Global solutions for semilinear Klein-Gordon equation in FLRW spacetimes

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Abstract

We consider waves, which obey the semilinear Klein-Gordon equation, propagating in the Friedmann-Lemaître-Roberson-Walker spacetimes. The equations in the de Sitter and Einstein-de Sitter spacetimes are the important particular cases. We show the global in time existence in the energy class of solutions of the Cauchy problem.

Keywords

Klein-Gordon equation · Einstein-de Sitter universe · de Sitter universe · global solutions

1 Introduction

In this article we consider the Klein-Gordon equation in the spacetimes belonging to some family of the Friedmann-Lemaître-Roberson-Walker spacetimes (FLRW spacetimes). In the FLRW spacetime, one can choose coordinates so that the metric has the form

\[ ds^2 = -dt^2 + a^2(t)d\sigma^2. \] (1.1)

where \( \delta_{ij} \) is the Kronecker symbol and \( \ell = \frac{1}{n\gamma} \). The equation of state \( p = (\gamma - 1)\mu \) (equation for the pressure) implies that in order to have a non-negative pressure for a positive density, it must be assumed that \( \gamma \geq 1 \) for the physical space with \( n = 3 \) (see [3] p.124). The spacetime with \( \gamma = 1 \) and \( n = 3 \) is called the Einstein-de Sitter universe. We reveal in this paper the significance of the restriction \( \gamma \geq 1 \) on the range of \( \ell \); in fact, we show that it is closely related to the non-growth of the energy and to the existence in the energy space of the global in time solution of the Cauchy problem for the Klein-Gordon equation. Another important spacetime, the so-called de Sitter spacetime, is also a member of that family and it will be discussed as well.

In quantum field theory the matter fields are described by a function \( \psi \) that must satisfy equations of motion. In the case of a massive scalar field, the equation of motion is the Klein-Gordon equation generated by the metric \( g \):

\[
\frac{1}{\sqrt{|g(x)|}} \frac{\partial}{\partial x^i} \left( \sqrt{|g(x)|} g^{ik}(x) \frac{\partial \psi}{\partial x^k} \right) = m^2 \psi + V'_{\psi}(x, \psi).
\] (1.2)

In physical terms this equation describes a local self-interaction for a scalar particle. A typical example of a potential function would be \( V(\phi) = \phi^4 \).

To motivate our approach, we first consider the covariant Klein-Gordon equation in the metric (1.1), which can be written in the global coordinates as follows

\[
\psi_{tt} - t^{-\ell} \Delta \psi + \frac{n\ell}{2t} \psi_t + m^2 \psi + V'_{\psi}(x, t, \psi) = 0.
\]
Here \( x \in \mathbb{R}^n, t \in \mathbb{R} \), and \( \Delta \) is the Laplace operator on the flat metric, \( \Delta := \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2} \).

We study the Cauchy problem with the data prescribed at some positive time \( t_0 \)
\[
\psi(t_0, x) = \psi_0, \quad \psi_t(t_0, x) = \psi_1,
\]
and we look for the solution defined for all values of \( t \in [t_0, \infty) \) and \( x \in \mathbb{R}^n \). We change the unknown function \( \psi = t^{-\frac{4}{n-2}}u \), then for the new function \( u = u(t, x) \) we obtain the equation
\[
u_{tt} - t^{-\ell} \Delta u + M^2(t)u + t^{n\ell/4}V'_\psi(x, t, t^{-\frac{4}{n-2}}u) = 0
\]
with the “effective” (or “curved mass”)
\[
M^2_{EdS}(t) := m^2 - \frac{n\ell(n\ell - 4)}{16t^2}.
\]
It is easily seen that for the range \((0, \frac{4}{n-2})\) of the parameter \( \ell \) the curved mass is positive while its derivative is non-positive. This is crucial for the non-increasing property of the energy and in the derivation of the energy estimate.

Let \((V, g)\) be smooth pseudo Riemannian manifold of dimension \( n + 1 \) and \( V = \mathbb{R} \times S \) with \( S \) an \( n \)-dimensional orientable smooth manifold, and \( g \) be a FLRW metric. We restrict our attention to the case of \( n \geq 3 \) and to the spacetime with the line element
\[
ds^2 = -dt^2 + a^2(t) \sigma.
\]
Then we consider an expanding universe that means that \( \dot{a}(t) > 0 \). For the metric with \( \dot{a}(t) > 0 \) we define the norm
\[
\| \psi \|_{X(t)} := \| \psi_t \|_{L^\infty([t_0, \infty) ; L^2(S))} + \| a^{-1}(\cdot) \nabla_\sigma \psi \|_{L^\infty([t_0, \infty) ; L^2(S))} + \| M(\cdot) \psi \|_{L^\infty([t_0, \infty) ; L^2(S))} + \| \sqrt{\dot{a}} a^{-3} \nabla_\sigma \psi \|_{L^2([t_0, \infty) \times S)},
\]
where \( 0 < t_0 < t \leq \infty \) and \( M(t) > 0 \) is a curved mass defined by:
\[
M^2(t) = m^2 + \left( \frac{n}{2} - \frac{n^2}{4} \right) \left( \dot{a}(t) \right)^2 - \frac{n}{2} \frac{\ddot{a}(t)}{a(t)}.
\]
Hence, in the classification suggested in [14], mass \( m \) is large if the metric \( g \) is a de Sitter metric \(-dt^2 + e^{2t} dx^2\), \( x \in \mathbb{R}^n \). Here and henceforth \( \dot{a}(t) \) denotes the derivative with respect to time. In order to describe admissible nonlinearities we make the following definition.

**Condition (L).** The function \( F = F(s, u), \ F : S \times \mathbb{R} \rightarrow \mathbb{R} \) is said to be Lipschitz continuous in \( u \), if there exist \( \alpha \geq 0 \) and \( C > 0 \) such that
\[
|F(s, u) - F(s, v)| \leq C|u - v|(\|u\|^\alpha + |v|^\alpha) \quad \text{for all } u, v \in \mathbb{R}, \ s \in S.
\]
The main result of this paper is the following theorem.

**Theorem 1.1.** Assume that \( n = 3, 4 \) and that the metric \( g = -dt^2 + a^2(t) \sigma \). Suppose also that \( m > 0 \) and that there is a positive number \( c_0 \) such that the real-valued positive function \( a = a(t) \) satisfies
\[
a(t) > 0, \quad \dot{a}(t) > 0 \quad \text{for all } \quad t \in [t_0, \infty),
\]
\[
M(t) > c_0 > 0, \quad \dot{M}(t) \leq 0 \quad \text{for all } \quad t \in [t_0, \infty).
\]
Consider the Cauchy problem for the equation (1.2) with the derivative of potential function \( V'_\psi(s, t, \psi) = -\Gamma(t) F(s, \psi) \) such that \( F \) is a Lipschitz continuous, \( F(s, 0) = 0 \) for all \( s \in S \), and either
\[
|\Gamma(t)| \leq c \frac{\dot{a}(t)}{a(t)} \quad \text{for all } t \in [t_0, \infty),
\]
or, there is $\alpha_0$ such that

$$
\int_0^\infty \left( \frac{a(t)}{a(t)} \right)^{\frac{n-\alpha}{n-\alpha_0}} \left| \Gamma(t) \right|^{\frac{4}{n-\alpha_0}} \, dt < \infty, \quad 0 < \alpha_0 < \frac{4}{n}.
$$

(1.10)

If $\frac{4}{n} \leq \alpha \leq \frac{2}{n-2}$, then for every $\psi_0 \in H^{(1)}(S)$ and $\psi_1 \in L^2(S)$, sufficiently small initial data, $\|\psi_0\|_{H^{(1)}(S)} + \|\psi_1\|_{L^2(S)}$, the problem (1.2), (1.3) has a unique solution $\psi \in C([t_0, \infty); H^{(1)}(S)) \cap C^1([t_0, \infty); L^2(S))$ and its norm $\|a(t)\tilde{\Gamma}\psi\|_{X^{(\infty)}}$ is small.

Condition (1.8) for the norm of solutions of the equation implies that the energy of solution is non-increasing. In the next theorem the local existence is stated with the less restrictive conditions and with the estimate for the lifespan.

**Theorem 1.2** Suppose that $m > 0$ and that there is a positive number $c_0$ such that the real-valued positive function $a = a(t)$ satisfies (1.7), (1.8). Consider the Cauchy problem for the equation (1.2) with the derivative of potential function $V'(s, t, \psi) = -\Gamma(t)F(s, \psi)$ such that $F$ is a Lipschitz continuous, $F(s, 0) = 0$ for all $s \in S$.

If $0 \leq \alpha \leq \frac{2}{n-2}$, then for every $\psi_0 \in H^{(1)}(S)$ and $\psi_1 \in L^2(S)$, the problem (1.2), (1.3) has a unique solution $\psi \in C([t_0, T_1); H^{(1)}(S)) \cap C^1([t_0, T_1); L^2(S))$.

Denote by $C_{a, \Gamma, c_0}(T)$ and $C_{a, \Gamma, c_0}(r)$ the function

$$
C_{a, \Gamma, c_0}(T) := \left( \int_0^T \left( \frac{\alpha(t)}{a(t)} \right)^{\frac{n-\alpha}{n-\alpha_0}} \left| \Gamma(t) \right|^{\frac{4}{n-\alpha_0}} \, dt \right)^{\frac{4-n_0}{4}} \quad 0 < \alpha_0 < \frac{4}{n}.
$$

and its inverse, respectively. Then the lifespan of the solution can be estimated as follows

$$
T_1 - t_0 \geq CC_{a, \Gamma, c_0}^{-1}(\|\psi_0\|_{H^{(1)}(S)} + \|\psi_1\|_{L^2(S)}).
$$

If the nonlinear term has an energy conservative potential function, then in the next theorem we establish the existence of the global solution for large initial data.

**Theorem 1.3** Suppose that all conditions of Theorem 1.2 on $n$, $\alpha$, and $a = a(t)$, are satisfied, and additionally,

$$
\frac{2}{n} \frac{a(t)}{a(t)} V'(t, s, a^{-n/2}(t)w) + 2V(t, s, a^{-n/2}(t)w) - a^{-n/2}(t)wV''(s, a^{-n/2}(t)w) \leq 0 \quad (1.11)
$$

for all $(t, s, w) \in [t_0, \infty) \times S \times \mathbb{R}$.

Then for every $\psi_0 \in H^{(1)}(\mathbb{R}^n)$ and $\psi_1 \in L^2(\mathbb{R}^n)$, the problem (1.2), (1.3) has a unique solution $\psi \in C([t_0, \infty); H^{(1)}(\mathbb{R}^n)) \cap C^1([t_0, \infty); L^2(\mathbb{R}^n))$ and its norm $\|a(t)\tilde{\Gamma}\psi\|_{X^{(\infty)}}$ is finite.

