$\mathcal{PT}$ symmetric interpretation of effective potentials

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Abstract. Conventional systems in equilibrium should have convex effective potentials. $\mathcal{PT}$ symmetry applies to systems which are in between open and closed systems. A $\mathcal{PT}$ symmetric interpretation can allow some non-convex effective potentials to be entirely physical. The one-loop effective potentials of the Higgs field in the Standard Model and the gravitino condensate in dynamically broken supergravity are conventionally unstable at large field values. In the specially simple case of space-independent and time-independent fields, the effective potentials are governed by $\mathcal{PT}$-symmetric quantum mechanics. The $\mathcal{PT}$-symmetric reinterpretation of these models at a quantum-mechanical level eliminates these instabilities and involves unusual semi-classical analysis involving many Riemann sheets. This interpretation is based on a combination of numerical analysis and semi-classical asymptotics.

1. Introduction

Quantum field theory currently gives us the preferred setting for the formulation of fundamental interactions for physics on the smallest scales. A feature of theories of fundamental interactions is symmetry which is implemented through Lagrangians that remain unchanged under the symmetry operation. However the the fundamental theories often require that this symmetry is broken by the ground state of the system.

Quantum field theory is an extremely complicated mathematical and physical construct. The dynamics takes place in an infinite-dimensional space and this means that analytic and numerical calculations are very difficult. Various approximation methods have been designed to make progress in understanding quantum field theory. One approach of historical significance is the use of an effective potential [1, 2], which represents an infinite-dimensional theory by a function $V_{\text{eff}}(\varphi)$ of a field variable $\varphi$, called the mean field. The effective potential is then used to explore possible vacua of the quantum field theory.

It is commonplace to consider non-convex effective potentials as intrinsically unphysical: arguments based on thermodynamical equilibrium imply convexity[3, 4]. Effective potentials have long been used to study symmetry breaking [1, 2] in field theory and much is known about the structure of such effective theories. The renormalized effective potential for a four-dimensional conformally invariant theory of a scalar field $\varphi$ interacting with fermions and gauge fields has the form $\Gamma[\varphi] = \varphi^4 f \left[ \log \left( \varphi^2 / \mu^2 \right) , g \right]$, where $\mu$ is a mass scale and $g$ denotes the coupling constants in the theory [5, 6]. Different theories are distinguished by the function $f$. The large-$\varphi$ behavior of the effective action determines the stability of the vacuum state. Such effective potentials are typically non-convex globally.
1.1. Calculation of effective actions

In quantum field theory the effective action $\Gamma[\bar{\varphi}]$ is the Legendre transform of the generating function of connected Green's functions [4] where $\bar{\varphi}$ is the mean field. Both $\Gamma[\bar{\varphi}]$ and $\bar{\varphi}$ are c-numbers. In the absence of quantum effects $\Gamma[\bar{\varphi}]$ coincides with the classical action. More generally

$$\Gamma[\bar{\varphi}] = \int_{t_1}^{t_2} dt \int_{\Sigma} d\sigma_x L_{\text{eff}}(t, \vec{x})$$

(1)

where $\Sigma$ is a bounded region of $\mathbb{R}^3$ (assuming that we are in a 4-dimensional space-time), $d\sigma_x$ is the volume element in $\Sigma$, $t$ is time and $L_{\text{eff}}$ is the local effective Lagrangian depending on the mean field $\bar{\varphi}$. From the principle of stationary action [7] we have

$$\Gamma[\bar{\varphi} + \delta \bar{\varphi}] = \Gamma[\bar{\varphi}] .$$

(2)

