RATIONAL LIMIT CYCLES OF ABEL EQUATIONS

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Abstract. We deal with Abel equations \( \frac{dy}{dx} = A(x)y^2 + B(x)y^3 \), where \( A(x) \) and \( B(x) \) are real polynomials. We prove that these Abel equations can have at most two rational limit cycles and we characterize when this happens. Moreover we provide examples of these Abel equations with two nontrivial rational limit cycles.

1. Introduction and statement of the results. We study the Abel equations

\[ \frac{dy}{dx} = A(x)y^2 + B(x)y^3, \tag{1} \]

where \( x \in [0,1] \) and \( y \) are real variables and \( A(x) \) and \( B(x) \) are polynomials. The limit cycles of these equations have been intensively investigated mainly when the functions \( A(x) \) and \( B(x) \) are periodic (see for instance [1, 2, 3, 4, 5, 6, 9, 11, 14, 15, 17, 18, 19, 20, 21, 23, 24, 25, 26]), and also when \( A(x) \) and \( B(x) \) are polynomial (see for instance [10, 12, 13, 16, 22]). Here we are interested in the rational limit cycles of equation (1) when the functions \( A(x) \) and \( B(x) \) are polynomials.

A periodic solution of equation (1) is a solution \( y(x) \) defined in the closed interval \([0,1]\) such that \( y(0) = y(1) \), note that without loss of generality we are assuming that the period is 1. We say that a limit cycle is a periodic solution isolated in the set of periodic solutions of a differential equation (1).

The limit cycle is called a polynomial limit cycle if the periodic solution \( y(x) \) is a polynomial in the variable \( x \). In particular the authors of [16] proved that any polynomial limit cycle of the differential equation (1) is of the form \( y = c \) with \( c \in \mathbb{R} \), and that if a polynomial limit cycle exists with \( c \neq 0 \), then no other polynomial limit cycles can exist.

In this paper we study the existence of rational limit cycles for the differential equation (1), i.e. we want to consider limit cycles of the form \( y(x) = q(x)/p(x) \)

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where \( p, q \in \mathbb{R}[x] \) and \((p(x), q(x)) = 1\). As usual \( \mathbb{R}[x] \) denotes the set of all real polynomials in the variable \( x \). We will study only the rational limit cycles that are not polynomial limit cycles.

We recall that in [19] the authors provide examples of differential equations (1) having one or two rational limit cycles, and that these limit cycles are hyperbolic. In the present paper we prove that in fact the maximum number of rational limit cycles of equations (1) is two and that this bound can be reached with nonhyperbolic limit cycles.

Our main theorem is the following one.

**Theorem 1.1.** Differential equation (1) has at most two rational limit cycles and this bound is reached.

It was proved in [19] that the differential equation (1) with

\[
A(x) = -(x - 1)(8x^3 - 35x^2 + 24x - 3), \\
B(x) = (2 + x^2 - x)(x^2 - x + 1)(x - 4)(x + 1)(x - 3)(5 + 3x^2 - 12x)
\]

has the following two rational limit cycles

\[
p_1(x) = (x - 3)(x + 1)(x - 4)(x^2 - x + 1), \\
p_2(x) = (x - 3)(x + 1)(x - 4)(x^2 - x + 2).
\]

Moreover, they also prove that both are hyperbolic limit cycles (we recall that if we denote by \( y(x, x_0) \) the solution of the differential equation (1) such that \( y(0, x_0) = x_0 \). It holds that \( y(x, x_0) \) is a periodic solution of the differential equation (1) if it is a zero of the function \( \phi(x_0) = y(1, x_0) - x_0 \) and we say that it is a limit cycle if it is an isolated zero. When we have a simple isolated zero of \( \phi(x_0) \), i.e. \( \phi(x_0) = 0 \) and \( \phi'(x_0) \neq 0 \), then we say that \( y(x, x_0) \) is a hyperbolic limit cycle).

Theorem 1.1 is proved in section 2. The proof is as follows: first we show that system (1) has at most three rational limit cycles, and after we prove that in case system (1) has three periodic solutions it has a center because system (1) satisfies the so-called polynomial composition condition (see the proof of Theorem 1.1 for a precise definition). In particular this implies that the periodic solutions are not isolated and so they cannot have three rational limit cycles.

2. Proof of Theorem 1.1. We start with two auxiliary results.

**Lemma 2.1.** The rational function \( y = q(x)/p(x) \) with \( p(x) \) non-constant is a periodic solution of the differential equation (1) if and only if \( q(x) = c \in \mathbb{R} \setminus \{0\} \), \( p(0) = p(1) \) and \( p(x) \) has no zero in \([0, 1]\) and

\[
cB(x) + \frac{p(x)p'(x)}{c} + p(x)A(x) = 0. \tag{2}
\]

Lemma 2.1 is in fact Lemma 2 of [19]. We provide its proof in the present paper for completeness.

