Strategy Proof Mechanisms for Facility Location in Euclidean and Manhattan Space

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Abstract

We study the impact on mechanisms for facility location of moving from one dimension to two (or more) dimensions and Euclidean or Manhattan distances. We consider three fundamental axiomatic properties: anonymity which is a basic fairness property, Pareto optimality which is one of the most important efficiency properties, and strategy proofness which ensures agents do not have an incentive to mis-report. We also consider how well such mechanisms can approximate the optimal welfare. Our results are somewhat negative. Moving from one dimension to two (or more) dimensions often makes these axiomatic properties more difficult to achieve. For example, with two facilities in Euclidean space or with just a single facility in Manhattan space, no mechanism is anonymous, Pareto optimal and strategy proof. By contrast, mechanisms on the line exist with all three properties. We also show that approximation ratios may increase when moving to two (or more) dimensions. All our impossibility results are minimal. If we drop one of the three axioms (anonymity, Pareto optimality or strategy proofness) multiple mechanisms satisfy the other two axioms.

Introduction

The facility location problem captures many real world problems such as locating hospitals, telephone exchanges, ambulances, post offices or playgrounds. Beyond geographical domains, there are a range of other real world problems modeled by facility location problems such as choosing tax-rates, room temperatures, or members of a committee. The facility location problem has attracted researchers from a range of areas including AI, operations research and social choice (e.g. (Drezner and Hamacher 2002; Escoffier et al. 2013; Procaccia and Tennenholtz 2013; Feigenbaum and Sethuraman 2015; Golowich, Narasimhan, and Parkes 2018)). Our goal here is to design strategy proof mechanisms that elicit the true locations of a set of agents and use this information to locate one or more facilities to serve the agents fairly and efficiently. In particular, we look to minimize the total or maximum distance of the agents from the facility serving them. These are respectively an utilitarian or egalitarian welfare objective.

In previous work on mechanism design for facility location problems, researchers have often limited their attention to the one dimensional problem, locating facilities on a line (e.g. (Procaccia and Tennenholtz 2013)). This captures a number of real world settings such as locating distribution centres along a motorway, or sewage plants along a river. However, many real world problems have two (or more) dimensions, and use metrics such as Euclidean or Manhattan distances. This focus on one dimensional problems has been justified as a starting point to consider more complex settings (e.g., trees and networks), and as it may provide insight into these more complex settings (e.g. lower bounds for 1-d problem are inherited by 2-d problem). Our contribution here is to suggest caution in such arguments. We show, for example, that many Pareto optimal and strategy proof mechanisms do not lift from 1-d to 2-d space. The extra freedom provided by an additional dimension can make it harder to achieve desirable axiomatic properties like Pareto optimality and strategy proofness.

Formal Background

We have \( n \) agents located in 2-d space, and want to locate \( m \) facilities to serve all \( n \) agents. Agent \( i \) is at location \((x_i, y_i)\). Distances are either Euclidean or Manhattan: \( d((x, y), (u, v)) = \sqrt{(x - u)^2 + (y - v)^2} \) or \( d((x, y), (u, v)) = |x - u| + |y - v| \). The extension of the problem to higher dimensions is straightforward, while the restriction to 1-d space simply requires setting all \( y \)-coordinates to zero. In this case, we suppose agents are ordered so that \( x_1 \leq \ldots \leq x_n \). An agent is served by the nearest facility, and a solution is a location \((u_j, v_j)\) for each facility \( j \). We let \( a_i \in [1, m] \) be the facility serving agent \( i \). We consider an utilitarian welfare objective of the total distance, \( \sum_{i=1}^{n} d((x_i, y_i), (u_{a_i}, v_{a_i})) \) and an egalitarian welfare objective of the maximum distance, \( \max_{i=1}^{n} d((x_i, y_i), (u_{a_i}, v_{a_i})) \). The goal is to locate facilities to minimize one of these two distances.

We consider a number of mechanisms for the facility location problem. Given a fixed order over the agents, the SD mechanism is a serial dictatorship that allocates the first facility to the location of the first agent, and subsequent facilities to the next agent in order at a new location. In the 1-d problem, the PERCENTILE mechanism is a family of mechanisms with parameters \( p_1 \) to \( p_m \) that locate facility...
every facility located at \( x_1 + |p_j(n-1)| \) for \( j \in [1, m] \). For instance, with \( m = 1 \), the LEFTMOST mechanism has \( p_1 = 0 \), the MEDIAN mechanism has \( p_1 = \frac{1}{2} \), and the RIGHTMOST mechanism has \( p_1 = 1 \). And, with \( m = 2 \), the ENDPOINT mechanism, which locates facilities at the left and rightmost agents, has \( p_1 = 0 \) and \( p_2 = 1 \). The \( d \)-dimensional PERCENTILE mechanism picks an orthogonal set of \( d \) axes and applies a PERCENTILE mechanism to each axis. For example, the 2-dimensional MEDIAN mechanism picks an orthogonal set of axes, and applies the MEDIAN mechanism to each axis.

