Ahlfors problem for polynomials

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Abstract. We present a conjecture that the asymptotics for Chebyshev polynomials in a complex domain can be given in terms of the reproducing kernels of a suitable Hilbert space of analytic functions in this domain. It is based on two classical results due to Garabedian and Widom. To support this conjecture we study the asymptotics for Ahlfors extremal polynomials in the complement to a system of intervals on $\mathbb{R}$, arcs on $\mathbb{T}$, and the asymptotics of the extremal entire functions for the continuous counterpart of this problem.

Bibliography: 35 titles.

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§ 1. Introduction

Starting with the works of Chebyshev, Markov, Bernstein, Akhiezer, Widom, . . . many explicit asymptotics for the best uniform approximation have been found and become classical; see the book [2], for example, especially the addendum in it, and also the survey paper [27]. For problems dealing with approximation on the real axis there is really a very broad range of results, including, for instance, approximations with varying weights [18], see also [29], or approximations of quite exotic functions, see [11], which nonetheless have applications in computational mathematics [34]. There is no doubt that we owe this to the Chebyshev alternation theorem, which gives a complete description of the structure of generalized polynomials with the least deviation from zero on the real axis.

Despite some new results of the highest level (see [30] and [15], for example), not much is known in the complex plane even in the most classical setting. We do not have asymptotics for Chebyshev polynomials in finitely connected domains bounded by smooth arcs (or even in simply connected domains, for instance, in the complement of a spiral curve).

The main goal of this paper is to give some reasons to support the following hypothesis.

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Conjecture 1.1. Asymptotics for Chebyshev polynomials in a complex domain $\mathcal{D}$ can be given in terms of the reproducing kernels of a suitable Hilbert space of analytic functions in $\mathcal{D}$.

The main motivation for this conjecture was given by the following two classical results.

1. Widom proved that if the boundary $\partial \mathcal{D}$ of the domain $\mathcal{D}$ consists of finitely many (disjoint) smooth Jordan curves, then asymptotics for the Chebyshev polynomials on $\mathcal{D}$ are represented in terms of $H^\infty$-extremal functions in $\mathcal{D}$ (see [32], Theorem 8.3).

2. In turn, in 1949 Garabedian expressed certain extremal properties of uniformly bounded analytic functions in $\mathcal{D}$, which form $H^\infty_\mathcal{D}$, in terms of reproducing kernels [13]. Actually, Garabedian called his paper “Schwarz’s lemma and the Szegő kernel”, paying tribute to Ahlfors “to whom the most far-reaching work on Schwarz’s lemma is due” (see [13], the first paragraph on page 2). Based on this, by the Ahlfors problem we mean the following task.

Problem 1.1. Let $H^\infty_\mathcal{D}$ be a collection of uniformly bounded analytic functions in a domain $\mathcal{D}$. Find

$$\mathcal{A}_\mathcal{D}(z) = \sup\{|w'(z)| : \|w\|_{H^\infty_\mathcal{D}} \leq 1, w(z) = 0\}, \quad z \in \mathcal{D}.$$ 

Recall that for a compact set $E = \partial \mathcal{D}$ the proper defined value $A_\mathcal{D}(\infty)$ is called the analytic capacity of $E$. Garabedian showed that $A_\mathcal{D}(z) = k_{S_2}(z, z; \mathcal{D})$, where $k_{S_2}(z, z_0; \mathcal{D})$ is the Szegő reproducing kernel in $\mathcal{D}$. This is a generalization of the classical Schwarz lemma: if $\mathcal{D}$ is the right half-plane, Re $\lambda > 0$, then

$$A_\mathcal{D}(\Lambda_0) = \left(\frac{\lambda - \lambda_0}{\lambda + \lambda_0}\right)'_{\lambda = \lambda_0} = \frac{1}{\lambda_0 + \overline{\lambda_0}} = k_{S_2}(\lambda_0, \lambda_0; \mathcal{D}).$$

Having in mind the conformal invariance of our conjecture, we will study the following three problems simultaneously, which are naturally related to the Ahlfors problem.

Problem 1.2. Let $E_J$ be a real compact set consisting of $g + 1$ nondegenerate intervals, $E_J = [b_0, a_0] \setminus \bigcup_{j=1}^g (a_j, b_j)$. Let $\mathcal{P}_n(E_J)$ be the collection of polynomials of degree $n$ bounded in absolute value by 1 on $E_J$. Define

$$A_n(z; E_J) = \sup\{|P'(z)| : P \in \mathcal{P}_n(E_J), P(z) = 0\}, \quad z \in \mathbb{C} \setminus E_J.$$ 

Find asymptotics for $A_n(z; E_J)$ as $n \to \infty$.

Problem 1.3. Let $E_T$ be a system of arcs, $E_T = \mathbb{T} \setminus \{e^{iz} : z \in \bigcup_{j=0}^g (a_j, b_j)\}$. Let $\mathcal{P}_n(E_T)$ be the collection of polynomials of degree $n$ bounded in absolute value by 1 on $E_T$. Define

$$A_n(\zeta; E_T) = \sup\{|P'(\zeta)| : P \in \mathcal{P}_n(E_T), P(\zeta) = 0\}, \quad \zeta = e^{iz} \in \mathbb{C} \setminus E_T.$$ 

Find asymptotics for $A_n(\zeta; E_T)$ as $n \to \infty$. 
Problem 1.4. Let \( E_S = \mathbb{R}_+ \setminus \bigcup_{j=1}^g (a_j, b_j) \). Let \( \mathcal{E}_\ell(E_S) \) be the collection of entire functions \( F(z) \) of order \( 1/2 \), of exponential type at most \( \ell \) and bounded in absolute value by 1 on \( E_S \), that is,

\[
|F(z)| \leq C(\ell') e^{\ell' \sqrt{|z|}} \quad \forall \ell' > \ell, \quad |F(z)| \leq 1, \quad z \in E_S.
\]

Define

\[
A_\ell(z; E_S) = \sup \{|F'(z)| : F \in \mathcal{E}_\ell(E_S), \quad F(z) = 0\}, \quad z \in \mathbb{C} \setminus E_S.
\]

Find asymptotics for \( A_\ell(z; E_S) \) as \( \ell \to \infty \).

We point out that, in fact, Widom’s Theorem 8.3 in [32], which we mentioned, requires extremal properties of multivalued \( H_\infty \)-functions in \( \mathcal{D} \) (but with single-valued absolute value). That is, to rewrite his result in terms of reproducing kernels it is necessary to generalize Problem 1.1 slightly (for the exact setting see Problem 2.1 in §2.2), and find a counterpart of Garabedian’s theorem, which was done later by Abrahamse [1]. All these, including a proper definition of the Szegő kernels are given in the preliminary §2. Now, we formulate our solution of Problems 1.2–1.4 for simply connected domains (this, in itself, is quite significant).

**Theorem 1.1.** Let \( E_S = \mathbb{R}_+ \), that is, \( \mathcal{D} = \{z = -\lambda^2 : \text{Re} \lambda > 0\} \). Then

\[
\Upsilon(\lambda) := \lim_{\ell \to \infty} e^{-\ell \text{Re} \lambda} A_\ell(-\lambda^2; \mathbb{R}_+) 2|\lambda| = \frac{1}{\lambda + \lambda_0} \frac{2\sqrt{\lambda \sqrt{\lambda}}}{(\sqrt{\lambda} + \sqrt{\lambda})^2}, \tag{1.1}
\]

and also

\[
\Upsilon(\lambda) = \lim_{n \to \infty} \left| \frac{\lambda - 1}{\lambda + 1} \right|^n A_n(z; [-2, 2]) \left| \frac{dz}{d\lambda} \right| = \lim_{n \to \infty} \left| \frac{\lambda - \lambda_0}{\lambda + \lambda_0} \right|^n A_n(\zeta; E_T) \left| \frac{d\zeta}{d\lambda} \right|,
\]

where

\[
z = 2 \frac{\lambda^2 + 1}{\lambda^2 - 1}, \quad \text{Re} \lambda > 0, \quad \zeta = \frac{z - iy_0}{z + iy_0}, \quad y_0 \in \mathbb{R}_+,
\]

and

\[
\lambda_0^2 = \frac{i y_0 - 2}{i y_0 + 2}, \quad \text{Re} \lambda_0 > 0, \quad E_T = \left\{ \zeta = \frac{z - iy_0}{z + iy_0} : z \in [-2, 2] \right\}.
\]

**Remark 1.1.** Two remarks concerning Theorem 1.1.

(i) **Universality.** While the first (exponential) term in the asymptotics depends on the setting of the problem, the second term \( \Upsilon(\lambda) \) is a conformally invariant quantity. Note that generally in Problems 1.2–1.4 all three domains \( \mathcal{D} = \mathbb{C} \setminus E \), where \( E = E_J, E_T \) or \( E_S \) (with a suitable choice of parameters), are conformally equivalent. At the same time, they have certain features, in particular, they are related to different classes of operators quite famous in spectral theory: the so-called finite gap Jacobi matrices, CMV-matrices and 1-D Schrödinger operators. Recently, we added GMP-matrices [35] to this family. These are related to approximation by rational functions with a prescribed system of poles. There is no doubt that after a suitable choice of the exponential factor the limit of the least deviation in such
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A rational approximation will lead to the same function $\Upsilon(\lambda)$ (see [9], where this universality was demonstrated for the Chebyshev extremal problem).

(ii) 

Hilbert space structure.

Log-subharmonicity is a general property of upper envelopes of families of analytic functions (see [17], Lecture 7, and also [9]). This explains why the following matrix formed by partial derivatives is nonnegative:

$$
\begin{bmatrix}
\Upsilon(\lambda) & \partial \Upsilon(\lambda) \\
\bar{\partial} \Upsilon(\lambda) & \bar{\partial} \partial \Upsilon(\lambda)
\end{bmatrix} \geq 0.
$$

But the limit value $\Upsilon(\lambda)$ represents the diagonal of a certain reproducing kernel:

$$
k(\lambda, \lambda_0) := \frac{1}{\lambda + \lambda_0} \frac{2\sqrt{\lambda} \sqrt{\lambda_0}}{(\sqrt{\lambda} + \sqrt{\lambda_0})^2}, \quad \text{Re} \lambda > 0, \quad \text{Re} \lambda_0 > 0.
$$

Thus, for some reason we have an infinite number of inequalities of this sort:

$$
[\bar{\partial}^m \partial^n \Upsilon(\lambda)]_{n, m=0}^N \geq 0 \quad \text{for all } N \in \mathbb{Z}_+.
$$

We cannot comment on the appearance of this structure in the given context and leave this as an open problem. We point out that the resulting kernel $k(\lambda, \lambda)$ is only collinear to the Szegő kernel on the real axis, $\Upsilon(\lambda) = \frac{1}{2} k_{Sz}(\lambda, \lambda), \lambda \in \mathbb{R}_+.$

Now we outline the structure of the paper and comment on its other results.

The preliminary section (§2.1) contains statements which are known at least on a folklore level. To work with multivalued functions in a multi-connected domain $D$, we prefer to use a universal covering, $D \simeq \mathbb{C}_+ / \Gamma$, where $\Gamma$ is a discrete subgroup of $SL_2(\mathbb{R})$. We introduce multivalued complex Green’s and Martin functions, define their characters and make a connection with conformal mappings on so-called comb-domains.

The prime form is a standard object in the algebraic approach to the theory of reproducing kernels on Riemann surfaces (see [12], Ch. II, and [22], Ch. III,b). The language of Hilbert spaces of automorphic forms $A^2_1(\Gamma, \alpha)$ is probably much easier for specialists in analysis. We use this to define the Szegő kernel as the reproducing kernel of this Hilbert space of analytic functions, Definition 2.2. In fact, it is not that important whether we work with Hardy spaces of character-automorphic functions or forms. But the relation between the unitary character $\beta \in \Gamma^*$ in the character automorphic Ahlfors Problem 2.1 and its solution (Theorem 2.2) looks particularly simple in the second version, $\alpha^2 = \beta$. Note that the extremal character $\alpha$ here is defined up to a square root of unity (or half a period in another terminology) $j$, $j^2 = 1_{\Gamma^*}$. We demonstrate that Garabedian’s case of the trivial character, $\beta = 1_{\Gamma^*}$, in which the extremal character does not depend on $\lambda_0$, Theorem 2.3, is an exception. In fact, in general the half-period varies with $\lambda_0$, $j = j(\lambda_0)$.

An interrelation between the Ahlfors and Abel-Jacobi inversion problems has already been noted in [13]. While the theory for orthogonal polynomials [3] and the spectral theory for finite gap Jacobi matrices, which is related to it, leads to the classical Abel-Jacobi inversion problem [22], the character automorphic $H^\infty$-extremal problem requires a certain modification, Proposition 2.3. We clarify this in §2.3.

Our asymptotic relation for $A_n(z; E_j)$ for real $z$, Theorem 3.3, is Widom’s Theorem 11.5 [32] with a specific weight, see also [6]. But the asymptotics of
\( A_n(\zeta; E_T), \zeta \in \mathbb{T}, \) require a certain varying weight in this reduction. Instead, in this symmetric case we solve Problems 1.2–1.4 in a unified way: using Chebyshev’s alternation theorem we represent extremal functions in terms of the corresponding comb functions. We then use a simple relation (3.10) between the Martin/Green’s functions of the given domain and its \( \ell/n \)-regular extension. Thus, in this case Conjecture 1.1 is confirmed using the Szegő character automorphic reproducing kernel.

