{-1}d/dt \) and \( \tau \) the Lebesgue integral, we get the classical Gohberg-Krein theorem (see [PR, §4(a)]).

The proof consists of putting together different notions from noncommutative geometry in such a way that deformations can be applied.

Remark 1. The assumption that \( \tau \) is invariant under the \( \mathbb{R}^n \)-action, i.e. that \( \tau \circ \alpha_t = \tau \) for all \( t \in \mathbb{R}^n \), precisely ensures that the weight \( \hat{\tau} \) on \( \pi_\alpha(A \times_\alpha \mathbb{R}^n)'' \) dual to \( \tau \) (see Definition 2) is a trace. We have modified the approach of [Le], [PR], [CGPRS] slightly by allowing \( A \) to be represented on Hilbert spaces other than the GNS space of \( \tau \). The motivation for this is that if the group \( \alpha \) does not preserve \( \tau \), it might still be unitarily implemented and the construction carries through. In that case one would have to use very different \( K \)-theoretical notions since \( \hat{\tau} \) is not a trace [An3], ending up with something like the “modular” spectral triples considered recently; see e.g. [CNNR]. We will not pursue this generalization here but with the present setup it may be possible to manage nontracial weights as well.

The final equality in Theorem 1 can be used to relate the Chern character \( \text{Ch}_\tau \) also to the index of Toeplitz operators with invertible functions \( f : \mathbb{S}^{n-1} \to A \) on the \((n−1)\)-sphere as symbols; see Theorem 1.

In addition to their purely mathematical relevance, it is worth mentioning that Rieffel deformations have great potential for physical applications (see e.g. [L], [Mu]). We conclude this paper with a brief discussion about this.

2 Setup and Proof

2.1 Notation

Unless otherwise specified, \( A, \alpha, \tau \) are as in the introduction, i.e. \( \alpha \) is a strongly continuous automorphic action of \( \mathbb{R}^n \) on a separable \( C^* \)-algebra \( A \) with smooth domain \( A \), where \( n = 1, 2, 3, \ldots \), and \( \tau \) is a faithful norm lower semicontinuous \( \alpha \)-invariant trace on \( A \) with dense domain \( \text{Dom}(\tau) \). We write \( \delta = (\delta_1, \ldots, \delta_n) \) for the generator of \( \alpha \).

The Rieffel deformation of \( A \) (resp. \( A_\Theta \)) for a fixed skew-symmetric \( n \times n \)-matrix \( \Theta \) is denoted \( A_\Theta \) (resp. \( A_\Theta \)) and defined in Section 2.2. We assume that the completion of \( A_\Theta \) in the norm obtained as in [Rie] coincides with \( A_\Theta \). The crossed products \( B := A \times_\alpha \mathbb{R}^n \) and \( B_\Theta := A_\Theta \times_\alpha \mathbb{R}^n \) are the smooth ones (see Section 2.2), while \( B := A \times_\delta \mathbb{R}^n \) and \( B_\Theta := A_\Theta \times_\delta \mathbb{R}^n \) are \( C^* \)-crossed products.

The space of Schwartz functions \( f : \mathbb{R}^n \to A \) is denoted \( S^A(\mathbb{R}^n) \) (with algebra or Hilbert \( A \)-module structure specified when necessary). The Hilbert space \( \mathcal{H} \) on which \( A \) and \( B \) act is introduced in Section 2.4. In Section 2.7 we introduce the \( * \)-algebra \( \mathcal{C} = \text{Dom}(\tau)^2 \cap A \).

The von Neumann algebra \( \mathcal{N} \) is defined as the bicommutant \( B'' \) of \( B \) in \( \mathcal{B}(\mathcal{H}) \), and the semifinite trace on \( \mathcal{N} \) dual to \( \tau \) is denoted \( \hat{\tau} \). The ideal of \( \hat{\tau} \)-compact operators in \( \mathcal{N} \) is \( \mathcal{K}_{\mathcal{N}} \) or \( \mathcal{K}(\mathcal{N}, \hat{\tau}) \). We denote by \( N(T) \) the projection onto the kernel of an operator \( T \).
The identity in \( M_N(\mathbb{C}) \) is written \( \mathbb{1}_N \). The identity on \( H \) is \( \mathbb{1} \). Sometimes we suppress in formulas the tensor product \( \otimes \) of an algebra with \( M_N(\mathbb{C}) \).

The \( C^* \)-algebra of compact operators on an infinite-dimensional separable Hilbert space is denoted by \( K \). The multiplier algebra of a \( C^* \)-algebra \( D \) is \( \mathcal{M}(D) \), the corona is \( \mathcal{Q}(D) := \mathcal{M}(D)/D \) and the minimal unitization is \( D^\sim := D \oplus \mathbb{C} \). We let \( S^k A \) be the \( k \)-fold iterated suspension of \( A \).

We will sometimes write \( \sqrt{-1} \) for the imaginary unit \( i \in \mathbb{C} \) for no other reason than to avoid confusing \( i \) with something else.

### 2.2 Rieffel Deformations in Three Ways

Here we will describe three different pictures of the same deformation and how to pass from one to another.

Rieffel’s deformation quantization \cite{Rie} starts with the data of (i) a Fréchet or \( C^* \)-algebra \( A \), (ii) an automorphic action \( \alpha \) of \( \mathbb{R}^n \) on \( A \), and (iii) a skew-symmetric \( n \times n \) matrix \( \Theta \). From this is constructed another algebra \( A_{\Theta} \) with a product \( \times_{\Theta} \) defined by

\[
a \times_{\Theta} b := \int_{\mathbb{R}^n \times \mathbb{R}^n} \alpha_{\Theta}(a) \alpha_v(b) e^{2\pi i u \cdot v} \, du \, dv, \quad \forall a, b \in A_{\Theta},
\]

and the norm of \( A_{\Theta} \) can be defined via a representation of \( A_{\Theta} \) on the Hilbert \( A \)-module \( S^A(\mathbb{R}^n) \) of \( A \)-valued Schwartz functions on \( \mathbb{R}^n \), where the \( A \)-valued inner product is given by

\[
(f, g)_A := \int_{\mathbb{R}^n} f(s)^* g(s) \, ds, \quad \forall f, g \in S^A(\mathbb{R}^n).
\]

**Remark 2.** In the case of \( C^* \)-algebras the action is automatically isomorphic since each \( \alpha_v \) is assumed to be a \( * \)-automorphism. In the case of Fréchet algebras one has to assume that the action is isometric (although it is possible to do with slightly less than that; see \cite{LW}). We shall consider the case of a strongly continuous action \( \alpha \) of \( \mathbb{R}^n \) by \( * \)-automorphisms on a \( C^* \)-algebra \( A \). So the action is also automatically isometric on the Fréchet subalgebra \( A \) of smooth elements. From now on \( A \) is the smooth subalgebra and the action is defined on the \( C^* \)-algebra \( A \).

Consider the smooth crossed product \( B := A \rtimes_\alpha \mathbb{R}^n \). Recall \cite{ENN} that this is the Fréchet \( * \)-algebra \( S^A(\mathbb{R}^n) \) with an \( \alpha \)-twisted convolution product (same as in a \( C^* \)-algebraic crossed product using \( \alpha \))

\[
(f \star g)(u) := \int_{\mathbb{R}^n} f(v) \alpha_v(g(u - v)) \, dv,
\]

and the involution \( f^*(u) := \alpha_u(f(-u))^* \). The algebras \( A \) and \( B \) are embedded in the multiplier algebra of \( B \) by letting \( x \in A \) act as \( \pi_\alpha(x) \), where

\[
\pi_\alpha(x)(\xi)(v) := \alpha_{-v}(x)\xi(v), \quad \forall \xi \in S^A(\mathbb{R}^n)
\]

as usual. On the same space \( S^A(\mathbb{R}^n) \) acts the Rieffel-deformed algebra \( A_{\Theta} \) (the deformation of \( A \) using action \( \alpha \) and matrix \( \Theta \)), via

\[
\pi_\Theta(x)(\xi) := \alpha(x) \times_{\Theta} \xi, \quad x \in A_{\Theta}, \xi \in S^A(\mathbb{R}^n),
\]

where \( \alpha(x) : \mathbb{R}^n \to A \) is the function \( \alpha(x)(v) := \alpha_{-v}(x) \).
Remark 3. Note that for $\Theta = 0$ (the zero matrix) we have (rewriting the Rieffel product slightly using the Fourier transform)

\[
\hat{\pi}^\Theta(a)\xi(t) = \int_{\mathbb{R}^n} \alpha_{\Theta^{-1}}(a)\tilde{\xi}(\xi)e^{2\pi i\xi t} \, ds
= \alpha_{-1}(a) \int_{\mathbb{R}^n} \tilde{\xi}(\xi)e^{2\pi i\xi t} \, ds
= \alpha_{-1}(a)\xi(t)
= (\hat{\pi}_\alpha(a)\xi)(t).
\]

That is,

\[
\hat{\pi}^\Theta = \hat{\pi}_\alpha,
\]

so that when we discuss $\pi^\Theta$ with general skew-symmetric $\Theta$ we automatically include the case $\pi_\alpha$.

We also have the $C^*$-algebraic crossed product $B := A \rtimes_\alpha \mathbb{R}^n$ which acts on $L^2(\mathbb{R}^n, \delta)$ if $A \subset B(\delta)$. Let $u \to \lambda_u$ be a unitary implementation of $\alpha$ in $L^2(\mathbb{R}^n, \delta)$,

\[
\pi_\alpha(\alpha_u(x)) = \lambda_u^\pi \pi^\Theta(x)\lambda_u = e^{iu\cdot D} \pi_\alpha(x)e^{-iuv}D, \quad x \in A, u \in \mathbb{R}^n
\]

where $D = (D_1, \ldots, D_n)$ are the generators of $u \to \lambda_u$ and $u\cdot D := u_1D_1 + \cdots + u_nD_n$. Now let $\Phi : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{T}$ be a 2-cocycle and define $\Phi(u, D_1, \ldots, D_n) = \lambda(\Phi_u)$ by functional calculus. We can take

\[
\lambda(\Phi_u) = e^{iu \cdot \Theta D}
\]

for some $n \times n$ matrix $\Theta$.

Denote by $\hat{\alpha} : \mathbb{R} \to \text{Aut}(B)$ the dual action. Kasprzak noticed [Kas] that the action defined by

\[
\hat{\alpha}^\Theta_u(x) := \lambda(\Phi_u)^*\hat{\alpha}_u(x)\lambda(\Phi_u), \quad \forall x \in B
\]

satisfies the same defining properties as the original dual action $\hat{\alpha}$. By Landstad’s theory of crossed products [P, Sec.7.8] the fixed-point subalgebra of $M(B)$ for $\hat{\alpha}^\Theta$ is isomorphic to a $C^*$-algebra $A_\Theta$ such that

\[
A_\Theta \rtimes_\alpha \mathbb{R}^n = B = A \rtimes_\alpha \mathbb{R}^n,
\]

where $\alpha^\Theta$ is the same action as $\alpha$ but on a different algebra (namely on $A_\Theta$ instead of $A$). In [Kas] the algebra $A_\Theta$ was called the “Rieffel deformation” of $A$. Several workers subsequently tried to elucidate the relation to Rieffel’s deformation by actions of $\mathbb{R}^n$ [HM], [Sa], [BNS]. Finally it was shown in [Ne] that the deformed algebra of Rieffel’s, which we denote also by $A_\Theta$, satisfies (4) (up to isomorphism). The isomorphism also extends to multiplier algebras.

We will use extensively the explicit isomorphism $B \ni f \to f^\Theta \in B_\Theta$ which underlies (4). For $f \in \mathcal{S}^A(\mathbb{R}^n)$ it is given by [Ne]

\[
f^\Theta(u) := \int_{\mathbb{R}^n} \alpha_{\Theta u}(f(v))e^{2\pi iuv} \, dv.
\]

We denote by $\tilde{\pi}_\alpha := \pi_\alpha \times \lambda$ and $\tilde{\pi}^\Theta := \pi^\Theta \times \lambda$ the representations of $B$ and $B_\Theta$ induced by $\pi_\alpha$ and $\pi^\Theta$ respectively. The important relation is [Ne]

\[
\tilde{\pi}^\Theta(f) = \tilde{\pi}_\alpha(f^\Theta), \quad \forall f \in \mathcal{S}^A(\mathbb{R}^n)
\]
where $f$ on the left-hand side is viewed as an element of $\mathcal{B}_\Theta$ and on the right-hand side as $f \in B$.

