The Born rule for probabilities of measurement results is deduced from the set of five assumptions. The assumptions state that: (a) the state vector fully determines the probabilities of all measurement results; (b) between measurements, any quantum system is governed by that part of standard quantum mechanics, which does not refer to measurements; (c) probabilities of measurement results obey the rules of the classical theory of probability; (d) no information transfer is possible without interaction; (e) if two spin-1/2 particles are in the entangled state $\sqrt{\lambda - 1}|\uparrow\uparrow\rangle + \sqrt{\lambda}|\downarrow\downarrow\rangle$, and one of spins is measured by the Stern–Gerlach apparatus, then the state of the other spin after this measurement will be either $|\uparrow\rangle$ or $|\downarrow\rangle$, corresponding to the measurement result. No one of these assumptions can be omitted. The method of the derivation is based on Zurek’s idea of environment-assisted invariance [PRL 90, 120404 (2003)], though taken in rather modified form. Entanglement plays a crucial role in our approach (like in Zurek’s one), so the probabilities are first considered in the case when the measured quantum system is entangled with some environment. Probabilities in pure states appear then as a particular case. The method of the present paper can be applied to both ideal and non-ideal measuring devices, irrespective to their constructions and principles of operation.

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I. INTRODUCTION

The Born rule, stating that a probability is proportional to the squared absolute value of a complex amplitude, seems to stay apart from other fundamental concepts of quantum mechanics, as an independent postulate. On the other hand, no alternatives to the Born rule were invented so far. This suggests that the Born rule may actually be deduced from the rest of the quantum mechanics’ corpus. There are many attempts to derive the Born rule, none of them is generally accepted by now. Born himself simply postulated his law for probabilities [1]. Gleason [2] proved a theorem about measures in a Hilbert space, that can be considered as a justification of the Born rule. Everett, in his famous paper devoted to the “relative-state” (or “many-worlds”) interpretation of quantum mechanics [3], argued that frequencies of events of spins is measured by the Stern–Gerlach apparatus, then the state of the other spin after this measurement will be either $|\uparrow\rangle$ or $|\downarrow\rangle$, corresponding to the measurement result. No one of these concepts of quantum mechanics, as an independent postulate, seems to stay apart from other fundamental concepts of quantum mechanics, as an independent postulate.

Recently, Zurek [8] suggested a phenomenon of “environment-assisted invariance” (i.e. environment-assisted invariance) as a tool to derive the Born rule. The key idea of environment invariance is that if two parts of a quantum system are entangled, then some unitary transformations acting on the first subsystem can be completely undone by applying corresponding “countertransformations” to the second subsystem. Then the state of the first subsystem is “environment-assisted invariance” under such transformations, which means that they do not change probabilities of any measurements’ outcomes. Here starts the way to the Born’s rule. The following discussion [4, 6, 11] had demonstrated a variety of opinions on how exactly can one arrive to the Born’s rule from the environment-assisted invariance, and what additional assumptions one needs for that.

The aim of the present work is to make the arguments of Zurek more stronger. For this purpose, we introduce a minimal set of five assumptions (Assumptions 1–5 of Subsection II C) sufficient for the derivation of the Born rule. They are: three basic statements about quantum mechanics (Assumptions 1–3), the statement that there is no signalling without interaction (Assumption 4), and a proposition about the value of the state vector after the measurement (Assumption 5). No one of these assumptions can be omitted, as will be shown in Section II.

There are two features of the Zurek’s derivation [8] and its versions by other authors [4, 6, 11], that can be considered as its restrictions. The first one is treating the probability $P$ of the measurement result as a function of the state vector $|\Psi\rangle$ related to the system under measurement, and of the eigenvector $|\varphi_n\rangle$ related to the given (n-th) measurement result:

$$P = P(|\Psi\rangle, |\varphi_n\rangle).$$

(1)

There are at least two implicit assumptions in this: (a) that any measuring device is fully characterized (from the point of view of probabilities) by the set of eigenvectors of the corresponding observable; and (b) that if observables of two measuring devices have the same eigenvector $|\varphi_n\rangle$, then probabilities of obtaining the result corresponding to $|\varphi_n\rangle$ are the same for both devices. (These assumptions are also necessary for obtaining the Born rule from the Gleason’s theorem.) These assumptions can be easily justified after the Born rule has already been established, but it is not clear how to justify them without using the
Born rule. Herbut [11] avoids this problem by considering an ideal measurement of a special type. However our goal is to develop an approach suitable for any measuring device. For this reason, we adhere to a more “low-level” view in the present paper—namely, we consider the probability $\mathcal{P}$ as a function of the state vector $|\Psi\rangle$, the measuring apparatus $\mathcal{A}$, and the number $n$ of the measurement result, without any references to observables as operators:

$$\mathcal{P} = \mathcal{P}_n(|\Psi\rangle, \mathcal{A}). \quad (2)$$

One more advantage of such “low-level” point of view is the possibility of considering a broader class of measuring devices: not only “ideal” devices described by operators, but also ones described by POVMs (positive operator-valued measures) [12].

The second problem is that Zurek’s analysis is restricted by only those states in which the measured system is entangled with some other quantum system (an “environment”), and the Schmidt decomposition of the state vector contains only eigenvectors of the measured observable. Pure states, as well as entangled states of general type, stay beyond the consideration. More complete analysis was performed by Herbut [9], but he was forced to make an additional assumption (see the “sixth stipulation” in Ref. [9]). We show that it is possible to proceed without such special assumption.

We also got rid of the assumption of continuity of the probability, as a function of the state vector. The job of such assumption is made by Lemma 1 in the present paper.

Our approach is quite different in its form from Zurek’s one, but the crucial role of entanglement in derivation of the Born rule is conserved. Namely, the most part of our paper is devoted to entangled states; pure states appear only as a particular case.

For the sake of simplicity, we concentrate on measurements of the spin degree of freedom of a spin-1/2 particle. This is not a restriction of our method—we show in Subsection III C how to generalize the obtained results to other systems.

The paper is organized as follows. Section II contains a preliminary information: statement of the problem, description of the Stern–Gerlach apparatus that serves as a “reference” measuring device (Subsection II A), the language of quantum circuit diagrams (Subsection II B), and the list of assumptions (Subsection II C). Section III contains the derivation of the Born rule: a proof of linearity of functional dependence of the probability on the spin polarization vector (Subsection III A—this is the main part of the paper), a proof of the Born rule for measuring the spin projection (Subsection III B), and for measuring the particle coordinate (Subsection III C). In Section IV the question of necessity of each assumption is examined. Closing remarks are gathered in Section V.

II. STATEMENT OF THE PROBLEM

Let us consider probabilities $\mathcal{P}_\uparrow(S, \mathcal{A})$ and $\mathcal{P}_\downarrow(S, \mathcal{A})$ of the outcomes $+1/2$ and $−1/2$ when the vertical projection of the spin is measured by means of some apparatus $\mathcal{A}$ for some spin-1/2 particle in a quantum state $S$. There are two questions:

Question 1. Do these probabilities really depend on the apparatus $\mathcal{A}$, on its principle of operation, construction, etc.?

Question 2. What is the mathematical expression for the functions $\mathcal{P}_\uparrow(S, \mathcal{A})$, $\mathcal{P}_\downarrow(S, \mathcal{A})$ with a fixed $\mathcal{A}$?

As we have no a priori answer to the first question, we will proceed with the following strategy. First, we select some “reference” measurement device $\mathcal{A}_0$ for which it is possible to postulate an additional property (see Assumption 5 below), due to simplicity of the device. We choose the “Stern–Gerlach apparatus” described in Subsection II A as a “reference” device. Then, in Subsection III A we will consider thought experiments with a system of several entangled spins, when one spin is measured by the “reference” apparatus $\mathcal{A}_0$, and another spin is measured by some arbitrary apparatus $\mathcal{A}$. Analysis of these experiments gives the opportunity to use the knowledge about the apparatus $\mathcal{A}_0$ for making conclusions about the apparatus $\mathcal{A}$. On this way we will manage to answer Questions 1 and 2.

In this section, we describe the “Stern–Gerlach apparatus” for measuring the vertical component of the spin (Subsection II A), introduce notations for probabilities of different results of experiments (Subsection II B), make necessary Assumptions and consider some consequences of them (Subsection II C). The detailed discussion of the set of Assumptions will be presented later, in Section IV.

A. Stern–Gerlach apparatus

When a beam of spin-1/2 atoms (initially unpolarized) passes through the area of inhomogeneous magnetic field, it splits into two beams, each consists of spin-polarized atoms. This is the famous Stern–Gerlach experiment. One can use this effect also to measure the spin of a single atom, just by throwing it into the field and catching it at two possible exit trajectories (see Fig. 1). The gradient of the magnetic field is chosen here to be parallel to the vertical axis, therefore the apparatus is able to distinguish spin-up $|\uparrow\rangle$ and spin-down $|\downarrow\rangle$ states. We place two ideal detectors of atoms in the output of the apparatus, one of them catches everything that goes on the upper trajectory, another catches everything on the lower trajectory. When a detector catches an atom, it immediately reports this event. Later, this construction will be called the “Stern–Gerlach apparatus”.

