Long-time dynamics of a stochastic density dependent predator-prey model with Holling II functional response and jumps

Olga Borysenko\textsuperscript{a,0}, Oleksandr Borysenko\textsuperscript{b,1}

\textsuperscript{a}Department of Mathematical Physics, National Technical University of Ukraine, 37, Prosp.Peremohy, Kyiv, 03056, Ukraine
\textsuperscript{b}Department of Probability Theory, Statistics and Actuarial Mathematics, Taras Shevchenko National University of Kyiv, Ukraine, 64 Volodymyrska Str., Kyiv, 01601 Ukraine

olga_borisenko@ukr.net (Olg. Borysenko), odb@univ.kiev.ua (O. Borysenko)

Abstract The existence and uniqueness of a global positive solution is proven for the system of stochastic differential equations describing a nonautonomous stochastic density dependent predator-prey model with Holling-type II functional response disturbed by white noise, centered and non-centered Poisson noises. Sufficient conditions are obtained for stochastic ultimate boundedness, stochastic permanence, non-persistence in the mean, weak persistence in the mean and extinction of a population densities in the considered stochastic predator-prey model.

Keywords Stochastic Predator-Prey Model, Predator Density Dependence, Holling-type II functional response, Global Solution, Stochastic Ultimate Boundedness, Stochastic Permanence, Extinction, Non-Persistence in the Mean, Weak Persistence in the Mean

2010 MSC 92D25, 60H10, 60H30

\textsuperscript{1}Corresponding author.
1 Introduction

The deterministic autonomous Rosenzweig-Mac’Arthur model ([1]) is a generalization of Voltera-V erhulst model in that the linear functional response is replaced by Holling II functional response. This model has a form

\[
\begin{align*}
    dx_1(t) &= x_1(t) \left( a_1 - bx_1(t) - \frac{cx_2(t)}{1 + mx_1(t)} \right) dt, \\
    dx_2(t) &= x_2(t) \left( -a_2 + \frac{\kappa cx_1(t)}{1 + mx_1(t)} \right) dt, \\
\end{align*}
\]

where \(x_1(t)\) and \(x_2(t)\) are the prey and predator population densities at time \(t\), respectively; \(a_1 > 0\) is the growth rate of prey \(x_1\); \(b > 0\) measures the strength of competition among individuals of species \(x_1\); \(c\) is the maximum ingestion rate; \(m > 0\) is the half-saturation; \(a_2 > 0\) is the death rate of predator \(x_2\), and \(\kappa > 0\) is the conversion factor.

In the paper [2] the stochastic version of model (1) is considered in the following form

\[
\begin{align*}
    dx_1(t) &= x_1(t) \left( a_1 - bx_1(t) - \frac{cx_2(t)}{1 + mx_1(t)} \right) dt + \sigma_1 x_1(t) dw_1(t), \\
    dx_2(t) &= x_2(t) \left( -a_2 + \frac{\kappa cx_1(t)}{1 + mx_1(t)} \right) dt + \sigma_2 x_2(t) dw_2(t), \\
\end{align*}
\]

where \(w_1(t)\) and \(w_2(t)\) are mutually independent Wiener processes. In [2] the authors proved that there is a unique positive solution to the system (2), deduced the conditions that there is a stationary distribution of the system, which implies that the system is permanent, and obtained the sufficient conditions for the system that is going to be extinct.

Population systems may suffer abrupt environmental perturbations, such as epidemics, fires, earthquakes, etc. So it is natural to introduce Poisson noises into the population model for describing such discontinuous systems. There are considerable evidences in nature that predator species may be density dependent. So we need to take into account levels of predator density dependence.

In this paper, we consider the non-autonomous density dependent predator-prey model with Holling-type II functional response, disturbed by white noise and jumps generated by centered and non-centered Poisson measures. So, we take into account not only “small” jumps, corresponding to the centered Poisson measure, but also the “large” jumps, corresponding to the non-centered Poisson measure. This model is driven by the system of stochastic differential equations

\[
\begin{align*}
    dx_i(t) &= x_i(t) \left[ (-1)^{i-1} \left( a_i(t) - \frac{c_i(t)x_{3-i}(t)}{1 + m(t)x_i(t)} \right) - b_i(t)x_i(t) \right] dt \\
    &+ \sigma_i(t)x_i(t) dw_i(t) + \int_{\mathbb{R}} \gamma_i(t,z)x_i(t^-) \nu_1(dt,dz) + \int_{\mathbb{R}} \delta_i(t,z)x_i(t^-) \nu_2(dt,dz), \\
    x_i(0) &= x_{i0} > 0, \quad i = 1, 2. 
\end{align*}
\]
where \( x_1(t) \) and \( x_2(t) \) are the prey and predator population densities at time \( t \), respectively, \( b_2(t) > 0 \) is the predator density dependence rate, \( c_2(t) = \kappa c_1(t) \), \( w_i(t), i = 1, 2 \) are independent standard one-dimensional Wiener processes, \( \nu_i(t, A), i = 1, 2 \) are independent Poisson measures, which are independent on \( w_i(t), i = 1, 2 \), \( \tilde{\nu}_i(t, A) = \nu_i(t, A) - t\Pi_i(A), E[\nu_i(t, A)] = t\Pi_i(A), i = 1, 2, \Pi_i(A), i = 1, 2 \) are finite measures on the Borel sets \( A \) in \( \mathbb{R} \).

To the best of our knowledge, there have been no papers devoted to the dynamical properties of the stochastic predator-prey model (3), even in the case of centered Poisson noise. It is worth noting that the impact of centered and non-centered Poisson noises to the stochastic non-autonomous logistic model, to the stochastic two-species mutualism model and to the stochastic predator-prey model with modified version of Leslie-Gower term and Holling-type II functional response is studied in the papers [3] – [6].

In the following we will use the notations \( X(t) = (x_1(t), x_2(t)) \), \( X_0 = (x_{10}, x_{20}) \), \( |X(t)| = \sqrt{x_1^2(t) + x_2^2(t)} \), \( \mathbb{R}^2_+ = \{ X \in \mathbb{R}^2 : x_1 > 0, x_2 > 0 \} \),

\[
\alpha_i(t) = a_i(t) + \int_{\mathbb{R}} \delta_i(t, z)\Pi_2(dz),
\]

\[
\beta_i(t) = \frac{\sigma_i^2(t)}{2} + \int_{\mathbb{R}} \left[ \gamma_i(t, z) - \ln(1 + \gamma_i(t, z)) \right] \Pi_1(dz) - \int_{\mathbb{R}} \ln(1 + \delta_i(t, z))\Pi_2(dz),
\]

\( i = 1, 2 \). For the bounded, continuous functions \( f(t), f_i(t), t \in [0, +\infty), i = 1, 2 \), let us denote

\[
 f_{\text{sup}} = \sup_{t \geq 0} f(t), f_{\text{inf}} = \inf_{t \geq 0} f(t), f_{1\text{sup}} = \sup_{t \geq 0} f_1(t), f_{1\text{inf}} = \inf_{t \geq 0} f_1(t), i = 1, 2,
\]

\[
 f_{\text{max}} = \max\{ f_{1\text{sup}}, f_{2\text{sup}} \}, f_{\text{min}} = \min\{ f_{1\text{inf}}, f_{2\text{inf}} \}.
\]

We prove that the system (3) has a unique, positive, global (no explosion in a finite time) solution for any positive initial value, and that this solution is stochastically ultimately bounded. The sufficient conditions for stochastic permanence, non-persistence in the mean, weak persistence in the mean and extinction of solution are derived.

