Friction and inertia for a mirror in a thermal field

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INTRODUCTION

Objects which scatter a quantum field are submitted to a radiation pressure. When two scatterers are present in vacuum, a mean force, the so-called Casimir force [], results for each of them. A recent discussion and references may be found in []. For a scatterer alone in vacuum, the force of zero mean value still has quantum fluctuations [], which are associated through fluctuation-dissipation relations with a motional force [].

Using the techniques of quantum field theory, Fulling and Davies have computed the motional force for a perfectly reflecting mirror in the vacuum state of a scalar field in a two-dimensional spacetime. A linear approximation (first order expansion in the mirror’s displacement) leads from their result [] to a force proportional to the third time derivative of the mirror’s position

\[ \delta F_0(t) = \frac{\hbar}{6\pi c^2} \delta q'''(t) \] (1a)

This force vanishes for a uniform velocity, as well as for a uniform acceleration. These properties are related to spatial symmetries of the vacuum: vacuum fields are invariant under the action of Lorentz boosts [] while they appear to a uniformly accelerating observer as thermal fields in its own frame[]. Then, no friction force, proportional to the velocity, nor inertial force, proportional to the acceleration, appear in the radiative reaction of vacuum fields upon the moving mirror.

In a thermal field, the motional force becomes []

\[ \delta F_T(t) = \delta F_0(t) - \lambda_T \delta q'(t) \] (1b)

\[ \lambda_T = \frac{2\pi T^2}{3\hbar c^2} \] (1c)

The temperature is measured as an energy \((k_B = 1)\).

In a more elaborate treatment, the mirror is described by reflection and transmission amplitudes obeying unitarity and causality requirements [], and the mirror is supposed to become transparent above a reflection cutoff \(\omega_C\), much smaller in reduced units than the mirror’s mass \(m\)

\[ \hbar \omega_C \ll mc^2 \] (2)

It is then possible (neglecting the recoil effect) to compute the susceptibility function describing the motional force in a linear approximation []. In contrast with the unphysical model of a perfect mirror, the susceptibility now obeys the expected causality and stability properties [][]. The associated dispersion relations set limits on the possible values of the mirror’s mass \(m\). We will not consider this problem in the present letter, although it will certainly have to be included in a complete treatment of inertia corrections.

Here, we shall show that the expression computed for the motional susceptibility corresponds to a non vanishing inertial force for a partially transmitting mirror scattering a thermal field. In conformity with the particular results stated above, the associated mass correction goes to zero in the limiting cases of perfect reflection or zero temperature.

LINEAR RESPONSE THEORY AT THE QUASISTATIC LIMIT

In a first order expansion in the mirror’s displacement \(\delta q\) (linear approximation), the motional force \(\delta F_T\) can be written in terms of a susceptibility \(\chi_T\)
\[ \delta F_T(t) = \int d\tau \chi_T(\tau) \delta q(t - \tau) \quad (3a) \]
\[ \delta F_T[\omega] = \chi_T[\omega] \delta q[\omega] \quad (3b) \]

where we denote for any function \( f \)
\[ f(t) = \int \frac{d\omega}{2\pi} f[\omega] e^{-i\omega t} \]

In contrast with the usual models of quantum Brownian motion lying upon a linear coupling between mirror and fields, the mirror is here coupled to a quantity, the radiation pressure, quadratic in the fields. Then, the susceptibility depends upon temperature and can be written in terms of the reflection and transmission amplitudes \( r \) and \( s \) as:
\[ \chi_T[\omega] = \frac{i\hbar}{2c^2} \int \frac{d\omega'}{2\pi} \omega' (\omega - \omega') \alpha[\omega', \omega - \omega'] \times (\varepsilon_T[\omega'] + \varepsilon_T[\omega - \omega']) \quad (4a) \]
\[ \alpha[\omega, \omega'] = 1 + r[\omega] r[\omega'] - s[\omega] s[\omega'] \quad (4b) \]
\[ \varepsilon_T[\omega] = \coth \frac{\hbar \omega}{2T} \quad (4c) \]

