Spectral design for matrix Hamiltonians: different methods of constructing of a matrix intertwining operator

Andrey V Sokolov

Department of Theoretical Physics, Saint-Petersburg State University, Ulianovskaya ul., 1, Petrodvorets, 198504 Saint-Petersburg, Russia

E-mail: avs_avs@rambler.ru

Received 19 July 2014, revised 12 November 2014
Accepted for publication 9 December 2014
Published 4 February 2015

Abstract
We study relations describing the intertwining of $n \times n$ matrix, in general non-Hermitian, one-dimensional Hamiltonians by $n \times n$ matrix linear differential operators with nondegenerate coefficients of $d/dx$ in the highest degree. Some methods for constructing $n \times n$ matrix intertwining operators of the first order of general form are proposed and their interrelations are examined. We construct, as an example, a $2 \times 2$ matrix Hamiltonian of general form intertwined by an operator of the first order with the Hamiltonian with zero matrix potential. It is shown that one can add, for the final $2 \times 2$ matrix Hamiltonian with respect to the initial matrix Hamiltonian, with the help of an intertwining operator of first order, up to two bound states for different energy values, or up to two bound states described by vector-eigenfunctions for the same energy value, or up to two bound states described by a vector-eigenfunction and an associated vector-function for the same energy value.

Keywords: intertwining operator, matrix non-Hermitian Hamiltonian, spectrum, supersymmetry

1. Introduction

There are two main areas in which matrix models with supersymmetry are applied in quantum mechanics: multichannel scattering and spectral design in the description of the motion of spin particles in external fields. The simplest cases of such models are considered, for example, in [1–15] and systematic studies of them are contained in [16–30] (see also the recent reviews [31, 32] and references therein). The authors of [16] investigate the
intertwining of matrix Hermitian Hamiltonians by $n \times n$ first-order and $2 \times 2$ second-order matrix differential operators and the corresponding supersymmetric algebras. The main result of [17] is the set of formulas that provide us with the opportunity to construct for a given $n \times n$ matrix, in general non-Hermitian, Hamiltonian a new $n \times n$ matrix Hamiltonian and an $n \times n$ matrix linear differential operator of arbitrary order with the identity matrix coefficient of $d/dx$ of the highest degree that intertwines these Hamiltonians.

There are some shortcomings of the results of [17]. Firstly, the formulas of [17] are built in terms of a basis in a subspace that is invariant with respect to the initial Hamiltonian, i.e. an $n \times n$ matrix intertwining operator of the $N$th order and the corresponding new Hamiltonian are constructed in terms of columns of an $n \times nN$ matrix-valued solution $\Psi(x)$ of the equation

$$H_+ \Psi = \Psi \Lambda,$$

(1)

where $H_+$ and $\Lambda$ are respectively the initial Hamiltonian and the $nN \times nN$ constant matrix. It was shown in [23] that one can get any $n \times n$ matrix intertwining operator of arbitrary order with arbitrary nondegenerate matrix coefficient of $d/dx$ in the highest degree and the corresponding new Hamiltonian with the help of a matrix-valued solution of (1) such that $\Lambda'$ for this solution is a matrix of normal (Jordan) form. The columns of the solution $\Psi(x)$ in this case are obviously formal vector-eigenfunctions and formal associated vector-functions of the Hamiltonian $H_+$, where the word ‘formal’ emphasizes that these vector-functions are not necessarily normalizable. It seems easier to find formal vector-eigenfunctions and formal associated vector-functions of the Hamiltonian $H_+$ and to construct the matrix $\Psi(x)$ from these vector-functions than to look for a matrix solution $\Psi(x)$ of general form for (1) as was proposed in [17]. Hence, the method for constructing a matrix intertwining operator and the corresponding new matrix Hamiltonian in terms of formal vector-eigenfunctions and associated vector-functions of the initial Hamiltonian $H_+$ offered in [23] (see also the partial case of this method based on just the use of formal vector-eigenfunctions, in [18]) allows us to simplify without loss of generality the procedure proposed in [17].

Secondly, the formulas of [17] are unnecessarily complicated since they contain quasi-determinants, introduced in [33]. The significantly simpler formulas in terms of the usual determinants for constructing a matrix intertwining operator with the identity matrix coefficient of $d/dx$ in the highest degree and the corresponding new Hamiltonian were derived in a rather sophisticated way in [18]. But the formulas of [18] were obtained only for the partial case, where all columns of $\Psi(x)$ are formal vector-eigenfunctions of $H_+$ and the intertwined Hamiltonians are Hermitian. It should be emphasized that applying only formal vector-eigenfunctions of $H_+$ as columns in $\Psi(x)$ results in significant narrowing of the set of intertwining operators obtained even in the case where $H_+$ is Hermitian. The formulas that provide us with the opportunity to build, with the help of the usual determinants, any $n \times n$ matrix intertwining operator of arbitrary order with arbitrary constant nondegenerate matrix coefficient of $d/dx$ in the highest degree for a given $n \times n$ matrix, in general non-Hermitian, initial Hamiltonian $H_+$ and the corresponding new matrix Hamiltonian were obtained in a simple way in [23]. In the partial case of [18], the formulas of [23] indicated correspond to the formulas of [18]. A detailed analysis of some more shortcomings of [17] and [18] can be found in [23].

The paper [19] contains the formulas that allow us to construct any $n \times n$ matrix differential intertwining operator of the first order with arbitrary nondegenerate matrix coefficient of $d/dx$ in terms of the $n \times n$ matrix-valued solution $\Psi(x)$ of equation (1) for the case where the Hamiltonian $H_+$ is Hermitian. Also, the author of [19] considers the corresponding supersymmetry algebra for the case where the aforementioned coefficient of $d/dx$ is the identity matrix and both intertwined Hamiltonians are Hermitian, builds $n \times n$ matrix
differential intertwining operators of higher orders from chains of first-order \( n \times n \) matrix differential intertwining operators, and investigates in detail \( n \times n \) matrix intertwining operators of the second order obtained in this way.

The generalization of results from the paper [18] to the case of a degenerate matrix coefficient of an intertwining operator of \( d/dx \) in the highest degree is considered in [20]. The author of [21] builds \( n \times n \) matrix differential intertwining operators of the second order for Hermitian matrix Hamiltonians with all real-valued elements in their potentials in terms of two \( n \times n \) matrix-valued solutions \( \Psi_1(x) \) and \( \Psi_2(x) \) of equation (1) for the matrices \( \Lambda_1 \) and \( \Lambda_2 \), respectively, in its right-hand side of the form

\[
\Lambda_1 = E_1 I_n, \quad \Lambda_2 = E_2 I_n, \quad E_1, E_2 \in \mathbb{C},
\]

where \( I_n \) is the identity matrix of the \( n \)th order. Also the corresponding polynomial supersymmetry algebra of the second order is constructed and different applications of the results obtained are examined in [21].

The author of [22] proposes studying a supersymmetry generated by two \( n \times n \) matrix, in general non-Hermitian, Hamiltonians \( H_+ \) and \( H_- \) and two \( n \times n \) matrix differential operators \( Q_N^+ \) and \( Q_N^- \) of the same order \( N \) with constant coefficients proportional to the identity matrix of \( (d/dx)^N \) that intertwine \( H_+ \) and \( H_- \) in opposite directions and such that the products \( Q_N^+ Q_N^- \) and \( Q_N^- Q_N^+ \) are the same polynomials with matrix coefficients of \( H_+ \) and \( H_- \), respectively. Moreover, the operators \( Q_N^+ \) and \( Q_N^- \) are supposed to be related to one another by some unnatural operation which is not, in general, either a transposition or a Hermitian conjugation. Hence, intertwining of \( H_+ \) and \( H_- \) by one of the operators \( Q_N^+ \) and \( Q_N^- \) does not lead, in general, to the intertwining of \( H_+ \) and \( H_- \) by another of the operators \( Q_N^+ \) and \( Q_N^- \) even if the Hamiltonians \( H_+ \) and \( H_- \) are both symmetric with respect to transposition or Hermitian conjugation. Thus, the intertwining operators \( Q_N^+ \) and \( Q_N^- \) generate independent—in general—restrictions on the system in question. In addition, in [22] there is neither a proof of existence of the system considered for arbitrary \( n \) and \( N \), nor any general method for constructing this system. Only for the case \( n = N = 2 \) does the author find the general form of \( H_+, H_-, Q_N^+ \) and \( Q_N^- \) under the additional assumption that \( H_+, H_- \) and all coefficients of the operators \( Q_N^+ \) and \( Q_N^- \) are Hermitian.

The paper [23] contains, in addition to the formulas for constructing an arbitrary matrix intertwining operator and the corresponding new matrix Hamiltonian (see above), results on the existence, for an arbitrary \( n \times n \) matrix intertwining operator of the order \( N \) with arbitrary nondegenerate matrix coefficient of \( (d/dx)^N \), an \( n \times n \) matrix differential operator of different—in general—order \( N' \) that intertwines the same Hamiltonians in the opposite directions and on the corresponding polynomial supersymmetry algebra. Earlier, the case of two scalar differential operators of different—in general—orders that intertwine two scalar differential operators of partial form in the opposite directions was considered in [34]. Also there are given in [23] the criteria of minimizability [35, 36] and of reducibility [37–42] for a matrix intertwining operator.

Some supersymmetric matrix models with shape invariance are investigated in [24–28]. Most of the papers mentioned above on the matrix case are devoted in fact to the case of one spatial variable. The cases of two and three spatial variables are considered in [1, 4, 5, 25, 29, 30].

The purpose of this paper is (i) to derive some methods for constructing an arbitrary \( n \times n \) matrix first-order intertwining operator with arbitrary constant nondegenerate matrix coefficient of \( d/dx \) and the corresponding new matrix Hamiltonian in the case where both intertwined Hamiltonians are, in general, non-Hermitian, (ii) to investigate the interrelations of
these methods and (iii) to demonstrate the capabilities of these methods for the spectral design of matrix Hamiltonians. The present paper is organized as follows. Section 2 contains basic definitions and notation. Section 3 is devoted to the derivation of some methods for constructing any \( n \times n \) matrix first-order intertwining operator with arbitrary constant non-degenerate matrix coefficient of \( \frac{d}{dx} \) and the corresponding new matrix Hamiltonian. Namely, we present the matrix superpotential method and one more method in section 3.1, the method of transformation vector-functions in section 3.2 and the transformation matrix method in section 3.3. Also we examine in section 3 the interrelations of these methods. The advantages of the methods described with respect to the methods described earlier in different papers are the following: (i) the methods of section 3.1 are new; (ii) the advantages of the method of section 3.2, compared with the analogous method of [18], are the use of associated vector-functions, which allows us to significantly extend the set of Hamiltonians constructed, and the use of Hamiltonians that are in general non-Hermitian; (iii) the advantage of the method of section 3.3 in comparison with the methods of [17, 19] is the use, without loss of generality, of a transformation matrix of simpler form constructed from only formal vector-eigenfunctions and associated vector-functions, and not an arbitrary solution of (1). At the end of section 3 we present a sufficient condition that provides \( PT \)-symmetry (see, on \( PT \)-symmetry and its significance for non-Hermitian Hamiltonians, for example [43–45] and references therein) for the final matrix Hamiltonian. Section 4 includes a brief description of the generalization of the method of transformation vector-functions to the case of a matrix intertwining operator of arbitrary order. In section 5 we present three examples that demonstrate the capabilities of the methods of section 3 for the spectral design of matrix Hamiltonians. It is shown that one can add, for the final \( 2 \times 2 \) matrix Hamiltonian with respect to the initial \( 2 \times 2 \) matrix Hamiltonian, with the help of a \( 2 \times 2 \) first-order matrix intertwining operator, up to two bound states for different energy values (section 5.1), or up to two bound states described by vector-eigenfunctions for the same energy value (section 5.2), or up to two bound states described by a vector-eigenfunction and an associated vector-function for the same energy value (section 5.3). In the conclusions, we itemize some problems which could be considered in future papers.

2. Basic definitions and notation

2.1. Intertwining relation

Let us consider two matrix Hamiltonians defined on the entire axis, of Schrödinger form:

\[
H_+ = -I_n \frac{\partial^2}{\partial x^2} + V_+(x), \quad H_- = -I_n \frac{\partial^2}{\partial x^2} + V_-(x), \quad \partial \equiv \frac{d}{dx},
\]

where \( I_n \) is the identity matrix of the \( n \)th order, \( n \in \mathbb{N} \), and \( V_+(x) \) and \( V_-(x) \) are square \( n \times n \) matrices, all of whose elements are sufficiently smooth and, in general, complex-valued functions. These Hamiltonians are supposed to be intertwined by a matrix linear differential operator \( Q_N \), such that

\[
Q_N H_+ = H_- Q_N, \quad Q_N = \sum_{j=0}^{N} X_j^-(x) \partial^j,
\]

where \( X_j^-(x), \quad j = 0, \ldots, N \), are also square \( n \times n \) matrices, all of whose elements are sufficiently smooth and, in general, complex-valued functions. The operator \( Q_N \) in this case is called an intertwining operator.
It follows from (2) (see [23]) that

\[ X_N^N = \text{Const} \quad \text{and} \quad X_N^N V_4(x) = -2X_{N-1}^N(x) + V_-(x)X_N^- . \quad (3) \]

We shall suppose below that \( \det X_N^- \neq 0 \). In this case one can find from (3) the matrix potential \( V_-(x) \) in terms of \( V_+(x) \) and \( X_{N-1}^- \):

\[ V_-(x) = X_N^- V_4(x)(X_N^-)^{-1} + 2X_{N-1}^- (x)(X_N^-)^{-1}. \quad (4) \]

2.2. The structure of intertwining operator kernel and transformation vector-functions

In view of (2), the kernel of the intertwining operator \( Q_N^- \) is an invariant subspace for the Hamiltonian \( H_+ \):

\[ H_+, \ker Q_N^- \subset \ker Q_N^- . \]

Therefore, for any basis \( \Phi_1^-, \ldots, \Phi_d^- \) in the kernel of \( Q_N^- \), \( d = \dim \ker Q_N^- = nN \) (see [46]; the number \( d \) for \( \det X_N^- = 0 \) can be different from \( nN \)), there exists a constant square \( d \times d \) matrix \( T^+ \) such that

\[ H_+, \Phi_i^- = \sum_{j=1}^d T_{ij}^+ \Phi_j^- , \quad i = 1, \ldots, d. \quad (5) \]

Let us note that the Wronskian of all elements of any basis in \( \ker Q_N^- \) does not vanish on the entire axis.

One can construct from the elements of the basis \( \Phi_1^-, \ldots, \Phi_d^- \), as from columns, the \( n \times d \) matrix-valued solution

\[ \Psi(x) = (\Phi_1^-(x), \ldots, \Phi_d^- (x)) \]

of equation (1), and the matrix \( \Lambda \) from (1) is interrelated with the matrix \( T^+ \) by the obvious equality

\[ \Lambda = (T^+)\dagger . \]

In the what follows, the matrix \( T \) of an intertwining operator is defined as a matrix which is constructed for the operator in the same way as the matrix \( T^+ \) is constructed for \( Q_N^- \). In this case, we do not specify the basis in the kernel of the intertwining operator in which the matrix \( T \) is chosen if we are concerned only with spectral characteristics of the matrix, or, what comes to the same, spectral characteristics of the restriction of the corresponding Hamiltonian to the kernel of the intertwining operator considered (cf (5)).

A basis in the kernel of an intertwining operator in which the matrix \( T \) of this operator has a normal (Jordan) form is called a canonical basis. Elements of a canonical basis are called transformation vector-functions.

If a Jordan form of the matrix \( T \) of an intertwining operator contains block(s) of order higher than 1, then the corresponding canonical basis contains not only formal vector-eigenfunction(s) of the corresponding Hamiltonian but also its formal associated vector-function(s) which are defined as follows (see [46]).

A vector-function \( \Phi_{m,i} (x) \) is called a formal associated vector-function of \( \text{ith order} \) of an \( n \times n \) matrix Hamiltonian \( H = -I_n \partial^2 + V(x) \) for a spectral value \( \lambda_m \) if

\[ (H - \lambda_m I_n)^{i+1} \Phi_{m,i} \equiv 0 \quad \text{and} \quad (H - \lambda_m I_n)^i \Phi_{m,i} \not\equiv 0 , \]

\[ (H - \lambda_m I_n)^{i+1} \Phi_{m,i} \equiv 0 \quad \text{and} \quad (H - \lambda_m I_n)^i \Phi_{m,i} \not\equiv 0 , \]

\[ (H - \lambda_m I_n)^{i+1} \Phi_{m,i} \equiv 0 \quad \text{and} \quad (H - \lambda_m I_n)^i \Phi_{m,i} \not\equiv 0 , \]
where the term ‘formal’ emphasizes that the vector-function \( \Phi_{m,i}(x) \) is not necessarily normalizable (does not necessarily belong to \( L^2(\mathbb{R}, \mathbb{C}^n) \)). In particular, a formal associated vector-function of zero order \( \Phi_{m,0}(x) \) is a formal vector-eigenfunction of \( H \).

A finite or infinite set of vector-functions \( \Phi_{m,i}(x) \), \( i = 0, 1, 2, ... \), is called a chain of formal associated vector-functions of an \( n \times n \) matrix Hamiltonian \( H = -L_n \partial^2 + V(x) \) for a spectral value \( \lambda_m \) if

\[
H \Phi_{m,0} = \lambda_m \Phi_{m,0}, \quad \Phi_{m,0}(x) \not\equiv 0, \quad (H - \lambda_m L_n) \Phi_{m,i} = \Phi_{m,i-1}, \quad i = 1, 2, 3, ....
\]

It is evident that \( \Phi_{m,i}(x) \) in this case is a formal associated vector-function of \( i \)th order of the Hamiltonian \( H \) for the spectral value \( \lambda_m, i = 0, 1, 2, ... \).