Finally, using Kolmogorov’s theorem on the uniqueness of the polynomial of best approximation in the Haar system (see [2], §48), we show how to move \( z_0 \) in the complex plane. We get a reduction of the extremal problem to a generalization of the modified Abel-Jacobi inversion problem; see Problem 4.1. As we have already mentioned, the quantity \( \Upsilon(z_0, \beta) \), which is responsible for the asymptotics, is universal, but is not collinear to the Szegő reproducing kernel if \( \text{Im} z_0 \neq 0 \) (see Theorem 4.2 and also Remark 4.4).

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\section*{§2. Preliminaries. Ahlfors problem for character-automorphic functions and modified Abel-Jacobi inversion}

\subsection*{2.1. Comb-domains and elements of potential theory.} The following comb-domains are standard objects in the spectral theory of reflectionless operators [20], see also [10] and [8]. Let

\begin{align*}
\Pi_J &= \{ \vartheta = \xi + i\eta: 0 < \xi < \pi, \eta > 0 \} \setminus \bigcup_{j=1}^{g} \{ \vartheta = \omega_j + i\eta, \eta \in (0, h_j) \}, \\
\Pi_T &= \mathbb{C}_+ \setminus \bigcup_{j=0}^{g} \bigcup_{m \in \mathbb{Z}} \{ \vartheta = \omega_j + 2\pi m + i\eta, \eta \in (0, h_j) \}
\end{align*}

and

\begin{align*}
\Pi_S &= \{ \vartheta = \xi + i\eta: \xi > 0, \eta > 0 \} \setminus \bigcup_{j=1}^{g} \{ \vartheta = \omega_j + i\eta, \eta \in (0, h_j) \}
\end{align*}

(see Figure 1). In the case of \( \Pi_J \), \( \omega_j \in (0, \pi) \), in the case of \( \Pi_T \), \( \omega_0 = 0 \) and \( \omega_j \in (0, 2\pi) \) for \( j = 1, \ldots, g \). We map \( \mathbb{C}_+ \) conformally onto one of the corresponding combs making normalizations

\begin{align*}
\tau_J(b_0) &= 0, \quad \tau_J(a_0) = \pi, \quad \tau_J(\infty) = \infty, \quad (2.1) \\
\tau_T(iy) &\simeq iy, \quad y \to \infty, \quad \tau_T(0) = 0, \quad (2.2) \\
\tau_S(-x) &\simeq i\sqrt{x}, \quad x \to \infty, \quad \tau_S(0) = 0. \quad (2.3)
\end{align*}

Note that in the case (2.2) we automatically have \( \tau_T(z + 2\pi) = \tau(z) + 2\pi \) [8], that is, \( e^{i\tau_T} \) is well defined as a function of \( \zeta = e^{iz} \in \mathbb{D} \). In the cases (2.1) and (2.3) we get the gaps \( (a_j, b_j) \) as the preimages of the corresponding vertical slits, \( j = 1, \ldots, g \). In the same way, in the case (2.2) we get a system of arcs \( \{ \zeta = e^{iz}: z \in (a_j, b_j) \}_{j=0}^{g} \), which form the complement of \( E_T \).
If $\tau$ denotes one of the maps (2.1)–(2.3), then $\text{Im} \, \tau(z)$ can be extended through the gaps to a single valued harmonic function in $D$. Moreover,

$$\text{Im} \, \tau_J(z) = G(z, \infty), \quad \text{Im} \, \tau_T(z) = G(e^{iz}, 0) + G(e^{iz}, \infty) \quad \text{and} \quad \text{Im} \, \tau_S(z) = M(z),$$

(2.4)

where $G(z, z_0)$ denotes the Green’s function with respect to $z_0$ in the corresponding domain and $M(z)$ stands for the Martin function with respect to infinity. The function $e^{i\tau(z)}$ can be extended to $D$ by the symmetry principle as a multivalued function. Such functions become single-valued on a universal covering.

We fix $D$ in the form $C \setminus E_S$. Recall that for an arbitrary $\overline{C} \setminus E_T$ or $\overline{C} \setminus E_J$ we can always find a suitable conformally equivalent domain $\mathcal{D}$ of the above form. A universal covering $z = z(\lambda), \lambda \in C_+$, also corresponds to a conformal mapping. For a given $E_S$ there exists a system of half-discs $D_j^+$ such that the conformal mapping

$$C_+ \to \mathcal{F}_+ = \{\lambda = \xi + i\eta: \xi > 0, \eta > 0\} \setminus \bigcup_{j=1}^{g} D_j^+, \quad \lambda(-x) \approx i\sqrt{x}, \quad x \to \infty,$$

transforms the negative half-axis into the imaginary half-axis in the $\lambda$-plane and the gaps $(a_j, b_j)$ into the boundary of the half-discs $\partial D_j^+$, see Figure 1. The inverse map $z = z(\lambda)$ can be extended to the whole of the upper half-plane by a system of reflections. Indeed, let $\gamma_j$, acting in the $\lambda$-plane, correspond to the double reflection with respect to the negative half-axis and the gap $(a_j, b_j)$ in the $z$-plane. This is a linear fractional transformation, which maps $\partial D_j^+$ onto $-\partial D_j^+$, and we have $z(\gamma_j(\lambda)) = z(\lambda)$. In this case, the system $\{\gamma_j\}_{j=1}^g$ represents a generator of the
Fuchsian group $\Gamma$ and $\mathcal{F} = \mathcal{F}_+ \cup -\mathcal{F}_+ \cup i\mathbb{R}_+$ is a fundamental domain for the action of $\Gamma$ on $\mathbb{C}_+$ such that $\mathbb{C}_+/\Gamma \cong \mathbb{C} \setminus E_S$ (to $\overline{\mathbb{C}} \setminus E_J$ or $\overline{\mathbb{C}} \setminus E_T$, respectively).

Conformal mappings on comb-domains are partial cases of Schwarz-Christoffel transformations. Using the classical formula, say,

$$\tau_J(z) = i \int_{b_0}^{z} \prod_{j=1}^{g} \frac{z - c_j}{\sqrt{(z - a_j)(z - b_j)}} \frac{dz}{\sqrt{(z - a_0)(z - b_0)}},$$

where $c_j \in (a_j, b_j)$ corresponds to the top $\omega_j + ih_j$ of the slit.

Now we extend $e^{i\tau_S(z)}$ along the generator $\gamma_{\bar{g}}$ of the fundamental group in $\mathcal{D}$, see Figure 2, which corresponds to the action of $\gamma_j$ on the universal covering. As a result, we obtain

$$e^{i\tau_S(z(\gamma_j(\lambda)))} = \alpha_S(\gamma_j)e^{i\tau_S(z(\lambda))}, \quad \alpha_S(\gamma_j) = e^{2\omega_j i}.$$  

This system of multipliers given on the generators forms an element $\alpha_S(\gamma)$, $\gamma \in \Gamma$, of the group of unitary characters $\Gamma^*$. Similar relations generate the characters $\alpha_J$ and $\alpha_T$.

The function $e^{i\tau_J(z(\lambda))}$ is called the complex Green’s function of the group $\Gamma$, and can be represented as the Blaschke product $b_\infty(\lambda) = e^{i\tau_J(z(\lambda))}$ along the orbit $z^{-1}(\infty)$. Generally, for $z_0 = z(\lambda_0)$ we have

$$b_{z_0}(\lambda) = \prod_{\gamma \in \Gamma} \frac{\lambda - \gamma(\lambda_0)}{\lambda - \gamma(\lambda_0)} e^{i\psi_\gamma}, \quad \text{where} \quad e^{i\psi_\gamma} := \frac{i - \gamma(\lambda_0)}{i - \gamma(\lambda_0)} \left| \frac{i - \gamma(\lambda_0)}{i - \gamma(\lambda_0)} \right|.$$  

We point out that

$$- \log |b_{z_0}(\lambda)| = G(z(\lambda), z_0).$$

The character generated by $b_{z_0}$ is denoted by $\mu_{z_0}$, in particular, $\mu_{\infty} = \alpha_J$.

In the domain $\mathcal{D} = \overline{\mathbb{C}} \setminus E_T$ we have, see (2.4),

$$G(\zeta(\lambda), 0) = - \log |\zeta(\lambda)| + \text{Im} \tau_T(\zeta(\lambda)) \frac{2}{2}, \quad \text{where} \quad \zeta(\lambda) = \frac{b_0(\lambda)}{b_\infty(\lambda)}.$$  

That is,

$$b_0(\lambda) = \sqrt{\zeta(\lambda)e^{i\tau_T(\zeta(\lambda))}} \quad \text{and} \quad \mu_0^2 = \alpha_T, \quad \mu_0(\gamma_j) = \mu_{\infty}(\gamma_j) = e^{i\omega_j}.$$
Finally, we should mention the well-known relation between the $\omega_j$ and the harmonic measures. Let $\omega(z, F)$ be the harmonic measure of the set $F \subset E$ in the domain $\mathcal{D} = \mathbb{C} \setminus E$ with respect to $z \in \mathcal{D}$. In $\overline{\mathbb{C}} \setminus E_J$ we have

$$\omega_k = \pi \omega(\infty, E_J^k), \quad E_J^k = E_J \cap [b_0, a_k],$$

and

$$\omega_k = 2\pi \omega(\infty, E_T^k) = 2\pi \omega(0, E_T^k), \quad E_T^k = E_T \cap \{e^{iz} : z \in [0, a_k]\},$$

in $\overline{\mathbb{C}} \setminus E_T$.

We discuss the relationship between the Martin function in $\mathcal{D}$ and the complex Martin function of the group $\Gamma$ in the next subsection.

2.2. The Szegő kernel on the universal covering and Ahlfors problem. Let $H^p$ be the standard Hardy space in $\mathbb{C}_+$, 

$$\|f\|^p = \frac{1}{2\pi} \int_{\mathbb{R}} |f(\xi)|^p \, d\xi, \quad f \in H^p$$

(with a proper modification for $p = \infty$). For a fixed character $\alpha \in \Gamma^*$ we introduce

$$H^p(\alpha) = \{f(\lambda) : f \in H^p, f(\gamma(\lambda)) = \alpha(\gamma)f(\lambda)\}.$$ 

Lemma 2.1. If $H^2(\alpha) \neq \{0\}$ for some $\alpha$, then

$$\sum_{\gamma \in \Gamma} \gamma'(\xi) < \infty$$

for almost all $\xi \in \mathbb{R}$.

Proof. In fact, if there is a measurable fundamental set $E = \partial \mathcal{F} \cap \mathbb{R}$ for the action of $\Gamma$ on $\mathbb{R}$ and a positive automorphic function $f \in L^1$, $f \circ \gamma = f$, nonvanishing almost everywhere, then

$$\frac{1}{2\pi} \int_{\mathbb{R}} f(\xi) \, d\xi = \frac{1}{2\pi} \int_{\mathcal{F}} \left\{ \sum_{\gamma \in \Gamma} \gamma'(\xi) \right\} f(\xi) \, d\xi < \infty,$$

and we get (2.7).

For a reason that will be clear in a moment we consider $\Gamma$ as a subgroup of $\text{SL}_2(\mathbb{R})$ (see Remark 2.1). Condition (2.7) guarantees that the following series converges in the upper half-plane

$$\sum_{\gamma \in \Gamma} \text{Im} \gamma(\lambda) = \sum_{\gamma \in \Gamma} \frac{\text{Im} \lambda}{|\gamma_{21}\lambda + \gamma_{22}|^2}, \quad \gamma = \begin{bmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{21} & \gamma_{22} \end{bmatrix}, \quad \gamma_{ij} \in \mathbb{R}, \quad \det \gamma = 1.$$

(2.8)

In other words the mass-point measure supported on the orbit of infinity \{\xi_\gamma = -\gamma_{22}/\gamma_{21} = \gamma^{-1}(\infty)\} with masses $\sigma_\gamma = 1/\gamma_{21}^2$ satisfies the condition

$$\sum_{\gamma \in \Gamma, \gamma \neq 1} \frac{\sigma_\gamma}{1 + \xi_\gamma^2} = \sum_{\gamma \in \Gamma, \gamma \neq 1} \frac{1}{\gamma_{21}^2 + \gamma_{22}^2} < \infty.$$
Thus, the corresponding function
\[ M(\lambda) = \lambda + \sum_{\gamma \in \Gamma, \gamma \neq 1} \frac{1 + \lambda \xi_\gamma}{\xi_\gamma - \lambda} - \frac{\sigma_\gamma}{1 + \xi_\gamma^2} \]
is well defined and has positive imaginary part given by (2.8) and, furthermore, \( \text{Im} M(\gamma(\lambda)) = \text{Im} M(\lambda) \). That is, \( e^{i M(\lambda)} \) is the complex Martin function of the group \( \Gamma \) with respect to infinity, \( \text{Im} M(\lambda) = M(z(\lambda)) \) and for \( z(\lambda) \simeq \lambda^2 \), \( \lambda = i \eta \), \( \eta \to \infty \), we have
\[ \lambda \frac{M(\lambda) - M(0)}{d\lambda} = \tau_S(z(\lambda)) = -i \int_{0}^{z(\lambda)} \prod_{j=1}^{g} \frac{z - c_j}{(z - a_j)(z - b_j)} \frac{dz}{2\sqrt{-z}}. \] (2.9)

The Blaschke product along the trajectories corresponding to the critical points
\[ W(\lambda) = \prod_{j=1}^{g} b_{c_j}(\lambda) \]
is called the \textit{Widom function}. Note that convergence of this product is called the \textit{Widom condition for the given group} (domain). Such Fuchsian groups were studied by Pommerenke [24]. Widom's condition obviously holds for a complement of a system of intervals.

