Probing the Icy Shell Structure of Ocean Worlds with Gravity–Topography Admittance

Ryunosuke Akiba1, Anton I. Ermakov2, and Burkhard Militzer1,3

1 Department of Earth and Planetary Science, University of California, Berkeley, CA 94720, USA; ryusterakiba@berkeley.edu
2 Space Sciences Laboratory, University of California, Berkeley, CA 94720, USA
3 Department of Astronomy, University of California, Berkeley, CA 94720, USA

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Abstract

The structure of the icy shells of ocean worlds is important for understanding the stability of their underlying oceans as it controls the rate at which heat can be transported outward and radiated to space. Future spacecraft exploration of the ocean worlds (e.g., by NASA’s Europa Clipper mission) will allow for higher-resolution measurements of gravity and shape than currently available. In this paper, we study the sensitivity of gravity–topography admittance to the structure of icy shells in preparation for future data analysis. An analytical viscous relaxation model is used to predict admittance spectra given different shell structures determined by the temperature-dependent viscosity of a tidally heated, conductive shell. We apply these methods to the ocean worlds of Europa and Enceladus. We find that admittance is sensitive to the mechanisms of topography support at different wavelengths and estimate the required gravity performance to resolve transitions between these mechanisms. We find that measuring admittance would be complementary to ice-penetrating radar in constraining shell thickness as it would cover the cases for which a direct ice–ocean interface detection is less favorable with radar, i.e., for warmer and thicker shells. Finally, we find that admittance may be used to constrain the tidal dissipation within the icy shell, which would be complementary to a more demanding measurement of the tidal phase lag.

Unified Astronomy Thesaurus concepts: Ocean planets (1151); Europa (2189); Saturnian satellites (1427); Tides (1702)

1. Introduction

The existence of subsurface oceans within icy satellites is of great interest due to their potential habitability. In this paper, we focus on two such ocean worlds: Jupiter’s moon Europa and Saturn’s moon Enceladus. The presence of a global subsurface ocean on Europa has been inferred from observations by Galileo of an induced magnetic field. A global salty ocean is believed to be the conductive fluid causing the induced magnetic field (Khurana et al. 1998). The presence of liquid water in Enceladus was inferred from observations of water vapor plumes (Porco et al. 2006). Thomas et al. (2016) inferred the global nature of the ocean from observations of the large amplitudes of the physical libration.

Considering life as we know it, the presence of large amounts of liquid water, essential elements from chondritic material at the base of the ocean, energy from tides, and radiogenic sources as well as chemical gradients make ocean worlds promising astrobiology targets (Hand et al. 2009). A major consideration for assessing habitability is the persistence and stability over geologic timescales of a liquid-water ocean to allow for life to develop and be sustained. The structure of the overlying icy shell, its thickness, heat transport mechanism, and the spatial distribution of tidal heating are particularly important for understanding the overall heat budget and the long-term survivability of the ocean.

For Europa, the structure of the icy shell has been studied using the classical Airy isostasy model and observations of topography (e.g., Kadel et al. 2000), flexural analysis (e.g., Billings & Kattenhorn 2005; Nimmo et al. 2011), and the tidal-convective equilibrium model (e.g., Moore 2006), but large uncertainty remains for the mean shell thickness. For Enceladus, shell thickness was constrained by gravity, shape, and libration data (e.g., Hemingway & Mittal 2019). However, for both Europa and Enceladus, the properties of the icy shell that govern heat transport, such as viscosity and temperature profile, remain poorly constrained.

In this paper, we explore the use of gravity–topography admittance to study the icy shells of the ocean worlds Europa and Enceladus. Gravity–topography admittance $Z_n$ is defined as a wavelength-dependent ratio of gravity to topography spectral amplitudes:

$$Z_n = \frac{\text{gravity amplitude}}{\text{shape amplitude}}$$

at spherical harmonic degree $n$ [mGal km$^{-1}$],

where gravity and shape data are obtained from spacecraft observations. Gravity–topography admittance can be computed from the spherical harmonic expansion coefficients of gravity and shape up to the degree of the lower-resolution data set. Typically, the accuracy and resolution of admittance are limited by the quality of the gravity data set.

Current observations of the gravity and shape of Europa and Enceladus from the Galileo and Cassini spacecraft, respectively, are limited to long-wavelength measurements. In terms of spherical harmonic expansions, gravity fields have been measured up to spherical harmonic degree 2 for Europa (spatial scale of 4900 km) (Anderson et al. 1998; Gomez Casajus et al. 2021) and up to degree 3 for Enceladus (Iess et al. 2014) (spatial scale of 528 km). Nimmo et al. (2007) made ellipsoidal fits of Europa’s shape, and Enceladus’ shape has been mapped up to spherical harmonic degree 16 (Tajeddine et al. 2017) (spatial scale of 100 km). Future missions such as NASA’s upcoming Europa Clipper will deliver higher-resolution measurements of...
the gravity and shape, allowing the determination of gravity–topography admittance over a larger range of spatial scales. The Europa Clipper mission may resolve the gravity up to degree 10 (Park et al. 2011). Our goal is to prepare for these future higher-resolution data by exploring the sensitivity of gravity–topography admittance to various aspects of the icy shell structure. We define sensitivity as a partial derivative of a measured quantity (i.e., admittance) with respect to the quantity of scientific interest (e.g., shell thickness). As we will discuss later in the manuscript, the sensitivity itself changes depending on the assumed shell structure.

The gravity–topography admittance bears clues to the icy shell structure, specifically to topography support mechanisms. The dominant topography support mechanism could vary depending on the spatial scale. The Airy isostasy model, used extensively for Earth (e.g., Watts 2001), describes a surface topography supported by a buoyancy force arising due to the topography of a density–discontinuity interface (i.e., a crustal root) located at a certain depth called “depth of compensation.” Topographic loads can also be supported by elastic and viscous stresses within the body. Because the interface between the icy shell and ocean is a phase boundary, melting and freezing can produce nonhydrostatic basal topography, inducing a flow throughout the shell (Cadek et al. 2019). These different topography support mechanisms influence the amplitudes of topography at different wavelengths at the surface and the base of the shell, which affect the moon’s gravity field and, therefore, are reflected in the admittance spectrum.

In summary, the goals of the paper are

1. to provide an algorithm for computing the gravity–topography admittance suitable for icy shells with large gradients of viscosity; and
2. to explore the sensitivity of gravity–topography admittance to the viscosity profile, shell tidal heating, and shell thickness.

### 2. Methods

In our exploration of the sensitivity of admittance to the icy shell structure, we follow the process laid out in Figure 1. We model Europa and Enceladus with three layers, placing an icy shell on top of a liquid-water ocean overlying a solid mantle. For each choice of shell thickness, the density of the mantle and the thickness of the ocean are computed to satisfy the total mass, radius, and moment of inertia factor of the moon as given in Table 1. In addition, the viscoelastic parameters shown in Table 1 are assigned to each layer and are used in the tidal heating model. A temperature profile is found by solving the conductive heat equation with tidal heating as a source term. Because tidal heating depends on a temperature-dependent ice rheology, we compute the temperature profile iteratively by alternating calculations of tidal heating and heat conduction to arrive at a steady-state solution. We use this converged temperature profile to determine the shell’s viscosity profile. The viscosity profile is then used in a viscous relaxation model to find the shape amplitudes at the surface and the base of the icy shell, treating the ice–ocean boundary as either a material or a phase boundary. The material boundary assumes no mass flows across the ice–ocean boundary, while a phase boundary describes a dynamic equilibrium between radial flow of relaxing ice and phase transitions, which keep the boundary shape constant. The viscous relaxation model is equivalent to a viscoelastic approach when considering the asymptotic state, which is unaffected by the initial elastic response (Beuthe 2021). Finally, the relation between the shell’s surface and base amplitude at each spherical harmonic degree is used to obtain the expected gravity–topography admittance spectrum. We will now proceed with a detailed description of these steps.

#### Figure 1.
Flowchart for our exploration of the icy shell structure.

#### Table 1
Model Parameters

| Parameter                          | Europa      | Enceladus   |
|------------------------------------|-------------|-------------|
| Radius (km)                        | 1560.8 ± 0.3 | 252.24 ± 0.2 |
| Ice shell density (kg m$^{-3}$)    | 920         |             |
| Ocean density (kg m$^{-3}$)        | 1050        |             |
| Poisson’s ratio of the shell       | 0.33        |             |
| Poisson’s ratio of the mantle      | 0.33        |             |
| Shear modulus of the shell (GPa)   | 3.3         |             |
| Shear modulus of the mantle (GPa)  | 40          |             |
| Bulk modulus of the ocean (GPa)    | 2.15        |             |
| Viscosity of the mantle (Pa s)     | ∞           |             |

Note. We use the central values in our models.
2.1. Assumptions of Ice Rheology

The rheologic model of ice governs its response to applied stresses. We assume that the icy shell flows viscously on geologic timescales. The properties of ices under conditions relevant to icy moons are poorly understood, largely due to the difficulty in reproducing low-stress, frequency, and temperature conditions with laboratory experiments (≤0.1 MPa; Tobie et al. 2003). The flow mechanisms of diffusion creep and grain boundary sliding (GBS) are most relevant for the low stress and strain rates, low temperatures, and small grain sizes expected in icy moons (Goldsbay & Kohlstedt 2001). The diffusion creep flow mechanism, extrapolated to these low stresses from experimental data and described by Goldsbay & Kohlstedt (2001), results in a Newtonian flow, in which the strain rate is proportional to stress. GBS has lower activation energy compared to diffusion creep and, unlike diffusion creep, is non-Newtonian, which introduces a stress dependence of viscosity. Diffusion creep likely dominates at the conditions and grain sizes in the icy shell (Moore 2006). Thus, we assume viscosity is independent of stress and the flow described by the diffusion creep deformation mechanism is Newtonian (e.g., Tobie et al. 2003; Showman & Han 2004; Mitri & Showman 2005). Under these assumptions, the temperature-dependent viscosity of pure water ice is described by

$$\eta(T) = \eta_{\text{melt}} \exp \left( \frac{E_a}{R T_{\text{m}} (T - 1)} \right),$$

(2)

where $E_a$ is the activation energy of diffusion creep, taken to be 59.4 kJ mol$^{-1}$ (Goldsbay & Kohlstedt 2001); $T_m$ is the melting temperature of ice, taken to be 273 K; and $R = 8.314$ J mol$^{-1}$ K$^{-1}$ is the universal gas constant. The grain size dependency is included within the viscosity at the melting point $\eta_{\text{melt}}$. The grain size is estimated to be in the range from 0.1 to 1 mm (Kirk & Stevenson 1987; McKinnon 1999), corresponding to $\eta_{\text{melt}}$ between 10$^{13}$ and 10$^{15}$ Pa.s.

There are several simplifications in our relaxation modeling. We do not model solid-state convection within the shell. Convection, if it occurs, is most likely confined to the bottom part of the shell and is less likely to occur in thinner shells (e.g., McKinnon 1999). The inferred large amplitude of the basal shell topography at Enceladus argues against convection as convection would likely rapidly relax this topography (Hemingway & Mittal 2019). In addition, the rheology of ice at the base of the shell near the melting point can be influenced by premelting and partial melting that would reduce viscosity, enhancing the flow (Tobie et al. 2003). Additionally, impurities within the ice, such as salts and silicates, can affect grain sizes and increase viscosity (Barr & Showman 2009).

