HOW TO COUNT THE STATES OF EXTREMAL BLACK HOLES IN $\mathcal{N} = 8$ SUPERGRAVITY

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Abstract. $\mathcal{N} = 8$ supergravity has a rich spectrum of black holes charged under the 56 $U(1)$ gauge fields of the theory. Duality predicts that the entropy of these black holes is related to the quartic invariant of the $E(7,7)$ group. We verify this prediction in detail by constructing black holes that correspond to supersymmetric bound states of 2-branes at angles and 6-branes. The general bound state contains an arbitrary number of branes rotated relative to each other, and we derive the condition for these rotations to preserve supersymmetry. The microscopic bound state degeneracy matches the black hole entropy in detail. The entire 56 charge spectrum of extremal black holes in $\mathcal{N} = 8$ supergravity can be displayed as the orbit under duality of a five parameter generating solution. We exhibit a new generating configuration consisting of D3-branes at angles and discuss its entropy.

1. Introduction

The basic technique for counting the states of extremal black holes in string theory is to represent the black hole as a bound state of p-brane solitons. The horizon area of the resulting supergravity solution is related to an entropy by the Bekenstein-Hawking formula $S = A/4G_N$. At weak coupling, p-branes charged under the Ramond-Ramond fields of Type II string theory are easily quantized as D-branes [1]. This fact was exploited in the seminal paper of Strominger and Vafa who showed that for some black holes
corresponding at weak coupling to systems of D-branes, the collective coordinate degeneracy of the bound state can be easily calculated [2]. If the bound state in question is a BPS state, this degeneracy can be extrapolated back to strong coupling since the representation theory of supersymmetry protects it from corrections. To summarize, the program for counting the states of black holes amounts to treating the black hole as a bound state of solitons, quantizing the solitons at weak coupling, and extrapolating the resulting collective coordinate degeneracy back to strong coupling via an appeal to supersymmetry.

**Figure 1.** How to count black hole states

In the present paper we apply this method of counting states (see Figure 1) to supersymmetric black holes in $N = 8$ supergravity, the low energy effective theory of Type II string theory compactified on a 6-torus. In the bulk of the paper we will work with IIA string theory which has 0-, 2-, 4- and 6-brane solitons. Charged black holes can be constructed by wrapping these branes on the 6-torus. We are interested in supersymmetric black holes because the weak-to-strong coupling transformation used in Fig. 1 generally preserves only the degeneracy of states that are annihilated by some of the supercharges. Such supersymmetric black holes are also extremal because they will satisfy a BPS mass formula relating the mass of the black hole to the charges it carries.

We begin in Sec. 2 by discussing the general prediction for the entropy of extremal black holes in terms of the quartic invariant of the $E(7,7)$ duality group of $N = 8$ supergravity. We want to construct extremal black holes out of bound states of branes. So, in Sec. 3, we show how to create general supersymmetric bound states of 2-branes at angles and 6-branes wrapped on the internal 6-torus. These are T-dual to bound states of 4-
branes at angles and 0-branes. Sec. 4 finds the classical black hole solutions to $N = 8$ supergravity corresponding to such branes at angles. Their classical entropy $S = A/4G_N$ precisely matches the prediction given by the $E_7$ quartic invariant in Sec. 2. Then, in Sec. 5, we count the degeneracy of the microscopic bound state of D-branes and demonstrate detailed agreement with the classical entropy.

In general, extremal black holes in $N = 8$ supergravity can carry 56 different $U(1)$ charges. As discussed in Sec. 2, this space of black holes can be generated by dualities from a 5-parameter generating solution [3]. The known generating solution of Cvetic and Tseytlin [4] contains NS 5-branes and fundamental strings and it is difficult to count its states microscopically. On the other hand, all treatments in the literature of four dimensional black hole entropy using D-branes correspond to 4 parameter generating solutions (for example, see [5, 6]). In the discussion concluding this paper, we show how a 5 parameter generating solution containing only D-branes can be constructed by intersecting 3-branes at angles. This solution can be dualized into a system of 4-branes, 2-branes and 0-branes, and clearly displays in D-brane language the final parameter missing from the discussion of extremal black hole entropy in $N = 8$ supergravity. Counting the states of these black holes introduces some interesting new features.

