A bifurcation analysis of some McKean-Vlasov equations

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Abstract

We study the long-time behavior of some McKean-Vlasov stochastic differential equations used to model the evolution of large populations of interacting agents. We give conditions ensuring the local stability of an invariant probability measure. Lions derivatives are used in a novel way to obtain our stability criteria. We obtain results for non-local McKean-Vlasov equations on $\mathbb{R}^d$ and for McKean-Vlasov equations on the torus where the interaction kernel is given by a convolution. On $\mathbb{R}^d$, we prove that the location of the roots of a holomorphic function determines the stability. On the torus, our stability criterion involves the Fourier coefficients of the interaction kernel. In both cases, we prove the convergence in the Wasserstein metric $W_1$.

Keywords McKean-Vlasov SDE, Long-time behavior, Mean-field interaction, Lions derivative

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1 Introduction

We are interested in the long-time behavior of the solutions of a class of McKean-Vlasov stochastic differential equations (SDE) of the form:

$$\begin{align*}
    dX_t^\nu &= \mathcal{V}(X_t^\nu, \mu_t)dt + \sigma dW_t, \\
    \mu_t &= \mathcal{L}(X_t^\nu), \\
    \mu_0 &= \nu.
\end{align*}$$

(1.1)

In this equation, $(W_t)_{t \geq 0}$ is a standard $\mathbb{R}^d$-valued Brownian motion, $\sigma$ is a deterministic matrix, and $\nu$ is the law of the initial condition $X_0^\nu$, assumed to be independent of $(W_t)_{t \geq 0}$.

McKean-Vlasov equations appear naturally as the limit $N \to \infty$ of the following particle system $(X_{i,N}^t)_{t \geq 0}$, solution of

$$\begin{align*}
    \forall i \in 1, \ldots, N, \quad dX_{i,N}^t &= \mathcal{V}(X_{i,N}^t, \mu_t^N)dt + \sigma dW_{i,N}^t, \\
    \mu_t^N &= \frac{1}{N} \sum_{j=1}^N \delta_{X_{j,N}^t},
\end{align*}$$

(1.2)

where $\mu_t^N$ is the empirical measure $\mu_t^N = \frac{1}{N} \sum_{j=1}^N \delta_{X_{j,N}^t}$ and $(W_{i,N}^t)_{t \geq 0}$ are $N$ independent standard Brownian motions. We refer to [41] for an introduction to this topic.

Such particle systems and their mean-field counterparts are used in a wide range of applications such as plasma physics [21, 31], fluid mechanics [25], astrophysics [26], and others.

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stars or galaxies \([46]\), bio-sciences (to understand the collective behavior of animals \([8]\)), neuroscience (to model assemblies of neurons, such as integrate and fire neurons \([24, 24]\) or FitzHugh–Nagumo neurons \([37]\)), opinion dynamics \([17]\) and economics \([10]\).

In these applications, one important question concerns the long-time behavior of the solutions. As such, the ergodic properties of McKean-Vlasov equations \([1, 11]\) have been studied in many different contexts and approaches.

Two families of assumptions are known to ensure that \(1.1\) admits a unique, globally attractive invariant probability measure. The first type of assumption deals with kernels given by

\[
\mathcal{V}(x, \mu) = -\nabla V(x) - \nabla W*\mu(x),
\]

where \(V, W\) have suitable convexity properties. The first results in this direction were obtained in \([3, 4]\) in dimension one. In larger dimensions, \([39, 45]\) proved the convergence uniformly in-time of a suitable particle system towards the mean-field equation. As such, they obtained the ergodicity of the McKean-Vlasov equation from the ergodicity of the particle system. These uniformly in-time propagation of chaos arguments have been used wisely; see for instance \([18, 30]\) for recent results in this direction. These results have also been obtained by using functional inequalities \([5, 11]\): the idea is to define a measured valued functional (known as the entropy or free energy), which only decreases along the trajectories of the solution of \(1.1\).

The second kind of assumption involves weak enough interactions. When the dependence of \(V\) with respect to the measure is sufficiently weak, one expects global stability because this situation can be seen as a perturbation of the case without interactions. As such, it is possible to extend techniques from ergodic Markov processes to the case of weak interactions. This includes, for instance, coupling techniques \([22, 2, 22, 23]\) or Picard iterations in suitable spaces \([15]\).

It is also well-known that, in general, such global stability results cannot hold because \(1.1\) may have multiple invariant probability measures and periodic solutions \([40, 33, 44]\). These examples motivate the current question of the paper, namely the study of the local stability of a given invariant probability measure of \(1.1\). That is, being given \(\nu_\infty\) an invariant probability measure of \(1.1\), we address the following question:

| Is there exist an open neighborhood of \(\nu_\infty\) such that for all initial conditions \(\nu\) within this neighborhood, the law of \(X^\nu_t\) converges to \(\nu_\infty\), as \(t\) goes to infinity? If so, for which metric does the convergence hold, and what is the rate of convergence? |

Such local stability results can be obtained via partial differential equation (PDE) techniques, using that the marginals of the non-linear process solve a non-linear PDE (the Fokker-Planck equation). The strategy is to linearize the non-linear PDE around \(\nu_\infty\), to study the existence of a spectral gap for the linear equation in appropriate Banach spaces, and to use perturbation techniques to obtain the convergence for the non-linear PDE. We refer to \([32, 35]\) for an overview of these techniques. When the non-linear PDE admits a gradient flow structure, it is also possible to study the local stability of an invariant probability measure using functional inequalities; see \([43, 12]\).

Our approach differs from these two methods on several points. We do not rely on the non-linear Fokker-Planck PDE but instead, use the stochastic version \(1.1\). Our strategy is to derive the interaction kernel with respect to the initial probability measure, in the neighborhood of \(\nu_\infty\). There are several notions of derivation with respect to probability measures (see \([10]\)): we use here the Lions derivatives. We denote by \(\mathcal{P}_2(\mathbb{R}^d)\) the set of probability measures on \(\mathbb{R}^d\) having a second moment. For all \(x \in \mathbb{R}^d\) and \(t \geq 0\), we consider the function

\[
\mathcal{P}_2(\mathbb{R}^d) \ni \nu \mapsto \mathcal{V}(x, \mathcal{L}(X^\nu_t)) =: \nu^x_t(\nu) \in \mathbb{R}^d,
\]

where \(X^\nu_t\) is the solution of \(1.1\) starting with \(\nu\) at time 0. We prove that under suitable assumptions, this function is Lions differentiable at \(\nu_\infty\), meaning that for all \(\nu \in \mathcal{P}_2(\mathbb{R}^d)\), we
have
\[ \mathcal{V}(x, \mathcal{L}(X_t^\nu)) = \mathcal{V}(x, \nu_\infty) + \mathbb{E}\partial_v v_t^\nu(\nu_\infty)(X_0) \cdot (X - X_0) + o((\mathbb{E}|X - X_0|^2)^{1/2}). \]

In this equation, \( X, X_0 \) are any random variables defined on the same probability space, with laws equal to \( \nu \) and \( \nu_\infty \). We write \( \mathbb{E}(X - X_0|X_0) = k(X_0) \), where \( k \) is a deterministic function from \( \mathbb{R}^d \) to \( \mathbb{R}^d \). As such, the function \( k \) encodes the correlations between the two initial conditions \( X \) and \( X_0 \). It follows from the Cauchy–Schwarz inequality that \( \mathbb{E}|k(X_0)|^2 \leq \mathbb{E}|X - X_0|^2 < \infty \). Therefore, \( k \in L^2(\nu_\infty) \). We define the linear operator \( \Omega_t : L^2(\nu_\infty) \to L^2(\nu_\infty) \) by
\[ \Omega_t(k) := x \mapsto \mathbb{E}\partial_v v_t^\nu(\nu_\infty)(X_0) \cdot k(X_0). \]

The fact that \( \Omega_t(k) \in L^2(\nu_\infty) \) for all \( k \in L^2(\nu_\infty) \) is not trivial and follows from our assumptions on the function \( \mathcal{V} \). So we have (recall that \( \nu = \mathcal{L}(X) \) and \( \mathbb{E}(X - X_0|X_0) = k(X_0) \))
\[ \mathcal{V}(x, \mathcal{L}(X_t^\nu)) = \mathcal{V}(x, \nu_\infty) + \Omega_t(k)(x) + o((\mathbb{E}|X - X_0|^2)^{1/2}). \]

Our spectral conditions under which we prove that \( \nu_\infty \) is locally stable can be stated in terms of the decay of the function \( t \mapsto \Omega_t \), as \( t \) goes to infinity. We show that the integrability of this function on \( \mathbb{R}_+ \) implies the stability of \( \nu_\infty \). In addition, the decay of \( t \mapsto \Omega_t \) as \( t \) goes to infinity gives precisely the rate of convergence of \( \mathcal{L}(X_t^\nu) \) towards \( \nu_\infty \), in Wasserstein metrics.

Crucial to our analysis, we provide an explicit integral equation to compute this function \( \Omega_t \). To do so, we consider the linear process \( (Y_t^x)_{t \geq 0} \) associated with \( \mathcal{L} \) and \( \nu_\infty \), defined as the solution of
\[ dY_t^x = \mathcal{V}(Y_t^x, \nu_\infty)dt + \sigma dW_t, \]

starting from \( \mathcal{L}(Y_0^x) = \nu \). We define similarly for \( x \in \mathbb{R}^d \) and \( t \geq 0 \) the function \( \mathcal{P}_t(\mathbb{R}^d) \ni \nu \mapsto u_t^\nu(\nu) := \mathcal{V}(x, \mathcal{L}(Y_t^x)) \). Under our assumptions, \( u_t^\nu \) is Lions differentiable at \( \nu_\infty \), and we can define
\[ \forall k \in L^2(\nu_\infty), \quad \Theta_t(k) := x \mapsto \mathbb{E}\partial_v u_t^\nu(\nu_\infty)(X_0) \cdot k(X_0). \]

We prove the following key relation between \( \Theta_t \) and \( \Omega_t \)
\[ \forall t \geq 0, \quad \Omega_t(k) = \Theta_t(k) + \int_0^t \Theta_{t-s}(\Omega_s(k))ds. \]

That is, \( \Omega \) is a solution of a Volterra integral equation whose kernel is given by \( \Theta \); in the language of integral equations, \( \Omega \) is the resolvent of \( \Theta \). This relation is helpful because it is easier to get estimates on \( u_t^\nu \), which involves a linear Markov process, rather than getting estimates on \( v_t^\nu \), which involves the solution of the McKean-Vlasov equation \( \mathcal{L} \).

In particular, this relation allows deducing the decay properties of \( \Omega \) from properties of \( \Theta \). We obtain our stability results for the Wasserstein \( W_1 \) metric. To avoid technical issues, we consider two simplified scenarios.

First, in Section 2 we assume that the function \( \mathcal{V}(x, \mu) \) is given by \( \mathcal{V}(x, \mu) = b(x) + \int_{\mathbb{R}^d} f(y)\mu(dy) =: b(x) + \mu(f) \), for some smooth functions \( b, f : \mathbb{R}^d \to \mathbb{R}^d \). In that way, the non-local and non-linear part of the equation \( \mu(f) \) is clearly separated from drift \( b(x) \).