After passing the area of inhomogeneous magnetic field, the spin-up and spin-down parts of the atom’s wavefunction are fully separated in space. This property of the Stern–Gerlach apparatus makes it a proper candidate
to the role of the “reference” measuring device. One can rely on this separation for justifying Assumption 5.

An obvious consequence of the spatial separation mentioned above is that, if the atom’s spin is up, then no part of its wavefunction reaches the lower detector (assuming that the gradient of the z-component of the field is directed downwards). In other words, if the atom is in the pure spin-up state $|\uparrow\rangle$, it will for certain be caught by the upper detector. Analogously, the atom in the pure spin-down state $|\downarrow\rangle$ will be caught by the lower detector with the probability 1. Thus, we will say that the measurement outcome is “spin up” if the atom is caught by the upper detector; otherwise (if the atom is caught by the lower detector) the outcome is “spin down”.

### B. Notations for probabilities

We will make extensive use of thought experiments consisting in unitary transformations and measurements on systems of several spins prepared initially in some states. The natural way for describing such experiments is the language of quantum circuit diagrams. This is a simple example of such diagram:

$$|\uparrow\rangle \xrightarrow{U} \begin{array}{c} \text{upper detector} \end{array},$$

which means that the spin was prepared in the state $|\uparrow\rangle$, then the unitary operator $U$ was applied to the spin, and finally the measurement was performed under this spin. We reserve the sign $\begin{array}{c} \text{upper detector} \end{array}$ for the measurement by the Stern–Gerlach apparatus; other measurements will be denoted as $\begin{array}{c} \text{lower detector} \end{array}$.

To denote probabilities of measurement outcomes we will put the diagrams into square brackets, and indicate the particular outcome inside the measurement sign. For example, the following record:

$$\begin{bmatrix} |\uparrow\rangle \xrightarrow{U} \end{bmatrix} \begin{array}{c} \uparrow \end{array},$$

means the probability of the “spin-up” outcome when the spin, after preparing in the state $|\uparrow\rangle$ and applying the operator $U$, was measured by the Stern–Gerlach apparatus.

The probability of the “spin-down” outcome in the same experiment is denoted as

$$\begin{bmatrix} \downarrow \end{bmatrix}.$$

An analogous notation holds for the case of more than one spin. For example, this record:

$$\begin{bmatrix} |\Psi\rangle \xrightarrow{\downarrow} \begin{array}{c} \uparrow \end{array} \end{bmatrix}$$

denotes the probability that, for the system of two spins prepared in the state $|\Psi\rangle$, measurement of the first spin gives the result “spin up” and successive measurement of the second spin gives the result “spin down”.

### C. Assumptions

We start from three very basic assumptions.

- **Assumption 1.** A state vector of a quantum system provides the full information about probabilities of all outcomes of measurements on this system and on its subsystems.
- **Assumption 2.** Between measurements, a quantum system obeys the rules of the non-measurement part of the quantum mechanics.
- **Assumption 3.** Probabilities of measurement results obey the rules of usual classical theory of probability.

Some consequences of these assumptions can be expressed in the language of quantum circuit diagrams introduced in the previous Subsection. First, the very statement that these notations for probabilities have any sense is a consequence of Assumption 1. Then, this Assumption guarantees that the probability has not been changed when an additional system is added into consideration, as in the following example:

$$\begin{bmatrix} |\psi\rangle \xrightarrow{\uparrow} \begin{array}{c} \uparrow \end{array} \end{bmatrix} = \begin{bmatrix} |\varphi\rangle \xrightarrow{\uparrow} \begin{array}{c} \uparrow \end{array} \end{bmatrix},$$

where $|\psi\rangle$ and $|\varphi\rangle$ are arbitrary state vectors.

The second Assumption tells, in particular, that any quantum gates are described by unitary (norm-conserving) operators; that independent action of different quantum gates on different subsystems is expressed by the tensor product of corresponding operators.

The third Assumption allows for doing some arithmetics with probabilities. First, the “spin-up” and “spin down” outcomes form an exhaustive and mutually exclusive set of events, so the sum of their probabilities is unity:

$$\begin{bmatrix} |\psi\rangle \xrightarrow{\uparrow} \begin{array}{c} \uparrow \end{array} \end{bmatrix} + \begin{bmatrix} |\psi\rangle \xrightarrow{\downarrow} \begin{array}{c} \downarrow \end{array} \end{bmatrix} = 1.$$
Let us consider a system of two spins prepared in some state $|\Psi\rangle$. If the state of the first spin, after the measurement the second spin with the result $a$, is known to be $|\varphi_a\rangle$, then

$$\left[|\Psi\rangle\right]_a = \left[|\Psi\rangle\right]_a \cdot \left[|\varphi_a\rangle\right]_b.$$  

(5)

Here the left hand side is the probability that the measurement of the 2nd spin gives the result $a$ and the subsequent measurement of the 1st spin gives the result $b$. The first factor of the right hand side is the probability of the result $a$ when only the 2nd spin is measured; the second factor plays the role of the conditional probability of the result $b$ of the 1st spin’s measurement provided that the measurement of the 2nd spin gave the result $a$. Therefore Eq. (5) is simply an expression of the multiplicative rule for classical probabilities of successive events. (In this example, as well as in some examples below, the first spin is measured by some arbitrary device, and the second spin is measured by the Stern–Gerlach device.)

Another property of classical probability is causality: a probability of some event cannot depend on what happens after the event, for example:

$$\left[|\Psi\rangle\right]_a = \left[|\Psi\rangle\right]_a \cdot \left[|\varphi_a\rangle\right]_U,$$  

(6)

where $U$ is any operation.

Does Eq. (6) remain valid if the operation $U$ is performed before the measurement? Common sense says yes, because there is no interaction between the two spins, so any manipulations with the second spin cannot affect the probabilities of events that happens with the first spin. Let us formulate this statement in a more general form:

- **Assumption 4.** No information transfer is possible between non-interacting systems.

If some manipulations with one system could cause change of probabilities associated with another system, then one could use them for signalling without interaction. So, “no information transfer” implies also that any action on one system cannot change probabilities of other system’s measurement results. For example,

$$\left[|\Psi\rangle\right]_a = \left[|\Psi\rangle\right]_a \cdot \left[|\varphi_a\rangle\right]_U.$$  

(7)

Also the measurement of one system does not change probabilities for another system, if the measurement result is unknown:

$$\left[|\Psi\rangle\right]_a = \left[|\Psi\rangle\right]_a \cdot \left[|\varphi_a\rangle\right]_a.$$  

(8)

According to the additive rule of the probability theory, the diagram in the right hand side can be expanded as

$$\left[|\Psi\rangle\right]_a = \left[|\Psi\rangle\right]_a \cdot \left[|\varphi_a\rangle\right]_a.$$  

(9)

Now let us postulate one property of the Stern–Gerlach apparatus. When the atom is eaten by a detector of the Stern–Gerlach apparatus, its state apparently cannot be further traced. But, if a system of several spins is under consideration, it is important to know the state of other spins after measurement of one spin. Let us consider the simplest case of two spins in the state

$$|S_\lambda\rangle = \sqrt{1-\lambda} |\uparrow\rangle + \sqrt{\lambda} |\downarrow\rangle,$$  

(10)

where $\lambda \in [0,1]$. It is natural to suppose that, if one spin is found to be up, the other will be also up; otherwise (one spin down), the other spin will be down. So we accept the following statement:

- **Assumption 5.** If a system of two spin-1/2 particles was in the state $\sqrt{1-\lambda} |\uparrow\rangle + \sqrt{\lambda} |\downarrow\rangle$, where $0 \leq \lambda \leq 1$, and one of the spins is measured by the Stern–Gerlach apparatus, then the state of the other spin after the measurement will be $|\uparrow\rangle$ in the case of the “spin-up” measurement result, and $|\downarrow\rangle$ in the case of the “spin-down” result.

Though Assumption 5 looks like collapse, it also makes sense in no-collapse interpretations of quantum mechanics. In the Everettian interpretation, states $|\uparrow\rangle$ and $|\downarrow\rangle$ of the remaining spin after the measurement have the meaning of states relative to the measurement results $\uparrow$ and $\downarrow$.

Using Eq. (10), one can express Assumption 5 in the language of quantum circuit diagrams:

$$\left[|S_\lambda\rangle\right]_\uparrow = \left[|S_\lambda\rangle\right]_\uparrow \cdot |\uparrow\rangle,$$  

(11)

$$\left[|S_\lambda\rangle\right]_\downarrow = \left[|S_\lambda\rangle\right]_\downarrow \cdot |\downarrow\rangle,$$  

(12)

where $-\langle a |$ can be any measurement, or even a combination of unitary transformations and measurements. Then, substituting Eqs. (9), (11), (12) into (8), one can get the following relation:

$$\left[|S_\lambda\rangle\right]_\uparrow = \left[|S_\lambda\rangle\right]_\uparrow \cdot |\uparrow\rangle,$$  

$$\left[|S_\lambda\rangle\right]_\downarrow = \left[|S_\lambda\rangle\right]_\downarrow \cdot |\downarrow\rangle.$$  

(13)

This equation will be used in Subsection III A.

