String theory and the Taniyama-Shimura conjecture

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The worldsheet of the string theory, which consisting of 26 free scalar fields in Minkowski space, is two dimensional conformal field theory. If we denote the two dimension conformal field theory by elliptic curve and denote the partition function of string theory by modular form, then the relation between conformal field theory and the string theory can be represented as the Taniyama-Shimura conjecture. Moreover, it also can be generalized to the $F$-theory.

PACS numbers: 11.25.Hf, 11.25.Mj, 11.25.-w

I. INTRODUCTION

The Langlands program of number theory, was first proposed more or less by Robert Langlands in the late 1960s. It is a kind of unified scheme for many results in algebra geometry and arithmetic geometry, which ranging from quadratic reciprocity, which suggested by Gauss, to modern mathematics results such as Faltings’ proof of Mordell conjecture [1] and Wiles proof of Fermats last theorem [2], which is considered as a special case of the Langlands program. So it does not depend on the Riemannian metric. To see this, one can writes the action as,

\[ I = \left\{ V, Q \right\} + \frac{i}{4\pi} \int_X Tr F_A \wedge F_A, \]

where $V$ is a gauge-invariant function of the fields, and $\Psi$ is given by

\[ \Psi = \frac{\theta}{2\pi} + \frac{t^2 - 1}{t^2 - 1} - \frac{4i\pi}{c^2}. \]

As we know that the Taniyama-Shimura conjecture [8] plays a cure role in Wiles’ proof of Fermat’s conjecture [2] [10], which is part of the Langlands program. So there is a natural question to ask can we geometrize the Taniyama-Shimura conjecture? In the following work, we show that it may shed light on this issue. In the next two sections, conformal field theory and string theory are briefly introduced. Section IV is devote to the Taniyama-Shimura conjecture. The summary is shown in the last section.

II. CONFORMAL FIELD THEORY

Suppose we have a 2D conformal field theory [11]. Then there is a representation of left-moving and right-moving Virasoro algebras $L$ with central charges $c$ and $\bar{c}$,
and it can write as
\[ [L_m, L_n] = (n - m) L_{m+n} + \frac{c}{12} (n^3 - n) \delta_{n+m}. \]  
(6)

In particular, the spectrum will be assumed discrete. The most useful quantities, which we can associate to, is the partition function. We denote \( q = e^{2\pi i \tau} \), with \( \tau = \theta + i\beta \),

and the partition function is defined as,
\[ Z_{CFT} = \text{Tr} q^{L_0 - \frac{c}{24}} e^{2\pi i \tau} P, \]  
(9)

where the Hamiltonian \( H \) is
\[ H = L_0 + \bar{L}_0 - \frac{(c + \bar{c})}{24}, \]  
(10)

and the momentum \( P \) is:
\[ P = L_0 - \bar{L}_0 - \frac{(c + \bar{c})}{24}. \]  
(11)

The most importance of this observation is that we can now study the behavior of this theory under diffeomorphisms of the torus \( T^2 \). As we mention above that it can be interpreted as the path integral on a flat torus with modular parameter \( \tau \), which can write as,
\[ E = \mathbb{C}/(\tau \mathbb{Z}). \]  
(12)

Then one may note that the torus \( T^2 \) is isomorphism to the elliptic curve. In other word, the path integral on a flat torus \( T^2 \), can be considered on the elliptic curve.

\section{III. STRING THEORY}

That scalar field decomposes into a zero mode and an infinite number harmonic oscillator modes. Including the central charge, \( c = 1 \), the contribution from the oscillator modes is,
\[ \text{Tr} q^{L_0 - \frac{c}{24}} = \frac{1}{q^{24}} \prod_{i=1}^{\infty} \frac{1}{1 - q^i}. \]  
(13)

So, including both the zero mode and oscillators, we get the partition function for a single free scalar field
\[ Z_{\text{scalar}} = \frac{1}{\sqrt{(\text{Im}\tau)} (\bar{q} q)^{24}} \prod_{i=1}^{\infty} \frac{1}{1 - q^i} \prod_{i=1}^{\infty} \frac{1}{1 - \bar{q}^i}. \]  
(14)

In lightcone gauge, there are 24 oscillator modes and 26 zero modes. Finally, we integrate over the moduli space, then the partition function of string theory is,
\[ Z_{\text{string}} = \int \frac{1}{(\text{Im}\tau)} (\alpha \text{Im}\tau)^{13} (\bar{q} q) \left( \prod_{i=1}^{\infty} \frac{1}{1 - q^i} \right)^{24} \left( \prod_{i=1}^{\infty} \frac{1}{1 - \bar{q}^i} \right)^{24} \ d^2\tau. \]  
(15)

The function appearing in the string partition function is the Dedekind \( \eta \) function \([13]\), which it is,
\[ \eta(q) = q^{\frac{1}{24}} \sum_{n=1}^{\infty} (1 - q^{24})^{n}. \]  
(16)

It was studied long time ago by mathematicians who interested in the properties of functions under modular transformations. one may found that the eta-function satisfies the identities,
\[ \eta(\tau + 1) = \eta(\tau). \]  
(17)

\[ \eta \left( \frac{-1}{\tau} \right) = \eta(\tau). \]  
(18)

These two statements ensure that partition function of the scalar theory is a modular invariant function. In terms of \( \eta \), then the string partition function can be rewrite as,
\[ Z_{\text{string}} = \int \frac{1}{(\text{Im}\tau)^2} \left( \frac{1}{\sqrt{\text{Im}\tau}} \eta(q) \eta(\bar{q}) \right)^{24} \ d^2\tau. \]  
(19)

Since both the measure and the integrand are individually modular invariant, then one can prove the string partition is also a modular function.

