Towards a Direct Method for the Analyticity of
the Pressure for Certain Classical Unbounded
Spin Systems

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Abstract
The aim of this paper is to study direct methods for the analyticity
of the pressure for certain classical unbounded spin models. We provide
a representation in terms of the Witten Laplacian on one-forms of the
nth-derivative of the pressure as function of some order parameter t. The
technique involves the formula for the covariance introduced by B. Helffer
and J. Sjostrand.

1 Introduction
As already mentioned in [66], The methods for investigating critical phenomena
for certain physical systems took an interesting direction when powerful and so-
phisticated PDE techniques were introduced. The methods are generally based
on the analysis of suitable differential operators

\[ W^{(0)}_\Phi = \left( -\Delta + \frac{|\nabla \Phi|^2}{4} - \frac{\Delta \Phi}{2} \right) \]

and

\[ W^{(1)}_\Phi = -\Delta + \frac{|\nabla \Phi|^2}{4} - \frac{\Delta \Phi}{2} + \text{Hess}\Phi. \]

These are in some sense deformations of the standard Laplace Beltrami operator. They are commonly called Witten Laplacians, and were first introduced
by Edward Witten, [18] in 1982 in the context of Morse theory for the study
of topological invariants of compact Riemannian manifolds. In 1994, Bernard
Helffer and Johannes Sjöstrand [8] introduced two elliptic differential operators

\[ A^{(0)}_\Phi := -\Delta + \nabla \Phi \cdot \nabla \]

and

\[ A^{(1)}_\Phi := -\Delta + \nabla \Phi \cdot \nabla + \text{Hess}\Phi \]
sometimes called Helffer-Sjöstrand operators serving to get direct method for
the study of integrals and operators in high dimensions of the type that ap-
pear in Statistical Mechanics and Euclidean Field Theory. In 1996, Johannes
Sjöstrand [13] observed that these so-called Helffer-Sjöstrand operators were in
fact equivalent to Witten’s Laplacians. Since then, there have been significant
advances in the use of these Laplacians to study the thermodynamic behavior
of quantities related to the Gibbs measure $Z^{-1}e^{-\Phi}dx$.

Numerous techniques have been developed in the study of integrals asso-
ciated with the equilibrium Gibbs state for certain unbounded spins systems.
One of the most striking results is an exact formula for the covariance of two
functions in terms of the Witten Laplacian on one forms leading to sophisti-
cated methods for estimating the correlation functions of a random field. As
mentioned in [10], this formula is in some sense a stronger and more flexible
version of the Brascamp-Lieb inequality [1]. The formula may be written as
follow:

$$\text{cov}(f, g) = \int \left( A^{(1)}_{\Phi} - 1 \nabla f \cdot \nabla g \right) e^{-\Phi(x)} dx.$$  \hspace{1cm} (1)

We attempt in these notes, to study a direct method for the analyticity of
the pressure for certain classical convex unbounded spin systems. It is central
in Statistical Mechanics to study the differentiability or even the analyticity
of the pressure with respect to some distinguished thermodynamic parameters
such as temperature, chemical potential or external field. In fact the analytic
behavior of the pressure is the classical thermodynamic indicator for the absence
or existence of phase transition. The most famous result on the analyticity of
the pressure is the circle theorem of Lee and Yang [28]. This theorem asserts
the following: consider a $\{-1, 1\}$-valued spin system with ferromagnetic pair
interaction and external field $h$ and regard the quantity $z = e^h$ as a complex
parameter, then all zeroes of all partition functions (with free boundary condi-
tion), considered as functions of $z$ lie in the complex unit circle. This theorem
readily implies that the pressure is an analytic function of $h$ in the region $h > 0$
and $h < 0$. Heilmann [29] showed that the assumption of pair interaction is
necessary. A transparent approach to the circle theorem was found by Asano
[30] and developed further by Ruelle [31],[32], Slawny [33], and Gruber et al
[34]. Griffiths [35] and Griffiths-Simon [36] found a method of extending the
Lee-Yang theorem to real-valued spin systems with a particular type of a priory
measure. Newman [37] proved the Lee-Yang theorem for every a priory mea-
sure which satisfies this theorem in the particular case of no interaction. Dunlop
[38],[39] studied the zeroes of the partition functions for the plane rotor model.
A general Lee-Yang theorem for multicomponent systems was finally proved by
Lieb and Sokal [40]. For further references see Glimm and Jaffe [41].