The hyperbolic equations in the de Sitter spacetime have permanently bounded domain of influence. Nonlinear equations with a permanently bounded domain of influence were studied, in particular, in [13]. In that paper, there is given an example of such an equation, which has a blow-up solution for arbitrarily small data. Moreover, it was discovered in [13] that the time-oscillation of the metric, due to the parametric resonance, can cause the blowup phenomena for the wave map type nonlinearities even for the arbitrarily small data. On the other hand, in the absence of oscillations in the metric, Choquet-Bruhat [4] proved for small initial data the global existence and uniqueness of wave maps on the FLRW expanding universe with the metric $g = -dt^2 + R^2(t)\sigma$ and a smooth Riemannian manifold $(S, \sigma)$ of dimension $n \leq 3$, which has a time independent metric $\sigma$ and non-zero injectivity radius, and with $R(t)$ being a positive increasing function such that $1/R(t)$ is integrable on $[t_0, \infty)$. If the target manifold is flat, then the wave map equation reduces to a linear system. On the other hand, in the Einstein–de Sitter spacetime the domain of influence is not permanently bounded.

In the de Sitter space, that is, in the spacetime with the line element

$$
\text{d}s^2 = -\text{d}t^2 + e^{2t} \sum_{i,j=1,...,n} \delta_{ij} \text{d}x^i \text{d}x^j, \quad (1.12)
$$


the second author [15]-[16] studied the Cauchy problem for the semilinear equation
\[ \Box g u + m^2 u = f(u), \quad u(x, t_0) = \varphi_0(x) \in H^{(s)}(\mathbb{R}^n), \quad u_t(x, t_0) = \varphi_1(x) \in H^{(s)}(\mathbb{R}^n), \]
if \( s > n/2 \). In [15]-[16] a global existence of small data solutions of the Cauchy problem for the semilinear Klein-Gordon equation and systems of equations in the de Sitter spacetime is proved. It was discovered that unlike the same problem in the Minkowski spacetime, no restriction on the order of nonlinearity is required, provided that a physical mass of the field belongs to some set, \( m \in (0, \sqrt{n^2 - 1/2}) \cup \left[ n/2, \infty \right) \). It was also conjectured that \((\sqrt{n^2 - 1/2}, n/2)\) is a forbidden mass interval for the small data global solvability of the Cauchy problem for all \( \alpha \in (0, \infty) \). For \( n = 3 \) the interval \((0, \sqrt{2})\) is called the Higuchi bound in quantum field theory [6]. The proof of the global existence in [15]-[16] is based on the \( L^p - L^q \) estimates.

Baskin [2] discussed small data global solutions for the scalar Klein-Gordon equation on asymptotically de Sitter spaces, which are compact manifolds with boundary. More precisely, in [2] the following Cauchy problem is considered for the semilinear equation
\[ \Box_g u + m^2 u = f(u), \quad u(x, t_0) = \varphi_0(x) \in H^{(1)}(\mathbb{R}^n), \quad u_t(x, t_0) = \varphi_1(x) \in L^2(\mathbb{R}^n), \]
where mass is large, \( m^2 > n^2/4 \), \( f \) is a smooth function and satisfies conditions \( |f(u)| \leq c |u|^{\alpha+1}, \ |u| \cdot |f'(u)| \sim |f(u)|, \ f(u) - f'(u) \cdot u \leq 0, \ \int_0^1 f(v)dv \geq 0 \), and \( \int_0^1 f(v)dv \sim |u|^{\alpha+2} \) for large \( |u| \). It is also assumed that \( \alpha = \frac{2}{n-1} \). In Theorem 1.3 [2] the existence of the global solution for small energy data is stated. (For more references on the asymptotically de Sitter spaces, see the bibliography in [1], [12].)

Hintz and Vasy [7] considered semilinear wave equations of the form
\[(\Box_g - \lambda)u = f + g(u, du)\]
on a manifold \( M \), where \( q \) is typically a polynomial vanishing at least quadratically at \((0, 0)\), in contexts such as asymptotically de Sitter and Kerr-de Sitter spaces, as well as asymptotically Minkowski spaces. The linear framework in [7] is based on the \( b \)-analysis, in the sense of Melrose, introduced in this context by Vasy to describe the asymptotic behavior of solutions of linear equations. Hintz and Vasy have shown the small data solvability of suitable semilinear wave and Klein-Gordon equations.

Nakamura [8] considered the Cauchy problem for the semi-linear Klein-Gordon equations in de Sitter spacetime with \( n \leq 4 \), that is, with the line element (1.12). The nonlinear term is of power type for \( n = 3, 4 \), or of exponential type for \( n = 1, 2 \). For the power type semilinear term with \( \frac{2}{n} \leq \alpha \leq \frac{2}{n-2} \) Nakamura [8] proved the existence of global solutions in the energy class.

This paper is organized as follows. In Section 2 we derive the energy estimates. Then, in Section 3, we give the estimate for the nonlinear term. The completion of the proof of Theorems 1.1,1.2,1.3 is given in Section 4. To illustrate our results we discuss some examples in that section.

## 2 The linear Klein-Gordon equation

### 2.1 FLRW universe. The effect of the damping term

The goal of the transformation that has been used in [14] and will be used in this subsection is twofold; on one hand, it reduces the damping term and makes possible to prove non increase property of the energy and, on the other hand, it provides the nonlinear term with some weight function, which is decreasing in time. In fact, this is a particular case of the Liouville transformation that is used to study boundedness, stability and asymptotic behavior of solutions of the second order differential equations.

The line element in the theorems for the FLRW spacetime implies
\[ g_{00} = g^{00} = -1, \quad g_{0j} = g^{0j} = 0, \quad g_{ij} = a^2(t)\delta_{ij}(x), \quad |g| = a^{2n}(t) |\det \delta(x)|, \quad g^{ij} = a^{-2}(t)\delta^{ij}(x), \]
\( i, j = 1, 2, \ldots, n \), where \( \delta^{ij}(x)\delta_{jk}(x) = \delta_{ik} \) (Kronecker symbol) and the metric \( \sigma \) in the local chart is given by \( \delta_{ik}(x) \).
The linear covariant Klein-Gordon equation in that background is \( \Box_g \psi = m^2 \psi - f \) and in the coordinates this equation can be written as follows

\[
\psi_{tt} - \frac{1}{a^2(t) \sqrt{\text{det} \delta(x)}} \sum_{i,j=1}^{n} \frac{\partial}{\partial x^i} \left( \sqrt{\text{det} \delta(x)} \frac{\partial}{\partial x^j} \psi \right) + n \frac{\dot{a}(t)}{a(t)} \psi_t + m^2 \psi = f.
\]

In order to eliminate the damping term \( n \frac{\dot{a}(t)}{a(t)} \psi_t \) we introduce the new unknown function \( \psi = b(t)u \), then the equation for \( u \) is

\[
u_{tt} - \frac{1}{a^2(t) \sqrt{\text{det} \delta(x)}} \sum_{i,j=1}^{n} \frac{\partial}{\partial x^i} \left( \sqrt{\text{det} \delta(x)} \frac{\partial u}{\partial x^j} \right) + \left( 2 \frac{\dot{b}(t)}{b(t)} + n \frac{\dot{a}(t)}{a(t)} \right) u + \frac{m^2}{b(t)} u = \frac{1}{b(t)} f.
\]

We look for the function \( b = b(t) \) such that the following equation is fulfilled:

\[
2 \frac{\dot{b}(t)}{b(t)} + n \frac{\dot{a}(t)}{a(t)} = 0.
\]

In particular, we can choose \( b(t) = a^{-\frac{2}{n}}(t) \).

Consequently, the dumping term vanished while the coefficient of the term with \( u \) of the equation (2.1) became

\[
2 \frac{\dot{b}(t)}{b(t)} + n \frac{\dot{a}(t)}{a(t)} = \left( n \frac{\dot{a}(t)}{a(t)} \right)^2 - \frac{n \ddot{a}(t)}{2 a(t)}.
\]

Hence, the Klein-Gordon equation for the function \( u = u(x,t) \) can be written as follows:

\[
u_{tt} - \frac{1}{a^2(t) \sqrt{\text{det} \delta(x)}} \sum_{i,j=1}^{n} \frac{\partial}{\partial x^i} \left( \sqrt{\text{det} \delta(x)} \frac{\partial u}{\partial x^j} \right) + \left( m^2 + \left( n \frac{\dot{a}(t)}{a(t)} \right)^2 - \frac{n \ddot{a}(t)}{2 a(t)} \right) u = \frac{1}{b(t)} f.
\]

We denote the coefficient of the last equation by

\[
c(t) := m^2 + \left( n \frac{\dot{a}(t)}{a(t)} \right)^2 - \frac{n \ddot{a}(t)}{2 a(t)}.
\]

For the FLRW spacetime with \( a(t) = t^{\ell/2}, \ell \leq \frac{4}{n} \), and

\[ g_{00} = g^{00} = -1, g_{0j} = g^{0j} = 0, g_{ij} = t^\ell \delta_{ij}(x), \ |g| = t^{\ell n} |\text{det} \delta(x)|, \ g^{ij} = t^{-\ell} \delta^{ij}(x), \]

\( i,j = 1, \ldots, n \), in accordance with [3] p.124, we have

\[
c(t) = m^2 - \frac{n \ell (n \ell - 4)}{16 \ell^2} > 0, \quad \dot{c}(t) = \frac{n \ell (n \ell - 4)}{8 \ell^3} \leq 0 \quad \text{for} \quad m > 0, \ \ell \leq \frac{4}{n}.
\]

The last inequalities suggest the assumptions on \( a(t), m, \) and \( n \):

\[
c(t) = m^2 + \left( n \frac{\dot{a}(t)}{a(t)} \right)^2 - \frac{n \ddot{a}(t)}{2 a(t)} > 0 \quad \text{for all large} \ t \quad \text{and} \quad m > 0, \]
\[
\dot{c}(t) = \frac{d}{dt} \left( \left( n \frac{\dot{a}(t)}{a(t)} \right)^2 - \frac{n \ddot{a}(t)}{2 a(t)} \right) \leq 0 \quad \text{for all large} \ t.
\]
Thus, in order to study the equation (1.2) we can first consider the following linear equation

$$u_{tt} - \frac{1}{a^2(t)} \Delta_x u + M^2(t)u = \frac{1}{b(t)} f,$$

with the curved mass $M(t)$, $M^2(t) = c(t)$, of (1.6), and derive in the next subsection for the solutions the energy estimates. Here $\Delta_x$ is a Laplace-Beltrami operator in the metric $\sigma$.

### 2.2 Energy estimate

In this subsection we show that for the large physical mass $m$ the expansion property of the de Sitter metric leads, via transformation (2.2), to the dissipative effect for the Klein-Gordon equation.

