PULLBACK ATTRACTORS FOR GENERALIZED
EVOLUTIONARY SYSTEMS

ALEXEY CHESKIDOV AND LANDON KAVLIE

Abstract. We give an abstract framework for studying nonautonomous PDEs, called a generalized evolutionary system. In this setting, we define the notion of a pullback attractor. Moreover, we show that the pullback attractor, in the weak sense, must always exist. We then study the structure of these attractors and the existence of a strong pullback attractor. We then apply our framework to both autonomous and nonautonomous evolutionary systems as they first appeared in earlier works by Cheskidov, Foias, and Lu. In this context, we compare the pullback attractor to both the global attractor (in the autonomous case) and the uniform attractor (in the nonautonomous case). Finally, we apply our results to the nonautonomous 3D Navier-Stokes equations on a periodic domain with a translationally bounded force. We show that the Leray-Hopf weak solutions form a generalized evolutionary system and must then have a weak pullback attractor.

1. Introduction

Uniform attractors have been extensively studied since their introduction by Haraux as the minimal compact set which attracts all the trajectories starting from a bounded set uniformly with respect to the initial time [17]. One approach that has been widely accepted uses the tools developed by Chepyzhov and Vishik [7], [8]. Their method involves the use of a time symbol and a family of processes. The uniform attractor, under sufficient conditions is fibered over the symbol space into “kernel sections.” These kernel sections, however, do not have classical attraction properties. The attraction is in a pullback sense, letting the initial time go to minus infinity [6]. The concept of a pullback attractor originated in the work of Crauel, Flandoli, Kloeden, and Schmalfuss [14], [20]. For more information on pullback attractors, see the books of Kloeden and Rasmussen [19], Carvalho et al [4], and Cheban [5].

The question of uniqueness for the 3D Navier-Stokes equations (NSEs) is still unresolved. Even so, many frameworks exist for studying the asymptotic dynamics of a system evolving according to the 3D NSEs, without assuming uniqueness of the solutions. For a comparison between two canonical frameworks, see Caraballo et al [3]. In their paper, they compare the framework of multivalued semiflows used by Melnik and Valero [23] to the framework of generalized semiflows developed by Ball [1]. The first approach involves a set-valued function with some inclusion properties that account for the lack of uniqueness. The approach used by Ball involves defining trajectories in the phase space, keeping in mind that there may be more than one trajectory starting from a single starting value. Ball’s construction requires certain
assumptions about trajectories including the ability to concatenate them which are still not known for the Leray-Hopf weak solutions to the 3D NSEs.

In their paper [9], Cheskidov and Foias introduced the concept of an evolutionary system allowing them to construct a framework based on known results for the Leray-Hopf weak solutions to the 3D NSEs. Moreover, they defined the concept of a global attractor in this setting. Cheskidov later added the idea of a trajectory attractor for an evolutionary system [11]. In order to deal with nonautonomous systems, Cheskidov and Lu introduced the concept of a nonautonomous evolutionary system and have applied it to the 3D NSEs and certain reaction-diffusion equations ([12], [10]). In this paper, we introduce the idea of a generalized evolutionary system, a generalization of the previous concepts where we remove the ability to “shift trajectories.” In this setting, we explore the existence of pullback attractors as well as their relationship to the 3D NSEs. We also explore the relationship between pullback attractors for generalized evolutionary systems, global attractors for autonomous evolutionary systems, and uniform attractors for nonautonomous evolutionary systems.

As in the autonomous case, several frameworks exist for studying the nonautonomous dynamical systems without uniqueness. We describe two of the those theories. The first is the theory of trajectory attractors ([20], [13], [8], [27]). This approach studies trajectories as points in the “space of trajectories” (such as $C([0, \infty); L^2)$). Then, one studies the attraction properties in this space of trajectories. One can then try to project down to the phase space by using an evaluation map and study the attraction properties of these projections. The second technique is the use of multivalued processes ([18], [29]).

Although our framework follows trajectories which exist in a “space of trajectories,” our framework differs from the framework of trajectory attractors. We use trajectories only to follow the evolution of points in the phase space. Thus, we develop attraction properties only in the phase space. Our framework also differs from the classical framework of multivalued processes in that we follow individual trajectories taking a single point to another single point in the phase space. If one were to union together all of the possible ending points from a given starting point, one could define a multivalued process from individual trajectories. Also different from either of these frameworks, we exploit, simultaneously, both the strong and weak topologies on our space. To our knowledge, this gives the first proof of the existence of a weak pullback attractor for the Leray-Hopf weak solutions of the 3D NSEs using the known properties of the Leray-Hopf weak solutions. The outline of our paper is given below.

First, in Section 2, we define a generalized evolutionary system $\mathfrak{E}$, a pullback attractor $\mathfrak{A}(t)$, and the concept of a pullback omega-limit set $\Omega(A,t)$. Here, $\bullet$ represents either $w$ or $s$ to represent the fact that we are considering our phase space $X$ with two topologies, the weak and the strong topology, respectively. We then show the existence of the weak pullback attractor $\mathfrak{A}_w(t)$ and give a characterization of the strong pullback attractor $\mathfrak{A}_s(t)$ (if it exists) in terms of pullback omega-limits. Moreover, we relate the concept of a generalized evolutionary system to the classical theory of processes as given in [3]. We follow this with a short section of worked examples in Section 3. Here, we apply our framework to both abstract and physical situations to illustrate the possible relationships between the weak and strong pullback attractors.
In Section 4, we introduce the concept of pullback asymptotic compactness. We then show that in this setting, the weak pullback attractor $A_w(t)$ is, in fact, a strongly compact strong pullback attractor $A_s(t)$. This generalizes classical results using asymptotic compactness in the autonomous setting. Next, in Section 5, we add the assumption that each family of trajectories is compact in $C([s, \infty); X_w)$. This is true of the Leray-Hopf solutions for the nonautonomous 3D NSEs under certain assumptions. We introduce the ideas of pullback invariance, pullback semi-invariance, and pullback quasi-invariance. We then show that the family of weak omega-limit sets $\Omega_w(A, t)$ is pullback quasi-invariant. This allows us to give our first characterization of the weak pullback attractor in terms of complete trajectories. That is, the weak pullback attractor is the maximal pullback invariant and maximal pullback quasi-invariant subset of the phase space. Moreover, we have a weak pullback tracking property (Theorem 5.8). If $E$ satisfies the property of being pullback asymptotically compact, then we have that the strong pullback attractor $A_s(t)$ is the maximal pullback invariant and maximal pullback quasi-invariant set in the phase space. Moreover, we have a characterization of $A_s(t)$ in terms of complete trajectories as well as a strong pullback tracking property (Theorem 5.9).

For Section 6, we add other assumptions to our generalized evolutionary system which are known for Leray-Hopf weak solutions to the nonautonomous 3D NSEs with an appropriate forcing term. Namely, an energy inequality and strong convergence a.e. for a weakly convergent sequence of trajectories. Under these assumptions, along with the compactness in $C([s, \infty); X_w)$ from Section 5 and the assumption that complete trajectories are strongly continuous (that is, $E((−\infty, \infty)) \subseteq C((−\infty, \infty); X_s)$), we can show that the generalized evolutionary system is actually pullback asymptotically compact. This is a generalization of the corresponding result in [11]. Earlier results of this kind can be found in [1] and [25].

The next section, Section 7 relates results already discovered for evolutionary systems given in [9], [11], and [10] to our new generalized framework. We begin by recalling the basic definitions for an autonomous evolutionary system (as given in [9] and [11]) and a nonautonomous evolutionary system (as given in [12] and [10]). In the autonomous case, we prove that the global attractor $A_\bullet$ for an autonomous evolutionary system exists if and only if the pullback attractor $A_s(t)$ exists. Moreover, we have that for each $t$, $A_\bullet = A_s(t)$. In the nonautonomous case, we show that the $d_\bullet$-uniform attractor (if it exists) $A_{\Sigma}^{\bullet}$ always contains the union of all the pullback Omega-limits. That is, in Theorem 7.18 we show that

\[ \bigcup_{\sigma \in \Sigma} \Omega_\sigma^\bullet(X, t_0) \subseteq A_{\Sigma}^{\bullet}, \]

where $\Omega_\sigma^\bullet(X, t_0)$ is the $d_\bullet$-pullback omega limit set at an arbitrary fixed time $t_0$ for the generalized evolutionary system $E_\sigma$ with the fixed symbol $\sigma \in \Sigma$. As before, using the existence of the weak uniform and weak pullback attractors, Corollary 7.20 gives us that

\[ \bigcup_{\sigma \in \Sigma} A_s^\bullet(X, t_0) \subseteq A_{\Sigma}^{\bullet}. \]

On the other hand, with additional assumptions on the nonautonomous evolutionary system $E_{\Sigma}$ and each generalized evolutionary system $E_\sigma$ for $\sigma \in \Sigma$ fixed, we use known characterizations of the weak uniform attractor $A_{\Sigma}^w$ and the weak pullback
attractor $\mathcal{A}_w(t)$ to say in Theorem 7.23 that

$$\mathcal{A}_w^w = \bigcup_{\sigma \in \Sigma} \mathcal{A}_w^w(0).$$

Finally, we add additional assumptions to ensure asymptotic compactness of $E_{\Sigma}$ and each $E_\sigma$. This then guarantees the existence of a strongly compact strong uniform attractor for $E_{\Sigma}$ and a strongly compact strong pullback attractor for each $E_\sigma$. In this case, as shown in Theorem 7.24,

$$\mathcal{A}_{\Sigma}^s = \mathcal{A}_w^w = \bigcup_{\sigma \in \Sigma} \mathcal{A}_\sigma^w(0) = \bigcup_{\sigma \in \Sigma} \mathcal{A}_\sigma^s(0).$$

This harkens back to the classical theory given in [8] which shows that the uniform attractor, under certain assumptions, is fibered. Moreover, the sections of the uniform attractor have attraction properties which are in a pullback sense. For a discussion on the relationship between pullback and uniform attractors using the framework of multivalued processes, we refer the reader to [2].

Finally, Section 8 closes with an application of our setup to Leray-Hopf weak solutions of the nonautonomous 3D NSEs. We use a periodic setup along with a translationally bounded force. In this setting, we prove that Leray solutions (which are continuous at the starting time) converge in a to an absorbing ball in $L^2$ which is weakly compact. Using this, we show that the Leray-Hopf weak solutions form a generalized evolutionary system. Therefore, by the previously-developed theory, there exists a weak pullback attractor. In fact, we have that

$$\mathcal{A}_w(t) = \{ u(t) : u \text{ is a complete bounded solution} \}.$$  

This generalizes the results Foias and Temam [16]. Moreover, if the force is assumed to be normal, and if we add the assumption that complete trajectories, Leray-Hopf solutions which exist for all $t \in (-\infty, \infty)$, are strongly continuous, in $C((-\infty, \infty); X_s)$, then the three properties listed above apply and the given generalized evolutionary system is pullback asymptotically compact. In this case, the weak attractor is a strongly compact, strong pullback attractor.

2. Generalized Evolutionary System

2.1. Preliminaries. We start with the setup as it first appeared in [9]. So, let $(X, d_\bullet(\cdot, \cdot))$ be a metric space with a metric $d_\bullet$ known as the strong metric on $X$. Let $d_w$ be another metric on $X$ satisfying the following conditions:

1. $X$ is $d_w$-compact.
2. If $d_w(u_n, v_n) \to 0$ as $n \to \infty$ for some $u_n, v_n \in X$ then $d_\bullet(u_n, v_n) \to 0$ as $n \to \infty$.

As justified by property (2), we will call $d_w$ the weak metric on $X$. Denote by $\overline{A}$ the closure of the set $A \subseteq X$ in the topology generated by $d_\bullet$. Note that any strongly compact set ($d_\bullet$-compact) is also weakly compact ($d_w$-compact), and any weakly closed set ($d_w$-closed) is also strongly closed ($d_\bullet$-closed).

Let $C([a, b]; X_\bullet)$, where $\bullet = s$ or $w$, be the space of $d_\bullet$- continuous $X$-valued functions on $[a, b]$ endowed with the metric

$$d_{C([a, b]; X_\bullet)}(u, v) := \sup_{t \in [a, b]} d_\bullet(u(t), v(t)).$$

Let also \( C([a, \infty); X_\bullet) \) be the space of all \( d_\bullet \)-continuous \( X \)-valued functions on \([a, \infty)\) endowed with the metric

\[
d_C([a, \infty); X_\bullet)(u, v) := \sum_{n \in \mathbb{N}} 1 \sup_{t \in [a, a+n]} \frac{\|d(u(t), v(t))\|}{2^n 1 + \sup_{t \in [a, a+n]} \|d(u(t), v(t))\|}.\]

and for each \( I \) be called a generalized evolutionary system if the following conditions are satisfied:

Let

\[
\mathcal{T} := \{ I \subset \mathbb{R} : I = [T, \infty) \text{ for some } T \in \mathbb{R} \} \cup \{-\infty, \infty\},
\]

for each \( I \in \mathcal{T} \), let \( \mathcal{F}(I) \) denote the set of all \( X \)-valued functions on \( I \).

**Definition 2.1.** A map \( \mathcal{E} \) that associates to each \( I \in \mathcal{T} \) a subset \( \mathcal{E}(I) \subset \mathcal{F}(I) \) will be called a generalized evolutionary system if the following conditions are satisfied:

1. \( \mathcal{E}(\tau, \infty) \neq \emptyset \) for each \( \tau \in \mathbb{R} \).
2. \( \{u(\cdot)\}_{I_1} : u(\cdot) \in \mathcal{E}(I_1) \} \subset \mathcal{E}(I_2) \) for each \( I_1, I_2 \in \mathcal{T} \) with \( I_2 \subset I_1 \).
3. \( \mathcal{E}((-\infty, \infty)) = \{u(\cdot) : u(\cdot) \in \mathcal{E}([T, \infty)) \forall T \in \mathbb{R}\} \).

We will refer to \( \mathcal{E}(I) \) as the set of all trajectories on the time interval \( I \). Trajectories in \( \mathcal{E}((-\infty, \infty)) \) are called complete. Next, for each \( t \geq s \in \mathbb{R} \) and \( A \subset X \), we define the map

\[
P(t, s) : \mathcal{P}(X) \to \mathcal{P}(X),
\]

\[
P(t, s)A := \{u(t) : u(s) \in A, u \in \mathcal{E}([s, \infty))\}.
\]

We get, for each \( t \geq s \geq r \in \mathbb{R} \) and \( A \subset X \)

\[
P(t, r)A \subset P(t, s)P(s, r)A.
\]

We will also study generalized evolutionary systems endowed with the following properties:

A1 \( \mathcal{E}([s, \infty)) \) is compact in \( C([s, \infty); X_\bullet) \) for each \( s \in \mathbb{R} \).

A2 (Energy Inequality) Let \( X \) be a set in some Banach space \( H \) satisfying the Radon-Riesz Property (see below) with norm \( |\cdot| \) so that \( d_\bullet(x, y) = |x - y| \)

for each \( x, y \in X \), and assume that \( d_\bullet \) induces the weak topology on \( X \). Assume that for each \( \epsilon > 0 \) and each \( s \in \mathbb{R} \) there is a \( \delta := \delta(\epsilon, s) \) so that for every \( u \in \mathcal{E}([s, \infty)) \) and \( t > s \in \mathbb{R} \)

\[
|u(t)| \leq |u(t_0)| + \epsilon
\]

for \( t_0 \) a.e. in \((t - \delta, t)\).

