Multi-version Coding with Side Information

Ramy E. Ali, Viveck R. Cadambe, Jaime Llorca and Antonia M. Tulino

Abstract

In applications of storage systems to modern key-value stores, the stored data is highly dynamic due to frequent updates from the system write clients. The multi-version coding problem has been formulated to study the cost of storing dynamic data in asynchronous distributed storage systems. In this problem, previous work considered a completely decentralized system where a server is not aware of which versions of the data are received by the other servers. In this paper, we relax this assumption and study a system where a server may acquire side information of the versions propagated to some other servers. In particular, we study a storage system with \( n \) servers that store \( \nu \) totally ordered independent versions of a message. Each server receives a subset of these \( \nu \) versions that defines the state of that server. Assuming that the servers are distributed in a ring, a server is aware of which versions have been received by its \( h \)-hop neighbors. If the server is aware of the states of \((n - 2)\) other servers, we show that this side information can result in a better storage cost as compared with the case where there is no side information. Through an information-theoretic converse, we identify scenarios where, even if the server is aware of the states of \((n - 3)/2\) other servers, the side information may not help in improving the worst-case storage cost beyond the case where servers have no side information.

I. INTRODUCTION

The primary role of distributed storage systems is to ensure the availability of the data. By replicating or encoding data across multiple distributed servers, users can access the data even if some servers fail or are too slow to respond. However, large-scale distributed storage systems face new challenges when the stored data is dynamic, as in most modern applications, where data is frequently updated by the system write clients. Due to the unpredictable nature of the propagation delays of the updates, the system is inherently asynchronous, in the sense that all servers storing the data may not receive the updates at the same time. In such settings, ensuring that a user that connects to the system can get the latest possible update is a problem of critical relevance. The notion that the latest version of the data must be accessible despite the frequent updates is known as consistency in distributed computing [2], [3], [4]. Consistent storage systems are prevalent in modern digital applications such as reservation systems, social networks, financial transactions, and multi-player gaming. In fact, there are many cloud-based consistent data storage services such as Amazon Dynamo [3] and Apache Cassandra [5].

The design of consistent storage systems has been studied extensively in the distributed computing literature [2], [6]. Recently, the multi-version coding framework [7] has been proposed to study the storage costs of ensuring consistency in asynchronous distributed storage systems from an information-theoretic point of view. In the multi-version coding problem, a distributed storage system of \( n \) servers that store \( \nu \) totally ordered versions of a message of length \( K \) bits is studied. The higher ordered versions are interpreted as later versions, and lower ordered versions as earlier versions. The data versions may not propagate to all the servers due to the inherent asynchrony of the system. Specifically, each server receives an arbitrary subset of these \( \nu \) versions. We refer to the subset of versions received by a server as the state of the server. The encoding is decentralized, i.e., each server does not know the state of any other server and encodes the versions that it receives in a distributed manner.
The decoding requirements of multi-version coding have been chosen to reflect quorum-based distributed algorithms that provide consistent data storage services, e.g., see [7], [9] and references therein. Specifically, for a multi-version code that tolerates \( v \) crash failures, any version that has propagated to at least \( c_W \leq (n - f) \) servers is dubbed a complete version, and the goal at the decoder is to connect to an arbitrary subset of \( c_R \leq (n - f) \) servers and decode the latest complete version - the complete version with the highest order - or a later version. We refer the reader to [8] for a tutorial-like description of the multi-version coding problem, and the connections to distributed systems theory and practice.

We notice that for a decoder that connects to \( c_R \) servers, any complete version is present at least at \( c = c_W + c_R - n \) servers among these \( c_R \) servers; therefore, a simple singleton-bound like argument indicates that the worst-case storage cost of a server is at least \( K/c \). In multi-version coding, since no server is aware of the complete versions received by other servers, servers may have to store information corresponding to multiple versions. In fact, reference [7] has provided code constructions and a converse result that showed that the amount of information to be stored is at least \( \frac{\nu K}{c + \nu - 1} - o(K) = \left( \frac{\nu}{c} - \frac{\nu(\nu - 1)}{c^2} + o\left( \frac{1}{c^2} \right) \right) K \). That is, the result of [7] implies that there is a significant cost to be paid for the decentralized nature and the asynchrony in the system, in applications where consistency is required.

