RAMIFICATION FILTRATION AND DIFFERENTIAL FORMS

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ABSTRACT. Let $L$ be a complete discrete valuation field of prime characteristic $p$ with finite residue field. Denote by $\Gamma_L^{(v)}$ the ramification subgroups of $\Gamma_L = \text{Gal}(L^{sep}/L)$. We consider the category $\mathcal{M}_{\Gamma_L}^{Lie}$ of finite $\mathbb{Z}_p[\Gamma_L]$-modules $H$, satisfying some additional (Lie)-condition on the image of $\Gamma_L$ in $\text{Aut}_\mathbb{Z}_p H$. In the paper it is proved that all information about the images of the groups $\Gamma_L^{(v)}$ in $\text{Aut}_\mathbb{Z}_p H$ can be explicitly extracted from some differential forms $\tilde{\Omega}[N]$ on the Fontaine etale $\phi$-module $M(H)$ associated with $H$. The forms $\tilde{\Omega}[N]$ are completely determined by a canonical connection $\nabla$ on $M(H)$. In the case of fields $L$ of mixed characteristic, which contain a primitive $p$-th root of unity, we show that a similar problem for $\mathbb{F}_p[\Gamma_L]$-modules also admits a solution. In this case we use the field-of-norms functor to construct the corresponding $\phi$-module together with the action of the Galois group of a cyclic extension $L_1$ of $L$ of degree $p$. Then our solution involves the characteristic $p$ part (provided by the field-of-norms functor) and the condition for a “good” lift of a generator of $\text{Gal}(L_1/L)$. Apart from the above differential forms the statement of this condition uses the power series coming from the $p$-adic period of the formal group $\mathbb{G}_m$.

INTRODUCTION

Let $L$ be a complete discrete valuation field with finite residue field of characteristic $p$. Let $\Gamma_L$ be the absolute Galois group of $L$. Let $\{\Gamma_L^{(v)}\}_{v \geq 0}$ be the filtration of $\Gamma_L$ by the ramification subgroups. This filtration provides $\Gamma_L$ with additional structure and allows us to introduce various classes of infinite field extensions (arithmetically profinite, deeply ramified etc.), which play an important role in modern arithmetic algebraic geometry. For $\Gamma_L$-modules $H$, the evaluation of $v_0(H) \in \mathbb{Q}$ such that $\Gamma_L^{(v)}$ act trivially on $H$ for $v > v_0(H)$, provides us with good estimates for discriminants of the fields of definition of $h \in H$. Such estimates are used very often to answer various number theoretic questions.

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However, an explicit description of the structure of the ramification filtration for a very long time was known only at the level of the Galois groups of abelian field extensions. At the time when the structure of the Galois group $\Gamma_L$ was completely described (the case of the maximal $p$-extensions – Shafarevich [21], Demushkin [13], and the general case – Janssen-Wingberg [16]) it became clear that $\Gamma_L$ is a very weak invariant of the field $L$. The situation cardinaly changed later when it was established (Mochizuki [19], author [4]) that taking $\Gamma_L^{(v)}$ into account gives us absolute invariant of the field $L$. However, in order to work with this invariant we still need to know an explicit description of the filtration by the groups $\Gamma_L^{(v)}$.

I.R. Shafarevich always paid attention to this problem, for example, cf. Introduction to [17]. His motivation was the following: for every prime number $p$ there is only one such filtration and we know almost nothing about its structure. In 1990’s the author developed a nilpotent version of the Artin-Schreier theory and obtained an explicit description of the ramification filtration modulo the subgroup of $p$-th commutators of $\Gamma_L$. Such description was obtained, first, in the characteristic $p$ case, [1, 2, 3], and then developed in the mixed characteristic case, [7, 8, 9]. These results play a crucial role in this paper, where we study the images of the ramification subgroups $\Gamma_L^{(v)}$ in the group of automorphisms $\text{Aut}_{\mathbb{Z}_p} H$ of finite $\mathbb{Z}_p[\Gamma_L]$-modules $H$. Our main result states that this arithmetic structure can be completely described (under some additional condition) in purely geometric properties of Fontaine’s etale $\phi$-modules $M(H)$.

More precisely, if $L = p$ we construct differential forms $\tilde{\Omega}[N]$, $N \in \mathbb{N}$, on an extension of scalars of $M(H)$, and specify the way how the image of the ramification filtration in $\text{Aut}_{\mathbb{Z}_p} H$ can be recovered from these forms. Note that the definition of $\tilde{\Omega}[N]$ depends only on a natural connection constructed on $M(H)$. If char$L = 0$ we assume that $L$ contains a $p$-th root of unity $\zeta_1 \neq 1$ and restrict ourselves to the case of Galois $\mathbb{F}_p$-modules. Then we apply the field-of-norms functor to reduce the situation to the characteristic $p$ case and use a characterization of “good” lifts of automorphisms of our cyclic field extension of $L$ from [7, 8]. This characterization uses again the differential forms $\tilde{\Omega}[N]$ and a power series coming from the $p$-adic period of the formal group $\mathbb{G}_m$.

By our opinion this result establishes quite interesting link between the Galois theory of local fields and very popular area of $D$-modules, lifts of Frobenius, Higgs vector bundles etc.

1. Statement of the main result

1.1. General notation. Everywhere in the paper $p$ is a fixed prime number. If $E_0$ is a field then $E_0^{sep}$ is its separable closure in some algebraic closure $E_0^{alg}$ of $E_0$. If $E$ is a field such that $E_0 \subset E \subset E_0^{sep}$
set $\Gamma_E = \text{Gal}(E_0^{\text{sep}}/E)$. The field $E_0^{\text{sep}}$ will be considered as a left $\Gamma_{E_0}$-module, i.e. for any $\tau_1, \tau_2 \in \Gamma_{E_0}$ and $o \in E_0^{\text{sep}}$, $(\tau_1 \tau_2) o = \tau_1(\tau_2 o)$. If $\text{char}E_0 = p$ we set for any $a \in E_0^{\text{sep}}$, $\sigma(a) = a^p$.

If $V$ is a module over a ring $R$ then $\text{End}_R V$ is the $R$-algebra of $R$-linear endomorphisms of $V$. We always consider $V$ as a left $\text{End}_R V$-module, i.e. if $l_1, l_2 \in \text{End}_R V$ and $v \in V$ then $(l_1 l_2)v = l_1(l_2(v))$. We also consider $\text{End}_R V$ as a Lie $R$-algebra with the Lie bracket $[l_1, l_2] = l_1 l_2 - l_2 l_1$. If $S$ is an $R$-module then we often denote by $V_S$ the extension of scalars $V \otimes_R S$.

1.2. **Functorial system of lifts to characteristic 0.** Suppose $K_0 = k_0((t_0))$ is the field of formal Laurent series in a (fixed) variable $t_0$ with coefficients in a finite field $k_0$ of characteristic $p$. The uniformiser $t_0$ provides a $p$-basis for any field extension $E$ of $K_0$ in $K_0^{\text{sep}}$, i.e. the set $\{1, t_0, \ldots, t_0^{p-1}\}$ is a $p$-basis of $E$. We use this $p$-basis to construct a compatible system of lifts $O(E)$ of the fields $E$ to characteristic 0.

This is a special case of the construction of lifts from [11]; it can be explained as follows.

For all $M \in \mathbb{N}$, set $O_M(E) = W_M(\sigma^{M-1}E)[t_0]$, where $\bar{t}_0 = [t_0]$ is the Teichmuller representative of $t_0$ in the ring of Witt vectors $W_M(E)$. The rings $O_M(E)$ are the lifts of $E$ modulo $p^M$, i.e. they are flat $\mathbb{Z}/p^M$-algebras such that $O_M(E) \otimes_{\mathbb{Z}/p^M} \mathbb{Z}/p = E$. Note that the system

$$\{O_M(E) \mid M \in \mathbb{N}, K_0 \subset E \subset K_0^{\text{sep}}\}$$

is functorial in $M$ and $E$. In particular, if $E/K_0$ is Galois then there is a natural action of $\text{Gal}(E/K_0)$ on $O_M(E)$ and $O_M(E)^{\text{Gal}(E/K_0)} = O_M(K_0)$. The morphisms $W_M(\sigma)$ induce $\sigma$-linear morphisms on $O_M(E)$ which will be denoted again by $\sigma$. In particular, $\sigma(\bar{t}_0) = \bar{t}_0^\sigma$.

Introduce the lifts of the above fields $E$ to characteristic 0 by setting $O(E) = \varprojlim_M O_M(E)$. Then $O_M(E) = O(E)/p^M$ and $O(E)[1/p]$ is a complete discrete valuation field with uniformiser $p$ and the residue field $E$. Clearly, we have the induced morphism $\sigma$ on each $O(E)$. Also, if $E/K_0$ is Galois then $\text{Gal}(E/K_0)$ acts on $O(E)$ and $O(E)^{\text{Gal}(E/K_0)} = O(K_0)$. Notice that $O(K_0) = \varprojlim_M W_M(k_0)((\bar{t}_0))$ is the completion of the ring of formal Laurent series $W(k)((\bar{t}_0))$.

