Exact controllability of stochastic differential equations with multiplicative noise

V. Barbu

Al.I. Cuza University and Romanian Academy, Iasi, Romania

L. Tubaro

University of Trento, Italy

Abstract

One proves that the $n$-D stochastic controlled equation $dX + A(t)X dt = \sigma(X) dW + B(t)u dt$, where $\sigma \in \text{Lip}((\mathbb{R}^n, L(\mathbb{R}^d, \mathbb{R}^n)))$, $A(t) \in L(\mathbb{R}^n)$ and $B(t) \in L(\mathbb{R}^m, \mathbb{R}^n)$ is invertible, is exactly controllable with high probability in each $y \in \mathbb{R}^n$, $\sigma(y) = 0$ on each finite interval $(0, T)$. An application to approximate controllability to stochastic heat equation is given. The case where $B \in L(\mathbb{R}^m, \mathbb{R}^n)$, $1 \leq m < n$ and the pair $(A, B)$ satisfies the Kalman rank condition is also studied.

Keywords: stochastic equation, controllability, feedback controller

2010 MSC: 60H10, 60H15, 93B05, 93B52

1. Introduction

Consider the stochastic $n$-D differential equation

$$dX + A(t)X dt = \sigma(X) dW + B(t)u dt, \quad t \geq 0$$

$$X(0) = x \in \mathbb{R}^n,$$  (1)

where $\sigma: \mathbb{R}^n \to L(\mathbb{R}^d, \mathbb{R}^n)$; $A(t) \in L(\mathbb{R}^n)$, $B(t) \in L(\mathbb{R}^m, \mathbb{R}^n)$, $t \in [0, T]$, are assumed to satisfy the following hypotheses

(i) $y \in \mathbb{R}^n$, $\sigma \in \text{Lip}(\mathbb{R}^n, L(\mathbb{R}^d, \mathbb{R}^n))$, $\sigma(y) = 0$.

(ii) $A, B \in C(\mathbb{R}^+; L(\mathbb{R}^n, \mathbb{R}^n))$ and for some $\gamma > 0$

$$B(t)B^\ast(t) \geq \gamma^2 I, \quad \forall t \in [0, \infty).$$  (2)

(iii) $\sigma(X) dW(t) = \sum_{j=1}^d \sigma_j(X) d\beta_j(t)$, $t \geq 0$ where $\{\beta_j\}_{j=1}^d$ is a system of independent Brownian motions in the probability space $\{\Omega, \mathcal{F}, \mathbb{P}\}$.

We denote by $\{\mathcal{F}_t\}_{t \geq 0}$ the filtration corresponding to $\{\beta_j\}_{j=1}^d$ and by $X^u$ the solution to (1).

The problem we address here is the following

Problem 1. Given $x, y \in \mathbb{R}^n$ find an $(\mathcal{F}_t)_{t \geq 0}$-adapted controller $u \in L^2((0, T) \times \Omega; \mathbb{R}^m)$ such that

$$X^u(0) = x, \quad X^u(T) = y.$$  (3)

The main result of this work, Theorem 2.1 below, amounts to saying that, under hypotheses (i)–(iii), Problem 1 has a solution $u^\ast$ in a sense to be made precise later on and moreover the controller $u^\ast$ can be found in a feedback form $u^\ast = \Phi^\ast(X)$.

As regards the literature on exact controllability of equation (1), the works [10] should be primarily cited. In particular, in the recent work [10], it is solved the above exact controllability problem in the special case where $\sigma$ is linear and $B \equiv B(t)$ satisfies the condition (2).

With respect to above mentioned papers the main novelty of this work is the exact controllability of equation via a new controllability approach to (1) by designing a
feedback controller \(u^*\) of relay type which steers with high probability \(x\) in \(y\) in the time \(T\). This constructive approach allowed to solve the controllability problem for control systems \(\textbf{[1]}\) with Lipschitzian volatility term \(\sigma\).