Now $f^\Theta$ is the Fourier transform of $v \to \alpha_{\Theta v}(\hat{f}(v))$ so in the spectral representation of the $D_b$'s, the operator $\tilde{\pi}_\alpha(f^\Theta)$ acts as multiplication by the function $v \to \pi_\alpha(\alpha_{\Theta v}(\hat{f}(v)))$. So

$$\tilde{\pi}_\alpha(f^\Theta) = \int_{\mathbb{R}^n} \pi_\alpha((f^\Theta(u))) e^{-2\pi i u \cdot D} du$$

$$= \int_{\mathbb{R}^n} e^{i\Theta v \cdot D} \pi_\alpha(\hat{f}(v)) e^{-i\Theta v \cdot D} e^{-2\pi i u \cdot v} du$$

and if $E^D(v)$ is the spectral measure of $D$ then we can (formally) write

$$\tilde{\pi}_\alpha(f^\Theta) = \int_{\mathbb{R}^n \times \mathbb{R}^n} \pi_\alpha((f^\Theta(u))) e^{-2\pi i u \cdot v} dE^D(v) du$$

$$= \int e^{i\Theta v \cdot D} \pi_\alpha(\hat{f}(v)) e^{-i\Theta v \cdot D} dE^D(v).$$

For elements $a \in A$ we have the function $\alpha(a) : \mathbb{R}^n \to A$ defined by $\alpha(a)(s) := \alpha_{\cdot a}(a)$, and we can define $\alpha(a)^\Theta$ using the same formula as for $f \in S^A(\mathbb{R}^n)$ above. That amounts to considering the multiplier $a^\Theta$ of the Hilbert $A$-module $S^A(\mathbb{R}^n)$ defined by

$$\langle \eta|a^\Theta \xi\rangle_A := \int_{\mathbb{R}^n} \eta(s)^* (\alpha(a) \times_{\Theta} \xi)(s) dt$$

$$= \int_{\mathbb{R}^n \times \mathbb{R}^n} \eta(s)^* \alpha_{\cdot-t \Theta a}(a) e^{2\pi i s \cdot \xi} \xi(s) dt ds, \quad \forall \eta, \xi \in S^A(\mathbb{R}^n),$$

where $\langle \cdot|\cdot \rangle_A$ is the inner product on $S^A(\mathbb{R}^n)$ defined in the beginning of this section and $\xi$ denotes the Fourier transform of $\xi$. When $\Theta = 0$, this operator $a^\Theta$ is just multiplication by the function $\alpha(a)$ introduced above. Extending $\pi_\alpha$ to $\mathcal{M}(A)$ we have

$$\pi_\alpha(a^\Theta) = \int_{\mathbb{R}^n} e^{i\Theta v \cdot D} a e^{-i\Theta v \cdot D} dE^D(v).$$

A deformation of concrete operators on Hilbert space (keeping the operator multiplication unchanged) was introduced in [BLS] under the name “warped convolution”.

**Observation 1.** The operator $\pi_\alpha(a^\Theta)$ is nothing but the warped convolution of $a \in A$ using generators $D$ and matrix $\Theta$.

Thus the notion of warped convolution extends to the crossed product. It is therefore very useful to consider $\pi_\alpha(A)$ instead of $A$ so that we can use the isomorphism $B_\Theta \cong B$ etc. We would probably not obtain something like the results of this paper without viewing the operator $\pi_\alpha(a^\Theta)$ as a multiplier of the crossed product.

From [BLS] we then get that $\pi_\alpha((a^\Theta) = \pi^\Theta(a)$, so that this relation extends beyond what is shown for $B$ in [Ne]. Note that this gives

$$\pi_\alpha(a^\Theta b^\Theta) = \pi_\alpha(a^\Theta) \pi_\alpha(b^\Theta) = \pi^\Theta(a) \pi^\Theta(b) = \pi^\Theta(a \times_{\Theta} b) = \pi_\alpha((a \times_{\Theta} b)^\Theta).$$

We can also deduce (7) from $\langle \eta|a^\Theta b^\Theta \xi\rangle = \langle \eta|(a \times_{\Theta} b)^\Theta \xi\rangle$, which is clear from the definition (5).
We now take the opportunity to digress on an even clearer transition between the Kasprzak and Rieffel approaches (and the warped convolution; this will not be relevant for the rest of the paper until we come to the applications). We consider warped convolution for unbounded operators $M$ acting on $L^2(\mathbb{R}^n, \mathcal{H})$ and denote by $M^\Theta$ the resulting operator; for the proof of the following, see [An1, An2, Mu].

**Theorem 2.** Let $Y_1, \ldots, Y_n$ denote the generators of the unitary group implementing the dual action $\hat{\alpha}$ on $B$. Then for all $j \in \{1, \ldots, n\}$,

$$Y_j^\Theta = Y_j + \sum_{k=1}^{n} \Theta_{jk} D_k.$$

This is the Kasprzak approach (for $\mathbb{R}^n$) because changing the dual action as done there corresponds exactly to the addition of the terms $\Theta_{jk} D_k$ (here we use that $[Y_j, D_k] = \sqrt{-1}\delta_{jk}$, which implies that the cocycle intertwining the unitary groups generated by $Y_j$ and $Y_j^\Theta$ is simply $\mathbb{R}^n \ni s \mapsto e^{i\Theta D}$) so that we get the action (3).

**Remark 4.** Rieffel showed that the $K$-theories of $A$ and $A_{\Theta}$ are isomorphic. However, there is no explicit description for the generators of $K_\ast(A_{\Theta})$ even when the generators of $K_\ast(A)$ are known.

### 2.3 The Dirac Operator

From now on, the $C^*$-algebra $A$, the action $\alpha$ and the trace $\tau$ are as in 2.1.

Let $A$ be identified with its image $A \subset B(\mathcal{H})$ in some a faithful representation such that $\alpha$ is unitarily implemented (for example, this always happens if $A''$ is in standard form [Ta2, Chap. IX, §1]). The Hilbert space $\mathcal{H}$ determines a representation $\pi_\alpha: A \rightarrow B(L^2(\mathbb{R}^n, \mathcal{H}))$ defined by the same formula (2) as in the last section, and we again denote by $\tilde{\pi}_\alpha$ the induced representation of the crossed product. The von Neumann algebra $N := \tilde{\pi}_\alpha(A \rtimes_\alpha \mathbb{R}^n)'\prime$ is independent of the choice of $\mathcal{H}$, up to isomorphism [Ta2, Chap.X, Thm1.7], and we fix such a $\mathcal{H}$ and the corresponding $\pi_\alpha$.

It turns out that the $K$-theoretical constructions that we want require the crossed product $B = A \rtimes_\alpha \mathbb{R}^n$ to be represented not on $L^2(\mathbb{R}^n, \mathcal{H})$ but rather on an amplification thereof. Write

$$N := \begin{cases} 2^{n/2} & \text{if } n \text{ is even,} \\ 2^{(n-1)/2} & \text{if } n \text{ is odd.} \end{cases}$$

The $n$-dimensional complex Clifford algebra $\mathbb{C}_n$ can then be identified with

$$\mathbb{C}_n \cong \begin{cases} M_N(\mathbb{C}) & \text{if } n \text{ is even,} \\ M_N(\mathbb{C}) \oplus M_N(\mathbb{C}) & \text{if } n \text{ is odd.} \end{cases}$$

The irreducible representation of $\mathbb{C}_n$ for even $n$ is on $\mathbb{C}^N$. For odd $n$ there are two irreducible representations, given by sending the first respectively the second $M_N(\mathbb{C})$-summand in $\mathbb{C}_n$ to the fundamental representation of $M_N(\mathbb{C})$ on $\mathbb{C}^N$. We therefore consider the Hilbert space

$$\mathcal{H} := \mathbb{C}^N \otimes L^2(\mathbb{R}^n, \mathcal{H}).$$
From $\pi = \pi_\alpha, \pi_\Theta$ we obtain representations of $A, A_\Theta$ on $\mathcal{H}$ by sending $a \in A$ to

$$\pi(a) \equiv 1_N \otimes \pi(a).$$

The selfadjoint generators $D_1, \ldots, D_n$ of the unitary group implementing $\alpha$ on $L^2(\mathbb{R}^n, \delta)$ can be used to define the Dirac operator (the tensor product implicit)

$$\slashed{D} := \sum_{k=1}^n \gamma^k D_k,$$

where $\gamma^1, \ldots, \gamma^n$ are hermitian $N \times N$ matrices representing the generators of $C_n$ on $\mathbb{C}^N$, satisfying therefore the Clifford relations $\gamma^j \gamma^k + \gamma^k \gamma^j = 2 \delta^{jk}$. For example,

$$\slashed{D} = -D_1$$

if $n = 1$,

$$\slashed{D} = \begin{pmatrix} 0 & iD_1 + D_2 \\ -iD_1 + D_2 & 0 \end{pmatrix}$$

if $n = 2$,

$$\slashed{D} = \begin{pmatrix} -iD_3 & iD_1 + D_2 \\ -iD_1 + D_2 & iD_3 \end{pmatrix}$$

if $n = 3$,

and for both $n = 2$ and $n = 3$, the $\gamma^k$s are the Pauli matrices.

For even $n$ we can always find a grading operator $\Gamma$ on $\mathcal{H}$ such that $\Gamma \pi_\alpha(a) = \pi_\alpha(a) \Gamma$ for all $a \in A$ and $\Gamma \slashed{D} = -\slashed{D} \Gamma$. In the example $n = 2$ we can take $\Gamma$ to be the third Pauli matrix. We write

$$\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$$

for even $n$, with $\mathcal{H}_\pm$ the $\pm 1$-eigenspace of the grading operator $\Gamma$.

Let

$$M_N(B) := M_N(\mathbb{C}) \otimes B$$

be regarded as a subalgebra of $B(\mathcal{H})$ (thus omitting $\pi_\alpha$ from the notation).

In the next lemma we use the principal value integral on $\mathbb{R}^n$ for a distribution kernel $K$, defined by

$$\text{P.V.} \int_{\mathbb{R}^n} K(t) \, dt := \lim_{\varepsilon \to 0^+} \int_{|t| > \varepsilon} K(t) \, dt.$$

We also write $|D| := \sqrt{D_1^2 + \cdots + D_n^2}$.

**Lemma 1.** For $k = 1, \ldots, n$, define

$$R_k := D_k |D|^{-1},$$

as the operator on $\mathcal{H}$ with Fourier multiplier $\mathbb{R}^n \ni p \to p_k / |p|$. Then the commutators $[R_k, \pi_\alpha(a)]$ belong to $M_N(B)$ for all $a \in A$, since

$$[R_k, \pi_\alpha(a)] = \frac{c_n}{\sqrt{-1}} \text{P.V.} \int_{\mathbb{R}^n} \pi_\alpha \left( \alpha_t(a) - a \right) e^{-2\pi i a \cdot \cdot \cdot D_k} dt / |t|^{n+1},$$

where $c_n := \Gamma((n+1)/2) / \pi^{(n+1)/2}$.
Proof. Except for being tensored with the identity on $\mathcal{H} \otimes \mathbb{C}^N$, the operator $R_k$ is the integral operator on $L^2(\mathbb{R}^n)$ with kernel given by [St, §III.1]

$$K_k(t) := \frac{c_n}{\sqrt{-1}} \frac{t_k}{|t|^{n+1}}, \quad \forall t = (t_1, \ldots, t_n) \in \mathbb{R}^n,$$

which is homogeneous of degree $-n$. Hence $R_k$ defines a bounded operator $\mathcal{H}$. We can therefore calculate as in [PR, Lemma 3.2] that for $\xi \in \mathcal{S}^\alpha(\mathbb{R}^n)$ and for $a \in A$ (the dense subalgebra of $\alpha$-smooth elements),

$$([R_k, \pi_\alpha(a)]\xi)(s) = \text{P.V.} \int_{\mathbb{R}^n} (K_k(t)(\pi_\alpha\xi)(s-t) - \alpha_-\pi_\alpha(R_k\xi)(s)) \, dt = \text{P.V.} \int_{\mathbb{R}^n} (K_k(t)(\alpha_-\pi_\alpha(a) - \alpha_-\pi_\alpha(a))\xi(s-t) dt = \text{P.V.} \int_{\mathbb{R}^n} \pi_\alpha(K_k(t)(\alpha_\pi(a) - a)) e^{-2\pi it \cdot D}\xi(s) dt,$$

and we have to prove that $f_k(t) := K_k(t)(\alpha_\pi(a) - a)$ belongs to $L^1(\mathbb{R}^n, A)$. Since $f_k$ vanishes at infinity, we need only worry about local integrability in a neighborhood of $0 \in \mathbb{R}^n$. However, for each $j = 1, \ldots, n$,

$$\lim_{t \to 0} f_k(t) = \frac{c_n}{\sqrt{-1}} \lim_{t \to 0} \frac{t_k}{|t|^{n+1}} (\alpha_\pi(a) - a) = \frac{c_n}{\sqrt{-1}} \lim_{t \to 0} \frac{t_k t_j}{|t|^{n+1}} \delta_j(a) \lim_{t \to 0} \frac{t_k}{|t|^{n+1}},$$

and the function $t \to t_k t_j/|t|^{n+1}$ is homogeneous of degree $-n + 1 > -n$, hence locally integrable at $|t| \leq 1$ (as seen by changing to polar coordinates). This shows that $f_k$ is integrable, and therefore that $[R_k, \pi_\alpha(a)]$ is an element of $M_N(B)$ when $a$ is in $A$. The result follows from density of $A$ in $A$. \hfill \Box

Corollary 1. Let $P := \mathcal{D}^\alpha(\mathbb{R}^n)$ denote the projection on $\mathcal{H}$ corresponding to the nonnegative spectrum of the Dirac operator (8). Then for all $a \in A$,

$$[P, \pi_\alpha(a)] = \frac{c_n}{2\sqrt{-1}} \text{P.V.} \sum_{k=1}^n \gamma^k \int_{\mathbb{R}^n} \pi_\alpha(\alpha_\pi(a) - a) e^{2\pi it \cdot D} \frac{t_k dt}{|t|^{n+1}}$$

belongs to $M_N(B)$.

Proof. Let $R := \mathcal{D}|D|^{-1}$ denote the phase of the Dirac operator $\mathcal{D}$, defined as the integral operator with kernel the fundamental solution of $\mathcal{D}$ (see e.g. [BL]). Then

$$R = \mathcal{D}|D|^{-1} = \sum_{k=1}^n \gamma^k R_k,$$

where the $R_k$’s are defined in Lemma 1. Now note that $P = (1 + R)/2$. \hfill \Box
2.4 The Toeplitz Algebra

To obtain a relation between the $\hat{\tau}$-index and Chern characters (both to be defined later) we use the Toeplitz extension of $A$ by $B = A \rtimes_\alpha \mathbb{R}^n$. This was considered in [Le] for the case $n = 1$ only so we have to extend the theory to higher dimensions.