While the multi-version coding problem opens the door to information-theoretic analysis of consistent data storage systems, its assumptions are arguably too conservative. Specifically, in multi-version coding, the storage cost is made on a worst-case assumption, assuming independent versions of the data, servers receive a completely arbitrary subset of versions (i.e., complete asynchrony), and that each server is completely unaware of the versions received by any other server in the system (i.e., completely decentralized). However, in real-world distributed systems, these assumptions can be somewhat relaxed, as it may be possible to make finer predictions of the version arrivals to the servers, and the servers may opportunistically obtain side information of the states of some other servers based on the network topology. In this case, a server that receives the latest complete version simply stores it with a maximum distance separable (MDS) code of dimension \( c \). Therefore, in the completely centralized setting, the storage cost is \( K/c \), which in contrast with the decentralized setting does not grow with \( \nu \). Here, we provide results that depart from the two extreme points - the completely centralized and synchronous setting where classical erasure coding-based bounds and constructions suffice, and the multi-version coding setting which is completely decentralized and asynchronous - and bridge the gap between them.

Specifically, we assume that each server is aware of the states of its \( h \)-hop neighbors assuming that the \( n \) servers are distributed in a ring. In Section III we provide code constructions that show that for the simple case where the server is aware of the states of \((n - 2)\) other servers, this side information can reduce the worst-case storage cost significantly as compared with the case where servers do not share their states; specifically, the storage cost is \( \left( \frac{1}{c} + \frac{2(\nu - 1)}{c^2} + o\left( \frac{1}{c^2} \right) \right) K \). In Section IV we provide information-theoretic converse results for the case of \( \nu = 2 \) and identify a curious outcome of these converses. Specifically, we identify a scenario where each server is aware of the versions received by \((n - 3)/2\) other servers, and yet the side information does not help in improving the worst-case storage cost.

The remainder of this paper is organized as follows. In Section II we formulate the multi-version coding problem with side information. In Section III we provide our code constructions. In Section IV we provide lower bounds on the worst-case storage costs. Finally, we conclude the paper in Section V.
II. System Model

In this section, we present the system model. We start with some notation. We use boldface for vectors. For a positive integer \(i\), we denote by \(\mathbf{i}\) the set \(\{1, 2, \ldots, i\}\). For any set of ordered indices \(S = \{s_1, s_2, \ldots, s_{|S|}\} \subseteq \mathbb{Z}\), where \(s_1 < s_2 < \cdots < s_{|S|}\), and for any ensemble of variables \(\{X_i : i \in S\}\), the tuple \((X_{s_1}, X_{s_2}, \ldots, X_{s_{|S|}})\) is denoted by \(X_S\). We use \(\log(.)\) to denote the logarithm to the base 2 and \(H(.)\) to denote the binary entropy function. We use the notation \([2^K]\) to denote the set of \(K\)-length binary strings.

We study a storage system of \(n\) servers where the objective is to store \(\nu\) independently totally ordered versions of a message of length \(K\) bits. We denote the set of servers by \(S\). The state of that server is aware of the states of its \(\mathcal{H}\) servers. We design the system to tolerate \(c\) failures. A version that has been received by at least \(c_W \leq (n - f)\) servers is referred to as a complete version. In state \(S = \{S(0), S(1), \ldots, S(n - 1)\} \in \mathcal{P}([\nu])^n\), where \(\mathcal{P}([\nu])\) denotes the power set of \([\nu]\). We assume that the servers are distributed in a ring and that every server is aware of the states of its \(h\)-hop neighbors. That is, server \(i\) is aware of the states of servers \(\mathcal{H}_i = \{i - h, \ldots, i + h\}\), where the addition is modulo \(n\). We denote the states of these servers by \(S(\mathcal{H}_i) = \{S(i - h), \ldots, S(i - 1), S(i + 1), S(i + 2), \ldots, S(i + h)\}\).

The server stores a symbol \(\{q\}\) based on the versions that it receives \(W_{S(i)}\) and the side information \(S(\mathcal{H}_i)\).

We denote the set of servers that have received version \(u \in [\nu]\) in state \(S \in \mathcal{P}([\nu])^n\) by \(A_S(u)\). We design the system to tolerate \(f\) failures. A version that has been received by at least \(c_W \leq (n - f)\) servers is referred to as a complete version. In state \(S \in \mathcal{P}([\nu])^n\), we denote the set of complete versions by \(C_S = \{u \in [\nu] : |A_S(u)| \geq c_W\}\) and the latest among them by version \(L_S := \max C_S\). The decoder connects to any \(c_R \leq (n - f)\) servers and decodes a version \(u\), where \(u \geq L_S\). We notice that among these \(c_R\) servers, any complete version is present at least at \(c := c_W + c_R - n\) servers. We next define the multi-version code with side information formally.