2. The main result

**Theorem 2.1.** Assume that hypotheses (i)–(iii) hold. Let \(x, y \in \mathbb{R}^n\) and \(T > 0\) be arbitrary but fixed. Then, for each \(\rho > 0\), there is an \((\mathcal{F}_t)_{t \geq 0}\)-adapted controller \(u^* \in L^\infty((0,T] \times \Omega; \mathbb{R}^m)\) such that if

\[
\tau = \inf\{t \geq 0 : |X^u(t) - y| = 0\},
\]

we have

\[\mathbb{P}(\tau \leq T) \geq 1 - (\rho y)^{-1} (|y| + (1 - e^{-C^*T})^{-1} |x-y|)\] (5)

for some \(\eta, C^* > 0\) independent of \(\rho, x\) and \(y\). Moreover, the controller \(u^*\) is expressed in the feedback form

\[u^*(t) \in -\rho \text{sign}(B^*(t)(X(t) - y)), \quad t \in (0, T).\] (6)

Here \(\text{sign} : \mathbb{R}^n \to \mathbb{R}^n\) is the multivalued mapping

\[
\text{sign} y = \begin{cases} \frac{y}{|y|} & \text{if } y \neq 0 \\ \{ \theta \in \mathbb{R}^n : |\theta| \leq 1 \} & \text{if } y = 0. \end{cases}
\] (7)

In a few words the idea of the proof is to show that the corresponding closed loop stochastic system

\[dX(t) + A(t)X(t)dt + \rho B(t)\text{sign}(B^*(t)(X(t) - y))dt \geq \sigma(X)dW,\]

\[X(0) = x\] (8)

is well posed that is, it has a unique absolutely continuous solution, and that if \(\tau\) is the stopping time defined by \(\textbf{[1]}\) then \(\textbf{[3]}\) holds. By \(\textbf{[3]} - \textbf{[4]}\) we see that \(u^*\) is a relay controller given by

\[
\begin{cases}
    u^*(t) = -\rho U(X(t)) |U(X(t))| & \text{on } \{(t, \omega) | U(X(t)) \neq 0\} \\
    |u^*(t)| \leq \rho & \text{on } \{(t, \omega) | U(X(t)) = 0\}
\end{cases}
\]

where \(U(X(t)) = B^*(t)(X(t) - y)\). Though \(u^*\) is not explicitly defined on \(G = \{(t, \omega) | U(X(t)) = 0\}\), it is however an \(\mathcal{F}_t\)-adapted controller multivalued process which is uniquely defined on \(G^c\), i.e. the complement of \(G\).

**Theorem 2.1** amounts to saying that under assumptions (i)–(iii), system \(\textbf{[1]}\) is exactly controllable to each \(y \in \sigma^{-1}(0)\) with high probability for \(\rho\) large enough. In particular one has exact null controllability if \(\sigma(0) = 0\).

We shall denote by the same symbol \(|\cdot|\) the norm in the Euclidean spaces \(\mathbb{R}^n\) and \(L(\mathbb{R}^n, \mathbb{R}^m) = \mathbb{R}^{nm}\). For \(n = m\) we simply write \(L(\mathbb{R}^n, \mathbb{R}^n) = L(\mathbb{R}^n)\).

3. Proof of Theorem 2.1

We have

**Proposition 3.1.** Let \(0 < T < \infty\). There is a unique strong solution \(X \in L^2(\Omega; C([0,T]; \mathcal{L}(\mathbb{R}^n)))\) to \(\textbf{[8]}\). More precisely, there are \(X \in L^2(\Omega; C([0,T]; \mathcal{L}(\mathbb{R}^n)))\) and an \((\mathcal{F}_t)_{t \geq 0}\)-adapted process \(\xi : [0, T] \to \mathcal{L}(\mathbb{R}^n)\) such that \(\xi \in L^\infty((0,T) \times \Omega; \mathcal{L}(\mathbb{R}^n))\) and

\[\xi(t) \in \text{B}(t)(\text{sign}(B^*(t)(X(t) - y))), \quad \text{a.e. in } (0,T) \times \Omega\] (9)

\[dX(t) + A(t)X(t)dt + \rho \xi(t)dt = \sigma(X)dW,\]

\[X(0) = x\] (10)

We shall prove Proposition \(\textbf{3.1}\) at the end of this section and now we use it to prove Theorem \(\textbf{2.1}\). The proof is based on some extinction type arguments already developed in a different context in \(\textbf{[2]}\) and \(\textbf{[3]}\), pag. 68]. (In the following we shall write \(A\) instead of \(A(t)\).)

