A SEMIDISCRETE SCHEME FOR EVOLUTION EQUATIONS WITH MEMORY

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Abstract. We introduce a new mathematical framework for the time discretization of evolution equations with memory. As a model, we focus on an abstract version of the equation

$$\partial_t u(t) + \int_0^\infty g(s) \Delta u(t-s) \, ds = 0$$

with Dirichlet boundary conditions, modeling hereditary heat conduction with Gurtin-Pipkin thermal law. Well-posedness and exponential stability of the discrete scheme are shown, as well as the convergence to the solutions of the continuous problem when the time-step parameter vanishes.

1. Introduction. Let $\langle H, \langle \cdot, \cdot \rangle, \| \cdot \| \rangle$ be a separable real Hilbert space, and let

$$A : D(A) \subseteq H \rightarrow H$$

be a strictly positive selfadjoint linear operator, where $D(A) \subseteq H$ stands for compact embedding. Hence, the inverse $A^{-1}$ is a compact operator. For $t > 0$, we consider the abstract evolution equation with memory

$$\dot{u}(t) + \int_0^\infty g(s) Au(t-s) \, ds = 0,$$  \hspace{1cm} (1.1)

where the $\dot{\cdot}$ stands for derivative with respect to the time variable $t$. The unknown $u = u(t)$ is understood to be an assigned datum for $t \leq 0$ (the so-called initial past history of $u$), where it need not solve the problem. The kernel $g : [0, \infty) \rightarrow [0, \infty)$ is a convex summable function of unitary total mass, having the explicit form

$$g(s) = \int_s^\infty \mu(y) \, dy,$$

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where $\mu : \mathbb{R}^+ \doteq (0, \infty) \to [0, \infty)$ is a (nonnull) nonincreasing absolutely continuous summable function of total mass
\[ g(0) = \int_0^\infty \mu(s) \, ds \doteq \kappa > 0. \]
Besides, $\mu$ is supposed to satisfy for almost every $s > 0$ and some $\delta_0 > 0$ the Dafermos condition (see [4])
\[ \mu'(s) + \delta_0 \mu(s) \leq 0, \quad (1.2) \]
which implies in particular the exponential decay of $\mu$ at infinity.

**Remark 1.1.** Actually, in more generality and with no changes in the proofs, one could take $\mu$ only piecewise absolutely continuous with a finite number of jumps (or even an infinite number of jumps, provided that they do not accumulate anywhere on the line). In this case, $\mu$ can be supposed to be right-continuous (or left continuous).

**Remark 1.2.** For a smooth bounded domain $\Omega \subset \mathbb{R}^3$, the choice $H = L^2(\Omega)$ and $A = -\Delta$ with $\mathcal{D}(A) = H^2(\Omega) \cap H^1_0(\Omega) \subset L^2(\Omega)$ leads to the integrodifferential equation
\[ \partial_t u(t) - \int_0^\infty g(s) \Delta u(t - s) \, ds = 0, \quad (1.3) \]
with Dirichlet boundary conditions, which serves as a model for heat propagation in a rigid isotropic homogeneous conductor with hereditary memory of Gurtin-Pipkin type [10]. Actually, it can be viewed as a fully hyperbolic (nonlocal) relaxation of the classical heat equation, formally recovered from (1.3) when memory kernel collapses at the Dirac mass at $0^+$. It is worth noting that in (1.3) the temperature evolution is influenced by the past history of the temperature itself. This feature may be regarded as a more realistic description of physical reality [7, 15, 16, 20].

The study of numerical schemes for the discretization of PDEs is an extremely fruitful field of modern Mathematics, particularly relevant for concrete applications. Nowadays, the literature on the subject is huge, covering a large variety of topics both in the deterministic and in the stochastic framework. Equations with memory have been widely investigated also from the numerical viewpoint: several problems concerning well-posedness, error estimates and asymptotic stability of numerical solutions have been addressed (see e.g. [5, 11, 12, 13, 14, 19, 21, 22] and references therein). The importance of the discretization of equations with memory comes from their wide applicability in the study of a number of phenomena in several areas, ranging from Physics, Biology, Population Dynamics, and many more. Since these equations are much harder than their counterparts without memory (for instance, they are structurally nonlocal), having at disposal robust discrete schemes is possibly the only effective way to obtain significant results in view of concrete applications. Nevertheless, all the studies made so far deal with Volterra-type equations of the form
\[ \dot{u}(t) + \int_0^t g(s) Au(t - s) \, ds = 0. \quad (1.4) \]
The latter is just a particular instance of (1.1), corresponding to a null initial past history of the state variable $u$ (namely, $u \equiv 0$ for negative times). It should be noted that equation (1.1) is much more difficult to handle than (1.4). Indeed, a first thought in order to tackle (1.1) could be to fix just a particular initial past
history of $u$, and then regard $\int_t^\infty g(s)Au(t-s)\,ds$ as a time-dependent source term. In which case, the system becomes nonautonomous. On the other hand, one would like to study the problem for different initial past histories of $u$, playing in this fashion the role of initial data. This however implies to take into account, besides the evolution of the variable $u$ itself, the evolution of its past history as well. The strategy, first proposed by C.M. Dafermos in his pioneering paper [4], consists in introducing an auxiliary variable $\eta$ accounting for the (integrated) past history of $u$, whose evolution takes place in a suitable Hilbert space $\mathcal{M}_0$ called memory space. Within this approach, one can construct a solution semigroup acting on an extended phase space and exploit the powerful theory of dynamical systems. This allows to successfully address several issues regarding existence and uniqueness, continuous dependence, regularity and asymptotic behavior of solutions.

In this paper, we introduce a mathematical framework for the discretization of the evolution equation (1.1). To the best of our knowledge, this is the first attempt to discretize an equation with memory with infinite delay. And indeed, the difficulties in the continuous framework reflect in the discrete setting, making the problem intriguing and highly nontrivial. The first challenge, which is not encountered in the discretization of (1.4), consists in finding the discrete counterpart of the continuous memory space $\mathcal{M}_0$ and the additional variable $\eta$. Secondly, one needs to set up a well-posed numerical scheme which, in addition, has to be robust as the discretization parameters converge to zero. Here, we propose an implicit method where only the time variable is discretized, thus allowing the exploitation of one’s favorite spatial discretization when performing numerics. It is worth mentioning that our scheme does not rely on the linear structure of (1.1). Hence, it is conceivable that the framework developed in the present paper can be adapted to discretize nonlinear variants of the model. This task will be possibly the object of future projects. After the functional setting and a brief overview of the continuous case, we deal with the discretization of the memory space $\mathcal{M}_0$. Then, we introduce the semidiscrete scheme and we show its well-posedness, which translates into the existence of a discrete semigroup of solutions acting on a suitable extended discrete phase space. Next, we prove that the discrete semigroup decays exponentially to zero. Such an asymptotic behavior reflects the longterm properties of the continuous semigroup generated by (1.1), which decays exponentially to zero as well. In the last part of the work, we deal with the convergence of the discrete solutions to the continuous ones as the time-step parameter vanishes. This passage is essential in order to ensure that the scheme provides a good approximation of the continuous equation.

2. Functional setting. For $r \in \mathbb{R}$, we introduce the compactly nested family of Hilbert spaces (the index $r$ will be omitted whenever zero)

$$H^r = \mathcal{D}(A^{\frac{r}{2}}), \quad \langle u, v \rangle_r = \langle A^\frac{r}{2}u, A^\frac{r}{2}v \rangle, \quad \|u\|_r = \|A^\frac{r}{2}u\|.$$

In particular, we have the compact embedding $H^1 \Subset H$, along with the Poincaré inequality

$$\lambda_1 \|u\|^2 \leq \|u\|_1^2, \quad \forall u \in H^1,$$

where $\lambda_1 > 0$ is the first eigenvalue of $A$. If $r > 0$, it is understood that $H^{-r}$ denotes the completion of the domain, so that $H^{-r}$ is the dual space of $H^r$. Accordingly, the symbol $\langle \cdot, \cdot \rangle$ also stands for duality product between $H^r$ and $H^{-r}$. Next, we
consider the memory spaces of square summable $H^{r+1}$-valued functions on $\mathbb{R}^+$ with respect to the measure $\mu(s)ds$ (again, $r$ will be omitted whenever zero)

$$\mathcal{M}_0 = L^2_\mu(\mathbb{R}^+; H^{r+1})$$

endowed with the weighted inner product and norm

$$\langle \eta, \xi \rangle_{\mathcal{M}_0} = \int_0^\infty \mu(s)\langle \eta(s), \xi(s) \rangle_{H^{r+1}} ds, \quad \|\eta\|_{\mathcal{M}_0} = \left(\int_0^\infty \mu(s)\|\eta(s)\|_{H^{r+1}}^2 ds\right)^{1/2}.$$ 

Due to the Poincaré inequality (2.1), we have the continuous (but not compact) inclusion $\mathcal{M}_0 \subset \mathcal{M}_0^{-1}$. Finally, we define the phase space of our problem

$$\mathcal{H}_0 = H \times \mathcal{M}_0,$$

normed by

$$\|(u, \eta)\|_{\mathcal{H}_0}^2 = \|u\|^2 + \|\eta\|^2_{\mathcal{M}_0}.$$ 

In the last part of the paper, we will also make use of the “asymmetric” energy spaces

$$\mathcal{V}_0 = H^r \times \mathcal{M}_0^{-1}, \quad (2.2)$$

equipped with the natural product norm.

