Chaining Mutual Information and Tightening Generalization Bounds

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Abstract

Bounding the generalization error of learning algorithms has a long history, that yet falls short in explaining various generalization successes including those of deep learning. Two important difficulties are (i) exploiting the dependencies between the hypotheses, (ii) exploiting the dependence between the algorithm’s input and output. Progress on the first point was made with the chaining method, originating from the work of Kolmogorov and used in the VC-dimension bound. More recently, progress on the second point was made with the mutual information method by Russo and Zou ’15. Yet, these two methods are currently disjoint. In this paper, we introduce a technique to combine chaining and mutual information methods, to obtain a generalization bound that is both algorithm-dependent and that exploits the dependencies between the hypotheses. We provide an example in which our bound significantly outperforms both the chaining and the mutual information bounds. As a corollary, we tighten Dudley inequality under the knowledge that a learning algorithm chooses its output from a small subset of hypotheses with high probability; an assumption motivated by the performance of SGD discussed in Zhang et al. ’17.

1 Introduction

Understanding the generalization phenomenon in machine learning has been a central question for many years, revived in the recent years with the success and mystery of deep learning: why do neural nets generalize well, although they operate in a classically overparametrized setting? In particular, classical generalization bounds do not explain this phenomenon. Even simpler instances of successful machine learning problems and algorithms are not properly explained with current generalization bounds, e.g. [1]. This paper aims at deriving tighter generalization bounds for learning algorithms by combining ideas from information theory and from high dimensional probability.

Generalization bounds have evolved throughout the years, starting from the basic union bound over the hypothesis set, the refined union bound, VC-dimension and Rademacher complexity [2]; and algorithm dependent bounds such as PAC-Bayesian bounds [3], uniform
stability [4], compression bounds [5], and, recently, and most related to our work, the mutual information bound [6].

We highlight some of the key limitations of current bounds with two pitfalls:

A. Ignoring the dependencies between the hypotheses. Consider the following example (which we refer to as Example I): an algorithm observes \( G^2 = (G_1, G_2) \), where \( G_1 \) and \( G_2 \) are two independent standard normal random variables; the hypothesis set \( H = \{ h_t : t \in T \} \) consists of functions \( h_t(G^2) = \langle t, G^2 \rangle \), where \( T = \{ t \in \mathbb{R}^2 : \| t \|_2 = 1 \} \). Suppose the algorithm is designed to choose the hypothesis which achieves \( \max_{t \in T} h_t(G^2) \).

It is clear that \( h_t(G^2), t \in T \) are all zero mean random variables, therefore the expected generalization error is \( \mathbb{E}[\max_{t \in T} h_t(G^2)] \). Since \( H \) consists of infinite number of hypotheses, the union bound (or equivalently the maximal inequality) over the hypothesis set is doomed to failure. However, the fact is that we are not dealing with infinite number of independent random variables: the random variables \( h_t(G^2) \) and \( h_s(G^2) \) are actually quite dependent on each other when \( t \) and \( s \) are close (say, in Euclidean distance). Note that PAC-Bayesian bounds, compression bounds and bounds based on uniform stability also do not exploit the dependencies between the hypotheses as they are not based on any metric on the hypothesis set.

To exploit the dependencies, the powerful technique of chaining has been developed in high dimensional probability in order to obtain uniform bounds on random processes, and has proven successful in a variety of problems including statistical learning. More specifically, chaining is the method for proving the tightest generalization bound using VC-dimension [7], [8]. Originating from the work of Kolmogorov in 1934 (see [7, p. 149]) and later developed by Dudley, Fernique, Talagrand and many others [9], the basic idea of chaining is to first describe the dependencies between the hypotheses by a metric \( d \) on the set \( T \), then to discretize \( T \) and to approximate the maximal value (\( \max_{t \in T} h_t(G^2) \)) by approximating the maxima over successively refined finite discretizations, using union bounds in each step, and by introducing the notion of \( \epsilon \)-nets and covering numbers [10]. For instance, with this method, one can prove the finite upper bound \( \mathbb{E}[\max_{t \in T} h_t(G^2)] \leq 19.0353 \). Even for many examples of finite hypothesis sets, chaining is known to give far tighter bounds than the union bound [7]. This method of refined discretizations has also shown its strength in proving the celebrated law of the iterated logarithm [11], and in studying random tournaments in combinatorics [12], among others. Here we state a fundamental result which is based on the chaining method.

For a metric space \( (T, d) \), let \( N(T, d, \epsilon) \) denote the covering number of \( (T, d) \) at scale \( \epsilon \). For the definitions of \( \epsilon \)-net and covering number, see Definition 8 in subsection 5.3, and for the definition of separable subgaussian processes see Definitions 1 and 2.

**Theorem 1** (Dudley). Assume that \( \{ X_t \}_{t \in T} \) is a separable subgaussian process on the bounded metric space \( (T, d) \). Then

\[
\mathbb{E}[\sup_{t \in T} X_t] \leq 6 \sum_{k \in \mathbb{Z}} 2^{-k} \sqrt{\log N(T, d, 2^{-k})}.
\]  (1)

B. Ignoring the dependence between the algorithm input (data) and output. Generalization bounds based on Rademacher complexity and VC-dimension only depend on the hypothesis set and not on the algorithm, effectively rendering them too pessimistic. Recent experimental findings in [13] have shown that in the over-parameterized regime of deep neural nets, such complexity measures give vacuous bounds for the generalization
error. A possible explanation for that failure is as follows: if \( \mathcal{H} = \{ h_t : t \in T \} \) denotes the hypothesis set and for every \( t \in T, X_t \) denotes the generalization error of hypothesis \( h_t \) and \( W \) denotes the index of the chosen hypothesis by the algorithm, then to upper bound the expected generalization error \( \mathbb{E}[X_W] \), one uses

\[
\mathbb{E}[X_W] \leq \mathbb{E}[\sup_{t \in T} X_t],
\]

and aims at upper bounding \( \mathbb{E}[\sup_{t \in T} X_t] \) with these bounds, hence giving a uniform bound over the generalization errors of the entire hypothesis set. That is while all we need to control is the generalization error of the specific hypothesis \( W \) which the algorithm picks as its output which can be much smaller, i.e. inequality (2) can be loose (see also [15]). In other words, such bounds are not taking into account the input-output relation of the algorithm, and uniform bounding seems to be too stringent for this application. Consider the following example (which we refer to as Example II): let \( X_1, X_2, ..., X_n \) be standard normal random variables and assume that the algorithm output is index \( W \). Therefore the expected generalization error is \( \mathbb{E}[X_W] \) and the goal is to upper bound it. By the maximal inequality (or equivalently the union bound), we have

\[
\mathbb{E} \left[ \sup_{1 \leq i \leq n} X_i \right] \leq \sqrt{2 \log n},
\]

where (3) is asymptotically tight if \( X_i, i = 1, 2, ..., n \) are independent (see [10, Chapter 2]). But what if the algorithm is always more likely to choose \( W \) among a small subset of \( \{1, 2, ..., n\} \)? Then \( \mathbb{E}[X_W] \) could be much smaller than the right side of (3), as the chances of having an outlier value is smaller. Or, if the choice of \( W \) is not dependent on the data, then \( \mathbb{E}[X_W] = 0 \). Interestingly, to explain this phenomenon and to obtain tighter upper bounds on \( \mathbb{E}[X_W] \) an important information theoretic measure appears: the mutual information. This was originally proposed in the key paper of Russo and Zou [6] and then generalized in [16, 17], and in [18] for infinite number of hypotheses:

**Theorem 2.** [6,18] Let \( \{ X_t \}_{t \in T} \) be a random process and \( T \) an arbitrary set. Assume that \( X_t \) is \( \sigma^2 \)-subgaussian and \( \mathbb{E}[X_t] = 0 \) for every \( t \in T \), and let \( W \) be a random variable taking values on \( T \). Then

\[
|\mathbb{E}[X_W]| \leq \sqrt{2\sigma^2 I(W; \{X_t\}_{t \in T})}.
\]

In Example II, instead of using (2) and (3), one can have the tighter upper bound

\[
\mathbb{E}[X_W] \leq \sqrt{2 I(W; X_1, ..., X_n)}.
\]

For example, if the algorithm chooses \( W \) among \( \{1, 2, ..., \lfloor \log n \rfloor \} \) with probability \( 1 - o(1) \), then (5) implies

\[
\mathbb{E}[X_W] \leq \sqrt{2((1-o(1)) \log \log n + o(1) \log(n - \log n) + 1) \ll \sqrt{2 \log n}.
\]

However, this method does not give a finite bound for Example I, since

\[
I(\arg\max_{t \in T} h_t(G^2); \{h_t(G^2)\}_{t \in T}) = \infty.
\]

Similarly, as discussed in [19], the mutual information bound for perturbed SGD or any iterative algorithm which adds degenerate noise in each iteration blows up, and the authors leave finding strategies for analyzing generalization error of such algorithms as an open direction.

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This paper. By combining the ideas of the chaining method and the mutual information method, in this paper we obtain a chained mutual information bound which takes into account the dependencies between hypotheses as well as the dependence between output and input of the algorithm. When applied to the two aforementioned simple examples (Examples I and II), our bound yields the better bound between the classical chaining and classical mutual information bounds. More importantly, we provide examples for which our bound outperforms both of the previous bounds significantly: in Example 1 we provide a family of examples where the chaining method gives a relatively large constant, the mutual information bound blows up, but our bound tends towards zero. We also discuss how our new bound gives a possible direction to explain the phenomena described in [19] (see Remark 3) and [14] (see section 4).