### 2. Black Hole Entropy in $N = 8$ Supergravity

The no hair theorem for black holes says that a non-rotating black hole is completely characterized by its mass and charges. The mass of a supersymmetric black hole is further related to its charges by a BPS bound. It follows from this that the area and entropy of an extremal black hole in $N = 8$ supergravity should be completely specified by its charges under the 56 $U(1)$ fields of the theory. $N = 8$ supergravity has a duality group $- E(7,7) -$ that mixes up these charges and dresses them with the moduli associated with the shape of the internal $T^6$ at infinity. However, the entropy, $S = A/4G_N$, should be invariant under duality since we do not expect the degeneracy of supersymmetric states to change in a dual description [7, 8, 9]. So the entropy of a supersymmetric black hole is expected to be a duality invariant.

The charges of $N = 8$ supergravity rotate in the 56 dimensional antisymmetric tensor representation of the $E(7,7)$ duality symmetry. These charges are associated with various solitons wrapped on the internal $T^6$ - for example, there are 15 charges coming from 2-branes wrapped on different cycles and 15 from 4-branes. The integral quantized charges are generally "dressed" by moduli scalar fields that represent the shape of the asymptotic

\[1\text{The useful reference for the remainder of this section is Cremmer and Julia [10].}\]
internal torus and therefore parametrize inequivalent vacua. The moduli parametrize the coset $E(7,7)/SU(8)$ and therefore $SU(8)$ is the part of the duality group that mixes the charges in a nontrivial way.

For our purposes it is convenient to work in the $SO(8)$ formalism where the 56 charges are assembled in an 8 by 8 antisymmetric tensor whose indices rotate in the vector representation of $SO(8) \in SU(8)$. Then the charge matrix is:

$$z_{ij} = \frac{1}{\sqrt{2}}(x_{ij} + iy_{ij})$$  \hspace{1cm} (1)

For solutions that contain only Ramond-Ramond charges associated with D-branes, the $x$ and $y$ variables are directly related to wrapped branes. Taking $q_{ij}$ and $p_{ijkl}$ to be the charges of 2-branes and 4-branes wrapped on the $(ij)$ 2-cycle and the $(ijkl)$ 4-cycle, $x_{ij} = \epsilon_{ijklmn}p_{klmn}/\sqrt{2}$ and $y_{ij} = q_{ij}/\sqrt{2}$ for $i, j \leq 6$. Finally, if $Q_0$ and $Q_6$ are the 0-brane and 6-brane charges in the system, $x^{78} = Q_0/\sqrt{2}$ and $y^{78} = -Q_6/\sqrt{2}$.

By the above arguments, the area of a black hole must be given in terms of a duality invariant constructed from the charges. After accounting for dimensions, this was identified in [9] as $A = 4\pi\sqrt{J_4}$ where $J_4$ is the quartic invariant [10]:

$$-J_4 = x^{ij}y_{jk}x^{kl}y_{li} - \frac{x^{ij}y_{ij}x^{kl}y_{kl}}{4} + \frac{\epsilon_{ijklmnop}}{96}(x^{ij}x^{kl}x^{mn}x^{op} + y^{ij}y^{kl}y^{mn}y^{op})$$

This gives a prediction from duality for the area of four dimensional black holes in string theory. In this paper we will construct rather complicated bound states made from arbitrary numbers of 2-branes and 6-branes and show that the above prediction is verified in detail.

2.1. GENERATING SOLUTIONS

Since a supersymmetric black hole is completely specified by its charges, and duality mixes up these charges, the entire 56 dimensional spectrum of black holes can be generated by duality from a much smaller space of generating solutions. In fact, the generating solution must have five independent parameters [3]. To see this it is convenient to rewrite the charge matrix in $SU(8)$ formalism [10]:

$$Z_{AB} = -\frac{1}{4}z_{ij}(\Gamma^{ij})_{AB}$$  \hspace{1cm} (3)

where $\Gamma^{ij}$ are generators of $SO(8)$. Now $Z$ transforms under $SU(8)$ as $Z \rightarrow UZU^T$. The $SU(8)$ transformations can be used to skew diagonalize
$Z$ giving four complex eigenvalues:

$$Z = \begin{pmatrix}
\lambda_1 e^{i\theta_1 \tau} & 0 & 0 & 0 \\
0 & \lambda_2 e^{i\theta_2 \tau} & 0 & 0 \\
0 & 0 & \lambda_3 e^{i\theta_3 \tau} & 0 \\
0 & 0 & 0 & \lambda_4 e^{i\theta_4 \tau}
\end{pmatrix}; \quad \tau = \begin{pmatrix}
0 & 1 \\
-1 & 0
\end{pmatrix} \quad (4)$$

We can perform further $SU(8)$ transformations that eliminate the phase of the eigenvalues in first three blocks, by adding compensating phases in the last block. So, in general, we can diagonalize to get 1 complex and 3 real skew eigenvalues for a total of 5 parameters. Turning this around, in order to obtain the full spectrum of black holes by duality transformations of a generating solution, the latter must have five parameters.