Let \(\varphi \in C^2(\mathbb{R}^+)\) be such that \(\varphi_r(x) = \frac{x}{r}\) for \(0 \leq r \leq \epsilon, \varphi_r(x) = 1 + \epsilon\) for \(r \geq 2\epsilon\) and \(|\varphi''(r)| \leq \frac{C}{\epsilon}, \forall r \in \mathbb{R}^+\). We set \(\Phi_r(x) = \varphi_r(|X|), \forall X \in \mathbb{R}^n\). We have \(\nabla \Phi_r(x) = \varphi''(|X|) \text{sign} X, \nabla^2 \Phi_r(x) = 0\) for \(|X| \leq \epsilon\) and \(|X| \geq 2\epsilon, |\nabla^2 \Phi_r(x)| \leq \frac{C}{\epsilon}\). We apply Itô’s formula in \(\textbf{[10]}\) to func-
tion \( t \to \Phi_{t}(X(t) - y) \). We get
\[
\begin{align*}
    & d\Phi_{t}(X(t) - y) \\
    & + \langle A(X(t)) - A(y), \nabla\Phi_{t}(X(t) - y) \rangle dt \\
    & + \rho \langle \xi(t), B^{*}(t)\nabla\Phi_{t}(X(t) - y) \rangle dt \\
    & + \frac{1}{2} \sum_{i,j=1}^{n} \alpha_{ij}(\nabla^{2}\Phi_{t}(X(t) - y)_{ij}) dt \\
    & + \langle \sigma(X(t)) dW, \nabla\Phi_{t}(X(t) - y) \rangle
\end{align*}
\]
where \( \alpha_{ij} = \sum_{l=1}^{d} \sigma_{il}\sigma_{lj} \). We note that for \( \varepsilon \to 0 \)
\( \Phi_{t}(X(t) - y) \to |X(t) - y|, \nabla\Phi_{t}(X(t) - y) \to \eta(t) \in \text{sign}(X(t) - y), |\nabla\Phi_{t}(X(t) - y)| \leq 1 + \varepsilon, \) and because \( \sigma(y) = 0 \) we have also \( |\alpha_{ij}(\nabla^{2}\Phi_{t}(X(t) - y)_{ij})| \leq C_{2} \varepsilon \) for all \( t \geq 0 \).

On the other hand, by (2) it follows that there is \( \gamma > 0 \) such that
\[
|B^{*}(t)(X(t) - y)| \geq \gamma |X(t) - y|
\]  
(11)
We note also that
\[
|\alpha_{ij}(t)| \leq C_{2} |X(t) - y|^{2}.
\]
Integrating on \( (s, t) \subset (0, \infty) \) we get
\[
\Phi_{t}(X(t) - y) + \rho \int_{s}^{t} \langle \xi(r), B^{*}(t)\nabla\Phi_{t}(X(r) - y) \rangle dr \\
\Phi_{s}(X(s) - y) + |A|(1 + \varepsilon) \int_{s}^{t} (|y| + |X(r) - y|) dr \\
+ C_{2} \varepsilon + \int_{s}^{t} (\sigma(X(r)) dW, \nabla\Phi_{t}(X(r) - y)).
\]
Taking into account that
\[
B^{*}(t) \nabla\Phi_{t}(X(r) - y) \to B^{*}(t)\eta(r),
\]
with \( \eta(r) \in \text{sign}(X(r) - y) \) and that
\[
\langle \xi(r), B^{*}(t)\eta(r) \rangle = |B^{*}(t)(X(r) - y)| |X(r) - y|^{-1} \mathbb{1}_{|X(r) - y| \neq 0},
\]
by (11) we get for \( \varepsilon \to 0 \)
\[
|X(t) - y| + \rho \gamma \int_{s}^{t} \mathbb{1}_{|X(r) - y| \neq 0} dr
\leq |X(s) - y| + C^{*} \int_{s}^{t} |X(r) - y| dr + C^{*}(t - s) |y| \\
+ \int_{s}^{t} (\sigma(X(r)) dW, \text{sign}(X(r) - y))).
\]
where \( C^{*} \) is independent of \( x, y \) and \( \rho \). Hence
\[
e^{-C^{*} t}|X(t) - y| + \rho \gamma \int_{s}^{t} e^{-C^{*} r} 1_{|X(r) - y| \neq 0} dr
\leq e^{-C^{*} s}|X(s) - y| + (1 - e^{-C^{*}(t-s)}) |y|
+ \int_{s}^{t} e^{-C^{*} r} (|\sigma(X(r)) dW|, \text{sign}(X(r) - y))), \quad 0 \leq s \leq t < \infty
\]  
(12)
In particular, (12) implies that the process
\[
t \to e^{-C^{*} t}|X(t) - y|
\]
is a \( \mathcal{F}_{t} \geq 0 \)-supermartingale that is,
\[
\mathbb{E}(e^{-C^{*} t}|X(t) - y| | \mathcal{F}_{s}) \leq e^{-C^{*} s}|X(s) - y|, \quad \forall t \geq s.
\]
This yields \( |X(t) - y| = 0, \forall t \geq \tau \), where \( \tau \) is defined by (1).

