Bounds on the decay of the auto-correlation in phase ordering
dynamics.

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Abstract

We obtain bounds on the decay exponent $\lambda$ of the autocorrelation function in phase ordering dynamics (defined by $\lim_{t_2 \gg t_1} \langle \phi(r, t_1) \phi(r, t_2) \rangle \sim L(t_2)^{-\lambda}$). For non-conserved order parameter, we recover the Fisher and Huse inequality, $\lambda \geq d/2$. If the order parameter is conserved we also find $\lambda \geq d/2$ if $t_1 = 0$. However, for $t_1$ in the scaling regime, we obtain $\lambda \geq d/2 + 2$ for $d \geq 2$ and $\lambda \geq 3/2$ for $d = 1$. For the one-dimensional scalar case, this, in conjunction with previous results, implies that $\lambda$ is different for $t_1 = 0$ and $t_1 \gg 1$. In 2-dimensions, our extensive numerical simulations for a conserved scalar order parameter show that $\lambda \approx 3$ for $t_1 = 0$ and $\lambda \approx 4$ for $t_1 \gg 1$. These results contradict a recent conjecture that conservation of order parameter requires $\lambda = d$. Quenches to and from the critical point are also discussed.

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Phase separation dynamics proceeds when a system is quenched from its high temperature, homogeneous phase to a low temperature, inhomogeneous phase (where several phases coexist in equilibrium). Due to its simple description yet rich behavior, phase ordering dynamics has greatly enhanced our understanding of non-equilibrium processes [1]. At late times, the spatial distribution of domains can be described by a single time-dependent length, $L(t)$ which typically grows algebraically in time, $L(t) \sim t^{1/z}$. This results in a scale invariant equal-time correlation function $C(r, t)$. More recently it has been realized that the unequal-time correlation function is scale covariant. In particular, the asymptotic decay of the two-time autocorrelation function, $C(r, t_1, t_2) = \langle \phi(r, t_1)\phi(0, t_2) \rangle$ defines an independent exponent $\lambda$, via $\lim_{t_1 \ll t_2} C(0, t_1, t_2) \sim (L(t_1)/L(t_2))^\lambda$. This exponent bears no apparent relation to the growth exponent $z$ and so its value provides a sensitive test for approximate theories of phase ordering kinetics [2–8]. Although the autocorrelation function has been studied extensively for non-conserved order parameter dynamics [2,4–10], there has been hardly any work on conserved dynamics. However in a recent Letter, Majumdar et al. have shown numerically and analytically that that $\lambda = 1$ for $m = 1, d = 1$ and $t_1 = 0$ where $m$ is the number of components in the order parameter [11]. It has been further argued [11,12] that the conservation of order parameter demands that $\lambda = d$ for all $m$.

In this Letter, we obtain lower bounds on the decay exponent $\lambda$. For non-conserved order parameters, $\lambda \geq d/2$ independent of $t_1$, consistent with a general argument of Fisher and Huse [3]. For conserved order parameters, we also obtain $\lambda \geq d/2$ for $t_1 = 0$ (assuming the quench is from a high temperature phase). However, for $t_1$ in the scaling regime, we find that $\lambda \geq d/2 + 2$ for $d \geq 2$ and $\lambda \geq 3/2$ for $d = 1$. This difference arises from the small $k$ behavior of the scattering intensity $S(k, t_1)$. In conjunction with the exact result for $\lambda$, for the 1-dimensional scalar model, with $t_1 = 0$ [11], we conclude that for $d = 1$, $\lambda$ depends on whether $t_1 = 0$ or $t_1 \gg 1$. To carry out the investigation in higher dimensions, we perform an extensive numerical integration of the Cahn-Hilliard equation (see Eq. (4) below) in $d = 2$. We find that $\lambda \approx 3$ for $t_1 = 0$ and $\lambda \approx 4$ for $t_1$ in the scaling regime. This is inconsistent with the recent conjecture that $\lambda = d$ [11,12]. We discuss why this conjecture
fails. We also derive bounds on $\lambda$ for quenches to and from the critical point. Our results easily extend to vector order parameters.

We begin by obtaining the lower bounds on $\lambda$. The equal point auto-correlation $C(t_1, t_2) \equiv C(0, t_1, t_2)$ is related to the $k$ space auto-correlation $S(k, t_1, t_2)$ by

$$C(t_1, t_2) = \int d\mathbf{k} \langle \delta \phi_k(t_1) \delta \phi_{-k}(t_2) \rangle = \int d\mathbf{k} S(k, t_1, t_2).$$

Here $\phi(r, t)$ is the order parameter at point $r$ and time $t$ and $\delta \phi(r, t) \equiv \phi(r, t) - m_0$ with $m_0 = V^{-1} \int d\mathbf{r} \phi(r, t)$ and the fourier transform $\delta \phi_k(t) \equiv V^{-1/2} \int d\mathbf{r} e^{-i\mathbf{k} \cdot \mathbf{r}} \delta \phi(r, t)$. The angular brackets indicate an average over initial conditions. For a critical quench $\langle m_0 \rangle = 0$ and $\langle m_0^2 \rangle$ is $O(V^{-1})$, whereas for an off-critical quench $\langle m_0 \rangle$ is $O(1)$.

