SPECTRAL REPRESENTATION OF ONE-DIMENSIONAL LIOUVILLE BROWNIAN MOTION AND LIOUVILLE BROWNIAN EXCURSION

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ABSTRACT. In this paper we apply the spectral theory of linear diffusions to study the one-dimensional Liouville Brownian Motion and Liouville Brownian excursions from a given point. As an application we estimate the fractal dimensions of level sets of one-dimensional Liouville Brownian motion as well as various probabilistic asymptotic behaviours of Liouville Brownian motion and Liouville Brownian excursions.

1. INTRODUCTION

Liouville Brownian motion (LBM) was introduced by Garban, Rhodes and Vargas [15] and by Berestycki [6] as a way of understanding better the geometry of two-dimensional Liouville quantum gravity (LQG). Roughly speaking, planar Liouville Brownian motion is the time-change of a planar Brownian motion by the additive functional whose Revuz measure with respect to Lebesgue is the so-called Liouville measure

$$\mu_\gamma(dz) = e^{\gamma h(x)} dx, \ x \in D,$$

where $\gamma \geq 0$ is a given parameter, $D$ is a regular planar domain and $h$ is a Gaussian free field (GFF) on $D$ with certain boundary conditions. As GFFs are defined as random distributions or Gaussian processes on a certain space of measures which does not contain Dirac masses, $h(x)$ is not well-defined for individual points $x \in D$. Therefore certain smooth approximations of $h$ are needed to define the measure $\mu_\gamma$ rigorously. This was done by Duplantier and Sheffield in [10] for $\gamma \in [0,2)$ by using circle averages around given points. The resulting measure $\mu_\gamma$ is a random measure on $D$ carried by a random fractal set whose fractal dimension is $2 - \gamma^2/2$.

Such random fractal measures with “log-Gaussian” densities obtained via a limiting procedure have a long history. The study was initiated by Mandelbrot [21] in the 1970s to analyse the energy dissipation phenomenon in fully developed turbulence. The mathematically rigorous foundation of these random measures was built later by Kahane [18] in 1985,
which now is referred as the Gaussian multiplicative chaos (GMC) theory. For a historical review of GMC and its relation to GFF and LQG, see for example the survey paper [23] of Rhodes and Vargas and the lecture notes [8] of Berestycki.

The study of planar Liouville Brownian motion was carried out in [14, 23, 1] with a focus on the regularity of the transition density function of LBM (so-called Liouville heat kernels). In this paper we shall continue the study but mainly focus on the case of one-dimensional Liouville Brownian motion, defined as a generalized linear diffusion process with natural scale function and speed measure ν, where ν is a boundary Liouville measure on \( \mathbb{R} \) obtained from a GFF on the upper half-plane with Neumann boundary conditions.

The advantage of studying the one-dimensional case is that there exists in the literature a fully developed theory on the probabilistic interpretation of linear diffusions in terms of their scale functions and speed measures, namely the spectral theory of linear diffusions (see [11] for example). With the help of the spectral theory of linear diffusions, we are able to estimate various probabilistic asymptotic behaviours of one-dimensional LBM as well as that of Liouville Brownian excursions (LBE) from a given point. For example in Theorem 4.1 we calculate the Hausdorff and packing dimension of the level sets of the one-dimensional LBM, and in Theorem 4.7 and 4.8 we study the asymptotic behaviours of the lifetime of the excursion under the Liouville Brownian excursion measure.

The rest of the paper is organized as follows: in Section 2 we give a brief review of one-dimensional Brownian motion and Brownian excursions, then we define the one-dimensional LBM and LBE from a given point via time-change of additive functionals obtained from boundary Liouville measures; in Section 3.2 we give a brief review of Krein’s spectral theory of strings and list the spectral representation of LBM and LBE using the spectral theory of excursions of linear diffusions developed in [28, 25]; in Section 4 we study various probabilistic asymptotic behaviours of LBM and LBE.

2. ONE DIMENSIONAL LIOUVILLE BROWNIAN MOTION AND LIOUVILLE BROWNIAN EXCURSION

2.1. Brownian motion and Brownian excursion. Let \( W = C(\mathbb{R}_+, \mathbb{R}) \) denote the Wiener space consisting of continuous functions from \( \mathbb{R}_+ \) to \( \mathbb{R} \). We regard \( W \) as a complete separable metric space. Let \( W \) denote its Borel \( \sigma \)-field. Let \( P_{\text{BM}}^0 \) be the Wiener measure on \( (W, \mathcal{W}) \), under which
the canonical process $w = \{w(t)\}_{t \geq 0}$ is a one-dimensional Brownian motion starting from 0. For $x \in \mathbb{R}$ let $\mathbf{P}^x_{BM}(\cdot)$ denote the measure $\mathbf{P}^0_{BM}(\cdot + x)$, that is the law of the one-dimensional Brownian motion starting from $x$.

Let $\{L(t,x)\}_{t \geq 0, x \in \mathbb{R}}$ denote the joint-continuous version of the local time of the Brownian motion under $\mathbf{P}^0_{BM}$. For any bounded continuous function $f$ on $\mathbb{R}$ one has

$$\int_0^t f(w(s)) \, ds = 2 \int_{\mathbb{R}} f(y) L(t,x) \, dx$$

for $\mathbf{P}^0_{BM}$-almost every $w \in W$. For $\ell \geq 0$ let

$$\tau(\ell) = \inf\{t \geq 0 : L(t,0) > \ell\}$$

be the right-continuous inverse of the local time at 0. For $\ell \geq 0$ such that $\tau(\ell -) < \tau(\ell)$, we may define the path of the excursion at $\ell$ as

$$e^{(\ell)}(t) = \begin{cases} |w(\tau(\ell -) + t)| & \text{if } 0 \leq t \leq \tau(\ell) - \tau(\ell -), \\ 0 & \text{if } t > \tau(\ell) - \tau(\ell -). \end{cases}$$

The excursion $e^{(\ell)}$ takes values in the subspace $E \subset W$ consisting of continuous paths $e : [0,\infty) \to [0,\infty)$ such that if $e(t_0) = 0$ for some $t_0 > 0$ then $e(t) = 0$ for all $t > t_0$. Let $\mathcal{E}$ denote its Borel $\sigma$-field. By Ito's excursion theory, there exists a $\sigma$-finite measure $\mathbf{n}_{BE}$ on $(E,\mathcal{E})$ such that under $\mathbf{P}^0_{BM}$, the point measure

$$\sum_{\ell \geq 0 : \tau(\ell -) < \tau(\ell)} \delta_{e^{(\ell)}(\cdot)}(dsde)$$

is a Poisson measure on $\mathbb{R}_+ \times E$ with intensity

$$ds \otimes \mathbf{n}_{BE}(de).$$

The measure $\mathbf{n}_{BE}$ is called Ito’s excursion measure of the Brownian motion. Here we present the following four descriptions of $\mathbf{n}_{BE}$ listed in [13]. For more details on Brownian excursions, see [22, Chapter XII] for example.

For $x > 0$ let $\mathbf{Q}^x_{BM}$ denote the law of the one-dimensional Brownian motion starting form $x$ and absorbed at 0. For $x \geq 0$ let $\mathbf{P}^x_{3B}$ denote the law of the 3-dimensional Bessel process starting from $x$. Let $\mathbf{W}^x_{3B}$ denote the law of the path obtained by piecing together two independent $\mathbf{P}^0_{3B}$-process up to their first hitting time to $x$ (the second one runs backwards in time). These measures may be all considered to be defined on $(E,\mathcal{E})$. For $e \in E$ let $M(e) = \max_{t \geq 0} e(t)$ denote the maximum of $e$ and let $\zeta(e) = \inf\{t > 0 : e(t) = 0\}$ denote the lifetime of $e$, with the convention that $\inf\emptyset = \infty$. 

\[ e^{(\ell)}(t) = \begin{cases} |w(\tau(\ell -) + t)| & \text{if } 0 \leq t \leq \tau(\ell) - \tau(\ell -), \\ 0 & \text{if } t > \tau(\ell) - \tau(\ell -). \end{cases} \]

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\[ e^{(\ell)}(t) = \begin{cases} |w(\tau(\ell -) + t)| & \text{if } 0 \leq t \leq \tau(\ell) - \tau(\ell -), \\ 0 & \text{if } t > \tau(\ell) - \tau(\ell -). \end{cases} \]
(i) We have \( n_{BE}(M = 0) = 0 \) and for every bounded continuous functional \( F \) on \( E \) supported by \( \{ M > x \} \) for some \( x > 0 \),
\[
n_{BE}(F) = \lim_{\epsilon \to 0^+} \frac{1}{\epsilon} Q_{BM}^\epsilon(F).
\]

(ii) Under \( n_{BE} \) the excursion process \( e = \{ e(t) \}_{t \geq 0} \) is a strong Markov process with transition kernel \( Q_{BM}^x(e(t) \in dy) \) and entrance law \( \frac{1}{x} \mathbb{P}_{BM}^0(e(t) \in dx) \). In particular for each positive stopping time \( \tau \) and every measurable set \( \Gamma \),
\[
n_{BE}(e(\tau + \cdot) \in \Gamma) = \int_{(0,\infty)} \frac{1}{x} \mathbb{P}_{BM}^0(e(\tau) \in dx)Q_{BM}^x(\Gamma).
\]

(iii) For every measurable set \( \Gamma \),
\[
n_{BE}(\Gamma) = \int_0^\infty W_{3B}^x(\Gamma) \frac{dx}{x^2}.
\]

This means that \( n_{BE}(M \in dx) = \frac{dx}{x^2} \) and the law of \( n_{BE} \) conditioned on \( M = x \) is \( W_{3B}^x \).

