We study the interface effects of quark-hadron mixed phase in compact stars. The properties of nuclear matter are obtained based on the relativistic-mean-field model. For the quark phase, we adopt perturbation model with running quark masses and coupling constant. At certain choices of parameter sets, it is found that varying the quark-hadron interface tension will have sizable effects on the radii (ΔR ≈ 600 m) and tidal deformabilities (ΔΛ/Λ ≈ 50%) of hybrid stars. These provide possibilities for us to constrain the quark-hadron interface tension with future gravitational wave observations as well as the ongoing NICER mission.

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I. INTRODUCTION

With the first observation of gravitational waves from the binary neutron star merger event GW170817 [1, 2], astrophysics has entered the multi-messenger era. Combined with the electromagnetic observations of the transient counterpart AT2017gfo and short gamma ray burst GRB170817A [3], the dimensionless combined tidal deformability of the corresponding compact stars is constrained within 279 ≤ Λ ≤ 720 at 90% confidence level [1–6]. Meanwhile, the recent measurements of neutron stars’ radii indicate that their values lie at the lower end of 10–14 km range [7–15]. In light of the precise mass measurements of the two-solar-mass pulsars PSR J1614-2230 (1.928 ± 0.017 M⊙) [16, 17] and PSR J0348+0432 (2.01 ± 0.04 M⊙) [18], we have by far the most stringent constraint on the equation of states (EoS) of dense matter, which have been examined extensively in the past year [19–30].

The situation is even more exciting in the coming years. As the implementation of the planned detector upgrades, the sensitivity of gravitational wave observation may be improved by several times, which enables us to observe postmerger signals and constrain neutron stars’ radii to higher accuracy (on the order of a few hundred meters) [31, 32]. As the X-ray pulse profiles currently being measured by the NICER mission to an unprecedented accuracy [33], a precise measurement on neutron stars’ masses and radii is likely to take place in the near future [34–38]. Meanwhile, pulsars that are more massive than PSR J0348+0432 may be expected, e.g., PSR J2215+5135 (2.27±0.15 M⊙) [39]. Thus, the perspective for future pulsar observations provide opportunities to constrain the properties of dense matter to an unprecedented accuracy.

At large energy densities, hadronic matter (HM) is expected to undergo a deconfinement phase transition. For vanishing chemical potentials, a crossover was observed at the critical temperature $T_c \approx 170$ MeV [40]. Similar cases were also expected to occur in dense matter, where the transition between HM and quark matter (QM) is a smooth crossover [41–51]. More traditionally, one expects a first-order phase transition from HM to QM [52]. In such cases, a distinct interface between quark and hadronic matter are formed. Adopting the Maxwell construction, the properties of hybrid stars with a strong first-order phase transition and their relevance to gravitational wave observations were investigated [25–27]. The existence of third family solutions for hybrid stars were examined as well [29, 30]. It was found that a sharp phase transition will lead to small tidal deformabilities and induce discontinuities in the relation between tidal deformability and gravitational mass [27]. Meanwhile, a significant deviation from the empirical relation between the dominant postmerger gravitational wave frequency $f_{\text{peak}}$ and the radius/tidal deformability of a star at a given mass was observed if a strong first-order phase transition occurs [32]. All those features can be served as distinct signals for a strong first-order phase transition in the forthcoming gravitational wave observations.

Nevertheless, the Maxwell construction for the quark-hadron mixed phase (MP) is only valid if the surface tension $\sigma$ exceeds the critical value $\sigma_c$ [53]. In fact, depending on the values of $\sigma$, MP exhibits various structures [54]. The MP consists of point-like HM and QM when the surface tension $\sigma$ is zero, which is consistent with the Gibbs construction [55]. If the surface tension value is moderate, the finite-size effects become important and the geometrical structures such as droplet, rod, slab, tube, and bubble start to appear [54, 56–61]. The
sizes of the geometrical structures increase with the surface tension and will approach to the limit of Maxwell construction scenarios at $\sigma > \sigma_{\text{c}}$, i.e., bulk separation of quark and hadron phases, which suggests the nonexistence of MP inside hybrid stars.

Such kind of structural differences due to the quark-hadron interface effects are expected to affect many physical processes in hybrid stars. For example, the coherent scattering of neutrinos off the QM droplets may greatly enhance the neutrino opacity of the core [62]. Due to the relaxation of charge neutrality condition, the emergence of hyperons may be hindered [63], which prevents a fast cooling via the hyperon Urca processes [64–66]. Despite that the maximum mass of hybrid stars varies little with respect to the structural differences, it was found that their radii are more affected [54]. Similar cases were found in Ref. [67], where the robustness of third family solutions for hybrid stars was examined against the formation of pasta structures in the MP. Adopting MRE method, larger surface tension ($\sigma$) for color-flavor locked phase may be much larger, e.g., $\approx 300 \text{ MeV}/\text{fm}^2$ [86].

