Global solvability and stability to a nutrient-taxis model with porous medium slow diffusion

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Abstract

In this paper, we study a nutrient-taxis model with porous medium slow diffusion

\[
\begin{aligned}
    u_t &= \Delta u^m - \chi \nabla \cdot (u \nabla v) + \xi uv - \rho u, \\
    v_t - \Delta v &= -u + \mu v(1 - v),
\end{aligned}
\]

in a bounded domain $\Omega \subset \mathbb{R}^3$ with zero-flux boundary condition. It is shown that for any $m > 1$, the problem admits a global weak solution for any large initial datum. We divide the study into three cases, (i) $\xi \mu = 0, \rho \geq 0$; (ii) $\xi \mu \rho > 0$; (iii) $\xi \mu > 0, \rho = 0$. In particular, for Case (i) and Case (ii), the global solutions are uniformly bounded. Subsequently, the large time behavior of these global bounded solutions are also discussed. The methods and results of this paper are also applicable for the coupled nutrient-taxis-Stokes system. Important progresses for the special case $\xi = \mu = \rho = 0$ with $m > \frac{7}{6}$, $m > \frac{8}{7}$ and $m > \frac{9}{8}$ have been carried out respectively by \cite{19, 16, 20}, but leave a gap for $1 < m \leq \frac{9}{8}$. Our result of Case (i) fills this gap.

Keywords: Nutrient-Taxis Model, Porous Medium Diffusion, Global Solvability, Stability.

1 Introduction

In this paper, we consider the following nutrient-taxis model involving food-supported proliferation

\[
\begin{aligned}
    u_t &= \Delta u^m - \chi \nabla \cdot (u \nabla v) + \xi uv - \rho u, \\
    v_t - \Delta v &= -uv + \mu_v(1 - v), \\
    (\nabla u^m - \chi u \nabla v) \cdot \mathbf{n}_{|\partial \Omega} &= \left. \frac{\partial v}{\partial \mathbf{n}} \right|_{\partial \Omega} = 0, \\
    u(x, 0) &= u_0(x), v(x, 0) = v_0(x), \quad x \in \Omega,
\end{aligned}
\]

where $m > 1$, $Q = \Omega \times \mathbb{R}^+$, $\Omega \subset \mathbb{R}^3$ is a bounded domain, and the boundary $\partial \Omega$ is appropriately smooth, $u, v$ represent the bacteria cell density, the concentration of nutrient respectively, $\chi > 0$ is the
sensitivity coefficient of aggregation induced by the concentration changes of nutrient, the appearance of $\xi uv$ implies that the cell proliferation relies on the availability of nutrient resource $v$, and $\xi \geq 0$ is the conversion rate (growth yield) of consumed nutrient to bacterial growth, $-\rho u$ ($\rho \geq 0$) is the linear degradation of the bacteria cells, $-vu$ and $\mu v(1-v)$ with $\mu \geq 0$ represent the consumption and reproduction of nutrients, respectively. In addition to the above biological explanation, this model is often used to describe the prey-taxis phenomenon involving Lotka-Volterra type interaction, see for example [2, 10, 9].

Colonies of bacteria growing on the surface of thin agar plates show varieties of morphological patterns in response to surrounding environmental conditions, such as the nutrient concentration, the solidity of an agar medium and temperature. Based on experimental observations, Kawasaki et al. [8] proposed the following reaction-diffusion model for bacterial aggregation patterns on the surface of thin agar plates

$$
\begin{align*}
\frac{du}{dt} &= \nabla \cdot (D_u \nabla u) + \xi f(u, v), \\
\frac{dv}{dt} &= D_v \Delta v - f(u, v),
\end{align*}
$$

where, $D_u$ and $D_v$ are the diffusion coefficients of the bacterial cells and nutrient, respectively. In recent years, bacterial chemotaxis has attracted much attention due to its critical role in pattern formation. To explore the aggregation effects caused by such chemotactic patterns, Leyva et. al [11] took nutrient chemotactic term into the above model, and developed the following model

$$
\begin{align*}
\frac{du}{dt} &= \nabla \cdot (uv \nabla u) - \nabla \cdot (u^2v \nabla v) + uv, \\
\frac{dv}{dt} &= \Delta v - uv.
\end{align*}
$$

Recently, Winkler [17] considered a simplified form of this model, that is

$$
\begin{align*}
\frac{du}{dt} &= \Delta u - \chi \nabla \cdot (u \nabla v) + \xi uv - \rho u, \\
\frac{dv}{dt} &= \Delta v - uv + \mu v(1-v),
\end{align*}
$$

(1.2)

where, the author included the possibility of linear degradation in the cell population, and the reproduction of chemoattractant through either linear or logistic mechanism. For this model, Winkler obtained the existence of global weak solutions, and further proved that under some assumptions on these coefficients, each of these solutions becomes eventually smooth and stabilizes toward a spatially homogeneous equilibrium. When $\xi = \rho = \mu = 0$, this model (1.2) is reduced to the following form

$$
\begin{align*}
\frac{du}{dt} &= \Delta u - \chi \nabla \cdot (u \nabla v), \\
\frac{dv}{dt} &= \Delta v - uv.
\end{align*}
$$

(1.3)

In 2012, Tao and Winkler [14] showed the existence of global weak solutions in three dimensions, for which, they also proved that, after some time $T$, these weak solutions become smooth and go to constant equilibria in the large time limit. While if the cell mobility is described by a nonlinear function of the cells density, for example, the porous medium diffusion, then the model becomes

$$
\begin{align*}
\frac{du}{dt} &= \Delta u^m - \chi \nabla \cdot (u \nabla v), \\
\frac{dv}{dt} &= \Delta v - uv.
\end{align*}
$$

(1.3)

If the fluid velocity is considered into this model, the system (1.3) becomes a classical chemotaxis-Stokes system, which has also been studied by many authors. However, the known theory researches for the chemotaxis model and the coupled chemotaxis-Stokes model are almost parallel. So, in what follows, we no longer distinguish between the chemotaxis model and the coupled chemotaxis-Stokes model when introducing the known results. For the two dimensional case of (1.3), the global solvability and boundedness of weak solutions are established completely for any $m > 1$ in [15]. While in
three dimensional space, the research of (1.3) is rather tortuous. The first effort to this 3-D problem is due to the work by Di Francesco et al. [3], in which, they obtained the existence of global bounded weak solutions for $m$ in some finite interval, namely $m \in \left(\frac{2 + \sqrt{17}}{12}, 2\right]$ (approximating to $(1.8109, 2]$); it was Tao and Winkler [16], in 2013, who established the global existence of locally bounded weak solutions with $m$ belonging to the infinite interval $(\frac{8}{7}, +\infty)$. Afterwards, Winkler [19] supplemented the uniform boundedness of solutions for the case $m > \frac{7}{6}$; recently, Winkler [20] further improved this result to the case $m > \frac{9}{7}$ with $\Omega$ being a convex domain. However, it still leave a gap to the case $m \in (1, \frac{9}{7})$.

In the present paper, we pay our attention to the global existence and uniform boundedness of weak solutions for the system (1.1). We divide the research into three cases according to the nonnegative coefficients $\rho, \xi, \mu$, that is

\[(i) \, \xi \mu = 0, \rho \geq 0; \quad (ii) \, \xi \mu \rho > 0; \quad (iii) \, \xi \mu > 0, \rho = 0.\]

We show that for any $m > 1$, this problem admits a global weak solution for any large initial datum and any nonnegative coefficients $\rho, \xi, \mu$. In particular, the solution is uniformly bounded for Cases (i) and (ii), while for Case (iii), the solution is just locally bounded on time $t$ since the $L^1$-norm of $u$ depends on $t$. It is worthy of noticing that for a special case of (i), that is the case $\xi = \rho = \mu = 0$, the system (1.1) is reduced to (1.3), our research supplements the gap of [16 19 20] for $m \in (1, \frac{9}{7})$.

