Lyapunov-type inequalities for quasilinear elliptic equations with Robin boundary condition

Ülkü Dinlemez Kantar and Tülay Özden

Abstract
The aim of this study is to prove Lyapunov-type inequalities for a quasilinear elliptic equation in $\mathbb{R}^2$. Also the lower bound for the first positive eigenvalue of the boundary value problem is obtained.

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1 Introduction
In [1], Lyapunov proved that, if $p(x)$ is a nonnegative and continuous function and $u(x) \in C(I, \mathbb{R})$, a necessary condition for the following boundary value problem:

$$
\begin{aligned}
&u''(x) + p(x)u(x) = 0, \quad u(x) \neq 0, \forall x \in I, \\
&u(a_1) = 0 = u(b_1),
\end{aligned}
$$

(1.1)

to have nontrivial solutions is

$$
\frac{4}{b_1 - a_1} \leq \int_{a_1}^{b_1} p(x) \, dx,
$$

(1.2)

where $I = [a_1, b_1]$.

Since Lyapunov’s study, because the inequality of (1.2) plays a key role for the qualitative properties, such as oscillatory and disconjugacy etc., of differential equations’ solutions, several authors focused on the inequality of (1.2). Those authors improved and generalized the inequality of (1.2) in $\mathbb{R}$. In this work the literature of the one-dimensional case is not studied in detail but it is listed in the references for the interested reader. See [1–17] and the references cited therein.

In addition to studies in $\mathbb{R}$, several authors [18–24] have extended the inequality of (1.2) in $\mathbb{R}^n$ recently. To the best of our knowledge, it was extended by Cañada, Montero, and Villegas [19] for the first time. In [19] Cañada et al. considered the linear elliptic problem.

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as follows:

$$\begin{cases} -\Delta u = a(x)u, & x \in \Omega, \\ \partial u/\partial n = 0, & x \in \partial \Omega, \end{cases}$$

(1.3)

where $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain with $N \geq 2$ and the function $a : \Omega \to \mathbb{R}$ belongs to the set

$$\Lambda = \left\{ a \in L^{N/2}(\Omega) \setminus \{0\} : \int_\Omega a(x) \, dx \geq 0 \text{ and (1.3) has a nontrivial solution} \right\}$$

(1.4)

if $N \geq 3$,

$$\Lambda = \left\{ a : \Omega \to \mathbb{R} \text{ s.t. } \exists q \in (1, \infty] \text{ with } a \in L^q(\Omega) \setminus \{0\} : \int_\Omega a(x) \, dx \geq 0 \right\}$$

and (1.3) has a nontrivial solution

(1.5)

if $N = 2$, we define

$$\beta_q := \inf_{a \in A \cap L^q(\Omega)} \|a\|_{L^q(\Omega)}, \quad 1 \leq q \leq \infty.$$

(1.6)

Their main result is as follows.

**Theorem A** The following statements hold.

1. If $N = 2$ then $\beta_q > 0 \iff 1 < q \leq \infty$. If $N \geq 3$ then $\beta_q > 0 \iff \frac{2}{N} \leq q \leq \infty$.
2. If $\frac{2}{N} < q \leq \infty$ then $\beta_q$ is attained. In this case, any function $a \in A \cap L^q(\Omega)$ from which $\beta_q$ is attained has one of the following forms:
   i. $a(x) = \lambda_1$ if $p = \infty$, where $\lambda_1$ is the first strictly positive eigenvalue of (1.3).
   ii. $a(x) = |u(x)|^{2/(p-1)}$, if $\frac{2}{N} < q < \infty$, where $u$ is a solution of the problem as follows:

$$\begin{cases} -\Delta u = |u(x)|^{2/(p-1)}u, & x \in \Omega, \\ \partial u/\partial n = 0, & x \in \partial \Omega. \end{cases}$$

(1.7)

3. The map $\left( \frac{2}{N}, \infty \right) \to \mathbb{R}$, $p \mapsto \beta_p$, is continuous and the map $\left[ \frac{2}{N}, \infty \right) \to \mathbb{R}$, $p \mapsto |\Omega|^{\frac{1}{2}} \beta_p$, is strictly increasing.
4. The limits $\lim_{p \to \infty} \beta_p$ and $\lim_{p \to \left( \frac{2}{N} \right)^+} \beta_p$ always exist and take the values
   i. $\lim_{p \to \infty} \beta_p = \beta_\infty$, if $N \geq 2$,
   ii. $\lim_{p \to \left( \frac{2}{N} \right)^+} \beta_p \geq \beta_\frac{2}{N} > 0$, if $N \geq 3$, $\lim_{p \to 1^+} \beta_p = 0$, if $N = 2$.

