Abstract
The generalisation of continuous orthogonal polynomial ensembles from random matrix theory to the $q$-lattice setting is considered. We take up the task of initiating a systematic study of the corresponding moments of the density from two complementary viewpoints. The first requires knowledge of the ensemble average with respect to a general Schur polynomial, from which the spectral moments follow as a corollary. In the case of little $q$-Laguerre weight, a particular $3\phi_2$ basic hypergeometric polynomial is used to express density moments. The second approach is to study the $q$-Laplace transform of the un-normalised measure. Using integrability properties associated with the $q$-Pearson equation for the $q$-classical weights, a fourth-order $q$-difference equation is obtained, generalising a result of Ledoux in the continuous classical cases.

Keywords
$q$-Pearson pair · Schur polynomials · Askey scheme · Orthogonal polynomials · $q$-Moments in random matrices

Mathematics Subject Classification
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1 Introduction

1.1 Special properties of moments for classical ensembles with unitary symmetry

An ensemble of $N \times N$ complex Hermitian random matrices $\{H\}$ is said to have unitary symmetry if the probability density function (PDF) has the invariance property $P(U^{-1}HU) = P(H)$ for all unitary matrices $U \in U(N)$. A structurally simple example is when, up to proportionality, $P(H)$ has this invariance and is given by $\prod_{i=1}^{N} w(x_i)$, where $\{x_i\}_{i=1}^{N}$ are the eigenvalues of $H$ and $w(x)$ is referred to as the weight function. In this circumstance, up to normalisation, the corresponding eigenvalue PDF is given by (see [20, Prop. 1.3.4])

$$\prod_{i=1}^{N} w(x_i) \prod_{1 \leq j < k \leq N} |x_k - x_j|^2.$$  \hfill (1.1)

The class of PDFs of the form (1.1) are closely related to orthogonal polynomials. Thus let $\{p_k(x)\}_{k=0,1,...}$, where each $p_k(x)$ is a monic polynomial of degree $k$, have the orthogonality

$$\int_I w(x) p_j(x) p_k(x) \, dx = h_j \delta_{j,k},$$  \hfill (1.2)

where $I$ denotes the support of $w(x)$, and $h_j > 0$ is the normalisation. It is a standard result in random matrix theory (see [20, Prop. 5.1.2]) that

$$\rho(k),N(x_1, \ldots, x_k) = \det[K_N(x_j, x_l)]_{j,l=1}^{k},$$  \hfill (1.3)

where

$$K_N(x, y) = \left( w(x)w(y) \right)^{1/2} \sum_{j=0}^{N-1} \frac{p_j(x)p_j(y)}{h_j}.$$  \hfill (1.4)

Here $\rho(k),N$ denotes the $k$-point correlation function, obtained by integrating a suitably normalised version of (1.1) over all variables except $x_1, \ldots, x_k$. In the case $k = 1$ this corresponds to the eigenvalue density, and we read off from (1.3) and (1.4) that

$$\rho(1),N(x) = w(x) \sum_{j=0}^{N-1} \frac{(p_j(x))^2}{h_j}.$$  \hfill (1.5)

A distinguished class of weight functions have the property that their logarithmic derivative can be written in the particular rational function form [11]

$$\frac{d}{dx} \log w(x) = \frac{a_0 + a_1 x}{b_0 + b_1 x + b_2 x^2},$$  \hfill (1.6)
for some polynomials of degree less than or equal to one in the numerator and two in the denominator, as indicated. Requiring too that \( w(x) \) has all moments finite leaves just three possibilities, up to affine scalings,

\[
w^{(G)}(x) = e^{-x^2}, \quad w^{(L)}(x) = x^a e^{-x} \chi_{x>0}, \quad w^{(J)}(x) = x^a (1-x)^b \chi_{0<x<1}, \quad (1.7)
\]

referred to as the Gaussian, Laguerre and Jacobi weights, respectively. The notation \( \chi_A \) used in (1.7) is the indicator function of the condition \( A \), which takes on the value one for \( A \) true, and zero otherwise. For the parameters in the Laguerre and Jacobi case we require \( a, b > -1 \) for the weights to be normalisable. Note that the corresponding orthogonal polynomials in (1.2) are, up to normalisation, the classical Hermite, Laguerre and Jacobi polynomials [56]. The corresponding (classical) random matrix ensembles with eigenvalue PDF (1.1) are given the names GUE, LUE and JUE, respectively, with the first letter corresponding to the weight, and UE denoting unitary ensemble. Here unitary is used in the context of the symmetry noted in the first paragraph.

For a non-negative integer \( k \), the moments associated with the spectral density are defined by

\[
m_{k,N} = \int_I x^k \rho_{(1),N}(x) \, dx. \quad (1.8)
\]

There are special properties associated with the moments in the case of the classical ensembles. Consider for example the GUE. It is a long established result in the applications of random matrices, due to Harer and Zagier [32], that the moments (1.8) have combinatorial and topological significance. This comes about through the large \( N \) terminating expansion

\[
\frac{2^k}{N^{1+2k}} m_{2k,N}^{(G)} = \sum_{g=0}^{\lfloor k/2 \rfloor} \frac{c(g; k)}{N^{2g}}, \quad (1.9)
\]

It was shown in [32] that the coefficients \( c(g; k) \) can be specified as the number of pairings of the edges of a \( 2k \)-gon dual to a map on a surface of genus \( g \). The leading coefficient in (1.9) is given by

\[
c(0; k) = \frac{1}{k+1} \binom{2k}{k}, \quad (1.10)
\]

which is the \( k \)-th Catalan number.

An analogous expansion to (1.9) holds for the LUE with Laguerre parameter \( a = \alpha N \) [14, 16]. The former of these references involves the so-called double monotone Hurwitz numbers, which have both combinatorial and topological significance. The recent work [28] considers the coefficients in the \( 1/N^2 \) expansion of the moments for the ensemble (1.1) in the case of the JUE with Jacobi parameters \( a = \alpha_1 N, b = \alpha_2 N \). Unlike the Gaussian and Laguerre cases, this expansion no longer terminates. It is
shown that the coefficients can be expressed in terms of triple monotone Hurwitz numbers.

Another significant feature of the moments of the GUE and LUE is that they permit evaluations in terms of hypergeometric polynomials. Thus for the GUE [59, Eq. (4.33)]

$$
\frac{2^k}{N} \int_{-\infty}^{\infty} |x|^{2k} \rho_{(1),N}^{(G)}(x) \, dx = \frac{\Gamma(2k+1)}{\Gamma(k+1)} \, \, _2F_1\left(\frac{-k,1-N}{2} \Big| 2\right). \tag{1.11}
$$

This result in fact remains valid for complex moments \( \text{Re}k > -1/2 \) [15]. In the case of the LUE, it is known [15, Eq. 4.11]

$$
m_{k,N}^{(L)} = N(N+a) \frac{(k+a)!}{(1+a)!} \, 3F_2\left(\frac{1-k,2+k,1-N}{2,2+a} \Big| 1\right), \tag{1.12}
$$

which also extends to continuous \( k \). For the JUE case, an analogous hypergeometric expression for differences of consecutive moments is also known [15, Eq. 4.19].

### 1.2 q-classical ensembles with unitary symmetry and their moments

It is well known that the classical polynomials admit \( q \)-generalisations catalogued according to the Askey scheme; see e.g. [34, 39]. Distinguishing the weight functions for classical \( q \)-orthogonal polynomials is that they satisfy the \( q \)-Pearson equation

$$
D_q(\sigma(x)w(x)) = \tau(x)w(x), \quad D_q f(x) = \frac{f(x) - f(qx)}{(1-q)x}, \tag{1.13}
$$

where it is required that \( \sigma(x) \) be a polynomial of degree less than or equal to 2, and \( \tau(x) \) be a polynomial of degree less than or equal to 1. Note that with \( D_q \) replaced by a derivative with respect to \( x \), (1.13) is equivalent to (1.6). Our interest in the present article is in the moments of the density corresponding to the probability density function (1.1) with \( w(x) \) being given by a classical \( q \)-weight, and the support of the measure being appropriately chosen.

It would seem that the first study of this type [5, 49] was for the particular deformation of (1.1) from the real line to the unit circle with eigenvalue PDF proportional to

$$
\prod_{l=1}^{N} \vartheta_3(e^{i\theta_l}; q) \prod_{1 \leq j < k \leq N} |e^{i\theta_k} - e^{i\theta_j}|^2. \tag{1.14}
$$

Here

$$
\vartheta_3(z; q) = \sum_{n=-\infty}^{\infty} q^{n^2} z^n, \quad |q| < 1 \tag{1.15}
$$
is a Jacobi theta function. This came about through the study of a certain solvable $U(N)$ lattice gauge theory in two dimensions.

Let the eigenvalue density associated with (1.14) be denoted $\rho^{(RS)}_{(1),N}(q)$. Here the superscript $(RS)$ indicates that (1.14) relates to the Rogers–Szegö polynomials in the circular analogue of (1.2). Define the moments

$$m_{k,N}^{(RS)}(q) = \int_0^{2\pi} e^{ik\theta} \rho^{(RS)}_{(1),N}(\theta) \, d\theta, \quad k \in \mathbb{Z}. \quad (1.16)$$

Then we have from [5, 49] that

$$m_{k,N}^{(RS)}(q) = -\left(-q\right)^{k/2} 2\Phi_1\left(\frac{q^{2k}q^{-2k}}{q^2}; q^{2N+2}\right). \quad (1.17)$$

Here the special function on the RHS refers to Heine’s $q$-generalisation of the Gauss hypergeometric function,

$$2\Phi_1\left(\frac{a_1}{b_1}, \frac{a_2}{b_1}; q, z\right) := \sum_{n=0}^{\infty} \frac{(a_1; q)_n(a_2; q)_n}{(q; q)_n(b_1; q)_n} z^n, \quad (1.18)$$

with

$$(u; q)_n := (1 - u)(1 - qu) \cdots (1 - q^{n-1}u). \quad (1.19)$$

Moreover it is noted in [5, 49] that with the scaling $q = e^{-\lambda/N}$, the moments permit a $1/N^2$ expansion (recall (1.9))

$$\left. \frac{1}{N} m_{k,N}^{(RS)}(q) \right|_{q = e^{-\lambda/N}} = \mu_{k,0}^{(RS)}(\lambda) + \frac{1}{N^2} \mu_{k,2}^{(RS)}(\lambda) + \frac{1}{N^4} \mu_{k,4}^{(RS)}(\lambda) + \cdots \quad (1.20)$$

with

$$\mu_{k,0}^{(RS)}(\lambda) = -\frac{(-1)^k}{2\lambda k} 2F_1\left(-k, \frac{1}{1} \left| e^{-2\lambda}\right.\right). \quad (1.21)$$

Attracting attention in the recent literature [9, 22] is the particular example of (1.1) proportional to

$$\prod_{l=1}^{N} w^{(SW)}(u_l; q) \prod_{1 \leq j < k \leq N} (u_k - u_j)^2, \quad u_l \in \mathbb{R}^+, \quad (1.22)$$

where

$$w^{(SW)}(u; q) = \frac{k}{\sqrt{\pi}} e^{-k^2(\log u)^2}, \quad q = e^{-1/(2k^2)}, \quad (1.23)$$
which is (one form of) the Stieltjes–Wigert weight from the theory of orthogonal polynomials [53, 56]. Note that for the weight (1.23),

$$\int_0^\infty u^n w^{(SW)}(u; q) \, du = q^{-(n+1)^2/2}. \tag{1.24}$$

The ensemble (1.22) turns out to be closely related to (1.14). In particular, it was shown in [22, Prop. 1.1] that the moments $m^{(SW)}_{k,N}(q)$ of the spectral density corresponding to (1.22) have, for $k \geq 1$, the evaluation

$$\frac{1}{N} q^N m^{(SW)}(q) = -\frac{1}{N} \frac{(-q^{-1/2})^k}{1 - q^{-k}} 2\phi_1 \left( \frac{q^k - q^{-k}}{q^{-1}} \bigg| q^{-1}; q^{-N-1} \right) \tag{1.25}$$

(cf. (1.17)). Furthermore, scaling $q = e^{-\lambda/N}$, the same $1/N^2$ expansion of the RHS results as in (1.20), but with $-2\lambda$ replaced by $\lambda$ throughout.

