Casimir stress for concentric spheres in de Sitter space

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Abstract

The Casimir stress on two concentric spherical shell in de Sitter background for massless scalar field is calculated. The scalar field satisfies Dirichlet boundary conditions on the spheres. The metric is written in conformally flat form to make maximum use of Minkowski space calculations. Then the Casimir stress is calculated for inside and outside of the shell with different backgrounds. This model may be used to study the effect of the Casimir stress on the dynamics of the domain wall formation in inflationary models of early universe.

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1 Introduction

The Casimir effect is one of the most interesting manifestations of nontrivial properties of the vacuum state in quantum field theory[1,2]. Since its first prediction by Casimir in 1948[3] this effect has been investigated for different fields having different boundary geometries[4-7]. The Casimir effect can be viewed as the polarization of vacuum by boundary conditions or geometry. Therefore, vacuum polarization induced by a gravitational field is also considered as Casimir effect.

Casimir effect for spherical shells in the presence of the electromagnetic fields has been calculated several years ago[9, 10]. A recent simplifying account of it for the cases of electromagnetic and scalar fields with both Dirichlet and Neumann boundary conditions on sphere is given in[11]. The dependence of Casimir energy on the dimension of space for electromagnetic and scalar fields with Dirichlet boundary conditions in the presence of a spherical shell is discussed in[12, 13]. The Casimir energy for odd and even space dimensions and different fields, including the spinor field, and all the possible boundary conditions have been considered in[14]. There it is explicitly shown that although the Casimir energy for interior and exterior of a spherical shell are both divergent, irrespective of the number of space dimensions, the total Casimir energy of the shell remains finite for the case of odd space dimensions. Heat kernel coefficients and zeta function of the Laplace operator on a D-dimensional ball with different boundary conditions, both of them useful tools to calculate Casimir energies, have been calculated in[15], [16]. More recently a new method has been developed for the scalar Casimir stress on the D-dimensional sphere, in[24]. In this reference the regularized vacuum expectation values for the scalar field energy-momentum tensor inside and outside a spherical shell and in the region between two concentric spheres have been calculated. Of some interest are cases where the field is confined to the inside of a spherical shell. This is sometimes called the bag boundary condition. The application of Casimir effect to the bag model is considered for the case of massive scalar field[17] and the Dirac field[18]. We use the renormalization procedure in the above cases for our problem.

Casimir effect in curved space-time has not been studied extensively. Casimir effect for spherical boundary in curved space-time is considered in[19, 20] where the Casimir energy for half of $S^3$ and $S^2$ with Dirichlet and Neumann boundary conditions for massless conformal scalar field is calculated analytically using all the existing methods. Casimir effect in the presence of a general relativistic domain wall is considered in[21] and a study of the relation between trace anomaly and the Casimir effect can be found in[22]. Casimir effect may have interesting implications for the early universe. It has been shown, e.g., in[23] that a closed Robertson-Walker space-time in which the only contribution to the stress tensor comes from Casimir energy of a scalar field is excluded. In inflationary models, where the dynamics of bubbles may play a major role, this dynamical Casimir effect has not yet been taken into account. Let us mention that in[25] we have investigated the Casimir effect of a massless scalar field with Dirichlet boundary condition in spherical shell having different vacuua inside and outside which represents a bubble in early universe with false/true vacuum inside/outside. In this reference the sphere have zero thickness. In the present paper we shall extend our analysis to the spherical shell with nonvanishing thickness.

Our aim is to calculate the Casimir stress on two concentric spherical shell with constant comoving radius having different vacuua inside and outside in de Sitter space. This config-
uration is corresponding to a spherical symmetry domain wall with thickness. We assume the inner and outer regions of thick shell are in a vacuum corresponding to degenerate vacuum in domain wall configuration. This example is similar to our recent study of planar cosmic domain wall [26].

The organization of the paper is as follows. In section two we consider two concentric sphere in flat space-time, we shall rely on the result of [24] and consider vacuum stress on each single spheres. In section three we obtain renormalized Casimir energy for each single sphere in a de Sitter space. We use a procedure similar to that of our previous work [25], then we calculate total stress on the sphere due to the boundary conditions and gravitational vacuum polarization. In last section we conclude and summarize the results.

