Fekete’s lemma for componentwise subadditive functions of two or more real variables

Silvio Capobianco∗†

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Abstract

We prove an analogue of Fekete’s subadditivity lemma for functions of several real variables which are subadditive in each variable taken singularly. This extends both the classical case for subadditive functions of one real variable, and a result in a previous paper by the author. The arguments follows those of Chapter 6 in E. Hille’s 1948 textbook.

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1 Introduction

A real-valued function $f$ defined on a semigroup $(S, \cdot)$ is subadditive if

$$f(x \cdot y) \leq f(x) + f(y)$$

for every $x, y \in S$. Examples of subadditive functions include the absolute value of a complex number, the ceiling of a real number, and the length of a word over an alphabet. Subadditive functions are involved in several phenomena, such as the definition of entropy for subshifts [8]. If $S$ is the additive group of either the positive integers or the positive reals, then Fekete’s lemma [3] (see also [5, Theorem 6.6.1]) states that:

$$\lim_{x \to \infty} \frac{f(x)}{x} = \inf_{x > 0} \frac{f(x)}{x}.$$ (2)

∗Department of Software Science, Tallinn University of Technology. silvio@cs.ioc.ee silvio.capobianco@taltech.ee

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If \( S = S_1 \times S_2 \) is a direct product of two semigroups, another option can be considered: that the function be subadditive in each variable, however given the other. That is, instead of requiring \( f(x_1y_1, x_2y_2) \leq f(x_1, x_2) + f(y_1, y_2) \) for every \( x_1, x_2, y_1 \) and \( y_2 \), we demand that:

1. \( f(x_1y_1, x_2) \leq f(x_1, x_2) + f(y_1, x_2) \) for every \( x_1, x_2, \) and \( y_1 \); and
2. \( f(x_1, x_2y_2) \leq f(x_1, x_2) + f(x_1, y_2) \) for every \( x_1, x_2, \) and \( y_2 \).

The two requirements, even together, are not equivalent to subadditivity as a function defined on the product semigroup: see Example 3.3.

The aim of this paper is to prove a generalization to arbitrary dimension of the following statement:

**Proposition 1.1.** Let \( f \) be a function of two positive real variables \( x, y \) which is subadditive in each of them, however given the other. For every \( \varepsilon > 0 \), there exists \( R > 0 \) such that, if both \( x > R \) and \( y > R \), then:

\[
\frac{f(x, y)}{xy} < \inf_{x,y>0} \frac{f(x, y)}{xy} + \varepsilon.
\]

A similar statement for functions defined on \( d \)-tuples of positive integers (instead of reals) was proved in [1]. The argument presented there, however, relies on a boundedness property which comes for free in the integer setting, but must be proved in the new one. This is done by adapting the proof of [5, Theorem 6.4.1].

The paper is organized as follows. Section 2 provides the theoretical background. In Section 3, we introduce componentwise subadditivity and explain how it is different from subadditivity in the product semigroup. In Section 4, we adapt the argument from [5, Theorem 6.4.1] to prove that componentwise subadditive functions of \( d \) real variables are bounded on compact subsets of \( \mathbb{R}^d \). In Section 5, we state, prove, and discuss the main theorem: boundedness will have a crucial role in the proof. Section 6 contains a short discussion on how the Ornstein-Weiss lemma [6, 9] relates to Fekete’s lemma.

## 2 Background

Throughout the paper, the sets and the functions in the hypotheses of the claims are presumed to be Borel measurable.

We denote by \( \mathbb{R} \), \( \mathbb{R}_+ \), and \( \mathbb{R}_- \) the sets of real numbers, positive real numbers, and negative real numbers, respectively. Similarly, we denote by
The sets of integers, positive integers, and negative integers, respectively. All these sets are considered as additive semigroups (groups in the case of $\mathbb{R}$ and $\mathbb{Z}$). If $m$ and $n$ are integers and $m \leq n$ we denote the slice 
\{m, m+1, \ldots, n-1, n\} = [m, n] \cap \mathbb{Z}$ as $[m:n]$.

If the sets $X$ and $Y$ where the variable $x$ and the expression $E(x)$ take values are irrelevant or clear from the context, we denote by $x \mapsto E(x)$ the function that associates to each value $\mathbb{T}$ taken by $x$ the value $E(\mathbb{T})$. For example, $x \mapsto 1$ is the constant function that always takes value 1.

A directed set is a partially ordered set $U = (U, \leq)$ with the following additional property: for every $u, v \in U$, there exists $w \in U$ such that $u \leq w$ and $v \leq w$. Every totally ordered set is a directed set, and so is the family of decompositions

\[ x = \{a = x_0 < x_1 < \ldots < x_n = b\} \]

of the compact interval $[a, b]$ with the partial order $x \preceq y$ iff for every $i$ there exists $j$ such that $x_i = y_j$. A function $f$ defined on $U$ is also called a net on $U$. If $Y$ is the codomain of $f$, a subnet of $f$ is a net $g : V \to Y$ on a directed set $V = (V, \preceq)$ together with a function $\phi : V \to U$ such that:

1. $f \circ \phi = g$; and
2. for every $u \in U$ there exists $v \in V$ such that, if $z \in V$ and $z \succeq v$, then $\phi(z) \succeq u$.

For example, a subsequence $\{x_{n_k}\}_{k \geq 1}$ of a sequence $\{x_n\}_{n \geq 1}$ of real numbers is a subnet, with $V = U = \mathbb{Z}_+$ and $\phi(k) = n_k$.

