Abstract

The maximum density still life problem (MDSLP) is a hard constraint optimization problem based on Conway’s game of life. It is a prime example of weighted constrained optimization problem that has been recently tackled in the constraint-programming community. Bucket elimination (BE) is a complete technique commonly used to solve this kind of constraint satisfaction problem. When the memory required to apply BE is too high, a heuristic method based on it (denominated mini-buckets) can be used to calculate bounds for the optimal solution. Nevertheless, the curse of dimensionality makes these techniques unpractical for large size problems. In response to this situation, we present a memetic algorithm for the MDSLP in which BE is used as a mechanism for recombining solutions, providing the best possible child from the parental set. Subsequently, a multi-level model in which this exact/metaheuristic hybrid is further hybridized with branch-and-bound techniques and mini-buckets is studied. Extensive experimental results analyze the performance of these models and multi-parent recombination. The resulting algorithm consistently finds optimal patterns for up to date solved instances in less time than current approaches. Moreover, it is shown that this proposal provides new best known solutions for very large instances.

1. Introduction

The game of life was proposed by John H. Conway in the 60s. Afterwards, it was divulged by Martin Gardner in his Scientific American columns (Gardner, 1970, 1971, 1983). The game is played on an infinite checkerboard in which the only player places checkers on some of its squares. Each square on the board is called a cell and has eight neighbors: the eight cells that share one or two corners with it. A cell is alive if there is a checker on it, and is dead otherwise. The contents of the board evolve iteratively, in such a way that the state at time $t$ determines the state at time $t + 1$ according to three simple rules (see Fig. 1):

1. If a cell has exactly two living neighbors, then its state remains the same in the next iteration. This is called the life constraint.

2. If a cell has exactly three living neighbors, then it is alive in the next iteration. This is called the birth constraint.

3. If a cell has fewer than two or more than three living neighbors, then it is dead in the next iteration. These are called the death by isolation and death by over-crowding constraints respectively.
As it can be seen, the game of life is defined in terms of simple rules, but these can nevertheless generate incredibly complicated patterns and dynamics, and hence, it has attracted the interest of many scientists.

One challenging constraint optimization problem based on the game of life is the maximum density still life problem (MDSLP). In order to introduce this problem, let us define a stable pattern (also called a still life) as a board configuration that does not change through time, and let the density of a region be its percentage of living cells. The MDSLP in an $n \times n$ grid consists of finding a still life of maximum density. Elkies (1998) has shown that, for infinite boards, the maximum density is $1/2$ (for finite size, no exact formula is known). In this paper, we are concerned with the MDSLP and finite patterns, that is, finding maximal $n \times n$ still lifes.

The MDSLP is very hard to solve, and though it has not been proven to be NP-hard to the best of our knowledge, no polynomial-time algorithm for it is known. Our interest in this problem is manifold. Firstly, it must be noted that the patterns resulting in the game of life are very interesting. For example, by clever placement of the checkers and adequate interpretation of the patterns, it is possible to create a Turing-equivalent computing machine (Berlekamp, Conway, & Guy, 1982). From a more applied point of view, it is interesting to consider that many aspects of discrete dynamical systems have been developed or illustrated by examples in the game of life (Gardner, 1971, 1983). In this sense, finding stable patterns can be regarded as a mathematical abstraction of a standard issue in discrete systems control. Finally, the MDSLP is a prime example of weighted constrained optimization problem. As such, it constitutes an excellent test bed for different optimization techniques. Indeed, the problem has been included in the CSPLib\(^1\) repository. A dedicated web page\(^2\) maintains up-to-date results.

The MDSLP has been tackled using different approaches. Bosch and Trick (2002) compared different formulations for the MDSLP using integer programming (IP) and constraint programming (CP). Their best results were obtained with a hybrid algorithm mixing the two approaches. They were able to solve the cases for $n = 14$ and $n = 15$ in about 6 and 8 days of CPU time respectively. Smith (2002) used a pure constraint programming

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1. http://www.csplib.org
2. http://www.ai.sri.com/~nysmith/life
Table 1: Best experimental results reported in (Bosch & Trick, 2002) (CP/IP), (Larrosa & Morancho, 2003) (BE) and (Larrosa et al., 2005) (HYB-BE) for solving the MDSLP.

| n  | opt | CP/IP | BE       | HYB-BE |
|----|-----|-------|----------|--------|
| 12 | 68  | 11536 | 1638     | 1      |
| 13 | 79  | 12050 | 13788    | 2      |
| 14 | 92  | $5 \times 10^5$ | $10^5$ | 2      |
| 15 | 106 | $7 \times 10^5$ | 58      |        |
| 16 | 120 |       |          | 7      |
| 17 | 137 |       |          | 1091   |
| 18 | 153 |       |          | 2029   |
| 19 | 171 |       |          | 56027  |
| 20 | 190 |       |          | $2 \times 10^5$ |

approach to undertake the problem and proposed a formulation of the problem as a constraint satisfaction problem with 0-1 variables and non-binary constraints. Its dual graph translation into a binary constraint satisfaction problem was also considered. Surprisingly, it was proven that, although the dual representation has as many variables as the original one and very large domains, its performance was much better. However, only instances up to $n = 10$ could be solved. The best results for this problem were reported by Larrosa and Morancho (2003) and Larrosa, Morancho, and Niso (2005), showing the usefulness of bucket elimination (BE), an exact technique based on variable elimination and commonly used for solving constraint satisfaction problems described in detail in Section 2.4. Their basic approach could solve the problem for $n = 14$ in about $10^5$ seconds. Further improvements pushed the solvability boundary forward to $n = 20$ in about twice as much time. Recently, Cheng and Yap (2005, 2006) have tackled the problem via the use of ad-hoc global case constraints, but their results are comparable to IP/CP hybrids, and thus lie far from the ones obtained previously by Larrosa et al.

Table 1 resumes experimental results for current approaches used to tackle the MDSLP. The first column contains the problem size. The second column shows the optimal solution as the number of dead cells. Remaining columns report times in seconds by the hybrid IP/CP algorithm of Bosch and Trick (2002), by the BE approach of Larrosa and Morancho (2003) and by the BE/search hybrid of Larrosa et al. (2005). Although different computational platforms may have been used for these experiments, the trends are very clear and give a pristine indication of the potential of the different approaches. In this sense, note that all of these techniques applied to the MDSLP are exact approaches that are inherently limited for increasing problem sizes, and whose capabilities as anytime algorithms are unclear.

To tackle this problem, we recently proposed the use of hybrid methods combining exact and metaheuristic approaches. Particularly, in (Gallardo, Cotta, & Fernández, 2006a) we considered the hybridization of BE with evolutionary algorithms (an stochastic population-based search method) endowed with tabu search (a local search method). The resulting algorithm was a memetic algorithm (MA; see Section 2.2) that used BE as a mechanism for
recombining solutions, providing the best possible child from the parental set. Experimental tests indicated that the algorithm provided optimal or near-optimal results at an acceptable computational cost. Afterwards, in (Gallardo, Cotta, & Fernández, 2006b) we studied expanded multi-level models in which our previous hybrid algorithm was further hybridized with a branch-and-bound derivative, namely beam search (BS). Studies about the influence that variable clustering and multi-parent recombination have on the performance of the algorithm were also conducted. Results indicated that variable clustering was detrimental for this problem but also that multi-parent recombination improves the performance of the algorithm. To the best of our knowledge, these are the only heuristic approaches that have been applied to this problem to date.

This paper includes and extends our previous research on this problem. As new contributions, we have redone all experiments using an improved implementation of the bucket elimination crossover operator, described in Section 3.2. Additionally, we present a more extensive experimental analysis of our BS/MA hybrid described in (Gallardo et al., 2006b), analyzing the sensitivity of its parameters. We also propose a new hybrid algorithm that uses the technique of mini-buckets (MB) (Dechter, 1997) to further improve the lower bounds of the partial solutions that are considered in the BS part of the hybrid algorithm. This new algorithm is obtained from the hybridization, at different levels, of complete solving techniques (BE), incomplete deterministic methods (BS and MB) and stochastic algorithms (MAs). An experimental analysis shows that this new proposal consistently finds optimal solutions for MDSLP instances up to \( n = 20 \) in considerably less time than all the previous approaches reported in the literature. Finally, in order to test the scalability of our approach, this novel hybrid algorithm has been run on very large instances of the MDSLP for which optimal solution are currently unknown. Results were very successful, as the algorithm performed at the state-of-the-art, providing solutions that are equal or better to the best ones reported to date in the literature.