Throughout this paper, we assume that

\[
\begin{align*}
&u_0 \in L^\infty(\Omega), \nabla u_0^m \in L^2(\Omega), v_0 \in W^{2,\infty}(\Omega), \\
&u_0, v_0 \geq 0, \\
&\partial \Omega \in C^{2,\alpha}. \\
\end{align*}
\]

In what follows, we give the existence results. For Cases (i) and (ii), we have

**Theorem 1.1** Assume (H), $m > 1$. If (i) $\xi \mu = 0$ with $\rho \geq 0$, or (ii) $\xi \mu \rho > 0$, the problem (1.1) admits a nonnegative global bounded weak solution $(u, v)$ with $u \in \mathcal{X}_1$, $v \in \mathcal{X}_2$, where

\[
\mathcal{X}_1 = \{u \in L^\infty((0, \infty); L^2(\Omega)), (u_t^{\frac{m+1}{m}})_t, \nabla u^{\frac{m+1}{m}} \in L^2_{loc}([0, \infty); L^2(\Omega))\},
\]

\[
\mathcal{X}_2 = \{v \in L^\infty((0, \infty); W^{1,\infty}(\Omega)); v_t, \Delta v \in L^p_{loc}([0, \infty); L^p(\Omega)) \text{ for any } p > 1\},
\]

such that

\[
\sup_{t \in (0, +\infty)} (\|u(\cdot, t)\|_{L^\infty} + \|v(\cdot, t)\|_{W^{1,\infty}}) \leq M_1,
\]

\[
\sup_{t \in (0, +\infty)} \int_\Omega |\nabla u^m|^2 \, dx + \sup_{t \in (0, +\infty)} \|u_t^{\frac{m+1}{m}}\|_{W^{1,1}(Q(t))} \leq M_2,
\]

\[
\sup_{t \in (0, +\infty)} \|v\|_{W^{2,1}_p(Q(t))} \leq M_3 \quad \text{for any } p > 1.
\]

Here $Q(t) = \Omega \times (t, t+1)$, $M_i$ ($i = 1, 2, 3$) are constants depending only on $\xi, \chi, \rho, \mu, \Omega, u_0, v_0$.

For the case (iii), the global solution of (1.1) is locally bounded, which can be stated as follows.

**Theorem 1.2** Assume (H), $m > 1$. If (iii) $\xi \mu > 0$, and $\rho = 0$, the problem (1.1) admits a nonnegative local bounded weak solution $(u, v)$ with $u \in \tilde{\mathcal{X}}_1$, $v \in \tilde{\mathcal{X}}_2$, where

\[
\tilde{\mathcal{X}}_1 = \{u \in L^\infty_{loc}((0, +\infty); L^\infty(\Omega)); \nabla u^m \in L^\infty_{loc}((0, +\infty); L^2(\Omega)), (u_t^{\frac{m+1}{m}})_t, \nabla u^{\frac{m+1}{m}} \in L^2_{loc}([0, +\infty); L^2(\Omega))\},
\]

\[
\tilde{\mathcal{X}}_2 = \{v \in L^\infty_{loc}((0, +\infty); W^{1,\infty}(\Omega)); v_t, \Delta v \in L^p_{loc}([0, +\infty); L^p(\Omega)) \text{ for any } p > 1\}.
\]
such that for any } T > 0,\end{equation}

\[ \sup_{t \in (0, T)} (\|u(\cdot, t)\|_{L^\infty} + \|v(\cdot, t)\|_{L^\infty}) \leq \tilde{M}_1(T), \]

\[ \sup_{t \in (0, T)} \int_{\Omega} |\nabla u^m|^2 \, dx + \|u^m\|_{W^{1,1}(Q_T)} \leq \tilde{M}_2(T), \]

\[ \|v\|_{W^{1,1}(Q_T)} \leq \tilde{M}_3(T) \text{ for any } p > 1. \]

Here } Q_T = \Omega \times (0, T), \tilde{M}_i(T) (i = 1, 2, 3) \text{ are constants depending only on } T, \xi, \chi, \rho, \mu, \Omega, u_0, v_0. \]

On the basis of establishing the global solvability, we further consider the large time behavior of the global solutions. We only consider the cases (i) and (ii), since the global solutions are bounded uniformly for the two cases.

Theorem 1.3 Assume (H), } m > 1, \text{ and } u_0 \neq 0, v_0 \neq 0. \text{ Let } (u, v) \text{ be the global bounded solution obtained above. Then we have}

(i) When } \mu = 0, \rho = 0, \xi \geq 0, \text{ then}

\[ \lim_{t \to \infty} \|v\|_{L^\infty} = 0, \quad \lim_{t \to \infty} \|u - A\|_{L^p} = 0 \quad \text{for any } p > 1, \]

where } A = \frac{1}{|\Omega|} \int_{\Omega} (u_0 + \xi v_0) \, dx > 0; \]

(ii) When } \mu = 0, \rho > 0, \xi \geq 0, \text{ there exists a constant } B \text{ with } 0 < B < \frac{1}{|\Omega|} \int_{\Omega} v_0 \, dx, \text{ such that}

\[ \lim_{t \to \infty} \|v - B\|_{L^\infty} = 0, \quad \lim_{t \to \infty} \|u\|_{L^p} = 0 \quad \text{for any } p > 1; \]

(iii) When } \mu > 0, \rho > 0, 0 \leq \xi < \rho, \text{ then}

\[ \lim_{t \to \infty} \|v - 1\|_{L^\infty} = 0, \quad \lim_{t \to \infty} \|u\|_{L^p} = 0 \quad \text{for any } p > 1. \]

Remark 1.1 We claim that, our methods are also applicable for the following coupled system

\[
\begin{aligned}
& n_t + u \cdot \nabla n = \Delta n^m - \chi \nabla \cdot (n \cdot \nabla c) + \xi n c - \rho n, \quad (x, t) \in Q, \\
& c_t + u \cdot \nabla c - \Delta c = -cn + \mu c(1 - c), \quad (x, t) \in Q, \\
& u_t + \nabla P = \Delta u + n \nabla \varphi, \quad (x, t) \in Q, \\
& \text{div } u = 0, \quad (x, t) \in Q, \\
& (\nabla n^m - \chi n \cdot \nabla c) \cdot v|_{\partial \Omega} = \frac{\partial c}{\partial \nu}|_{\partial \Omega} = 0, \quad u|_{\partial \Omega} = 0,
\end{aligned}
\]

and for any } m > 1, \text{ the similar results of Theorem } 1.1-1.3 \text{ can also be proved.

2 Preliminaries

We first give some notations, which will be used throughout this paper. 

Notations: } } || \cdot ||_{L^p} = || \cdot ||_{L^p(\Omega)}, \ Q_1(t) = \Omega \times (t, t + 1), \ Q_T := Q_T(0) = \Omega \times (0, T). 

Next, we give the definition of weak solutions.
**Definition 2.1** (u, v) is called a weak solution of (1.1), if u ≥ 0, v ≥ 0 and u ∈ X, v ∈ W^{1,0}_2(Q_T) for any T > 0, such that

\[
- \int_{Q_T} u \varphi dx dt - \int_\Omega u(x,0) \varphi(x,0) dx + \int_{Q_T} (\nabla u^m - \chi u \nabla \varphi) \nabla \varphi dx dt = \int_{Q_T} (\xi uv - \rho u) \varphi dx dt,
\]

\[
- \int_{Q_T} v \varphi dx dt - \int_\Omega v(x,0) \varphi(x,0) dx + \int_{Q_T} \nabla v \nabla \varphi dx dt + \int_{Q_T} uv \varphi dx dt = \mu \int_{Q_T} v(1-v) \varphi dx dt,
\]

for any \( \varphi \in C^\infty(\overline{Q_T}) \) with \( \varphi(x, T) = 0 \), where \( X = \{ u \in L^2(Q_T); \nabla u^m \in L^2(Q_T) \} \).

Before going further, we list some important lemmas, which will be used throughout this paper. Firstly, by [5, 6], we give the following two lemmas.

**Lemma 2.1** Let \( T > 0, \tau \in (0, T), \sigma \geq 0, a > 0, b \geq 0, \) and suppose that \( f : [0, T) \rightarrow [0, \infty) \) is absolutely continuous, and satisfies

\[ f'(t) + af^{1+\sigma}(t) \leq h(t), t \in \mathbb{R}, \tag{2.1} \]

where \( h \geq 0, h(t) \in L^1_{\text{loc}}([0, T]) \) and

\[ \int_{t-\tau}^t h(s) ds \leq b, \text{ for all } t \in [\tau, T). \]

Then

\[ \sup_{t \in (0, T)} f(t) + a \sup_{t \in (\tau, T)} \int_{t-\tau}^t f^{1+\sigma}(s) ds \leq b + 2 \max\{f(0) + b + a\tau, \frac{b}{a\tau} + 1 + 2b + 2a\tau\}. \tag{2.2} \]

**Lemma 2.2** Assume that \( u_0 \in W^{2,p}(\Omega) \), and \( f \in L^p_{\text{loc}}((0, +\infty); L^p(\Omega)) \) with

\[ \sup_{t \in (\tau, +\infty)} \int_{t-\tau}^t \|f\|^p_{L^p} ds \leq A, \]

where \( \tau > 0 \) is a fixed constant. Then the following problem

\[
\begin{dcases}
    u_t - \Delta u + u = f(x,t), \\
    \frac{\partial u}{\partial n}\bigg|_{\partial \Omega} = 0, \\
    u(x,0) = u_0(x)
\end{dcases} \tag{2.3}
\]

admits a unique solution \( u \) with \( u \in L^p_{\text{loc}}((0, +\infty); W^{2,p}(\Omega)), u_t \in L^p_{\text{loc}}((0, +\infty); L^p(\Omega)) \) with

\[ \sup_{t \in (\tau, +\infty)} \int_{t-\tau}^t (\|u\|^p_{W^{2,p}} + \|u_t\|^p_{L^p}) ds \leq AM \frac{e^{p\tau}}{e^{\frac{p\tau}{2}} - 1} + M e^{\frac{p\tau}{2}} \|u_0\|^p_{W^{2,p}}, \tag{2.4} \]

where \( M \) is a constant independent of \( \tau \).