If $U = (U, \leq)$ is a directed set and $f : U \to \mathbb{R}$ is a function, the lower limit and the upper limit of $f$ in $U$ are the quantities

\[ \liminf_{u \to U} f(u) = \sup_{u \in U} \inf_{v \succeq u} f(v), \]

\[ \limsup_{u \to U} f(u) = \inf_{u \in U} \sup_{v \succeq u} f(v), \]

respectively. The following chain of equalities holds:

\[ \inf_{u \in U} f(u) \leq \liminf_{u \to U} f(u) \leq \limsup_{u \to U} f(u) \leq \sup_{u \in U} f(u). \]

Moreover, if $V = (V, \preceq)$ is a directed set and $g : V \to \mathbb{R}$ is a subnet of $f$, then:

\[ \liminf_{u \to U} f(u) \leq \liminf_{v \to V} g(v) \leq \limsup_{v \to V} g(v) \leq \limsup_{u \to U} f(u). \]
If \( \liminf_{u \to \mathcal{U}} f(u) \leq \limsup_{u \to \mathcal{U}} f(u) \), their common value \( L \) is called the limit of \( f \) in \( \mathcal{U} \), and we say that \( f \) converges to \( L \) in \( \mathcal{U} \). This is equivalent to the following: for every \( \varepsilon > 0 \) there exists \( u_\varepsilon \in U \) such that \( |f(u) - L| < \varepsilon \) for every \( u \geq u_\varepsilon \). In this case, every subnet of \( f \) also converges to \( L \).

The ordered product of a family \( \{(X_i, \leq_i)\}_{i \in I} \) of ordered sets is the ordered set \( (X, \leq) \) where \( X = \prod_{i \in I} X_i \) and the product ordering \( \leq \) is defined as:

\[
x \leq y \iff x_i \leq y_i \quad \forall i \in I.
\]

If each \( (X_i, \leq_i) \) is a directed set, then so is \( (X, \leq) \). For \( d \geq 2 \) and \( w \in \{0, 1\}^d \) we define the octant \( \mathbb{R}_w \) denoted by \( w \) as the directed set \( \mathcal{R}_w = \prod_{i=1}^d (X_i, \leq_i) \) where \( (X_i, \leq_i) = (\mathbb{R}_+, \leq) \) if \( w_i = 0 \) and \( (X_i, \leq_i) = (\mathbb{R}_-, \geq) \) if \( w_i = 1 \). In particular, the main octant of \( \mathbb{R}^d \), corresponding to \( w = 0^d \), is the directed set:

\[
\mathcal{R}_+^d = \left( \mathbb{R}_+^d, \leq \right).
\]

Note that, if \( f : \mathbb{R}_+^d \to \mathbb{R} \) is a net on \( \mathbb{R}_+^d \) and \( \{x_{i,n}\}_{n \geq 1}, i \in [1:d] \), are divergent sequences of positive reals, then \( g(n) = f(x_{1,n}, \ldots, x_{d,n}) \) is a subnet of \( f \): consequently, if \( f \) converges to \( L \in \mathbb{R} \) in \( \mathcal{R}_+^d \), then \( g(n) \) converges to \( L \) for \( n \to \infty \).

### 3 Componentwise subadditivity

In the literature, subadditivity is most often studied in functions of a single variable, which sometimes may be vector rather than scalar. But in some cases, it is of interest to consider functions of \( d \) independent variables, which are subadditive when considered as depending on only one of those.

**Definition 3.1.** Let \( S_1, \ldots, S_d \) be semigroups, let \( S = \prod_{i=1}^d S_i \), and let \( f : S \to \mathbb{R} \). Given \( i \in [1:d] \), we say that \( f \) is subadditive in \( x_i \) independently of the other variables if, however given \( x_j \in S_j \) for every \( j \in [1:d] \setminus \{i\} \), the function \( x_i \mapsto f(x_1, \ldots, x_i, \ldots, x_d) \) is subadditive on \( S_i \). We say that \( f \) is *componentwise subadditive* if it is subadditive in each variable independently of the others.

**Example 3.2.** Let \( A \) be a finite set with \( a \geq 2 \) elements. The *translation* by \( v \in \mathbb{Z}^2 \) is the function \( \sigma_v : A^{\mathbb{Z}^2} \to A^{\mathbb{Z}^2} \) defined by \( \sigma_v(c)(x) = c(x + v) \) for every \( x \in \mathbb{Z}^2 \). Let \( T \) be a transformation of \( A^{\mathbb{Z}^2} \) into itself which commutes with translations. Given two positive integers \( n_1, n_2 \), an *orphan* of sides \( n_1 \)

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1 By analogy with the four quadrants in the 2D real plane, there will be eight octants in the 3D real space: hence the chosen name.
and $n_2$ for $T$ is a function $p : [1:n_1] \times [1:n_2] \rightarrow A$ for which, however given $c : \mathbb{Z}^2 \rightarrow A$, the restriction of $T(c)$ to $[1:n_1] \times [1:n_2]$ is different from $p$. Let $\text{Out}(n_1, n_2)$ be the number of non-orphans for $T$ of sides $n_1$ and $n_2$: then

$$f(n_1, n_2) = \log_2 \text{Out}(n_1, n_2) \quad \text{for every } n_1, n_2 \in \mathbb{R}_+$$  \hspace{1cm} (9)

is componentwise subadditive, because every non-orphan of sides $n_1 + m_1$ and $n_2$ can be obtained by joining a non-orphan of sides $n_1$ and $2$ with a non-orphan of sides $m_1$ and $n_2$, and similar for non-orphans of sides $n_1$ and $n_2 + m_2$. This works because $T$ commutes with translations.

Componentwise subadditivity is very different from subadditivity with respect to the operation of the product semigroup. Already with $d = 2$, if $f : S_1 \times S_2 \rightarrow \mathbb{R}$ is subadditive, then for every $x_1, y_1 \in S_1$ and $x_2, y_2 \in S_2$ we have:

$$f(x_1 y_1, x_2 y_2) \leq f(x_1, x_2) + f(y_1, y_2),$$  \hspace{1cm} (10)

while if $f$ is componentwise subadditive, then for every $x_1, y_1 \in S_1$ and $x_2, y_2 \in S_2$ we have the more complex upper bound:

$$f(x_1 y_1, x_2 y_2) \leq f(x_1, x_2) + f(x_1, y_2) + f(y_1, x_2) + f(y_1, y_2).$$  \hspace{1cm} (11)

If $f$ is nonnegative, then (10) implies (11), which however is weaker than the conditions of Definition 3.1 if $f$ is nonpositive, then (11) implies (10). But, in general, neither implies the other.