The invariant phase that the $SU(8)$ transformations are unable to get rid of is associated with the presence of electric and magnetic dual objects at the same time. It is difficult to turn on such dual charges supersymmetrically and so the only five parameter generating solution available at present is that of [4]. The bulk of this paper deals with solutions carrying pure electric or magnetic charge; under duality these transform into a 55 parameter family of solutions, one short of the desired total. The conclusion discusses a new five parameter generating configuration made purely from D3-branes at angles.

3. Supersymmetry and Branes at Angles

In the previous section we discussed the prediction from duality for the area of extremal black holes in $N = 8$ supergravity: $A = 4\pi \sqrt{J_4}$. To test this prediction we will construct complicated black holes containing many microscopic branes and compute their area. We begin, in this section, on the microscopic side (the upper right corner of Fig. 1) by identifying the most general supersymmetric bound states of branes that can be made from 2-branes and 6-branes. This can be done using the techniques of [11, 12].

Working in lightcone frame, $Q$ and $\tilde{Q}$, the two supercharges of Type II string theory, are 16 component chiral $SO(8)$ spinors. A Dp-brane imposes the projection $[13, 11]$$^3$:

$$Q \pm \Omega_p(\gamma)\tilde{Q} = 0 \quad (5)$$

$^2$For example, consider a system with 6-branes, and 2-branes on the (12), (34) and (56) cycles. Then if we could add 4-brane charge on the (3456) cycle, $z_{12}$ would be complex. The resulting phase in $Z$ could not be removed by duality.

$^3$We take $\Gamma_{11}\tilde{Q} = +\tilde{Q}$.
Here Ω is the volume form of the brane \( \Omega_p(\gamma) = \frac{1}{(p+1)!} \epsilon_{i_0 \cdots i_p} \gamma^{i_0} \cdots \gamma^{i_p} \), and is normalized to be a projection. The ± signs in Eq. 5 distinguish between branes and anti-branes or, equivalently, between opposite orientations.4

In order to study simultaneous projections imposed by many 2-branes we introduce complex coordinates \((z_1, z_2, z_3)\) on the 6-torus, related to the real coordinates via \(z_\mu = (y_{2\mu-1} + iy_{2\mu})/\sqrt{2}, \mu = 1, 2, 3.\) The corresponding complexified Gamma matrices are \(\Gamma_\mu = (\gamma^{2\mu-1} + i\gamma^{2\mu})/2\) and their complex conjugates \(\bar{\Gamma}_\mu = (\gamma^{2\mu-1} - i\gamma^{2\mu})/2.\) Take \(z_4\) and \(\bar{z}_4\) to be complex coordinates for the remaining two dimensions in lightcone frame and define \(\Gamma_4\) and \(\bar{\Gamma}_4\) accordingly. These complexified matrices obey a Clifford algebra and so define a Fock basis

\[ |n_1, n_2, n_3, n_4\rangle \]

on which \(\Gamma_\mu (\bar{\Gamma}_\mu)\) and \(\Gamma_4 (\bar{\Gamma}_4)\) act as annihilation (creation) operators. Specifically:

\[ \bar{\Gamma}_\mu \Gamma_\mu |n_1, n_2, n_3, n_4\rangle = n_\mu |n_1, n_2, n_3, n_4\rangle \]

where the \(n_\mu\) and \(n_4\) take values 0 and 1.

3.1. 2-BRANES AT ANGLES AND 6-BRANES

Using the above complex notation, a 2-brane wrapped on the \((y_1, y_2)\) cycle imposes the projection:

\[ \gamma^0 Q = \gamma^1 \gamma^2 \tilde{Q} = -i(2\bar{\Gamma} \Gamma - 1)\tilde{Q} \]

Consider a set of 2-branes rotated relative to this reference brane on the 6-torus. The \(i\)th brane is rotated by some \(R_i \in SO(6)\) and imposes the projection:

\[ \gamma^0 Q = (R_i \gamma)^1 (R_i \gamma)^2 \tilde{Q} \]

where \(R_i\) is in the fundamental representation of \(SO(6).\) 5 We can also write this as:

\[ \gamma^0 Q = -i S_{(R_i)} (2\Gamma \Gamma - 1) S^\dagger_{(R_i)} \tilde{Q} \]