The Lee-Yang theorem and its variants depend on the ferromagnetic character
of the interaction. There are various other way of proving the infinite differ-
entiability or the analyticity of the pressure for (ferromagnetic and non ferro-
magnetic) systems at high temperatures, or at low temperatures, or at large
external fields. Most of these take advantage of a sufficiently rapid decay of cor-
relations and /or cluster expansion methods. Here is a small sample of relevant
references. Bricmont, Lebowitz and Pfister [42], Dobroshin [43], Dobroshin and Sholsman [44],[45], Dumeau et al [46],[47],[48], Glimm and Jaffe [41],[49], Israel [50], Kotecky and Preiss [51], Kunz [52], Lebowitz [53],[54], Malyshev [55], Malyshev and Milnos [56] and Prakash [57]. M. Kac and J.M. Luttinger [58] obtained a formula for the pressure in terms of irreducible distribution functions.

In this present study, we propose a new way of analyzing the analyticity of the pressure for certain unbounded models through a representation by means of the Witten Laplacians of the remainder of the Taylor series expansion. The methods known up to now rely on complicated indirect arguments.

2 Towards the analyticity of the Pressure

Let Λ be a finite domain in \( \mathbb{Z}^d \) \( (d \geq 1) \) and consider the Hamiltonian of the phase space given by,

\[
\Phi(x) = \Phi_\Lambda(x) = \frac{x^2}{2} + \Psi(x), \quad x \in \mathbb{R}^\Lambda. \tag{2}
\]

where

\[
|\partial^\alpha \nabla \Psi| \leq C_\alpha, \quad \forall \alpha \in \mathbb{N}^{\Lambda}, \tag{3}
\]

\[
\text{Hess}\Phi(x) \geq \delta_0, \quad 0 < \delta_0 < 1. \tag{4}
\]

Let \( g \) be a smooth function on \( \mathbb{R}^\Gamma \) with lattice support \( S_g = \Gamma \). We identified \( \tilde{g} \) defined on \( \mathbb{R}^\Lambda \) by

\[
\tilde{g}(x) = g(x_\Gamma) \quad \text{where } x = (x_i)_{i \in \Lambda} \text{ and } x_\Gamma = (x_i)_{i \in \Gamma} \tag{5}
\]

and satisfying

\[
|\partial^\alpha \nabla g| \leq C_\alpha \quad \forall \alpha \in \mathbb{N}^{\Gamma}. \tag{6}
\]

Under the additional assumptions that \( \Psi \) is compactly supported in \( \mathbb{R}^\Lambda \) and \( g \) is compactly supported in \( \mathbb{R}^\Gamma \), it was proved in [66] (see also [8]) that the equation

\[
\begin{cases}
-\Delta f + \nabla \Phi \cdot \nabla f = g - \langle g \rangle \\
(f)_{L^2(\mu)} = 0
\end{cases}
\]

has a unique smooth solution satisfying \( \nabla^k f(x) \to 0 \) as \( |x| \to \infty \) for every \( k \geq 1 \).

Recall also that \( \nabla f \) is a solution of the system

\[
(-\Delta + \nabla \Phi \cdot \nabla) \nabla f + \text{Hess}\Phi \nabla f = \nabla g \quad \text{in } \mathbb{R}^\Lambda. \tag{7}
\]

As in [66] and [8], these assumptions will be relaxed later on.

