D-affinity and Frobenius morphism on quadrics

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ABSTRACT: We compute decomposition of Frobenius push-forwards of line bundles on quadrics into a direct sum of line bundles and spinor bundles. As an application we show when the Frobenius push-forward gives a tilting bundle and we apply it to study D-modules on quadrics.

Introduction

Let $X$ be a smooth projective variety. A coherent sheaf $E \in \text{Coh}_X$ is called quasi-exceptional if $\text{Ext}^i(E,E) = 0$ for all $i > 0$. A coherent sheaf $E \in \text{Coh}_X$ is called tilting if it is quasi-exceptional, $E$ Karoubian generates the derived category $D^b(X)$ and the algebra $\text{Hom}_X(E,E)$ has finite global dimension.

Let $Q_n$ be an $n$-dimensional quadric defined over an algebraically closed field $k$ of characteristic $p > 2$. Assume for simplicity that $n \geq 3$. Let $F: Q_n \to Q_n$ be the absolute Frobenius morphism and let $F^s$ be the composition of $s$ absolute Frobenius morphisms.

THEOREM 0.1. Let $s$ be a positive integer. Then $F^*_s \mathcal{O}_{Q_n}$ is a tilting bundle if and only if one of the following holds:

1. $s = 1$ and $p > n$,
2. $s = 2$, $n = 4$ and $p = 3$,
3. $s \geq 2$, $n$ is odd and $p \geq n$.

The above theorem is a summary of Corollaries 4.6, 4.7 and 4.8. In case of $s = 1$ and $n = 3, 4$ the theorem (see also Corollary 4.3) gives the main result of [Sa1]. In case of $s = 1$ A. Samokhin in [Sa2] proved a related result for $p \gg 0$ but using a completely different method.

In fact, we prove a much stronger result than stated above: we determine the decomposition of Frobenius push-forwards of line bundles on quadrics. In case of projective spaces one can easily compute the corresponding decomposition using the Horrocks splitting criterion (see, e.g.,
We use a similar strategy in the case of quadrics although it is much more difficult in carrying out as we need to prove some non-trivial vanishing and non-vanishing theorems for cohomology of Frobenius pull-backs of spinor bundles.

Let $X$ be a smooth projective variety defined over an algebraically closed field $k$ of characteristic $p > 0$. Let $X^{(s)}$ denote the $s$-th Frobenius twist of $X$, i.e., $X$ with the $k$-structure defined by the fiber product of $X$ over the $s$-th Frobenius morphism on $k$. Then the sheaf $\mathcal{D}_X$ of $k$-algebras of differential operators admits the so called $p$-filtration

$$
\mathcal{D}_X^1 \subset \mathcal{D}_X^2 \subset \ldots \subset \mathcal{D}_X^s = \text{Hom}_{X^{(s)}}(\mathcal{O}_X, \mathcal{O}_X) \subset \ldots \subset \mathcal{D}_X = \lim_{\leftarrow s} \mathcal{D}_X^s.
$$

Let us also set $\mathcal{D}_X^s = \text{Hom}_{X^{(s)}}(\mathcal{O}_X, \mathcal{O}_X)$. By definition $\mathcal{D}_X^s$ is an algebra, finite dimensional over $k$. By [HKR, Proposition 3.4] the above theorem implies the following corollary:

**Corollary 0.2.** If one of the conditions 1, 2, 3 of Theorem 0.1 holds then there is a canonical triangulated equivalence between the bounded derived category $D^b(\text{Coh}(\mathcal{D}_Q^n))$ of coherent $\mathcal{D}_Q^n$-modules and the bounded derived category $D^b(\mathcal{D}_Q^n - \text{mod})$ of $\mathcal{D}_Q^n$-modules.

This is an analogue of the derived localization theorem of Beilinson and Bernstein. Note that contrary to some expectations this equivalence does not hold in any characteristic if $n$ is even and $s \geq 3$.

Let us recall that a variety $X$ is called $D$-quasi-affine if any $\mathcal{O}_X$-quasi-coherent $\mathcal{D}_X$-module is $\mathcal{D}_X$-generated by its global sections. $X$ is $D$-affine if it is $D$-quasi-affine and $H^i(\mathcal{D}_Q^n) = 0$ for $i > 0$.

In characteristic zero Beilinson and Bernstein (see [BB]) proved that flag varieties for semisimple algebraic groups are $D$-affine. In [Ha] Haastert proved that in positive characteristic the projective space and the full flag variety for $\text{SL}_3$ are $D$-affine but no other examples of smooth projective $D$-affine varieties are known. We prove that all quadrics are $D$-quasi-affine (see Proposition 1.7). Together with Theorem 0.1 this implies the following corollary:

**Corollary 0.3.** If $n$ is odd and $p \geq n$ then $Q_n$ is $D$-affine.

Let us note that contrary to [AK, Proposition], non-vanishing of $H^i(\mathcal{D}_Q^n)$ for even $n$ and $s \geq 3$ does not imply that $H^i(\mathcal{D}_Q^n) \neq 0$ (there is an error in the last two lines of proof of [AK, Proposition] as the natural inclusions $\mathcal{D}^s \to \mathcal{D}^r$ do not agree with the maps from [AK, Lemma]). In particular, our results do not immediately imply that other quadrics are not $D$-affine.

We do not consider the case of characteristic 2 although our methods apply also in this case. We skip it as this case needs a separate, quite long treatment. Let us just note that on the $3$-dimensional quadric already $F^4_i \mathcal{O}_Q$ is not quasi-exceptional (but $F^i_i \mathcal{O}_Q$ is quasi-exceptional if $i \leq 3$).

The structure of the paper is as follows. In Section 1 we give a few interpretations of spinor bundles, compute some related cohomology and show some dualities. In Section 2 we study
Hilbert functions of some algebras that occur to describe decompositions of Frobenius push-forwards of line bundles on a quadric. In Section 3 we use these results to prove some vanishing and non-vanishing theorems for Frobenius pull-backs of spinor bundles. In Section 4 we use them for computation of decompositions of Frobenius push-forwards of line bundles on a quadric and we prove Theorem 1.1.

1 Spinor bundles

1.1 Spinor bundles via representation theory

Let $k$ be an algebraically closed field of characteristic $p > 2$. Let us recall that Spin$(n+2)$, $n \geq 1$, is a connected, semi-simple, simply connected $k$-group, isogenous to SO$(n+2)$.

We fix a maximal torus $T \subset$ Spin$(n+2)$. If $n = 2m + 1$ or $n = 2m$ then Spin$(n+2)$ has $m + 1$ simple roots $\alpha_1, \ldots, \alpha_{m+1}$. Let us recall that a smooth $n$-dimensional quadric $Q_n \subset \mathbb{P}^N$, $N = n + 1$, is a homogeneous space Spin$(n+2)/P(\alpha_1)$.

If $n = 2m + 1$ then the Dynkin diagram of Spin$(n+2)$ is of type $B_{m+1}$. If $n = 2m$ then the Dynkin diagram is of type $A_1 \times A_1$ for $m = 1$, $A_3$ for $m = 2$ and $D_{m+1}$ if $m \geq 3$.

Let $\lambda_t \in X^*(T)$ be the fundamental weights defined by $2\langle \lambda_t, \alpha_j \rangle / \langle \alpha_i, \alpha_j \rangle = \delta_{ij}$.

For $n = 2m$ there are two spin representations of the Levi quotient of $P(\alpha_1)$ (which is of the same type as Spin$(n)$) with highest weight $\lambda_m$ and $\lambda_{m+1}$. They are of dimension $2^{m-1}$. If $n = 2m + 1$ then there is one spin representation of the Levi quotient of $P(\alpha_1)$ with highest weight $\lambda_{m+1}$. It is of dimension $2^m$.

Duals of vector bundles on $Q_n$ associated to the principal $P(\alpha_1)$-bundle Spin$(n+2) \to$ Spin$(n+2)/P(\alpha_1)$ via these representations are called spinor bundles and denoted by $\Sigma$ if $n$ is odd and $\Sigma_-$ and $\Sigma_+$ if $n$ is even. The determinant of any spinor bundle on $Q_n$, $n \geq 3$ is isomorphic to $\mathcal{O}_{Q_n}(-2\lceil \frac{m+1}{2} \rceil)$. Let us note the following useful isomorphisms:

\[
\begin{align*}
\Sigma^* &\simeq \Sigma(1) & \text{if } n = 2m + 1, \\
\Sigma_-^* &\simeq \Sigma_-(1) & \text{if } n = 4m, \\
\Sigma_+^* &\simeq \Sigma_+(1) & \text{if } n = 4m + 2.
\end{align*}
\]

We leave it to the reader to verify these isomorphisms using the well known computation of the centre of the Spin group.

Let us recall that a vector bundle $E$ on a smooth $n$-dimensional hypersurface $X = (f = 0) \subset \mathbb{P}^{n+1}$ = Proj $S$ is called arithmetically Cohen-Macaulay (ACM for short), if it has no intermediate cohomology, i.e., $H^i(X, E(t)) = 0$ for $0 < i < n$ and all $t$. A vector bundle $E$ is ACM if and only if the corresponding graded $S/(f)$-module is maximal Cohen-Macaulay.

Let us also recall that a vector bundle is called strongly slope (semi)stable if all its Frobenius pull-backs are slope (semi)stable.

**Theorem 1.1.** A spinor bundle $\Sigma$ on $Q_n \subset \mathbb{P}^N$ is a strongly slope stable ACM bundle. Moreover, $h^0(Q_n, \Sigma(t)) = 0$ for $t \leq 0$ and $h^0(Q_n, \Sigma(t)) = 2 \lceil \frac{N}{2} \rceil (t+n-1)$ for $t \geq 1$. 




Proof. Let us first note that spin representations are irreducible (this follows from [Ja, Part II, Corollary 5.6]). Therefore by [Bi, Theorem 2.1] spinor bundles are slope stable. Strong slope stability follows in a standard way from inequality \( \mu_{\max}(\Omega_{Q_n}) < 0 \).

The fact that spinor bundles are ACM can be proven in the same way as [Ot, Theorem 2.3]. The last part of the theorem will be obvious later (use sequences (1), (2) and (3)).

### 1.2 Spinor bundles via matrix factorization

Theorem 1.1 implies that spinor bundles correspond to irreducible maximal Cohen–Macaulay modules on an affine cone over the quadric (or equivalently to indecomposable matrix factorizations of the equation of the quadric). Below we give an explicit construction of spinor bundles using matrix factorization.