In many facility location problems, facilities may also have capacity constraints (e.g. [Brimberg et al. 2001; Aziz et al. 2019, 2020]). A telephone exchange can connect at most a fixed number of houses, a kindergarten has places for only a given number of children, a doctor’s practice can serve just a limited number of patients, etc. Therefore, we also consider an extension of the facility location problem that includes capacity constraints. This extension was previously studied in [Aziz et al. 2019, 2020]. In this capacitated setting, the \( j \)-th facility can serve up to \( c_j \) agents. We assume that \( n \leq \sum_{j=1}^{m} c_i \) so that every agent can be served. One special setting we consider is when there is no spare capacity (i.e. \( \sum_{j=1}^{m} c_j = n \)). Another special setting we consider is when facilities are identical (i.e. \( c_i = c_j \) for all \( i < j \)). A solution in the capacitated setting is now both a location \((u_j, v_j)\) for each facility \( j \), and an assignment of agents to facilities such that the capacity constraint on each facility is not exceeded. Agents no longer have to be served by their nearest facility. Let \( N_j \) denote the set of agents assigned to facility \( j \), i.e., \( N_j = \{ i | a_i = j \} \). Then the capacity constraints ensure \(|N_j| \leq c_j\) for all \( j \in [1, m] \). The goal in the capacitated setting is to locate facilities and assign agents to facilities to minimize either the total or maximum distance.

We consider three desirable axiomatic properties of mechanisms for facility location: anonymity, strategy proofness and Pareto optimality. Anonymity is a simple but fundamental fairness property that requires all agents to be treated the same. Pareto optimality is a simple efficiency property that is one of the most fundamental normative properties in the whole of economics. It demands that we cannot make one agent better off without making other agents worse off. Finally, strategy proofness is a fundamental game theoretic property that ensures there is no incentive for agents to act strategically by mis-reporting their location.

More formally, a mechanism is anonymous iff permuting the agents does not change the solution. It is not hard to see that the SD mechanism is not anonymous, but that any PERCENTILE mechanism is. A mechanism is Pareto optimal iff it returns solutions that are always Pareto optimal. A solution is Pareto optimal iff there is no other solution in which one agent travels a strictly shorter distance, and every other agent travels the same or shorter distance to the facility serving them. For example, the SD mechanism is Pareto optimal. A mechanism is strategy proof iff agents cannot improve the solution by mis-reporting their location (i.e. no agent can mis-report and reduce the distance to travel to the facility serving them). For example, the \( d \)-dimensional PERCENTILE mechanism is strategy proof with any parameters (Theorem 3 in [Sui, Boutilier, and Sandholm 2013]).

Finally, we will consider how well a mechanism approximates the optimal possible welfare. A mechanism achieves an approximation ratio \( \rho \) for the total (maximum) distance iff the total (maximum) distance in any solution it returns is at most \( \rho \) times the optimal. In this case, we say that the mechanism \( \rho \)-approximates the optimal total (maximum) distance. For example, in 1-d space, any PERCENTILE mechanism such as the LEFTMOST mechanism 2-approximates the optimal maximum distance (Procaccia and Tennenholtz 2013).

One Facility in Euclidean Space

We begin with one of the simplest possible settings: a single uncapacitated facility in Euclidean space. This is a setting in which several insightful results are already known about strategy proof mechanisms.

Anonymity and Pareto Optimality

With an odd number of agents, a mechanism for locating a single facility in 2-dimensional Euclidean space is anonymous, Pareto optimal and strategy proof iff it is a 2-dimensional MEDIAN mechanism ([Peters et al. 1992]). For two dimensions and an even number of agents, or for three or more dimensions and an odd or even number of agents, no mechanism is anonymous, Pareto optimal and strategy proof ([Peters et al. 1992]).

We can consider dropping in turn anonymity, Pareto optimality and strategy proofness. If we drop anonymity, there are multiple mechanisms for Euclidean space that are Pareto optimal and strategy proof. For instance, any SD mechanism is Pareto optimal and strategy proof but not anonymous. Similarly, if we drop Pareto optimality, there are multiple mechanisms for Euclidean space that are anonymous and strategy proof but not Pareto optimal. For instance, the 2-dimensional LEFTMOST and the 2-dimensional RIGHTMOST mechanisms are both anonymous and strategy proof, but not Pareto optimal. Finally, if we drop strategy proofness, there are multiple mechanisms for Euclidean space that are anonymous and Pareto optimal but not strategy proof. For instance, consider a mechanism that locates the facility at the location of the agent with smallest \( x \) coordinate, tie-breaking by the smallest \( y \) coordinate. This mechanism is anonymous and Pareto optimal but not strategy proof. We conclude therefore that, with an odd number of agents in Euclidean space, anonymity, Pareto optimality and strategy proofness are a minimal combination of axioms characterizing the 2-dimensional MEDIAN mechanism.