2 Scalar Casimir stress for concentric spheres in flat space-time

We consider two concentric spherical shells with zero thickness and with radii \(a\) and \(b\), \(a < b\). Consider now the Casimir force due to fluctuation of a free massless scalar field satisfying Dirichlet boundary conditions on the spherical shells in Minkowski space-time. The vacuum force per unit area of the inner sphere is given by [24]

\[
F_a(a, b) = F(a) - P_a(a, b, a),
\]

where \(F(a)\) is the force per unit area of a single sphere with radius \(a\), and \(P_a(a, b, a)\) is due to the existence of the second sphere (interaction force). In a similar manner vacuum force acting on per unit area of outer sphere is

\[
F_b(a, b) = F(b) + P_b(a, b, b).
\]

The vacuum force per unit area of a single sphere is the sum of Casimir forces \(F_{in}\) and \(F_{out}\) for inside and outside of the shell.

\[
F(a) = F_{in}(a) + F_{out}(a), \quad F(b) = F_{in}(b) + F_{out}(b).
\]

Just as described in [13, 25], Casimir force inside and outside of the shell are divergent individually. In flat space when we add interior and exterior forces to each other, divergent parts will cancel each other out. Interaction forces \(P_a(a, b, a)\) and \(P_b(a, b, b)\) are finite and for Dirichlet boundary condition, are given by

\[
P_a(a, b, a) = \frac{-1}{8\pi^2 a^3} \sum_{l=0}^{\infty} (2l + 1) \int_0^\infty dz \frac{K_{\nu}^{(b)}(bz) / K_{\nu}^{(a)}(az)}{K_{\nu}^{(a)}(az) I_{\nu}^{(b)}(bz) - K_{\nu}^{(b)}(bz) I_{\nu}^{(a)}(az)},
\]

\[
P_b(a, b, b) = \frac{-1}{8\pi^2 b^3} \sum_{l=0}^{\infty} (2l + 1) \int_0^\infty dz \frac{I_{\nu}^{(a)}(az) / I_{\nu}^{(b)}(bz)}{K_{\nu}^{(a)}(az) I_{\nu}^{(b)}(bz) - K_{\nu}^{(b)}(bz) I_{\nu}^{(a)}(az)},
\]

where \(I_{\nu}\) and \(K_{\nu}\) are modified Bessel function, and can be deduced from Eq.(5.15) in Ref. [7] These quantities are always negative.
3 Scalar Casimir stress for concentric spheres in de Sitter space

Consider now the system of two concentric spheres in de Sitter space. To make the maximum use of the flat space calculation we use the de Sitter metric in conformally flat form

$$ds^2 = \frac{\alpha^2}{\eta^2} [d\eta^2 - \sum_{i=1}^{3} (dx^i)^2],$$

where $\eta$ is the conformal time:

$$-\infty < \eta < 0.$$  

The relation between parameter $\alpha$ and cosmological constant $\Lambda$ is given by

$$\alpha^2 = \frac{3}{\Lambda}.$$  

(8)

Under conformal transformation in four dimensions, the vacuum forces inside and outside for a single sphere with zero thickness are given by (see [25])

$$\tilde{F}_{in} = \frac{\eta^2}{\alpha_{in}^2} F_{in} = \frac{\Lambda_{in} \eta^2}{3} F_{in},$$

$$F_{out} = \frac{\eta^2}{\alpha_{out}^2} F_{out} = \frac{\Lambda_{out} \eta^2}{3} F_{out}.$$  

(10)

In this case when we add interior and exterior forces to each other, the Casimir force becomes divergent. To obtain $\tilde{F}$, we use a procedure similar to that of [25]. First of all we consider that the inside region of the sphere with radius $a$ and outside region of the sphere with radius $b$ have cosmological constant $\Lambda$ and the region between two spheres have cosmological constant $\Lambda'$. Therefore we have [25]

$$\tilde{E}_{in}(a) = \frac{\eta^2 \Lambda}{6a} (c_1 + \frac{c'_1}{\varepsilon}), \quad \tilde{E}_{out}(a) = \frac{\eta^2 \Lambda'}{6a} (c_2 - \frac{c'_1}{\varepsilon})$$

$$\tilde{E}_{in}(b) = \frac{\eta^2 \Lambda'}{6b} (c_1 + \frac{c'_1}{\varepsilon}), \quad \tilde{E}_{out}(b) = \frac{\eta^2 \Lambda}{6b} (c_2 - \frac{c'_1}{\varepsilon})$$

(11)

(12)

where $c_1 = 0.008873, c_2 = -0.003234, c'_1 = 0.001010$ and $\varepsilon$ is cutoff parameter. As we see, each of the energies for in-and outside of the shells are divergent.