3. The continuous case. Following the Dafermos scheme [4], we introduce the auxiliary variable

$$\eta^t(s) = \int_s^0 u(t - y)dy,$$

containing all the information on the (integrated) past history of the variable $u$ and satisfying the “boundary condition”

$$\eta^t(0) = 0. \quad (3.1)$$

Then, integrating by parts, we rewrite (1.1) in the form

$$\dot{u}(t) + \int_0^\infty \mu(s)A\eta^t(s) ds = 0. \quad (3.2)$$

Since $u = u(t)$ is assigned for $t \leq 0$, we can interpret

$$\eta^0(s) = \int_s^0 u(-y)dy$$

as an initial datum. Accordingly, when $s > t$, we have the equality

$$\eta^t(s) = \eta^0(s - t) + \int_0^t u(t - y)dy.$$

This leads us to the following definition of (weak) solution.

**Definition 3.1.** Let $T > 0$ be given, and let $z = (w, \xi) \in \mathcal{H}_0$ be a fixed vector. A function

$$(u(t), \eta^t) \in C([0, T], \mathcal{H}_0)$$

is a solution to problem (1.1) on the time-interval $[0, T]$ with initial datum $z$ if:

(i) $u(0) = w$ and $\eta^0 = \xi$. 

(ii) The function $\eta$ is given by
\[ \eta^s(t) = \begin{cases} \int_0^s u(t-y)dy, & 0 < s \leq t, \\ \eta^0(s-t) + \int_0^t u(t-y)dy, & s > t. \end{cases} \] (3.3)

(iii) The function $u$ solves (3.2) in the weak sense, i.e.
\[ \langle \dot{u}(t), \varphi \rangle + \int_0^\infty \mu(s)\langle \eta^t(s), \varphi \rangle_1 ds = 0, \]
for every test function $\varphi \in H^1$ and almost every $t \in [0,T]$.

**Remark 3.2.** The fact that $\eta$ fulfills the representation formula (3.3) can be equivalently stated by saying that $\eta$ is a weak solution to the linear equation on $M_0$
\[ \dot{\eta} = T\eta + u, \] (3.4)
where $T$ is the infinitesimal generator of the right-translation semigroup on $M_0$. Namely, the operator with domain
\[ D(T) = \{ \eta \in M_0 : \eta' \in M_0 \text{ and } \eta(0) = 0 \} \]
acting as
\[ T = -\eta', \quad \forall \eta \in D(T), \]
where the *prime* denotes the distributional derivative with respect to the internal variable $s$ of the function $\eta$ (see [9]). Indeed, such a linear equation usually appears in the definition of solutions in problems with memory. However, the alternative Definition 3.1 adopted in this paper, and firstly introduced in [2], presents several advantages. In particular, it is more flexible when performing regularization schemes (since the variable $\eta$, which belongs to a space of functions with values in a Hilbert space, is automatically regularized via $u$). Besides, and this is important in view of our purposes, it can be discretized in a very natural way.

Well-posedness and asymptotic stability of the problem above have been analyzed in several works (see e.g. [1, 3, 6, 8, 15] and references therein). In particular, the following results have been established.

- For every $T > 0$ and every $z \in H_0$ there exists a unique weak solution $(u(t), \eta^t)$ on $[0,T]$ with initial datum $z$, in the sense of Definition 3.1. Hence, the problem generates a linear $C_0$-semigroup
\[ S_0(t) : H_0 \to H_0 \quad \text{acting as} \quad S_0(t)z = (u(t), \eta^t). \]

- Assuming the Dafermos condition (1.2), the semigroup $S_0(t)$ is exponentially stable, namely, there exist $\nu > 0$ and $C \geq 1$ such that
\[ \|S_0(t)z\|_{H_0} \leq Ce^{-\nu t}\|z\|_{H_0}, \quad \forall z \in H_0. \]
In fact, as shown in [3], the Dafermos condition (1.2) is only sufficient in order for exponential stability to occur.
4. Discretization of the memory space. Throughout the paper we agree to denote
\[ N = \{0,1,2,3,\ldots\} \quad \text{and} \quad N^+ = \{1,2,3,\ldots\}. \]
For an arbitrarily given \( \tau > 0 \), we introduce the nonnegative nonincreasing (and vanishing at infinity) sequence
\[ \mu_k = \mu_k(\tau) = \frac{1}{\tau} \int_{k\tau}^{(k+1)\tau} \mu(s) \, ds, \quad k \in \mathbb{N}. \]  
(4.1)
Note that
\[ \tau \sum_{k=0}^{\infty} \mu_k = \int_{0}^{\infty} \mu(s) \, ds = \kappa. \]  
(4.2)
Then, we consider the discrete memory space
\[ \mathcal{M}_\tau = \left\{ \eta = \{\eta_k\}_{k \in \mathbb{N}^+} : \eta_k \in H^1 \text{ and } \tau \sum_{k=1}^{\infty} \mu_k \|\eta_k\|_1 < \infty \right\} \]
endowed with the weighted inner product and norm
\[ \langle \eta, \xi \rangle_{\mathcal{M}_\tau} = \tau \sum_{k=1}^{\infty} \mu_k \langle \eta_k, \xi_k \rangle_1 \quad \text{and} \quad \|\eta\|_{\mathcal{M}_\tau} = \left( \tau \sum_{k=1}^{\infty} \mu_k \|\eta_k\|_1^2 \right)^{\frac{1}{2}}. \]
Finally, we define the discrete phase space
\[ \mathcal{H}_\tau = H \times \mathcal{M}_\tau, \]
normed by
\[ \|(u, \eta)\|_{\mathcal{H}_\tau}^2 = \|u\|^2 + \|\eta\|^2_{\mathcal{M}_\tau}. \]
A word of warning. Along the paper, the Young and Poincaré inequalities, as well as the continuous and the discrete Hölder inequalities, will be used several times, often without explicit mention. In particular, recalling (4.2), for all \( \eta \in \mathcal{M}_\tau \) we have the estimate
\[ \tau \sum_{k=1}^{\infty} \mu_k \|\eta_k\|_1 \leq \tau \left( \sum_{k=1}^{\infty} \mu_k \right)^{\frac{1}{2}} \left( \sum_{k=1}^{\infty} \mu_k \|\eta_k\|_1^2 \right)^{\frac{1}{2}} \leq \sqrt{\kappa} \|\eta\|_{\mathcal{M}_\tau}. \]
We conclude the section by showing that the sequence \( \mu_k \) fulfills a discrete version of the Dafermos condition (1.2).

Lemma 4.1. Defining the positive constant
\[ \delta_\tau = e^{\delta_0 \tau} - 1, \]
the inequality
\[ \frac{\mu_k - \mu_{k-1}}{\tau} + \delta_\tau \mu_k \leq 0 \]  
holds for every \( k \in \mathbb{N}^+ \).

Proof. From the very definition of the sequence \( \mu_k \), we infer that
\[ (1 + \tau \delta_\tau) \mu_k = \frac{e^{\delta_0 \tau}}{\tau} \int_{k\tau}^{(k+1)\tau} \mu(s) \, ds = \frac{e^{\delta_0 \tau}}{\tau} \int_{(k-1)\tau}^{k\tau} \mu(\tau + s) \, ds. \]
It is also apparent that (1.2) is equivalent to
\[ \mu(t + s) \leq e^{-\delta_0 t} \mu(s) \]
for every \( t \geq 0 \) and almost every \( s > 0 \). Hence,

\[
\frac{e^{\delta_0 \tau}}{\tau} \int_{(k-1)\tau}^{k\tau} \mu(\tau+s)ds \leq \frac{1}{\tau} \int_{(k-1)\tau}^{k\tau} \mu(s)ds = \mu_{k-1},
\]

and the conclusion follows.

\[ \square \]

**Remark 4.2.** Since \( \delta_0 \to 0 \) as \( \tau \to 0 \), the discrete Dafermos condition (4.3) collapses into (1.2) when \( \tau \) vanishes.