Total Distance

With a single facility in Euclidean space, the total distance is minimized by locating the facility at the geometric median. A mechanism locating a single facility at the geometric median is anonymous and Pareto optimal, but not strategy proof ([Goel and Hann-Caruthers 2020]). Consider, for instance, agents at \((0, 0), (0, 2), (12, 0)\) and \((12, 2)\). The agent at \((12, 0)\) has an incentive to mis-report their location as \((12, 2)\) to move the geometric median from \((6, 1)\) to \((12, 2)\) which is nearer to \((12, 0)\). Thus, a strategy proof
mechanism in Euclidean space cannot always locate a single facility so as to minimize the total distance. We contrast this with the 1-d case where the \textsc{Median} mechanism is strategy proof and returns the optimal total distance.

As in (Procaccia and Tennenholtz 2013), we might ask: since a strategy proof mechanism in Euclidean space cannot be optimal with respect to the total distance, how well can it approximate the optimal total distance? We prove next that no strategy proof mechanism can be better than an $\alpha$-approximation for some $\alpha$ strictly greater than 1.

**Theorem 1** In Euclidean space, there exists an $\alpha > 1$ such that no deterministic and strategy proof mechanism for locating a single facility is an $\alpha$-approximation or better of the optimal total distance.

**Proof:** Assume the opposite. For any $\alpha > 1$, there exists a deterministic and strategy proof mechanism that is an $\alpha$-approximation of the optimal total distance. Consider four agents at $(0, 0), (0, 2), (12, 0)$ and $(12, 2)$. The optimal placement of the facility is at the geometric median: $(6, 1)$. The total distance of the four agents from the geometric median is $4\sqrt{37}$. Suppose the facility moves one unit distance from the geometric median. Then the total distances of the agents from the facility increases to at least $2(\sqrt{50} + \sqrt{20})$. This smallest total distance occurs when the facility is at $(5, 1)$ or $(7, 1)$. The ratio between these two total distances is $\frac{5\sqrt{21}}{2\sqrt{23}}$. This is strictly between 1.0003 and 1.0004. Set $\alpha = 1.0003$. Then the mechanism must locate the facility within one unit distance of the geometric median, and a distance of over five units from $(12, 0)$.

Suppose now the agent at $(12, 0)$ mis-reports their location as $(12, 2)$. The geometric median moves to $(12, 2)$. The total distance of the reported positions of the four agents from $(12, 2)$ is $12 + 2\sqrt{37}$ which is approximately 24.166. If the mechanism locates the facility more than two units distance from $(12, 2)$ then algebraic reasoning shows that the total distance of the facility from the reported positions of the agents increases from $12 + 2\sqrt{37}$ to more than 24.18. The ratio between these total distances is greater than 1.0005. Hence, if the mechanism is an $\alpha$-approximation for $\alpha = 1.0003$, then the facility must be located less than two units distance from $(12, 2)$. But the facility in this case is strictly less than four units distance from $(12, 0)$. Hence the agent at $(12, 0)$ has an incentive to mis-report their location as $(12, 2)$. This contradicts the assumption that the mechanism is strategy proof. Therefore the initial assumption that it is an $\alpha$-approximation for $\alpha = 1.0003$ must be false.

This result shows that there is a lower bound on the approximation ratio of any deterministic and strategy proof mechanism in Euclidean space. It leaves open how large this approximation ratio must be. For an odd number of agents and 2-dimensional Euclidean space, the 2-dimensional \textsc{Median} mechanism provides an upper bound on the best possible approximation ratio for the total distance. In particular, the 2-dimensional \textsc{Median} mechanism returns an $\frac{\sqrt{2\sqrt{n^2 + 1}}}{n + 1}$-approximation of the optimal total distance (Goel and Hann-Caruthers 2020). By letting $n$ tend to infinity, we conclude that $\alpha \leq \sqrt{2}$. We contrast this with the 1-d setting in which the \textsc{Median} mechanism is always strategy proof and optimal, locating the facility to minimize the total distance.

**Maximum Distance**

In Euclidean space, bounds on the performance of strategy proof mechanisms in computing the optimal maximum distance are less well studied. It is again easy to see that no strategy proof mechanism can return the optimal solution. For instance, the \textsc{OneCentre} mechanism, which locates the facility at the centre of the smallest enclosing circle, returns an optimal solution for agents on the Euclidean plane. The smallest enclosing circle can be found in linear time (Megiddo 1982). While this mechanism is anonymous and Pareto optimal, it is not strategy proof. Consider, for instance, two agents at $(0, 0)$ and one at $(0, 1)$. The rightmost agent can mis-report their location as $(0, 2)$ to achieve a better outcome. In fact, it follows from the 1-d setting (Theorem 3.2 of (Procaccia and Tennenholtz 2013), that no deterministic and strategy proof mechanism can do better than 2-approximate the optimal maximum distance in Euclidean space.

With an odd number of agents on the Euclidean plane, the only mechanism that is anonymous, Pareto optimal and strategy proof is the 2-dimensional \textsc{Median} mechanism (Peters et al. 1992). We now prove that this mechanism 2-approximates the optimal maximum distance. To show this, we need a lemma that this mechanism locates the facility on or within the smallest enclosing circle. To ensure the median is unambiguously defined and the mechanism is strategy proof, we restrict ourselves to an odd number of agents.