**Definition 1** (Multi-version code with side information). A \((n, c_W, c_R, \nu, h, 2^K, q)\) multi-version code with side information consists of the following:

- **encoding functions**

  \[ \varphi^{(i)}_{S(\mathcal{H}_i)} : [2^K]^{\mathcal{S}(i)} \times \mathcal{P}([\nu])^{(2h + 1)} \to [q], \]

  for every \(i \in [n]\) and every \(S(\mathcal{H}_i) \subseteq \mathcal{P}([\nu])^{(2h + 1)}\),

- **decoding functions**

  \[ \psi^{(T)}_S : [q]^{c_R} \to [2^K] \cup \{\text{NULL}\}, \]

  that satisfy the following

  \[ \psi^{(T)}_S \left( \varphi^{(t_1)}_{S(\mathcal{H}_{t_1})}, \ldots, \varphi^{(t_{c_R})}_{S(\mathcal{H}_{t_{c_R}})} \right) = \begin{cases} W_m & \text{for some } m \geq L_S, \\ \text{NULL} & \text{if } C_S \neq \emptyset, \\ \text{if } C_S = \emptyset, \end{cases} \]

  for every possible system state \(S \in \mathcal{P}([\nu])^n\), every \(W_{[\nu]}\) and every set of servers \(T \subseteq [n]\), where \(T = \{t_1, t_2, \ldots, t_{c_R}\}\), such that \(t_1 < t_2 < \cdots < t_{c_R}\).

In our model, the side information is instantaneously available at the neighboring nodes. In reality, the system evolves with time and our model is applicable in a system where all neighboring servers that have a version \(u \in [\nu]\) receive it within a limited time window that is known a priori and the side information can be obtained in a much shorter time than the given time window.
Since the server is aware of the states of \((2h + 1)\) servers, including its own state, the server allocates memory resources to each version that it receives based on this side information. The objective that we consider is minimizing the per-server worst-case storage cost that we define next.

**Definition 2** (Storage cost). The storage cost of a \((n, c_W, c_R, \nu, h, 2^K, q)\) multi-version code is equal to \(\alpha = \log q\) bits.

Reference [7] has shown that the storage cost for the case where the versions are independent and servers do not share their states, i.e., \(h = 0\), is lower-bounded as follows
\[
\log q \geq \frac{\nu}{c + \nu - 1} K - \frac{\log (\nu^{(c + \nu - 1)}/(c + \nu - 1))}{(c + \nu - 1)}.
\]
In addition, it has provided a code construction with storage cost of
\[
\alpha = \max \left\{ \frac{\nu}{c} - \left( \frac{\nu - 1}{tc} \right), 1 \right\} K,
\]
where
\[
t = \begin{cases} 
\left\lceil \frac{c - 1}{\nu} \right\rceil + 1 & \text{if } c \geq (\nu - 1)^2, \\
\left\lceil \frac{c}{\nu - 1} \right\rceil & \text{if } c < (\nu - 1)^2,
\end{cases}
\]
which is asymptotically tight for the case where \(\nu|\nu - 1\).

### III. Coding With Side Information

In this section, we provide code constructions that show that the side information can help in reducing the worst-case storage cost for the case where \(c_W = 2h + 1 = n - 1\) and arbitrary \(c_R \leq (n - 1)\). In this case, each server is aware of the states of all servers except one server and the system tolerates one failure. Although knowing the states of \((n - 1)\) servers may be hard to realize in a practical system, studying this case is an important step toward understanding the role of side information in general. If a server is aware of the states of all servers in the system, then it can identify which version is the latest complete version and a storage cost of \(K/c\) can be achieved. As the servers get less side information, more servers may not be able to decide which version is the latest complete version. The main idea of our achievable schemes is that given a side information of the states of \((n - 2)\) other servers, most servers can decide which version is the latest complete version correctly.

We begin with some notations. Consider the \(i\)-th server, where \(i \in \{0, 1, \ldots, n-1\}\), and a system state \(S \in \mathcal{P}([\nu])^n\). The \(i\)-th server stores \(\alpha_{i,u}^{S(H_i)}\) bits of version \(u\) in this state, where \(u \in [\nu]\). The worst-case storage cost is then given by
\[
\alpha = \max_{i,S} \sum_{u=1}^{\nu} \alpha_{i,u}^{S(H_i)}.
\]
We next provide our first construction for the case where we have two versions.

