

Q − Φ criticality in the extended phase space of (n+1)-dimensional RN-AdS black holes

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Abstract. In order to achieve a deeper understanding of gravity theories, it is important to further investigate the thermodynamic properties of black hole at the critical point, besides the phase transition and critical behaviors. In this paper, by using Maxwell's equal area law, we choose T, Q, Φ as the state parameters and study the phase equilibrium problem of general (n+1)-dimensional RN-AdS black holes thermodynamic system. The boundary of the two-phase coexistence region and its isotherm and isopotential lines are presented, which may provide theoretical foundation for studying the phase transition and phase structure of black hole systems.

PACS. 04.70.Dy Quantum aspects of black holes, evaporation, thermodynamics

1 Introduction

In the past analysis, the cosmological constant in AdS space-time and the state parameter-pressure in general thermodynamic system are always parallelized as

\[ A = \frac{n(n-1)}{2l^2}, \quad P = \frac{n(n-1)}{16\pi l^2} \]

(1)

and the corresponding thermodynamics volume for black hole thermodynamic system expresses as

\[ V = \left( \frac{\partial M}{\partial P} \right)_{S,Q,J} \]

(2)

where T, P, and V are the variable state parameters. This traditional method has been extensively applied to the characterization of general thermodynamic systems, and also to the establishing of complete simulation for AdS Space-time black hole thermodynamic liquid/gas system. This proved that at the phase transition point, the second-order phase transition exists in AdS Space-time black holes and the thermodynamics agree very well with Ehrenfest equation. At present there are two common methods to investigate the critical phenomenon of AdS Space-time Black Holes in the literature. The first is to study the thermodynamics and state space geometry of black holes, which found that the black hole phase transition point meets the requirements of thermodynamic second-order phase transition. The second approach is to turn to Maxwell’s equal area law and discuss the critical behavior of AdS Space-time black hole system, which have also proved the existence of second-order phase transition at Black Hole phase transformation point.

Despite of the promising results obtained about thermodynamic properties of AdS Space-time black holes, from theoretical point of view, there should be critical behaviors and phase transition process, if taking black holes in AdS Space-time as thermodynamic systems. However, the statistical mechanics background of Black Holes as thermodynamic systems is still unknown, which makes it very meaningful to study the relations between different thermodynamic properties of AdS Space-time black holes. Moreover, such study will contribute to a deeper understanding of the thermodynamic properties of black hole (entropy, temperature, and heat capacity), as well as the completion of self-consistent geometry theory of black hole thermodynamics.

The P − V phase diagram of AdS Space-time black holes was analyzed in Ref. [10], which implied the existence of a mechanical unstable region when the black hole temperature is low. In this region with partial negative pressure, pressure increases together with volume \( \partial M/\partial P > 0 \) in the isotherm, which is similar to the result obtained from the P − V phase diagram of van der Waals-Maxwell liquid/gas. A possible solution to the famous Maxwell’s equal area law, which was extensively applied to the study of AdS Space-time black holes thermodynamic
system \cite{49,51}. The yielded $T-P$ curve for system biphasic equilibrium and the slope expression of biphasic equilibrium curve showed that Ads Space-time black hole has the second-order phase transition and can be in a two phase coexistence state. We remark here that, except the phase transition point, other phase transitions in Ads Space-time black hole are all of first order. However, all the above results were derived on the base of the precondition of invariance of electric charge. As is well known to everyone, the parameters describing the charged Ads Space-time Black Holes thermodynamic system are not only related to state parameters ($T, P, V$), but also electromagnetic parameters like charge and electric potential. In this paper, we will use Maxwell’s equal area law to study the thermodynamic properties of general $(n + 1)$-dimensional RN-AdS black holes. More specifically, the discussion of the following two problems is the main motivation of our analysis: For black hole thermodynamic system with constant $P$ and $V$, is there still second-order phase transition when taking $(T, Q, \Phi)$ as state parameters? If so, is the critical point still the same as that when $(T, P, V)$ are taken as state parameters? In the second part, we give a brief introduction of $(n + 1)$-dimensional RN-AdS black hole. In the third part, we apply Maxwell’s equal area law in $(n + 1)$-dimensional RN-AdS black hole thermodynamic system, and obtain the relationship between different parameters and the boundary of two phase coexistence region. The last part is the conclusion.