There is an equivalent formulation of the result (1.25) which replaces the continuous integral giving an average over the probability density function by a Jackson $q$-integral. First recall that the Jackson integral with terminals 0 to $\infty$ is defined by

$$\int_0^\infty f(x) \, dq \, x = (1 - q) \sum_{k=-\infty}^{\infty} f(q^k) q^k. \tag{1.26}$$

By introducing the quantity

$$c_q = (-q, -1, q; q)_\infty, \quad (\alpha, \beta, \gamma; q)_\infty := \prod_{l=0}^{\infty} (1 - \alpha q^l)(1 - \beta q^l)(1 - \gamma q^l), \tag{1.27}$$

we know from [10] that with

$$w^{(SW)}(x; q) = \frac{1}{(1 - q)\sqrt{q} c_q} x^{-1/2} e^{((\log x)/(2 \log q))^2}, \tag{1.28}$$

we have

$$\int_0^\infty w^{(SW)}(x; q)(q^{-1/2} x)^n \, dq \, x = q^{-(n+1)^2/2}. \tag{1.29}$$

This is precisely the same as for (1.24), which is permitted since the quadratic exponential order of the rate of increase of the moments means they do not uniquely determine a weight function. As a consequence, the ensemble specified by replacing each $w^{(SW)}(u_l; q)$ by $w^{(SW)}(u_l; q)$, and with the $u_l$ restricted to the $q$-lattice in the sense of (1.26), we have the moments of this particular Jackson integral ensemble when multiplied by $q^{-k}$ are also specified by (1.25).
1.3 Moments via Schur averages in the $q$ case

The method used to derive (1.25) in [22] made use of knowledge of a more general result, namely the closed-form evaluation of [19]

$$\langle s_\kappa(x_1, \ldots, x_N) \rangle^{(SW)}$$

where $s_\kappa$ denotes the Schur polynomial indexed by the partition $\kappa$,

$$s_\kappa(x_1, \ldots, x_N) := \frac{\det[x_k^{N-j+1} + j-1]_{j,k=1}^N}{\det[x_k^{j-1}]_{j,k=1}^N}$$

and the average $\langle \cdot \rangle^{(SW)}$ is taken over PDF (1.1) with weight function specified by (1.23). In what follows, we use $\langle \cdot \rangle$ to denote an average with respect to (1.1) for generic weights and add proper superscripts if a specific weight is taken. Its relevance is seen from the identity (see e.g. [43])

$$\sum_{j=1}^N x_j^k = \sum_{r=0}^{\min(k-1,N-1)} (-1)^r s_{(k-r,1^r)}(x_1, \ldots, x_N),$$

where $(k-r, 1^r)$ denotes the partition with largest part $\kappa_1 = k-r$, $r$ parts ($r \leq N-1$) equal to 1 and the remaining parts equal to 0, and thus

$$\int_0^\infty x^k \rho(1)_N(x) d_q x = \sum_{r=0}^{\min(k-1,N-1)} (-1)^r \langle s_{(k-r,1^r)}(x_1, \ldots, x_N) \rangle.$$ (1.33)

This is the same strategy as used in [5] to derive (1.17).

In Sect. 2 we will discuss in detail particular $q$-weights which give evaluations of $\langle s_\kappa \rangle$ simple enough that $\langle \sum_{j=1}^N x_j^k \rangle$ as implied by (1.33) is in a structured form suitable for further analysis. For example, as already remarked the evaluation (1.25) permits computation of the $N \to \infty$ limit, scaled by requiring $q = e^{-\lambda/N}$. In Sect. 2.1 we revisit the case of Stieltjes–Wigert weight, specifically in the form (1.28), and show that the generating function in $N$ of the moments $m_{k,N}^{(SW)}$ admit the simple product form (2.8), a result which was conjectured recently in [9]. In Sect. 2.2 we consider the discrete $q$-Hermite weight, which in a certain scaling, limits to the Gaussian weight in (1.7) for $q \to 1^-$. The corresponding probability density function (1.1) is supported on the $q$-lattice corresponding to the Jackson integral

$$\int_a^b f(x) d_q x := \int_0^b f(x) d_q x - \int_0^a f(x) d_q x$$ (1.34)
with
\[ \int_0^a f(x) \, dq \ x := (1 - q) \alpha \sum_{n=0}^{\infty} q^n f(\alpha q^n) \] (1.35)
in the case \( a = -1, b = 1 \). The moments in this case have also been the subject of some previous literature [44, 57, 58]. As a new result, we use knowledge of the generating function in \( N \) of \( m_{k,N}(d-qH) \), obtained in the recent work [44], and rederived by our own working below, to deduce the explicit form of the leading term in the corresponding scaled \( 1/N^2 \) expansion; see (2.45). In Sect. 2.3 the little \( q \)-Laguerre weight is considered. The corresponding ensemble (1.1) is supported on the \( q \)-lattice implied by (1.34) with \( a = 0, b = 1 \). Upon a certain scaling, this reclaims the Laguerre weight in (1.7) for \( q \to 1^- \). Our main result is a closed-form evaluation of the moments in terms of a particular \( 3\phi_2 \) basic hypergeometric function. This is presented in Proposition 2.12.

### 1.4 Integrable structures associated with \( q \)-orthogonal polynomial weights

Combining (1.8) and (1.5) shows that the exponential generating function of the moments can be expressed as a sum according to
\[ \int_I e^{tx} \rho(1),N(x) \, dx = \sum_{j=0}^{N-1} \frac{1}{h_j} \int_I e^{tx} (p_j(x))^2 \, d\mu, \quad d\mu = w(x) \, dx. \] (1.36)

In the works [40, 41] Ledoux undertook a study of integrability properties associated with the integral in the summand on the RHS of (1.36), in the classical continuous cases, and also classical discrete cases. This is equivalent to studying
\[ \int_I e^{tx} \left( \rho(1),N(x) - \rho(1),N-1(x) \right) \, dx, \] (1.37)
which is the exponential generating function for the difference \( m_{k,N} - m_{k,N-1} \). Making essential use of the Pearson equation, Ledoux derived a fourth-order differential (difference) equation for \( \int_I e^{tx} (p_j(x))^2 \, d\mu \) in the classical continuous (discrete) cases, respectively. This methodology will be revised in Sect. 3. In Sect. 4, with the meaning of the integral, and also the exponential function \( e^{tx} \), appropriately adjusted to the \( q \) setting as in Sect. 2, the classical \( q \) analogues of Ledoux’s integrability results are obtained. Thus, the \( q \) case as described in Sect. 1.2 is missed from consideration in [40, 41]. Specifically, in Sect. 4 the approach of [40] is used to derive a fourth-order \( q \)-differential equation satisfied by the \( q \)-Laplace transform of the un-normalised measure \((p_n(x; q))^2 \, dq \mu\). Here \( dq \mu \) denotes the appropriately weighted \( q \)-lattice corresponding to classical \( q \)-orthogonal polynomials \( \{p_n(x; q)\} \). Crucial to our considerations is the \( q \)-Pearson equation (1.13).

In [40, 41], knowledge that (1.37) satisfies a fourth-order differential (difference) equation was not used in obtaining explicit formulas for \( \{m_{k,N}\} \) in the classical
cases. Instead, for the classical continuous weights, use was made of the fact that the derivative with respect to \( t \) of the LHS of (1.36) is then simply related to

\[
\int_I e^{tx} p_N(x) p_{N-1}(x) \, d\mu. \tag{1.38}
\]

The methods used in [40] to study (1.37) applied to (1.38) then lead to a second-order linear recurrence for the even moments in the case of the classical Gaussian weight (obtained first in [32]), and a second-order linear recurrence for the moments in the case of the classical Laguerre weight (obtained first in [31]). In the case of classical discrete weights on a linear lattice, there is no special property associated with the discrete (or continuous) derivative of the LHS of (1.36). Instead in [41] special properties of the shifted \( k \)-th moment

\[
\int x(x-1) \cdots (x-k+1) (p_j(x))^2 \, d\mu \tag{1.39}
\]

for the Charlier and Meixner classical discrete weights, together with recurrences used in the derivation of the fourth-order difference equation relating to (1.37) as applies for all the discrete classical weights, were used to obtain formulas in these special cases. This working was extended to the Krawtchouk weight in the recent work [13], and moreover the evaluations have been identified in terms of hypergeometric polynomials.

With these points in mind, in Sect. 5 use is made of the \( q \)-Pearson equation to study the moments of the measure \( (H_N(x; q))^2 d_q \mu \), where \( H_N(x; q) \) denotes the normalised discrete \( q \)-Hermite polynomials and \( d_q \mu \) denotes the \( q \)-Hermite weight supported on the appropriate \( q \)-lattice, studied from the viewpoint of the Schur average (1.30) in Sect. 2. This is of particular interest for its combinatorial meaning in terms of rook placements [58].

## 2 Averages over Schur polynomials

### 2.1 The Stieltjes–Wigert weight (1.28)

Let \( \langle \tilde{\mathrm{SW}} \rangle \) refer to an average with respect to the weight (1.28) in (1.1) on the \( q \)-lattice in the sense of (1.26). Let \( \langle \mathrm{SW} \rangle \) refer to an average with respect to the weight (1.23) in (1.1). Our interest is in the moments of the density, \( m_{k, N}^{\langle \tilde{\mathrm{SW}} \rangle} \), which we calculate via the identity (1.33).

With \( s_k \) the Schur polynomial (1.31), according to (1.24) and (1.29) we have that

\[
\langle s_k \rangle^{\langle \tilde{\mathrm{SW}} \rangle} = \langle s_k \rangle^{\langle \mathrm{SW} \rangle}. \tag{2.1}
\]

The exact form of the average on the RHS is known from [22, Prop. 3.1], [54, Appendix A.2] as well as the earlier work [19]. Reading off from these references then gives

\[
\langle s_k \rangle^{\langle \tilde{\mathrm{SW}} \rangle} = q^{-\frac{1}{2}} \sum_{i=1}^{N} \kappa_i^2 \prod_{1 \leq j < k \leq N} \frac{1 - q^{-\kappa_j - j - \kappa_k + k}}{1 - q^{-(k-j)}}. \tag{2.1}
\]

\[ \text{Springer} \]
As shown in [22, Coroll. 3.3 and Prop. 1.1], specialising to \( \kappa = (k - r, 1^r) \) and making use of (1.33) then implies that \((q^{Nk}/N)m_{k,N}^{\text{(SW)}}\) is given by (1.25), or equivalently

\[
\frac{1}{N} q^{(N-1/2)k} m_{k,N}^{\text{(SW)}} = -\frac{1}{N} \frac{(-q^{-1/2})^k}{1-q^{-k}} p_k^{(lq^{-J})}(q^{-N}; 1, q|q^{-1}).
\]  

(2.2)

Here \( \{p_k^{(lq^{-J})}\}_{k=0,1,...} \) denote little \( q \)-Jacobi polynomials, specified in terms of the \( q \)-hypergeometric function (1.18) by

\[
p_n^{(lq^{-J})}(x; a, b|q) = 2\phi_1\left(q^{-n}, abq^{n+1}|q; qx\right).
\]  

(2.3)

There is a further equivalent form of \( m_{k,N}^{\text{(SW)}} \). This involves the \( q \)-binomial coefficient

\[
\left[ \begin{array}{c} n \\ l \end{array} \right]_q = \frac{(1 - q^n)(1 - q^{n-1}) \cdots (1 - q^{n-l+1})}{(1 - q^l)(1 - q^{l-1}) \cdots (1 - q)}
\]  

(2.4)

for \( n, l \) non-negative integers. Thus from [22, Eq. (3.15) combined with (3.14)], we read off

\[
q^{(N-1/2)k} m_{k,N}^{\text{(SW)}} = \sum_{r=0}^{k-1} (-1)^r q^{-(k-r)^2/2-r/2} \left[ \begin{array}{c} N+k-r-1 \\ k \end{array} \right]_{q^{-1}} \left[ \begin{array}{c} k-1 \\ r \end{array} \right]_{q^{-1}}
\]

\[
= q^{-kN-(k^2-2k)/2} \sum_{r=0}^{k-1} (-1)^r q^{(r^2+r)/2} \left[ \begin{array}{c} N+k-r-1 \\ k \end{array} \right]_{q} \left[ \begin{array}{c} k-1 \\ r \end{array} \right]_{q},
\]  

(2.5)

where the final equality follows from the general property of the \( q \)-binomial coefficients

\[
\left[ \begin{array}{c} n \\ l \end{array} \right]_{q^{-1}} = q^{-(n-l)t} \left[ \begin{array}{c} n \\ l \end{array} \right]_{q}.
\]  

(2.6)

The utility of (2.5) is that it allows for the computation of the generating function with respect to \( N \) (cf. [9, Eq. (2.56)]).

**Proposition 2.1** Introduce the generating function

\[
G_k^{\text{(SW)}}(z) = \sum_{N=1}^{\infty} (q^{2k}z)^N m_{k,N}^{\text{(SW)}}.
\]  

(2.7)

For integers \( k \geq 1 \) we have

\[
G_k^{\text{(SW)}}(z) = q^{-(k^2-2k)/2} \frac{z}{1-z} \frac{(q^{k+1}z; q)_{k-1}}{(qz; q)_k}.
\]  

(2.8)
Proof Simple manipulation gives

\[
\sum_{N=1}^{\infty} z^N \binom{N + k - r - 1}{k} = \sum_{N=1}^{\infty} z^N \binom{N + k - r - 1}{N - r - 1} = z^{r+1} \sum_{N=0}^{\infty} z^N \binom{N + k}{N}.
\]

(2.9)

This last sum can be evaluated [46, Eq. 17.2.38] to give

\[
\frac{z^{r+1}}{(z; q)_{k+1}}.
\]

(2.10)

Thus

\[
G_{k}^{(\tilde{SW})}(z) = \frac{zq^{-(k^2-2k)/2}}{(z; q)_{k+1}} \sum_{r=0}^{k-1} (-z)^r q^{(r^2+r)/2+k-1} \binom{k-1}{r}.
\]

Performing the sum according to the \(q\)-generalisation of the simple binomial expansion [46, Eq. 17.2.35] gives (2.8).

