3D GLASS - Acoustic Solar Modeling of Background Flows

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ABSTRACT

We present a 3-dimensional (3D) numerical solver of the linearized compressible Euler equations (3D GLASS – GLobal Acoustic Solar Simulation), used to model acoustic oscillations throughout the solar interior. The governing equations are solved in conservation form on a fully global spherical mesh (0 ≤ φ ≤ 2π, 0 ≤ θ ≤ π, 0 ≤ r ≤ R⊙) over a background state generated by the standard Solar Model S. We implement an efficient pseudo-spectral computational method to calculate the contribution of the compressible material derivative dyad to internal velocity perturbations, computing oscillations over arbitrary 3D background velocity fields. This model offers a foundation for a “forward-modeling” approach, using helioseismology techniques to explore various regimes of internal mass flows. We demonstrate the efficacy of the numerical method presented in this paper by reproducing observed solar power spectra, showing rotational splitting due to solid body rotation, and applying local helioseismology techniques to measure travel times created by a simple model of single-cell meridional circulation.

Keywords: Solar Physics, Computational Fluid Dynamics, 3D Solar Modeling

1. INTRODUCTION

Since the first measurements of 5-minute solar oscillations nearly 60 years ago (Leighton et al. 1962; Claverie et al. 1979), helioseismology has evolved into a rich and diverse field using surface observations to probe solar depths, inferring solar parameters and dynamics. Comprehensive overviews of helioseismology can be found in Christensen-Dalsgaard (2002); Di Mauro (2003); Gizon & Birch (2005). The global nature of these oscillations has allowed for some of the most precise measurements of the solar interior that are currently available and have become an indispensable tool in measuring internal solar rotation (Duvall et al. 1984; Schou et al. 1998) and meridional circulation (Giles et al. 1997; Zhao & Kosovichev 2004).

The nature of Solar rotation is a complex problem and carries with it important information on the dynamic structure of the Sun. Accurate modeling of internal rotation is vital to understanding effects that rapid rotation rates and angular momentum transport can play in internal mixing of elements and hence, the evolution of solar structure. Observations of the solar surface show a pattern of differential rotation (Snodgrass & Ulrich 1990) which varies widely in its period at the equator (∼ 24 days) to near the poles (∼ 30 days). This pattern of differential rotation is mimicked throughout the convective interior (upper 30% of the solar radius), with a slight maximum in the velocity of subsurface layers (∼ 0.95R⊙). These rates begin to converge at the tachocline (Kosovichev 1996) where the differential rotation is coupled to a solid core rotating at rate of ∼ 430 nHz. Even though helioseismology has

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made unprecedented strides in modeling interior solar dynamics, many unanswered questions remain. Inversion results near the poles show some discrepancies between the Global Oscillation Network Group (GONG) and the Michelson Doppler Imager (MDI) (Schou et al. 2002) data pipelines as well as newer HMI (Howe et al. 2011) observations. There are also conflicting measurements of the solar core’s rotation rate using low-degree modes (l = 1 − 4) – ranging from significantly lower rates (BiSON, Chaplin et al. (1996)) to much higher ones (IRIS, Lazrek et al. (1996)). Recent measurements of g-modes have also implied rotation rates more than twice as fast as previous estimates (Fossat et al. 2017).

Perpendicular to global rotation, we see strong poleward flows (20 m s\(^{-1}\)) in each hemisphere (Hathaway 1996). These meridional mass flows operate as mechanisms that distribute angular momentum and magnetic flux throughout the convective interior (Hathaway et al. 2003). Techniques in local helioseismology use perturbations in acoustic travel times to observe these flows (Gizon & Birch 2005), however it becomes more difficult to clearly resolve structures at greater depths. There is still no consensus on the nature and location of the return flow of meridional circulation cells – while commonly thought to sit in the relatively deep region at the base of the tachocline (Giles 1999), new techniques and measurements have begun to question this assumption (Mitra-Kraev & Thompson 2007; Hathaway 2012). The structure of these circulation cells has also been put to doubt, with recent measurements inferring more than just a single cell per hemisphere (Zhao et al. 2013), cf. Gizon et al. (2020).