2 general $(n + 1)$-dimensional RN-AdS black holes For general $(n + 1)$-dimensional RN-AdS black holes, the space-time metric can be written as \cite{32}

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2d\Omega_{n-1}^2,$$

where $d\Omega_{n-1}$ is the metric of the associated $(n + 1)$ dimensional base manifold and

$$f(r) = k - \frac{8\Gamma(n/2)M}{(n-1)\pi^{n/2}r^{n-2}} + \frac{Q^2}{r^{2n-4}} + \frac{r^2}{l^2},$$

Here $k = 1, 0, -1$ respectively corresponds to the sphere, plane and hyperbola symmetric cases. If denoting $r_+$ as the position of black hole horizon satisfying $f(r_+) = 0$, one can straightforwardly obtain the value of $r_+$, with the mass of the black hole within the event horizon radius

$$M = \frac{(n-1)\pi^{\frac{n-2}{2}}r_+^{n-2}(2Q^2r_+^4 + l^2r_+^2k + r_+^{2n+2})}{8\Gamma(n/2)},$$

Correspondingly, the Hawking temperature, entropy and potential of the black hole could be obtained as

$$T = \frac{f'(r_+)}{4\pi} = \frac{1}{4\pi} \left[ \frac{k(n-2)}{r_+} - (n-2)Q^2r_+^{3-2n} - \frac{2r_+A}{n-1} \right],$$

$$S = \int_0^{r_+} \frac{\partial_r M(r_+, Q)}{T} = \frac{\pi r_+^{n-1}}{2\Gamma(\frac{n}{2})},$$

$$\Phi = \left( \frac{\partial M}{\partial Q} \right)_S = \frac{(n-1)\pi^{\frac{n-2}{2}}Qr_+^{2-n}}{4\Gamma(\frac{n}{2})},$$

When taking $k = 1$, from Eq. (6), we will get

$$Q^{\frac{2}{n-1}} = -\frac{n-1}{4A} A \left( 4\pi T - \sqrt{16\pi^2 T^2 - \frac{8A(n-2)}{n-1}} \left( A^{2n-4} - 1 \right) \right),$$

where $A = \left( \frac{4\Gamma(n/2)}{(n-1)\pi^{n/2}} \right)^{1/(n-2)}$. From the derivative of Eq. (9)

$$\left( \frac{\partial Q}{\partial \Phi} \right)_T = 0,$$

$$\left( \frac{\partial^2 Q}{\partial \Phi^2} \right)_T = 0,$$

we will obtain the following expression

$$(n-2)(2n-3)A^{2n-2} - (n-2)A^2 - \frac{2A}{n-1} Q^{2/(n-2)} = 0,$$

$$(n-1)(2n-3)A^{2n-4} - 1 = 0,$$

Combining Eqs. (9), (12) and (13), one can obtain the three quantities under critical condition

$$\Phi_c = \frac{\pi^{n/2-1}}{4\Gamma(n/2)} \sqrt{\frac{(n-1)}{(2n-3)}},$$

$$T_c = \frac{(n-2)}{\pi(2n-3)} \left( -2A \right)^{1/2},$$

$$Q_c = \left( \frac{(n-2)^2}{2A} \right)^{(n-2)/2} \left( (n-1)(2n-3) \right)^{-1/2}.$$

The critical values for $Q, \Phi, T$ have been given in Eqs.(14)-(16). According to these critical values, the critical ratio is given by

$$\rho_c = \sqrt{A^2 - \frac{\pi}{2} \left( (n-1)\pi^{n/2} \left( \frac{(n-2)^2}{A} \right)^{n/2} \right)}.$$

(17)

Obviously, not like the P-V criticality, it depends on $A$ and $n$.

And, when taking $n = 3, 4, 5, 6$ and $A = -1$, one can get the $Q - \Phi$ graphs at different temperature from the above equations, as well as different critical temperatures under different space-time dimension, respectively $T_{c3} = 0.150053, T_{c4} = 0.180063, T_{c5} = 0.192925$ and $T_{c6} = 0.20007$. Fig. 1 shows the $Q - \Phi$ diagram at different critical temperature $T_c$, from which one can see that $Q - \Phi$ curve intersect with x-axis at $\Phi = \Phi_c$. It is apparent that when the temperature $T > T_c$, the special region
with \( \left( \frac{\partial Q}{\partial \Phi} \right)_T < 0 \) indeed exist in the \( Q-\Phi \) diagram, which does not satisfy the requirements of thermodynamic stability in the process of black hole evolution. And in Fig. 2, we also plot the \( Q-\Phi \) curves at different values of \( \Lambda \). It is shown that the \( Q-\Phi \) criticality nearly unchanges. Only the position of the critical point changes.