**Proof.** For the reverse implication we note that if \( q(x) = c \in \mathbb{R} \setminus \{0\}, p(0) = p(1), p(x) \) has no zero in \([0, 1]\) and equality (2) holds then it is clear that the rational function \( y = c/p(x) \) is a periodic solution of equation (1).
For the direct implication we note that if \( y(x) = q(x)/p(x) \) is a periodic solution of equation (1) then \( p(x) \neq 0 \) for \( x \in [0, 1] \). Let \( g(x, y) = p(x) y - q(x) \). Then

\[
0 = f r a c d g(x, y) dx |_{g(x,y)=0} = p'(x) y + p(x) \frac{dy}{dx} - q'(x) \\
= p'(x) y + p(x) (A(x)y^2 + B(x)y^3) - q'(x).
\]

Note that \( g(x, y) \) is irreducible, so there exists a polynomial \( k(x, y) \) so that

\[
p'(x) y + p(x) (A(x)y^2 + B(x)y^3) - q'(x) = k(x, y) g(x, y).
\] (3)

Since the highest degree in \( y \) in the left-hand side is 3 and the highest degree in \( y \) in \( g(x, y) \) is 1 we get that the highest degree in \( y \) in \( k(x, y) \) is 2 and so it can be written as \( k(x, y) = k_0(x) + k_1(x)y + k_2(x)y^2 \), where \( k_0, k_1, k_2 \in \mathbb{R}[x] \). Comparing the coefficients of \( y^0, y^1, y^2 \) and \( y^3 \) in (3) we get

\[
q'(x) = k_0(x) q(x),
\]

\[
p'(x) = k_0(x) p(x) - k_1(x) q(x),
\]

\[
p(x) A(x) = k_1(x) p(x) - k_2(x) q(x),
\]

\[
p(x) B(x) = k_2(x) p(x).
\] (4)

From the first relation we get that \( q(x) | q'(x) \). This implies that \( q(x) \) is a constant that we denote by \( c \), that is, \( q(x) = c \in \mathbb{R} \). If \( c = 0 \) then \( y = q(x)/p(x) = 0 \). This is not possible and so \( c \neq 0 \). Moreover, \( y = q(x)/p(x) = c/p(x) \) is a periodic solution, then \( p(0) = p(1) \). From the second relation we get that \( k_1(x) = -p'(x)/c \) and from the fourth relation we obtain that \( k_2(x) = B(x) \). Substituting \( k_1(x) \) and \( k_2(x) \) in the third relation we get (2) and the direct inclusion is proved.

Note that it is not restrictive to take \( c = 1 \) and consider all rational limit cycles of the form \( y = 1/p(x) \) with \( p(x) \) satisfying \( p(0) = p(1) \) with \( p(x) \) having no zero in \([0, 1]\) and satisfying (2).

From (2) we must have that \( B(x) \) is multiple of \( p(x) \) and so \( B(x) = p(x) r(x) \) for some polynomial \( r(x) \). Therefore, (2) becomes

\[
r(x) + p'(x) + A(x) = 0 \quad \text{and so} \quad p(x) = \kappa - \int (A(s) + r(s)) ds, \quad \kappa \in \mathbb{R}.
\] (5)

Assume that equation (1) has two rational limit cycles, \( y(x) = 1/p_1(x) \) and \( y(x) = 1/p_2(x) \) with \( p_1(x), p_2(x) \in \mathbb{R}[x] \) \( \setminus \mathbb{R} \). Denote by \( q(x) = (p_1(x), p_2(x)) \), i.e. the maximum common divisor of the polynomials \( p_1(x) \) and \( p_2(x) \), and consequently

\[
p_1(x) = q(x) r_1(x), \quad p_2(x) = q(x) r_2(x)
\] (6)

with \( q(x), r_1(x) \in \mathbb{R}[x] \) and \( (r_1(x), r_2(x)) = 1 \). Note that in view of the above observation we must have that

\[
B(x) = q(x) r_1(x) r_2(x) r_3(x)
\] (7)

for some \( r_3(x) \in \mathbb{R}[x] \).