If $\bar{\varphi}$ is independent of time (a natural requirement since $\bar{\varphi}$ determines the ground state), then

$$\Gamma[\bar{\varphi}] = -(t_2 - t_1) \int_{\Sigma} d\sigma_x V_{\text{eff}}$$

(3)

where $V_{\text{eff}}$ is the effective potential. For a space $\Sigma$ which is homogeneous and isotropic we can take $\bar{\varphi}$ to be independent of $\vec{x}$ and the effective action satisfies

$$\Gamma[\bar{\varphi}] = -(t_2 - t_1) \Delta_{\Sigma} V_{\text{eff}}$$

(4)

where $\Delta_{\Sigma}$ is the volume of $\Sigma$. This is a very simple situation and it would seem that all the complexities of quantum field theory have melted away. However, when we consider the stability of ground states, we will have to transition through states which are not ground states (in the sense of $\bar{\varphi}$ being $\vec{x}$ independent). The effective action $\Gamma[\bar{\varphi}]$ has a classical part and a quantum part:

$$\Gamma[\bar{\varphi}] = \Gamma^{[0]}[\bar{\varphi}] + \sum_{r=1}^{\infty} \Gamma^{[r]}[\bar{\varphi}] .$$

(5)

The classical part $\Gamma^{[0]}[\bar{\varphi}]$ is obtained from tree diagrams of the quantum field theory; the remaining quantum part can be obtained from loop diagrams of the quantum field theory (with $r$ denoting the number of loops) and can only be calculated approximately. For a simple self-interacting scalar field theory with

$$\Gamma^{(0)} = -(t_2 - t_1) \int_{\Sigma} d\sigma_x \left[ \frac{1}{2} \partial^i \bar{\varphi} \partial_i \bar{\varphi} + \frac{1}{2} m^2 \bar{\varphi}^2 + U(\bar{\varphi}) \right],$$

(6)

$U(\bar{\varphi})$ denoting the self-interaction potential, $\Gamma^{[1]}[\bar{\varphi}]$ is a measure for zero point energy in field space, in analogy with the zero-point energy familiar for harmonic oscillators. In particular, in the equation of motion for the field $\varphi$

$$\Box \varphi + m^2 \varphi + U'(\varphi) = 0$$

we can substitute $\varphi = \bar{\varphi} + \delta$ and linearise in $\delta$. The resulting equation is

$$\Box \delta + m^2 \delta + U''(\bar{\varphi}) \delta = 0$$

(7)

whose normal modes are parametrized by

$$\delta_n(t, \vec{x}) = \exp(-i E_n t) d_n(\vec{x}) .$$

(8)
Consequently
\[
\left(-\nabla^2 + U''(\bar{\phi})\right) d_n(\bar{x}) = \epsilon_n d_n(\bar{x})
\]  
(9)
with \( E_n^2 = \epsilon_n + m^2 \). In general the mean field is \( \bar{x} \) dependent and \( \Gamma[\bar{\phi}] \) is a functional and not a function. Consequently information on the solutions of the eigenvalue problem of (9) is limited. If the \( \epsilon_n \) can be determined then
\[
\Gamma^{[1]}[\bar{\phi}] = -(t_2 - t_1) \sum_n \frac{\hbar}{2} \epsilon_n
\]  
(10)
which is the analogue of zero-point energy for the harmonic oscillator. In general this sum is difficult to evaluate.

However, in this paper, we will consider only configurations where \( \bar{\varphi} \) is independent of space-time, i.e. both the extrema of \( \Gamma[\bar{\varphi}] \) and the paths connecting the extrema are space-time independent. The expression in (10) can then be calculated [4] and the stability of ground states established using quantum mechanics.

1.2. Two examples of symmetry breaking in high energy physics

1.2.1. The Standard Model of particle physics The quantum mechanics of interactions of elementary particles is now described by a Standard Model. The symmetry group of the model has a part for the electro-weak symmetry, \( SU(2)_L \times U(1) \), and a part for quantum chromodynamics \( SU(3)_c \). Apart from matter and gauge fields it has spin 0 fields, the Higgs bosons \( \phi^+ \) and \( \phi^0 \) which allow the breaking of the electro-weak symmetry. The resultant electro-weak gauge fields are the massive \( W^\pm \) and \( Z^0 \) bosons and a massless photon field. From the Higgs bosons the massive scalar field \( \phi = Re\phi_0 \) survives.

