Some Symmetric Identities for Degenerate Carlitz-type \((p, q)\)-Euler Numbers and Polynomials

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Abstract: In this paper we define the degenerate Carlitz-type \((p, q)\)-Euler polynomials by generalizing the degenerate Euler numbers and polynomials, degenerate Carlitz-type \(q\)-Euler numbers and polynomials. We also give some theorems and exact formulas, which have a connection to degenerate Carlitz-type \((p, q)\)-Euler numbers and polynomials.

Keywords: degenerate Euler numbers and polynomials; degenerate \(q\)-Euler numbers and polynomials; degenerate Carlitz-type \((p, q)\)-Euler numbers and polynomials

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1. Introduction

Many researchers have studied about the degenerate Bernoulli numbers and polynomials, degenerate Euler numbers and polynomials, degenerate Genocchi numbers and polynomials, degenerate tangent numbers and polynomials (see [1–7]). Recently, some generalizations of the Bernoulli numbers and polynomials, Euler numbers and polynomials, Genocchi numbers and polynomials, tangent numbers and polynomials are provided (see [6,8–13]). In this paper we define the degenerate Carlitz-type \((p, q)\)-Euler polynomials and numbers and study some theories of the degenerate Carlitz-type \((p, q)\)-Euler numbers and polynomials.

Throughout this paper, we use the notations below: \(\mathbb{N}\) denotes the set of natural numbers, \(\mathbb{Z}\) denotes the set of integers, \(\mathbb{Z}_+ = \mathbb{N} \cup \{0\}\) denotes the set of nonnegative integers. We remind that the classical degenerate Euler numbers \(E_n(\lambda)\) and Euler polynomials \(E_n(x, \lambda)\), which are defined by generating functions like (1) and (2) (see [1,2])

\[
\frac{2}{(1+\lambda t)^{\frac{1}{\lambda}}+1} = \sum_{n=0}^{\infty} \frac{E_n(\lambda)}{n!} t^n, \\
(1)
\]

and

\[
\frac{2}{(1+\lambda t)^{\frac{1}{\lambda}}+1} (1+\lambda t)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty} \frac{E_n(x, \lambda)}{n!} t^n, \\
(2)
\]

respectively.

Carlitz [1] introduced some theories of the degenerate Euler numbers and polynomials. We recall that well-known Stirling numbers of the first kind \(S_1(n, k)\) and the second kind \(S_2(n, k)\) are defined by this (see [2,7,14])

\[
(x)_n = \sum_{k=0}^{n} S_1(n, k)x^k \text{ and } x^n = \sum_{k=0}^{n} S_2(n, k)(x)_k,
\]
respectively. Here \((x)_n = x(x - 1) \cdots (x - n + 1)\). The numbers \(S_2(n, m)\) is like this
\[
\sum_{n=m}^{\infty} S_2(n, m) \frac{t^n}{n!} = \frac{(e^t - 1)^m}{m!}.
\]
We also have
\[
\sum_{n=m}^{\infty} S_1(n, m) \frac{t^n}{n!} = \frac{(\log(1 + t))^m}{m!}.
\]
The generalized falling factorial \((x|\lambda)_n\) with increment \(\lambda\) is defined by
\[
(x|\lambda)_n = \prod_{k=0}^{n-1} (x - \lambda k)
\]
for positive integer \(n\), with \((x|\lambda)_0 = 1\); as we know,
\[
(x|\lambda)_n = \sum_{k=0}^{n} S_1(n, k) \lambda^{n-k} x^k.
\]
\((x|\lambda)_n = \lambda^n x|1\lambda)_n\) for \(\lambda \neq 0\). Clearly \((x|0)_n = x^n\). The binomial theorem for a variable \(x\) is
\[
(1 + \lambda t)^x = \sum_{n=0}^{\infty} (x|\lambda)_n \frac{t^n}{n!}.
\]
The \((p, q)\)-number is defined as
\[
[n]_{p,q} = \frac{p^n - q^n}{p - q} = p^{n-1} + p^{n-2} q + p^{n-3} q^2 + \cdots + p^2 q^{n-3} + pq^{n-2} + q^{n-1}.
\]
We begin by reminding the Carlitz-type \((p, q)\)-Euler numbers and polynomials (see [9–11]).

