M, B AND CO₁ ARE RECOGNISABLE BY THEIR PRIME GRAPHS

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Abstract. The prime graph, or Gruenberg–Kegel graph, of a finite group $G$ is the graph $\Gamma(G)$ whose vertices are the prime divisors of $|G|$, and whose edges are the pairs $\{p, q\}$ for which $G$ contains an element of order $pq$. A finite group $G$ is recognisable by its prime graph if every finite group $H$ with $\Gamma(H) = \Gamma(G)$ is isomorphic to $G$. By a result of Cameron and Maslova, every such group must be almost simple, so one natural case to investigate is that in which $G$ is one of the 26 sporadic simple groups. Existing work of various authors answers the question of recognisability by prime graph for all but three of these groups, namely the Monster, M, the Baby Monster, B, and the first Conway group, Co₁. We prove that these three groups are recognisable by their prime graphs.

1. Introduction

The Gruenberg–Kegel graph of a finite group $G$, introduced in an unpublished 1975 manuscript of Karl W. Gruenberg and Otto H. Kegel, is the labelled graph $\Gamma(G)$ whose vertices are the prime divisors of $|G|$, and whose edges are the pairs $\{p, q\}$ for which $G$ contains an element of order $pq$. It is also called the prime graph of $G$, and we use this name here for brevity. A finite group $G$ is recognisable by its prime graph if every finite group $H$ with $\Gamma(H) = \Gamma(G)$ is isomorphic to $G$. More generally, $G$ is $k$-recognisable by its prime graph if there are precisely $k$ pairwise non-isomorphic groups with the same prime graph as $G$. If no such $k$ exists, then $G$ is unrecognisable by its prime graph.

The question of recognisability of various groups by their prime graphs has attracted significant interest. We refer the reader to the recent article of Cameron and Maslova [3] for an up-to-date review of the literature (and several new results). Much work has focused on simple groups, and this is justified by [3, Theorem 1.2], which states, in particular, that if $G$ is $k$-recognisable by its prime graph for some $k$, then $G$ is almost simple, i.e. $G_0 \leq G \leq \text{Aut}(G_0)$ for some non-abelian simple group $G_0$. It is therefore natural to ask, in particular, which of the 26 sporadic simple groups are recognisable by their prime graphs. As summarised in Table 1 this question has been answered for all but three of these groups: the Monster, M, the Baby Monster, B, and the first Conway group, Co₁. The purpose of this note is to settle these remaining cases. We prove the following theorem.

Theorem 1.1. M, B and Co₁ are recognisable by their prime graphs.

2. Proof of Theorem 1.1 — Co₁

Recall that $|\text{Co}_1| = 2^{21} \cdot 3^9 \cdot 5^4 \cdot 72 \cdot 11 \cdot 13 \cdot 23$. The vertex set of $\Gamma(\text{Co}_1)$ is therefore $\{2, 3, 5, 7, 11, 13, 23\}$, and we see from the ATLAS [4] that $\Gamma(\text{Co}_1)$ has two connected components, one of which is the isolated vertex 23. We also recall from the ATLAS [4] that $\text{Co}_1$ contains both $\text{Co}_3$ and $(A_7 \times L_2(7)) : 2$ as maximal subgroups. (Here we follow the ATLAS [4] and use Artin’s notation “L” for “PSL”, e.g. $L_2(7) = \text{PSL}_2(7)$.)

Our argument begins along the lines of Kondrat’ev’s proofs [9, 10], which in turn rely on earlier work of Hagie [7]. Suppose that $G$ is a finite group with $\Gamma(G) = \Gamma(\text{Co}_1)$. Then [7, Theorem 3] implies that $G / F(G) \cong \text{Co}_1$, and that the prime divisors of the Fitting
In particular, such an element does not other than $23$ on every faithful irreducible module in characteristic $p$. Proposition 2.1. Let $V$ be a faithful irreducible module for $G$ in any characteristic other than $23$. Then every element of order $23$ in $G$ fixes some point of $V$.