**Lemma 1** With 2-dimensional Euclidean space and an odd number of agents, the 2-dimensional \textsc{Median} mechanism locates the facility on the circumference of or within the smallest enclosing circle.

**Proof:** Suppose that there are $2k + 1$ agents, the facility is at $(x, y)$, and the centre of the smallest enclosing circle is $(x', y')$. There are two cases. In the first, $x \geq x'$. Let the circumference of the smallest enclosing circle go through $(x, y' + c)$ and $(x, y' - c)$. There are therefore $k + 1$ agents with $x$ coordinates greater than or equal to $x$. These $k + 1$ agents have a $y$ coordinate that is smaller than or equal to $y' + c$ as they are on or within the smallest enclosing circle. The median $y$ coordinate of the agents cannot then be larger than $y' + c$. By a similar argument, the median $y$ coordinate of the agents cannot also be smaller than $y' - c$. Hence, the facility lies on or within the smallest enclosing circle. In the second case, $x \leq x'$. This case is symmetric to the first.

**Theorem 2** With 2-dimensional Euclidean space and an odd number of agents, the 2-dimensional \textsc{Median} mechanism 2-approximates the optimal maximum distance.

**Proof:** By the previous lemma, the facility is located on or within the smallest enclosing circle. The worst case is when the facility is located on the circumference of this circle and there is an agent diametrically opposite. In this case, this agent is twice the optimal maximum distance away from the facility.
It follows that the 2-dimensional MEDIAN mechanism is optimal in terms of strategy proof mechanisms for approximating the optimal maximum distance. No deterministic and strategy proof mechanism has a better approximation ratio even when restricted to the line (Theorem 3.2 of [Procaccia and Tennenholtz 2013]). With an odd number of agents and a single facility, moving from the 1-d to the 2-d setting changes little. The MEDIAN mechanism on the line achieves the same (and optimal) 2-approximation of the maximum distance as the 2-dimensional MEDIAN mechanism does on the Euclidean plane.

Multiple Facilities in Euclidean Space

We next consider strategy proof mechanisms for locating two or more uncapatcitated facilities in Euclidean space. There is less known about this setting than about two or more facilities in one dimensional Euclidean space, or about a single facility in an Euclidean space of two or more dimensions. Our results here are somewhat negative. We first prove a strong impossibility theorem: when locating two or more facilities in Euclidean space, we cannot simultaneously achieve anonymity, Pareto optimality and strategy proofness.

**Theorem 3** With $m$ uncapacitated facilities ($m \geq 2$), $n$ agents ($n \geq m + 3$) and Euclidean space with two or more dimensions, no mechanism for facility location is anonymous, Pareto optimal and strategy proof.

**Proof:** We consider just two dimensional Euclidean space. The proof for three or more dimensions merely needs to restrict agents to a plane. Suppose we have 2 facilities and $n$ agents with $n \geq 5$. Either $n = 2k + 2$ or $n = 2k + 1$. Put $2k$ agents at $(0, 0)$ and the remaining $n - 2k$ agents (which is one or two at most) at $(100, 100)$. The unique Pareto optimal mechanism locates the facilities at $(0, 0)$ and $(100, 100)$ serving the agents at each location. Suppose the agents at $(100, 100)$ are fixed but those at $(0, 0)$ are allowed to move in the box bounded between $(0, 0)$ and $(1, 1)$. Then essentially we have a single facility location problem on the remaining $2k$ agents that locates the leftmost facility somewhere in the convex hull of the reported locations of the $2k$ agents. Theorem 4.1 in [Peters et al 1992] demonstrates that no mechanism is anonymous, Pareto optimal and strategy proof for an even number of agents in two dimensional Euclidean space. This proof continues to hold in the Euclidean plane with just an odd (or just an even) number of agents. It is again somewhat disappointing then that, when we move from one facility to two facilities in the Euclidean plane, we can no longer simultaneously achieve these desirable axiomatic properties.

We next consider dropping in turn anonymity, Pareto optimality and strategy proofness. If we drop anonymity, there are multiple mechanisms for locating two or more facilities in Euclidean space that are Pareto optimal and strategy proof but not anonymous. For example, any SD mechanism is Pareto optimal and strategy proof but not anonymous. Similarly, if we drop Pareto optimality, there are multiple mechanisms that are anonymous and strategy proof but not Pareto optimal (e.g. any multi-dimensional PERCENTILE mechanism with parameters $p_1 = p_2$ for all $i$ and $j$). And if we drop strategy proofness, there are multiple mechanisms that are anonymous and Pareto optimal but not strategy proof.

Consider, for example, the mechanism that picks some orthogonal set of coordinate axes, then orders agents by $x$-coordinate, tie-breaking by $y$-coordinate, and locates the first facility at the position of the first agent in this order, and subsequent facilities at the position of the next agent in the order at a new location. We therefore conclude that anonymity, Pareto optimality and strategy proofness are a minimal combination of incompatible axioms for locating two or more facilities in Euclidean space.