First we consider the solution $u = u(t, x)$ of the equation without source term, $f = 0$,

$$u_{tt} - \frac{1}{a^2(t)} \Delta_x + M^2(t)u = 0.$$

For a Riemannian manifold $(S, \sigma)$ we denote by $\nabla_\sigma$ the covariant gradient and by $d\mu_\sigma$ the volume element in the metric $\sigma$. The Sobolev space $W^p_2(S)$ is a Banach space with the norm

$$\|u\|_{W^p_2(S)} := \left( \int_S |\partial^{\alpha} u|^p d\mu_\sigma \right)^{1/p}, \quad 1 \leq p < \infty.$$

We define the energy of the solution $u = u(t, s)$ by

$$E(t) := \frac{1}{2} \|u_t\|_{L^2(S)}^2 + \frac{1}{2} a^{-2}(t) \|\nabla_\sigma u\|_{L^2(S)}^2 + \frac{1}{2} M^2(t) \|u\|_{L^2(S)}^2.$$  \hfill (2.3)

Then

$$\frac{d}{dt} E(t) = \frac{1}{2} (a^{-2}(t))_t \|\nabla_\sigma u\|_{L^2(S)}^2 + \frac{1}{2} (M^2(t))_t \|u\|_{L^2(S)}^2 \leq 0.$$

The integration gives

$$E(t) - E(t_0) = \int_{t_0}^t \left[ \frac{1}{2} (a^{-2}(\tau))_\tau \|\nabla_\sigma u\|_{L^2(S)}^2 + \frac{1}{2} (M^2(\tau))_\tau \|u\|_{L^2(S)}^2 \right] d\tau = E(t_0).$$

In particular, due to the assumptions on $a(t)$ and $M(t)$ we obtain $E(t) \leq E(t_0)$, that is,

$$\|u_t\|_{L^2(S)} + a^{-2}(t) \|\nabla_\sigma u\|_{L^2(S)} + M^2(t) \|u\|_{L^2(S)}^2 \leq \|u_t(t_0)\|_{L^2(S)} + a^{-2}(t_0) \|\nabla_\sigma u(t_0)\|_{L^2(S)} + M^2(t_0) \|u(t_0)\|_{L^2(S)}^2.$$

We also have $0 < M_0 \leq M(t) \leq M_1 < \infty$ for all $t \geq t_0$ with some constants $M_0$, $M_1$.

Consider now the solution $u$ of the equation

$$u_{tt} - a^{-2}(t) \Delta_x u + M^2(t)u = g$$  \hfill (2.4)

with the source term $g$.

**Proposition 2.1** Assume that conditions (1.7) and (1.8) are fulfilled. Then, the solution $u = u(t, s)$ of the equation (2.4) satisfies the following estimate

$$\|u_t\|_{L^\infty([t_0, t] \times S)} + \|a^{-1}(t) \nabla u(x, t)\|_{L^\infty([t_0, t] \times S)} + \|M(t)u(x, t)\|_{L^\infty([t_0, t] \times S)} + \|M(t_0)u(x, t)\|_{L^\infty([t_0, t] \times S)} \leq c \left( \|u_t(t_0, \cdot)\|_{L^2(S)} + \|a^{-1}(t_0) \nabla u(t_0, \cdot)\|_{L^2(S)} + \|M(t_0)u(t_0, \cdot)\|_{L^2(S)} + \|g(t, \cdot)\|_{L^1([t_0, t] \times S)} \right)$$

for all $t > t_0$. 


Proof. For the energy $E(t)$ (2.3) of the solution $u = u(t, s)$ of (2.4) we have

$$\frac{d}{dt} E(t) = \frac{1}{2} (a^{-2}(t))_t \| \nabla \sigma u \|_{L^2(S)}^2 + \frac{1}{2} (M^2(t))_t \| u \|_{L^2(S)}^2 + \int_S (\partial_t u) g d\mu_\sigma. $$

Integration gives

$$E(t) = \int_{t_0}^t \left[ \frac{1}{2} (a^{-2}(\tau))_\tau \| \nabla \sigma u \|_{L^2(S)}^2 + \frac{1}{2} (M^2(\tau))_\tau \| u \|_{L^2(S)}^2 \right] d\tau
- \int_{t_0}^t \left[ (a^{-2}(\tau))_\tau \| \nabla \sigma u \|_{L^2(S)}^2 + (M^2(\tau))_\tau \| u \|_{L^2(S)}^2 \right] d\tau
\leq \| u(t) \|_{L^2(S)}^2 + a^{-2}(t) \| \nabla \sigma u(t) \|_{L^2(S)}^2 + M(t) \| u(t) \|_{L^2(S)}^2
+ 2 \max_{t_0 \leq \tau \leq t} \| u(t) \|_{L^2(S)} \int_{t_0}^t \| g(\tau) \|_{L^2(S)} d\mu_\sigma d\tau.
$$

It follows that for every $\varepsilon > 0$ the following inequality

$$\| u(t) \|_{L^2(S)}^2 + a^{-1}(t) \| \nabla \sigma u \|_{L^2(S)} + M(t) \| u \|_{L^2(S)}
+ \int_{t_0}^t \left[ (a^{-2}(\tau))_\tau \| \nabla \sigma u \|_{L^2(S)}^2 + (M^2(\tau))_\tau \| u \|_{L^2(S)}^2 \right] d\tau
\leq c \left( \| u(t) \|_{L^2(S)} + a^{-1}(t) \| \nabla \sigma u(t) \|_{L^2(S)} + M(t) \| u(t) \|_{L^2(S)}
+ \varepsilon \max_{t_0 \leq \tau \leq t} \| u(t) \|_{L^2(S)} \right),$$

is fulfilled. In fact, we obtain

$$\| u(t) \|_{L^2(S)}^2 + a^{-1}(t) \| \nabla \sigma u \|_{L^2(S)} + M(t) \| u \|_{L^2(S)}
+ \left( \| \sqrt{a^{-2}} \nabla \sigma u \|_{L^2([t_0,t] \times S)}^2 + \| \sqrt{|M^2(t)|} \| u \|_{L^2([t_0,t] \times S)}^2 \right)^{1/2}
\leq c \left( \| u(t) \|_{L^2(S)} + a^{-1}(t) \| \nabla \sigma u(t) \|_{L^2(S)} + M(t) \| u(t) \|_{L^2(S)}
+ \varepsilon \max_{t_0 \leq \tau \leq t} \| u(t) \|_{L^2(S)} \right),$$

and, consequently, for sufficiently small $\varepsilon > 0$ we have

$$\| u(t) \|_{L^2(S)} + a^{-1}(t) \| \nabla \sigma u \|_{L^2(S)} + M(t) \| u \|_{L^2(S)}
+ \| \sqrt{a^{-2}} \nabla \sigma u \|_{L^2([t_0,t] \times S)} + \| \sqrt{|M^2(t)|} \| u \|_{L^2([t_0,t] \times S)}
\leq c \left( \| u(t) \|_{L^2(S)} + a^{-1}(t) \| \nabla \sigma u(t) \|_{L^2(S)} + M(t) \| u(t) \|_{L^2(S)} + \| g \|_{L^1([t_0,t] \times L^2(S))} \right).$$

The proposition is proved. □
Corollary 2.2 Under condition of the proposition we have
\[ \|u\|_{X(t)} + \|\sqrt{t} u\|_{L^2([t_0,t] \times S)} \leq c \left( \|u(t_0,\cdot)\|_{L^2(S)} + a^{-1}(t_0)\|\nabla u(t_0,\cdot)\|_{L^2(S)} \right. 
\]
\[ \left. + M(t_0)\|u(t_0,\cdot)\|_{L^2(S)} + \|g(x,\tau)\|_{L^1([t_0,t];L^2(S))} \right) 
\]
for all \( t > t_0 \). In particular,
\[ \|u\|_{X(t)} \leq c \left( \|u(t_0,\cdot)\|_{L^2(S)} + a^{-1}(t_0)\|\nabla u(t_0,\cdot)\|_{L^2(S)} + M(t_0)\|u(t_0,\cdot)\|_{L^2(S)} \right. 
\]
\[ \left. + c\|g(x,\tau)\|_{L^1([t_0,t];L^2(S))} \right) 
\]
for all \( t > t_0 \).

3 Estimate of self-interaction term

The next lemma is a simple consequence of the Gagliardo-Nirenberg inequality (see, e.g., [10], p.22 and Lemma 8.2 [11]) and we give detailed proof for the sake of the self-completeness of this paper.

Lemma 3.1 If \( |F(s,\varphi)| \leq C|\varphi|^{n+1} \) for all \((s,\varphi) \in S \times \mathbb{R} \), then the inequality
\[ \|F(s,\varphi(s))\|_{L^2(S)} \leq C \|\nabla \varphi(s)\|_{L^2(S)}\|\varphi(s)\|_{L^2(S)}^{\alpha+1-n} \] (3.1)
holds provided that \( n \geq 3 \) and \( 0 < \alpha \leq \frac{2}{n-2} \). Moreover, if \( F(s,\varphi) \) is Lipschitz continuous in \( \varphi \), then
\[ \|F(s,\varphi_1(s)) - F(s,\varphi_2(s))\|_{L^2(S)} \leq C \max_{\theta = \varphi_1,\varphi_2} \left( \|\nabla (\theta(s))\|_{L^2(S)} \|\nabla (\varphi_1(s) - \varphi_2(s))\|_{L^2(S)} \right)^{\alpha+1-n} \] (3.2)

Proof. First we consider the case of \( S = \mathbb{R}^n \), that is, if \( |F(x,\varphi)| \leq C|\varphi|^{n+1} \) for all \((x,\varphi) \in \mathbb{R}^{n+1} \), then the inequality
\[ \|F(x,\varphi(x))\|_{L^2(\mathbb{R}^n)} \leq C \|\nabla \varphi(x)\|_{L^2(\mathbb{R}^n)}\|\varphi(x)\|_{L^2(\mathbb{R}^n)}^{n+1-n} \] (3.3)
holds provided that \( n \geq 3 \) and \( 0 < \alpha \leq \frac{2}{n-2} \). Moreover, if \( F(x,\varphi) \) is Lipschitz continuous in \( \varphi \), then
\[ \|F(x,\varphi_1(x)) - F(x,\varphi_2(x))\|_{L^2(\mathbb{R}^n)} \leq C \max_{\psi = \varphi_1,\varphi_2} \left( \|\nabla \psi(x)\|_{L^2(\mathbb{R}^n)} \|\nabla (\varphi_1(x) - \varphi_2(x))\|_{L^2(\mathbb{R}^n)} \right)^{\alpha+1-n} \] (3.4)