A chain \( \Psi_{m,l}^{+}(x), l = 0, 1, 2, ... \), of formal associated vector-functions of the Hamiltonian \( H_+ \) for a spectral value \( \lambda_m \), in view of the equalities

\[
(H_+ - \lambda_m L_n) Q_N \Psi_{m,l}^{+} = Q_N (H_+ - \lambda_m L_n) \Psi_{m,l}^{+} = Q_N \Psi_{m,l-1}^{+},
\]

that arise due to (2), is mapped by \( Q_N \) into a chain of formal associated vector-functions of the same spectral value \( \lambda_m \), with the possible exception of some number of vector-functions \( Q_N \Psi_{m,l_0}^{-} \) with lower numbers which can be identical zeros. It is clear in view of (6) that if \( Q_N \Psi_{m,l_0}^{-} \equiv 0 \) for some \( l_0 \), then \( Q_N \Psi_{m,l}^{-} \equiv 0 \) for any \( l < l_0 \), and if \( Q_N \Psi_{m,l_0}^{-} \not\equiv 0 \) for some \( l_0 \), then \( Q_N \Psi_{m,l}^{-} \not\equiv 0 \) for any \( l > l_0 \). Thus, if \( l_0 \) is a minimal number such that \( Q_N \Psi_{m,l_0}^{-} \not\equiv 0 \), then one can represent the chain of formal associated vector-functions of \( H_- \) arising in the form

\[
\Psi_{m,l}^{+}(x) = Q_N \Psi_{m,l_0}^{-}(x), \quad l = 0, 1, 2, ....
\]

3. Methods for constructing a first-order matrix intertwining operator

3.1. The matrix superpotential method and one more method

Let us consider the case where two \( n \times n \) matrix Hamiltonians \( H_+ \) and \( H_- \) are intertwined by a first-order \( n \times n \) matrix differential operator

\[
Q_1^- = X_1^- \partial + X_0^- (x),
\]

such that

\[
Q_1^- H_+ = H_- Q_1^-.
\]

In view of section 2.1, we know that the matrix coefficient \( X_1^- \) is a constant nondegenerate matrix. Thus, we can rewrite the equality (7), with the help of a multiplication of it from the left by \((X_1^-)^{-1}\), in the form

\[
\left((X_1^-)^{-1} Q_1^-\right) H_+ = \left((X_1^-)^{-1} H_- X_1^-\right)\left((X_1^-)^{-1} Q_1^-\right).
\]

It follows from the latter equality that two \( n \times n \) matrix Hamiltonians

\[
H_+ = -L_n \partial^2 + V_+(x), \quad \tilde{H}_- = -L_n \partial^2 + \tilde{V}_-(x), \quad \tilde{V}_-(x) = (X_1^-)^{-1} V_-(x) X_1^-,
\]

are intertwined by the first-order \( n \times n \) matrix differential operator

\[
\tilde{Q}_1^- = L_n \partial + \tilde{X}_0^- (x), \quad \tilde{X}_0^- (x) = (X_1^-)^{-1} X_0^- (x),
\]

\[
\tilde{Q}_1^- H_+ = H_- \tilde{Q}_1^-.
\]
such that
\[ \tilde{\mathcal{Q}}_i^{-} H_+ = \tilde{H}_- \tilde{\mathcal{Q}}_i^{-}. \] (8)

Now we shall look for a general solution of the intertwining relation (8). This solution can be found (see below) in the form of a parameterization of the potentials \( V_+ (x) \) and \( \tilde{V}_- (x) \) and of the superpotential \( \tilde{X}_0^- (x) \) in terms of \( n^2 \) arbitrary scalar functions which are, in general, complex-valued. After obtaining this solution, a general solution of the intertwining relation (7) can be restored with the help of the following obvious relations:
\[ V_+ (x) = V_+ (x), \quad V_- (x) = X_1^- \tilde{V}_- (x) (X_1^-)^{-1}, \quad X_0^- (x) = X_1^- \tilde{X}_0^- (x), \] (9)
with arbitrary nondegenerate \( n \times n \) matrix \( X_1^- \).

Intertwining relation (8) is equivalent to two equations:
\[ V_+ (x) = -2 \tilde{X}_0^- (x) + \tilde{V}_- (x), \quad V_+ (x) + \tilde{X}_0^- (x) V_+ (x) = - \tilde{X}_0^- (x) + \tilde{V}_- (x) \tilde{X}_0^- (x). \] (10)
It follows from the first of these equations that \( V_+ (x) \) and \( \tilde{V}_- (x) \) can be represented in the form
\[ V_+ (x) = V_0^- (x) - \tilde{X}_0^- (x), \quad V_- (x) = V_0^- (x) + \tilde{X}_0^- (x), \] (11)
with some unknown \( n \times n \) matrix-valued function \( V_0^- (x) \). This function, by virtue of the second equation in (10), satisfies the equation
\[ V_0^- (x) = \left[ V_0^- (x), \tilde{X}_0^- (x) \right] + \tilde{X}_0^- (x) \tilde{X}_0^- (x) + \tilde{X}_0^- (x) \tilde{X}_0^- (x) \tilde{X}_0^- (x). \]
The latter equation, after the change
\[ V_0^- (x) = U_0^- (x) + \left( \tilde{X}_0^- (x) \right)^2, \] (12)
transforms into
\[ U_0^- (x) = \left[ U_0^- (x), \tilde{X}_0^- (x) \right], \] (13)
where \( U_0^- (x) \) is a new unknown \( n \times n \) matrix-valued function.

A general solution of the equation (13) can be constructed in various ways. One of these ways is the following. One can consider all \( n^2 \) elements of the matrix superpotential \( \tilde{X}_0^- (x) \) as arbitrary complex-valued—in general—parameterizing functions. Then the equation (13) is a system of \( n^2 \) linear first-order ordinary differential equations with respect to elements of the matrix \( U_0^- (x) \). The general solution of this system is parameterized via \( n^2 \) arbitrary functions (elements of \( \tilde{X}_0^- (x) \)) and \( n^2 \) arbitrary complex—in general—constants.

Another way to find a general solution of the system (13) is to take all \( n^2 \) elements of the matrix \( U_0^- (x) \) as arbitrary complex-valued—in general—parameterizing functions. Then equation (13) is a system of \( n^2 \) linear algebraic equations (SLAE) with respect to elements of the matrix superpotential \( \tilde{X}_0^- (x) \). This SLAE is degenerate in general, and conditions for its compatibility lead to restrictions on the elements of the matrix \( U_0^- (x) \) and, consequently, to a decrease in the number of independent functions among the elements of the matrix \( U_0^- (x) \). Nevertheless, the total number of independent parameterizing functions is again equal to \( n^2 \) due to the appearance of free variables and to the obvious feature that the number of compatibility conditions is equal to the number of free variables appearing. Thereby, the general solution of the SLAE (13) is parameterized using \( n^2 \) arbitrary functions (independent elements of \( U_0^- (x) \) and free variables).

The latter of the two methods of parameterization of the general solution of (13) described above is more suitable than the former, since the latter method, in contrast to the
former one, leads to explicit parameterizing formulas. Two more ways of constructing a
general solution of (13) will be presented in the following two subsections.

Thus, a general solution of the intertwining relation (8) is given, in view of (11) and (12),
by the formulas

\[
V_x(x) = U_0(x) + \left(\tilde{X}_0^{-}(x)\right)^2 - \tilde{X}_0^{-}(x), \quad \tilde{V}_-(x) = U_0(x) + \left(\tilde{X}_0^{-}(x)\right)^2 + \tilde{X}_0^{-}(x),
\]

(14)

where \(U_0(x)\) and \(\tilde{X}_0^{-}(x)\) are found in one of the ways described. Hence, a general solution of
the intertwining relation (7) is given by (9) together with (14).

It is evident that in view of (14), the Hamiltonians \(H_+\) and \(\tilde{H}_-\) can be represented in the
form

\[
H_+ = \tilde{Q}^+_1 \tilde{Q}^+_1 + U_0(x), \quad \tilde{H}_- = \tilde{Q}^-_1 \tilde{Q}^+_1 + U_0(x), \quad \tilde{Q}^+_1 = -I_0 \partial + \tilde{X}_0^{-}(x).
\]

Moreover, the intertwining relation (8) for these Hamiltonians is provided by the condition

\[
\left[ U_0(x), \tilde{Q}^-_1 \right] = 0,
\]

(16)

which is equivalent to the equation (13).

The intertwining of the Hamiltonians \(H_+\) and \(\tilde{H}_-\) by the operator \(\tilde{Q}^+_1\),

\[
H_+ \tilde{Q}^+_1 = \tilde{Q}^+_1 \tilde{H}_-,
\]

(17)

is equivalent (in the case where this intertwining takes place) to the condition

\[
\left[ U_0(x), \tilde{Q}^+_1 \right] = 0.
\]

(18)

The latter condition is equivalent, in turn, in view of (16), to the equality

\[
\left[ U_0(x), \partial \right] = 0,
\]

(19)

i.e. to independence of all elements of the matrix \(U_0(x)\) from \(x\).

By virtue of (9) and (15), a general solution of the intertwining relation (7) can be
represented in the form

\[
H_+ = Q^+_1 Q^-_1 + U_0(x), \quad H_- = Q^-_1 Q^+_1 + U(x), \quad U(x) = X^-_1 U_0(x)(X^-_1)^{-1},
\]

(20)

\[
Q^-_1 \equiv X^-_1 \partial + X^-_0(x) = X^-_1 \tilde{Q}^-_1, \quad X^-_0(x) = X^-_1 \tilde{X}_0^{-}(x),
\]

(21)

\[
Q^+_1 \equiv X^+_1 \partial + X^+_0(x) = \tilde{Q}^+_1 (X^-_1)^{-1}, \quad X^+_0(x) = -\tilde{X}_0^{-}(x)(X^-_1)^{-1}.
\]

(22)

The intertwining relation (7) for the Hamiltonians constructed, \(H_+\) and \(H_-\), is valid due to the
relation

\[
Q^-_1 U_0(x) = U(x) Q^-_1
\]

which follows from (20), (22) and (18). It is easy to see that the intertwining

\[
H_+ Q^+_1 = Q^+_1 H_-
\]

is equivalent to the relation

\[
U_0(x) Q^+_1 = Q^+_1 U(x),
\]

which is equivalent, in turn, by (21) and (22), to (21) and, consequently, to (19). The latter is
obviously equivalent to the independence of \(U_0(x)\) and \(U(x)\) from \(x\).
3.2. The method of transformation vector-functions

Let us consider $H_\ast$ as a known initial $n \times n$ matrix Hamiltonian, and let $\Phi_l^\ast(x)$, $l = 1, \ldots, n$, be a set of formal associated vector-functions of $H_\ast$ such that

$$H_\ast \Phi_l = \lambda_l \Phi_l + \sigma_l \Phi_{l+1}, \quad \sigma_l = \begin{cases} 1, & \text{if } \Phi_l^\ast(x) \text{ is not a formal vector–eigenfunction}, \\ 0, & \text{if } \Phi_l^\ast(x) \text{ is a formal vector–eigenfunction}, \end{cases}$$

(23)

$\Phi_l^\ast(x) \equiv \begin{pmatrix} \varphi_{11}^\ast(x) \\ \varphi_{12}^\ast(x) \\ \vdots \\ \varphi_{mn}^\ast(x) \end{pmatrix}$, $l = 1, \ldots, n$, $\Phi_{n+1}^\ast(x) \equiv \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$,

(24)

where $\lambda_l$ is the spectral value of $H_\ast$ corresponding to $\Phi_l^\ast(x)$, $l = 1, \ldots, n$, and

$$\lambda_{l+1} = \lambda_l \quad \text{if} \quad \sigma_l = 1, \quad l = 1, \ldots, n - 1.$$  (25)

We shall suppose that the Wronskian of these vector-functions

$$W(x) \equiv \begin{vmatrix} \varphi_{11}^\ast(x) & \varphi_{12}^\ast(x) & \cdots & \varphi_{mn}^\ast(x) \\ \varphi_{21}^\ast(x) & \varphi_{22}^\ast(x) & \cdots & \varphi_{2n}^\ast(x) \\ \vdots & \vdots & \ddots & \vdots \\ \varphi_{n1}^\ast(x) & \varphi_{n2}^\ast(x) & \cdots & \varphi_{nn}^\ast(x) \end{vmatrix}$$

(26)

does not vanish on the entire axis. In this case we can consider all $n^2$ elements of the matrix potential $V_\varphi(x)$ of the Hamiltonian $H_\ast$ as functions that implicitly parameterize vector-functions $\Phi_l^\ast(x)$, $l = 1, \ldots, n$, and, consequently, the $n \times n$ matrix superpotential $X_\ast(x)$, from the intertwining operator $Q_\ast = X_\ast \partial + X_\ast (x)$ and the $n \times n$ matrix potential $V_\ast(x)$ of the final Hamiltonian $H_\ast$, which will be constructed below in terms of $\Phi_l^\ast(x)$, $l = 1, \ldots, n$.

It is possible also to suppose that the Hamiltonian $H_\ast$ is not known initially and that the vector-functions (24) are arbitrary vector-functions with complex-valued—in general—components such that the Wronskian (26) does not vanish on the entire axis. In this case one can choose arbitrarily constants $\lambda_l \in \mathbb{C}$ and $\sigma_l \in \{0, 1\}$, $l = 1, \ldots, n$, such that the conditions (25) are valid, and thereafter find the only $n \times n$ matrix potential

$$V_\varphi(x) \equiv \| v_i^\varphi (x) \|$$

of the Hamiltonian $H_\ast$ such that the relations (23) hold, with the help of a solving of the following SLAE:

$$\begin{cases} v_{i1}^\varphi \varphi_{11}^\ast + v_{i2}^\varphi \varphi_{12}^\ast + \cdots + v_{in}^\varphi \varphi_{in}^\ast = \varphi_{1l}^\ast + \lambda_l \varphi_{i1}^\ast + \sigma_l \varphi_{2l}^\ast, \\ v_{i1}^\varphi \varphi_{21}^\ast + v_{i2}^\varphi \varphi_{22}^\ast + \cdots + v_{in}^\varphi \varphi_{2n}^\ast = \varphi_{2l}^\ast + \lambda_2 \varphi_{i2}^\ast + \sigma_2 \varphi_{3l}^\ast, \quad l = 1, \ldots, n, \\ v_{i1}^\varphi \varphi_{n1}^\ast + v_{i2}^\varphi \varphi_{n2}^\ast + \cdots + v_{in}^\varphi \varphi_{nn}^\ast = \varphi_{nl}^\ast + \lambda_n \varphi_{in}^\ast + \sigma_n \varphi_{n+1,l}^\ast, \end{cases}$$

which are equivalent to (23). Any of these SLAE possesses only one solution, due to the fact that $W(x)$ does not vanish on the entire axis, and, thus, the elements of $V_\varphi(x)$ can be found with the help of Cramer formulas:
\[ v^+_l = \frac{1}{W} \begin{pmatrix} \varphi_{11}^- & \varphi_{12}^- & \ldots & \varphi_{1, l-1}^- + \lambda_1 \varphi_{1, l-1}^+ + \sigma_1 \varphi_{2l}^- & \varphi_{1, l+1}^- & \ldots & \varphi_{1n}^- \\ \varphi_{21}^- & \varphi_{22}^- & \ldots & \varphi_{2, l-1}^- + \lambda_2 \varphi_{2, l-1}^+ + \sigma_2 \varphi_{3l}^- & \varphi_{2, l+1}^- & \ldots & \varphi_{2n}^- \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \varphi_{nl}^- & \varphi_{n, l-1}^- & \ldots & \varphi_{n, l-1}^- + \lambda_n \varphi_{n, l-1}^+ + \sigma_n \varphi_{nl+1}^- & \varphi_{n, l+1}^- & \ldots & \varphi_{ln}^- \end{pmatrix}, \]

\[ l, j = 1, \ldots, n. \] (27)

In this case one can consider all \( n^2 \) components of \( \Phi_l^- (x) \), \( l = 1, \ldots, n \), as parameterizing functions. Then the elements of \( V_+ (x) \) are parameterized with these components explicitly with the help of (27), and explicit parameterizations of \( X_0^- (x) \) and \( V_- (x) \) in terms of the components considered will be presented below. Thus, the parameterization in terms of the elements of \( V_+ (x) \), since the former is explicit and the latter is implicit.

Let us now construct an auxiliary operator \( \tilde{Q}_1^- \), operators \( Q_1^- \) and \( Q_1^+ \), and Hamiltonian \( H_- \), and thereafter check that \( H_+ \) and \( H_- \) are intertwined by \( Q_1^- \).