where \(S_{(R_i)}\) is the spinor representation of the rotation. The Fock space elements \(|n_1, n_2, n_3\rangle \otimes |n_4\rangle\) which form a basis for the spinors \(\tilde{Q}\) are eigenstates of the \(\Gamma\)-matrix projections in Eq. 7 and Eq. 9: \(-i(2\bar{\Gamma} \Gamma - 1)|n_1 n_2 n_3\rangle \otimes |n_4\rangle = i(1 - 2n_4)|n_1 n_2 n_3\rangle \otimes |n_4\rangle.\) So there are simultaneous solutions of Eq. 7 and all the Eqs. 9 for each \(i,\) if there exist some \(\tilde{Q}\) which are singlets under all the rotations: \(S_{(R_i)} \tilde{Q} = S^\dagger_{(R_i)} \tilde{Q} = \tilde{Q}.\)

Given such a collection of \(R_i \in SO(6)\) that leave some \(\tilde{Q}\) invariant, the set of products of \(R_i\) and their inverses form a subgroup of \(SO(6).\)

4We will take the torus to have unit moduli. See [11] for extensions to a general torus.
5Of course, only a discrete subgroup of \(SO(6)\) is allowed so that winding numbers remain finite.
So the problem of finding supersymmetric relative rotations is reduced to one of finding subgroups of $SO(6)$ that leave some $\tilde{Q}$ invariant. Examining the explicit Fock basis in Eq. 6, it is clear that $|n_4\rangle$ is inert under $SO(6)$ rotations so that $\tilde{Q}$ decomposes as $16 \rightarrow 8_0 + 8_1$. The 8 indicates the 8 dimensional spinor representation of $SO(6)$ and the subscripts indicate the eigenvalue of $n_4$ in each representation. The largest subgroup of $SO(6)$ under which spinors transform as singlets is $SU(3)$, with the decomposition $8 \rightarrow 1 + 3 + \bar{3} + 1$. So an arbitrary collection of 2-branes related by $SU(3)$ rotations is supersymmetric. In fact, the branes can be related by $U(3)$ rotations because the $U(1)$ factor in $U(3) = SU(3) \times U(1)$ cancels between $S(R_i)$ and $S^\dagger(R_i)$ in Eq. 9. Concisely, a collection of 2-branes may be wrapped supersymmetrically on arbitrary (1, 1) cycles relative to some complex structure. Further global rotations of the entire configuration by $SO(6)/U(3)$ rotations may be used to turn on 2-brane charges on arbitrary 2-cycles. It is argued in [12] that the general supersymmetric bound state of 2-branes on a 6-torus has this form.

We can determine the amount of supersymmetry surviving the presence of $U(3)$ rotated D2-branes by looking for $U(3)$ invariant spinors $\tilde{Q}$. Given the reference configuration Eq. 7 and the Fock basis discussed above, it is readily shown that the $U(3)$-invariant spinors are $\tilde{Q} = \{000 \otimes |n_4\rangle, |111 \otimes |n_4\rangle\}$ where $n_4 = \{0, 1\}$. These four solutions give the equivalent of $N = 1, d = 4$ supersymmetry. D6-branes can be added without breaking any additional supersymmetry so long as we pick the orientation associated with the minus sign in Eq. 5. The 6-brane imposes the condition:

$$\gamma^0 Q = -i(2\bar{\Gamma}^1 \Gamma^1 - 1)(2\bar{\Gamma}^2 \Gamma^2 - 1)(2\bar{\Gamma}^3 \Gamma^3 - 1)\tilde{Q}$$

which is solved by the same spinors that survive the presence of the 2-branes. T-duality of the entire 6-torus converts the 6-branes into 0-branes. Furthermore, 2-branes on some (1, 1) cycle turn into 4-branes on the dual (2, 2) cycle. So we learn that 4-branes wrapped on arbitrary (2, 2) cycles are supersymmetric since they are related by relative $U(3)$ rotations, and 0-branes may be bound to them without breaking supersymmetry.

4. Classical Solutions for Branes at Angles

Having identified supersymmetric microscopic configurations of 2-branes and 6-branes, we would like to find the corresponding classical solutions. These will turn out to be four dimensional extremal black holes, giving us an opportunity to test the $E(7, 7)$ invariant prediction for black hole entropy.

