On Emery-Kivelson line and universality of Wilson ratio of spin anisotropic Kondo model

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Abstract

Yuval-Anderson’s scaling analysis and Affleck-Ludwig’s Conformal Field Theory approach are applied to the $k$ channel spin anisotropic Kondo model. Detailed comparisons with the available Emery-Kivelson’s Abelian Bosonization approaches are made. It is shown that the EK line exists for any $k$, although it can be mapped to free fermions only when $k = 1$ or 2. The Wilson ratio is universal if $k = 1$ or 2, but not universal if $k > 2$. The leading low temperature correction to the electron resistivity is not affected by the spin anisotropy for any $k$. A new universal ratio for $k > 2$ is proposed to compare with experiments.
In the general multichannel Kondo model, a magnetic impurity with spin $s$ couples to $k$ degenerate bands of spin 1/2 conduction electrons by Heisenberg exchange interaction. When $k \leq 2s$ (underscreening and complete-screening), the electron-impurity system can be described by a local Fermi liquid at very low temperatures. However, when $k > 2s$ (overscreening), Nozières and Blandin [1] showed that the low temperature physics, being controlled by a non-trivial zero temperature fixed point, is not described by Fermi liquid behavior. Bethe ansatz [2], Boundary conformal field theory (BCFT) [3], Numerical renormalization group (NRG) [4], Bosonization [5] etc. have been utilized to investigate the nature of this non-fermi liquid fixed point. By using BCFT approach, Affleck and Ludwig (AL) identified the non-trivial fixed point symmetry as Affine Kac-Moody algebra $\widehat{SU}_k(2) \times \widehat{SU}_2(k) \times U(1)$. Near the fixed point, they classified all the possible perturbations according to the representation theory of the underlying KM algebra at the fixed point. AL only considered isotropic case, although they showed [3] that spin anisotropy is always irrelevant for $s = 1/2$ in the sense that no relevant operators will appear by allowing spin anisotropy. BCFT is very elegant and applicable for any channel. The symmetry is also explicitly demonstrated in this approach. But the relation between the boundary operators of BCFT and the original scaling variables is not transparent and it is hard to apply CFT near the weak coupling fixed line.

Emery and Kivelson (EK) [5,6], using Abelian Bosonization, found an alternative solution to $k = 2, s = 1/2$ anisotropic Kondo model. By using a canonical transformation $U = e^{iS_z \Phi_s(0)}$ and refermionization, they located a exactly solvable line (EK line) which is analogous to the Toulouse line [7] for the ordinary single channel Kondo model. They also found one leading irrelevant operator $S_z \partial_x \Phi_s$ away from the EK line. By doing perturbative calculations around the EK line with this operator, they recovered the generic low temperature behaviors of the impurity specific heat and susceptibility. Sengupta and Georges [10], using EK’s method, calculated the Wilson ratio independently. Later EK’s approach was extended to the 4 channel case [4] and applied to the electron assisted tunneling of a heavy particle between two sites in a metal [11]. EK’s approach can be applied easily, but it is lim-
ited to special values of $k$. The symmetry is hidden and there is no systematic classification of all the possible operators in this approach.

The two channel Kondo model has been argued to describe uranium based heavy fermion systems and tunneling of a non-magnetic impurity between two sites in a metal [12], $k = 1, 2$ and $k = 3$ magnetic Kondo models have also been proposed to interpret $Ce^{3+}$ based alloys [13]. It is believed that the general spin anisotropy of the form in Eq.1 is ubiquitous in all the experimental systems mentioned above, therefore it is important to investigate its effects in detail.

