Reliable Error Estimates for Optimal Control of Linear Elliptic PDEs with Random Inputs

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Abstract. We discretize a risk-neutral optimal control problem governed by a linear elliptic partial differential equation with random inputs using a Monte Carlo sample-based approximation and a finite element discretization, yielding finite dimensional control problems. We establish an exponential tail bound for the distance between the finite dimensional problems' solutions and the risk-neutral problem's solution. The tail bound implies that solutions to the risk-neutral optimal control problem can be reliably estimated with the solutions to the finite dimensional control problems. Numerical simulations illustrate our theoretical findings.

Key words. PDE-constrained optimization, sample average approximation, Monte Carlo sampling, uncertainty quantification, stochastic programming, finite element discretization, error estimates, sparse controls, large deviations

AMS subject classifications. 90C15, 90C60, 35Q93, 35R60, 49M25, 49N10, 65M60, 65C05

1. Introduction

Many applications in science and engineering require the solution of optimization problems constrained by physics-based models, such as those modeled using partial differential equations (PDEs). Optimization problems with PDE constraints include, for instance, tidal-stream array optimization [27], optimal design of permanent magnet synchronous machines [2], and topology design of cantilever structures [70]. When parameters in PDEs are uncertain, we obtain a parameter-dependent objective function. The solutions to the parameterized PDE-constrained optimization may depend significantly on the particular parameter choices. In such cases, we may be interested in obtaining a decision resilient to uncertainty. We model the parameter vector as random vector and seek a decision performing best on average. The task is formulated as a risk-neutral PDE-constrained optimization problem. Its objective function is defined as the expected value of the parameter-dependent objective function.

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Risk-neutral PDE-constrained optimization problems are challenging both from a theoretical and practical perspective. Their feasible sets are often infinite dimensional, high-dimensional random vectors result in high-dimensional integrals, and PDEs commonly lack closed-form solutions. For numerical simulations, risk-neutral control problems must generally be discretized and expectations be approximated. We combine a Monte Carlo sample-based approximation of expectations with finite dimensional approximations of the decision and PDE solution spaces to obtain a finite dimensional optimization problem, ready to be solved using state-of-the-art solution methods for PDE-constrained optimization. We derive error bounds on the distance between solutions to the finite dimensional problem and that to the risk-neutral problem, which are valid with high probability. When formulating the risk-neutral optimal control problem, we allow for a particular nonsmooth control regularization, as it appears in control placement applications such as the design of tidal-stream renewable energy farms [20].

We consider the optimal control problem governed by an elliptic PDE

$$\min_{u \in U_{\text{ad}}} \left( \frac{1}{2} \mathbb{E} \left[ \| S(u, \xi) - y_d \|_{L^2(D)}^2 + (\alpha/2) \| u \|_{L^2(D)}^2 + \gamma \| u \|_{L^1(D)} \right] \right), \quad (1.1)$$

where $y_d \in L^2(D)$, $D \subset \mathbb{R}^d$ is a convex, polygonal, bounded domain, $\alpha > 0$ and $\gamma \geq 0$ are regularization parameters, and for each $(u, \xi) \in L^2(D) \times \Xi$, $y = S(u, \xi) \in H^1_0(D)$ solves the parameterized elliptic

$$\int_D \kappa(\xi) \nabla y \cdot \nabla v \, dx = \int_D u v \, dx \quad \text{for all } v \in H^1_0(D). \quad (1.2)$$

Here, $\xi$ is a random element mapping from a probability space to a complete, separable metric space $\Xi$. The space $\Xi$ is equipped with its Borel sigma-field. For the numerical simulations presented in section 6, $\Xi$ is a closed subset of $\mathbb{R}^p$. The space $L^2(D)$ is the space of square integrable functions defined on $D$ and $L^1(D)$ is that of integrable functions. Moreover, $H^1_0(D)$ is the Sobolev space of $L^2(D)$-functions with square integrable weak derivatives and zero boundary traces. We require the random diffusion coefficient $\kappa : \Xi \to C^1(\bar{D})$ be essentially bounded and assume that there exist $\kappa_{\min} > 0$ such that $\kappa_{\min} \leq \kappa(\xi)(x)$ for all $(\xi, x) \in \Xi \times \bar{D}$. The terms $\kappa(\xi)$, $\nabla y$, $\nabla v$, and $u$ in (1.2) are evaluated at $x \in D$, but we omit writing these evaluations. Throughout the text, we omit similar evaluations in other equations as well. The inner product of a real Hilbert space $H$ is denoted by $\langle \cdot, \cdot \rangle_H$ and its norm by $\| \cdot \|_H = (\langle \cdot, \cdot \rangle_H)^{1/2}$. The feasible set $U_{\text{ad}}$ is defined by

$$U_{\text{ad}} = \{ u \in L^2(D) : 1 \leq u \leq u \text{ a.e. in } D \}, \quad \text{where } 1 \leq 0 \leq u \text{ and } 1, u \in \mathbb{R}. \quad (1.3)$$

We discretize the control problem (1.1) using the sample average approximation (SAA) [37, 61] and a finite element discretization [32, 66]. To approximate the expected value in (1.1), let $\xi^1, \xi^2, \ldots$ be defined on a complete probability space $(\Omega, \mathcal{F}, P)$ and be independent identically distributed $\Xi$-valued random elements, each having the same distribution as that of $\xi$. For a discretization parameter $h \in (0, 1)$, let $Y_h$ and $U_h$ be finite dimensional subspaces, such as finite element spaces, of the state space $H^1_0(D)$ and the control space $L^2(D)$, respectively.

Now, we can formulate the discretized SAA problem

$$\min_{u_h \in U_{\text{ad}, h}} \frac{1}{2N} \sum_{i=1}^N \| S_h(u_h, \xi_i) - y_d \|_{L^2(D)}^2 + (\alpha/2) \| u_h \|_{L^2(D)}^2 + \gamma \| u_h \|_{L^1(D)}, \quad (1.4)$$
where \( N \in \mathbb{N} \) is the sample size, \( U_{ad,h} = U_{ad} \cap U_h \) and for each \((u, \xi) \in L^2(D) \times \Xi, y_{\xi,h} = S_h(u, \xi) \in Y_h \) solves the discretized PDE

\[
\int_D \kappa(\xi) \nabla y_{\xi,h} : \nabla \mathbf{v}_h \,dx = \int_D u \mathbf{v}_h \,dx \quad \text{for all } \mathbf{v}_h \in Y_h. \tag{1.5}
\]

Whereas the optimal control problem (1.1) is infinite dimensional and its objective function involves a potentially high-dimensional integral, the discretized SAA problem (1.4) is a finite dimensional optimization problem with the expectation approximated by the sample average. The discretized SAA problem can be efficiently solved using semismooth Newton methods [44, 62, 67].

We analyze the accuracy and reliability of solutions to the discretized SAA problem (1.4) as approximate solutions to the infinite dimensional PDE-constrained problem (1.1). The manuscript’s main novelty is the derivation of an error estimate allowing us to conclude that (depending on the discretization parameter \( h \) and sample size \( N \)), the finite dimensional sample-based solutions are close to the infinite dimensional problem’s solution with extremely high probability. Let \( u^* \) be the solution to (1.1) and \( u_{h,N}^* \) be that to (1.4). We establish the exponential tail bound, our main contribution,

\[
\Pr(||u_{h,N}^* - u^*||_{L^2(D)} \geq c_1 h + c_2 \varepsilon) \leq 2 \exp(-\varepsilon^2 N/2) \quad \text{for all } \varepsilon > 0, \tag{1.6}
\]

where \( c_1, c_2 \in (0, \infty) \) are deterministic, problem-dependent parameters, which we explicitly derive within our error estimation. The constants \( c_1 \) and \( c_2 \) depend, for example, on the regularization parameter \( \alpha > 0 \) and deterministic characteristics of the random diffusion coefficient. The exponential tail bound (1.6) implies that for each \( 0 < \delta < 1 \), with a probability of at least \( 1 - \delta \),

\[
||u_{h,N}^* - u^*||_{L^2(D)} < c_1 h + c_2 \sqrt{2 \ln(2/\delta) / N}. \tag{1.7}
\]

This estimate essentially expresses the fact that the random vector \( u_{h,N}^* \) is “close” to the solution \( u^* \) to (1.1) with high probability, provided that \( 0 < \delta \ll 1 \). Since the inverse of \( \delta \in (0, 1) \) only appears logarithmically in (1.7), we can choose a small “error probability” \( \delta \), say \( \delta = 10^{-12} \). Therefore, we may conclude that the solution to the infinite dimensional (1.1) can be reliably estimated via solutions to the discretized SAA problem (1.4). We refer to the exponential tail bound (1.6) as reliable error estimate.

Establishing an exponential tail bound, such as (1.6), is one approach to analyzing the accuracy of the discretized SAA problem’s solutions. Another approach is to establish an expectation bound, providing an estimate on the expected distance between the SAA solutions and the true one. Expectation bounds can be combined with Tschebyshev’s inequality to obtain tail bounds. For example, suppose that there exists positive constants \( \tilde{c}_1 \) and \( \tilde{c}_2 \) such that for all \( h \in (0, 1) \) and \( N \in \mathbb{N} \), we have the expectation bound

\[
\mathbb{E}[||u_{h,N}^* - u^*||_{L^2(D)}] \leq \tilde{c}_1 h^2 + \tilde{c}_2 / N. \tag{1.8}
\]

Combining (1.8) with Tschebyshev’s inequality, we obtain the polynomial tail bound

\[
\Pr(||u_{h,N}^* - u^*||_{L^2(D)} \geq \varepsilon) \leq \varepsilon^{-2} (\tilde{c}_1 h^2 + \tilde{c}_2 / N) \quad \text{for all } \varepsilon > 0. \tag{1.9}
\]

Comparing this tail bound with that in (1.6), we obtain a nonexponential decay of the tail probability in (1.9) as a function of the sample size \( N \). Moreover, the discretization
parameter $h$ appears on the right-hand side in (1.9). As a result, the expectation bound (1.8) may not imply an exponential tail bound such as that in (1.6). On the other hand, the tail bound (1.6) implies bounds on all finite moments of $\|u_{h,N}^* - u^*\|_{L^2(D)}$ (see (B.2)). In particular, the exponential tail bound (1.6) implies the expectation bound
\[
\mathbb{E}[\|u_{h,N}^* - u^*\|_{L^2(D)}] \leq c_1 h + c_2 \sqrt{\pi / \sqrt{N}}; \tag{1.10}
\]
see appendix A. Whereas (1.7) expresses the fact that $u_{h,N}^*$ is “close” to $u^*$ with a probability of at least $1 - \delta$, the expectation bounds in (1.8) and (1.10) imply that $u_{h,N}^*$ is “close” to $u^*$ on average. The validity of the expectation bound (1.8) may require fewer assumptions on the random diffusion coefficient than those needed to derive (1.6). In particular, while we require $\kappa: \Xi \to C^1(\bar{D})$ be essentially bounded and the existence of a constant $\kappa_{\min} > 0$ such that $\kappa_{\min} \leq \kappa(\xi)(x)$ for all $(\xi, x) \in \Xi \times D$, the expectation bound (1.8) may be established for more general random diffusion coefficients, such as lognormal random diffusion coefficients. Moreover, the constants $c_1$ and $c_2$ in (1.8) may be smaller than $c_1$ and $c_2$ in (1.6) even if the exponential tail bound (1.6) holds true. We further comment on our choice of random diffusion coefficients in section 5.5.