Using the Cauchy-Schwartz inequality, we find

$$C(t_1, t_2) \leq \int d\mathbf{k} \langle \delta \phi_k(t_1) \delta \phi_{-k}(t_1) \rangle^{1/2} \langle \delta \phi_k(t_2) \delta \phi_{-k}(t_2) \rangle^{1/2},$$

$$\sim \int d\mathbf{k} S(k, t_1)^{1/2} S(k, t_2)^{1/2},$$

(1)

where $S(k, t) = S(k, t, t)$.

Now assume $t_2$ to be in the scaling regime with $t_2 > t_1$. At late times, the scattering is due to the sharp interfaces or defects. The $k$ modes, $\delta \phi_k$ at times $t_1$ and $t_2$ will be uncorrelated when the interfaces move a distance greater than $2\pi/k$ so that $S(k, t_1, t_2)$ decreases rapidly for $k (L(t_2) - L(t_1)) \gg 1$. The upper limit of the integral over $k$ in Eq. (1) can then be cut off at $2a\pi/L(t_2)$ where $a$ is a constant of $O(1)$ \textsuperscript{[13]}. For $L(t_2) \gg L(t_1)$, only the small $k$ behavior of $S(k, t_1)$ contributes to the integral. Assume that $\lim_{k \to 0} S(k, t_1) \sim k^\beta (\beta \geq 0)$. For quenches to zero temperature, $S(k, t_2)$ will have the scaling form $S(k, t_2) = L(t_2)^d f(kL(t_2))$. Substituting into Eq. (1) (with the appropriate limits of integration), gives

$$\lim_{t_2 \gg t_1} C(t_1, t_2) \sim L(t_2)^{-\lambda} \leq L(t_2)^{d/2} \int_0^{2a\pi/L(t_2)} dk \int_0^{(d+\beta)/2} \int dk^d f(kL(t_2)),$$

$$\sim L(t_2)^{-(d+\beta)/2}.$$

This immediately gives a lower bound on $\lambda$,

$$\lambda \geq \frac{\beta + d}{2}.\]
The argument just presented, holds for conserved and nonconserved, scalar and vector order parameters.

We now consider specific dynamical scenarios. Let $T_I$ and $T_F$ be the temperatures of the initial and final states respectively. We first focus on quenches from the high temperature phase ($T_I = \infty$) to zero temperature ($T_F = 0$). Since the initial state is disordered, $\lim_{k \to 0} S(k,0) \sim k^0$. In the absence of a conservation law, $\lim_{k \to 0} S(k,t_1) \sim k^0$ for both $t_1 = 0$ and $t_1$ in the scaling regime. Therefore $\beta = 0$ and

$$\lambda \geq d/2.$$  \hfill (2)

This inequality was also obtained by Fisher and Huse using general scaling arguments \cite{2} and is consistent with all results to date \cite{1,4,5,7,8,9}. For $t_1 = 0$, conservation of the order parameter does not affect this inequality since $\beta = 0$ for $t_1 = 0$. However, if $t_1$ is in the scaling regime, then $\lim_{k \to 0} S(k,t) \sim k^4$ for $d \geq 2$ \cite{14} and $\beta = 4$. For $d = 1$, the dynamics is dominated by noise and Majumdar et al. find that $\lim_{k \to 0} S(k,t) \sim k^2$ \cite{11}, so that $\beta = 2$ for $d = 1$. Therefore, for $t_1$ in the scaling regime,

$$\lambda \geq \begin{cases} \frac{d}{2} + 2 & \text{if } d \geq 2, \\ \frac{3}{2} & \text{if } d = 1. \end{cases}$$  \hfill (3)

These bounds suggest that the asymptotic exponent may depend on whether $t_1$ is, or is not in the scaling regime but do not rule out that the exponent is independent of $t_1$. However for $d = 1$, Majumdar et al. find analytically and numerically that $\lambda = 1$ for $t_1 = 0$, while we find that $\lambda \geq 3/2$ for $t_1$ in the scaling regime \cite{15}.

For vector fields (with $m$, the number of components of the order parameter, $> 2$), an argument analogous to Ref. \cite{14}, gives the same $\lim_{k \to 0} S(k,t) \sim k^4$. This is supported by an extensive numerical integration of the Cahn-Hilliard equation \cite{16}. Therefore the lower bounds on $\lambda$ derived above are valid even for vector order parameters with $m > 2$.

Quenches from the critical point ($T_I = T_c, T_F = 0$) lead to long-range correlations of the initial configurations. In this case, $\lambda \geq d/2$ no longer holds. More generally if $S(k,0) \sim k^{-\sigma}$ we obtain $\lambda \geq (d - \sigma)/2$. (For critical dynamics $\sigma = 2 - \eta$ where $\eta$ is the static critical
exponent). This is consistent with the result of Bray et al. who found that, for nonconserved order parameter, \( \lambda = (d - \sigma)/2 \) for \( \sigma \) greater than a critical value \( \sigma_c \) [6].

Analysis of the bounds on the autocorrelation exponent for quenches to the critical point \((T_I = \infty, T_F = T_c)\), has to start afresh from Eq. (I). Since \( t_2 \) is in the critical point scaling regime, the correlation function has the following scaling form, \( S(k; t_2) \sim k^{-2+\eta}f_c(kL(t_2)) \). Substituting this form into Eq. (I) gives \( \lambda \geq (2d - 2 + \eta + \beta)/2 \). Therefore when \( t_1 = 0 \) we get \( \lambda \geq (2d - 2 + \eta)/2 \). When \( t_1 \) is also in the scaling regime, the bound on \( \lambda \) depends on the behaviour of the scaling function \( f_c(kL(t_1)) \) as \( kL(t_1) \to 0 \). For nonconserved systems \( \lim_{x \to 0} f_c(x) \to \text{const.} \), or \( \beta = -2 + \eta \) leading to \( \lambda \geq d - 2 + \eta \).

