The Dirichlet Obstruction in AdS/CFT

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Abstract The obstruction for a perturbative reconstruction of the five-dimensional bulk metric starting from the four-dimensional metric at the boundary, that is, the Dirichlet problem, is computed in dimensions $6 \leq d \leq 10$ and some comments are made on its general structure and, in particular, on its relationship with the conformal anomaly, which we compute in dimension $d = 8$. 
1 Introduction

One of the more interesting problems in the celebrated AdS/CFT correspondence conjectured by Maldacena ([10]) in the language introduced by Witten ([15]; cf. also the reviews [1], [2], [12]) is the decoding of the hologram. In the minimal gravitational setting this amounts to recover the metric in the bulk space from the metric at the boundary (which is what mathematicians call a Dirichlet problem).

To be specific, in the framework of the geometric approach to holography in its Poincaré form (that is, when the d-dimensional manifold $M_d$ is represented as Penrose’s conformal infinity of another manifold $B_{d+1}$ with one extra holographic dimension), there is a privileged system of coordinates such that the boundary $\partial B_{d+1} \sim M_d$ is located at the value $\rho = 0$ of the holographic coordinate, namely

$$ds^2 = g^{(B)}_{\mu\nu}dx^\mu dx^\nu = \frac{-l^2d\rho^2}{4\rho^2} + \frac{1}{\rho}h_{ij}(x, \rho)dx^idx^j$$  \hspace{1cm} (1)

(The normalization corresponds to a cosmological constant $\lambda \equiv \frac{d-2}{2d}R \equiv \frac{d(d-1)}{2l^2}$ when $h_{ij} = \eta_{ij}$). It has been already mentioned that the boundary condition is of the Dirichlet type, i.e.

$$h_{ij}(x, \rho = 0) = g_{ij}(x)$$  \hspace{1cm} (2)

where $g_{ij}$ is an appropriate metric on $M_d$.

In the basic mathematical work by Fefferman and Graham (FG) ([6]), it is proved that there is a formal power series solution to Einstein’s equations with a cosmological constant as above. This power series gives, in principle, a complete solution to the Dirichlet problem, at least in the vicinity of the boundary. When $d \in 2\mathbb{Z}$ there is, however, an obstruction, which in four dimensions is the Bach tensor and in higher dimensions is a new tensor, whose specific form is unknown, which will be called in the sequel the FG tensor, $Z^{d}_{ij}$, with conformal weight $(d-2)/2$. When this tensor does not vanish, there are logarithmic terms
which appear in the expansion starting at the order \( \rho^{d/2} \)

\[
h_{ij} = g_{ij} + h_{ij}^{(1)} \rho + \ldots + h_{ij}^{(d/2)} \rho^{d/2} + \tilde{h}_{ij}^{(d/2)} \rho^{d/2} \log \rho + \ldots
\]

(3)

Even the term \( h_{ij}^{(d/2)} \) is not completely determined; as shown in [7], Einstein’s equations only give its trace as well as its covariant derivative. On the other hand, the coefficient of the logarithmic term is given in terms of the local anomaly \( a_d \) by (cf. [7]).

\[
\tilde{h}_{ij}^{d/2} = -\frac{4}{d\sqrt{g} \delta g_{ij}} \int dx_d a_d
\]

(4)
i.e., it is the energy momentum tensor of the integrated anomaly.

The object of the present paper is to analyze this problem in detail, using in particular some codimension two techniques, also due to FG originally. We shall find general expressions for the obstructions, recursive relationships between them, and explicit formulas for the eight dimensional conformal anomaly.

## 2 Einstein’s equations for \( B_{d+1} \) and \( \lambda \neq 0 \)

Einstein’s equations \(^2\) corresponding to a cosmological constant \( \lambda = \frac{d(d-1)}{2l^2} \): read

\[
R_{\mu \nu} = \frac{d}{l^2} g^{(B) \mu \nu}
\]

(5)

Which reduce in the FG coordinates to:

\[
\rho \left( 2h'' - 2h'h^{-1}h' + tr[h^{-1}h']h' \right)_{ij} + l^2 R[h]_{ij} - (d-2)h'_{ij} - tr[h^{-1}h']h_{ij} = 0
\]

(6)

\[
(h^{-1})^{kl}(\nabla_{i}h)[h]_{kl} - \nabla_{i}[h]h'_{ik} = 0
\]

(7)

\[
tr[h^{-1}h''] - \frac{1}{2}tr[h^{-1}h'h^{-1}h'] = 0
\]

(8)

We shall consistently denote by \( g_{ij} \equiv h_{ij}(\rho = 0) \) the (Penrose) boundary metric and from now on latin indices will be raised and lowered using this metric\(^3\). Equations (6) and (8)

\[^2\text{We use the conventions } R_{\nu \alpha \beta} = 2\partial_{[\alpha} \Gamma_{\beta]}^{\mu} \nu + 2\Gamma_{[\alpha \Gamma_{\beta]}^{\mu} \nu} \text{ and } R_{\mu \nu} = R_{\mu \nu}^{\alpha}.\]

\[^3\text{We shall denote by } h^{(n)} = g^{ij}h^{(n)}_{ij}.\]
(the ones needed for our purposes) can now be expanded in a formal power series of which we give the first two orders.