If take expectation \( \mathbb{E} \) in (12), we obtain, for \( s = 0 \),
\[
e^{-C^{*} t}|X(t) - y| + \rho \gamma \int_{0}^{t} e^{-C^{*} r} \mathbb{P}(\tau > r) dr
\leq |x - y| + (1 - e^{-C^{*} t}) |y|.
\]
Hence, for \( t = T \) we get
\[
\mathbb{P}(\tau > T)
\leq C^{*} \rho \gamma ((1 - e^{-C^{*} T})^{-1}|x - y| + |y|)
\]  
(13)

Proof of Proposition [4]

Let \( F_{\lambda}(t) \in C([0, T]; \mathbb{R}^{n}) \) be the Yosida approximation of \( F(t, X) = \rho B(t)\text{sign}(B^{*}(t)(X - y)) \), that is (see [1, pag. 97])
\[
F_{\lambda}(t) = \frac{1}{\lambda} (I - (I + \lambda F(t))^{-1}), \quad \lambda > 0
\]  
(14)
We note that the operator \( F(t) \) is \( m \)-accretive in the space \( \mathbb{R}^{n} \times \mathbb{R}^{n} \). Since the \( F_{\lambda}(t) \) are Lipschitz for \( t \in [0, T] \), the equation
\[
dX_{\lambda} + A(t) X_{\lambda} dt + F_{\lambda}(t, X_{\lambda}) dt = \sigma(X_{\lambda}) dW
\]  
(15)
has for each \( T > 0 \) a unique solution
\[
X_{\lambda} \in L^{2}(\Omega; C([0, T]; \mathbb{R}^{n})).
\]
Taking into account that for each \( \lambda > 0 \)
\[
F_{\lambda}(t, X) \in F(t, I + \lambda F(t))^{-1} X, \quad \forall X \in \mathbb{R}^{n}
\]  
(16)
\[
|F_{\lambda}(t, X)| \leq C \rho, \quad \forall X \in \mathbb{R}^{n}, \quad \lambda > 0
\]  
(17)
and that $X \to F(t, X)$ is monotone in $\mathbb{R}^n$, we get, via
the Burkholder-Gundy-Davis inequality, the estimate
\[
\mathbb{E} \sup_{t \in [0, T]} |X_\lambda(t)|^2 \leq C, \forall \lambda > 0
\]
and
\[
\mathbb{E} \sup_{t \in [0, T]} |X_\lambda(t) - X_\mu(t)| \leq C \mathbb{E} \int_0^t (\lambda F_\lambda(r, X_\lambda(r))^2 + \mu |F_\mu(r, X_\mu(r))|^2) \, dr
\leq C (\lambda + \mu), \forall \lambda, \mu > 0.
\]
Hence, there is
\[
X = \lim_{\lambda \to 0} X_\lambda \text{ in } L^2(\Omega; C([0, T]; \mathbb{R}^n))
\] (18)
and
\[
\xi = \lim_{\lambda \to 0} F_\lambda(t, x_\lambda) \text{ in } L^\infty((0, T) \times \Omega; \mathbb{R}^n)
\] (19)
Since by (11) and (15)
\[
(I + \lambda F(t))^{-1} X_\lambda(t) \to X(t) \text{ in } L^2(\Omega; C([0, T]; \mathbb{R}^n)),
\]
for $\lambda \to 0$, it follows by (10), (17) and the maximal monotonicity of $F(t): \mathbb{R}^n \to \mathbb{R}^n$ that
\[
\xi(t) \in F(t, x(t)), \text{ a.e. in } (0, T) \times \Omega.
\]
Hence $X$ is a solution to (9)-(10) as claimed. The uniqueness is immediate by monotonicity of the mapping $F(t)$ but we omit the details.