First consider $p = 3$. An element of order $23$ in $G$ normalises a subgroup of order $2^{11}$ (see [4]), so it follows from [6] Theorem 3.4.4 that such an element has a fixed point on every faithful $F$-module, for every field $F$ of characteristic not equal to $2$ or $3$. In particular, such an element does not act fixed-point freely on any faithful irreducible $F$-module defined over a field $F$ of characteristic $3$. Therefore, $O_3(G)$ must be trivial.
Now consider \( p = 2 \). Suppose that \( V \) is a faithful irreducible module for \( \text{Co}_3 \) in characteristic 2, and let \( \chi \) be the corresponding Brauer character. Consider a maximal subgroup \( \text{Co}_3 \) of \( \text{Co}_1 \). According to the 2-modular character table of \( \text{Co}_3 \), which is available in \texttt{GAP} [2, 5], the only irreducible module for \( \text{Co}_3 \) in characteristic 2 on which elements of order 23 act fixed-point freely is the module, call it \( W \), of dimension 22. Therefore, each of the \( k \) composition factors of the restriction \( V \downarrow \text{Co}_3 \) of \( V \) to \( \text{Co}_3 \) is isomorphic to \( W \), and so the restriction of \( \chi \) to \( \text{Co}_3 \) is \( k \) times the Brauer character of \( W \). Now consider a maximal subgroup \( H = (A_7 \times L_2(7)) : 2 \) of \( \text{Co}_1 \). Note that there are exactly 8 conjugacy classes of elements of odd order in \( \text{Co}_3 \) that intersect both \( \text{Co}_3 \) and \( H \). These classes are listed in Table 2 together with the corresponding values of the hypothesised 2-modular Brauer character \( \chi \) and the corresponding classes in \( \text{Co}_3 \) and \( H \). (Note that the class fusions can be readily determined in \texttt{GAP} [2, 5] by applying the functions \texttt{FusionConjugacyClasses} and \texttt{ClassNames} to the relevant ordinary character tables.)

Now, consider that the restriction of \( \chi \) to \( H \) must be a linear combination of the irreducible 2-modular Brauer characters for \( H \), with non-negative integer coefficients. Using the 2-modular character table of \( H \), which is also available in \texttt{GAP} [2, 5], and comparing with the final column of Table 2 we obtain a system of equations for the multiplicities of the characters appearing in this restriction. There are 8 equations in 16 unknowns: one equation for each of the 8 conjugacy classes of \( H \) appearing in Table 2 and one variable for each of the irreducible 2-modular Brauer characters for \( H \). Explicitly, if we label these variables \( x_1, \ldots, x_{16} \) according to the numbering of the characters for \( H \) used in \texttt{GAP} [2, 5], then we obtain the linear system with augmented matrix

\[
\begin{bmatrix}
1 & 8 & 6 & 14 & 20 & 6 & 24 & 24 & 36 & 84 & 120 & 8 & 64 & 48 & 112 & 160 & 22k \\
1 & 2 & -1 & -1 & 6 & 6 & 6 & -6 & -6 & 8 & 16 & 8 & -8 & -8 & 4k & 6k \\
1 & 8 & 6 & 14 & 20 & - & - & - & -1 & -8 & -6 & -14 & -20 & -2k & -2k & -2k \\
1 & -4 & 3 & 2 & -4 & - & - & - & -1 & 4 & -3 & -2 & 4 & -2k & -2k & -2k \\
1 & 2 & -1 & -1 & - & - & - & -1 & -2 & - & 1 & 1 & -2k & -2k & -2k & -2k \\
1 & 1 & -1 & -1 & 6 & 3 & 3 & -6 & -6 & 8 & 8 & -8 & -8 & k & k & k \\
1 & 1 & -1 & -1 & -1 & -4 & 3 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & k \\
1 & 1 & -1 & -1 & - & - & - & -1 & -1 & 1 & 1 & 1 & -2 & -2 & -2 & -2 & -2 & -2
\end{bmatrix},
\]

where \( \cdot \) denotes 0. As explained below, this system has no non-negative integer solutions \((x_1, \ldots, x_{16})\) for any positive integer \( k \), so the hypothesised module \( V \) for \( \text{Co}_1 \) does not exist. Therefore, \( O_2(G) \) is also trivial, so \( F(G) \) is trivial and hence \( G \cong \text{Co}_1 \), as claimed.