In this paper, fixing the impurity spin $s = 1/2$, we extend the scaling analysis of YA near the weak coupling fixed line to the multi-channel Kondo model and AL’s approach near the intermediate coupling fixed point to the spin anisotropic Kondo model. We also make detailed comparisons among YA’s, AL’s and EK’s approaches. We find 1-1 mapping between the boundary operators in CFT and those in EK’s solution, therefore establish the relation between the scaling parameters in the two approaches. We define the EK line as a special line in parameter space where the impurity part of Kondo system completely decouples from the uniform external magnetic field, hence $\chi_{imp} \equiv 0$ [14]. The EK line defined in this way coincides with the conventional EK line if $k = 2$ and the decoupling line [6] if $k = 1$. We show that the EK line is a natural property for any $k$ channel Kondo model, although this line can be mapped to free theory only when $k = 1, 2$ [15]. The reason that only one leading irrelevant operator was found in EK’s approach to 2 (1) channel Kondo model is due to first order (second order) KM null states [17], therefore the Wilson ratio is universal for the general spin anisotropic 2 (1) channel-Kondo model Eq.1. However, the Wilson ratio is not universal for $k > 2$ due to two or more leading irrelevant operators with the same scaling dimension. It is shown that the general spin anisotropy does not affect the leading low temperature correction to the electron resistivity for any $k$. A new universal ratio for $k > 2$ is proposed to compare with experiments.

The three-dimensional Kondo problem can be mapped to a one dimensional problem [3]. Following closely the notation of AL, we write the following $k$ channel general spin
anisotropic Kondo Hamiltonian in a uniform magnetic field

\[ H = \frac{v_F}{2\pi} \int_{-\infty}^{\infty} dx i \psi_{i\alpha L}^\dagger(x) \frac{d\psi_{i\alpha L}(x)}{dx} + 2\pi \sum_{a,b=x,y,z} \lambda_{K}^{ab} J_{L}^{a}(0) S^{b} + h \int dx J_{L}^{z}(x) + S^{z} \] (1)

where \( J_{L}^{a}(x) = \frac{1}{2} \psi_{i\alpha L}^\dagger \sigma_{\alpha\beta}^{a} \psi_{i\beta L}(x) \) is the spin current of the conduction electrons, \( S^{a} \ (a = x, y, z) \) is the impurity spin [16]. In most of this paper, we limit our discussions to XXZ (\( U(1) \times Z_{2} \sim O(2) \)) case.

**Weak coupling analysis:** We extend the YA’s scaling equations [8] near the weak coupling fixed line to the \( k \) channel case [18]:

\[
\frac{dQ}{dl} = -\frac{1}{2}(kQ - 1)\lambda^{2} + \cdots
\]

\[
\frac{d\lambda}{dl} = [1 - (k - 1)Q^{2} - (Q - 1)^{2}]\lambda + \cdots
\] (2)

where \( Q = \frac{\delta}{\pi}, \delta = 2\delta_{+}, \delta_{+} = -\delta_{-} = \tan^{-1}(\frac{\pi\lambda_{K}}{4}) \).

Defining \( Q = \frac{1}{k} + q \). It is easy to see that, near the fixed point \((q_{0}, 0)\), there is only one relevant direction. The Kondo scale is given by \( T_{K} \sim D\lambda^{k/(1-(q_{0})^{2})} \). The special line \( q = 0 \) where phase shift \( \delta_{+} = \frac{\pi}{2k} \) will be discussed later.

**Intermediate coupling analysis:** In order to locate the intermediate-coupling fixed point, we set \( \lambda_{K}^{ab} = \lambda_{K}\delta_{ab}, h = 0 \); then the Hamiltonian Eq.(4) has global \( SU(2) \times SU(k) \times U(1) \) symmetry. Fourier transforming Eq.(4) on a finite line segment \(-l \leq x \leq l \) of length \( 2l \), the spin part of the Hamiltonian which contains the Kondo interaction becomes

\[
H_{s} = \frac{v_F\pi}{l} \left( \sum_{n=-\infty}^{\infty} \frac{1}{2+k} : \vec{J}_{-n} \cdot \vec{J}_{n} : + \lambda_{K} \sum_{n=-\infty}^{\infty} \vec{J}_{n} \cdot \vec{S} \right)
\] (3)

where \( \vec{J}_{n} \)s are the Fourier modes of the spin current operator: \( \vec{J}_{n} = \frac{1}{2\pi} \int_{-l}^{l} dx e^{inx/l} \vec{J}_{L}(x) \). They obey the \( SU(2) \) level \( k \) KM commutation relations

\[
[J_{n}^{a}, J_{m}^{b}] = i\epsilon^{abc} J_{n+m}^{c} + \frac{k}{2} \delta_{n+m,0} \delta^{ab} \delta_{n+m,0}
\] (4)