Let

\[
\Phi^t_\Lambda(x) = \Phi(x) - tg(x), \tag{8}
\]

where \( x = (x_i)_{i \in \Lambda} \), and assume additionally that \( g \) satisfies

\[
\text{Hess}g \leq C. \tag{9}
\]

3
We consider the following perturbation
\[ \theta_{\Lambda}(t) = \log \left( \int dx e^{-\Phi_{\Lambda}(x)} \right). \] (10)

Denote by
\[ Z_t = \int dx e^{-\Phi_t(\Lambda)(x)} \] (11)
and
\[ \langle \cdot \rangle_{t,\Lambda} = \frac{\int \cdot dx e^{-\Phi_{\Lambda}(x)}}{Z_t}. \] (12)

3 Parameter Dependency of the Solution

From the assumptions made on \( \Phi \) and \( g \), it is easy to see that there exists \( T > 0 \) such that or every \( t \in [0, T) \), \( \Phi^t_{\Lambda}(x) \) satisfies all the assumptions required for the solvability, regularity and asymptotic behavior of the solution \( f(t) \) associated with the potential \( \Phi^t_{\Lambda}(x) \). Thus, each \( t \in [0, T) \) is associated with a unique \( C^\infty \)-solution, \( f(t) \) of the equation
\[
\begin{cases}
A^{(0)}_{\Phi^t_{\Lambda}} f(t) = g - \langle g \rangle_{L^2(\mu)} \\
\langle f(t) \rangle_{L^2(\mu)} = 0.
\end{cases}
\]

Hence,
\[ A^{(1)}_{\Phi^t_{\Lambda}} v(t) = \nabla g \] (13)
where \( v(t) = \nabla f(t) \). Notice that the map
\[ t \mapsto v(t) \]
is well defined and
\[ \{ v(t) : t \in [0, T) \} \]
is a family of smooth solutions on \( \mathbb{R}^\Lambda \) satisfying
\[ \partial^\alpha v(t) \to 0 \quad \text{as} \quad |x| \to \infty \quad \forall \alpha \in \mathbb{N}^{||\Lambda||} \quad \text{and for each} \quad t \in [0, T) \]
and corresponding to the family of potential
\[ \{ \Phi^t_{\Lambda} : t \in [0, T) \} . \] (14)

Let us now verify that \( v \) is a smooth function of \( t \in (0, T) \). We need to prove that for each \( t \in (0, T) \), the limit
\[ \lim_{\varepsilon \to 0} \frac{v(t + \varepsilon) - v(t)}{\varepsilon} \]
exists. Let
\[ v^\varepsilon(t) = \frac{v(t + \varepsilon) - v(t)}{\varepsilon}. \]
We use a technique based on regularity estimates to get a uniform control of \( v^\varepsilon(t) \) with respect to \( \varepsilon \).

With \( \varepsilon \) small enough, we have

\[
0 = -\Delta \left( \frac{v(t+\varepsilon) - v(t)}{\varepsilon} \right) + \frac{\nabla \Phi^t \cdot \nabla v(t+\varepsilon) - \nabla \Phi^t \cdot \nabla v(t)}{\varepsilon} + \frac{\text{Hess} \Phi^t + \varepsilon v(t+\varepsilon) - \text{Hess} \Phi^t v(t)}{\varepsilon}.
\]

Equivalently,

\[
-\Delta \left( \frac{v(t+\varepsilon) - v(t)}{\varepsilon} \right) + \frac{\nabla \Phi^t \cdot \nabla [v(t+\varepsilon) - v(t)]}{\varepsilon} + \text{Hess} \Phi^t + \varepsilon \left( \frac{v(t+\varepsilon) - v(t)}{\varepsilon} \right)
= - \left( \frac{\text{Hess} \Phi^t + \varepsilon - \text{Hess} \Phi^t}{\varepsilon} \right) v(t) - \left( \frac{\nabla \Phi^t + \varepsilon - \nabla \Phi^t}{\varepsilon} \right) \cdot \nabla v(t)
\]
and

\[
-\Delta v^\varepsilon(t) + \nabla \Phi^t + \varepsilon \cdot \nabla v^\varepsilon(t) + \text{Hess} \Phi^t + \varepsilon v^\varepsilon(t)
= - \left( \frac{\text{Hess} \Phi^t + \varepsilon - \text{Hess} \Phi^t}{\varepsilon} \right) v(t) - \left( \frac{\nabla \Phi^t + \varepsilon - \nabla \Phi^t}{\varepsilon} \right) \cdot \nabla v(t).
\]