There is only one \( n \times n \) matrix linear differential operator \( \tilde{Q}_1^- \) of the form

\[ \tilde{Q}_1^- = \frac{1}{W(x)} \begin{pmatrix} \varphi_{11}^- (x) & \varphi_{12}^- (x) & \ldots & \varphi_{1, l-1}^- (x) & \varphi_{1l}^- (x) \\ \varphi_{21}^- (x) & \varphi_{22}^- (x) & \ldots & \varphi_{2, l-1}^- (x) & \varphi_{2l}^- (x) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \varphi_{nl}^- (x) & \varphi_{n, l-1}^- (x) & \ldots & \varphi_{n, l-1}^- (x) & \varphi_{nl}^- (x) \\ P_1 & P_2 & \ldots & P_n & I_n \end{pmatrix}, \]

\[ \Phi = \varphi_l, \ \forall \ \Phi (x) = \begin{pmatrix} \varphi_1 (x) \\ \varphi_2 (x) \\ \vdots \\ \varphi_l (x) \end{pmatrix}, \ l = 1, \ldots, n, \] (28)

where during the calculation of the determinant, in each of its terms, the corresponding one of the operators \( P_1, \ldots, P_n, I_n \partial \) must be placed in the last position. It is not hard to see, in view of (28), that the \( l \)th column of the matrix \( X_0^- (x) \) is equal to

\[ -\frac{1}{W(x)} \begin{pmatrix} \varphi_{11}^- (x) & \varphi_{12}^- (x) & \ldots & \varphi_{1, l-1}^- (x) & \varphi_{1l}^- (x) \\ \varphi_{21}^- (x) & \varphi_{22}^- (x) & \ldots & \varphi_{2, l-1}^- (x) & \varphi_{2l}^- (x) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \varphi_{nl}^- (x) & \varphi_{n, l-1}^- (x) & \ldots & \varphi_{n, l-1}^- (x) & \varphi_{nl}^- (x) \\ \varphi_{11}^- (x) & \varphi_{12}^- (x) & \ldots & \varphi_{1, l-1}^- (x) & \varphi_{1l}^- (x) \end{pmatrix}, \ l = 1, \ldots, n. \] (29)

Using the operator \( \tilde{Q}_1^- \) and arbitrary nondegenerate matrix \( X^-_0 \), one can construct the operators \( Q_1^- \) and \( Q_1^+ \) with the help of the formulas (21) and (22) with \( Q_1^+ = -I_n \partial + \tilde{X}_0^- (x) \), represent the Hamiltonian \( H_+ \) in the form

\[ H_+ = Q_1^+ Q_1^- + U_0 (x), \quad U_0 (x) = V_+ (x) - \left( \tilde{X}_0^- (x) \right)^2 + \tilde{X}_0^- (x) \] (30)
(cf (14) and (20)), and build a new Hamiltonian of Schrödinger form:
\[
H_\equiv -i_n \partial^2 + V_- (x) = Q^-_1 Q^+_1 + U (x), \quad U (x) = X^-_1 U_0 (x) (X^-_1)^{-1},
\]
\[
V_- (x) = X^-_1 \left[ (X^-_0 (x))^2 + X^-_0 (x) + U_0 (x) \right] (X^-_1)^{-1} = X^-_1 V_0 (x) + 2X^-_0 (x) (X^-_1)^{-1}
\]
\[
= X^-_1 V_0 (x) (X^-_1)^{-1} + 2X^-_0 (x) (X^-_1)^{-1}
\]
(31)
(cf (4) and (20)).

We shall check now that the Hamiltonians \(H_+\) and \(H_-\) are intertwined by \(Q^-_1\) in accordance with (7). This intertwining is, in view of (30) and (31), equivalent to the condition
\[
Q^-_1 U_0 (x) - U (x) Q^-_1 = 0.
\]
(32)
The left-hand part of (32), by virtue of (21), (30) and (31), is an \(n \times n\) matrix-valued function, and the following chain is valid due to the construction of \(Q^-_1\) and to (23) and (30):
\[
\left[ Q^-_1 U_0 (x) - U (x) Q^-_1 \right] \Phi_l = \left[ Q^-_1 H_+ - Q^-_1 Q^+_1 Q^-_1 - U (x) Q^-_1 \right] \Phi_l
\]
\[
= Q^-_1 H_+ \Phi_l = Q^-_1 \left[ \lambda_I \Phi_l + \sigma_l \Phi_{l+1} \right]
\]
\[
= 0, \quad l = 1, \ldots, n.
\]
Thus, in view of the fact that the Wronskian \(W(x)\) of the vector-functions \(\Phi_l (x), l = 1, \ldots, n\), does not vanish on the entire axis, we have that the condition (32) holds and, consequently, the operator \(Q^-_1\) intertwines the Hamiltonians \(H_+\) and \(H_-\).

Let us note that the condition that the Wronskian \(W(x)\) does not vanish on the entire axis provides existence and smoothness (absence of pole(s)) for all matrix-valued functions \(V_0 (x)\), \(V_- (x), X^-_0 (x), U_0 (x)\) and \(U(x)\) considered in this subsection and also for the coefficients \(X^-_0 (x)\) and \(X^-_1 (x)\) of the operators \(Q^-_1\) and \(Q^+_1\) (see (21) and (22)).

All objects of this subsection coincide with the objects of the previous subsection denoted in the same way if one chooses the vector-functions \(\Phi_l (x), l = 1, \ldots, n\), in this subsection as elements of a canonical basis in the kernel of the intertwining operator \(Q^-_1\) from the previous subsection. This statement is valid in view of the fact that a matrix linear first-order differential operator with a fixed nondegenerate matrix coefficient of \(\partial\) is specified uniquely by a basis in its kernel. Thus, any solution of the intertwining (7) with non-degenerate matrix coefficient \(X^-_1\) can be constructed also by the method proposed in this subsection, and a general solution of the equation (13) can be presented in the form of the explicit parameterization of \(U_0 (x)\) and \(X^-_0 (x)\) using the \(n^2\) components of the vector-functions \(\Phi^-_l (x), l = 1, \ldots, n\), and the constants \(\lambda_l\) and \(\sigma_l\), \(l = 1, \ldots, n\), with the help of the formulas (27), (29) and (30).

3.3. The method of transformation vector-functions versus the transformation matrix method

Using the transformation vector-functions \(\Phi^-_l (x), l = 1, \ldots, n\) of section 3.2, one can construct the matrix
\[
\Phi^- (x) = \begin{pmatrix}
\varphi^-_{11} (x) & \varphi^-_{12} (x) & \cdots & \varphi^-_{1n} (x) \\
\varphi^-_{21} (x) & \varphi^-_{22} (x) & \cdots & \varphi^-_{2n} (x) \\
\vdots & \vdots & \ddots & \vdots \\
\varphi^-_{n1} (x) & \varphi^-_{n2} (x) & \cdots & \varphi^-_{nn} (x)
\end{pmatrix}
\]
(33)
This matrix, the Hamiltonian \(H_+\) and the matrix \(T^+_1\), i.e the matrix \(T\) of the intertwining operator \(Q^-_1\) in the basis \(\Phi^-_l (x), l = 1, \ldots, n\), are interrelated (see section 2.2) by the equality
which is equivalent to equalities (23). With the help of the matrix \( \Phi^{-}(x) \), one can represent the intertwining operator \( \tilde{Q}_{1}^{\pm} \) in the form

\[
\tilde{Q}_{1}^{-} = I_{n} \partial - \Phi^{-}(x)(\Phi^{-}(x))^{-1}
\]

\[
\Rightarrow Q_{1}^{-} = X_{1}^{-:\partial} \left[ I_{n} \partial - \Phi^{-}(x)(\Phi^{-}(x))^{-1} \right],
\]

where (35) holds due to the following chain:

\[
\begin{bmatrix}
I_{n} \partial - \Phi^{-}(x)(\Phi^{-}(x))^{-1}
\end{bmatrix} \Phi^{-}(x) = 0
\]

\[
\Rightarrow \begin{bmatrix}
I_{n} \partial - \Phi^{-}(x)(\Phi^{-}(x))^{-1}
\end{bmatrix} \Phi_{l}^{-}(x) = 0,
\]

\[
\Rightarrow \ker \left[ I_{n} \partial - \Phi^{-}(x)(\Phi^{-}(x))^{-1} \right] = \ker \tilde{Q}_{1}^{-}.
\]

Thus, there is another formula for finding the matrix \( X_{0}^{-}(x) \):

\[
X_{0}^{-}(x) = -\Phi^{-}(x)(\Phi^{-}(x))^{-1}.
\] (37)

The equalities (35) and (36) for the corresponding partial cases were found earlier, in [2, 18, 19].

One can represent the Hamiltonians \( H_{+} \) and \( H_{-} \) with the help of the matrix \( \Phi^{-}(x) \) in the form

\[
H_{+} = \tilde{Q}_{1}^{+} \tilde{Q}_{1}^{-} + \Phi^{-}(x) \left( T_{1}^{+} \right)^{T} \left( \Phi^{-}(x) \right)^{-1},
\] (38)

\[
H_{-} = \tilde{Q}_{1}^{+} \tilde{Q}_{1}^{-} + \Phi^{-}(x) \left( T_{1}^{+} \right)^{T} \left( \Phi^{-}(x) \right)^{-1}
\] (39)

\[
\Rightarrow H_{+} = Q_{1}^{+} Q_{1}^{-} + \Phi^{-}(x) \left( T_{1}^{+} \right)^{T} \left( \Phi^{-}(x) \right)^{-1},
\]

\[
H_{-} = Q_{1}^{-} Q_{1}^{+} + X_{1}^{-} \Phi^{-}(x) \left( T_{1}^{+} \right)^{T} \left( \Phi^{-}(x) \right)^{-1} \left( X_{1}^{-} \right)^{-1},
\]

where (38) and (39) hold due to the equalities (15), (34) and

\[
\begin{bmatrix}
\tilde{Q}_{1}^{+} \tilde{Q}_{1}^{-} + \Phi^{-}(x) \left( T_{1}^{+} \right)^{T} \left( \Phi^{-}(x) \right)^{-1}
\end{bmatrix} \Phi^{-}(x) = \Phi^{-}(x) \left( T_{1}^{+} \right)^{T},
\]

and to the facts that the Wronskian \( W(x) \equiv \det \Phi^{-}(x) \) does not vanish on the entire axis and the right-hand part of (38) is a matrix Hamiltonian of Schrödinger form. The formulas (38) and (39) were obtained earlier for the corresponding partial cases in [18, 19].

It follows from (15) and (38) that

\[
U_{0}(x) = \Phi^{-}(x) \left( T_{1}^{+} \right)^{T} \left( \Phi^{-}(x) \right)^{-1}.
\] (40)

Hence, the spectrum of the matrix \( U_{0}(x) \) does not depend on \( x \). Moreover, since the vector-functions \( \Phi_{l}^{-}(x) \), \( l = 1, \ldots, n \), constitute a canonical basis in \( \ker Q_{1}^{-} \) and thereby the matrix \( T_{1}^{+} \) is of normal (Jordan) form, we have that a normal (Jordan) form of \( U_{0}(x) \) coincides with \( T_{1}^{+} \) up to possible permutation of Jordan blocks. In the particular case where all vector-functions \( \Phi_{l}^{-}(x) \), \( l = 1, \ldots, n \), are formal vector-eigenfunctions of \( H_{a} \) for the same spectral value \( \lambda_{0} = \lambda_{1} = \cdots = \lambda_{n} \), the matrix \( U_{0}(x) \) obviously takes the form

\[
U_{0}(x) = \lambda_{0} I_{n}.
\]
Thus, in view of the results of section 3.2, any solution of the intertwining (7) with nondegenerate matrix $X_1^{-}$ can be constructed also in terms of a matrix of the form (33), and a general solution of the equation (13) can be presented in the form of the explicit parameterization of $U_0(x)$ and $\tilde{X}_0^{-}(x)$ using the $n^2$ components of the vector-functions $\Phi_l^{-}(x)$, $l = 1, \ldots, n$, and constants $\lambda_l$ and $\sigma_l$, $l = 1, \ldots, n$, with the help of the formulas (37) and (40).

If all vector-functions $\Phi_l^{-}(x)$ are PT-symmetric and all numbers $\lambda_l$ and all elements of $X_1^{-}$ are real, then $Q_1^{-}$, in view of (36), is PT-antisymmetric and $H_+$ and $H_-$, by virtue of (4) and (27), are PT-symmetric. If $H_+$ is known and PT-symmetric, then it is sufficient obviously, for PT-symmetry of all $\Phi_l^{-}(x)$, that all of the $\lambda_l$ are real and that the vector-function $\Phi_l^{-}(x)$ of highest order from any chain of the set $\Phi_l^{-}(x)$, $l = 1, \ldots, n$, is PT-symmetric.

4. Constructing a higher order matrix intertwining operator: the method of transformation vector-functions

It is possible to build chains of first-order matrix intertwining operators with the help of the formulas of section 3, and also higher order matrix intertwining operators in the form of products of elements of such chains. Results of this type can be found in, for example, [18, 19], and in a more general form in remark 1 of [47]. But the way of constructing higher order matrix intertwining operators indicated is rather restricted, since [23] for any $n \geq 2$ and $N \geq 2$ there are $n \times n$ matrix intertwining operators of the $N$th order that cannot be represented in the form of products of matrix intertwining operators of the lower orders. We present below a method that generalizes the method of section 3.2 and allows one to construct any $n \times n$ matrix intertwining operator of arbitrary order $N$ with arbitrary constant nondegenerate matrix coefficient of $\partial^N$ and the corresponding final matrix Hamiltonian in terms of transformation vector-functions.

Let us consider $H_+$ as a known initial $n \times n$ matrix Hamiltonian, and let $\Phi_l^{-}(x)$, $l = 1, \ldots, nN$, $N \in \mathbb{N}$, be a set of formal associated vector-functions of $H_+$ such that the formulas (23) and (24) hold for any $l = 1, \ldots, nN$, $\Phi_{nN+1}(x) \equiv 0$, the condition (25) is valid for any $l = 1, \ldots, nN - 1$ and the Wronskian of these vector-functions

$$W(x) \equiv \begin{vmatrix}
\begin{array}{cccccc}
\phi_{11}^- & \cdots & \phi_{1n}^- & \phi_{11}^- & \cdots & \phi_{1n}^- \\
\phi_{21}^- & \cdots & \phi_{2n}^- & \phi_{21}^- & \cdots & \phi_{2n}^- \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
\phi_{nN,1}^- & \cdots & \phi_{nN,n}^- & \phi_{nN,1}^- & \cdots & \phi_{nN,n}^-
\end{array}
\end{vmatrix}$$

does not vanish on the entire axis. There is only one $n \times n$ matrix linear differential operator of the $N$th order $Q_N^{-}$ with arbitrary nondegenerate constant $n \times n$ matrix coefficient $X_N^{-}$ of $\partial^N$ whose kernel contains all vector-functions $\Phi_l^{-}(x)$, $l = 1, \ldots, nN$, and, moreover, one can find this operator with help of the following obvious formula:
where $P_1,...,P_n$ are the same projection operators as in (28) and during the calculation of the determinant (41), in each of its terms, the corresponding one of the operators $P_1,...,P_n, P_1\partial,...,P_n\partial, P_1\partial^{N-1},...,P_n\partial^{N-1}, L_0\partial^N$ must be placed in the last position. It follows from (41) that the $l$th column of the matrix coefficient $X_j^-(\chi)$ of $Q_N^-$ (see (2)) is equal to

$$
\begin{pmatrix}
\varphi_{11}^- & \ldots & \varphi_{1n}^- & \varphi_{11}^- & \ldots & \varphi_{1n}^- & \quad (\varphi_{11}^-)^{(N-1)} & \ldots & (\varphi_{1n}^-)^{(N-1)} & (\Phi_1^-)^{(N)} \\
\varphi_{21}^- & \ldots & \varphi_{2n}^- & \varphi_{21}^- & \ldots & \varphi_{2n}^- & \quad (\varphi_{21}^-)^{(N-1)} & \ldots & (\varphi_{2n}^-)^{(N-1)} & (\Phi_2^-)^{(N)} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \quad \vdots & \vdots & \vdots & \vdots \\
\varphi_{nN,1}^- & \ldots & \varphi_{nN,n}^- & \varphi_{nN,1}^- & \ldots & \varphi_{nN,n}^- & \quad (\varphi_{nN,1}^-)^{(N-1)} & \ldots & (\varphi_{nN,n}^-)^{(N-1)} & (\Phi_{nN}^-)^{(N)} \\
\end{pmatrix} (41)
$$

The operator $Q_N^-$ intertwines [23] the initial Hamiltonian $H_\chi$ with some new $n \times n$ matrix Hamiltonian of Schrödinger form, $H_\chi \equiv -L_0\partial^2 + V_-(\chi)$, according to (2), and the potential $V_-(\chi)$ of $H_\chi$ can be found with the help of (4) and (42) with $j = N - 1$.

It should be emphasized that the condition of the Wronskian $W(\chi)$ is nonvanishing on the entire axis guarantees, in view of (4), (41) and (42), existence of $Q_N^-$ and smoothness (absence of pole(s)) for the matrix-valued functions $X_j^- \chi(x), X_j_{-1}^- \chi(x)$ and $V_-(\chi)$. The partial case of the representation of $Q_N^-\Phi$ for arbitrary $n$-dimensional vector-function $\Phi(\chi)$, with the help of (41), and of the representation of $V_-(\chi)$, with the help of (4) and (42), with $j = N - 1$, when all vector-functions $\Phi_l^- \chi(x)$, $l = 1, ..., N$, are formal vector-eigenfunctions of the Hamiltonian $H_\chi$, and $X_N^- = L_0$ was found in [18].

The fact that any $n \times n$ matrix intertwining operator of arbitrary order $N$ with arbitrary nondegenerate constant matrix coefficient of $\partial^N$ can be obtained by the method presented in this section is a corollary of the facts that (i) for any operator of this type there is a canonical basis in its kernel, whose Wronskian does not vanish on the entire axis, and (ii) an $n \times n$ matrix linear differential operator of the order $N$ with a given nondegenerate constant matrix coefficient of $\partial^N$ is uniquely determined by a basis in its kernel.
5. Examples: the $n = 2, N = 1$ case

In this section we present some examples of the construction of $2 \times 2$ matrix linear differential intertwining operators of the first order $Q^{-1}_1$ and of the new $2 \times 2$ matrix Hamiltonians $H_-$ of Schrödinger form corresponding to them, with the help of the methods of section 3. Also, we demonstrate by dint of these examples the utility of these methods for the spectral design of matrix Hamiltonians. As the initial $2 \times 2$ matrix Hamiltonian $H_*$ we shall use the Hamiltonian of Schrödinger form with zero $2 \times 2$ matrix potential $V_0(x)$:

$$H_*= -I_2 \partial_x^2, \quad V_*(x) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}. \quad (43)$$

Vector-eigenfunctions for the continuous spectrum of the new Hamiltonians $H_-$ can be straightforwardly calculated in trivial way:

$$\Psi_{i,0}^* = \frac{e^{k_i x}}{k_i}, \quad \Psi_{i,0} = \frac{e^{-k_i x}}{k_i}, \quad \Psi_{i,1}^* = \frac{0}{k_i}, \quad \Psi_{i,1} = \frac{0}{k_i}, \quad \kappa \in \mathbb{R},$$

so we shall seek only normalizable vector-eigenfunctions and associated vector-functions of these Hamiltonians.

