EQUIDISTRIBUTION OF POLYNOMIAL MAPS ON LOCALLY
COMPACT GROUPS

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Abstract. We formulate and prove two generalizations of Weyl's classical equidistribution theorem: The first theorem applies to any polynomial map from a locally compact amenable group to a compact abelian group. The second theorem applies to any polynomial map from a countable group to a compact abelian group.

1. Equidistribution for polynomial maps

Weyl's classical equidistribution theorem says that for any polynomial $P$ with real coefficients, the sequence $([P(n)])_{n=1}^\infty$ is either uniformly distributed in $[0,1]$ or periodic. There are numerous classical and more recent generalizations and refinements of Weyl's equidistribution theorem. For instance, Bergelson and Leibman obtained an analog of Weyl's equidistribution theorem in finite characteristic [1].

The purpose of this note is to formulate and prove the following generalization of Weyl's classical theorem:

Theorem 1.1. Let $P : \Gamma \to G$ be a polynomial map from a locally compact metrizable amenable group $\Gamma$ to a compact metrizable abelian group $G$. Then:

1. The set $P(\Gamma)$, which is the closure of the image of $\Gamma$ under $P$, is a coset of a closed subgroup $H$ of $G$.
2. The function $P : \Gamma \to G$ is well distributed with respect to the Haar measure on $\overline{P(\Gamma)}$.

Our proof of Theorem 1.1 is based on a minor adaptation of the classical van der Corput trick.

If we restrict ourselves to polynomial mappings from a countable discrete group $\Gamma$ to a compact metrizable abelian group $G$, there is a meaningful formulation of the result that does not require amenability of $\Gamma$.

Theorem 1.2. Let $P : \Gamma \to G$ be a polynomial map from a countable discrete group $\Gamma$ to a compact metrizable abelian group $G$. Then:

1. The $\Gamma$-orbit closure of $P$ in $\subseteq G^\Gamma$ is a coset of a closed subgroup of $G^\Gamma$.
2. There is a unique $\Gamma$-invariant measure on $\overline{P(\Gamma)}$, which is Haar measure on $\overline{P(\Gamma)}$.

At first glance, it might seem surprising that equidistribution results can be obtained under such general assumptions on the group $\Gamma$. This seeming generality is somewhat deceiving: Liebman has shown that any polynomial map taking values in a nilpotent group factors through a nilpotent quotient [3]. Liebman's result allows us to easily derive Theorem 1.2 from Theorem 1.1, despite its seemingly increased generality (no amenability assumption).

In the proof of both results we do use the fact that the “range” group $G$ is abelian, since the proof involves Weyl’s equidistribution criterion via the group of characters. It is natural to anticipate a nilpotent extension: Leibman’s paper provides evidence that nilpotent
groups are “the natural framework” for results about polynomial maps. Green and Tao’s seminal work on polynomial orbits on nilmanifolds [2] provides a plausible approach to such generalization. In the case of degree 1 polynomials, the analog of Theorem 1.1 is easy. We provide a short proof for this result in Section 5.

Finally, a remark about the compactness assumption of the “target” group G: If G is not assumed to be compact, already the statement that the closure of the image of a polynomial map is a subgroup fails. Indeed, already in the simplest example of the quadratic polynomial \( \Gamma = G = \mathbb{Z} \) and \( P(n) = n^2 \) we have \( P(\mathbb{Z}) = \{0, 1, 2, \ldots\} \), which is not a subgroup.

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2. Definitions and conventions

Let us quickly introduce some notation and clarify definitions of various terms in the statement of Theorem 1.1.

We use multiplicative notation for the group operation in a locally compact second countable topological group \( \Gamma \), whenever we do not assume \( \Gamma \) is commutative. We let \( m_\Gamma \) denote left Haar measure on \( \Gamma \). For a compact group, we always assume that the Haar measure is normalized to be a probability measure. Otherwise, when \( \Gamma \) is not compact, \( m_\Gamma(\Gamma) = +\infty \) and we assume some (arbitrary) choice of normalization for \( m_\Gamma \).

