The polynomial algorithm for graphs’ isomorphism testing

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Abstract

A polynomial algorithm for graphs’ isomorphism testing will be constructed in assumption that there exists a corresponding polynomial algorithm for graphs with trivial automorphism group.

In this paper we use the term “graph” for any partition of the cartesian square $V^2$ of the set of $n$ vertices $V$. In other words, the graph $G(V)$ denotes a color digraph with color vertices.

It is known [1] that the graphs’ isomorphism problem is equivalent to the problem of determination of orbits of graph’s automorphism group. We will construct the polynomial algorithm for latter problem.

1 Lemma

Let $A, B$ be permutation groups on $V$, and $\text{Orb}(A), \text{Orb}(B)$ be corresponding systems of orbits on $V$ then a partition $Q = \text{Orb}(A) \cup \text{Orb}(B)$ is the system of orbits $\text{Orb}(C)$ of group $C$ generated by subgroups $A, B$ (where the union of partitions means the union of its classes containing non-empty intersection).

Proof

1. As any permutation $a \in A$ keeps partition $\text{Orb}(A)$ and any permutation $b \in B$ keeps partition $\text{Orb}(B)$ then any product of permutations from $A, B$ keeps partition $Q$.

2. Let $U$ be a class of $Q$ and points $x, y \in U$ then there exists a sequence of classes $V_1, W_1, V_2, W_2, V_3, \ldots$ so that $V_i \in \text{Orb}(A), W_i \in \text{Orb}(B)$, any two of nearby classes have non-empty intersection and points $x, y$ are located in the utmost classes. It follows that there exists a product of elements from $A, B$ that transforms $x$ in $y$.

2 Algorithm

Let $A$ be a polynomial algorithm that colors vertices of the graph $G(V)$ in ordered colors and thus determines an ordered partition on the set $V$. Let a graph $AG$ obtained from graph $G$ by action of algorithm $A$ has the same automorphism group as graph $G$. Let also this algorithm recognize graphs with trivial automorphism group so that it colors every vertex of such graph in its unique color.

Now we can construct a polynomial algorithm that finds the orbits of automorphism group of any graph $G(V)$.

1. Obtain graphs $R = AG, R(x_1) = AG(x_1), R(x_1, x_2) = AG(x_1, x_2), \ldots, R(x_1, x_2, \ldots, x_k) = AG(x_1, x_2, \ldots, x_k)$ consecutively, by a consecutive coloring of vertices $x_1, \ldots, x_k$ from the set $V$, until the graph $R(x_1, \ldots, x_k)$ with regular automorphism group from the initial graph $G$ is obtained.
It means that the graph $R(x_1, \ldots, x_k)$ has a non-trivial automorphism group, but for any other vertex $x_{k+1}$ a graph $G(x_1, \ldots, x_{k+1})$ obtained from the graph $R(x_1, \ldots, x_k)$ by coloring a vertex $x_{k+1}$ has the trivial automorphism group.

Of course, if $G$ has trivial automorphism group, the algorithm stops immediately.

2. By testing the graphs $R(x_1, \ldots, x_{k+1}) = AG(x_1, \ldots, x_{k+1})$ on isomorphism, construct the system of orbits of the group $Aut(G(x_1, \ldots, x_k))$.

It is possible to iterate steps 1 and 2 taking every time a new sequence of fix-vertices so as to obtain a new automorphic partition (a system of orbits of some subgroup of $Aut(G)$) on $V$. By union of automorphic partitions we will every time (according to Lemma) obtain a larger and larger automorphic partition until we will have obtained a system of orbits of automorphism group of initial graph $G$.

We can notice that the larger automorphic partition is the more restricted is the possibility for the choice of fix-vertices, because all vertices in the same orbit are identical for such choice.

By finding the isomorphic graphs we find not only automorphic partitions, but also automorphisms. So by constructing the system of orbits of $Aut(G)$ we also obtain a defining system of this group.

3 How to select fix-vertices

At first we can see that it is sufficient to find at most $n - 1$ automorphic partitions, having non-trivial intersection, in order to obtain the system of orbits of $Aut(G)$ as union of these partitions. More precise, if a system of orbits $Q$ of some subgroup of $Aut(G)$ is already found then it is sufficient to discover further at most $|Q| - 1$ corresponding automorphic partitions.

It can be used different strategies for separation of fix-vertices. One of these can be based on a partial order of set $V$ that orders vertices by its number of previous fixations.

The main problem is to verify whether an obtained system of orbits $Q$ is $Orb(Aut(G))$. Let $O_1, O_2 \in Q$. Now we will give the verifying algorithm that verifies whether there exists an automorphism of graph $G$ connecting these two orbits (suborbits). It is sufficient to search a corresponding automorphism for any pair of vertices $o_1 \in O_1$ and $o_2 \in O_2$.

1. By fixation of vertices, obtain the graph $R(x_1, \ldots, x_k) = AG(x_1, \ldots, x_k)$ relative to that the vertices $o_1, o_2$ are equivalent (have the same color), but for any other vertex $x_{k+1}$ the graph $R(x_1, \ldots, x_{k+1}) = AG(x_1, \ldots, x_{k+1})$ distinguishes the vertices $o_1$ and $o_2$.

2. Obtain the graphs $T(o_1) \equiv R(x_1, \ldots, x_k, o_1)$ and $T(o_2) \equiv R(x_1, \ldots, x_k, o_2)$.

If connecting automorphism for $o_1$ and $o_2$ exists then the graphs $T(o_1)$ and $T(o_2)$ are isomorphic and, accordingly to the result of the previous step, have the empty intersection of the same color classes.