By [18], we also have the following lemma.

**Lemma 2.3** Suppose that \( h \in C^2(\mathbb{R}) \), then for all \( \varphi \in C^2(\overline{\Omega}) \) fulfilling \( \frac{\partial \varphi}{\partial \mathbf{n}} = 0 \) on \( \partial \Omega \), we have

\[
\int_\Omega h'(\varphi) |\nabla \varphi|^2 \Delta \varphi dx + \frac{2}{3} \int_\Omega h(\varphi) |\Delta \varphi|^2 dx = 2 \int_\Omega h(\varphi) |D^2 \varphi|^2 dx - \frac{1}{3} \int_\Omega h''(\varphi) |\nabla \varphi|^4 dx - \frac{1}{3} \int_{\partial \Omega} h(\varphi) \frac{\partial |\nabla \varphi|^2}{\partial \mathbf{n}} ds, \tag{2.5}\]

and

\[
\int_\Omega \frac{|\nabla \varphi|^4}{\varphi^5} dx \leq (2 + \sqrt{N})^2 \int_\Omega \varphi |D^3 \ln \varphi|^2 dx. \tag{2.6}\]
By [12], we have

**Lemma 2.4** Assume that \( \Omega \) is bounded and let \( \omega \in C^2(\overline{\Omega}) \) satisfy \( \frac{\partial \omega}{\partial y} \big|_{\partial \Omega} = 0 \). Then we have

\[
\frac{\partial |\nabla \omega|^2}{\partial y} \leq 2\kappa |\nabla \omega|^2 \quad \text{on } \partial \Omega,
\]

where \( \kappa > 0 \) is an upper bound for the curvatures of \( \Omega \).

### 3 Boundedness and Global Existence of Weak Solutions

We first consider the approximate problems given by

\[
\begin{cases}
    u_{\varepsilon} = \Delta(\varepsilon u_{x} + u_{x}^m) - \chi \nabla \cdot (u_{x} \cdot \nabla v_{x}) + \xi u_{x}v_{x} - \rho u_{x} - \varepsilon u_{x}^2, \quad (x, t) \in Q, \\
    v_{\varepsilon} - \Delta v_{x} = -v_{x}u_{x} + \mu v_{x}(1 - v_{x}), \quad (x, t) \in Q, \\
    \frac{\partial u_{x}}{\partial n} \big|_{\partial \Omega} = \frac{\partial v_{x}}{\partial n} \big|_{\partial \Omega} = 0, \\
    u_{x}(x, 0) = u_{\varepsilon 0}(x), \quad v_{x}(x, 0) = v_{\varepsilon 0}(x), \quad x \in \Omega.
\end{cases}
\]

where \( u_{\varepsilon 0}, v_{\varepsilon 0} \in C^{2+\alpha}(\overline{\Omega}) \) with \( \frac{\partial u_{\varepsilon 0}}{\partial n} \big|_{\partial \Omega} = 0, \frac{\partial v_{\varepsilon 0}}{\partial n} \big|_{\partial \Omega} = 0, \|u_{\varepsilon 0}\|_{L^\infty} + ||u_{\varepsilon 0}||_{L^2} + ||v_{\varepsilon 0}||_{W^{2,\infty}} \leq M_0 \), and

\[
u_{\varepsilon 0} \rightarrow u_0, v_{\varepsilon 0} \rightarrow v_0, \quad \text{uniformly}.
\]

According to the arguments in [6], each of these problems is globally solvable in the classical sense.

**Lemma 3.1** Assume (H) holds. Then for any \( \varepsilon > 0 \), the problem (1.1) admits a unique nonnegative classical solution \( (u_{\varepsilon}, v_{\varepsilon}) \in C^{2+\alpha,1+\frac{\alpha}{2}}(\overline{Q}) \).

Using this lemma, we show the global existence of solutions for the problem (1.1). For this purpose, we show some a priori estimates of solutions. In what follows, we let \( C, C_i, \tilde{C} \) denote some different constants, which are independent of \( \varepsilon \), and if no special explanation, these constants depend at most on \( \Omega, \chi, \xi, \rho, \mu, u_0, v_0 \).

We first give the following lemma

**Lemma 3.2** Let \( (u_{\varepsilon}, v_{\varepsilon}) \) be the solution of (3.1), then we have

\[
\sup_{t \in (0, \infty)} \|v(\cdot, t)\|_{L^\infty} \leq C_1, \tag{3.2}
\]

\[
\sup_{t \in (0, \infty)} \int_{\Omega} u_{\varepsilon} dx \leq C_2, \quad \text{if } \xi \mu = 0, \text{ or } \xi \mu \rho > 0, \tag{3.3}
\]

\[
\int_{\Omega} u_{\varepsilon}(x, t) dx \leq C_3(1 + t), \quad \text{if } \xi \mu > 0 \text{ and } \rho = 0, \tag{3.4}
\]

where \( C_1, C_2 \) and \( C_3 \) are independent of \( \varepsilon \).

**Proof.** Firstly, consider the initial value problem of the following ODE

\[
\begin{cases}
y'(t) = \mu y(1 - y), \quad t > 0, \\
y(0) = M_0,
\end{cases}
\]
It is easy to obtain that \( 0 \leq y \leq \max \{ 1, M_0 \} \). By comparison lemma, we obtain that for any \( t > 0, \)
\[
\| v(\cdot, t) \|_{L^\infty} \leq y(t) \leq \max \{ 1, \| v_0 \|_{L^\infty} \}.
\]
Integrating the first equation and the second equation respectively, we obtain
\[
\frac{d}{dt} \int_\Omega u_e dx + \rho \int_\Omega u_e dx + \varepsilon \int_\Omega u_e^2 dx = \xi \int_\Omega u_e v_e dx
\]
and
\[
\xi \frac{d}{dt} \int_\Omega v_e dx = -\xi \int_\Omega v_e u_e dx + \xi \mu \int_\Omega v_e(1 - v_e) dx.
\]
Adding up the above two equalities, we obtain
\[
\frac{d}{dt} \int_\Omega (u_e + \xi v_e) dx + \rho \int_\Omega (u_e + \xi v_e) dx \leq \rho \xi \int_\Omega v_e dx + \xi \mu \int_\Omega v_e(1 - v_e) dx.
\]
When \( \xi \mu = 0 \), it is easy to obtain
\[
\int_\Omega (u_e + \xi v_e) dx \leq \int_\Omega (u_{e0} + \xi v_{e0}) dx.
\]
When \( \xi \mu > 0 \), we obtain
\[
\frac{d}{dt} \int_\Omega (u_e + \xi v_e) dx + \rho \int_\Omega (u_e + \xi v_e) dx \leq C,
\]
if \( \rho > 0 \), by a direct calculation, we obtain (3.3); while if \( \rho = 0 \), then
\[
\int_\Omega (u_e + \xi v_e) dx \leq \int_\Omega (u_{e0} + \xi v_{e0}) dx + Ct.
\]
The proof is complete. \( \square \)

From this lemma, we see that for the cases (i) \( \xi \mu = 0, \rho \geq 0 \) and (ii) \( \xi \mu > 0 \), the \( L^1 \)-norm of \( u_e(\cdot, t) \) is uniformly bounded on \( t \). However, for the case (iii) \( \xi \mu > 0 \) and \( \rho = 0 \), the \( L^1 \)-norm of \( u_e(\cdot, t) \) depends on \( t \). In what follows, we only show the energy estimates independent of time \( t \) for the cases (i) and (ii). For the case (iii), the similar energy estimates also hold, but depend on time \( t \).