In the framework of supervised statistical learning, $\mathcal{X}$ is the instances domain, $\mathcal{Y}$ is the labels domain and $\mathcal{Z} = \mathcal{X} \times \mathcal{Y}$ denotes the examples domain. Furthermore, $\mathcal{H} = \{h_w : w \in \mathcal{W}\}$ is the hypothesis set where the hypotheses are indexed by an index set $\mathcal{W}$, and there is a nonnegative loss function $\ell : \mathcal{H} \times \mathcal{Z} \to \mathbb{R}^+$. A learning algorithm receives the training set $S = (Z_1, Z_2, ..., Z_n)$ of $n$ examples with i.i.d. random elements drawn from $\mathcal{Z}$ with distribution $\mu$. Then it picks an element $h_W \in \mathcal{H}$ as the output hypothesis according to a random transformation $P_W | S$ (thus, we are allowing randomized algorithms). For any $w \in \mathcal{W}$, let

$$L_\mu(w) \triangleq \mathbb{E}[\ell(h_w, Z)], \quad Z \sim \mu$$

(8)
denote the statistical (or population) risk of hypothesis $h_w$. For a given training set $S$, the empirical risk of hypothesis $h_w$ is defined as

$$L_S(w) \triangleq \frac{1}{n} \sum_{i=1}^{n} \ell(h_w, Z_i),$$

(9)

and the generalization error of hypothesis $h_w$ (dependent on the training set) is defined as

$$\text{gen}(w) \triangleq L_\mu(w) - L_S(w).$$

(10)

Averaging with respect to the joint distribution $P_{S,W} = \mu^{\otimes n} P_{W|S}$, we denote the expected generalization error and the expected absolute value of generalization error by

$$\text{gen}(\mu, P_{W|S}) \triangleq \mathbb{E}[L_\mu(w) - L_S(w)],$$

(11)

and

$$\text{gen}^+(\mu, P_{W|S}) \triangleq \mathbb{E}[[L_\mu(w) - L_S(w)]]$$

(12)

respectively. Our purpose is to find upper bounds on $\text{gen}(\mu, P_{W|S})$ and $\text{gen}^+(\mu, P_{W|S})$.

If $\mathcal{N}$ is a set, then $X_N \triangleq \{X_i : i \in \mathcal{N}\}$ denotes a random process indexed by the elements of $\mathcal{N}$. Let $\mathbf{0}$ denote the identically zero function. In this paper, all logarithms are in natural base and all information theoretic measures are in nats. $H(X)$ denotes the Shannon entropy of a discrete random variable $X$, and $h(Y)$ denotes the differential entropy of an absolutely continuous random variable $Y$.

2 Main results

Assume that $\{X_t\}_{t \in T}$ is a random process with the index set $T$. In the chaining method, we impose a metric $d$ on $T$ which describes the dependencies between the random variables. The widely used subgaussian processes capture this notion and they arise in many applications:
Definition 1 (Subgaussian process). The random process \( \{X_t\}_{t \in T} \) on the metric space \((T, d)\) is called subgaussian if \( \mathbb{E}[X_t] = 0 \) for all \( t \in T \) and

\[
\mathbb{E}[e^{\lambda(X_t - X_s)}] \leq e^{\frac{1}{2} \lambda^2 d^2(t, s)} \quad \text{for all} \quad t, s \in T, \lambda \geq 0.
\] (13)

For example, based on the Azuma-Hoeffding inequality, \( \{\text{gen}(w)\}_{w \in \mathcal{W}} \) is a subgaussian process with the metric

\[
d(\text{gen}(w), \text{gen}(v)) \triangleq \frac{||\ell(h_w, \cdot) - \ell(h_v, \cdot)||_{\infty}}{\sqrt{n}},
\] (14)

regardless of the choice of distribution \( \mu \) on \( Z \).

The following is a technical assumption which holds in almost all cases of interest:

Definition 2 (Separable process). The random process \( \{X_t\}_{t \in T} \) is called separable if there is a countable set \( T_0 \subseteq T \) such that \( X_t \in \lim_{s \rightarrow t} X_s \) for all \( t \in T \) a.s., where \( x \in \lim_{x \rightarrow t} x_n \) means that there is a sequence \( s_n \rightarrow t \) such that \( x_{s_n} \rightarrow x \).

For example, if \( t \rightarrow X_t \) is continuous a.s., then \( X_t \) is a separable process [7].

Our main results rely on the notion of increasing sequence of \( \epsilon \)-partitions of the metric space \((T, d)\):

Definition 3 (Increasing sequence of \( \epsilon \)-partitions). We call a partition \( \mathcal{P} = \{A_1, A_2, \ldots, A_m\} \) of the set \( T \) an \( \epsilon \)-partition of the metric space \((T, d)\) if for all \( i = 1, 2, \ldots, m \), \( A_i \) can be contained within a ball of radius \( \epsilon \). A sequence of partitions \( \{\mathcal{P}_k\}_{k=m}^\infty \) of a set \( T \) is called an increasing sequence if for for all \( k \geq m \) and each \( A \in \mathcal{P}_{k+1} \), there exists \( B \in \mathcal{P}_k \) such that \( A \subseteq B \). For any such sequence and any \( t \in T \), let \( [t]_k \) denote the unique set \( A \in \mathcal{P}_k \) such that \( t \in A \).

For a bounded metric space \((T, d)\), let \( k_1(T) \) be an integer such that \( 2^{-(k_1(T))} \geq \text{diam}(T) \). We have the following upper bounds on \( \text{gen}(\mu, P_{W|S}) \) and \( \text{gen}^+(\mu, P_{W|S}) \) based on the mutual information between the training set \( S \) and the discretized output of the learning algorithm, where each of these mutual information terms is multiplied by a exponentially decreasing weight \( 2^{-k} \), in which the exponent measures how finely the output \( W \) of the learning algorithm is discretized.

Theorem 3. Assume that \( \{\text{gen}(w)\}_{w \in \mathcal{W}} \) is a separable subgaussian process on the bounded metric space \((W, d)\). Let \( \{\mathcal{P}_k\}_{k=k_1(W)}^\infty \) be an increasing sequence of partitions of \( \mathcal{W} \), where for each \( k \geq k_1(W) \), \( \mathcal{P}_k \) is a \( 2^{-k} \)-partition of \((W, d)\).

(a)

\[
\text{gen}(\mu, P_{W|S}) \leq 3\sqrt{2} \sum_{k=k_1(W)}^\infty 2^{-k} \sqrt{I([W]_k; S)},
\] (15)

(b) If \( 0 \in \{\ell(h_w, \cdot) : w \in W\} \), then

\[
\text{gen}^+(\mu, P_{W|S}) \leq 3\sqrt{2} \sum_{k=k_1(W)}^\infty 2^{-k} \sqrt{I([W]_k; S)} + \log 2.
\] (16)
Remark 1. Based on the general definition of mutual information with partitions ([20, p. 252]), we have \( I(W; S) = \sup_k I([W]_k; S) \) therefore \( I([W]_k; S) \to I(W; S) \) as \( k \to \infty \).

Theorem 3 is stated in the context of statistical learning. The more general counterpart in the context of random processes is:

**Theorem 4.** Assume that \( \{X_t\}_{t \in T} \) is a separable subgaussian process on the bounded metric space \((T,d)\). Let \( \{P_k\}_{k=k_1(T)}^\infty \) be an increasing sequence of partitions of \( T \), where for each \( k \geq k_1(T) \), \( P_k \) is a \( 2^{-k} \)-partition of \((T,d)\).

(a) 
\[
E[X_W] \leq 3\sqrt{2} \sum_{k=k_1(T)}^\infty 2^{-k} \sqrt{I([W]_k; X_T)}.
\]

(b) For any arbitrary \( t_0 \in T \),
\[
E[|X_W - X_{t_0}|] \leq 3\sqrt{2} \sum_{k=k_1(T)}^\infty 2^{-k} \sqrt{I([W]_k; X_T)} + \log 2.
\]

Note that in Theorem 4 if we let \( T \triangleq W \) and \( X_w \triangleq \text{gen}(w) \) for all \( w \in W \), then for each \( k \geq k_1(T) \), due to the Markov chain \( X_T = \{\text{gen}(w)\}_{w \in W} \leftrightarrow S \leftrightarrow W \leftrightarrow [W]_k \),

\[
X_T = \{\text{gen}(w)\}_{w \in W} \leftrightarrow S \leftrightarrow W \leftrightarrow [W]_k,
\]

and the data processing inequality, we have \( I([W]_k; X_T) \leq I([W]_k; S) \). Therefore Theorem 3 follows from Theorem 4. The proof of Theorem 4 and the etymology of “chaining mutual information” is given in Section 3.

Remark 2. For random processes other than subgaussian processes, where the tail of increments are controlled by a function \( \psi \), similar results can be derived from Theorem 12 in subsection 5.4.

Both Theorem 3 and Theorem 4 capture the dependencies between the hypotheses by utilizing a metric \( d \), and they are algorithm dependent as the mutual information between the algorithm’s discretized output and its input appears in their bounds. Now, to demonstrate the power of Theorem 4 and to compare it with the existing results in the literature, consider the following example:

**Example 1.** Let \( T \) be an arbitrary subset of \( \mathbb{R}^n \), and \( G^n \triangleq (G_1, ..., G_n) \sim \mathcal{N}(0, I_n) \) be a standard normal random vector in \( \mathbb{R}^n \). The canonical Gaussian process is defined as \( \{X_t\}_{t \in T} \), where

\[
X_t \triangleq \langle t, G^n \rangle \text{ for all } t \in T.
\]

Note that \( \{X_t\}_{t \in T} \) is a subgaussian process on the metric space \((T,d)\), where \( d \) is the Euclidean distance.