The rest of this paper is organized as follows. In Section 2, we prove the existence of the unique global positive solution to the system (3) and derive some auxiliary results. In Section 3, we prove the stochastic ultimate boundedness of the solution to the system (3), obtaining conditions under which the solution is stochastically permanent. The sufficient conditions for non-persistence in the mean, weak persistence in the mean and extinction of the solution are derived.

2 Existence of global solution and some auxiliary lemmas

Let \( (\Omega, \mathcal{F}, P) \) be a probability space, \( w_i(t), i = 1, 2, t \geq 0 \) are independent standard one-dimensional Wiener processes on \( (\Omega, \mathcal{F}, P) \), and \( \nu_i(t, A), i = 1, 2 \) are independent Poisson measures defined on \( (\Omega, \mathcal{F}, P) \) independent on \( w_i(t), i = 1, 2 \). Here \( E[\nu_i(t, A)] = t\Pi_i(A), i = 1, 2, \tilde{\nu}_i(t, A) = \nu_i(t, A) - t\Pi_i(A), i = 1, 2, \)
The coefficients of the system $T > \tau$ is not true, there are constants $\exp c \tau$ hence, there is any initial value $X$ on $[0, \tau]$. From the Itô's formula we derive that the process $X$ is a unique, positive local solution to the system $(\xi_1(t), \xi_2(t))$. Let us consider the system of stochastic differential equations

$$d\xi_i(t) = \left( -1 \right)^{i-1} \left( a_i(t) - \frac{c_i(t) \exp(\xi_{3-i}(t))}{1 + m(t) \exp(\xi_1(t))} \right) - b_i(t) \exp(\xi_i(t))$$

$$- \beta_i(t) \right) dt + \sigma_i(t) dw_i(t) + \int_{\mathbb{R}} \ln(1 + \gamma_i(t,z)) \nu_1(t, dz)$$

$$+ \int_{\mathbb{R}} \ln(1 + \delta_i(t,z)) \nu_2(t, dz), \quad \xi_i(0) = \ln x_{i0}, \ i = 1, 2. \quad (4)$$

The coefficients of the system $(4)$ are local Lipschitz continuous. So, for any initial value $(\xi_1(0), \xi_2(0))$ there exists a unique local solution $\Xi(t) = (\xi_1(t), \xi_2(t))$ on $[0, \tau_e)$, where $\sup_{t < \tau_e} |\Xi(t)| = +\infty$ (cf. Theorem 6, p.246, [7]). Therefore, from the Itô’s formula we derive that the process $X(t) = (\exp(\xi_1(t)), \exp(\xi_2(t)))$ is a unique, positive local solution to the system $(3)$. To show this solution is global, we need to show that $\tau_e = +\infty$ a.s. Let $n_0 \in \mathbb{N}$ be sufficiently large for $x_{i0} \in [1/n_0, n_0]$, $i = 1, 2$. For any $n \geq n_0$ we define the stopping time

$$\tau_n = \inf \left\{ t \in [0, \tau_e) : X(t) \notin \left( \frac{1}{n}, n \right) \times \left( \frac{1}{n}, n \right) \right\}.$$ 

It is easy to see that $\tau_n$ is increasing as $n \to +\infty$. Denote $\tau_\infty = \lim_{n \to \infty} \tau_n$, whence $\tau_\infty \leq \tau_e$ a.s. If we prove that $\tau_\infty = +\infty$ a.s., then $\tau_e = +\infty$ a.s. and $X(t) \in \mathbb{R}_+^2$ a.s. for all $t \in [0, +\infty)$. So we need to show that $\tau_\infty = +\infty$ a.s. If it is not true, there are constants $T > 0$ and $\varepsilon \in (0, 1)$, such that $P\{\tau_\infty < T\} > \varepsilon$. Hence, there is $n_1 \geq n_0$ such that

$$P\{\tau_n < T\} > \varepsilon, \quad \forall n \geq n_1. \quad (5)$$

For the non-negative function $V(X) = \sum_{i=1}^{2} k_i (x_i - 1 - \ln x_i)$, $X = (x_1, x_2)$,
\(x_i > 0, \, k_i > 0, \, i = 1, 2\) by the Itô’s formula we obtain
\[
dV(X(t)) = \sum_{i=1}^{2} k_i \left\{ (-1)^{i-1} (x_i(t) - 1) \left( a_i(t) - \frac{c_i(t) x_{3-i}(t)}{1 + m(t) x_1(t)} \right) - b_i(t)x_i(t)(x_i(t) - 1) + \beta_i(t) + \int_{\mathbb{R}} \delta_i(t, z)x_i(t)\Pi_2(dz) \right\} dt
\]
\[+ \sum_{i=1}^{2} k_i \left\{ (x_i(t) - 1)\sigma_i(t)dw_i(t) + \int_{\mathbb{R}} [\gamma_i(t, z)x_i(t^-) - \ln(1+\gamma_i(t, z))]\tilde{\nu}_1(dt, dz) + \int_{\mathbb{R}} [\delta_i(t, z)x_i(t^-) - \ln(1+\delta_i(t, z))]\tilde{\nu}_2(dt, dz) \right\}.
\]

Let us consider the function
\[
f(t, x_1, x_2) = -k_1b_1(t)x_1^2 + k_1 \left( \alpha_1(t) + b_1(t) \right)x_1 + k_1(\beta_1(t) - a_1(t)) -k_2b_2(t)x_2^2 + k_2 \left( -a_2(t) + b_2(t) + \int_{\mathbb{R}} \delta_2(t, z)\Pi_2(dz) \right)x_2 + k_2(\beta_2(t) + a_2(t))
\]
\[+ \frac{k_2(x_2 - 1)c_2(t)x_1 - k_1(x_1 - 1)c_1(t)x_2}{1 + m(t)x_1}, \, x_i > 0, \, i = 1, 2.
\]

