Global existence of a unique solution and a bimodal travelling wave solution for the 1D particle-reaction-diffusion system

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Abstract
In recent years, a mathematical model for collective motions of a self-propelled material have been introduced on the basis of the experiments and numerical analysis for the model have made progress in theoretical understanding of the mechanism of collective motions. On the other hand, there are few mathematically rigorous studies for the mathematical model. In this study, to provide mathematical justification for the mathematical model and its numerical analysis, we show the global existence of a unique solution of the mathematical model for a self-propelled motion on \( \mathbb{R} \). Moreover, we give sufficient conditions for the existence of a non-trivial travelling solution on \( \mathbb{R} \), which we call a bimodal travelling wave solution.

1. Introduction
To understand basic mechanisms of collective motions observed in, for example, a flock of birds, a school of fish and bacteria [1], many experiments about motions of self-propelled materials, which are inanimate systems independent of characteristics of each organism, have been conducted [2]. One of those experiments has treated a camphor disk as a self-propelled material and reported that the motion of a single camphor disk depends on the domain shape [3] and the motion of the camphor depends on the camphor shape [4]. Another experiment has used a camphor boat that is asymmetric motion material, reported the appearance of oscillatory motions by two camphor boats [5], and the production of the self-oscillatory motions by the single camphor boat [6]. Other than those above, droplets have also been investigated by many researchers as an example of a deformable material. For example, the preceding studies [7–9] have treated oil droplets, which are driven by changing the surface tension of the liquid surface, and there are droplets whose movement are induced by the asymmetry of the contact angle, which is due to the chemical gradient of the glass substrate [10, 11] or the adsorption of the surface-active agent molecules on the glass substrate [12–15]. Moreover, in those experimental systems, the appearance of collective motions by self-propelled materials have also been reported: the collective motion by camphor boats on the water surface [16, 17], the traffic jam like motion by the camphor boats in the annular water channel [18, 19] and the self-assembly motion by the oil droplets [20]. In recent years, to understand the mechanism of the self-propelled motions, mathematical modelling for the experimental systems have been introduced and numerically studied with the help of the computer-aided bifurcation analysis. For instance, as for a self-propelled material moving in a one-dimensional (1D) channel without deformation, the camphor disk with constant velocity in an infinite channel has been numerically analyzed in [21] and the bifurcation analysis has been applied to the case of a finite channel [22]. Regarding the 2D problem, the bifurcation analysis for a rotational motion of the camphor disk in the water surface [23] and for the elliptic camphor particle in the 2D channel [24, 25] has been reported. In this paper, we focus on the 1D mathematical model: the simplest 1D model is described as follows.
\[
\frac{d^2x_i}{dt^2} = G(u) - \frac{dx_i}{dt},
\]
for \( t > 0 \) and \( x \in \mathbb{R} \), where \( x_i = x_i(t) \) and \( u = u(x, t) \) represent the position of a self-propelled material and the surface concentration of the surface-active agent molecule layer supplied from the self-propelled material, respectively. The constants \( \rho, \mu, d_u \) and \( k \) denote the area density of the self-propelled material, the viscosity coefficient, the diffusion coefficient of the surface-active agent molecules, and the combined rate of sublimation and dissolution of the surface-active agent, respectively. The functions \( G \) and \( F_S \) represent the driving force to the self-propelled material and the supply of the surface-active agent molecules from the self-propelled material, respectively: refer to [26] for examples of the functions \( F_S \) and \( G \). In this paper, we consider the following functions:

\[
G(u) = \frac{\gamma(u(x_i(t) + r, t) - \gamma(u(x_i(t) - r, t))}{2r}, \quad F_S(x) = \begin{cases} k_0 S_0 & |x| \leq r, \\ 0 & |x| > r, \end{cases}
\]
where \( k_0 \) is the supply rate and \( S_0 \) is the surface-active agent molecule density in self-propelled material. The function \( \gamma \in C(0, \infty) \) denotes the surface tension on the water surface, that is, a function of the surface concentration of the surface-active agent molecule layer. We assume that \( \gamma \) is a strictly decreasing function, for instance, given by

\[
\gamma(u) = \frac{a^m}{a^m + u^m}(\gamma_0 - \gamma_1) + \gamma_1,
\]
where \( a > 0, m \in \mathbb{N} \) and \( \gamma_0, \gamma_1 > 0 \) are the surface tension of the pure water surface and the surface tension of the critical micelle concentration of the surface-active agent, respectively. Then, the appearance of a traffic jam like motion by multiple self-propelled boats [27] and a collective motion by multiple self-propelled disks [28] on the annular channel has been reported by applying the bifurcation theory to the following mathematical model derived from (1.1):

\[
\frac{d^2x_i}{dt^2} = \frac{\gamma(u(x_i^i(t) + r, t) - \gamma(u(x_i^i(t) - r, t))}{2r} - \frac{dx_i^i}{dt}, \quad \frac{\partial u}{\partial t} = d_u \frac{\partial^2u}{\partial x^2} - ku + \Sigma_{i=1}^{N} F_S(x - x_i^i(t)),
\]
where \( x_i^i \) represents the position of \( i \)-th surface-active agent disk. Without loss of generality, we set \( x_1^i(t) < x_2^i(t) < \cdots < x_N^i(t) \) and assume

\[
\min_{1 \leq i < N - 1} |x_i^{i+1}(t) - x_i^i(t)| > 2r,
\]
for any \( t > 0 \), which is the non-collision condition among self-propelled materials. Note that there are several preceding studies about mathematical analysis for the motions by the two camphor disks [29], the reduction of analyzing the jam motion of the camphor disks [30], its mathematical justification [31] and the analysis of the jam motion by using the reduction equation [32].

For the mathematical model (1.3) with (1.2), we have found a stable solution (as shown in figure 1), in which two self-propelled materials move at asymmetrical positions, under the periodic boundary conditions given in [29] and shown its existence with mathematical rigor in [26].

Although the preceding results mentioned above treat solutions of the mathematical model and mathematical analysis for them is important to understand the phenomenon, the existence of a solution of the model (1.1) or (1.3) has not been shown [31–33]. To guarantee the validity of the mathematical model and the preceding results about the mathematical analysis for the model, we show the global existence of a unique solution to the self-propelled model (1.3) on \( \mathbb{R} \). In addition, we show the existence and non-existence of the non-trivial travelling wave solution for two self-propelled materials, which is a special solution describing the collective motion.

This paper is organized as follows. In section 2, we introduce the dimensionless model of (1.3) and state our main results consisting of the global existence of a unique solution and the existence of a bimodal travelling solution, whose proofs are given in section 3 and 4, respectively. Section 5 is devoted to concluding remarks and some future problems. 5 provides some useful lemmas utilized for the proof in section 4.

2. Main results

To analyze the mathematical model (1.3) theoretically, we consider the dimensionless model of (1.3): introduce the following dimensionless variables:
Then, we find that (1.3) is equivalent to
\[
\frac{\tilde{r}}{\rho} \frac{d^2 \tilde{x}_i}{dt^2} = \frac{\tilde{\gamma}(\tilde{u}(\tilde{x}_i + \tilde{t}) - \tilde{\gamma}(\tilde{u}(\tilde{x}_i - \tilde{t}))}{2\tilde{r}} - \frac{\tilde{\mu}}{\rho} \frac{d\tilde{x}_i}{dt},
\]
\[
\frac{\partial \tilde{u}}{\partial \tilde{t}} = \frac{\partial^2 \tilde{u}}{\partial \tilde{x}^2} - \tilde{u} + \sum_{i=1}^{N} F(\tilde{x} - \tilde{x}_i),
\]
where \(\tilde{r} = \rho k d_0 / \gamma(0), \tilde{\gamma} = \gamma / \gamma(0), \tilde{\mu} = \mu d_0 / \gamma(0)\) and \(F\) is defined by
\[
F(x) = \begin{cases} 
1 & |x| \leq r, \\
0 & |x| > r.
\end{cases}
\]
(2.1)

For simplicity of notation, we use the original variables and parameters in the dimensionless model (2.1):
\[
\frac{d^2 x_i}{dt^2} = \frac{\gamma(u(x_i + r, t)) - \gamma(u(x_i - r, t))}{2r} - \frac{\mu}{\rho} \frac{dx_i}{dt},
\]
\[
\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} - u + \sum_{i=1}^{N} F(x - x_i).
\]
(2.3)

Our interest is the global solvability of the system (2.3) and, in this paper, we treat the solutions of (2.3) satisfying \(x_i \in C^2(0, \infty)\) and \(u \in C((0, \infty); C_0(\mathbb{R}))\) with the initial condition,
\[
x_i(0) = X_{0,i}, \quad \frac{dx_i}{dt}(0) = V_{0,i}, \quad i = 1, \ldots, N, \quad \text{for} \quad i = 1, \ldots, N,
\]
(2.4)
\[
u(\cdot, 0) = u_0(\cdot) \quad \text{in} \quad C_0(\mathbb{R}) \cap L^1(\mathbb{R}),
\]
(2.5)

where \(C_0(\mathbb{R})\) denotes the space of continuous functions vanishing at infinity equipped with the uniform norm. Of course, any \(f \in C_0(\mathbb{R})\) satisfies \(\sup_{x \in \mathbb{R}} |f(x)| < \infty\). More precisely, we define a weak solution for (2.3)–(2.5) as follows.