Consider a canonical Gaussian process where \( n = 2 \) and \( T = \{t \in \mathbb{R}^2 : \|t\|_2 = 1\} \). The process \( \{X_t\}_{t \in T} \) can be reparameterized according to the phase of each point \( t \in T \): the random variable \( X_t \) can also be denoted as \( X_\phi \), where \( \phi \in [0, 2\pi) \) is the phase of \( t \). In other
words, \( \phi \) is the unique number in \([0, 2\pi)\) such that \( t = (\sin \phi, \cos \phi) \). As such, we will assume the indices are in the phase form in the following.

Let the relation between the input \( X_T \) of an algorithm and its output \( W \) be as
\[
W \triangleq \operatorname{argmax}_{\phi \in [0, 2\pi)} X_\phi \oplus Z \pmod{2\pi},
\]
where the noise \( Z \) is independent from \( X_T \), and has an atom with probability mass \( \epsilon \) on 0 and \( 1 - \epsilon \) probability is uniformly distributed on \((-\pi, \pi)\). Note that since \( Z \) has a singular (degenerate) part, \( h(Z) = -\infty \).

Due to symmetry, \( W \) has uniform distribution over \([0, 2\pi)\). But we have
\[
\begin{align*}
    I(W; X_T) &= h(W) - h(W|X_T) \\
    &= \log 2\pi - h(\operatorname{argmax}_{\phi \in [0, 2\pi)} X_\phi \oplus Z|X_T) \\
    &= \log 2\pi - h(Z|X_T) \\
    &= \log 2\pi - h(Z) \\
    &= \infty.
\end{align*}
\]

Therefore the mutual information between the input \( X_T \) and output \( W \) of the algorithm is \( \infty \), hence the upper bound on \( \mathbb{E}[X_W] \) due to the mutual information method (see Theorem 2) blows up:
\[
\mathbb{E}[X_W] \leq \sqrt{2I(W; X_T)} = \infty.
\]

Note that \( 2^{-(-2)} \geq \text{diam}(T) = 2 \). Therefore let \( k_1(T) \leftarrow -1 \) and for all integers \( k \geq -1 \), define
\[
\mathcal{P}_k \triangleq \left\{ \left[ 0, \frac{2\pi}{2^k+2} \right], \left[ \frac{2\pi}{2^k+2}, \frac{2\pi}{2^k+2} \right], \ldots, \left[ \frac{(2^{k+2} - 1)\,2\pi}{2^k+2}, \frac{2\pi}{2^k+2} \right] \right\}.
\]

It is clear that \( \{\mathcal{P}_k\}_{k=-1}^\infty \) is an increasing sequence of partitions of \( T \). Furthermore, for each \( k \geq -1 \), the length of the arc of each set in \( \mathcal{P}_k \) is \( \delta_k \triangleq \frac{2\pi}{2^k+2} < 2^{-k} \). Thus each \( \mathcal{P}_k \) is a \( 2^{-k} \)-partition of \((T, d)\) and \( |\mathcal{P}_k| = 2^{k+2} \) (see Figure 1).

Now by using the classical chaining method (see Theorem 1) to upper bound \( \mathbb{E}[X_W] \) by upper bounding \( \mathbb{E}[\sup_{\phi \in [0, 2\pi)} X_\phi] \) and ignoring the algorithm, we get
\[
\begin{align*}
    \mathbb{E}[X_W] &\leq \mathbb{E}[\sup_{\phi \in [0, 2\pi)} X_\phi] \\
    &\leq 3\sqrt{2} \sum_{k=-1}^\infty 2^{-k} \sqrt{\log 2^{k+2}} \\
    &= 19.0352. \quad (27)
\end{align*}
\]

On the other hand, for every \( k \geq -1 \) we have
\[
\begin{align*}
    I([W]|k; X_T) &= H([W]|k) - H([W]|k|X_T) \\
    &= \log 2^{k+2} - H(\operatorname{argmax}_{\phi \in [0, 2\pi)} X_\phi \oplus Z|k|X_T) \\
    &= \log 2^{k+2} - H(\epsilon + \frac{1-\epsilon}{2^{k+2}}, \frac{1-\epsilon}{2^{k+2}}, \ldots, \frac{1-\epsilon}{2^{k+2}}).
\end{align*}
\]

\(^1\)The exact value of the bound of Theorem 1 is slightly smaller, since with our partitions we are using a rough approximate for the covering numbers. For example, at scale \( 2^{-(-1)} \), the covering number is 1, while we have used partition \( \mathcal{P}_{-1} \) with \( |\mathcal{P}_{-1}| = 2 \) sets.
Therefore, based on Theorem 4 (chained mutual information method), we have

\[
E[X_W] \leq 3\sqrt{2} \sum_{k=-\infty}^{\infty} 2^{-k} \sqrt{I(W_k; X_T)}
\]

(35)

\[
= 3\sqrt{2} \sum_{k=-\infty}^{\infty} 2^{-k} \sqrt{\log 2^{k+2} - H \left( \epsilon + \frac{1-\epsilon}{2^{k+2}}, \frac{1-\epsilon}{2^{k+2}}, \ldots, \frac{1-\epsilon}{2^{k+2}} \right)}
\]

(36)

Numerical values of the right side of (36) for different values of \(\epsilon\) are given in Table 1 (CMI bound). Note that indeed \(I(W_k; X_T) \to I(W; X_T) = \infty\) as \(k \to \infty\). However, the slow rate of that convergence and the existence of the \(2^{-k}\) term makes the sum not only finite, but very small. In fact, as \(\epsilon \to 0\), the right side of (36) tends to 0 as well.

It is interesting to note that for this toy example, the exact values of \(E[\sup_{\phi \in [0, 2\pi]} X_\phi]\) and \(E[X_W]\) can be computed. As \(\sup_{\phi \in [0, 2\pi]} X_\phi\) has a Rayleigh distribution, we have \(E[\sup_{\phi \in [0, 2\pi]} X_\phi] = \sqrt{\frac{2}{\pi}} = 1.253\ldots\). Since the noise \(Z\) is independent from \(X_T\), the effect of its continuous part cancels out, and we have \(E[X_W] = \epsilon \sqrt{\frac{2}{\pi}}\). See Table 1.
which has a degenerate part, causing the mutual information bound to blow up. Similarly, of the random variables of

Note that knowing the value of $N_k$ for every $k \geq k_1(T)$, consider $P_k = \{A_1, A_2, ..., A_m\}$. Since $P_k$ is a $2^{-k}$-partition of $(T, d)$, by definition there exists a set (or a multiset) $N_k \triangleq \{a_1, a_2, ..., a_m\} \subseteq T$ and a mapping $\pi_{N_k} : T \rightarrow N_k$ such that $\pi_{N_k}(t) = a_i$ if $t \in A_i$, and further $d(t, \pi_{N_k}(t)) \leq 2^{-k}$, for all $i = 1, 2, ..., m$. Therefore $N_k$ is a $2^{-k}$-net and $\pi_{N_k}$ is its associated mapping. It is also clear that for an arbitrary $t_0 \in T$, $N_{k_0} \triangleq \{t_0\}$ is a $2^{-(k_1(T)-1)}$-net. Note that for any integer $n \geq k_1(T)$ we can write

$$X_W = X_{t_0} + \sum_{k=k_1(T)}^n (X_{\pi_{N_k}(W)} - X_{\pi_{N_{k-1}}(W)}) + (X_W - X_{\pi_{N_n}(W)}).$$

(37)

Since by the definition of subgaussian processes the process is centered, we have $E[X_{t_0}] = 0$. Thus

$$E[X_W] = E[X_{t_0}] - E[X_{t_0} - X_{\pi_{N_n}(W)}] = \sum_{k=k_1(T)}^n E[X_{\pi_{N_k}(W)} - X_{\pi_{N_{k-1}}(W)}].$$

(38)

For every $k \geq k_1(T)$, $\{X_{\pi_{N_k}(t)} - X_{\pi_{N_{k-1}}(t)}\}_{t \in T}$ is a subgaussian process with at most $|N_k||N_{k-1}|$ distinct terms, hence it is a finite process. Based on the triangle inequality,

$$d(\pi_{N_k}(t), \pi_{N_{k-1}}(t)) \leq d(t, \pi_{N_k}(t)) + d(t, \pi_{N_{k-1}}(t)) \leq 3 \times 2^{-k}. \quad \text{(39)}$$

Note that knowing the value of $(\pi_{N_k}(W), \pi_{N_{k-1}}(W))$ is enough to determine which one of the random variables of $\{X_{\pi_{N_k}(t)} - X_{\pi_{N_{k-1}}(t)}\}_{t \in T}$ is chosen according to $W$. Therefore $(\pi_{N_k}(W), \pi_{N_{k-1}}(W))$ is playing the role of the random index, and since $X_{\pi_{N_k}(t)} - X_{\pi_{N_{k-1}}(t)}$ is $d^2(\pi_{N_k}(t), \pi_{N_{k-1}}(t))$-subgaussian, based on Theorem 3 and an application of data processing

| $\epsilon$ | $\frac{1}{2n}$ | $\frac{1}{3n}$ | $\frac{1}{4n}$ | $\frac{1}{5n}$ | $\frac{1}{10n}$ | $\frac{1}{20n}$ | $\frac{1}{40n}$ |
|------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|
| $2\sqrt{I(W; X_T)}$ | $\infty$ | $\infty$ | $\infty$ | $\infty$ | $\infty$ | $\infty$ | $\infty$ |
| Chaining bound | 19.0352 | 19.0352 | 19.0352 | 19.0352 | 19.0352 | 19.0352 | 19.0352 |
| CMI Bound | 1.1013 | 0.7507 | 0.5709 | 0.4612 | 0.2364 | 0.1204 | 0.0610 |
| $E[X_W]$ | 0.0626 | 0.0417 | 0.0313 | 0.0250 | 0.0125 | 0.0062 | 0.0031 |

**Remark 3.** Note that in Example 1 there exists an independent additive noise term $Z$ which has a degenerate part, causing the mutual information bound to blow up. Similarly, as discussed in [19], the mutual information bound for perturbed SGD or any iterative algorithm which adds degenerate noise in each iteration blows up. Example 1 suggests that combining the mutual information method with the chaining method as in our bound could give tight generalization bounds for such algorithms as well.