The set of five Assumptions listed here provides the minimal background for derivation of the Born rule.
III. PROOF OF THE BORN RULE

A. Probability and spin polarization

We will consider an arbitrary measuring device, treating it as a black box which accepts a spin-1/2 particle on the input, and gives some classical information on the output. It is supposed that the measuring device interacts only with the spin degrees of freedom of the particle under measurement. For simplicity, we consider only one bit of information on the output—namely, we will say that the device either “clicks” of “not clicks” when a particle comes into it. This simplification does not lead to any loss of generality when we are interested in probabilities of measurement outcomes. Indeed, one can select some particular outcome and consider the appearance of this outcome as “clicking”.

Let \( \mathcal{P}_{\text{click}}(\ket{\Psi}) \) be the probability of “clicking” the detector when it interacts with the spin:

\[
\mathcal{P}_{\text{click}}(\ket{\Psi}) = \begin{bmatrix} \Psi \end{bmatrix}^\dagger \begin{bmatrix} -\text{click} \end{bmatrix},
\]

where \( \ket{\Psi} \) is some state vector of the composite system “spin + environment”. An environment is any quantum system separated from the measured particle, or even the whole Universe except the particle and the measuring device. The single wire (—) in this diagram refers to the spin, and the double wire (---) refers to the environment.

Any state vector of this composite system can be represented in the form of Schmidt decomposition:

\[
\ket{\Psi} = c_1 |a_1\rangle |b_1\rangle + c_2 |a_2\rangle |b_2\rangle,
\]

where \( c_1 \) and \( c_2 \) are non-negative real numbers; \( |a_1\rangle \) and \( |a_2\rangle \) are two mutually orthogonal unit vectors of the spin; \( |b_1\rangle \) and \( |b_2\rangle \) are any two mutually orthogonal unit vectors of the environment:

\[
c_1 \geq 0, \quad c_2 \geq 0, \quad \langle a_k|a_l\rangle = \delta_{kl}.
\]

Let two vectors \( \ket{\Psi'} \) and \( \ket{\Psi''} \) have Schmidt decompositions with the same \( c_1, c_2, |a_1\rangle \) and \( |a_2\rangle \):

\[
\ket{\Psi'} = c_1 |a_1\rangle |b'_1\rangle + c_2 |a_2\rangle |b'_2\rangle,
\]

\[
\ket{\Psi''} = c_1 |a_1\rangle |b''_1\rangle + c_2 |a_2\rangle |b''_2\rangle.
\]

Then, there exists a unitary transformation \( \mathbf{U} \) acting on the environment degrees of freedom, which maps the set \( \{ |b'_1\rangle, |b'_2\rangle \} \) of unit vectors into the set \( \{ |b''_1\rangle, |b''_2\rangle \} \):

\[
\mathbf{U} |b'_1\rangle = |b''_1\rangle, \quad \mathbf{U} |b'_2\rangle = |b''_2\rangle.
\]

It follows from Eqs. (17) - (19) that

\[
\begin{bmatrix} \Psi' \end{bmatrix} = \begin{bmatrix} \Psi'' \end{bmatrix}.
\]

Using this transformation, it is easy to prove that

\[
\mathcal{P}_{\text{click}}(\ket{\Psi'}) = \mathcal{P}_{\text{click}}(\ket{\Psi''}).
\]

Indeed, let us apply Eq. (7), and then Eq. (20):

\[
\mathcal{P}_{\text{click}}(\ket{\Psi'}) = \begin{bmatrix} \Psi' \end{bmatrix}^\dagger \begin{bmatrix} -\text{click} \end{bmatrix} = \begin{bmatrix} \Psi' \end{bmatrix}^\dagger \begin{bmatrix} \mathbf{U} \end{bmatrix}^\dagger \begin{bmatrix} -\text{click} \end{bmatrix} = \begin{bmatrix} \Psi'' \end{bmatrix}^\dagger \begin{bmatrix} -\text{click} \end{bmatrix} = \mathcal{P}_{\text{click}}(\ket{\Psi''}).
\]

Now we introduce the spin polarization vector \( \mathbf{p} = (p_x, p_y, p_z) \) defined (for the normalized state vector \( \ket{\Psi} \)) as follows:

\[
\mathbf{p}(\ket{\Psi}) = \langle \Psi | \mathbf{\hat{\sigma}} \otimes \mathbf{I}_E | \Psi \rangle,
\]

where \( \mathbf{\hat{\sigma}} = (\hat{\sigma}_x, \hat{\sigma}_y, \hat{\sigma}_z) \) is the vector of Pauli matrices, and \( \mathbf{I}_E \) is the identity matrix in the environment state space. The set of possible values of \( \mathbf{p} \) is the ball \( |\mathbf{p}| \leq 1 \) (the Bloch ball).

One can easily see that states \( \ket{\Psi'} \) and \( \ket{\Psi''} \) defined by Eqs. (17) and (18) have the same spin polarization. The converse statement is also true: if two normalized state vectors \( \ket{\Psi_1} \) and \( \ket{\Psi_2} \) have the same spin polarization, then there exist Schmidt decompositions for these vectors with the same \( c_1, c_2, |a_1\rangle \), and \( |a_2\rangle \). Taking Eq. (21) into account, we arrive to the conclusion that, if two state vectors have equal spin polarization, they have equal probability of “clicking” the detector:

\[
\mathbf{p}(\ket{\Psi_1}) = \mathbf{p}(\ket{\Psi_2}) \implies \mathcal{P}_{\text{click}}(\ket{\Psi_1}) = \mathcal{P}_{\text{click}}(\ket{\Psi_2}).
\]

In other words, \( \mathcal{P}_{\text{click}}(\ket{\Psi}) \) is a function of \( \mathbf{p}(\ket{\Psi}) \). We denote this function as \( F_{\text{click}}(\mathbf{p}) \):

\[
\mathcal{P}_{\text{click}}(\ket{\Psi}) = F_{\text{click}}(\mathbf{p}(\ket{\Psi})).
\]

Then we will formulate and prove three statements about the function \( F_{\text{click}}(\mathbf{p}) \).

**Lemma 1.** For any two vectors \( \mathbf{p}_0 \) and \( \mathbf{p}_1 \) in the Bloch ball \( (|\mathbf{p}_0| \leq 1, |\mathbf{p}_1| \leq 1) \) and any number \( \lambda \in [0, 1] \), the value \( F_{\text{click}}((1 - \lambda)\mathbf{p}_0 + \lambda\mathbf{p}_1) \) lies between \( F_{\text{click}}(\mathbf{p}_0) \) and \( F_{\text{click}}(\mathbf{p}_1) \):

\[
F_{\text{click}}(\mathbf{p}_0) \leq F_{\text{click}}((1 - \lambda)\mathbf{p}_0 + \lambda\mathbf{p}_1) \leq F_{\text{click}}(\mathbf{p}_1)
\]

or

\[
F_{\text{click}}(\mathbf{p}_0) \geq F_{\text{click}}((1 - \lambda)\mathbf{p}_0 + \lambda\mathbf{p}_1) \geq F_{\text{click}}(\mathbf{p}_1).
\]

**Lemma 2.** For any two vectors \( \mathbf{p}_0 \) and \( \mathbf{p}_1 \) in the Bloch ball \( (|\mathbf{p}_0| \leq 1, |\mathbf{p}_1| \leq 1) \)

\[
F_{\text{click}}\left(\frac{\mathbf{p}_0 + \mathbf{p}_1}{2}\right) = \frac{F_{\text{click}}(\mathbf{p}_0) + F_{\text{click}}(\mathbf{p}_1)}{2}.
\]

**Lemma 3.** For any two vectors \( \mathbf{p}_0 \) and \( \mathbf{p}_1 \) in the Bloch ball \( (|\mathbf{p}_0| \leq 1, |\mathbf{p}_1| \leq 1) \) and any number \( \lambda \in [0, 1] \),

\[
F_{\text{click}}((1 - \lambda)\mathbf{p}_0 + \lambda\mathbf{p}_1) = (1 - \lambda)F_{\text{click}}(\mathbf{p}_0) + \lambda F_{\text{click}}(\mathbf{p}_1).
\]
Proof of Lemma 1. It is always possible to find such unit vectors $|\psi_0\rangle$ and $|\psi_1\rangle$ in a state space of two spins that

$$p(|\psi_0\rangle) = p_0, \quad p(|\psi_1\rangle) = p_1,$$

where the second spin plays the role of the environment. According to Eq. (24),

$$P_{\text{click}}(|\psi_0\rangle) = F_{\text{click}}(p_0),$$
$$P_{\text{click}}(|\psi_1\rangle) = F_{\text{click}}(p_1).$$

