Superconvergence of both the Crouzeix–Raviart and Morley elements

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Abstract In this paper, a new method is proposed to prove the superconvergence of both the Crouzeix–Raviart and Morley elements. The main idea is to fully employ equivalences with the first order Raviart–Thomas element and the first order Hellan–Herrmann–Johnson element, respectively. In this way, some special conformity of discrete stresses is explored and superconvergence of mixed elements can be used to analyze superconvergence of nonconforming elements. Finally, the superconvergence of one and a half order by postprocessing is proved for both nonconforming elements.

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1 Introduction

The superconvergence analysis is well studied for conforming finite elements, see [9, 27], and as well as mixed finite elements [5] of second order problems. We also refer interested readers to [14] for some superconvergence results of locally discontinuous Galerkin method. For triangular mixed elements, Douglas et al. [17] proved superconvergence for the displacement variable on general triangulations, see also [1]. Brandts [3, 4] proved superconvergence for the stress variable on uniform triangula-
tions for the first and second order Raviart–Thomas elements [36], respectively. For superconvergence in rectangular and quadrilateral mixed finite element methods, see [11,18] and [19], respectively. However, in the case of nonconforming finite elements, due to the reduced continuity of trial and test functions, it becomes much more difficult to establish superconvergence properties and related asymptotic error expansions. For the Crouzeix–Raviart element, see [7,8], some gradient recovery postprocessing techniques are applied in the posteriori error analysis and probably show superconvergence in numerical results. There are several superconvergence results on rectangular elements. In [12,38], for the Wilson element [13], the superconvergence estimate of the gradient error on the centers of elements was obtained. The essential point employed therein is that the Wilson element space can be split into a conforming part and a nonconforming part. Thanks to the superconvergence estimate of the consistency error, some superconvergence results of the nonconforming rotated $Q_1$ element [35] and its variants were derived, see [20,26,33]. As for the plate bending problems, the superconvergence results of mixed finite elements [6] can be found in [24,25] for Ciarlet–Raviart elements on rectangular meshes and [15] for the displacement variables by the Hellan–Herrmann-Johnson element. In the case of nonconforming finite elements, in [10], Chen first established the supercloseness of the corrected interpolation of the incomplete biquadratic element [37] on uniform rectangular meshes. By using similar corrected interpolations as in [10], Mao et al. [31] first proved superconvergence of one and a half order for the Morley element [34] and the incomplete biquadratic nonconforming element on uniform rectangular meshes. Based on the equivalence to the Stokes equations and a superconvergence result of Ye [40] on the Crouzeix–Raviart element [16], Huang et al. [21] derived the superconvergence for the Morley element, which was postprocessed by projecting the finite element solution to another finite element space on a coarser mesh [39]. In [29,30], there are some results for the second order convergence by the Adini element.

In this paper, a new method is proposed to derive the superconvergence for nonconforming finite elements. The main idea is to explore some conformity of discrete stresses produced by nonconforming methods. Note that such conformity can not be obtained within original formulations for nonconforming elements. Fortunately, for the Crouzeix–Raviart element of the Poisson problem and the Morley element of the plate bending problem, it can be deduced by using the equivalences with the first order Raviart–Thomas element [32] and the first order Hellan–Herrmann-Johnson element [1], respectively. More precisely, based on these equivalences, we can translate the problem of superconvergence of nonconforming elements to the problem of superconvergence of mixed elements. Note that mixed elements are conforming methods within mixed formulations. This enables us to use superconvergence of mixed elements to establish superconvergence of nonconforming elements. In this way, it is able to overcome the main difficulty caused by nonconformity for the superconvergence analysis of nonconforming finite elements. In particular, one and a half order superconvergence by postprocessing is proved for both aforementioned two nonconforming elements on uniform triangulations. As a byproduct, the superconvergence is established for the Hellan–Herrmann-Johnson element which is somehow missing in literature. Numerical tests are provided to demonstrate theoretical results.
The remaining paper is organized as follows. Section 2 proposes the Poisson problem and the corresponding nonconforming and mixed finite elements. Section 3 presents the superconvergence result for the Raviart–Thomas element and proves the superconvergence result for the Crouzeix–Raviart element. Section 4 proposes the plate bending problem and the corresponding nonconforming and mixed finite elements. Section 5 proves the superconvergence result for the Hellan–Herrmann–Johnson element and the Morley element. Section 6 presents some numerical tests.

2 The Poisson problem and its Crouzeix–Raviart element

Throughout this paper, let $\Omega \subset \mathbb{R}^2$ be a polygonal domain. We recall some notations for Sobolev spaces (see [13]). For a subdomain $G$ of $\Omega$, let $P_m(G)$ be the space of polynomials of degree less than or equal to $m$ over $G$. $H^s(G)$ denotes the classical Sobolev space with norm $\| \cdot \|_{s,G}$ and the seminorm $| \cdot |_{s,G}$. $W^{k,\infty}(G)$ denotes the classical Sobolev space with norm $\| \cdot \|_{k,\infty,G}$ and the seminorm $| \cdot |_{k,\infty,G}$.