4. The case of linear multiplicative noise

Consider here the equation
\[
dX + A(t)X \, dt = \sum_{i=1}^d \sigma_i(X) \, d\beta_i + B(t) u(t) \, dt
\]
(20)
with the final target $X(T) = y$, where $B(t)$ satisfies assumption (ii) and $\sigma_i \in L(\mathbb{R}^n)$.

Let $\Gamma \in C([0, T]; L(\mathbb{R}^n))$ be the solution to equation
\[
d\Gamma(t) = \sum_{i=1}^d \sigma_i \Gamma(t) \, d\beta_i, \quad t \geq 0, \quad \Gamma(0) = I.
\]
(21)
By the substitution $X(t) = \Gamma(t)y(t)$ one transforms via
Itô’s formula equation (20) into random differential equation
\[
\frac{dy}{dt}(t) + \Gamma^{-1}(t)A(t)\Gamma(t) y(t) = \Gamma^{-1}(t)B(t)u(t).
\]
(22)
In (22) we take $u$ the feedback controller
\[
u(t) = -\hat{\rho} \text{sign} \left( (B(t)\Gamma^{-1}(t))^\ast(y(t) - y_T) \right), \quad t \geq 0
\]
(23)
where $y_T = \Gamma^{-1}(T)X_T$. Arguing as in the proof of Proposition 3.1 it follows that (22) has (for each $\omega \in \Omega$) unique absolutely continuous solution $y$ with $\frac{dy}{dt} \in L^2(0, T; \mathbb{R}^n)$.

We note that if $y$ is an $(\mathcal{F}_t)_{t \geq 0}$-adapted solution to (22)-(23) then $X = \Gamma(t)y(t)$ is the solution to closed loop system (20) with feedback control
\[
u(t) = -\hat{\rho} \text{sign} \left( (B(t)\Gamma^{-1}(t))^\ast(\Gamma^{-1}(t)X(t) - \Gamma(t)\Gamma^{-1}(T)X_T) \right).
\]
(24)
We have

Theorem 4.1. Let $T > 0$, $x \in \mathbb{R}^n$ and $X_T \in \mathcal{F}_T \cap L^2(\Omega)$ be arbitrary but fixed. Then there is $\hat{\rho} \in \mathcal{F}_T \cap L^2(\Omega)$ such that the feedback controller (23) steers $x$ in $y_T$, in time $T$, with probability one.

Proof. If multiply equations (22)-(23) by $y(t) - y_T$ we get by (2) that
\[
\frac{1}{2} \frac{d}{dt} |y(t) - y_T|^2 + \hat{\rho} \gamma C_1^\ast |y(t) - y_T| \leq C_2^\ast (|y(t) - y_T| + |y_T|) |y(t) - y_T|,
\]
(25)
a.e. $t \in (0, T)$, where
\[
(C_1^\ast)^{-1} = \sup\{||\Gamma(t)||^{-1}|_{L(\mathbb{R}^n)}: t \in [0, T]\}, \quad C_2^\ast = \sup\{||\Gamma^{-1}(t)A(t)\Gamma(t)||_{L(\mathbb{R}^n)}: t \in [0, T]\}.
\]
By (25) it follows that if $\hat{\rho} \gamma C_1^\ast > C_2^\ast|y_T|$ then the function
\[
t \to e^{-C_2^\ast t} |y(t) - y_T| + (\hat{\rho} \gamma C_1^\ast - C_2^\ast|y_T|)(C_2^\ast)^{-1}(1 - e^{-C_2^\ast t})
\]
is monotonically decreasing and so $y(T) - y_T = 0$ if $\hat{\rho}$ is taken in such a way that
\[
(\hat{\rho} \gamma C_1^\ast - C_2^\ast|y_T|)(C_2^\ast)^{-1}(1 - e^{-C_2^\ast T}) \geq |x - y_T|.
\]
Then Theorem 3.1 follows for
\[
\hat{\rho} = (\gamma C_1^\ast)^{-1}(\gamma C_1^\ast|y_T| + |x - y_T|).
\]
It should be noted that since \( \hat{\rho} \) is not \( F_0 \)-measurable, the solution \( y \) to system (22)–(25) is not \((\mathcal{F}_t)_{t \geq 0}\) adapted and so it is not equivalent with (20)–(22). This happens however if \( A(t) \) and \( B(t) \) commute with \( \sigma_i \), because in this case \( C^*_i \), \( i = 1, 2 \) are deterministic and so can be chose \( \hat{\rho} \).