The proof of the inequality (3.3) is very similar to the one of (3.4) with \( \varphi_2 = 0 \) and it does not require for \( F \) to be Lipschitz continuous in \( \varphi \), therefore we prove only the last one. Let \( 1/p_1 + 1/q_1 = 1 \), then by property of \( F \) and by Hölder’s inequality we have the following inequality
\[ \|F(x,\varphi_1(x)) - F(x,\varphi_2(x))\|_{L^2(\mathbb{R}^n)}^2 \leq C \int_{\mathbb{R}^n} |\varphi_1(x) - \varphi_2(x)|^2 (|\varphi_1(x)| + |\varphi_2(x)|)^{2\alpha} \, dx \]
\[ \leq C \left( \int_{\mathbb{R}^n} |\varphi_1(x) - \varphi_2(x)|^{2p_1} \, dx \right)^{\frac{1}{p_1}} \left( \int_{\mathbb{R}^n} (|\varphi_1(x)| + |\varphi_2(x)|)^{2q_1} \, dx \right)^{\frac{1}{q_1}}. \]
Denote \( \varphi(x) = \varphi_1(x) - \varphi_2(x) \) and \( \psi(x) = |\varphi_1(x)| + |\varphi_2(x)| \), then
\[
\|F(x, \varphi_1(x)) - F(x, \varphi_2(x))\|_{L^2(\mathbb{R}^n)} \leq C\|\varphi\|_{L^{2p_1}(\mathbb{R}^n)}\|\psi\|_{L^{2q_1}(\mathbb{R}^n)}^{\frac{\alpha}{2}}.
\]
Now we use the Gagliardo-Nirenberg inequality (see [10], p.22 and Lemma 8.2 [11])
\[
\|\varphi(x)\|_{L^r(\mathbb{R}^n)} \leq C\|\varphi(x)\|_{L^2(\mathbb{R}^n)}^{\frac{1}{r_2}}\|\nabla \varphi(x)\|_{L^2(\mathbb{R}^n)}^{\frac{\alpha}{2}}\]
with \( 0 \leq \theta \leq 1 \). More precisely, we set
\[
r_1 = 2p_1, \quad \frac{1}{2p_1} = \frac{1}{2} - \frac{\vartheta_1}{n}, \quad r_2 = 2a q_1, \quad \frac{1}{2a q_1} = \frac{1}{2} - \frac{\vartheta_2}{n},
\]
and write
\[
\|\varphi(x)\|_{L^{2p_1}(\mathbb{R}^n)} \leq C\|\varphi(x)\|_{L^2(\mathbb{R}^n)}^{\frac{1}{r_2}}\|\nabla \varphi(x)\|_{L^2(\mathbb{R}^n)}^{\frac{\alpha}{2}},
\]
\[
\|\psi(x)\|_{L^{2q_1}(\mathbb{R}^n)} \leq C\|\psi(x)\|_{L^2(\mathbb{R}^n)}^{\frac{1}{r_2}}\|\nabla \psi(x)\|_{L^2(\mathbb{R}^n)}^{\frac{\alpha}{2}}.
\]
If we choose
\[
p_1 = \alpha + 1, \quad q_1 = \frac{\alpha + 1}{\alpha}, \quad \vartheta_1 = \frac{n \alpha}{2(\alpha + 1)} = \vartheta_2,
\]
then
\[
\|\varphi(x)\|_{L^{2p_1}(\mathbb{R}^n)} \leq C\|\varphi(x)\|_{L^2(\mathbb{R}^n)}^{\frac{1}{r_2}}\|\nabla \varphi(x)\|_{L^2(\mathbb{R}^n)}^{\frac{\alpha}{2}}\]
and
\[
\|\psi(x)\|_{L^{2q_1}(\mathbb{R}^n)} \leq C\|\psi(x)\|_{L^2(\mathbb{R}^n)}^{\frac{1}{r_2}}\|\nabla \psi(x)\|_{L^2(\mathbb{R}^n)}^{\frac{\alpha}{2}}.
\]
Hence
\[
\|F(x, \varphi_1(x)) - F(x, \varphi_2(x))\|_{L^2(\mathbb{R}^n)} \leq C\|\varphi(x)\|_{L^2(\mathbb{R}^n)}^{\frac{1}{r_2}}\|\nabla \varphi(x)\|_{L^2(\mathbb{R}^n)}^{\frac{\alpha}{2}}\|\psi(x)\|_{L^2(\mathbb{R}^n)}^{\frac{1}{r_2}}\|\nabla \psi(x)\|_{L^2(\mathbb{R}^n)}^{\frac{\alpha}{2}}.
\]
Thus, (3.4) is proven.

Next we turn to the case of manifold \( S \). Let \( U \subseteq \mathbb{R}^n \) be a local chart with local coordinates \( x \in U \). If \( \psi \in C_0^\infty(U) \), \( \psi \geq 0 \), then due to (3.3) we have
\[
\|\psi(x)F(x, \varphi(x))\|_{L^2(U)} \leq C\|\psi(x)\|_{L^2(U)}\|\varphi(x)\|_{L^2(U)}^{a+1} \leq C\|\nabla \varphi(x)\|_{L^2(U)}^{\frac{\alpha}{2}}\|\varphi(x)\|_{L^2(U)}^{a+1} \leq C\|\nabla \varphi(x)\|_{L^2(U)}^{\frac{\alpha}{2}}\|\varphi(x)\|_{L^2(U)}^{a+1}.
\]
Let \( \{\psi_j\} \) be a locally finite partition of unity on \( S \) subordinated to the cover \( \{U_j\} \). Then
\[
\|F(x, \varphi(x))\|_{L^2(S)} = \sum_j \|\psi_j(x)F(x, \varphi(x))\|_{L^2(U_j)} \leq C \sum_j \|\nabla \varphi(x)\|_{L^2(U_j)}^{\frac{\alpha}{2}}\|\varphi(x)\|_{L^2(U_j)}^{a+1}.
\]
Since \( \frac{\alpha}{2} \geq \alpha \), we have \( \alpha + 1 - \frac{\alpha}{2} \geq 0 \), and we obtain
\[
\|F(x, \varphi(x))\|_{L^2(S)} \leq C \left( \sum_j \|\nabla \varphi(s)\|_{L^2(U_j)}^{\frac{\alpha}{2}} \|\varphi(s)\|_{L^2(U_j)} \right)^{a+1} \leq C \|\nabla \varphi(s)\|_{L^2(S)}^{\frac{\alpha}{2}}\|\varphi(s)\|_{L^2(S)}^{a+1}.
\]
Proposition 3.2

If $T \psi H$, hence (3.1) is proven. Now for $F(x, \varphi_1(x)) - F(x, \varphi_2(x))$ \(L^2(U)\)

\[
\|F(s, \varphi_1(s)) - F(s, \varphi_2(s))\|_{L^2(S)} \leq C \sum_j \left( \max_{\theta = \varphi_1, \varphi_2} \|\nabla \theta(x)\|_{L^2(U)} \|\nabla (\varphi_1(x) - \varphi_2(x))\|_{L^2(U)} \right)^{\alpha_0^+} \frac{\alpha_0}{\alpha_0^+}
\]

where (3.4) has been used. If \{\psi_j\} is a locally finite partition of unity on \(S\) subordinated to the cover \{U_j\}, then

\[
\|F(s, \varphi_1(s)) - F(s, \varphi_2(s))\|_{L^2(S)} = \sum_j \|\psi_j(x)(F(x, \varphi_1(x)) - F(x, \varphi_2(x)))\|_{L^2(U_j)} \leq C \sum_j \left( \max_{\theta = \varphi_1, \varphi_2} \|\nabla \theta(x)\|_{L^2(U_j)} \|\nabla (\varphi_1(x) - \varphi_2(x))\|_{L^2(U_j)} \right)^{\alpha_0^+} \frac{\alpha_0}{\alpha_0^+}
\]

proves (3.2). Lemma is proven.

The next proposition gives the estimate of the self-interaction term, which is transformed by the reduction of the damping term \(n \dot{a}(t)/a(t)\psi_1\) of the equation. It is also connected with the energy of the linear equation. Define the norm

\[
\|\Phi \|_{X(t)} := \|\Phi(t, x, t)\|_{L^\infty([t_0, t]; L^2(\mathbb{R}^n))} + \|a^{-1}(t) \nabla_x \Phi(t, x, t)\|_{L^\infty([t_0, t]; L^2(\mathbb{R}^n))}
\]

where \(0 < t_0 < t \leq \infty\).

Proposition 3.2 If \(|F(x, \varphi)| \leq C|\varphi|^{\alpha+1}\) for all \((x, \varphi) \in \mathbb{R}^{n+1}\), then the inequalities

\[
\|a^{-\frac{\varphi}{n}}(t) \Gamma(t) F(x, a^{-\frac{\varphi}{n}}(t) u)\|_{L^1([t_0, T]; L^2(\mathbb{R}^n))} \leq C_{\mu, a, \Gamma}(T) \|u(x, t)\|_{L^\infty([t_0, T]; L^2(\mathbb{R}^n))} \times \|a^{-\frac{\varphi}{n}}(t) \nabla_x u(x, t)\|_{L^\infty([t_0, T]; L^2(\mathbb{R}^n))} \|\sqrt{aa^{-3} \nabla_x u(x, t)}\|_{L^2([t_0, T] \times \mathbb{R}^n)}^{\alpha_0^+} \frac{\alpha_0}{\alpha_0^+}
\]

hold for all \(T \in (t_0, \infty)\), with the functions \(a(t)\) and \(\Gamma(t)\) satisfying conditions (1.9) or (1.10) of Theorem 1.1. Here \(\frac{2}{n} \leq \alpha\). The function \(C_{\mu, a, \Gamma}(T)\) is such that

\[
C_{\mu, a, \Gamma}(T) = \text{(3.5)}
\]
\[
\begin{align*}
\text{For the positive function } & a(t) = \frac{\text{const}}{t^{\frac{1}{\alpha}}}, \quad \text{if } (1.9) \quad \text{and } \alpha_0 = \frac{4}{n}, \\
& \left( \int_{t_0}^{T} \left( \frac{a(t)}{a(t)} \right)^{\frac{4}{n\alpha_0}} \| \Gamma(t) \|^{\frac{1}{4n\alpha_0}} \, dt \right)^{\frac{4n\alpha_0}{4}} , \quad \text{if } (1.9)\text{ or } (1.10) \text{ and } \alpha_0 < \frac{4}{n}.
\end{align*}
\]