The effective potential has been calculated (on taking into account the contribution of the top quark). A generic form for the effective potential is
\[
V^{(1)}_{\text{eff}}(\phi) = \mu^2 \phi^2 + \lambda \phi^4 - \phi^4 \ln(\phi^2)
\]  
(11)
where \( \lambda > 0 \). Clearly the large \( \phi \) behaviour would indicate that the vacuum is unstable. One way to avoid this instability is to invoke new physics at the Planck scale can be invoked[8, 9]. We will tackle this instability using \( PT \) symmetry.

1.2.2. Gravitino condensation in supergravity The desire to go beyond the Standard Model and incorporate gravitational forces led to a local version of supersymmetry known as supergravity (SUGRA). Supergravity contains quarks, leptons, Higgsinos, squarks, sleptons, Higgses, gauge bosons and gauginos. Supergravity also contains gravitons and gravitinos. The gravitino \( \psi^\mu \) (a spin 3/2 fermion) plays the role of the gauge field for supersymmetry. Spontaneous symmetry breaking of supersymmetry has been attempted through non-vanishing gravitino condensates \( \langle \psi_\mu \psi^\mu \rangle \equiv \phi \). The effective potential describing such a condensate [10] has a large \( \phi \) form \( V^{(2)}_{\text{eff}}(\phi) = -\phi^4 \ln(\phi^2) \). This potential is not only unbounded below but is complex. Such behaviour of potentials is regarded as unphysical in the conventional quantum mechanical framework.

1.3. One-dimensional effective action

Of course, \( V_{\text{eff}}(\bar{\phi}) \), where \( \bar{\phi} \) is a c-number, represents a severe approximation because it is a one-dimensional approximation to an infinite-dimensional structure. This is the kind of approach that is used in the mini-superspace analysis of quantum cosmology [11].
As an illustration of a one-dimensional analysis of a difficult high-dimensional problem, let us consider a nuclear fission process [12]. A fission process can be understood as a bag of several hundred nucleons that split into two smaller bags. Of course, the bag of nucleons undergoes violent quantum fluctuations and oscillations. Thus, a quantitative description of such a complicated process, either numerically or analytically, is almost intractable. However, we can approximate the process of fission but considering a bag of nucleons in static equilibrium. We then stretch this bag adiabatically into a dumbbell shape and plot the static potential of such a configuration as a one-dimensional function of the distance between the two lobes of the dumbbell. Having obtained a one-dimensional potential function, we can then make a one-dimensional semi-classical WKB approximation of the tunneling amplitude through the potential barrier. This calculation gives a reasonably accurate estimate the fission lifetime of a heavy nucleus.

It is in this spirit that we view an effective-potential approach in quantum field theory as a useful one-dimensional approximation to the full theory. The effective potential in quantum field theory is difficult to calculate and the standard perturbative method that is used is the loop expansion. Interestingly, when the effective potential for the Higgs field in the standard model is calculated in the one-loop approximation, $V_{\text{eff}}(\phi)$ becomes unbounded below for large values of the Higgs field. Many authors have taken this fact as an indication that the Higgs vacuum is unstable [13].

In this paper our approach will be to treat the effective potential as a function of the one-dimensional variable $\phi_c$ and to examine this potential using the techniques of quantum mechanics. We will argue that while an upside-down potential may seem to suggest that the vacuum state is unstable, this may not actually be so. In particular, we will use the techniques of $\mathcal{PT}$-symmetric quantum mechanics to show that some quantum field theories, whose effective potentials suggest that the vacuum state of a theory is unstable, may actually possess a stable vacuum state.

The emergence of $\mathcal{PT}$-symmetry in effective potentials may imply that renormalisation, due to the infinite number of degrees of freedom inherent in field theory, necessitates a $\mathcal{PT}$-symmetric formulation. For the exactly soluble Lee model of quantum field theory, it has been demonstrated in [14] that this occurs.

2. $\mathcal{PT}$ Quantum Mechanics

The position operator $\hat{\phi}$ for a quantum mechanical system transforms under parity $\mathcal{P}$ as

$$\mathcal{P} \hat{\phi} \mathcal{P} = -\hat{\phi}. \quad (12)$$

Under time reversal $\mathcal{T}$ the position operator transforms as

$$\mathcal{T} \hat{\phi} \mathcal{T} = \hat{\phi}. \quad (13)$$

Owing to the antilinearity of $\mathcal{T}$ we also have $\mathcal{T} i \mathcal{T} = -i$.