Now, we add two more spins into the environment, and construct a four-spin state vector $|\Psi_\lambda\rangle$ as follows:

$$|\Psi_\lambda\rangle = \sqrt{1-\lambda} |\psi_0\rangle |\uparrow\uparrow\rangle + \sqrt{\lambda} |\psi_1\rangle |\downarrow\downarrow\rangle.$$  

(28)

Let us calculate the spin polarization for the state $|\Psi_\lambda\rangle$:

$$p(|\Psi_\lambda\rangle) = \langle \Psi_\lambda | \hat{S} \otimes I_3 | \Psi_\lambda \rangle$$
$$= (1-\lambda) \langle \psi_0 | \hat{S} \otimes I_1 | \psi_0 \rangle + \lambda \langle \psi_1 | \hat{S} \otimes I_1 | \psi_1 \rangle$$
$$= (1-\lambda)p_0 + \lambda p_1$$

($I_1$ and $I_3$ being identity matrices for one spin and three spins). Therefore, according to Eq. (24),

$$P_{\text{click}}(|\Psi_\lambda\rangle) = F_{\text{click}}((1-\lambda)p_0 + \lambda p_1).$$

(29)

On the other hand, it is possible to evaluate the probability $P_{\text{click}}(|\Psi_\lambda\rangle)$ using the fact that (according to Assumption 4) applying the Stern–Gerlach measurement to the 4th spin does not change this probability. To do so, we first note that three-spin state vectors $|\psi_0\rangle |\uparrow\rangle$ and $|\psi_1\rangle |\downarrow\rangle$ are mutually orthogonal unit vectors. Consequently it is possible to map them into the basis vectors $|\uparrow\uparrow\rangle$ and $|\downarrow\downarrow\rangle$ by some unitary transformation $V$:

$$V |\psi_0\rangle |\uparrow\rangle = |\uparrow\uparrow\rangle, \quad V |\psi_1\rangle |\downarrow\rangle = |\downarrow\downarrow\rangle.$$  

(30)

The inverse transformation $V^{-1}$ maps vectors $|\uparrow\uparrow\rangle$ and $|\downarrow\downarrow\rangle$ into $|\psi_0\rangle |\uparrow\rangle$ and $|\psi_1\rangle |\downarrow\rangle$:

$$V^{-1} |\uparrow\uparrow\rangle = |\psi_0\rangle |\uparrow\rangle, \quad V^{-1} |\downarrow\downarrow\rangle = |\psi_1\rangle |\downarrow\rangle.$$  

(31)

Let us apply the operator $V$ to 1st, 2nd and 3rd spins of a system of four spins in the state $|\Psi_\lambda\rangle$:

$$V|\Psi_\lambda\rangle = \sqrt{1-\lambda} (V|\psi_0\rangle |\uparrow\rangle |\uparrow\rangle) + \sqrt{\lambda} (V|\psi_1\rangle |\downarrow\rangle |\downarrow\rangle)$$
$$= \sqrt{1-\lambda} |\uparrow\uparrow\uparrow\rangle + \sqrt{\lambda} |\downarrow\downarrow\rangle$$
$$= |\uparrow\uparrow\rangle \left( \sqrt{1-\lambda} |\uparrow\rangle + \sqrt{\lambda} |\downarrow\rangle \right) = |\uparrow\uparrow\rangle |\uparrow\rangle |\downarrow\rangle.$$  

(32)

where the two-spin vector $|S_\lambda\rangle$ is defined by Eq. (10). The transformation rule (32) can be expressed also in a symbolic form:

$$|\Psi_\lambda\rangle \begin{bmatrix} V \end{bmatrix} = |\uparrow\rangle |\uparrow\rangle |\downarrow\rangle.$$  

(33)

Now everything is ready for evaluation the quantity $P_{\text{click}}(|\Psi_\lambda\rangle)$ (numbers above equality signs denote which rule or equation is used at the given step):

$$P_{\text{click}}(|\Psi_\lambda\rangle) = F_{\text{click}}((1-\lambda)p_0 + \lambda p_1).$$

(34)

Proof of Lemma 2. Repeating all the calculations of the proof of Lemma 1 for $\lambda = 1/2$, one can get as a particular case of Eq. (36) that

$$F_{\text{click}}((1-\lambda)p_0 + \lambda p_1) = a_{1/2} F_{\text{click}}(p_0) + (1-a_{1/2}) F_{\text{click}}(p_1).$$

(37)
Applying Lemma 2 to the vectors $x$ we see that
\[
\text{where } a_{1/2} = \left[\left(\frac{\left\lfloor \frac{1}{2} \right\rfloor}{\sqrt{2}} + \frac{\left\lfloor \frac{1}{2} \right\rfloor}{\sqrt{2}}\right) - \frac{1}{2}\right].
\] (38)

Note that the quantity $a_{1/2}$ is independent of $p_0$ and $p_1$. Therefore, Eq. (37) remains valid (with the same value of $a_{1/2}$) if we swap $p_0$ and $p_1$:
\[
F_{\text{click}}\left(\frac{p_1 + p_0}{2}\right) = a_{1/2} F_{\text{click}}(p_1) + (1 - a_{1/2}) F_{\text{click}}(p_0).
\] (39)

Adding Eqs. (37) and (39), one can obtain
\[
2 F_{\text{click}}\left(\frac{p_0 + p_1}{2}\right) = F_{\text{click}}(p_0) + F_{\text{click}}(p_1),
\]

Q. E. D.

Remark. It follows from Eqs. (37) and (39) that $a_{1/2} = 1/2$, i.e.
\[
\left[\left(\frac{\left\lfloor \frac{1}{2} \right\rfloor}{\sqrt{2}} + \frac{\left\lfloor \frac{1}{2} \right\rfloor}{\sqrt{2}}\right) - \frac{1}{2}\right] = \frac{1}{2}.
\] (40)

Proof of Lemma 3. If $F_{\text{click}}(p_0) = F_{\text{click}}(p_1)$ then, according to Lemma 1,
\[
F_{\text{click}}((1 - \lambda)p_0 + \lambda p_1) = F_{\text{click}}(p_0) = F_{\text{click}}(p_1),
\]

that proves Lemma 3 for this case.

Consider the opposite case, $F_{\text{click}}(p_0) \neq F_{\text{click}}(p_1)$. Let us introduce the notations $p_x$ and $f(x)$:
\[
p_x = (1 - x) p_0 + x p_1,
\] (41)
\[
f(x) = \frac{F_{\text{click}}(p_x) - F_{\text{click}}(p_0)}{F_{\text{click}}(p_1) - F_{\text{click}}(p_0)},
\] (42)

where $x$ is any number in the range $[0, 1]$. By definitions, $f(0) = 0$, $f(1) = 1$.

(43)

Let us consider two numbers $x_1 \in [0, 1]$ and $x_2 \in [0, 1]$. Applying Lemma 2 to the vectors $p_{x_1}$ and $p_{x_2}$, one can see that
\[
f\left(\frac{x_1 + x_2}{2}\right) = \frac{f(x_1) + f(x_2)}{2}.
\] (44)

Then, one can apply Lemma 1 to the vectors $p_0$ and $p_1$ taking $\lambda = x_2$. This gives the inequality
\[
0 \leq f(x_2) \leq 1.
\]

If $x_1 \leq x_2$, one can apply Lemma 1 to vectors $p_0$ and $p_{x_2}$ taking $\lambda = x_1/x_2$, with the following result:
\[
0 \leq x_1 \leq x_2 \leq 1 \Rightarrow 0 \leq f(x_1) \leq f(x_2) \leq 1.
\] (45)

Therefore the function $f(x)$ is monotonically increasing in the range $x \in [0, 1]$.

Let us prove that $f(x) \equiv x$ on the basis of Eqs. (43), (44), (45). First, we apply Eq. (44) with $x_1 = 0$, $x_2 = 1$:
\[
f\left(\frac{1}{2}\right) = \frac{f(0) + f(1)}{2} = \frac{1}{2}.
\]

Then, Eq. (44) can be applied once again with $x_1 = 0$, $x_2 = 1/2$, and with $x_1 = 1/2$, $x_2 = 1$:
\[
f\left(\frac{1}{4}\right) = \frac{f(0) + f(1/2)}{2} = \frac{1}{4},
\]
\[
f\left(\frac{3}{4}\right) = \frac{f(1/2) + f(1)}{2} = \frac{3}{4},
\]

and so on. As a result, we get the equality
\[
f\left(\frac{p}{2^k}\right) = \frac{p}{2^k}
\] (46)

for any $k = 0, 1, 2, \ldots$, and any $p = 0, 1, \ldots, 2^k$.