**Lemma 2.2.** The following equalities hold

\[
r_3(x) = q'(x) \quad \text{and} \quad r_1(x) - r_2(x) = c \in \mathbb{R}.
\] (8)
Proof. Note that in view of (5) we have
\[ q(x)r_1(x) = \kappa_0 - \int (A(s) + r_2(s)r_3(s)) \, ds, \]
\[ q(x)r_2(x) = \kappa_1 - \int (A(s) + r_1(s)r_3(s)) \, ds, \] (9)
with \( \kappa_0, \kappa_1 \in \mathbb{R} \). Hence,
\[ q'(x)r_1(x) + q(x)r'_1(x) = - A(x) - r_2(x)r_3(x), \]
\[ q'(x)r_2(x) + q(x)r'_2(x) = - A(x) - r_1(x)r_3(x), \]
and so
\[ q'(x)(r_1(x) - r_2(x)) + q(x)(r_1(x) - r_2(x))' = (r_1(x) - r_2(x))r_3(x), \]
which gives
\[ q(x)(r_1(x) - r_2(x))' = (r_1(x) - r_2(x))(r_3(x) - q'(x)). \]
Hence
\[ \frac{(r_1(x) - r_2(x))'}{r_1(x) - r_2(x)} = \frac{r_3(x)}{q(x)} - \frac{q'(x)}{q(x)}. \]
Therefore
\[ r_1(x) - r_2(x) = \kappa_2 \frac{1}{q(x)} \exp \left( \int \frac{r_3(s)}{q(s)} \, ds \right), \] (10)
for some \( \kappa_2 \in \mathbb{R} \). Let \( H(x) = (q(x), r_3(x)) \). Then
\[ \frac{r_3(x)}{q(x)} = \frac{H(x)r_3(x)}{H(x)q(x)} = \frac{\bar{r}_3(x)}{\bar{q}(x)}. \]
If \( \deg(\bar{r}_3(x)) > \deg(\bar{q}(x)) \) then we make the Euclidean division and we get
\[ \bar{r}_3(x) = r_4(x)\bar{q}(x) + r_5(x) \]
where \( \deg(r_5(x)) < \deg(\bar{q}(x)) \). Therefore we have
\[ \frac{\bar{r}_3(x)}{\bar{q}(x)} = r_4(x) + \frac{r_5(x)}{\bar{q}(x)}. \]
Integrating we get
\[ r_1(x) - r_2(x) = \kappa_2 \frac{1}{q(x)} \exp(\int r_4(x)) \exp \left( \int \frac{r_5(x)}{\bar{q}(x)} \right). \] (11)
The first factor in (11) cannot cancel with the second factor of (11) and this gives a contradiction because \( r_1(x) - r_2(x) \) is a polynomial. So \( \deg(\bar{r}_3(x)) \leq \deg(\bar{q}(x)) \). In this case we consider two cases: \( \bar{q}(x) \) is not-square free or \( \bar{q}(x) \) is square-free. In the first case using the affine transformation \( x \mapsto x + \alpha \) with \( \alpha \in \mathbb{C} \) (if necessary) we can write \( \bar{q}(x) = x^\mu r(x) \) where \( \mu > 1 \) and \( r(0) \neq 0 \). Moreover \( \bar{r}_3(0) \neq 0 \) because \( \bar{r}_3(x) \) and \( \bar{q}(x) \) are coprime. If we develop \( \bar{r}_3(x)/\bar{q}(x) \) in simple fractions of \( x \) we obtain
\[ \frac{\bar{r}_3(x)}{\bar{q}(x)} = \frac{c_\mu}{x^\mu} + \frac{c_{\mu-1}}{x^{\mu-1}} + \cdots + \frac{c_1}{x} + \frac{\alpha(x)}{r(x)}. \]
where $\alpha(x)$ is a polynomial with $\deg(\alpha(x)) < \deg(r(x))$ and $c_i \in \mathbb{C}$ for $i = 1, \ldots, \mu$. Note that $c_\mu = r(0)/\vec{q}(0) \neq 0$. Integrating we get

$$r_1(x) - r_2(x) = \kappa_2 \frac{1}{\vec{q}(x)} \exp\left(c_\mu \frac{1}{1 - \mu x^\mu - 1}\right) \exp\left(\int \left(\frac{c_{\mu-1}}{x^\mu} + \cdots + \frac{c_1}{x} + \frac{\alpha(x)}{r(x)}\right) dx\right).$$ (12)

The first exponential factor cannot cancel with any part of the second exponential factor and we get to a contradiction with the fact that $r_1(x) - r_2(x)$ is a polynomial. So $\bar{q}(x)$ is square-free. Then we have that

$$\int \frac{r_3(s)}{\bar{q}(s)} ds = \kappa_3 \log h(x), \quad \kappa_3 \in \mathbb{R}, \quad h(x) \in \mathbb{R}[x] \setminus \{0\}. \quad (13)$$

Therefore

$$\frac{\bar{r}_3(x)}{\bar{q}(x)} = \kappa_3 \frac{h'(x)}{h(x)},$$

where $h(x)$ is square-free, and so $h'(x)$ and $h(x)$ are coprime. Hence $\bar{q}(x) = h(x)$ and $\bar{r}_3(x) = \kappa_3 \bar{q}(x)$. Writing $H(x) = (q(x), r_3(x))$ we get $q(x) = H(x)h(x)$. From (10) and (13) we have

$$r_1(x) - r_2(x) = \kappa_2 \frac{1}{H(x)} \bar{q}(x)^{\kappa_3-1}.$$

Since $r_1(x) - r_2(x)$ must be a polynomial and $\kappa_3 \neq 0$, it follows that $H(x) = \bar{q}(x)^\ell$ for some $0 \leq \ell \leq \kappa_3 - 1$. Hence,

$$r_3(x) = \kappa_3 \bar{q}(x)^\ell \bar{q}'(x) \quad \text{and} \quad r_1(x) - r_2(x) = \kappa_2 \bar{q}(x)^{\kappa_3-1-\ell}. \quad (14)$$

On the other hand, doing a change of variables of the form $Y = \beta y$ where $\beta^2 = \text{sign}(\kappa_3)\kappa_3$, the Abel equation (1) becomes

$$\frac{dY}{dx} = \frac{A(x)}{\beta} Y^2 + \frac{B(x)}{\beta^2} Y^3 = \overline{A}(x) Y^2 + \overline{B}(x) Y^3. \quad (15)$$

Since $B(x) = \bar{q}(x)^{2\ell} r_1(x) r_2(x) \kappa_3 \overline{q}(x)$, then $\overline{B}(x) = \pm \bar{q}(x)^{2\ell} r_1(x) \bar{q}'(x)$. In what follows we shall work with the Abel equation (15).