(iv) For every measurable set \( \Gamma \),
\[
n_{BE}(\Gamma) = \int_0^\infty \mathbb{P}_{3B}^0(e_t \in \Gamma | e(t) = 0)p_{3B}(t,0,0)dt.
\]

Here \( e_t(\cdot) = e(t \wedge \cdot) \) is the path of \( e \) stopped at \( t \), and \( p_{3B}(t,0,0) = (2\pi t)^{-\frac{3}{2}} \) is obtained from the transition probability density \( p_{3B}(t,x,y) \) of \( \mathbb{P}_{3B}^\epsilon \) with respect to its speed measure \( y^2dy \) evaluated at \( x = y = 0 \). Since \( \mathbb{P}_{3B}^0(\zeta(e) = t | e(t) = 0) = 1 \), this description means that \( n_{BE}(\zeta \in dt) = p_{3B}(t,0,0)dt \) and the law of \( n_{BE} \) conditioned on \( \zeta = t \) is \( \mathbb{P}_{3B}^0(e_t \in \cdot | e(t) = 0) \).

2.2. Gaussian multiplicative chaos and Liouville quantum gravity. Let \( D \subset \mathbb{R}^k \) be a domain. Let \( K(x,y) \) be a nonnegative definite kernel of the form
\[
-\log|x-y| + g(x,y),
\]
where \( g \) is continuous over \( \overline{D} \times \overline{D} \). Let
\[
\mathcal{M}_+ = \left\{ \sigma\text{-finite measure } \rho \text{ on } D \text{ with } \int_D \int_D K(x,y)\rho(dx)\rho(dy) < \infty \right\}
\]
and let \( \mathcal{M} \) be the set of the signed measures of the form \( \rho = \rho_+ - \rho_- \), where \( \rho_+,\rho_- \in \mathcal{M}_+ \). Let \( h = \{ h(\rho) \}_{\rho \in \mathcal{M}} \) be a centered Gaussian process indexed by \( \mathcal{M} \) with covariance function
\[
\text{Cov}(h(\rho),h(\rho')) = \int_D \int_D K(x,y)\rho(dx)\rho'(dy).
\]
The process \( h \) is called a Gaussian field on \( D \) with covariance kernel \( K \). Let \( \theta \) be a smooth mollifier and for \( \epsilon > 0 \) let \( \theta_{\epsilon}(x) = e^{-k\theta(x/\epsilon)} \). Let \( \theta_{\epsilon}(x) = \frac{x}{\epsilon} \).
\( h \ast \theta_\epsilon(x) \) be the smooth approximation of \( h \). Then for \( \epsilon > 0 \) we may define a random measure

\[
\mu_{\gamma, \epsilon}(dx) = e^{\gamma h_\epsilon(x) - \frac{\epsilon^2}{2} E(h_\epsilon(x))} dx, \quad x \in D,
\]

where \( \gamma \geq 0 \) is a given parameter. When \( \gamma < \sqrt{2k} \), the sequence of measures \( \mu_{\gamma, \epsilon} \) converges weakly in probability to a limiting measure \( \mu_{\gamma} \) called a Gaussian multiplicative chaos measure on \( D \) (See [7] for example for an elementary proof).

Here we shall focus on the case when \( D = \mathbb{H} \) is the upper half-plane and consider the boundary Liouville measures on \( \mathbb{R} \) defined as follows:

let \( g(x, y) = -\log |x - \bar{y}| \), then \( h = h_f \) is the Gaussian free field on \( \mathbb{H} \) with Neumann boundary conditions. For \( x \in \mathbb{R} \) and \( \epsilon > 0 \) let \( \rho_{x, \epsilon} \) denote the Lebesgue measure on the semi-circle \( \{ y \in \mathbb{H} : |y - x| = \epsilon \} \) in \( \mathbb{H} \) normalized to have mass 1. Let \( \gamma \in [0, \sqrt{2}) \) be fixed. For \( n \geq 1 \) define

\[
\nu_n(dx) = 2^{-n^2} e^{\frac{n^2}{2} (\rho_{x, \epsilon^2})} dx, \quad x \in \mathbb{R}.
\]

Then almost surely \( \nu_n \) converge weakly to a non-trivial measure \( \nu \) as \( n \to \infty \). The measure \( \nu \) is called the boundary Liouville measure on \( \mathbb{H} \) with parameter \( \gamma \).

2.3. One-dimensional Liouville Brownian motion. We assume that the GFF \( h \) and the Brownian motion are independent of each other. Let \( \nu \) be an instance of the boundary Liouville measure on \( \mathbb{R} \) with parameter \( \gamma \) as constructed in Section 2.2. Define

\[
A_\nu(t) = \int_{\mathbb{R}} L(t, x) \nu(dx), \quad t \geq 0.
\]

Then \( A_\nu = \{ A_\nu(t) \}_{t \geq 0} \) forms an additive functional of the Brownian motion. Let

\[
\tau_\nu(t) = \inf\{ s \geq 0 : A_\nu(s) > t, t \geq 0 \}
\]

be its right-continuous inverse. For \( w \in W \) let

\[
w_\nu(t) = w(\tau_\nu(t)), \quad t \geq 0
\]

denote the time-change of \( w \) by \( \tau_\nu \). Define a probability measure on \( W \) by

\[
P^x_\nu(\cdot) = P^x_{BM}(w_\nu \in \cdot).
\]

Then \( P^x_\nu \) is the law of the one-dimensional Liouville Brownian motion with respect to the boundary Liouville measure \( \nu \). In other words, one-dimensional Liouville Brownian motion is a generalized linear diffusion.
process on $\mathbb{R}$ with natural scale function and speed measure $\nu$. Its joint-continuous transition density $p_\nu(t; x, y)$ is given by

$$P^x_\nu(w(t) \in B) = \int_B p_\nu(t; x, y) \nu(\text{d}y)$$

for $t > 0$, $x \in \mathbb{R}$ and $B \in \mathcal{B}(\mathbb{R})$. The process $\{L_\nu(t, x) = L_\nu(t, x)\}_{t \geq 0, x \in \mathbb{R}}$ is the joint-continuous local time of the Liouville Brownian motion under $P_\nu$, that is, for any bounded continuous function $f$ on $\mathbb{R}$ one has

$$\int_0^t f(w(s)) \text{d}s = 2 \int_{\mathbb{R}} f(y) L_\nu(t, x) \nu(\text{d}x)$$

for $P_\nu$-almost every $w \in W$.

