Multipole Hair of Schwarzschild-Tangherlini Black Holes

Matthew S. Fox

Harvey Mudd College,
Claremont, California 91711, USA

E-mail: msfox@g.hmc.edu

ABSTRACT: We study the field of a point charge that is slowly lowered into an $n+1$ dimensional Schwarzschild-Tangherlini black hole. For an electric charge, we find that if $n > 3$, then countably infinite nonzero multipole moments manifest to outside observers as the charge approaches the event horizon. This suggest the final state of the black hole is not characterized by a Reissner-Nordström-Tangherlini (RNT) geometry. Instead, for odd $n$, the final state either possesses a degenerate horizon, undergoes a discontinuous topological transformation during the infall of the charge, or both. For even $n$, the final state is not guaranteed to be asymptotically flat.

KEYWORDS: Black Holes, Black Holes in String Theory

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1 Introduction

The properties of four-dimensional black holes are rigidly constrained. For instance, all stationary and asymptotically-flat black hole solutions to the Einstein-Maxwell equations are topologically spherical and unique up to the choice of three asymptotic observables: mass, electric charge, and angular momentum [1–7]. This is Wheeler’s famous “no-hair theorem” (NHT) [8].

Higher dimensional black holes are less constrained, largely for two reasons.\(^\dagger\) One, the rotation group $\text{SO}(n)$ permits $\left\lfloor \frac{n}{2} \right\rfloor$ independent angular momenta. Accordingly, the rotational degrees of freedom of black holes in $n + 1$ dimensional spacetime become progressively more complex as $n$ increases [10, 11]. Furthermore, black holes with fixed masses in $n \geq 5$ spatial dimensions may have arbitrarily large angular momentum [12]. Two, Hawking’s theorem on the topology of black holes [5] does not directly generalize to higher dimensions because his proof relies on the Gauss-Bonnett theorem. Although topological restrictions exist for higher dimensional black holes [13–15], a hyperspherical topology is not the only option [11, 16]. As a result, extended black $p$-branes are not precluded in higher dimensional spacetimes [11, 16, 17]. These results imply that the uniqueness theorems for four-dimensional black holes do not immediately extend to higher dimensions.

However, if restricted to solutions with hyperspherical topology and non-degenerate horizons, then the Schwarzschild-Tangherlini (ST) black hole [18] is the unique static and asymptotically-flat vacuum solution to the higher dimensional Einstein equations [10, 19–21]. Furthermore, the higher dimensional Reissner-Nordström (RN-Tangherlini, or simply RNT) black hole is the unique static and asymptotically-flat electrovac solution to the

\(^\dagger\)See Ref. [9] for a separate and less heuristic perspective.
higher dimensional Einstein-Maxwell equations [22, 23]. Non-uniqueness is most apparent in the context of stationary black hole solutions [11, 12, 16, 17].

For four-dimensional black holes, Wheeler’s NHT implies that the process of slowly\(^2\) lowering an electric point charge of strength \(q\) into a Schwarzschild black hole of mass \(M\) results in a RN black hole of mass \(M\) and charge \(q\). Furthermore, the resulting black hole does not possess unconserved charges like electric multipole moments (excluding the monopole) as these are “hair” for the black hole. The details of this process can be found in Ref. [24].

That in four dimensions the slow infall of an electric charge into a Schwarzschild black hole results in only one type of black hole — the RN black hole — may be viewed as a corollary of the uniqueness theorem for RN black holes. In the same way, the uniqueness of RNT black holes ostensibly implies that a sufficiently slow infall of an electric charge into a ST black hole results in a unique final state — the RNT black hole. If this is the final state, then, due to the hyperspherical symmetry of RNT spacetime, all electric multipole moments (except the monopole) necessarily vanish as the charge approaches the event horizon.

Following the analyses of Refs. [24] and [25], we prove the contrary: if an electric point charge falls slowly into a ST black hole, then the final state acquires countably infinite nonzero multipole moments. Depending on the spatial dimension \(n\), these multipole anisotropies need not even be finite. This suggests the resulting black hole is not RNT in nature, and, depending on \(n\), brings about the possibility of destruction of the horizon. At the same time, we analyze the effect of lowering a non-electric point source whose field obeys Laplace’s equation into a ST black hole. In this case, we find (in agreement with a result by Persides [25] for the \(n = 3\) case) that the final state acquires countably infinite nonzero multipole moments, not all of which are finite.

In this paper, we employ the metric signature \((- + \cdots +)\) and work in the natural system of units in which \(c = G = 1\). We also adopt the following notation: \(S^{n+1}\) is \(n + 1\) dimensional ST spacetime, \(R^n\) is \(n\) dimensional Euclidean space, \(C\) is the complex plane, \(Z^+\) is the set of positive integers, \(Z^* \equiv Z^+ \cup \{0\}\) is the set of nonnegative integers, and \(S^{n-1}\) is the unit \(n - 1\) sphere.

2 Schwarzschild-Tangherlini Geometry

The \(n + 1\) dimensional ST black hole is described by the \(n + 1\) dimensional ST spacetime, \(S^{n+1}\). In this spacetime, there exists a chart \((U^{n+1}, \psi)\) (the ST chart) with map \(\psi \equiv (t, r, \varphi) : U^{n+1} \subseteq S^{n+1} \to R^{n+1}\) that reduces to the canonical four-dimensional Schwarzschild map \((t, r, \theta, \phi) : U^{3+1} \subseteq S^{3+1} \to R^4\) when \(n = 4\). In this way, the ST chart is a direct generalization of the Schwarzschild chart. In the ST chart, the coordinates \(t : U^{n+1} \to R\) and \(r : U^{n+1} \to R\) retain the meaning (outside the event horizon) of “time as measured by an asymptotic observer” and “circumferential radius as measured by an asymptotic observer.” The angular functions \((\theta, \phi)\), however, are generalized to the hyperspherical coordinates \(\varphi \equiv (\varphi_1, \ldots, \varphi_{n-1})\), where \(\varphi_i : U^{n+1} \to [0, \pi]\) for \(i = 1, \ldots, n - 2\) and \(\varphi_{n-1} : U^{n+1} \to [0, 2\pi]\).