**Construction 1.** We construct a code as follows for the case where \(c_W = 2h + 1 = n - 1\) and \(n\) is even.
\[
\alpha = \frac{c + 2}{c^2} K
\]
and
\[
\alpha_{i,2}^{S(H_i)} = \begin{cases} 
K/c & |A_S(2) \cap H_i| \geq (n - 2), \\
0 & \text{otherwise}.
\end{cases}
\]
\[
\alpha_{i,1}^{S(H_i)} = \alpha - \alpha_{i,2}^{S(H_i)},
\]
where \(i \in \{0, 1, \ldots, n-1\}\).
Theorem 1. Construction 1 is a \((n, c_W = n - 1, c_R \leq (n - 1), \nu = 2, h = n/2 - 1, 2^K, q)\) multi-version code with side information with a worst-case storage cost of

\[
\frac{c + 2}{c^2} K, \tag{7}
\]

where \(n\) is even.

**Proof.** Consider any state \(S \in \mathcal{P}([\nu])^n\). We show that the latest complete version in this state, \(L_S\), is decodable. Specifically, we consider the following cases.

- **Case 1.** If \(L_S = 2\), we show that version 2 is decodable. Since version 2 is complete, then at least \(c_W\) servers have it. The decoder connects to any \(c_R\) servers. Among these \(c_R\) servers, at least \(c\) servers have received this version. Each of these \(c\) servers observes at least \((n - 2)\) servers that received version 2. Therefore, each of these servers stores \(K/c\) of version \(L_S\). Hence, version 2 is decodable.

- **Case 2.** If \(L_S = 1\), we show that version 1 is decodable. The decoder connects to any \(c_R\) servers. Among these \(c_R\) servers, at least \(c\) servers have received version 1. Since \(L_S = 1\), then version 2 is received by at most \(c_W - 1\) servers. In this case, at most two servers observe \((n - 2)\) servers that have received version 2. According to the construction, each of these two servers will store \(K/c\) of version 2 and \((\alpha - K/c)\) of version 1. Therefore, the storage allocation of version 2 is at least

\[
2(\alpha - K/c) + (c - 2)\alpha = K.
\]

Therefore, version 1 can be recovered in this state.

We notice that the storage cost of Construction 1 is strictly less than \(\frac{2}{c+1} K\) for \(c \geq 4\). We provide a lower bound on the storage cost in Theorem 3, where we show that the storage cost must be at least \(\frac{2}{2c-1} K = \left(\frac{1}{c} + \frac{0.5}{c^2} + o\left(\frac{1}{c^2}\right)\right) K\).

We next provide our second construction for any number of versions \(\nu\). In state \(S \in \mathcal{P}([\nu])^n\), we denote the latest version that server \(i\) receives that is at least received by \((n - 2)\) servers in the neighborhood of server \(i\) by \(L_{S(i)} := \max \{u \in [\nu] : |\mathcal{A}_S(u) \cap \mathcal{H}_i| \geq (n - 2)\}\).

**Construction 2.** We construct a code as follows for the case where \(c_W = 2h + 1 = n - 1, c_R \leq (n - 1)\) and \(n\) is even.

\[
\alpha = \frac{1}{c - 2(\nu - 1)} K. \tag{8}
\]

and

\[
\alpha_{i,j}^{S(H_i)} = \begin{cases} 
\frac{1}{c - 2(\nu - 1)} K & \text{if } j = L_{S(i)}; \\
0 & \text{otherwise},
\end{cases} \tag{9}
\]

where \(i \in \{0, 1, \ldots, n - 1\}\) and \(c \geq 2\nu - 1\).

**Theorem 2.** Construction 2 is a \((n, c_W = n - 1, c_R \leq (n - 1), \nu, h = n/2 - 1, 2^K, q)\) multi-version code with side information with a worst-case storage cost of

\[
\frac{1}{c - 2(\nu - 1)} K, \tag{10}
\]

where \(c \geq 2\nu - 1\), and \(n\) is even.