An effective potential $V(\hat{\phi})$ is $\mathcal{PT}$ symmetric if

$$\mathcal{PT} V(\hat{\phi}) \mathcal{PT} = V(\hat{\phi}). \quad (14)$$

It is easy to see that

$$\mathcal{PT} V_{\text{eff}}^{(1)} \mathcal{PT} = V_{\text{eff}}^{(1)}$$
$$\mathcal{PT} V_{\text{eff}}^{(2)} \mathcal{PT} = V_{\text{eff}}^{(2)}$$

and so $V_{\text{eff}}^{(1)}$ and $V_{\text{eff}}^{(2)}$ are $\mathcal{PT}$ symmetric[15].
We consider two theories [16] of current interest: (i) a theory of dynamical breaking of gravity via a gravitino condensate field $\varphi$, where [17, 10]

$$\Gamma[\varphi] \propto -\varphi^4 \log(i\varphi) \quad (\varphi \text{ large}); \quad (15)$$

(ii) the Standard Model of particle physics for which $\varphi$ is the Higgs field [8] and

$$\Gamma[\varphi] \propto -\varphi^4 \log(\varphi^2) \quad (\varphi \text{ large}). \quad (16)$$

Evidently, radiative corrections and renormalization can lead to effective potentials that suggest that the theory is unstable (has complex energy levels).

An early observation that renormalization can cause instability was made by Källén and Pauli [18], who showed that renormalizing the Lee model [19] makes the Hamiltonian complex and that the S-matrix becomes nonunitary because of ghost states. However, a $\mathcal{PT}$-symmetric analysis tames the apparent instabilities of the Lee model; the ghost states disappear, energies are real, and the S-matrix becomes unitary [14]. $\mathcal{PT}$-symmetric quantum theory also repairs the ghost problem in the Pais-Uhlenbeck model [20], the illusory instability of the double-scaling limit of $O(N)$-symmetric $\varphi^4$ theory [21, 22], and difficulties associated with the complex Hamiltonian for timelike Liouville field theory [23].

This article examines three $\mathcal{PT}$-symmetric quantum-mechanical Hamiltonians associated with the two problematic quantum field theories above. (As we have noted, $\mathcal{P}$ denotes parity reflection $x \rightarrow -x$, $p \rightarrow -p$; $\mathcal{T}$ denotes time reversal $x \rightarrow x$, $p \rightarrow -p$, $i \rightarrow -i$ [15]. In order to connect to the standard nomenclature of quantum mechanics, we have replaced $\hat{\varphi}$ by $x$.) The first Hamiltonian,

$$H = p^2 + x^4 \log(ix), \quad (17)$$

is a toy model we developed to study logarithmic $\mathcal{PT}$-symmetric theories. We show that the spectrum of this complex $\mathcal{PT}$-symmetric Hamiltonian is discrete, real, and positive. The second Hamiltonian,

$$H = p^2 - x^4 \log(ix), \quad (18)$$

is the quantum-mechanical analog of (15). We show that the spectrum of this complex and apparently unstable Hamiltonian is also discrete, real, and positive, and this suggests that there is no instability in the supergravity theory of inflation in Ref. [10]. The third Hamiltonian,

$$H = p^2 - x^4 \log(x^2), \quad (19)$$

is motivated by the renormalized effective potential for the Higgs model (16). We show that the ground-state energy of this Hamiltonian is real and positive, and this suggests the intriguing possibility that, contrary to earlier work [24], the Higgs vacuum may be stable.