**Lemma 3.3** Let \((u_e, v_e)\) be the solution of (3.1). Then for Cases (i) and (ii), we have
\[
\sup_{t \in (0, +\infty)} \int_\Omega \left( \frac{|\nabla v_e|^2}{v_e} + u_e \ln u_e \right) dx + \sup_{t \in (0, +\infty)} \int_\Omega \left( \varepsilon u_e^2 \ln(1 + u_e) + \varepsilon \left| \nabla u_e \right|^2 \right) dx
\]
\[
+ \sup_{t \in (0, +\infty)} \int_\Omega \left( v_e |D^2 \ln v_e|^2 + \left| \nabla u_e \right|^2 + \frac{u_e}{v_e} |\nabla v_e|^2 + \frac{|\nabla v_e|^4}{v_e^3} + u_e^{m+4} \right) dx \leq C,
\]
where \( C \) is independent of \( \varepsilon \).

**Proof.** Using the second equation of (3.1), we obtain
\[
\frac{1}{2} \frac{d}{dt} \int_\Omega \frac{|\nabla v_e|^2}{v_e} dx = \int_\Omega \frac{\nabla v_e}{v_e} \cdot \nabla v_e dx - \frac{1}{2} \int_\Omega \frac{\nabla v_e|^2}{v_e^2} v_e dx
\]
\[
= - \int_\Omega \frac{\Delta v_e}{v_e} \left( \frac{\nabla v_e|^2}{v_e^2} - 1 \right) - \frac{1}{2} \int_\Omega \frac{|\nabla v_e|^2}{v_e^2} v_e dx
\]
\[
= - \int_\Omega \frac{\Delta v_e}{v_e} \left( \Delta v_e - v_e u_e + \mu v_e(1 - v_e) \right) dx + \frac{1}{2} \int_\Omega \frac{|\nabla v_e|^2}{v_e^2} \left( \Delta v_e - v_e u_e + \mu v_e(1 - v_e) \right) dx
\]
\[
= - \int_\Omega \frac{|\Delta v_e|^2}{v_e} + \frac{1}{2} \int_\Omega \frac{|\nabla v_e|^2}{v_e} \Delta v_e dx - \int_\Omega \left( \nabla u_e \cdot \nabla v_e + \frac{3\mu}{2} v_e |\nabla v_e|^2 + \frac{u_e}{v_e} |\nabla v_e|^2 - \mu \frac{1}{v_e} |\nabla v_e|^2 \right) dx.
\]
Noticing that

\[ \int_{\Omega} v_e |D^2 \ln v_e|^2 \, dx = \int_{\Omega} \left( \frac{|D^2 v_e|^2}{v_e} + \frac{|\nabla v_e|}{v_e^3} - 2 \frac{\nabla v_e \nabla^2 v_e \nabla v_e}{v_e^2} \right) \, dx \]

Using Lemma 2.3 and combining with the above equality, we see that

Substituting the above inequality into (3.9), and using Lemma 3.2, we see that

\[ \int_{\Omega} \left( \frac{|D^2 v_e|^2}{v_e} + \frac{|\nabla v_e|}{v_e^3} - \frac{|\nabla v_e|^2}{v_e^2} \Delta v_e - 2 \frac{|\nabla v_e|^4}{v_e^4} \right) \, dx \]

Multiplying the first equation of (3.1) by 1/2, and integrating it over \( \Omega \), we obtain

Using (3.2), and it implies that

\[ \frac{d}{dt} \int_{\Omega} u_e \ln u_e \, dx + \int_{\Omega} \frac{\nabla u_e}{u_e} \, dx + \int_{\Omega} \left( \int_{\Omega} \frac{\nabla u_e}{u_e} \right) \, dx + \int_{\Omega} \rho u_e \ln(1 + u_e) \, dx + \varepsilon \int_{\Omega} u_e^2 \ln(1 + u_e) \, dx \]

Substituting the above inequality into (3.9), and using Lemma 3.2, we see that

\[ \frac{d}{dt} \int_{\Omega} u_e \ln u_e \, dx + \int_{\Omega} \frac{\nabla u_e}{u_e} \, dx + \int_{\Omega} \left( \frac{\nabla u_e}{u_e} \right)^2 \, dx + \int_{\Omega} \rho u_e \ln(1 + u_e) \, dx + \varepsilon \int_{\Omega} u_e^2 \ln(1 + u_e) \, dx \]
Combining (3.11) with (3.7), we obtain

\[
\frac{d}{dt} \int_\Omega \left( \frac{1}{2} \frac{|\nabla v_e|^2}{v_e^2} + \frac{1}{\chi} u_e \ln u_e \right) dx + \int_\Omega v_e |D^2 \ln v_e|^2 dx + \frac{\varepsilon}{\chi} \int_\Omega \frac{|\nabla u_e|^2}{u_e} dx + \frac{2}{m \chi} \int_\Omega |\nabla u_e|^2 dx \\
+ \frac{1}{\chi} \int_\Omega \rho u_e \ln(1 + u_e) dx + \varepsilon \int_\Omega u_e^2 \ln(1 + u_e) dx + \int_\Omega \left( \frac{3\mu}{2} |\nabla v_e|^2 + \frac{1}{2} \rho u_e^2 |\nabla v_e|^2 \right) dx \\
\leq \frac{1}{2} \int_{\partial \Omega} \frac{1}{v_e} \frac{\partial}{\partial n} |\nabla v_e|^2 ds + \frac{\mu}{2} \int_\Omega \frac{1}{v_e} |\nabla v_e|^2 dx + C.
\]

(3.12)

By the boundary trace embedding theorem [11] and Lemma 2.4, we see that

\[
\frac{1}{2} \int_{\partial \Omega} \frac{1}{v_e} \frac{\partial}{\partial n} |\nabla v_e|^2 ds \leq \kappa \int \frac{1}{v_e} |\nabla v_e|^2 ds = \kappa \int |v_e^\frac{1}{2} \nabla \ln v_e|^2 ds \\
\leq \eta_1 \kappa \int |D(v_e^{\frac{1}{2}} \nabla \ln v_e)|^2 dx + C_{\eta_1} \int v_e |\nabla \ln v_e|^2 dx \\
\leq \eta_1 \kappa \int \left( \frac{1}{2} v_e^{-\frac{1}{2}} \nabla v_e \nabla \ln v_e + v_e^{\frac{1}{2}} D^2 \ln v_e \right)^2 dx + C_{\eta_1} \int |\nabla v_e|^2 dx \\
\leq 2\eta_1 \kappa \int |v_e| D^2 \ln v_e|^2 dx + C_{\eta_1} \int |\nabla v_e|^2 dx \\
\leq 10\eta_1 \kappa \int v_e |D^2 \ln v_e|^2 dx + C_{\eta_1} \int |\nabla v_e|^2 dx
\]

(3.13)

for any sufficiently small \( \eta_1 > 0 \). By (3.2) and (2.6), for any sufficiently small \( \eta_2 > 0 \), we obtain

\[
\int_\Omega \frac{|\nabla v_e|^2}{v_e} dx \leq \eta_2 \int_\Omega \frac{|\nabla v_e|^4}{v_e^2} dx + C_{\eta_2} \\
\leq 16\eta_2 \int_\Omega v_e |D^2 \ln v_e|^2 dx + C_{\eta_2}.
\]

(3.14)

Combining (3.12), (3.13) and (3.14), we obtain

\[
\frac{d}{dt} \int_\Omega \left( \frac{1}{2} \frac{|\nabla v_e|^2}{v_e^2} + \frac{1}{\chi} u_e \ln u_e \right) dx + \frac{1}{2} \int_\Omega v_e |D^2 \ln v_e|^2 dx + \frac{\varepsilon}{\chi} \int_\Omega \frac{|\nabla u_e|^2}{u_e} dx + \frac{2}{m \chi} \int_\Omega |\nabla u_e|^2 dx \\
+ \frac{1}{\chi} \int_\Omega \rho u_e \ln(1 + u_e) dx + \varepsilon \int_\Omega u_e^2 \ln(1 + u_e) dx + \int_\Omega \left( \frac{3\mu}{2} |\nabla v_e|^2 + \frac{1}{2} \rho u_e^2 |\nabla v_e|^2 \right) dx \leq C.
\]

(3.15)

By (2.6), Lemma 2.1 and (3.10), we complete the proof. \( \square \)

**Lemma 3.4** Assume (i) or (ii), and let \((u_e, v_e)\) be the solution of (3.1). Then for any \( r > 0 \), we have

\[
\sup_{t \in (0, +\infty)} \|u_e(\cdot, t)\|_{L^{r+1}}^{r+1} + \sup_{t \in (0, +\infty)} \int_t^{t+1} \|\nabla u_e(\cdot, s)\|_{L^2}^2 ds \leq C_r, \quad \text{for} \quad r > 0,
\]

(3.16)

and

\[
\sup_{t \in (0, +\infty)} \|v_e\|_{W^{2,1}_r} \leq \tilde{C}_r, \quad \text{for} \quad r > 0,
\]