As a special case of Knörrer’s periodicity theorem (see [Kn, Theorem 3.1]) we get the following theorem:

**THEOREM 1.2.** Any ACM bundle on a smooth projective quadric defined over an algebraically closed field of characteristic \( p \neq 2 \) is a direct sum of line bundles and twisted spinor bundles.

If \( n \leq 2 \) then the Picard group is not generated by \( \mathcal{O}_{Q_n}(1) \) and any ACM bundle on \( Q_n \) is isomorphic to a direct sum of line bundles \( \mathcal{O}_{Q_n}(i) \) and spinor bundles twisted by some \( \mathcal{O}_{Q_n}(i) \). If \( n \geq 3 \) then any direct sum of line bundles and twisted spinor bundles is ACM.

Let us set \( \varphi_0 = \psi_0 = (x_0) \) and let us define inductively pairs of matrices

\[
\varphi_{m+1} = \left( \begin{array}{cc} \varphi_m & x_{2m+1}I_{2^m \times 2^m} \\ x_{2m+2}I_{2^m \times 2^m} & -\psi_m \end{array} \right)
\]

and

\[
\psi_{m+1} = \left( \begin{array}{cc} \psi_m & x_{2m+1}I_{2^m \times 2^m} \\ x_{2m+2}I_{2^m \times 2^m} & -\varphi_m \end{array} \right).
\]

Then the pair \((\varphi_m, \psi_m)\) is a matrix factorization of \( x_0^2 + x_1x_2 + \ldots + x_{2m-1}x_{2m} \).

Let us set \( \varphi_0' = (x_1), \psi_0' = (x_2) \) and as above define inductively pairs of matrices

\[
\varphi_{m+1} = \left( \begin{array}{cc} \varphi_m' & x_{2m+1}I_{2^m \times 2^m} \\ x_{2m+2}I_{2^m \times 2^m} & -\psi_m' \end{array} \right)
\]

and

\[
\psi_{m+1} = \left( \begin{array}{cc} \psi_m' & x_{2m+1}I_{2^m \times 2^m} \\ x_{2m+2}I_{2^m \times 2^m} & -\varphi_m' \end{array} \right).
\]

Then the pair \((\varphi_m', \psi_m')\) is a matrix factorization of \( x_1x_2 + \ldots + x_{2m-1}x_{2m} \).

Let \( i : Q_n \hookrightarrow \mathbb{P}^N \) be the above defined embedding of a quadric \((n = 2m \text{ or } n = 2m + 1)\). Then we have the following short exact sequences of sheaves on \( \mathbb{P}^N \):

\[
0 \to \mathcal{O}_{\mathbb{P}^N}^{2m+1} (-1) \to \mathcal{O}_{\mathbb{P}^N}^{2m+1} \to i_*\Sigma(1) \to 0,
\]

(1)
if \( n = 2m + 1 \), and
\[
0 \to \mathcal{O}_{P_N}^{2m}(-1)^{\varphi_m} \to \mathcal{O}_{P_N}^{2m} \to i_* \Sigma_-(1) \to 0, \tag{2}
\]
\[
0 \to \mathcal{O}_{P_N}^{2m}(-1)^{\varphi_m} \to \mathcal{O}_{P_N}^{2m} \to i_* \Sigma_+(1) \to 0. \tag{3}
\]
if \( n = 2m \).
Using the above description we get the following short exact sequences of vector bundles:
\[
0 \to \Sigma \to \mathcal{O}_{Q_n}^{2m+1} \to \Sigma(1) \to 0, \tag{4}
\]

if \( n = 2m + 1 \), and
\[
0 \to \Sigma_+ \to \mathcal{O}_{Q_n}^{2m} \to \Sigma_+(1) \to 0, \tag{5}
\]
\[
0 \to \Sigma_- \to \mathcal{O}_{Q_n}^{2m} \to \Sigma_-(1) \to 0. \tag{6}
\]
if \( n = 2m \). It should be noted that the above explicit presentations allow for computer calculations of cohomology groups of Frobenius pull backs of spinor bundles.

**Lemma 1.3.** For any spinor bundles \( \Sigma_1, \Sigma_2 \) on \( Q_n \) we have \( H^1(Q_n, \Sigma_1 \otimes \Sigma_2(t)) = 0 \) if \( t \neq 0 \). Moreover,
\[
h^1(Q_n, \Sigma_1 \otimes \Sigma_2) = \begin{cases} 
1 & \text{if } n = 2m + 1, \\
1 & \text{if } n = 4m \text{ and } \Sigma_1 \not\simeq \Sigma_2, \\
0 & \text{if } n = 4m \text{ and } \Sigma_1 \simeq \Sigma_2, \\
1 & \text{if } n = 4m + 2 \text{ and } \Sigma_1 \simeq \Sigma_2, \\
0 & \text{if } n = 4m + 2 \text{ and } \Sigma_1 \not\simeq \Sigma_2.
\end{cases}
\]

In particular, for any \( 0 < i < n \) there exist spinor bundles \( \Sigma_1, \Sigma_2 \) such that \( \text{Ext}^i(\Sigma_1(i), \Sigma_2) \neq 0 \).

**Proof.** As spinor bundles on a quadric are strongly stable we see that \( \text{Hom}(\Sigma_1, \Sigma_2(t)) = 0 \) if \( t < 0 \) or if \( t = 0 \) and \( \Sigma_1 \) and \( \Sigma_2 \) are not isomorphic. This remark, together with sequences (4), (5), (6), imply the second part of the lemma.

To prove the first assertion note that by Lemma 1.6 for \( s = 0 \) there exist spinor bundles \( \Sigma_1, \Sigma_2 \) such that
\[
H^1(Q_n, \Sigma_1 \otimes \Sigma_2(t)) \simeq H^1(Q_n, \Sigma_1 \otimes \Sigma_2(-t))^s.
\]
So it is sufficient to note that by Theorem 1.1 sequences (4), (5), (6) imply that \( H^1(\Sigma_1 \otimes \Sigma_2(t)) \) is a quotient of \( H^0(\Sigma'_1 \otimes \Sigma_2(t + 1)) \) for some spinor bundle \( \Sigma'_1 \). This last cohomology group vanishes for \( t < 0 \) as we can write it as \( \text{Hom}(\Sigma''_2, \Sigma_2(t)) \) for some spinor bundle \( \Sigma''_2 \).

The last part of the lemma follows from isomorphisms
\[
\text{Ext}^i(\Sigma_1(i), \Sigma_2) \simeq H^i(\Sigma'_1 \otimes \Sigma_2(-i)) \simeq H^i(\Sigma'_1 \otimes \Sigma_2)
\]
for some spinor bundle \( \Sigma'_1 \).

**Corollary 1.4.** Let \( E \) be an ACM bundle on \( Q_n \) and let \( 0 < i < n \) be a fixed integer. If for all spinor bundles \( \Sigma \) on \( Q_n \) we have \( H^i(Q_n, E \otimes \Sigma(t)) = 0 \) for all \( t \in \mathbb{Z} \) then \( E \) is a direct sum of line bundles.

**Proof.** Using sequences (4), (5), (6) and Theorem 1.1 we see that \( H^i(E \otimes \Sigma(t)) = H^1(E \otimes \Sigma'(t + i - 1)) \) for some spinor bundle \( \Sigma' \). Then the required assertion follows from Theorem 1.2 and Lemma 1.3. \( \square \)
1.3 Some dualities

For a non-negative integer \( s \) let us set \( q = p^s \) and \( d_{n,s} = (n - 1) \frac{q-1}{2} \).

**Lemma 1.5.** Let \( \Sigma \) be a spinor bundle on \( Q_n \). Then for all \( 0 < i < n \) there exists some spinor bundle \( \tilde{\Sigma} \) such that for all integers \( j \) we have the following duality

\[
H^i(Q_n, (F^s)^* \Sigma (d_{n,s} - (i - 1)q + j)) \simeq H^i(Q_n, (F^s)^* \tilde{\Sigma} (d_{n,s} - (i - 1)q - j - 1))^*.
\]

**Proof.** Let us first prove the lemma for \( i = 1 \). Taking Frobenius pull backs of sequences (4), (5), (6) and twisting by \( \mathcal{O}_{Q_n}(j) \) we get the following isomorphisms

\[
H^1(Q_n, (F^s)^* \Sigma (q + j)) \simeq H^{i+1}(Q_n, (F^s)^* \Sigma_1(j)),
\]

for some spinor bundle \( \Sigma_1 \), \( 0 < i < n - 1 \) and all integers \( j \). Hence we get

\[
H^1(Q_n, (F^s)^* \Sigma_1(j)) \simeq H^2(Q_n, (F^s)^* \Sigma_1(j - q)) \simeq \ldots \simeq H^{n-1}(Q_n, (F^s)^* \Sigma_{n-2}(j - (n - 2)q))
\]

for some spinor bundles \( \Sigma_1, \ldots, \Sigma_{n-2} \). Now using the Serre duality we have

\[
H^{n-1}(Q_n, (F^s)^* \Sigma_{n-2}(j - (n - 2)q)) \simeq H^1(Q_n, (F^s)^* \tilde{\Sigma}((n - 1)(q - 1) - j - 1))^*
\]

for some spinor bundle \( \tilde{\Sigma} \), which proves the lemma for \( i = 1 \).

In general, there exist some spinor bundles \( \Sigma_1, \tilde{\Sigma}_1 \) and \( \tilde{\Sigma} \) such that

\[
H^i(Q_n, (F^s)^* \Sigma(j)) \simeq H^1(Q_n, (F^s)^* \Sigma_1(j + (i - 1)q)) \simeq H^1(Q_n, (F^s)^* \tilde{\Sigma}_1(2d_{n,s} - (i - 1)q - j - 1))^*
\]

\[
\simeq H^1(Q_n, (F^s)^* \tilde{\Sigma}(2d_{n,s} - 2(i - 1)q - j - 1))^*.
\]

\[\square\]

From the proof of the lemma it is clear that we can easily determine dependence of \( \tilde{\Sigma} \) on \( \Sigma \) but we need to consider some cases depending on \( n \pmod{4} \). More precisely, \( \tilde{\Sigma} = \Sigma \) if \( n \) is odd or \( n \) is divisible by 4 and \( \tilde{\Sigma} \) is the opposite spinor bundle otherwise (at least for \( i = 1 \)).