**Definition 1.** For \(0 < q < p \leq 1\) and \(h \in \mathbb{Z}\), the Carlitz-type \((p, q)\)-Euler polynomials \(E_n(p,q)(x)\) and \((h, p, q)\)-Euler polynomials \(E_n^{(h)}(p,q)(x)\) are defined like this
\[
\sum_{n=0}^{\infty} E_n(p,q)(x) \frac{t^n}{n!} = [2]_q \sum_{m=0}^{\infty} (-1)^m q^m e^{[m+x]_{p,q} t},
\]
\[
\sum_{n=0}^{\infty} E_n^{(h)}(p,q)(x) \frac{t^n}{n!} = [2]_q \sum_{m=0}^{\infty} (-1)^m q^m p^{hm} e^{[m+x]_{p,q} t},
\]
respectively (see [9–11]).

Now we make the degenerate Carlitz-type \((p, q)\)-Euler number \(E_n(p,q)(\lambda)\) and \((p, q)\)-Euler polynomials \(E_n(p,q)(x, \lambda)\). In the next section, we introduce the degenerate Carlitz-type \((p, q)\)-Euler numbers and polynomials. We will study some their properties after introduction.

2. **Degenerate Carlitz-Type \((p, q)\)-Euler Polynomials**

In this section, we define the degenerate Carlitz-type \((p, q)\)-Euler numbers and polynomials and make some of their properties.
Definition 2. For $0 < q < p \leq 1$, the degenerate Carlitz-type $(p, q)$-Euler numbers $E_{n,p,q}(\lambda)$ and polynomials $E_{n,p,q}(x, \lambda)$ are related to the generating functions

$$F_{p,q}(t, \lambda) = \sum_{n=0}^{\infty} E_{n,p,q}(\lambda) \frac{t^n}{n!} = [2]_q \sum_{m=0}^{\infty} (-1)^m q^m (1 + \lambda t) \frac{[m]_{p,q}}{\lambda},$$

and

$$F_{p,q}(t, x, \lambda) = \sum_{n=0}^{\infty} E_{n,p,q}(x, \lambda) \frac{t^n}{n!} = [2]_q \sum_{m=0}^{\infty} (-1)^m q^m (1 + \lambda t) \frac{[m + x]_{p,q}}{\lambda},$$

respectively.

Let $p = 1$ in (4) and (5), we can get the degenerate Carlitz-type $q$-Euler number $E_{n,q}(x, \lambda)$ and $q$-Euler polynomials $E_{n,q}(x, \lambda)$ respectively. Obviously, if $p = 1$, then we have

$$E_{n,p,q}(x, \lambda) = E_{n,q}(x, \lambda), \quad E_{n,p,q}(\lambda) = E_{n,q}(\lambda).$$

When $p = 1$, we have

$$\lim_{q \to 1} E_{n,p,q}(x, \lambda) = E_{n}(x, \lambda), \quad \lim_{q \to 1} E_{n,p,q}(\lambda) = E_{n}(\lambda).$$

We see that

$$(1 + \lambda t) \frac{[x + y]_{p,q}}{\lambda} = e^{\frac{[x + y]_{p,q}}{\lambda} \log(1 + \lambda t)} = \sum_{n=0}^{\infty} \left( \frac{[x + y]_{p,q}}{\lambda} \right)^n \frac{\log(1 + \lambda t)^n}{n!} \sum_{n=0}^{\infty} S_1(n, m) \lambda^{-m} [x + y]_{p,q}^{m} \frac{t^n}{n!}.$$

By (5), it follows that

$$\sum_{n=0}^{\infty} E_{n,p,q}(x, \lambda) \frac{t^n}{n!} = [2]_q \sum_{m=0}^{\infty} (-1)^m q^m (1 + \lambda t) \frac{[m + x]_{p,q}}{\lambda}$$

$$= [2]_q \sum_{m=0}^{\infty} (-1)^m q^m$$

$$= \sum_{n=0}^{\infty} \sum_{l=0}^{n} \sum_{i=0}^{l} S_1(n, l) \lambda^{n-l} \frac{i^j \left( \frac{(-1)^{i} p^{l+m}(l-i) q^{l+m}}{(p-q)^l} \lambda^{l} \right)}{1 + q^{l+1} p^{l-j}} \frac{t^n}{n!}.$$