Due to the ambiguities in estimating the values of $\sigma$, in this work we consider the possibilities of constraining $\sigma$ with pulsar observations in the multi-messenger era. In particular, we study the interface effects of quark-hadron mixed phase in hybrid stars. It is found that the maximum mass, tidal deformabilities, and radii of hybrid stars increase with $\sigma$. These provide possibilities for us to constrain the quark-hadron interface tension with future gravitational wave observations as well as the ongoing NICER mission. The paper is organized as follows. In Sec. II, we present our theoretical framework, where the properties of nuclear matter and quark matter were obtained. The properties of their mixed phases and the interface effects are investigated in Sec. III, where both the Gibbs and Maxwell constructions are adopted and examined for the properties of hybrid stars in Sec. III A. As an example, adopting certain choices of parameters, the geometrical structures in hybrid stars are investigated in Sec. III B, which verifies our findings in Sec. III A. Our conclusion is given in Sec. IV.

II. THEORETICAL FRAMEWORK

A. Nuclear matter

In the mean field approximation, for infinite nuclear matter, the Lagrangian density of relativistic-mean-field model [87] is given as

$$\mathcal{L} = \sum_{i=n,p} \bar{\psi}_i \left[ i \gamma^\mu \partial_\mu - \gamma^0 \left[ g_\omega \omega + g_\rho \tau_3 \rho \right] - m_i \right] \psi_i - m_i \psi_i - \frac{1}{2} m_\sigma^2 \sigma^2 + \frac{1}{2} m_\omega^2 \omega^2 + \frac{1}{2} m_\rho^2 \rho^2 + \sum_{i=e,\mu} \bar{\psi}_i \left[ i \gamma^\mu \partial_\mu - m_i \right] \psi_i. \tag{1}$$

Three types of mesons are included to describe the interactions between nucleons, i.e., the isoscalar-scalar meson $\sigma$, isoscalar-vector meson $\omega$, and isovector-vector meson $\rho$. Note that the coupling constants $g_\sigma$, $g_\omega$, and $g_\rho$ are density dependent, which were obtained in accordance with the self-energies of Dirac-Brueckner calculations of nuclear matter [88], i.e.,

$$g_{\sigma,\omega}(n) = g_{\sigma,\omega}(n_0) a_{\sigma,\omega} \frac{1 + b_{\sigma,\omega} (n/n_0 + d_{\sigma,\omega})^2}{1 + c_{\sigma,\omega} (n/n_0 + d_{\sigma,\omega})^2}, \tag{2}$$

$$g_\rho(n) = g_\rho(n_0) \exp \left[ -a_\rho (n/n_0 - 1) \right]. \tag{3}$$

Here $n$ is the baryon number density and $n_0$ the saturation density of nuclear matter.

For the effective $N-N$ interactions, we adopt the covariant density functional TW99 [88], which is consistent with all seven constraints related to symmetric nuclear matter, pure neutron matter, symmetry energy, and its derivatives [89]. Carrying out a standard variational procedure, one obtains the energy density $E^H$, chemical potential $\mu_i$, and pressure $P^H$ at given particle number densities $n_i$. The energy density is determined by

$$E^H = \sum_i \epsilon_i (n_i, m^*_i) + \sum_{\phi=\sigma,\omega,\rho} \frac{1}{2} m^2_{\phi} \phi^2, \tag{4}$$

where $\epsilon_i$ is the kinetic energy density of free Fermi gas at given Fermi momentum $n_i$ and effective mass $m^*_i = m_i + g_\phi \mu_i$. Note that the effective masses remain the same for leptons, i.e., $m^*_{e,\mu} \equiv m_{e,\mu}$. The number density for particle type $i$ is given by $n_i = n_i^\phi / 3\pi^2$, while the
chemical potentials for baryons $\mu_i$ and leptons $\mu_{e,\mu}$ are
\begin{align}
\mu_i &= g_i \omega + g_i \tau_3 \rho + \Sigma^R + \sqrt{\nu_i^2 + m_i^2}, \\
\mu_{e,\mu} &= \sqrt{\nu_{e,\mu}^2 + m_{e,\mu}^2},
\end{align}
with $\Sigma^R$ being the “rearrangement” term, which is introduced to maintain thermodynamic self-consistency with density dependent coupling constants [90]. The pressure is obtained with
\[ P^H = \sum_i \mu_i n_i - E^H. \]

Based on Eqs. (4-7), the EoS for nuclear matter is obtained, which gives the saturation density $n_0 = 0.153$ fm$^{-3}$, saturation energy $E^H_0/n_0 - m_N = -16.25$ MeV, incompressibility $K = 240.2$ MeV and symmetry energy $E_{sym} = 32.77$ MeV. A detailed contour figure for the energy per baryon $\varepsilon = E^H/n$ of neutron star matter in $\beta$-equilibrium is presented in Fig. 1 as a function of the chemical potentials of baryons $\mu_b = \mu_n$ and electrons $\mu_e$, in obtaining which we have disregarded the local charge neutrality condition.