We will see that very similar working to that in the above proof can be used to compute the generating function with respect to \(N\) for the moments corresponding to (1.1) with the discrete \(q\)-Hermite weight to be considered in the next subsection. If we start with (2.2) instead of (2.5) a functional form distinct from (2.8) results.

Proposition 2.2 As an alternative to the evaluation (2.8) we have

\[
G_{k}^{(\tilde{SW})}(z) = z \sum_{s=0}^{k} b_s^{(k)} \frac{q^r}{1 - zq^s},
\]

(2.11)

where

\[
b_s^{(k)} = (-1)^s q^{-(1/2)k^2+k-s(s+1)/2} \frac{(q; q)_{2k-s-1}}{(q; q)_s ((q; q)_{k-s})^2}.
\]

(2.12)

Proof Starting from (2.2), simple manipulation shows

\[
q^{2Nk} m_{k,N}^{(\tilde{SW})} = \sum_{s=0}^{k} b_s^{(k)} q^{Ns},
\]

(2.13)

where \(b_s^{(k)}\) is given by (2.12). Substituting in the definition (2.7), the result (2.11) follows by performing a geometric series. \(\square\)
Notice that (2.11) corresponds to the partial fractions expansion of (2.8). Also, in keeping with the large $N$ expansion

$$\left. \frac{1}{N} m_{k,N}(q) \right|_{q = e^{-\lambda/N}} = \mu_{k,0}^{(SW)}(\lambda) + \frac{1}{N^2} \mu_{k,2}^{(SW)}(\lambda) + \frac{1}{N^4} \mu_{k,4}^{(SW)}(\lambda) + \cdots \quad (2.14)$$

where

$$\mu_{k,0}^{(SW)}(\lambda) = \frac{(-1)^k}{\lambda k} 2 F_1(-k, k, 1; e^{-\lambda}), \quad (2.15)$$

as remarked upon in the text below (1.25), we can check from (2.12) that $b_s^{(k)}|_{q \to q^{-1}} = -b_s^{(k)}$ and furthermore that $(1/N)b_s^{(k)}|_{q = e^{-\lambda/N}}$ has a well-defined limit. We remark too that with the scaled density

$$\rho_{(1),0}^{(SW)}(x; \lambda) := \lim_{N \to \infty} \frac{1}{N} \rho_{(1,N)}^{(SW)}(x)|_{q = e^{-\lambda/N}}, \quad (2.16)$$

characterised by its relationship to $\mu_{k,0}^{(SW)}(\lambda)$ in (2.28),

$$\mu_{k,0}^{(SW)}(\lambda) = \int_0^\infty \lambda^k \rho_{(1),0}^{(SW)}(x; \lambda) \, dx, \quad (2.17)$$

it was shown in [22] how knowledge of the explicit functional form (2.15) can be used to deduce that

$$\rho_{(1),0}^{(SW)}(x; \lambda) = \frac{1}{\pi \lambda x} \arctan \left( \sqrt{\frac{4e^{\lambda}x - (1 + x)^2}{1 + x}} \right) \chi_{z_- < x < z_+}, \quad (2.18)$$

where $z_\pm = -z \pm (z^2 - 1)^{1/2}$ with $z = 1 - 2e^{\lambda}$.

### 2.2 The discrete $q$-Hermite weight

Making use of the notation used in (1.27), the discrete $q$-Hermite weight is

$$w_{(d-qH)}(x) = \frac{(q x, -q x; q)_{\infty}}{(q, -1, -q; q)_{\infty}}. \quad (2.19)$$

This is supported on the $q$-lattice corresponding to (1.34) with $a = -1$ and $b = 1$. Let an average with respect to (1.1) defined by the Jackson integral (1.34) in each variable with $a = -1$ and $b = 1$ and with weight (2.19) be denoted $(\cdot)^{(d-qH)}$.

**Proposition 2.3** Consider a partition $\kappa = (\kappa_1, \ldots, \kappa_N)$. Require that the number of $j$ such that $\kappa_j + N - j$ is even must equal $[(N + 1)/2]$, while the number of $j$ such that $\kappa_j + N - j$ is odd must equal $[N/2]$. Let these $j$ be denoted $\{j_i^{(N+1)/2}\}_{i=1}^{(N+1)/2}$ and 

\[ \begin{array}{c}
\end{array} \]
\{j_l^o\}_{l=1}^{[N/2]}$, respectively, and ordered from smallest to biggest within each set. In the case of the empty partition, denote these same \(j\) by \(\{j_l^o(0) = 2l - 1\}_{l=1}^{[(N+1)/2]}\) and \(\{j_l^o(0) = 2l\}_{l=1}^{[N/2]}\). Let \(S_0\) equal the number of \(\kappa_j\) that are odd. We have

\[
\langle s_\kappa \rangle^{(d-qH)} = (-1)^{S_0/2} \prod_{l=1}^{[(N+1)/2]} \frac{(q; q^2)^{(\kappa_j^o+N-j_l^o)/2}}{(q; q^2)^{(N-j_l^o(0))/2}} \times \prod_{1 \leq k < l \leq [(N+1)/2]} \frac{q^{\kappa_j^o+N-j_l^o} - q^{\kappa_j^o+N-j_k^o}}{q^{N-j_k^o(0)} - q^{N-j_l^o(0)}} \times \prod_{l=1}^{[N/2]} \frac{(q; q^2)^{(\kappa_j^o+N+1-j_l^o)/2}}{(q; q^2)^{(N+1-j_l^o(0))/2}} \prod_{1 \leq k < l \leq [N/2]} \frac{q^{\kappa_j^o+N-j_k^o} - q^{\kappa_j^o+N-j_l^o}}{q^{N-j_k^o(0)} - q^{N-j_l^o(0)}}.
\]

(2.20)

If the requirement relating to the parity of \(\kappa_j + N - j\) does not hold, then \(\langle s_\kappa \rangle^{(d-qH)} = 0\).

**Proof** Denote

\[
I_\kappa^{(d-qH)} := \frac{1}{N!} \int_{-1}^{1} d_q x_1 w^{(d-qH)}(x_1) \cdots \int_{-1}^{1} d_q x_N w^{(d-qH)}(x_N) s_\kappa(x_1, \ldots, x_N)
\times \prod_{1 \leq j < k \leq N} (x_k - x_j)^2.
\]

(2.21)

Making use of the Vandermonde determinant identity

\[
\det[x_j^{k-1}]_{j,k=1}^{N} = \prod_{1 \leq j < k \leq N} (x_k - x_j)
\]

(2.22)

to substitute for the product of differences in (2.21), and of (1.31) to substitute for the Schur polynomial, we see

\[
I_\kappa^{(d-qH)} = \frac{1}{N!} \int_{-1}^{1} d_q x_1 w^{(d-qH)}(x_1) \cdots \int_{-1}^{1} d_q x_N w^{(d-qH)}(x_N)
\times \det[x_j^{\kappa_{N+1-k}+k-1}]_{j,k=1}^{N} \det[x_j^{k-1}]_{j,k=1}^{N}
= \det \left[ \int_{-1}^{1} w^{(d-qH)}(x)x^{\kappa_{N+1-j}+j+k-2} d_q x \right]_{j,k=1}^{N},
\]

(2.23)

where the second equality follows from Andréief’s identity \([4, 21]\).

The final determinant in (2.23) consists of the moments of the discrete \(q\)-Hermite weight, which we know are given by (see e.g. [48])

\[
\int_{-1}^{1} w^{(d-qH)}(x)x^k d_q x = (1 - q) \left( \frac{1 + (-1)^k}{2} \right) (q; q^2)^{k/2}.
\]

(2.24)
In particular, this tells us that along each row of the determinant every second entry is zero. Interchange of rows and columns can therefore bring the determinant to the block form

$$\det \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix},$$

(2.25)

where the rows in $A_1$ have $\kappa_{N+1-j} + j - 1$ even, while the rows in $A_2$ have $\kappa_{N+1-j} + j - 1$ odd. Furthermore, the number of columns in $A_1$ is $[(N + 1)/2]$, while the number of columns in $A_2$ is $[N/2]$. For (2.25) to be non-zero we require $A_1$ and $A_2$ to be square. After replacing $j$ by $N + 1 - j$ it follows that the number of $j$ such that $\kappa_j + N - j$ is even (odd) must equal $[(N + 1)/2] ([N/2])$. Let these $j$ be denoted as in the statement of the proposition.

The block form of the determinant (2.25) implies the factorisation

$$I_k^{(d-qH)} = (-1)^{S_0/2} (1 - q)^N \det A_1 \det A_2$$

(2.26)

where

$$A_1 = \left[ (q; q^2)^{(\kappa_j + N-j+2(k-1))/2} \right]_{j,k=1}^{((N+1)/2)}$$

$$A_2 = \left[ (q; q^2)^{(\kappa_j + N+1-j+2(k-1))/2} \right]_{j,k=1}^{[N/2]}$$

(2.27)

and $S_0$ denotes the number of parts $\kappa_j$ that are odd. Note that the factor $(-1)^{S_0/2}$ is due to it being necessary to undertake $S_0/2$ row interchanges.

Now, for general $a_j \in \mathbb{Z}_{\geq 0}$

$$(q; q^2)_{a_j+k-1} = (q; q^2)_{a_j} (q^{1+2a_j}; q^2)_{k-1}$$

(2.28)

and thus

$$\det \left[ (q; q^2)_{a_j+k-1} \right]_{j,k=1}^n = \prod_{j=1}^n (q; q^2)_{a_j} \det \left[ (q^{1+2a_j}; q^2)_{k-1} \right]_{j,k=1}^n.$$  

(2.29)

The determinant on the RHS has the known evaluation [47]

$$\prod_{1 \leq k < l \leq n} (q^{1+2a_k} - q^{1+2a_l}) \prod_{k=0}^{n-1} q^{2k(n-1-k)}.$$  

(2.30)

It follows that both determinants in (2.27) can be evaluated explicitly. Doing this, and normalising to unity for the empty partition, (2.20) is obtained. \qed

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From the definition of the Jackson integral (1.34), for any $a > 0$

$$\int_{-1}^{1} f(x) \, dq \, x = \frac{1}{a} \int_{-a}^{a} f \left( \frac{x}{a} \right) \, dq \, x$$  \hspace{1cm} (2.31)$$
as is familiar in the continuous case. Moreover, for the discrete Hermite weight (2.19) it is well known that (see [3] for details of the required working)

$$\lim_{q \to 1-} \frac{1}{a} w^{(d-qH)} \left( \frac{x}{a} \right) \left|_{a=(1-q^2)^{-1/2}} \right. = \frac{1}{\sqrt{\pi}} e^{-x^2}.$$  \hspace{1cm} (2.32)$$
The significance of (2.31) and (2.32) in the context of averages $\langle \cdot \rangle^{(d-qH)}$ is that they imply

$$\lim_{q \to 1-} \left( 1 - q^2 \right)^{-|\kappa|/2} \langle s_\kappa \rangle^{(d-qH)} = \langle s_\kappa \rangle^{(G)},$$  \hspace{1cm} (2.33)$$
where the average on the RHS is with respect to (1.1) with the Gaussian weight $w^{(G)}(x)$ from (1.7). Applying (2.33) then allows for the computation of $\langle s_\kappa \rangle^{(G)}$.

**Corollary 2.4** Let the parity of the parts of $\kappa$ be restricted as in Proposition 2.3, and furthermore make use of the notation therein. We have

$$\langle s_\kappa \rangle^{(G)} = (-1)^{S_{0}/2} \prod_{l=1}^{[(N+1)/2]} \frac{(\kappa_{j_l^e} + N - j_l^e - 1)!!}{(N - j_l^e(0) - 1)!!} \times \prod_{1 \leq k < l \leq [(N+1)/2]} \frac{\kappa_{j_l^e} - \kappa_{j_k^e} + j_k^e}{j_l^e - j_k^e} \times \prod_{l=1}^{[N/2]} \frac{\kappa_{j_l^o} + N - j_l^o - 1)!!}{(N - j_l^o(0))!!} \prod_{1 \leq k < l \leq [N/2]} \frac{\kappa_{j_l^o} - \kappa_{j_k^o} + j_k^o}{j_l^o - j_k^o}.$$  \hspace{1cm} (2.34)$$

**Remark 2.5** The evaluation of $\langle s_\kappa \rangle^{(G)}$, albeit in a different but equivalent form to that on the RHS of (2.34), was first obtained by Di Francesco and Itzykson [17]. Their derivation made essential use of the character interpretation of the Schur polynomials. Subsequently, but without the details being given, it was noted in [37] that a direct derivation along the lines of that implied in the proof of Proposition 2.3 could be given.

We now specialise (2.34) to partitions of the form $\kappa = (2k - r, 1^r)$.

**Corollary 2.6** (Venkataramana [57, Th. 4, with the leading sign and exponent of the factor of $q$ corrected]) For $r$ a non-negative integer, $0 \leq r \leq 2k$, we have

$$\langle s_{(2k-r, (1)^r)} \rangle^{(d-qH)} = (-1)^{[(r+1)/2]} q^{[(r/2)+1]} \left[ N + 2k - r - 1 \right]_{2k}^{q} \left[ k - 1 \right]_{[r/2]} \left( q; q^2 \right)_k.$$  \hspace{1cm} (2.35)$$
**Proof** Our task is to simplify (2.34) in the case of $\kappa_1 = 2k - r, \kappa_2 = 1, \ldots, \kappa_{r+1} = 1$. For convenience we suppose that $N$ is even; the working in the case $N$ odd is analogous, and leads to the same result.