Choosing complex coordinates $z^j = (x^{2j-1} + ix^{2j})/\sqrt{2}$ on the $T^6$, the Kähler form and volume of the asymptotic torus are $k = i \sum_{j=1}^{3} dz^j \wedge d\bar{z}^j$.
and Vol($T^6$) = \( \int_{T^6} d\text{Vol} = \int_{T^6} k \wedge k \wedge k / 3! \). The volume is set to 1 by taking the asymptotic moduli to be unity.\(^6\) Following the previous section, we consider a supersymmetric collection of 2-branes wrapped on (1, 1) cycles. Thinking of each brane as being $U(3)$ rotated relative to the given complex structure, the $j$th brane is characterized by the (1, 1) volume form $\omega_j = i (R_{(j)})^1_1 (R_{(j)})^1_2 dz^J \wedge d\bar{z}^K$. Such wrapped branes produce a pressure on the geometry causing it to deform between the position of the branes and the flat space at infinity.

Remarkably, the full solution can be understood in terms of a single (1, 1) form $\omega$ characterizing the ensemble of branes: $\omega = \sum_j X_j \omega_j$ Here the $X_j$ are harmonic functions in the transverse space $X_j = P_j / r$ and $P_j$ is the charge of the $j$th 2-brane.\(^7\) Since the classical solutions will only depend on $\omega$, many different microscopic configurations of branes will have the same macroscopic solution - this is a reflection of the no-hair theorem. It is also useful to define the intersection numbers $C_{ij} = (1 / \text{Vol}(T^6)) \int_{T^6} k \wedge \omega_i \wedge \omega_j$ and $C_{ijk} = (1 / \text{Vol}(T^6)) \int_{T^6} \omega_i \wedge \omega_j \wedge \omega_k$. Then $C_{ijk}$ is proportional to the number of points at which a T-dual collection of 4-branes intersect on the 6-torus. This connection will be used in Sec. 5 in computing the degeneracy of the solutions constructed in this section.

A classical solution corresponding to a collection of 6-branes and 2-branes at angles on a 6-torus is completely described in terms of the metric, the dilaton, the $RR$ 3-form gauge field and the $RR$ 7-form gauge field. The solution in string metric is:

$$
\begin{align*}
\text{ds}^2 &= \left( F_2 F_6 \right)^{\frac{3}{4}} dx_+^2 + \left( F_2 F_6 \right)^{-\frac{1}{4}} \left[ -dt^2 + (h_{\mu \bar{\nu}} \, dz^\mu \wedge d\bar{z}^\nu + h_{\mu \nu} \, d\bar{z}^\mu \wedge dz^\nu) \right] \\
A_{(3)} &= \frac{1}{F_2} dt \wedge K \\
e^{-2\Phi} &= \sqrt{\frac{F_3^3}{F_2}}
\end{align*}
$$

where the 2-form $K$ is:

$$
K \equiv \ast \frac{(k + \omega) \wedge (k + \omega)}{2!}
$$

and is simply proportional to the internal Kähler metric in the presence of 2-branes:

$$
G = ig_{\mu \bar{\nu}} \, dz^\mu \wedge d\bar{z}^\nu = \frac{i}{\sqrt{F_2 F_6}} h_{\mu \bar{\nu}} \, dz^\mu \wedge d\bar{z}^\nu \equiv \frac{K}{\sqrt{F_2 F_6}}
$$

\(^6\)We follow the conventions of [12] for the normalization of forms, wedge products and Hodge dual.

\(^7\)In general, we could separate the branes in the transverse space by choosing $X_j = P_j / |r - \bar{r}_j|$ but will not do so here since we are interested in four-dimensional black holes.
The functions $F_2$ and $F_6$ have simple expressions:

$$F_2 = \frac{f_0 (k + \omega)^3}{3\text{Vol}(T^6)} = 1 + \sum_i X_i + \sum_{i<j} X_i X_j C_{ij} + \sum_{i<j<k} X_i X_j X_k C_{ijk}$$

$$F_6 = 1 + \frac{Q_6}{r} \tag{15}$$

Here $dx^2_\perp = dx^2_7 + dx^2_8 + dx^2_9$ refers to the noncompact part of the space.

In [12] it is explicitly shown that this solution satisfies the spacetime equations of motion and is supersymmetric.\(^8\) The four dimensional Einstein metric is related to the string metric by

$$e^{-2\Phi} 4 = e^{-2\Phi} \sqrt{\text{det} g_{\text{int}}} = \sqrt{F_6^3/F_2^2/F_6^3} = 1.$$ So the Einstein metric is:

$$ds_4^2 = (F_2 F_6)^{-1/2} (-dt^2) + (F_2 F_6)^{1/2} (dr^2 + r^2 d\Omega^2)$$

which describes a black hole with horizon at $r = 0$. The area of the sphere at radius $r$ is

$$A = 4\pi r^2 (F_2 F_6)^{1/2}.$$ In the limit $r \to 0$ this gives the area:

$$A = 4\pi (Q_6 \sum_{i<j<k} P_i P_j P_k C_{ijk})^{1/2} \tag{16}$$

In order to compare this with the prediction from duality it is convenient to project the brane charges $P_j$ onto the different 2-cycles of the torus. Defining a basis for $(1,1)$ forms $\Omega^{ab} = idz^a \wedge d\bar{z}^b$, the charge on the $(ab)$ cycle is given by

$$q_{ab} = \sum_i P_i \alpha_{iab} \omega_i = \sum_{ab} \alpha_{iab} \Omega^{ab}.$$ In terms of this charge matrix, the area in Eq. 16 can be rewritten as:

$$A = 4\pi \sqrt{Q_6 \det q} \tag{17}$$

Now consider the prediction from duality that $A = 4\pi \sqrt{J_4}$. Comparing with Eq. 2 for $J_4$, all the $x_{ij}$ vanish since there are no 4-branes or 0-branes. Then, $y_{ij}$ for $i, j \leq 6$ represent 2-brane charges on different cycles while $y_{78}$ is the 6-brane charge. Transforming the $J_4$ symbol into complex coordinates as in this section, it becomes

$$J_4 = Q_6 \det q$$

matching Eq. 17 for an arbitrary number of 2-branes at angles in the presence of 6-branes.

5. Counting the States of Our Black Holes

The first step towards a microscopic understanding of the entropy of our black holes is to understand how the physical charges $P_i$ and $Q_6$ are related to the wrapping numbers of branes on different cycles. Following [12], the

\(^8\) There is an extensive literature on intersecting branes. See [14], for example. Classical branes at angles are discussed, among other articles, in [15]. More references are provided in [12].
quantization condition for 6-branes is \( Q_6 = (l_s g/4\pi)N_6 \) where \( N_6 \) is the number of times the 6-branes wrap the entire torus and \( l_s = 2\pi\sqrt{\alpha'} \) is the string length. Similarly, if \( P_j \) and \( \omega_j \) are the physical charge and volume form characterizing the jth 2-brane, we have \( P_j\omega_j = (l_s g/4\pi)M_jv_j \) where \( M_j \) is the wrapping number and \( v_j \) is the element of integral cohomology characterizing the cycle on which the brane is wrapped.

We then define \( N_{ijk} = \frac{1}{\text{vol}(T^6)} \int_{T^6} v_i \wedge v_j \wedge v_k \). T-duality of the entire 6-torus turns 2-branes wrapped on \( v_i \) into 4-branes wrapped on \( *v_i \). Three 4-branes on a 6-torus generically intersect at a point and the \( N_{ijk} \) are integers counting the number of intersection points. Using the quantized charges we find that

\[
\sum_{i<j<k} P_i P_j P_k C_{ijk} = (l_s g/4\pi)^3 \sum_{i<j<k}(M_i M_j M_k)N_{ijk}
\]

Finally, the entropy of the black hole is given by \( S = 4\pi \sqrt{N_6} \sum_{i<j<k} (M_i M_j M_k)N_{ijk} \).

Putting everything together we find that:

\[
S = 2\pi \sqrt{N_6 \sum_{i<j<k} (M_i M_j M_k)N_{ijk}} \tag{18}
\]

We will now explain the microscopic origin of this entropy.

To count the states of our black holes it is easiest to dualize the entire 6-torus so that the 6-branes become 0-branes, and the 2-branes become 4-branes wrapped on the cycles \( *v_j \). Of course, the entropy in Eq. 18 remains the same, with \( N_{ijk} \) counting the number of points at which the ith, jth and kth 4-branes intersect. In fact, it is convenient to lift the system into 11 dimensions and view the 4-branes as M-theory 5-branes intersecting along the 11th circle. Imagine making the torus small and the 11th circle big. Then the mutual intersection of any triple of 5-branes is a string that has \((0, 4)\) supersymmetry [6, 16, 12, 17]. Putting momentum along this effective string would induce 0-brane charge from the 10 dimensional perspective. However, since we want to preserve supersymmetry we can only introduce momentum in the left-moving direction. From the D-brane point of view, this means that only one of the two signs in Eq. 5 is allowed, as observed for the 6-branes at the end of Sec. 3.

So the entropy of the black hole arises from the number of ways in which \( N_6 \) units of left-moving momentum can be distributed amongst the effective intersection strings. It was argued heuristically in [6, 16, 18] that the left moving, non-supersymmetric sector of the effective string has a central charge \( c = 6 \). As shown more carefully in [17] there are indeed 6 bosons in the left moving sector, corresponding to the position of the effective string on the \( T^6 \). Actually, the authors of [17] count the states of black holes by moving away from the singular limit of intersecting branes. Instead they consider a single M5-brane wrapped on \( S^1 \) times a complicated 4-cycle in a Calabi-Yau 3-fold and count its fluctuations. Their formulae do not directly apply to tori which introduce some new features. So we will
not use their techniques here and will remain in the intersecting brane limit following [12].

