Effects of chemical reaction, viscosity, thermal conductivity, heat source, radiation/absorption, on MHD mixed convection nano-fluids flow over an unsteady stretching sheet by HAM and numerical method

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Abstract

Investigations are performed for further observations of heat and mass transfer in magneto-hydrodynamic mixed-convectional nano-fluid flow with the assumption of variable viscosity and thermal-conductivity over an unsteady stretching sheet. Base fluid is Carboxy-methyl cellulose (CMC) water as a carrier fluid with different nano-particles such as TiO₂ (Titanium), Ag (Silver), Al₂O₃ (Aluminum), and Cu (Copper). Flow contains different physical parameters, such as heat source, chemical reaction effect, Schmidt number, and radiation absorptions effects are observed to be significant in the presences of magnetic field. Obtained equations are solved by numerically BVP4C-package (shooting method) and analytically by BVPh2.0-package (Homotopy Analysis Method “HAM”). Interested physical quantities are, viscosity-parameter \( A \), Thermal-conductivity parameter \( N \), Thermocapillary-number \( M \), Hartmann-number \( Ma \), Prandtl-number \( Pr \), 4-nano-particles \( f \), temperature Grashof number \( Gr_t \), and mass Grashof number \( Gr_c \) are the focus to the velocity, temperature, and solute concentration profiles. It is concluded that, Solute concentration of \( Al₂O₃ \)-water has higher than the other 3-nano-fluids. Mass flux, heat flux, and Skin friction of fluids are direct functions of magnetic force, while inverse function of temperature. Magnetic force also decreased the speed of fluids and hence mass flux reduced which implies that, the temperature reduces. \( Gr_t \) has also inverse relation with mass flux, heat flux, and skin friction, while direct relation with the speed of fluids. Similarly, \( Gr_c \) has inverse relation with \(-n(0), -\theta'(0), \) and \(-f''(0)\), but direct relation with \( u(1) \). Different results are shown in graphical and tabular form.

Keywords

Viscosity parameter, variable thermal-conductivity, thermocapillary number, magnetic force, unsteady stretching surface, Grashof number, Sherwood number, chemical reaction effect, heat source, radiation absorptions

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Introduction

Non-Newtonian fluids are the important class of fluids, where the connection between shear stress and shear rate is non-linear therefore, they are much complicated and not possible to have a single constitutive relation. Large number of non-Newtonian fluids can be found in the literature, for example nano-fluid, Micro-polar fluid, Maxwell fluid, Jeffrey fluid, Casson fluid, second grade fluid, etc. The nano-fluid has been examined widely by the scientists and researchers due to its complicity. Nano-particles are the nano-sized powders consisting of heterogeneous mixture of metallic polymeric or nonmetallic polymeric containing in a base fluid to obtained the best heat transfer rate in many problem. A large number of application based on the nano-fluid due to the requirement of heat transfer to the problem, therefore researchers give full consideration and produce a large number of problems on nano-fluid. Similarly, on manufacturing side large number of applications such as, wire coating, daily life uses equipments, foodstuff processing, polymer extrusion, drawing of plastic sheets, metal extrusion, continuous casting, etc, the nano-fluids plays an important rule over an unsteady free surface flow. Wang took the unsteady flow by considering thin film with time dependent stretching surface. Another scientist, Jou and Tzeng took the concept of Khanafer to understand the effects of heat transfer under the enclosure nano-fluid for various parameters. Palm et al., heat transfer phenomena is adjusted to the cooling system of flow with the introduce of nano-size particles to the laminar forced-convection flow, and solution of the nano-fluid is done with the help of numerical method. Circular tube considered by Zeinali Heris et al. and pass laminar flow of nano-fluid (Al$_2$O$_3$ water) under forced convection and the temperature of the wall kept constant. Experimental results for suitable parameters of heat transfers are noted. Buoyancy forces are considered by Oztop and Abu-Nada for the analysis of heat transfer during nano-fluid flows under the condition that the vertical opposite side walls are flush heaters and solution is discussed by numerical method. Heterogeneous mixture (nano-particles) is considered by Wang and Mujumdar, in the base fluid flow under the force convection, this experimental observation is performed on the heat transfer. Meanwhile, Ho et al. assumed, that conductivity and viscosity are variables in the presence of free convection, and flow is observed in square area filled with aluminum (nano-fluid-water) while, numerical method is used for discussion. Bazhlekova et al. took second grade fluid Rayleigh-Stokes model. Chen et al. took second grade fluid and observed there five cases in the micro-tube volume inlet flow with the help of Laplace transform method (technique). Wang et al. discussed, second grade fluid with the help of fractional derivatives and solutions are presented with the of numerical and analytical methods. The experts, Dehghan and Abbaszadeh took Rayleigh-Stokes model for second grade liquids and numerical technique is used for solution by applying finite element method. Experimental work of Fotukian and Nasr Esfahany discussed turbulent flow of convective nano-fluid in a tube for heat transfer, and observed that dilute CuO/water become more dilute (less than 0.24/100) then pressure profile dropped, and when CuO particles increased in water then heat flux increased. Porous inclined plates are chosen by Goyal and Kumari for the observation of many physical properties to the MHD convected flow and results are made for different parameters by analytical method. Vertical heated plates are considered by Ghalambaz et al. for the nano-size fluid flow with the saturated Darcy porous medium and solved by Numerical method. Further, Haroun et al. added unsteady MHD stretching/shrinking sheet and Viscus dissipation term to the work of Ghalambaz et al. Some relevant and recent work (from literature) can be seen in Lin et al. and Bibi et al., and the corresponding references are there. A special type of particles called GDN (Ginger-derived nano-particles) are discussed by Zhuang et al. in the mouse model and found that mice verses alcohol induced liver damages are protected. Some more physical based experts in the same area and their derivation can be seen in resent literature. Hayat et al. considered Copper nano-particles due to its importance in engineering and industry applications for both theoretical and experimental areas. Khan et al. presented copper nano-fluids study in the presence of MHD with base fluid is water in two parallel plates. Further, lecturer survey for copper nano-particles can be seen in Toghraie et al., Arqub and Shawagfeh, Abu Arqub, and their references are cited in them.

In the above survey, it is noted that further meaningful models exists for investigation of nano-fluids with the possible physical properties. For this purpose, new nano-particles along with new physical properties are considered for the improvement of sufficient heat and mass transfer. Aluminum, Titanium, Silver, and copper are the considered particles with the most important physical properties, mixed convection, radiation and...
absorption, Schmidt number, chemical reaction, variables viscosity, thermal-conductivity, and MHD thin film flow. The surface tension is directly dependent on temperature. Different relations are derived between nano-particles with several important physical properties, for the system of partial differential equations describing MHD flow. Momentum profile along with heat and mass transfers of a thin fluid film on an unsteady stretching nonuniform-surface in the presence of external magnetic field are observed. A two dimensional model, nonlinear partial differential equations (PDEs) are found. Considered transformations enable reductions of the model to a coupled system of nonlinear ordinary-differential equations (ODEs). Analytic solution is presented by HAM by HAM package and numerical solution is presented by shooting package, are shown in figures and tables.

