TOWARD A CONNECTION BETWEEN THE ORIENTED MATROID THEORY AND SUPERSYMMETRY

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Abstract

We considered the possibility that the oriented matroid theory is connected with supersymmetry via the Grassmann-Plucker relations. The main reason for this, is that such relations arise in both in the chirotopes definition of an oriented matroid, and in maximally supersymmetric solutions of eleven- and ten-dimensional supergravity theories. Taking this observation as a motivation, and using the concept of a phirotope, we propose a mechanism to implement supersymmetry in the context of the oriented matroid theory.

Keywords: oriented matroid theory, maximal supersymmetry in supergravity, chirotopes

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1.- Introduction

It has been shown [1]-[6] through the chirotope concept that oriented matroid theory [7] is related to several sectors of $M$-theory, including Chern-Simons theory, supergravity, string theory, and $p$-branes physics. These connections motivated a recent proposal [8] of considering the oriented matroid theory as the underlying mathematical framework for $M$-theory. But due the expected fermionic structure of $M$-theory is almost impossible to avoid wondering about supersymmetry in this scenario. In fact, until now all connections between matroids and $M$-theory have been realized in the bosonic sector. The main reason for this development has to do with a technical reason; as far as we know, mathematically oriented matroid theory has not been linked to supersymmetry.

In this work, we argue that the phirotope concept [9] which is a generalization of the chirotope concept (see Ref. [7]) may provide the route for such a link. As it is known, the chirotope concept determines one possible definition of an oriented matroid. In the case of realizable oriented matroid the chirotope can be defined in terms of the Grassmann-Plucker relations over the real ones. It turns out that these ideas can be transferred from real to complex structure and in this case the chirotope is called phirotope [9] (see also Refs. [10] and [11]). As we shall explain, the phirotope is an alternating function over the complex one that satisfies a generalized Grassmann-Plucker relations.

Another source of motivation for the present work arose when we observed that surprisingly, the Grassmann-Plucker relations have played a very important role in a related subject: the classification of maximally supersymmetric solutions of ten- and eleven-dimensional supergravity theories. In fact, Figueroa-O’Farrill and Papadopoulos [12]-[13] had shown that maximal supersymmetry in eleven-dimensional (ten-dimensional) supergravity leads to a quadratic condition for the four-form $F$ (five-form $F$) which implies the Grassmann-Plucker relations for $F$. From this result they conclude that the AdS solutions, the Hpp-waves, and flat space solutions exhaust the maximally supersymmetric solutions of eleven-dimensional supergravity. Similar conclusions hold in the case of ten-dimensional supergravity.

Here, we claim that looking at the $F$ field as a chirotope the Figueroa-O’Farrill-Papadopoulos’ construction can be understood as a link between the oriented matroid theory and the maximal supersymmetry in ten- and eleven-dimensional supergravities.

Just for introducing some notation and definitions in section 2 we briefly review the notions of chirotope and phirotope. In section 3, we summarize
the maximally supersymmetric solutions of eleven-dimensional supergravity constrains for the $F$ field and compare them with the chirotope concept via the Grassmann-Plucker relations. In section 4, we present a possible definition of a superphirotope. Finally, in section 5, we make some last comments.

2.- Chirotope and Phirotope concepts

Let us start introducing the completely antisymmetric symbol

\[ \varepsilon^{a_1...a_d} \in \{-1, 0, 1\}. \tag{1} \]

Here the indices $a_1, ..., a_d$ run from 1 to $d$. This is a $d$-rank tensor which values are +1 or −1 depending on even or odd permutations of $\varepsilon^{12...d}$, respectively. Moreover, $\varepsilon^{a_1...a_d}$ takes the value 0 unless $a_1...a_d$ are all different. Let $v^i_a$ be any $d \times n$ matrix over some field $F$, where the index $i$ takes values in the set $E = \{1, ..., n\}$. Consider the object

\[ \Sigma^{i_1...i_d} = \varepsilon^{a_1...a_d} v_{i_1}^{a_1} \cdots v_{i_d}^{a_d}, \tag{2} \]

which can also be written as

\[ \Sigma^{i_1...i_d} = \det(v^{i_1}, ..., v^{i_d}). \tag{3} \]

Using the $\varepsilon$-symbol property

\[ \varepsilon^{a_1...[a_d} \varepsilon^{b_1...b_d]} = 0. \tag{4} \]

It is not difficult to prove that $\Sigma^{i_1...i_d}$ satisfies the so-called Grassmann-Plucker relations, namely

\[ \Sigma^{i_1...[i_d} \Sigma^{j_1...j_d]} = 0. \tag{5} \]

The brackets in the indices of (4) and (5) mean completely antisymmetrized.