Given $f \in L^2(\Omega)$, the Poisson model problem finds $u \in H^1_0(\Omega)$ such that

$$\nabla u, \nabla v = (f, v) \quad \text{for all } v \in H^1_0(\Omega).$$

By introducing an auxiliary variable $\sigma := \nabla u$, the problem can be formulated as the following equivalent mixed problem which seeks $(\sigma, u) \in H(\text{div}, \Omega) \times L^2(\Omega)$ such that

$$(\sigma, \tau) + (u, \text{div} \tau) = 0 \quad \text{for any } \tau \in H(\text{div}, \Omega),$$

$$(\text{div} \sigma, v) = (-f, v) \quad \text{for any } v \in L^2(\Omega).$$

Suppose that $\tilde{\Omega}$ is covered by uniform shape regular triangulations $T$ consisting of triangles in two dimensions. $T$ is said to be uniform if any two adjacent triangles of $T$ form a parallelogram. $h$ denotes the diameter of the element $K \in T$. Let $E$ denote the set of edges of $T$, and $E(\partial \Omega)$ denote the set of all the boundary edges. Given $e \in E$, let $v_e$ be the unit normal vector of $e$ and $[\cdot]$ be jumps of piecewise functions over $e$, namely

$$[v] := v|_K^+ - v|_K^-$$

for piecewise functions $v$ and any two elements $K^+$ and $K^-$ which share the common edge $e$. Note that $[\cdot]$ becomes traces of functions on $e$ for boundary edges $e$. Throughout the paper, an inequality $A \lesssim B$ replaces $A \leq CB$ with some multiplicative mesh–size independent constant $C > 0$.

The Crouzeix–Raviart element [16] space over $T$ is defined by

$$W_{\text{CR}} := \left\{ v \in L^2(\Omega) : v|_K \in P_1(K) \text{ for each } K \in T, \int_E [v]ds = 0 \text{ for all } e \in E(\Omega) \right\},$$

$$V_{\text{CR}} := \left\{ v \in W_{\text{CR}} : \int_E vds = 0 \text{ for all } e \in E(\partial \Omega) \right\}.$$
The Crouzeix–Raviart element method of Problem (2.1) finds \( u_{CR} \in V_{CR} \) such that
\[
(\nabla_{NC} u_{CR}, \nabla_{NC} v) = (f, v) \quad \text{for all } v \in V_{CR}.
\] (2.3)

To analyze the superconvergence of the Crouzeix–Raviart element, we introduce the first order Raviart–Thomas element \([36]\) whose shape function space is
\[
RT(K) := (P_0(K))^2 + x P_0(K) \quad \text{for any } K \in T.
\]

Then the corresponding global finite element space reads
\[
RT(T) := \{ \tau \in H(\text{div}, \Omega) : \tau|_K \in RT(K) \text{ for any } K \in T \}. \quad (2.4)
\]

To get a stable pair of space, the piecewise constant space is proposed to approximate the displacement, namely,
\[
U_{RT}(T) := \{ v \in L^2(\Omega) : v|_K \in P_0(K) \text{ for any } K \in T \}. \quad (2.5)
\]

The Raviart–Thomas element method of Problem (2.2) seeks \( (\sigma_{RT}, u_{RT}) \in RT(T) \times U_{RT}(T) \) such that
\[
(\sigma_{RT}, \tau) + (u_{RT}, \text{div} \tau) = 0 \quad \text{for any } \tau \in RT(T),
\]
\[
(\text{div} \sigma_{RT}, v) = (-f, v) \quad \text{for any } v \in U_{RT}(T). \quad (2.6)
\]

Given \( K \in T \) and \( f \in L^2(K) \), define \( f_K = \frac{1}{|K|} \int_K f \, dx \). Given \( f \in L^2(\Omega) \), define the piecewise constant projection \( \Pi_0 f \) by
\[
(\Pi_0 f)|_K = f_K.
\]

Because of the definition of \( U_{RT}(T) \), \( f \) in the second equation of (2.6) can be replaced by \( \Pi_0 f \). We define the auxiliary method: Find \( \bar{u}_{CR} \in V_{CR} \) such that
\[
(\nabla_{NC} \bar{u}_{CR}, \nabla_{NC} v) = (\Pi_0 f, v) \quad \text{for all } v \in V_{CR}.
\] (2.7)

Note that this method differs from (2.3) only by the presence of the projection in the right hand side. Marini \([32]\) proved its equivalence to the Raviart–Thomas element method (2.6):
\[
\sigma_{RT}|_K = \nabla \bar{u}_{CR}|_K - \frac{f_K}{2}(x - \text{Mid}(K)) \quad x \in K \text{ for any } K \in T, \quad (2.8)
\]

where \( \text{Mid}(K) \) denotes the center of \( K \).

Subtracting (2.7) from (2.3) with \( v = u_{CR} - \bar{u}_{CR} \) yields that
\[
(\nabla_{NC}(u_{CR} - \bar{u}_{CR}), \nabla_{NC}(u_{CR} - \bar{u}_{CR})) = (f - \Pi_0 f, u_{CR} - \bar{u}_{CR})
\]
\[
= (f - \Pi_0 f, u_{CR} - \bar{u}_{CR} - \Pi_0(u_{CR} - \bar{u}_{CR})).
\]
Hence, the Poincaré inequality from [23] yields
\[ \| \nabla_{NC}(u_{CR} - \bar{u}_{CR}) \|_{0, \Omega} \leq \frac{h^2}{j_{1,1}^{1/2}} | f |_{1, \Omega}. \] (2.9)
where \( j_{1,1} = 3.8317 \) denotes the first positive root of the Bessel function of the first kind.

3 Superconvergence analysis of the Crouzeix–Raviart element

In this section, we first present the superconvergence result of the Raviart–Thomas element by Brandts [3]. Then, based on this result and the equivalence (2.8), we derive the superconvergence result of the Crouzeix–Raviart element.

3.1 The superconvergence result of the Raviart–Thomas element

We introduce a result on Sobolev spaces in the following lemma, which describes the behavior of functions near the boundary. Define \( \Omega_h \) as the subset of points in \( \Omega \) having distance less than \( h \) from the boundary:
\[ \Omega_h = \{ x \in \Omega : \exists y \in \partial \Omega, \text{dist}(x, y) \leq h \}. \]
Then we have the following result, see [3,28].