At the surface, applying Equation (2) at the equilibrium surface temperatures of Europa (92 K) or Enceladus (59 K) implies that ice has a viscosity on the order of 10$^{30}$ and 10$^{30}$ Pa.s, respectively. Such high viscosities will prevent the relaxation of surface topography on geologic timescales. Passey (1983) and Bland et al. (2012) find that craters on Enceladus are morphologically relaxed despite these low surface temperatures. Passey (1983) determined a range of surface viscosities on Enceladus between 10$^{24}$ and 10$^{25}$ Pa.s to explain the relaxation of the observed craters and suggested that an insulating layer at the surface could increase the effective surface temperature and thus decrease viscosity. On Europa, Showman & Han (2004) argue that fractures in ice and brittle deformation could affect ice rheology at the surface and parameterize this more complex rheology by also imposing an upper viscosity bound. We follow this simplification of a bounded viscosity used by Showman & Han (2004) and Cadek et al. (2017) for our viscous relaxation model and consider values $\eta_{\text{bound}} = 10^{22}$–$10^{25}$ Pa.s. We compute tidal heating (Section 2.2) and the temperature structure of the shell (Section 2.3) using unbounded viscosity, and the resulting viscosity profile is subsequently set to $\eta_{\text{bound}}$, where $\eta(t) > \eta_{\text{bound}}$, to compute the relaxation of the shell (Section 2.4). Finally, we note that surface viscosity may have varied over time following the thermal evolution of the satellite.

2.2. The Effect of Tidal Heating

As an icy moon orbits its parent planet, tidal forces cause a periodic deformation of the moon resulting in heat production. In this paper, we introduce tidal heating within the icy shell through its influence on the temperature profile and, thus, the viscosity profile through Equation (2). We use the Maxwell rheology to model the viscoelastic deformation of a spherically symmetric, layered body under the influence of tidal potential. For a chosen shell thickness and viscosity profile, our layered model of a moon is described by the viscoelastic and orbital parameters listed in Table 1. While the mantle and ocean are treated as single layers with constant viscoelastic parameters, the icy shell is broken up into 80 layers prescribed by the viscosity profile. The degree 2 tidal potential expanded to first order in eccentricity for a tidally locked satellite is used to set the surface boundary condition for potential. This potential drives the deformation of the viscoelastic shell causing tidal heating. We follow Tobie et al. (2005) to compute a radial profile of the surface-averaged volumetric tidal dissipation rate, $h_\text{tidal}(r)$. We ignore the lateral variation of tidal heating because it is small compared to the radial variation. We use $h_\text{tidal}(r)$ as a source term in the heat conduction equation in the following Section 2.3. A full description of the tidal dissipation model and the computation of $h_\text{tidal}(r)$ is given in Appendix A.

Tidal dissipation is maximized in the portion of the shell where Maxwell time, defined as the ratio of viscosity to the elastic shear modulus, is close to the forcing period (e.g., Tobie et al. 2003). The viscosity for which this occurs can be expressed as $\eta_{\text{Maxwell}} = \mu / \omega$. For the tidal forcing at Europa’s orbital frequency $\omega = 2.0 \times 10^{-7}$ rad s$^{-1}$ and taking the elastic shear modulus $\mu = 3.3$ GPa, maximum dissipation occurs at a viscosity $\eta_{\text{Maxwell}} = 1.6 \times 10^{14}$ Pa.s. For Enceladus, a higher orbital frequency $\omega = 5.3 \times 10^{-7}$ rad s$^{-1}$ leads to maximum dissipation at a lower viscosity $\eta_{\text{Maxwell}} = 6.2 \times 10^{13}$ Pa.s. As discussed in Section 2.1, the viscosity at the base of the shell $\eta_{\text{melt}}$ ranges from 10$^{13}$ to 10$^{15}$ Pa.s covering a range of grain sizes. Since $\eta_{\text{Maxwell}}$ is close to $\eta_{\text{melt}}$, we expect that our tidal heating model will produce maximum tidal dissipation near the base of the shell. In summary, the amount of tidal heating and the shell temperature profile depend strongly on the assumed melt viscosity.

2.3. Temperature Structure

Conductive heat transport through a tidally heated shell defines its temperature profile, which controls its viscosity profile. Because tidal heating depends on the viscosity profile, we need to solve for tidal dissipation and temperature profile...
simultaneously. In a steady state, assuming that heat transport dominates in the radial direction, the temperature and tidal dissipation satisfy the 1D conductive heat equation in spherical coordinates:

\[
\frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \kappa(T) \frac{\partial T}{\partial r} \right) + h_{\text{tide}}(r) = 0,
\]

(3)

where the source term \( h_{\text{tide}}(r) \) is the surface-averaged volumetric tidal dissipation rate found in Section 2.2 and thermal conductivity for pure water ice is given by \( \kappa(T) = 0.4685 + 488.12/T \text{ in W m}^{-1} \text{K}^{-1} \) (Hobbs 2010).

Once the tidal dissipation term \( h_{\text{tide}}(r) \) is known, Equation (3) can be solved numerically as a boundary value problem with temperatures set at the surface and base of the shell. The temperature at the base is assumed to be \( T_b = 273 \text{ K} \) and the surface temperature \( T_s \) is calculated as an equilibrium temperature using a fast-rotator approximation, which gives \( T_s = 92 \text{ K} \) for Europa and \( T_s = 59 \text{ K} \) for Enceladus (see Appendix B). For the range of viscosity values \( \eta_{\text{melt}} \) considered, Tobie et al. (2003) and Roberts & Nimmo (2008) show for Europa and Enceladus, respectively, that the tidal heating rate at the base is larger than the heat flux from radiogenic heating in the silicate mantle. However, the tidal heat production in the rocky mantles can be significant if the rocky mantle is loosely consolidated (Roberts 2015). In our modeling, however, we ignore mantle dissipation and the heat flux from the ocean for simplicity, considering only the tidal heat produced within the icy shell. Thus, the mantle is assumed to have an infinite viscosity and, thus, deforms only elastically.

To solve for a steady-state solution of the heat conduction equation, we converge the tidal dissipation and temperature profiles with the following iterative approach:

**Step 1:** The solution to the boundary value problem of Equation (3) excluding the tidal heating term \( h_{\text{tide}}(r) = 0 \) is set as an initial guess for the temperature profile.

**Step 2:** Temperature profile obtained is used to find the viscosity profile with Equation (2), and tidal dissipation \( h_{\text{tide}}(r) \) is computed.

**Step 3:** Then, the full Equation (3) is used with \( h_{\text{tide}}(r) \), and a new temperature profile is found as the solution to the boundary value problem.

Steps 2 and 3 are repeated until the temperature profile has converged or the temperature at any point within the shell has exceeded 273 K. Temperatures above the melting point in portions of the shell would cause different flow mechanisms to dominate due to partial melting, which would invalidate our ice rheology assumption (Equation (2)). Thus, we exclude such temperature profiles from the relaxation modeling.

### 2.4. Viscous Relaxation Model

Given a shell structure, we model the viscous relaxation of the icy shell as an incompressible fluid in a spherical shell with self-gravitation following the methodology of Hager & Clayton (1989). The icy shell is described by a sequence of layers of constant density and viscosity as prescribed by the viscosity profile. We have tested how the relaxation study results depend on the number of layers and found that 80 layers are needed to arrive at a converged solution. An incompressible Stokes flow along with Poisson’s equation for gravitational potential is formulated as a system of six linear differential equations for a spherically symmetric shell (Hager & Clayton 1989). The state vector is given as radial functions of radial and poloidal velocities, and the radial normal stress and poloidal shear stress as coefficients of vector spherical harmonics. Gravitational potential perturbation and its derivative are given as coefficients of scalar spherical harmonics. Given boundary conditions at the surface and base of the shell, the system of equations can be solved with a propagator matrix method (Gantmacher & Brenner 2005). Thus, our layered model can be expressed as a single linear system that relates the state vector at the ocean–ice interface to the state vector at the ice–outer-space interface. A full description of the viscous relaxation model and the implementation of boundary conditions (Section 2.4.1) are given in Appendix C.

#### 2.4.1. Boundary Conditions

The two boundaries of the icy shell are the ice–outer-space interface at the surface of the moon and the ice–ocean interface at the base of the shell. Under the formulation of Hager & Clayton (1989), the shapes of the interfaces are represented as loads applied at the artificial spherical boundaries \( r = R \) and \( r = R - D \), for the mean radius \( R \) and shell thickness \( D \). The amplitude of this topography is assumed to be small compared to its wavelength \( 2\pi R/\sqrt{n(n+1)} \approx 2\pi R/n \) for a spherical harmonic degree \( n \). The loads generated by topography create perturbations in radial stress and gravity potential. The surface ice–outer-space boundary is a material boundary. Thus, the radial velocity at the boundary directly affects the surface topography. Finally, free–slip boundary conditions are imposed at both boundaries, setting shear stress to zero.

At the bottom ice–ocean boundary, two types of boundary conditions are considered as illustrated in Figure 2. These two boundary conditions are end-members with regard to allowing or not allowing phase transitions at the bottom of the shell. A bottom material boundary condition, which we will refer to as “bottom material boundary,” assumes that no material crosses the water–ice interface through phase transitions. A bottom material boundary is similar to the boundary condition described by Hager & O’Connell (1979) for an isostatic rebound problem and “constant load” described by Beuthe (2021). With this material condition, the radial component of the flow at the base of the icy shell will create or remove basal topography. The two interfaces will approach hydrostatic equilibrium over time. On the other hand, a bottom phase boundary condition, or “bottom phase boundary,” allows for a flow across the ice–ocean interface. Under this boundary condition, described by Cadek et al. (2019) to model the shell of Enceladus, the surface nonhydrostatic topography is maintained in dynamic equilibrium by the flow at the base of the shell that is balanced by phase transitions (melting or freezing) given by their Equation (5):

\[
\hat{n} \cdot v_L p_1 = \hat{n} \cdot (q_1 - q_2),
\]

(4)

where \( \hat{n} \) is a unit vector normal to the interface, \( v \) is the velocity of ice flow, \( L \) is the latent heat of fusion for ice, \( p_1 \) is the density of the shell, and \( q_1 \) and \( q_2 \) are heat fluxes from the icy shell and ocean, respectively. Our bottom phase boundary is the same as the “constant shape” described by Beuthe (2021). The bottom phase boundary allows us to model the relaxation of the shell when the heat flux at the base is sufficient to sustain the dynamic equilibrium. The bottom material boundary is
applicable where there is insufficient heating to balance the flow, thus leading to relaxation. For a realistic icy shell, we expect the boundary condition to transition from one to another depending on where tidal heating is present, with the bottom material boundary dominating at higher degrees.

We solve for the Stokes flow using each of these boundary conditions with the goal of obtaining the asymptotic shape of the icy shell at spherical harmonic degree $n$ for the calculation of gravity–topography admittance (Section 2.5). In the following analysis, we ignore the tidal and rotational bulges that would contribute primarily to degrees 2 and 4. Thus, in practice, the hydrostatic contributions to the shape and gravity spherical harmonic coefficients need to be subtracted prior to modeling viscous relaxation. We define “shape” as this deviation of the body from hydrostatic equilibrium. The shape of the icy shell at the surface and base are expanded in spherical harmonics at radii $R$ and $R - D$, and their spherical harmonic coefficients $h_t$ and $h_b$ are referred to as shape coefficients or amplitudes, where the dependence on $n$ is implied. The shape of the ice–ocean boundary is described by the shape ratio $w_n = -h_b/h_t$. The negative sign in the definition of $w_n$ is a matter of convention, and we use it to make $w_n$ positive for the conventional Airy isostasy case.