Order $\rho^0$.

- $(d - 2)h^{(1)}_{ij} + h^{(1)}g_{ij} - l^2 R[g]_{ij} = 0 \quad (9)$
- $2h^{(2)} - \frac{1}{2} h^{(1)kl} h^{(1)kl} = 0 \quad (10)$

Order $\rho^1$.

- $2(d - 4)h^{(2)}_{ij} + \left(2h^{(2)} - h^{(1)kl} h^{(1)kl}\right) g_{ij} + 2h^{(1)}_{im} h^{(1)jm} - \frac{l^2}{2} \left[ \nabla_k \nabla_j h^{(1)}_{ij} + \nabla_k \nabla_j h^{(1)}_{ik} - \nabla^2 h^{(1)}_{ij} - \nabla_i \nabla_j h^{(1)} \right] = 0 \quad (11)$
- $6h^{(3)} + h^{(1)k} h^{(1)l} h^{(1)n} h^{(1)m} - 4h^{(1)kl} h^{(2)kl} = 0 \quad (12)$

Explicit expressions up to order $\rho^4$ can be found in the Appendix B.

### 3 The Dirichlet obstruction in d=4 and d=6

Equations (9) and (10) give $h^{(1)}_{ij}$ and $h^{(2)}$

- $h^{(1)}_{ij} = \frac{l^2}{d - 2} \left( R_{ij} - \frac{R}{2(d - 1)} g_{ij} \right) \equiv l^2 A_{ij} \quad (13)$
- $h^{(1)} = l^2 \frac{R}{2(d - 1)} = l^2 A \quad (14)$
- $h^{(2)} = \frac{1}{4} h^{(1)kl} h^{(1)kl} = \frac{l^4}{4(d - 2)^2} \left( R^{kl} R_{kl} + \frac{4 - 3d}{4(d - 1)^2} R^2 \right) = \frac{l^4}{4} A^{kl} A_{kl} \quad (15)$

If we now remember the definition of the Bach tensor (using the conventions in $[2]$)

- $B_{ij} \equiv 2\nabla^k \nabla_{[k} A_{i]j} + A^{kl} W_{ijkl}$

(where $W_{ijkl}$ is the Weyl tensor), and use it in equation (11), we get

- $(d - 4)(d - 4)h^{(2)}_{ij} - h^{(1)kl} h^{(1)kl} + l^4 B_{ij} = 0 \quad (17)$
conveying the fact that, in \( d = 4 \), if \( B_{ij} \neq 0 \) the expansion is inconsistent. Note that, had we introduced the logarithmic term, a new coefficient in the expansion of the metric comes into play and we get a chance to make Einstein’s equations vanish up to second order, as FG claim.

On the other hand, if the Bach tensor vanishes, the only further information of \( h^{(2)} \) given up to this order is the value of its covariant derivative, which can be obtained from equation (7).

It is interesting to point out that there is a very simple relationship between the Bach tensor and the four-dimensional anomaly.

The vanishing of the Bach tensor \( \Pi \) is the equation of motion corresponding to the lagrangian:

\[
L \equiv I_4 \equiv \frac{1}{64} \sqrt{g} W_{\alpha \beta \gamma \delta} W^{\alpha \beta \gamma \delta} = \frac{1}{64} \sqrt{g} (R^{\alpha \beta \gamma \delta} R_{\alpha \beta \gamma \delta} - 2R^{\alpha \beta} R_{\alpha \beta} + \frac{1}{3} R^2) \tag{18}
\]

But precisely the anomaly (modulo local counterterms) for conformal invariant matter (5) is given by a combination of the Euler characteristic density, \( E_4 \) plus this conformal invariant lagrangian, \( I_4 \). In four dimensions the Euler characteristic density, is given explicitly by

\[
E_4 = \frac{1}{64} \epsilon^{abcd} R_{ab\mu\nu} R_{cd\rho\sigma} \epsilon^{\mu\nu\rho\sigma} \equiv \frac{1}{64} (R^{\alpha \beta \gamma \delta} R_{\alpha \beta \gamma \delta} - 4R^{\alpha \beta} R_{\alpha \beta} + R^2) \tag{19}
\]

and is a topological invariant, that is, independent of the metric. The vanishing of the Bach tensor is then the condition for the integrated anomaly to be topological, that is, independent of the metric, which according to the previous equation (\( \Pi \)) is the condition for

\[
\tilde{h}^{d/2}_{ij} = 0 \tag{20}
\]

in which case there is no obstruction for a perturbative solution to the Dirichlet problem.

The fact that

\[
Z_{ij}^{(d)} \sim \tilde{h}^{d/2}_{ij} \tag{21}
\]
gives (using (4)) the FG tensor as the variational derivative of the conformal anomaly with respect to the background metric. Although both quantities are unknown in the general case, this is a useful constraint.