The SAA problem (1.4) is a canonical approximation of the control problem (1.1) and is obtained via two types of approximations: finite dimensional discretizations of the control and state spaces, and random approximations of expectations using sample averages. To analyze these approximations, we make use of error analyses developed in the literature: our derivation of the exponential tail bound (1.6) is inspired by the error analyses for deterministic control problems developed in [6, 13, 48, 69], makes use of stability and error estimates derived in [1, 14, 45, 46], and uses an exponential tail bound (a large deviation-type bound) for $L^2(D)$-valued random vectors established in [56, 57]. Large deviation-type results and tools are well-established in the literature on stochastic programming [26, 53, 64] and on uncertainty quantification with differential equations [17, 65].

A comprehensive analysis of the SAA approach with a focus on finite dimensional problems is provided in [61, Chap. 5]. The current text is partly based on results established in the author’s dissertation [49]. The SAA method is analyzed in [60, 50] as applied to infinite dimensional risk-neutral optimal control of elliptic PDEs with random inputs, but without considering state and control discretizations. Expectation bounds for unconstrained risk-neutral elliptic control problems with and without state and control discretization and $\gamma = 0$ are derived in [45] (see also [46]). Besides deriving qualitative and quantitative stability for infinite dimensional risk-neutral optimization problems, Monte Carlo approximations and finite dimensional discretizations of risk-neutral elliptic control problems with $\gamma = 0$ are studied in [33] using probability metrics. Our error estimation approach differs from that in [33, sect. 6] in that, for example, we estimate $\|u_{h,N}^* - u^*\|_{L^2(D)}$ without using the error decomposition $\|u_{h,N}^* - u^*\|_{L^2(D)} \leq \|u_{h,N}^* - u_N^*\|_{L^2(D)} + \|u_N^* - u^*\|_{L^2(D)}$, where $u_N^*$ is the solution to an infinite dimensional SAA problem. Monte Carlo sampling provides one approach to approximating the expected value in risk-neutral PDE-constrained optimization problems. Alternative approximation approaches are, for example, quasi-Monte Carlo sampling [29, 28], stochastic collocation and sparse grids [39, 63], and tensor-based methods [21, 22]. Risk-averse control of elliptic equations with uncertain fractional exponents is considered in [5] and error estimates for sample-based and finite element approximations are derived. Solution methods for PDE-constrained optimization under uncer-
tainty include, for example, stochastic gradient methods [23, 45, 46] and inexact trust-region methods [22, 40].

The rest of the manuscript is organized as follows. We provide further notation in section 2 and formulate assumptions on the control problem (1.1) in section 3. In section 3 we also introduce the state and control discretization of (1.1) as a set of two assumptions, which allows us to avoid to formally define finite element spaces. The reliable error estimate is stated in section 4 and established in section 5. We recast the PDEs in (1.2) and (1.5) as linear systems in section 5. To establish the error estimate, we demonstrate that the solution to (1.1) has square integrable weak derivatives, allowing us to approximate it using a quasi-interpolation operator. We present numerical illustrations in section 6. The manuscript is concluded with section 7.

2. Preliminaries and further notation

If not specified otherwise, relations between random elements are supposed to hold with probability one. Metric spaces are equipped with their Borel sigma-field. Let $\Lambda$ be a real Banach space. The norm of $\Lambda$ is denoted by $\| \cdot \|_\Lambda$. Let $\Lambda_1$ and $\Lambda_2$ be real Banach spaces. The space of linear, bounded operators mapping from $\Lambda_1$ to $\Lambda_2$ is denoted by $\mathcal{L}(\Lambda_1, \Lambda_2)$. We define $\Lambda^* = \mathcal{L}(\Lambda, \mathbb{R})$. The dual pairing between $\Lambda^*$ and $\Lambda$ is denoted by $\langle \cdot, \cdot \rangle_{\Lambda^*, \Lambda}$. If $\Lambda$ is a real, reflexive Banach space, then we identify $(\Lambda^*)^*$ with $\Lambda$ and write $(\Lambda^*)^* = \Lambda$. The adjoint operator of $Y \in \mathcal{L}(\Lambda_1, \Lambda_2)$ is denoted by $Y^* \in \mathcal{L}(\Lambda_2^*, \Lambda_1^*)$. Let $(\Theta, \mathcal{A}, \mu)$ be a probability space. An operator-valued mapping $Y : \Theta \rightarrow \mathcal{L}(\Lambda_1, \Lambda_2)$ is called uniformly measurable if there exist a sequence of simple mappings $Y_k : \Theta \rightarrow \mathcal{L}(\Lambda_1, \Lambda_2)$ with $Y_k(\theta) \rightarrow Y(\theta)$ in $\mathcal{L}(\Lambda_1, \Lambda_2)$ as $k \rightarrow \infty$ for all $\theta \in \Theta$. Let $D \subset \mathbb{R}^d$ be a bounded domain. We define the Sobolev spaces $H^p(D)$ with $p \in \{1, 2\}$ as the spaces of $L^2(D)$-functions with square integrable weak derivatives up to order $p$ and $L^\infty(D)$ as the space of essentially bounded functions defined on $D$. Furthermore, we define the seminorm $|y|_{H^1(D)} = \| \nabla y \|_{L^2(D)^d}$ for $y \in H^1(D)$, where $\nabla y$ is the weak gradient of $y$. The norm of $H^1(D)$ is given by $\| y \|_{H^1(D)} = (\| y \|_{L^2(D)}^2 + |y|_{H^1(D)}^2)^{1/2}$. We define $H^{-1}(D) = H_0^1(D)^*$. We identify $L^2(D)$ with its dual and write $L^2(D) = L^2(D)^*$. Friedrichs’ constant $C_D > 0$ of the domain $D \subset \mathbb{R}^d$ is defined by $C_D = \sup_{v \in H_0^1(D) \setminus \{0\}} \| v \|_{L^2(D)} / \| v \|_{H^1(D)}$. Since $D$ is bounded, $C_D < \infty$ [32, Thm. 1.13]. Let $H$ be a real Hilbert space. For a convex, lower semicontinuous, proper function $\chi : H \rightarrow (-\infty, \infty]$, the proximity operator $\text{prox}_\chi : H \rightarrow H$ of $\chi$ is defined by

$$\text{prox}_\chi(v) = \arg \min_{w \in H} \chi(w) + (1/2)\| v - w \|_H^2; \quad (2.1)$$

see [8, Def. 12.23]. We define the indicator function $I_{H_0} : H \rightarrow [0, \infty]$ of $H_0 \subset H$ by $I_{H_0}(v) = 0$ if $v \in H_0$ and $I_{H_0}(v) = \infty$ otherwise.

3. Assumptions on the optimal control problem

We state assumptions on the domain, the random diffusion coefficient, and the state and control space discretization.
3.1. Domain and random diffusion coefficient

We impose conditions on the domain $D$ and the random diffusion coefficient $\kappa$.

**Assumption 3.1.** (a) The domain $D \subset \mathbb{R}^d$ with $d \in \{1,2,3\}$ is bounded, convex and polyhedral.

(b) The random diffusion coefficient $\kappa : \Xi \to C^1(\bar{D})$ is (strongly) measurable and there exists $\kappa_{\min}, \kappa_{\max} \in (0,\infty)$ with $\kappa_{\min} \leq \kappa(\xi)(x) \leq \kappa_{\max}$ for all $(\xi,x) \in \Xi \times \bar{D}$ and $\kappa_{\max,1} \in (0,\infty)$ with $\|\kappa(\xi)\|_{C^1(\bar{D})} \leq \kappa_{\max,1}$ for all $\xi \in \Xi$.

Assumption 3.1 allows us to establish higher regularity of the PDE solution using results established in [1]. While it may be possible to derive reliable error estimates in case $\kappa : \Xi \to C^1(\bar{D})$ for some $t \in (0,1]$, we assume that Assumption 3.1 (b) holds true. Assumption 3.1 (b) is violated if $\kappa$ is a log-normal random diffusion coefficient [14].

3.2. Discretization of state and control space

We introduce the discretization for the state space $H^1_0(D)$ and for the control space $L^2(D)$.

**Assumption 3.2.** For each $h \in (0,1)$, $Y_h$ is a finite dimensional subspace of $H^1_0(D)$. For a constant $C_Y > 0$ independent of $h \in (0,1)$,

$$\inf_{v_h \in Y_h} \|v - v_h\|_{H^1(D)} \leq C_Y h \|v\|_{H^2(D)} \quad \text{for all } v \in H^1_0(D) \cap H^2(D) \text{ and } h \in (0,1). \quad (3.1)$$

Let Assumption 3.2 hold. Since $Y_h$ is a finite dimensional subspace of $H^1_0(D)$, $Y_h$ is closed [41, Thm. 3.2-4]. Hence $Y_h$ is a Hilbert space. Assumption 3.2 is satisfied if Assumption 3.1 (a) holds true and $Y_h$ is the space of piecewise linear finite elements defined on a certain regular meshes of $\bar{D}$ with zero boundary conditions (cf. [1, Lem. 4.3]). The following assumption is based on [69, Assumption 4.2] and [16, Assumption 3.3].

**Assumption 3.3.** For each $h \in (0,1)$, there exists $n_h \in \mathbb{N}$ and $\phi^j_h \in L^\infty(D)$ with $\phi^j_h \geq 0$ a.e. in $D$, $\|\phi^j_h\|_{L^\infty(D)} = 1$ for $j = 1,\ldots,n_h$ and $\sum_{j=1}^{n_h} \phi^j_h = 1$ a.e. in $D$.

Let Assumptions 3.1 (a) and 3.3 hold. For each $h \in (0,1)$, we define $U_h$ as the linear span of $\{\phi^j_h : j = 1,\ldots,n_h\}$. Since $\phi^j_h \in L^\infty(D)$ and $D$ is bounded, we have $\phi^j_h \in L^2(D)$. Hence $U_h$ is a subspace of $L^2(D)$ [41, p. 56]. Since $U_h$ is finite dimensional, it is complete [41, Thm. 3.2-4].

When Assumption 3.1 (a) is fulfilled, $d \in \{2,3\}$, and $U_h$ is given by either piecewise constant or piecewise linear finite elements defined on regular meshes of $\bar{D}$ defined by triangles if $d = 2$ and tetrahedra if $d = 3$, then Assumption 3.3 is fulfilled [16, Rem. 3.1]. Moreover, if Assumption 3.1 (a) holds, $d = 1$, and $U_h$ is given by piecewise constant finite elements defined on intervals, then Assumption 3.3 holds true.