We note that the presence of mountains and valleys perturbs the equipotential surface. We define the term “topography” as the elevation difference between the body’s shape and the perturbed equipotential surface. The topographic ratio defined by Equation (16) in Čadek et al. (2019) describes the ratio of topographies, which is distinct from our shape ratio $w_n$. For convenience, in our work, we use the term “topography” to describe undulations of the interfaces or topographic loads that drive viscous flow within the shell.

Solving for the Stokes flow under the described boundary conditions requires two different approaches. For the bottom material boundary, the shapes of the interfaces are functions of time and eventually relax to hydrostatic equilibrium. Applying the bottom material boundary, the propagator matrix solution of the Stokes flow equations (Hager & Clayton 1989) can be rearranged as a $2 \times 2$ system of ordinary differential equations for radial velocities $\frac{dh_t}{dt}$ and $\frac{dh_b}{dt}$ (see Appendix C for details). The solution for the shape amplitudes $h(t) = [h_t(t), h_b(t)]^T$ is given as a sum of two decaying exponentials describing the
time evolution of the interfaces:

$$h(t) = X e^{-\gamma_1 t} + Y e^{-\gamma_2 t},$$

where $X = [X_1, X_2]^T$, $Y = [Y_1, Y_2]^T$ are the eigenvectors of the system, and $\gamma_1 = 1/\tau_1$, $\gamma_2 = 1/\tau_2$ are the characteristic decay times. We set $\tau_1 < \tau_2$ so that the second term in Equation (5) corresponds to the longer decay time. Two modes arise from Equation (5), which are described by Hager & O’Connell (1979) as symmetric and antisymmetric modes of relaxation. In the symmetric mode, the two interfaces move in the same direction. For example, if a mountain were placed on the surface and allowed to relax, a crustal root would initially form at the base of the icy shell. Conversely, in the antisymmetric mode, the two interfaces move in opposite directions. In this mode of relaxation, the height of the mountain and the crustal root will decrease with time, approaching hydrostatic equilibrium. We obtain this asymptotic state from the eigenvector $[Y_1, Y_2]^T$ that corresponds to the longer decay time $\gamma_2$. The degree-dependent shape ratio is given by $w_n = -Y_n / Y_1$. Although we use this example of a mountain at the surface, the asymptotic state of the shell (i.e., $w_n$) is the same regardless of the location of the initial load.

For the bottom phase boundary, we do not solve for a time-dependent solution of the interfaces. Instead, we find the asymptotic state after relaxation where the base of the shell is in a dynamic equilibrium between the ice flow and phase transitions. In this state, the radial velocity of the surface interface is set to zero to maintain the shape of the surface. However, the radial velocity at the base can be nonzero as long as the flow across the bottom interface is balanced by phase transitions satisfying Equation (4). By setting the surface shape $h_s$ to unity, the propagator matrix solution to the system of Hager & Clayton (1989) can be rearranged into a system of four equations with four unknowns: bottom shape $h_b$, bottom radial velocity, bottom poloidal velocity, and surface poloidal velocity. Thus, the shape ratio $w_n$ is equal to $-h_b$. We note that with an observed value of $h_b$, assuming that the shell is in this dynamic equilibrium, we can find the bottom radial velocity and, thus, the heat flux from Equation (4). Our method of applying the bottom phase boundary is similar to the spectral method used by Čakš et al. (2019).  

2.5. Gravity–Topography Admittance

Finally, we compute the gravity–topography admittance using the shape amplitudes of viscously relaxed icy shells. We follow the approach Ermakov et al. (2017) used to model the admittance of Ceres. Assuming the shape amplitude is small relative to its wavelength, we use a mass-sheet approximation to get a linear relationship between gravity and shape. Summing up the contributions to gravity from the shape of the surface and bottom boundaries of the shell and dividing by surface shape, we find an expression for admittance $Z_n$ at degree $n$ for shape ratio $w_n$:

$$Z_n = \frac{GM}{R^3} \frac{3(n+1)}{2n+1} \left[ \frac{\rho_1}{\bar{\rho}} - \frac{\Delta \rho}{\bar{\rho}} \left( \frac{R - D}{R} \right)^{n+2} w_n \right],$$

where $\bar{\rho}$ is the mean density of the moon, $\rho_1$ is the density of the icy shell, $\rho_2$ is the ocean density, and $\Delta \rho = \rho_2 - \rho_1$ is the ice–ocean density contrast. See Appendix D for the derivation. Thus, with a degree-dependent shape ratio $w_n$ obtained from the viscous relaxation model, we find the admittance spectrum for a given shell structure. As $w_n$ is an asymptotic ratio, the admittance spectrum found from our model is one that is approached by the shell as it relaxes viscously either to hydrostatic equilibrium or the phase boundary dynamic equilibrium. Thus, in general, our models are agnostic to the origin of topography and initial elastic response.

We have two points of comparison for admittance. First, in the classical case of Cartesian Airy isostatic compensation, the amplitude of the bottom shape is related to the amplitude at the surface through the ratio of the shell density to the density contrast, leading to $w_n = \rho_1 / \Delta \rho$. Airy-compensated values of admittance can inform us of the topography supported by buoyancy forces acting on the bottom topography, or isostatic roots. We note, however, that it has been argued by Ermakov et al. (2017), Čakš et al. (2019), and Beuthe (2021) that the accuracy of the Airy-compensation model degrades when the shell thickness is a significant fraction of body radius. The second limiting case is that of uncompensated surface topography supported by viscous (or elastic) stresses within the shell, for which there is no corresponding bottom topology $h_b = 0$. Therefore, $w_n = 0$ for uncompensated surface topography. The admittance for the uncompensated topography provides an upper bound for admittance for a given density of a shell. Uncompensated values can indicate that topography is supported by stresses within the shell. We note, however, that Airy admittance does not necessarily provide a lower bound on admittance.

3. Results

3.1. The Effect of the Bottom Boundary Condition

We begin by studying the effects of bottom material boundary and bottom phase boundary on admittance. A model of Europa with a 30 km shell is created for a range of simple viscosity profiles, where viscosity is assumed to decrease exponentially with depth. We vary the viscosity contrast between the top and the bottom of the shell ($\eta_b / \eta_i$). We will apply a more complete treatment of temperature-dependent viscosity later in Section 3.3. Figure 3 compares admittance spectra for the two boundary conditions. We observe that for the uniform-viscosity case ($\eta_b / \eta_i = 1$) shown in purple, the two boundary conditions produce admittance spectra that are nearly identical to each other and to the Airy isostasy case shown as the dotted curve.

However, as the viscosity gradient steepens, it becomes harder to transmit a stress perturbation between the top and the bottom of the shell, and the two boundary conditions start to diverge at high spherical harmonic degrees. The bottom phase boundary produces lower admittance values compared to those of the bottom material boundary. Lower admittance values correspond to larger values of the shape ratio $w_n$. While for the material boundary condition basal lateral flow effectively relaxes crustal roots (making $w_n$ smaller), a larger crustal root (i.e., larger $w_n$) is required to maintain surface topography for the bottom phase boundary. Consequently, a larger basal radial flow requires larger heat flux variations at high degrees to counterbalance viscous relaxation.

The bottom phase boundary can be justified if a sufficient heat source is present at the appropriate wavelength and magnitude to support the observed surface topography. This would be difficult to reconcile with tidal heating, which is
expected to have contributions mostly at low degrees. The tidal potential is dominated by the degree 2 component. The higher-degree terms decrease quickly in amplitude. For example, for Europa, the degree 3 term is 1/430 the degree 2 term (Sabadini et al. 2016). Therefore, tidal heating could drive phase transitions only at low degrees, unless there is significant heterogeneity within the shell or in the heat flow from the ocean.

### 3.2. The Effect of the Viscosity Gradient within the Shell

Here, we explore the effect of the shell viscosity structure on admittance using exponential viscosity profiles of variable steepness. To illustrate the effect of the viscosity profile, Figure 4 shows the velocity field comparing the ice flow within a uniform- and gradient-viscosity shell for the bottom material and phase boundaries. The flow for the bottom material boundary is shown as it approaches the asymptotic state. The flow within the icy shell of Enceladus is shown for illustrative purposes, as Enceladus’ shell is thicker compared to its radius relative to Europa’s shell. The flow pattern for the four cases shown is qualitatively similar for Europa.

For uniform-viscosity shells, the flow is quasi-uniform in amplitude throughout the shell beneath the topographic load. (Figures 4(A) and (B)). On the other hand, a viscosity gradient in the shell causes a strong lateral flow concentrated at the low-viscosity base for both types of bottom boundary conditions (Figures 4(D) and (C)). For the bottom material boundary, this lateral flow rapidly relaxes the basal topography compared to the relaxation of the high-viscosity near-surface layers. For the bottom phase boundary, there is net freezing at the base of the shell associated with the topographic load at the pole. This freezing is balanced by upward radial flow such that the bottom interface remains stationary.

We observe that as the viscosity gradient steepens, admittance values approach high, uncompensated values at progressively lower degrees for bottom material boundary (Figure 3(A)). Uncompensated values indicate that there is little basal topography relative to surface topography. Thus, surface topography is not supported by buoyancy as in the Airy isostasy case. Instead, viscous stresses within the shell provide the dominant topography support mechanism. For very large viscosity contrasts (the red bounded and unbounded curves for the bottom material boundary; Figure 3(A)), the high relative viscosity at the surface impedes the relaxation of surface topography even at large scales while the bottom topography relaxes, leading to uncompensated admittance values at low degrees. For the bottom phase boundary, we find that admittance spectra do not vary with the steepness of the viscosity gradient (Figure 3(B)) compared to the bottom material boundary case, especially at the lowest degrees. This observation also holds for admittance spectra of Enceladus and is consistent with the observation of similar shape amplitudes of the bottom interfaces seen in Figures 4(B) and (D).

We note that admittance spectra are only affected by the viscosity contrast $\eta_b/\eta_s$ and not the actual values of $\eta_b$ or $\eta_s$, which, instead, affect the timescales of relaxation, with higher viscosities increasing the time it takes to reach the asymptotic state of topography. The choice of the top viscosity bound affects the admittance spectra, with higher values for $\eta_{bound}$ increasing the viscosity contrast, causing admittance to approach uncompensated values at lower degrees.