Another important thermodynamical quantity is the heat capacity \( C_Q \) at constant charge, which measures the stability against small perturbation and could be defined as

\[
C_Q = T \left( \frac{\partial S}{\partial T} \right)_Q = - \frac{\left( 2(-1 + n)^2 \pi^{1+\frac{2}{n}} r^2 Q \frac{1+n}{2+n} \frac{1}{r_T^{2-n}} \right)_T}{\Delta},
\]

Fig. 1. \( Q-\Phi \) diagram under isothermal conditions when the parameter \( n \) is fixed at \( n = 3, 4, 5, 6 \), respectively.

Fig. 2. \( Q-\Phi \) diagram under isothermal conditions when the parameter \( n = 3 \) is fixed and \( \Lambda = -0.5, -1, -2, -3 \), respectively.
where
\[
\Delta = \Gamma \left( \frac{n}{2} \right) (-2n^3 \left( (Qr_+^{-2-n})^{\frac{1}{n}} \right)^{2n} + n^2 \left( (Qr_+^{-2-n})^{\frac{1}{n}} + 9 \left( (Qr_+^{-2-n})^{\frac{1}{n}} \right)^{2n} \right) - n \Gamma \left( \frac{n}{2} \right) 3 \left( (Qr_+^{-2-n})^{\frac{1}{n}} + 13 \left( (Qr_+^{-2-n})^{\frac{1}{n}} \right)^{2n} \right) + 2 \Gamma \left( \frac{n}{2} \right) \left( (Qr_+^{-2-n})^{\frac{1}{n}} + 3 \left( (Qr_+^{-2-n})^{\frac{1}{n}} \right)^{2n} \right) + 2 \Gamma \left( \frac{n}{2} \right) Q^{-1} \frac{1}{n} \frac{1}{2 \pi n} A_i \right)
\]

For comparison, in Fig. 3 we show the heat capacity \( C_Q \) changing with \( r_+ \) and \( Q \), fixing \( A = -1 \) and \( A = -0.5 \) at \( n = 3 \). As can be seen from Fig. 3(a), for small value of \( Q \), with the increase of \( r_+ \), the heat capacity first goes to positive infinity at \( r_+ = r_1 \), then increases from negative infinity to a finite negative value and goes back to negative infinity at \( r_+ = r_2 \). Finally, it will decrease from positive infinity to a finite positive value, and then monotonically increases to infinity at \( r_+ = \infty \). From the above analysis, it is quite evident that the heat capacity \( C_Q \) is positive for \( r_+ < r_1 \) and \( r_+ > r_2 \), while negative for \( r_2 < r_+ < r_1 \). Note that \( r_1 \) represents the transition point where a small stable black hole with \( C_Q > 0 \) changes to an intermediate unstable one with \( C_Q < 0 \). \( r_2 \) corresponds to the transition point at which an intermediate unstable black hole changes to a large stable one.

On the other hand, for large value of \( Q \), we find the divergent behavior vanishes and the black hole will stay in a stable phase. In order to obtain a better understanding of the divergent behaviors of the heat capacity \( C_Q \), we illustrate in Fig. 4(a) the divergent point in the \((Q, r_+)\) plane. We emphasize \( T < 0 \) represents a non-black hole case which will not be discussed here. For \( Q < Q_{c3} \), there exist two divergent points at \( r_1 \) and \( r_2 \); for \( Q = Q_{c3} \), the two divergent points coincide with each other at \( r_+ = r_{c3} \); while the divergent point disappears at \( Q > Q_{c3} \). Therefore, \( Q_{c3} \) represents a critical phase transition point, which corresponds to a local maxima along this divergent curve. Moreover, this critical value also varies with the parameter \( A \), as can be seen from both Fig. 3(b) and Fig. 4(b).

### 3 Phase equilibrium and Maxwell’s equal area law

For general \((n+1)\)-dimensional RN-AdS black holes with constant temperature, the \( Q - \Phi \) curve shows an unstable region with \((2H)_T < 0 \). This is similar to Ads Space-time black holes with \((T, P, V)\) as state parameters. However, these problems were all solved when taking the gas to liquid phase transition \[31\,50\,55\]. In this section, we will study the famous Maxwell’s equal area law in Van der Waals equation, and apply it to the phase transition of general \((n+1)\)-dimensional RN-AdS black holes. Here we choose \((T, Q, \Phi)\) to be the state parameters and obtain the two phase coexistence region boundary. In this region, isotherm of general \((n+1)\)-dimensional RN-AdS black holes is replaced by isopotential line. The maximum of coexistence curve is the so-called critical point, at which Maxwell’s equal area law is no longer applicable.