One of the best approaches we have to gain insight into the nature of these problems is by using forward-modeling to test various theoretical and computational models of such flow structures. To that end we demonstrate the efficacy of a new pseudo-spectral global acoustic algorithm – developed to compute stochastically excited oscillations over a variety of 3-dimensional background velocity fields. Such fully global solar simulations have the potential to offer insights on the hydrodynamic nature of the solar interior, from modeling g-modes and their interaction with the rotating solar core to simulating various patterns of meridional circulation and torsional oscillations.

This paper is organized as follows: in §2 we describe the mathematical background and computational set-up of the model. In §3 we detail the specific numerical methods we used to compute our governing equations. In §4 we reproduce the solar p-mode spectrum and frequency splitting due to solid body rotation. We continue our validation by reproducing measurements of a simple single-cell model of meridional circulation using local helioseismology techniques in §5. Finally, we discuss future plans for the model and state our concluding remarks in §6.

2. MODEL DESCRIPTION

2.1. Mathematical Background

In this section we present the mathematical formulation and computational set-up for the 3D GLASS (GLobal Acoustic Solar Simulation) code – a 3-dimensional numerical solver of the linearized compressible Euler equations, used to model acoustic oscillations throughout the solar interior. For the sake of simplicity we assume the adiabatic approximation, where the time-scale of heat transfer is much smaller than the period of oscillations and is therefore neglected in the conservation of energy (Christensen-Dalsgaard 2003). The governing equations are solved on a fully global spherical mesh, 0 ≤ φ ≤ 2π, 0 ≤ θ ≤ π, 0 ≤ r ≤ R⊙, in the Cauchy conservation form; they are enumerated as follows:

\[
\frac{\partial \rho'}{\partial t} + \nabla \cdot (\rho' \mathbf{u}') = 0 ,
\]

\[
\frac{\partial \mathbf{u}'}{\partial t} + \nabla : (\mathbf{M}' \mathbf{u} + \rho' \mathbf{u}' \mathbf{u}') = -\nabla^2 (\rho') - \nabla \cdot (\rho' \mathbf{g}) + \nabla \cdot \mathbf{S}' \mathbf{r} ,
\]

\[
\frac{\partial \rho'}{\partial t} = -\frac{\Gamma_1 \rho}{\rho} \left( \nabla \cdot \mathbf{u}' + \rho' \mathbf{u}' \cdot \mathbf{u}' - \frac{\rho'}{\rho} \mathbf{u} \cdot \nabla \rho + \rho' \mathbf{u}' \cdot \frac{N^2}{g} \right). \tag{3}
\]

In Eqns. (1)-(3), we solve for oscillations in the potential field (φ) independently, and solenoidal contributions are discarded, here Υ is defined as the divergence of the momentum field M (Υ = ∇ · M = ∇ · ρu = ∇\(^2\)φ). ρ, u, p, g are the density, velocity, pressure and gravity terms respectively. Γ\(_1\) is the adiabatic ratio, constant in time in the adiabatic approximation. The governing equations are linearized - split into a base flow (tilde) and a perturbation from that base flow (prime), and only the 1\(^{st}\) order correlation terms are considered; the base values are derived from a theoretical background state – the standard solar model S (Christensen-Dalsgaard et al. 1996). N\(^2\) is the Brunt-Väisälä frequency.
oscillations. The requirement for properly simulating effects of differential rotation and meridional circulation on the frequency of solar background velocities exert on the conservation of momentum (Eq. 2) in the acoustic regime. This is an important spectrum of the source is generated in frequency space with a Gaussian function centered at $\mu$ ($r > R$) effects throughout most of the model interior, as we move towards the surface ($c_s$) throughout the large variations in sound speed ($c_s$). To mimic a stochastic excitation of our modes (Woodard 1984), we multiply our temporal Gaussian function by a set of random numbers at each frequency interval ($f_s = 1/\Delta t$); subsequently applying a Fourier transform to produce a randomized oscillating function for our source, of which a unique one is created for every quantum number ($l$) in our spherical harmonic decomposition.

The material derivative ($\nabla : (M^\prime u + \rho \bar{u} u')$) is solved in its conservation form in order to fully account for effects that background velocities exert on the conservation of momentum (Eq. 2) in the acoustic regime. This is an important requirement for properly simulating effects of differential rotation and meridional circulation on the frequency of solar oscillations.