Let Assumption 3.3 be satisfied and let $h \in (0,1)$. Following [12, Def. 2.2], [16, eqns. (10) and (11)] and [69, p. 868], let us define $\pi^j_h : L^1(D) \to \mathbb{R}$ and the quasi-interpolation operator $\mathcal{I}_h : L^1(D) \to U_h$ by

$$\pi^j_h[u] = (\phi^j_h , u)_{L^2(D)}/(\phi^j_h , 1)_{L^2(D)} \quad \text{and} \quad \mathcal{I}_h u = \sum_{j=1}^{n_h} \pi^j_h[u] \phi^j_h. \quad (3.2)$$

Assumption 3.3 and Hölder’s inequality imply that $\pi^j_h$ and $\mathcal{I}_h$ are well-defined.
Assumption 3.4. For a constant $C_U > 0$ independent of $h \in (0, 1)$, we have for all $h \in (0, 1)$ and every $u \in H^1(D)$,
\[
\|u - I_h u\|_{L^2(D)} \leq C_U h |u|_{H^1(D)} \quad \text{and} \quad \|u - I_h u\|_{H^1(D)} \leq C_U h^2 |u|_{H^1(D)}.
\]

According to [16, Lems. 4.3 and 4.4], Assumption 3.4 is fulfilled if [16, Assumption 3.3] is satisfied. If Assumption 3.1 (a) holds and $d \in \{2, 3\}$, then [16, Rem. 3.1] ensures that [16, Assumption 3.3] is fulfilled if $U_h$ is defined by piecewise constant finite elements defined on regular meshes of $\bar{D}$ defined by triangles if $d = 2$ and tetrahedra if $d = 3$, for example. We summarize properties of the quasi-interpolation operator $I_h$ which we use to establish the reliable error estimate.

Lemma 3.5. If Assumptions 3.1 (a) and 3.3 hold and $h \in (0, 1)$, then (a) $\|I_h u\|_{L^1(D)} \leq \|u\|_{L^1(D)}$ for each $u \in L^1(D)$, and (b) $I_h u \in U_{ad,h}$ for all $u \in U_{ad}$.

Proof. (a) A proof based on a computation in [69, p. 870] can be found in [49, Lem. 3.3.6].

(b) Fix $u \in U_{ad}$. We have $I_h u \in U_{ad}$, as $D$ is bounded. Using Assumption 3.3, (1.3), and (3.2), it follows that $1 \leq \tau_h |u| \leq u$. Using Assumption 3.3 once more, we have $I_h u \in U_{ad}$. Combined with $U_{ad,h} = U_{ad} \cap U_h$, we obtain $I_h u \in U_{ad,h}$.

\[\square\]

4. Reliable error estimates

Theorem 4.1 states the reliable error estimate. Let $u^*$ be the solution to the risk-neutral problem (1.1) and let $u_{h,N}^*$ be the solution to the discretized SAA problem (1.4). We demonstrate in Lemma 5.4 the existence of solutions and the measurability of $u_{h,N}^*$. We define the problem-dependent parameter
\[C^* = \frac{C_1}{\kappa_{\min}} \|u^*\|_{L^2(D)} + \|y_d\|_{L^2(D)}.\]

Theorem 4.1. Suppose that Assumptions 3.1–3.4 are fulfilled. Let $\epsilon > 0$ and let $\delta, h \in (0, 1)$. If $N \geq 2 \ln(2/\delta)/\epsilon^2$, then with a probability of at least $1 - \delta$,
\[
\|u_{h,N}^* - u^*\|_{L^2(D)} < (1/\alpha) h \left(8C_{U1}^1/c_1^2 + 17C_U \alpha + \frac{16C_4 C_5}{\kappa_{\min}}\right) \|u^*\|_{H^1(D)}
\]
\[+ \frac{32/\alpha}{\epsilon} \left(\frac{C_3}{\kappa_{\min}} \right) \] (4.2a)
\[+ \frac{4/\alpha}{\epsilon} \left(\frac{C_3}{\kappa_{\min}} \right) \] (4.2b)
\[+ \frac{32/\alpha}{\epsilon} \left(\frac{C_3}{\kappa_{\min}} \right) \] (4.2c)
\[+ \frac{32/\alpha}{\epsilon} \left(\frac{C_3}{\kappa_{\min}} \right) \] (4.2d)

Theorem 4.1 is established in section 5. Theorem 4.1 yields the exponential tail bound in (1.6) with $c_1$ and $c_2$ being problem-dependent parameters. Since $U_{ad,h}$ is bounded, we can establish a bound on $\|u^*\|_{L^2(D)}$ using (1.3). An upper bound on $\|u^*\|_{H^1(D)}$ can be derived using Lemmas 5.1 and 5.6. The deterministic parameters in the error estimate (4.2) are as follows: $C_D > 0$ is Friedrichs’ constant of the domain $D$, the parameters $\kappa_{\min}$, $\kappa_{\max}$, and $\kappa_{\max,1}$ are finite by Assumption 3.1, but depend on characteristics of the random diffusion coefficient $\kappa$, $C_H > 0$ is a constant appearing in an $H^2(D)$-regularity estimate of the PDE solution (see Lemma 5.7), $C_Y > 0$ is defined in Assumption 3.2 and depends on the state space discretization, and $C_U > 0$ is defined in Assumption 3.4 and depends on the control space discretization.
5. Properties of the control problem and proof of the reliable error estimates

To establish the reliable error estimate, we define certain mappings. We define $J : H^1_0(D) \to \mathbb{R}$ and $\tilde{J} : H^1_0(D) \times \Xi \to \mathbb{R}$ by

$$J(y) = (1/2)\|y - y_d\|_{L^2(D)}^2 \quad \text{and} \quad \tilde{J}(u, \xi) = (1/2)\|S(u, \xi) - y_d\|_{L^2(D)}^2.$$  

Furthermore, let us define $F : L^2(D) \to \mathbb{R}$, $\hat{F}_N : L^2(D) \to \mathbb{R}$ and $\hat{F}_{h,N} : L^2(D) \to \mathbb{R}$ by

$$F(u) = \mathbb{E}[\tilde{J}(u, \xi)] + (\alpha/2)\|u\|_{L^2(D)}^2 \quad \text{and} \quad \hat{F}_N(u) = \frac{1}{N} \sum_{i=1}^N \tilde{J}(u, \xi^i) + (\alpha/2)\|u\|_{L^2(D)}^2 ,$$

$$\hat{F}_{h,N}(u) = \frac{1}{N} \sum_{i=1}^N J(S_h(u, \xi^i)) + (\alpha/2)\|u\|_{L^2(D)}^2.$$  

Since $\xi^1, \xi^2, \ldots$ are defined on the common probability space $(\Omega, \mathcal{F}, P)$, we can consider the functions $\hat{F}_N$ and $\hat{F}_{h,N}$ as mappings defined on $L^2(D) \times \Xi$, but we often omit the second argument.

To address measurability issues and the existence of solutions, and to compute derivatives, we express the PDEs (1.2) and (1.5) as linear systems. The linear elliptic PDE (1.2) can be written equivalently as $A(\xi)\eta_\xi = Bu$, where $A : \Xi \to \mathcal{L}(H^1_0(D), H^{-1}(D))$ and $B : L^2(D) \to H^{-1}(D)$ are defined by

$$\langle A(\xi)y, v \rangle_{H^{-1}(D), H^1_0(D)} = \int_D \kappa(\xi) \nabla y \cdot \nabla v \, dx \quad \text{and} \quad \langle Bu, v \rangle_{H^{-1}(D), H^1_0(D)} = \langle u, v \rangle_{L^2(D)}.$$  

Owing to Assumption 3.1, $A$ and $B$ are well-defined and $\|B\|_{\mathcal{L}(L^2(D), H^1_0(D))} \leq C_D$. We compute the gradients of $F$, $\hat{F}_N$, and $\hat{F}_{h,N}$ using the adjoint approach. Therefore, we state the adjoint equation to the linear elliptic PDE (1.2). For each $(u, \xi) \in L^2(D) \times \Xi$, let the adjoint state $z_\xi = z(u, \xi) \in H^1_0(D)$ be the solution to the parameterized adjoint equation

$$\int_D \kappa(\xi) \nabla z_\xi \cdot \nabla v \, dx = -\int_D (S(u, \xi) - y_d)v \, dx \quad \text{for all} \quad v \in H^1_0(D). \quad (5.1)$$  

The discretized PDE (1.5) can be expressed as $A_h(\xi)\eta_{\xi,h} = B_h u$, where $A_h : \Xi \to \mathcal{L}(Y_h, Y^*_h)$ and $B_h : L^2(D) \to Y^*_h$ are defined by

$$\langle A_h(\xi)y_h, v_h \rangle_{Y^*_h, Y_h} = \int_D \kappa(\xi) \nabla y_h \cdot \nabla v_h \, dx \quad \text{and} \quad \langle B_h u, v_h \rangle_{Y^*_h, Y_h} = \langle u, v_h \rangle_{L^2(D)}.$$  

Assumptions 3.1 (a) and 3.2 ensure that $A_h$ and $B_h$ are well-defined. Finally, we state the discretized adjoint equation. For each $(u, \xi) \in L^2(D) \times \Xi$, let the discretized adjoint state $z_{\xi,h} = z_h(u, \xi) \in Y_h$ be the solution to the discretized adjoint equation

$$\int_D \kappa(\xi) \nabla z_{\xi,h} \cdot \nabla v_h \, dx = -\int_D (S_h(u, \xi) - y_d)v_h \, dx \quad \text{for all} \quad v_h \in Y_h. \quad (5.2)$$
5.1. Stability estimates

We formulate stability estimates for the states and adjoint states, and demonstrate the uniform measurability of the random operator $A$ and its inverse. The stability estimates are well-known (see, e.g., [14]) and allow us to establish the reliable error estimate (4.2) with problem-dependent parameters made explicit. Measurability of inverses of measurable operator-valued mappings between two real separable Banach spaces is a classical topic (see, e.g., [30, p. 192]). Under Assumption 3.1, we demonstrate the uniform measurability of $A$ and its inverse.

**Lemma 5.1.** If Assumption 3.1 is satisfied, then the following statements hold.

(a) For each $ξ ∈ Ξ$, $A(ξ)$ is self-adjoint and has a bounded inverse. The mapping $A$ and its inverse are uniformly measurable.

(b) For all $(u, ξ) ∈ L^2(D) × Ξ$,

$$|S(u, ξ)|_{L^2(D)} ≤ (C_D/κ_{min})∥u∥_{L^2(D)} \text{ and } ∥S(u, ξ)∥_{L^2(D)} ≤ (C_D^2/κ_{min})∥u∥_{L^2(D)}.$$

(c) For all $(u, ξ) ∈ L^2(D) × Ξ$, we have $z(u, ξ) = -S(S(u, ξ) - y_d, ξ)$,

$$|z(u, ξ)|_{H^1(D)} ≤ (C_D/κ_{min})∥u∥_{L^2(D)} + ∥y_d∥_{L^2(D)}, \text{ and } ∥z(u, ξ)∥_{H^1(D)} ≤ (C_D/κ_{min})(C_D^2/κ_{min})∥u∥_{L^2(D)} + ∥y_d∥_{L^2(D)}.$$  

Proof. (a) Fix $ξ ∈ Ξ$. The computations in [32, p. 62] imply that $A(ξ)$ is self-adjoint. Assumption 3.1 and the Lax–Milgram lemma imply that $A(ξ)$ has a bounded inverse with $∥A(ξ)^{-1}∥_{L^2(H^{-1}(D), L^2(D))} ≤ 1/κ_{min}$.