As an example, we shall begin with the six dimensional case. As in $d = 4$, the obstruction comes out taking the limit $d \rightarrow 6$ in the left hand side of the first equation in (47), that is:

$$H_{ij} = \frac{1}{4}(\frac{1}{2}B_{kl}A^{kl} + A^k_i A_m A^m_k)g_{ij} - B^m_{(i} A^m_{j)} + \frac{1}{8} AB_{ij} - \frac{1}{4} AA_{im} A^m_j$$

$$+ \frac{1}{8} \nabla_k \left(-\frac{1}{2} DB_{ij}^k + 2 A_{l(i} C_{j)}^{kl} - 3 A^k_l D A_{ij}^l + A^{kl} \nabla_l A_{ij} \right)$$

$$+ \frac{1}{4} A^{kl} \nabla_i \nabla_j A_{kl} - \frac{1}{4} \nabla_l D A_{ij}^l + \frac{1}{2} \nabla_i A_{ik} C^{ij}$$  \hspace{0.5cm} (22)

where $C_{ijk} = 2 \nabla_{[i} A_{jk]}$ is the Cotton tensor. Then, according to our previous reasoning, this obstruction should coincide with the metric variation of the anomaly.

To apply a first check to this idea, let us recall the expression for the six-dimensional holographic anomaly that has been proposed in (8):

$$A \sim \frac{1}{2} RR_{ij} R_{ij} - \frac{3}{50} R^3 - R^{ij} R^{kl} R_{ijkl} + \frac{1}{5} R^{ij} \nabla_i \nabla_j R - \frac{1}{2} R^{ij} \Box R_{ij} + \frac{1}{20} R \Box R$$  \hspace{0.5cm} (23)

which can also be written as:

$$A \sim A^{ij} (B_{ij} + 2 A_{ik} A^k_j - 3 A A_{ij} + A^2 g_{ij})$$  \hspace{0.5cm} (24)

We are interested in the metric variation, that is,

$$\tilde{H}_{ij} \equiv \frac{\delta}{\delta g^{ij}} \int \sqrt{g} A$$  \hspace{0.5cm} (25)

yielding:

$$\tilde{H}_{ij} = -\frac{1}{2} B_{kl} A_{kl} g_{ij} + \frac{1}{2} AB_{ij} + 3 A_{k(i} B_{j)}^k - A^k_i A^{kl} W_{j}^{kl}_{ijkl} + 2 (A^{kl} A - A^{ks} A^i_s) W_{kl}$$

$$+ A^{kl} A_{kl} A_{ij} + 5 A A_{ik} A^k_j - 6 A_{kl} A^k_i A^k_j - (A_{kl} A^{kl} A - A^k_i A_s A_{kl}) g_{ij} + \frac{1}{4} \nabla_k D B_{ij}^k$$

$$+ \nabla^2 (A A_{ij} - A_i^k A_j^k) - \nabla_k \nabla_l (A^{kl} A_{ij} - A_i^k A_j^l) - \frac{1}{2} \nabla_i \nabla_j (A^{kl} A_{kl} + A^2)$$

$$- \nabla^k (A^l_i C_{jkl} + A^l_i C_{jkl}) + \nabla_i (A^{kl} C_{l(ij)}) + \nabla (i A^{kl} C_{l}) + 2 \nabla (i A^{kl} \nabla_k A_{lj})$$  \hspace{0.5cm} (26)
As a matter of fact, the obstruction given by \((22)\), can be manipulated in order to prove, after some calculation, that both tensors \(-4H\) and \(\tilde{H}\) are identical.

4 A general codimension-two approach

In order to work out higher dimensions, it proves useful to put at work the codimension two approach of \((6)\), reviewed in \([2]\). In this approach, the conformal invariants of the manifold \(M_n\) are obtained from diffeomorphism invariants of the so-called ambient space, \(A_{n+2}\) with two extra dimensions, one spacelike and the other timelike (cf. the analysis of diffeomorphisms that reduce to Weyl transformations on the boundary (PBH) in \([3]\)).

If the ambient space is to have lorentzian signature, then the boundary is necessarily euclidean. Then, of the two extra (holographic) coordinates, one is timelike, and the other, spacelike. They will be denoted by \((t, \rho)\), respectively.

In the references just cited it is proven that one can choose coordinates in such a way that the ambient metric reads

\[
ds^2 = g^{(A)}_{\mu\nu} dx^\mu dx^\nu = \frac{t^2}{l^2} h_{ij}(x, \rho) dx^i dx^j + \rho dt^2 + t dt \rho
\]

A power expansion in the variable \(\rho\) can now be performed:

\[
h_{ij}(x, \rho) = g_{ij}(x) + \sum_{a=1}^{\infty} \rho^a h^{(a)}_{ij}(x)
\]

\((g_{ij}\) is the boundary metric tensor, and it is assumed that the dimension of this space is even).

The ambient space has to be Ricci-flat (this is a necessary ingredient of the FG construction, which also guarantees that it is an admissible string background). This means that

\[
^{(A)} R_{ij} = \rho \left(2h'' - 2h'h^{-1}h' + tr[h^{-1}h']h'\right)_{ij} + l^2 R[h]_{ij} - (d - 2)h'_{ij} - tr[h^{-1}h']h_{ij} = 0
\]
which is identical to the equation (6), as has been proved in detail in [3]. It is remarkable that the Ricci flatness condition in $A_{n+2}$ is indeed equivalent to Einstein’s equations with an appropriate cosmological constant in the associated bulk space, $B_{n+1}$.