This simplified setting permits us to introduce the main ideas and tools. Our main result, Theorem 2.2, states that the stability of an invariant probability measure is determined by the location of the roots of a holomorphic function associated with the dynamics. When all the roots lie on the left-half plane, stability holds.

Second, in Section 3 we consider a McKean-Vlasov equation on the torus \( \mathbb{T}^d := (\mathbb{R}/(2\pi \mathbb{Z}))^d \), with an interaction kernel given by a convolution: \( \mathcal{V}(x, \mu) = -\int_{\mathbb{T}^d} \nabla W(x - y)\mu(dy) \), where \( W \) is a smooth function from \( \mathbb{T}^d \) to \( \mathbb{R} \). We moreover assume that \( \sigma = \sqrt{2 \beta^{-1}} I_d \) for some \( \beta > 0 \), where \( I_d \) is the identity matrix. This setting covers many interesting models; see [12].
We study the stability of the uniform probability measure \( U(dx) := \frac{dx}{(2\pi)^d} \). Our second main result, Theorem 3.1, states that when \( \inf_{n \in \mathbb{Z}^d \setminus \{0\}} |n|^2 (\beta + \tilde{W}(n)) > 0 \), \( \tilde{W}(n) \) being the \( n \)-th Fourier coefficient of \( W \), then \( U \) is locally stable for the \( W_1 \) metric.

In both cases, we use the strategy described above with the Lions derivatives, and the criteria we obtain are optimal: violations of the criteria occur exactly at bifurcation points. The strategy presented in this work also applies to mean-field models of noisy integrate-and-fire neurons. In an unpublished preliminary version of this work [14], we study the stability of the stationary solutions of such a mean-field model of noisy neurons. In addition, the existence of periodic solutions via Hopf bifurcations is studied in [16]. For the sake of clarity, we restrict here ourselves to a diffusive setting.

Finally, we mention an important open problem concerning the long-time behavior of the particle system (1.2). On the one hand, general conditions are known to ensure that the particle system is ergodic. On the other hand, numerical studies show that this particle system can have a metastable behavior in the sense that the convergence of the empirical measure \( \mu_N \) towards its invariant state can be very slow when \( N \) is large. The locally stable invariant probability measures of the non-linear equation (1.1) are good candidates to be metastable states of the particle system (1.2). Characterizing those metastable states in quantitative terms is a challenging mathematical question. Recent partial results have been obtained in this direction [7, 13, 19, 14], and we hope to progress on this question in future works.

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2 Non-local McKean-Vlasov equations on \( \mathbb{R}^d \)

Let \( \mathcal{P}_1(\mathbb{R}^d) \) be the space of probability measures on \( \mathbb{R}^d \) with a finite first moment. We consider the following McKean-Vlasov equation on \( \mathbb{R}^d \):

\[
\begin{align*}
\text{d}X^\nu_t &= b(X^\nu_t)\text{d}t + Ef(X^\nu_t)\text{d}t + \sigma\text{d}W_t, \\
\text{where} \quad (W_t)_{t \geq 0} \quad \text{is a d-dimensional standard Brownian motion,} \\
\sigma \in \mathcal{M}_d(\mathbb{R}) \quad \text{is a constant d \times d matrix with} \quad \det \sigma > 0 \quad \text{and} \\
b,f : \mathbb{R}^d \rightarrow \mathbb{R}^d \quad \text{are deterministic functions.}
\end{align*}
\]

2.1 Main result

Assumption 2.1. We assume that:

1. The function \( b \) is Lipschitz continuous, and there exists \( \beta > 0 \) and \( R \geq 0 \) such that

\[
\forall x,y \in \mathbb{R}^d, \quad |x - y| \geq R \implies (x - y) \cdot (b(x) - b(y)) \leq -\beta |x - y|^2.
\]

2. The function \( f \in C^2(\mathbb{R}^d;\mathbb{R}^d) \) with \( ||\nabla f||_{\infty} + ||\nabla^2 f||_{\infty} < \infty \).

Let \( \nu_\infty \) be an invariant probability measure of (2.1). Denote by \( \alpha \in \mathbb{R}^d \) the interaction term under \( \nu_\infty \):

\[
\alpha := \nu_\infty(f) := \int_{\mathbb{R}^d} f(y)\nu_\infty(\text{d}y).
\]
Each invariant probability measure of (2.1) is characterized by its associated vector \( \alpha \), and we sometimes denote by \( \nu^\infty_\alpha \) such invariant probability measure to emphasize the dependence on \( \alpha \). We give a sufficient condition ensuring that \( \nu^\infty \) is locally stable for the McKean-Vlasov dynamics (2.1). Consider \( (Y^\alpha_t)_{t \geq 0} \) the solution of the linear SDE
\[
dY^\alpha_t = b(Y^\alpha_t)dt + \sigma(Y^\alpha_t)dB_t.
\]
Note that \( \nu^\infty = \nu^\infty_\alpha \) is also an invariant probability measure of this linear SDE, and the assumptions above imply that \( \nu^\infty_\alpha \) is the unique invariant probability measure of (2.3). Key to our analysis is the following family of matrices \( \Theta_t \) defined for all \( t \geq 0 \):
\[
\Theta_t := \int \nabla y \mathbb{E}_y f(Y^\alpha_t) \nu^\infty_\alpha(dy).
\]
The notation \( \mathbb{E}_y f(Y^\alpha_t) \) means that the initial condition of (2.3) is set to be \( y \in \mathbb{R}^d \) (that is \( Y^\alpha_0 = y \)). In addition, \( \nabla y \mathbb{E}_y f(Y^\alpha_t) \) is the Jacobian matrix of \( y \mapsto \mathbb{E}_y f(Y^\alpha_t) \). A result of Eberle [22] (see below) implies that:
\[
\exists \kappa_+ > 0, \quad \sup_{t \geq 0} \| \Theta_t \| e^{\kappa_+ t} < \infty,
\]
where \( \| \cdot \| \) is any norm on the matrices \( M_d(\mathbb{R}) \). Let \( I_d \) be the \( d \times d \) identity matrix and let \( \hat{\Theta}(z) \) be the Laplace transform of \( \Theta_t \), defined by
\[
\forall z \in \mathbb{C} \text{ with } \Re(z) > -\kappa_+, \quad \hat{\Theta}(z) := \int_0^\infty e^{-zt} \Theta_t dt.
\]
Our main result is:
\[\text{Theorem 2.2.} \quad \text{Consider } \nu^\infty \text{ an invariant probability measure of (2.1). Assume Assumption (2.1) holds and let } \alpha \text{ be given by (2.2). Define the “abscissa” of the rightmost zeros of } \det \left( I_d - \hat{\Theta}(z) \right): \]
\[
- \lambda' := \sup \{ \Re(z) \mid z \in \mathbb{C}, \quad \det \left( I_d - \hat{\Theta}(z) \right) = 0 \}. \tag{2.5}
\]
Assume that \( \lambda' > 0 \). Then \( \nu^\infty \) is locally stable: there exists \( C, \epsilon > 0 \) and \( \lambda \in (0, \lambda') \) such that for all \( \nu \in \mathcal{P}_1(\mathbb{R}^d) \) with \( W_1(\nu, \nu^\infty) < \epsilon \), it holds that
\[
\forall t \geq 0, \quad W_1(\mathcal{L}(X^\nu_t), \nu^\infty) \leq CW_1(\nu, \nu^\infty) e^{-\lambda t}. \tag{2.6}
\]
\[\text{2.2 Remarks and examples}\]
We now make some remarks on Theorem 2.2.

Gradients bounds
We denote by \( (Y^{\alpha,\delta}_t) \) the solution of (2.3) with initial condition \( Y^{\alpha,\delta}_0 = x \). Under assumptions (2.1) Theorem 1 in [22] applies: there exists \( \kappa_+ > 0 \) and \( C > 1 \) such that for all \( x, y \in \mathbb{R}^d \) and all \( t \geq 0 \),
\[
W_1(\mathcal{L}(Y^{\alpha,\delta}_t), \mathcal{L}(Y^{\alpha,\delta}_y)) \leq C e^{-\kappa_+ t}|x - y|. \tag{2.6}
\]
We deduce from this inequality the following gradient bound:
\[
\forall y \in \mathbb{R}^d, \quad \| \nabla y \mathbb{E}_y f(Y^\alpha_t) \| \leq C \| \nabla f \| e^{-\kappa_+ t}. \tag{2.7}
\]
and so \( \sup_{t \geq 0} \| \Theta_t \| e^{\kappa_+ t} < \infty \). In particular, the function \( z \mapsto \det \left( I_d - \hat{\Theta}(z) \right) \) is well defined and holomorphic on the half-plane \( \Re(z) > -\kappa_+ \), and so its zeros are isolated. Note that it is possible to obtain gradient bounds similar to (2.7) under less restrictive assumptions on \( b \) and \( \sigma \); see for instance [42].
**Case of weak interactions**

One way to check that the condition \( \lambda' > 0 \) is verified (\( \lambda' \) given by (8)) is to compute the \( L^1 \) norm of \( \Theta_t \):

**Lemma 2.3.** Assume that \( \int_0^\infty \|\Theta_t\|dt < 1 \), where \( \|A\|^2 := \sum_{i,j=1}^d |A_{ij}|^2 \) is the Frobenius norm of matrices. Then \( \lambda' > 0 \) and so \( \nu_\infty \) is locally stable.

**Proof.** By assumption, there exists \( \delta > 0 \) small enough such that

\[
\int_0^\infty e^{\delta t}\|\Theta_t\|dt < 1.
\]

For \( R(z) \geq -\delta \), it holds that \( \|\hat{\Theta}(z)\| \leq \int_0^\infty e^{-R(z) t}\|\Theta_t\|dt < 1 \). We deduce that \( I_d - \hat{\Theta}(z) \) is invertible for \( R(z) \geq -a \), with inverse given by \( \sum_{k \geq 0}(\hat{\Theta}(z))^k \). So \( \lambda' \geq \delta > 0 \).

Note that the Frobenius norm can be replaced by any sub-multiplicative norm of matrices in this argument. This assumption is typically satisfied if the non-linear part in (2.1) is weak enough. Consider for instance the case \( b(x) = -\nabla V(x) \) for some uniformly strongly convex function \( V \):

\[
\exists \kappa_* > 0, \quad \nabla^2 V(x) \geq \kappa_* I_d, \quad \forall x \in \mathbb{R}^d.
\]

Then, for any Lipschitz smooth test function \( g : \mathbb{R}^d \to \mathbb{R} \), it holds that

\[
|\partial_{y_i}E_y g(Y^t)| \leq \|\partial_{y_i} g\|_\infty e^{-\kappa_* t}.
\]

So the matrix \( \Theta_t = (\Theta^{i,j}_t)_{i,j \in \{1 \ldots d\}} \) satisfies:

\[
|\Theta^{i,j}_t| \leq \|\partial_{y_i} f^j\|_\infty e^{-\kappa_* t},
\]

and so

\[
\sum_{i,j=1}^d \left( \|\partial_{y_i} f^j\|_\infty \right)^2 < (\kappa_*)^2.
\]

then \( \int_0^\infty \|\Theta_t\|dt < 1 \) and so \( \lambda' > 0 \). Therefore, by Theorem 2.2, any invariant probability measure of (2.1) is locally stable.