In what follows we write $A$ and $B$ for $\pi_\alpha(A)$ and $\pi_\alpha(B)$ to simplify notation. We let $P$ denote the spectral projection of $\mathcal{D}$ corresponding to the interval $[0, +\infty)$. 

**Definition 1.** The Toeplitz algebra $\mathcal{T}$ is the $C^*$-subalgebra of $B(\mathcal{H})$ generated by $M_N(B)$ together with elements of the form

$$T_a := P\pi_\alpha(a)P$$

for $a \in A$.

The following two side-remarks serve only to clarify the relationship between different known approaches to a “Toeplitz extension” of $A$ by the crossed product $B$, as well as how this is related to the invertibility of the Dirac operator.

**Remark 5.** The kernel of the “symbol map”

$$\mathcal{T} \to A, \quad T_a \to a$$

contains the ideal (the “semicommutator ideal”) generated by $\{T_aT_b - T_{ab}|a, b \in A\}$. By generalizing the proof in [Le, Prop. 3.3] one can show that the reverse inclusion holds. However, what we want from the Toeplitz extension is that it should represent the Thom element in $KK^1(A, B)$. So we must obtain an extension of $A$ by $M_N(B)$ instead.

To do so, the approach of several authors [Ji], [Le] has been to replace $P$ by a smoothened version $P$ by a smoothened version $P$ by a smoothened version $P$ by a smoothened version $P$ by a smoothened version $P$ by a smoothened version $P$ by a smoothened version $P$ by a smoothened version $P$ by a smoothened version $P$ by a smoothened version $P$ by a smoothened version $P$ by a smoothened version $P$ by a smoothened version $P$ by a smoothened version $P$ by a smoothened version $P$ by a smoothened version $P$ by a smoothened version $P$ by a smoothened version $P$ by a smoothened version $P$ by a smoothened version $P$ by a smoothened version $P$ by a smoothened version $P$ by a smoothened version $P$ by a smoothened version $P$ by a smoothened version $P$ by a smoothened version $P$ by a smoothened version $P$ by a smoothened version $P$ by a smoothened version $P$ by a smoothened version $P$ by a smoothened version $P$ by a smoothened version $P$ by a smoothened version $P$ by a smoothened version $P$ by a smoothened version $P$ by a smoothened version $P$ by a smoothened version $P$ by a smoothened version $P$ by a smoothened version $P$ by a smoothened version $P$ by a smoothened version $P$ by a smoothened version $P$ by a smoothened version $P$ by a smoothened version $P$ by a smoothened version $P$ by a smoothened version $P$ by a smoothened version $P$ by a smoothened version $P$ by a smoothened version $P$ by a smoothened version $P$ by a smoothened version $P$ by a smoothened version $P$ by a smoothened version $P$ by a smoothened version $P$ by a smoothened version $P$ by a smoothened version $P$ by a smoothened version $P$ by a smoothened version $P$ by a smoothened version $P$ by a smoothened version $P$ by a smoothened version $P$ by a smoothened version $P$ by a smoothened version $P$ by a smoothened version $P$ by a smoothened version $P$ by a smoothened version $P$ by a smoothened version $P$ by a smoothened version $P$ by a smoothened version $P$ by a smoothened version $P$ by a smoothened version $P$ by a smoothened version $P$ by a smoothened version $P$ by a smoothened version $P$ by a smoothened version $P$ by a smoothened version $P$ by a smoothened version $P$ by a smoothened version $P$ by a smoothened version $P$ by a smoothened version $P$ by a smoothened version $P$ by a smoothened version $P$ by a smoothened version $P$ by a smoothened version $P$ by a smoothened version $P$ by a smoothened version $P$ by a smoothened version $P$ by a smoothened version $P$ by a smoothened version $P$ by a smoothened version $P$ by a smoothened version $P$ by a smoothened version $P$ by a smoothened version $P$ by a smoothened version $P$ by a smoothened version $P$ by a smoothened version $P$ by a smoothened version $P$ by a smoothened version $P$ by a smoothened version $P$ by a smoothened version $P$ by a smoothened version $P$ by a smoothened version $P$ by a smoothened version $P$ by a smoothened version $P$ by a smoothened version $P$ by a smoothened version $P$ by a smoothened version $P$ by a smoothened version $P$ by a smoothened version $P$ by a smoothened version $P$ by a smoothened version $P$ by a smoothened version $P$ by a smoothened version $P$ by a smoothened version $P$ by a smoothened version $P$ by a smoothened version $P$ by a smoothened version $P$ by a smoothened version $P$ by a smoothened version $P$ by a smoothened version $P$ by a smoothened version $P$ by a smoothened version $P$ by a smoothened version $P$ by a smoothened version $P$ by a smoothened version $P$ by a smoothened version $P$ by a smoothened version $P$ by a smoothened version $P$ by a smoothened version $P$ by a smoothened version $P$ by a smoothened version $P$ by a smoothened version $P$ by a smoothened version $P$ by a smoothened version $P$ by a smoothened version $P$ by a smoothened version $P$ by a smoothened version $P$ by a smoothened version $P$ by a smoothened version $P$ by a smoothened version $P$ by a smoothened version $P$ by a smoothened version $P$ by a smoothened version $P$ by a smoothened version $P$ by a smoothened version $P$ by a smoothened version $P$ by a smoothened version $P$ by a smoothened version $P$ by a smoothened version $P$ by a smoothened version $P$ by a smoothened version $P$ by a smoothened version $P$ by a smoothened version $P$ by a smoothened version $P$ by a smoothened version $P$ by a smoothened version $P$ by a smoothened version $P$ by a smoothened version $P$ by a smoothened version $P$ by a smoothened version $P$ by a smoothened version $P$ by a smoothened version $P$ by a smoothened version $P$ by a smoothened version $P$ by a smoothened version $P$ by a smoothened version $P$ by a smoothe
because four anticommuting variables are needed to linearize the energy-momentum relation \( E = \sqrt{p^2 + m^2} \) when \( m \neq 0 \). The operator in (9), where \( \mathcal{D} \) is defined in (8) for \( n = 3 \), has a gap \([-m, +m]\) in the spectrum. Hence, using (9) the smoothened version \( h(\mathcal{D}) \) (cf. Remark 5) and the true projection \( P \) would actually coincide, provided \( \varepsilon < m \). Another interesting observation is that (9) is unitarily equivalent to the operator obtained from a general construction [CGRS, Def. 3.9], by doubling-up to \( 2 \times 2 \) matrices, of an invertible representative of the operator defining a KK-class coming from a spectral triple. Thus the addition of a mass term has both a physical and technical motivation, yet we will not need any of that in this paper.

**Proposition 1.** There is a short exact sequence

\[
0 \longrightarrow M_N(B) \longrightarrow \mathcal{T} \longrightarrow A \Theta \longrightarrow 0.
\]

**Proof.** Using Corollary 1 we can repeat the proof in [Le, Prop. 3.1(1)] to show that \( M_N(B) \) is an ideal in \( \mathcal{T} \). The quotient of \( \mathcal{T} \) by \( M_N(B) \) identifies with a subalgebra of \( A \). So to obtain the above sequence it suffices to show that \( T_a / \in M_N(B) \) for all \( a \in A \). Since \([P,\pi_\alpha(a)]\) is in \( M_N(B) \) by Corollary 1,

\[
T_a \equiv P\pi_\alpha(a) \mod M_N(B),
\]

so it is enough to show that \( P\pi_\alpha(x) \) is not in \( M_N(B) \). Then it is enough to show that \( R_k\pi_\alpha(a) / \in B \) for some \( k \). For that we can follow [Le, Prop. 3.3].

Now back to the deformation to treat the general case of \( \Theta \neq 0 \) at the same time. We define \( T_\Theta a \) just as \( T_a \) but with \( \pi_\Theta \) instead of \( \pi_\alpha \). So

\[
T_\Theta a := P\pi_\Theta(a)P, \quad \forall a \in A,
\]

and the Toeplitz algebra \( \mathcal{T}_\Theta \) is generated by

\[
\{T_\Theta a | a \in A\} \cup M_N(B_\Theta).
\]

Let \( \mathcal{N} := B'' \) be the closure of \( B \) in the weak operator topology on \( B(\mathcal{H}) \). We say that an operator \( T \in \mathcal{N} \) is Fredholm relative to an ideal \( J \subset \mathcal{N} \) if \( T \) is invertible modulo \( J \). The notion carries over to ideals in \( M_N(\mathcal{N}) \).

**Lemma 2.** The operator \( T_\Theta a \in \mathcal{T}_\Theta \) is Fredholm (as an operator on \( PH \)) relative to \( M_N(B_\Theta) \) iff \( a \) is invertible in \( A^- \).

**Proof.** The short exact sequence

\[
0 \longrightarrow M_N(B_\Theta) \longrightarrow \mathcal{T}_\Theta \longrightarrow A_\Theta \longrightarrow 0
\]

is obtained using \( T_\Theta a = T_a / \Theta \) for \( a \in A \). It follows that if \( T_\Theta a \) is invertible modulo \( M_N(B_\Theta) \) then \( a \) is invertible. Conversely, if \( u \in A^- \) is invertible then, since \([P,\pi_\alpha(u)] \in M_N(B_\Theta)\) by Lemma 1, we get

\[
(P\pi_\alpha(u)P)(P\pi_\alpha(u^{-1})P) \equiv P \mod M_N(B),
\]

and similarly for \( u \leftrightarrow u^{-1} \). Now \( P \) is the identity in \( PHP \). We can then define a \( K_\Theta(B_\Theta) \)-valued index for \( M_N(B_\Theta) \)-relative Fredholm operators in \( \mathcal{T}_\Theta \). On the von Neumann algebra \( \mathcal{N} \) there is a semifinite trace [H1], [H2].
Remark 7. Hence, with \( F \) of two elements in \( \text{Dom}(\tilde{\varphi}) \) where \( \varphi \) is the operator-valued weight from \( N \) to \( N^\alpha = \pi_\alpha(A)'' \) given by

\[
I_\varphi(x^*x) := \int_{\mathbb{R}^n} \tilde{\alpha}_\rho(x^*x) \, d\rho, \quad \forall x \in N.
\]

For \( f \in L^1(\mathbb{R}^n, A) \cap L^2(\mathbb{R}^n, \mathcal{F}) \), the important formula is

\[
\hat{\tau}(\pi_\alpha(f^* \pi_\alpha(f))) = \tau((f|f)_A),
\]

where \( \langle \cdot, \cdot \rangle_A \) is defined in (1). We will only use \( \hat{\tau} \) tensored by the matrix trace \( \text{Tr} \) on \( M_N(\mathbb{C}) \) so we write \( \hat{\tau} := \text{Tr} \otimes \hat{\tau} \).

We refer to [H3], [H4], [Ta2, Chap. IX] for the notion of operator-valued weights.

Remark 7. If \( \mathcal{F} = \mathcal{H}_\tau \) is the GNS space of \( \tau \) then we can also define \( \hat{\tau} \) as the semifinite weight on \( N' \) corresponding to the Hilbert algebra \( L^1(\mathbb{R}^n, \text{Dom}(\tau)) \cap L^2(\mathbb{R}^n, \mathcal{H}_\tau) \) (embedded in \( N \) via \( \tilde{\pi}_\alpha \)).

Remark 8. Recall that we assume \( \tau \) to be invariant under the \( \mathbb{R}^n \)-action. If not, say if \( \tau(\alpha_t(a)) = \tau(\rho^t a) \) for all \( a \in A, t \in \mathbb{R} \) for some positive invertible operator \( \rho \) affiliated to \( A' \), then the modular automorphism group \( \sigma^{\hat{\tau}} \) of \( \hat{\tau} \) is nontrivial, namely

\[
\sigma^{\hat{\tau}}_t(x) = x, \quad \sigma^{\hat{\tau}}_t(e^{2\pi isD}) = \pi_\alpha(\rho^{it}) e^{2\pi isD}, \quad \forall x \in N', s, t \in \mathbb{R}^n,
\]

so \( \hat{\tau} \) is not a trace in this case. This is why we assume that \( \alpha \) preserves \( \tau \), but our approach should generalize slightly to allow some non-modularity, something that we leave to the future; cf. Remark 1.

Let \( K_N = K(N, \hat{\tau}) \) denote the ideal of \( \hat{\tau} \)-compact operators. To apply the induced homomorphism \( \hat{\tau}_* : K_0(K_N) \rightarrow \mathbb{R} \) to elements of \( K_0(B) \) we need that the elements are represented by \( \hat{\tau} \)-finite differences of projections.

