On the discrete Safronov–Dubovskii coagulation equations: Well-posedness, mass conservation, and asymptotic behavior

Mashkoor Ali1,2 | Pooja Rai2 | Ankik Kumar Giri2

1Jindal Global Business School, O. P. Jindal Global University, Sonipat, India
2Department of Mathematics, Indian Institute of Technology Roorkee, Roorkee, India

Correspondence
Mashkoor Ali, Jindal Global Business School, O. P. Jindal Global University, Sonipat 131001, Haryana, India.
Email: mashkoor.ali@jgu.edu.in

Funding information
MA would like to thank the University Grant Commission (UGC), India, for granting the Ph.D. fellowship through grant no. 416611.

1 | INTRODUCTION

The discrete coagulation equation is an infinite set of ordinary differential equations for the dynamics of cluster growth that describes the mechanism allowing clusters to undergo coagulation as the only event. We restrict ourselves to binary coagulation, which means two clusters combine to form a bigger one. It is assumed that the clusters are fully identified by their size/volume (a positive integer). A cluster of size \( i \) (or \( i \)-cluster) is made of \( i \) identical elementary particles known as monomers. The discrete coagulation equation describes the evolution of the concentration \( \xi_i(t) \), \( i \in \mathbb{N}/\{0\} \) of clusters of size \( i \) (or \( i \)-mers) at time \( t \geq 0 \) and can be written as the nonlinear nonlocal equation of the form

\[
\frac{d\xi_i}{dt} = \frac{1}{2} \sum_{j=1}^{i-1} \gamma_{i,j-1} \xi_{i-j} - \sum_{j=1}^{\infty} \gamma_{i,j} \xi_j,
\]

\[
\xi_i(0) = \xi_i^{in} \geq 0,
\]

for \( i \geq 1 \). Here \( \gamma_{i,j} \) is the coagulation rate at which clusters of size \( i \) merge with the clusters of size \( j \) to form the larger clusters. It is assumed that \( \gamma_{i,j} \) is nonnegative and symmetric, that is, \( 0 \leq \gamma_{i,j} = \gamma_{j,i} \forall i, j \geq 1 \). The first term on the right-hand side of (1.1) accounts for the formation of \( i \)-clusters by binary coalescence of smaller ones, while the second term accounts for their depletion through coagulation with other clusters. In earlier studies [1, 2], Smoluchowski initially introduced a system of mathematical equations of the form (1.1) and (1.2) which is later referred to as Smoluchowski coagulation equation describing the coagulation of colloids moving in a Brownian motion. The system of equations given by (1.1) and (1.2) has been extensively studied in the presence of the fragmentation term, using two different techniques. If the primary focus is on the effects of strong coagulation, such as gelation, it becomes necessary to impose assumptions...
that ensure the coagulation term dominates over other processes. In such cases, the analysis requires the use of weak compactness arguments and working with weak solutions, as demonstrated in previous works [3–8]. Alternatively, if the process is driven by the linear fragmentation part, it allows for more flexibility in selecting the fragmentation and transport parts, resulting in strong, classical solutions obtained within the framework of the semigroup theory. This has been shown in previous works [9–11]. This discrete system and its continuous counterpart have received considerable attention in the mathematics and physics literature in recent years; due to the enormous number of works devoted to them, we refer to the classical review [12] and the more recent [13] for an overview of the topic.

In this article, we are mainly concerned with coagulation models with disperse systems, and the application of such models can be found in astrophysics (formation of stars and planets), chemistry (reacting polymers), meteorology (formation of clouds), and physics (growth of gas bubbles in solids). Dubovski [14] looked at a dispersed system and introduced a model known as the Safronov–Dubovski coagulation model, in which only binary collisions between particles can happen simultaneously, and the mass of each particle is also assumed to be proportional to some \( m_0 \). Collisions between particles of mass \( im_0 \) and \( jm_0 \) cause particles to grow in the system. Particles with mass \( im_0 \) will be referred to as \( i \)-mers, with \( m_0 \) being the mass of the smallest particle in the system. In this model, a collision between an \( i \)-mer and a \( j \)-mer causes the \( j \)-mer to split into \( j \) monomers if \( j \leq i \). Hence, we have another characterization of the coagulation process that leads to the balanced equation, also known as the discrete Safronov–Dubovski coagulation equations (DSDCE) is of the form

\[
\frac{d\xi_i(t)}{dt} = \xi_{i-1}(t) \sum_{j=1}^{i-1} j\gamma_{i-1,j} \xi_j(t) - \xi_i(t) \sum_{j=1}^{i-1} j\gamma_{i,j} \xi_j(t) - \xi_i(t) \sum_{j=i}^{\infty} \gamma_{i,j} \xi_j(t) \xi_j(t),
\]

\[
\xi_i(0) = \xi_{i}^{\text{in}} \geq 0,
\]

for \( i \geq 1 \). Note that Equations (1.3)–(1.4) are a nonlinear initial value problem that describes the dynamics of evolution of the concentration \( \xi_i(t), i \in \mathbb{N} \) of clusters of size \( i \) at time \( t \geq 0 \). The coagulation kernel, \( \gamma_{i,j} \) (with \( i \neq j \)), specifies the rate at which \( i \)-mers collide with \( j \)-mers. The first sum in (1.3) describes the \( i \)-mer introduction into the system as a result of collisions between \((i-1)\)-mers and monomers produced from fragmented \( j \)-mers. If \( i = 1 \), the initial sum is zero. The second sum represents the loss (or decay) of \( i \)-mers due to monomer merging. The first and second terms are multiplied by \( j \) to demonstrate that the collision involves exactly \( j \) monomers. The third sum represents the decay of \( i \)-mers due to fragmentation caused by collisions with bigger particles.

Dubovski [15] obtained the coagulation model DSDCE (1.3) and (1.4) and from the continuous model originally proposed by Oort and Hulst [16] and later reformulated by Safronov [17], which is given as

\[
\frac{\partial f}{\partial t} = -\frac{\partial}{\partial x} \left[ f(x,t) \int_{a(x)}^{\infty} y a(x,y) f(y,t) dy \right] - f(x,t) \int_{x}^{\infty} a(x,y) f(y,t) dy.
\]

The distribution function \( f(x,t) \) denotes the distribution of particle of size \( x \in (0, \infty) \) at time \( t \geq 0 \) and the coagulation kernel \( a(x,y) \), satisfying \( a(x,y) = a(y,x) \geq 0, \forall x,y \in (0, \infty)^2 \), determines the rate at which particles of size \( x \) and \( y \) coalesce. Using the classical weak-L^1 compactness technique, the existence of weak solutions to (1.5), with suitable initial data, has been shown in earlier studies [18, 19], whereas the self-similar solutions have been discussed in previous works [20–22].