Assuming the temperature is higher than the critical temperature \((T_0 > T_c)\), we set the x-axis of the boundary of two-phase region to be \( \Phi_2^{1/(n-2)} = \Phi_2 \) and \( \Phi_1^{1/(n-2)} = \Phi_1 \), while the y-axis to be \( Q_1^{1/(n-2)} = Q_0 \). From Maxwell’s equal area law, we have

\[
\tilde{Q}_0(\tilde{\Phi}_2 - \tilde{\Phi}_1) = \int \tilde{Q} d\tilde{\Phi},
\]

Taking \( A = \tilde{A} \Phi \) and \( A = \left( \frac{4r_0(n/2)}{(n-1)\pi^{n/2}} \right)^{1/(n-2)} \), Eq. (9) will transform into

\[
\tilde{Q} = -\frac{n-1}{4A} \tilde{\Phi} \left( 4\pi T - \sqrt{16n^2T^2 - \frac{8(n-2)}{n-1} (\tilde{A}^2n^{-4} - 1)} \right)
\]

and the above equation can be rewritten as

\[
\tilde{Q} = -\frac{n-1}{4A} 4\pi T \tilde{\Phi} + \frac{n-1}{4A} \tilde{A} \Phi \sqrt{B - C \Phi^{2n-4}},
\]

where \( B = 16n^2T^2 + \frac{8(n-2)}{n-1} \), \( C = \frac{8(n-2)}{n-1} \).

Now again, considering a special case with \( n = 3 \), Eq.(20) turns into

\[
Q_0(\Phi_2 - \Phi_1) = -\frac{\pi T}{A} \Phi_2 - \frac{1}{3A} (4\pi^2 T^2 + \Lambda - \Phi_2^2)^{\frac{1}{2}}
\]

\[
+ \frac{\pi T}{A} \Phi_1 + \frac{1}{3A} (4\pi^2 T^2 + \Lambda - \Phi_1^2)^{\frac{1}{2}}
\]

and

\[
Q_0 = \left( \frac{-2\pi T}{A} + \frac{1}{A} \sqrt{4\pi^2 T^2 + \Lambda - \Phi_2^2} \Phi_2, \right.
\]

\[
Q_0 = \left( \frac{-2\pi T}{A} + \frac{1}{A} \sqrt{4\pi^2 T^2 + \Lambda - \Phi_1^2} \Phi_1 \right)
\]

From the above Eq. (24), one can easily obtain the following expressions

\[
0 = -2\pi T(\Phi_2 - \Phi_1)
\]

\[
+ \Phi_2 \sqrt{4\pi^2 T^2 + \Lambda - \Phi_2^2} - \Phi_1 \sqrt{4\pi^2 T^2 + \Lambda - \Phi_1^2}
\]

\[
2Q_0 = - \frac{2\pi T}{A} (\Phi_2 + \Phi_1)
\]

\[
+ \frac{\Phi_2}{A} \sqrt{4\pi^2 T^2 + \Lambda - \Phi_2^2} + \frac{\Phi_1}{A} \sqrt{4\pi^2 T^2 + \Lambda - \Phi_1^2}
\]
From the above formulae, one can see that the value of \( x_0 = \frac{c}{\Lambda} \) from Eq. (15), the combination of Eq. (25) and (27) provides \( x_0 = \frac{3}{\Lambda} \).

If letting \( x = \frac{\Phi_1}{\Phi_2}, T = \chi T_c, \) and \( T_c = \sqrt{2n/3} \) from Eq. (15), the combination of Eq. (25) and (27) provides us

\[ 0 = -\frac{2\pi}{3} \chi (1-x) + \sqrt{\frac{8}{9}} \chi^2 - 1 + \frac{9}{2} \chi^2 - x \sqrt{\frac{8}{9}} \chi^2 - 1 + x^2 \Phi_2^2, \] 

(28)

\[ (1-x) \left( \sqrt{\frac{8}{9}} \chi^2 - 1 + \frac{9}{2} \chi^2 - x^2 \Phi_2^2 \right) \]

(29)

From the above formulae, one can see that the value of \( x \) and \( \Phi_2 \) is independent of \( A \). For a fixed \( \chi, \) i.e., a fixed \( T_0, \) we can get a certain value for \( x \) and \( \Phi_2 \) from Eq. (28) and (29).

Substituting Eq. (24) into Eq. (21), one can obtain similar formula for high-dimensional space-time

\[ \dot{Q} = -\frac{n - 1}{\sqrt{2n}} \Phi_2 \]

\[ \left( 2(n-2)\chi \frac{n-2}{(2n-3)} + \frac{n-2}{n-1} (\Lambda \Phi_2)^{2n-4} - 1 \right). \]

(30)

Note we can get the \( \dot{Q}_{\theta} \) in coexistence region from the above equation. Fig. 5 shows the \( Q - \Phi \) line on the background of isotherms at different temperature. When the temperature is lower than \( T_c, \) the isopotential line will replace the curve which does not meet the requirements of thermodynamic stability. The numerical values of \( \chi, x, \Phi_1, \Phi_2, T_0, \) and \( Q_0 \) at different space-time dimensions are also explicitly illustrated in Table 1.