It remains to verify that the aforementioned linear system has no non-negative integer solutions. It suffices to consider \( k = 1 \) and check that there are no non-negative rational solutions. When \( k = 1 \), the augmented matrix given above is row equivalent to

\[
\begin{bmatrix}
1 & -2 & -1 & 2 & -2 \\
1 & 1 & -1 & -1 & -1 \\
-1 & -1 & -1 & -1 \\
-1 & -1 & -1 & -1 & -1 \\
-2 & -2 & 3/2 & 3/2 & 3/2 \\
-3 & 1 & 1 & 1 & 1 \\
-3 & 1 & 1 & 1 & 1 \\
3/2 & 1/2 & 1/2 & 1/2 & 1/2 \\
1 & 1 & 1 & 1 & 1 \\
3/2 & 3/2 & 3/2 & 3/2 & 3/2
\end{bmatrix}.
\]

Given that all of \( x_1, \ldots, x_{16} \) should be non-negative, the final row immediately implies that \( x_{10} = x_{11} = x_{15} = x_{16} = 0 \), and then the fourth row yields \( x_4 = x_5 = 0 \). The first row then implies that \( x_{12} = x_1 + 2 \), and in particular \( x_{12} \geq 2 \). Putting \( x_{12} \geq 2 \) and \( x_{11} = x_{16} = 0 \) into the second-last row then forces \( x_8 + x_9 + 2x_{13} + x_{14} \leq 0 \), so all of \( x_8 \), \( x_9 \), \( x_{13} \) and \( x_{14} \) must also vanish. Looking again at the second-last row, it follows that \( x_{12} = 2 \). However, the fifth row then implies that \( x_5 = -3 < 0 \).
The page contains a mathematical discussion about the properties of groups and modules. It includes a table and several theorems and proofs. Here is a structured summary:

1. **Classification of Elements**
   - The page begins with a discussion on the classification of elements of order 4 in a group \( H \), specifically involutions. The notation \( \rho \) is used to denote the union of these classes, and \( C \) represents the set of elements admitting a faithful irreducible module.

2. **Module Theory**
   - It is mentioned that elements of order 41 act fixed-point freely on a module. The page notes that the normal subgroup of order 3 divides the highest power of 3 dividing the order of the group, and that the module restricts to the subgroup \( O_{\rho} \).

3. **Theorem Proof**
   - The proof of Theorem 1.1 involves the 3-modular character table of the orthogonal group \( O_{\rho} \). It is noted that the prime graph of the group \( G \) has three isolated vertices, namely 41, 59, and 71. The page uses GAP to check these properties.

4. **Table**
   - A table is provided showing the conjugacy classes of odd-order elements in \( Co_1 \) that intersect both of the maximal subgroups \( Co_3 \) and \( (A_7 \times L_2(7)) : 2 \). The table includes columns for class names, sizes, and character values.

The page continues with additional mathematical content, including more proofs and discussions on the properties of these groups.
to the class 4B in $M$, because this is the only class of elements of order 4 in $M$ that square to elements of 2A. Next, we define a homomorphism $\varphi : H \to H/O_3(H) \cong O^-_8(3).2_3$ using the LMGRadicalQuotient function in MAGMA [1], and check that there exist classes in $C$ that project under $\varphi$ to the classes 4A and 4D in $O^-_8(3).2_3$. Let us fix labels 4A’ and 4D’ for these two classes, so that 4A’ projects to 4A and 4D’ projects to 4D. (Note that two of the classes in $C$ project to 4A, but the following argument does not depend on which one we choose to work with. The final class projects to 4E.)