Moreover, if, additionally, \( F(x, \varphi) \) is Lipschitz continuous in \( \varphi \), then
\[
\| a^{-\frac{3}{2}}(t) \| F(x, a^{-\frac{3}{2}}(t)u) - F(x, a^{-\frac{3}{2}}(t)v) \|_{L^1([t_0, T]; L^2(\mathbb{R}^n))} \leq C_{\mu, a, \Gamma}(T) \max_{u=v, v} \left( \| w(x, t) \|_{L^{\infty}([t_0, T]; L^2(\mathbb{R}^n))} \| u - v \|_{L^{\infty}([t_0, T]; L^2(\mathbb{R}^n))} \right)^{\alpha + 1 - \frac{2\alpha}{n}}
\]
\[
\times \left( \| a^{-1}(t) \| \nabla w(x, t) \|_{L^{\infty}([t_0, T]; L^2(\mathbb{R}^n))} \| a^{-1}(t) \| \nabla u(x, t) - v(x, t) \|_{L^2(\mathbb{R}^n)} \right)^{\frac{2 \alpha}{n \alpha_0}}.
\]
\[
\leq \text{const} C_{\mu, a, \Gamma}(T) \max_{u=v, v} \| w \|_{X(t)} \| u - v \|_{X(t)} .
\]

**Proof.** We prove only the second part of the proposition since the proof of the first one is very similar, but it does not appeal to the above mentioned additional condition on \( F \). Denote \( b(t) = a^{-\frac{3}{2}}(t), \tilde{\alpha} := \frac{n \alpha_0}{2(\alpha + 1)}. \)

From (3.4) we derive
\[
\| b^{-1}(t) \| F(x, b(t)u) - F(x, b(t)v) \|_{L^1([t_0, T]; L^2(\mathbb{R}^n))} \leq \int_{t_0}^{T} b^{-1}(t) \| F(x, b(t)u) - F(x, b(t)v) \|_{L^2(\mathbb{R}^n)} \, dt
\]
\[
\leq \int_{t_0}^{T} b^\alpha(t) \| \nabla w(x, t) \|_{L^2(\mathbb{R}^n)} \| \nabla (u(x, t) - v(x, t)) \|_{L^2(\mathbb{R}^n)} \tilde{\alpha}^{(\alpha + 1)}
\]
\[
\times \left( \| w(x) \|_{L^2(\mathbb{R}^n)} \| u(x, t) - v(x, t) \|_{L^2(\mathbb{R}^n)} \right)^{(1 - \tilde{\alpha})(\alpha + 1)} \, dt .
\]

For the positive function \( \mu = \mu(t) \) this leads to the following estimate
\[
\| b^{-1}(t) \| F(x, b(t)u) - F(x, b(t)v) \|_{L^1([t_0, T]; L^2(\mathbb{R}^n))} \leq \int_{t_0}^{T} \mu(t) b^\alpha(t) \| \Gamma(t) \|_{L^1([t_0, T]; L^2(\mathbb{R}^n))}
\]
\[
\max_{u=v, v} \left( \| \mu^{-b_1}(t) \| \nabla w(x, t) \|_{L^2(\mathbb{R}^n)} \| \mu^{-b_1}(t) \| \nabla (u(x, t) - v(x, t)) \|_{L^2(\mathbb{R}^n)} \right)^{\tilde{\alpha}(\alpha + 1)}
\]
\[
\times \left( \| w(x) \|_{L^2(\mathbb{R}^n)} \| u(x, t) - v(x, t) \|_{L^2(\mathbb{R}^n)} \right)^{(1 - \tilde{\alpha})(\alpha + 1)} \, dt ,
\]
where we have denoted
\[
b_1 = \frac{1}{\tilde{\alpha}(\alpha + 1)} = \frac{2}{n \alpha}.
\]

Now we set
\[
\mu(t)b^\alpha(t) | \Gamma(t) | \leq \text{const} .
\]

and consider two cases: (B) (1.9) and \( \alpha_0 = \frac{\alpha}{n} ; \) (U) \( 0 < \alpha_0 < \frac{\alpha}{n} \) and (1.9) or (1.10).

In the first case (B) due to (1.9) we obtain
\[
\mu(t)b^\alpha(t) | \Gamma(t) | \leq \text{const} .
\]
Therefore,
\[ \| b^{-1}(t) \Gamma(t) (F(x, b(t)u) - F(x, b(t)v)) \|_{L^1([t_0, T]; L^2(\mathbb{R}^n))} \]
\[ \leq \int_{t_0}^{T} \max_{w=u,v} \left( \| \mu^{-\frac{n}{m}}(t) \nabla w(x, t) \|_{L^2(\mathbb{R}^n)} \| \mu^{-\frac{n}{m}}(t) \nabla (u(x, t) - v(x, t)) \|_{L^2(\mathbb{R}^n)} \right) \frac{\alpha}{\alpha + \frac{n}{m}} dt \]
\[ \times \left( \| w(x) \|_{L^\infty((t_0, T); L^2(\mathbb{R}^n))} \| u(x, t) - v(x, t) \|_{L^2(\mathbb{R}^n)} \right)^{\alpha + 1 - \frac{n}{m}} \]
\[ \leq \max_{w=u,v} \left( \| w(x) \|_{L^\infty((t_0, T); L^2(\mathbb{R}^n))} \| u(x, t) - v(x, t) \|_{L^2(\mathbb{R}^n)} \right)^{\alpha + 1 - \frac{n}{m}} \]
\[ \times \int_{t_0}^{T} \max_{w=u,v} \left( \| \mu^{-\frac{n}{m}}(t) \nabla w(x, t) \|_{L^2(\mathbb{R}^n)} \| \mu^{-\frac{n}{m}}(t) \nabla (u(x, t) - v(x, t)) \|_{L^2(\mathbb{R}^n)} \right) \frac{\alpha}{\alpha + \frac{n}{m}} dt . \]

Consider the last integral, where \( \mu(t) \) is replaced with its definition (3.9). After long but simple transformations we arrived at
\[ \int_{t_0}^{T} \max_{w=u,v} \left( \| a(t)^{-\frac{n}{m}}(t) \nabla w(x, t) \|_{L^2(\mathbb{R}^n)} \| a(t)^{-\frac{n}{m}}(t) \nabla (u(x, t) - v(x, t)) \|_{L^2(\mathbb{R}^n)} \right) \frac{\alpha}{\alpha + \frac{n}{m}} dt \]
\[ = \int_{t_0}^{T} \max_{w=u,v} \left( \| \sqrt{a(t)} a(t)^{-\frac{n}{m}}(t) \nabla w(x, t) \|_{L^2(\mathbb{R}^n)} \right) \frac{\alpha}{\alpha + \frac{n}{m}} dt \]
\[ \times \max_{w=u,v} \left( \| \sqrt{a(t)} a(t)^{-\frac{n}{m}}(t) \nabla (u(x, t) - v(x, t)) \|_{L^2(\mathbb{R}^n)} \right) \frac{\alpha}{\alpha + \frac{n}{m}} dt \]
\[ \leq \max_{w=u,v} \left( \| a(t)^{-\frac{n}{m}}(t) \nabla w(x, t) \|_{L^2(\mathbb{R}^n)} \frac{\alpha}{\alpha + \frac{n}{m}} \right) \frac{\alpha}{\alpha + \frac{n}{m}} \]
\[ \times \max_{w=u,v} \left( \| \sqrt{a(t)} a(t)^{-\frac{n}{m}}(t) \nabla (u(x, t) - v(x, t)) \|_{L^2(\mathbb{R}^n)} \right) \frac{\alpha}{\alpha + \frac{n}{m}} dt . \]

Thus,
\[ \| b^{-1}(t) \Gamma(t) (F(x, b(t)u) - F(x, b(t)v)) \|_{L^1([t_0, T]; L^2(\mathbb{R}^n))} \]
\[ \leq \max_{w=u,v} \left( \| w(x) \|_{L^\infty((t_0, T); L^2(\mathbb{R}^n))} \| u(x, t) - v(x, t) \|_{L^2(\mathbb{R}^n)} \right)^{\alpha + 1 - \frac{n}{m}} \]
\[ \times \max_{w=u,v} \left( \| a(t)^{-\frac{n}{m}}(t) \nabla w(x, t) \|_{L^2(\mathbb{R}^n)} \| a(t)^{-\frac{n}{m}}(t) \nabla (u(x, t) - v(x, t)) \|_{L^2(\mathbb{R}^n)} \right) \frac{\alpha}{\alpha + \frac{n}{m}} \]
\[ \times \int_{t_0}^{T} \max_{w=u,v} \left( \| \sqrt{a(t)} a(t)^{-\frac{n}{m}}(t) \nabla w(x, t) \|_{L^2(\mathbb{R}^n)} \right) \frac{\alpha}{\alpha + \frac{n}{m}} dt . \]

We set here \( \alpha_0 = \frac{4}{n} \leq \alpha \) and by means of the condition \( \alpha \leq \frac{2}{n+2} \) and by the Hölder inequality with
\[ p_2 = \frac{4(\alpha + 1)}{n\alpha_0}, \quad q_2 = \frac{4(\alpha + 1)}{n\alpha_0}, \]
derive for the integral of the last inequality the following estimate
\[ \int_{t_0}^{T} \max_{w=u,v} \left( \| \sqrt{a(t)} a(t)^{-\frac{n}{m}}(t) \nabla w(x, t) \|_{L^2(\mathbb{R}^n)} \right) \frac{\alpha}{\alpha + \frac{n}{m}} dt \]
\[ \leq \max_{w=u,v} \left( \| \sqrt{a(t)} a(t)^{-\frac{n}{m}}(t) \nabla w(x, t) \|_{L^2([t_0, T] \times \mathbb{R}^n)} dt \right) \frac{\alpha}{\alpha + \frac{n}{m}} \]
\[ \times \left( \| \sqrt{a(t)} a(t)^{-\frac{n}{m}}(t) \nabla (u(x, t) - v(x, t)) \|_{L^2([t_0, T] \times \mathbb{R}^n)} dt \right) \frac{\alpha}{\alpha + \frac{n}{m}} . \]
This implies (3.6) and (3.7).