Similarly as above we have the following duality:

**Lemma 1.6.** Let \( \Sigma_1 \) and \( \Sigma_2 \) be spinor bundles on \( Q_n \) (possibly equal). Then for all \( 0 < i < n \) there exists some spinor bundles \( \tilde{\Sigma}_1 \) and \( \tilde{\Sigma}_2 \) such that for all integers \( j \) we have the following duality

\[
H^i(Q_n, \Sigma_1 \otimes (F^s)^* \Sigma_2(d_{n,s} - (i - 1)q + j)) \simeq H^i(Q_n, \tilde{\Sigma}_1 \otimes (F^s)^* \tilde{\Sigma}_2(d_{n,s} - (i - 1)q - j))^*.
\]

**Proof.** The proof of the lemma is similar to that of Lemma [1,5] and so we just sketch it for \( i = 1 \). We can easily show the following isomorphisms

\[
H^1(Q_n, \Sigma_1 \otimes (F^s)^* \Sigma_2(j)) \simeq \ldots \simeq H^{n-1}(Q_n, \Sigma_1' \otimes (F^s)^* \Sigma_2'(j - (n - 2)q))
\]

for some spinor bundles. Finally, using the Serre duality we have

\[
H^{n-1}(Q_n, \Sigma_1' \otimes (F^s)^* \Sigma_2'(j - (n - 2)q)) \simeq H^1(Q_n, \tilde{\Sigma}_1 \otimes (F^s)^* \tilde{\Sigma}_2((n - 1)(q - 1) - j))^*.
\]

\[\square\]

Similarly as above one can easily find \( \tilde{\Sigma}_1 \) and \( \tilde{\Sigma}_2 \) corresponding to \( \Sigma_1 \) and \( \Sigma_2 \).
1.4 D-quasi-affinity of quadrics

Let $L$ be a line bundle on a smooth projective variety $X$ and let $\mathcal{D}_X(L)$ be the $\mathcal{O}_X$-bimodule of differential operators from $L$ to $L$ (see [Ha, 1.1]). $X$ is called $D(L)$-quasi-affine if any $\mathcal{O}_X$-quasi-coherent $\mathcal{D}_X(L)$-module is $\mathcal{D}_X(L)$-generated by its global sections. We say that $X$ is $D(L)$-affine if it is $D(L)$-quasi-affine and for any $\mathcal{O}_X$-quasi-coherent $\mathcal{D}_X(L)$-module $\mathcal{M}$ we have $H^i(X, \mathcal{M}) = 0$ for $i > 0$.

**Proposition 1.7.** Let $j$ be a non-negative integer. Then any smooth projective quadric $Q_n$ is $D(\mathcal{O}_{Q_n}(j))$-quasi-affine.

**Proof.** The proof is analogous to the proof of [Ha, Satz 3.1]. By [Ha, Proposition 2.3.3] it is sufficient to show that for any integer $t$ the module $\mathcal{D}(\mathcal{O}_{Q_n}(j)) \otimes \mathcal{O}_{Q_n}(-t)$ is $\mathcal{D}(\mathcal{O}_{Q_n}(j))$-generated by its global sections. Therefore by [Ha, Proposition 2.3.4] it is sufficient to show that for any positive integer $t$ there exists $s_0$ such that for every $s \geq s_0$ $(F^s_*(\mathcal{O}_{Q_n}(j + t)))^*$ is globally generated as an $\mathcal{O}_{Q_n}$-module. Let us take as $s_0$ any integer such that $p^{s_0} > j + t$. By Theorem 1.2 we can write

$$F^s_*(\mathcal{O}_{Q_n}(j + t)) \simeq \bigoplus \mathcal{O}_{Q_n}(a_i) \oplus \bigoplus \Sigma_i(b_i)$$

for some spinor bundles $\Sigma_i$ and some integers $a_i$ and $b_i$. Hence we need to show that all $a_i$ and $b_i$ are non-negative.

Note that $\mathcal{O}_{Q_n}(a_i) \hookrightarrow F^s_*(\mathcal{O}_{Q_n}(j + t))$ gives rise to a non-zero map $(F^s)^*(\mathcal{O}_{Q_n}(a_i)) = \mathcal{O}_{Q_n}(p^s a_i) \hookrightarrow \mathcal{O}_{Q_n}(j + t) \subset \mathcal{O}_{Q_n}(p^{s_0} - 1)$, so $a_i \leq 0$. Similarly, $\Sigma_i(b_i) \hookrightarrow F^s_*(\mathcal{O}_{Q_n}(j + t))$ gives rise to a non-zero map $(F^s)^*\Sigma_i(p^s b_i) \hookrightarrow \mathcal{O}_{Q_n}(j + t) \subset \mathcal{O}_{Q_n}(p^{s_0} - 1)$. This gives rise to a section of $(F^s)^*\Sigma_i(p^{s_0} - 1 - p^s b_i)$. Now it is sufficient to show that $H^0((F^s)^*\Sigma_i(-1)) = 0$ as then $b_i \leq 0$. But this can be easily shown by restricting to quadrics of lower dimension and induction on the dimension $n$ (cf. the proof of Theorem 3.3). \qed

2 Hilbert functions of some algebras

In this section we study Hilbert functions of some finite dimensional algebras in positive characteristic. Their geometric meaning will become clear in Sections 3 and 4 (see the proof of Corollary 3.3 and Theorem 4.4).

**Proposition 2.1.** Let $k$ be a field of characteristic $p > 2$. Let $0 \leq e < p$ be an integer. Then for any $d \leq (N + 1) \cdot \frac{p^e - 1}{p - 1} - e$ the $d$th grading of the ideal $(x_0^p, \ldots, x_N^p) : (\sum_{i=0}^N x_i^2)^e$ of $k[x_0, \ldots, x_N]$ is contained in $(x_0^p, \ldots, x_N^p, \sum_{i=0}^N x_i^2)^d$.

**Proof.** We need to prove that if for some homogeneous polynomial $h \in k[x_0, \ldots, x_N]_d$ with $d + e \leq (N + 1) \cdot \frac{p^e - 1}{p - 1}$ we have

$$\left(\sum_{i=0}^N x_i^2\right)^e h = \sum_{i=0}^N g_i x_i^p$$

(7)
for some homogeneous polynomials \( g_i \in k[x_0, \ldots, x_N] \) then
\[
h \in (x_0^p, \ldots, x_N^p, \sum_{i=0}^N x_i^2).
\]

We can assume that \( h = h_0 + h_1 x_0 + \ldots + h_{p-1} x_0^{p-1} \), where \( h_0, \ldots, h_{p-1} \) are polynomials in \( x_1, \ldots, x_N \). Moreover, we can also assume that all \( h_i \)'s are of degree less than \( p \) in each variable. Let us set \( y = \sum_{i=1}^N x_i^2 \). To simplify exposition we divide the proof into two cases: first we deal with the “even” part of \( h \) and later we deal with the “odd” part of \( h \).

Let us set
\[
W_{2j} = \binom{e}{j} h_0 + \binom{e}{j-1} y h_2 + \ldots + \binom{e}{0} y^j h_{2j}
\]
for \( j = 0, \ldots, \frac{p-1}{2} \). Note that in \( \sum_{i=0}^N g_i x_0^i \) treated as a polynomial in \( x_0 \), coefficients at \( x_i^j \) for \( i < p \) are in the ideal \( (x_1^p, \ldots, x_N^p) \). Comparing coefficients in (7) at even powers of \( x_0 \) we get therefore the following equalities:
\[
y^{e-j} W_{2j} \equiv 0 \pmod{(x_1^p, \ldots, x_N^p)}
\]
for \( j = 0, \ldots, \frac{p-1}{2} \) (note that if \( j > e \) then all the terms of \( W_{2j} \) are divisible by \( y^{j-e} \) and we still have a polynomial on the left hand side of the above equality).

**Lemma 2.2.** We have
\[
y^j h_{2j} = (-1)^j \left( \frac{e + j - 1}{e - 1} \right) W_0 + (-1)^{j-1} \left( \frac{e + j - 2}{e - 1} \right) W_2 + \ldots + (-1)^0 \left( \frac{e - 1}{e - 1} \right) W_{2j}.
\]

**Proof.** Let \( a_0, \ldots, a_{p-1} \) be arbitrary elements in a fixed ring \( R \). Consider the following equality of formal power series in \( R[[x]] \):
\[
a_0 + a_1 x + \ldots + a_{p-1} x^{p-1} = (1 + x^2)^{-e} \left( (1 + x^2)^e (a_0 + a_1 x + \ldots + a_{p-1} x^{p-1}) \right).
\]
Now use expansion
\[
(1 + x^2)^{-e} = \sum_{j \geq 0} \binom{-e}{j} x^{2j} = \sum_{j \geq 0} (-1)^j \binom{e + j - 1}{j} x^{2j}
\]
and multiply this by expansion of the product \( (1 + x^2)^e (a_0 + a_1 x + \ldots + a_{p-1} x^{p-1}) \). Then comparing appropriate coefficients of the product we compute \( a_j \). The assertion in the lemma is a special case of such equality when comparing coefficients at even powers of \( x \).

**Lemma 2.3.** If \( j \geq e - \frac{p-1}{2} \) then \( W_{2j} \) is a linear combination of \( W_{2j+2}, \ldots, W_{p-1} \) and \( y^{e-j} h_{p-1-2j}, \ldots, y^{e-j} h_{p-1} \).
Proof. Set \( m = \frac{p - 1}{2} - j \). By Lemma 2.2 we have
\[
\sum_{k \leq j} (-1)^k \left( \frac{e + s - 1 - k}{e - 1} \right) W_{2k} = y^s h_s - \sum_{k > j} (-1)^k \left( \frac{e + s - 1 - k}{e - 1} \right) W_{2k}
\]
for \( s = m, \ldots, m + j \). The determinant of this linear system of equations is up to sign equal to
\[
\begin{vmatrix}
\left( \frac{e - 1 + m}{e - 1} \right) & \cdots & \left( \frac{e - 1 + m - j}{e - 1} \right) \\
\vdots & \ddots & \vdots \\
\left( \frac{e - 1 + m + j}{e - 1} \right) & \cdots & \left( \frac{e - 1 + m}{e - 1} \right)
\end{vmatrix}
\]
where the first equality follows by inductive subtracting columns, and the second equality follows by inductive subtracting rows. The last determinant can be computed and by (IT) it is equal to
\[
\prod_{i=0}^{j} \frac{(e - j + m - 1 + i)! i!}{(m - 1 + i)! (e - i)!}.
\]
It is non-zero when \( e + m - 1 \leq p - 1 \), which is equivalent to \( j \geq e - \frac{p + 1}{2} \). In this case we can solve the above linear system of equations obtaining the required assertion. \( \square \)

**Lemma 2.4.** If \( e - \frac{p - 1}{2} \leq j \leq \frac{p - 1}{2} \) then there exists a polynomial \( W'_{2j} \) such that \( W_{2j} - y^{p - e + j} W'_{2j} \) lies in the ideal \((x_1^p, \ldots, x_N^p)\).