Those equations are written for arbitrary dimension, but it is clear just by looking at the equivalent form (9), etc., that for any even dimension ($d = 2n$) the equations are inconsistent unless the tensor that is not multiplied by $(d-2n)$ vanishes, so that it is natural to call that tensor the obstruction. Let us insist that even when the obstruction vanishes there is an indetermination in the expansion from $h^{(n)}$ on.

We turn our attention now to tensors defined in the ambient space. For instance, the non vanishing components of the Riemann tensor are

$$R_{ijkl} = \frac{t^2}{\rho^2} \left( R_{ijkl}(h) + \frac{1}{\rho^2} \left[ -h_{ik} \partial_\rho h_{jn} + h_{in} \partial_\rho h_{jk} - h_{jn} \partial_\rho h_{ki} + h_{jk} \partial_\rho h_{in} \right] \right)$$

$$\left. + \rho \frac{1}{\rho^2} \left[ \partial_\rho h_{ik} \partial_\rho h_{jn} - \partial_\rho h_{in} \partial_\rho h_{jk} \right] \right)$$

$$(30)$$

If we take the $\rho \to 0$ limit we obtain

$$R_{\rho ij} = \frac{t^2}{\rho^2} \left( \nabla_k \partial_\rho h_{ij} - \nabla_j \partial_\rho h_{ik} \right)$$

$$R_{\rho k i j} = \frac{t^2}{\rho^2} \left( \partial_\rho \partial_\rho h_{ij} - \frac{1}{2} h^{kl} \partial_\rho h_{kj} \partial_\rho h_{li} \right)$$

$$\text{If } d \neq 4, \text{ we can define } Z^{(4)}_{ij} \text{ as}$$

$$R_{\rho ij} = -\frac{t^2}{2} \frac{Z^{(4)}_{ij}}{2 - d}$$

$$(31)$$

where $W$ and $C$ are the Weyl and Cotton tensors.

If $d \neq 4$, we can define $Z^{(4)}_{ij}$ as

$$R_{\rho ij} = -\frac{t^2}{2} \frac{Z^{(4)}_{ij}}{2 - d}$$

$$\text{and using the equation (11) for } h^{(2)}_{ij} \text{ we establish that}$$

$$Z^{(4)}_{ij} = \frac{1}{2!^d} \left( h^{(1)}_{kl} h^{(1)}_{ij} g_{ij} - 4 h^{(1)}_{ik} h^{(1)}_{jk} + 2 \nabla_k \nabla_{(i} h^{(1)}_{j)} - \nabla^2 h^{(1)}_{ij} \right)$$

$$- \nabla_{(i} \nabla_{j)} h^{(1)} - (d - 4) h^{(1)}_{ik} h^{(1)}_{(i} \right) = -\frac{1}{2} B_{ij}$$

$$(32)$$
where $B_{ij}$ is the Bach tensor (in any dimension). Then $Z^{(4)}_{ij}$ is precisely what we have called obstruction in $d = 4$.

The four-dimensional obstruction has then been obtained afresh in a natural geometric way from the ambient Riemann tensor. A natural question is whether this could be generalized to higher dimensions.

Let us consider a component of the ambient tensor

$$\nabla_{\alpha_1} \cdots \nabla_{\alpha_n} R_{\lambda\mu\sigma}$$

with $n$ indices fixed: $\alpha_1 = \ldots = \alpha_n = \rho$, and take the $\rho \to 0$ limit. This leads to:

$$\begin{align*}
(\nabla_{\rho})^n R_{\rho\rho} & = -\frac{l^2}{2} l^{2n+2} Z^{(4+2n)}_{ij} \\
(\nabla_{\rho})^n R_{ijkl} & = -l^2 l^{2n-2} W^{(4+2n)}_{ijkl} \\
(\nabla_{\rho})^n R_{ijk\rho} & = \frac{l^2}{2} l^{2n} C^{(4+2n)}_{ijk}
\end{align*}$$

(35)

where $W^{(4+2n)}$ and $C^{(4+2n)}$ are generalizations of the Weyl ($W = W^{(4)}$) and Cotton ($C = C^{(4)}$) tensors.

Performing the same calculation as before for $n = 1, \ldots$, we obtain the obstructions for higher dimensions (the results are presented in the appendix C). These tensors yield the Dirichlet obstruction in the corresponding dimension.

From the general expression (34) we obtain the relationship

$$\begin{align*}
(\nabla_{\rho})^{n+1} R_{\rho\rho} |_{\rho=0} &= \partial_{\rho}(\nabla_{\rho})^n R_{\rho\rho} |_{\rho=0} - 2 \Gamma^k_{\rho i}(\nabla_{\rho})^n R_{\rho j k} |_{\rho=0} \\
&= \partial_{\rho} \nabla_{\rho} \cdots \nabla_{\rho} R_{\lambda\mu\sigma} |_{\rho=0}
\end{align*}$$

(36)

(36) which relates $Z^{(p)}$ to $Z^{(p-2)}$, suggesting a recurrent computation. Unfortunately this is not straightforward because we need to know the general tensor $\nabla_{\rho} \cdots \nabla_{\rho} R_{\lambda\mu\sigma}$ not only in the $\rho \to 0$ limit, but also for general $\rho$. Nevertheless, using the fact that $Z^{(4)}$ is traceless, it can be shown inductively from (36) that $Z^{(4+2n)}$ is traceless as well. Moreover, as

$$\nabla_{\rho}^{n+1} R_{\rho\rho} = \partial_{\rho}^{n+1} R_{\rho\rho} = 0$$

(37)
we see that the property of $Z^{(4+2n)}$ being tracefree is equivalent (in the $\rho \to 0$ limit) to the trace equations for $h^{(n+2)}$ given by the equation (8).