Repeating the previous computations starting with the Abel equation (15) and since $\kappa_3 \geq 0$ we have that $\kappa_3 = 1$ and so $\ell = 0$ which yields $H(x) = 1$. We thus arrive to equation (14) which now writes

$$s_3(x) = q'(x) \quad \text{and} \quad s_1(x) - s_2(x) = \kappa_2.$$

This concludes the proof of the lemma.

\[\square\]

Note that from (6), (7) and Lemma 2.2 we have that

$$B(x) = q(x) q'(x) s_1(x) s_2(x). \quad (16)$$

**Proof of Theorem 1.1.** Assume that equation (1) has three rational limit cycles, $y_1 = 1/p_1(x)$ and $y_2 = 1/p_2(x)$ and $y_3 = 1/p_3(x)$ with $p_1, p_2, p_3 \in \mathbb{R}[x] \setminus \mathbb{R}$. Denote by $q_1(x) = (p_1(x), p_2(x))$, $q_2(x) = (p_1(x), p_3(x))$ and $q_3(x) = (p_2(x), p_3(x))$. In view of Lemma 2.2 we have

$$p_1(x) = q_1(x) s_1(x) = q_2(x) s_2(x),$$

$$p_2(x) = q_1(x)(s_1(x) + c_1) = q_3(x) s_3(x),$$

$$p_3(x) = q_2(x)(s_2(x) + c_2) = q_3(x)(s_3(x) + c_3), \quad (17)$$

$$p_1(x) = q_1(x) s_1(x) = q_2(x) s_2(x),$$

$$p_2(x) = q_1(x)(s_1(x) + c_1) = q_3(x) s_3(x),$$

$$p_3(x) = q_2(x)(s_2(x) + c_2) = q_3(x)(s_3(x) + c_3), \quad (17)$$
for some polynomials $s_1(x), s_2(x), s_3(x)$ and constants $c_1, c_2, c_3 \in \mathbb{R} \setminus \{0\}$. Hence, we get

$$
\begin{align*}
&\frac{p_2(x) - p_1(x)}{s_2(x)} = q_1(x)c_1, \ c_1 \in \mathbb{R}, \\
&\frac{p_3(x) - p_1(x)}{s_2(x)} = q_2(x)c_2, \ c_2 \in \mathbb{R}, \\
&\frac{p_3(x) - p_2(x)}{s_2(x)} = q_3(x)c_3, \ c_3 \in \mathbb{R},
\end{align*}
$$

and so

$$
q_2(x)c_2 = q_1(x)c_1 + q_3(x)c_3. \tag{18}
$$

We consider two situations.

*Case 1: $q_1(x)$ and $s_2(x)$ are coprime.* Note that from (17) we have that $q_1(x)s_1(x) = q_2(x)s_2(x)$, and then from (18) we get

$$
\frac{q_1(x)s_1(x)c_2}{s_2(x)} = q_2(x)c_2 = q_1(x)c_1 + q_3(x)c_3.
$$

In particular there exists $T(x) \in \mathbb{R}[x]$ so that

$$
q_3(x) = q_1(x)T(x),
$$

and consequently

$$
\frac{s_1(x)c_2}{s_2(x)} = c_1 + T(x)c_3,
$$

which yields

$$
s_1(x) = \frac{s_2(x)}{c_2}(c_1 + T(x)c_3).
$$

Therefore from (17) we get

$$
q_2(x)s_2(x) = q_1(x)s_1(x) = q_1(x)\frac{s_2(x)}{c_2}(c_1 + T(x)c_3),
$$

and so

$$
q_2(x) = q_1(x)\frac{c_1 + T(x)c_3}{c_2}.
$$

Hence we have

$$
\begin{align*}
p_1(x) &= q_1(x)s_1(x) = q_1(x)\frac{s_2(x)}{c_2}(c_1 + T(x)c_3), \\
p_2(x) &= q_1(x)(s_1(x) + c_1) = q_1(x)\left(\frac{s_2(x)}{c_2}(c_1 + T(x)c_3) + c_1\right), \\
p_3(x) &= q_2(x)(s_2(x) + c_2) = q_1(x)\frac{c_1 + T(x)c_3}{c_2}(s_2(x) + c_2).
\end{align*}
\tag{19, 20, 21}
$$

We consider two subcases.