A3 (Strong Convergence a.e.) Let \( u, u_n \in \mathcal{E}([s, \infty)) \) be so that \( u_n \to u \) in

\( C([s, t]; X_\bullet) \) for some \( s \leq t \in \mathbb{R} \). Then, \( u_n(t_0) \xrightarrow{d_\bullet} u_n(t_0) \) for a.e. \( t_0 \in [s, t] \).

**Remark 2.2.** A Banach space \( H \) with norm \(|\cdot|\) satisfies the Radon-Riesz property if \( x_n \to x \) in norm if and only if \( x_n \to x \) weakly and

\[
\lim_{n \to \infty} |x_n| = |x|.
\]

Often, \( X \) will be a closed, bounded subset of a separable, reflexive Banach space. By the Troyanski Renorming Theorem, we can assume that our norm makes \( H \) a locally uniformly convex space, at which point the Radon-Riesz property is satisfied.

To see how this relates back to the classical setting, let \( H \) be a separable, reflexive Banach space, which we call the phase space. Let \( S(\cdot, \cdot) \) be a process on
Moreover, if we know that \( A_1 \) holds (that is, for each \( s \) for each \( t \), \( S(t, s) = S(t, r)S(r, s) \) for any \( t \geq r \geq s \). A trajectory on \( H \) is a mapping \( u: [s, \infty) \to H \) so that \( u(t) = S(t, s)u(s) \) for each \( t \geq s \). A set \( X \subseteq H \) will be called absorbing if, for each \( s \in \mathbb{R} \) and \( B \subseteq H \) bounded, there is \( t_0 := t_0(B, s) \) so that for \( t \geq t_0 \), \( S(t, s)B \subseteq X \).

If there exists a closed absorbing ball \( X \), then we call the process \( S \) dissipative.

If \( S \) is dissipative, and we can ensure that it is dissipative arbitrarily far in the past (that is, for each \( s \in \mathbb{R} \), there is a trajectory \( u: [s, \infty) \to X \), then studying the asymptotic pullback dynamics of \( S \) on \( H \) amounts to studying the asymptotic pullback dynamics of \( S \) on \( X \). That is, using the definition of the pullback attractor \( \Omega \), one can show that if \( X \) has a pullback attractor \( \mathcal{A} \) (by restricting \( S \) to \( X \)), then \( \mathcal{A} \) is a pullback attractor for \( H \). Note that since \( H \) is a separable reflexive Banach space, both the strong and weak topologies on \( X \) are metrizable.

We define a generalized evolutionary system on \( X \) by

\[
\mathcal{A}([s, \infty)) := \{ u(\cdot) : u(t) = S(t, s)u(s), u(t) \in X \ \forall t \geq s \}.
\]

In particular, this also gives us the following characterization for each \( t \gg s \in \mathbb{R} \) and \( A \subseteq X \)

\[
P(t, s)A = S(t, s)A.
\]

As we will see later, by Theorem \( 2.10 \) and Theorem \( 2.11 \) that the weak pullback attractor exists for \( \mathcal{A} \) and

\[
\mathcal{A}_w(t) = \Omega_w(X, t) = \bigcap_{s \leq r \leq s} \bigcup_{t \geq s} S(t, r)X^w.
\]

Moreover, if we know that \( A_1 \) holds (that \( \mathcal{A}([s, \infty)) \) is compact in \( C([s, \infty); X_w) \) for each \( s \in \mathbb{R} \)), then we get, using Theorem \( 5.8 \) that

\[
\mathcal{A}_w(t) = \{ u(t) : u \in \mathcal{A}((\infty, \infty)) \}.
\]

Finally, if we also have that \( A_2 \) and \( A_3 \) also hold and complete trajectories are strongly continuous \( \mathcal{A}((\infty, \infty)) \subseteq C((\infty, \infty); X) \), then by Corollary \( 6.3 \) \( \mathcal{A} \) possesses a strongly compact, strong pullback attractor \( \mathcal{A}_p(t) \). In fact, by Corollary \( 5.13 \)

\[
\mathcal{A}_p(t) = \mathcal{A}_w(t) = \{ u(t) : u \in \mathcal{A}((\infty, \infty)) \}.
\]

### 2.2. Pullback Attracting Sets, \( \Omega \)-limits, and Pullback Attractors.

Let \( \mathcal{A} \) be a fixed generalized evolutionary system on a metric space \( X \). For \( A \subseteq X \) and \( r > 0 \), denote \( B_*(A, r) := \{ x \in X : d_*(x, A) < r \} \), where

\[
d_*(x, A) := \inf_{a \in A} d_*(x, a), \quad \bullet = s, w.
\]

A family of sets \( A(t) \subseteq X, t \in \mathbb{R} \) (uniformly) pullback attracts a set \( B \subseteq X \) in the \( d_* \)-metric (\( \bullet = s, w \)) if for any \( \epsilon > 0 \), there exists an \( s_0 := s_0(B, \epsilon, t) \in \mathbb{R} \) so that for \( s \leq s_0 \),

\[
P(t, s)B \subseteq B_*(A(t), \epsilon).
\]

**Definition 2.3.** A family of sets \( A(t) \subseteq X, t \in \mathbb{R} \) are \( d_* \)-pullback attracting (\( \bullet = s, w \)) if they pullback attract \( X \) in the \( d_* \)-metric.
Definition 2.4. A family of sets $\mathcal{A}(t) \subseteq X$ is the $d_*$-pullback attractor of $X$ if for each $t$, $\mathcal{A}(t)$ is $d_*$-closed, $d_*$-pullback attracting and $\mathcal{A}(t)$ is minimal with respect to these properties.

Next, we define the concept of the pullback $\Omega_*$-limit.

Definition 2.5. For each $A \subseteq X$ and $t \in \mathbb{R}$, we define the pullback $\Omega$-limit ($\bullet = s$, $w$) of $A$ as

$$\Omega_*(A, t) := \bigcap_{s \leq t} \bigcup_{r \leq s} P(t, r)A.$$ 

Equivalently, we have that $x \in \Omega_*(A, t)$ if there exist sequences $s_n \to -\infty$, $s_n \leq t$, $x_n \in P(t, s_n)A$ so that $x_n \to x$ in the $d_*$-metric. We now present some basic properties of $\Omega_*$.

Lemma 2.6. Let $A \subseteq X$ and $t \in \mathbb{R}$. Then,

1. $\Omega_*(A, t)$ is $d_*$-closed ($\bullet = s$, $w$).
2. $\Omega_*(A, t) \subseteq \Omega_w(A, t)$.
3. If $\Omega_w(A, t)$ is strongly compact and uniformly, strongly, pullback attracts $A$, then $\Omega_*(A, t) = \Omega_w(A, t)$.

Proof. Part 1 is obvious from the definition. For part 2, let $x \in \Omega_w(A, t)$. Then, there exists sequences $s_n \leq t$, $s_n \to -\infty$ and $x_n \in P(t, s_n)A$ with $x_n \xrightarrow{d_*} x$. But then, $x_n \xrightarrow{d_w} x$ and $x \in \Omega_w(A, t)$.

Now, suppose that $\Omega_w(A, t)$ is strongly compact and $d_*$-pullback attracts $A$. Let $x \in \Omega_w(A, t)$. Then, by definition, there are sequences $s_n \leq t$, $s_n \to -\infty$ and $x_n \in P(t, s_n)A$ with $x_n \xrightarrow{d_*} x$. Since $\Omega_w(A, t)$ $d_*$-pullback attracts $A$, there exists a sequence $y_n \in \Omega_w(A, t)$ with $d_*(x_n, y_n) \to 0$ as $n \to \infty$. Because $d_*(x_n, y_n) \to 0$, $d_*(x_n, y_n) \to 0$. Since $\Omega_w(A, t)$ is compact, there is some subsequence $y_{n_k} \xrightarrow{d_*} y$ for some $y \in \Omega(A, t)$. But then, $x_{n_k} \xrightarrow{d_*} y$ which means that $x_{n_k} \xrightarrow{d_*} y$. Thus, $y = x$ which means that $x_{n_k} \xrightarrow{d_*} x$. That is, $x \in \Omega_*(A, t)$. \hfill $\Box$

Lemma 2.7. Let $A(t)$ be a family of $d_*$-closed, $d_*$-pullback attracting sets ($\bullet = s$, $w$). Then $\Omega_*(X, t) \subseteq A(t)$.

Proof. Let $x \in \Omega_*(X, t)$. Then, there exist sequences $s_n \leq t$, $s_n \to -\infty$, and $x_n \in P(t, s_n)A$ with $x_n \xrightarrow{d_*} x$. Since $A(t)$ is $d_*$-pullback attracting, there exists $a_n \in A(t)$ with $d_*(x_n, a_n) \to 0$ as $n \to \infty$. But, $x_n \xrightarrow{d_*} x$ which gives us that $a_n \xrightarrow{d_*} x$. Since $A(t)$ is $d_*$-closed, $x \in A(t)$. \hfill $\Box$

Now, we are ready to show that if the $d_*$-pullback attractor exists, then it is unique.

Theorem 2.8. If the pullback attractor $\mathcal{A}(t)$ exists ($\bullet = s$, $w$), then

$$\mathcal{A}(t) = \Omega_*(X, t).$$

Proof. By the above lemmas, $\Omega_*(X, t) \subseteq \mathcal{A}(t)$. Now, let $x \in \mathcal{A}(t) \setminus \Omega_*(X, t)$. Then, there exists $\epsilon > 0$ and $s_0 \leq t$ so that for $s \leq s_0$,

(1) $P(t, s)X \cap B_*(x, \epsilon) = \emptyset$. 

Otherwise, for each \( n \) and \( t - n \leq 0 \) there exists \( s_n \leq t - n \) with
\[
 x_n \in \mathcal{P}(t, s_n)X \cap B(x, 1/n) \neq \emptyset.
\]
But, then \( x_n \xrightarrow{d_*} x \), and \( x \in \Omega_\bullet(X, t) \). This is a contradiction. Thus, (1) holds. In this case, \( \mathcal{A}_\bullet(t) \setminus B(x, \epsilon) \) is a strict subset of \( \mathcal{A}_\bullet(t) \) which is \( d_* \)-closed \( d_* \)-pullback attracting. This contradicts the definition of \( \mathcal{A}_\bullet(t) \). \( \square \)

An immediate consequence of Theorem 2.8 and Lemma 2.7 is the following:

**Corollary 2.9.** The pullback attractor \( \mathcal{A}_\bullet(t) \) exists if and only if \( \Omega_\bullet(X, t) \) is a \( d_* \)-pullback attracting set.

Next, we study the structure of \( \Omega_w(A, t) \) for some \( A \subseteq X \) and \( t \in \mathbb{R} \).

**Theorem 2.10.** Let \( A \subseteq X \) be such that for each \( t \in \mathbb{R} \) and \( r \leq t \), there is some \( u \in \mathcal{U}([r, \infty)) \) with \( u(t) \in A \). Then, \( \Omega_w(A, t) \) is a nonempty, weakly compact set. Moreover, \( \Omega_w(A, t) \) weakly pullback attracts \( A \).

**Proof.** Due to the assumptions on \( A \), we have that \( \mathcal{P}(t, r)A \neq \emptyset \). Also, due to the fact that \( X \) is weakly compact, we have that
\[
 W(s) := \bigcup_{r \leq s} \mathcal{P}(t, r)A
\]
is nonempty and weakly compact for each \( s \leq t \). Moreover, for \( s_0 \leq s_1 \leq t \), \( W(s_0) \subseteq W(s_1) \). Thus, by Cantor’s intersection theorem,
\[
 \Omega_w(A, t) = \bigcap_{s \leq t} W(s)
\]
is a nonempty weakly compact set.

To see that \( \Omega_w(A, t) \) weakly pullback attracts \( A \), suppose for contradiction that it doesn’t. Then, there exists some \( \epsilon > 0 \) and a sequence \( s_n \to -\infty \), \( s_n \leq t \) with
\[
 \mathcal{P}(t, s_n)A \cap B_w(\Omega_w(A, t), \epsilon^c) \neq \emptyset.
\]
Therefore,
\[
 K_n := \bigcup_{r \leq s_n} \mathcal{P}(t, r)A \cap B_w(\Omega_w(A, t), \epsilon^c) \neq \emptyset.
\]
Passing to a subsequence if necessary and reindexing, we can assume that the \( s_n \)'s are monotonically decreasing. Thus, we get a decreasing sequence of nonempty weakly compact sets. Again, by Cantor’s intersection theorem, we have that \( x \in \cap_n K_n \neq \emptyset \). That is,
\[
 x \in \bigcap_{s_n \leq t} \bigcup_{r \leq s_n} \mathcal{P}(t, r)A^w = \Omega_w(A, t)
\]
This contradicts the definition of the \( K_n \)'s. \( \square \)

Using the above results, we have the following:

**Theorem 2.11.** Every generalized evolutionary system possesses a weak pullback attractor \( \mathcal{A}_w(t) \). Moreover, if the strong pullback attractor \( \mathcal{A}_s(t) \) exists, then \( \mathcal{A}_w(t) = \mathcal{A}_s(t) \).
Since \( d(t, b) \neq 0 \), we conclude that \( \mathcal{A}_s(t) \subseteq \mathcal{A}_w(t) \). Thus, we find that \( \mathcal{A}_s(t) \) is strongly pullback attracting, there is a sequence \( x_n \in \mathcal{A}_s(t) \) which contradicts the choice of \( x_n \). In addition, \( \mathcal{A}_s(t) \) is weakly closed and weakly pullback attracting. Thus, by Theorem 2.7, \( \mathcal{A}_w(t) \) is weakly closed. Therefore, by Theorems 2.9 and 2.8, \( \mathcal{A}_w(t) = \Omega_w(X, t) \). As we will see in Section 7, this turns \( \Omega_w \) into a generalized evolutionary system by letting \( \mathcal{A}_w(t) \) be the set of all \( x \in X \) such that \( x \) is strongly pullback attracting, there is a sequence \( x_n \in \mathcal{A}_s(t) \) which contradicts the choice of \( x_n \). In addition, \( \mathcal{A}_s(t) \) is weakly closed and weakly pullback attracting. Thus, by Theorem 2.7, \( \mathcal{A}_w(t) \) is weakly closed. Therefore, by Theorems 2.9 and 2.8, \( \mathcal{A}_w(t) = \Omega_w(X, t) \). As we will see in Section 7, this turns \( \Omega_w \) into a generalized evolutionary system by letting \( \mathcal{A}_w(t) \) be the set of all \( x \in X \) such that \( x \) is strongly pullback attracting, there is a sequence \( x_n \in \mathcal{A}_s(t) \) which contradicts the choice of \( x_n \). In addition, \( \mathcal{A}_s(t) \) is weakly closed and weakly pullback attracting. Thus, by Theorem 2.7, \( \mathcal{A}_w(t) \) is weakly closed. Therefore, by Theorems 2.9 and 2.8, \( \mathcal{A}_w(t) = \Omega_w(X, t) \).
for each $t \in \mathbb{R}$. In particular, the weak and strong pullback attractors are not equal.

