Bohr type inequality for Cesáro and Bernardi integral operator on simply connected domain

VASUDEVARAO ALLU\textsuperscript{1,*} and NIRUPAM GHOSH\textsuperscript{2}

\textsuperscript{1}School of Basic Sciences, Indian Institute of Technology Bhubaneswar, Argul, Bhubaneswar 752 050, India
\textsuperscript{2}Statistics and Mathematics Unit, Indian Statistical Institute, 8th Mile, Mysore Road, Bangalore 560 059, India
*Corresponding Author. E-mail: avrao@iitbbs.ac.in

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Abstract. In this article, we study the Bohr type inequality for Cesáro operator and Bernardi integral operator acting on the space of analytic functions defined on a simply connected domain containing the unit disk $\mathbb{D}$.

Keywords. Analytic functions; Bohr radius; Cesáro operator; Bernardi integral.

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1. Introduction

In 1914, Bohr [5] proved that if the power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ converges in the unit disk $\mathbb{D} := \{ z \in \mathbb{C} : |z| < 1 \}$ and $|f(z)| \leq 1$ for all $z \in \mathbb{D}$, then the majorant series $M_f(r) = \sum_{n=0}^{\infty} |a_n| r^n \leq 1$ for all $|z| = r < 1/6$. Later M. Riesz, I. Schur and N. Wiener independently established the best possible constant as $1/3$ (see [27], [28]). Recently, in 2022, Paulsen et al. [23] provided a simple and interesting proof of Bohr inequality. The idea of Bohr type inequalities in the higher dimension case has been introduced by Boas and Khavinson [6]. Later Aizenson [3] studied the Bohr type inequalities on Reinhardt domain and on complete circular domain (see [24]). Moreover, Paulsen and Singh [21,22] have considered similar problems for Hardy spaces or for more abstract spaces. Over the past two decades, there has been significant interest on the Bohr type inequalities. In this context, we suggest the reader to refer articles [1,2,4,8,11,13,15] and the references therein.

Let $\Omega$ be a proper simply connected domain containing $\mathbb{D}$ and $\mathcal{H}(\Omega)$ be the class of analytic functions on $\Omega$. Let

$$\mathcal{B}(\Omega) = \{ f \in \mathcal{H}(\Omega) : |f(z)| \leq 1 \text{ for } z \in \Omega \}.$$ 

In 2010, Fournier and Ruscheweyh [9] extended the notion of Bohr radius for the family $\mathcal{B}(\Omega)$. They also defined the positive real number $R_\Omega \in (0, 1)$ given by
\[ R_{\Omega} = \sup \left\{ r \in (0, 1) : M_f(r) \leq 1 \text{ for } f(z) = \sum_{n=0}^{\infty} a_n z^n \in B(\Omega), \ z \in \mathbb{D} \right\}, \]

where \( M_f(r) = \sum_{n=0}^{\infty} |a_n| r^n \) with \( |z| = r \) is the majorant series associated with \( f \in B(\Omega) \) in \( \mathbb{D} \).

Besides the Bohr radius, there is a notion of Rogosinski radius [19, 25] which is described as follows: Let \( f(z) = \sum_{n=0}^{\infty} a_n z^n \in B(\mathbb{D}) \) and the corresponding partial sum of \( f \) be defined by \( S_N(z) = \sum_{n=0}^{N-1} a_n z^n \). Then, for every \( N \geq 1 \), we have \( |S_N(z)| < 1 \) in the disk \( |z| < 1/2 \) and the radius 1/2 is sharp. Motivated by Rogosinski radius, Kayumov and Ponnusamy [12] have considered the Bohr–Rogosinski sum as

\[ R_{\Omega}^f := |f(z)| + \sum_{n=N}^{\infty} |a_n| |z|^n \]

for \( f \in B(\mathbb{D}) \) and have defined the Bohr–Rogosinski radius as the largest number \( r > 0 \) such that \( R_{\Omega}^f \leq 1 \) for \( |z| < r \). For a significant and extensive research in the direction of Bohr–Rogosinski radius, we referred to [11, 14, 17] and the references therein.