First consider a situation with \( N_6 \gg \sum N_{ijk} M_i M_j M_k \). Each of the \( N_{int} = \sum N_{ijk} M_i M_j M_k \) effective strings has 6 bosons in its leftmoving sector. So the problem is simply to distribute \( N_6 \) units of momentum amongst \( 6N_{int} \) bosons. This is identical to the problem of computing the density of states of a string with central charge \( c_{eff} = 6N_{int} \) and for large \( N_6 \) this can be read off from \([19]\): \( d(N_6) = \exp 2\pi \sqrt{N_6 c_{eff}/6} \). Taking the logarithm we find the entropy:

\[
S = 2\pi \sqrt{N_6 \sum_{i<j<k} (M_i M_j M_k) N_{ijk}}
\]  

which exactly matches Eq. 18. Here we assumed that at a given intersection point of the ith, jth and kth branes there are \( M_i M_j M_k \) effective strings singly wound around the 11th circle. We can relax the condition \( N_6 \gg \sum N_{ijk} M_i M_j M_k \) by including the multiply wound strings arising from multiply wound 5-branes.

The easiest way to deal with this is to observe that a given triple intersection of 5-branes with wrapping numbers \( M_i, M_j \) and \( M_k \) will be described by a \((0,4)\) supersymmetric sigma model on the orbifold target:

\[
\mathcal{M} = \frac{(T^6)^{M_i M_j M_k}}{S(M_i M_j M_k)}
\]  

We orbifold by the symmetric group \( S(M_i M_j M_k) \) to account for symmetry under exchange of the 5-branes.\(^9\) We are interested in putting momentum on the leftmoving side where there are 6 bosons characterizing the position on \( T^6 \). In our case, each triplet of 5-branes intersects in \( N_{ijk} \) locations giving rise to \( N_{ijk} \) effective strings. So from the M-theory perspective, our solutions are described in the small torus limit by \( N_{tot} = \sum_{i<j<k} N_{ijk} \) effective strings, each propagating on an orbifold like Eq. 20. The appropriate total effective conformal field theories have central charges that are the sum of contributions from many effective strings, and the resulting degeneracy exactly matches our Eq. 18 following \([20, 21]\).

### 6. Discussion: 5 Parameter Generating Solution

So far we have shown that duality prediction for the entropy of a black hole is verified in detail for general supersymmetric systems of 2-branes.

\(^9\) One might have naively supposed that the appropriate orbifold group would be \( S(M_i) S(M_j) S(M_k) \). This would account for the exchange symmetry of each kind of 5-brane. However, the analysis of \([2, 20]\) indicates that the appropriate group is \( S(M_1 M_2 M_3) \). Indeed, this is the orbifold that is consistent with T-duality.
and 6-branes or 4-branes and 0-branes. Classical solutions containing arbitrary numbers of branes at angles have the appropriate horizon area and it is possible to count the states microscopically. However, as discussed in Sec. 2.1, the orbit under duality of the configurations discussed so far does not account for the full 56 charge spectrum of four dimensional black holes. One way to see this is to observe that the entropy in Eq. 17 is characterized by four invariant parameters - \( Q_6 \) and the 3 real eigenvalues of the charge matrix \( q \). Equivalently, in the notation of Sec. 2, we have studied configurations in which the charge matrix \( z_{ij} \) is either pure real or pure imaginary.

The additional invariant phase in the charge matrix that is absent in our solutions is associated with the presence of electric-magnetic dual pairs at the same time. It is therefore natural to expect that the desired generating solution can be constructed solely out of self-dual 3-branes.\(^{10}\) Due to lack of space, the corresponding classical solutions and the counting of states will be discussed elsewhere. Here we show how 3-branes at angles can be a convenient generating configuration and derive the duality prediction for the entropy.