3.3. The Heat Equation. For our next example, let $X$ be the unit ball in $L^2(\mathbb{R})$. The strong metric on $X$ is the metric induced by the norm on $L^2(\mathbb{R})$. That is, for any $f, g \in L^2(\mathbb{R})$,

$$d_s(f, g) := \|f - g\|_{L^2(\mathbb{R})} = \left(\int_{\mathbb{R}} |f(x) - g(x)|^2 \, dx\right)^{1/2}.$$

For the weak metric, we first choose any countable dense subset $\phi_n$ for $L^2(\mathbb{R})$ for $n \in \mathbb{N}$. For example, one could use wavelets as an orthonormal basis, as is explained in [21]. Then, the weak metric on $X$ is given by

$$d_w(f, g) := \sum_{k \in \mathbb{N}} \frac{1}{2^k} \frac{|(f, \phi_k) - (g, \phi_k)|}{1 + |(f, \phi_k) - (g, \phi_k)|}.$$ 

Now, consider the heat equation on $X$. That is, for some starting time $s \in \mathbb{R}$,

$$\begin{cases} u_t = u_{xx} \\ u(s) = f(x) \end{cases}$$

for some $f \in X$. Then, using the Fourier transform,

$$\hat{f}(\xi) := \int_{\mathbb{R}} f(x) e^{ix \cdot \xi} \, dx,$$

we find that a solution to (2) is given by

$$\hat{u}(\xi, t) = e^{\xi^2 (s-t)} \hat{f}(\xi).$$

Note that by Plancherel’s theorem, we may work exclusively in Fourier space. Define a generalized evolutionary system on $X$ via

$$\mathcal{E}(\{s, \infty\}) := \{u : u \text{ is a solution to (2)}\}.$$ 

We will see that the weak pullback attractor $\mathcal{A}_w(t)$ is given by the single point $\{0\}$ for each $t$. On the other hand, the strong pullback attractor $\mathcal{A}_s(t)$ does not exist.

To see this, we first note that for fixed $t \in \mathbb{R}$, $\|u(t)\|_{L^2(\mathbb{R})} \to 0$ as $s \to -\infty$. This gives us the candidate weak and strong pullback attractor $\{0\}$. In the weak metric, this is the pullback attractor. On the other hand, by Theorem 2.11, if the strong pullback attractor exists, it must be the case that $\mathcal{A}_w(t) = \mathcal{A}_s(t) = \{0\}$. So, the only possibility for the strong pullback attractor is $\mathcal{A}_s(t) = \{0\}$. However, we fail to have uniform convergence in the strong metric. By definition, if $\{0\}$ was the strong pullback attractor, then, for any $\epsilon > 0$, there is an $s_0 \leq t$ with

$$P(t, s)X \subseteq B_\epsilon(\{0\}, \epsilon)$$

for each $s \leq s_0$. So, let $\epsilon := 1/2$ and let $s_0 \leq t$ be given. Then, consider $f \in L^2(\mathbb{R})$ with $\|f\|_{L^2(\mathbb{R})} = 1$ and $\text{supp}(\hat{f}) \subseteq \{\xi : 2^j - 1 \leq |\xi| \leq 2^j + 1\}$ for some $j \in \mathbb{Z}$ to be
determined later. Then, we see that
\[ \|u(\xi, t)\|_{L^2(\mathbb{R})}^2 = \int_{\mathbb{R}} e^{2\xi^2(s_0 - t)} f(\xi)^2 d\xi \]
\[ = \int_{2^{j-1} \leq |\xi| \leq 2^{j+1}} \exp(2\xi^2(s_0 - t)) f(\xi)^2 d\xi \]
\[ \geq \exp(2 \cdot 2^{2j-2}(s_0 - t)) \int_{2^{j-1} \leq |\xi| \leq 2^{j+1}} f(\xi)^2 d\xi \]
\[ = \exp(2^{2j-1}(s_0 - t)) \|f\|_{L^2(\mathbb{R})}^2 \]
\[ = \exp(2^{2j-1}(s_0 - t)). \]
Therefore, we find that
\[ \|u(\xi, t)\|_{L^2(\mathbb{R})} \geq \exp(2^{2j-2}(s_0 - t)). \]
This is greater than or equal to \( \epsilon := 1/2 \) provided that
\[ j \leq \frac{1}{2} \left( \log_2 \left( \frac{\ln(2)}{t - s_0} \right) + 2 \right). \]
Therefore, the convergence to 0 is not uniform, and no strong pullback attractor exists.

3.4. A Phase Space that is not Weakly Compact. Finally, a simple example showing the importance of the compactness of \( X \) in the weak topology. Let \( X := \mathbb{R} \) with the weak and strong metrics both given by \( d_w(x, y) := |x - y| \) for any \( x, y \in \mathbb{R} \). For each \( s \in \mathbb{R} \), define
\( \mathcal{E}([s, \infty)) := \{ u(t) := t - s \}. \)
Then, we have that for some \( t \in \mathbb{R} \) and some \( s \leq t \)
\[ P(t, s)X = \{ u(t) : u(s) \in X, u \in \mathcal{E}([s, \infty)) \} = \{ t - s \}. \]
But, as \( s \to -\infty \), the limit does not exist. Thus, the weak and strong pullback attractors do not exist.

4. Existence of a Strong Pullback Attractor

**Definition 4.1.** A generalized evolutionary system is pullback asymptotically compact if for any \( t \in \mathbb{R} \), \( s_n \to -\infty \) with \( s_n \leq t \), and any \( x_n \in P(t, s_n)X \), the sequence \( \{x_n\} \) is relatively strongly compact.

**Theorem 4.2.** Let \( \mathcal{E} \) be pullback asymptotically compact. Let \( A \subseteq X \) be so that for each \( t \in \mathbb{R} \) and \( r \leq t \), there is some \( u \in \mathcal{E}([r, \infty)) \) with \( u(t) \in A \). Then, \( \Omega_w(A, t) \) is a nonempty strongly compact set which strongly pullback attracts \( A \). Moreover, \( \Omega_w(A, t) = \Omega_w(A, t) \).

**Proof.** By Theorem 2.8, our assumptions on \( A \) imply that \( \Omega_w(A, t) \neq \emptyset \). We will see that \( \Omega_w(A, t) \) strongly pullback attracts \( A \). Suppose it does not. Then, there is some \( \epsilon > 0 \) and a sequence \( s_n \to -\infty \) with
\[ x_n \in P(t, s_n)A \cap B_u(\Omega_w(A, t), \epsilon) \neq \emptyset. \]
Since \( X \) is pullback asymptotically compact, this sequence has a convergent subsequence. After passing to a subsequence and dropping a subindex, we have that \( x_n \xrightarrow{d_w} x \). Therefore, by the equivalent definition of \( \Omega_w(A, t) \),
Therefore, if \( B \) complete trajectory \( u(4) \) We say that \( B \) We say that \( s \) Moreover, we introduce the following variation of the mapping \( P \) set with \( \Omega \) by Theorem 2.8 and Corollary 2.9, the strong pullback attractor \( A \). Proof. By Theorem 4.2, \( \Omega \) gives us that \( x \) is a subsequence \( \{ x_{n_k} \} \) with \( x_{n_k} \xrightarrow{d} y \) for some \( y \in X \). But then, \( x_{n_k} \xrightarrow{d} y \) which gives us that \( x = y \) and thus, \( x_n \xrightarrow{d} x \). That is, \( x \in \Omega_{s}(A, t) \) and \( \Omega_{s}(A, t) = \Omega_{w}(A, t) \).

Finally, we establish the strong compactness of \( \Omega_{s}(A, t) \). So, let \( \{ x_n \} \) be any sequence in \( \Omega_{s}(A, t) \). By the equivalent definition of \( \Omega_{s}(A, t) \), there is a corresponding sequence \( \{ s_k^n \} \) for each \( x_n \) with \( s_k^n \to -\infty \), \( s_k^n \leq t \), and \( x_k^n \in P(t, s_k^n)A \) so that \( x_k^n \xrightarrow{d} x_n \). Letting \( y_n, x_n \) be the diagonals of these families, we have that \( d_s(y_n, x_n) \to 0 \) as \( n \to \infty \).

By pullback asymptotic compactness, \( \{ y_n \} \) is relatively strongly compact. Hence, \( \{ x_n \} \) is also relatively strongly compact. Since \( \Omega_{s}(A, t) \) is closed, the limit of this subsequence lies in \( \Omega_{s}(A, t) \) giving us that \( \Omega_{s}(A, t) \) is compact.

Using this result, we have the following existence result for strong pullback attractors.

**Theorem 4.3.** If a generalized evolutionary system \( \mathcal{E} \) is pullback asymptotically compact, then \( \mathcal{A}_{w}(t) \) is a strongly compact strong pullback attractor.

**Proof.** By Theorem 4.2 \( \Omega_{s}(X, t) \) is strongly compact strong pullback attracting set with \( \Omega_{s}(X, t) = \Omega_{w}(X, t) = \mathcal{A}_{w}(t) \), the weak pullback attractor. Therefore, by Theorem 2.8 and Corollary 2.9 the strong pullback attractor \( \mathcal{A}_{s}(t) \) exists and \( \mathcal{A}_{s}(t) = \Omega_{s}(X, t) = \mathcal{A}_{w}(t) \).

5. **Invariance and Tracking Properties**

Now, we assume that \( \mathcal{E} \) satisfies \( \text{A1}\) That is,

\( \text{A1}\) \( \mathcal{E}([s, \infty)) \) is compact in \( C([s, \infty); X_w) \) for each \( s \in \mathbb{R} \).

Moreover, we introduce the following variation of the mapping \( P \): for \( A \subseteq X \) and \( s \leq t \in \mathbb{R} \)

\[ \tilde{P}(t, s)A := \{ u(t) : u(s) \in A, u \in \mathcal{E}((-\infty, \infty)) \}. \]

**Definition 5.1.** We say that a family of sets \( \mathcal{B}(t) \subseteq X \) is pullback semi-invariant if for each \( s \leq t \in \mathbb{R} \),

\[ \tilde{P}(t, s)\mathcal{B}(s) \subseteq \mathcal{B}(t). \]

We say that \( \mathcal{B}(t) \) is pullback invariant if for \( s \leq t \in \mathbb{R} \),

\[ \tilde{P}(t, s)\mathcal{B}(s) = \mathcal{B}(t). \]

We say that \( \mathcal{B}(t) \) is pullback quasi-invariant if for each \( b \in \mathcal{B}(t) \), there is some complete trajectory \( u \in \mathcal{E}((-\infty, \infty)) \) with \( u(t) = b \) and \( u(s) \in \mathcal{B}(s) \) for each \( s \leq t \).

Note that if \( \mathcal{B}(t) \) is pullback quasi-invariant, then for each \( s \leq t \),

\[ \mathcal{B}(t) \subseteq \tilde{P}(t, s)\mathcal{B}(s) \subseteq P(t, s)\mathcal{B}(s). \]

Therefore, if \( \mathcal{B}(t) \) is pullback quasi-invariant and pullback semi-invariant, then \( \mathcal{B}(t) \) is pullback invariant.
Theorem 5.2. Let $\mathcal{E}$ be a generalized evolutionary system satisfying $[A1]$ Then, $\Omega_w(A, t)$ is pullback quasi-invariant for each $A \subseteq X$.

Proof. Let $x \in \Omega_w(A, t)$. Then, there are sequences $s_n \to -\infty$ with $s_n \leq t$ and $x_n \in P(t, s_n)A$ so that $x_n \xrightarrow{w} x$. Note that by passing to a subsequence, we can assume without loss of generality that $s_n$ is a monotonically decreasing sequence. Since $x_n \in P(t, s_n)A$, there is some $u_n \in \mathcal{E}([s_n, \infty))$ with $x_n = u_n(t)$, $u_n(s_n) \in A$. Using $[A1]$ $\mathcal{E}([s_n, \infty))$ is compact in $C([s_n, \infty); X_w)$. Moreover, by the definition of $\mathcal{E}$,

$$\{u|_{[s_1, \infty)} : u \in \mathcal{E}([s_n, \infty))\} \subseteq \mathcal{E}([s_1, \infty)).$$

Thus, using compactness on $\{u_n|_{[s_1, \infty)}\}$, we can pass to a subindex and drop a subsequence obtaining $u^1 \in \mathcal{E}([s_n, \infty))$ so that

$$u_n|_{[s_1, \infty)} \to u^1 \text{ in } C([s_1, \infty); X_w).$$

Repeating the argument above with our subsequence, we can find another subsequence, which, after dropping another subindex, gives us some $u^2 \in \mathcal{E}([s_2, \infty))$ with

$$u_n|_{[s_2, \infty)} \to u^2 \text{ in } C([s_2, \infty); X_w).$$

Note that, by construction, $u^2|_{[s_1, \infty)} = u^1$. Continuing, inductively, we get $u^k \in \mathcal{E}([s_k, \infty))$ with

$$u_n|_{[s_k, \infty)} \to u^k \text{ in } C([s_k, \infty); X_w).$$

and $u^k|_{[s_{k-1}, \infty)} = u^{k-1}$. A standard diagonalization process gives us some subsequence of $u_n$ and $u \in \mathcal{E}((-\infty, \infty))$ so that $u|_{[-T, \infty)} \in \mathcal{E}([-T, \infty))$ and $u_n \to u$ in $C([-T, \infty); X_w)$ for any $T > 0$. That is, $u \in \mathcal{E}((-\infty, \infty))$, by definition.

Note that $u(t) = x$, by construction. Now, let $s \leq t$. Then, $u_n(s) \xrightarrow{w} u(s)$. By definition, since $u_n(s_n) \in A$, we then get that $u_n(s_n) \in P(s, s_n)A$ for $s_n \leq s$ (n sufficiently large). Hence, $u(s) \in \Omega_w(A, s)$ and $\Omega_w(A, t)$ is pullback quasi-invariant.

This characterization of $\Omega_w(A, t)$ gives us the following important consequences.

Corollary 5.3. Let $\mathcal{E}$ be a generalized evolutionary system satisfying $[A1]$ Let $A \subseteq X$ be such that $\Omega_w(A, t) \subseteq A$ for each $t \in \mathbb{R}$. Then, $\Omega_w(A, t) = \Omega_w(A, t)$.

Proof. By Theorem 5.2 we have that $\Omega_w(A, t)$ is pullback quasi-invariant. Thus, by $[H]$, we have that $\Omega_w(A, t) \subseteq P(t, s)\Omega_w(A, s)$ for each $s \leq t$. By assumption, $\Omega_w(A, t) \subseteq A$, thus $\Omega_w(A, t) \subseteq P(t, s)A$ for each $s \leq t$. Therefore, $\Omega_w(A, t) \subseteq \Omega_w(A, t)$. On the other hand, by Lemma 2.6 $\Omega_w(A, t) \subseteq \Omega_w(A, t)$. That is, $\Omega_w(A, t) = \Omega_w(A, t)$.

The following result is a direct result of Corollary 5.3 and Theorem 2.8.

Corollary 5.4. Let $\mathcal{E}$ be a generalized evolutionary system satisfying $[A1]$ Then, if the strong pullback attractor $\mathcal{A}_w(t)$ exists, $\mathcal{A}_w(t) = \mathcal{A}_w(t)$, the weak pullback attractor.

In fact, we get a new characterization of pullback invariance for a weakly closed set $A$.

Theorem 5.5. Let $\mathcal{E}$ be a generalized evolutionary system satisfying $[A1]$ Then, for a family of weakly closed subsets $\mathcal{B}(t) \subseteq X$, $\mathcal{B}(t)$ is pullback invariant if and only if $\mathcal{B}(t)$ is pullback semi-invariant and pullback quasi-invariant.
Proof. If \( \mathcal{B}(t) \) is pullback semi-invariant and pullback quasi-invariant, then by the definition of pullback semi-invariant and pullback quasi-invariant, \( \mathcal{B}(t) \) is pullback invariant.

For the other direction, assume \( \mathcal{B}(t) \) is pullback invariant. Then, we have that \( \mathcal{B}(t) \) is clearly pullback semi-invariant. To see pullback quasi-invariance, let \( b \in \mathcal{B}(t) \). Then, by pullback invariance, we can construct a monotonically decreasing sequence \( s_n \to -\infty \) and find \( u_n \in \mathcal{E}([s_n, \infty)) \) with \( u_n(s_n) \in \mathcal{B}(s_n) \) and \( u_n(t) = b \).