These three models all have a new $\mathcal{PT}$-symmetric structure that has not previously been examined, namely, the logarithm term in the Hamiltonian. We show that the $\mathcal{PT}$ symmetry of the Hamiltonians (17) and (18) is unbroken; that is, their spectra are entirely real. However, the $\mathcal{PT}$ symmetry of $H$ in (19) is broken; only the four lowest-lying states have real energy. Thus, while the ground state is stable, almost all other states in the theory are unstable. This is indeed what is observed in nature; almost all particles are unstable and there are only a few stable particles. This suggests the conjecture that the Higgs vacuum is stable as a consequence of $\mathcal{PT}$ symmetry and that the universe may be described by a Hamiltonian having a broken $\mathcal{PT}$-symmetry.

The structure of the paper is as follows: in the next section 3 we discuss the toy Hamiltonian (17) and demonstrate that the energy eigenvalues are real, as a consequence of the closed nature of the classical trajectories in the complex plane that characterise the solutions. In section 4
we repeat the analysis for the Hamiltonian (18) with similar conclusions, as far as the reality of the energy eigenvalues is concerned, again as a consequence of the closed trajectories, whilst in section 5 we study the mini-superspace approximation of the one-loop corrected Higgs potential. For the latter case there are differences as far as the classical trajectories are concerned, in that the latter are open. Nevertheless there are four real energy eigenvalues, which also suggests that the Higgs vacuum is stable. Our Conclusions are presented in section 6.

3. Analysis of the toy model Hamiltonian (17)

To analyze \( H \) in (17) we first locate the complex turning points of the particle trajectory. Next, we examine the complex classical trajectories on an infinite-sheeted Riemann surface and find that all these trajectories are closed. This shows that the energy levels are all real [25]. Last, we perform a WKB calculation of the eigenvalues and note that the results agree with a precise numerical determination of the eigenvalues.

The turning points for \( H \) in (17) satisfy the equation

\[
E = x^4 \log(ix),
\]

where \( E \) is the energy (which is dimensionless since it is scaled by the particle mass). We take \( E = 1.24909 \) because this is the numerical value of the ground-state energy obtained by solving the Schrödinger equation for the potential \( x^4 \log(ix) \) (see Table 1).

One turning point lies on the negative imaginary-x axis. To find this point we set \( x = -ir \) (\( r > 0 \)) and obtain the algebraic equation \( E = r^4 \log r \). Solving this equation by using Newton’s method, we find that the turning point lies at \( x = -1.39316i \). To find the other turning points we seek solutions to (20) in polar form \( x = re^{i\theta} \) (\( r > 0, \theta \) real). Substituting for \( x \) in (20) and taking the imaginary part, we obtain

\[
\log r = -(2k\pi + \theta + \pi/2) \cos(4\theta)/\sin(4\theta),
\]

where \( k \) is the sheet number in the Riemann surface of the logarithm. (We choose the branch cut to lie on the positive-imaginary axis.) Using (21), we simplify the real part of (20) to

\[
E = -r^4(2k\pi + \theta + \pi/2)/\sin(4\theta).
\]

We then use (21) to eliminate \( r \) from (22) and use Newton’s method to determine \( \theta \). For \( k = 0 \) and \( E = 1.24909 \), two \( \mathcal{PT} \)-symmetric (left-right symmetric) pairs of turning points lie at \( \pm 0.93803 - 0.38530i \) and at \( \pm 0.32807 + 0.75353i \). For \( k = 1 \) and \( E = 1.24909 \) there is a turning point at \(-0.53838+0.23100i\); the \( \mathcal{PT} \)-symmetric image of this turning point lies on sheet \( k = -1 \) at \( 0.53838 + 0.23100i \).

The turning points determine the shape of the classical trajectories. Two topologically different kinds of classical paths are shown in Figs. 1 and 2. All classical trajectories are closed and left-right symmetric, and this implies that the quantum energies are all real [25].