The paper is self-contained and structured as follows: Section 2 gives preliminary concepts that will be used in the rest of paper. Section 3 defines the MDSLP as a weighted constraint satisfaction problem, and shows how to solve it using BE. In Section 4, a MA for the MDSLP that uses BE as a recombination operator is presented and experimentally evaluated along with a hybrid multilevel algorithm that integrates the previous MA with Branch-and-Bound derivatives. Section 5 proposes and evaluates a novel hybrid algorithm that exploits the technique of mini-buckets. Finally, Section 6 presents conclusions and outlines future work.

2. Preliminaries

In this section, we briefly introduce concepts and techniques that will be used in the rest of paper. To this end, we first present beam search, a heuristic tree search algorithm derived from the branch and bound method. Subsequently, memetic algorithms are introduced. Finally, weighted constraint satisfaction problems are defined and the technique of bucket elimination – commonly used to solve them – is introduced. For the sake of notational simplicity, we use in this last subsection the notation of (Larrosa & Morancho, 2003; Larrosa et al., 2005).
2.1 Beam Search

Branch and bound (BB) (Lawler & Wood, 1966) is a general tree search method to solve combinatorial optimization problems. Tree search methods are constructive, in the sense that they work on partial solutions. In this way, tree search methods start with an empty solution that is incrementally extended by adding components to it. The way that partial solutions can be extended depends on the constraints imposed by the problem being solved. The solution construction mechanism maps the search space to a tree structure, in such a way that a path from the root of the tree to a leaf node corresponds to the construction of a solution. In order to efficiently explore this search tree, BB algorithms maintain an upper bound and estimate lower bounds for partially constructed solutions. Assuming a minimization problem, the upper bound corresponds to the cost of the best solution found so far. During the search process, a lower bound is computed for any partial solution generated, estimating the cost of the best solution that can be constructed by extending it. If this lower bound is greater than the current upper bound, solutions constructed by extending it will not lead to an improvement, and thus all nodes descending from it can be pruned from the search tree. Clearly, the capability of the algorithm for pruning the search tree depends on the existence of an accurate lower bound, that should additionally be computationally inexpensive in order to be practical.

Beam search (BS) (Barr & Feigenbaum, 1981) algorithms are incomplete derivatives of BB algorithms, and are thus heuristic methods. Essentially, BS works by extending every partial solution from a set B (called the beam) in at most $k_{ext}$ possible ways. Each new partial solution so generated is stored in a set $B'$. When all solutions in B have been processed, the algorithm constructs a new beam by selecting the best up to $k_{bw}$ (called the beam width) solutions from $B'$. Clearly, a way of estimating the quality of partial solutions, such as a lower bound, is needed for this.

One interesting peculiarity of BS is that it works by extending in parallel a set of different partial solutions in several possible ways. For this reason, BS is a particularly suitable tree search method to be used in a hybrid collaborative framework, as it can be used to provide periodically promising partial solutions to a population based search method such as a MA. Gallardo, Cotta, and Fernández (2007) have shown that this kind of hybrid algorithms can provide excellent results for some combinatorial optimization problems. We will subsequently present a hybrid tree search/memetic algorithm for the MDSLP based on this idea.

2.2 Memetic Algorithms

Evolutionary algorithms (EAs) are population-based metaheuristic optimization methods inspired by biological evolution (Bäck, 1996; Bäck, Fogel, & Michalewicz, 1997). In order to explore the search space, the EA maintains a set of solutions known as the population of individuals ($\mu$ is used to denote the total number of individuals in the population). These are initialized usually in a random way across the search space, although an heuristic may also be used. After the initialization, three different phases are iteratively performed until a termination condition is reached: selection, reproduction and replacement. In the context of EAs, the objective function assigning values to each solution is termed a fitness function, and is used to guide the search by comparing the goodness of different individuals.
Figure 2: Pseudo code of a memetic algorithm (MA). Although different variants are possible with respect to this scheme, it broadly captures the algorithmic structure typically used in MAs.

Note that EAs are black box optimization procedures in the sense that no knowledge of the problem (apart from the fitness function) is used. The need to exploit problem-knowledge has been repeatedly shown in theory (Wolpert & Macready, 1997) and in practice (Davis, 1991) though (see also Culberson, 1998). Different attempts have been made to answer this need; Memetic algorithms (Moscato, 1999; Moscato & Cotta, 2003; Moscato, Mendes, & Cotta, 2004; Krasnogor & Smith, 2005) (MAs) are one of the most successful to date (Hart, Krasnogor, & Smith, 2005). As EAs, MAs are also population based metaheuristics. The main difference is that the components of the population (sometimes termed agents in the MAs terminology) are not passive entities. These agents are active entities that cooperate and compete in order to find improved solutions.

There are many possible ways to implement MAs. The most common implementation consists of combining an EA with a procedure to perform local search on some or all solutions.
in the population during the main generation loop (cf. Krasnogor & Smith, 2005). Fig. 2 shows the general outline of such a MA. It must be noted however that the MA paradigm does not simply reduce itself to this particular scheme and there are diverse places (e.g., population initialization, genotype to phenotype mapping, evolutionary operators, etc.) where problem specific knowledge can be incorporated. In this work, apart from using tabu search (Glover, 1989, 1990)(TS) as a local search procedure within the MA, we have designed an “intelligent” recombination operator that uses an exact technique (bucket elimination) in order to find the best solution that can be constructed from a set of parents without introducing implicit mutation (i.e., exogenous information).

2.3 Weighted Constraint Satisfaction Problems

A weighted constraint satisfaction problem (WCSP) (Schiex, Fargier, & Verfaillie, 1995; Bistarelli, Montanari, & Rossi, 1997) is a constraint satisfaction problem (CSP) in which preferences among solutions can be expressed. Formally, a WCSP can be defined by a tuple \((\mathcal{X}, \mathcal{D}, \mathcal{F})\), where \(\mathcal{D} = \{D_1, \ldots, D_n\}\) is a set of finite domains, \(\mathcal{X} = \{x_1, \ldots, x_n\}\) is a set of variables taking values from their finite domains (\(D_i\) is the domain of variable \(x_i\)) and \(\mathcal{F}\) is a set of cost functions (also called soft constraints or weighted constraints) used to declare preferences among possible solutions. Permitted assignments of variables receive finite costs that express their degree of preference (the lower the value the better the preference) and forbidden assignments receive cost \(\infty\). Note that each \(f \in \mathcal{F}\) is defined over a subset of variables, \(\text{var}(f) \subseteq \mathcal{X}\), called its scope. The objective function \(F\) is defined as the sum of all functions in \(\mathcal{F}\):

\[
F = \sum_{f \in \mathcal{F}} f
\]  

The assignment of value \(v_i \in D_i\) to variable \(x_i\) is noted \(x_i = v_i\). A partial assignment of \(m\) variables is a tuple \(t = (x_{i_1} = v_1, x_{i_2} = v_2, \ldots, x_{i_m} = v_m)\). A complete assignment of all variables with values in their domains that satisfies every soft constraint (i.e., with a finite valuation for \(F\)) represents a solution to the WCSP. The optimization goal is to find a solution that minimizes this objective function.

A WCSP instance is usually depicted by means of its constraint graph, which has one node for each variable \(x_i \in \mathcal{X}\), and one edge connecting any two nodes whose variables appear in the same scope of some cost function \(f \in \mathcal{F}\).

2.4 Bucket Elimination

Bucket elimination (BE) (Dechter, 1999) is a generic technique suitable for many automated reasoning and optimization problems and, in particular, for WCSP solving. The functioning of BE is based upon the following two operators over functions (Larrosa et al., 2005):

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3. Observe that the constraints are not weighted in the sense of having an external weight parameter assigned to them. Indeed, each of them has the same influence on the overall function value, as shown in Eq. (1). The reason they are called “weighted” is that the output of each function is not binary (satisfied vs. unsatisfied) but a numerical value when it is satisfied.
• the sum of two functions \( f \) and \( g \), denoted \( f + g \), is a new function with scope \( \text{var}(f) \cup \text{var}(g) \) which returns for each tuple the sum of costs of \( f \) and \( g \),
\[
(f + g)(t) = f(t) + g(t).
\] (2)

• The elimination of variable \( x_i \) from \( f \), denoted \( f \downarrow x_i \), is a new function with scope \( \text{var}(f) - \{x_i\} \) which returns for each tuple \( t \) the minimum cost extension of \( t \) to \( x_i \),
\[
(f \downarrow x_i)(t) = \min_{v \in D_i} \{f(t \cdot (x_i = v))\},
\] (3)

where \( t \cdot (x_i = v) \) means the extension of the assignment \( t \) with the assignment of value \( v \) to variable \( x_i \). Observe that when \( f \) is a unary function (i.e., it has arity one), a constant is obtained upon elimination of the only variable in its scope.