5. **The semidiscrete scheme.** The main goal of this paper is the analysis of a (semi) discrete version of the evolution problem (1.1). To this end, we first give the exact discrete formulation of the Definition 3.1 of solution.

5.1. **The scheme.** For an arbitrarily given time-step parameter \( \tau > 0 \), we partition the plane \((t, s)\) into a square grid with cells of the form

\[
[n\tau, (n+1)\tau] \times [k\tau, (k+1)\tau], \quad (n, k) \in \mathbb{N} \times \mathbb{N}^+,
\]

where \( n \) and \( k \) play the roles of the time variable \( t \geq 0 \) and the “internal” variable \( s > 0 \), respectively. Next, we sample the state variable \((u, \eta)\), making use of an implicit approximation scheme. More precisely, given any initial datum

\[
z^0 = (u^0, \eta^0) \in \mathcal{H}_\tau,
\]

we plan to solve recursively the equation

\[
\frac{u^{n+1} - u^n}{\tau} + \tau \sum_{k=1}^{\infty} \mu_{k-1} A\eta_{k}^{n+1} = 0,
\]

where

\[
\eta_{k}^{n+1} = \begin{cases} 
\tau \sum_{j=0}^{k-1} u^{n+1-j}, & k \leq n + 1, \\
\eta_{k-n-1}^{0} + \tau \sum_{j=0}^{n} u^{n+1-j}, & k > n + 1.
\end{cases}
\]

Here and in what follows, we will always assume \( n \in \mathbb{N} \) and \( k \in \mathbb{N}^+ \), unless otherwise specified. In fact, equality (5.1) is to be correctly interpreted in the weak form

\[
\frac{1}{\tau} \langle u^{n+1} - u^n, \varphi \rangle + \tau \sum_{k=1}^{\infty} \mu_{k-1} \langle \eta_{k}^{n+1}, \varphi \rangle = 0,
\]

for every test function \( \varphi \in H^1 \).

Extending the definition of \( \eta_k^n \) for \( k = 0 \) as

\[
\eta_0^n = 0, \quad \forall n \in \mathbb{N},
\]

the following lemma holds.

**Lemma 5.1.** The representation formula (5.2) is satisfied if and only if the identity

\[
\eta_{k}^{n+1} = \eta_{k-1}^{n} + \tau u^{n+1}
\]

is verified for all \( n \in \mathbb{N} \) and all \( k \in \mathbb{N}^+ \).

The proof of the lemma is actually nothing but a direct calculation, and is left to the reader. It is worth observing that the (formal) position (5.4) actually reflects the boundary condition (3.1) of the continuous model.
Remark 5.2. Equality (5.5) above can be rewritten as
\[ \eta^{n+1}_{k} - \eta^{n}_{k} \tau = - \frac{\eta^{n}_{k} - \eta^{n}_{k-1}}{\tau} + u^{n+1}_{k}, \tag{5.6} \]
which is nothing but the classical first-order upwind scheme \[\text{[17]}\] for the advection equation (3.4). Besides, due to the definition of \( \mu_{k} \), the discrete equation (5.1) is the backward Euler approximation of the integrodifferential equation (3.2), the integral being computed by the rectangle method.

5.2. Well-posedness. The first step is showing that the scheme is well posed.

Theorem 5.3. For any given initial datum \( z^{0} = (u^{0}, \eta^{0}) \in H_{\tau} \), equations (5.1)-(5.2) define a unique sequence \((u^{n}, \eta^{n}) \in H_{\tau}\). Moreover,
\[ u^{n} \in H^{1}, \quad \forall n \in \mathbb{N}^{+}. \]

Proof. Assume that \((u^{n}, \eta^{n})\) has been defined up to \( m \in \mathbb{N} \). Then, substituting (5.5) into (5.1) and exploiting (4.2), we obtain
\[ (1 + \tau^{2} \kappa A)u^{n+1} = u^{n} - \tau^{2} \sum_{k=1}^{\infty} \mu_{k-1} A \eta^{m}_{k-1}. \]

It is immediate to verify that the right-hand side of the identity above belongs to the space \( H^{-1} \). Indeed, due to (2.1) and (5.4),
\[ \left\| u^{n} - \tau^{2} \sum_{k=1}^{\infty} \mu_{k-1} A \eta^{m}_{k-1} \right\|_{-1} \leq \frac{1}{\sqrt{\lambda_{1}}} \left\| u^{n} \right\|_{1} + \tau^{2} \sum_{k=1}^{\infty} \mu_{k} \left\| \eta^{m}_{k} \right\|_{1}, \]
\[ \leq \frac{1}{\sqrt{\lambda_{1}}} \left\| u^{n} \right\|_{1} + \tau \sqrt{\kappa} \left\| \eta^{m} \right\|_{M_{\tau}}, \]
where the latter estimate follows from the (discrete) Hölder inequality. Therefore
\[ u^{n+1} = (1 + \tau^{2} \kappa A)^{-1} \left[ u^{n} - \tau^{2} \sum_{k=1}^{\infty} \mu_{k-1} A \eta^{m}_{k-1} \right] \in H^{1}. \]

At this point, we merely define \( \eta^{m+1} \) through (5.5) and we apply Lemma 5.1. It remains to show that \( \eta^{m+1} \) belongs to \( M_{\tau} \). Indeed, since the sequence \( \mu_{k} \) is nonincreasing, and using again (5.4), we infer that
\[ \tau \sum_{k=1}^{\infty} \mu_{k} \| \eta^{m}_{k-1} \|_{1}^{2} + \tau u^{m+1} \|_{1}^{2} \leq 2 \tau \sum_{k=1}^{\infty} \mu_{k} \| \eta^{m}_{k-1} \|_{1}^{2} + 2 \tau \sum_{k=1}^{\infty} \mu_{k} \| \tau u^{m+1} \|_{1}^{2} \]
\[ \leq 2 \| \eta^{m} \|_{M_{\tau}}^{2} + 2 \tau^{2} \kappa \| u^{m+1} \|_{1}^{2} < \infty, \]
meaning that \( \eta^{m+1} \) has finite norm in \( M_{\tau} \).

5.3. Basic energy estimate. The natural energy
\[ E^{n} = \| u^{n} \|^{2} + \| \eta^{n} \|_{M_{\tau}}^{2}, \]
associated with (5.1)-(5.2) fulfills a certain inequality, crucial for our purposes. Defining the nonnegative quantity
\[ \Gamma[\eta^{n}] = \sum_{k=1}^{\infty} [\mu_{k-1} - \mu_{k}] \| \eta^{n}_{k} \|_{1}^{2}, \tag{5.7} \]
the following theorem holds.
Theorem 5.4. For any given initial datum \( z^0 = (u^0, \eta^0) \in \mathcal{H}_\tau \) and all \( n \in \mathbb{N} \),
\[
E^{n+1} + \tau \Gamma[\eta^{n+1}] \leq E^n. \tag{5.8}
\]
In particular, the sequence \( E^n \) is nonincreasing.

Proof. Choosing \( \varphi = 2\tau u^{n+1} \) in (5.3) (recall that \( u^{n+1} \in H^1 \)), and using the identity
\[
2\langle u^{n+1} - u^n, u^{n+1} \rangle = \|u^{n+1}\|^2 + \|u^{n+1} - u^n\|^2 - \|u^n\|^2,
\]
we get
\[
\|u^{n+1}\|^2 + \|u^{n+1} - u^n\|^2 - \|u^n\|^2 + 2\tau^2 \sum_{k=1}^{\infty} \mu_{k-1} \langle \eta^{n+1}_k, u^{n+1} \rangle_k = 0.
\]
In light of (5.4) and (5.5), the last term can be rewritten as
\[
2\tau^2 \sum_{k=1}^{\infty} \mu_{k-1} \langle \eta^{n+1}_k, u^{n+1} \rangle_k = 2\tau^2 \sum_{k=1}^{\infty} \mu_{k-1} \|\eta^{n+1}_k - \eta^{n}_k - 1\|^2_m + \tau \Gamma[\eta^{n+1}] - \|\eta^n\|^2_m
\]
Hence,
\[
E^{n+1} + \tau \Gamma[\eta^{n+1}] + P^n = E^n,
\]
where
\[
P^n = \|u^{n+1} - u^n\|^2 + \tau \sum_{k=1}^{\infty} \mu_{k-1} \|\eta^{n+1}_k - \eta^{n}_k - 1\|^2_m.
\]
Neglecting the term \( P^n \geq 0 \), the conclusion follows. \( \square \)

Theorem 5.4 produces an immediate corollary.