In the second case (U) with \( \alpha_0 < \frac{4}{n} \) we use (3.8) with \( \frac{1}{p_1} + \frac{1}{q_1} = 1, b_1 = \frac{2}{n\alpha}, \) and \( \tilde{\alpha} = \frac{n\alpha}{2(\alpha+1)}. \) We choose \( \alpha_0, p_1, q_1 \) such that
\[
\frac{1}{p_1} = 1 - \frac{n\alpha_0}{4} > 0, \quad \frac{1}{q_1} = \frac{n\alpha_0}{4} > 0, \quad 0 < \alpha_0 < \frac{4}{n},
\]
and then apply Hölder inequality:
\[
\| b^{-1}(t) \Gamma(t) (F(x, b(t)u) - F(x, b(t)v)) \|_{L^1([t_0, T]; L^2(\mathbb{R}^n))} \\
\leq C \left( \int_{t_0}^{T} |\mu(t)b^n(t)\Gamma(t)|^{p_1} \ dt \right)^{1/p_1} \\
\times \left( \int_{t_0}^{T} \max_{w=u,v} \left( \|\mu^{-b_1}(t)\nabla w(x, t)\|_{L^2(\mathbb{R}^n)} \|\mu^{-b_1}(t)\nabla (u(x, t) - v(x, t))\|_{L^2(\mathbb{R}^n)} \right) \right)^{(1-\tilde{\alpha})(\alpha+1)q_1} \\
\times \left( \|w(x)\|_{L^2(\mathbb{R}^n)} \|u(x, t) - v(x, t)\|_{L^2(\mathbb{R}^n)} \right)^{(1-\tilde{\alpha})(\alpha+1)q_1} dt^{1/q_1}.
\]
Next we use definition of the function \( C_{b, \Gamma, \alpha_0}(T) \) (3.5) that reads
\[
C_{b, \Gamma, \alpha_0}(T) = \left( \int_{t_0}^{T} |\mu(t)b^n(t)\Gamma(t)|^{p_1} \ dt \right)^{1/p_1}.
\]
Hence we obtain the estimate
\[
\| b^{-1}(t) \Gamma(t) (F(x, b(t)u) - F(x, b(t)v)) \|_{L^1([t_0, T]; L^2(\mathbb{R}^n))} \\
\leq CC_{b, \Gamma, \alpha_0}(T) \\
\times \left( \int_{t_0}^{T} \max_{w=u,v} \left( \|\mu^{-b_1}(t)\nabla w(x, t)\|_{L^2(\mathbb{R}^n)} \|\mu^{-b_1}(t)\nabla (u(x, t) - v(x, t))\|_{L^2(\mathbb{R}^n)} \right) \right)^{(1-\tilde{\alpha})(\alpha+1)} \\
\times \left( \|w(x)\|_{L^2(\mathbb{R}^n)} \|u(x, t) - v(x, t)\|_{L^2(\mathbb{R}^n)} \right)^{(1-\tilde{\alpha})(\alpha+1)q_1} dt^{1/q_1}.
\]
It follows
\[
\| b^{-1}(t) \Gamma(t) (F(x, b(t)u) - F(x, b(t)v)) \|_{L^1([t_0, T]; L^2(\mathbb{R}^n))} \\
\leq CC_{b, \Gamma, \alpha_0}(T) \\
\times \max_{w=u,v} \left( \|w(x)\|_{L^\infty([t_0, T]; L^2(\mathbb{R}^n))} \|u(x, t) - v(x, t)\|_{L^\infty([t_0, T]; L^2(\mathbb{R}^n))} \right)^{(1-\tilde{\alpha})(\alpha+1)} \\
\times \left( \int_{t_0}^{T} \max_{w=u,v} \left( \|\mu^{-b_1}(t)\nabla w(x, t)\|_{L^2(\mathbb{R}^n)} \right) \right)^{(1-\tilde{\alpha})q_1} dt^{1/q_1}.
\]
Next we estimate the integral of the above inequality:
\[
\left. \int_{t_0}^{T} \max_{w=u,v} \left( \|\mu^{-b_1}(t)\nabla w(x, t)\|_{L^2(\mathbb{R}^n)} \|\mu^{-b_1}(t)\nabla (u(x, t) - v(x, t))\|_{L^2(\mathbb{R}^n)} (1-\tilde{\alpha})q_1 \right) dt \right) \\
\leq \max_{w=u,v} \left( \|a^{-1}(t)\nabla w(x, t)\|_{L^\infty([t_0, T]; L^2(\mathbb{R}^n))} \right)^{(1-\tilde{\alpha})q_1}.$
\[ \times \| a^{-1}(t) \nabla (u(x, t) - v(x, t)) \|_{L^\infty([t_0, T]; L^2(\mathbb{R}^n))}^{\frac{1}{1+n\alpha q_1}} \]
\[ \times \int_{t_0}^T \max_{w=u,v} \left( \| \sqrt[\alpha]{a(t)} a(t)^{-3} \nabla w(x, t) \|_{L^\infty(\mathbb{R}^n)} \right) \]
\[ \times \| \sqrt[\alpha]{a(t)} a(t)^{-3} \nabla (u(x, t) - v(x, t)) \|_{L^\infty(\mathbb{R}^n)}^{\frac{n\alpha q_1}{1+n\alpha q_1}} \] dt.

It remains to estimate the last integral, and we set \( \alpha_0 = \frac{4}{nq_1} < \alpha \) and apply Hölder’s inequality with
\[ p_2 = \frac{4(\alpha + 1)}{\alpha_0 n\alpha q_1}, \quad q_2 = \frac{4(\alpha + 1)}{\alpha_0 nq_1}, \quad \alpha_0 = \frac{4}{nq_1} < \alpha. \]

Thus we obtain
\[ \int_{t_0}^T \max_{w=u,v} \left( \| \sqrt[\alpha]{a(t)} a(t)^{-3} \nabla w(x, t) \|_{L^\infty(\mathbb{R}^n)} \right) \]
\[ \times \| \sqrt[\alpha]{a(t)} a(t)^{-3} \nabla (u(x, t) - v(x, t)) \|_{L^\infty(\mathbb{R}^n)} \]
\[ \leq \max_{w=u,v} \left( \| \sqrt[\alpha]{a(t)} a(t)^{-3} \nabla w(x, t) \|_{L^\infty([t_0, T]; L^2(\mathbb{R}^n))} \right)^{\frac{n\alpha q_1}{1+n\alpha q_1}} \]
\[ \times \left( \| \sqrt[\alpha]{a(t)} a(t)^{-3} \nabla (u(x, t) - v(x, t)) \|_{L^\infty([t_0, T]; L^2(\mathbb{R}^n))} \right)^{\frac{n\alpha q_1}{1+n\alpha q_1}}. \]

To check (3.7) of the statement we just apply the inequality from Problem 78[9]. Proposition is proven. \( \square \)

In the special case of the de Sitter spacetime, \( a(t) = \exp(\Gamma t) \) and \( \Gamma = \text{const} \), the last proposition implies results of Lemma 3.1 [8].

**Proposition 3.3** If \( |F(x, \varphi)| \leq C|\varphi|^{\alpha+1} \) for all \((s, \varphi) \in S \times \mathbb{R}\), then the inequalities
\[ \| a^\alpha(t) \Gamma(t) F(s, a^{-\frac{\alpha}{3}}(t) u) \|_{L^2([t_0, T]; L^2(S))} \]
\[ \leq C_{u,a,\Gamma}(T) \| u \|_{L^\infty([t_0, T]; L^2(S))}^{\alpha+1} \]
\[ \times \| a^{-1}(t) \nabla x u \|_{L^\infty([t_0, T]; L^2(S))}^{\frac{n(\alpha-a_0)}{2(\alpha+1)}}, \]

(3.10) hold for all \( T \in (t_0, \infty) \), with the functions \( a(t) \) and \( \Gamma(t) \) satisfying conditions (1.9) or (1.10) of Theorem 1.1. Here \( \frac{2}{\alpha} \leq \alpha \). The function \( C_{u,a,\Gamma}(T) \) is defined in (3.5).

Moreover, if, additionally, \( F(s, \varphi) \) is Lipschitz continuous in \( \varphi \), then
\[ \| a^\alpha(t) \Gamma(t) \left( F(s, a^{-\frac{\alpha}{3}}(t) u) - F(s, a^{-\frac{\alpha}{3}}(t) v) \right) \|_{L^1([t_0, T]; L^2(S))} \]
\[ \leq C_{u,a,\Gamma}(T) \max_{u,v} \left( \| u \|_{L^\infty([t_0, T]; L^2(S))} \right)^{\alpha+1} \]
\[ \times \max_{u,v} \left( \| a^{-1}(t) \nabla x u \|_{L^\infty([t_0, T]; L^2(S))} \right)^{\frac{n(\alpha-a_0)}{2}}, \]
\[ \times \max_{u,v} \left( \| \sqrt[\alpha]{a(t)} a(t)^{-3} \nabla (u(x, t) - v(x, t)) \|_{L^\infty([t_0, T]; L^2(\mathbb{R}^n))} \right)^{\frac{n\alpha q_1}{1+n\alpha q_1}} \]
\[ \times \left( \| \sqrt[\alpha]{a(t)} a(t)^{-3} \nabla (u(x, t) - v(x, t)) \|_{L^\infty([t_0, T]; L^2(\mathbb{R}^n))} \right)^{\frac{n\alpha q_1}{1+n\alpha q_1}}, \]

(3.12) (3.13)
**Proof.** Let \( U \subseteq \mathbb{R}^n \) be a local chart with local coordinates \( x \in U \). If \( \varphi \in C_0^\infty(U) \), then due to Proposition 3.2 we have
\[
\| \varphi(x)a^{3}(t)\Gamma(t)F(x,a^{-3}(t)u)\|_{L^1([t_0,T];L^2(\mathbb{R}^n))} \\
\leq C_{\mu,a,\Gamma}(T)\|u(x,t)\|_{L^\infty((t_0,T);L^2(U))}^{\alpha+1-\frac{n\alpha}{2}} \\
\times\|a^{-1}(t)\nabla_x u(x,t)\|_{L^\infty((t_0,T);L^2(U))}^{\frac{n\alpha}{2}}\|\sqrt{aa^{-3}}\nabla_x u(x,t)\|_{L^2((t_0,T)\times U)}^{\frac{n\alpha}{2}}.
\]
Let \( \{ \psi_j \} \) be a locally finite partition of unity on \( S \) subordinated to the cover \( \{ U_j \} \). Then
\[
\|a^{3}(t)\Gamma(t)F(x,a^{-3}(t)u)\|_{L^1([t_0,T];L^2(S))} \\
= \sum_j \|\psi_j a^{3}(t)\Gamma(t)F(x,a^{-3}(t)u)\|_{L^1([t_0,T];L^2(U_j))} \\
\leq C_{\mu,a,\Gamma}(T)\|u(x,t)\|_{L^\infty((t_0,T);L^2(U_j))}^{\alpha+1-\frac{n\alpha}{2}} \\
\times\|a^{-1}(t)\nabla_x u(x,t)\|_{L^\infty((t_0,T);L^2(U_j))}^{\frac{n\alpha}{2}}\|\sqrt{aa^{-3}}\nabla_x u(x,t)\|_{L^2((t_0,T)\times U_j)}^{\frac{n\alpha}{2}}.
\]
This proves (3.10) and (3.11). To prove (3.12) and (3.13) we apply Proposition 3.2
\[
\|a^{3}(t)\Gamma(t)\left(F(s,a^{-3}(t)u) - F(s,a^{-3}(t)v)\right)\|_{L^1([t_0,T];L^2(S))} \\
= \sum_j \|\psi_j a^{3}(t)\Gamma(t)\left(F(s,a^{-3}(t)u) - F(s,a^{-3}(t)v)\right)\|_{L^1([t_0,T];L^2(U_j))} \\
\leq C_{\mu,a,\Gamma}(T)\max_{u,v} \left(\|u\|_{L^\infty((t_0,T);L^2(U_j))}^{\frac{n\alpha}{2}}\|u - v\|_{L^\infty((t_0,T);L^2(U_j))}\right)^{\alpha+1-\frac{n\alpha}{2}} \\
\times\left(\|a^{-1}(t)\nabla_x u\|_{L^\infty((t_0,T);L^2(U_j))}^{\frac{n\alpha}{2}}\|a^{-1}(t)\nabla_x (u - v)\|_{L^\infty((t_0,T);L^2(U_j))}\right)^{\frac{n\alpha}{2}} \\
\times\left(\|\sqrt{aa^{-3}}\nabla_x u\|_{L^2((t_0,T)\times U_j)}^{\frac{n\alpha}{2}}\|\sqrt{aa^{-3}}\nabla_x (u - v)\|_{L^2((t_0,T)\times U_j)}^{\frac{n\alpha}{2}}\right)^{\frac{n\alpha}{2}}.
\]
This completes the proof of (3.12) and (3.13). Proposition is proven. \(\square\)
4 Completion of the proof of theorems. Examples

