Consistent Quantum Expansion Around Soliton Solutions

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ABSTRACT

I show that a standard application of the semiclassical techniques to 1+1 dimensional field theories, as originally discussed by Dashen, Hasslacher and Neveu, explicitly violates the Poincare algebra. This problem is traced to the incorrect regularization of the ultraviolet divergences and can be resolved by using a different regularization. I further show that in the case of the doublet solutions in the sine-Gordon theory the semiclassical treatment given by Dashen, Hasslacher and Neveu leads to ambiguous results which depend on the choice of the renormalization counterterm. I discuss a consistent weak coupling expansion which does not suffer from this problem.

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1. Introduction

The procedure of calculating quantum corrections to classical soliton solutions was developed in the mid seventies [1,2]. The formalism is based on the assumption that the quantum corrections are small compared to the classical solution. The basic idea can be explained by taking the simple example of 1+1 dimensional sine-Gordon theory. This theory admits both time independent and time dependent soliton solutions. For simplicity I consider first the time independent classical solution \( \phi_{cl}(x) \). The quantum corrections can be calculated by expanding the field \( \phi \) as,

\[
\phi(x,t) = \phi_{cl}(x-Z(t)) + \sum_n q_n(t) \psi_n(x-Z(t))
\]

Here I have introduced the collective coordinate \( Z \) and the fluctuation coordinates \( q_n \). The quantum hamiltonian can then be obtained easily by introducing any set of orthonormal coordinates \( Q_n \) and making a change of variables to the set \( Z,q_n \).

The formalism described above has been applied to the 1+1 dimensional sine-Gordon as well as the \( \phi^4 \) field theory by several authors. This application, however, did not respect Poincare invariance or multiplicative renormalizability. It turns out that these difficulties arise because of incorrect regularization and because of the fact that certain reasonable semi-classical techniques, valid for finite number of degrees of freedom, fail in quantum field theory.

The fact the Poincare algebra is not satisfied can be easily checked by evaluating the matrix elements of the commutator of momentum with the boost generator between one soliton state. Explicit calculation shows that this is not equal to the \(-i\) times the energy of the soliton, if we follow the procedure described in Ref. [1]. The problem is traced to the incorrect choice of regularization to regulate the ultraviolet divergent sum over the frequencies. The regularization used in Ref. [1] is,

\[
A_1 = \sum_{n=-N}^{N} \tilde{\omega}_n - \sum_{n=-N}^{N} \omega_n
\]

where \( \omega_n = \sqrt{k_n^2 + m^2} \) are the frequencies in the vacuum sector and \( \tilde{\omega}_n = \sqrt{q_n^2 + m^2} \) are the frequencies in the soliton sector, \( m \) being the meson mass. The problem is easily corrected by choosing a different regulator such as,

\[
A_2 = \sum_{n=-\infty}^{\infty} [\tilde{\omega}_n f(t,q_n) - \omega_n f(t,k_n)]
\]

where \( f(t,q_n) \) is a regulating function such that it goes to 1 as \( t \) goes to 0 and falls off sufficiently fast for large \( |q_n| \) so as to regulate the two sums. The fact that some regulators fail to give reasonable answers is somewhat disturbing but not too surprising. The same problem occurs if we calculate the Casimir energy between two parallel plates in QED. The result given by regulator in Eq. 2 is infinite and has the wrong sign. The correct result is obtained by using the regulator in Eq. 3.

We next examine the classical doublet solutions of the sine-Gordon equation. The semi-classical analysis of these states is rather interesting since the quantum
expansion used by Dashen, Hasslacher and Neveu (DHN) in Ref. [1] seems to yield exact answers. However as we have argued above, the regularization used in Ref. [1] violates Poincare invariance and therefore it is somewhat surprising that it yields exact results. A careful analysis of the calculation of quantum corrections to the doublet states, however, reveals that the exact answer is obtained only by a specific choice of the mass counterterm. A different choice of the counterterm gives a wrong answer if the same procedure is used as the one used in Ref. [1] for the doublet solutions, thereby leading to a breakdown of multiplicative renormalizability.

To discuss the problem in more detail I briefly review the semi-classical technique applied to the sine-Gordon doublet states in Ref. [1]. The authors consider $G(E)$, defined as,

$$G(E) = Tr \left[ \frac{1}{E - H} \right] = iTr \int dT \exp[i(E - H)T] \tag{4}$$

where $\phi(x, t = 0) = \phi(x, t = T)$, $\phi$ being in the classical doublet solution. By making an expansion around the classical doublet solution and keeping only the leading order corrections to the classical solution DHN show that

$$G(E) = \int \frac{dM}{2\pi i} G^0(M, E) \tilde{G}(M) \tag{5}$$

$$\tilde{G}(M) = -\left(-\frac{i}{2\pi}\right)^{1/2} \sum_{l} \int_{0}^{\infty} \tau d\tau \sqrt{l} \left[\frac{d\tilde{S}}{d\tau} \right]^{1/2} \left|d^2\tilde{S}/d\tau^2\right|^{1/2} \exp[i(l\tilde{S} + M\tau)] \tag{6}$$

where $\tau$ is the time period for one cycle. DHN perform the integration over $\tau$ by using the stationary phase approximation which yields,

$$M = -\frac{\partial \tilde{S}}{\partial \tau} \tag{7}$$

$$\tilde{S} = S_{cl}(\phi) - \frac{1}{2} \sum_{i} \nu_{i} + S_{ct}(\phi) \tag{8}$$

where $S_{ct}$ is the mass counterterm in the action. The poles in $G(E)$ are then found to be at,

$$W(M_N) = S_{cl}(\phi) - \frac{1}{2} \sum_{i} \nu_{i} + S_{ct}(\phi) + E\tau(M) = 2N\pi \tag{9}$$

where $N=1,2,\ldots\leq 8\pi/\gamma$, $\gamma$ is a function of the mass and the coupling constant and is defined in [1].

However as shown in [3], the above procedure does not yield unique results for $M_2/M_1$. The exact result was obtained in [1] by a specific choice of mass counterterm. If we choose a different counterterm and follow the same procedure as outlined above we get a wrong answer [3].

This problem can be corrected if the integration over the time period in equation 6 is performed by applying the stationary phase approximation only to classical action. The quantum corrections are then calculated by making a weak coupling expansion around this stationary point. As shown in [3] the result obtained
for $M_n/M_j$ is independent of the choice of counterterm to order $\lambda^3$, $\lambda$ being the coupling constant, and agree with the standard perturbation theory result to this order. As expected, the result does not agree with the higher order two loop result.

In conclusion we have discussed some difficulties with a standard application of the semiclassical techniques to 1+1 dimensional field theories. We have shown that although some regularization techniques violate Poincare algebra it is possible to choose one which does not suffer from this problem. We have also argued that in order to get unambiguous results it is necessary to make a weak coupling expansion around the classical solution. The alternate procedure used in [1], which seems to go beyond the weak coupling expansion, fails to give unambiguous results in field theory.

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6. References

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