\(^2\)By “slowly” we mean “slow enough that our static considerations remain valid.”
In the ST chart, the metric $g$ of $\mathbb{S}^{n+1}$ possesses the line element
\[ g(\mathrm{d}\psi, \mathrm{d}\psi) = -\left(1 - \frac{2M}{r^{n-2}}\right) \mathrm{d}t^2 + \left(1 - \frac{2M}{r^{n-2}}\right)^{-1} \mathrm{d}r^2 + r^2 \gamma(\mathrm{d}\varphi, \mathrm{d}\varphi), \quad (2.1) \]
where $\gamma$ is the metric of $\mathbb{S}^{n-1}$ with line element
\[ \gamma(\mathrm{d}\varphi, \mathrm{d}\varphi) = \mathrm{d}\varphi_1^2 + \sum_{i=2}^{n-1} \prod_{j=1}^{i-1} \sin^2 \varphi_j \mathrm{d}\varphi_i^2. \quad (2.2) \]

The value $M$ in Eq. (2.1) is a constant related to the physical mass $M$ of the black hole by
\[ M \equiv \frac{8\pi M}{(n-1)\Omega_{n-1}}, \quad (2.3) \]
where $\Omega_{n-1} \equiv 2\pi^{n/2}/\Gamma(n/2)$ is the volume of $\mathbb{S}^{n-1}$. The singular nature of Eq. (2.1) at $r = r_s \equiv (2M)^{1/(n-2)}$ (the ST radius) is an artifact of the choice of chart (an appropriate diffeomorphism will transform it away). However, the singular nature at $r = 0$ is a genuine curvature singularity (the Kretschmann scalar is infinite there). The locus of points for which $r = r_s$ constitute the event horizon of the black hole and the singular point for which $r = 0$ is the singularity.

Of interest to us is the effect of the geometry (2.1) on the form of Laplace’s equation. Using the abstract index notation, the Laplacian $\Delta$ on a general $n+1$ dimensional spacetime with metric $g$ is defined by [26]
\[ \Delta \equiv \frac{1}{\sqrt{|\det g|}} \partial_i \left( \sqrt{|\det g|} g^{ij} \partial_j \right), \quad (2.4) \]
where vertical bars $| \cdot |$ denote absolute value and Latin indices run over the spatial components from 1 to $n$ corresponding to (in hyperspherical coordinates) $r, \varphi_1, \ldots, \varphi_{n-1}$, respectively. The Laplacian on $\mathbb{S}^{n+1}$ in the ST chart, $\Delta_{\mathbb{S}^{n+1}}$, is thus
\[ \Delta_{\mathbb{S}^{n+1}} = \frac{1}{r^{n-1}} \partial_r \left[ r^{n-1} \left(1 - \frac{r_s^{n-2}}{r^{n-2}}\right) \partial_r \right] + \frac{1}{r^2} \Delta_{\mathbb{S}^{n-1}}, \quad (2.5) \]
where $\Delta_{\mathbb{S}^{n-1}}$ is the Laplacian on $\mathbb{S}^{n-1}$ in hyperspherical coordinates (the hyperspherical Laplacian).

### 2.1 Hyperspherical Harmonics

The eigenfunctions of the hyperspherical Laplacian constitute the higher dimensional generalization of the canonical spherical harmonics on $\mathbb{S}^2$. These eigenfunctions are the hyperspherical harmonics. Specifically, an $n$ dimensional hyperspherical harmonic of degree $k \in \mathbb{Z}^*$ is a map $Y_k : \mathbb{S}^{n-1} \to \mathbb{C}$ satisfying
\[ \Delta_{\mathbb{S}^{n-1}} Y_k(\varphi) = -k(k + n - 2) Y_k(\varphi), \quad (2.6) \]
among other conditions [27]. Indeed, for the case $n = 3$, Eq. (2.6) reduces to the equation $\Delta_{\mathbb{S}^2} Y_k(\varphi) = -k(k + 1) Y_k(\varphi)$, which is familiar from quantum mechanics.
Importantly, if \( k \neq l \), then the functions \( Y_k(\varphi) \) and \( Y_l(\varphi) \) can be chosen to be orthogonal over \( S^{n-1} \) with respect to the inner product [28]

\[
\langle Y_k, Y_l \rangle = \int_{S^{n-1}} \hat{Y}_k(\varphi) Y_l(\varphi) \, d\Omega_{n-1} = 0, \tag{2.7}
\]

where a hat denotes complex conjugation and \( d\Omega_{n-1} \) is the natural volume form on \( S^{n-1} \).

For fixed \( n \geq 3 \), the degree of a hyperspherical harmonic completely determines the number of hyperspherical harmonics of the same degree that are linearly independent to it. With this in mind, we denote by \( \Gamma_k \) the number of linearly independent hyperspherical harmonics of degree \( k \). For \( n \geq 3 \), a combinatorial argument [27, 28] proves

\[
\Gamma_k = \frac{(2k + n - 2)(n + k - 3)!}{k!(n-2)!}. \tag{2.8}
\]

Using the Gram-Schmidt orthonormalization procedure, one can then produce an orthonormal set of hyperspherical harmonics \( \{Y^i_k\}_{i=1}^{\Gamma_k} \) satisfying

\[
\langle Y^i_k, Y^j_l \rangle = \int_{S^{n-1}} \hat{Y}^i_k(\varphi) Y^j_l(\varphi) \, d\Omega_{n-1} = \delta_{k,l} \delta_{i,j}, \tag{2.9}
\]

where \( \delta_{k,l} \) is the Kronecker delta. These functions constitute an orthonormal basis for all square-integrable functions on \( S^{n-1} \) [27]. Thus, the hyperspherical harmonics obey the completeness relation

\[
\sum_{k \geq 0} \sum_{l=1}^{\Gamma_k} \hat{Y}^i_k(\vartheta) Y^j_l(\varphi) = \delta^{n-1}(\varphi - \vartheta), \tag{2.10}
\]

where \( \vartheta : U^{n+1} \rightarrow S^{n-1} \) is a hyperspherical coordinate and \( \delta \) is the Dirac delta function.

### 2.2 Poisson’s Equation

Consider now a real-valued test field \( \Phi : U^{n+1} \rightarrow \mathbb{R} \), i.e., a real scalar field weak enough that the geometry is unaffected by it. Let \( \Phi \) satisfy the d’Alembert wave equation,

\[
\Box \Phi \equiv \nabla^\mu \nabla_\mu \Phi = \Omega_{n-1} f(t, r, \varphi), \tag{2.11}
\]

where \( f : U^{n+1} \rightarrow \mathbb{R} \) is a well-behaved function, \( \nabla \) is the covariant derivative with respect to the metric \( g \) on \( S^{n+1} \), and Greek indices run from 0 to \( n \) corresponding to \( t, r, \varphi_1, \ldots, \varphi_{n-1} \), respectively.

