SOME REMARKS ON TOROIDAL MORPHISMS

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1. Introduction

This note contains some results related to the definition of toroidal morphisms over a field $k$ of characteristic zero. In [2] this notion was defined by requiring that the base change of the morphism to an algebraic closure of $k$ is toroidal. The notion of a toroidal morphism $f$ over an algebraically closed field was introduced long before by several authors, see e.g. [1] and [5]. Roughly the definition requires that for each closed point $x$ of the source of $f$ one can choose formal toric coordinates at $x$ and formal toric coordinates at $f(x)$, such that in these coordinates the morphism is given by monomials. When $k$ is not algebraically closed, there is the natural question whether this remains true over the base field $k$ itself instead of over an algebraic closure of $k$. In this note we show that the answer is yes for toroidal morphisms between strict toroidal embeddings if the residue field $k(x)$ of $x$ equals $k$ or if $k(x)$ is algebraically closed. This is implied by Proposition 3.3 below which is actually a stronger statement. An easy counterexample (Remark 3.4) shows that the condition on $k(x)$ cannot be omitted. Proposition 3.3 can be proved using Kato’s paper [10], adapting the argument in section 3.13 of [10]. However we preferred to provide a self-contained proof which does not use logarithmic geometry. Proposition 3.3 in the special case of nonsingular toroidal embeddings is used in [7] for applications of toroidalization to model theory. Proposition 3.3 also implies that in the definition of toroidal morphisms, as formulated in [2], we can replace the completions by henselizations. This also holds for the definition of toroidal embeddings, see Remark 2.4 below.

We will use without further mentioning the terminology and notation of [2], in particular we refer to [2] for the notions of toroidal embeddings, strict toroidal embeddings, and toroidal morphisms. Moreover, $k$ will always denote a field of characteristic zero, except that the material in 1.1 up to 2.5 remains valid for any field $k$. 

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2. Toroidal Embeddings

In [2] an open embedding of algebraic varieties \( U \subset X \) over \( k \) is defined to be toroidal if its base change to an algebraic closure of \( k \) is a toroidal embedding. Nevertheless we have the following proposition.

**Proposition 2.1.** Let \( U \subset X \) be a strict toroidal embedding of varieties over \( k \), and \( x \) a closed point of \( X \). Then there exists an affine toric variety \( V \) over \( k \) and an étale \( k \)-morphism \( \varphi \) from an open neighborhood of \( x \) in \( X \) to \( V \), such that locally at \( x \) (for the Zariski topology) we have \( U = \varphi^{-1}(T) \), where \( T \) is the big torus of \( V \).

This is proved in [12], page 195, when \( k \) is algebraically closed. However the proof remains valid in the general case due to Lemma 2.3 below.

**Definition 2.2.** We call \((V,\varphi)\), with \( V \) and \( \varphi \) as in Proposition 2.1, an étale chart for \( U \subset X \) at \( x \). Note that there always exists an étale chart for \( U \subset X \) at \( x \) such that \( \varphi(x) \) belongs to the closed orbit of \( V \).

**Lemma 2.3.** Let \( X \) be a normal algebraic variety over \( k \) and \( D \) a strict Weil divisor on \( X \), i.e. the irreducible components of \( D \) are normal. Let \( x \) be a closed point of \( X \) and \( \tilde{O}_{X,x} \) the completion of a strict henselization of \( O_{X,x} \). Let \( y \in \tilde{O}_{X,x} \) and assume that the Weil divisor \( \text{div}(y) \) of \( y \) on \( \text{Spec}(\tilde{O}_{X,x}) \) is supported on the preimage \( \tilde{D} \) of \( D \) in \( \text{Spec}(\tilde{O}_{X,x}) \). Then there exists \( z \in O_{X,x} \) such that \( y/z \) is a unit in \( \tilde{O}_{X,x} \).