## 3 Proof outline

Here we provide an outline of the proof of Theorem 4. As noted in Section 2, Theorem 3 follows from Theorem 3.

For an arbitrary $k \geq k_1(T)$, consider $P_k = \{A_1, A_2, ..., A_m\}$. Since $P_k$ is a $2^{-k}$-partition of $(T, d)$, by definition there exists a set (or a multiset) $N_k \triangleq \{a_1, a_2, ..., a_m\} \subseteq T$ and a mapping $\pi_{N_k} : T \rightarrow N_k$ such that $\pi_{N_k}(t) = a_i$ if $t \in A_i$, and further $d(t, \pi_{N_k}(t)) \leq 2^{-k}$, for all $i = 1, 2, ..., m$. Therefore $N_k$ is a $2^{-k}$-net and $\pi_{N_k}$ is its associated mapping. It is also clear that for an arbitrary $t_0 \in T$, $N_{k_0} \triangleq \{t_0\}$ is a $2^{-(k_1(T)-1)}$-net. Note that for any integer $n \geq k_1(T)$ we can write

$$X_W = X_{t_0} + \sum_{k=k_1(T)}^n (X_{\pi_{N_k}(t)} - X_{\pi_{N_{k-1}}(t)}) + (X_W - X_{\pi_{N_n}(t)}).$$

(37)

Since by the definition of subgaussian processes the process is centered, we have $E[X_{t_0}] = 0$. Thus

$$E[X_W] = E[X_{t_0}] - E[X_{t_0} - X_{\pi_{N_n}(t)}] = \sum_{k=k_1(T)}^n E[X_{\pi_{N_k}(t)} - X_{\pi_{N_{k-1}}(t)}].$$

(38)

For every $k \geq k_1(T)$, $\{X_{\pi_{N_k}(t)} - X_{\pi_{N_{k-1}}(t)}\}_{t \in T}$ is a subgaussian process with at most $|N_k||N_{k-1}|$ distinct terms, hence it is a finite process. Based on the triangle inequality,

$$d(\pi_{N_k}(t), \pi_{N_{k-1}}(t)) \leq d(t, \pi_{N_k}(t)) + d(t, \pi_{N_{k-1}}(t)) \leq 3 \times 2^{-k}. \quad \text{(39)}$$

Note that knowing the value of $(\pi_{N_k}(W), \pi_{N_{k-1}}(W))$ is enough to determine which one of the random variables of $\{X_{\pi_{N_k}(t)} - X_{\pi_{N_{k-1}}(t)}\}_{t \in T}$ is chosen according to $W$. Therefore $(\pi_{N_k}(W), \pi_{N_{k-1}}(W))$ is playing the role of the random index, and since $X_{\pi_{N_k}(t)} - X_{\pi_{N_{k-1}}(t)}$ is $d^2(\pi_{N_k}(t), \pi_{N_{k-1}}(t))$-subgaussian, based on Theorem 3 and an application of data processing
inequality, we have
\[
\mathbb{E} \left[ X_{\pi_{\mathcal{N}_k}(W)} - X_{\pi_{\mathcal{N}_{k-1}}(W)} \right] \leq 3\sqrt{2} \times 2^{-k} \sqrt{I(\pi_{\mathcal{N}_k}(W), \pi_{\mathcal{N}_{k-1}}(W); X_T)}.
\] (40)

Note the chain of mutual information terms in right side of (40). Since \( \{\mathcal{P}_k\}_{k=k_1(T)}^\infty \) is an increasing sequence of partitions, for any \( t \in T \), knowing \( \mathcal{N}_k(t) \) will uniquely determine \( \mathcal{N}_{k-1}(t) \). Therefore
\[
I(\pi_{\mathcal{N}_k}(W), \pi_{\mathcal{N}_{k-1}}(W); X_T) = I(\pi_{\mathcal{N}_k}(W); X_T)
\] (41)
\[
= I([W]_k; X_T)
\] (42)
The rest of the proof follows from the definition of separable processes (see Definition 2).

For more details, see proof of Theorem 11 in subsection 5.4.

4 Additional result: small set property

In this section, we state a result which can be obtained from the chained mutual information method:

It is known that for linear models, the stochastic gradient descent (SGD) algorithm always converges to a solution with small norm [4]. Inspired by this observation, we tighten Dudley inequality (Theorem 1), given the fact that the output \( W \) of the algorithm chooses a hypothesis from a set with small covering numbers, with high probability:

**Theorem 5** (Small set property). Assume that \( \{X_i\}_{i \in T} \) is a separable subgaussian process on the bounded metric space \((T, d)\). Let \( \{T_1, T_2\} \) be a partition of \( T \) and assume that \( W \) is a random variable taking values on \( T \) with \( \mathbb{P}[W \in T_1] = \alpha \). Then we have
\[
\mathbb{E}[X_W] \leq 6 \sum_{k=k_1(T)}^\infty 2^{-k} \sqrt{\alpha \log N(T_1, d, 2^{-k}) + (1 - \alpha) \log N(T_2, d, 2^{-k}) + H(\alpha)}.
\] (43)

**Proof.** For each \( k \geq k_1(T) \), let \( \mathcal{N}_k^{(1)} \) and \( \mathcal{N}_k^{(2)} \) be minimal \( 2^{-k} \)-nets for \( T_1 \) and \( T_2 \), respectively. It is clear that \( \mathcal{N}_k \equiv \mathcal{N}_k^{(1)} \cup \mathcal{N}_k^{(2)} \), is a \( 2^{-k} \)-net for \( T \). Let
\[
\pi_{\mathcal{N}_k}(t) \equiv \begin{cases} 
\pi_{\mathcal{N}_k^{(1)}}(t) & \text{if } t \in T_1, \\
\pi_{\mathcal{N}_k^{(2)}}(t) & \text{if } t \in T_2.
\end{cases}
\]

Based on Theorem 11 and Remark 9, we have
\[
\mathbb{E}[X_W] \leq 3\sqrt{2} \sum_{k=k_1(T)}^\infty 2^{-k} \left( H(\pi_{\mathcal{N}_k}(W)) + H(\pi_{\mathcal{N}_{k-1}}(W)) \right)^{\frac{1}{2}}
\]
\[
\leq 3\sqrt{2} \sum_{k=k_1(T)}^\infty 2^{-k} \left( \alpha \log |\mathcal{N}_k^{(1)}| + (1 - \alpha) \log |\mathcal{N}_k^{(2)}| 
\right. \\
+ \alpha \log |\mathcal{N}_{k-1}^{(1)}| + (1 - \alpha) \log |\mathcal{N}_{k-1}^{(2)}| + 2H(\alpha) \right)^{\frac{1}{2}}
\]
\[
\leq 3\sqrt{2} \sum_{k=k_1(T)}^\infty 2^{-k} \left( \alpha \log |\mathcal{N}_k^{(1)}|^2 + (1 - \alpha) \log |\mathcal{N}_k^{(2)}|^2 + 2H(\alpha) \right)^{\frac{1}{2}}
\]
$$\leq 6 \sum_{k=k_1(T)}^{\infty} 2^{-k} \left( \alpha \log |N_k^{(1)}| + (1 - \alpha) \log |N_k^{(2)}| + H(\alpha) \right)^{\frac{1}{2}}$$

$$= 6 \sum_{k=k_1(T)}^{\infty} 2^{-k} \left( \alpha \log N(T_1, d, 2^{-k}) + (1 - \alpha) \log N(T_2, d, 2^{-k}) + H(\alpha) \right)^{\frac{1}{2}} \quad (44)$$

**Remark 4.** One can upper bound the right side of (43) by replacing $N(T_2, d, 2^{-k})$ with $N(T, d, 2^{-k})$.

## 5 Formal results

In subsection 5.2 which deals with finite random processes and which serves as the basic foundation of chaining, the known results of maximal inequality (Theorem 1) and its improvement via mutual information (Theorem 7) are reviewed. Then we give a condition for a random process in Corollary 3 which for the result of Theorem 7 can be improved by upper and lower bounding $\mathbb{E}[X_W]$. The aforementioned results concern $\mathbb{E}[X_W]$; in Theorem 8 we obtain inequalities for the tail behavior of $X_W$.

In the next step of building upon the results of subsection 5.2 to be able to handle infinite processes, in subsection 5.3 we introduce the notion of $\epsilon$-nets (see Definition 8) and its related definitions, and in Theorem 9 we upper bound $\mathbb{E}[X_W]$ for Lipschitz processes (see Definition 7) using mutual information. This is the strengthened version of the so-called $\epsilon$-net argument, with the usage of mutual information. Remark 8 discusses upper bounding $|\mathbb{E}[X_W]|$ for Lipschitz processes.