Lemma 3. Let \( \text{Dom}(\tau)^2 \) be the \( * \)-subalgebra of \( A \) consisting of finite sums of products of two elements in \( \text{Dom}(\tau) \). Then

\[
\pi_\alpha(a)(1 + B)^{-1} \in K_N, \quad \forall a \in \text{Dom}(\tau)^2.
\]

Hence, with \( F_B := B(1 + B)^{-1/2} \),

\[
\pi_\alpha(b)(F_B, \pi_\alpha(a)) \in K_N, \quad \forall a, b \in \text{Dom}(\tau)^2.
\]

Proof. The first part will be proven in Lemma 6. We indicate how to extract the statement involving \( F_B \) from results in [CPRS2, 3.4] (for the unital case, see [CP, Prop. 2.4]). By [CPRS2, Prop. 3.18] we can assume that \( \text{Dom}(\tau)^2 \) is complete with respect to the \( \delta-\varphi \)-topology discussed there (or just consider the completion instead), as doing so will not change the summability condition involving \( B \). Now if \( \text{Dom}(\tau)^2 \) is complete in the \( \delta-\varphi \)-topology then it is stable under the holomorphic functional calculus by op. cit., and this property in turn implies the desired summability condition on \( F_B \), using [CPRS2, Prop. 3.12].

\[\square\]
Remark 9. Note that for $n$ even, $F_B$ decomposes in $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$ as
\[
F_B = \begin{pmatrix} 0 & F_- \\ F_+ & 0 \end{pmatrix}
\]
with $F_- = (F_+)^*$. In case $n = 1$ it is proved in [Le] that for unitary $u \in A$ (here $A$ is assumed to be unital and $\alpha$-smooth), the $\tau$-Breuer Fredholm index of $T_u \in \mathcal{T}$ is given by
\[
\text{Index}_\tau(T_u) = -\frac{1}{2\pi i} \tau(u^* \delta(u))
\]
where $\delta$ is the generator of $\alpha$. In [PR] this was extended to nonunital $A$ and semifinite $\tau$. With the construction of this section we can attempt a generalization to higher dimensions $n$ and deformed algebras $A_{\Theta}$.

Remark 10. Replacing $A$ by $M_n(A)$ for any $n$ one obtains a similar Toeplitz extension with the same formula for $u \in M_n(A)$, tensoring $\tau$ with the standard trace on $M_n(\mathbb{C})$. Thus we know from [Le] that for $n = 1$ we obtain a map from the odd $K$-theory of $A$ to the even $K$-theory of $B$. The map $\partial_1: [u] \to \text{Index}(T_u)$, where $\text{Index}(T_u)$ is the $K_0(B)$-valued index defined below, is an explicit description of Connes’ “Thom isomorphism” [Co] in odd dimensions. For even $n$, the fact that $\pi^n(a)$ is even for the decomposition $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$ for all $a \in A$ shows that $T_u$ has zero Fredholm index (when it can be defined). The Thom isomorphism is best viewed as a Kasparov product by a $KK$-element $[M_N(B), \pi_n, F_B]$ defined by the Dirac operator, both in even and odd dimension, as we discuss next.

2.5 Extensions and Thom Elements

Here we discuss why index theory naturally appears when considering crossed products. We fix $n$ odd.

Recall that we tensored $B := A \rtimes_\alpha \mathbb{R}^n$ with the full matrix algebra $M_N(\mathbb{C})$ to obtain the Toeplitz extension. The resulting algebra $M_N(B) = M_N(\mathbb{C}) \otimes B$ has the same $K$- and $KK$-theory as $B$. Let us briefly review the fact [Bla, 17.6.4] that
\[
KK^1(A, M_N(B)) \cong KK^1(A, B) \cong \text{Ext}(A, B)^{-1},
\]
where $\text{Ext}(A, B)^{-1}$ is the group if invertible elements in the semigroup $\text{Ext}(A, B)$ of extensions of $A$ by $B$. Consider the “corona algebra” of $B$,
\[
\mathcal{Q}(B) := \mathcal{M}(B)/B.
\]
An element of $KK^1(A, B)$ can be described as a pair $(\psi, P)$ where $\psi: A \to \mathcal{M}(B \otimes K)$ is a homomorphism and $P$ is a projection in $\mathcal{M}(B \otimes K)$ for which $[\psi(A), P] \subset B \otimes K$. It determines an extension $t_\psi: A \to \mathcal{Q}(B \otimes K)$ by
\[
t_\psi(a) := q(P\psi(a)P),
\]
where $q: \mathcal{M}(B \otimes K) \to \mathcal{Q}(B \otimes K)$ is the quotient map. This provides an isomorphism from $KK^1(A, B)$ to the group of invertibles in $\text{Ext}(A, B)$.

Now consider the case $\psi = \pi_\alpha := 1_N \otimes \pi_\alpha$ (where $\pi_\alpha$ again denotes the usual embedding of $A$ into the multiplier algebra of the crossed product $B$); we shall also
be interested in the case $\psi = \pi^\Theta$. We take the projection $P$ to be the spectral projection of our Dirac operator $\mathcal{D}$ (8) corresponding to $[0, +\infty)$. The extension defined by

\[ t_\alpha(a) := q(P \pi_\alpha(a) P) \]

is then the smooth Toeplitz extension from Proposition 1, where one notes that $T \subset M(M_N(B))$. In the picture of $KK^1(A, B)$ as a set of classes of Kasparov modules $(E, \varphi, F)$, the triple defined by the above extension is $(M_N(B), \pi_\alpha, 2P - \|I\|)$, or rather $(M_N(B), \pi_\alpha, 2P - \|I\|) \oplus (M_N(B), \pi_\alpha, \|I\| - 2P)$ representing a class $t_\alpha$ in $KK^1(A, B)$.

Now with $A_\Theta$ we obtain an element $t^{\Theta}_n$ of $KK^1(A_\Theta, B_\Theta)$ from $a \to q(P \pi^\Theta(a) P)$. Following [FS] we call $t_n$ and $t^{\Theta}_n$ Thom elements.

**Remark 11.** When $n$ is even, the bounded transform $F_\mathcal{D} := \mathcal{D}(1 + \mathcal{D}^2)^{-1/2}$ of the Dirac operator defines an element in $KK^1(A, B)$ instead of $KK^1(A, B)$. We can still obtain an extension by exploiting periodicity, but for calculating the Thom isomorphism one can view it as an element of $KK(A, B)$ directly; see Proposition 4.

In [KNR] a spectral flow was constructed for certain “von Neumann spectral triples” relative to a pair $(\mathcal{N}, J)$ where $J$ is an ideal in a von Neumann algebra $\mathcal{N}$. There the construction of an element in $KK^1(A, J)$ enabled the $K_0(J)$-valued spectral flow to be obtained by applying the connecting map $\partial : K_1(\mathcal{N}/J) \to K_0(J)$. For unitaries $u \in A$, the spectral flow $\text{Sf}((\mathcal{D}, u^* \mathcal{D} u))$ was shown to equal the index of the Toeplitz operator $PuP$, where $P$ is the positive eigenprojection of $\mathcal{D}$. We will elaborate on this in our special case.

Let then $\mathcal{N} := B''$ where $B := A \rtimes \alpha \mathbb{R}^n$ as above. Since $B$ is a subalgebra of $\mathcal{N}$ we can relate our element $t_\alpha \in KK^1(A, B)$ to the constructions in [KNR]. In the end we want to apply the homomorphism $\tilde{\tau} : \mathcal{K}_0(\mathcal{N}) \to \mathbb{R}$ induced by $\tau$ and hence we need Lemma 3.

Recall that the spectral flow of a path $[0, 1] \ni t \to \mathcal{D}_t$ of unbounded operators is defined to be the spectral flow of the corresponding path $t \to F_t$ of bounded transforms $F_t := \mathcal{D}_t (1 + \mathcal{D}_t^2)^{-1/2}$ [BCPRSW].

**Proposition 2.** Suppose $n$ is odd. Let $\text{Dom}(\tau)^2$ be as in Lemma 3 and let $u$ be a unitary in the minimal unitization $\text{Dom}(\tau)^2 \oplus \mathbb{C}$. Consider the path $\mathcal{D}_t := (1 - t)\mathcal{D} - u^* \mathcal{D} u$ where $\mathcal{D}$ is given in (8). Identify $u$ with $\pi^\Theta(u)$ in $M_N(\mathbb{C}) \otimes M(B)$. Finally, write $P$ for the positive spectral projection of $\mathcal{D}$. Then

\[ \text{Sf}((\mathcal{D}, u^* \mathcal{D} u)) = \tilde{\tau}_\alpha(\partial[q(PuP)]) \]

\[ = \tilde{\tau}(N(PuP + 1 - P)) - \tilde{\tau}(N(Pu^*P + 1 - P)) \]

\[ = \text{Index}_\tau(PuP), \]

where $q : \mathcal{N} \to \mathcal{N}/K_N$ is the quotient map, $\partial : K_1(\mathcal{N}/K_N) \to K_0(K_N)$ is the isomorphism and $N(T)$ denotes the projection onto the kernel of an operator $T$.

**Proof.** We know from Lemma 2 that $P \pi^\Theta(u) P$ is Fredholm relative to $M_N(B_\Theta)$. The fact that we can apply $\tilde{\tau}_\alpha$ follows from Lemma 3 and [KNR, Thm. 6.4]. The equalities follow as in the proof of [KNR, Thm 6.9] modified to the nonunital setting, i.e. we do not have $(1 + \mathcal{D}^2)^{-1/2} \in \mathcal{K}_\mathcal{N}$ but only $\pi_\alpha(u)(\|I\| + \mathcal{D}^2)^{-1/2} \in \mathcal{K}_\mathcal{N}$. The only extra result that we need is [CGPRS, Cor. 4.4].
Moreover, we shall see Section 2.7 that the above data provides a spectral triple 
\((\text{Dom}(\tau)^2, \mathcal{H}, \mathcal{D})\) in the sense of Connes’ [Co4], albeit in the generalized sense [CPRS] 
that trace and compactness conditions are defined relative to the semifinite von 
Neumann algebra \(N\). As stressed in [KNR], since the usual \(C^*\)-algebra of compact 
operators \(K\) is replaced by \(K_M\) the triple \((\text{Dom}(\tau)^2, \mathcal{H}, \mathcal{D})\) does not represent an 
element of the \(K\)-homology \(K^0(A)\) but rather an element \(KK^1(A, B)\), as we have 
also seen here. The spectral flow can be considered as a pairing \([u], [\text{Dom}(\tau)^2, \mathcal{H}, \mathcal{D}]\) 
between the \(K\)-theory class defined by \(u \in A\) and the \(KK\)-class defined by the 
spectral triple \((\text{Dom}(\tau)^2, \mathcal{H}, \mathcal{D})\). For the precise meaning of this we refer to [CGRS].

2.6 Pseudodifferential Operators and a Kasparov Product

In [Co1], a space \(\Psi^0(A)\) of zero-order pseudodifferential operators was introduced, 
fitting into the short exact sequence

\[
0 \longrightarrow A \xrightarrow{\alpha} \mathbb{R}^n \longrightarrow \Psi^0(A) \xrightarrow{\sigma} C(\mathbb{S}^{n-1}) \otimes A \longrightarrow 0,
\]

where \(\mathbb{S}^{n-1}\) is a space of half-rays in \(\mathbb{R}^n\) (see [Co1] and also [DHK]). From this exact 
sequence derives the formula

\[
\text{Index}(T) = \partial_1 \circ \beta^{-1}(\sigma_T),
\]

(11)

where \(\sigma_T\) is the element of \(C(\mathbb{S}^{n-1}) \otimes A\) corresponding to \(T \in \Psi^0(A)\) via the above 
exact sequence (\(\sigma_T\) assumed to be invertible), \(\beta : K(A) \to K(C_0(\mathbb{S}^{n-1} \setminus \{pt\}) \otimes A)\) 
is an isomorphism, and \(\partial_1\) is Connes’ Thom isomorphism [Co].

Explicitly, since the Thom homomorphism is given by Kasparov product with the 
Thom element \(t_\alpha \in KK^1(A, B)\) [FS], the index \(\text{Index}(T) \in K_0(B)\) of an operator 
\(T \in \Psi^0(A)\) is given as a Kasparov product between \(K_0(\mathbb{S}^{n-1}A^+)\) and \(KK^1(A, B)\),

\[
\text{Index}(T) = ([\sigma_T] \otimes \mathbb{C}[\beta_{n-1}]) \otimes_A t_\alpha,
\]

where \([\beta_{n-1}] \in KK(\mathbb{S}^{n-1}, \mathbb{C})\) is the Bott element (this is due to [FS]). We shall use 
the fact from Section 2.5 that the Thom element can be described using a Toeplitz 
extrusion. Let us show that if \(u \in A\) is invertible then we can find a similar formula 
for \(\text{Index}(T_u)\).

**Proposition 3.** For \(n\) odd and a unitary \(u \in A^*_\alpha\), there exists an element \([\rho_u]\) of 
\(KK(C_0(\mathbb{R}), A) \cong K_1(A)\) such that the index of the Toeplitz operator \(T_u^\Theta = P_u^{\Theta}(u)P\) 
is given by

\[
\text{Index}(T_u^\Theta) = [\rho_u] \otimes_A t_\alpha \in K_0(B_{\Theta}).
\]

**Proof.** To a unitary \(u \in A\) there corresponds a homomorphism \(\rho_u : C_0(\mathbb{R})^\sim \to \mathcal{M}(A)\) 
which takes \(z - 1\) to \(u - 1\) under the identification of \(K_1(A)\) with \(KK(C_0(\mathbb{R}), A)\).

The Kasparov product of the class of \(\rho_u\) with the Thom element \(t_\alpha = [\pi_{N}(B), \pi_{\alpha}, 2P - I]\) 
is given by

\[
[\rho_u] \otimes_A t_\alpha = [M_N(B), (\pi_{\alpha} \circ \rho_u), 2P - I],
\]

which is an element of \(KK^1(C_0(\mathbb{R}), B)\). Identifying \(KK^1(C_0(\mathbb{R}), B)\) with \(K_1(Q(B \otimes K))\) 
(see [Bla, Prop. 17.5.7]) we get

\[
[\rho_u] \otimes_A t_\alpha = [q(P\pi_{\alpha}(u)P - I + P)].
\]
where $q : M(B \otimes K) \to Q(B \otimes K)$ is the quotient map. The connecting homomorphism $\delta_1 : K_1(Q(B \otimes K)) \to K_0(B)$ is an isomorphism [Bla, Prop. 12.2.3] and the image of $[p] \otimes_A t_\alpha$ under this map is (see [Bla, §8.3.2])
\[ \delta_1([p] \otimes_A t_\alpha) = [N(P\pi_\alpha(u)P - I + P)] - [N(P\pi_\alpha(u^*)P - I + P)], \]
i.e. the $K_0(B)$-valued index of the Toeplitz operator $P\pi_\alpha(u)P$ as an operator on $PH$. Now this is in a form that we can easily use for deformations. Simply replace $t_\alpha$ by $t_\alpha^\Theta$.