A natural question arises: “Can we extend the Bohr type inequality for certain complex integral operators defined on various function spaces?” The idea has been initiated for the classical Cesáro operator in [16, 17] and for Bernardi integral operator in [18] in the case of unit disk \( \mathbb{D} \). In [16–18], the Bohr type and Bohr–Rogosinski type inequalities for Cesáro operator and Bernardi integral operator defined on \( B(\mathbb{D}) \) have been studied.

Cesáro operator and its various generalizations have been extensively studied. For example, the boundedness and compactness of the Cesáro operator on different function spaces have been well studied. In the classical setting, for an analytic function \( f(z) = \sum_{n=0}^{\infty} a_n z^n \) on the unit disk \( \mathbb{D} \), the Cesáro operator is defined by [10] (see also [7, 26]) as

\[ C f(z) := \sum_{n=0}^{\infty} \frac{1}{n+1} \left( \sum_{k=0}^{n} a_k \right) z^n = \int_{0}^{1} \frac{f(tz)}{1-tz} dt. \quad (1) \]

It is not difficult to show that for \( f \in B(\mathbb{D}) \),

\[ |C f(z)| = \left| \sum_{n=0}^{\infty} \frac{1}{n+1} \left( \sum_{k=0}^{n} a_k \right) z^n \right| \leq \frac{1}{r} \ln \frac{1}{1-r} \quad \text{for } |z| = r. \]

In 2020, Kayumov et al. [16] established the following Bohr type inequality for Cesáro operator.

**Theorem 1 [16].** If \( f \in B(\mathbb{D}) \) and \( f(z) = \sum_{n=0}^{\infty} a_n z^n \), then

\[ C f(r) = \sum_{n=0}^{\infty} \frac{1}{n+1} \left( \sum_{k=0}^{n} |a_k| \right) r^n \leq \frac{1}{r} \ln \frac{1}{1-r} \]

for \( |z| = r \leq R \), where \( R = 0.5335 \ldots \) is the positive root of the equation

\[ 2x = 3(1 - x) \ln \frac{1}{1-x}. \]

The number \( R \) is the best possible.
For an analytic function \( f(z) = \sum_{n=m}^{\infty} a_n z^n \) on the unit disk \( \mathbb{D} \), the Bernardi integral operator (see [20]) is defined by
\[
L_\beta f(z) := \frac{1 + \beta}{z^\beta} \int_0^z f(\xi) \xi^{\beta-1} d\xi = (1 + \beta) \sum_{n=m}^{\infty} \frac{a_n}{\beta + n} z^n,
\]
where \( \beta > -m \) and \( m \geq 0 \) is an integer. It is worth mentioning that for each \( |z| = r \in [0, 1) \), the integral representation for the Bernardi integral operator yields the following for \( f \in B(\mathbb{D}) \):
\[
|L_\beta f(z)| = \left| (1 + \beta) \sum_{n=m}^{\infty} \frac{a_n}{\beta + n} z^n \right| \leq (1 + \beta) \frac{r^m}{m + \beta},
\]
which is equivalent to the following expression:
\[
\left| \sum_{n=m}^{\infty} \frac{a_n}{\beta + n} z^n \right| \leq \frac{r^m}{m + \beta}.
\]
Recently, Kumar and Sahoo [18] have studied the following Bohr type inequality for the Bernardi integral operator.

**Theorem 2 [18].** Let \( \beta > -m \). If \( f(z) = \sum_{n=m}^{\infty} a_n z^n \in B(\mathbb{D}) \), then
\[
\sum_{n=m}^{\infty} \frac{|a_n|}{\beta + n} |z|^n \leq \frac{r^m}{m + \beta}
\]
for \( |z| = r \leq R(\beta) \). Here \( R(\beta) \) is the positive root of the equation
\[
\frac{x^m}{m + \beta} - 2 \sum_{n=m+1}^{\infty} \frac{x^n}{n + \beta} = 0
\]
that cannot be improved.