These lower bounds on \( \lambda \) of course do not fix the value of the exponent. As previously mentioned, exact analytical and numerical computations on the 1-dimensional scalar model have been carried out for the case when \( t_1 = 0 \). In higher dimensions, however, the empirical results are not very conclusive [7]. We therefore compute the asymptotic value of \( \lambda \) by numerically integrating the Cahn-Hilliard equation in two-dimensions,

\[
\frac{\partial \phi(r, t)}{\partial t} = \nabla^2 \mu(r, t),
\]

where \( \mu = -\phi + \phi^3 - \nabla^2 \phi \). We used an Euler discretization with \( \delta t = 0.1 \) and \( \delta x = 1.09 \) and periodic boundary conditions. We discretize the Laplacian as

\[
\nabla^2 \phi_{i,j} = \frac{1}{\delta x^2} \frac{\sqrt{2}}{1 + \sqrt{2}} \left[ \frac{1}{2} \sum_{n.n.n.} + \sum_{n.n.} - 6 \right] \phi_{i,j}.
\]

This choice decreases lattice anisotropy effects and allows a larger \( \delta t \) before the onset of the checkerboard instability [8]. This dynamical equation is solved subject to random initial conditions which are uncorrelated and uniformly distributed between \(-0.05\) and \(0.05\) (the initial state is disordered). Decreasing \( \delta t \) has no effect on the numerical results. Increasing \( \delta x \) to \(1.32\) results in pinning effects which lead to a slower decay of the autocorrelation function at late times (even though the effect on the single-time behavior is less apparent).

We have used the interfacial area density as a measure of the characteristic lengthscale \( L(t) \). Operationally, this is defined as \((2 \delta x n_x n_y)/n_{opp}\), where \( n_x n_y \) is the total number of
lattice sites and $n_{\text{opp}}$ is the number of sites with a nearest neighbor with $\phi$ of opposite sign. We recover the standard result that $L(t)$ grows as $t^{1/3}$ for all $t > 400$. Other measures of the characteristic lengthscale, such as the first zero of the real space correlation function, also behave in the same manner (for $t > 400$).

Fig. 1 shows $C(t_1, t)$ vs. $L(t)$ for $t_1 = 0$ and $t$ between 100 and 12800 for three lattice sizes $n = n_x = n_y = 64$ (3084 initial conditions), $n = 256$ (1120 initial conditions) $n = 1024$ (42 initial conditions). Concentrating on the largest lattice size, $C(0, t) \sim L^{-3.7}$ for approximately two decades of $t$ in its decay. There is a crossover to a slower decay at late times with $C(0, t) \sim L^{-3.0}$. At extremely late times there is an indication of an even slower decay, which we attribute to finite size effects.

To emphasize the asymptotic trend, Fig. 2 shows $L^3 C(0, t)$ vs. $L(t)$ for the same data. Here it is clearer that the slower late time decay occurs at earlier times for smaller lattices, indicating finite size effects. The importance of the finite size effects was initially surprising since, for single time quantities, finite size effects only become important when $L(t)$ is of order of the lattice dimension, $L_0$. Thus the usual length scales extracted from single-time quantities were identical for $n = 256$ and $n = 1024$. However, since $C(0, t)$ decays rapidly with $t$, any small systematic effect becomes increasingly relevant as $t$ increases. Clearly finite size effects on $C(0, t)$ can be important (though not necessarily so) when the spread in $C(0, t)$ is of the same order as $C(0, t)$. The spread in $C(0, t)$ decreases as $L_0^{-d/2}$ and, based on our simulations, depends only weakly on $L(t)$. Hence finite size effects can become important when $C(0, t) \sim L(t)^{-\lambda} \sim L_0^{-d/2}$, i.e., much earlier than for single-time quantities.

From Fig. 3, $C(0, t)$ for $n = 64$ first shows significant differences from the $n = 256$ result at $C \approx 2.5 \times 10^{-4}$ or $L \approx 18$. Thus we expect the $n = 1024$ data to be free of finite size effects down to $C \approx 2.5 \times 10^{-4}/16 \approx 1.5 \times 10^{-5}$ or $L \approx 50$. Therefore we make the preliminary conclusion that the true asymptotic value of $\lambda$ is approximately 3. Finally note that the late time result for $n = 256$ is consistent with $\lambda = d = 2$. However, comparing this with the result for $n = 1024$ indicates that this regime is due to the finite size of the lattice.

Fig. 3 shows $C(t_1, t_2)$ vs. $L(t_2)/L(t_1)$ for $t_1 = 100, 200$ and 400. Although we cannot rule
out a further slower decay, our result is consistent with $\lambda \approx 4$ for $t_1$ in the scaling regime. Hence, we find that, for the $d = 2$, conserved, scalar model, $\lambda \approx 3$ for $t_1 = 0$ and $\lambda \approx 4$, for $t_1$ in the scaling regime. To pin down these values more precisely would require simulations on larger lattices and for larger values of $t_1$ and $t_2$.

