Quasi-exactly solvable quartic: real algebraic spectral locus

Alexandre Eremenko and Andrei Gabrielov

Department of Mathematics, Purdue University, West Lafayette, IN 47907, USA
E-mail: eremenko@math.purdue.edu and agabriel@math.purdue.edu

Received 17 October 2011
Published 12 April 2012
Online at stacks.iop.org/JPhysA/45/175205

Abstract
We describe the real quasi-exactly solvable spectral locus of the PT-symmetric quartic using the Nevanlinna parametrization.

PACS numbers: 02.30.Hq, 03.65.Sq
Mathematics Subject Classification: 81Q05, 34M60, 34A05

Following Bender and Boettcher [3], we consider the eigenvalue problem in the complex plane

\[ w'' + (\zeta^4 + 2b\zeta^2 + 2iJ\zeta + \lambda)w = 0, \quad w(t e^{\pm\pi i/3}) \to 0, \quad t \to +\infty, \]  

where \( J \) is a positive integer. This problem is quasi-exactly solvable [3]: there exist \( J \) elementary eigenfunctions \( w = p_n(\zeta) \exp(-i\zeta^3/3 - ib\zeta) \), where \( p_n \) is a polynomial of degree \( n = J - 1 \).

When \( b \) is real, the problem is PT-symmetric. By the change of the independent variable \( z = i\zeta \), (1) is equivalent to

\[ -y'' + (z^4 - 2bz^2 + 2Jz)y = \lambda y, \quad y(t e^{\pm\pi i/3}) \to 0, \quad t \to +\infty. \]  

Polynomial \( h \) in the exponent of an elementary eigenfunction \( y(z) \) is \( h(z) = z^3/3 - bz \). The spectral locus \( Z_f \) is defined as

\[ \{ (b, \lambda) \in \mathbb{C}^2 : \exists y \neq 0, \text{ satisfying } (2) \}. \]

The real spectral locus \( Z_f(\mathbb{R}) \) is \( Z_f \cap \mathbb{R}^2 \). The quasi-exactly solvable spectral locus \( Z_f^{QES} \) is the set of all \( (b, \lambda) \in Z_f \) for which there exists an elementary solution \( y \) of (2). This is a smooth irreducible algebraic curve in \( \mathbb{C}^2 \) [1, 2]. In this paper, we describe \( Z_f^{QES}(\mathbb{R}) = Z_f^{QES} \cap \mathbb{R}^2 \). We prove the result announced in [6].

**Theorem 1.** For \( n \geq 0 \), \( Z_f^{QES}(\mathbb{R}) \) consists of \([n/2] + 1 \) disjoint analytic curves \( \Gamma_{n,m} \), \( 0 \leq m \leq [n/2] \) (analytic embeddings of \( \mathbb{R} \) to \( \mathbb{R}^2 \)).

For \((b, \lambda) \in \Gamma_{n,m}\), the eigenfunction has \( n \) zeros, with \( n - 2m \) of them being real.

If \( n \) is odd, then \( b \to +\infty \) on both ends of each curve \( \Gamma_{n,m} \). If \( n \) is even, then the same holds for \( m < n/2 \), but on the ends of \( \Gamma_{n,n/2} \), we have \( b \to -\infty \).

If \((b, \lambda) \in \Gamma_{n,m}\), \( (b, \mu) \in \Gamma_{n,m+1} \) and \( b \) is sufficiently large, then \( \mu > \lambda \).
This theorem establishes the main features of $Z_{QES}^{J}(R)$ which can be seen in the computer-generated figure in [3]. Similar results were proved in [5] for two other PT-symmetric eigenvalue problems.

Our theorem parametrizes all polynomials $P$ of degree 4 with the property that the differential equation $y'' + Py = 0$ has a solution with $n$ zeros, with $n - 2m$ of them being real [10, 7, 5].