For a compact abelian group \( G \), we usually use additive notation for the group operation. The one exception to this convention is the case where \( G = S^1 = \{z \in \mathbb{C}^* : |z| = 1\} \) is the unit sphere in \( \mathbb{C} \), in which case the group operation is the multiplication of complex numbers, for which we use multiplicative notation. Also, for a compact abelian group \( G \) we let \( \hat{G} \) denote the group of characters of \( G \), or the Pontryagin dual group, namely the continuous homomorphism from \( G \) to \( S^1 \).

If \( G \) is a compact abelian group, \( H \subset G \) is a closed subgroup and \( C = v + H \) is a coset of \( H \) (where \( v \in G \) is some element of the group), by “the Haar measure on \( C \)” we mean the unique probability measure on \( C \) which is invariant to translation by elements of \( H \). Equivalently, the Haar measure on the coset \( C \) is the pushforward of the Haar measure on the closed subgroup \( H \) via the map \( x \mapsto x + v \) from \( H \) to \( C \).

In the following, \( \Gamma \) will denote a locally compact second countable topological group and \( G \) a compact abelian group.

Definition 2.1. For subset \( F_0, F \subset \Gamma \) and \( \gamma \in \Gamma \) we denote:

\[
\partial_\gamma F = \gamma F \triangle F \quad \text{and} \quad \partial_{F_0} F = \bigcup_{\gamma \in F_0} \partial_\gamma F.
\]

For \( \gamma \in \Gamma \) let \( \sigma_\gamma : G^\Gamma \to G^\Gamma \) be given by \( \sigma_\gamma(\Phi)(x) := \Phi(\gamma^{-1} x) \).

A sequence \( (F_n)_{n=1}^\infty \) of measurable subset of \( \Gamma \) having finite non-zero Haar measure is called a Følner sequence if for every \( \gamma \in \Gamma \) it holds that:

\[
\lim_{n \to \infty} \frac{m_\Gamma(\partial_\gamma F_n)}{m_\Gamma(F_n)} = 0.
\]

Definition 2.2. For a function \( P : \Gamma \to G \) and \( \gamma \in \Gamma \), let \( \Delta_\gamma P : \Gamma \to G \) denote the discrete derivative of \( P \) with respect to \( \gamma \):

\[
\Delta_\gamma P(x) = P(\gamma x) - P(x) \quad \text{for} \quad x \in \Gamma.
\]
Theorem 1.1 is based on two lemmas.

**Definition 2.3.** A continuous function $P : \Gamma \to G$ is called a *Polynomial map of degree at most* $d - 1$ if

$$\Delta_{\gamma_1} \ldots \Delta_{\gamma_d} P = 0 \quad \forall \gamma_1, \ldots, \gamma_d \in \Gamma.$$  

A polynomial map of degree 0 is a constant map from $\Gamma$ to $G$.

**Definition 2.4.** Let $\Gamma$ be a locally compact amenable group.

For $\phi \in L^\infty(\Gamma)$ and a measurable set $F \subset \Gamma$ with $0 < m_F(F) < \infty$, we denote $A_F(\phi) := \frac{1}{m_F(F)} \int_F \phi(x) dm_F(x)$.

We say that $f \in L^\infty(\Gamma)$ has *mean* $z_0 \in \mathbb{C}$ if for any Følner sequence $(F_n)_{n=1}^\infty$ in $\Gamma$ the following holds:

$$\lim_{n \to \infty} A_{F_n}(f) = z_0.$$  

Let $\phi : \Gamma \to X$ be a Borel measurable function on a metrizable space $X$, and let $\mu$ be a Borel probability measure on $X$. We say that $\phi$ is *equidistributed with respect to* $\mu$ if for every continuous function $f : X \to \mathbb{C}$ the mean of $f \circ \phi : \Gamma \to \mathbb{C}$ is $\int f dm$.

In other words, a function $\phi : \Gamma \to X$ is equidistributed with respect to a probability measure $\mu$ on $X$ if for any Følner sequence $(F_n)_{n=1}^\infty$ in $\Gamma$ the measures $\frac{1}{m_F(F_n)} \int_{F_n} \delta_{\phi(\gamma)} dm_\gamma(\gamma)$ converge weak-* to $\mu$.