To begin, we observe that the contribution from the single products in (2.35), the first involving $\{j^{2}\}$ and the second involving $\{j^{0}\}$, comes from the first (second) product for $r$ odd (even). This contribution is seen to equal

$$\frac{(q; q^2)^N_{2+k-\lfloor r \rfloor}}{(q; q^2)^N_{2-\lfloor r \rfloor}}. \quad (2.36)$$

It is similarly true that the contribution from the double products in (2.35), the first involving $\{j^{2}\}$ and the second involving $\{j^{0}\}$, comes from the first (second) product for $r$ odd (even). After simplification, this contribution is seen to equal

$$q^{\lfloor r/2 \rfloor} \frac{(q^2; q^2)^N_{2+k-\lfloor r/2 \rfloor-1}}{(q^2; q^2)^N_{2-\lfloor r/2 \rfloor-1}} \left( q^2 \right)_{k}. \quad (2.37)$$

According to (2.34) there is also a factor

$$(-1)^{\lfloor (r+1)/2 \rfloor}. \quad (2.38)$$

Multiplying together (2.36), (2.37), (2.38) and simplifying gives (2.35). \hfill \Box

**Remark 2.7** The proof of (2.35) given in [57] is different to the one above, relying on the properties of particular multivariate discrete $q$-Hermite polynomials; for generalisations of the latter in the setting of Macdonald polynomial theory involving parameter $(q, t)$, see [7]. The average of a Macdonald polynomial in the latter setting has been conjectured in [42]. The case $q = t$ corresponds to the Schur average in Proposition 2.3, giving a character-based formula which is a $q$-generalisation of the evaluation of (2.34) in [17]. In the recent work [44], specialising the latter to partitions $\kappa = (2k - r, (1)^{r})$ has been shown to give a formula equivalent to (2.35).

Substituting (2.35) in (1.33) gives for the moments of the density of the discrete $q$-Hermite ensemble [57]

$$m_{2k,N}^{(d-qH)} = \sum_{r=0}^{2k-1} (-1)^{r+\lfloor (r+1)/2 \rfloor} q^{\lfloor r/2 \rfloor} (q^2)_{k} \left( q^2 \right)_{k}; \quad (2.39)$$

cf. (2.5). For example, with $k = 1$
\[ m_{2,N}^{(d-qH)} = (q; q^2) \left( \frac{[N+1]}{2} + \frac{N}{2} \right) \]

\[ = \frac{1}{1-q^2} \left( q^{2N} (q + q^{-1}) - q^N (q + 2 + q^{-1}) + 2 \right). \quad (2.40) \]

It has recently been observed [44] that (2.39) simplifies upon forming the generating function with respect to \( N \).

**Proposition 2.8** (Morozov, Popolitov and Shakirov [44, Eq. (4-7) with \( \lambda = zq^{2m}, m = k \)) Define

\[ G_{2k}^{(d-qH)}(z) = \sum_{N=1}^{\infty} z^N m_{2k,N}^{(d-qH)}. \quad (2.41) \]

For integers \( k \geq 1 \) we have

\[ G_{2k}^{(d-qH)}(z) = \frac{z(1+z)}{1-z} \frac{(qz)^2; q^2}{(qz; q)_{2k}} (q^2)_k. \quad (2.42) \]

**Proof** This can be established by working analogous to that used in the derivation of Proposition 2.1. Thus, according to (2.9) and (2.10) with \( k \) replaced by \( 2k \) we have

\[ \sum_{N=1}^{\infty} z^N \left[ \frac{N+2k-r-1}{2k} \right] = \frac{z^{r+1}}{(z; q)_{2k+1}}. \]

Hence

\[ G_{2k}^{(d-qH)}(z) = \frac{(q; q^2)_k}{(z; q)_{2k+1}} \sum_{r=0}^{2k-1} z^{r+1} (-1)^{r+\lceil (r+1)/2 \rceil} q^{\lfloor (r/2)\rfloor + \lfloor (r/2)\rfloor + 1} \left[ \frac{k-1}{\lfloor r/2 \rfloor} \right] q^2 \]

\[ = z(1+z) \sum_{r=0}^{k-1} z^{2r} (-1)^r q^{r(r+1)} \left[ \frac{k-1}{r} \right] q^2, \]

where the second equality follows by breaking up the sum in the first equality according to the parity of \( r \). But according to the \( q \)-generalisation of the simple binomial expansion [46, Eq. 17.2.35], this last sum has the evaluation \( ((qz)^2; q^2)_{k-1} \), and (2.42) follows. \( \square \)

In keeping with the relationship between (2.13), (2.11) and (2.7) we can use (2.42) to deduce the coefficients in the expansion

\[ q^k m_{2k,N}^{(d-qH)} = \sum_{p=0}^{2k} c_p^{(k)} q^{pN} \quad (2.43) \]

(here the factor of \( q^k \) in the LHS is for later convenience).
Proposition 2.9 (Morozov, Popolitov and Shakirov [44, Equivalent to Eqns. (3–7) and (3–8)]) We have

\[ c^{(k)}_0 = \frac{2q^k}{1-q^{2k}}, \]
\[ c^{(k)}_1 = \begin{cases} 0, & k > 1, \\ \frac{1+q}{1-q}, & k = 1. \end{cases} \]

For \( p \geq 2 \) we have

\[ c^{(k)}_p = 0, \quad 2 \leq p < k \]
\[ c^{(k)}_p = (-1)^{k+p-1} \left( \frac{1+q^p}{1-q^p} \right) q^{p(p+1)/2-2pk+k(k-1)} \]
\[ \times \frac{(q^2; q^2)_{p-1}(q; q^2)_k}{(q^2; q^2)_{p-k}(q; q)_{p-1}(q; q)_{2k-p}}, \quad k \leq p \leq 2k. \]

Proof: Substituting (2.43) in (2.41) we have

\[ q^k G^{(d-qH)}_{2k}(z) = z \sum_{p=0}^{2k} c^{(k)}_p q^p \frac{1}{1-zq^p}. \]

Hence, with \( \text{Res}_{z=z_0} f(z) \) denoting the residue at \( z = z_0 \) of \( f(z) \),

\[ c^{(k)}_p = -q^{p+k} \text{Res}_{z=q^{-p}} G^{(d-qH)}_{2k}(z). \]

The product form of \( G^{(d-qH)}_{2k}(z) \) (2.42) allows for a simple computation of the residues, and the stated results follow. \( \square \)

It is simple to check from the results of Proposition 2.9 that \( c^{(k)}_p |_{q \rightarrow q^{-1}} = -c^{(k)}_p \).

We can also check that with \( q = e^{-\lambda/N} \), dividing \( c^{(k)}_p \) by \( N \) leads to a well-defined limit. Thus analogous to (1.20) the scaled moments possess a \( 1/N^2 \) expansion

\[ \frac{1}{N} m^{(d-qH)}_{k,N}(q) \big|_{q=e^{-\lambda/N}} = \mu^{(d-qH)}_{k,0}(\lambda) + \frac{1}{N^2} \mu^{(d-qH)}_{k,2}(\lambda) + \frac{1}{N^4} \mu^{(d-qH)}_{k,4}(\lambda) + \cdots. \]

(2.44)

Moreover, with \( x = q^{-\lambda} \), we read off from the results of Proposition 2.9 that

\[ \lambda \mu_{k,0}(\lambda) = \frac{2}{k} - \delta_{k,1}x + (-1)^{k-1} \sum_{\substack{p \geq k \\text{p} \neq 1}} (-1)^p \frac{2}{p} \frac{(2(p-1))!(2k-1)!!!}{(p-1)!(2k-p)!!} x^p. \]

(2.45)
However, it is not apparent if knowledge of this explicit expression can be used to compute the corresponding scaled density analogous to (2.16)–(2.18) in the Stieltjes–Wigert case.

**Remark 2.10** It follows from (2.33) that

$$\lim_{q \to 1^-} (1 - q^2)^{-|\kappa|/2} m^{(d-q^H)}_{2k,N} = m^{(G)}_{2k,N},$$  \hspace{1cm} (2.46)

where $m^{(G)}_{2k,N}$ refers to the moments of the density for the Gaussian case. The form (2.43) is not suitable for taking this limit, as the limit does not exist term-by-term. It is possible to set $q = 1$ in each term of (2.39). Considering separately the $r$ odd and $r$ even terms shows

$$m^{(G)}_{2k,N} = (2k - 1)!! \sum_{r=0}^{k-1} (-1)^r \left\{ \binom{N + 2k - 2r - 1}{2k} + \binom{N + 2k - 2r - 2}{2k} \right\} \binom{k - 1}{r}. \hspace{1cm} (2.47)$$

Another formula for which it is possible to directly set $q = 1$ is (2.42). Doing this we see [44, Eq. (3-4)]

$$G^{(d-q^H)}_{2k}(z) \bigg|_{q=1} = (2k - 1)!! \frac{z(1 + z)^k}{(1 - z)^{k+2}}. \hspace{1cm} (2.48)$$

Calculating the coefficient of $z^N$, which we do this using the formula for the product of two power series, then implies

$$m^{(G)}_{2k,N} = (2k - 1)!! \sum_{l=0}^{N-1} \frac{(k + 2)l}{l!} \binom{k}{N - 1 - l}. \hspace{1cm} (2.49)$$

However, neither (2.47) nor (2.49) reveal the known special function form [15, 59]

$$m^{(G)}_{2k,N} = 2^{-k} (2k - 1)!! N_2 F_1(-k, 1 - N, 2; 2), \hspace{1cm} (2.50)$$

which moreover is valid for continuous $k$ and satisfies the second-order recurrence [32]

$$(k + 1)m^{(G)}_{2k,N} = (2k - 1)m^{(G)}_{2k-2,N} + (k - 1/2)(k - 1)(k - 3/2)m^{(G)}_{2k-4,N}. \hspace{1cm} (2.51)$$

The open question along these lines is if $m^{(d-q^H)}_{2k,N}$ admits a $q$-special form analogous to (2.2), or satisfies a linear recurrence. In fact subsequent to the posting of our work on the arXiv, Cohen [12] has shown that $m^{(d-q^H)}_{2k,N}/[2k - 1]q^2!!$ can be written as a linear combination of two $3\phi_2$ hypergeometric polynomials. It is also shown that
(1 − λ)² ∑_{N=1}^{∞} N²λ^{N−1} m_{2k,N}^{(d−qH)} can be identified in terms of a particular q-Hahn polynomials, and so satisfies a 3-term recurrence.

2.3 The little q-Laguerre weight

We consider next the case of (1.1), supported on the q-lattice corresponding to (1.34) with \( a = 0, b = 1 \), and with the little q-Laguerre weight

\[
\omega^{(l-qL)}(x) = x^\alpha(qx; q)_\infty. \tag{2.52}
\]

Most important for present purposes is that the moments of this weight have the simple explicit form [48]

\[
\int_0^1 \omega^{(l-qL)}(x) x^k \, dq_x = (q; q)_\infty \frac{(q^\alpha; q)_\infty}{(q^{\alpha+1}; q)_\infty} (q^\alpha+1; q)_k. \tag{2.53}
\]

**Proposition 2.11** We have

\[
\langle s_\kappa \rangle^{(l-qL)} = \prod_{j=1}^{N} \frac{(q^\alpha; q)_\kappa \alpha + 1}{(q^\alpha; q)_N} \prod_{1 \leq j < l \leq N} \frac{q^{\kappa_j + N - j} - q^{\kappa_l + N - l}}{q^{N - j} - q^{N - l}}. \tag{2.54}
\]

**Proof** Denote

\[
I_k^{(l-qL)} := \frac{1}{N!} \int_0^1 d_q x_1 \omega^{(l-qL)}(x_1) \cdots \int_0^1 d_q x_N \omega^{(l-qL)}(x_N) s_\kappa(x_1, \ldots, x_N)
\]

\[
\times \prod_{1 \leq j < k \leq N} (x_k - x_j)^2. \tag{2.55}
\]

Analogous to (2.23) we have

\[
I_k^{(l-qL)} = \det \left[ \int_0^1 \omega^{(l-qL)}(x) x^{\kappa_N + N - j + j + k - 2} \, dq_x \right]_{j,k=1}^N
\]

\[
= \left( \frac{(q; q)_\infty}{(q^{\alpha+1}; q)_\infty} \right)^N \prod_{j=1}^{N} \frac{(q^\alpha; q)_{\kappa_N + j - 1 + j - 1}}{q^{N - j} - q^{N - l}} \det \left[ (q^{\alpha+\kappa_N + N - j + j}; q)_{k-1} \right]_{j,k=1}^N. \tag{2.56}
\]

Using the determinant evaluation noted in (2.30), and normalising to unity for the empty partition, gives (2.53).

\[ \square \]