**B. Quark matter**

At ultra-high densities, the properties of quark matter can be obtained with perturbative QCD (pQCD), which are often extrapolated to lower density regions [91, 92]. Similarly, here we adopt pQCD to the order of $\alpha_s$ and investigate the properties of quark matter [93], while the non-perturbative contributions are treated with phenomenological approaches. The pQCD thermodynamic potential density is given by
\[ \Omega^p = \sum_i (\omega^0_i + \omega^1_i \alpha_s), \]
with
\begin{align}
\omega^0_i &= -\frac{m_i^4}{4\pi^2} \left[ u_i v_i \left( u_i^2 - \frac{5}{2} \right) + \frac{3}{2} \ln(u_i + v_i) \right], \\
\omega^1_i &= \frac{m_i^4}{2\pi^2} \left[ \left[ 6 \ln \left( \frac{\Lambda}{m_i} \right) + 4 \right] [u_i v_i - \ln(u_i + v_i)] \\
&+ 3[u_i v_i - \ln(u_i + v_i)]^2 - 2v_i^4 \right],
\end{align}
where $u_i \equiv \mu_i/m_i$ and $v_i \equiv \sqrt{u_i^2 - 1}$ with $\mu_i$ and $m_i$ being the chemical potential and mass for particle type $i$, respectively. By solving the $\beta$-function and $\gamma$-function [94] and neglecting higher order terms, the running coupling constant and quark masses read [93]
\[ \alpha_s(\Lambda) = \frac{1}{\beta_0 L} \left( 1 - \frac{\beta_1 \ln L}{\beta_0^2} \right), \]
\[ m_i(\Lambda) = \hat{m}_i \alpha_s^{\frac{3}{4}} \left[ 1 + \left( \frac{\gamma_1}{\beta_0} - \frac{\beta_1}{\beta_0^2} \right) \alpha_s \right]. \]

Here $L = 2 \ln \left( \frac{\Lambda}{E_{MS}} \right)$ with $E_{MS} = 376.9$ MeV being the MS renormalization point, while the invariant quark masses are $\hat{m}_u = 3.8$ MeV, $\hat{m}_d = 8$ MeV, and $\hat{m}_s = 158$ MeV according to the results obtained by Particle Data Group [95]. Note that $\beta_0 = \frac{1}{12}(11 - \frac{2}{3} N_f)$ and $\beta_1 = \frac{1}{1057}(102 - \frac{38}{3} N_f)$ for the $\beta$-function while $\gamma_0 = 1/\pi$ and $\gamma_1 = \frac{1}{16\pi^2} \left( \frac{292}{3} - \frac{20}{9} N_f \right)$ for the $\gamma$-function. At present, it is not clear how the renormalization scale $\Lambda$ evolves with the chemical potentials of quarks, where many possibilities exist [93, 96]. In this work, we adopt the following formalism:
\[ \Lambda = C \sum_{i=u,d,s} \frac{\mu_i}{3}, \]
with $C = 1 \sim 4$ [91].

To incorporate the non-perturbative effects, we introduce an extra bag constant $B$ to take into account the energy difference between the physical and perturbative vacua. According to various studies, it was found that the bag constant can vary with state variables, e.g., the temperature [97, 98], chemical potentials of quarks [99], density [100] and even magnetic field [101]. The bag constant at vanishing chemical potentials is found to be around 455 MeV fm$^{-3}$ according to QCD sum-rule [102], while carrying out fits to light hadron spectra suggests $B \approx 50$ MeV fm$^{-3}$ [103]. At larger chemical potentials, comparing Eq. (7) with the pQCD calculations to the order of $\alpha_s^2$ [91], an increasing difference on the thermodynamic potential density is observed. At the same time, the dynamic equilibrium condition at the critical temperature of deconfinement phase transition demands
$B \approx 400 \text{ MeV fm}^{-3}$ [104], indicating a large bag constant value at high energy density. On combination of those values, similar to Ref. [100, 105], we adopt the following parametrization of $B$, i.e.,

$$B = B_{\text{QCD}} + (B_0 - B_{\text{QCD}}) \exp \left[ -\left( \sum_i \frac{\mu_i - 930}{\Delta \mu} \right)^4 \right],$$

(13)

which gives $B = B_0 = 50 \text{ MeV fm}^{-3}$ at $\mu_u + \mu_d + \mu_s = 930 \text{ MeV}$. The width parameter $\Delta \mu$ and $B_{\text{QCD}}$ are left undetermined and to be fixed later. Note that adopting smaller $B_{\text{QCD}}$ reduces the maximum mass of hybrid stars and shrinks the parameter space for $\Delta \mu$ and $C$ in light of the observational mass of PSR J0348+0432 [18], which is indicated in Fig. 5.

