Constraint likelihood method: generalization for colored noise.

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Abstract. We present a method for detection and reconstruction of gravitational wave signals with a network of interferometric detectors. The method is based on the constraint maximization of the likelihood ratio functional. We describe the method for the cases of white and colored detector noise.

1. Introduction
Several gravitational wave (GW) detectors are now operating around the world, including both laser interferometers [1, 2, 3, 4] and resonant mass detectors [5]. Coherent analysis of data from such a network of detectors can benefit both the detection of GW signals and estimation of signal parameters. Networks of detectors are particularly important for searches of gravitational wave burst signals. These are defined to be broadband signals that may come either from unanticipated sources or from sources for which no reliable theoretical prediction exists for signal waveforms. Potential astrophysical sources of burst signals are stellar core collapse in Supernovae [6], mergers of binary neutron star or black hole systems [7] and Gamma Ray Burst progenitors [8].

Several papers [9, 10, 11] in the GW literature have explored formal approaches to constructing coherent network analysis algorithms for burst GW signals. Klimenko et al [12] have recently proposed a modification of the standard likelihood approach [10, 13, 14] by introducing constraints imposed on the GW signal waveforms. This approach yields a new class of detection and reconstruction methods called constraint likelihood methods.

In this paper we generalize the results obtained in [12] for the case of white Gaussian noise to the case of colored detector noise. The rest of the paper is organized as follows. Section 2 lays out much of the basic notation and conventions used in the paper. In Section 3, we provide an overview of the constrained likelihood method. Section 4 describes the constrained likelihood method for the case of colored detector noise.

2. Likelihood analysis of gravitational wave data
In this section, we present a brief overview of the standard likelihood approach to the detection and reconstruction of gravitational wave burst signals using a network of detectors. Consider a network consisting of $K$ detectors with arbitrary detector alignment. Let the time series from the $k$th detector be $x_k$ represented by a column vector containing $N$ samples. The detector noise is described by the covariance matrix $C_k$. By definition, $C_k$ is an $N \times N$ real, symmetric and
positive definite matrix. We will define an inner product between two time series \( x_k \) and \( y_k \) as
\[
\langle x_k, y_k \rangle_k = x_k^T C_k^{-1} y_k.
\] (1)

Assuming that the noise in each detector is Gaussian and stationary, the logarithmic likelihood ratio is given by
\[
L = \sum_{k=1}^{K} \left( -\frac{1}{2} \langle x_k - \xi_k, x_k - \xi_k \rangle_k + \frac{1}{2} \langle x_k, x_k \rangle_k \right),
\] (2)

where \( \xi_k \) are the detector responses to a gravitational wave. In the likelihood ratio the detector responses are calculated for a fixed direction to the source and the corresponding light travel time shifts are introduced to each detector data stream. The detector response is given by
\[
\xi_k = F_{+k} h_+ + F_{\times k} h_\times,
\] (3)

where \( h_+ \) and \( h_\times \) are the GW polarization components expressed in the wave frame. In this coordinate frame a gravitational wave propagates in the direction of the \( z \) axis pointing to the center of Earth, and the \( x \) and \( y \) axes may have an arbitrary orientation.

To characterize the angular and strain sensitivity of the network, we introduce the network response matrix
\[
G = \begin{pmatrix} g_{++} & g_{+\times} \\ g_{+\times} & g_{\times\times} \end{pmatrix}.
\] (4)

The matrices \( g_{++}, g_{\times\times} \) and \( g_{+\times} \) are the network antenna patterns
\[
g_{ab} = \sum_{k=1}^{K} F_{ak} F_{bk} C_k^{-1},
\] (5)

where indexes \( a \) and \( b \) can be either \(+\) or \(\times\). We also define the network data matrices \( X_+ \) and \( X_\times \) which combine the output time series \( x_k \) from individual detectors
\[
X_+ = \sum_{k=1}^{K} F_{+k} C_k^{-1} x_k, \quad X_\times = \sum_{k=1}^{K} F_{\times k} C_k^{-1} x_k.
\] (6)