**Governing equations**

Unsteady flow emerges from a narrow-slit which lying on the origin of the non-rotating frame of reference under the influence of large number of physical properties. Here thickness of the slit is denoted by \( h(x, t) \) over a plane sheet with the effect of pressure profile depending upon velocity. Mass transfer is also focus of this study which are inter related with heat transfer. Infinite stretching sheet is considered along in one direction (\( x \)-axis) while the magnetic force (\( B = B_0/(1-\alpha t)^{1/2} \)) is considered in normal direction (\( y \)-axis). Geometry of the two-dimensional nano-fluid flow is given in Figure 1. The very basic equation in fluid dynamics are:

\[
\frac{\partial \rho_{nf}}{\partial t} + \text{div}(\rho_{nf} \mathbf{V}) = 0.
\]

When the fluid is independent of the density then the fluid flow will be incompressible and \( \rho_{nf} \) will be constant therefore the reduce equation will get the form as

\[
\text{div}(\mathbf{V}) = 0.
\]

The momentum equation is:

\[
\rho_{nf} \left( \frac{\partial \mathbf{V}}{\partial t} + \mathbf{V} \cdot \nabla \mathbf{V} \right) = - \nabla P + \nabla \cdot \mathbf{\tau} + \rho_{nf} f,
\]

where the acceleration time’s derivative is on left hand side, that is actually the total rate of change of velocity vector with respect to time, while in the above equation the other side (right hand side) expresses the addition of body forces. Surface forces are \( \nabla \cdot \mathbf{\tau} \). Cauchy tensor is denoted by \( \mathbf{\tau} \) and can be expressed as,

\[
\mathbf{\tau} = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{pmatrix}
\]

where \( \sigma_{xx}, \sigma_{yy} \) denote normal stresses and \( \sigma_{xy}, \sigma_{yx} \) are the shear stresses. Momentum equation in \( x, y \) directions along \( u \) and \( v \) components are given by

\[
\rho_{nf} \left( \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right) = - \frac{\partial P}{\partial x} + \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \rho_{nf} f_x,
\]

\[
\rho_{nf} \left( \frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} \right) = - \frac{\partial P}{\partial y} + \frac{\partial \sigma_{yx}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \rho_{nf} f_y,
\]

as the gravitational force called body force therefore, the change of gravity will effect the principle of \( f_x, f_y \) respectively. Energy equation is:

\[
(p c_p)_{nf} \frac{D T}{D t} = \frac{D P}{D t} + \nabla \cdot (k_{nf} \nabla T) + \Phi,
\]

where \( \frac{D}{D t} \) is the material derivative, \( c_p \) the specific heat, \( \rho_{nf} \) is the constant density, \( T \) the fluid temperature, \( k_{nf} \) the thermal conductivity, and \( \Phi \) is the dissipation function define as

\[
\Phi = tr(S.L),
\]

here the tensor \( S \) denotes shear stress and \( L \) indicates velocity gradient. The diffusion equation in three-dimensional (in various forms of documentation) as

\[
\frac{\partial C}{\partial t} = D \left( \frac{\partial^2 C}{\partial x^2} + \frac{\partial^2 C}{\partial y^2} \right) = D \nabla^2 C = D \frac{\partial^2 C}{\partial x^2} + \text{HOTs},
\]

which is the very basic equation in fluid dynamics of a concentration equation. The following governing equations are:

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \quad (1)
\]
\[
\rho_\text{nf} \left( \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right) + \frac{\partial P}{\partial x} = \mu_\text{nf} \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 v}{\partial y^2} \right) + \frac{\partial \rho_\text{nf} \partial v}{\partial y} + \frac{\partial \rho_\text{nf} \partial u}{\partial x} \\
+ \frac{\partial \rho_\text{nf} \partial u}{\partial x} \frac{\partial \rho_\text{nf} \partial v}{\partial y} + \beta_1 \rho_\text{nf} g(T - T_0) + \beta_2 \rho_\text{nf} g(C - C_0) - \sigma B^2 u,
\]

\[
\rho_\text{nf} \left( \frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} \right) + \frac{\partial P}{\partial y} = \mu_\text{nf} \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right) + \frac{\partial \mu_\text{nf} \partial u}{\partial x} + \frac{\partial \mu_\text{nf} \partial v}{\partial y} \\
+ \frac{\partial \mu_\text{nf} \partial v}{\partial x} \frac{\partial \mu_\text{nf} \partial u}{\partial y} + \beta_2 \rho_\text{nf} u(T - T_0) + \beta_2 \rho_\text{nf} g(C - C_0),
\]

The following scaling is used for the governing equations with boundary conditions to transform into their dimensionless form:

\[
x = x^* L, \quad y = y^* \hat{\delta}, \quad u = u^* U, \quad v = v^* U \hat{\delta}, \quad T = T^* (T_s - T_0) + T_0,
\]

Vertical length is denoted by \( \hat{\delta} \) and horizontal is denoted by \( L \). The relation \( \xi << 1 \) represents aspect ratio, sheet surface temperature is denoted by \( T_0 \) while fluid surface temperature is \( T_s \). Using (6) (dimensionless scaling) to (1)-(5) and remove asterisk we get:

\[
\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0,
\]

\[
\rho_\text{nf} \left( \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right) + \frac{\partial P}{\partial x} = \mu_\text{nf} \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) + \frac{\partial \rho_\text{nf} \partial u}{\partial x} + \frac{\partial \rho_\text{nf} \partial v}{\partial y} + \frac{\beta_1 \rho_\text{nf} g(T_s - T_0)}{\rho_\text{nf}} \Phi + \frac{L \sigma B^2 u}{U}.
\]
\[ T_s \] denote surface (temperature) and \( T_0 \) denote sheet (temperature), where \( T_{\text{ref}} \) denote constant reference (temperatures) \( \forall t<1/\alpha \). Converting PDEs to ODEs, below given transformations are to be used:

\[
\begin{align*}
  u &= \frac{bx}{1-\alpha t} f'(\eta), \quad v = -\sqrt{\frac{\nu f b}{1-\alpha t}} \beta f(\eta), \\
  \eta &= \sqrt{\frac{b}{\nu f(1-\alpha t)}} \beta^2 \gamma y, \\
  \beta &= \sqrt{\frac{b}{\nu f(1-\alpha t)}} \gamma h(t), \quad P = P_0 - P_{\text{ref}} \frac{b^2 \gamma^2}{2(1-\alpha t)} f'(\eta), \\
  \theta(\eta) &= \frac{T - T_0}{T_s - T_0}, \quad \Phi(\eta) = \frac{C - C_0}{C_s - C_0}, \\
  \kappa_f &= \kappa_f(1 + N \theta), \quad \mu_f = \mu_f(1 + A \theta),
\end{align*}
\]

where \( f \) and \( \Phi \) are the new variable represents, derivatives of the functions with the new variable \( \eta \) where, the non-dimensional \( Y = \beta^2 \) represents film-thickness, \( S = \alpha/b, \quad Ma = L \delta B^2_0/Ub_0, \quad Gr_r = \beta_b T_{\text{ref}}, \quad Gr_r = \beta_b C_{\text{ref}}, \) are the time dependent parameter, magnetic number, thermal Grashof number, and mass Grashof number. Further, \( Q = \frac{LQ_0(1-\alpha t)}{U(b \gamma n h_s)} \), \( S_c = \frac{\nu_f}{D_0}, \quad R_i = \frac{L(R(1-\alpha t)R(1-\alpha t))}{R(b_0 C_{\text{ref}})} \), \( Pr = \frac{\nu_f}{\alpha_0}, \quad M = \frac{\delta \tau T_0 b}{\mu_f \sqrt{b_0 f}}, \) and \( Q_i = \frac{LQ_0 C_{\text{ref}}(1-\alpha t)}{U(b \gamma n h_s)} \) are heat source, Schmidt number, chemical reaction parameter, Prandtl number, Thermocapillary number, and radiation absorption number, respectively.

\[ \text{HAM}^{34} \text{ is used for the result of equations (18)–(20) and initial guesses are:} \]

\[
\begin{align*}
  \theta_0(\eta) &= 1, \\
  f_0(\eta) &= \eta \left( \frac{M \theta_0(\eta)}{4} \eta^2 - \frac{M \theta_0(\eta)}{4} \eta + 1 \right), \\
  \Phi_0(\eta) &= 1.
\end{align*}
\]

Auxiliary linear operators are \((E_\theta = \partial^2/\partial \eta^2), \quad (E_\Phi = \partial^2/\partial \eta^2)\) as:

\[
\begin{align*}
  &E_\theta [E_1 + E_2 \eta + E_3 \eta^2 + E_4 \eta^3] = 0, \\
  &E_\Phi [E_1 + E_2 \eta] = 0, \\
  &E_\Phi [E_1 + E_2 \eta] = 0,
\end{align*}
\]

\( E_i \) are integral constants.