3. Proof of the equidistribution theorem

The proof of Theorem 1.1 is based on two lemmas.

The first says that if a function into $\mathbb{S}^1$ has some constant but non-trivial directional derivative, then that function must have zero mean:

**Lemma 3.1.** Let $\Phi : \Gamma \to \mathbb{S}^1$ be a measurable function. If there exists $\gamma \in \Gamma$ and $z_0 \in \mathbb{S}^1 \setminus \{1\}$ such that $\Delta_\gamma \Phi(x) = z_0$ for all $x \in \Gamma$ then $\Phi$ has mean zero.

**Proof.** Suppose $z_0 \in \mathbb{S}^1 \setminus \{1\}$ and that $\Delta_\gamma \Phi(x) = z_0$ for all $x \in \Gamma$. By definition,

$$\Phi(\gamma x) = \Delta_\gamma \Phi(x) \Phi(x) = z_0 \Phi(x).$$

So for every $k \in \mathbb{Z}$, $\Phi(\gamma^k x) = z_0^k \Phi(x)$. Because $|\Phi(x)| \leq 1$, for any compact set $F \subset \Gamma$ we have

$$\left| \int_F \Phi(x) dm_F(x) - \int_F \Phi(\gamma x) dm_F(x) \right| \leq m_F(\partial_\gamma(F)).$$

Substituting $\Phi(\gamma x) = z_0 \Phi(x)$ it follows that

$$\left| \int_F \Phi(x) dm_F(x) - \int_F z_0 \Phi(x) dm_F(x) \right| < m_F(\partial_\gamma(F)).$$

Using $z_0 \neq 1$, we have:

$$\left| \int_F \Phi(x) dm_F(x) \right| < \frac{1}{|1 - z_0|} m_F(\partial_\gamma(F)).$$
Lemma 3.2

Let \((F_n)_{n=1}^\infty\) be a Følner sequence in \(\Gamma\). For each \(n \in \mathbb{N}\) we have:

\[
\left| \frac{1}{m_\Gamma(F_n)} \int_{F_n} \Phi(x) \, dm_\Gamma(x) \right| < \frac{1}{|1 - z_0|} \frac{m_\Gamma(\partial_\gamma(F_n))}{m_\Gamma(F_n)}.
\]

Taking \(n \to \infty\) we conclude that \(\Phi\) has mean zero.

The second lemma is a version of the well-known van der Corput Lemma:

**Lemma 3.2** (The van der Corput Lemma for maps from amenable groups to compact abelian groups). Let \(\Phi : \Gamma \to G\) be a function such that \(\Delta_\gamma \Phi : \Gamma \to G\) is equidistributed with respect to the Haar measure on \(G\) for any \(\gamma \in \Gamma \setminus \{1\}\). Then \(\Phi\) is equidistributed with respect to the Haar measure on \(G\).

The proof of Lemma 3.2 is based on the following version of the van der Corput inequality:

**Lemma 3.3** (The Van der Corput inequality for complex valued functions on amenable groups). Let \(\Gamma\) be a discrete group, and let \(F, F_0 \subset \Gamma\) be Borel subsets of \(\Gamma\) having finite positive Haar measure. Let \(\phi : \Gamma \to \mathbb{C}\) be a function with \(|\phi(g)| \leq 1\) for all \(g \in \Gamma\). Then

\[
|A_F(\phi)| \leq \frac{1}{m_\Gamma(F_0)} \sqrt{\int_{F_0} \int_{F_0} A_F(\sigma_{\gamma_2}^{-1}(\Delta_{\gamma_1^{-1} \gamma_2} \phi)) \, dm_\Gamma(\gamma_1) \, dm_\Gamma(\gamma_2) + \frac{m_\Gamma(\partial_{F_0} F)}{m_\Gamma(F)}} \quad (1)
\]

The proof here is essentially an imitation of the classical one. The presentation here is closely inspired by the post on Terry Tao’s blog “The van der Corput trick, and equidistribution on nilmanifolds” from 2008:

**Proof.** Observe that for any \(f : \Gamma \to \mathbb{C}\) with \(|f| \leq 1\) and any \(F_0, F \in \Gamma, \gamma \in F_0\) we have

\[
|A_F(f) - A_F(\sigma_\gamma f)| \leq \frac{m_\Gamma(\partial_{F_0} F)}{m_\Gamma(F)}. \quad (2)
\]

Taking \(f = \phi\), averaging over \(\gamma \in F_0\) and using the triangle inequality we obtain

\[
|A_F(\phi) - A_F(\frac{1}{m_\Gamma(F_0)} \int_{F_0} (\sigma_\gamma \phi) \, dm_\Gamma(\gamma)| \leq \frac{m_\Gamma(\partial_{F_0} F)}{m_\Gamma(F)}.
\]

Applying the triangle inequality again and the Cauchy-Schwarz inequality we get:

\[
|A_F(\phi)| \leq \left[ A_F\left( \left| \frac{1}{m_\Gamma(F_0)} \int_{F_0} \sigma_\gamma \phi \, dm_\Gamma(\gamma) \right|^2 \right) \right]^{\frac{1}{2}} + \frac{m_\Gamma(\partial_{F_0} F)}{m_\Gamma(F)}.
\]

Expanding

\[
\left| \frac{1}{m_\Gamma(F_0)} \int_{F_0} \sigma_\gamma \phi \, dm_\Gamma(\gamma) \right|^2 = \frac{1}{m_\Gamma(F_0)^2} \left( \int_{F_0} \int_{F_0} \sigma_{\gamma_1} \phi \sigma_{\gamma_2} \phi \, dm_\Gamma(\gamma_1) \, dm_\Gamma(\gamma_2) \right),
\]

Using that \(\sigma_{\gamma_1} \phi \sigma_{\gamma_2} \phi = \sigma_{\gamma_2}^{-1}(\Delta_{\gamma_1^{-1} \gamma_2} \phi)\) we get:

\[
|A_F(\phi)| \leq \frac{1}{m_\Gamma(F_0)} \sqrt{\int_{F_0} \int_{F_0} A_F(\sigma_{\gamma_2}^{-1}(\Delta_{\gamma_1^{-1} \gamma_2} \phi)) \, dm_\Gamma(\gamma_1) \, dm_\Gamma(\gamma_2) + \frac{m_\Gamma(\partial_{F_0} F)}{m_\Gamma(F)}}
\]

□
One final ingredient that we need is the well known criterion of Weyl for equidistribution:

**Lemma 3.4** (Weyl’s equidistribution criterion). A function measurable function $\Phi : \Gamma \to G$ from a locally compact amenable group to a compact metrizable abelian group $G$ is equidistributed if and only if $\chi \circ \Phi$ has mean zero for every non-trivial character $\chi \in \hat{G}$.

Indeed, this follows since any continuous function $f : G \to \mathbb{C}$ can be uniformly approximated by characters.

**Proof of Lemma 3.2.** Let $\Phi : \Gamma \to G$ be as in the statement. By Weyl’s equidistribution criterion, it suffices to prove that $\phi := \chi \circ \Phi$ has mean zero for any non-trivial character $\chi \in \hat{G}$. Since $\Delta_\gamma(\chi \circ \Phi) = \chi \circ \Delta_\gamma \Phi$, the result follows from Equation (1).

**Definition 3.5.** Given $\Phi : \Gamma \to G$, let

$$\ker(\Phi) = \{ \gamma \in \Gamma : \Delta_\gamma \Phi(x) = 0 \ \forall x \in \Gamma \}.$$

The following easy lemma justifies the name $\ker(\Phi)$:

**Lemma 3.6.** For any continuous map $\Phi \in \Gamma \to G$, we have that $\ker(\Phi)$ is a closed normal subgroup of $\Gamma$. The function $\Phi$ factors through a map $\overline{\Phi}$ from the quotient of $\Gamma / \ker(\Phi)$.