**Lemma 3.1** For \( v \in H^s(\Omega) \), where \( 0 \leq s \leq \frac{1}{2} \), we have
\[ \| v \|_{0, \Omega_h} \lesssim h^s \| v \|_{s, \Omega}. \]
Given \( q \in (H^1(\Omega))^2 \), define the interpolation operator \( \Pi_{RT} q \in \RT(T) \) by
\[ \int_e (\Pi_{RT} q - q)^T v_e ds = 0 \quad \text{for all } e \in \mathcal{E}. \]
Brandts gave the following superconvergence result of the Raviart–Thomas element, see [3, Theorem 3.2].

**Theorem 3.2** Let \( \sigma \in (H^2(\Omega))^2 \) and \( \sigma_{RT} \) be the solutions of (2.2) and (2.6), respectively. There holds that
\[ \| \sigma_{RT} - \Pi_{RT} \sigma \|_{0, \Omega} \lesssim h^\frac{3}{2} \left( \| \sigma \|_{\frac{3}{2}, \Omega} + h^\frac{1}{2} | \sigma |_{2, \Omega} \right). \]
Furthermore, a post-processing mechanism was proposed in [3], which when applied to the projection \( \Pi_{RT} q \) of a function \( q \in (H^2(\Omega))^2 \), will improve its approximation property. Given \( q_h \in \RT(T) \), define function \( K_h q_h \in (W_{CR})^2 \) as follows (see also Fig. 1).
Fig. 1 Post-processing a function $q_h \in RT(\mathcal{T})$

- Given $e \in \mathcal{E}(\Omega)$, suppose that $e = K_1 \cap K_2$ and $P$ denotes the center of $e$. Let

$$K_hq_h(P) = \frac{1}{2} \left( q_h|_{K_1}(P) + q_h|_{K_2}(P) \right).$$

- Given $e \in \mathcal{E}(\partial \Omega)$ and $e \subset \partial K$, there exists at least one $\tilde{K} \in \mathcal{T}$ such that $N = K \cup \tilde{K}$ is a parallelogram. The straight line through the center $P$ of $e$ and the center $N_c$ of the parallelogram intersects the boundary of $N$ in another point $\tilde{P}$. Define

$$K_hq_h(P) = 2K_hq_h(N_c) - K_hq_h(\tilde{P}).$$

Brandts [3] proved that the vector $K_h \Pi_{RT} q$ is a higher order approximation of $q$ than $\Pi_{RT} q$ itself.

**Theorem 3.3** Suppose $q \in (H^2(\Omega))^2$, then there holds that

$$\|q - K_h \Pi_{RT} q\|_{0,\Omega} \lesssim h^2 |q|_{2,\Omega}.$$

Combining Theorems 3.2 and 3.3 concludes that the post-processing operator $K_h$ also improves the order of approximation of $\sigma_{RT}$.

**Corollary 3.4** Let $\sigma \in (H^2(\Omega))^2$ and $\sigma_{RT}$ be the solutions of (2.2) and (2.6), respectively. There holds that

$$\|\sigma - K_h \sigma_{RT}\|_{0,\Omega} \lesssim h^\frac{3}{2} \left( \|\sigma\|_{3,\Omega} + h^\frac{1}{2} |\sigma|_{2,\Omega} \right).$$

### 3.2 The superconvergence result of the Crouzeix–Raviart element

**Theorem 3.5** Let $u \in H^3(\Omega)$ and $u_{CR}$ be the solutions of (2.1) and (2.3), respectively. Further, suppose that $f \in W^{1,\infty}(\Omega)$, then we have

$$\|\nabla u - K_h \nabla u_{CR}\|_{0,\Omega} \lesssim h^\frac{3}{2} \left( \|u\|_{3,\Omega} + h^\frac{1}{2} |u|_{3,\Omega} + h^\frac{1}{2} |f|_{1,\infty,\Omega} \right). \tag{3.1}$$
Proof Using the equivalence equality (2.8), for $e = K_1 \cap K_2$ and the center $P$ of $e$, there holds that

$$|K_h(\nabla_{\nu} \tilde{u}_{CR} - \sigma_{RT})(P)| = \left| \frac{f_{K_1}}{4}(P - \text{Mid}(K_1)) + \frac{f_{K_2}}{4}(P - \text{Mid}(K_2)) \right|$$

Since $K_1$ and $K_2$ form a parallelogram, we have $P - \text{Mid}(K_1) = \text{Mid}(K_2) - P$. This yields that

$$|K_h(\nabla_{\nu} \tilde{u}_{CR} - \sigma_{RT})(P)| = \frac{1}{4}|(f_{K_1} - f_{K_2})(P - \text{Mid}(K_1))| \lesssim h^2|f|_{1,\infty,\Omega}.$$ 

Suppose that $\phi_i$, $1 \leq i \leq 3$ denote the nodal basis functions on $K$ of $W_{CR}$. Hence, by the definition of $K_h$ and scaling arguments, there holds that

$$\|K_h(\nabla_{\nu} \tilde{u}_{CR} - \sigma_{RT})\|_{0,K}^2 \lesssim h^4|f|_{1,\infty,\Omega}^2 \sum_{i=1}^3 \|\phi_i\|_{0,K}^2 \lesssim h^6|f|_{1,\infty,\Omega}^2.$$ 