Now let us consider some value $x \in [0, 1]$ and some natural number $k$, and introduce two quantities
\[
x_- = \frac{\lfloor 2^k x \rfloor}{2^k}, \quad x_+ = \frac{\lceil 2^k x \rceil}{2^k},
\] (47)

where the square brackets denote the integer part. According to Eq. (40),
\[
f(x_-) = x_-, \quad f(x_+) = x_+.
\] (48)

Since
\[
0 \leq x_- \leq x \leq x_+ \leq 1,
\] (49)
and $f(x)$ is a monotonically increasing function, then
\[
f(x_-) \leq f(x) \leq f(x_+),
\] (50)
or, taking Eq. (48) into account,
\[
x_- \leq f(x) \leq x_+.
\] (51)

One can see from Eqs. (49) and (51) that both $x$ and $f(x)$ are bound within the range $[x_-, x_+]$. Therefore
\[
|f(x) - x| \leq x_+ - x_- = 2^{-k}.
\] (52)

As the number $k$ can be arbitrarily large, Eq. (52) actually means that
\[
|f(x) - x| = 0.
\]

So $f(x) = x$ for any $x \in [0, 1)$. This equality is valid also for $x = 1$, due to Eq. (48). Therefore $f(\lambda) = \lambda$, i.e.
\[
\frac{F_{\text{click}}((1 - \lambda)p_0 + \lambda p_1) - F_{\text{click}}(p_0)}{F_{\text{click}}(p_1) - F_{\text{click}}(p_0)} = \lambda.
\]
This can be done in four steps: find that
expression for the probability

(iii) for any \( p = (p_x, p_y, p_z) \) in the tetrahedron \( OABC \) one can find that

\[
F_{\text{click}}(p) = \alpha_x p_x + \alpha_y p_y + \alpha_z p_z + \beta \equiv \alpha \cdot p + \beta \quad (53)
\]

by applying Lemma 3 with parameters

\[
p_0 = \frac{p_z, p_y, 0}{1 - p_z}, \quad p_1 = (0, 0, 1), \quad \lambda = p_z;
\]

(iv) finally, Eq. (53) can be established for an arbitrary point \( p \) in the Bloch ball by drawing a line that passes through the point \( p \) and intersects the tetrahedron \( OABC \), choosing two different points \( q_1 \) and \( q_2 \) lying on this line and belonging the tetrahedron, and applying Lemma 3 with parameters

\[
p_0 = p, \quad p_1 = q_2, \quad \lambda = \frac{|p - q_1|}{|p - q_2|}.
\]

Eq. (53) proves that the function \( F_{\text{click}}(p) \) is linear. Substituting it into Eq. (24), one can get the most general expression for the probability \( P_{\text{click}} \) of the “click” event, as a function of the state vector \(|\Psi\rangle\):

\[
P_{\text{click}}(|\Psi\rangle) = \alpha \cdot p(|\Psi\rangle) + \beta, \quad (54)
\]

where a vector \( \alpha = (\alpha_x, \alpha_y, \alpha_z) \) and a number \( \beta \) are characteristics of the measuring device.

Let us consider so far only pure states of the combined system “spin + environment”. Now we will generalize the results to the case of mixed states. Let the state \( S \) be a statistical mix of pure states \(|\Psi_k\rangle\) taken with probabilities \( P_k \). The polarization vector \( p(S) \) in this state is a weighted average of spin polarizations in the states \(|\Psi_k\rangle\):

\[
p(S) = \sum_k P_k p(|\Psi_k\rangle). \quad (55)
\]

The probability \( P_{\text{click}}(S) \) of the “click” event in the state \( S \) can be found by the law of total probability:

\[
P_{\text{click}}(S) = \sum_k P_k P_{\text{click}}(|\Psi_k\rangle). \quad (56)
\]

Substitution of Eq. (53) into Eq. (56) gives

\[
P_{\text{click}}(S) = \alpha \cdot \left( \sum_k P_k p(|\Psi_k\rangle) \right) + \beta, \quad (57)
\]

that is,

\[
P_{\text{click}}(S) = \alpha \cdot p(S) + \beta. \quad (58)
\]

So the probability \( P_{\text{click}} \) is a linear function of the spin polarization, not only for pure states, but for mixed states as well.

It was mentioned in the beginning of this Subsection, that any outcome of any measuring device (interacting

\[
F_{\text{click}}((1 - \lambda)p_0 + \lambda p_1) = (1 - \lambda)F_{\text{click}}(p_0) + \lambda F_{\text{click}}(p_1),
\]

Q.E.D.

Using Lemma 3, it is easy to prove that the function \( F_{\text{click}}(p) \) is linear. For this, we choose four reference points \( O, A, B, C \) in the Bloch ball (see Fig. 2) corresponding to the following vectors of spin polarization:

\[
p_O = (0, 0, 0), \quad p_A = (1, 0, 0),
\]

\[
p_B = (0, 1, 0), \quad p_C = (0, 0, 1),
\]

and introduce four parameters

\[
\alpha_x = F_{\text{click}}(p_A) - F_{\text{click}}(p_O),
\]

\[
\alpha_y = F_{\text{click}}(p_B) - F_{\text{click}}(p_O),
\]

\[
\alpha_z = F_{\text{click}}(p_C) - F_{\text{click}}(p_O),
\]

\[
\beta = F_{\text{click}}(p_O).
\]

Let us show that the probability \( F_{\text{click}}(p) \) for any \( p \) can be expressed through the parameters \( \alpha_x, \alpha_y, \alpha_z, \beta \). This can be done in four steps:

(i) for any \( p = (p_x, 0, 0) \) in the line segment \( OA \) one can find that

\[
F_{\text{click}}(p) = \alpha_x p_x + \beta
\]

by applying Lemma 3 with parameters

\[
p_0 = (0, 0, 0), \quad p_1 = (1, 0, 0), \quad \lambda = p_x;
\]

(ii) for any \( p = (p_x, p_y, 0) \) in the triangle \( OAB \) one can find that

\[
F_{\text{click}}(p) = \alpha_x p_x + \alpha_y p_y + \beta
\]

by applying Lemma 3 with parameters

\[
p_0 = \frac{(p_x, 0, 0)}{1 - p_y}, \quad p_1 = (0, 1, 0), \quad \lambda = p_y;
\]

(iii) for any \( p = (p_x, p_y, p_z) \) in the tetrahedron \( OABC \) one can find that

\[
F_{\text{click}}(p) = \alpha_x p_x + \alpha_y p_y + \alpha_z p_z + \beta \equiv \alpha \cdot p + \beta \quad (53)
\]

by applying Lemma 3 with parameters

\[
p_0 = \frac{(p_x, p_y, 0)}{1 - p_z}, \quad p_1 = (0, 0, 1), \quad \lambda = p_z;
\]

(iv) finally, Eq. (53) can be established for an arbitrary point \( p \) in the Bloch ball by drawing a line that passes through the point \( p \) and intersects the tetrahedron \( OABC \), choosing two different points \( q_1 \) and \( q_2 \) lying on this line and belonging the tetrahedron, and applying Lemma 3 with parameters

\[
p_0 = p, \quad p_1 = q_2, \quad \lambda = \frac{|p - q_1|}{|p - q_2|}.
\]

Eq. (53) proves that the function \( F_{\text{click}}(p) \) is linear. Substituting it into Eq. (24), one can get the most general expression for the probability \( P_{\text{click}} \) of the “click” event, as a function of the state vector \(|\Psi\rangle\):

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\]

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We considered so far only pure states of the combined system “spin + environment”. Now we will generalize the results to the case of mixed states. Let the state \( S \) be a statistical mix of pure states \(|\Psi_k\rangle\) taken with probabilities \( P_k \). The polarization vector \( p(S) \) in this state is a weighted average of spin polarizations in the states \(|\Psi_k\rangle\):

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\]

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\]

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\]

that is,

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P_{\text{click}}(S) = \alpha \cdot p(S) + \beta. \quad (58)
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So the probability \( P_{\text{click}} \) is a linear function of the spin polarization, not only for pure states, but for mixed states as well.

It was mentioned in the beginning of this Subsection, that any outcome of any measuring device (interacting

\[
F_{\text{click}}((1 - \lambda)p_0 + \lambda p_1) = (1 - \lambda)F_{\text{click}}(p_0) + \lambda F_{\text{click}}(p_1),
\]

Q.E.D.
with the spin) can play the role of the “click”. Therefore the results of this Subsection are valid for any outcome. We summarize them in the following Theorem.

**Theorem 1.** It follows from the five Assumptions listed in Subsection [II.C](#) that, for any measuring device interacting to the spin degree of freedom of a spin-1/2 particle, the probability of any measurement result is a linear function of the spin polarization vector of this particle.

This Theorem is the central result of the present paper. In the next two Subsections, we will use it for proving the Born rule for the spin projection measurement, as well as for the measurement of the particle coordinate.