The main aim of this paper is to find the sharp Bohr type inequality for the Cesáro operator and Bernardi integral operator for functions in the class \( B(\Omega_\gamma) \), where
\[
\Omega_\gamma := \left\{ z \in \mathbb{C} : \left| z + \frac{\gamma}{1 - \gamma} \right| < \frac{1}{1 - \gamma} \right\} \quad \text{for } 0 \leq \gamma < 1.
\]
Clearly, the unit disk \( \mathbb{D} \) is always a subset of \( \Omega_\gamma \). In 2010, Fournier and Ruscheweyh [9] extended the Bohr’s inequality for functions in \( B(\Omega_\gamma) \).

The following lemma by Evdoridis et al. [8] plays a crucial rule to prove our main results.

**Lemma 3 [8].** For \( \gamma \in [0, 1) \), let
\[
\Omega_\gamma := \left\{ z \in \mathbb{C} : \left| z + \frac{\gamma}{1 - \gamma} \right| < \frac{1}{1 - \gamma} \right\},
\]
and let \( f \) be an analytic function in \( \Omega_\gamma \), bounded by 1, with the series representation \( f(z) = \sum_{n=0}^{\infty} a_n z^n \) in \( \mathbb{D} \). Then
\[
|a_n| \leq \frac{1 - |a_0|^2}{1 + \gamma} \quad \text{for } n \geq 1.
\]
2. Main results

We state and prove our first main result.

**Theorem 4.** For $0 \leq \gamma < 1$, let $f \in B(\Omega)_{\gamma}$ with $f(z) = \sum_{n=0}^{\infty} a_n z^n$ in $\mathbb{D}$. Then we have

$$C_f(r) = \sum_{n=0}^{\infty} \frac{1}{n+1} \left( \sum_{k=0}^{n} |a_k| \right) r^n \leq \frac{1}{r} \ln \frac{1}{1-r} \text{ for } |z| = r \leq R_\gamma,$$

where $R_\gamma$ is the positive root of

$$(3 + \gamma)(1 - x) \ln \frac{1}{1 - x} = 2x.$$

The number $R_\gamma$ is the best possible.

**Proof.** Let $|a_0| = a < 1$. A simple computation of the Cesáro operator in (1) shows that

$$C_f(r) = \sum_{n=0}^{\infty} \frac{1}{n+1} \left( \sum_{k=0}^{n} |a_k| \right) r^n \leq \frac{1}{r} \ln \frac{1}{1-r} \text{ for } |z| = r \leq R_\gamma,$$

Using Lemma 3 in (3), we obtain the following estimation for the Cesáro operator:

$$C_f(r) \leq \frac{a}{r} \ln \frac{1}{1-r} + \frac{1 - a^2}{1+\gamma} \sum_{n=1}^{\infty} \frac{n}{n+1} r^n.$$

Let

$$P_{\gamma,r}(a) = \frac{a}{r} \ln \frac{1}{1-r} + \frac{1 - a^2}{1+\gamma} \left( \frac{1}{1-r} - \frac{1}{r} \ln \frac{1}{1-r} \right).$$

Then differentiation of $P_{\gamma,r}$ twice with respect to $a$ shows that

$$P''_{\gamma,r}(a) = \frac{-2}{1+\gamma} \left( \frac{1}{1-r} - \frac{1}{r} \ln \frac{1}{1-r} \right) \leq 0$$

for all $a \in [0, 1)$ and for all $r \in [0, 1)$. Therefore, $P'_{\gamma,r}$ is a decreasing function and hence we obtain
\[
P'_{\gamma,r}(a) \geq P'_{\gamma,r}(1) = \frac{1}{r(1-r)(1+\gamma)} \left(-2r + (3 + \gamma)(1-r) \ln \frac{1}{1-r}\right) \geq 0
\]  

(5)

for all \( r \leq R_{\gamma} \). Thus, \( P_{\gamma,r}(a) \) is increasing for \( r \leq R_{\gamma} \) and for all \( \gamma \in [0, 1) \) and hence

\[
P_{\gamma,r}(a) \leq P_{\gamma,r}(1) = \frac{1}{r} \ln \frac{1}{1-r}
\]

for all \( r \leq R_{\gamma} \).