Hölder’s inequality implies that the mapping $φ : C^1(\bar{D}) → L^2(H^1_0(D), H^{-1}(D))$ defined by $⟨φ(κ)y, v⟩_{L^2(H^1_0(D), H^{-1}(D))} = \int_D κ∇y · ∇v dx$ is (Lipschitz) continuous. Since $A = φ ◦ κ$, the uniform measurability of $A$ follows from the strong measurability of $κ$ and the composition rule [34, Cor. 1.1.11]. The set, $O$, of invertible maps in $L^2(H^1_0(D), H^{-1}(D))$ is open [4, Thm. 5.8] and $O ⊃ T → T^{-1}$ is continuous. Hence $A^{-1}$ is the composition of a continuous function and $A$. Hence $A^{-1}$ is uniformly measurable [34, p. 7].

(b) The first bound follows from [14, eq. (2.1)]. The second bound is implied by the first one and Friedrichs’ inequality.

(c) Using (1.2) and (5.1), we obtain $z(u, ξ) = -S(S(u, ξ) - y_d, ξ)$. Combined with part (b), we obtain the first stability estimate. Friedrichs’ inequality implies $∥z(u, ξ)∥_{H^1(D)} ≤ (C_D + 1)|z(u, ξ)|_{H^1(D)}$. Hence the first stability estimate implies the second one. □

Lemma 5.2 establishes results similar to those in Lemma 5.1, but for the discretized PDEs.

**Lemma 5.2.** If Assumptions 3.1 and 3.2 hold and $h ∈ (0, 1)$, then the following statements hold.

(a) For each $ξ ∈ Ξ$, $A_h(ξ)$ is self-adjoint and has a bounded inverse. The mapping $A_h$ and its inverse are uniformly measurable.

(b) For all $(u, ξ) ∈ L^2(D) × Ξ$,

$$|S_h(u, ξ)|_{H^1(D)} ≤ (C_D/κ_{min})∥u∥_{L^2(D)} \text{ and } ∥S_h(u, ξ)∥_{L^2(D)} ≤ (C_D^2/κ_{min})∥u∥_{L^2(D)}.$$

(c) For all $(u, ξ) ∈ L^2(D) × Ξ$, we have $z_h(u, ξ) = -S_h(S_h(u, ξ) - y_d, ξ)$.

Proof. The assertions can be established using arguments similar to those used in the proof of Lemma 5.1. □
5.2. Differentiability, existence of solutions, and optimality conditions

We establish the continuous differentiability of \( F, \hat{F}_N \) and \( \hat{F}_{h,N} \) using the adjoint approach and [46, Lem. 2.1], comment on the existence of solutions and state first-order necessary optimality conditions. These basic facts are used in subsequent sections and are essential for our proof of Theorem 4.1.

Lemma 5.3. Let Assumptions 3.1–3.3 hold. Then, the following statements hold true.

(a) The mappings \( S, S_h, z, \) and \( z_h \) are Carathéodory mappings.
(b) For each \( u \in L^2(D), \ E[z(u, \xi)] \in H^1_0(D). \)
(c) The functions \( F, \hat{F}_N \) and \( \hat{F}_{h,N} \) are strongly convex with parameter \( \alpha \) and continuously differentiable. For all \( u \in L^2(D), \nabla F(u) = \alpha u - B^*E[z(u, \xi)], \)
\[
\nabla \hat{F}_N(u) = \alpha u - B^* \left[ \frac{1}{N} \sum_{i=1}^N z(u, \xi^i) \right], \quad \text{and} \quad \nabla \hat{F}_{h,N}(u) = \alpha u - B_{h}^* \left[ \frac{1}{N} \sum_{i=1}^N z_h(u, \xi^i) \right].
\]
(d) For each \( u \in L^2(D), \ B^*v = v \) for all \( v \in H^1_0(D) \) and \( B_{h}^*v_h = v_h \) for all \( v_h \in Y_h. \)
(e) The function \( \hat{F}_{h,N} \) is a Carathéodory function.

Proof. (a) We verify the statements for \( S \) and \( z \) only. We have \( S(u, \xi) = A(\xi)^{-1}Bu \) and \( z(u, \xi) = -S(S(u, \xi) - y_d, \xi) \); see Lemma 5.1. Lemma 5.1 further ensures the uniform measurability of \( A^{-1}. \) Combined with composition rule [34, Prop. 1.1.28], we conclude that \( S(u, \cdot) = A(\cdot)^{-1}Bu \) and hence \( z(u, \cdot) \) are measurable for each \( u \in L^2(D). \) For each \( \xi \in \Xi, \) Lemma 5.1 also ensures that \( S(\cdot, \xi) \) and \( z(\cdot, \xi) \) are continuous.

(b) Lemma 5.1 and part (a) ensure that \( z(u, \cdot) \) is measurable and \( E[|z(u, \xi)|_{H^1_0(D)}] < \infty. \) Hence \( E[z(u, \xi)] \in H^1_0(D) \) [34, p. 14].

(c) The statements for \( \hat{F}_N \) and \( \hat{F}_{h,N} \) are a consequence of the adjoint approach [32, sect. 1.6.2], the fact that \( U_h \) and \( Y_h \) are Hilbert spaces, and Lemmas 5.1 and 5.2. The Gâteaux differentiability of \( F \) and the gradient formula are implied by [46, Lem. 2.1] (see also [46, p. 989] and [24, Lem. C.3]) as well as the fact that \( E[B^*z(u, \xi)] = B^*E[z(u, \xi)] \) for all \( u \in L^2(D). \) This identity is implied by part (b), the linearity and boundedness of \( B^* \), and the definition of the Bochner integral [34, eq. (1.2)]. Using Lemmas 5.1 and 5.2 and the dominated convergence theorem, we can show the continuous differentiability of \( F. \) The strong convexity is a result of the control regularization \( (\alpha/2)|| \cdot ||^2_{L^2(D)} \) and the fact that \( U_h \subset L^2(D) \) is a Hilbert space.

(d) Since the adjoint operator \( B^* \) of \( B \) is given by \( (B^*v, u)_{L^2(D)} = (v, u)_{L^2(D)} \) for \( u \in L^2(D) \) and \( v \in H^1_0(D) \) (cf. [32, p. 62]), we obtain the first identity. We recall that \( L^2(D) \) is identified with \( L^2(D)^* \) and \( (Y^*_h)^* \) is identified with \( y_h. \) Let us define \( t_h \in L^\infty(Y_h, L^2(D)) \) by \( (\psi_h, u)_{L^2(D)} = (\psi_h, u)_{L^2(D)}. \) Since \( Y_h \) is a subspace of \( L^2(D) \), we have \( t_h = B_{h}^* \) (cf. [9, p. 21]). Hence \( (t^*_h)^* = B_{h}^* \). Combined with \( (t^*_h)^* = t_h \) (cf. [4, p. 390]), we have \( t_h = B_{h}^* \).

(e) The mapping \( (u, \xi) \mapsto J(S_h(u, \xi)) \) is a composition of the continuous function \( H^1_0(D) \ni y \mapsto J(y) \) with the Carathéodory mapping \( (u, \xi) \mapsto S_h(u, \xi) \in Y_h. \) Since \( Y_h \subset H^1_0(D) \) is a closed subspace, the function \( \hat{F}_{h,N} \) is a Carathéodory function.

Lemma 5.4 establishes the existence of solutions using standard arguments.

Lemma 5.4. If Assumptions 3.1–3.3 hold, then (a) the control problem (1.1) has a unique solution \( u^* \), (b) for each \( \omega \in \Omega, \) the discretized SAA problem (1.4) has a unique solution \( u_{h,N}(\omega), \) and (c) \( u_{h,N} \) is measurable.
Proof. (a)–(b) The existence and uniqueness can be established using [9, Lem. 2.33].
(c) Since \( \hat{F}_{h,N} \) is a Carathéodory function (see Lemma 5.3), \((\Omega, F, P)\) is complete, and
\( \| \cdot \|_{L^1(D)} \) is continuous, \( u^*_{h,N} : \Omega \to U_h \) is measurable [7, Thm. 8.2.11]. Since \( U_h \) is a closed subspace of \( L^2(D) \), \( u^*_{h,N} : \Omega \to L^2(D) \) is also measurable. \( \Box \)

The next lemma provides consequences of first-order necessary optimality conditions.

**Lemma 5.5.** If Assumptions 3.1–3.3 hold, then

\[
(\nabla F(u^*), u^*_{h,N} - u^*)_{L^2(D)} + \gamma \| u^*_{h,N} \|_{L^1(D)} - \gamma \| u^* \|_{L^1(D)} \geq 0,
\]

\[
(\nabla \hat{F}_{h,N}(u^*_{h,N}), \mathcal{I}_h u^* - u^*_{h,N})_{L^2(D)} + \gamma \| \mathcal{I}_h u^* \|_{L^1(D)} - \gamma \| u^*_{h,N} \|_{L^1(D)} \geq 0.
\]

**Proof.** Lemma 5.3 ensures that \( F \) and \( \hat{F}_{h,N} \) are continuously differentiable and convex. Lemma 3.5 ensures \( \mathcal{I}_h u^* \in U_{ad,h} \) and \( U_{ad,h} = U_h \cap U_{ad} \) yields \( u^*_{h,N} \in U_{ad} \). Hence the inequalities follow from the optimality conditions derived in [35, Thm. 4.42]. \( \Box \)

### 5.3. Regularity of the solution

We show that the solution \( u^* \) to (1.1) has square integrable weak derivatives and provide a bound on the solution’s weak derivatives. Our derivation is based on standard arguments used to establish regularity of solutions to deterministic PDE-constrained optimization problems [10, 13, 18, 25, 43, 66, 69].

**Lemma 5.6.** If Assumption 3.1 holds, then \( u^*, \nabla F(u^*) \in H^1(D) \), and

\[
\| u^* \|_{H^1(D)} \leq (1/\alpha) \| \mathbb{E}[z(u^*, \xi)] \|_{H^1(D)} + (|\xi| + u) \| 1 \|_{L^2(D)}.
\]

**Proof.** Using Lemma 5.3 and the optimality of \( u^* \), we obtain the optimality condition \( u^* = \text{prox}_{\psi/\alpha}((1/\alpha)B^* \mathbb{E}[z(u^*, \xi)]) \) (cf. [44, p. 2092]), where \( \psi(\cdot) = \gamma \cdot \| \cdot \|_{L^1(D)} + I_{U_{ad}}(\cdot) \). Lemma 5.3 yields \( \mathbb{E}[z(u^*, \xi)] \in L^1(D) \). Combined with \( B^* v = v \) valid for all \( v \in H^1_0(D) \) and Lemma C.1, we obtain \( u^* \in H^1(D) \) and the stability estimate. Lemma 5.3 further ensures \( \nabla F(u^*) \in H^1(D) \). \( \Box \)

### 5.4. Basic error estimates

We state basic error estimates which are direct consequences of those derived in [1, 15] (see also [28]).