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**Bucket Elimination for a WCSP \((X, D, F)\)**

```python
function BE(X, D, F)
1: for i := n downto 1 do
2:   Bi := \{f \in F \mid x_i \in \text{var}(f)\}
3:   gi := (\sum_{f \in Bi} f) \downarrow x_i
4:   F := (F \cup \{gi\}) - Bi
5: end for
6: t := \emptyset
7: for i := 1 to n do
8:   v := arg\min_{a \in D_i} \{((\sum_{f \in Bi} f)(t \cdot (x_i = a)))\}
9:   t := t \cdot (x_i = v)
10: end for
11: return (F, t)
end function
```

Figure 3: The general template, adapted from Larrosa and Morancho (2003), of bucket elimination for a WCSP \((X, D, F)\).

Without loss of generality, let us assume a lexicographical ordering for the variables in \( X \), i.e., \( o = (x_1, x_2, \cdots, x_n) \). Fig. 3 shows a pseudo-code of the BE algorithm for solving a WCSP instance, that returns the optimal cost in \( F \) and one optimal assignment in \( t \). Observe that, in a first phase, BE eliminates one variable \( x_i \in X \) in each iteration of the loop comprising lines 1-5. This is done by computing firstly the bucket \( B_i \) of variable \( x_i \) as the set of all cost functions in \( F \) having \( x_i \) in their scope. Then, a new function \( g_i \) is defined as the sum of all these functions in \( B_i \) in which variable \( x_i \) has been eliminated. Finally, \( F \) is updated by removing the functions involving \( x_i \) (i.e., those in \( B_i \)) and adding the new function that does not contain \( x_i \). The consequence is that \( x_i \) does not exist in \( F \) but the value of the optimal cost is preserved. The elimination of \( x_1 \) produces an empty scope function (i.e., a constant) which is the optimal cost of the problem. Then, in lines 6-10, BE
generates an optimal assignment of variables by considering these in the order imposed by $o$: this is done by starting from an empty assignment $t$ and assigning to $x_i$ the best value regarding the extension of $t$ with respect to the sum of functions in $B_i$ ($\arg\min_a \{f(a)\}$ denotes the value of $a$ producing minimum $f(a)$).

Note that BE has exponential space complexity because, in general, the result of summing functions or eliminating variables cannot be expressed intensionally by algebraic expressions and, as a consequence, intermediate results have to be collected extensionally in tables. To be precise, the complexity of BE depends on the problem structure (as captured by its constraint graph $G$) and the ordering $o$. According to Larrosa and Morancho (2003), the complexity of BE along ordering $o$ is time $\Theta(Q \times n \times d^{w^*(o)+1})$ and space $\Theta(n \times d^{w^*(o)})$, where $d$ is the largest domain size, $Q$ is the cost of evaluating cost functions (usually assumed $\Theta(1)$), and $w^*(o)$ is the induced width of the graph along ordering $o$, which describes the largest clique created in the graph by bucket elimination, and which corresponds to the largest scope of a function recorded by the algorithm. Although finding the optimal ordering is NP-hard (Arnborg, 1985), heuristics and approximation algorithms have been developed for this task (check Dechter, 1999 for details).

3. The Maximum Density Still Life Problem

According to the definition of MDSLP and the three rules of the game, it is easy to see that each cell in a still life must satisfy the following conditions:

- If the cell is alive, it must have two or three neighbors.
- If the cell is dead, it will have either more than three or less than three neighbors.

Note that finite still lifes are not allowed to produce new living cells outside the grid, and hence stability conditions must hold in the cells surrounding the $n \times n$ square, that are assumed to be dead. This can equally be achieved by requiring further that:

- if the cell is at the boundary of the $n \times n$ square, it must not be part of a sequence of three consecutive living cells in the direction of the boundary.

Fig. 4 shows some maximum density still lifes for small values of $n$.

![Figure 4: Maximum density still lifes for $n \in \{3, 4, 5\}$.

The constraints and objectives of the MDSLP are formalized in the following subsections in which we follow a similar notation to the one used in (Larrosa & Morancho, 2003; Larrosa et al., 2005).
3.1 Problem Formulation

To state the problem formally, let \( r \) be an \( n \times n \) binary matrix, such that \( r_{ij} \in \{0, 1\}, 1 \leq i, j \leq n \) (\( r_{ij} = 0 \) if cell \((i, j)\) is dead, and 1 otherwise). In addition, let \( \mathcal{N}(r, i, j) \) be the set comprising the neighborhood of cell \( r_{ij} \):

\[
\mathcal{N}(r, i, j) = \{ r_{(i+x)(j+y)} \mid x, y \in \{-1, 0, 1\} \wedge x^2 + y^2 \neq 0 \wedge 1 \leq (i + x), (j + y) \leq \|r\| \} \tag{4}
\]

where \( \|r\| \) denotes the number of rows (or columns) of square matrix \( r \), and let the number of living neighbors for cell \( r_{ij} \) be noted as \( \eta(r, i, j) \):

\[
\eta(r, i, j) = \sum_{c \in \mathcal{N}(r, i, j)} c \tag{5}
\]

According to the rules of the game, let us also define the following predicate that checks whether cell \( r_{ij} \) is stable:

\[
S(r, i, j) = \begin{cases} 
2 \leq \eta(r, i, j) \leq 3, & r_{ij} = 1 \\
\eta(r, i, j) \neq 3, & r_{ij} = 0. 
\end{cases} \tag{6}
\]

In order to check boundary conditions, we will further denote by \( \tilde{r} \) the \( (n+2) \times (n+2) \) matrix obtained by embedding \( r \) in a frame of dead cells:

\[
\tilde{r}_{ij} = \begin{cases} 
r_{(i-1)(j-1)}, & 2 \leq i, j \leq n + 1 \\
0, & \text{otherwise}. 
\end{cases} \tag{7}
\]

The maximum density still life problem for an \( n \times n \) board, MDSLP\((n)\), can now be stated as finding an \( n \times n \) binary matrix \( r \), such that

\[
\sum_{1 \leq i, j \leq n} (1 - r_{ij}) \text{ is minimal,} \tag{8}
\]

subject to

\[
\bigwedge_{1 \leq i, j \leq n+2} S(\tilde{r}, i, j). \tag{9}
\]

3.2 The MDSLP as a Weighted Constraint Satisfaction Problem

As shown by Larrosa and Morancho (2003) and Larrosa et al. (2005), the MDSLP fits nicely within the framework of WCSPs. To this end, an \( n \times n \) board configuration can be represented by an \( n \)-dimensional vector \((r_1, r_2, \ldots, r_n)\). Each vector component encodes (as a binary string) a row, so that the \( j \)-th bit of row \( r_i \) (noted \( r_{ij} \)) signifies the state of the \( j \)-th cell of the \( i \)-th row (a value of 1 represents a live cell and a value of 0 a dead cell).

Two functions over rows will be useful to describe the constraints that must be satisfied by a valid configuration. The first one,

\[
\text{zeroes}(a) = \sum_{1 \leq i \leq n} (1 - a_i), \tag{10}
\]
returns the number of dead cells in a row (i.e., the number of zeroes in binary string \( a \)). The second one,

\[
Adjs(a) = Adjs'(a, 1, 0) \tag{11}
\]

\[
Adjs'(a, i, l) = \begin{cases} 
 l, & i > n \\
 Adjs'(a, i + 1, l + 1), & a_i = 1 \\
 \max(l, Adjs'(a, i + 1, 0)), & a_i = 0,
\end{cases}
\]

computes the maximum number of adjacent living cells in row \( a \). We also introduce a ternary predicate, \( Stable(r_{i−1}, r, r_{i+1}) \), that takes three consecutive rows in a board configuration and is satisfied if, and only if, all cells in the central row are stable (i.e., all cells in \( r \) will remain unchanged in the next iteration):

\[
Stable(a, b, c) = \bigwedge_{1 \leq i \leq n} S(a, b, c, i) \tag{12}
\]

\[
S(a, b, c, i) = \begin{cases} 
 2 \leq \eta(a, b, c, i) \leq 3, & b_i = 1 \\
 \eta(a, b, c, i) \neq 3, & b_i = 0 \\
 \end{cases}
\]

\[
\eta(a, b, c, i) = \sum_{\max(1, i−1) \leq j \leq \min(n, i+1)} (a_j + b_j + c_j) - b_i,
\]

where \( \eta(a, b, c, i) \) is the number of living neighbors of cell \( b_i \), assuming \( a \) and \( c \) are the rows above and below row \( b \).

