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The Ganikhodjaev Model of ABO Blood Groups

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Abstract. In 2010, N. Ganikhodjaev proposed the models of ABO and Rh blood groups of Malaysian people. Based on some numerical simulations, it was showed that the evolution of ABO blood groups of Malaysian people has a unique stable equilibrium. In this paper, we analytically prove that the Ganikhodjaev model of ABO blood groups has a unique fixed point.

1. Introduction
A blood group provides an ideal opportunity for the study of human variation without cultural prejudice. It can be easily classified for many different genetically inherited blood typing systems. We rarely take blood types into consideration in selecting mates. A few people know their own type today and no one did prior to 1900. As a result, differences in blood type frequencies around the world are most likely due to other factors than social discrimination. ABO blood group is the classification of human blood based on the inherited properties of red blood cells (erythrocytes) as determined by the presence or absence of the alleles A and B, which are carried on the surface of the red cells. Persons may thus have group A (which is carrying only A allele), group B (which is carrying only B allele), group O (which is neither carrying A allele nor B allele), and group AB (which is carrying both A and B alleles). The ABO blood group system has been discovered by the Austrian scientist Karl Landsteiner, who found three different blood types in 1900. He was awarded the Nobel Prize in Medicine in 1930 for his work. When we donate blood or have surgery, a small sample is usually taken in advance for at least ABO and Rh systems typing. We have learned a good deal about how common each of the ABO blood types is around the world. It is quite clear that the distribution patterns are complex. About 21% of all people in the world share the A blood group. The highest frequencies of A are found in small, unrelated populations, especially the Blackfoot Indians of Montana (30-35%), the Australian Aborigines (many groups are 40-53%). Overall in the world, the B blood group is the rarest ABO blood group. Only 16% of humanity have it. Note that it is highest in Central Asia and lowest among the indigenous peoples of the America and Australia. The O blood group is very common around the world. About 63% of humans share it. The group O is particularly high in frequency among the indigenous populations of Central and South America, where it approaches 100%. The rest of people in the world are sharing AB blood group.

A quadratic stochastic operator which was first studied by Bernstein [1] is a primary source for investigations of dynamical properties of population genetics [6, 7] in which it describes a distribution of a species for the next generation if the current distribution of these species was given. It also has a fascinating application in physics, economics (see [17]), and control system
Let \( \|x\|_1 = \sum_{k=1}^{m} |x_k| \) be a norm of a vector \( x = (x_1, \ldots, x_m) \in \mathbb{R}^m \). We say that \( x \geq 0 \) (resp. \( x > 0 \)) if \( x_k \geq 0 \) (resp. \( x_k > 0 \)) for all \( k = 1, m \). Let \( S^{m-1} = \{ x \in \mathbb{R}^m : \|x\|_1 = 1, x \geq 0 \} \) be the \((m-1)\)-dimensional standard simplex. An element of the simplex \( S^{m-1} \) is called a stochastic vector. Recall that a square matrix \( P = (p_{ij})_{i,j=1}^{m} \) is called stochastic if every row is a stochastic vector. A square stochastic matrix \( P = (p_{ij})_{i,j=1}^{m} \) is called positive if \( p_{ij} > 0, \forall i, j = 1, m \). A cubic matrix \( P = (p_{ijk})_{i,j,k=1}^{m} \) is called stochastic if \( \sum_{k=1}^{m} p_{ijk} = 1, \ p_{ijk} \geq 0, \forall i, j, k = 1, m \). Every cubic stochastic matrix is associated with a quadratic stochastic operator \( Q : S^{m-1} \to S^{m-1} \) as follows

\[
(Q(x))_k = \sum_{i,j=1}^{m} x_i x_j p_{ijk}, \quad \forall k = 1, m.
\]  

(2.1)

Let \( \text{Fix}(Q) = \{ x \in S^{m-1} : Q(x) = x \} \) be a fixed point set. Due to Brouwer’s theorem, \( \text{Fix}(Q) \neq \emptyset \). A fixed point of the quadratic stochastic operator is an equilibrium of the system.