We have estimate
\[
f(t, x_1, x_2) \leq \sum_{i=1}^{2} k_i \left( -b_i\inf x_i^2 + (\alpha_i\sup + b_i\sup)x_i \right) + k_1c_1\sup x_2 + k_1(\beta_1\sup - a_1\inf) + k_2(\beta_2\sup + a_2\sup) + \frac{c_1(t)x_1x_2(2k_2\kappa - k_1)}{1 + m(t)x_1}
\]

So for \(k_2 = k_1/\kappa\) there is a constant \(L(k_1, k_2) > 0\), such that \(f(t, x_1, x_2) \leq L(k_1, k_2)\). From (6) we obtain by integrating
\[
V(X(T \wedge \tau_n)) \leq V(X_0) + L(k_1, k_2)(T \wedge \tau_n)
\]
\[+ \sum_{i=1}^{2} k_i \left\{ \int_{0}^{T \wedge \tau_n} (x_i(t) - 1)\sigma_i(t)dw_i(t) + \int_{0}^{T \wedge \tau_n} [\gamma_i(t, z)x_i(t^-) - \ln(1+\gamma_i(t, z))]\tilde{\nu}_1(dt, dz) + \int_{0}^{T \wedge \tau_n} [\delta_i(t, z)x_i(t^-) - \ln(1+\delta_i(t, z))]\tilde{\nu}_2(dt, dz) \right\}.
\]

Taking expectations we derive from (7)
\[
E[V(X(T \wedge \tau_n))] \leq V(X_0) + L(k_1, k_2)T.
\]
Set $\Omega_n = \{\tau_n \leq T\}$ for $n \geq n_1$. Then by (5), $P(\Omega_n) = P(\tau_n \leq T) > \varepsilon$, for all $n \geq n_1$. Note that for every $\omega \in \Omega_n$ at least one of $x_1(\tau_n, \omega)$ and $x_2(\tau_n, \omega)$ equals either $n$ or $1/n$. So

$$V(X(\tau_n)) \geq \min\{k_1, k_2\} \min \left\{n - 1 - \ln n, \frac{1}{n} - 1 + \ln n \right\}.$$ 

From (8) it follows

$$V(X_0) + L(k_1, k_2)T \geq E[1_{\Omega_n} V(X(\tau_n))]$$

$$\geq \varepsilon \min\{k_1, k_2\} \min \left\{n - 1 - \ln n, \frac{1}{n} - 1 + \ln n \right\},$$

where $1_{\Omega_n}$ is the indicator function of $\Omega_n$. Letting $n \to \infty$ leads to the contradiction $\infty > V(X_0) + L(k_1, k_2)T = \infty$. This completes the proof of the theorem. \hfill \Box

**Lemma 1.** The density of the population $x_i(t), i = 1, 2$ obeys

$$\limsup_{t \to \infty} \frac{x_i(t)}{t} \leq 0, \quad i = 1, 2 \quad a.s.$$ 

**Proof.** By the Itô’s formula we have for $i = 1, 2$

$$e^t \ln x_i(t) - \ln x_{i0} = \int_0^t e^s \left\{ \ln x_i(s) + (-1)^{i-1} \left[ a_i(s) - \frac{c_i(s)x_3 - i(s)}{1 + m(s)x_1(s)} \right] \right. - b_i(s)x_i(s) - \frac{\sigma_i^2(s)}{2} + \int_{\mathbb{R}} \left[ \ln(1 + \gamma_i(s, z)) - \gamma_i(s, z) \right] \Pi_1(dz) \bigg\} ds + \psi_i(t), \quad (9)$$

where

$$\psi_i(t) = \int_0^t e^s \sigma_i(s)dw_i(s) + \int_0^t e^s \ln(1 + \gamma_i(s, z)) \tilde{\nu}_1(ds, dz)$$

$$+ \int_{\mathbb{R}} \int_0^t e^s \ln(1 + \delta_i(s, z)) \nu_2(ds, dz), \quad i = 1, 2.$$ 

By virtue of the exponential inequality ([3], Lemma 2.2) we have

$$P \left\{ \sup_{0 \leq t \leq T} \zeta_i(\mu, t) > \beta \right\} \leq e^{-\mu \beta}, \quad \forall 0 < \mu \leq 1, \quad \beta > 0, \quad i = 1, 2$$

where

$$\zeta_i(\mu, t) = \psi_i(t) - \frac{\mu}{2} \int_0^t e^{2s} \sigma_i^2(s)ds - \frac{1}{\mu} \int_0^t \left[ (1 + \gamma_i(s, z))^{\mu e^s} - 1 \right] (1 + \gamma_i(s, z))^{\mu e^s} - 1 \right] \Pi_1(dz)ds$$

$$- \mu e^s \ln(1 + \gamma_i(s, z)) \right] \Pi_1(dz)ds - \frac{1}{\mu} \int_0^t \left[ (1 + \delta_i(s, z))^{\mu e^s} - 1 \right] \Pi_2(dz)ds,$$
$i = 1, 2$. Choosing $T = k\tau, k \in \mathbb{N}, \tau > 0, \mu = e^{-k\tau}, \beta = \theta e^{k\tau} \ln k, \theta > 1$ we get

$$P \left\{ \sup_{0 \leq t \leq k\tau} \zeta_i(\mu, t) > \theta e^{k\tau} \ln k \right\} \leq \frac{1}{k^\theta}, i = 1, 2.$$ 

By the Borel-Cantelli lemma for almost all $\omega \in \Omega$, there is a random integer $k_0(\omega)$, such that for all $\forall k \geq k_0(\omega)$ and $0 \leq t \leq k\tau$

$$\psi_i(t) \leq \frac{1}{2e^{k\tau}} \int_0^t e^{2s} \sigma_i^2(s) ds + e^{k\tau} \int_0^t \int_\mathbb{R} \left[(1 + \gamma_i(s, z))e^{s-k\tau} - 1 \right] \Pi_1(dz) ds + e^{k\tau} \int_0^t \int_\mathbb{R} \left[(1 + \delta_i(s, z))e^{s-k\tau} - 1 \right] \Pi_2(dz) ds + \theta e^{k\tau} \ln k, \ i = 1, 2. \quad (10)$$

Applying the inequality $x^r \leq 1 + r(x-1), \forall x \geq 0, 0 \leq r \leq 1$ with $x = 1 + \gamma_i(s, z)$, $r = e^{s-k\tau}$, then with $x = 1 + \delta_i(s, z)$, $r = e^{s-k\tau}$, we derive from (10) the estimates

$$\psi_i(t) \leq \frac{1}{2e^{k\tau}} \int_0^t e^{2s} \sigma_i^2(s) ds + \int_0^t \int_\mathbb{R} \left[e^s [\gamma_i(s, z) - \ln(1 + \gamma_i(s, z))] \Pi_1(dz) ds \right] ds + \int_0^t \int_\mathbb{R} \left[e^s \delta_i(s, z) \Pi_2(dz) ds + \theta e^{k\tau} \ln k, \ i = 1, 2. \quad (11)$$