**Example 3.3.** Let $f(x_1, x_2) = \sqrt{x_1 x_2}$ for every $x_1, x_2 \in \mathbb{R}_+$. Then $f$ is componentwise subadditive: for example, for every two $x_1, y_1 \in \mathbb{R}_+$ it is $x_1 + y_1 < (\sqrt{x_1} + \sqrt{y_1})^2$, thus also, for every $x_2 \in \mathbb{R}_+$,

$$\sqrt{(x_1 + y_1) x_2} < \sqrt{(\sqrt{x_1} + \sqrt{y_1})^2 \sqrt{x_2}} = \sqrt{x_1 x_2} + \sqrt{y_1 x_2}.$$  

However, $f$ is not subadditive on $\mathbb{R}_+^2$: if $x_1 = y_2 = 1$ and $x_2 = y_1 = 2$, then

$$f(x_1 + y_1, x_2 + y_2) = \sqrt{(1 + 2) \cdot (2 + 1)} = 3$$  

is strictly larger than

$$f(x_1, x_2) + f(y_1, y_2) = \sqrt{1 \cdot 2} + \sqrt{2 \cdot 1} = 2\sqrt{2} = 2.82\ldots$$

**Example 3.4.** The function (9) of Example 3.2 is not, in general subadditive. For example, if $T$ is surjective, then it has no orphans, so:

$$f(n_1 + m_1, n_2 + m_2) = (n_1 + m_1)(n_2 + m_2)$$

$$> n_1 n_2 + m_1 m_2$$

$$= f(n_1, n_2) + f(m_1, m_2).$$
The following observation is crucial for the next sections.

**Proposition 3.5.** Let \( w = w_1 \ldots w_d \) be a binary word of length \( d \) and let \( f : \mathbb{R}_w \to \mathbb{R} \). For every \( i \in [1:d] \) let \( x_{w,i} = (-1)^{w_i} x_i \). The following are equivalent:

1. \( f(x_1, \ldots, x_d) \) is componentwise subadditive in \( \mathbb{R}_w \).

2. \( f(x_{w,1}, \ldots, x_{w,d}) \) is componentwise subadditive in \( \mathbb{R}^d_+ \).

The same holds if \( \mathbb{R}_w \) and \( \mathbb{R}^d_+ \) are replaced with \( \mathbb{Z}_w = \mathbb{R}_w \cap \mathbb{Z}^d \) and \( \mathbb{Z}^d_+ \), respectively.

4 Componentwise subadditive functions of \( d \) real variables are bounded on compacts

In [1] we prove the following:

**Theorem 4.1** (Fekete’s lemma in \( \mathbb{Z}^d_+ \); [1, Theorem 1]). Let \( U = (\mathbb{Z}^d_+, \leq \Pi) \) and let \( f : \mathbb{Z}^d_+ \to \mathbb{R} \) be componentwise subadditive. Then:

\[
\lim_{(x_1, \ldots, x_d) \to U} \frac{f(x_1, \ldots, x_d)}{x_1 \cdot \ldots \cdot x_d} = \inf_{x_1, \ldots, x_d \in \mathbb{Z}^d_+} \frac{f(x_1, \ldots, x_d)}{x_1 \cdot \ldots \cdot x_d}.
\] (12)

We try to reuse the argument from [1] to prove Proposition 1.1. Fix \( s, t > 0 \). Every \( x > 0 \) large enough has a unique writing \( x = qs + r \) with \( q \) positive integer and \( r \in [s, 2s) \), and every \( y > 0 \) large enough has a unique writing \( y = mt + p \) with \( m \) positive integer and \( p \in [t, 2t) \). By subadditivity,

\[
\frac{f(x, y)}{x \cdot y} \leq \frac{q}{x} \cdot \frac{f(s, y)}{y} + \frac{1}{x} \cdot f(r, y)
\]

\[
\leq \frac{q}{x} \cdot \frac{m}{y} \cdot f(s, t)
\]

\[+ \frac{q}{x} \cdot \frac{1}{y} \cdot f(s, p) + \frac{1}{x} \cdot \frac{m}{y} \cdot f(r, t)
\]

\[+ \frac{1}{x} \cdot \frac{1}{y} \cdot f(r, p)
\]

Consider the four summands on the right-hand side of the last inequality. By construction, \( \lim_{x \to +\infty} q/x = 1/s \) and \( \lim_{y \to +\infty} m/y = 1/t \): therefore, the first summand converges to \( f(s, t)/st \) for \((x, y) \to \mathbb{R}^2_+\).

Now, by [5, Theorem 6.4.1], a subadditive function of one positive real variable is bounded in every compact subset of \( \mathbb{R}_+ \). Then \( p \mapsto f(s, p) \) is
bounded on \([t, 2t]\) and \(r \mapsto f(r, t)\) is bounded on \([s, 2s]\): consequently, the second and third summand vanish for \((x, y) \to \mathbb{R}^2_+\).

But the fourth summand presents a problem. What we know, is that \(x \mapsto f(x, y)\) is bounded in \([s, 2s]\) for every \(y \in [t, 2t]\), and \(y \mapsto f(s, y)\) is bounded in \([t, 2t]\) for every \(x \in [s, 2s]\): this is a priori less than \(f\) being bounded in \([s, 2s] \times [t, 2t]\), which is what we actually need to show that the fourth summand vanishes when \(x\) and \(y\) both grow arbitrarily large!

Can we infer boundedness in \([s, 2s] \times [t, 2t]\) from boundedness in \([s, 2s] \times \{y\}\) for every \(y \in [t, 2t]\) and in \({x}\) \times \([t, 2t]\) for every \(x \in [s, 2s]\)? In general, \(\text{no}\).