In general, the main problem in the nonlinear operator theory is to study the asymptotic behavior of the nonlinear operator. This problem was not fully finished even in the class of quadratic stochastic operators (for more details see [5]). In [10], a special class of the nonlinear operators was studied as a generalization of a logistic mapping into the higher dimension. A fixed point set and an omega limiting set of the quadratic stochastic operators defined on the finite dimensional simplex were deeply studied in the series of papers [2, 4, 12, 15]. Ergodicity and chaotic dynamics of the quadratic stochastic operators on the finite dimensional simplex were studied in [9, 11, 13]. In [5, 8], it was given a long self-contained exposition of the recent achievements and open problems in the theory of the quadratic stochastic operators.

We recall some necessary notions and concepts which will be used in this paper.

The important tool to study a system of polynomial equations is a resultant.

Let us consider two polynomials

\[
f(x) = c_0 x^4 + c_1 x^3 + c_2 x^2 + c_3 x + c_4.
\]  

(2.2)

\[
g(x) = d_0 x^3 + d_1 x^2 + d_2 x + d_3.
\]  

(2.3)

The Sylvester matrix, denoted by \( \text{Syl}(f, g) \), is defined as follows:

\[
\text{Syl}(f, g, x) = \begin{bmatrix}
c_0 & 0 & 0 & d_0 & 0 & 0 & 0 \\
c_1 & c_0 & 0 & d_1 & d_0 & 0 & 0 \\
c_2 & c_1 & c_0 & d_2 & d_1 & d_0 & 0 \\
c_3 & c_2 & c_1 & d_3 & d_2 & d_1 & d_0 \\
c_4 & c_3 & c_2 & c_0 & d_3 & d_2 & d_1 \\
0 & c_4 & c_3 & 0 & d_3 & d_2 & d_1 \\
0 & 0 & c_4 & c_3 & 0 & d_3 & d_2 \\
n & 0 & 0 & 0 & 0 & d_3 & d_2
\end{bmatrix}
\]
The resultant of two polynomials \( f \) and \( g \), denoted by \( \text{Res}(f, g) \), is the determinant of the Sylvester matrix \( \text{Res}(f, g) = \det(\text{Syl}(f, g)) \). After simple calculation, we have the following

\[
\det(\text{Syl}(f, g)) = c_0c_2^3d_1^4 - 4c_0c_2^2d_0d_1^2d_2 + 4c_0c_2^2d_0^2d_1d_2 + 2c_0c_2^2d_0^2d_2^2 + c_0c_1c_4d_0d_1^3d_2 \\
- 5c_0c_3c_4d_0d_1d_2d_3 + 4c_0c_3c_2d_0d_2^2d_3 - 5c_0c_1c_3d_0d_1d_2^3 - 4c_0c_3c_4d_1d_2^3d_3 \\
+ 2c_0c_4d_1^3d_2^3 + 3c_0c_1c_4d_1^2d_2d_3 - 2c_0c_2c_4d_1^3d_2^3 + c_0^2c_4d_2^2 - c_0c_1c_4d_1d_2^3 \\
+ c_0c_2c_4d_1^2d_2^3 - c_0c_3c_1d_1^3d_2 + 3c_0c_3c_4d_0d_1d_2^3 - 2c_0c_2c_4d_1d_2^3 \\
+ 4c_0c_2c_4d_1^2d_2d_3 - 3c_0c_2c_4d_1d_2^3d_3 - c_0^2c_3d_2^2d_3 + 3c_0^2c_3d_1d_2d_3^2 - 3c_0^2c_3d_0d_3^2 \\
+ c_0c_1c_3d_1^2d_2^3 - 2c_0c_1c_3d_1d_2d_3^2 - c_0c_2c_3d_0d_1d_2^2d_3 + c_0c_3d_1^3d_2d_3 \\
- 3c_0c_3d_0d_1d_2d_3^2 + 3c_0c_3d_0d_2d_3^2 - c_0c_1c_3d_0d_2d_3^2 + c_0^2c_3d_1^3d_3 \\
- 2c_0c_4d_1^3d_3^2 - c_0^2c_3d_1^2d_3^2 + c_0c_1c_4d_1d_2d_3^2 + c_1^2c_4d_0d_3^2 \\
+ 3c_0c_1c_2d_0d_3^3 + c_0c_2^2d_1d_3^3 - 2c_0c_2d_0d_2d_3^2 + c_0c_1c_4d_0d_2d_3 + c_1^2c_4d_0d_3^2 \\
- 3c_0c_4d_0d_1d_2d_3 + 3c_0c_4d_0d_2d_3^2 - c_0c_2c_4d_0d_1d_2^3 + 2c_1c_2c_4d_0d_1d_3^2 + c_1c_3c_4d_0d_2d_2 \\
- c_1c_3c_1d_0d_1d_3 + 3c_1c_3c_0d_1d_2d_3 - 3c_1c_3c_0d_2d_3^2 + c_2c_4d_0d_2d_3^2 - 2c_0c_1c_4d_0d_2d_2 \\
- c_1c_1d_0d_2d_3^2 - c_1c_2c_4d_0d_1d_3^2 + c_2c_3c_4d_0d_2d_2 + c_0^2c_3d_0d_3^2 - c_0c_2d_0d_1d_3^2 \\
- 3c_1c_2c_3d_2d_3^2 + c_2c_2d_0d_2d_3^2 - c_0c_3c_1d_0d_2d_3 + 2c_1c_3c_2d_0d_2d_3 - 3c_1d_3d_0d_3^2 + c_0^2c_3d_0d_2d_2 \\
- c_0^2c_3d_1d_3d_2 - c_2c_3c_4d_0d_2d_3 + c_2c_3d_0d_2d_3^2 - 2c_2c_3d_0d_2d_3^2 - c_2c_3d_0d_2d_3^2 \\
- c_2c_3d_0d_2d_3^2 - c_2c_3d_0d_2d_3^2 - c_3c_4d_0d_0d_1 - c_3c_4d_0d_0d_1^2 + c_3c_4d_0d_0d_1 + c_3^3d_0d_3 + c_3^3d_0^3.
\]