**Definition 2.1.** Let \(\mu > 0\) and \(\gamma \in C[0, \infty)\) be Lipschitz continuous. For a given \(u_0 \in C_0(\mathbb{R}) \cap L^1(\mathbb{R})\), functions \(u \in L^1_{loc}((0, \infty); L^1(\mathbb{R})) \cap C((0, \infty); C_0(\mathbb{R}))\) and \(x_i = (x^1_i, \ldots, x^N_i) \in (C^2(0, \infty))^N\) are called a weak solution of (2.3)–(2.5) provided that

(i) \(x_i \in C^2(0, \infty)\) satisfies
\[
\frac{d^2 x_i}{dt^2} = -\frac{\mu}{\rho} \frac{dx_i}{dt} + \frac{\gamma(u(x_i + r, t)) - \gamma(u(x_i - r, t))}{2r},
\]
(2.6)

with the initial condition (2.4) for \(i = 1, \ldots, N\),

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**Figure 1.** (a) The trajectory of the asymptotic solution \(x^1_c\) and \(x^2_c\) corresponding to an asymmetrically rotating motion of two camphor disks. The solid and dashed lines show the trajectories of two camphor disks. (b) The profile of \(u\) corresponds to it. Both solutions are obtained by the numerical computation for (2.8) with \(\mu = 0.015, r = 0.5, L = 600, m = 2\) and \(a = 0.05\).
(ii) \( u \in L^1_{\text{loc}}((0, \infty); L^1(\mathbb{R})) \cap C((0, \infty); C_0(\mathbb{R})) \) satisfies
\[
\int_{\mathbb{R}} \varphi(x, 0) u_0(x) dx + \int_0^\infty \int_{\mathbb{R}} \left( \frac{\partial \varphi}{\partial t} + \frac{\partial^2 \varphi}{\partial x^2} - \varphi \right) u + \varphi \sum_{i=1}^N F(x - x_i^j) dx dt = 0,
\]
for any \( \varphi \in C_0^\infty(\mathbb{R} \times (0, \infty)) \).

One of our main results is the following theorem about the global existence of a unique solution to (2.3)–(2.5) in the sense of definition 2.1.

**Theorem 2.1.** Suppose \( \mu > 0 \) and that \( \gamma \in C[0, \infty) \) is Lipschitz continuous. Then, for any \( u_0 \in C_0(\mathbb{R}) \cap L^1(\mathbb{R}) \), there exist a unique solution \((u, x_c)\) of (2.3)–(2.5) in the sense of definition 2.1.

Another interest of this study is the existence of travelling wave solutions of (2.3), for which it is often effective to consider the moving coordinate system. We introduce the variables \( z = x - ct \) and \( z^j_i(t) = x^j_i(t) - ct \), in which \( c > 0 \) denotes the uniform velocity. Then, (2.3) on the moving coordinate is described by
\[
\rho \frac{dz^j_i}{dt} = \gamma(u(z^j_i(t) + r, t)) - \gamma(u(z^j_i(t) - r, t)) - \mu \frac{dz^j_i}{dt} - \mu c, \quad i = 1, \ldots, N,
\]
\[
\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial z^2} + \epsilon \frac{\partial u}{\partial z} - u + \sum_{i=1}^N F(z - z^j_i(t)).
\]

Then, travelling wave solutions of (2.3)–(2.5) are characterized by a stationary solution of (2.8).

**Definition 2.2.** For a given constant \( c > 0 \), \( \{Z^j_i\}_{i=1}^N \subseteq \mathbb{R}^N \) and \( U \in C_0^1(\mathbb{R}) \) are called a travelling wave solution for (2.3)–(2.5) with a uniform velocity \( c > 0 \) provided that they satisfy
\[
0 = \frac{\gamma(U(Z^j_i + r)) - \gamma(U(Z^j_i - r))}{2r} - \mu c, \quad i = 1, \ldots, N,
\]
\[
0 = U'' + cU' - U + \sum_{i=1}^N F(z - Z^j_i), \quad z \in \mathbb{R}.
\]

It is obvious that, standing wave solution, i.e. in the case of \( c = 0 \), exists with \( N = 1 \). On the other hand, there does not exist any standing wave solution with \( N \geq 2 \) because of the spatial asymmetry of \( u \) and the condition (1.4). We shall see this result in detail later with \( N = 2 \). In the case of \( c > 0 \), it is easily seen that the travelling wave solution with \( N = 1 \) exists too as with the standing wave solution with \( N = 1 \). In contrast these cases, it is not obvious that a travelling solution with \( N \geq 2 \) exists or not. In this thesis, the case of \( N = 2 \) will be discussed. We call a travelling solution for (2.3) with \( N = 2 \) a bimodal travelling wave solution. Note that travelling wave solutions should satisfy \(|Z^i_{i+1} - Z^j_i| > 2r\) for \( i = 1, \ldots, N - 1 \) in terms of the assumption (1.4). Another main result is the following theorem about the existence of a bimodal travelling wave solution.

**Theorem 2.2.** Suppose \(|Z^j_i - Z^j_{i+1}| > 2r\) and that \( \gamma \in C[0, \infty) \) is a Lipschitz continuous and strictly decreasing function satisfying \( \gamma > 0 \). Then, the following statements hold for (2.3) with \( N = 2 \):

1. For any \( \mu \in \mathbb{R} \), there is no bimodal travelling wave solution with \( c = 0 \).
2. Suppose \( \gamma \in C^1(0, \infty) \) and that \( \gamma' \) is strictly increasing. Then, for any \( \mu > 0 \), there is no bimodal travelling wave solution with \( c > 0 \).
3. Suppose that \( \gamma \in C^2[0, \infty) \) satisfies
\[
\frac{1 + 4r^2}{2r^2} \gamma'(0) < \gamma''(0) < \gamma'(0) < 0.
\]

Then, there exists a bimodal travelling wave solution for sufficiently large \( c > 0 \).

Theorem 2.2 insists that the existence of bimodal travelling wave solutions are closely related to the shape of \( \gamma \). Indeed, for a smooth function \( \gamma \), a travelling wave solution exists under the condition that \( \gamma \) has a concave part. In the preceding study [26], the result similar to theorem 2.2 has been shown for the case of the finite interval, whose length is \( L > 0 \), with the periodic boundary condition. Although, we find that the (2.10) coincides with the \( L \to \infty \) limit of the corresponding result in [26], this extension is not obvious in terms of the mathematical analysis. One of the main purposes of the present study is to clarify a condition for the existence of bimodal travelling solutions of (2.3) and show consistency with the result in [26].
3. Proof of the existence of initial value problem

We state the outline of the proof of Theorem 2.1. For a given $X_c(t) = \{X_i^c(t)\}_{i=1}^N$, (2.7) has a unique solution represented by

$$U[X_c](x, t) = e^{-t} \int_\mathbb{R} H(x - y, t) u_0(y) \, dy + e^{-t} \int_0^t \int_\mathbb{R} H(x - y, t - \tau) e^{\int_0^\tau} F(y - X_i^c(\tau)) \, dy \, d\tau,$$

where $H(x, t) \equiv (4\pi t)^{1/2}e^{-x^2/(4t)}$ denotes the Gaussian kernel, and it is rewritten by

$$U[X_c](c, t) = (\widehat{H}(c) * u_0(c)) + \int_0^t (\widehat{H}(c, t - \tau) * F[X_c](c, \tau)) \, d\tau,$$

in which

$$\widehat{H}(x, t) \equiv e^{-t}H(x, t), \quad F[X_c](x, t) \equiv \sum_{i=1}^N F(x - X_i^c(t)).$$

Substituting this formula into (2.6), we find

$$\frac{d^2X_i^c(t)}{dt^2} = -\mu \frac{dX_i^c(t)}{dt} + \frac{\gamma(U[X_c](X_i^c(t) + r, t)) - \gamma(U[X_c](X_i^c(t) - r, t))}{2r},$$

for $i = 1, \cdots, N$. We prove the existence of a unique solution $X_c$ of (3.1) by applying Picard’s iterative method. Consider the following system, which is equivalent to (3.1):

$$X_i^c(t) = X_i^0 + \int_0^t V_i^c(\tau) \, d\tau,$$

$$V_i^c(t) = V_i^0 + \int_0^t -\mu V_i^c(\tau) + \frac{\gamma(U[X_c](X_i^c(\tau) + r, \tau)) - \gamma(U[X_c](X_i^c(\tau) - r, \tau))}{2r} \, d\tau,$$

where $V_i^c(t)$ is an auxiliary function corresponding to the derivative of $X_i^c(t)$, and define the map $P$ by

$$P: \left(\begin{array}{c} X_i^c(t) \\ V_i^c(t) \end{array} \right) \mapsto \left(\begin{array}{c} X_i^0 + \int_0^t V_i^c(\tau) \, d\tau \\ V_i^0 + \int_0^t -\mu V_i^c(\tau) + \frac{\gamma(U[X_c](X_i^c(\tau) + r, \tau)) - \gamma(U[X_c](X_i^c(\tau) - r, \tau))}{2r} \, d\tau \end{array} \right).$$

Using the Banach fixed point theorem, we prove the existence of a fixed point $X^*_c$ of the map $P$ in a proper functional space so that $(X^*_c, \ U[X^*_c])$ is a unique solution of (3.1).