In the case where both \( \Phi \) and \( f \) are time-independent, Eq. (2.11) reduces to Poisson’s equation,

\[
\Delta_{S^{n+1}} \Phi(r, \varphi) = \Omega_{n-1} f(r, \varphi). \tag{2.12}
\]

Importantly, the equation of motion for the field of an electric point charge in the vicinity of a ST black hole assumes the form of Eq. (2.12). To see this, recall that in a general curved spacetime, Maxwell’s equations can be expressed as [26]

\[
\Omega_{n-1} j^\nu = \frac{1}{\sqrt{\det g}} \partial_\mu \left( \sqrt{\det g} F^{\mu\nu} \right), \tag{2.13}
\]
where \( F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu \) are the components of the Faraday tensor and \( A_\mu \) are the components of the \( n + 1 \) potential. In particular, in an appropriate gauge, \( V \equiv A_0 \) is the electric potential. Assuming no magnetic fields \( (A_i = 0) \) and static electric fields (time-independent \( V \)), Eq. (2.13) vanishes trivially for \( \nu = 1, \ldots, n + 1 \). However, for \( \nu = 0 \), Eq. (2.13) reduces to the nontrivial expression

\[
\Omega_{n-1} j^0 = g^{00} \Delta_{\mathbb{S}^{n+1}} V - \frac{g^{00} r_s^{n-2} (n-2)}{r^{n-1}} \partial_r V. \tag{2.14}
\]

As \( V(r, \varphi) \) is time-independent (by assumption), so is the physical source \( j^0(r, \varphi) \). Hence, the electrostatic problem reduces to solving Poisson’s equation with the effective source

\[
f(r, \varphi) = g^{00} j^0(r, \varphi) + \frac{r_s^{n-2} (n-2)}{\Omega_{n-1} r^{n-1}} \partial_r V(r, \varphi). \tag{2.15}
\]

Consequently, to understand Eq. (2.12) is to understand electrostatics in \( \mathbb{S}^{n+1} \).

3 Radial Equation

We consider first Poisson’s equation (2.12) in the absence of sources, \( f(r, \varphi) = 0 \). In this case, Eq. (2.12) reduces to Laplace’s equation,

\[
\Delta_{\mathbb{S}^{n+1}} \Phi(r, \varphi) = 0. \tag{3.1}
\]

We look for solutions to Eq. (3.1) of the form

\[
\Phi(r, \varphi) = \sum_{k \geq 0} R_k(r) Y_k(\varphi). \tag{3.2}
\]

Using Eq. (2.5) and the eigenfunction relation (2.6), one deduces that, for each \( k \in \mathbb{Z}^* \), \( R_k(r) \) must satisfy

\[
R_k'' + \frac{(n-1)r_s^{n-2} - r_s^{n-2}}{r(r_s^{n-2} - r_s^{n-2})} R_k' - \frac{k(k + n - 2)r_s^{n-4}}{r_s^{n-2} - r_s^{n-2}} R_k = 0, \tag{3.3}
\]

where a prime denotes differentiation with respect to \( r \).

Crucially, had we instead focused on the electrostatic model with source (2.15) and assumed an absence of physical sources \( (j^0 = 0) \), then the associated radial equation would be

\[
R_k'' + \frac{(n-1)(r_s^{n-2} - r_s^{n-2})}{r(r_s^{n-2} - r_s^{n-2})} R_k' - \frac{k(k + n - 2)r_s^{n-4}}{r_s^{n-2} - r_s^{n-2}} R_k = 0. \tag{3.4}
\]

Evidently, the only difference between Eqs. (3.3) and (3.4), and thus the consequence of the additional effective sourcing term in Eq. (2.15), is the factor of \( n - 1 \) on \( r_s^{n-2} \) in the numerator of the \( R_k' \) coefficient. Thus, to capture both a general test field \( \Phi \neq V \) and the static electric potential \( \Phi = V \) in this analysis, we introduce into Eq. (3.3) the piecewise function

\[
\sigma(\Phi) \equiv \begin{cases} 
1 & \text{if } \Phi \neq V \text{ (non-electrostatic field)}, \\
-1 & \text{if } \Phi = V \text{ (electrostatic field)}. 
\end{cases} \tag{3.5}
\]
This way, the general differential equation capturing the radial behavior of both fields is

\[ R''_k + \frac{(n-1)r^{n-2} - \sigma r_s^{n-2}}{r(r^{n-2} - r_s^{n-2})} R'_k - \frac{k(k + n - 2)r^{n-4}}{r^{n-2} - r_s^{n-2}} R_k = 0. \] (3.6)

For later convenience, we abbreviate the polynomial coefficients to

\[ P_k(r, r_s) \equiv \frac{(n-1)r^{n-2} - \sigma r_s^{n-2}}{r(r^{n-2} - r_s^{n-2})}, \] (3.7)

and

\[ Q_k(r, r_s) \equiv -\frac{k(k + n - 2)r^{n-4}}{r^{n-2} - r_s^{n-2}}. \] (3.8)

Note that the differential equation (3.6) is invariant under the exchange \( k \leftrightarrow -(k + n - 2) \). Hence, given a solution, a second solution follows by swapping \( k \leftrightarrow -(k + n - 2) \). Of course, one must check if this second solution is linearly independent of the first.

The differential equation (3.6) has three nonessential singularities at \( r = 0, r = r_s, \) and \( r = \infty \). When \( r_s = 0 \), the singularities are \( r = 0 \) and \( r = \infty \), and two independent solutions are \( r^k \) and \( r^{-(k+n-2)} \). Evidently, these solutions are valid for all \( r \in (0, \infty) \). When \( r_s \neq 0 \), we substitute \( \rho \equiv \left( \frac{r}{r_s} \right)^{n-2} \) into Eq. (3.6), which becomes

\[ \rho^2(\rho - 1)\ddot{R}_k + \left( \frac{n - 1 - \sigma}{n - 2} \right) \rho \dot{R}_k + \frac{k(k + n - 2)}{(n - 2)^2} R_k = 0, \] (3.9)

where a dot denotes differentiation with respect to \( \rho \). This is a special case of the hypergeometric differential equation. The equation has three nonessential singularities at \( \rho = 0, \rho = 1, \) and \( \rho = \infty \) corresponding to \( r = \infty, r = r_s, \) and \( r = 0 \), respectively.

We shall solve the differential equation (3.9) around \( \rho = 0 \) for two reasons. One, after transitioning back to the Schwarzschild chart, Frobenius’ method [29] guarantees a convergent solution for all \( r \in (r_s, +\infty) \), which is the desired region of study. Two, only by solving around \( \rho = 0 \) is the physically meaningful limit \( r_s \to 0^+ \) \([M \to 0^+ \text{ in Eq. (2.3)}]\) well-defined in the solution. To understand the second point, first note that the structure of Eq. (3.6) is such that

\[ \lim_{r_s \to 0^+} \lim_{r \to \infty} r r P_k(r, r_s) = \lim_{r \to \infty} \lim_{r_s \to 0^+} r r P_k(r, r_s), \] (3.10)

and

\[ \lim_{r_s \to 0^+} \lim_{r \to \infty} r^2 Q_k(r, r_s) = \lim_{r \to \infty} \lim_{r_s \to 0^+} r^2 Q_k(r, r_s). \] (3.11)

for all \( k \in \mathbb{Z}^* \). That these two pairs of limits commute implies the indicial equation around \( r = +\infty \) for Eq. (3.6) does not change as \( r_s \to 0^+ \). Accordingly, the form of the solutions is the same for all \( r_s \geq 0 \). This is obviously crucial if the solutions to Eq. (3.6) with \( r_s \neq 0 \) are to reduce to \( r^k \) and \( r^{-(k+n-2)} \) in the limit \( r_s \to 0^+ \). Incidentally, this does not happen if the differential equation is solved around either \( r = 0 \) or \( r = r_s \). For these solutions, the limit \( r_s \to 0^+ \) is generally undefined because the indicial equation around either \( r = 0 \) or \( r = r_s \) does not commute in the sense of Eqs. (3.10) and (3.11). Hence, the forms of the
\( r_s \neq 0 \) solutions around either \( r = 0 \) or \( r = r_s \) do not match those associated with the \( r_s = 0 \) solutions. As a result, the \( r_s \neq 0 \) solutions do not in general reduce to the \( r_s = 0 \) solutions as \( r_s \to 0^+ \).