A realizable chirotope $\chi$ is defined as

\[ \chi^{i_1...i_d} = \text{sign} \Sigma^{i_1...i_d}. \tag{6} \]

From the point of view of exterior algebra one finds that there is a close connection between Grassmann algebra and a chirotope. Let us denote by $\wedge_d R^n$ the $\binom{n}{d}$-dimensional real vector space of alternating $d$-forms on $R^n$. An element $\Sigma$ in $\wedge_d R^n$ is said to be decomposable if
\[
\Sigma = v_1 \wedge v_2 \wedge \ldots \wedge v_d, \quad (7)
\]

for some \(v_1, v_2, \ldots, v_d \in \mathbb{R}^n\). It is not difficult to see that (7) can be written as

\[
\Sigma = \frac{1}{r!} \Sigma^{i_1 \ldots i_d} e_{i_1} \wedge e_{i_2} \wedge \ldots \wedge e_{i_d}, \quad (8)
\]

where \(e_{i_1}, e_{i_2}, \ldots, e_{i_d}\) are 1-form bases in \(\mathbb{R}^n\) and \(\Sigma^{i_1 \ldots i_d}\) is given in (3). This shows that \(\Sigma^{i_1 \ldots i_d}\) can be identified with an alternating decomposable \(d\)-form.

In order to define non-realizable chirotopes it is convenient to write the expression (5) in an alternative form

\[
\sum_{k=1}^{d+1} s_k = 0, \quad (9)
\]

where

\[
s_k = (-1)^k \Sigma^{i_1 \ldots i_d-1 j_k} \Sigma^{j_1 \ldots j_k \ldots j_{d+1}}, \quad (10)
\]

Here, \(j_{d+1} = i_d\) and \(j_k\) establish the notation for omitting this index. Thus, in general for any \(d\)-rank chirotope \(\chi : E^d \to \{-1, 0, 1\}\) and

\[
s_k = (-1)^k \chi^{i_1 \ldots i_d-1 j_k} \chi^{j_1 \ldots j_k \ldots j_{d+1}}, \quad (11)
\]

for \(k = 1, \ldots, d+1\) there exist \(r_1, \ldots, r_{d+1} \in \mathbb{R}^+\) such that

\[
\sum_{k=1}^{d+1} r_k s_k = 0. \quad (12)
\]

It is clear that (9) is a particular case of (12). Therefore, there are chirotopes that may be non-realizable. Moreover, this definition of a chirotope is equivalent to various others (see Ref. [7] for details), but the present one is more convenient for a generalization to the complex structure setting.

The generalization of a chirotope to a phirotope is straightforward. A function \(\varphi : E^d \to S^1 \cup \{0\}\) on all \(d\)-tuples of \(E = \{1, \ldots, n\}\) is called a \(d\)-rank phirotope if (a) \(\varphi\) is alternating and (b) for

\[
\omega_k = (-1)^k \varphi^{i_1 \ldots i_d-1 j_k} \varphi^{j_1 \ldots j_k \ldots j_{d+1}} = 0, \quad (13)
\]

for \(k = 1, \ldots, d+1\) there exist \(r_1, \ldots, r_{d+1} \in \mathbb{R}^+\) such that

\[
\sum_{k=1}^{d+1} r_k \omega_k = 0. \quad (14)
\]
In the case of a realizable phirotope we have
\[ \Omega_{i_1 \ldots i_d} = \omega(\det(u^{i_1}, \ldots, u^{i_d})), \] (15)
where \( \omega(z) \in S^1 \cup \{0\} \) and \((u^{i_1} \ldots u^{i_d})\) are a set of complex vectors in \( C^d \). We observe that one of the main differences between a chirotope and a phirotope is that the image of a phirotope is no longer a discrete set (see Ref. [9] for details).