**Welfare Bounds**

We now consider how well strategy proof mechanisms can approximate the optimal welfare when locating multiple facilities in Euclidean space. For instance, can we bound the approximation ratio for the maximum or total distance?

When locating two facilities on the line, the only deterministic, anonymous and strategy proof mechanism with a bounded approximation ratio for the total distance is the ENDPOINT mechanism (Theorem 3.1 in [Potakis and Tzamos 2013]). Recall that the ENDPOINT mechanism is an instance of the PERCENTILE mechanism with $m = 2$, $p_1 = 0$ and $p_2 = 1$. The ENDPOINT mechanism also provides a 2-approximation of the optimal maximum distance. Again, this is optimal as no deterministic and strategy proof mechanism can have a better approximation ratio (Corollary 4.4 of [Procaccia and Tennenholtz 2013]).
In fact, the ENDPOINT mechanism is not just optimal in terms of the approximation ratio of the maximum distance. It is the only PERCENTILE mechanism for locating two uncapacitated facilities on the line with a bounded approximation ratio for the maximum distance (Anonymous 2021).

We therefore consider next welfare bounds for multi-dimensional PERCENTILE mechanism when locating two or more facilities in Euclidean space. It is known that in Euclidean space the multi-dimensional PERCENTILE mechanism for two or more uncapacitated facilities has an unbounded approximation ratio for the total distance (Theorem 4 in (Sui, Boutilier, and Sandholm 2013)). Can we do better in approximating the maximum distance? This is certainly the case in 1-dimension where we can approximate the maximum distance better than the total distance. Specifically, with any mechanism on the line that is deterministic and strategy proof, the optimal approximation ratio for the total distance is \( n - 2 \), but for the maximum distance is just 2 (Fotakis and Tzamos 2013). We prove next that, in Euclidean space, multi-dimensional PERCENTILE mechanisms cannot do better at approximating the maximum distance than they can at approximating the total distance.

**Theorem 4** No multi-dimensional PERCENTILE mechanism for two (or more) facilities in two (or more) dimensional Euclidean space bounds the approximation ratio for the maximum distance.

**Proof:** The maximum distance, \( d_{\text{max}} \), is lower bounded by the following identity relating it to \( d_{\text{total}} \), the total distance: \( d_{\text{max}} \geq \frac{d_{\text{total}}}{n} \). Hence, if the total distance is unbounded (as demonstrated by Theorem 4 in (Sui, Boutilier, and Sandholm 2013)), then the maximum distance is too. Rather than call upon this identity, we will give an explicit proof that the approximation ratio for the maximum distance is unbounded since the proof provides insight into why the approximation ratio can be unbounded, and how the PERCENTILE mechanism can go wrong. Note that each case in this proof is also an example of how PERCENTILE mechanisms do not bound the approximation ratio for the total distance.

We consider two facilities in two dimensions. The proof easily generalizes to more facilities and more dimensions. There are three cases. In the first case, neither of the two facilities is at a location corresponding to parameters \((1, 1)\). That is, neither of the two facilities is at \((x_{\text{max}}, y_{\text{max}})\) where \(x_{\text{max}} \) and \(y_{\text{max}} \) are the maximum reported \(x \) and \(y \) coordinates. Let \( p_{\text{max}} \) be the largest parameter less than 1, and \( n = \lceil \frac{1}{p_{\text{max}}} \rceil \). Consider \( n - 1 \) agents at \((0, 0)\) and one at \((1, 1)\). The optimal maximum distance is zero with a facility at \((0, 0)\) and one at \((1, 1)\). However, the PERCENTILE mechanism returns a solution with both facilities at \((0, 0)\), and maximum distance of \(\sqrt{2} \) in the second case. One of the facilities is at a location corresponding to parameters \((1, 1)\) and the other is at a location corresponding to \((p_1, p_2)\) where at least one \(p_i\) is non-zero. Let \( p_{\text{max}} \) be the maximum of \(p_1\) and \(p_2\), and \( n = \lceil \frac{3}{p_{\text{max}}} \rceil \). Consider \( n - 1 \) agents at \((1, 1)\) and one at \((0, 0)\). The optimal maximum distance is zero with a facility at \((0, 0)\) and one at \((1, 1)\). However, the PERCENTILE mechanism returns a solution with one facility at \((1, 1)\), and the other not at \((0, 0)\). This solution has a maximum distance of \(\sqrt{2} \). In the third case, one of the facilities is at a location corresponding to parameters \((0, 0)\) and the other is at a location corresponding to parameters \((1, 1)\). Consider one agent at \((0, 1)\) and the other at \((1, 0)\). The optimal maximum distance is zero with a facility at each agent, but the PERCENTILE mechanism returns a solution with a maximum distance of 1.

### Capacitated Facilities

We next consider the impact of capacity constraints on facility location in Euclidean space. We might hope to design mechanisms with stronger axiomatic properties in the capacitated setting than the uncapacitated setting for at least two reasons. The first reason is that adding capacity limits requires us to allocate agents to particular facilities, and this may reduce the opportunity for agents to be strategic. For example, in a general metric space, the randomized PROPORTIONAL mechanism is not strategy proof with three or more agents but becomes so when we make it winner imposing (Fotakis and Tzamos 2010).