Summing over all triangles $K \in T$ gives that

$$\|K_h(\nabla_{\nu} \tilde{u}_{CR} - \sigma_{RT})\|_{0,\Omega} \lesssim h^2|f|_{1,\infty,\Omega}. \quad (3.2)$$

Since $\nabla_{\nu} \tilde{u}_{CR} - \nabla_{\nu} u_{CR}$ is a piecewise constant, the inverse estimate and (2.9) yield that

$$\|K_h(\nabla_{\nu} \tilde{u}_{CR} - \nabla_{\nu} u_{CR})\|_{0,\Omega} \lesssim \|\nabla_{\nu}(\tilde{u}_{CR} - u_{CR})\|_{0,\Omega} \lesssim h^2|f|_{1,\Omega} \lesssim h^2|f|_{1,\infty,\Omega}. \quad (3.3)$$

The triangle inequality plus Corollary 3.4, (3.2) and (3.3) complete the proof. \qed

4 The plate bending problem and its Morley element

Given $f \in L^2(\Omega)$, the plate bending model problem finds $u \in H^2_0(\Omega)$ such that

$$(\nabla^2 u, \nabla^2 v) = (f, v) \quad \text{for all } v \in H^2_0(\Omega). \quad (4.1)$$

Given any space $V$, we define $(V)_s^4$ as follows:

$$(V)_s^4 := \{ \tau = (\tau_{ij}), 1 \leq i, j \leq 2 : \tau_{ij} \in V, \tau_{12} = \tau_{21} \}.$$ 

Given $K \in T$, $\nu$ denotes the unit outward normal to $\partial K$ and $t$ the unit tangent to $\partial K$. Given $\tau \in (H^1(K))_s^4$, we set

$$M_{vv}(\tau) = v^T \tau v,$$

$$M_{vt}(\tau) = v^T \tau t.$$
By introducing an auxiliary variable \( \sigma := \nabla^2 u \), the mixed formulation of (4.1) seeks \((\sigma, u) \in S \times D\), see [22],

\[
(\sigma, \tau) + \sum_{K \in T} -(\tau, \nabla^2 u)_{L^2(K)} + \int_{\partial K} M_{vv}(\tau) \frac{\partial u}{\partial v} ds = 0 \quad \text{for any } \tau \in S,
\]

\[
\sum_{K \in T} -(\sigma, \nabla^2 v)_{L^2(K)} + \int_{\partial K} M_{vv}(\sigma) \frac{\partial v}{\partial v} ds = (-f, v) \quad \text{for any } v \in D,
\]

where

\[
S = \left\{ \tau \in (L^2(\Omega))^4_s : \tau|_K \in (H^1(K))^4_s \text{ for all } K \in T, \right. \\
\left. \text{and } M_{vv}(\tau) \text{ is continuous across interior edges} \right\},
\]

\[
D = \left\{ v \in H^1_0(\Omega) : v|_K \in H^2(K) \text{ for all } K \in T \right\}.
\]

The Morley element space [34] \( V_M \) over \( T \) is defined by

\[
V_M := \left\{ v \in L^2(\Omega) : v|_K \in P_2(K) \text{ for each } K \in T, v \text{ is continuous at each interior vertex and vanishes on each boundary vertex, } \int_e \left[ \frac{\partial v}{\partial v_e} \right] ds = 0 \right. \\
\left. \text{for all } e \in E(\Omega), \text{ and } \int_e \frac{\partial v}{\partial v_e} ds = 0 \text{ for all } e \in E(\partial \Omega) \right\}.
\]

The Morley element method of Problem (4.1) finds \( u_M \in V_M \) such that

\[
\left( \nabla_{NC}^2 u_M, \nabla_{NC}^2 v \right) = (f, v) \quad \text{for all } v \in V_M.
\]  

(4.3)

To analyze the superconvergence of the Morley element, we introduce the first order Hellan–Herrmann–Johnson element [22]. Define

\[
HHJ(T) = \left\{ \tau \in S : \tau|_K \in (P_0(K))^4_s \text{ for any } K \in T \right\},
\]

\[
U_{HHJ}(T) = \left\{ v \in H^1_0(\Omega) : v|_K \in P_1(K) \text{ for any } K \in T \right\}.
\]

The first order Hellan–Herrmann–Johnson element of Problem (4.2) finds \((\sigma_{HHJ}, u_{HHJ}) \in HHJ(T) \times U_{HHJ}(T)\) such that

\[
(\sigma_{HHJ}, \tau) + \sum_{K \in T} \int_{\partial K} M_{vv}(\tau) \frac{\partial u_{HHJ}}{\partial v} ds = 0 \quad \text{for any } \tau \in HHJ(T),
\]

\[
\sum_{K \in T} \int_{\partial K} M_{vv}(\sigma_{HHJ}) \frac{\partial v}{\partial v} ds = (-f, v) \quad \text{for any } v \in U_{HHJ}(T).
\]  

(4.4)
Given \( v \in H^2_0(\Omega) \cup V_M \), define the interpolation operator \( \Pi_D : H^2_0(\Omega) \cup V_M \rightarrow U_{HHJ}(T) \) by
\[
\Pi_D v(z) = v(z) \text{ for each vertex } z \text{ of } T. \tag{4.5}
\]
Hence, we introduce the auxiliary method: The modified Morley element finds \( \bar{u}_M \in V_M \) such that
\[
\left( \nabla^2_{\mathcal{NC}} \bar{u}_M, \nabla^2_{\mathcal{NC}} v \right) = (f, \Pi_D v) \text{ for all } v \in V_M. \tag{4.6}
\]
Arnold et al. [1] proved the following equivalence between the Hellan–Herrmann–Johnson element and the modified Morley element:
\[
\sigma_{HHJ} = \nabla^2_{\mathcal{NC}} \bar{u}_M, u_{HHJ} = \Pi_D \bar{u}_M, \tag{4.7}
\]
and moreover
\[
\left\| \nabla^2_{\mathcal{NC}} (u_M - \bar{u}_M) \right\|_{0,\Omega} \lesssim h^2 \| f \|_{0,\Omega}. \tag{4.8}
\]

5 Superconvergence analysis of the Morley element

In this section, following the similar arguments for the Raviart–Thomas element in [3], we prove the superconvergence result of the Hellan–Herrmann–Johnson element. Then, based on this result and the equivalence (4.7), we derive the superconvergence result of the Morley element.