**Proposition 4.** For $n$ even and a projection $e \in A_\Theta$, there exists an element $[p] \otimes_A t_\alpha^\Theta \in K_0(B)$ of $KK(C_0(\mathbb{R}), SA) \cong K_0(A)$ such that the index of the operator $\pi^\Theta(e)F_+\pi^\Theta(e) : eH_+ \to eH_-$ is given by
\[ \text{Index}(\pi^\Theta(e)F_+\pi^\Theta(e)) = [p] \otimes_{SA} t_\alpha^\Theta \in K_0(B_\Theta) \]
where $F_+ : H_+ \to H_-$ is as in Remark 9.

**Proof.** Note that the Bott map $\beta : K_0(A) \to K_1(SA)$ takes the class of a projection $e \in A^\ast$ to the homotopy class of an invertible map $\beta_e : \mathbb{R} \to A^\ast$. The map $\beta_e$ in turn determines, as in the proof Proposition 3, a homomorphism $\rho_e : C_0(\mathbb{R})^\ast \to M(SA)$, with
\[ (\rho_e(\beta_e))(z) = (\beta_e - 1)(z) = ze + 1 - e - 1 = (z - 1)e, \]
and $\rho_e$ defines an element of $KK(C_0(\mathbb{R}), SA)$. The Thom element $t_\alpha^\Theta = [M_N(B), \pi^\Theta, F]$ belongs in the even case to $KK(A, B) \cong KK(SA, SB)$. The Kasparov product
\[ [p] \otimes_{SA} t_\alpha^\Theta \in K_0(B_\Theta), \]
is an element of $KK(C_0(\mathbb{R}), SB)$. By $KK(C_0(\mathbb{R}), SB) \cong KK(\mathbb{C}, B)$ there exists a homomorphism $\rho^\Theta_e : \mathbb{C} \to M(B \otimes K)$ such that $\rho^\Theta_e(1) = (\pi^\Theta \otimes 1) \circ \rho_e$, which is seen [Lee, Thm. 3.14] to imply that $\rho^\Theta_e(1) = \pi^\Theta(e)$. Consider the identification
\[ [p] \otimes_{SA} t_\alpha^\Theta \in KK(\mathbb{C}, B). \]

Among the data specifying an element of $KK(\mathbb{C}, B)$ is a homomorphism from $\mathbb{C}$ to $M(B \otimes K)$, but such a map is determined by the projection in $M(B \otimes K)$ that $1 \in \mathbb{C}$ is mapped to. So an element of $KK(\mathbb{C}, B)$ can be described by an operator $T \in M(B \otimes K)$ and a projection $p \in M(B \otimes K)$ such that $pT^*pT - p$ and $pT^*p - p$ are in $B \otimes K$. As we have seen, in our case $p = \pi^\Theta(e)$. It is fruitful to describe this element in terms of the corona $Q(B \otimes K)$. Let $q : M(B \otimes K) \to Q(B \otimes K)$ be the quotient map again. Using
\[ (\pi^\Theta(e)F_+\pi^\Theta(e))(\pi^\Theta(e)F_+^{-1}\pi^\Theta(e)) = \pi^\Theta(e) \mod B \otimes K, \]
we note that $q(\pi_\alpha(e)F_+\pi_\alpha(e) - 1 + \pi_\alpha(e))$ is a unitary in $Q(B \otimes K)$. This unitary represents the class $[p] \otimes_{SA} t_\alpha$ under the identification of $KK(C_0(\mathbb{R}), SB) \cong KK^1(C_0(\mathbb{R}), B) \cong KK(\mathbb{C}, B)$ with $K_1(Q(B \otimes K))$. Under the isomorphism $\delta_1 : K_1(Q(B \otimes K)) \to K_0(B)$, the $K_0(B)$-class is, as in [Bla, §8.3.2], given by
\[ \delta_1([p] \otimes_{SA} t_\alpha^\Theta) = [N(\pi^\Theta(e)F_+\pi^\Theta(e) - 1 + \pi^\Theta(e))] - [N(\pi^\Theta(e)F_+^{-1}\pi^\Theta(e) - 1 + \pi^\Theta(e))], \]
as asserted.
2.7 Numerical Index for $\Theta = 0$

In this section and the next section we will prove a formula in the spirit of [Le], [PR], [CGPRS] for the $\hat{\tau}$-index of Toeplitz operators $T^\Theta u = P\pi^\Theta(u)P$. We begin here with the case $\Theta = 0$ so that, according to Remark 3, we have $\pi^\Theta = \pi_\alpha$.

A powerful approach to the case $n = 1$ given in the recent paper [CGPRS] allows us to make a very slim presentation (not reprinting all details). We summarize and point out what makes it so easy to generalize their proof to higher dimensions.

We want to apply the general version of the local index formula [CGRS] by finding a nonzero $*$-algebra $C \subset A$ for which it applies relative to $(N, \hat{\tau})$. For that we need both a suitable smoothness property of elements $a \in C$ with respect to $\mathcal{D}$, as well as a $(\hat{\tau}, \mathcal{D})$-integrability condition on $\pi_\alpha(a)$. The required notion of smoothness is the following.

**Definition 3.** Define the derivation $\delta_{\mathcal{D}}$ on $N$ by

$$\delta_{\mathcal{D}}(T) := \frac{1}{2}([I + \mathcal{D}^2]^{1/2}, T)$$

whenever $TH_\infty \subset H_\infty := \bigcap_{k \geq 0} \text{Dom}(\mathcal{D}^k)$. We say that a $*$-subalgebra $\mathcal{D} \subset A$ is quantum smooth if $\pi_\alpha(a)$ and $[\mathcal{D}, \pi_\alpha(a)]$ belong to the domain of $(\delta_{\mathcal{D}})^k$ for all $k$ and all $a \in \mathcal{D}$.

As expected, in our setting the smoothness with respect to $\mathcal{D}$ is tightly related to the smoothness with respect to the $\mathbb{R}^n$-action $\alpha$. In fact, we get as in Proposition 3.12 of [CGPRS] that if an element $a \in A$ is smooth for the generator $\delta$ of $\alpha$ then $\pi_\alpha(a)$ is smooth for $\delta_{\mathcal{D}}$. Hence the $\alpha$-smooth subalgebra $A \subset A$ can be used to define a quantum smooth spectral triple $(A, H, \mathcal{D})$ as in Definition 3. We will not be able to use all of $A$ since we also need an integrability condition, as we discuss next.

Remember that $A$ is acting on a Hilbert space $\mathcal{H}$ and that

$$\pi_\alpha : A \to L^2(\mathbb{R}^n, \mathcal{H})$$

is defined in terms of $\mathcal{H}$. Since the dual trace $\hat{\tau}$ on $N$ is defined in terms of the Hilbert algebra (see Remark 7)

$$\mathfrak{A}_\tau := L^2(\mathbb{R}^n, H_\tau) \cap L^1(\mathbb{R}^n, \text{Dom}(\tau)),$$

it is natural to want to have $\pi_\alpha$ as a representation on $L^2(\mathbb{R}^n, H_\tau)$, and this was the approach in [CGPRS]. The action $\alpha$ is then required to preserve the trace, or else it will not have a unitary implementation. However, the dual trace can also be described (see Definition 2) as the composition of $\tau, \pi_\alpha^{-1}$ and the operator-valued weight $I_\mathcal{H}$ and this can be used as follows. (Again, we do assume that $\alpha$ preserves $\tau$ in this paper but we aim for some flexibility in the choice of $\mathcal{H}$ that could be useful in the future.)

**Lemma 4.** Let as before $N = \tilde{\pi}_\alpha(B)''$ be the von Neumann algebra generated by the crossed product $B$, where $\pi_\alpha$ is defined for the Hilbert space $\mathcal{H}$ on which $A$ is represented. Then the domain and $L^p$-spaces of the semifinite trace $\hat{\tau}$ on $N$ dual to $\tau$ are related to $\tau$ in exactly the same way as if we define $\hat{\tau}$ on the weak closure of $B$ in $L^2(\mathbb{R}^n, H_\tau)$, where $H_\tau$ is the GNS space of $\tau$. 

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Proof. For any choice of $\pi_\alpha: A \to L^2(\mathbb{R}^n, \mathcal{H})$ (i.e. for any choice of $\mathcal{H}$) we have $N = \pi_\alpha(\mathfrak{A}_\tau)^{''}$ for the von Neumann algebra $N = \pi_\alpha(\mathfrak{A}_\tau)^{''} \subset B(L^2(\mathbb{R}^n, \mathcal{H}))$, where $\mathfrak{A}_\tau$ is the left Hilbert algebra which completely defines $\hat{\tau}$. So it does not matter which such $N$ we regard $\hat{\tau}$ as a trace on, since $\pi_\alpha$ is always a faithful representation.

Again we use $\pi_\alpha(a)$ to denote $I_2 \otimes \pi_\alpha(a)$ for $a \in A$ and we write $\hat{\tau}$ for $\text{Tr} \otimes \hat{\tau}$ where $\text{Tr}$ is the matrix trace on $M_{2N}(\mathbb{C})$.

**Proposition 5.** For $a$ in the intersection of the domains $\text{Dom}(\delta_k)$ of the generators $\delta_k$ of $\alpha$, we have

$$[\mathcal{D}, \pi_\alpha(a)] = \frac{1}{2\pi i} \sum_{k=1}^{n} \gamma_k \pi_\alpha(\delta_k(a)).$$

**Proof.** This is seen as in [CGPRS, Prop. 3.3] using

$$[\mathcal{D}, \pi_\alpha(a)] = \sum_{k=1}^{n} \gamma_k [D_k, \pi_\alpha(a)].$$

Namely, if $\xi$ is in the domain of $D_k$ and $a$ is in the domain of $\delta_k$ then

$$(D_k \pi_\alpha(a) \xi)(t) = \frac{1}{2\pi i} \frac{\partial}{\partial t_k} (\alpha_t(a) \xi(t)) = \frac{1}{2\pi i} \alpha_{-t}(\delta_k(a)) \xi(t) + \frac{1}{2\pi i} \alpha_{-t}(a) \frac{\partial}{\partial t_k} \xi(t),$$

while $(\pi_\alpha(a) D \xi) = (2\pi i)^{-1} \partial \xi(t)/\partial t_k$. Thus

$$[D_k, \pi_\alpha(a)] = \frac{1}{2\pi i} \alpha_{-t}(\delta_k(a)) \xi(t),$$

and $\pi_\alpha(a) \xi$ is in the domain of $D$.

**Remark 12.** The appearance of the $\gamma_k$'s in $[\mathcal{D}, \pi_\alpha(a)]$ shows that only a product of $n$ such commutators has graded nonzero trace [BGV, Prop. 3.21].

Let $\mathcal{M} := A''$ be the weak closure of $A$ in its original representation. Then $\tau$ extends to a normal trace $\hat{\tau}$ on $\mathcal{M}$ with the same GNS space as $\tau$. The following lemma is the counterpart of [CGPRS, Lemma 3.4].

**Lemma 5.** Let $h \in L^\infty(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ and let $a \in \mathcal{M}$ such that $a^* a$ is in $\text{Dom}(\hat{\tau})$. If we define

$$x(t) := ab(t),$$

then $\tilde{\pi}_\alpha(x) \in \mathcal{N}$ is $\hat{\tau}$-Hilbert-Schmidt and

$$\hat{\tau}(\tilde{\pi}_\alpha(x)^* \tilde{\pi}_\alpha(x)) = \tau(a^* a) \int_{\mathbb{R}^n} |h(t)|^2 \, dt.$$  

**Proof.** We write $\tilde{\pi}_\alpha(x) = \int \pi_\alpha(a) h(s) e^{-is \cdot D} \, ds$ so that

$$\dot{\pi}_p(\tilde{\pi}_\alpha(x)) = \int_{\mathbb{R}^n} \pi_\alpha(a) h(s) e^{-ip \cdot s} e^{-is \cdot D} \, ds, \quad \forall p \in \mathbb{R}^n.$$
Since \( \hat{\tau} = \check{\tau} \circ \pi^{-1}_\alpha \circ I_\alpha \), the assumptions on \( x \) give
\[
\hat{\tau}(\check{\pi}_\alpha(x^* \check{\pi}_\alpha(x))) = \check{\tau} \circ \pi^{-1}_\alpha \left( \int_{\mathbb{R}^n} \check{\alpha}_p(\check{\pi}_\alpha(x^* x)) \, dp \right)
\]
\[
= \check{\tau} \circ \pi^{-1}_\alpha \left( \int_{\mathbb{R}^n \times \mathbb{R}^n} \pi_\alpha(a^* a) |h(s)|^2 e^{-isp} e^{-isD} \, ds \, dp \right)
\]
\[
= \check{\tau} \circ \pi^{-1}_\alpha \circ \pi_\alpha(a^* a) |h(0)|^2
\]
\[
= \check{\tau}(a^* a) \int_{\mathbb{R}^n} |h(t)|^2 \, dt
\]
where we used Plancherel’s formula in the last line.

\[\square\]

**Corollary 2.** Let \( s > n \) and define
\[
\varphi_s(T) := \hat{\tau}((1 + \mathcal{B}^2)^{-s/4} T (1 + \mathcal{B}^2)^{-s/4})
\]
for all \( T \in \mathcal{N}_+ \). Then the restriction of \( \varphi_s \) to \( \pi_\alpha(A)' \), viewed as a subalgebra of \( \mathcal{N} \), is proportional to \( \check{\tau} \circ \pi^{-1}_\alpha \).