Let $\rho$ denote the restriction to $H$ of the 3-modular Brauer character corresponding to the hypothesised module $V$ for $M$. Then $\rho$ must be a linear combination with non-negative integer coefficients of the characters of the aforementioned 8-, 56- and 104-dimensional modules for $H$. Label these characters and their associated coefficients by $\rho_d$ and $x_d$, respectively, for each $d \in \{8, 56, 104\}$. Since both of the classes 4A’ and 4D’ in $H$ belong to the same class (4B) in $M$, we must have $\rho(4A’) = \rho(4D’)$. Since they project under the homomorphism $\varphi$ to the classes 4A and 4D in $O^-_8(3).2_3$, and (as explained above) the 3-modular character tables of $H$ and $O^-_8(3).2_3$ coincide, we must have

$$\rho_8'(4A)x_8 + \rho_{56}'(4A)x_{56} + \rho_{104}'(4A)x_{104} = \rho_8'(4D)x_8 + \rho_{56}'(4D)x_{56} + \rho_{104}'(4D)x_{104},$$

where $\rho_d'$ denotes the 3-modular Brauer character of $O^-_8(3).2_3$ corresponding (via $\varphi$) to $\rho_d$. Since the classes 4A and 4D in $O^-_8(3).2_3$ lie in $O^-_8(3)$, and (as explained above) the 8-, 56- and 104-dimensional representations of $O^-_8(3)$ extend to $O^-_8(3).2_3$, we can read off the character values in the above equation from the 3-modular character table of $O^-_8(3)$ (which is available in GAP [2][5]) to deduce that

$$-6x_8 - 26x_{56} - 38x_{104} = -2x_8 + 2x_{56} - 2x_{104}, \quad \text{i.e.} \quad 4x_8 + 28x_{56} + 36x_{104} = 0.$$ 

Given that this equation has no (non-trivial) non-negative integer solutions $(x_8, x_{56}, x_{104})$, the module $V$ does not exist. Therefore, $O_3(G)$ is trivial, and so $G \cong M$.

4. Proof of Theorem 1.1 — B

Recall that $|B| = 2^{31} \cdot 3^{13} \cdot 5^6 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23 \cdot 31 \cdot 47$. We see from the ATLAS [4] that the prime graph $\Gamma(B)$ has two isolated vertices, namely 31 and 47, and that all of its other vertices lie in a connected component. We also recall that from the ATLAS [4] that $B$ contains amongst its maximal subgroups the groups $L_2(31), 47 : 23$ and $2^{1+22}.Co_2$, and a group of shape $[2^{30}]L_5(2)$ (an extension of $L_5(2)$ by a certain group of order $2^{30}$.)

The strategy here is similar to that of the preceding two sections, but the argument is a little more involved. Suppose that $G$ is a finite group with $\Gamma(G) = \Gamma(B)$. Then [7] Theorem 3] implies that $G/F(G) \cong B$ and that $F(G)$ is either trivial or a 2-group. We must show that $F(G)$ is trivial, so it suffices to show that $O_2(G)$ is trivial. Supposing towards a contradiction that $O_2(G)$ is non-trivial, we infer that $B$ must admit a faithful irreducible module $V$ in characteristic 2 on which every element of order 31 or 47 acts fixed-point freely. Let $\chi$ denote the corresponding Brauer character. We first consider restricting $V$ to the maximal subgroups $L_2(31)$ and $[2^{30}]L_5(2)$ of $B$, which contain elements of order 31, and the maximal subgroup $47 : 23$, which contains elements of order 47. This allows us to deduce the values of $\chi$ on various conjugacy classes that intersect the maximal subgroup $2^{1+22}.Co_2$. Since $2^{1+22}.Co_2$ has relatively few classes of odd-order elements, it turns out that we can impose sufficiently many constraints to establish the non-existence of $V$.