**Proof.** This is well known. Since we could not find a good reference, we include a proof. Because the natural morphism \( \text{Spec}(\tilde{O}_{X,x}) \to \text{Spec}(O_{X,x}) \) is flat, it induces a morphism \( \tau \) from the group of Weil divisors on \( \text{Spec}(O_{X,x}) \) to the group of Weil divisors on \( \text{Spec}(\tilde{O}_{X,x}) \), cf. [1] Proposition 21.10.6. Moreover \( \tau \) restricts to a bijection between the divisors supported on \( D \) and those supported on \( \tilde{D} \), because the ideal in \( O_{X,x} \) of any irreducible component of \( D \) containing \( x \) generates a prime ideal in \( \tilde{O}_{X,x} \), since these components are normal. Thus there exists a Weil divisor \( W \) on \( \text{Spec}(O_{X,x}) \) such that \( \tau(W) = \text{div}(y) \). The morphism \( \tau \) induces an injection on the groups of divisor classes, indeed this follows easily by adapting the proof of Proposition 16 in section 1.10 of [4]. Hence there exists \( z \in O_{X,x} \) such that \( \text{div}(z) = W \). Thus \( \tau(\text{div}(z)) = \text{div}(y) \) and \( y/z \) is a unit in \( \tilde{O}_{X,x} \).

**Remark 2.4.** Proposition 2.1 directly implies that in the definition of strict toroidal embeddings of varieties over \( k \), as formulated in [2], we can replace the completions by henselizations. Hence this also holds
for toroidal embeddings that are not necessarily strict. Indeed, for any toroidal embedding $U_X \subset X$ over $k$, and any closed point $x$ of $X$, there exists an étale morphism $f : X' \to X$ onto a neighborhood of $x$ such that $f^{-1}(U) \subset X'$ is a strict toroidal embedding. This assertion follows from Lemma 2.5 below with $A$ the strict henselization of $\mathcal{O}_{X,x}$.

**Lemma 2.5.** Let $A$ be an excellent normal henselian local ring, and $P$ a prime ideal of height 1 in $A$. Let $\hat{A}$ be the completion of $A$. Assume for each height one prime ideal $P'$ of $\hat{A}$, with $P$ contained in $P'$, that $\hat{A}/P$ is normal. Then $A/P$ is normal.

**Proof.** Because $A/P$ is excellent, henselian and integral, its completion is also integral, by Corrolaire 18.9.2 in [9] (or by Artin’s Approximation Theorem [3] if $A$ is moreover the strict henselization of the local ring of a closed point on a variety over $k$). Thus $PA$ is a prime ideal of $\hat{A}$. Since $PA$ has height one (because $A$ and $\hat{A}$ are catenary), our assumption implies that $\hat{A}/PA$ is normal. Thus the completion of $A/P$ is normal. By faithfully flat descent this implies that $A/P$ is normal (see e.g. Remark 2.24 in chapter 1 of [13]). □

**Definition 2.6.** Let $U_X \subset X$ be a strict toroidal embedding of varieties over $k$. Denote by $D := X \setminus U_X$ the toroidal divisor. The sheaf of logarithmic differential 1-forms on $X$ is defined as the sheaf of $\mathcal{O}_X$-modules $\Omega^1_X(\log D) := j_*(\Omega^1_{X_0}(\log D \cap X_0))$, where $j : X_0 \to X$ is any nonsingular open subscheme of $X$ with codimension $\geq 2$ such that $D \cap X_0$ is nonsingular.

The sheaf $\Omega^1_X(\log D)$ of $\mathcal{O}_X$-modules is locally free: as basis in a neighborhood of a closed point $x$ of $X$ one can take $\frac{dx_1}{x_1}, \ldots, \frac{dx_m}{x_m}$, where $x_1, \ldots, x_m$ are the pullbacks, to the function field $K(X)$ of $X$, of the elements of a basis for the $\mathbb{Z}$-module of characters of the big torus of $V$, for any étale chart $(V, \varphi)$ for $U_X \subset X$ at $x$. Indeed, this follows from Proposition 15.5 in [6] (the assumption there that $k = \mathbb{C}$ is not necessary).

3. Logarithmically smooth morphisms

**Definition 3.1.** Let $U_X \subset X$ and $U_B \subset B$ be strict toroidal embeddings of varieties over $k$, and $x$ a closed point of $X$. Denote the toroidal divisors by $D_X := X \setminus U_X$, $D_B := B \setminus U_B$. Let $f : X \to B$ be a dominant $k$-morphism mapping $U_X$ into $U_B$. The morphism $f$ is called logarithmically smooth at $x$ (with respect to $U_X \subset X$ and $U_B \subset B$) if the sheaf of $\mathcal{O}_X$-modules

$$\Omega^1_X(\log D_X)/f^*(\Omega^1_B(\log D_B))$$
is locally free at \( x \). This is equivalent with the condition that the fiber at \( x \) of this sheaf has dimension \( \dim X - \dim B \) as vector space over the residue field \( k(x) \) of \( X \) at \( x \). (Note that we required \( f \) to be dominant.)