### Nusselt number, Skin-friction coefficient, and Sherwood number

Some interesting, physical numbers are \( N_{tu}, \quad C_{fu}, \quad \text{and} \quad S_{ku} \), are:

\[
\begin{align*}
  N_{tu} &= \frac{q_w}{k_f} (T_s - T_0)^{-1}, \quad C_{fu} = \frac{2 \tau_w}{\rho_f U^2} \bar{a}, \quad h_s = \frac{q_m}{D} (C_s - C_0)^{-1} \text{respectively,}
\end{align*}
\]

Where \( q_w \) (heat transfer from sheet), \( \tau_w \) (shearing-stress at surface of wall), and \( q_m \) (mass-flux at surface of wall) respectively, as shown below:

\[
\begin{align*}
  \tau_w &= - \mu_f \left( \frac{\partial u}{\partial \eta} \right)_{\eta = 0}, \\
  q_w &= - \kappa_f \left( \frac{\partial T}{\partial \eta} \right)_{\eta = 0}, \\
  q_m &= - D \left( \frac{\partial C}{\partial \eta} \right)_{\eta = 0}.
\end{align*}
\]
Where, $\mu_{nf}$ is viscosity and $\kappa_{nf}$ is thermal conductivity of the nano-fluid. With the use of mention transformation, we obtained as:

$$N_u = - \frac{\kappa_{nf}}{\kappa_f} \beta^{-\frac{1}{2}} Re_z \theta'(0),$$

$$C_p = \frac{-2(1 + A \theta)}{(1 - \phi) \beta} \beta^{-\frac{1}{2}} Re_z \phi''(0),$$

$$Sh_x = - \beta^{-1} Re_z \Psi'(0).$$

The expression $Re_x = x U_x / v$ is a local Reynolds number.

### Optimal-convergence control parameters

Before the discussion of the problem by HAM, here we check the validity of the method and therefore first present error analysis. For this purpose, we draw Figure 2 and constructed Tables 2 to 4. Minimum error $10^{-40}$ is chosen for the HAM-package (BVPPh2.0) during solution procedure. Plus point of the package is the

Figure 2. Using HAM package where $M = 1, R_1 = 1, Q_1 = 1, Pr = 0.8, Ma = 1, Gr_1 = 0.01, Gr_2 = 0.01, P_{eff} = 1, Sc = 1, Y = 0.48721, \kappa_f = 0.6130, (c_p)_f = 4179.0, \rho_f = 8933, S = 0.4, (c_p)_s = 385.0, Q = 0.1, \kappa_s = 401, \rho_f = 997.1,$ and $A = 1.1$ for observing different errors.
auxiliary parameters, that is $h_f \neq 0$, $h_\theta \neq 0$, $h_\phi \neq 0$ are available for the predication of the solution region, during solution algorithm. These auxiliary numbers are used for the average residual error to their optimal values introduced by Liao\textsuperscript{35} as:

$$
\hat{\varepsilon}_m^f = (k + 1)^{-1} \sum_{j = 0}^{k} \left[ \frac{N_f}{\sum_{j = 0}^{m} F(\eta) \sum_{j = 0}^{m} \Theta(\eta) \sum_{j = 0}^{m} \Psi(\eta)} d\eta \right]_{\eta = \beta_\eta}^{d\eta} \nonumber
$$

$$
\hat{\varepsilon}_m^\theta = (k + 1)^{-1} \sum_{j = 0}^{k} \left[ \frac{N_\theta}{\sum_{j = 0}^{m} F(\eta) \sum_{j = 0}^{m} \Theta(\eta) \sum_{j = 0}^{m} \Psi(\eta)} d\eta \right]_{\eta = \beta_\eta}^{d\eta} \nonumber
$$

$$
\hat{\varepsilon}_m^\phi = (k + 1)^{-1} \sum_{j = 0}^{k} \left[ \frac{N_\phi}{\sum_{j = 0}^{m} F(\eta) \sum_{j = 0}^{m} \Theta(\eta) \sum_{j = 0}^{m} \Psi(\eta)} d\eta \right]_{\eta = \beta_\eta}^{d\eta} \nonumber
$$

By Liao\textsuperscript{35}

$$
(\hat{\varepsilon}_m') = (\hat{\varepsilon}_m^f) + (\hat{\varepsilon}_m^\theta) + (\hat{\varepsilon}_m^\phi).
$$

The expression $\varepsilon_m'$ is used to represent, the summation of square residual error, for the fixed value of $k = 20$, and $\delta_\eta = 0.50$. A package (Mathematica package BVPh2.0) was introduced by Zhao\textsuperscript{36} to minimize the total average squared residual error. Different values of parameters are considered to observe errors for arbitrary order of approximation.

The value of $A$ (viscosity-parameter) is increased and the remaining all parameters are kept constants, as $M = 1$, $R_1 = 1$, $Q_1 = 1$, $Pr = 0.8$, $Ma = 1$, $Gr_c = 0.01$, $Gr_t = 0.01$, $P_{ref} = 1$, $Sc = 1$, $Y = 0.48721$, $\kappa_f = 0.6130$, $(cp)_f = 4179.0$, $\rho_s = 8933$, $S = 0.4$, $(cp)_s = 385.0$, $Q = 0.1$, $k_\tau = 401$, $\rho_f = 997.1$, $N = 1.0$, and $\phi = 0.2$. Figure 2 shows that, maximum average squared residual-error decreased as the order of approximation increased for the corresponding value of $A = 0.3.1$. In Figure 2, $f$, $\theta$, and $\Phi$ are also discussed for different values of $A$.

Nano-fluid properties\textsuperscript{12,37} are listed in the Table 1, where water is base fluid and the other nano-fluid particles are $Cu$, $Ag$, $Al_2O_3$, and $TiO_2$. Convergence control-parameter with the minimum-values of total averaged squared residual error versus different orders of approximation where, $M = 1$, $R_1 = 1$, $Q_1 = 1$, $Pr = 0.800$, $Ma = 1$, $Gr_c = 0.01$, $Gr_t = 0.01$, $P_{ref} = 1$, $Sc = 1$, $Y = 0.099721$, $\kappa_f = 0.6130$, $(cp)_f = 4179.0$, $\rho_s = 8933$, $S = 0.4$, $(cp)_s = 385.0$, $Q = 0.1$, $\kappa_\tau = 401$, $\rho_f = 997.1$, $N = 1.0$, $A = 1.1$, and $\phi = 0.2$ as shown in Table 2. Optimal values are the self selection of the package (BVPh2.0) therefore, different orders of approximation to the individual average-squared