When \( \lambda_{K} = \lambda_{K}^{*} = \frac{2}{2\pi k} \) to be identified as the intermediate coupling fixed point, the global symmetry is enlarged to a local KM symmetry \( \widetilde{SU}_{k}(2) \times \widetilde{SU}_{2}(k) \times U(1) \), the interacting spin Hamiltonian of Eq.(3) is exactly the same as free spin Hamiltonian when written in terms of
the shifted current operators $\vec J_n = \vec J_n + \vec S$ which obey the same KM algebra Eq.3 as the free operators.

Near the fixed point, the Hamiltonian can be written as the fixed point Hamiltonian plus possible perturbations:

$$H = H^* + \sum \lambda_i O_i(0)$$  \hspace{1cm} (5)

We can classify all the possible perturbations $O_i$ in the physical problem according to the representation theory of the underlying KM algebra at the fixed point. The possible operators which can take us away from the fixed point should also respect the global symmetry of the problem. According to the fusion rule [3], for the overscreened case, the spin-1 (adjoint) representation is always allowed, its first order KM descendants have nine Cartesian tensor operators of rank two $\mathcal{J}_a \phi^b$ which can be decomposed into the irreducible spherical tensors under $SU(2)$ with angular momentum $j = 0, 1, 2$.

\begin{align*}
T^0_0 &= \vec J_{-1} \cdot \vec \phi \\
T^1_a &= L_{-1} \phi^a \\
T^2_{\pm 1} &= \mathcal{J}_{-1} \phi^1 - \mathcal{J}_{-1} \phi^2 \pm i(\mathcal{J}_{-1} \phi^2 + \mathcal{J}_{-1} \phi^1) \\
T^2_0 &= \mp(\mathcal{J}_{-1} \phi^3 + \mathcal{J}_{-1} \phi^2) \\
T^2_2 &= \mathcal{J}_{-1} \phi^3 - \frac{1}{2}(\mathcal{J}_{-1} \phi^1 + \mathcal{J}_{-1} \phi^2)
\end{align*}  \hspace{1cm} (6)

where $L_{-1}$ is the Virasoro lowering operator.

In the following, we will discuss $k > 2$, $k = 2$ and $k = 1$ respectively.

$k > 2$ case: For the isotropic $SU(2)$ case, AL identified $T^0_0 = \vec J_{-1} \cdot \vec \phi$ as the only leading irrelevant operator whose scaling dimension is $1 + \Delta$ ($\Delta = \frac{2}{2+k}$ is the scaling dimension of $\vec \phi$), the corresponding coupling constant $\lambda_0$ has R. G. eigenvalue $-\Delta < 0$, therefore is irrelevant.

For $U(1) \times Z_2$ case, the second operator $T^2_0$ is also allowed by all the symmetry of the model. We regroup the two boundary operators as:

$$O = \lambda_0 T^0_0 + \lambda_2 T^2_0 = \alpha_1 O_1 + \alpha_2 O_2$$
\[ O_1 = \left( \frac{k}{2} - 1 \right)T_0^0 - \left( \frac{k}{2} + 2 \right)T_0^2 \]
\[ O_2 = T_0^0 + 2T_0^2 = 3J_{1}^3\phi^3 \] (7)

We can easily extend Eq.(3.29) of Ref. [3] to

\[
\exp(-\beta f_{imp}(T, \lambda_0, \lambda_2, h)) = \exp(-\beta f_{imp}(T, 0, 0, 0)) \langle \exp\left\{ \frac{3k}{2} \alpha_2 \int_{-\beta/2}^{\beta/2} d\tau O_1(\tau) \right\} \rangle_T \
+ \alpha_1 \int_{-\beta/2}^{\beta/2} d\tau O_1(\tau) + \alpha_2 \int_{-\beta/2}^{\beta/2} d\tau O_2(\tau) \rangle_T 
\] (8)