**Remark 3.2.** Clearly, if \( f : X \to B \) is toroidal with respect to \( U_X \subset X \) and \( U_B \subset B \), then \( f \) is logarithmically smooth at each closed point of \( X \). The converse is also true, this follows from Theorem 3.5 and Proposition 3.12 in [10], and section 8.1 of [11]. However this converse is also implied by Proposition 3.3 below, which is a stronger assertion. Proposition 3.3 can be proved, adapting the argument in section 3.13 of [10]. Because this argument is phrased in the framework of logarithmic structures on schemes, we give below an elementary self-contained proof of Proposition 3.3 which does not use logarithmic geometry.

**Proposition 3.3.** Let \( U_X \subset X \) and \( U_B \subset B \) be strict toroidal embeddings of varieties over \( k \), and \( x \) a closed point of \( X \). Let \( f : X \to B \) be a dominant \( k \)-morphism mapping \( U_X \) into \( U_B \). Set \( b := f(x) \). Let \((V_B, \varphi_B)\) be an étale chart for \( U_B \subset B \) at \( b \). Assume that \( f \) is logarithmically smooth at \( x \) with respect to \( U_X \subset X \) and \( U_B \subset B \). Assume also that \( k(x) = k \) or that \( k(x) \) is algebraically closed.

Then there exist

1. an étale \( k \)-morphism \( \pi : X' \to X \),
2. a closed point \( x' \) on \( X' \) with \( \pi(x') = x \) and \( k(x') = k(x) \),
3. an étale chart \((V_{X'}, \varphi_{X'})\) for \( U_{X'} := \pi^{-1}(U_X) \subset X' \) at \( x' \),
4. a toric morphism \( g : V_{X'} \to V_B \),
5. a translation \( t : V_B \to V_B \) by a \( k \)-rational point on the big torus of \( V_B \),

such that the following diagram of rational maps commutes

\[
\begin{array}{ccc}
X' & \xrightarrow{\varphi_{X'}} & V_{X'} \\
\downarrow \pi & & \downarrow g \\
X & \xrightarrow{f} & V_B \\
\end{array}
\]

If \((V_X, \varphi_X)\) is any étale chart for \( U_X \subset X \) at \( x \) such that \( \varphi_X(x) \) belongs to the closed orbit of \( V_X \), then we can choose \( V_{X'} = V_X \), with \( \varphi_{X'}(x') \) in the orbit of \( \varphi_X(x) \). And when \( k(x) \) is algebraically closed we can moreover take for \( t \) the identity.

**Proof.** The proof consists of several steps.

Some reductions. Let \((V_X, \varphi_X)\) be an étale chart for \( U_X \subset X \) at \( x \) such that \( \varphi_X(x) \) belongs to the closed orbit of \( V_X \). Replacing \( B \) and \( X \) by suitable open subvarieties we may suppose that \( \varphi_B \) and \( \varphi_X \) are
defined and étale everywhere. We can assume that \( \dim B = \dim X \)
by replacing \( f : X \rightarrow B \) by \( f \times (h \circ \varphi_X) : X \rightarrow B \times \mathbb{A}^{\dim X - \dim B} \),
with \( h : V_X \rightarrow \mathbb{A}^{\dim X - \dim B} \) a general enough toric morphism so that \( f \times (h \circ \varphi_X) \) is still dominant and logarithmically smooth at \( x \).

Choose a basis \( c_1, \ldots, c_n \) for the \( \mathbb{Z} \)-module of characters on the big torus \( T_B \) of \( V_B \). We will denote \( \varphi_B^*(c_i) \) again by \( c_i \), for \( i = 1, \ldots, n \).