Corollary 5.5. The linear map \( S_\tau : \mathcal{H}_\tau \to \mathcal{H}_\tau \) acting as
\[
S_\tau(u^0, \eta^0) = (u^1, \eta^1)
\]
is bounded (hence continuous).

As a consequence, the implicit scheme (5.1)-(5.2) generates a discrete semigroup
\[
S^n_\tau : \mathcal{H}_\tau \to \mathcal{H}_\tau
\]
acting as
\[
S^n_\tau z^0 = S_\tau \circ S_\tau \circ \cdots \circ S_\tau z^0._{n\text{-times}}
\]

6. Exponential stability. The aim of this section is to prove the exponential decay of the discrete semigroup \( S^n_\tau \), and to discuss the result in the limit \( \tau \to 0 \).
6.1. **Exponential stability of the discrete semigroup.** Our main theorem reads as follows.

**Theorem 6.1.** There exists \( q_\tau < 1 \) such that the inequality

\[
\| S_\tau^n z^0 \|_{\mathcal{H}_\tau} \leq 2q_\tau^n \| z^0 \|_{\mathcal{H}_\tau}
\]

(6.1)

holds for every \( z^0 \in \mathcal{H}_\tau \) and all \( n \in \mathbb{N} \).

The proof requires the introduction of the auxiliary energy-type functional

\[
\Psi^n = -2\tau \sum_{k=1}^{\infty} \mu_k \langle u^n_k, \eta^n_k \rangle.
\]

Observe that, on account of (2.1),

\[
|\Psi^n| \leq \sqrt{\frac{\kappa}{\lambda_1}} E^n. 
\]

(6.2)

**Lemma 6.2.** For every \( n \in \mathbb{N} \), the functional \( \Psi^n \) satisfies the inequality

\[
\Psi^{n+1} + \tau \mathcal{A} \| u^{n+1} \|^2 \leq \Psi^n + \frac{T\mu_0}{6\lambda_1} \Gamma[\eta^{n+1}] + 2\tau \mathcal{A} \| \eta^n \|_{\mathcal{M}_\tau} \left[ \| \eta^{n+1} \|_{\mathcal{M}_\tau} + \sqrt{\tau \Gamma[\eta^{n+1}]} \right].
\]

**Proof.** By means of direct calculations, we get the identity

\[
\Psi^{n+1} = \Psi^n - 2\tau \sum_{k=1}^{\infty} \mu_k \langle u^{n+1}_k - u^n_k, \eta^n_k \rangle + 2\tau \sum_{k=1}^{\infty} \mu_k \langle u^{n+1}_k, \eta^n_k - \eta^{n+1}_k \rangle.
\]

Invoking (5.1), the first term on the right-hand side can be written as

\[
-2\tau \sum_{k=1}^{\infty} \mu_k \langle u^{n+1}_k - u^n_k, \eta^n_k \rangle = 2\tau^2 \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} \mu_{j-1} \langle \eta^{n+1}_j, \eta^n_k \rangle \leq 2\tau^2 \mathcal{A} \| \eta^n \|_{\mathcal{M}_\tau} \sum_{j=1}^{\infty} \mu_{j-1} \| \eta^{n+1}_j \|_1 \\
\leq 2\tau \mathcal{A} \| \eta^n \|_{\mathcal{M}_\tau} \left[ \| \eta^{n+1} \|_{\mathcal{M}_\tau} + \sqrt{\tau \Gamma[\eta^{n+1}]} \right].
\]

Moreover, as \( \mu_k \downarrow 0 \), we infer from (5.4)-(5.5) that

\[
2\tau \sum_{k=1}^{\infty} \mu_k \langle u^{n+1}_k, \eta^n_k - \eta^{n+1}_k \rangle = -2\tau \mathcal{A} \| u^{n+1} \|^2 + 2\tau \sum_{k=1}^{\infty} \langle \mu_{k-1} - \mu_k \rangle \langle u^{n+1}_k, \eta^n_k \rangle \\
\leq -2\tau \mathcal{A} \| u^{n+1} \|^2 + 2\tau \sqrt{\frac{\mu_0}{\lambda_1}} \| u^{n+1} \| \sqrt{\Gamma[\eta^{n+1}]} \\
\leq -\tau \mathcal{A} \| u^{n+1} \|^2 + \frac{T\mu_0}{6\lambda_1} \Gamma[\eta^{n+1}].
\]

Collecting the estimates above, the proof is finished. \( \square \)

At this point, we introduce the positive quantity

\[
\varepsilon = \varepsilon(\tau) = \min \left\{ \frac{1}{2} \sqrt{\frac{\lambda_1}{\kappa}}, \frac{\kappa \lambda_1}{\mu_0}, \frac{\delta_0}{4\varepsilon}, \frac{\delta_0}{32\varepsilon \kappa \varepsilon^0 \tau} \right\}.
\]

(6.3)

Next, we consider the functional

\[
\Lambda^n = E^n + \varepsilon \Psi^n.
\]
Note that, due to (6.2) and the fact that $\varepsilon \leq \frac{1}{2} \sqrt{\frac{\lambda_1}{\kappa}}$,
\begin{equation}
\frac{1}{2} E^n \leq \Lambda^n \leq 2 E^n.
\end{equation}

**Lemma 6.3.** The inequality
\begin{equation}
\left[1 + \frac{\varepsilon \tau \kappa}{2}\right] \Lambda^{n+1} \leq \left[1 + \frac{\varepsilon \tau \kappa}{4}\right] \Lambda^n
\end{equation}
holds for every $n \in \mathbb{N}$.

**Proof.** Invoking (5.8) and Lemma 6.2, together with the constraint $\varepsilon \leq \frac{\kappa \lambda_1}{4 \mu_0}$, we have
\begin{align*}
\Lambda^{n+1} + \tau \Gamma[\eta^{n+1}] + \varepsilon \tau \kappa \|u^{n+1}\|^2 \\
\leq \Lambda^n + \frac{\varepsilon \tau \mu_0}{\kappa \lambda_1} \Gamma[\eta^{n+1}] + 2 \varepsilon \tau \kappa \|\eta^n\|_{\mathcal{M}_r} \left[\|\eta^{n+1}\|_{\mathcal{M}_r} + \sqrt{\tau \Gamma[\eta^{n+1}]}\right] \\
\leq \Lambda^n + \frac{\tau}{2} \Gamma[\eta^{n+1}] + \frac{\varepsilon \tau \kappa}{4} \|\eta^{n+1}\|_{\mathcal{M}_r}^2 + \frac{4 \tau \kappa^2 e^\delta_\tau}{} \|\eta^n\|_{\mathcal{M}_r}^2.
\end{align*}
In addition, a closer look at (5.7) tells us that, within the discrete Dafermos condition (4.3),
\begin{align*}
\Gamma[\eta^{n+1}] \geq \delta_\tau \|\eta^{n+1}\|_{\mathcal{M}_r}^2.
\end{align*}
Thus, inequality (6.5), along with the conditions $\delta_0 \leq \delta_\tau$, $\varepsilon \leq \frac{\delta_0}{4 \tau}$ and $\varepsilon \leq \frac{\delta_0}{4 \kappa \varepsilon \tau}$, yield
\begin{align*}
\Lambda^{n+1} + \varepsilon \tau \kappa E^{n+1} \leq \Lambda^{n+1} + \frac{\tau \delta_0}{4} \|\eta^{n+1}\|_{\mathcal{M}_r}^2 + \varepsilon \tau \kappa \|u^{n+1}\|^2 \leq \Lambda^n + \frac{\varepsilon \tau \kappa}{8} E^n.
\end{align*}
Exploiting (6.4), the claim follows. \qed

We have now all the ingredients to prove Theorem 6.1.