The energy for each single sphere is as follows:

$$\tilde{E}(a) = \frac{\eta^2}{6a} [(c_1 \Lambda + c_2 \Lambda') + \frac{c'_1}{\varepsilon}(\Lambda - \Lambda')],$$

$$\tilde{E}(b) = \frac{\eta^2}{6b} [(c_1 \Lambda' + c_2 \Lambda) + \frac{c'_1}{\varepsilon}(\Lambda' - \Lambda)].$$

(13)

(14)

To renormalize the above Casimir energy we use a procedure similar to that of our previous paper [25]. We consider the classical energy for each sphere. The classical energy of a sphere immersed in a cosmological background, as we are considering, may be written as

$$E_{class} = PR^2 + \sigma R^2 + FR + K + \frac{\hbar}{R}.$$  

(15)
In this way classical energy of spherical shell is determined by above parameter, where \( R \) is radius of spheres. Therefore total energy of each sphere is given by

\[
\tilde{\mathcal{E}}(a) = \tilde{\mathcal{E}}(a) + \mathcal{E}_{\text{class}}(a), \quad \tilde{\mathcal{E}}(b) = \tilde{\mathcal{E}}(b) + \mathcal{E}_{\text{class}}(b).
\] (16)

The renormalization can be achieved now by shifting the parameter \( h \) in \( E_{\text{class}} \) by an amount which cancels the divergent contribution. For inner and outer spheres we have

\[
h \to h + \frac{\eta^2 c_1'}{6} (\Lambda' - \Lambda),
\] (17)

\[
h \to h + \frac{\eta^2 c_1'}{6} (\Lambda - \Lambda').
\] (18)

After the renormalization we obtain for the Casimir energies

\[
\tilde{\mathcal{E}}_{\text{ren}}(a) = \frac{\eta^2}{6a} (c_1 \Lambda + c_2 \Lambda'),
\] (19)

\[
\tilde{\mathcal{E}}_{\text{ren}}(b) = \frac{\eta^2}{6b} (c_1 \Lambda' + c_2 \Lambda).
\] (20)

Now we use the following relation for the stress on the shell

\[
F = \frac{-1}{4\pi a^2} \frac{\partial \mathcal{E}}{\partial a}.
\] (21)

Then the stresses on the each single shell due to the boundary condition are given by

\[
\frac{\tilde{F}(a)}{A} = \frac{-1}{4\pi a^2} \frac{\partial \tilde{\mathcal{E}}(a)}{\partial a} = \frac{\eta^2}{24\pi a^4} (c_1 \Lambda + c_2 \Lambda'),
\] (22)

\[
\frac{\tilde{F}(b)}{B} = \frac{-1}{4\pi b^2} \frac{\partial \tilde{\mathcal{E}}(b)}{\partial b} = \frac{\eta^2}{24\pi b^4} (c_1 \Lambda' + c_2 \Lambda).
\] (23)

Under conformal transformation, interaction forces are given by

\[
\tilde{P}_a(a, b, a) = \frac{\eta^2 \Lambda'}{3} P_a(a, b, a), \quad \tilde{P}_b(a, b, b) = \frac{\eta^2 \Lambda'}{3} P_b(a, b, b).
\] (24)

Therefore the total stress on the spheres due to boundary conditions are obtained

\[
\frac{F_a(a, b)}{A} = \frac{F(a)}{A} - \tilde{P}_a(a, b, a) = \frac{\eta^2}{24\pi a^4} (c_1 \Lambda + c_2 \Lambda') - \frac{\eta^2}{3} P_a(a, b, a),
\] (25)

\[
\frac{F_b(a, b)}{B} = \frac{F(b)}{B} + \tilde{P}_b(a, b, b) = \frac{\eta^2}{24\pi b^4} (c_1 \Lambda' + c_2 \Lambda) + \frac{\eta^2}{3} P_b(a, b, b).
\] (26)

Now we consider the pure effect of vacuum polarization due to gravitational field without any boundary conditions. The renormalized stress tensor for massless scalar field in de sitter space is given by [8, 27]

\[
\langle T_{\mu\nu} \rangle = \frac{1}{960\pi^4} \delta_{\mu\nu}.
\] (27)

Now, the effective pressure created by above gravitational part is different for different parts of space-time:

\[
P_{\text{in}}^g(a) = \frac{-\Lambda^2}{8640\pi^2}, \quad P_{\text{out}}^g(a) = \frac{-\Lambda'^2}{8640\pi^2}.
\] (28)
\[ P^g_{\text{in}}(b) = -\frac{\Lambda^2}{8640\pi^2} \quad P^g_{\text{out}}(b) = -\frac{\Lambda^2}{8640\pi^2}. \] (29)

Therefore, gravitational pressures over spheres, are given by
\[ P^g(a) = -\frac{1}{8640\pi^2}(\Lambda^2 - \Lambda'^2), \quad P^g(b) = -\frac{1}{8640\pi^2}(\Lambda'^2 - \Lambda^2). \] (30)