It is not hard to see that for the Hamiltonian (43) there is the following complete set of linearly independent formal eigenfunctions and associated first-order vector-functions for the spectral value $\lambda = -k_i^2 \neq 0$:

$$\Psi_{i,0}^* = \begin{pmatrix} e^{k_i x} \\ 0 \end{pmatrix}, \quad \Psi_{i,0} = \begin{pmatrix} e^{-k_i x} \\ 0 \end{pmatrix}, \quad \Psi_{i,1}^* = \begin{pmatrix} 0 \\ e^{k_i x} \end{pmatrix}, \quad \Psi_{i,1} = \begin{pmatrix} 0 \\ e^{-k_i x} \end{pmatrix}, \quad (44)$$

These vector-functions will be used below for constructing the intertwining operators and new Hamiltonians.

We adopt the following notation in this section: $\lambda_1$ and $\lambda_2$ are eigenvalues of the matrix $T$ of the intertwining operator $Q^{-1}_1$, and $g^{-1}_1$ is the geometric multiplicity of the eigenvalue $\lambda_1$. Also, we suppose that the matrix coefficient $X^{-1}_1$ of $\partial_x$ in the operator $Q^{-1}_1$ is equal to the identity matrix: $X^{-1}_1 = I_2$.

5.1. The $\lambda_1 \neq \lambda_2$ subcase: adding up to two bound states for different energy values

In this subcase the general form of the transformation vector-functions $\Phi_1^-(x)$ and $\Phi_2^-(x)$ is the following, in view of (44):

$$\Phi_1^-(x) = \begin{pmatrix} C_1 e^{k_1 x} + C_2 e^{-k_1 x} \\ C_1 e^{k_1 x} + C_4 e^{-k_1 x} \end{pmatrix}, \quad \Phi_2^-(x) = \begin{pmatrix} C_3 e^{k_2 x} + C_6 e^{-k_2 x} \\ C_5 e^{k_2 x} + C_8 e^{-k_2 x} \end{pmatrix}, \quad (45)$$

where $C_1, \ldots, C_8$ are arbitrary complex—in general—constants and we assume without the loss of generality that $C_1 = 1$. The remaining constants $C_2, \ldots, C_8$ are chosen such that the Wronskian $W(x)$ of the vector-functions $\Phi_1^-(x)$ and $\Phi_2^-(x)$,
\[ W(x) = \left[ C_7 - C_3 C_5 \right] e^{(k_1 + k_2) x} + \left[ C_8 - C_3 C_6 \right] e^{(k_1 - k_2) x} \\
+ \left[ C_2 C_7 - C_4 C_5 \right] e^{-(k_1 - k_2) x} + \left[ C_2 C_8 - C_4 C_6 \right] e^{-(k_1 + k_2) x} \] 

(46)

does not vanish on the real axis. The operators \( Q_1^- \) and \( Q_1^+ \), the matrix \( U_0(x) \) and the new Hamiltonian \( H_- \) take the following forms:

\[ Q_1^\pm = \mp i \partial \frac{1}{W(x)} \left[ \begin{array}{c}
\left( k_1 C_7 - k_2 C_3 C_5 \right) \\
\left( k_1 C_7 - k_2 C_3 C_7 \right) \\
\left( k_2 C_7 - k_1 C_3 C_5 \right)
\end{array} \right] e^{(k_1 + k_2) x} \\
+ \left[ \begin{array}{c}
(k_1 C_8 + k_2 C_3 C_6) \\
(k_1 + k_2) C_3 C_8 \\
-k_2 C_8 - k_1 C_3 C_6
\end{array} \right] e^{(k_1 - k_2) x} \\
+ \left[ \begin{array}{c}
-k_1 C_2 C_7 - k_2 C_4 C_5 \\
-(k_1 + k_2) C_4 C_7 \\
-k_2 C_7 + k_1 C_4 C_5
\end{array} \right] e^{-(k_1 - k_2) x} \\
+ \left[ \begin{array}{c}
-k_1 C_2 C_8 + k_2 C_4 C_6 \\
-(k_1 - k_2) C_4 C_8 \\
-k_2 C_8 + k_1 C_4 C_6
\end{array} \right] e^{-(k_1 + k_2) x} \] 

(47)

\[ U_0(x) = \frac{1}{W(x)} \left[ \begin{array}{c}
\left( k_1^2 C_4 C_6 - k_2^2 C_7 \right) \\
\left( k_1^2 C_4 C_7 - k_2^2 C_6 \right) \\
\left( k_2^2 C_4 C_5 - k_2^2 C_7 \right)
\end{array} \right] e^{(k_1 + k_2) x} \\
+ \left[ \begin{array}{c}
-k_1^2 C_3 C_6 - k_2^2 C_8 \\
-(k_1^2 - k_2^2) C_3 C_8 \\
-k_2^2 C_3 C_6 - k_2^2 C_8
\end{array} \right] e^{(k_1 - k_2) x} \\
+ \left[ \begin{array}{c}
-k_1^2 C_4 C_5 - k_2^2 C_7 \\
-(k_1^2 - k_2^2) C_4 C_7 \\
-k_2^2 C_4 C_5 - k_2^2 C_7
\end{array} \right] e^{-(k_1 - k_2) x} \\
+ \left[ \begin{array}{c}
-k_1^2 C_4 C_6 - k_2^2 C_8 \\
-(k_1^2 - k_2^2) C_4 C_8 \\
-k_2^2 C_4 C_6 - k_2^2 C_8
\end{array} \right] e^{-(k_1 + k_2) x} \] 

(48)

\[ H_- = -i \partial^2 - \frac{4}{W^2(x)} \left[ \begin{array}{c}
C_3 \left[ k_1 \Delta_2 - k_2 \left( \delta_2 - 2C_3 C_5 C_6 \right) \right] \\
C_5 \left[ k_1 \Delta_2 C_3 + k_2 \left( \delta_2 C_3 - 2C_2 C_5 C_6 \right) \right] \\
C_5 \left[ k_1 \Delta_2 C_3 + k_2 \left( \delta_2 C_3 - 2C_2 C_5 C_6 \right) \right]
\end{array} \right] e^{2k_1 x} \\
+ \left[ \begin{array}{c}
C_3 \left[ k_2 \Delta_1 C_5 - k_1 \left( \delta_1 C_5 - 2C_2 C_7 \right) \right] \\
C_5 \left[ k_2 \Delta_1 C_7 + k_1 \left( \delta_1 C_7 - 2C_3 C_4 C_5 \right) \right]
\end{array} \right] e^{2k_2 x} \]

16
\[\Delta_1 = C_4 - C_2 C_3, \quad \Delta_2 = C_3 C_8 - C_6 C_7, \quad \delta_1 = C_4 + C_2 C_3, \quad \delta_2 = C_3 C_8 + C_6 C_7. \quad (49)\]

such that

\[H_+ = Q_+^\dagger Q_-^\dagger + U_0(x), \quad H_- = Q_-^\dagger Q_+^\dagger + U_0(x), \quad Q_-^\dagger H_+ = H_- Q_+^\dagger. \quad (50)\]

For the spectral values \(\lambda_1\) and \(\lambda_2\) of the Hamiltonian \(H_-\), one can easily construct formal vector-eigenfunctions with the help of the following formulas:

\[\Psi_1^+(x) = Q_+^\dagger \left( \begin{array}{c} e^{k_1 x} \\ 0 \end{array} \right), \quad \Psi_2^+(x) = Q_+^\dagger \left( \begin{array}{c} e^{-k_1 x} \\ 0 \end{array} \right).\]

\[\Psi_3^+(x) = Q_+^\dagger \left( \begin{array}{c} 0 \\ e^{k_1 x} \end{array} \right), \quad \Psi_4^+(x) = Q_+^\dagger \left( \begin{array}{c} 0 \\ e^{-k_1 x} \end{array} \right).\]

\[\Psi_5^+(x) = Q_+^\dagger \left( \begin{array}{c} e^{k_2 x} \\ 0 \end{array} \right), \quad \Psi_6^+(x) = Q_+^\dagger \left( \begin{array}{c} e^{-k_2 x} \\ 0 \end{array} \right).\]

\[\Psi_7^+(x) = Q_+^\dagger \left( \begin{array}{c} 0 \\ e^{k_2 x} \end{array} \right), \quad \Psi_8^+(x) = Q_+^\dagger \left( \begin{array}{c} 0 \\ e^{-k_2 x} \end{array} \right).\]

\[H \cdot \Psi_i^+ = \lambda_i \Psi_i^+, \quad i = 1, 2, 3, 4, \quad H \cdot \Psi_j^+ = \lambda_j \Psi_j^+, \quad j = 5, 6, 7, 8, \quad (51)\]

and the resulting expressions can be found in the full version of this paper [48]. Only six of these vector-functions are linearly independent, in view of the fact that the vector-functions \(\Phi_1^-(x)\) and \(\Phi_2^-(x)\) (see (45)) form a canonical basis in the kernel of \(Q_-^\dagger\). The latter leads to the relations

\[\Psi_1^+(x) + C_2 \Psi_2^+(x) + C_3 \Psi_3^+(x) + C_4 \Psi_4^+(x) = 0, \quad (52)\]

\[C_5 \Psi_5^+(x) + C_6 \Psi_6^+(x) + C_7 \Psi_7^+(x) + C_8 \Psi_8^+(x) = 0.\]

1 Some formulas are partially omitted here and below by recommendation of a referee in order to reduce the total size of the paper. The full versions of all these formulas can be found in [48].
It follows from the results of [23] that in the subcase $\lambda_1 \neq \lambda_2$ considered, there is a linear differential operator of third order, $Q^+_3$, with the coefficient $I_2$ of $\partial^3$, that intertwines the Hamiltonians $H_+$ and $H_-$ in the opposite direction: $Q^+_3H_- = H_+Q^+_3$, and six linearly independent vector-functions from the set (51) form a canonical basis in the kernel of $Q^+_3$, providing an opportunity to construct $Q^+_3$ explicitly with the help of (41).

The formal vector-eigenfunctions $\Psi^+_9(x)$ and $\Psi^+_10(x)$ of the Hamiltonian $H_-$ that are linearly independent of (51), for the spectral values $\lambda_1$ and $\lambda_2$ respectively, can be found in the form

$$\Psi^+_9(x) = Q^-_1 \begin{pmatrix} -\frac{x}{2k_1}e^{k_1x} + C_2\frac{x}{2k_1}e^{-k_1x} \\ -C_3\frac{x}{2k_1}e^{k_1x} + C_4\frac{x}{2k_1}e^{-k_1x} \end{pmatrix},$$

$$\Psi^+_10(x) = Q^-_1 \begin{pmatrix} -C_5\frac{x}{2k_2}e^{k_2x} + C_6\frac{x}{2k_2}e^{-k_2x} \\ -C_7\frac{x}{2k_2}e^{k_2x} + C_8\frac{x}{2k_2}e^{-k_2x} \end{pmatrix},$$

$$H_-\Psi^+_9 = \lambda_1\Psi^+_9, \quad H_-\Psi^+_10 = \lambda_2\Psi^+_10.$$  \hspace{1cm} (53)

since

$$ \begin{pmatrix} H_+ - \lambda_1I_2 \\ H_+ - \lambda_2I_2 \end{pmatrix} = \begin{pmatrix} -\frac{x}{2k_1}e^{k_1x} + C_2\frac{x}{2k_1}e^{-k_1x} \\ -C_3\frac{x}{2k_1}e^{k_1x} + C_4\frac{x}{2k_1}e^{-k_1x} \end{pmatrix} = \Phi^+_1(x),$$

$$ \begin{pmatrix} H_+ - \lambda_1I_2 \\ H_+ - \lambda_2I_2 \end{pmatrix} = \begin{pmatrix} -C_5\frac{x}{2k_2}e^{k_2x} + C_6\frac{x}{2k_2}e^{-k_2x} \\ -C_7\frac{x}{2k_2}e^{k_2x} + C_8\frac{x}{2k_2}e^{-k_2x} \end{pmatrix} = \Phi^+_2(x),$$

the vector-functions $\Phi^+_1(x)$ and $\Phi^+_2(x)$ belong to the kernel of $Q^-_1$, and a chain of associated vector-functions of the Hamiltonian $H_+$ is mapped (see section 2.2) by the intertwining operator $Q^-_1$ into a chain of associated vector-functions of the Hamiltonian $H_-$ for the same spectral value (some first terms of the chain can be mapped by $Q^-_1$ into zeros).

Analysis of the vector-functions (51) and (53) leads to the following results:

(1) if

$$\text{Re } k_1 \text{ Re } k_2 > 0, \quad (C_7 - C_3C_5)(C_2C_8 - C_4C_6) \neq 0,$$

then for each of the eigenvalues $\lambda_1$ and $\lambda_2$ there is only (up to a constant factor) one normalizable vector-eigenfunction of the Hamiltonian $H_-$. 


\[ \Psi_{11}^{-}(x) = \Psi_{1}^{+}(x) + C_3 \Psi_{3}^{+}(x) = -C_2 \Psi_{2}^{+}(x) - C_4 \Psi_{4}^{+}(x) = \frac{1}{W(x)} \]

\[
\times \left\{ \begin{array}{c}
k_2 \Delta_1 C_5 - k_1 (\delta_1 C_5 - 2C_2 C_7) \\
k_2 \Delta_1 C_7 + k_1 (\delta_1 C_7 - 2C_3 C_5)
\end{array} \right\} e^{k_2 x}
\]

\[
- \left\{ \begin{array}{c}
k_2 \Delta_1 C_6 + k_1 (\delta_1 C_6 - 2C_2 C_8) \\
k_2 \Delta_1 C_8 - k_1 (\delta_1 C_8 - 2C_3 C_4 C_6)
\end{array} \right\} e^{-k_2 x}
\]

\[ \Psi_{12}^{-}(x) = C_5 \Psi_{3}^{+}(x) + C_7 \Psi_{7}^{+}(x) = -C_6 \Psi_{6}^{+}(x) - C_8 \Psi_{8}^{+}(x) = \frac{1}{W(x)} \]

\[
\times \left\{ \begin{array}{c}
k_1 \Delta_2 - k_2 (\delta_2 - 2C_3 C_5 C_6) \\
k_1 \Delta_2 C_3 + k_2 (\delta_2 C_3 - 2C_7 C_8)
\end{array} \right\} e^{k_1 x}
\]

\[
+ \left\{ \begin{array}{c}
k_1 \Delta_2 C_2 + k_2 (\delta_2 C_2 - 2C_4 C_5 C_6) \\
k_1 \Delta_2 C_4 - k_2 (\delta_2 C_4 - 2C_2 C_7 C_8)
\end{array} \right\} e^{-k_1 x}
\]

\[ H_{-} \Psi_{11}^{-} = \lambda_1 \Psi_{11}^{-}, \quad H_{-} \Psi_{12}^{-} = \lambda_2 \Psi_{12}^{-}, \quad \Psi_{1}^{\pm}(x), \ \Psi_{2}^{\pm}(x) \in \ker Q_{3}^{\pm}; \quad (54)
\]

(2) if

\[
\text{Re} \ k_1 > \text{Re} \ k_2 > 0, \quad C_7 - C_3 C_5 = C_5 (C_4 - C_2 C_3) = 0, \\
(C_8 - C_3 C_6)(C_2 C_8 - C_4 C_6) \neq 0,
\]

or

\[
\text{Re} \ k_1 > 2 \text{Re} \ k_2 > 0, \quad C_7 - C_3 C_5 = 0, \quad (C_8 - C_3 C_6)(C_2 C_8 - C_4 C_6) \neq 0
\]

or

\[
\text{Re} \ k_1 > \text{Re} \ k_2 > 0, \quad C_2 C_8 - C_4 C_6 = C_4 - C_2 C_3 = 0, \\
(C_7 - C_3 C_5)(C_2 C_7 - C_4 C_5) \neq 0,
\]

or

\[
\text{Re} \ k_1 > 2 \text{Re} \ k_2 > 0, \quad C_2 C_8 - C_4 C_6 = 0, \quad (C_7 - C_3 C_5)(C_2 C_7 - C_4 C_5) \neq 0
\]

or

\[
\text{Re} \ k_1 > 2 \text{Re} \ k_2 > 0, \quad C_7 - C_3 C_5 = C_2 C_8 - C_4 C_6 = 0, \\
(C_8 - C_3 C_6)(C_2 C_7 - C_4 C_5) \neq 0,
\]

then for the eigenvalue \( \lambda_1 \) there is only (up to a constant factor) one normalizable vector-eigenfunction \( \Psi_{11}^{\pm}(x) \) of the Hamiltonian \( H_{-} \) and for the spectral value \( \lambda_2 \) there is no normalizable vector-eigenfunction of \( H_{-} \).