The differential equation (3.9) is solved by first noting that around \( \rho = 0 \), the indicial equation has roots

\[
\chi_k^\pm \equiv \frac{1 \pm \sqrt{1 + \frac{4k}{n-2}}}{2}.
\]  

(3.12)

Clearly, \( \chi_k^+ > \chi_k^- \) and \( \chi_k^+ - \chi_k^- \in \mathbb{Z}^* \) if and only if \( (n - 2) \mid k \), where \( \mid \) means “divides.” One solution to Eq. (3.9) is then of the form

\[
R_k^{(\alpha)}(r, r_s; \sigma) \equiv \sum_{m \geq 0} \alpha_{k,m} \rho^{m+\chi_k^+},
\]  

(3.13)

and a second follows from the exchange \( k \leftrightarrow -(k + n - 2) \),

\[
R_k^{(\tilde{\alpha})}(r, r_s; \sigma) \equiv \sum_{m \geq 0} \tilde{\alpha}_{k,m} \rho^{m-\chi_k^-}.
\]  

(3.14)

In writing the second solution, we utilized the useful relation

\[
\chi_{-(k+n-2)}^\pm = -\chi_k^\pm.
\]  

(3.15)

In Eqs. (3.13) and (3.14), \( \{\alpha_{k,m}\}_{m \geq 0} \) and \( \{\tilde{\alpha}_{k,m}\}_{m \geq 0} \) are \( k \)- and \( m \)-dependent sequences of real numbers for which \( \alpha_{k,0} \neq 0 \) and \( \tilde{\alpha}_{k,0} \neq 0 \) for all \( k \in \mathbb{Z}^* \). The barred sequence is related to the unbarred sequence via the conjugation \( k \leftrightarrow -(k + n - 2) \), i.e., \( \tilde{\alpha}_{k,m} = \alpha_{-(k+n-2),m} \).

Substituting Eq. (3.13) into Eq. (3.9) establishes that each \( \alpha_{k,m} \) must satisfy

\[
\tilde{\alpha}_{k,m} \equiv \frac{\alpha_{k,m}}{\alpha_{k,0}} = \frac{(\chi_k^+)m \left( \frac{n-1-\sigma}{n-2} + \chi_k^- \right)m}{m! (2\chi_k^-)^m},
\]  

(3.16)

where, for all \( x \in \mathbb{R} \), \( (x)_m \equiv x(x+1) \cdots (x+m-1) \) is the Pochhammer symbol defined such that \( (x)_0 = 1 \). Accordingly, from Eq. (3.15), the barred sequence satisfies

\[
\tilde{\alpha}_{k,m} \equiv \frac{\tilde{\alpha}_{k,m}}{\tilde{\alpha}_{k,0}} = \frac{(-\chi_k^-)_m \left( \frac{n-1-\sigma}{n-2} - \chi_k^+ \right)_m}{m! (-2\chi_k^-)_m}.
\]  

(3.17)

Importantly, the unbarred sequence \( \{\alpha_{k,m}\}_{m \geq 0} \) never terminates while the barred sequence \( \{\tilde{\alpha}_{k,m}\}_{m \geq 0} \) terminates if and only if there exists an \( N_k \in \mathbb{Z}^* \) such that

\[
N_k = \min \left\{ z \in \mathbb{Z}^* : (-\chi_k^-)_{z+1} = 0 \quad \text{or} \quad \left( \frac{n-1-\sigma}{n-2} - \chi_k^+ \right)_{z+1} = 0 \right\}.
\]  

(3.18)

As \( \sigma \in \{1, n-1\} \), such an \( N_k \) exists if and only if \( (n-2) \mid k \), in which case Eq. (3.18) implies \( N_k = \frac{k}{n-2} \). With this in mind, we define the piecewise function

\[
\Lambda_k \equiv \begin{cases} 
\frac{k}{n-2} & \text{if } (n-2) \mid k, \\
+\infty & \text{otherwise}.
\end{cases}
\]  

(3.19)
Two solutions to Eq. (3.6) are then
\[ R_k^{(\alpha)}(r, r_s; \sigma) = r_s^{-(k+n-2)} \sum_{m=0}^{\Lambda_3} \tilde{\alpha}_{k,m} \left( \frac{r_s}{r} \right)^{k+m+1}(n-2) \]  
(3.20)
and
\[ R_k^{(\bar{\alpha})}(r, r_s; \sigma) = r_s^k \sum_{m=0}^{\Lambda_3} \tilde{\alpha}_{k,m} \left( \frac{r}{r_s} \right)^{k-m-2} \]  
(3.21)
Here, we have fixed \( \alpha_{k,0} = r_s^{-(k+n-2)} \) and \( \tilde{\alpha}_{k,0} = r_s^k \) so that \( R_k^{(\alpha)} \to r_s^{-(k+n-2)} \) and \( R_k^{(\bar{\alpha})} \to r_s^k \) as \( r_s \to 0^+ \), as desired. Crucially, both solutions are absolutely convergent on \((r_s, +\infty)\) by Frobenius’ method [29]. The linear independence of the solutions follows from the fact that, asymptotically,
\[ R_k^{(\alpha)}(r, r_s; \sigma) \sim r_s^{-(k+n-2)} \]  
(3.22)
and
\[ R_k^{(\bar{\alpha})}(r, r_s; \sigma) \sim r_s^k. \]  
(3.23)
Thus, assuming \( n \geq 3 \), the Wronskian,
\[ W \left[ R_k^{(\alpha)}, R_k^{(\bar{\alpha})} \right] = R_k^{(\alpha)} R_k^{(\bar{\alpha})} - R_k^{(\alpha)} R_k^{(\bar{\alpha})} \sim (2k + n - 2)r_s^{-(n-1)}, \]  
(3.24)
is nonzero asymptotically for all \( k \in \mathbb{Z}^* \). Hence, for \( n \geq 3 \), the Wronskian is nonzero on \((r_s, +\infty)\) for all \( k \in \mathbb{Z}^* \), so the solutions (3.20) and (3.21) are linearly independent on \((r_s, +\infty)\) for all \( k \in \mathbb{Z}^* \). For later use, we compute the general (non-asymptotic) Wronskian to be\(^3\)
\[ W \left[ R_k^{(\alpha)}, R_k^{(\bar{\alpha})} \right] = \frac{E_k}{r^{\sigma} \left( r_s^{n-2} - r_s^{n-2} \right)^{1-\frac{\sigma-1}{n-2}}}, \]  
(3.25)
where \( E_k \) is a constant. Equating Eq. (3.25) in the limit of large \( r \) with its asymptotic value in Eq. (3.24) gives \( E_k = 2k + n - 2 \). Importantly, the Wronskian is finite for all \( r \in (r_s, +\infty) \), regardless of \( \sigma \). However, as \( r \to r_s^+ \), the Wronskian is divergent when \( \sigma = 1 \) and finite \((= E_k/r_s^{n-1})\) when \( \sigma = n - 1 \). These statements hold true for all \( k \in \mathbb{Z}^* \) when \( n \geq 3 \). We now study the behavior of the solutions (3.20) and (3.21) as \( r \to r_s^+ \).