*Subcase 1.1: Assume that $T(x)$ and $s_2(x) + c_2$ are coprime.* Then the maximum common divisor between $p_2(x)$ and $p_3(x)$ is $q_1(x)$. Indeed, we will show that

$$
r_1(x) = \frac{s_2(x)}{c_2}(c_1 + T(x)c_3) + c_1
$$

and

$$
r_2(x) = (c_1 + T(x)c_3)(s_2(x) + c_2)
$$

are coprime. Note that if $x^*$ is a zero of $c_1 + T(x)c_3$ then we have that $r_2(x^*) = 0$ but $r_1(x^*) = c_1 \neq 0$. Moreover, if $\hat{x}$ is a solution of $s_2(x) + c_2 = 0$ then $r_2(\hat{x}) = 0$.
but \( r_1(\hat{x}) = -(c_1 + T(\hat{x})c_3) + c_1 = T(\hat{x})c_3 \neq 0 \). Therefore, using \( p_1(x) \) and \( p_2(x) \) from (16) and (19) we can write
\[
B(x) = q_1(x)q'_1(x)\frac{s_2(x)}{c_2} (c_1 + T(x)c_3) \left( \frac{s_2(x)}{c_2} (c_1 + T(x)c_3) + c_1 \right),
\]
and from \( p_1(x) \) and \( p_3(X) \) we can write
\[
B(x) = q_1(x)q'_1(x) \left( \frac{s_2(x)}{c_2} (c_1 + T(x)c_3) + c_1 \right) \frac{c_1 + T(x)c_3}{c_2} (s_2(x) + c_2),
\]
and so
\[
s_2(x) = s_2(x) + c_2,
\]
which is not possible because \( c_2 \neq 0 \).

**Subcase 1.2:** Assume that \( T(x) \) and \( s_2(x) + c_2 \) are not coprime. Write
\[
T(x) = \alpha_1(x)\alpha_2(x), \quad s_2(x) + c_2 = \alpha_1(x)\alpha_3(x),
\]
where \( \alpha_2, \alpha_3 \in \mathbb{R}[x] \) and \( \alpha_1(x) \in \mathbb{R}[x] \setminus \mathbb{R} \). Then
\[
p_3(x) = q_1(x)\alpha_1(x)\alpha_3(x)\frac{c_1 + T(x)c_3}{c_2},
\]
\[
p_2(x) = q_1(x)\frac{\alpha_1(x)}{c_2}(c_1\alpha_3(x) + s_2(x)\alpha_2(x)c_3).
\]
We note that the maximum common divisor between \( p_2(x) \) and \( p_3(x) \) is \( q_1(x)\alpha_1(x) \).

To do so, we will show that
\[
r_3(x) = \alpha_3(x)(c_1 + T(x)c_3) \quad \text{and} \quad r_4(x) = c_1\alpha_3(x) + s_2(x)\alpha_2(x)c_3
\]
are coprime. If \( x^* \) is a zero of \( \alpha_3(x) \) then \( r_3(x^*) = 0 \) but \( r_4(x^*) = s_2(x^*)\alpha_2(x^*)c_3 = -c_2\alpha_2(x^*)c_3 \). Since \( \alpha_2(x) \) and \( \alpha_3(x) \) are coprime, we get that \( \alpha_2(x^*) \neq 0 \), and then \( r_4(x^*) \neq 0 \). Moreover, if \( c_1 + T(\hat{x})c_3 = 0 \) then \( r_4(\hat{x}) = c_1 \neq 0 \). So \( r_3(x) \) and \( r_4(x) \) are coprime.

From \( p_1(x) \), \( p_2(x) \), (16) and (19) we get
\[
B(x) = q_1(x)q'_1(x)\frac{s_2(x)}{c_2} (c_1 + T(x)c_3) \frac{\alpha_1(x)}{c_2}(c_1\alpha_3(x) + s_2(x)\alpha_2(x)c_3). \tag{22}
\]
Note that from \( p_2(x) \), \( p_3(x) \), (16) and (19) we have
\[
B(x) = q_1(x)\frac{c_2}{c_2} \alpha_1(x)(q_1(x)\alpha_1(x))'\alpha_3(x)(c_1 + T(x)c_3)(c_1\alpha_3(x) + s_2(x)\alpha_2(x)c_3). \tag{23}
\]
Comparing (22) with (23) we obtain
\[
\alpha_3(x)(q_1(x)\alpha_1(x))' = q'_1(x)(\alpha_1(x)\alpha_3(x) - c_2),
\]
\[
i.e. \quad -c_2q'_1(x) = -\alpha_3(x)q_1(x)\alpha'_1(x),
\]
which is not possible unless either \( \alpha_3(x) = 0 \) or \( \alpha'_1(x) = 0 \), but then \( q_1(x) \) would be constant, a contradiction. In short, Case 1 is not possible.