Coming back to Equations (1.3) and (1.4), several results are available dealing with the solutions’ existence, uniqueness, and mass conservation property. Existence of weak solutions to (1.3) and (1.4) has been discussed in Bagland [23], when the coagulation kernels satisfy \( \lim_{j \to \infty} \frac{\gamma_{i,j}}{j^2} = 0, i,j \geq 1 \) and \( \xi_{i}^{\text{in}} \in L_1 \). Also, a connection between (1.3) and (1.5) has been established by a suitable sequence of solutions to (1.3). In Davidson [24], the global existence of the classical solutions was demonstrated when the coagulation kernel satisfies \( j\gamma_{i,j} \leq M, j \leq i \) and for the kernel of the form \( \gamma_{i,j} \leq C_i h_1 h_j \) with an additional assumption \( \frac{h_j}{j} \to 0 \). Furthermore, the mass conservation property for \( \gamma_{i,j} \leq C_i h_1 h_j \) for \( h_1 \leq 1 \) and uniqueness for bounded kernels, that is, \( \gamma_{i,j} \leq C_i, i,j \geq 1 \) have been discussed. Recently, in Das and Saha [25], global existence and mass conservation property of solutions have been established for \( \gamma_{i,j} \leq (1+i+j)^{\alpha} \) where \( \alpha \in [0,1] \), whereas uniqueness of solutions is shown under the condition that \( \gamma_{i,j} \leq C_i \alpha^k \) for \( i \leq j, k \leq 2 \). Kaushik and Kumar [26] establish the existence, uniqueness, and mass conservation of (1.3) and (1.4) when dealing with an unbounded kernel in the form of \( \min\{i,j\} \gamma_{i,j} \leq (i+j) \) for all \( i,j \geq 1 \) in the weighted \( l_1 \) space. In the same space, recently, in Ali and Giri [27],
the existence of (1.3) and (1.4) is proved for the coagulation coefficients of a multiplicative type, which are defined as follows:

\[ \gamma_{i,j} = \theta_i \theta_j + \kappa_{i,j}. \]

Moreover, these coefficients satisfy the following conditions:

\[ \inf_{i \geq 1} \frac{\theta_i}{i} = B > 0, \quad \text{and} \quad \kappa_{i,j} \leq A \theta_i \theta_j \quad \text{for each} \quad i, j \geq 1 \quad (A \geq 0). \]

Next, we define the moments of the concentration \( \xi = (\xi_i(t))_{i \geq 1} \) of order \( m \geq 0 \) as

\[ M_m(\xi(t)) := M_m(t) = \sum_{i=1}^{\infty} i^m \xi_i(t), \quad (1.6) \]

where the zeroth \((m = 0)\) and the first \((m = 1)\) moments denote the total number of particles and total mass of particles in the system, respectively. Observe that since particles are neither created nor destroyed in the reactions described by (1.1) and (1.3), the total mass is expected to be conserved through the time evolution. Because the DSDCE (1.3) and (1.4) only account for coagulation processes, the total number of particles (which is nothing but the \( l_1 \)-norm of \( \xi = (\xi_i)_{i \geq 1} \)) is supposed to decrease to zero as time increases to infinity which is shown in Section 6. The current article improves on the results obtained in Das and Saha [25], as several flaws were identified and stated below. The proof of the local existence theorem demonstrated that the truncated mass conservation law holds, which is incorrect. Furthermore, the finiteness of the second moment of initial data has been employed to prove the existence and uniqueness of the solution, while the first moment of initial data is assumed to be finite. In fact, the uniqueness of the solution is examined for the coagulation kernels that do not overlap with the one for which the existence of a solution is shown, as far as unbounded kernels are concerned. Last but not the least, the authors, in Das and Saha [25], claimed to have established the existence of a global classical solution to (1.3) and (1.4) in the sense of the definition given for mild solution. Hence, the novelty of our work is that we have improved these results in several ways. We have refer the local existence result from Bagland [23, Lemma 13] and proved the existence of global classical solution to (1.3) and (1.4) in a weighted \( l_1 \) space without assuming the finiteness of the second moment. Furthermore, the uniqueness investigated for the same class of coagulation kernels as the one used to prove its existence.

The content of the paper is organized as follows. In Section 2, we introduce the space and state the main theorem. Section 3 outlines the finite-dimensional systems of ordinary differential equations approximating (1.3), and the propagation of the moments of their solutions is also explored. The proof of the existence and mass conservation property of solutions are discussed in Section 4. The uniqueness of solutions is examined in Section 5 which is followed by the continuous dependence on initial data discussed in Section 6. Finally, the large-time behavior of solutions is investigated in Section 7.

2 | MAIN RESULTS

To begin with, we introduce some notations and specify what we mean by a solution to (1.3) and (1.4). The mathematical study of (1.3) and (1.4) requires to take into account suitable spaces. Following the usual works related to the coagulation fragmentation area, we will consider the Banach spaces

\[ \Xi_\lambda = \left\{ \xi = (\xi_i)_{i \geq 1} \in \mathbb{R}^N, \sum_{i=1}^{\infty} i^\lambda |\xi_i| < \infty \right\}, \quad \lambda \geq 0, \quad (2.1) \]

with the norm defined by

\[ \|\xi\|_\lambda = \sum_{i=1}^{\infty} i^\lambda |\xi_i| \]

and their positive cones

\[ \Xi_\lambda^+ = \{ \xi = (\xi_i)_{i \geq 1} \in \Xi_\lambda : \xi_i \geq 0 \quad \text{for each} \quad i \geq 1 \}. \]

Let us now define the notions of solutions to (1.3) and (1.4) that we will consider.
Definition 2.1. Let $T \in (0, +\infty]$ and $\varepsilon^\text{in} = (\varepsilon^\text{in}_i)_{i \geq 1}$ be a sequence of nonnegative real numbers. A solution $\xi = (\xi_i)_{i \geq 1}$ to (1.3) and (1.4) on $[0, T]$ is a sequence of nonnegative continuous functions satisfying for each $i \geq 1$ and $t \in (0, T)$

(a) $\xi_i \in C([0, T))$, $\sum_{i=1}^{\infty} \|\gamma_i \xi_i\|_i \in L^1(0, t)$,
(b) there holds

$$\xi_i(t) = \xi^\text{in}_i + \int_0^1 \left[ \xi_{i-1} - \sum_{j=1}^{i-1} j \gamma_{i-j} \xi_j - \sum_{j=1}^{i} \gamma_{i-j} \xi_j - \sum_{j=1}^{\infty} \gamma_{i-j} \xi_j \right] \, dt.$$  

In order to show the existence, uniqueness, and mass conservation property of solutions to (1.3) and (1.4), assume that the collision kernel $\gamma_{i,j}$ is nonnegative and symmetric, that is,

$$0 \leq \gamma_{i,j} = \gamma_{j,i}, i, j \geq 1,$$  

(2.2)

and satisfies the following growth condition

for all $i, j \geq 1, \gamma_{i,j} \leq A(i + j)$,  

(2.3)

where $A$ is a positive constant.

Our existence result then reads as follows.

Theorem 2.1. Assume that the coagulation rate $\gamma_{i,j}$ satisfies assumptions (2.2) and (2.3) and $\varepsilon^\text{in} = (\varepsilon^\text{in}_i)_{i \geq 1} \in R^*_+$. Then there is at least one solution $\xi$ to (1.3) and (1.4) satisfying

$$\|\varepsilon(t)\|_1 = \|\varepsilon^\text{in}\|_1, \quad t \in [0, +\infty).$$  

(2.4)

In other words, the mass density of the solution $\xi$ is conserved through time evolution.

3 APPOXIMATING SYSTEMS

We demonstrate the existence of solutions to (1.3) and (1.4) by taking the limit of solutions to the truncated finite-dimensional system of (1.3) and (1.4). To be more specific, for $k \geq 2$, let us consider the following truncated system of $k$ ordinary differential equations,

$$\frac{d^\varepsilon_k(t)}{dt} = \sum_{j=1}^{k} j \gamma_{i-j} \varepsilon_j(t) - \varepsilon_i(t) \sum_{j=1}^{k} j \gamma_{i-j} \varepsilon_j(t) - \sum_{j=1}^{k} \gamma_{i-j} \varepsilon_j(t) \varepsilon_i(t),$$  

(3.1)

$$\varepsilon_k(0) = \varepsilon^\text{in} \geq 0,$$  

(3.2)

for $1 \leq i \leq k$. Next, we require following existence result from Bagland [23, Lemma 13] for (3.1) and (3.2).