So from (9) and (11) we get for $i = 1, 2$

$$e^t \ln x_i(t) \leq \ln x_{i0} + \int_0^t e^s \left\{ \ln x_i(s) + (-1)^{i-1} \left[a_i(s) - \frac{c_i(s)x_{3-i}(s)}{1 + m(s)x_1(s)} \right] \right\} ds + \theta e^{k\tau} \ln k 
\leq \ln x_{i0} + \int_0^t e^s [\ln x_i(s) - b_{i\inf}x_i(s) + K_i] ds + \theta e^{k\tau} \ln k 
\leq \ln x_{i0} + L(e^t - 1) + \theta e^{k\tau} \ln k, \forall k \geq k_0(\omega), 0 \leq t \leq k\tau,$$

for some constant $L > 0$, where $K_1 = \alpha_{1\sup}, K_2 = \sup_{t \geq 0} \int_{\mathbb{R}} \delta_2(t, z) \Pi_2(dz) + c_{2\sup}/m_{\inf}$.

So for any $-(k-1)\tau \leq t \leq k\tau$, $\forall k \geq k_0(\omega)$ we have

$$\frac{\ln x_i(t)}{\ln t} \leq e^{-t} \ln x_{i0} \ln t + \frac{L}{\ln t} (1 - e^{-t}) + \frac{\theta e^{k\tau} \ln k}{e^{(k-1)\tau} \ln (k-1)\tau}, i = 1, 2 \text{ a.s.}$$
Therefore
\[
\limsup_{t \to \infty} \frac{\ln x_i(t)}{\ln t} \leq \theta e^\tau, \quad i = 1, 2, \quad \forall \theta > 1, \quad \forall \tau > 0, \quad \text{a.s.}
\]

If \( \theta \downarrow 1, \tau \downarrow 0 \), then we obtain
\[
\limsup_{t \to \infty} \frac{\ln x_i(t)}{\ln t} \leq 1, \quad i = 1, 2 \quad \text{a.s.}
\]

So
\[
\limsup_{t \to \infty} \frac{\ln x_i(t)}{t} \leq 0, \quad i = 1, 2 \quad \text{a.s.}
\]

\[\Box\]

**Lemma 2.** Let \( p > 0 \). Then for any initial value \( x_{i0} > 0 \), \( i = 1, 2 \) we have
\[
\limsup_{t \to \infty} E[x^p_i(t)] \leq K_i(p), \quad i = 1, 2,
\]
where \( K_i(p) > 0, i = 1, 2 \) are some constants depending on \( p \).

**Proof.** Let \( \tau_n \) be the stopping time defined in Theorem 1. Applying the Itô’s formula to the process \( V(t, x_i(t)) = e^{t}x^p_i(t), \quad i = 1, 2, p > 0 \), we obtain for \( i = 1, 2 \)
\[
V(t \wedge \tau_n, x_i(t \wedge \tau_n)) = x^p_{i0} + e^s x^p_i(s) \left\{ 1 \right. + \left. p \left[ (-1)^{i-1} \left( a_i(s) - \frac{c_i(s)x_{3-i}(s)}{1 + m(s)x_1(s)} \right) - b_i(s)x_i(s) \right] + \frac{p(p-1)\sigma^2_i(s)}{2} + \int_\mathbb{R} \left[ (1 + \gamma_i(s, z))^p - 1 - p\gamma_i(s, z) \right] \Pi_1(dz) \right. \\
\left. + \int_\mathbb{R} \left[ (1 + \delta_i(s, z))^p - 1 \right] \Pi_2(dz) \right\} ds + \int_0^{t \wedge \tau_n} e^{s} x^p_i(s) \sigma_i(s) dw_i(s) \right.
\]
\[
\left. + \int_0^{t \wedge \tau_n} e^{s} x^p_i(s)^{-1} [(1 + \gamma_i(s, z))^p - 1] \tilde{\nu}_1(ds, dz) \\
\left. + \int_0^{t \wedge \tau_n} e^{s} x^p_i(s)^{-1} [(1 + \delta_i(s, z))^p - 1] \tilde{\nu}_2(ds, dz). \right. \tag{12}
\]
Under Assumption 1 there are a constants $K_i(p) > 0, i = 1, 2$, such that
\begin{align*}
 &e^s x_i^p(s) \left\{ 1 + p \left[ (-1)^{i-1} \left( a_i(s) - \frac{c_i(s)x_{3-i}(s)}{1 + m(s)x_1(s)} \right) - b_i(s)x_i(s) \right] \right.
 + \frac{p(p-1)\sigma_i^2(s)}{2} + \int_{\mathbb{R}} \left[ (1 + \gamma_i(s,z))^p - 1 - p\gamma_i(s,z) \right] \Pi_1(dz) \\
 &\left. + \int_{\mathbb{R}} \left[ (1 + \delta_i(s,z))^p - 1 \right] \Pi_2(dz) \right\} \leq e^s K_i(p) \tag{13}
\end{align*}

From (12) and (13), taking the expectation, we obtain
\[ E[V(t \wedge \tau_n, x_i(t \wedge \tau_n))] \leq x_{i0}^p + K_i(p)e^t, \ i = 1, 2. \]

If $n \to \infty$, then we get
\[ e^t E[x_i^p(t)] \leq x_{i0}^p + K_i(p)e^t, \ i = 1, 2. \]

Hence $\limsup_{t \to \infty} E[x_i^p(t)] \leq K_i(p), \ i = 1, 2.$ \hfill \Box

**Lemma 3.** If $p_{2inf} > 0$, where $p_2(t) = -a_2(t) - \beta_2(t)$, then for any initial value $x_{20} > 0$, the predator population density $x_2(t)$ has the property that
\[ \limsup_{t \to \infty} E \left[ \left( \frac{1}{x_2(t)} \right)^\theta \right] \leq K(\theta), \ 0 < \theta < 1, \tag{14} \]

**Proof.** For the process $U(t) = 1/x_2(t)$ by the Itô’s formula we have
\begin{align*}
 U(t) &= U(0) + \int_0^t U(s) \left[ a_2(s) - \frac{c_2(s)x_1(s)}{1 + m(s)x_1(s)} + b_2(s)x_2(s) + \sigma_2^2(s) \right. \\
 &\quad + \int_{\mathbb{R}} \frac{\gamma_2(s,z)}{1 + \gamma_2(s,z)} \Pi_1(dz) \left. \right] ds \\
 &\quad - \int_0^t U(s^-) \frac{\gamma_2(s,z)}{1 + \gamma_2(s,z)} \nu_1(ds,dz) - \int_0^t U(s^-) \frac{\delta_2(s,z)}{1 + \delta_2(s,z)} \nu_2(ds,dz).
\end{align*}