### Table 1. Nano-fluid properties with base fluid\textsuperscript{12,37}

| Properties | Basefluid | Cu(Copper) | Ag(Silver) | Al$_2$O$_3$(AluminumOxide) | TiO$_2$(Titaniumoxide) |
|------------|-----------|------------|------------|----------------------------|-------------------------|
| $\sigma \times 10^5$ (m$^2$/s) | 1.470 | 1163.10 | 1738.60 | 131.70 | 30.70 |
| $c_p$(J/kgK) | 4179.0 | 385.0 | 235.0 | 765.0 | 686.20 |
| $\kappa$(W/mK) | 0.6130 | 401.0 | 429.0 | 40.0 | 8.95380 |
| $\rho$(kg/m$^3$) | 997.10 | 8933.0 | 10500.0 | 3970.0 | 4250.0 |
| $\mu \times 10^6$(kg/ms) | 8.910 | $\ldots$ | $\ldots$ | $\ldots$ | $\ldots$ |
| $\beta \times 10^6$(K$^{-1}$) | 21.0 | 1.670 | 1.890 | 0.850 | 0.90 |

### Table 2. Convergence control-parameter with the total averaged squared residual error values, versus different orders of approximation.

| Order of approximation | $h_f$ | $h_\theta$ | $h_\phi$ | $\varepsilon_m'$ |
|------------------------|-------|------------|----------|-----------------|
| 2                      | $-0.460411$ | $-0.430624$ | $-0.965284$ | $1.48757 \times 10^{-6}$ |
| 3                      | $-0.499334$ | $-0.454133$ | $-0.938005$ | $7.72690 \times 10^{-8}$ |
| 4                      | $-0.493808$ | $-0.452610$ | $-1.015860$ | $3.03819 \times 10^{-10}$ |
| 5                      | $-0.498697$ | $-0.438797$ | $-0.941023$ | $2.05422 \times 10^{-13}$ |
| 6                      | $-0.472116$ | $-0.428209$ | $-1.010350$ | $-1.38402 \times 10^{-13}$ |
Table 3. Arbitrary values of auxiliary-parameters with averaged squared-residual errors.

| m  | $a_n^L$ | $a_n^H$ | $a_n^φ$ |
|----|---------|---------|---------|
| 2  | 2.66957×10^{-6} | 2.09127×10^{-8} | 6.3358×10^{-7} |
| 3  | 3.85138×10^{-7} | 1.59909×10^{-10} | 3.65529×10^{-9} |
| 5  | 1.17367×10^{-11} | 5.67685×10^{-15} | 1.23852×10^{-13} |
| 7  | 4.11432×10^{-16} | 2.21808×10^{-19} | 4.24669×10^{-18} |
| 9  | 1.40877×10^{-20} | 8.8595×10^{-24} | 1.46423×10^{-22} |
| 15 | 4.9034×10^{-32} | 7.3665×10^{-35} | 4.78401×10^{-33} |
| 19 | 4.60673×10^{-32} | 7.60763×10^{-35} | 4.89186×10^{-33} |
| 25 | 4.60673×10^{-32} | 8.02919×10^{-35} | 4.93908×10^{-33} |
| 30 | 4.60673×10^{-32} | 8.02919×10^{-35} | 4.93908×10^{-33} |
| 40 | 4.83848×10^{-32} | 8.13727×10^{-35} | 4.89186×10^{-33} |

Table 4. Convergency table for “HAM” in term of the values of $-f''(0)$, $-θ'(0)$, and $-Φ'(0)$.

| m  | $f''(0)$ | $θ'(0)$ | $Φ'(0)$ |
|----|----------|---------|---------|
| 1  | 0.00966790472703494239 | 0.17550245069278184543 |
| 5  | 0.0067877683169273017 | 0.165503496694357119 |
| 10 | 0.0067874895054950691 | 0.1655035125571259250 |
| 15 | 0.0067874895055680346 | 0.1655035125649807218 |
| 20 | 0.0067874895055680346 | 0.1655035125649807011 |
| 25 | 0.0067874895055680346 | 0.1655035125649807011 |
| 30 | 0.0067874895055680346 | 0.1655035125649807011 |
| 35 | 0.0067874895055680346 | 0.1655035125649807011 |
| 40 | 0.0067874895055680346 | 0.1655035125649807011 |

residual-errors shown in Table 3, for $M = 1$, $R_1 = 1$, $Q_1 = 1$, $Pr = 0.800$, $Ma = 1$, $Gr_c = 0.01$, $Gr_t = 0.01$, $P_{ref} = 1$, $Sc = 1$, $Y = 0.098721$, $κ_f = 0.6130$, $(c_p)_f = 4179.0$, $ρ_s = 8933$, $S = 0.4$, $(c_p)_s = 385.0$, $Q = 0.1$, $κ_s = 401$, $ρ_f = 997.1$, $N = 10$, $φ = 0.04$, and $A = 1.1$. In Table 4, 15th order of approximation gives 20th decimal places accuracy of $(-f''(0), -θ'(0))$ for $M = 1$, $R_1 = 1$, $Q_1 = 1$, $Pr = 0.7$, $Ma = 1$, $Gr_c = 0.01$, $Gr_t = 0.01$, $P_{ref} = 1$, $Sc = 1$, $Y = 0.098721$, $κ_f = 0.6130$, $(c_p)_f = 4179.0$, $ρ_s = 8933$, $S = 0.3$, $(c_p)_s = 385.0$, $Q = 0.1$, $κ_s = 401$, $ρ_f = 997.1$, $N = 1.1$, $φ = 0.04$, and $A = 1.1$.

Results and discussion

Investigation is made for the flow of nano-fluid over a stretching sheet, in which CMC-water has base-fluid with the involved nano-particles of Titanium, Aluminum, Silver, and Copper. Interesting results are obtained by BVP4C and HAM. Meaning full results of $-\Phi'(0)$, $Y$, $θ(1)$, $f''(0)$, and $-θ'(0)$ due to the variations of $h_f$, $h_θ$, and $h_θ$ shown in Table 5. While, Tables 6 to 15 are made for the observations of $Al_2O_3$, $Ag$, $Ma$, $Cu$, $Gr_c$, $φ$, $Gr_t$, $TiO_2$, and $Pr$ on $-Φ'(0)$, $-f''(0)$, $-θ'(0)$, $Y$, and $θ(1)$.

In Table 6, nano-particles are varying while, the remaining parameters are fixed and analysis is performed on free temperature ($θ(1)$), skin friction ($-f''(0)$), temperature flux ($-θ'(0)$), and mass flux ($-Φ'(0)$). Here when we increased nano-particles that is $Al_2O_3$, $Cu$, $TiO_2$, $Ag$ and reducing the film thickness, $-f''(0)$ and $-θ'(0)$ are increased, while $θ(1)$ and $-Φ'(0)$ are decreased. This shows that, increased in nano-particles, sheet fraction, and heat flux are increased but, free temperature, and mass flux are decreased. Table 7 is about the effect of thermal-Grashof number, in which $Gr_r$ is increased the corresponding $θ(1)$ and $-Φ'(0)$ are decreased while, $-f''(0)$ and $-θ'(0)$ are increased. Concentration-Grashof number has direct relation with nano-fluids, because heat transfer provides dominant buoyancy-forces to the flows. Table 8, shows that as $Gr_r$ is increased and thickness of the sheet is reduced, $θ(1)$ and $-Φ'(0)$ are decreased, meanwhile $-f''(0)$ and $-θ'(0)$ are more quantities. Table 9 represents, the effect of $A$, the more viscosity of the fluid are the increased of $-θ'(0)$ and $-f''(0)$, meanwhile the decreased of $-Φ'(0)$ and $θ(1)$. Increased of $N$ are the increased of $-θ'(0)$ and
Table 5. Variations of \( h^\prime \), \( h'' \), and \( \tilde{h}_0 \) on \( \eta^\prime - \Phi(0) \), \( \eta^\prime' - \Phi(0) \), \( \eta''(0) \), and \( \eta^\prime(0) \) by using 20th order HAM (BVP2.0 Mathematica package), where \( M = 1, R_1 = 1, Q_1 = 1, P_r = 0.800, M_a = 1, G_\tau = 0.01, P_{mf} = 1, S = 1, \kappa_\tau = 0.6130, (c_{p f})_f = 4179.0, \rho_f = 8933, S = 0.4, (c_{pf})_f = 385.0, Q = 0.1, \kappa_s = 401, \rho_f = 997.1, N = 1.0, and A = 1.1.