It can also be written (\( \tilde{\xi}_T \) and \( \xi_T \) real functions of \( \omega \))
\[ \chi_T[\omega] = \tilde{\xi}_T[\omega] + i\xi_T[\omega] \]

The dissipative part \( \xi_T \) is the commutator of the radiation pressure force operator \( F \), and is related to the stationary correlation function \( C_T \) computed for a motionless mirror
\[ \xi_T(t) = \frac{\langle [F(t), F(0)] \rangle}{2\hbar} = \frac{C_T(t) - C_T(-t)}{2\hbar} \]
\[ C_T(t) = \langle F(t) F(0) \rangle - \langle F \rangle^2 \]

The dispersive part \( \tilde{\xi}_T \) can be deduced from the dissipative one \( \xi_T \) through dispersion relations.

In order to give a precise evaluation of the inertia corrections, we now introduce a quasistatic expansion of the motional force (3)
\[ \delta F_T(t) = - (\lambda_T \delta q'[t] + \mu_T \delta q''[t]) + \ldots \]
\[ \chi_T[\omega] = i\omega \lambda_T + \omega^2 \mu_T + \ldots \]
\[ \lambda_T = - i \chi_T'[0] \quad \mu_T = \chi_T''[0] \quad (5c) \]

\( \chi_T[0] \), which would describe a position dependent static force, vanishes since the mean radiation pressure is zero in a thermal state. The coefficients \( \lambda_T \) and \( \mu_T \) are respectively associated with viscous and inertial forces; we are not interested in higher order quasistatic coefficients.

The functions \( \xi_T \) and \( C_T \) are connected through a fluctuation-dissipation relation
\[ C_T[\omega] = \frac{2\hbar}{1 - e^{-\omega/\hbar}} \xi_T[\omega] \]

At the low frequency limit where (\( \lambda_T \) and \( \mu_T \) are real)
\[ \xi_T[\omega] \approx \omega \lambda_T \quad \text{for} \quad \omega \to 0 \]
\[ C_T[\omega] \approx \frac{2T}{\omega} \xi_T[\omega] \approx 2T \lambda_T \quad \text{for} \quad \omega \to 0 \]

Einstein’s relation between the viscosity coefficient \( \lambda_T \) and the momentum diffusion coefficient is recovered (one has only supposed \( \lambda_T \neq 0 \)).

**EVALUATION OF THE QUASISTATIC COEFFICIENTS**

We now evaluate the two coefficients \( \lambda_T \) and \( \mu_T \), first in the vacuum state, then in a thermal state.

At zero temperature, the function \( \varepsilon_T \) appearing in equations (4) coincides with the sign function (\( \varepsilon(\omega) = \frac{\omega}{|\omega|} \)), and the known result is recovered
\[ \chi[0][\omega] = \frac{i\hbar}{c^2} \int_0^\infty \frac{d\omega'}{2\pi} \omega'(\omega - \omega') \alpha[\omega', \omega - \omega'] \]

At a non zero temperature, we write the ‘smoothed’ sign function \( \tilde{\varepsilon}_T \) as the sum of the sign function (\( \varepsilon(\omega) = \frac{\omega}{|\omega|} \)), and of a correction involving the mean number \( n_T \) of thermal photons per mode
\[ \varepsilon_T[\omega] = \varepsilon(\omega) + \delta \varepsilon_T[\omega] \]
\[ \delta \varepsilon_T[\omega] = 2\varepsilon(\omega) n_T[\omega] \]
\[ n_T[\omega] = \frac{1}{e^{\omega/\hbar} - 1} \]

We then deduce
\[ \chi_T[\omega] = \chi[0][\omega] + \delta \chi_T[\omega] \]
\[ \delta \chi_T[\omega] = \frac{2i\hbar}{c^2} \int_0^\infty \frac{d\omega'}{2\pi} \omega' n_T[\omega'] \times (\alpha[\omega', \omega - \omega'] + (\omega + \omega') \alpha[-\omega', \omega + \omega']) \]