(2.4)

**Theorem 2.1** ([16]). The resultant of two polynomials with coefficients in an integral domain is zero if and only if they have a common root in an algebraically closed field containing the coefficients.

We now recall Sturm’s theorem which expresses the number of distinct real roots of a polynomial located in an interval in terms of the number of changes of signs of the values of the Sturm’s sequence at the end points of the interval.

The Sturm sequence of a polynomial \( p(x) \) is the following sequence of polynomials of decreasing degree:

\[
p_0(x) := p(x), \\
p_1(x) := p'(x), \\
p_2(x) := \text{rem}(p_0(x), p_1(x)) = p_1(x)q_0(x) - p_0(x), \\
p_3(x) := \text{rem}(p_1(x), p_2(x)) = p_2(x)q_1(x) - p_1(x), \\
\vdots
\]

\[
0 = \text{rem}_{m-1}(x), p_m(x).
\]

where \( \text{rem}(p_1, p_j) \) and \( q_i \) are the remainder and the quotient of the polynomial long division of \( p_i \) by \( p_j \) and \( m \) is the minimal number of polynomial divisions (never greater than \( \deg(p) \)) needed to obtain a zero remainder. That is, successively take the remainders with polynomial division and change their signs. Since \( \deg(p_{i+1}) < \deg(p_i) \) for \( 0 \leq i < m \), the algorithm terminates. The final polynomial, \( p_m \), is the greatest common divisor of \( p \) and its derivative. If \( p \) is square free, it shares no roots with its derivative, hence \( p_m \) will be a non-zero constant polynomial. A sequence \( p_0, p_1, p_2, \ldots, p_m \) is called the canonical Sturm chain. Let \( \sigma(\xi) \) be the number of sign changes (ignoring zeroes) in the sequence \( p_0(\xi), p_1(\xi), p_2(\xi), \ldots, p_m(\xi) \).
Theorem 2.2 (Sturm’s Theorem, [16]). The number of real zeros of \( p(x) \) in \((\alpha, \beta)\) is \( \sigma(\alpha) - \sigma(\beta) \).