By comparing the coefficients $\frac{t^n}{n!}$ in the above equation, we have the following theorem.
Theorem 1. For $0 < q < p \leq 1$ and $n \in \mathbb{Z}_+$, we have

$$
\mathcal{E}_{n,p,q}(x, \lambda) = [2]_q \sum_{l=0}^{n} \sum_{j=0}^{i} \frac{S_1(n,l)\lambda^{n-l}(1)(-1)^j q^m p^{x(l-j)}}{(p-q)^{j}} \frac{1}{1+q^{l+1}p^{l-j}}
$$

$$
= [2]_q \sum_{m=0}^{\infty} \sum_{l=0}^{n} \frac{S_1(n,l)\lambda^{n-l}(1)m^q m^p [x+m]_p}{1+q^{l+1}p^{l-j}}
$$

We make the degenerate Carlitz-type $(p,q)$-Euler number $\mathcal{E}_{n,p,q}(\lambda)$. Some cases are

$$
\mathcal{E}_{0,p,q}(\lambda) = 1,
$$

$$
\mathcal{E}_{1,p,q}(\lambda) = \frac{[2]_q}{(p-q)(1+pq)} - \frac{[2]_q}{(p-q)(1+q^2)},
$$

$$
\mathcal{E}_{2,p,q}(\lambda) = - \frac{[2]_q\lambda}{(p-q)(1+pq)} - \frac{[2]_q}{(p-q)^2(1+p^2q)} + \frac{[2]_q\lambda}{(p-q)(1+q^2)} - \frac{2[2]_q}{(p-q)^2(1+pq^2)} + \frac{[2]_q\lambda}{(p-q)^3(1+p^3q)} - \frac{2[2]_q\lambda^2}{3[2]_q}\frac{\lambda^2}{(p-q)^2(1+p^2q)} + \frac{3[2]_q\lambda}{(p-q)^3(1+p^3q)} - \frac{6[2]_q\lambda}{3[2]_q}\frac{\lambda^2}{(p-q)^2(1+p^2q)^2} - \frac{3[2]_q\lambda}{3[2]_q}\frac{\lambda^2}{(p-q)^3(1+p^3q)^2} - \frac{2[2]_q\lambda^2}{(p-q)^2(1+q^2)} + \frac{3[2]_q\lambda}{(p-q)^3(1+p^3q)} - \frac{3[2]_q}{(p-q)^2(1+p^2q)^2} + \frac{3[2]_q}{(p-q)^3(1+p^3q)^2}.
$$

We use $t$ instead of $\frac{e^{\lambda t} - 1}{\lambda}$ in (5), we have

$$
\sum_{m=0}^{\infty} E_{m,p,q}(x) \frac{\mu^m}{m!} = \sum_{n=0}^{\infty} \mathcal{E}_{n,p,q}(x, \lambda) \left( \frac{e^{\lambda t} - 1}{\lambda} \right)^n \frac{1}{n!}
$$

$$
= \sum_{n=0}^{\infty} \mathcal{E}_{n,p,q}(x, \lambda) \lambda^{-n} \sum_{m=n}^{\infty} S_2(m,n) \frac{\lambda^m}{m!} \frac{\mu^m}{m!}
$$

$$
= \sum_{m=0}^{\infty} \left( \sum_{n=0}^{m} \mathcal{E}_{n,p,q}(x, \lambda) \lambda^{m-n} S_2(m,n) \right) \frac{\mu^m}{m!}.
$$

Thus we have the following theorem.

Theorem 2. For $m \in \mathbb{Z}_+$, we have

$$
E_{m,p,q}(x) = \sum_{n=0}^{m} \mathcal{E}_{n,p,q}(x, \lambda) \lambda^{m-n} S_2(m,n).
$$
Use $t$ instead of $\log(1 + \lambda t)^{1/\lambda}$ in (3), we have
\[
\sum_{n=0}^{\infty} E_{n,p,q}(x) \left( \log(1 + \lambda t)^{1/\lambda} \right)^n \frac{1}{n!} = \left[ 2 \right]_q \sum_{m=0}^{\infty} \frac{(-1)^m q^m (1 + \lambda t)^{m-x}}{\lambda} = \sum_{m=0}^{\infty} E_{m,p,q}(x,\lambda) \frac{t^m}{m!},
\] (9)
and
\[
\sum_{n=0}^{\infty} E_{n,p,q}(x) \left( \log(1 + \lambda t)^{1/\lambda} \right)^n \frac{1}{n!} = \sum_{m=0}^{\infty} \left( \sum_{n=0}^{m} E_{n,p,q}(x) \lambda^{m-n} S_1(m,n) \right) \frac{t^m}{m!}.
\] (10)
Thus we have the below theorem from (9) and (10).