### 2.2. Radial Grid

The radial grid is spaced evenly with respect to acoustic travel time ($\int 1/c_s dr$) in order to compute acoustic oscillations across the large variations in sound speed ($c_s$) throughout the model. While this grid is effective at capturing effects throughout most of the model interior, as we move towards the surface ($r > 0.99 R_\odot$) pressure and density scale heights begin to drop off faster than sound speed. In order to resolve the effect of the Brunt-Väisälä frequency ($N^2$), we switch to a logarithmic pressure grid spaced evenly in $\ln(p)$ (Hanasoge et al. 2006). Above the model surface ($r > R_\odot$), we implement a thin isothermal buffer layer with constant grid spacing up to $r = 1.001 R_\odot$.

### 2.3. Boundary Conditions

The boundary layers are solved as simple reflective walls with a zero velocity perturbation condition ($u' = 0$). To avoid non-physical surface reflections from affecting our solution, we implement a buffer layer by placing a damping factor ($\sigma$) into our governing equations:

$$\frac{\partial \rho'}{\partial t} = -\Upsilon' - \sigma \rho', \quad \frac{\partial \Upsilon'}{\partial t} = -\nabla^2 p' + \Upsilon' - \sigma \Upsilon', \quad \frac{\partial \rho'}{\partial t} = \mathcal{P} - \sigma \rho'. \quad (5)$$

To avoid precision errors from our damping term, it is computed implicitly in our time-discretization scheme using the integrating factor method. Damping is initiated at the model surface ($R_\odot$) and steadily increased into the atmospheric layers, mimicking the escape of acoustic oscillations above the cutoff frequency (> 6 mHz).

### 3. NUMERICAL METHOD

The time-discretization scheme used in governing Eqns. (1)-(3) is the 2nd order accurate ($O(h^2)$) Adams-Bashforth method – used to compute the effect of external contributions of background fields to our conservation of momentum and conservation of energy (Eqns. 2 - 3) considerations, denoted by $\Upsilon$ and $\mathcal{P}$ in Eq. (5) respectively. To advance our continuity and conservation of momentum forward in time ($\rho' (\Upsilon'), \Upsilon' (\rho')$), (Eqns. 1 - 2), we use the implicit 1st order accurate ($O(h)$) backward Euler method – helping to maintain the stability of our solution.

Spatial differentiation in the radial direction also employs a 2nd order accurate central finite-difference scheme when computing first and second order derivatives. These schemes, however, can become 1st order accurate when used on non-uniform grids, such as the one considered here (§2.2).

Differentiation tangent to the sphere is done in the pseudo-spectral regime through spherical harmonic decomposition.
using the Libsharp spherical harmonic library (Reinecke & Seljebotn 2013). Tangential resolution is defined by the maximum allowable quantum number \( l_{\text{max}}(r) \) at each radial point. This value controls the mesh size of our Gauss-Legendre grid, containing \( N_\phi = 3l_{\text{max}} \) azimuthal grid points and \( N_\theta = 3l_{\text{max}}/2 \) latitudinal grid points. These values are chosen to avoid aliasing during spherical harmonic decomposition. The azimuthal mesh points \( (N_\phi) \) are spaced at even intervals between \( 0 < \phi < 2\pi \), while the latitudinal points \( (N_\theta) \) are placed at the roots of the corresponding Legendre Polynomial between \( 0 < \theta < \pi \). To avoid oversampling at high latitudes the Libsharp library natively implements polar optimisation, or the “reduced Gauss-Legendre grid”.

To compute the tangential components of the divergence we use vector and tensor spherical harmonic bases (VSH and TSH) to calculate our terms spectrally, as defined in §3.1 and §3.2 respectively.

### 3.1. Vector Spherical Harmonics

Using the “pure-spin” vector spherical harmonic (VSH) components defined by Thorne (1980), we can expand an arbitrary vector field \( \mathbf{E} \) into the following linearly independent basis.

\[
\mathbf{E} = \sum_{m=-l_{\text{max}}}^{l_{\text{max}}} \sum_{l=|m|}^{l_{\text{max}}} \left( E_{lm}^r(r) \mathbf{Y}_{lm} + E_{lm}^{(1)}(r) \mathbf{\Psi}_{lm} + E_{lm}^{(2)}(r) \Phi_{lm} \right),
\]

where \( E_{lm}^r \) is the radial vector component, and \( E_{lm}^{(1)}, E_{lm}^{(2)} \) are components tangential to the surface of the 2-sphere. Transformations between the spherical coordinate basis and our VSH basis is achieved by a set of functions defined in Novak et al. (2010), and computed using recurrence relations of the spherical harmonic \( (Y_{lm}) \). We define the divergence of our vector field \( \mathbf{E} \) in Eq. (6) in our new basis with Eq. (7).

\[
\nabla \cdot \mathbf{E} = \sum_{m=-l_{\text{max}}}^{l_{\text{max}}} \sum_{l=|m|}^{l_{\text{max}}} \left( \frac{dE_{lm}^r}{dr} + \frac{2}{r} E_{lm}^r - \frac{l(l+1)}{r} E_{lm}^{(1)} \right) Y_{lm}
\]

The radial derivative is computed using the finite-difference method described in §3.