Suppose that $(b, \lambda) \in Z_{j}^{QES}(R)$. Then the corresponding eigenfunction $y$ of (2) can always be chosen to be real. Let $y_{1}$ be a real solution of the differential equation in (2) normalized by $y_{1}(x) \to 0$ as $x \to +\infty$, $x \in R$. Then $y_{1}$ is linearly independent of $y$. Consider the meromorphic function $f = y/y_{1}$. This function has no critical points in $C$, and the only singularities of $f^{-1}$ are six logarithmic branch points. A meromorphic function in $C$ with no critical points and whose inverse has finitely many logarithmic singularities is called a Nevanlinna function. All Nevanlinna functions $f$ arise from differential equations $y'' + Py = 0$, where $P$ is a polynomial by the above construction: $f$ is a ratio of two linearly independent solutions of the differential equation.

Consider the sectors

$$S_{j} = \{v e^{\theta} : t > 0, |\theta - \pi j/3| < \pi/6\}, \quad j = 0, \ldots, 5.$$  

The subscript $j$ in $S_{j}$ will always be understood as a residue modulo 6. Function $f$ has asymptotic values $\infty, 0, c, 0, \bar{c}, 0$ in the sectors $S_{0}, \ldots, S_{5}$, where $c \in \bar{C}$. It is known that $f$ must have at least three distinct asymptotic values [9], so $c \not= 0, \infty$. Function $f$ is defined up to multiplication by a non-zero real number, so we can always assume that $c = e^{i\beta}$, $0 \leq \beta \leq \pi$, where the points 0 and $\pi$ can be identified. The asymptotic value $c$ is called the Nevanlinna parameter. There is a simple relation between $c$ and the Stokes multipliers [11, 8].

The sectors $S_{j}$ correspond to logarithmic singularities of the inverse function $f^{-1}$. Thus, $f^{-1}$ has six logarithmic singularities that lie over four points if $c \not= \bar{c}$, or over three points if $c = \bar{c}$.

The map $(b, \lambda) \mapsto \beta \pmod{\pi}$, $Z_{j}^{QES}(R) \to R$ is analytic and locally invertible [11, 2], so $\beta$ can serve as a local parameter on the real QES spectral locus. To obtain a global parametrization, one needs suitable charts on $Z_{j}^{QES}(R)$, where this map is injective.

To recover $f$, one has to know the asymptotic value $c$ and one more piece of information, a certain cell decomposition of the plane described below. Once $f$ is known, $b$ and $\lambda$ are found from the formula

$$\frac{f'''}{f'} - \frac{3}{2} \left( \frac{f''}{f'} \right)^{2} = -2(c^{4} - 2bc^{2} + 2cz - \lambda). \quad (3)$$

Now we describe, following [4], the cell decompositions needed to recover $f$ from $c$. Suppose first that $c \not\in R$.

Consider the cell decomposition $\Phi$ of the Riemann sphere $\bar{C}$ shown by solid lines in the left part of figure 1. It consists of one vertex at the point 2, three edges (loops $\gamma_{c}$, $\gamma_{e}$ and $\gamma_{\infty}$ around non-zero asymptotic values) and four faces (cells of dimension 2). The faces are labeled by the asymptotic values 0, $c$, $\bar{c}$, $\infty$. Label 0 is not shown in the picture. The face labeled 0 is the unbounded region in the picture. (The point 1 in the figure is neither a label, nor a part of the cell decomposition. It will be needed, together with the dashed line $L$, for the limit at $\beta = 0$.) As

$$f : C \setminus f^{-1}(0, \infty, c, \bar{c}) \to \bar{C} \setminus \{0, \infty, c, \bar{c}\}$$

is a covering map, the cell decomposition $\Phi$ pulls back to a cell decomposition $\Psi$ of the plane.

Examples of $\Psi$ are shown with solid lines in figures 2 and 4 (left). The faces of $\Psi$ are labeled with the same labels as their images. Non-zero labels of bounded faces are omitted in
the picture. The reader can restore them from the condition that labels around a vertex must
be in the same cyclic order as in figure 1 (left, solid lines). The labeled cell decomposition \( \Psi \) defines \( f \) up to a pre-composition with an affine map of \( \mathbb{C} \). Two cell decompositions define the same \( f \) if they can be obtained from each other by a homeomorphism of the plane preserving orientation and the labels. Such cell decompositions are called equivalent.