(3) if

\[
\text{Re} \ k_1 > \text{Re} \ k_2 > 0, \quad C_7 - C_3 C_5 = C_8 - C_3 C_6 = 0, \\
(C_2 C_7 - C_4 C_5)(C_2 C_8 - C_4 C_6) \neq 0
\]
or
\[ \text{Re } k_1 > \text{Re } k_2 > 0, \quad C_2 C_7 - C_4 C_5 = C_2 C_8 - C_4 C_6 = 0, \]
\[ (C_7 - C_3 C_5)(C_8 - C_3 C_6) \neq 0, \]
then for the eigenvalue \( \lambda_2 \) there is only (up to a constant factor) one normalizable vector-eigenfunction \( \Psi_{12}(x) \) of the Hamiltonian \( H_- \) and for the spectral value \( \lambda_1 \) there is no normalizable vector-eigenfunction of \( H_- \);

(4) if
\[ 2 \text{Re } k_2 \geq \text{Re } k_1 > \text{Re } k_2 > 0, \quad C_7 - C_3 C_5 = 0, \]
\[ C_5(C_4 - C_2 C_3)(C_8 - C_3 C_6)(C_2 C_8 - C_4 C_6) \neq 0 \]
or
\[ 2 \text{Re } k_2 \geq \text{Re } k_1 > \text{Re } k_2 > 0, \quad C_2 C_8 - C_4 C_6 = 0, \]
\[ (C_4 - C_2 C_3)(C_7 - C_3 C_5)(C_2 C_7 - C_4 C_5) \neq 0 \]
or
\[ 2 \text{Re } k_2 \geq \text{Re } k_1 > \text{Re } k_2 > 0, \quad C_7 - C_3 C_5 = C_2 C_8 - C_4 C_6 = 0, \]
\[ (C_8 - C_3 C_6)(C_2 C_7 - C_4 C_5) \neq 0 \]
or
\[ C_7 - C_3 C_5 = C_8 - C_3 C_6 = C_2 C_7 - C_4 C_5 = 0, \quad C_2 C_8 - C_4 C_6 \neq 0 \]
or
\[ C_7 - C_3 C_5 = C_8 - C_3 C_6 = C_2 C_8 - C_4 C_6 = 0, \quad C_2 C_7 - C_4 C_5 \neq 0 \]
or
\[ C_7 - C_3 C_5 = C_2 C_7 - C_4 C_5 = C_2 C_8 - C_4 C_6 = 0, \quad C_8 - C_3 C_6 \neq 0 \]
or
\[ C_8 - C_3 C_6 = C_2 C_7 - C_4 C_5 = C_2 C_8 - C_4 C_6 = 0, \quad C_7 - C_3 C_5 \neq 0, \]
then there is no normalizable vector-eigenfunction of the Hamiltonian \( H_- \) for the spectral values \( \lambda_1 \) and \( \lambda_2 \).

There are some interesting partial situations in which the formulas obtained become significantly simpler. Descriptions of these situations can be found in the full version of this paper [48]. In general, the formulas (45)–(49) and (54) can be simplified with the help of a similarity transformation for \( \Delta_1 \neq 0 \Leftrightarrow C_4 \neq C_2 C_3 \) as follows:

\[
C^{-1}\Phi_1^+(x) = \begin{pmatrix} e^{k_1x} \\ e^{-k_1x} \end{pmatrix}, \quad C^{-1}\Phi_2^+(x) = \begin{pmatrix} \tilde{C}_5 e^{k_2x} + \tilde{C}_6 e^{-k_2x} \\ \tilde{C}_7 e^{k_5x} + \tilde{C}_8 e^{-k_5x} \end{pmatrix},
\]
\[
W(x) = \tilde{C}_7 e^{(k_1+k_2)x} + \tilde{C}_8 e^{(k_1-k_2)x} - \tilde{C}_5 e^{-(k_1-k_2)x} - \tilde{C}_6 e^{-(k_1+k_2)x} = \frac{1}{\Delta_1} W(x),
\]
\[ C^{-1}Q_{\pm} C = \mp \mathcal{I}_2 \partial - \begin{pmatrix} k_1 & 0 \\ 0 & -k_1 \end{pmatrix} - \frac{1}{W(x)} \left( (k_1 + k_2) \begin{pmatrix} \tilde{C}_6 e^{-(k_1+k_2)x} & -\tilde{C}_6 e^{(k_1-k_2)x} \\ -\tilde{C}_7 e^{-(k_1-k_2)x} & \tilde{C}_7 e^{(k_1+k_2)x} \end{pmatrix} \right) \\
+ (k_1 - k_2) \begin{pmatrix} \tilde{C}_5 e^{-(k_1-k_2)x} & -\tilde{C}_5 e^{(k_1+k_2)x} \\ -\tilde{C}_6 e^{-(k_1+k_2)x} & \tilde{C}_6 e^{(k_1-k_2)x} \end{pmatrix} \right], \]

\[ C^{-1}U_0(x) C = -k_1^2 \mathcal{I}_2 - \left( \frac{k_1^2 - k_2^2}{W(x)} \right) \times \left( \begin{array}{c} \tilde{C}_5 e^{-(k_1-k_2)x} + \tilde{C}_6 e^{-(k_1+k_2)x} - \tilde{C}_5 e^{(k_1+k_2)x} - \tilde{C}_6 e^{(k_1-k_2)x} \\ \tilde{C}_7 e^{-(k_1-k_2)x} + \tilde{C}_8 e^{-(k_1+k_2)x} - \tilde{C}_7 e^{(k_1+k_2)x} - \tilde{C}_8 e^{(k_1-k_2)x} \end{array} \right), \]

\[ C^{-1}H_{\pm} C = -\mathcal{I}_2 \partial^2 - \frac{4}{W^2(x)} \left[ 2k_2^2 \begin{pmatrix} \tilde{C}_5 \tilde{C}_6 e^{-2k_1x} & -\tilde{C}_5 \tilde{C}_6 \\ -\tilde{C}_7 \tilde{C}_8 e^{2k_1x} \end{pmatrix} \right] \\
+ k_2 \left( (k_1 - k_2) \tilde{C}_5 \tilde{C}_6 - (k_1 + k_2) \tilde{C}_5 \tilde{C}_7 \right) \begin{pmatrix} 1 & -e^{-2k_1x} \\ -e^{-2k_1x} & 1 \end{pmatrix} \\
+ k_1 k_2 \left[ \begin{array}{cc} [\tilde{C}_6 e^{k_2x} - \tilde{C}_6 e^{-k_2x}] & [-\tilde{C}_5 e^{k_2x} + \tilde{C}_6 e^{-k_2x}] \\ [\tilde{C}_7 e^{k_2x} + \tilde{C}_8 e^{-k_2x}] & [\tilde{C}_5 e^{k_2x} - \tilde{C}_6 e^{-k_2x}] \end{array} \right] \times \left[ \begin{array}{cc} [\tilde{C}_6 e^{k_2x} + \tilde{C}_6 e^{-k_2x}] & [-\tilde{C}_5 e^{k_2x} + \tilde{C}_6 e^{-k_2x}] \\ [\tilde{C}_7 e^{k_2x} + \tilde{C}_8 e^{-k_2x}] & [\tilde{C}_5 e^{k_2x} - \tilde{C}_6 e^{-k_2x}] \end{array} \right] \\
+ k_1^2 \left[ \begin{array}{cc} [\tilde{C}_5 e^{k_2x} + \tilde{C}_6 e^{-k_2x}] & [-\tilde{C}_5 e^{k_2x} + \tilde{C}_6 e^{-k_2x}] \\ [\tilde{C}_7 e^{k_2x} + \tilde{C}_8 e^{-k_2x}] & [\tilde{C}_5 e^{k_2x} - \tilde{C}_6 e^{-k_2x}] \end{array} \right] \times \left[ \begin{array}{cc} [\tilde{C}_5 e^{k_2x} + \tilde{C}_6 e^{-k_2x}] & [-\tilde{C}_5 e^{k_2x} + \tilde{C}_6 e^{-k_2x}] \\ [\tilde{C}_7 e^{k_2x} + \tilde{C}_8 e^{-k_2x}] & [\tilde{C}_5 e^{k_2x} - \tilde{C}_6 e^{-k_2x}] \end{array} \right]. \]

\[ C^{-1}W_{11}^+(x) = \frac{1}{W(x)} \left[ \begin{pmatrix} (k_1 - k_2) \tilde{C}_5 \ e^{k_1x} & -(k_1 + k_2) \tilde{C}_6 \ e^{-k_1x} \\ -(k_1 - k_2) \tilde{C}_5 \ e^{-k_1x} & (k_1 + k_2) \tilde{C}_6 \ e^{k_1x} \end{pmatrix} \right] \]

\[ C^{-1}W_{12}^+(x) = \frac{1}{W(x)} \left[ \begin{pmatrix} (k_1 + k_2) \tilde{C}_6 \tilde{C}_7 - (k_1 - k_2) \tilde{C}_5 \tilde{C}_8 \\ 2k_2 \tilde{C}_7 \tilde{C}_8 \end{pmatrix} \tilde{C}_6 \ e^{k_1x} \right] \\
- \begin{pmatrix} 2k_2 \tilde{C}_7 \tilde{C}_8 \tilde{C}_6 \\ (k_1 + k_2) \tilde{C}_6 \tilde{C}_7 - (k_1 - k_2) \tilde{C}_5 \tilde{C}_8 \tilde{C}_6 \ e^{-k_1x} \end{pmatrix}. \]
\[ C = \begin{pmatrix} 1 & C_2 \\ C_3 & C_4 \end{pmatrix}, \quad C^{-1} = \frac{1}{\Delta_1} \begin{pmatrix} C_4 & -C_2 \\ -C_3 & 1 \end{pmatrix}, \quad \det C = \Delta_1, \]

\[ \tilde{C}_5 = -\frac{C_2 C_7 - C_4 C_5}{\Delta_1}, \quad \tilde{C}_6 = -\frac{C_2 C_6 - C_4 C_8}{\Delta_1}, \]

\[ \tilde{C}_7 = \frac{C_7 - C_3 C_5}{\Delta_1}, \quad \tilde{C}_8 = \frac{C_8 - C_3 C_6}{\Delta_1}, \]

and for \( \Delta_1 = 0 \Leftrightarrow C_4 = C_2 C_3 \) as follows:

\[ C^{-1} \Phi^+ (x) = \begin{pmatrix} e^{k_{1,x}} + C_2 e^{-k_{1,x}} \\ 0 \end{pmatrix}, \quad C^{-1} \Phi^- (x) = \begin{pmatrix} C_3 e^{k_{2,x}} + C_6 e^{-k_{1,x}} \\ C_7 e^{k_{2,x}} + C_8 e^{-k_{1,x}} \end{pmatrix}, \]

\[ \tilde{W} (x) = \begin{pmatrix} e^{k_{1,x}} + C_2 e^{-k_{1,x}} \end{pmatrix} \begin{pmatrix} C_3 e^{k_{2,x}} + C_6 e^{-k_{1,x}} \\ C_7 e^{k_{2,x}} + C_8 e^{-k_{1,x}} \end{pmatrix} = -\frac{1}{\alpha} \tilde{W} (x), \]

\[ C^{-1} Q \xi C = \mp l_2 \partial \]

\[ -k_1 \begin{pmatrix} e^{k_{2,x}} - C_2 e^{-k_{1,x}} \\ e^{k_{2,x}} + C_2 e^{-k_{1,x}} \end{pmatrix} \begin{pmatrix} C_5 e^{k_{2,x}} + C_6 e^{-k_{1,x}} \\ C_7 e^{k_{2,x}} + C_8 e^{-k_{1,x}} \end{pmatrix} \]

\[ \begin{pmatrix} 0 \\ k_2 \frac{C_7 e^{k_{2,x}} - C_8 e^{-k_{1,x}}}{C_7 e^{k_{2,x}} + C_8 e^{-k_{1,x}}} \end{pmatrix}, \]

\[ C^{-1} U_0 (x) C = -\begin{pmatrix} k_1^2 - (k_1^2 - k_2^2) \frac{C_5 e^{k_{2,x}} + C_6 e^{-k_{1,x}}}{C_7 e^{k_{2,x}} + C_8 e^{-k_{1,x}}} \\ 0 \end{pmatrix}, \]

\[ C^{-1} H \xi C = -l_2 \partial^2 - \begin{pmatrix} 8k_1^2 C_2 \left[ e^{k_{1,x}} + C_2 e^{-k_{1,x}} \right]^2 - 8k_2^2 C_2 \left[ e^{k_{1,x}} + C_2 e^{-k_{1,x}} \right] \left[ C_7 e^{k_{2,x}} + C_8 e^{-k_{1,x}} \right] \\ 0 \end{pmatrix} - \begin{pmatrix} 8k_2^2 C_2 \left[ e^{k_{1,x}} + C_2 e^{-k_{1,x}} \right] \left[ C_7 e^{k_{2,x}} + C_8 e^{-k_{1,x}} \right] \\ 8k_2^2 \frac{C_7 e^{k_{2,x}} + C_8 e^{-k_{1,x}}}{C_7 e^{k_{2,x}} + C_8 e^{-k_{1,x}}} \end{pmatrix} \]

\[ + \begin{pmatrix} 0 \\ 4k_1 k_2 \left( C_5 \tilde{C}_8 - C_6 \tilde{C}_7 \right) \left[ e^{k_{1,x}} - C_2 e^{-k_{1,x}} \right] - 4k_1^2 \left( C_5 \tilde{C}_8 + C_6 \tilde{C}_7 \right) \end{pmatrix}, \]

\[ \left. \begin{pmatrix} k_1 \frac{C_7 e^{k_{2,x}} - C_8 e^{-k_{1,x}}}{C_7 e^{k_{2,x}} + C_8 e^{-k_{1,x}}} \\ k_2 \frac{C_7 e^{k_{2,x}} - C_8 e^{-k_{1,x}}}{C_7 e^{k_{2,x}} + C_8 e^{-k_{1,x}}} \end{pmatrix} \right)^2. \]
\[
C^{-1} \Phi^+_1(x) = \begin{pmatrix}
\frac{2k_1C_2}{e^{k_1x} + C_5e^{-k_1x}} \\
0
\end{pmatrix},
\]
\[
C^{-1} \Phi^+_2(x) = \begin{pmatrix}
\frac{k_2(C_5\tilde{C}_8 + C_6\tilde{C}_7)}{C_7e^{k_1x} + \tilde{C}_6e^{-k_1x}} & -k_1(C_5\tilde{C}_8 - C_6\tilde{C}_7)\left[e^{k_1x} + C_2e^{-k_1x}\right] \\
\frac{2k_2\tilde{C}_7\tilde{C}_8}{C_7e^{k_1x} + \tilde{C}_6e^{-k_1x}} & \frac{2e^{k_1x} + C_2e^{-k_1x}}{C_7e^{k_1x} + \tilde{C}_6e^{-k_1x}}
\end{pmatrix},
\]
\[
C = \begin{pmatrix}
\frac{1}{C_3} & 0 \\
-\alpha & C_3
\end{pmatrix}, \quad C^{-1} = \begin{pmatrix}
\frac{1}{C_3} & 0 \\
-\alpha & C_3
\end{pmatrix}, \quad \det C = -\alpha, \quad \tilde{C}_7 = -\frac{1}{\alpha}(C_7 - C_3C_5),
\]
\[
\tilde{C}_8 = -\frac{1}{\alpha}(C_8 - C_3C_6), \quad \alpha \in \mathbb{C}, \quad \alpha \neq 0,
\]

where \( \tilde{W}(x) \) is the Wronskian of \( C^{-1}\Phi^+_1(x) \) and \( C^{-1}\Phi^+_2(x) \). It is evident that the representations and the intertwining (50) transform trivially into the analogous formulas for the Hamiltonians \( C^{-1}H_1\mathcal{C} = H_+ = -\partial^2 \) and \( C^{-1}H_\mathcal{C} \), for the matrix \( C^{-1}U_0(x)\mathcal{C} \) and for the operators \( C^{-1}Q_1\mathcal{C} \) and \( C^{-1}Q_2\mathcal{C} \), that \( C^{-1}\Psi^+_1(x) \) and \( C^{-1}\Psi^+_2(x) \) are vector-eigenfunctions (sometimes formal) of the Hamiltonian \( C^{-1}H_\mathcal{C} \) for the same eigenvalues \( \lambda_1 = -k_1^2 \) and \( \lambda_2 = -k_2^2 \) respectively, and that \( C^{-1}\Phi^+_1(x) \) and \( C^{-1}\Phi^+_2(x) \) are transformation vector-functions corresponding to conversion of the Hamiltonian \( C^{-1}H_+\mathcal{C} = H_+ \) to the Hamiltonian \( C^{-1}H_\mathcal{C} \) with the help of the intertwining operator \( C^{-1}Q_i\mathcal{C} \). For \( k_1, k_2 > 0, C_2 = 1, C_5 = C_6 \in \mathbb{R} \) and \( \tilde{C}_7 = \tilde{C}_8 > 0 \), the vector-functions (55) and (60), the operators (58) and (59) and the Wronskian (56) are obviously \( PT \)-symmetric, the operators (57) are \( PT \)-antisymmetric and the spectrum of \( C^{-1}H_\mathcal{C} \) is real.