As the first step of the proof, we introduce the functional space in which we show $P$ is a contraction mapping. Let $\phi \in C[0, \infty)$ be a positive function. Then, the functional space $C^\alpha_0[0, \infty) \equiv \{f \in C[0, \infty) \mid \|f\|_\alpha < \infty\}$ with $\|f\|_\alpha \equiv \sup_{t \in [0, \infty)} |e^{-\alpha t}f(t)|$ is a Banach space. Indeed, for any Cauchy sequence $\{f_n\} \subset C^\alpha_0[0, \infty)$, there exists $F \in C[0, \infty)$ such that $\|f_n - F\|_{L^\infty[0, \infty)} \to 0$ as $n \to \infty$. Thus, setting the function $f \in C^\alpha_0[0, \infty)$ by $f = F/\phi$, we find $\|f_n - f\|_\alpha \to 0$ as $n \to \infty$. Thus, $\{f_n\}$ is a convergent sequence in $C^\alpha_0[0, \infty)$. In what follows, we consider the functional space $C^\alpha_0(\mathbb{R}, \infty)$ for $\phi(t) = e^{-\alpha t}, \alpha > 0$ and, for convenience, we introduce the following notation:

**Definition 3.1.** For any $f \in C[0, \infty)$, we define

$$\|f\|_\alpha \equiv \sup_{t \in [0, \infty)} |e^{-\alpha t}f(t)|,$$

and $C^\alpha_0 \equiv \{f \in C[0, \infty) \mid \|f\|_\alpha < \infty\}$.

In addition to the above definition, we use the notations,

$$\|X(t)\| = \max_{1 \leq i \leq N} |X_i(t)|, \quad \|X\|_\alpha = \max_{1 \leq i \leq N} \|X_i^\alpha\|_\alpha,$$

for any $X = (X_1, \cdots, X_N) \in (C^\alpha_0)^N$. Note that we omit the domain in the $L^p$ norm when it is given by $\mathbb{R}$. 
Next, to show that $P$ is a contraction mapping, we introduce maps $P_1$ and $P_2$ on $C_\alpha$:

$$
P_1: V \mapsto \int_0^t V(\tau) d\tau, \quad P_2: V \mapsto \int_0^t -\mu V(\tau) d\tau, \quad \mu > 0.
$$

Then, the maps $P_1$ and $P_2$ are continuous on $C_\alpha$. More precisely, we have

$$
\|P_1 V - P_1 V'\|_\alpha \leq \frac{1}{\alpha} \|V - V'\|_\alpha, \quad \|P_2 V - P_2 V'\|_\alpha \leq \frac{\mu}{\alpha} \|V - V'\|_\alpha,
$$

for any $V, V' \in C_\alpha$, since it follows that

$$
\left| e^{-\alpha t} \left( \int_0^t V(\tau) d\tau - \int_0^t V'(\tau) d\tau \right) \right| \leq \|V - V'\|_\alpha \int_0^t e^{-\alpha(t-\tau)} d\tau \leq \frac{1}{\alpha} \|V - V'\|_\alpha.
$$

We also define the map $P_3$ on $(C_\alpha)^N$ by

$$
P_3: X_c \mapsto \left( \frac{1}{2\tau} \int_0^t (\gamma(U[X_c](X'_c(\tau) + r, \tau)) - \gamma(U[X_c](X'_c(\tau) - r, \tau))) d\tau \right)_{1 \leq i \leq N}.
$$

Before deriving an estimate for $P_3$, we show the following lemma.

**Lemma 3.1.** For any $X_c, X'_c \in (C_\alpha)^N$, we have

$$
\left\| \frac{\partial U[X_c]}{\partial x}(t, t) \right\|_{L^\infty} \leq \frac{e^{-t}}{(\pi t)^{\frac{1}{2}}} \|u_0\|_{L^\infty} + N,
$$

and

$$
\left\| U[X_c](, t) - U[X'_c](, t) \right\|_{L^\infty} \leq \frac{Ne^{\alpha t}}{(1 + \alpha)^2} \|X_c - X'_c\|_\alpha.
$$

**Proof.** Note that the derivative of $U[X_c](x, t)$ with respect to $x$ is expressed by

$$
\frac{\partial U[X_c]}{\partial x}(x, t) = \left( \frac{\partial H}{\partial x}(t, t) * u_0(t) \right) + \int_0^t \left( \frac{\partial H}{\partial x}(t, t - \tau) * F[X_c](, \tau) \right) d\tau.
$$

The first term in the right-hand side is estimated by Young’s inequality:

$$
\left\| \frac{\partial H}{\partial x}(x, t) * u_0(t) \right\|_{L^\infty} \leq e^{-t} \left\| \frac{\partial H}{\partial x}(x, t) \right\|_{L^1} \|u_0\|_{L^\infty} \leq \frac{e^{-t}}{(\pi t)^{\frac{1}{2}}} \|u_0\|_{L^\infty}.
$$

The second term is estimated by

$$
\left\| \int_0^t \frac{\partial H}{\partial x}(x, t - \tau) * F[X_c](, \tau) d\tau \right\|_{L^\infty} \leq \int_0^t e^{-|t-\tau|} \left\| \frac{\partial H}{\partial x}(x, t - \tau) \right\|_{L^1} \|F[X_c](, \tau)\|_{L^\infty} \leq \frac{Ne^{\alpha t}}{(1 + \alpha)^2} \|X_c\|_\alpha.
$$

where $\Gamma(t)$ denotes the Gamma function. Thus, we obtain

$$
\left\| \frac{\partial U[X_c]}{\partial x}(x, t) \right\|_{L^\infty} \leq \frac{e^{-t}}{(\pi t)^{\frac{1}{2}}} \|u_0\|_{L^\infty} + N.
$$

We show the continuity of $U[X_c]$ with respect to $X_c$. Note that

$$
|U[X_c](x, t) - U[X'_c](x, t)| \leq \int_0^t (H(\cdot, t - \tau) * (F[X_c](, \tau) - F[X'_c](, \tau))) d\tau
$$

$$
\leq \int_0^t \int_\mathbb{R} H(x - y, t - \tau) \sum_{i=1}^N |\chi_{A(x)}(y) - \chi_{A(x')}(y)| 1_{X_c(\cdot, t) - r} \chi_{X'_c(\cdot, t) + r}(y) dy d\tau,
$$

where $\chi_A(x)$ is the indicator function, that is, $\chi_A(x) = 1$ for $x \in A$ and $\chi_A(x) = 0$ for $x \not\in A$. Using the notations: $A \triangle B \equiv (A \cup B) \setminus (A \cap B)$ and $I^r_i(t) \equiv \chi_{X'_c(\cdot, t) - r} \chi_{X'_c(\cdot, t) + r}$, we find

$$
|\chi_{X_c(\cdot, t) - r} \chi_{X'_c(\cdot, t) + r}(x) - \chi_{A(x)}(x)| = 1_{I^r_i(t) \in I^r_i(t)(x)},
$$

and

$$
|I^r_i(t) \triangle I^r_i(t)| = \begin{cases} 4r, & I^r_i(t) \cap I^r_i(t) = \emptyset, \\ 2|X'_c(t) - X''_c(t)|, & I^r_i(t) \cap I^r_i(t) \neq \emptyset. \end{cases}
$$
Owing to the Hölder inequality, we have
\[ |I_1(t) \cap I_2(t)| = 2r, \]
and thus, it follows from the Hölder inequality that
\[ |U[X_c](x, t) - U[X_c'](x, t)| \leq \frac{N}{\sqrt{\pi}} \int_0^t \frac{e^{-(t-\tau)}}{(t-\tau)^{\frac{1}{2}}} \|X_c(\tau) - X_c'(\tau)\| d\tau, \]
and we obtain
\[ \|U[X_c](c, t) - U[X_c'](c, t)\|_{\infty} \leq \frac{N\epsilon}{\sqrt{\pi}} \|X_c - X_c'\|_{\alpha} \int_0^1 \frac{e^{-(1+\alpha)(t-\tau)}}{(t-\tau)^{\frac{1}{2}}} \tau^{-\frac{1}{2}} e^{-\tau} d\tau \]
\[ \leq \frac{N\epsilon}{\sqrt{\pi}(1+\alpha)^{\frac{1}{2}}} \|X_c - X_c'\|_{\alpha}, \]
in which we have used \( \Gamma(1/2) = \sqrt{\pi} \) in the last inequality.

We now show the continuity of \( P_2 \) on \((C_0)^N\).

**Lemma 3.2.** For any \( X_c, X'_c \in (C_0)^N \), there exist constants \( q > 2 \) and \( C(q) > 0 \) such that
\[ \|P_3X_c - P_3X'_c\|_{\alpha} \leq \frac{M}{\alpha^{1/4}} \left( \frac{2N}{\alpha^{1/4}} + C(q)\|u_0\|_{\infty}^q \right) \|X_c - X'_c\|_{\alpha}. \]

**Proof.** For \( X_c, X'_c \in (C_0)^N \), it follows from lemma 3.1 that
\[ |\gamma(U[X_c](X_c'(t) + r, t)) - \gamma(U[X'_c](X'_c(t) + r, t))| \leq M|U[X_c](X_c'(t) + r, t) - U[X'_c](X'_c(t) + r, t)| \]
\[ + M|U[X_c'](X_c'(t) + r, t) - U[X'_c'](X'_c(t) + r, t)| \]
\[ \leq M \|X_c - X'_c\|_{\alpha} \left[ \frac{N\epsilon}{(1+\alpha)^{\frac{1}{2}}} + \left( \frac{e^{-t}}{(\pi t)^{\frac{1}{2}}} \|u_0\|_{\infty} + N\right) \epsilon \right], \]
and thus
\[ \left\| \int_0^t \gamma(U[X_c](X_c'(\tau) + r, \tau)) - \gamma(U[X'_c](X'_c(\tau) + r, \tau)) d\tau \right\|_{\alpha} \]
\[ \leq M \|X_c - X'_c\|_{\alpha} \sup_{r \geq 0} \left[ \frac{N}{\alpha} \left( \frac{1}{(1+\alpha)^{\frac{1}{2}}} + 1 \right) \int_0^t e^{-\alpha(\tau-\tau)} d\tau + \|u_0\|_{\infty} \int_0^t \frac{e^{-\alpha(\tau-\tau)} d\tau}{(\pi t)^{\frac{1}{2}}} \right] \]
\[ = M \|X_c - X'_c\|_{\alpha} \left[ \frac{N}{\alpha} \left( \frac{1}{(1+\alpha)^{\frac{1}{2}}} + 1 \right) + \|u_0\|_{\infty} \frac{\sqrt{\pi}}{\sqrt{\pi}} \int_0^t \tau^{-\frac{1}{2}} e^{-\frac{\tau}{\alpha^{\frac{1}{2}}}} d\tau \right]. \]