In the last step, in section 5.4, we loosen the “almost sure” Lipschitz condition of the dependencies of the random variables of a process to a “in probability” condition, defined as subgaussian processes (see Definition 9). After reviewing the classical chaining result of Dudley Theorem (Theorem 10), we combine the mutual information method and the chaining method in Theorem 11 for subgaussian processes, and in Theorem 12 for more general processes.

### 5.1 Preliminaries

**Definition 4** (Cumulant generating function). Let $X$ be a random variable. The cumulant generating function of $X$ is defined as $\Lambda_X(\lambda) \overset{\Delta}{=} \log \mathbb{E}[e^{\lambda X}]$ for all $\lambda \in \mathbb{R}$.

The following lemma is a well known fact about the cumulant generating function:

**Lemma 1.** Let $X$ be a random variable. Then its cumulant generating function $\Lambda_X$ is convex and $\Lambda'_X(0) = \mathbb{E}[X]$.

An important and widely used class of random variables is the class of subgaussian random variables:

**Definition 5** (Subgaussian random variables). The random variable $X$ is called $\sigma^2$-subgaussian if $\mathbb{E}[e^{\lambda(X - \mathbb{E}X)}] \leq e^{\frac{\lambda^2\sigma^2}{2}}$ for all $\lambda \in \mathbb{R}$. In particular, if $X$ is $\sigma^2$-subgaussian and $\mathbb{E}[X] = 0$, then its cumulant generating function satisfies $\Lambda_X(\lambda) \leq \frac{\lambda^2\sigma^2}{2}$ for all $\lambda \in \mathbb{R}$. The constant $\sigma^2$ is called the variance proxy.
We will use the notion of Legendre dual, defined as follows, in our bounds.

**Definition 6 (Legendre dual).** For a convex function \( \psi : \mathbb{R}_+ \to \mathbb{R} \), the Legendre dual \( \psi^* : \mathbb{R} \to \mathbb{R} \) is defined as

\[
\psi^*(x) \triangleq \sup_{\lambda \geq 0} \{ \lambda x - \psi(\lambda) \} \quad \text{for all } x \in \mathbb{R}.
\]  

(45)

For a proof of the next lemma see [2] p. 115):

**Lemma 2 (Legendre dual properties).** Let \( \psi : \mathbb{R}_+ \to \mathbb{R} \) be a convex function and \( \psi(0) = \psi'(0) = 0 \). Then \( \psi^*(x) \) is a convex, strictly increasing, nonnegative and unbounded function for \( x \geq 0 \), and \( \psi^*(0) = 0 \). Therefore its inverse \( \psi^{*-1}(y) \) is well defined for \( y \geq 0 \).

Recall from Definition 5 that if \( X \) is \( \sigma^2 \)-subgaussian and \( \mathbb{E}[X] = 0 \) then \( \Lambda_X(\lambda) \leq \frac{\lambda^2 \sigma^2}{2} \). The following lemma gives the Legendre dual of \( \psi(\lambda) \triangleq \frac{\lambda^2 \sigma^2}{2} \).

**Lemma 3.** Let \( \psi(\lambda) \triangleq \frac{\lambda^2 \sigma^2}{2} \) for all \( \lambda \geq 0 \). Then \( \psi^{*-1}(x) = \sqrt{2\sigma^2 x} \) for all \( x \in \mathbb{R} \).

The following is the well-known Chernoff bound:

**Lemma 4 (Chernoff).** Let \( X \) be a random variable, and \( \psi \) be a function such that \( \Lambda_X(\lambda) \leq \psi(\lambda) \) for all \( \lambda \geq 0 \). Then

\[
\mathbb{P}[X \geq x] \leq e^{-\psi^*(x)} \quad \text{for all } x \in \mathbb{R}.
\]  

(46)

The variational representation of relative entropy is a useful information theoretic tool:

**Theorem 6 (Variational representation of relative entropy).** Let \( X \) and \( Y \) be random variables taking values on \( \mathcal{A} \) with distributions \( P_X \) and \( P_Y \), respectively. Then

\[
D(P_X \| P_Y) = \max_{f \in \mathcal{F}} \left\{ \mathbb{E} [f(X)] - \log \mathbb{E} [e^{f(Y)}] \right\},
\]  

(47)

where the maximum is with respect to \( \mathcal{F} = \{ f : \mathcal{A} \to \mathbb{R} \text{ s.t. } \mathbb{E}[e^{f(Y)}] < \infty \} \), and is achieved by \( f^*(a) = i_{X \| Y}(a) \).

### 5.2 Finite processes (random vectors)

In this section we consider a random process \( \{X_t\}_{t \in T} \) where \( T \) is a finite set. The following is a well known result (see [10] Theorem 2.5):

**Proposition 1 (Maximal inequality).** Let \( \{X_t\}_{t \in T} \) be a random process and \( T \) a finite set. Assume that \( \Lambda_{X_t}(\lambda) \leq \psi(\lambda) \) for all \( \lambda \geq 0 \) and \( t \in T \), where \( \psi \) is convex and \( \psi(0) = \psi'(0) = 0 \). Then

\[
\mathbb{E} [\sup_{t \in T} X_t] \leq \psi^{*-1}(\log |T|).
\]  

(48)

In particular, if \( X_t \) is \( \sigma^2 \)-subgaussian and \( \mathbb{E}[X_t] = 0 \) for every \( t \in T \), then

\[
\mathbb{E} [\sup_{t \in T} X_t] \leq \sqrt{2\sigma^2 \log |T|}.
\]  

(49)

**Remark 5.** Note that based on Lemma 1, for all \( t \in T \), the condition \( \Lambda_{X_t}(\lambda) \leq \psi(\lambda) \) for all \( \lambda \geq 0 \) and \( \psi'(0) = 0 \) implies that \( \mathbb{E}[X_t] = 0 \).
Proposition 2. If in addition to the assumptions of Proposition 7, we assume that \( \Lambda_{X_t}(-\lambda) \leq \psi(\lambda) \) for all \( \lambda \geq 0 \) and \( t \in T \), then we have
\[
\mathbb{E}[\sup_{t \in T} |X_t|] \leq \psi^{-1}(\log(2|T|)).
\] (50)
In particular, if \( X_t \) is \( \sigma^2 \)-subgaussian and \( \mathbb{E}[X_t] = 0 \) for every \( t \in T \), then
\[
\mathbb{E}[\sup_{t \in T} |X_t|] \leq \sqrt{2\sigma^2 \log(2|T|)}.
\] (51)

Proof. Apply Proposition 7 on the random process \( \{X_t\}_{t \in T} \cup \{-X_t\}_{t \in T} \).

The next result bounds \( \mathbb{E}[X_W] \), where \( W \) is a random variable taking values on \( T \):

Theorem 7. \cite{12, 10} Let \( \{X_t\}_{t \in T} \) be a random process and \( T \) a finite set. Assume that \( \Lambda_{X_t}(\lambda) \leq \psi(\lambda) \) for all \( \lambda \geq 0 \) and \( t \in T \), where \( \psi \) is convex and \( \psi(0) = \psi'(0) = 0 \), and let \( W \) be a random variable taking values on \( T \). Then
\[
\mathbb{E}[X_W] \leq \psi^{-1}(I(W; X_T)).
\] (52)
In particular, if \( X_t \) is \( \sigma^2 \)-subgaussian and \( \mathbb{E}[X_t] = 0 \) for every \( t \in T \), then
\[
\mathbb{E}[X_W] \leq \sqrt{2\sigma^2 I(W; X_T)}.
\] (53)
Based on Lemma 2, \( \psi^{-1} \) is an increasing function. Therefore one can replace \( I(W; X_T) \) with any larger quantity in the right side of (52). For example,
\[
\mathbb{E}[X_W] \leq \psi^{-1}(I(W; X_T)) \leq \psi^{-1}(H(W)).
\] (54)
Since \( W \) takes values on \( T \), we have \( H(W) \leq \log |T| \). Therefore the right side of (52) is not larger than the right side of (53).

Based on Lemma 2 the right side of (52) is zero if and only if \( I(W; X_T) = 0 \), i.e. \( W \) is independent of \( X_T \). In this case, (52) turns into an equality: based on Remark 5 we have \( \mathbb{E}[X_t] = 0 \) for all \( t \in T \), hence \( \mathbb{E}[X_W] = \mathbb{E}[\mathbb{E}[X_W | W]] = 0 \).

Now, by adding an assumption, we prove upper and lower bounds for \( \mathbb{E}[X_W] \), and an upper bound for \( \mathbb{E}[|X_W|] \). We should mention that the proof of part (b) of the following proposition is similar to the proof of Theorem 4 in \cite{18}.

Proposition 3. If in addition to the assumptions of Theorem 7, we assume that \( \Lambda_{X_t}(-\lambda) \leq \psi(\lambda) \) for all \( \lambda \geq 0 \) and \( t \in T \), then we have
(a)
\[
|\mathbb{E}[X_W]| \leq \psi^{-1}(I(W; X_T)),
\] (55)
(b)
\[
\mathbb{E}[|X_W|] \leq \psi^{-1} \left( I(W; X_T) + \log 2 \right).
\] (56)
Proof.