Lemma 3.1. For each $k \geq 2$, there exists a unique nonnegative solution $\varepsilon^k = (\varepsilon^k_i)_{1 \leq i \leq k} \in C^1([0, T], R^k)$ to systems (3.1) and (3.2). Moreover, we have

$$\sum_{i=1}^{k} i \varepsilon^k_i(t) \leq \sum_{i=1}^{k} i \varepsilon^\text{in}_i, \quad t \in [0, \infty).$$  

(3.3)

Furthermore, if $(\psi_i) \in R^k$, there holds

$$\sum_{i=1}^{k} \psi_i \frac{d^\varepsilon_k}{dt} = \sum_{i=1}^{k-1} j \psi_{i+1} \gamma_{i+1} \varepsilon^k_{j} - \sum_{i=1}^{k} j \psi_{j} \gamma_{i-j} \varepsilon^k_{i} \varepsilon^k_{j}.$$  

(3.4)
We first introduce some notation. We denote by \( G_1 \) the set of nonnegative and convex functions \( G \in C^1([0, +\infty)) \cap W^{2,\infty}_{\text{loc}}(0, +\infty) \) such that \( G(0) = 0, G'(0) \geq 0 \) and \( G' \) is a concave function. We next denote by \( G_{1,\infty} \) the set of functions \( G \in G_1 \) satisfying, in addition,

\[
\lim_{\zeta \to +\infty} G'(\zeta) = \lim_{\zeta \to +\infty} \frac{G(\zeta)}{\zeta} = +\infty. \tag{3.5}
\]

**Remark 3.1.** It is clear that \( \zeta \mapsto \zeta^p \) belongs to \( G_1 \) if \( p \in [1, 2] \) and to \( G_{1,\infty} \) if \( p \in (1, 2] \).

Next, we recall the following inequality from Banasiak et al. [13, Proposition 7.1.9].

**Lemma 3.2.** For \( G \in G_1 \) and \( i, j \geq 1 \), there holds

\[
(i + j) (G(i + j) - G(i) - G(j)) \leq 2 (iG(j) + jG(i)). \tag{3.6}
\]

We may now state and prove the main result of this section.

**Proposition 3.1.** Consider \( T \in (0, +\infty) \) and \( G \in G_1 \). There exists a constant \( \kappa(T) \) depending only on \( A, G, \|\xi^\text{in}\|_{\xi, i} \) and \( T \) such that, for each \( k \geq 3 \), the solution \( \xi^k \) to (3.1) and (3.2) given by Lemma 3.1 satisfies

\[
\sum_{i=1}^{k} G(i) \xi^k_i(t) \leq \kappa(T) \sum_{i=1}^{k} G(i) \xi^\text{in}_i, \quad t \in [0, T]. \tag{3.7}
\]

**Proof.** For \( k \geq 3 \) and \( t \in [0, T] \), we put

\[
M^k_G = \sum_{i=1}^{k} G(i) \xi^k_i(t).
\]

We infer from (2.3) and (3.4) that

\[
\frac{dM^k_G}{dt} \leq A \sum_{i=1}^{k} \sum_{j=1}^{i} (i + j) [j((G(i + 1) - G(i)) - G(j)] \xi^k_i \xi^k_j.
\]

Now using the convexity of \( G \), we obtain

\[
j(G(i + 1) - G(i)) - G(j) = jG\left(\frac{1}{j}(i + j) + \frac{j-1}{j}i\right) - jG(i) - G(j)
\]

\[
\leq j\left[\frac{1}{j}G(i + j) + \frac{j-1}{j}G(i)\right] - jG(i) - G(j)
\]

\[
\leq G(i + j) - G(i) - G(j).
\]

The above inequality and (3.6) now yield

\[
\frac{dM^k_G}{dt} \leq 2A \sum_{i=1}^{k} \sum_{j=1}^{i} [iG(j) + jG(i)] \xi^k_i \xi^k_j \leq 4A \|\xi^\text{in}\|_{\xi, i} M^k_G,
\]

which yields (3.7) by the Gronwall lemma. \( \square \)

Next, we recall the following lemma from Bagland [23, Lemma 14] which provides the time equicontinuity of \((\xi^k_i)_{k \geq i}\).

**Lemma 3.3.** Let \( i \geq 1 \). There exists a constant \( \sigma_i \), depending only upon \( \|\xi^\text{in}\|_{\xi, i} \), and \( i \) such that, for each \( k \geq i \)

\[
\left|\frac{d\xi^k_i}{dt}\right| \leq \sigma_i, \quad t \in [0, +\infty). \tag{3.8}
\]
Now we are in a position to prove Theorem 2.1. As a prelude to this, we first recall a refined version of the de la Vallee–Poussin theorem for integrable functions [13, Theorem 7.1.6].

**Theorem 4.1.** Let \( (\Sigma, \mathcal{A}, \nu) \) be a measure space and consider a function \( \zeta \in L^1(\Sigma, \mathcal{A}, \nu) \). Then there exists a function \( G \in \mathcal{G}_{1,\infty} \) such that
\[
G(|\zeta|) \in L^1(\Sigma, \mathcal{A}, \nu).
\]

**Proof of Theorem 2.1.** We next apply Theorem 4.1, with \( \Sigma = \mathbb{N} \) and \( \mathcal{A} = 2^{\mathbb{N}} \), the set of all subsets of \( \mathbb{N} \). Defining the measure \( \nu \) by
\[
\nu(J) = \sum_{i \in J} \varepsilon_i, \quad J \subset \mathbb{N},
\]
the condition \( \varepsilon^{in} \equiv \Xi_1^+ \) ensures that \( \zeta \mapsto \zeta \) belongs to \( L^1(\Sigma, \mathcal{A}, \nu) \). By Theorem 4.1, there is thus a function \( G_0 \in \mathcal{G}_{1,\infty} \) such that \( \zeta \mapsto G_0(\zeta) \) belongs to \( L^1(\Sigma, \mathcal{A}, \nu) \), that is,
\[
G_0 = \sum_{i=1}^{\infty} G_0(i) \varepsilon^{in}_i < \infty. \tag{4.1}
\]

In the following, we denote by \( C_0 \) be a positive constant depending only on \( A, \|\varepsilon^{in}\|_1, G_0, \) and \( G_0 \). The dependence of \( C_0 \) on any additional parameters will be explicitly stated. By (3.3) and (3.8), we infer that the sequence \( (\varepsilon_i^k)_{k \geq 1} \) is bounded in \( W^{1,1}(0, T) \) for each \( i \geq 1 \) and \( T \in (0, +\infty) \). Using the Helly theorem [28, pp. 372–374], we can conclude that there is a subsequence of \( (\varepsilon_i^k)_{k \geq 1} \) (which we still refer to as \( (\varepsilon_i^k)_{k \geq 1} \)) and a sequence \( \xi = (\xi_i)_{i \geq 1} \) of functions of locally bounded variation such that
\[
\lim_{k \to +\infty} \varepsilon_i^k(t) = \xi_i(t) \tag{4.2}
\]
for each \( i \geq 1 \) and \( t \geq 0 \). Clearly, \( \xi_i(t) \geq 0 \) for \( i \geq 1 \) and \( t \geq 0 \) and it follows from (4.2) and (3.3) that \( \xi(t) \in \Xi_1^+ \) with
\[
\|\xi(t)\|_1 \leq \|\varepsilon^{in}\|_1, \quad t \geq 0. \tag{4.3}
\]

In addition, since \( G_0 \) belongs to \( \mathcal{G}_{1,\infty} \), we can deduce from (4.1) and Proposition 3.1 that for every \( t \geq 0 \) and \( k \geq 3 \), there holds
\[
\sum_{i=1}^{k} G_0(i) \varepsilon_i^k(t) \leq C_0(T). \tag{4.4}
\]

Consider now \( T \in (0, +\infty) \) and \( q \geq 2 \). By (4.4), we have for \( k \geq q + 1 \) and \( t \in [0, T] \)
\[
\sum_{i=1}^{q} G_0(i) \varepsilon_i^k(t) \leq C_0(T).
\]

Due to (4.2) we may pass to the limit as \( k \to \infty \) in the last estimate, which implies that they both remain valid with \( \varepsilon_i^k \) being replaced by \( \xi_i \). We next allow \( q \to \infty \) and get
\[
\sum_{i=1}^{\infty} G_0(i) \xi_i(t) \leq C_0(T), \quad t \in [0, T]. \tag{4.5}
\]