\textbf{Theorem 2.1} (Pommerenke). The function \( M'(\lambda) \) is holomorphic in the upper half-plane with zeros at \( \{z^{-1}(c_j)\}_{j=1}^{g} \); see (2.9). Moreover, it is a function of bounded characteristic in the upper half-plane (see Remark 2.2) such that
\[ \frac{d M(\lambda)}{d\lambda} = \sum_{\gamma \in \Gamma} \frac{1}{(\gamma_{21} \lambda + \gamma_{22})^2} = \sum_{\gamma \in \Gamma} \gamma'(\lambda) = W(\lambda) \varpi(\lambda)^2, \] (2.10)
where \( \varpi \) is an outer function. In addition, its boundary values on the real axis satisfy
\[ \frac{d M(\lambda)}{d\lambda} = |\varpi(\lambda)|^2 = \sum_{\gamma \in \Gamma} \frac{1}{|\gamma_{21} \lambda + \gamma_{22}|^2} \geq 1, \quad W(\lambda) \varpi(\lambda) = \varpi(\lambda). \] (2.11)

For an analytic function in \( \mathbb{C}_+, \gamma \in \Gamma \), we write
\[ f|_{[\gamma]} = \left[ f(\gamma(\lambda)) \right]_{\gamma_{21} \lambda + \gamma_{22}}, \quad \gamma = \begin{bmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{21} & \gamma_{22} \end{bmatrix}. \] (2.12)
Let \( \alpha_W \in \Gamma^* \) be the character of the Widom function \( W \). Then, see (2.10),
\[ (M(\gamma(\lambda)))' = M'(\lambda) \quad \Rightarrow \quad \varpi|_{[\gamma]} = \nu(\gamma) \varpi, \] (2.13)
where \( \nu \) denotes a certain fixed root of the character \( \alpha_W^{-1} \); \( \nu^2 = \alpha_W^{-1} \).

\textbf{Remark 2.1.} Generally, a square root of a character is defined up to a half-period, that is, up to a character \( j \in \Gamma^* \) such that \( j^2 = 1_{\Gamma^*} \). Note that in the definition (2.12)
it is essential that $\Gamma$ be considered as a subgroup of $\text{SL}_2(\mathbb{R})$, but not as a Fuchsian group, where $\pm \gamma$ generates the same transform. In fact, such a passage is also defined up to a choice of a half-period. Indeed, any of the groups
\begin{equation}
\Gamma_1 = \{j(\gamma)\gamma : \gamma \in \Gamma\} \subset \text{SL}_2(\mathbb{R}), \quad j \in \Gamma^*, \quad j^2 = 1_{\Gamma^*},
\end{equation}
generates the same group of linear fractional transforms. Note that the group under consideration is freely generated.

**Remark 2.2.** Recall that a function $f(z)$ is of bounded characteristic in $\mathbb{C}_+$ if it is a quotient of two bounded analytic functions and it is of Smirnov class, or of Nevanlinna class $\mathcal{N}_+$ in another terminology (see [14], Ch. II, §5), if its denominator is an outer function. Note that functions of this class obey a maximum principle of high generality. For instance, if $f \in \mathcal{N}_+$ and its boundary values satisfy $f \in L^2$, then $f \in H^2$.

**Definition 2.1.** For $\alpha \in \Gamma^*$ the space $A^2_1(\Gamma, \alpha)$ is formed of those analytic functions $f$ in $\mathbb{C}_+$ that satisfy the following three conditions:

(i) $f$ is of Smirnov class;
(ii) $f[\gamma] = \alpha(\gamma)f$ for all $\gamma \in \Gamma$;
(iii) $\|f\|^2 = \|f\|^2_{A^2_1(\Gamma, \alpha)} = \frac{1}{2\pi} \int_E |f(\lambda)|^2 d\lambda < \infty$.

**Proposition 2.1.** The following map $f \mapsto \varpi f$ sets a unitary correspondence between $H^2(\alpha)$ and $A^2_1(\Gamma, \nu \alpha)$.

**Proof.** Let $f \in H^2(\alpha)$. Then
\begin{equation}
\|f\|^2 = \frac{1}{2\pi} \int_E |f(\lambda)|^2 \sum_{\gamma \in \Gamma} \gamma'(\lambda) d\lambda = \frac{1}{2\pi} \int_E |f(\lambda) \varpi(\lambda)|^2 d\lambda.
\end{equation}
Since $f \in H^2$ and $\varpi$ is outer, we have $f \varpi \in \mathcal{N}_+$. The property (ii) follows from (2.13). Conversely, if $g \in A^2_1(\Gamma, \nu \alpha)$, then $f = g / \varpi$ is in the standard $L^2$ and of Smirnov class. Thus, it belongs to $H^2$. The ratio of two forms (see (ii)) generates a function with character $\alpha$.

The point evaluation functional is bounded in $H^2$. By $k^{\alpha}_{\lambda}\alpha(\lambda) = k^{\alpha}(\lambda, \lambda_0)$ we denote the reproducing kernel in $H^2(\alpha)$, $\langle f, k^{\alpha}_{\lambda0} \rangle = f(\lambda_0)$ for all $f \in H^2(\alpha)$.

**Definition 2.2.** We call the reproducing kernel of the space $A^2_1(\Gamma, \alpha)$ the Szegő kernel (corresponding to the given group $\Gamma$ and its character $\alpha$),
\begin{equation}
k_S^{\alpha}(\lambda, \lambda_0; \Gamma, \alpha) = k^{\alpha}_{S}(\lambda, \lambda_0) = k^{\alpha \nu^{-1}}(\lambda, \lambda_0) \varpi(\lambda) \overline{\varpi(\lambda_0)}.
\end{equation}
We generalize the Ahlfors Problem (Problem 1.1) slightly.

**Problem 2.1.** For $\lambda_0 \in \mathbb{C}_+$ find
\[ \mathcal{A}(\lambda_0, \beta) = \sup \{|w'(\lambda_0)| : w \in H^\infty(\beta), \|w\| \leq 1, w(\lambda_0) = 0\}. \]
Theorem 2.2. A solution of Problem 2.1 is given in terms of the Szegő kernels
\[ \mathcal{A}(\lambda_0, \beta) = \min_{\alpha^2 = \beta} k_{Sz}^{\alpha}(\lambda_0, \lambda_0) = k_{Sz}^{\alpha(\lambda_0)}(\lambda_0, \lambda_0). \tag{2.16} \]
If \( \beta = 1_{\Gamma^*} \) the above minimum is attained on the same half-period \( j \in \Gamma^* \) for all \( \lambda_0 \in \mathbb{C}_+ \), that is (see (2.14)) the Garabedian formula holds:
\[ \mathcal{A}(\lambda_0, 1_{\Gamma^*}) = \mathcal{A}(\lambda_0, 1_{\Gamma^*}) = k_{Sz}(\lambda_0, \lambda_0; \Gamma_j, 1_{\Gamma^*}). \tag{2.17} \]
Generally, the extremal character \( \alpha(\lambda_0) \), \( \alpha(\lambda_0)^2 = \beta \), depends on \( \lambda_0 \).

Definition 2.3. In what follows the choice of the group \( \Gamma \subset \text{SL}_2(\mathbb{R}) \) is assumed to match with the extremal half-period in the Ahlfors problem, see Remark 2.1. That is (cf. (2.17))
\[ \mathcal{A}(\lambda_0, 1_{\Gamma^*}) = k_{Sz}(\lambda_0, \lambda_0; \Gamma, 1_{\Gamma^*}) = k_{Sz}(\lambda_0, \lambda_0). \]

As was mentioned in Garabedian’s original paper [13], Problem 1.1 is the simplest version of Nevanlinna-Pick type problems for multiply connected domains (Riemann surfaces). They were studied later by Abrahamse [1] and in many subsequent papers (see [5] and [16], for example). The statement below is an easy consequence of Abrahamse’s theorem. Note that a similar statement can be found in what is an essentially much more general situation in [31].

Proposition 2.2. The following equality holds:
\[ \sup_{w \in H^\infty(\beta), \|w\| \leq 1} |w(\lambda_0)| \leq \inf_{\alpha \in \Gamma^*} k_{Sz}^{\alpha\beta}(\lambda_0, \lambda_0) = \frac{k_{Sz}^{\alpha_0\beta}(\lambda_0, \lambda_0)}{k_{Sz}^{\alpha_0}(\lambda_0, \lambda_0)} \]
\[ \alpha_0 = \alpha_0(\lambda_0). \tag{2.18} \]
Moreover, an extremal function \( w_{\lambda_0, \beta}(\lambda) \) is a Blaschke product and
\[ w_{\lambda_0, \beta}(\lambda) w_{\lambda_0, \beta}(\lambda_0) = \frac{k_{Sz}^{\alpha_0\beta}(\lambda, \lambda_0)}{k_{Sz}^{\alpha_0}(\lambda_0, \lambda_0)}. \tag{2.19} \]

Note that one side of the statement deals with the trivial estimate
\[ |w(\lambda_0) k_{Sz}^{\alpha}(\lambda_0, \lambda_0)|^2 = |\langle w k_{Sz}^{\alpha} \rangle_{\lambda_0} k_{Sz}^{\alpha}(\lambda_0, \lambda_0)|^2 \]
\[ \leq \|k_{Sz}^{\alpha} \|_{\lambda_0}^2 \|k_{Sz}^{\alpha}(\lambda_0, \lambda_0) \|_{\lambda_0}^2 = k_{Sz}^{\alpha}(\lambda_0) k_{Sz}^{\alpha}(\lambda_0). \tag{2.20} \]
Also, due to the nature of formulae (2.18) and (2.19), it does not matter whether we use Szegő kernels or the reproducing kernels of character automorphic Hardy spaces.

Proof of (2.16) in Theorem 2.2. Let \( v \) be an extremal function for Problem 2.1. Then \( v = b_{z_0} w \), where \( z_0 = z(\lambda_0) \) and \( w = w_{\lambda_0, \beta_{=z_0}} \) is the extremal function from (2.19).

Let \( L^2_{d\lambda|E} \) be the space of square-integrable functions on \( E \) with respect to the measure \( d\lambda \). For an arbitrary character \( \alpha \in \Gamma^* \) the space \( A^2_{\alpha}(\Gamma, \alpha) \) forms a closed subspace of it. Recall that in the classical case the orthogonal complement to \( H^2 \) in \( L^2 \) can be described as the set of conjugates of holomorphic functions
\[ L^2 \ominus H^2 = \{ f(z) = \overline{g(z)} : g \in H^2 \} = \overline{H^2}. \]
Similarly, the orthogonal complement to $A^2_{1}(\Gamma, \alpha)$ is given by
\[
L^2_{d\lambda|E} \ominus A^2_{1}(\Gamma, \alpha) = A^2_{1}(\Gamma, \alpha^{-1})
\]
(see [33]). Due to this duality we have
\[
b_{z_0}(\lambda) \frac{k^{\alpha}_{S_z}(\lambda, \lambda_0)}{\| k^{\alpha}_{S_z, \lambda_0} \|} = k^{\alpha^{-1}}_{S_z}(\lambda, \lambda_0) \frac{\| k^{\alpha^{-1}}_{S_z, \lambda_0} \|}{\| k^{\alpha^{-1}}_{S_z, \lambda_0} \|}, \quad \lambda \in \mathbb{R},
\]
and
\[
k^{\alpha^{-1}}_{S_z}(\lambda_0, \lambda_0) k^{\alpha}_{S_z}(\lambda_0, \lambda_0) = |b'_{z_0}(\lambda_0)|^2.
\]
Using $|w(\lambda)| = 1$ on $\mathbb{R}$ we obtain both (2.19) and the dual representation
\[
w(\lambda)w(\lambda_0) = \frac{k^{\alpha_0}_S(\lambda, \lambda_0)}{k^{\alpha_0}_S(\lambda_0, \lambda_0)} = \frac{k^{\alpha_0^{-1}}_{S_z}(\lambda, \lambda_0)}{k^{\alpha_0^{-1} \mu_{z_0}}_S(\lambda, \lambda_0)}.
\]
Generically, a solution of the problem is a unique Blaschke product of $g$ complex Green’s functions (see Theorem 2.3 and [1], [16]), that is, $\alpha_0 = \alpha_0^{-1} \beta^{-1}$ and for this character
\[
|v'(\lambda_0)|^2 = |b'_{z_0}(\lambda_0)|^2 |w(\lambda_0)|^2 = |b'_{z_0}(\lambda_0)|^2 \frac{k^{\alpha_0}_S(\lambda_0, \lambda_0)}{k^{\alpha_0^{-1} \mu_{z_0}}_S(\lambda_0, \lambda_0)}.
\]
By (2.21), we have
\[
|b'_{z_0}(\lambda_0)|^2 \frac{k^{\alpha_0^{-1} \beta}_S(\lambda_0, \lambda_0)}{k^{\alpha_0^{-1} \mu_{z_0}}_S(\lambda_0, \lambda_0)} = |b'_{z_0}(\lambda_0)|^2 \frac{k^{\alpha}_S(\lambda_0, \lambda_0)}{k^{\alpha^{-1} \mu_{z_0}}_S(\lambda_0, \lambda_0)} = k^{\alpha}_S(\lambda_0, \lambda_0)^2
\]
for an arbitrary $\alpha$ such that $\alpha^2 = \beta$. Due to (2.18), we obtain (2.16) with $\alpha(\lambda_0) = \alpha_0^{-1}$. In degenerate cases the formula holds by continuity, although the choice for $\alpha_0$ is not unique.