Set $O_{\text{sep}} = O(K_0^{\text{sep}})$.

The system of lifts $O(E)$, $E \subset K_0^{\text{sep}}$, can be extended to the system of lifts of all extensions of $K_0$ in $K_0^{\text{alg}}$. Indeed, note that $K_0^{\text{alg}} = \bigcup_{n \geq 0} K_0(t_n)^{\text{sep}}$, where $t_n^p = t_0$. Then $t_n$ gives the $p$-basis $\{1, t_n, \ldots, t_n^{p-1}\}$ for all separable extensions $E$ of $K_0(t_n)$ in $K_0^{\text{alg}}$ and we obtain (as earlier) the corresponding lifts $O(E)$. The system of lifts $O(E)$ is functorial in $E \subset K_0^{\text{alg}}$ (use that any separable extension $E_n$ of $K_0(t_n)$ appears uniquely as the composite $E_n K_0(t_n)$, where $E_n/K_0$ is separable). In particular, consider $K_0^{\text{ad}} = \bigcup_{N \in \mathbb{N}} K_0^{\text{ur}}(t_0^{1/N})$. Then for the
above defined lift of $K_0^{rad}$ we have $O(K_0^{rad}) = \bigcup_{N \in \mathbb{N}} O(K_0^{ur})[t_0^{-1/N}]$, where $K_0^{ur} = k_0((t_0))$ is the maximal unramified extension of $K_0$.

1.3. **Equivalence of the categories of $p$-groups and Lie algebras**, \cite{18}. Let $L$ be a finitely generated Lie $\mathbb{Z}_p$-algebra of nilpotent class $< p$, i.e. the ideal of $p$-th commutators $C_p(L)$ of $L$ is equal to zero. Let $A$ be an enveloping algebra of $L$. Then the elements of $L \subset A$ generate the augmentation ideal $J$ of $A$. There is a morphism of $\mathbb{Z}_p$-algebras $\Delta : A \to A \otimes A$ uniquely determined by the condition $\Delta(l) = l \otimes 1 + 1 \otimes l$ for all $l \in L$. Then the set $\exp(L)$ mod $J^p$ is identified with the group of all "diagonal elements modulo degree $p^v$ consisting of $a \in 1 + J$ mod $J^p$ such that $\Delta(a) \equiv a \otimes a \bmod(J \otimes 1 + 1 \otimes J)^p$.

In particular, there is a natural embedding $L \subset A/J^p$ and the identity

$$\exp(l_1) \cdot \exp(l_2) \equiv \exp(l_1 \circ l_2) \bmod J^p$$

induces the Campbell-Hausdorff composition law

$$(l_1, l_2) \mapsto l_1 \circ l_2 = l_1 + l_2 + \frac{1}{2}[l_1, l_2] + \ldots, \quad l_1, l_2 \in L.$$  

This composition law provides the set $L$ with a group structure. We denote this group by $G(L)$. Clearly $G(L) \simeq \exp(L) \bmod J^p$.

With the above notation the functor $L \mapsto G(L)$ determines the equivalence of the categories of finitely generated $\mathbb{Z}_p$-Lie algebras and profinite $p$-groups of nilpotence class $< p$. Note that a subset $I \subset L$ is an ideal in $L$ iff $G(I)$ is a normal subgroup in $G(L)$.

1.4. **(Lie)-condition.** For any finite field extension $K$ of $K_0$ in $K_0^{sep}$, let $\text{MIG}_K$ be the category of finitely generated $\mathbb{Z}_p$-modules $H$ with continuous left action of $\Gamma_K$. Each element $h \in H$ is defined over some finite extension $K(h)$ of $K$. In some sense the family of these fields determines "arithmetic" properties of $H$. More detailed information about the fields $K(h)$ can be obtained from the knowledge of the images of the ramification subgroups in upper numbering $\Gamma_K^{(v)}$, $v > 0$, in $\text{Aut}_{\mathbb{Z}_p} H$. For example, the minimal number $v_0(H) \in \mathbb{Q}$ such that all $\Gamma_K^{(v)}$ with $v > v_0(H)$ act trivially on $H$ provides us with upper estimates for the discriminants of the fields of definition of $h \in H$, cf. \cite{4}.

Let $H_0 \in \text{MIG}_{K_0}$ and let $\pi_{H_0} : \Gamma_{K_0} \to \text{Aut}_{\mathbb{Z}_p} H_0$ be the group homomorphism which determines the $\Gamma_{K_0}$-module structure on $H_0$. Consider the full subcategory $\text{MIG}_{K_0}^{\text{Lie}}$ in $\text{MIG}_{K_0}$ which consists of modules $H_0$ satisfying the following condition:

**(Lie)** The image $I(H_0) : = \pi_{H_0}(\mathcal{I}) \subset \text{Aut}_{\mathbb{Z}_p}(H_0)$ of the wild inertia subgroup $\mathcal{I} \subset \Gamma_{K_0}$ appears in the form $\exp(L(H_0))$, where $L(H_0) \subset \text{End}_{\mathbb{Z}_p} H_0$ is a Lie subalgebra such that $L(H_0)^p = 0$.

The condition $L(H_0)^p = \{l_1 \ldots l_p \mid l_1, \ldots, l_p \in L(H_0)\} = 0$ (the product is taken in $\text{End}_{\mathbb{Z}_p} H_0 \supset L(H_0)$) implies that $L(H_0)$ is a finitely
generated nilpotent \( \mathbb{Z}_p \)-algebra Lie of nilpotence class \(< p \). This gives the group isomorphism \( \exp : G(L(H_0)) \simeq I(H_0) \). Note that any normal subgroup of \( I(H_0) \) appears in the form \( \exp G(J) \), where \( J \) is a Lie ideal of \( L(H_0) \).

**Remark 1.1.** If \( pH_0 = 0 \) and \( \dim_{\mathbb{F}_p} H_0 \leq p \) then \( H_0 \) is of the Lie type.

1.5. **The first main result: the characteristic \( p \) case.** Suppose \( H_0 \in \mathbb{G}^{\text{Lie}}_{H_0} \). Our target is to determine for all \( v > 0 \), the images \( \pi_{H_0}(\Gamma_0^{(v)}) \) of the ramification subgroups \( \Gamma_0^{(v)} \) via an explicit construction of the ideals \( L(H_0)^{(v)} \subset L(H_0) \) such that \( \exp(L(H_0)^{(v)}) = \pi_{H_0}(\Gamma_0^{(v)}) \).

Our approach uses Fontaine’s “analytical” description of the Galois modules \( H_0 \in \mathbb{G}^{\text{Lie}}_{H_0} \) in terms of etale \( (\phi, O(K_0)) \)-modules \( M(H_0) \). A geometric nature of \( M(H_0) \) is supported by the existence of an analogue of the classical connection \( \nabla : M(H_0) \longrightarrow M(H_0) \otimes_{O(K_0)} \Omega^1_{O(K_0)} \), cf. [12]. (This map is uniquely characterized by the condition \( \nabla \cdot \phi = (\phi \otimes \phi) \cdot \nabla \).) The required information about the behaviour of ramification subgroups can be then extracted from some differential forms

\[
\tilde{\Omega}[N] \in M(H_0)_{O(K_p^{\text{ad}})} \otimes_{O(K_0)} \Omega^1_{O(K_0)}
\]

The construction of these differential forms is given completely in terms of the above connection \( \nabla \) and can be explained as follows.

Let \( K \subset K_0^{\text{sep}} \) be a fixed tamely ramified finite extension of \( K_0 \) such that \( \pi_{H_0}(\Gamma_K) = I(H_0) \). Then \( H := H_0|_{\Gamma_K} \) can be described via an etale \( (\phi, O(K)) \)-module \( M(H) = M(H_0) \otimes_{O(K_0)} O(K) \). Recall that \( M(H) = (H \otimes_{\mathbb{Z}_p} O_{\text{sep}})^{\Gamma_K} \) is a finitely generated \( O(K) \)-module with a \( \sigma \)-linear morphism \( \phi : M(H) \longrightarrow M(H) \) such that the image \( \phi(M(H)) \) generates \( M(H) \) over \( O(K) \). This allows us to identify the elements of \( H \) with a set of \( O_{\text{sep}}^{\rho} \)-solutions of a suitable system of equations with coefficients in \( O(K) \).

We establish below the construction of \( M(H) \) by introducing a \( \mathbb{Z}_p \)-linear embedding \( \mathcal{F} : H \longrightarrow M(H) \) which induces by extension of scalars the identification \( H_{O(K)} \simeq M(H) \). (We denote this identification by the same symbol \( \mathcal{F} \).)