**Proof.** Let $X_\Phi$ denote the closure of $\{ \sigma_\gamma(\Phi) \} : \gamma \in \Gamma$ in the space of continuous functions from $\Gamma$ to $G$, with the topology of uniform convergence on compact sets. The map $\gamma \mapsto \sigma_\gamma$ is a continuous homomorphism from $\Gamma$ to the group of homeomorphisms of $X_\Phi$, and $\ker(\Phi)$ is exactly the kernel of this homomorphism.

**Proof of Theorem 1.1.** We prove the theorem by induction on the degree $d$ of the polynomial map $P$. The basis for the induction is the trivial case $d = 0$: Indeed, if $d = 0$, then $P$ is constant, and $P(\Gamma)$ is a singleton, which is a coset of the trivial group, and $P$ is obviously well distributed on this (trivial) coset.

Now suppose $P : \Gamma \to G$ is a polynomial map of degree $d \geq 1$. Let $G_0$ denote the closed group generated by all elements of the form $\Delta_\gamma P(x)$ where $\gamma, x \in \Gamma$. Clearly $P$ takes values in $P(1_\Gamma) + G_0$. By subtracting a constant from $P$ we can assume without loss of generality that $P(1_\Gamma) = 0$, and under this assumption our goal is to prove that $P$ is equidistributed on $G_0$. With the above reductions in mind, our goal is to show that $P$ is well distributed on $G_0$. By Weyl’s equidistribution criterion it suffices to show that for any character $\chi$ that does not vanish on $G_0$, $\chi \circ P : \Gamma \to \mathbb{C}$ has mean zero. Clearly, a character $\chi \in \hat{G}$ vanishes on $G_0$ if and only if $\chi(\Delta_\gamma P(x)) = 0$ for every $\gamma, x \in \Gamma$. Let $\chi$ be a character that does not vanish on $G_0$. For every $\gamma \in \Gamma$, $\chi \circ \Delta_\gamma P = \Delta_\gamma (\chi \circ P)$ is a polynomial map of degree less than $d$. Thus, by the induction hypothesis, $\chi \circ \Delta_\gamma P$ is equidistributed on the closure of $\chi \circ \Delta_\gamma P(\Gamma)$, which is a coset of closed subgroup of $S^1$. If $\chi \circ \Delta_\gamma P$ is equidistributed on a coset of a non-trivial subgroup of $S^1$, it has mean zero. Thus, by Lemma 3.2 we are done unless there exists $\gamma \in \Gamma \setminus \{1_\Gamma\}$ for which $\chi \circ \Delta_\gamma P$ is supported (and equidistributed) on a coset of the trivial group, namely, $\Delta_\gamma (\chi \circ P)$ is constant. By Lemma 3.6 we can replace $\Gamma$ with $\Gamma / \ker(\chi \circ P)$, thus assuming $\ker(\chi \circ P)$ is the trivial subgroup. So suppose there exists $\gamma \in \Gamma \setminus \{1_\Gamma\}$ for which $\chi \circ \Delta_\gamma P$ is identically equal to a constant $z_0 \neq 1 \chi \circ P$. By assumption $\gamma \neq 1_\Gamma$ and $\ker(\chi \circ P)\{1_\Gamma\}$, so $z_0 \neq 1$. It follows directly from Lemma 3.1 that $\chi \circ P$ has mean zero.
4. Polynomials into locally compact groups with a compact image

Leibman [3, Proposition 1.21] has shown that a polynomial map from a countable group $\Gamma_1$ to a countable torsion free group $\Gamma_2$ that takes finitely many values must be constant. The following is a direct generalization of Leibman’s observation to the locally-compact setting:

**Proposition 4.1.** Let $P : \Gamma_1 \to \Gamma_2$ be a polynomial map between locally compact groups $\Gamma_1$ and $\Gamma_2$. Suppose that the only subgroup of $\Gamma_2$ with compact closure is the trivial group. If $P(\Gamma_1)$ is a compact subset of $\Gamma_2$ then $P$ is constant.