**Lemma 5.7.** Let Assumptions 3.1 and 3.2 hold and let \( h \in (0,1) \). Then, the following statements hold.

(a) There exists \( C_{H^2} > 0 \) such that \( \| S(u, \xi) \|_{H^r(D)} \leq C_{H^2} C_1 \| u \|_{L^2(D)} \) and \( S(u, \xi) \in H^2(D) \) for all \( (u, \xi) \in L^2(D) \times \Xi, \)

\[
C_1 = (\kappa_{\text{max}}/\kappa_{\text{min}}^4)^{1/2}.
\]

(b) \( |S(u, \xi) - S_h(u, \xi)|_{H^1(D)} \leq C_{Y} C_{H^2} C_2 h \| u \|_{L^2(D)} \) for all \( (u, \xi) \in L^2(D) \times \Xi, \)

\[
C_2 = (\kappa_{\text{max}}^2/\kappa_{\text{min}}^2)^{1/2} = C_1 (\kappa_{\text{max}}/\kappa_{\text{min}})^{1/2}.
\]
(c) \( \| S(u, \xi) - S_h(u, \xi) \|_{L^2(D)} \leq C_S^2 C_{h^2} C_3 h^2 \| u \|_{L^2(D)} \) for all \( (u, \xi) \in L^2(D) \times \Xi \), where
\[
C_3 = (\kappa_{\max}^{7/2} / \kappa_{\min}^{7/2}) \kappa_{\max,1}^4 = \kappa_{\max} C_1 C_2.
\]

(d) \( \| \nabla \hat{F}_{h,N}(u^*) - \nabla F_N(u^*) \|_{L^2(D)} \leq C_4 h^2 \left( (C_S^2 / \kappa_{\min}) \| u^* \|_{L^2(D)} + \| y_d \|_{L^2(D)} \right) \), where
\[
C_4 = 2 C_S^2 C_{h^2} (\kappa_{\max}^{7/2} / \kappa_{\min}^{7/2}) \kappa_{\max,1}^4 = 2 C_S^2 C_{h^2} C_3.
\]

**Proof.** (a) Applying [1, Thm. 3.1], we obtain the statements.
(b) The assertion follows from the proof of [15, Thm. 3.9]. Céa’s lemma used in the proof of [15, Thm. 3.9] remains valid under our assumptions because \( Y_h \) is a closed subspace of \( H_0^1(D) \).
(c) The assertion follows from the arguments used in the proof of [1, Thm. 4.4], but we apply Assumption 3.1 instead of [1, Lem. 4.3] to establish the assertion. The Galerkin orthogonality used in the proof of [1, Thm. 4.4] is valid under our assumptions because \( Y_h \) is a closed subspace of \( H_0^1(D) \).
(d) Using Lemmas 5.1–5.3, we find that for all \( v \in L^2(D) \),
\[
(\nabla \hat{F}_{h,N}(u^*) - \nabla F_N(u^*), v)_{L^2(D)} = \frac{1}{N} \sum_{i=1}^{N} \left( S_h(S_h(u^*, \xi_i^*) - y_d, \xi_i) - S(S(u^*, \xi_i^*) - y_d, \xi_i), v \right)_{L^2(D)}.
\]
We separately estimate \( S_h(S_h(u^*, \xi) - y_d, \xi) - S(S(u^*, \xi) - y_d, \xi) \) and \( S_h(S_h(u^*, \xi) - y_d, \xi) - S_h(S(u^*, \xi) - y_d, \xi) \) for \( \xi \in \Xi \). Using part (c) and Lemma 5.2, we find that
\[
\| S_h(S_h(u^*, \xi) - y_d, \xi) - S_h(S(u^*, \xi) - y_d, \xi) \|_{L^2(D)} \leq \left( \frac{C_S^2}{\kappa_{\min}} \right) \| S(h(u^*, \xi) - S(u^*, \xi) \|_{L^2(D)} \leq \left( \frac{C_S^2}{\kappa_{\min}} \right) C_S^2 C_{h^2} C_3 h^2 \| u^* \|_{L^2(D)}.\]

Using part (c) and Lemma 5.1, we further have
\[
\| S_h(S(u^*, \xi) - y_d, \xi) - S(S(u^*, \xi) - y_d, \xi) \|_{L^2(D)} \leq C_S^2 C_{h^2} C_3^2 h^2 \| S(u^*, \xi) - y_d \|_{L^2(D)}.\]

The triangle inequality and Lemma 5.1 also yield
\[
\| S(u^*, \xi) - y_d \|_{L^2(D)} \leq \left( \frac{C_S^2}{\kappa_{\min}} \right) \| u^* \|_{L^2(D)} + \| y_d \|_{L^2(D)}.\]

Putting together the pieces, we obtain the assertion. 

We collect further basic error estimates based on those established in [45, 46, 69].

**Lemma 5.8.** Let Assumptions 3.1–3.4 hold and let \( h \in (0, 1) \). Then the following statements hold.

(a) We have \( \| \nabla \hat{F}_{h,N}(I_h u^*) - \nabla \hat{F}_{h,N}(u^*) \|_{L^2(D)} \leq \left( \alpha + \frac{C_S^2}{\kappa_{\min}} \right) \| I_h u^* - u^* \|_{L^2(D)}. \)

(b) It holds that \( \| \nabla F(u^*), I_h u^* - u^* \|_{L^2(D)} \| \leq C_S h^2 \| \nabla F(u^*) \|_{H^1(D)} \| u^* \|_{H^1(D)}. \)

**Proof.** (a) The estimate can be established using computations similar to those used in [45, Lem. 3.5] (see also [46, pp. 987 and 989, and Lem. 2.2]).
(b) Since \( H^1(D) \hookrightarrow L^2(D) \hookrightarrow H_0^1(D)^* \) is a Gelfand triple [66, p. 147], the embedding \( L^2(D) \hookrightarrow H^1(D)^* \) is given by \( \langle v, w \rangle_{H^1(D)^*, H^1(D)} = \langle v, w \rangle_{L^2(D)} \) for all \( v \in L^2(D) \) and \( w \in H^1(D) \) [32, Rem. 1.17]. Combined with \( u^*, \nabla F(u^*) \in H^1(D) \) (see Lemma 5.6), and \( I_h u^* \in U_{ad,h} \subset L^2(D) \) (see Lemma 3.5), we have
\[
\| (\nabla F(u^*), I_h u^* - u^*) \|_{L^2(D)} \leq \| \nabla F(u^*) \|_{H^1(D)} \| I_h u^* - u^* \|_{H^1(D)}.
\]

Together with Assumption 3.4, we obtain the estimate. 

\[\square\]
Proposition 5.9 establishes an upper bound on $\alpha \|I_h u^* - u_{h,N}^*\|^2_{L^2(D)}$.

**Proposition 5.9.** If Assumptions 3.1–3.4 hold, then

$$
\alpha \|I_h u^* - u_{h,N}^*\|^2_{L^2(D)} \leq \left( (\nabla \hat{F}_{h,N}(I_h u^*) - \nabla \hat{F}_{h,N}(u^*), I_h u^* - u_{h,N}^*)_{L^2(D)} + (\nabla \hat{F}_{h,N}(u^*), I_h u^* - u_{h,N}^*)_{L^2(D)} 
\right)
$$

(5.4)

**Proof.** The proof is inspired by the arguments used in [48, Thm. 5.2]. Adding the inequalities in (5.3) and using $\|I_h u^*\|_{L^1(D)} \leq \|u^*\|_{L^1(D)}$ (see Lemma 5.3) yields

$$
0 \leq (\nabla F(u^*), u_{h,N}^* - u^*)_{L^2(D)} + (\nabla \hat{F}_{h,N}(u_{h,N}^*), I_h u^* - u_{h,N}^*)_{L^2(D)}.
$$

(5.5)

Since $\hat{F}_{h,N}$ is strongly convex with parameter $\alpha > 0$ and Gâteaux differentiable on $L^2(D)$ (see Lemma 5.3), we have

$$
\alpha \|u_{h,N}^* - I_h u^*\|^2_{L^2(D)} \leq (\nabla \hat{F}_{h,N}(I_h u^*) - \nabla \hat{F}_{h,N}(u_{h,N}^*), I_h u^* - u_{h,N}^*)_{L^2(D)}.
$$

Adding this inequality and the estimate (5.5), we conclude that

$$
\alpha \|I_h u^* - u_{h,N}^*\|^2_{L^2(D)} \leq \left( (\nabla \hat{F}_{h,N}(I_h u^*) - \nabla \hat{F}_{h,N}(u^*), I_h u^* - u_{h,N}^*)_{L^2(D)} + (\nabla \hat{F}_{h,N}(u^*), I_h u^* - u_{h,N}^*)_{L^2(D)} 
\right)
$$

(5.4)

$$
= (\nabla \hat{F}_{h,N}(I_h u^*) - \nabla F(u^*), I_h u^* - u_{h,N}^*)_{L^2(D)} + (\nabla \hat{F}_{h,N}(u^*), I_h u^* - u_{h,N}^*)_{L^2(D)}
$$

$$
+ (\nabla F(u^*), I_h u^* - u_{h,N}^*)_{L^2(D)}.
$$

Manipulating the terms in the right-hand side, we obtain (5.4). \qed

The gradients in (5.4) are evaluated at the deterministic controls $u^*$ and $I_h u^*$, but not at the random control $u_{h,N}^*$. This fact is used for our error estimation. Furthermore, under the hypotheses of Proposition 5.9, the estimate (5.4) yields

$$
\alpha \|I_h u^* - u_{h,N}^*\|^2_{L^2(D)} \leq \left( \|\nabla \hat{F}_{h,N}(I_h u^*) - \nabla F(u^*)\|_{L^2(D)} \|I_h u^* - u_{h,N}^*\|_{L^2(D)} + (\nabla F(u^*), I_h u^* - u_{h,N}^*)_{L^2(D)} 
\right)
$$

(5.6)

Hence $\|I_h u^* - u_{h,N}^*\|_{L^2(D)}$ can be estimated using bounds on $\|\nabla \hat{F}_{h,N}(I_h u^*) - \nabla F(u^*)\|_{L^2(D)}$ and $\|(\nabla F(u^*), I_h u^* - u_{h,N}^*)_{L^2(D)}\|^{1/2}$.