5.2. The \( \lambda_1 = \lambda_2, g_1 = 2 \) subcase: adding up to two bound states described by vector-eigenfunctions for the same energy value

In this subcase the formulas (45)–(54) are still valid with \( k_1 = k_2 \) and we assume, additionally to assuming the condition \( C_1 = 1 \), without the loss of generality that \( C_5 = 0 \), since the latter condition can be achieved in any case by making a change in the canonical basis in the kernel of \( Q_1\mathcal{C} \): using \( \Phi^+_1(x) \) and \( \Phi^+_2(x) = C_3\Phi^+_1(x) \) instead of \( \Phi^+_1(x) \) and \( \Phi^+_2(x) \). Thus, the formulas (45)–(54) take, in the subcase considered, the following simpler form:

\[
\Phi^+_1(x) = \begin{pmatrix}
C_1e^{k_1x} + C_2e^{-k_1x} \\
C_3e^{k_1x} + C_4e^{-k_1x}
\end{pmatrix}, \quad \Phi^+_2(x) = \begin{pmatrix}
C_5e^{k_1x} + C_6e^{-k_1x} \\
C_7e^{k_1x} + C_8e^{-k_1x}
\end{pmatrix}, \quad C_1 = 1, \quad C_5 = 0,
\]
\[
H_i\Phi^+_i = \tilde{\lambda}\Phi^+_i, \quad i = 1, 2, \quad \tilde{\lambda} = \lambda_1 = \lambda_2 = -k^2 \neq 0, \quad k = k_1 = k_2,
\]
\[
W(x) = C_7e^{2k_1x} + \left(C_8 - C_3C_6 + C_2C_7\right) + \left[C_2C_8 - C_4C_6\right]e^{-2k_1x},
\]
\[ Q^+_i = x I_2 \partial - \frac{k}{W(x)} \left\{ \left[ C_7 e^{2kx} - \left( C_2 C_8 - C_4 C_6 e^{-2kx} \right) \right] I_2 \right. \]
\[ + \left( \left( C_8 + C_3 C_6 - C_2 C_7 \right) - 2C_6 \right) \]
\[ \left. \left( 2C_3 C_8 - C_4 C_7 \right) - \left( C_8 + C_3 C_6 - C_2 C_7 \right) \right\} \] (63)
\[ U_0(x) = -k^2 I_2, \] (64)
\[ H_- = -I_2 \partial^2 - \frac{8k^2}{W^2(x)} \left[ C_7 \left( \frac{C_2 C_7 - C_3 C_6}{C_8} \right) e^{2kx} + 2C_7 \left( C_2 C_8 - C_4 C_6 \right) I_2 \right. \]
\[ + \left( C_2 C_8 - C_4 C_6 \right) \left( \frac{C_8}{C_3 C_8 - C_4 C_7} - \frac{C_6}{C_2 C_7 - C_3 C_6} \right) e^{-2kx} \right\} \] (65)
\[ \Psi_i^+(x) = Q_i^{-1} \left( \begin{array}{c} e^{kx} \\ 0 \end{array} \right), \quad \Psi_0^+(x) = Q_i^{-1} \left( \begin{array}{c} e^{-kx} \\ 0 \end{array} \right), \] (66)
\[ \Psi_i^+(x) = Q_i^{-1} \left( \begin{array}{c} 0 \\ e^{kx} \end{array} \right), \quad \Psi_0^+(x) = Q_i^{-1} \left( \begin{array}{c} 0 \\ e^{-kx} \end{array} \right), \] (67)
\[ \Psi_i^0(x) = Q_i^{-1} \left( \begin{array}{c} \frac{x}{2k} e^{kx} + C_2 \frac{x}{2k} e^{-kx} \\ -C_3 \frac{x}{2k} e^{kx} + C_4 \frac{x}{2k} e^{-kx} \end{array} \right), \quad \Psi_0^0(x) = Q_i^{-1} \left( \begin{array}{c} C_5 \frac{x}{2k} e^{-kx} \\ -C_7 \frac{x}{2k} e^{kx} + C_8 \frac{x}{2k} e^{-kx} \end{array} \right), \] (68)
\[ H_\lambda \Psi^+_i = \lambda \Psi^+_i, \quad i = 1, \ldots, 12, \]
\[ \Psi_1^+(x) + C_2 \Psi_2^+(x) + C_3 \Psi_3^+(x) + C_4 \Psi_4^+(x) = 0, \]
\[ C_6 \Psi_6^+(x) + C_7 \Psi_7^+(x) + C_8 \Psi_8^+(x) = 0, \] (69)

where the constants \( C_2, C_3, C_4, C_6, C_7 \) and \( C_8 \) are chosen such that the Wronskian (62) does not have real zeros. Moreover, the relations (50), in accordance with the results of section 3, can be supplemented with the additional intertwining relation with the operator \( Q_i^+ \) as follows:

\[ H_+ = Q_i^+ Q_i^- + U_0(x), \quad H_- = Q_i^- Q_i^+ + U_0(x), \]
\[ Q_i^- H_+ = H_- Q_i^+, \quad Q_i^+ H_- = H_+ Q_i^- \] (70)

There are only two linearly independent vector-functions in the set \( \Psi_1^+(x), \Psi_2^+(x), \Psi_3^+(x) \) and \( \Psi_4^+(x) \), in view of the fact that the vector-functions \( \Phi_1^-(x) \) and \( \Phi_2^-(x) \) (see (61)) form a canonical basis in the kernel of \( Q_i^- \). The corresponding relations between the vector-functions \( \Psi_1^+(x), \Psi_2^+(x), \Psi_3^+(x) \) and \( \Psi_4^+(x) \) are expressed by the formulas (69). It is not hard to check
that two of these vector-functions that are linearly independent form a canonical basis in the
kernel of the intertwining operator $Q_1^+$ and that these two vector-functions make up, together
with the vector-functions $\Psi_{10}^+(x)$ and $\Psi_{10}^-(x)$, a complete set of linearly independent formal
vector-eigenfunctions of the Hamiltonian $H_-$ for the spectral value $\lambda$. In addition, the vector-
functions $\Psi_{11}^+(x)$ and $\Psi_{11}^-(x)$, as linear combinations of $\Psi_{11}^+(x)$, $\Psi_{11}^-(x)$, $\Psi_{11}^+(x)$ and $\Psi_{11}^-(x)$, belong to the kernel of $Q_1^+$.

Analysis of the vector-functions (66), (67) and (68) leads to the following results:

(1) if

$$\text{Re } k \neq 0, \quad C_7(C_2C_8 - C_4C_6) \neq 0,$$

then for the eigenvalue $\lambda$ there are only two linearly independent normalizable vector-
eigenfunctions of the Hamiltonian $H_-$:

$$\Psi_{11}^+(x), \quad \Psi_{11}^-(x) \quad \text{or} \quad \Psi_{12}^+(x), \quad \Psi_{12}^-(x);$$

(2) if

$$\text{Re } k \neq 0, \quad C_7 = 0, \quad (C_8 - C_3C_6 + C_2C_7)(C_2C_8 - C_4C_6) \neq 0$$

or

$$\text{Re } k \neq 0, \quad C_2C_8 - C_4C_6 = 0, \quad (\lvert C_2 \rvert + \lvert C_4 \rvert)C_7(C_8 - C_3C_6 + C_2C_7) \neq 0,$$

then for the eigenvalue $\lambda$ there is only (up to a constant factor) one normalizable vector-
eigenfunction $\Psi_{11}^+(x)$ of the Hamiltonian $H_-$;

(3) if

$$\text{Re } k \neq 0, \quad C_2 = C_4 = 0, \quad C_7(C_8 - C_3C_6) \neq 0,$$

then for the eigenvalue $\lambda$ there is only (up to a constant factor) one normalizable vector-
eigenfunction $\Psi_{12}^+(x)$ of the Hamiltonian $H_-$;

(4) if

$$C_7 = C_2C_8 - C_4C_6 = 0, \quad C_8 - C_3C_6 + C_2C_7 \neq 0$$

or

$$C_7 = C_8 - C_3C_6 + C_2C_7 = 0, \quad C_2C_8 - C_4C_6 \neq 0$$

or

$$C_8 - C_3C_6 + C_2C_7 = C_2C_8 - C_4C_6 = 0, \quad C_7 \neq 0,$$

then for the spectral value $\lambda$ there is no normalizable vector-eigenfunction of the Hamiltonian $H_-$.  

It follows from (63), in view of (4), that the potential of the new Hamiltonian $H_-$ can be
reduced, with the help of a similarity transformation produced by a constant nondegenerate
$2 \times 2$ matrix, either to a diagonal form or to an upper triangular form with equal diagonal
elements. Let us consider these situations in more detail.

If the determinant of the matrix from the last term of (63) is nonzero:

$$4C_6(C_3C_8 - C_4C_7) - (C_8 + C_3C_6 - C_2C_7)^2 \neq 0, \quad (71)$$
then the formulas (61)–(65) and (68) for \( C_0 \neq 0 \) can be simplified as follows:

\[
\Phi_1^-(x) = C^{-1}\left\{ \Phi_1^-(x) + \frac{1}{C_6} \left[ \frac{2(C_2C_8 - C_4C_6)}{C_8 + C_2C_7 - C_3C_6 - \Delta} - C_2 \right] \Phi_2^-(x) \right\} = \left( e^{kx} + \tilde{C}_2e^{-kx} \right) \nonumber \\
\Phi_2^-(x) = C^{-1}\left[ \frac{\Delta - C_8 + C_2C_7 + C_3C_6 \Phi_2^-(x) - C_6C_7 \Phi_1^-(x)}{\Delta} \right] = \left( 0, C_7e^{kx} + \tilde{C}_8e^{-kx} \right) \nonumber \\
\]

\[
W(x) = [ e^{kx} + \tilde{C}_2e^{-kx}][ C_7e^{kx} + \tilde{C}_8e^{-kx}] = W(x), \nonumber \\
\]

\[
C^{-1}Q\tilde{C}C = \mp I_2\partial - k \begin{pmatrix} e^{kx} - \tilde{C}_2e^{-kx} & 0 \\ 0 & \frac{C_7e^{kx} - \tilde{C}_8e^{-kx}}{C_7e^{kx} + \tilde{C}_8e^{-kx}} \end{pmatrix}, \nonumber \]

\[
C^{-1}U_0(x)C = -k^2I_2, \quad C^{-1}H_C = -I_2\partial^2 - \begin{pmatrix} \frac{8k^2\tilde{C}_2}{[ e^{kx} + \tilde{C}_2e^{-kx}]^2} & 0 \\ 0 & \frac{8k^2C_7\tilde{C}_8}{[ C_7e^{kx} + \tilde{C}_8e^{-kx}]^2} \end{pmatrix}, \quad (72) \nonumber \\
\]

\[
\Psi_1^+(x) = \frac{1}{2kC_2}C^{-1}\left\{ \Psi_1^+(x) + \frac{1}{C_6} \left[ \frac{2(C_2C_8 - C_4C_6)}{C_8 + C_2C_7 - C_3C_6 - \Delta} - C_2 \right] \Psi_2^+(x) \right\} = \left( \frac{e^{kx} + \tilde{C}_2e^{-kx}}{C_7e^{kx} + \tilde{C}_8e^{-kx}} \right), \nonumber \]

\[
\Psi_2^+(x) = \frac{1}{2kC_7C_8}C^{-1}\left[ \frac{\Delta - C_8 + C_2C_7 + C_3C_6 \Psi_2^+(x) - C_6C_7 \Psi_1^+(x)}{\Delta} \right] = \left( \frac{1}{C_7e^{kx} + \tilde{C}_8e^{-kx}} \right), \nonumber \]

\[
C = \begin{pmatrix} C_3C_8 - C_4C_7 & 0 \\ C_8 - C_2C_7 & 1 \end{pmatrix}, \quad C^{-1} = \begin{pmatrix} 1 & 0 \\ -C_3C_8 + C_4C_7 & C_8 - C_2C_7 \end{pmatrix}, \quad \det C = 1, \quad \tilde{C}_2 = \frac{2(C_2C_8 - C_4C_6)}{C_8 + C_2C_7 - C_3C_6 - \Delta}, \nonumber \]

\[
\tilde{C}_8 = \frac{1}{2} \left[ C_8 + C_2C_7 - C_3C_6 - \Delta \right], \quad \Delta = \sqrt{(C_8 + C_3C_6 - C_2C_7)^2 - 4C_6(C_3C_8 - C_4C_7)}, \quad (73) \nonumber \\
\]
and for $C_0 = 0$ as follows:

$$
\Phi_1^- (x) = \mathcal{C}^{-1} \left[ \Phi_1^- (x) - \frac{C_4 - C_2 C_3}{C_8 - C_2 C_7} \Phi_2^- (x) \right] = \begin{pmatrix} e^{kx} + C_2 e^{-kx} \\ 0 \end{pmatrix},
$$

$$
\Phi_2^- (x) = \mathcal{C}^{-1} \Phi_2^- (x) = \begin{pmatrix} 0 \\ C_2 e^{kx} + C_8 e^{-kx} \end{pmatrix}.
$$

$$
\bar{W} (x) = \left[ e^{kx} + C_2 e^{-kx} \right] \left[ C_7 e^{kx} + C_8 e^{-kx} \right] = W (x),
$$

$$
\mathcal{C}^{-1} Q_1^+ \mathcal{C} = \pi I_2 \partial - k \begin{pmatrix} \frac{e^{kx} - C_2 e^{-kx}}{e^{kx} + C_2 e^{-kx}} & 0 \\ 0 & \frac{C_2 e^{kx} - C_8 e^{-kx}}{C_7 e^{kx} + C_8 e^{-kx}} \end{pmatrix}.
$$

$$
\mathcal{C}^{-1} U_0(x) \mathcal{C} = -k^2 I_2, \quad \mathcal{C}^{-1} H \mathcal{C} = -I_2 \partial^2 - \begin{pmatrix} 8k^2 C_2 \\ 0 \\ 0 \\ 8k^2 C_7 C_8 \end{pmatrix}, \quad \text{(74)}
$$

$$
\Psi_1^+ (x) = \frac{1}{2k C_2} \mathcal{C}^{-1} \left[ \Psi_1^+ (x) - \frac{C_4 - C_2 C_3}{C_8 - C_2 C_7} \Psi_2^+ (x) \right] = \begin{pmatrix} 1 \\ 0 \end{pmatrix},
$$

$$
\Psi_2^+ (x) = \frac{1}{2k C_7 C_8} \mathcal{C}^{-1} \Psi_2^+ (x) = \begin{pmatrix} 0 \\ \frac{1}{C_7 e^{kx} + C_8 e^{-kx}} \end{pmatrix}.
$$

$$
\mathcal{C} = \begin{pmatrix} C_3 C_8 - C_4 C_7 & 0 \\ C_4 C_8 - C_3 C_7 & 1 \end{pmatrix}, \quad \mathcal{C}^{-1} = \begin{pmatrix} 1 \\ -C_3 C_8 - C_4 C_7 \\ C_4 C_8 - C_3 C_7 \\ 1 \end{pmatrix}, \quad \det \mathcal{C} = 1,
$$

where $\bar{W} (x)$ is the Wronskian of $\Phi_1^- (x)$ and $\Phi_2^- (x)$, and the root (73) has arbitrary value such that $C_8 + C_2 C_7 - C_3 C_6 - \Delta \neq 0$ (this condition can be satisfied due to (71)). It is evident here and in what follows below in section 5.2 that the representations and the intertwings (70) transform trivially to the analogous formulas for the Hamiltonians $\mathcal{C}^{-1} H_\mathcal{C} = H_\mathcal{C} = -\partial^2$ and $\mathcal{C}^{-1} H_\mathcal{C}$, for the matrix $\mathcal{C}^{-1} U_0(x) \mathcal{C}$, and for the intertwining operators $\mathcal{C}^{-1} Q_1^+ \mathcal{C}$ and $\mathcal{C}^{-1} Q_2^+ \mathcal{C}$, that $\Psi_1^+ (x)$ and $\Psi_2^+ (x)$ are vector-eigenfunctions (sometimes formal) of the Hamiltonian $\mathcal{C}^{-1} H \mathcal{C}$ for the same eigenvalue $\lambda = -k^2$, and that $\Phi_1^- (x)$ and $\Phi_2^- (x)$ are transformation vector-functions corresponding to conversion of the Hamiltonian $\mathcal{C}^{-1} H \mathcal{C} = H_\mathcal{C}$ to the Hamiltonian $\mathcal{C}^{-1} H \mathcal{C}$ with the help of the intertwining operator $\mathcal{C}^{-1} Q_1^+ \mathcal{C}$. It follows from (72) and (74) that any of two diagonal elements of the potential of the reduced Hamiltonian (72) or (74) is either zero or the potential of Pöschl and Teller.
For
\[ 4C_6(C_3C_8 - C_4C_7) - (C_8 + C_3C_6 - C_2C_7)^2 = 0, \quad |C_6| + |C_3C_8 - C_4C_7| \neq 0 \quad (75) \]
the formulas (61)–(65) and (68) convert into the following:
\[
\Phi^{-}_{1}(x) = C^{-1} \left[ \sqrt{C_6} + \sqrt{C_7} \sqrt{C_2C_8 - C_4C_6} \sqrt{C_3C_8 - C_4C_7} \right] \Phi^{-}_{1}(x)
- \frac{C_4 \sqrt{C_6^*} + C_4 \sqrt{C_3C_8 - C_4C_7^*}}{|C_6| + |C_3C_8 - C_4C_7|} \Phi^{-}_{2}(x) = \left( -\sqrt{C_4} \sqrt{C_7 e^{kx} + \tilde{C}_4 e^{-kx}} \right)
\]
\[
\Phi^{-}_{2}(x) = C^{-1} \left[ \sqrt{C_6} + C_3 \sqrt{C_3C_8 - C_4C_7^*} \right] \Phi^{-}_{2}(x) - \frac{C_7 \sqrt{C_3C_8 - C_4C_7^*}}{|C_6| + |C_3C_8 - C_4C_7|} \Phi^{-}_{1}(x)
= \left( \tilde{C}_7 e^{kx} + \tilde{C}_4 \sqrt{C_7 e^{kx} - \tilde{C}_4 e^{-kx}} \right),
\]
\[
\Psi^{+}_{1}(x) = \frac{\sqrt{\alpha}}{2k \sqrt{C_7 \sqrt{C_2C_8 - C_4C_6}}},
C^{-1} \left[ \sqrt{C_6} \sqrt{\Psi^{+}_{1}(x) + \sqrt{C_2C_3 - C_4C_7^*}} \right]
= \left( \tilde{C}_7 e^{kx} + \tilde{C}_4 e^{-kx} \right).
\]
\[ \Psi^+_2(x) = \frac{\sqrt{\alpha}}{2k \sqrt{C_7} \sqrt{C_2 C_8 - C_4 C_6}} C^{-1} \psi_{12}^+(x) \]
\[ \times \left[ \sqrt{C_2 C_3 - C_4 + 2 \sqrt{C_7} \frac{C_2 \sqrt{C_6^*} + C_4 \sqrt{C_3 C_8 - C_4 C_7^*}}{|C_6| + |C_3 C_8 - C_4 C_7|}} \right] \psi^*_1(x) \]
\[ - \left( \sqrt{C_6} + 2 \sqrt{C_7} \frac{C_2 C_8 - C_4 C_6 \sqrt{C_3 C_8 - C_4 C_7}}{|C_6| + |C_3 C_8 - C_4 C_7|} \right) \psi^*_1(x) \]
\[ = \left( \sqrt{C_7} e^{i \xi} - \sqrt{\bar{C}_4} e^{-i \xi} \right) \left( \sqrt{C_7} e^{i \xi} + \sqrt{\bar{C}_4} e^{-i \xi} \right)^{-1} \]
\[ \sqrt{C_7} e^{i \xi} + \sqrt{\bar{C}_4} e^{-i \xi} \]
\[ \sqrt{C_2 C_8 - C_4 C_6} = C_2 \sqrt{C_7} + \sqrt{C_6} \sqrt{C_2 C_3 - C_4}, \]
\[ \sqrt{C_3 C_8 - C_4 C_7} = C_3 \sqrt{C_6} + \sqrt{C_7} \sqrt{C_2 C_3 - C_4}, \]
and \( \tilde{W} (x) \) is the Wronskian of \( \Phi_1^- (x) \) and \( \Phi_2^- (x) \). The possibility of defining the roots \( \sqrt{C_6}, \sqrt{C_7}, \sqrt{C_2 C_8 - C_4 C_6}, \sqrt{C_3 C_8 - C_4 C_7} \) and \( \sqrt{C_2 C_3 - C_4} \) such that the relations (76) hold is provided by the first of the conditions (75).