**Example 4.2** (suggested by [10]). Let \(h : \mathbb{R}_+ \to \mathbb{R}\) be such that \(h(t)\) is the denominator of the representation of \(t\) as an irreducible fraction if \(t\) is rational, and 0 if \(t\) is irrational. Then \(f : \mathbb{R}^2_+ \to \mathbb{R}\) defined by \(f(x, y) = \text{min}(h(x), h(y))\) satisfies the following conditions:

1. for every \(x \in [1, 2]\), the function \(y \mapsto f(x, y)\) is bounded in \([1, 2]\); 
2. for every \(y \in [1, 2]\), the function \(x \mapsto f(x, y)\) is bounded in \([1, 2]\).

However, \(f\) is not bounded in \([1, 2] \times [1, 2]\), because \(h(1 + 1/n) = n\) for every \(n \in \mathbb{Z}_+\). On the other hand, \(h(1) = 1\) and \(h(1/\pi) = h(1 − 1/\pi) = 0\), so \(f\) is neither subadditive nor componentwise subadditive in \(\mathbb{R}^2_+\).

We could overcome this issue if a result of boundedness such as the one in [5, Theorem 6.4.1] held for componentwise subadditive functions. Luckily, it is so, and we can follow the same idea of Hille’s proof. Given \(f : \mathbb{R}^d_+ \to \mathbb{R}\) and \(t_1, \ldots, t_d \in \mathbb{R}_+\), let:

\[
V_{t_1, \ldots, t_d, k} = \{(x_1, \ldots, x_d) \in \mathbb{R}^d_+ | 0 < x_i < t_i \forall i \in [1:d], f(x_1, \ldots, x_d) \geq k\}.
\]

**(13)**

**Lemma 4.3.** Let \(f : \mathbb{R}^d_+ \to \mathbb{R}\) be componentwise subadditive. For every \(t_1, \ldots, t_d \in \mathbb{R}_+\), the set

\[
W_{t_1, \ldots, t_d} = V_{t_1, \ldots, t_d, \frac{f(t_1, \ldots, t_d)}{2^d}}
\]

has measure at least \(t_1 \cdots t_d/2^d\).

**Proof.** For every \(i \in [1:d]\), given \(x_i \in (0, t_i]\), let \(y_i^{(0)} = x_i\) and \(y_i^{(1)} = t_i − x_i\). By repeatedly applying subadditivity, once in each variable, we arrive at:

\[
f(t_1, \ldots, t_d) \leq \sum_{w \in \{0, 1\}^d} f\left(y_1^{(w_1)}, \ldots, y_d^{(w_d)}\right).
\]

**(15)**
For example, for $d = 2$ we have:

$$f(t_1, t_2) \leq f(x_1, t_2) + f(t_1 - x_1, t_2)$$
$$\leq f(x_1, x_2) + f(x_1, t_2 - x_2) + f(t_1 - x_1, x_2) + f(t_1 - x_1, t_2 - x_2).$$

For (15) to hold, at least one of the $2^d$ summands on the right-hand side must be no smaller than $f(t_1, \ldots, t_d)/2^d$: for it to hold for every $x_i \in (0, t_i)$ for every $i \in [1: d]$, the set $W_{t_1, \ldots, t_d}$ must have at least measure $t_1 \cdots t_d/2^d$. Note that we can conclude so because $f$ is measurable.

**Theorem 4.4.** Let $f : \mathbb{R}^d_+ \to \mathbb{R}$ be componentwise subadditive. Then $f$ is bounded in every compact subset of $\mathbb{R}^d_+$.

**Proof.** It is sufficient to prove the thesis for every compact hypercube of the form $H = [a, b]^d$ with $0 < a < b$. We proceed by contradiction, following the argument from [5, Theorem 6.4.1].

First, suppose that $f$ is unbounded from above in $H$. Then for every $n \geq 1$ and $i \in [1: d]$ there exists $x_{i,n} \in [a, b]$ such that $f(x_{1,n}, \ldots, x_{d,n}) \geq 2^d n$. Let $W_{x_{1,n}, \ldots, x_{d,n}}$ be defined by (14). By construction, for every $n \geq 1$ we have

$$W_{x_{1,n}, \ldots, x_{d,n}} \subseteq V_{b, \ldots, b, n},$$

and by Lemma 4.3 the left-hand side has measure at least

$$\frac{x_{1,n} \cdots x_{d,n}}{2^d} \geq \left(\frac{a}{2}\right)^d.$$

Now, the sets $V_{b, \ldots, b, n}$ are measurable and form a nonincreasing sequence, so their intersection $V$ is measurable and has measure at least $(a/2)^d$: in particular, it cannot be empty. But for $(x_1, \ldots, x_d) \in V$ it must be $f(x_1, \ldots, x_d) \geq n$ for every $n \geq 1$: which is impossible.

Next, suppose that $f$ is unbounded from below in $H$. Then for every $n \geq 1$ and $i \in [1: d]$ there exists $x_{i,n} \in [a, b]$ such that $f(x_{1,n}, \ldots, x_{d,n}) \leq -n$: we may assume that $\lim_{n \to \infty} x_{i,n} = x_i \in [a, b]$ exists for every $i \in [1: d]$. Let $s = \min(a, 1)$, $t = b + 4$, and $J = [s, t]^d$: then every point $(z_1, \ldots, z_d)$ where each $z_i$ belongs to either $[a, b]$ or $[1, 4]$ belongs to $J$. Let now $y_i \in [1, 4]$ for every $i \in [1: d]$ and

$$M = \sup\{f(z_1, \ldots, z_d) \mid (z_1, \ldots, z_d) \in J\},$$

which is a real number because of the previous point. By applying subadditivity in each variable, for such $y_1, \ldots, y_d$ and $n$ we obtain

$$f(y_1 + x_{1,n}, \ldots, y_d + x_{d,n}) \leq (2^d - 1)M - n.$$
because \(-n\) is an upper bound for \(f(x_{1,n}, \ldots, x_{d,n})\) and \(M\) is an upper bound for the other \(2^n - 1\) summands. For example, for \(d = 2\) we have:

\[
f(y_1 + x_{1,n}, y_2 + x_{2,n}) \leq f(y_1, y_2) + f(y_1, x_{2,n}) + f(x_{1,n}, y_2) + f(x_{1,n}, x_{2,n}) \leq 3M - n.
\]

But for every \(n\) such that \(|x_{i,n} - x_i| \leq 1\) it is \([x_i + 2, x_i + 3] \subseteq [x_{i,n} + 1, x_{i,n} + 4]\):
calling

\[
K = \prod_{i=1}^{d} [x_i + 2, x_i + 3] \subseteq J,
\]

for every \(n\) large enough every element of \(K\) can be written in the form \((y_1 + x_{1,n}, \ldots, y_d + x_{d,n})\) for suitable \(y_1, \ldots, y_d \in [1, 4]\). For every \((z_1, \ldots, z_d) \in K\) it must then be \(f(z_1, \ldots, z_d) \leq (2^d - 1)M - n\) for every \(n\) large enough: which is impossible.

