Private Multi-File Retrieval From Distributed Databases

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Abstract—Suppose there are $N$ distributed databases each storing a full set of $M$ independent files. A user wants to retrieve $r$ out of the $M$ files without revealing the identity of the $r$ files. When $r = 1$ it is the classic problem of private information retrieval (PIR). In this paper we study the problem of private multi-file retrieval (PMFR) which covers the case of general $r$. We first prove an upper bound on the capacity of PMFR schemes which indicates the minimum possible download size per unit of retrieved files. Then we design a general PMFR scheme which happens to attain the upper bound when $r \geq \frac{M}{2}$, thus achieving the optimal communication cost. As $r$ goes down we show the trivial approach of executing $r$ independent PIR instances achieves the near optimal communication cost. Comparing with the capacity-achieving PIR schemes, our PMFR scheme reduces the number of subpackages needed for each file from $N^M$ to $N^2$, which implies a great reduction of implementation complexity.

I. INTRODUCTION

Private information retrieval (PIR) schemes allow a user to retrieve records from public databases while revealing no information about the identity of the retrieved records. According to the pioneer work by Chor et al. [4], the problem of PIR can be formulated as $N$ distributed databases each storing the full set of $M$ independent files $W_1, ..., W_M$ and a user who wants to retrieve one of the files, say $W_i$, $i \in \{1, ..., M\}$, while keeping the index $i$ private from any database. Namely, after several rounds of communication between the user and the $N$ databases, the user gets file $W_i$ while any individual database learns nothing about the index $i$. In addition to extensive applications in cryptographic protocols, PIR is also closely related to coding theory and theoretical computer science, e.g., locally decodable code [12] and one-way function [2].

A key issue in PIR is reducing communication cost. In the initial setting of PIR where each file is one bit long, communication cost is measured by the total number of bits transferred from the user to the databases (i.e. query size or upload size) and bits from the databases to the user (i.e. answer size or download size). By now, the most efficient PIR schemes require communication cost $M^{O(\log_2 r)}$ [5], [6]. Determining the limits on communication cost remains an open problem. However, it is more common in application scenarios that the size of each file is much larger than $M$ and $N$, thus the query size is negligible compared to the answer size. As a result, communication cost can be measured by taking only the answer size (i.e. download size) into account. Specifically, define the rate of a PIR scheme as the ratio between the size of the retrieved file and the answer strings, and define the capacity as the supremum of the rate over all PIR schemes. Obviously, the capacity describes the minimum possible download size per unit of retrieved files. Recently, much work has been done on determining the capacity of PIR in various cases. Sun and Jafar first proved in [7] the capacity of PIR for replicated databases is $\frac{1}{1+\frac{1}{2r}+\frac{1}{M}}$. Then they derived the capacity for PIR with colluding databases [3] and PIR with symmetric privacy [9]. For distributed databases storing MDS coded files, the capacity of PIR in some cases has also been settled [1], [11]. A drawback of these capacity-achieving schemes is that each file has to be divided into a large number (e.g. $N^M$ in [7]) of subpackages to guarantee privacy. By using random interference, some schemes [3], [10] sacrifice a bit of capacity for much easier implementation.

On the other hand, due to various scenarios of information retrieval in real life, it is necessary to study some variants of the PIR model. For example, robust PIR was proposed in [8] to model the situation where some databases fail to respond and the user can still retrieve the file based on answers from the remaining databases. In this paper we consider a variant of PIR, called private multi-file retrieval (PMFR). The only difference from the PIR model is that, in PMFR it allows a user to retrieve more than one file, say $r$ files, $1 \leq r \leq M$, privately from $N$ distributed databases. It is fairly common in real life that a user is interested in multiple files from a public database and tends to keep all his selections secret. Although one can achieve this by executing $r$ independent PIR schemes, our PMFR scheme needs lower communication cost. For example, set $N = 2, M = 3$ and $r = 2$. Note that independent executions of PIR keep the rate of individual schemes. The highest rate (i.e. capacity-achieving rate) of a PIR scheme in this case is $\frac{1}{1+\frac{1}{2}+\frac{1}{3}} = \frac{3}{7}$, while our PMFR scheme (as displayed in Table I) achieves rate $\frac{1}{7}$ saving 28.6% in communication cost. Additionally, our scheme needs much less number of subpackages for each file than the PIR scheme, so it implies an easier implementation in some sense.