**Proof of Theorem 6.1.** On account of Lemma 6.3, we get
\begin{equation}
\Lambda^{n+1} \leq q^2 \Lambda^n,
\end{equation}
where
\begin{equation}
q_\tau = \sqrt{1 - \frac{\varepsilon \tau \kappa}{4 + 2 \varepsilon \tau \kappa}} < 1.
\end{equation}
Hence, iterating the inequality above,
\begin{align*}
\Lambda^n \leq q^2 \Lambda^{n-1} \leq \cdots \leq q^2 \Lambda^0.
\end{align*}
Making use of (6.4), we arrive at (6.1). \qed

**6.2. Passage to the limit.** Here we show that the conclusion of Theorem 6.1 is consistent with the exponential decay of the continuous semigroup $S_0(t)$, formally (at this stage) obtained by letting $\tau \to 0$. Namely, we want to render precise the following statement: the fact that
\begin{equation}
\|S_0^n\| \leq 2 q^n
\end{equation}
implies, in the limit $\tau \to 0$, that
\begin{equation}
\|S_0(t)\| \leq 2 e^{-\omega t}
\end{equation}
for some $\omega > 0$. The symbols $\| \cdot \|$ denote the operator norms in the respective spaces.
Proposition 6.4. There exists a constant \( \omega > 0 \) independent of \( \tau > 0 \) such that, for every fixed \( t \geq 0 \), we have
\[
\limsup_{\tau \to 0} \| S^{n_\tau}_\tau \| \leq 2e^{-\omega t},
\]
where \( n_\tau \in \mathbb{N} \) is the unique integer satisfying
\[
t \leq \tau n_\tau < t + \tau.
\]

Proof. For simplicity, we proceed within the further assumption
\[
\mu(0) = \lim_{s \to 0^+} \mu(s) < \infty,
\]
ensuring in particular that the function \( \varepsilon(\tau) \) defined in (6.3) satisfies
\[
\bar{\varepsilon} = \lim_{\tau \to 0} \varepsilon(\tau) > 0.
\]
Nevertheless, the result is valid even when \( \mu \) is not bounded about zero (see Remark 6.5 below). Owing to Theorem 6.1, the inequality
\[
\| S^{n_\tau}_\tau \| \leq 2q^n_\tau
\]
holds, where \( q_\tau \) is given by (6.6). Since \( \tau n_\tau \to t \) as \( \tau \to 0 \), invoking (6.7) we get
\[
\lim_{\tau \to 0} q^n_\tau = e^{-\omega t}
\]
where
\[
\omega = \frac{\bar{\varepsilon} \kappa}{8} > 0.
\]
The result is proved. \( \square \)

Remark 6.5. When the memory kernel \( \mu \) is unbounded about zero, we have
\[
\lim_{\tau \to 0} \mu_0(\tau) = \infty,
\]
implying that \( \bar{\varepsilon} = 0 \). In this situation, when proving Theorem 6.1, one need to replace the original kernel \( \mu \) with the truncated one
\[
\rho(s) = \mu(s_*)\chi_{(0,s_*)}(s) + \mu(s)\chi_{(s_*,\infty)}(s),
\]
for some \( s_* > 0 \) such that
\[
\int_0^{s_*} \mu(s) \, ds \leq \frac{\kappa}{4}.
\]
In this way, in the definition of \( \varepsilon(\tau) \), the quantity \( \mu_0(\tau) \) is replaced by a certain function of \( \tau \) that converges to \( \mu(s_*) > 0 \) as \( \tau \to 0 \). Then one repeats exactly the previous argument. We leave the full details to the interested reader.

7. Convergence. Finally, we show that the solutions of the discrete model converge to the solutions of the continuous one when the time-step parameter \( \tau > 0 \) vanishes. In other words, for every fixed \( t \geq 0 \), we want to prove the convergence (in some suitable sense)
\[
S^{n_\tau}_\tau \to S(t) \quad \text{as} \quad \tau \to 0,
\]
where, as before, \( t \leq \tau n_\tau < t + \tau \). To this end, we need to introduce the discretization operator \( \mathbb{D}_\tau \) acting on \( \mathcal{M}_0 \) as
\[
\mathbb{D}_\tau(\xi) = \{\xi_k\}_{k \in \mathbb{N}^+} \quad \text{where} \quad \xi_k = \frac{1}{\tau} \int_{(k-1)\tau}^{k\tau} \xi(s) \, ds.
\]
In the next Lemma 7.2 we will see that \( \mathbb{D}_\tau \) maps \( \mathcal{M}_0 \) into \( \mathcal{M}_\tau \).
7.1. Statement of the result. Let $z = (w, \xi) \in H_0$ be a fixed vector. Setting $u_0^0 = w$ and $\eta_0^0 = D_\tau(\xi)$, we consider the solution $(u^n, \eta^n)$ to the discrete scheme (5.1)-(5.2) corresponding to the initial datum $(u^0, \eta^0) \in H_\tau$.

Then, calling $\Lambda(t) = \max(1 - |t|, 0)$, we introduce the linear interpolating function

$$u_\tau(t) = \sum_{k=0}^{\infty} u^k \Lambda \left( \frac{t - k}{\tau} \right), \quad t \geq 0,$$

and we define

$$\eta_\tau^r(s) = \begin{cases} 
\int_0^s u_\tau(t - y) dy, & 0 < s \leq t, \\
\xi(s - t) + \int_0^t u_\tau(t - y) dy, & s > t. 
\end{cases} \quad (7.1)$$

Recalling the definition of the spaces $V_0^r$ given in (2.2), the main result of the section reads as follows.

**Theorem 7.1.** Let $T > 0$ be arbitrarily given, and let $(u(t), \eta^t) \in C([0, T], H_0)$ be the (unique) solution to the continuous problem (3.2)-(3.3) on the time-interval $[0, T]$ with initial datum $z = (w, \xi)$. Then, the pair $(u_\tau, \eta_\tau)$ belongs to the space $C([0, T], V_\tau^{-r})$ and, for every fixed $r > 0$, we have the convergence as $\tau \to 0$

$$(u_\tau, \eta_\tau) \to (u, \eta) \quad \text{in} \quad C([0, T], V_0^{-r}).$$

Besides, we also have the convergence

$$(u_\tau, \eta_\tau) \overset{w^*}{\to} (u, \eta) \quad \text{in} \quad L^\infty(0, T; V_0).$$

The remaining of the paper is devoted to the proof of Theorem 7.1. In what follows, $T > 0$ is (arbitrary but) fixed.

7.2. Uniform bounds. The first step is proving estimates on $u_\tau$ which are independent of the time-step parameter $\tau$. We first need a result on the operator $D_\tau$ introduced earlier.

**Lemma 7.2.** The discretization operator $D_\tau$ maps $M_0$ into $M_\tau$ (in fact, onto), and

$$\|D_\tau(\xi)\|_{M_\tau} \leq \|\xi\|_{M_0}.$$  

**Proof.** Making use of (4.1) and the definition of $\xi_k$,

$$\|D_\tau(\xi)\|_{M_\tau}^2 = \sum_{k=1}^{\infty} \int_{k\tau}^{(k+1)\tau} \|\mu(s)\xi_k\|^2 ds \leq \frac{1}{\tau} \sum_{k=1}^{\infty} \int_{k\tau}^{(k+1)\tau} \mu(s) \int_{(k-1)\tau}^{k\tau} \|\xi(\sigma)\|^2 d\sigma ds.$$ 

Since in the last formula above, for every fixed $k$, the variable $s$ belongs to $(k\tau, (k+1)\tau]$ and $\mu$ is nonincreasing, it is clear that

$$\mu(s) \leq \mu(k\tau), \quad \text{for} \quad k \geq 1.$$
Thus, for all \( s \in (k\tau, (k+1)\tau] \), owing once more to the fact that \( \mu \) is nonincreasing, we obtain

\[
\mu(s) \int_{(k-1)\tau}^{k\tau} ||\xi(\sigma)||^2_1 \, d\sigma \leq \int_{(k-1)\tau}^{k\tau} \mu(k\tau)||\xi(\sigma)||^2_1 \, d\sigma \leq \int_{(k-1)\tau}^{k\tau} \mu(s)||\xi(\sigma)||^2_1 \, d\sigma.
\]

Hence,

\[
\frac{1}{\tau} \sum_{k=1}^{\infty} \int_{k\tau}^{(k+1)\tau} \mu(s) \int_{(k-1)\tau}^{k\tau} ||\xi(\sigma)||^2_1 \, d\sigma \, d\sigma \leq \sum_{k=1}^{\infty} \int_{(k-1)\tau}^{k\tau} \mu(s)||\xi(\sigma)||^2_1 \, d\sigma = ||\xi||^2_{L^0},
\]

and the conclusion follows.

**Lemma 7.3.** The function \( u_\tau \) is bounded in the space \( L^\infty(0, T; H) \cap W^{1,\infty}(0, T; H^{-1}) \), and the bound is uniform w.r.t. \( \tau > 0 \).