From Theorem 4.4 and Proposition 3.5 follows:

**Corollary 4.5.** Let \(w \in \{0, 1\}^d\) and let \(f : \mathbb{R}_w \to \mathbb{R}\) be componentwise subadditive. Then \(f\) is bounded in every compact subset of \(\mathbb{R}_w\).

In turn, Corollary 4.5 allows us to prove:

**Theorem 4.6.** Let \(f : \mathbb{R}^d \to \mathbb{R}\) be componentwise subadditive. Then \(f\) is bounded in every compact subset of \(\mathbb{R}^d\).

**Proof.** We give the proof for \(d = 3\): the ideas for arbitrary \(d \geq 1\) are similar. Let \(I = [-1/2, 1/2]\) and \(U = [-3/2, -1/2] \cup [1/2, 3/2]\).

We start by showing that \(f\) is bounded in every compact subset of the set

\[
Z_{00} = \{(x, y, z) \in \mathbb{R}^3 \mid x > 0, y > 0\},
\]

which is the union of the open octants \(\mathbb{R}_{000}\) and \(\mathbb{R}_{001}\) together with the “quadrant”

\[
D_{00} = \{(x, y, z) \in \mathbb{R}^3 \mid x > 0, y > 0, z = 0\}.
\]

To do this, it will suffice to show that \(f\) is bounded in every set of the form \(H = [a, b] \times [a, b] \times I\). Let \(V = [a, b] \times [a, b] \times U\): if \((x, y, z) \in H\), then \((x, y, z - 1)\) and \((x, y, z + 1)\) are both in \(V\). Let \(T\) and \(t\) be an upper bound and a lower bound for \(f\) in \(V\), respectively: then for every \((x, y, z) \in H\),

\[
f(x, y, z) \leq f(x, y, z - 1) + f(x, y, 1) \leq 2T
\]
and
\[ f(x, y, z) \geq f(x, y, z + 1) - f(x, y, 1) \geq t - T. \]

By similar arguments, \( f \) is bounded in every subset of \( \mathbb{R}^3 \) which is the union of two adjacent octants and the corresponding “quadrant”.

We now show that \( f \) is bounded in every compact subset of the “upper demispace”
\[ Z_0 = \{(x, y, z) \in \mathbb{R}^3 \mid z > 0\}. \]

To do so, it will suffice to show that \( f \) is bounded in every set of the form
\[ L = I \times I \times [a, b] \text{ with } 0 < a < b. \]
Let \( W = U \times U \times [a, b] \) and let \( S \) and \( s \) be an upper bound for \( f \) in \( W \), respectively: then for every \((x, y, z) \in L\),
\[ f(x, y, z) \leq f(x - 1, y, z) + f(1, y, z) \leq f(x - 1, y - 1, z) + f(x - 1, 1, z) + f(1, y - 1, z) + f(1, 1, z) \leq 4S \]
and
\[ f(x, y, z) \geq f(x + 1, y, z) - f(1, y, z) \geq f(x + 1, y + 1, z) - f(x + 1, 1, z) - f(1, y, z) \geq s - 2S. \]

By similar reasoning, \( f \) is bounded in each “demispace”.

To conclude the proof, we only need to show that \( f \) is bounded in \( K = I \times I \times I \). Let \( E = U \times U \times U \) and let \( M \) and \( m \) be an upper bound and a lower bound for \( f \) in \( E \), respectively: then for every \((x, y, z) \in K\),
\[ f(x, y, z) \leq f(x - 1, y, z) + f(1, y, z) \leq f(x - 1, y - 1, z) + f(x - 1, 1, z) + f(1, y - 1, z) + f(1, 1, z) + f(x - 1, z - 1) + f(x - 1, 1, 1) + f(1, y - 1, z - 1) + f(1, y - 1, 1) + f(1, 1, z - 1) + f(1, 1, 1) \leq 8M \]
and

\[
f(x, y, z) \geq f(x + 1, y, z) - f(1, y, z)
\]
\[
\geq f(x + 1, y + 1, z) - f(x + 1, 1, z) - f(1, y, z)
\]
\[
\geq f(x + 1, y + 1, z + 1) - f(x + 1, y + 1, 1)
\]
\[
\quad - f(x + 1, 1, z) - f(1, y, z)
\]
\[
\geq m - 3M.
\]

Note that the argument of Lemma 4.3 also works if \( f \) is subadditive, rather than componentwise subadditive. In this case, however, the denominator in (14) and in the thesis is 2 rather than \( 2^d \). A more complex variant of it can then be stated, where \( f \) is a function of \( k \) variables \( x_i \), each taking values in an octant of the form \( \mathbb{R}^d_{+} \): and the denominator would then be \( 2^k \). This, in turn, generalizes Theorem 4.4 to the case of componentwise functions of \( k \) variables, the \( i \)th of which takes values in \( \mathbb{R}^d_{+} \).

5 Fekete’s lemma for componentwise subadditive functions of \( d \) real variables

We can now state and prove the main result of this paper.

**Theorem 5.1** (Fekete’s lemma in \( \mathbb{R}^d_{+} \)). Let \( d \geq 1 \) and let \( f : \mathbb{R}^d_{+} \to \mathbb{R} \) be componentwise subadditive. Then:

\[
\lim_{(x_1, \ldots, x_d) \to \mathbb{R}^d_{+}} \frac{f(x_1, \ldots, x_d)}{x_1 \cdots x_d} = \inf_{x_1, \ldots, x_d \in \mathbb{R}^d_{+}} \frac{f(x_1, \ldots, x_d)}{x_1 \cdots x_d}. \tag{16}
\]

The proof is similar to that of [1, Theorem 1], with an important change. We report it entirely for convenience of the reader.