3- Maximal supersymmetry in eleven-dimensional supergravity and the chirotope concept

In Ref. [12] Figueroa-O’Farril and Papadopoulos showed that maximal supersymmetry in eleven-dimensional supergravity implies the two conditions
\[ F_{M[L_1 L_2 L_3 F_{L_4 L_5 L_6 L_7}] = 0 } \] (16)
and
\[ F_{M[P_1 P_2 P_3 F_{Q_1 Q_2 Q_3} N] = 0 } \] (17)
for the 4-form field strength \( F = dA \) in eleven dimensions. Moreover, from (16) and (17) they showed that \( F \) is parallel and decomposable. This last property means that \( F \) satisfies the Grassmann-Plucker relations
\[ F_{MP_1 P_2 [P_3 F_{Q_1 Q_2 Q_3} N] = 0 } \] (18)
Thus, according to the discussion of the previous section we discover that (18) establishes that \( F \) is a realizable 4-rank chirotope with a ground set \( E = \{1, \ldots, 11\} \). This in turn means that maximal supersymmetry in eleven-dimensional supergravity is related to oriented matroid theory. Similar conclusion can be obtained for the case of ten-dimensional supergravity (see section 5). Hence, from the chirotope concept we have found a link between supersymmetry and the oriented matroid theory. Therefore, one should expect a generalization of oriented matroid theory which would include supersymmetry. But in order to develop this idea it turns out more convenient to consider a complex structure, and this means that we need to focus on the superphirotope notion rather than on the superchirotope concept which should arise as a particular case of the former.
4. Superphirotepe

The main goal of this section is to outline a possible supersymmetrization of a phirotepe. Because of convenience we shall call superphirotepe such a supersymmetric phirotepe. Inspired in supebrane theory we find that one way to define a superphirotepe, which assures supersymmetry, is as follows. First, we need to locally consider the expressions (13)-(15) in the sense that \( \phi^{i_1\ldots i_d}(\xi) \) is a local phirotepe if

\[
\omega_k = (-1)^k \phi^{i_1\ldots i_{d-1} j_k}(\xi) \phi^{j_1\ldots j_{d+1}}(\xi),
\]

for \( k = 1, \ldots, d + 1 \) there exist \( r_1, \ldots, r_{d+1} \in R^+ \) such that

\[
\sum_{k=1}^{d+1} r_k \omega_k(\xi) = 0.
\]

In the case of a realizable local phirotepe we have

\[
\Omega^{i_1\ldots i_d}(\xi) = \omega(\det(u^{i_1}(\xi), \ldots, u^{i_d}(\xi)),
\]

where \( \xi = (\xi^1, \ldots, \xi^d) \) are local coordinates of some \( d \)-dimensional manifold \( B \). The vectors \( v^{i_1}(\xi), \ldots, v^{i_d}(\xi) \) can be thought as vectors in the tangent space \( T_\xi(B) \) at \( \xi \). One can assume that the possibility of considering the expressions (19)-(21) in a local context may be justified in principle by the so-called matroid bundle notion \[14]-[17]. It is known that the projective variety of decomposable forms is isomorphic to the Grassmann variety of \( d \)-dimensional linear subspaces in \( R^n \). In turn, the Grassmann variety is the classifying space for vector bundle structures. Taking these ideas as a motivation, MacPherson \[14\] developed the combinatorial differential manifold concept. The matroid bundle concept \[15]-[17\] arises as a generalization of the MacPherson proposal. Roughly speaking, a matroid bundle is a structure in which at each point of the differentiable manifold an oriented matroid is attached as a fiber (see \[14]-[17\] for details).

Now, let us consider a supermanifold \( B \) parametrized by the local coordinates \( (\xi, \theta) \) where \( \theta \) are elements of the odd Grassmann algebra (anticommuting variables). We shall now consider the supersymmetric prescription

\[
v^i \rightarrow \pi^i = v^i - i\theta \gamma^i \partial \theta.
\]

Here, \( \gamma^i \) are elements of a Clifford algebra. Using (22) we can generalize (21) in the form
\[ \Psi^{i_1 \ldots i_d}(\xi, \theta) = \omega(\det(\pi^{i_1}(\xi, \theta), \ldots, \pi^{i_d}(\xi, \theta))). \tag{23} \]

Here, the symbol \( \det \) means the superdeterminant. One should expect that (23) satisfies a kind of supersymmetric Grassmann-Plucker relations. It is not difficult to see that up to total derivative (23) is invariant under the global supersymmetric transformations

\[ \delta \theta = \epsilon \tag{24} \]

and

\[ \delta v^{i_1} = i\bar{\epsilon}\gamma^i \partial \theta, \tag{25} \]

where \( \epsilon \) is a constant complex spinor parameter.