In the canonical ensemble with fixed charge, the potential, which is also the free energy of the system, presents the thermodynamic behavior of a system in a standard approach. However, in our analysis we will consider an extended phase space. According to first law of black hole thermodynamics and the interpretation of \( M \) (total mass of black hole) [55, 56] as \( H \) (the black hole enthalpy) [55,56], the Gibbs free energy of black hole can be written...
The simulated isothermal phase transition by isobars and the boundary of two-phase coexistence region for RN-AdS black hole. The boundary of the two-phase equilibrium region is denoted by dotted dashed curve with $n = 3$, and $\chi = 1$ (black), $\chi = 1.004$ (blue), $\chi = 1.008$ (green), $\chi = 1.012$ (red). The area enclosed by yellow dotted lines represents the two-phase coexistence region.

Table 1. State quantities at phase transition points, considering different space-time dimensions ($A = 1$).

| $n$ | $\chi$ | $x$ | $\phi_1$ | $\phi_2$ | $Q_n$ | $T_n$ |
|-----|--------|-----|---------|---------|------|-----|
| 1   | 1      | 0.482848 | 0.408248 | 0.288675 | 0.150053 |
| 1.002 | 0.802808 | 0.364321 | 0.453809 | 0.286363 | 0.150353 |
| 1.004 | 0.732417 | 0.346572 | 0.471990 | 0.284047 | 0.150653 |
| 1.006 | 0.682245 | 0.333122 | 0.488237 | 0.241726 | 0.150953 |
| 4   | 1      | 0.693876 | 0.663875 | 0.516839 | 0.180048 |
| 4.001 | 0.643326 | 0.414251 | 0.749933 | 0.503992 | 0.180423 |
| 4.004 | 0.533619 | 0.433306 | 0.813887 | 0.491562 | 0.180784 |
| 4.006 | 0.460886 | 0.398923 | 0.865339 | 0.479109 | 0.181144 |
| 6   | 1      | 0.534718 | 0.660438 | 1.235115 | 0.195925 | 0.200270 |
| 6.001 | 0.508734 | 0.566763 | 1.386630 | 0.778500 | 0.200470 |

As

$$G = M - TS - Q\Phi$$

$$\frac{1}{8\Gamma(\frac{2}{n})}\pi^{n-2}n^{-2} \left( \left( Q_r + \frac{2n}{(n-1)} \right) \right)$$

$$\left( (n-2) \left( (Q_r + \frac{2n}{(n-1)}) \right) - (Q_r^{n-2}) \right)$$

$$\left( \left( Q_r^{n-2} \right) \right)$$

$$\left( \left( Q_r^{n-2} \right) \right)$$

Here, according to Eq. (9), $r_+$ is a function of charge and temperature, $r_+ = r_+ (Q, T)$. In Fig. 6-7, we plot the change of the free energy $G$ with $T$ and $Q$, for fixed $A = 1$ and different space-time dimensions. The existence of "swallow tail" behavior is clearly revealed, which indicates that the small-large black hole phase transition occurring in the system is of the first order.

And in Fig.8 and Fig.9, we also plot the $G-T$ curves at the same dimension and different values of $A$. It is shown that the $G-T$ criticality nearly unchanges. Only the position of the critical point changes.

From Fig. 6-7, we find that general $(n+1)$-dimensional RN-AdS black holes thermodynamic systems have typical characteristics of Van’s system gas/liquid phase transition. If we do not treat the cosmological constant as a thermodynamic variable and consider the non-extended phase space, black hole mass $M$ now should be the internal energy of the system, the Gibbs free energy is defined as the following form [29]

$$G = M - TS - Q\Phi$$

$$\frac{1}{8\Gamma(\frac{2}{n})}\pi^{n-2}n^{-2} \left( \left( Q_r + \frac{2n}{(n-1)} \right) \right)$$

$$\left( (n-2) \left( (Q_r + \frac{2n}{(n-1)}) \right) - (Q_r^{n-2}) \right)$$

$$\left( \left( Q_r^{n-2} \right) \right)$$

In Fig. 6-7 we plot the change of the free energy $G$ with $T$ and $Q$, for fixed $A = 1$ and different space-time dimensions. Fig. 10 reveals the existence of "swallow tail" behavior of the free energy. However, due to a distinct definition for Gibbs free energy, we fail to detect the "swallow tail" in Fig. 11, which is different from the case shown in Fig. 7.