Combining both the pQCD results in Eq. (7) and parameterized bag constant in Eq. (13), the thermodynamic potential density for quark matter is obtained with $\Omega^Q = \Omega^{\text{pt}} + \omega^Q_0/3 + B$, including the contributions of electrons. Based on the basic thermodynamic relations, the particle number density is $n_i = -\frac{\partial \Omega}{\partial \mu_i}$, and energy density of quark matter

$$E^Q = \Omega^{\text{pt}} + \frac{1}{3} \omega^Q_0 + B + \sum_i \mu_i n_i.$$  

(14)

The pressure takes negative values of the thermodynamic potential density, i.e., $P^Q = -\Omega^Q$.

C. Approximate the EoSs of HM and QM

For matter inside compact stars, to reach the lowest energy, particles will undergo weak reactions until the $\beta$-equilibrium condition is fulfilled, i.e.,

$$\mu_i = B_i \mu_b - q_i \mu_e,$$

(15)

where $B_i$ ($B_p = B_n = 1$, $B_u = B_d = B_s = 1/3$, and $B_{\mu} = B_{\bar{\mu}} = 0$) is the baryon number and $q_i$ ($q_p = 1$, $q_n = 0$, $q_u = 2/3$, $q_d = q_s = -1/3$ and $q_e = q_{\mu} = -1$) the charge of particle type $i$. Note that the chemical potential of neutrinos is set to zero since they can leave the system freely.

To simplify our calculation, it is convenient to approximate the pressures and energy densities of HM and QM by expanding them with respect to $\mu_e$, i.e.,

$$P(\mu_b, \mu_e) = P_0(\mu_e) - \frac{1}{2} \partial n^\prime_{ch}(\mu_b) [\mu_e - \mu_e(\mu_b)]^2,$$

$$E(\mu_b, \mu_e) = E_0(\mu_b) + E^\prime(\mu_b) [\mu_e - \mu_e(\mu_b)]$$

$$+ \frac{1}{2} E^\prime\prime(\mu_b) [\mu_e - \mu_e(\mu_b)]^2.$$  

(16)

Here $P_0$, $E_0$, and $\mu_e(\mu_b)$ is the pressure, energy density, and electron chemical potential obtained by fulfilling the local charge neutrality condition $n_{ch} = \sum q_i n_i = 0$. We have adopted prime notion to represent taking derivatives with respect to $\mu_e$ at $\mu_e = \mu_e(\mu_b)$, i.e.,

$$n^\prime_{ch} = \frac{\partial n_{ch}}{\partial \mu_e}, \quad E^\prime = \frac{\partial E}{\partial \mu_e}, \quad E^\prime\prime = \frac{\partial^2 E}{\partial \mu_e^2}.$$  

Note that $n^\prime_{ch}$ is related to the Debye screening length with $\lambda_D \equiv (4\pi \alpha n_{ch}^0)^{-1/2}$. Based on Eqs. (16-17) and basic thermodynamic relations, we have

$$n_{ch}(\mu_b, \mu_e) = -\frac{\partial P}{\partial \mu_e} \bigg|_{\mu_b} = \frac{1}{\mu_b} n^\prime_{ch}(\mu_e - \mu_e(\mu_b)),$$

$$n(\mu_b, \mu_e) = (E + \mu_e n_{ch} + P)/\mu_b.$$  

(18)

(19)

The obtained properties of HM and QM in Sec. II A and Sec. II B are then well reproduced by Eqs. (16-19). As an example, in Fig. 2 we plot the relative deviations of energy per baryon for nuclear matter with $\Delta \varepsilon = \varepsilon^{\text{cal}} - \varepsilon^{\text{fit}}$, which lies within 1%.

III. MIXED PHASE AND INTERFACE EFFECTS

A. The Gibbs and Maxwell constructions

To investigate the effects of quark-hadron interface on the properties of MP and compact stars, we consider two extreme cases, i.e., the Gibbs construction at $\sigma \to 0$ and the Maxwell construction at $\sigma > \sigma_c$. In both cases, at a given baryon chemical potential $\mu_b$, the dynamic stability condition needs to be satisfied, i.e.,

$$P^H = P^Q.$$  

(20)

In principle, leptons are free to move throughout the quark-hadron interface, then the chemical potentials of electrons in each phase become the same, i.e., $\mu^H_e = \mu^Q_e$, which is the case for the Gibbs construction.
Maxwell construction, the scale of MP is much larger than the Debye screening length $\lambda_D$, so that the local charge neutrality condition is effectively restored due to Coulomb repulsion. Thus, for the two types of phase construction schemes, we have

- **Gibbs**: $\mu^Q_e = \mu^Q_q$, $(1 - \chi)n^{H}_\text{ch} + \chi n^{Q}_\text{ch} = 0$; (21)
- **Maxwell**: $\mu^H_e \neq \mu^Q_q$, $n^{H}_\text{ch} = 0$, $n^{Q}_\text{ch} = 0$. (22)

Here the quark fraction $\chi \equiv V^Q/V$ with $V^Q$ being the volume occupied by quarks and $V$ the total volume. Based on Eqs. (16-19), Eqs. (20-22) can be solved analytically at given $\mu_0$. Then the properties of MP can be obtained.