Similarly, one can generalize the superphirotope to the non-representable case by assuming that if

\[ \omega_k = (-1)^k \varphi^{i_1 \ldots i_d} j_k(\xi, \theta) \varphi^{j_1 \ldots j_{d+1}}(\xi, \theta), \tag{26} \]

for \( k = 1, \ldots, d + 1 \) there exist \( r_1, \ldots, r_{d+1} \in \mathbb{R}^+ \) such that

\[ \sum_{k=1}^{d+1} r_k \omega_k(\xi, \theta) = 0. \tag{27} \]

Of course, in the case that the complex structure is projected to the real structure one should expect that the superphirotope is reduced to the super-chirotope.

5. Final remarks

With the superphirotope \( \Psi^{i_1 \ldots i_d}(x, \theta) \) at hand one may consider a possible partition function

\[ Z = \int D\Psi \exp(iS), \tag{28} \]

where

\[ S = \frac{1}{2} \int d^d \xi d\theta (\lambda^{-1} \Psi^{i_1 \ldots i_d}(\xi, \theta) \Psi_{i_1 \ldots i_d}(\xi, \theta) - \lambda T_d^2) \tag{29} \]

is a Schild type action for a superphirotope. Here, \( \lambda \) is a Lagrange multiplier and \( T_d \) is the \((d - 1)\)-phirotope tension. Moreover, in a more general context the action may have the form
The advantage of the actions (29) and (30) is that duality is assured in an automatic way. In fact, in oriented matroid theory duality is a main subject in the sense that any chirotope has an associated dual chirotope. This means that a theory described in the context of an oriented matroid automatically contents a duality symmetry. Therefore, with our prescription we are assuring not only the supersymmetry for the action (29) or (30) but also the duality symmetry.

The action (29) can be related to an ordinary super p-brane by assuming that $\Psi_{i_1 \ldots i_d}(\xi, \theta)$ is a closed $d$-form because in that case we can write

$$\pi^i_a = \partial_a x^i - i \theta \gamma^i \partial_a \theta.$$  

(31)

Here, the coordinates $x^i$ are the $p$-brane bosonic coordinates. It is worth mentioning that the closedness of the bosonic sector of $\Psi_{i_1 \ldots i_d}(\xi, \theta)$ is a constraint of Nambu-Poisson geometry which has been related to oriented matroid theory (see Ref. [18] for details).

It may be interesting for further research to consider the action (29) from the point of view of a superfield formalism instead of using the prescription (31). In this case one may consider a supersymmetrization in the form $\pi^i_a(\xi, \theta) = \partial_a X^i$, with $X^i$ as a scalar superfield admitting a finite expansion in terms of $\theta$. For instance, in four dimensions we have

$$X^i(\xi, \theta) = x^i(\xi) + i \theta \psi^i(\xi) + \frac{i}{2} \theta \theta B^i(\xi).$$  

(32)

Here, $\psi^i$ is a Majorana spinor field and $B^i$ is an auxiliary field. By substituting (32) into (23) one should expect a splitting of (29) in several terms containing the variables $x^i(\xi), \psi^i(\xi)$ and $B^i(\xi)$. The important thing is that using the prescription (32) supersymmetry becomes evident in the sense that the algebra of supersymmetry transformations is closed off the mass-shell (see Ref. [19]).

Although in section 3 we focused on eleven-dimensional supergravity similar arguments can be applied to the case of ten-dimensional supergravity. Specifically, by studying maximal supersymmetry in IIB supergravity Figueroa-O’Farril and Papadopoulos [12] used the vanishing of the curvature of the supercovariant derivative to derive the analogue Grassmann-Plucker formula

$$F_{LP_1P_2P_3}[P_4F^L_{Q_1Q_2Q_3Q_4}] = 0,$$  

(33)
for the five-form $F_{LP_1P_2P_3P_4}$. Moreover, in Ref. [13] is proved that (33) implies that

$$F = G + *G,$$

(34)

where $G$ is a decomposable five-form and $*G$ denotes the ten-dimensional dual of $G$. This means that $G$ and $*G$ satisfy the Grassmann-Plucker relations and therefore can be identified with a 5–rank chirotope. Thus, we conclude that not only maximal supersymmetry in eleven dimensional supergravity implies a link between supersymmetry and the oriented matroid theory via the chirotope concept but also maximal supersymmetry in ten-dimensional supergravity.

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