Our numerical results for $d = 2$ and the lower bound on $\lambda$ for $t_1$ in the scaling regime (Eq. (3)) are inconsistent with the recent conjecture (Refs. [11][12]) that conservation of order parameter requires that $\lambda = d$ in all cases. We believe the apparent inconsistency is because the argument leading to $\lambda = d$, incorrectly applies a scaling analysis to the non-scaling, $k = 0$ mode. To be explicit, we briefly review the argument of Ref. [11] (the argument presented in Ref. [12] is similar). The Cahn-Hilliard equation (Eq. (4)) in $k$ space is

$$\frac{\partial \phi_k(t)}{\partial t} = D(k, t),$$

where $D(k, t) = k^2 \mu_k(t)$. Define $\tilde{S}(k, 0, t) \equiv \langle \phi_k(0) \phi_{-k}(t) \rangle = S(k, 0, t) + V \delta_{k, 0} \langle m_0^2 \rangle$. The formal solution to $\tilde{S}(k, 0, t)$ is

$$\tilde{S}(k, 0, t) = \tilde{S}(k, 0, 0) \exp \left( \int_0^t dt' \frac{\gamma(L(t'), kL(t'))}{t'} \right),$$

where $\gamma(L, kL(t)) \equiv t \langle D(k, t) \phi_{-k}(t) \rangle / \tilde{S}(k, 0, t)$. In the scaling regime, $\gamma(L, kL) = \gamma(kL)$ and we obtain (in the limit $t \gg 0$),

$$\int_0^t dt' \frac{\gamma(kL(t'))}{t'} = z \int_{kL(0)}^{kL(t)} dx \frac{\gamma(x)}{x},$$

$$= z \gamma(0) \log \left( \frac{L(t)}{L(0)} \right) + z \gamma_1(kL(t)),$$

where $L(t) \sim t^{1/z}$ and $\gamma_0 = \lim_{x \to 0^+} \gamma(x)$. The result is

$$\tilde{S}(k, 0, t) = \tilde{S}(k, 0, 0) \left( \frac{L(t)}{L(0)} \right)^{\gamma_0 z} F_1(kL(t)).$$

Since $\int d\mathbf{k} \tilde{S}(k, 0, t) = \langle \phi(r, 0) \phi(r, t) \rangle \sim L(t)^{-\lambda} = L(t)^{-d+\gamma_0 z}$ or $\lambda = d - \gamma_0 z$. However, since $\tilde{S}(k = 0, 0, t)$ is constant in time (conservation law!), it is argued that $\gamma_0$ must vanish and hence $\lambda = d$.

Our contention is that, even though the relation $\lambda = d - \gamma_0 z$ with $\gamma_0 = \lim_{x \to 0^+} \gamma(x)$ holds, the conclusion that, due to conservation of order parameter, $\gamma_0$ necessarily vanishes
does not. This is because the relation $\lambda = d - \gamma_0 z$ is based on scaling, which only holds for $k > 0$, while the vanishing of $\gamma_0$ is based on the conservation of order parameter which only holds for $k = 0$. Therefore the conclusion $\gamma_0 = 0$ is invalid since it applies a scaling argument to the nonscaling $k = 0$ behavior.

To see this more clearly, consider the quasi-static scattering intensity $\tilde{S}(k, t) = \tilde{S}(k, t, t)$. For large $t$, $\tilde{S}$ is given by $\tilde{S}(k, t) = L(t)^d f(kL(t)) + V \delta_{k,0} \langle m_0^2 \rangle$. Global conservation of order parameter requires that $\tilde{S}(k = 0, t) = \text{constant}$, while the locally conservative dynamics requires that $f(0) = 0$. Hence global conservation leads to a discontinuity at $k = 0$ for both $\tilde{S}(k, t_1, t_2)$ and $\gamma(L, kL)$ [19]. The singularities in $\gamma(x)$ and $\tilde{S}(k, t_1, t_2)$ at $k = 0$ can be removed by choosing the initial distribution so that $\langle m_0^2 \rangle = 0$. However, in this case, $\tilde{S}(0, t_1, t_2)$ vanishes independent of the value of $\gamma_0$, so that the application of the conservation law does not fix the value of $\gamma_0$. (A. J. Bray has made a similar argument proving $\lambda = d$ does not necessarily hold [20]).

Having provided useful lower bounds on $\lambda$, we now ask whether it is possible to bound $\lambda$ from above? Unfortunately, we have not been able to provide useful upper bounds. However, we note that the bound (Eq. (3)) as well as our numerical results violate the upper bound conjectured by Fisher and Huse, $\lambda \leq d$ [2]. As they originally noted, this conjecture contains many assumptions. Moreover, in as much as their argument is aimed at the decay of the magnetization, their conjecture has validity only when the order parameter is not conserved and when $t_1 = 0$ (so that $S(k, t_1) \sim k^0$).

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FIGURES

FIG. 1. The autocorrelation function $C(0,t)$ for $n = 64$ (3084 initial conditions), 256 (1120 initial conditions) and 1024 (42 initial configurations). Lines corresponding to $C(0,t) \sim L^{-3}$ and $C(t) \sim L^{-2}$ are shown for comparison.