By replacing multiple edges of the 1-skeleton of \( \Psi \) with single edges and removing
the loops, we obtain a simpler cell decomposition \( T \) whose 1-skeleton is a tree, which we
denote by the same letter \( T \). The cell decomposition \( \Psi \) is uniquely recovered from its tree
\( T \) embedded in the plane, [4]. The faces of \( T \) are asymptotic to the sectors \( S_j \), and the labels
are the asymptotic values in \( S_j \). Two faces with a common edge cannot have the same label.
The cell decomposition \( T \) is invariant under the reflection in the real axis, with simultaneous
interchange of \( c \) and \( \overline{c} \). It is easy to classify all possible embedded planar trees \( T \) with labeled
faces that satisfy these properties. They depend on two integer parameters \( k \) and \( l \) with \( l \geq 0 \).
These trees form two families, \( \{X_{k,l}, k \geq 0, l \geq 0\} \) and \( \{X_{k,l}, k < 0, l \geq 0\} \), as shown in
figure 3. Integers \( |k| \) and \( l \) are the numbers of edges between ramification vertices, as shown
in figure 3.
Cell decompositions in figure 2 (solid lines) correspond to the trees $X_{0,1}$ and $X_{1,1}$. Cell decomposition in the left part of figure 4 (solid lines) corresponds to the tree $X_{-1,1}$ in the right part of figure 4.

Parameters of the trees $X_{k,l}$ can be interpreted as follows:

$$k^- := \min\{-k, 0\}$$

is the number of real zeros of $f$, and $2l$ is the number of non-real zeros. So the total number of zeros is $n = 2l + k^-$. Functions $f$ corresponding to the trees $X_{k,l}$, $k \geq 0$, have $2l$ zeros, none of them real. Zeros of the eigenfunction $\gamma$ coincide with those of $f$.

For given $n$, the number of trees $X_{k,l}$ with $k < 0$, $2l - k = n$ is

$$(n + 1)/2 \text{ when } n \text{ is odd, and } n/2 \text{ when } n \text{ is even.} \quad (4)$$

Every tree $X_{k,l}$ and every $\beta \in (0, \pi)$ define a meromorphic function $f$ satisfying (3) with $J = 2l + k^- + 1$ and some $(b, \lambda)$ depending on $\beta$, $k$ and $l$. This follows from a result of Nevanlinna [9]; see also [4]. From this function $f$, the coordinates of a point $(b, \lambda)$ on the
real QES spectral locus are recovered from the Schwarz equation (3). Thus, we have a map $F : (T, \beta) \mapsto (b, \lambda)$ which we call the Nevanlinna map. This map is of highly transcendental nature: construction of $f$ from $T$ and $\beta$ involves the uniformization theorem. We refer to [4, 5, 9] for details.

Each of the trees from our classification defines a chart of $Z^\text{QES}_J(\mathbb{R})$. To obtain the global parametrization of $Z^\text{QES}_J(\mathbb{R})$, we only have to find out how these charts are pasted together. We will see that the boundaries of our charts correspond to the values $c = \pm 1$.

**Proof of theorem 1.** We begin with the charts $X_{k,l}$, $k < 0$. We show that in these charts the limits as $\beta \to 0$, $\pi$ do not belong to the spectral locus. This is proved by the arguments similar to those in [5, theorem 4.1].

**Lemma 1.** For $k < 0$ and $l \geq 0$, the limit of the Nevanlinna map is
\[
\lim_{\beta \to 0} F(X_{k,l}, \beta) = \infty.
\]

**Proof.** When $\beta \to 0$, we have $c \to 1$, $\pi \to 1$. Suppose by contradiction that $F(X_{k,l}, \beta)$ has a limit $(b_0, \lambda_0)$. Then there is a limit function $f_0$, a solution of the Schwarz equation (3) with the parameters $b_0$ and $\lambda_0$. The meromorphic function $f_0$ has three asymptotic values, 0, 1, $\infty$, and we are going to find the corresponding cell decomposition. Let $\Phi_1$ be the cell decomposition of the Riemann sphere with one vertex at the point 2 and two loops, $\gamma_\infty$ and $L$ (see figure 1, left). Let $\Psi_1 = f^{-1}(\Phi_1)$.