**On the existence and uniqueness of the invariant measures**

The existence of an invariant probability measure of (2.1) does not follow directly from our assumptions. Consider for instance \( d = 1 \), \( b(x) = -x \), \( \sigma \equiv 1 \) and \( f(x) = \kappa x \) for some \( \kappa > 1 \). This satisfies all our assumptions and (2.1) does not have any invariant probability measure because for all \( \nu \), \( \mathbb{E}_\nu X^\nu_t = e^{(\kappa-1)t}\mathbb{E}_\nu X^\nu_0 \to \infty \) as \( t \to \infty \). However, if \( f \) is bounded, then there exists at least one invariant probability measure. This follows from the Brouwer fixed point theorem, see Corollary 2.18. The uniqueness of the invariant probability measure of (2.1) does not hold in general; see for instance the example below. This example also shows that the condition \( \lambda' > 0 \) is required.

**Case with no noise (\( \sigma \equiv 0 \))**

The case \( \sigma \equiv 0 \) would require special treatment and is not included in Theorem 2.2. It is however instructive to see that criterion (2.5) is equivalent to the classical stability criterion of deterministic dynamical systems in \( \mathbb{R}^d \). Indeed, when \( \sigma \equiv 0 \), the invariant measures are of the form \( \delta_{x_*} \) for some \( x_* \in \mathbb{R}^d \). A simple computation shows that:

\[
\Theta_t = \nabla f(x_*)e^{t \nabla b(x_*)}.
\]
We deduce that for all $z \in \mathbb{C}$ with $\Re(z) > -\kappa_*$,
\[
\hat{\Theta}(z) = \nabla f(x_*) \int_0^\infty e^{-t(zI_d - \nabla b(x_*))}dt = \nabla f(x_*)(zI_d - \nabla b(x_*))^{-1}.
\]

Now, if $\det \left( I_d - \hat{\Theta}(z) \right) = 0$, there exists $y \in \mathbb{R}^d \setminus \{0\}$ such that $y = \nabla f(x_*)(zI_d - \nabla b(x_*))^{-1}y$. Setting $x = (zI_d - \nabla b(x_*))^{-1}y$, we have $x \neq 0$ and
\[
xx = (\nabla f(x_*) + \nabla b(x_*))x.
\]
So $z$ is an eigenvalue of $\nabla b(x_*) + \nabla f(x_*)$. Conversely, if $z \in \mathbb{C}$ is an eigenvalue of this matrix then $\det \left( I_d - \hat{\Theta}(z) \right) = 0$. So criterion \eqref{eq:EigenvalueCondition} is equivalent to the fact that the Jacobian matrix of the vector field $b + f$ at the point $x_*$ has all its eigenvalues in the left-half plane $\Re(z) < 0$.

A simple explicit example

We close this section with a simple explicit example showing that the criteria $\lambda' > 0$ is sharp. Consider for $J \in \mathbb{R}$ the following McKean-Vlasov SDE on $\mathbb{R}$:
\[
dX_t = -X_t dt + J \cos(X_t)dt + \sqrt{2}dW_t.
\]
(2.8)
The associated linear process $(Y_t^\alpha)$ is solution of the Ornstein–Uhlenbeck SDE:
\[
dY_t^\alpha = -Y_t^\alpha dt + \alpha dt + \sqrt{2}dW_t.
\]
This linear process admits a unique invariant probability measure given by $\nu_\alpha^R = \mathcal{N}(\alpha, 1)$, such that if $G$ is a standard Gaussian random variable, $E \cos(Y_t^\alpha \nu_\alpha^R) = E \cos(\alpha + G) = \frac{\cos(\alpha)}{\sqrt{\pi}}$.

We deduce that the invariant probability measures of \eqref{eq:MckeanVlasov} are $\{\mathcal{N}(\alpha, 1) \mid \alpha \in \mathbb{R}, \tan(\alpha) = \cos(\alpha)\}$. Let $\alpha \in \mathbb{R}$ such that $\frac{\tan(\alpha)}{\alpha} = \cos(\alpha)$. We find:
\[
\forall t \geq 0, \quad \Theta_t = J \int \frac{d}{dy} E_y \cos(Y_t^\alpha \nu_\alpha^R)(dy) = -J \frac{1}{\sqrt{\pi}} e^{-t^2} \sin(\alpha).
\]
So, for $\Re(z) > -1$, $\hat{\Theta}(z) = -J \frac{1}{\sqrt{\pi}} e^{-z/2} \sin(\alpha)$ and the equation $\hat{\Theta}(z) = 1$ has a unique solution $z = -J e^{-1/2} \sin(\alpha) - 1$. This root is strictly negative if and only if $J \sin(\alpha) > -\sqrt{\pi}$. We deduce by Theorem \ref{thm:EigenvalueCondition} that $\nu_\alpha^R$ is locally stable provided that $J \sin(\alpha) > -\sqrt{\pi}$. Recall that $\alpha \sqrt{\pi} = J \cos(\alpha)$. So among all the invariant probability measures of \eqref{eq:MckeanVlasov}, the (locally) stable ones are the $\mathcal{N}(\alpha, 1)$ with
\[
\alpha \tan(\alpha) > -1.
\]

2.3 Lions derivatives and an integrated sensitivity formula

In this section, we introduce two functions of probability measures and explain how the behaviors of their Lions derivatives are related to the long-time behavior of \eqref{eq:MckeanVlasov}. Before we proceed, we introduce some new notation.
Notations
For $x \in \mathbb{R}^d$, we denote by $|x|$ its Euclidean norm. For $X$ an $\mathbb{R}^d$-valued random variable with a second moment, we denote by $||X||_2 = \sqrt{\mathbb{E}|X|^2}$ its $L^2$ norm. Let $L^2(\nu_\infty)$ be the space of measurable functions $k : \mathbb{R}^d \to \mathbb{R}^d$ such that $||k||_{L^2(\nu_\infty)} := \int |k(x)|^2 \nu_\infty(dx) < \infty$.

We denote by $\mathcal{H} = \mathbb{R}^d$ the $d$-dimensional subspace of constant functions, equipped with $||h||_{L^2(\nu_\infty)} = |h|$. Given $I$ a closed interval of $\mathbb{R}_+$, we denote by $C(I; \mathcal{H})$ the space of continuous functions from $I$ to $\mathcal{H}$. Let $a \in C(\mathbb{R}_+; \mathcal{H})$. Consider $Y^\alpha_{t,s}$ the solution of the following linear non-homogeneous $\mathbb{R}^d$-valued SDE:

$$dY^\alpha_{t,s} = b(Y^\alpha_{t,s})dt + a_tdt + \sigma dW_t,$$

where at time $s$, the initial condition of $Y$ has law $\nu$. Note that $Y^\alpha_{t,0}$ is a solution of (2.1) provided that $a$ satisfies the following closure equation:

$$\forall t \geq 0, \quad a_t = \mathbb{E} f(Y^\alpha_{t,0}).$$

When the initial condition is taken at $s = 0$, we write $Y^\alpha_{t,0} := Y^\alpha_{t,s}$ to simplify the notation. When $a \equiv \alpha$ is constant in time, the SDE is time-homogeneous, and so it holds that $Y^\alpha_{t,s} = Y^\alpha_{t,s}$. Finally, for $y \in \mathbb{R}^d$ and a test function, we sometimes write $E_y g(Y^\alpha_{t,s}) := E_y g(Y^\alpha_{t,s})$.

2.3.1 Lions derivative for the linear process
Let $\alpha \in \mathbb{R}^d$ satisfying $\mathbb{E} f(Y^\alpha_{t,0})$. For $t \geq 0$, we let:

$$u^\alpha_t(\nu) := \mathbb{E} f(Y^\alpha_{t,0}).$$

By the Markov property, we have

$$u^\alpha_t(\nu) = \int_{\mathbb{R}^d} \mathbb{E} f(Y^\alpha_{t,s}) \nu(dy),$$

where $\mathbb{E} f(Y^\alpha_{t,s}) := \mathbb{E} f(Y^\alpha_{t,s})$: the function $\nu \mapsto u^\alpha_t(\nu)$ is linear with respect to $\nu$. The function $y \mapsto \mathbb{E} f(Y^\alpha_{t,s})$ is differentiable with a bounded derivative. It follows from Section 5.2.2 that the function $\nu \mapsto u^\alpha_t(\nu)$ is continuously Lions differentiable, with

$$\partial_\nu u^\alpha_t(\nu)(y) := \nabla_y \mathbb{E} f(Y^\alpha_{t,s}).$$

The Lions differentiability of $\nu \mapsto u^\alpha_t(\nu)$ at $\nu_\infty$ means that for $X_0 \sim \nu_\infty$ and $K$ two random variables defined on a same probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with $||K||_2 := \mathbb{E}[|K|^2] < \infty$, we have for $\nu = \mathcal{L}(X_0 + K)$:

$$u^\alpha_t(\nu) = \alpha + \mathbb{E} \partial_\nu u^\alpha_t(\nu_\infty)(X_0) \cdot K + o_t(||K||_2).$$

We used here that $u^\alpha_t(\nu_\infty) = \alpha$ and wrote $o_t(\cdot)$ to emphasize that the error depends a priori on $t$. For such $K \in L^2(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^d)$, there exists a deterministic measurable function $k : \mathbb{R}^d \to \mathbb{R}^d$ such that

$$\mathbb{P}(d\omega) \text{ p.s., } \quad \mathbb{E}(K|X_0) = k(X_0).$$

It follows from $\mathbb{E}[|K|^2] < \infty$ and from the Cauchy-Schwarz inequality that $k \in L^2(\nu_\infty)$. With this notation, it holds that

$$\mathbb{E} \partial_\nu u^\alpha_t(\nu_\infty)(X_0) \cdot K = \int_{\mathbb{R}^d} \partial_\nu u^\alpha_t(\nu_\infty)(y) \cdot k(y) \nu_\infty(dy) =: \Theta_t(k).$$

(2.12)
This serves us as the definition of \( k \mapsto \Theta_t(k) \), viewed as a linear operator from \( L^2(\nu_\infty) \) to \( \mathcal{H} \). In particular, we have
\[
\forall k \in L^2(\nu_\infty), \quad \Theta_t(k) = \lim_{\varepsilon \to 0} \frac{u_0^\varepsilon(\mathcal{L}(X_0 + \varepsilon k(X_0))) - \alpha}{\varepsilon}, \quad \text{where} \quad X_0 \sim \nu_\infty.
\]

**Remark 2.4.** Our assumption on \( \sigma \) ensures that \( \nu_\infty \) has a density, so Brenier’s theorem applies \cite[Th. 5.20]{B}: for any \( \nu \in \mathcal{P}_d(\mathbb{R}^d) \), there exists a \((-\infty,\infty] \)-valued lower semi-continuous proper convex function \( \varphi \) on \( \mathbb{R}^d \) that is almost everywhere differentiable and such that
\[
W_2(\nu, \nu_\infty)^2 = \int_{\mathbb{R}^d} |\nabla \varphi(y) - y|^2 \nu_\infty(\mathrm{d}y).
\]

So an optimal coupling between \( \nu \) and \( \nu_\infty \) is obtained by setting
\[
X_0 \sim \nu_\infty, \quad X = X_0 + k(X_0), \quad \text{with} \quad k(y) = \nabla \varphi(y) - y.
\]

In addition, it holds that \( W_2(\nu, \nu_\infty) = \|k\|_{L^2(\nu_\infty)} \).