Let \( w(t) \) be the unique \( C^\infty \)--solution of the system

\[
-\Delta w(t) + \nabla \Phi^t \cdot \nabla w(t) + \text{Hess} \Phi^t w(t) = \text{Hess} g(v(t)) - \nabla g \cdot \nabla v(t).
\] (15)

Combining the last two systems above, we get

\[
-\Delta \left[ w(t) - v^\varepsilon(t) \right] + \nabla \Phi^t \cdot \nabla \left[ w(t) - v^\varepsilon(t) \right] + \text{Hess} \Phi^t \left[ w(t) - v^\varepsilon(t) \right]
= \text{Hess} g(v(t)) - \nabla g \cdot \nabla v(t) + \left( \frac{\text{Hess} \Phi^t + \varepsilon - \text{Hess} \Phi^t}{\varepsilon} \right) v(t)
+ \left( \frac{\nabla \Phi^t + \varepsilon - \nabla \Phi^t}{\varepsilon} \right) \cdot \nabla v(t) + (\text{Hess} \Phi^t + \varepsilon - \text{Hess} \Phi^t) v^\varepsilon(t).
\] (16)

Now using the unitary transformation \( U_{\Phi^t} \), we get

\[
\left( -\Delta + \frac{\nabla \Phi^t}{4} + \frac{\Delta \Phi^t}{2} \right) \left( w(t) - v^\varepsilon(t) \right) e^{-\Phi^t/2}
+ \text{Hess} \Phi^t \left( w(t) - v^\varepsilon(t) \right) e^{-\Phi^t/2}
= o_\varepsilon(1)e^{-\Phi^t/2} + [(\nabla \Phi^t + \varepsilon - \nabla \Phi^t) \cdot \nabla v^\varepsilon(t)
+ (\text{Hess} \Phi^t + \varepsilon - \text{Hess} \Phi^t) v^\varepsilon(t)]e^{-\Phi^t/2}
\] (17)

Next, we propose to estimate the last two terms of the right hand side of this equation.

5
Again using the unitary transformation \( U_{\Phi, t} \), we reduce the system

\[-\Delta v^\varepsilon(t) + \nabla \Phi^{t+\varepsilon} \cdot \nabla v^\varepsilon(t) + \text{Hess} \Phi^{t+\varepsilon} v^\varepsilon(t)\]

\[-\Delta v^\varepsilon(t) + \nabla \Phi^t \cdot \nabla v^\varepsilon(t)\]

into

\[( -\Delta + \frac{|\nabla \Phi^{t+\varepsilon}|^2}{4} - \frac{\Delta \Phi^{t+\varepsilon}}{2} ) v^\varepsilon + \text{Hess} \Phi^{t+\varepsilon} v^\varepsilon =
- \left( \frac{\text{Hess} \Phi^{t+\varepsilon} - \text{Hess} \Phi^t}{\varepsilon} \right) v(t) e^{-\Phi^{t+\varepsilon}/2}
- \left( \frac{\nabla \Phi^{t+\varepsilon} - \nabla \Phi^t}{\varepsilon} \right) \cdot \nabla v(t) e^{-\Phi^{t+\varepsilon}/2}\]

where \( v^\varepsilon = v^\varepsilon(t) e^{-\Phi^{t+\varepsilon}/2} \). Taking scalar product with \( v^\varepsilon \) on both sides of this last equality and integrating, we get

\[
\left\| \partial_t + \frac{\nabla \Phi^{t+\varepsilon}}{2} \right\|^2_{L^2} + \int \text{Hess} \Phi^{t+\varepsilon} v^\varepsilon \cdot v^\varepsilon dx =
- \int \left[ \left( \frac{\text{Hess} \Phi^{t+\varepsilon} - \text{Hess} \Phi^t}{\varepsilon} \right) v(t) e^{-\Phi^{t+\varepsilon}/2} \right] \cdot v^\varepsilon dx
- \int \left[ \frac{\nabla \Phi^{t+\varepsilon} - \nabla \Phi^t}{\varepsilon} \right] \cdot \nabla v(t) e^{-\Phi^{t+\varepsilon}/2} \cdot v^\varepsilon dx\]