It is easy to construct $\Psi_1$ from the original cell decomposition $\Psi$. First, removing pre-images of $\gamma_\infty$ and $\gamma_3$, we obtain the cell decomposition $\Psi_\infty$, the pre-image of the loop around $\infty$ in $\Phi$. It is shown with the bold solid lines in figure 4.

Next, for each vertex $v$ of $\Psi$ consider the path $L_v$ consisting of the edge of $f^{-1}(\gamma_3)$ starting at $v$ and ending at some vertex $v'$, followed by the edge of $f^{-1}(\gamma_\infty)$ starting at $v'$ and ending at some vertex $v''$. Then the edge of $f^{-1}(L)$ from $v$ to $v''$ is homotopic to $L_v$ in the complement of $\Psi_\infty$. The new edges are shown with dashed lines in figure 4. The resulting cell decomposition is equivalent to $\Psi_1$.

Let $V$ be the set of the vertices of $\Psi$ contained in the boundary of the sector $S_3$. It is connected to the rest of the vertices of $\Psi$ only at one vertex ($v'$ in figure 4, left) which is also at the boundary of both sectors $S_2$ and $S_4$. The dashed line replacing the edges of $\Psi$ that connect $v'$ to the two adjacent vertices of $V$ ($v$ and $v''$ in figure 4, left), goes from $v$ to $v''$. All other dashed lines connect the vertices of $V \setminus \{v'\}$ with the other vertices from the same set. Hence $V \setminus \{v'\}$ is the set of vertices of a connected component of the 1-skeleton of the cell decomposition $\Psi_1$. This contradicts our assumption that $\Psi_1 = f_0^{-1}(\Phi_1)$, since the 1-skeleton of $f_0^{-1}(\Phi_1)$ must be connected. This contradiction proves the lemma. □

**Remark.** Consider all meromorphic functions with no critical points and at most six asymptotic values. These functions $f$ are defined by their asymptotic values and cell decompositions. Assume that one vertex $v_0$ of $\Psi$ is placed at $z = 0$ and normalize so that $f'(0) = 1$. The class of normalized functions obtained in this way is compact [12]. Let $f_\nu \to f_0$ be a converging sequence. The 1-skeletons of the corresponding cell decompositions $\Psi(\nu)$ converge to the 1-skeleton of the cell decomposition $\Psi(0)$ as embedded graphs with a marked vertex. If two asymptotic values collide in the limit, then one has to use the procedure described in the proof of lemma 1: replacing two loops by one loop. The limiting cell decomposition obtained in

---

1 Here, $\infty$ refers to a point added to the $(b, \lambda)$-plane in the one-point compactification.

2 Uniform convergence on compact subsets in the plane, with respect to the spherical metric in the target sphere.
Lemma 3 suggests that the eigenvalue problem (2) tends to a harmonic oscillator when $c \to 1$, the fact we will later prove by different arguments.

Lemma 2. For $k < 0$ and $l \geq 0$,
\[
\lim_{\beta \to \pi} F(X_{k,l}, \beta) = \infty.
\]

Proof. When $\beta \to \pi$, we have $c \to -1$, $c' \to -1$. Suppose by contradiction that $F(X_{k,l}, \beta)$ has a limit $(b_0, \lambda_0)$. Then there is a limit function $f_0$, a solution of the Schwarz equation (3) with the parameters $b_0$ and $\lambda_0$. The meromorphic function $f_0$ has three asymptotic values, $0$, $-1$, $\infty$, and we are going to find the corresponding cell decomposition.

To do this, it is convenient to choose another cell decomposition $\Phi_1'$ of the Riemann sphere, shown in the right part of figure 1 (solid lines). When $c \to -1$, $\Phi_1'$ collapses to $\Phi_1'$ where the two loops $\gamma'_{\infty}$ and $\gamma'_{c}$ are replaced with a single loop $L'$ around $-1$ (dashed line in figure 1, right).

We need the transition formula from $\Psi_1 = f^{-1}(\Phi_1)$ to $\Psi_1' = f^{-1}(\Phi'_1)$. This formula is obtained by combining the two decompositions (see figure 5) and expressing the loops of $\Phi'_1$ in terms of the loops of $\Phi$.