As a result of (2.3) and (4.3), we get that, for each \( i \geq 1 \),
\[
\sum_{j=1}^{\infty} \gamma_{ij} \xi_j \in L^1(0, T). \tag{4.6}
\]
We now claim that for each \( i \geq 1 \), there holds

\[
\lim_{k \to +\infty} \left| \sum_{j=1}^{k} Y_{ij} \xi_{ij}^{k} - \sum_{j=1}^{\infty} Y_{ij} \xi_{ij} \right|_{L^1(0,T)} = 0. \tag{4.7}
\]

Let us consider now \( i \geq 1 \) and \( q \geq i+1 \). Using (3.3), (4.2), and (4.3), along with the Lebesgue-dominated convergence theorem, we obtain that

\[
\lim_{k \to +\infty} \left| \sum_{j=1}^{q} Y_{ij} \xi_{ij}^{k} - \sum_{j=1}^{\infty} Y_{ij} \xi_{ij} \right|_{L^1(0,T)} = 0. \tag{4.8}
\]

Also, we infer from (2.3), (3.3), and (4.4) that for each \( k \geq q + 1 \),

\[
\left| \sum_{j=q+1}^{k} Y_{ij} \xi_{ij}^{k} \right|_{L^1(0,T)} \leq \| \varepsilon_{ij}^{\infty} \|_1 \left| \sum_{j=q+1}^{k} j \xi_{ij}^{k} \right|_{L^1(0,T)} \leq C_0(i, T) \sup_{j \geq q} \frac{j}{G_0(j)} \left| \sum_{j=q+1}^{k} G_0(j) \xi_{ij}^{k} \right|_{L^1(0,T)},
\]

and hence, applying (4.4), we have

\[
\left| \sum_{j=q+1}^{k} Y_{ij} \xi_{ij}^{k} \right|_{L^1(0,T)} \leq C_0(i, T) \sup_{j \geq q} \frac{j}{G_0(j)}. \tag{4.9}
\]

Similarly, (2.3), (4.3), and (4.5) entail that

\[
\left| \sum_{j=q+1}^{\infty} Y_{ij} \xi_{ij} \right|_{L^1(0,T)} \leq C_0(i, T) \sup_{j \geq q} \frac{j}{G_0(j)}. \tag{4.10}
\]

Combining (4.8)–(4.10), we obtain

\[
\lim \sup_{k \to +\infty} \left| \sum_{j=1}^{k} Y_{ij} \xi_{ij}^{k} - \sum_{j=1}^{\infty} Y_{ij} \xi_{ij} \right|_{L^1(0,T)} \leq C_0(i, T) \sup_{j \geq q} \frac{j}{G_0(j)},
\]

for every \( q \geq i + 1 \). Recalling that \( G_0 \) belongs to \( G_{1, \infty} \), we can observe that the right-hand side of the above inequality converges to zero as \( q \to +\infty \); as a result, we obtain (4.7). With the help of (3.3), (4.2), (4.3), and (4.7), we can easily verify that \( \xi_i \) satisfies Definition 2.1(b) for each \( i \geq 1 \). By making use of (4.6), the continuity of \( \xi_i \) then follows, and we have thus shown that \( \xi = (\xi_i) \) is a solution to (1.3) and (1.4) on \([0, +\infty)\). In order to complete the proof of Theorem 2.1, it remains to prove that (2.4) holds true. Let \( t \in (0, +\infty) \). For \( k \geq q \geq 3 \), we have (3.3) that

\[
\| \xi(t) \|_1 - \| \varepsilon_{ij}^{\infty} \|_1 \leq \sum_{i=1}^{q} \| \xi_{ij}^{k}(t) - \xi_{ij}(t) \|_1 + \sum_{i=k+1}^{\infty} \| i \xi_{ij}^{k}(t) \|_1 + \sum_{i=q+1}^{k} \| i \xi_{ij}^{k}(t) \|_1 + \sum_{i=q+1}^{\infty} i \xi_{ij}(t).
\]

Subsequently, it can be deduced from (4.4) and (4.5) that

\[
\| \xi(t) \|_1 - \| \varepsilon_{ij}^{\infty} \|_1 \leq \sum_{i=1}^{q} \| \xi_{ij}^{k}(t) - \xi_{ij}(t) \|_1 + \sum_{i=k+1}^{\infty} \| i \xi_{ij}^{k}(t) \|_1 + C_0(T) \sup_{i \geq q} \frac{i}{G_0(i)}.
\]

Since \( \varepsilon_{ij}^{\infty} \in \Xi_{ij}^+ \), we can infer from (4.2) that

\[
\| \xi(t) \|_1 - \| \varepsilon_{ij}^{\infty} \|_1 \leq C_0(T) \sup_{i \geq q} \frac{i}{G_0(i)}.
\]

By noting that \( G_0 \in G_{1, \infty} \), it follows that \( \| \xi(t) \|_1 = \| \varepsilon_{ij}^{\infty} \|_1 \), and the proof of Theorem 2.1 is complete. \( \square \)
Next, we deduce that the solution constructed in Theorem 2.1 is, in fact, first-order differentiable.

**Corollary 4.1.** Assume that assumptions (2.2) and (2.3) are fulfilled. Let \( \xi^\text{in} \in \Xi^+_1 \) and consider the solution \( \xi = (\xi_i)_{i \geq 1} \) to (1.3) and (1.4) on \([0, +\infty)\) given by Theorem 2.1. Then \( \xi_i \) is continuously differentiable on \([0, +\infty)\) for each \( i \in \mathbb{N} \).

**Proof.** For \( \tau_1, \tau_2 \in [0, T] \) and \( q \geq 1 \),

\[
\|\xi(\tau_1) - \xi(\tau_2)\|_1 \leq \sum_{i=1}^{q} i^2 |\xi_i(\tau_1) - \xi_i(\tau_2)| + \sum_{i=q+1}^{\infty} i(\xi_i(\tau_1) + \xi_i(\tau_2)) \\
\leq \sum_{i=1}^{q} i|\xi_i(\tau_1) - \xi_i(\tau_2)| + \sup_{i \geq q} \frac{i}{i G(i)} \sum_{j=q+1}^{\infty} G(i)(\xi_i(\tau_1) + \xi_i(\tau_2)).
\]

As \( \xi_i \) is a continuous function for \( i \in \{1, 2, \cdots, q\} \), we can infer from the above inequality that

\[
\lim_{\tau_1 \to \tau_2} \sup_{\tau_1 \to \tau_2} \|\xi(\tau_2) - \xi(\tau_1)\|_1 \leq 2C_0(T) \sup_{i \geq q} \frac{i}{G_0(i)}.
\]

Since \( G_0 \in G_1 \), recalling (4.5), we take the limit as \( q \to +\infty \) to obtain

\[
\lim_{\tau_1 \to \tau_2} \|\xi(\tau_2) - \xi(\tau_1)\|_1 = 0.
\]

Next, for \( 0 \leq \tau_1 \leq \tau_2 < T \) and \( i \geq 1 \)

\[
\left| \sum_{j=1}^{\infty} \gamma_{i,j} \xi_j(\tau_2) - \sum_{j=1}^{\infty} \gamma_{i,j} \xi_j(\tau_1) \right| \leq 2A \|\xi(\tau_2) - \xi(\tau_1)\|_1
\]

from which the time continuity of \( \sum_{j=1}^{\infty} \gamma_{i,j} \xi_j \) follows. It is evident from this that the right-hand side of (1.3) is continuous in time, implying the continuity of the derivative of \( \xi_i \). Consequently, this ensures the existence of the classical solution. \( \square \)

We end this section by demonstrating that when the initial data belong to a certain suitable class and has a finite moment, there exists at least one solution to Equations (1.3) and (1.4) that maintains the same property for all times.