(3.17)

where \( C_r, \tilde{C}_r \) depend on \( r \), and are independent of \( \varepsilon \).
Multiplying the first equation of (3.1) by \( u_e^r \) for any \( r > 0 \), and integrating it over \( \Omega \), we obtain

\[
\frac{1}{r + 1} \frac{d}{dt} \int_{\Omega} u_e^{r+1} dx + \epsilon r \int_{\Omega} u_e^{r-1} |\nabla u_e|^2 dx + rm \int_{\Omega} u_e^{m+r-2} |\nabla u_e|^2 dx + \int_{\Omega} u_e^{r+1} dx + \epsilon \int_{\Omega} u_e^{r+2} dx
\]

\[
= \xi \int_{\Omega} u_e^{r+1} v_e dx + r \chi \int_{\Omega} u_e^{r} \nabla v_e \nabla u_e dx + \int_{\Omega} u_e^{r+1} dx
\]

\[
\leq C \int_{\Omega} u_e^{r+1} dx + \frac{rm}{4} \int_{\Omega} u_e^{m+r-2} |\nabla u_e|^2 dx + C \int_{\Omega} u_e^{r+2} dx.
\] (3.18)

By Gagliardo-Nirenberg interpolation inequality, we see that

\[
C ||u_e||^r_{L^{r+1}} = C ||u_e||^\frac{m}{r} ||u_e^\frac{m}{r^2} + \frac{2(\frac{1}{r} - 1)}{m} ||\nabla u_e||^\frac{m}{r^2} + C_2 ||u_e||^{r+1}_{L^{r+1}}
\]

\[
\leq C_3(1 + ||\nabla u_e||^\frac{m}{r^2} + \frac{6r}{m^2 + 3r - 1}) \leq \frac{mr}{(m + r)^2} ||\nabla u_e||^r_{L^r} + C_4.
\]

Substituting this inequality into (3.18), we obtain

\[
\frac{1}{r + 1} \frac{d}{dt} \int_{\Omega} u_e^{r+1} dx + \frac{rm}{2} \int_{\Omega} u_e^{m+r-2} |\nabla u_e|^2 dx + \int_{\Omega} u_e^{r+1} dx
\]

\[
\leq C \int_{\Omega} u_e^{r+2-m} |\nabla v_e|^2 dx + C_4.
\] (3.19)

Taking \( r = m - 1 \), and noticing that \( u_e|\nabla v_e|^2 \leq ||v_e||_{L^\infty} \frac{m}{r} \nabla |\nabla v_e|^2 \) then combining (3.19) with (3.5) yields

\[
\sup_{t \in (0, +\infty)} ||u_e||_{L^m} + \sup_{t \in (0, +\infty)} \int_t^{t+1} \int_{\Omega} u_e^{m-3} |\nabla u_e|^2 dx ds \leq C.
\] (3.20)

Multiplying the first equation of (3.1) by \( v_e \), and multiplying the second equation of (3.1) by \( u_e \), and adding the two equalities, we obtain

\[
\frac{d}{dt} \int_{\Omega} u_e v_e dx + (\epsilon + 1) \int_{\Omega} \nabla u_e \nabla v_e dx + \int_{\Omega} \nabla u_e^{m} \nabla v_e dx + \int_{\Omega} (\rho u_e v_e + \xi u_e v_e^2 - \mu u_e v_e (1 - v_e)) dx
\]

\[
- \chi \int_{\Omega} u_e |\nabla v_e|^2 dx = - \int_{\Omega} v_e u_e^2 dx.
\]

Integrating this equality from \( t \) to \( t + 1 \) for any \( t \geq 0 \), and using Lemma 3.2 and Lemma 3.3, we obtain

\[
\int_t^{t+1} \int_{\Omega} v_e u_e^2 dx ds \leq -(\epsilon + 1) \int_t^{t+1} \int_{\Omega} \nabla u_e \nabla v_e dx ds - \int_t^{t+1} \int_{\Omega} \nabla u_e^{m} \nabla v_e dx ds + C.
\] (3.21)

Integrating (3.8) from \( t \) to \( t + 1 \), and using Lemma 3.3 it is easy to see that

\[
- \int_t^{t+1} \int_{\Omega} \nabla u_e \nabla v_e dx \leq C.
\]

By (3.5) and (3.20), we also have

\[
- \int_t^{t+1} \int_{\Omega} \nabla u_e^{m} \nabla v_e dx ds \leq \int_t^{t+1} \int_{\Omega} u_e^{2m-3} |\nabla u_e|^2 dx ds + \int_t^{t+1} \int_{\Omega} u_e |\nabla v_e|^2 dx ds \leq C.
\]

Substitute the above two inequalities into (3.21), we obtain

\[
\sup_{t \in (0, +\infty)} \int_t^{t+1} \int_{\Omega} v_e u_e^2 dx ds \leq C.
\] (3.22)
By this inequality and Lemma 2.2, we further have

$$\sup_{t \in (0, +\infty)} \int_{t}^{t+1} \int_{\Omega} \left( \frac{|\partial v_{e}|^2}{\partial t} + |\Delta v_{e}|^2 \right) dx dt \leq C. \quad (3.23)$$

Next, we go back to estimate the first term on the right hand side of the inequality (3.19). For any $p \in (\frac{4}{3}, 2)$, using Hölder’s inequality and Gagliardo-Nirenberg interpolation inequality, we have

$$\int_{\Omega} u_{e}^{r+2-m} |\nabla v_{e}|^2 dx \leq \|\nabla v_{e}\|_{L^p}^2 \|u_{e}\|_{L^{p^*}}^{r+2-m}$$

$$\leq C \|v_{e}\|_{L^\infty}^{\frac{8-2p}{4-p}} \|\nabla v_{e}\|_{L^p}^2 \|u_{e}\|_{L^{p^*}}^{r+2-m} + C \|v_{e}\|_{L^\infty}^2 \|u_{e}\|_{L^{p^*}}^{r+2-m}$$

$$\leq \delta \|u_{e}\|_{L^{\frac{m(p+2-m)}{3-p}}} + C_{\delta} (1 + \|\Delta v_{e}\|_{L^2}^2) \quad (3.24)$$

for any sufficiently small constant $\delta > 0$. Next, we pay our attention to this term $\delta \|u_{e}\|_{L^{\frac{m(p+2-m)}{3-p}}}$. Using Gagliardo-Nirenberg interpolation inequality, and recalling (3.20), we have

$$\delta \|u_{e}\|_{L^{\frac{m(p+2-m)}{3-p}}} = \delta \|u_{e}\|_{L^{p^*}}^{\frac{2(p+2-m)}{3-p} - \frac{m(p+2-m)}{3-p}} \leq \delta C \|u_{e}\|_{L^{p^*}}^{\frac{2(p+2-m)}{3-p} - \frac{m(p+2-m)}{3-p}} \|\nabla u_{e}\|_{L^2}^{\frac{6(p+2-m) - 6m(p-1)}{(2m+3r)(3-p)}}$$

$$+ \delta C \|u_{e}\|_{L^{p^*}}^{\frac{2(p+2-m)}{3-p} - \frac{m(p+2-m)}{3-p}}$$

where $p^* = \frac{2(p+2-m)}{(3-p)(m+r)} - \frac{6p(r+2-m) - 6m(p-1)}{(2m+3r)(3-p)}$. We take $p = \frac{3m+9r}{6(r+1)-4m}$, then

$$\frac{6p(r+2-m) - 6m(p-1)}{(2m+3r)(3-p)} = 2.$$  

Substituting (3.25) into (3.24) with $p = \frac{3m+9r}{6(r+1)-4m}$, clearly for any $r \geq 4m, m > 1$, we have $p \in (\frac{3}{2}, 2)$, then we obtain

$$\int_{\Omega} u_{e}^{r+2-m} |\nabla v_{e}|^2 dx \leq \|\nabla v_{e}\|_{L^p}^2 \|u_{e}\|_{L^{p^*}}^{r+2-m}$$

$$\leq C_{\delta} \|\nabla u_{e}\|_{L^2}^2 + C_{\delta} (1 + \|\Delta v_{e}\|_{L^2}^2) \quad (3.26)$$

for any sufficiently small constant $\delta > 0$. Combining (3.26) with (3.19), we finally have

$$\frac{1}{r+1} \frac{d}{dt} \int_{\Omega} u_{e}^{r+1} dx + \frac{rm}{4} \int_{\Omega} u_{e}^{m+r-2} |\nabla u_{e}|^2 dx + \int_{\Omega} u_{e}^{r+1} dx \leq C \|\Delta v_{e}\|_{L^2}^2 + C \quad (3.27)$$

for any $r \geq 4m$. By (3.23), we finally obtain (3.16) for any $r \geq 4m$. By Sobolev inequality, we further have (3.16) for any $r > 0$. while, (3.17) is a direct result of (3.16) by Lemma 2.2.