By the Cauchy-Schwarz inequality
\[
|\mathbb{E}_{\nu} \partial u_t^0(\nu_\infty)(X_0) \cdot k(X_0)| \leq \sqrt{\mathbb{E}_{\nu} |\nabla u_t^0(\nu_\infty)(X_0)|^2}\|k\|_{L^2(\nu_\infty)}.
\]

So \( \Theta_t \) is a bounded linear operator from \( L^2(\nu_\infty) \) to \( \mathcal{H} \). In particular, the restriction of \( \Theta_t \) to \( \mathcal{H} \) can be represented by a \( d \times d \) matrix, which is precisely given by \ref{fg}

### 2.3.2 Integrated sensitivity formula

The goal of this section is to prove the following “integrated sensitivity” formula:

**Proposition 2.5.** Let \( a, h \in C(\mathbb{R}^d, \mathcal{H}) \) and \( \nu \in \mathcal{P}_d(\mathbb{R}^d) \). Let \( g \in C^2(\mathbb{R}^d) \) with \(|\nabla g||_\infty + |\nabla^2 g||_\infty < \infty \). It holds that for all \( t \geq 0 \),
\[
\mathbb{E}_\nu g(Y_{t}^{a+h,\nu}) - \mathbb{E}_\nu g(Y_{t}^{a,\nu}) = \int_0^t \int_{\mathbb{R}^d} \nabla_y g(Y_{t,\theta}^{a}) \cdot \mathcal{L} (Y_{\theta}^{a+h,\nu})(Y_{\theta}^{a+h,\nu})(d\theta)(d\theta).
\]

**Proof of Proposition 2.5**

Let \( u_{t,\nu}^g := \mathbb{E}_g(Y_{t}^{a,\nu}) \). Using \ref{2.3}, it suffices to prove that
\[
\mathbb{E}_\nu g(Y_{t}^{a+h,\nu}) - \mathbb{E}_\nu g(Y_{t}^{a,\nu}) = \int_0^t \mathbb{E}_{\nu} \partial u_t^g(\mathcal{L}(Y_{\theta}^{a+h,\nu}))(Y_{\theta}^{a+h,\nu}) \cdot h_\theta d\theta.
\]

Given \( h \in C(\mathbb{R}^d, \mathcal{H}) \), we write for all \( u \geq 0 \):
\[
h_\theta(u) := 1_{\{u \leq \theta\}} h_u. \tag{2.13}
\]

The proof of Proposition \ref{2.5} follows from

**Lemma 2.6.** The function \( \theta \mapsto \mathbb{E}_\nu g(Y_{t}^{a+h,\nu}) \) is differentiable for all \( \theta \in (0, t) \) and
\[
\frac{d}{d\theta} \mathbb{E}_\nu g(Y_{t}^{a+h,\nu}) = \mathbb{E}_{\nu} \partial u_t^g(\mathcal{L}(Y_{\theta}^{a+h,\nu}))(Y_{\theta}^{a+h,\nu}) \cdot h_\theta. \tag{2.14}
\]
Cormier

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The notations are illustrated on Figure 1. We have by the Markov property satisfied by $Y$ at time $\theta + \delta$

$$E(g(Y_{\theta+\delta}^{a+h|\theta+\delta}|)) - E(g(Y_{\theta}^{a+h|\theta}|)) = E(g(Y_{\theta}^{a+h|\theta}|)) - E(g(Y_{\theta+\delta}^{a+\mu_{\theta+\delta}^1})).$$

By definition of the Lions derivative at the point $\mu_{\theta+\delta}^1$ we have

$$E(g(Y_{t,\theta+\delta}^{a,\mu_{\theta+\delta}^1}) - E(g(Y_{t,\theta+\delta}^{a,\mu_{\theta+\delta}^1})) = E\partial_u u_{t,\theta+\delta}(\mu_{\theta+\delta}^1)(Y_{\theta+\delta}^{1} - Y_{\theta+\delta}^{0}) + o(||Y_{\theta+\delta}^{1} - Y_{\theta+\delta}^{0}||_2).$$

(2.15)

By Lemma 2.7(a) below, it holds that $o(||Y_{\theta+\delta}^{1} - Y_{\theta+\delta}^{0}||_2) = o(\delta)$ as $\delta$ goes to zero. We now approximate $Y_{t,\theta+\delta}^{1}$ and $Y_{t,\theta+\delta}^{2}$ by a one-step Euler scheme:

$$Y_{\theta+\delta}^{1} \approx \hat{Y}_{\theta+\delta}^{1} := Y_{\theta} + (b(Y_{\theta}) + a_{\theta})\delta + \sigma \cdot (W_{\theta+\delta} - W_{\theta})$$

(2.16)

$$Y_{\theta+\delta}^{2} \approx \hat{Y}_{\theta+\delta}^{2} := Y_{\theta} + (b(Y_{\theta}) + a_{\theta} + h_{\theta})\delta + \sigma \cdot (W_{\theta+\delta} - W_{\theta})$$

Note that $\hat{Y}_{\theta+\delta}^{2} - \hat{Y}_{\theta+\delta}^{1} = h_{\theta}\delta$. The one-step Euler scheme has an error in $L^2$ norm of size $o(\delta)$ (see Lemma 2.7(b) below)

$$||Y_{\theta+\delta}^{1} - \hat{Y}_{\theta+\delta}^{1}||_2 + ||Y_{\theta+\delta}^{2} - \hat{Y}_{\theta+\delta}^{2}||_2 = o(\delta),$$

so (2.15) gives

$$E(g(Y_{t,\theta+\delta}^{a,\mu_{\theta+\delta}^1}) - E(g(Y_{t,\theta+\delta}^{a,\mu_{\theta+\delta}^1})) = \delta E\partial_u u_{t,\theta+\delta}(\mu_{\theta+\delta}^1)(Y_{\theta+\delta}^{1}) \cdot h_{\theta} + o(\delta).$$
Finally, one has
\[ |E\partial_u u^{\alpha}_{t\theta,\delta}(\mu_{\theta+\delta})(Y^{1}_{\theta+\delta})\cdot h_\theta - E\partial_u u^{\alpha}_{t\theta}(\mu_\theta)(Y_\theta)\cdot h_\theta| \leq |E\partial_u u^{\alpha}_{t\theta,\delta}(\mu_{\theta+\delta})(Y^{1}_{\theta+\delta})\cdot h_\theta - E\partial_u u^{\alpha}_{t\theta}(\mu_\theta)(Y_\theta)\cdot h_\theta| + |E\partial_u u^{\alpha}_{t\theta,\delta}(\mu_\theta)(Y_\theta)\cdot h_\theta - E\partial_u u^{\alpha}_{t\theta}(\mu_\theta)(Y_\theta)\cdot h_\theta| =: A_1 + A_2. \]

Lemma 2.8(a) gives
\[ A_1 \leq C(t)E[Y^{2}_{\theta+\delta} - Y^{2}_{\theta}]|h_\theta| \leq C\sqrt{\delta} \sup_{\theta \in [0, t]} |h_\theta|. \]

Let \( \epsilon > 0 \) be fixed. Lemma 2.8(b) yields for \( \delta \) small enough:
\[ A_2 \leq \epsilon \sup_{\theta \in [0, t]} |h_\theta|. \]

Altogether, we find that
\[ E\gamma(Y^{\alpha+\delta}_{t\theta}, \nu) - E\gamma(Y^{\alpha}_{t\theta}, \nu) = \delta E\partial_u u^{\alpha}_{t\theta,\delta}(\mu_\theta)(Y_\theta)\cdot h_\theta + o(\delta). \]

This ends the proof.

We used the following classical estimates:

**Lemma 2.7.** We have, with the notations introduced in the proof of Lemma 2.8,
\[ \begin{align*}
2.7(a) & \text{ it holds that } E[Y^{2}_{\theta+\delta} - Y^{2}_{\theta}] \leq C(t) \sup_{\theta \in [0, t]} |h_\theta|^2 \delta^2. \\
2.7(b) & \text{ the Euler scheme } \text{2.10} \text{ satisfies } E[Y^{1}_{\theta+\delta} - Y^{1}_{\theta}]^2 + E[Y^{2}_{\theta+\delta} - Y^{2}_{\theta}]^2 = o(\delta^2), \text{ as } \delta \text{ goes to zero.} \\
2.7(c) & \text{ it holds that } E[Y^{1}_{\theta+\delta} - Y^{1}_{\theta}]^2 \leq C(t)\delta. \end{align*} \]

We also used the following regularity results on \( \partial_u u^{\alpha}_{t\theta}(\nu)(y) = \nabla_y E\gamma(Y^{\alpha}_{t\theta}, \nu) \). The proofs follow easily from the stochastic representation of \( y \mapsto \nabla_y E\gamma(Y^{\alpha}_{t\theta}, \nu) \); in particular this function has a bounded derivative (because \( g \) has a bounded derivative, see [28, Th. 7.18]).

**Lemma 2.8.** Let \( T > 0, a \in C([0, T], \mathcal{H}) \) be fixed. It holds that
\[ \begin{align*}
2.8(a) & \text{ the Lions derivative is Lipschitz continuous: there exists a constant } C(T) \text{ such that any square-integrable variables } Z, Z', \\
& \quad \sup_{0 \leq s \leq t \leq T} \mathbb{E} \left| \partial_u u^{\alpha}_{t\theta}(\mathcal{L}(Z))(Z) - \partial_u u^{\alpha}_{t\theta}(\mathcal{L}(Z'))(Z') \right|^2 \leq C(T)\mathbb{E}|Z - Z'|^2. \\
2.8(b) & \text{ the function } s \mapsto \partial_u u^{\alpha}_{t\theta}(\mathcal{L}(Z))(Z) \text{ is continuous: for all } \epsilon > 0 \text{ there exists } \delta > 0 \text{ such that} \\
& \quad \forall s, s' \in [0, t], \quad |s - s'| \leq \delta \implies \mathbb{E} \left| \partial_u u^{\alpha}_{t\theta}(\mathcal{L}(Z))(Z) - \partial_u u^{\alpha}_{t\theta}(\mathcal{L}(Z))(Z') \right|^2 < \epsilon. \end{align*} \]

As a first application of Proposition 2.5 we have

**Remark 2.9.** It is also possible to prove Proposition 2.5 without using Lions derivatives. Fix \( t > 0 \) and define for \( s \in (0, t) \)
\[ (s, y) \mapsto \phi(s, y) := E\gamma(Y^{\alpha}_{t\theta}). \]
Let $\mathcal{L}_t^\alpha \psi := (b + a_t) \cdot \nabla \psi + \frac{1}{2} \sum_{i,j=1}^{d} \sigma^i \sigma^j \partial_{x_i} \partial_{x_j} \psi$ be the infinitesimal generator associated to $Y^t$. It holds that $\phi \in C^{1,2}([0, t] \times \mathbb{R}^d)$ with
$$\frac{\partial}{\partial s} \phi(s, y) = -\mathcal{L}_s^\alpha \phi(s, y).$$

So, by Itô’s lemma,
$$\mathbb{E} \phi(s, Y_s^{a+h, \nu}) = \mathbb{E} \phi(0, Y_0^{a+h, \nu}) - \int_0^s \mathbb{E} \mathcal{L}_s^\alpha \phi(\theta, Y_\theta^{a+h, \nu}) d\theta + \int_0^s \mathbb{E} \mathcal{L}_s^\alpha \phi(\theta, Y_\theta^{a+h, \nu}) d\theta$$
$$= \mathbb{E} \phi(0, Y_0^{a+h, \nu}) + \int_0^s \mathbb{E} \nabla_y \phi(\theta, Y_\theta^{a+h, \nu}) \cdot h d\theta$$

We used that $\mathcal{L}_0^\alpha \psi - \mathcal{L}_0^\alpha \psi = \nabla \psi \cdot h$. Using the definition of $\phi$, we find:
$$\mathbb{E} \phi(s, Y_s^{a+h, \nu}) = \mathbb{E} g(Y_s^{a, \nu}) + \int_0^s \int_{\mathbb{R}^d} \nabla_y \mathbb{E}_y g(Y_s^{a, \nu}) \cdot h d\theta.$$  

Finally, we let $s$ converges to $t$ and find the stated formula.