**5.5. Exponential tail bounds for SAA gradients**

We establish an exponential tail bound for $\nabla \hat{F}_{h,N}(u^*) - \nabla F(u^*)$ using the large deviation-type bound established in [56, Thm. 3] (see also [57, Thm. 3.5]). Since $\nabla \hat{F}_{h,N}(u^*) - \nabla F(u^*) = B^* (\mathbb{E}[z(u^*, \xi)] - (1/N) \sum_{i=1}^N z(u^*, \xi_i))$, the accuracy and reliability of $\nabla \hat{F}_{h,N}(u^*)$ as an estimator for $\nabla F(u^*)$ is determined by that of the adjoint state’s sample average. Let $\tau$ be a positive scalar. Suppose that $Z_1, Z_2, \ldots$ are independent, mean-zero $L^2(D)$-valued random vectors defined on a common probability space such that for each $i \in \mathbb{N}$, with probability one, $\|Z_i\|_{L^2(D)} \leq \tau$. Then for each $N \in \mathbb{N}$,

$$
\text{Prob}(\|Z_1 + \cdots + Z_N\|_{L^2(D)} \geq \epsilon) \leq 2 \exp\left(\frac{-\epsilon^2}{2N \tau^2}\right) \quad \text{for all } \epsilon > 0; \tag{5.7}
$$

see [56, Thm. 3] and [57, Thm. 3.5]. Using (5.7), we establish a tail bound for SAA gradients.
Lemma 5.10. Let \( \delta \in (0, 1) \) and let \( \epsilon > 0 \). If Assumption 3.1 holds and \( N \geq 2 \ln(2/\delta)/\epsilon^2 \), then with a probability of at least \( 1 - \delta \),

\[
\| \nabla \hat{F}_N(u^*) - \nabla F(u^*) \|_{L^2(D)} < \epsilon \tau, \quad \text{where} \quad \tau = \frac{2C^2}{\kappa_{\min}} \frac{(C^2)}{\kappa_{\min}} ||u^*||_{L^2(D)} + ||y_t||_{L^2(D)}.
\]

Proof. The random vectors \( Z_i = \nabla_u \hat{f}(u^*, \xi_i) - \nabla F(u^*) \) are independent because \( \xi_1, \xi_2, \ldots \) are independent identically distributed \( \Xi \)-valued random elements and \( \nabla_u \hat{f}(u^*, \cdot) \) is measurable. Lemmas 5.1 and 5.3 ensure that each \( Z_i \) is integrable and has zero mean. Using Lemmas 5.1 and 5.3 and Friederichs' inequality, we have for each \( \xi \in \Xi \),

\[
\| \nabla_u \hat{f}(u^*, \xi) - \nabla F(u^*) \|_{L^2(D)} = \| B^*z(u^*, \xi) - B^*\mathbb{E}[z(u^*, \xi)] \|_{L^2(D)} \\
\leq C_D \| z(u^*, \xi) \|_{H^1(D)} + C_D \| \mathbb{E}[z(u^*, \xi)] \|_{H^1(D)} \\
\leq \frac{2C^2}{\kappa_{\min}} \frac{(C^2)}{\kappa_{\min}} ||u^*||_{L^2(D)} + ||y_t||_{L^2(D)} = \tau.
\]

Hence for each \( i \in \mathbb{N} \), with probability one, \( \| Z_i \|_{L^2(D)} \leq \tau \). Using (5.7) and \( N \geq 2 \ln(2/\delta)/\epsilon^2 \), we obtain the assertion. \( \square \)

The exponential tail bound (5.7) remains valid for \( L^2(D) \)-valued random vectors other than essentially bounded ones. If \( q > 0 \) and \( Z_1, Z_2, \ldots \) are independent, mean-zero \( L^2(D) \)-valued random vectors such that for each \( i \in \mathbb{N} \), \( \mathbb{E}[\exp(q^{-2}\| Z_i \|_{L^2(D)}^2)] \leq \epsilon \), then (5.7) holds true with \( \tau \) replaced by a constant proportional to \( q \); see [58, Thm. 3] and [50, Thm. 1]. While the condition \( \mathbb{E}[\exp(q^{-2}\| Z_i \|_{L^2(D)}^2)] \leq \epsilon \) allows for a larger class of random vectors than essentially bounded ones, the random vectors \( Z_i \) considered in the proof of Lemma 5.10 generally violate this condition if the random field \( \kappa \) is lognormal (see, e.g., [49, p. 87]). This fact has been our main motivation to work with Assumption 3.1 (b).

In the proof of Lemma 5.10, we applied (5.7) to obtain a large deviation-type bound for SAA gradients. This bound depends on the constant \( \tau \) defined in Lemma 5.10 which provides an almost sure upper bound on \( \| \nabla_u \hat{f}(u^*, \xi) - \nabla F(u^*) \|_{L^2(D)} \). As an alternative to using (5.7), we may use the Bernstein-type inequality [58, Cor. 1] to analyze the exponential tail behavior of SAA gradients. We refer the reader to [47, pp. 24 and 26] for discussions on Bernstein’s inequality for real-valued random variables.

### 5.6. Proof of reliable error estimates

In this section, we establish Theorem 4.1.

**Proof of Theorem 4.1.** Lemma 5.6 ensures \( u^* \in H^1(D) \). Combined with the triangle inequality, \( ||u^*||_{H^1(D)} \leq ||u^*||_{H^1(D)} \) and Assumption 3.4, we have

\[
||u^*_{h,N} - u^*||_{L^2(D)} \leq C_U \hat{u} \| u^* \|_{H^1(D)} + ||u^*_{h,N} - \hat{I}_h u^* \|_{L^2(D)},
\]

(5.8)

where \( \hat{I}_h \) is defined in (3.2). The first addend in (5.8) will contribute to the term in (4.2a).

In the following steps, we derive a bound on \( ||u^*_{h,N} - \hat{I}_h u^* \|_{L^2(D)} \) using Lemmas 5.1, 5.7, 5.8 and 5.10 and Proposition 5.9. Our derivations use the inequality \((\rho_1 \rho_2)^{1/2} \leq (2\rho_1 \rho_2)^{1/2} \leq \rho_1 + \rho_2 \) valid for all \( \rho_1, \rho_2 \geq 0 \). Using the Cauchy–Schwarz inequality, we
find that
\[
|\nabla \hat{F}_{h,N}(u^*) - \nabla \hat{F}_N(u^*) - I_h u^* - u_{h,N}^*|_{L^2(D)}^{1/2} \\
\leq (4/a^{1/2}) \|I_h u^* - u_{h,N}^*\|_{L^2(D)} + (a^{1/2}/4) \|I_h u^* - u_{h,N}^*\|_{L^2(D)},
\]
and
\[
|\nabla \hat{F}_N(u^*) - \nabla F(u^*)|_{L^2(D)}^{1/2} \\
\leq (4/a^{1/2}) \|\nabla \hat{F}_N(u^*) - \nabla F(u^*)\|_{L^2(D)} + (a^{1/2}/4) \|I_h u^* - u_{h,N}^*\|_{L^2(D)}.
\]
Furthermore, we have
\[
|\nabla \hat{F}_N(I_h u^*) - \nabla \hat{F}_N(u^*) - I_h u^* - u_{h,N}^*|_{L^2(D)}^{1/2} \\
\leq (4/a^{1/2}) \|\nabla \hat{F}_N(u^*) - \nabla F(u^*)\|_{L^2(D)} + (a^{1/2}/4) \|I_h u^* - u_{h,N}^*\|_{L^2(D)}.
\]
Combined with (5.4), we find that
\[
(1/4) \|u_{h,N} - I_h u^*\|_{L^2(D)} \leq (4/a) \|\nabla \hat{F}_N(u^*) - \nabla \hat{F}_N(u^*)\|_{L^2(D)} \\
+ (4/a) \|\nabla \hat{F}_N(u^*) - \nabla F(u^*)\|_{L^2(D)} \\
+ (4/a) \|\nabla \hat{F}_N(u^*) - \nabla F(u^*)\|_{L^2(D)} \\
+ (a^{1/2}/4) \|\nabla \hat{F}_N(u^*) - \nabla F(u^*)\|_{L^2(D)}^{1/2}.
\]
Using Lemma 5.8 and assumption 3.4, we obtain
\[
\|\nabla \hat{F}_N(u^*) - \nabla F(u^*)\|_{L^2(D)} \leq C_4 h \langle \alpha + \frac{C^2}{k_{\min}} \rangle \|u^*\|_{H^1(D)}.
\]
This estimate will contribute to the term in (4.2a). Using Lemma 5.7, we obtain the bound
\[
\|\nabla \hat{F}_N(u^*) - \nabla \hat{F}_N(u^*)\|_{L^2(D)} \leq C_4 h^2 \langle \frac{C^2}{k_{\min}} \rangle \|u^*\|_{L^2(D)} + \|y_d\|_{L^2(D)} = C_4 h^2 C^*. \tag{5.11}
\]
on the second addend in the right-hand side in (5.9). This error contribution will result in (4.2d). The constant $C^*$ is defined in (4.1).

We must yet derive bounds on the third and fourth term in the right-hand side of (5.9). Since $N \geq 2 \ln(2/\delta)/\epsilon^2$, Lemma 5.10 ensures with a probability of at least $1 - \delta$,
\[
\|\nabla \hat{F}_N(u^*) - \nabla F(u^*)\|_{L^2(D)} \leq 2C_3 \|u^*\|_{L^2(D)} + \|y_d\|_{L^2(D)} \leq 2C_3 \|C^* \|_{L^2(D)} \epsilon. \tag{5.12}
\]
This error contribution will result in (4.2b). Lemma 5.8 yields
\[
\nabla \nabla (F(u^*) - I_h u^* - u_{h,N}^*)|_{L^2(D)}^{1/2} \leq C_4 h \|\nabla F(u^*)\|_{H^1(D)} \|u^*\|_{H^1(D)}^{1/2} \tag{5.13}
\]
Using Lemmas 5.1, 5.3 and 5.6, we find that
\[
\|\nabla F(u^*)\|_{H^1(D)} \leq \alpha \|u^*\|_{H^1(D)} + \|\Xi (z(u^*, \xi))\|_{H^1(D)} \\
\leq \alpha \|u^*\|_{H^1(D)} + \|\Xi (z(u^*, \xi))\|_{H^1(D)} + \|y_d\|_{L^2(D)} \\
= \alpha \|u^*\|_{H^1(D)} + \frac{C_{D+1}C_0}{k_{\min}} \|u^*\|_{L^2(D)} + \|y_d\|_{L^2(D)}.
\]
Hence
\[
\|\nabla F(u^*)\|_{H^1(D)}^{1/2} \leq \|u^*\|_{H^1(D)} + (1/\alpha) \|\nabla F(u^*)\|_{H^1(D)} \\
\leq 2 \|u^*\|_{H^1(D)} + \frac{C_{D+1}C_0}{\alpha k_{\min}} C^*. \tag{5.14}
\]
This error contribution will result in (4.2c) and contribute to (4.2a). Combining (5.8), (5.9), (5.10), (5.11), (5.12), (5.13), and (5.14), we obtain (4.2).
6. Numerical illustrations

We illustrate the theoretical results derived in Theorem 4.1 (see also (1.6)) empirically on two instances of the optimal control problem (1.1). Furthermore, we empirically verify the expectation bound (1.10), which is implied by (1.6). For each instance, we present graphical illustrations of an approximate solution to the control problem (1.1) (called “reference solution”) and of solutions to a nominal problem and to an “expected diffusion” problem associated with (1.1).