Observing \( h \) acquires no anomalous dimension to any loops and applying finite size scaling lead to the following scaling form of the impurity free energy at low temperature

\[
- \frac{\delta f_{imp}(T, \alpha_1, \alpha_2, h)}{T} = TQ_{imp}(h/T, \alpha_1 T^{\Delta}, \alpha_2 T^{\Delta}) 
\]
\[
= T\{ A[\frac{9}{8}k(k-2)(\frac{k}{2} + 2)\alpha_1^2 + \frac{9}{2}k\alpha_2^2]T^{2\Delta} + G(h\frac{3k}{2} \alpha_2 T^{(\Delta-1)}) \} (1 + O(T)) 
\] (9)

where \( \delta f_{imp}(T, \alpha_1, \alpha_2, h) = f_{imp}(T, \alpha_1, \alpha_2, h) + T \log \sqrt{2} \) and \( G \) takes the following asymptotic form

\[
G(s) = \begin{cases} 
\sim s^2 & s \to 0 \\
\sim s^{1/(1-\Delta)} & s \to \infty 
\end{cases} 
\] (10)

From the above scaling equation, it is easy to see

\[
C_{imp} \sim \frac{9}{8}k(k-2)(\frac{k}{2} + 2)\alpha_1^2 + \frac{9}{2}k\alpha_2^2 T^{2\Delta} 
\]
\[
\chi_{imp} \sim (\frac{3k}{2} \alpha_2)^2 T^{2\Delta-1} 
\] (11)

The Wilson ratio \( R = \frac{T\chi_{imp}}{C_{imp}} \) is not a universal constant in contrast to the isotropic case.

The \( h \) dependent part of the zero temperature impurity free energy is given by :

\[
f_{imp}(0, \alpha_1, \alpha_2, h) \sim (\frac{3k}{2} \alpha_2 h)^{1/(1-\Delta)} \quad h \to 0 
\] (12)

Therefore the impurity susceptibility is given by \( \chi_{imp} = \frac{m_{imp}}{h} \sim h^{\frac{2\Delta-1}{1-\Delta}} \) at \( T = 0 \).

If \( \alpha_2 = 0 \), the impurity part of the system decouples from the external magnetic field, \( \chi_{imp} \) vanishes to the leading order. This is exactly the feature of the EK line in the \( k > 2 \) case.
Extending EK’s approach to the four channel Kondo model, Fabrizio and Gogolin [9] located the EK line which, unlike the $k = 2$ case, cannot be transformed to a Hamiltonian for free fermions by refermionization. They managed to find two orthogonal operators with scaling dimension $4/3$, one contributes only to $C_{\text{imp}}$, another contributes to both $C_{\text{imp}}$ and $\chi_{\text{imp}}$. Their results are totally consistent with ours. By symmetry we can identify $O_2 = J^{-3}_1\phi^3$ with $S_2\partial_2\Phi_s$ found by them.

If $\alpha_1 = \alpha_2 = 0$, the next leading irrelevant operators belong to the conformal tower of identity operator with scaling dimension 2. Similar to Eq.8, they can be classified as

\begin{align}
Q^0_0 &= J^a_{-1} J^a_{-1} I \\
Q^1_a &= J^a_{-2} I = \partial J^a \\
Q^2_{\pm 2} &= [\mathcal{J}^1_{-1} \mathcal{J}^1_{-1} - \mathcal{J}^2_{-1} \mathcal{J}^2_{-1} \pm i(\mathcal{J}^1_{-1} \mathcal{J}^1_{-1} + \mathcal{J}^2_{-1} \mathcal{J}^2_{-1})] I \\
Q^2_{\pm 1} &= \mp[\mathcal{J}^3_{-1} \mathcal{J}^3_{-1} + \mathcal{J}^2_{-1} \mathcal{J}^2_{-1} \mp i(\mathcal{J}^2_{-1} \mathcal{J}^2_{-1} + \mathcal{J}^3_{-1} \mathcal{J}^3_{-1})] I \\
Q^2_0 &= [\mathcal{J}^3_{-1} \mathcal{J}^3_{-1} - \frac{1}{2}(\mathcal{J}^1_{-1} \mathcal{J}^1_{-1} + \mathcal{J}^2_{-1} \mathcal{J}^2_{-1})] I
\end{align}