### 3.3. Tidal Heating Influence on Viscosity

We proceed from simple viscosity gradients to more realistic viscosity profiles by including a temperature-dependent viscosity and tidal heating. We incorporate the effect of tidal heating on a conductive temperature profile of the icy shell using a steady-state solution obtained from the iterative method described in Section 2.3. Figure 5 shows the effect of including tidal heating on the temperature (Figure 5(A)), viscosity (Figure 5(B)), and volumetric tidal dissipation rate (Figure 5(C)) for two choices of $\eta_{melt}$. As mentioned in Section 2.2, tidal dissipation is maximized if the forcing period is near the Maxwell time of the material being deformed. The viscosity for which this occurs for Europa ($\eta_{Maxwell} \approx 1.6 \times 10^{14}$ Pa s) is shown by the dotted line in Figure 5(B). The viscosity profiles of the 30 km shells modeled with two values of $\eta_{melt}$ cross $\eta_{Maxwell}$ near the base of the shell.
For $\eta_{\text{melt}} = 10^{13}$ Pa s, this intersection occurs at $\approx 78\%$ of the shell’s depth, which corresponds to a maximum in tidal dissipation as expected. For the case with $\eta_{\text{melt}} = 10^{14}$ Pa s, maximum dissipation occurs closer to the base of the shell. This causes tidal dissipation to be lower throughout the shell, which is seen in the temperature profile in Figure 5(A), where $\eta_{\text{melt}} = 10^{14}$ Pa s causes the temperature profile to be closer to the no tidal heating case. Although not shown in Figure 5, $\eta_{\text{melt}} = 10^{15}$ Pa s further decreases the influence of tidal heating, and the temperature profile becomes nearly identical to the case without tidal heating. In summary, the choice of $\eta_{\text{melt}}$ and thus the assumption of the grain size of ice, has a large effect on tidal heating and, therefore, the temperature structure of the shell.

### 3.4. Shell Thickness

Finally, we use our complete model with temperature-dependent viscosity and tidal dissipation to study the sensitivity of admittance to the shell thickness. Predictions of Europa’s shell thickness vary from local estimates of several kilometers to more than 30 km as a global average (Billings & Kattenhorn 2005). We consider mean thicknesses between $D = 8$ and 35 km. Depending on the viscosity at the base $\eta_{\text{melt}}$, the amount of tidal heating for thick shells causes temperature profiles to not converge with our iterative method described in Section 2.3. For $\eta_{\text{melt}} = 10^{14}$ Pa s, a temperature profile did not converge above a shell thickness of 32 km due to temperatures exceeding the melting point in our iterative method.
Figure 5. Effects of tidal heating on temperature and viscosity in a 30 km shell of Europa. (A) Conductive temperature profiles converged from the iterative process for a melting point viscosity $\eta_{\text{melt}} = [10^{13}, 10^{14}]$ Pa s and for a case without tidal heating ($\eta_{\text{melt}} = 0$). (B) Unbounded temperature-dependent viscosity profiles calculated from the temperature profiles of (A) with Equation (2). The viscosity at which Maxwell time is crossed is shown by the red dotted line. The case without tidal heating is shown for both choices of $\eta_{\text{melt}}$. (C) Surface-averaged volumetric tidal dissipation rate is shown for $\eta_{\text{melt}} = [10^{13}, 10^{14}]$ Pa s. Maximum tidal dissipation can be seen occurring at the depth where the viscosity profile crosses $\eta_{\text{Maxwell}}$.

Figure 6. Admittance spectra and viscosity profiles for tidally heated shells of Europa with different shell thicknesses. (A) Admittance for the bottom material boundary (solid curve) and the Airy isostasy (dotted curve) are shown for different shell thicknesses with solid lines. Admittance spectra assuming uncompensated topography are shown with a dashed curve. (B) Admittance spectra for the bottom phase boundary (dotted-dashed curve) are shown for different shell thicknesses. (C) Viscosity profiles are plotted against depth normalized to shell thickness, $D$. A value of $\eta_{\text{melt}} = 10^{14}$ Pa s is chosen, and a viscosity bound of $\eta_{\text{bound}} = 10^{24}$ Pa s is imposed.

The shell thickness has a strong effect on the admittance spectrum, which is illustrated in Figure 6. The bottom material boundary condition is used in Figure 6(A), and the bottom phase boundary condition is used in Figure 6(B). We observed that increasing shell thickness lowers viscosity at relative depths (Figure 6(C)) as thicker shells generate more heat from tidal dissipation. Even though a thinner shell reaches a higher maximum volumetric dissipation value, its overall dissipation is lower. For a conductive shell, the thinner the shell, the steeper its temperature and viscosity profiles. Thus, for a thinner shell, a smaller region at the base of the shell ends up being close to the Maxwell time, where tidal dissipation is maximized, reducing total dissipation in the shell.

To understand the behavior of admittance and topography support mechanisms at different wavelengths, we apply Jeffreys’ theorem. It states that the minimum stress difference required to support a surface load is $\approx 1/3$ of the load and is concentrated in a region with dimensions comparable to the width of the load (Melosh 2011). Thus, a surface topographic load samples the shell beneath it to a depth comparable to its wavelength. A short-wavelength load feels only the near-surface viscosity value and thus we expect topography to be supported by stresses within the shell as the wavelength becomes smaller. A load with wavelength comparable to the shell thickness samples deeper into the shell and is sensitive to its viscosity profile. For a load with a wavelength much longer than the shell thickness, the shell acts essentially as a membrane. Thus, such a load is not sensitive to the viscosity profile.

As the shell thickness increases, admittance for the bottom material boundary (Figure 6(A)) approaches uncompensated values at progressively lower degrees or longer wavelengths. For thicker shells, it is difficult to propagate buoyancy-produced stresses from the base of the shell to support surface topography, causing topography at progressively longer wavelengths to be supported by stresses within the shell. Wavelengths much longer than the thickness of the shell do not sense the viscosity variation throughout the shell, thus yielding an Airy-like admittance. This is seen for the 8 km shell at low degrees ($n < 15$) and at the lowest degrees for a 16 km shell (Figure 6(A)). For shorter wavelengths, admittance is affected by the steep viscosity profile resulting from a tidally heated, conductive shell and approaches uncompensated values.

For the bottom phase boundary (Figure 6(B)), admittances are closer to the Airy isostasy admittance values, especially at low degrees. As the shell thickness increases, the degree to which admittance deviates from the Airy isostasy becomes progressively lower. We observe, in general, smaller absolute differences of admittance versus shell thickness for the bottom phase boundary.

Comparing two types of bottom boundary conditions (Figures 6(A) and (B)), we observe that as we increase the
shell thickness, predicted admittances diverge at progressively lower degrees. For example for a 32 km thick shell (red curve in Figure 6), the bottom material boundary predicts uncompensated values past degree 4, while the bottom phase boundary predicts nearly compensated topography. Thus, a dip in the measured admittance—characteristic of the bottom phase boundary—could suggest that the topography is supported by strong basal viscous flow counterbalanced by phase transitions.

Finally, we can get a sense of how the sensitivity of admittance to the shell thickness changes by looking at solid colored curves in Figure 6(A) for \( n < 10 \). We focus on these low degrees as these are the degrees that will be measured by Europa Clipper. We observe that as the shell thickness decreases, the admittance becomes progressively less sensitive to the shell thickness, approaching the compensated limit.

4. Discussion

4.1. Topography Support Mechanism

Admittance can provide insight into the topography support mechanism. High, uncompensated values of admittance can indicate that surface topography is supported by viscous or elastic stresses within the shell. Low admittance could indicate that topography is supported by buoyancy, as predicted from the Airy isostasy model. Alternatively, low admittance could indicate that surface topography is supported by basal gravitational relaxation flow, which is balanced by phase transitions such that the shape of the interfaces does not change. In this case, the heat flux pattern controls the rate of basal melting and freezing. Further study on the distribution of basal heat flux is needed to understand the wavelengths at which the bottom phase boundary could be applicable. If tidal heating in the shell is small, for example, due to high values of \( \eta_{\text{melt}} \), the influence of radiogenic and tidal heating from the mantle and ocean (Roberts 2015; Rekier et al. 2019; Rovira-Navarro et al. 2019) may become more important. Ocean circulation patterns resulting from mantle heat sources and salinity gradients could also affect the distribution of heat at the base of the shell (Kang et al. 2020, 2021).

We consider the behavior of the shell on viscous timescales. However, on shorter timescales, the elastic response becomes important, and topography can be supported by elastic stresses. Further considerations of the age and origin of topographic features at different wavelengths would be needed for a viscoelastic approach.

4.2. Ice Rheology

Temperature-dependent viscosity for a tidally heated conductive shell yields a steep viscosity gradient within the shell. The depth at which the Maxwell time is crossed influences the distribution of tidal dissipation, which, in turn, determines the extent of the high-temperature, low-viscosity region at the base of the shell. On Europa, tidal dissipation is concentrated at the base of the shell, where viscosity is low and is close to \( \eta_{\text{Maxwell}} \). For the bottom material boundary condition, we find that the low viscosity at the base causes rapid relaxation of the bottom topography by lateral flow, leading to uncompensated surface topography.

Ice deformation mechanisms are difficult to study in the laboratory, and different non-Newtonian flow mechanisms such as GBS may be more appropriate when considering solid-state convection (Barr & Showman 2009). Although our analytical viscous relaxation model does not allow for convection, we expect that convection at the base of the shell would effectively relax the shorter-wavelength bottom topography, resulting in uncompensated values of admittance.

In addition to the flow mechanism, the grain size of ice has a large effect on the viscosity profile, tidal dissipation, and thus the temperature structure of the shell. The viscosity at the base of the shell at the melting temperature depends on this poorly constrained grain size of ice. For Europa and Enceladus, the viscosity at which Maxwell time is crossed, \( \eta_{\text{Maxwell}} \), lies within the range of melt viscosities \( \eta_{\text{melt}} = 10^{13} - 10^{15} \) Pa s typically considered. We find that tidal heating has little influence on temperature structure when \( \eta_{\text{melt}} > \eta_{\text{Maxwell}} \). If \( \eta_{\text{melt}} < \eta_{\text{Maxwell}} \), the influence of \( \eta_{\text{melt}} \) on temperature profile is amplified depending on the extent of the low-viscosity region and the relative depth at which \( \eta_{\text{Maxwell}} \) is crossed. The melting temperature \( T_{\text{melt}} \) also changes with pressure and affects the value of \( \eta_{\text{melt}} \) and the temperature gradient. However, changes in grain size likely have a larger effect on \( \eta_{\text{melt}} \), thus we can approximate different melting temperatures with the range of \( \eta_{\text{melt}} \) considered.

The diffusion creep temperature-dependent viscosity predicts high viscosities at the surfaces of Europa and Enceladus, implying the very slow relaxation of surface topography, leading to uncompensated values of admittance. Properties of shallow subsurface ice, such as fracturing, porosity, and impurities, affect deformation mechanisms, viscosity, and thermal conductivity. An insulating layer, considered by Bland et al. (2012) in their study of crater relaxation on Enceladus, could raise the shallow subsurface temperature and, thus, lower viscosities near the surface. In addition, isolated periods of warming may have occurred in the past (Bland et al. 2012). Such warming episodes could have facilitated viscous relaxation, affecting admittance. Thus, measuring admittance and comparing the inferred viscosity profile to predictions from the crater relaxation study could validate the notion of past heating episodes on Enceladus.