Now we will essentially specify the second statement of Theorem 2.2 for finitely connected Denjoy domains. Note that the general formula for the analytic capacity in Denjoy domains is a well-known result of Pommerenke’s (see [23], and also [26], § 8.8).

**Theorem 2.3.** Let $\mathcal{D} = \mathbb{C} \setminus E_S$ and
\[
\Omega(z) = \frac{1}{\sqrt{-z}} \prod_{j=1}^{g} \sqrt{\frac{z-a_j}{z-b_j}}.
\]
Then
\[
k_S(z_0, \lambda_0) = \frac{\text{Im} \Omega(z_0)}{2 \text{Im} z_0 |\Omega(z_0)|} \left| \frac{dz}{d\lambda}(\lambda_0) \right|, \quad z_0 = z(\lambda_0).
\]
In addition, the extremal Ahlfors function of Problem 1.1 is of the form
\[
w_{z_0, \mathcal{D}}(z) = \frac{z - z_0}{z - \overline{z_0}} \frac{\Omega(z) - \Omega(z_0)}{\Omega(z) + \Omega(z_0)}.
\]
First we prove the following lemma.

**Lemma 2.2.** Let $H^2_\Omega$ be the space of Smirnov class functions $F(z)$ in $\mathcal{D}$ with the scalar product
\[
\|F\|_{H^2_\Omega}^2 = \frac{1}{\pi} \int_E \frac{|F(x+i0)|^2 + |F(x-i0)|^2}{2} \Im \Omega(x) \, dx = \frac{1}{2\pi i} \int_E |F(x)|^2 \Omega(x) \, dx.
\]

Then the reproducing kernel of this space is of the form
\[
K_\Omega(z, z_0) = \frac{-\Omega^{-1}(z) + \Omega^{-1}(\overline{z_0})}{2(z - \overline{z_0})}.
\] \hspace{1cm} (2.26)

**Proof.** Note that $\Omega(x) = -\Omega(x)$ if $x \in E$ and $\Omega(\overline{z}) = \Omega(z)$ for $z \in \mathcal{D}$. For $F \in H^2_\Omega$ we can apply the Cauchy theorem
\[
\langle F, K_\Omega(\cdot, z_0) \rangle = \frac{1}{2\pi i} \int_E \frac{\Omega^{-1}(x) + \Omega^{-1}(z_0)}{2(x - z_0)} F(x) \Omega(x) \, dx = F(z_0).
\]

**Proof of Theorem 2.3.** According to Proposition 2.1 the space $A^2_\Omega(\Gamma)$ can be interpreted as the collection of functions $f \varpi$, where $f \in H^2(\nu^{-1})$. In turn, to $f \in H^2(\nu^{-1})$ we associate a multivalued function $F$ in $\mathcal{D}$ such that $F(z(\lambda)) = f(\lambda)$. Note that $|F(z)|$ is single valued in $\mathcal{D}$, and the scalar product, by (2.11) and (2.15), has the form
\[
\|f\|_{H^2(\nu^{-1})}^2 = \frac{1}{\pi} \int_E \frac{|F(x+i0)|^2 + |F(x-i0)|^2}{2} \prod_{j=1}^g \frac{x-c_j}{\sqrt{(x-a_j)(x-b_j)}} \, dx.
\]

Let $\phi_\Omega$ be the multivalued outer function in $\mathcal{D}$ given by
\[
\phi_\Omega^{-2} = \frac{1}{2\mathcal{W}} \prod_{j=1}^g \frac{z-c_j}{z-a_j}.
\] \hspace{1cm} (2.27)

Note that
\[
|\phi_\Omega|^{-2} = \frac{1}{2} \prod_{j=1}^g \frac{x-c_j}{x-a_j}, \quad x \in E, \quad \text{and} \quad \varpi \circ \gamma = \nu^{-1}(\gamma) \phi_\Omega;
\]
at this point we fix the half-period for $\Gamma$, see Definition 2.3, so that the function $\phi_\Omega$ and the form $\varpi$ generate mutually inverse characters. Since
\[
\|f\|_{H^2(\nu^{-1})}^2 = \frac{1}{\pi} \int_E \frac{|(F/\phi_\Omega)(x+i0)|^2 + |(F/\phi_\Omega)(x-i0)|^2}{2} \Im \Omega(x) \, dx
\]
for $g \in A^2_\Omega(\Gamma)$ we obtain $g = G(z(\lambda))\phi_\Omega \varpi$, where $G \in H^2_\Omega$. Furthermore,
\[
k_{\mathcal{D}}(\lambda, \lambda_0) = K_\Omega(z(\lambda), z(\lambda_0))\phi_\Omega(z(\lambda))\varpi(\lambda)\phi_\Omega(z(\lambda_0))\varpi(\lambda_0).
\] \hspace{1cm} (2.28)

By (2.10) and (2.27) we have
\[
\varpi^2 \phi_\Omega^2 = -i \frac{1}{\mathcal{W}} \prod_{j=1}^g \frac{z-c_j}{\sqrt{(z-a_j)(z-b_j)}} \frac{4}{2\sqrt{z}} \frac{dz}{d\lambda} \prod_{j=1}^g \frac{z-a_j}{z-c_j} = -i \Omega(z) \frac{dz}{d\lambda}.
\]
Thus (2.26) and (2.28) imply (2.24).
It remains to explain the extremal property of the fixed half-period. Consider the following family of functions:

$$\Omega_{\varepsilon_1,\ldots,\varepsilon_g}(z) = \Omega(z) \prod_{k=1}^{g} \left( \frac{z - b_k}{z - a_k} \right)^{(1-\varepsilon_k)/2}, \quad \varepsilon_k = \pm 1.$$  

Similarly to Lemma 2.2, we have $2^g$ Hilbert spaces $H^2_{\Omega_{\varepsilon_1,\ldots,\varepsilon_g}}$ with the reproducing kernels

$$K_{\Omega_{\varepsilon_1,\ldots,\varepsilon_g}}(z, z_0) = \frac{-\Omega_{\varepsilon_1,\ldots,\varepsilon_g}^{-1}(z) + \Omega_{\varepsilon_1,\ldots,\varepsilon_g}^{-1}(z_0)}{2(z - \bar{z}_0)}.$$  

We define the collection of outer functions

$$\varphi^2_{\varepsilon_1,\ldots,\varepsilon_g} = \prod_{k=1}^{g} \left( \frac{z - b_k}{z - a_k} \right)^{(1-\varepsilon_k)/2},$$

whose characters (for the already fixed $\Gamma$) form all possible $2^g$ half-periods on this group. In this case

$$A^2_1(\Gamma, j) = \{ g = G(z(\lambda)) \varphi \Omega_{\varepsilon_1,\ldots,\varepsilon_g} : G \in H^2_{\Omega_{\varepsilon_1,\ldots,\varepsilon_g}} \}.$$  

Thus, we have to compare the values

$$k_{\text{Sz}}(\lambda_0, \lambda_0; \Gamma, j) = \frac{\text{Im} \Omega_{\varepsilon_1,\ldots,\varepsilon_g}(z_0)}{2 \text{Im} z_0 |\Omega_{\varepsilon_1,\ldots,\varepsilon_g}(z_0)|} \left| \frac{dz}{d\lambda}(\lambda_0) \right|$$

for all half-periods $j$ to choose the minimal one. We use the exponential representation

$$\Omega_{\varepsilon_1,\ldots,\varepsilon_g}(z) = C \exp \left\{ \int_{\mathbb{R}} \frac{1 + xz}{x - z} \chi_{\varepsilon_1,\ldots,\varepsilon_g}(x) \frac{dx}{1 + x^2} \right\}, \quad C > 0,$$

$$\chi_{\varepsilon_1,\ldots,\varepsilon_g}(x) := \frac{1}{\pi} \arg \Omega_{\varepsilon_1,\ldots,\varepsilon_g}(x),$$

due to which

$$\frac{\text{Im} \Omega_{\varepsilon_1,\ldots,\varepsilon_g}(z)}{|\Omega_{\varepsilon_1,\ldots,\varepsilon_g}(z)|} = \sin(I_{\varepsilon_1,\ldots,\varepsilon_g}(z)), \quad I_{\varepsilon_1,\ldots,\varepsilon_g}(z) = \int_{\mathbb{R}} \frac{\text{Im} z}{|x - z|^2} \chi_{\varepsilon_1,\ldots,\varepsilon_g}(x) \, dx.$$  

The minimal value in the last expression corresponds to the minimum between the two extreme values

$$\min_{\varepsilon_k = \pm 1} \sin(I_{\varepsilon_1,\ldots,\varepsilon_g}(z)) = \min\{ \sin(I_+(z)), \sin(I_-(z)) \}, \quad I_{\pm}(z) := I_{\varepsilon_k = \pm 1, \forall k}(z).$$  

For these two we have

$$\sin(I_-(z)) - \sin(I_+(z)) = 2 \sin \left\{ \frac{1}{2} \int_{\mathbb{R}} \frac{\text{Im} z \, dx}{|x - z|^2} \right\} \cos \left\{ \frac{1}{2} \int_{\mathbb{R}} \frac{\text{Im} z \, dx}{|x - z|^2} \right\}.$$  

Since $\int_{\mathbb{R}} \frac{\text{Im} z \, dx}{|x - z|^2} = \pi$ we get $\sin(I_+(z)) < \sin(I_-(z))$. That is, the configuration $\{ \varepsilon_k = 1 \text{ for all } k \}$ corresponds to the global minimum.
Note that this extremal choice of $\varepsilon_k$ corresponds to the exceptional case when

$$\text{Re} \Omega_{\varepsilon_1, \ldots, \varepsilon_g}(z) \geq 0$$

in the upper half-plane and therefore for all $z \in \mathcal{D}$. Therefore, $\Omega(z) - \Omega(z_0)$ has $g + 1$, that is, the maximal possible number of zeros, $\{z_k\}_{k=0}^g$, in $\mathcal{D}$. Respectively,

$$\frac{\Omega(z(\lambda)) - \Omega(z_0)}{\Omega(z(\lambda)) + \Omega(z_0)} = \prod_{k=0}^g b_{z_k}(\lambda),$$

and we obtain (2.25).

To describe reproducing kernels in the general case we introduce the following notation and definitions (see [22], Ch. III, a, and also [28]).

**Definition 2.4.** Let $\mathcal{R}$ denote the hyperelliptic Riemann surface

$$\mathcal{R} = \left\{ P = (z, \Omega) : \Omega^2 = -\frac{1}{z} \prod_{j=1}^g \frac{z - a_j}{z - b_j} \right\},$$

and let $\overline{\mathcal{R}}$ be its compactification. The upper sheet means the collection of points $\{P = (z, \Omega) : \text{Re} \Omega > 0\}$ and we identify it with the domain $\mathcal{D} = \mathbb{C} \setminus E_S$, where $\Omega = \Omega(z)$ is well defined. Thus, for a generic point on $\mathcal{R}$ we can write $(z, 1)$, $z \in \mathcal{D}$, having in mind a point on the upper sheet and $(z, -1)$, $z \in \mathcal{D}$, for a point on the lower sheet. Note that $(a_j, \pm 1)$ ($(b_j, \pm 1)$, respectively) denotes the same point on $\mathcal{R}$. We call the collection of points of the form $D = \{(x_j, \varepsilon_j) : x_j \in [a_j, b_j], \varepsilon_j = \pm 1\}_{j=1}^g$ a divisor. Topologically, they form a $g$-dimensional torus $D_{\mathcal{R}}$. To the given $D$ we associate a multivalued function in $\mathcal{D}$ (a generalization of (2.27))

$$\phi_D(z(\lambda)) = \sqrt{\frac{1}{2} \prod_{j=1}^g \frac{(z(\lambda) - x_j)b_{x_j}(\lambda))}{(z(\lambda) - c_j)b_{c_j}(\lambda))} \prod_{j=1}^g b_{x_j}^{(1+\varepsilon_j)/2}(\lambda)},$$

which can be extended to $\mathcal{R}$. In this case $D$ corresponds to the zeros of $\phi_D$. Note that poles correspond to the divisor $\{(c_j, -1)\}_{j=1}^g$. The character generated by this function is denoted by $\alpha_D$.