Note that we obtain by choosing $g = f$ and $a = \alpha \in \mathbb{R}^d$, Proposition 2.5 gives:
$$\mathbb{E} f(Y_t^{a+h, \nu}) - \mathbb{E} f(Y_t^{a, \nu}) = \int_0^t \mathbb{E} \partial_{\nu \alpha} (\mathcal{L}(Y_\theta^{a, \nu})) (Y_\theta^{a+h, \nu}) \cdot h d\theta.$$  

Recall that $\Theta_t(h)$ is defined by 2.12. When $\nu = \nu_\infty$ and when $h$ is small, we obtain:
$$\mathbb{E} f(Y_t^{a+h, \nu_\infty}) - \alpha \approx \int_0^t \int_{\mathbb{R}^d} \nabla_y \mathbb{E}_y f(Y_\theta^{a, \nu}) \cdot h \nu_\infty d\theta = \int_0^t \Theta_{\nu \alpha} (h \nu) d\theta.$$  

More precisely, the Fréchet derivative of the function $C(\mathbb{R}_+: \mathbb{R}^d) \ni h \mapsto \mathbb{E} f(Y_t^{a+h, \nu_\infty})$ at $h = 0$ is given by a convolution between $\Theta$ and $h$. This observation is crucially used in the next section.

### 2.3.3 Lions derivative for the non-linear process

Let $x \in \mathbb{R}^d$, $t \geq 0$ be fixed. Consider now $\nu_t : \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}^d$
$$\nu \mapsto \nu_t(\nu) := \mathbb{E} f(X_t^\nu),$$  

where $(X_t^\nu)$ denotes the solution of the non-linear SDE starting at $t = 0$ with law $\nu$. We define the linear operator $\Omega_t$ from $L^2(\nu_\infty)$ to $H$ by taking the following Dyson-Phillips series
$$\Omega_t(k) := \sum_{i \geq 1} \Theta_t^{\otimes i}(k),$$  

where the linear operators $\Theta_t^{\otimes i}$ are defined recursively by
$$\forall t \geq 0, \quad \Theta_t^{\otimes (i+1)}(k) = \int_0^t \Theta_{t-s} (\Theta_{s}^{\otimes i}(k)) ds, \quad \text{and} \quad \Theta_t^{\otimes 1}(k) = \Theta_t(k).$$  

So the operators $\Omega_t$ and $\Theta_t$ satisfy the following Volterra integral equation:
$$\forall k \in L^2(\nu_\infty), \quad \Omega_t(k) = \Theta_t(k) + \int_0^t \Theta_{t-s} (\Omega_s(k)) ds$$  

$$= \Theta_t(k) + \int_0^t \Omega_{t-s} (\Theta_s(k)) ds$$  

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The series (2.18) converges uniformly on any compact \([0, T]\) for \(T > 0\). Note that for all \(t \geq 0\),
\[ \alpha \overset{\text{def}}{=} \nu_\infty(f) = v_t(\nu_\infty). \]

We have:

**Proposition 2.10.** Let \(X_0, K\) be two square-integrable random variables defined on the same probability space such that \(\mathcal{L}(X_0) = \nu_\infty\). Let \(\nu = \mathcal{L}(X_0 + K)\) and write \(\mathbb{E}(K|X_0) = k(X_0)\) for some \(k \in L^2(\nu_\infty)\). It holds that for all \(t \geq 0\):
\[ v_t(\nu) = \alpha + \Omega_t(k) + a_t(||K||_2), \]
as \(||K||_2\) goes to zero.

**Remark 2.11.** In other words, the function \(v_t\) is Lions differentiable at the point \(\nu_\infty\) and its derivative is given by \(\Omega_t\). As a direct consequence, it holds that
\[ \forall k \in L^2(\nu_\infty), \quad \Omega_t(k) = \lim_{\epsilon \to 0} \frac{\nu_t(\mathcal{L}(X_0 + \epsilon k(X_0))) - \alpha}{\epsilon}. \]

Again, the restriction of \(\Omega_t\) to \(\mathcal{H}\) can be represented by a \(d \times d\) matrix (that we also denote by \(\Omega_t\)). From (2.10), we deduce that the \(d \times d\) matrices \(\Theta_t\) and \(\Omega_t\) are linked by the following convolution Volterra integral equation (the dots represent a matrix/matrix product)
\[ \forall t \geq 0, \quad \Theta_t = \Theta_0 + \int_0^t \Theta_{t-s} \cdot \Omega_s ds = \Theta_t + \int_0^t \Omega_{t-s} \cdot \Theta_s ds. \]

**Proof of Proposition 2.10** Let \(T > 0\) and \(\nu \in \mathcal{P}_2(\mathbb{R}^d)\) be fixed. Let \(X_0, K\) be random variables such that
\[ \mathcal{L}(X_0) = \nu_\infty, \quad \mathcal{L}(X_0 + K) = \nu. \]
We write \(\mathbb{E}(K|X_0) = k(X_0)\) for some \(k \in L^2(\nu_\infty)\). It holds that \(W_2(\nu, \nu_\infty) \leq ||k||_{L^2(\nu_\infty)} \leq \sqrt{\mathbb{E}|K|^2}\). We define all \(t \in [0, T]\):
\[ h_t^* := v_t(\nu) - \alpha. \]

Note that \(h_t^* \in \mathcal{H}\) is a function of the initial condition \(\nu\). The closure equation (2.10) gives
\[ h_t^* = \mathbb{E}f(Y_t^{\alpha + h^*, \nu}) - \alpha \]
\[ = \left( \mathbb{E}f(Y_t^{\alpha + h^*, \nu}) - \mathbb{E}f(Y_t^{\alpha, \nu}) \right) + \left( \mathbb{E}f(Y_t^{\alpha, \nu}) - \alpha \right) \]
\[ =: A_1 + A_2. \]

First, by definition of \(\Theta_t\) and \(k\), we have
\[ A_2 = \Theta_t(k) + o_T(||K||_2). \]

Using Eq. (2.17), it holds that
\[ A_1 = \int_0^t \mathbb{E}\partial_k u_t(\mathcal{L}(Y_0^{\alpha + h^*, \nu})) (Y_0^{\alpha + h^*, \nu}) \cdot h_\theta d\theta. \]

Using that \(f\) is Lipschitz and Lemma (2.12) below, we have:
\[ \sup_{t \in [0, T]} |h_t^*| \leq C_T W_1(\nu, \nu_\infty) \leq C_T W_2(\nu, \nu_\infty). \]
So, using Lemma 2.8 we deduce that

\[ A_1 = \int_0^t \mathbb{E} \partial_\theta u_1^n (L(Y_\theta^{\alpha, \nu}))(Y_\theta^{\alpha, \nu}) \cdot h_\theta^* d\theta + o_T \left( \sup_{t \in [0, T]} |h_\theta^*| \right) \]

\[ = \int_0^t \Theta_{t-\theta}(h_\theta^*) d\theta + o_T \left( \sup_{t \in [0, T]} |h_\theta^*| \right). \]

Using again that \( \sup_{t \in [0, T]} |h_\theta^*| \leq C_T W_2(\nu, \nu_\infty) \leq C_T ||K||_2 \), we find that

\[ h_\theta^* = \Theta_t(k) + \int_0^t \Theta_{t-\theta}(h_\theta^*) d\theta + o_T(||K||_2). \]

That is, there exists a function \( \varepsilon_T : \mathbb{R}_+ \rightarrow \mathbb{R}_+ \) with \( \lim_{\theta \to 0} \varepsilon_T(y)/y = 0 \) such that

\[ \left| h_\theta^* - \Theta_t(k) - \int_0^t \Theta_{t-\theta}(h_\theta^*) d\theta \right| \leq \varepsilon_T(||K||_2). \]

To end the proof, it suffices to iterate this estimate: by induction, it holds that for all \( n \geq 1 \)

\[ \left| h_\theta^n - \sum_{i=1}^n \Theta_\theta^{\circ i}(k) - \int_0^t \Theta_{t-\theta}^{\circ n}(h_\theta^*) d\theta \right| \]

\[ \leq \frac{\varepsilon_T}{n!} \left[ \sup_{\theta \in [0, T]} ||\Theta_\theta|| \sup_{\theta \in [0, T]} |h_\theta^*| \right]^n. \]

The series on the right-hand side is converging. In addition, we have

\[ \left| \int_0^t \Theta_{t-\theta}^{\circ n}(h_\theta^*) d\theta \right| \leq \frac{\varepsilon_T}{n!} \left[ \sup_{\theta \in [0, T]} ||\Theta_\theta|| \sup_{\theta \in [0, T]} |h_\theta^*| \right]^n. \]

So, letting \( n \) go to infinity, we find that there exists a constant \( C_T \) such that

\[ |h_\theta^* - \Theta_t(k)| \leq C_T \varepsilon_T(||K||_2). \]

This ends the proof.

\[ \square \]

**Lemma 2.12.** Let \( T > 0 \). There exists a constant \( C_T \) such that for all \( \mu_1, \mu_2 \in \mathcal{P}_1(\mathbb{R}^d) \)

\[ \forall t \in [0, T], \quad W_1(L(X_t^{\mu_1}), L(X_t^{\mu_2})) \leq C_T W_1(\mu_1, \mu_2). \]

**Proof.** Consider \((X_t^{\mu_1}, X_t^{\mu_2})\) the solutions of (2.1) coupled with the same Brownian motion. The initial conditions \((X_0^{\mu_1}, X_0^{\mu_2})\) are chosen such that \( \mathbb{E}|X_0^{\mu_1} - X_0^{\mu_2}| = W_1(\mu_1, \mu_2) \). Let \( \mu_1 := L(X_t^{\mu_1}) \) and \( \mu_2 := L(X_t^{\mu_2}) \). From (2.1) and Assumption 2.1 we have

\[ \mathbb{E}|X_t^{\mu_1} - X_t^{\mu_2}| \leq \mathbb{E}|X_0^{\mu_1} - X_0^{\mu_2}| + \mathbb{E} \int_0^t |b(X_s^{\mu_1}) - b(X_s^{\mu_2}) + Ef(X_s^{\mu_1}) - Ef(X_s^{\mu_2})| ds. \]

The functions \( f \) and \( b \) are Lipschitz, so there exists a constant \( L \) such that

\[ \mathbb{E}|X_t^{\mu_1} - X_t^{\mu_2}| \leq \mathbb{E}|X_0^{\mu_1} - X_0^{\mu_2}| + L \int_0^t \mathbb{E}|X_s^{\mu_1} - X_s^{\mu_2}| ds. \]

By Grönwall’s inequality, we deduce that

\[ W_1(L(X_t^{\mu_1}), L(X_t^{\mu_2})) \leq \mathbb{E}|X_t^{\mu_1} - X_t^{\mu_2}| \leq e^{LT} \mathbb{E}|X_0^{\mu_1} - X_0^{\mu_2}| = e^{LT} W_1(\mu_1, \mu_2). \]

\[ \square \]
2.4 Spectral assumption

We now comment on the spectral assumption \( \mathcal{L} \) involving the Laplace transform of \( \Theta \). In view of Proposition \ref{prop:bounds} if \( \nu_\infty \) is locally stable, one expects \( \Omega_t \) to decay to zero at \( t \) goes to infinity. In addition, if the convergence to \( \nu_\infty \) occurs at an exponential rate, we expect that \( \Omega_t \) also converges to zero at the same exponential rate. We prove here that our spectral condition \( \lambda' > 0 \) is indeed equivalent to the decay of \( \Omega_t \) at rate \( e^{-\lambda't} \).