By expanding the inner products in the expression for the likelihood ratio in Eq. 2, with this new notations the likelihood functional can be written as
\[
L = \left( X_+^T \quad X_\times^T \right) \begin{pmatrix} h_+ \\ h_\times \end{pmatrix} - \frac{1}{2} \begin{pmatrix} h_+^T & h_\times^T \end{pmatrix} G \begin{pmatrix} h_+ \\ h_\times \end{pmatrix}.
\] (7)

The equations for the GW waveforms are obtained by variation of the likelihood functional over two unknown functions \( h_+ \) and \( h_\times \)
\[
\begin{pmatrix} X_+ \\ X_\times \end{pmatrix} = G \begin{pmatrix} h_+ \\ h_\times \end{pmatrix}.
\] (8)

The maximum likelihood ratio (MLR) statistic \( L_{\text{max}}(\hat{h}_+, \hat{h}_\times) \) is defined as the maximum value of the likelihood functional given by the solutions \( \hat{h}_+ \) and \( \hat{h}_\times \) of Eq. 8.
3. White detector noise

The covariance matrix of white Gaussian and stationary noise is $C = \sigma^2 I$ where $\sigma^2$ is the variance of the noise and $I$ is the identity matrix. In this case the network antenna pattern matrices are

$$ g_{ab} = g_{ab} I, \quad g_{ab} = \sum_{k=1}^{K} \frac{F_{ak} F_{bk}}{\sigma_k^2}, $$

(9)

where $a, b = +, \times$. The network response matrix depends on three parameters $g_{+\times}, g_{\times\times}$ and $g_{+\times}$. However, it can be shown that the likelihood functional depends only on the eigenvalues of the matrix $G$. Indeed, we have a freedom to select an arbitrary wave frame by applying a rotation around $z$ axis. The rotation induces the transformation of the GW polarizations and the detector antenna patterns, but the detector responses and the likelihood ratio remain invariant. We can always select such a wave frame, called below the dominant polarization frame (DP), where $g_{+\times} = 0$ and the ratio $\epsilon = g_{\times\times}/g_{+\times}$ is positive definite. In this frame, the network matrix takes a diagonal form and the likelihood functional can be written as $L = L_1 + L_2$:

$$ L_1 = X_1^T h_1 - \frac{g}{2} h_1^T h_1, \quad L_2 = X_2^T h_1 - \frac{\epsilon g}{2} h_2^T h_2, $$

(10)

where $X_1, X_2$ are the network data matrices and $h_1, h_2$ are the GW polarization components calculated in the DP frame. We will distinguish them from the data matrices $X_{+}, X_{\times}$ and polarizations $h_{+}, h_{\times}$ defined for an arbitrary wave frame. The MLR statistic for each component is

$$ L_{max1} = \frac{1}{2} \sum_{i=1}^{N} \frac{X_1[i]}{g}, \quad L_{max2} = \frac{1}{2} \sum_{i=1}^{N} \frac{X_2[i]}{\epsilon g} $$

(11)

where the sums are taken over the data samples. The parameter $g = g_{++}$ characterizes the overall network sensitivity to the gravitational waves. The ratio $\epsilon$ (network alignment factor) shows the relative sensitivity of the network to the GW components $h_1$ and $h_2$. From Eq. 11 it follows that the maximum likelihood ratio statistic satisfies

$$ L_{max} \approx \frac{g}{2} (\langle h_{w1}^2 \rangle + \epsilon \langle h_{w2}^2 \rangle) $$

(12)

where $||h_{w1}|| >$ and $||h_{w2}||$ are the sum-square energies carried by each GW component. Therefore, to contribute equally into the detection statistic, the $h_{w2}$ wave should carry $1/\epsilon$ times more energy than the $h_{w1}$ wave. The alignment factor reflects also the angular alignment of the detectors. For co-aligned detectors $\epsilon = 0$ and the $h_{w2}$ component of the GW signal can not be detected. Examples of $g$ and $\epsilon$ calculated as functions of sky coordinates for several network configurations are shown in [12].