**Choosing character bases.** Choose a basis \( z_1, \ldots, z_r, z_{r+1}, \ldots, z_m \)
for the \( \mathbb{Z} \)-module of characters on the big torus \( T_X \) of \( V_X \) such that \( z_{r+1}, \ldots, z_m \) form a basis for the \( \mathbb{Z} \)-module of characters on \( T_X \) that are defined and not vanishing at \( \varphi_X(x) \). We will denote \( \varphi_X^*(z_i) \) again by \( z_i \), for \( i = 1, \ldots, m \). Because \( \dim B = \dim X \) we have \( m = n \).

From Lemma 3.6 below it follows that for \( j = 1, \ldots, n = m \) we can write
\[
(1) \quad f^*(c_j) = u_j z_1^{e_{j,1}} z_2^{e_{j,2}} \cdots z_n^{e_{j,n}},
\]
with the \( u_j \) suitable units in \( \mathcal{O}_{X,x} \). Moreover we can choose \( e_{j,r+1}, \ldots, e_{j,n} \)
arbitrarily if we adapt the \( u_j \) to these choices, since \( z_{r+1}, \ldots, z_n \) are units in \( \mathcal{O}_{X,x} \).

**Changing coordinates by Hensel’s Lemma.** Let \( J \) be the logarithmic jacobian matrix of \( f \), i.e. the square matrix consisting of the coefficients expressing \( f^*(\frac{dx}{c_j}) \), \( j = 1, \ldots, n \), as \( \mathcal{O}_{X,x} \)-linear combinations of \( \frac{dx}{z_i} \), \( i = 1, \ldots, n \). We denote by \( J(x) \) the square matrix over \( k(x) \) obtained from \( J \) by evaluation at \( x \in X \). Because \( f \) is logarithmically smooth at \( x \) we have \( \det J(x) \neq 0 \). From (1) and Lemma 3.6 it follows that the first \( r \) columns of \( J(x) \) equal the first \( r \) columns of the matrix \( E := (e_{j,i})_{j,i=1,\ldots,n} \). Thus the first \( r \) columns of \( E \) are linearly independent. Since the last \( n-r \) columns of \( E \) can be chosen arbitrarily, we can choose these such that \( \det(E) \neq 0 \).

For \( j = 1, \ldots, n \) we set \( \lambda_j := 1 \) if \( k(x) \) is algebraically closed. Otherwise \( k(x) = k \), by our assumption on \( k(x) \), and then we set \( \lambda_j := u_j(x) \in k \).

Hence for \( j = 1, \ldots, n \) we can write
\[
(2) \quad f^*(c_j) = \lambda_j w_j z_1^{e_{j,1}} z_2^{e_{j,2}} \cdots z_n^{e_{j,n}},
\]
with the \( w_j \) suitable units in \( \mathcal{O}_{X,x} \). Moreover \( w_j(x) = 1 \) when \( k \) is not algebraically closed.

Since \( \det(E) \neq 0 \), it follows from Hensel’s Lemma that there exist units \( \epsilon_1, \ldots, \epsilon_n \) in the henselization of \( \mathcal{O}_{X,x} \) such that for \( j = 1, \ldots, n \)
\[
(3) \quad w_j = \epsilon_1^{e_{j,1}} \epsilon_2^{e_{j,2}} \cdots \epsilon_n^{e_{j,n}}.
\]

We will use the change of coordinates \( z_i \leftarrow \epsilon_i z_i \), but to make this precise we have to go to an étale extension \( X' \) of \( X \) and use a new étale chart \( (V_{X'}, \varphi_{X'}) \).
Construction of $\pi : X' \to X$. There exists an étale morphism $\pi : X' \to X$ and a closed point $x'$ on $X'$, with $\pi(x') = x$ and $k(x') = k(x)$, such that $\epsilon_1, \ldots, \epsilon_n$ are units in $\mathcal{O}_{X', x'}$. We may even assume that $X'$ is affine and that $\epsilon_1, \ldots, \epsilon_n$ are units in the coordinate ring of $X'$. Note that $(V_X, \varphi_X \circ \pi)$ is an étale chart at $x'$ for the toroidal embedding $U_{X'} := \pi^{-1}(U_X) \subset X'$, but we will need another chart $(V_{X'}, \varphi_{X'})$.