We first consider $V \downarrow L_2(31)$. By the 2-modular character table of $L_2(31)$, the only irreducible modules for $L_2(31)$ in characteristic 2 on which elements of order 31 act fixed-point freely are the two modules of dimension 15. Let $\chi_2$ and $\chi_3$ denote the corresponding Brauer characters, for consistency with the numbering in the character table as given in
GAP [2 5]. Then there exist non-negative integers $k_2$ and $k_3$ (not both zero) such that
\begin{equation}
\chi|_{L_2(31)} = k_2 \chi_2 + k_3 \chi_3.
\end{equation}
Moreover, $\chi_2$ and $\chi_3$ vanish on all elements of order 3, 5 and 15 in $L_2(31)$, and all such elements belong to the conjugacy classes 3B, 5B and 15B in B, respectively, so
\begin{equation}
\chi(3B) = \chi(5B) = \chi(15B) = 0.
\end{equation}

Next, we claim that $k_2 = k_3$. To prove this, consider one of the two conjugacy classes of elements of order 31 in $L_2(31)$, say the class 31A, which belongs to the class 31A in B. Checking the values of $\chi_2$ and $\chi_3$ on 31A elements in $L_2(31)$, we infer that the value of $\chi$ on 31A elements in B must be
\begin{equation}
\chi(31A) = k_2 c + k_3 \overline{c} \quad \text{where} \quad c = \frac{-1 + \sqrt{-31}}{2}
\end{equation}
and $\overline{c}$ denotes the complex conjugate of $c$. Now consider $V \downarrow H$, where $H = [2^{30}] \cdot L_5(2)$. There are 16 irreducible 2-modular Brauer characters for $H$. Label them $\rho_1, \ldots, \rho_{16}$ in accordance with the numbering in the 2-modular character table of $H$ in GAP [2 5], and let $x_1, \ldots, x_{16}$ denote the corresponding multiplicities in the restriction of $\chi$ to $H$. The requirement that all elements of order 31 in $H$ act fixed-point freely on $V$ implies that $x_i = 0$ for all $i \in \{1, 6, 11, 12, 13, 14, 15, 16\}$. In particular, we have
\begin{equation}
\chi|_H = \sum_{i \in I} x_i \rho_i \quad \text{where} \quad I = \{2, 3, 4, 5, 7, 8, 9, 10\}.
\end{equation}

Note that the remaining possibilities for the composition factors of $V \downarrow H$ are the two modules of dimension 5, corresponding to $i \in \{2, 3\}$, the two modules of dimension 10, corresponding to $i \in \{4, 5\}$, and the four modules of dimension 40, corresponding to $i \in \{7, 8, 9, 10\}$. Now, consider the class 31A in $H$. Elements in this class also belong to the class 31A in B, so we can use [1] to obtain an expression for $\chi(31A)$ in terms of the known values of the characters $\rho_i$, $i \in I$, on 31A elements of $H$, and the unknown multiplicities $x_i$, $i \in I$. The values of the $\rho_i$ on 31A elements of $H$ are all integer linear combinations of primitive 31st roots of unity, and so too are the complex numbers $c$ and $\overline{c}$ in [5]. Hence, equating the right-hand sides of (3) and (4) (with the latter applied to 31A elements of $H$), we obtain an equality between two integer linear combinations of primitive 31st roots of unity; one involving the unknowns $k_2$ and $k_3$, the other involving the unknowns $x_i$, $i \in I$. Since the primitive 31st roots of unity are linearly independent over $\mathbb{Z}$, we can regard this equality as a system of 30 linear equations (one per root) in the eight unknowns $x_i$, $i \in I$. It turns out that only six of these equations are distinct; explicitly, we obtain the linear system with augmented matrix
\[
\begin{bmatrix}
1 & \cdots & 1 & \cdots & 1 & \cdots & 2 & k_2 \\
\cdots & 1 & 1 & 2 & 1 & 1 & k_3 \\
\cdots & 1 & \cdots & 2 & 2 & 1 & 2 & k_2 \\
\cdots & 1 & 2 & 1 & 1 & 1 & k_2 \\
\cdots & 1 & 2 & 1 & 1 & 1 & k_2 \\
\cdots & 1 & \cdots & 2 & 2 & 2 & 1 & k_3 \\
\cdots & 1 & \cdots & 2 & 2 & 2 & 1 & k_3 \\
\cdots & 1 & \cdots & 1 & \cdots & 1 & \cdots & k_3 - k_2 \\
\end{bmatrix}
\sim
\begin{bmatrix}
1 & \cdots & \cdots & \cdots & \cdots & 1 & 1 & k_3 \\
\cdots & 1 & \cdots & \cdots & \cdots & 1 & 1 & k_2 \\
\cdots & \cdots & 1 & \cdots & \cdots & 1 & -1 & k_2 \\
\cdots & \cdots & 1 & \cdots & \cdots & 1 & -1 & k_3 \\
\cdots & \cdots & \cdots & 1 & \cdots & 1 & 1 & k_2 - k_3 \\
\cdots & \cdots & \cdots & \cdots & 1 & 1 & 1 & k_3 - k_2 \\
\end{bmatrix},
\]
where $\cdot = 0$ and $\sim$ denotes row equivalence. Since the $x_i$ must be non-negative integers, it follows that
\begin{equation}
x_7 = x_8 = x_9 = x_{10} = 0 \quad \text{and} \quad x_2 = x_3 = x_4 = x_5 = k_2 = k_3.
\end{equation}
In particular, $k_2 = k_3$ as claimed.