(6)

Therefore, the desired inequality (2) follows (6).

Now we show that the radius \( R_{\gamma} \) cannot be improved. In order to prove the sharpness of the result, we consider the function \( G : \Omega_{\gamma} \to \mathbb{D} \) defined by \( G(z) = (1-\gamma)z + \gamma \) and \( \psi : \mathbb{D} \to \mathbb{D} \) defined by

\[
\psi(z) = \frac{a-z}{1-a\gamma}
\]

for \( a \in (0, 1) \). Then \( f_{\gamma} = \psi \circ G \) maps \( \Omega_{\gamma} \) univalently onto \( \mathbb{D} \). A simple computation shows that

\[
f_{\gamma}(z) = \frac{a - \gamma - (1-\gamma)z}{1-a\gamma - a(1-\gamma)z} = A_0 - \sum_{n=1}^{\infty} A_n z^n, \quad z \in \mathbb{D},
\]

where \( a \in (0, 1) \),

\[
A_0 = \frac{a - \gamma}{1-a\gamma} \quad \text{and} \quad A_n = \frac{1-a^2}{a(1-a\gamma)} \left( \frac{a(1-\gamma)}{1-a\gamma} \right)^n.
\]

(7)

For a given \( \gamma \in [0, 1) \), let \( a > \gamma \). Then the Cesáro operator on \( f_{\gamma} \) shows that

\[
C_{f_{\gamma}}(r) = \sum_{n=0}^{\infty} \frac{1}{n+1} \left( \sum_{k=0}^{n} |A_k| \right) r^n
\]

(8)

\[
= \frac{A_0}{r} \ln \frac{1}{1-r} + \sum_{n=1}^{\infty} \frac{1}{n+1} \left( \sum_{k=1}^{n} |A_k| \right) r^n.
\]

By substituting \( A_0 \) and \( A_n \) for \( n \geq 1 \) in (8), we obtain

\[
C_{f_{\gamma}}(r) = \frac{a - \gamma}{r(1-a\gamma)} \ln \frac{1}{1-r} + \sum_{n=1}^{\infty} \frac{1}{n+1} \left( \sum_{k=1}^{n} \frac{1-a^2}{a(1-a\gamma)} \left( \frac{a(1-\gamma)}{1-a\gamma} \right)^k \right) r^n
\]

\[
= \frac{a - \gamma}{r(1-a\gamma)} \ln \frac{1}{1-r} + \frac{(1-a^2)(1-\gamma)}{(1-a\gamma)^2} \sum_{n=1}^{\infty} \frac{1}{n+1} \left( \sum_{k=1}^{n} \left( \frac{a(1-\gamma)}{1-a\gamma} \right)^{k-1} \right) r^n
\]
\[ a - \gamma \frac{\ln \frac{1}{1-r}}{1-a\gamma} + \frac{(1+a)(1-\gamma)}{(1-a\gamma)} \sum_{n=1}^{\infty} \frac{1}{n+1} \left( 1 - \frac{a^n(1-\gamma)^n}{(1-a\gamma)^n} \right) r^n. \]  

(9)