As in Theorem 5.2, we can find a subsequence which, after dropping a subindex, is so that \( u_n \to u \) for some \( u \in \mathcal{E}((0, \infty)) \) in the sense of \( C([-T, \infty); X_\omega) \) for each \( T > 0 \). Moreover, \( u(t) = b \) and \( u(s) \in \mathcal{B}(s) \) for each \( s \leq t \) since each \( \mathcal{B}(s) \) is weakly closed. Therefore, \( \mathcal{B}(t) \) is pullback quasi-invariant. \( \square \)

Let \( \mathcal{I}(t) \) be a family of subsets of \( X \) given by

\[ \mathcal{I}(t) := \{ u(t) : u \in \mathcal{E}((0, \infty)) \} \]

Then, \( \mathcal{I}(t) \) is both pullback semi-invariant and pullback quasi-invariant. Moreover, \( \mathcal{I}(t) \) contains every pullback quasi-invariant and every pullback invariant set. Thus, by Theorem 5.2,

\[ \Omega_w(A, t) \subseteq \mathcal{I}(t) \]

for each \( t \in \mathbb{R} \) and each \( A \subseteq X \).

Now, we will show that \( \Omega_w(A, t) \) contains all the asymptotic behavior (as the initial time goes to \( -\infty \)) of every trajectory starting in \( A \), provided \( \mathcal{A} \) holds.

**Theorem 5.6** (Weak pullback tracking property). Let \( \mathcal{E} \) be a generalized evolutionary system satisfying \( \mathcal{A} \) and let \( A \subseteq X \). Then, for each \( \epsilon > 0 \) and each \( t \in \mathbb{R} \), there is some \( s_0 := s_0(\epsilon, t) \leq t \) so that for \( s' < s_0 \) and \( u \in \mathcal{E}([s', \infty)) \) with \( u(s') \in A \) satisfies

\[ d_{C([s', \infty); X_\omega]}(u, v) < \epsilon \]

for some \( v \in \mathcal{E}((0, \infty)) \) with \( v(s) \in \Omega_w(A, s) \) for each \( s \leq t \).

**Proof.** For contradiction, suppose not. Then, there exists \( \epsilon > 0 \) and sequences \( s_n \leq t \) with \( s_n \to -\infty \), \( u_n \in \mathcal{E}([s_n, \infty)) \) with the property that \( u_n(s_n) \in A \) and

\[ d_{C([s_n, \infty); X_\omega]}(u_n, v) \geq \epsilon \]

for each \( n \) and each \( v \in \mathcal{E}((0, \infty)) \) with \( v(s) \in \Omega_w(A, s) \) for \( s \leq t \). As in the proof of Theorem 5.2, we find a \( u \in \mathcal{E}((0, \infty)) \) and a subsequence which after dropping the subindex can be written as \( u_n \) with \( u_n \to u \) in \( C([-T, \infty); X_\omega) \) for each \( T > 0 \). In particular, \( u(s) \in \Omega_w(A, s) \) for each \( s \leq t \). In particular, for large enough \( n \), \( d_{C([s_n, \infty); X_\omega]}(u_n, u) < \epsilon \) which contradicts (3). \( \square \)

**Theorem 5.7** (Strong pullback tracking property). Let \( \mathcal{E} \) be a pullback asymptotically compact generalized evolutionary system satisfying \( \mathcal{A} \) and let \( A \subseteq X \). Then, for each \( \epsilon > 0 \), \( t \in \mathbb{R} \), and \( T > 0 \), there is some \( s_0 := s_0(\epsilon, t, T) \leq t \) so that for \( s' < s_0 \) and each \( u \in \mathcal{E}([s', \infty)) \) with \( u(s') \in A \), we have

\[ d_u(u(\hat{s}), v(\hat{s})) < \epsilon \]

for each \( \hat{s} \in [s', s'+T] \) and some \( v \in \mathcal{E}((0, \infty)) \) so that \( v(s) \in \Omega_w(A, s) \) for each \( s \leq t \).

**Proof.** Again, suppose not. Then, there is some \( \epsilon > 0 \), \( T > 0 \), and sequences \( s_n \leq t \) with \( s_n \to -\infty \), \( u_n \in \mathcal{E}([s_n, \infty)) \) so that \( u_n(s_n) \in A \) and

\[ \sup_{\hat{s} \in [s_n, s_n + T]} d_u(u_n(\hat{s}), v(\hat{s})) \geq \epsilon \]
for each \( n \) and each \( v \in \mathcal{E}((-\infty, \infty)) \) with \( v(s) \in \Omega_n(A, s) \) for each \( s \leq t \).

By Theorem 5.6 there exists a sequence \( v_n \in \mathcal{E}((-\infty, \infty)) \) with \( v_n(s) \in \Omega_n(A, s) \) for \( s \leq t \) so that

\[
\lim_{n \to \infty} \sup_{s \in [s_n, s_n + T]} d_w(u_n(s), v_n(s)) = 0.
\]

Using the pullback asymptotic compactness of \( \mathcal{E} \) we get that \( \Omega_n(A, t') = \Omega_{n'}(A, t') \) for all \( t' \in \mathbb{R} \). Then, by (6), there is a sequence \( s_n \in [s_n, s_n + T] \) so that

\[
d_w(u_n(s_n), v_n(s_n)) \geq \epsilon/2.
\]

Again, using the pullback asymptotic compactness of \( \mathcal{E} \), the sequences \( \{u_n(s_n)\} \) and \( \{v_n(s_n)\} \) have convergent subsequences. So, passing to a subsequence and dropping a subindex, we have that \( u_n(s_n) \xrightarrow{d_w} x \), \( v_n(s_n) \xrightarrow{d_w} y \) for some \( x, y \in X \).

Using the pullback asymptotic compactness of \( \mathcal{E} \), the sequences \( \{u_n(s_n')\} \) and \( \{v_n(s_n')\} \) have convergent subsequences. So, passing to a subsequence and dropping a subindex, we have that \( u_n(s_n') \xrightarrow{d_w} x \), \( v_n(s_n') \xrightarrow{d_w} y \) for some \( x, y \in X \).

By (7), \( x = y \) contracting (8).

Next, we use the above tracking properties to show that the weak pullback attractor \( \mathcal{A}_w(t) = \mathcal{I}(t) \) if the generalized evolutionary system \( \mathcal{E} \) satisfies \( \mathcal{A}_1 \).

**Theorem 5.8.** Let \( \mathcal{E} \) be a generalized evolutionary system satisfying \( \mathcal{A}_1 \). Then, the weak pullback attractor \( \mathcal{A}_w(t) = \mathcal{I}(t) \), and \( \mathcal{A}_w(t) \) is the maximal pullback quasi-invariant and maximal pullback invariant subset of \( X \). Moreover, for each \( \epsilon > 0 \) and \( t \in \mathbb{R} \) there is some \( s_0 := s_0(\epsilon, t) \leq t \) so that for \( s' < s_0 \) and every trajectory \( u \in \mathcal{E}([s', \infty)) \) has

\[
d_{C([s', \infty); X_w]}(u, v) < \epsilon
\]

for some complete trajectory \( v \in \mathcal{E}((-\infty, \infty)) \).

**Proof.** Since \( \mathcal{A}_w(t) = \Omega_{w}(X, t) \) and \( \Omega_{w}(X, t) \) is pullback quasi-invariant by Theorem 5.2 we have by \( \mathcal{A}_1 \) that \( \mathcal{A}_w(t) \subseteq \mathcal{I}(t) \). For the other inclusion, let \( u(t) \in \mathcal{I}(t) \). Suppose \( u(t) \notin \mathcal{A}_w(t) \). Then, since \( \mathcal{A}_w(t) \) is weakly closed, there is some \( \epsilon > 0 \) and \( B_w(u(t), \epsilon) \) so that \( \mathcal{A}_w(t) \cap B_w(u(t), \epsilon) = \emptyset \). By the weak pullback tracking property on \( \mathcal{A}_w(t) = \Omega_{w}(X, t) \), there is some \( s' \) so that for any \( \hat{u} \in \mathcal{E}([s', \infty)) \),

\[
d_{C([s', \infty); X_w]}(\hat{u}, v) < \epsilon
\]

for some \( v \in \mathcal{E}((-\infty, \infty)) \) with \( v(s) \in \Omega_{w}(X, s) \) for each \( s \leq t \). In particular, \( u \in \mathcal{E}([s', \infty)) \) for each \( s' < t \). Thus, there is some \( v \in \mathcal{E}((-\infty, \infty)) \) so that \( v(t) \in \Omega_{w}(X, t) \) and

\[
d_w(u(t), v(t)) \leq d_{C([s', \infty); X_w]}(u, v) < \epsilon.
\]

This contradicts that \( \mathcal{A}_w(t) \cap B_w(u(t), \epsilon) = \emptyset \). The rest of the theorem follows from Theorem 5.6.

Putting together this result as well as Theorem 4.3 and Theorem 5.7 we have the following corollary.

**Corollary 5.9.** Let \( \mathcal{E} \) be a pullback asymptotically compact generalized evolutionary system satisfying \( \mathcal{A}_1 \). Then, the strong pullback attractor \( A_s(t) = \mathcal{I}(t) \) and \( A_s(t) \) is the maximal pullback invariant and maximal pullback quasi-invariant set. Moreover, for each \( \epsilon > 0 \), \( t \in \mathbb{R} \), and \( T > 0 \), there is some \( s_0 := s_0(\epsilon, t, T) \leq t \) so that for \( s' < s_0 \), every trajectory \( u \in \mathcal{E}([s', \infty)) \) satisfies

\[
d_w(u(s), v(s)) < \epsilon
\]

for each \( s \in [s', s' + T] \) and some complete trajectory \( v \in \mathcal{E}((-\infty, \infty)). \)
6. Energy Inequality

In this section, we assume that our generalized evolutionary system satisfies properties \( A_2 \) and \( A_3 \). That is, for \( A_2 \) we let \( X \) be a set in some Banach space \( H \) satisfying the Radon-Riesz Property with norm \( |·| \) so that \( d_w(x, y) = |x - y| \) for each \( x, y \in X \), and assume that \( d_w \) induces the weak topology on \( X \). Assume that for each \( \epsilon > 0 \) and each \( s \in \mathbb{R} \) there is a \( \delta := \delta(\epsilon, s) \) so that for every \( u \in \mathcal{E}([s, \infty)) \) and \( t > s \in \mathbb{R} \) that

\[ |u(t)| \leq |u(t_0)| + \epsilon \]

for \( t_0 \) a.e. in \((t - \delta, t)\). For \( A_3 \) we assume that if \( u, u_n \in \mathcal{E}([s, \infty)) \) with \( u_n \to u \) in \( C([s, t]; X_w) \) for some \( s \leq t \in \mathbb{R} \), then \( u_n(t_0) \xrightarrow{d_w} u(t_0) \) for a.e. \( t_0 \in [s, t] \).

**Theorem 6.1.** Let \( \mathcal{E} \) be a generalized evolutionary system satisfying \( A_2 \) and \( A_3 \). Let \( u_n \in \mathcal{E}([s, \infty)) \) be so that \( u_n \to u \) in \( C([s, t]; X_w) \) for some \( u \in \mathcal{E}([s, \infty)) \). If \( u(t) \) is strongly continuous at some \( t^* \in (s, t) \), then \( u_n(t^*) \xrightarrow{d_w} u(t^*) \).

**Proof.** By \( A_3 \) there is a set \( E \subset [s, t] \) of measure zero so that \( u_n(t_0) \xrightarrow{d_w} u(t_0) \) on \([s, t]\setminus E \). Let \( \epsilon > 0 \). By the energy inequality \( A_2 \) and the strong continuity of \( u(t) \), there is some \( t_0 \in [s, t] \setminus E \) so that

\[ |u_n(t^*)| \leq |u_n(t_0)| + \epsilon/2, \quad |u(t)| \leq |u(t^*)| + \epsilon/2, \]

for each \( n \). Taking the upper limit, we then have that

\[ \limsup_{n \to \infty} |u_n(t^*)| \leq \limsup_{n \to \infty} |u_n(t_0)| + \epsilon/2 = |u(t_0)| + \epsilon/2 \leq |u(t^*)| + \epsilon. \]

Letting \( \epsilon \to 0 \), we have that

\[ \limsup_{n \to \infty} |u_n(t^*)| \leq |u(t^*)|. \]

Since \( u_n(t^*) \xrightarrow{d_w} u(t^*) \), by assumption, we know that \( \liminf_{n \to \infty} |u_n(t^*)| \geq |u(t^*)| \). Thus, \( \lim_{n \to \infty} |u_n(t^*)| = |u(t^*)| \), and, using the Radon-Riesz property, we have that \( u_n(t^*) \xrightarrow{d_w} u(t^*) \).

**Theorem 6.2.** Let \( \mathcal{E} \) be a generalized evolutionary system satisfying \( A_1 \), \( A_2 \), and \( A_3 \). If \( \mathcal{E}((-\infty, \infty)) \subset C((-\infty, \infty); X_s) \), then \( \mathcal{E} \) is pullback asymptotically compact.

**Proof.** Let \( s_n \to -\infty \), \( s_n \leq t \) for some \( t \in \mathbb{R} \) and \( x_n \in P(t, s_n)X \). Since \( X \) is weakly compact, we can pass to a subsequence and drop a subindex to assume that \( x_n \xrightarrow{w} x \) for some \( x \in X \).

Next, since \( x_n \in P(t, s_n)X \), there is some \( u_n \in \mathcal{E}([s_n, \infty)) \) with \( u_n(t) = x_n \) for each \( n \). Using \( A_1 \) and the usual diagonalization process, we can pass to a subsequence and drop a subindex to find that \( u_n \to u \) in \( C((-\infty, \infty); X_w) \) for some \( u \in \mathcal{E}((-\infty, \infty)) \subset C((-\infty, \infty); X_s) \). Since \( u \) is strongly continuous at \( t \), Theorem \( 6.1 \) implies that \( u_n(t) = x_n \xrightarrow{d_w} x = u(t) \). Therefore, \( \mathcal{E} \) is pullback asymptotically compact.

Together with Theorem \( 4.3 \), we have the following:

**Corollary 6.3.** Let \( \mathcal{E} \) be a generalized evolutionary system satisfying \( A_1 \), \( A_2 \), and \( A_3 \). If every complete trajectory is strongly continuous, then \( \mathcal{E} \) possesses a strongly compact, strong pullback attractor \( \mathcal{A}_\mathcal{E}(t) \).
In fact, following the proofs of Theorem 6.2 and Theorem 6.4 we have the following generalization.

**Theorem 6.4.** Let \( \mathcal{E} \) be a generalized evolutionary system satisfying \( A1, A2 \) and \( A3 \). Let \( A \subseteq X \) be such that for each \( s \in \mathbb{R} \) there exists some \( u \in \mathcal{E}([s, \infty)) \) with \( u(t) \in A \). Assume that \( u \) is strongly continuous at \( t \) for each \( u \in \mathcal{E}((\omega, \infty)) \) with \( u(t) \in \Omega_\omega(A, t) \). Then, \( \Omega_\omega(A, t) \) is a nonempty, strongly compact set that strongly pullback attracts \( A \). Moreover, \( \Omega_\omega(A, t) = \Omega_w(A, t) \).