Now let \( \tilde{B} \) be a (unique) \( O(K) \)-linear operator on \( M(H) \) such that for any \( m \in \mathcal{F}(H) \), \( \nabla(m) = \tilde{B}(m)d\ell/\ell \). Then for every \( N \in \mathbb{Z}_{\geq 0} \) we introduce the differential forms

\[
\tilde{\Omega}[N] = \phi^N \tilde{B}\phi^{-N} d\ell/\ell \in \text{End}M(H)_{O(K_p^{\text{ad}})} \otimes_{O(K)} \Omega^1_{O(K)}
\]

Now we can use the identification \( \mathcal{F} : H_{O(K)} \simeq M(H) \) to obtain the corresponding differential forms \( \Omega[N] \) on \( \text{End}(H)_{O(K_p^{\text{ad}})} \) and to verify that

\[
\Omega[N] \in L(H)_{O(K_p^{\text{ad}})} \otimes_{O(K)} \Omega^1_{O(K)} = L(H_0)_{O(K_p^{\text{ad}})} \otimes_{O(K_0)} \Omega^1_{O(K_0)}.
\]
Remark 1.2. Our differential forms will usually appear in the form\[ \Omega = F \cdot d\bar{r}_0/i_0, \] where \( F \in L(H)_{K_0^{ram}} \). Then we set by definition\[ (id_{L(H)} \otimes \sigma)\Omega = (id_{L(H)} \otimes \sigma)F \cdot d\bar{r}_0/i_0. \]

Our first main result can be stated as follows.

Theorem 1.1. Suppose \( H_0 \in \mathcal{M}^{\text{Lie}}_{K_0} \) is finite. Then there is \( N_0(H_0) \in \mathbb{N} \) such that for any (fixed) \( N \geq N_0(H_0) \) the following property holds:

if \( (id_{L(H)} \otimes \sigma^{-N})\Omega[N] = \sum_{r \in \mathbb{Q}} \bar{r}_0^{-1}l_r d\bar{r}_0/i_0 \),

then the ideal \( L(H_0)^{(e)} \) is the minimal ideal in \( L(H_0) \) such that for all \( r \geq v \), \( l_r \in L(H_0)_{W(k_0)} \).

Corollary 1.2. If \( v_0(H_0) = \max \{ r \mid l_r \neq 0 \} \) then the ramification subgroups \( \Gamma^{(e)}_{K_0} \) act trivially on \( H_0 \) iff \( v > v_0(H_0) \).

Remark 1.3. The construction of \( \Omega[N] \) almost does not depend on the choice of the tamely ramified finite field extension \( K \) of \( K_0 \). It depends essentially on the choice of the uniformising element \( t_0 \) in \( K_0 \) and a compatible system of \( \alpha(k) \in W(k) \), where \( [k : k_0] < \infty \), such that the trace of \( \alpha(k) \) in the field extension \( W(k)[1/p]/K_0 \) equals 1.

Remark 1.4. If \( H_0 \) is not \( p \)-torsion Theorem 1.1 can be applied to the factors \( H_0/p^M \) and our result describes the structure of the images of \( L(H_0)^{(e)} \) in all \( L(H_0)/p^M \).

1.6. The second main result: the mixed characteristic case. Let \( E_0 \) be a finite field extension of \( \mathbb{Q}_p \) with residue field \( k_0 \) and a uniformizing element \( \pi_0 \). Assume that \( E_0 \) contains a \( p \)-th primitive root of unity \( \zeta_1 \). Consider the category \( \mathcal{M}^{\text{Lie}}_{E_0,1} \) of finitely generated \( \mathbb{F}_p[\Gamma_{E_0}] \)-modules which satisfy a direct analog of the Lie condition from Sec. 1.4.

Take the infinite arithmetically profinite field extension \( \widetilde{E}_0 \) obtained from \( E_0 \) by joining all \( p \)-power roots of \( \pi_0 \). Then the theory of the field-of-norms functor \( X \) provides us with the complete discrete valuation field of characteristic \( p \), \( X(\widetilde{E}_0) = K_0 \), which has the same residue field and the uniformizing element \( t_0 \) obtained from the \( p \)-power roots of \( \pi_0 \). The functor \( X \) also provides us with the identification of Galois groups \( \Gamma_{K_0} = \Gamma_{\widetilde{E}_0} \subset \Gamma_{E_0} \).

If \( H_{E_0} \in \mathcal{M}^{\text{Lie}}_{E_0,1} \) then we obtain \( H_0 := H_{E_0}|_{\Gamma_{K_0}} \in \mathcal{M}^{\text{Lie}}_{K_0} \). As earlier, take a finite tamely ramified extension \( K \) of \( K_0 \) (it corresponds to a unique tamely ramified extension \( E \) of \( E_0 \) with uniformizer \( \pi \) such that \( \pi^{e_0} = \pi_0 \), where \( e_0 \) is the ramification index of \( E/E_0 \), and construct \( (\phi, O(K)) \)-module \( M(H) \). This module inherits the action of \( \text{Gal}(E(\sqrt[p]{\pi})/E) = \langle \tau_0 \rangle^{Z/p} \). (Here \( \tau_0 \) is such that \( \tau_0(\sqrt[p]{\pi}) = \zeta_1^{1/p} \). This situation was considered in all details in the papers \[7, 8\]. In particular, in those papers we gave a characterization of the “good” lifts \( \tau_0 \).
of \( \tau_0 \). By definition, \( \hat{\tau}_0 \in \Gamma_E \) is “good” if its restriction to \( H_{E_0} \) belongs to the image of the ramification subgroup \( \Gamma_E^{(e^*)} \). Here \( e^* := pe/(p - 1) \) and \( e = e(E/Q_p) \) is the ramification index of \( E/Q_p \). (This makes sense because \( \tau_0 \in \text{Gal}(E(\sqrt[p]{\pi}/E)^{(e^*)}) \). Note that the field-of-norms functor is compatible with ramification filtrations on \( \Gamma_{E_0} \) and \( \Gamma_{K_0} \). Therefore, the knowledge of "good lifts" \( \hat{\tau}_0 \) together with Theorem 1.1 gives a complete description of the image of the ramification filtration of \( \Gamma_{E_0} \) in \( \text{Aut} H_{E_0} \).

In \([8]\) we proved that the action of \( \hat{\tau}_0 \) appears from an action of a formal group scheme of order \( p \). As a result, the lift \( \hat{\tau}_0 \) is completely determined by the value \( d_{\hat{\tau}_0}(0) \in L(H) \) of its differential at 0, and we can use the characterization of differentials of “good ” lifts from \([8]\), Theorem 5.1.

Namely, let us first specify our \( p \)-th root of unity

\[
\zeta_1 = 1 + \sum_{j \geq 0} [\beta_j] \pi^{(e^*/p)+j} \mod p
\]

(here all \([\beta_j]\) are the Teichmuller representatives of elements from the residue field of \( E \)). Then we introduce the power series \( \omega(t) \in O(K) \) such that

\[
1 + \sum_{j \geq 0} \beta_j^p e^{e^*+pj} = \bar{\exp}(\omega(t)^p),
\]

(here \( \bar{\exp} \) is the truncated exponential). The series \( \omega(t)^p \) is a kind of approximation of the \( p \)-adic period of the formal multiplicative group, which appears usually in explicit formulas for the Hilbert symbol, e.g. \([6]\). In other words, we obtain another geometric condition characterizing "arithmetic" of the \( \Gamma_{E_0} \)-module \( H_{E_0} \).

**Theorem 1.3.** The lift \( \hat{\tau}_0 \) is “good” iff

\[
d_{\hat{\tau}_0}(0) \equiv \sum_{m \geq 0} \text{Res} \left( \omega(t)^{m+1} \Omega[m] \right) \mod L(H)^{(e^*)}_k.
\]

**Remark 1.5.** Notice that the power series \( \omega(t)^p \) has non-zero coefficients only for the powers \( t^{e^*+pj} \) and all these exponents \( \geq e^* \). Therefore, the differential forms \( \Omega[m] \) contribute to the right hand side only via the images of \( F^{0}_{e^*+pj,-m} t^{-(e^*+pj)} \) in \( L(H)_k \). But for \( m \gg 0 \), these images belong to the images of the ramification ideals \( L_k^{(e^*)} \) and, therefore, disappear modulo \( L(H)^{(e^*)}_k \), and the sum in the right hand side is, as a matter of fact, finite.

2. \( \phi \)-MODULE \( M(H) \)

2.1. **Specification of** \( \log \pi_H : \Gamma_K \rightarrow G(L(H)) \). As earlier, \( H_0 \in M\Gamma_{K_0}^{\text{Lie}}, K \) is a finite tamely ramified extension of \( K_0 \) in \( K_0^{\text{sep}} \) such that \( \pi_{H_0}(\Gamma_K) = I(H_0), H = H_0|_{\Gamma_K} \). Set \( L(H) = L(H_0), \pi_H = \pi_{H_0}|_{\Gamma_K} \).
Consider the continuous group epimorphism $l_H := \log(\pi_H) : \Gamma_K \rightarrow G(L(H))$. Since the $p$-group $G(L(H))$ has nilpotence class $< p$ this epimorphism can be described in terms of the covariant version of the nilpotent Artin-Schreier theory from [2]. Namely, there are $e \in L(H)_O(K)$ and $f \in L(H)_O^{sep}$ such that $(\id_{L(H)} \otimes \sigma)(f) = e \circ f$ and for any $\tau \in \Gamma_K$, $l_H(\tau) = (-f) \circ (\id_{L(H)} \otimes \tau)f$.