**Proof.** Let \( t \in [0, T] \) be arbitrarily fixed, and let \( n = n(t, \tau) \in \mathbb{N} \) be such that

\[
t \in [n\tau, (n+1)\tau).
\]

It is immediate to check that, being \( u_\tau(t) \) a convex combination of \( u^n \) and \( u^{n+1} \),

\[
||u_\tau(t)||^2 \leq \max \{||u^n||^2, ||u^{n+1}||^2\} \leq \max \{E^n, E^{n+1}\}.
\]

Since \( E^n \) is nonincreasing, by Lemma 7.2 we get

\[
||u_\tau(t)||^2 \leq E^0 = ||w||^2 + ||D_\tau(\xi)||^2_{L^0} \leq ||w||^2 + ||\xi||^2_{L^0},
\]

implying that \( u_\tau \) is bounded in \( L^\infty(0, T; H) \) uniformly with respect to \( \tau \). Moreover, for \( t \neq n\tau \), the time derivative of \( u_\tau \) reads

\[
\dot{u}_\tau(t) = \frac{u^{n+1} - u^n}{\tau}.
\]

As a consequence, exploiting (5.1) together with (4.2), (5.7) and the discrete Hölder inequality,

\[
||\dot{u}_\tau(t)||^2_1 = \tau^2 \sum_{k=1}^{\infty} \mu_{k-1} A \eta_k^{n+1} \leq \tau \sum_{k=1}^{\infty} \mu_{k-1} ||\eta_k^{n+1}||^2_1
\]

\[
\leq \tau \sum_{k=1}^{\infty} ||\eta_k^{n+1}||^2_{L^\infty} + \tau \Gamma||\eta^{n+1}||^2_{L^0}.
\]

Invoking (5.8) and Lemma 7.2, the right-hand side can be estimated as

\[
\tau ||\eta^{n+1}||^2_{L^\infty} + \tau \Gamma||\eta^{n+1}||^2_{L^0} \leq \tau E^0 \leq \tau \|w\|_2 + \tau ||\xi||^2_{L^0},
\]

meaning that \( \dot{u}_\tau \) is bounded in \( L^\infty(0, T; H^{-1}) \) uniformly with respect to \( \tau \).

**7.3. The limit functions.** In light of Lemma 7.3 and the classical Simon-Aubin compact embedding, there exist a positive sequence \( \tau_n \to 0 \) and

\[
v \in L^\infty(0, T; H) \cap W^{1,\infty}(0, T; H^{-1})
\]

such that, for every fixed \( r > 0 \),

\[
u_{\tau_n} \to v \quad \text{in} \quad C([0, T], H^{-r}), \quad (7.2)
\]

and

\[
u_{\tau_n} \to \dot{v} \quad \text{in} \quad L^\infty(0, T; H^{-1}). \quad (7.3)
\]
The next step is passing (7.1) to the limit. To this end, calling
\[ \zeta_t(s) = \begin{cases} \int_0^s v(t-y)dy, & 0 < s \leq t, \\ \xi(s-t) + \int_0^t v(t-y)dy, & s > t, \end{cases} \]
the following holds.

**Lemma 7.4.** The functions \( \eta_{\tau_n} \) and \( \zeta \) belong to the space \( \mathcal{C}([0,T],\mathcal{M}_0^{-1}) \) and, for every fixed \( r > 0 \),
\[ \eta_{\tau_n} \to \zeta \quad \text{in} \quad \mathcal{C}([0,T],\mathcal{M}_0^{-r-1}). \] (7.4)

**Proof.** Since \( u_{\tau_n} \in L^\infty(0,T;H) \), and so \( u_{\tau_n} \in L^\infty(0,T;\mathcal{M}_0^{-1}) \) (viewed as a constant function of the variable \( s \)), the representation formula (7.1) is completely equivalent to the assertion that the function \( \eta_{\tau_n} \) is the (unique) mild solution in the sense of Pazy [18] to the linear problem
\[ \begin{aligned} \dot{\eta}_{\tau_n} &= T\eta_{\tau_n} + u_{\tau_n}, \\ \eta_{\tau_n}(0) &= \xi, \end{aligned} \]
where now \( T \) is the infinitesimal generator of the right-translation semigroup on \( \mathcal{M}_0^{-1} \) (see [9]). Accordingly,
\[ \eta_{\tau_n} \in \mathcal{C}([0,T],\mathcal{M}_0^{-1}). \]

The same argument shows that
\[ \zeta \in \mathcal{C}([0,T],\mathcal{M}_0^{-1}) \]
as well. In order to prove (7.4), for every fixed \( t \in [0,T] \), we estimate
\[ \begin{align*} \| \eta_{\tau_n}^t - \zeta^t \|^2_{\mathcal{M}_0^{-r-1}} &= \int_0^\infty \mu(s)\| \eta_{\tau_n}^t(s) - \zeta^t(s) \|^2_{-r} ds \\ &= \int_0^t \mu(s)\left\| \int_0^s [u_{\tau_n}(t-y) - v(t-y)]dy \right\|^2_{-r} ds \\ &\quad + \int_t^\infty \mu(s)\left\| \int_0^t [u_{\tau_n}(t-y) - v(t-y)]dy \right\|^2_{-r} ds \\ &\leq \left[ \int_0^t s^2\mu(s) ds + t^2 \int_t^\infty \mu(s) ds \right] \| u_{\tau_n} - v \|^2_{\mathcal{C}([0,T],H^{-r})}. \end{align*} \]
Due to the Dafermos condition (1.2), the function \( s^2\mu(s) \) is summable. In particular, there exists a structural constant \( c > 0 \) such that
\[ \int_0^t s^2\mu(s) ds + t^2 \int_t^\infty \mu(s) ds \leq c(1 + T^2). \]

Finally, appealing to (7.2),
\[ \| \eta_{\tau_n} - \zeta \|_{\mathcal{C}([0,T],\mathcal{M}_0^{-r-1})} \leq \sqrt{c(1 + T^2)} \| u_{\tau_n} - v \|_{\mathcal{C}([0,T],H^{-r})} \to 0, \]
yielding the conclusion. \( \square \)
7.4. Identifying the limit. We now show that \((v, \zeta)\) coincides with the solution \((u, \eta)\) of the continuous problem \((3.2)-(3.3)\) with initial datum \(z = (w, \xi)\).

**Lemma 7.5.** The equality \((v(t), \zeta^t) = (u(t), \eta^t)\) holds.

An auxiliary result is needed, stated again as a lemma.

**Lemma 7.6.** As \(\tau \to 0\), we have the convergence
\[
\dot{u}_\tau + \int_0^\infty \mu(s)A\eta_{\tau_n}(s)\,ds \to 0 \quad \text{in} \quad L^\infty(0,T; H^{-2}).
\]

The proof of Lemma 7.6 is rather long and technical, and it is postponed to the final section of this work.

**Proof of Lemma 7.5.** Let \(r > 0\) be arbitrarily fixed. Exploiting (7.3), it is apparent to see that
\[
\dot{u}_{\tau_n} \rightharpoonup \dot{v} \quad \text{in} \quad L^\infty(0,T; H^{-r-2}).
\]  
(7.5)

We now claim that
\[
\int_0^\infty \mu(s)A\eta_{\tau_n}(s)\,ds \rightharpoonup \int_0^\infty \mu(s)A\zeta(s)\,ds \quad \text{in} \quad L^\infty(0,T; H^{-r-2}).
\]  
(7.6)

Indeed, for every test function \(\psi \in L^1(0,T; H^{r+2})\), making use of Lemma 7.4 we get
\[
\left| \int_0^T \int_0^\infty \mu(s)\langle A\eta_{\tau_n}(s) - A\zeta(s), \psi(t) \rangle \,ds \,dt \right| \leq \int_0^T \|\psi(t)\|_{r+2} \int_0^\infty \mu(s)\|\eta_{\tau_n}(s) - \zeta(s)\|_{-r} \,ds \,dt \leq \sqrt{K} \int_0^T \|\psi(t)\|_{r+2} \|\eta_{\tau_n} - \zeta\|_{M_{0}^{-r-1}} \,dt \leq \sqrt{K} \|\eta_{\tau_n} - \zeta\|_{L^1(0,T; M_{0}^{-r-1})} \|\psi\|_{L^1(0,T; H^{r+2})} \to 0.
\]

At this point, collecting (7.6) and Lemma 7.6, we readily obtain the convergence
\[
\dot{u}_{\tau_n} \rightharpoonup \dot{v} - \int_0^\infty \mu(s)A\zeta(s)\,ds \quad \text{in} \quad L^\infty(0,T; H^{-r-2}).
\]

Therefore, by comparison in (7.5), we end up with the equality
\[
\dot{v} + \int_0^\infty \mu(s)A\zeta(s)\,ds = 0.
\]

In particular, due to the regularity of \(v\) and \(\zeta\), the identity
\[
\langle \dot{v}(t), \varphi \rangle + \int_0^\infty \mu(s)\langle \zeta^s(s), A\varphi \rangle \,ds = 0
\]
is satisfied for every test function \(\varphi \in H^2\) and almost every \(t \in [0,T]\). Moreover, since \(u_{\tau_n}(0) = w\), invoking (7.2) we also get
\[
v(0) = w.
\]

