Quantum dynamics of PT-symmetrically kicked particle confined in a 1D box

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We study quantum particle dynamics in a box and driven by PT-symmetric, delta-kicking complex potential. Such dynamical characteristics as the average kinetic energy as function of time and quasi-energy at different values of the kicking parameters. Breaking of the PT-symmetry at certain values of the non-Hermitian kicking parameter is shown. Experimental realization of the model is also discussed.

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I. INTRODUCTION

PT-symmetric quantum systems attracted much attention during past two decades after the discovery of the fact that non-Hermitian, but PT-symmetric system can have a set of eigenstates with real eigenvalues \cite{1}. In other words, self-adjointness of the Hamiltonian is not necessary condition for being the eigenvalues real. Currently quantum physics of PT-symmetric such systems has become rapidly developing topic of contemporary physics and great progress is made in the study of different aspects of such systems (see, e.g., papers \cite{2}-\cite{23} for review of recent developments on the topic). These studies allowed to construct complete theory of PT-symmetric quantum systems, including PT-symmetric field theory \cite{8, 15}. Experimental realization of such systems was also subject for extensive research. The latter has been done mainly in optics \cite{24-27}. Some other PT-symmetric systems are discussed recently in the literature \cite{30, 31}. PT-symmetric relativistic system are also studied in \cite{19, 20}. General condition for PT-symmetry in quantum systems has been derived in terms of so-called CPT-symmetric inner product \cite{3, 10, 15}. Similarly to the case of Hermiticity, PT-symmetry in quantum systems can be introduced either through the complex potential, or by imposing proper boundary conditions, which provide such symmetry via the CPT-inner product \cite{3, 15}. Different types of complex potentials providing PT-symmetry in Hamiltonian have been considered in \cite{10, 15, 29, 30}. PT-symmetric particle-in-box system, where the box boundary conditions provide PT-symmetry of the system, have been studied in \cite{14, 21-23}. Certain progress is also done in nonlinear extension of PT-symmetric systems \cite{26, 28}.

In this paper we consider quantum particle confined in a 1D box and driven by a PT-symmetric, delta-kicking potential with the focus on the role of non-Hermitian parameter on such characteristics as average kinetic, total energy and quasienergy. Here we mention that some time ago, both the classical and the quantum dynamics of systems interacting with a delta-kicking potential have been extensively studied in the context of nonlinear dynamics and quantum chaos theory \cite{38-43}. Kicked quantum particle dynamics in a box have been also considered in \cite{35, 37}. For kicked systems, the classical dynamics is characterized by diffusive growth of the average kinetic energy as a function of time, while for corresponding quantum systems such growth suppressed (except the special cases of so-called quantum resonances). The latter is called quantum localization of classical chaos \cite{38-42}. The dynamics of kicked nonrelativistic system is governed by single parameter, product of the kicking strength and kicking period.

We note that earlier, PT-symmetrically kicked systems have been considered in the Refs. \cite{29, 30} in the context of quantum chaos theory. In \cite{29} PT-symmetrically kicked rotor is studied by developing one-parameter scaling theory for non-Hermitian parameter and focusing on the gain, loss effects. In \cite{30} PT-symmetrically kicked quantum rotor is studied by analyzing quasienergy spectrum and evolution of the momentum distribution at different values of the non-Hermitian parameter. Here we consider PT-symmetrically kicked confined system, by focusing on the role of confinement and non-Hermitian part of the kicking potential. Usual way for creating of kicked quantum system is confining of the system in a standing wave cavity. PT-symmetric analog of such system could be realized in a cavity with the losses. Another option, putting the system in a transverse beam propagation inside a passive optical resonator with combined phase and loss gratings, was discussed, e.g., \cite{30}. An optical waveguide which is driven by PT-symmetric optical field can be considered as another version of the model we are going to treat. This paper is organized as follows. In the next section we briefly recall Hermitian counterpart of our system, quantum particle confined in a 1D box and driven by delta-kicking potential. In section III we consider similar system with PT-symmetric delta-kicks. Section IV presents some concluding remarks.
II. KICKED QUANTUM PARTICLE DYNAMICS IN A BOX

Hermitian counterpart of the system we are going to study, is a quantum particle confined in one-dimensional box of size $L$ and driven by external delta-kicking potential given by

$$U(x,t) = \epsilon \cos\left(\frac{2\pi x}{\mu}\right) \sum_l \delta(t - lT),$$

where $\mu$, $\epsilon$ and $T$ are the wavelength, kicking strength and period, respectively. Such system was considered earlier in the context of quantum chaos theory e.g., in [35, 36] and described by the following time-dependent Schrödinger equation:

$$i \frac{\partial}{\partial t} \Psi(x,t) = \left[ -\frac{1}{2} \frac{d^2}{dx^2} + U(x,t) \right] \Psi(x,t), \quad (1)$$