4.3. Comparing Europa to Enceladus

We now apply our methods to a model of Enceladus and compare it to Europa. We assume a shell thickness of 21 km for Enceladus, taking the central value from Hemingway & Mitall (2019), and compare it to Europa with a 30 km shell using melt viscosity \( \eta_{\text{melt}} = 10^{14} \) Pa s for both ocean worlds. Importantly, Enceladus has a thicker shell in proportion to its radius compared to Europa, which leads to a larger difference between the admittance computed from viscous relaxation-based models and the Cartesian Airy isostatic compensation model. We show a comparison of admittance spectra in Figure 7. Because the wavelength is inversely proportional to degree, the same degree on Enceladus corresponds to a shorter wavelength than on Europa. Thus, we expect admittance spectra of bottom material boundary for Enceladus to approach uncompensated values at a lower spherical harmonic degree than for Europa due to shorter wavelengths at low degrees for Enceladus sampling the viscosity gradient of the shell. Comparing the admittance of the bottom phase boundary, we see that admittance for Enceladus deviates from the Airy isostasy case at low degrees more than for Europa, perhaps making it easier to distinguish the topography support mechanism from future observations.
For tidal heating, the key differences between the two moons are orbital frequency and surface temperature. The higher orbital frequency of Enceladus causes $\eta_{\text{Maxwell}} = 6.2 \times 10^{13}$ Pa s to be lower than that of Europa. Thus, in Figure 7 with $\eta_{\text{melt}} = 10^{14}$ Pa s, $\eta_{\text{melt}} > \eta_{\text{Maxwell}}$ for Enceladus while $\eta_{\text{melt}} < \eta_{\text{Maxwell}}$ for Europa. A lower surface temperature of Enceladus leads to a steeper viscosity gradient within Enceladus’ shell. This reduces the thickness of the layer where the viscosity is close to $\eta_{\text{Maxwell}}$, decreasing the influence of tidal heating on temperatures near the base. As a result, tidal heating does not affect the shell temperature profile on Enceladus as strongly as it does for Europa.

For the bottom phase boundary, Figure 7 shows lower values of admittance compared to the bottom material boundary at low degrees for our Enceladus model. In general, Airy-compensated values of admittance only hold for low spherical harmonic degrees or for wavelengths much greater than the shell thickness. Thus, we reach the same conclusion as Čadek et al. (2019) that the Airy isostasy model can be applied for long wavelengths but may be inaccurate at short wavelengths for the bottom phase boundary.

### 4.4. Sensitivity of Future Gravity and Shape Data to the Icy Shell Structure

In this subsection, we characterize what can be achieved with future data at Europa and Enceladus by comparing various shell structure end-members to expected mission performance. To access the accuracy of admittance recovery, two items are needed. First, one needs to estimate the accuracy of the gravity coefficients determination, which can be done either with a simplified approach of Bills & Ermakov (2019) or with a full mission simulation using covariance analysis. Second, a global shape model or, at least, an estimate of the shape power spectrum is required. A global shape model is available for Enceladus (Tajeddine et al. 2017) but is not currently available for Europa to our knowledge. Thus, we can only approximately judge admittance recovery at Europa based on the expected resolution of the gravity field.

Park et al. (2011) conducted a covariance analysis study simulating the Europa Clipper flyby mission and found that Europa’s gravity field may be resolved up to degree 10. A transition between low (Airy-compensation-like) admittance values to higher, uncompensated values marks the shift of the dominant topography support mechanism from buoyancy to viscous stresses within the shell. Thus, we can pose the question: Under what conditions does this transition occur at $n < 10$? We find that the transition is captured in admittance spectra at $n < 10$ for shell thicknesses greater than $\approx 24$ km for $\eta_{\text{melt}} = 10^{14}$ Pa s (Figure 6(A)). If a lower melt viscosity is assumed ($\eta_{\text{melt}} = 10^{13}$ Pa s, not shown in Figure 6(A)), the tidal heating is stronger within the shell and the admittance transition is captured for shell thicknesses greater than $\approx 15$ km. Thus, we conclude that admittance measurements at low degrees would provide an efficient way to constrain the shell thickness if the shell is thick and warm. Extending the capability of capturing the transition of the topography support mechanism to higher degrees will improve the ability to constrain the thickness of thinner and colder shells.

Ermakov et al. (2021) presented a covariance analysis for Enceladus orbiter missions. Ermakov et al. (2021) studied the recovery of Enceladus’ gravity field and tides simulating one month of continuous radio tracking for a single orbiter with radio tracking to Earth and a GRAIL-like dual spacecraft with intersatellite tracking. The ranging accuracy was assumed to be $10^{-7}$ km s$^{-1}$ for the single spacecraft case and $10^{-9}$ km s$^{-1}$ for the dual spacecraft case. These accuracies are typical for $X$-band and $Ka$-band ranging, respectively. We used the covariance analysis by Ermakov et al. (2021) and the Enceladus shape model by Tajeddine et al. (2017) to estimate the performance of two orbiter mission configurations in recovering admittance. The gravity error rms spectra for the two orbiter configurations are found from the diagonal elements (i.e., variances $\sigma_{c_m}^2$ and $\sigma_{s_m}^2$) of the gravity covariance matrix to be

$$M_{\text{rms}} = \sqrt{\frac{\sum_{m=0}^{n_0}(\sigma_{c_m}^2 + \sigma_{s_m}^2)}{2n + 1}}. \quad (7)$$

The gravity error rms spectra for the two orbiter configurations are shown in Figure 8(B) and correspond to blue and yellow curves in Figure 6 in Ermakov et al. (2021). In order to match Ermakov et al. (2021), we present the error in the radial
gravitational acceleration coefficient $M_n^{gs}g(n+1)$, where $g$ is surface gravity. For comparison, we show the gravity error rms from two currently available gravity field models by Iess et al. (2014) for degrees 2 and 3. The variance of admittance at degree $n$ is given by

$$\sigma_z^2 = D_n^\text{T}C_nD_n,$$

where $D_n$ is a vector of partial derivatives of degree-$n$ admittance with respect to gravity coefficients:

$$D_n = \frac{\hat{h}_n \, GM}{V_n^g \, R^3} \, (n+1).$$

Here, $V_n^g$ is the shape variance spectrum (see Equation (D3) in Appendix D for definition), $\hat{h}_n = [\hat{A}_{n0}, \hat{A}_{n1}, \hat{B}_{n1}, \ldots \hat{A}_{nn}, \hat{B}_{nn}]$ is a vector of normalized shape coefficients from the shape model of Tajeddine et al. (2017), and $C_n$ is the degree $n$ submatrix of the covariance matrix from Ermakov et al. (2021). Note that Tajeddine et al. (2017) provided unnormalized shape coefficients. The shape is assumed to be error free, thus the error comes solely from gravity. The admittance error $\sigma_z$ is shown for the single and dual spacecraft configurations in Figure 8(A). Excluding covariances by looking at the diagonal elements of $C_n$, we find the correlations between recovered coefficients contribute little to the admittance error for both orbiter configurations. Similarly, admittance error is shown for the gravity error for the two gravity field models from Iess et al. (2014).

The ability to distinguish between different end-members of the shell structure using admittance can be used to place a requirement on the gravity error. A gravity error rms spectrum is estimated from an admittance error requirement $\sigma_z$ by

$$M_n^{gs} = \sigma_z \sqrt{\frac{V_n^g \, GM}{R^3} \, (n+1)},$$

where we assume gravity error is uniform across all orders for each degree and there are no correlations between coefficients of different orders. A range of admittance errors from 1 to 30 mGal km$^{-1}$ is shown as gravity error rms spectra in Figure 8(B). An admittance error requirement is satisfied if the gravity error rms of a given spacecraft mission configuration is below that for the desired admittance error represented by colored curves in Figure 8(B). For the model of Enceladus shown in Figure 7, the difference in admittance due to the boundary conditions is from $\approx20$ to 30 mGal km$^{-1}$ at degrees of up to 10. As can be seen in Figure 8(B), both mission configurations studied in Ermakov et al. (2021) would yield an admittance accuracy smaller than the expected difference between the two boundary conditions. Thus, such mission configurations would provide sufficient accuracy to distinguish between two types of behavior of the shell-ocean interface.

The use of admittance may augment ice-penetrating radar in constraining shell thickness. One of the goals of Europa Clipper’s Radar for Europa Assessment and Sounding: Ocean to Near-surface (REASON) instrument is to detect the ice–ocean interface. However, the attenuation of radio waves within the ice limits the detection of the ice–ocean interface in the case of a thick and warm shell. Kalousová et al. (2017) find that direct detection of Europa’s ocean with REASON may be possible up to 15 km for a conductive shell or in an area of cold downwelling within a convective shell. In the case where the icy shell of Europa is thicker than 15 km, admittance can be used to estimate shell thickness by looking at the transition from low, compensated values to high, uncompensated values that occur at $n<10$, where admittance measurements are possible. Thus, this makes admittance measurements complementary to radar, which is more suitable for thin and cold shells.

4.5. The Effect of a Rocky Interior Topography

The topography of the rocky interior can contribute to the gravity signal and, thus, affect values of admittance. In the results presented thus far, we assumed the mantle was hydrostatic and the only nonhydrostatic contributions to the gravity signal came from the shell surface and bottom interfaces. In order to understand the contribution to admittance spectra from the rocky interior, we conducted Monte Carlo modeling of gravity–topography admittance for a range of rocky interior topography power laws.

Assuming the internal structure of Enceladus from Section 4.3, we use the measured topography from Tajeddine et al. (2017) and compute the ice–ocean interface using the shape ratio $w_e$ for the bottom material boundary condition. We
model the rocky interior topography with a power law given by

\[ \text{rms}_n = \alpha n^{-\beta}, \]  

where \( \beta \) is either 2 or 3. We choose \( \alpha \) to correspond to an \( \text{rms}_2 \) of 10 or 100 m. A Monte Carlo approach is used to find admittance, including the gravity contribution from the pseudorandom rocky interior topography following the prescribed power law (Equation (11)). The total gravity is found as the sum of the gravity contributions from topography at each interface (see Appendix D, Equation (D9)). Total admittance is found with the spherical harmonic coefficients of the surface shape and total gravity using Equation (D5).

Figure 9 shows the result from the Monte Carlo modeling of admittance given a rocky interior topography power law with \( \beta = 2 \) ((A), (B)) and \( \beta = 3 \) ((C), (D)), and \( \text{rms}_2 = 10 \text{ m} \) ((A), (C)) and \( \text{rms}_2 = 100 \text{ m} \) ((B), (D)). The black dotted curve corresponds to the reference admittance spectrum with a hydrostatic rocky interior. The blue curve and orange region show the distribution of admittance spectra assuming a nonhydrostatic rocky interior. We observe that a rocky interior topography with an \( \text{rms}_2 \) of 10 m has little effect on admittance spectra. However, for \( \text{rms}_2 \) of 100 m with \( \beta = 2 \), we see large deviations from the reference model. The deviation is more pronounced at degrees where the observed surface topography is small. With a faster-decaying power law (\( \beta = 3 \)), the contribution of the rocky interior to gravity decays rapidly and admittance values approach the reference model for \( n > 6 \).

A large rocky interior contribution to admittance adds an additional source of error to the measurement error discussed in Section 4.4 in constraining admittance for the shell. Care will need to be taken to consider a rocky interior contribution to gravity when analyzing future data.