**Theorem 3.** For $m \in Z_+$, we have
\[
E_{m,p,q}(x,\lambda) = \sum_{n=0}^{m} E_{n,p,q}(x) \lambda^{m-n} S_1(m,n).
\]
We have the degenerate Carlitz-type $(p, q)$-Euler polynomials $E_{n,p,q}(x,\lambda)$. some cases are
\[
E_{0,p,q}(x,\lambda) = 1,
E_{1,p,q}(x,\lambda) = \frac{[2]_q p^x}{(p-q)(1+pq)} - \frac{[2]_q q^x}{(p-q)(1+q^2)},
E_{2,p,q}(x,\lambda) = -\frac{2[2]_q \lambda p^x}{(p-q)(1+pq)} + \frac{[2]_q q^x p^2}{(p-q)^2(1+p^2 q)} + \frac{[2]_q \lambda q^x}{(p-q)(1+q^2)}
- \frac{2[2]_q q^x p^2}{(p-q)^2(1+pq^2)} + \frac{[2]_q q^x}{(p-q)^2(1+q^3)},
E_{3,p,q}(x,\lambda) = \frac{2[2]_q \lambda^2 p^x}{(p-q)(1+pq)} - \frac{3[2]_q \lambda q^x p^2}{(p-q)^2(1+p^2 q)} + \frac{2[2]_q \lambda q^x}{(p-q)^2(1+p^3 q)}
- \frac{6[2]_q q^x p^2}{(p-q)^2(1+pq^2)} + \frac{3[2]_q p^2 q^2}{(p-q)^3(1+p^2 q^2)}
- \frac{2[2]_q \lambda q^x}{(p-q)^2(1+q^2)} + \frac{3[2]_q p^2 q^2}{(p-q)^3(1+pq^2)} - \frac{2[2]_q \lambda q^x}{(p-q)^2(1+q^2)}.
\]
We introduce a $(p, q)$-analogue of the generalized falling factorial $(x|\lambda)_n$ with increment $\lambda$. The generalized $(p, q)$-falling factorial $([x]_{p,q}|\lambda)_n$ with increment $\lambda$ is defined by
\[
([x]_{p,q}|\lambda)_n = \prod_{k=0}^{n-1} ([x]_{p,q} - \lambda k)
\]
for positive integer $n$, where $([x]_{p,q}|\lambda)_0 = 1$. 
By (4) and (5), we get

\[ -[2]_q (-1)^n q^n \sum_{l=0}^{\infty} (-1)^l q^l (1 + \lambda t) \frac{[l + n]_{p,q}}{\lambda} \]

\[ + [2]_q \sum_{l=0}^{\infty} (-1)^l q^l (1 + \lambda t) \frac{[l]_{p,q}}{\lambda} \]

\[ = [2]_q \sum_{l=0}^{n-1} (-1)^l q^l (1 + \lambda t) \frac{[l]_{p,q}}{\lambda}. \]

Hence we have

\[ (-1)^{n+1} q^n \sum_{m=0}^{\infty} \mathcal{E}_{m,p,q}(n, \lambda) \frac{t^m}{m!} + \sum_{m=0}^{\infty} \mathcal{E}_{m,p,q}(\lambda) \frac{t^m}{m!} \]

\[ = \sum_{m=0}^{\infty} \left( [2]_q \sum_{l=0}^{n-1} (-1)^l q^l ([l]_{p,q}|\lambda)_m \right) \frac{t^m}{m!}. \]

By comparing the coefficients of \( \frac{t^m}{m!} \) on both sides of (11), we have the following theorem.