Now using the uniform strict convexity on the left hand side and Cauchy-Schwarz on the right hand side, we obtain

\[
\| v^\varepsilon \|_{B^0} \leq C \quad \text{for small enough } \varepsilon. \quad (21)
\]

We then deduce that

\[
\left( -\Delta + \frac{|\nabla \Phi^{t+\varepsilon}|^2}{4} \right) v^\varepsilon = \tilde{q}_\varepsilon\]

where

\[
\tilde{q}_\varepsilon = - \left( \frac{\text{Hess} \Phi^{t+\varepsilon} - \text{Hess} \Phi^t}{\varepsilon} \right) v(t) e^{-\Phi^{t+\varepsilon}/2} - \left( \frac{\nabla \Phi^{t+\varepsilon} - \nabla \Phi^t}{\varepsilon} \right) \cdot \nabla v(t) e^{-\Phi^{t+\varepsilon}/2}
+ \frac{\Delta \Phi^{t+\varepsilon}}{2} V^\varepsilon - \text{Hess} \Phi^{t+\varepsilon} V^\varepsilon\]

is bounded in \( B^\theta \) uniformly with respect to \( \varepsilon \) for \( \varepsilon \) small enough.

Now taking scalar product with \( v^\varepsilon \) on both sides of (28) and integrating by parts, we obtain

\[
\| \nabla v^\varepsilon \|_{L^2}^2 + \left\| \frac{\nabla \Phi^{t+\varepsilon}}{2} \cdot V^\varepsilon \right\|_{L^2}^2 \leq \| \tilde{q}_\varepsilon \|_{L^2} \| V^\varepsilon \|_{L^2} \quad (24)
\]
It follows that $V^\varepsilon$ is uniformly bounded with respect to $\varepsilon$ in $B_{\Phi t}^1$ for $\varepsilon$ small enough.

Next, observe that
\[
\left( -\Delta + \frac{\nabla \Phi t^2}{4} \right) V^\varepsilon = \hat{q}_\varepsilon 
\] (25)

where
\[
\hat{q}_\varepsilon = \tilde{q}_\varepsilon - \frac{\nabla \Phi t^2 - \nabla \Phi t^2}{4} V^\varepsilon + \frac{\nabla \Phi t^2 - \nabla \Phi t^2}{2} \cdot \nabla \Phi t^2 V^\varepsilon
\] (26)
is uniformly bounded in $B^0$ with respect to $\varepsilon$ for small enough $\varepsilon$. Using regularity, it follows that for small enough $\varepsilon$, $V^\varepsilon$ is uniformly bounded in $B_{\Phi t}^2$ with respect to $\varepsilon$. This implies that $\hat{q}_\varepsilon$ is uniformly bounded in $B_{\Phi t}^1$ for $\varepsilon$ small enough.

It is then clear that for small enough $\varepsilon$, the right hand side of (23) is $O(\varepsilon)$ in $B^0$ and consequently, using the same argument as above, we get that for small enough $\varepsilon$, $V^\varepsilon$ is uniformly bounded in $B_{\Phi t}^k$ for any $k$.

Again, we can continue by a bootstrap argument to consequently get that for small enough $\varepsilon$, $V^\varepsilon$ is uniformly bounded in $B_{\Phi t}^2$ with respect to $\varepsilon$.

Proposition 1 Under the above on $\Phi$ and $g$, there exists $T > 0$ so that for each $t \in (0, T)$, $v^\varepsilon(t)$ converges to $w(t)$ in $C^\infty$.

Remark 2 The proposition establishes that $v(t)$ is differentiable in $t$ and $\frac{d}{dt} v(t)$ is given by the unique $C^\infty$ solution $w(t)$ of the system

\[-\Delta w(t) + \nabla \Phi t \cdot \nabla w(t) + \text{Hess} \Phi \cdot w(t) = \text{Hess} g(t) - \nabla g \cdot \nabla v(t).\] (27)

Iterating this argument, we easily get that, $v(t)$ is smooth in $t \in (0, T)$.