Owing to the Hölder inequality, we have
\[ \int_0^t \tau^{-\frac{1}{2}} e^{-\frac{\tau}{\alpha^{\frac{1}{2}}}} d\tau \leq \left( \int_0^t \tau^{-\frac{1}{2}} d\tau \right)^{\frac{1}{2}} \left( \int_0^t e^{-\frac{\tau}{\alpha^{\frac{1}{2}}} d\tau} \right)^{\frac{1}{2}} \]
\[ \leq (\alpha q)^{\frac{1}{2}} \left( \int_0^\infty \tau^{-\frac{1}{2}} e^{-\tau} d\tau \right)^{\frac{1}{2}} = C(q)\alpha^{-\frac{1}{2}}, \]
for \( 1 < p < 2 \) and \( q > 2 \) satisfying \( 1/p + 1/q = 1 \). Hence, we obtain
\[ \|P_3X_c - P_3X'_c\|_{\alpha} \leq \frac{M}{\alpha^{1/4}} \left( \frac{2N}{\alpha^{1/4}} + C(q)\|u_0\|_{\infty} \right) \|X_c - X'_c\|_{\alpha}. \]

On the basis of the continuity of \( P_1, P_2 \) and \( P_3 \), we show that the map \( P \) is a contraction mapping on \((C_0)^{2N}\).

Let \( X_c \equiv (X_c, V_c) \in (C_0)^N \times (C_0)^N = (C_0)^{2N} \). For any \( X_c, X'_c \in (C_0)^{2N} \), we have
with the boundary and decay conditions:

\[ P_{i} V_{c}^{i} - P_{i} V_{c}^{i} = P_{i} V_{c}^{i} - P_{i} V_{c}^{i} \]

It follows from (3.4) and lemma 3.2 that there exist constants \( q > 2 \) and \( C(q, N) > 0 \) such that

\[ \| P_{X} - P_{X} \|_{a} \leq \frac{C(q, N)}{\alpha^{1/q}} \| X - X \|_{a}. \quad (3.5) \]

Since we have \( C(q, N)\alpha^{-1/q} < 1 \) for sufficiently large \( \alpha > 0 \), \( P \) is a contraction mapping on \((C_{0})^{2N}\). Thus, there exists a unique fixed point \( X_{c} = (X_{c}^{1}, V_{c}^{1}) \in (C_{0})^{2N}\) of the map \( P \). By this, \( X_{c}^{1} \) satisfies (3.2) and (3.3), which concludes that \((X_{c}^{1}, U[X_{c}^{1}])\) is a unique solution to (2.6)–(2.7).

4. Proof of theorem 2.2

We first reformulate the system (2.9) with \( N = 2 \):

\[ 0 = \frac{\gamma(U(Z_{c}^{i} + r)) - \gamma(U(Z_{c}^{i} - r))}{2r} - \mu c_{i}, \quad i = 1, 2, \quad (4.1) \]

\[ 0 = U^{\alpha} + cU^{\prime} - U + F(z - Z_{c}^{i}) + F(z - Z_{c}^{i}). \quad (4.2) \]

As for (4.2), considering the function \( F \) given by (2.2), we find that (4.2) is rewritten by

\[ 0 = U^{\alpha} + cU^{\prime} - U, \quad z \in (-\infty, Z_{c}^{1} - r) \cup (Z_{c}^{1} + r, Z_{c}^{2} - r) \cup (Z_{c}^{2} + r, \infty), \]

\[ 0 = U^{\alpha} + cU^{\prime} - U + 1, \quad z \in (Z_{c}^{1} - r, Z_{c}^{1} + r) \cup (Z_{c}^{2} - r, Z_{c}^{2} + r). \]

Note that we have \( \delta \equiv Z_{c}^{1} - Z_{c}^{2} - 2r > 0 \) and that the solution \( U \in C^{1}(\mathbb{R}) \) satisfies \( U(z) \to 0 \) as \( z \to \pm \infty \).

Then, it is sufficient to consider the following system:

\[ 0 = U^{\alpha}, \quad 0 = U^{\alpha} + cU^{\prime} - U_{0,1}, \quad z \in (-\infty, Z_{c}^{1} - r), \]

\[ 0 = U^{\alpha} + cU^{\prime} - U_{1,2} + 1, \quad z \in (Z_{c}^{1} - r, Z_{c}^{1} + r), \]

\[ 0 = U^{\alpha} + cU^{\prime} - U_{1,2} - U_{1,2}, \quad z \in (Z_{c}^{1} + r, Z_{c}^{2} - r), \]

\[ 0 = U^{\alpha} + cU^{\prime} - U_{1,2} + 1, \quad z \in (Z_{c}^{1} - r, Z_{c}^{2} + r), \]

\[ 0 = U^{\alpha} + cU^{\prime} - U_{2,3} + 1, \quad z \in (Z_{c}^{2} - r, \infty), \]

with the boundary and decay conditions:

\[ U_{0,1}(Z_{c}^{1} - r) = U_{1}(Z_{c}^{1} - r), \quad U_{1}(Z_{c}^{1} + r) = U_{1,2}(Z_{c}^{1} + r), \]

\[ U_{1}(Z_{c}^{1} - r) = U_{2}(Z_{c}^{1} - r), \quad U_{2}(Z_{c}^{1} + r) = U_{2,3}(Z_{c}^{1} + r), \]

\[ U_{0,1}(Z_{c}^{2} - r) = U_{1,2}(Z_{c}^{2} - r), \quad U_{1,2}(Z_{c}^{2} + r) = U_{2,3}(Z_{c}^{2} + r), \]

\[ \lim_{z \to -\infty} U_{0,1}(z) = \lim_{z \to \infty} U_{2,3}(z) = 0. \quad (4.3) \]

Owing to the translational symmetry of the system, (4.1) and (4.2) is equivalent to the following system:

\[ 0 = \gamma(U_{i}(2r)) - \gamma(U_{i}(0)) - \mu c, \quad i = 1, 2, \quad (4.4) \]

\[ 0 = U_{i}^{\alpha} + cU_{i}^{\prime} - U_{i} + 1, \quad z \in (0, 2r), \quad i = 1, 2, \]

\[ 0 = U_{i}^{\alpha} + cU_{i,2}^{\prime} - U_{i,2}^{\prime} - U_{i,2}, \quad z \in (0, d), \]

\[ 0 = U_{i}^{\alpha} + cU_{i,2}^{\prime} - U_{i,2}, \quad z \in (0, \infty), \quad (4.5) \]

with (4.3) and the boundary conditions:

\[ U_{0,1}(0) = U_{0,1}(0), \quad U_{0,2}(2r) = U_{0,2}(0), \quad U_{1,2}(0) = U_{1,2}(d), \quad U_{2,3}(2r) = U_{2,3}(0), \]

\[ U_{0,2}(0) = U_{0,2}(0), \quad U_{1,2}(2r) = U_{1,2}(0), \quad U_{1,2}(0) = U_{1,2}(d), \quad U_{2,3}(2r) = U_{2,3}(0), \quad (4.6) \]
Next, we construct a solution of (4.5). By a classical theory, a solution of (4.5) is expressed by

\[
U_i(x) = a_i^1 \exp(x \lambda_+) + a_i^2 \exp(x \lambda_-) + 1,
\]
\[
U_0(x) = a_0^2 \exp(x \lambda_+) + a_0^1 \exp(x \lambda_-) + 1,
\]
\[
U_{0,1}(x) = b_0^0 \exp(x \lambda_+) + b_0^1 \exp(x \lambda_-),
\]
\[
U_{1,2}(x) = b_1^0 \exp(x \lambda_+) + b_1^1 \exp(x \lambda_-),
\]
\[
U_{2,3}(x) = b_2^0 \exp(x \lambda_+) + b_2^1 \exp(x \lambda_-),
\]

where \( \lambda_+ \) are the functions of \( c > 0 \) defined by

\[
\lambda_\pm(c) = \phi(c) \pm \theta(c), \quad \phi(c) = -\frac{c}{2}, \quad \theta(c) = \frac{\sqrt{c^2 + 4}}{2}.
\]