**Proposition 4.1.** Consider \( \xi^\text{in} \in \Xi^+_1 \) and assume that there is \( G \in G_1 \) such that

\[
\sum_{i=1}^{\infty} G(i) \xi^\text{in}_i < +\infty. \tag{4.11}
\]

Then under assumptions (2.2) and (2.3), there is at least one solution \( \xi \) to (1.3) and (1.4) on \([0, +\infty)\) satisfying (2.4) for each \( T \in (0, +\infty) \) and such that

\[
\sup_{t \in [0, T]} \sum_{i=1}^{\infty} G(i) \xi_i(t) < +\infty. \tag{4.12}
\]

**Proof.** We only need to show that the solution constructed in the proof of Theorem 2.1 enjoys the additional property (4.12). But, as \( G \in G_1 \), (4.12) follows at once from Proposition 3.1 and (4.2). \( \square \)

## 5 | UNIQUENESS OF CLASSICAL SOLUTION

We establish the following identity before proving the uniqueness theorem for the solutions in the space \( \Xi^+_1 \).
Lemma 5.1. Let \( T \in (0, +\infty) \) and \( \xi = (\xi_i)_{i \geq 1} \in \Xi^+ \) be a solution of (1.3) and (1.4). Furthermore, suppose that \((\Phi_i)_{i \geq 1}\) be a sequence decaying sufficiently rapidly. Then, for all \( t \in [0, T] \) and \( q > 1 \), we have

\[
\frac{d}{dt} \sum_{i=1}^{q} \Phi_i \xi_i(t) = \sum_{i=1}^{q} j \Phi_{i+1} Y_{i,j} \xi_i(t) \xi_j(t) - \sum_{i=1}^{q} (j \Phi_i + \Phi_j) Y_{i,j} \xi_i(t) \xi_j(t) - \sum_{i=1}^{q} \Phi_j Y_{i,j} \xi_i(t) \xi_j(t),
\]

(5.1)

where

\[
P_1 = \{(i, j) : 1 \leq i \leq q, 1 \leq j \leq i\},
\]

\[
P_2 = \{(i, j) : 1 \leq i \leq q, 1 \leq j \leq i\},
\]

\[
P_3 = \{(i, j) : i \geq q + 1, 1 \leq j \leq q\}.
\]

Proof. From (1.3), we have

\[
\frac{d \xi_i(t)}{dt} = \xi_{i-1}(t) \sum_{j=1}^{i-1} j Y_{i-1,j} \xi_j(t) - \xi_i(t) \sum_{j=1}^{i} j Y_{i,j} \xi_j(t) - \sum_{j=m}^{\infty} j Y_{i,j} \xi_j(t).
\]

Multiplying by \( \Phi_i \) in the above equation and then taking summation from \( i = 1 \) to \( i = q \) on both sides, we obtain

\[
\frac{d}{dt} \sum_{i=1}^{q} \Phi_i \xi_i(t) = \sum_{i=1}^{q} \sum_{j=1}^{i-1} j \Phi_i Y_{i-1,j} \xi_j(t) \xi_j(t) - \sum_{i=1}^{q} \sum_{j=1}^{i} j \Phi_i Y_{i,j} \xi_j(t) \xi_j(t) - \sum_{i=1}^{q} \Phi_i Y_{i,j} \xi_j(t) \xi_j(t).
\]

(5.2)

On changing the order of summation in the first and the last terms as the r.h.s. to (5.2), we get

\[
\frac{d}{dt} \sum_{i=1}^{q} \Phi_i \xi_i(t) = \sum_{j=1}^{q} \sum_{i=j+1}^{q-1} j \Phi_i Y_{i-1,j} \xi_j(t) \xi_j(t) - \sum_{i=1}^{q-1} \sum_{i=1}^{j} j \Phi_i Y_{i,j} \xi_j(t) \xi_j(t) - \sum_{i=1}^{q} \sum_{j=q+1}^{i} j \Phi_i Y_{i,j} \xi_j(t) \xi_j(t) - \sum_{i=1}^{q} \sum_{j=q+1}^{i} j \Phi_i Y_{i,j} \xi_j(t) \xi_j(t).
\]

(5.3)

Next, in the first summation, we replace \( i-1 \) with \( i \), and in the last two summations on the right-hand side, we exchange \( i \) and \( j \) and use the symmetry of \( Y_{i,j} \) in (5.3), to obtain

\[
\frac{d}{dt} \sum_{i=1}^{q} \Phi_i \xi_i(t) = \sum_{j=1}^{q-1} \sum_{i=j}^{q-1} j \Phi_i Y_{i,j} \xi_j(t) \xi_j(t) - \sum_{i=1}^{q-1} \sum_{i=1}^{j} j \Phi_i Y_{i,j} \xi_j(t) \xi_j(t) - \sum_{i=1}^{q} \sum_{j=q+1}^{i} j \Phi_i Y_{i,j} \xi_j(t) \xi_j(t) - \sum_{i=1}^{q} \sum_{j=q+1}^{i} j \Phi_i Y_{i,j} \xi_j(t) \xi_j(t).
\]

(5.4)

Again, by using a change in the order of summations in the first summation on the right-hand side of (5.4), we have

\[
\frac{d}{dt} \sum_{i=1}^{q} \Phi_i \xi_i(t) = \sum_{i=1}^{q-1} \sum_{i=1}^{j} j \Phi_i Y_{i,j} \xi_j(t) \xi_j(t) - \sum_{i=1}^{q} \sum_{i=1}^{j} j \Phi_i Y_{i,j} \xi_j(t) \xi_j(t) - \sum_{i=1}^{q} \sum_{j=q+1}^{i} j \Phi_i Y_{i,j} \xi_j(t) \xi_j(t) - \sum_{i=1}^{q} \sum_{j=q+1}^{i} j \Phi_i Y_{i,j} \xi_j(t) \xi_j(t).
\]

(5.5)

which clearly implies that (5.1) holds.

Now we are in a position to give the proof of the uniqueness of the solutions.
Theorem 5.1. Consider \( \zeta^{in} \in \Xi^+ \) and assume that the coagulation kernel satisfies assumption (2.2). Assume further that there are \( \delta \in [0, 1] \) and \( A_\delta > 0 \) such that

\[
\sum_{i=1}^{\infty} i^{1+\delta} \tau_i^{in} < +\infty \text{ and } \gamma_{ij} \leq A(i^\delta + j^\delta), \; i, j \geq 1. \tag{5.6}
\]

Then there is one and only one solution \( \xi \) to (1.3) and (1.4) on \([0, +\infty)\) satisfying (2.4), and for each \( T \in (0, +\infty), \)

\[
\sup_{t \in [0, T]} \sum_{i=1}^{\infty} i^{1+\delta} \xi_i(t) < +\infty. \tag{5.7}
\]

Proof. As \( \zeta \mapsto \zeta^{1+\delta} \) belongs to \( G_1 \), the existence of solution to (1.3) and (1.4) on \([0, +\infty)\) satisfying (2.4) and (5.7) follows from Proposition 4.1.