Since these solutions are Gaussian hypergeometric functions, Gauss’ hypergeometric theorem [30] proves
\[ \lim_{r \to r_s^+} R_k^{(\alpha)}(r, r_s; \sigma) = \frac{\Gamma(2\chi^+_k)\Gamma\left(\frac{\sigma-1}{n-2}\right)}{\Gamma(\chi^+_k)\Gamma\left(\frac{n+k+\sigma-3}{n-2}\right)} r_s^{(k+n-2)}. \]  
(3.26)
Assuming \( n \geq 3 \), this limit diverges if \( \sigma = 1 \) and converges if \( \sigma = n - 1 \). On the other hand,
\[ \lim_{r \to r_s^+} R_k^{(\bar{\alpha})}(r, r_s; \sigma) = r_s^k \left\{ \sum_{m=0}^{\Lambda_3} \tilde{\alpha}_{k,m} \left[ \frac{\Gamma(-2\chi^+_k)\Gamma\left(\frac{\sigma-1}{n-2}\right)}{\Gamma(\chi^+_k)\Gamma\left(\frac{n+k+\sigma-3}{n-2}\right)} \right] \right\} \text{ if } (n-2) \mid k, \]  
(3.27)
\( \text{otherwise.} \)

\(^3\)This was derived using Abel’s identity [29], not the canonical definition (3.24). Of course, both methods must yield the same result.
We study this limit in the two possible cases.

First, suppose \((n - 2) \nmid k\). Then, \(n > 3\) and \(\Lambda_k = +\infty\), and \(\chi_k^- \equiv \frac{k}{n-2} \notin \mathbb{Z}^+\). If \(\sigma = 1\), then Eq. (3.27) diverges. If \(\sigma = n - 1\), then Eq. (3.27) converges if and only if \(2\chi_k^- \notin \mathbb{Z}^+\). Since \((n - 2) \nmid k\), \(2\chi_k^- \notin \mathbb{Z}^+\) if and only if \(n \neq 2(d_k + 1)\), where \(d_k \in \mathbb{Z}^+\) is a divisor of \(k\).

Now suppose \((n - 2) \mid k\). Then, \(\Lambda_k = \frac{k}{n-2} \in \mathbb{Z}^+\). As a result, Eq. (3.27) converges for all \(k \in \mathbb{Z}^+\) when \(n \geq 3\), regardless of \(\sigma\). In fact, if \(k \neq 0\) and \(\sigma = n - 1\), then

\[
\sum_{m=0}^{\Lambda_k} \tilde{\alpha}_{k,m} = -\frac{(-\Lambda_k - 1)\Lambda_{k+1}(-\Lambda_k)\Lambda_{k+1}}{(\Lambda_k + 1)!(-2\Lambda_k)\Lambda_{k+1}} = 0, \tag{3.28}
\]

where equality to zero follows from \((-\Lambda_k)\Lambda_{k+1} = 0\). Using \(R_0^{(\sigma)}(r_s, r_s; n-1) = 1\), Eqs. (3.27) and (3.28) show that, for electrostatic fields \((\sigma = n - 1)\),

\[
\lim_{r \to r_s^+} R_k^{(\sigma)}(r, r_s; n-1) = \begin{cases} 1 & \text{if } k = 0, \\ 0 & \text{if } k \in \mathbb{Z}^+ \text{ and } (n - 2) \mid k, \\ \Gamma(-2\chi_k^-) r_s^{k - \chi_k^-} & \text{otherwise.}
\end{cases} \tag{3.29}
\]

These convergence and divergence properties constitute the origin of the electric multipole hair on ST black holes. They also indicate why \(n = 3\) is special: only with this dimension does \((n - 2) \mid k\), and thus is \(R_k^{(\sigma)}(r, r_s; n-1) = 0\) as \(r \to r_s^+\), for all \(k \in \mathbb{Z}^+\).

Finally, we compute the derivative of \(R_k^{(\sigma)}(r, r_s; \sigma)\) as \(r \to r_s^+\). The derivative properties of hypergeometric functions imply

\[
\lim_{r \to r_s^+} R_k^{(\sigma)}(r, r_s; \sigma) = -(n - 2) r_s^{-\chi_k^- + k + n - 3} \left[ \tilde{\alpha}_{k,1} + \sum_{m=0}^{\tilde{\beta}_{k,m}} \frac{\Gamma(2\chi_k^+) \Gamma\left(\frac{n-1}{n-2}\right) \chi_k^+}{\Gamma(\chi_k^+) \Gamma\left(\frac{n+k+\sigma-3}{n-2}\right)} \right], \tag{3.30}
\]

where the sequence \(\{\tilde{\beta}_{k,m}\}_{m \geq 0}\) is defined by

\[
\tilde{\beta}_{k,m} \equiv \frac{(\chi_k^+ + 1)m \left(\frac{n-1}{n-2} + \chi_k^+\right)}{m! (2\chi_k^+ + 1)m}. \tag{3.31}
\]

The same methods used to obtain Eqs. (3.26) and (3.27) show that the \(\tilde{\beta}_{k,m}\) sum in Eq. (3.30) diverges for all \(k \in \mathbb{Z}^+\), regardless of \(\sigma\). Consequently, \(R_k^{(\sigma)}(r, r_s; \sigma)\) diverges as \(r \to r_s^+\) for all \(k \in \mathbb{Z}^+\) and \(\sigma \in \{1, n - 1\}\) when \(n \geq 3\).