**Case 2:** \( q_1(x) \) and \( s_2(x) \) are not coprime. We write
\[
q_1(x) = R_1(x)R_2(x), \quad s_2(x) = R_1(x)R_3(x)
\]
with \( R_1(x), R_2(x), R_3(x) \in \mathbb{R}[x] \) and \( R_1(x) \notin \mathbb{R} \).

We consider two different subcases.

**Subcase 2.1:** \( R_3(x) = R \in \mathbb{R} \). So \( s_2(x) = R_1(x)R \) and \( q_1(x) = R_2(x)s_2(x)/R \).

We also consider two cases
From (6) we have $q_1(x)s_1(x) = q_2(x)s_2(x)$ and so $q_2(x) = R_2 s_1(x)/R$. Then
\[
\begin{align*}
p_1(x) &= \frac{R_2}{R} s_1(x)s_2(x), \\
p_2(x) &= \frac{R_2}{R} s_2(x)(s_1(x) + c_1), \\
p_3(x) &= \frac{R_2}{R} s_1(x)(s_2(x) + c_2).
\end{align*}
\]
From $p_1(x)$, $p_2(x)$, (16) and (19) we get
\[
B(x) = \left(\frac{R_2}{R}\right)^2 s_2(x)s_1(x)(s_1(x) + c_1)s'_2(x),
\]
and from $p_1(x)$, $p_3(x)$, (16) and (19) we obtain
\[
B(x) = \left(\frac{R_2}{R}\right)^2 s_2(x)s_1(x)s'_1(x)(s_2(x) + c_2),
\]
and so
\[
s'_2(x)(s_1(x) + c_1) = s'_1(x)(s_2(x) + c_2),
\]
which yields
\[
\frac{s'_2(x)}{s_2(x) + c_2} = \frac{s'_1(x)}{s_1(x) + c_1}
\]
and integrating
\[
\log(s_2(x) + c_2) = \kappa + \log(s_1(x) + c_1), \quad \kappa \in \mathbb{R}
\]
and so
\[
s_2(x) + c_2 = \kappa_1(s_1(x) + c_1), \quad \kappa_1 = e^\kappa \in \mathbb{R}^+.
\]
Hence
\[
\begin{align*}
p_1(x) &= \frac{R_2}{R} s_1(x)(\kappa_1(s_1(x) + c_1) - c_2), \quad (24) \\
p_2(x) &= \frac{R_2}{R} (\kappa_1(s_1(x) + c_1) - c_2)(s_1(x) + c_1), \quad (25) \\
p_3(x) &= \frac{R_2}{R} s_1(x)\kappa_1(s_1(x) + c_1), \quad (26)
\end{align*}
\]
and from (16) we obtain
\[
B(x) = \left(\frac{R_2}{R}\right)^2 \kappa_1 s_1(x)s'_1(x)(s_1(x) + c_1)(\kappa_1(s_1(x) + c_1) - c_2).
\]
Doing the rescaling $Y = \beta y$, we can assume that the constant $(R_2/R)^2\kappa_1 = 1$ and the constants $R_2/R$ and $\kappa_1$ in the expression of $p_1(x)$ is one (see the proof of Lemma 2.2). It follows from (24) that if there exists three rational periodic solutions, then there must exist a polynomial $s_1(x)$ and two different constants $c_1, c_2 \in \mathbb{R} \setminus \{0\}$ such that
\[
B(x) = s_1(x)s'_1(x)(s_1(x) + c_1)(s_1(x) + c_1 - c_2).
\]
Moreover the three periodic solutions can be chosen as
\[
\begin{align*}
p_1(x) &= s_1(x)(s_1(x) + c_1 - c_2), \\
p_2(x) &= (s_1(x) + c_1 - c_2)(s_1(x) + c_1), \\
p_3(x) &= s_1(x)(s_1(x) + c_1).
\end{align*}
\]
2.1.2: $R_2(x) \in \mathbb{R}[x] \setminus \mathbb{R}$. Since $R_3(x) = R$ we have $s_2(x) = R_1(x)R$ and $q_1(x) = R_2(x)s_2(x)/R$. Since $q_1(x)s_1(x) = q_2(x)s_2(x)$ we get $q_2(x) = R_2(x)s_1(x)/R$. From (18) we have

$$\frac{R_2(x)s_1(x)c_2}{R} = R_2(x)s_2(x)c_1 + q_3(x)c_3,$$

and so

$$q_3(x) = \frac{R_2(x)}{c_3} \left(x_1(x)c_2 - \frac{c_1s_2(x)}{R}\right).$$

In short

$$p_1(x) = \frac{R_2(x)}{R}s_1(x)s_2(x),$$

$$p_2(x) = \frac{R_2(x)}{R}s_2(x)(s_1(x) + c_1),$$

$$p_3(x) = \frac{R_2(x)}{R}s_1(x)(s_2(x) + c_2).$$

We consider two cases:

2.1.2.1: $s_1(x)$ and $s_2(x)$ are coprime. In this case the maximum common divisor between $p_2(x)$ and $p_1(x)$ is $R_2(x)$ and so from (16) we get

$$B(x) = \frac{R_2(x)}{R^2}s_2(x)(R_2(x)s_2(x)')s_1(x)(s_1(x) + c_1)$$

$$= \frac{R_2(x)}{R^2}s_2(x)s_1(x)R'_2(x)(s_1(x) + c_1)(s_2(x) + c_2),$$

and so

$$(R_2(x)s_2(x))' = R'_2(x)(s_2(x) + c_2),$$

that is

$$R'_2(x)s_2(x) + R_2(x)s'_2(x) = R'_2(x)s_2(x) + R'_2(x)c_2,$$

which yields

$$\frac{R'_2(x)}{R_2(x)} = \frac{s'_2(x)}{c_2}.$$ 

Hence $R_2(x) = e^{c_2 s_2(x)}$ which is not possible.