| \( \phi \) | \( h^\prime \) | \( h'' \) | \( \tilde{h}_0 \) | \( \eta^\prime(0) \) | \( \eta''(0) \) | \( \eta^\prime(0) \) | \( -\Phi(0) \) |
|---|---|---|---|---|---|---|---|
| 0.15 | -0.434883 | -0.462200 | -0.945527 | 0.098721 | -3.07867 | 1.00479 | -0.00027683 | 0.165503 |
| 0.035 | 0.509409 | -0.415644 | -0.971426 | 0.088721 | -3.07584 | 1.00436 | -0.00073536 | 0.149368 |
| 0.439945 | -0.466722 | -0.950890 | 0.078721 | -3.07300 | 1.00392 | -0.00157037 | 0.133097 |
| 0.451238 | -0.460377 | 1.011950 | 0.068721 | -3.07014 | 1.00347 | -0.0045869 | 0.116688 |
| 0.507471 | -0.416417 | -0.96934 | 0.098721 | -3.07358 | 1.00271 | -0.00355292 | 0.165505 |
| 0.448828 | -0.456356 | -0.96059 | 0.088721 | -3.07123 | 1.00247 | -0.00324491 | 0.149369 |
| 0.503311 | -0.452872 | -0.974448 | 0.078721 | -3.06888 | 1.00222 | -0.00292572 | 0.133098 |
| 0.501097 | -0.430809 | -0.978096 | 0.068721 | -3.06652 | 1.00196 | -0.00259517 | 0.116688 |

Table 6. For, \( M = 1, R_1 = 1, Q_1 = 1, P_r = 0.800, M_a = 1, G_\tau = 0.01, P_{mf} = 1, S = 1, \kappa_\tau = 0.6130, (c_{p f})_f = 4179.0, \rho_f = 8933, S = 0.4, (c_{pf})_f = 385.0, Q = 0.1, \kappa_s = 401, \rho_f = 997.1, N = 1.0, A = 1.1 \) and varying \( \phi \), the comparisons between analytical HAM, and numerical BVP4C package of \( -\Phi(0), f''(0), \theta(1), \text{and } -\theta'$ \).

| Particles | \( \phi \) | \( \eta^\prime(0) \) | \( \eta''(0) \) | \( \eta^\prime(0) \) | \( -\Phi(0) \) |
|---|---|---|---|---|---|
| \( Cu \) | 0.04 | 0.0987 | -3.07774 | 1.00771 | -0.0101 | 0.16550 |
| 0.08 | 0.0887 | -3.07610 | 0.0623 | -0.0080 | 0.14936 |
| 0.12 | 0.0787 | -3.07370 | 0.0499 | -0.0063 | 0.13309 |
| 0.16 | 0.0687 | -3.07077 | 0.0393 | -0.0049 | 0.11668 |
| \( Ag \) | 0.04 | 0.0987 | -3.07855 | 0.0764 | -0.0011 | 0.16550 |
| 0.08 | 0.0887 | -3.07742 | 0.0611 | -0.0008 | 0.14936 |
| 0.12 | 0.0787 | -3.07527 | 0.0484 | -0.0005 | 0.13309 |
| 0.16 | 0.0687 | -3.07238 | 0.0378 | -0.0003 | 0.11668 |
| \( Al_2O_3 \) | 0.04 | 0.0987 | -3.07491 | 0.0772 | -0.0010 | 0.16550 |
| 0.08 | 0.0887 | -3.07152 | 0.0624 | -0.0008 | 0.14936 |
| 0.12 | 0.0787 | -3.06823 | 0.0496 | -0.0005 | 0.13309 |
| 0.16 | 0.0687 | -3.06509 | 0.0394 | -0.0002 | 0.11668 |
| \( TiO_2 \) | 0.04 | 0.0987 | -3.07520 | 0.0783 | -0.0012 | 0.16550 |
| 0.08 | 0.0887 | -3.07198 | 0.0642 | -0.0004 | 0.14936 |
| 0.12 | 0.0787 | -3.06878 | 0.0520 | -0.0006 | 0.13309 |
| 0.16 | 0.0687 | -3.06566 | 0.0416 | -0.0004 | 0.11668 |

Table 7. For, \( M = 1, R_1 = 1, Q_1 = 1, P_r = 0.800, M_a = 1, G_\tau = 0.01, P_{mf} = 1, S = 1, \kappa_\tau = 0.6130, (c_{p f})_f = 4179.0, \rho_f = 8933, S = 0.4, (c_{pf})_f = 385.0, Q = 0.1, \kappa_s = 401, \phi = 0.04, \rho_f = 997.1, N = 1.0, A = 1.1 \) and varying \( G_\tau \), the comparisons between analytical HAM, and numerical BVP4C package of \( -\Phi(0), f''(0), \theta(1), \text{and } -\theta'$ \) with Titanium nano-fluid.

| \( G_\tau \) | \( \eta^\prime(0) \) | \( \eta''(0) \) | \( \eta^\prime(0) \) | \( -\Phi(0) \) |
|---|---|---|---|---|
| 0.03 | 0.098721 | -3.07519 | 1.00783 | -0.010273 | 0.165503 |
| 0.06 | 0.088721 | -3.07276 | 1.00714 | -0.009386 | 0.149368 |
| 0.09 | 0.078721 | -3.07030 | 1.00642 | -0.008466 | 0.133097 |
| 0.12 | 0.068721 | -3.06782 | 1.00568 | -0.007512 | 0.116688 |
| 0.15 | 0.058721 | -3.06531 | 1.00492 | -0.006524 | 0.100138 |
| 0.18 | 0.048721 | -3.06277 | 1.00414 | -0.005501 | 0.083445 |
| 0.21 | 0.038721 | -3.06020 | 1.00334 | -0.004443 | 0.066608 |
| 0.24 | 0.028721 | -3.05761 | 1.00251 | -0.003349 | 0.049623 |
Table 8. For $M = 1, R_1 = 1, Q_1 = 1, Pr = 0.800, Ma = 1, Gr_t = 0.01, P_{ref} = 1, S = 1, \kappa_t = 0.6130, (c_p s) = 4179.0, \rho_s = 8933$, $S = 0.4, (c_p s) = 385.0, Q = 0.1, \kappa_s = 401, \phi = 0.04, \rho_f = 997.1, N = 1.0, A = 1.1$ and varying $Gr_c$, the comparisons between analytical HAM, and numerical BVP4C package of $-\Phi(0), f'(0), Y, \theta(1)$, and $-\theta'(0)$ with Titanium nano-fluid.

| $Gr_c$ | $Y$       | HAM          | BVP4C          |
|-------|-----------|--------------|----------------|
|       | $f'(0)$   | $\theta(1)$ | $-\theta'(0)$ |
| 0.03  | 0.098721  | -3.07520     | 1.00783       |
| 0.06  | 0.088721  | -3.07278     | 1.00713       |
| 0.09  | 0.078721  | -3.07033     | 1.00642       |
| 0.12  | 0.068721  | -3.06784     | 1.00568       |
| 0.15  | 0.058721  | -3.06533     | 1.00492       |
| 0.21  | 0.038721  | -3.06022     | 1.00334       |
| 0.24  | 0.028721  | -3.05762     | 1.00251       |