7. **Pullback Attractors for Evolutionary Systems**

7.1. **Autonomous Case.** We will begin with the definitions and major results for autonomous evolutionary systems as given in \[11]. Note that \( X \) has the same structure as it had in Section 2. That is, \( X \) is endowed with two metrics \( d_s \) known as the strong metric and \( d_w \) known as the weak metric so that \( X \) is \( d_w \)-compact and every \( d_w \)-convergent sequence is also \( d_w \)-convergent.

**Definition 7.1.** \[11\] A map \( \mathcal{E} \) that associates to each \( I \in \mathcal{F} \) a subset \( \mathcal{E}(I) \subseteq \mathcal{F}(I) \) will be called an evolutionary system if the following conditions are satisfied:

1. \( \mathcal{E}((0, \infty)) \neq \emptyset \).
2. \( \mathcal{E}(I + s) = \{u(\cdot) : u(\cdot + s) \in \mathcal{E}(I)\} \) for all \( s \in \mathbb{R} \).
3. \( \{u(\cdot)|_{I_2} : u(\cdot) \in \mathcal{E}(I_2)\} \subseteq \mathcal{E}(I_2) \) for all pairs \( I_1, I_2 \in \mathcal{F} \), so that \( I_2 \subseteq I_1 \).
4. \( \mathcal{E}((\omega, \infty)) = \{u(\cdot) : u(\cdot)|_{[T, \infty)} \in \mathcal{E}([T, \infty)) \forall T \in \mathbb{R} \} \).

As with a generalized evolutionary system, \( \mathcal{E}(I) \) is referred to as the set of all trajectories on the time interval \( I \). Trajectories in \( \mathcal{E}((\omega, \infty)) \) are known as complete. Let \( \mathcal{P}(X) \) be the set of all subsets of \( X \). For each \( t \geq 0 \), the map

\[
R(t) : \mathcal{P}(X) \to \mathcal{P}(X)
\]

is defined by

\[
R(t)A := \{u(t) : u(0) \in A, u \in \mathcal{E}([0, \infty))\}
\]

for \( A \subseteq X \).

By the definitions of \( \mathcal{E} \) and \( R(t) \), \( R \) has the following property for each \( A \subseteq X \), \( t, s \geq 0 \),

\[
R(t + s)A \subseteq R(t)R(s)A.
\]

**Definition 7.2.** \[11\] A set \( \mathcal{A}_* \subseteq X \) is a \( d_* \)-global attractor \((* = s \text{ or } w)\) if \( \mathcal{A}_* \) is a minimal set which is

1. \( d_* \)-closed.
2. \( d_* \)-attracting: for any \( B \subseteq X \) and any \( \epsilon > 0 \), there is a \( t_0 := t_0(B, \epsilon) \) so that

\[
R(t)B \subseteq B_*(\mathcal{A}_*, \epsilon) := \{u : \inf_{x \in \mathcal{A}_*} d_*(u, x) < \epsilon\}
\]

for all \( t \geq t_0 \).

**Definition 7.3.** \[11\] The \( \omega_* \)-limit \((* = s \text{ or } w)\) of a set \( A \subseteq X \) is

\[
\omega_*(A) := \bigcap_{T \geq 0} \bigcup_{t \geq T} R(t)A.
\]

Equivalently, \( x \in \omega_*(A) \) if there exist sequences \( t_n \to \infty \) as \( n \to \infty \) and \( x_n \in R(t_n)A \), such that \( x_n \to x \) as \( n \to \infty \).

To extend the notion of invariance from a semiflow to an evolutionary system, the following mapping is used for \( A \subseteq X \) and \( t \in \mathbb{R} \):

\[
R(t)A := \{u(t) : u(0) \in A, u \in \mathcal{E}((\omega, \infty))\}.
\]
Definition 7.4. \( [11] \) A set \( A \subseteq X \) is positively invariant if for each \( t \geq 0 \),
\[
R(t)A \subseteq A.
\]
We say that \( A \) is invariant if for each \( t \geq 0 \),
\[
\hat{R}(t)A = A.
\]
A is quasi-invariant if for every \( a \in A \), there exists a complete trajectory \( u \in \mathcal{E}((\infty, \infty)) \) with \( u(0) = a \) and \( u(t) \in A \) for all \( t \in \mathbb{R} \).

Definition 7.5. \( [11] \) The evolutionary system \( \mathcal{E} \) is asymptotically compact if for any \( t_k \to \infty \) and any \( x_k \in R(t_k)X \), the sequence \( \{x_k\} \) is relatively strongly compact.

Here are other assumptions that are imposed on \( \mathcal{E} \).

B1. \( \mathcal{E}([0, \infty)) \) is a compact set in \( C([0, \infty); X_w) \).

B2. Assume that \( X \) is a set in some Banach space \( H \) satisfying the Radon-Riesz property with the norm denoted by \( |\cdot| \), so that \( d_w(x, y) = |x - y| \) for \( x, y \in X \) and \( d_w \) induces the weak topology on \( X \). Assume also that for any \( \epsilon > 0 \), there exists \( \delta := \delta(\epsilon) \), such that for every \( u \in \mathcal{E}([0, \infty)) \) and \( t > 0 \),
\[
|u(t)| \leq |u(t_0)| + \epsilon,
\]
for \( t_0 \) a.e. in \( (t - \delta, t) \).

B3. Let \( u, u_n \in \mathcal{E}([0, \infty)) \), be so that \( u_n \to u \) in \( C([0, T]; X_w) \) for some \( T > 0 \). Then, \( u_n(t) \to u(t) \) strongly a.e. in \( [0, T] \).

Theorem 7.6. \( [11] \) Let \( \mathcal{E} \) be an evolutionary system. Then,
1. If the \( \mathcal{E} \)-global attractor \( \mathcal{A}_\mathcal{E} \) exists, then \( \mathcal{A}_\mathcal{E} = \omega(X) \).
2. The weak global attractor \( \mathcal{A}_w \) exists.

Furthermore, if \( \mathcal{E} \) satisfies \( [21] \) then
3. \( \mathcal{A}_w = \omega_w(X) = \omega(X) = \{u_0: u_0 = u(0) \text{ for some } u \in \mathcal{E}((\infty, \infty))\} \).
4. \( \mathcal{A}_w \) is the maximal invariant and maximal quasi-invariant set.
5. (Weak uniform tracking property) For any \( \epsilon > 0 \), there exists \( t_0 := t_0(\epsilon) \), so that for any \( t > t_0 \), every trajectory \( u \in \mathcal{E}([0, \infty)) \) satisfies \( d_w([t, t_0); X_w](u, v) < \epsilon \), for some complete trajectory \( v \in \mathcal{E}((-\infty, \infty)) \).

Theorem 7.7. \( [11] \) Let \( \mathcal{E} \) be an asymptotically compact evolutionary system. Then,
1. The strong global attractor \( \mathcal{A}_\mathcal{E} \) exists, it is strongly compact, and \( \mathcal{A}_\mathcal{E} = \mathcal{A}_w \).

Furthermore, if \( \mathcal{E} \) satisfies \( [21] \) then
2. (Strong uniform tracking property) for any \( \epsilon > 0 \) and \( T > 0 \), there exists \( t_0 := t_0(\mathcal{E}, T) \), so that for any \( t^* > t_0 \), every trajectory \( u \in \mathcal{E}([0, \infty)) \) satisfies \( d_w(u(t), v(\tau)) < \epsilon \), for all \( \tau \in [t, t^* + T] \), for some complete trajectory \( v \in \mathcal{E}((-\infty, \infty)) \).

Theorem 7.8. \( [11] \) Let \( \mathcal{E} \) be an evolutionary system satisfying \( [B1, B2] \) and \( [B3] \) and so that every complete trajectory is strongly continuous. Then, \( \mathcal{E} \) is asymptotically compact.

Now, we consider the existence of the pullback attractor in the context of an evolutionary system. Note that every evolutionary system is also a generalized evolutionary system. In this case, we have the following relationship for the set functions \( P \) and \( R \): Let \( s \leq t \in \mathbb{R} \), and let \( A \subseteq X \), then
\[
P(t, s)A = P(t - s, 0)A = R(t - s)A.
\]
The following results can also be easily verified:

1. A set $A$ is $d_s$-attracting if and only if the family of sets $A(t) := A$ for all $t \in \mathbb{R}$ is $d_s$-pullback attracting.

2. $\mathcal{E}$ is asymptotically compact if and only if $\mathcal{E}$ is pullback asymptotically compact.

3. $\mathcal{E}$ satisfies \textbf{B1} if and only if $\mathcal{E}$ satisfies \textbf{A1}

4. $\mathcal{E}$ satisfies \textbf{B2} if and only if $\mathcal{E}$ satisfies \textbf{A2}

5. $\mathcal{E}$ satisfies \textbf{B3} if and only if $\mathcal{E}$ satisfies \textbf{A3}

Moreover, for $B \subseteq X$ and $B(t) := B$ for all $t \in \mathbb{R}$, we have that the following invariance relations:

6. $B$ is positively invariant if and only if $B(t)$ is pullback semi-invariant.

7. $B$ is invariant if and only if $B(t)$ is pullback invariant.

8. If $B$ is quasi-invariant, then $B(t)$ is pullback quasi-invariant.

This gives us the following characterization of the $\omega_s$-limit and $\Omega_s$-limit sets.

**Theorem 7.9.** Let $\mathcal{E}$ be an evolutionary system. Let $t \in \mathbb{R}$ and $A \subseteq X$ then,

$$
\Omega_s(A, t) = \omega_s(A) \quad (\bullet = s, w).
$$

**Proof.** Let $x \in \Omega_s(A, t)$. Then, there exist sequences $s_n \to -\infty$, $s_n \leq t$, and $x_n \in P(t, s_n)A$ so that $x_n \xrightarrow{d_s} x$. Then, $(t - s_n) \to \infty$ for $(t - s_n) \geq 0$, and $x_n \in P(t, s_n)A = P(t - s_n, 0)A = R(t - s_n)A$. That is, $x \in \omega_s(A)$.

Now, let $x \in \omega_s(A)$. Then, there exist sequences $t_n \to \infty$, $t_n \geq 0$, and $x_n \in R(t_n)A$ so that $x_n \xrightarrow{d_s} x$. But, $t_n = t - (t - t_n)$. Therefore,

$$
x_n \in R(t_n)A = R(t - (t - t_n))A = P(t - (t - t_n), 0)A = P(t, t - t_n)A
$$

for $t - t_n \leq t$ and $t - t_n \to \infty$. That is, $x \in \Omega_s(A, t)$. \hfill \Box

Using this result as well as Theorem 7.6, Theorem 2.8 and Theorem 2.11, we have the following corollary.

**Theorem 7.10.** Let $\mathcal{E}$ be an evolutionary system. Then, the weak global attractor $\mathcal{A}_w$ and the weak pullback attractor $\mathcal{A}_w(t)$ exist, and $\mathcal{A}_w = \mathcal{A}_w(t)$ for each $t \in \mathbb{R}$. Moreover, the strong global attractor $\mathcal{A}_s$ exists if and only if the strong pullback attractor $\mathcal{A}_s(t)$ exists, and $\mathcal{A}_s = \mathcal{A}_s(t)$.

**Proof.** Using Theorem 7.6 we know that the weak global attractor $\mathcal{A}_w$ exists, and $\mathcal{A}_w = \omega_w(X)$. By Theorem 2.8 and Theorem 2.11, the weak pullback attractor $\mathcal{A}_w(t)$ exists and $\mathcal{A}_w(t) = \Omega_w(X, t)$. By Theorem 7.9 we have that

$$
\mathcal{A}_w = \omega_w(X) = \Omega_w(X, t) = \mathcal{A}_w(t).
$$

Now, suppose the strong global attractor $\mathcal{A}_s$ exists. Then, as in the above section, we have that

$$
\mathcal{A}_s = \omega_s(X) = \Omega_s(X, t).
$$

But, then $\Omega_s(X, t)$ is $d_s$-attracting which means that it is $d_s$-pullback attracting. Therefore, the strong pullback attractor $\mathcal{A}_s(t)$ exists, and $\mathcal{A}_s(t) = \mathcal{A}_s$. An analogous argument shows that if the strong pullback attractor $\mathcal{A}_s(t)$ exists, then the strong global attractor $\mathcal{A}_s$ exists and $\mathcal{A}_s = \mathcal{A}_s(t)$. \hfill \Box

Furthermore, if $\mathcal{E}$ is asymptotically compact, then $\mathcal{E}$ is pullback asymptotically compact, and the strong global attractor $\mathcal{A}_s$ and strong pullback attractor $\mathcal{A}_s(t)$ both exist, and $\mathcal{A}_s(t) = \mathcal{A}_s$. 

7.2. Nonautonomous Case. The modern theory of uniform attractors (using a symbol space across which attraction is uniform) was first introduced by Chepyzhov and Vishik. They applied this framework to the 2D Navier-Stokes equations with an appropriate forcing term. For more information on this theory, see [8]. Later, a weak uniform attractor was proved for the 3D Navier-Stokes equations by Kapustyan and Valero [18]. Using the framework of evolutionary systems, Cheskidov and Lu added a structure theorem and tracking properties of the uniform attractor [10]. We follow the closely-related framework in [10] to compare the structure of the weak uniform attractor to the pullback attractor.

Following the theory of [8] the concept of symbols is introduced. So, let $\Sigma$ be a parameter set and $\{T(s)\}_{s \geq 0}$ be a family of operators acting on $\Sigma$ satisfying $T(s)\Sigma = \Sigma$, for each $s \geq 0$. Any element $\sigma \in \Sigma$ will be called a (time) symbol and $\Sigma$ will be called the (time) symbol space. For instance, in many applications $\{T(s)\}$ is the translation semigroup and $\Sigma$ is the translation family of time dependent items of the system being considered or its closure in some appropriate topological space.

**Definition 7.11.** [10] A family of maps $\mathcal{E}_\sigma$, $\sigma \in \Sigma$ which for each $\sigma \in \Sigma$ associates to each $I \in \mathcal{F}$ a subset $\mathcal{E}_\sigma(I) \subseteq \mathcal{F}(I)$ will be called a nonautonomous evolutionary system if the following conditions are satisfied:

1. $\mathcal{E}_\sigma([\tau, \infty)) \neq \emptyset$ for each $\tau \in \mathbb{R}$.
2. $\mathcal{E}_\sigma(I + s) = \{u(\cdot) : u(s + s) \in \mathcal{E}_{T(s)}(I)\}$ for each $s \geq 0$.
3. $\{u(\cdot)|_{I_2} : u(\cdot) \in \mathcal{E}_\sigma(I_1)\} \subseteq \mathcal{E}_\sigma(I_2)$ for each $I_1, I_2 \in \mathcal{F}$, $I_2 \subseteq I_1$.
4. $\mathcal{E}_\sigma((\sigma, \infty)) = \{u(\cdot) : u(\cdot) \in \mathcal{E}_\sigma([\tau, \infty))\} \forall \tau \in \mathbb{R}$.

Analogous to our previous definitions, $\mathcal{E}_\sigma(I)$ is called the set of all trajectories with respect to the symbol $\sigma$ on the time interval $I$. Trajectories in $\mathcal{E}_\sigma((\sigma, \infty))$ are called complete with respect to $\sigma$. Note that if we fix any symbol $\sigma \in \Sigma$ in a nonautonomous evolutionary system $\mathcal{E}_\sigma$, then we obtain a generalized evolutionary system. On the other hand, if we let $\Sigma := \mathbb{R}$ with $T(s)t := t + s$, the translation semigroup, as well as

$$\mathcal{E}_\sigma([T, \infty)) := \{u(\cdot) : u(\cdot - t) \in \mathcal{E}([T + t, \infty))\},$$

we obtain a nonautonomous evolutionary system from a given generalized evolutionary system.