**Theorem 4.** For \( n \in \mathbb{Z}_+ \), we have

\[ \sum_{l=0}^{n-1} (-1)^l q^l ([l]_{p,q}|\lambda)_m = \frac{(-1)^{n+1} q^n \mathcal{E}_{m,p,q}(n, \lambda) + \mathcal{E}_{m,p,q}(\lambda)}{[2]_q}. \]

We get that

\[ \frac{[x + y]_{p,q}}{(1 + \lambda t)} \]

\[ \frac{p^y[x]_{p,q}}{(1 + \lambda t)} \]

\[ = \frac{q^y[y]_{p,q}}{(1 + \lambda t)} \]

\[ = \sum_{m=0}^{\infty} (p^y[x]_{p,q}|\lambda)_m \frac{t^m}{m!} \frac{q^y[y]_{p,q}}{(1 + \lambda t)} \]

\[ = \sum_{m=0}^{\infty} (p^y[x]_{p,q}|\lambda)_m \frac{t^m}{m!} \sum_{l=0}^{\infty} \left( \frac{q^y[y]_{p,q}}{\lambda} \right)^l \frac{\log(1 + \lambda t)^l}{l!} \]

\[ = \sum_{m=0}^{\infty} (p^y[x]_{p,q}|\lambda)_m \frac{t^m}{m!} \sum_{l=0}^{\infty} \left( \frac{q^y[y]_{p,q}}{\lambda} \right)^l \frac{\log(1 + \lambda t)^l}{l!} \sum_{k=0}^{\infty} \lambda^k k! \]

\[ = \sum_{m=0}^{\infty} \left( \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \left( \frac{n}{k} \right) (p^y[x]_{p,q}|\lambda)_n \lambda^{k-1} q^x[y]_{p,q} S_1(k, l) \right) \frac{t^m}{m!}. \]
Theorem 5. For 0 < q < p ≤ 1 and n ∈ Z_+, we have

\[ E_{n,p,q}(x, \lambda) = [2]_q \sum_{m=0}^{\infty} (-1)^m q^m (1 + \lambda t)^{m+x} \frac{[m+x]}{p,q} \]

= \[2]_q \sum_{m=0}^{\infty} (-1)^m q^m \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{n}{k} (p^m[x]_{p,q}|\lambda)_{n-k}\lambda^{k-l}q^x[m]_{p,q}S_1(k,l) \frac{t^n}{n!} \]

By comparing the coefficients of \( \mu_n \) in the above equation, we have the theorem below.

Theorem 5. For 0 < q < p ≤ 1 and n ∈ Z_+, we have

\[ E_{n,p,q}(x, \lambda) = [2]_q \sum_{m=0}^{\infty} (-1)^m q^m (p^m[x]_{p,q}|\lambda)_{n-k}\lambda^{k-l}q^xS_1(k,l) \]

3. Symmetric Properties about Degenerate Carlitz-Type \((p, q)\)-Euler Numbers and Polynomials

In this section, we are going to get the main results of degenerate Carlitz-type \((p, q)\)-Euler numbers and polynomials. We also make some symmetric identities for degenerate Carlitz-type \((p, q)\)-Euler numbers and polynomials. Let \(w_1\) and \(w_2\) be odd positive integers. Remind that \([xy]_{p,q} = [x]_{p^r,q^r}[y]_{p,q}\) for any \(x, y \in \mathbb{C}\).

By using \(w_1 x + \frac{w_1 x}{w_2}\) instead of \(x\) in Definition 2, use \(p\) by \(p^{w_2}\), use \(q\) by \(q^{w_2}\) and use \(\lambda\) by \(\frac{\lambda}{[w_2]_{p,q}}\), respectively, we can get

\[ \sum_{n=0}^{\infty} \left(2\right)^{\frac{w_1}{w_2}} \left[w_2\right]_{p,q}^{\frac{w_2}{w_2}} \sum_{I=0}^{\infty} (-1)^l q^{w_2 l} E_{n,p^{w_2},q^{w_2}} \left(w_1 x + \frac{w_1 x}{w_2}, \frac{\lambda}{[w_2]_{p,q}}\right) \frac{t^n}{n!} \]

= \[2\]_q \sum_{I=0}^{\infty} (-1)^l q^{w_2 l} \sum_{n=0}^{\infty} E_{n,p^{w_2},q^{w_2}} \left(w_1 x + \frac{w_1 x}{w_2}, \frac{\lambda}{[w_2]_{p,q}}\right) \frac{[w_2]_{p,q}}{n!} \]

= \[2\]_q \sum_{I=0}^{\infty} (-1)^l q^{w_2 l} \sum_{n=0}^{\infty} (1-n) q^{w_2 n} \boxed{[w_1 x + \frac{w_1 x}{w_2} + n]_{p^{w_2},q^{w_2}}} \]

\times \left(1 + \lambda t\right) \frac{\lambda}{[w_2]_{p,q}} \frac{t^n}{n!}.
Thus, we have the following theorem from (13) and (14).