For convenience, we use the notation \( \exp(x) \equiv e^x \) throughout this section and appendix. It follows from (4.6) that

\[
a_i^1 + a_i^1 + 1 = b_i^0 + b_i^1,
\]
\[
a_i^1 \lambda_+ + a_i^1 \lambda_- = b_i^0 \lambda_+ + b_i^0 \lambda_-,
\]
\[
a_i^1 \exp(2r \lambda_+) + a_i^1 \exp(2r \lambda_-) + 1 = b_i^1 + b_i^1,
\]
\[
a_i^1 \lambda_+ \exp(2r \lambda_+) + a_i^1 \lambda_- \exp(2r \lambda_-) = b_i^1 \lambda_+ + b_i^1 \lambda_-,
\]
\[
a_i^2 + a_i^2 + 1 = b_i^1 \exp(d \lambda_+) + b_i^1 \exp(d \lambda_-),
\]
\[
a_i^2 \lambda_+ + a_i^2 \lambda_- = b_i^1 \lambda_+ \exp(d \lambda_+) + b_i^1 \lambda_- \exp(d \lambda_-),
\]
\[
a_i^2 \exp(2r \lambda_+) + a_i^2 \exp(2r \lambda_-) + 1 = b_i^2 + b_i^2,
\]
\[
a_i^2 \lambda_+ \exp(2r \lambda_+) + a_i^2 \lambda_- \exp(2r \lambda_-) = b_i^2 \lambda_+ + b_i^2 \lambda_-.
\]

(4.7)

For convenience, we introduce the following notations:

\[
a^i \equiv \begin{pmatrix} a_{i1}^1 \\ a_i^1 \end{pmatrix}, \quad b^i \equiv \begin{pmatrix} b_{i1}^0 \\ b_i^0 \end{pmatrix}, \quad e \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad P \equiv \begin{pmatrix} 1 & 1 \\ \lambda_+ & \lambda_- \end{pmatrix}, \quad Q(x) \equiv \begin{pmatrix} \exp(x \lambda_+) & 0 \\ 0 & \exp(x \lambda_-) \end{pmatrix}.
\]

Then, (4.7) is rewritten by

\[
Pa^i + e = Pb^i, \quad PQ(2r)a^i + e = Pb^i, \quad Pa^2 + e = PQ(d)b^i, \quad PQ(2r)a^2 + e = Pb^2,
\]

that is,

\[
a^1 = b^0 - P^{-1}e, \quad b^i = Q(2r)a^1 + P^{-1}e,
\]
\[
a^2 = Q(d)b^1 + P^{-1}e, \quad b^2 = Q(2r)a^2 + P^{-1}e.
\]

(4.8)

Thus, considering \( Q(p)Q(q) = Q(p + q) \), we find

\[
b^2 = Q(4r + d)b^0 + (I - Q(2r))Q(2r + d) - Q(4r + d))P^{-1}e
\]
\[
= Q(4r + d)b^0 + (I - Q(2r))(I + Q(d)Q(2r))P^{-1}e,
\]

which yields

\[
b^0 = Q(-(4r + d))b^2 + (I - Q(-2r))(I + Q(-d)Q(-2r))P^{-1}e.
\]

Since the condition (4.3) yields \( b^0 = b_0^2 = 0 \), we obtain

\[
b_0^0 = U_+(1 + \exp(-d \lambda_+) \exp(-2r \lambda_+)), \quad b_0^2 = U_-(1 + \exp(d \lambda_-) \exp(2r \lambda_-)),
\]

where

\[
U_+(c) = \frac{1}{2 \theta(c)} \left( 1 - \exp(\mp 2r \lambda_+(c)) \right).
\]

(4.9)

Then, it follows from (4.8) that

\[
\begin{pmatrix} a_1^1 \\ a_1^2 \end{pmatrix} = \begin{pmatrix} U_+(1 + \exp(-d \lambda_+) \exp(-2r \lambda_+)) + \frac{\lambda_+}{2 \theta} \\ -\frac{\lambda_+}{2 \theta} \end{pmatrix}, \quad \begin{pmatrix} b_1^1 \\ b_1^2 \end{pmatrix} = \begin{pmatrix} U_+ \exp(-d \lambda_+) \\ U_- \exp(-2r \lambda_-) \end{pmatrix},
\]
\[
\begin{pmatrix} a_2^2 \\ a_2^2 \end{pmatrix} = \begin{pmatrix} \frac{\lambda_+}{2 \theta} \exp(-2r \lambda_+) \\ \exp(-2r \lambda_-) \end{pmatrix} U_+ - \frac{\lambda_+}{2 \theta} \exp(-2r \lambda_-).
\]
Thus, the functions $U_i$ and $U_{i,+}$ with $\alpha'$ and $\beta'$, which are uniquely determined for any given constants $(\alpha, \beta)$, satisfy (4.5). In particular, $U_1$ and $U_2$ are explicitly given by
\[
U_1(x) = \exp(\lambda_+ x)(1 + \exp(-d\lambda_+)\exp(-2r\lambda_+))U_+
+ \frac{\lambda_+}{2\theta}(1 - \exp(x\lambda_-)) - \frac{\lambda_-}{2\theta}(1 - \exp(x\lambda_+)),
\]
\[
U_2(x) = \exp(x - 2r\lambda_-)(1 + \exp(d\lambda_-)\exp(2r\lambda_-))U_-
+ \frac{\lambda_+}{2\theta}(1 - \exp((x - 2r)\lambda_-)) - \frac{\lambda_-}{2\theta}(1 - \exp((x - 2r)\lambda_+)).
\]

Thus, we obtain
\[
U_1(0) = U_1(1 + \exp(-d\lambda_+)\exp(-2r\lambda_+)), \quad U_1(2r) = U_1(\exp(-d\lambda_+) + U_+),
U_2(0) = U_+ + U_-. \exp(d\lambda_-), \quad U_2(2r) = U_2(1 + \exp(d\lambda_-)\exp(2r\lambda_-)).
\]

To make it clear that these quantities depend on the parameters $(\alpha, \beta)$, we use the following notations:
\[
U_{pf}(\alpha, \beta) \equiv U_2(2r) = U_+(\alpha + U_. \exp(d\lambda_-)),
U_{pr}(\alpha, \beta) \equiv U_2(0) = U_+(\alpha + U_. \exp(d\lambda_-)),
U_f(\alpha, \beta) \equiv U_1(2r) = U_1(\exp(-d\lambda_+) + U_.),
U_f(\alpha, \beta) \equiv U_1(0) = U_1(1 + \exp(-d\lambda_+)\exp(-2r\lambda_+)),
\]
and, for later use, we introduce the following functions:
\[
\Delta U_p(\alpha, \beta) \equiv U_{pf}(\alpha, \beta) - U_{pr}(\alpha, \beta), \quad \Delta U_f(\alpha, \beta) \equiv U_f(\alpha, \beta) - U_0(\alpha, \beta).
\]

We are now ready to prove theorem 2.2. To show the (non-)existence of a solution to (2.9), we see whether the constructed solution $(U_1, U_2)$ to (4.5) satisfies (4.4) or not.

4.1. Proof of theorem 2.2.1

Note that (4.4) requires
\[
\gamma(U_{pf}(0, \beta)) - \gamma(U_{pf}(0, \beta)) = 0, \quad \gamma(U_{pr}(0, \beta)) - \gamma(U_{pr}(0, \beta)) = 0.
\]

Since $\gamma$ is strictly decreasing, (4.11) is equivalent to $\Delta U_p(0, \beta) = \Delta U_f(0, \beta) = 0$. For the case of $\alpha = 0$, it follows from $\lambda_+ = -\lambda_- = -1$ that
\[
U_{pf}(0, \beta) = U_0(\beta) = U_0(1 + \exp(-d\exp(-2r)),
U_{pr}(0, \beta) = U_0(\beta) = U_0(1 + \exp(-d)),
\]
where $U_0 \equiv U_0(\alpha) = (1 - \exp(-2r)) / 2 > 0$. Thus, we obtain
\[
\Delta U_p(0, \beta) = -\Delta U_f(0, \beta) = -U_0 \exp(-d)(1 - \exp(-2r)) < 0,
\]
and conclude that there exists no solution of (2.9) with $\alpha = 0$.

4.2. Proof of theorem 2.2.2

Note that it follows from lemma 1(ii) that $\Delta U_p < \Delta U_f$. For the case that $\gamma$ is a linear function, however, (4.14) is equivalent to $\Delta U_p = \Delta U_f$, and thus there exists no solution. For the case that $\gamma'$ is strictly increasing, since $\gamma$ is strictly decreasing, (4.4) yields $\Delta U_p < 0$ and $\Delta U_f < 0$, that is, $U_{pf} < U_{pr}$ and $U_{pr} < U_f$. We first consider the case of $U_{pf} < U_{pr} \leq 0$. Then, (4.14) is rewritten by
\[
\int_{U_{pf}}^{U_{pr}} -\gamma'(U)dU = \int_{U_f}^{U_0} -\gamma'(U)dU.
\]

Since $\gamma'$ is strictly increasing, we find $\gamma'(U_{pf}(r, \alpha, \beta)) \leq \gamma'(U_{pr}(r, \alpha, \beta))$ and
\[
\min_{U \in [U_{pf}, U_{pr}]} (-\gamma'(U)) \leq \gamma'(U_{pf}) \leq \gamma'(U_{pr}), \quad \max_{U \in [U_f, U_0]} (-\gamma'(U)) = -\gamma'(U_{pr}).
\]