Table 9. Analysis by analytical HAM (BVPh2.0 Mathematica package), of the quantities $Y$, $-\Phi(0), \theta(1), f'(0)$, and $-\theta'(0)$ are listed, when $M = 1, R_1 = 1, Q_1 = 1, Pr = 0.800, Ma = 1, Gr_t = 0.01, P_{ref} = 1, S = 1, \kappa_t = 0.04, \phi = 0.04, N = 1.0, A = 1.1, Gr_c = 0.01$, and varying $A$ with Copper nano-fluid.

| $A$   | $Y$       | HAM          |
|-------|-----------|--------------|
|       | $f'(0)$   | $\theta(1)$ |
| 0.4   | 0.098721  | 1.00771      |
| 0.8   | 0.088721  | 1.00702      |
| 1.2   | 0.078721  | 1.00632      |
| 1.6   | 0.068721  | 1.00559      |
| 2.0   | 0.058721  | 1.00485      |
| 2.4   | 0.048721  | 1.00408      |
| 2.8   | 0.038721  | 1.00328      |
| 3.2   | 0.028721  | 1.00247      |

Table 10. Analysis by analytical HAM (BVPh2.0 Mathematica package), of the quantities $Y$, $-\Phi(0), \theta(1), f'(0)$, and $-\theta'(0)$ are listed, when $M = 1, R_1 = 1, Q_1 = 1, Pr = 0.800, Ma = 1, Gr_t = 0.01, P_{ref} = 1, S = 1, \kappa_t = 0.04, A = 1.1$, and varying $N$ with Copper nano-fluid.

| $N$   | $Y$       | HAM          |
|-------|-----------|--------------|
|       | $f'(0)$   | $\theta(1)$ |
| 0.2   | 0.098721  | 1.01154      |
| 0.5   | 0.088721  | 1.00842      |
| 0.8   | 0.078721  | 1.00632      |
| 1.1   | 0.068721  | 1.00480      |
| 1.4   | 0.058721  | 1.00364      |
| 1.7   | 0.048721  | 1.00247      |
| 2.0   | 0.038721  | 1.00197      |
| 2.3   | 0.028721  | 1.00135      |

$-f''(0)$, but decreased of $-\Phi(0)$ and $\theta(1)$ as shown in Table 10. Further, Table 11 represents the effects of $Q$. Increased of $Q$ and decreased of $Y$, implies that decreased are seen in $-\Phi(0), -\theta(0)$, and $-f''(0)$, but increased is seen in $\theta(1)$. Similarly, increased in $Q_1$ is a reason for the decreased of $-\Phi(0), -\theta(0)$, and $-f''(0)$,
while increased of $\theta(1)$ as displayed in Table 12. As $Pr$ is increased the reciprocal effects are seen on $-f''(0)$ and $-\Phi'(0)$, while direct effects are seen on $\theta(1)$ and $-\theta'(0)$ as shown in Table 13. Addition of Table 14 is about $R_1$. Increased of $R_1$ and decreased of $Y$, implies that increased are seen in $-f''(0)$, $-\Phi'(0)$, and $-\theta'(0)$ while decreased is seen in $\theta(1)$. Addition of the effect $Ma$ on $\theta(1)$, $-f''(0)$, $-\Phi'(0)$, and $-\theta'(0)$ are shown in

### Table 11. Analysis by analytical HAM (BVPh2.0 Mathematica package), of the quantities $Y$, $-\Phi'(0)$, $\theta(1)$, $f''(0)$, and $-\theta'(0)$ are listed, when $M = 1$, $R_1 = 1$, $Q_i = 1$, $Pr = 0.800$, $Ma = 1$, $Gr_c = 0.01$, $P_{ref} = 1$, $Sc = 1$, $S = 0.4$, $N = 0.8$, $\phi = 0.04$, $A = 1.1$, $Gr_c = 0.01$, and varying $Q$ with Copper nano-fluid.

| $Q$  | $Y$   | $f''(0)$ | $\theta(1)$ | $\Phi'(0)$ | $-\theta'(0)$ |
|------|-------|----------|--------------|-------------|---------------|
| 0.01 | 1.098721 | -3.32508 | 0.997383     | 0.366996    | 1.33412       |
| 0.05 | 1.088721 | -3.25685 | 1.004740     | 0.021483    | 1.32511       |
| 0.09 | 1.078721 | -3.26356 | 1.012040     | 0.006368    | 1.31606       |
| 0.13 | 1.068721 | -3.27022 | 1.019280     | -0.008588   | 1.30699       |
| 0.17 | 1.058721 | -3.27682 | 1.02644      | -0.023432   | 1.29788       |
| 0.21 | 1.048721 | -3.28335 | 1.03354      | -0.038142   | 1.28873       |
| 0.25 | 1.038721 | -3.28982 | 1.04056      | -0.052710   | 1.27955       |
| 0.29 | 1.028721 | -3.29622 | 1.04751      | -0.067131   | 1.27034       |

### Table 12. Analysis by analytical HAM (BVPh2.0 Mathematica package), of the quantities $Y$, $-\Phi'(0)$, $\theta(1)$, $f''(0)$, and $-\theta'(0)$ are listed, when $M = 1$, $R_1 = 1$, $Q = 0.1$, $Pr = 0.800$, $Ma = 1$, $Gr_c = 0.01$, $P_{ref} = 1$, $Sc = 1$, $S = 0.4$, $N = 0.8$, $\phi = 0.04$, $A = 1.1$, $Gr_c = 0.01$, and varying $Q_i$ with Copper nano-fluid.

| $Q_i$ | $Y$   | $f''(0)$ | $\theta(1)$ | $\Phi'(0)$ | $-\theta'(0)$ |
|-------|-------|----------|--------------|-------------|---------------|
| 0.01  | 1.098721 | -3.15089 | 0.914014     | 0.233960    | 1.33632       |
| 0.05  | 1.088721 | -3.15470 | 0.918850     | 0.222673    | 1.32736       |
| 0.09  | 1.078721 | -3.15848 | 0.923641     | 0.211491    | 1.31836       |
| 0.13  | 1.068721 | -3.16220 | 0.928387     | 0.200415    | 1.30932       |
| 0.17  | 1.058721 | -3.16589 | 0.933086     | 0.189445    | 1.30025       |
| 0.21  | 1.048721 | -3.16952 | 0.937739     | 0.178354    | 1.29114       |
| 0.25  | 1.038721 | -3.17311 | 0.942346     | 0.167831    | 1.28200       |
| 0.29  | 1.028721 | -3.17666 | 0.946905     | 0.157189    | 1.27282       |

### Table 13. Analysis by analytical HAM (BVPh2.0 Mathematica package), of the quantities $Y$, $-\Phi'(0)$, $\theta(1)$, $f''(0)$, and $-\theta'(0)$ are listed, when $M = 1$, $R_1 = 1$, $Q = 0.1$, $Q_i = 0.10$, $Ma = 1$, $Gr_c = 0.01$, $P_{ref} = 1$, $Sc = 1$, $S = 0.4$, $N = 0.8$, $\phi = 0.04$, $A = 1.1$, $Gr_c = 0.01$, and varying $Pr$ with Copper nano-fluid.