Then by the Itô’s formula we derive for $0 < \theta < 1$
\begin{align*}
 (1 + U(t))^\theta &= (1 + U(0))^\theta + \int_0^t \theta(1 + U(s))^\theta - 2 \left\{ (1 + U(s))U(s) \right. \\
 &\quad \times \left[ a_2(s) - \frac{c_2(s)x_1(s)}{1 + m(s)x_1(s)} + b_2(s)x_2(s)\sigma_2^2(s) + \int_{\mathbb{R}} \frac{\gamma_2^2(s,z)}{1 + \gamma_2(s,z)} \Pi_1(dz) \right] \\
 &\left. - \int_0^t U(s^-) \frac{\gamma_2(s,z)}{1 + \gamma_2(s,z)} \nu_1(ds,dz) - \int_0^t U(s^-) \frac{\delta_2(s,z)}{1 + \delta_2(s,z)} \nu_2(ds,dz). \right\}
\end{align*}
\[
+ \frac{1}{\theta} \int_{\mathbb{R}} (1 + U(s))^2 \left( \frac{1 + U(s) + \gamma_2(s, z)}{1 + \gamma_2(s, z)(1 + U(s))} \right)^\theta - 1 \right) \\
+ \theta(1 + U(s)) \frac{U(s)\gamma_2(s, z)}{1 + \gamma_2(s, z)} \Pi_1(dz) \\
+ \frac{1}{\theta} \int_{\mathbb{R}} (1 + U(s))^2 \left( \frac{1 + U(s) + \delta_2(s, z)}{(1 + \delta_2(s, z))(1 + U(s))} \right)^\theta - 1 \right] \Pi_2(dz) \right) ds \\
- \int_0^t \theta(1 + U(s))^{\theta - 1} U(s)\sigma_2(s)dw_2(s) \\
+ \int_0^t \left[ \left( 1 + \frac{U(s^-)}{1 + \gamma_2(s, z)} \right)^\theta - (1 + U(s^-))^{\theta} \right] \bar{\nu}_1(ds, dz) \\
+ \int_0^t \left[ \left( 1 + \frac{U(s^-)}{1 + \delta_2(s, z)} \right)^\theta - (1 + U(s^-))^{\theta} \right] \bar{\nu}_2(ds, dz) = (1 + U(0))^{\theta} \\
+ \theta(1 + U(s))^{\theta - 2} J(s) ds - I_{1, \text{stoch}}(t) + I_{2, \text{stoch}}(t) + I_{3, \text{stoch}}(t), \quad (15)
\]

where \( I_{j, \text{stoch}}(t), j = 1, 3 \) are corresponding stochastic integrals in (15). Under Assumption 1 there exist constants \( |K_1(\theta)| < \infty, |K_2(\theta)| < \infty \) such, that for the process \( J(t) \) we have the estimate

\[
J(t) \leq (1 + U(t))U(t) \left[ a_2(t) + b_2(t)U^{-1}(t) + \sigma_2^2(t) \right. \\
+ \int_{\mathbb{R}} \gamma_2(s, z)\Pi_1(dz) + \frac{\theta - 1}{2} U^2(s)\sigma_2^2(s) \\
+ \frac{1}{\theta} \int_{\mathbb{R}} (1 + U(s))^2 \left( \frac{1}{1 + \gamma_2(s, z)} + \frac{1}{1 + U(s)} \right)^\theta - 1 \right] \Pi_1(dz) \\
+ \frac{1}{\theta} \int_{\mathbb{R}} (1 + U(s))^2 \left( \frac{1}{1 + \delta_2(s, z)} + \frac{1}{1 + U(s)} \right)^\theta - 1 \right] \Pi_2(dz)
\]
\[\leq U^2(t) \left[ a_2(t) + \frac{\sigma_z^2(t)}{2} + \int \gamma_2(t, z) \Pi_1(dz) + \frac{\theta}{2} \sigma_z^2(t) \right.\]
\[+ \frac{1}{\theta} \int \left[ (1 + \gamma_2(t, z))^{-\theta} - 1 \right] \Pi_1(dz) + \frac{1}{\theta} \int \left[ (1 + \delta_2(t, z))^{-\theta} - 1 \right] \Pi_2(dz) + K_1(\theta)U(t) + K_2(\theta) = -K_0(t, \theta)U^2(t) + K_1(\theta)U(t) + K_2(\theta).\]

Here we used the inequality \((x + y)\theta \leq x^{\theta} + \theta x^{\theta-1} y, 0 < \theta < 1, x, y > 0,\) in the first integral term for \(x = (1 + \gamma_2(s, z))^{-1}, y = (1 + U(s))^{-1},\) and then in the second integral term for \(x = (1 + \delta_2(s, z))^{-1}, y = (1 + U(s))^{-1}.\) Due to

\[
\lim_{\theta \to 0^+} \left[ \frac{\theta}{2} \sigma_i^2(t) + \frac{1}{\theta} \int [(1 + \gamma_i(t, z))^{-\theta} - 1] \Pi_1(dz) \right.
\[+ \frac{1}{\theta} \int [(1 + \delta_i(t, z))^{-\theta} - 1] \Pi_2(dz) + \int \ln(1 + \gamma_i(t, z)) \Pi_1(dz)\]
\[\left. + \int \ln(1 + \delta_i(t, z)) \Pi_2(dz) \right] = \lim_{\theta \to 0^+} \Delta(\theta, t) = 0,
\]

and the condition \(p_{2, \inf} > 0\) we can choose a sufficiently small \(0 < \theta < 1\) so that

\[K_0(\theta) = \inf_{t \geq 0} K_0(t, \theta) = \inf_{t \geq 0} [p_2(t) - \Delta(\theta, t)] > 0\]

is satisfied. So from (15) and the estimate for \(J(t)\) we derive

\[d \left[ (1 + U(t))^\theta \right] \leq \theta(1 + U(t))^{\theta-2} [-K_0(\theta)U^2(t) + \int \left[ \left( 1 + \frac{U(t)}{1 + \gamma(t, z)} \right)^\theta - \left( 1 + \frac{U(t)}{1 + \delta(t, z)} \right)^\theta \right] \Pi_2(dt, dz).\]