Let $T(z) = z \prod_{j=1}^g (z - a_j)(z - b_j)$ and $U_D(z) = \prod_{j=1}^g (z - x_j)$ so that

$$\frac{U_D(z)}{\sqrt{-T(z)}} = \Omega(z) \prod_{j=1}^g \frac{z - x_j}{z - a_j}$$

has positive imaginary part in the upper half-plane of the upper sheet. Let

$$m_D^\pm(z) = m_\pm(z) := \frac{-\sqrt{-T(z)} \pm V_D(z)}{U_D(z)},$$

(2.30)
where the polynomial $V_D(z)$, $V_D(0) = 0$, of degree $g$ is uniquely defined by the condition that on $\mathcal{D}$ the function $m_+(z)$ has poles exactly at points forming the divisor $D$ (and, by construction, at infinity). We point out that both functions have positive imaginary values in the upper half-plane, and $m_+^{D}(x) = -m_-(x)$, $x \in E_S$.

**Theorem 2.4.** For an arbitrary character $\alpha \in \Gamma^*$ there exists a unique divisor $D$ such that $\alpha = \alpha_D$ with the character generated by $\phi_D$ (2.29). Let $H^2_{m_+^D}$ be the space of meromorphic functions $F(z)$ in $\mathcal{D}$ such that $F(z(\lambda))\phi_D(z(\lambda))$ is of Smirnov class equipped with the scalar product

$$
\|F\|_{m_+^D}^2 = \frac{1}{\pi} \int_{E_S} \frac{|F(x + i0)|^2 + |F(x - i0)|^2}{2} \text{Im} \frac{U_D(x)}{\sqrt{-T(x)}} \, dx.
$$

Then the reproducing kernel of this space is of the form

$$K_{m_+^D}(z, z_0) = \frac{m_+^{D}(z) - m_+^{D}(z_0)}{2(z - \overline{z}_0)}, \quad z_0 \neq x_k. \tag{2.31}
$$

Consequently, the reproducing kernel of the space $A^2_{\Gamma}(\Gamma, \alpha \nu)$ is of the form

$$k_{S_z}(\lambda, \lambda_0; \Gamma, \alpha \nu) = K_{m_+^D}(z(\lambda), z(\lambda_0))\overline{\varphi}(\lambda)\overline{\phi_D}(z(\lambda_0))\overline{\varphi}(\lambda_0).
$$

**Proof of Theorem 2.4 and Theorem 2.2.** The existence and uniqueness of $D$ for the given $\alpha$ follows from the Abel-Jacobi inversion theorem. Since the poles of $m_+^D$ complement each other in $\mathcal{D}$ (see (2.30)) by the Cauchy theorem we get

$$\langle F, K_{m_+^D}(\cdot, z_0) \rangle = \frac{1}{2\pi i} \int_{E_S} \frac{-m_-^{D}(x) - m_+^{D}(z_0)}{2(x - z_0)} F(x) \frac{U_D(x)}{\sqrt{-T(x)}} \, dx = F(z_0).
$$

It remains to prove the last statement of Theorem 2.2. Consider an elliptic case. By (2.22) we are interested in the reproducing kernel $k_{S_z}^p(\lambda, \lambda_0)$ such that

![Figure 3. The graph of $m_+ = m_+^{D}(z)$ on the complement to $E$. Change by a half-period is required if $z_0 \in (x_*, 0)$.](image-url)
\( \alpha^2 = \beta \) and the corresponding \( K_{m_+^D}(z, z_0) \) has a zero in the gap \((a_1, b_1)\) (on the upper sheet). We use the representation (2.31), assuming that \( x_1 \in (a_1, b_1) \) is a pole of \( m_+^D \). Note that in this case \( m_+^D(a_1) > 0 \) and \( m_+^D(b_1) < 0 \). Therefore (see a sketch of a graph of \( m_+^D(z) \) in Figure 3), provided \( z_0 < x_* < 0 \) (the corresponding zero belongs to the lower sheet). Thus, in this range a change of the half-period is required (an extremal function corresponds to another divisor \( D \)).

2.3. The Ahlfors problem and Abel-Jacobi inversion. Recall that Abelian differentials on the Riemann surface \( R \) (see Definition 2.4) form a \( g \)-dimensional linear space. We will fix a basis in this space in the form

\[
\frac{d\mathbf{w}_k}{\sqrt{-T(z)}} = Q(z)dz - T(z), \quad \deg Q = g - 1, \quad \int_{a_j}^{b_j} d\mathbf{w}_k = \frac{1}{2}\delta_{k,j}, \quad k, j = 1, \ldots, g, \tag{2.32}
\]

where \( \delta_{k,j} \) is the Kronecker symbol and integration is along an interval on the upper sheet. Note that

\[
2 \Re \int_0^z d\mathbf{w}_k = \omega(z, E_k) \quad \text{in} \ D \tag{2.34}
\]

Theorem 2.5. Fix a base point \( D_0 = \{a_j\}_{j=1}^g \) in the collection of divisors \( D_R \) on \( R \) and define the map from \( D_R \) to the standard real torus \( \mathbb{R}^g/\mathbb{Z}^g \) by

\[
\alpha_k(D) = \sum_{j=1}^g \int_{a_j}^{(x_j, \varepsilon_j)} d\mathbf{w}_k, \quad D = \{(x_j, \varepsilon_j)\}_{j=1}^g \in D_R, \quad \alpha = \{\alpha_k\}_{k=1}^g \in \mathbb{R}^g/\mathbb{Z}^g, \tag{2.33}
\]

with integration along the interval on the lower or upper sheet depending on \( \varepsilon_j \). This map is one-to-one.

Evidently, (2.33) can be rewritten as

\[
\alpha_k = \sum_{j=1}^g \frac{\varepsilon_j}{2} \omega(x_j, E_k) \quad \text{mod } 1. \tag{2.34}
\]

In the context of the Chebyshev extremal problems, we have a quite similar, but actually different inversion problem. For a fixed \( \beta \) and \( x_0 < 0 \) the extremal function from Proposition 2.2 has the form of the Blaschke product

\[
w_{x_0, \beta}(\lambda) = \prod b_{x_j}(\lambda), \tag{2.35}
\]
and for its character we get
\[ \beta_k = \sum_{j=1}^{g} \omega(x_j, E_k) \mod 1. \] (2.36)

Thus, the relation between \( \beta \) and the divisor \( \{(x_j, 1)\}_{j=1}^{g} \) can not be reduced exactly to the standard Abel-Jacobi inversion (Theorem 2.5). Indeed, directly from (2.36) we can only get
\[ \alpha_k = \sum_{j=1}^{g} \frac{1}{2} \omega(x_j, E_k) \mod 1, \]
where \( 2\alpha_k = \beta_k \) is still defined up to a half-period. If one of the solutions \( \{\hat{\alpha}_k\} \) is fixed, the whole collection is of the form \( \{\hat{\alpha}_k + (1 - \delta_k)/4\}_{k=1}^{g} \), \( \delta_k = \pm 1 \). In this way, to find the \( x_j \), we have to solve (2.33) for all \( 2^g \) possible collections \( \{\delta_k\}_{k=1}^{g} \) and then choose among all \( 2^g \) solutions the divisor \( D = \{(x_j, \varepsilon_j)\}_{j=1}^{g} \) with \( \varepsilon_j = 1 \), see (2.34).

Therefore, in our context Theorem 2.5 has to be replaced by the following claim.

**Proposition 2.3.** Let \( I_j \) be the interval \([a_j, b_j]\) with identification of the endpoints and equipped with the corresponding topology of the unit circle. Let \( \mathcal{I} = \bigotimes_{j=1}^{g} I_j \) be the topological \( g \)-dimensional torus. Then the map \( \mathcal{I} \rightarrow \mathbb{R}^g/\mathbb{Z}^g \) given by
\[ \beta_k(\mathcal{X}) = \sum_{j=1}^{g} \omega(x_j, E_k), \quad \mathcal{X} = \{x_j\}_{j=1}^{g} \in \mathcal{I}, \quad \beta = \{\beta_k\}_{k=1}^{g} \in \mathbb{R}^g/\mathbb{Z}^g, \] (2.37)
is a homeomorphism.

**Proof.** We fix \( x_0 \) in the complementary gap and for the given \( \beta \) get an extremal Blaschke product \( w_{x_0, \beta} \) in the form (2.35). Due to (2.36) we have existence in (2.37).

To get uniqueness, we note that any Blaschke product \( w_{\beta}(\lambda) \) with character \( \beta \), of the form (2.35) represents the inner part of a reproducing kernel \( k_{\alpha_0}^{\alpha_{\beta}}(\lambda, \lambda_0) \), \( x_0 = z(\lambda_0) \), for a certain \( \alpha_0 \in \Gamma^* \). (We note that the position of the zeros of reproducing kernels is directly connected with the Abel-Jacobi inversion (2.33); see Theorem 2.4.) Since an outer part of a reproducing kernel is also a reproducing kernel we get
\[ k_{\alpha_0}^{\alpha_{\beta}}(\lambda, \lambda_0) = w_{\beta}(\lambda) \frac{1}{w_{\beta}(\lambda_0)} k_{\alpha_0}^{\alpha_{\beta}}(\lambda, \lambda_0). \]

Thus, see (2.20),
\[ |w_{\beta}(\lambda_0)|^2 = \inf_{\alpha \in \Gamma^*} \frac{k_{\alpha_0}^{\alpha_{\beta}}(\lambda_0, \lambda_0)}{k_{\alpha_0}^{\alpha_{\beta}}(\lambda_0, \lambda_0)} = \frac{k_{\alpha_0}^{\alpha_{\beta}}(\lambda_0, \lambda_0)}{k_{\alpha_0}^{\alpha_{\beta}}(\lambda_0, \lambda_0)}. \]

As the extremal function is unique (in the normalization \( w_{\beta}(\lambda_0) > 0 \)) we get uniqueness in (2.37) (up to identification of the gap endpoints).

### § 3. Asymptotics: the real case

**Definition 3.1.** We say that the comb \( \Pi_J \) (\( \Pi_S \)) is \( n \)-regular (\( \ell \)-regular) if \( \frac{n}{\pi} \omega_k \in \mathbb{Z} \) for all \( k \) (respectively, \( \frac{\ell}{\pi} \omega_k \in \mathbb{Z} \)).
The Chebyshev alternation theorem and more general Markov's correction method allow us to reveal the structure of the extremal polynomial or entire function in the real case. It can then be represented in terms of a conformal mapping on a suitable regular comb domain; see the survey [27].

**Theorem 3.1.** Let \( x \in \mathbb{R} \setminus E_J \) \((x \in \mathbb{R} \setminus E_S)\). For a given \( n \) \((\ell)\) there exists an \( n\)-regular \((\ell\)-regular\) comb \( \widetilde{\Pi}_{J,n,x} \) \((\widetilde{\Pi}_{S,\ell,x})\) such that the extremal function of Problem 1.2 \((\text{Problem } 1.4)\) is given in terms of the corresponding conformal mapping

\[
P_{n,x}(z) = \cos n\tilde{T}_{J,n,x}(z) \quad (F_{\ell,x}(z) = \cos \ell\tilde{T}_{S,\ell,x}(z)) \quad (3.1)
\]

and

\[
A_n(x, E_J) = |P'_{n,x}(x)| \quad (A_\ell(x, E_S) = |F'_{\ell,x}(x)|).
\]

In this case, the set \( \widetilde{E}_{J,n,x} \) \((\widetilde{E}_{S,\ell,x})\), which corresponds to the base of the regular comb, contains the initial set \( E_J \) \((E_S)\) and, on the other hand, for all \( k = 0, \ldots, n \) \((k \in \mathbb{Z}_+)\) the original set contains the preimage of at least one of possibly two different points of the form \( \pi k/n \pm 0 \) \((\pi k/\ell \pm 0)\) on the base of the regular comb.

We will prove a counterpart of this theorem related to Problem 1.3 \((\text{Problem } 1.4)\) \((\text{for the Pell equation approach see [19]}\)). A comb \( \Pi_T \) is called \( n\)-regular if \( \frac{n}{2\pi} \omega_j \in \mathbb{Z} \) for all \( j \), see Figure 1.

**Theorem 3.2.** Let \( e^{ix} \in \mathbb{T} \setminus E_T \). For a given \( n \) there exists an \( n\)-regular comb \( \widetilde{\Pi}_{T,n,e^{ix}} \) such that the extremal function of Problem 1.3 \( \text{is given in terms of the corresponding conformal mapping}\)

\[
P_{n,e^{ix}}(e^{ix}) = e^{inz/2} \cos \frac{n}{2} \tilde{T}_{T,n,e^{ix}}(z)
\]

by \( A_n(e^{ix}, E_T) = |P'_{n,e^{ix}}(e^{ix})|\).