### 2.4. Liouville Brownian excursion

Fix $a \in \mathbb{R}$. For $x \geq 0$ denote by

$$m_{\nu, a, +}(x) = \nu([a, a + x]) \text{ and } m_{\nu, a, -}(x) = \nu([a - x, a]).$$

Define

$$A_{\nu, a, \pm}(t) = \int_{(0, \infty)} L(t, a \pm x) \text{d}m_{\nu, a, \pm}(x), \ t \geq 0.$$  

Then $A_{\nu, a, \pm}$ forms an additive functional of the Brownian motion. Let

$$\tau_{\nu, a, \pm}(t) = \inf\{s \geq 0 : A_{\nu, a, \pm}(s) > t\}, \ t \geq 0$$

be the right-continuous inverse of $A_{\nu, a, \pm}$. For $w \in W$ let

$$w_{\nu, a, \pm}(t) = w(\tau_{\nu, a, \pm}(t)), \ t \geq 0$$

be the time change of $w$ by $\tau_{\nu, a, \pm}$. For $x > 0$ define a probability measure on $W$ by

$$P^x_{\nu, a, \pm}(\cdot) = P^{a \pm x}_{BM}(\pm(w_{\nu, a, \pm} - a) \in \cdot).$$

We shall also use the same notation $L(t, x)$ to denote the joint-continuous version of the local time of the Brownian motion/excursion under $Q^x_{BM}$, $P^x_{3B}$, $W^x_{3B}$ and $n_{BE}$ on $E$. For $e \in E$ let

$$A_{\nu, a, \pm}(t) = \int_{(0, \infty)} L(t, x) \text{d}m_{\nu, a, \pm}(x), \ t \geq 0,$$

and

$$\tau_{\nu, a, \pm}(t) = \begin{cases} \inf\{s \geq 0 : A_{\nu, a, \pm}(s) > t\} & \text{if } 0 \leq t < A_{\nu, a, \pm}(\zeta); \\ \zeta & \text{if } t \geq A_{\nu, a, \pm}(\zeta), \end{cases}$$

as well as

$$e_{\nu, a, \pm}(t) = e(\tau_{\nu, a, \pm}(t)), \ t \geq 0.$$
Define the measures on $E$ by

$$Q_{v,a,\pm}^x (\cdot) = Q_{BM}^x (e_{v,a,\pm} \in \cdot),$$

$$Q_{h-v,a,\pm}^x (\cdot) = P_{3B}^x (e_{v,a,\pm} \in \cdot),$$

$$W_{v,a,\pm}^x (\cdot) = W_{BM}^x (e_{v,a,\pm} \in \cdot),$$

$$n_{v,a,\pm} (\cdot) = n_{BE} (e_{v,a,\pm} \in \cdot).$$

We have

(i) $P_{v,a,\pm}^x$ is the law of the generalized linear diffusion with natural scale function and speed measure $dm_{v,a,\pm}(x)$ on $\mathbb{R}_+$ starting from $x$ and with 0 as an instantaneously reflecting boundary.

(ii) $Q_{v,a,\pm}^x$ is the law of the generalized linear diffusion with natural scale function and speed measure $dm_{v,a,\pm}(x)$ on $\mathbb{R}_+$ starting from $x$ and absorbed at 0.

(iii) $Q_{h-v,a,\pm}^x$ is the law of the generalized linear diffusion with natural scale function and speed measure $dm_{v,a,\pm}(x)$ on $\mathbb{R}_+$ starting from $x$ and conditioned never hit 0. Indeed $Q_{h-v,a,\pm}^x$ is Doob's $h$-transform of $Q_{v,a,\pm}^x$ with $h(x) = x$. It is therefore the law of the generalized linear diffusion with speed measure $x^2dm_{v,a,\pm}(x)$ and scale function $-1/x$.

(iv) $W_{v,a,\pm}^x$ is the law of the following process: consider two independent $Q_{h-v,a,\pm}^0$-processes until they first hit $x$ and splice the two paths together (the second one runs backward in time).

Finally, by applying [13, Theorem 2.5], we have the following descriptions of the Itô's excursion measure $n_{v,a,\pm}$.

**Theorem 2.1.**

(i) We have $n_{v,a,\pm}(M = 0) = 0$ and for every bounded continuous functional $F$ on $E$ supported by $\{M > x\}$ for some $x > 0$,

$$n_{v,a,\pm} (F) = \lim_{\epsilon \to 0^+} \frac{1}{\epsilon} Q_{v,a,\pm}^x (F).$$

(ii) Under $n_{v,a,\pm}$ the excursion process $(e(t))_{t \geq 0}$ is a strong Markov process with the transition kernel $Q_{v,a,\pm}^x (e(t) \in dy)$ and the entrance law $\frac{1}{x} Q_{h-v,a,\pm}^0 (e(t) \in dx)$. In particular for each positive stopping time $\tau$ and every measurable set $\Gamma$,

$$n_{v,a,\pm}(e(\tau + \cdot) \in \Gamma) = \int_{(0,\infty)} \frac{1}{x} Q_{h-v,a,\pm}^0 (e(\tau) \in dx) Q_{v,a,\pm}^x (\Gamma).$$

(iii) For every measurable set $\Gamma$,

$$n_{v,a,\pm} (\Gamma) = \int_0^{\infty} W_{v,a,\pm}^x (\Gamma) \frac{dx}{x^2}.$$
This means that \( n_{v,a,\pm}(M \in dx) = \frac{dx}{x} \) and the law of \( n_{v,a,\pm} \) conditioned on \( M = x \) is \( W^x_{v,a,\pm} \).

3. Spectral representation of Liouville Brownian motion and Liouville Brownian excursion

3.1. Krein's spectral theory of strings. This section is based on \([19]\). Let \( \mathcal{M} \) be the set of non-decreasing right-continuous functions \( m : [0, \infty) \to [0, \infty] \) with \( m(0^-) = 0 \) and \( m(\infty) = \infty \). Each \( m \in \mathcal{M} \) represents the mass distribution of a string. For \( m \in \mathcal{M} \) let \( l = \sup\{x \geq 0 : m(x) < \infty\} \) denote the length of \( m \). For \( \lambda \in \mathbb{C} \) let \( \phi(x, \lambda) \) and \( \psi(x, \lambda) \) be the unique solution of the following integral equations on \([0, l]\) respectively:

\[
\phi(x, \lambda) = 1 + \lambda \int_{(0,x]} (x - y) \phi(y, \lambda) \, dm(y);
\]

\[
\psi(x, \lambda) = x + \lambda \int_{(0,x]} (x - y) \psi(y, \lambda) \, dm(y).
\]

The functions \( \phi \) and \( \psi \) have the following explicit expressions: let

\[
\phi_0(x) = 1; \quad \phi_{n+1}(x) = \int_{(0,x]} (x - y) \phi_n(y) \, dm(y) \quad \text{for} \ n \geq 0,
\]

\[
\psi_0(x) = x; \quad \psi_{n+1}(x) = \int_{(0,x]} (x - y) \psi_n(y) \, dm(y) \quad \text{for} \ n \geq 0,
\]

then

\[
\phi(x, \lambda) = \sum_{n=0}^{\infty} \phi_n(x) \lambda^n; \quad \psi(x, \lambda) = \sum_{n=0}^{\infty} \psi_n(x) \lambda^n.
\]

For each fixed \( x \in [0, l] \), \( \phi(x, \cdot) \) and \( \psi(x, \cdot) \) are real entire functions, i.e., they are entire functions of \( \lambda \) and they take real values if \( \lambda \in \mathbb{R} \). Set

\[
h(\lambda) = \int_0^l \frac{dx}{\varphi(x, \lambda)^2} = \lim_{x \uparrow l} \frac{\psi(x, \lambda)}{\varphi(x, \lambda)}.
\]

The function \( h \) is called Krein's correspondence of the string \( m \).

Let \( \mathcal{H} \) be the set of functions \( h : (0, \infty) \to \mathbb{C} \) such that \( h(\lambda) \) can be extended to a homomorphic function on \( \mathbb{C} \setminus (-\infty, 0] \) such that \( \text{Im} h(\lambda) \leq 0 \) for \( \lambda \in \mathbb{C} \) with \( \text{Im} \lambda > 0 \) and \( h(\lambda) > 0 \) for \( \lambda > 0 \). Introduce the topology on \( \mathcal{M} \) such that \( m_n \to m \) if and only if \( m_n(x) \to m(x) \) on every continuous point of \( m \), and the topology on \( \mathcal{H} \) such that \( h_n \to h \) if and only if \( h_n(\lambda) \to h(\lambda) \) for every \( \lambda > 0 \).

**Theorem 3.1** (Krein's correspondence). \( \mathcal{M} \) and \( \mathcal{H} \) are compact metric spaces and Krein's correspondence \( m \in \mathcal{M} \mapsto h \in \mathcal{H} \) defines a homeomorphism. Moreover, \( h \in \mathcal{H} \) has a unique representation

\[
h(\lambda) = c + \int_{(0,\infty)} \frac{\sigma(\xi)}{\lambda + \xi},
\]
where \( c = \inf\{x > 0 : m(x) > 0\} \) and \( \sigma \) is a non-negative Borel measure on \([0, \infty)\) with \( \int_{(0,\infty)} \sigma(d\xi) < \infty \).