**Proof of Theorem 5.1.** For every \( i \in \{1, \ldots, d\} \) and \( x_1, \ldots, x_d \in \mathbb{R}_{+} \), if \( x_i = qt + r \) with \( q \in \mathbb{Z}_{+} \) and \( r, t \in \mathbb{R}_{+} \), then:

\[
f(x_1, \ldots, x_i, \ldots, x_d) \leq q \cdot f(x_1, \ldots, t, \ldots, x_d) + f(x_1, \ldots, r, \ldots, x_d).
\]

Fix \( t_1, \ldots, t_d \in \mathbb{R}_{+} \). For every \( i \in [1:d] \) and \( x_i \geq 2t_i \) there exist unique \( q_i \in \mathbb{Z}_{+} \) and \( r_i \in [t_i, 2t_i) \) such that \( x_i = q_i t_i + r_i \). For every \( i \in [1:d] \) let
\(y_i^{(0)} = r_i\) and \(y_i^{(1)} = t_i\): by repeatedly applying subadditivity, once per each variable, we find:

\[
f(x_1, \ldots, x_d) \leq \sum_{w \in \{0,1\}^d} q_1^{w_1} \cdots q_d^{w_d} \cdot f \left( y_1^{(w_1)}, \ldots, y_d^{(w_d)} \right) .
\] (17)

Now, on the right-hand side of (17), each occurrence of \(f\) has \(k\) arguments chosen from the \(t_i\)’s and \(d-k\) chosen from the \(r_i\)’s, is multiplied by the \(q_i\)’s corresponding to the \(t_i\)’s, and is bounded from above by the constant

\[
M = \sup \{ f(y_1, \ldots, y_d) \mid y_i \in [t_i, 2t_i] \forall i \in [1:d] \},
\]

which exists because of Theorem 4.4. Such boundedness is crucial for the proof, and was ensured for free in the case of positive integer variables from [1], but had to be proved for positive real variables.

By dividing both sides of (17) by \(x_1 \cdots x_d\) we get:

\[
\frac{f(x_1, \ldots, x_d)}{x_1 \cdots x_d} \leq \frac{q_1 \cdots q_d}{x_1 \cdots x_d} f(t_1, \ldots, t_d) + M \cdot \sum_{w \in \{0,1\}^d \setminus \{1^d\}} q_1^{w_1} \cdots q_d^{w_d} x_1 \cdots x_d .
\] (18)

where \(1^d\) is the binary word of length \(d\) where all the bits are 1.

By construction, \(\lim_{x_i \to \infty} q_i / x_i = 1/t_i\). Given \(\varepsilon > 0\), choose \(x_1^{(\varepsilon)}, \ldots, x_d^{(\varepsilon)} \in \mathbb{R}_+\) such that, if \(x_i \geq x_i^{(\varepsilon)}\) for each \(i \in [1:d]\), then the following hold:

1. \(\frac{f(x_1, \ldots, x_d)}{x_1 \cdots x_d} < \frac{f(t_1, \ldots, t_d)}{t_1 \cdots t_d} + \frac{\varepsilon}{2^d};\)

2. \(\frac{q_1^{w_1} \cdots q_d^{w_d}}{x_1 \cdots x_d} < \frac{\varepsilon}{M \cdot 2^d}\) for every \(w \in \{0,1\}^d \setminus \{1^d\}\).

This is possible because if \(w \neq 1^d\) then at least one of the \(q_i^{w_i}\) equals 1. For such \(x_1, \ldots, x_d\) it is:

\[
\frac{f(x_1, \ldots, x_d)}{x_1 \cdots x_d} < \frac{f(t_1, \ldots, t_d)}{t_1 \cdots t_d} + \varepsilon .
\]

As \(\varepsilon > 0\) is arbitrary, it must be:

\[
\limsup_{(x_1, \ldots, x_d) \to \mathbb{R}_+^d} \frac{f(x_1, \ldots, x_d)}{x_1 \cdots x_d} \leq \frac{f(t_1, \ldots, t_d)}{t_1 \cdots t_d} .
\]
But the $t_i$’s are also arbitrary, hence:

$$\limsup_{(x_1,\ldots,x_d) \to \mathbb{R}_+^d} \frac{f(x_1,\ldots,x_d)}{x_1 \cdots x_d} \leq \inf_{t_1,\ldots,t_d \in \mathbb{R}_+} \frac{f(t_1,\ldots,t_d)}{t_1 \cdots t_d} \leq \liminf_{(x_1,\ldots,x_d) \to \mathbb{R}_+^d} \frac{f(x_1,\ldots,x_d)}{x_1 \cdots x_d},$$

which completes the proof.

From Theorem 5.1 and Proposition 3.5 follows:

**Theorem 5.2.** Let $d \geq 1$, let $w, w' \in \{0,1\}^d$ and let $f : \mathbb{R}_w \to \mathbb{R}$ be a subadditive function.

1. If $w$ contains evenly many 1s, then:

$$\lim_{(x_1,\ldots,x_d) \to \mathbb{R}_w} \frac{f(x_1,\ldots,x_d)}{x_1 \cdots x_d} = \inf_{(x_1,\ldots,x_d) \in \mathbb{R}_w} \frac{f(x_1,\ldots,x_d)}{x_1 \cdots x_d} \quad (19)$$

is not $+\infty$, but can be $-\infty$.

2. If $w$ contains oddly many 1s, then:

$$\lim_{(x_1,\ldots,x_d) \to \mathbb{R}_w} \frac{f(x_1,\ldots,x_d)}{x_1 \cdots x_d} = \sup_{(x_1,\ldots,x_d) \in \mathbb{R}_w} \frac{f(x_1,\ldots,x_d)}{x_1 \cdots x_d} \quad (20)$$

is not $-\infty$, but can be $+\infty$.