By the Itô’s formula and (16) we have

\[d \left[ e^{\lambda t}(1 + U(t))^\theta \right] = \lambda e^{\lambda t}(1 + U(t))^\theta dt + e^{\lambda t}d \left[ (1 + U(t))^\theta \right] \leq e^{\lambda t} \theta (1 + U(t))^{\theta-2} \left[ -K_0(\theta) - \frac{\lambda}{\theta} \right] U^2(t) + \left( K_1(\theta) + \frac{2\lambda}{\theta} \right) U(t) + K_2(\theta) + \frac{\lambda}{\theta} dt - \theta e^{\lambda t}(1 + U(t))^{\theta-1} U(t) \sigma_2(t) dw_2(t)\]
\[ + e^{\lambda t} \int_{\mathbb{R}} \left[ \left( 1 + \frac{U(t)}{1 + \gamma_2(t, z)} \right)^\theta - (1 + U(t))^\theta \right] \tilde{\nu}_1(dt, dz) \]
\[ + e^{\lambda t} \int_{\mathbb{R}} \left[ \left( 1 + \frac{U(t)}{1 + \delta_2(t, z)} \right)^\theta - (1 + U(t))^\theta \right] \tilde{\nu}_2(dt, dz). \]  
(17)

Let us choose \( \lambda = \lambda(\theta) > 0 \) such, that \( K_0(\theta) - \lambda/\theta > 0 \). Then there is a constant \( K > 0 \), such that
\[
(1 + U(t))^{\theta - 2} \left[ - \left( K_0(\theta) - \frac{\lambda}{\theta} \right) U^2(t) \right. \\
+ \left. \left( K_1(\theta) + \frac{2\lambda}{\theta} \right) U(t) + K_2(\theta) + \frac{\lambda}{\theta} \right] \leq K. \]  
(18)

Let \( \tau_n \) be the stopping time defined in the Theorem 1. Then by integrating (17), using (18) and taking the expectation we obtain
\[
E \left[ e^{\lambda(t \wedge \tau_n)} (1 + U(t \wedge \tau_n))^\theta \right] \leq \left( 1 + \frac{1}{x_{20}} \right)^\theta + \frac{\theta}{\lambda} K (e^{\lambda t} - 1). \]

Letting \( n \to \infty \) leads to the estimate
\[
e^t E \left[ (1 + U(t))^\theta \right] \leq \left( 1 + \frac{1}{x_{20}} \right)^\theta + \frac{\theta}{\lambda} K (e^{\lambda t} - 1). \]  
(19)

From (19) we obtain
\[
\limsup_{t \to \infty} E \left[ \left( \frac{1}{x_2(t)} \right)^\theta \right] = \limsup_{t \to \infty} E \left[ U^\theta(t) \right] \\
\leq \limsup_{t \to \infty} E \left[ (1 + U(t))^\theta \right] \leq \frac{\theta}{\lambda(\theta)} K,
\]
this implies (14).

3 The long time behaviour

Definition 1. ([8]) The solution \( X(t) \) to the system (3) will be said stochastically ultimately bounded, if for any \( \varepsilon \in (0, 1) \), there is a positive constant \( \chi = \chi(\varepsilon) > 0 \), such that for any initial value \( X_0 \in \mathbb{R}^2_+ \), the solution to the system (3) has the property that
\[
\limsup_{t \to \infty} P \{ |X(t)| > \chi \} < \varepsilon.
\]

In what follows in this section we will assume that Assumption 1 holds.

Theorem 2. The solution \( X(t) \) to the system (3) is stochastically ultimately bounded for any initial value \( X_0 \in \mathbb{R}^2_+ \).
Proof. From Lemma 2 we have estimate
\[
\limsup_{t \to \infty} E[x_i(t)] \leq K_i, \quad i = 1, 2. \tag{20}
\]
For \( X = (x_1, x_2) \in \mathbb{R}_+^2 \) we have \(|X| \leq x_1 + x_2\), therefore, from (20) \( \limsup_{t \to \infty} E[|X(t)|] \leq L = K_1 + K_2 \). Let \( \chi > L/\varepsilon, \forall \varepsilon \in (0, 1) \). Then applying the Chebyshev inequality yields
\[
\limsup_{t \to \infty} P\{ |X(t)| > \chi \} \leq \frac{1}{\chi} \limsup_{t \to \infty} E[|X(t)|] \leq \frac{L}{\chi} < \varepsilon.
\]

The property of stochastic permanence is important since it means the long-time survival in a population dynamics.

Definition 2. The population density \( x(t) \) will be said stochastically permanent if for any \( \varepsilon > 0 \), there are positive constants \( H = H(\varepsilon), h = h(\varepsilon) \) such that
\[
\liminf_{t \to \infty} P\{ x(t) \leq H \} \geq 1 - \varepsilon, \quad \liminf_{t \to \infty} P\{ x(t) \geq h \} \geq 1 - \varepsilon,
\]
for any initial value \( x(0) = x_0 > 0 \).

Theorem 3. If \( p_2 \inf > 0 \), where \( p_2(t) = -a_2(t) - \beta_2(t) \), then for any initial value \( x_{20} > 0 \), the predator population density \( x_2(t) \) is stochastically permanent.

Proof. From Lemma 2 we have estimate
\[
\limsup_{t \to \infty} E[x_2(t)] \leq K.
\]
Thus for any given \( \varepsilon > 0 \), let \( H = K/\varepsilon \), by virtue of Chebyshev’s inequality, we can derive that
\[
\limsup_{t \to \infty} P\{ x_2(t) \geq H \} \leq \frac{1}{H} \limsup_{t \to \infty} E[x_2(t)] \leq \varepsilon.
\]
 Consequently \( \liminf_{t \to \infty} P\{ x_2(t) \leq H \} \geq 1 - \varepsilon \).

From Lemma 3 we have the estimate
\[
\limsup_{t \to \infty} E\left[ \left( \frac{1}{x_2(t)} \right)^\theta \right] \leq K(\theta), \quad 0 < \theta < 1.
\]
For any given \( \varepsilon > 0 \), let \( h = (\varepsilon/K(\theta))^{1/\theta} \), then by Chebyshev’s inequality, we have
\[
\limsup_{t \to \infty} P\{ x_2(t) < h \} \leq \limsup_{t \to \infty} P\left\{ \left( \frac{1}{x_2(t)} \right)^\theta > h^{-\theta} \right\}
\]
\[
\leq h^{\theta} \limsup_{t \to \infty} E\left[ \left( \frac{1}{x_2(t)} \right)^\theta \right] \leq \varepsilon.
\]
 Consequently \( \liminf_{t \to \infty} P\{ x_2(t) \geq h \} \geq 1 - \varepsilon. \)
If the predator is absent, i.e. \( x_2(t) \equiv 0 \) a.s., then the equation for prey population is the non-autonomous stochastic logistic differential equation. The sufficient conditions for the stochastic permanence of the solution to the stochastic logistic differential equation are obtained in [6]:

**Theorem 4.** If the predator is absent, i.e. \( x_2(t) \equiv 0 \) a.s., and \( p_1(0) = a_1(0) - \beta_1(0) > 0 \), where \( p_1(t) = a_1(t) - \beta_1(t) \), then for any initial value \( x_{10} > 0 \), the prey population density \( x_1(t) \) is stochastically permanent.