The WKB quantization condition is a complex path integral on the principal sheet of the logarithm (\( k = 0 \)). On this sheet a branch cut runs from the origin to \( +i\infty \) on the imaginary axis; this choice of branch cut respects the \( \mathcal{PT} \) symmetry of the configuration. The integration path goes from the left turning point \( x_L \) to the right turning point \( x_R \) [26]:

\[
(n + \frac{1}{2}) \pi \sim \int_{x_L}^{x_R} dx \sqrt{E - V(x)} \quad (n >> 1).
\]

If the energy is large (\( E_n \gg 1 \)), then from (21) with \( k = 0 \) we find that the turning points lie slightly below the real axis at \( x_R = re^{i\theta} \) and at \( x_L = re^{-\pi - i\theta} \) with

\[
\theta \sim -\pi/(8 \log r) \quad \text{and} \quad r^4 \log r \sim E.
\]
Figure 1. [Color online] Three nested closed classical paths $x(t)$ (red dashed curves) on the principal sheet (sheet 0) of the Riemann surface plane for energy $E = 1.24909$ and initial conditions $x(0) = -0.6i, -0.7i, -0.8i$ (black dots). The paths do not cross the branch cut on the positive-imaginary axis (solid black line) so they remain on sheet 0. The paths are $\mathcal{PT}$ symmetric (left-right symmetric). Turning points at $\pm 0.938 - 0.385i$ (small circles) cause the paths to turn around in the right-half and left-half plane.

Figure 2. [Color online] Closed classical path for energy $E = 1.24909$. The path begins at $x = -i$ (black dot) on sheet 0 of the logarithmic Riemann surface, moves to the right as a red dashed curve, and visits sheets $-1$ and 1 before returning to its starting point. The path is shown as a solid blue curve on sheet 1 and a dotted purple line on sheet $-1$. Turning points (small circles) at $x = \pm 0.938 - 0.385i, \pm 0.538 + 0.231i, \pm 0.328 + 0.754i$, and $-1.393i$ determine the shape of the path. The total path is $\mathcal{PT}$ symmetric (left-right symmetric).

We choose the path of integration in (23) to have a constant imaginary part so that the path is a horizontal line from $x_L$ to $x_R$. Since $E$ is large, $r$ is large and thus $\theta$ is small. We obtain the simplified approximate quantization condition

$$
(n + \frac{1}{2}) \pi \sim r^3 \log r \int_{-1}^{1} dt \sqrt{1 - t^4},
$$

which leads to the WKB approximation for $n \gg 1$:

$$
\frac{E_n}{[\log(E_n)]^{1/3}} \sim \left[ \frac{\Gamma(7/4)(n + 1/2)\sqrt{\pi}}{\Gamma(5/4)\sqrt{2}} \right]^{4/3}.
$$

To test the accuracy of (26) we have computed numerically the first 13 eigenvalues by solving
the Schrödinger equation for (17). Some of these eigenvalues are listed in Table 1. The accuracy of this WKB approximation increases smoothly with increasing n.

| n  | Numerical value of $E_n$ | $[\log(E_n)]^{1/3}$ | WKB prediction | % error  |
|----|--------------------------|----------------------|----------------|----------|
| 0  | 1.24909                  | 2.06161              | 0.54627        | 73.5028% |
| 3  | 13.7383                  | 9.96525              | 7.31480        | 26.5969% |
| 6  | 31.6658                  | 20.9458              | 16.6979        | 20.2804% |
| 9  | 52.9939                  | 33.4674              | 27.6956        | 17.2463% |
| 12 | 76.9748                  | 47.1776              | 39.9324        | 15.3573% |

Table 1. Eigenvalues of the Hamiltonian in (17) compared with the WKB approximation in (26).

4. Analysis of the supergravity model Hamiltonian (18)

The classical trajectories for the Hamiltonian (18) are plotted in Figs. 3 and 4. Like the classical trajectories for the Hamiltonian (17), these trajectories are closed, which implies that all the eigenvalues for $H$ in (18) are real.

Figure 3. [Color online] Three nested classical trajectories for $H$ in (18) with $E = 2.07734$. The trajectories begin at $-0.8i, -i, -1.2i$ (black dots), do not cross the branch cut on the positive-imaginary axis (solid black line), and are closed.