4.1 Integral equation. Proof of theorems.

Now we consider the Cauchy problem for the equation

$$\Box_g \psi = m^2 \psi + V'_\psi(t, x, \psi).$$

It can be written as an equation for $u = a^{-\frac{t}{T}}(t)\psi$:

$$u_{tt} - \frac{1}{a^2(t)}\Delta_\sigma u + M^2(t)u = -\frac{1}{b(t)}V'_\psi(t, x, b(t)u),$$

(4.1)

with $b(t) = a^{-\frac{t}{T}}(t)$ and with the curved mass (1.6). Here $\Delta_\sigma$ is Laplace-Beltrami operator in the metric $\sigma$.

By applying the solution operator for the Cauchy problem for the equation (4.1), the problem can be rewritten as an integral equation for the function $\Phi = \Phi(t, s)$:

$$\Phi(t, s) = \Phi_0(t, s) - G[b^{-1}(t)V'_\psi(t, x, b(t)\Phi)](t, s).$$

(4.2)

Here $\Phi_0(t, s)$ is a given function. For the numbers $R > 0$ and $T > t_0$ we define the complete metric space

$$X(R, T) := \{\Phi \in C([t_0, T]; H_{(1)}(S) \cap C^1([t_0, T]; L^2(S)) \mid \|\Phi\|_{X(R)} \leq R\}$$

with the metric

$$d(\Phi_1, \Phi_2) := \|\Phi_1 - \Phi_2\|_{X(T)}.$$

Denote by $C_{\mu, a, \Gamma}^{(-1)}(r)$ the function inverse to $C_{\mu, a, \Gamma}(T)$, which is given by the second case of (3.5), then $C_{\mu, a, \Gamma}^{(-1)}(0) = t_0$. The following theorem guarantees local and global solvability of the integral equation (4.2).

**Theorem 4.1** (i) Assume that conditions of Theorem 1.3 are satisfied. Then for every $\Phi_0 \in X(R, T)$ there exist $T_1 > t_0$, $R_1 > 0$, and the unique (local) solution $\Phi \in X(R_1, T)$ of the equation (4.2). The life span $T_1 - t_0 > 0$ can be estimated from below as follows: there is $C$ such that for every $R_1 > R$,

$$T_1 - t_0 \geq C \min \left\{ C_{\mu, a, \Gamma}^{(-1)} \left( \frac{R_1 - R}{\epsilon_0 R} \right), C_{\mu, a, \Gamma}^{(-1)} \left( \frac{1}{\epsilon_0 R_1^2} \right) \right\}.$$

(ii) Assume that conditions of Theorem 1.1 are satisfied, then there is $\epsilon_0 > 0$ such that for every given function $\Phi_0 \in X(\epsilon, T)$ with small norm

$$\|\Phi_0\|_{X(T)} \leq \epsilon < \epsilon_0,$$

$0 < T \leq \infty$, the integral equation (4.2) has a unique solution $\Phi \in X(2\epsilon, T)$ and

$$\|\Phi\|_{X(T)} \leq 2\epsilon.$$

**Proof.** Consider the mapping

$$S[\Phi](t, s) := \Phi_0(t, s) + G[b^{-1}(t)\Gamma(t)F(b(t)\Phi)](t, s).$$

Due to (4.2) for every $T_1 \in (t_0, T]$ we have

$$\|S[\Phi]\|_{X(T_1)} \leq \|\Phi_0\|_{X(T_1)} + \|G[b^{-1}(t)\Gamma(t)F(b(t)\Phi)]\|_{X(T_1)}.$$

(4.3)

Meanwhile, according to Corollary 2.2 and Proposition 3.2, we derive

$$\|G[b^{-1}(t)\Gamma(t)F(b(t)\Phi)]\|_{X(T_1)} \leq C_{\mu, a, \Gamma}(T_1)\|\Phi\|_{X(T_1)}^{n+1},$$

(4.4)

with the function $C_{\mu, a, \Gamma}(T_1)$ (3.5). For the local existence ($T_1 < \infty$) we choose the second case of (3.5), while for the global existence ($T_1 = \infty$) the first case can be used as well.
In order to prove local solvability, we claim that for some $R_1 > R$ the operator $S$ maps $X(R_1, T_1)$, onto itself and that $S$ is a contraction provided that $T_1 - t_0$ is sufficiently small. Indeed, for $\Phi(x, t) \in X(R_1, T_1)$ inequalities (4.3) and (4.4) imply
\[
\|S[\Phi]\|_{X(T_1)} \leq R + c_0C_{\mu,a,\Gamma}(T_1)\|\Phi\|_{X(T_1)}^{\alpha+1} \leq R + c_0C_{\mu,a,\Gamma}(T_1)R_1 < R_1
\]
provided that $T_1 - t_0$ is sufficiently small since $\lim_{t_1 \to t_0} C_{\mu,a,\Gamma}(T_1) = 0$. To prove that $S$ is a contraction we write
\[
\|S[\Phi] - S[\Psi]\|_{X(T_1)} = \|G[b^{-1}(t)\Gamma(t) (F(b(t)\Phi) - F(b(t)\Psi))]|_{\|X(T_1)} \leq c_0C_{\mu,a,\Gamma}(T_1) \|b^{-1}(t)\Gamma(t) (F(b(t)\Phi) - F(b(t)\Psi))|_{L^1([t_0,T_1];L^2(S))} \leq c_0C_{\mu,a,\Gamma}(T_1) \max_{\Omega = \Phi, \Psi} \|\Omega\|_{X(T_1)} \|\Phi - \Psi\|_{X(T_1)}
\]
and then choose $T_1$ such that $c_0C_{\mu,a,\Gamma}(T_1)R_1^\alpha < 1$ due to $\lim_{t_1 \to t_0} C_{\mu,a,\Gamma}(T_1) = 0$.

In order to prove global solvability ($T_1 = T = \infty$), we are going to prove that for the given $\Phi_0(x, t) \in X(\varepsilon_0, \infty)$, the operator $S$ maps $X(\varepsilon, \infty)$, $\varepsilon_0 < \varepsilon$, into itself and that $S$ is a contraction provided that $\varepsilon_0$ and $\varepsilon$ are sufficiently small. Thus,
\[
\|S[\Phi]\|_{X(\infty)} \leq \|\Phi_0\|_{X(\infty)} + c_0C_{\mu,a,\Gamma}(\infty)\|\Phi\|_{X(\infty)}^{\alpha+1},
\]
and the operator $S$ maps the space $X(\varepsilon, \infty)$ into itself provided that
\[
c_0C_{\mu,a,\Gamma}(\infty)\|\Phi\|_{X(\infty)}^{\alpha+1} \leq C_0\varepsilon_0e^{\alpha+1}, \quad \varepsilon_0 + c_0C_{\mu,a,\Gamma}(\infty)e^{\alpha+1} \leq \varepsilon.
\]
To prove that $S$ is a contraction we consider
\[
\|S[\Phi](x, t) - S[\Psi]\|_{X(\infty)} = \|G[b^{-1}(t)\Gamma(t) (F(b(t)\Phi) - F(b(t)\Psi))]|_{\|X(\infty)} \leq c_0C_{\mu,a,\Gamma}(\infty) \|b^{-1}(t)\Gamma(t) (F(b(t)\Phi) - F(b(t)\Psi))|_{L^1([t_0,T_1];L^2(S))} \leq c_0C_{\mu,a,\Gamma}(\infty) \max_{\Omega = \Phi, \Psi} \|\Omega\|_{X(\infty)} \|\Phi - \Psi\|_{X(\infty)}
\]
and choose $c_0C_{\mu,a,\Gamma}(\infty) \max_{\Omega = \Phi, \Psi} \|\Omega\|_{X(\infty)} < 1$. Theorem is proven.

**Proof of Theorem 1.1.** We have to prove that the function $\Phi_0(x, t)$, generated by the Cauchy problem (1.3) for the linear equation without source, belongs to $X(\varepsilon, T)$ and that it has sufficiently small norm. Indeed, according to Corollary 2.2, we have the following estimate
\[
\|\partial_t \Phi_0\|_{L^\infty([t_0,t];L^2(S))} + \|\alpha^{-1}(\cdot)\nabla \Phi_0\|_{L^\infty([t_0,t];L^2(S))} + \|M(\cdot)\Phi_0\|_{L^\infty([t_0,t];L^2(S))}
\]
\[
+ \|\sqrt{\alpha a^{-3}} \nabla \Phi_0\|_{L^2([t_0,t]^{\times S})} + \|\sqrt{\alpha} \Phi_0\|_{L^2([t_0,t]^{\times S})} \leq c \left( \|\varphi_1\|_{L^2(S)} + \|\alpha^{-1}(t_0)\|\nabla \varphi_0\|_{L^2(S)} + M(t_0)\|\varphi_0\|_{L^2(S)} \right)
\]
for the solution of the linear problem without source, consequently,
\[
\|\Phi_0\|_{X(T)} \leq c \left( \|\varphi_1\|_{L^2(S)} + \|\alpha^{-1}(t_0)\|\nabla \varphi_0\|_{L^2(S)} + M(t_0)\|\varphi_0\|_{L^2(S)} \right).
\]
It remains to set the right hand side of the last inequality sufficiently small by the proper choice of the initial functions and to apply the statement (ii) of Theorem 4.1. Theorem is proven.