### 3.2. Tensor Spherical Harmonics

The tensor spherical harmonic (TSH) basis is built on groups 0 and 2 of the irreducible representation of rotation in \( \text{SO}(3) \) \( (D^0, D^2) \) where \( t = D^0 + D^1 + D^2 \), which correspond to the trace and the symmetric traceless tensor respectively (Mathews 1962). This basis is coupled with spin-0 and spin-2 spherical harmonics to form 6 irreducible representations of a symmetric tensor, which can be rearranged into the “pure-spin tensor harmonics” defined by Zerilli (1970) and Thorne (1980). We can use these components to expand an arbitrary symmetric tensor \( t \) into the following orthogonal basis (Novak et al. 2010).

\[
t = \sum_{m=-l_{\text{max}}}^{l_{\text{max}}} \sum_{l=|m|}^{l_{\text{max}}} \left( T_{lm}^0 \mathbf{T}_{lm}^0 + E_{lm}^1 \mathbf{T}_{lm}^1 + B_{lm}^1 \mathbf{T}_{lm}^1 + T_{lm}^0 \mathbf{T}_{lm}^0 + T_{lm}^2 \mathbf{T}_{lm}^2 + E_{lm}^2 \mathbf{T}_{lm}^2 + B_{lm}^2 \mathbf{T}_{lm}^2 \right),
\]

where \( T_{lm}^0 \) is the fully radial component \( (t^{rr}) \) and \( T_{lm}^0 \) is the transverse \( (t^{\theta\theta}, t^{\phi\phi}) \) portion of the trace. \( E_{lm}^1 \) and \( B_{lm}^1 \) represent the mixed radial/transverse components \( (t^{r\theta}, t^{r\phi}) \), and \( E_{lm}^2 \) and \( B_{lm}^2 \) are symmetric transverse traceless components. We can use this basis to solve for the divergence of a tensor in spherical coordinates, where

\[
\nabla \cdot t = \begin{cases} (\nabla \cdot t)_{lm}^1 Y_{lm} = \left( \sum_{m=-l_{\text{max}}}^{l_{\text{max}}} \sum_{l=|m|}^{l_{\text{max}}} \left[ \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 T_{lm}^1) - \frac{1}{r} (l(l+1) T_{lm}^1 + T_{lm}^0) \right] Y_{lm} \right), \\
(\nabla \cdot t)_{lm}^{(1)} Y_{lm} = \left( \sum_{m=-l_{\text{max}}}^{l_{max}} \sum_{l=|m|}^{l_{max}} \left[ -l(l+1) \left( \frac{1}{r^3} \frac{\partial}{\partial r} (r^3 E_{lm}^1) + \frac{1}{r} \left( T_{lm}^1 \right)^2 - (l-1)(l+2) E_{lm}^2 \right) \right] Y_{lm} \right), \\
(\nabla \cdot t)_{lm}^{(2)} Y_{lm} = \left( \sum_{m=-l_{\text{max}}}^{l_{max}} \sum_{l=|m|}^{l_{max}} \left[ -l(l+1) \left( \frac{1}{r^3} \frac{\partial}{\partial r} (r^3 B_{lm}^1) - \frac{(l-1)(l+2)}{r} B_{lm}^2 \right) \right] Y_{lm} \right).
\end{cases}
\]

Coordinate transformations between the spherical, VSH and TSH bases is done using relations defined in Novak et al. (2010). We can plug the results of this computation into Eq. (7) to compute the divergence dyad \( (\nabla \cdot t) \) in our material derivative (Eq. 2).
4. MODEL VALIDATION

In order to validate the mathematical set-up (§2) and computational techniques (§3) described in this paper, we start by matching the generated power spectrum from our model with a theoretical prediction computed using the standard solar model S (§4.1). We test the computation of our material derivative (Eq. 2) in two simple regimes of background velocity flows: solid body rotation (§4.2) and a single-cell model of meridional rotation (§5).