FIG. 2. $L^3C(0,t)$ vs. $L(t)$ for same data as Fig. 1. Lines corresponding to $C(0,t) \sim L^{-3}$ and $C(t) \sim L^{-2}$ are shown for comparison. These results emphasize that the slower decay at late times is due to finite size effects.

FIG. 3. $C(t_1,t_2)$ vs. $L(t_2)/L(t_1)$ for $n = 1024$ and $t_1 = 100, 200$ and 400. Lines corresponding to $C(t_1,t_2) \sim L^{-4}$ and $C(t_1,t_2) \sim L^{-2}$ are shown for comparison.
Bounds on the decay of the auto-correlation in phase ordering dynamics.

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We obtain bounds on the decay exponent $\lambda$ of the autocorrelation function in phase ordering dynamics (defined by $\lim_{t_1 \to\infty} \langle \phi(\mathbf{r}, t_1) \phi(\mathbf{r}, t_2) \rangle \sim L(t_1)^{-\lambda}$). For non-conserved order parameter, we recover the Fisher and Huse inequality, $\lambda \geq d/2$. If the order parameter is conserved we also find $\lambda \geq d/2$ if $t_1 = 0$. However, for $t_1$ in the scaling regime, we obtain $\lambda \geq d/2 + 2$ for $d \geq 2$ and $\lambda \geq 3/2$ for $d = 1$. For the one-dimensional scalar case, this, in conjunction with previous results, implies that $\lambda$ is different for $t_1 = 0$ and $t_1 \gg 1$. In 2-dimensions, our extensive numerical simulations for a conserved scalar order parameter show that $\lambda \approx 3$ for $t_1 = 0$ and $\lambda \approx 4$ for $t_1 \gg 1$. These results contradict a recent conjecture that conservation of order parameter requires $\lambda = d$. Quenches to and from the critical point are also discussed.

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Phase separation dynamics proceeds when a system is quenched from its high temperature, homogeneous phase to a low temperature, inhomogeneous phase (where several phases coexist in equilibrium). Due to its simple description yet rich behavior, phase ordering dynamics has greatly enhanced our understanding of non-equilibrium processes [1]. At late times, the spatial distribution of domains can be described by a single time-dependent length, $L(t)$ which typically grows algebraically in time, $L(t) \sim t^{1/2}$. This results in a scale invariant equal-time correlation function $C(r, t)$. More recently it has been realized that the unequal-time correlation function is scale covariant. In particular, the asymptotic decay of the two-time autocorrelation function, $C(r, t_1, t_2) = \langle \phi(\mathbf{r}, t_1) \phi(\mathbf{r}, t_2) \rangle$ defines an independent exponent $\lambda$, via $\lim_{t_1 \to\infty} C(0, t_1, t_2) \sim (L(t_1)/L(t_2))^\lambda$. This exponent bears no apparent relation to the growth exponent $z$ and so its value provides a sensitive test for approximate theories of phase ordering kinetics [2–8]. Although the autocorrelation function has been studied extensively for non-conserved order parameter dynamics [2,4–10], there has been hardly any work on conserved dynamics. However in a recent Letter, Majumdar et al. have shown numerically and analytically that for $\lambda = 1$ for $m = 1$, $d = 1$ and $t_1 = 0$ where $m$ is the number of components in the order parameter [11]. It has been further argued [11,12] that the conservation of order parameter demands that $\lambda = d$ for all $m$.

In this Letter, we obtain lower bounds on the decay exponent $\lambda$. For non-conserved order parameters, $\lambda \geq d/2$ independent of $t_1$, consistent with a general argument of Fisher and Huse [2]. For conserved order parameters, we also obtain $\lambda \geq d/2$ for $t_1 = 0$ (assuming the quench is from a high temperature phase). However, for $t_1$ in the scaling regime, we find that $\lambda \geq d/2 + 2$ for $d \geq 2$ and $\lambda \geq 3/2$ for $d = 1$. This difference arises from the small $k$ behavior of the scattering intensity $S(k, t_1)$. In conjunction with the exact result for $\lambda$, for the 1-dimensional scalar model, with $t_1 = 0$ [11], we conclude that for $d = 1$, $\lambda$ depends on whether $t_1 = 0$ or $t_1 \gg 1$.

To carry out the investigation in higher dimensions, we perform an extensive numerical integration of the Cahn-Hilliard equation (see Eq. (4) below) in $d = 2$. We find that $\lambda \approx 3$ for $t_1 = 0$ and $\lambda \approx 4$ for $t_1$ in the scaling regime. This is inconsistent with the recent conjecture that $\lambda = d$ [11,12]. We discuss why this conjecture fails. We also derive bounds on $\lambda$ for quenches to and from the critical point. Our results easily extend to vector order parameters.

We begin by obtaining the lower bounds on $\lambda$. The equal point auto-correlation $C(t_1, t_2) \equiv C(0, t_1, t_2)$ is related to the $k$ space auto-correlation $S(k, t_1, t_2)$ by

$$C(t_1, t_2) = \frac{1}{2\pi} \int d\mathbf{k} \delta \phi_k(t_1) \delta \phi_{-k}(t_2).$$

Here $\phi(\mathbf{r}, t)$ is the order parameter at point $\mathbf{r}$ and time $t$ and $\delta \phi(\mathbf{r}, t) \equiv \phi(\mathbf{r}, t) - m_0$ with $m_0 = V^{-1} \int d\mathbf{r} \phi(\mathbf{r}, t)$ and the Fourier transform $\delta \phi_k(t) \equiv V^{-1/2} \int d\mathbf{r} e^{-i\mathbf{k}\cdot\mathbf{r}} \delta \phi(\mathbf{r}, t)$. The angular brackets indicate an average over initial conditions. For a critical quench $\langle m_0 \rangle = 0$ and $\langle m_0^2 \rangle$ is $O(V^{-1})$, whereas for an off-critical quench $\langle m_0 \rangle$ is $O(1)$.