The MDSLP can now be formulated as a WCSP using \( n \) cost functions \( f_i \), \( i \in \{1 . . n\} \). Accordingly, \( f_n \) is binary with scope the last two rows of the board (\( \text{var}(f_n) = \{r_{n−1}, r_n\} \)) and is defined as:

\[
f_n(a, b) = \begin{cases} 
 \infty, & \neg Stable(a, b, c) \\
 \infty, & Adjs(b) > 2 \\
 \text{zeroes}(b), & \text{otherwise}.
\end{cases} \tag{13}
\]

The first line checks that all cells in row \( r_n \) are stable, whereas the second one checks that no new cells are produced below the \( n \times n \) board. Note that any pair of rows representing an unstable configuration is assigned a cost of \( \infty \), whereas a stable one is assigned its number of dead cells (to be minimized).

For \( i \in \{2 . . n−1\} \), corresponding \( f_i \) cost functions are ternary with scope \( \text{var}(f_i) = \{r_{i−1}, r_i, r_{i+1}\} \) and are defined as:

\[
f_i(a, b, c) = \begin{cases} 
 \infty, & \neg Stable(a, b, c) \\
 \infty, & a_1 = b_1 = c_1 = 1 \\
 \infty, & a_n = b_n = c_n = 1 \\
 \text{zeroes}(b), & \text{otherwise}.
\end{cases} \tag{14}
\]

In this case, boundary conditions are checked to the left and right of the board. As regards cost function \( f_1 \), it is binary with scope the first two rows of the board (\( \text{var}(f_1) = \{r_1, r_2\} \)) and is specified similarly to \( f_n \):

\[
f_1(b, c) = \begin{cases} 
 \infty, & \neg Stable(\theta, b, c) \\
 \infty, & Adjs(b) > 2 \\
 \text{zeroes}(b), & \text{otherwise}.
\end{cases} \tag{15}
\]
3.3 Solving the MDSLP with BE

According to the formulation of the MDSLP as a WSCP introduced in Section 3.2, the corresponding constraint graph has a sequential structure, in which an arbitrary row is linked to the two rows above and below it. Due to this simple structure, it is easy to find an optimal elimination order for BE, and variables can be eliminated starting with the last one and proceeding in decreasing order. Fig. 5 shows the resulting algorithm. Function BE takes two parameters: \( n \), the size of the instance to be solved, and \( D \), the domain for each variable (row) in the solution. If domain \( D \) is set to \( \{0, \ldots, 2^n-1\} \) (i.e., a set containing all possible rows) the function implements an exact method that returns the optimal solution for the problem instance (as the number of dead cells) and a vector corresponding to the rows of that solution. The algorithm starts by eliminating the last variable \( r_n \), whose bucket is \( B_n = \{f_n, f_{n-1}\} \), the only cost functions containing \( r_n \) in their scopes. In lines 1-3, \( B_n \) is used to compute a new cost function, \( g_n(a, b) \), with scope \( \{r_{n-2}, r_{n-1}\} \), that represents the cost of the best extension of \((r_{n-2} = a, r_{n-1} = b)\) to the removed variable \( r_n \). At this point, the bucket of the next variable, \( r_{n-1} \), is \( B_{n-1} = \{g_n, f_{n-1}\} \), that can be used to compute a new cost function, \( g_{n-1}(a, b) \) with scope \( \{r_{n-3}, r_{n-2}\} \) representing the cost of the best extension of \((r_{n-3} = a, r_{n-2} = b)\) to the removed variables \( r_n \) and \( r_{n-1} \). This process can be iterated (lines 4-8) to eliminate variables up to \( r_3 \). Optimal values for variables \( r_1 \) and \( r_2 \) can be calculated using an exhaustive search (line 9). At this time, the optimal cost can be calculated and the optimal values for remaining variables can be set in increasing order using their bucket and variables assigned beforehand (lines 11-14).

Note that the space complexity of the algorithm, when used as an exact method, is \( \Theta(n \times 2^n) \), due to the memory required to store extensionally \( n \) cost functions \( g_i \) having each \( 2^n \times 2^n \) entries. The time complexity is \( \Theta(n^2 \times 2^{3n}) \) due to lines 4-8, as finding the minimum of \( 2^n \) alternatives, being the computation of each one \( \Theta(n) \), has to be repeated \( \Theta(n \times 2^n) \) times. On the other hand, a basic search-based solution to the problem could be implemented with worst case time complexity \( \Theta(2^n n^2) \) and polynomial space. Observe that the time complexity of BE is therefore an exponential improvement over basic search algorithms, although its high space complexity makes the approach impractical for large instances.

One interesting optimization, presented by Larrosa and Morancho (2003), allows reducing the complexity of the algorithm. In the following, we assume that \( n \) is even, although a similar reasoning can be used if the size of the board is odd. The optimization avoids the computation needed to eliminate variables \( r_1, r_2, \ldots, r_{\frac{n}{2}-1} \), as a result of the symmetry of the problem. In this way, the algorithm starts by eliminating variables \( r_n, r_{n-1}, \ldots, r_{\frac{n}{2}+2} \). Observe that, at this point, cost functions \( g_n, g_{n-1}, \ldots, g_{\frac{n}{2}+2} \) have been computed. At this point, the order to eliminate remaining variables can be changed to \( r_1, r_2, \ldots, r_{\frac{n}{2}-1} \). The elimination of \( r_1 \) would produce \( g_1 \) with scope \( \{r_1, r_2\} \), but this computation can be avoided, as it is the same to eliminate \( r_1 \) or to rotate the board by 180 degrees and eliminate variable \( r_n \), i.e.:

\[
g_1(a, b) = g_n(\overline{b}, \overline{a}),
\]

where \( \overline{r} \) denotes the reflection value of the binary string \( r \). Moreover, if the board is vertically reflected, an equivalent problem is obtained, so it follows that

\[
g_n(\overline{b}, \overline{a}) = g_n(b, a),
\]
function \text{BE}(n, D)
1: \text{for } a, b \in D \text{ do}
2: \quad g_n(a, b) := \min_{c \in D} \{f_{n-1}(a, b, c) + f_n(b, c)\}
3: \text{end for}
4: \text{for } i := n - 1 \text{ downto } 3 \text{ do}
5: \quad \text{for } a, b \in D \text{ do}
6: \quad \quad g_i(a, b) := \min_{c \in D} \{f_{i-1}(a, b, c) + g_{i+1}(b, c)\}
7: \quad \text{end for}
8: \text{end for}
9: \quad (r_1, r_2) := \text{argmin}_{a,b \in D} \{g_3(a, b) + f_1(a, b)\}
10: \quad \text{opt} := g_3(r_1, r_2) + f_1(r_1, r_2)
11: \text{for } i := 3 \text{ to } n - 1 \text{ do}
12: \quad r_i := \text{argmin}_{c \in D} \{f_{i-1}(r_{i-2}, r_{i-1}, c) + g_{i+1}(r_{i-1}, c)\}
13: \text{end for}
14: \quad r_n := \text{argmin}_{c \in D} \{f_{n-1}(r_{n-2}, r_{n-1}, c) + f_n(r_{n-1}, c)\}
15: \text{return } (\text{opt}, (r_1, r_2, \ldots, r_n))
\text{end function}

Figure 5: Bucket elimination for the MDSLP.

and hence
\[ g_1(a, b) = g_n(b, a). \] (18)

The optimized algorithm is obtained by applying the same reasoning to the rest of the variables. Larrosa and Morancho (2003) and Larrosa et al. (2005) have used this method\(^4\) to solve the MDSLP up to size 14. The fourth column in Table 1 reproduces their results, obtained with a 2GHz Pentium IV machine with 2Gb of memory. Notice the limitations of the approach: the \(n = 15\) instance could not be solved due to space restraints. In Section 4.1, we will show how BE can be embedded in a MA with reduced complexity in order to implement a smart recombination operator.

4. A Multi-Level Memetic/Exact Hybrid Algorithm for the MDSLP

WCSPs are very amenable for being undertaken with evolutionary metaheuristics. Obviously, the quality of the results will greatly depend on how well knowledge of the problem is incorporated into the search mechanism. Our final goal is to present an algorithmic model based on the hybridization of MAs with exact techniques at two levels: within the MA (as an embedded operator), and outside it (in a cooperative model). Firstly, we will focus in the next subsection on the first level of hybridization, that incorporates an exact tech-

\(^4\) Actually, an instance of the algorithm in which each variable is allowed to take values in the whole computation domain. This is not our case as it will be shown in next sections.
nique (namely BE) within the MA as an embedded recombination operator. Subsequently, we will proceed to a second level of hybridization, in which the MA cooperates with a branch-and-bound based beam search algorithm.

4.1 A Memetic Algorithm with BE for the MDSLP

In this subsection we describe a MA for the MDSLP that uses tabu search (TS) as a local search operator and BE as an optimal recombination operator. Before detailing these two components, let us describe the basic underlying evolutionary algorithm (EA).