We next make use of (1.32) to deduce from (2.53) the moments of the corresponding density.
Proposition 2.12 Let the \( q \)-hypergeometric function \( {}_3\phi_2 \) have its usual meaning. For \( k \) a positive integer we have

\[
m^{(l-qL)}_{k,N} = A_{k,N} {}_3\phi_2\left(q^{-(k-1)}q^{-(N-1)}q^{-(\alpha+N-1)}\left| q; q^{-k}\right.\right),
\]

where

\[
A_{k,N} := \frac{(q; q)_{N+k-1+\alpha}(q; q)_{N+k-1}}{(q; q)_{N-1}(q; q)_k(q; q)_{N+\alpha-1}}.
\]

**Proof** According to (1.32) we want to simplify (2.53) for partitions \( \kappa = (k-r, 1^r) \). Noting that

\[
\prod_{1 \leq j < l \leq N} \frac{q^{k_j+N-j} - q^{k_l+N-l}}{q^{N-j} - q^{N-l}} = q^{\sum_{l=1}^{N}(l-1)k_l} \prod_{1 \leq j < l \leq N} \frac{1 - q^{k_j - k_l + l-j}}{1 - q^{l-j}},
\]

we see from (2.1) that the contribution from this factor in (2.53) is, up to the prefactors, and up to mapping \( q \leftrightarrow q^{-1} \), the same as for the Stieltjes–Wigert weight. Thus we read off from the first equality in (2.5) that the contribution of (2.59) is

\[
q^{r(r+1)/2 \left[ \binom{N+k-r-1}{k} \right] q^{\left[ \binom{k-1}{r} \right] q}} = q^{r(r+1)/2} \frac{(q; q)_{k-1}}{(q; q)_k(q; q)_r(q; q)_{N-r}(q; q)_{k-r}}.
\]

where on the RHS we have separated terms of the form \( (q; q)_s \) for some \( s \) independent of \( r \). For the remaining factor in (2.53), which involves a single product, the contribution can be written

\[
\frac{(q; q)_{k-r+N-1+\alpha}}{(q; q)_{N+\alpha-1}} \prod_{l=1}^{r} (1 - q^{\alpha+N-l}) = \frac{(q; q)_{k-r+N-1+\alpha}}{(q; q)_{N+\alpha-1}} (-1)^r q^{r(\alpha+N)-r(r+1)/2} (q^{-\alpha+N+1}; q)_r.
\]

For \( r \) a non-negative integer, we have the general formula

\[
(q; q)_{m-r} = (-1)^r q^{r(r+1)/2} q^{-r(m+1)} \frac{(q; q)_m}{(q^{m-r}; q)_r},
\]

which we recognise as closely related to the manipulation used to obtain the equality in (2.61). This formula can be used to rewrite the terms in the product of (2.60) and (2.61) of the functional form \( (q; q)_{m-r} \). Thus we have
(q; q)_{k-r+N-1+\alpha} \frac{(q; q)_{N+k-1+r}}{(q; q)_{N-1}(q; q)_{k-r-1}}

= q^{-r(k+N+\alpha)} \frac{(q; q)_{N+k-1+\alpha} (q; q)_{N+k-1}}{(q; q)_{N-1}(q; q)_{k-r-1}}

\times \frac{(q^{-k}; q)_r (q^{-N-1}; q)_r}{(q^{-N+k+\alpha-1}; q)_r (q^{-N+k-1}; q)_r}. \quad (2.63)

The terms in (2.60) and (2.61) excluding those on the LHS of (2.63) are

\begin{equation}
(-1)^r q^{r(\alpha+N)} \frac{(q; q)_{k-1}(q^{-(\alpha+N)+1}; q)_r}{(q; q)_k(q; q)_r(q; q)_{N+\alpha-1}}.
\end{equation} \quad (2.64)

Multiplying together (2.64) and the RHS of (2.63) gives us a structured evaluation of \( \langle s_{(k-r,1)} \rangle^{(l-qL)} \). Forming the sum as required by (1.32), and specifying \( A_{k,N} \) by (2.58), shows

\begin{equation}
m_{k,N}^{(l-qL)} = A_{k,N} \sum_{r=0}^{k-1} \frac{(q^{-k}; q)_r (q^{-N-1}; q)_r (q^{-\alpha+N-1}; q)_r q^{-rk}}{(q^{-N+k+\alpha-1}; q)_r (q^{-N+k-1}; q)_r}. \quad (2.65)
\end{equation}

In standard \( q \)-hypergeometric series notation, this is (2.57).

From the identity [46, Eq. 17.3.2] and corresponding limit

\( (x; q)_\infty = \sum_{n=0}^{\infty} q^{n(n-1)/2} (-x)^n (q; q)_n \), \quad \lim_{q \to 1^-} (x(1-q); q)_\infty = e^{-x}, \quad (2.66)

the definition (2.52) of the little \( q \)-Laguerre weight, and (2.31) with \( a = 1/(1-q) \), we see that

\begin{equation}
\lim_{q \to 1^-} (1-q)^{-|k|} \langle s_k \rangle^{(l-qL)} = \langle s_k \rangle^{(L)}.
\end{equation} \quad (2.67)

Here the average on the RHS is with respect to (1.1) with the Laguerre weight \( w^{(L)}(x) \) from (1.7). It follows from (1.32) that the moments in the little \( q \)-Laguerre ensemble are likewise related to those in the Laguerre ensemble by

\begin{equation}
\lim_{q \to 1^-} (1-q)^{-|k|} m_{k,N}^{(l-qL)} = m_{k,N}^{(L)}.
\end{equation} \quad (2.68)

Computing this limit from (2.57) shows

\begin{equation}
m_{k,N}^{(L)} = \frac{(N + k - 1 + \alpha)! (N + k - 1)!}{(N - 1)! k! (N + \alpha - 1)!} 3F_2 \left( \begin{array}{c} -(k-1), -(N-1), -(\alpha+N-1) \\ -N+k-1, -(N+k+\alpha-1) \end{array} \right) \left| 1 \right), \quad (2.69)
\end{equation}

which agrees with a known result [15, displayed equation below (4.16)]. After transformation, (2.69) can be recognised as an example of a particular continuous dual
Hahn polynomial of degree $N - 1$ \cite{15}, which in fact remains true for continuous $k$. In the case of $k$ a non-negative integer, it can be recognised as a continuous Hahn polynomial of degree $k - 1$, and the three-term recurrence of this class of orthogonal polynomial implies the known three-term recurrence for the Laguerre ensemble moments \cite{31},

$$(k + 2)m^{(L)}_{k+1,N} = (2k + 1)(2N + \alpha)m^{(L)}_{k,N} + (k - 1)(k^2 - \alpha^2)m^{(L)}_{k-1,N}. \quad (2.70)$$

In the little $q$-Laguerre case, the form (2.57) does not reveal identification with a $q$-orthogonal polynomial, and the question as to whether $\{m^{(L-qL)}_{k,N}\}$ satisfies a linear recurrence remains open.

In the little $q$-Laguerre case, the form (2.57) does not reveal identification with a $q$-orthogonal polynomial, and the question as to whether $\{m^{(L-qL)}_{k,N}\}$ satisfies a linear recurrence remains open.

Inspection of (2.57) shows the moments $m^{(L-qL)}_{k,N}$ permit the expansion

$$m^{(L-qL)}_{k,N} = \sum_{s_1=0}^{2k} \sum_{s_2=0}^{2k} c_{s_1,s_2} q^{s_1\alpha + s_2 N}, \quad (2.71)$$

where the coefficients $\{c_{s_1,s_2}\}$ are Laurent polynomials independent of $\alpha, N$. However, unlike the analogous expansion in the Stieltjes–Wigert and discrete $q$-Hermite cases (2.13) and (2.43), we have no access to a general explicit evaluation of these coefficients. Equivalently, we are not able to find a closed form for the generating function $\sum_{N,\alpha=0}^{\infty} w_N^\alpha m^{(L-qL)}_{k,N}$. Small values of $k$ (2.57) can be used for this purpose, and indicates that $q^{-1/2}c_{s_1,s_2}$ is a series in $(q^{-n/2} - q^{n/2})_{n=1,2,...}$ and thus the scaled coefficients $(q^{-1/2}/N)c_{s_1,s_2} |_{q=e^{\lambda/N}}$ admit a $1/N^2$ expansion analogous to (2.14) and (2.44).

### 3 Pearson equation and Ledoux’s results

In this section, we revise Ledoux’s results in \cite{40, 41} on the consequences of the Pearson equation in the continuous and linear discrete classical cases, for analysing the integrability properties of (1.37). To begin, we give a brief introduction to the Pearson equation and its consequences in random matrix theory and related topics, and then treat the continuous case and linear discrete case separately.

#### 3.1 The Pearson equation and applications

The structured differential relation for the weight function

$$(\sigma(x)w(x))' = \tau(x)w(x) \quad (3.1)$$

is to be referred to as the Pearson equation in the continuous case. Here it is assumed that $\sigma(x)w(x)$ decays upon approaching the boundary of the support of $w(x)$. As is consistent with (1.6) — this is the original form of the Pearson differential equation \cite{50} — the classical weights are when $\sigma$ and $\tau$ are polynomials of degrees less than 2 and
1, respectively. The Pearson equation has been generalised to discrete measures/non-uniform lattices; see for example [34, 39, 45]. It was proved that for the discrete classical orthogonal polynomials, the weight function \( w(s) \) should satisfy a discrete analogue of the Pearson equation

\[
\frac{\Delta}{\Delta x(s - \frac{1}{2})} [\sigma(s) w(s)] = \tau(s) w(s),
\]

where \( \Delta \) is the forward difference operator corresponding to the lattice, and \( x(s) \) denotes the position of the lattice. Interesting examples include the linear discrete lattice and exponential discrete lattice. The former one induces the so-called discrete orthogonal polynomials, whose weight functions satisfy a linear discrete equation [45]

\[
\frac{w(x + 1)}{w(x)} = \frac{\sigma(x) + \tau(x)}{\sigma(x + 1)},
\]

and the latter one corresponds to the \( q \)-orthogonal polynomials, and their weight functions satisfy a \( q \)-difference equation (1.13).

In addition to its applications in the classification of classical orthogonal polynomials, the Pearson equation has seen a number of applications in the contexts of complex analysis and random matrix theory. One class of application is the evaluation of classical beta integrals [51] and Barnes and Ramanujan-type integrals on the \( q \)-linear lattice [52]. Applications in random matrix theory up to the year 2010 have been discussed in the monograph [20]. The most prominent is to connect classical orthogonal polynomials to particular skew orthogonal polynomials. This result was used to evaluate the Christoffel–Darboux kernel for orthogonal and symplectic ensembles with classical weights, and to show in particular that they are rank-one perturbations of the kernel for the corresponding unitary ensemble [1]. Subsequently the \( q \)-Pearson equation has been used to generalise this theory, for the symplectic ensembles, to the classical \( q \)-cases [25, 55]. Differential identities that follow the Pearson equation in the classical cases [20, Sec. 5.4] have found application in the derivation of formulas for the structure function — that is the covariance of the linear statistics \( \sum_{j=1}^{N} \exp(i k_1 \lambda_j) \), \( \sum_{j=1}^{N} \exp(-i k_2 \lambda_j) \) — in the Gaussian and Laguerre unitary ensembles of random matrix theory [23, 24]. Another recent development has been in relation to analysing the partition function of particular two-component log-gas systems on a line with charges +1 and +2 by making use of the Pearson pair in the classical cases [26].

### 3.2 Continuous case

The continuous orthogonal polynomials \( \{ P_n(x) \}_{n \in \mathbb{N}} \), orthogonal with respect to the weight function \( w(x) \), the Pearson pair \( (\sigma, \tau) \) in (3.1) induces a second-order differential equation

\[
\sigma(x) P''_N(x) + \tau(x) P'_N(x) + \lambda_N P_N(x) = 0,
\]
where $\lambda_N$ is a constant independent of $x$. Since $\sigma(x)w(x)$ in the Pearson pair is required to decay fast at the ends of supports, it follows that for arbitrary $f, g \in L^2(d\mu, [a, b])$ there is an integration by part formula

$$\int_a^b \sigma(x)f'(x)g(x)d\mu = -\int_a^b \sigma(x)f(x)g'(x)d\mu - \int_a^b \tau(x)f(x)g(x)d\mu, \quad (3.4)$$

where $d\mu := w(x)dx$ and $[a, b]$ is the support of $d\mu$.