3. The Ganikhodjaev Model of ABO Blood Groups

During the period 2007-2008, Professor N. Ganikhodjaev collected the current blood group distributions of Malaysian people from more than 10 000 randomly chosen families from the states of Pahang and Kuala Lumpur. According to statistics, the current blood distribution in Malaysia was as follows: 20% from A; 29.3% from B; 8.4% from AB; 42.3% from O. The mathematical model of the transmission of ABO blood groups was described as the quadratic stochastic operator acting on 3D simplex (see [3, 4]).

\[
\begin{align*}
\dot{x}_A &= 0.91x_A^2 + 0.6x_Ax_B + 0.92x_Ax_AB + 0.98x_Ax_O + 0.01x_B^2 + 0.20x_Bx_Ax_B + 0.02x_Bx_O + 0.09x_{AB}^2 + 0.38x_Ax_BO + 0.01x_O^2 \\
\dot{x}_B &= 0.01x_A^2 + 0.76x_Ax_B + 0.32x_Ax_AB + 0.02x_Ax_O + 0.94x_B^2 + 1.26x_Bx_Ax_B + 1.06x_Bx_O + 0.06x_{AB}^2 + 0.38x_Ax_BO + 0.01x_O^2 \\
\dot{x}_{AB} &= 0.01x_A^2 + 0.50x_Ax_B + 0.72x_Ax_AB + 0.02x_Ax_O + 0.01x_B^2 + 0.44x_Bx_Ax_B + 0.02x_Bx_O + 0.84x_{AB}^2 + 0.40x_Ax_BO + 0.01x_O^2 \\
\dot{x}_O &= 0.07x_A^2 + 0.14x_Ax_B + 0.04x_Ax_AB + 0.98x_Ax_O + 0.04x_B^2 + 0.10x_Bx_Ax_B + 0.90x_Bx_O + 0.01x_{AB}^2 + 0.84x_Ax_BO + 0.97x_O^2
\end{align*}
\]

Theorem 3.1. The quadratic stochastic operator given above has a unique fixed point which belongs to \( \Omega = \{ x \in S^3 : 0.08 < x_A < 0.15, 0.4 < x_B < 0.6, 0.03 < x_{AB} < 0.09, 0.2 < x_O < 0.4 \} \).

Proof. For the convenience, we use \( x = (x_A, x_B, x_{AB}, x_O) = (x_1, x_2, x_3, x_4) \). It is easy to see that any fixed point of the quadratic stochastic operator given above lies inside of the simplex, i.e., \( x_1, x_2, x_3, x_4 > 0 \). We now want to show that the following system of equations

\[
\begin{align*}
x_1 &= 0.91x_1^2 + 0.01x_2^2 + 0.09x_3^2 + 0.01x_4^2 + 0.6x_1x_2 + 0.92x_1x_3 + 0.98x_1x_4 + 0.2x_2x_3 + 0.02x_2x_4 + 0.38x_3x_4. \\
x_2 &= 0.01x_1^2 + 0.94x_2^2 + 0.06x_3^2 + 0.01x_4^2 + 0.76x_1x_2 + 0.32x_1x_3 + 0.02x_1x_4 + 1.26x_2x_3 + 1.06x_2x_4 + 0.38x_3x_4. \\
x_3 &= 0.01x_1^2 + 0.01x_2^2 + 0.84x_3^2 + 0.01x_4^2 + 0.5x_1x_2 + 0.72x_1x_3 + 0.02x_1x_4 + 0.44x_2x_3 + 0.02x_2x_4 + 0.43x_3x_4. \\
x_4 &= 0.07x_1^2 + 0.04x_2^2 + 0.01x_3^2 + 0.97x_4^2 + 0.14x_1x_2 + 0.04x_1x_3 + 0.98x_1x_4 + 0.10x_2x_3 + 0.92x_2x_4 + 0.84x_3x_4.
\end{align*}
\]

has a unique solution. Since \( x_4 = 1 - x_1 - x_2 - x_3 \), we obtain that

\[
\begin{align*}
x_1 &= -0.06x_1^2 - 0.28x_2^2 - 0.38x_1x_2 - 0.42x_1x_3 - 0.18x_2x_3 + 0.96x_1 + 0.36x_3 + 0.01 \\
x_2 &= -0.11x_1^2 - 0.31x_2^2 - 0.30x_1x_2 - 0.06x_1x_3 - 0.16x_2x_3 + 1.04x_2 + 0.36x_3 + 0.01 \\
x_3 &= 0.45x_3^2 - 0.48x_1x_2 + 0.32x_1x_3 + 0.04x_2x_3 + 0.38x_3 + 0.01
\end{align*}
\] (3.1)

Let \( x_1 = x \) be a variable and \( x_2 = a \), \( x_3 = b \) be parameters.