Organization and Contribution. In Section II we formally describe the model of PMFR. The privacy requirement is given in an information-theoretic sense. In the next two sections, we respectively prove an upper bound on the capacity of PMFR and design a scheme with rate $\frac{r N}{M + 1(M - 1)}$. It turns out this rate attains the upper bound proved in Section III when $r \geq \frac{M}{2}$.
Thus for privately retrieving a majority of the files, our scheme is optimal with respect to the communication cost, and the rate of this scheme actually defines the capacity of PMFR in this case. Comparing with executing a PIR scheme in $r$ independent instances each for one desired file, our PMFR scheme improves the capacity by at least a factor of $1 + \frac{1}{N}$. Moreover, our scheme reduces the number of subpackages needed for each file form $N^M$ to $N^2$. Conclusions of this paper are given in Section V.

II. MODEL OF PMFR

Suppose there are $N$ distributed databases, denoted as $DB_1, ..., DB_N$, each storing all the $M$ files, i.e. $W_1, ..., W_M$. In general, we assume these files are independent and of the same size, i.e.

$$H(W_1, ..., W_M) = \sum_{i=1}^{M} H(W_i) = ML,$$

where $H(\cdot)$ denotes the entropy function and $H(W_i) = L$ for $1 \leq i \leq M$. Denote by $[M]$ the set $\{1, ..., M\}$. For a set $R \subseteq [M]$ with $|R| = r$, $1 \leq r \leq M$, let $W_R$ denote the subset $\{W_i \mid i \in R\}$. A private multi-file retrieval (PMFR) scheme allows a user to privately retrieve $r$ files, say $W_R$, from the $N$ databases. Note $r$ is a public parameter known by the user and each database. For simplicity, in this paper we use the abbreviation PMFR instead of the more precise one PrFR. Specifically, a PMFR scheme consists of two phases:

- **Query phase.** Given the subset $R$ and some random resources $S$, the user computes $Q(R, S) = (Q_R^1, ..., Q_R^N)$, and sends $Q_R^i$ to $DB_i$ for $1 \leq i \leq N$. Note that $S$ and $R$ are private information only known by the user himself.

- **Response phase.** The $i$th database, $1 \leq i \leq N$, at receiving $Q_R^i$, computes $A_i(Q_R^i, W_M) = A_i^R$ and sends $A_i^R$ to the user.

Also, the following two conditions must be satisfied:

1. **correctness:** $H(W_R|A^R_1, ..., A^R_N, S, R) = 0$.
2. **privacy:** $\forall 1 \leq i \leq N, I(R; Q^i_R, A^R_i, W_M) = 0$

In this paper we use the capital letters to denote sets as well as the random variables taking values over the corresponding sets. The correctness condition means the user can definitely determine the files $W_R$ from all the information he is allowed to possess, and the privacy condition means any individual database knows nothing about $R$ from all the information it obtains in the scheme.

For simplicity of notations, we denote $Q = \{Q_i^R \mid 1 \leq i \leq N, R \subseteq [M] \text{ and } |R| = r\} \cup \{S, R\}$, then the correctness condition implies

$$\forall R \subseteq [M] \text{ and } |R| = r, \ H(W_R|A^R_{[N]}, Q) = 0 \quad (2)$$

In the next section, we will also use some obvious equalities, such as $\forall n \in [N], \forall R \subseteq [M]$ with $|R| = r$,

$$H(A^R_n|W_M, Q^R_n) = 0$$

$$I(Q; W_M) = 0 \quad (4)$$

Note that $Q$ means the queries are generated independently of specific information of the files.