4.1.1 Representation and Fitness Calculation

The natural representation of MDSLP solutions is their binary encoding. Accordingly, a configuration for an \( n \times n \) board will be represented as a binary \( n \times n \) matrix \( r \). Clearly, infeasible solutions can be represented, since not all such binary matrices will correspond to stable patterns. One way to deal with this scenario is using penalty-based fitness functions. To be precise, the fitness (to be minimized) of a configuration \( r \) is defined as:

\[
f(r) = \sum_{i=1}^{n} \sum_{j=1}^{n} (1 - r_{ij}) + K \sum_{i=1}^{n+2} \sum_{j=1}^{n+2} [\tilde{r}_{ij}\phi_1(\eta(\tilde{r}, i, j)) + (1 - \tilde{r}_{ij})\phi_0(\eta(\tilde{r}, i, j))].
\] (19)

Recall that stability is not only required within the \( n \times n \) board, but also in its immediate neighborhood, and this is taken into account by working with \( \tilde{r} \), the \( (n+2) \times (n+2) \) binary matrix obtained by embedding \( r \) in a frame of dead cells, as defined in (7). \( K \) is a constant, \( \eta(r, i, j) \) is the number of live neighbors of cell \( r_{ij} \), and \( \phi_0, \phi_1 : \mathbb{N} \rightarrow \mathbb{N} \) are two functions (to be used with dead or alive cells respectively), that take the number of alive neighbors of a cell, and return a penalty depending on how many of them should be flipped to have a stable configuration, defined as:

\[
\phi_0(\eta) = \begin{cases} 
0, & \eta \neq 3 \\
K' + 1, & \text{otherwise}
\end{cases}
\] (20)

\[
\phi_1(\eta) = \begin{cases} 
0, & 2 \leq \eta \leq 3 \\
K' + 2 - \eta, & \eta < 2 \\
K' + \eta - 3, & \eta > 3,
\end{cases}
\] (21)

where \( K' \) is another constant. The first double sum in (19) corresponds to the basic quality measure for feasible solutions, i.e., its number of dead cells. With respect to the last term, it represents the penalty for infeasible solutions. The strength of penalization is controlled by constants \( K \) and \( K' \). The values we have chosen for them (\( K = n^2 \) and \( K' = 5n^2 \)) ensure that given any two solutions, the one that violates less constraints is preferred; if two solutions violate the same number of constraints, the one whose overall degree of violation (i.e., distance to feasibility) is lower is preferred. Finally, if the two solutions are feasible, the penalty term is null and the solution with the higher number of live cells is better.
4.1.2 A Local Improvement Strategy Based on Tabu Search

The fitness function defined above provides a stratified notion of gradient that can be exploited by a local search strategy. Moreover, notice that the function is quite decomposable, since interactions among variables are limited to adjacent cells in the board. Thus, whenever a configuration is modified, the new fitness can be computed only considering the cells located in adjacent positions to changed cells. To be precise, assume that cell \((i, j)\) is modified in solution \(r\), resulting in solution \(s\); the new fitness \(f(s)\) can be computed as:

\[
f(s) = f(r) + K \left[ \Delta f_1(r_{ij}, \eta(r, i, j)) + \sum_{c \in N(r, i, j)} \Delta f_2(c, \eta(c), r_{ij}) \right],
\]

and functions \(\Delta f_1\) and \(\Delta f_2\) are defined as:

\[
\Delta f_1(c, \eta) = \begin{cases} 
0, & \eta = 2 \\
(-1)^{(1-c)\phi_0(\eta)}, & \eta = 3 \\
(-1)^{c\phi_1(\eta)}, & \text{otherwise}
\end{cases}
\]

\[
\Delta f_2(c', \eta, c) = (1 - c')\Delta f_{2,0}(\eta, c) + c'\Delta f_{2,1}(\eta, c)
\]

\[
\Delta f_{2,0}(\eta, c) = \begin{cases} 
K' + 1, & (\eta = 2 \land c = 0) \lor (\eta = 4 \land c = 1) \\
-(K' + 1), & \eta = 3 \\
0, & \text{otherwise}
\end{cases}
\]

\[
\Delta f_{2,1}(\eta, c) = \begin{cases} 
K' + 1, & (\eta = 2 \land c = 1) \lor (\eta = 3 \land c = 0) \\
-(K' + 1), & (\eta = 1 \land c = 0) \lor (\eta = 4 \land c = 1) \\
1, & (\eta = 1 \land c = 1) \lor (\eta \geq 4 \land c = 0) \\
-1, & (\eta = 0) \lor (\eta \geq 5 \land c = 1) \\
0, & \text{otherwise}
\end{cases}
\]

Using this efficient fitness re-computation mechanism, our local search strategy explores the neighborhood \(N(r) = \{ s \mid \text{Hamming}(r, s) = 1 \}\), i.e., the set of solutions obtained by flipping exactly one cell in the configuration. This neighborhood comprises \(n^2\) configurations, and it is fully explored in order to select the best neighbor. In order to escape from local optima, a tabu-search scheme is used: up-hill moves are allowed, and after flipping a cell, it is put in the tabu list for a number of iterations (randomly drawn from \([n/2, 3n/2]\) to hinder cycling in the search). Thus, it cannot be modified in the subsequent iterations unless the aspiration criterion is fulfilled. In this case, the aspiration criterion is improving the best solution found in that run of the local search strategy. The whole process is repeated until a maximum number of iterations is reached, and the best solution found is returned.

4.1.3 Optimal recombination with BE

Recall that the fitness function that we have defined is able to evaluate any representable configuration (feasible or not), and hence, the binary representation used turns out to be freely manipulable. With this setting, any standard recombination operator for binary strings could be used in principle. For example, the two-dimensional version of single-point crossover (2D-SPX), depicted in Fig. 6, could be employed. Although such a blind operator
is feasible from a computational point of view, it would perform poorly, as it would behave like a macromutation operation. In order to achieve a sensible recombination of information, we can resort to BE.

Even though the performance of BE as an exact method for the MDSLP was better than basic search-based approaches, it was shown in Section 3.3 that the corresponding time and space complexity were still very high, making it unsuitable for large instances. In the following, we explain how BE can be used to implement an intelligent recombination operator for the MDSLP. Such operator will explore the dynastic potential (Radcliffe, 1994) (possible children) of the solutions being recombined, providing the best solution that can be constructed without introducing implicit mutation, i.e., exogenous information (cf. Cotta & Troya, 2003). Moreover, we will show that this operator is tractable from a computational point of view.

For this purpose, let \( x = (x_1, x_2, \cdots, x_n) \) and \( y = (y_1, y_2, \cdots, y_n) \) be two board configurations for an \( n \times n \) instance of the MDSLP. Our operator will calculate the best configuration that can be obtained by combining rows in \( x \) and \( y \) without introducing information not present in any of the parents. This can be achieved by restricting the domain of variables in BE to take values corresponding to the rows of the configurations being recombined. Using the optimized version of the BE algorithm, the recombination operator becomes \( \text{BE-Opt}(n, \{x_1, x_2, \cdots, x_n, y_1, y_2, \cdots, y_n\}) \), so that the result returned by this invocation to the algorithm is the best possible recombination.

\[
\begin{array}{c}
\text{Random Column} \\
\downarrow \\
\text{Random Row} \\
\rightarrow \\
A_1 \quad A_2 \quad A_3 \quad A_4 \times B_1 \quad B_2 \quad B_3 \quad B_4 = A_1 \quad B_2 \quad B_3 \quad A_4
\end{array}
\]

Figure 6: Blind recombination operator for the MDSLP.

In order to analyze the time complexity for this recombination operator, the critical part of the algorithm is the execution of lines 4-8 in Fig. 5. In this case, line 6 has complexity \( O(n^2) \) (finding the minimum of at most \( 2n \) alternatives, the computation of each being \( \Theta(n) \)). Line 6 has to be executed \( n/2 \times 2n \times 2n \) times at most (recall the optimization), making a global complexity of \( O(n^5) = O(|x|^{2.5}) \), where \( |x| \in \Theta(n^2) \) is the size of solutions. Notice also that the recombination procedure can be readily made to further exploit the symmetry of the problem, extending variable domains to column values in addition to row values. The complexity bounds remain the same in this case.