In general it follows for system (20)–(25) with \( \hat{\rho} = \rho \) and \( y_T \) deterministic, a result similar to that in Theorem 2.1.

Namely, by (25) it follows as above (see (12)) that

\[
\mathbb{E}(e^{-C^*_T t}|y(t) - y_T]) + \rho \gamma \mathbb{E} \int_0^t e^{-C^*_T r} \mathbb{P}(r > \tau) \, dr \leq \mathbb{E}|x - y_T| + (1 - e^{-C^*_T T}) |y_T|
\]

and therefore \( \mathbb{P}(\tau > T) \leq 1 - (\rho \gamma)^{-1}(|x - y_T| + |y_T|) \) for some \( \eta > 0 \).

**Remark 4.2.** Clearly Theorem 1.1 extends to Lipschitzian mappings \( A(t): \mathbb{R}^n \to \mathbb{R}^n \).

Consider now system (1) where \( A \in \mathcal{L}(\mathbb{R}^n) \), \( B \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^n) \), \( 1 \leq m < n \) is time dependent and is satisfied the Kalman rank condition

\[
\text{rank}||B, AB, \ldots, A^{n-1}B|| = n
\]

Assume also that \( d = 1 \), \( \sigma_1 = \sigma \) and

\[
\sigma^k = \sigma, \quad \forall k \geq 2
\]

\[
\sigma(\mathbb{R}^d) \subset B(\mathbb{R}^m)
\]

for some \( \sigma \in \mathbb{R} \).

We have

**Theorem 4.3.** Let \( T > 0 \) and \( x \in \mathbb{R}^n \) be arbitrary but fixed. Then under hypotheses (27)–(29) there is an \((\mathcal{F}_t)_{t \geq 0}\)-adapted controller \( u \in L^2((0, T) \times \Omega; \mathbb{R}^n) \) which steers \( x \) in origin, in time \( T \), with probability one.

**Proof.** By the transformation \( X(t) = \Gamma(t)y(t) \) one reduces (1) to the random system (22), that is

\[
\frac{dy}{dt}(t) + \exp(-\beta(t) \sigma + \eta \frac{1}{2} \sigma^2) A \exp(\beta(t) \sigma - \eta \frac{1}{2} \sigma^2) = B u(t)
\]

\[
y(0) = x
\]

because \( \Gamma(t) = \exp(\beta(t) \sigma - \eta \frac{1}{2} \sigma^2) \).

Taking into account hypothesis (27), we can rewrite (29) as

\[
\frac{dy}{dt} + A y = B u - \sigma D(t) y + \sigma D_1(t) u
\]

\[
y(0) = x.
\]

where \( D_1(t) = \sum_{k=1}^{\infty} \frac{1}{k!} (\beta(t) - \frac{m a^2}{2})^k \sigma^{k-1} \) and

\[
D(t) = \sum_{k=1}^{\infty} \frac{1}{k!} (-\beta(t) + \frac{1}{2} t a^2)^k \sigma^{k-1} A \sum_{k=1}^{\infty} \frac{1}{k!} (\beta(t) - \frac{1}{2} t a^2) \sigma^k
\]

Now by Kalman’s condition (20), we know that there is a deterministic controller \( \bar{u} \in L^2(0, T; \mathbb{R}^m) \) such that

\[
\frac{d\bar{y}}{dt} + A \bar{y} = B \bar{u}, \quad t \in (0, T)
\]

\[
\bar{y}(0) = x, \quad \bar{y}(T) = 0.
\]

Since \( B^{-1} \in (B(\mathbb{R}^m), \mathbb{R}^n) \), it follows by (29) that

\[
\frac{d\bar{y}}{dt} + \Gamma^{-1} A \Gamma \bar{y} = \Gamma^{-1} B \bar{u}(t) + \sigma D_1(t) \bar{y}(t) + \sigma D_1(t) \bar{\bar{u}}
\]

\[
\Gamma^{-1} B \bar{u}(t) + B^{-1} (\sigma D(t) \bar{y}(t) + \sigma D_1(t) \bar{\bar{u}}) = \Gamma^{-1} B \bar{u}(t)
\]

\[
\bar{y}(0) = x, \quad \bar{y}(T) = 0.
\]

This means that \((\bar{y}, u = \bar{u}(t) + B^{-1}(\sigma D(t) \bar{y}(t) + D_1(t) \bar{\bar{u}}))\) satisfies system (30) and \( \bar{y}(T) = 0 \). The controller \( u \) is obviously \((\mathcal{F}_t)_{t \geq 0}\)-adapted and so \((X(t) = \Gamma^{-1}(t)\bar{y}(t), u(t))\) satisfies system (11) and \( X(T) = 0 \) \( \mathcal{P} \)-a.s.