Before considering the instances, we discuss the setting used for numerical results. Let $D = (0,1)^d$ with $d \in \{1,2\}$. We chose piecewise constant finite elements defined on a regular triangulation of $[0,1]^d$ with $1/h \in \mathbb{N}$ being the number of cells in each direction to discretize the control space $L^2(D)$ and piecewise linear finite elements with zero boundary conditions defined on the same triangulation to discretize the state space $H^1_0(D)$. We refer to $h$ as “mesh width.” The discretization ensures that the discretized feasible set $U_{ad,h} = U_{ad} \cap U_h$ can be expressed as $U_{ad,h} = \{ \sum_{j=1}^{n_h} u_j \phi^j_h \in U_h : u_j \in \mathbb{R}, 1 \leq u_j \leq u, j = 1, \ldots, n_h \}$ [69, Lem. 4.4].

To solve the SAA problems, we used dolfin-adjoint [52, 19] with FEniCs [3, 42] and a semismooth Newton-CG method [44, 62, 67] applied to a normal map reformulation of the first-order optimality conditions for (1.4). The implementation adapts that of Moola’s NewtonCG [54]. We chose the zero control as an initial point for each SAA problem.

To empirically verify the expectation bound in (1.10), we approximate the expectation $\mathbb{E}[\|u_{h,N}^* - u^*\|_{L^2(D)}]$ using 48 samples. To demonstrate the reliable error estimate, we exploit the fact that it implies a bound on the Luxemburg norm of $u_{h,N}^* - u^*$. Let $H$ be a real, separable Hilbert space. We define the Luxemburg norm $\| \cdot \|_{L^p(H)}$ of a random vector $Z : \Omega \to H$ by

$$\|Z\|_{L^p(H)} = \inf_{\tau > 0} \left\{ \tau : \mathbb{E}[\psi(\|Z\|_H/\tau)] \leq 1 \right\} = \inf_{\tau > 0} \left\{ \tau : \mathbb{E}[e^{\|Z\|_H^2/\tau^2}] \leq 2 \right\},$$  

(6.1)

with the Young function $\psi(x) = e^{x^2} - 1$. The Orlicz space $L^p(\Omega; H) = L^p(\|\cdot\|_H)(\Omega; H)$ consists of all random vectors mapping from $\Omega$ to $H$ with finite Luxemburg norm; see [38, Chap. 6]. The exponential tail bound (1.6) implies

$$\|u_{h,N}^* - u^*\|_{L^p(\Omega; L^2(D))} \leq 3\sqrt{2}(c_1 h + (c_2/\sqrt{N}));$$  

(6.2)

see appendix B. We approximate the expectation in (6.1) using the same samples used to estimate $\mathbb{E}[\|u_{h,N}^* - u^*\|_{L^2(D)}]$. The simulations used to compute the SA solutions were performed on PACE’s Phoenix cluster [55] with Dual Intel Xeon Gold 6226 2.7 GHz CPUs.

Besides those for reference solutions, we provide visualizations of the solution to the nominal problem

$$\min_{u_h \in U_{ad,h}} (1/2) \| S_h(u_h, \mathbb{E}[\xi]) - y_d \|_{L^2(D)}^2 + (\alpha/2)\|u_h\|_{L^2(D)}^2 + \gamma \|u_h\|_{L^1(D)};$$  

(6.3)

and for the solution to the “expected diffusion” problem

$$\min_{u_h \in U_{ad,h}} (1/2) \| G_h(u_h) - y_d \|_{L^2(D)}^2 + (\alpha/2)\|u_h\|_{L^2(D)}^2 + \gamma \|u_h\|_{L^1(D)};$$  

(6.4)
where for each \( u \in L^2(D) \), \( w_h = G_h(u) \in Y_h \) solves
\[
\int_D \mathbb{E}[\kappa(\xi)] \nabla w_h \cdot \nabla v_h \, dx = \int_D u v_h \, dx \quad \text{for all } \; v_h \in Y_h.
\]
For the random diffusion coefficients considered in sections 6.1 and 6.2, the expected value \( \mathbb{E}[\kappa(\xi)(x)] \) can be computed explicitly for each \( x \in D \). We use this fact for the numerical solution of (6.4). To graphically visualize controls, we interpolate the piecewise constant controls to the discretized state space.

### 6.1. One-dimensional equation

We consider an instance of the control problem (1.1) with \( d = 6 \), \( \alpha = 0.001 \), \( \gamma = 0.01 \), \( \Gamma = -3 \), \( u = 3 \), \( y_d(x) = -1/2 \) if \( x \in (1/4,3/4) \) and \( y_d(x) = 1 \) otherwise. We choose \( \Xi = [-1,1]^d \) and
\[
\kappa(\xi)(x) = \exp \left( \sigma \xi_1 \cos(1.1\pi x) + \sigma \xi_2 \cos(1.2\pi x) + \sigma \xi_3 \sin(1.3\pi x) + \sigma \xi_4 \sin(1.4\pi x) \right),
\]
where \( \sigma > 0 \) is a parameter; cf. [45, eq. (8.1)]. The random variables \( \xi_1, \ldots, \xi_4 \) are independent, each with \([-1,1] \)-uniform distribution. The probability distribution of \( \xi \) was approximated by a discrete uniform distribution supported on the grid points of a uniform mesh of \( \Xi \) using 32 grid points in each direction, yielding a discrete distribution with \( 32^4 = 2^{20} \) scenarios. Samples for the SAA problems were generated from this discrete distribution. The reference solution \( u^* = u_{h^*,N^*} \) to (1.1) was computed with \( h^* = 2^{-10} \) and \( N^* = (1/h^*)^2 = 2^{20} \).

The reference solutions for \( \sigma \in \{1,2\} \) are depicted in Figure 1a. The solution to the nominal problem (6.3) and those to the “expected diffusion” problems with \( h = h^* \) and \( \sigma \in \{1,2\} \) are depicted in Figure 1b. The convergence rates depicted in Figure 2a for \( \sigma = 1 \) are close to the theoretical rates provided by (1.10) and (6.2). For \( \sigma \in \{1,2\} \), Figures 2b and 2c depict 48 samples, and the empirical mean and Luxemburg norm of
\[
\| \nabla \hat{F}_{h,N}(\mathcal{I}u^*) - \nabla F(u^*) \|_{L^2(D)},
\]
which approximates \( \| \nabla \hat{F}_{h,N}(\mathcal{I}u^*) - \nabla F(u^*) \|_{L^2(D)} \) in (5.6). The convergence rates depicted in Figures 2b and 2c are close to the expected rates. Since the term in (6.6) does not depend on the SAA solutions \( u_{h,N}^* \), it is computationally cheaper to evaluate than \( \| u_{h,N}^* - u^* \|_{L^2(D)} \).

### 6.2. Two-dimensional equation

We consider an instance of problem (1.1) with \( d = 6 \), \( \alpha = 0.001 \), \( \gamma = 0.01 \), \( \Gamma = -6 \), \( u = 6 \), \( y_d(x) = -1 \) if \( x \in (1/4,3/4)^2 \) and \( y_d(x) = 1 \) otherwise (cf. [68, p. 49]). We chose \( \Xi = [-1,1]^d \) and the random diffusion coefficient (cf. [45, eq. (8.1)])
\[
\kappa(\xi)(x) = \exp \left( \xi_1 \cos(1.1\pi x_1) + \xi_2 \cos(1.2\pi x_1) + \xi_3 \sin(1.3\pi x_2) + \xi_4 \sin(1.4\pi x_2) \right).
\]
The random variables \( \xi_1, \ldots, \xi_4 \) are independent, each with \([-1,1] \)-uniform distribution. The small number of random variables allows us to obtain an accurate reference solution to (1.1). For our setting, Assumptions 3.1–3.4 hold true. The probability distribution of \( \xi \) was approximated by a discrete uniform distribution supported on the
grid points of a uniform mesh of \( \Xi \) using 12 grid points in each direction, yielding a discrete distribution with \( 144^2 = 20736 \) scenarios. Samples for the SAA problems were generated from this discrete distribution. The reference solution \( u^* = u_{h^*, N^*}^* \) to (1.1) was computed with \( h^* = 1/144 \) and \( N^* = 144^2 \).

Figure 3 depicts the reference solution, the solution to the nominal problem (6.3) and that to the “expected diffusion” problem (6.4) with \( h = h^* \). The controls depicted in Figures 3b and 3c differ from the reference solution depicted in Figure 3a.

Figure 4a illustrates the theoretical bounds (1.10) and (6.2) empirically. The convergence rates shown in Figure 4 were computed using least squares. We used the mesh widths \( h \in \{1/8, 1/12, 1/16, 1/24, 1/36, 1/48, 1/72\} \) with corresponding sample sizes \( N \in \{8^2, 12^2, \ldots, 72^2\} \) to perform the simulations depicted in Figure 4a. While these mesh widths are not equidistant on a binary logarithmic scale, they are multiples of 1/144, which facilitates the computation of the errors \( \|u_{h, N}^* - u^*\|_{L^2(D)} \). The convergence rates depicted in Figure 4a are close to the theoretical rates. For a fixed mesh width \( h \), Figure 4b depicts the error’s empirical Luxemburg norm and mean as a function of the sample size \( N \). Moreover, for a fixed sample size \( N \), Figure 4c shows the error’s Luxemburg norm and mean as a function of the inverse mesh width \( 1/h \). The simulation output depicted in Figure 4 may suggest that small errors cannot be obtained by keeping either the mesh width or the sample size fixed.

7. Discussion

The numerical solution of risk-neutral optimal control problems governed by PDEs with random inputs require both approximations of expected values and space discretizations. We have applied the SAA method and a Galerkin-type approximation to a risk-neutral linear elliptic PDE-constrained optimization problem and derived an exponential tail bound for the distance between the solution to the finite dimensional SAA
problem and that to the risk-neutral program. The exponential tail bound implies that SAA solutions are close to the risk-neutral problem’s solution with high probability. The bound can be used to balance errors caused by space discretizations and Monte Carlo sample-based approximations. Our main tool for deriving the exponential tail bound is a large deviation-type bound derived in [56, 57]. We performed numerical simulations illustrating our theoretical error bound.