4.6. Sensitivity of Admittance to Tidal Dissipation

A measurement of the tidal phase lag, or equivalently, the imaginary part of the tidal Love number \( k_2 \) can provide a constraint on the total dissipation within the satellite

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Figure 9. Comparison of admittance spectra of Enceladus for rocky interior topographies of different amplitudes. The model of Enceladus is the same as the one used in Figure 7. The black dashed curve corresponds to the reference admittance spectrum of Enceladus with a bottom material boundary and a hydrostatic mantle. The median rocky interior admittance \( Z_{\text{med}} \) found from the Monte Carlo modeling is shown by the blue curve, and the 5%–95% confidence interval is shown in orange. Admittances are given for rocky interior topography power laws with \( \beta = 2 \) (panels (A) and (B)) and \( \beta = 3 \) (panels (C) and (D)). Power laws are computed with an \( \text{rms}_2 \) of 10 m (panels (A) and (C)) and \( \text{rms}_2 \) of 100 m (panels (B) and (D)), and corresponding values of \( \alpha \) are shown.
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Figure 10. A comparison of how real and imaginary parts of the tidal Love number \( k_2 \) as well as admittance for Europa \((Z_3 \text{ and } Z_{10})\) depend on the shell thickness \( D = 5–30 \text{ km} \) and melt viscosity \( \eta_{melt} = 10^{12–10^{15}} \text{ Pa s} \) using the material boundary condition at the base of the shell. Contours of \( \text{Re}(k_2) \) (dashed) are plotted in panels (A) and (C). Contours of \( \text{Im}(k_2) \) (dotted) are plotted in panels (B) and (D). Filled contours of admittance at degree 3 are shown in panels (A) and (B). Filled contours of admittance at degree 10 are shown in panels (C) and (D). A viscosity bound of \( 10^{14} \text{ Pa s} \) was applied. Combinations of thickness and \( \eta_{melt} \), where a temperature profile did not converge according to our criteria (Section 2.3), are shown in gray.

Thus, measuring admittance and \( \text{Re}(k_2) \) would allow constraining \( \text{Im}(k_2) \) and, therefore, constraining the magnitude of dissipation within the icy shell.

We note that the values of \( \text{Im}(k_2) \) are smaller than the estimated uncertainty of \( k_2 \) recovery (Park et al. 2011; Verma & Margot 2018) assuming \( \text{Im}(k_2) \) is measured to the same precision as \( \text{Re}(k_2) \). Thus, directly constraining the magnitude of tidal dissipation within Europa’s shell will be challenging from the Europa Clipper data. Park et al. (2011) showed that the gravity field of Europa can be estimated to degree 10. More recently, Verma & Margot (2018) showed that only degrees 3 and 4 of the gravity field can be estimated. The global shape can be estimated from fitting limb profiles or from building a geodetic network of reference points, which has been done previously for Enceladus using Cassini flyby data (Nimmo et al. 2011; Tajeddine et al. 2017). Thus, because measuring the gravity field, the shape, and, therefore, the gravity-topography admittance at low degrees is less demanding than measuring \( \text{Im}(k_2) \), admittance measurements would likely allow the magnitude of tidal dissipation within Europa’s shell to be constrained prior to the \( \text{Im}(k_2) \) measurement becoming available.

(Peale et al. 1979). However, this measurement is challenging and so far has been achieved only for Earth (Ray et al. 2001), the Moon (Williams et al. 2014), and Mars (Bills et al. 2005), which have abundant data sets. Early future data for ocean worlds will likely be more limited. Park et al. (2011) estimated that Europa Clipper will enable the determination of \( k_2 \) of Europa with an uncertainty of 0.009. A more recent study by Verma & Margot (2018) is more pessimistic and predicts that Europa’s \( k_2 \) can be determined with an uncertainty of \( \approx 0.05–0.06 \) depending on the Europa Clipper trajectory and the selection of ground-based radio-tracking assets. We computed the real and imaginary parts of Europa’s Love numbers (see Appendix A for details) and explored their dependence on the shell thickness and viscosity profile. Figure 10 shows how \( \text{Re}(k_2) \) and \( \text{Im}(k_2) \) depend on the shell thickness and melt viscosity \( \eta_{melt} \) for a model of Europa with viscosity bound \( \eta_{bound} = 10^{24} \text{ Pa s} \). We observe that \( \text{Re}(k_2) \) depends primarily on the shell thickness. On the other hand, \( \text{Im}(k_2) \), and therefore total dissipation, varies strongly depending on both shell thickness and \( \eta_{melt} \). In addition, Figure 10 shows the gravity-topography admittance at degrees 3 and 10 as filled contours. It can be seen that the admittance contours are, in general, not parallel to the contours of \( \text{Re}(k_2) \) or \( \text{Im}(k_2) \).
Finally, if tidal dissipation is estimated from both \( \text{Im}(k_2) \) and admittance, the potential disagreement between these estimates could indicate either a violation of our modeling assumptions, such as the choice of \( \eta_{\text{bound}} \) and the rheology of ice near the cold surface; the choice of the bottom boundary condition; the assumptions of uniformity of tidal heating distribution and conductive heat transport; or that significant heating occurs in the rocky mantle or the ocean.

5. Conclusions

We simulated the viscous relaxation of the icy shells of ocean worlds to test the sensitivity of gravity–topography admittance to the structure of the shell. Our models show that the behavior of the bottom interface as a material or phase interface strongly influences the admittance spectrum. At degrees where tidal heating can drive melting and freezing, the base of the shell may behave as a phase boundary and cause admittance to be small, near compensated values. Otherwise, a lack of a heat source would render a material boundary condition more applicable. If the base is treated as a material boundary, low viscosities cause the relaxation of basal topography by lateral flow faster than the relaxation of higher-viscosity surface topography, leading to uncompensated topography. For the bottom material boundary, the larger the viscosity gradient within the shell, the more the admittance spectrum deviates from the Cartesian Airy isostasy model. In the case of Enceladus, with an extreme viscosity gradient due to low surface temperature, the bottom material boundary predicts an effectively uncompensated admittance spectrum.

A transition between topography support mechanisms may be resolved with future measurements, allowing us to probe the shell structure of ocean worlds. The viscosity structure of the shell is controlled by the rheology of ice. In particular, the poorly constrained grain size of ice influences the tidal heat production and the extent of the low-viscosity region at the base of the shell, affecting the temperature profile. If admittance is resolved up to degree 10 for Europa, we will capture the transition for shells thicker than 24 km when \( \eta_{\text{melt}} = 10^{14} \text{ Pa s} \) and for shells thicker than 15 km when \( \eta_{\text{melt}} = 10^{15} \text{ Pa s} \). Measurements of admittance at high degrees \((n > 10)\) would enable the thickness of thinner and/or colder shells to be constrained.

Because admittance is better suited for constraining the shell thickness for thicker and warmer shells, it may augment the shell thickness determination from the ice-penetrating radar, which is better suited for thinner and colder shells. Thus, a combination of radar and gravity measurements in future ocean world missions may improve the robustness of the measurement strategy.

We also find that the admittance measurement can be used to constrain the magnitude of the tidal dissipation within the icy shell. Such a measurement would be complementary to a demanding measurement of the imaginary part of the tidal Love number, which is proportional to the total tidal dissipation. Measuring admittance in addition to \( \text{Im}(k_2) \) can be used to separate the shell heating from the mantle and ocean heating. Knowledge of heating distribution within ocean worlds would allow for a better understanding of the heat budget and stability of their oceans, which is critical for their long-term habitability.

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Appendix A

Tidal Heating

We compute the volumetric tidal heat production rate using a spherically symmetric internal structure model and a Maxwell model of viscoelasticity. Our modeling follows the steps of Takeuchi & Saito (1972), who define six radial functions \( y_i(r) \) to describe tidal field flow. These functions are radial multipliers of the corresponding fields expanded in spherical harmonics. The index \( n \) refers to a spherical harmonic degree. We restrict our analysis to degree 2 tides and, for now on, we will omit the dependence on \( n \). The functions \( y_i \) describe radial and tangential displacements \((y_1, y_3)\) and radial and tangential stresses \((y_2, y_4)\) expanded in vector spherical harmonics. The gravitational potential \( y_6 \) is expanded in scalar spherical harmonics. \( y_6 \) contains the radial derivative of the gravitational potential and is formulated in the following way by Takeuchi & Saito (1972) to simplify the surface boundary condition:

\[
y_6(r) = \frac{dy_5(r)}{dr} - 4\pi G\rho y_1(r) + \frac{n + 1}{r} y_3(r),
\]

where \( \rho \) is the density of the layer. Note that the definition for \( y_6(r) \) is different from another commonly used notation of Alterman et al. (1959). These six radial functions are found by solving a system of linear differential equations within layers of constant density and viscoelastic parameters:

\[
\frac{dy(r)}{dr} = T_{\text{tidal}} y(r),
\]

where \( T_{\text{tidal}} \) is a 6 \( \times \) 6 matrix given by Equation (82) in Takeuchi & Saito (1972). The surface boundary condition at \( r = R \) for surface radius \( R \) is given by vanishing stresses \((y_2(R) = y_4(R) = 0)\) and \( y_6(R) = (2n + 1)/R \) describing the potential and its derivative continuity. Across the liquid–solid interfaces, which occur at the ice–ocean and ocean–mantle boundaries for our three-layer model, additional boundary conditions are required. At the center of the planet \( r = 0 \), displacements are zero \((y_1(0) = y_3(0) = 0)\) and the gravitational potential perturbation is also zero \((y_6(0) = 0)\). The system of differential equations becomes singular at \( r = 0 \). An analytical solution at the surface of a small homogeneous sphere is used as a starting solution of the radial functions \( y_i(r) \) (see Takeuchi & Saito 1972 and Martens 2016 for more details). This starting solution at \( r_0 \ll R \), consisting of three linearly independent solutions, is then propagated through successive layers by solving Equation (A2) with an eighth-order Runge–Kutta numerical integrator scheme using variable normalization from Martens (2016). Finally, the three independent solutions at the surface are linearly weighted to satisfy the surface boundary
condition and combined to form the six radial functions $y_i(r)$. The complex-valued tidal Love number $K_2$ is found to be

$$y_5(R) - 1.$$ 

To find the radial distribution of tidal heating, we follow Tobie et al. (2005). Tidal heating is driven by the periodic tidal potential. The degree 2 tidal potential for a tidally locked satellite on an eccentric orbit, computed to the first order in eccentricity and evaluated on the surface of the satellite, is given by Moore & Schubert (2000):

$$\Phi = R^2 \omega^2 e \left[ -\frac{3}{2} P_{2,0}(\cos \theta) \cos \omega t \\
+ \frac{1}{4} P_{2,2}(\cos \theta) (3 \cos \omega t \cos 2\phi + 4 \sin \omega t \sin 2\phi) \right],$$  

(A3)

where $\omega$ is the orbital (and rotational) frequency, $e$ is the eccentricity, $\theta$ is colatitude, $\phi$ is longitude, $t$ is time, and $P_{2,0}(x)$ and $P_{2,2}(x)$ are unnormalized associated Legendre functions.

Our internal structure model consists of three main layers. The solid icy shell and mantle are described by the elastic shear modulus $\mu$, Poisson’s ratio $\nu$, and viscosity $\eta$, while the ocean is described by the bulk modulus $K$ following the values given in Table 1. The viscosity profile in the shell is broken up into 80 constant viscosity layers. The viscoelastic Maxwell rheology moduli used in solving for $\eta_i$ are given by the complex-valued Lamè’s first parameter $\tilde{\lambda}$ and the complex shear modulus $\tilde{\mu}$ in terms of their real-valued counterparts along with the bulk modulus $\kappa$ and viscosity $\eta$:

$$\tilde{\lambda} = \frac{\omega N_\kappa + \mu \kappa / \eta}{\omega I + \mu / \eta},$$  

(A4)

$$\tilde{\mu} = \frac{\omega \mu I}{\omega I + \mu / \eta}.$$  

(A5)

We then use the radial functions $y_i(r)$ to find the sensitivity parameter $H_p$ introduced by Tobie et al. (2005) to describe the radial sensitivity to the shear modulus (Equation (33) of Tobie et al. 2005) as it enables the radial distribution of the volumetric dissipation rate to be computed. A simplified version of $H_p$ is given by Equation (25) in Beuthe (2013).