The function \( \chi[0][\omega] \) scales as \( \omega^3 \) at low frequencies so that the coefficients \( \chi[0][0] \) and \( \chi[0][0] \) vanish. A straightforward calculation thus leads to the following expression of the viscosity coefficient \( \lambda_T \) defined by equations (5)
\[ \lambda_T = \frac{2\hbar}{c^2} \int_0^\infty \frac{d\omega}{2\pi} n_T[\omega] \partial[\omega^3 a[\omega]] \quad (6a) \]
\[ a[\omega] = 1 + r[\omega] r[-\omega] - s[\omega] s[-\omega] \quad (6b) \]

\(^1\)Note that \( \mu_T \) has not the same definition as in ref. [3].
This expression may be transformed, first by integrating by parts, then by using the fact that \( n_T[\omega] \) is a function of \( \frac{\omega}{T} \) only
\[
-\omega \partial_\omega n_T[\omega] = T \partial_T n_T[\omega]
\]
One eventually obtains
\[
\lambda_T = \frac{2T}{c^2} \frac{dA}{dT} 
\]
\[
A(T) = \int_0^\infty \frac{d\omega}{2\pi} \hbar \omega n_T[\omega] a[\omega] 
\]
In a similar manner, we get the mass correction \( \mu_T \) defined by equations (5)
\[
\mu_T = \frac{\hbar}{c^2} \int_0^\infty \frac{d\omega}{2\pi} n_T[\omega] \partial_\omega (\omega^2 b[\omega]) 
\]
\[
b[\omega] = i (r[\omega] r'[-\omega] + r[\omega] r'[-\omega]) 
\]
\[
- i (s'[\omega] s[-\omega] + s[\omega] s'[-\omega]) 
\]
that is
\[
\mu_T = \frac{T}{c^2} \frac{dB}{dT} 
\]
\[
B(T) = \int_0^\infty \frac{d\omega}{2\pi} \hbar \omega n_T[\omega] b[\omega] 
\]
At this point, we want to emphasize that the two coefficients \( \lambda_T \) and \( \mu_T \) result from the radiative reaction upon the mirror of the thermal fields only. There is no viscous nor inertial force for a partially transmitting mirror in vacuum, as well as for a perfect one, in agreement with the spatial symmetries discussed in the introduction. It is worth noting that the vacuum fields may be reintroduced in expressions (6) and (7) by replacing \( n_T[\omega] \) by \( \frac{1}{2} + n_T[\omega] \) without changing the resulting values in equations (6a) and (7a), where the vacuum contribution \( \frac{1}{2} \) leads to a null integral. Alternatively, the term \( \frac{1}{2} \) contributes to equations (6d) and (7d), but its contributions, being temperature independent, do not affect the resulting values of entropy-like expressions (6c) and (7c). The foregoing discussion shows that the expressions of \( \lambda_T \) and \( \mu_T \) could as well have been obtained from a description of vacuum fluctuations as ‘zero-point fields’\(^2\) This is not true for the complete expression of the susceptibility: when replacing \( n_T[\omega] \) by \( \frac{1}{2} \) in the thermal contribution \( \delta \chi_T[\omega] \) to susceptibility, one obtains an expression which differs from the vacuum contribution \( \chi_0[\omega] \). The quasistatic coefficients of higher order, especially the coefficient \( \chi'''[0] \) which describes the radiative reaction of vacuum, are not correctly obtained without fully accounting for the quantum character of vacuum fluctuations. The difference between a quantum and a classical description of vacuum radiation pressure is also discussed in ref. [14].