Theorem 1 can be formulated also in the language of the reduced density matrix \( \hat{\rho} \) of the measured spin, which is connected to the spin polarization \( p \) as follows:

\[
\hat{\rho} = \frac{1 + \hat{\sigma} \cdot \rho}{2} = \frac{1}{2} \left( \begin{array}{c} 1 + p_z \ - p_y \ - i p_y \\ p_x + i p_y \ 1 - p_z \end{array} \right).
\] (59)

Namely, Theorem 1 states that the probability is a linear function of the real and imaginary parts of the density matrix \( \hat{\rho} \). It is easy to find an explicit form for this linear function. One can see that Eq. (59) can be presented in the following equivalent form:

\[
P_{\text{click}}(\mathcal{S}) = \text{Tr} \left( \hat{M}_{\text{click}} \hat{\rho}(\mathcal{S}) \right),
\] (60)

where \( \hat{M}_{\text{click}} \) is a non-negative self-adjoint operator:

\[
\hat{M}_{\text{click}} = \alpha \cdot \hat{\sigma} + \beta \equiv \left( \begin{array}{c} \beta + \alpha_z \ & \alpha_x - i \alpha_y \\ \alpha_x + i \alpha_y \ & \beta - \alpha_z \end{array} \right),
\] (61)

and the density matrix \( \hat{\rho}(\mathcal{S}) \) is given by Eq. (59).

**B. Proof of the Born rule for spin-1/2 projection measurement**

Now we consider measurement of a spin-1/2 particle by some device having two possible outcomes: ↑ and ↓. According to Theorem 1, the probabilities \( P_\uparrow \) and \( P_\downarrow \) of these outcomes are linear functions of the spin polarization \( p \). In particular,

\[
P_\uparrow = \alpha_\uparrow \cdot p + \beta_\uparrow,
\] (62)

where the vector \( \alpha_\uparrow \) and the scalar \( \beta_\uparrow \) are characteristics of the measuring device. Let us denote as \( n \) the unit vector directed along \( \alpha_\uparrow \):

\[
n = \alpha_\uparrow / |\alpha_\uparrow|.
\] (63)

As the polarization vector \( p \) is bound in the Bloch ball \(|p| \leq 1\), then the maximal value \( P_\text{max} \) of the probability \( P_\uparrow \) is reached when the vector \( p \) is directed along \( \alpha_\uparrow \) and has its maximal length \(|p| = 1\), i.e. when \( p = n \). Therefore,

\[
P_\text{max} = \alpha_\uparrow \cdot n + \beta_\uparrow = |\alpha_\uparrow| + \beta_\uparrow.
\] (64)

Analogously, the minimal value \( P_\text{min} \) of the probability \( P_\uparrow \) is reached when \( p = -n \):

\[
P_\text{min} = \alpha_\uparrow \cdot (-n) + \beta_\uparrow = -|\alpha_\uparrow| + \beta_\uparrow.
\] (65)

Since the maximal and minimal values of \( P_\uparrow \) are reached at the maximal spin polarization \(|p| = 1\), then they correspond to some pure states of the spin. Let us denote corresponding unit spinors as \(|\varphi_\uparrow\rangle\) and \(|\varphi_\downarrow\rangle\):

\[
p(|\varphi_\uparrow\rangle) = n, \quad p(|\varphi_\downarrow\rangle) = -n
\] (66)

(see Fig. 3). Remembering the definition (22) of the vector \( p \), one can rewrite Eq. (60) as

\[
\langle \varphi_\uparrow | \hat{\sigma} | \varphi_\uparrow \rangle = n, \quad \langle \varphi_\downarrow | \hat{\sigma} | \varphi_\downarrow \rangle = -n.
\] (67)

Note that \(|\varphi_\uparrow\rangle\) and \(|\varphi_\downarrow\rangle\) are orthogonal to each other:

\[
\langle \varphi_\uparrow | \varphi_\downarrow \rangle = 0,
\] (68)

because they correspond to the opposite spin directions.

Using Eqs. (63), (64), (65), (67), one can express parameters \( \alpha_\uparrow \) and \( \beta_\uparrow \) via \( P_\text{max} \), \( P_\text{min} \), and \(|\varphi_\uparrow\rangle\):

\[
\alpha_\uparrow = \frac{P_\text{max} - P_\text{min}}{2} \langle \varphi_\uparrow | \hat{\sigma} | \varphi_\uparrow \rangle,
\] (69)

\[
\beta_\uparrow = \frac{P_\text{max} + P_\text{min}}{2}.
\] (70)

With these expressions for \( \alpha_\uparrow \) and \( \beta_\uparrow \), Eq. (62) takes the following form:

\[
P_\uparrow = \frac{P_\text{max} - P_\text{min}}{2} \langle \varphi_\uparrow | \hat{\sigma} | \varphi_\uparrow \rangle \cdot p + \frac{P_\text{max} + P_\text{min}}{2}.
\] (71)

Finally, the quantity \( \langle \varphi_\uparrow | \hat{\sigma} | \varphi_\uparrow \rangle \cdot p \) can be expressed via the spin density matrix \( \hat{\rho} \) using Eq. (59):

\[
\langle \varphi_\uparrow | \hat{\sigma} | \varphi_\uparrow \rangle \cdot p = 2 \langle \varphi_\uparrow | \hat{\rho} | \varphi_\uparrow \rangle - 1.
\] (72)
Substituting this into Eq. (71), we get the following result:

\[ \mathcal{P}_\uparrow = (\mathcal{P}_{\uparrow \text{max}} - \mathcal{P}_{\uparrow \text{min}}) \langle \varphi_\uparrow | \hat{P} | \varphi_\uparrow \rangle + \mathcal{P}_{\uparrow \text{min}}. \]  

(73)

Analogous expression can be obtained for the probability \( \mathcal{P}_\downarrow \):

\[ \mathcal{P}_\downarrow = (\mathcal{P}_{\downarrow \text{max}} - \mathcal{P}_{\downarrow \text{min}}) \langle \varphi_\downarrow | \hat{P} | \varphi_\downarrow \rangle + \mathcal{P}_{\downarrow \text{min}}, \]

where

\[ \mathcal{P}_{\downarrow \text{max}} = 1 - \mathcal{P}_{\uparrow \text{min}}, \quad \mathcal{P}_{\downarrow \text{min}} = 1 - \mathcal{P}_{\uparrow \text{max}}. \]

(75)

If the spin is in some pure state \(| \psi \rangle \), the density matrix is equal to

\[ \hat{\rho} = | \psi \rangle \langle \psi | \]

(the state vector \(| \psi \rangle \) is assumed to be normalized). Correspondingly, Eqs. (73) and (74) for pure states take the form

\[ \mathcal{P}_\uparrow = (\mathcal{P}_{\uparrow \text{max}} - \mathcal{P}_{\uparrow \text{min}}) | \langle \varphi_\uparrow | \psi \rangle |^2 + \mathcal{P}_{\uparrow \text{min}}, \]

(76)

\[ \mathcal{P}_\downarrow = (\mathcal{P}_{\downarrow \text{max}} - \mathcal{P}_{\downarrow \text{min}}) | \langle \varphi_\downarrow | \psi \rangle |^2 + \mathcal{P}_{\downarrow \text{min}}. \]

(77)

It is clearly seen from Eqs. (73)--(77) that if (and only if)

\[ \mathcal{P}_{\uparrow \text{max}} = 1 \quad \text{and} \quad \mathcal{P}_{\uparrow \text{min}} = 0, \]

(78)

then the probabilities \( \mathcal{P}_\uparrow \) and \( \mathcal{P}_\downarrow \) obey the Born rule—namely, probabilities for any state \( S \) characterized by the spin density matrix \( \hat{\rho} \) are

\[ \mathcal{P}_\uparrow(S) = \langle \varphi_\uparrow | \hat{P} | \varphi_\uparrow \rangle, \quad \mathcal{P}_\downarrow(S) = \langle \varphi_\downarrow | \hat{P} | \varphi_\downarrow \rangle, \]

(79)

and probabilities for any pure state \(| \psi \rangle \) of the spin are

\[ \mathcal{P}_\uparrow(\psi) = | \langle \varphi_\uparrow | \psi \rangle |^2, \quad \mathcal{P}_\downarrow(\psi) = | \langle \varphi_\downarrow | \psi \rangle |^2. \]

(80)

So the necessary and sufficient condition for the Born rule is the existence of such two eigenvectors \(| \varphi_\uparrow \rangle \) and \(| \varphi_\downarrow \rangle \), that

\[ \mathcal{P}_\uparrow(\varphi_\uparrow) = 1 \quad \text{and} \quad \mathcal{P}_\downarrow(\varphi_\downarrow) = 1, \]

(81)

i.e. that the measurement results are predictable for these states: \( \uparrow \) for \(| \varphi_\uparrow \rangle \), and \( \downarrow \) for \(| \varphi_\downarrow \rangle \) with probability 1. According to Eq. (73), the eigenvectors are necessarily orthogonal to each other. One can thus say that the two eigenvectors define the spin projection which is measured by the given device.

The Stern–Gerlach apparatus is an example of a device that measures the vertical projection of the spin. As was mentioned in Subsection II A if the spin is in the pure state \(| \uparrow \rangle \) before the measurement, then the measurement result should be necessary “spin up”. Analogously, the measurement of the spin in the state \(| \downarrow \rangle \) gives the result “spin down” with probability 1. Hence the vectors \(| \uparrow \rangle \) and \(| \downarrow \rangle \) are eigenvectors for the Stern–Gerlach apparatus: \( \mathcal{P}_\uparrow(\uparrow) = \mathcal{P}_\downarrow(\downarrow) = 1 \).