Then we define the rate $R$ of a PMFR scheme as

$$R = \frac{H(W_R)}{\sum_{i=1}^{N} H(A^R_i)} = \frac{rL}{\sum_{i=1}^{N} H(A^R_i)}.$$

From the privacy condition, we have $I(R; A^R_i) = 0$ which implies $\sum_{i=1}^{N} H(A^R_i)$ is independent of the set $R$. Therefore, the definition of rate $R$ is independent of the selection of set $R$. Furthermore, we define the capacity of PMFR, denoted by $C_{\text{PMFR}}$, as the supremum of the rate over all PMFR schemes. It is clear that $C_{\text{PMFR}}$ describes (by its reciprocal) the minimum download size per unit of retrieved files.

III. UPPER BOUND OF PMFR CAPACITY

In this section, we derive an upper bound on $C_{\text{PMFR}}$ (i.e. Theorem 4). All notations are defined as in Section II. The three lemmas (i.e. Lemma 1 to 3) progressively induce the proof of Theorem 4.

**Lemma 1.** For any $R \subseteq [M]$ with $|R| = r$, and for any subset $T \subseteq [M]$ and any $n \in [N]$,

$$H(A^R_n|W_T, Q^R_n) = H(A^R_n|W_T, Q).$$

**Proof.** Obviously, $H(A^R_n|W_T, Q^R_n) \geq H(A^R_n|W_T, Q)$. On the other hand,

$$H(A^R_n|W_T, Q^R_n) - H(A^R_n|W_T, Q) = I(A^R_n; Q|Q^R_n|W_T, Q^R_n) \leq I(A^R_n; W_M|T; Q|Q^R_n|W_T, Q^R_n)$$

$$= I(W_M|T; Q|Q^R_n|W_T, Q^R_n) + I(A^R_n; Q|Q^R_n|W_M, Q^R_n)$$

$$= I(W_M; Q|Q^R_n|Q^R_n) - I(W_T; Q|Q^R_n|Q^R_n) (5) = 0$$

(6)

where (5) comes from the fact $I(A^R_n; Q|Q^R_n|W_M, Q^R_n) = 0$ induced from (3) and the chain rule for mutual information, and (6) follows from (4).

**Lemma 2.** For any $R, R' \subseteq [M]$ with $|R| = |R'| = r$, and for any subset $T \subseteq [M]$ and any $n \in [N]$,

$$H(A^R_n|W_T, Q) = H(A^{R'}_n|W_T, Q).$$

**Proof.** From the privacy condition, we particularly have

$$I(R; A^R_n, Q^R_n, W_T) = 0 \quad \text{and} \quad I(R; Q^R_n, W_T) = 0.$$ As a result, it holds $H(A^{R'}_n|Q^R_n, W_T) = H(A^{R'}_n|Q^R_n, W_T)$ and $H(Q^R_n|W_T) = H(Q^R_n|W_T)$. Thus, $H(A^{R'}_n|Q^R_n, W_T) = H(A^{R'}_n|Q^R_n, W_T)$. At last, we finish the proof by using Lemma 1.

**Lemma 3.** For any $R, R' \subseteq [M]$ with $|R| = |R'| = r$, and for any subset $T \subseteq [M]$,

$$H(A^{R'}_{[N]}|W_T, Q) \geq \frac{1}{N} \left( H(A^{R'}_{[N]}|W_T \cup R', Q) + |R' \setminus T|L \right).$$
Proof. Since $H(A^R_{[N]}|W_T,Q) \geq H(A^R_n|W_T,Q)$ for any $n \in [N]$, then $H(A^R_{[N]}|W_T,Q) \geq \frac{1}{N} \sum_{n=1}^{N} H(A^R_n|W_T,Q)$. Combining with Lemma 2 we have

$$H(A^R_{[N]}|W_T,Q) \geq \frac{1}{N} \sum_{n=1}^{N} H(A^R_n|W_T,Q) \geq \frac{1}{N} H(A^R_n|W_T,Q) = \frac{1}{N} \left( H(A^R_{[N]}|W_T,L) + I(A^R_{[N]};W_R|W_T,Q) \right)$$