Further simplification of (9) shows that

\[ C_{f,\gamma}(r) = a - \gamma \frac{\ln \frac{1}{1-r}}{1-a\gamma} + \frac{(1+a)(1-\gamma)}{r(1-a\gamma)} \left( \frac{1}{1-r} - \frac{1+a}{ar} \ln \left( \frac{1}{1 - \frac{(1-\gamma)ar}{1-a\gamma}} \right) \right) \]

\[ = \frac{1}{r} \ln \frac{1}{1-r} + \frac{(1-a)}{(1-a\gamma)} \frac{2r + (3+\gamma)(1-r) \ln(1-r)}{r(1-r)} + D_{a,\gamma}(r), \]

where

\[ D_{a,\gamma}(r) = \frac{(3-a) - \gamma(1+a)}{1-a\gamma} - 2 \frac{(1-a)}{(1-a\gamma)(1-r)} \]

\[ - \frac{1+a}{ar} \ln \left( \frac{1}{1 - \frac{(1-\gamma)ar}{1-a\gamma}} \right) \]

\[ = \sum_{n=1}^{\infty} \left( \frac{(3-a) - \gamma(1+a)}{1-a\gamma} - 2 \frac{(1-a)}{(1-a\gamma)} - \frac{a^n(1+a)(1-\gamma)^{n+1}}{(1-a\gamma)^{n+1}} \right) r^n \]

\[ = O((1-a)^2) \text{ as } a \to 1. \]

From (5), we obtain \((-2r + (3+\gamma)(1-r) \ln \frac{1}{1-r}) \geq 0 \text{ for all } r \leq R_{\gamma}\) and hence

\[ \frac{2r + (3+\gamma)(1-r) \ln(1-r)}{r(1-r)} > 0 \text{ for } r > R_{\gamma}. \]

These two facts show that the number cannot be improved. \qed

Remark 5. Since for \(\gamma = 0\), the domain \(\Omega_{\gamma}\) reduces to the unit disk \(\mathbb{D}\). Theorem 1 is a direct consequence of Theorem 4 when \(\gamma = 0\).

In the next result, we study the Bohr type inequality for Bernardi integral operator for the class of analytic functions defined on \(\Omega_{\gamma}\).

**Theorem 6.** For \(0 \leq \gamma < 1\), let \(f \in \mathcal{B}(\Omega_{\gamma})\) with \(f(z) = \sum_{n=0}^{\infty} a_n z^n\) in \(\mathbb{D}\). Then for \(\beta > 0\),

\[ \sum_{n=0}^{\infty} \frac{|a_n|}{n + \beta} r^n \leq \frac{1}{\beta} \text{ for } r \leq R_{\gamma,\beta}. \]
where $R_{\gamma, \beta}$ is the positive root of

$$\frac{1}{\beta} = \frac{2}{1 + \gamma} \sum_{n=1}^{\infty} \frac{r^n}{n + \beta}.$$ 

The number $R_{\gamma, \beta}$ is the best possible.

Proof. Let $|a_0| = a < 1$. Then

$$\sum_{n=0}^{\infty} \frac{|a_n|}{n + \beta} r^n = \frac{a}{\beta} + \sum_{n=1}^{\infty} \frac{|a_n|}{n + \beta} r^n. \quad (10)$$

In view of Lemma 3 and (10), we obtain

$$\sum_{n=0}^{\infty} \frac{|a_n|}{n + \beta} r^n \leq \frac{a}{\beta} + \frac{1 - a^2}{1 + \gamma} \sum_{n=1}^{\infty} \frac{r^n}{n + \beta}.$$

Let

$$\Phi_{\gamma, \beta}(a) = \frac{a}{\beta} + \frac{1 - a^2}{1 + \gamma} \sum_{n=1}^{\infty} \frac{r^n}{n + \beta}.$$

Then differentiation of $\Phi_{\gamma, \beta}$ twice with respect to $a$ shows that

$$\Phi''_{\gamma, \beta}(a) = -\frac{2}{1 + \beta} \sum_{n=1}^{\infty} \frac{r^n}{n + \gamma} \leq 0$$

for all $a \in [0, 1]$ and for all $r \in [0, 1)$. This implies that $\Phi'_{\gamma, \beta}$ is decreasing and

$$\Phi'_{\gamma, \beta}(a) \geq \Phi'_{\gamma, \beta}(1) = \left( \frac{1}{\beta} - \frac{2}{1 + \gamma} \sum_{n=1}^{\infty} \frac{r^n}{n + \beta} \right) \geq 0 \quad (11)$$