**Remark 4.4.** One might suspect that the controller \( u \) steering \( x \) in origin can be found in feedback form but this problem is open.

**5. An example**

Consider the controlled \( n \)-order stochastic differential equation

\[
X^{(n)}(t) + \sum_{i=1}^{n} a_i X^{(i-1)}(t)
\]

\[
= \sigma_0(X, X', \ldots, X^{(n-1)}) W + u(t)
\]

\[
\{X^{(k)}(0)\}_{k=0}^{n-1} = x \in \mathbb{R}^n
\]

where \( a_i \in \mathbb{R}, \sigma_0(x_1, x_2, \ldots, x_n) = \sum_{i=1}^{n} b_i x_i \) and \( W \) is a Wiener process in 1-D.
A typical example is the stochastic harmonic oscillator
\[ \dot{X} + aX \, dt + bX \, dt = \sigma_0 \, W \]
\[ X(0) = X_0, \quad \dot{X}(0) = X_1. \]
Equation (31) is viewed as the stochastic differential system
\[ dX + A \, X \, dt = B \, u \, dt + \sigma(X) \, dW \]
where \( X = (X_t)_{t=1}^N, X_t = X^{(i-1)}, X(0) = x, \)
\[ \sigma = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}, \]
\[ A = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ a_1 & a_2 & a_3 & \cdots & a_n \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}. \]
Clearly assumptions (26)-(28) hold and so by Theorem 4.3 it follows that, for each \( t \in \mathbb{R}^n \), there is an \((\mathcal{F}_t)_{t \geq 0}\)-adapted feedback controller \( u^*(t) \) such that \( X^{(i-1)}(T) = 0 \) for \( i = 1, 2, \ldots, n - 1 \).

6. Approximate controllability of stochastic heat equation

Consider the stochastic equation
\[ dX - \Delta X \, dt = \sum_{j=1}^d X e_j \, d\beta_j + \mathbb{I}_{\Omega_0} \, u \, dt, \]
\[ (t, \xi) \in (0, T) \times \Omega \]
\[ X(0, \xi) = x(\xi), \quad \xi \in \Omega \]
\[ X(t, \xi) = 0, \quad \forall (t, \xi) \in (0, T) \times \partial \Omega. \] (32)
Here \( d \geq 1, \Omega \subset \mathbb{R}^n \) is a bounded and open domain with smooth boundary \( \partial \Omega \), \( \Omega_0 \) is an open subset of \( \Omega \) and \( \{e_j\}_{j=1}^d \) is an orthonormal base in \( L^2(\Omega) \), given by
\[ -\Delta e_j = \lambda_j e_j \quad \text{in} \quad \Omega, \quad e_j = 0 \quad \text{on} \quad \partial \Omega. \]
The controller \( u: (0, \infty) \rightarrow L^2(\Omega) \) is an \((\mathcal{F}_t)_{t \geq 0}\)-adapted process.

We set \( X = \sum_{i=1}^N X_i \, e_i, \quad \tilde{u} = \sum_{i=1}^N u_i \, e_i \) and approximate (32) by the \( N \)-D differential equation
\[ dX - A \, X \, dt = \sum_{j=1}^d \sigma_j(X) \, d\beta_j + B \, u \, dt, \]
\[ X(0) = 0 \]
where
\[ X = \{X_i\}_{i=1}^N, \quad u = \{u_i\}_{i=1}^N, \]
\[ B = \left( \int_{\Omega_0} e_i \, e_j \, d\xi \right)_{i,j=1}^N, \]
\[ A = \text{diag}(\lambda_i)_{i=1}^N, \quad \sigma_j(X) = \left( \sum_{k=1}^N (e_k \, e_j)_2 X_k^N \right)_{i=1}^N, \]
\[ (\cdot, \cdot)_2 \] is the scalar product in \( L^2(\Omega) \).