Thus, considering lemma 1(ii), we obtain
\[
\int_{U_{pf}}^{U_{pr}} -\gamma'(U)dU \geq -\gamma'(U_{pf})(U_{pf} - U_{pr}) > -\gamma'(U_{pf})(U_{pr} - U_f)
\geq -\gamma'(U_{pr})(U_{pr} - U_f) \geq \int_{U_f}^{U_0} -\gamma'(U)dU,
\]
which contradicts (4.12). Next, we consider the case of $U_{pr} - U_{ff} > 0$. Then, (4.14) is rewritten by
\[
\int_{U_{ff}}^{U_{f}} - \gamma'(U)dU = \int_{U_{pr}}^{U_{f}} - \gamma'(U)dU. \tag{4.13}
\]
Owing to lemma 1(iii), the left-hand side of (4.13) is positive. If $U_{ir} \leq U_{pr}$, then the right-hand side of (4.13) is not positive. Thus, it is sufficient to consider the case of $U_{ir} > U_{pr}$. Note that
\[
\min_{U \in \{U_{fr}, U_{fc}\}} (-\gamma'(U)) = -\gamma'(U_{fr}), \quad \max_{U \in \{U_{fr}, U_{fc}\}} (-\gamma'(U)) = -\gamma'(U_{fc}),
\]
and it follows from $U_{fr} - U_{ff} > 0$ that $-\gamma'(U_{fr}) \leq -\gamma'(U_{fc})$. Since lemma 1(ii) yields $U_{ir} - U_{pr} < U_{ir} - U_{fr}$, we obtain
\[
\int_{U_{fr}}^{U_{f}} - \gamma'(U)dU \geq \int_{U_{fr}}^{U_{f}} - \gamma'(U_{fr})(U_{fr} - U_{ff}) - \gamma'(U_{fr})(U_{ir} - U_{fr}) > -\gamma'(U_{fr})(U_{ir} - U_{fr}) \geq -\gamma'(U_{pr})(U_{ir} - U_{pr}) + \int_{U_{pr}}^{U_{f}} - \gamma'(U)dU,
\]
which contradicts (4.13). Thus, there is no solution of (2.9) provided that $\gamma'$ is strictly increasing.

### 4.3. Proof of theorem 2.2–3
In order to show theorem 2.2–3, we introduce the following function:
\[
\Gamma(c, d) \equiv \gamma(U_{fr}(c, d)) - \gamma(U_{ir}(c, d)) - (\gamma(U_{fr}(c, d)) - \gamma(U_{pr}(c, d))). \tag{4.14}
\]
We investigate the properties of $\Gamma(c, d)$ in the limits of $d \to 0$ and $d \to \infty$. Note that it follows from (4.10) that
\[
\lim_{d \to 0} U_{pr}(c, d) = U_{r}(c)(1 + \exp(2r \lambda_{s}(c))),
\]
\[
\lim_{d \to 0} U_{pr}(c, d) = U_{r}(c) + U_{c}(c),
\]
\[
\lim_{d \to 0} U_{fr}(c, d) = U_{r}(c) + U_{c}(c),
\]
\[
\lim_{d \to 0} U_{fr}(c, d) = U_{r}(c)(1 + \exp(-2r \lambda_{s}(c))). \tag{4.15}
\]
Then, we easily confirm that
\[
\Gamma_{d}(c) \equiv \lim_{d \to 0} \Gamma(c, d) = 2\gamma(U_{r}(c)) - [\gamma(U_{r}(c)(1 + \exp(-2r \lambda_{s}(c))) + \gamma(U_{c}(c)(1 + \exp(2r \lambda_{s}(c))))].
\]
Regarding the limit $d \to \infty$ limit of $\Gamma(c, d)$, we have the following lemma.

**Lemma 4.1.** For sufficiently large $d > 0$, the sign of $\Gamma(c, d)$ coincides with that of
\[
\Gamma_{\infty}(c) \equiv \gamma'(U_{r}(c))\exp(2r \lambda_{s}(c)) - \gamma'(U_{r}(c)),
\]
provided that $\Gamma_{\infty}(c) \neq 0$.

**Proof.** Note that lemma 1(iii) gives $U_{fr} < U_{ff}$ and lemma A.2(ii) implies $U_{pr} < U_{ir}$ for sufficiently large $d$. Then, it follows from the mean value theorem that there exist constants $U^{*} \in (U_{fr}, U_{ff})$ and $U^{**} \in (U_{pr}, U_{ir})$ such that
\[
\gamma(U_{fr}) - \gamma(U_{fr}) = \gamma'(U^{*})(U_{fr} - U_{fr}), \quad \gamma(U_{fr}) - \gamma(U_{fr}) = \gamma'(U^{**})(U_{fr} - U_{fr}).
\]
Thus, we obtain
\[
\Gamma(c, d) = \gamma'(U^{*})(U_{fr} - U_{pr}) - \gamma'(U^{**})(U_{fr} - U_{pr})
\]
\[
= (U_{fr} - U_{pr}) \left[ \frac{\gamma'(U^{*})U_{fr} - U_{pr}}{U_{fr} - U_{pr}} - \gamma'(U^{**}) \right].
\]
Considering $U_{pr} < U_{ir}$, we find that the sign of $\Gamma(c, d)$ corresponds to that of
\[
\gamma'(U^{*}) \frac{U_{fr} - U_{pr}}{U_{fr} - U_{pr}} - \gamma'(U^{**}).
\]
Note that it follows from (4.10) that
\[
\lim_{d \to \infty} U_{pr}(c, d) = \lim_{d \to \infty} U_{fr}(c, d) = U_{r}(c), \quad \lim_{d \to \infty} U_{pr}(c, d) = \lim_{d \to \infty} U_{fr}(c, d) = U_{r}(c),
\]
which implies $U^*(d) \to U_-$ and $U^{**}(d) \to U_+$ in the $d \to \infty$ limit. Since we have

$$U_f - U_{pf} = \frac{\exp(-d\lambda_+)}{2\theta}[-\lambda_-(1 - \exp(-2r \lambda_-))$$

$$- \lambda_+ \exp(2r \lambda_-) \exp(d(-\lambda_+ + \lambda_+))(1 - \exp(2r \lambda_-))],$$

$$U_i - U_{pi} = \frac{\exp(-d\lambda_+)}{2\theta}[-\lambda_-(1 - \exp(-2r \lambda_-)) \exp(-2r \lambda_+$$

$$- \lambda_+ \exp(d(-\lambda_+ + \lambda_+))(1 - \exp(2r \lambda_-))],$$

it follows that

$$\lim_{d \to \infty} \frac{U_f(c, d) - U_{pf}(c, d)}{U_i(c, d) - U_{pi}(c, d)}$$

$$= \lim_{d \to \infty} \frac{\lambda_-(1 - \exp(-2r \lambda_-)) + \lambda_+(1 - \exp(2r \lambda_-)) \exp(d(-\lambda_+ + \lambda_+)) \exp(2r \lambda_-)}{\lambda_-(1 - \exp(-2r \lambda_-)) \exp(-2r \lambda_+) + \lambda_+ \exp(d(-\lambda_+ + \lambda_+))(1 - \exp(2r \lambda_-))}$$

$$= \exp(2r \lambda_+).$$

Summarizing the above estimates, we obtain

$$\lim_{d \to \infty} \left[ \gamma'(U^*) U_f(c, d) - U_{pf}(c, d) - \gamma'(U^{**}) U_i(c, d) - U_{pi}(c, d) \right] = \gamma'(U_-) \exp(2r \lambda_-) - \gamma'(U_+) = \Gamma_\infty.$$

Since $\Gamma(c, d)$ is continuous for $d$, the sign of $\Gamma(c, d)$ coincides with that of $\Gamma_\infty(c)$ for sufficiently large $d$ when $\Gamma_\infty(c) \neq 0$ holds.

Suppose that $\Gamma_d(c_0) \Gamma_\infty(c_0) < 0$ for a constant $c_0 > 0$. Then, the intermediate value theorem implies that there exists a constant $d_0 > 0$ satisfying (4.14), that is, $\Gamma(c_0, d_0) = 0$. Since we have lemma 4(i) and $\gamma$ is strictly decreasing, there exists a constant $M > 0$ such that (4.4) holds. To see the existence of a constant $c_0 > 0$ satisfying $\Gamma_d(c_0) \Gamma_\infty(c_0) < 0$, we show the following lemma.

**Lemma 4.2.**

(i) Suppose $\gamma'(0) - \gamma''(0) = 0$. Then, for sufficiently large $c > 0$, the sign of $\gamma'(0) - \gamma''(0)$ coincides with that of $\Gamma_\infty(c)$.

(ii) Suppose that

$$\frac{1 + 4r^2}{8r^2} \gamma'(0) - \gamma''(0) < 0.$$

Then, we have $\Gamma_0(c) < 0$ for sufficiently large $c > 0$.