Let \( \xi \) and \( \eta \) be two distinct solutions of (1.3) and (1.4) having initial condition \( \xi_i(0) = \eta_i(0) \) for all \( i \geq 1 \). Let \( \pi(t) = \xi(t) - \eta(t) \), for \( t \in [0, T] \). Then we consider the following function as

\[
H(t) = \sum_{i=1}^{q} i|\xi_i(t) - \eta_i(t)| = \sum_{i=1}^{q} i|i\pi_i(t)|
\]

\[
\frac{d\pi_i}{dt} = \sum_{j=1}^{i-1} j\gamma_{i-1,j} [\xi_{i-1}(t)\xi_j(t) - \eta_{i-1}(t)\eta_j(t)] - \sum_{j=1}^{i} j\gamma_{i,j} [\xi_i(t)\xi_j(t) - \eta_i(t)\eta_j(t)] - \sum_{j=i+1}^{\infty} \gamma_{i,j} [\xi_i(t)\xi_j(t) - \eta_i(t)\eta_j(t)]. \tag{5.8}
\]

Now using Equation (5.1) with \( \Phi_i = i \text{ sgn}(\pi_i(t)) \) for \( \xi_i \) and \( \eta_i \) and taking the difference, we have

\[
\sum_{i=1}^{q} i|\xi_i(t) - \eta_i(t)| = \int_{0}^{t} \left[ \sum_{i=1}^{q-1} \sum_{j=1}^{i} (i+1) \text{sgn}(\pi_{i+1}(s)) j\gamma_{i,j} [\xi_i(s)\xi_j(s) - \eta_i(s)\eta_j(s)] - \sum_{i=1}^{q} \sum_{j=1}^{i} (ji \text{ sgn}(\pi_i(s)) + j \text{ sgn}(\pi_j(s))\gamma_{i,j} [\xi_i(s)\xi_j(s) - \eta_i(s)\eta_j(s)]
\]

\[
- \sum_{i=q+1}^{\infty} \sum_{j=1}^{q} j \text{ sgn}(\pi_j(s))\gamma_{i,j} [\xi_i(s)\xi_j(s) - \eta_i(s)\eta_j(s)] \right] ds. \tag{5.9}
\]

Since

\[\xi_i(t)\xi_j(t) - \eta_i(t)\eta_j(t) = \xi_i(t)\pi_j(t) + \eta_j(t)\pi_i(t),\]

the identity (5.9) can be written as

\[
\sum_{i=1}^{q} i|\xi_i(t) - \eta_i(t)| = \int_{0}^{t} \left[ \sum_{i=1}^{q-1} \sum_{j=1}^{i} (i+1) j \text{ sgn}(\pi_{i+1}(s))\gamma_{i,j} [\xi_i(s)\pi_j(s) + \eta_j(s)\pi_i(s)]
\]

\[
- \sum_{i=1}^{q} \sum_{j=1}^{i} (ji \text{ sgn}(\pi_i(s)) + j \text{ sgn}(\pi_j(s))\gamma_{i,j} [\xi_i(s)\pi_j(s) + \eta_j(s)\pi_i(s)]
\]

\[
- \sum_{i=q+1}^{\infty} \sum_{j=1}^{q} j \text{ sgn}(\pi_j(s))\gamma_{i,j} [\xi_i(s)\pi_j(s) + \eta_j(s)\pi_i(s)] \right] ds := I_1 + I_2 + I_3. \tag{5.10}
\]
Now, we estimate the terms $I_1$, $I_3$, and $I_3$ separately. Let us first consider the term $I_1$ as

$$I_1 = \int_0^t \sum_{i=1}^{q-1} \sum_{j=1}^i (i + 1) j \sgn(\pi_{i+1}(s)) \gamma_{i,j} \left[ \xi_i(s) \pi_j(s) + \eta_j(s) \pi_i(s) \right] ds$$

Similarly, we have

$$I_2 = -\sum_{i=1}^q \sum_{j=1}^i j \sgn(\pi_j(s)) \gamma_{i,j} \left[ \xi_i(s) \pi_j(s) + \eta_j(s) \pi_i(s) \right] ds$$

Adding (5.11) and (5.12) and using (6.6), we get

$$I_1 + I_2 \leq 2 \int_0^t \sum_{i=1}^q \sum_{j=1}^i j \gamma_{i,j} \xi_i(s) \pi_j(s) + 2 \int_0^t \sum_{i=1}^q \sum_{j=1}^i j \gamma_{i,j} \eta_j(s) \pi_i(s) ds$$

$$\leq 2A \int_0^t \sum_{i=1}^q \sum_{j=1}^i j^{1+\delta} \xi_i(s) \pi_j(s) ds + 2A \int_0^t \sum_{i=1}^q \sum_{j=1}^i j^{1+\delta} \eta_j(s) \pi_i(s) ds$$

Recalling (5.7), we deduce that

$$I_1 + I_2 \leq 4A(M_{1+\delta}(T) + M_1(0)) \int_0^t \sum_{i=1}^q i |\pi_i(s)| ds.$$

Similarly, $I_3$ can be estimated as

$$I_3 = -\int_0^t \sum_{i=q+1}^\infty \sum_{j=1}^q j \sgn(\pi_j(s)) \gamma_{i,j} \left[ \xi_i(s) \pi_j(s) + \eta_j(s) \pi_i(s) \right] ds$$

Using (5.11), (5.12), and (5.14) in (5.10), we obtain

$$\sum_{i=1}^q i |\pi_i(t)| \leq 4A(M_{1+\delta}(T) + M_1(0)) \int_0^t \sum_{i=1}^q i |\pi_i(s)| ds + A(M_{1+\delta}(T) + M_1(0)) \int_0^t \sum_{i=q+1}^\infty i |\xi_i(s) + \eta_i(s)| ds.$$
Therefore, using (5.7), we may pass to the limit as \( q \to \infty \) in (5.15); we obtain
\[
\sum_{i=1}^{\infty} \frac{1}{|\pi_i(t)|} \leq 4A(M_{1+\delta}(T) + M_1(0)) \int_0^t \sum_{i=1}^{\infty} \frac{1}{|\pi_i(s)|} ds.
\]
Since \( \pi_i(0) = 0 \), then, by the application of Gronwall’s lemma, we conclude that
\[
\sum_{i=1}^{\infty} \frac{1}{|\pi_i(t)|} = 0, \forall t \in [0, T];
\]
hence, \( \pi_i = 0 \), for all \( i \geq 1 \) and \( t \in [0, T] \), which proves the uniqueness. \( \square \)

### 6 CONTINUOUS DEPENDENCE ON INITIAL DATA

Concerning the continuous dependence relative to the initial condition, we prove the following result.

**Proposition 6.1.** If assumptions (2.2) and (5.6) hold and if \( \xi \) and \( \eta \) are solutions to (1.3) in \( \Xi^+_1 \) satisfying \( \xi(0) = \xi^{in} \) and \( \eta(0) = \eta^{in} \) then, for each \( t \in [0, T] \), there is a positive constant \( Y(T, M_1(0), M_{1+\delta}(T)) \) such that
\[
\|\xi - \eta\|_1 \leq Y(T, M_1(0), M_{1+\delta}(T))\|\xi^{in} - \eta^{in}\|_1.
\]

**Proof.** Defining \( \pi(t) = \xi(t) - \eta(t) \) and using the same estimates as in the proof of Theorem 5.1 to obtain
\[
\sum_{i=1}^{q} \frac{1}{|\pi_i(t)|} \leq \sum_{i=1}^{q} \frac{1}{|\pi_i(0)|} + 4A(M_{1+\delta}(T) + M_1(0)) \int_0^t \sum_{i=1}^{q} \frac{1}{|\pi_i(s)|} ds
\]
\[
+ A(M_{1+\delta}(T) + M_1(0)) \int_0^t \sum_{i=q+1}^{\infty} \frac{1}{|\pi_i(s) + \eta_i(s)|} ds.
\]
As a consequence, by making \( q \to \infty \) and using the similar arguments as in the proof of uniqueness, we get
\[
\sum_{i=1}^{\infty} \frac{1}{|\pi_i(t)|} \leq \sum_{i=1}^{\infty} \frac{1}{|\pi_i(0)|} + 4A(M_{1+\delta}(T) + M_1(0)) \int_0^t \sum_{i=1}^{\infty} \frac{1}{|\pi_i(s)|} ds.
\]
By using Gronwall’s lemma and then taking supremum over \( t \), we obtain (6.1). \( \square \)

### 7 ASYMPTOTIC BEHAVIOR OF SOLUTIONS

In this section, we investigate the behavior of the solutions to (1.3) and (1.4) as \( t \to \infty \) and here we follow the proof from Carr and Da Costa [5, Theorem 4.3].