Let \( z^i = (x^{2i-1} + ix^{2i})/\sqrt{2} \) be complex coordinates for a 6-torus. Consider any collection of 3-branes wrapped on cycles \( \text{Re}(\tilde{z}^1, \tilde{z}^2, \tilde{z}^3) \), with \( \tilde{z}^i = R^i_j z^j \) where \( R \in SU(3) \). Such a collection of 3-branes is supersymmetric [11]. We consider diagonal \( SU(3) \) rotations \( R = \text{Diag}(e^{-i\theta_1}, e^{-i\theta_2}, e^{-i\theta_3}) \) that rotate the branes separately on the (12), (34) and (56) tori. The generating configuration has the following branes at angles:

| Wrapping Numbers | \( \theta_1 \) | \( \theta_2 \) | \( \theta_3 \) |
|------------------|----------------|----------------|----------------|
| \( N_1 \)        | 0              | 0              | 0              |
| \( N_2 \)        | 0              | \( \pi/2 \)    | \( -\pi/2 \)   |
| \( N_3 \)        | \( -\pi/2 \)   | 0              | \( \pi/2 \)    |
| \( N_4 \)        | \( \theta \)   | \( -\theta \)  | 0              |

Here \( \theta_i \) is the angle on the \((x^{2i-1}, x^{2i})\) torus and \( N_j \) are the wrapping numbers on the rotated 3-cycles. In order that all wrapping numbers are finite, \( \cot \theta = p/q \) is rational. If \( N(stu) \) is the induced 3-brane charge on the \((stu)\) cycle we have: \( N(135) = N_1 + N_4q^2 \), \( N(146) = -N_2 \), \( N(236) = -N_3 \), \( N(245) = -N_4q^2 \), \( N(145) = -pqN_4 \), \( N(235) = pqN_4 \).

\(^{10}\)In writing this section I have benefitted from Finn Larsen’s unpublished notes concerning the \( E_7 \) quartic invariant and black holes.
To show that this configuration displays the five parameters required in a generating solution, we T-dualize on the 1, 3 and 5 cycles. The first three sets of branes turn into 0-branes and 4-branes, and the last set turns into 4-branes with worldvolume fluxes turned on. The worldvolume fluxes induce additional 0-brane and 2-brane charges via the Chern-Simons couplings on the 4-brane \[22\]. The net charges can be read off from the \(N(stu)\) 3-brane charges. We have \(Q_0 = N_1 + N_4 p^2\) 0-branes, and \(-N_4 q^2\) 4-branes on the (1234) cycle. There are \(-N_3\) and \(-N_2\) 4-branes on the (1256) and (3456) cycles with \(-N_4 pq\) and \(N_4 pq\) 2-branes on the dual (34) and (12) cycles. Assembling these charges into the \(J_4\) invariant in Eq. 2 we find:

\[
J_4 = Q_0 Q_2 Q_3 Q_4 - \frac{Q_2^2 \cot^2 \theta}{4} (Q_2 + Q_3)^2
\]

where \(Q_2 = N_2\), \(Q_3 = N_3\), \(Q_4 = N_4 q^2\) and \(\cot \theta = p/q\). The second term in this equation arises from the mixing of the real and imaginary parts of the charges \(z_{ij}\). This was not probed by the configurations studied in earlier sections.

To verify that this is a five parameter generating configuration we transform to the \(SU(8)\) representation of the charge matrix following Eq. 3. It is helpful to choose variables \(P_L = Q_3 - Q_2\), \(P_R = -(Q_3 + Q_2)\), \(Q_L = Q_0 - Q_4\), \(Q_R = Q_0 + Q_4\) and \(\beta = Q_4 \cot \theta\). Then the \(SU(8)\) form of the charge matrix is:

\[
Z_{12} = \frac{1}{4}(Q_L - P_L - 2i\beta) \quad ; \quad Z_{56} = \frac{1}{4}(Q_R + P_R) \quad (22)
\]

\[
Z_{34} = \frac{1}{4}(Q_L + P_L + 2i\beta) \quad ; \quad Z_{78} = \frac{1}{4}(Q_R - P_R) \quad (23)
\]

The resulting \(J_4\) invariant is:

\[
J_4 = \left[ \left( \frac{Q_R^2 - Q_L^2}{4} \right) \left( \frac{P_R^2 - P_L^2}{4} \right) - \left( \frac{\beta^2 P_R^2}{4} \right) \right]
\]

Written this way, the charge matrix and \(J_4\) invariant are precisely the same as those of the NS-NS, five parameter generating solution discussed in [4, 3]. This shows that the classical solution for the configuration of 3-branes at angles in Table 1 is also a generating solution.

A definitive way of counting the states of this generating configuration is to extend the analysis of [17] to tori. The 4-branes then become M5-branes wrapped on a circle and the 2-brane charges become flux on the 5-brane. We hope to report on this elsewhere. Counting the states of our generating configuration accounts, after duality, for the entropy of the entire 56 parameter space of extremal black holes in \(N = 8\) supergravity.
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