for $r \leq R_{\gamma, \beta}$. Hence $\Phi_{\gamma, \beta}(a)$ is increasing for $r \leq R_{\gamma, \beta}$. Therefore for all $a \in [0, 1]$,

$$\Phi_{\gamma, \beta}(a) \leq \Phi_{\gamma, \beta}(1) = \frac{1}{\beta} \quad \text{for } r \leq R_{\gamma, \beta}$$

and hence

$$\sum_{n=0}^{\infty} \frac{|a_n|}{n + \beta} r^n \leq \frac{1}{\beta} \quad \text{for } r \leq R_{\gamma, \beta}.$$

We now show that $R_{\gamma, \beta}$ cannot be improved. In order to do this, consider the function

$$f_{\gamma}(z) = \frac{a - \gamma - (1 - \gamma)z}{1 - a\gamma - a(1 - \gamma)z} = A_0 - \sum_{n=1}^{\infty} A_n z^n, \quad z \in \mathbb{D},$$
where $a \in (0, 1)$, and $A_n(n \geq 0)$ are given by (7). For a given $\gamma \in [0, 1)$, let $a > \gamma$. Then for $\gamma \in [0, 1)$ and $\beta \geq 1$, we have

$$
\sum_{n=0}^{\infty} \frac{|A_n|}{n+\beta} r^n = \frac{A_0}{\beta} + \sum_{n=1}^{\infty} \frac{|A_n|}{n+\beta} r^n = \frac{a-\gamma}{(1-a\gamma)\beta} + \sum_{n=1}^{\infty} \frac{1-a^2}{a(1-a\gamma)} \left( \frac{a(1-\gamma)}{1-a\gamma} \right)^n \frac{r^n}{n+\beta}.
$$

By a simple computation, from (12), we obtain

$$
\sum_{n=0}^{\infty} \frac{|A_n|}{n+\beta} r^n = \frac{1}{\beta} - (1-a) \left( \frac{1}{\beta} - \frac{2}{1+\gamma} \sum_{n=1}^{\infty} \frac{r^n}{n+\beta} \right) + M_{a,\gamma,\beta}(r),
$$

where

$$
M_{a,\gamma,\beta}(r) = -\frac{1}{\beta} + \frac{a-\gamma}{(1-a\gamma)\beta} + \frac{(1-a^2)}{a(1-a\gamma)} \sum_{n=1}^{\infty} \left( \frac{a(1-\gamma)}{1-a\gamma} \right)^n \frac{r^n}{n+\beta} + (1-a) \left( \frac{1}{\beta} - \frac{2}{1+\gamma} \sum_{n=1}^{\infty} \frac{r^n}{n+\beta} \right) = (a-1) \left( \frac{\gamma(a+1)}{1-a\gamma} \right) + \frac{2}{1+\gamma} \sum_{n=1}^{\infty} \frac{r^n}{n+\beta} + \frac{(1-a^2)}{a(1-a\gamma)} \sum_{n=1}^{\infty} \left( \frac{a(1-\gamma)}{1-a\gamma} \right)^n \frac{r^n}{n+\beta}.
$$

Letting $a \to 1$, we obtain

$$
M_{a,\gamma,\beta}(r) = O((1-a)^2).
$$

Further from (11), we obtain $\left( \frac{1}{\beta} - \frac{2}{1+\gamma} \sum_{n=1}^{\infty} \frac{r^n}{(n+\beta)} \right) \geq 0$ for all $r \leq R_{\gamma,\beta}$. Therefore,

$$
\frac{1}{\beta} - \frac{2}{1+\gamma} \sum_{n=1}^{\infty} \frac{r^n}{n+\beta} < 0 \quad \text{for} \quad r > R_{\gamma,\beta}.
$$

These two facts show that $R_{\gamma,\beta}$ cannot be improved. \qed

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Conflict of interest
The authors declare that there are no conflicts of interest regarding the publication of this paper.

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