4.1 The Bianchi identities

Let us now study the Bianchi identity in the ambient space $A_{n+2}$

$$\nabla_{[\mu} R_{\nu\lambda]\sigma\tau} = 0 \quad (38)$$

If we contract, for instance, $\mu$ and $\tau$, and remember that the ambient space is Ricci-flat, it reduces to

$$\nabla_\mu R^\mu_{\nu\lambda\sigma} = \nabla_\lambda R_{\sigma\nu} - \nabla_\sigma R_{\lambda\nu} = 0 \quad (39)$$

In the particular case $\nu = i$, $\lambda = \rho$, $\sigma = j$, one gets close to the boundary (using (31)) the identity

$$B_{ij} = \nabla^k C_{kij} + A^{kl} W_{kijl} \quad (40)$$

which is the Bach tensor (i.e. $Z^{(4)}$) in terms of $A$, $C$, and $W$.

We can go further and assume that this is a general feature, that is, the obstruction could be expressed in terms of the generalization of the Weyl and Cotton tensors.

Starting from the identity

$$\nabla_\rho \nabla_\mu R^\mu_{\nu ij} = 0 \quad (41)$$

we can perform a calculation along similar lines. The result however is more complicated than expected:

$$Z^{(6)}_{ij} = \frac{1}{2} \left[ \nabla^k C^{(6)}_{kij} + A^{kl} W^{(6)}_{kij} \right] - \frac{1}{2} A^{kl} \nabla_k C_{lij} - \frac{1}{2} A^{kl} A_i^s W_{ij}^s - \frac{1}{d-4} A_i^s B_{js}$$

$$+ \frac{1}{d-1} A B_{ij} + \frac{1}{2} C_{ikl} C^{lk}_{ij} + \frac{1}{2} C_{ikl} C^{lk}_{ij} - \frac{1}{2} A^{kl} W_{kijl} \right] \quad (42)$$

5 The $d = 8$ Holographic Anomaly

As a by-product of this computation, the holographic anomaly in $d = 8$ can be evaluated. This anomaly, when integrated, would give rise to $Z^{(8)}$ upon metric variation. It should
correspond to the anomaly of a conformal invariant quantum field theory and, as such, expressible in terms of the Euler characteristic $E_8$ and a basis of conformal invariants, $W_8$. We are not aware, however, of any existing explicit construction of such a basis, so that we express our result in terms of Riemann tensors, their derivatives, and contractions thereof.

This yields:

$$a^{(8)} = \frac{1}{32} \left\{ \frac{1}{54} R_{ij} R_{ij} R_{kl} R_{kl} - \frac{11}{756} R^2 R_{ij} R_{ij} + \frac{17}{18522} R^4 + \frac{1}{27} R_{kiij} R_{kl} R^p R^j_p \\
- \frac{1}{18} R R_{kjij} R^k l R^j - \frac{1}{12} R_{kpiq} R^{piq} R^k R_{ij} - \frac{13}{5292} R^2 \nabla^2 R - \frac{43}{5292} R R^{ij} \nabla_i \nabla_j R \\
+ \frac{7}{216} R R^{ij} \nabla^2 R_{ij} + \frac{1}{126} R^{ij} R_{ij} \nabla^2 R - \frac{1}{27} R^{ij} R_{kl} \nabla_i \nabla_j R_{kl} + \frac{1}{27} R^{ij} R_{kl} \nabla_k \nabla_i \nabla_j R_{jl} \\
+ \frac{1}{189} R^{il} R^{lj} \nabla_i \nabla_j R + \frac{1}{36} R^{ij} R_{kl} \nabla^2 R_{kiij} - \frac{1}{42} R^{ij} R_{kiij} \nabla^k \nabla^l R + \frac{1}{12} R^{ij} R_{kiij} \nabla^2 R_{kl} \\
- \frac{1}{5292} R \nabla_k R \nabla^k R + \frac{1}{216} R \nabla_k R_{ij} \nabla^k R^{ij} + \frac{1}{1008} R \nabla^2 \nabla^2 R + \frac{1}{168} R^{ij} \nabla^2 \nabla_i \nabla_j R \\
- \frac{1}{72} R^{ij} \nabla^2 \nabla^2 R_{ij} - \frac{1}{882} R^{ij} \nabla_i R \nabla_j R + \frac{1}{189} R^{ij} \nabla_i R_{jk} \nabla^k R + \frac{1}{54} R^{ij} \nabla_i R_{kl} \nabla_j R^{kl} \\
+ \frac{1}{27} R^{ij} \nabla_k R_{il} \nabla^l R^{kj} - \frac{1}{27} R^{ij} \nabla_i R_{kl} \nabla^k R_{lj} + \frac{1}{18} R^{ij} \nabla_p R_{klj} \nabla^p R_{kl} + \frac{1}{3528} \nabla^2 R \nabla^2 R \\
- \frac{1}{784} \nabla_i \nabla_j R \nabla^i \nabla^j R + \frac{1}{168} \nabla_i \nabla_j R \nabla^2 R_{ij} - \frac{1}{144} \nabla^2 R \nabla^2 R_{ij} \right\} \quad(43)$$