In this case, the set \( \widetilde{E}_{T,n,e^{ix}} \), which corresponds to the base of the regular comb, contains the initial set \( E_T \). On the other hand, at least one of possibly two different points \( e^{i\tilde{T}_{T,n,e^{ix}}^{-1}(2\pi k/\pm 0)} \) \((2\pi k/\pm 0)\) belongs to \( E_T \) for all \( k = 0, \ldots, n - 1 \).

**Proof.** Without loss of generality we assume that \( x = x_0 \in (a_0, b_0) \). Let \( P(\zeta) = P_{n,e^{ix_0}}(\zeta) \) be an extremal polynomial. We represent it in the form

\[
P(e^{iz}) = e^{inz/2} F(z). \quad (3.2)
\]

Clearly \( F(z) \) is a periodic entire function, \( F(z + 2\pi) = (-1)^n F(z) \), and \( F(x_0) = 0 \). Since an extremal polynomial is given up to multiplication by a unimodular constant, we can assume that \( F'(x_0) > 0 \). Then we can substitute \( F(z) \) by its symmetric part \( \frac{1}{2}(F(z) + \overline{F(z)}) \) and we still get an extremal polynomial \( P(e^{iz}) \). Thus, we can assume that \( F(z) \) is real on the real axis in the representation (3.2).

We claim that \( F(z) \) does not have complex zeros, or, equivalently, \( P(\zeta) \) does not have zeros in \( \mathbb{C} \setminus \mathbb{T} \). Indeed, let \( F(z_0) = 0 \), \( \text{Im } z_0 > 0 \). Note that \( F(\overline{z}_0) = 0 \). Consider

\[
Q(\zeta) = P(\zeta) \left(1 - \delta \frac{(\zeta - e^{ix_0})(1 - \zeta e^{-ix_0})}{(\zeta - e^{iz_0})(1 - \zeta e^{-iz_0})}\right), \quad \delta > 0.
\]
Note that $Q(\zeta)$ is a polynomial of degree $n$ such that
\[
Q(e^{ix_0}) = 0 \quad \text{and} \quad Q'(e^{ix_0}) = P'(e^{ix_0}). \tag{3.3}
\]
At the same time, for $\zeta \in \mathbb{T}$
\[
\frac{(\zeta - e^{ix_0})(1 - \zeta e^{-ix_0})}{(\zeta - e^{i\tau_0})(1 - \zeta e^{-i\tau_0})} = \frac{|\zeta - e^{ix_0}|^2}{|\zeta - e^{i\tau_0}|^2},
\]
that is, for a sufficiently small but positive $\delta$ we have
\[
\max_{\zeta \in E_T} |P(\zeta)| > \max_{\zeta \in \bar{E}_T} |Q(\zeta)|. \tag{3.4}
\]
Thus $P(\zeta)$ was not an extremal polynomial, a contradiction.

In a similar way we prove that all zeros of $F(z)$ are simple. Now we prove that between two (necessarily real) consecutive zeros of this function, say $z_1$ and $z_2$, there is a point $y \in (z_1, z_2)$ such that $|F(y)| = 1$ and $e^{iy} \in E_T$. Assuming that on the contrary \(\{e^{iy} \mid y \in (z_1, z_2)\} \cap E_T = \emptyset\), or that $\max_{y \in (z_1, z_2) \cap E_T} |F(y)| < 1$, we define the polynomial
\[
Q(\zeta) = e^{inz/2}G(z), \quad \text{where } G(z) = F(z) \left(1 - \frac{\sin^2((z - x_0)/2)}{\sin((z - z_1)/2) \sin((z - z_2)/2)}\right).
\]
On the period we have to consider three regions: $I_1$ is a union of small neighbourhoods of points $z_1$ and $z_2$; the interval $I_2 = (z_1 + \varepsilon, z_2 - \varepsilon)$; and the remaining set $I_3$. On $I_1$, $|G(z)|$ is strictly less than one if $\delta$ is small. In $I_2$ the factor in brackets is greater than one, but there is no restriction on $|G(y)|$ if no points $e^{iy}$, $y \in I_2$, belong to $E_T$. In the second case, $\max_{z \in I_2} |F(z)|$ is a fixed value, which is less than one. So, a small $\delta > 0$ can be chosen such that the product $|G(z)|$ is still less than one. On the remaining part $I_3$, $\max |G(z)| < 1$ due to the chosen correction factor. Since (3.3) and (3.4) hold, we get a contradiction. Note that the case $z_1 = x_0$ ($z_2 = x_0$) requires special consideration, but it is basically the same.

We can refer to general theorems in [20] and [27], or, having in mind the periodicity of $F(z)$, just to count the number of $\pm 1$ points (including multiplicity) on a period to conclude that all such points are real. Thus
\[
\tilde{\tau}_T(z) = \frac{2}{n} \cos^{-1} F(z), \quad \tilde{\tau}_T(x_0) = \frac{\pi}{n},
\]
is well defined in the upper half-plane. Looking at the boundary behaviour we conclude that this is a conformal mapping on a suitable comb $\Pi_T$. Moreover, this comb is $n$-regular, according to our definition.

Since $|F(z)| \leq 1$ for $e^{iz} \in E_T$ and generally $|F(z)| \leq 1$ if and only if $\tilde{\tau}_T(z) \in \mathbb{R}$, we obtain
\[
E_T \subset \bar{E}_T = \{e^{iz} : \tilde{\tau}_T(z) \in \mathbb{R}\}.
\]
The zeros of $F(z)$ correspond to $z_k$ such that $\tilde{\tau}_T(z_k) = (\pi + 2\pi k)/n$, $k \in \mathbb{Z}$. Therefore, between each consecutive pair $(z_{k-1}, z_k)$ there is a point $e^{iy} \in E_T$ such that $F(y) = \pm 1$. If the boundary of the domain $\Pi_T$ contains a slit with the base
at $\tilde{\omega}_k = 2\pi k/n$ then $y$ corresponds either to the left- or right-hand limit point. Otherwise, this $y$ corresponds to a single point $\tilde{\omega}_k$ on the boundary of $\tilde{\Pi}_T$.

Conversely, if we have an $n$-regular comb $\tilde{\Pi}_T$ and $\tilde{\tau}_T(z)$ is the comb function with the normalization $\tilde{\tau}_T(x_0) = \pi/n$, then

$$F(z) = \cos \frac{n}{2} \tilde{\tau}_T(z)$$

is an analytic function in the upper half-plane and is real valued on the real axis. Being extended by the symmetry principle in the lower half-plane, it represents an entire function of exponential type $n/2$. Also $F(z + 2\pi) = (-1)^n F(z)$. Thus $P(\zeta)$ of the form (3.2) is a polynomial of degree $n$. Every set, which contains one of possibly two different points $e^{i\tilde{\tau}_T(2\pi k/n\pm 0)}$ for all $k = 0, \ldots, n-1$, forms the so called maximal Chebyshev set of alternation; see [27], for example. By the Chebyshev theorem $P(\zeta)$ is an extremal polynomial on an arbitrary $E_T$ containing the given set of alternation.

**Remark 3.1.** The set $\tilde{E}_T = \tilde{E}_{T,n,e^i\tau_0}$ represents an ‘extension’ of the set $E_T$,

$$\tilde{E}_T = E_T \cup \left\{ e^{iz}; \ z \in \bigcup_{j=0}^{n}[u_j,v_j]\right\}, \quad [u_j,v_j] \subset [a_j,b_j].$$

A simple analysis shows that there are the following three possibilities of a proper extension in the gap $(a_j,b_j)$: for a suitable $k_j \in \mathbb{Z}$

(a) $[u_j,v_j]$ is an internal subinterval: there are two slits, $\tilde{h}_{k_j} > 0$ and $\tilde{h}_{k_j+1} > 0$, and

$$\tilde{\tau}_T(a_j) = \frac{2\pi k_j}{n} - 0 \quad \text{and} \quad \tilde{\tau}_T(b_j) = \frac{2\pi (k_j+1)}{n} + 0;$$

(b) a one-sided extension, say, $u_j = a_j$, $v_j < b_j$: there is a slit, $\tilde{h}_{k_j+1} > 0$, and

$$\tilde{\tau}_T(b_j) = \frac{2\pi (k_j+1)}{n} + 0 \left( \tilde{\tau}_T(v_j) = \frac{2\pi (k_j+1)}{n} - 0 \right) \quad \text{and} \quad \tilde{\tau}_T(a_j) \geq \frac{2\pi k_j}{n} + 0;$$

(c) the gap is completely closed, that is, $u_j = a_j$ and $v_j = b_j$:

$$\frac{2\pi k_j}{n} + 0 \leq \tilde{\tau}_T(a_j) < \tilde{\tau}_T(b_j) \leq \frac{2\pi (k_j+1)}{n} - 0.$$

Using (2.6) the harmonic measure at the origin of each additional arc $\{e^{iz}; \ z \in [u_j,v_j]\}$ in $\tilde{D} = \overline{C} \setminus \tilde{E}_T$ is not more than $1/n$. That is, the length of each additional arc tends to zero as $n \to \infty$. Thus, $(E_T$ is fixed and $x_0 \in (a_0,b_0))$ for sufficiently large $n$ case (c) is not possible, and we always have case (a) for the chosen gap $(a_0,b_0)$, since $x_0 \in [u_0,v_0]$.

Now we can pass to the limit in $n$. 

Theorem 3.3. Provided that $z_0 = z(\lambda_0)$ is real, solutions of Problems 1.2–1.4 are given by

\[
\lim_{n \to \infty} \left\{ e^{-nG_{z_0,\alpha}} \left| \frac{dz}{d\lambda} (\lambda_0) \right| A_n(z_0, E_\gamma) - \frac{1}{2} \mathcal{A}(\lambda_0, \alpha_{j,n}) \right\} = 0, \tag{3.5}
\]

\[
\lim_{n \to \infty} \left\{ e^{-nG_{e^{iz_0},\alpha}} \left| \frac{dz}{d\lambda} (\lambda_0) \right| A_n(e^{iz_0}, E_T) - \frac{1}{2} \mathcal{A}(\lambda_0, \alpha_{T,n}) \right\} = 0, \tag{3.6}
\]

\[
\lim_{\ell \to \infty} \left\{ e^{-\ell M(z_0)} \left| \frac{dz}{d\lambda} (\lambda_0) \right| A_\ell(z_0, E_S) - \frac{1}{2} \mathcal{A}(\lambda_0, \alpha_{S,\ell}) \right\} = 0, \tag{3.7}
\]

where $G(z, z_0), G(\zeta, \zeta_0)$ and $M(z)$ are the Green’s and Martin functions in the corresponding domains, and the characters are defined on the generators $\gamma_j \in \Gamma$ by

\[
\alpha_{j,n}(\gamma_j) = e^{2i\omega_j n}, \quad \alpha_{T,n}(\gamma_j) = e^{i\omega_j n} \quad \text{and} \quad \alpha_{S,\ell}(\gamma_j) = e^{2i\omega_j \ell},
\]

with $\omega_j$ corresponding to the combs $\Pi_J, \Pi_T$ and $\Pi_S$ respectively (see Figure 1).

Proof. We give a proof for entire functions, other cases are quite similar. Let $F_{\ell,x_0}(z), \ell > 0, x_0 < 0$, be the extremal function, see Theorem 3.1. Using the compactness of $\Gamma^*$ we choose a convergent sequence

\[
\beta = \beta(\{\ell_k\}) = \lim_{k \to \infty} \alpha_{S,\ell_k}. \tag{3.8}
\]

Using the compactness of the family

\[
\{F_{\ell,x_0}(z(\lambda))e^{i\ell \mathcal{M}(\lambda)}\}_{\ell > 0},
\]

we choose an arbitrary subsequence of $\{\ell_k\}$ (but keep the same notation) so that the following limit exists:

\[
w(\lambda) = w(\lambda; \{\ell_k\}) = \lim_{k \to \infty} F_{\ell_k,x_0}(z(\lambda))e^{i\ell_k \mathcal{M}(\lambda)}. \tag{3.9}
\]

Now, consider the Martin function $\widetilde{M}(z) = \widetilde{M}_{\ell,x_0}(z) = \text{Im} \, \widetilde{\tau}_{S,\ell,x_0}$ of the domain $C \setminus \widetilde{E}_{S,\ell,x_0}$ in the original domain $\mathcal{D}$. It is harmonic in the complement to the additional intervals $\bigcup_{j=0}^{\eta} [u_j, v_j] = \widetilde{E}_{S,\ell,x_0} \setminus E_S$. It is continuous in the whole of $\mathcal{D}$, but its normal derivative has jumps on the union of these intervals. Recall that $\widetilde{\tau}(x) = \widetilde{\tau}_{S,\ell,x_0}(x)$ is real valued on $\widetilde{E}_{S,\ell,x_0}$. By the Cauchy-Riemann equations we have

\[
\frac{\partial \widetilde{M}}{\partial y}(x) = \frac{\partial \widetilde{\tau}(x)}{\partial x}, \quad x \in \widetilde{E}_{S,\ell,x_0} \setminus E_S.
\]