**Proof.** The short proof is nothing but a direct adaptation of Leibman’s proof to this more general setting: The proof is by induction on the degree of $P$. If $\text{deg}(P) = 0$, then $P$ is constant, and the claim is trivial. If $\text{deg}(P) = 1$, then $P$ is of the form $P(\gamma) = h(\gamma)\gamma_0$, for some continuous homomorphism $h : \Gamma_1 \to \Gamma_2$ and $\gamma_0 \in \Gamma$. So assuming that $P(\Gamma_1)$ is compact and that the only subgroup of $\Gamma_2$ with compact closure is the trivial group, we conclude in this case that $P(\Gamma_1)$ is a coset of the trivial group, in other words that $P$ is constant. Now assume the conclusion of the lemma holds for polynomial maps of degree at most $d$ for some $d \geq 1$. Suppose $\text{deg}(P) = d + 1$. For any $\gamma_1 \in \Gamma_1$, we have that $\Delta_{\gamma_1} P(\Gamma_1) \subseteq P(\Gamma_1)(P\Gamma_1)^{-1}$. Hence, if $P(\Gamma_1)$ has compact closure so does $\Delta_{\gamma_1} P(\Gamma_1)$. By the induction hypothesis, since $\text{deg}(\Delta_{\gamma_1} P) \leq d$, we have that $\Delta_{\gamma_1} P(\Gamma_1)$ must in this case be a singleton, under the assumption that the only subgroup of $\Gamma_2$ with compact closure is the trivial group. This means that $\text{deg}(P) = 1$, a contradiction. $\square$

Proposition 4.1 shows for instance that any bounded polynomial map from a locally compact group $\Gamma$ into $\mathbb{R}$ or $\mathbb{C}$ must be constant.

5. Equidistribution of homomorphisms into non-abelian compact groups

**Theorem 5.1.** Let $\Phi : \Gamma \to G$ be a continuous homomorphism from a locally compact metrizable amenable group $\Gamma$ to a compact metrizable group $G$. Then $G_\Phi := \Phi(\Gamma)$, the closure of the image of $\Gamma$ under $\Phi$, is a closed subgroup of $G$, and $\Phi$ is equidistributed with respect to Haar measure on $G_\Phi$.

**Proof.** The image of a group under a homomorphism is always a group. Let $(F_n)_{n=1}^{\infty}$ be a Følner sequence in $\Gamma$, and let $\mu$ be a probability measure on $X$ which is an accumulation point of $\mu_n := \frac{1}{m(F_n)} \int_{F_n} \delta_{F(\gamma)} dm(\gamma)$. Our goal is to prove that $\mu$ is equal to Haar measure on $G_\Phi$. Since $\mu_n(G_\Phi) = 1$, it is clear that the topological support of the measure $\mu$ is equal to $G_\Phi$. It thus suffices to prove that $\mu$ is invariant with respect to translation by elements of $G_\Phi$. Let $f \in C(G)$ be a continuous function on $G$. We need to prove that

$$\int f(g) \mu(g) = \int f(gh) \mu(g) \quad \text{for all } g \in G_F,$$

Since the function $h \mapsto \int f(gh) d\mu(g)$ from $G$ to $\mathbb{C}$ is continuous, it suffices to prove the above for a dense subset of $G_\Phi$. Specifically, it is enough to show that

$$\int f(g) d\mu(g) = \int f(g(\Phi(\gamma))) \mu(g) \quad \text{for all } \gamma \in \Gamma.$$

To prove this, it suffices to prove that

$$\lim_{n \to \infty} \left| \int f(g) d\mu_n(g) - \int f(g(\Phi(\gamma))) d\mu_n(g) \right| = 0.$$
Indeed,
\[ \left| \int f(g)d\mu_n(g) - \int f(g(F(\gamma)))d\mu_n(g) \right| \leq \|f\|_{C(G)} \cdot \frac{m_T(F_n \Delta F_n \Phi(\gamma))}{m_T(F_n)} \to 0 \text{ as } n \to \infty. \]

6. Unique ergodicity of orbit closures of polynomial maps

In this section we prove Theorem 1.2 and recall the relevant definitions.