Adopting both the Gibbs and Maxwell constructions, we investigate the properties of MP at various parameter sets with $C = 2 \sim 3.5$ and $\Delta \mu = 770 \sim 1000$ MeV. Note that the deconfinement phase transition occurs at densities smaller than 0.09 fm$^{-3}$ if $C \gtrsim 3.5$, while at $\Delta \mu \lesssim 770$ MeV and $B_{\text{QCD}} = 400$ MeV fm$^{-3}$ the velocity of sound in QM may exceeds the speed of light, which are excluded in our calculation. By solving Eq. (20) and (22), the densities of nuclear matter $n^{H}_T$ and quark matter $n^{Q}_T$ on the occurrence of deconfinement phase transition can be obtained at given $C$, $\Delta \mu$, and smaller $B_{\text{QCD}}$, which are presented in Figs. 3 and 4. Since the energy per baryon of QM decreases if we adopt larger $C$, $\Delta \mu$, and smaller $B_{\text{QCD}}$, the transition density $n^{H}_T$ decreases accordingly. The density jump $n^{Q}_T - n^{H}_T$ is increasing with $B_{\text{QCD}}$ and decreasing with $C$ and $\Delta \mu$, where a large $n^{Q}_T - n^{H}_T$ indicates a strong first-order phase transition. At $C \gtrsim 2.8$, we find varying $\Delta \mu$ or $B_{\text{QCD}}$ does not affect the transition densities $n^{H}_T$ and $n^{Q}_T$, while $n^{Q}_T$ decreases slightly with $C$.

Finally, based on the EoSs of NM, QM, and MP, we solve the Tolman-Oppenheimer-Volkov (TOV) equation

$$\frac{dP}{dr} = -\frac{GME (1 + P/E)(1 + 4\pi r^3P/M)}{r^2 - 2GM/r}$$

with subsidiary condition

$$\frac{dM(r)}{dr} = 4\pi Er^2.$$  

Here the gravity constant is taken as $G = 6.707 \times 10^{-45}$ MeV$^{-2}$. Note that at subsaturation densities, uniform nuclear matter becomes unstable and geometrical structures emerge. In such cases, we adopt the EoS presented in Refs. [106–108] at $n < 0.08$ fm$^{-3}$. The mass $M$ and radius $R$ of a compact star are obtained at given centre pressure. In Fig. 5 we present the maximum mass $M_{\text{max}}$ of hybrid stars obtained with the Maxwell construction. It is found that $M_{\text{max}}$ decreases with $\Delta \mu$ and
The parameter space shrinks if we adopt smaller \( B_{QCD} \). At fixed \( \Delta \mu \) and \( B_{QCD} \), the obtained maximum mass decreases with \( C \) at \( C \lesssim 2.8 \). This is mainly due to the softening of EoSs with the occurrence of deconfinement phase transition. For larger \( C \), as indicated in Fig. 3, QM appears at densities smaller than \( 2 n_0 \). In such cases, the core of a hybrid star is comprised almost entirely of QM, which has a similar parameter dependence on \( C \) as a strange star \( [115] \), i.e., the corresponding \( M_{\text{max}} \) is increasing with \( C \). For our calculation to be consistent with the observational mass of PSR J0348+0432 \( (2.01 \pm 0.04 \, M_\odot) \) \( [18] \), smaller \( \Delta \mu \), \( C \), and larger \( B_{QCD} \) are favored, i.e., the lower left regions in Fig. 5 with \( M_{\text{max}} \geq 1.97 \, M_\odot \). This area in the parameter space shrinks if we adopt smaller \( B_{QCD} \). Note that introducing the Gibbs construction will result in a different maximum mass. In Fig. 6 we present the variations on the maximum mass of hybrid stars \( \Delta M_{\text{max}} = M_{\text{max}}^\text{Maxwell} - M_{\text{max}}^\text{Gibbs} \) caused by introducing the Gibbs and Maxwell constructions. It is found that the difference is larger and positive at smaller \( C \), while \( \Delta M_{\text{max}} \) becomes negative and approaches to its minimum at \( C \approx 2.8 \). Nevertheless, \( \Delta M_{\text{max}} \) is positive for the cases with \( M_{\text{max}} > 1.97 \, M_\odot \), where the obtained \( M_{\text{max}}^\text{Maxwell} \) is larger than \( M_{\text{max}}^\text{Gibbs} \). In general, we find that the difference is insignificant \( (|\Delta M_{\text{max}}| \lesssim 0.08 \, M_\odot) \) comparing with the masses of hybrid stars.