It follows from the first equation that

\[
x^2 + \left( \frac{19}{3}a + 7b + \frac{2}{3} \right)x + \frac{14}{3}b^2 - 6b + 3ab - \frac{1}{6} = 0
\] (3.2)
and it follows from the second and third equations that
\[ x = \frac{0.11a^2 + 0.31b^2 + 0.16ab - 0.04a - 0.36b - 0.01}{-0.30a - 0.06b}, \]  
(3.3)

\[ x = \frac{-0.45b^2 + 0.62b - 0.04ab - 0.01}{0.48a + 0.32b}. \]  
(3.4)

By equalizing equations (3.3) and (3.4), we get that
\[ 0.0528a^3 + 0.0722b^3 + 0.1a^2b + 0.0626ab^2 - 0.0192a^2 - 0.078b^2 + 0.0004ab - 0.0078a - 0.0038 = 0. \]  
(3.5)

By substituting (3.4) into (3.2), we obtain that
\[ \frac{778}{1875}a^2b^2 - 0.7368ab^3 - \frac{9892}{3000}b^4 + 1.2944ab^2 + 0.1204b^3 + 0.5696a^3b + 0.4896a^2b 
+ \frac{353}{3750}ab + 0.4862b^2 - 0.0688a^2 - 0.0032a - \frac{109}{7500}b + 0.0001 = 0. \]  
(3.6)

Consequently, the system of equations (3.1) has a solution \((x_1, x_2, x_3)\) if and only if the system of equations (3.5) and (3.6) must have a solution \((a, b)\) in which \(x_2 = a, \ x_3 = b\) and \(x\) is defined by (3.4). Note that we are looking for all possible solutions \((a, b)\) of the system of equations (3.5) and (3.6) in which \(0 < a, b < 1, 0 < a + b < 1\) such that
\[ 0 < \frac{-0.45b^2 + 0.62b - 0.04ab - 0.01}{0.48a + 0.32b} < 1, \]  
(3.7)

\[ 0 < a + b + \frac{-0.45b^2 + 0.62b - 0.04ab - 0.01}{0.48a + 0.32b} < 1. \]  
(3.8)

We define two polynomials
\[ h_1(y) = \alpha_0y^4 + \alpha_1y^3 + \alpha_2y^2 + \alpha_3y + \alpha_4, \]  
(3.9)
\[ h_2(y) = \beta_0y^3 + \beta_1y^2 + \beta_2y + \beta_3 \]  
(3.10)

where
\[ \alpha_0 = \frac{-9892}{3000}, \quad \alpha_1 = -0.7368a + 0.1204, \quad \alpha_2 = \frac{1}{3}1.2448a^2 + 1.2944a + 0.4862, \]  
\[ \alpha_3 = 0.5696a^3 + 0.4896a^2 + \frac{1}{3}0.2824a - \frac{1}{3}0.0436, \quad \alpha_4 = -0.0688a^2 - 0.0032a + 0.0001, \]  
\[ \beta_0 = 0.0722, \quad \beta_1 = 0.0626a - 0.078, \quad \beta_2 = 0.1a^2 + 0.0004a - 0.0038, \]  
\[ \beta_3 = 0.0528a^3 - 0.0192a^2 - 0.078a. \]

The simple observation: a pair \((a, b)\) is a solution of the system of equations (3.5) and (3.6) if and only if \(y = b\) is a common root of the two polynomials defined by (3.9) and (3.10).