**Proof.** Consider any state \(S \in \mathcal{P}([\nu])^n\). We show that the latest complete version in this state, version \(L_S\), is decodable. Since version \(L_S\) is complete, then it must have been received by at least \(c_W = n - 1\) servers.
Consider any other version \( u \) such that \( u < L_S \). A decoder that connects to any \( c_R \) servers will have at least \( c \) servers storing version \( L_S \) and none of these servers store version \( u \) according to the construction. Consider any other version \( u > L_S \). Version \( u \) is received by at most \( c_W - 1 \) servers, since it is not a complete version. Suppose that version \( u \) is received by \( c_W - 1 \) servers. At most two servers out of these \( c_W - 1 \) servers will observe \( (n - 2) \) servers having version \( u \). Each of these two servers stores \( \alpha \) of version \( u \) if it is the latest version that they have. Since there are at most \( (\nu - 1) \) versions that are not equal to \( L_S \), at least \((c - 2(\nu - 1))\) servers will store at least \( \frac{1}{c-2(\nu-1)}K \) of version \( L_S \). Therefore, the storage allocation of version \( L_S \) is at least
\[
(c - 2(\nu - 1))\frac{1}{c - 2(\nu - 1)}K = K.
\]

Hence, version \( L_S \) is decodable in this state. \( \square \)

**Remark 1.** Construction \( \square \) has a strictly better storage cost as compared with Construction \( \square \) for the case where \( \nu = 2 \).

We notice that the storage cost of Construction 2 is strictly less than \( \frac{\nu}{c+\nu-1}K \) for \( c > 2\nu \).

## IV. LOWER BOUNDS ON THE STORAGE COSTS

In this section, we provide our impossibility results for the case where we have two independent versions. We start with the case where \( c_W = 2h + 1 = n - 1 \) and \( c_R \leq (n - 1) \) in Theorem 3 and provide a lower bound on the storage cost. Later on in this section, we study the case where \( c_W = c_R \) and \( h \leq (n - c)/4 \) in Theorem 4 An important outcome of Theorem 4 is that a side information of \((n - 3)/2 \) servers may not help in improving the worst-case storage cost. We begin with the case where where \( c_W = 2h + 1 = n - 1 \) and \( c_R \leq (n - 1) \).

**Theorem 3.** A \((n, c_W = n - 1, c_R \leq (n - 1), \nu = 2, h = n/2 - 1, 2^K, q)\) multi-version code with side information, where \( n \) is even, must satisfy
\[
\log q \geq \frac{2}{2c - 1}K,
\]
for the case where the versions are independent.

**Proof.** We construct two states as shown in Fig. 1 and Fig. 2, with different decoding requirements, such that server \( n/2 - 2 \) observes the same side information in both states and hence it cannot differentiate between these two states. We notice that server \( n/2 - 2 \) is not aware of the state of server \( n - 2 \). We next specify these states, where only server \( n - 2 \) changes its state from state 1 to state 2.

1) **State** \( S_1 \). In this state, \( c_W = (n - 1) \) servers have received \( W_2 \). Therefore, the decoder must be able to decode this version by connecting to any \( \nu = c + 1 \) servers. In particular, we have
\[
A_{S_1}(1) = A_{S_1}(2) = \{0, 1, \cdots, n - 2\}.
\]

Suppose that the decoder connects to the following set of servers
\[
\mathcal{R}_1 = \{i_1, i_2, \cdots, i_{c-1}\} \cup \{n/2 - 2, n - 1\},
\]
where \( \{i_1, i_2, \cdots, i_{c-1}\} \subset N \) such that \( \{i_1, i_2, \cdots, i_{c-1}\} \cap \{n/2 - 2, n - 1\} = \emptyset \).

Denote the value stored at the \( i \)-th server in this state by \( X_i \), which is given by
\[
X_i = \varphi_{S_1(H_i)}^{(i)}(W_{S_1(i)}, S_1(H_i)),
\]
where \( i \in N \). Since server \( n - 1 \) does not have any version, we must have
\[
H(W_2|X_1, X_2, \cdots, X_{i-1}, X_{n/2-2}) = 0.
\]
Therefore, we have
\[
\sum_{j=1}^{c-1} H(X_{ij}|W_1) + H(X_{n/2-2}|W_1) \\
\geq H(X_{i_1}, \ldots, X_{i_{c-1}}, X_{n/2-2}|W_1) \\
= I(X_{i_1}, \ldots, X_{i_{c-1}}, X_{n/2-2}; W_2|W_1) \\
= K. \tag{15}
\]

2) **State** $S_2$. In this state $c_W$ servers have $W_1$ and $(c_W - 1)$ servers have $W_2$. Therefore, the decoder
can decode either $W_1$ or $W_2$ by connecting to any $c_R = c + 1$ servers. Specifically, we have

$$A_S(1) = \{0, 1, \cdots, n - 2\},$$
$$A_S(2) = \{0, 1, \cdots, n - 3\}. \quad (16)$$