Let us therefore write $k = k_2 = k_3$. Note first that (1) then reads $\chi|_{L_2(31)} = k(\chi_2 + \chi_3)$, and so the value of $\chi$ on the identity class 1A of B is
\begin{equation}
\chi(1A) = 30k,
\end{equation}
Table 3. Some conjugacy classes of odd-order elements in B that intersect the maximal subgroup \(2^{1+22}.\text{Co}_2\), the corresponding classes in \(2^{1+22}.\text{Co}_2\), and the corresponding values of the 2-modular Brauer character \(\chi\) of the hypothesised module \(V\) for B in Section 4, per (2), (6), (8), (9) and (10).

\[
\begin{array}{ccc}
\text{B} & 2^{1+22}.\text{Co}_2 & \chi \\
1A & 1A & 30k \\
3A & 3B & 6k \\
3B & 3A & 0 \\
5B & 5A & 0 \\
7A & 7A & 2k \\
15B & 15B, 15C & 0 \\
23A & 23A & 0 \\
23B & 23B & 0 \\
\end{array}
\]

because (as noted above) the modules for \(L_2(31)\) corresponding to the characters \(\chi_2\) and \(\chi_3\) both have dimension 15. Now apply (5) also to (4). This yields

\[
\chi|_H = k(\rho_2 + \rho_3 + \rho_4 + \rho_5). \tag{7}
\]

In other words, \(V\) restricts to \(H\) as \(k\) copies of each of the two 5-dimensional and two 10-dimensional modules for \(H\). Next, consider either of the classes 7A and 7B in \(H\), both of which belong to the class 7A in B. By (7) and the 2-modular character table of \(H\),

\[
\chi(7A) = k \left( \frac{3 + \sqrt{-7}}{2} + \frac{3 - \sqrt{-7}}{2} + \frac{-1 + \sqrt{-7}}{2} + \frac{-1 - \sqrt{-7}}{2} \right) = 2k. \tag{8}
\]

Similarly, the class 3A in \(H\) belongs to the class 3A in B, so

\[
\chi(3A) = k(2 + 2 + 1 + 1) = 6k. \tag{9}
\]

Finally, we claim that

\[
\chi(23A) = \chi(23B) = 0. \tag{10}
\]

This is easily seen by considering the restriction of \(V\) to \(47 : 23\). Since elements of order 47 in B must act fixed-point freely on \(V\), the only possible composition factors of \(V \downarrow 47 : 23\) are the two modules of dimension 23, and the Brauer characters of these modules vanish on all elements of order 23 in \(47 : 23\). In particular, \(H\) must satisfy (10).