By the unique continuation property of eigenfunctions \( e_j \), it follows that \( \text{det} B = 0 \), which implies (33). Then, by Theorem 4.3 for each \( N \in \mathbb{N} \), equation (32) is exactly controllable on \([0, T]\) in the sense of (41)-(42). Taking into account that,
\[ |x - \sum_{i=1}^N (x, e_i)_2 e_i|_2 \rightarrow 0 \quad \text{as} \quad N \rightarrow \infty, \]
\[ E(\sup\{|X(t)|^2, \quad t \in [0, T]}) \rightarrow 0 \quad \text{as} \quad N \rightarrow \infty, \] we get the following controllability result.

**Theorem 6.1.** Let \( x \in L^2(\Omega) \) and \( T > 0 \) be arbitrary but fixed. Then for each \( \varepsilon > 0 \) there is an \((\mathcal{F}_t)_{t \geq 0}\)-adapted controller \( u_\varepsilon \in L^2((0, T) \times \Omega; L^2(\Omega)) \) such that
\[ P(|X^{u_\varepsilon}(t)|^2 < \varepsilon, \quad \forall t \geq T) \rightarrow 1 - \varepsilon. \] (34)

**Remark 6.2.** In 1-D a similar result was established by a different method in [8]. It turns out (see [4]) that, under the above assumptions, there is an \((\mathcal{F}_t)_{t \geq 0}\)-adapted controller \( u \) which steers \( x \) into a linear subspace of \( L^2(\Omega; \mathcal{O}) \). However, it remains an open problem the exact null controllability. (For other partial results to exact null controllability, see [3], [10].)

7. Conclusion

Under hypotheses (i)-(iii), the stochastic differential equation (1) is exactly controllable to any \( y \in \sigma^{-1}(0) \).
by a stochastic feedback controller \( u \) which is explicitly designed. In the special case of stochastic equations with linear multiplicative noise the controllability set \( \{ y = X^u(T) \} \) is all \( \mathbb{R}^n \). Moreover if the pair \((A, B)\) satisfies the Kalman rank condition and \( \sigma(R^d) \subset B(R^m) \) then the system (1) is exactly null controllable. As application the approximate controllability of stochastic heat equation with multiplicative Wiener noise was given.

**Acknowledgments**

The authors thank the Mathematics Department of University of Trento for the financial support. V. Barbu was supported by the grant of Romanian Ministry of Research and Innovation CNCS-UEFISCDI, DN-III-D4-DCE-2016-0011

**References**

[1] V. Barbu, *Nonlinear Differential Equations of Monotone Type in Banach Spaces*, Springer 2010

[2] V. Barbu, S. Bonaccorsi, L. Tubaro, Stochastic differential equations with variable structure driven by multiplicative Gaussian noise and sliding mode dynamic, *Math. Control Signals Systems* 28 (2016), no. 3, Art. 26, 28 pp.

[3] V. Barbu, G. da Prato, M. Röckner, *Stochastic Porous Media Equations*, Lecture Notes in Mathematics 2163, Springer, 2016

[4] V. Barbu, A. Rascanu, G. Tessitore, Carleman estimates and controllability of stochastic heat equations with multiplicative noise, *Appl. Math. Optim.*, 5 (2003), 1-20.

[5] M. Erhardt, W. Kliemann, Controllability of linear stochastic systems, *Systems & Control Letters* 2 (1982/83), 145-153

[6] D. Goreac, A Kalman type condition for stochastic approximate controllability, *C.R. Math. Acad. Sci. Paris* 346 (2008), 183-188

[7] F. Liu, S. Peng, On controllability for stochastic control systems when the coefficient is time invariant, *J. Systems Sci. Complex* 23 (2010), 270-278

[8] Q. Lü, Some results on the controllability of forward stochastic heat equations with control on the drift, *J. Funct. Anal.*, 260 (2011), 832-851.

[9] S. Tang, X. Zhang, Null controllability for forward and backward stochastic parabolic equations, *SIAM J. Control Opt.*, 48 (2009), 2191-2216.

[10] Y. Wang, D. Yang, J. Yong, Z. Yu, Exact controllability of linear stochastic differential equations and related problems, *Mathematical Control and Related Fields*, 7 (2017), 305-345