4 Green’s Function

For sake of clarity, we write \(r \equiv (r, \varphi)\) and henceforth suppress all dependencies on \(r_s\). We assume that if \(\sigma = 1\) (non-electrostatic field) or \(\sigma = n - 1\) (electrostatic fields), then the source function \(f(r)\) in Eq. (2.12) or \(j^0(r)\) in Eq. (2.15), respectively, is zero for sufficiently large \(r\). Furthermore, we impose the Dirichlet boundary condition \(\Phi \sim r^{-(n-2)}\). The

\(^{4}\)Otherwise \((n = 3), (n - 2) \mid k\) for all \(k \in \mathbb{Z}^+\), so the case we are considering never applies.
two situations we examine are when \( f(\mathbf{r}) \) and \( j^0(\mathbf{r}) \) are nonzero only at a singular point \( p \in \mathbb{S}^{n+1} \), for which \( \mathbf{r}(p) = r_p \equiv (r_p, \phi_p) \) is constant. We shall assume \( r_p > r_s \) until stated otherwise. In each of these scenarios, the scalar field is generated by a stationary point source outside the black hole. Depending on the nature of the field, the source function is either

\[
f(\mathbf{r}) = \delta^n(\mathbf{r} - \mathbf{r}_p) \quad (\sigma = 1),
\]

or

\[
f(\mathbf{r}) = g_{00}\delta^n(\mathbf{r} - \mathbf{r}_p) + \frac{r_s^{n-2}(n-2)}{\Omega_{n-1}r_{n-1}} \partial_r \Phi(\mathbf{r}) \quad (\sigma = n-1),
\]

where we have used Eq. (2.15) for the effective source in the \( \sigma = n-1 \) case. Here, the normalizations of the \( \delta \) functions are chosen so that

\[
\Omega_{n-1} \int \delta(r - r_p) r^{n-1} dr = 1
\]

and

\[
\int_{\mathbb{S}^{n-1}} \delta^{n-1}(\varphi - \varphi_p) d\Omega_{n-1} = 1.
\]

For each source, the solution to the respective Dirichlet problem posed by Eq. (2.12) is the Green’s function \( G(\mathbf{r}, \mathbf{r}_p; \sigma) \). To find the Green’s function, we first write

\[
\delta^n(\mathbf{r} - \mathbf{r}_p) = \delta(r - r_p) \sum_{k \geq 0} \sum_{l=1}^{\Gamma_k} \hat{Y}^l_k(\varphi_p) Y^l_k(\varphi),
\]

where we have employed the completeness relation (2.10). Next, we propose the ansatz

\[
G(\mathbf{r}, \mathbf{r}_p; \sigma) = \sum_{k \geq 0} \sum_{l=1}^{\Gamma_k} Z^l_k(\varphi_p) R_k(r, \mathbf{r}_p; n - 1) Y^l_k(\varphi).
\]

Here, \( \{Z^l_k\}_{l=1}^{\Gamma_k} \) is a set of undecided, complex-valued functions on \( \mathbb{S}^{n-1} \). At this point, it is clearest to consider a particular \( \sigma \) model and derive the Green’s function for that case.

### 4.1 Electrostatic Source (\( \sigma = n-1 \))

In this case, the source function is Eq. (4.2). Substituting Eq. (4.6) into Eq. (2.12) establishes that \( Z^l_k(\varphi_p) = \hat{Y}^l_k(\varphi_p) \) and that \( R_k(r, \mathbf{r}_p; n - 1) \) satisfies

\[
\frac{d}{dr} \left[ r^{n-1} \left(1 - \frac{r_s^{n-2}}{r^{n-2}}\right) R_k\right] = -r^{n-3}k(k + n - 2)R_k - (n-2)r_s^{n-2}R_k' = g_{00}\Omega_{n-1}r^{n-1}\delta(r - r_p).
\]

This is identical to Eq. (3.4) for all \( r \neq r_p \). Therefore, \( R_k(r, \mathbf{r}_p; n - 1) \) is a linear combination of both \( R_k^{(\alpha)} \) and \( \hat{R}_k^{(\alpha)} \),

\[
R_k(r, \mathbf{r}_p; n - 1) = \begin{cases} A_k R_k^{(\alpha)}(r; n - 1) + \hat{A}_k R_k^{(\hat{\alpha})}(r; n - 1) & \text{if } r > r_p, \\ B_k R_k^{(\alpha)}(r; n - 1) + \hat{B}_k R_k^{(\hat{\alpha})}(r; n - 1) & \text{if } r < r_p, \end{cases}
\]

where
As $\Phi = V \sim r^{-(n-2)}$, we require $A_k = 0$ since $R^{(\alpha)}(r; n - 1) \sim r^k$ by Eq. (3.23). We determine $B_k$ by requiring the Lorentz scalar

$$\frac{1}{2} F_{\mu \nu} F^{\mu \nu} = (\partial_r V)^2 + \left( 1 - \frac{r_s^{n-2}}{r^{n-2}} \right)^{-1} (\partial_{\varphi_i} V)(\partial^{\varphi_i} V)$$

(4.9)

to be finite as $r \to r_s^+$ when $r_p > r_s$. As we saw in Eq. (3.30), $R^{(\alpha)}$ is divergent for all $k \geq 0$ as $r \to r_s^+$. Hence, we are forced to set $B_k = 0$ so that the divergence of $\partial_r V$ is suppressed. Finally, we require that the solution be continuous at $r = r_p$. We conclude that

$$R_k(r, r_p; n - 1) = C_k R^{(\alpha)}_k(r_>; n - 1) R^{(\alpha)}_k(r_<; n - 1),$$

(4.10)

where $C_k \equiv A_k / R^{(\alpha)}_k(r_p; n - 1) = B_k / R^{(\alpha)}_k(r_p; n - 1)$ is a constant and $r_\equiv \equiv \min\{r, r_p\}$ while $r_\geq \equiv \max\{r, r_p\}$. At $r = r_p$, the function $R_k(r, r_p; n - 1)$ is continuous (by design), though its first derivative is not. Integrating Eq. (4.13) over the interval $(r_p - \epsilon, r_p + \epsilon)$ of radius $\epsilon > 0$ and using the Wronskian (3.25), we determine the value of $C_k$ to be

$$C_k = -\frac{1}{2k + n - 2}.$$  

(4.11)
Combining this with Eq. (4.6), we obtain the Green’s function

$$G(r, r_p; n - 1) = -\sum_{k \geq 0} \sum_{l=1}^{\Gamma_k} R^{(\alpha)}_k(r_>; n - 1) R^{(\alpha)}_k(r_<; n - 1) \frac{Y_l(\varphi_p)Y_l(\varphi)}{2k + n - 2}.$$  

(4.12)

Next, we consider a non-electrostatic point source.