Using this lemma, we further have

**Lemma 3.5** Assume (i) or (ii), and let $(u_{e}, v_{e})$ be the solution of (3.1). Then

$$\sup_{t \in (0, +\infty)} \|v_{e}(\cdot, t)\|_{W^{1,\infty}} \leq C, \quad (3.28)$$

$$\sup_{t \in (0, +\infty)} \|u_{e}(\cdot, t)\|_{L^{\infty}} \leq C, \quad (3.29)$$

where the constants $C$ are independent of $\varepsilon$. 

**Proof.** By $t$-anisotropic embedding theorem, that is $W^{2,1}_p \hookrightarrow C^{0,\frac{\alpha}{p}}$ for any $0 < \alpha \leq 2 - \frac{2}{p}$, we obtain (3.28) by (3.17). By (3.28), and similarly to the proof of in [6], we can obtain the $L^\infty$ estimate of $u_e$ by using Moser iteration method.

□

**Lemma 3.6** Assume (i) or (ii), and let $(u_e, v_e)$ be the solution of (3.1). Then
\[
\sup_{t \in (0, +\infty)} \int_\Omega |\nabla u_e|^2 dx + \epsilon \sup_{t \in (0, +\infty)} \int_0^t \int_\Omega \left| \frac{\partial u_e}{\partial t} \right|^2 dxds + \sup_{t \in (0, +\infty)} \int_0^t \int_\Omega u_{e}^{m-1} \left| \frac{\partial u_e}{\partial t} \right|^2 dxds \leq C, \tag{3.30}
\]
where $C$ is independent of $\epsilon$.

**Proof.** Multiplying the first equation of (3.1) by $\frac{\partial (\epsilon u_e + u_{e}^m)}{\partial t}$, and integrating it over $\Omega$ gives
\[
\frac{1}{2} \frac{d}{dt} \int_\Omega |\nabla (\epsilon u_e + u_{e}^m)|^2 dx + \epsilon \int_\Omega \left| \frac{\partial u_e}{\partial t} \right|^2 dx + m \int_\Omega u_{e}^{m-1} \left| \frac{\partial u_e}{\partial t} \right|^2 dx
\leq -\chi \int_\Omega \nabla \cdot (u_e \nabla v_e) dx + \int_\Omega (\xi u_e v_e - \rho u_e - \epsilon u_e^2) \frac{\partial (\epsilon u_e + u_{e}^m)}{\partial t} dx
\leq m\chi^2 \int_\Omega |\nabla (u_e \nabla v_e)|^2 dx + \epsilon \chi^2 \int_\Omega |\nabla (u_e \nabla v_e)|^2 dx + \frac{m}{2} \int_\Omega \left| \frac{\partial u_e}{\partial t} \right|^2 dx
+ \epsilon \int_\Omega (\xi u_e v_e - \rho u_e - \epsilon u_e^2) dx + m \int_\Omega u_{e}^{m-1} (\xi u_e v_e - \rho u_e - \epsilon u_e^2) dx + \frac{m}{2} \int_\Omega u_{e}^{m-1} \left| \frac{\partial u_e}{\partial t} \right|^2 dx,
\]
recalling (3.28) and (3.29), we further have
\[
\frac{1}{2} \frac{d}{dt} \int_\Omega |\nabla (\epsilon u_e + u_{e}^m)|^2 dx + \frac{\epsilon}{2} \int_\Omega \left| \frac{\partial u_e}{\partial t} \right|^2 dx + m \frac{1}{2} \int_\Omega u_{e}^{m-1} \left| \frac{\partial u_e}{\partial t} \right|^2 dx + \int_\Omega |\nabla (\epsilon u_e + u_{e}^m)|^2 dx
\leq m\chi^2 \int_\Omega |\nabla (u_e \nabla v_e)|^2 dx + \epsilon \chi^2 \int_\Omega |\nabla (u_e \nabla v_e)|^2 dx + \int_\Omega |\nabla (\epsilon u_e + u_{e}^m)|^2 dx + C
\leq C \int_\Omega (|\nabla u_e|^2 + |\Delta v_e|^2) dx + C \epsilon \int_\Omega \frac{|\nabla u_e|^2}{u_e} dx + C.
\]
Recalling (3.5), (3.27), and using Lemma 2.1, we obtain
\[
\sup_{t \in (0, +\infty)} \int_\Omega |\nabla (\epsilon u_e + u_{e}^m)|^2 dx + \epsilon \sup_{t \in (0, +\infty)} \int_0^t \int_\Omega \left| \frac{\partial u_e}{\partial t} \right|^2 dxds + \sup_{t \in (0, +\infty)} \int_0^t \int_\Omega u_{e}^{m-1} \left| \frac{\partial u_e}{\partial t} \right|^2 dxds \leq C.
\]
and (3.30) is obtained.

Completely similar to the proof as Lemma 3.3-Lemma 3.6, for the case $\xi \mu > 0$ and $\rho = 0$, we also have

**Lemma 3.7** Assume (iii) $\xi \mu > 0$ and $\rho = 0$, and let $(u_e, v_e)$ be the solution of (3.1). Then for any $T > 0$,
\[
\sup_{t \in (0, T)} (||u_e(\cdot, t)||_{L^\infty} + ||\nabla u_e^m||_{L^2} + ||v_e(\cdot, t)||_{W^{1,\infty}}) + \int_\Omega \left( \epsilon ||\nabla u_e||^2 + ||\nabla u_e^m||^2 + \left| \frac{\partial u_e^m}{\partial t} \right|^2 \right)dxdt \leq C, \tag{3.31}
\]
and
\[
||v_e||_{W^{2,1}_p (Q_T)} \leq C_p, \text{ for any } p \in (1, +\infty), \tag{3.32}
\]
where the constants $C_p$ and $C$ depend on $T$, $C_p$ depends on $p$, and both of them are independent of $\epsilon$. 
4 Large Time Behavior of Solutions

To investigate the large time behavior of solutions of the problem (1.1), we need the following two lemmas [7].

**Lemma 4.1** Assume that \( f \geq 0, f(t) \in L^1(T, \infty) \) for some constant \( T > 0 \), and

\[
|f(t) - f(s)| \leq A(t - s) \quad (\text{or} \quad \geq -A(t - s)), \quad \text{for any} \quad t > s > T.
\]

Then

\[
\lim_{t \to \infty} f(t) = 0.
\]

**Lemma 4.2** Assume that \( f(t), g(t) \geq 0, \lim_{t \to \infty} g(t) = 0, \) and \( f(t) \in L^1(T, +\infty) \) with some constant \( T \geq 0 \). Let \( F(t) = f(t) - g(t) \), and that

\[
F(t) - F(s) \geq -A(t - s), \forall t > s > T,
\]

then

\[
\lim_{t \to \infty} f(t) = 0.
\]

Firstly for the case \( \mu = 0 \), we have the following lemma.

**Lemma 4.3** Assume that \( \mu = 0 \). Let \((u, v)\) with \( u \in X_1, v \in X_2 \) be the global solution. Then we have

\[
\int_0^\infty \int_{\Omega} (|\Delta v|^2 + |\nabla v|^2 + uv + u^{m+r-2} |\nabla u|^2 + \rho u)dxdt \leq C_r, \quad \text{for any} \quad r > (m-2)_+,
\]

(4.1)

where \( C_r \) is a constant depends on \( r \) and the initial datum \((u_0, v_0)\). In particular,

\[
\lim_{t \to \infty} \int_{\Omega} (|\nabla v|^2 + \rho u)dx = 0.
\]