A second reason we might do better in the capacitated setting in Euclidean space is that, with multiple uncapacitated facilities, irrespective of whether the total number of agents is even or odd, one facility might need to serve an even number of agents, and no mechanism in Euclidean space is anonymous, Pareto optimal and strategy proof with an even number of agents (Peters et al. 1992). On the other hand, in the capacitated setting, we can limit ourselves to situations in which the capacity of each facility is an odd number. Recall that in the Euclidean plane with an odd number of agents, there exists an unique mechanism to locate a single uncapacitated facility that is anonymous, Pareto optimal and strategy proof (Peters et al. 1992).

Unfortunately, this hope is not realized. We prove another strong impossibility theorem: no mechanism for locating two or more capacitated facilities in Euclidean space cannot simultaneously be anonymous, Pareto optimal, and strategy proof even limited to facilities with an odd capacity.

**Theorem 5** With two or more facilities of capacity 3 or larger and Euclidean space with two or more dimensions, no mechanism for capacitated facility location is anonymous, Pareto optimal and strategy proof.

**Proof:** Suppose that such a mechanism exist. We can assume that the facilities have the same size and that none of their capacity is spare since there is already no mechanism that is anonymous, Pareto optimal and strategy proof in the one dimensional setting. When we have two facilities with two or more different sizes, or we have spare capacity (Anonymous 2021). We consider also just two dimensional Euclidean space. The proof for three or more dimensions merely needs to restrict agents to a two dimensional plane.

We suppose that we have two facilities of capacity \(c\). There are two cases. If \(c\) is even, say \(c = 2k\) with \(k > 1\), we consider \(2k\) agents in the box between \((0, 0)\) and \((1, 1)\), and another \(2k\) agents at \((100, 100)\). Any anonymous and Pareto optimal mechanism locates one facility at \((100, 100)\)
serving the agents at this location, and the other in the box between (0, 0) and (1, 1). Suppose the agents at (100, 100) are fixed. Then essentially we have a single facility location problem on the remaining 2k agents. Theorem 4.1 in Peters et al. [1992] demonstrates that no mechanism is anonymous, Pareto optimal and strategy proof for an even number of agents in two dimensional Euclidean space. This proof continues to hold if the agents are limited to a small box such as between (0, 0) and (1, 1). Note that an agent cannot profitably mis-report their location as (100, 100). To ensure anonymity and Pareto optimality, the mechanism must then locate both facilities at (100, 100) which puts the facility serving this agent at a greater distance. A sincere report locates the facility within the convex hull of the leftmost 2k agents which is nearer to their true location than (100, 100). Hence, there is no mechanism for two facilities with even capacity in the Euclidean plane that is anonymous, and Pareto optimal and strategy proof.

If c is odd, say \( c = 2k + 1 \) with \( k \geq 1 \), we consider \( 2k + 1 \) agents at (0, 0), and another \( 2k + 1 \) agents at (100, 100). Any anonymous and Pareto optimal mechanism locates one facility at (0, 0) and the other at (100, 100). Suppose the agents at (100, 100) are fixed. Then essentially we have a single facility location problem on the remaining \( 2k + 1 \) agents. Theorem 3.1 in Peters et al. [1992] demonstrates that any mechanism in this setting that is anonymous, Pareto optimal and strategy proof is a multi-dimensional MEDIAN mechanism. Whatever orthogonal coordinate axes we take, this puts the leftmost facility at (0, 0). Suppose one agent at (0, 0) is moved along the diagonal towards the \( 2k + 1 \) agents at (100, 100). The two facilities remain immobile at (0, 0) and (100, 100). Consider now the agent arriving at (100, 100). By anonymity over the \( 2k + 2 \) agents now at (100, 100) and Pareto optimality, both facilities must be at (100, 100). This contradicts the leftmost facility remaining at (0, 0). This contradiction means that our initial assumption, that an anonymous, Pareto optimal and strategy proof mechanism exists, is false. For \( m \) facilities with \( m > 2 \), we consider an additional \( (m - 2)c \) agents at (200, 200).}

An interesting open case is four agents and two identical facilities with no space capacity. Is there an anonymous, Pareto optimal and strategy proof mechanism for this setting?

For a single agent, we locate the facility at the agent. For two agents, consider a mechanism which sets each coordinate of the facility to be the max (or the min) of the two reported coordinates of the two agents. This mechanism is anonymous and strategy proof. To show Pareto optimality, there are two cases. Either the facility is at the location of one of the agents which is clearly Pareto optimal, or it is at an empty vertex of the bounding hypercube around the two facilities. Moving along one coordinate direction towards one agent reduces the Manhattan distance to this agent, but increases the Manhattan distance to the other agent. Hence, the solution is Pareto optimal.