Finally, the radial profile of the surface-averaged volumetric tidal dissipation rate averaged over the moon’s orbit is computed using Equation (37) of Tobie et al. 2005, accounting for the sign correction made by Beuthe (2013):

$$h_{\text{tide}}(r) = \frac{21}{10} \frac{\omega^5 R^4 \kappa^2}{r^2} H_p \text{Im}(\tilde{\mu}).$$  

(A6)

Our tidal dissipation code was benchmarked by comparing our $y_i(r)$ profiles with values from Figures C1 and C2 of Kamata et al. (2015).

**Appendix B**  
**Surface Temperature**

The surface temperature of our spherically symmetric moon is given by the uniform equilibrium temperature using a fast-rotator approximation:

$$T_{eq} = \left[ \frac{F(1 - A)}{4\sigma} \right]^{1/4},$$  

(B1)

where $F$ is the solar irradiance, $A$ is the Bond albedo, and $\sigma$ is the Stefan–Boltzmann constant. For Europa, we use the solar irradiance at Jupiter of 50.26 W m$^{-2}$ and Bond albedo 0.68 (Grundy et al. 2007) and find the surface temperature $T_1 \approx 92$ K. For Enceladus, we use the solar irradiance at Saturn of 14.82 W m$^{-2}$ and a Bond albedo of 0.81 (Spencer et al. 2006) and find the surface temperature $T_1 \approx 59$ K.

**Appendix C**  
**Stokes Flow with Self-gravitation**

The viscous relaxation of the icy shell is modeled by solving for the Stokes flow of an incompressible fluid in a spherical shell with self-gravitation. Similarly to the tidal heating computation, a spherically symmetric icy shell is broken up into layers of constant density $\rho$ and viscosity $\eta$, where the viscosity is set by the temperature profile found in Section 2.3. The governing equations are

$$\nabla \cdot \mathbf{v} = 0,$$

$$\nabla^2 V = -4\pi G \rho,$$

$$0 = -\nabla p + \eta \nabla^2 \mathbf{v} + F,$$  

(C1)

where $\mathbf{v}$ is the velocity, $V$ is the potential, $p$ is the pressure, $G$ is the universal gravitational constant, and $F$ is the external force, equal to gravity in our case. To solve for the Stokes flow, we follow Hager & Clayton (1989), who define six radial functions $y_i^n(r)$, where $n$ is the spherical harmonic degree (Equations (4.11)–(4.18)). These functions are not to be confused with the $y$ functions discussed above for the tidal heating problem. The radial and poloidal velocities ($y_1^0(r)$ and $y_2^0(r)$), radial normal stress ($y_3^0(r)$), and poloidal shear stress ($y_4^0(r)$) are the coefficients of the corresponding vector fields expanded in vector spherical harmonics. The gravitational potential perturbation ($y_5^0(r)$) and its radial derivative ($y_6^0(r)$) are coefficients of scalar spherical harmonics. The decoupling of the system of equations for the velocities and stresses from that of the potential is achieved by introducing a new set of variables $u_i$ and $v_i$. Again, we will drop the dependence on degree $n$ for convenience. The components of the state vectors $\mathbf{u}$ and $\mathbf{v}$ are given in terms of the $y_i$ variables as

$$u_1 = y_1,$$

$$u_2 = y_2,$$

$$u_3 = r y_3 / \eta_0 + \rho y_5 / \eta_0,$$

$$u_4 = r y_4 / \eta_0,$$

$$v_1 = \rho_0 r y_6 / \eta_0,$$

$$v_2 = \rho_0 r^2 y_7 / \eta_0,$$

(C2)

where $\rho$ is the local density, $\rho_0$ is a reference density, and $\eta_0$ is a reference viscosity chosen close to $\eta_{\text{bound}}$. The radial derivatives for $u_i$ and $v_i$ can be written in matrix form:

$$\frac{du_i}{dv} = \mathbf{J}^{\text{Stokes}}_{i,\text{bound}} \mathbf{u},$$

$$\frac{dv_i}{dv} = \mathbf{K}^{\text{Stokes}}_{i,\text{bound}} \mathbf{v},$$  

(C3)

where $\mathbf{v} = \ln(r/R)$, $R$ is the mean radius of the moon, $\mathbf{J}^{\text{Stokes}}_{i,\text{bound}}$ is a $4 \times 4$ matrix, and $\mathbf{K}^{\text{Stokes}}_{i,\text{bound}}$ is a $2 \times 2$ matrix given by Equations (4.33) and (4.34) in Hager & Clayton (1989). These matrices depend only on the viscosity $\eta$ within a layer and the spherical harmonic degree $n$. We note that the same system of
Equations (C1) is solved by Hager & O’Connell (1979) as a nearly identical coupled 6 × 6 system.

With a choice of boundary conditions at the surface and base of the icy shell, a solution can be propagated from the base to the surface analytically using the propagator matrix method (Equation (4.43) for the u system and Equation (4.44) for the v system in Hager & Clayton 1989).

A free-slip boundary condition is imposed at both boundaries, setting poloidal shear stresses and, correspondingly, \( u_{4b} \) and \( u_{4b} \) to zero. To account for self-gravity in the boundary conditions for \( u_4 \), we need to solve the second equation of Equations (C3) for \( v_1 \) at the surface and the base of the shell. We solve the v system given the shape amplitudes of the top and bottom interfaces: \( h_t \) and \( h_b \), respectively. We get the following expressions for \( v_{1t} \) and \( v_{1b} \):

\[
v_{1t} = \frac{4\pi G \rho_0 R}{\eta_0 (2n + 1)} \left[ R h_t \rho_1 + r_b h_b \Delta \rho (r_b / R)^{n+1} \right],
\]

\[
v_{1b} = \frac{4\pi G \rho_0 \rho_b}{\eta_0 (2n + 1)} \left[ R h_t \rho_1 (R / r_b)^{n} + r_b h_b \Delta \rho \right], \tag{C4}
\]

where \( r_b = R - D \) for shell thickness \( D \); and \( \rho_1 \) and \( \rho_2 \) are the densities of the icy shell and the ocean, respectively, with a density contrast of \( \Delta \rho = \rho_2 - \rho_1 \). Using the solution for \( v_{1t} \) and \( v_{1b} \), we find the boundary conditions for \( u_3 \) at the surface (\( u_{3,1} \)) and base (\( u_{3,2} \)) using Equations (4.53) and (4.54) in Hager & Clayton 1989). The boundary conditions for \( u_3 \) and \( u_{3b} \) are given as functions of the shape amplitudes \( h_t \) and \( h_b \) as

\[
\begin{align*}
    u_{3t} &= u_3(R) = -\frac{\rho_1 R g_1}{\eta_0} + \frac{4\pi G \rho_1^2 R^2}{\eta_0 + 2\eta_0 n} h_t \\
    &+ \left( \frac{4\pi G \rho_1 \Delta \rho \rho_1^2 (r_b / R)^n}{\eta_0 + 2\eta_0 n} \right) h_b, \\
    u_{3b} &= u_3(R - D) = -\frac{4\pi G \rho_1 \Delta \rho \rho_1^2 (r_b / R)^n}{\eta_0 + 2\eta_0 n} h_t \\
    &+ \frac{\Delta \rho \rho_1 g_2}{\eta_0} - \frac{4\pi G \rho_1 \Delta \rho \rho_1^2 r_b^2}{\eta_0 + 2\eta_0 n} h_b, \tag{C5}
\end{align*}
\]

where \( g_1 \) and \( g_2 \) are the gravitational accelerations at the top and bottom boundaries, respectively. We can write these two equations in matrix form:

\[
\begin{bmatrix}
    u_{3t} \\
    u_{3b}
\end{bmatrix} = \begin{bmatrix} Q_t & W_t \end{bmatrix} \begin{bmatrix} h_t \\
    h_b
\end{bmatrix}. \tag{C6}
\]

The propagator matrix solution for the \( u \) system in Equation (C3) is

\[
\begin{bmatrix}
    u_{1t} \\
    u_{2t} \\
    u_{3t} \\
    u_{4t}
\end{bmatrix} = P \cdot \begin{bmatrix} u_{1b} \\
    u_{2b} \\
    u_{3b} \\
    u_{4b}
\end{bmatrix}, \tag{C7}
\]

where \( u_t \) and \( u_b \) are the state vectors at the surface and the base, respectively. \( P \) is the propagator matrix, which is found as the product of layer-wise propagator matrices for each layer of uniform viscosity between the bottom (\( k \)) boundary, \( r_b = r_k = R - D \), and the surface, \( r_1 = R \):

\[
P = \begin{bmatrix} P_{1,1} & P_{1,2} & P_{1,3} & P_{1,4} \\
    P_{2,1} & P_{2,2} & P_{2,3} & P_{2,4} \\
    P_{3,1} & P_{3,2} & P_{3,3} & P_{3,4} \\
    P_{4,1} & P_{4,2} & P_{4,3} & P_{4,4}
\end{bmatrix}
= P_1(r_1, r_2) \cdot P_2(r_2, r_3) \cdot \ldots \cdot P_{k-1}(r_{k-1}, r_k). \tag{C8}
\]

The propagator matrix for the \( i \)th layer is given by the following matrix exponential:

\[
P_i(r_i, r_{i+1}) = \exp \left( \int_{r_i}^{r_{i+1}} \tilde{\kappa}_i \ln \left( \frac{r_i}{r_{i+1}} \right) \right).
\]

For the bottom material boundary, we solve for a time-evolving system illustrated on the left side of Figure 2. Thus, the radial flow velocities \( u_{1t} \) and \( u_{1b} \) are unknowns. We rearrange Equation (C7) and use the boundary condition for \( u_3 \) from Equation (C5) to express a system of linear differential equations for the flow velocities \( u_1 \) at the top and the base of the shell:

\[
\begin{bmatrix} dh_t \\
    dh_b
\end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\
    1 & 0 & 0 & 0 \\
    0 & 1 & 0 & 0 \\
    0 & 0 & 1 & 0
\end{bmatrix} \begin{bmatrix} u_{1t} \\
    u_{1b}
\end{bmatrix}
= \begin{bmatrix} Q_t & W_t \\
    Q_b & W_b
\end{bmatrix} \begin{bmatrix} h_t \\
    h_b
\end{bmatrix} \tag{C10}
\]

where \( Q_t, Q_b, W_t, \) and \( W_b \) are the elements of the \( u_3 \) matrix from Equation (C6).

From Equation (C10), we find the eigenvalues \( \gamma_1, \gamma_2 \) and eigenvectors \( X, Y \) of the matrix \( Z \). We then get the time evolution of \( h \) as a sum of two decaying exponentials, resulting in Equation (5). The degree-dependent shape ratio \( w_n = -h_b / h_t \) needed to compute the admittance converges to the ratio of the elements of the eigenvector \( Y \), which corresponds to the longer relaxation time \( \tau_2 \).