Finally, if

\[ 4C_6 (C_3 C_8 - C_4 C_7) - (C_8 + C_3 C_6 - C_2 C_7)^2 = 0, \quad C_6 = C_3 C_8 - C_4 C_7 = 0, \]

then

\[ W (x) = C_7 \left[ e^{ix} + C_2 e^{-ix} \right]^2 \Rightarrow C_7 \neq 0, \]
\[ Q_1^z = + I_2 \partial \partial_k - k \frac{e^{ix}}{e^{ix} + C_2 e^{-ix}} I_2, \quad U_0 (x) = -k^2 I_2, \]
\[ H_\gamma = -I_2 \partial \partial_k - \frac{8k^2 C_2}{\left( e^{ix} + C_2 e^{-ix} \right)^2} I_2, \]

(77)
\[ \Phi_1^- (x) = \left( \frac{1}{C_7} \right) \left[ e^{ix} + C_2 e^{-ix} \right], \quad \Phi_2^- (x) = C_7 \left( \frac{1}{1} \right) \left[ e^{ix} + C_2 e^{-ix} \right], \]
\[ \Psi_{11}^+ (x) = 2kC_2 \left( \frac{1}{C_7} \right) \left[ e^{ix} + C_2 e^{-ix} \right], \quad \Psi_{12}^+ (x) = 2kC_2 C_7 \left( \frac{1}{1} \right) \left[ e^{ix} + C_2 e^{-ix} \right], \]

and it is possible to use the vector-functions

\[ \tilde{\Phi}_1^- (x) = \Phi_1^- (x) - \frac{C_1}{C_7} \Phi_2^- (x) = \left( \begin{array}{c} e^{ix} + C_2 e^{-ix} \\ 0 \end{array} \right), \]
\[ \tilde{\Phi}_2^- (x) = \frac{1}{C_7} \Phi_2^- (x) = \left( \begin{array}{c} 0 \\ e^{ix} + C_2 e^{-ix} \end{array} \right) \]
as transformation vector-functions instead of \( \Phi_1^- (x) \) and \( \Phi_2^- (x) \), and the vector-functions

\[ \tilde{\Psi}_1^+ (x) = \frac{1}{2kC_2} \left[ \Psi_{11}^+ (x) - \frac{C_1}{C_7} \Psi_{12}^+ (x) \right] = \left( \begin{array}{c} 1 \\ e^{ix} + C_2 e^{-ix} \end{array} \right), \]
\[ \tilde{\Psi}_2^+ (x) = \frac{1}{2kC_2 C_7} \Psi_{12}^+ (x) = \left( \begin{array}{c} 0 \\ 1 \frac{1}{e^{ix} + C_2 e^{-ix}} \end{array} \right) \]
as vector-eigenfunctions (formal for \( C_2 = 0 \) and normalizable for \( C_2 \neq 0 \)) instead of \( \Psi_{11}^+ (x) \) and \( \Psi_{12}^+ (x) \). One can see that both diagonal elements of the potential of the new Hamiltonian (77) are either zeros, for \( C_2 = 0 \), or the identical potentials of Pöschl and Teller, for \( C_2 \neq 0 \).

5.3. The \( \lambda_1 = \lambda_2, g_1 = 1 \) subcase: adding up to two bound states described by vector-eigenfunctions and associated vector-function for the same energy value

In this subcase the general form of the transformation vector-functions \( \Phi_1^- (x) \) and \( \Phi_2^- (x) \) is the following, in view of (44):
\[ \Phi_{1}^{\pm}(x) = \begin{cases} -C_{1} \frac{x}{2k} e^{kx} + C_{2} \frac{x}{2k} e^{-kx} + C_{3} e^{kx} + C_{4} e^{-kx} \\ -C_{1} \frac{x}{2k} e^{kx} + C_{2} \frac{x}{2k} e^{-kx} + C_{3} e^{kx} + C_{4} e^{-kx} \end{cases} \]

\[ \Phi_{2}^{\pm}(x) = \begin{cases} C_{1} e^{kx} + C_{2} e^{-kx} \\ C_{3} e^{kx} + C_{4} e^{-kx} \end{cases} \]

\[ H_{+} \Phi_{1}^{\pm} = \lambda \Phi_{1}^{\mp}, \quad H_{-} \Phi_{2}^{\pm} = \lambda \Phi_{2}^{\mp}, \quad \lambda = -k^{2} \neq 0, \quad (78) \]

where \( C_{1}, \ldots, C_{8} \) are arbitrary complex—in general—constants and we assume without the loss of generality that \( C_{1} = 1 \) and \( C_{3} = 0 \) (the latter condition can be achieved in any case by making a change in the canonical basis in the kernel of \( Q_{1}^{-} \): using \( \Phi_{1}^{-}(x) - C_{3} \Phi_{2}^{-}(x) \) and \( \Phi_{2}^{-}(x) \) instead of \( \Phi_{1}^{-}(x) \) and \( \Phi_{2}^{-}(x) \)). The remaining constants \( C_{2}, C_{4}, C_{6}, C_{7} \) and \( C_{8} \) are chosen such that the Wronskian \( W(x) \) of the vector-functions \( \Phi_{1}^{-}(x) \) and \( \Phi_{2}^{-}(x) \),

\[ W(x) = -C_{7} e^{2kx} - \left[ C_{2} C_{8} - C_{4} C_{6} \right] e^{-2kx} - \frac{1}{k} \left[ C_{4} - C_{2} C_{3} \right] x - \left[ C_{8} + C_{2} C_{7} - C_{3} C_{6} \right], \quad (79) \]

does not vanish on the entire axis. The operators \( Q_{1}^{-} \) and \( Q_{1}^{+} \), the matrix \( U_{0}(x) \), and the new Hamiltonian \( H_{-} \) take the following form:

\[ Q_{1}^{\pm} = \mp i \hbar \partial - \frac{1}{W(x)} \left\{ k C_{7} e^{2kx} - k \Delta_{28} e^{-2kx} + \frac{1}{2k} \Delta_{1} I_{2} \right\} I_{2} + \frac{1}{2k} M_{1} e^{2kx} - \frac{1}{2k} M_{2} e^{-2kx} + M_{3} x + k M_{4}, \quad (80) \]

\[ U_{0}(x) = -k^{2} I_{2} + \frac{1}{W(x)} \]

\[ \times \left( \left[ e^{kx} + C_{2} e^{-kx} \right] \left[ C_{3} e^{kx} + C_{4} e^{-kx} \right] - \left[ e^{kx} + C_{2} e^{-kx} \right]^{2} \right) \]

\[ \equiv -k^{2} I_{2} + \frac{1}{W(x)} \left[ M_{1} e^{2kx} + M_{2} e^{-2kx} + M_{3} \right]. \quad (81) \]

\[ H_{-} = -i \hbar \partial^{2} + \frac{2}{W(x)} \left\{ \left[ -2k \left[ \Delta_{1} x + k \left( C_{8} + \Delta_{27} \right) \right] \left[ C_{7} e^{2kx} + \Delta_{28} e^{-2kx} \right] \right. \right. \]

\[ + 2 \Delta_{1} \left[ C_{7} e^{2kx} - \Delta_{28} e^{-2kx} \right] - 8k^{2} C_{7} \Delta_{28} + \frac{\Delta_{1}^{2} C_{7} \Delta_{28}}{2k^{2}} \right\} I_{2} \]

\[ - \left[ \frac{1}{k} \Delta_{1} e^{2kx} + \left( C_{8} + \Delta_{27} \right) - \frac{1}{2k^{2}} \Delta_{1} \right] e^{2kx} + 4 \Delta_{28} \right\} M_{1} \]

\[ - \left[ \frac{1}{k} \Delta_{1} e^{-2kx} + \left( C_{8} + \Delta_{27} \right) + \frac{1}{2k^{2}} \Delta_{1} \right] e^{-2kx} + 4 C_{7} \right\} M_{2} \]

\[ + \left[ 2k^{2} \left[ C_{7} e^{2kx} - \Delta_{28} e^{-2kx} \right] - \left[ C_{7} e^{2kx} + \Delta_{28} e^{-2kx} \right] \right\} M_{3} \]

\[ + 2k^{2} \left[ C_{7} e^{2kx} - \Delta_{28} e^{-2kx} \right] M_{4} \right\}, \quad (82) \]
\[
M_1 = \begin{pmatrix} C_1 & -1 \\ C_2 & 2 \\ C_3 & -C_3 \end{pmatrix}, \quad M_2 = \begin{pmatrix} C_2C_4 & -C_2^2 \\ C_2 & -C_2C_4 \end{pmatrix},
\]
\[
M_3 = \begin{pmatrix} C_4 + C_2C_3 & -2C_2 \\ 2C_2C_4 & -C_4 - C_2C_3 \end{pmatrix}, \quad M_4 = \begin{pmatrix} C_8 - \Delta_{27} & -2C_6 \\ 2\Delta_{38} & -\left[ C_8 - \Delta_{27} \right] \end{pmatrix},
\]

\[
\Delta_1 = C_4 - C_2C_3, \quad \Delta_{27} = C_2C_7 - C_3C_6, \quad \Delta_{28} = C_2C_8 - C_4C_6,
\]
\[
2\Delta_{38}M_1 + 2C_7M_2 - (C_8 + \Delta_{27})M_3 + \Delta_1M_4 = 0,
\]
such that

\[
H_+ = Q_1^+Q_1^- + U_0(x), \quad H_- = Q_1^-Q_1^+ + U_0(x), \quad Q_1^-H_+ = H_-Q_1^+.
\]

For the spectral value \( \lambda \) of the Hamiltonian \( H_- \) one can easily construct formal vector-eigenfunctions and formal associated vector-functions of the first order:

\[
\Psi_{1,0}^+(x) = Q_1^\lambda e^{\lambda x}, \quad \Psi_{2,0}^+(x) = Q_1^\lambda e^{-\lambda x},
\]
\[
\Psi_{3,0}^+(x) = Q_1^\lambda \begin{pmatrix} 0 \\ e^{\lambda x} \\ 0 \end{pmatrix}, \quad \Psi_{4,0}^+(x) = Q_1^\lambda \begin{pmatrix} 0 \\ e^{-\lambda x} \\ 0 \end{pmatrix},
\]
\[
\Psi_{1,1}^+(x) = Q_1^\lambda \begin{pmatrix} -xe^{\lambda x} \\ 0 \\ 2k \end{pmatrix}, \quad \Psi_{2,1}^+(x) = Q_1^\lambda \begin{pmatrix} xe^{-\lambda x} \\ 0 \\ 2k \end{pmatrix},
\]
\[
\Psi_{3,1}^+(x) = Q_1^\lambda \begin{pmatrix} 0 \\ -xe^{\lambda x} \\ 2k \end{pmatrix}, \quad \Psi_{4,1}^+(x) = Q_1^\lambda \begin{pmatrix} 0 \\ xe^{-\lambda x} \\ 2k \end{pmatrix},
\]

\[
H_+\Psi_{1,0}^+ = \lambda \Psi_{1,0}^+, \quad (H_- - \lambda I_2)\Psi_{1,1}^+ = \Psi_{1,0}^+, \quad i = 1, 2, 3, 4,
\]

only six of which are linearly independent in view of the fact that the vector-functions \( \Phi_1^-(x) \) and \( \Phi_2^-(x) \) (see (78)) form a canonical basis in the kernel of \( Q_1^- \). The latter leads to the relations

\[
\Psi_{1,0}^+(x) + C_2\Psi_{2,0}^+(x) + C_3\Psi_{3,0}^+(x) + C_4\Psi_{4,0}^+(x) = 0,
\]
\[
\Psi_{1,1}^+(x) + C_2\Psi_{2,1}^+(x) + C_3\Psi_{3,1}^+(x) + C_4\Psi_{4,1}^+(x)
\]
\[+ C_6\Psi_{5,0}^+(x) + C_7\Psi_{5,0}^+(x) + C_8\Psi_{5,0}^+(x) = 0.
\]

It follows from the results of [23] that in the subcase \( \lambda_1 = \lambda_2, g_1 = 1 \) considered, there is a linear differential operator of third order, \( Q_3^+ \), with the coefficient \( I_2 \) of \( \partial^3 \), that intertwines the Hamiltonians \( H_+ \) and \( H_- \) in the opposite direction: \( Q_3^+H_+ = H_-Q_3^+ \), and six linearly independent vector-functions from the set (84) form a canonical basis in the kernel of \( Q_3^+ \), providing an opportunity to construct \( Q_3^+ \) explicitly with the help of (41).

A formal vector-eigenfunction \( \Psi_{5,0}^+(x) \) of the Hamiltonian \( H_- \) that is linearly independent of (84) for the spectral value \( \lambda \) can be found in the form

\[J. \text{Phys. A: Math. Theor. 48 (2015) 085202 A V Sokolov}\]
\[ \Psi_{5,0}^+(x) = Q_1^-(x) = \frac{1}{8k^2} \left( x^2 - \frac{x}{k} \right) e^{kx} + \frac{C_2}{8k^2} \left( x^2 + \frac{x}{k} \right) e^{-kx} + \frac{C_6}{2k} \left( x + \frac{1}{2k} \right) e^{-kx} \]

\[ H_+ \Psi_{5,0}^+ = \lambda \Psi_{5,0}^+ \]

since

\[ (H_+ - \lambda I_2) \]

\[ \times \left( \frac{1}{8k^2} \left( x^2 - \frac{x}{k} \right) e^{kx} + \frac{C_2}{8k^2} \left( x^2 + \frac{x}{k} \right) e^{-kx} + \frac{C_6}{2k} \left( x + \frac{1}{2k} \right) e^{-kx} \right) \]

\[ = \Phi_1^-(x), \]

the vector-function \( \Phi_1^-(x) \) belongs to the kernel of \( Q_1^- \), and a chain of formal associated vector-functions of the Hamiltonian \( H_+ \) is mapped (see section 2.2) by the operator \( Q_1^- \) into a chain of formal associated vector-functions of the Hamiltonian \( H_- \) (some first terms of the chain can be mapped by \( Q_1^- \) into zeros).