**EXPRESSION IN TERMS OF THE REFLECTION PROBABILITIES AND PHASE SHIFTS**

In order to interpret the expressions obtained for \( \lambda_T \) and \( \mu_T \), we introduce the modulus and phase of the scattering coefficients. Using the unitarity of the scattering matrix
\[
|s[\omega]|^2 + |r[\omega]|^2 = 1 
\]
\[
\begin{align*}
 s[\omega] r[\omega]^* + r[\omega] s[\omega]^* &= 0 
\end{align*}
\]
and the reality of the scattering functions (written in the time domain)
\[
\begin{align*}
 s[-\omega] &= s[\omega]^* \\
 r[-\omega] &= r[\omega]^* 
\end{align*}
\]
one deduces that these modulus and phases may be written in terms of only two functions
\[
\begin{align*}
 |r[\omega]|^2 &= R[\omega] \\
 |s[\omega]|^2 &= 1 - R[\omega] \\
 \frac{s[\omega]}{r[\omega]} &= e^{i\Delta[\omega]} \\
 \frac{r[\omega]}{s[\omega]} &= e^{-i\Delta[\omega]} 
\end{align*}
\]
The function \( R \) is the reflection probability while \( \Delta \) is the sum of the two phase shifts associated with the scattering matrix: \( e^{i\Delta[\omega]} \) is precisely the determinant \( s[\omega]^2 - r[\omega]^2 \) of the scattering matrix. We will also define the scattering delays for fields around frequency \( \omega \): more precisely, \( 2\tau \) will be the sum of the two time delays associated with the scattering matrix
\[
2\tau[\omega] = \Delta'[\omega] 
\]
It turns out that the viscosity coefficient \( \lambda_T \) depends only upon the reflection probability
\[
a[\omega] = 2 R[\omega] 
\]
whereas the mass correction \( \mu_T \) also depends upon the phase shifts
\[
b[\omega] = 2 (1 - 2R[\omega]) \tau[\omega] 
\]
In particular, the behaviour at low frequencies of \( a \) and \( b \), which will play a dominant role at the low temperature limit, is determined by the parameters \( R_0 = R[\omega \to 0] \) and \( \tau_0 = \tau[\omega \to 0] \).

A simple model fulfilling the requirements of unitarity, causality and high frequency transparency corresponds to the lorentzian scattering functions.
\[ r[\omega] = \frac{-1}{1 - i\omega \tau_0} \quad s[\omega] = \frac{-i\omega \tau_0}{1 - i\omega \tau_0} \]
\[ R[\omega] = \frac{1}{1 + \omega^2 \tau_0^2} \quad \tau[\omega] = \frac{\tau_0}{1 + \omega^2 \tau_0^2} \]

The parameter \( \tau_0 \), defined as the time delay evaluated at frequencies lower than the reflection cutoff of the mirror, also appears as the inverse of this cutoff.

**DISCUSSION OF THE VISCOSITY COEFFICIENT**

Collecting equations (6), we are now able to give a simple interpretation of the quantity

\[ A(T) = \int_0^\infty \frac{d\omega}{2\pi} 2\hbar \omega n_T[\omega] R[\omega] \]

This is indeed the energy flux associated with the two thermal fields coming onto the mirror from the left and from the right sides (\( \frac{1}{2\pi} \hbar \omega n_T[\omega] \) is the energy flux in the band \( d\omega \) for one propagation direction and the factor 2 stands for the two input fields) integrated over the reflection bandwidth \( R[\omega] \) of the mirror. It is easily shown that \( A(T) \) increases with \( T \), so that \( \lambda_T \) is positive and corresponds effectively to a damping force. Simple expressions are obtained at the low and high temperature limits.

At the high temperature limit (\( T \gg \hbar \omega_C \) where \( \omega_C \) is the reflection cutoff), \( n_T[\omega] \) can be replaced by its classical approximation \( (\hbar \omega n_T[\omega] \approx T) \) and the incident energy flux is the product of the temperature by the bandwidth

\[ A(T) = 2T \Omega_C \quad \Omega_C = \int_0^\infty \frac{d\omega}{2\pi} R[\omega] \]

\( \Omega_C \) is of the order of the reflection bandwidth; \( \Omega_C = \frac{2\pi}{4} \) for the lorentzian model. As \( A(T) \) is a linear function of \( T \), it follows from equation (6a) that

\[ \lambda_T = \frac{2A(T)}{c^2} \]

This result can also be understood in a simple manner. For each photon of energy \( \hbar \omega \) reflected by the mirror moving with a uniform velocity \( v \), there is a net momentum transfer \( \frac{\hbar \omega}{c} \) to the mirror, as a consequence of the Doppler effect. The viscosity coefficient \( \lambda_T \) is therefore \( \frac{2A}{c^2} \), where \( A \) is the energy of all photons impinging on the mirror per unit time.