4. Colored detector noise

The covariance matrix for stationary noise implies $C_{ij} = \phi(|i - j|)$ where $\phi$ is the autocovariance function of the noise. Thus, the vector $x = Cy$ is the convolution of $y$ with the autocovariance function $\phi$. Hence, in the Fourier domain the vectors $x$ and $y$ will be related as

$$ \tilde{x}[f] = \tilde{y}[f]S[f], $$

(13)

where $\tilde{x}[f], \tilde{y}[f]$ and $S[f]$ are the Fourier components of $x, y$ and the power spectral density of the noise at the $f$th frequency bin respectively.
We can re-write the linear equations 8 in the Fourier domain

\[
\begin{pmatrix}
\tilde{X}_+ \\
\tilde{X}_x
\end{pmatrix} = \mathbf{H} \begin{pmatrix}
\tilde{h}_+ \\
\tilde{h}_x
\end{pmatrix},
\]  

(14)

where,

\[
\mathbf{H} = \begin{pmatrix}
\text{diag} \left( \sum_{k=1}^{K} \frac{F^2_{k,k}}{S_k[f]} \right) & \text{diag} \left( \sum_{k=1}^{K} \frac{F_{k,k}F_{x,k}}{S_k[f]} \right) \\
\text{diag} \left( \sum_{k=1}^{K} \frac{F_{x,k}F_{k,k}}{S_k[f]} \right) & \text{diag} \left( \sum_{k=1}^{K} \frac{F^2_{x,k}}{S_k[f]} \right)
\end{pmatrix}.
\]  

(15)

and \(\text{diag}(a_i)\) denotes a \(N \times N\) matrix with elements \(a_1, a_2, \ldots, a_N\) on the diagonal. The \(S_k\) is the power spectral density of the noise in the \(k\)th detector. As follows from Eq. 14 and Parseval’s theorem, the likelihood functional in the Fourier domain is

\[
\mathcal{L} = \left( \tilde{X}_+^T \tilde{X}_x^T \right) \begin{pmatrix}
\tilde{h}_+ & 0 \\
0 & \tilde{h}_x
\end{pmatrix} \frac{1}{2} \left( \tilde{h}_+^T \tilde{h}_x^T \right) \mathbf{H} \begin{pmatrix}
\tilde{h}_+ \\
\tilde{h}_x
\end{pmatrix}.
\]  

(16)

Let us now consider the matrix \(\mathcal{H}[f]\) defined as,

\[
\mathcal{H}[f] = \begin{pmatrix}
\sum_{k=1}^{K} \frac{F^2_{k,k}}{S_k[f]} & \sum_{k=1}^{K} \frac{F_{k,k}F_{x,k}}{S_k[f]} \\
\sum_{k=1}^{K} \frac{F_{x,k}F_{k,k}}{S_k[f]} & \sum_{k=1}^{K} \frac{F^2_{x,k}}{S_k[f]}
\end{pmatrix}.
\]  

(17)

This matrix consists on the non-zero elements from rows \(f\) and \(f+N\) of the matrix \(\mathbf{H}\). Similar to the case of white noise we can diagonalize the matrix \(\mathcal{H}[f]\) by rotation of the wave frame. By introducing the rotation operator \(\mathcal{U}[f]\) we can find \(\mathcal{H}[f]\) in the dominant polarization frame

\[
\mathcal{U}[m] \mathcal{H}[m] \mathcal{U}[m]^T = g[m] \begin{pmatrix} 1 & 0 \\ 0 & \epsilon[m] \end{pmatrix},
\]  

(18)

where \(g[f]\) and \(\epsilon[f]\) are the network sensitivity and the alignment factor for the frequency bin \(f\). As one can see, for colored noise the selection of the dominant polarization frame is different for different frequency bins. Therefore, we define a global rotational operator \(\mathbf{U}\)

\[
\mathbf{U} = \begin{pmatrix}
\text{diag}(\mathcal{U}_{1,1}[f]) & \text{diag}(\mathcal{U}_{1,2}[f]) \\
\text{diag}(\mathcal{U}_{2,1}[f]) & \text{diag}(\mathcal{U}_{2,2}[f])
\end{pmatrix}
\]  

(19)

where \(\mathcal{U}_{m,n}[f]\) is the element in row \(m\) and column \(n\) of the matrix \(\mathcal{U}[f]\). Then

\[
\mathbf{U} \mathbf{H} \mathbf{U}^T = \begin{pmatrix}
\text{diag}(g[f]) & 0 \\
0 & \text{diag}(g[f] \epsilon[f])
\end{pmatrix},
\]  