Let \( \mathcal{L} := L^2(\nu_\infty); \mathcal{H} \) be the set of bounded linear operators from \( L^2(\nu_\infty) \) to \( \mathcal{H} \), equipped with the operator norm:

\[
\forall G \in \mathcal{L}, \quad ||G|| := \sup_{||k||_{L^2(\nu_\infty)} \leq 1} |G(k)|.
\]

For \( \lambda \in \mathbb{R} \), we denote by \( L^1(\mathcal{L}) \) the following \( L^1 \) weighted space

\[
L^1(\mathcal{L}) := \{ \kappa : \mathbb{R} \to \mathcal{L}, \quad \int_0^\infty e^{\lambda t}||\kappa_t||dt < \infty \}.
\]

We equip \( L^1(\mathcal{L}) \) with \( ||\kappa||_{L^1} := \int_0^\infty e^{\lambda t}||\kappa_t||dt \). Similarly, let:

\[
L^\infty(\mathcal{L}) := \{ \kappa : \mathbb{R} \to \mathcal{L}, \quad \sup_{t \geq 0} e^{\lambda t}||\kappa_t|| < \infty \}, \quad ||\kappa||_\infty := \sup_{t \geq 0} e^{\lambda t}||\kappa_t||.
\]

Proposition 2.13. Let \( \kappa_* > 0 \) such that \( \ref{prop:bounds} \) holds. Under Assumption \ref{ass:stab}, we have

\begin{enumerate}[(a)]
\item For all \( \lambda < \kappa_* \), it holds that \( \Theta \in L^1(\mathcal{L}) \cap L^\infty(\mathcal{L}) \).
\item For \( z \in \mathbb{C} \) with \( \Re(z) > -\kappa_* \), let \( \hat{\Theta}(z) := \int_0^\infty e^{-zt}\Theta_t dt \in M_\mathbb{R}(\mathbb{C}) \) be the Laplace transform of the matrices \( \Theta_t \) \( t \geq 0 \), given by \( \ref{eq:laplace} \). Consider the abscissa of the first zero of \( z \mapsto \det(I_d - \hat{\Theta}(z)) \)

\[
\lambda' := -\sup\{ \Re(z) \mid \Re(z) > -\lambda^* \text{ and } \det(I_d - \hat{\Theta}(z)) = 0 \}.
\]

Then for all \( \lambda < \lambda' \), it holds that \( \Omega \in L^1(\mathcal{L}) \cap L^\infty(\mathcal{L}) \).
\item Conversely, assume that there exists \( \lambda \) such that \( \Omega \in L^1(\mathcal{L}) \). Then \( \lambda' \geq \lambda \).
\end{enumerate}

Proof. First, for \( k \in L^2(\nu_\infty) \), provided that \( \mathcal{L}(X_0) = \nu_\infty \), one has

\[
\Theta_t(k) = \mathbb{E}\partial_\nu u^\infty_t(\nu_\infty)(X_0) \cdot k(X_0).
\]

Applying the Cauchy-Schwarz inequality and \( \ref{prop:bounds} \), we deduce that there exists \( C > 0 \):

\[
||\partial_\nu u^\infty_t(\nu_\infty)(X_0) \cdot k(X_0)|| \leq C||k||_{L^2(\nu_\infty)} e^{-\kappa_* t}.
\]

This proves \( \ref{prop:bounds} \). Second, let \( \lambda < \kappa_* \). We apply \cite[Ch. 2, Th. 4.1]{ref1} with the matrices \( r(t) = e^{\lambda t}\Omega_t \) and \( k(t) = -e^{\lambda t}\Theta_t \). By assumption, we have \( \det(I_d + \hat{k}(z)) \neq 0 \) for \( \Re(z) \geq 0 \), and so \( r \in L^1(\mathbb{R}^+; M_\mathbb{R}(\mathbb{R})) \). This shows that \( t \mapsto e^{\lambda t}\Omega_t \in L^1(\mathbb{R}^+; M_\mathbb{R}(\mathbb{R})) \). Let \( k \in L^2(\nu_\infty) \).

There exists a constant \( C > 0 \) such that for all \( t \geq 0 \),

\[
\Theta_t(k) \in \mathcal{H} \quad \text{and} \quad ||\Theta_t(k)|| \leq Ce^{-\kappa_* t||k||_{L^2(\nu_\infty)}}.
\]

Using that \( \Theta_k(k) \in \mathcal{H} \) and \( \ref{prop:bounds} \), we find that

\[
||\Omega_t(k)|| \leq Ce^{-\kappa_* t||k||_{L^2(\nu_\infty)}} + C||k||_{L^2(\nu_\infty)} \int_0^t e^{-\kappa_* s||\Omega_t(s)||} ds.
\]
Finally, Fubini yields $\int_0^\infty e^{\lambda t} \int_0^t e^{-\lambda s} |\Omega_{t-s}| ds dt = \int_0^\infty e^{-(\lambda - \nu_t) s} \int_0^\infty |\Omega_u| e^{\lambda u} du ds < \infty$.

So, $\Omega \in L^1_\infty(\mathcal{L})$. Similarly, $\Omega \in L^\infty(\mathcal{L})$. This shows [2.13(b)]

Third, by assumption, we have $t \mapsto \Omega e^{\lambda t} \in L^1(\mathbb{R}^d ; M_d(\mathbb{R}))$, for some $\lambda \in \mathbb{R}$. Let $z \in \mathbb{C}$ with $\Re(z) > -\lambda$. From (2.10), we have

$$e^{-zt} \Omega_t = e^{-zt} \Theta_t + \int_0^t e^{-z(t-s)} \Theta_{t-s} \cdot e^{-zs} ds.$$ Integrating from $t = 0$ to $\infty$, we find $\Omega(z) = \Theta(z) + \hat{\Theta}(z) \cdot \hat{\Theta}(z)$. That is, we have

$$(I_d - \hat{\Theta}(z))(I_d + \hat{\Theta}(z)) = I_d,$$

and so $\det(I_d - \hat{\Theta}(z)) \det(I_d + \hat{\Theta}(z)) = 1$. We deduce that

$$\Re(z) > -\lambda \implies \det(I_d - \hat{\Theta}(z)) \neq 0.$$ So $\lambda' \geq \lambda$: this ends the proof of [2.13(c)].

\[ \square \]

### 2.5 Proof of Theorem 2.2

Control of the non-linear interactions

For all $t \geq 0$ and $\nu \in \mathcal{P}(\mathbb{R}^d)$, we define

$$\phi^\nu_t = \mathbb{E}(Y^\nu_t) - \alpha.$$ Recall that $(X^\nu_t)$ denotes the solution of the McKean-Vlasov equation (2.1). We have

**Proposition 2.14.** For all $T > 0$, there is a constant $C_T$ such that for all $t \in [0, T]$ and all $\nu \in \mathcal{P}(\mathbb{R}^d)$:

$$\left| \mathbb{E}(X^\nu_t) - \alpha - \phi_t^\nu - \int_0^t \Omega_{t-s} (\phi_s^\nu) ds \right| \leq C_T(W_1(\nu, \nu_\infty))^2.$$ 

**Proof.** The argument is similar to the proof of Proposition 2.10. Let $h^\nu_t := \mathbb{E}(X^\nu_t) - \alpha$. We write

$$h^\nu_t = \mathbb{E}(Y_t^{\alpha + h^\nu_t, \nu}) - \alpha = \mathbb{E}(Y_t^{\alpha + h^\nu_t, \nu}) - \mathbb{E}(Y_t^{\alpha, \nu}) + \phi^\nu_t.$$ By Proposition 2.10 it holds that

$$\mathbb{E}(Y_t^{\alpha + h^\nu_t, \nu}) - \mathbb{E}(Y_t^{\alpha, \nu}) = \int_0^t \int_{\mathbb{R}^d} \nabla_y \mathbb{E}_y f(Y_{t-s}^\alpha) \cdot h^\nu_s \mathcal{L}(Y_s^{\alpha + h^\nu_t, \nu})(dy) ds$$

$$= \int_0^t \Theta_{t-s} (h^\nu_s) ds + R_t,$$

where

$$R_t = \int_0^t \int_{\mathbb{R}^d} \nabla_y \mathbb{E}_y f(Y_{t-s}^\alpha) \cdot h^\nu_s \mathcal{L}(Y_s^{\alpha + h^\nu_t, \nu} - \nu_\infty)(dy) ds.$$ We let $G_t(y) = \nabla_y \mathbb{E}_y f(Y_t^\alpha)$, such that

$$R_t = \int_0^t (\mathbb{E}G_{t-s}(X^\nu_s) - G_{t-s}(\nu_\infty)) \cdot h^\nu_s d\theta.$$
Because $f \in C^2(\mathbb{R}^d)$ with $||\nabla^2 f||_{\infty} + ||\nabla f||_{\infty} < \infty$, there exists a constant $C_T$ such that for all $t \in [0, T]$, $|\nabla_y G_t(y)| \leq C_T$. By Lemma 2.12, we have the apriori estimate $W_1(L(X_t^\nu), L(X_T^\nu)) \leq C_T W_1(\nu, \nu_\infty)$. Therefore, we deduce that $|h_t^\nu| \leq C_T W_1(\nu, \nu_\infty)$ and that $|EG_t\nu(X_t^\nu) - G_t\nu(\nu_\infty)| \leq C_T W_1(\nu, \nu_\infty)$. So

$$\forall t \in [0, T], \quad |R_t| \leq C_T(W_1(\nu, \nu_\infty))^2.$$

Therefore, we have for another constant $C_T$:

$$\forall t \in [0, T], \quad |h_t^\nu - \varphi_t^\nu - \int_0^t \Theta_{t-s}(h_s^\nu)ds| \leq C_T(W_1(\nu, \nu_\infty))^2.$$

By iterating this estimate as in the proof of Proposition 2.10, we obtain the result. $\square$

**Control of the Wasserstein distance**

**Proposition 2.15.** There is a constant $C > 0$ such that for all $t \geq 0$ and $\nu \in C([0, t]; \mathcal{H})$, it holds that

$$W_1(L(Y_t^{\alpha+h,\nu}), L(Y_t^{\alpha,\nu})) \leq C \int_0^t e^{-\kappa_s(t-\theta)}|h_\theta|d\theta.$$