5.1 The superconvergence result of the Hellan–Herrmann–Johnson element

First we introduce the interpolation operator \( \Pi_{HHJ} : S \rightarrow HHJ(T) \) as in [6]:
\[
\int_e M_{vv}(\Pi_{HHJ} \tau) ds = \int_e M_{vv}(\tau) ds \text{ for all } e \in \mathcal{E}. \tag{5.1}
\]
Moreover if \( \tau \in (H^1(\Omega))^d \),
\[
\left\| \tau - \Pi_{HHJ} \tau \right\|_{0,\Omega} \lesssim h |\tau|_{1,\Omega}. \tag{5.2}
\]
An integration by parts yields that the following Green’s formulae holds for any \( \tau \in (H^1(K))^4 \) and \( v \in H^2(K) \),
\[
\int_K \tau : \nabla^2 v dx = -\int_K \text{div} \tau \cdot \nabla v dx + \int_{\partial K} M_{vv}(\tau) \frac{\partial v}{\partial n} ds + \int_{\partial K} M_{vt}(\tau) \frac{\partial v}{\partial t} ds. \tag{5.3}
\]
We have the following result.

**Lemma 5.1** Let \( \sigma \) and \( \sigma_{HHJ} \) be the solutions of (4.2) and (4.4), respectively. Then
\[
(\sigma_{HHJ} - \sigma, \sigma_{HHJ} - \Pi_{HHJ} \sigma) = 0. \tag{5.4}
\]
Proof Let \( \tau \in \text{HHJ}(T) \), \( v \in \text{UHHJ}(T) \) in (4.2) and (4.4), which, together with (5.3), yield that

\[
(\sigma_{\text{HHJ}} - \sigma, \tau) = -\sum_{K \in T} \int_{\partial K} M_{vv}(\tau) \frac{\partial (u_{\text{HHJ}} - u)}{\partial v} ds - \sum_{K \in T} (\tau, \nabla^2 u)_{L^2(K)}
\]

and

\[
\sum_{K \in T} \int_{\partial K} M_{vv} (\sigma_{\text{HHJ}} - \sigma) \frac{\partial v}{\partial v} ds = 0. \tag{5.6}
\]

By the definition of \( \Pi_D u \) in (4.5), since \( M_{vv}(\tau) \) is constant on each edge of \( K \), a combination of (5.5) and (5.3) leads to

\[
(\sigma_{\text{HHJ}} - \sigma, \tau) = \sum_{K \in T} \int_{\partial K} M_{vv}(\tau) \frac{\partial (u_{\text{HHJ}} - \Pi_D u)}{\partial t} ds
\]

and

\[
\sum_{K \in T} \int_{\partial K} M_{vv} (\sigma_{\text{HHJ}} - \sigma) \frac{\partial v}{\partial v} ds = 0. \tag{5.7}
\]

Thanks to the definition of \( \Pi_{\text{HHJ}} \) in (5.1), substituting \( \tau = \sigma_{\text{HHJ}} - \Pi_{\text{HHJ}} \sigma, v = u_{\text{HHJ}} - \Pi_D u \) into (5.6) and (5.7), respectively, yields that

\[
(\sigma_{\text{HHJ}} - \sigma, \sigma_{\text{HHJ}} - \Pi_{\text{HHJ}} \sigma) = -\sum_{K \in T} \int_{\partial K} M_{vv}(\sigma_{\text{HHJ}} - \Pi_{\text{HHJ}} \sigma) \frac{\partial (u_{\text{HHJ}} - \Pi_D u)}{\partial v} ds
\]

and

\[
\sum_{K \in T} \int_{\partial K} M_{vv}(\sigma_{\text{HHJ}} - \sigma) \frac{\partial (u_{\text{HHJ}} - \Pi_D u)}{\partial v} ds = 0.
\]

This completes the proof. \( \square \)

**Lemma 5.2** Let \( N \) be a parallelogram forming by two triangles \( K_1, K_2 \). Then for all \( r \in (P_1(N))^4 \), we have that

\[
\int_N (r - \Pi_{\text{HHJ}} r) dx = 0.
\]

**Proof** We may assume that \( N \) is centered around the origin and, since \( r = \Pi_{\text{HHJ}} r \) whenever \( r \) is constant, take \( r \in (P_1(N))^4 \) zero at the origin and thus odd. But then \( \Pi_{\text{HHJ}} r \) is odd as well, which completes the proof. \( \square \)

We recall some notations in [3]. Denote a parallelogram consisting of two triangles sharing a side with normal \( f_i \) by \( N_{f_i}, (i = 1, 2, 3) \). For each \( i = 1, 2, 3 \), the domain \( \Omega \) can be partitioned into parallelograms \( N_{f_i} \) and some resulting boundary triangles.
which we denote by \( T_{fi} \). For an example of the definitions and notations concerning the triangulations, see Fig. 2.