For three agents, consider a mechanism which sets each coordinate to be the median of the three reported coordinates of the three agents. This mechanism is anonymous and strategy proof. To show Pareto optimality, there are two cases. Either the facility is at the location of one of the agents which is clearly Pareto optimal, or the facility is in the interior of the bounding hypercube around the three facilities. Moving along one coordinate direction towards one agent reduces the Manhattan distance to this agent, but increases the Manhattan distance to the other agent. Hence, the solution is Pareto optimal.

Theorem 6 For three or fewer agents in two or more dimensions, there exist anonymous, strategy proof and Pareto optimal mechanisms for locating a single facility with Manhattan distances.

Proof: For a single agent, we locate the facility at the agent. For two agents, consider a mechanism which sets each coordinate of the facility to be the max (or the min) of the two reported coordinates of the two agents. This mechanism is anonymous and strategy proof. To show Pareto optimality, there are two cases. Either the facility is at the location of one of the agents which is clearly Pareto optimal, or it is at an empty vertex of the bounding hypercube around the two facilities. Moving along one coordinate direction towards one agent reduces the Manhattan distance to this agent, but increases the Manhattan distance to the other agent. Hence, the solution is Pareto optimal.

For three agents, consider a mechanism which sets each coordinate to be the median of the three reported coordinates of the three agents. This mechanism is anonymous and strategy proof. To show Pareto optimality, there are two cases. Either the facility is at the location of one of the agents which is clearly Pareto optimal, or the facility is in the interior of the bounding hypercube around the three facilities. Moving along one coordinate direction towards one agent reduces the Manhattan distance to this agent, but increases the Manhattan distance to the other agent. Hence, the solution is Pareto optimal.

Theorem 7 For four or more agents in two or more dimensions, there is no anonymous, strategy proof and Pareto optimal mechanism for locating a single facility with Manhattan distances.

Proof: To show the result for four or more agents in two or more dimensions, we suppose agents are limited to the x-y plane. Suppose we have three agents at (0, 2), (1, 0) and
Consider a mechanism that locates a facility at the maximum of each coordinate axis. This puts the facility at \((2, 2)\). This is Pareto dominated by the solution which locates the facility at \((1, 1)\). Similarly, the mechanism that locates a facility at the minimum of each coordinate axis puts the facility at \((0, 0)\). This is also Pareto dominated by the solution which locates the facility at \((1, 1)\). If we rotate these three agents 90 degrees, we can also demonstrate that any mechanism that puts the facility at the maximum of one coordinate axis and the minimum of the other can return Pareto dominated solutions.

We now consider a four agent problem made up of these three agents and one other. By Theorem 2 in (Barber, Gul, and Stacchetti 1993), any strategy proof mechanism in two dimensions is a coordinate-wise combination of strategy proof mechanisms. By (Moulin 1980), it must be a coordinate-wise combination of generalized median voting schemes. We can shift our three voters in the box bounded by \((0, 0)\) and \((2, 2)\) so that they are well away from any “phantom” voters. Hence, any strategy proof mechanism for our four agents picks out the \(j\)th ranked \(x\)-coordinate and the \(k\)th ranked \(y\)-coordinate for some \(j, k \in [1, 4]\). Suppose \(j = 1\) and \(k = 2\). Then place the fourth agent at \((3, 0)\). Our strategy proof mechanism then locates the facility at the minimum \(x\)-coordinate and minimum \(y\)-coordinate of the original three agents. That is, it locates the facility at \((0, 0)\). This is Pareto dominated by the mechanism that locates the facility at \((1, 1)\). Similar constructions show that any other parameters \(j\) and \(k\) also return solutions that are Pareto dominated. Hence, no mechanism for four agents in two dimensions is anonymous, strategy proof and Pareto optimal. The proof extends to five or more agents by placing additional agents at appropriate boundary positions.

A similar impossibility result for locating multiple facilities in Manhattan space quickly follows: there is no anonymous, strategy proof and Pareto optimal mechanism for locating \(m\) uncapturable facilities \((m \geq 2)\) and \(m+3\) or more agents in Manhattan space.

We next consider dropping in turn anonymity, Pareto optimality and strategy proofness. If we drop anonymity, there are multiple mechanisms for locating one or more facilities in Manhattan space that are Pareto optimal and strategy proof but not anonymous. For example, any SD mechanism is Pareto optimal and strategy proof but not anonymous. Similarly, if we drop Pareto optimality, there are multiple mechanisms for Manhattan space that are anonymous and strategy proof but not Pareto optimal (e.g. any multidimensional \(p\)-PERCENTILE mechanism with parameters \(p_i = 0\) for all \(i\)). And if we drop strategy proofness, there are multiple mechanisms that are anonymous and Pareto optimal but not strategy proof. Consider again the mechanism that orders agents by \(x\)-coordinate, tie-breaking by \(y\)-coordinate, and locates the first facility at the position of the first agent in this order, and subsequent facilities at the position of the next agent in the order at a new location. We therefore conclude that anonymity, Pareto optimality and strategy proofness are a minimal combination of incompatible axioms for locating one or more facility in two or more dimensional Manhattan space.