It could be easily verified that $l_H$ is a group homomorphism. Indeed, $l_H(\tau_1 \tau_2) = (-f) \circ (\id_{L(H)} \otimes \tau_1 \tau_2)f = (-f) \circ (\id_{L(H)} \otimes \tau_1)f \circ (-f) \circ (\id_{L(H)} \otimes \tau_2)f = l_H(\tau_1) \circ l_H(\tau_2)$, because $(\id_{L(H)} \otimes \tau_1)l_H(\tau_2) = l_H(\tau_2)$.

**Notation.** We will use below the following notation: $\sigma_H = \id_H \otimes \sigma$ and $\sigma_{L(H)} = \id_{L(H)} \otimes \sigma$. For example, if $u = \sum_{\alpha} h_{\alpha} \otimes o_{\alpha}$, where all $h_{\alpha} \in H$ and $o_{\alpha} \in O(K)$ then $\sigma_H(u) = \sum_{\alpha} h_{\alpha} \otimes \sigma(o_{\alpha})$. Or, if $X$ is a linear operator on $L(H)_O(K)$ then $\sigma_{L(H)}X$ is also a linear operator such that $\sigma_{L(H)}X \sum_{\alpha} h_{\alpha} \otimes o_{\alpha} = \sum_{\alpha} \sigma_H(X(h_{\alpha}))(1 \otimes o_{\alpha})$.

In addition, $\mathcal{X} := X \cdot \sigma_H$ is a unique sigma linear operator such that $\mathcal{X}|_H = X|_H$, and we have the following identity $\sigma_H \cdot X = \sigma_{L(H)}X \cdot \sigma_H$. If there is no risk of confusion we will use just the notation $\sigma$.

**Remark 2.1.** Originally we developed in [2] the contravariant version of the nilpotent Artin-Schreier theory, cf. the discussion in [5]. The contravariant version uses similar relations $\sigma_{L(H)}(f) = f \circ e$ and the map $l_H$ was defined via $\tau \mapsto (\id_{L(H)} \otimes \tau)f \circ (-f)$. In this case $l_H$ determines the group homomorphism from $\Gamma_K$ to the opposite group $G^0(L(H))$ (this group is isomorphic to $G(L(H))$ via the map $g \mapsto g^{-1}$). The results from the papers [11, 2, 3] were obtained in terms of the contravariant version, but the results from [7, 8, 9, 10] used the covariant version. We can easily switch from one theory to another via the automorphism $-\id_{L(H)}$.

We consider $O^{sep}$ as a left $\Gamma_K$-module via the action $o \mapsto \tau(o)$ with $o \in O^{sep}$ and $\tau \in \Gamma_K$. This corresponds to our earlier agreement about the left action of the elements of $\Gamma_K$ as endomorphisms of $O^{sep}$. As a result we obtain the left $\Gamma_K$-module structure on $H_{O^{sep}}$ by the use of the (left) action of $\Gamma_K$ on $H$ via $h \mapsto l_H(\tau)(h)$.

Because $L(H)_{O^{sep}} \subset \text{End}_{O^{sep}}(H_{O^{sep}})$ we can introduce for any $h \in H$, $\mathcal{F}(h) := \exp(-f)(h) \in H_{O^{sep}}$.

**Proposition 2.1.** For any $h \in H$, $\mathcal{F}(h) \in (H_{O^{sep}})^{\Gamma_K}$.

**Proof.** Suppose $f = \sum_{\alpha} l_{\alpha} \otimes o_{\alpha}$, where all $l_{\alpha} \in L(H)$ and $o_{\alpha} \in O^{sep}$. [Omit proof due to length limitations]
If $\tau \in \Gamma_K$ then
\[
\tau(\mathcal{F}(h)) = (\tau \otimes \text{id}_{O_{\text{sep}}}) \left( \exp \left( -\sum_{\alpha} l_{\alpha} \otimes \tau(o_{\alpha}) \right)(h) \right) =
\]
\[
(\tau \otimes \text{id}_{O_{\text{sep}}}) \left( \exp \left( (\text{id}_{L(H)} \otimes \tau)(-f) \right)(h) \right) =
\]
\[
(\tau \otimes \text{id}_{O_{\text{sep}}}) \left( \exp \left( (-l_H(\tau)) \circ (-f) \right)(h) \right) =
\]
\[
(\tau \otimes \text{id}_{O_{\text{sep}}}) \left( \exp(-l_H(\tau)) \cdot \exp(-f) \right)h =
\]
\[
(\tau \otimes \text{id}_{O_{\text{sep}}})(\pi_H(\tau^{-1}) \otimes \text{id}_{O_{\text{sep}}} ) \mathcal{F}(h) =
\]
\[
= (\tau \cdot \pi_H(\tau^{-1}) \otimes \text{id}_{O_{\text{sep}}}) \mathcal{F}(h) = \mathcal{F}(h).
\]
\[\Box\]

The elements $e$ and $f$ are not determined uniquely by $l_H$. A pair $e' \in L(H)_{O(K)}$ and $f' \in L(H)_{O_{\text{sep}}}$ give the same group epimorphism $l_H$ iff there is $x \in L(H)_{O(K)}$ such that $e' = \sigma(x) \circ e \circ (-x)$ and $f' = x \circ f$.

2.2. Special choice of $e \in L(H)_{O(K)}$. We can always assume (by replacing, if necessary, $K$ by its finite unramified extension) that a uniformiser $t$ in $K$ is such that $t^{e_0} = t_0$, where $e_0$ is the ramification index for $K/K_0$. Then $O(K) = \varprojlim_M W_M(k)((\bar{t}))$, where $\bar{t}^{e_0} = \bar{t}_0$ and $\bar{t}$ is the Teichmuller representative of $t$. We denote by $k$ the residue field of $K$ and fix a choice of $\alpha_0 = \alpha(k) \in W(k)$ such that its trace in the field extension $W(k)[1/p]/\mathbb{Q}_p$ equals 1.

Let $\mathbb{Z}^+(p) := \{a \in \mathbb{N} \mid \gcd(a, p) = 1\}$ and $\mathbb{Z}^0(p) = \mathbb{Z}^+(p) \cup \{0\}$.

**Definition.** An element $e \in L(H)_{O(K)}$ is **special** if $e = \sum_{a \in \mathbb{Z}^0(p)} \bar{t}^{-a} l_{a0}$, where $l_{00} \in \alpha_0 L(H)$ and for all $a \in \mathbb{Z}^+(p)$, $l_{a0} \in L(H)_{W(k)}$.

**Lemma 2.2.** Suppose $e \in L(H)_{O(K)}$. Then there is $x \in L(H)_{O(K)}$ such that $\sigma(x) \circ e \circ (-x)$ is **special**.

**Proof.** Let $L$, Use induction on $s$ to prove lemma modulo the ideals of $s$-th commutators $C_s(L(H)_{O(K)})$.

If $s = 1$ there is nothing to prove.

Suppose lemma is proved modulo $C_s(L(H)_{O(K)})$.

Then there is $x \in L(H)_{O(K)}$ such that
\[
\sigma(x) \circ e \circ (-x) = \sum_{a \in \mathbb{Z}^0(p)} \bar{t}^{-a} l_{a0} + l,
\]
where $l \in C_s(L(H)_{O(K)})$. Using that
\[
(2.1) \quad O(K) = (\sigma - \text{id}_{O(K)}) O(K) \oplus (\mathbb{Z}_p \alpha_0) \oplus \left( \sum_{a \in \mathbb{Z}^+(p)} W(k) \bar{t}^{-a} \right)
\]
we obtain the existence of \( x_s \in C_s(L(H)_{O(K)}) \) such that
\[
l = \sigma(x_s) - x_s + \sum_{a \in \mathbb{Z}^p(p)} t^{-a} l_a,
\]
where \( l_0 \in \alpha_0 L \) and all remaining \( l_a \in L_{W(k)} \). Then we can take \( x' = x - x_s \) to obtain our statement modulo \( C_{s+1}(L(H)_{O(K)}) \). □

**Lemma 2.3.** Suppose \( e \in L_{O(K)} \) is special and \( x \in L_{O(K)} \). Then the element \( \sigma(x) \circ e \circ (-x) \) is special iff \( x \in L \) (or, equivalently, if \( \sigma x = x \)).

**Proof.** Use relation cf. [12],
\[
\text{Ad}(\exp(X)) \exp(Y) = \exp \left( \sum_{n \geq 0} \frac{1}{n!} \text{ad}^n(X)(Y) \right) \mod(\text{deg} p),
\]
where \( \text{Ad}(U)(V) = UVU^{-1} \) and \( \text{ad}(U)(V) = [U, V] \). Indeed, if \( X = x \in L(H) \) and \( Y = e \) then \( \sum_{n \geq 0} \text{ad}^n(x)(e)/n! \) is also special.

When proving the inverse statement we can use induction modulo the ideals \( C_s(L(H))_{O(K)}, s \geq 1 \), as follows.