Now we are ready for the following:

4 Formula for $\theta^{(n)}(t)$

For an arbitrary suitable function $f(t) = f(t, w)$
\[
\frac{\partial}{\partial t} < f(t) >_{t, \Lambda} = < f'(t) >_{t, \Lambda} + \text{cov}(f, g). \] (28)

Hence,
\[
\frac{\partial}{\partial t} < f(t) >_{t, \Lambda} = < f'(t) >_{t, \Lambda} + A^{(1)}_{g_f} (\nabla f) \cdot \nabla g_{t, \Lambda}. \] (29)

Let
\[
A_g f := A^{(1)}_{g_f} (\nabla f) \cdot \nabla g. \] (30)
Thus,
\[
\frac{\partial}{\partial t} < f(t) >_{t, \Lambda} = < \left( \frac{\partial}{\partial t} + A_g \right) f >_{t, \Lambda}. \tag{31}
\]

The linear operator \( \frac{\partial}{\partial t} + A_g \) will be denoted by \( H_g \).

\[
\theta'_\Lambda(t) = < g >_{t, \Lambda} = < \left( \frac{\partial}{\partial t} + A_g \right)^0 g >_{t, \Lambda} = < H_g^0 g >_{t, \Lambda};
\]

\[
\theta''_\Lambda(t) = \frac{\partial}{\partial t} < g >_{t, \Lambda} = < A_g^{-1} (\nabla g) \cdot \nabla g >_{t, \Lambda} = < \left( \frac{\partial}{\partial t} + A_g \right) g >_{t, \Lambda}.
\]

\[
\theta'''_\Lambda(t) = \frac{\partial}{\partial t} < A_g^{-1} (\nabla g) \cdot \nabla g >_{t, \Lambda} = < \left( \frac{\partial}{\partial t} + A_g \right) A_g^{-1} g >_{t, \Lambda}.
\]

By induction it is easy to see that
\[
\theta^{(n)}_\Lambda(t) = < \left( \frac{\partial}{\partial t} + A_g \right)^{n-1} g >_{t, \Lambda} \quad (\forall n \geq 1).
\]

Next, we propose to find a simpler formula for \( \theta^{(n)}_\Lambda(t) \) that only involves \( A_g \).

\[
H_g g = A_g^{-1} (\nabla g) \cdot \nabla g = A_g g.
\]

\[
H^2_g g = \frac{\partial}{\partial t} \nabla f \cdot \nabla g + \left( A_g^{-1} \nabla \left( A_g^{-1} \nabla g \right) \cdot \nabla g \right) \cdot \nabla g. \tag{32}
\]

where \( f \) satisfies the equation
\[
\nabla f = A_g^{-1} (\nabla g). \tag{33}
\]
With $v(t) = \nabla f$, as before, we get
\[
\frac{\partial}{\partial t} \nabla f \cdot \nabla g = A^{(1)}_{q^t}^{-1} (\text{Hess} g v(t) - \nabla g \cdot \nabla v(t)) \cdot \nabla g
\]
and $H^2_g$ becomes
\[
H^2_g = A^{(1)}_{q^t}^{-1} \left[ (\text{Hess} g v(t) - \nabla g \cdot \nabla v(t)) + \nabla \left( A^{(1)}_{q^t}^{-1} (\nabla g) \cdot \nabla g \right) \right] \cdot \nabla g
= A^{(1)}_{q^t}^{-1} 2 \nabla (A_g g) \cdot \nabla g
= 2 A^2_g g.
\]

**Proposition 3** If

\[
\theta_{\Lambda}(t) = \log \left[ \int dx e^{-\Phi'(x)} \right]
\]

where
\[
\Phi'(x) = \Phi_{\Lambda}(x) - tg(x)
\]
is as above then $\theta_{\Lambda}^{(n)}(t)$, the $n$th derivative of $\theta_{\Lambda}(t)$ is given by the formula
\[
\theta_{\Lambda}'(t) = < g >_{t,\Lambda},
\]
and for $n \geq 1$
\[
\theta_{\Lambda}^{(n)}(t) = (n-1)! < A^{-1}_g g >_{t,\Lambda}.
\]

**Proof.** We have already established that
\[
\theta_{\Lambda}^{(n)}(t) = < H^{-1}_g g >_{t,\Lambda} \quad \text{for} \quad n \geq 1.
\]