**Definition 3.** The solution \( X(t) = (x_1(t), x_2(t)) \), \( t \geq 0 \) to the system (3) will be said extinct if for every initial data \( X_0 \in \mathbb{R}_+^2 \), we have \( \lim_{t \to \infty} x_i(t) = 0 \) a.s., \( i = 1, 2 \).

**Theorem 5.** If \( \bar{q}_i^* = \limsup_{t \to \infty} \frac{1}{t} \int_0^t q_i(s)ds < 0 \), where

\[
q_1(t) = a_1(t) - \beta_1(t), \quad q_2(t) = -a_2(t) + \frac{c_2(t)}{m(t)} - \beta_2(t)
\]

then the solution \( X(t) \) to the system (3) with the initial condition \( X_0 \in \mathbb{R}_+^2 \) will be extinct.

**Proof.** By the Itô’s formula, we have

\[
\ln x_i(t) = \ln x_{i0} + \int_0^t \left\{ \left( -1 \right)^{i-1} \left( a_i(s) - \frac{c_i(s)x_{3-i}(s)}{1 + m(s)x_1(s)} \right) - \beta_i(s) - b_i(s)x_i(s) \right\} ds + M_i(t) \leq \ln x_{i0} + \int_0^t q_i(s)ds + M_i(t), \quad i = 1, 2 \tag{21}
\]

where the martingale

\[
M_i(t) = \int_0^t \sigma_i(s)dw_i(s) + \int_0^t \int_\mathbb{R} \ln(1 + \gamma_i(s,z))\tilde{\nu}_1(ds,dz) \\
+ \int_0^t \int_\mathbb{R} \ln(1 + \delta_i(s,z))\tilde{\nu}_2(ds,dz), \quad i = 1, 2 \tag{22}
\]

has quadratic variation (Meyer’s angle bracket process)

\[
\langle M_i, M_i \rangle(t) = \int_0^t \sigma_i^2(s)ds + \int_0^t \int_\mathbb{R} \ln^2(1 + \gamma_i(s,z))\Pi_1(dz)ds \\
+ \int_0^t \int_\mathbb{R} \ln^2(1 + \delta_i(s,z))\Pi_2(dz)ds \leq Kt, \quad i = 1, 2.
\]
Then the strong law of large numbers for local martingales ([9]) yields \( \lim_{t \to \infty} M_i(t)/t = 0, \ i = 1, 2 \) a.s. Therefore, from (21) we obtain

\[
\limsup_{t \to \infty} \frac{\ln x_i(t)}{t} \leq \limsup_{t \to \infty} \frac{1}{t} \int_0^t q_i(s) ds < 0, \quad \text{a.s.}
\]

So \( \lim_{t \to \infty} x_i(t) = 0, \ i = 1, 2 \) a.s. \( \Box \)

**Definition 4.** ([10]) The solution \( X(t) = (x_1(t), x_2(t)), \ t \geq 0 \) to the system (3) will be said non-persistent in the mean if for every initial data \( X_0 \in \mathbb{R}_+^2 \), we have \( \lim_{t \to \infty} \frac{1}{t} \int_0^t x_i(s) ds = 0 \) a.s., \( i = 1, 2 \).

**Theorem 6.** If \( \bar{q}_i^* = 0, \ i = 1, 2, \) then the solution \( X(t) = (x_1(t), x_2(t)) \) to the system (3) with the initial value \( X_0 \in \mathbb{R}_+^2 \) will be non-persistent in the mean.

**Proof.** From the first equality in (21) we have

\[
\ln x_i(t) \leq \ln x_{i0} + \int_0^t q_i(s) ds - b_i \inf_t \int_0^t x_i(s) ds + M_i(t), \ i = 1, 2,
\]

(23)

where martingales \( M_i(t), i = 1, 2 \) are defined in (22). From the definition of \( \bar{q}_i^*, i = 1, 2 \) and the strong law of large numbers for \( M_i(t), i = 1, 2 \) it follows, that \( \forall \varepsilon > 0, \exists t_0 \geq 0, \) and \( \exists \Omega_\varepsilon \subset \Omega, \ P(\Omega_\varepsilon) \geq 1 - \varepsilon \) such that

\[
\frac{1}{t} \int_0^t q_i(s) ds \leq \bar{q}_i^* + \frac{\varepsilon}{2}, \quad \frac{M_i(t)}{t} \leq \frac{\varepsilon}{2}, \ i = 1, 2, \ \forall t \geq t_0, \ \omega \in \Omega_\varepsilon.
\]

So, from (23) we derive

\[
\ln x_i(t) - \ln x_{i0} \leq t(\bar{q}_i^* + \varepsilon) - b_i \inf_t \int_0^t x_i(s) ds
\]

\[
= t\varepsilon - b_i \inf_t \int_0^t x_i(s) ds, \ i = 1, 2, \ \forall t \geq t_0, \ \omega \in \Omega_\varepsilon.
\]

(24)

Let \( y_i(t) = \int_0^t x_i(s) ds, i = 1, 2, \) then from (24) we have for \( i = 1, 2 \)

\[
\ln \left( \frac{dy_i(t)}{dt} \right) \leq \varepsilon t - b_i \inf_t y_i(t) + \ln x_{i0}, \ \forall t \geq t_0, \ \omega \in \Omega_\varepsilon.
\]

Therefore

\[
e^{b_i \inf_t y_i(t)} \frac{dy_i(t)}{dt} \leq x_{i0} e^{\varepsilon t}, \ i = 1, 2, \ \forall t \geq t_0, \ \omega \in \Omega_\varepsilon.
\]
By integrating the last inequality from \( t_0 \) to \( t \) we obtain
\[
e^{b_{i \inf} y_i(t)} \leq \frac{b_{i \inf} x_{i0}}{\varepsilon} \left( e^{\varepsilon t} - e^{\varepsilon t_0} \right) + e^{b_{i \inf} y_i(t_0)}, \quad i = 1, 2, \forall t \geq t_0, \ \omega \in \Omega_\varepsilon.
\]

So
\[
y_i(t) \leq \frac{1}{b_{i \inf}} \ln \left[ e^{b_{i \inf} y_i(t_0)} + \frac{b_{i \inf} x_{i0}}{\varepsilon} \left( e^{\varepsilon t} - e^{\varepsilon t_0} \right) \right], \quad i = 1, 2, \forall t \geq t_0, \ \omega \in \Omega_\varepsilon,
\]
and therefore
\[
\limsup_{t \to \infty} \frac{1}{t} \int_0^t x_i(s)ds \leq \frac{\varepsilon}{b_{i \inf}}, \quad i = 1, 2, \ \forall \omega \in \Omega_\varepsilon.
\]