(a) Apply Theorem 7 to the process \( \{-X_t\}_{t \in T} \), while noting that \( \Lambda_{-X_t}(\lambda) = \Lambda_{X_t}(-\lambda) \) for all \( \lambda \geq 0 \) and \( t \in T \), and \( I(W; -X_T) = I(W; X_T) \), since mutual information is invariant to one-to-one functions.
(b) Define the random process \( \mathbf{X} = \{ X_{t,w} \}_{t \in T} \) such that
\[
X_{t,w} = \begin{cases} 
X_t & t \in T, w = 0 \\
-X_t & t \in T, w = 1 
\end{cases}
\]
and let \( R \) be a random variable taking values on \( \{ 0, 1 \} \) such that
\[
R = \begin{cases} 
0 & \text{if } X_W \geq 0 \\
1 & \text{if } X_W < 0 
\end{cases}
\]

Based on Theorem \[\cite{3}\] applied on the random process \( \mathbf{X} \) and random variables \( W \) and \( R \), and based on the chain rule of entropy, we get
\[
\mathbb{E}[|X_W|] = \mathbb{E}[X_{W,R}] 
\leq \psi^* - 1 \left( \log |T| + u \right) 
\leq e^{-u/2} \] for all \( u \geq 0 \).

**Corollary 1.** If \( T \) is a finite set, \( \psi(\lambda) \triangleq \frac{\lambda^2}{2\sigma^2} \), and for all \( t \in T \), \( X_t \) is \( \sigma^2 \)-subgaussian and \( \mathbb{E}[X_t] = 0 \), then the conditions of Theorem \[\cite{3}\] is satisfied, and \( \mathbb{E}[X_W] \) can be improved to
\[
\mathbb{E}[X_W] \leq \sqrt{2\sigma^2 I(W; X_T)},
\]
as was shown in \[\cite{6}\].

The previous results concerned \( \mathbb{E}[\sup_{t \in T} X_t] \) and \( \mathbb{E}[X_W] \). We now state a result for estimating the tail probability of \( \sup_{t \in T} X_t \):

**Proposition 4.** \[\cite{7}\] Let \( \{ X_t \}_{t \in T} \) be a random process and \( T \) a finite set. Assume that \( \Lambda_{X_t}(\lambda) \leq \psi(\lambda) \) for all \( \lambda \geq 0 \) and \( t \in T \), where \( \psi \) is convex and \( \psi(0) = \psi'(0) = 0 \). Then
\[
\mathbb{P} \left[ \sup_{t \in T} X_t \geq \psi^{-1}(\log |T| + u) \right] \leq e^{-u} \text{ for all } u \geq 0.
\]

In particular, if \( X_t \) is \( \sigma^2 \)-subgaussian and \( \mathbb{E}[X_t] = 0 \) for every \( t \in T \), then
\[
\mathbb{P} \left[ \sup_{t \in T} X_t \geq \sqrt{2\sigma^2 \log |T| + x} \right] \leq e^{-\frac{x^2}{2\sigma^2}} \text{ for all } x \geq 0.
\]

We estimate the tail probability of \( X_W \) in the following theorem:
Theorem 8. Let \( \{X_t\}_{t \in T} \) be a random process and \( T \) a finite set. Assume that \( \Delta_{X_t}(\lambda) \leq \psi(\lambda) \) for all \( \lambda \geq 0 \) and \( t \in T \), where \( \psi \) is convex and \( \psi(0) = \psi'(0) = 0 \), and let \( W \) be a random variable taking values on \( T \). Then for all \( u \geq 0 \),
\[
\mathbb{P} \left[ X_W \geq \psi^{-1}(I(W;X_T) + u) \right] 
\leq \min \left\{ \frac{I(W;X_T) + \log \left( 2 - e^{-I(W;X_T) - u} \right)}{I(W;X_T) + u}, e^{\log |T| - I(W;X_T) - u} \right\},
\]
(69)

In particular, if \( X_t \) is \( \sigma^2 \)-subgaussian and \( \mathbb{E}[X_t] = 0 \) for every \( t \in T \), then for all \( x \geq 0 \),
\[
\mathbb{P} \left[ X_W \geq \sqrt{2\sigma^2 I(W;X_T)} + x \right] 
\leq \min \left\{ \frac{I(W;X_T) + \log \left( 2 - e^{-I(W;X_T) - \frac{x^2}{2\sigma^2}} \right)}{I(W;X_T) + \frac{x^2}{2\sigma^2}}, e^{\log |T| - I(W;X_T) - \frac{x^2}{2\sigma^2}} \right\},
\]
(70)

Proof. Analogous to the proof of Theorem 7 in [6]. [16], we invoke the variational representation of relative entropy (Theorem 6) in our proof.

Define \( n \triangleq |T| \) and without loss of generality, let \( T \triangleq \{1, 2, \ldots, n\} \). Note that
\[
\mathbb{P} \left[ X_W \geq \psi^{-1}(I(W;X_T) + u) \right] 
= \sum_{i=1}^{n} \mathbb{P} \left[ X_W \geq \psi^{-1}(I(W;X_T) + u) \middle| W = i \right] \mathbb{P} \left[ W = i \right] 
= \sum_{i=1}^{n} \mathbb{P} \left[ X_i \geq \psi^{-1}(I(W;X_T) + u) \middle| W = i \right] \mathbb{P} \left[ W = i \right].
\]
(71)

Define
\[
f(a) \triangleq \zeta \mathbb{1}_{\{a \geq \psi^{-1}(I(W;X_T) + u)\}},
\]
(73)
where \( \zeta > 0 \) is an arbitrary real number. Choose an arbitrary \( 1 \leq i \leq n \), and define random variable \( X \) such that \( P_X = P_{X_i | W = i} \). We have
\[
\zeta \mathbb{P} \left[ X_i \geq \psi^{-1}(I(W;X_T) + u) \middle| W = i \right] 
= \mathbb{E}[f(X)] 
\leq D(P_X \parallel P_{X_i}) + \log \mathbb{E}[e^{f(X_i)}] \]
(74)
\[
= D(P_{X_i | W = i} \parallel P_{X_i}) + \log \mathbb{E}[e^{f(X_i)}] \]
(75)
\[
= D(P_{X_i | W = i} \parallel P_{X_i}) + \log \left( e^\zeta \mathbb{P} \left[ X_i \geq \psi^{-1}(I(W;X_T) + u) \right] + \mathbb{P} \left[ X_i < \psi^{-1}(I(W;X_T) + u) \right] \right) \]
(76)
\[
= D(P_{X_i | W = i} \parallel P_{X_i}) + \log \left( e^\zeta \mathbb{P} \left[ X_i \geq \psi^{-1}(I(W;X_T) + u) \right] + 1 \right) \]
(77)
\[
\leq D(P_{X_i | W = i} \parallel P_{X_i}) + \log \left( (e^\zeta - 1) \mathbb{P} \left[ X_i \geq \psi^{-1}(I(W;X_T) + u) \right] + 1 \right) \]
(78)
\[
\leq D(P_{X_i | W = i} \parallel P_{X_i}) + \log \left( (e^\zeta - 1) e^{-I(W;X_T) - u} + 1 \right) \]
(79)
\[
\leq D(P_{X_T | W = i} \parallel P_{X_T}) + \log \left( (e^\zeta - 1) e^{-I(W;X_T) - u} + 1 \right),
\]
(80)
where (75) is based on Theorem 6, (79) is based on Lemma 4 and (80) is based on the data processing inequality for relative entropy. Therefore

\[
P[X_i \geq \psi^*-1(I(W;X_T)+u) \mid W = i] \leq \frac{1}{\zeta} \left(D(P_{X_T \mid W=i} || P_{X_T}) + \log \left((e^\zeta - 1)e^{-I(W;X_T)-u} + 1\right)\right).
\]  

(81)

Since \(i\) was chosen arbitrarily, (81) holds for all \(i = 1, 2, \ldots, n\). Thus, based on (71) and (72) we have

\[
P[X \geq \psi^*-1(I(W;X_T)+u)] \leq \frac{1}{\zeta} \left(\sum_{i=1}^{n} D(P_{X_T \mid W=i} || P_{X_T})P[W = i] + \log \left((e^\zeta - 1)e^{-I(W;X_T)-u} + 1\right)\right).
\]  

(82)

Since (83) holds for arbitrary \(\zeta > 0\), we can infimize the right side of (83) over \(\zeta\) to obtain

\[
P[X \geq \psi^*-1(I(W;X_T)+u)] \leq \inf_{\zeta > 0} \left\{ \frac{1}{\zeta} \left(\sum_{i=1}^{n} D(P_{X_T \mid W=i} || P_{X_T})P[W = i] + \log \left((e^\zeta - 1)e^{-I(W;X_T)-u} + 1\right)\right) \right\}.
\]  

(84)

Now, we upper bound the right side of (84) by choosing \(\zeta \leftarrow I(W;X_T) + u\), to get

\[
P[X \geq \psi^*-1(I(W;X_T)+u)] \leq \frac{I(W;X_T) + \log \left(2 - e^{-I(W;X_T)-u}\right)}{I(W;X_T) + u},
\]  

(85)

which is one of the terms in the right side of (69). To prove the other upper bound in (69), note that

\[
P[X \geq \psi^*-1(I(W;X_T)+u)] = \sum_{i=1}^{n} P[X \geq \psi^*-1(I(W;X_T)+u), W = i]
\]  

(86)

\[
= \sum_{i=1}^{n} P[X_i \geq \psi^*-1(I(W;X_T)+u), W = i]
\]  

(87)

\[
\leq \sum_{i=1}^{n} P[X_i \geq \psi^*-1(I(W;X_T)+u)]
\]  

(88)

\[
\leq ne^{-I(W;X_T)-u}
\]  

(89)

\[
= e^{\log |T| - I(W;X_T)-u},
\]  

(90)

where (89) is based on Lemma 4.

For the subgaussian case, note that

\[
\psi^*(\log |T| + u) = \sqrt{2\sigma^2(\log |T| + u)}
\]  

(91)

\[
\leq \sqrt{2\sigma^2 \log |T| + \sqrt{2\sigma^2 u}},
\]  

(92)

therefore, based on (85) and (90), we get (70).
Note that our upper bound in (85) is slightly stronger than Lemma 4.1 in [1], and our method of proving (85) shows that Lemma 4.1 in [1] is a corollary of the well known variational representation of relative entropy (Theorem 6).