The formulas, using the notations in figure 5, are
\[
\gamma'_{\infty} = \alpha \beta, \quad \gamma'_{\infty} = \beta \alpha, \quad \gamma'_{c} = \beta \gamma, \beta^{-1}, \quad \gamma'_{c} = \alpha^{-1} \gamma \alpha.
\]
Here, the product should be read left to right. Similar formulas were obtained in [5] in the proof of theorem 4.1. The application of these transition formulas to the cell decomposition $\Psi$ of type $X_{-1,1}$ is illustrated in figures 6 and 7. In figure 6, the circles denote the vertices of $\Psi'$ (pre-images of the vertex of $\Phi'$) and the dotted lines correspond to the pre-images of $\gamma'_{c}$ and $\gamma'_{c'}$ determined from (5). The pre-images of $\gamma_{\infty}$ and $\gamma'_{\infty}$ coincide. They are shown with the bold solid line.

The same arguments as in lemma 1 show that the 1-skeleton of the degeneration $\Psi'_{-1}$ of $\Psi'$ as $\beta \to \pi$ (shown with dotted lines in figure 7) is not connected. This proves the lemma.

Lemmas 1 and 2 show that for $k < 0$, the charts $X_{k,m}$ with $2m - k = n$ cover connected components of $Z_{n+1}^{QES}(\mathbb{R})$, each parametrized by $\beta \in (0, \pi)$. We call these components $\Gamma_{n,m}$. These are simple disjoint analytically embedded curves in $\mathbb{R}^2$. 

![Figure 5. Two cell decompositions of figure 1 combined.](image)
Figure 6. Transition formulas (5) applied to the cell decomposition $\Psi$ in figure 4.

When $\beta \to 0$, $\pi$, we must have $b \to \pm \infty$. We will show below that $b \to +\infty$ on both ends of $\Gamma_{n,m}$ when $k < 0$.

When $n$ is odd (that is $J$ is even), these curves $\Gamma_{n,m}$ constitute the whole spectral locus $Z_{n+1}^{QES}(\mathbb{R})$. 

Figure 7. Cell decomposition $\Psi'$ (solid lines) corresponding to the cell decomposition $\Psi$ in figure 4.
Now consider the part of the spectral locus covered by the charts $X_k, l, k \geq 0$. This part is present only when $n = 2l$ is even.

**Lemma 3.** For $k \geq 0$ and $l \geq 0$, we have
\[
\lim_{\beta \to \pi} F(X_k, l, \beta) = \lim_{\beta \to 0} F(X_{k+1}, l, \beta)
\]
and
\[
\lim_{\beta \to 0} F(X_0, l, \beta) = \infty.
\]

**Proof of lemma 3.** This is similar to the arguments in lemmas 1 and 2. Computation is illustrated in figures 2 and 8–10.

In the left part of figure 8, we use $\Psi$ from the right part of figure 2. It corresponds to the tree $X_{1,1}$ in figure 8, right.

In figure 9, the circles denote the vertices of $\Psi'$ (pre-images of the vertex of $\Phi'$) and the dotted lines correspond to the pre-images of $\gamma'_c$ and $\gamma'_d$ determined from (5). The pre-images of $\gamma_\infty$ and $\gamma'_\infty$ coincide. They are shown with the bold solid line.

Removing the pre-images of $\gamma'_c$ and $\gamma'_d$ (thin solid lines in figure 9) and the vertices of $\Psi$, we obtain the cell decomposition $\Psi'$ shown in figure 10 (left) corresponding to the tree $X_{2,1}$ (right).

Thus, for $k \geq 0$, the cell decomposition $\Psi'_{k-1}$ of the plane obtained from $X_{k, l}$ in the limit $\beta \to \pi$ as the pre-image of $\Phi'_{k-1}$ is equivalent to the cell decomposition $\Psi_1$ obtained from $X_{k+1, l}$ in the limit $\beta \to 0$ as the pre-image of $\Phi_1$. Since $\Phi'_{-1} = -\Phi_1$, the Nevanlinna theory implies that the corresponding functions $f$ and $f'$ satisfy $f' = -f$. (The symbol $f'$ here should not be confused with the derivative.) Hence these two functions correspond to the same point of $Z^0_{\text{JES}}(\mathbb{R})$.