2.1.2.2: $s_1(x)$ and $s_2(x)$ are not coprime. In this case we write

$$s_1(x) = \kappa(x)s_1(x), \quad s_2(x) = \kappa(x)s_2(x),$$

with $\kappa(x), s_1(x), s_2(x) \in \mathbb{R}[x]$ with $\kappa(x) \notin \mathbb{R}$. Then

$$p_1(x) = \frac{R_2(x)}{R}\kappa^2(x)s_1(x)s_2(x),$$

$$p_2(x) = \frac{R_2(x)}{R}\kappa(x)s_2(x)(\kappa(x)s_1(x) + c_1),$$

$$p_3(x) = \frac{R_2(x)}{R}\kappa(x)s_1(x)(\kappa(x)s_2(x) + c_2).$$

Then

$$B(x) = \frac{R_2(x)}{R^2}\kappa(x)(R_2(x)\kappa(x))'s_2(x)(\kappa(x)s_1(x) + c_1)s_1(x)(\kappa(x)s_2(x) + c_2)$$

$$= \frac{R_2(x)}{R^2}\kappa(x)s_2(x)(R_2(x)\kappa(x)s_2(x))'(\kappa(x)s_1(x) + c_1)\kappa(x),$$

and so

$$(R_2(x)\kappa(x))'(\kappa(x)s_2(x) + c_2) = \kappa(x)(R_2(x)\kappa(x)s_2(x))'.$$
which yields
\[(R_2(x)\kappa(x))'c_2 = (R_2(x)\kappa(x))\kappa(x)s_2(x).\]
This is not possible because the left hand side has less degree than the right hand side. In summary, Subcase 2.1.2 is not possible.

**Subcase 2.2:** \(R_2(x) \in \mathbb{R}[x] \setminus \mathbb{R}.\) We have \(q_1(x) = R_1(x)R_2(x)\) and \(s_2(x) = R_1(x)R_3(x).\) Then
\[\frac{R_2(x)s_1(x)c_2}{R_3(x)} = R_1(x)R_2(x)c_1 + q_3(x)c_3.\]
Since \(R_2(x)\) and \(R_3(x)\) are coprime there exists \(T(x) \in \mathbb{R}[x]\) so that
\[q_3(x) = R_2(x)T(x),\]
and so
\[\frac{s_1(x)c_2}{R_3(x)} = R_1(x)c_1 + T(x)c_3,\]
which yields \(s_1(x) = R_4(x)R_3(x).\) Therefore, from \(p_1(x)\) in (6) we get
\[q_2(x)s_2(x) = q_1(x)s_1(x) = R_1(x)R_2(x)R_4(x) = q_2(x)R_1(x)R_3(x)\]
and so
\[q_2(x) = R_2(x)R_4(x).\]
Hence we have
\[p_1(x) = q_1(x)s_1(x) = R_1(x)R_2(x)R_3(x)R_4(x),\]
\[p_2(x) = q_1(x)(s_1(x) + c_1) = R_1(x)R_2(x)(R_3(x)R_4(x) + c_1),\]
\[p_3(x) = q_2(x)(s_2(x) + c_2) = R_2(x)R_4(x)(R_1(x)R_3(x) + c_2).\]
We consider two cases.

2.2.1: \(R_1(x)\) and \(R_4(x)\) are coprime. We have
\[B(x) = R_1(x)R_2(x)(R_1(x)R_2(x))'(R_3(x)R_4(x))(R_3(x)R_4(x) + c_1)\]
\[= R_2(x)R_2'(x)R_1(x)R_4(x)(R_3(x)R_4(x) + c_1)(R_1(x)R_3(x) + c_2),\]
and so
\[(R_1(x)R_2(x))'R_3(x) = R_2'(x)(R_1(x)R_3(x) + c_2),\]
which yields
\[R_1'(x)R_2(x)R_3(x) = c_2R_2'(x).\]
This is not possible because the right hand side has less degree than the left hand side.