**Remark 6.** If the assumptions of Proposition 3 hold, then by applying Theorem 8 on \{-X_t\}_{t \in T}, it is straightforward to obtain analogous lower tail bounds for \(X_W\).

### 5.3 Lipschitz processes and the \(\epsilon\)-net argument

The generalization of the maximal inequality (Proposition 1) to random processes with infinite number of random variables is not useful, since its upper bound blows up. But in many applications, there exists some dependence structure between the random variables of the random process which can be exploited to give better bounds. In this section we define Lipschitz structure and mention the \(\epsilon\)-net argument. Then we show how to tighten that by using mutual information.

**Definition 7** (Lipschitz process). The random process \(\{X_t\}_{t \in T}\) is called Lipschitz for a metric \(d\) on \(T\) if there exists a random variable \(C\) such that \(|X_t - X_s| \leq Cd(t, s)\) for all \(t, s \in T\).

Here we give the definitions of \(\epsilon\)-net and covering number \(N(T, d, \epsilon)\):

**Definition 8** (\(\epsilon\)-net and covering number). Let \(d\) be a metric on the set \(T\).

(a) A finite set \(N\) is called an \(\epsilon\)-net for \((T, d)\) if there exists a function \(\pi_N\) which maps every point \(t \in T\) to \(\pi_N(t) \in N\) such that \(d(t, \pi_N(t)) \leq \epsilon\).

(b) The covering number for a metric space \((T, d)\) is the smallest cardinality of an \(\epsilon\)-net for that space, where we denote it by \(N(T, d, \epsilon)\). In other words,

\[
N(T, d, \epsilon) \triangleq \inf \{|N| : N \text{ is an } \epsilon\text{-net for } (T, d)\}. \tag{93}
\]

(c) An \(\epsilon\)-net \(N\) for the metric space \((T, d)\) is called minimal if \(|N| = N(T, d, \epsilon)\).

For Lipschitz processes, the following inequality usually gives better bounds than the maximal inequality (Proposition 1), and it is also referred to as the \(\epsilon\)-net argument:

**Proposition 5** (Lipschitz maximal inequality). Assume that \(\{X_t\}_{t \in T}\) is a Lipschitz process for the metric \(d\) on \(T\), and \(\Lambda_{X_t}(\lambda) \leq \psi(\lambda)\) for all \(\lambda \geq 0\) and \(t \in T\), where \(\psi\) is convex and \(\psi(0) = \psi'(0) = 0\). Then

\[
\mathbb{E}[\sup_{t \in T} X_t] \leq \inf_{\epsilon > 0} \left\{ \epsilon \mathbb{E}[C] + \psi^*(1) \left( \log N(T, d, \epsilon) \right) \right\}. \tag{94}
\]

For a proof of Proposition 5 see [7]. The following theorem tightens Proposition 5 by using the mutual information method:

**Theorem 9.** Assume that \(\{X_t\}_{t \in T}\) is a Lipschitz process for the metric \(d\) on \(T\), and \(\Lambda_{X_t}(\lambda) \leq \psi(\lambda)\) for all \(\lambda \geq 0\) and \(t \in T\), where \(\psi\) is convex and \(\psi(0) = \psi'(0) = 0\). If for all \(\epsilon > 0\), \(N_{\epsilon}\) is an \(\epsilon\)-net for \((T, d)\), then

\[
\mathbb{E}[X_W] \leq \inf_{\epsilon > 0} \left\{ \epsilon \mathbb{E}[C] + \psi^*(1)(I(\pi_{N_{\epsilon}}(W); X_{N_{\epsilon}})) \right\}, \tag{95}
\]

where the infimum is over all \(\epsilon > 0\) and all \(\epsilon\)-nets \(N_{\epsilon}\) of \((T, d)\).
Proof. We have \( X_W = (X_W - X_{\pi_N \epsilon}(W)) + X_{\pi_N \epsilon}(W) \). Therefore, based on Theorem 7 and Definition 7, we have

\[
E[X_W] = E[X_W - X_{\pi_N \epsilon}(W)] + E[X_{\pi_N \epsilon}(W)]
\]

\[
\leq E[|X_W - X_{\pi_N \epsilon}(W)|] + E[X_{\pi_N \epsilon}(W)]
\]

\[
\leq \epsilon E[C] + \psi^{-1}(I(\pi_N \epsilon(W); X_{\pi_N \epsilon}))
\]

(96)

(97)

(98)

Remark 7. Note that in the infimum in (95), for all \( \epsilon > 0 \) one can restrict \( N \epsilon \) to be a minimal \( \epsilon \)-net to conclude that the right side of (95) is no larger than the right side of (94), due to Lemma 2 and the following inequalities:

\[
I(\pi_N \epsilon(W); X_{\pi_N \epsilon}) \leq H(\pi_N \epsilon(W))
\]

\[
\leq \log N(T, d, \epsilon).
\]

(99)

(100)

Proposition 6. With the assumptions of Theorem 9, we have

\[
\inf_{\epsilon > 0} \left\{ \epsilon E[X_W] + \psi^{-1}(I(\pi_N \epsilon(W); X_{\pi_N \epsilon})) \right\} \leq \psi^{-1}(I(W; X_T)).
\]

(101)

Therefore the bound on \( E[X_W] \) given in Theorem 9 is no larger than the bound given in Theorem 7.

Proof. For all \( \epsilon > 0 \), based on the chain rule of mutual information (or the data processing inequality), we have

\[
I(\pi_N \epsilon(W); X_{\pi_N \epsilon}) \leq I(\pi_N \epsilon(W); X_T).
\]

(102)

Furthermore, the Markov chain \( \pi_N \epsilon(W) \leftrightarrow W \leftrightarrow X_T \) and the data processing inequality for mutual information yield

\[
I(\pi_N \epsilon(W); X_T) \leq I(W; X_T).
\]

(103)

Lemma 2 along with (102) and (103) conclude

\[
\epsilon E[C] + \psi^{-1}(I(\pi_N \epsilon(W); X_{\pi_N \epsilon})) \leq \epsilon E[C] + \psi^{-1}(I(W; X_T)).
\]

(104)

Letting \( \epsilon \to 0 \) completes the proof.

Remark 8. If in addition to the assumptions of Theorem 9 we have \( \Lambda_X(-\lambda) \leq \psi(\lambda) \) for all \( \lambda \geq 0 \) and \( t \in T \) (see Corollary 1 for an example), then similar to the proof of Proposition 3 we can prove

\[
|E[X_W]| \leq \epsilon E[C] + \psi^{-1}(I(\pi_N \epsilon(W); X_{\pi_N \epsilon})).
\]

(105)

5.4 Chaining mutual information

We loosen the “almost sure” Lipschitz condition of the dependencies of the random variables of a process to a “in probability” condition, defined as subgaussian processes:

Definition 9 (Subgaussian process). The random process \( \{X_t\}_{t \in T} \) on the metric space \( (T, d) \) is called subgaussian if \( E[X_t] = 0 \) for all \( t \in T \) and

\[
E[e^{\lambda(X_t - X_s)}] \leq e^{\frac{1}{2}\lambda^2 d^2(t, s)} \quad \text{for all} \quad t, s \in T, \lambda \geq 0.
\]

(106)
We now state a classical chaining result:

**Theorem 10** (Dudley). Assume that \( \{X_t\}_{t \in T} \) is a separable subgaussian process on the bounded metric space \((T, d)\). Then

\[
\mathbb{E}[\sup_{t \in T} X_t] \leq 6 \sum_{k \in \mathbb{Z}} 2^{-k} \sqrt{\log N(T, d, 2^{-k})}. \tag{107}
\]

By combining the mutual information method and the chaining method, we obtain the following result:

**Theorem 11.** Assume that \( \{X_t\}_{t \in T} \) is a separable subgaussian process on the bounded metric space \((T, d)\) and let \(k_0\) be an integer such that \(2^{-k_0} \geq \text{diam}(T)\). Let \(\{\mathcal{N}_k\}_{k=k_0+1}^\infty\) be a sequence of sets, where for each \(k > k_0\), \(\mathcal{N}_k\) is a \(2^{-k}\)-net for \((T, d)\). For an arbitrary \(t_0 \in T\), let \(\mathcal{N}_{k_0} \triangleq \{t_0\}\). Assume that \(W\) is a random variable which takes values on \(T\). We have

(a) Since \(2^{-k_0} \geq \text{diam}(T)\), we have \(N(T, d, 2^{-k_0}) = 1\), therefore \(\mathcal{N}_{k_0}\) is a \(2^{-k_0}\)-net for \((T, d)\).