4 Discussion
Taking general $(n+1)$-dimensional RN-AdS black holes as thermodynamic systems, the state equation of which is meaningless in some region. Using Maxwell’s equal area law (deduced from minimum Gibbs free energy theory) and taking phase transition into consideration, the meaningless region in the state equation no longer exists. Fig. 1 and Fig. 5 show that, when the system is at constant temperatures higher than the critical temperature, the $Q - \Phi$ curves are partially replaced by the isotherm and isopotential lines, which implies $Q$ and $T$ are invariants while potential $\Phi$ is changing. This region is a two-state coexistence region, where the phase transition is of first order according to Ehrenfest classification.
Fig. 6. Gibbs free energy versus $T$ for (a) $n = 3, \Lambda = -1$ and (b) $n = 4, \Lambda = -1$.

Fig. 7. Gibbs free energy versus $Q$ for (a) $n = 3, \Lambda = -1$ and (b) $n = 4, \Lambda = -1$.

Fig. 8. Gibbs free energy versus $T$ for $n = 3$, and $\Lambda = -0.5, -1, -2, -3$, respectively.
Fig. 9. Gibbs free energy versus $T = T_c$ for $n = 3$, and $\Lambda = -0.5, -1, -2, -3$, respectively.

Fig. 10. Gibbs free energy versus $T$ for (a) $n = 3, \Lambda = -1$ and for (b) $n = 4, \Lambda = -1$

Fig. 11. Gibbs free energy versus $Q$ for (a) $n = 3, \Lambda = -1$ and for (b) $n = 4, \Lambda = -1$

From the discussion above, we know that for general $(n + 1)$-dimensional RN-AdS black holes thermodynamic system, when taking $(T, Q, \Phi)$ as state parameters, the system shows similar phase transition characteristics to that of Van’s system. The position of the critical point is also the same as the case when taking $(T, P, V)$ as state parameters.

Moreover, by applying Maxwell’s equal area law to phase transition behaviors of thermodynamic system, we have derived both the position of critical point and the two phase coexistence region, which make it possible to obtain a more clear understanding of the phase transition process of such systems [32].

Taking Ads black hole as a thermodynamic system, it was found that the phase transition of various Ads black holes are similar to that of the Vander waals-Maxwell gas liquid [1, 34, 36]. Therefore, we can find some observable systems (Vander waals gas) similar to the Ads and ds background black holes. Considering the similarities they share in the thermodynamic properties, we may work backward and investigate other properties of black holes, such as phase transition and critical behaviors. This study will further contribute to a deeper understanding of
black hole entropy, temperature, and thermal capacity, as well as the completion of self-consistent black hole thermodynamics.