The bottom phase boundary condition is different in that we solve for a dynamic equilibrium illustrated on the right side of Figure 2. Thus, the topography amplitude vector \( h \) is constant in time. This gives us the condition that the radial velocity at the surface (\( u_{1t} \)) is zero. Although the topography at the base \( h_b \) is constant, radial velocity (\( u_{1b} \)) can be nonzero because the flow across the boundary is assumed to be balanced by phase transitions (melting or freezing). Similar to the bottom material boundary, \( u_{3t} \) and \( u_{3b} \) are taken as functions of \( h_t \) and \( h_b \) using Equation (C6), and a free-slip boundary condition gives \( u_{3t} = u_{3b} = 0 \). We again rearrange the terms in the propagator matrix solution (Equation (C7)), expressing a vector of
unknowns in terms of the known quantities:

\[
\begin{pmatrix}
p_{1,3} W_b & p_{1,1} & 0 & p_{1,2} \\
p_{2,3} W_b & p_{2,1} & -1 & p_{2,2} \\
p_{3,3} W_b - W_t & p_{3,1} & 0 & p_{3,2} \\
p_{3,3} W_b & p_{3,1} & 0 & p_{3,2}
\end{pmatrix}
\begin{pmatrix}
h_b \\
u_{b1} \\
u_{b2} \\
u_{b2}
\end{pmatrix}
= \begin{pmatrix}
0 & -p_{1,3} \\
0 & -p_{2,3} \\
1 & -p_{3,3} \\
0 & -p_{3,3}
\end{pmatrix}
\begin{pmatrix}
Q_t h_t \\
Q_b h_t
\end{pmatrix}.
\] (C11)

We set \(h_t = 1\) to get four equations with four unknowns and solve for \(h_b, u_{b1}, u_{b2},\) and \(u_{b2}\). Finally, we find the shape ratio \(w_n = -h_b/h_t = -h_{b0}\), which we use in computing the gravity–topography admittance.

### Appendix D: Gravity–Topography Admittance

In this section, we present how the gravity–topography admittance is (1) computed from the shape and gravity data and (2) modeled using the degree-dependent shape ratio \(w_n\) found in the previous subsection. Our approach is similar to Ermakov et al. (2017) in their study of Ceres’ admittance. The measured shape of the icy moon can be expressed as a spherical harmonic expansion (Wieczorek 2007):

\[
r(\theta, \phi) = R \left[ 1 + \sum_{n=1}^{\infty} \sum_{m=0}^{n} \left( \tilde{A}_{nm} \cos(m\phi) + \tilde{B}_{nm} \sin(m\phi) \right) \tilde{P}_{nm}(\cos(\theta)) \right],
\] (D1)

where \(\theta\) is the colatitude, \(\phi\) is the longitude, \(R\) is the mean radius, \(\tilde{A}_{nm}\) and \(\tilde{B}_{nm}\) are normalized coefficients of the spherical harmonic expansion, and \(\tilde{P}_{nm}\) are normalized associated Legendre functions for spherical harmonic degree \(n\) and order \(m\). In order to consider deviations in shape from the rotational and tidal bulges, the hydrostatic contribution to the shape must be subtracted (e.g., Less et al. 2014) before applying our analysis. This procedure has the greatest effect on modifying the coefficients at degrees 2 and 4. We will denote the coefficients of the processed shape as \(\tilde{A}_{nm}'\) and \(\tilde{B}_{nm}'\). Similarly, the gravitational potential is expanded in spherical harmonics:

\[
U(r, \theta, \phi) = \frac{GM}{r} \left[ 1 + \sum_{n=2}^{\infty} \sum_{m=0}^{n} \left( \frac{R_0}{r} \right)^n C_{nm} \cos(m\phi) + S_{nm} \sin(m\phi) \right] \tilde{P}_{nm}(\cos(\theta)),
\] (D2)

where \(R_0\) is the reference radius, \(r\) is the observation radius, and \(C_{nm}\) and \(S_{nm}\) are the normalized spherical harmonic coefficients of the gravitational potential. Note that the reference radius is not necessarily equal to the mean radius \(R\). Again, the hydrostatic contribution to gravity must be subtracted, and the modified coefficients are denoted \(\tilde{C}_{nm}'\) and \(\tilde{S}_{nm}'\). The variance spectrum of the shape \(V_n^\alpha\) and gravity \(V_n^g\), and the cross-variance spectrum of gravity and shape \(V_n^{g\alpha}\) are found from the spherical harmonic coefficients in Equations (D1) and (D2) as

\[
V_n^\alpha = \sum_{m=0}^{n} (\tilde{A}_{nm}'^2 + \tilde{B}_{nm}'^2),
\]
\[
V_n^g = \sum_{m=0}^{n} (\tilde{C}_{nm}'^2 + \tilde{S}_{nm}'^2),
\]
\[
V_n^{g\alpha} = \sum_{m=0}^{n} (\tilde{A}_{nm}' \tilde{C}_{nm}' + \tilde{B}_{nm}' \tilde{S}_{nm}').
\] (D3)

Finally, the coefficient Root-Mean-Square (rms) spectra can be found as:

\[
\text{rms}_r^\alpha = \sqrt{\frac{V_n^\alpha}{2n + 1}},
\]
\[
\text{rms}_r^g = \sqrt{\frac{V_n^g}{2n + 1}}.
\]

We are interested in finding the gravity–topography admittance between radial gravitational acceleration and the shape. Thus, we need to differentiate Equation (D2) with respect to \(r\), which results in simply multiplying each term in Equation (D2) by \((n + 1)/r\). The gravity–topography admittance \(Z_n\) in terms of the gravity–shape cross-variance and shape variance is given as

\[
Z_n = \frac{V_n^{g\alpha}}{V_n^\alpha} \frac{GM}{R^3} (n + 1),
\] (D4)

This expression is known to produce a more accurate admittance estimate in the case when gravity data is more noisy than the shape data (McKenzie 1994), which is typically the case for ocean worlds. It is fully equivalent to

\[
Z_n = \sqrt{\frac{V_n^{g\alpha}}{V_n^\alpha}} R_n^{g\alpha} \cdot \frac{GM}{R^3} (n + 1),
\] (D5)

where \(R_n^{g\alpha} = V_n^{g\alpha}/\sqrt{V_n^\alpha V_n^g}\) is the gravity–shape correlation coefficient. We note that in the viscous relaxation models presented in this paper, the gravity and shape are always “in phase.” Thus, the correlation is either 1 or \(-1\). Because our model concerns small-amplitude topography, it is spectrally pure. That is, the topography harmonic of degree \(n\) and order \(m\) produces the gravity perturbation only of the same degree and order. Thus, Equation (D5) can be simply written in terms of shape and gravity coefficients:

\[
Z_n = \frac{\sigma}{h} \cdot \frac{GM}{R^3} (n + 1),
\] (D6)

where \(\sigma = [\tilde{C}_{nm}', \tilde{S}_{nm}']\) and \(h = [\tilde{A}_{nm}', \tilde{B}_{nm}']\) are degree-\(n\) coefficients of gravity and shape, respectively.

In order to compute the model admittance, we need to find the relationship between the gravity and shape spherical harmonic coefficients. Assuming a homogeneous spherical body with topography small in amplitude relative to its wavelength (a “mass sheet” approximation), we can use the first-order term of Equation (10) in Wieczorek & Phillips (1998) to get the linear relationship between gravity and shape coefficients:

\[
\sigma = \frac{3}{2n + 1} \left( \frac{R}{R_{vol}} \right)^3 \left( \frac{R}{R_0} \right)^n h,
\] (D7)

where \(R_{vol}\) is the radius of the volume-equivalent sphere.
For a body consisting of multiple layers, the topography at the interfaces associated with density changes generates perturbations in gravity. We only consider topography at the surface and the base of the icy shell assuming the ocean floor has no topography. Thus, the total gravity includes contributions only from the upper two interfaces. The gravity contribution of each layer needs to be weighted by fractional mass of that layer \((\rho_i/\bar{\rho})(n_i/R)^n\), where \(\bar{\rho}\) is the mean density of the body. Because we seek the gravity coefficients referenced to the mean radius of the body \((R_0 = R\) in Equation (D2)), the contributions to gravity from the \(i\)th layer must be upward propagated by \((n_i/R)^n\) to find the contribution to gravity at the surface. Finally, we reference the topography coefficients to the mean radius \(R\) of the body in Equation (D1). Thus, the topography at the base of the shell \(h_b\) needs to be upscaled by \((R/\sqrt{R-D})\). In summary, we find the total gravity coefficient at degree \(n\) to be

\[
\sigma = \frac{3}{2n+1} \cdot h_s \cdot \left(\frac{\rho_1}{\bar{\rho}}\right) + \frac{3}{2n+1} \cdot h_b \times \left(\frac{R}{R-D}\right)^{\frac{n+2}{n+1}} \prod_{i=1}^{N} \frac{\rho_i}{\bar{\rho}} \cdot \left(\frac{R-D}{R}\right)^{\frac{n+2}{n+1}}.
\]

(D8)

In general, we can write the total gravity coefficient at degree \(n\) as a sum of the contributions from topography at each interface:

\[
\sigma = \sum_{i=1}^{N} \frac{3}{2n+1} \cdot h_i \cdot \left(\frac{R_i}{R}\right)^{\frac{n+2}{n+1}} \prod_{j=1}^{i-1} \frac{\rho_j}{\bar{\rho}} \cdot \left(\frac{R-D}{R}\right)^{\frac{n+2}{n+1}} \cdot w_n.
\]

(D9)

where \(N\) is the number of layers. In this derived expression, we ignored the factor \((R/R_0)^3\) as the difference between \(R_0\) and \(R\) is small for a small-radius planet, which is one of our assumptions anyway. Plugging in \(h_b = -w_0 h_s\), dividing by the surface shape coefficients \(h_s\), and multiplying by \(GMR^{-3}(n+1)\) to find the ratio of radial gravitational acceleration to shape coefficients (in mGal km\(^{-1}\)), the gravity–topography admittance simplifies to

\[
Z_n = \frac{GM}{R^3} \cdot \frac{3(n+1)}{(2n+1)} \cdot \left(\frac{\rho_1}{\bar{\rho}} - \frac{\Delta \rho}{\bar{\rho}} \left(\frac{R-D}{R}\right)^{\frac{n+2}{n+1}} w_n\right).
\]

(D10)

The shape ratio \(w_n\) is found given the viscosity profile and boundary conditions described in Appendix C. Plugging in \(w_n = \rho_1/\Delta \rho\) into Equation (D10), we recover the admittance for the standard Cartesian Airy-compensation model:

\[
Z_n = \frac{GM}{R^3} \cdot \frac{3(n+1)}{(2n+1)} \cdot \frac{\rho_1}{\bar{\rho}} \cdot \left[1 - \left(\frac{R-D}{R}\right)^{\frac{n+2}{n+1}}\right].
\]

(D11)

If \(w_n = 0\), we recover the admittance for an uncompensated topography:

\[
Z_n = \frac{GM}{R^3} \cdot \frac{3(n+1)}{(2n+1)} \cdot \frac{\rho_1}{\bar{\rho}}.
\]

(D12)

ORCID iDs

Ryunosuke Akiba  https://orcid.org/0000-0002-2681-3195
Anton I. Ermakov  https://orcid.org/0000-0002-7020-7061
Burkhard Militzer  https://orcid.org/0000-0002-7092-5629

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