Assume the lemma is proved modulo \( C_s(L(H))_{O(K)} \). Then using the IF part we can assume that \( x \in C_s(L(H))_{O(K)} \). Therefore, \( e + \sigma(x) - x \) is special modulo \( C_{s+1}(L(H))_{O(K)} \), i.e.
\[
\sigma(x) - x \in \alpha_0 C_s(L) + \sum_{a \in \mathbb{Z}^+} t^{-a} C_s(L)_{W(k)}
\]
modulo \( C_{s+1}(L(H))_{O(K)} \). By (2.1) this implies the congruence
\[
\sigma(x) \equiv x \mod C_{s+1}(L(H))_{O(K)}
\]
i.e. \( x \in C_s(L(H)) \mod C_{s+1}(L(H))_{O(K)} \).

The lemma is proved. □

2.3. **Construction of the \( \phi \)-module \( M(H) \).** Note that \( \pi_H = \exp(l_H) \), and therefore for all \( \tau \in \Gamma_K \), it holds
\[
\pi_H(\tau) = \exp(-f) \cdot \exp(\text{id}_{L(H)} \otimes \tau) f,
\]
where \( f \in L(H)_{O_{sep}} \subset \text{End}_{O_{sep}}(H_{O_{sep}}), \sigma_{L(H)}(f) = e \circ f \) and \( \exp f \in \text{Aut}_{O_{sep}}(H_{O_{sep}}) \).

Let \( \text{MF}^\phi_K \) be the category of etale \( \phi \)-modules over \( O(K) \). Recall that its objects are \( O(K) \)-modules of finite type \( M \) together with a \( \sigma \)-linear morphism \( \phi : M \rightarrow M \) such that its \( O(K) \)-linear extension \( \phi_{O(K)} : M \otimes_{O(K),\sigma} O(K) \rightarrow M \) is isomorphism. The correspondence \( H \mapsto M(H) := (H \otimes_{\mathbb{Z}_p} O_{sep})^K \), where \( \phi : M(H) \rightarrow M(H) \) comes from the action of \( \sigma \) on \( O_{sep} \), determines the equivalence of the categories \( \text{MF}^\phi_K \) and \( \text{MF}^\text{et}_K \).
Consider the $\mathbb{Z}_p$-linear embedding $\mathcal{F} : H \to H_{O_{sep}}$ from Sec.2.1. Let $M(H) = \mathcal{F}(H)_{O(K)}$. By extension of scalars we obtain natural isomorphisms (use that $O(K)$ and $O_{sep}$ are flat $\mathbb{Z}_p$-modules):

$$\mathcal{F} \otimes \text{id}_{O_{sep}} : H_{O_{sep}} \simeq M(H)_{O_{sep}}$$

$$\mathcal{F} \otimes \text{id}_{O(K)} : H_{O(K)} \simeq M(H),$$

which will be denoted for simplicity just by $\mathcal{F}$.

Note that by Prop.2.1, $M(H) = (H_{O_{sep}})^{1_K}$.

The $O(K)$-module $M(H)$ is provided with the $\sigma$-linear morphism $\phi : M(H) \to M(H)$ uniquely determined for all $h \in H$ via

$$\phi(\mathcal{F}(h)) = \exp(-\sigma)(h) = (\exp(-\sigma)\exp(-e))(h) = \mathcal{F}(\exp(-e)(h)).$$

Consider the $O(K)$-linear operator $A := \exp(-e) \in \exp(L(H)_{O(K)}) \subset \text{Aut}_{O(K)} H_{O(K)}$. Then $\mathcal{A} := A \cdot \sigma_H$ will be a unique $\sigma$-linear operator on $L(H)_{O(K)}$ such that $\mathcal{A}|_H = A|_H$. Clearly, for any $u \in H_{O(K)}$,

$$\phi(\mathcal{F}(u)) = \mathcal{F}(\mathcal{A}(u)),$$

and $M(H)$ is etale $\phi$-module associated with the $\mathbb{Z}_p[\Gamma_K]$-module $H$.

For example, suppose $pH = 0$ and $\{h_i \mid 1 \leq i \leq N\}$ is $\mathbb{F}_p$-basis of $H$. Then $\{\mathcal{F}(h_i) \mid 1 \leq i \leq N\}$ is a $K$-basis for $M(H)$. If $A(h_i) = \exp(-e)(h_i) = \sum_j a_{ij}h_j$ with all $a_{ij} \in K$ then $\phi(\mathcal{F}(h_i)) = \sum_j a_{ij}\mathcal{F}(h_j)$, and $((a_{ij}))$ appears as the corresponding “Frobenius matrix”.

It can be easily seen also that if $\{h_i \mid 1 \leq i \leq N\}$ is a minimal system of $\mathbb{Z}_p$-generators in $H$ then $\{\mathcal{F}(h_i) \mid 1 \leq i \leq N\}$ is a minimal system of $O(K)$-generators in $M(H)$.

2.4. The connection $\nabla$ on $M(H)$. The $(\phi, O(K))$-module $M(H)$ can be provided with a connection $\nabla : M(H) \to M(H) \otimes_{O(K)} \Omega^1_{O(K)}$, [11]. This is an additive map uniquely determined by the properties:

a) for any $o \in O(K)$ and $m \in M(H)$, $\nabla(mo) = \nabla(m)o + m \otimes d(o)$;

b) $\nabla \cdot \phi = (\phi \otimes \phi) \cdot \nabla$.

By a), $\nabla$ is uniquely determined by its restriction to $\mathcal{F}(H)$ (use that $M(H) = \mathcal{F}(H)_{O(K)}$). Let $\tilde{B}$ be a unique $O(K)$-linear operator on $M(H)$ such that for any $m \in \mathcal{F}(H)$, $\nabla(m) = \tilde{B}(m)\,dt/t$. Consider the $O(K)$-linear operator $B \in \text{End}_{O(K)}$ such that for all $u \in H_{O(K)}$, $\tilde{B}(\mathcal{F}(u)) = \mathcal{F}(B(u))$. Obviously, $\tilde{B}$ and $B$ can be recovered one from another.

Note that for any $\mathbb{Z}_p$-module $\mathcal{C}$, the elements $c \in C_{O(K)}$ appear uniquely in the form $c = \sum n_c \otimes n^a$ with all $c_n \in C_{W(K)}$. Therefore, the map $\text{id}_C \otimes \partial_t : C_{O(K)} \to C_{O(K)}$ such that $c \mapsto \sum n_c \otimes n^n$ is well-defined. If $\mathcal{C} \subset C_1$ is an embedding of $\mathbb{Z}_p$-modules then we have $(\text{id}_C \otimes \partial_t)|_{C_{O(K)}} = \text{id}_C \otimes \partial_t$. 

RAMIFICATION FILTRATION AND DIFFERENTIAL FORMS
With the above notation:

1) for all \( m \in M(H) \), \( \nabla(m) = (\tilde{B} + \text{id}_{F(H)} \otimes \partial_{\tilde{t}})(m)d\tilde{t}/\tilde{t} \).
2) for all \( u \in H_{O(K)} \) and \( X \in \text{End} H_{O(K)} \),
\[ (\text{id}_H \otimes \partial_{\tilde{t}})(X(u)) = (\text{id}_{\text{End} H} \otimes \partial_{\tilde{t}})(X)(u) + X((\text{id}_H \otimes \partial_{\tilde{t}})u) \).

**Proposition 2.4.** Let \( C = -(\text{id}_{\text{End} H} \otimes \partial_{\tilde{t}})A \cdot A^{-1} \) and let for any \( n \geq 1 \), \( D^{(n)} = A \cdot \sigma_{\text{End} H}(A) \cdot \ldots \cdot \sigma_{\text{End} H}^{n{-1}}(A) \). Then
\[ B = \sum_{n \geq 0} p^n \text{Ad}(D^{(n)})\sigma_{\text{End} H}^n(C) . \]

**Remark 2.2.** In Sect.4 we will prove that \( C, B \in L(H)_{O(K)} \subset \text{End} H_{O(K)} \). In particular, \( \sigma_{\text{End} H} C = \sigma_{L(H)} C \) and the correspondence \( u \mapsto (B + \text{id}_{L(H)} \otimes \partial_{\tilde{t}})(u)d\tilde{t}/\tilde{t} \) gives a connection on \( L(H)_{O(K)} \).

**Proof.** Indeed, for any \( u \in H_{O(K)} \), it holds
\[ (\nabla \cdot \phi)(\mathcal{F}(u)) = (\nabla \cdot \mathcal{F} \cdot \mathcal{A})(u) = ((\tilde{B} + \text{id}_{F(H)} \otimes \partial_{\tilde{t}}) \cdot \mathcal{F} \cdot \mathcal{A})(u)d\tilde{t}/\tilde{t} \]
\[ = (\mathcal{F} \cdot (B + \text{id}_H \otimes \partial_{\tilde{t}}) \cdot \mathcal{A})(u)d\tilde{t}/\tilde{t} . \]

On the other hand, \( (\phi \otimes \phi)(\nabla(\mathcal{F}(u))) = \phi(\mathcal{F}(B + \text{id}_H \otimes \partial_{\tilde{t}})u)\phi(d\tilde{t}/\tilde{t}) = p(\mathcal{F} \cdot \mathcal{A} \cdot (B + \text{id}_H \otimes \partial_{\tilde{t}}))(u)d\tilde{t}/\tilde{t} . \)

Equivalently, we have the following identity on \( H_{O(K)} \),
\[ (B \cdot A + (\text{id}_H \otimes \partial_{\tilde{t}}) \cdot A) \cdot \sigma_H = pA \cdot \sigma_H \cdot (B + \text{id}_H \otimes \partial_{\tilde{t}}) . \]

Rewrite this equality as follows
\[ (B \cdot A - pA \cdot \sigma_{\text{End} H}(B)) \cdot \sigma_H = (- (\text{id}_H \otimes \partial_{\tilde{t}}) \cdot A \cdot \sigma_H \cdot (\text{id}_H \otimes p\partial_{\tilde{t}}) \cdot \sigma_H^{-1}) \cdot \sigma_H . \]

Notice that that on \( H_{O(K)} \) we have \( \sigma_H \cdot (\text{id}_H \otimes p\partial_{\tilde{t}}) \cdot \sigma_H^{-1} = \text{id}_H \otimes \partial_{\tilde{t}} \).