Using the Cauchy-Schwartz inequality, we find

$$C(t_1, t_2) \leq \frac{1}{2\pi} \int d\mathbf{k} \delta \phi_k(t_1) \delta \phi_{-k}(t_1)^{1/2} \delta \phi_k(t_2) \delta \phi_{-k}(t_2)^{1/2},$$

$$= \int d\mathbf{k} S(k, t_1)^{1/2} S(k, t_2)^{1/2}. \quad (1)$$

where $S(k, t) = S(k, t, t)$.

Now assume $t_2$ to be in the scaling regime with $t_2 > t_1$. At late times, the scattering is due to the sharp interfaces or defects. The $k$ modes, $\delta \phi_k$ at times $t_1$ and $t_2$
will be uncorrelated when the interfaces move a distance greater than $2\pi/k$ so that $S(k,t_1,t_2)$ decreases rapidly for $k(l(t_2) - L(t_1)) \gg 1$. The upper limit of the integral over $k$ in Eq. 1 can then be cut off at $2a\pi/L(t_1)$ where $a$ is a constant of $O(1)$ [13]. For $L(t_2) \gg L(t_1)$, only the small $k$ behavior of $S(k,t_1)$ contributes to the integral. Assume that $\lim_{k \to 0} S(k,t_1) \sim k^\beta$ ($\beta > 0$). For quenches to zero temperature, $S(k,t_2)$ will have the scaling form $S(k,t_2) = L(t_2)^d f(kL(t_2))$. Substituting into Eq. (1) (with the appropriate limits of integration), gives

$$\lim_{t_2 \to t_1} C(t_1,t_2) \sim L(t_2)^{-\lambda} \leq L(t_2)^{d/2} \int_0^{2a\pi/L(t_1)} dk \, k^{d-1} L(t_2)^\beta f(kL(t_2)).$$

$$\sim L(t_2)^{-\lambda_{2/2}}.$$

This immediately gives a lower bound on $\lambda$,

$$\lambda \geq \frac{\beta + d}{2}.$$

The argument just presented, holds for conserved and nonconserved, scalar and vector order parameters.

We now consider specific dynamical scenarios. Let $T_I$ and $T_F$ be the temperatures of the initial and final states respectively. We first focus on quenches from the high temperature phase ($T_I = \infty$) to zero temperature ($T_F = 0$). Since the initial state is disordered, $\lim_{k \to 0} S(k,0) \sim k^0$. In the absence of a conservation law, $\lim_{k \to 0} S(k,t_1) \sim k^0$ for both $t_1 = 0$ and $t_1$ in the scaling regime. Therefore $\beta = 0$ and

$$\lambda \geq \frac{d}{2}. \tag{2}$$

This inequality was also obtained by Fisher and Huse using general scaling arguments [2] and is consistent with all results to date [4,5,7-10]. For $t_1 = 0$, conservation of the order parameter does not affect this inequality since $\beta = 0$ for $t_1 = 0$. However, if $t_1$ is in the scaling regime, then $\lim_{k \to 0} S(k,t) \sim k^4$ for $d \geq 2$ [14] and $\beta = 4$. For $d = 1$, the dynamics is dominated by noise and Majumdar et al. find that $\lim_{k \to 0} S(k,t) \sim k^2$ [11], so that $\beta = 2$ for $d = 1$. Therefore, for $t_1$ in the scaling regime,

$$\lambda \geq \frac{\beta + d}{2} \text{ if } d \geq 2,$$

$$\lambda \geq \frac{3}{2} \text{ if } d = 1. \tag{3}$$

These bounds suggest that the asymptotic exponent may depend on whether $t_1$ is, or is not in the scaling regime but do not rule out that the exponent is independent of $t_1$. However for $d = 1$, Majumdar et al. find analytically and numerically that $\lambda = 1$ for $t_1 = 0$, while we find that $\lambda \geq 3/2$ for $t_1$ in the scaling regime [15].

For vector fields (with $m$, the number of components of the order parameter, $> 2$), an argument analogous to Ref. [14], gives the same limit $\lim_{k \to 0} S(k,t) \sim k^4$. This is supported by an extensive numerical integration of the Cahn-Hilliard equation [16]. Therefore the lower bounds on $\lambda$ derived above are valid even for vector order parameters with $m > 2$.

Quenches from the critical point ($T_I = T_c, T_F = 0$) lead to long-range correlations of the initial configurations. In this case, $\lambda \geq d/2$ no longer holds. More generally if $S(k,0) \sim k^{-\sigma}$ we obtain $\lambda \geq (d - \sigma)/2$. (For critical dynamics $\sigma = 2 - \eta$ where $\eta$ is the static critical exponent). This is consistent with the result of Bray et al. who found that, for nonconserved order parameter, $\lambda = (d - \sigma)/2$ for $\sigma$ greater than a critical value $\sigma_c$ [6].