2.2.2: \(R_1(x)\) and \(R_4(x)\) are not coprime. We write
\[R_1(x) = R(x)\hat{R}_1(x), \quad R_4(x) = R(x)\hat{R}_4(x)\]
where \(R(x), \hat{R}_1(x), \hat{R}_4(x) \in \mathbb{R}[x]\) with \(R(x) \notin \mathbb{R}.\) Note that
\[p_1(x) = q_1(x)s_1(x) = R_2^2(x)\hat{R}_1(x)R_2(x)R_3(x)\hat{R}_4(x),\]
\[p_2(x) = q_1(x)(s_1(x) + c_1) = R(x)\hat{R}_1(x)R_2(x)(R_3(x)R(x)\hat{R}_4(x) + c_1),\]
\[p_3(x) = q_2(x)(s_2(x) + c_2) = R_2(x)R(x)\hat{R}_4(x)(R(x)\hat{R}_1(x)R_3(x) + c_2).\]
Then
\[B(x) = (R(x)\hat{R}_1(x)R_2(x))'R(x)\hat{R}_1(x)R_2(x)R(x)R_3(x)\hat{R}_4(x)(R_3(x)R(x)\hat{R}_4(x) + c_1)\]
\[= R(x)R_2(x)(R(x)R_2(x))'\hat{R}_1(x)\hat{R}_4(x)(R_3(x)R_4(x) + c_1)(R(x)\hat{R}_1(x)R_3(x) + c_2),\]
and so

\[(R(x)\hat{R}_1(x)R_2(x))'(R(x)R_3(x)) = (R(x)R_2(x))'(R(x)\hat{R}_1(x)R_3(x) + c_2),\]

that is

\[(R(x)R_2(x))'(R(x)R_3(x)) + (R(x)R_2(x))\hat{R}_1(x)R(x)R_3(x)
\]

\[= (R(x)R_2(x))'(R(x)\hat{R}_1(x)R_3(x) + c_2(R(x)R_2(x))'),\]

which yields

\[R(x)R_2(x)(\hat{R}_1(x))'R(x)R_3(x) = c_2(R(x)R_2(x))'.\]

This is not possible because the right hand side has less degree than the left hand side. So subcase 2.2 is not possible.

In short from (24) there are at most two rational periodic solutions and if they exist then there must exist a polynomial \(S(x) = s_1(x)\) and two different constants \(c_1, c_2 \in \mathbb{R} \setminus \{0\}\) such that from Lemma 2.1 we have \(S(0) = S(1)\), and \(S(x) \neq 0\), \(S(x) + c_1 \neq 0\) and \(S(x) + c_2 \neq 0\) for \(x \in [0, 1]\), and

\[B(x) = S(x)S'(x)(S(x) + c_1)(S(x) + c_2). \quad (28)\]

Moreover the three possible limit cycles can be chosen as

\[p_1(x) = S(x)(S(x) + c_1), \quad p_2(x) = (S(x) + c_1)(S(x) + c_2), \quad p_3(x) = S(x)(S(x) + c_2). \quad (29)\]

Note that \(y_i(x) = 1/p_i(x)\) for \(i = 1, 2, 3\), satisfy (2) with \(c = 1\) and that \(p_i(0) = p_i(1)\) and \(p_i(x) \neq 0\) for \(x \in [0, 1]\). Hence, in view of Lemma 2.1, the three solutions \(y_1, y_2, y_3\) are three periodic solutions of (1).

Note that in this case it follows from (2), (28) and (29) that

\[A(x) = \frac{B(x) + p(x)p'(x)}{p(x)} = -(c_1 + c_2 + 3S(x))S'(x).\]

We define

\[\bar{B}(x) = \int_0^x B(s) \, ds, \quad \bar{A}(x) = \int_0^x A(s) \, ds.\]

We say that equation (1) satisfies the polynomial composition condition if there exists a polynomial \(A(x), B(x)\) and \(W(x)\) such that

\[\bar{A}(x) = \hat{A}(W(x)), \quad \bar{B}(x) = \hat{B}(W(x)),\]

and \(W(0) = W(1)\). It was proved in [7, 8] that if system (1) satisfies the polynomial composition condition then equation (1) has a center at the pair \(\{0, 1\}\) in the sense that \(y(0) = y(1)\) for any solution \(y(x)\) of equation (1) with initial condition \(y(0)\) sufficiently small. In this way the periodic solutions are not isolated.

We will show that the Abel equation (1) satisfies the polynomial composition condition with

\[\hat{A}(z) = -(c_1 + c_2)z + \frac{3z^2}{2}, \quad \hat{B}(z) = \frac{1}{4}z^4 + \frac{c_1 + c_2}{3}z^3 + \frac{c_1c_2}{2}z^2, \quad W(x) = S(x).\]

Indeed

\[\bar{A}(x) = \int_0^x A(s) \, ds = -\int_0^x (c_1 + c_2 + 3S(s))S'(s) \, ds = -(c_1 + c_2)S(x) - \frac{3S(x)^2}{2}\]
and

\[
\mathcal{B}(x) = \int_0^x B(s) \, ds = \int_0^x \left( S^3(s) + (c_1 + c_2) S^2(s) + c_1 c_2 S(s) \right) S'(s) \, ds.
\]

Therefore, the solutions are not isolated and so they cannot be limit cycles of (1). This concludes the proof of the theorem.

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