\section{IV. TANIYAMA-SHIMURA CONJECTURE}

The Shimura-Taniyama conjecture has provided a important role of much works in arithmetic geometry over the last few decades. The Taniyama-Shimura conjecture, since known as the modularity theorem, is an important conjecture (and now theorem) which connects topology and number theory, arising from several problems proposed by Taniyama and Shimura. It is observed by Frey [14] that the rational solutions of Fermat curves can be used to construct certain special types of semi-stable elliptic curves. Moreover, Jean-Pierre Serre suggested that this type Frey elliptic curve could not be modular. So, prove the Shimura-Taniyama conjecture would therefore finally prove Fermats last theorem. Ribet’s proof [15] of Frey’s conjecture, which provided key motivation for Wiles to prove the Shimura-Taniyama conjecture. Then
it indicates that the Fermat's theorem is true. More recently the work of Wiles and Taylor-Wiles has been extended to the full Shimura-Taniyama theorem [10], removing the requirement of semi-stability.

Let \( E \) be an elliptic curve which has integer coefficients \( a, b, c, d \). Let \( N \) be the \( j \)-conductor of \( E \), let \( a_n \) be the number appearing in the \( L \)-function of \( E \). Then the Taniyama-Shimura conjecture says that there exists a modular form of weight two and level \( N \) whose eigenvalue under the Hecke operators and has a Fourier series \( \sum a_n q^n \).

Generally, the conjecture says that every rational elliptic curve is a modular form [17]. More formally, the conjecture suggests, for every elliptic curve, it writes,

\[
y^2 = ax^3 + bx^2 + cx + d,
\]
over the rationals, then there exist nonconstant modular functions \( f(z) \) and \( g(z) \) of the same level \( N \) which can be written as,

\[
[f(z)]^2 = a [g(z)]^3 + b [g(z)]^2 + cg(z) + d.
\tag{21}
\]

It means that every elliptic curve can be uniformized by modular function. In geometry, it is shown that every point on the upper half plane or modular function can attach to a torus. Here one should note that conformal field theory can be considered as elliptic curve and the partition function of string theory \( Z_{\text{string}} \) is modular invariant function, so we can rewrite Eq. (21) as,

\[
Z_{\text{string}}^2 = aZ_{\text{string}'}^3 + bZ_{\text{string}'}^2 + cZ_{\text{string}'} + d,
\tag{22}
\]

where \( Z_{\text{string}'} \) is a modular function which has same level \( N \) with \( Z_{\text{string}} \). Then, the relation between conformal field theory and string theory can be represented as Taniyama-Shimura conjecture. Actually, in physics, conformal field theory is the worldsheet of string theory.

In fact, there is something similar arise in the F-theory [18–20]. It is obtained by compactificating the type IIB string in which the complex coupling varies over the torus [21]. And the action of type IIB string is \( SL(2, Z) \) invariant [22]. If we regard the torus as the elliptic curve, then the F-theory raise naturally in the Taniyama-Shimura conjecture. The action of Type IIB has strong-weak coupling duality [23], the modular parameter is

\[
\tau = i \exp^{-\phi} + C_0.
\tag{23}
\]

In terms of the axio-dilaton, this is,

\[
\tau \to -\frac{1}{\tau}.
\tag{24}
\]

The axionic shift symmetry is,

\[
C_0 = C_0 + 1,
\tag{25}
\]

which acts as,

\[
\tau \to \tau + 1.
\tag{26}
\]

Then, these generate the \( SL(2, Z) \) duality group

\[
\tau = \frac{a \tau + b}{c \tau + d} \quad \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in SL(2, Z).
\tag{27}
\]

So the partition function of type IIB string can be considered as a modular function. By the result of Taniyama-Shimura conjecture, we can write elliptic fibration over type IIB space-time in Weierstrass form,

\[
Z^2_{\text{typeIIB}} = aZ_{\text{typeIIB}}^3 + bZ_{\text{typeIIB}}^2 + cZ_{\text{typeIIB}} + d,
\tag{28}
\]

where \( Z_{\text{typeIIB}} \) is the partition function of type IIB string, and \( Z_{\text{typeIIB}} \) is a modular function which has the same level \( N \) with \( Z_{\text{typeIIB}} \). In physics, it is shown that the elliptic curve or the torus \( T^2 \) can be attached to every point of type IIB space-time, which interprets the modular parameter as the value of axio-dilaton. And it is similar to the Sen limit [21].

V. SUMMARY

In summary, we introduce the partition function of conformal field theory and string theory, which both of them are modular function. And the conformal field theory can be considered as elliptic curve. In fact the conformal field theory is the propagator of the string theory. It is suggested that the relation between conformal theory and string theory can be represented as Taniyama-Shimura conjecture. F-theory is obtained by compactificating the Type IIB string on the two-torus. In this case, we suggest that it also may be described by Taniyama-Shimura conjecture. The Alday-Gaiotto-Tachikawa (AGT) correspondence [24, 25] hold that the partition function of 4D super Yang-Mills is dual to the 2D conformal field theory. Then it is natural to ask the question: Can we generalize the AGT correspondence by Taniyama-Shimura conjecture? If it is true, then it may indicate that the conjecture is an universal principle in physics. Actually, Taniyama-Shimura conjecture is essentially a modular theorem. The modular function is consider as very high symmetry function. As we know, most of the partition function of string theory, supergravity theory and super Yang-Mills theory are also satisfied some symmetry. This may be the reason that modular theorem arises in physics.

Acknowledgment

Part of the work was done when the first author Jing Zhou visited the Yau Mathematical Science Center. Jing Zhou thanks for discussing with Si Li, Hai Lin and Xun Chen. This work is partly supported by the National Science Foundation of China under Contract Nos. 11775118, 11535005.
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