The wave function, $\Psi(x,t)$ fulfills the box boundary conditions given by

$$\Psi(0,t) = \Psi(L,t). \quad (2)$$

Exact solution of Eq.(1) can be obtained within the single kicking period [36, 38] by expanding the wave function, $\Psi(x,t)$ in terms of the complete set of the eigenfunctions of the unperturbed system as

$$\Psi(x,t) = \sum_n A_n(t) \psi_n(x) \quad (3)$$

where $\psi_n(x) = \sqrt{2/L} \sin\left(\pi nx/L\right)$. Eqs.(1) and (3) lead to quantum mapping for the wave function amplitudes, $A_n(t)$ which is given by

$$A_n(t + T) = \sum_l A_l(t) U_{ln} e^{-iE_l T}, \quad (4)$$

where

$$U_{ln} = \int_0^L \psi^*_n(x) e^{-i\epsilon \cos(2\pi x/\mu)} \psi_l(x) dx.$$
where $E_n$ are given by Eq.5. Fig.2 presents plots of
the average kinetic energy, $\langle E_k(t) \rangle$ at different values
of the kicking strength, $\varepsilon$ for fixed kicking period $T$. Unlike
the kicked rotor $\langle E_k(t) \rangle$ grows during
some initial time and suppression with the subsequent decrease
occurs for large enough number of kick ($N = t/T$). For very large number of kicks one can observe periodic
or quasi-periodic time-dependence of $\langle E_k(t) \rangle$. Such behavior in some kicked quantum systems have been discussed in [10]. Another feature
of kicked quantum particle confined in a box is the absence of quantum resonance. It should be noted
that the dynamics of kicked particle confined in a box depends on two factors, such as interaction with the
kicking force and bouncing of particle from the box walls. Depending on the sign of of cosine in the kicking
potential, the kicking force can be attractive and repulsive. When the kicking potential is repulsive
particle gains the energy, while in case of attractive potential it losses its energy. Therefore depending on which
area in the box, i.e. on the area where the kicking force is positive or negative, acceleration or deceleration
of the particle may occur. Very important factor is "synchronization" of the kicking force and bouncing
of particle from the box wall. It also may cause acceleration and deceleration of the particle.

III. $PT$–SYMMETRICALLY KICKED QUANTUM PARTICLE A ONE DI DIMENSIONAL BOX

$PT$-symmetric analog of the above system can be constructed by adding into the kicking potential an imaginary part. Then $PT$-symmetric kicking

potential can be written as

$$V_{PT}(x,t) = f(t) \left[ \varepsilon \cos \left( \frac{2\pi x}{\mu} \right) + i\gamma \sin \left( \frac{2\pi x}{\mu} \right) \right],$$

(10)

where $\varepsilon$ and $T$ are the kicking strength and period, respectively, $\gamma \geq 0$ is the non-Hermitian parameter that measures the strength of the imaginary part of the potential and $f(t) = \sum \delta(t - iT).$ The dynamics of the system is governed by the following time-dependent Schrödinger equation:

$$i\frac{\partial}{\partial t} \Psi(x,t) = H_{PT}\Psi(x,t),$$

(11)

where $H_{PT}$ is the Schrödinger operator containing potential $U_{PT}$. The same boundary conditions as those in Eq.2. Exact solution of Eq.11 can be obtained similarly to the case of Hermitian counterpart and one gets quantum mapping for the evolution of the amplitude, $A_n(t)$ within the one kicking period, $T$:

$$A_n(t + T) = \sum_{l} A_l(t)V_{ln} e^{-iE_l T},$$

(12)

where

$$V_{ln} = \int \psi_n^*(x)e^{-i\varepsilon \cos(2\pi x/\mu) \gamma \sin(2\pi x/\mu)} \psi_l(x) dx$$

and $E_l = (\pi l/L)^2$. The evolution operator corresponding to Eq.12 can be written as

$$\hat{U}_{PT} = \exp(-i\frac{\partial^2}{2\partial x^2}) \exp(-i\beta V(x)) \exp(-i\frac{\partial^2}{2\partial x^2}),$$

(14)

where

$$\beta = \frac{\pi T}{\mu^2}.$$

For a quantum systems with complex $PT$-symmetric potentials, the norm conservation is broken, i.e., the amplitudes, $A_n(t)$ do not fulfill Eq.10.
IV. CONCLUSIONS

We studied quantum dynamics of a particle confined in a 1d box and driven by PT-symmetric, delta-kicking potential. Different characteristics of the dynamics, such as the time-dependence of the average kinetic energy, quasienergy and the average total energy are analyzed using the exact solution of the time-dependent Schrödinger equation for single kicking period. It is found that no unbound acceleration in PT-symmetric quantum regime is possible, as the average kinetic energy is the periodic or quasi-periodic in time. However, in PT-symmetrically driven system the gain of energy and acceleration are more intensive than those for the Hermitian counterpart. The above model can be realized in different versions using optical systems where it is possible to create PT-symmetric kicking potential. Such kicking field could be realized e.g., in an optical cavity with losses and gains. Confining an optical pulse in such cavity would be a version for our model. Another option is considering a PT-symmetric periodic optical structure, e.g., array of optical waveguides driven by laser field.

In the absence of external perturbation such system is described by the Helmholtz equation with periodic boundary condition, which is an analog of the box boundary condition. Therefore the driven waveguide array can be considered as an analog of the above model.

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