The tidal deformability can be estimated with

\[
\Lambda = \frac{2 k_2}{3} \left( \frac{R}{GM} \right)^5,
\]

where \( k_2 \) is the second Love number \( [109–111] \). For hybrid stars with \( M = 1.36 \, M_\odot \), it is found that both tidal deformability \( \Lambda_{1.36} \) and radius \( R_{1.36} \) are insensitive to the choices of \( B_{QCD} \) according to our calculation. We thus take \( B_{QCD} = 400 \, \text{MeV fm}^{-3} \) and present the obtained \( \Lambda_{1.36} \) with the Gibbs construction at the centre panel of Fig. 7, which is decreasing with \( C \) but insensitive to \( \Delta \mu \) and \( B_{QCD} \). If we assume the mass ratio \( m_2/m_1 = 1 \) for the binary neutron star merger event GW170817, combined with the measured chirp mass \( \mathcal{M} = (m_1 m_2)^{3/5} (m_1 + m_2)^{-1/5} = 1.186 \pm 0.001 \, M_\odot \) \( [2] \), we then have \( m_1 = m_2 = 1.362 \, M_\odot \). The dimensionless combined tidal deformability \( \tilde{\Lambda} = \Lambda_1 = \Lambda_2 \approx \Lambda_{1.36} \) with the constraint \( 279 \leq \Lambda \leq 720 \) \([1–6]\). In fact, the obtained \( \Lambda \) may deviate slightly from \( \Lambda_{1.36} \) for other mass ratios as indicated in Fig. 13, while the variations are insignificant. In such cases, the region in the parameter space centered at \( C \approx 3.2 \) can be excluded since \( \Lambda \lesssim 279 \). To show the interface effects on the properties of hybrid stars at \( M = 1.36 \, M_\odot \), we compare the radii and tidal deformabilities of hybrid stars obtained based on the Gibbs and Maxwell constructions. The variations on \( \Lambda_{1.36} \) and \( R_{1.36} \) are presented in the top and bottom panels of Fig. 7, where \( \Delta R = R_{\text{Maxwell}} - R_{\text{Gibbs}} \) and \( \Delta \Lambda/\Lambda = \Lambda_{\text{Maxwell}}/\Lambda_{\text{Gibbs}} - 1 \). At certain choice of parameters, e.g., \( C \approx 2.7 \), the interface effects on the properties of hybrid stars become sizable. It is found that the radius of a hybrid star at \( M = 1.36 \, M_\odot \) may vary up to 600 m, which is within the capability of the NICER mission \( [33] \) or gravitational wave observations \( [31, 32] \). Meanwhile, the relative variations on the tidal deformability may even reach 50\%, which can be distinguished by future gravitational wave observations.

To further examine the interface effects on more mas-
sive hybrid stars, in Fig. 8 we present the variations of
tidal deformability with $M = 1.4$, 1.6, and 1.8 $M_\odot$ obtained
at $B_{QCD} = 400$ MeV fm$^{-3}$. It is found that the region
with large $\Delta \Lambda$ in the parameter space varies with the
mass of hybrid stars, where the centre shifts to smaller $C$
as $M$ increases. This is mainly due to the fact that the
interface effects become important when deconfinement
phase transition starts to take place at the centre of the
star, which is around the densities indicated in Figs. 3
and 4. Similar cases are expected for the radii of hybrid
stars as well as adopting other values of $B_{QCD}$.

In summary, based on the results indicated in Figs. 3-
8, the parameter $C$ can be constrained and is likely small
($\lesssim 3$) according to the expected hadron-quark transition
density in heavy-ion collision phenomenology, the
observational mass of PSR J0348+0432 [18], and the
lower limit of the dimensionless combined tidal deform-
bility [3]. In such cases, the interface effects play
important roles for the radii and tidal deformabilities of hybrid
stars as indicated in Figs. 7 and 8. With the planed up-
grades on gravitational wave detectors [31, 32], the on-
going NICER mission [33], and the mass measurements
of massive pulsars [39], we may have a good chance to
constrain simultaneously the parameters $C$, $\Delta \mu$, $B_{QCD}$,
as well as the quark-hadron interface tension in the near
future with the accurately measured masses, radii, and
tidal deformabilities of pulsars.

B. Geometrical structures

Since the emergence of geometrical structures is in-
evitable if the interface tension $\sigma < \sigma_c$, it is necessary to
investigate the interface effects on those structures and
consequently on the properties of MP. To construct the
geometrical structures of MP, we employ a Wigner-Seitz
approximation and assume spherical symmetry, i.e., only
the droplet and bubble phases are considered.

As was done in Refs. [112–115] but neglecting the con-
tributions of gravity, the internal structure of the Wigner-
Seitz cell is determined by minimizing the mass, which is
consistent with the constancy of chemical potentials

$$\bar{\mu}_i = \mu_i(r) + q_i \varphi(r) = \text{constant},$$

with the electric potential $\varphi(r)$ determined by

$$r^2 \frac{d^2 \varphi}{dr^2} + 2r \frac{d \varphi}{dr} + 4\pi \alpha \sigma n_{ch}(r) = 0.$$  

Here $\alpha = 1/137$ is the fine-structure constant. Since
the $\beta$-equilibrium condition is fulfilled, the local chemical
potentials are determined by Eq. (15) with a constant $\mu_0$ and space dependent $\mu_e(r) = \varphi(r) + \mu_e$. With the
linearization adopted in Eq. (18), Eq. (27) can be solved
analytically and gives

$$\varphi^I = \frac{C^I}{r} \sinh \left( \frac{r}{\lambda_D^I} \right) + \varphi_0^I,$$

$$\varphi^O = \frac{C^O}{r (R_W + \lambda_D^O)} \left[ \sinh(\tilde{r}) \lambda_D^O + \cosh(\tilde{r}) R_W \right] + \varphi_0^O$$

with $\tilde{r} \equiv (r - R_W)/\lambda_D^O$.