**Proof.** The proof uses a similar coupling argument that in Proposition 2.5. We consider

$$\forall \theta \in (0, t), \quad G_t(\theta) := W_1(L(Y_t^{\alpha+h|\theta}), L(Y_t^{\alpha,\nu}),$$

where $h_{|\theta}$ is given by (2.13). As in the proof of Proposition 2.5 let $\mu_{\theta+\delta}^2 := L(Y_{t,\theta+\delta}^{\nu+h})$ and $\mu_{\theta+\delta} := L(Y_{t,\theta+\delta}^{\alpha+h|\theta})$ (the notations are summarized in Figure 1). By the triangular inequality satisfied by $W_1$, we have

$$|G_t(\theta + \delta) - G_t(\theta)| \leq W_1(L(Y_t^{\alpha+h|\theta+\delta}), L(Y_t^{\alpha+h|\theta}))$$

$$= W_1(L(Y_{t,\theta+\delta}^{\alpha+h|\theta}), L(Y_{t,\theta+\delta}^{\alpha+h|\theta}))$$

$$\leq Ce^{-\kappa_s(t-(\theta+\delta)})W_1(\mu_{\theta+\delta}^2, \mu_{\theta+\delta}).$$

We used (2.3) to obtain the last inequality. By Grönwall’s inequality, there exists a constant $C$ such that for all $\delta < 1$:

$$W_1(\mu_{\theta+\delta}^2, \mu_{\theta+\delta}) \leq E|Y_{\theta+\delta,\theta}^{\alpha+h} - Y_{\theta+\delta,\theta}^{\alpha}| \leq C \int_\theta^{\theta+\delta} |h_u|du.$$

Therefore, we deduce that

$$\forall \theta \in (0, t), \quad \limsup_{\delta \downarrow 0} \frac{1}{\delta} |G_t(\theta + \delta) - G_t(\theta)| \leq Ce^{-\kappa_s(t-\theta)}|h_\theta|,$$

Using that $\theta \mapsto h_\theta$ is uniformly continuous on $[0, t]$, we deduce that the inequality above is uniform with respect to $\theta$, and so we have $G_t(\theta) \leq \int_0^t e^{-\kappa_s(t-\theta)}|h_\theta|d\theta$, as stated. $\square$
Proof of Theorem 2.2

Combining the two results above as well as Proposition 2.13(b), we obtain

**Lemma 2.16.** Let $\lambda \in (0, \lambda')$ such that $\Omega \in L^1(\mathcal{L})$. There exists a constant $C_\lambda$ such that for all $T > 0$, there is a constant $C_T$ such that for all $\nu \in P(\mathbb{R}^d)$ and for all $t \in [0, T]$: \[ W_1(\mathcal{L}(X^\nu_t), \nu) \leq C_\lambda e^{-\lambda t} W_1(\nu, \nu_T) + C_T W_1(\nu, \nu_T)^2. \]

Importantly, the constant $C_\lambda$ above does not depend on $T$. We deduce the proof of Theorem 2.2 by following the argument of [7, Proposition 5.2].

**Proof of Theorem 2.2.** We choose $T$ large enough such that $C_\lambda e^{-\lambda T} \leq \frac{1}{4}$. We choose $\epsilon > 0$ small enough such that \[ W_1(\nu, \nu_T) \leq \epsilon \implies C_T W_1(\nu, \nu_T)^2 \leq \frac{1}{4} W_1(\nu, \nu_T). \]

Therefore we have, by induction, provided that $\alpha$.

\[ W_1(\mathcal{L}(X^\nu_{kT}), \nu) \leq (1/2)^k W_1(\nu, \nu_T). \]

We write $t = kT + s$ for some $s \in [0, T)$. Using Lemma 2.12 there exists a constant $C$ such that \[ W_1(\mathcal{L}(X^\nu_t), \nu) \leq (1/2)^k W_1(\nu, \nu_T) \leq C e^{-\lambda T} W_1(\nu, \nu_T), \]

where $c := \frac{\log(2)}{T}$. This ends the proof of Theorem 2.2.

\[ \Box \]

2.6 Bifurcation analysis

Our result also provides some information on the number of the invariant probability measures of 2.1. For $\alpha \in \mathbb{R}^d$, we denote by $\nu^\alpha_T$ the unique invariant probability measure of $(Y^\alpha_t)$, solution of 2.3. Recall that $f_\alpha(t)$ is given by 2.2.

**Proposition 2.17.** The function $\alpha \mapsto \nu^\alpha_T(f)$ is differentiable with \[ \nabla \alpha \nu^\alpha_T(f) = \int_0^\infty f_\alpha(t)dt = \tilde{\Theta}_\alpha(0). \]

**Proof.** We have for all $T \geq 0$,

\[ \nu^\alpha_T(f) = \left[ E f(Y^\alpha_T) - E f(Y^\alpha_T) \right] + \left[ E f(Y^\alpha_T) - E f(Y^\alpha_T) \right] \]

\[ = A + B. \]

We have $|A| \leq C e^{-c_1 T} W_1(\nu^\alpha_T, \nu^\alpha_T)$, and this term can be made arbitrarily small by choosing $T$ sufficiently large. In addition, using Proposition 2.5 we have

\[ B = \int_0^T \int_{\mathbb{R}^d} \nabla \alpha \tilde{\Theta}_\alpha(t) \cdot \epsilon \mathcal{L}(\Theta^\alpha_T(\nu^\alpha_T))(dy)dt. \]

It follows that $|B| \leq C|\epsilon|$. Letting $T \to \infty$ proves that $\alpha \mapsto \nu^\alpha_T(f)$ is Lipschitz continuous.

A refinement of the previous argument shows that the function $\alpha \mapsto \nu^\alpha_T(f)$ is $C^1$ with the stated derivative. Define $G_1(y) := \nabla \| \|_\epsilon f(Y^\alpha_T)$. We have $B = \int_0^T \int_{\mathbb{R}^d} \epsilon G_{T-\theta}(Y^\alpha_T(\nu^\alpha_T)) \cdot dyd\theta$ and, by Girsanov’s theorem, provided that $|\epsilon|^2 T < 1$, we have:

\[ \left| \epsilon \nabla \rho_{T-\theta}(Y^\alpha_T(\nu^\alpha_T)) - \epsilon \nabla \rho_{T-\theta}(Y^\alpha_T(\nu^\alpha_T)) \right| \leq C e^{-c_1(T-\theta)} |\epsilon| \sqrt{T}. \]
Overall there exists a constant $C$ such that
\[
\left| \nu_{\alpha}^{n+1}(f) - \nu_{\alpha}^{n}(f) - \int_{0}^{\infty} \int_{\mathbb{R}^{d}} \nabla_y E_y f(Y_{\alpha}^{n}(y)) \cdot \nu_{\alpha}^{n}(dy) du \right| \leq C|\epsilon^{-n}\nu + |\epsilon|\sqrt{T}|.
\]
We choose $T = |\epsilon|^{-1/2}$ and let $|\epsilon|$ goes to zero: the right-hand term is a $o(|\epsilon|)$ and so $\alpha \rightarrow \nu_{\alpha}^{\infty}(f)$ is differentiable.

**Corollary 2.18.** Assume that $f$ is bounded. Then (2.1) has at least one invariant probability measure.

**Proof.** Let $K := \{\alpha \in \mathbb{R}^{d}, |\alpha| \leq ||f||_{\infty}\}$ and consider the function $F : K \rightarrow K$ defined by $F(\alpha) := \nu_{\alpha}^{\infty}(f)$.

By Proposition 2.17, $\Psi$ is continuous. So the Brouwer fixed point theorem implies the existence of an $\alpha \in K$ such that (2.2) holds with $\nu_{\infty} = \nu_{\alpha}^{\infty}$. So $\nu_{\alpha}^{\infty}$ is an invariant probability measure of (2.1). \hfill $\Box$

## 3 McKean-Vlasov of convolution type on the torus

Let $\beta > 0$. We consider the following McKean-Vlasov equation on the torus $T^{d} := (\mathbb{R}/2\pi\mathbb{Z})^{d}$:
\[
dx_{\tau} = -\int \nabla W(X_{\tau} - y)\mathcal{L}(X_{\tau})(dy) dt + \sqrt{2\beta^{-1}}dB_{t},
\]
with initial condition $\mathcal{L}(X_{0}^{\tau}) = \nu \in \mathcal{P}(T^{d})$. Here $(B_{t})$ is a Brownian motion on $T^{d}$. This equation generalizes the Kuramoto model $[1, 6, 36, 27]$, for which $d = 1$ and $W = -\kappa \cos$ for some constant $\kappa \geq 0$. We refer to [12] for a detailed presentation of the model as well as a study of the bifurcations of (3.1). We study the local stability of the uniform probability measure, using the same strategy than in Section 2.

### 3.1 Main result

We write the interaction kernel $W$ in Fourier:
\[
W(x) = \sum_{n \in \mathbb{Z}^{d}} \tilde{W}(n)e^{i n \cdot x}, \quad x \in T^{d},
\]
where $n \cdot x = \sum_{i=1}^{d} n_{i}x_{i}$. We write $|n|^{2} = n \cdot n$. The Fourier coefficients of $W$ are given by
\[
\tilde{W}(n) = \frac{1}{(2\pi)^{d}} \int_{\mathbb{R}^{d}} W(y)e^{-i n \cdot y} dy, \quad n \in \mathbb{Z}^{d}.
\]
We assume that $W \in C^{3}(T^{d})$ and that $\sum_{n \in \mathbb{Z}^{d}} |n|^{2} |\tilde{W}(n)| < \infty$. The uniform probability measure
\[
U(dx) := \frac{dx}{(2\pi)^{d}}
\]
is an invariant probability measure of (3.1) and

**Theorem 3.1.** Assume that
\[
\lambda' := \inf_{n \in \mathbb{Z}^{d}(0)} |n|^{2} \left(\beta^{-1} + \Re(\tilde{W}(n))\right) > 0.
\]
Then $U(dx) = \frac{dx}{(2\pi)^{d}}$ is locally stable: there exists $\lambda \in (0, \lambda')$, $\epsilon > 0$ and $C > 1$ such that for all $\nu \in \mathcal{P}(T^{d})$ with $W_{1}(\nu, U) < \epsilon$, it holds that:
\[
\forall t \geq 0, \quad W_{1}(\mathcal{L}(X_{\tau}^{\nu}), U) \leq CW_{1}(\nu, U)e^{-\lambda t}.
\]

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Remark 3.2. When $W$ is even, [12] studies the existence of bifurcations of the invariant probability measures of (3.1), provided that there exists $n \in \mathbb{Z}^d \setminus \{0\}$ such that $\beta^{-1} + R\bar{W}(n) = 0$. The criterion (3.3) is sharp: we prove that the uniform measure is stable up to the first bifurcation. Note that we do not require here $W$ to be even.

3.2 Proof

To simplify the notations, we first assume that $d = 1$. We discuss the case $d > 1$ afterwards, most of the arguments being the same. The proof is divided into the following steps. We write $\sigma := \sqrt{2\beta^{-1}}$.