After this work was completed, we found a previous proposal for $a^{(8)}$ in the context of the AdS/CFT correspondence ([11]), although from a slightly different approach. It is worth noting that, by construction, this expression depends only on the coefficients $h^n$ which in turn are given in terms of $h^1$ (due to the recurrence relation stemming from the equations of motion). Then, as has been already remarked in [13], the holographic anomaly vanishes in a Ricci-flat background, and this is a general property of the conformal anomaly at any dimension obtained by the holographic setup.
6 Conclusions

The higher dimensional \(6 \leq d \leq 10\) Dirichlet obstruction \(Z^{(d)}\) have been computed in several ways, which we believe are equivalent.

The use of the so called *ambient space*, with a codimension two associated space-time has proved very fruitful both in computing the obstructions themselves as well as in uncovering unexpected relationships between the obstructions in different dimensions.

We have also analyzed the relationship of the generic obstruction to the generic conformal anomaly (for conformally invariant matter), along the lines of [7], although here clearly we have only scratched the surface of a deep problem, which deserves further study.

The full expression of the eight dimensional holographic anomaly has been derived in terms of geometrical quantities. It would be most interesting to check ([14]) the expressions of the holographic anomaly in dimensions six and eight. This would imply, in particular, to verify that the total coefficient of all independent Riemann contractions vanishes, due to the basic property of the holographic anomaly that it is zero for Einstein spaces.

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A Bach-flat spaces

An interesting question is the characterization of those spaces for which there is no obstruction. In \(d = 4\) dimensions, this is equivalent to find Bach-flat spaces. It is known (cf. [9]) that all conformally Einstein spaces are in this class, but there are other solutions besides those (cf. [13]), and a complete characterization is not known to us.
The non-asymptotically flat solutions found by Schmidt are a generalization of well-known Kasner solutions, with metric

$$ds^2 = dt^2 - \sum_{i=1}^{3} a_i^2 dx_i^2$$  \hfill (44)$$

with Hubble parameters $h_i = a_i^{-1} da_i/dt$, $h = \Sigma h_i$ and anisotropy parameters $m_i = h_i - h/3$

If we denote $r = (h_1^2 + h_1 h_2 + h_2^2)^{1/2}$ and $p = h_1/r$, then the vanishing of Bach’s tensor is equivalent to:

$$3d^2(pr)/dt^2 = 8pr^3 + 4c, \quad c = \text{const.}, \quad p^2 \leq 4/3,$$ \hfill (45)

$$9(dp/dt)^2 r^4 = [2r^2 - (dr/dt)^2 - 4r^4](4r^2 - 3p^2r^2).$$ \hfill (46)

Each solution of this system of equations with $dp/dt \neq 0$ represents a non asymptotically flat metric enjoying vanishing Bach tensor.

**B Higher order equations of motion**

In this appendix can be found the equations of motion (6) and (8), from order $\rho^2$ to order $\rho^4$. $R^{(n)}_{ij}$ corresponds to the power expansion of the Ricci tensor $R_{ij}[h]$, and we employ the notation $D h^{(n)}_{ab} = \nabla_a h^{(n)} b + \nabla_b h^{(n)} a - \nabla^c h^{(n)} ab$.

**Order $\rho^2$**

- $3(d-6)h^{(3)}_{ij} + \left(3h^{(3)} - 3h^{(1)}_{kl}h^{(2)}_{kl} + h^{(1)}_{s}h^{(1)}_{i}h^{(1)}_{k}\right)g_{ij} + 8h^{(2)}m_{(i}h^{(1)}_{j)m} - 2h^{(1)}_{il}h^{(1)}_{jm}h^{(1)}_{mk} - h^{(1)}_{lj}h^{(2)}_{ij} - l^2 R^{(2)}_{ij} = 0$  \hfill (47)

- $12h^{(4)} - 9h^{(1)}_{kl}h^{(3)}_{kl} - 4h^{(2)}_{kl}h^{(2)}_{kl} + 7h^{(1)}_{ik}h^{(1)}_{j}h^{(2)}_{ij} - 3h^{(1)}_{ij}h^{(1)}_{k}h^{(1)}_{l}h^{(1)}_{i} = 0$  \hfill (48)