Now it suffices to prove $I(A^R_{[N]};W_R|W_T,Q) = |R'| \setminus |T|L$. Actually,

$$I(A^R_{[N]};W_R|W_T,Q) = H(W_R|W_T,Q) - H(W_R|A^R_{[N]},W_T,Q) = H(W_R|W_T,Q) - H(W_R|W_T,Q) - H(W_R|W_T,Q)$$

$$= H(W_R|W_T,Q) - H(W_R|W_T,Q) = H(W_R|W_T,Q) - H(W_T|Q) = H(W_T|W_T) - H(W_T|W_T) = |R'| \setminus |T|L$$

where (7) follows from (2), the former equality in (8) is from (4) and the latter equality is due to (1). \qed

Theorem 4.

$$C_{\text{PNSM}} = \min \left\{ \frac{rN(|W|)}{M - r[\frac{M}{4}] + rN + \cdots + rN[\frac{M}{4}]}, \right\}$$

Proof. Suppose $M = ar + b, 0 \leq b < r$. Because $1 \leq r \leq M$, it has $a \geq 1$. Choose a subset $R_1 \subseteq R, R_2 \subseteq M$ such that $|R_1| = r$ for $1 \leq i \leq a$ and $R_i \cap R_j = \emptyset$ for $i \neq j$. Without loss of generality, we can set $R_1 = R$. Denote $D_i = \cup_{j=1}^a R_j$ for $1 \leq i \leq a$. Next, we prove

$$H(A^R_{[N]}|Q) \geq \frac{(b + r \sum_{i=1}^{a} N^i)L}{N^a}.$$ (9)

First, we have

$$H(W_R) = H(W_R|Q) - H(W_R|A^R_{[N]}) = H(W_R|W_T, A^R_{[N]}) = H(A^R_{[N]}|Q) - H(A^R_{[N]}|W_T, Q),$$ (11)

where (10) comes from (2) and (4). From Lemma 3 it follows $H(A^R_{[N]}|W_T, Q) = H(A^R_{[N]}|W_R, Q) \geq \frac{1}{N} H(A^R_{[N]}|W_D, Q) + rL$. Then by recursively using Lemma 3 we finally have

$$H(A^R_{[N]}|W_R, Q) \geq \frac{1}{N} H(A^R_{[N]}|W_D, Q) = \frac{1}{N} H(A^R_{[N]}|W_D, Q) + rL \sum_{i=1}^{a-1} \frac{1}{N^i} \cdot$$ (12)

If $b = 0$, then $D_a = [M]$, and $H(A^R_{[N]}|W_D, Q) = 0$ due to (4). Combining (11) and (12), it has

$$H(A^R_{[N]}|Q) = H(W_R) + H(A^R_{[N]}|W_R, Q) \geq rL + rL \sum_{i=1}^{a-1} \frac{1}{N^i} \cdot$$

Note the last equality holds because $b = 0$ in this case.

If $1 \leq b < r$, choose set $D_{a+1} = [M] \setminus D_a$ and $|D_{a+1}| = r$. Based on (12), using another time of Lemma 3 we get

$$\frac{1}{N} H(A^R_{[N]}|W_R, Q) \geq \frac{1}{N} H(A^R_{[N]}|W_R, Q) + bL \sum_{i=1}^{a} \frac{1}{N^i} \cdot$$

Again from (5) it has $H(A^R_{[N]}|W_R, Q) = 0$. Then (9) follows immediately.

Finally for any PMFR scheme, we know its rate $R = \frac{H(W_R)}{H(W_M)}$, $\frac{rL}{H(A^R_{[N]}|Q)}$. Combining with (9),

$$\mathcal{R} \leq \frac{rN^a}{b + r \sum_{i=1}^{a} N^i} \cdot$$

Then the theorem follows from the definition of $C_{\text{PNSM}}$. \qed

IV. PMFR SCHEME CONSTRUCTION

In this section, we design a general PMFR scheme. To illustrate how our scheme work generally, we begin with an example of the scheme at specific values of $N, M$ and $r$.