One interesting property of the described operator is that it can be generalized to recombine any number of board configurations like \( \text{BE-Opt}(n, \bigcup_{x \in S} \{x_i \mid i \in \{1 \ldots n\}\}) \), where \( S \) is a set comprising the solutions to be recombined. In this situation, the time complexity is \( O(k^3n^5) \) (line 6 is \( O(kn^2) \), and it is executed \( O(k^2n^3) \) times), where \( k = |S| \) is the number of configurations being recombined. This multi-parental capability will be explored in the rest of the paper.
4.1.4 Experimental Results

In order to evaluate the usefulness of the described hybrid recombination operator, a set of experiments for problem sizes from $n = 12$ up to $n = 20$ has been realized (recall that optimal solutions to the MDSLP are known up to $n = 20$). The experiments were performed using a steady-state evolutionary algorithm ($\text{popsize} = 100$, $p_m = 1/n^2$, $p_X = 0.9$, binary tournament selection). With the aim of maintaining diversity, duplicated individuals were not allowed in the population. Algorithms were run until an optimal solution was found or a time limit was exceeded. This time limit was set to 3 minutes for problem instances of size 12 and was gradually incremented by 60 seconds for each size increment. For each algorithm and each instance size, 20 independent executions were run. All the experiments in this paper have been performed in a Pentium IV PC (2400MHz and 512MB of main memory) under SuSE Linux.

The base algorithm used is a MA using 2D-SPX for recombination, and endowed with tabu search for local improvement ($\text{maxiter} = n^2$). This algorithm is termed MA$_{TS}$, and has been shown to be capable of finding feasible solutions systematically, solving to optimality instances with $n < 15$ (see MA$_{TS}$ in Fig. 7). Although the performance of the algorithm degrades for larger instances, it provides distributions for the solutions whose average relative distance to the optimum is less than 5.29% in all cases. This contrasts with the case of plain EAs, which are incapable of finding even a feasible solution in most runs (Gallardo et al., 2006a).

![Figure 7: Relative distances to optimum for different algorithms for sizes ranging from 12 up to 20. Each box summarizes 20 runs. In this and in all subsequent figures, boxes comprise the second and third quartiles of the distribution (i.e., the inner 50%), an horizontal line marks the median, a plus sign indicates the mean, and circles indicate results further from the median than 1.5 times the interquartile-distance.](image)

MA$_{TS}$ is firstly compared with MAs endowed with BE for performing recombination. Since the use of BE for recombination has a higher computational cost than a simple blind
recombination, and there is no guarantee that recombining two infeasible solutions will result in a feasible solution, we have defined three variants of the MAs:

- In the first one, denoted MA-BE, BE is always used to perform recombination.
- In the second one, termed MA-BE$_{1F}$, we require that at least one of the parents is feasible in order to apply BE; otherwise blind recombination is used.
- In the last one, identified as MA-BE$_{2F}$, we require the two parents to be feasible, thus being more restrictive in the application of BE.

By evaluating these variants, we intend to explore the computational tradeoffs involved in the application of BE as an embedded component of the MA. For these algorithms, mutation was performed prior to recombination in order to take advantage of good solutions provided by BE. Fig. 7 shows the empirical performance of the different algorithms evaluated (as the relative distance to the optimum). Results show that MA-BE improves significantly over MA$_{TS}$ and can find better solutions. MA-BE$_{2F}$ can find slightly better solutions than MA-BE on smaller instances ($n \in \{13, 15, 16\}$), but on larger instances the winner is MA-BE. It seems that the effort saved not recombining unfeasible solutions does not further improve the performance of the algorithm. Note also that, for larger instances, MA-BE$_{1F}$ is better than MA-BE$_{2F}$. This correlates well with the fact that BE is used more frequently in the former than in the latter.

As mentioned in Section 4.1.3, the optimal recombination scheme we use can be readily extended to multi-parent recombination (Eiben, Raue, & Ruttkay, 1994): an arbitrary number of solutions can contribute their constituent rows for constructing a new solution. Additional experiments were done to explore the effect of this capability of MA-BE. Fig. 8 shows the results obtained by MA-BE for a different number of parents being recombined (arities 2, 4, 8 and 16). For $arity = 2$, the algorithm was able to find the optimum solution for all instances except for $n = 18$ and $n = 20$ (the relative distance to the optimum for the best solution found is less than 1.04% in these cases). Executions with $arity = 4$ cannot find optimum solutions for the remaining instances, but note that the distribution improves in some cases. Clearly, the performance of the algorithm deteriorates when combining more than 4 parents due to the higher computational cost. Variable clustering could be used to alleviate this higher computational cost, but this results in performance degradation since the more coarse granularity of the information pieces hinders information mixing (Cotta & Troya, 2000; Gallardo et al., 2006b).

### 4.2 A Beam Search/MA Hybrid Algorithm

In this subsection, we describe a hybrid tree search/memetic algorithm for the MDSLP. This algorithm combines, in a collaborative way, a BS algorithm and a MA. As noted before, BS works by extending in parallel a set of different partial solutions in several possible ways, and thus can be used to provide promising partial solutions to a population based search method such as a MA. The goal is to exploit the capability of BS for identifying probably good regions of the search space, and the strength of the MA for exploring these, synergistically combining these two different approaches.

The proposed hybrid algorithm, that executes BS and the MA in an interleaved way, is depicted in Fig. 9. In the pseudo-code, a (possible partial) solution for an $n \times n$ instance
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Figure 8: Relative distances to optimum for different arities for MA-BE for sizes ranging from 12 up to 20. Each box summarizes 20 runs.

is represented by a vector of rows \( s = (r_1, r_2, \ldots, r_i), \ i \leq n \), where rows are encoded as binary strings, \( s \cdot (r_i = v) \) stands for the extension of partial solution \( s \) by assigning value \( v \) to its \( i \)-th row, and \( \overline{v} \) denotes the reflection value of the binary string \( v \). The hybrid algorithm, Hybrid(\( n, k_{bw}, k_{MA} \)), constructs a search tree, such that its leaves consist of all possible board configurations of size \( n \times n \) that can be generated using solely symmetric rows (this symmetry constraint was imposed to keep the branching factor, \( k_{ext} \), of the BS at a manageable level for the range of instance sizes considered), and internal nodes at level \( i \) represent partially specified (up to the \( i \)-th row) board configurations. This tree is incompletely traversed in a breadth-first way using a BS algorithm with beam width \( k_{bw} \) (i.e., maintaining only the best \( k_{bw} \) nodes at each level of the tree). For the beam selection (line 10), a simple quality measure is defined for partial solutions, whose value is either \( \infty \) if the partial configuration is unstable, or its number of dead cells otherwise. The algorithm starts (line 2) with a totally unspecified solution (i.e., a solution with 0 rows). Initially, only the BS part of the algorithm is executed. During each iteration of the BS (lines 3-17), a new row is added to every solution in the beam (line 7). The interleaved execution of the MA starts only when partial solutions in the beam have at least \( k_{MA} \) rows (line 11). For each iteration of the BS, the best popsize solutions in the beam are selected (using the quality measure described above) to initialize the population of the MA (line 12). Since these are partial solutions, they must be first converted into full solutions, e.g., by completing remaining rows randomly. After running the MA, its solution is used to update the incumbent solution (\( sol \)), and this process is repeated until the search tree is exhausted.

4.2.1 Experimental Results

Experiments were conducted to evaluate the hybrid algorithm (BS-MA-BE). The methodology was the same as Section 4.1.4 (20 executions were performed for each algorithm and instance size), but arities for the MA where in \{2, 3, 4\}. The setting of parame-
Hybrid algorithm for the MDSLP

```
function Hybrid (n, k_{bw}, k_{MA})
1:    sol := ∞
2:    B := { () }
3:    for i := 1 to n do
4:        B' := {} 
5:            for s ∈ B do
6:                for r := 0 to 2^{\lceil n/2 \rceil} − 1 do
7:                    B' := B' ∪ \{s \cdot (r_i = r or \overline{r})\}
8:                end for
9:            end for
10:        B := select best k_{bw} nodes from B'
11:   if (i ≥ k_{MA}) then
12:        initialize MA population with best popsize nodes from B'
13:        run MA
14:        sol := min (sol, MA solution)
15:    end if
16: end for
17: return sol
end function
```

**Figure 9:** Hybrid algorithm for the MDSLP.

ters was $k_{bw} = 2000$ (preliminary tests indicated that this value was reasonable), and $k_{MA} \in \{0.3 \cdot n, 0.5 \cdot n, 0.75 \cdot n\}$, i.e., the best 2000 nodes were kept on each level of the BS algorithm, and 30%, 50% or 75% of the levels of the BS tree were initially descended before starting to run the MA. With respect to termination conditions, each execution of the MA within the hybrid algorithm consisted of 1000 generations, and no time limits were imposed for the hybrid algorithms that were run for $n$ iterations of the BS.