Note that for any integer \(n > k_0\) we can write

\[
X_W = X_{t_0} + \sum_{k=k_0+1}^n (X_{\pi_{\mathcal{N}_k}(W)} - X_{\pi_{\mathcal{N}_{k-1}}(W)}) + (X_W - X_{\pi_{\mathcal{N}_0}(W)}). \tag{110}
\]

Since by the definition of subgaussian processes the process is centered, we have \(\mathbb{E}[X_{t_0}] = 0\). Thus

\[
\mathbb{E}[X_W] = \mathbb{E}[X_{t_0} + \sum_{k=k_0+1}^n (X_{\pi_{\mathcal{N}_k}(W)} - X_{\pi_{\mathcal{N}_{k-1}}(W)}) + (X_W - X_{\pi_{\mathcal{N}_0}(W)})]. \tag{111}
\]

Note that for every \(k > k_0\), \(\{X_{\pi_{\mathcal{N}_k}(t)} - X_{\pi_{\mathcal{N}_{k-1}}(t)}\}_{t \in T}\) is a subgaussian process with at most \(|\mathcal{N}_k| |\mathcal{N}_{k-1}|\) distinct terms, hence it is a finite process. Based on triangle inequality,

\[
d(\pi_{\mathcal{N}_k}(t), \pi_{\mathcal{N}_{k-1}}(t)) \leq d(t, \pi_{\mathcal{N}_k}(t)) + d(t, \pi_{\mathcal{N}_{k-1}}(t)) \leq 3 \times 2^{-k}. \tag{112}
\]

Note that knowing the value of \((\pi_{\mathcal{N}_k}(W), \pi_{\mathcal{N}_{k-1}}(W))\) is enough to determine which one of the random variables of \(\{X_{\pi_{\mathcal{N}_k}(t)} - X_{\pi_{\mathcal{N}_{k-1}}(t)}\}_{t \in T}\) is chosen according to \(W\).

Therefore \((\pi_{\mathcal{N}_k}(W), \pi_{\mathcal{N}_{k-1}}(W))\) is playing the role of the random index, and since \(X_{\pi_{\mathcal{N}_k}(t)} - X_{\pi_{\mathcal{N}_{k-1}}(t)}\) is \(d^2(\pi_{\mathcal{N}_k}(t), \pi_{\mathcal{N}_{k-1}}(t))\)-subgaussian, based on Theorem 7 we have

\[
\mathbb{E}
\left[ X_{\pi_{\mathcal{N}_k}(W)} - X_{\pi_{\mathcal{N}_{k-1}}(W)} \right] \leq 3\sqrt{2} \times 2^{-k} \left( I(\pi_{\mathcal{N}_k}(W), \pi_{\mathcal{N}_{k-1}}(W); \{X_{\mathcal{N}_k}(t) - X_{\mathcal{N}_{k-1}(t)}\}_{t \in T}) \right)^{\frac{1}{2}}. \tag{113}
\]
Based on the chain rule of mutual information, adding random variables to one side of mutual information does not decrease its value. Thus

\[
\mathbb{E}[X_{\pi_{N_k}(W)} - X_{\pi_{N_{k-1}}(W)}] \leq 3\sqrt{2} \times 2^{-k} \left( I(\pi_{N_k}(W), \pi_{N_{k-1}}(W); X_{N_k} - X_{N_{k-1}}) \right)^{\frac{3}{2}}. \quad (114)
\]

Based on (111) and by using (114) for each \( k = k_0 + 1, \ldots, n \), we conclude

\[
\mathbb{E}[X_W] - \mathbb{E}[X_W - X_{\pi_{N_n}(W)}] \leq \sum_{k=k_0+1}^n 3\sqrt{2} \times 2^{-k} \left( I(\pi_{N_k}(W), \pi_{N_{k-1}}(W); X_{N_k} - X_{N_{k-1}}) \right)^{\frac{3}{2}}. \quad (115)
\]

Note that \( |\mathbb{E}[X_W - X_{\pi_{N_n}(W)}]| \leq \mathbb{E}[\sup_{t \in T} (X_t - X_{\pi_{N_n}(t)})] \), and since the process is separable, we have

\[
\lim_{n \to \infty} \mathbb{E}[\sup_{t \in T} (X_t - X_{\pi_{N_n}(t)})] = 0, \quad (116)
\]

(see proof of Theorem 5.24 in (7)). Hence

\[
\lim_{n \to \infty} \mathbb{E}[X_W - X_{\pi_{N_n}(W)}] = 0. \quad (117)
\]

Based on (115) and (117), we get

\[
\mathbb{E}[X_W] \leq 3\sqrt{2} \sum_{k=k_0+1}^\infty 2^{-k} \left( I(\pi_{N_k}(W), \pi_{N_{k-1}}(W); X_{N_k} - X_{N_{k-1}}) \right)^{\frac{3}{2}}. \quad (118)
\]

By further upper bounding the right side of (118), we obtain

\[
\mathbb{E}[X_W] \leq 3\sqrt{2} \sum_{k=k_0+1}^\infty 2^{-k} \left( I(\pi_{N_k}(W), \pi_{N_{k-1}}(W); X_{N_k} - X_{N_{k-1}}) \right)^{\frac{3}{2}}
\]

\[
\leq 3\sqrt{2} \sum_{k=k_0+1}^\infty 2^{-k} \left( I(\pi_{N_k}(W), \pi_{N_{k-1}}(W); X_{N_k} - X_{N_{k-1}}, X_{N_{k-1}}) \right)^{\frac{3}{2}}
\]

(119)

\[
= 3\sqrt{2} \sum_{k=k_0+1}^\infty 2^{-k} \left( I(\pi_{N_k}(W), \pi_{N_{k-1}}(W); X_{N_k \cup N_{k-1}}) \right)^{\frac{3}{2}}
\]

(120)

\[
\leq 3\sqrt{2} \sum_{k=k_0+1}^\infty 2^{-k} \left( I(\pi_{N_k}(W), \pi_{N_{k-1}}(W); X_T) \right)^{\frac{3}{2}},
\]

(121)

where (119) and (121) follow from the chain rule of mutual information, and (120) follows from the fact that mutual information is invariant to one-to-one functions.

(b) From (110) we conclude that

\[
|X_W - X_{t_0}| \leq \sum_{k=k_0+1}^n |X_{\pi_{N_k}(W)} - X_{\pi_{N_{k-1}}(W)}| + |X_W - X_{\pi_{N_n}(W)}|. \quad (122)
\]

Hence

\[
\mathbb{E}[|X_W - X_{t_0}|] - \mathbb{E}[|X_W - X_{\pi_{N_n}(W)}|] \leq \sum_{k=k_0+1}^n \mathbb{E}[|X_{\pi_{N_k}(W)} - X_{\pi_{N_{k-1}}(W)}|]. \quad (123)
\]
The rest of the proof is similar to previous part, with the difference of instead of using Theorem 7 to obtain (113), we use Proposition 3 (b) with \( \psi(\lambda) \triangleq \lambda^2 \sigma^2 \) to obtain

\[
\mathbb{E} \left[ |x_{\pi_{N_k}(w)} - x_{\pi_{N_{k-1}}(w)}| \right] \leq 3\sqrt{2} \times 2^{-k} \left( I(\pi_{N_k}(W), \pi_{N_{k-1}}(W); \{X_{N_k(t)} - X_{N_{k-1}(t)}\}_{t \in T}) + \log 2 \right)^{\frac{1}{2}}. 
\] (124)

\[
\text{Remark 9. Note that for all } k > k_0,

\[
I(\pi_{N_k}(W), \pi_{N_{k-1}}(W); X_T) \leq H(\pi_{N_k}(W)) + H(\pi_{N_{k-1}}(W)) - \log |N_k| + \log |N_{k-1}| \leq 2 \log |N_k|. 
\] (128)

Therefore, if we assume that for each \( k > k_0, \mathcal{N}_k \) is a minimal \( 2^{-k} \)-net for \((T,d)\), then we have replaced the Hartley entropy in Dudley inequality (Theorem 10) with Shannon entropy (because \( \log |N_k| = \log N(T,d,2^{-k}) \)) and further with mutual information.

For random processes other than subgaussian processes, where the tail of increments are controlled by a function \( \psi \), we have the following result whose proof is similar to the proof of Theorem 11:

**Theorem 12.** Assume that \( \{X_t\}_{t \in T} \) is a separable process defined on the bounded metric space \((T,d)\), with \( \mathbb{E}[X_t] = 0 \) for all \( t \in T \) and

\[
\log \mathbb{E} \left[ e^{\frac{\lambda(x_t - x_s)}{d(t,s)}} \right] \leq \psi(\lambda) \text{ for all } t, s \in T, \lambda \geq 0, 
\] (129)

where \( \psi \) is convex and \( \psi(0) = \psi'(0) = 0 \). Let \( k_0 \) be an integer such that \( 2^{-k_0} \geq \text{diam}(T) \) and \( \{\mathcal{N}_k\}_{k=k_0+1}^{\infty} \) be a sequence of sets, where for each \( k > k_0, \mathcal{N}_k \) is a \( 2^{-k} \)-net for \((T,d)\). For an arbitrary \( t_0 \) \( \in T \), let \( \mathcal{N}_{k_0} \triangleq \{t_0\} \). Assume that \( W \) is a random variable which takes values on \( T \). We have

(a)

\[
\mathbb{E}[X_W] \leq 3\sqrt{2} \sum_{k=k_0+1}^{\infty} 2^{-k} \psi^{*-1} \left( I(\pi_{N_k}(W), \pi_{N_{k-1}}(W); X_T) \right). 
\] (130)

(b)

\[
\mathbb{E}[|X_W - x_{t_0}|] \leq 3\sqrt{2} \sum_{k=k_0+1}^{\infty} 2^{-k} \psi^{*-1} \left( I(\pi_{N_k}(W), \pi_{N_{k-1}}(W); X_T) + \log 2 \right). 
\] (131)

### 6 Conclusion

We combined ideas from information theory and from high dimensional probability to obtain a generalization bound that takes into account both the dependencies between the hypotheses and the dependence between the input and the output of a learning algorithm. We showed
on an example that our chained mutual information bound significantly outperforms previous bounds and gets close to the true generalization error. Inspired by the findings of [13] on SGD, we provided a corollary of our bound which tightens Dudley inequality when the learning algorithm chooses its output from a small subset of hypotheses with high probability.

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