A. An Example

Suppose $N = 2, M = 3$ and $r = 2$. Let $q$ be a prime power with $q > 3$. Denote the files by $W_i = (w_{i1}, w_{i2}, w_{i3}, w_{i4}) \in \mathbb{F}_q^4, 1 \leq i \leq 3$. Namely, each file is divided into 4 subpackages and all the operations are done over these subpackages.

Without loss of generality, suppose the user wants to retrieve $W_1$ and $W_2$. In the query phase, the user randomly selects matrices $S_1, S_2 \in \mathbb{F}^{4 \times 4}$ and $S_3 \in \mathbb{F}^{2 \times 4}$, where $S_1$ and $S_2$ are of rank 4 and $S_3$ is of rank 2. For simplicity we denote

$$(a_1, a_2, a_3, a_4)^T = S_1 W_1^T, \quad (b_1, b_2, b_3, b_4)^T = S_2 W_2^T$$ (13)

and $(c_1, c_2)^T = S_3 W_3^T$, where $a_1, b_1, c_1 \in \mathbb{F}_q^4$ and “$\sim$” means the transpose. Let $a, b, \gamma$ be three distinct nonzero elements in $\mathbb{F}_q$. Then we describe the scheme by listing in Table 1 the answers given by each database at the user’s queries.

| DB1 | DB2 |
|-----|-----|
| $a_1, b_1, c_1$ | $a_2, b_2, c_2$ |
| $a_3 + b_3 + c_2$ | $a_4 + b_4 + c_1$ |
| $\alpha a_3 + \beta b_3 + \gamma c_2$ | $\alpha a_4 + \beta b_4 + \gamma c_1$ |

Table 1

The correctness condition is satisfied. First, the user directly obtains $a_1, a_2, b_1, b_2$ and $c_1, c_2$ from the first line of the answers listed in Table 1 Then substituting the values of $c_1, c_2$ into the last two lines, the user gets a system of linear equations on $a_3, a_4, b_3, b_4$, which is obviously solvable. Finally from (13), the user can recover $W_1$ and $W_2$ by multiplying the inverses of $S_1$ and $S_2$ respectively.

The privacy condition is also satisfied, because the query to each database presents a symmetric form with respect to the indices of files. Later in Section [IV.B] we will give a formal expression of the query and prove the privacy condition generally.
It can be seen the retrieved file size is $8|F_q|$, and the answer size is $10|F_q|$, so the ratio is $\frac{5}{2}$. Here we use $|F_q|$ to denote the size of an element uniformly chosen from $F_q$. Actually the query to each database can be represented by a $5 \times 12$ matrix over $F_q$, so the total query size is $120|F_q|$, which is negligible compared to the file size and the answer size when $l$ is very large.

**B. General Construction**

Let $q > M$ be a prime power. Suppose each file is an $N^2$-dimensional row vector over $F_q$, denoted by $W_j \in F_q^{N^2}$, $1 \leq j \leq M$. Without loss of generality, suppose the user wants to retrieve files $W_1, ..., W_r$, where $1 \leq r \leq M$. Our PMFR scheme generally works as follows.

The user randomly selects matrices $S_1, ..., S_r \in F_q^{N^2 \times N^2}$ and $S_{r+1}, ..., S_M \in F_q^{N^2 \times N^2}$, where $S_1, ..., S_r$ are of rank $N^2$ and $S_{r+1}, ..., S_M$ of rank $N$, i.e., the rows in each matrix $S_i$ are linearly independent. Then define for $r+1 \leq j \leq M$,

$$T_j = \begin{pmatrix} 
S_j \\
PS_j \\
\vdots \\
PN^{-1}S_j 
\end{pmatrix}, \text{ where } P = \begin{pmatrix} 
0 & I_{N-1} & 0 & 0 & \cdots & 0 \\
1 & 0 & \cdots & 0 & \cdots & 0 
\end{pmatrix} \in F_q^{N \times N}$$

is a permutation matrix. Here $I_{N-1}$ means the identity matrix of size $N - 1$. Denote

$$X_j = (x_1^j, ..., x_{N^2}^j) = \begin{cases} 
W_j S_j^r, & 1 \leq j \leq r \\
W_j T_j^r, & r+1 \leq j \leq M 
\end{cases}$$