(13)

We regroup the two boundary operators $Q^0_0, Q^2_0$ as :

\begin{align}
P &= \beta_1 P_1 + \beta_2 P_2 \\
P_1 &= (k - 1)Q^0_0 - (k + 2)Q^2_0 \\
P_2 &= Q^0_0 + 2Q^2_0 = 3\mathcal{J}^3_{-1} \mathcal{J}^3_{-1} I
\end{align}

(14)

Although the operators $P_1$ and $P_2$ are Virasoro descendants, therefore have non-vanishing expectation values at finite temperature, it still can be shown that $P_1$, running along the EK line, contributes only to $C_{\text{imp}} \sim T$; $P_2$, running away from the EK line, contributes to both $C_{\text{imp}} \sim T$ and $\chi_{\text{imp}} \sim \text{const.}$

We can continue this process to any level of KM towers and locate the EK line where $\alpha_2 = \beta_2 = \cdots = 0$, hence $\chi_{\text{imp}} \equiv 0$. The coefficients $\alpha_1, \alpha_2, \beta_1, \beta_2 \cdots$ all depend on $q_0$ where we start at high energy scale. The precise relation may be obtained by Bethz ansatz solution [14]. $\chi_{\text{imp}}$ has also been shown [19] to vanish at the special line $q_0 = 0$ found at the weak coupling analysis, therefore if $q_0 = 0$, then $\alpha_2 = \beta_2 = \cdots = 0$. 

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If we apply a local magnetic field to the system, the boundary operator $hS^z = h(\phi^3 + J^3 + L_{-1}\phi^3 + \cdots)$ is added to the fixed point Hamiltonian. Unlike in the uniform field, this boundary operator is totally independent of other boundary operators, therefore we always have $\chi_{\text{imp}} \sim h^{2\Delta-1}$ at $T = 0$, non-vanishing even at the EK line.

In the isotropic case, as shown by AL [20,21], the first order perturbation in $T_0^2$ gives the leading low temperature correction to the electron resistivity. Because $T^2_q(q = 0, \pm 1, \pm 2)$ carry spin 2, the first order perturbations in these operators vanish, therefore the leading low temperature correction to the electron resistivity is not affected by the general spin anisotropy in Eq.1

$$\rho(T) \sim \rho(0)(1 + \lambda_0 T^\Delta) \sim \rho(0)(1 + [(k/2 - 1)\alpha_1 + \alpha_2] T^\Delta)$$ (15)

Eqs.11, 15 can be used to get rid of the two independent parameters $\alpha_1, \alpha_2$, therefore a new universal ratio can be formed. Other transport properties like thermopower and thermal conductivity can also be calculated, more universal ratios can be formed [22].

The predictions of this section may be tested by the possible experimental realization of the 3 channel Kondo model proposed in Ref. [13].