The parameters used in this validation are as follows: the surface resolution of the model is set by a spherical harmonic degree of $l_{max} = 200$, corresponding to $N_\phi = 600$ longitudinal and $N_\theta = 450$ latitudinal mesh points. These values are chosen in order to fully resolve the acoustic modes that intersect the base of the tachocline (a depth of $\sim 0.70 R_\odot$) – letting us infer flows from any point in the convective interior using local helioseismology techniques (Gizon & Birch 2005). Our simulation is run for a period of 65 hours model time. This is too short to properly resolve flow velocities on the order of those seen in meridional circulation ($\sim 14 \text{ m s}^{-1}$). In order to achieve a desirable signal-to-noise ratio in our validation of meridional circulation measurements, we increase the magnitude of our meridional velocity by a factor of 36, to a maximum of $500 \text{ m s}^{-1}$ – simulating an SNR that would be observed over $\sim 9.5 \text{ yrs}$, approaching the decade minimum that is estimated to be needed in order to resolve deep meridional flows (Braun & Birch 2008).

4.1. Power Spectrum

Acoustic oscillations on the solar surface, can be decomposed into eigenmodes, representing standing waves throughout the solar interior. These modes can be conceptualized as a combination of spherical harmonics ($Y_{lm}$) with a frequency dependent radial order ($\xi_n$),

$$f(r, \theta, \phi, t) = \sum_{n,l} \xi_n(r) Y_{lm}(\theta, \phi) e^{i\omega t}. \quad (10)$$

The power spectrum of these modes can be visualized using an $l - \nu$ diagram, showing continuous radial modes throughout the solar interior as a function of their frequency ($\omega = 2\pi \nu$) and spherical harmonic degree ($l$), Fig. 1. The eigenmodes are excited in the frequency range determined by our source function ($S$, see §2.1). By setting negative values of the Brunt-Väisälä frequency to zero we remove the convective instabilities that act as sources of acoustic perturbations normally seen below 2 mHz. We see an agreement in the structure of the eigenmodes generated by our model to theoretical calculations from the model S (Christensen-Dalsgaard et al. 1996), denoted by the dashed blue lines in Fig. 1.

Figure 1. An $l - \nu$ diagram, showing the power spectrum of p-modes sampled 20 km above the model surface. Blue dashed lines represent theoretical predictions made by the standard solar model S (Christensen-Dalsgaard et al. 1996).
4.2. Rotation

In order to test the computation of our material derivative (Eq. 2), we implement a simple model of solid body rotation in a non-rotating reference frame by defining our background velocity term as

$$\pi_\phi = r\Omega_0,$$

which we set to the solar rotation rate of $\Omega_0/2\pi = 430$ nHz. This simple azimuthal velocity flow field creates easily detectable rotational splitting in the structure of our eigenmodes. Rotation will shift up the frequency of prograde modes and shift down the frequency of retrograde modes as a function of their azimuthal order ($m$), for our simple case we can estimate this shift to be $\delta \nu \approx m\Omega_0/2\pi$. In order to visualize this shift we can use an $m - \nu$ diagram of the power spectrum for spherical harmonic degree $l = 180$ (Fig. 2), where our simulated modes reproduce the expected tilt shown by the blue dashed line.

Figure 2. An $m - \nu$ diagram, showing the power spectrum of acoustic oscillation sampled 30 km above the model surface, for spherical harmonic degree $l = 180$. Blue dashed lines represent theoretical predictions of frequency splitting ($\delta \nu \approx m\Omega_0/2\pi$).

5. MERIDIONAL CIRCULATION

The computation of our material derivative (Eq. 2) can be tested in a regime of global meridional flows by using a simple single-cell model of meridional circulation as our background velocity term ($\mathbf{u}$, Fig. 4). We chose the model described and tested by Hartlep et al. (2013), recreating their measurements as a simple validation of our computational techniques.