Denote the value stored at the $i$-th server in this state by $Y_i$, which is given by

$$Y_i = \phi_{S_2(H_i)}(W_{S_2(H_i)}, S_2(H_i)), \quad (17)$$

where $i \in \mathbb{N}$. We notice that server $n/2 - 2$ observes the same side information as in state $S_1$, hence we have

$$Y_{n/2 - 2} = X_{n/2 - 2}. \quad (18)$$

The decoder connects to the following set of servers

$$R_2 = \{l_1, l_2, \cdots, l_{c - 2}\} \cup \{n/2 - 2, n - 2, n - 1\}, \quad (19)$$

where $\{l_1, l_2, \cdots, l_{c - 2}\} \subset \{0, 1, \cdots, n - 1\}$ such that $\{l_1, l_2, \cdots, l_{c - 2}\} \cap \{n/2 - 2, n - 2, n - 1\} = \emptyset$.

We next consider the following cases based on which version is decoded in this state.

a) In order to decode $W_2$, we must have

$$H(W_2|Y_{l_1}, Y_{l_2}, \cdots, Y_{l_{c - 2}}, X_{n/2 - 2}) = 0. \quad (20)$$

Therefore, we have

$$(c - 1) \log q \geq \sum_{j=1}^{c-2} H(Y_j) + H(X_{n/2 - 2})$$

$$\geq \sum_{j=1}^{c-2} H(Y_j|W_1) + H(X_{n/2 - 2}|W_1)$$

$$\geq H(Y_{l_1}, \cdots, Y_{l_{c - 2}}, X_{n/2 - 2}|W_1)$$

$$= I(Y_{l_1}, \cdots, Y_{l_{c - 2}}, X_{n/2 - 2}; W_2|W_1)$$

$$= K, \quad (21)$$

hence in this case we have

$$\log q \geq \frac{K}{c - 1}. \quad (22)$$

b) In order to decode $W_1$, we must have

$$H(W_1|Y_{l_1}, \cdots, Y_{l_{c - 2}}, X_{n/2 - 2}, Y_{n - 2}) = 0. \quad (23)$$

Therefore, we have

$$\sum_{j=1}^{c-2} H(Y_j|W_2) + H(X_{n/2 - 2}|W_2) + H(Y_{n - 2}|W_2)$$

$$\geq H(Y_{l_1}, \cdots, Y_{l_{c - 2}}, X_{n/2 - 2}, Y_{n - 2}|W_2)$$

$$= I(Y_{l_1}, \cdots, Y_{l_{c - 2}}, X_{n/2 - 2}, Y_{n - 2}; W_1|W_2)$$

$$= K. \quad (24)$$
Moreover, since $H(X_i|\mathbf{W}_{[2]}) = 0$, $i \in \mathcal{N}$ and $\mathbf{W}_1$ and $\mathbf{W}_2$ are independent, we have

$$I(X_i;\mathbf{W}_{[2]}) = H(X_i)$$

$$= H(\mathbf{W}_1) + H(\mathbf{W}_2)$$

$$- H(\mathbf{W}_1|X_i) - H(\mathbf{W}_2|\mathbf{W}_1,X_i)$$

$$= H(X_i) - H(X_i|\mathbf{W}_1)$$

$$+ H(X_i) - H(X_i|\mathbf{W}_2),$$

(25)

for $i \in \mathcal{N}$. Similarly, we also have

$$H(Y_i) = H(Y_i|\mathbf{W}_1) + H(Y_i|\mathbf{W}_2),$$

(26)

for $i \in \mathcal{N}$. Therefore, we have

$$\sum_{j=1}^{c-2} H(Y_{i_j}) + H(X_{n/2-2})$$

$$+ H(Y_{n-2}) \geq K + H(X_{n/2-2}|\mathbf{W}_1)$$

(27)

which implies that

$$c \log q \geq K + H(X_{n/2-2}|\mathbf{W}_1).$$

(28)

We notice that (28) and (15) imply the following bound in this case

$$\log q \geq \frac{2K}{2c-1}.$$  

(29)

Therefore, the storage cost is lower-bounded as follows

$$\log q \geq \min \left\{ \frac{K}{c-1}, \frac{2K}{2c-1} \right\}$$

$$= \frac{2K}{2c-1}.$$  

(30)

We next provide a converse that shows that the side information may not always help even if the server is aware of the states of $(n-3)/2$ other servers for the case where we have two independent versions.