| $Pr$  | $Y$   | $f''(0)$ | $\theta(1)$ | $\Phi'(0)$ | $-\theta'(0)$ |
|-------|-------|----------|--------------|-------------|---------------|
| 0.5   | 1.098721 | -3.26501 | 1.00956      | 0.002931    | 1.33379       |
| 1.0   | 1.088721 | -3.27219 | 1.01875      | 0.005566    | 1.32473       |
| 1.5   | 1.078721 | -3.27915 | 1.02760      | 0.007890    | 1.31565       |
| 2.0   | 1.068721 | -3.28590 | 1.03615      | 0.009893    | 1.30653       |
| 2.5   | 1.058721 | -3.29247 | 1.04443      | 0.011567    | 1.29739       |
| 3.0   | 1.048721 | -3.29889 | 1.05247      | 0.012899    | 1.28822       |
| 3.5   | 1.038721 | -3.30515 | 1.06028      | 0.013910    | 1.27902       |
| 4.0   | 1.028721 | -3.31130 | 1.06789      | 0.014572    | 1.26979       |
The number of particles increased in the base fluid, the buoyancy force is increased, which implies that the resistance force become stronger. Comparatively, Al₂O₃-water is warm than TiO₂-water and TiO₂-water is warm than Cu-water and similarly Cu-water is warm than Ag-water as shown in Figure 3(c). As we know that, the thermal-conductivity of aluminum is more than, titanium, and titanium is more than, copper and copper is more than silver, see in Table 1, where temperature is decreased as A is increased. Clearly, the comparisons of solute-concentration of Al₂O₃, Ag, TiO₂, and Cu as shown in Figure 3(d). Increased of N, all nano-particles are less azimuthal-velocities shown in Figure 3(a). Initially all particles flow are increased and than decelerated shown in Figure 3(b). It is happened due to the number of particles increased in the base fluid, the buoyancy force is increased. Excellent agreement are observed between two methods (HAM and BVP4C) for different physical number shown in Tables 6 to 8.

As this observation is also about mass transfer and the parameters involved in this model effected the concentration of nano-fluid. Heat, mass and momentum equations are discussed by HAM. Figures 3 to 10, are drawn for arbitrary parameters with nano-particles to observe the influence of azimuthal-velocity, velocity, solute-concentration, temperature, and their rate of changes. Increasing the quantity of A (viscosity) for the particles Cu, Ag, TiO₂, and Al₂O₃, in base fluid the azimuthal-velocity of all the particles are decreased. The azimuthal-velocity of Ag-water is increased than Cu-water and the azimuthal-velocity of Cu-water is increased than Al₂O₃-water and Al₂O₃-water is increased than TiO₂-water nano-fluid given in Figure 3(a). Initially all particles flow are increased and than decelerate shown in Figure 3(b).

Table 15. Analysis by analytical HAM (BVPh2.0 Mathematica package), of the quantities $\Phi$, $\Phi'$, $\theta(1)$, $\Phi''(0)$, and $\Phi''(0)$ are listed, when $M = 1$, $Q_i = 0.1$, $Q = 0.1$, $Pr = 0.800$, $Ma = 1$, $Gr = 0.01$, $Pr_{ref} = 1$, $R_i = 1$, $S = 0.4$, $N = 0.8$, $\phi = 0.04$, $A = 1.1$, $Gr_c = 0.01$, and varying $R_i$ with Copper nano-fluid.

| Ma | $\Phi''(0)$ | $\Phi''(0)$ | $\Phi''(0)$ | $\Phi''(0)$ | $\Phi''(0)$ |
|----|-------------|-------------|-------------|-------------|-------------|
| 0.3 | 1.098721 | -3.29503 | 1.03672 | -0.035139 | 0.936098 |
| 0.6 | 1.088721 | -3.28071 | 1.02561 | -0.016776 | 1.11284 |
| 0.9 | 1.078721 | -3.26870 | 1.01654 | -0.001812 | 1.26619 |
| 1.2 | 1.068721 | -3.25844 | 1.00903 | 0.010563 | 1.40138 |
| 1.5 | 1.058721 | -3.24955 | 1.00274 | 0.020922 | 1.52214 |
| 1.8 | 1.048721 | -3.24176 | 0.99741 | 0.029665 | 1.63117 |
| 2.1 | 1.038721 | -3.23482 | 0.99286 | 0.037136 | 1.73048 |
| 2.4 | 1.028721 | -3.22864 | 0.98899 | 0.043489 | 1.82162 |
Figure 3. Using HAM package where $N = 1$, $\phi = 0.1$, $Gr_c = 0.03$, $Y = 8.05671$, $P_{ref} = 1$, $Gr_c = 0.02$, $S = 0.2$, $Ma = 1$, $Pr = 1$, $Q = 1$, $Q_1 = 1$, $Sc = 1$, $R_t = 1$, $M = 1$ and varying $A$ for the effects of $f$, $f'$, $\theta$ and $\Phi$.

Figure 4. Using HAM package where $A = 1$, $Gr_c = 0.02$, $\phi = 0.1$, $R_t = 1$, $Y = 8.05671$, $P_{ref} = 1$, $Gr_c = 0.03$, $S = 0.2$, $Ma = 1$, $Pr = 1$, $Q = 1$, $Q_1 = 1$, $Sc = 1$, $M = 1$ and varying $N$ for the effects of $f$, $f'$, $\theta$ and $\Phi$. 
in Figure 4(d). In Figure 5(a) and (b) thermal-Grashof number is direct functions of azimuthal-velocity and velocity. Also increase of $Grt$, the temperature of the nano-fluid raised as in Figure 5(c). It mens that, the more quantity of $Grt$ will lead to the less buoyancy force and in a response the fluid flow become more faster. Mass-Grashof (concentration-Grashof) number is observed in Figure 6. It is seen in Figure 6(a) and (b), that $Grc$ is increased the reciprocal impacts are seen on azimuthal-velocities and velocity profiles for different nano-fluid flows and in a results fluids temperature decreased as in Figure 6(c). This happened, due to the stronger buoyancy-forces and flows decelerate. When the unsteady number $S$ is increased, velocity and azimuthal-velocity are also increased, but solute concentration and temperature are decreased see in Figure 7. Magnetic-force effects are seen in Figure 8 for different nano-particles. As $Ma$ is increased, the azimuthal-velocities and the velocity profiles are decreased as in Figure 8(a) and (b), which implies that the mass-flux decreased and temperature become low as in Figure 8(c) and (d). Further, as magnetic-force provides Lorentz-force and increase of Lorentz-force means the more resistance to the flow and in a result velocities reduced. Prandtl-number effects are seen in Figure 9. By the increased of $Pr$ the reciprocal effects on the azimuthal-velocity and velocity profiles are seen, meanwhile the temperature of the flow increased as in Figure 9(a) to (c). Meanwhile, Solute concentration is not affected by $Pr$, see in Figure 9(d). Increasing heat source number $Q$, azimuthal-velocity and velocity decreased, and the fluid flows become hot see in Figure 10(a) to (c). Solute concentration is not affected by $Q$, see in Figure 10(d).

**Concluding remarks**

In this problem, new physical properties with new nano-particles along with base fluid are considered for the improvement of sufficient heat and mass transfer. The considered particles are Aluminum, Titanium, Silver, and copper with the most important physical properties, radiation-absorption, mixed-convection, Schmidt number, variables viscosity and thermal-
Figure 6. Using HAM package where $A = 0.5, P_{ef} = 1, N = 1, Q = 1, \phi = 0.1, Y = 8.05671, Gr_c = 0.03, S = 0.2, Ma = 1, Pr = 1, Q_1 = 1, Sc = 1, R_1 = 1, M = 1$ and varying $Gr_c$ for the effects of $f, f', \theta$ and $\Phi$.