### 4.2 Non-Electrostatic Source ($\sigma = 1$)

In this case, the source function is Eq. (4.1). Substituting Eq. (4.6) into Eq. (2.12) establishes that $Z^{(\alpha)}_k(\varphi_p) = Y^{(\alpha)}_k(\varphi_p)$ and that $R_k(r, r_p; 1)$ satisfies

$$\frac{d}{dr} \left[ r^{n-1} \left( 1 - \frac{r_s^{n-2}}{r^{n-2}} \right) R_k \right] - r^{n-3} k(k + n - 2) R_k = \Omega_{n-1} r^{n-1} \delta(r - r_p).$$

(4.13)

Similar to before, this is identical to Eq. (3.3) for all $r \neq r_p$. Therefore, $R_k(r, r_p; 1)$ is a linear combination of both $R^{(\alpha)}_k$ and $R^{(\alpha)}_k$. We demand that $R_k(r, r_p)$ be finite as $r \to +\infty$, and also at $r = r_s$ when $r_p > r_s$. However, contrary to the electrostatic case, finiteness of the field on the horizon is impossible to enforce for all $k \geq 0$ when $n > 3$. This follows from Eq. (3.27), which says that $R^{(\alpha)}_k(r; 1)$ diverges as $r \to r_s^+$ whenever $(n - 2) \nmid k$. Clearly, this condition is satisfied for countably infinite $k$ when $n > 3$. As a result, it is expected that infinite fields appear at the event horizon when $n > 3$, even if the point source generating those fields is arbitrarily far away. We shall address this point in the next section. For now, we will assume the form of $R_k(r, r_p; 1)$ is the same as in the electrostatic case,\(^5\)

$$R_k(r, r_p; 1) = D_k R^{(\alpha)}_k(r_>; 1) R^{(\alpha)}_k(r_<; 1),$$

(4.14)

\(^5\)This assumption is also motivated by the fact that Eq. (4.14) is the most general form for $R_k(r, r_p; 1)$ when $n = 3$ [25]. Because the conclusions drawn in our analysis ought to apply in the $n = 3$ case, the form of $R_k(r, r_p; 1)$ for general $n$ must reduce to the $n = 3$ solution when $n = 3$. From this perspective, Eq. (4.14) seems the only possible form of the solution.
where $D_k$ is a constant, and study the limiting behavior as $r_p \rightarrow r^+_s$ in the final Green’s function. Integrating Eq. (4.13) over the interval $(r_p - \epsilon, r_p + \epsilon)$ determines that $D_k = C_k$. Thus, the Green’s function is

$$G(r, r_p; 1) = -\sum_{k \geq 0} \sum_{l=1}^{\Gamma_k} \frac{R_k^{(\alpha)}(r >; 1) R_k^{(\beta)}(r <; 1) \vec{Y}_k^l(\varphi_p) Y_k^l(\varphi)}{2k + n - 2}.$$  

(4.15)

The essential properties of the Green’s functions (4.12) and (4.15) are surveyed in the next section.

5 Multipole Hair and Discussion

We consider now the case where the only source is a stationary point charge of strength $q$ located at $r_p = (r_p, \varphi_p)$, where $r_p > r_s$ (outside the event horizon). The solution $\Psi(r, r_p; \sigma)$ to Eq. (2.12) that behaves appropriately is obviously $\Psi(r, r_p; \sigma) = qG(r, r_p; \sigma)$, since in this case $f(r, \varphi) = q\delta^0(r - r_p)$ or $j^0(r, \varphi) = q\delta^0(r - r_p)$, respectively, depending on the nature of the field.

In this analysis, we shall examine the behavior of the fields at points $r = (r, \varphi)$ for which $r > r_p$ as $r_p \rightarrow r^+_s$. Physically, this limit corresponds to a “slow fall” of the point source into the event horizon of the black hole. We assume the fall is slow enough such that our static considerations remain valid. In the following, the multipole moments are identified relative to the monopole term, which in $n+1$ dimensional spacetime is asymptotic to $r^{-(\mu_k + n - 2)}$. Accordingly, terms asymptotic to $r^{-(\mu_k + n - 2)}$ characterize the $\mu_k$-pole moment.

For the $n = 3$ case, Persides [25] found that a non-electrostatic charge affords the black hole nonzero, but finite, multipole moments. On the other hand, Cohen and Wald [24] found that the spacetime approaches the Reissner-Nordström geometry for any observer outside the event horizon, and hence that all electric multipole moments vanish, with the exception of the monopole. Consequently, when $n = 3$, observers outside the event horizon detect multipole moments once $r_p = r_s$ if the test field is not electrostatic in nature. The conclusion is markedly different when $n > 3$. Here, regardless of the nature of the point charge being lowered, the final ST black hole exhibits countably infinite nonzero multipole moments. Furthermore, for both sources, there exist spatial dimensions $n > 3$ in which a nonzero number of the multipole moments are of infinite strength. These conclusions follow immediately from the Green’s functions (4.12) and (4.15), but we prove them explicitly below. In doing so, we frequently reference the set

$$\Lambda \equiv \{k \in \mathbb{Z}^+: (n-2) \nmid k\} \cup \{0\}. \quad (5.1)$$

As $n \geq 3$, $\Lambda = \{0\}$ if and only if $n = 3$. Additionally, $1 \notin \Lambda$ for all $n > 3$.

Consider first the case of lowering a non-electrostatic point charge of strength $q$ into a ST black hole. An observer outside the horizon measures the field

$$\Psi(r, r_p; 1) |_{r_p=r_s} = -q \sum_{k \geq 0} \sum_{l=1}^{\Gamma_k} \sum_{m \geq 0} \frac{\tilde{\alpha}_{k,m} R_k^{(\alpha)}(r_s>; 1) \vec{Y}_k^l(\varphi_p) Y_k^l(\varphi)}{r_s^{2k+n-2}(2k + n - 2)} \frac{(r_s)}{r}^{k+(m+1)(n-2)}. \quad (5.2)$$

6We emphasize that $\{0\}$ is not the empty set. Rather, it is the singleton with element 0.
If \( n = 3 \), then Eq. (3.27) implies \( R_k^{(\tilde{a})}(r_s; 1) \) is finite and in general nonzero for all \( k \in \mathbb{Z}^* \). This implies Persides’ result [25] — namely, that the multipole moments of the field for a non-electrostatic point source falling into a Schwarzschild black hole do not vanish. In particular, the multipole moments are all finite. If \( n > 3 \), however, then \( R_k^{(\tilde{a})}(r_s; 1) \), and hence all \( \mu_k \)-pole moments for which \( \mu_k \) is congruent to \( k \) modulo \( n - 2 \), is infinite for all \( k \in \Lambda\setminus\{0\} \). By Eq. (3.27), the \( \mu_k \)-pole moments are guaranteed to be finite if and only if \((n - 2) \mid k \), i.e., if and only if \((n - 2) \mid \mu_k \).

The behavior of the field at the horizon as the source approaches the horizon can be determined by merely swapping \( r \) and \( r_p \) in Eq. (5.2) and taking the limit \( r_p \to r^+_s \). It is clear that \( \Psi \) diverges in this limit if \( n \geq 3 \). This divergence brings about the possibility of destruction of the horizon.