Construction of the chart $(V_{X'}, \varphi_{X'})$. By multiplicativity, the assignment $z_i \mapsto \epsilon_i$ extends uniquely to a homomorphism $\epsilon$ from to group of characters of $T_X$ to $\Gamma(\mathcal{O}_{X'}, X')^\times$. Set $V_{X'} := V_X$ and let $\varphi_{X'} : X' \to V_{X'} = V_X$ be the unique $k$-morphism with
\begin{equation}
\varphi_{X'}^*(z) = \epsilon(z)(\varphi_X \circ \pi)^*(z),
\end{equation}
for each character $z$ of $T_X$ (note that the characters of $T_X$ that are regular on $V_X$ generate the coordinate ring of $V_X$). Note that $\varphi_{X'}(x')$ belongs to the orbit of $\varphi_X(x)$ under the action of $T_X$. We show below that the pair $(V_{X'}, \varphi_{X'})$ is an étale chart at $x'$ for $U_{X'} \subset X'$.

Construction of translation $t$ and toric morphism $g$. Let $t : V_B \to V_B$ be the translation by the $k$-rational point of $T_B$ on which the character $c_j$ takes the value $\lambda_j^{-1}$, for $j = 1, \ldots, n$.

Finally, let $g : V_{X'} \to V_B$ be the toric rational map defined by
\begin{equation}
g^*(c_j) = z_1^{e_{j,1}} z_2^{e_{j,2}} \cdots z_n^{e_{j,n}},
\end{equation}
for $j = 1, \ldots, n$. We show below that $g$ is regular at each point of $V_{X'}$, i.e. $g$ is a morphism. From (4), (3), and (2) it follows that
\begin{equation}
(t \circ \varphi_B) \circ f \circ \pi = g \circ \varphi_{X'}.
\end{equation}
Thus the diagram in 3.3 is indeed commutative.

The rational map $g$ is regular on $V_{X'}$. To prove this it suffices to show that $g^*(c)$ is regular on $V_{X'}$, for each character $c$ of $T_B$ that is regular on $V_B$. From (6) it follows that $(g \circ \varphi_{X'})^*(c)$ is regular on $X'$, hence by (4) also $(\varphi_X \circ \pi)^*(g^*(c))$ is regular on $X'$. Thus $g^*(c) \in \mathcal{O}_{V_{X'}, \varphi_{X}(x)}$ because the homomorphism $\mathcal{O}_{V_{X'}, \varphi_{X}(x)} \to \mathcal{O}_{X', x'}$ induced by $\varphi_X \circ \pi : X' \to V_{X'} = V_X$ is faithfully flat, since $\varphi_X \circ \pi$ is étale. Moreover $g^*(c)$ is a character of $T_X$, hence its divisor on $V_X$ is supported on $V_X \setminus T_X$. Because $\varphi_X(x)$ belongs to the closed orbit of $V_X$, all irreducible components of $V_X \setminus T_X$ contain $\varphi_X(x)$. Since we know already that $g^*(c)$ is regular at $\varphi_X(x)$, we conclude that $g^*(c)$ is regular at each point of $V_{X'}$.

The pair $(V_{X'}, \varphi_{X'})$ is an étale chart at $x'$ for $U_{X'} \subset X'$. For this it suffices to show that $\varphi_{X'} : X' \to V_{X'} = V_X$ is étale at $x'$, because $\epsilon(z)$ in formula (4) is a unit in $\Gamma(\mathcal{O}_{X'}, X')$. Since $f$ is logarithmically smooth at
Lemma 3.5. Let $x$, formula (6) implies that $g \circ \varphi_{X'}$ is logarithmically smooth at $x'$. Since $\dim V_B = \dim V_{X'}$, this implies (by the definition of logarithmically smooth) that $\varphi_{X'}$ is logarithmically smooth at $x'$ with respect to $U_{X'} \subset X'$ and $T_X \subset V_X$. Hence Lemma 3.8 below (with $X, \rho, \psi$ replaced by $X', \varphi_X \circ \pi, \varphi_{X'}$) implies that $\varphi_{X'}$ is étale at $x'$.

This terminates the proof of Proposition 3.3. □

Remark 3.4. Note that the assumption on $k(x)$ in the statement of Proposition 3.3 is always satisfied if $k = \mathbb{R}$. The following counterexample shows that we cannot omit this assumption in Proposition 3.3.