3. If $w$ contains evenly many 1s, $w'$ differs from $w$ in only one coordinate, and $f$ is defined and componentwise subadditive in all of $\mathbb{R}^d$, then

$$\lim_{(x_1,\ldots,x_d) \to \mathbb{R}_w} \frac{f(x_1,\ldots,x_d)}{x_1 \cdots x_d} \leq \lim_{(x_1,\ldots,x_d) \to \mathbb{R}_w} \frac{f(x_1,\ldots,x_d)}{x_1 \cdots x_d} \quad : (21)$$

in particular, both limits are finite.

Note that for $d = 1$ we recover precisely [5, Theorem 6.6.1]. To prove Theorem 5.2 we make use of the following result, whose proof we leave to the reader.

**Lemma 5.3.** Let $S$ be a semigroup and $f : S \to \mathbb{R}$ be a subadditive function. If $S$ is a monoid with identity $e$, then $f(e) \geq 0$. If, in addition, $S$ is a group, then $f(x) + f(x^{-1}) \geq 0$ for every $x \in S$. 

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Proof of Theorem 5.2. For \((x_1, \ldots, x_d) \in \mathcal{R}_w\) and \(i \in [1:d]\) let \(x_{w,i}\) be defined as in Proposition 3.5. If \(w\) contains evenly many 1s, then \(x_1 \cdots x_d = x_{w,1} \cdots x_{w,d}\) and:

\[
\lim_{x \to \mathcal{R}_w} \frac{f(x_1, \ldots, x_d)}{x_1 \cdots x_d} = \lim_{x \to \mathcal{R}_w} \frac{f(x_{w,1}, \ldots, x_{w,d})}{x_{w,1} \cdots x_{w,d}} = \inf_{x_{w} \in \mathbb{R}_+^d} \frac{f(x_{w,1}, \ldots, x_{w,d})}{x_{w,1} \cdots x_{w,d}} = \inf_{x \in \mathcal{R}_w} \frac{f(x_1, \ldots, x_d)}{x_1 \cdots x_d}.
\]

If \(w\) contains oddly many 1s, then \(x_1 \cdots x_d = -x_{w,1} \cdots x_{w,d}\) and:

\[
\lim_{x \to \mathcal{R}_w} \frac{f(x_1, \ldots, x_d)}{x_1 \cdots x_d} = -\lim_{x \to \mathcal{R}_w} \frac{f(x_{w,1}, \ldots, x_{w,d})}{x_{w,1} \cdots x_{w,d}} = -\inf_{x_{w} \in \mathbb{R}_+^d} \frac{f(x_{w,1}, \ldots, x_{w,d})}{x_{w,1} \cdots x_{w,d}} = \sup_{x \in \mathcal{R}_w} \frac{f(x_1, \ldots, x_d)}{x_1 \cdots x_d}.
\]

Suppose now \(w\) has evenly many 1s and \(w'\) differs from \(w\) only in component \(i\), and \(f\) is defined and componentwise subadditive in \(\mathbb{R}^d\). By Lemma 5.3 for every \(x_1, \ldots, x_d \in \mathbb{R},\)

\[
f(x_1, \ldots, x_i, \ldots, x_d) + f(x_1, \ldots, -x_i, \ldots, x_d) \geq 0.
\]

Then:

\[
\lim_{x \to \mathcal{R}_w} \frac{f(x_1, \ldots, x_d)}{x_1 \cdots x_d} - \lim_{x' \to \mathcal{R}_{w'}} \frac{f(x',1, \ldots, x'_d)}{x'_1 \cdots x'_d} = \lim_{x \to \mathcal{R}_w} \frac{f(x_1, \ldots, x_i, \ldots, x_d)}{x_1 \cdots x_d} + \lim_{x \to \mathcal{R}_w} \frac{f(x_1, \ldots, -x_i, \ldots, x_d)}{x_1 \cdots x_d} = \lim_{x \to \mathcal{R}_w} \frac{f(x_1, \ldots, x_i, \ldots, x_d) + f(x_1, \ldots, -x_i, \ldots, x_d)}{x_1 \cdots x_d}
\]

is nonnegative. \(\square\)

As every subnet of a convergent net converges to the same limit, we get:
Corollary 5.4. Let $f : \mathbb{R}^d_+ \to \mathbb{R}$ be componentwise subadditive. For every $i \in [1 : d]$ let $\{x_{i,n}\}_{n \geq 1}$ be a divergent sequence of positive real numbers. Then:

$$
\lim_{n \to \infty} \frac{f(x_{1,n}, \ldots, x_{d,n})}{x_{1,n} \cdots x_{d,n}} = \inf_{x_{1,\ldots,x_d} > 0} \frac{f(x_1, \ldots, x_d)}{x_1 \cdots x_d}, \quad (22)
$$

and also

$$
\inf_{n \geq 1} \frac{f(x_{1,n}, \ldots, x_{d,n})}{x_{1,n} \cdots x_{d,n}} = \inf_{x_{1,\ldots,x_d} > 0} \frac{f(x_1, \ldots, x_d)}{x_1 \cdots x_d}. \quad (23)
$$

In particular,

$$
\lim_{n \to \infty} \frac{f(n, \ldots, n)}{n^d} = \inf_{n \geq 1} \frac{f(n, \ldots, n)}{n^d} = \inf_{x_{1,\ldots,x_d} > 0} \frac{f(x_1, \ldots, x_d)}{x_1 \cdots x_d}. \quad (24)
$$

Sketch of proof. We only remark that (23) follows from (16) and:

$$
\inf_{x_{1,\ldots,x_d} > 0} \frac{f(x_1, \ldots, x_d)}{x_1 \cdots x_d} \leq \inf_{n \geq 1} \frac{f(x_{1,n}, \ldots, x_{d,n})}{x_{1,n} \cdots x_{d,n}} \leq \liminf_{n \geq 1} \frac{f(x_{1,n}, \ldots, x_{d,n})}{x_{1,n} \cdots x_{d,n}}.
$$

The exponent $d$ in Corollary 5.4 suggests that componentwise subadditivity of $f$ does not imply subadditivity of $n \mapsto f(n, \ldots, n)$. This is indeed the case, and provides further evidence that Theorems 5.1 and 5.2 are not special cases of [5, Theorem 6.6.1].