Step 1. Because $\nabla W$ is Lipschitz, the equation (3.1) has a unique path-wise solution satisfying the following apriori estimate:

$$\forall T > 0, \exists C_T: \forall \nu, \mu \in \mathcal{P}(T), \sup_{t \in [0, T]} W_t(\mathcal{L}(X^\nu_t), \mathcal{L}(X^\mu_t)) \leq C_T W_0(\nu, \mu).$$

Step 2. We define for $\nu \in \mathcal{P}(T)$, $x \in \mathbb{T}$ and $t \geq 0$:

$$h_t^\nu(x) := -E\nabla W (x - X^\nu_t).$$

Recall that $U(dx) = \frac{dx}{\pi}$. Because $h_t^\nu \equiv 0$, we have, by Step 1:

$$||h_t^\nu||_\infty = \sup_{x \in \mathbb{T}} |h_t^\nu(x) - h_t^\nu(x)| \leq C_T ||\nabla^2 W||_\infty W_0(\nu, U).$$

In addition, $x \mapsto h_t^\nu(x)$ is differentiable and:

$$||\nabla h_t^\nu||_\infty = \sup_{x \in \mathbb{T}} |\nabla h_t^\nu(x) - \nabla h_t^\nu(x)| \leq C_T ||\nabla^3 W||_\infty W_0(\nu, U).$$

Step 3. We now use that there exists $C > 1$ and $\kappa_* > 0$ such that

$$\forall x, y \in \mathbb{T}, \forall t \geq 0, \quad W_0(\mathcal{L}(x + \sigma B_t), \mathcal{L}(y + \sigma B_t)) \leq Ce^{-\kappa_* t} |x - y|.$$ 

We refer to [34], Prop. 4. We define for all $t \geq 0$, $x \in \mathbb{T}$ and $\nu \in \mathcal{P}(\mathbb{T})$:

$$\phi_t^\nu(x) := -E\nabla W (x - X_0^\nu - \sigma B_t),$$

where $X_0^\nu$ is independent of $B_t$ and has law $\nu$. Because $||\nabla^2 W||_\infty < \infty$, by the preceding result, there exists a constant $C > 0$ such that:

$$||\phi_t^\nu||_\infty \leq Ce^{-\kappa_* t} W_0(\nu, U).$$

Step 4. We let $\mathcal{H} := L^\infty(\mathbb{T}; \mathbb{T})$. For $h \in C(\mathbb{R}_+; \mathcal{H})$ and $\nu \in \mathcal{P}(\mathbb{T})$, we consider $(Y_t^{h, \nu})$ the solution of the following linear non-homogeneous SDE:

$$dY_t^{h, \nu} = ht(Y_t^{h, \nu})dt + \sigma dB_t,$$

starting with $\mathcal{L}(Y_0^{h, \nu}) = \nu$. Let $g \in C^2(\mathbb{T})$. The integrated sensibility formula of Section 2.3.2 writes, in this context:

$$Eg(Y_t^{h, \nu}) - Eg(Y_t^{0, \nu}) = \int_0^t \int_{\mathbb{T}} \nabla \phi_t g(y + \sigma B_t, \nu) \cdot h_t(y) \mathcal{L}(Y_t^{h, \nu})(dy) \, dt.$$
Step 5. We let for $k \in L^2(\mathbb{T})$ and $x \in \mathbb{T}$:

$$\Theta_t(k)(x):= -\int_\mathbb{T} \nabla_y \mathbb{E}_y \nabla W(x - y - \sigma B_t) \cdot h(y) \frac{dy}{2\pi}.$$  

Using that $\mathbb{E}e^{i \sigma B_t} = e^{-\frac{\sigma^2 t}{2}} = e^{-\frac{n^2 t}{\beta}}$, we find that the Fourier series of $\Theta_t(k)(x)$ is:

$$\Theta_t(k)(x) = -\sum_{n \in \mathbb{Z}} n^2 \hat{W}(n) \hat{k}(n) e^{-\frac{n^2 t}{\beta}} e^{inx}.$$  

So $\Theta_t$ is diagonal in the Fourier basis $(e^{inx})_{n \in \mathbb{Z}}$ and $\hat{\Theta_t}(k)(n) = -n^2 \hat{W}(n) e^{-\frac{n^2 t}{\beta}} \hat{k}(n)$. In addition, we have:

$$||\Theta_t(k)||_\infty \leq C_0 e^{-t/\beta} ||k||_\infty,$$

where $C_0 := \sum_{n \in \mathbb{Z}} n^2 |\hat{W}(n)| < \infty$.

Step 6. We then define $\Omega_t(k)$ to be the unique solution of the Volterra integral equation:

$$\forall t \geq 0, \quad \Omega_t(k) = \Theta_t(k) + \int_0^t \Theta_{t-s}(\Omega_s(k)).$$  

Again, $\Omega_t$ is diagonal in the Fourier basis:

$$\Omega_t(k)(x) = -\sum_{n \in \mathbb{Z}} n^2 \hat{W}(n) \exp \left(-n^2 t \left[\beta^{-1} + \hat{W}(n)\right]\right) \hat{k}(n) e^{inx}.$$  

Let $\lambda'$ be given by (3.3). We have:

$$||\Omega_t(k)||_\infty \leq C_0 e^{-\lambda' t} ||k||_\infty.$$  

So, under the condition $\lambda' > 0$, $(\Omega_t)$ decays at an exponential rate toward zero.

Step 7. Let $x \in \mathbb{T}$ be fixed. We now apply Step 4 with $g(y) := -\nabla W(x - y)$, and with $h_t(y) := h_t^x(y)$, where $h_t^x$ is defined in Step 2. Note that with this choice, $Y_t^{h,x} = X_t^x$ and so $\mathbb{E}g(Y_t^{h,x}) = h_t^x(x)$. Similarly, $\mathbb{E}g(Y_t^{d,x}) = \phi_t^x(x)$, where $\phi_t^x(x)$ is defined in Step 3. Therefore, we have:

$$h_t^x(x) - \phi_t^x(x) = \int_0^t \int_\mathbb{T} \nabla^2 W(x - y - \sigma B_{t-s}) \cdot h_s^x(y) \mathcal{L}(X_s^x)(dy) d\theta$$

$$= \int_0^t \Theta_{t-s}(h_s^x)(x) d\theta + R_t(x),$$

where

$$R_t(x) := \int_0^t \mathbb{E} \left[G_{t,s}^x(X_s^x) - G_{t,s}^x(X_{t-s}^x)\right] d\theta,$$

$$G_{t,s}^x(y) := \mathbb{E} \nabla^2 W(x - y - \sigma B_{t-s}) \cdot h_s^x(y).$$

Using the apriori estimates of Step 2, we deduce that there exists a constant $C_T$ such that for all $0 \leq \theta \leq t \leq T$:

$$||\nabla_y G_{t,s}^x(y)|| \leq C_T W_1(\nu, U).$$

Using Step 1, we conclude that $R_t(x) \leq C_T(W_1(\nu, U))^2$. To summarize, we have proven that for all $T > 0$, there exists a constant $C_T$ such that for all $\nu \in \mathcal{P}(\mathbb{T})$ and for all $t \in [0, T]$:

$$\left| h_t^x(x) - \phi_t^x(x) - \int_0^t \Theta_{t-s}(h_s^x)(x) d\theta \right| \leq C_T(W_1(\nu, U))^2.$$
**Step 8.** By iterating the last inequality of Step 7, we obtain that for all $T > 0$, there exists a constant $C_T$ such that
\[ \left| h_t^\nu(x) - \phi_t^\nu(x) - \int_0^t \Omega_t - \theta(\phi_s^\nu)(x) d\theta \right| \leq C_T (W_t(\nu, U))^2. \]

**Step 9.** We prove that there exists a constant $C > 0$ such that for all $t > 0$, for all $\nu \in \mathcal{P}(T)$, it holds that
\[ W_t(\mathcal{L}(Y_t^{h,\nu}), \mathcal{L}(Y_t^{0,\nu})) \leq C \int_0^t e^{-\kappa_s(t-\theta)} \|h_\theta\|_\infty d\theta. \]

The proof is similar to the proof of Proposition 2.13, it uses the estimate of Step 3.

**Step 10.** We fix $\lambda \in (0, \min(\kappa_*, \lambda'))$. Using Step 8, Step 6, and Step 3, we deduce that there exists $C_\lambda > 0$ such that for all $T > 0$, there is a constant $C_T$ such that for all $t \in [0, T]$ and $\nu \in \mathcal{P}(T)$:
\[ \|h_t^\nu\|_\infty \leq C_T (W_t(\nu, U))^2 + C_{\lambda} W_t(\nu, \mu) e^{-\lambda t}. \]

Let $h_t(x) := h_t^\nu(x)$. Using that $X_t^\nu = Y_t^{h,\nu}$, we have:
\[ W_t(\mathcal{L}(X_t^\nu), U) \leq W_t(\mathcal{L}(Y_t^{h,\nu}), \mathcal{L}(Y_t^{0,\nu})) + W_t(\mathcal{L}(Y_t^{0,\nu}), U). \]

By Step 3, we have
\[ W_t(\mathcal{L}(Y_t^{0,\nu}), U) \leq C e^{-\kappa_\lambda t} W_t(\nu, \mu). \]

By Step 9, we have
\[ W_t(\mathcal{L}(Y_t^{h,\nu}), \mathcal{L}(Y_t^{0,\nu})) \leq C \int_0^t e^{-\kappa_\lambda (t-\theta)} \|h_\theta\|_\infty d\theta. \]

Altogether, we deduce that there is a constant $C_\lambda$ such that for all $T > 0$, there exists $C_T > 0$ such that for all $t \in [0, T]$, for all $\nu \in \mathcal{P}(T)$, we have:
\[ W_t(\mathcal{L}(X_t^\nu), U) \leq C_{\lambda} W_t(\nu, U) e^{-\lambda t} + C_T (W_t(\nu, U))^2. \]

The proof of Theorem 3.1 is deduced from this estimate, exactly as we did at the end of Section 2.5. This ends the proof for $d = 1$.

The case $d > 1$ is similar; the only differences are in the expressions of $\Theta_t$ and $\Omega_t$ of Steps 5 and 6. Given $n \in \mathbb{Z}^d$, we denote by $P(n)$ the $d \times d$ matrix defined by $P_i(n) = \langle n, n_j \rangle_{i,j \in \{1, \ldots, d\}}$. We find that for all $k \in L^2(\mathbb{T}^d ; \mathbb{P}^d)$ and for all $x \in \mathbb{T}^d$,
\[ \Theta_t(k)(x) = - \sum_{n \in \mathbb{Z}^d} e^{i n : x} \tilde{W}(n) e^{-\frac{|n|^2}{2}} P(n) \hat{k}(n), \]
and
\[ \Omega_t(k)(x) = - \sum_{n \in \mathbb{Z}^d} e^{i n : x} \tilde{W}(n) e^{-\frac{|n|^2}{2}} P(n) e^{-i \tilde{W}(n) P(n)} \hat{k}(n). \]

The eigenvalues of $P(n)$ are $|n|^2$ (of order 1) and zero (of order $d - 1$). In addition, it holds that for $\theta \in \mathbb{R}$,
\[ (e^{i \theta P(n)})_{i,j} = \delta_{i,j} + \frac{|n_i n_j|}{|n|^2} (e^{i \theta |n|^2} - 1). \]

Therefore, the estimates of Steps 5 and 6 still hold in dimension $d > 1$. This ends the proof.
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