The study [40] took up the task of finding a differential equation satisfied by the (un-normalised) probability density $P^2_N(x)d\mu$, or more particularly its Laplace transform (for earlier works relating to studies of the differential equation satisfied by products of classical orthogonal polynomials see [33] and references therein). Denote the Laplace transform of the density as

$$\phi(s) := \int e^{sx} P^2_N(x)d\mu,$$

then by making advantage of the integration by part formula (3.4) and taking $f = e^{sx}$ and $g = P^2_N(x)$, $(P^2_N(x))'$ and $(P'_N(x))^2$, one can find

$$s \int e^{sx} \sigma(x)P^2_N(x)d\mu = -\int e^{sx} \tau(x)P^2_N(x)d\mu - \int e^{sx} \sigma(x)(P^2_N(x))'d\mu, \quad (3.5a)$$

$$s \int e^{sx} \sigma(x)P_N(x)P'_N(x)d\mu = -\int e^{sx} \sigma(x)(P'_N(x))^2d\mu + \lambda_N \int e^{sx} P^2_N(x)d\mu, \quad (3.5b)$$

$$s \int e^{sx} \sigma(x)(P'_N(x))^2d\mu = \int e^{sx} \sigma(x)(P'_N(x))^2d\mu + 2\lambda_N \int e^{sx} P_N(x)P'_N(x)d\mu. \quad (3.5c)$$

To characterise the differential equation satisfied by such Laplace transform, one can introduce a differential operator $g$ corresponding to a polynomial $g(x) = \sum_{i=0}^k g_i x^i$, such that

$$g(\phi) = \int e^{sx} g(x) P^2_N(x)d\mu.$$

By some simple calculation, one verifies that $g = \sum_{i=0}^k g_i \partial^i_x$. Moreover, such a differential operator $g$ satisfies the Leibniz rule

$$g(s^k \phi) = \sum_{i=0}^k \binom{k}{i} s^i g^{(k-i)}(\phi), \quad (3.6)$$
where $g^{(i)}$ is the operator corresponding to $g^{(i)}(x)$. Therefore, one can define operators $B$ and $A$ related to $\sigma(x)$ and $\tau(x)$ via

$$B \phi(s) = \int e^{sx} \sigma(x) P_N^2(x) d\mu, \quad A \phi(s) = \int e^{sx} \tau(x) P_N^2(x) d\mu,$$

allowing equations (3.5a)-(3.5c) to be rewritten into the forms

$$(sB + A) \phi = -2B \int e^{sx} P_N(x) P'_N(x) d\mu, \quad (3.7a)$$

$$sB \int e^{sx} P_N(x) P'_N(x) d\mu = -B \int e^{sx} (P'_N(x))^2 + \lambda N \phi, \quad (3.7b)$$

$$(sB - A) \int e^{sx} (P'_N(x))^2 d\mu = 2\lambda N \int e^{sx} P_N(x) P'_N(x) d\mu. \quad (3.7c)$$

Acting $sB$ on equation (3.7c) and substituting $sB \int e^{sx} P_N(x) P'_N(x) d\mu$ by (3.7b) yield

$$sB(sB - A) \int e^{sx} (P'_N(x))^2 d\mu = -2\lambda N B \int e^{sx} (P'_N(x))^2 + 2\lambda N^2 \phi.$$

By using (3.6), we can rewrite this as

$$(s^2B + sB^{(1)} - sA + 2\lambda N) B \int e^{sx} (P'_N(x))^2 d\mu = 2\lambda N^2 \phi.$$

Moreover, from (3.7a) and (3.7b), one knows that

$$(s^2B + sA + 2\lambda N) \phi = 2B \int e^{sx} (P'_N(x))^2 d\mu,$$

thus acting $(s^2B + sB^{(1)} - sA + 2\lambda N)$ on the above equation, one finally gets

$$(s^2B + sB^{(1)} - sA + 2\lambda N)(s^2B + sA + 2\lambda N) \phi = 4\lambda N^2 \phi.$$

**Proposition 3.1** [40, Corollary 3.2] The Laplace transform of the (un-normalised) probability density $P_N^2(x) w(x)$ satisfies the differential equation

$$(s^4M_4 + s^3M_3 + s^2M_2 + sM_1) \phi = 0, \quad (3.8)$$

where $\{M_i\}_{i=1}^4$ are defined by

$$M_1 = B^{(2)}A - A^{(2)}B - A^{(1)}A + 2\lambda N B^{(1)},$$

$$M_2 = 3B^{(2)}B + 2B^{(1)}A + 4\lambda N B - A^2 - 2A^{(1)}B,$$

$$M_3 = 3B^{(1)}B, \quad M_4 = B^2.$$
We give explicit formulas in the appendix.

### 3.3 Linear discrete case

There are many settings involving (1.1) defined on a linear discrete lattice \( \{ x \in \mathbb{Z} \} \) in different physically significant models; see for example [29, 30, 35, 36].

Ledoux [41] generalised the considerations revised above for the continuous classical weights to the case of discrete classical weights on a linear lattice. Starting with a discrete Pearson equation (3.2), analogous to (3.3) it can be deduced that the discrete orthogonal polynomials \( \{ P_N(x) \}_{N=0}^{\infty} \) satisfy the second-order difference equation [34]

\[
[\sigma(x) + \tau(x)] P_N(x + 1) - [2\sigma(x) + \tau(x) - \lambda_N] P_N(x) + \sigma(x) P_N(x - 1) = 0.
\]

Also, with the assumption that the product \( \sigma(x)w(x) \) vanishes at the end points of the support, we can check from (3.2) that the equations

\[
\sum_{x \in \mathbb{Z}} \psi(x) \sigma(x) w(x) = \sum_{x \in \mathbb{Z}} \psi(x + 1) \sigma(x + 1) w(x + 1) = \sum_{x \in \mathbb{Z}} \psi(x + 1) w(x) (\sigma(x) + \tau(x))
\]

are valid for any summable function \( \psi(x) \). Therefore, if we set \( A(x) = \sigma(x) \) and \( B(x) = \sigma(x) + \tau(x) \), the above equations can be equivalently written as [41, Eqs. (11),(12)]

\[
PB P_N(x + 1) = (A + B + C_N) P_N(x) - AP_N(x - 1), \quad (3.9a)
\]

\[
\int A f(x) d\mu = \int B f(x + 1) d\mu, \quad \forall f \in L^1(\mathbb{R}). \quad (3.9b)
\]

Equation (3.9b) is fundamental in deriving a difference equation for the Laplace transform of the (un-normalised) measure \( P^2_N d\mu \), where \( d\mu \) is the discrete measure defined in linear lattice. Since the derivation of the differential equation satisfied by the Laplace transform in discrete case is similar to the continuous case, we omit the details here.

It should be noted that a general boundary condition was considered in [8]. The weight function \( w(x) \) is not defined on the whole real line, but rather supported by a union of several disjoint intervals with condition

\[
\frac{w(x)}{w(x - 1)} = \frac{\phi_+(x)}{\phi_-(x)},
\]

where \( \phi^\pm(x) \) vanishes at the ends of each support and are polynomials of degree at most \( d \). This condition was applied to compute the loop equation (or the so-called Schwinger–Dyson equation) for several discrete models.
A closely related loop equation formalism has been applied recently in the study of a $q$-boxed plane partition model relating to an example of (1.1) on an exponential measure [18]. This is part of our motivation, in addition to its relevance to moments, to consider whether $q$-orthogonal polynomial ensembles satisfy such an integration by part formula, and whether it can be applied to formulate a $q$-difference equation for the Laplace transform of the density.

4 Laplace transform of discrete $q$-measure and its $q$-difference equation

In this part, we consider discrete orthogonal polynomials (or so-called $q$-orthogonal polynomials) defined on the exponential lattice $\{q^n \mid n \in \mathbb{Z}\}$. We call a family of $q$-orthogonal polynomials $\{P_N(x; q)\}_{N=0}^{\infty}$ classical, with respect to the weight function $\rho(x; q)$, if they satisfy the orthogonal relation

$$\int_a^b P_N(x; q)P_M(x; q)\rho(x; q)d_qx = h_N\delta_{N,M}, \quad h_N > 0,$$

where the Jackson’s $q$-integral is given as (1.34). Moreover, the weight function $\rho(x; q)$ satisfies a $q$-Pearson equation (1.13) with degrees of $\sigma(x)$ and $\tau(x)$ being, at most, of 2 and 1, respectively. From [2, 45], one knows that classical $q$-orthogonal polynomials satisfy the following second-order difference equation

$$\sigma(x)D_qD_{q^{-1}}P_N(x) + \tau(x)D_qP_N(x) + C_NP_N(x) = 0, \quad (4.1)$$

where $C_N$ is a constant independent of $x$. Moreover, regarding with the Pearson pair $(\sigma, \tau)$, one can state the following proposition.

**Proposition 4.1** The $q$-Pearson equation (1.13) induces the $q$-integration by parts formula

$$\int_a^b \sigma(x)D_qf(x)g(x)d_q\mu = -\int_a^b \tau(x)f(qx)g(qx)d_q\mu$$

$$-\int_a^b \sigma(x)f(qx)D_qg(x)d_q\mu, \quad (4.2)$$

where $d_q\mu = \rho(x; q)d_qx$ and $f(x), g(x)$ are two arbitrary $q$-summable functions.

**Proof** Let us start with the general integration by parts formula given by [34]

$$\int_a^b \sigma(x)D_qf(x)g(x)d_q\mu = f(x)g(x)\sigma(x)\rho(x)|_a^b$$

$$-\int_a^b f(qx)D_q(g(x)\sigma(x)\rho(x))d_qx.$$
Since for classical $q$-weight functions, $\sigma(x)\rho(x)$ vanishes at the end points of support [2, eq. (3.6)], we know the first term on the right-hand side vanishes. In relation to the second term on the right-hand side, we observe from the $q$-Pearson equation (1.13) that

$$D_q(g(x)\sigma(x)\rho(x)) = D_q g(x)\sigma(x)\rho(x) + g(qx)\tau(x)\rho(x),$$

thus completing the proof.

** Remark 4.2** With $q \mapsto q^{-1}$, the above integration by part formula can be written

$$\int_a^b \sigma(x) D_{q^{-1}} f(x) g(x) dq \mu = -q \int_a^b \tau(x) f(x) g(x) dq \mu - \int_a^b \sigma(x) f(q^{-1}x) D_{q^{-1}} g(x) dq \mu.$$  (4.3)

To obtain a $q$-analogue of the linear difference equation (3.9b), let us introduce a polynomial function

$$T(x) = \sigma(x) - (1 - q)x\tau(x)$$  (4.4)

and observe

$$\int_a^b \sigma(x) f(x) dq \mu = q \int_a^b T(x) f(qx) dq \mu.$$  (4.5)

One can define a $q$-Laplace transform of the un-normalised measure $P^2_N(x) dq \mu$

$$\phi(\lambda) := \int_a^b e_q(\lambda x) P^2_N(x) dq \mu,$$  (4.6)

where $e_q(x) = \sum_{k=0}^{\infty}((1 - q)x)^k/(q; q)_k$; cf. (2.66). This $q$-exponential function satisfies

$$D_{q, x} e_q(\lambda x) = \lambda e_q(\lambda x), \quad D_{q^{-1}} e_q(\lambda x) = \lambda e_q(q^{-1}\lambda x) = \lambda \Lambda^{-1} e_q(\lambda x),$$  (4.7)

where $\Lambda$ is a shift operator such that $\Lambda \phi(\lambda) = \phi(q\lambda)$. Therefore, for an arbitrary polynomial function $R(x) = a_n x^n + \cdots + a_0$, there exists a corresponding $q$-difference operator $\mathcal{R}(D_{q, \lambda}) = a_n D_{q, \lambda}^n + \cdots + a_0$ such that

$$\mathcal{R}(D_{q, \lambda}) \phi(\lambda) = \int_a^b e_q(\lambda x) R(x) P^2_N(x) dq \mu.$$  (4.8)

Moreover, for arbitrary $n, m \in \mathbb{N}$, we have

$$\int_a^b e_q(q^n \lambda x) R(q^m x) P^2_N(x) dq \mu = \Lambda^{n-m} \mathcal{R}(q^m D_{q, \lambda}) \phi(\lambda),$$
providing us with an analogue of (3.6).

**Proposition 4.3** For a polynomial \( R(x) = a_n x^n + \cdots + a_0 \), the corresponding \( q \)-difference operator \( \mathcal{R}(D_q, \lambda) \) satisfies

\[
\mathcal{R}(D_q, \lambda)(\lambda^m \phi) = \sum_{i=0}^{m} \lambda^{m-i} \binom{m}{i}_q \mathcal{R}_m^{(i)} \phi, \quad m \in \mathbb{Z}_{\geq 0},
\]

where \( \mathcal{R}_m^{(i)} \) is the operator corresponding to \( \Lambda^{m-i} D_q^i R(x) \).