**Theorem 5.3** Let \( \sigma \in (H^2(\Omega))^4 \) and \( \sigma_{HHJ} \) be the solutions of (4.2) and (4.4), respectively. Then

\[
\| \sigma_{HHJ} - \Pi_{HHJ} \sigma \|_{0, \Omega} \lesssim h^{3/2} (\| \sigma \|_{2, \Omega} + h^{1/2} |\sigma|_{2, \Omega}).
\]

**Proof** First because of (5.4), we find that

\[
(\sigma_{HHJ} - \Pi_{HHJ} \sigma, \sigma_{HHJ} - \Pi_{HHJ} \sigma) = (\sigma_{HHJ} - \Pi_{HHJ} \sigma, \sigma - \Pi_{HHJ} \sigma).
\]

Let \( \tau_{fi} \in (P_0(K))^4 \), \( 1 \leq i \leq 3 \) denote the basis functions, i.e., \( M_{fi,fi}(\tau_{fi}) = \delta_{ij} \).

Then we have the following decomposition:

\[
(\sigma_{HHJ} - \Pi_{HHJ} \sigma, \sigma - \Pi_{HHJ} \sigma) = \sum_{K \in T} \int_K (\sigma_{HHJ} - \Pi_{HHJ} \sigma) : (\sigma - \Pi_{HHJ} \sigma) dx
\]

\[
= \sum_{K \in T} \int_K \sum_{i=1}^{3} M_{fi,fi}(\sigma_{HHJ} - \Pi_{HHJ} \sigma) \tau_{fi} : (\sigma - \Pi_{HHJ} \sigma) dx
\]

\[
= \sum_{i=1}^{3} I_i
\]

where

\[
I_i = \sum_{K \in T} \int_K M_{fi,fi}(\sigma_{HHJ} - \Pi_{HHJ} \sigma) \tau_{fi} : (\sigma - \Pi_{HHJ} \sigma) dx.
\]

Since \( M_{fi,fi}(\sigma_{HHJ} - \Pi_{HHJ} \sigma) \) is continuous and constant on \( N_{fi} \), and since \( \tau_{fi} \) is constant on \( N_{fi} \), rewriting the sum \( I_i \) as a sum over parallelogram \( N_{fi} \), boundary triangles \( T_{fi} \), we find:
\[ |I_i| \leq \sum_{N_{fi}} \left| M_{fi,fi} (\sigma_{\text{HHJ}} - \Pi_{\text{HHJ}} \sigma) \tau_{fi} : \int_{N_{fi}} (\sigma - \Pi_{\text{HHJ}} \sigma) dx \right| \\
+ \sum_{T_{fi}} \left| \int_{T_{fi}} M_{fi,fi} (\sigma_{\text{HHJ}} - \Pi_{\text{HHJ}} \sigma) \tau_{fi} : (\sigma - \Pi_{\text{HHJ}} \sigma) dx \right|. \quad (5.8) \]

Denote \( \partial \Omega_{fi} \) the union of the boundary triangle \( T_{fi} \). In bounding (5.8) we use the Cauchy–Schwarz inequality and the estimate
\[ |M_{fi,fi} (\sigma_{\text{HHJ}} - \Pi_{\text{HHJ}} \sigma) \tau_{fi}| \lesssim h^{-1} \| \sigma_{\text{HHJ}} - \Pi_{\text{HHJ}} \sigma \|_{0,N_{fi}}. \]

which results in
\[ |I_i| \lesssim h^{-1} \| \sigma_{\text{HHJ}} - \Pi_{\text{HHJ}} \sigma \|_{0,\Omega} \left( \sum_{N_{fi}} \left| \int_{N_{fi}} (\sigma - \Pi_{\text{HHJ}} \sigma) dx \right|^2 \right)^{\frac{1}{2}} \\
+ \| \sigma_{\text{HHJ}} - \Pi_{\text{HHJ}} \sigma \|_{0,\partial \Omega_{fi}} \| \sigma - \Pi_{\text{HHJ}} \sigma \|_{0,\partial \Omega_{fi}}. \quad (5.9) \]

Define the linear functional \( F \) on \( (H^2(N_{fi}))_s^4 \) by
\[ F(\tau) = \int_{N_{fi}} (\tau - \Pi_{\text{HHJ}} \tau) dx, \tau \in (H^2(N_{fi}))_s^4. \]

For this functional, the Cauchy–Schwarz inequality and (5.2) yield:
\[ |F(\tau)| \lesssim h^3 |\tau|_{2,N_{fi}} \quad \text{for all } \tau \in (H^2(N_{fi}))_s^4. \quad (5.10) \]

Combining (5.9), (5.2) and (5.10), we conclude that
\[ |I_i| \lesssim \| \sigma_{\text{HHJ}} - \Pi_{\text{HHJ}} \sigma \|_{0,\Omega} (h^2 |\sigma|_{2,\Omega} + h |\sigma|_{1,\partial \Omega_{fi}}). \]

Lemma 3.1 implies that
\[ |\sigma|_{1,\partial \Omega_{fi}} \leq |\sigma|_{1,\Omega_h} \lesssim h^\frac{3}{2} \| \sigma \|_{\frac{3}{2},\Omega}. \]

This completes the estimate of \( |I_i| \).
\[ \Box \]
We use a similar post-processing mechanism as in Sect. 3 and still denote the post-processing operator as \( K_h \). Thus given \( \tau_h \in \text{HHJ}(T) \), \( K_h \tau_h \in (W_{CR})^4 \) is similar defined as in Sect. 3. Following the idea of [3, Theorem 5.1], we can prove the following result.