The results of this Subsection are summarized in the following Theorem:

**Theorem 2.** If three conditions (i)–(iii) are satisfied:

(i) five Assumptions of Subsection II C are fulfilled,

(ii) an apparatus \( A \) interacts with the spin degree of freedom of a spin-1/2 particle and has two possible classical outcomes \( \uparrow \) and \( \downarrow \),

(iii) there are such two states \( S_\uparrow \) and \( S_\downarrow \) of the spin, that the results of measurements by the apparatus \( A \) are predictable for these states: \( \uparrow \) for \( S_\uparrow \) and \( \downarrow \) for \( S_\downarrow \) with probability 1, then

(a) \( S_\uparrow \) and \( S_\downarrow \) are pure states of the spin, and they can be characterized by normalized state vectors \(| \varphi_\uparrow \rangle \) and \(| \varphi_\downarrow \rangle \) correspondingly,

(b) the vectors \(| \varphi_\uparrow \rangle \) and \(| \varphi_\downarrow \rangle \) are orthogonal to each other,

(c) when the spin is measured by the apparatus \( A \), the probabilities \( \mathcal{P}_\uparrow \) and \( \mathcal{P}_\downarrow \) of the outcomes \( \uparrow \) and \( \downarrow \) obey the Born rule (80) for any pure state \(| \psi \rangle \) of the spin, and obey its generalization (81) for any mixed state.

Based on Theorem 2, we can answer the Questions asked in the beginning of Section II.

**Answer to Question 1.** For different devices \( A_1 \) and \( A_2 \) that measure the same spin projection (i.e. there are such two states \(| \varphi_\uparrow \rangle \) and \(| \varphi_\downarrow \rangle \) that \( \mathcal{P}_\uparrow(| \varphi_\uparrow \rangle, A_1) = \mathcal{P}_\uparrow(| \varphi_\uparrow \rangle, A_2) = 1 \) and \( \mathcal{P}_\downarrow(| \varphi_\downarrow \rangle, A_1) = \mathcal{P}_\downarrow(| \varphi_\downarrow \rangle, A_2) = 1 \), probabilities are the same for any state \( S \):

\[ \forall S \quad \mathcal{P}_\uparrow(S, A_1) = \mathcal{P}_\uparrow(S, A_2), \quad \mathcal{P}_\downarrow(S, A_1) = \mathcal{P}_\downarrow(S, A_2). \]

**Answer to Question 2.** If an apparatus \( A \) obeys the conditions of Theorem 2, then it measures some projection of the spin-1/2, and the probabilities \( \mathcal{P}_\uparrow(S, A) \) and \( \mathcal{P}_\downarrow(S, A) \) are given by Eq. (73) for any state \( S \), and by (80) for pure states. In a more general case of arbitrary apparatus interacting with the spin degree of freedom (for spin 1/2), the probability of any measurement result is a linear function of the spin polarization.

**C. Proof of the Born rule for coordinate measurement**

We considered so far only measurements of the spin degree of freedom, and only in the case of spin 1/2. Now we will show that the obtained results can be generalized to measurements of other systems.

Let \( \{ | \varphi_0 \rangle, | \varphi_1 \rangle, | \varphi_2 \rangle, \ldots \} \) be an orthonormal basis in the state space of some quantum object \( O \). Let there be a measuring device that, being applied to \( O \), always “clicks” when the object \( O \) is in the state \(| \varphi_1 \rangle \), and never “clicks” when the object \( O \) is in the state \(| \varphi_0 \rangle \):

\[ \mathcal{P}_{\text{click}}(| \varphi_0 \rangle) = 0, \quad \mathcal{P}_{\text{click}}(| \varphi_1 \rangle) = 1, \]

(82)

where \( \mathcal{P}_{\text{click}} \) denotes the probability of “clicking”. The question is: what is the probability of “clicking” when
the quantum object is in some superposition of states $|\varphi_0\rangle$ and $|\varphi_1\rangle$:

$$P_{\text{click}}(c_0|\varphi_0\rangle + c_1|\varphi_1\rangle) = ?$$  \hspace{1cm} (83)

Here $c_0$ and $c_1$ are complex numbers, $|c_0|^2 + |c_1|^2 = 1$.

As in Subsection III A, one needs consideration of a larger system to answer this question. Let an “environment” $E$ be a large enough quantum system containing at least three spin-1/2 particles. We will consider a composite system $O + E$. Each state vector of the composite system can be represented as

$$|\varphi_0\rangle|x_0\rangle + |\varphi_1\rangle|x_1\rangle + |\varphi_2\rangle|x_2\rangle + \ldots ,$$  \hspace{1cm} (84)

where $|x_0\rangle$ etc. are state vectors of $E$. Among all state vectors of the system $O + E$, we select the subspace $S$ of vectors containing only contributions of $|\varphi_0\rangle$ and $|\varphi_1\rangle$, i.e. having the form

$$|\varphi_0\rangle|x_0\rangle + |\varphi_1\rangle|x_1\rangle$$  \hspace{1cm} (85)

with arbitrary state vectors $|x_0\rangle, |x_1\rangle$ of the “environment”. One can see that the subspace $S$ is formally equivalent to the space $S$ of state vectors of the system “spin-1/2 + environment”. Indeed, each vector of $S$ can be represented as

$$|\psi\rangle = |\varphi_0\rangle|x_0\rangle + |\varphi_1\rangle|x_1\rangle,$$  \hspace{1cm} (86)

and it is clearly seen from Eqs. (85) and (86) that there is one-to-one correspondence between vectors of $S$ and vectors of $S$.

As a result of the similarity between vector spaces $S$ and $S$, one can introduce an “isospin” $\hat{s}$ (an observable having all formal properties of the spin 1/2) on $S$. Namely, the vectors $|\varphi_0\rangle|x\rangle$ and $|\varphi_1\rangle|x\rangle$ (with any $|x\rangle$) can be considered as eigenvectors of $\hat{s}$ with the corresponding eigenvalues $-1/2$ and $+1/2$. Then, one can define the isospin “polarization vector” $\hat{p}$ for any state $S$ of $S$, as a quantity equal to the spin polarization $p$ for the corresponding state $S$ of $S$. More explicitly, the components $(\hat{p}_x, \hat{p}_y, \hat{p}_z)$ of $\hat{p}$ are:

$$\hat{p}_x = 2 \text{Re} \langle x | x_0 \rangle,$$

$$\hat{p}_y = 2 \text{Im} \langle x | x_0 \rangle,$$

$$\hat{p}_z = \langle x | x_1 \rangle - \langle x_0 | x_0 \rangle.$$

Now, all the derivations of Subsection III A can be literally repeated for the system $O + E$ (with the restricted vector state $S$), replacing the spin polarization $p$ with the “isospin polarization” $\hat{p}$. As a result, one can see that the probability $P_{\text{click}}$ is a linear function of $\hat{p}$ for all state vectors of $S$. Then, all considerations of Subsection III B can be applied to the state vectors of $S$ (with $P_{\text{click}}, \hat{p}, |\varphi_0\rangle$ and $|\varphi_1\rangle$ instead of $P, p, |\varphi_0\rangle$ and $|\varphi_1\rangle$). So, for a pure state

$$|\psi\rangle = c_0|\varphi_0\rangle + c_1|\varphi_1\rangle$$  \hspace{1cm} (87)

of the system $O$, we obtain the result

$$P_{\text{click}}(|\psi\rangle) = |\langle \varphi_1 | \psi \rangle|^2 = |c_1|^2 ,$$  \hspace{1cm} (88)

as an analog of Eq. (80).

Let us illustrate this idea on the example of measurement of the particle position. We suppose for simplicity that the particle has no internal degrees of freedom, and its wavefunction $\psi$ is one-dimensional: $\psi = \psi(x)$. As a measuring device, we consider an ideal detector that catches everything within the range $[x_1, x_2]$ of the $x$-coordinate, and beeps when it catches a particle. So the probability of the beep can be interpreted as a probability of finding a particle in the coordinate range $[x_1, x_2]$. We denote this probability (for the wavefunction $\psi(x)$) as $P(|\psi\rangle)$.

We postulate two obvious properties of the detector:

1. If the wavefunction of the particle vanishes for all $x \in [x_1, x_2]$, then the detector never beeps ($P = 0$).
2. If the wavefunction vanishes for all $x$ outside the range $[x_1, x_2]$, then the detector always beeps ($P = 1$).