**Proof.** From the Clifford relations we get
\[
\mathcal{B}^2 = \sum_{k=1}^n I \otimes D_k^2,
\]
and so if \( h_s(t) := (1 + |t|^2)^{-s/2} \) then by Lemma 5 we have for each positive \( a \) in the domain of \( \tau \) that
\[
\varphi_s(\pi_\alpha(a)) = \hat{\tau}((1 + \mathcal{B}^2)^{-1/2} \pi_\alpha(a))
\]
\[
= \check{\tau}(h_s(D) \pi_\alpha(a))
\]
\[
= 2^n \tau(a) \int_{\mathbb{R}^n} |h_s(t)|^2 \, dt.
\]
It follows that more generally that \( \varphi_s = \|h_s\|_2^2 \hat{\tau} \) holds on \( \text{Dom}(\hat{\tau}) \subset \mathcal{M}_+ \). That \( \varphi_s(a) = +\infty \) whenever \( \check{\tau}(a) = +\infty \) can be seen as in Corollary 3.5 of [CGPRS].

\[\square\]

**Remark 13.** The fact that \( \mathcal{B}^2 \) is just a sum of squares \( I \otimes D_k^2 \) makes the results for \( n = 1 \) carry over easily to general \( n \).

Let \( \delta_\tau \) denote the restriction of \( \delta \) to \( \text{Dom}(\tau) \).

**Lemma 6.** If \( a = bc \in \text{Dom}(\delta_\tau) \) is a product of two elements \( b, c \) in \( \text{Dom}(\delta_\tau) \) (or more generally a finite sum of such products) then
\[
\check{\pi}_\alpha(a)(1 + \mathcal{B}^2)^{-s/2} \in L^1(\mathcal{N}, \hat{\tau}), \quad \forall s > n.
\]

**Proof.** The proof of Lemma 3.6 in [CGPRS] carries over using Prop. 5.

\[\square\]
This also proves Lemma 3, as we promised.

Denoting by $\text{Dom}(\delta_\tau)^2$ the $*$-algebra of elements $a$ as in Lemma 6, the preceding statements show that $(\text{Dom}(\delta_\tau)^2, \mathcal{H}, \mathcal{D})$ is a semifinite spectral triple relative to $(\mathcal{N}, \mathcal{F})$. For the local index formula we also need some smoothness for the action $\alpha$, and thus we cannot use all of $\text{Dom}(\delta_\tau)^2$ in general. By the remarks in the beginning of the section, the best we can have is to use the triple

$$(\mathcal{C}, \mathcal{H}, \mathcal{D}),$$

where $\mathcal{C} := \mathcal{A} \cap \text{Dom}(\tau)^2$.

**Proposition 6.** For $n$ odd and a unitary $u = 1 + a \in \mathcal{C}^\sim$ with $a \in \mathcal{C}$, we have

$$\text{Index}_\tau(T_u) = -\frac{2^n}{(2\pi i)^n n!} \tau \left( (u^* \delta(u))^n \right).$$

**Proof.** Lemma 6 shows that $(\mathcal{A}, \mathcal{H}, \mathcal{D})$, hence $(\mathcal{C}, \mathcal{H}, \mathcal{D})$, has spectral dimension $n$. Due to the appearance of the Clifford matrices in commutators with $\mathcal{D}$, the only nonvanishing residue cocycle $\phi_k : \mathcal{A} \otimes \mathcal{A}^\otimes k \to \mathbb{C}$ for $(\mathcal{C}, \mathcal{H}, \mathcal{D})$ (see [CGRS, §4.1.1]) is

$$\phi_n(a_0, a_1, \ldots a_n) := \frac{1}{n!} \text{Res}_{s=n} \tau(\pi_a(a_0)[\mathcal{D}, \pi_a(a_1)] \cdots [\mathcal{D}, \pi_a(a_n)](1 + |t|^2)^{-s/2}).$$

It follows that $(\mathcal{C}, \mathcal{H}, \mathcal{D})$ has isolated spectral dimension $n$. Hence, for a unitary $u = 1 + a \in \mathcal{C}^\sim$ with $a \in \mathcal{C}$, the local index formula [CGRS, Thm.4.33] gives

$$\text{Index}_\tau(T_u) = -\phi_n(u^*, u, \ldots, u^*, u),$$

while Lemma 5 shows that

$$\phi_n(u^*, u, \ldots, u^*, u) = \frac{2^n}{(2\pi i)^n n!} \tau \left( (u^* \delta(u))^n \right) \text{Res}_{s=n} \int_{\mathbb{R}^n} (1 + |t|^2)^{-s/2} dt$$

(the factors of $1/2\pi i$ come from Proposition 5). Now since $\delta$ is a derivation, $uu^{-1} = 1$ gives

$$\delta(u^{-1}) = -u^{-1} \delta(u) u^{-1},$$

and so

$$u^*(\delta(u)\delta(u^*))^{(n-1)/2} \delta(u) = u^* \left( \delta(u^*) u^* \delta(u) u^* \right)^{(n-1)/2} \delta(u) = (u^* \delta(u))^n,$$

from which

$$\phi_n(u^*, u, \ldots, u^*, u) = \frac{2^n}{(2\pi i)^n n!} \tau \left( (u^* \delta(u))^n \right) \text{Res}_{s=n} \int_{\mathbb{R}^n} (1 + |t|^2)^{-s/2} dt.$$

**Proposition 7.** For $n$ even and a projection $e \in \mathcal{C}$, we have

$$\text{Index}_\tau(\pi_a(e)F+ \pi_a(e)) = \frac{2^n}{(2\pi i)^n n!} \tau \left( (e\delta(e) \delta(e))^n \right).$$
Proof. From [CPRS2, Thm.7.1, Thm.8.1] we have

$$\text{Index}_\tau(eF, e) = \phi_n(e, \ldots, e),$$

and the expression from Lemma 5,

$$\phi_n(e, \ldots, e) = \frac{2^n}{(2\pi i)^n} \tau(e^\delta(e) \cdots e^\delta(e)) \text{Res}_{s=n} \int_{\mathbb{R}^n} (1 + |t|^2)^{-s/2} dt,$$

can rearranged using $e^\delta(e)^{n-1} = (e^\delta(e))^n$, which follows from the idempotency of $e$.

2.8 Numerical Index for $\Theta \neq 0$

For nonzero $\Theta$ we have additional problems because we need to view not only $\pi_\alpha(a_\Theta)$ as a multiplier of $B_\Theta$ but also $\sigma^\Theta$ defines only a multiplier of the Hilbert $A$-module $S^A(\mathbb{R}^n)$. There is a general construction to extend traces to multipliers of a given Hilbert module (see [LN, §1]). Applying this to the present case means that we have to replace $\tau(a_\Theta)$ for $a \in A$ by

$$\tilde{\tau}(a_\Theta) := \sup_{\mathcal{I}} \sum_{\phi \in \mathcal{I}} \tau(\langle \phi | a_\Theta \phi \rangle_A),$$

where the supremum is taken over all finite subsets $\mathcal{I}$ of $S^A(\mathbb{R}^n)$ for which it holds $\sum_{\phi \in \mathcal{I}} \phi \phi^* \leq 1$, where $\phi \phi^*$ is regarded as a compact operator on $S^A(\mathbb{R}^n)$. We denote by $\tilde{\tau}$ this extension of $\tau$ to the $C^*$-algebra $\mathcal{L}_A(S^A(\mathbb{R}^n))$ of adjointable operators on $S^A(\mathbb{R}^n)$. Note that this also extends the trace $\bar{\tau} : A_+^{\prime} \to [0, +\infty]$.

**Lemma 7.** The $\alpha$-smooth subalgebra has an approximate identity $\{e_k\}_k$ consisting of positive elements of $A$. Moreover, $\{e_k\}_k$ is a bounded approximate identity also for the deformed product $\times_\Theta$ on $A$ for any $\Theta$.

Proof. This is [Rie, Props. 2.17, 2.18].

**Lemma 8.** Let $\tilde{\tau}$ be the extension of $\tau$ to the multiplier algebra of $S^A(\mathbb{R}^n)$ as above. Then for all $a, b \in A$ we have

$$\tilde{\tau}(a_\Theta b_\Theta) = \tau(a \times_\Theta b).$$

Proof. Let $\{e_k\}_k$ be a bounded approximate identity for $A$ and let $\{f_k\}_k$ be an approximate identity for $S(\mathbb{R}^n)$, where the latter implies that $f_k \geq 0$ and $\int f_k(0) = 1$ for all $k$. Then we get an approximate identity $\{\phi_k\}_k$ for $S^A(\mathbb{R}^n)$ from

$$\phi_k(t) := f_k(t) e_k, \quad \forall t \in \mathbb{R}^n.$$

For all $a \in A$,

$$\lim_k \langle \phi_k | a_\Theta \phi_k \rangle = \lim_k \int_{\mathbb{R}^n} |f_k(t)|^2 e_k a \times_\Theta e_k dt$$

$$= \lim_k \int_{\mathbb{R}^n} |f_k(t)|^2 e_k \alpha_{\Theta} \alpha(a) \alpha_\Theta(e_k) e^{2\pi i s \cdot v} ds dv dt$$

$$= \lim_k \int f_k(0)^2 e_k (a \times_\Theta e_k)$$

$$= \langle a | \epsilon \rangle$$
where we used Lemma 7 in the last line. Hence we get a back when we apply \( \hat{\tau} \) to \( \alpha^\omega \), even if \( \tau \) is not invariant under the \( \mathbb{R}^n \)-action. On the other hand, for \( a, b \in \mathcal{A} \) the same calculation shows that

\[
\lim_{k} \langle \phi_k \mid a^\Theta b^\Theta \phi_k \rangle = \lim_{k} \langle \phi_k \mid (a \times_\Theta b)^\Theta \phi_k \rangle = a \times_\Theta b.
\]

\[\Box\]

Remark 14. Note that \( \tau(a \times_\Theta b) \) is not equal to \( \tau(ab) \) even if \( \alpha \) preserves the trace. This fact should not be confused with the identity [Rie, Prop. 3.6]

\[
\int_{\mathbb{R}^n} (f \times_\Theta g)(t) \, dt = \int_{\mathbb{R}^n} f(t)g(t) \, dt,
\]

which holds for all \( f, g \in \mathcal{S}^4(\mathbb{R}^n) \).

Lemma 9. Let \( x(t) := ah(t) \) with \( h \in L^\infty(\mathbb{R}^n) \cap L^2(\mathbb{R}^n) \) and \( \alpha \)-smooth \( a \in \mathcal{A}^\alpha \) with \( a^* a \in \text{Dom}(\hat{\tau}) \). Then \( \hat{\tau}^\Theta(x) \in \mathcal{N}_\Theta \) is \( \hat{\tau} \)-Hilbert-Schmidt and

\[
\hat{\tau}^\Theta(x) = \tau(a^* \times_\Theta a) \int_{\mathbb{R}^n} |h(t)|^2 \, dt.
\]

Proof. If \( h \) is in \( \mathcal{S}(\mathbb{R}^n) \) then \( x \) is in \( \mathcal{S}^4(\mathbb{R}^n) \) and we can define \( x^\Theta \) explicitly. This is again an element of \( \mathcal{S}^4(\mathbb{R}^n) \) and hence \( \pi^\Theta(x^* \pi^\Theta(x) \) is \( \hat{\tau} \)-traceable. For general \( h \in L^\infty(\mathbb{R}^n) \cap L^2(\mathbb{R}^n) \) we have \( x \in L^2(\mathbb{R}^n, \mathcal{A}) \) and we define \( x^\Theta \) by approximation with Schwartz functions. In this way \( x^\Theta \) is again in \( L^2(\mathbb{R}^n, \mathcal{A}) \) and hence \( \tilde{\pi}^\alpha(x^\Theta) = \pi^\Theta(x) \) is Hilbert-Schmidt for \( \hat{\tau} \). Proceeding as in Lemma 5 one obtains the formula. \( \Box \)

In the same way one can show that the other results of the preceding section generalizes as we replace \( \tau \) by its extension \( \hat{\tau} \) to \( \mathcal{L}_A(\mathcal{S}^4(\mathbb{R}^n)) \). This gives the local formulae in Theorem 1.

2.9 Chern Character for Dynamical Systems

Recall that \( \mathcal{A} \) is the dense \( * \)-subalgebra of \( \mathcal{A} \) smooth under the action \( \alpha \). Let \( \Xi^\infty \) be a finite projective module over \( \mathcal{A} \). Then there is an idempotent \( e \in \mathcal{M}_m(\mathcal{A}) \) for which \( \Xi^\infty = e\mathcal{A}^m \); we shall assume \( m = 1 \). We write \( \Omega \) for the \( \mathbb{Z} \)-graded differential algebra with \( \Omega^k = \bigwedge^k \mathbb{R}^n \) for all \( k \geq 0 \). Let \( \delta \) denote the infinitesimal generator of \( \alpha \) and define a differential \( d : \mathcal{A} \rightarrow \Omega^1 \otimes \mathcal{A} \) by [Co1],[Co2]

\[
(da)(x) = \delta_x(a) := \lim_{t \to 0} \frac{\alpha_{at}(a) - a}{t}, \quad \forall x \in \mathbb{R}^n, a \in \mathcal{A},
\]

with \([0,1] \ni t \rightarrow \gamma^x_t \) a path in \( \mathbb{R}^n \) such that \( x \) is tangent to \( \gamma^x_0 \). Thus, if \( x_1, \ldots, x_n \) is a basis for \( \mathbb{R}^n \), one has

\[
da = \sum_{k=1}^n dx^k \otimes \delta_k(a) \in \Omega^1 \otimes \mathcal{A},
\]

where \( \delta_k \) is the infinitesimal generator of \( \alpha \) corresponding to the \( k \)th coordinate. (The Hilbert space \( \mathcal{H} \) considered in the previous sections can also be identified with
Ω ⊗ L^2(ℝ^n, δ) and Ω ≃ C_n acts by left Clifford multiplication on Ω, representing
\(dx^k\) as \(\sqrt{-1}\gamma^k\).