Analysis of the bounds on the autocorrelation exponent for quenches to the critical point ($T_I = \infty, T_F = T_c$), has to start afresh from Eq. (1). Since $t_2$ is in the critical point scaling regime, the correlation function has the following scaling form, $S(k,t_2) \sim k^{-2+\eta} f(kL(t_2))$. Substituting this form into Eq. (1) gives $\lambda \geq (2d - 2 + \eta + \beta)/2$. Therefore when $t_1 = 0$ we get $\lambda \geq (2d - 2 + \eta)/2$. When $t_1$ is also in the scaling regime, the bound on $\lambda$ depends

![FIG. 1. The autocorrelation function $C(0,t)$ for $n = 844$ (3084 initial conditions) 256 (1120 initial conditions) and 1024 (42 initial configurations). Lines corresponding to $C(0,t) \sim L^{-7}$ and $C(t) \sim L^{-5}$ are shown for comparison.](image1)

![FIG. 2. $L^3 C(0,t)$ vs. $L(t)$ for some data as Fig. 1. Lines corresponding to $C(0,t) \sim L^{-7}$ and $C(t) \sim L^{-2}$ are shown for comparison. These results emphasize that the slower decay at late times is due to finite size effects.](image2)
on the behaviour of the scaling function $f_r(kL(t_1))$ as $kL(t_1) \to 0$. For nonconserved systems $\lim_{x \to 0} f_r(x) \to \text{const.}$, or $\beta = -2 + \eta$ leading to $\lambda \geq d - 2 + \eta$.

These lower bounds on $\lambda$ of course do not fix the value of the exponent. As previously mentioned, exact analytical and numerical computations on the 1-dimensional scalar model have been carried out for the case when $t_1 = 0$. In higher dimensions, however, the empirical results are not very conclusive [17]. We therefore compute the asymptotic value of $\lambda$ by numerically integrating the Cahn-Hilliard equation in two-dimensions,

$$\frac{\partial \phi(r, t)}{\partial t} = \nabla^2 \mu(r, t),$$

where $\mu = -\phi + \phi^3 - \nabla^2 \phi$. We used an Euler discretization with $\delta t = 0.1$ and $\delta x = 1.09$ and periodic boundary conditions. We discretize the Laplacian as

$$\nabla^2 \phi_{i,j} = \frac{1}{\delta x^2} \left[ \frac{1}{2} \sum_{n.n.n} + \sum_{n.n} - 6 \right] \phi_{i,j}.$$  

This choice decreases lattice anisotropy effects and allows a larger $\delta t$ before the onset of the checkerboard instability [18]. This dynamical equation is solved subject to random initial conditions which are uncorrelated and uniformly distributed between $-0.05$ and $0.05$ (the initial state is disordered). Decreasing $\delta t$ has no effect on the numerical results. Increasing $\delta x$ to 1.32 results in pinning effects which lead to a slower decay of the autocorrelation function at late times (even though the effect on the single-time behavior is less apparent).

We have used the interfacial area density as a measure of the characteristic length scale $L(t)$. Operationally, this is defined as $(2\delta x n_x n_y)/n_{opp}$, where $n_x n_y$ is the total number of lattice sites and $n_{opp}$ is the number of sites with a nearest neighbor with $\phi$ of opposite sign. We recover the standard result that $L(t)$ grows as $t^{1/3}$ for all $t > 400$. Other measures of the characteristic length scale, such as the first zero of the real space correlation function, also behave in the same manner (for $t > 400$).

Fig. 1 shows $C(t_1, t)$ vs. $L(t)$ for $t_1 = 0$ and $t$ between 100 and 12800 for three lattice sizes $n = n_x = n_y = 64$ (3084 initial conditions), $n = 256$ (1120 initial conditions) and $n = 1024$ (42 initial conditions). Concentrating on the largest lattice size, $C(0, t) \sim L^{-3.7}$ for approximately two decades of $t$ in its decay. There is a crossover to a slower decay at late times with $C(0, t) \sim L^{-3.8}$. At extremely late times there is an indication of an even slower decay, which we attribute to finite size effects.

To emphasize the asymptotic trend, Fig. 2 shows $L^3 C(0, t)$ vs. $L(t)$ for the same data. Here it is clearer that the slower late time decay occurs at earlier times for smaller lattices, indicating finite size effects. The importance of the finite size effects was initially surprising, since for single time quantities, finite size effects only become important when $L(t)$ is of order of the lattice dimension, $L_0$. Thus the usual length scales extracted from single-time quantities were identical for $n = 256$ and $n = 1024$. However, since $C(0, t)$ decays rapidly with $t$, any small systematic effect becomes increasingly relevant as $t$ increases. Clearly finite size effects on $C(0, t)$ can be important (though not necessarily so) when the spread in $C(0, t)$ is of the same order as $C(0, t)$. The spread in $C(0, t)$ decreases as $L_0^{-d/2}$ and, based on our simulations, depends only weakly on $L(t)$. Hence finite size effects can become important when $C(0, t) \sim L(t)^{-\lambda} \sim L_0^{-d/2}$, i.e., much earlier than for single-time quantities. From Fig. 2, $C(0, t)$ for $n = 64$ first shows significant differences from the $n = 256$ result at $C \approx 2.5 \times 10^{-4}$ or $L \approx 18$. Thus we expect the $n = 1024$ data to be free of finite size effects down to $C \approx 2.5 \times 10^{-4}/16 \approx 1.5 \times 10^{-5}$ or $L \approx 50$. Therefore we make the preliminary conclusion that the true asymptotic value of $\lambda$ is approximately 3. Finally note that the late time result for $n = 256$ is consistent with $\lambda = d = 2$. However, comparing this with the result for $n = 1024$ indicates that this regime is due to the finite size of the lattice.