In what follows, $\Gamma$ will be a discrete countable group and $G$ a compact abelian group. The space $G^\Gamma$ of functions from $\Gamma$ to $G$ is equipped with the product topology, which makes it into a compact topological space. It is furthermore a compact abelian group, with respect to the pointwise addition in $G$. The group $\Gamma$ acts on $G^\Gamma$ by homomorphisms (the shift action). This action is furthermore an algebraic action: Each element of $\Gamma$ acts as continuous automorphism of the group $G^\Gamma$. A polynomial map $P : \Gamma \to G$ is by definition an element of $G^\Gamma$. We denote by $\Gamma P$ the closure in $G^\Gamma$ of the orbit of $P$ under the action of $\Gamma$.

Any function $\Phi : \Gamma \to G$ naturally defines a function $\Phi^\Gamma : \Gamma \to G^\Gamma$ by
\[ \Phi^\Gamma(\gamma) = \sigma_\gamma(\Phi) \quad \gamma \in \Gamma. \]

Note that for any $\Phi \in G^\Gamma$
\[ \Gamma \Phi = \Phi^\Gamma(\Gamma) = \{ \sigma_\gamma(\Phi) : \gamma \in \Gamma \}. \]

The following lemma is a straightforward but crucial observation:

**Lemma 6.1.** If $P : \Gamma \to G$ is a polynomial map of degree $d$ then $P^\Gamma : \Gamma \to G^\Gamma$ is a polynomial map of degree $d$.

The proof of Theorem 1.2 can be completed as follows:

**Proof of Theorem 1.2.** Let $P : \Gamma \to G$ be a polynomial map, where $\Gamma$ is a discrete countable group and $G$ is a compact abelian group.

By Leibman’s result [2, Proposition 3.21] there exists a nilpotent group $\Gamma_0$, a homomorphism $h : \Gamma \to \Gamma_0$ and a polynomial map $P_0 : \Gamma_0 \to G$ such that $P = P_0 \circ h$. By Lemma 6.1 the map $P_0^{\Gamma_0} : \Gamma_0 \to G^{\Gamma_0}$ is also a polynomial map. Since nilpotent groups are amenable, It follows from Theorem 1.1 that $P_0^{\Gamma_0}(\Gamma_0)$ is a coset of a closed subgroup of $G^{\Gamma_0}$, and that $P_0^{\Gamma_0}$ is well distributed with respect to Haar measure on this coset. By subtracting a constant we can assume without loss of generality that $P(1_\Gamma) = 0$. Under this assumption $P(1_\Gamma) = 0$ and the coset $\overline{P_0^{\Gamma_0}(\Gamma_0)}$ is actually a closed subgroup. But from the definition of $P_0^{\Gamma_0}$ it follows directly that $\overline{P_0^{\Gamma_0}(\Gamma_0)}$ is precisely equal to $\Gamma_0 P_0$. The orbit closure $\Gamma_0 P_0$ is clearly invariant under the action of $\Gamma_0$, and since this is a subgroup. By uniqueness of the Haar measure, the Haar measure on the subgroup $\Gamma_0 P_0$ is indeed $\Gamma_0$-invariant. Using the ergodic theorem (for actions of nilpotent group), for any $\Gamma_0$ invariant measure $\mu$ on $\overline{P_0^{\Gamma_0}(\Gamma_0)}$ one can find a Følner sequence $(F_n)_{n=1}^\infty$ in $\Gamma_0$ such that
\[ \lim_{n \to \infty} \frac{1}{m_{\Gamma_0}(F_n)} \sum_{\gamma_0 \in F_n} \delta_{P_0^{\Gamma_0}(\gamma)} = \mu, \]
where the convergence is weak*.

But the fact that $P_0^{\Gamma_0}$ is equidistributed with respect to Haar measure on $\overline{P_0^{\Gamma_0}(\Gamma_0)}$ implies that any such $\mu$ is equal to Haar measure on $\overline{P_0^{\Gamma_0}(\Gamma_0)}$. 
The last step of the proof is to “lift” everything from $\Gamma_0$ to $\Gamma$ via the homomorphism $h$. \hfill \Box

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