The unique Borel measure \( \sigma \) is called the spectral measure of \( m \). From the functional analysis point of view, \( \sigma \) is the unique measure on \([0, \infty)\) such that for \( f \in L^2([0, l), dm) \),

\[
\|f\|_{L^2([0, l), dm)} = \|\hat{f}\|_{L^2([0, \infty), \sigma)},
\]

where

\[
\hat{f}(\lambda) = \int_{0}^{l} f(x)\varphi(x, \lambda) \, dm(x)
\]

is the generalized Fourier transform.

Note that the right-continuous inverse \( m^* : t \geq 0 \mapsto \inf\{x > 0 : m(x) > t\} \) also belongs to \( \mathcal{M} \) with length \( l^* = m(\infty-) \). It is called the dual string of \( m \). Its Krein's correspondence is given by \( h^*(\lambda) = \frac{1}{\lambda \rho(\lambda)} \), which also has a unique representation

\[
h^*(\lambda) = c^* + \int_{(0,\infty)} \frac{\sigma^*(d\xi)}{\lambda + \xi},
\]

where \( c^* = m(0) \) and \( \sigma^* \) is a non-negative Borel measure supported on \([0, \infty)\) with \( \int_{(0,\infty)} \frac{\sigma^*(d\xi)}{\lambda + \xi} < \infty \). The measure \( \sigma^* \) is called the spectral measure of the dual string of \( m \).

3.2. Spectral representation of Liouville Brownian motion. Throughout this Section and Section 3.3 let \( \nu \) be a Borel measure on \( \mathbb{R} \) satisfying

(A1) \( \nu \) has no atoms; \( 0 < \nu([a, b]) < \infty \) for all \( -\infty < a < b < \infty \); \( \nu([0, x]) \to \infty \) and \( \nu([-x, 0]) \to \infty \) as \( x \to \infty \).

In particular an instance of the boundary Liouville measure constructed in Section 2.2 satisfies (A1).

Let \( \mathcal{M}_c \subset \mathcal{M} \) be the set of functions \( m : [0, \infty] \mapsto [0, \infty) \) that are continuous, strictly increasing functions with \( m(0) = 0 \) and \( \sup\{x : m(x) < \infty\} = \infty \). For \( x \geq 0 \) define \( m_{v,+}(x) = \nu([0, x + \xi]) \) and \( m_{v,-}(x) = \nu([-\xi, 0]) \). Then by (A1) both \( m_{v,+} \) and \( m_{v,-} \) belong to \( \mathcal{M}_c \). Let \( \varphi_{v,\pm}(x, \lambda) \) and \( \psi_{v,\pm}(x, \lambda) \) be the unique solutions of the integral equations

\[
\varphi_{v,\pm}(x, \lambda) = 1 + \lambda \int_{(0,\infty)} (x - y)\varphi_{v,\pm}(y, \lambda) \, dm_{v,\pm}(y);
\]

\[
\psi_{v,\pm}(x, \lambda) = x + \lambda \int_{(0,\infty)} (x - y)\psi_{v,\pm}(y, \lambda) \, dm_{v,\pm}(y),
\]

and let

\[
h_{v,\pm}(\lambda) = \int_{0}^{\infty} \frac{dx}{\varphi_{v,\pm}(x, \lambda)} = \lim_{x \to \infty} \frac{\psi_{v,\pm}(x, \lambda)}{\varphi_{v,\pm}(x, \lambda)}.
\]
be the Krein’s correspondence of $m_{v,\pm}$. Let $\sigma_{v,\pm}$ be the spectral measure of $m_{v,\pm}$, that is the unique non-negative Borel measure on $[0,\infty)$ with $\int_{(0,\infty)} \frac{\sigma_{v,\pm}(d\xi)}{1+\xi} < \infty$ such that

$$h_{v,\pm}(\lambda) = \int_0^\infty \frac{\sigma_{v,\pm}(d\xi)}{1+\xi}. $$

Let $h_v$ be the Krein’s correspondence of $m_v = m_{v,+} + m_{v,-}$, which satisfies

$$\frac{1}{h_v(\lambda)} = \frac{1}{h_{v,+}(\lambda)} + \frac{1}{h_{v,-}(\lambda)}. $$

Let $\sigma_v$ be the spectral measure of $m_v$, that is the unique non-negative Borel measure on $[0,\infty)$ with $\int_{(0,\infty)} \frac{\sigma_v(d\xi)}{1+\xi} < \infty$ such that

$$h_v(\lambda) = \int_0^\infty \frac{\sigma_v(d\xi)}{1+\xi}. $$

Define

$$\varphi_v(x,\lambda) = \begin{cases} \varphi_{v,+}(x,\lambda) & \text{if } x \geq 0; \\ \varphi_{v,-}(-x,\lambda) & \text{if } x < 0; \end{cases}$$

$$\psi_v(x,\lambda) = \begin{cases} \psi_{v,+}(x,\lambda) & \text{if } x \geq 0; \\ -\psi_{v,-}(-x,\lambda) & \text{if } x < 0. \end{cases}$$

For $\lambda > 0$ the $\lambda$-resolvent operator $G^\lambda_v$ of the LBM is defined as

$$G^\lambda_v f(x) = E^x_v \left( \int_0^\infty e^{-\lambda t} f(w(t)) \, dt \right)$$

for any bounded continuous function $f$ on $\mathbb{R}$. We have the following spectral representation:

$$G^\lambda_v f(x) = \int_\mathbb{R} g^\lambda_v(x,y) f(y) \nu(dy),$$

where the $\lambda$-resolvent kernel $g^\lambda_v(x,y)$ is given by

$$g^\lambda_v(x,y) = h_v(\lambda)(\varphi_v(x,\lambda) + h_{v,+}(\lambda)^{-1}\psi_v(x,\lambda))(\varphi_v(y,\lambda) - h_{v,-}(\lambda)^{-1}\psi_v(y,\lambda)).$$

Many probabilistic quantities of the LBM are related to the $\lambda$-resolvent kernel $g^\lambda_v(x,y)$. For example

(i) It is the Laplace transform of the transition density $p_v(t;x,y)$:

$$g^\lambda_v(x,y) = \int_0^\infty e^{-\lambda t} p_v(t;x,y) \, dt. $$

(ii) The right-continuous inverse of the local time $\ell_{v,0}(t) = \inf\{s \geq 0 : L_v(s,0) > t\}$ at 0 is a Lévy subordinator, whose Lévy exponent is given by

$$E^0_v \left( e^{-\lambda \ell_{v,0}(t)} \right) = e^{-t/g^\lambda_v(0,0)} = e^{-t/h_v(\lambda)}. $$
(iii) For $a \in \mathbb{R}$ let $H_a = \inf\{t > 0 : w(t) = a\}$ denote the first hitting time at $a$. Then for $a, b \in \mathbb{R}$ we have

$$E_v^a(e^{-\lambda H_b}) = \frac{g_v^\lambda(a, b)}{g_v^\lambda(b, b)}.$$ 

3.3. **Spectral representation of Liouville Brownian excursion.** Fix $a \in \mathbb{R}$. Recall the strings $m_{v, a, +}(x) = \nu([a, a + x])$ and $m_{v, a, -}(x) = \nu([a - x, a])$ for $x \geq 0$. Note that both $m_{v, a, +}$ and $m_{v, a, -}$ belong to $\mathcal{M}_c$. Let $\varphi_{v, a, \pm}(x, \lambda)$ and $\psi_{v, a, \pm}(x, \lambda)$ be the unique solutions of the integral equations

$$\varphi_{v, a, \pm}(x, \lambda) = 1 + \lambda \int_{[0, x]} (x - y) \varphi_{v, a, \pm}(y, \lambda) \, dm_{v, a, \pm}(y);$$

$$\psi_{v, a, \pm}(x, \lambda) = x + \lambda \int_{[0, x]} (x - y) \psi_{v, a, \pm}(y, \lambda) \, dm_{v, a, \pm}(y),$$

and let

$$h_{v, a, \pm}(\lambda) = \int_0^\infty \frac{\mathrm{d}x}{\varphi_{v, a, \pm}(x, \lambda)} = \lim_{x \to \infty} \frac{\psi_{v, a, \pm}(x, \lambda)}{\varphi_{v, a, \pm}(x, \lambda)}$$

be the Krein's correspondence of $m_{v, a, \pm}$. Let $\sigma_{v, a, \pm}$ be the spectral measure of $m_{v, a, \pm}$. Let $m_{v, a, \pm}^* : t \geq 0 \mapsto \inf\{s \geq 0 : m_{v, a, \pm}(s) > t\}$ denote the dual string of $m_{v, a, \pm}$ and let $h_{v, a, \pm}^*$ denote its Krein's correspondence. We have

$$h_{v, a, \pm}^*(\lambda) = \frac{1}{\lambda h_{v, a, \pm}(\lambda)}.$$ 

Let $\sigma_{v, a, \pm}^*$ be the spectral measure of $m_{v, a, \pm}^*$. We have the following spectral representations of LBM with different boundary conditions:

1. Let $p_{v, a, \pm}(t; x, y)$ be the joint-continuous transition density of the generalized linear diffusion with natural scale function and speed measure $dm_{v, a, \pm}(x)$ on $\mathbb{R}_+$, and with $0$ as an instantaneously reflecting boundary, that is, for $t > 0$, $x, y \in [0, \infty)$ and $B \in \mathcal{B}(\mathbb{R}_+)$

$$P_{v, a, \pm}^t(w(t) \in B) = \int_B p_{v, a, \pm}(t; x, y) \, dm_{v, a, \pm}(y).$$

Then

$$p_{v, a, \pm}(t; x, y) = \int_{(0, \infty)} e^{-t \lambda} \varphi_{v, a, \pm}(x, -\lambda) \varphi_{v, a, \pm}(y, -\lambda) \sigma_{v, a, \pm}(d\lambda).$$

The associated $\lambda$-resolvent kernel $g_{v, a, \pm}^\lambda(x, y)$ is given by

$$g_{v, a, \pm}^\lambda(x, y) = \int_{(0, \infty)} e^{-t \lambda} p_{v, a, \pm}(t; x, y) \, dt$$

$$= \int_{(0, \infty)} \frac{\varphi_{v, a, \pm}(x, -\xi) \varphi_{v, a, \pm}(y, -\xi)}{\lambda + \xi} \sigma_{v, a, \pm}(d\xi).$$
(2) Let \( q_{v,a,\pm}(t; x, y) \) be the joint-continuous transition density of the generalized linear diffusion with natural scale function and speed measure \( dm_{v,a,\pm}(x) \) on \( \mathbb{R}_+ \), and with 0 as an absorbing boundary, that is, for \( t > 0, x, y \in (0, \infty) \) and \( B \in \mathcal{B}(\mathbb{R}_+) \),

\[
Q_{v,a,\pm}^x(e(t) \in B) = \int_B q_{v,a,\pm}(t; x, y) \, dm_{v,a,\pm}(y).
\]

Then

\[
q_{v,a,\pm}(t; x, y) = \int_{(0, \infty)} e^{-t\lambda} \psi_{v,a,\pm}(x, -\lambda) \psi_{v,a,\pm}(y, -\lambda) \lambda \sigma_{v,a,\pm}^*(d\lambda).
\]

The associated \( \lambda \)-resolvent kernel \( \hat{g}_{v,a,\pm}^\lambda(x, y) \) is given by

\[
\hat{g}_{v,a,\pm}^\lambda(x, y) = \int_{(0, \infty)} e^{-t\lambda} q_{v,a,\pm}(t; x, y) \, dt = \int_{(0, \infty)} \frac{\psi_{v,a,\pm}(x, -\xi) \psi_{v,a,\pm}(y, -\xi)}{\lambda + \xi} \xi \sigma_{v,a,\pm}^*(d\xi).
\]

(3) Let \( q_{h-v,a,\pm}(t; x, y) \) be the joint-continuous transition density of the generalized linear diffusion with natural scale function and speed measure \( dm_{v,a,\pm}(x) \) on \( \mathbb{R}_+ \), and conditioned never hit 0, that is, for \( t > 0, x, y \in (0, \infty) \) and \( B \in \mathcal{B}(\mathbb{R}_+) \),

\[
Q_{h-v,a,\pm}^x(e(t) \in B) = \int_B q_{h-v,a,\pm}(t; x, y) \, dm_{v,a,\pm}(y).
\]

Then

\[
q_{h-v,a,\pm}(t; x, y) = \frac{q_{v,a,\pm}(t; x, y)}{xy} = \int_{(0, \infty)} e^{-t\lambda} \frac{\psi_{v,a,\pm}(x, -\lambda) \psi_{v,a,\pm}(y, -\lambda)}{x} \frac{\lambda \sigma_{v,a,\pm}^*(d\lambda)}{y}.
\]

The associated \( \lambda \)-resolvent kernel \( \tilde{g}_{v,a,\pm}^\lambda(x, y) \) is given by

\[
\tilde{g}_{v,a,\pm}^\lambda(x, y) = \int_{(0, \infty)} e^{-t\lambda} q_{h-v,a,\pm}(t; x, y) \, dt = \int_{(0, \infty)} \frac{\psi_{v,a,\pm}(x, -\xi) \psi_{v,a,\pm}(y, -\xi)}{xy(\lambda + \xi)} \xi \sigma_{v,a,\pm}^*(d\xi).
\]

(4) The partial derivative of \( q_{v,a,\pm}(t; x, y) \) at \( y = 0 \),

\[
\pi_{v,a,\pm}(t; x) = \lim_{y \to 0^+} \frac{q_{v,a,\pm}(t; x, y)}{y} = \int_{(0, \infty)} e^{-t\lambda} \psi_{v,a,\pm}(x, -\lambda) \lambda \sigma_{v,a,\pm}^*(d\lambda),
\]

is the density of the first hitting time \( H_0 = \inf\{t > 0 : w(t) = 0\} \) under \( Q_{v,a,\pm}^x \), that is

\[
Q_{v,a,\pm}^x(H_0 \in dt) = \pi_{v,a,\pm}(t; x) \, dt.
\]
In particular

\[ Q_{v,a,\pm}^x(H_0 > t) = \int_0^\infty e^{-t\lambda} \pi_{v,a,\pm}(t;x) \sigma_{v,a,\pm}^*(d\lambda). \]

It also defines an entrance law: for \( t, s > 0 \) and \( y \in (0, \infty) \),

\[ \int_{(0,\infty)} \pi_{v,a,\pm}(t;x) q_{v,a,\pm}(s;x,y) \, dm_{v,a,\pm}(x) = \pi_{v,a,\pm}(t+s;y). \]

(5) For \( t > 0 \) let \( G_t = \sup\{s \leq t : w(s) = 0\} \) and \( D_t = \inf\{s \geq t : w(s) = 0\} \). Then for \( u < t < v \) and \( x > 0 \),

\[ P_{v,a,\pm}^0(G_t \in du, w(t) \in dx, D_t \in dv) = p_{v,a,\pm}(u,0)\pi_{v,a,\pm}(t-u;x)\pi_{v,a,\pm}(v-t;x) \, du \, dv \, dm_{v,a,\pm}(x). \]

(6) The partial derivative of \( \pi_{v,a,\pm}(t;x) \) at \( x = 0 \),

\[ n_{v,a,\pm}(t) = \lim \frac{\pi_{v,a,\pm}(t;x)}{x} = \int_{(0,\infty)} e^{-t\lambda} \lambda \sigma_{v,a,\pm}^*(d\lambda), \]

is the density of the Lévy measure of the Lévy subordinator \( \{\ell_{v,a,\pm}(t) := \inf\{s \geq 0 : L(\tau_{v,a,\pm}(s),0) > t\}\}_{t \geq 0} \) under \( P_{v,a,\pm}^0 \), that is

\[ E_{v,a,\pm}^0(e^{-\lambda \ell_{v,a,\pm}(t)}) = e^{-t h_{v,a,\pm}(\lambda)}, \]

where the Lévy exponent \( 1/h_{v,a,\pm}(\lambda) \) takes the form

\[ \frac{1}{h_{v,a,\pm}(\lambda)} = \int_0^\infty (1 - e^{-t\lambda}) n_{v,a,\pm}(t) \, dt. \]

Let \( Q_{v,a,\pm}^{x,t,0} \) denote the law of the \( Q_{h-v,a,\pm}^{0,t} \)-process pinned at 0 with lifetime \( t \). Alternatively \( Q_{v,a,\pm}^{0,t,0} \) is the weak limit of the law of the Markovian bridge \( Q_{v,a,\pm}^{x,t,y} \) as \( y \to 0+ \), \( x \to 0+ \) (see [12] for example). By applying the results in [28, 25], we have the following spectral representation of the Ito’s excursion measure \( n_{v,a,\pm} \).