As a result, the right hand side equals \(- (\text{id}_{\text{End} H} \otimes \partial_{\tilde{t}})(A) \cdot \sigma_H \).

From \( \sigma_H |_{H} = \text{id}_H \) it follows that by restriction on \( H \) we have
\[ B \cdot A - pA \cdot \sigma_{\text{End} H}(B) = -(\text{id}_{\text{End} H} \otimes \partial_{\tilde{t}})A . \]

By \( O(K) \)-linearity this identity holds on the whole \( H_{O(K)} \).

As a result, our identity appears in the form
\[ (\text{id}_H - p\text{Ad}(A) \cdot \sigma_{\text{End} H})B = -(\text{id}_H \otimes \partial_{\tilde{t}})(A) \cdot A^{-1} . \]

It remains to recover \( B \) using that
\[ (\text{id}_H - p\text{Ad}(A) \cdot \sigma_{\text{End} H})^{-1} = \sum_{n \geq 0} p^n \text{Ad}(D^{(n)}) \cdot \sigma_{\text{End} H}^n \cdot . \]

\[ \square \]

3. Ramification filtration modulo \( p \)-th commutators

Recall that \( K = k((t)) \subset K^p_{\text{res}}, \pi_H(\Gamma_K) = \exp(L(H)) = I(H) \subset \text{Aut}_{Z_p}(H) \). Note also that \( O(K) = W(k)((t)) \), where \( t^{e_0} = \tilde{t}_0 \) and \( e_0 \) is the ramification index of \( K/K_0 \).
3.1. Lie algebra $\mathcal{L}$ and identification $\eta_{<p}$. Let $K_{<p}$ be the maximal $p$-extension of $K$ in $K_{0}^{sep}$ with the Galois group of nilpotent class $< p$. Then $G_{<p} := \text{Gal}(K_{<p}/K) = \lim_{\rightarrow} \Gamma_{K}/\Gamma_{K}^{p^{n}} C_{p}(\Gamma_{K})$.

Let $\tilde{W}_{(k)}$ be a profinite free Lie $W(k)$-algebra with the set of topological generators $\{D_{0} \cup \{D_{an} \mid a \in \mathbb{Z}^{+}(p), n \in \mathbb{Z}/N_{0}\}$. Let $\mathcal{L}_{W(k)} = \tilde{W}_{(k)}/C_{p}(\tilde{W}_{(k)})$, where $C_{p}(\tilde{W}_{(k)})$ is the ideal of $p$-th commutators. Define the $\sigma$-linear action on $\mathcal{L}_{W(k)}$ via $D_{an} \mapsto D_{a,n+1}$ and $D_{0} \mapsto D_{0}$, denote this action by the same symbol $\sigma$, and set $\mathcal{L} = \mathcal{L}_{W(k)}|_{\sigma = \text{id}}$.

Fix $\alpha_{0} = \alpha_{0}(k) \in W(k)$ such that the trace of $\alpha_{0}$ in the field extension $W(k)[1/p]/\mathbb{Q}_{p}$ equals 1. For any $n \in \mathbb{Z}/N_{0}$, set $D_{0n} = (\sigma^{n} \alpha_{0})D_{0}$.

We are going to apply the profinite version of the covariant nilpotent Artin-Schreier theory to the Lie algebra $\mathcal{L}$ and the special element $e_{<p} = \sum_{a \in \mathbb{Z}^{0}(p)} i^{-a}D_{a0} \in \mathcal{L} \hat{\otimes} O(K)$. In other words, if we fix

$$f_{<p} \in \{ f \in \mathcal{L} \hat{\otimes} O_{sep} \mid \sigma_{\mathcal{L}}(f) = e \circ f \} \neq \emptyset,$$

then the map $\eta_{<p} := \pi_{<p}(e_{<p})$ given by the correspondence $\tau \mapsto (-f_{<p}) \circ (\text{id}_{\mathcal{L}} \otimes \tau)f_{<p}$ induces the group isomorphism $\tilde{\eta}_{<p} : \Gamma_{<p} \simeq G(\mathcal{L})$.

The following property is an easy consequence of the above construction.

**Proposition 3.1.** Suppose $e \in L(H)_{O(K)}$ is special and given with notation from the definition from Sect. Then the map $\log \pi_{H} : G_{<p} \rightarrow G(L(H))$ is given via the correspondences $D_{a0} \mapsto l_{a0}$ (and $D_{an} \mapsto \sigma_{L(H)}^{n}(l_{a0})$) for all $a \in \mathbb{Z}^{0}(p)$.

3.2. The ramification ideals $\mathcal{L}^{(v)}$. For $v \geq 0$, denote by $G_{<p}^{(v)}$ the image of $\Gamma_{K}^{(v)}$ in $G_{<p}$. Then $\tilde{\eta}_{<p}(G_{<p}^{(v)}) = G(\mathcal{L}^{(v)})$, where $\mathcal{L}^{(v)}$ is an ideal in $\mathcal{L}$. The images $\mathcal{L}^{(v)}(M)$ of the ideals $\mathcal{L}^{(v)}$ in the quotients $\mathcal{L}/p^{M}\mathcal{L}$ for all $M \in \mathbb{N}$ were explicitly described in [3]. By going to the projective limit on $M$ this description can be presented as follows.

**Definition.** Let $\bar{n} = (n_{1}, \ldots, n_{s})$ with $s \geq 1$. Suppose there is a partition $0 = i_{0} < i_{1} < \cdots < i_{r} = s$ such that for $i_{j} < u \leq i_{j+1}$, it holds $n_{u} = m_{j+1}$ and $m_{1} > m_{2} > \cdots > m_{r}$. Then set

$$\eta(\bar{n}) = \frac{1}{(i_{1} - i_{0})! \cdots (i_{r} - i_{r-1})!}.$$

If such a partition does not exist we set $\eta(\bar{n}) = 0$.

For $s \in \mathbb{N}$, $\bar{a} = (a_{1}, \ldots, a_{s}) \in \mathbb{Z}^{0}(p)^{s}$ and $\bar{n} = (n_{1}, \ldots, n_{s}) \in \mathbb{Z}^{s}$, set

$$[D_{\bar{a},\bar{n}}] = [\ldots [D_{a_{1}n_{1}}, D_{a_{2}n_{2}}], \ldots, D_{a_{s}n_{s}}].$$
For $\alpha \geq 0$ and $N \in \mathbb{Z}_{\geq 0}$, introduce $F_{\alpha,-N}^0 \in L_W(k)$ such that
\[
F_{\alpha,-N}^0 = \sum_{1 \leq s < p} a_1 \eta(\bar{n}) p^{n_1} [D_{\bar{a},\bar{n}}].
\]
Here $n_1 \geq 0$, all $n_i \geq -N$ and $\gamma(\bar{a}, \bar{n}) = a_1 p^{n_1} + a_2 p^{n_2} + \cdots + a_s p^{n_s}$.

Note that the non-zero terms in the above expression for $F_{\alpha,-N}^0$ can appear only if $n_1 \geq n_2 \geq \ldots \geq n_s$ and $\alpha$ has at least one presentation in the form $\gamma(\bar{a}, \bar{n})$.

Denote by $T^{(v)}[N]$ the minimal closed ideal in $L$ such that its extension of scalars $T^{(v)}[N]_W(k)$ contain all $F_{\alpha,-N}^0$ with $\alpha \geq v$.

Our result from [3] about explicit generators of the ideal $L^{(v)}$ can be stated in the following form.

**Theorem 3.2.** For any $v > 0$ and $M \in \mathbb{N}$, there is $\bar{N}(v, M) \in \mathbb{N}$ such that if $N \geq \bar{N}(v, M)$, then the images of the ideals $L^{(v)}$ and $T^{(v)}[N]$ in $L/p^M$ coincide.

### 3.3. Some relations.

Let $A(L)$ be the enveloping algebra of $L$ and $\tilde{A}(L) = A(L)/J(L)^p$, where $J(L)$ is the augmentation ideal in $A(L)$. Note that there is a natural embedding of $\mathbb{Z}_p$-modules $L \subset \tilde{A}(L)$.

Let $A_{<p} = \exp(-e_{<p}) \in \tilde{A}(L)_{O(K)}$ and $C_{<p} = -(\partial_{\tilde{A}(L)} \otimes \partial_l) A_{<p} \cdot A_{<p}^{-1}$.