**Order $\rho^3$**

- $4(d-8)h^{(4)}_{ij} + \left(4h^{(4)} - 4h^{(3)}_{kl}h^{(1)}_{kl} - 2h^{(2)}_{kl}h^{(2)}_{kl} + 4h^{(2)}_{kl}h^{(1)}_{kn}h^{(1)}_{l} - h^{(1)}_{kl}h^{(1)}_{n}h^{(1)}_{m}h^{(1)}_{l} \right)g_{ij} - 2h^{(1)}_{ij}h^{(3)}_{ij} + 12h^{(3)}_{k(i}h^{(1)}_{j)} + (-2h^{(2}}$
\[ \begin{align*}
&+h^{(1)kl} h^{(1)kl} h^{(2)ij} + 8h^{(2)}_{ik} h^{(2)k}_{ij} - 8h^{(2)l}_{ij} h^{(1)kl} - 2h^{(2)kl} h^{(1)ik} h^{(1)jl} \\
&2h^{(1)ik} h^{(1)jl} h^{(1)kn} h^{(1)n} - l^2 R^{(3)}ij = 0
\end{align*} \]

- \[ 20h^{(5)} - 16h^{(4)kl} h^{(1)kl} - 14h^{(3)kl} h^{(2)kl} + 13h^{(3)ij} h^{(1)k}_{ij} + 12h^{(2)ij} h^{(2)j} h^{(1)k}_{i}
- 11h^{(1)ij} h^{(1)k}_{ij} h^{(1)l} h^{(1)m}_{i} \]

\[ \text{Order } \rho^4 \]

- \[ 5(d - 10)h^{(5)}ij + \left( 5h^{(5)} - 5h^{(4)kl} h^{(1)kl} - 5h^{(3)kl} h^{(2)kl} + 5h^{(3)kl} h^{(1)k} h^{(1)n} h^{(1)n} \right) g_{ij} \\
+ 5h^{(2)} h^{(2)k}_{k} h^{(1)ln} - 5h^{(1)kl} h^{(1)_k n} h^{(2)}_{ml} + h^{(1)kl} h^{(1)_k n} h^{(1)n} h^{(1)m} h^{(1)p} h^{(1)pt} \]

- \[ -3h^{(1)} h^{(4)}ij + 16h^{(1)k(i)} h^{(1)j)} k + \left( -4h^{(2)} + 2h^{(1)kl} h^{(1)kl} \right) h^{(3)ij} + 24h^{(3)ij} h^{(2)j} h^{(1)kl} \\
- 12h^{(3)_k(i)} h^{(1)j} h^{(1)kl} - 2h^{(3)kl} h^{(1)ik} h^{(1)jl} + \left( -3h^{(3)} + 3h^{(2)kl} h^{(1)kl} \\
- h^{(1)kn} h^{(1)l} h^{(1)n} h^{(1)kl} + 8h^{(1)kn} h^{(1)l} h^{(1)n} i h^{(1)j} h^{(1)jl} - 2h^{(1)kn} h^{(1)n} h^{(1)m} h^{(1)ml} h^{(1)n} i h^{(1)jl} \\
- l^2 R^{(4)}ij = 0 \]

- \[ 30h^{(6)} + \ldots = 0 \]

The corresponding coefficients of the \( R[h] \) expansion are

\[ R^{(2)}ij = \frac{1}{2} \left[ \nabla_k \left( Dh^{(2)}ij h^{(1)k}_{ij} \right) - \nabla_i \left( Dh^{(2)}ij h^{(1)k}_{ij} \right) \right] \\
+ \frac{1}{2} \left( Dh^{(1)kl} h^{(1)l} - Dh^{(1)kl} h^{(1)j} \right) \]

\[ R^{(3)}ij = \frac{1}{2} \left[ \nabla_k \left( Dh^{(3)}ij h^{(1)k}_{ij} \right) - \nabla_i \left( Dh^{(3)}ij h^{(1)k}_{ij} \right) \right] \\
- \frac{1}{2} \left( Dh^{(1)kl} h^{(1)l} - Dh^{(1)kl} h^{(1)j} \right) \]

\[ R^{(4)}ij = \frac{1}{2} \left[ \nabla_k \left( Dh^{(4)}ij h^{(1)k}_{ij} \right) - \nabla_i \left( Dh^{(4)}ij h^{(1)k}_{ij} \right) \right] \]
\begin{align*}
- \nabla_i \left( \nabla_j h^{(4)} - h^{(3)} k_l D h^{(1)}_{kj} - h^{(1)} k_l D h^{(2)}_{kj} - h^{(2)} k_l D h^{(3)}_{kj} \right) \\
+ [h^{(1)km} h^{(2)}_{lm} + h^{(2)km} h^{(1)}_{lm} - h^{(1)km} h^{(1)ln} D h^{(2)}_{kj}]
\end{align*}
\begin{align*}
&= \frac{1}{2} \left( D h^{(3)}_{ij} D h^{(1)}_{kl} + D h^{(1)}_{ij} D h^{(3)}_{kl} - 2 D h^{(3)}_{k(i} D h^{(1)}_{j)l} + D h^{(2)}_{ij} D h^{(2)}_{kl} \\
&+ D h^{(2)}_{ik} D h^{(2)}_{jl} \right) + \frac{1}{2} \left( h^{(2)k} - h^{(1)ln} \right) \left( 2 D h^{(1)}_{k(i} D h^{(1)}_{j)m} \\
&- D h^{(1)}_{ij} D h^{(1)}_{km} - D h^{(1)}_{ij} D h^{(1)km} \right) + \frac{1}{2} h^{(1)}_{i} \left( 2 D h^{(2)}_{k(i} D h^{(2)}_{j)m} - D h^{(2)}_{ij} D h^{(2)}_{km} - D h^{(2)}_{ij} D h^{(2)km} \right) + \frac{1}{2} h^{(1)}_{i} \left( 2 D h^{(1)}_{k(i} D h^{(2)}_{j)m} - D h^{(1)}_{ij} D h^{(2)}_{km} - D h^{(1)}_{ij} D h^{(2)km} \right) \right) \tag{55}
\end{align*}

C The Dirichlet obstructions in higher dimensions

Following the procedure explained in section 4, we present in this appendix the obstructions in higher dimension. We employ the Ricci expansion coefficients \( R^{(n)}_{ij} \) defined in the previous Appendix.