Example 5.5. The function $f(x_1, x_2) = x_1 \cdot x_2$ is clearly componentwise subadditive in the first quadrant of the real plane. However, $g(n) = f(n, n) = n^2$ is not subadditive on the positive integers.

A real-valued function defined on a semigroup $(S, \cdot)$ is superadditive if it satisfies $f(x \cdot y) \geq f(x) + f(y)$ for every $x, y \in S$. As $f$ is superadditive if and only if $-f$ is subadditive, an analogue of Theorem 5.1 holds for componentwise superadditive functions. If $f$ is superadditive in some variables and subadditive in other variables, however, Fekete’s lemma does not hold.

Example 5.6. The function $f : \mathbb{R}^2_+ \to \mathbb{R}$ defined by $f(x_1, x_2) = x_1^2 \sqrt{x_2}$ is superadditive in $x$ and subadditive in $y$, and $f(x_1, x_2)/x_1x_2 = x_1/\sqrt{x_2}$. But $\lim_{(x_1, x_2) \to \mathbb{R}^2_+} f(x_1, x_2)/x_1x_2$ does not exist, because for every $y, R > 0$ there exist $x_1, x_2 > R$ such that $x_1/\sqrt{x_2} = y$. 

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As a final remark, the following statement appears in the literature as an extension to arbitrary dimension of [5, Theorem 6.1.1]:

**Proposition 5.7** (cf. [7, Theorem 16.2.9]). Let \( f : \mathbb{R}^d \to \mathbb{R} \) be subadditive in the variable \( x \in \mathbb{R}^d \). Then for every \( x \in \mathbb{R}^d \) the following limit exists:

\[
L_x = \lim_{t \to +\infty} \frac{f(tx)}{t}.
\]

This, however, is not so much an extension than a corollary. If \( f : \mathbb{R}^d \to \mathbb{R} \) satisfies \( f(x + y) \leq f(x) + f(y) \) for every \( x, y \in \mathbb{R}^d \), then obviously \( g_x(t) = f(tx) \) satisfies \( g_x(s + t) \leq g_x(s) + g_x(t) \) for every \( s, t > 0 \); and \( L_x \) is simply the limit of \( g_x(t)/t \) according to [5, Theorem 6.1.1]. On the other hand, Theorem 5.2 is an extension.

6 A comparison with the Ornstein-Weiss lemma

A group \( G \) is amenable if there exist a directed set \( U = (U, \preceq) \) and a net \( \{F_x\}_{x \in U} \) of finite nonempty subsets of \( G \) such that

\[
\lim_{x \to U} \frac{|gF_x \setminus F_x|}{|F_x|} = 0 \quad \forall g \in G.
\]

(25)

A net such as in (25) is called a (left) Følner net, from the Danish mathematician Erling Følner who introduced them in [4]. Every abelian group (in particular, \( \mathbb{R}^d \)) is amenable: for a proof, see [2, Chapter 4].

**Proposition 6.1** (Ornstein-Weiss lemma; cf. [9]). Let \( G \) be an amenable group, let \( \mathcal{P} \mathcal{F}(G) \) be the set of the finite subsets of \( G \), and let \( f : \mathcal{P} \mathcal{F}(G) \to \mathbb{R} \) be subadditive with respect to union (that is, \( f(A \cup B) \leq f(A) + f(B) \) for every \( A, B \in \mathcal{P} \mathcal{F}(G) \)) and satisfy \( f(A) = f(gA) \) for every \( A \in \mathcal{P} \mathcal{F}(G) \) and \( g \in G \). Then for every directed set \( U = (U, \preceq) \) and every left Følner net \( \mathcal{F} = \{F_x\}_{x \in U} \) in \( G \),

\[
L = \lim_{x \to U} \frac{f(F_x)}{|F_x|}
\]

(26)

exists, and does not depend on the choice of \( U \) and \( \mathcal{F} \).

The Ornstein-Weiss lemma says that on amenable groups a notion of asymptotic average is well defined. A detailed proof of Proposition 6.1 is given by F. Krieger in [6].

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Although the sets $F_n = [1:n]^d$ form a Følner sequence for $\mathbb{Z}^d$, the Ornstein-Weiss lemma is not a generalization of Fekete’s lemma! Even with $d = 1$, if $f: \mathbb{Z} \to \mathbb{R}$ is subadditive, the “natural” conversion

$$g(A) = \begin{cases} f(|A|) & \text{if } A \neq \emptyset, \\ 0 & \text{otherwise,} \end{cases}$$

(27)

where $|A|$ is the number of elements of $A$, is not, in general, subadditive on $\mathcal{P}\mathcal{F}(\mathbb{Z})$, for at least two reasons. First, $|A \cup B|$ needs not equal $|A| + |B|$. Next, while invariance by translations is essential in the Ornstein-Weiss lemma, a translate of a subadditive function needs not be subadditive.

**Example 6.2.** The function $f(n) = n \mod 2$ is subadditive on $\mathbb{Z}$, because if the left-hand side of (1) is 0, then the right-hand side is either 0 or 2, and if the former is 1, then the latter is 1 too. However, (27) is not subadditive on $\mathcal{P}\mathcal{F}(\mathbb{Z})$, because if $U = \{1,2\}$ and $V = \{2,3\}$, then $g(U \cup V) = 1$ and $g(U) = g(V) = 0$. Observe that $h(n) = f(n + 1)$ is not subadditive on $\mathbb{Z}$, because $h(1) = 0$ but $h(2) = 1$.

The situation is even worse with $d \geq 2$, because in the Ornstein-Weiss lemma the subadditive function $f$ needs to be defined on every finite subset of $\mathbb{Z}^d$: not only those which are products of slices. In other words, the $d$ integer variables cannot be considered independent, which is the idea behind componentwise subadditivity.

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