Since for any non-negative integer \( n \) and odd positive integer \( w_1 \), there is the unique non-negative integer \( r \) such that \( n = w_1 r + j \) with \( 0 \leq j \leq w_1 - 1 \). So this can be written as

\[
[2]_{q^{w_1}} [2]_{q^{w_2}} \sum_{i=0}^{w_3-1} (-1)^i q^{w_1 i} \sum_{n=0}^{\infty} (-1)^n q^{aw_2 n} \times (1 + \lambda t) \frac{[w_1 w_2 x + w_1 i + n w_2]_{p, q}}{\lambda}.
\]

\[
= [2]_{q^{w_1}} [2]_{q^{w_2}} \sum_{i=0}^{w_3-1} (-1)^i q^{w_1 i} \sum_{j=0}^{w_2 - 1} \sum_{r=0}^{\infty} (-1)^{w_1 r + j} q^{w_2 (w_1 r + j)} \times (1 + \lambda t) \frac{[w_1 w_2 x + w_1 i + (w_1 r + j) w_2]_{p, q}}{\lambda}.
\]

\[
= [2]_{q^{w_1}} [2]_{q^{w_2}} \sum_{i=0}^{w_3-1} \sum_{j=0}^{w_2 - 1} \sum_{r=0}^{\infty} (-1)^i (-1)^r (-1)^{w_1 i} q^{w_2 r} q^{w_2 j} \times (1 + \lambda t) \frac{[w_1 w_2 x + w_1 i + w_1 w_2 r + w_2 j]_{p, q}}{\lambda}.
\]

We have the below formula using the above formula

\[
\sum_{n=0}^{\infty} \left[2\right]_{q^{w_2}} [w_2]_{p, q}^{w_3-1} \sum_{i=0}^{w_2 - 1} (-1)^i q^{w_1 i} E_{n, p, q}^{w_2, q} \left( w_1 x + \frac{w_1 i}{w_2} \right) \left( \frac{\lambda}{[w_2]_{p, q}} \right) \frac{t^n}{n!}.
\]

\[
= [2]_{q^{w_1}} [2]_{q^{w_2}} \sum_{i=0}^{w_3-1} \sum_{j=0}^{w_2 - 1} \sum_{r=0}^{\infty} (-1)^{i}(-1)^r (-1)^{w_1 i} q^{w_2 r} q^{w_2 j} \times (1 + \lambda t) \frac{[w_1 w_2 x + w_1 i + w_1 w_2 r + w_2 j]_{p, q}}{\lambda}.
\]

(13)

From a similar approach, we can have that

\[
\sum_{n=0}^{\infty} \left[2\right]_{q^{w_2}} [w_1]_{p, q}^{w_3-1} \sum_{i=0}^{w_1 - 1} (-1)^i q^{w_2 i} E_{n, p, q}^{w_1, q} \left( w_2 x + \frac{w_2 i}{w_1} \right) \left( \frac{\lambda}{[w_1]_{p, q}} \right) \frac{t^n}{n!}.
\]

\[
= [2]_{q^{w_1}} [2]_{q^{w_2}} \sum_{i=0}^{w_3-1} \sum_{j=0}^{w_2 - 1} \sum_{r=0}^{\infty} (-1)^i (-1)^r (-1)^{w_2 i} q^{w_1 r} q^{w_1 j} \times (1 + \lambda t) \frac{[w_1 w_2 x + w_2 i + w_1 w_2 r + w_1 j]_{p, q}}{\lambda}.
\]

(14)

Thus, we have the following theorem from (13) and (14).
Theorem 6. Let $w_1$ and $w_2$ be odd positive integers. Then one has

$$
[2]_{q^w}[w_2]^n_{p,q} \sum_{i=0}^{w_1-1} (-1)^i q^{w_1i} E_{n,p^{w_2},q^{w_2}} \left( w_1 x + \frac{w_1 i}{w_2}, \frac{\lambda}{[w_2]_{p,q}} \right) = [2]_{q^w}[w_1]^{n}_{p,q} \sum_{j=0}^{w_1-1} (-1)^j q^{w_1j} E_{n,p^{w_1},q^{w_1}} \left( w_2 x + \frac{w_2 j}{w_1}, \frac{\lambda}{[w_1]_{p,q}} \right).
$$