Let $X = \text{Spec}(\mathbb{Q}[x, y, y^{-1}]/(y^2 - x + 1))$, $U_X = X \setminus V(x)$, where $V(x)$ denotes the locus of $x = 0$, $B = \text{Spec}(\mathbb{Q}[z])$, $U_B = B \setminus V(z)$, and let $f : X \rightarrow B$ be given by $z = yx^4$. Let $b = (0) \in B$ and $a$ the unique point in $X$ with $f(a) = b$. The morphism $f$ is logarithmically smooth at $a$ with respect to $U_X \subset X$ and $U_B \subset B$. However the conclusion in Proposition 3.3 (with $x$ replaced by $a$) does not hold. Indeed, there does not exist a unit $u$ in the henselization of $\mathcal{O}_{B, b}$, and a unit $v$ in the henselization of $\mathcal{O}_{X, a}$, such that $zu = (xv)^4$. Otherwise $yu = v^4$, and taking values at $a$ we see that then $\sqrt{-1}$ could be written as $\alpha \beta^4$, with $\alpha \in \mathbb{Q}$ and $\beta \in \mathbb{Q}(\sqrt{-1})$. However, this is impossible.

Lemma 3.5. Let $U_X \subset X$ be a strict toroidal embedding of varieties over $k$ and let $x$ be a closed point of $X$. Let $(V, \varphi)$ be an étale chart for $U_X \subset X$ at $x$. Let $y \in \mathcal{O}_{X,x}$ and assume that the divisor of $y$ is supported on $X \setminus U_X$ in some Zariski neighborhood of $x$ in $X$. Then there exists a character $c$ of the big torus $T$ of $V$ such that $y/\varphi^*(z)$ is a unit in $\mathcal{O}_{X,x}$.

Proof. This is very well known. From Lemma 2.3 with $X, D$ replaced by $V, V \setminus T$, it follows that there exists a unit $z$ in $\mathcal{O}_{V, \varphi(x)}$ such that $y/\varphi^*(z)$ is a unit in $\mathcal{O}_{X,x}$. The ideal generated by $z$ in $\mathcal{O}_{V, \varphi(x)}$ is invariant under the action of $T$, hence it is generated by characters of $T$. Since it is principal, Nakayama’s Lemma yields that it is generated by one of these characters (cf. section 3.3 of [8]). □

Lemma 3.6. Let $U_X \subset X$ be a strict toroidal embedding of varieties over $k$, and $x$ a closed point of $X$. Let $(V, \varphi)$ be an étale chart of $U_X \subset X$ at $x$, and set $v := \varphi(x)$. Denote by $T$ the big torus of $V$, and set $D_V := V \setminus T, D_X := X \setminus U_X$. Choose a basis $z_1, \ldots, z_r, z_{r+1}, \ldots, z_m$ for the $\mathbb{Z}$-module of characters on $T$, such that $z_{r+1}, \ldots, z_m$ form a basis for the $\mathbb{Z}$-module of characters on $T$ that are defined and not vanishing at $v$. Let $y \in \mathcal{O}_{X,x}$. Consider $dy$ and $\frac{dz_i}{z_i}$, for $i = 1, \ldots, m$, as elements...
af \( (\Omega^1_X(\log D_X))_x \), and write
\[
dy = \sum_{i=1}^m a_i \frac{dz_i}{z_i},
\]
with \( a_i \in \mathcal{O}_{X,x} \), for \( i = 1, \ldots, m \). Then \( a_i(x) = 0 \), for \( i = 1, \ldots, r \).

**Proof.** Since \( \varphi \) is étale at \( x \), the \( \mathcal{O}_{X,x} \)-module \( \Omega^1_{X,x} \) is generated by the pullbacks under \( \varphi \) of the elements of \( \Omega^1_{V,v} \). Thus we may suppose that \( X = V \), \( \varphi = \text{id} \), and \( x = v \). We may also assume that \( y \) is a character of \( T \) that is regular on \( V \), since these generate the coordinate ring of \( V \). Moreover, replacing \( V \) by a suitable open toric subvariety (on which \( z_{r+1}, \ldots, z_m \) are regular), we may also suppose that \( v \) belongs to the closed orbit of \( V \).