At the low temperature limit \( T \ll \hbar \omega_C \), the reflection probability can be replaced by its low-frequency value \( R_0 \) so that one gets \( A \) in terms of the energy flux coming onto the mirror integrated over all frequencies

\[ A(T) = \frac{R_0 \pi T^2}{6\hbar} \]

\( A(T) \) is now a quadratic function of \( T \), and equation (6a) implies that \( \lambda_T \) is twice the value expected from the interpretation in terms of Doppler shifts

\[ \lambda_T = \frac{4A(T)}{c^2} = \frac{R_0 2\pi T^2}{3\hbar c^2} \]

It is well known that the interpretation in terms of Doppler shifts is too naive: taken seriously, it would lead to a viscous damping of the mirror in vacuum. This paradox has already been elucidated [6]: the Lorentz transformation affects the field amplitudes as well as the field frequencies; when this is taken into consideration, it turns out that the vacuum fields effectively obey Lorentz invariance and do not contribute to damping. The results obtained in the present paper show that the modification of the field amplitudes also plays a role at non zero temperature. This role appears to be negligible at high temperatures, while it amounts to double the coefficient evaluated from Doppler shifts at low temperatures.

**DISCUSSION OF THE INERTIAL FORCE**

The expressions (7) giving the mass correction \( \mu_T \) can be discussed along the same lines. The quantity

\[ B(T) = \int_0^\infty \frac{d\omega}{2\pi} 2\hbar \omega n_T[\omega] (1 - 2R[\omega]) \tau[\omega] \]

is an integral over frequency of the thermal energy flux, with a weight function proportional to the time delays.

For a perfect mirror, the phase shifts do not depend upon frequency, so that the mass correction vanishes \( (\tau[\omega] = 0) \) at any temperature, in consistency with the results stated in the introduction (see eqs 1). For a partially transmitting mirror however, the phase shifts are frequency dependent in order to obey causality and high frequency transparency and the scattering delays do not vanish. In a two-mirror configuration, the Casimir energy can be expressed in terms of the phase shifts [6]: it appears as a finite part of the incident field energy stocked because of the time delays. The quantity \( B \) would have the same interpretation of an incident energy stocked because of the time delays for a mirror having a small reflexion probability at all frequencies: the function \( (1 - 2R[\omega]) \) could be replaced by 1 in its expression. This is not always the case, since the function \( (1 - 2R[\omega]) \) can even change its sign with frequency. This prevents to give a simple interpretation of \( B \) as a stocked energy.

At the high temperature limit, one gets

\[ B(T) = T \Delta S \]
\[ \Delta S = \int_0^\infty \frac{d\omega}{2\pi} (1 - 2R[\omega]) 2\tau[\omega] \]
\[ \mu_T = \frac{T dB}{c^2 dT} = \frac{B(T)}{c^2} \]
For instance, the model of lorentzian scattering coefficients leads to the value $B = 0$. This differs from the expression $\frac{T}{2}$ of the stocked energy computed from the same expressions with $(1 - 2R[\omega])$ replaced by 1.

At the low temperature limit, the functions $R$ and $\tau$ can be replaced by their low frequency values

$$B(T) = \left(\frac{1}{R_0} - 2\right)\tau_0 A(T) = (1 - 2R_0) \frac{\tau_0 \pi T^2}{6h}$$

$A$ is the integrated energy flux, discussed in the previous section. One deduces

$$\mu_T = \frac{2B(T)}{c^2}$$

For a mirror perfectly reflecting at low frequencies ($R_0 = 1$), the mass correction is negative. As the low temperature expression is valid for $T \ll \hbar\omega_C$, one checks that the mass correction remains smaller than the mirror’s mass, using eq.(2) and noting that the delays are of the order of the inverse of the cutoff frequency.

Viscous drag due to blackbody radiation has often been discussed, particularly as a potential effect of the cosmic microwave background radiation (see for instance ref. [15]). It follows from the foregoing results that the background radiation may also affect the inertial properties of scatterers.

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