We end with some discussion about welfare bounds. In particular, we consider welfare bounds for multi-dimensional \(p\)-PERCENTILE mechanism when locating one or more facility in Manhattan space. Recall that such mechanisms are anonymous and strategy proof. We suppose the mechanism chooses axes that align with those used to compute Manhattan distances. We first observe that lower bounds on approximation ratios for facility location in Manhattan space can be inherited from lower bounds on approximation ratios for facility location on the line (e.g. those bounds in (Procaccia and Tennenholtz 2013)). Hence, with a single facility, no deterministic and strategy proof mechanism is better than a 2-approximation for the maximum Manhattan distance. In fact, with a single facility, the multi-dimensional \(p\)-MEDIAN mechanism is optimal with respect to both the total and the maximum Manhattan distances. No deterministic and strategy proof mechanism provides better approximation ratios.

**Theorem 8** The multi-dimensional \(p\)-MEDIAN mechanism locates a single facility to minimize the total Manhattan distance, and to 2-approximate the optimal maximum Manhattan distance.

**Proof:** With Manhattan distances, the total distance is minimized by minimizing each coordinate individually. The \(p\)-MEDIAN mechanism does this along each coordinate. For the maximum Manhattan distance, each coordinate at worst 2-approximates its contribution to the overall Manhattan distance.

When locating more facilities in Manhattan space, approximation becomes more challenging. In particular, with two or more uncapturable facilities in two or more dimensions, no multi-dimensional \(p\)-PERCENTILE mechanism has a bounded approximation ratio for the total Manhattan distance (Theorem 4 in (Sui, Boutilier, and Sandholm 2013)). It follows quickly that no multi-dimensional \(p\)-PERCENTILE mechanism has a bounded approximation ratio for the maximum Manhattan distance (using the same argument as in the proof of Theorem 4).

We contrast these results with the 1-d setting. With a single facility, approximation ratios are the same in 1-d as 2-d Manhattan space. With two facilities on the line, the ENDPOINT mechanism has bounded approximation ratios for the total and maximum distances. When we move to 2-d Manhattan space, the approximation ratios become unbounded. With three or more facilities, approximation ratios are unbounded whether we are in 1-d or 2-d Manhattan space. In general, moving from 1-d to 2-d Manhattan space, tends to increase approximation ratios for the total or maximum distance. We also contrast results for Manhattan space with those for Euclidean space. With a single facility, the 2-dimensional \(p\)-MEDIAN mechanism is optimal with respect to the total Manhattan distance but is just a primary approximation of the optimal Euclidean distance. In other settings we observe similar approximation ratios for Manhattan as Euclidean distances (e.g. with two or more facilities and multi-dimensional \(p\)-PERCENTILE mechanisms).
Other Mechanisms for Multiple Facilities

A number of specific mechanisms have been proposed with good axiomatic properties that will locate multiple facilities in two or higher dimensional space. With two facilities and any type of metric space, Lu et al. prove that the randomized PROPORTIONAL mechanism is strategy proof and 4-approximates the total distance (2010). With three or more facilities, they note that this mechanism is no longer strategy proof. With any fixed number $m$ of facilities ($m \geq 1$) and any type of metric space, Fotakis and Tzamos prove that the winner imposing version of the PROPORTIONAL mechanism is strategy proof and achieves an approximation ratio for the total distance of at most $4m$ (Fotakis and Tzamos 2010). Finally, with $n-1$ facilities and $n$ agents and any type of metric space, Escoffier et al. prove that the randomized INVERSELY PROPORTIONAL mechanism is strategy proof and an $n/2$-approximation for the total distance and an $n$-approximation for the maximum distance (Escoffier et al. 2011).

Conclusions

We have studied the impact on strategy proof mechanisms for facility location when moving from one dimension to two (or more) dimensions with Euclidean or Manhattan distances. We considered two additional axiomatic properties: anonymity which is a fundamental fairness property, and Pareto optimality which is a fundamental efficiency property. We also consider how well such mechanisms can approximate the optimal welfare. Our results are somewhat negative. Moving from one dimension to two (or more) dimensions often makes these axiomatic properties more difficult to achieve. For example, with two facilities in Euclidean space or with just a single facility in Manhattan space, no mechanism is anonymous, Pareto optimal and strategy proof. By contrast, in one dimension when locating up to two facilities on the line, there exists mechanisms that are anonymous, Pareto optimal and strategy proof. As a second example, with two facilities in Euclidean space no PERCENTILE mechanism has a bounded approximation ratio for the total or maximum distance. By contrast, when locating up to two facilities on the line, there exists PERCENTILE mechanisms that bound these approximation ratios. Indeed, such mechanisms can have optimal approximation ratios. All our impossibility results are minimal. If we drop one of the three axioms (anonymity, Pareto optimality or strategy proofness) multiple mechanisms satisfy the other two axioms. For example, when locating two facilities in Euclidean space, there exist multiple mechanisms that are anonymous and Pareto optimal, that are anonymous and strategy proof, and that are Pareto optimal and strategy proof.

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