The proof of (7) is similar to that of lemma 1. This completes the proof of the lemma. □
Now we continue the proof of theorem 1.

For even $n = 2l$, charts $X_{k,l}$, $k \geq 0$ parametrize segments of one curve in the real QES spectral locus, and we call this curve $\Gamma_{n,n/2}$. We parametrize the curve $\Gamma_{n,n/2}$ by the real line, so that the number $k$ decreases as the parameter $t$ increases. Thus the chart $X_{0,n/2}$ corresponds...
Figure 11. The tree corresponding to $Y_3$.

to parameter values $t > t_0$. When the parameter $t$ on $\Gamma_{n,n/2}$ tends to $+\infty$, the asymptotic value $c = \exp(i\beta)$ tends to 1. On the other hand, when $t \to -\infty$ on $\Gamma_{n,n/2}$, the asymptotic value $c$ does not have a limit; it oscillates, passing each point of the unit circle infinitely many times.

The curves $\Gamma_{n,m}$ are disjoint. Indeed, different cell decompositions give different functions $f$. This proves the first two statements of theorem 1.

Now we deal with the asymptotic behavior of our curves $\Gamma_{n,m}$. We use the rescaling of (2) as in [6]. The QES spectral locus is defined by a polynomial equation $Q_{n+1}(b, \lambda) = 0$ which is of degree $n + 1$ in $\lambda$. So on a ray $b > b_0$, there are $n + 1$ branches $\lambda_j(b)$. In [6, equation (25)], we found that all $\lambda_j$ have asymptotics $\lambda_j(b) \sim b^2 + O(\sqrt{b})$, $b \to -\infty$, and as $b \to +\infty$, each QES eigenfunction $y_j$ tends to some eigenfunction $Y_\ell$ of the harmonic oscillator

$$-
Y'' + 4z^2Y = \mu Y, \quad Y(it) \to 0, \quad t \to \pm\infty. \quad (8)$$

The eigenvalues of this harmonic oscillator are $\mu_\ell = 2(2\ell + 1)$, $\ell = 0, 1, \ldots$.

Only one of the eigenfunctions $y_j$ can tend to a given $Y_\ell$, and the corresponding eigenvalue satisfies

$$\lambda_j(b) = b^2 + (\mu_\ell - 2\ell + o(1))\sqrt{b}, \quad b \to +\infty.$$

It follows that all $\lambda_j$ are real. The graph of each $\lambda_j$ is a part of a curve $\Gamma_{n,m}$, and each $\Gamma_{n,m}$ has only two ends.

Now we consider the degeneration of the $X_{0,1}$ chart with $l \geq 0$, the chart which parametrizes the right end of $\Gamma_{n,n/2}$, $n = 2l$. On the end of $\Gamma_{n,n/2}$, where $t \to -\infty$ in the parametrization described after lemma 3, there are infinitely many points $\Gamma_{n,n/2}(l_k)$ which belong to the real QES locus, and where the asymptotic value $c$ is real. It was proved in [6] that these are exactly those points where $Z_{jQES}^n(R)$ crosses the non-quasi-exactly solvable part of $Z_j(R)$, and these points correspond to $b_k \to -\infty$.

So only on one end of $\Gamma_{n,n/2}$ (where $t \to +\infty$) we can have $b \to +\infty$. On the other hand, each $\Gamma_{n,m}$, $m < n/2$ contains at most two graphs of $\lambda_j$. According to (4), the total number of these graphs $\lambda_j$ is $n + 1$, and the total number of curves $\Gamma_{n,m}$ is $(n + 1)/2$ when $n$ is odd and $n/2 + 1$ when $n$ is even. It follows that, when $n$ is odd, each $\Gamma_{n,m}$ contains two graphs of $\lambda_j$. When $n$ is even, each $\Gamma_{n,m}$ except one contains two graphs of $\lambda_j$, while the exceptional component $\Gamma_{n,n/2}$ contains one graph of $\lambda_j$. 

Thus $b \to +\infty$ as $c \to \pm 1$ in the $X_{k,1}$-charts with $k < 0$, which proves the third statement of theorem 1. To prove the last statement, we study zeros of the eigenfunctions as $b \to +\infty$.