**Proof of Theorem 1.2.** We just repeat the above argument and then apply the statement (i) of Theorem 4.1. Theorem is proven.
Proof of Theorem 1.3. With the potential function $V(t, x, u)$, we define the energy

$$E_V(t) := \frac{1}{2} \left\{ \|u_t\|_{L^2(S)}^2 + a^{-2}(t)\|\nabla_x u\|_{L^2(S)}^2 + M^2(t)\|u\|_{L^2(S)}^2 + b^{-2}(t) \int_S V(t, s, b(t)u(t, s)) \, d\mu_s \right\}.$$  

Then we have

$$\frac{d}{dt} E_V(t) = \frac{1}{2}(a^{-2}(t))_t \|\nabla_x u\|_{L^2(S)}^2 + \frac{1}{2}(M^2(t))_t \|u\|_{L^2(S)}^2$$

$$+ \frac{1}{2} \int_S b^{-2}(t) \left\{ V_t(t, s, b(t)u(t, s)) - 2b^{-1}(t)\dot{b}(t)V(t, s, b(t)u(t, s)) \right\} \, d\mu_s.$$  

Assumption (1.11) implies

$$V_t(t, s, b(t)u(t, s)) - 2b^{-1}(t)\dot{b}(t)V(t, s, b(t)u(t, s)) + \dot{b}(t)u(t, x)V_\psi(s, b(t)u(t, s)) \leq 0$$

for all $t \in [t_0, \infty)$, $s \in S$, $w \in \mathbb{R}$. The integration gives

$$E_V(t) - \int_{t_0}^t \left\{ \frac{1}{2}(a^{-2}(\tau))_\tau \|\nabla_x u\|_{L^2(S)}^2 + \frac{1}{2}(M^2(\tau))_\tau \|u\|_{L^2(S)}^2 \right\} \, d\tau$$

$$- \frac{1}{2} \int_{t_0}^t \int_S b^{-2}(\tau) \left\{ V_t(\tau, s, b(\tau)u(\tau, s)) - 2b^{-1}(\tau)\dot{b}(\tau)V(\tau, s, b(\tau)u(\tau, s)) \right\} \, d\mu_s \, d\tau = E_V(t_0).$$  

In particular, due to the assumption (1.11) and assumptions on $a = a(t)$ and $M = M(t)$, we obtain

$$E_V(t) - \int_{t_0}^t \left\{ \frac{1}{2}(a^{-2}(\tau))_\tau \|\nabla_x u\|_{L^2(S)}^2 + \frac{1}{2}(M^2(\tau))_\tau \|u\|_{L^2(S)}^2 \right\} \, d\tau \leq E_V(t_0).$$

Since $E_V(t)$ does not blow up in finite time, the local solution can be extended globally for all $t \geq t_0$. Theorem is proven. 

4.2 Examples

To make examples more transparent, in this subsection we can restrict them to the case of the manifold $S$ with the single global chart, that is $S = \mathbb{R}^n$. In that case the Laplace-Beltrami operator $\Delta_\sigma$ in the metric $\sigma$ can be simplified and the equation (1.2) can be rewritten as follows

$$\psi_{tt} - \frac{1}{a^2(t)\sqrt{\sigma(x)}} \frac{\partial}{\partial x} \left( \sigma(x) \sigma^{ik}(x) \frac{\partial \psi}{\partial x^k} \right) + n \frac{\dot{a}(t)}{a(t)} \psi_t + m^2 \psi = -V_\psi(t, x, \psi),$$

(4.5)

where the coefficients $\sigma^{ik}(x)$, $i, k = 1, 2, \ldots, n$, belong to the space $B^1(\mathbb{R}^n)$ of the functions with the uniformly bounded derivatives. Moreover, the symmetric form $\sigma^{ik}(x)\xi_i \xi_k$ is positive: $\sigma^{ik}(x)\xi_i \xi_k \geq \text{const} > 0$ for all $x$, $\xi \in \mathbb{R}^n$, $|\xi| = 1$.

Example 4.2 Consider the equation (1.2) with $V_\psi(x, \psi) = \Gamma(t)F(s, \psi)$, $a(t) = t^\ell$, $\ell > 0$:

$$\psi_{tt} - t^{-\ell} \Delta_\sigma \psi + \frac{n\ell}{2t} \psi_t + m^2 \psi = \Gamma(t)F(s, \psi).$$

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The number $\ell = 4/3$ for $n = 3$ coincides with the Einstein-de Sitter exponent. We make change $\psi = t^{-4/3}u$, then

$$u_{tt} - t^{-4} \Delta u + M^2(t)u = t^{n/4} \Gamma(t) F(s, t^{-4/3}u)$$

with the curved mass $M^2_{EdS}(t)$ is given by (1.4). If $n\ell = 4$ and, in particular, in the case of the Einstein-de Sitter spacetime with $n = 3$ and $l = 4/3$ the curved mass coincides with the physical mass, and the equation is

$$u_{tt} - t^{-4/3} \Delta u + m^2 u = t^n \Gamma(t) F(s, t^{-1}u) .$$

The condition (1.8) means $n\ell \leq 4$. For the condition (1.9) we obtain

$$|\Gamma(t)| \leq ct^{-1} \text{ for all } t \geq t_0 .$$

For (1.10) we have

$$\int_{t_0}^{\infty} t^{1/n\alpha_0} |\Gamma(t)|^{\frac{4}{4-n\alpha_0}} \, dt < \infty ,$$

where $0 < n\alpha_0 < 4$. If $\Gamma(t) = t^\gamma$, then the last integral is convergent if $\gamma < -1$.

**Example 4.3** Consider a spacetime with the scale function

$$a(t) = \exp(\beta H t^\beta) , \quad H, \beta \in \mathbb{R} ,$$

and

$$c(t) = m^2 - \frac{1}{4} \beta H n t^{\beta - 2} \left( \beta H n t^\beta + 2 \beta - 2 \right) ,$$

$$\dot{c}(t) = -\frac{1}{2} (\beta - 1) \beta H n t^{\beta - 3} \left( \beta + \beta H n t^\beta - 2 \right) .$$

If $H$ is positive, then it can be regarded as the Hubble constant. For the condition (1.7) we have to set $\beta H > 0$. For the condition (1.8) we consider two cases:

1. $\beta > 0$. Both conditions of (1.8) are fulfilled only if $\beta = 1$ and $0 < H < \frac{2m}{n}$. That is a case of the de Sitter metric and the large mass in the classification of [14]. For $\beta = 1$ the condition (1.9) reads

$$|\Gamma(t)| \leq \text{const} \text{ for all } t \geq t_0 .$$

The condition (1.10) for $\Gamma(t) = t^\gamma$ implies $\gamma < -1 + \frac{1}{4} \alpha_0 n$.

The condition (1.11) for the self-interaction $V'_\psi(x, \psi) = -\Gamma(t) \mu(x) |\psi|^\alpha \psi$ and $\beta = 1$ is satisfied if $\dot{\Gamma}(t) \leq \frac{\alpha_0 H}{2} \Gamma(t)$, that is $0 \leq \Gamma(t) \leq C \exp(\frac{\alpha_0 H}{2} t)$, and $0 < c_0 \leq \mu(x) \leq c_1$. We do not know if the last condition on $\Gamma(t)$ is a necessary restriction for the case of the energy conservative potentials.

2. $\beta < 0$. In this case $H < 0$. Both conditions of (1.8) are fulfilled for $\beta < 0$ and $H < 0$.

For the condition (1.9) we obtain

$$|\Gamma(t)| \leq c t^{\beta - 1} \text{ for all } t \geq t_0$$

while for (1.10) we have

$$\int_{t_0}^{\infty} t^{(1-\beta)\frac{n\alpha_0}{4-n\alpha_0}} |\Gamma(t)|^{\frac{4}{4-n\alpha_0}} \, dt < \infty ,$$

where $0 < n\alpha_0 < 4$. If $\Gamma(t) = t^\gamma$, then for the convergence of the last integral we need $\gamma < -1 + \frac{1}{4} \beta n\alpha_0$.

Hence, $\gamma$ must decay with the rate constant:

$$\gamma \leq \beta - 1 \quad \text{or } \quad \gamma < \frac{n\alpha_0}{4} \beta - 1 .$$

We do not know if the last condition is a necessary restriction.
Example 4.4 Consider the following example of the scale function

\[ a(t) = t^\ell \exp(Ht^\beta), \quad \ell, H, \beta \in \mathbb{R}, \]

and

\[ \dot{a}(t) = t^{\ell-1} \left( \beta H t^\beta + \frac{\ell}{2} \right) \exp(Ht^\beta), \]

\[ c(t) = m^2 - \frac{n}{4\ell^2} \left( \beta^2 H^2 n^2 \beta + 2 \beta H t^\beta \left( \beta + \frac{\ell}{2} \right) \right) > 0 \text{ for } t \geq t_0, \]

\[ \dot{c}(t) = -\frac{n}{2t^3} \left( (\beta - 1) \beta^2 H^2 n \beta + (\beta - 2) \beta H t^\beta \left( \beta + \frac{\ell}{2} \right) \right) \leq 0 \text{ for } t \geq t_0. \]

Here \( t_0 \) is sufficiently large number. The case of \( \ell = 0 \) coincides with Example 4.3, while the case of \( \beta = 0 \) or \( H = 0 \) coincides with Example 4.2.

For the conditions (1.7)–(1.10) we consider two separate cases.

1. \( \beta > 0 \). In this case the condition \( \dot{a}(t) > 0 \) of (1.7) implies \( H > 0 \).
   The condition \( c(t) > 0 \) of (1.8) is satisfied for \( \beta < 1 \) or \( \beta = 1 \) & \( m^2 - \frac{1}{4}(Hn)^2 > 0 \), while the condition \( \dot{c}(t) \leq 0 \) implies \( \beta > 1 \) or \( \beta = 1 \) and \( H = 0 \). Thus, (1.8) is satisfied only for \( \beta = 1 \) and \( H = 0 \), which coincides with Example 4.2.

2. \( \beta < 0 \). In this case the assumption \( \dot{a}(t) > 0 \) of (1.7) implies \( \ell > 0 \). Both conditions of (1.8) are satisfied for \( 0 \leq \ell \leq \frac{4}{n} \). Now, if \( \ell > 0 \) for the condition (1.9) we obtain

\[ |\Gamma(t)| \leq ct^{-1} \quad \text{for all } t \geq t_0, \]

and for (1.10) we have

\[ \int_{t_0}^\infty \left( \frac{t}{\beta H t^\beta + \frac{\ell}{2}} \right)^{\frac{n}{4\ell n_0}} |\Gamma(t)|^{\frac{n}{4\ell n_0}} dt < \infty, \]

where \( 0 < n_0 < 4 \). If \( |\Gamma(t)| = c \), then for the convergence of the last integral we need \( \gamma < -1 \). Hence, the condition for the decay rate constant \( \gamma \) coincides with the condition in Example 4.2.

Finally, we note here that the results of this paper are applicable to the equation (4.5) with \( x \)-dependent coefficient, while the results of [13],[15],[16],[8] are restricted to the equation with \( x \)-independent coefficients.

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