For every $t \geq \tau$, $\tau \in \mathbb{R}$, $\sigma \in \Sigma$, the map

$$R_\sigma(t, \tau) : \mathcal{P}(X) \to \mathcal{P}(X)$$

is defined by

$$R_\sigma(t, \tau)A := \{u(\cdot) : u(\tau) \in A, u \in \mathcal{E}_\sigma([\tau, \infty))\} \text{ for } A \subseteq X.$$

By the assumptions on $\mathcal{E}_\sigma$ for each $\sigma \in \Sigma$, it is found that

$$R_\sigma(t, \tau)A \subseteq R_\sigma(t, s)R_\sigma(s, \tau)A$$

for each $A \subseteq X$, $t \geq s \geq \tau \in \mathbb{R}$. Using the following Lemma, one can reduce a nonautonomous evolutionary system to an evolutionary system.

**Lemma 7.12.** [10] Let $\tau_0 \in \mathbb{R}$ be fixed. Then, for any $\tau \in \mathbb{R}$ and $\sigma \in \Sigma$, there exists at least one $\sigma' \in \Sigma$ so that

$$\mathcal{E}_\sigma([\tau, \infty)) = \{u(\cdot) : u(\cdot + \tau - \tau_0) \in \mathcal{E}_{\sigma'}([\tau, \infty))\}.$$
Thus, it is found that for $A \subseteq X$, $\tau \in \mathbb{R}$ and $t \geq 0$,

$$
\bigcup_{\sigma \in \Sigma} R_{\sigma}(t, 0) A = \bigcup_{\sigma \in \Sigma} R_{\sigma}(t + \tau, \tau) A.
$$

Moreover, defining

$$
\delta_{\Sigma}(I) := \bigcup_{\sigma \in \Sigma} \delta_{\sigma}(I)
$$

for $I \in \mathcal{T}$, then $\delta_{\Sigma}$ defines an autonomous evolutionary system. Also, for $A \subseteq X$, $t \geq 0$, the map $R_{\Sigma}(t) : \mathcal{P}(X) \to \mathcal{P}(X)$ is given by

$$
R_{\Sigma}(t) A := \bigcup_{\sigma \in \Sigma} R_{\sigma}(t, 0) A.
$$

Let $\omega_{w}^{\Sigma}(A)$ be the corresponding omega-limit set for $\delta_{\Sigma}$. That is,

$$
\omega_{w}^{\Sigma}(A) := \bigcap_{T \geq 0} \bigcup_{t \geq T} R_{\Sigma}(t) = \bigcap_{T \geq 0} \bigcup_{t \geq T} R_{\sigma}(t, 0).
$$

**Definition 7.13.** [10] For the autonomous evolutionary system $\delta_{\Sigma}$, we denote its $d_{w}$-global attractor (if it exists) by $\omega_{\Sigma}$. We call $\omega_{\Sigma}$ the $d_{w}$-uniform attractor for $\delta_{\Sigma}$.

The following results for $\delta_{\Sigma}$ are then attained using Theorem 7.6.

**Theorem 7.14.** [10] Let $\delta_{\Sigma}$ be a nonautonomous evolutionary system. Then, if the $d_{w}$-uniform attractor exists, then $\omega_{\Sigma} = \omega_{w}^{\Sigma}(X)$. Also, the weak uniform attractor $\omega_{w}^{\Sigma}$ exists.

Here are additional assumptions imposed on $\delta_{\Sigma}$.

C1 $\delta_{\Sigma}([0, \infty))$ is precompact in $C([0, \infty); X_{w})$.

C2 Assume that $X$ is a set in some Banach space $H$ satisfying the Radon-Riesz property with the norm denoted by $|.|$, such that $d_{w}(x, y) := |x - y|$ for all $x, y \in X$ and $d_{w}$ induces the weak topology on $X$. Assume also that for any $\epsilon > 0$, there exists $\delta := \delta(\epsilon)$, so that for each $u \in \delta_{\Sigma}([0, \infty))$ and $t > 0$,

$$
|u(t)| \leq |u(t_{0})| + \epsilon,
$$

for $t_{0}$ a.e. in $(t - \delta, t)$.

C3 Let $u_{k} \in \delta_{\Sigma}([0, \infty))$ be so that $u_{k}$ is $d_{C([0, T]; X_{w})}$-Cauchy sequence in $C([0, T]; X_{w})$ for some $T > 0$. Then, $u_{k}(t)$ is $d_{w}$-Cauchy for a.e. $t \in [0, T]$.

Next, the closure of the evolutionary system $\delta_{\Sigma}$ is introduced. This is given by $\bar{\delta}$ defined as follows:

$$
\bar{\delta}([\tau, \infty)) := \overline{\delta_{\Sigma}([\tau, \infty))}^{C([\tau, \infty); X_{w})}
$$

for each $\tau \in \mathbb{R}$. This is an evolutionary system. Let $\bar{\omega}_{w}(A)$ and $\bar{\omega}_{w}$ be the corresponding omega-limit set and global attractor for $\bar{\delta}$, respectively. Then, $\bar{\delta}$ has the following properties:

**Lemma 7.15.** [10] If $\delta_{\Sigma}$ satisfies $C1$, then $\bar{\delta}$ satisfies $B1$. Moreover, if $\delta_{\Sigma}$ satisfies $C2$ and $C3$, then $\bar{\delta}$ satisfies $B2$ and $B3$.

**Theorem 7.16.** [10] Assume that $\delta_{\Sigma}$ satisfies $C1$. Then, the weak uniform attractor exists by Theorem 7.14. Also,

1. $\omega_{\Sigma}^{\varepsilon} = \omega_{w}^{\Sigma}(X) = \bar{\omega}_{w}(X) = \bar{\omega}_{w} = \{u_{0} \in X : u_{0} = u(0)\}$ for some $u \in \bar{\delta}((\sigma, \infty))$. 


Theorem 7.17. Let $\mathcal{E}$ be a nonautonomous evolutionary system with symbol space $\Sigma$ and shift operators $T(s) : \Sigma \to \Sigma$ for each $s \geq 0$. Then, we have that $P_\sigma(t,s) = R_\sigma(t,s)$ for all $t \geq s$. We can use property (2) in Definition 7.14 to obtain the following identity for any $\sigma \in \Sigma$, $t \geq r \in \mathbb{R}$, $s > 0$ and $A \subseteq X$,

$$R_\sigma(t + s, r + s)A = R_{T(s)}(t, r)A.$$  

Using this fact, we have can say that

$$\Omega^\sigma(A, t + s) = \Omega^{T(s)}(A, t)$$

for any $\sigma \in \Sigma$, $t \in \mathbb{R}$, and $A \subseteq X$. Thus, we get that

$$\bigcup_{\sigma \in \Sigma} \bigcup_{t \in \mathbb{R}} \Omega^\sigma(A, t) = \bigcup_{\sigma \in \Sigma} \Omega^\sigma(A, t_0).$$

for any fixed $t_0 \in \mathbb{R}$. Now, we have can draw the following relationship between $\bigcup_{\sigma \in \Sigma} \Omega^\sigma(A, t_0)$ and the uniform attractor $\mathcal{A}_w^\Sigma$ (if it exists).

Theorem 7.18. Let $\mathcal{E}$ be a nonautonomous evolutionary system. Then, if the $d_*$-uniform attractor $\mathcal{A}_w^\Sigma$ exists, we have that

$$\bigcup_{\sigma \in \Sigma} \Omega^\sigma(X, t_0) \subseteq \mathcal{A}_w^\Sigma$$

for any fixed $t_0 \in \mathbb{R}$.

Remark 7.19. Note that $x \in \omega^\Sigma(A)$ if and only if there exist sequences $\sigma_n \in \Sigma$, $t_n \geq 0$ with $t_n \to \infty$, and $x_n \in R_{\sigma_n}(t_n, 0)A$ with $x_n \xrightarrow{d_*} x$. Similarly, if $x \in \bigcup_{\sigma \in \Sigma} \Omega^\sigma(A, t_0)$ then there exist sequences $\sigma_n \in \Sigma$, $s_n \in \mathbb{R}$, $s_n \leq t_0$ with $s_n \to -\infty$, and $x_n \in R_{\sigma_n}(t_0, s_n)A$ so that $x_n \xrightarrow{d_*} x$.

Proof. Let $x \in \bigcup_{\sigma \in \Sigma} \Omega^\sigma(X, t_0)$. Then, there exist sequences $\sigma_n \in \Sigma$, $s_n \to -\infty$ with $s_n \leq t_0$ and $x_n \in R_{\sigma_n}(t_0, s_n)X$ with $x_n \xrightarrow{d_*} x$. Without loss of generality, we can pass to a subsequence and assume that $s_n \leq 0$ for each $n$. Using the fact that
\[ R_{\sigma_n}(t_0, s_n)X = R_{\sigma'_n}(t - s_n, 0)X \text{ for any } \sigma'_n \text{ so that } T(-s_n)\sigma'_n = \sigma_n, \text{ we see that } \]
\[ x \in \omega^S(X) \text{ by Remark 7.19} \text{. Thus, } \]
\[ \bigcup_{\sigma \in \Sigma} \Omega^\sigma(X, t_0) \subseteq \omega^S(X) = \mathcal{A}^S \]
by Theorem 7.14 \( \square \)

Using this result, Theorem 7.16, Theorem 2.9, and Theorem 2.11, we get the following corollary.

\textbf{Corollary 7.20.} Let \( \mathcal{E}_\sigma \) be a nonautonomous evolutionary system. Then, the weak uniform attractor \( \mathcal{A}^w_\Sigma \) exists. Similarly, for each \( \sigma \in \Sigma \), the induced generalized evolutionary system, there exists a weak pullback attractor \( \mathcal{A}^w_\sigma(t) \). Moreover,
\[ \bigcup_{\sigma \in \Sigma} \mathcal{A}^w_\sigma(t) = \mathcal{A}^w_\Sigma \]
for any fixed \( t_0 \in \mathbb{R} \).

Combining Theorem 7.18 with Theorem 7.17, the second half of Theorem 7.16, Theorem 4.3, and the following Lemma 7.21, we get a similar embedding of the union of the strong pullback attractors within the strong uniform attractor. But first, we need to know that the asymptotic compactness of \( \mathcal{E}_\Sigma \) guarantees the pullback asymptotic compactness of each \( \mathcal{E}_\sigma \). This is given in the following lemma.

\textbf{Lemma 7.21.} Let \( \mathcal{E}_\Sigma \), the induced autonomous evolutionary system from the nonautonomous evolutionary system \( \mathcal{E}_\sigma \), be asymptotically compact. Then, for each fixed \( \sigma \in \Sigma \), the induced generalized evolutionary system \( \mathcal{E}_\sigma \) is pullback asymptotically compact.

\textbf{Proof.} Let \( \sigma \in \Sigma \) be fixed. Let \( s_n \leq t \) be so that \( s_n \to -\infty \). Also, let \( x_n = R_{\sigma}(t, s_n)X \). Then, using (10), we get that
\[ x_n \in \bigcup_{\sigma \in \Sigma} R_{\sigma}(t, s_n)X = \bigcup_{\sigma \in \Sigma} R_{\sigma}(t - s_n, 0)X = R_{\Sigma}(t - s_n)X \]
for \( t - s_n \to \infty \). Thus, \( x_n \) has a convergent subsequence by the asymptotic compactness of \( \mathcal{E}_\Sigma \) \( \square \)

\textbf{Theorem 7.22.} Let \( \mathcal{E}_\Sigma \), the induced autonomous evolutionary system from the nonautonomous evolutionary system \( \mathcal{E}_\sigma \), be asymptotically compact or let \( \mathcal{E}_\Sigma \) satisfy \( C_1 \), \( C_2 \) and \( C_3 \) with complete trajectories strongly continuous. Then, the weak uniform attractor \( \mathcal{A}^w_\Sigma \) is the strongly compact strong uniform attractor \( \mathcal{A}^s_\Sigma \). Also, for each fixed \( \sigma \in \Sigma \), the weak pullback attractor \( \mathcal{A}_w^\sigma(t) \) is a strongly compact strong pullback attractor \( \mathcal{A}_s^\sigma(t) \). Moreover,
\[ \bigcup_{\sigma \in \Sigma} \mathcal{A}_w^\sigma(t_0) = \bigcup_{\sigma \in \Sigma} \mathcal{A}_s^\sigma(t_0) \subseteq \mathcal{A}^w_\Sigma = \mathcal{A}^s_\Sigma \]
for any fixed \( t_0 \in \mathbb{R} \).

Unfortunately, the reverse inclusion is not known in full generality at this time. However, with the inclusion of some fairly weak assumptions, we can prove the reverse inclusion. To this end, let \( \mathcal{E}_\sigma \) be a nonautonomous evolutionary system. Assume that \( \mathcal{E}_\Sigma \), the induced evolutionary system from \( \mathcal{E}_\sigma \) satisfies \( C_4 \). Assume that for a fixed \( \sigma \in \Sigma \) that \( \mathcal{E}_\sigma([s, \infty)) \) is closed in \( C([s, \infty); X_w) \) for each \( s \in \mathbb{R} \). Then, \( \mathcal{E}_\sigma \)
satisfies $\mathcal{A}^\sigma_w(t)$ the $d_w$-pullback attractor for the generalized evolutionary system $\mathcal{E}_\sigma$ (if it exists). Using Theorem 7.10 and Theorem 5.3, we get the following.

**Theorem 7.23.** Let $\mathcal{E}_\sigma$ be a nonautonomous evolutionary system. Let $\mathcal{E}_\Sigma$, the induced evolutionary system satisfy $\mathcal{C}_t$. Assume that for any $\sigma \in \Sigma$ and any $s \in \mathbb{R}$ that $\mathcal{E}_\sigma([s, \infty))$ is closed in $C([s, \infty); X_w)$. Then the weak uniform attractor $\mathcal{A}_w^\Sigma$ is given by

$$\mathcal{A}_w^\Sigma = \bigcup_{\sigma \in \Sigma} \mathcal{A}_w^\sigma(0)$$

where $\mathcal{A}_w^\sigma(t)$ is the weak pullback attractor for the generalized evolutionary system $\mathcal{E}_\sigma$. Moreover, the weak uniform tracking property holds.