The eigenfunction $Y_\ell$ of (8) corresponding to the eigenvalue $\mu_\ell$ has exactly $\ell$ zeros on $i\mathbb{R}$ and no other zeros in $\mathbb{C}$. One of these zeros is real iff $\ell$ is odd.

The trees corresponding to $Y_\ell$ (see figure 11) are constructed similarly to those corresponding to $y$, using the two loop cell decomposition of the sphere, consisting of $\gamma_{\infty}$ and the dashed loop in figure 1, left.

For general results on the convergence of Nevanlinna functions like our $f$ we refer to [12].

When $b \to +\infty$, each QES eigenvalue $\lambda(b)$ must tend to some $\mu_\ell$, and the corresponding QES eigenfunction tends to $Y_\ell$. Suppose that $\lambda(b) \in \Gamma_{n,m}$ with $m < n/2$. Then the tree corresponding to $\lambda(b)$ is $X_{k,m}$, $k < 0$, $2m - k = n$. From the arguments in the proofs of lemmas 1 and 2 (see figures 4 and 7), degeneration of the cell decomposition $\Psi$ corresponding to such a tree has a connected component with $2m$ bounded faces when $\beta \to 0$ and with $2m + 1$ bounded faces when $\beta \to \pi$. This implies that the corresponding eigenfunction can only converge to $Y_{2m}$ as $\beta \to 0$ and to $Y_{2m+1}$ as $\beta \to \pi$.

If $n$ is odd, these curves constitute the whole QES locus. If $n$ is even, there is one more branch $\lambda(b)$ of the QES locus for large positive $b$, the right end of $\Gamma_{n,n/2}$ corresponding to the tree $X_0,n/2$. From the proof of lemma 3 (see figure 2, left) degeneration of the cell decomposition $\Psi$ corresponding to such a tree has a connected component with $n$ bounded faces when $\beta \to 0$. This implies that the corresponding eigenfunction can only converge to $Y_n$.

So the ordering of the ends of the curves $\Gamma_{n,m}$ corresponds to the natural ordering of the first $n + 1$ eigenvalues of the harmonic oscillator. This completes the proof.

Acknowledgment

Both authors are supported by NSF grant DMS-1067886.

References

[1] Alexandersson P and Gabrielov A 2012 On eigenvalues of the Schrödinger operator with a complex-valued polynomial potential *Comput. Method. Funct. Theory* 12 119–44
[2] Bakken I 1977 A multiparameter eigenvalue problem in the complex plane *Am. J. Math.* 99 1015–44
[3] Bender C and Boettcher S 1998 Quasi-exactly solvable quartic potential *J. Phys. A: Math. Gen.* 31 L273–7
[4] Eremenko A and Gabrielov A 2009 Analytic continuation of eigenvalues of a quartic oscillator *Commun. Math. Phys.* 287 431–57
[5] Eremenko A and Gabrielov A 2011 Singular perturbation of polynomial potentials in the complex domain with applications to PT-symmetric families *Moscow Math. J.* 11 473–503
[6] Eremenko A and Gabrielov A 2011 Quasi-exactly solvable quartic: elementary integrals and asymptotics *J. Phys. A: Math. Theor.* 44 312001
[7] Eremenko A and Merenkov A 2005 Nevanlinna functions with real zeros *Illinois J. Math.* 49 1093–110
[8] Masoero D 2010 Y-system and deformed thermodynamic Bethe ansatz *Lett. Math. Phys.* 94 151–64
[9] Nevanlinna R 1932 Über Riemannsche Flächen mit endlich vielen Windungspunkten *Acta Math.* 58 295–373
[10] Shin K C 2010 All cubic and quartic polynomials $P$ for which $f'' + P(z)f = 0$ has a solution with infinitely many real zeros and at most finitely many non-real zeros, *Proc. 2010 Spring Southeastern Sectional Meeting* (Lexington, KY, 27–28 March 2010) Abstracts AMS 1057-34-26
[11] Shibuya Y 1975 Global Theory of a Second Order Linear Ordinary Differential Equation with a Polynomial Coefficient (Amsterdam: North-Holland)
[12] Volkovyski L 1948 Converging sequences of Riemann surfaces *Mat. Sb.* 23 361–82