(4.2)
Proof. Integrating the second equation of (1.1) over \(\Omega\), we see that
\[
\frac{d}{dt} \int_{\Omega} v dx + \int_{\Omega} uv dx = 0,
\]
integrating this equality from 0 to \(\infty\), we obtain
\[
\int_{0}^{\infty} \int_{\Omega} uvdxdt \leq \int_{\Omega} v_{0} dx. \tag{4.3}
\]
Multiplying the second equation of (1.1) by \(v\), we have
\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega} v^2 dx + \frac{1}{2} \int_{\Omega} \vert \nabla v \vert^2 dx + \int_{\Omega} v^2 ud x = 0,
\]
it integrating this equality from 0 to \(\infty\), we obtain
\[
\int_{0}^{\infty} \int_{\Omega} (\vert \nabla v \vert^2 + v^2 u) dxdt \leq \int_{\Omega} \vert v_{0} \vert^2. \tag{4.4}
\]
Multiplying the second equation of (1.1) by \(\Delta v\), and using Young’s inequality, we obtain
\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega} \vert \nabla v \vert^2 dx + \frac{1}{2} \int_{\Omega} \vert \Delta v \vert^2 dx \leq \int_{\Omega} \vert u \vert_{L^{\infty}} \int_{\Omega} \vert v \vert^2 dx,
\]
integrating this equality from 0 to \(\infty\), we obtain
\[
\int_{0}^{\infty} \int_{\Omega} \vert \nabla v \vert^2 dxdt \leq \int_{\Omega} \vert v_{0} \vert^2 + \sup_{t} \int_{\Omega} \vert u \vert_{L^{\infty}} \int_{0}^{\infty} \int_{\Omega} \vert v \vert^2 dxdt. \tag{4.6}
\]
Integrating the first equation of (1.1) over \(\Omega\), we obtain
\[
\frac{d}{dt} \int_{\Omega} u dx + \rho \int_{\Omega} u dx = \xi \int_{\Omega} uv dx, \tag{4.7}
\]
which implies
\[
\int_{\Omega} u(x, t) dx + \rho \int_{0}^{\infty} \int_{\Omega} u dxdt = \xi \int_{0}^{\infty} \int_{\Omega} uv dxdt + \int_{\Omega} u_{0} dx \leq C. \tag{4.8}
\]
Multiplying the first equation of (1.1) by \(u^r\) for any \(r > (m - 2)_{+}\), we obtain
\[
\frac{1}{r + 1} \frac{d}{dt} \int_{\Omega} u^{r+1} dx + mr \int_{\Omega} u^{m+r-2} \vert \nabla u \vert^2 dx + \rho \int_{\Omega} u^{r+1} dx = \chi r \int_{\Omega} u' \nabla u \nabla v dx + \xi \int_{\Omega} u^{r+1} v dx
\]
\[
\leq \frac{mr}{2} \int_{\Omega} u^{m+r-2} \vert \nabla u \vert^2 dx + C \int_{\Omega} u^{r+2-m} \vert \nabla v \vert^2 dx + \xi \int_{\Omega} u^{r+1} v dx,
\]
\[
\leq \frac{mr}{2} \int_{\Omega} u^{m+r-2} \vert \nabla u \vert^2 dx + C \int_{\Omega} \vert \nabla v \vert^2 dx + C \int_{\Omega} uv dx,
\]
by (4.3) and (4.4) and the boundedness of \(u\) and \(v\), we obtain
\[
\int_{0}^{\infty} \int_{\Omega} u^{m+r-2} \vert \nabla u \vert^2 dx dt \leq C. \tag{4.9}
\]
Combining (4.3), (4.4), (4.6), (4.8) and (4.9), (4.1) is proved.
On the other hands, by (4.5) and (4.7), and using the boundedness of $u$ and $v$, we also have

$$\frac{d}{dt} \int_{\Omega} (|\nabla v|^2 + \rho u) dx \leq C,$$

which implies that

$$\int_{\Omega} (|\nabla v(x,t)|^2 + \rho u(x,t)) dx - \int_{\Omega} (|\nabla v(x,s)|^2 + \rho u(x,s)) dx \leq C(t-s).$$

Hence, by Lemma 4.1 and the inequality (4.1), we obtain (4.2). The proof is complete. \(\square\)

**Lemma 4.4** Assume that $\mu = 0$, $\rho = 0$, $\xi \geq 0$, $u_0 \neq 0$. Let $(u, v)$ with $u \in X_1, v \in X_2$ be the global solution. Then we have

$$\lim_{t \to \infty} \|v\|_{L^\infty} = 0, \quad \lim_{t \to \infty} \|u - A\|_{L^p} = 0, \quad \text{for any } p > 1. \quad (4.10)$$

where $A = \frac{1}{|\Omega|} \int_{\Omega} (u_0 + \xi v_0) dx > 0$.

**Proof.** We denote

$$a(t) = \frac{1}{|\Omega|} \int_{\Omega} u(x,t) dx, \quad b(t) = \frac{1}{|\Omega|} \int_{\Omega} v(x,t) dx.$$

It is easy to see that

$$a'(t) = \frac{\xi}{|\Omega|} \int_{\Omega} uv dx \geq 0, \quad b'(t) = -\frac{1}{|\Omega|} \int_{\Omega} uv dx \leq 0,$$

$$(a + \xi b)'(t) = 0,$$

which means that

$$a(t) + \xi b(t) \equiv A.$$

Note that $a(t)$ is monotonically increasing and bounded above, $b(t)$ is monotonically decreasing and bounded below, then there exists two constants $a^* > 0$ and $b^* \geq 0$ such that $a^* + b^* = A$, and

$$\lim_{t \to \infty} a(t) = a^*, \quad \lim_{t \to \infty} b(t) = b^*.$$

By Poincaré inequality, we have

$$\|v - b(t)\|_{L^2} \leq C\|\nabla v\|_{L^2},$$

and note that

$$\|v - b^*\|_{L^2} \leq \|v - b(t)\|_{L^2} + |\Omega|^{1/2}|b(t) - b^*|,$$

combining with (4.2), we have

$$\lim_{t \to \infty} \|v - b^*\|_{L^2} = 0.$$

and we further have

$$\lim_{t \to \infty} \|v - b^*\|_{L^\infty} = 0 \quad (4.11)$$

since

$$\|v - b^*\|_{L^\infty} \leq C_1\|v - b^*\|_{L^2}^{2/3}\|\nabla v\|_{L^\infty}^{1/3} + C_2\|v - b^*\|_{L^2}.$$
Next, we show \( b^* = 0 \). Suppose to the contrary, that is \( b^* > 0 \). By (1.6), we also have \( v \in C^{\alpha, \frac{2}{\alpha}}(Q) \) for some positive constant \( \alpha > 0 \), then there exists \( T_0 > 0 \), such that

\[
\nu(x, t) > \frac{b^*}{2}, \quad \text{for any } t > T_0.
\]

Then we have

\[
b'(t) = -\frac{1}{|\Omega|} \int_{\Omega} uv dx \leq -\frac{b^*}{2|\Omega|} \int_{\Omega} udx = -\frac{b^* a(t)}{2} \leq -\frac{a^* b^*}{2} < 0,
\]

which implies that \( b(t) \to 0 \), that is \( b^* = 0 \). It is a contradiction. The first equality of (4.10) is proved.

It also implies that \( a^* = A \), that is

\[
\lim_{t \to \infty} a(t) = A.
\]

Next, we show the second equality of (4.10). Denote

\[
a^m(t) = \frac{1}{|\Omega|} \int_{\Omega} u^m(x, t) dx,
\]

then we have

\[
a^m(t) = \frac{1}{|\Omega|} \int_{\Omega} u^m dx \geq \left( \frac{1}{|\Omega|} \int_{\Omega} udx \right)^m = a(t)^m.
\]

By (4.1) with \( r = m \) and Poincaré inequality, we also have

\[
\int_0^\infty \int_{\Omega} |u^m - \alpha^m|^2 dx dt \leq \int_0^\infty \int_{\Omega} |\nabla u^m|^2 dx dt \leq C.
\]

By a direct calculation, and using (1.4), (1.5), we obtain

\[
\begin{align*}
\frac{d}{dt} \int_{\Omega} |u^m - \alpha^m|^2 dx &= \frac{d}{dt} \int_{\Omega} u^{2m} dx - 2|\Omega| \alpha^m(\alpha^m)' \\
&= 2m^2(m - 1)\alpha^m \int_{\Omega} u^{2m-2} |\nabla u|^2 dx - 2(2m - 1) \int_{\Omega} u^{m-1} |\nabla u|^2 dx - (2m - 1)\chi \int_{\Omega} u^{2m} \Delta v dx \\
&\quad + 2(m - 1)\chi \alpha^m \int_{\Omega} u^m \Delta v dx + 2m\xi \int_{\Omega} u^{2m} v dx - 2m\alpha^m \xi \int_{\Omega} u^m v dx - 2mp \int_{\Omega} u^{2m} dx + 2mp\alpha^m \int_{\Omega} u^m dx \\
&\geq -C - \int_{\Omega} |\Delta v|^2 dx.
\end{align*}
\]

Combining with (4.5), we obtain

\[
\frac{d}{dt} \int_{\Omega} (|u^m - \alpha^m|^2 - |\nabla v|^2) dx \geq -C,
\]

which implies that for any \( t > s > 0 \),

\[
\int_{\Omega} (|u^m(x, t) - \alpha^m(t)|^2 - |\nabla v(x, t)|^2) dx - \int_{\Omega} (|u^m(x, s) - \alpha^m(s)|^2 - |\nabla v(x, s)|^2) dx \geq -C(t - s).
\]