**Proof** Let us first prove that

\[
D_q^n(\lambda f) = [n]_q D_q^{n-1} f + q^n \lambda D_q^n f, \quad [n]_q = \frac{1 - q^n}{1 - q} \tag{4.9}
\]

holds for arbitrary function \( f \). Making use of induction, we assume that (4.9) is true and then

\[
D_q^{n+1}(\lambda f) = D_q([n]_q D_q^{n-1} f + q^n \lambda D_q^n f)
\]

\[
= [n]_q D_q^n f + q^n D_q^n f + q^{n+1} \lambda D_q^{n+1} f
\]

\[
= [n + 1]_q D_q^n f + q^{n+1} \lambda D_q^{n+1} f.
\]

Taking \( f = \lambda^{m-1} \phi, m \geq 1 \), and using \( q \)-binomial formula the result follows. \( \square \)

For example, if we consider \( R(x) \) as a second-order polynomial, we have

\[
\mathcal{R}(\lambda \phi) = \lambda \mathcal{R}_1^{(0)} \phi + \mathcal{R}_1^{(1)} \phi,
\]

\[
\mathcal{R}(\lambda^2 \phi) = \lambda^2 \mathcal{R}_2^{(0)} \phi + \lambda(1 + q) \mathcal{R}_2^{(1)} \phi + \mathcal{R}_2^{(2)} \phi,
\]

\[
\mathcal{R}(\lambda^3 \phi) = \lambda^3 \mathcal{R}_3^{(0)} \phi + \lambda^2 (1 + q + q^2) \mathcal{R}_3^{(1)} \phi + \lambda(1 + q + q^2) \mathcal{R}_3^{(2)} \phi
\]

since \( \mathcal{R}_3^{(3)} = 0 \). Specifically, if we denote \( R(x) = r_2 x^2 + r_1 x + r_0 \), then we have

\[
\mathcal{R}_1^{(0)} = r_2 q^2 D_q^2 + r_1 q D_q + r_0, \quad \mathcal{R}_1^{(1)} = r_2 (1 + q) D_q + r_1,
\]

\[
\mathcal{R}_2^{(0)} = r_2 q^4 D_q^4 + r_1 q^2 D_q + r_0, \quad \mathcal{R}_2^{(1)} = r_2 q(1 + q) D_q + r_1, \quad \mathcal{R}_2^{(2)} = r_2 (1 + q),
\]

\[
\mathcal{R}_3^{(0)} = r_2 q^6 D_q^6 + r_1 q^3 D_q + r_0, \quad \mathcal{R}_3^{(1)} = r_2 q^2 (1 + q) D_q + r_1, \quad \mathcal{R}_3^{(2)} = r_2 (1 + q).
\]

For later use, we denote \( \mathcal{A}, \mathcal{B} \) and \( \mathcal{T} \) as operators corresponding to \( \tau(x), \sigma(x) \) and \( T(x) \) in the sense of (4.8). Now taking \( f = e_q(\lambda x) \) and \( g = P_N^2(\phi) \), and using formulas (4.3) and (4.5), one can get
\[(\lambda,\Lambda^{-1}B + qA)\phi = -\Lambda^{-1}B \int_0^\infty e_q(\lambda x) P_N D_q^{-1} P_N d_q \mu \]

\[-qT \int_0^\infty e_q(\lambda x) P_N D_q P_N d_q \mu. \tag{4.10}\]

Since \(qT \int_0^\infty e_q(\lambda x) P_N D_q P_N d_q \mu = B \int_0^\infty e_q(q^{-1}\lambda x) P_N (q^{-1}x) D_{q^{-1}} P_N d_q \mu,\) and by using formula (4.2) with \(f = e_q(\lambda x)\) and \(g = P_N (q^{-1}x) D_{q^{-1}} P_N,\) one has

\[\lambda T \int_0^\infty e_q(\lambda x) P_N D_q P_N d_q \mu = C_N \phi - q^{-1}B \int_0^\infty e_q(\lambda x)(D_{q^{-1}} P_N)^2 d_q \mu. \tag{4.11}\]

Similarly, by taking \(f = e_q(\lambda x)\) and \(g = P_N D_q^{-1} P_N,\) one has

\[\lambda B \int_0^\infty e_q(\lambda x) P_N D_{q^{-1}} P_N d_q \mu = C_N \phi - q^{-1}B \int_0^\infty e_q(\lambda x)(D_{q^{-1}} P_N)^2 d_q \mu. \tag{4.12}\]

Use of formula (4.2) is made to evaluate the last term in the above equation. Thus by taking \(f = e_q(\lambda x)\) and \(g = (D_{q^{-1}} P_N)^2\) in the integration by parts formula, one has

\[\lambda B \int_0^\infty e_q(\lambda x)(D_{q^{-1}} P_N)^2 d_q \mu = \Lambda A \int_0^\infty e_q(\lambda x) D_q P_N D_{q^{-1}} P_N d_q \mu \]

\[+ C_N \Lambda \int_0^\infty e_q(\lambda x) P_N D_{q^{-1}} P_N d_q \mu + C_N \Lambda \int_0^\infty e_q(\lambda x) P_N D_q P_N d_q \mu. \tag{4.13}\]

Here we remark that when we take the continuum limit \(q \to 1,\) then equations (4.10)-(4.13) coincide with equations (3.7a)-(3.7c). However, these equations have not yet been closed — more relations are needed. In this regard, one can notice that

\[\sigma D_{q^{-1}} P_N (D_q P_N - D_{q^{-1}} P_N) = \sigma (q - 1)x D_{q^{-1}} P_N D_q D_{q^{-1}} P_N \]

\[= (1 - q)x D_{q^{-1}} P_N (\tau D_q P_N + C_N P_N),\]

which results in

\[T \int_0^\infty e_q(\lambda x) D_{q^{-1}} P_N D_q P_N d_q \mu \]

\[= B \int_0^\infty e_q(\lambda x)(D_{q^{-1}} P_N)^2 d_q \mu + C_N \mathcal{F} \int_0^\infty e_q(\lambda x) P_N D_{q^{-1}} P_N d_q \mu, \tag{4.14}\]

where \(\mathcal{F}\) is the \(q\)-difference operator corresponding to \((1- q)x\) satisfying \(T = B - A \mathcal{F}.\)

Therefore, we end up with five equations (4.10)-(4.14)) for \(\phi, \psi_1, \psi_2, \psi_3\) and \(\psi_4,\)
where
\[
\psi_1 := \int_0^\infty e_q(\lambda x) P_N D_q^{-1} P_N d_q \mu, \quad \psi_2 := \int_0^\infty e_q(\lambda x) P_N D_q P_N d_q \mu, \\
\psi_3 := \int_0^\infty e_q(\lambda x)(D_q^{-1} P_N)^2 d_q \mu, \quad \psi_4 := \int_0^\infty e_q(\lambda x) D_q^{-1} P_N D_q P_N d_q \mu.
\]

We will now show how one can eliminate the latter four terms to get an equation for \(\phi\). By acting operator \(T \Lambda^{-1} - A\) on both sides of (4.13) and making use of (4.14), one has
\[
(q^{-1}T \Lambda^{-1} - A)B \psi_3 = C_N B \psi_1 + C_N T \psi_2.
\]
By substituting Eq. (4.11) and (4.12) into the above equation, one has
\[
(\lambda q^{-1}T \Lambda^{-1} - \lambda A + 2q^{-1}C_N)B \psi_3 = C_N^2 (1 + \Lambda) \phi. \tag{4.15}
\]
Meanwhile, by substituting Eqs. (4.11) and (4.12) into Eq. (4.10), one finds that
\[
q(q\lambda^2 B + q\lambda A + 2C_N \Lambda) \phi = (1 + \Lambda)B \psi_3.
\]
Thus, together with Eq. (4.15), we have
\[
(\lambda T \Lambda^{-1} - q\lambda A + 2C_N)(1 + \Lambda)^{-1}(q\lambda^2 B + q\lambda A + 2C_N \Lambda) \phi = C_N^2 (1 + \Lambda) \phi. \tag{4.16}
\]
By noting that for \(i \in \mathbb{Z}\), there holds a commutative formula
\[
(1 + \Lambda)^{-1} \lambda^i = \lambda^i (1 + q^i \Lambda)^{-1},
\]
and degree of \(T\) and \(A\) are no greater than 2 and 1, respectively, we can state the following result.

**Theorem 4.4** The \(q\)-Laplace transform of the un-normalised measure \(P_N^2 d_q \mu\) on the exponential lattice satisfies
\[
(\lambda^4 \mathcal{M}_4 + \lambda^3 \mathcal{M}_3 + \lambda^2 \mathcal{M}_2 + \lambda \mathcal{M}_1 + \mathcal{M}_0) \phi = 0, \tag{4.17}
\]
where
\[
\mathcal{M}_4 = q^{-1}T_3^{(0)} \Lambda^{-1} (1 + q^2 \Lambda)^{-1} B, \\
\mathcal{M}_3 = (q^{-1}(1 + q + q^2)T_3^{(1)} \Lambda^{-1} - q^2 A_2^{(0)})(1 + q^2 \Lambda)^{-1} B + T_2^{(0)}(1 + q \Lambda)^{-1} A, \\
\mathcal{M}_2 = (q^{-1}(1 + q + q^2)T_3^{(2)} \Lambda^{-1} - q^2(1 + q)A_2^{(1)} + 2qC_N)(1 + q^2 \Lambda)^{-1} B \\
+ ((1 + q)T_2^{(1)} - q^2 A_1^{(0)} \Lambda)(1 + q \Lambda)^{-1} A + 2C_N T_1^{(0)}(1 + \Lambda)^{-1}.
\]
\( \mathcal{M}_1 = (T_2^{(2)} - q^2 A_1^{(1)} \Lambda + 2qC_N \Lambda)(1 + q \Lambda)^{-1} \Lambda + 2C_N(T_1^{(1)} - qA\Lambda)(1 + \Lambda)^{-1}, \)
\( \mathcal{M}_0 = C_N^2(4(1 + \Lambda)^{-1} \Lambda - (1 + \Lambda)). \)

Here \( \Lambda \) is the shift operator satisfying \( \Lambda \phi(x) = \phi(\lambda x) \), \( C_N \) is a particular coefficient in the \( q \)-difference equation (4.1) and \( A, B \) and \( T \) are \( q \)-difference operators related to \( \tau(x), \sigma(x) \) and \( T(x) \) in the sense of (4.8), with \( \sigma(x), \tau(x) \) specified as the Pearson pair and \( T(x) = \sigma(x) - (1 - q)\tau(x) \).

5 Application in the case of the discrete \( q \)-Hermite weight

Let \( \{p_n^{(d-qH)}(x; q)\}_{n \in \mathbb{N}} \) denote the monic \( q \)-orthogonal polynomials with respect to the discrete \( q \)-Hermite weight (2.19). It has been shown in [58] that up to proportionality, the moments of the corresponding un-normalised measure \( (p_n^{(d-qH)}(x; q))^2 d_q \mu \) have combinatorial meaning in terms of rook placements. This motivates giving particular attention to this case.

Our first result of interest is that, according to Theorem 4.4, the \( q \)-Laplace transform of the measure satisfies the fourth-order \( q \)-difference (4.17). Regarding the explicit form of the quantities in the latter, we note that the Pearson pair in the case of the discrete \( q \)-Hermite weight is given by
\[
(\sigma, \tau) = (x^2 - 1, (1 - q)^{-1}x). \tag{5.1}
\]
Thus the \( q \)-difference operators \( A, B, T \) are specified by in the sense of (4.8) with the corresponding polynomials. (4.8) with
\[
A(x) = (1 - q)^{-1}x, \quad B(x) = x^2 - 1, \quad T(x) = -1.
\]
Also, for the constant \( C_N \) in (4.1), equating like powers of \( x^N \) shows
\[
C_N = -q^{-(N-1)} \frac{1 - q^N}{(1 - q)^2}. \tag{5.2}
\]
We notice that when \( N = 0 \) the formula (5.2) gives \( C_0 = 0 \). Each of the operators \( \mathcal{M}_i \) for \( i = 0, 1, 2 \) in (4.17) then simplify relative to the general case. The \( q \)-difference equation is then simple enough for each term to be written out. Let the corresponding \( q \)-Laplace transform be denoted by \( \phi_0(\lambda) \), so that
\[
\phi_0(\lambda) := \int_{-1}^{1} e_q(\lambda x) w^{(d-qH)}(x) d_q x. \tag{5.3}
\]
Noting in the definition of \( e_q(x) \) as a sum below (4.6) that all terms are positive, as is \( w^{(d-qH)}(x) \), shows that the order of the sum and the integral can be interchanged.
Computing the resulting integral using (2.24) then gives
\[ \phi_0(\lambda) = \sum_{k=0}^{\infty} \frac{(1 - q)^{2k+1}\lambda^{2k}}{(q^2; q^2)_k} = (1 - q)e_q^2 \left( \frac{\lambda^2}{(1 + q)^2} \right). \] (5.4)

The simplicity of (5.4) allows for an independent verification of the \( q \)-difference equation.

**Proposition 5.1** The \( q \)-difference Laplace transform \( \phi_0(\lambda) \) satisfies the \( q \)-difference equation

\[
\begin{align*}
&\left( \lambda^4 q^{-1} \Lambda^{-1}(1 + q^2 \Lambda)^{-1}(D_q^2 - 1) \\
&+ \lambda^3 (q^4 (1 - q)^{-1} D_q (1 + q^2 \Lambda)^{-1} (D_q^2 - 1) - (1 - q)^{-1} (1 + q \Lambda)^{-1} D_q) \\
&+ \lambda^2 (q^2 (1 + q)(1 - q)^{-1} (1 + q^2 \Lambda)^{-1} (D_q^2 - 1) \\
&+ q^2 (1 - q)^{-2} D_q \Lambda (1 + q \Lambda)^{-1} D_q) \\
&+ \lambda q^2 (1 - q)^{-2} \Lambda (1 + q \Lambda)^{-1} D_q \right) \phi_0(\lambda) = 0.
\end{align*}
\] (5.5)

**Proof** The Pearson pair with respect to the discrete \( q \)-Hermite polynomials is given by
\[ (\sigma, \tau) = (x^2 - 1, (1 - q)^{-1}x), \] (5.6)

and the corresponding operators in the sense of (4.8) are
\[ A(x) = (1 - q)^{-1}x, \quad B(x) = x^2 - 1, \quad T(x) = -1, \]
while the constant is given by \( C_0 = 0 \). By substituting these into Proposition 4.4, we obtain the explicit formula for those difference operators and thus the \( q \)-difference equation.