Figure 7. Using HAM package where $Gr_c = 0.03, A = 0.5, P_{ef} = 1, N = 1, Q = 1, \phi = 0.1, Y = 8.05671, Gr_c = 0.03, S = 0.2, Ma = 1, Pr = 1, Q_1 = 1, Sc = 1, R_1 = 1, M = 1$ and varying $S$ for the effects of $f, f', \theta$ and $\Phi$. 
Figure 8. Using HAM package where $Gr_c = 0.03, A = 0.5, P_{ref} = 1, N = 1, Q = 1, \phi = 0.1, Y = 8.05671, Gr_t = 0.03, S = 0.2, Pr = 1, Q_1 = 1, Sc = 1, R_1 = 1, M = 1$ and varying $M_a$ for the effects of $f, f', \theta$ and $\Phi$.

Figure 9. Using HAM package where $Gr_c = 0.03, A = 0.5, P_{ref} = 1, N = 1, Q = 1, \phi = 0.1, Y = 8.05671, Gr_t = 0.03, Ma = 1, S = 0.2, Q_1 = 1, Sc = 1, R_1 = 1, M = 1$ and varying $Pr$ for the effects of $f, f', \theta$ and $\Phi$. 
conductivity, chemical reaction, and MHD thin film flow. Here surface tension is directly dependent on temperature. We derive relations between different nanoparticles with several important physical properties, for the system of partial differential equations describing MHD flow. Heat transfer, mass transfer, and momentum profile in a thin liquid film on an unsteady stretching nonuniform-surface in the presence of external magnetic field are considered. A two-dimensional, highly nonlinear model is found and we use a couple of the admitted physical properties to construct similarity transformations for the said model. These transformations enable reductions of the model to a coupled system of nonlinear ordinary differential equations. We present analytic solution by HAM package and numerical by shooting package, for the obtained systems of ordinary differential equations are shown in figures and tables. Excellent agreement are archived between both the method. The following investigation are observed as:

1. By the increasing of $A$, azimuthal-velocity of all the particles are decreased and $Ag$-water is more then $Cu$-water. $Cu$-water is more then $Al_2O_3$-water and $Al_2O_3$-water is more than $TiO_2$-water.

2. Velocity-profiles of all particles are increased for increasing $A$, up to some extent and than decelerate. The temperature of $Al_2O_3$-water is higher than $TiO_2$-water, $TiO_2$-water is higher than $Cu$-water, where $Cu$-water is higher than $Ag$-water. The fact is, aluminum thermal-conductivity is higher than titanium, titanium is higher than copper, where copper is higher than silver.

3. Increasing of $N$, the azimuthal-velocity of all nano-fluids are reduced, but velocities of all nano-fluids are accelerated, up to $\eta = 0.65$ and then decelerated.

4. Change of $N$, the temperature of $Al_2O_3$-water is higher than $TiO_2$-water, and $Cu$-water. As aluminum thermal conductivity is higher than titanium, titanium is higher than copper and silver. With the increase of $N$ the temperature increased.

5. The number of particles increased in the base fluid, the buoyancy force is increased, which implies that the resistance force become stronger.

6. The more quantity of $Gr_t$ will lead to the less buoyancy force and in a response the fluid flow become more faster.

7. Increase of nano-particles, mass flux increased.

Figure 10. Using HAM package where $A = 0.1, N = 1, \phi = 0.1, Gr_t = 0.02, Y = 8.05671, Pr_{ref} = 1, Gr_c = 0.03, S = 0.2, Ma = 1, pr = 1, Q_1 = 1, Sc = 1, R_1 = 1, M = 1$ and varying $Q$ for the effects of $f, f^0, \theta$ and $\Phi$. 

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8. As $Ma$ is increased, the azimuthal-velocities and the velocity profiles are decreased, which implies that the mass-flux decreased and temperature become low. Further, as the magnetic-force provides Lorentz-force and increase of Lorentz-force means the more resistance to the flow and in a result velocities reduced.

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**Author contributions**

All authors participated in the analysis of the results and manuscript coordination. All authors read and approved the final manuscript.

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**Appendix**

**Notations**

| Symbol | Description |
|--------|-------------|
| \( A \) | viscosity parameter |
| \( Gr_t \) | thermal Grashof number |
| \( Q \) | heat source parameter |
| \( S_c \) | Schmidt number |
| \( B \) | magnetic field \( \left( k g^{1/2} m^{-1/2} s^{-1} \right) \) |
| \( Re \) | Reynolds number |
| \( u, v \) | \( x \) and \( y \) components of velocity \( (m.s^{-1}) \) |
| \( Pr \) | Prandtl number \( (\nu/\kappa) \) |
| \( b \) | positive constant \( (s^{-1}) \) |
| \( x, y \) | spatial Cartesian coordinates \( (m) \) |
| \( S \) | unsteadiness parameter \( (a/b) \) |
| \( Ma \) | Hartmann number \( (U_0 B_0^2/\mu \rho b) \) |
| \( U_s \) | stretching surface velocity \( (m.s^{-1}) \) |
| \( M \) | thermocapillary number |
| \( g \) | gravitational acceleration \( (m.s^{-2}) \) |
| \( T_0 \) | temperature at the stretching sheet \( (K) \) |
| \( L \) | characteristic height scale \( (m) \) |
| \( T_{ref} \) | reference temperature \( (K) \) |
| \( U \) | surface velocity \( (m.s^{-1}) \) |
| \( T_s \) | temperature at the surface of fluid \( (K) \) |
| \( T \) | temperature \( (K) \) |
| \( \beta \) | thermal expansion coefficient \( (K^{-1}) \) |
| \( h(t) \) | liquid film thickness \( (m) \) |

**Subscripts**

| Symbol | Description |
|--------|-------------|
| \( s \) | at the surface of fluid |
| \( ref \) | reference value |
| \( 0 \) | at the stretching sheet |
| \( nf \) | nanofluid |
| \( f \) | base fluid |
| \( N \) | thermal conductivity parameter |
| \( Gr_c \) | concentration Grashof number |
| \( Q_l \) | radiation absorption parameter |
| \( R_l \) | chemical reaction parameter |
| \( D \) | Brownian diffusion coefficient |

**Greek symbols**

| Symbol | Description |
|--------|-------------|
| \( \phi \) | nano-size particles |
| \( \eta \) | similarity variable \( \left( \frac{2\pi \beta \Omega^{-1}}{(1-\alpha)(\eta^2)} \right) \) |
| \( \alpha \) | positive constant \( (s^{-1}) \) |
| \( \delta \) | positive constant \( (K^{-1}) \) |
| \( \rho \) | density \( (kg.m^{-3}) \) |
| \( \beta \) | dimensionless film thickness |
| \( \sigma \) | surface tension \( (kg.s^{-2}) \) |
| \( \dot{\sigma} \) | electrical conductivity \( (m^{-2}.s) \) |
| \( \kappa \) | thermal conductivity \( (W.m^{-1}.K^{-1}) \) |
| \( \mu \) | viscosity \( (kg.m^{-1}.s^{-1}) \) |
| \( \nu \) | dimensionless film thickness |

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*Note: The above notations and Greek symbols are based on the provided text and are intended to represent the general symbols used in fluid dynamics and heat transfer.*
| Symbol | Explanation |
|--------|-------------|
| $\alpha_{nf}$ | thermal diffusivity of nanofluid |
| $\mu_{nf}$ | viscosity of nanofluid |
| $\theta$ | dimensionless temperature |
| $\delta$ | characteristic length scale ($m$) |
| $\sigma_0$ | surface tension at sheet ($kg.s^{-2}$) |
| $\phi$ | nano-size particles |

* superscript, dimensionless variable