The Grassmannian connection on the module \(eA\) is given by

\[\nabla^e := e de.\]

In particular, for all \(ξ \in eA\) and \(x \in \mathbb{R}^n\),

\[\nabla^e_x(ξ) = e δ_x(ξ).\]

The curvature of \(\nabla^e\) is then

\[R^e = e(de ∧ de).\]

In [Co1] was defined the Chern character of the projection \(e \in A\) as

\[\text{Ch}_τ(e) := \sum_{k=0}^{n} \frac{1}{k!(2\pi i)^k} τ^k(R_e, \ldots, R_e) ∈ Z^*(\mathbb{R}^n; \mathbb{R}),\]

where \(τ^k : \Omega^k ⊗ A → Ω^k\) is the cocycle defined by

\[τ^k(ω_1 ⊗ a_1, \ldots, ω_k ⊗ a_k) := τ(a_1 \cdots a_k)ω_1 ∧ \cdots ∧ ω_k.\]

Here \(τ : A → \mathbb{C}\) is assumed to be a finite trace which is invariant under \(α\) [Co1]. It is possible to define other connections \(\nabla\) on the module \(Ξ^∞\) and their corresponding curvatures \(R\). The invariance of \(τ\) ensures that \(τ^k(R, \ldots, R)\) is independent of the choice of connection. The fact that \(\text{Ch}_τ(e)\) depends only on the \(K\)-theory class of \(Ξ^∞\) does not require that \(τ\) is \(α\)-invariant. This is because the class of the projection \(e\) determines the class of \(Ξ^∞ = eA\).

We denote by \([\text{Ch}_τ(e)]\) the cohomology class of \(\text{Ch}_τ(e)\) in \(H^*(\mathbb{R}^n; \mathbb{R})\). Since \(e\) defines an element of \(K_0(A)\) and \([e] → [\text{Ch}_τ(e)]\) is a homomorphism, we get a map

\[\text{Ch}_τ : K_0(A) → H^*(\mathbb{R}^n; \mathbb{R})\]

playing the role of the usual Chern character. The map \(\text{Ch}_τ\) is defined also on \(K_1\): for a unitary \(u \in A\),

\[\text{Ch}_τ(u) := \sum_{k=0}^{n} \frac{(-1)^k}{k!(2\pi i)^k} τ^k(u^{-1} du, \ldots, u^{-1} du).\]

**Remark 15.** It is possible to define a connection \(\nabla^u\) also from a unitary \(u \in A\), setting

\[\nabla^u := d + u^* du.\]

Considering the path of connections \(\nabla^u_s := d + u^*_s du\) for \(s ∈ [0, 1]\) one has \(\nabla^0_s = d\) and \(\nabla^1_s = \nabla^u\). One gets a result analogous to the classical [TWZ, §2] that \(\text{Ch}_τ(u)\) is obtained as the Chern-Simons form of path \(s → \nabla^u_s\). Indeed, having in mind integral formulae for spectral flow [CP], [CP1], this is the statement of Theorem 1 (c.f. [Ge]).

Applying the dual trace \(\hat{τ}\) to the formula (11), Connes notices that this gives

\[\hat{τ}(\text{Index}(T)) = \langle ([\text{Ch}_τ ◦ β^{-1})(σT)], [\mathbb{R}^n] \rangle,\] (12)
where the right-hand side is the pairing between the cohomology class of \( (\text{Ch}_r \circ \beta^{-1})(\sigma_T) \) and the homology class \([\mathbb{R}^n]\) represented by the top-degree form in \( \Omega^n \).

Let us recall the brief argument for (12) given in [Co2]. First of all, Connes proved in [Co] that for \( n = 1 \),

\[
\hat{\tau}(\partial_1[u]) = -\frac{1}{2\pi i} \tau(u^* \delta(u))
\]

(13)

for unitaries \( u \in \mathcal{A} \), where \( \partial_1 : K_1(A) \rightarrow K_0(B) \) is, as before, the Thom isomorphism. Now this is precisely \( \text{Ch}_r(u) = \langle (\text{Ch}_r(u)), [\mathbb{R}] \rangle \). The result for general \( n \) follows from suspension and naturality of the Thom isomorphism together with the properties of the Chern character. We could take a similar approach for \( A_\theta \). However, we want to consider Toeplitz operators \( T_n \), and these have symbol \( a \) in \( A \) itself. Using the Toeplitz extension we can do without the map \( \beta \) and things simplify further.

We have proven the formula (13) for traces \( \tau \) not necessarily finite. Moreover, using the map \( u \rightarrow u^\Theta \) the situation is, in symbols, that

\[
[\rho_u] \otimes A_\theta t_\alpha^\Theta = [\rho_u^\Theta] \otimes_A t_\alpha.
\]

As we have seen, it is possible to apply \( \hat{\tau} \) on \( \text{Index}(T_u^\Theta) \) since \( T_u^\Theta \) is \( \hat{\tau} \)-Fredholm. We see that this gives the same as evaluating \([\text{Ch}_r(u^\Theta)]\) against \([\mathbb{R}^n]\) (here we use the trace \( \hat{\tau} \) introduced in Section 2.8).

**Proposition 8.** For \( n \) odd and invertible \( u \in \mathcal{C}^\infty \),

\[
\hat{\tau}(\text{Index}(T_u^\Theta)) = \langle [\text{Ch}_r(u^\Theta)], [\mathbb{R}^n] \rangle,
\]

and for \( n \) even and projection \( e \in \mathcal{C} \),

\[
\hat{\tau}(\text{Index}(eF_u e)) = \langle [\text{Ch}_r(e^\Theta)], [\mathbb{R}^n] \rangle.
\]

The proof of Theorem 1 is complete.

Now consider \( C(\mathbb{S}^{n-1}) \otimes A \), where \( \mathbb{S}^{n-1} \) denotes the unit sphere in \( \mathbb{R}^n \), and define \( \mathcal{T}^{(n)} \) to be the Toeplitz algebra generated by \( B \) and the operators \( T_f := P\pi_\alpha(f)P \) for \( f \in C(\mathbb{S}^{n-1}, A) \). From the same reasoning as in Proposition 1 we get the exact sequence

\[ 0 \rightarrow M_N(B) \rightarrow \mathcal{T}^{(n)} \rightarrow C(\mathbb{S}^{n-1}) \otimes A \rightarrow 0. \]

**Theorem 3.** For \( n \) odd and invertible \( f \in C^{\infty}(\mathbb{S}^{n-1}, \mathcal{C}^\infty) \),

\[
\hat{\tau}(\text{Index}(T_f^\Theta)) = \langle [\text{Ch}_r \circ \beta^{-1}(f^\Theta)], [\mathbb{R}^n] \rangle.
\]

**Proof.** View \( f \) as a function on \( \mathbb{R}^n \) vanishing on \( \mathbb{R}^n \setminus \mathbb{S}^{n-1} \). If we write \( f \) as \( a \otimes u \) for some invertibles \( a \in C(\mathbb{S}^{n-1}) \) and \( u \in \mathcal{C}^\infty \) then it is seen that no integration or smoothness criteria are different from those that apply for \( u \) alone. Observing that

\[
[\partial_\alpha, \pi_\alpha(a \otimes u)] = \frac{1}{2\pi i} \sum_{k=1}^n \int_{\mathbb{R}^n} \gamma^k \left( \frac{\partial g}{\partial t_k}(t) \pi_\alpha(u) + g(t) \pi_\alpha(\delta_k(u)) \right) e^{-2\pi it \cdot D} dt
\]

\[
= \frac{1}{2\pi i} \sum_{k=1}^n \gamma^k \pi_\alpha(\left( \partial g/\partial t_k \otimes u + g \otimes \delta_k(u) \right),
\]

\[ 25 \]
we get
\[ \hat{\tau}(\text{Index}(T_f)) = -\frac{2^n}{(2\pi i)^n} \int_{S^{n-1}} \tau \left( (f^{-1}(t) \, df(t))^{n} \right) dt, \]
where we used the shorthand notation
\[ df = \sum_{k=1}^{n} \left( \frac{\partial f}{\partial t_k} + \delta_k \circ f \right). \]

The case of general \( \Theta \) follows from our standard arguments. \( \square \)

3 Outlook to Application in Physics

Example 1 (Callias’ index theorem). In Theorem we considered a function \( f : S^{n-1} \to A \) on the unit sphere in \( \mathbb{R}^n \) with values in a \( C^* \)-algebra \( A \). Such an \( f \in C^\infty(S^{n-1}, A) \) could be induced from bounded multipliers \( F \in C_b(\mathbb{R}^n, A) \subset \mathcal{M}(\mathbb{R}^n) \) which are nonvanishing and homogeneous of degree 0 for \( |t| \to \infty \). Taking \( A = \mathbb{M}_m(\mathbb{C}) \) (the full matrix algebra for some integer \( m \geq n \)) with the trivial action \( \alpha = \iota \) (so \( \delta \equiv 0 \)), we have \( \mathbb{M}_m(\mathbb{C}) = C_0(\mathbb{R}^n) \otimes \mathbb{M}_m(\mathbb{C}). \) The Dirac operator \( D \) defined in (8) describes a quantum system an internal degree of freedom and elements of \( A \) correspond to additional interactions, mass terms or chemical potentials. Mathematically, Theorem 1 recovers, in this simple setting, Callias’ index formula [Ca]. We obtain it as a spectral flow and a Kasparov product with the element defined by \( D \) (another approach to such results in the classical setting is known [Bu], for general manifolds). Since we can also replace \( M_m(\mathbb{C}) \) by any (separable) \( C^* \)-algebra \( A \), we can also have interactions with quantized fields \( F : \mathbb{R}^n \to A. \)

Example 2 (Quantum Hall effect). Consider the commutative \( C^* \)-algebra \( A = C_0(\Omega) \) where (\( \Omega, \mu \)) is a probability measure space. Let \( \tau(f) := \int_{\Omega} f(t) \, d\mu(t) \) be the trace given by integration on (\( \Omega, \mu \)) (note that \( \tau \) is here finite on all of \( A \) instead of just \( A \) compared to the general case). Let \( \alpha \) be an action of \( \mathbb{R}^n \) on \( A = C_0(\Omega) \). Suppose \( X_1, X_2, \ldots, X_n \) are generators of a unitary group implementing \( \alpha \) in \( L^2(\mathbb{R}^n \times \Omega, \mu) \). On the crossed product \( L^\infty(\Omega, \mu) \rtimes_{\alpha} \mathbb{R}^n \) there is a weight dual to \( \tau \). We let an element \( x \in L^\infty(\Omega, \mu) \rtimes \mathbb{R}^n \) be written as
\[ x \sim \int_{\mathbb{R}^n} x(p) e^{ip \cdot X}, \]
where each \( x(p) : \Omega \to \mathbb{C} \) belongs to \( L^\infty(\Omega, \mu) \). Then the dual weight is given by (see e.g. [Kos])
\[ \hat{\tau}(x) = \tau(x(0)) = \int_{\Omega} (x(0))(\omega) \, d\mu(\omega). \]

It is also possible to consider a crossed product \( \mathcal{N} := L^\infty(\Omega, \mu) \rtimes_{\alpha, B} \mathbb{R}^n \) twisted by a 2-cocycle \( (u,v) \to e^{u \cdot Bv} \) on \( \mathbb{R}^n \); the dual trace construction works in this case as well [Su]. The \( C^* \)-algebra of interest is then a subalgebra \( B = C_0(\Omega; B) \subset \mathcal{N} \) with the same product.

Let \( X \) denote the Dirac operator formed as in (8) from the generators \( X_1, \ldots, X_n \) of the unitary group implementing \( \alpha \) on \( \mathcal{N} \). Take \( n \) odd, for example. Then for a
unitary \( u \in \mathcal{A} \) we get a formula for the spectral flow \( \text{Sf}(\mathcal{X}, u^* \mathcal{X} u) \) from Theorem 1 in case \( \Theta = 0 \)

\[
\text{Sf}(\mathcal{X}, u^* \mathcal{X} u) = -\frac{2^n}{(2\pi i)^{n+1}} \sum_{\epsilon} (-1)^{\epsilon} \tau \left( \prod_{k=1}^{n} u^{-1} \sqrt{-1}[X_{\epsilon(k)}, u] \right)
\]

\[
= -\frac{2^n}{(2\pi i)^{n+1}} \sum_{\epsilon} (-1)^{\epsilon} \prod_{k=1}^{n} \int_{\Omega} u^{-1}(\omega) \sqrt{-1}[X_{\epsilon(k)}, u](\omega) d\mu(\omega),
\]

the sum running over all permutations \( \epsilon \) of \( \{1, \ldots, n\} \) with sign \( (-1)^{\epsilon} \).