On the other hand, the validity of (5.5) can be checked directly. By denoting
\[ f_k = \frac{(1 - q)^{2k+1}}{(q^2; q^2)_k}, \quad [k]_q = \frac{1 - q^k}{1 - q}, \]
and taking \( \phi_0(\lambda) \) in terms of the series form from (5.4) and substituting it into the \( q \)-difference equation, one can compute the coefficients of \( \lambda^{2k} \) as

\[
\begin{align*}
[\lambda^{2k}] &= \frac{1}{1 + q^{2k-2}} \\
&\times \left( -q^{-2k+3} [2k - 2]_q [2k - 3]_q f_{k-1} + q^{-2k+3} f_{k-2} - \frac{[2k - 2]_q}{1 - q} f_{k-1} \right) \\
+ \frac{1}{1 + q^{2k}} \left( -q^4 [2k]_q [2k - 1]_q [2k - 2]_q f_k + q^4 [2k - 2]_q f_{k-1} \right)
\end{align*}
\]
\[-q^2 \frac{1+q}{1-q} ([2k]_q[2k-1]_q f_k - f_{k-1}) \]
\[-q^{2k+2} \frac{[2k]_q[2k-1]_q}{(1-q)^2} f_k - q^{2k+1} [2k]_q f_k \]
\[= q^{2k-1} \frac{(1-q^{2k}) f_k}{(1-q)^4(1+q^{2k})} q^4(1-q^{2k-2}) + q^2(1-q^2) \]
\[-q^3 (1-q^{2k-1}) - q^2 (1-q) \] = 0,

as required. \(\square\)

We now turn to the derivation of an explicit formula for the moments of \((p_N^{(d-qH)}(x; q))^2 d_q \mu\). Different from [38, Sec. 3.28], we consider orthonormal polynomials \(\{H_\ell(x; q)\}_\ell \in \mathbb{N}\) satisfying
\[\int_{-1}^{1} H_m(x; q) H_n(x; q) d_q \mu = \delta_{n,m},\]
where \(d_q \mu\) is specified by the corresponding weight function of discrete \(q\)-Hermite polynomials given by (2.19). It is known that the normalised discrete \(q\)-Hermite polynomials have the generating function [39]
\[(t^2; q^2)_{\infty} = \sum_{n=0}^{\infty} q^{\frac{1}{2}(n-1)} \sqrt{1-q} H_n(x; q) t^n.\]
The generating function implies the \(q\)-difference equation
\[H_n(qx; q) = H_n(x; q) - (1-q^n) d_n^{-1} x H_{n-1}(x; q), \quad d_n = q^{(n-1)/2}(1-q^n)^{1/2}, \quad (5.7)\]
and the three-term recurrence relation
\[x H_n(x; q) = d_{n+1} H_{n+1}(x; q) + d_n H_{n-1}(x; q).\]
Combining these two equations leads to
\[H_n(qx; q) = q^n H_n(x; q) - q^{-n+1} d_n d_{n-1} H_{n-2}(x; q), \quad (5.8)\]
which is useful for deriving a particular linearisation formula.

**Proposition 5.2** The shifted normalised discrete \(q\)-Hermite polynomial \(H_n(q^k x; q)\) is expressed in terms of the un-shifted normalised polynomials according to
\[H_n(q^k x; q) = \sum_{l=0}^{\min(k, \lfloor \frac{k}{2} \rfloor)} \left((-1)^l q^{(k-l)(n-2l)+l(l-n)} \left[\begin{array}{c} k \\ k-l \end{array}\right] q^{2l-1} \prod_{i=0}^{l-1} d_{n-i}\right) H_{2l}(x; q).\]
Proof By induction using (5.8), a combinatorial formula for $H_n(q^k x; q)$ is

$$
H_n(q^k x; q) = \sum_{l=0}^{\min(k, \lceil \frac{n}{2} \rceil)} \left( (-1)^l q^{l(n-l-n)} \sum_{p \in \mathcal{P}_l} q^{\sum_{i=1}^l p_i(n-2i+2)} \prod_{i=0}^{2l-1} d_{n-i} \right) H_{n-2l}(x; q),
$$

where $\mathcal{P}_l$ denotes all the $l$-partitions $(p_1, p_2, \ldots, p_l)$ of $m$ such that $\sum_{i=1}^l p_i = m$. This summation can be interpreted as a generating function for paths between two fixed points in the $\mathbb{Z}^2$ lattice, which is explained below. Denote

$$
\hat{f}_{l,k} = \sum_{p \in \mathcal{P}_l} q^{\sum_{i=1}^l p_i(n-2i+2)} := q^{k(n-2l+2)} \hat{f}_{l,k}^l,
$$

and consider the lattice $\mathbb{Z} \times 2\mathbb{Z}$ with edges pointing rightward and downward, one can view $\hat{f}_{l,k}^l$ as the generating function of paths from the point $(0, 2(l-1))$ to the point $(k, 0)$, and one obtains the recurrence relation

$$
q^{2k} \hat{f}_{l,k}^{l-1} + \hat{f}_{k-1}^l = \hat{f}_{l,k}^l, \quad \hat{f}_{1,1}^1 = 1, \quad \hat{f}_{1,2}^2 = 1 + q^2, \quad \hat{f}_{2,2}^2 = 1.
$$

The recurrence relation is solved in terms of the $q$-binomial formula and

$$
\hat{f}_{l,k}^l = \left\lfloor \frac{l - 1 + k}{k} \right\rfloor q^2,
$$

thus completing the proof. \qed

We are now in a position to deduce an evaluation of the moments of the normalised measure $H_N^2(x; q) d_q \mu$.

**Proposition 5.3** Define

$$
\tilde{M}_{k,N} := \int_{-1}^{1} \prod_{i=0}^{k-1} (1 - q^{-2i} x^2) H_N^2(x; q) d_q \mu, \quad \tilde{m}_{2p,N} := \int_{-1}^{1} x^{2p} H_N^2(x; q) d_q \mu.
$$

(5.9)

One has

$$
\tilde{M}_{k,N} = q^k \sum_{l=0}^{k} q^{2kN+l(4l-4k-2N-1)} \left\lfloor \frac{k}{k-l} \right\rfloor^2 \frac{(q; q)_N}{q^{2l} (q; q)_{N-2l}}
$$

(5.10)

and

$$
\tilde{m}_{2p,N} = \sum_{i=0}^{p} (-1)^i q^{i(i-1)} \left\lfloor \frac{p}{i} \right\rfloor q^2 \tilde{M}_{i,N}.
$$

(5.11)
Proof From the integration by parts formula (4.5), which we know is a consequence of the Pearson equation, one notices that for the measure $d_q \mu$ of the discrete $q$-Hermite polynomials,

$$ q \int_{-1}^{1} f(qx) d_q \mu = \int_{-1}^{1} (1 - x^2) f(x) d_q \mu. $$

Hence

$$ \int_{-1}^{1} \prod_{i=0}^{k-1} (1 - q^{2i} x^2) H_N^2(x; q) d_q \mu = q^k \int_{-1}^{1} H_N^2(q^k x; q) d_q \mu $$

$$ = q^k \sum_{l=0}^{\min(k, \frac{N}{2})} q^{2(k-l)(N-2l)+2l(l-N)} \left[ \frac{k}{k-l+1} \right] \prod_{i=0}^{2l-1} d_{N-i}^2, \quad (5.12) $$

which when substituted in the definition of $\tilde{M}_{k,N}$ from (5.9), and upon recalling the definition of $d_n$ from (5.7), gives (5.10).

By recognising the fact that

$$ x^{2p} = \sum_{i=0}^{p} (-1)^i q^{i(i-1)} \left[ \begin{array}{c} p \\ i \end{array} \right] q^{2} \prod_{j=0}^{i-1} (1 - q^{-2j} x^2), $$

we see from the definition of $\tilde{m}_{2p,N}$ in (5.9) that the evaluation (5.11) follows as a corollary of (5.10). \hfill \Box

**Remark 5.4** From the definition of the spectral density moments $m_{2k,N}^{(d-qH)}$ implied by substituting (1.5) in (1.8), and the definition of $\tilde{m}_{2p}$ from (5.9) we have

$$ \tilde{m}_{2p,N} = m_{2p,N+1}^{(d-qH)} - m_{2p,N}^{(d-qH)}. \quad (5.13) $$

Substituting for the RHS according to (2.40) gives a formula for $\tilde{m}_{2p,N}$ involving the difference of two single sums rather than a double sum as in (5.11). For low orders we have used computer algebra to check that both expressions are identical, but do not have a general proof.

The expression for $\tilde{m}_{2p,N}$ as implied by (5.13) is already reported in [57, Th. 5]. Previously, using combinatorial methods based on a rook placement interpretation of $\{\tilde{m}_{2p,N}\}$ yet another distinct expression was obtained [58, Eq. (5.26)]. Another relevant point is that $q$-difference equation (4.17) for the $q$-Laplace transform implies a corresponding linear recurrence for the moments, analogous to what is known in the continuous classical cases [40].

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Appendix

Here we rederive known differential equations satisfied by the Laplace transform of the un-normalised measure \((P_N(x))^2 d\mu\) in the classical continuous cases using the Pearson pair formalism of Sect. 3.2. The classical continuous cases correspond to the weights (1.7), distinguished by having all moments finite, and the Cauchy weight \((1 + x^2)^{-a}\), \(a > 1/2\), where the number of finite moments depends on \(a\). The corresponding Pearson pairs are

\[
(\sigma, \tau) = \begin{cases} 
(1, -2x) & \text{Hermite case,} \\
(x, a + 1 - x) & \text{Laguerre case,} \\
(x(1 - x), a + 1 - (a + b + 2)x) & \text{Jacobi case,} \\
(1 + x^2, 2(1 - a)x) & \text{Cauchy case}
\end{cases}
\]

According to the theory of Sect. 3.2, these imply the differential operators

\[
\begin{align*}
A_H &= -2 \frac{d}{ds}, \quad A_H^{(1)} = -2, \quad B_H = 1, \quad B_H^{(1)} = B_H^{(2)} = 0, \quad \lambda_N^{(H)} = 2N, \\
A_L &= a + 1 - \frac{d}{ds}, \quad A_L^{(1)} = -1, \quad B_L = \frac{d}{ds}, \quad B_L^{(1)} = 1, \quad B_L^{(2)} = 0, \quad \lambda_N^{(L)} = N, \\
A_J &= a + 1 - (a + b + 2) \frac{d}{ds}, \quad A_J^{(1)} = -a - b - 2, \\
B_J &= \frac{d}{ds} - \frac{d^2}{ds^2}, \quad B_J^{(1)} = 1 - 2 \frac{d}{ds}, \\
B_J^{(2)} &= -2, \quad \lambda_N^{(J)} = N(N + a + b + 1), \\
A_C &= 2(1 - a) \frac{d}{ds}, \quad A_C^{(1)} = 2(1 - a), \quad B_C = 1 + \frac{d^2}{ds^2}, \quad B_C^{(1)} = 2 \frac{d}{ds}, \quad B_C^{(2)} = 2, \\
\lambda_N^{(C)} &= -N(N - 2a + 1)
\end{align*}
\]

in the four cases, respectively. Application of Proposition 3.1 then gives the differential equations of the Laplace transforms of the un-normalised measure. These are

\[
4s \Phi'' + 4 \Phi' - (s^3 + (8N + 4)s) \Phi = 0,
\]

in Hermite case

\[
(s^3 - s) \Phi'' + (3s^2 + 2(2N + a + 1)s - 1) \Phi' + (s(1 - a^2) + a + 1 + 2N) \Phi = 0,
\]

in the Laguerre case

\[
\begin{align*}
&\quad \quad s^3 \Phi^{(4)} + (6s^2 - 2s^3) \Phi''' \\
&\quad + \left( s^3 - 9s^2 + [6 - 4N(N + a + b + 1) - (a + b)(a + b + 2)]s \right) \Phi'' \\
&\quad + \left( 3s^2 + [4N(N + a + b + 1) + 2(a + 1)(a + b) - 6]s \right.
\end{align*}
\]
\[-(a + b)(a + b + 4) - 4N(N + a + b + 1) \Phi' + \left( (1 - a^2)s + (a + b)(a + 1) + 2N(N + a + b + 1) \right) \Phi = 0\]

in the Jacobi case, and
\[
\lambda^3 \Phi^{(4)} + 6\lambda^2 \Phi^{(2)}'' + \left( 2\lambda^3 + [4a(1 - a) + 6 - 4N(N - 2a + 1)]\lambda \right) \Phi'' + \left( 6\lambda^2 + 4[a(1 - a) - N(N - 2a + 1)] \right) \Phi' + \left( \lambda^3 + [2 - 4N(N - 2a + 1) + 4a]\lambda \right) \Phi = 0,
\]

in the Cauchy case. The first three cases were considered in [40] and the Cauchy one was recently shown in [6]; see also [27] in relation to the latter.

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