We have derived reliable error estimates for a basic model problem with $L^1(D)$-norm control regularization. Our derivation of the reliable error estimate (1.6) relies on Assumption 3.1 (b), which is violated for lognormal random fields. For a lognormal field, we conjecture that an exponential tail bound similar to that in (1.6) does not hold. However, even for lognormal fields, the estimate (5.4) can be used to establish bounds on $\mathbb{E}[\|u^*_h - u^*\|^2_{L^2(D)}]$ in terms of the sample size $N$ and discretization parameter $h$. Our results may be extended in several ways. For example, rather than discretizing the control space using finite element functions, the variational discretization approach [31] can be considered. In order to numerically solve the discretized SAA problems, we implicitly assumed that bilinear forms in the discretized PDEs can be computed exactly. However, without structural assumptions on $\kappa$, the random diffusion coefficient $\kappa$ must be approximated using, for example, projection or interpolation. Furthermore, our error analysis exploits that realizations of the random diffusion coefficient are continuously differentiable. The error analysis may be extended to allow for rougher random diffusion coefficients, but the convergence with respect to the space discretization may then be slower. Moreover, our error analysis relies on the linearity of the elliptic

Figure 2: For the problem considered in section 6.1, realizations of the error $\|u^*_h - u^*\|_{L^2(D)}$ and their empirical mean and Luxemburg norm as a function of the inverse mesh width $1/h$ with $N = (1/h)^2$ and $\sigma = 1$ (a), and realizations of $\|\nabla F_h^*(I_hu^*) - \nabla F_h^*(u^*)\|_{L^2(D)}$ and their empirical mean and Luxemburg norm as a function of the inverse mesh width $1/h$ with $N = (1/h)^2$ and $\sigma = 1$ (b) and $\sigma = 2$ (c). Here $u^* = u^*_{h^*,N^*}$ with $h^* = 2^{-10}$ and $N^* = 2^{20}$ is the reference solution, and $\nabla F_h^*(u^*)$ is the reference gradient. For subfigures (b) and (c), the first empirical estimates were excluded from the computation of the least squares fits. The parameter $\sigma$ refers to that used in the random field (6.5).
We show that the exponential tail bound (1.6) implies the Luxemburg norm bound

\[ E[\|u_h^* - u^*\|_{L^2(D)}] = \int_0^\infty \text{Prob}(\|u_h^* - u^*\|_{L^2(D)} \geq r)dr. \]

Combined with (1.6), \( h, c_1, c_2 > 0 \) and \( 2 \int_0^\infty \exp(-x^2/(2\sigma^2))dx = \sqrt{2\pi\sigma^2} \) with \( \sigma^2 = c_2^2/N > 0 \), we find that

\[ E[\|u_h^* - u^*\|_{L^2(D)}] \leq c_1 h + \int_0^\infty \text{Prob}(\|u_h^* - u^*\|_{L^2(D)} \geq c_1 h + r)dr \leq c_1 h + \int_0^\infty 2\exp(-r^2N/(2c_2^2))dr = c_1 h + c_2 \sqrt{2\pi}/\sqrt{N}. \]

Hence the exponential tail bound (1.6) implies the expectation bound (1.10).

**B. Tail bound implies bound on Luxemburg norm**

We show that the exponential tail bound (1.6) implies the Luxemburg norm bound provided in (6.2) and as a consequence we obtain bounds on all finite moments of \( \|u_h^* - u^*\|_{L^2(D)} \). Rewriting (1.6) and defining \( Z = \|u_h^* - u^*\|_{L^2(D)} \), we have

\[ \text{Prob}(|Z| \geq c_1 h + (c_2/\sqrt{N})\varepsilon) \leq 2\exp(-\varepsilon^2/2) \quad \text{for all} \quad \varepsilon > 0. \]  

We have \( h, c_1, c_2 > 0 \). If \( \varepsilon \geq 1 \), then \( |Z| \geq c_1 h \varepsilon + (c_2/\sqrt{N})\varepsilon \) yields \( |Z| \geq c_1 h + (c_2/\sqrt{N})\varepsilon \). We also have \( 2\exp(-\varepsilon^2/2) > 1 \) for all \( \varepsilon \in [0,1] \). Hence (B.1) implies

\[ \text{Prob}(|Z| \geq (c_1 h + (c_2/\sqrt{N})\varepsilon)\varepsilon) \leq 2\exp(-\varepsilon^2/2) \quad \text{for all} \quad \varepsilon > 0. \]
(b) Fixed mesh width $h$.

Figure 4: For the problem considered in section 6.2, realizations of the error $\|u^*_{h,N} - u^*\|_{L^2(D)}$ and their empirical mean and Luxemburg norm as a function of the inverse mesh width $1/h$ with sample size $N = (1/h)^2$ (a), of the sample size for fixed mesh width (b), and of the inverse mesh width for fixed sample size (c). Here $u^* = u^*_{h,N}$ is the reference solution with $h^* = 1/144$ and $N^* = 144^2$. For subfigures (b) and (c), the last three empirical estimates were excluded from the computation of the least squares fit.

Combined with [11, Thm. 3.4 on p. 56] (used with $\varphi(x) = x^2$ [11, pp. 42 and 55], $C = 2$ and $D = \sqrt{2}(c_1 h + (c_2/\sqrt{N}))$) and eq. (6.1), we obtain the bound

$$\|u^*_{h,N} - u^*\|_{L^p(\Omega;L^2(D))} = \|Z\|_{L^p(\Omega;\mathbb{R})} \leq 3\sqrt{2}(c_1 h + (c_2/\sqrt{N})).$$

Hence the exponential tail bound (1.6) yields the Luxemburg norm bound in (6.2). Using [11, Lem. 3.4 on p. 58], we further have $E[\exp(\lambda|Z|)] \leq 2\exp(\lambda^2 q^2/4)$ for all $\lambda \in \mathbb{R}$, where $q = 3\sqrt{2}(c_1 h + (c_2/\sqrt{N}))$. Combined with the computations in [11, p. 7] (used with $\tau = q/\sqrt{2}$),

$$E[\|u^*_{h,N} - u^*\|_{L^2(D)}^p] \leq 3p^2(p/e)^{p/2}(c_1 h + (c_2/\sqrt{N}))^p \quad \text{for all} \quad p > 0. \quad (B.2)$$

C. Higher regularity of a proximity operator

We establish a higher regularity result of a certain proximity operator, which is essentially known [10, 13, 18, 25, 43, 66, 69]. It is used in Lemma 5.6 to establish higher regularity of the solution to (1.1). For $a, b \in L^2(D)$, we define $[a, b] = \{ u \in L^2(D) : a \leq u \leq b \text{ a.e. in } D \}$.

Lemma C.1. Let $D \subset \mathbb{R}^d$ be a bounded Lipschitz domain, and let $a, b \in H^1(D) \cap L^\infty(D)$ fulfill $a \leq b$ a.e. in $D$. We define $\psi : L^2(D) \to [0, \infty]$ by $\psi(v) = \mu \|v\|_{L^1(D)} + I_{[a,b]}(v)$ with $\mu \in [0, \infty)$. Let $\text{prox}_\psi : L^2(D) \to L^2(D)$ be the proximity operator of $\psi$ as defined in (2.1). Then $\text{prox}_\psi(v) \in H^1(D)$ for all $v \in H^1(D)$ and

$$\|\text{prox}_\psi(v)\|_{H^1(D)} \leq \|a\|_{H^1(D)} + \|b\|_{H^1(D)} + |v|_{H^1(D)} \quad \text{for all} \quad v \in H^1(D).$$

We prove Lemma C.1 using Lemma C.2. If $a, b \in L^2(D)$ with $a \leq b$ a.e. in $D$, then $\text{prox}_{[a,b]}$ equals the projection operator onto $[a, b]$.
Lemma C.2. Let hypotheses of Lemma C.1 hold, and let $v, w \in H^1(D)$. Then

(a) $\text{prox}_{I_{[a,b]}} (w) \in H^1(D)$ and $|\text{prox}_{I_{[a,b]}} (w)|^2_{H^1(D)} \leq |a|^2_{H^1(D)} + |b|^2_{H^1(D)} + |w|^2_{H^1(D)}$.

(b) $v - \text{prox}_{I_{[-\mu,\mu]}} (v) \in H^1(D)$ and $|v - \text{prox}_{I_{[-\mu,\mu]}} (v)|_{H^1(D)} \leq |v|_{H^1(D)}$.

Proof. In this proof, we omit writing evaluations at $x \in D$.

(a) We have $\text{prox}_{I_{[a,b]}} (w) \in H^1(D)$ [66, pp. 114–115]. Since $\text{prox}_{I_{[a,b]}} (w) = a$ if $w < a$, $\text{prox}_{I_{[a,b]}} (w) = w$ if $w \in [a, b]$, and $\text{prox}_{I_{[a,b]}} (w) = b$ otherwise, we obtain the estimate.

(b) We have $\text{prox}_{I_{[-\mu,\mu]}} (v) \in H^1(D)$ [66, pp. 114–115] and hence $v - \text{prox}_{I_{[-\mu,\mu]}} (v) \in H^1(D)$. Combined with $v - \text{prox}_{I_{[-\mu,\mu]}} (v) = v + \mu$ if $v < -\mu$, $v - \text{prox}_{I_{[-\mu,\mu]}} (v) = 0$ if $v \in [-\mu, \mu]$, and $v - \text{prox}_{I_{[-\mu,\mu]}} (v) = v - \mu$ otherwise, we obtain the estimate. \hfill \Box

Proof of Lemma C.1. The set $[a, b] \subset L^2(D)$ is nonempty, convex and closed [66, pp. 116–117]. Hence the indicator function $I_{[a,b]}$ is proper, convex and lower semicontinuous [9, Ex. 2.115]. Combined with the boundedness of $D$ and Hölder’s inequality, we find that $v \in L^1(D)$ and $\|v\|_{L^1(D)} \leq \|1\|_{L^1(D)} \|v\|_{L^2(D)} < \infty$ for each $v \in L^2(D)$. Hence $\psi$ is proper, convex and lower semicontinuous. Fix $v \in H^1(D)$. Since $L^2(D)$ is decomposable [59, p. 677] and $\text{prox}_{\psi}(v)$ is well-defined [8, p. 211], the theorem on the interchange of minimization and integration [59, Thm. 14.60] ensures for almost every $x \in D$,

$$\text{prox}_{\psi}(v)(x) = \text{prox}_{\mu|\cdot|+I_{[a(x),b(x)]}} (v(x)).$$

Similarly, for each $w \in L^2(D)$, we obtain for almost every $x \in D$,

$$\text{prox}_{I_{[-\mu,\mu]}} (w)(x) = \text{prox}_{I_{[-\mu,\mu]}} (w(x)) \text{ and } \text{prox}_{I_{[a(x),b(x)]}} (w(x)) = \text{prox}_{I_{[a(x),b(x)]}} (w)(x).$$

Since $\text{prox}_{\mu|\cdot|+I_{[t_1,t_2]}} (t_3) = \text{prox}_{I_{[t_1,t_2]}} (t_3 - \text{prox}_{I_{[-\mu,\mu]}} (t_3))$ for all $t \in \mathbb{R}^3$ [51, Ex. 3.2.9], we have

$$\text{prox}_{\psi}(v) = \text{prox}_{I_{[a,b]}} (v - \text{prox}_{I_{[-\mu,\mu]}} (v)). \quad (C.1)$$

Combined with Lemma C.2, it follows that $\text{prox}_{\psi}(v) \in H^1(D)$ and

$$|\text{prox}_{\psi}(v)|^2_{H^1(D)} \leq |a|^2_{H^1(D)} + |b|^2_{H^1(D)} + |v|^2_{H^1(D)}. \quad (C.2)$$

Using (C.1), we have $\text{prox}_{\psi}(v) \in [a, b]$. Since $a(x) \leq \text{prox}_{\psi}(v)(x) \leq b(x)$ ensures the estimate $|\text{prox}_{\psi}(v)(x)|^2 \leq b(x)^2 + a(x)^2$, we obtain

$$\|\text{prox}_{\psi}(v)\|^2_{L^2(D)} \leq \|a\|^2_{L^2(D)} + \|b\|^2_{L^2(D)}.$$

Combined with (C.2) and $(\rho_1 + \rho_2)^{1/2} \leq \rho_1^{1/2} + \rho_2^{1/2}$ valid for all $\rho_1, \rho_2 \geq 0$, we obtain the stability estimate. \hfill \Box

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