**Theorem 3.2.** The Ito’s excursion measure \( n_{v,a,\pm} \) has the following representation:

\[ n_{v,a,\pm}(de) = \int_0^\infty n_{v,a,\pm}(\zeta \in dt) Q_{v,a,\pm}^{0,t,0}(de), \]

where the law of the lifetime \( \zeta \) under \( n_{v,a,\pm} \) is equal to the Lévy measure of \( \ell_{v,a,\pm} \):

\[ n_{v,a,\pm}(\zeta \in dt) = n_{v,a,\pm}(t) \, dt. \]

Moreover, we have the following finite dimensional distribution: for \( 0 < t_1 < t_2 < \cdots < t_n \) and \( x_i > 0 \), \( i = 1, \ldots, n \),

\[ n_{v,a,\pm}(e(t_1) \in dx_1, e(t_2) \in dx_2, \ldots, e(t_n) \in dx_n) = \pi_{v,a,\pm}(t_1;x_1) d\pi_{v,a,\pm}(x_1) q_{v,a,\pm}(t_2-t_1;x_1,x_2) d\pi_{v,a,\pm}(x_2) \times \cdots \times q_{v,a,\pm}(t_n-t_{n-1};x_{n-1},x_n) d\pi_{v,a,\pm}(x_n). \]
In particular
\[ n_{v,a,z}(e(t) \in dx) = \pi_{v,a,z}(t;x) \, dm_{v,a,z}(x), \]
and it holds that
\[ n_{v,a,z}(\zeta > t) = \int_0^\infty n_{v,a,z}(e(t) \in dx) = \int_0^\infty \pi_{v,a,z}(t;x) \, dm_{v,a,z}(x). \]

**Remark 3.1.** Note that \( n_{v,a,z}(\zeta > t) \) also has the expression
\[ n_{v,a,z}(\zeta > t) = \int_t^\infty n_{v,a,z}(s) \, ds = \int_0^\infty e^{-\lambda t} \sigma_{v,a,z}^*(dt). \]

This yields the identity (see [25] Proposition 3)
\[ \int_0^t p_{v,a,z}(u;0,0) \, du \int_{t-u}^\infty n_{v,a,z}(v) \, dv = 1. \]

Recall that \( H_x = \inf\{t > 0 : w(t) = x\} \) is the first hitting time to \( x \) and denote by \( H_x^\alpha = \sup\{t > 0 : w(t) = x\} \) the last exit time from \( x \). Let \((\cdot)^\gamma\) denote the time reverse operator on \( E^0 = \{e \in E : e(0) = 0\} \), that is, for \( e \in E^0 \),
\[ e^\gamma(t) = e((\zeta - t)^\gamma), \quad t \geq 0. \]

Let \( \mathcal{E}^0 \) denote the Borel \( \sigma \)-field of \( E^0 \) and let \( \mathcal{E}^0_{(0,H_a)}, \mathcal{E}^0_{(H_a,H_x)} \) and \( \mathcal{E}^0_{(H_x,H_\zeta)} \) denote the sub \( \sigma \)-fields with respect to the corresponding time intervals (see [28] for more precise definitions). Let \( \theta_t(W)(\cdot) = w(t + \cdot) \) denote the left-shift operator. We have the following time reverse and first-entrance-last-exit decomposition of the excursion measure \( n_{v,a,z} \) from [28].

**Theorem 3.3.**

(i) For \( \Gamma \in \mathcal{E}^0 \) one has
\[ n_{v,a,z}(\Gamma^\gamma) = n_{v,a,z}(\Gamma). \]

(ii) For \( x > 0 \) and \( \Gamma_1 \in \mathcal{E}^0_{(0,H_x)}, \Gamma_2 \in \mathcal{E}^0_{(H_x,H_x^\alpha)}, \Gamma_3 \in \mathcal{E}^0_{(H_x,H_\zeta)} \) one has
\[ n_{v,a,z}(\Gamma_1 \cap \Gamma_2 \cap \Gamma_3) = \frac{1}{x} p_{h-v,a,z}(\Gamma_1) Q_{v,a,z}(\theta_{H_x}(\Gamma_2)) p_{h-v,a,z}(\Gamma_3). \]

In particular
\[ n_{v,a,z}(H_x \in dt_1 \cap [H_x^\alpha - H_x] \in dt_2 \cap [\zeta - H_x^\alpha] \in dt_3) = \frac{1}{x} p_{h-v,a,z}(H_x \in dt_1) Q_{v,a,z}(H_x^\alpha \in dt_2) p_{h-v,a,z}(H_x \in dt_3). \]

Consequently, \( n_{v,a,z}(H_x \in dt) = \frac{1}{x} p_{h-v,a,z}(H_x \in dt) \) and
\[ n_{v,a,z}(e^{-\lambda H_x}) = \frac{1}{x} p_{h-v,a,z}(e^{-\lambda H_x}) = \frac{1}{\psi_{v,a,z}(x,\lambda)}. \]
4. PROBABILISTIC ASYMPTOTIC BEHAVIOURS OF LIOUVILLE BROWNIAN MOTION AND LIOUVILLE BROWNIAN EXCURSIONS

As an application of the spectral representation in Section 3.2 and 3.3, we shall study the probabilistic asymptotic behaviours of LBM and LBE. Throughout this section let \( \nu \) be a Borel measure on \( \mathbb{R} \) satisfying (A1) as well as

(A2) Ergodicity: There exists a positive constant \( Z \) such that for every \( a \in \mathbb{R} \),

\[
\lim_{\eta \to \infty} \frac{\nu[a, a + \eta]}{\eta} = \lim_{\eta \to \infty} \frac{\nu[a - \eta, a]}{\eta} = Z.
\]

(A3) Multifractality: There exists an open interval \( I_\nu \), a family \( \{\nu_q : q \in I_\nu\} \) of Borel measures on \( \mathbb{R} \) and a family \( \{\alpha(q) : q \in I_\nu\} \) of positive reals such that for \( q \in I_\nu \), for \( \nu_q \)-almost every \( a \in \mathbb{R} \),

\[
\lim_{r \to 0} \frac{1}{\log r} \log \nu(a - r, a + r) = \alpha(q).
\]

Many stationary multifractal random measures have these properties, for example the log-infinitely divisible cascade measures constructed in [4, 2]. In particular, an instance of the boundary Liouville measure constructed in Section 2.2 satisfies (A2) and (A3):

(1) Since the boundary Liouville measure \( \nu \) is a stationary positive measure on \( \mathbb{R} \), that is \( \nu(x + \cdot) \) has the same law as \( \nu(\cdot) \) for any \( x \in \mathbb{R} \) and \( \nu(I) > 0 \) for any open interval \( I \), by Birkhoff ergodic theory there exists a positive random variable \( Z \) with finite mean such that almost surely for every \( a \in \mathbb{R} \),

\[
\lim_{\eta \to \infty} \frac{\nu[a, a + \eta]}{\eta} = \lim_{\eta \to \infty} \frac{\nu[a - \eta, a]}{\eta} = Z.
\]

(2) For \( q \in (-\frac{\sqrt{2}}{\gamma}, \frac{\sqrt{2}}{\gamma}) \) let \( \nu_q \) be the boundary Liouville measure with parameter \( q \gamma \) defined via the same GFF \( h \) as \( \nu \). In particular \( \nu_0 \) is the Lebesgue measure on \( \mathbb{R} \) and \( \nu_1 = \nu \). By the multifractal analysis of \( \nu \) (see [3], see also [23, Theorem 4.1] for a direct proof for positive \( q \)) we have that almost surely for \( \nu_q \)-almost every \( a \in \mathbb{R} \),

\[
\lim_{r \to 0} \frac{1}{\log r} \log \nu(a - r, a + r) = 1 + \frac{1}{2} \left( \frac{1}{2} - q \right) \gamma^2.
\]

As the first application we have the following theorem on the fractal dimensions of the level sets of LBM. For \( a \in \mathbb{R} \) denote by \( m_{\nu, a} = m_{\nu, a,+} + m_{\nu, a,-} \) and let

\[
V_{\nu, a}(r) = \int_0^r m_{\nu, a}(x) \, dx, \quad r \geq 0.
\]
Let $h_{v,a}$ be the Krein's correspondence of $m_{v,a}$, which satisfies
\[
\frac{1}{h_{v,a}(\lambda)} = \frac{1}{h_{v,a,+}(\lambda)} + \frac{1}{h_{v,a,-}(\lambda)}.
\]
From [19] we have for $\eta > 0$
\[
(4.3) \quad \frac{1}{4} h_{v,a}(1/\eta) \leq (V_{v,a})^{-1}(\eta) \leq 64 h_{v,a}(1/\eta).
\]

**Theorem 4.1.** For $v_q$-almost every $a \in \mathbb{R}$, for $P^a_q$-almost every $w \in W$,
\[
\dim_H \{ t \geq 0 : w(t) = a \} = \dim_P \{ t \geq 0 : w(t) = a \} = \frac{1}{1 + \alpha(q)}.
\]