**Proof.** Assume that the assertion of the lemma is false and choose the largest \( j \leq \frac{b - 1}{2} \) for which it fails. If \( j \geq e \) then we get a contradiction with (8) as we know that \( W_{2j} \equiv 0 \pmod{(x_1^p, \ldots, x_N^p)} \). Therefore \( j < e \). By assumption \( j \geq e - \frac{p - 1}{2} \) and for every \( m > j \) there exists a polynomial \( W'_{2m} \) such that \( W_{2m} \equiv y^{p - e + m} W'_{2m} \pmod{(x_1^p, \ldots, x_N^p)} \). By Lemma 2.3 there exist constants \( b_0, \ldots, b_{p - 1} \) such that
\[
W_{2j} = b_0 y^{\frac{p - 1}{2} - j} h_{p - 1 - 2j} + \ldots + b_j y^{\frac{p - 1}{2} - (p - e + j)} h_{p - 1 + b_j + 1} W_{2j + 2} + \ldots + b_{\frac{p - 1}{2}} y^{\frac{p - 1}{2} - j} W_{p - 1}.
\]
If \( 2j \leq e - \frac{p - 1}{2} \) then \( \frac{p - 1}{2} - j \geq p - e + j \) and for the polynomial
\[
W'_{2j} = b_0 y^{\frac{p - 1}{2} - j - (p - e + j)} h_{p - 1 - 2j} + \ldots + b_j y^{\frac{p - 1}{2} - (p - e + j)} h_{p - 1 + b_j + 1} W'_{2j + 2} + \ldots + b_{\frac{p - 1}{2}} y^{\frac{p - 1}{2} - j} W'_{p - 1},
\]
we have \( W_{2j} \equiv y^{p - e + j} W'_{2j} \pmod{(x_1^p, \ldots, x_N^p)} \). Therefore \( 2j > e - \frac{p - 1}{2} \). We have
\[
y^{e - j} W_{2j} \equiv 0 \pmod{(x_1^p, \ldots, x_N^p)}.
\]
and from equations (8) we see that \( y^{e - j} W_{2m} \equiv 0 \pmod{(x_1^p, \ldots, x_N^p)} \) for \( m > j \). Hence by (9) we have
\[
y^{e + \frac{p - 1}{2} - 2j} (b_0 h_{p - 1 - 2j} + \ldots + b_j y^j h_{p - 1}) \equiv 0 \pmod{(x_1^p, \ldots, x_N^p)}.
\]
But we know that \(0 \leq e + \frac{p-1}{2} - 2j < p\) and 
\[
\left( e + \frac{p-1}{2} - 2j \right) + \deg h_{p-1-2j} = d + e - \frac{p-1}{2} \leq N \frac{p-1}{2},
\]
so we can apply the induction assumption. Therefore there exists a polynomial \(P\) such that 
\[
b_0 h_{p-1-2j} + \ldots + b_j y^j h_{p-1} \equiv y^{p-e+j-(\frac{p-1}{2}-j)} P \pmod{(x_1^p, \ldots, x_N^p)}.
\]
If we multiply this equality by \(y^\frac{p-1}{2}-j\) and use equation (9) and we see that 
\[
W_{2j} \equiv y^{p-e+j} \left( P + b_{j+1} W_{2j+2} + \ldots + b_{p-1} y^\frac{p-1}{2}-j W_{p-1} \right) \pmod{(x_1^p, \ldots, x_N^p)}.
\]
This gives a contradiction with our choice of \(j\).  

**Lemma 2.5.** 
\[
\sum_{j=0}^{\frac{p-1}{2}} (-1)^j y^j h_{2j} = \left( e + \frac{p-1}{2} \right) W_0 + \left( e + \frac{p-1}{2} - 1 \right) W_2 + \ldots + \left( \frac{e}{e} \right) W_{p-1}.
\]

**Proof.** Use Lemma 2.2 and equality 
\[
\binom{n}{n} + \binom{n+1}{n} + \ldots + \binom{n+m}{n} = \binom{n+m+1}{n+1}.
\]

Since \(\left( e + \frac{p-1}{2} - j \right)\) vanishes if \(j < e - \frac{p-1}{2}\), we have 
\[
h_0 + h_2 x_0^2 + \ldots + h_{p-1} x_0^{p-1} \equiv \sum_{j=0}^{\frac{p-1}{2}} (-1)^j y^j h_{2j} = \sum_{j \geq e - \frac{p-1}{2}} \left( e + \frac{p-1}{2} - j \right) W_{2j} \pmod{(x_1^p, \ldots, x_N^p, \sum_{i=0}^{N} x_i^2)}.
\]
Now by Lemma 2.4 for any \(j \geq e - \frac{p-1}{2}\) there exists \(W_{2j}^{'j}\) such that \(W_{2j} \equiv y^{p-e+j} W_{2j}^{'j} \pmod{(x_1^p, \ldots, x_N^p)}\). But \(p - e + j \geq \frac{p+1}{2}\) and 
\[
y^\frac{p+1}{2} \equiv \pm x_0^{p+1} \equiv 0 \pmod{(x_1^p, \ldots, x_N^p, \sum_{i=0}^{N} x_i^2)}.
\]
Therefore \(h^{\text{even}} = h_0 + h_2 x_0^2 + \ldots + h_{p-1} x_0^{p-1}\) belongs to the ideal \((x_0^p, \ldots, x_N^p, \sum_{i=0}^{N} x_i^2)\).

In a similar way we deal with the remaining part of \(h\). First, let us set 
\[
W_{2j+1} = \binom{e}{j} h_1 + \binom{e}{j-1} y h_3 + \ldots + \binom{e}{0} y^j h_{2j+1}
\]
for \(j = 0, \ldots, \frac{p-3}{2}\). Comparing coefficients in (7) at odd powers of \(x_0\) we get 
\[
y^{e-j} W_{2j+1} \equiv 0 \pmod{(x_1^p, \ldots, x_N^p)}.
\]
for \(j = 0, \ldots, \frac{p-3}{2}\).
LEMMA 2.6. We have
\[ y^j h_{2j+1} = (-1)^j \left( \frac{e+j-1}{e} \right) W_1 + (-1)^{j-1} \left( \frac{e+j-2}{e} \right) W_3 + \ldots + (-1)^0 \left( \frac{e-1}{e} \right) W_{2j+1}. \]

LEMMA 2.7. If \( j \geq e - \frac{p+1}{2} \) then \( W_{2j+1} \) is a linear combination of \( W_{2j+3}, \ldots, W_{p-2} \) and \( y^{\frac{p-3}{2}j} h_{p-2-2j}, \ldots, y^{\frac{p-3}{2}h_{p-2}} \).

LEMMA 2.8. If \( e - \frac{p+1}{2} \leq j \leq \frac{p-3}{2} \) then there exists a polynomial \( W'_{2j+1} \) such that \( W_{2j+1} - y^{p-e+j} W'_{2j+1} \) lies in the ideal \( (x_1^p, \ldots, x_N^p) \).

LEMMA 2.9.
\[ \sum_{j=0}^{\frac{p-3}{2}} (-1)^j y^j h_{2j+1} = \left( e + \frac{p-3}{2} \right) W_1 + \left( e + \frac{p-3}{2} - 1 \right) W_3 + \ldots + \left( e \right) W_{p-2}. \]

Since \( e + \frac{p-3}{2} - j \) vanishes if \( j < e - \frac{p+1}{2} \), we have
\[ h_1 + h_3 x_0^2 + \ldots + h_{p-2} x_0^{p-3} = \sum_{j=0}^{\frac{p-3}{2}} (-1)^j y^j h_{2j+1} = \sum_{j \geq e - \frac{p+1}{2}} \left( e + \frac{p-3}{2} - j \right) W_{2j+1} \pmod{(x_1^p, \ldots, x_N^p, \sum_{i=0}^N x_i^2)}. \]

Now by Lemma 2.8 for any \( j \geq e - \frac{p+1}{2} \) there exists \( W'_{2j+1} \) such that \( W_{2j+1} - y^{p-e+j} W'_{2j+1} \pmod{(x_1^p, \ldots, x_N^p)} \). But \( p - e + j \geq \frac{p-1}{2} \) and
\[ y^{\frac{p-1}{2}} \equiv \pm x_0^{p-1} \pmod{(x_0^p, \ldots, x_N^p, \sum_{i=0}^N x_i^2)}. \]

Therefore \( h^{\text{odd}} = h_1 x_0 + h_3 x_0^3 + \ldots + h_{p-2} x_0^{p-2} \) belongs to the ideal \( (x_0^p, \ldots, x_N^p, \sum_{i=0}^N x_i^2) \).

Since \( h = h^{\text{even}} + h^{\text{odd}} \), this finishes proof of the proposition.

The following corollary of Proposition 2.1 is the main step in our proof of Theorem 3.4.

PROPOSITION 2.10. Let \( k \) be a field of characteristic \( p > 2 \). Let \( 0 \leq e < p \) be an integer. Then for any \( d \leq (N + 1) \cdot \frac{p-1}{2} - e \) we have
\[ ((x_0^p, \ldots, x_N^p) : (\sum_{i=0}^N x_i^2)^e)_d = (x_0^p, \ldots, x_N^p, (\sum_{i=0}^N x_i^2)^{p-e})_d. \]
in \( k[x_0, \ldots, x_N] \).
Proof. We need to prove that for every homogeneous degree \( d \) (with \( d + e \leq (N + 1) \cdot \frac{p-1}{2} \)) polynomial \( h \) such that

\[
\left( \sum_{i=0}^{N} x_i^2 \right)^{\epsilon} h \equiv 0 \pmod{(x_0^p, \ldots, x_N^p)}
\]

we have \( h \in (x_0^p, \ldots, x_N^p, (\sum_{i=0}^{N} x_i^2)^{p-e})_d \).

If \( e = 0 \) then the assertion is trivial so we can assume that \( e \geq 1 \).

The proof is by induction on \( N \). For \( N = 0 \) we have \( 2(d + e) \leq p - 1 \). But if \( h \neq 0 \) then counting degrees we get \( d + 2e \geq p \), a contradiction. Now assume that the assertion holds for polynomials in \( N \) variables (i.e., for \( N - 1 \)).