To find the eigenvalues of the complex Hamiltonian (18) we follow the procedure described in Ref. [15]; to wit, we obtain (18) as the parametric limit $\epsilon : 0 \to 2$ of the Hamiltonian $H = p^2 + x^2(ix)^{\epsilon} \log(ix)$, which has real positive eigenvalues when $\epsilon = 0$. As $\epsilon \to 2$, the Stokes wedges for the time-independent Schrödinger eigenvalue problem rotate into the complex-$x$ plane [15]. Thus, this procedure defines the eigenvalue problem for $H$ in (18) and specifies the energy levels. In Fig. 5 we plot the eigenvalues as functions of $\epsilon$. Note that this figure is topologically identical to Fig. 1 in Ref. [15] except that the ground-state energy diverges at $\epsilon = -2$ rather than at $\epsilon = -1$ (see Ref. [27]). This plot indicates that when $\epsilon < 0$ the $PT$ symmetry is broken, but that when $\epsilon \geq 0$ the $PT$ symmetry is unbroken (all real eigenvalues).

WKB theory gives a good approximation to the eigenvalues of $H$ in (18). We seek turning points for $H$ in polar form $x = re^{i\theta}$ and find that on the principal sheet of the Riemann surface a $PT$-symmetric pair of turning points lies at $\theta = -\pi/4 - \delta$ and $\theta = -3\pi/4 + \delta$. When $E \gg 1$, $\delta$
Figure 4. [Color online] Complex classical trajectory for $H$ in (18) with $E = 2.07734$. The trajectory begins at $0.9i$, crosses the branch cut on the positive-imaginary axis (solid black line), and visits three sheets of the Riemann surface but it is still closed and $\mathcal{PT}$ symmetric (left-right symmetric).

is small, $\delta \sim \pi/(16 \log r)$, and $r$ is large, $r^4 \log r \sim E$. The WKB calculation yields a formula for the eigenvalues that is identical to (26) except that there is no factor of $\sqrt{2}$ in the denominator. Thus, for large $n$ the $n$th eigenvalue of $H$ in (18) agrees approximately with the $n$th eigenvalue of $H$ in (17) multiplied by $2^{2/3}$. A numerical determination of the first six eigenvalues gives $2.07734, 7.9189, 15.4216, 24.0932, 33.7053$, and $44.1189$.

5. Analysis of the Higgs model Hamiltonian (19)
To make sense of the Hamiltonian (19) we again introduce a parameter $\epsilon$ and we define $H$ in (19) as the limit of $H = p^2 + x^2(ix)^\epsilon \log(x^2)$ as $\epsilon : 0 \to 2$. This case is distinctly different

Figure 5. [Color online] Energies of the Hamiltonian $H = p^2 + x^2(ix)^\epsilon \log(ix)$ plotted versus $\epsilon$. This Hamiltonian reduces to $H$ in (18) when $\epsilon = 2$. The energies are real when $\epsilon \geq 0$. 
Figure 6. [Color online] Eigenvalues of the Hamiltonian $H = p^2 + x^2(ix)^\epsilon \log(x^2)$, which reduces to $H$ in (19) when $\epsilon = 2$. There are just four real energies when $\epsilon = 2$.

Figure 7. [Color online] Classical path for the Hamiltonian $H = p^2 - x^4 \log(x^2)$. The initial point is $0.9i$ and the energy is $E = 1.10543$. The trajectory is not $PT$ symmetric. It makes bigger and bigger loops and does not close.

from that for $H$ in (18). Figure 6 shows that the $PT$ symmetry is broken for all $\epsilon \neq 0$. When $\epsilon = 2$, there are only four real eigenvalues: $E_0 = 1.1054311$, $E_1 = 4.577736$, $E_2 = 10.318036$, and $E_3 = 16.06707$. To confirm this result we plot a classical trajectory for $\epsilon = 2$ in Fig. 7. In contrast with Fig. 4, the trajectory is open and not left-right symmetric.