**Proof.** By (4.2), for $v_q$-almost every $a \in \mathbb{R}$ for every $\epsilon > 0$ there exists $r_{a,\epsilon} > 0$ such that
\[
r^{\alpha(q) + \epsilon} \leq v(a - r, a + r) \leq r^{\alpha(q) - \epsilon}, \forall r \leq r_{a,\epsilon}.
\]
This implies that
\[
(4.4) \quad \frac{1}{1 + \alpha(q)} r^{1+\alpha(q)+\epsilon} \leq V_{v,a}(r) \leq \frac{1}{1 + \alpha(q) - \epsilon} r^{1+\alpha(q)-\epsilon}, \forall r \leq r_{a,\epsilon}.
\]
Equivalently for $\eta > 0$ small enough
\[
(1 + \alpha(q) - \epsilon)^{1+\alpha(q)-\epsilon} \eta^{1+\alpha(q)-\epsilon} \leq (V_{v,a})^{-1}(\eta) \leq (1 + \alpha(q) + \epsilon)^{1+\alpha(q)+\epsilon} \eta^{1+\alpha(q)+\epsilon}.
\]
By (4.3) we deduce that there exist constants $0 < c_{a,\epsilon}, C_{a,\epsilon} < \infty$ such that for $\lambda$ large enough,
\[
(4.5) \quad c_{a,\epsilon} \lambda^{-\frac{1}{1+\alpha(q)+\epsilon}} \leq h_{v,a}(\lambda) \leq C_{a,\epsilon} \lambda^{-\frac{1}{1+\alpha(q)-\epsilon}}.
\]

The local time $L_v(t, a) = L(\tau_v(t), a)$ at $a$ is carried by the level set $\{ t \geq 0 : w(t) = a \}$ for $P^a_q$-almost every $w \in W$ and its the right-continuous inverse $\ell_{v,a}(t) := \inf\{ s \geq 0 : L_v(s, a) > t \}$ is a Lévy subordinator, whose Lévy exponent is given by
\[
E_v^a(e^{\lambda \ell_{v,a}(t)}) = e^{-t h_{v,a}(\lambda)}.
\]
By the general theory of fractal dimensions of images of Lévy subordinator (see [9] Chapter 5 for example), we have for $P^a_q$-almost every $w \in W$,
\[
\dim_H \{ t \geq 0 : w(t) = a \} = \liminf_{\lambda \to \infty} \frac{-\log h_{v,a}(\lambda)}{\log \lambda};
\]
\[
\dim_P \{ t \geq 0 : w(t) = a \} = \limsup_{\lambda \to \infty} \frac{-\log h_{v,a}(\lambda)}{\log \lambda}.
\]
By (4.5) with $\epsilon \to 0$ we get for $P^a_q$-almost every $w \in W$,
\[
\dim_H \{ t \geq 0 : w(t) = a \} = \dim_P \{ t \geq 0 : w(t) = a \} = \frac{1}{1 + \alpha(q)}.
\]
□
Remark 4.1. Theorem 4.1 is linked to work \cite{16} of Jackson on the Hausdorff dimension of the times that planar LBM spent on the thick points of the corresponding Gaussian free field. Theorem 4.1 estimates the size of the times that one-dimensional LBM spent at $\nu_q$-almost every $a$, whereas \cite{16} estimates the size of planar LBM spent in the support of $\mu_\gamma$. So, roughly speaking, Theorem 4.1 can be considered as a fiber version of the result in \cite{16} in dimension 1. Since in dimension 2 there does not exist the local time of BM/LBM at a given point, it seems difficult to derive an analogue of Theorem 4.1 in dimension two.

As the second application we shall estimate the asymptotic behaviours of the transition density $p_\nu(t; a, a)$ at a given point $a \in \mathbb{R}$. First note that $p_\nu(t; a, a)$ has the following spectral representation:

$$p_\nu(t; a, a) = \int_0^{\infty} e^{-t\lambda} \sigma_{\nu,a}(d\lambda),$$

where $\sigma_{\nu,a}$ be the spectral measure of $m_{\nu,a}$, that is the unique non-negative Borel measure on $[0, \infty)$ with $\int_{(0,\infty)} \frac{\sigma_{\nu,a}(d\xi)}{1+\xi} < \infty$ such that

$$h_{\nu,a}(\lambda) = \int_0^{\infty} \frac{\sigma_{\nu,a}(d\xi)}{\lambda + \xi}.$$

This yields the following lemma of Tomisaki \cite{26}.

**Lemma 4.1.** Let $\phi$ be a positive and non-increasing function on $(0, \delta)$ for some $\delta > 0$. Then

$$\int_0^{\delta} \phi(t) p_\nu(t; a, a) dt < \infty \Leftrightarrow \int_0^{(V_{\nu,a})^{-1}(\delta)} \phi(V_{\nu,a}(x)) dx < \infty.$$

We have the following result on the short term behaviour of $p_\nu(t; a, a)$.

**Theorem 4.2.** For $\nu_q$-almost every $a \in \mathbb{R}$, for any $\beta > \frac{1}{1+\alpha(q)}$,

$$\int_{0^+} t^{-\beta} p_\nu(t; a, a) dt = \infty,$$

and for any $\beta < \frac{1}{1+\alpha(q)}$,

$$\int_{0^+} t^{-\beta} p_\nu(t; a, a) dt < \infty.$$

In particular

$$\liminf_{t \to 0} \frac{\log p_\nu(t; a, a)}{-\log t} \leq 1 - \frac{1}{1+\alpha(q)} \leq \limsup_{t \to 0} \frac{\log p_\nu(t; a, a)}{-\log t}. \tag{4.6}$$
Proof. As a direct consequence of Lemma 4.1 and (4.4) we have that for any $\beta > \frac{1}{1+\alpha(q)}$,
\[
\int_{0+}^t t^{-\beta} p_\nu(t; a, a) \, dt = \infty,
\]
and for any $\beta < \frac{1}{1+\alpha(q)}$,
\[
\int_{0+}^t t^{-\beta} p_\nu(t; a, a) \, dt < \infty,
\]
which implies (4.6). \qed

Remark 4.2. By using Tauberian theorem it can be shown that if $\nu(a-r, a+r)$ is a regular varying function of $r$ as $r \to 0$ then the inequalities in (4.6) become an equality, see [3] for example for the case when $\nu$ is a Bernoulli measure on $[0, 1]$. However due to the multifractal nature of GMC measures, it is the case that for $\nu_q$ almost every $a$, $\nu(a-r, a+r)$ is not regular varying as $r \to 0$. So it is not clear whether the limit in (4.6) exists.

Remark 4.3. The short term behavior (4.6) is quite different comparing to [24, Corollary 2.1]. The reason is that in [24] the one-dimensional Liouville Brownian motion is defined as a linear diffusion with scale function $m_\nu$ and speed measure $d\nu$, and the corresponding transition density $p_*(t; x, x)$ is defined with respect to $\nu(dx)$ rather than $d\nu$. Therefore by change of variables it is straightforward to verify that for every $x \in \mathbb{R}$,
\[
\lim_{t \to 0} \frac{-\log p_*(t; x, x)}{\log t} = \frac{1}{2}.
\]
We also have the following long term behaviour of $p_\nu(t; a, a)$.

Theorem 4.3. For every $a \in \mathbb{R}$,
\[
\lim_{t \to \infty} \sqrt{2\pi t} p_\nu(t; a, a) = \frac{1}{\sqrt{Z}}.
\]

Proof. By the ergodicity (4.1), for each $x \geq 0$ we have
\[
m^{(\eta)}_{\nu, a, \pm}(x) := \frac{1}{\eta Z} m_{\nu, a, \pm}(\eta x) \to x \text{ as } \eta \to \infty.
\]
By change of variables it is easy to see that for constants $\eta, \xi > 0$ one has the following relation of the Krein’s correspondence:
\[
\frac{\xi}{\eta} m(\frac{x}{\eta}) \leftrightarrow \xi h(\eta \lambda).
\]
Since Krein’s correspondence is a homeomorphism, when $m^{(\eta)}_{\nu, a, \pm}(x) \to x$ as $\eta \to \infty$ for each $x \geq 0$, the corresponding generalized linear diffusion
process converges in law to one-dimensional Brownian motion. This implies that

$$\lim_{\eta \to \infty} \eta Z p_\nu(\eta^2 Z; a, a) = \frac{1}{\sqrt{2\pi t}}$$

In other words,

$$\lim_{t \to \infty} \sqrt{2\pi t} p_\nu(t; a, a) = \frac{1}{\sqrt{Z}}.$$

In the third application we shall study the first hitting/exit time of LBM. For $a \in \mathbb{R}$ recall that $H_a = \inf\{t > 0 : w(t) = a\}$ is the first hitting time at $a$. We have

**Theorem 4.4.** For $a \neq 0$ we have

$$\lim_{t \to \infty} \sqrt{2\pi t} P_\nu^0(H_a \geq t) = |a|Z.$$

**Proof.** Since

$$\lim_{\eta \to \infty} \frac{\nu[0, \eta]}{\eta} = \lim_{\eta \to \infty} \frac{\nu[-\eta, 0]}{\eta} = Z,$$

the result is a direct application of [27, Theorem 4] with $\alpha = \frac{1}{2}$ and $K(x) = Z$. \qed

**Remark 4.4.** Theorem 4.3 and 4.4 suggest that in long term one-dimensional LBM behaves exactly like one-dimensional Brownian motion.