**Theorem 5.4** Let \( \tau \in (H^2(\Omega))^4_s \). Then for \( K_h \Pi_{\text{HHJ}} \tau \in (W_{CR})^4_s \), we have

\[
\| \tau - K_h \Pi_{\text{HHJ}} \tau \|_{0,\Omega} \lesssim h^2 |\tau|_{2,\Omega}.
\]

**Proof** First, let \( r \in (P_1(\tilde{K}))^4_s \), where \( \tilde{K} \) is the union of \( K \) and the triangles sharing an edge with \( K \). Then, using the same arguments as in Lemma 5.2 we find that

\[
K_h \Pi_{\text{HHJ}} r = r \text{ on } K \quad \text{for all } r \in (P_1(\tilde{K}))^4_s. \tag{5.11}
\]

For all \( \tau \in (H^2(\Omega))^4_s \), since \( K_h \Pi_{\text{HHJ}} \tau \) is a linear function on \( K \), there holds that

\[
\| K_h \Pi_{\text{HHJ}} \tau \|_{0,\infty,K} \lesssim \| \Pi_{\text{HHJ}} \tau \|_{0,\infty,\tilde{K}}. \tag{5.12}
\]

Since the interpolation \( \Pi_{\text{HHJ}} \tau \) is constant on \( K \), and since the angles between the normals of the edges of \( K \) are bounded away from 0 and \( \pi(-\pi) \), we have

\[
\| \Pi_{\text{HHJ}} \tau \|_{0,\infty,K} \lesssim \sum_{j=1}^3 |M_{f_j f_j}(\Pi_{\text{HHJ}} \tau)| \lesssim \sum_{j=1}^3 \| M_{f_j f_j}(\tau) \|_{0,\infty,\partial K_j} \lesssim \| \tau \|_{0,\infty,K}, \tag{5.13}
\]

where \( \partial K_j \) denotes the corresponding edge of \( K \) with the normal \( f_j \). From (5.12) and (5.13), we conclude that

\[
\| K_h \Pi_{\text{HHJ}} \tau \|_{0,\infty,K} \lesssim \| \tau \|_{0,\infty,\tilde{K}},
\]

so that using (5.11), for all \( r \in (P_1(\tilde{K}))^4_s \)

\[
\| \tau - K_h \Pi_{\text{HHJ}} \tau \|_{0,K} \lesssim h \| \tau - K_h \Pi_{\text{HHJ}} \tau \|_{0,\infty,K} \lesssim h \| (I - K_h \Pi_{\text{HHJ}})(\tau - r) \|_{0,\infty,K} \lesssim h \| \tau - r \|_{0,\infty,\tilde{K}}.
\]

The interpolation theory in Sobolev spaces (see [13]) shows that

\[
\inf \{ r \in (P_1(\tilde{K}))^4_s : \| \tau - r \|_{0,\infty,\tilde{K}} \} \lesssim h |\tau|_{2,\tilde{K}},
\]

which yields

\[
\| \tau - K_h \Pi_{\text{HHJ}} \tau \|_{0,K} \lesssim h^2 |\tau|_{2,\tilde{K}}. \tag{5.14}
\]

Hence, squaring (5.14) and summing over all triangles \( K \in T \) complete the proof. \( \square \)

A combination of the superconvergence result and Theorem 5.4, concludes that the post-processing operator \( K_h \) also improves the order of approximations of \( \sigma_{\text{HHJ}} \).
Corollary 5.5 Let \( \sigma \in (H^2(\Omega))^d \) and \( \sigma_{HHJ} \) be the solutions of (4.2) and (4.4), respectively. There holds that

\[
\| \sigma - K_h \sigma_{HHJ} \|_{0, \Omega} \lesssim h^{3/2} \left( \| \sigma \|_{2, \Omega} + h^{1/2} |\sigma|_{2, \Omega} \right).
\] (5.15)

Proof The triangle inequality

\[
\| \sigma - K_h \sigma_{HHJ} \|_{0, \Omega} \leq \| \sigma - K_h \Pi_{HHJ} \sigma \|_{0, \Omega} + \| K_h (\Pi_{HHJ} \sigma - \sigma_{HHJ}) \|_{0, \Omega},
\]

combined with the boundedness of the operator \( K_h \) on \( HHJ(T) \) from (5.12) and the application of Theorems 5.3 and 5.4 complete the proof. \( \square \)

5.2 The superconvergence result of the Morley element

Theorem 5.6 Let \( u \in H^4(\Omega) \) and \( u_M \) be the solutions of (4.1) and (4.3), respectively. Then we have

\[
\left\| \nabla^2 u - K_h \nabla^2_{NC} u_M \right\|_{0, \Omega} \lesssim h^{3/2} \left( \left\| u \right\|_{2, \Omega} + h^{1/2} |u|_{4, \Omega} + h^{1/2} \| f \|_{0, \Omega} \right).
\] (5.16)

Proof The triangle inequality plus the equivalence (4.7) and the inverse estimate give that

\[
\left\| \nabla^2 u - K_h \nabla^2_{NC} u_M \right\|_{0, \Omega} \lesssim \left\| \nabla^2 u - K_h \nabla^2_{NC} \bar{u}_M \right\|_{0, \Omega} + \left\| K_h \left( \nabla^2_{NC} u_M - \nabla^2_{NC} \bar{u}_M \right) \right\|_{0, \Omega}
\lesssim \| \sigma - K_h \sigma_{HHJ} \|_{0, \Omega} + \left\| \nabla^2_{NC} u_M - \nabla^2_{NC} \bar{u}_M \right\|_{0, \Omega}.
\]

Thus (5.15) and (4.8) complete the proof. \( \square \)

We can only prove one and a half order superconvergence in Theorem 5.6. Under the same assumptions as in [31, Theorem 4.4], we give the following second order superconvergence result.