**Proof.**

(i) Note that

$$\Gamma_\infty(c) \equiv \exp(2r \lambda_+(c)) \gamma'(U_-(-c)) - \gamma'(U_+(c)) \exp(-2r \lambda_+(c)).$$

Since there exists a constant $U^* \in (U_-, U_+)$ such that

$$\gamma'(U_-) - \gamma'(U_+) = (U_+ - U_-) \gamma''(U^*),$$

we have

$$\gamma'(U_-) - \gamma'(U_+) \exp(-2r \lambda_+) = [\gamma'(U_-) - (U_+ - U_-) \gamma''(U^*)] - \gamma'(U_+) \exp(-2r \lambda_+$$

$$= \gamma'(U_+) (1 - \exp(-2r \lambda_+)) - (U_+ - U_-) \gamma''(U^*)$$

$$= (1 - \exp(-2r \lambda_+)) \left[ \gamma'(U_-) - \gamma''(U^*) \frac{U_+ - U_-}{1 - \exp(-2r \lambda_+)} \right].$$

Thus, the sign of $\Gamma_\infty(c)$ coincides with that of

$$\gamma'(U_+(c)) - \gamma''(U^*) \frac{U_+ - U_-}{1 - \exp(-2r \lambda_+)}.$$
Note that, taking the $c \to \infty$ limit, we have
\[ \theta(c) \to \infty, \quad \lambda_+(c) \to 0, \quad \lambda_-(c) \to -\infty, \quad \frac{-\lambda_-(c)}{2\theta(c)} \to 1, \quad U_\beta(r, c) \to 0, \] (4.16)
and $U_\alpha < U^* < U_\beta$ yields $U^*(c) \to 0$. For later use, we introduce the function $g(x) = (1 - \exp(-2rx))/x$ for $x > 0$, which satisfies
\[ \lim_{c \to \infty} g(\lambda_+(c)) = \lim_{c \to \infty} \frac{1 - \exp(-2r\lambda_+(c))}{\lambda_+(c)} = \lim_{c \to \infty} \frac{2r\lambda_+(c) + o(\lambda_+(c))}{\lambda_+(c)} = 2r. \] (4.17)
Then, it follows from $\lambda_+\lambda_- = 1$ and (4.9), that
\[ \lim_{c \to \infty} \frac{U_\alpha(c)}{1 - \exp(-2r\lambda_+(c))} = \lim_{c \to \infty} -\frac{\lambda_-(c)}{2\theta(c)} = 1, \]
\[ \lim_{c \to \infty} \frac{U_\beta(c)}{1 - \exp(-2r\lambda_+(c))} = \lim_{c \to \infty} \frac{1 - \exp(2r\lambda_-(c))}{2\theta(c)g(\lambda_+(c))} = 0, \] (4.18)
and thus
\[ \lim_{c \to \infty} \left[ \gamma'(U_\alpha(c)) - \gamma''(U^*) \frac{U_\alpha(c) - U_\beta(c)}{1 - \exp(-2r\lambda_+(c))} \right] = \gamma'(0) - \gamma''(0), \]
which concludes that if $\gamma'(0) - \gamma''(0) < 0$, the sign of $\Gamma_\infty(c)$ coincides with that of $\gamma'(0) - \gamma''(0)$ for sufficiently large $c > 0$.

We introduce the function $h(c) = U_\beta(c) - U_\alpha(c)\exp(-2r\lambda_+(c))$. Then, we have
\[ h = \left[ 1 - \exp(-2r\lambda_+) \right] \frac{U_\beta - U_\alpha\exp(-2r\lambda_+)}{1 - \exp(-2r\lambda_+)} \]
and it follows from (4.16) and (4.18) that
\[ \lim_{c \to \infty} \left[ \frac{U_\beta(c)}{1 - \exp(-2r\lambda_+(c))} - \frac{U_\alpha(c)\exp(-2r\lambda_+(c))}{1 - \exp(-2r\lambda_+(c))} \right] = -1, \]
which implies that $h(c) < 0$ holds for sufficiently large $c > 0$. Hence, we find from lemma A.2(ii) and (4.15) that $U_\beta(c, 0) < U_\alpha(c, 0)$ and
\[ U_\beta(c)(1 + \exp(2r\lambda_-(c))) < U_\alpha(c) + U_\beta(c)(1 + \exp(-2r\lambda_+(c))), \]
for sufficiently large $c > 0$. It follows from the mean value theorem that there exist constants $U^*$ and $U^{**}$ such that
\[ U_\alpha(1 + \exp(2r\lambda_-)) < U^* < U_\beta + U_\alpha < U^{**} < U_\alpha(1 + \exp(-2r\lambda_+)), \] (4.19)
and
\[ \gamma(U_\alpha + U_\beta) - \gamma(U_\alpha(1 + \exp(2r\lambda_-))) = (U_\alpha - U_\beta\exp(2r\lambda_-))\gamma'(U^*), \]
\[ \gamma(U_\alpha(1 + \exp(-2r\lambda_+)) - \gamma(U_\alpha + U_\beta) = (U_\alpha\exp(-2r\lambda_+) - U_\beta)\gamma'(U^{**}). \]
Thus, we obtain
\[ \Gamma_0(c) = (U_\alpha(c) - U_.(c)\exp(2r\lambda_-(c)))\gamma'(U^*) - (U_\alpha(c)\exp(-2r\lambda_+(c)) - U_.(c))\gamma'(U^{**}). \]
Note that
\[ \frac{U_\beta - U_\alpha\exp(2r\lambda_-)}{1 - \exp(-2r\lambda_+)} = \frac{1}{2\theta} \left( -\lambda_- - \frac{1 - \exp(-2r\lambda_-)}{g(\lambda_-)} \exp(2r\lambda_-) \right), \]
\[ \frac{U_\alpha\exp(-2r\lambda_+)}{1 - \exp(-2r\lambda_+)} = \frac{1}{2\theta} \left( -\lambda_+\exp(-2r\lambda_+) - \frac{1 - \exp(2r\lambda_+)}{g(\lambda_+)} \right), \]
and there exists a constant $U^{***} \in (U^*, U^{**})$ such that
\[ \gamma'(U^{**}) - \gamma'(U^*) = (U^{**} - U^*)\gamma''(U^{***}). \]
Then, we find
\[
\Gamma(c) = \frac{\eta_0(c)}{(1 - \exp(-2r\lambda_+))(U^{**} - U^{*})} \\
= \frac{1}{2\theta(U^{**} - U^{*})} \left[ \left( -\lambda_- - \frac{1 - \exp(2r\lambda_-)}{g(\lambda_-)} \exp(2r\lambda_-) \right) \gamma'(U^{*}) \\
- \left( -\lambda_- \exp(-2r\lambda_-) - \frac{1 - \exp(2r\lambda_-)}{g(\lambda_-)} \right) \gamma'(U^{**}) \right] \\
= \frac{1}{2\theta(U^{**} - U^{*})} \left[ -\lambda_-(\gamma'(U^{*}) - \exp(-2r\lambda_-)\gamma'(U^{**})) \\
- \frac{1 - \exp(2r\lambda_-)}{g(\lambda_-)}(\exp(2r\lambda_-)\gamma'(U^{*}) - \gamma'(U^{**})) \right] \\
= \frac{1}{2\theta(U^{**} - U^{*})} \left[ -\lambda_-(\gamma'(U^{*}) - \exp(-2r\lambda_-)\gamma'(U^{*}) + (U^{**} - U^{*})\gamma''(U^{**})) \\
- \frac{1 - \exp(2r\lambda_-)}{g(\lambda_-)}(\exp(2r\lambda_-)\gamma'(U^{*}) - (\gamma'(U^{*}) + (U^{**} - U^{*})\gamma''(U^{**}))) \right] \\
= \left[ g(\lambda_+) + \frac{(1 - \exp(2r\lambda_-))^2}{2\theta g(\lambda_+)} - \exp(-2r\lambda_+)\frac{-\lambda_-}{2\theta} \right] \gamma''(U^{**}) \\
+ \left[ 1 - \exp(2r\lambda_-) - \frac{2\theta g(\lambda_+)}{2\theta g(\lambda_+)} - \exp(-2r\lambda_+)\frac{-\lambda_-}{2\theta} \right] \gamma''(U^{**}).
\]

Owing to \((1 - \exp(-2r\lambda_-))(U^{**} - U^{*}) > 0\), the sign of \(\eta_0(c)\) coincides with that of \(\Gamma(c)\). Since it follows from (4.19) that
\[
0 < U^{**} - U^{*} < U_{\alpha}(1 + \exp(-2r\lambda_-)) - U_{\alpha}(1 + \exp(2r\lambda_-)) \equiv \Delta U^{*},
\]
and \(\gamma'(u) < 0\) for \(u > 0\), we have
\[
\tilde{\Gamma}(c) = \frac{g(\lambda_+(c))}{2\theta(c) U^{*}(c)} \gamma'(U^{*}) + \frac{(1 - \exp(2r\lambda_-))g(\lambda_+(c))}{2\theta(c) U^{*}(c)} \gamma'(U^{*}) \\
+ \left[ 1 - \exp(2r\lambda_-) - \frac{2\theta g(\lambda_+)}{2\theta g(\lambda_+)} - \exp(-2r\lambda_+)\frac{-\lambda_-}{2\theta} \right] \gamma''(U^{**}) \\
\tilde{\Gamma}_1(c) + \tilde{\Gamma}_2(c) + \tilde{\Gamma}_3(c).
\]

It follows from (4.16) and (4.18)
\[
\lim_{c \to \infty} 2\theta(c) U_{\alpha}(c) = \lim_{c \to \infty} g(\lambda_{\alpha}(c)) = 2r, \quad \lim_{c \to \infty} 2\theta(c) U_{\alpha}(c) = \lim_{c \to \infty} g(-\lambda_{-}(c)) = 0,
\]
which yields \(\lim_{c \to \infty} 2\theta(c) \Delta U^{*}(c) = 4r\). In addition, considering (4.19) and \(U^{*} < U^{***} < U^{**}\), we have \(\lim_{c \to \infty} U^{*}(c) = \lim_{c \to \infty} U^{***}(c) = 0\). Hence, we obtain
\[
\lim_{c \to \infty} \Gamma_1(c) = \frac{1}{2} \gamma'(0), \quad \lim_{c \to \infty} \Gamma_2(c) = \frac{1}{8r^2} \gamma''(0), \quad \lim_{c \to \infty} \Gamma_3(c) = -\gamma''(0),
\]
which yields
\[
\lim_{c \to \infty} \tilde{\Gamma}(c) \leq \frac{1}{8r^2} \gamma''(0) < 0.
\]
Since \(\eta_0(c)\) has the same sign as \(\tilde{\Gamma}(c)\), we conclude that \(\eta_0(c) < 0\) holds for sufficiently large \(c > 0\). \(\square\)

Since the assumption of theorem 2.2–3 gives
\[
\gamma'(0) - \gamma''(0) > 0, \quad \frac{1}{8r^2} \gamma''(0) - \gamma''(0) < 0,
\]
we have \(\eta_0(c) < 0 < \Gamma_\infty(c)\) for sufficiently large \(c > 0\). Thus, lemma 4.1 concludes that there exists a constant \(\mu > 0\) satisfying (4.4).
5. Conclusion

We showed the global existence and uniqueness of the weak solution for the model equations and gave sufficient conditions for the existence of the bimodal travelling solution. The numerical calculations for the sufficiently long interval (as shown in figure 2) have suggested that this solution is stable since if the perturbation is given to the distance between the two self-propelled materials, they continue to move back to the original state. By the use of computer-aided analysis, we can perform stability analysis in semi-rigorous in the same way as in [29]. However, it is difficult to evaluate the essential spectrum and the solution to the nonlinear equations satisfying the eigenvalues so that the rigorous stability analysis of the bimodal travelling solution has not been completed yet, which is future work.