The total stress on the spherical shells, is then given by
\[ P^{\text{tot}}(a) = \frac{\eta^2}{24\pi a^4}(c_1\Lambda + c_2\Lambda') - \frac{\eta^2\Lambda'}{3}P_a(a, b, a) - \frac{1}{8640\pi^2}(\Lambda^2 - \Lambda'^2), \] (31)
\[ P^{\text{tot}}(b) = \frac{\eta^2}{24\pi b^4}(c_1\Lambda' + c_2\Lambda) + \frac{\eta^2\Lambda'}{3}P_b(a, b, b) - \frac{1}{8640\pi^2}(\Lambda'^2 - \Lambda^2). \] (32)

In \( b \to \infty \) the outer sphere disappears. In this case one can see from large \( z \) behavior of \( I_\nu(z) \) and \( K_\nu(z) \), that interaction force \( P_a(a, b, a) \) vanishes. Therefore \( P^{\text{tot}}(a) \) is exactly the result of Ref. \([25]\) for total pressure on spherical shell with zero thickness.

In another limiting case, when \( a, b \to \) and \( b - a = l \), the system of two concentric sphere transform to the two parallel plates configuration with distance \( l \). In this limit the first term in Eqs. (31) and (32) vanishes. Using asymptotic formula for Bessel function one can show that interaction forces in Eqs. (31) and (32) change as standard result for the Casimir forces acting on parallel plates configuration (see section 10 of Ref [7]). These total pressures may be both negative or positive. To see the different possible cases, let us first assume \( \Lambda > \Lambda' \), noting that \( c_1 > c_2 \), then
\[ c_1\Lambda + c_2\Lambda' > 0, \] (33)
therefore \( \bar{F}(b) > 0 \). Now noting that \( P_a(a, b, a) < 0 \), second term in Eq. (31) is always positive, therefore the Casimir force on the inner shell is repulsive, but in this case the gravitational part \( P^g(a) \) is negative. Therefore the total pressure \( P^{\text{tot}}(a) \) may be either negative or positive. Given \( P^{\text{tot}}(a) > 0 \) initially, then the initial expansion of the shell leads to a change of the Casimir part of the pressure. This change, depending on the details of the dynamics of the shell, may be an increase or a decrease. Therefore, the initial expansion of the shell may end and a contraction phase begins. Given \( P^{\text{tot}}(a) < 0 \), there is an initial contraction which ends up at a minimum radius. For the case \( \Lambda > \Lambda' \) situation of the outer sphere is as follows:

Let us first assume
\[ c_1\Lambda' + c_2\Lambda > 0, \] (34)
then \( \bar{F}(b) > 0 \), the gravitational part \( P^g(b) \) is positive also, second term is negative. Therefore the total pressure \( P^{\text{tot}}(b) \) may be either negative or positive. In this case situation is similar to the inner sphere. Now consider the case
\[ c_1\Lambda' + c_2\Lambda < 0. \] (35)

In this case \( \bar{F}(b) < 0 \), the gravitational part is positive, interaction term is negative and above discussion for recent situation is also correct.

Now consider the case \( \Lambda < \Lambda' \) and also
\[ c_1\Lambda + c_2\Lambda' > 0. \] (36)
Similar to the previous for inner shell Casimir force is repulsive, the gravitational part is also repulsive, then the total pressure $P_{\text{tot}}(a)$ is always positive. Therefore the inner shell expands without any limitation.

Let us assume

$$c_1 \Lambda + c_2 \Lambda' < 0,$$

then $\frac{\bar{F}(a)}{A} < 0$, therefore the total pressure $P_{\text{tot}}(a)$ may be either negative or positive, and all cases of contraction, expansion may be appear.

Now we consider the situation of outer sphere, noting that $\Lambda < \Lambda'$, then $\frac{\bar{F}(b)}{B}$ is positive, interaction force and the gravitational part are negative. Then the total pressure $P_{\text{tot}}(b)$ may be repulsive or contractive.

4 Conclusion

We have considered two concentric spherical shells in de Sitter space with a massless scalar field, coupled conformally to the background, satisfying the Dirichlet boundary conditions. Our calculation shows that interaction force between two spheres in de Sitter space, similar to the flat space is negative, therefore, interaction forces between two spheres are attractive, similar to the parallel plate configuration. There is another similarity between interaction force in concentric spheres and boundary part force acting on parallel plates in de Sitter space, as one can see in our previous work [26], the boundary part pressure acting on plates depends only on the cosmological constant between the plates, which is like the case of interaction force between concentric spheres that as well depends on the cosmological constant in spherical layer region.

The final result for total pressure, which in this paper has been obtained, in the limit $b \rightarrow \infty$, corresponds to the result of [25] for spherical shell with zero thickness. In the limit $a, b \rightarrow \infty$ and $b - a = l$ corresponds to the result of [26] for parallel plates configuration with distance $l$. Total stress, which acts on the each single spheres shows that the detail dynamics of spherical symmetry domain wall depends on different parameters, and all cases of contraction and expansion may appear.

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