**Theorem 4.** A $(n,c_W,c_R = c_W, \nu = 2, h \leq (n-c)/4, 2K, q)$ multi-version code with side information, where $c \geq 3$, $n \geq \lceil 7c/3 \rceil + 4$ such that $(n-c)/4$ is an integer, must satisfy

$$\log q \geq \max \left\{ \lceil 2c/3 \rceil, \frac{2}{2c - \lceil 2c/3 \rceil} \right\} K,$$

for the case where the versions are independent.

**Proof.** We consider the following two states shown in Fig. 3 and Fig. 4. In both states, the first $c$ servers observe the same side information. That follows since these servers cannot see the states of servers \{$(n+3c)/4, \cdots, (3n+c-4)/4$\}. Therefore, they cannot differentiate between the two states.

1) **State $S_1$.** In this state, $c_W$ servers have $\mathbf{W}_2$. Therefore, the decoder must decode $\mathbf{W}_2$.

In particular, the set of servers that have $\mathbf{W}_2$ is given by

$$\mathcal{A}_{S_1}(2) = \{0, 1, \cdots, c-1\} \cup \{(n+3c)/4, \cdots, (3n+c-4)/4\}.$$

Also, the set of servers that have $\mathbf{W}_1$ is given by

$$\mathcal{A}_{S_1}(1) = \{0, 1, \cdots, (n+3c-4)/4\} \cup \{(3n+c)/4, \cdots, n-1\}.$$
Fig. 3: The system in state $S_1$ is shown. The edges between a node and the $h$-hop neighbors are not shown. In this state, $W_2$ must be decoded.

Denote the value stored at the $i$-th server in this state by $X_i$ which is given by

$$X_i = \varphi_{S_i(h_i)}^{(i)}(W_{S_i(i)}, S_1(h_i)),$$

where $i \in \mathcal{N}$. Suppose that the decoder connects to the set $\mathcal{R}_1 = \mathcal{A}_{S_1}(1)$ to decode $W_2$. Since $W_1$ and $W_2$ are independent, to decode $W_2$ in this state we require that

$$H(W_2|X_0, X_1, \ldots, X_{c-1}) = 0.$$ (31)

Fig. 4: The system in state $S_2$ is shown. The edges between a node and the $h$-hop neighbors are not shown. Servers $\{(n + 3c)/4, \ldots, (3n + c - 4)/4\}$ have not received any version. In this state, either $W_1$ or $W_2$ is decoded.
Therefore, we have
\[
\sum_{i=0}^{c-1} H(X_i|W_1) \geq H(X_0, X_1, \cdots, X_{c-1}|W_1) = I(X_0, X_1, \cdots, X_{c-1}; W_2|W_1) = K. \tag{32}
\]

2) **State** $S_2$. In this state, $c_W$ servers have $W_1$ and less than $c_W$ servers have $W_2$. Therefore, the decoder can either decode $W_1$ or $W_2$.

In particular, the set of servers that have $W_1$ is given by $A_{S_2}(1) = A_{S_1}(1)$ and the set of servers that have $W_2$ is given by $A_{S_2}(2) = \{0, 1, \cdots, c-1\}$.

Denote the value stored at the $i$-th server in this state by $Y_i$, which is given by
\[
Y_i = \varphi_{S_2(H_i)}^{(i)}(W_{S_2(i)}, S_2(H_i)),
\]
where $i \in \mathcal{N}$.

We observe that servers $\{0, 1, \cdots, c-1\}$ have the same side information in state 1 and state 2. Therefore, we have $Y_i = X_i$, $i \in \{0, 1, \cdots, c-1\}$.

Suppose that the decoder connects to the following set of servers and requires to decode $W_1$ or $W_2$
\[
R_2 = \{0, 1, \cdots, l\} \\
\cup \{(n + 3c)/4, \cdots, (3n + c - 4)/4\} \\
\cup \{c, c + 1, \cdots, 2c - l - 2\}, \tag{33}
\]
where $l \in \{0, 1, \cdots, c-1\}$ such that $2c - l - 2$ is at most $(n + 3c - 4)/4$. We next consider the following cases based on which version is decoded in that state.