In order to infer flows in the model interior using surface measurements, we apply a method of deep focusing (Zhao et al. 2009) – a local helioseismology technique (for an in-depth review, see Gizon & Birch (2005)) which uses travel times of acoustic waves to probe structures in solar and stellar interiors. This method consists of choosing two points on the model surface, separated by some angular distance ($\Delta$); the signals at these points are cross-correlated, measuring acoustic travel times of internal oscillations (p-modes). As these waves travel through the resonant cavity of the convective interior they are advected by internal mass flows. Sampling waves traveling in opposite directions results in travel time differences ($\delta \tau$) which provide a basis for inferring background velocities. The ray path approximation under the assumption of Fermat’s principle offers a simple relation between these values (Giles 1999).

$$\delta \tau = -2 \int_{\Gamma_0} \frac{\mathbf{u} \cdot \mathbf{n}}{c^2} ds.$$  

Travel time differences can be estimated along the unperturbed ray path ($\Gamma_0$).

We employ this technique to infer meridional velocities by measuring travel time differences between southward
and northward traveling waves ($\delta \tau_{NS}$). Using each pixel in our data set as a center point, we remap the surrounding 60° x 60° patch into azimuthal equidistant coordinates (Postel’s proejection) using cubic hermite splines. The remapped resolution is approximately 0.6° per pixel, the same spatial resolution as our Gauss-Legendre grid (§3). An illustration of our method can be seen in Fig. 3.

**Figure 3.** An illustration of pixel selection for our deep-focusing method. A 60° x 60° patch is remapped into azimuthal equidistant coordinates to a resolution of approximately 0.6° per pixel. Six concentric circles are selected at diameters $\Delta = 12°, 18°, 24°, 30°, 36°$ and $42°$. Pixels are chosen in 30° wide northern and southern sectors two pixels in width.

A series of six concentric great circles are drawn at diameters of $\Delta = 12°, 18°, 24°, 30°, 36°$ and $42°$. Sets of pixels (two pixels in width) are selected along the rings in 30° wide northern and southern sectors. The pixels in each sector are averaged together and the signals in opposing sectors are cross-correlated. This process is repeated for every grid point in our model ($N_{\phi}, N_{\theta}$) and the cross-correlated signal is averaged over every point in the longitude ($N_{\phi}$) and $\pm 3°$ ($\pm 5$ px. in $N_{\theta}$) in the latitude. To further smooth our data, we average the diameter of each great circle over $\pm 2.4°$ – travel times of the varying diameter distances are interpolated to an estimated time offset based on ray path theory (Giles 1999). The radial turning points of acoustic waves corresponding to each angular distance are $\sim 0.93, 0.89, 0.85, 0.81, 0.77, 0.72 \, R_{\odot}$ respectively, allowing us to probe the entirety of the convective interior. In Fig. 4 we show the meridional flow profile used in the model with dashed lines in the upper hemisphere corresponding to the ray paths for each great circle.

**Figure 4.** The latitudinal velocity ($\bar{u}_{\theta}$) of a single-cell model of meridional circulation (Hartlep et al. 2013). The dashed lines represent ray paths of acoustic oscillations (p-modes) between diameters of $\Delta = 12°, 18°, 24°, 30°, 36°$ and $42°$ with radial turning points at depths: $\sim 0.93, 0.89, 0.85, 0.81, 0.77, 0.72 \, R_{\odot}$ respectively.
5.1. Results

The travel time differences ($\delta \tau_{NS}$) for each ring diameter are plotted as a function of latitude in Fig. 5. These values are compared to theoretical travel time differences (dashed lines in Fig. 5) computed using the ray path approximation (Eq. 12) employing the standard solar model S (Christensen-Dalsgaard et al. 1996).

Figure 5. The N-S travel-time differences ($\delta t_{NS}$) as a function of latitude for six depths: ∼ 0.93, 0.89, 0.85, 0.81, 0.77, 0.72 R⊙ corresponding to travel distances of $\Delta = 12^\circ, 18^\circ, 24^\circ, 30^\circ, 36^\circ$ and $42^\circ$ respectively. The signal is averaged over ±3° in latitude and ±2.4° in travel distance.

The travel time differences for all ring diameters show solid agreement with theoretical predictions as well as the analysis of Hartlep et al. (2013). These results show a key validation of the numerical procedure used to compute the model as well as the deep focusing techniques used to analyze the data. The error ($\sigma_{NS}$) is calculated using a separate model with no background flows; this reference model uses an identical source function (S, Eq. 2) and analysis sequence, producing the same error profile we see for each ring diameter (Fig. 5). We compute this error as the standard deviation of travel time differences ($\delta \tau$) from zero in our reference model, taking the RMS over latitudinal grid points.

$$\sigma_{NS} = \sqrt{\frac{1}{N} \sum_{i=1}^{N} (\delta \tau_i)^2}.$$  \hspace{1cm} (13)

In order to increase the SNR in our measurements we apply a phase speed filter, defined by a Gaussian function with a width of $\sigma = 0.05v_p$, where $v_p = \omega/L$ is the phase speed (Nigam et al. 2007). After the application of this filter, the travel time differences display high levels precision but do show a latitude-independent systematic offset for different ring diameters. This offset seems to have a maximum of approximately ±1 s (Fig. 6).