Letting $\lambda \to 0$ in Theorem 6, we can immediately obtain the symmetric identities for Carlitz-type $(p,q)$-Euler polynomials (see [10])

$$
[2]_{q^w}[w_2]^n_{p,q} \sum_{i=0}^{w_1-1} (-1)^i q^{w_1i} E_{n,p^{w_2},q^{w_2}} \left( w_1 x + \frac{w_1 i}{w_2} \right) = [2]_{q^w}[w_1]^{n}_{p,q} \sum_{j=0}^{w_1-1} (-1)^j q^{w_1j} E_{n,p^{w_1},q^{w_1}} \left( w_2 x + \frac{w_2 j}{w_1} \right).
$$

It follows that we show some special cases of Theorem 6. Let $w_2 = 1$ in Theorem 6, we have the multiplication theorem for the degenerate Carlitz-type $(p,q)$-Euler polynomials.

Corollary 1. Let $w_1$ be odd positive integer. Then

$$
E_{n,p,q}(x,\lambda) = \frac{[2]_{q^w}[w_1]^{n}_{p,q} \sum_{j=0}^{w_1-1} (-1)^j q^j E_{n,p^{w_1},q^{w_1}} \left( x + j, \frac{\lambda}{w_1} \right)}{[2]_{q^w}[w_1]^{n}_{p,q}}.
$$

Let $p = 1$ in (15). This leads to the multiplication theorem about the degenerate Carlitz-type $q$-Euler polynomials

$$
E_{n,q}(x,\lambda) = \frac{[2]_{q^w}[w_1]^{n}_{q} \sum_{j=0}^{w_1-1} (-1)^j q^j E_{n,q^{w_1}} \left( x + j, \frac{\lambda}{w_1} \right)}{[2]_{q^w}[w_1]^{n}_{q}}.
$$

Giving $q \to 1$ in (16) induce to the multiplication theorem about the degenerate Euler polynomials

$$
E_{n}(x,\lambda) = w_1^{n} \sum_{j=0}^{w_1-1} (-1)^j E_{n} \left( x + i, \frac{\lambda}{w_1} \right).
$$

If $\lambda$ approaches to 0 in (17), this leads to the multiplication theorem about the Euler polynomials(see [15])

$$
E_{n}(x) = w_1^{n} \sum_{j=0}^{w_1-1} (-1)^j E_{n} \left( x + i \right).
$$

Let $x = 0$ in Theorem 6, then we have the following corollary.

Corollary 2. Let $w_1$ and $w_2$ be odd positive integers. Then it has

$$
[2]_{q^w}[w_2]^n_{p,q} \sum_{i=0}^{w_1-1} (-1)^i q^{w_1i} E_{n,p^{w_2},q^{w_2}} \left( w_1 i + w_2 j, \frac{\lambda}{[w_2]_{p,q}} \right) = [2]_{q^w}[w_1]^n_{p,q} \sum_{j=0}^{w_1-1} (-1)^j q^{w_1j} E_{n,p^{w_1},q^{w_1}} \left( w_1 i + w_2 j, \frac{\lambda}{[w_1]_{p,q}} \right).
$$

By Theorem 3 and Corollary 2, we have the below theorem.
**Theorem 7.** Let $w_1$ and $w_2$ be odd positive integers. Then

\[
\sum_{l=0}^{n} S_1(n,l) \lambda^{n-l} [w_2]^l_{p,q} \sum_{i=0}^{w_2-1} (-1)^i q^{w_1 i} E_{l,p^{w_2},q^{w_2}} \left( \frac{w_1}{w_2} i \right)
\]

\[
= \sum_{l=0}^{n} S_1(n,l) \lambda^{n-l} [w_1]^l_{p,q} \sum_{i=0}^{w_1-1} (-1)^i q^{w_2 i} E_{l,p^{w_1},q^{w_1}} \left( \frac{w_2}{w_1} i \right).
\]

We get another result by applying the addition theorem about the Carlitz-type $(p, q)$-Euler polynomials $E_{n,p,q}(x)$.