For $s \geq 1$, set $\bar{0}_s = (0, \ldots, 0)$.

**Proposition 3.3.** Let $D_{<p}^{(m)} := A_{<p} \cdot \sigma_{\tilde{A}(L)}(A_{<p}) \cdot \ldots \cdot \sigma_{\tilde{A}(L)}^{m-1}(A_{<p})$, where $m \geq 1$. Then we have the following relations:

\[(3.1)\quad C_{<p} = \sum_{s \geq 1, \bar{a}} a_1 \eta(\bar{0}_s) [D_{\bar{a},\bar{0}_s}]^{1-\gamma(\bar{a}, \bar{0}_s)};\]

\[(3.2)\quad B_{<p} := \sum_{n \geq 0} p^n \Ad(D^{(n)}_{<p})(\sigma_{\tilde{A}(L)}^n(C_{<p})) = \sum_{\alpha \geq 0} F_{\alpha,0}^0 t^{-\alpha};\]

\[(3.3)\quad \Ad \sigma_{\tilde{A}(L)}^{-m}(D_{<p}^{(m)})(B_{<p}) = \sum_{\alpha \geq 0} F_{\alpha,-m}^0 t^{-\alpha}.\]

**Proof.** For (3.1) use, cf. [12], theorem 4.22, to obtain
\[
d \exp(-e_{<p}) \cdot \exp(e_{<p}) = \sum_{k \geq 1} \frac{1}{k!} (-\text{ade}_{<p})^k (-d e_{<p})
\]
and note that $(-\text{ade}_{<p})^{k-1}(d e_{<p}) = (-1)^{k-1}[e_{<p}, \ldots, [e_{<p}, d e_{<p}], \ldots] = [\ldots [d e_{<p}, e_{<p}], \ldots, e_{<p}]$.
For (3.2) we need the following relation cf. [12], Sect. 4.4
\[ \exp(X) \cdot Y \cdot \exp(-X) = \sum_{n \geq 0} \frac{1}{n!} \text{ad}^n(X)(Y). \]

After applying this relation to the summand with \( m = 1 \) we obtain
\[ p \text{Ad} D_{<p}(\sigma_L C_{<p}) = \exp(-e_{<p}) \cdot \sigma_L(C_{<p}) \cdot \exp(e_{<p}) = \]
\[ = \sum_{s \geq 0} \eta(\tilde{0})_{s} (-1)^{s} \text{ad}^{s}(e_{<p})(C_{<p}) = \]
\[ \sum_{n \geq 0} \sum_{n_{1} = 1}^{n} \eta(\tilde{a})_{n_{1}} [D_{\tilde{a}, \tilde{a}}]^{t - \gamma(\tilde{a}, \tilde{a})}. \]

Repeating this procedure we obtain relation (3.2).

Similar calculations prove the remaining item (3.3).

\[ \square \]

4. PROOF OF THEOREM 1.21

Recall briefly what we’ve already achieved.

The field \( K = k((t)) \) is tamely ramified extension of \( K_0 = k_0((t_0)) \), where \( t_0 = t, H := H_0|_{\Gamma_K} \) and the corresponding group epimorphism \( \pi_H : \Gamma_K \longrightarrow I(H) \subset \text{Aut}_{\mathbb{Z}_p}H \) is such that \( I(H) = \exp(L(H)), \) where \( L(H) \subset \text{End}_{\mathbb{Z}_p}H \) and \( L(H)^p = 0. \)

Applying formalism of nilpotent Artin-Schreier theory we obtained a special \( e = \sum_{a \in \mathbb{Z}_p} e_t^{-a} \frac{1}{a!} \in L(H)_{O(K)} \) and \( f \in L(H)_{O_{K, \text{sep}}} \) such that
\[ \exp(\sigma_{L(H)}f) = \exp(e) \cdot \exp(f) \]
and for any \( \tau \in \Gamma_K, \) \( \tau H(\tau) = \exp(-f) \cdot \exp(\text{id}_{L(H)} \otimes \tau) f. \)

We used \( O(K) \)-linear operator \( F = \exp(-f) \) to introduce \( O(K) \)-module \( M(H) := F(H_{O(K)}). \) Let \( A = \exp(-e) \) and let \( A \) be a unique \( \sigma \)-linear operator on \( H_{O(K)} \) such that for any \( h \in H, A(h) = \sigma(h). \) Then the \( \sigma \)-linear \( \phi : M(H) \longrightarrow M(H) \) is such that for any \( u \in L(H)_{O(K)}, \) \( \phi(F(u)) = F(A(u)). \) As a result, we obtain the structure of etale \( (\phi, O(K)) \)-module on \( M(H) \) related to the \( \Gamma_K \)-module \( H. \)

Let \( \nabla \) be the connection on \( M(H) \) from Sect.2.1 and let \( B \) be the \( O(K) \)-linear operator on \( M(H) \) uniquely determined by the condition: for any \( m \in F(H), \) \( \nabla(m) = B(m) \frac{d\xi}{\xi}. \) Then for any \( u \in M(H), \)
\[ \nabla(u) = (\tilde{B} + \text{id}_{F(H)} \otimes \partial_t)(u) \frac{d\xi}{\xi} \]
and we introduce the differential forms
\[ \hat{\Omega}[N] = \phi^{N} B \phi^{-N} d\xi/\xi \in \text{End}_{M(H)_{O(K^{\text{rad}})}} \otimes \Omega^{1}_{O(K)}. \]

Finally, define the \( O(K) \)-linear operator \( B \) on \( H \) by setting for any \( u \in H_{O(K)}, \) \( F(B(u)) = B(F(u)), \) and transfer \( \hat{\Omega}[N] \) to \( \text{End}_{H_{O(K^{\text{rad}})}} \) in the following form
\[ \Omega[N] = \text{Ad}(A^{N})(B) \frac{d\xi}{\xi} \in \text{End}_{(H)_{O(K^{\text{rad}})}} \otimes O_{O(K)} \Omega^{1}_{O(K)}. \]
Remark 4.1. Obviously we have the following identification
\[ \text{End}(H)_{O(K')} \otimes_{O(K)} \Omega^1_{O(K)} = \text{End}(H_0)_{O(K_0')} \otimes_{O(K_0)} \Omega^1_{O(K_0)}. \]

Recall that \( A = A \cdot \sigma_H \), where \( A = \exp(-e) \).

Lemma 4.1. If \( \mathcal{D}^{(N)} = \sigma^{-N}_{\text{End}H}(A) \cdot \ldots \cdot \sigma^{-1}_{\text{End}H}(A) \) then
\[ (\sigma^{-N}_{\text{End}H} \cdot \text{Ad}(A^N))(B) = \text{Ad}\mathcal{D}^{(N)}(B), \]

Proof. Use induction on \( N \geq 0 \). If \( N = 0 \) there is nothing to prove. Suppose lemma is proved for \( N \geq 0 \). Then
\[
(\sigma^{-(N+1)}_{\text{End}H} \cdot \text{Ad}(A^{N+1}))(B) = \sigma^{-(N+1)}_{\text{End}H}(A \cdot \text{Ad}(A^N))(B) \cdot A^{-1} = \\
\sigma^{-(N+1)}_{\text{End}H}(A \cdot (\sigma^N_{\text{End}H} \cdot \text{Ad}(\mathcal{D}^{(N)}))(B)) \cdot A^{-1} = \\
\sigma^{-(N+1)}_{\text{End}H}(A \cdot \sigma_H \cdot (\sigma^N_{\text{End}H} \text{Ad}(\mathcal{D}^{(N)}))(B)) \cdot A^{-1} = \\
\sigma^{-(N+1)}_{\text{End}H}(A \cdot (\sigma^{N+1}_{\text{End}H} \text{Ad}(\mathcal{D}^{(N)}))(B)) \cdot A^{-1} = \\
\sigma^{-(N+1)}_{\text{End}H}(A) \cdot \text{Ad}(\mathcal{D}^{(N)})(B) \cdot \sigma^{-(N+1)}_{\text{End}H}(A^{-1}) = \text{Ad}(\mathcal{D}^{(N+1)})(B)
\]
The lemma is proved.

Under the projection log \( \tilde{\pi}_H : \mathcal{G}_{<p} \longrightarrow G(L(H)) \) we have:
\[
D_{an} \mapsto l_{an} = \sigma^e_{L(H)} l_{e0}, \quad e_{<p} \mapsto e, \quad f_{<p} \mapsto f, \quad A_{<p} \mapsto A, \quad C_{<p} \mapsto C,
\]
\[
B_{<p} \mapsto B \text{ and } \sigma^{-m}_{\mathcal{A}(L)} D_{<p} \mapsto \mathcal{D}^{(m)}.
\]

Remark 4.2. Because log \( \tilde{\pi}_H(\mathcal{L}_{<p}) = L(H) \) we obtain the statement from Remark 2.2.

As a result, our differential form appears as the image of \( \sum \mathcal{F}_0 \bar{f}^{-\alpha} \bar{d} \bar{f}/\bar{t} \).