Here the Wigner-Seitz cell is divided into the inner part
(I) and outer part (O), i.e., a small sphere with radius $R$
enclosed within a spherical shell with outer radius $R_W$. The
MP is at the droplet phase if we have QM located at the
inner part and HM in the outer part, and vice versa, the
MP is at the bubble phase. The electric fields $\varphi^I(r)$
($r < R$) and $\varphi^O(r)$ ($R < r \leq R_W$) and their derivatives
should match with each other at $r = R$, which determines
the parameters $C^I, \varphi_0^I, C^O$, and $\varphi_0^O$ at given $\mu^I_0$ and $\mu^O_0$. The radius $R$ is fixed based on the dynamic stability of the
quark-hadron interface, i.e.,

$$P^I(R) - 2 \frac{\sigma}{R} = P^O(R).$$

The Wigner-Seitz cell radius $R_W$ is obtained by mini-
imizing the energy per baryon $M/A$ at a given baryon
density number $n = A/V_W$ ($V_W = 4\pi R_W^3/3$), where the
total mass $M$ and baryon number $A$ are fixed with

$$M = \int_{V_W} \left[ E(r) + \frac{1}{8\alpha \pi} \left( \frac{d \varphi}{dr} \right)^2 \right] dV + 4\pi R^2 \sigma,$$

$$A = \int_{V_W} n(r) dV.$$
Note that analytical expressions can be obtained for $M$ and $A$ based on Eqs. (17) and (19).

The properties of MP and the corresponding geometrical structures can then be determined based on Eqs. (26-32). To show the interface effects on the properties of MP and hybrid stars, as an example, we take $C = 2.7$, $\Delta \mu = 800$ MeV, and $B_{\text{QCD}} = 400$ MeV fm$^{-3}$ is adopted.

In Fig. 10 we present the EoSs of nuclear matter (NM), QM, and MP in compact stars. According to Fig. 9, the onset density for QM increases with $\sigma$, which approaches to the transition densities $n_H^T$ obtained with the Maxwell construction. Similarly, the corresponding energy density for the occurrence of deconfinement phase transition increases with $\sigma$. As the emergence of QM in NM, the EoSs become softer, which will consequently affect the properties of hybrid stars.

Based on the EoSs indicated in Fig. 10, we solve the TOV equation (23) and obtain the structures of compact stars. In Fig. 11 we present the masses of compact stars as functions of radius (Left panel) and central baryon number density (Right panel), which are compared with the mass-radius relations of PSR J0348+0432 ($2.01 \pm 0.04$ $M_\odot$) [18] is indicated with the horizontal band.
pared with the observational mass of PSR J0348+0432 (2.01 ± 0.04 $M_\odot$) [18]. As QM starts to appear at the centre of hybrid stars, the maximum mass and radii become smaller. Comparing with the variations caused by introducing different C, $\Delta \mu$, and $B_{\text{QCD}}$, interface effects on the maximum mass of hybrid stars are insignificant, which is consistent with our findings in Fig. 6 by introducing both the Gibbs and Maxwell constructions. Nevertheless, the interface effects on hybrid stars’ radii are sizable, where the variations can be as large as 600 m. For hybrid stars with a given mass, the radius increases with $\sigma$.

The tidal deformabilities of compact stars corresponding to Fig. 11 can be obtained with Eq. (25), which are presented in Fig. 12. Based on the observations of the binary neutron star merger event GW170817 and its transient counterpart AT2017gfo and short gamma ray burst GRB170817A, the dimensionless combined tidal deformability is constrained within 279 $\leq \Lambda \leq$ 720 [1–6], which is a mass-weighted linear combination of tidal deformabilities $\Lambda_{\text{Gibbs}}$.

$$\Lambda = \frac{16}{13} \frac{(m_1 + 12m_2)m_1^4 \Lambda_1 + (m_2 + 12m_1)m_2^4 \Lambda_2}{(m_1 + m_2)^5}.$$  \hfill (33)

With the best measured chirp mass $M = (m_1m_2)^{3/5}(m_1 + m_2)^{-1/5} = 1.186 \pm 0.001$ $M_\odot$ [2], in Fig. 13 we present the obtained $\Lambda$ as functions of the mass ratio $m_2/m_1$. Unlike nucleon stars [117], a slight deviation ($|\Delta \Lambda| \lesssim 20$) is observed for hybrid stars as we vary $m_2/m_1$, which is insignificant comparing with the deviations caused by the interface effects. Similar cases are also observed for other choices of parameters, where the deviation on $\Lambda$ is insignificant. It is then convenient for us to assume $m_1 = m_2 = 1.362$ $M_\odot$ with $\Lambda_1 = \Lambda_2 = \Lambda$. The corresponding constraints are indicated in Fig. 12, which can not distinguish hybrid stars obtained at different $\sigma$. Nevertheless, the situation will likely be changed in the near future with the implementation of the planned detector upgrades for gravitational wave observations [31, 32].