The error function ($\sigma_{NS}$, Eq. 13) gives us a good foundation for measuring realization noise at various acoustic travel depths. We show the error as a function of travel distance ($\Delta$) in Fig. 7, along with travel time differences ($\delta \tau$) from our reference model without background flows for 5 separate latitudinal averages ($30^\circN - 50^\circN$, $10^\circN - 30^\circN$, $10^\circS - 10^\circN$, $10^\circS - 30^\circS$, $30^\circS - 50^\circS$) and compare them with the error in the unfiltered signal (Fig. 5). The cause of the apparent systematic error is unclear, however, the latitude-dependent offset profiles are strongly linked to the
Figure 6. The N-S travel-time differences ($\delta t_{NS}$) under the application of a Gaussian phase speed filter ($\sigma = 0.05v_p$) as a function of latitude for six depths: $\sim 0.93, 0.89, 0.85, 0.81, 0.77, 0.72$ R$_{\odot}$ corresponding to travel distances of $\Delta = 12^\circ, 15^\circ, 24^\circ, 30^\circ, 36^\circ$ and $42^\circ$ respectively. The signal is averaged over $\pm 3^\circ$ in latitude and $\pm 2.4^\circ$ in travel distance.

structure of our source ($S$, Eq. 2), with different random number seeds generating different error profiles. This effect deserves its own systematic investigation for varying parameters of source locations and structures.

Figure 7. The error in travel time differences ($\delta \tau$) as a function of travel distance ($\Delta$) for 5 latitudinal averages spanning $30^\circ N - 50^\circ N, 10^\circ N - 30^\circ N, 10^\circ S - 10^\circ N, 10^\circ S - 30^\circ S, 30^\circ S - 50^\circ S$. Error bars show the standard deviation of the measured offset ($\sigma_{NS}$, Eq. 13) across the entire latitude. Left) Error for data analyzed with a Gaussian phase speed filter ($\sigma = 0.05v_p$). Right) Error for unfiltered signal.

In Fig. 7, we see a similar systematic error structure in both the filtered and unfiltered signals. As we move towards greater depths, however, noise in the unfiltered signal grows significantly, concealing any potential offset. These
results may have interesting implications for measuring meridional flow structures at the base of the tachocline. The application of our phase speed filter seems to have preserved signal quality relatively evenly throughout the convection zone, offering encouraging results for probing flows deep in the solar interior.

6. DISCUSSION

We present a global linearized acoustic algorithm with new computational methods that will facilitate the testing and validation of local and global helioseismology techniques in diverse regimes of 3-dimensional flows. While helioseismology has been an indispensable tool in exploring interior dynamics on the Sun, it can have trouble resolving exact profiles of flow, especially at greater depths. Forward modeling offers the opportunity to test the impact of subtle differences generated by a variety of theoretical models of mean mass flows, forming a basis to better interpret observational oscillation data.

In future work, we plan to use this model to test differences in acoustic travel-time signatures between models of one and two-cell meridional circulation. Our investigation will explore the limit of resolving differences within the two-decade long window of current solar observations. These models will be used for a systematic analysis of realization noise in simulations of stochastic excitations of surface oscillations, for varying source profiles and measurement times. We also plan to create a new generation of models that include linear effects of magnetic field structures on acoustic oscillations. The numerical method demonstrated in this paper will be the basis for computing this effect.

The model is validated for two distinct profiles: rotation and meridional circulation. These two regimes are critical for understanding and simulating the distribution of angular momentum and magnetic flux that governs the solar magnetic cycle. To simulate these structures, we present an application of pseudo-spectral techniques which will be necessary components for future models to efficiently compute spherical harmonic resolutions of $l_{\text{max}} > 300$, previously considered to be too computationally expensive. These resolutions will be necessary to model local helioseismology techniques on sound speed perturbations due to small-scale structures on the solar surface, creating a link to linear effects of global perturbations.

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