**Theorem 8.** Let $w_1$ and $w_2$ be odd positive integers. Then we have

\[
\sum_{l=0}^{n} \sum_{k=0}^{l} \binom{l}{k} S_1(n,l) \lambda^{n-l} p^{w_1 w_2 x k} [2]^q_{p,q} \sum_{i=0}^{w_2-1} (-1)^i q^{w_1 i} E_{l,p^{w_2},q^{w_2}} (w_1 x) S_{l,k,p^{w_1},q^{w_1}} (w_2)
\]

\[
= \sum_{l=0}^{n} \sum_{k=0}^{l} \binom{l}{k} S_1(n,l) \lambda^{n-l} p^{w_1 w_2 x k} [2]^q_{p,q} \sum_{i=0}^{w_1-1} (-1)^i q^{w_2 i} E_{l,p^{w_1},q^{w_1}} (w_2 x) S_{l,k,p^{w_2},q^{w_2}} (w_1),
\]

where $S_{l,k,p,q}(w_1) = \sum_{i=0}^{w_1-1} (-1)^i q^{(l-k+1)i}/[i]^p_{q}$ is called as the $(p, q)$-sums of powers.

**Proof.** From (3), Theorems 3 and 6, we have

\[
[2]^q_{p,q} \sum_{i=0}^{w_2-1} (-1)^i q^{w_1 i} E_{n,p^{w_2},q^{w_2}} \left( w_1 x + \frac{w_1 i}{w_2} \right) \left( \frac{\lambda}{[w_2]_{p,q}} \right)^{n-l} S_1(n,l)
\]

\[
= [2]^q_{p,q} \sum_{i=0}^{w_1-1} (-1)^i q^{w_1 i} \sum_{l=0}^{n} E_{l,p^{w_2},q^{w_2}} \left( w_1 x + \frac{w_1 i}{w_2} \right) \left( \frac{\lambda}{[w_2]_{p,q}} \right)^{n-l} S_1(n,l)
\]

\[
= [2]^q_{p,q} \sum_{i=0}^{w_1-1} \sum_{k=0}^{l} \binom{l}{k} p^{w_1 w_2 x k} [2]^q_{p,q} \sum_{i=0}^{w_2-1} (-1)^i q^{w_1 (1-k)i} p^{w_1 w_2 x k}
\]

\[
\times E_{l-k,p^{w_2},q^{w_2}} (w_1 x) \left( \frac{[w_3]_{p,q}}{[w_2]_{p,q}} \right)^k \left( [i]^p \right)^{w_1 q^{w_2}}
\]

\[
= [2]^q_{p,q} \sum_{i=0}^{w_1-1} \sum_{k=0}^{l} \binom{l}{k} p^{w_1 w_2 x k} [2]^q_{p,q} \sum_{i=0}^{w_2-1} (-1)^i q^{w_2 (1-k)i} p^{w_1 w_2 x k}
\]

\[
\times E_{l-k,p^{w_2},q^{w_2}} (w_1 x) S_{l,k,p^{w_1},q^{w_1}} (w_2).
\]

Therefore, we induce that

\[
[2]^q_{p,q} \sum_{i=0}^{w_2-1} (-1)^i q^{w_1 i} E_{n,p^{w_2},q^{w_2}} \left( w_1 x + \frac{w_1 i}{w_2} \right) \left( \frac{\lambda}{[w_2]_{p,q}} \right)
\]

\[
= \sum_{l=0}^{n} \sum_{k=0}^{l} \binom{l}{k} S_1(n,l) \lambda^{n-l} p^{w_1 w_2 x k} [2]^q_{p,q} \sum_{i=0}^{w_1-1} (-1)^i q^{w_2 i} E_{l,p^{w_1},q^{w_1}} \left( \frac{w_2}{w_1} i \right)
\]

\[
\times E_{l-k,p^{w_2},q^{w_2}} (w_1 x) S_{l,k,p^{w_1},q^{w_1}} (w_2),
\]
and

$$
\binom{2}{q_2, \lambda} \left[ \binom{w_1}{n} \right]_{p,q} \sum_{j=0}^{w_1-1} (-1)^j q^{w_2 j} E_{n,p,q}^{\lambda j} \left( \frac{w_2 j}{w_1} \right) = \sum_{l=0}^{n} \sum_{k=0}^{l} \binom{l}{k} S_1 (n, l) \lambda^{n-l} p^{p_2 w_2 x k} \binom{2}{q_2, \lambda} \left[ \binom{w_1}{n} \right]_{p,q} \sum_{j=0}^{l-k} E_{l-k, p,q}^{\lambda j} \left( \frac{w_2 j}{w_1} \right) S_{l-k, p,q}^{\lambda j} \left( \frac{w_2 j}{w_1} \right).$$

(19)

By (18) and (19), we make the desired symmetric identity. □

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