Being \(\zeta^0 = \xi\), we infer that the couple \((v, \zeta)\) solves the continuous problem \((3.2)-(3.3)\) on the phase space
\[
H^{-1}_0 = H^{-1} \times M_{0}^{-1}.
\]
On the other hand, the unique solution \((u, \eta)\) to the continuous problem (3.2)-(3.3) is also the (unique) solution to (3.2)-(3.3) on the weaker phase space \(H_0^{-1}\). This readily implies the equality \((v, \zeta) = (u, \eta)\). \(\square\)

7.5. **Conclusion of the proof of Theorem 7.1.** Note that \(u_\tau \in C([0, T], H)\) by construction. Besides, arguing as in the proof of Lemma 7.4 (with \(\tau_n\) replaced by \(\tau\)), it is immediate to see that \(\eta_\tau \in C([0, T]; M_0^{-1})\), meaning that

\[(u_\tau, \eta_\tau) \in C([0, T], V_0).\]

In addition, making use of (7.2)-(7.4) and Lemma 7.5, we infer that

\[(u_{\tau_n}, \eta_{\tau_n}) \rightarrow (u, \eta) \text{ in } C([0, T], V_0^{-r}).\]

On the other hand, for any \(0 < \tau_n \rightarrow 0\), we can repeat the whole argument and conclude that \((u_{\tau_n}, \eta_{\tau_n}) \rightarrow (u, \eta) \in C([0, T], V_0^{-r})\) up to a subsequence. Hence, \((u, \eta)\) is the unique cluster point of \((u_\tau, \eta_\tau)\), which forces the convergence

\[(u_\tau, \eta_\tau) \rightarrow (u, \eta) \rightarrow C([0, T], V_0^{-r}) \quad (7.7)\]

as \(\tau \rightarrow 0\). Owing now to (7.1), the Poincaré inequality (2.1) and the monotonicity of \(\mu\), for every fixed \(t \in [0, T]\) we readily get

\[\|\eta_\tau\|_{M_0^{-1}}^2 \leq c(1 + T^2)\|u_\tau\|_{L^\infty(0,T; H)}^2 + c\|\xi\|_{M_0}^2,\]

where \(c > 0\) is a structural constant independent of \(t\) and \(\tau\). Invoking Lemma 7.3, we conclude that \((u_\tau, \eta_\tau)\) is bounded in \(L^\infty(0, T; V_0)\) uniformly with respect to \(\tau\). As a consequence, there exist a positive sequence \(\tau_n \rightarrow 0\) and a couple \((v, \zeta) \in L^\infty(0, T; V_0)\) such that

\[(u_{\tau_n}, \eta_{\tau_n}) \begin{array}{c} w^* \end{array} (v, \zeta) \quad \text{in } L^\infty(0, T; V_0).\]

On the other hand, (7.7) ensures that \((v, \zeta) = (u, \eta)\). Finally, arguing as above, we see that \((u, \eta)\) is the unique weak-\(^*\) cluster point of \((u_\tau, \eta_\tau)\), and the desired conclusion follows. \(\square\)

8. **Proof of Lemma 7.6.** For every \(n \in \mathbb{N}\), let us introduce the interval (depending on \(\tau\))

\[\mathcal{I}_n = [nt, (n + 1)t).\]

Let \(t \in [0, T]\) be arbitrarily fixed. Then

\[t \in \mathcal{I}_n \quad \text{for some } n = n(t, \tau).\]

Since for \(t \neq nt\)

\[\dot{u}_\tau(t) = \frac{u^{n+1} - u^n}{\tau},\]

an exploitation of equation (5.1) together with (4.1) yield the equality

\[\dot{u}_\tau(t) + \int_0^\infty \mu(s) A\eta^k(s) ds = -\tau \sum_{k=1}^\infty \mu_{k-1} A\eta_k^{n+1} + \int_0^\infty \mu(s) A\eta^k(s) ds = \sum_{k=1}^\infty \int_{(k-1)\tau}^{k\tau} \mu(s)(A\eta^k(s) - A\eta_k^{n+1}) ds.\]

Accordingly, the conclusion of Lemma 7.6 is attained if we show that

\[\sum_{k=1}^\infty \int_{(k-1)\tau}^{k\tau} \mu(s)\|A\eta^k(s) - A\eta_k^{n+1}\|_{-2} ds = \sum_{k=1}^\infty \int_{(k-1)\tau}^{k\tau} \mu(s)\|\eta^k(s) - \eta_k^{n+1}\| ds \rightarrow 0\]
as $\tau \to 0$, uniformly with respect to $t$. To this end, we split the series as

$$
\sum_{k=1}^{\infty} \int_{(k-1)\tau}^{k\tau} \mu(s) ||\eta^t_\tau(s) - \eta^{n+1}_k|| ds = f_t(\tau) + g_t(\tau) + h_t(\tau),
$$

where

$$
f_t(\tau) = \sum_{k=1}^{n} \int_{(k-1)\tau}^{k\tau} \mu(s) ||\eta^t_\tau(s) - \eta^{n+1}_k|| ds,
$$

$$
g_t(\tau) = \int_{n\tau}^{(n+1)\tau} \mu(s) ||\eta^t_\tau(s) - \eta^{n+1}_{n+1}|| ds,
$$

$$
h_t(\tau) = \sum_{k=n+2}^{\infty} \int_{(k-1)\tau}^{k\tau} \mu(s) ||\eta^t_\tau(s) - \eta^{n+1}_k|| ds,
$$

and we estimate separately the three terms. In what follows, $C > 0$ will denote a generic constant independent of $t, \tau$, and depending only on the $\mathcal{H}_0$-norm of the initial datum $z = (w, \xi)$ and on the structural quantities of the problem.

**Lemma 8.1.** We have the inequality

$$
f_t(\tau) \leq C\tau.
$$

**Proof.** Let $n > 0$ (otherwise the term $f_t(\tau)$ does not appear), and let $1 \leq k \leq n$ and $s \in \mathcal{J}_{k-1}$ be arbitrarily chosen. Being $s < n\tau \leq t$, from the representation formula (7.1) we learn that

$$
\eta^t_\tau(s) = \int_{t-s}^{t} u_\tau(y) dy = \int_{(n-k)\tau}^{n\tau} u_\tau(y) dy + \int_{n\tau}^{t} u_\tau(y) dy + \int_{t-s}^{(n-k)\tau} u_\tau(y) dy.
$$

As $k \leq n$, the first equation in (5.2) yields

$$
\eta^{n+1}_k = \tau \sum_{j=0}^{k-1} u^n_{n+1-j}.
$$

Combining the two relations above,

$$
\eta^t_\tau(s) - \eta^{n+1}_k = \int_{(n-k)\tau}^{n\tau} u_\tau(y) dy - \tau \sum_{j=0}^{k-1} u^n_{n+1-j} + \int_{n\tau}^{t} u_\tau(y) dy + \int_{t-s}^{(n-k)\tau} u_\tau(y) dy.
$$

Making use of Lemma 7.3 we readily see that

$$
\left\| \int_{n\tau}^{t} u_\tau(y) dy + \int_{t-s}^{(n-k)\tau} u_\tau(y) dy \right\| \leq C\tau.
$$

Moreover, since the first integral on the right-hand side can be explicitly calculated as

$$
\int_{(n-k)\tau}^{n\tau} u_\tau(y) dy = \frac{\tau}{2} \sum_{j=n-k}^{n-1} (u^j + u^{j+1}),
$$

appealing again to Lemma 7.3 we get

$$
\left\| \int_{(n-k)\tau}^{n\tau} u_\tau(y) dy - \tau \sum_{j=0}^{k-1} u^n_{n+1-j} \right\| = \left\| \tau(u^{n-k+1} - u^n) + \frac{\tau}{2} (u^{n-k} - u^n) \right\| \leq C\tau.
$$
We conclude that

\[ f_t(\tau) \leq C\tau \int_0^{\tau} \mu(s) \, ds \leq C\tau, \]

as claimed. \( \square \)

**Lemma 8.2.** We have the inequality

\[ g_t(\tau) \leq C\tau + C \left( \int_0^\tau \mu(s) \, ds \right)^{\frac{1}{2}}. \]

**Proof.** The argument is similar. Invoking (7.1), we find the control

\[ g_t(\tau) \leq \int_0^\tau \mu(s) \left\| u(r) \right\| \, ds + \int_0^{(n+1)\tau} \mu(s) \left\| \int_0^t u(r) \, dr - \eta_{n+1} \right\| \, ds \]

\[ + \int_0^{(n+1)\tau} \mu(s) \left\| \xi(s-t) \right\| \, ds. \]