Fig. 3 shows $C(t_1, t_2)$ vs. $L(t_2)/L(t_1)$ for $t_1 = 100, 200$ and 400. Although we cannot rule out a further slower decay, our result is consistent with $\lambda = d = 2$ in all cases. We believe the apparent inconsistency is because the argument leading to $\lambda = d$, incorrectly applies a scaling analysis to the non-scaling, $k = 0$ mode. To be explicit, we briefly review the argument of Ref. [11] (the argument presented in Ref. [12] is similar). The Cahn-Hilliard
equation (Eq. (4)) in k space is
\[
\frac{\partial \phi_k(t)}{\partial t} = D(k, t),
\]
where \(D(k, t) = k^2 \mu_k(t)\). Define \(\tilde{S}(k, 0, t) \equiv \langle \phi_k(0) \phi_{-\mathbf{k}}(t) \rangle = S(k, 0, t) + V \delta_k, \delta(m_0^2)\). The formal solution to \(\tilde{S}(k, 0, t)\) is
\[
\tilde{S}(k, 0, t) = \tilde{S}(k, 0, 0) \exp \left( \int_0^t dt' \frac{\gamma(L(t'), kL(t'))}{t'} \right),
\]
where \(\gamma(L(t), kL(t)) \equiv t \langle D(k, t) \phi_{-\mathbf{k}}(t) \rangle \). In the scaling regime, \(\gamma(L, kL) = \gamma(kL)\) and we obtain (in the limit \(t \gg 0\)),
\[
\int_0^t dt' \frac{\gamma(kL(t'))}{t'} = \int_{kL(0)}^{kL(t)} dx \frac{\gamma(x)}{x}
= \gamma_0(0) \log \left( \frac{L(t)}{L(0)} \right) + \gamma_1(kL(t)),
\]
where \(L(t) \sim t^{1/2}\) and \(\gamma_0 = \lim_{x \to 0^+} \gamma(x)\). The result is
\[
\tilde{S}(k, 0, t) = \tilde{S}(k, 0, 0) \left( \frac{L(t)}{L(0)} \right)^{\gamma_0 z} F_k(kL(t)).
\]
Since \(\int \mathbf{k} \tilde{S}(k, 0, t) = \langle \phi(\mathbf{r}, 0) \phi(\mathbf{r}, t) \rangle \sim L(t)^{-\lambda} = L(t)^{-d+\gamma_0 z}\) or \(\lambda = d - \gamma_0 z\). However, since \(\tilde{S}(k, 0, 0, t)\) is constant in time (conservation law!), it is argued that \(\gamma_0\) must vanish and hence \(\lambda = d\).

Our contention is that, even though the relation \(\lambda = d - \gamma_0 z\) with \(\gamma_0 = \lim_{x \to 0^+} \gamma(x)\) holds, the conclusion that, due to conservation of order parameter, \(\gamma_0\) necessarily vanishes does not. This is because the relation \(\lambda = d - \gamma_0 z\) is based on scaling, which only holds for \(k > 0\), while the vanishing of \(\gamma_0\) is based on the conservation of order parameter which only holds for \(k = 0\). Therefore the conclusion \(\gamma_0 = 0\) is invalid since it applies a scaling argument to the non-scaling \(k = 0\) behavior.

To see this more clearly, consider the quasi-static scattering intensity \(S(k, t) = S(k, t)\). For large \(t\), \(S\) is given by \(\tilde{S}(k, t) = L(t)^d f(kL(t)) + V \delta_k, \delta(m_0^2)\). Global conservation of order parameter requires that \(\tilde{S}(k, 0, t) = 0\), while the locally conservative dynamics requires that \(f(0) = 0\). Hence global conservation leads to a discontinuity at \(k = 0\) for both \(\tilde{S}(k, t_1, t_2)\) and \(\gamma(kL(t))\) [19]. The singularities in \(\gamma(x)\) and \(\tilde{S}(k, t_1, t_2)\) at \(k = 0\) can be removed by choosing the initial distribution so that \(\langle m_0^2 \rangle = 0\). However, in this case, \(\tilde{S}(0, t_1, t_2)\) vanishes independent of the value of \(\gamma_0\); so that the application of the conservation law does not fix the value of \(\gamma_0\). (A. J. Bray has made a similar argument proving \(\lambda = d\) does not necessarily hold [20]).

Having provided useful lower bounds on \(\lambda\), we now ask whether it is possible to bound \(\lambda\) from above? Unfortunately, we have not been able to provide useful upper bounds. However, we note that the bound (Eq. (3)) as well as our numerical results violate the upper bound conjectured by Fisher and Huse, \(\lambda \leq d - 2\) [2]. As they originally noted, this conjecture contains many assumptions. Moreover, in as much as their argument is aimed at the decay of the magnetization, their conjecture has validity only when the order parameter is not conserved and when \(t_1 = 1\) (so that \(\tilde{S}(k, t_1) \sim k^0\)).

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