**Theorem 7.1.** For \( T \in (0, +\infty) \) and \( \xi^{in} = (\xi^{in}_i)_{i \geq 1} \in \Xi^+_1 \), let \( \xi = (\xi_i)_{i \geq 1} \in \Xi^+_1 \) be a solution of (1.3) and (1.4), then there is \( \xi^{\infty} = (\xi^{\infty}_i)_{i \geq 1} \in \Xi^+_1 \) such that
\[
\lim_{t \to +\infty} \xi_i(t) = \xi^{\infty}_i, \ i \geq 1.
\]
Moreover, if \( \gamma_{il} > 0 \) for all \( i \geq 1 \), then we have
\[
\xi^{\infty}_i = 0, \ \text{for all} \ i \geq 1.
\]
Proof. Consider \( q \geq 1, \tau \geq 0 \) and \( s \geq \tau \). Using \( \Phi \equiv 1 \) into Lemma 5.1 we obtain that

\[
\frac{d}{dt} \sum_{i=1}^{q} \xi_i(t) = \sum_{i=1}^{q-1} \sum_{j=1}^{i} j \gamma_{i,j} \xi_i(t) \xi_j(t) - \sum_{i=1}^{q} \sum_{j=i+1}^{q} (j+1) \gamma_{i,j} \xi_i(t) \xi_j(t)
\]

\[
- \sum_{i=q+1}^{\infty} \sum_{j=1}^{q} \gamma_{i,j} \xi_i(t) \xi_j(t)
\]

\[
\leq - \sum_{i=1}^{q} \sum_{j=1}^{i} \gamma_{i,j} \xi_i(t) \xi_j(t) - \sum_{i=q+1}^{\infty} \sum_{j=1}^{q} \gamma_{i,j} \xi_i(t) \xi_j(t)
\]

\[
\leq - \frac{1}{2} \sum_{i=1}^{q} \sum_{j=1}^{q} \gamma_{i,j} \xi_i(t) \xi_j(t)
\]

\[
\leq 0.
\]

Taking integration with respect to \( t \) from \( \tau \) to \( s \), we have

\[
\sum_{i=1}^{q} \xi_i(s) \geq \sum_{i=1}^{q} \xi_i(\tau).
\]

(7.4)

Hence, for each \( q \geq 1 \), the function \( f_q : t \mapsto \sum_{i=1}^{q} \xi_i(t) \) is a nonincreasing and nonnegative function of time. Hence, there exists a positive constant \( \overline{f}_q \) such that

\[
f_q(t) \to \overline{f_q} \text{ as } t \to +\infty,
\]

and thus,

\[
\xi_q(t) \to \xi_q^\infty \text{ as } t \to +\infty,
\]

with \( \xi_1^\infty = f_1 \) and \( \xi_q^\infty = f_q - f_{q-1} \geq 0 \) for \( q \geq 2 \). Moreover, we conclude from \( \xi = (\xi_i)_{i \geq 1} \in \Xi_1^+ \) that

\[
\sum_{i=1}^{q} \xi_i(t) \leq \sup_{t \in [0, +\infty)} \|\xi(t)\|_1 < +\infty,
\]

for each \( q \geq 1 \) and \( t \in [0, +\infty) \). Hence,

\[
\sum_{i=1}^{\infty} \xi_i^\infty < +\infty.
\]

Now, we prove (7.2), that is, \( \xi_q^\infty = 0 \) for all \( q \geq 1 \). We first set \( q = 1 \) and \( \Phi \equiv 1 \) into Lemma 5.1 and after integration, we obtain

\[
\xi_1(t+\tau) - \xi_1(t) = - \int_{t}^{t+\tau} \left[ \sum_{k=1}^{\infty} \sum_{k=1}^{\infty} \gamma_{1,k} \xi_k(s) \right] ds
\]

\[
= \int_{t}^{t+\tau} \left[ -2 \xi_1^2(s) \gamma_{1,1} - \xi_1(s) \sum_{k=2}^{\infty} \gamma_{1,k} \xi_k(s) \right] ds
\]

\[
\leq - \int_{t}^{t+\tau} \gamma_{1,1} \xi_1^2(s) ds,
\]

for all \( t \geq 0 \) and \( \tau > 0 \). Now, letting \( t \to \infty \), we get

\[
\lim_{t \to \infty} \int_{t}^{t+\tau} \gamma_{1,1} \xi_1^2(s) ds = 0.
\]
and this implies $\xi_1^\infty = 0$, because otherwise there would exist a positive constant $0 < \theta_1 < \xi_1^\infty$ such that, for all sufficiently large $t$, $\xi_1(t) > \theta_1$ and

$$\lim_{t \to \infty} \int_t^{t+\tau} \gamma_{1,1}\xi_1^2(s)ds \geq \gamma_{1,1}\theta_1^2 \tau > 0,$$

a contradiction. Next, we set $q = 2$ into Lemma 5.1 and get

$$\xi_2(t + \tau) - \xi_2(t) = \int_t^{t+\tau} \left[ \xi_2^2(s)\gamma_{1,1} - \xi_2(s) \sum_{k=1}^2 k\gamma_{2,k}\xi_k(s) - \xi_2(s) \sum_{k=2}^\infty \gamma_{2,k}\xi_k(s) \right] ds$$

$$= \int_t^{t+\tau} \left[ \xi_2^2(s)\gamma_{1,1} - 3\xi_2^2(s)\gamma_{2,2} - \gamma_{1,2}\xi_1^2(s) - \xi_2(s) \sum_{k=3}^\infty \gamma_{2,k}\xi_k(s) \right] ds$$

$$\leq \int_t^{t+\tau} \left( \xi_2^2(s)\gamma_{1,1} - \xi_2^2(s)\gamma_{2,2} \right) ds.$$

Now, letting $t \to \infty$ and obtain

$$0 \leq \lim_{t \to \infty} \int_t^{t+\tau} (\xi_2^2(s)\gamma_{1,1} - \xi_2^2(s)\gamma_{2,2}) ds$$

and since $\xi_1(t) \to 0$ as $t \to \infty$ and $(\gamma_{1,1})_{t \geq 1} > 0$, we have

$$\lim_{t \to \infty} \int_t^{t+\tau} \gamma_{2,2}\xi_2^2(s)ds = 0,$$

and this implies $\xi_2^\infty = 0$, because otherwise there would exist a positive constant $0 < \theta_2 < \xi_2^\infty$ such that, for all sufficiently large $t$, $\xi_2(t) > \theta_2$ and

$$\lim_{t \to \infty} \int_t^{t+\tau} \xi_2^2(s)ds \geq \gamma_{2,2}\theta_2^2 \tau > 0,$$

a contradiction.

Proceeding by induction, assuming $\xi_1^\infty = \cdots = \xi_{q-1}^\infty = 0$, we prove $\xi_q^\infty = 0$:

$$0 = \lim_{t \to \infty} \xi_q(t + \tau) - \xi_q(t)$$

$$= \lim_{t \to \infty} \int_t^{t+\tau} \left( \xi_{q-1}(s) \sum_{k=1}^{q-1} k\gamma_{q-1,k}\xi_k(s) - \xi_q(s) \sum_{k=1}^q k\gamma_{q,k}\xi_k(s) - \xi_q(s) \sum_{k=q}^\infty \gamma_{q,k}\xi_k(s) \right) ds$$

$$\leq -\phi_{q,q} \lim_{t \to \infty} \int_t^{t+\tau} (\xi_q(t))^2 ds,$$

and the conclusion follows as before. Hence, the proof of Theorem 7.1 is completed. □

Finally, in the next proposition, we will show that the total number of particles goes to zero as time increases to infinity.