If \( e = p - 1 \) then the required assertion follows from Proposition 2.1. Otherwise, by Proposition 2.1 there exists a polynomial \( h' \) such that

\[
h \equiv \left( \sum_{i=0}^{N} x_i^2 \right)^{\epsilon} h' \pmod{(x_0^p, \ldots, x_N^p)}.
\]

Therefore

\[
\left( \sum_{i=0}^{N} x_i^2 \right)^{\epsilon + 1} h' \equiv \left( \sum_{i=0}^{N} x_i^2 \right)^{\epsilon} h \equiv 0 \pmod{(x_0^p, \ldots, x_N^p)}
\]

and now we can again apply Proposition 2.1, since \( e + 1 + \deg h' = e + \deg h \leq (N + 1) \cdot \frac{p-1}{2} \).

Continuing in this way till new \( e \) becomes \( p - 1 \), we get the required assertion. \( \square \)

Let us set \( S = k[x_0, \ldots, x_N] \), where char \( k = p \) and \( N = n + 1 \). Let \( q = p^s \) for some non-negative integer \( s \). Set \( I = (x_0^2 + \ldots + x_N^q, x_1^q, \ldots, x_N^q) \) and \( A = S/I \). Let us recall that

\[
\left( \frac{1-t^q}{1-t} \right)^N = \sum_{i=0}^{\infty} \alpha_{i,N} t^i,
\]

where

\[
\alpha_{i,N} = \sum_{j=0}^{N} (-1)^j \binom{N}{j} \binom{i - jq + N - 1}{N - 1},
\]

where we set \( \binom{a}{b} = 0 \) if \( a < b \) or \( b < 0 \). The Hilbert series of \( A \) can be computed as

\[
h_A(t) = \sum \dim A_i t^i = h_S(t)(1-t^q)(1-t^q)^N = (1+t) \left( \frac{1-t^q}{1-t} \right)^N.
\]

Therefore

\[
\dim A_i = \alpha_{i,N} + \alpha_{i-1,N}.
\]

Note that \( \dim A = 2q^N \) (e.g., by the Bezout’s theorem). The following lemma will be used in the proof of Lemma 2.15.
LEMMA 2.11. For every integer $i$ we have
\[
\sum_{j \in \mathbb{Z}} \dim A_{i+jq} = 2q^n.
\]

Proof. By definition
\[
(1 + t + \ldots + t^{q-1}) \sum_{i=0}^\infty \alpha_{i,N} t^i = \sum_{i=0}^\infty \alpha_{i,N} t^i.
\]
This gives $\alpha_{i,N} = \alpha_{i,n} + \ldots + \alpha_{i,q+1,n}$. Hence
\[
\sum_{j \in \mathbb{Z}} \alpha_{i+jq,N} = \sum_{j \in \mathbb{Z}} \alpha_{j,n} = q^n.
\]
Now the lemma follows from equality $\dim A_l = \alpha_{l,N} + \alpha_{l-1,N}$. \qed

Let us set $d_{N,s} = \frac{1}{2} n(q-1)$ and
\[
\gamma_N(i) = \frac{1}{2^N} \sum_{j \in \mathbb{Z}} (-1)^j \dim A_{d_{N,s} + i + jq}
\]
for $i \in \mathbb{Z}$. In principle, we could write the formulas for $\gamma_N(i)$ using formulas for $\dim A_j$, but the obtained formulas are rather useless and we need different formulas. Let us first define inductively some sequences of numbers and functions. Set $w_0 = 1$ and assume we have defined integers $w_0, \ldots, w_k$. Then we set
\[
F_k(i) = \sum_{j=0}^k (-1)^j w_{k-j} \binom{i+j}{2j+1}
\]
and
\[
w_{k+1} = \sum_{j=0}^k (-1)^j w_{k-j} \binom{q+1}{2j+2}.
\]
Similarly, set $u_0 = 0$ and assume we have defined $u_0, \ldots, u_k$. Then we set
\[
G_k(i) = (-1)^k \binom{i+k}{2k} + \sum_{j=0}^k (-1)^j u_{k-j} \binom{i+j}{2j+1}
\]
and
\[
u_{k+1} = (-1)^k \binom{q+1}{2k+1} + \sum_{j=0}^k (-1)^j u_{k-j} \binom{q+1}{2j+2}.
\]

LEMMA 2.12. We have $\gamma_N(0) = 0$, $\gamma_N(i) > 0$ for $i = 1, \ldots, q-1$, and $\gamma_N(q-i) = \gamma_N(i)$. Moreover, for $i = 1, \ldots, \frac{q-1}{2}$ we have
\[
\gamma_N(i) = \begin{cases} 
F_k(i) & \text{if } N = 2k+2, \\
G_k(i) & \text{if } N = 2k+1.
\end{cases}
\]
Proof. Using the same method as in the proof of Lemma 2.11 we get the following recursion:

\[ 2 \gamma_N(i) = \sum_{|j| \leq q - 1} \gamma_n(i + j). \]

Then one can see that

\[ \gamma_1(i) = \begin{cases} 0 & \text{if } i = 0, \\ 1 & \text{if } i = 1, \ldots, q - 1, \end{cases} \]

\[ \gamma_N(0) = 0, \gamma_N(q - i) = \gamma_N(i) \text{ for } i = 1, \ldots, q - 1 \]

\[ \gamma_N(i) = \sum_{j=1}^{i} \gamma_n \left( \frac{q + 1}{2} - j \right) \]

for \( i = 1, \ldots, \frac{q - 1}{2} \). Using this recursion and the formula

\[ \sum_{j=k}^{n} \binom{j}{k} = \binom{n+1}{k+1}, \]

one can easily prove the lemma by induction.

Set \( B = A/(x_0^p) \) and \( C = A/(I : x_0^p) \). Note that \( A \) and \( C \) are graded 0-dimensional Gorenstein rings (in fact, \( A \) is a complete intersection ring). From the definition of \( A \) one can easily see that \( \dim A_i = 0 \) if and only if \( i < 0 \) or \( i > N(q - 1) + 1 \). As \( C_i \subset A_{i+q} \), this implies that \( C_i = 0 \) if \( i < 0 \) or \( i > N(q - 1) + 1 - q = n(q - 1) \).

From now on we consider the case \( s = 1 \), i.e., \( q = p \).

\textbf{Proposition 2.13.} If \( p > 2 \) then \( C_d = B_d \) for \( d \leq d_N = \frac{1}{2} n(p - 1) \).

Proof. Since \( x_0^{2p} = (-\sum_{i \geq 1} x_i^2)^p \) in \( S \), we see that \( I + (x_0^p) \subset I + (I : x_0^p) \). Our assertion is equivalent to the fact that this inclusion is equality in gradings \( d \leq d_N \). Therefore it is sufficient to show that if \( g_0 \in (I : x_0^p)_d \), \( d \leq d_N \) then \( g_0 \in I + x_0^p \).

Assume that \( g_0 \in (I : x_0^p)_d \) for some \( d \leq d_N \). Then there exist some homogeneous polynomials \( g_i, i = 1, \ldots, N \) and \( h \), such that

\[ g_0 x_0^p = \sum_{i=1}^{N} g_i x_i^p + (\sum_{i=0}^{N} x_i^2) h. \]

Then by Proposition 2.10 we can write \( h \) as

\[ h = \sum_{i=0}^{N} x_i^p h_i + (\sum_{i=0}^{N} x_i^2)^{p-1} h', \]

for some homogeneous polynomials \( h_i, i = 0, \ldots, N \) and \( h' \). Then

\[ g_0 x_0^p = \sum_{i=1}^{N} (g_i + (\sum_{i=0}^{N} x_i^2) h_i) x_i^p + (\sum_{i=0}^{N} x_i^2) h_0 x_0^p + (\sum_{i=0}^{N} x_i^2 h') \]
and hence
\[(g_0 - x_0^p h_0 - (\sum_{i=0}^{N} x_i^2)h') x_0^p = \sum_{i=1}^{N} (g_i + (\sum_{i=0}^{N} x_i^2)h_i + x_i^p h') x_i^p.\]

Now comparing coefficients of both sides treated as polynomials in \(x_0\) we see that \(g_0 - x_0^p h_0 - (\sum_{i=0}^{N} x_i^2)h' \in (x_1^p, \ldots, x_N^p)\), which finishes the proof. \(\square\)

The above proposition allows us to compute the Hilbert functions of \(B\) and \(C\):

**Lemma 2.14.**
\[
\dim B_i = \begin{cases} 
\sum_{j \geq 0} (-1)^j \dim A_{i-jp} & \text{if } i \leq d_N + p, \\
\dim B_{2d_N-i} & \text{if } i \geq d_N + p.
\end{cases}
\]

In particular, \(B_i = C_i\) if \(i \leq d_N\) or \(i \geq d_N + p\). We also have \(\dim C_{2d_N-i} = \dim C_i\) for all integers \(i\). Moreover, \(B_i \neq 0\) (\(C_i \neq 0\)) if and only if \(0 \leq i \leq 2d_N = n(p-1)\).

**Proof.** Since we have short exact sequences
\[0 \to C_i \to A_{i+p} \to B_{i+p} \to 0,\]
Proposition 2.13 implies that
\[
\dim B_{i+p} - \dim A_{i+p} = -\dim C_i = -\dim B_i
\]
for \(i \leq d_N\). This allows us to compute \(\dim B_i\) for \(i \leq d_N + p\) and \(\dim C_i\) for \(i \leq d_N\). An easy calculation implies that \(\dim C_i\) is increasing for \(i \leq d_N\). Let us recall that \(C\) is Gorenstein and hence the Hilbert function of \(C\) is symmetric. Hence the above remark together with \(C_{2d_N+1} = 0\) imply that \(\dim C_i = \dim C_{2d_N-i}\).