$k = 2$ case: When $k = 2$, the coefficient of $\alpha_1$ in Eq.13 vanishes. The underlying reason for this is that the pure KM null states appear in the first order descendant for the $j = 1$ representation. Generally, for the spin $j$ representation of $\widehat{SU}_k(2)$, the pure KM null states [17] first appear at $(k - 2j + 1)'s$ order: $(\mathcal{J}^\pm_{-1})^{k-2j+1}|(j), j > = 0 \ (\mathcal{J}^\pm_{-1} = \mathcal{J}^\pm_{-1} \pm i\mathcal{J}^2_{1-1})$. For $j = 1$, it becomes $(\mathcal{J}^\pm_{-1})^{k-1}|(1), 1 > = 0$. If $k > 2$, the null states first appear at least at order 2, so the nine operators defined in Eq.6 are all independent of each other. However, for $k = 2$, there are 6 constraints: $\mathcal{J}^\pm_{-1}|1, \pm 1 > = 0$ plus cyclic permutations in x and y axis. It turns out the 6 constraints only give 5 independent equations: $T^2_q = 0, q = 0, \pm 1, \pm 2$, therefore there is only one leading irrelevant operator $T^0_0$ even for $U(1) \times Z_2$ case. Actually we can make an even stronger statement: for the general anisotropic 2-channel Kondo model Eq.14, there is only one leading irrelevant operator $T^0_0$. The EK line is given by $\alpha_2 = \beta_2 = 0$ which corresponds to $\delta_+ = \frac{\pi}{4}$ in the weak coupling analysis. The leading irrelevant operator along
this line is $P_1 = -3[\mathcal{J}_1^3 \mathcal{J}_1^3 - (\mathcal{J}_1^1 \mathcal{J}_1^1 + \mathcal{J}_1^2 \mathcal{J}_1^2)]I$ of Eq.14, the first order perturbation in $P_1$ gives $C_{\text{imp}} \sim T$. R. G. analysis of EK’s solution also show that the leading irrelevant operator along the EK line has scaling dimension 2 [18]. However, away from the EK line, $T_0^0 = 3\mathcal{J}_1^3 \phi^3$ which is Virasoro primary is the only leading irrelevant operator with scaling dimension $3/2$. By symmetry we can identify $T_0^0 = 3\mathcal{J}_1^3 \phi^3$ with $S_\pm \partial_x \Phi_s$ found by EK, therefore $\lambda_0 \sim \lambda_2 - 2\pi v_F$. This implies that $\lambda_0$ changes sign as it passes through the fixed point, therefore confirms the conjecture by AL [21] that the resistivity shows very different behaviors on the two sides of the fixed point. We can also identify $P_2 = 3\mathcal{J}_1^3 \mathcal{J}_1^3 I$ as $:(\partial_x \Phi_s)^2:$, therefore $\beta_2 \sim (\lambda_2 - 2\pi v_F)^2$.

$k = 1$ case: Only the $j = 0$ representation is allowed by the fusion rule in the ordinary 1-channel Kondo model [3]. For the spin 0 representation of $\widehat{SU}_k(2)$, the pure KM null states first appear at $(k + 1)$’s order: $(\mathcal{J}_1^+)^{k+1}|(0), 0 >= 0$. For $k > 1$, the null states first appear at least at order 3, so the nine operators defined in Eq.13 are all independent of each other. However, for $k = 1$, there are 6 constraints: $(\mathcal{J}_1^+)^2|(0), 0 >= 0$ plus cyclic permutations in $x$ and $y$ axis, they lead to $Q_0^2 = 0, q = 0, \pm 1, \pm 2$. There is only one leading irrelevant operator $Q_0^0$ for the general anisotropic 1-channel Kondo model Eq.1. The first order perturbation in $Q_0^0$ [18] leads to the generic fermi-liquid behaviors: $\chi_{\text{imp}} = \text{const}, C_{\text{imp}} \sim T, R \equiv 2$.

EK’s solution of the $O(2)$ one channel Kondo model [4] found a decoupling line which corresponds to $\delta_+ = \frac{\pi}{2}$ in the weak coupling analysis. R. G. analysis [18] near this line show that there is only one dimension 2 operator away from the fixed point. We can identify $Q_0^0 = 3\mathcal{J}_1^3 \mathcal{J}_1^3 I$ as $:(\partial_x \Phi_s)^2:$, therefore $\lambda_0 \sim (\lambda_2 - 4\pi v_F)^2$. This implies $\lambda_0$ is always positive, in contrast to the 2 channel case.

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changing operator [24]. I thank him for pointing out this connection to me. Maldocena and Ludwig [25] discussed the connection between CFT and Abelian Bosonization, but with different emphasize than this paper.
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