This result suggests that the Higgs vacuum is stable and that perhaps the real world is in a broken $PT$-symmetric regime. This possibility has interesting implications for the $C$ operator in $PT$-symmetric quantum theory. In an unbroken regime the $C$ operator, which is used to construct the Hilbert-space metric with respect to which the Hamiltonian is self-adjoint, commutes with the Hamiltonian and thus it cannot serve as the charge-conjugation operator in particle physics. However, in a broken $PT$ regime, the states of $H$ are not states of $C$, and thus $C$ may play the role of charge conjugation in particle physics [28].
6. Conclusions
In this article, we have discussed two types of field theoretic Hamiltonians that arise in interesting quantum field theory models as a result of quantum fluctuations. These are associated with either supergravity theories with dynamically broken local supersymmetry, or with the Higgs sector of the Standard Model of particle physics. The one-loop effective potential of such theories exhibit an upside-down behaviour or has imaginary parts, which in conventional treatment is interpreted as an instability of the quantum vacuum. Such behaviours arise in many other field theories.

In the approximation of a constant mean field (i.e. the mini-superspace approximation), both effective potentials have been shown to correspond to $\mathcal{P}\mathcal{T}$ symmetric theories, the Higgs one, though, being characterised by a broken $\mathcal{P}\mathcal{T}$ . In the former case, the classical trajectories characterising the solution of the one-dimensional Schroedinger equations are closed, leading to the reality of the entire set of the energy eigenvalues. In the Higgs case, the trajectories are open, and only the subset of the lowest eigenvalues are real. These results suggest that the vacuum in both cases is stable, but a detailed proof of such a conjecture constitutes a long term challenge. Related issues have been discussed in [29, 30]. In the future we will consider how our conclusions may be affected by inclusion of the term $\frac{1}{2}\partial_i\bar{\varphi}\partial_i\varphi$ in the effective potential.

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[1] Coleman S R and Weinberg E J 1973 Phys. Rev. D 7 1888
[2] Jackiw R 1974 Phys. Rev. D 9 1686
[3] Haymaker R W and Perez-Mercader J 1983 Phys. Rev. D 27 1948
[4] Iliopoulos J, Itzykson C and Martin A 1975 Rev. Mod. Phys. 47 165
[5] Meissner K A and Nicolai H 2007 Phys. Lett. B 648 312
[6] Meissner K A and Nicolai H 2009 Acta Phys. Polon. B 40 2737
[7] Toms D J 2007 The Schwinger Action Principle and Effective Action (New York: Cambridge University Press)
[8] Sher M 1989 Phys. Rept. 179, 273
[9] Isidori G, Ridolfi G and Strumia A 2001 Nucl. Phys. B 609 387
[10] Alexandre J, Houston N and Mavromatos N E 2013 Phys. Rev. D 88 125017 and 2015 Int. J. Mod. Phys. D 24 1541004
[11] Misner C W 1972 Minisuperspace in Magic Without Magic, ed. J R Klauder (Freeman, San Francisco); Misner C W 1991 Quantum cosmology and baby universes, Proc. 7th Jerusalem Winter School for Theoretical Physics (Jerusalem, Israel, 27 December 1989-4 January ) ed S R Coleman, J B Hartle, T Piran and S Weinberg (Singapore: World Scientific)
[12] Frankel S and Metropolis N 1947 Phys. Rev. 72 914 ; Foland W D and Present R D 1959 Phys. Rev. 113 613
[13] Ellis J 2015 J. Phys. Conf. Ser. 631, no. 1, 012001 [arXiv:1501.05418 [hep-ph]].
[26] Bender C M and Orszag S A 1978 *Advanced Mathematical Methods for Scientists and Engineers* (New York: McGraw Hill)

[27] The spectrum of $H = p^2 - ix$ is null; see I. Herbst 1979 Commun. Math. Phys. 64, 279. This is because there are precisely three Stokes wedges of angular opening 120°. If the solution to the Schrödinger equation vanishes exponentially in one wedge, it grows exponentially in the adjacent two wedges and thus no eigenvalue condition is possible. The branch cut allows the Hamiltonian $H = p^2 - ix \log(ix)$ to evade this constraint; there are infinitely many Stokes wedges on the Riemann surface.

[28] Bender C M and Mannheim P D 2010 Phys. Lett. A 374, 1616

[29] Guralnik G and Guralnik Z 2010 Annals Phys. 325 2486

[30] Nagao K and Nielsen H B 2011 Prog. Theor. Phys. 126 1021