It then only remains to prove that
\[
H^{-1}_g g = (n-1)! A^{-1}_g g \quad \text{for} \quad n \geq 1.
\]
The result is already established above for $n = 1, 2, 3, \ldots$ By induction, assume that
\[
H^{-1}_g g = (n-1)! A^{-1}_g g.
\]
if $n$ is replaced by $n \leq n$.
\[
H^n_g g = \left( \frac{\partial}{\partial t} + A_g \right) \left( (n-1)! A^{-1}_g g \right)
= (n-1)! \left( \frac{\partial}{\partial t} A^{-1}_g g + A^n_g g \right).
\]

Now
\[
A^{-1}_g g = \left[ A^{(1)}_{q^t}^{-1} \nabla (A^{-2}_g g) \right] \cdot \nabla g
= \nabla \varphi_n \cdot \nabla g
\]
where
\[ \nabla \varphi_n = \left[ A_{q,t}^{-1} \nabla (A_g^{n-2} g) \right]. \]

We obtain,
\[ \frac{\partial}{\partial t} \nabla \varphi_n = A_{q,t}^{-1} \left( \frac{\partial}{\partial t} \nabla A_g^{n-2} g + \text{Hess}_g \nabla \varphi_n - \nabla g \cdot \nabla (\nabla \varphi_n) \right). \]

We then have
\[ \frac{\partial}{\partial t} A_g^{n-1} g = \frac{\partial}{\partial t} \nabla \varphi_n \cdot \nabla g \]
\[ = \left[ A_{q,t}^{-1} \left( \frac{\partial}{\partial t} \nabla A_g^{n-2} g + \text{Hess}_g \nabla \varphi_n - \nabla g \cdot \nabla (\nabla \varphi_n) \right) \right] \cdot \nabla g \]
\[ = A_g \left[ \frac{\partial}{\partial t} A_g^{n-2} g + A_g (A_g^{n-2} g) \right] \]
\[ = A_g H_g (A_g^{n-2} g). \]
\[ = A_g H_g \left( \frac{1}{(n-2)!} H_g^{(n-2)} g \right) \quad \text{(from the induction hypothesis)} \]
\[ = \frac{1}{(n-2)!} A_g H_g^{(n-1)} g \]
\[ = \frac{1}{(n-2)!} A_g ((n-1)! A_g^{n-1} g) \quad \text{(still by the induction hypothesis)} \]
\[ = (n-1) A_g^n g. \]

Thus,
\[ H_g^n g = (n-1)! (n-1 + 1) A_g^n g \]
\[ = n! A_g^n g. \]

\[ \textbf{Proposition 4} \] If \( g(0) = 0 \), then the formula
\[ \theta_{\Lambda}^{(n)}(t) = (n-1)! < A_g^{n-1} g >_{t,\Lambda}, \quad n \geq 2 \]
still holds if we no longer require \( \Psi \) and \( g \) to be compactly supported in \( \mathbb{R}^\Lambda \).

\[ \textbf{Proof.} \] As in [8], consider the family cutoff functions
\[ \chi = \chi_\varepsilon \]
\[ (\varepsilon \in [0,1]) \text{ in } C_\infty(\mathbb{R}) \text{ with value in } [0,1] \text{ such that} \]
\[ \left\{ \begin{array}{ll}
\chi^{(k)}(t) \leq C_k \frac{\varepsilon}{|t|^k} & \text{for } |t| \leq \varepsilon^{-1} \\
\chi = 1 & \text{for } k \in \mathbb{N}
\end{array} \right. \]
We could take for instance
\[ \chi_\varepsilon(t) = f(\varepsilon \ln |t|) \]
for a suitable \( f \).

We then introduce
\[ \Psi_\varepsilon(x) = \chi_\varepsilon(|x|)\Psi, \quad x \in \mathbb{R}^\Lambda \]
and
\[ g_\varepsilon(x) = \chi_\varepsilon(|x|)g \quad x \in \mathbb{R}^\Gamma \]
One can check that both \( \Psi_\varepsilon(x) \) and \( g_\varepsilon(x) \) satisfies the assumptions made above on \( \Psi \) and \( g \).