**Proof.** For the closure of $\mathcal{E}_\Sigma$, $\mathcal{E}$, we have using Theorem 7.10 that the weak uniform attractor $\mathcal{A}_w^\Sigma$ is given by

$$\mathcal{A}_w^\Sigma = \{ u(0) : u \in \mathcal{E}((-\infty, \infty)) \}.$$

Also, by Theorem 5.3, we have that for each $\sigma \in \Sigma$,

$$\mathcal{A}_w^\sigma(t) = \mathcal{I}_\sigma(t) := \{ u(t) : u \in \mathcal{E}_\sigma((-\infty, \infty)) \}.$$

Thus, if we can show that

$$\{ u(0) : u \in \mathcal{E}((-\infty, \infty)) \} = \bigcup_{\sigma \in \Sigma} \{ u(0) : u \in \mathcal{E}_\sigma((-\infty, \infty)) \},$$

we are done. First, let $u \in \mathcal{E}((-\infty, \infty))$. Then, there are $u_n \in \mathcal{E}_{\sigma_n}((-\infty, \infty))$ for some $\sigma_n \in \Sigma$ with $u_n \to u$ in the sense of $C((-\infty, \infty); X_w)$. Then, $u_n(0) \to u(0)$. Therefore,

$$\{ u(0) : u \in \mathcal{E}((-\infty, \infty)) \} \subseteq \bigcup_{\sigma \in \Sigma} \{ u(0) : u \in \mathcal{E}_\sigma((-\infty, \infty)) \}.$$

For the other inclusion, let $u_n \in \mathcal{E}_{\sigma_n}((-\infty, \infty))$ be so that $u_n(0) \to x$ for some $x \in X$. Since $\mathcal{E}_\Sigma$ satisfies $\mathcal{C}_t$ there is a subsequence which we reindex as $u_n$ converging in the sense of $C([0, \infty); X_w)$ to some $u^0 \in \mathcal{E}([0, \infty))$. In particular, $u_n(0) \to x$. Hence, $u^0(0) = x$. Again, passing to another subsequence, we can find a subsequence and drop a subindex to obtain that $u_n \to u^1 \in \mathcal{E}([-1, \infty))$ in $C([-1, \infty); X_w)$. Note that then $u^1|_{[0, \infty)} = u^0$. By the usual diagonalization argument, we find a subsequence $u_n \to u$ for some $u \in \mathcal{E}((-\infty, \infty))$ in $C((-\infty, \infty); X_w)$ with $u(0) = x$. Therefore,

$$\{ u(0) : u \in \mathcal{E}((-\infty, \infty)) \} \supseteq \bigcup_{\sigma \in \Sigma} \{ u(0) : u \in \mathcal{E}_\sigma((-\infty, \infty)) \}.$$

\qed

Finally, we combine this with Lemma 7.21, Theorem 7.17, Theorem 7.16 and Theorem 4.3 to get the following Theorem.

**Theorem 7.24.** Let $\mathcal{E}_\sigma$ be a nonautonomous evolutionary system. Let $\mathcal{E}_\Sigma$, the induced evolutionary system satisfy $\mathcal{C}_t$. Assume that, for any $\sigma \in \Sigma$ and any $s \in \mathbb{R}$ that $\mathcal{E}_\sigma([s, \infty))$ is closed in $C([s, \infty); X_w)$. Moreover, assume that $\mathcal{E}_\Sigma$ is
asymptotically compact or that \( \mathcal{E}_\Sigma \) satisfies \( C_2 \) and \( C_3 \) with complete trajectories strongly continuous. Then, we have that

\[
\mathcal{A}_s^\Sigma = \mathcal{A}_w^\Sigma = \bigcup_{\sigma \in \Sigma} \mathcal{A}_\sigma^s(0) = \bigcup_{\sigma \in \Sigma} \mathcal{A}_\sigma^w(0).
\]

8. Application to the 3D Navier-Stokes Equations

The 3D space-periodic, incompressible Navier-Stokes Equations (NSEs) on the periodic domain \( \Omega := \mathbb{T}^3 \), the three-dimensional torus, are given as follows:

\[
\begin{aligned}
\frac{d}{dt} u - \nu \Delta u + (u \cdot \nabla) u + \nabla p &= f(t) \\
\nabla \cdot u &= 0
\end{aligned}
\]

where \( u \), the velocity and \( p \), the pressure, are unknowns; \( f(t) \) is a time-dependent forcing term; and \( \nu > 0 \) is the kinematic viscosity coefficient of the fluid. Assuming the initial condition, \( u_s := u(s) \), and the forcing term, \( f(t) \) for each \( t \), have the property that

\[
\int_\Omega u_s(x) dx = \int_\Omega f(x, t) dx = 0.
\]

we get that

\[
\int_\Omega u(x, t) dx = 0
\]

for all \( t \geq s \in \mathbb{R} \). We are now ready to set up the functional setting.

Denote by \( (\cdot, \cdot) \) and \(|\cdot|\) the \( L^2(\Omega)^3 \)-inner product and the \( L^2(\Omega)^3 \)-norm, respectively. Let \( L := \{ u \in [C^\infty(\Omega)]^3 : \int_\Omega u(x) dx = 0, \ \nabla \cdot u = 0 \} \).

Next, let \( H \) and \( V \) be the closures of \( L \) in \( L^2(\Omega)^3 \) and \( H^1(\Omega)^3 \), respectively. Denote by \( H_u \), the set \( H \) endowed with the weak topology.

Let \( P_\sigma : L^2(\Omega)^3 \to H \) be the \( L^2 \) orthogonal projection, known as the Leray projector. Let \( A := -P_\sigma \Delta = -\Delta \) be the Stokes operator with domain \( (H^2(\Omega))^3 \cap V \).

Note that the Stokes operator is a self-adjoint, positive operator with compact inverse. Let

\[
\|u\| := |A^{1/2} u|.
\]

Note that \( \|u\| \) is equivalent to the \( H^1 \) norm of \( u \) for \( u \in D(A^{1/2}) \) by the Poincaré inequality. Let \( (\cdot, \cdot) \) denote the corresponding inner product in \( H^1 \).

Next, denote by \( B(u, v) := P_\sigma (u \cdot \nabla v) \in V' \) for each \( u, v \in V \). This is a bilinear form with the following property:

\[
(B(u, v), w) = -(B(u, w), v)
\]

for each \( u, v, w \in V \).

We can now rewrite (11) as a differential equation in \( V' \). That is,

\[
\frac{d}{dt} u + \nu Au + B(u, u) = g
\]

for \( g := P_\sigma f \), and \( u \) is a \( V \)-valued function of time.

**Definition 8.1.** The function \( u : [T, \infty) \to H \) (or \( u : (-\infty, \infty) \to H \)) is a weak solution to (11) on \([T, \infty) \) (or \((-\infty, \infty) \)) if

1. \( \frac{d}{dt} u \in L^{1}_{loc}([T, \infty); V') \).
2. \( u \in C([T, \infty); H_\nu) \cap L^2_{loc}([T, \infty); V) \).

3. \( (\frac{du(t)}{dt}, \phi) + \nu ((u(t), \phi)) + (B(u(t), u(t)), \phi) = \langle g(t), \phi \rangle \) for a.e. \( t \in [T, \infty) \) and each \( \phi \in V \).

**Theorem 8.2** (Leray, Hopf). For each \( u_0 \in H \) and \( g \in L^2_{loc}(\mathbb{R}; V') \), there exists a weak solution of (11) on \([T, \infty)\) with \( u(T) = u_0 \), and for each \( t \geq t_0, t_0 \) a.e. in \([T, \infty)\) we have the following energy inequality:

\[
|u(t)|^2 + 2\nu \int_{t_0}^t |u(s)|^2 ds \leq |u(t_0)|^2 + 2 \int_{t_0}^t \langle g(s), u(s) \rangle ds.
\]

**Definition 8.3.** A weak solution to (11) satisfying (13) will be called a Leray-Hopf weak solution.

**Definition 8.4.** A weak solution to (11) on \([T, \infty)\) satisfying the energy inequality

\[
|u(t)|^2 + 2\nu \int_T^t \|u(s)\|^2 ds \leq |u(T)|^2 + 2 \int_T^t \langle g(s), u(s) \rangle ds
\]

for each \( t \geq T \) is called a Leray solution.

Leray solutions are special in that they are continuous at the starting time \( T \). Note that via the Galerkin method, one can prove the existence of Leray solutions for each \( u_0 \in H \) and \( T \in \mathbb{R} \).

Assume \( g \) is translationally bounded in \( L^2_{loc}(\mathbb{R}, V') \). That is,

\[
\|g\|_{L^2_v}^2 := \sup_{t \in \mathbb{R}} \int_t^{t+1} \|g(s)\|^2_{V'} ds < \infty.
\]

We will show that there exists a bounded set \( X \subset H \) which captures all of the asymptotic dynamics of Leray solutions with a translationally bounded force \( g \). We will be more precise with what this means in a moment. But first, we need a preliminary definition and a lemma. The lemma’s proof can be found in [8].

**Definition 8.5.** A function \( f(s) \) is almost everywhere equal to a monotonic non-increasing function on \([a, b]\) if \( f(t) \leq f(\tau) \) for any \( t, \tau \in [a, b]\setminus Q \) with \( \tau \leq t \) and the measure of \( Q \) is zero.

**Lemma 8.6.** Let \( f(s) \in L^1([a, b]) \). Then, the function \( f(s) \) is almost everywhere equal to a monotone non-increasing function on \([a, b]\) if and only if, for any \( \phi \in C^\infty_0((a, b)) \) with \( \phi(s) \geq 0 \), one has

\[
\int_a^b f(s)\phi'(s) ds \geq 0.
\]

So, let \( u \) be a Leray solution to (11) with \( g \) translationally bounded and starting time \( t_0 \). That is, a point where the energy inequality (13) is satisfied. Then, applying Young’s inequality followed by the Poincaré inequality, we find that

\[
|u(t)|^2 + \nu \lambda_1 \int_{t_0}^t |u(s)|^2 ds \leq |u(t_0)|^2 + \frac{1}{\nu} \int_{t_0}^t \|g(s)\|^2_{V'} ds.
\]

Let \( \phi \in C^\infty_0((t_0, \tau)) \) for \( \tau \geq t \). Then, the above inequality is equivalent to the following distributional inequality

\[
-\int_{t_0}^\tau |u(s)|^2 \phi'(s) ds + \nu \lambda_1 \int_{t_0}^\tau |u(s)|^2 \phi(s) ds \leq \frac{1}{\nu} \int_{t_0}^\tau \|g(s)\|^2_{V'} \phi(s) ds.
\]
Replacing $\phi$ with $e^{\nu \lambda_1 t} \phi \in C_0^\infty((t_0, \tau))$, we have that
\[
- \int_{t_0}^\tau |u(s)|^2 e^{\nu \lambda_1 s} \phi'(s) \, ds \leq \frac{1}{\nu} \int_{t_0}^\tau \|g(s)\|_V^2 e^{\nu \lambda_1 s} \phi(s) \, ds
\]
\[
= \frac{1}{\nu} \int_{t_0}^\tau \frac{d}{ds} \left( \int_0^s \|g(r)\|_V^2 e^{\nu \lambda_1 r} \, dr \right) \phi(s) \, ds
\]
\[
= -\frac{1}{\nu} \int_{t_0}^\tau \left( \int_0^s \|g(r)\|_V^2 e^{\nu \lambda_1 r} \, dr \right) \phi'(s) \, ds.
\]
Rearranging, and using Lemma 8.6, we get that $X$ is a closed absorbing ball in $\mathcal{V}$. Then, for $u, v \in X$, we have
\[
\phi(u, v) = \int_0^\tau \left( \int_0^s \|g(r)\|_V^2 e^{\nu \lambda_1 r} \, dr \right) \phi'(s) \, ds.
\]

Thus, we arrive at the following inequality
\[
|u(t)|^2 e^{\nu \lambda_1 t} - |u(t_0)|^2 e^{\nu \lambda_1 t_0} \leq \frac{1}{\nu} \int_{t_0}^t \|g(s)\|_V^2 e^{\nu \lambda_1 s} \, ds.
\]
It remains to estimate the right-hand side of (16). As in [22], we have that
\[
\int_{t_0}^t \|g(s)\|_V^2 e^{\nu \lambda_1 s} \, ds \leq \int_{t-1}^t \|g(s)\|_V^2 e^{\nu \lambda_1 s} \, ds + \int_{t-2}^{t-1} \|g(s)\|_V^2 e^{\nu \lambda_1 s} \, ds + \cdots
\]
\[
\leq e^{\nu \lambda_1 (t - 1)} (1 + e^{-\nu \lambda_1} + e^{-2\nu \lambda_1}) \sup_{t \in \mathbb{R}} \int_t^{t+1} \|g(s)\|_V^2 \, ds
\]
\[
\leq \frac{e^{\nu \lambda_1 t}}{1 - e^{-\nu \lambda_1}} \|g\|_{L_2^2}^2.
\]
Thus, we arrive at the following inequality
\[
|u(t)|^2 \leq |u(t_0)|^2 e^{\nu \lambda_1 (t_0 - t)} + \frac{1}{\nu} \frac{\|g\|_{L_2^2}^2}{1 - e^{-\nu \lambda_1}}.
\]
Let
\[
R := \frac{2\|g\|_{L_2^2}^2}{\nu(1 - e^{-\nu \lambda_1})}.
\]
Then, $X$ given by
\[
X := \{ u \in H : |u| \leq R \}
\]
is a closed absorbing ball in $H$ for Leray solutions. Moreover, $X$ is weakly compact and contains all of the asymptotic dynamics of Leray solutions by the above argument. Define the strong and weak distances on $X$, respectively, by
\[
d_s(u, v) := |u - v| \quad \text{and} \quad d_w(u, v) := \sum_{k \in \mathbb{Z}^3} \frac{1}{2^{|k|}} \frac{|\hat{u}_k - \hat{v}_k|}{1 + |\hat{u}_k - \hat{v}_k|}
\]
for $u, v \in H$ where $\hat{u}_k$ and $\hat{v}_k$ are the Fourier coefficients of $u$ and $v$, respectively. Note that the above weak metric $d_w$ induces the weak topology $H_w$ on $X$. Next, we define our generalized evolution system on $X$ by
\[
\mathcal{S}(T, \infty) := \{ u : u \text{ is a Leray–Hopf solution of (11) on } [T, \infty) \text{ and } u(t) \in X \text{ for } t \in [T, \infty) \},
\]
\[
\mathcal{S}^\prime(-\infty, \infty) := \{ u : u \text{ is a Leray–Hopf solution of (11) on } (-\infty, \infty) \text{ and } u(t) \in X \text{ for } t \in (-\infty, \infty) \}.
\]
Then, $\mathcal{S}$ satisfies the necessary properties in Definition 2.1 and forms a generalized evolutionary system on $X$. We must use Leray-Hopf solutions as our generalized evolutionary system since the restriction of a Leray solution may not be a Leray solution, but it is always a Leray-Hopf solution.
Figure 1. Possible Leray-Hopf weak solutions for the Navier-Stokes equations with zero forcing.

Note that an absorbing ball does not exist for the Leray-Hopf weak solutions. An absorbing ball is a bounded set $X \subset H$ so that for any $B \subset H$ bounded and any $s_0 \in \mathbb{R}$, there is some $\sigma := \sigma(B) \geq s_0$ so that if $u(s_0) \in B$, then $u(s) \in X$ for $s \geq \sigma$. This requires uniformity in $B$. However, Leray-Hopf solutions may have “jumps” at the starting point which can be as large as you like. Thus, even if you were to consider the bounded set $B := \{0\}$ for the Leray-Hopf solutions with $s_0 := 0$ in the autonomous case, you may not have such a structure as Figure 1 illustrates.

By Theorem 2.8, $E$ has a weak pullback attractor. Next, we will show that $E$ satisfies $A_1$ and $A_3$ under the current conditions, and $E$ satisfies $A_2$ under an additional assumption which we will name later. We start with a preliminary lemma.

Lemma 8.7. Let $u_n$ be a sequence of Leray-Hopf weak solutions of (11) on $[s, \infty)$, so that $u_n(t) \in X$ for all $t \geq s$ for some $s \in \mathbb{R}$. Then, there exists a subsequence $n_j$ so that $u_{n_j}$ converges to some $u$ in $C([t_1, t_2]; H_w)$. That is,

$$(u_{n_j}, v) \to (u, v)$$

uniformly on $[t_1, t_2]$ as $n_j \to \infty$ for all $v \in H$.

Proof. The major arguments in this lemma are classical. For more information, see [13], [28], and [24], among others.

First, using [13] as well as the definition of $\mathcal{E}$, we have that $u_n$ is uniformly bounded in $L^\infty([t_1, t_2]; H)$ and in $L^2((t_1, t_2); V)$. Thus, we use Alaoglu compactness theorem to find subsequences (which we will keep reindexing as $u_n$) which converge to some $u$ weak-* in $L^\infty((t_1, t_2); H)$ and weakly in $L^2((t_1, t_2); V)$.