Assume that \(i \geq d_N\). Then using once more the above short exact sequence we get
\[
\dim B_{i+p} - \dim A_{i+p} = -\dim C_i = -\dim B_{2d_N-i}.
\]
Finally, we can use the duality of the graded Gorenstein ring \(A\) to get
\[
\dim B_{i+p} = \dim A_{i+p} - \dim B_{2d_N-i} = \dim A_{2d_N-i} - \dim B_{2d_N-i} = \dim B_{2d_N-i-p}.
\]
The remaining part of the lemma follows from Proposition 2.13 and equality of dimensions \(\dim B_i = \dim C_i\) for \(i \geq d_N + p\). \(\square\)

**Lemma 2.15.** Let \(l\) be an integer. Then
\[
\sum_{i \in \mathbb{Z}} \dim B_{i+lp} = p^n + 2^n \gamma_N(l_0),
\]
where \(l_0\) is the unique integer such that \(0 \leq l_0 < p\) and \(l \equiv d_N + l_0 \pmod{p}\), and \(\gamma_N(\cdot)\) is as in Lemma 2.12.
Proof. Let us set \( l_1 = d_N + l_0 \). By Lemma 2.14 we have
\[
\sum_{i \in \mathbb{Z}} \dim A_{l_1 + ip} = \sum_{i \in \mathbb{Z}} (\dim B_{l_1 + ip} + \dim C_{l_1 + ip - p}) = 2 \sum_{i \in \mathbb{Z}} \dim B_{l_1 + ip} - (\dim B_{l_1} - \dim C_{l_1}).
\]
But
\[
\dim B_{l_1} - \dim C_{l_1} = \sum_{j \geq 0} (-1)^j \dim A_{l_1 - jp} + \sum_{j \geq 0} (-1)^{j+1} \dim A_{2d_N - l_1 - jp}
\]
\[
= \sum_{j \geq 0} (-1)^j \dim A_{l_1 - jp} + \sum_{j \leq -1} (-1)^j \dim A_{l_1 - jp} = \sum_{j \in \mathbb{Z}} (-1)^j \dim A_{l_1 - jp}.
\]
Therefore by Lemma 2.11
\[
\sum_{i \in \mathbb{Z}} \dim B_{l_1 + ip} = p^n + \frac{1}{2} \sum_{j \in \mathbb{Z}} (-1)^j \dim A_{l_1 - jp},
\]
which together with Lemma 2.12 proves the required equality.

3 Vanishing and non-vanishing theorems

In this section we prove some basic vanishing and non-vanishing theorems for cohomology of twisted Frobenius pull-backs of spinor bundles.

Let us set \( \psi_1 = \Omega_{\mathbb{P}^N}(1)|_{Q_n} \).

**Proposition 3.1.** For any spinor bundle \( \Sigma \) on \( Q_n \) we have
\[
h^1(Q_n, \psi_1 \otimes \Sigma(t)) = \begin{cases} 
0 & \text{if } t \neq 0, \\
2 \rk \Sigma = 2^{|V|} & \text{if } t = 0.
\end{cases}
\]

**Proof.** Every spinor bundle \( \Sigma \) is ACM and so it fits into the following short exact sequence of sheaves on \( \mathbb{P}^N \)
\[
0 \to \mathcal{O}_{\mathbb{P}^N}^{2^{|V|}} (-1) \to \mathcal{O}_{\mathbb{P}^N}^{2^{|V|}} \to i_* \Sigma(1) \to 0,
\]
(11)
where \( i : Q_n \hookrightarrow \mathbb{P}^N \) is the embedding (see (1), (2) and (3)). Tensoring this sequence with \( \Omega_{\mathbb{P}^N}(t) \) and using standard Bott formulas for cohomology of twists of \( \Omega_{\mathbb{P}^N}(t) \) on \( \mathbb{P}^N \) we get the result.

**Corollary 3.2.** If \( E \) is arithmetically Cohen-Macaulay on \( Q_n \) then it is a direct sum of line bundles if and only if
\[
\sum_{t \in \mathbb{Z}} h^1(E \otimes \psi_1(t)) = \rk E.
\]

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Proof. By Theorem 1.2 any ACM bundle on $Q_n$ is isomorphic to a direct sum of line bundles $\mathcal{O}_{Q_n}(i)$ and spinor bundles twisted by some $\mathcal{O}_{Q_n}(i)$. By Proposition 3.1 we see that

$$\sum_{t \in \mathbb{Z}} h^1(\Sigma(i) \otimes \psi_1(t)) = 2 \text{rk} \Sigma.$$ 

On the other hand

$$h^1(Q_n, \psi_1(t)) = \begin{cases} 0 & \text{if } t \neq -1, \\ 1 & \text{if } t = -1, \end{cases}$$

so if $\sum_{t \in \mathbb{Z}} h^1(E \otimes \psi_1(t)) = \text{rk} E$ then $E$ cannot contain any twists of spinor bundles as direct summands.

Let us recall that $d_N = (N-1)\frac{p-1}{2} = n\frac{p-1}{2}$.

Corollary 3.3. $F_*(\mathcal{O}_{Q_n}(t))$ is a direct sum of line bundles if and only if $t - d_N$ is divisible by $p$.

Proof. By [BM, Theorem 4] the graded $S$-module $\bigoplus_{t \in \mathbb{Z}} H^1(Q_n, F^*\psi_1(t - p))$ is isomorphic as a graded $S$-module to $B$. But $H^1(Q_n, F^*\psi_1(t + ip)) = H^1(Q_n, F_*(\mathcal{O}_{Q_n}(t)) \otimes \psi_1(i))$, so by the above corollary the fact that $F_*(\mathcal{O}_{Q_n}(t))$ is a direct sum of line bundles is equivalent to equality

$$\sum_{i \in \mathbb{Z}} \dim B_{t+ip} = p^n.$$ 

Now the required assertion follows from Lemma 2.15.

The following vanishing theorem allows to compute the decomposition of Frobenius push-forwards of line bundles:

Theorem 3.4. Let $\Sigma$ be a spinor bundle on $Q_n$. Then for $0 < i < n$ we have $H^i(Q_n, F^*\Sigma(t)) = 0$ if $t \leq d_N - ip$ or $t \geq d_N - (i - 1)p$.

Proof. By Lemma 1.5 it is sufficient to prove vanishing of $H^i(Q_n, F^*\Sigma(t))$ for $t \geq d_N - (i - 1)p$. By Corollary 3.3 and the projection formula we have

$$H^i(Q_n, F^*\Sigma(d_N + tp)) = H^i(Q_n, F_*(\mathcal{O}_{Q_n}(d_N)) \otimes \Sigma(t)) = 0$$

for any integer $t$. In particular, $H^i(Q_n, F^*\Sigma(d_N - (i - 1)p)) = 0$. Now the proof is by induction on $n$.

For $n = 2$, $\Sigma = \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(-1, 0)$ or $\Sigma = \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(0, -1)$ and in both cases it is easy to check the required assertion. Assume that the theorem holds for quadrics of dimension less than $n$. Let us recall that the restriction of the spin representation of $\text{Spin}(2m+1)$ to $\text{Spin}(2m)$ is the sum of the two spin representations of $\text{Spin}(2m)$. Similarly, the restriction of either spin representation of $\text{Spin}(2m)$ to $\text{Spin}(2m-1)$ is the spin representation. Therefore the restriction of a spinor bundle to a hypersurface quadric is either a spinor bundle or a direct sum of two spinor bundles. Using the long cohomology sequence for the short exact sequence

$$0 \to F^*\Sigma(t) \to F^*\Sigma(t+1) \to F^*\Sigma(t+1)|_{Q_{n-1}} \to 0$$

for any rational number $t$. In particular, $H^i(Q_n, F^*\Sigma(d_N - (i - 1)p)) = 0$. Now the proof is by induction on $n$.

For $n = 2$, $\Sigma = \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(-1, 0)$ or $\Sigma = \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(0, -1)$ and in both cases it is easy to check the required assertion. Assume that the theorem holds for quadrics of dimension less than $n$. Let us recall that the restriction of the spin representation of $\text{Spin}(2m+1)$ to $\text{Spin}(2m)$ is the sum of the two spin representations of $\text{Spin}(2m)$. Similarly, the restriction of either spin representation of $\text{Spin}(2m)$ to $\text{Spin}(2m-1)$ is the spin representation. Therefore the restriction of a spinor bundle to a hypersurface quadric is either a spinor bundle or a direct sum of two spinor bundles. Using the long cohomology sequence for the short exact sequence

$$0 \to F^*\Sigma(t) \to F^*\Sigma(t+1) \to F^*\Sigma(t+1)|_{Q_{n-1}} \to 0$$

for any rational number $t$. In particular, $H^i(Q_n, F^*\Sigma(d_N - (i - 1)p)) = 0$. Now the proof is by induction on $n$.

The following vanishing theorem allows to compute the decomposition of Frobenius push-forwards of line bundles:

Theorem 3.4. Let $\Sigma$ be a spinor bundle on $Q_n$. Then for $0 < i < n$ we have $H^i(Q_n, F^*\Sigma(t)) = 0$ if $t \leq d_N - ip$ or $t \geq d_N - (i - 1)p$.

Proof. By Lemma 1.5 it is sufficient to prove vanishing of $H^i(Q_n, F^*\Sigma(t))$ for $t \geq d_N - (i - 1)p$. By Corollary 3.3 and the projection formula we have

$$H^i(Q_n, F^*\Sigma(d_N + tp)) = H^i(Q_n, F_*(\mathcal{O}_{Q_n}(d_N)) \otimes \Sigma(t)) = 0$$

for any integer $t$. In particular, $H^i(Q_n, F^*\Sigma(d_N - (i - 1)p)) = 0$. Now the proof is by induction on $n$.

For $n = 2$, $\Sigma = \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(-1, 0)$ or $\Sigma = \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(0, -1)$ and in both cases it is easy to check the required assertion. Assume that the theorem holds for quadrics of dimension less than $n$. Let us recall that the restriction of the spin representation of $\text{Spin}(2m+1)$ to $\text{Spin}(2m)$ is the sum of the two spin representations of $\text{Spin}(2m)$. Similarly, the restriction of either spin representation of $\text{Spin}(2m)$ to $\text{Spin}(2m-1)$ is the spin representation. Therefore the restriction of a spinor bundle to a hypersurface quadric is either a spinor bundle or a direct sum of two spinor bundles. Using the long cohomology sequence for the short exact sequence

$$0 \to F^*\Sigma(t) \to F^*\Sigma(t+1) \to F^*\Sigma(t+1)|_{Q_{n-1}} \to 0$$

for any rational number $t$. In particular, $H^i(Q_n, F^*\Sigma(d_N - (i - 1)p)) = 0$. Now the proof is by induction on $n$.
and the induction assumption we see that for \( t \geq d_n - (i - 1)p - 1 \) we have a surjection

\[
H^i(Q_n, F^*\Sigma(t)) \to H^i(Q_n, F^*\Sigma(t + 1)) \to 0.
\]

In particular, vanishing of \( H^i(Q_n, F^*\Sigma(d_N - (i - 1)p)) \) implies vanishing of \( H^i(Q_n, F^*\Sigma(t)) \) for \( t \geq d_N - (i - 1)p \).