Analysis of the vector-functions (84) and (86) leads to the following results:

(1) if

\[ \text{Re } k \neq 0, \quad C_7 \{ C_2 C_8 - C_4 C_6 \} \neq 0, \]

then for the eigenvalue \( \lambda \) of the Hamiltonian \( H_- \) there is only (up to a constant factor) one normalizable vector-eigenfunction \( \Psi_{6,0}^+(x) \) and only (up to a constant factor and up to adding a vector-function proportional to \( \Psi_{6,0}^+(x) \)) one associated vector-function of the first order \( \Psi_{6,1}^+(x) \):

\[ \Psi_{6,0}^+(x) = \frac{1}{W(x)} \left( \frac{\Delta_1}{4k^2} - C_2 C_7 \right) e^{kx} - \left( \frac{C_8 + \frac{\Delta_2}{2k}}{2k} - 2kC_6 C_7 \right) e^{-kx} \]

\[ \Psi_{6,1}^+(x) = \frac{1}{W(x)} \left( \frac{\Delta_1}{4k^2} - C_2 C_7 \right) x e^{kx} + \left( \frac{C_8 + \frac{\Delta_2}{2k}}{2k} - 2kC_6 C_7 \right) x e^{-kx} \]

\[ + \left( \frac{\Delta_2}{2k} + \frac{C_4^2 C_7}{2k + 2kC_7\Delta_2} \right) e^{-kx} + \left( \frac{C_2 \frac{\Delta_1}{4k^2} + \Delta_2}{2k + 2kC_7\Delta_2} \right) x e^{-kx}, \]
\( H_- \Psi^+_{6,0} = \lambda \Psi^+_{6,0}, \quad (H_- - \lambda I_2) \Psi^+_{6,1} = \Psi^+_{6,0}, \quad \Psi^+_{6,0}(x), \Psi^+_{6,1}(x) \in \ker Q^+_2; \quad (87) \)

(2) if

\[
\text{Re } k \neq 0, \quad C_7 = C_4 - C_2 C_3 = 0, \quad (C_8 + C_2 C_7 - C_4 C_6)(C_2 C_8 - C_4 C_6) \neq 0
\]
or

\[
\text{Re } k \neq 0, \quad C_4 - C_2 C_3 = C_2 C_8 - C_4 C_6 = 0, \quad C_2 C_7(C_8 + C_2 C_7 - C_4 C_6) \neq 0
\]
or

\[
\text{Re } k = 0, \quad C_4 - C_2 C_3 \neq 0,
\]
then for the eigenvalue \( \lambda \) of the Hamiltonian \( H_- \) there is only (up to a constant factor) one normalizable vector-eigenfunction \( \Psi_{6,0}(x) \) and there is no normalizable associated vector-function of the first order;

(3) if

\[
\text{Re } k \neq 0, \quad C_2 = C_4 = 0, \quad C_7(C_8 - C_3 C_6) \neq 0,
\]
then for the eigenvalue \( \lambda \) of the Hamiltonian \( H_- \) there is only (up to a constant factor) one normalizable vector-eigenfunction

\[
\Psi^+_{6,1}(x)\bigg|_{C_3=C_i=0} = \left( \frac{2k C_6 C_7 - C_8 - C_3 C_6}{2k} - \frac{1}{C_7 e^{kx} + (C_8 - C_3 C_6)e^{-kx}} \right)
\]

\[
H_- \Psi^+_{6,1}\bigg|_{C_3=C_i=0} = \lambda \Psi^+_{6,1}\bigg|_{C_3=C_i=0}
\]

(cf (87)) and there is no normalizable associated vector-function of the first order;

(4) if

\[
\text{Re } k \neq 0, \quad C_7 = 0, \quad (C_4 - C_2 C_3)(C_2 C_8 - C_4 C_6) \neq 0
\]
or

\[
\text{Re } k \neq 0, \quad C_2 C_8 - C_4 C_6 = 0, \quad C_7(C_4 - C_2 C_3) \neq 0
\]
or

\[
\text{Re } k \neq 0, \quad C_7 = C_2 C_8 - C_4 C_6 = 0
\]
or

\[
\text{Re } k = 0, \quad C_4 - C_2 C_3 = 0,
\]
then for the eigenvalue \( \lambda \) of the Hamiltonian \( H_- \) there is no normalizable vector-eigenfunction.

For \( \Delta_1 \neq 0 \Leftrightarrow C_4 \neq C_2 C_3 \) the formulas (78)–(82) and (87) can be simplified with the help of a similarity transformation as follows:

\[
\tilde{\Phi}^-_{\Delta_1}(x) = C^{-1}\left[ \Phi^-_1(x) + \frac{C_2 C_7}{\Delta_1} \Phi^-_2(x) \right] = \left( \begin{array}{c} -\frac{x}{2k} e^{kx} + \tilde{C}_6 e^{-kx} \\ \frac{x}{2k} e^{-kx} + \tilde{C}_7 e^{kx} + \tilde{C}_8 e^{-kx} \end{array} \right).
\]
\[ \Phi^-_2(x) = C^{-1} \Phi^-_2(x) = \left( \begin{array}{c} e^{kx} \\ e^{-kx} \end{array} \right), \]

\[ W(x) = -C_7 e^{2kx} + C_6 e^{-2kx} - \frac{1}{k} \left[ x + kC_8 \right] = \frac{1}{\Delta_i} W(x), \]

\[ C^{-1}Q^+_2 C = xI_2 \sigma + \frac{k}{W(x)} \left\{ \left[ \begin{array}{c} C_7 e^{2kx} + C_6 e^{-2kx} + \frac{1}{2k^2} \end{array} \right] I_2 + 2 \left( \frac{1}{2k} \left[ x + kC_8 \right] - \left[ \begin{array}{c} \frac{e^{2kx}}{4k^2} + \tilde{C}_6 \\ \frac{e^{-2kx}}{4k^2} - \tilde{C}_7 e^{2kx} \end{array} \right] \right) \right\}, \]

\[ C^{-1}U_0(x) C = -k^2 I_2 + \frac{1}{W(x)} \left( \begin{array}{cc} 1 & -e^{2kx} \\ e^{-2kx} & -1 \end{array} \right), \]

\[ C^{-1}H \cdot C = -I_2 \sigma^2 + \frac{8k}{W^2(x)} \left\{ x + kC_8 \left[ \begin{array}{c} \frac{C_6 e^{-2kx}}{4k^2} \\ -\frac{e^{-2kx}}{4k^2} - \tilde{C}_7 e^{2kx} \end{array} \right] \right\} - k \left[ \begin{array}{c} \tilde{C}_7 e^{2kx} - \tilde{C}_6 e^{-2kx} \\ \frac{1}{4k^2} \end{array} \right] \left[ \begin{array}{c} \tilde{C}_6 \\ \tilde{C}_7 - \frac{1}{4k^2} \end{array} \right] \]

\[ + \left\{ \left[ \begin{array}{c} \tilde{C}_7 e^{kx} + \frac{e^{ks}}{4k^2} \\ \frac{e^{kx}}{4k^2} + \tilde{C}_6 e^{-ks} \end{array} \right] - \left[ \begin{array}{c} \frac{e^{kx}}{4k^2} + \tilde{C}_6 e^{-ks} \\ \frac{e^{-kx}}{4k^2} \end{array} \right] \right\} \right\}, \]

\[ \Psi^+_1(x) = C^{-1} \Psi^+_0(x) = \frac{2k}{W(x)} \left( \begin{array}{c} \frac{1}{4k^2} e^{kx} + \tilde{C}_6 e^{-kx} \\ -\tilde{C}_7 e^{kx} + \frac{1}{4k^2} e^{-kx} \end{array} \right) \]

35
\[ \varphi_{1,1}^+ (x) = C^{-1} \left[ \varphi_{0,1}^+ (x) - \frac{C_8 - C_2 C_7 - C_4 C_6}{2\Delta_1} \varphi_{0,0}^+ (x) \right] \]
\[ = \frac{2k}{W(x)} \left( \frac{x + k \tilde{C}_8}{2k} \left[ \frac{e^{kx}}{4k^2} - \tilde{C}_6 e^{-kx} \right] - \tilde{C}_6 \left[ \tilde{C}_7 e^{kx} + \frac{e^{-kx}}{4k^2} \right] \right) \]
\[ = \frac{2k}{W(x)} \left( \frac{x + k \tilde{C}_8}{2k} \left[ \frac{e^{kx}}{4k^2} + \tilde{C}_6 e^{-kx} \right] - \frac{x + k \tilde{C}_8}{2k} \left[ \tilde{C}_7 e^{kx} - \frac{e^{-kx}}{4k^2} \right] \right) \]

\[ C = \begin{pmatrix} 1 & C_2 \\ C_3 & C_4 \end{pmatrix}, \quad C^{-1} = \frac{1}{\Delta_1} \begin{pmatrix} C_4 & -C_2 \\ -C_3 & 1 \end{pmatrix}, \quad \det C = \Delta_1, \]

\[ \tilde{C}_6 = \frac{-\Delta_{28}}{\Delta_1}, \quad \tilde{C}_7 = \frac{C_7}{\Delta_1}, \quad \tilde{C}_8 = \frac{C_8 + \Delta_{27}}{\Delta_1}, \]

and if \( \Delta_1 = 0 \Leftrightarrow C_4 = C_2 C_3, \) then the formulas (78)–(82) and (87) can also be simplified:

\[ \varphi_1^- (x) = C^{-1} \varphi_1^- (x) = \begin{pmatrix} -\frac{x}{2k} e^{kx} + C_2 \frac{x}{2k} e^{-kx} + C_6 e^{-kx} \\ \tilde{C}_7 e^{kx} + \tilde{C}_8 e^{-kx} \end{pmatrix}. \]

\[ \varphi_2^- (x) = C^{-1} \varphi_2^- (x) = \begin{pmatrix} e^{kx} + C_2 e^{-kx} \\ 0 \end{pmatrix}. \] (88)

\[ \tilde{W}(x) = -\left[ \frac{e^{kx} + C_2 e^{-kx}}{e^{kx} + C_2 e^{-kx}} \left[ \tilde{C}_7 e^{kx} + \tilde{C}_8 e^{-kx} \right] \right] = -\frac{1}{\alpha} W(x), \] (89)

\[ C^{-1} Q_{\tilde{I}_2} C = \mp \tilde{I}_2 \partial - \begin{pmatrix} k \frac{e^{kx} - C_2 e^{-kx}}{e^{kx} + C_2 e^{-kx}} - \frac{1}{2k} \frac{e^{2kx} - C_2^2 e^{-2kx} + 4k (C_2 x + k C_6)}{e^{kx} + C_2 e^{-kx}} \left[ \tilde{C}_7 e^{kx} + \tilde{C}_8 e^{-kx} \right] \\ 0 \end{pmatrix}, \] (90)

\[ C^{-1} H_{\mathcal{I}_2} C = -\tilde{I}_2 \partial^2 - \begin{pmatrix} \frac{8k^2 C_2}{e^{kx} + C_2 e^{-kx}} - \frac{8k}{C_2 x + k C_6} \left[ \tilde{C}_7 e^{2kx} - C_2 \tilde{C}_8 e^{-2kx} \right] \\ 0 \end{pmatrix} \left[ \frac{\tilde{C}_7 e^{kx} + \tilde{C}_8 e^{-kx}}{\tilde{C}_7 e^{kx} + \tilde{C}_8 e^{-kx}} \right] \]
\[ + \begin{pmatrix} \frac{2k (\tilde{C}_8 + C_2 \tilde{C}_7)}{\tilde{C}_7 e^{kx} + \tilde{C}_8 e^{-kx}} + \frac{4C_2}{e^{kx} + C_2 e^{-kx}} \left[ \tilde{C}_7 e^{kx} + \tilde{C}_8 e^{-kx} \right] \\ 0 \end{pmatrix} \left[ \frac{\tilde{C}_7 e^{kx} + \tilde{C}_8 e^{-kx}}{\tilde{C}_7 e^{kx} + \tilde{C}_8 e^{-kx}} \right]^2. \]

\[ C^{-1} U_0 (x) C = \begin{pmatrix} -k^2 \frac{e^{kx} + C_2 e^{-kx}}{\tilde{C}_7 e^{kx} + \tilde{C}_8 e^{-kx}} \\ 0 \end{pmatrix}. \] (91)
\[ \Psi_{1,0}^+(x) = C^{-1}\Psi_{0,0}^+(x) = \begin{pmatrix} 2kC_2 e^{kx} + C_3 e^{-kx} \\ 0 \end{pmatrix}, \quad (92) \]

\[ \Psi_{1,1}^+(x) = C^{-1}\Psi_{0,1}^+(x) = \begin{pmatrix} \frac{4k^2C_6 \tilde{C}_7 - \tilde{C}_8}{2k} \\ \frac{2k \tilde{C}_7 e^{kx} + \tilde{C}_8 e^{-kx}}{\tilde{C}_8 e^{kx} + \tilde{C}_7 e^{-kx}} \end{pmatrix}, \quad (93) \]

\[ \left. \Psi_{1,1}^+(x) \right|_{C_3 = C_4 = 0} = C^{-1}\Psi_{0,1}^+(x) \bigg|_{C_3 = C_4 = 0} = \begin{pmatrix} 4k^2C_6 \tilde{C}_7 - \tilde{C}_8 \\ 2k \tilde{C}_7 e^{kx} + \tilde{C}_8 e^{-kx} \end{pmatrix}, \quad (94) \]

where \( \tilde{W}(x) \) is the Wronskian of \( \tilde{\Phi}_1(x) \) and \( \tilde{\Phi}_2(x) \). It is evident that the representations and the intertwining (83) transform trivially into the analogous formulas for the Hamiltonians \( C^{-1}H_{1,0}C = H_0 = -\partial^2 \) and \( C^{-1}H_{1,0}C \), for the matrix \( C^{-1}U_0(x)C \), and for the operators \( C^{-1}Q_0^+C \) and \( C^{-1}Q_0^-C \), that \( \Psi_{1,0}^+(x) \) and \( \Psi_{1,1}^+(x) \) for \( |C_2| + |C_4| \neq 0 \) are the vector-eigenfunction and associated vector-function of the first order (sometimes formal), respectively, of the Hamiltonian \( C^{-1}H_{1,0}C \) for the same eigenvalue \( \lambda = -k^2 \), that \( \tilde{\Psi}_{1,1}^+(x) \) for \( C_2 = C_4 = 0 \) is a vector-eigenfunction (sometimes formal) of the Hamiltonian \( C^{-1}H_{1,0}C \) for the same eigenvalue \( \lambda = -k^2 \), and that \( \tilde{\Phi}_1(x) \) and \( \tilde{\Phi}_2(x) \) are transformation vector-functions corresponding to the conversion of the Hamiltonian \( C^{-1}H_{1,0}C = H_0 \) to the Hamiltonian \( C^{-1}H_{1,0}C \) with the help of the intertwining operator \( C^{-1}Q_0^+C \). For \( k > 0 \), \( C_2 = 1 \), \( C_6 = 0 \) and \( \tilde{\Phi}_1 = \tilde{C}_7 \in \mathbb{R}, \tilde{\Phi}_2 = \tilde{C}_8 \neq 0 \), the vector-functions (88) and (92)–(94), the operators (91), and the Wronskian (89) are obviously \( PT \)-symmetric, the operators (90) are \( PT \)-antisymmetric, and the spectrum of \( C^{-1}H_{1,0}C \) is real.

### 6. Conclusions

In conclusion, we itemize some tasks which could be undertaken in future papers.

(1) To work out methods of spectral design for matrix Hamiltonians with the help of matrix intertwining operators of arbitrary order and, in particular, to find a criterion for transformation vector-functions that provides desired changes for the spectrum of the corresponding final matrix Hamiltonian with respect to the spectrum of an initial matrix.
Hamiltonian. It is possible to try, for this purpose, to generalize the index theorem and lemma 4 of [36, 49] to the matrix case.

(2) To investigate the (in)dependence of matrix differential intertwining operators using an approach analogous to that of [35] and, in particular, to define the notions of dependence and independence for these operators, to find a criterion of dependence for them, and to solve the questions of the maximal number of independent matrix differential intertwining operators and a basis of such operators.

(3) Using an approach analogous to that of [35, 50], to investigate in the matrix case properties of a minimal matrix differential hidden symmetry operator.

(4) To investigate the (ir)reducibility of matrix differential intertwining operators and, in particular, to classify irreducible and absolutely irreducible [23] matrix differential intertwining operators using an approach analogous to that of [37–42, 51–58].

Acknowledgments

The author is grateful to A A Andrianov for critical reading of this paper and valuable comments, to M V Ioffe for drawing attention to some papers on matrix models with supersymmetry, and to the organizers of PHHQP 14 for hospitality. This work was supported by the RFBR Grant 13-01-00136-a. The author acknowledges Saint-Petersburg State University for a research grant, 11.38.660.2013, and for a travel grant, 11.41.1418.2014.

References

[1] Andrianov A A and Ioffe M V 1988 Phys. Lett. B 205 507
[2] Amado R D, Cannata F and Dedonder J-P 1988 Phys. Rev. A 38 3797
[3] Amado R D, Cannata F and Dedonder J-P 1990 Int. J. Mod. Phys. A 5 3401
[4] Andrianov A A and Ioffe M V 1991 Phys. Lett. B 255 543
[5] Andrianov A A, Ioffe M V, Spiridonov V P and Vinet L 1991 Phys. Lett. B 272 297
[6] Cannata F and Ioffe M V 1992 Phys. Lett. B 278 399
[7] Cannata F and Ioffe M V 1993 J. Phys. A: Math. Gen. 26 L89
[8] Fukui T 1993 Phys. Lett. A 178 1
[9] Hau L V, Golovchenko J A and Burns M M 1995 Phys. Rev. Lett. 75 1426
[10] Sparenberg J-M and Baye D 1997 Phys. Rev. Lett. 79 3802
[11] de Lima Rodrigues R, da Silva Filho P B and Vaidya A N 1998 Phys. Rev. D 58 125023
[12] Das T K and Chakrabarti B 1999 J. Phys. A: Math. Gen. 32 2387
[13] Tkachuk V M and Roy P 1999 Phys. Lett. A 263 245 (arXiv:quant-ph/9905102)
[14] Ioffe M V, Kuru Ş, Negro J and Nieto L M 2006 J. Phys. A: Math. Gen. 39 6987 (arXiv:hep-th/0603005)
[15] Ferraro E, Messina A and Nikitin A G 2010 Phys. Rev. A 81 042108 (arXiv:0909.5543 [quant-ph])
[16] Andrianov A A, Cannata F, Nishnianidze D N and Ioffe M V 1997 J. Phys. A: Math. Gen. 30 5037 (arXiv:quant-ph/9707004)
[17] Goncharenko V M and Veselov A P 1998 J. Phys. A: Math. Gen. 31 5315
[18] Samsonov B F and Pecheritsin A A 2004 J. Phys. A: Math. Gen. 37 239 (arXiv:quant-ph/0307145)
[19] Suzko A A 2005 Phys. Lett. A 335 88
[20] Pecheritsin A A, Pupasov A M and Samsonov B F 2011 J. Phys. A: Math. Theor. 44 205305 (arXiv:1102.5255 [quant-ph])
[21] Pupasov-Maksimov A M Multichannel generalization of eigen-phase preserving supersymmetric transformations (arXiv:1301.4199 [math-ph])
[22] Tanaka T 2012 Mod. Phys. Lett. A 27 1250051 (arXiv:1108.0480 [math-ph])
[23] Sokolov A V 2013 Phys. Lett. A 377 655 (arXiv:1307.4449 [quant-ph])