Fig. 10 shows the results for different values of parameter $k_{MA}$. In order to better compare the distributions, the number of optimal solutions obtained by each algorithm (out of 20 executions) is shown above each box plot. For $k_{MA} = 0.3 \cdot n$, the performance of the resulting algorithm improves significantly over the original MA. Note that BS-MA-BE, using an arity of 2 parents, is able to find the optimum for all cases except for $n = 18$ (this instance is solved with $arity = 4$). All distributions for different instance sizes are significantly improved. For $n < 17$ and $arity \in \{2, 3, 4\}$, the algorithm consistently finds the optimum in all runs. For other instances, the solution provided by the algorithm is always within a 1.05% of the optimum, except for $n = 18$, for which the relative distance to the optimum for the worst solution is 1.3%. The other two charts show that, in general, the performance of the algorithm deteriorates with increasing values of the $k_{MA}$ parameter. This may be due to the low quality of the bounds used in the BS part.
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Figure 10: Relative distances to optimum for different arities for BS-MA-BE and $k_{MA} \in \{0.3 \cdot n, 0.5 \cdot n, 0.75 \cdot n\}$, for sizes ranging from 12 up to 20. Each box summarizes 20 runs. The numbers above each box indicate how many times the optimal solution was found.

Regarding execution times, Fig. 11 shows the distributions for the time (in seconds) to reach the best solution needed by the algorithms. Although BS-MA-BE requires more time than MA-BE, the time needed remains reasonable for these instances, and is always less than 2000 seconds. Note also how the execution time increases with the arity, as more time is needed by the MA to perform BE in the crossover operator. On the other hand, execution time decreases for larger values of $k_{MA}$ as the number of executions of the MA decreases, although, as we have already remarked, the quality of the solutions worsens.

To verify that the improved results of the hybrid algorithm were not only a consequence of the extended execution times, experiments for MA-BE were repeated with an increased time limit of 2800 seconds for each execution independently of the instance size. The results of these experiments are shown in Fig. 12. Clearly, the performance of MA-BE does not improve dramatically, and this provides evidence on the synergetic cooperation of BS and MA achieved by the hybrid algorithm.
Figure 11: Time (in seconds) to best solution for different arities for BS-MA-BE and $K_{\text{MA}} \in \{0.3 \cdot n, 0.5 \cdot n, 0.75 \cdot n\}$, for sizes ranging from 12 up to 20. Each box summarizes 20 runs.

5. A New Hybrid Algorithm Based on Mini-Buckets

In this section we present a novel hybrid algorithm based on the algorithm described in Section 4.2. This algorithm exploits the technique of Mini-buckets that is explained in the following.

5.1 Mini-Buckets

The main drawback of BE is that it requires exponential space to store functions extensionally. When this complexity is too high, the solution can be approximated using the technique of mini-buckets (MB) presented by Dechter (1997) (see also Detcher & Rish, 2003). Recall that, in order to eliminate variable $x_i$, with its corresponding bucket $B_i = \{f_1, \ldots, f_m\}$, BE calculates a new cost function

$$g_i = \left( \sum_{f \in B_i} f \right) \downarrow x_i$$  \hspace{1cm} (27)
whose time and space complexity increases with the arity of $g_i$, i.e., with the arity of the set $\bigcup_{f \in B_i} \text{var}(f) - \{x_i\}$. This complexity can be decreased by approximating the function $g_i$ with a set of smaller-arity functions. The basic idea is to partition bucket $B_i$ into $k$ so-called mini-buckets $B_{i1}, \ldots, B_{ik}$, such that the number of variables in the scope of each $B_{ij}$ is bounded by a parameter. Afterwards, a set of $k$ cost functions with the reduced arity sought can be defined as

$$g_{ij} = \left( \sum_{f \in B_{ij}} f \right) \downarrow x_i, j = 1 \ldots k,$$

and the required approximation to $g_i$ can be computed as their sum:

$$g'_i = \sum_{j=1}^{k} g_{ij} = \sum_{j=1}^{k} \left( \left( \sum_{f \in B_{ij}} f \right) \downarrow x_i \right)$$

Note that the minimization computed in $g_i$ by the $\downarrow$ operator has been migrated inside the sum. Since, in general, for any two non-negative functions $f_1(x)$ and $f_2(x)$, $\min_x (f_1(x) + f_2(x)) \geq \min_x f_1(x) + \min_x f_2(x)$, the following inequality holds

$$\left( \sum_{f \in B_i} f \right) \downarrow x_i \geq \sum_{j=1}^{k} \left( \left( \sum_{f \in B_{ij}} f \right) \downarrow x_i \right)$$

and, thus $g'_i$ is a lower bound on $g_i$. Therefore, if variable elimination is performed using approximated cost functions, it provides a lower bound for the optimal cost requiring less computation than BE. Notice that the described approach provides a family of under-estimating heuristic functions whose complexity and accuracy is parameterized by the maximum number of variables allowed in each mini-bucket.

Figure 12: Relative distances to optimum for different arities for MA-BE executed for 2800 seconds, for sizes ranging from 12 up to 20. Each box summarizes 20 runs.
5.2 Improving the Lower Bound Using Mini-Buckets

The simple quality measure for beam selection used in the algorithm in section 4.2 depends solely on the part of the solution that is already constructed. In this section, we will use the MB technique to compute a tight, yet computationally inexpensive, lower bound for the remaining part of the configuration with the aim of improving the performance of the BS part of the hybrid algorithm.

For this purpose, let us note that the MDSLP for an \( n \times n \) board can be formulated as an alternative WCSP, if we associate a different variable \( x_{ij} \) for each cell \((i, j)\) on the board. With this formulation, there are \( n^2 \) cost functions \( f_{ij}, 1 \leq i, j \leq n \). The scope of function \( f_{ij} \) is \( x_{ij} \) and all its neighborhood, and it returns \( \infty \) if the cell \((i, j)\) is unstable, 1 if cell \((i, j)\) is dead, and 0 otherwise. The following objective function

\[
F = \sum_{i=1}^{n} \sum_{j=1}^{n} f_{ij}
\]  

(31)

has to be minimized.

Note that the original formulation, introduced in Section 3.2, can be obtained from the present one by clustering all cost functions corresponding to row \( r_i \) into a single cost function \( f_i \). In the same way, let us cluster cost functions for the \( i \)-th row into \( M \) cost functions, \( f^1_i, f^2_i, \ldots, f^M_i \) of roughly the same arity \((\approx n/M)\), each one evaluating respectively one of the \( M \) segments of the row. To be precise,

\[
f^m_i = \sum_{j=1}^{\sum_{k=1}^{m-1} w_k} f_{ij}, \quad 1 \leq m \leq M
\]  

(32)

where \( w_n, 1 \leq n \leq M \), stands for the number of variables of each segment.

Using this formulation, BE would perform the elimination of all variables corresponding to the last row by computing a new \( g_n \) cost function as

\[
g_n = \left( \sum_{m=1}^{M} f^m_{n-1} + \sum_{m=1}^{M} f^m_{n} \right) \downarrow \{x_{n1}, x_{n2}, \ldots x_{nm}\},
\]  

(33)

whose bucket is \( B_n = \{f^1_{n-1}, f^2_{n-1}, \ldots, f^M_{n-1}, f^1_{n}, f^2_{n}, \ldots, f^M_{n}\} \). Applying mini-buckets, \( B_n \) can be partitioned into \( M \) buckets: \( B^m_n = \{f^m_{n-1}, f^m_{n}\}, 1 \leq m \leq M \), and a set of \( M \) cost functions to approximate \( g_n \) with reduced arity can be calculated as:

\[
g^m_n = (f^m_{n-1} + f^m_{n}) \downarrow x^m_i, \quad 1 \leq m \leq M
\]  

(34)

where

\[
x^1_i = \{x_{i1}, x_{i2}, \ldots, x_{i(1+w_1)}\}
\]  

(35)

\[
x^m_i = \{x_{i(\sum_{j=1}^{m-1} w_j)}, x_{i(1+\sum_{j=1}^{m-1} w_j)}, \ldots, x_{i(1+\sum_{j=1}^{m} w_j)}\}, \quad 1 < m < M
\]  

(36)

\[
x^M_i = \{x_{i(\sum_{j=1}^{M-1} w_j)}, x_{i(1+\sum_{j=1}^{M-1} w_j)}, \ldots, x_{in}\}
\]  

(37)
In this way, the number of variables in each meta-variable $x_i^m$, $1 \leq m \leq M$, is $n/M$ approximately. Because the scopes of $g_n^m$, $1 \leq m \leq M$, are $\{x_{m-2}^n, x_{n-1}^m\}$, their arities are approximately $1/M$ of the arity of $g_n$. The rest of the rows of the board can be processed in a similar way.