Due to Theorem 2.1, two polynomials (3.9) and (3.10) have a common root if and only if their resultant is zero. Let us calculate the resultant of two polynomials (3.9) and (3.10):
\[ p(a) := \text{Res}(h_1, h_2) = -0.00000688699757684925a^{12} - 0.0000227411808131865a^{11} \]  
\[ - 0.00003667390703892109a^{10} - 0.00004075705014039745a^9 \]  
\[ - 0.00002866410711261251a^8 - 0.0000663625825939163a^7 \]  
\[ + 0.0000584441237891252a^6 + 0.0000521421301017348a^5 \]  
\[ + 0.000016918649548385a^4 + 0.00002490702870395a^3 \]  
\[ + 0.0000001513772985017a^2 + 0.0000000024893985432a \]  
\[ + 0.00000000000121535388. \]  
(3.11)
We now want to show by means of Sturm’s Theorem that \( p \) has a unique root in \([0, 1]\).

Let us calculate the canonical Sturm chain of the polynomial (3.11)

\[
p_0(a) = -0.00000068869757684925a_{12} - 0.0000227411808131865a_{11} \\
-0.00003667390703892109a_{10} - 0.0000407570514039745a_9 \\
-0.00002866410711261251a_8 - 0.000066362582593916a_7 \\
+0.0000584441237891252a_6 + 0.0000502412301017348a_5 \\
+0.0000169186945948385a_4 + 0.0000024980702870395a_3 \\
+0.0000021537795017a^2 + 0.00000024893985432a \\
+0.00000000212535388,
\]

\[
p_1(a) = -0.000082643970922191a_{11} - 0.0002501529889450515a_{10} \\
-0.0003667390703892109a_9 - 0.00036681345126357705a_8 \\
-0.00022931285690090008a_7 - 0.0000465380781574141a_6 \\
+0.0000356647427347518a_5 + 0.0000260710650586735a_4 \\
+0.000067647783793544a_3 + 0.0000074942108611186a^2 \\
+0.000000302754597034a + 0.000000024893985433a,
\]

\[
p_2(a) = 0.00000037609142076049a_{10} + 0.00000177961548934657a_9 \\
+0.00000114334970673463a_8 - 0.0000024932363898563a_7 \\
-0.0000398743255665794a_6 - 0.0000223751938795324a_5 \\
-0.000005300869143829a_4 - 0.0000003217106723273a_3 \\
+0.0000000457010487098a^2 + 0.0000000046604785482a \\
+0.0000000000049305429,
\]

\[
p_3(a) = 0.00078225025051382988a_9 + 0.00134305877797089479a_8 \\
+0.00017140560958472293a_7 - 0.00095580681574198541a_6 \\
-0.00075689965182997732a_5 - 0.00002176031980386896a_4 \\
-0.00001982504788790518a^3 + 0.0000008604161052218a^2 \\
+0.00000014334804119511a + 0.0000000014344073614a,
\]

\[
p_4(a) = 0.00000088586724688229a_8 + 0.00000228216031456409a_7 \\
+0.0000022380550352612a_6 + 0.00000103574861505588a_5 \\
+0.00000020512611177995a_4 + 0.0000000384770335718a^3 \\
-0.000000032598406321a^2 - 0.0000000025757045472a \\
-0.00000000000241378479,
\]

\[
p_5(a) = 0.00007324940713243183a_7 + 0.00017225002007757511a_6 \\
+0.00015214345301715362a_5 + 0.00006535841671702326a_4 \\
+0.00001403217062030656a^3 + 0.00000138114851832599a^2 \\
+0.000004995584862444a + 0.000000003970537058a,
\]
We now calculate the number of sign changes $\sigma(0)$ and $\sigma(1)$ in the sequences $\{p_0(0), p_1(0), p_2(0), \cdots, p_{12}(0)\}$ and $\{p_0(1), p_1(1), p_2(1), \cdots, p_{12}(1)\}$, respectively.

By simple calculation

$$p_0(0) = 0.00000000000121535389,$$

$$p_1(0) = 0.000000000024893985433,$$

$$p_2(0) = 0.000000000049305429,$$

$$p_3(0) = 0.0000000014344073614,$$

$$p_4(0) = -0.000000000241378479,$$

$$p_5(0) = 0.0000000039705370558,$$

$$p_6(0) = 0.00000000349244959,$$

$$p_7(0) = -0.0000000058857113479,$$

$$p_8(0) = -0.00000000058857113479,$$

$$p_9(0) = -0.00000000033192915659,$$

$$p_{10}(0) = -0.00000000033192915659,$$

$$p_{11}(0) = 0.000000000000040999,$$

$$p_{12}(0) = 0.00000000000000874909.$$

we then obtain that $\sigma(0) = 4$. 