Exploiting the first equation in (5.2) and arguing as in the proof of Lemma 8.1 (the details are left to the reader), the first two integrals are controlled by

\[ \int_0^\tau \mu(s) \left\| u(r) \right\| \, ds + \int_0^{(n+1)\tau} \mu(s) \left\| \int_0^t u(r) \, dr - \eta_{n+1} \right\| \, ds \leq C\tau. \]

Next, since \( \mu \) is nonincreasing, we infer from the continuous Hölder inequality and the Poincaré inequality (2.1) that

\[ \int_0^{(n+1)\tau} \mu(s) \left\| \xi(s-t) \right\| \, ds \leq \left( \int_0^{(n+1)\tau} \mu(s) \, ds \right)^{\frac{1}{2}} \left( \int_0^{(n+1)\tau} \mu(s) \left\| \xi(s-t) \right\|^2 \, ds \right)^{\frac{1}{2}} \]

\[ \leq \left( \int_0^{\tau} \mu(s) \, ds \right)^{\frac{1}{2}} \left( \int_0^{\infty} \mu(s+t) \left\| \xi(s) \right\|^2 \, ds \right)^{\frac{1}{2}} \]

\[ \leq C \left( \int_0^{\tau} \mu(s) \, ds \right)^{\frac{1}{2}} \left\| \xi \right\|_{M_0}. \]

The proof is complete. \( \square \)

In order to estimate the last term \( h_t(\tau) \), we introduce the further quantity

\[ r_t(\tau) = \frac{1}{\tau} \sum_{k=n+2}^{\infty} \int_{(k-1)\tau}^{k\tau} \mu(s) \left\| \int_0^{(k-n-1)\tau} \left\| \xi(s-t) - \eta_k \right\| ds \right. \]

The following holds.

**Lemma 8.3.** We have the inequality

\[ h_t(\tau) \leq C\tau + r_t(\tau). \]

**Proof.** Making use of (7.1) and the second equation in (5.2), we obtain

\[ h_t(\tau) \leq \sum_{k=n+2}^{\infty} \int_{(k-1)\tau}^{k\tau} \mu(s) \left\| \int_0^t u(r) \, dr - \tau \sum_{j=0}^{n} u^{n+1-j} \right\| \, ds \]

\[ + \sum_{k=n+2}^{\infty} \int_{(k-1)\tau}^{k\tau} \mu(s) \left\| \xi(s-t) - \eta_{k-n-1} \right\| \, ds. \]
Arguing as in the proof of Lemma 8.1 (again, the details are left to the reader), we get
\[
\sum_{k=n+2}^{\infty} \int_{(k-1)\tau}^{k\tau} \mu(s) \left\| \int_0^t u_\tau(y)dy - \tau \sum_{j=0}^n u^{n+1-j} \right\| ds \leq C\tau.
\]

Finally, recalling that \( \eta^0 = \mathcal{D}_\tau(\xi) \), we readily see that the remaining term is less than or equal to \( r_\ell(\tau) \).

Collecting Lemmas 8.1-8.3, we arrive at
\[
\sum_{k=1}^{\infty} \int_{(k-1)\tau}^{k\tau} \mu(s) ||\eta_\tau^k(s) - \eta_\tau^{n+1}|| ds \leq C\tau + C \left( \int_0^\tau \mu(s) ds \right)^{\frac{1}{2}} + r_\ell(\tau).
\]

Since the first two terms above vanish as \( \tau \to 0 \), to complete the proof of Lemma 7.6 we just need to prove the final

**Lemma 8.4.** The limit
\[
\lim_{\tau \to 0} r_\ell(\tau) = 0
\]
holds uniformly with respect to \( t \).

**Proof.** Along the proof, it is convenient to highlight the dependence of \( r_\ell(\tau) \) on the second component \( \xi \) of the initial datum \( z \). Accordingly, we will write \( r_\ell(\tau, \xi) \), and we will denote by \( C_0 \) a generic constant independent of \( \xi \) (as well as of \( t, \tau \)). The proof requires a number of steps.

**Step 1.** We claim that
\[
r_\ell(\tau, \xi) \leq C_0 \| \xi \|_{\mathcal{M}_0}.
\]
To see that, we first estimate
\[
r_\ell(\tau, \xi) \leq \sum_{k=n+2}^{\infty} \int_{(k-1)\tau}^{k\tau} \mu(s) \| \xi(s-t) \| ds
\]
\[
+ \frac{1}{\tau} \sum_{k=n+2}^{\infty} \int_{(k-1)\tau}^{k\tau} \mu(s) \int_{(k-1)\tau}^{(k-2)\tau} \| \xi(\sigma) \| d\sigma ds.
\]
Being \( \mu \) nonincreasing, we readily see that
\[
\sum_{k=n+2}^{\infty} \int_{(k-1)\tau}^{k\tau} \mu(s) \| \xi(s-t) \| ds \leq \sum_{k=n+2}^{\infty} \int_{(k-1)\tau}^{k\tau} \mu(s) \| \xi(s) \| ds \leq C_0 \| \xi \|_{\mathcal{M}_0}.
\]
In addition, for all fixed \( k \geq n+2 \) and \( s \in \mathcal{I}_{k-1} \), exploiting once more the monotonicity of \( \mu \) we get
\[
\int_{(k-1)\tau}^{k\tau} \mu(s) \int_{(k-2)\tau}^{(k-1)\tau} \| \xi(\sigma) \| d\sigma ds \leq \tau \int_{(k-2)\tau}^{(k-1)\tau} \mu(\sigma) \| \xi(\sigma) \| d\sigma.
\]
Hence,
\[
\frac{1}{\tau} \sum_{k=n+2}^{\infty} \int_{(k-1)\tau}^{k\tau} \mu(s) \int_{(k-2)\tau}^{(k-1)\tau} \| \xi(\sigma) \| d\sigma ds \leq C_0 \| \xi \|_{\mathcal{M}_0},
\]
yielding the sought conclusion.

**Step 2.** We claim that¹
\[
r_\ell(\tau, \xi) \leq C_0 \tau \| \xi \|_{\mathcal{M}_0}, \quad \forall \xi \in \mathcal{D}(\mathbb{T}).
\]

¹The (dense) domain \( \mathcal{D}(\mathbb{T}) \) of the operator \( \mathbb{T} \) is defined in Remark 3.2.
To this end, given $k \geq n + 2$, we observe that, if $s \in J_{k-1}$ and $\sigma \in J_{k-n-2}$, then
\[ s \geq \max\{\sigma, s-t\}. \]

Therefore, by the monotonicity of $\mu$,
\[
G_k(s) = \mu(s) \int_{(k-n-1)\tau}^{(k-n-1)\tau} \|\xi(s-t) - \xi(\sigma)\| d\sigma \\
\leq \int_{(k-n-2)\tau}^{(k-n-2)\tau} \int_{\min\{\sigma, s-t\}}^{\max\{\sigma, s-t\}} \mu(\eta) \|\xi'(\eta)\| d\eta d\sigma.
\]

Moreover, it is easily seen that for such $s$ and $\sigma$
\[ \min\{\sigma, s-t\} \geq (k - n - 2)\tau \quad \text{and} \quad \max\{\sigma, s-t\} \leq (k - n)\tau. \]

Accordingly,
\[
G_k(s) \leq \tau \int_{(k-n-2)\tau}^{(k-n)\tau} \mu(\eta) \|\xi'(\eta)\| d\eta.
\]

We conclude that
\[
r_t(\tau, \xi) = \frac{1}{\tau} \sum_{k=n+2}^{\infty} \int_{(k-1)\tau}^{k\tau} G_k(s) ds \leq \tau \sum_{j=0}^{\infty} \int_{j\tau}^{(j+2)\tau} \mu(\eta) \|\xi'(\eta)\| d\eta \leq C_0 \tau \|\xi'\|_{M_0},
\]
as desired.

**Step 3.** Since $\mathcal{D}(\mathbb{T})$ is dense in $M_0$, for every $\varepsilon > 0$ there exists $\xi_\varepsilon \in \mathcal{D}(\mathbb{T})$ such that
\[ \|\xi - \xi_\varepsilon\|_{M_0} \leq \varepsilon. \]

It is also apparent that
\[ r_t(\tau, \xi) \leq r_t(\tau, \xi - \xi_\varepsilon) + r_t(\tau, \xi_\varepsilon). \]

Thus, exploiting the previous steps,
\[ r_t(\tau, \xi) \leq C_0 \varepsilon + C_0 \tau \|\xi'\|_{M_0}. \]

Letting $\tau \to 0$, from the arbitrariness of $\varepsilon > 0$ the conclusion follows.

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