Cost functions computed by the function $MB$ can be used to estimate a tight lower bound for a partial solution during the execution of the hybrid algorithm as follows: let $s = (r_1, r_2, \ldots, r_k)$ be a partial solution with $k$ rows for an $n \times n$ instance of the MDSLP. As defined, $g_{k+1}^k(r_{k-1}, r_k)$ returns the cost of the best extension to partial solution $s$ that can be attained in rows $k$ to $n$, considering only the first column. In a similar manner, $g_{m+1}^m(r_{m-1}, r_m)$ can be used to estimate the best extension considering only columns 2 to $n$ respectively. Hence, a lower bound for a partial solution can be computed as:

$$\text{lb}(r_1, r_2, \ldots, r_k) = \sum_{i=1}^{k-1} \sum_{j=1}^{n} f_{ij} + \sum_{m=1}^{n} g_{k+1}^m(r_{k-1}^m, r_k^m),$$

where the first sum corresponds to the part of the solution already assigned. This bound can be used to rank nodes for beam selection and the initialization of the MA population.

In the following subsection, we have experimented with setting $M = 3$, so that

$$(w_1, w_2, w_3) = \begin{cases} (n/3, n/3, n/3), & n \mod 3 = 0 \\ ([n/3], [n/3], [n/3]), & n \mod 3 = 1 \\ ([n/3], [n/3], [n/3]), & n \mod 3 = 2. \end{cases}$$

Observe that, for these settings, the space complexity of function $MB$ is $O(n \times 2^{2([\frac{n}{3}]+2)})$, whereas its time complexity is $O(n^2 \times 2^{3([\frac{n}{3}]+2)})$. When this complexity is still too high, the approach described in this subsection can be utilized to reduce it further, considering more than three clustered cost functions for each row of variables, although the resulting bounds would be less tight.

5.2.1 Experimental Results

Experiments were repeated for the hybrid algorithm equipped with the new lower bound, BS-MA-BE-MB. Fig. 13 shows the results of these experiments for values of $k_{MA} \in \{0.3 \cdot n, 0.5 \cdot n, 0.75 \cdot n\}$. The algorithm finds the optimum for all instances and arities and the relative distance to the optimum for the worst solution found is less than 1.05% in all cases. The best results are obtained with $arity = 4$, although this requires slightly more execution time. Note also how BS-MA-BE-MB is less sensitive to the setting of parameter $k_{MA}$, which means that execution times can be reduced considerably using a large value for this parameter (see Fig. 14). The particular combination of parameters $k_{MA} = 0.75 \cdot n$ and $arity = 4$ provides excellent results at a lower computational cost, as execution times are always below 570 seconds for $n \leq 20$. As a comparison, recall that the only approach in the literature that can solve these instances – described by Larrosa et al. (2005) – requires over 33 minutes for $n = 18$, 15 hours for $n = 19$ and 2 days for $n = 20$, and that other approaches are unaffordable for $n > 15$. Note however that these times correspond to a computational platform different to ours. In order to do a fairer comparison, we executed the algorithm of
Figure 13: Relative distances to optimum for different arities for BS-MA-BE-MB and \( k_{MA} \in \{0.3 \cdot n, 0.5 \cdot n, 0.75 \cdot n\} \), for sizes ranging from 12 up to 20. Each box summarizes 20 runs.

Figure 14: Time (in seconds) to best solution for different arities for BS-MA-BE-MB and \( k_{MA} = 0.75 \cdot n \), for sizes ranging from 12 up to 20. Each box summarizes 20 runs.

Larrosa et al.\textsuperscript{5} in our platform. In this case, it required 1867 seconds (i.e., more than 31 minutes) in order to solve the \( n = 18 \) instance, and more than 1 day and 18 hours to solve the \( n = 20 \) instance. These values are very close to the times reported by Larrosa et al. (2005), and hence indicate that the computational platforms are fairly comparable.

\textsuperscript{5} Available at \url{http://www.lsi.upc.edu/~larrosa/publications/LIFE-SOURCE-CODE.tar.gz}. Time for \( n = 19 \) could not be obtained as the code provided by Larrosa et al. can only be used with even sized instances.
Figure 15: Relative distances to best known solutions for different arities for BS-MA-BE-MB and $k_{MA} = 0.3 \cdot n$, for very large instances (i.e., sizes of 22, 24, 26, and 28). Each box summarizes 20 runs. Note the improvement of best known solutions for sizes 24 and 26.

Figure 16: New best known maximum density still lifes for $n \in \{24, 26\}$.

Table 2: Optimal solutions for the SMDLP.

| $n$ | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 |
|-----|----|----|----|----|----|----|----|----|----|
| opt | 68 | 79 | 92 | 106| 120| 137| 154| 172| 192|

| $n$ | 22 | 24 | 26 | 28 |
|-----|----|----|----|----|
| opt | 232| 276| 326| 378|

5.3 Results on Very Large Instances

As already mentioned, there is currently no approach available to tackle the MDSLSP for $n > 20$. Larrosa et al. (2005) tried their algorithm for $n = 21$ and $n = 22$, but they could
not solve any of those instances within a week of CPU). For these very large instances, only solutions to some relaxations of the problem are known. One of these relaxations, known as the symmetrical maximum density still life problem (SMDSLP), was proposed in (Bosch & Trick, 2002), and consists of considering only symmetric boards (either horizontally or vertically) which reduces the search space from $2^{2n^2}$ to $2^{n[n/2]}$.

The optimized version of BE algorithm (Section 2.4) can be used find vertically symmetric still lifes, by defining as the variable domain, a set that contains only symmetric values for rows,

$$D = \{r \text{ or } \overline{r} \mid r \in \{0 . . 2^{[n/2]} - 1\}\}.$$  

(40)

Larrosa and Morancho (2003) and Larrosa et al. (2005) used this algorithm to solve the SMDSLP for the instances considered so far in this paper (i.e., for $n \in \{12 . . 20\}$), as well as for very large instances (i.e., $n \in \{22, 24, 26, 28\}$). Results are summarized in Table 2, which shows for each instance size the optimal symmetrical solution (as the number of dead cells). Clearly, the costs of optimal symmetric still lifes are upper bounds for the MDSLP, that can additionally be observed to be very tight for $n \leq 20$. Results for $n > 20$ are currently the best known solutions for these instances.

We also run our algorithm (BS-MA-BE-MB) for these very large instances (i.e., $n \in \{22, 24, 26, 28\}$), and compare our results to symmetrical solutions for these instances. Results (displayed in Fig. 15 shows that our algorithm was able to find two new best known solutions for the MDSLP, namely for $n = 24$ and $n = 26$. There are 275 and 324 dead cells respectively in the new solutions. These solutions are pictured in Fig. 16. Incidentally, our algorithm could also find a solution with 325 dead cells for the $n = 26$ instance. For the other instances, our algorithm could reach the best known solutions consistently. Let us note that the computation of mini-Buckets for these very large instances was done by considering four clustered costs functions for variables in each row of the board, as the complexity when using three costs functions was still too high.

6. Conclusions

The MDSLP represents an excellent example of WCSP; its highly constrained nature is typical in many optimization scenarios. Furthermore, the algorithmic hardness of solving this problem illustrates the limitations of classical optimization approaches. For this reason, it is not surprising that this problem has attracted the interest of the constraint-programming community, and has been central in the development and assessment of sophisticated techniques such as bucket elimination (BE). However, the high space complexity of BE as an exact technique (Dechter, 1999), makes this approach impractical for large instances. In this work, we have presented several proposals for the hybridization of BE with memetic algorithms and beam search (BS), and showed that they represent very promising models. The experimental results have been very positive, solving to optimality large instances of the MDSLP. We have also studied the influence that multi-parent recombination have on the performance of the algorithm. The results indicate multi-parent recombination can help to improve the results obtained by previous approaches.

Among all our proposals, we must distinguish a new algorithm resulting from the hybridization, at different levels, of complete solving techniques (i.e., bucket elimination), incomplete deterministic methods (i.e., beam search and mini-buckets) and stochastic algo-
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algorithms (i.e., memetic algorithms), that empirically produces good-quality results, not only solving to optimality very large instances of the constrained problem in a relatively short time, but also providing new best known solutions in some large instances. This algorithm exploits the technique of the mini-buckets to compute tight yet computationally inexpensive lower bounds of the partial solutions that are considered in the BS part of the hybrid algorithm.

As future work, we plan to consider complete versions of the hybrid algorithm. This involves the use of adequate data structures to store not yet considered but promising branch-and-bound nodes. While the memory requirements will of course grow enormously with the size of the problem instance considered, it will be interesting to analyze the computational tradeoffs of the algorithm as an anytime technique.

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