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**References**

1. David Kubiznak, Robert B. Mann, P-V criticality of charged AdS black holes, JHEP 1207:033,(2012). arXiv:1205.0559
2. Brian P. Dolan, David Kastor, David Kubiznak, Robert B. Mann, Jennie Traschen, Thermodynamic Volumes and Isoperimetric Inequalities for de Sitter black holes, Phys. Rev. D. 87. 104017(2013). arXiv:1301.5926
3. Sharmila Gunasekaran, David Kubiznak, Robert B. Mann, Extended phase space thermodynamics for charged and rotating black holes and Born-Infeld vacuum polarization, JHEP 11:110(2012). arXiv:1208.6251
4. Antonia M. Frassino, David Kubiznak, Robert B. Mann, Fil Simovic, Multiple Reentrant Phase Transitions and Triple Points in Lovelock Thermodynamics. arXiv:1406.7015
5. David Kubiznak, Robert B. Mann, black hole Chemistry. arXiv:1404.2126
6. Natasha Altamirano, David Kubiznak, Robert B. Mann, Zeinab Sherkatghanad, Thermodynamics of rotating black holes and black rings: phase transitions and thermodynamic volume. arXiv:1401.2586
7. Natasha Altamirano, David Kubiznak, Robert B. Mann, Zeinab Sherkatghanad, Kerr-AdS analogue of triple point and solid/liquid/gas phase transition. Class. Quant. Grav. 32,042001 (2014). arXiv:1308.2672
8. Natasha Altamirano, David Kubiznak, Robert B. Mann, Reentrant Phase Transitions in Rotating AdS black holes. Phys. Rev. D 88, 101502 (2013) arXiv:1306.5756.
9. S. W. Hawking and Don N. Page, Thermodynamics of Black Holes in Anti-de Sitter Space. Commun. Math. Phys. 87, 577-588 (1983)
10. Ren Zhao, Hui-Hua Zhao, Meng-Sen Ma, Li-Chun Zhang, On the critical phenomena and thermodynamics of charged topological dilaton AdS black holes. European Physical Journal C. November 73,2645 (2013).
11. Ren Zhao, Mengsen Ma, Huaifan Li and Lichun Zhang, On Thermodynamics of charged and rotating asymptotically AdS black strings, Advances in High Energy Physics vol. (2013).
12. Meng-Sen Ma and Ren Zhao, Phase transition and entropy spectrum of the BTZ black hole with torsion. Phys. Rev. D 89,044005 (2014).
13. Ma, Meng-Sen; Liu, Fang; Zhao, Ren, Continuous phase transition and critical behaviors of 3D black hole with torsion. Class. Quantum Grav. 31,095001 (2014). arXiv:1403.0449
14. Shao-Wen Wei, Peng Cheng, and Yu-Xiao Liu, Analytical and exact critical phenomena of d-dimensional singly spinning Kerr-AdS black holes. PHYSICAL REVIEW D 93,084015(2016).
15. M. H. Dehghani, A. Sheykhi, and Z. Dayyani, Critical behavior of Born-Infeld dilaton black holes PHYSICAL REVIEW D 93,024022 (2016).
16. Peng Cheng, Shao-Wen Wei, Yu-Xiao Liu, Critical phenomena in the extended phase space of Kerr-Newman-AdS black holes. arXiv:1603.08694
17. Devin Hansen, David Kubiznak, and Robert B. Mann. Criticality and Surface Tension in Rotating Horizon Thermodynamics. arXiv:1604.06312
18. S. H. Hendi, and M. H. Vahidinia, Extended phase space thermodynamics and PCV criticality of black holes with nonlinear source. Phys. Rev. D 88,084045 (2013). arXiv:1212.6128
19. S.-W. Wei and Y.-X. Liu, Insight into the Microscopic Structure of an AdS black hole from Thermodynamical Phase Transition, Phys. Rev. Lett. 115, 111302 (2015). arXiv:1502.00380
20. Jie-Xiong Mo and Wen-Biao Liu, Ehrenfest scheme for P-V criticality of higher dimensional charged black holes, rotating black holes, and Gauss-Bonnet AdS black holes. PHYSICAL REVIEW D 89, 084057 (2014).
21. J. X. Mo and W. B. Liu, Ehrenfest scheme for P-V criticality in the extended phase space of black holes. Phys. Lett. B 727, 336 (2013).
22. Jie-Xiong Mo, Wen-Biao Liu, P-V Criticality of Topological black holes in Lovelock-Born-Infeld Gravity. Eur. Phys. J. C 74,2836(2014). arXiv:1401.0785
23. J. Xu, L.-M. Cao, and Y.-P. Hu, P-V criticality in the extended phase space of black holes in massive gravity, Phys. Rev.D 91, 124033 (2015). arXiv:1506.03578
24. Jie-Xiong Mo, Gu-Qiang Li, Wen-Biao Liu, Another novel Ehrenfest scheme for PCV criticality Of RN-AdS black holes, Phys. Lett. B 730,111(2014).
25. Jie-Xiong Mo, Xiao-Xiong Zeng, Gu-Qiang Li, Xin Jiang, Wen-Biao Liu, A unified phase transition picture of the charged topological black hole in Hoava-Lifshitz gravity. JHEP1310,056(2013). arXiv:1404.2397.
26. J. X. Mo, Ehrenfest scheme with the extended phase space of f(R) black holes. Europhys. Lett. 105,20003(2014).
27. Arindam Lala and Dibakar Roychowdhury, Ehrenfests scheme and thermodynamic geometry in Born-Infeld AdS black holes. PHYSICAL REVIEW D 86,084027 (2012).