Thus, in terms of the Green’s function $G(z, z_0)$ of $\mathcal{D}$ we obtain

\[
\widetilde{M}_{S,\ell,x_0}(z) = M(z) - \frac{1}{\pi} \int_{\widetilde{E}_{S,\ell,x_0} \setminus E} G(z, x) \, d\widetilde{\tau}_{S,\ell,x_0}(x). \tag{3.10}
\]

According to Remark 3.1, for a sufficiently large $\ell$, each additional interval is of the form (a) or (b) and we have

\[
\widetilde{\tau}_{S,\ell,x_0}(u_j) - \widetilde{\tau}_{S,\ell,x_0}(v_j) \begin{cases} = \frac{\pi}{\ell} & \text{in case (a)}, \\ \leq \frac{\pi}{\ell} & \text{in case (b)}. \end{cases} \tag{3.11}
\]
Also recall that as \( \ell \to \infty \) the system of intervals \([u_0^{(\ell)}, v_0^{(\ell)}]\), which depends on \( \ell \), shrinks to the point \( x_0 \). We again choose a subsequence, keeping the same notation, so that

\[
\lim_{m \to \infty} u_j^{(\ell_k)} = \lim_{m \to \infty} v_j^{(\ell_k)} = x_j, \quad j = 1, \ldots, g,
\]

for some \( x_j \in [a_j, b_j] \). Since \( G(z, x) \) is continuous in \( x \) and \( G(z, a_j) = G(z, b_j) = 0 \), by (3.10) and (3.11), we obtain

\[
\lim_{m \to \infty} \ell_k(\overline{M}_{\ell_k, x_0}(z) - M(z)) = -\sum_{j=0}^{g} G(z, x_j). \quad (3.12)
\]

Now we go back to (3.9). For \( z = z(\lambda) \), by (3.1), we have

\[
|w(\lambda)| = \lim_{k \to \infty} e^{\ell_k(\overline{M}_{\ell_k, x_0}(z) - M(z))} \frac{1 + e^{2i\ell_k(\overline{M}_{\ell_k, x_0}(\lambda))}}{2} = \frac{1}{2} \prod_{j=0}^{g} |b_{x_j}(\lambda)|;
\]

and \( \beta = \prod_{j=0}^{g} \mu_{x_j} \). Since \( \{x_j\} \) with suitable identifications (see Proposition 2.3) corresponds to the extremal function of Problem 2.1 for the given \( \beta \), we conclude that

\[
\lim_{k \to \infty} e^{-\ell_k M(z_0)} \left| \frac{dz}{d\lambda}(\lambda_0) \right| A_{\ell_k}(z_0, E_S) = \frac{1}{2} \mathcal{A}(\lambda_0, \beta)
\]

along the original sequence \( \{\ell_k\} \). Since \( \beta \) is an arbitrary character of the form (3.8), we get (3.7).

§ 4. Making it complex

To complement the proofs given in the previous section, here we discuss the extremal polynomials \( P_n(z) = P_{n, z_0}(z) \) of Problem 1.2. Obviously, \( e^{ibc}P_n(z) \), \( c \in \mathbb{R} \), is also an extremal polynomial. So, a dual setting of the problem is the following: we fix an arbitrary nonzero value of the derivative at \( z_0 \) and look for a polynomial \( \tilde{P}_n(z) \) with the smallest supremum norm \( \|\tilde{P}_n\| \) on \( E_J \). In this case, \( P_n(z) = \tilde{P}_n(z)/\|\tilde{P}_n\| \). Thus, due to the Kolmogorov criterion (see [2], for instance), \( P_n(z) \) is extremal if and only if

\[
\inf_{x \in E_J: \|P_n(x)\|=1} \Re(x - \bar{z}_0)^2 P_n(x) \overline{Q_{n-2}(x)} \leq 0 \quad (4.1)
\]

for an arbitrary polynomial \( Q_{n-2} \) of degree \( n - 2 \).

Let

\[
\frac{z - \overline{z}_0}{z - \bar{z}_0} P_n(z) = \Phi_n(z) + i\Psi_n(z) \quad \text{and} \quad Q_{n-2}(z) = \mathcal{X}_{n-2}(z) + i\mathcal{Y}_{n-2}(z)
\]

be the decompositions of the corresponding polynomials into real and imaginary parts. Then (4.1) is of the form

\[
\inf_{x \in E_J: \Phi_n(x)^2 + \Psi_n(x)^2 = 1} \left\{ \Phi_n(x)\mathcal{X}_{n-2}(x) + \Psi_n(x)\mathcal{Y}_{n-2}(x) \right\} \leq 0.
\]
From the symmetry properties it is enough to solve Problem 1.2 for \( z_0 \) in the upper half-plane \( \mathbb{C}_+ \). Evidently, in this case \( P_n(z) \) has all its zeros, except for \( z_0 \), in the lower half-plane and
\[
\left| \frac{i + \Psi_n(z)/\Phi_n(z)}{i - \Psi_n(z)/\Phi_n(z)} \right| \leq 1, \quad z \in \mathbb{C}_+.
\]
In other words \(-\Psi_n(z)/\Phi_n(z)\) has positive imaginary part in the upper half-plane. Due to the well-known property the zeros of these two polynomials interlace.

**Lemma 4.1.** Suppose that two real polynomials \( \Psi_n(z) \) and \( \Phi_n(z) \) are given, with \(-\text{Im}(\Psi_n(z)/\Phi_n(z)) \geq 0, z \in \mathbb{C}_+, \) and suppose that the set \( X = \{ x \in E_J: \Phi_n(x)^2 + \Psi_n(x)^2 = 1 \} \) coincides with the set of all zeros of \( \Phi_n(x) \). If
\[
\Phi_n(x)^2 + \Psi_n(x)^2 \leq 1, \quad x \in E_J,
\]
then
\[
P_n(z) = \frac{z - z_0}{\overline{z}_0} (\Phi_n(z) + i\Psi_n(z))
\]
is an extremal polynomial of Problem 1.2 for an arbitrary zero \( z_0 \) of the complex polynomial \( \Phi_n(z) + i\Psi_n(z) \).

**Proof.** In this case, using the Kolmogorov criterion we have to check that
\[
\inf_{x \in X} \Psi_n(x) \mathcal{Y}(x) \leq 0
\]
for an arbitrary real polynomial \( \mathcal{Y}(x) \), \( \deg \mathcal{Y} = n - 2 \). Assume that there exists \( \mathcal{Y}(x) \) which violates this property. Since the zeros of \( \Phi_n(z) \) and \( \Psi_n(z) \) interlace and \( \Psi_n(x)\mathcal{Y}(x) > 0 \) for all \( x \in X \), the polynomial \( \mathcal{Y}(z) \) has \( n - 1 \) zeros. That is, \( \mathcal{Y}(z) \) is identically equal to zero, and we get a contradiction.

Let \( \tilde{E}_{J,n} \) be an \( n \)-regular extension for the given set \( E_J \) and let \( \tilde{\tau}_n \) be the corresponding comb-function. We define associated polynomials \( \Phi_n \) and \( \Psi_n \) by
\[
e^{-in\tilde{\tau}_n(z)} = \cos n\tilde{\tau}_n(z) - i \sin n\tilde{\tau}_n(z) = \Psi_n(z) + \sqrt{\prod_{j=0}^{g} \frac{(z - u_j)(z - v_j)}{(z - a_j)(z - b_j)}} \Phi_n(z) \quad (4.2)
\]
and note that their zeros interlace.

**Theorem 4.1.** Assume that for the given extension \( \tilde{E}_{J,n} \)
\[
\tilde{\rho}_n^2 := -\sup_{x \in E_J} \frac{g}{j=0} \frac{(x - u_j)(x - v_j)}{(x - a_j)(x - b_j)} > 0.
\]
Let \( \mathcal{Z}_n(\rho) = \{ z_j \}_{j=0}^{g} \) be the set of points conjugate to the zeros of
\[
p_n(z, \rho) = \rho \Phi_n(z) + i\Psi_n(z), \quad \rho^2 < \tilde{\rho}_n^2. \quad (4.3)
\]
Then
\[
P_{n,z_j}(z) = \frac{z - z_j}{\overline{z}_j} p_n(z, \rho)
\]
is the Ahlfors polynomial with respect to \( z_j \in \mathcal{Z}_n(\rho) \) for the given set \( E_J \).
Remark 4.1. To keep all the zeros of \( p_n(z, \rho) \) defined in (4.3) in the lower half-plane we have to choose \( \rho < 0 \) if \( u_0 > a_0 \) and \( \rho > 0 \) if \( v_0 < b_0 \) (note that the leading coefficient of \( \Phi_n(z) \) is positive, because of the standard choice of the square root at infinity in (4.2)).

Proof of Theorem 4.1. We have

\[
\Psi_n(x)^2 + \rho^2 \Phi_n(x)^2 \leq \Psi_n(x)^2 - \prod_{j=0}^{g} \frac{(x - u_j)(x - v_j)}{(x - a_j)(x - b_j)} \Phi_n(x)^2 = 1
\]

for \( x \in E_J \). Therefore, we can use Lemma 4.1.

Lemma 4.2. Assume that along a subsequence

\[
\lim_{k \to \infty} u_m^{(n_k)} = \lim_{k \to \infty} v_m^{(n_k)} = x_m \in (a_m, b_m), \quad m = 0, \ldots, g.
\]

Let \( \mathcal{X} = \{ x_m \}_{m=0}^{g} \) and

\[
\tilde{\rho}^2 = \rho^2(\mathcal{X}) = - \sup_{x \in E_J} \frac{U_X^2(x)}{T(x)} > 0,
\]

where

\[
T(z) = \prod_{j=0}^{g} (z - a_j)(z - b_j) \quad \text{and} \quad U_X(z) = \prod_{j=0}^{g} (z - x_j).
\]

Then

\[
\lim_{k \to \infty} e^{i n_k \tau_J(z)} p_{n_k}(z, \rho) = \frac{1}{2} \prod_{j=0}^{g} b_{x_j}(\lambda) \left( \rho \frac{\sqrt{T(z)}}{U_X(z)} + i \right)
\]

for \( \rho^2 < \tilde{\rho}^2 \), \( \rho(b_0 - x_0) > 0 \) and \( z = z(\lambda) \).

Proof. We use a counterpart of (3.12),

\[
\lim_{n \to \infty} e^{i n_k \tau_J(z) - \tau_{J,n_k}(z)} = \prod_{j=0}^{g} b_{x_j}(\lambda),
\]

where \( z = z(\lambda) \). Therefore, by definition (4.2),

\[
\lim_{k \to \infty} e^{i n_k \tau(z)} p_{n_k}(z, \rho) = \lim_{k \to \infty} \left( i e^{i n_k (\tau_J(z) - \tau_{J,n_k}(z)))} + e^{i n_k (\tau_J(z) + \tau_{J,n_k}(z))} \right) + \rho \left( \prod_{j=0}^{g} \frac{(z - a_j)(z - b_j)}{(z - u_j^{(n_k)})(z - v_j^{(n_k)})} \right). \]

By (4.4) and (4.7) we get (4.6).

Recall that any character is uniquely defined by its values on a system of free generators, \( \beta_j = \beta(\gamma_j), \ j = 1, \ldots, g \) (see Figure 2). Thus, \( \beta = \prod_{j=0}^{g} \mu_{x_j} \) is equivalent to

\[
\sum_{j=0}^{g} \omega(x_j, E_k) = \beta_k,
\]

for \( x \in E_J \). Therefore, we can use Lemma 4.1.

Proof of Theorem 4.1. We have

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where

\[
T(z) = \prod_{j=0}^{g} (z - a_j)(z - b_j) \quad \text{and} \quad U_X(z) = \prod_{j=0}^{g} (z - x_j).
\]

Then

\[
\lim_{k \to \infty} e^{i n_k \tau_J(z)} p_{n_k}(z, \rho) = \frac{1}{2} \prod_{j=0}^{g} b_{x_j}(\lambda) \left( \rho \frac{\sqrt{T(z)}}{U_X(z)} + i \right)
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for \( \rho^2 < \tilde{\rho}^2 \), \( \rho(b_0 - x_0) > 0 \) and \( z = z(\lambda) \).

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\]

where \( z = z(\lambda) \). Therefore, by definition (4.2),

\[
\lim_{k \to \infty} e^{i n_k \tau(z)} p_{n_k}(z, \rho) = \lim_{k \to \infty} \left( i e^{i n_k (\tau_J(z) - \tau_{J,n_k}(z)))} + e^{i n_k (\tau_J(z) + \tau_{J,n_k}(z))} \right) + \rho \left( \prod_{j=0}^{g} \frac{(z - a_j)(z - b_j)}{(z - u_j^{(n_k)})(z - v_j^{(n_k)})} \right). \]

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\[
\sum_{j=0}^{g} \omega(x_j, E_k) = \beta_k,
\]

for \( x \in E_J \). Therefore, we can use Lemma 4.1.
where \( E_k = E^k_j \) (see (2.5)) and \( \omega(z, E_k) \) is the harmonic measure of \( E_k \) in \( \mathcal{D} \) with respect to \( z \in \mathcal{D} \), and according to Proposition 2.3 the sets \( \mathcal{J} \) and \( \Gamma^* \cong \mathbb{R}^g/\mathbb{Z}^g \) are homeomorphic.