We now complete the proof by considering \(V \downarrow K\), where \(K = 2^{1+22}\text{.Co}_2\). Table 3 summarises the supposed values of \(\chi\) on various conjugacy classes of \(B\), per (2), (6), (8), (9) and (10), and lists the corresponding classes in \(K\). Now, \(K\) has only 13 irreducible 2-modular Brauer characters. Denote the associated coefficients in \(\chi|_K\) by \(x_1, \ldots, x_{13}\), following the numbering in the 2-modular character table of \(K\) in \textit{GAP} [2, 5]. Using that character table and Table 3 we obtain a system of nine equations (one per \(K\)-class appearing in Table 3) in the 13 unknowns \(x_1, \ldots, x_{13}\), with augmented matrix

\[
\begin{bmatrix}
1 & 22 & 230 & 748 & 748 & 3520 & 5312 & 6802 & 8602 & 36938 & 83948 & 83948 & 156538 & 1835008 & 30k \\
1 & -5 & 14 & -8 & -8 & -44 & 20 & 43 & 43 & -79 & -22 & 100 & -128 & . \\
1 & 4 & 5 & 1 & 1 & 10 & 2 & -20 & -20 & -52 & -58 & -80 & -128 & 6k \\
1 & -3 & 5 & -2 & -2 & -5 & 12 & 2 & 2 & 13 & -2 & -12 & 8 & . \\
1 & 1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -3 & -3 & -3 & . & 2k \\
1 & -1 & a & \overline{a} & 1 & -a & -\overline{a} & 1 & -2 & . & 2 & . \\
1 & -1 & \overline{a} & a & 1 & -\overline{a} & -a & 1 & -2 & . & 2 & . \\
1 & -1 & b & \overline{b} & 1 & -1 & . & . & -2 & . & -1 & . \\
1 & -1 & \overline{b} & b & 1 & -1 & . & . & -2 & . & -1 & .
\end{bmatrix}
\]
where \( \cdot = 0 \) and

\[
a = \frac{-1 + \sqrt{-15}}{2}, \quad b = \frac{1 - \sqrt{-23}}{2}.
\]

This system has no non-negative integer solutions \((x_1, \ldots, x_{13})\) for any positive integer \(k\) (see below), so the hypothesised module \(V\) for \(B\) does not exist. Therefore, \(O_2(G)\) is trivial, so \(F(G)\) is trivial and hence \(G \cong B\), as claimed.

It remains to verify that the aforementioned linear system has no non-negative integer solutions. It suffices to consider \(k = 1\) and check that there are no non-negative rational solutions. When \(k = 1\), the augmented matrix given above is row equivalent to

\[
\begin{bmatrix}
1 & \cdots & \cdots & \cdots & \cdots & 12 & 20 & 34 & 468 & 19982/9315 \\
1 & \cdots & \cdots & \cdots & \cdots & 7 & 20 & 36 & 478 & 17422/9315 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \ddots & \ddots & \ddots & \ddots \\
1 & \cdots & \cdots & \cdots & \cdots & 23 & 35 & 49 & 645 & 1982/1035 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
1 & \cdots & \cdots & \cdots & \cdots & 23 & 35 & 49 & 645 & 1982/1035 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
1 & \cdots & \cdots & \cdots & \cdots & 16 & 21 & 27 & 376 & 3074/3105 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
1 & \cdots & \cdots & \cdots & \cdots & 1 & -2 & -1 & 1 & -7 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
1 & \cdots & \cdots & \cdots & \cdots & 1 & -2 & -1 & 1 & -7 \\
\end{bmatrix}
\]

Given that we need \(x_7 \geq 0\), the third-last row forces \(x_{10} < 1/16\), \(x_{11} < 1/21\) and \(x_{13} < 1/376\). Putting this and \(x_{12} \geq 0\) into the last row yields \(x_9 < 0\).

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