a) In order to decode $W_2$ in this state, we must have $H(W_2|X_0, X_1, \cdots, X_l) = 0$. Therefore, we have
\[
(l + 1) \log q \geq \sum_{i=0}^{l} H(X_i) \geq \sum_{i=0}^{l} H(X_i|W_1) \geq H(X_0, X_1, \cdots, X_l|W_1) = I(X_0, X_1, \cdots, X_l; W_2|W_1) = K. \tag{34}
\]

Therefore, decoding $W_2$ in this state implies that the worst-case storage cost is lower-bounded as follows
\[
\log q \geq \frac{1}{l+1} K. \tag{35}
\]

b) On the other hand, to decode $W_1$, we must have $H(W_1|X_0, \cdots, X_l, Y_c, \cdots, Y_{2c-l-2}) = 0$, since servers $\{(n + 3c)/4, \cdots, (3n + c - 4)/4\}$ have not received any version. Therefore, we have
\[
\sum_{i=0}^{l} H(X_i|W_2) + \sum_{i=c}^{2c-l-2} H(Y_i|W_2) \geq H(X_0, \cdots, X_l, Y_c, \cdots, Y_{2c-l-2}|W_2) = I(X_0, \cdots, X_l, Y_c, \cdots, Y_{2c-l-2}; W_1|W_2) = K. \tag{36}
\]
Moreover, since $H(X_i|W_{[2]}) = 0$, $i \in \mathcal{N}$ and $W_1$ and $W_2$ are independent, we have

\[
I(X_i; W_{[2]}) = H(X_i)
= H(W_1) + H(W_2)
- H(W_1|X_i) - H(W_2|W_1, X_i)
= H(X_i) - H(X_i|W_1)
+ H(X_i) - H(X_i|W_2).
\]

(37)

Therefore, we have

\[
H(X_i) = H(X_i|W_1) + H(X_i|W_2).
\]

(38)

We notice that (36), (38), imply the following

\[
\sum_{i=0}^{l} (H(X_i) - H(X_i|W_1)) + \sum_{i=c}^{2c-l-2} H(Y_i) \geq K.
\]

Therefore, we must have

\[
c \log q \geq K + \sum_{i=0}^{l} H(X_i|W_1)
\]

(39)

which implies with (32) that we have

\[
c \log q \geq 2K - \sum_{i=l+1}^{c-1} H(X_i|W_1)
\geq 2K - (c - l - 1) \log q.
\]

(40)

Hence, we have

\[
\log q \geq \frac{2}{2c - l - 1} K.
\]

(41)

Therefore, the storage cost is lower-bounded as follows

\[
\log q \geq \min \left\{ \frac{K}{l+1}, \frac{2K}{2c - l - 1} \right\}.
\]

(42)

Specifically, choosing $l = \lceil 2c/3 \rceil - 1$, we get

\[
\log q \geq \frac{1}{\lceil 2c/3 \rceil} K.
\]

(43)

Also, choosing $l = \lfloor 2c/3 \rfloor - 1$ results in the following

\[
\log q \geq \frac{2}{2c - \lfloor 2c/3 \rfloor} K.
\]

(44)

In general, the storage cost is lower-bounded as follows

\[
\log q \geq \max \left\{ \frac{1}{\lceil 2c/3 \rceil}, \frac{2}{2c - \lfloor 2c/3 \rfloor} \right\} K.
\]

(45)

**Corollary 4.1.** A $(n, c_W = (n + 3)/2, c_R = (n + 3)/2, \nu = 2, h \leq (n - 3)/4, 2^K, q)$ multi-version code with side information must satisfy

\[
\log q \geq \frac{K}{2}.
\]
The implication of Corollary 4.1 is that for $c = 3$ even if the server is aware of the states of $(n - 3)/2$ other servers, the side information does not help in reducing the worst-case storage cost. That follows since a worst-case storage cost of $K/2$ can be achieved in a distributed manner without sharing the states between the servers using the code construction proposed in [7]. The question of whether sharing the states can help or not in reducing the worst-case storage cost for the case where $c > 3$ remains an open question.

V. CONCLUSION

In this work, we have studied a storage system with $n$ servers that are distributed in a ring that store $\nu$ independent versions of a message. Assuming that server can share the side information of which versions it has received with its $h$-hop neighbors, we have constructed codes that show that this side information can be used to reduce the worst-case storage cost if the server is aware of the states of $(n - 2)$ other servers. We have also shown that even if the server is aware of the states of $(n - 3)/2$ other servers, the side information may not help in reducing the worst-case storage cost for the case of two versions. Our future work is to characterize the role of the side information in all regimes of $h$.

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