The above vanishing theorem implies that the Frobenius pull-back of the spinor bundle on \( Q_3 \) is very similar to an instanton bundle. More precisely, we have the following proposition:

**Proposition 3.5.** Let us set \( E = F^*\Sigma(\frac{N-1}{2}) \) on \( Q_3 \). Then \( E \) is the cohomology of the monad

\[
0 \to \mathcal{O}(-1)^b \to \Sigma^{b+1} \to \mathcal{O}^b \to 0,
\]

where \( b = h^1(E) = -\chi(E) \).

**Proof.** The proof is an application of Horrock’s killing technique and it is quite similar to the proof of [OS, Proposition 1.1]. We leave the details to the reader.

**Corollary 3.6.** Let \( \Sigma \) be a spinor bundle on \( Q_n \). Then for \( 0 < i < n \) we have \( H^i(Q_n, F^*\Sigma(t)) \neq 0 \) if \( d_N - ip < t < d_N - (i - 1)p \).

**Proof.** If \( H^i(Q_n, F^*\Sigma(t)) = 0 \) for some \( d_N - ip < t < d_N - (i - 1)p \), then by Theorem 3.4

\[
H^i(Q_n, F^*(\mathcal{O}_{Q_n}(t)) \otimes \Sigma(j)) = H^i(Q_n, F^*(\Sigma(t + jp))) = 0
\]

for all integers \( j \).

Let us note that if \( n \) is even then there exists an automorphism \( a : Q_n \to Q_n \) such that \( a^*\Sigma \simeq \Sigma \). Therefore cohomology groups of \( F^*\Sigma_+(t + jp) \) and \( F^*\Sigma_-(t + jp) \) are the same. In particular, the above vanishing holds for all spinor bundles on \( Q_n \) and we can apply Corollary 1.4. But then we get contradiction with Corollary 3.3.

**Theorem 3.7.** Let \( \Sigma_1, \Sigma_2 \) be spinor bundles on \( Q_n, n \geq 2 \). Then for any \( 0 < i < n \) we have

\[
H^i(Q_n, \Sigma_1 \otimes F^*\Sigma_2(t)) = 0 \text{ if } t \leq d_N - ip \text{ or } t \geq d_N - (i - 1)p + 1.
\]

**Proof.** For simplicity of notation let us consider only odd dimensional quadrics \( Q_n, n = 2m + 1 \) (the proof in the even dimensional case is essentially the same). As before we can easily reduce to the case \( i = 1 \) (the proof of Lemma 1.6 gives vanishing of higher intermediate cohomology groups). Let us note the following short exact sequences:

\[
0 \to F^*\Sigma(t - p) \to \mathcal{O}_{Q_n}(t - p)^{2^m + 1} \to F^*\Sigma(t) \to 0 \tag{12}
\]

and

\[
0 \to \Sigma(t - p) \to \mathcal{O}_{Q_n}(t - p)^{2^m + 1} \to \Sigma(t - p + 1) \to 0. \tag{13}
\]

Using the long cohomology sequences for appropriate twists we get the following exact sequences:

\[
0 \to H^0(\Sigma \otimes F^*\Sigma(t - p)) \to H^0(\Sigma(t - p))^{2^{m+1}} \to H^0(\Sigma \otimes F^*\Sigma(t)) \to H^1(\Sigma \otimes F^*\Sigma(t - p)) \to 0
\]
for all \( t \), and by Theorem 3.4
\[
0 \to H^0(\Sigma \otimes F^*\Sigma(t-p)) \to H^0(F^*\Sigma(t-p))^\oplus 2^{n+1} \to H^0(\Sigma \otimes F^*\Sigma(t-p+1)) \to H^1(\Sigma \otimes F^*\Sigma(t-p)) \to 0
\]
(14)
for \( t \leq d_N \). Using these sequences we get the following recurrence equation
\[
h^0(\Sigma \otimes F^*\Sigma(t)) = 2^{m+1}(h^0(\Sigma(t-p)) - h^0(F^*\Sigma(t-p))) + h^0(\Sigma \otimes F^*\Sigma(t-p+1))
\]
(15)
for \( t \leq d_N \). Let us first prove that
\[
h^0(\Sigma \otimes F^*\Sigma(t)) + h^0(\Sigma \otimes F^*\Sigma(t+1)) = 2^{m+1}h^0(F^*\Sigma(t))
\]
(16)
for \( t \leq d_N - 1 \). We prove it by induction on \( t \) starting with very negative \( t \) for which the equality is obvious. By (15) and the induction assumption for \( t \leq d_N - 1 \) the left hand side of (16) is equal to
\[
2^{m+1}(h^0(\Sigma(t-p)) - h^0(F^*\Sigma(t-p))) + h^0(\Sigma(t-p+1)) - h^0(\Sigma(t-p+1))
\]
\[
= 2^{m+1}(h^0(\Sigma(t-p)) - h^0(F^*\Sigma(t-p))) + h^0(\Sigma(t-p+1)).
\]
But sequences (12) and (13) imply that for \( t \leq d_N \)
\[
h^0(F^*\Sigma(t-p)) + h^0(F^*\Sigma(t)) = 2^{m+1}h^0(\mathcal{O}_{Q_n}(t-p)) = h^0(\Sigma(t-p)) + h^0(\Sigma(t-p+1)),
\]
which proves the required assertion.

Now let us note that (14) and (16) imply vanishing of \( H^1(\Sigma \otimes F^*\Sigma(t)) \) for \( t \leq d_N - p \). By Lemma 1.6 this implies vanishing \( H^1(\Sigma \otimes F^*\Sigma(t)) \) for \( t \geq d_N + 1 \).

**Corollary 3.8.** Let us set \( S_n = \Sigma \) if \( n \) is odd and \( S_n = \Sigma_+ \oplus \Sigma_- \) if \( n \) is even. Then for any \( 0 < i < n \) we have \( H^1(Q_n, S_n \otimes F^*S_n(t)) \neq 0 \) if \( d_N - ip + 1 \leq t \leq d_N - (i-1)p \).

**Proof.** As before it is sufficient to prove the corollary for \( i = 1 \). The proof is by induction on \( n \). For \( n = 2 \) the group \( H^1(S_2 \otimes F^*S_2(t)) \) contains \( H^1(\mathcal{O}_{P^1 \times P^1}(t, t-p-1)) \) as a direct summand and this cohomology group is non-zero if \( 0 \leq t \leq p-1 \).

Assume we know the statement in dimensions less than \( n \). Then we have an exact sequence
\[
H^1(S_n \otimes F^*S_n(t-1)) \to H^1(S_n \otimes F^*S_n(t)) \to H^1(S_n \otimes F^*S_n(t)|_{Q_{n-1}}) \to H^2(S_n \otimes F^*S_n(t-1)).
\]
This sequence and Theorem 3.7 imply that \( h^1(S_n \otimes F^*S_n(t-1)) \leq h^1(S_n \otimes F^*S_n(t)) \) if \( t \geq d_n + 1 \), so by Lemma 1.6 it is sufficient to prove that \( H^1(S_n \otimes F^*S_n(d_N)) \neq 0 \). But Lemma 1.6 implies also that
\[
H^2(S_n \otimes F^*S_n(d_N-p)) \simeq H^1(S_n \otimes F^*S_n(d_N)) \simeq H^1(S_n \otimes F^*S_n(d_N-p+1))^*,
\]
so the above sequence applied for \( t = d_N - p + 1 \) gives by the induction assumption non-vanishing of \( H^1(S_n \otimes F^*S_n(d_N)) \).
4 Decomposition of Frobenius push-forwards

If $E$ is an ACM bundle then $F_* E$ is also an ACM bundle. In particular, if $n \geq 3$ then Frobenius push-forwards of line bundles and twisted spinor bundles split into a direct sum of line bundles and twisted spinor bundles. In this section we study the corresponding decompositions.

Let $S_n$ be as in Corollary 3.8 Let $q = p^s$ for some non-negative integer $s$.

**Proposition 4.1.**

1. If $F^*_s(\mathcal{O}_{Q_n}(j))$ contains $\mathcal{O}_{Q_n}(-t)$ as a direct summand then $0 \leq tq + j \leq n(q - 1)$.

2. If $F^*_s(S_n(j))$ contains $\mathcal{O}_{Q_n}(-t)$ as a direct summand then $1 \leq tq + j \leq n(q - 1)$.

**Proof.** (1) If $F^*_s(\mathcal{O}_{Q_n}(j))$ contains $\mathcal{O}_{Q_n}(-t)$ as a direct summand then

$$0 \neq H^0(\mathcal{O}_{Q_n}) \subset H^0(F^*_s(\mathcal{O}_{Q_n}(j)) \otimes \mathcal{O}_{Q_n}(t)) = H^0(\mathcal{O}_{Q_n}(j + tq)),$$

which implies that $tq + j \geq 0$. Similarly,

$$0 \neq H^n(\mathcal{O}_{Q_n}) \subset H^n(F^*_s(\mathcal{O}_{Q_n}(j)) \otimes \mathcal{O}_{Q_n}(t - n)) = H^n(\mathcal{O}_{Q_n}(j + (t - n)q))$$

which implies that $tq + j \leq n(q - 1)$.

(2) If $F^*_s(S_n(j))$ contains $\mathcal{O}_{Q_n}(-t)$ as a direct summand then

$$0 \neq H^0(\mathcal{O}_{Q_n}) \subset H^0(F^*_s(S_n(j)) \otimes \mathcal{O}_{Q_n}(t)) = H^0(S_n(j + tq)),$$

which implies that $tq + j \geq 1$. Similarly, by the Serre duality

$$0 \neq H^n(\mathcal{O}_{Q_n}) \subset H^n(F^*_s(S_n(j)) \otimes \mathcal{O}_{Q_n}(t - n)) = H^n(S_n(j + (t - n)q)) = (H^0(S_n(n(q - 1) - tq - j)))^*$$

which implies that $tq + j \leq n(q - 1)$. □

**Proposition 4.2.**

1. $F_*(\mathcal{O}_{Q_n}(j))$ contains $\Sigma(-t)$ as a direct summand if and only if $d_N - p + 1 \leq tp + j \leq d_N - 1$. In this case, $F_*(\mathcal{O}_{Q_n}(j))$ contains $S_n(-t)$. In particular, $F_*(\mathcal{O}_{Q_n}(j))$ contains at most one twist of a spinor bundle.

2. $F_*(S_n(j))$ contains $\Sigma(-t)$ as a direct summand if and only if $d_N - p + 1 \leq tp + j \leq d_N$. In this case, $F_*(S_n(j))$ contains also $S_n(-t)$. In particular, $F_*(S_n(j))$ contains exactly one twist of a spinor bundle.