We now calculate the number of sign changes $\sigma(0)$ and $\sigma(1)$ in the sequences $\{p_0(0), p_1(0), p_2(0), \cdots, p_{12}(0)\}$ and $\{p_0(1), p_1(1), p_2(1), \cdots, p_{12}(1)\}$, respectively.
Again, by simple calculation

\[ p_0(1) = -0.00012934381117902623, \]
\[ p_1(1) = -0.0012734118358872744, \]
\[ p_2(1) = -0.00000597634286000726, \]
\[ p_3(1) = 0.00034758512315804309, \]
\[ p_4(1) = 0.00000654729152686279, \]
\[ p_5(1) = 0.00047846496898514639, \]
\[ p_6(1) = 0.0000000004390239328741, \]
\[ p_7(1) = 0.00004242169174538096, \]
\[ p_8(1) = 0.0000000000000000874909, \]
\[ p_9(1) = 0.00000000000000000874909, \]

we then obtain that \( \sigma(1) = 3 \).

Thus, due to Sturm’s Theorem 2.2, the number of roots of the polynomial (3.11) in \([0, 1]\) is \( \sigma(0) - \sigma(1) = 4 - 3 = 1 \). Thus, the polynomial (3.11) has a unique root \( a_0 \) in \([0, 1]\) which belongs to the interval \((0.4, 0.6)\). Approximate value of \( a_0 \) can be calculated by using some software like MAPLE. Moreover, \( b \) is a common root of the polynomials (3.9) and (3.10) such that the inequalities (3.7) and (3.8) are satisfied. Some calculations shows that the inequalities (3.7) and (3.8) are satisfied for only one common root of the polynomials (3.9) and (3.10). In other words, the inequalities (3.7) and (3.8) are satisfied for a unique pair \((a_0, b_0)\). By means of equation (3.4), we can find a unique value of \( x \). It means that the system of equations (3.1) has a unique solution \((x_0, a_0, b_0)\). Therefore, a point \((x_0, a_0, b_0, 1 - x_0 - a_0 - b_0)\) is a unique fixed point of the quadratic stochastic operator. This completes the proof.

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References

[1] Bernstein S 1942 Annals of Mathematical Statistics 13 53-61
[2] Ganikhodjaev N 2001 Dokl. Math. 321-323.
[3] Ganikhodjaev N, Daoud J and Usmanova M 2010 J Applied Science 10(6) 1748-1754
[4] Ganikhodjaev N, Saburov M and Jamilov U 2013 App. Math. Info. Sci. 7(5) 1721-1729
[5] Ganikhodzhaev R, Mukhamedov F and Rozikov U 2011 Inf. Dim. Anal. Quan. Prob. Rel. Top. 14(2) 279-335
[6] Kesten H 1970 Adv. App. Prob. 2 1-82
[7] Lyubich Yu I 1992 Mathematical structures in population genetics (Springer–Verlag)
[8] Mukhamedov F and Ganikhodjaev N 2015 Lecture Notes Math 2133 (Berlin: Springer)
[9] Mukhamedov F and Embong A F 2015 J. Inequal. Appl. 2015 (2015) 226
[10] Mukhamedov F and Saburov M 2014 Bull. Malay. Math. Soc. Soc 37 59-64
[11] Saburov M 2015 Annals of Functional Analysis 6(4) 247-254
[12] Saburov M 2016 Mathematical Notes 99 (4) 635-639
[13] Saburov M 2016 Nonlinear Analysis: Theo Method Appl 143 105-119
[14] Saburov M and Saburov Kh 2016 Inter Journal of Sys Sci 47(10) 2449-2457
[15] Saburov M and Yusof N A 2016 Linear Algebra and Appl 507 153-157
[16] Sturmfels B 2002 Solving systems of polynomial equations (Texas)
[17] Ulam S 1960 A collection of mathematical problems (New-York, London)