Consider now the case of lowering an electrostatic point charge of strength \( q \) into a ST black hole. Eq. (3.29) implies \( R_k^{(\tilde{a})}(r_s; n - 1) \) is nonzero if and only if \( k \in \Lambda \). Therefore, an observer outside the horizon at \( r = (r, \varphi) \) measures the field

\[
\Psi(r, r_p; n - 1)|_{r_p=r_s} = -q \sum_{k \in \Lambda} \sum_{l=1}^{\Gamma_k} \sum_{m \geq 0} \tilde{a}_{k,m} R_k^{(\tilde{a})}(r_s; n - 1) \hat{Y}_k^l(\varphi) Y_l^m(\varphi) \left( \frac{r_p}{r} \right)^{2k+n-2}.
\]

If \( n = 3 \), then \( \Lambda = \{0\} \). Furthermore, \( \tilde{a}_{0,m} = 0 \) for all \( m \in \mathbb{Z}^* \). Hence, the field \( \Psi \) only has a monopole term. This implies Cohen and Wald’s result [24] — namely, that the multipole moments of the field for an electrostatic point charge (except the monopole) vanish as the charge approaches the event horizon of a Schwarzschild black hole. If \( n > 3 \), however, then there exists a \( k \in \Lambda \setminus\{0\} \) for which \( R_k^{(\tilde{a})}(r_s; n - 1) \neq 0 \). As \( \tilde{a}_{k,0} \neq 0 \) for \( k > 0 \), it follows that \( \Psi \) has a \( k \)-pole moment. But \( \tilde{a}_{k,m} \neq 0 \) for \( k > 0 \) and all \( m \in \mathbb{Z}^* \). Therefore, the existence of a single \( k \)-pole moment (excluding the monopole) implies the existence of countably infinite multipole moments — namely, again, all \( \mu_k \)-pole moments for which \( \mu_k \) and \( k \) are congruent modulo \( n - 2 \). It follows that ST black holes acquire countably infinite electric multipole moments from infalling, electrically-charged matter.

Interestingly, if there exists a \( k \in \mathbb{Z}^+ \) with divisor \( d_k \) such that \( n = 2(d_k + 1) \), which is true if and only if \( n > 3 \) is even, then, by Eq. (3.29) and the analysis thereafter, there exists a \( k \)-pole moment (and hence a countably infinite set of \( \mu_k \)-pole moments) of infinite strength. Therefore, in even dimensions \( n > 3 \), \( \Psi \) diverges globally (i.e., is everywhere infinite). However, if \( n > 3 \) is odd, then all nonzero multipole moments, and hence \( \Psi \), are everywhere finite. Incidentally, that \( 1 \notin \Lambda \) for all \( n > 3 \) implies the final state does not possess a dipole. This is reminiscent of the “no-dipole-hair theorem” of Ref. [31].

As before, the behavior of the field at the horizon as the source approaches the horizon can be determined by swapping \( r \) and \( r_p \) in Eq. (5.3) and taking the limit \( r_p \to r^+_s \). It is easy to see using Eqs. (3.26) and (3.27) that the field is infinite at the horizon if \( n > 3 \) is even. Otherwise, the field is well-behaved and finite at the horizon. As before, this divergence in even dimensions brings about the possibility of destruction of the horizon.

The conclusion that the final state of the ST black hole possesses countably infinite electric multipole moments presents a paradox. We expect the final state to be RNT in nature due to the uniqueness of the RNT solution among all non-degenerate, topologically hy-
perspherical, static, asymptotically-flat, electrovac solutions to the Einstein-Maxwell equations. However, the RNT black hole is hyperspherically symmetric, so it cannot possess electric multipole anisotropies. We conclude that the final state is not RNT in nature. In particular, the final state is not a non-degenerate, topologically hyperspherical, static, asymptotically-flat, electrovac solution to the Einstein-Maxwell equations. One (or more) of these characterizations must not apply to the final state, so to render it different from the RNT spacetime.\footnote{Note that the final state necessarily obeys the Einstein-Maxwell equations since all results in this paper have derived from these equations.}

Staticity and, at least for odd dimensions \( n > 3 \), asymptotic flatness can be assured, however. Staticity follows from the observation that our analysis never concerned itself with the rate at which the charge is lowered into the black hole. Thus, the lowering rate (the only non-static phenomenon in this study) can be assumed arbitrarily close to zero. For asymptotic flatness, we note that the source of the global divergence of \( \Psi \) as \( r_p \to r_s^+ \) for even \( n > 3 \) is the factor of \( R_k^{(g)}(r_p; n - 1) \) in Eq. (5.3). That this factor is independent of \( r \) and \( \varphi \) implies \( \partial_t \Psi \) also diverges globally as \( r_p \to r_s^+ \) for even \( n > 3 \). Since the energy content of the field is related to \( \partial_i \Psi \), the global divergence of \( \partial_i \Psi \) suggests \( \Psi \) may influence the asymptotic geometry. In particular, asymptotic flatness of the final state is not guaranteed for even \( n > 3 \). Conversely, for odd \( n > 3 \), \( \partial_i \Psi \) is everywhere finite in the horizon limit. Asymptotic flatness can then be assured by merely tuning the strength \( q \) of the charge to a value small enough (though nonzero) such that the geometry is unaffected by it. For odd dimensions, therefore, our initial assumption that the electric field does not influence the local spacetime geometry holds well as \( r_p \to r_s^+ \) for \( q \) sufficiently small. The influence of \( \Psi \) on the asymptotic geometry must then be particularly negligible, thereby preserving asymptotic flatness. At least for odd dimensions, these considerations guide us to the question of how a static and asymptotically-flat black hole can exhibit electric multipole anisotropies.

Assuming the horizon of the final state is both non-degenerate and homeomorphic to \( S^{n-1} \), then uniqueness of the RNT black hole forces the final state to be RNT spacetime. However, as we have remarked, RNT spacetime is hyperspherically symmetric, and the final state is not. As the final state (in odd dimensions) is static and asymptotically-flat, we are lead to the conclusion that one (or both) of the remaining assumptions (non-degenerate horizon and hyperspherical topology) is incorrect. If the horizon is degenerate, then the final state would be a counterexample to the expected non-degeneracy of static black hole solutions to the higher dimensional Einstein-Maxwell equations [32, 33]. Moreover, the final state would have a degenerate and necessarily ahyperspherical horizon in order to generate the multipole anisotropies. On the other hand, if the final state is not topologically hyperspherical, suggesting that infalling electric charges induce discontinuous topological transformations to the horizon, then the uniqueness of the RNT solution is invalidated. This would presumably allow for a topologically- and geometrically-ahyperspherical solution to characterize the final state, which is necessary for it to possess the multipole fields. While both these mechanisms ostensibly resolve the paradox (and are not immediately mutually
exclusive), uncovering their exact details warrants further investigation.

We conclude with a comment on even dimensions $n > 3$. As, in this case, $\partial_i \Psi$ diverges in the horizon limit, the global spacetime geometry may be altered in a significant way. Thus, asymptotic flatness of the final state is not guaranteed. It follows that the non-degeneracy and/or hyperspherical topology of the horizon need not be violated (though are not immediately precluded from being violated) to generate the multipole anisotropies. This is because relaxing the assumption of asymptotic flatness is enough to render the ST solution non-unique among all possible non-degenerate, topologically hyperspherical, and static solutions to the Einstein equations [20, 21]. It is conceivable, therefore, that in the even dimensional case, the absence of asymptotic flatness in the final state accounts for the multipolar structure of the electric potential. Of course, as in the odd dimensional case, the exact details of this possibility require a more in-depth analysis, on which we hope to report soon.

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