Based on Figs. 11 and 12, in Fig. 14 we present the evolution of maximum mass, tidal deformability, and radius of hybrid stars as functions of the surface tension $\sigma$, with given masses $M = 1.36, 1.4, 1.6, 1.8$ $M_\odot$ and $M_{\text{max}}$. For the cases with $\sigma = 0$, we adopt the results obtained with the Gibbs construction. It is found that $M_{\text{max}}, \Lambda$, and $R$ increase with $\sigma$. The results essentially interpolate between two types of values, i.e., as a function of $\sigma$ that connects the results obtained with the Gibbs construction at $\sigma \to 0$ and the Maxwell construction at $\sigma > \sigma_c$, where we have found $\sigma_c = 79.12$ MeV/fm$^2$. In principle, the corresponding function can be obtained by fitting to our results with certain assumption on its form, e.g., [53, Eq. (14)], which we intend to do in our future works. According to Fig. 14, we find the variations on the maximum mass, tidal deformability, and radius of hybrid stars are up to $\Delta M_{\text{max}} \approx 0.02$ $M_\odot$, $\Delta R \approx 600$ m, and $\Delta \Lambda/\Lambda \approx 50\%$, respectively. Even though $M_{\text{max}}$ increases little with $\sigma$, sizable changes are observed for $\Lambda$ and $R$, which is within the capability of the NICER mission or future gravitational wave observations [31, 32].

Finally, our investigations in Fig. 10–14 suggests that the radii and tidal deformabilities of hybrid stars are monotonous functions of $\sigma$, which approach to the scenarios of Gibbs construction at $\sigma \to 0$ and Maxwell construction at $\sigma > \sigma_c$. For other choices of parameter sets, we expect similar trends, where the critical surface ten-
The critical surface tension $\sigma_c$ can be well reproduced with [59]

$$
\sigma_c = \frac{(\mu_B^0 - \mu_Q^0)^2}{8\pi\alpha \left(\lambda_Q + \lambda_B^\alpha\right)}.
$$

The obtained results for $\sigma_c$ are presented in Fig. 15, where we have observed similar trends as in Fig. 3. This indicates correlations between the critical surface tension $\sigma_c$ and the thermodynamic quantities of MP, which was pointed out in [53, Fig. 5]. To show this explicitly, in Fig. 16 we present the obtained critical surface tension as a function of the chemical potential $\mu_T$ on the occurrence of deconfinement phase transition. It is found that $\sigma_c$ (in MeV/fm$^2$) increases linearly with $\mu_T$ (in MeV) and can be well approximated with $\sigma_c = 0.23(\mu_T - 930) + 19$, where the coefficients depend on the EoSs of HM and QM. In such cases, if the deconfinement phase transition occurs at large $\mu_T$, the emergence of geometrical structures may be inevitable in hybrid stars since typical estimations suggest $\sigma < \sigma_c$ with $\sigma < \approx 30$ MeV/fm$^2$ [74–82].

**IV. CONCLUSION**

We investigate the interface effects of quark-hadron mixed phase in hybrid stars. The properties of nuclear matter are obtained based on RMF model. For the $N$-$N$ interactions, we adopt the covariant density functional TW99 [88], which is consistent with all seven constraints related to symmetric nuclear matter, pure neutron matter, symmetry energy, and its derivatives [89]. For the quark phase, we adopt perturbation model by expanding the pQCD thermodynamic potential density to the order of $\alpha_s$ [93]. A parameterized bag constant is introduced by comparing with pQCD calculations to the order of $\alpha_s^2$ [91] as well as incorporating informations from QCD sum-rule [102] and light hadron mass spectra [103]. Since
the mixed phases obtained with the Gibbs and Maxwell constructions correspond to the two limits of the quark-hadron interface tension, i.e., \( \sigma \to 0 \) for the Gibbs construction and \( \sigma > \sigma_c \) for the Maxwell construction, we investigate the interface effects by comparing the results obtained by those two phase construction schemes. It is found that the quark-hadron interface has sizable effects on the radii (\( \Delta R \approx 600 \) m) and tidal deformabilities (\( \Delta \Lambda / \Lambda \approx 50\% \)) of hybrid stars for certain choices of parameters. This is then confirmed by considering the geometrical structures of the mixed phase with a specific choice of parameters, where for larger \( \sigma \) the sizes (\( \Delta M_{\text{max}} \approx 0.02 \), \( \Delta R \approx 600 \) m, and \( \Delta \Lambda / \Lambda \approx 50\% \)), respectively. This provides possibilities for us to constrain the quark-hadron interface tension with future gravitational wave observations [31, 32] as well as the NICER mission [33].

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