For $a < b$ let

$$H_{a,b} = \inf\{t > 0 : w(t) \not\in (a, b)\} = H_a \wedge H_b$$

denote the first exit time from $(a, b)$. Define

$$C_{\nu,a,b} = \frac{1}{b - a} \int_a^b (b - x)(x - a) \nu(dx).$$

Then we have

**Theorem 4.5.** If $\lambda < 1/C_{\nu,a,b}$, then

$$\mathbb{E}_\nu^0(e^{\lambda H_{a,b}}) < \infty.$$

**Proof.** This can be easily deduced by using the Kac formula: for $n \geq 1$,

$$\mathbb{E}_\nu^0(H_{a,b}^n) = n \int_a^b \frac{(b - x \vee 0)(x \wedge 0 - a)}{b - a} \mathbb{E}_\nu^0(H_{a,b}^{n-1}) \nu(dx) = n! G_{\nu}^n 1(0),$$

where $G_{\nu}$ is the Green operator

$$G_{\nu} f(x) = \int_a^b \frac{(b - y \vee x)(y \wedge x - a)}{b - a} f(y) \nu(dy).$$

See [20] Lemma 1.3] for example. \qed
Theorem 4.5 indicates that $A = -\frac{d}{dy} \frac{d}{dx}$, as a self-adjoint, non-negative definite operator on the Hilbert space $L^2((a, b), \nu)$, has a spectra gap. Indeed let $\lambda_{v,a,b}$ denote the smallest eigenvalue of $A$ on $L^2((a, b), \nu)$, and denote by
\[
\tilde{C}_{v,a,b} = \sup_{x \in (a,0]} (x-a)\nu((x,0]) \vee \sup_{x \in [0,b)} (b-x)\nu([0,x]).
\]
Then we have the following result of Katoni [19, Theorem 3, Appendix I] as an extension of the theorem of Kac and Krein [17].

**Theorem 4.6.**
\[
\tilde{C}_{v,a,b} \leq \lambda_{v,a,b}^{-1} \leq 4\tilde{C}_{v,a,b}.
\]

In the last application we study the asymptotic behaviours of the lifetime of LBE. Recall that $n_{v,a,\pm}(t)$ is the density of the inverse local time $\ell_{v,a,\pm}(t)$, which is also the density of the lifetime $\zeta$ under the excursion measure $m_{v,a,\pm}$, that is
\[
n_{v,a,\pm}(\zeta \in dt) = n_{v,a,\pm}(t) dt.
\]

First we present the asymptotic behaviour of $n_{v,a,\pm}(\zeta > t)$ as $t \to \infty$.

**Theorem 4.7.** For $a \in \mathbb{R}$ we have
\[
\lim_{t \to \infty} 2\sqrt{2\pi t^3} n_{v,a,\pm}(t) = \frac{1}{\sqrt{Z}}.
\]
Consequently
\[
\lim_{t \to \infty} \sqrt{2\pi t} n_{v,a,\pm}(\zeta > t) = \frac{1}{\sqrt{Z}}.
\]

**Proof.** Similar as in the proof of Theorem 4.3 for each $x \geq 0$,
\[
m_{v,a,\pm}^{(\eta)}(x) := \frac{1}{\eta Z} m_{v,a,\pm}(\eta x) \to x \text{ as } \eta \to \infty.
\]
This implies that the generalized diffusion process on $\mathbb{R}_+$ with natural scale function and speed measure $dm_{v,a,\pm}^{(\eta)}(x)$, and with 0 as an instantaneously reflecting boundary converges in law to the one-dimensional reflected Brownian motion as $\eta \to \infty$. Therefore the corresponding local time
\[
\ell_{v,a,\pm}^{(\eta)}(t) := \frac{1}{\eta^2 Z} \ell_{v,a,\pm}(t\eta)
\]
converges in law to the $\frac{1}{2}$-stable Lévy subordinator as $\eta \to \infty$. Since
\[
E_{v,a,\pm}^0 \left( e^{-\lambda \frac{1}{\eta^2 Z} \ell_{v,a,\pm}(t\eta)} \right) = \exp \left( -t\eta \int_0^\infty (1 - e^{-s\lambda \frac{1}{\eta^2 Z}}) n_{v,a,\pm}(s) ds \right)
\]
\[
= \exp \left( -t \int_0^\infty (1 - e^{-u\lambda}) \eta^3 Z n_{v,a,\pm}(u\eta^2 Z) du \right),
\]
we get that
\[ \eta^3 Z \cdot n_{\nu, a, \pm}(u \eta^2 Z) \to \frac{1}{2\sqrt{2\pi}u^3} \text{ as } \eta \to \infty. \]
In other words,
\[ \lim_{t \to \infty} 2\sqrt{2\pi} t^{3} \cdot n_{\nu, a, \pm}(t) = \frac{1}{\sqrt{Z}}. \]
Consequently
\[ \lim_{t \to \infty} \sqrt{2\pi} t n_{\nu, a, \pm}(\zeta > t) = \frac{1}{\sqrt{Z}}. \]

Now we present the asymptotic behaviour of \( n_{\nu, a, \pm}(\zeta > t) \) as \( t \to 0 \).

**Theorem 4.8.** For \( \nu_q \)-almost every \( a \in \mathbb{R} \), for any \( \beta > \frac{1}{1 + \alpha(q)} \),
\[ \int_{0^+} t^{\beta - 1} n_{\nu, a, \pm}(\zeta > t) \, dt < \infty, \]
and for any \( \beta < \frac{1}{1 + \alpha(q)} \),
\[ \int_{0^+} t^{\beta - 1} n_{\nu, a, \pm}(\zeta > t) \, dt = \infty. \]
In particular,
\[ \liminf_{t \to 0^+} \frac{\log n_{\nu, a, \pm}(\zeta > t)}{-\log t} \leq \frac{1}{1 + \alpha(q)} \leq \limsup_{t \to 0^+} \frac{\log n_{\nu, a, \pm}(\zeta > t)}{-\log t}. \]

**Proof.** We have, for \( \lambda > 0 \),
\[ \frac{1}{\lambda} h_{\nu, a, \pm}(\lambda) = \int_{0^+} e^{-\lambda t} n_{\nu, a, \pm}(\zeta > t) \, dt. \]
Therefore for any \( \delta > 0 \) and \( \beta > 0 \)
\[ \int_{\delta}^{\infty} \frac{1}{\lambda^{\beta + 1} h_{\nu, a, \pm}(\lambda)} \, d\lambda = \int_{\delta}^{\infty} \lambda^{-\beta} \int_{0^+} e^{-\lambda t} n_{\nu, a, \pm}(\zeta > t) \, dt \, d\lambda = \int_{\delta}^{\infty} \lambda^{-\beta} e^{-\lambda} \int_{0^+} t^{\beta - 1} n_{\nu, a, \pm}(\zeta > t) \, dt. \]
This implies that
\[ \int_{\delta}^{\infty} \frac{1}{\lambda^{\beta + 1} h_{\nu, a, \pm}(\lambda)} \, d\lambda < \infty \iff \int_{0^+} t^{\beta - 1} n_{\nu, a, \pm}(\zeta > t) \, dt < \infty. \]
By (4.5) this yields that for any \( \beta > \frac{1}{1 + \alpha(q)} \),
\[ \int_{0^+} t^{\beta - 1} n_{\nu, a, \pm}(\zeta > t) \, dt < \infty, \]
and for any $\beta < \frac{1}{1 + a(\zeta)}$,
\[
\int_{0+} t^{\beta - 1} n_{\nu, a, \pm}(\zeta > t) \, dt = \infty,
\]
which implies (4.7). □

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