Let $\alpha_1, ..., \alpha_M$ be distinct nonzero elements in $F_q$ and define

$$B = \begin{pmatrix} 
\alpha_1 & 1 & \cdots & \alpha_M \\
\vdots & \vdots & \ddots & \vdots \\
\alpha_1^{-1} & \alpha_2^{-1} & \cdots & \alpha_M^{-1} 
\end{pmatrix} \in F_q^{r \times M}.$$ 

Then upon receiving queries from the user, the answers of each database are listed in Table 2. Specifically, the answers

| $DB_1$ | $DB_i$, $2 \leq i \leq N$ |
| --- | --- |
| $x_1^1, x_2^1, \ldots, x_{N^2}^1$ | $x_1^i, x_2^i, \ldots, x_{N^2}^i$ |
| $B(x_{N+1}^1, \ldots, x_{N^2+1}^1)^\tau$ | $B(x_{N+1}^i, \ldots, x_{N^2+1}^i)^\tau$ |
| $\vdots$ | $\vdots$ |
| $B(x_{N-1}^1, x_{N+1}^1, \ldots, x_{N^2+1}^1)^\tau$ | $B(x_{N-1}^i, x_{N+1}^i, \ldots, x_{N^2+1}^i)^\tau$ |

of $DB_1$, $1 \leq i \leq N$, include the $M$ items $x_1^1, \ldots, x_M^1$ and $r(N-1)$ linear combinations of which every $r$ combinations are denoted by $B(x_{N+1}^1, \ldots, x_{N^2+1}^1)^\tau$, $1 \leq k \leq N - 1$. We demonstrate correctness and privacy of this scheme.

First note that due to the construction of $T_j$, $r+1 \leq j \leq M$, the values of $x_{N+1}^j, \ldots, x_{N^2}^j$ are repetitions of the values of $x_1^j, \ldots, x_{N^2}^j$, which can be directly obtained from the first line of the answers listed in Table 2. So the linear combinations $B(x_{N+1}^j, \ldots, x_{N^2+1}^j)^\tau$ induces a solvable system of linear equations on $x_{N+1}^j, \ldots, x_{N^2+1}^j$. Thus for $k$ running from 1 to $N - 1$ and $i$ from 1 to $N$, the user can obtain $X_1, ..., X_r$.

Because the non-singular matrices $S_1, ..., S_r$ are known by the user, he can finally recover the files $W_1, ..., W_r$, thus the correctness condition is satisfied.

To demonstrate the privacy condition, we next give a formal expression of the query given to each database. Denote $X = (X_1, ..., X_M)$ which is an $MN^2$-dimensional row vector over $F_q$. Then from Table 2 the answers given by $DB_i$, $1 \leq i \leq N$, are composed of $M + r(N - 1)$ elements in $F_q$. Lining these elements along the order in which they appear in Table 2 results in a vector $A_i \in F_q^{M+r(N-1)}$. It can be seen

$$A_i = \begin{pmatrix} 
I_M \otimes e_i \\
B \otimes e_{N+1} \\
\vdots \\
B \otimes e_{(N-1)+N} 
\end{pmatrix} X^\tau,
$$

where $e_i$ is the vector in $F_q^{N^2}$ with 1 in the $i$th coordinate and zeros elsewhere, $\otimes$ denotes the Kronecker product, and $W = (W_1, ..., W_M) \in F_q^{M \times N^2}$.