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Data availability statements

All data that support the findings of this study are included within the article (and any supplementary files).

Appendix Auxiliary lemmas and their proofs

We show some useful lemmas for the proof of theorem 2.2. Recall the following functions:

\[ U_{pf}(c, d) = U_-(c)(1 + \exp(d\lambda_-(c))\exp(2r\lambda_-(c))), \]
\[ U_{pr}(c, d) = U_+(c) + U_-(c)\exp(d\lambda_-(c)), \]
\[ U_{lf}(c, d) = U_-(c)\exp(-d\lambda_+(c)) + U_+(c), \]
\[ U_{lr}(c, d) = U_+(c)(1 + \exp(-d\lambda_+(c))\exp(-2r\lambda_+(c))), \]

and

\[ \Delta U_p(c, d) = U_{pf}(c, d) - U_{pr}(c, d), \quad \Delta U_l(c, d) = U_{lf}(c, d) - U_{lr}(c, d). \]

For any \( c > 0 \), \( \lambda_-(c) \) satisfy \( \lambda_-(c) < -\lambda_+(c) < 0 \) and \( \lambda_+(c)\lambda_-(c) = -1 \), and \( U_\pm(c) \) are given by

\[ U_\pm(c) = \frac{1}{2\theta(c)} \left( 1 - \exp(\pm2r\lambda_\pm(c)) \right) > 0, \]
with \( \theta(c) > 0 \). For later use, we introduce the following function:

\[
\Delta U(r) = \Delta U(r; c) \equiv U_r(c) - U_r(c) = \frac{-1}{2\theta(c)} \left( \frac{1 - \exp(2r\lambda_-(c))}{\lambda_-(c)} + 1 - \exp(-2r\lambda_+(c)) \right).
\]

For any fixed \( c > 0 \), \( \Delta U(r) \) is negative for \( r > 0 \). Indeed, we have \( \theta(\Delta U)'(r) = \exp(2r\lambda_-) - \exp(-2r\lambda_+) < 0 \) and \( \Delta U(r) \to 0 \) as \( r \to 0 \). Then, the following two lemmas hold.

**Lemma 1.** For any \( c, d > 0 \), \( (U_{pf}, U_{pr}, U_{df}, U_{df}) \) satisfy

(i) \( \Delta U_p(c, d) < 0 \),

(ii) \( \Delta U_p(c, d) < \Delta U_i(c, d) \),

(iii) \( U_{pf}(c, d) < U_{df}(c, d) \).

**Proof.** The claim directly follows from the following calculations:

\[
(i) \quad \Delta U_p = U_r(1 + \exp(d\lambda_-)\exp(2r\lambda_-)) - (U_r + U_\pm \exp(d\lambda_-)) \\
= -U_\pm \exp(d\lambda_-)(1 - \exp(2r\lambda_-)) + \Delta U < 0.
\]

\[
(ii) 2\theta(\Delta U_p - \Delta U_i)(c, d) \\
= \left( \Delta U - \lambda_- \exp(d\lambda_-)(1 - \exp(2r\lambda_-))^2 \right) - \left( \Delta U - \lambda_- \exp(-d\lambda_-)(1 - \exp(-2r\lambda_-))^2 \right) \\
= \lambda_- \exp(-d\lambda_-)(1 - \exp(-2r\lambda_-))^2 - \lambda_- \exp(d\lambda_-)(1 - \exp(2r\lambda_-))^2 < 0.
\]

\[
(iii) U_{pf} - U_{df} = U_r(1 + \exp(d\lambda_-)\exp(2r\lambda_-)) - (U_r \exp(-d\lambda_-) + U_\pm) \\
= U_r \exp(-d\lambda_-)\exp(2r\lambda_-) - U_r \exp(-d\lambda_-) \\
< \exp(-d\lambda_-)\Delta U < 0.
\]

**Lemma Appendix A.2.** Let \( h(c) = U_r(c) - U_i(c)\exp(-2r\lambda_+(c)) \). Then, for any \( c > 0 \), there exist constants \( d^a(c) > 0 \) and \( d^{**}(c) > 0 \) such that

(i) \( \Delta U_i(c, d) < 0 \) holds if and only if

\[
d > \begin{cases} 
0 & \text{if } h(c) \leq 0, \\
d^a(c) & \text{if } h(c) > 0.
\end{cases}
\]

(ii) \( U_{pf}(c, d) < U_{df}(c, d) \) holds if and only if

\[
d > \begin{cases} 
0 & \text{if } h(c) \leq 0, \\
d^{**}(c) & \text{if } h(c) > 0.
\end{cases}
\]

**Proof.**

(i) Since we have

\[
\Delta U_i = U_r \exp(-d\lambda_+) + U_\pm - (U_r(1 + \exp(-d\lambda_-)\exp(-2r\lambda_+))) \\
= U_r \exp(-d\lambda_+)\exp(-2r\lambda_+) + \Delta U,
\]

\( \Delta U_i(c, d) \) is strictly decreasing for \( d > 0 \) and satisfies

\[
\lim_{d \to 0} \Delta U_i(c, d) = U_r(c) - U_i(c)\exp(-2r\lambda_+(c)) = h(c), \\
\lim_{d \to \infty} \Delta U_i(c, d) = \Delta U < 0.
\]

If we have \( h(c) \leq 0 \) for fixed \( c > 0 \), \( \Delta U_i(c, d) < 0 \) holds for any \( d > 0 \). In the case of \( h(c) > 0 \), there exists a constant \( d^a(c) > 0 \) such that \( \Delta U_i(c, d^a(c)) = 0 \) and \( \Delta U_i(c, d) < 0 \) holds for any \( d > d^a \). Note that \( d^a(c) \) is explicitly expressed by

\[
d^a(c) = -\frac{1}{\lambda_+(c)} \log \left( \frac{U_r(c) - U_i(c)}{U_i(c)(1 - \exp(-2r\lambda_+(c)))} \right).
\]

We consider the following function:
\[ f_l(d) \equiv 2d(U_{pr}(c, d) - U_r(c, d)) \]

\[ = \lambda_r \exp(-d\lambda_r)\exp(-2r\lambda_r)(1 - \exp(-2r\lambda_r)) + \lambda_r \exp(d\lambda_r)(1 - \exp(2r\lambda_r)). \]

Then, \( f_l'(d) \) is given by

\[ f_l'(d) = \exp(-d\lambda_r)\exp(-2r\lambda_r)(1 - \exp(-2r\lambda_r)) - \exp(d\lambda_r)(1 - \exp(2r\lambda_r)) \]

\[ = \exp(-d\lambda_r)[\exp(-2r\lambda_r)(1 - \exp(-2r\lambda_r)) - \exp(d\lambda_r)(1 - \exp(2r\lambda_r))] \]

\[ = \exp(-d\lambda_r)f_2(d), \]

and it follows from \( \lambda_r - \lambda_+ < 0 \) that \( f_2(d) \) is a strictly increasing function satisfying

\[ \lim_{d \to 0} f_2(d) = \exp(-2r\lambda_+)(1 - \exp(-2r\lambda_+)) - (1 - \exp(2r\lambda_+)) < 0, \]

\[ \lim_{d \to \infty} f_2(d) = \exp(-2r\lambda_+)(1 - \exp(-2r\lambda_+)) > 0. \]

Hence, there exists a constant \( d_0 > 0 \) such that \( f_l'(d) \) is monotonically decreasing for \( 0 < d < d_0 \) and monotonically increasing for \( d > d_0 \). Considering

\[ \lim_{d \to 0} f_l'(d) = \lambda_r \exp(-2r\lambda_+)(1 - \exp(-2r\lambda_+)) + \lambda_+(1 - \exp(2r\lambda_+)) = 2\theta(c)h(c), \]

\[ \lim_{d \to \infty} f_l'(d) = 0, \]

we find that, for any \( c > 0 \) satisfying \( h(c) \leq 0 \), \( f_l(d) \) is negative for \( d > 0 \) and thus

\[ f_l(d) = 2d(U_{pr}(c, d) - U_r(c, d)) < 0 \]

holds for any \( d > 0 \). In the case of \( h(c) > 0 \), there exists a constant \( d^{**}(c) > 0 \) such that \( f_l(d^{**}(c)) = 0 \) and \( f_l(d) = 2d(U_{pr}(c, d) - U_r(c, d)) < 0 \) for any \( d > d^{**} \). Then, \( d^{**}(c) \) is given by

\[ d^{**}(c) = \frac{1}{\lambda_+(c) + \lambda_r(c)} \log\left( \frac{\exp(-2r\lambda_r(c))U_1(c)}{U_1(c)} \right). \]

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