(20)

where \(\mathbf{0}\) denotes the zero matrix. We define the dominant polarization frame in which the GW components and the network data vectors are represented as,

\[
\begin{pmatrix}
\tilde{h}_1 \\
\tilde{h}_2
\end{pmatrix} = \mathbf{U} \begin{pmatrix}
\tilde{h}_+ \\
\tilde{h}_x
\end{pmatrix}, \quad \begin{pmatrix}
\tilde{X}_1 \\
\tilde{X}_2
\end{pmatrix} = \mathbf{U} \begin{pmatrix}
\tilde{X}_+ \\
\tilde{X}_x
\end{pmatrix}
\]  

(21)

The expressions for the likelihood functionals \(\mathcal{L}_1\) and \(\mathcal{L}_2\) can be now written as,

\[
\mathcal{L}_1 = \sum_{f=1}^{N} \left( \tilde{X}_1[f] \tilde{h}_1[f] - \frac{g[f] \epsilon[f]}{2} |\tilde{h}_1[f]|^2 \right),
\]  

(22)

\[
\mathcal{L}_2 = \sum_{f=1}^{N} \left( \tilde{X}_2[f] \tilde{h}_2[f] - \frac{g[f] \epsilon[f]}{2} |\tilde{h}_2[f]|^2 \right),
\]  

(23)
where we explicitly sum over the frequency bins. The maximum likelihood ratio statistics for both terms can be trivially obtained from variation of the likelihood functionals

\[ L_{\text{max}1} = \frac{1}{2} \sum_{f=1}^{N} \frac{|\tilde{X}_1[f]|^2}{g[f]}, \quad L_{\text{max}2} = \frac{1}{2} \sum_{f=1}^{N} \frac{|\tilde{X}_2[f]|^2}{\epsilon[f]g[f]} \]. \tag{24}

5. Constraint likelihood

As follows from Eq. 3 and Eq. 6, in the presence of a gravitational wave \((\tilde{h}_{w1}, \tilde{h}_{w2})\), the network data vectors are

\[ \tilde{X}_1 = \tilde{n}_1 + \text{diag}(g[f])\tilde{h}_{w1}, \quad \tilde{X}_2 = \tilde{n}_2 + \text{diag}(\epsilon[f]g[f])\tilde{h}_{w2}. \] \tag{25}

Hence, the estimators for the GW waveforms are affected by the detector noise \(\tilde{n}_1\) and \(\tilde{n}_2\)

\[ \hat{h}_1[f] = \frac{\tilde{n}_1}{g[f]} + \tilde{h}_{w1}[f], \quad \hat{h}_2[f] = \frac{\tilde{n}_2}{g[f]\epsilon[f]} + \tilde{h}_{w2}[f]. \] \tag{26}

In average \(|\tilde{n}_2|^2 \approx \epsilon|\tilde{n}_1|^2\) and the relative error in the reconstruction of the second component is larger by a factor of \(1/\sqrt{\epsilon}\). Consequently, for sky directions where \(\epsilon \ll 1\), the reconstructed solution for the GW component \(h_2\) can be highly un-physical. In this case, assuming that both components carry on average the same energy, one should reduce the effect of the noisier term \(L_{\text{max}2}\) by imposing constraints on the second GW component \(h_2\). The simplest way is to require \(h_2 = 0\), which impose the hard constraint and yield the MLR statistic \(L_{\text{hard}} = L_{\text{max}1}\). Another possible approach is to equalize the noise amplitudes in the measurements of \(\hat{h}_1[f]\) and \(\hat{h}_1[f]\) by multiplying the second waveform by \(\sqrt{\epsilon}\), which results in the soft constraint MLR statistic:

\[ L_{\text{soft}} = \frac{1}{2} \sum_{f=1}^{N} \frac{|\tilde{X}_1[f]|^2 + |\tilde{X}_2[f]|^2}{g[f]}. \] \tag{27}

Both constraints are discussed in details in [12] in the case of the white Gaussian detector noise.

6. Conclusion

We have presented a method for detection and reconstruction of gravitational waves with an arbitrary network of interferometric detectors. It is based on the constraint maximization of the likelihood ratio functional. Constraints limit the parameter space of the GW waveforms, excluding the waveforms to which the detectors may not be sensitive. The constraint maximum likelihood ratio statistics is obtained for the case of white and colored Gaussian detector noise.

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