For simplicity, denote $A_i = Q_i X^\tau$. See Figure 1 for the specific expression of $Q_i$. The privacy condition equivalently means for $1 \leq i \leq N$, $Q_i$ reveals no information about the query set $\{1, ..., r\}$. Actually, because $DB_i$ does not know the matrices $S_1, ..., S_M$, all it observes from $Q_i$ is that for each file index $j \in [M]$, $N$ linearly independent coefficient vectors, i.e., $s_1^j, s_2^j, ..., s_{N-1+1}^j$ for $1 \leq j \leq r$ and $s_1^j, s_2^j, ..., s_{N-1+1}^j$ for $r+1 \leq j \leq M$, are contained in $Q_i$. Actually, we have $Q_i N - 1$ of the vectors are sequentially multiplied by 1, $\alpha_j, ..., \alpha_j^{-1}$ for some nonzero element $\alpha_j \in F_q$. Due to this symmetric form of $Q_i$ with respect to the file indices, the privacy condition evidently holds.

**C. Optimality When $r \geq \frac{M}{4}$**

It can be seen the rate of our PMFR scheme is

$$R = \frac{rN^2}{N(M + r(N - 1))} = \frac{rN}{M + r(N - 1)}.$$ 

We will show it meets the upper bound proved in Theorem 4 with equality when $r \geq \frac{M}{4}$. Actually, when $r = \frac{M}{4}$, $\lceil \frac{M}{4} \rceil = 2$ and by Theorem 4 $C_{\text{PFSR}} \leq \frac{rN^2}{rN + r(N - r)} = \frac{rN}{r + r(N - r)}$. Since $r = M - r$ for $r = \frac{M}{4}$, it immediately follows $C_{\text{PFSR}} \leq \frac{M}{M + r(N - 1)}$. When $r > \frac{M}{4}$, then $\lceil \frac{M}{4} \rceil = 1$ and by Theorem 3 $C_{\text{PFSR}} \leq \frac{M}{M + r(N - 1)}$. Since $C_{\text{PFSR}}$ is defined to be the supremum of the rate over all PMFR schemes, our scheme attaining the upper bound of $C_{\text{PFSR}}$ means our scheme achieves the highest possible rate, thus is optimal with respect to the communication cost when retrieving a majority of the files.

According to (7), $C_{\text{PFSR}} = \frac{1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{2}}}}{2}$. This is the best rate one can expect for the private retrieval of $M$ files by
executing $M$ independent PIR instances each for one desired file. When $r \geq \frac{M}{2}$,

$$\frac{rN}{M + r(N-1)} = \frac{rN(1 + \frac{1}{N} + \cdots + \frac{1}{N^{r-1}})}{M + r(N-1)} \geq 1 + \frac{1}{N} + \cdots + \frac{1}{N^{r-1}}$$

(14)

where (14) is due to the condition $M - r \leq r$. Thus comparing with PIR, our PMFR scheme improves the rate at least by a factor of $1 + \frac{1}{N}$ for $M \geq 4$.

When $r < \frac{M}{2}$, our PMFR scheme does not have the optimal rate. On the other hand, it can be seen the gap between $C_{\text{PR}}$ and the upper bound in Theorem 4 shrinks fast as $r$ goes down. For example, suppose $r \mid M$, then by Theorem 4

$$C_{\text{PMFR}} \leq \frac{N^M}{N + N^2 + \cdots + N^M} = \left(1 + \sum_{k=1}^{r-1} \frac{1}{N^{k}}\right)C_{\text{PR}}.$$

It means improvement on the rate of PMFR schemes over $C_{\text{PR}}$ is essentially limited in the case of $r < \frac{M}{2}$.

However, comparing with the $C_{\text{PMFR}}$-achieving scheme in [7] which divides each file into $N^M$ subpackages, our scheme only needs $N^2$ subpackages, greatly reducing implementation complexity in some sense.

V. CONCLUSION

In this paper, we investigate the problem of private multi-file retrieval (PMFR) from distributed databases. An upper bound on the capacity of PMFR schemes is derived, which indicates the minimum download size per unit of retrieved files. A general scheme attaining this upper bound when retrieving a majority of files is designed. Comparing with the trivial approach of executing multiple independent PIR instances each for one desired file, our scheme not only has advantage in communication cost but also greatly reduces the number of subpackages needed for each file. There remain many issues worthy of further study in the scope of PMFR, such as privacy against colluding databases, private retrieval from coded databases, and even how to keep the number of retrieved files (i.e., $r$) secret from the databases.

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