For \( n = 1 \) the six-dimensional \((d = 6)\) obstruction is given by the tensor

\begin{align*}
l^6 Z^{(6)}_{ij} &= \left( 2 h^{(1)kl} h^{(2)kl} - h^{(2)kl} h^{(1)ln} h^{(1)mk} \right) g_{ij} - 16 h^{(2)k(i} h^{(1)j)} + 4 h^{(1)kl} h^{(1)}_{ik} h^{(1)jk} \\
&+ 2 h^{(1)k} h^{(2)ij} + 12 R^{(2)}_{ij} + (d - 6) \left( -4 h^{(2)k(i} h^{(1)}_{j)} + h^{(1)kl} h^{(1)}_{ik} h^{(1)}_{jk} \right) \tag{56}
\end{align*}

For \( n = 2 \) we get

\begin{align*}
l^8 Z^{(8)}_{ij} &= \left( 2 h^{(1)kl} h^{(3)kl} + 4 h^{(2)kl} h^{(2)kl} - 10 h^{(2)kl} h^{(1)ln} h^{(1)mk} \right) g_{ij} + 12 h^{(1)}_{ij} h^{(3)}_{ij} \\
&+ \left( 12 h^{(2)} - 6 h^{(1)kl} h^{(1)kl} \right) h^{(2)ij} - 72 h^{(3)k(i} h^{(1)}_{j)} - 48 h^{(2)ij} h^{(2)}_{ij} \\
&+ 48 h^{(2)k(i} h^{(1)}_{j)l} h^{(1)kl} + 12 h^{(2)kl} h^{(1)}_{ik} h^{(1)}_{jl} - 12 h^{(1)km} h^{(1)ln} h^{(1)}_{ik} h^{(1)}_{jl} \\
&+ 6 l^2 R^{(3)}_{ij} + (d - 8) \left( -18 h^{(3)k(i} h^{(1)}_{j)} - 8 h^{(2)ij} h^{(2)}_{ij} + 4 h^{(2)kl} h^{(1)}_{ik} h^{(1)}_{jl} \\
&+ 10 h^{(2)k(i} h^{(1)}_{j)l} h^{(1)kl} - 3 h^{(1)km} h^{(1)ln} h^{(1)}_{ik} h^{(1)}_{jl} \right) \tag{57}
\end{align*}
Finally, for $n = 3$ the result is

\[
I^{10} Z_{ij}^{(10)} = \left( 24 h_{kl} h^{(4)kl} + 36 h_{kl} h_{m}^{(3) kl} - 42 h_{kl} h_{lm} h^{(1)mk} - 48 h_{kl} h_{m} h_{n} h^{(2) nk} \right) g_{ij} \\
+ 54 h_{kl} h_{m} h_{n} h^{(1) nk} - 12 h_{kl} h_{m} h_{n} h^{(1) np} h^{(1) pk} g_{ij} \\
+ 72 h_{kl} h_{m} h_{n} h^{(2) np} h^{(1) pk} - 24 h_{kl} h_{m} h_{n} h^{(3) np} h^{(1) pk} \\
- 576 h_{kl} h_{m} h_{n} h^{(3) np} h^{(1) pk} + 288 h_{kl} h_{m} h_{n} h^{(3) np} h^{(3) pk} \right) g_{ij} \\
- 24 h_{kl} h_{m} h_{n} h^{(3) np} h^{(3) pk} \right) g_{ij} \\
+ 192 h_{kl} h_{m} h_{n} h^{(3) np} h^{(1) pk} - 192 h_{kl} h_{m} h_{n} h^{(3) np} h^{(1) pk} \\
- 96 h_{kl} h_{m} h_{n} h^{(3) np} h^{(1) pk} + 48 h_{kl} h_{m} h_{n} h^{(3) np} h^{(1) pk} \\
\right) g_{ij} \\
+ (d - 10) \left( -96 h_{kl} h_{m} h_{n} h^{(3) np} h^{(1) pk} + 48 h_{kl} h_{m} h_{n} h^{(3) np} h^{(1) pk} \\
+ 21 h_{kl} h_{m} h_{n} h^{(3) np} h^{(1) pk} + 28 h_{kl} h_{m} h_{n} h^{(3) np} h^{(1) pk} \\
- 37 h_{kl} h_{m} h_{n} h^{(3) np} h^{(1) pk} + 29 h_{kl} h_{m} h_{n} h^{(3) np} h^{(1) pk} \\
+ 12 h_{kl} h_{m} h_{n} h^{(3) np} h^{(1) pk} \\
\right) g_{ij} \\
(58) 
\]

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