**Proof.** (1) If $F_*(\mathcal{O}_{Q_n}(j))$ contains $\Sigma(-t)$ as a direct summand then by symmetry (see the proof of Corollary 3.6) it also contains $S_n(-t)$. By Lemma 1.3 this happens if and only if $H^1(Q_n, F^*\Sigma(j + tp)) = H^1(F_*(\mathcal{O}_{Q_n}(j)) \otimes \Sigma(t)) \neq 0$ so the assertion follows from Theorem 3.4 and Corollary 3.6.

(2) If $F_*(S_n(j))$ contains $\Sigma(-t)$ as a direct summand then by symmetry it also contains $S_n(-t)$. By Lemma 1.3 this happens if and only if $H^1(Q_n, S_n \otimes F^*S_n(tp + j)) = H^1(F_*(S_n(j)) \otimes S_n(t)) \neq 0$ so the assertion follows from Theorem 3.7 and Corollary 3.8 □

**Corollary 4.3.** For any line bundle $\mathcal{L}$ on $Q_n$, $n \geq 3$, the bundle $F_*\mathcal{L}$ is quasi-exceptional.
\textit{Proof.} The assertion follows from Lemma 2.13 and Propositions 4.1 and 4.2.

Let us fix an integer \(0 \leq j < p\). By Propositions 4.1 and 4.2 we can write

\[
F_*(\mathcal{O}_{Q_n}(dN + j)) = \bigoplus \mathcal{O}_{Q_n}(-t)^{a_t} \oplus S_n(1)^b,
\]

where \(b = 0\) if \(j = 0\) and \(a_t = 0\) if \(|tp + j| > dN\).

\textbf{Theorem 4.4.} If \(|tp + j| \leq dN\) then \(0 < a_t = \dim C_{dN+tp+j}\). Moreover, \(b = 2^{[N/2]}\gamma_N(j)\), where \(\gamma_N(\cdot)\) is as in Lemma 2.12.

\textit{Proof.} By Proposition 3.1 we have

\[
h^1(F_*(\mathcal{O}_{Q_n}(dN + j)) \otimes \psi_1(t-1)) = \begin{cases} a_0 + 2^{[n/2]+1}b & \text{if } t = 0, \\ a_t & \text{if } t \neq 0. \end{cases}
\]

On the other hand, we have

\[
h^1(F_*(\mathcal{O}_{Q_n}(dN + j)) \otimes \psi_1(t-1)) = h^1(F^*\psi_1(dN + (t-1)p + j)) = \dim B_{dN+tp+j},
\]

so \(a_t = \dim B_{dN+tp+j} = \dim C_{dN+tp+j}\) for \(t \neq 0\). Comparing ranks in the decomposition we get

\[
\sum_{t \in \mathbb{Z}} a_t + 2^{[n/2]}b = p^n.
\]

Therefore

\[
\sum \dim B_{dN+tp+j} = \sum a_t + 2^{[n/2]}b = 2^{[n/2]}b + p^n
\]

and Lemma 2.15 implies that \(b = 2^{[N/2]}\gamma_N(j)\). By the proof of Lemma 2.15

\[
a_0 = \dim B_{dN+j} - 2^N\gamma_N(j) = \dim B_{dN+j} - (\dim B_{dN+j} - \dim C_{dN+j}) = \dim C_{dN+j},
\]

which finishes the proof.

Since \(F_*(\mathcal{O}_{Q_n}(j+tp)) \simeq F_*(\mathcal{O}_{Q_n}(j)) \otimes \mathcal{O}_{Q_n}(t)\), the above theorem gives the decomposition for all Frobenius push forwards of line bundles on \(Q_n\) \((n \geq 3)\).

We can also compute the decomposition of \(F_*(S_n(j))\) along the following lines. By \([13]\) and \([16]\) we have an exact sequence

\[
0 \rightarrow H^1(S_n \otimes F^*S_n(t)) \rightarrow H^1(F^*S_n(t)) \oplus 2^{[N/2]} \rightarrow H^1(\Sigma \otimes F^*S_n(t+1)) \rightarrow H^2(S_n \otimes S_n(t))
\]

for \(t \leq dN - 1\). By Theorem 3.7 the last cohomology group vanishes for \(t \geq dN - p + 1\). By the same theorem and the proof of Corollary 3.8 we also have an exact sequence

\[
0 \rightarrow H^1(S_n \otimes F^*S_n(dN - p + 1)) \rightarrow H^1(S_n \otimes F^*S_n(dN - p + 1)|_{Q_{n-1}}) \rightarrow H^1(S_n \otimes F^*S_n(dN - p + 1))^* \rightarrow 0.
\]

Together with Lemmas 1.3 and 1.6 this is sufficient to determine the required decomposition. By induction, this also gives decomposition of \(F_*(\mathcal{O}_{Q_n}(j))\). We skip the actual computation as it is long and it will not be used in the following.
COROLLARY 4.5. $F^i_s(\mathcal{O}_{Q_n}(j))$ contains $\mathcal{O}_{Q_n}(-t)$ as a direct summand if and only if $0 \leq tq + j \leq n(q - 1)$.

Proof. By Proposition 4.1 we need only to show that some line bundles appear in the decomposition. The proof is by induction on $m$. For $s = 1$ the required assertion follows from Theorem 4.4. Assume that $0 \leq tp^s + j \leq n(p^s - 1)$ for some $t$. Then $0 \leq tp^s + j/p \leq n(p^s - 1) + n - n/p$. Therefore there exists an integer $l$ such that $0 \leq tp^s - l \leq n(p^s - 1)$ and $-j/p \leq l \leq n - (n + j)/p$. Then $\mathcal{O}_{Q_n}(-t)$ is a direct summand of $(F^m)^*_s(\mathcal{O}_{Q_n}(-l))$ and $\mathcal{O}_{Q_n}(-l)$ is a direct summand of $F_s(\mathcal{O}_{Q_n}(j))$.

COROLLARY 4.6. $F_s\mathcal{O}_{Q_n}$ is a tilting bundle if and only if $p > n$.

Proof. By Theorem 4.4, $F_s\mathcal{O}_{Q_n}$ contains $\mathcal{O}_{Q_n}(-i)$ as a direct summand if and only if $0 \leq ip \leq n(p - 1)$. Moreover, $F_s\mathcal{O}_{Q_n}$ contains at most one twist of $S_n$. If $p < n$ then $i \leq n - 2$ as $n(p - 1) < p(n - 1)$. This implies that $F_s\mathcal{O}_{Q_n}$ Karoubian generates a proper subcategory of $D^b(Q_n)$ generated by at most $(n - 1)$ line bundles and one twist of spinor bundles. If $p = n$ then $F_s\mathcal{O}_{Q_n}$ is a direct sum of line bundles so it does not generate $D^b(Q_n)$. On the other hand, if $p > n$ then $F_s\mathcal{O}_{Q_n}$ contains as direct summands $\mathcal{O}_{Q_n}, \ldots, \mathcal{O}_{Q_n}(-n + 1)$ and one twist of $S_n$, so it is tilting.

COROLLARY 4.7. If $n = 2m$, $m \geq 2$ and $s \geq 2$ then $F^i_s\mathcal{O}_{Q_n}$ is quasi-exceptional only if $m = 2$, $p = 3$ and $s = 2$. In this case $F^2_s\mathcal{O}_{Q_n}$ is also tilting.

Proof. First, let us assume that $m > 2$ or $p > 3$. Note that $F_s\mathcal{O}_{Q_n}$ or $F_s\mathcal{O}_{Q_n}(-1)$ contain $S_n(-m + [m/p])$ as a direct summand. Set $l_0 = [n - n/p]$. By assumption $l_0 > m$ so $F_s\mathcal{O}_{Q_n}(-l_0)$ contains $S_n(-m)$ as a direct summand. Note that $F_s\mathcal{O}_{Q_n}$ contains $\mathcal{O}_{Q_n}, \mathcal{O}_{Q_n}(-1)$ and $\mathcal{O}_{Q_n}(-l_0)$ as direct summands, so $F^2_s\mathcal{O}_{Q_n}$ contains as direct summands both $S_n(-m + [m/p])$ and $S_n(-m)$. Then Lemma 4.3 implies that $F^2_s\mathcal{O}_{Q_n}$ is not quasi-exceptional. Since $F^2_s\mathcal{O}_{Q_n}$ contains as a direct summand $F^2_s\mathcal{O}_{Q_n}$ it is also not quasi-exceptional.

Now assume that $m = 2$ and $p = 3$. Then $F^2_s\mathcal{O}_{Q_4}$ contains as direct summands only $\mathcal{O}_{Q_4}, \ldots, \mathcal{O}_{Q_4}(-3)$ and $S_4(-1)$. But $F_s(\mathcal{O}_{Q_4}(-3))$ contains $S_4(-2)$ as a direct summand and hence $F^2_s\mathcal{O}_{Q_4}$ is not quasi-exceptional for $s \geq 3$.

COROLLARY 4.8. Assume that $n = 2m + 1$ and $s \geq 2$. If $p \geq n$ then $F^i_s\mathcal{O}_{Q_n}$ is a tilting bundle. If $p < n$ then $F^i_s\mathcal{O}_{Q_n}$ is not quasi-exceptional.

Proof. It is easy to see that if $p \geq n$ then $F_s(\mathcal{O}_{Q_n}(-j))$ for $0 \leq j \leq 2m$ contain as direct summands only line bundles and $\Sigma(-m)$. Moreover, $F_s(\Sigma(-m))$ contains as direct summands only line bundles and $\Sigma(-m)$. This allows easily to check that $F^i_s\mathcal{O}_{Q_n}$ is a tilting bundle.

Now assume that $p < n$. Let us note that by Corollary 4.5 $F_s\mathcal{O}_{Q_n}$ always contains $\mathcal{O}_{Q_n}(-t)$ for $0 \leq t \leq m$, as $0 \leq mp \leq n(p - 1) = mp + (m + 1)(p - 2) + 1$. But by Proposition 4.2, $F_s(\mathcal{O}_{Q_n}(-m))$ contains $\Sigma(-m)$ as a direct summand and $F_s(\mathcal{O}_{Q_n}(-m + \frac{p - 1}{2}))$ contains $\Sigma(-m + 1)$ as a direct summand. This implies that $F^2_s\mathcal{O}_{Q_n}$, and hence $F^i_s\mathcal{O}_{Q_n}$, are not quasi-exceptional.

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