Theorem 5.7 Under the assumption of Theorem 5.6, and further suppose that \( \nabla^3 u|_{\partial \Omega} = 0 \), then we have

\[
\| \nabla^2 u - K_h \nabla^2_{NC} u_M \|_{0, \Omega} \lesssim h^{3/2} \left( \| u \|_{4, \Omega} + \| f \|_{0, \Omega} \right).
\]

Proof We reconsider the estimate of the second term on the right hand of (5.8) in Theorem 5.3. Since \( \nabla^3 u|_{\partial \Omega} = 0 \), i.e., \( \nabla \sigma|_{\partial \Omega} = 0 \), the Poincaré inequality and scaling arguments show that

\[
\int_{T_{f_i}} (\sigma - \Pi_{HHJ} \sigma) dx \lesssim h^2 |\sigma|_{1, T_{f_i}} \lesssim h^3 |\sigma|_{2, T_{f_i}}.
\]
Hence, this results in superconvergence of second order as follows:

\[ \| \sigma_{\text{HHJ}} - \Pi_{\text{HHJ}} \sigma \|_{0, \Omega} \lesssim h^2 |\sigma|_{2, \Omega}. \]

Thus this completes the proof. \( \square \)

6 Numerical tests

In this section, we present two numerical tests to confirm some of the theoretical superconvergence analyses in the previous sections.

6.1 The Poisson problem

Suppose domain \( \Omega \) is a square, see Fig. 3. Consider the following Poisson problem

\[-\Delta u = f \text{ in } \Omega\]

with \( u \in H^1_0(\Omega) \). The exact solution is

\[ u(x_1, x_2) = \sin \pi x_1 \sin \pi x_2. \]

The results are reported in Fig. 4 and Table 1, where we see that the post-processing error \( \| \nabla u - K_h \nabla_{\text{NC}u_{\text{CR}}} \|_{0, \Omega} \) by the Crouzeix–Raviart element is much smaller than the error \( \| \nabla u - \nabla_{\text{NC}u_{\text{CR}}} \|_{0, \Omega} \). And it also can be seen that the superconvergence of one and a half order by the Crouzeix–Raviart element in Theorem 3.5 is verified by the numerical results. However, Table 1 shows that the convergence rate of \( \| \nabla u - K_h \nabla_{\text{NC}u_{\text{CR}}} \|_{0, \Omega} \) is \( O(h^2) \) and this indicates that the order proved in Theorem 3.5 may be suboptimal.
Table 1 Convergence of the Crouzeix–Raviart element

| Number of elements | $\|\nabla u - \nabla_{NC} u_{CR}\|_{0,\Omega}$ | Rate | $\|\nabla u - K_h \nabla_{NC} u_{CR}\|_{0,\Omega}$ | Rate |
|--------------------|---------------------------------|------|---------------------------------|------|
| $8 \times 4$      | $6.4104E-01$                    |      | $2.2880E-01$                    |      |
| $16 \times 8$     | $3.2395E-01$                    | 0.9847 | $5.1669E-02$                    | 2.1467 |
| $32 \times 16$    | $1.6241E-01$                    | 0.9961 | $1.2286E-02$                    | 2.0723 |
| $64 \times 32$    | $8.1259E-02$                    | 0.9990 | $2.9936E-03$                    | 2.0370 |
| $128 \times 64$   | $4.0636E-02$                    | 0.9998 | $7.3852E-04$                    | 2.0192 |
| $256 \times 128$  | $2.0319E-02$                    | 0.9999 | $1.8337E-04$                    | 2.0098 |

6.2 The plate bending problem

In this example, we validate the superconvergence of the Morely element on the uniform parallelogram meshes, see Fig. 5. Consider the following plate bending problem

$$\Delta^2 u = f \quad \text{in } \Omega$$

with $u \in H^2_0(\Omega)$. The exact solution is

$$u(x_1, x_2) = (x_1 - \sqrt{3}x_2)^2(x_1 - \sqrt{3}x_2 - 2)^2 x_2^2 \left(\frac{\sqrt{3}}{2} - x_2\right)^2.$$
Superconvergence of both the Crouzeix–Raviart and Morley elements

Fig. 5 Parallelogram domain with uniform triangulations

Fig. 6 Convergence of the Morley element

Table 2 Convergence of the Morley element

| Number of elements | $\|\nabla^2 u - \nabla_{NC}^2 u_M\|_{0, \Omega}$ | Rate | $\|\nabla^2 u - K_h\nabla_{NC}^2 u_M\|_{0, \Omega}$ | Rate |
|--------------------|---------------------------------|------|---------------------------------|------|
| $8 \times 4$       | 1.2599E+00                      |      | 7.6681E−01                      |      |
| $16 \times 8$      | 8.5516E−01                      | 0.5591| 2.7553E−01                      | 1.4766|
| $32 \times 16$     | 4.6008E−01                      | 0.8943| 7.3946E−02                      | 1.8977|
| $64 \times 32$     | 2.3428E−01                      | 0.9736| 1.8627E−02                      | 1.9891|
| $128 \times 64$    | 1.1768E−01                      | 0.9934| 4.6311E−03                      | 2.0080|
| $256 \times 128$   | 5.8909E−02                      | 0.9983| 1.1506E−03                      | 2.0090|

In this subsection, cf. Table 1. This gives an indication of superconvergence on uniform parallelogram meshes as well, which is coincided with the theoretical results in Theorem 5.6. In addition, the one and a half order of superconvergence may be also suboptimal for the Morely element.
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