Deforming with a matrix \( \Theta \) to incorporate the effect of some external interaction this becomes

\[
\text{Sf}^\Theta(\mathcal{X}, u^* \mathcal{X} u) = -\frac{2^n}{(2\pi i)^{n+1}} \sum_{\epsilon} (-1)^{\epsilon} \prod_{k=1}^{n} \int_{\Omega} u^{*}(\omega) \times \Theta \sqrt{-1}[X_{\epsilon(k)}, u](\omega) d\mu(\omega).
\]

The reader may recognize that what we are discussing here is the setting of the extremely elegant and successful formulation of the integral quantum Hall effect using noncommutative geometry, due to Bellissard et al. [BES]. It has been realized [KeR] that this is related to the more recent magnetic pseudodifferential calculus of [LM]. Twisted crossed product are very similar to Rieffel deformation but still different [BeM]. Using the results of this paper we can reproduce the quantum Hall algebra and the operators whose Fredholm indices give the quantized conductance, modulo the distinction between crossed products and Rieffel deformation.

Note however the fact that Rieffel’s procedure can be applied to a noncommutative algebra. Hence we can start with the above mentioned algebra \( C_0(\Omega; \tilde{B}) \), describing a background magnetic field, and use a deformation to realize more complicated interactions.

In [BES] the \( X_k \)'s play the role of position operators, generators of momentum translations, and a spectral triple is defined using \( \mathcal{X} \). Most prominently, the index of the bounded transform of \( \mathcal{X} \) (compressed with the Fermi projection) has been used to calculate the Hall conductivity when \( n = 2 \). In [PLB], this was generalized to the construction of a spectral triple from \( \mathcal{X} \) to any even \( n \geq 1 \), and for even \( n \) only. The index of Toeplitz operators \( PuP \) was used in [ASS] for a mathematical formulation of physical processes, including the integer Hall effect. Theorem 1 opens up a possibility to do so for more complicated interactions. We also see that the \( X_k \)'s appear also here, and even though the approaches [BES] and [ASS] seem very different at first, the distinction mainly comes from ”even versus odd” (although it may be argued that for \( \Theta = 0 \) there is no need to summon the \( X_k \)'s in the odd case).

For even dimensions the odd pairing can be used by considering \( \mathcal{X} \otimes A \) (see Section 2.9), which is also what is done in [ASS] for \( n = 2 \) (see also [Go]).

During the final preparations of this paper we became aware of a very recent paper [PS] showing the relevance of the Bellissard approach also to odd case. We can recover an analogue of their result using \( A = C(\Omega, \mu) \). We say ”analogue” again since the Rieffel deformation is slightly different from the twisted crossed product.

**Example 3 (\( \kappa \)-Minkowski space).** The Lebesgue integral \( \tau \) defines a trace on \( S(\mathbb{R}^2) \). A certain star-product \( \ast_{\kappa} \) put on a subalgebra \( \mathcal{A} \) of \( S(\mathbb{R}^2) \) leads to the noncommutative space called “\( \kappa \)-Minkowski space”. This can be described as a Rieffel deformation of \( \mathcal{A} \) [MS, §6] using an action which does not leave \( \tau \) invariant. It is a beautiful
The fact that $\tau$ is not a trace on $A_\Theta = (A, \ast)$ but rather a KMS weight with respect to a group of automorphisms of $A$ (we recommend [Ma] for details).

One reason why we are more attracted to the use of Rieffel deformation than twisted crossed products is the direct relation between Rieffel deformation to interactions as they are usually described in quantum physics [An1], [An2]. If the $D_k$'s are position operators then the dual action $\hat{\alpha} : \mathbb{R}^n \to \text{Aut}(\mathcal{N})$, implemented by a unitary group $e^{iv \cdot P}$, can be interpreted as the group of spacetime translations. Thus $P = (P_1, \ldots, P_n)$ are the energy-momenta. Performing a Rieffel deformation gives that the $P_k$'s are changed by a term coming from the $D_k$'s as we saw in Theorem 2. So the deformation is like adding an external term to the energy or to the momenta, interpreted suitably as coming from the interaction with another quantum system. Note that, by choice of gauge, a transient external electric field can be incorporated either via a potential energy term added to the Hamiltonian, or via an external vector potential term added to $P_1, \ldots, P_n$ [BoGK]. We know that either of these can be obtained from Rieffel deformation [An2].

On the other hand, if we have an action $\alpha$ generated by the momenta, so that $(D_1, \ldots, D_n) = (P_1, \ldots, P_n)$, then $\mathcal{D} = \gamma^k P_k$ is a Dirac operator in the physical sense, and the positive projection $P$ singles out the states of positive energy. The spectral flow between $\mathcal{D}$ and $\gamma^* \mathcal{D} \gamma$ is then like the amount of charge transferred due to the operation $\gamma$. This could be any real number, although it may be possible to obtain further restrictions on its possible values in specific examples.

In particular, if $A$ is the Fréchet algebra $L^1$ of all trace operators for the canonical operator trace, and if we use algebraic $K_1$ [CT],[CoT, Prop.7.2.3], then this spectral flow is an integer. However, to prove Theorem 1 we used topological $K$-theory (in which $K_1$ of $L^1$ is trivial) and Kasparov’s $KK$-theory (which is only defined for $C^*$-algebras). It would be interesting to carry over the proof using Cuntz’s bivariant $K$-theory so that also algebras like $L^1$ and $L^{1,\infty}$ are included and some reliance on the norm closure $A$ can be dropped. For the even case there is already no problem in letting $\tau$ be the operator trace on any $C^*$-algebra $A \subset \mathcal{B}(\mathcal{H})$.

Use of such $\mathcal{D}$ is not limited to condensed matter physics. In fact, (Lorentzian) spectral triples have been used to define a CAR algebra when the field operators act as multiplication operators with a Moyal product (i.e. the special kind of Rieffel product when the initial algebra $A$ is commutative) [BV],[V]. The relevant algebra is thus a Rieffel deformation of a commutative algebra like $S(\mathbb{R}^n)$ and the present paper strongly suggests that Connes’-type pairings can be used also in this setting.

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4 References

[An1] Andersson A. Operator deformations in quantum measurement theory. Lett. Math. Phys. Vol 104, Issue 4, pp. 415-430 (2014).
[CP2] Carey A, Phillips J. Spectral flow in Fredholm modules, eta invariants and the JLO cocycle. K-Theory. Vol 31, pp. 135-194 (2004).

[CPRS] Carey A, Phillips J, Rennie A, Sukochev F. The local index formula in semifinite von Neumann algebras I: Spectral flow. Adv. Math. Vol 202, pp. 451-516 (2006).

[CPRS2] Carey A, Phillips J, Rennie A, Sukochev F. The local index formula in noncommutative geometry revisited. In Noncommutative Geometry and Physics (pp. 3-36). Singapore: World Scientific. (2013)

[CPS] Carey A, Phillips J, Sukochev F. Spectral flow and Dixmier traces. arXiv:math/0205076v1 (2002).

[Co] Connes A. An analogue of the Thom isomorphism for crossed products of a C*-algebra by an action of R. Adv. Math. Vol 39, pp. 31-55 (1981).

[Co1] Connes A. Cyclic cohomology and the transverse fundamental class of a foliation. In: Geometric methods in operator algebras. Kyoto, Pitman Res. Notes in Math. Vol. 123, pp. 52-144 (1986).

[Co2] Connes A. C*-algebras and Differential Geometry. arXiv:hep-th/0101093v1 (2001).

[Co4] Connes A. Noncommutative geometry. Academic Press, Inc. (1994).

[Co5] Connes A. Noncommutative differential geometry. Publications mathematiques de V.H.E.S. Vol 62, pp. 41-144(1985)

[CoT] Cortiñas G, Thom A. Comparison between algebraic and topological K-theory of locally convex algebras. Adv. Math. Vol 218, Issue 1, pp. 266-307 (2008).

[CT] Cuntz J, Thom A. Algebraic K-theory and locally convex algebras. Math. Ann. Vol 334, pp. 339-371 (2006).

[DHK] Douglas RG, Hurder S, Kaminker J. The longitudinal cocycle and the index of Toeplitz operators. J. Func. Anal. Vol 101, pp. 120-144 (1991).

[DS] Durhuus B, Sitarz A. Star product realizations of κ-Minkowski space. J. Noncomm. Geom. Vol 3, Issue 7, pp. 605-645 (2011).

[ENN] Elliott G, Natsume T, Nest R. Cyclic cohomology for one-parameter smooth crossed products. Acta Math. Vol 160, Issue 3-4, pp. 285–305 (1988).

[FS] Fack T, Skandalis G. Connes’ analogue of the Thom isomorphism for the Kasparov groups. Invent. Math. Vol 64, pp. 7-14 (1981).

[Ge] Getzler E. The odd Chern character in cyclic homology and spectral flow. Topology. Vol 32, Issue 3, pp. 489-507 (1993).

[Go] Goffeng M. Index formulas and charge deficiencies on the Landau levels. J. Math. Phys. Vol 51 (2010).

[H1] Haagerup U. On the dual weight for crossed products of von Neumann algebras I. Math. Scand. Vol 43, pp. 99-118 (1978).

[H2] Haagerup U. On the dual weight for crossed products of von Neumann algebras II. Math. Scand. Vol 43, pp. 119-140 (1978).

[H3] Haagerup, Operator valued weights in von Neumann algebras I. J. Funct. Anal. Vol 32, pp. 175-206 (1979).

[H4] Haagerup, Operator valued weights in von Neumann algebras II. J. Funct. Anal. Vol 33, pp. 339-361 (1980).
[HM] Hannabuss K, Mathai V. Noncommutative principal torus bundles via parametrised strict deformation quantization. Lett. Math. Phys. Vol 102, Issue 1, pp. 107-123 (2012).

[Ji] Ji R. On the smoothed Toeplitz extensions and K-Theory. Proceedings of the American Mathematical Society. Vol. 109, Issue 1 (1990).

[KNR] Kaad J, Nest R, Rennie A. KK-Theory and spectral flow in von Neumann algebras. J. K-Theory Vol 10, pp. 241-277 (2012).

[Kas] Kasprzak P. Rieffel deformation via crossed products. J. Funct. Anal. Vol 257, Issue 5, pp.1288-1332 (2009).

[KeR] Kellendonk J, Richard S. Topological boundary maps in physics; General theory and applications. In: Perspectives in Operator Algebras and Mathematical Physics, 105-121 (2008).

[Kos] Kosaki H. Type III factors and index theory. Notes based on lectures given at Seoul National University (1998).

[LN] Laca M, Neshveyev S. KMS states of quasi-free dynamics on Pimsner algebras. J. Funct. Anal. Vol 211, pp. 457-482 (2004).

[L] Lechner G. Deformations of operator algebras and the construction of quantum field theories. In “XVIth International Congress on Mathematical Physics” (Proceedings of the ICMP 2009 in Prague), Exner (Ed.), World Scientific (2009).

[Lee] Lee HH. A note on Kasparov product and duality. arXiv:0712.1842v2 (2010).

[LW] Lechner G, Waldmann S. Strict deformation quantization of locally convex algebras and modules. arXiv:1109.5950 (2011).

[LM] Lein M, Măntoiu M, Richard S. Magnetic pseudodifferential operators with coefficients in C*-algebras. Publ. RIMS Kyoto Univ. Vol 46, pp. 755-788, (2010).

[Le] Lesch M. On the index of the infinitesimal generator of a flow. J. Operator Theory. Vol 25, pp. 73-92 (1991).

[Ma] Matassa M. A modular spectral triple for κ-Minkowski space. J. Geom. Phys. Vol 76C, pp. 136-157 (2014).

[MS] Mercati F, Sitarz A. κ-Minkowski differential calculi and star product. In: Proceedings of Science CNCFG2010:030 (2010).

[Mu] Much A. Quantum mechanical effects from deformation theory. J. Math. Phys. 55, 022302 (2014).

[Ne] Neshveyev S. Smooth crossed product of Rieffel’s deformations. Lett. Math. Phys. Vol 104, Issue 3, pp. 361-371 (2014).

[P] Pedersen P. C*-algebras and their automorphism groups. Academic Press (1979).

[PR] Phillips J, Raeburn I. An index theorem for Toeplitz operators with noncommutative symbol space. Journal of functional analysis. Vol 120, pp. 239-263 (1994).

[PLB] Prodan E, Leung B, Bellissard J. The non-commutative n-th Chern number (n ≥ 1). J. Phys. A: Math. Theor. 46 485202 (2013).

[PS] Prodan E, Schulz-Baldes H. Non-commutative odd Chern numbers and topological phases of disordered chiral systems. arXiv:1402.5002v1 (2014).

[Rie] Rieffel MA. Deformation quantization for actions of \( \mathbb{R}^d \). Mem. Amer. Math. Soc. 106(506) (1993).
[Sa] Sangha A. KK-fibrations arising from Rieffel deformations. arXiv:1109.5968v1 (2011).

[Su] Sutherland CE. Cohomology and extensions of von Neumann algebras I. Pub., RIMS. Kyoto Univ. Vol 16, pp. 105-133 (1980).

[St] Stein EM. Singular integrals and differentiability properties of functions. Princeton University Press, Princeton (1970).

[Ta2] Takesaki M. Theory of operator algebras II. Springer (2003).

[Th] Thaller B. The Dirac Equation. Springer-Verlag, Amsterdam (1992).

[TWZ] Tradler T, Wilson S, Zeinalian M. An elementary differential extension of odd K-theory. J. K-Theory. Vol 12, Issue 2, pp. 331-361 (2013).

[V] Verch R. Quantum Dirac field on Moyal-Minkowski spacetime - Illustrating quantum field theory over Lorentzian spectral geometry. Acta Phy. Pol. B. Proceedings supplement. Vol 4, Issue 3, pp. 507-527 (2011).