Next, using the fact that $A: V \to V'$ is continuous using the assignment

$$\langle Av, \phi \rangle := \left( A^{1/2} v, A^{1/2} \phi \right) = ((v, \phi)),$$

we get that $Au_n$ is uniformly bounded in $L^2((t_1, t_2); V')$. Thus, we can extract a subsequence and relable as $u_n$ so that $Au_n$ converges to $Au$ weakly in $L^2((t_1, t_2); V')$.

A classical estimate gives us that

$$\|B(u, u)\|_{V'} \leq C |u|^{1/2} \|u\|^{3/2}$$
for any \( u \in V \) and \( C \) a constant. Thus, using the fact that \( u_n \) is uniformly bounded in both \( L^\infty((t_1, t_2); H) \) and \( L^2((t_1, t_2); V) \), we see that \( B(u_n, u_n) \) is uniformly bounded in \( L^{4/3}((t_1, t_2); V') \) using the following estimates:

\[
\|B(u_n, u_n)\|^{4/3}_{L^{4/3}((t_1, t_2); V')} \leq C \int_{t_1}^{t_2} |u_n(s)|^{2/3} \|u_n(s)\|^2 ds
\]

\[
\leq C \|u_n\|^{2/3}_{L^\infty((t_1, t_2); H)} \|u_n\|^2_{L^{4/3}((t_1, t_2); V')}.
\]

Since \( Au_n, B(u_n, u_n), \) and \( g \) are uniformly bounded sequences in \( L^{4/3}((t_1, t_2); V') \), we use (12) to say that so is \( \frac{d}{dt} u_n \). So, we have that \( B(u_n, u_n) \) converges weakly in \( L^{4/3}((t_1, t_2); V') \) and \( \frac{d}{dt} u_n \) converges weakly in \( L^{4/3}((t_1, t_2); V') \). Moreover, a standard compactness argument gives us that then \( u_n \) converges strongly to \( u \) in \( L^2((t_1, t_2); H) \). Using the strong convergence of \( u_n \) as well as the uniform bound of \( u_n \) in \( L^\infty((t_1, t_2); H) \), a well known result shows that \( B(u_n, u_n) \) converges weakly to \( B(u, u) \) in \( L^{4/3}((t_1, t_2); V') \).

Passing to the limit gives us that

\[
\frac{d}{dt} u + \nu Au + B(u, u) = g
\]

in \( V' \). Now, take the inner product with \( v \in V \) and integrate from \( t \) to \( t + h \) where \( t_1 \leq t < t + h \leq t_2 \). We get that

\[
(u(t + h) - u(t), v) = -\nu \int_t^{t+h} ((u(r), v)) dr - \int_t^{t+h} \langle B(u(r), u(r)), v \rangle dr + \int_t^{t+h} \langle f(r), v \rangle dr.
\]

Taking the absolute value and using Cauchy-Swartz followed by Hölder’s inequality, we find that

\[
|(u(t + h) - u(t), v)| \leq \nu \left(\int_t^{t+h} \|u(r)\|^2 dr\right)^{1/2} \left(\int_t^{t+h} \|v\|^2 dr\right)^{1/2}
\]

\[
+ \left(\int_t^{t+h} \|B(u(r), u(r))\|^{4/3} dr\right)^{3/4} \left(\int_t^{t+h} \|v\|^4 dr\right)^{1/4}
\]

\[
+ \left(\int_t^{t+h} \|f(r)\|^2_{V'} dr\right)^{1/2} \left(\int_t^{t+h} \|v\|^2 dr\right)^{1/2}.
\]

Using the above uniform bounds then gives us that

\[
\lim_{h \to 0} (u(t + h) - u(t), v) = 0.
\]

Since \( V \) is dense in \( H \), \( u \in C([t_2, t_1], H_w) \), and we are done. \( \square \)

**Theorem 8.8.** The generalized evolutionary system \( \mathcal{E} \) satisfies \( A1 \) and \( A3 \).

**Proof.** Let \( u_n \) be a sequence in \( \mathcal{E}([s, \infty)) \) for some \( s \in \mathbb{R} \). Then, repeatedly using Lemma 8.7, there is a subsequence which we reindex as \( u_n \) that converges to some \( u^1 \in C([s, s + 1]; H_w) \). A further subsequence converges to some \( u^2 \in C([s, s_2]; H_w) \) with \( u^1(t) = u^2(t) \) on \([s, s + 1]\). Continuing this diagonalization process, we get that there is some subsequence \( u_{n_k} \) converging to \( u \in C([s, \infty), H_w) \). Note that the convergence in Lemma 8.7 gives us that the energy inequality

\[
|u_n(t)|^2 + 2\nu \int_{t_0}^{t} \|u_n(s)\|^2 ds \leq |u_n(t_0)|^2 + 2 \int_{t_0}^{t} \langle g(s), u_n(s) \rangle ds
\]
converges as well to

\[ |u(t)|^2 + 2\nu \int_{t_0}^t \|u(s)\|^2 \, ds \leq |u(t_0)|^2 + 2\int_{t_0}^t \langle g(s), u(s) \rangle \, ds \]

for \( t \geq t_0 \) and \( t_0 \) a.e. in \([s, \infty]\). That is, \( u \in \mathcal{E}(\infty) \), and \( A_1 \) is proven.

For \( A_3 \) let \( u_n \in \mathcal{E}(\infty) \) for some \( s \in \mathbb{R} \) and let \( u_n \to u \in \mathcal{E}(\infty) \) in \( C([s, t]; H) \). Then, as we saw in Lemma 8.7, \( u_n \) is uniformly bounded in \( L^2([s, t]; V) \) for any \( T \geq s \) and \( \frac{1}{\nu} u_n \) is uniformly bounded in \( L^{4/3}([s, t]; V') \) giving us that \( u_n \to u \) strongly in \( L^2([s, t]; H) \). In particular, we have

\[ \int_s^t |u_n(r) - u(r)|^2 \, dr \to 0 \]
as \( n \to \infty \). Thus, \( |u_n(t_0)| \to |u(t_0)| \) a.e. on \([s, t]\).

Therefore, using \( A_1 \) we have the following results for \( \mathcal{E} \).

**Theorem 8.9.** The weak global attractor for \( \mathcal{E} \), \( \mathcal{A}_w(t) \), is the maximal pullback quasi-invariant and maximal pullback invariant subset of \( X \). Also,

\[ \mathcal{A}_w(t) = \mathcal{A}(t) = \{ u(t) : u \in \mathcal{E}(\infty) \} \]

Moreover, \( \mathcal{E} \) satisfies the weak pullback tracking property and if the strong pullback attractor \( \mathcal{A}_s(t) \) exists, \( \mathcal{A}_w(t) = \mathcal{A}_s(t) \).

Next, we assume that \( g \) is normal in \( L^2_{\text{loc}}(\mathbb{R}; V') \). That is, the following definition introduced in \([22]\):

**Definition 8.10.** Let \( Y \) be a Banach space. We say that a function \( \phi \in L^2_{\text{loc}}(\mathbb{R}; Y) \) is normal in \( L^2_{\text{loc}}(\mathbb{R}; Y) \) if, for any \( \epsilon > 0 \), there exists a \( \delta := \delta(\epsilon) \), so that

\[ \sup_{t \in \mathbb{R}} \int_t^{t+\delta} \|\phi(s)\|_Y^2 \, ds \leq \epsilon. \]

This leads us to the following result.

**Theorem 8.11.** The generalized evolutionary system \( \mathcal{E} \) with normal forcing term \( g \) satisfies \( A_2 \).

**Proof.** Let \( u \in \mathcal{E}(\infty) \) for some \( s \in \mathbb{R} \). Let \( \epsilon > 0 \). Then, using the Leray-Hopf energy inequality \([13]\), we get that

\[ |u(t)|^2 \leq |u(t_0)|^2 + \frac{1}{\nu} \int_{t_0}^t \|g(s)\|^2_Y \, ds \]

from \([14]\) for all \( s \leq t_0 \leq t \), \( t_0 \) a.e. in \([s, \infty]\). Putting this together with the normality of \( g \), there is some \( \delta > 0 \) so that for \( t_0 \) a.e. in \( (t - \delta, t) \), we have that

\[ |u(t)|^2 \leq |u(t_0)|^2 + \epsilon, \]

and \( A_2 \) follows. \( \square \)

Using this, we now have that \( \mathcal{E} \) is pullback asymptotically compact, assuming that complete trajectories are strongly continuous. Thus, we can deduce the following results for \( \mathcal{E} \).
Theorem 8.12. Suppose the generalized evolutionary system $\mathcal{E}$ has a normal forcing term $g$. Also, suppose that $\mathcal{E}((\infty, \infty)) \subseteq C((\infty, \infty); X)$. Then, $\mathcal{E}$ has a strongly compact, strong pullback attractor $\mathcal{A}_s(t)$. Also, the strong and weak pullback attractors coincide giving us that $\mathcal{A}_s(t) = \mathcal{A}_w(t) = \mathcal{I}(t) = \{u(t) : u \in \mathcal{E}((\infty, \infty))\}$.

That is, $\mathcal{A}_s(t)$ is the maximal pullback invariant and maximal pullback quasi-invariant set. Finally, $\mathcal{E}$ has the strong pullback attracting property.

Proof. This is a direct consequence of Corollary [5.9] Theorem [5.7] and Theorem [6.2].

References

[1] J. M. Ball. Continuity properties and global attractors of generalized semiflows and the Navier-Stokes equations. *J. Nonlinear Sci.*, 7(5):475–502, 1997.

[2] T. Caraballo, J. A. Langa, V. S. Melnik, and J. Valero. Pullback attractors of nonautonomous and stochastic multivalued dynamical systems. *Set-Valued Anal.*, 11(2):153–201, 2003.

[3] T. Caraballo, P. Marín-Rubio, and J. C. Robinson. A comparison between two theories for multi-valued semiflows and their asymptotic behaviour. *Set-Valued Anal.*, 11(3):297–322, 2003.

[4] Alexandre N. Carvalho, José A. Langa, and James C. Robinson. *Attractors for infinite-dimensional non-autonomous dynamical systems*, volume 182 of *Applied Mathematical Sciences*. Springer, New York, 2013.

[5] David N. Cheban. *Global attractors of non-autonomous dissipative dynamical systems*, volume 1 of *Interdisciplinary Mathematical Sciences*. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2004.

[6] V. Chepyzhov and M. Vishik. A Hausdorff dimension estimate for kernel sections of nonautonomous evolution equations. *Indiana Univ. Math. J.*, 42(3):1057–1076, 1993.

[7] V. V. Chepyzhov and M. I. Vishik. Attractors of nonautonomous dynamical systems and their dimension. *J. Math. Pures Appl. (9)*, 73(3):279–333, 1994.

[8] Vladimir V. Chepyzhov and Mark I. Vishik. *Attractors for equations of mathematical physics*, volume 49 of *American Mathematical Society Colloquium Publications*. American Mathematical Society, Providence, RI, 2002.

[9] A. Cheskidov and C. Foias. On global attractors of the 3D Navier-Stokes equations. *J. Differential Equations*, 231(2):714–754, 2006.

[10] A. Cheskidov and S. Lu. Uniform global attractors for the nonautonomous 3D Navier-Stokes equations. *ArXiv e-prints*, December 2012.

[11] Alexey Cheskidov. Global attractors of evolutionary systems. *J. Dynam. Differential Equations*, 21(2):249–268, 2009.

[12] Alexey Cheskidov and Songsong Lu. The existence and the structure of uniform global attractors for nonautonomous reaction-diffusion systems without uniqueness. *Discrete Contin. Dyn. Syst. Ser. S*, 2(1):55–66, 2009.

[13] Peter Constantin and Ciprian Foias. *Navier-Stokes equations*. Chicago Lectures in Mathematics. University of Chicago Press, Chicago, IL, 1988.

[14] Hans Crauel and Franco Flandoli. Attractors for random dynamical systems. *Probab. Theory Related Fields*, 100(3):365–393, 1994.

[15] Franco Flandoli and Björn Schmalfuß. Weak solutions and attractors for three-dimensional Navier-Stokes equations with nonregular force. *J. Dynam. Differential Equations*, 11(2):355–398, 1999.

[16] Ciprian Foias and Roger Temam. The connection between the Navier-Stokes equations, dynamical systems, and turbulence theory. In *Directions in partial differential equations (Madison, WI, 1985)*, volume 54 of *Publ. Math. Res. Center Univ. Wisconsin*, pages 55–73. Academic Press, Boston, MA, 1987.

[17] Alain Haraux. *Systèmes dynamiques dissipatifs et applications*, volume 17 of *Recherches en Mathématiques Appliquées [Research in Applied Mathematics]*. Masson, Paris, 1991.
[18] A. V. Kapustyan and J. Valero. Weak and strong attractors for the 3D Navier-Stokes system. *J. Differential Equations*, 240(2):249–278, 2007.

[19] Peter E. Kloeden and Martin Rasmussen. *Nonautonomous dynamical systems*, volume 176 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2011.

[20] Peter E. Kloeden and Björn Schmalfuß. Nonautonomous systems, cocycle attractors and variable time-step discretization. *Numer. Algorithms*, 14(1-3):141–152, 1997. Dynamical numerical analysis (Atlanta, GA, 1995).

[21] P. G. Lemarié-Rieusset. *Recent developments in the Navier-Stokes problem*, volume 431 of *Chapman & Hall/CRC Research Notes in Mathematics*. Chapman & Hall/CRC, Boca Raton, FL, 2002.

[22] Songsong Lu, Hongqing Wu, and Chengkui Zhong. Attractors for nonautonomous 2D Navier-Stokes equations with normal external forces. *Discrete Contin. Dyn. Syst.*, 13(3):701–719, 2005.

[23] Valery S. Melnik and José Valero. On attractors of multivalued semi-flows and differential inclusions. *Set-Valued Anal.*, 6(1):83–111, 1998.

[24] James C. Robinson. *Infinite-dimensional dynamical systems*. Cambridge Texts in Applied Mathematics. Cambridge University Press, Cambridge, 2001. An introduction to dissipative parabolic PDEs and the theory of global attractors.

[25] Ricardo M. S. Rosa. Asymptotic regularity conditions for the strong convergence towards weak limit sets and weak attractors of the 3D Navier-Stokes equations. *J. Differential Equations*, 229(1):257–269, 2006.

[26] George R. Sell. Global attractors for the three-dimensional Navier-Stokes equations. *J. Dynam. Differential Equations*, 8(1):1–33, 1996.

[27] George R. Sell and Yuncheng You. *Dynamics of evolutionary equations*, volume 143 of *Applied Mathematical Sciences*. Springer-Verlag, New York, 2002.

[28] Roger Temam. *Navier-Stokes equations*. AMS Chelsea Publishing, Providence, RI, 2001. Theory and numerical analysis, Reprint of the 1984 edition.

[29] Dmitry Vorotnikov. Asymptotic behavior of the non-autonomous 3D Navier-Stokes problem with coercive force. *J. Differential Equations*, 251(8):2209–2225, 2011.

Department of Mathematics, Statistics, and Computer Science, University of Illinois at Chicago, 322 Science and Engineering Offices (M/C 249), 851 S. Morgan Street, Chicago, Illinois 60607-7045 USA

E-mail address: acheskid@uic.edu

Department of Mathematics, Statistics, and Computer Science, University of Illinois at Chicago, 322 Science and Engineering Offices (M/C 249), 851 S. Morgan Street, Chicago, Illinois 60607-7045 USA

E-mail address: lkavli2@uic.edu