**Proposition 7.1.** For $T \in (0, +\infty)$ and $\xi^\infty = (\xi^\infty_j)_{j \geq 1} \in \Xi^+$. Let $\xi = (\xi_j)_{j \geq 1} \in \Xi^+$ be a solution of (1.3) and (1.4). Further, assume that for some $\zeta > 0$,

$$\gamma_{i,j} \geq \zeta, \quad \text{and} \quad i, j \geq 1. \quad (7.5)$$

Then

$$\lim_{t \to \infty} \sum_{i=1}^\infty \xi_i(t) = 0.$$

**Proof.** From (7.3), it follows that

$$\sum_{i=1}^q \xi_i(t) + \frac{1}{2} \int_s^t \sum_{i=1}^q \sum_{j=1}^q \gamma_{i,j} \xi_j(t) \xi_j(t) dt \leq \sum_{i=1}^q \xi_i(s).$$
Now, the growth conditions (2.3) and (7.5) allow to pass to the limit as \( q \to \infty \) in the above equality; we thus obtain
\[
\sum_{i=1}^{\infty} \xi_i(\tau) + \frac{1}{2} \int_{\tau}^{s} \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \eta_{ij}(t) \xi_j(t) dt \leq \sum_{i=1}^{\infty} \xi_i(s),
\]
which implies that
\[
\int_{\tau}^{s} \left( \sum_{i=1}^{\infty} \xi_i(t) \right)^2 dt \leq \frac{2}{\zeta} \sum_{i=1}^{\infty} \xi_i(s).
\]
Next, we deduce from the previous estimate (with \( s = 0 \) and \( \tau = +\infty \)) that
\[
\int_{0}^{\infty} \left( \sum_{i=1}^{\infty} \xi_i(t) \right)^2 dt \leq \frac{2}{\zeta} \sum_{i=1}^{\infty} \xi_{in} \leq \frac{2}{\zeta} \sum_{i=1}^{\infty} \xi_{in} < +\infty.
\]
Recalling (7.4), we realize that the total number of particles \( M_0 \) is a nonincreasing and nonnegative function of time which also belongs to \( L^2(0, +\infty) \). Therefore, we obtain
\[
\lim_{t \to \infty} \sum_{i=1}^{\infty} \xi_i(t) = 0.
\]

**AUTHOR CONTRIBUTIONS**
All the authors have contributed equally.

**ACKNOWLEDGEMENTS**
This work is partially supported by the Department of Science and Technology (DST), India-Deutscher Akademischer Austauschdienst (DAAD) within the Indo-German joint project DST/INT/DAAD/P-18/2019. The authors would like to thank Prof. Philippe Laurençot for helpful discussion.

**CONFLICT OF INTEREST STATEMENT**
The authors declare that they have no conflict of interests.

**ORCID**

Mashkoor Ali [https://orcid.org/0009-0006-1591-3373](https://orcid.org/0009-0006-1591-3373)

**REFERENCES**
1. M. V. Smoluchowski, *Drei vortrage uber diffusion, brownsche bewegung und koagulation von kolloidteilichen*, Zeitschrift für Physik 17 (1916), 557–585.
2. M. V. Smoluchowski, *Versuch einer mathematischen Theorie der Koagulationskinetik kolloider Lösungen*, Z. Phys. Chem. 92 (1917), 129–168.
3. J. M. Ball and J. Carr, *The discrete coagulation-fragmentation equations: existence, uniqueness, and density conservation*, J. Stat. Phys. 61 (1990), 203–234.
4. J. Carr, *Asymptotic behaviour of solutions to the coagulation-fragmentation equations I. The strong fragmentation case*, Proc. Roy. Soc. Edinburgh 121A (1992), 231–244.
5. J. Carr and F. P. Da Costa, *Asymptotic behaviour of solutions to the coagulation-fragmentation equations. II. Weak fragmentation*, J. Stat. Phys. 77 (1994), 89–123.
6. F. P. Da Costa, *Existence and uniqueness of density conserving solutions to the coagulation fragmentation equations with strong fragmentation*, J. Math. Anal. Appl. 192 (1995), 892–914.
7. Ph. Laurençot, *Global solutions to the discrete coagulation equations*, Mathematika 46 (1999), no. 2, 433–442.
8. Ph. Laurençot, *The discrete coagulation equations with multiple fragmentation*, Proc. Edinb. Math. Soc. 45 (2002), no. 1, 67–82.
9. J. Banasiak, *Global classical solutions of coagulation fragmentation equations with unbounded coagulation rates*, Nonlinear Anal. Real World Appl. 13 (2012), 91–105.
10. J. Banasiak and W. Lamb, The discrete fragmentation equation: semigroups, compactness and asynchronous exponential growth, Kinet. Relat. Models 5 (2012), 223–236.
11. J. Banasiak, Analytic fragmentation semigroups and classical solutions to coagulation fragmentation equations—a survey, Acta Math. Sin. 35 (2019), 83–104.
12. R. L. Drake, A general mathematical survey of the coagulation equation, in topics in current aerosol research, part 2 international reviews in aerosol physics and chemistry, Pergamon Press, Oxford, 1972, pp. 203–376.
13. J. Banasiak, W. Lamb, and Ph. Laurençot, Analytic methods for coagulation-fragmentation models, Vol. 1 and 2, CRC Press, Boca Raton, 2019.
14. P. B. Dubovskiǐ, Structural stability of dispersive systems and finite nature of a coagulation front, J. Experim. Theor. Phys. 89 (1999), no. 2, 384–390.
15. P. B. Dubovskiǐ, A ‘triangle’ of interconnected coagulation models, J. Phys. A: Math. Gen. 32 (1999), no. 5, 781–793.
16. J. H. Oort and H. C. Van de Hulst, Gas and smoke in interstellar space, Bull. Astronom. Inst. Netherlands 10 (1946), 187–210.
17. V. S. Safronov, Evolution of the protoplanetary cloud and formation of the earth and the planets, 1972. Israel Program for Scientific Translations.
18. P. K. Barik, P. Rai, and A. K. Giri, Mass-conserving weak solutions to Oort–Hulst–Safronov coagulation equation with singular rates, J. Differ. Equ. 11 (2022), no. 5, 1125–1138.
19. M. Lachowicz, Ph. Laurençot, and D. Wrzosek, On the Oort–Hulst–Safronov coagulation equation and its relation to the Smoluchowski equation, SIAM J. Math. Anal. 34 (2003), 1399–1421.
20. V. Bagland and Ph. Laurençot, Self-similar solutions to the Oort–Hulst–Safronov coagulation equation, SIAM J. Math. Anal. 39 (2007), 345–378.
21. Ph. Laurençot, Convergence to self-similar solutions for a coagulation equation, Z. Angew. Math. Phys. 56 (2005), 398–411.
22. Ph Laurençot, Self-similar solutions to a coagulation equation with multiplicative kernel, Phys. D 222 (2006), 80–87.
23. V. Bagland, Convergence of a discrete Oort–Hulst–Safronov equation, Math. Methods Appl. Sci. 28 (2005), no. 13, 1613–1632.
24. J. Davidson, Existence and uniqueness theorem for the Safronov–Dubovskii coagulation equation, Z. Angew. Math. Phys. 65 (2014), no. 4, 757–766.
25. A. Das and J Saha, On the global solutions of discrete Safronov–Dubovskii aggregation equation, Z. Angew. Math. Phys. 72 (2021), 183.
26. S. Kaushik and R. Kumar, Uniqueness and mass conservation for Safronov–Dubovskii coagulation equation, Acta Appl. Math. 179 (2022), 10.
27. M. Ali and A. K. Giri, Global existence of solutions to the discrete Safronov–Dubovskii coagulation equations and failure of mass-conservation, J. Math. Anal. Appl. 519 (2023), no. 1, 126755.
28. A. N. Kolmogorov and S. V. Fomin, Introductory real analysis, (Prentice-Hall, Englewood Cliffs NJ, 1970.)