Let \( A \) be the set of characters of \( T \) that belong to the group generated by \( z_1, \ldots, z_r \) and are regular on \( V \). Let \( B \) be the group generated by \( z_{r+1}, \ldots, z_m \). Then \( V_0 := \text{Spec} \, k[A] \) is toric and \( T_0 := \text{Spec} \, k[B] \cong (\mathbb{G}_m)^{m-r} \). Moreover, since \( v \) belongs to the closed orbit of \( V_0 \), we have an isomorphism \( V \cong V_0 \times T_0 \) induced by \( a \otimes b \mapsto \text{ab} \) for \( a \in A, b \in B \), and the closed orbit of \( V_0 \) consists of only one point.

Because the lemma is trivial when \( V \) is a torus, we may assume that \( r = m \), and hence that the closed orbit of \( V \) consists of only one point, namely \( v \). This implies that each nontrivial character of \( T \), that is regular on \( V \), vanishes at \( v \). Thus \( y(v) = 0 \), because we may assume that \( y \) is such a character. Writing \( y = z_1^{e_1} z_2^{e_2} \cdots z_m^{e_m} \), with \( e_1, \ldots, e_m \in \mathbb{Z} \), we have \( \frac{dy}{y} = \sum_{i=1}^m e_i \frac{dz_i}{z_i} \). This terminates the proof of the lemma. \( \square \)

**Remark 3.7.** Assume the notation of Lemma 3.6 and denote by \( C \) the orbit of \( v \) under the action of \( T \). Then in \( \Omega^1_{\varphi^{-1}(C),x} \) we have the equality \( dy = \sum_{i=r+1}^m a_i \frac{dz_i}{z_i} \). This follows e.g. from the argument in the proof of Lemma 3.6.

The following lemma is (modulo the terminology of logarithmic geometry) a special case of Proposition 3.8 in [10].

**Lemma 3.8.** Let \( V \) be an affine toric variety over \( k \) and denote its big torus by \( T \). Let \( X \) be an algebraic variety over \( k \), \( x \) a closed point of \( X \), and \( \rho : X \to V \) an étale \( k \)-morphism. In particular, \( (V, \rho) \) is an étale chart at \( x \) of the toroidal embedding \( U_X := \rho^{-1}(T) \subset X \).

Let \( \psi : X \to V \) be a dominant \( k \)-morphism from \( X \) to \( V \), mapping \( U_X \) into \( T \). Assume the following two conditions.

1. For each character \( c \) of \( T \), we have that \( \rho^*(c)/\psi^*(c) \) is a unit in \( \mathcal{O}_{X,x} \).
The morphism \( \psi \) is logarithmically smooth at \( x \) with respect to \( U_X \subset X \) and \( T \subset V \).

Then \( \psi \) is étale at \( x \).

Proof. Instead of relying on Proposition 3.8 in [10], we give a self-contained proof that does not use logarithmic geometry. Condition (1) implies that \( \psi(x) \) belongs to the \( T \)-orbit of \( \rho(x) \) in \( V \). Hence, replacing \( V \) by a suitable open toric subvariety and \( X \) by a suitable open subvariety, we may assume that \( \rho(x) \) and \( \psi(x) \) belong to the closed orbit of \( V \). We denote the closed orbit of \( V \) by \( C \) and set \( v := \rho(x) \in C, w := \psi(x) \in C \).

To show that \( \psi \) is étale at \( x \), it suffices to prove that \( \psi \) is unramified at \( x \), because \( \psi \) is dominant with integral source and normal target (see e.g. Theorem 3.20 in chapter 1 of [13]). Hence it suffices to prove the following two claims.

- **Claim 1.** The ideal of \( \psi^{-1}(C) \) in \( \mathcal{O}_{X,x} \) is generated by elements \( \psi^*(c) \) with \( c \) in the ideal of \( C \) in \( \mathcal{O}_{V,w} \).

- **Claim 2.** The morphism \( \psi|_{\psi^{-1}(C)} : \psi^{-1}(C) \to V \) is unramified at \( x \).