**Theorem 4.2.** Assume that \( \alpha_J \) is in a generic position, that is, \( \text{clos}\{\alpha^n_J\}_{n \in \mathbb{Z}} = \Gamma^* \). Then there is an open set \( \mathcal{V} = \mathcal{V}_1 \times \mathcal{V}_2 \subset \Gamma^* \times \mathbb{C}_+ \) such that

\[
\lim_{k \to \infty} e^{-n_k G(z_0, \infty)} A_{J,n_k}(z_0) = Y(z_0, \beta) := \frac{1}{2 \text{Im} z_0} \exp\left\{ - \sum_{j=0}^{g} G(x_j, z_0) \right\},
\]

where \( \{\beta, z_0\} \in \mathcal{V}, \beta = \lim_{k \to \infty} \alpha^{n_k}_J, \) is related to \( \{x_0, \ldots, x_g, \rho\} \) by (4.8) and

\[
- i\rho = \frac{U_x(z_0)}{\sqrt{T(z_0)}}.
\]

**Proof.** We follow the line of the proof of Theorem 3.3. According to Lemma 4.2, we fix a subsequence \( n_k \) such that \( \alpha^{n_k}_J \to \beta \in \Gamma^* \). Comparing characters in (4.6), we have (4.8). We say that \( x_0 > a_0 \) is regular for the given \( \beta \) if for a solution of the system (4.8) we have \( x_j \in (a_j, b_j) \). If so, we can choose an interval \( I \) around \( x_0 \) such that all the points in this interval are regular (\( x_j \in (a_j, b_j) \) depends continuously on \( x_0 \)). Moreover,

\[
\inf_{x_0 \in I} \rho^2(\{x_j\}_{j=0}^{g}) > 0.
\]

Going back to (4.6), for a sufficiently small \( \rho^2, \tilde{\rho}_* < \rho < 0 \), we have a unique solution \( z_0 = z_0(x_0, \rho) \) of equation (4.10),

\[
z_0 \approx x_0 - i\rho \frac{\sqrt{T(x_0)}}{U_x(x_0)}.
\]

To summarize: for an open set of characters \( \beta \in \mathcal{V}_1 \subset \Gamma^* \) equations (4.8) and (4.10) set up a one-to-one correspondence \( z_0 = z_0(x_0, \rho) \) on an open set \( \mathcal{V}_2 = I \times (\tilde{\rho}_*, 0); \mathcal{V}_2 \subset \mathbb{C}_+ \) is defined as the image of \( \mathcal{V}_2 \).

Finally, we note that for an expression of the form

\[
g(z) = \frac{z - z_0}{z - \overline{z}_0} f(z), \quad f(z_0) = 0,
\]

we have

\[
|g'(z_0)| = \frac{|f(z_0)|}{2 \text{Im} z_0}.
\]

Thus, to get (4.9), we use (4.6) and

\[
\frac{\sqrt{T(z_0)}}{U_x(z_0)} = \frac{\sqrt{T(\overline{z}_0)}}{U_x(\overline{z}_0)} = - \frac{i}{\rho}.
\]

We now represent (4.10) in terms of potential theory.
Lemma 4.3. Let $\omega_{C^+}(z, F)$ be the harmonic measure of $F \subset \mathbb{R}$ at $z \in C^+$ in the upper half-plane. Then (4.10) implies

$$\omega_{C^+} \left( z_0, \bigcup_{j=0}^g [a_j, x_j] \right) = \omega_{C^+} \left( z_0, \bigcup_{j=0}^g [x_j, b_j] \right).$$

(4.11)

Moreover, (4.11) means that $\sqrt{TT(0)} / U_X(z_0)$ assumes a purely imaginary value.

Proof. Due to the integral representation

$$\frac{\sqrt{T(z)}}{U_X(z)} = iC \exp \left\{ \frac{1}{2} \int_{\gamma(a_j, x_j)} \frac{1 + xz}{x - z} dx \right\},$$

the required condition $\arg(\sqrt{T(z_0)} / U_X(z_0)) = \pm \pi/2$ (depending on sign of $C$) corresponds to

$$\int_{\gamma(a_j, x_j)} \frac{\text{Im } z}{|x - z|^2} dx = \int_{\gamma(x_j, b_j)} \frac{\text{Im } z}{|x - z_0|^2} dx,$$

that is, to (4.11).

Theorem 4.2 reduces the asymptotics in the complex polynomial Ahlfors problem to the following generalized Abel-Jacobi inversion problem (cf. §2.3, particularly Proposition 2.3).

Problem 4.1. For fixed $\beta \in \Gamma^*$ and $z \in C^+$ solve the system

$$\sum_{j=0}^g \omega(x_j, E_k) = \beta_k, \quad k = 1, \ldots, g,$$

$$\sum_{j=0}^g \left( \int_{a_j}^{x_j} + \int_{b_j}^{x_j} \right) \frac{dx}{|x - z|^2} = 0,$$

where $x_j \in [a_j, b_j]$.

Remark 4.2. We substitute these $x_j$ in (4.9) to define the function $\Upsilon(\lambda, \beta) = Y(z(\lambda), \beta)|dz/d\lambda|$ responsible for the required asymptotics (it is generally speaking multivalued).

Proposition 4.1. Problem 4.1 is locally solvable.

Proof. We will check that the Jacobian of the system (4.12), (4.13) does not vanish. To the polynomial $U_X(z)$ we associate the polynomial $V_X(z)$ of the form

$$V_X(z) = \sum_{j=0}^g \frac{\sqrt{T(x_j)}}{U_X(x_j)} \frac{U_X(z)}{z - x_j}.$$
By this definition
\[ m_{\chi}(z) := \det \begin{bmatrix} 1 & \ldots & 1 \\ x_0^{g-1} & \ldots & x_g^{g-1} \\ \sqrt{T(x_0)} & \ldots & \sqrt{T(x_g)} \\ z - x_0 & \ldots & z - x_g \end{bmatrix} = \prod_{0 \leq k < j \leq g} (x_k - x_j) \frac{V_{\chi}(z)}{U_{\chi}(z)}. \]

For this reason the Jacobian
\[ \det \begin{bmatrix} 1 & \ldots & 1 \\ x_0^{g-1} & \ldots & x_g^{g-1} \\ \sqrt{T(x_0)} & \ldots & \sqrt{T(x_g)} \\ z - x_0 & \ldots & z - x_g \end{bmatrix} = m_{\chi}(z) - \overline{m_{\chi}(z)} \]
does not vanish for all \( z \in \mathbb{C}_+ \) and \( x_j \neq x_k \) for \( j \neq k \).

**Proposition 4.2.** In the elliptic case, \( g = 1 \), Problem 4.1 is globally solvable, but for some \( \beta \) the solution is not unique.

**Proof.** We map \( \mathcal{D} \) to the complement of the system of two arcs \( E_T \) such that \( z_0 \rightarrow 0 \). Let
\[ E_T = \{ \zeta = e^{ix} : x \in [0, 2\pi) \setminus (a_0, b_0) \cup (a_1, b_1) \}, \quad 0 = a_0 < b_0 < a_1 < b_1 < 2\pi. \]

Since the harmonic measure \( \omega_{\mathbb{C}_+}(z_0, F) \) corresponds to the Lebesgue measure on \( \mathbb{T} \), (4.11) corresponds to
\[ x_0 + x_1 = \frac{b_0 + a_1 + b_1}{2} := c. \quad (4.14) \]

Without loss of generality we assume that \( b_0 > b_1 - a_1 \). As soon as \( x_1 \in (a_1, b_1) \), by (4.14) \( x_0 \) runs in the interval \((\xi_-, \xi_+), \xi_+ = c - a_1, \xi_- = c - b_1 \), and we have
\[ \omega(e^{i\xi_+}, E_2) \leq \omega(e^{i\xi_0}, E_2) + \omega(e^{ix_1}, E_2) \leq 1 + \omega(e^{i\xi_-}, E_2), \quad E_2 = E \setminus E_T^1. \]

Since \( \omega(e^{i\xi_+}, E_2) < \omega(e^{i\xi_-}, E_2) \), (4.8) is solvable for all \( \beta \in \mathbb{R}/\mathbb{Z} \), but in the range \((\omega(e^{i\xi_+}, E_2), \omega(e^{i\xi_-}, E_2)) \) the solution is not unique.

**Remark 4.3.** Numerical experiments show that even in the elliptic case a certain bifurcation in the shape of extremal polynomials may occur. In Figure 4,a we have the system of two arcs on \( \mathbb{T} \). Recall that \( \zeta_0 = 0 \) in this case and the values \( e^{ix_0} \) and \( e^{ix_1} \) are shown in the gaps. We vary the parameter \( \beta \),
\[ \beta = \omega(e^{i(c-x_1)}, E_2) + \omega(e^{ix_1}, E_2) \mod 1 \]
(the vertical axis in Figure 4, b) in the range, where the solution \( x_1 \) (on the horizontal axis) of Problem 4.1 is unique. At a certain moment the value \( \rho \) violates the condition \( \rho^2 \leq \tilde{\rho}^2 \); see (4.5) and Figure 4, c, where we plot the function \( T(\zeta)/U^2_{\chi}(\zeta), \zeta \in E_T \) (the horizontal line corresponds to the level \(-1/\rho^2\)).
Figure 4. a) the position of $e^{ix_0}$ and $e^{ix_1}$ in the complex plane; b) the dependence of $\beta$ on $e^{ix_1}$; c) the graph of $T(\zeta)/U_X(\zeta)$.

Remark 4.4. As was demonstrated in Remark 4.3 our function $Y(z, \beta)$ describes the asymptotics (4.9) for Problems 1.2–1.4 only in a certain neighbourhood of the real axis. Note, however, that an analytic reproducing kernel is uniquely defined by its diagonal

$$k(\lambda_1, \lambda_2) = \sum \frac{1}{n! \cdot m!} (\partial^n \overline{\partial}^m k)(\lambda_0, \lambda_0)(\lambda_1 - \lambda_0)^n(\overline{\lambda}_2 - \overline{\lambda}_0)^m.$$ (4.15)

Therefore, if Conjecture 1.1 is correct, our Theorem 4.2, in fact, gives a complete information on the asymptotics due to analytic continuation. Indeed, Theorem 3.3 provides asymptotics on the whole real axis in the $z$-plane, $\Upsilon(\lambda, \beta) = \overline{A}^{1/2}(\lambda, \beta)$, $z(\lambda) \in \mathbb{R}$. To use (4.15) it is enough to have an extension of $\overline{\mathcal{A}}^{1/2}(\lambda, \beta)$ given by $\Upsilon(\lambda, \beta)$ in an arbitrary small neighbourhood, $z(\lambda) \in V_2 \subset \mathbb{C}_+$,

$$k^{\alpha(\beta, \lambda_0)}(\lambda_1, \lambda_2) = \sum \frac{1}{n! \cdot m!} (\partial^n \overline{\partial}^m \Upsilon)(\lambda_0, \beta)(\lambda_1 - \lambda_0)^n(\overline{\lambda}_2 - \overline{\lambda}_0)^m, \quad z(\lambda_0) \in \mathbb{R}.$$

Recall condition (1.2) with respect to these partial derivatives.

Proposition 4.3. For simply connected domains Problem 4.1 is uniquely solvable. Moreover,

$$\Upsilon(\lambda) = Y(z(\lambda)) \left| \frac{dz}{d\lambda} \right| = \frac{2\sqrt{\lambda} \sqrt{\overline{\lambda}}}{(\lambda + \overline{\lambda})(\sqrt{\lambda} + \sqrt{\overline{\lambda}})^2}. \quad (4.16)$$

Proof of Proposition 4.3 and Theorem 1.1. Let $\mathcal{D} = \mathbb{C} \setminus \mathbb{R}_+$. In this case

$$\frac{Ux_0(z_0)}{\sqrt{T(z_0)}} = \frac{-z_0 + x_0}{\sqrt{-z_0}} = -i\rho \quad \Rightarrow \quad \text{Re} \sqrt{-z_0} + x_0 \frac{\text{Re} \sqrt{-z_0}}{|z_0|} = 0,$$

thus we get a unique $x_0 = -|z_0|$. By (4.9), we have

$$Y(z_0) = \frac{1}{|z_0 - \overline{z_0}|} \left| \frac{\sqrt{-z_0} - \sqrt{|z_0|}}{\sqrt{-z_0} + \sqrt{|z_0|}} \right|.$$
We consider the right half-plane as the universal covering. Let $z_0 = -\lambda^2$. Then

$$Y(z_0) = \left| \frac{1}{(\lambda - \overline{\lambda})(\lambda + \overline{\lambda})} \frac{\lambda - |\lambda|}{\lambda + |\lambda|} \right| = \frac{1}{(\lambda + \overline{\lambda})(\sqrt{\lambda} + \sqrt{\lambda})^2}.$$ 

Since $|z'(\lambda)| = 2|\lambda|$, we get (4.16).

Since this solution is global, the arguments in the proof of Theorem 4.2 are applicable to all $z_0 \in \mathbb{C}_+$ and we have (1.1).

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