It remains to notice that when getting back to the field \( K_0 \), we have \( d\bar{t}/\bar{t} = e_0^{-1} \bar{d}\bar{t}/\bar{t}_0, \ \bar{t}^{-\alpha} = \bar{t}^{-\alpha/e_0}, \ \Gamma_K^{(\alpha)} = \Gamma_K^{(\alpha/e_0)} \) and \( \pi_H|_{\mathcal{I}} = \pi_{H_0}|_{\mathcal{I}}. \)

Theorem 1.1 is proved.

Remark 4.3. a) The conjugacy class of the differential form \( \Omega[N] \) does not depend on a choice of a special form for \( e \).

b) It would be very interesting to verify whether our results could be established in the case of \( \Gamma_K \)-modules which do not satisfy the Lie condition, e.g. for the \( \Gamma_K \)-module from [15] (the case \( n = p \) in the notation of that paper).
5. Mixed characteristic

Let \( E_0 \) be a complete discrete valuation field of characteristic 0 with finite residue field \( k_0 \) of characteristic \( p \). Let \( \bar{E}_0 \) be an algebraic closure of \( E_0 \) and for any field \( E \) such that \( E_0 \subset E \subset \bar{E}_0 \), set \( \text{Gal}(\bar{E}_0/E) = \Gamma_E \). Suppose that \( E_0 \) contains a primitive \( p \)-th root of unity \( \zeta_1 \).

We are going to develop an analog of the above characteristic \( p \) theory in the context of finite \( \mathbb{F}_p[\Gamma_{E_0}] \)-modules \( H_{E_0} \) satisfying an analogue of the Lie condition from Sect. 2.1.

If \( \pi_{H_{E_0}} : \Gamma_{E_0} \to \text{Aut}_{\mathbb{F}_p}(H_{E_0}) \) determines a \( \Gamma_{E_0} \)-action on \( H_{E_0} \) then there is a Lie \( \mathbb{F}_p \)-subalgebra \( L(H_{E_0}) \subset \text{End}_{\mathbb{F}_p}(H_{E_0}) \) such that \( L(H_{E_0})^p = 0 \) and \( \exp(L(H_{E_0})) = \pi_{H_{E_0}}(I) \), where \( I \) is the wild ramification subgroup in \( \Gamma_{E_0} \).

**Remark 5.1.** Contrary to the characteristic \( p \) case we restrict ourselves to the Galois modules killed by \( p \) because the theory from [7, 8] is developed recently only under that assumption.

Fix a choice of a uniformising element \( \pi_0 \) in \( E_0 \).

Let \( \bar{E}_0 = E_0(\{\pi_0(n) \mid n \in \mathbb{Z}_{\geq 0}\}) \subset \bar{E}_0 \), where \( \pi_0(0) = \pi_0 \) and for all \( n \in \mathbb{N} \), \( \pi_0(n)^p = \pi_0(n-1) \). The field-of-norms functor \( X \) provides us with:

- a complete discrete valuation field \( X(\bar{E}_0) = K_0 \) of characteristic \( p \) with residue field \( k_0 \) and a fixed uniformizer \( t_0 = \lim \pi_0(n) \);

- an identification of \( \Gamma_{K_0} = \text{Gal}(K_0^{\text{sep}}/K_0) \) with \( \Gamma_{\bar{E}_0} \subset \Gamma_{E_0} \).

Let \( E \) be a finite tamely ramified extension of \( E_0 \) in \( \bar{E}_0 \) such that \( \pi_{H_{E_0}}(\Gamma_{E}) = I(H_0) \). By replacing \( E \) with a suitable finite unramified extension we can assume that \( E \) has uniformiser \( \pi \) such that \( \pi^{e_0} = \pi_0 \). Let \( k \) be the residue field of \( E \).

It is easy to see that the field \( \tilde{E} := E\bar{E}_0 \) appears in the form \( E(\{\pi(n) \mid n \geq 0\}) \), where \( \pi(0) = \pi \), \( \pi(n)^p = \pi(n-1) \) and for all \( n \), \( \pi(n)^{e_0} = \pi_0(n) \). In particular, \( K := X(\tilde{E}) = k((t)) \), where \( t = \lim \pi(n) \) is uniformiser such that \( t^{e_0} = t_0 \).

Let \( G_{<p} = \Gamma_{K}/\Gamma_{E_0}^pC_{p}(\Gamma_{K}) \) and \( \Gamma_{<p} = \Gamma_{E}/\Gamma_{E_0}^pC_{p}(\Gamma_{E}) \). According to [7, 8] we have the following natural exact sequence

\[ G_{<p} \to \Gamma_{<p} \to \langle \tau_0 \rangle^{Z/p} \to 1 \]

where \( \tau_0 \in \text{Gal}(E(\pi^{(1)})/E) \) is such that \( \tau_0(\pi^{(1)}) = \zeta_1\pi^{(1)} \). We can use the identification \( \eta_{<p} : G_{<p} \simeq G(\mathcal{L}) \) from Sect. 4.1 obtained via the special element \( e_{<p} \) and the corresponding \( f_{<p} \) such that \( \sigma_{L(H)}(f_{<p}) = e_{<p} \circ f_{<p} \).

Then we can use the equivalence of categories from Sect. [4.3] to identify \( \Gamma_{<p} \) with \( G(L) \) where \( L \) is a profinite Lie \( \mathbb{F}_p \)-algebra included into the following exact sequence

\[ (5.1) \quad \mathcal{L} \to L \to \mathbb{F}_p\pi_0 \to 0. \]
When studying the structure of (5.1) in [8] we proved that $\tau_0$ can be replaced by a suitable $h_0 \in \text{Aut} K$. More precisely, suppose
\[ \zeta_1 \equiv 1 + \sum_{j \geq 0} [\beta_j] \pi_0^{(e_0^+/p) + j} \text{ mod } p \]
with Teichmüller representatives $[\beta_j]$ of $\beta_j \in k$ and $e^* = e p/(p - 1)$, where $e$ is the ramification index for $E/\mathbb{Q}_p$. Then $h_0$ can be defined as follows: $h_0|_k = \text{id}_k$ and
\[ h_0(t) = t \left( 1 + \sum_{j \geq 0} \beta_j^p t^{e^* + p j} \right) = t \exp(\omega(t)^p), \]
where $\exp$ is the truncated exponential and $\omega(t) \in t^{e^*/p} k[[t]]^*$. This allowed us to apply formalism of the nilpotent Artin-Schreier theory to specify “good” lifts $\tau_{<p}$ of $\tau_0$ to $L$ (what is equivalent to specifying ”good lifts” $h_{<p}$ of $h_0$).

In particular, we obtained in [7, 8] the following description of the image $\bar{L} \subset L$ of $\mathcal{L}$ from exact sequence (5.1). Introduce the weight $c$ of the lift $\tau_{<p}$ to $K_{<p}$ then it is uniquely determined by $c = c(h_{<p}) \in L_K$ such that
\[ (\text{id}_{\mathcal{L}_{<p}} \otimes h_{<p}) f = c \circ (\text{Ad}(h_{<p}) \otimes \text{id}_{K_{<p}}) f. \]
This allowed us to describe the corresponding action of the group $(h_{<p})^{\mathbb{Z}/p}$ on $f$ as an action of an infinitesimal group scheme of order $p$. The differential of this action is given by the ”linear part” $c_1 \in \bar{L}_K$ of $c$ which could be described by a suitable recurrent procedure. Finally, we proved that $c_1(0) \in \bar{L}_k$ (where $c_1 = \sum_{n \in \mathbb{Z}} c_1(n) t^n$ with all $c_1(n) \in \bar{L}_k$) is an absolute invariant of the lift $h_{<p}$.

Consider the expansion $\omega(t)^p = \sum_{j \geq 0} A_j t^{e^* + p j}$, $A_j \in k$.

Denote by $\bar{L}^{(e^*)} \subset \bar{L}$ the image of the ramification subgroup $\mathcal{L}^{(e^*)}$ in $L$, cf. (5.1). Then Theorem 5.1 from [8] states:

- the lift $\tau_{<p}$ of $\tau_0$ is “good” iff the value $c_1(0) \in \mathcal{L}_k$ of the differential $d\tau_{<p}$ at $0$ satisfies the following congruence
\[ c_1(0) \equiv \sum_{j \geq 0} \sum_{i \geq 0} \sigma^i(A_j \mathcal{F}_{e^* + p j, -i}^0) \text{ mod } \bar{L}_k^{(e^*)}. \]

It remains to note that for $i, j \geq 0$, $\mathcal{F}_{e^* + p j, -i}^0 \in \bar{L}_k^{(e^*)}$ and the right-hand double sum contains only finitely many non-zero terms modulo $\bar{L}_k^{(e^*)}$. It can be rewritten also in the following form
\[ \sum_{i,j \geq 0} \text{Res} \left( \sigma^i(A_j t^{e^* + p j} \cdot \sigma^{-i} \Omega_{<p}^{[i]} ) \right), \]
and the image of this expression in $L(H)_k$ equals

$$\sum_{i \geq 0} \text{Res} \left( \sigma^{i+1} \omega(t) \cdot \Omega[i] \right).$$

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