Now consider the equation
\[ -\Delta f_\varepsilon + \nabla \Phi_t \cdot \nabla f_\varepsilon = g_\varepsilon - < g_\varepsilon >_{t, \Lambda}. \]
which implies
\[ (-\Delta + \nabla \Phi_t \cdot \nabla) \otimes v_\varepsilon + \text{Hess} \Phi_t^\varepsilon v_\varepsilon = \nabla g_\varepsilon \]
where
\[ v_\varepsilon = \nabla f_\varepsilon \]
It was proved in [8] that \( v_\varepsilon = A_{\Phi_t}^{(1)^{-1}} \nabla g_\varepsilon \) converges in \( C^\infty \) to \( A_{\Phi_t}^{(1)^{-1}} \nabla g \) as \( \varepsilon \to 0 \).

**Remark 5** If we denote by \( R_n \) the remainder of the Taylor series expansion of the pressure \( P_\Lambda(t) \), given by
\[ P_\Lambda(t) = \frac{\theta_\Lambda(t)}{\Lambda} \]
we have
\[ R_n = \frac{P_\Lambda^{(n+1)}(t_n)}{(n+1)!} \]
\[ = \frac{< A_{\Phi_t}^n g >_{t, \Lambda}}{(n+1)\Lambda} \bigg|_{t=t_n}. \]
If \( \Phi \) and \( g \) are such that \( < A_{\Phi_t}^n g >_{t, \Lambda} \) is uniformly bounded with respect to \( n \) and does not grow faster than \( \Lambda \), we automatically get the analyticity of the pressure in the thermodynamic limit.

## 5 Some Consequences of the Formula for \( nth \)–Derivative of the Pressure.

In the following, we shall additionally assume that
\[ \nabla g(0) = 0, \quad \text{and} \]
\[ \nabla \Phi_t^\varepsilon(0) = 0 \quad \text{for all} \ t \in [0, T). \]
When $n = 1$, we recall that $A^0_g g = g,
\theta'(t) = < g >_{t, \Lambda}
and if
v(t) = \nabla f = A^{(1)}_{q^*} \nabla g,
then we have

\left( -\Delta + \nabla \Phi^t \cdot \nabla \right) \otimes v(t) + \text{Hess} \Phi^t v(t) = \nabla g

and as in [8] $v(t)$ is a solution of the equation

\begin{equation}
g = < g >_{t, \Lambda} + v(t) \cdot \nabla \Phi^t - \text{div} v(t). \tag{39}
\end{equation}

Using the assumptions above, we have

\theta'(t) = \begin{cases} < g >_{t, \Lambda} \\
\text{div} v(t)(0).
\end{cases}

Similarly, the formula

\theta^{(n)}(t) = (n-1)! < A^{n-1}_g g >_{t, \Lambda},

implies that

\theta^{(n)}(t) = (n-1)! \text{div} v_n(t)(0),

where

v_n(t) = A^{(1)}_{q^*} \nabla \left( A^{n-1}_g g \right).

**Remark 6** The idea of representing $\theta'(t)$ in terms of $\text{div} v(t)(0)$ is due to Helffer and Sjöstrand [8] in the context of proving the exponential convergence of the thermodynamic limit in the one dimensional case.

We conclude these notes by a discussion about the potential contribution of this results towards solving the two dimensional dipole gas problem. The dipole gas and other gases of particles interacting through Coulomb forces are very important statistical systems. In particular, for dipole gas, the lack of screening is well known [59], and the analyticity of the pressure in the high temperature and low activity region has been proved in an indirect way, by means of renormalization group methods (see [60] and [61]).

A direct proof of the analyticity of the pressure based on estimating the coefficients of the Mayer (Taylor) series is still an open problem. The close relationship between this model and the Coulomb gas in the Kostelitz-Thouless phase ($\beta > 8\pi$), go along with the non-existence of any proof for the analyticity of the pressure in the Coulomb gas. Indirect arguments are attempted in [62], [63] and [64]. We believe that after a suitable regularization of the Coulomb potential at short distances to assure stability, we can fit the problem into the framework of
the model discussed above and get an estimate of the coefficients of the Mayer series through our formula for the $nth$ derivative of the pressure.

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