It is easy to see that any normalized wavefunction $\psi(x)$ can be represented as

$$\psi(x) = c_0 \varphi_0(x) + c_1 \varphi_1(x),$$  \hspace{1cm} (89)

where $c_0$ and $c_1$ are some numbers, $|c_0|^2 + |c_1|^2 = 1$, and $\varphi_0(x)$ and $\varphi_1(x)$ are such normalized wavefunctions that

$$\forall x \in [x_1, x_2] : \varphi_0(x) = 0 ,$$  \hspace{1cm} (90)

$$\forall x \notin [x_1, x_2] : \varphi_1(x) = 0 ,$$  \hspace{1cm} (91)

as illustrated in Fig. 4. Due to the postulated properties of the detector, we have:

$$P[\varphi_0] = 0 \quad \text{and} \quad P[\varphi_1] = 1.$$  \hspace{1cm} (92)
So we can use Eq. (88), and get

\[ P[\psi] = |c_1|^2. \]  

(93)

Now let us find \( c_1 \). First, we represent \( \psi(x) \) as the sum \( \Phi_0(x) + \Phi_1(x) \), where

\[ \Phi_0(x) = (1 - m(x)) \psi(x), \]  

(94)

\[ \Phi_1(x) = m(x) \psi(x), \]  

(95)

and

\[ m(x) = \begin{cases} 1, & \text{if } x \in [x_1, x_2], \\ 0, & \text{otherwise}. \end{cases} \]  

(96)

Then, we obtain \( \varphi_0(x) \) and \( \varphi_1(x) \) by normalization of the functions \( \Phi_0(x) \) and \( \Phi_1(x) \):

\[ \varphi_0(x) = \Phi_0(x)/c_0, \quad \varphi_1(x) = \Phi_1(x)/c_1, \]  

(97)

where \( c_0 \) and \( c_1 \) are norms of the functions \( \Phi_0(x) \) and \( \Phi_1(x) \):

\[ c_0 = \left( \int_{-\infty}^{+\infty} |\Phi_0(x)|^2 \, dx \right)^{1/2} = \left( \int_{-\infty}^{x_1} |\psi(x)|^2 \, dx + \int_{x_1}^{+\infty} |\psi(x)|^2 \, dx \right)^{1/2}, \]  

(98)

\[ c_1 = \left( \int_{-\infty}^{+\infty} |\Phi_1(x)|^2 \, dx \right)^{1/2} = \left( \int_{x_1}^{x_2} |\psi(x)|^2 \, dx \right)^{1/2}. \]  

(99)

It is easy to check that the functions \( \varphi_0(x) \) obey the properties (90), (91), and that the equality (93) is satisfied.

Finally, substitution of Eq. (97) into Eq. (93) gives the following answer for the probability of finding a particle in the range \([x_1, x_2]\) of \( x \)-coordinate:

\[ P[\psi] = \int_{x_1}^{x_2} |\psi(x)|^2 \, dx. \]  

(100)

This is the Born rule in its integral form. Thus, the five Assumptions of Subsection III C provide enough background for derivation the Born rule.

IV. NECESSITY OF EACH ASSUMPTION

It is shown above that the set of five Assumptions given in Subsection III C is sufficient for the derivation of the Born rule. The next question is: are these Assumptions necessary for the derivation of the Born rule?

We will prove by contradiction that each Assumption is indeed necessary. Namely, for each Assumption we will find a counterexample—an expression for the probability that differs from the Born rule and, at the same time, agrees with all the Assumptions except the given one. This will prove the necessity of the given Assumption.

Let \( P_{\uparrow}^{(0)}(S, A) \) and \( P_{\downarrow}^{(0)}(S, A) \) be “true” probabilities (obeying the Born rule) of the “spin-up” and “spin-down” outcomes when the spin in the state \( S \) is measured by the apparatus \( A \). Then, we introduce the quantities

\[ P_{\uparrow}^{(1)}(S, A, x) = \begin{cases} 1, & \text{if } P_{\uparrow}^{(0)}(S, A) > x, \\ 0, & \text{otherwise}, \end{cases} \]  

(101)

\[ P_{\downarrow}^{(1)}(S, A, x) = 1 - P_{\uparrow}^{(1)}(S, A, x), \]  

(102)

where \( x \) is a random number associated with the given measurement event and uniformly distributed in the range \([0, 1]\). Let us consider the quantities \( P_{\uparrow}^{(1)} \) and \( P_{\downarrow}^{(1)} \) as probabilities of “up” and “down” outcomes. One can easily see that they agree with all Assumptions except the first one. At the same time, they do not obey the Born rule. So, the expressions (101), (102) provide a counterexample proving that Assumption 1 is necessary.

Now we consider a case when Assumption 2 is violated. Let the evolution of some collection of \( N \) spins is described not by unitary (conserving the scalar product) operators, but by operators conserving a “modified scalar product”

\[ \langle \Psi| \hat{\Phi} \rangle = \langle \Psi| \hat{A} \otimes \hat{A} \otimes \cdots \otimes \hat{A} |\Psi\rangle \]  

(103)

of any \( N \)-spin state vectors \( |\Psi\rangle \) and \( |\Phi\rangle \), where \( \hat{A} \) is some positive self-adjoint operator in the spin state space. If Assumptions 1, 3, 4, 5 remain in force, then, instead of Theorem 1, one can deduce that the probability of any outcome is a linear function of the “modified spin polarization”

\[ \frac{\langle \Psi| \hat{\sigma} \otimes I_N |\Psi\rangle}{\langle \Psi| \hat{\sigma} |\Psi\rangle}. \]  

(104)

Then, instead of the Born rule (90), one can get the following probabilities:

\[ P_{\uparrow}^{(2)}(|\psi\rangle) = \frac{|\langle \varphi_{\uparrow} |\hat{A} |\psi\rangle|^2}{\langle \psi |\hat{A} |\psi\rangle}, \quad P_{\downarrow}^{(2)}(|\psi\rangle) = \frac{|\langle \varphi_{\downarrow} |\hat{A} |\psi\rangle|^2}{\langle \psi |\hat{A} |\psi\rangle}, \]  

(105)

where vectors \( |\varphi_{\uparrow}\rangle \) and \( |\varphi_{\downarrow}\rangle \) obey the relations

\[ \langle \varphi_{\uparrow} |\hat{A} |\varphi_{\uparrow}\rangle = \langle \varphi_{\downarrow} |\hat{A} |\varphi_{\downarrow}\rangle = 1, \]  

\[ \langle \varphi_{\uparrow} |\hat{A} |\varphi_{\downarrow}\rangle = 0. \]

Hence, necessity of Assumption 2 is proven by a counterexample provided by Eq. (105).

The next expressions can serve as a counterexample proving the necessity of the remaining three Assumptions:

\[ P_{\uparrow}^{(3)}(S, A) = \left( 3 - 2 P_{\uparrow}^{(0)}(S, A) \right) \left( P_{\uparrow}^{(0)}(S, A) \right)^2, \]  

(106)

\[ P_{\downarrow}^{(3)}(S, A) = \left( 3 - 2 P_{\downarrow}^{(0)}(S, A) \right) \left( P_{\downarrow}^{(0)}(S, A) \right)^2. \]  

(107)
The quantities $\mathcal{P}^{(3)}_\uparrow$ and $\mathcal{P}^{(3)}_\downarrow$, considered as probabilities of “up” and “down” measurement results, obey Eqs. (3), (4), (6), (7), but violate Eq. (13). This fact can be interpreted in three different ways. First, if we accept Assumptions 1,2,4,5, then violation of Eq. (13) means violation of either the multiplication rule (5), or the addition rule (9); both rules are contained in Assumption 3. The second option is to keep Assumptions 1,2,3,5, and attribute the violation of Eq. (13) to violation of Eq. (8), which means rejection of the Assumption 4. The third possibility is to accept Assumptions 1,2,3,4, and reject Assumption 5; in this case Eq. (13) is no more relevant, therefore the probabilities $\mathcal{P}^{(3)}_\uparrow$ and $\mathcal{P}^{(3)}_\downarrow$ are in accordance with the first four Assumptions.

Thus, if we sacrifice Assumption 3, or 4, or 5, then the probabilities (106), (107) would satisfy the remaining Assumptions but disagree with the Born rule. So Eqs. (106) and (107) provide a counterexample demonstrating necessity of assumptions 3,4,5.

Now it is proven that each of the five Assumptions is indeed necessary for the derivation of the Born rule.

V. CONCLUSIONS

In this paper, the Born rule is deduced from five very simple and reasonable assumptions given in Subsection II.C. Each of these assumptions is necessary, as shown in Section IV. Unlike previous studies, our approach treats measuring devices as black boxes (no analysis of their operation is required), and allows to consider non-ideal devices (which are not described by operators). This advantage makes it possible to get more than the Born rule—namely, we show that probabilities of outcomes of any (ideal or non-ideal) measurement on the spin $\frac{1}{2}$ are linear functions of the spin polarization vector (Theorem 1). This result is enough for associating a positive-operator valued measure (POVM) to any measurement on the spin $\frac{1}{2}$, see Eq. (60). The Born rule comes in the particular case of ideal measurements, considered in Theorem 2. Though we are concentrated on spin-$\frac{1}{2}$ measurements, our results can be easily generalized, as illustrated in Subsection III.C for the case of the position measurement.

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