**Proof of Claim 1.** The ideal of \( C \) in the coordinate ring of \( V \) is generated by the characters of \( T \) that are defined and vanishing at \( v \) (or equivalently at \( w \), because \( w \) belongs to the orbit of \( v \)). Hence condition (1) implies that \( \psi^{-1}(C) = \rho^{-1}(C) \) locally at \( x \). Thus, again by condition (1), in order to prove Claim 1, it suffices to show that the ideal of \( \rho^{-1}(C) \) in \( \mathcal{O}_{X,x} \) is generated by the elements \( \rho^*(c) \) with \( c \) running over all characters of \( T \) that are defined and vanishing at \( v \). But this follows directly from the fact that \( \rho \) is étale at \( x \), because these characters generate the ideal \( I \) of \( C \) in \( \mathcal{O}_{V,v} \), and because \( \mathcal{O}_{V,v}/I \) is normal and hence \( \rho^*(I) \) is prime.

**Proof of Claim 2.** Clearly it suffices to show that the morphism \( \psi^{-1}(C) \to C \) induced by \( \psi \) is étale at \( x \). We will show this by using the jacobian criterium. Note that \( C \) is smooth, and that \( \psi^{-1}(C) \) is smooth at \( x \), because \( \psi^{-1}(C) = \rho^{-1}(C) \) locally at \( x \) (as we saw in the proof of Claim (1)). Choose a basis \( z_1, \ldots, z_r, z_{r+1}, \ldots, z_m \) for the \( \mathbb{Z} \)-module of characters on \( T \), such that \( z_{r+1}, \ldots, z_m \) form a basis for the \( \mathbb{Z} \)-module of characters on \( T \) that are defined and not vanishing at \( v \) (or equivalently at \( w \)). Note that \( z_{r+1}, \ldots, z_m \) are uniformizing parameters for \( C \) at \( v \), and also at \( w \). Put

\[
x_i = \rho^*(z_i), \quad x'_i = \psi^*(z_i),
\]

for \( i = 1, \ldots, m \). Note that \( x_{r+1}, \ldots, x_m \) are uniformizing parameters for \( \psi^{-1}(C) \) at \( x \), because \( \psi^{-1}(C) = \rho^{-1}(C) \) locally at \( x \), and because \( \rho \)
is étale. Thus, by the jacobian criterium, we have to prove that
\[
\det \left( \frac{\partial x'_i}{\partial x_j}(x) \right)_{i,j=r+1,...,m} \neq 0.
\]

From condition (1) it follows that there are units $\epsilon_i$ in $\mathcal{O}_{X,x}$ such that for $i = 1, \ldots, m$
\[
x'_i = \epsilon_i x_i.
\]
Hence
\[
\psi^* \left( \frac{dz_i}{z_i} \right) = \frac{d(\epsilon_i x_i)}{\epsilon_i x_i} = \frac{d\epsilon_i}{\epsilon_i} + \frac{dx_i}{x_i}.
\]

Let $J$ be the logarithmic jacobian matrix of $\psi$, i.e. the square matrix of the coefficients expressing $\psi^* \left( \frac{dz_i}{z_i} \right)$, $i = 1, \ldots, m$, as $\mathcal{O}_{X,x}$-linear combinations of $\frac{dx_j}{x_j}$, $j = 1, \ldots, m$. We denote by $J(x)$ the square matrix over $k(x)$ obtained from $J$ by evaluation at $x \in X$. Because of condition (2), we have that det $J(x) \neq 0$. Applying Lemma 3.6 (with $y$ replaced by $\epsilon_i$) and using the fact that the $\epsilon_i$ are units, together with (5), we see that the matrix formed by the last $m - r$ rows and the first $r$ columns of $J(x)$ is zero. Hence the submatrix $J_0$ of $J$, formed by the last $m - r$ rows and the last $m - r$ columns, satisfies det($J_0(x)$) $\neq 0$. Note that for $i = r + 1, \ldots, m$, the $x_i$, and hence also the $x'_i$, are units in $\mathcal{O}_{X,x}$.

Hence $\left( \frac{\partial x'_i}{\partial x_j}(x) \right)_{i,j=r+1,...,m}$ can be obtained from $J_0(x)$ by multiplying the $i$-th row of $J_0(x)$ by $x'_i(x) \neq 0$ and dividing the $j$-th column of $J_0(x)$ by $x_j(x) \neq 0$ (see Remark 3.7). This yields (3) and terminates the proof of the lemma. □

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