28. Shao-Wen Wei and Yu-Xiao Liu, Critical phenomena and thermodynamic geometry of charged Gauss-Bonnet AdS black holes. PHYSICAL REVIEW D 87,044014 (2013). arXiv:1209.1707
29. Jishnu Suresh, R. Tharanath, Nijo Varghese, V. C. Kuriakose, Phase transitions and Geometrotthermodynamics of Regular black holes. General Relativity and Gravitation, April 47:46(2015). arXiv:1406.3916
30. Chao Niu, Yu Tian, Xiaoning Wu, Critical Phenomena and Thermodynamic Geometry of RN-AdS black holes. Phys. Rev. D 85,024017,(2012). arXiv:1104.3066
31. Meng-Sen Ma, Thermodynamics and phase transition of black hole in asanymptotically safe gravity, Physics Letters B 735,45C50(2014).
32. S. H. Hendi, Z. Armanfard, Extended phase space thermodynamics. arXiv:1510.06557
33. S.-Q. Lan, J.-X. Mo, and W.-B. Liu, A note on Maxwells equal area law for d-dimensional RN-AdS black hole. Advances in High Energy Physics,(2014).
34. E. Spallucci and A. Smailagic, Maxwells equal area law for black hole thermodynamics. Class. Quantum Grav., 30,36 (2015).
35. A. Belhaj, M. Chabab, H. El Moumni, K. Masmar, M. B. Sedra, Maxwell’s equal-area law for Gauss-Bonnet Anti-de Sitter black holes. The European Physical Journal C, 75, 71(2015). arXiv:1305.3379
36. T. K. Saini, S. K. Singh, A. K. Singh, and N. K. Saini, A note on Maxwells equal area law for gauge field. Eur. Phys. J. C 75, 419 (2015). arXiv:1503.07658
37. J. X. Mo, Ehrenfest scheme for the extended phase space of f(R) black holes. Phys. Rev. D 89,044002(2014). arXiv:1311.7299
38. De-Cheng Zou, Shao-Jun Zhang, and Bin Wang, Critical behavior of Born-Infeld AdS black holes in the extended phase space thermodynamics. Phys. Rev. D 90,044057(2014). arXiv:1402.2837
39. S.-W. Wei and Y.-X. Liu, Triple points and phase diagrams in the extended phase space of charged Gauss-Bonnet black holes in AdS space, Phys. Rev. D 90,044063(2014). arXiv:1404.5194
40. Mohammad Bagher Jahani Poshteh, Behrouz Mirza, The phase transition, critical behavior, and critical exponents of Myers-Perry black holes. Phys. Rev. D 88,024005(2013). arXiv:1306.4516
41. Wei Xua, Liu Zhao, Critical phenomena of static charged AdS black holes in conformal gravity. Physics Letters B 736,214C220(2014).
42. Roberto Emparan, Clifford V. Johnson, Robert C. Myers, Surface Terms as Counterterms in the AdS/CFT Correspondence. Phys. Rev. D 60, 104001 (1999). arXiv:hep-th/9903235
43. Brian P. Dolan, Pressure and volume in the first law of black hole thermodynamics. Class. Quantum Grav. 28 (2011) 235027. arXiv:1106.6260v3
44. Yunqi Liu, De-Cheng Zou, and Bin Wang, Signature of the Van der Waals like small-large charged AdS black hole phase transition in quasinormal modes, JHEP 09,179(2014). arXiv:1405.2644
45. Li-Chun Zhang and Ren Zhao, The universal Ehrenfest scheme on black holes. Modern Physics Letters A, 30,36 (2015).
46. Zixu Zhao, Jiliang Jing, Ehrenfest scheme for complex thermodynamic systems in full phase space. JHEP 11,037(2014). arXiv:1405.2640
47. Parthapratim Pradhan, P-V Criticality in Conformal Gravity holography in four Dimensions. arXiv:1603.07750
48. Parthapratim Pradhan, Thermodynamic Products in Extended Phase Space. International Journal of Modern Physics D, Vol. 26 (2017) 1750010. arXiv:1603.07748
49. Andrew Chamblin, Roberto Emparan, Clifford V. Johnson, Robert C. Myers, Charged AdS Black Holes and Catastrophic Holography. Phys.Rev. D60 (1999) 064018.arXiv:hep-th/9902170
50. Jun-Xin Zhao, Meng-Sen Ma, Li-Chun Zhang, Hai-Hun Zhao, Ren Zhao. The equal area law of asymptotically AdS black holes in extended phase space. Astrophysics and Space Science 352: 763-768(2014).
51. Lichun Zhang, Hui Hua Zhao, Ren Zhao and Mengsen Ma, Equal area laws and latent heat for d-dimensional RN-AdS black hole. Advances in High Energy Physics,(2014).
52. Andrew Chamblin, Roberto Emparan, Clifford V. Johnson, Robert C. Myers, Charged AdS Black Holes and Catastrophic Holography. Phys.Rev. D60 (1999) 064018.arXiv:hep-th/9902170
53. E. Spallucci and A. Smailagic, Maxwells equal area law for black hole phase transition, Eur. Phys. J. C 75, 419 (2015). arXiv:1503.07658
54. A. Belhaj, M. Chabab, H. El Moumni, K. Masmar, M. B. Sedra, Maxwell’s equal-area law for Gauss-Bonnet Anti-de Sitter black holes. The European Physical Journal C, 75, 71(2015). arXiv:1312.2162
55. Brian P. Dolan, Black holes and Boyle’s law theorem of the cosmological constant. arXiv:1408.4023
56. Brian P. Dolan, Holography, Thermodynamics and Fluctuations of Charged AdS Black Holes. Phys.Rev.D60:104026,1999. arXiv:hep-th/9904197