Since \( \varepsilon > 0 \) is arbitrary and \( x_i(t) > 0, i = 1, 2 \) a.s., we have for \( i = 1, 2 \)
\[
\lim_{t \to \infty} \frac{1}{t} \int_0^t x_i(s)ds = 0 \text{ a.s.}
\]

**Definition 5.** ([10]) The population \( x_i(t), i = 1, 2 \) will be said weakly persistent in the mean if for every initial data \( x_{i0} > 0, i = 1, 2, \) we have
\[
\limsup_{t \to \infty} \frac{1}{t} \int_0^t x_i(s)ds > 0, \text{ a.s.}, \quad i = 1, 2
\]

**Theorem 7.** If \( \bar{p}_2^* > 0 \), where \( p_2(t) = -a_2(t) - \beta_2(t) \), then the predator population density \( x_2(t) \) with the initial condition \( x_{20} > 0 \) will be weakly persistence in the mean
\[
\bar{x}_2^* = \limsup_{t \to \infty} \frac{1}{t} \int_0^t x_2(s)ds > 0 \text{ a.s.}
\]

**Proof.** If the assertion of theorem is not true, then \( P\{\omega \in \Omega \mid \bar{x}_2^* = 0\} > 0 \). From the first equality in (21) we get
\[
\frac{1}{t}(\ln x_2(t) - \ln x_{20}) + \frac{1}{t} \int_0^t b_2(s)x_2(s)ds = \frac{1}{t} \int_0^t p_2(s)ds + \frac{1}{t} \int_0^t \frac{c_2(s)x_1(s)}{1 + m(s)x_1(s)}ds + \frac{M_2(t)}{t} \geq \frac{1}{t} \int_0^t p_2(s)ds + \frac{M_2(t)}{t},
\]

where martingale \( M_2(t) \) is defined in (22). For \( \forall \omega \in \{\omega \in \Omega \mid \bar{x}_2^* = 0\} \) in virtue of the strong law of large numbers for martingale \( M_2(t) \) we have
\[
\limsup_{t \to \infty} \frac{\ln x_2(t)}{t} \geq \bar{p}_2^* > 0.
\]
Therefore
\[
\mathbb{P} \left\{ \omega \in \Omega \mid \limsup_{t \to \infty} \frac{\ln x_2(t)}{t} > 0 \right\} > 0.
\]

But from Lemma 1 we have
\[
\mathbb{P} \left\{ \omega \in \Omega \mid \limsup_{t \to \infty} \frac{\ln x_2(t)}{t} \leq 0 \right\} = 1.
\]

This is a contradiction. \( \square \)

**Theorem 8.** If \( \bar{p}_1^* > 0 \) and \( \bar{q}_2^* < 0 \), then the prey population density \( x_1(t) \) with initial condition \( x_{10} > 0 \) will be weakly persistence in the mean
\[
\bar{x}_1^* = \limsup_{t \to \infty} \frac{1}{t} \int_0^t x_1(s)ds > 0 \text{ a.s.}
\]

**Proof.** Let \( \mathbb{P}\{\bar{x}_1^* = 0\} > 0 \). From the first equality in (21) we get
\[
\frac{1}{t}(\ln x_1(t) - \ln x_{10}) + \frac{1}{t} \int_0^t b_1(s)x_1(s)ds = \frac{1}{t} \int_0^t p_1(s)ds \frac{1}{t} \int_0^t c_1(s)x_2(s)ds + \frac{M_1(t)}{t} \geq \frac{1}{t} \int_0^t p_1(s)ds - \frac{\sup c_1(s) x_2(s)}{\inf m_1} \int_0^t x_2(s)ds + \frac{M_1(t)}{t} \tag{25}
\]

where martingale \( M_1(t) \) is defined in (22). In virtue of the strong law of large numbers for martingale \( M_1(t) \) and the result of Theorem 5 for the predator population density \( x_2(t) \) we obtain from (25)
\[
\limsup_{t \to \infty} \frac{\ln x_1(t)}{t} \geq \bar{p}_1^* > 0
\]

for \( \forall \omega \in \{ \omega \in \Omega \mid \bar{x}_2^* = 0 \} \). Therefore
\[
\mathbb{P} \left\{ \omega \in \Omega \mid \limsup_{t \to \infty} \frac{\ln x_1(t)}{t} > 0 \right\} > 0.
\]

But from Lemma 1 we have
\[
\mathbb{P} \left\{ \omega \in \Omega \mid \limsup_{t \to \infty} \frac{\ln x_1(t)}{t} \leq 0 \right\} = 1.
\]

Therefore we have a contradiction. \( \square \)

**References**

[1] Iannelli, M., Pugliese, A.: An Introduction to Mathematical Population Dynamics, Springer, (2014).
[2] Liu, Zh., Shi, N., Jiang, D., Ji, Ch.: The asymptotic behavior of a stochastic predator-prey system with Holling II functional response, Abstract and Applied Analysis, vol. 2012, article ID 801812, (14 p.), (2012).

[3] Borysenko, O.D., Borysenko, D.O.: Persistence and extinction in stochastic nonautonomous logistic model of population dynamics, Theory of Probability and Mathematical Statistics, 2(99), 63–70, (2018).

[4] Borysenko, O.D., Borysenko, D.O.: Asymptotic behavior of the solution to the nonautonomous stochastic logistic differential equation, Theory of Probability and Mathematical Statistics, 2(101), 40–48, (2019).

[5] Borysenko, Olga, Borysenko, Oleksandr: Stochastic two-species mutualism model with jumps, Modern Stochastics: Theory and Applications, 7(1), 1–15, (2020), https://doi.org/10.15559/20-VMSTA150.

[6] Borysenko, Olga, Borysenko, Oleksandr: Long-time behavior of a nonautonomous stochastic predator-prey model with jumps, Modern Stochastics: Theory and Applications, 8(1), 17–39, (2021), https://doi.org/10.15559/21-VMSTA173.

[7] Gikhman, I.I., Skorokhod, A.V.: Stochastic Differential Equations and its Applications, Kyiv, Naukova Dumka, (1982). (In Russian)

[8] Li, X., Mao, X.: Population Dynamical Behavior of Non-Autonomous Lotka-Volterra Competitive System with Random Perturbation, Discrete & Continuous Dynamical Systems, 24, 523 – 545, (2009).

[9] Lipster, R.: A strong law of large numbers for local martingales, Stochastics, 3, 217–228, (1980).

[10] Liu, M., Wanga, K.: Persistence and extinction in stochastic non-autonomous logistic systems, Journal of Mathematical Analysis and Applications 375, 443–457, (2011).