Using (4.2), (4.14) and Lemma 4.2, we obtain

\[
\lim_{t \to \infty} \int_{\Omega} |u^m(x, t) - \alpha^m(t)|^2 dx = 0.
\]
Noticing that $\frac{a_\alpha}{a_{\alpha+1}} \geq \frac{a_{\alpha-1}}{a_{\alpha}}$, and using (4.13), we have
\begin{align}
\frac{a(t)^{2m-2}}{\|u - a(t)\|_{L^2}} \leq \frac{\alpha^{2m-2}(t)}{\|u - \alpha(t)\|_{L^2}} \leq \int_{\Omega} |u - \alpha(t)|^2 dx \leq \int_{\Omega} |u - a(t)|^2 dx.
\end{align}

Combining (4.12), (4.13) and (4.16), we obtain
\begin{align}
|a(t) - \alpha(t)|^2 = \left| \frac{1}{|\Omega|} \int_{\Omega} (u - \alpha) dx \right|^2 \leq \int_{\Omega} |u - \alpha|^2 dx \to 0, \quad \text{as} \quad t \to \infty.
\end{align}

Therefore, we have
\begin{align}
\|u - A\|_{L^2} \leq \|u - \alpha(t)\|_{L^2} + |\Omega|^\frac{1}{2} |\alpha(t) - a(t)| + |\Omega|^\frac{1}{2} |\alpha(t) - A| \to 0, \quad \text{as} \quad t \to \infty.
\end{align}

If $p < 2$, clearly, we have
\begin{align}
\|u - A\|_{L^p} \leq \|u - A\|_{L^2};
\end{align}

for any $p > 2$, we see that
\begin{align}
\|u - A\|_{L^p} \leq \|u - A\|_{L^2}^\frac{2}{p} \leq \|u - A\|_{L^2}^\frac{2}{p} \leq C \|u - A\|_{L^2}^\frac{2}{p},
\end{align}

and the proof is complete.

**Lemma 4.5** Assume that $\mu = 0, \rho > 0, \xi \geq 0$, and $v_0 \neq 0$. Let $(u, v)$ with $u \in X_1, v \in X_2$ be the global solution. Then there exists a constant $B \in (0, \frac{1}{|\Omega|} \int_{\Omega} v_0 dx)$, such that
\begin{align}
\lim_{t \to \infty} \|v - B\|_{L^\infty} = 0, \quad \lim_{t \to \infty} \|u\|_{L^p} = 0 \quad \text{for any} \quad p > 1.
\end{align}

**Proof.** Note that $b(t)$ is monotonically decreasing and bounded below, where $b(t)$ is defined as the proof of Lemma 4.3. Then there exists $B \geq 0$ such that
\begin{align}
\lim_{t \to \infty} b(t) = B.
\end{align}

Similar to the proof of Lemma 4.4, we have
\begin{align}
\lim_{t \to \infty} \|v - B\|_{L^\infty} = 0.
\end{align}

In what follows, we show that $B > 0$. Let’s consider the following problem
\begin{align}
\begin{cases}
\tilde{v}_t - \Delta \tilde{v} + M\tilde{v} = 0, \\
\frac{\partial \tilde{v}}{\partial n} = 0, \\
\tilde{v}(x, 0) = v_0(x).
\end{cases}
\end{align}

By [4], there exists $\Gamma_0 > 0$, such that
\begin{align}
\tilde{v}(x, t) = e^{-Mt}e^{\Delta t}v_0 \geq e^{-Mt}\Gamma_0 \int_{\Omega} v_0(x) dx, \quad \text{for any} \quad t \geq 1,
\end{align}

where $\{e^{\Delta t}\}_{t \geq 0}$ is the Neumann heat semigroup. Taking $M \geq \sup \|u(\cdot, t)\|_{L^\infty}$, and by comparison, we have
\begin{align}
v(x, t) \geq \tilde{v}(x, t) \geq e^{-Mt}\Gamma_0 \int_{\Omega} v_0(x) dx, \quad \text{for any} \quad t \geq 1.
\end{align}
Multiplying the second equation of (1.1) by \( \frac{1}{v} \), integrating it over \( \Omega \times (1, t) \), and combining with (4.1) and (4.22), we obtain
\[
\int_{\Omega} \ln v(x, t) dx = \int_{\Omega} \ln v(x, 1) dx + \int_{1}^{t} \int_{\Omega} \frac{\vert \nabla v \vert^2}{v^2} dx ds - \int_{1}^{t} \int_{\Omega} u dx ds \geq -C
\]  
for some positive constant \( C \). Recalling (4.20), it implies that \( B > 0 \).

Next, we show the second limit equality. By (4.2), we see that
\[
\lim_{t \to \infty} \| u \|_{L^1} = 0
\]
since \( \rho > 0 \). Thus, we further have
\[
\| u \|_{L^p} \leq \| u \|_{L^1}^{\frac{1}{p}} \| u \|_{L^\infty}^{\frac{p-1}{p}} \to 0, \text{ as } t \to \infty
\]
for any \( p > 1 \), and this lemma is proved. \( \square \)

Next, we turn our attention to discuss the asymptotic behavior of solutions to the problem (1.1) in the case of \( \mu > 0 \).

**Lemma 4.6** Assume that \( \mu > 0, \rho > 0, 0 \leq \xi < \rho, \) and \( v_0 \neq 0 \). Let \((u, v)\) with \( u \in X_1, v \in X_2 \) be the global solution. Then
\[
\lim_{t \to \infty} \| v - 1 \|_{L^\infty} = 0, \quad \lim_{t \to \infty} \| u \|_{L^p} = 0 \quad \text{for any } p > 1. \tag{4.24}
\]

**Proof.** Firstly, similar to the proof of (4.22), there exists \( \delta > 0 \) such that
\[
v(x, 1) > \delta,
\]
for any \( x \in \Omega \) since \( \int_{\Omega} v_0 dx > 0 \). Let
\[
F(t) = \int_{\Omega} (u + \xi(v - 1 - \ln v)) dx.
\]
Then we have
\[
F'(t) + \int_{\Omega} \frac{\vert \nabla v \vert^2}{v^2} dx + \mu \int_{\Omega} (v - 1)^2 dx = (\xi - \rho) \int_{\Omega} u dx \leq 0.
\]
Noting that \( F(t) \geq 0 \), then we further have
\[
\int_{1}^{t} \int_{\Omega} \frac{\vert \nabla v \vert^2}{v^2} dx dt + \mu \int_{1}^{t} \int_{\Omega} (v - 1)^2 dx dt \leq F(1) \leq C. \tag{4.25}
\]
Multiplying the second equation of (1.1) by \( v - 1 \), we obtain
\[
\frac{d}{dt} \int_{\Omega} \vert v - 1 \vert^2 dx + 2 \int_{\Omega} \vert \nabla v \vert^2 dx + 2\mu \int_{\Omega} v(v - 1)^2 dx \leq C,
\]
which means that
\[
\int_{\Omega} \vert v(x, t) - 1 \vert^2 dx - \int_{\Omega} \vert v(x, s) - 1 \vert^2 dx \leq C(t - s), \quad \text{for any } 1 \leq s < t.
\]
Hence, by Lemma 4.1 and (4.25), we have
\[
\lim_{t \to \infty} \| v(\cdot, t) - 1 \|_{L^2} = 0,
\]
and moreover,
\[ ||v(\cdot, t) - 1||_{L^\infty} \leq C_1 ||v(\cdot, t) - 1||_{L^2}^2 ||\nabla v||_{L^\infty}^3 + C_2 ||v(\cdot, t) - 1||_{L^2} \to 0, \quad \text{as} \quad t \to \infty. \]

Note that \( v \) is H"older continuous since \( v \in W^{2,1}_p \) for \( p > 5 \), which implies that for any \( \varepsilon \in (0, \rho - \xi) \), there exists \( T > 0 \) such that
\[ \xi v(x, t) - \rho < -\varepsilon, \quad \text{for all} \quad t \geq T \quad \text{and} \quad x \in \Omega. \]

Then for any \( t \geq T \), we have
\[ \frac{d}{dt} \int_{\Omega} u dx = \int_{\Omega} (\xi v - \rho) u dx \leq -\varepsilon \int_{\Omega} u dx, \]
which implies that
\[ \int_{\Omega} u(x, t) dx \leq e^{-\varepsilon(t-T)} \int_{\Omega} u(x, T) dx \leq Ce^{-\varepsilon t}, \quad \text{for any} \quad t \geq T, \]
and we further have
\[ \lim_{t \to \infty} ||u||_{L^p} = 0 \quad \text{for any} \quad p > 1. \]

The proof is complete. \( \square \)

Then Theorem 1.3 is a direct result of Lemma 4.4–Lemma 4.6

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