3. Using the whole algorithm from beginning, find a system of orbits of automorphism groups of graphs $T(o_1)$ and $T(o_2)$ and examine these graphs on isomorphism.

It is clear that, in order to finish the verifying algorithm, we need to examine some tree of graphs.

It is clear that the number of levels of consecutive pairs of vertices (the same as $o_1, o_2$) of such tree is smaller than $\log_2 n$, because the graphs $T(o_1), T(o_2)$ halve the set $V \setminus \{x_1, \ldots, x_k\}$. And hence the total number of graphs of this tree is smaller than $n$.

This consideration is sufficient for polynomiality of whole algorithm.

There is one more property of the base tree of whole algorithm that ascertains the polynomiality of the described algorithm: the total number of graphs of the base tree, that need to
be investigated, has a non-principal difference for various cases and is by order not greater than \( n^2 \), because

- the greater the number of levels of the tree, the smaller the number of branches of every level (as for complete graph in the limit, that requires algorithm \( A \) for about \( n^2 \) graphs )

and vice versa,

- the greater the number of branches that levels of tree have, the smaller the number of levels of the tree (as for graph with regular automorphism group in the limit, that gives \( n \) graphs for testing of their pairs on isomorphism).

The complexity of algorithm \( A \) is obviously greater than the complexity of graphs isomorphism testing for graphs with trivial automorphism group. Hence the case “more levels” has greater complexity than the case “more branches”. It follows that the whole algorithm has its greatest complexity for complete graph and this complexity is equal \( n^2 |A| \), where \( |A| \) is complexity of algorithm \( A \).

### 4 Conclusion

#### 4.1 Algorithm \( A \)

The sequence of graph stabilization algorithms can be constructed in the way B. Weisfeiler described in [1]. We will denote such algorithms as \( A_k \). The algorithm \( A_k \) stabilizes the graph \( G \) by stabilizing the structure on \( V^k \), generated by this graph. B. Weisfeiler examined the algorithm \( A_2 \) and began to examine the algorithm \( A_3 \).

The given algorithm puts up a question whether there exists a natural \( l \) so that for all \( k \geq l \) algorithm \( A_k \) recognizes the graphs with trivial automorphism group. To author an existence of counter-example for \( l = 3 \) is unknown, i.e. an existence of graph with trivial automorphism group in that any two isomorphic triangles have the same number of spanning isomorphic quadrangles.

#### 4.2 Perspectives

The algorithms \( A_k \) present a combinatorial direction of investigation of graphs’ symmetries. This direction for \( k > 2 \) cannot have a two-place algebraic interpretation. It is obviously more difficult but substantially stronger for considered problem than a generalization of \( A_2 \) on the algorithms acting on \( V^2 \) that leads to conception of distance-regular graph systematically described in [2].

On the way to solve the graphs isomorphism problem Author has discovered the original combinatorial objects, that was, most likely, not investigated earlier. These combinatorial objects present the transformation of principal properties of graph symmetries on the set \( C_k \) of \( k+1 \) classes from \( V^k \) that leads to conception of being \( C_k \) assembled or non-assembled in the subset of \( V^{k+1} \). The simplest example of this representation on \( V^2 \) we obtain in the next matrix notation:

1. Assembled case:

\[
C_2' = \left\{ \begin{bmatrix} 12 \\ 45 \end{bmatrix}, \begin{bmatrix} 23 \\ 56 \end{bmatrix}, \begin{bmatrix} 31 \\ 64 \end{bmatrix} \right\} = \begin{bmatrix} 123 \\ 456 \end{bmatrix}
\]

2. Non-assembled case:

\[
C_2'' = \left\{ \begin{bmatrix} 12 \\ 45 \end{bmatrix}, \begin{bmatrix} 23 \\ 56 \end{bmatrix}, \begin{bmatrix} 34 \\ 61 \end{bmatrix} \right\} \neq \begin{bmatrix} 123 \\ 456 \end{bmatrix}
\]
Both sets $C'$ and $C''$ have the same projection on the set $V$ that consists of subsets

$$\left[\begin{array}{c} 1 \\ 4 \end{array}\right], \left[\begin{array}{c} 2 \\ 5 \end{array}\right], \left[\begin{array}{c} 3 \\ 6 \end{array}\right].$$

The investigation of these objects for greater dimensions shows the way of the problem solution.

### 4.3 Related problems

The existence of the polynomial algorithm of graphs’ isomorphism testing allows to suggest the existence of a full polynomial invariant of a graph and then the existence of a full polynomial invariant of any finite group, because any finite group $G$ has a special for this group graph representation as a partition of the set $G^2$ generated by the left (right) action of group $G$ on the set $G^2$. Indeed, this graph can be simplified to a $d + 1$ color digraph (where one color is empty) with geometrical structure of many-dimensional tore by dimension $d$ being equal to a number of group’s generators. The canonization of group’s generating system can allow to identify a group by full invariant of its graph.

On the other hand, the existence of the polynomial algorithm of graphs’ isomorphism testing shows a possible existence of the full polynomial invariant of a partition of $V^l$. It gives a different approach for group identification as a permutation representation of minimal degree, acting on $V$, and then as automorphism group of its system of orbits on $V^l$ for minimal possible $l$; where for symmetric group $l = 1$, for graph automorphism group $l = 2$, for alternating group $l = n - 1$.

It can be seen that described symmetries are equivalent to symmetry of the set of left (right) cosets of cyclic subgroups of group $G$ generated by the group’s generators. This shows that considered approach is different from the group symmetry investigation through the structure of the set of its normal subgroups.

It should be said further if above considerations are correct then it follows that an independent natural full invariant of an abstract group is needed to exist. An investigation of invariants of described symmetrical objects generated by these groups gives the way for obtaining this full invariant.

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### References

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