Here, we also note that in the study Cañada et al. they proved that the relation between the $p$ and $\frac{2}{N}$ plays a crucial role. They also considered the equation in (1.3) with zero Dirichlet boundary condition. They presented similar inequalities at their study. Then others established Lyapunov-type inequalities for different equations with boundary conditions. For more information about the studies in $\mathbb{R}^n$, the interested reader can refer to [18–24] and the references cited therein.
The aim of this paper is to prove a Lyapunov-type inequality for the two-dimensional quasilinear elliptic problem as follows:

\[
\begin{cases}
-(\Phi_p(u_{xy}))_y = r(x,y)\Phi_p(u), \\
u(a_1,y) = 0 = u(b_1,y), \\
u_x(a_2) = 0 = u_x(b_2),
\end{cases} 
\tag{1.8}
\]

where \( \Omega = [a_1, b_1] \times [a_2, b_2] \) and \( r(x,y) \) is a measurable function on \( \Omega \), and \( \Phi_p(u(x,y)) = |u(x,y)|^{p-2}u(x,y) \) for \( p > 1 \). In addition to this, we note that by a solution of the problem (1.8), we mean that \( u(x,y) \in W^{3,p}(\Omega) \) in that

\[
W^{3,p}(\Omega) = \{ u : u, u_x, u_{xy} and u_{xxy} \in L^p(\Omega) \}.
\tag{1.9}
\]

As usual, \( L^p(\Omega) \) is a space of Lebesgue measurable functions.

2 Main results

Now, we give a key lemma as a proof of our main conclusions.

**Lemma 1** Assume that \( u(x,y) \in W^{3,p}(\Omega) \), it satisfies the boundary conditions in (1.8) and \( u(x,y) \neq 0 \) for \( \forall(x,y) \in \Omega^0 \). Then

\[
\left(4|u(x,y)|^p / (b_1 - a_1)^{p-1}(b_2 - a_2)^{p-1}\right) \leq \int_{a_1}^{b_1} \int_{a_2}^{b_2} |u_{xy}|^p \, dy \, dx,
\tag{2.1}
\]

\[
\left(2^p / (b_2 - a_2)^{p-1}\right) \int_{a_1}^{b_1} |u_x|^p \, dx \leq \int_{a_1}^{b_1} \int_{a_2}^{b_2} |u_{xy}|^p \, dy \, dx,
\tag{2.2}
\]

hold, respectively, where \( \Omega^0 \) is the set of all interior points of \( \Omega \).

**Proof** Let \( (x,y) \in \Omega \). Since \( u(x,y) \) satisfies the boundary conditions in (1.8), it is easy to see

\[
u(x,y) = \int_{a_1}^{x} \int_{a_2}^{y} u_{xx} \, ds \, dt,
\]

taking the absolute value, we obtain

\[
|u(x,y)| \leq \int_{a_1}^{x} \int_{a_2}^{y} |u_{xx}| \, ds \, dt.
\tag{2.3}
\]

Similarly, we get

\[
|u(x,y)| \leq \int_{a_1}^{x} \int_{y}^{b_2} |u_{xx}| \, ds \, dt,
\tag{2.4}
\]

\[
|u(x,y)| \leq \int_{x}^{b_1} \int_{a_2}^{y} |u_{xx}| \, ds \, dt,
\tag{2.5}
\]

and

\[
|u(x,y)| \leq \int_{x}^{b_1} \int_{y}^{b_2} |u_{xx}| \, ds \, dt.
\tag{2.6}
\]
Adding (2.3)-(2.6), we have

$$4|u(x,y)| \leq \int_{a_1}^{b_1} \int_{a_2}^{b_2} |u_{xy}| \, dy \, dx.$$  (2.7)

Then, applying Hölder’s inequality

$$\int_{a_1}^{b_1} |f(t)g(t)| \, dt \leq \left( \int_{a_1}^{b_1} |f(t)|^q \, dt \right)^{1/q} \left( \int_{a_1}^{b_1} |g(t)|^p \, dt \right)^{1/p}$$  (2.8)

to the right-hand side of (2.7), we get

$$(4|u(x,y)|)^p \leq (b_1 - a_1)^{p-1} \int_{a_1}^{b_1} \left[ \int_{a_2}^{b_2} |u_{xy}| \, dy \right]^p \, dx.$$  (2.9)

Applying Hölder’s inequality to the right hand side of (2.9) again, we obtain

$$\frac{(4|u(x,y)|)^p}{(b_1 - a_1)^{p-1}(b_2 - a_2)^{p-1}} \leq \int_{a_1}^{b_1} \int_{a_2}^{b_2} |u_{xy}|^p \, dy \, dx.$$  (2.10)

Thereby, the proof of (2.1) is completed.

Similarly we have

$$|u_s(x,y)| \leq \int_{a_2}^{b_2} |u_{sx}| \, ds$$  (2.11)

and

$$|u_s(x,y)| \leq \int_{a_2}^{b_2} |u_{sx}| \, ds.$$  (2.12)

Adding (2.10) and (2.11), we get

$$2|u_s(x,y)| \leq \int_{a_2}^{b_2} |u_{sx}| \, ds.$$  (2.13)

Applying Hölder’s inequality to the right hand side of (2.12) and integrating from $a_1$ to $b_1$, we have

$$\frac{2^p}{(b_2 - a_2)^{p-1}} \int_{a_1}^{b_1} |u_s| \, dx \leq \int_{a_1}^{b_1} \int_{a_2}^{b_2} |u_{xy}|^p \, dy \, dx.$$  (2.14)

Consequently, the proof of (2.2) is completed.

**Theorem 1** If $u(x,y) \in W^{3,p} (\Omega)$ is a nontrivial solution of the problem (1.8), then the following inequality:

$$2^{2p+1}/(b_1 - a_1)^p (b_2 - a_2)^p \leq \int_{a_1}^{b_1} \int_{a_2}^{b_2} |r(x,y)|^q \, dy \, dx$$  (2.15)

holds, where $q$ is the Hölder conjugate of $p$. 

□
Proof Let \( u(x,y) \in W^{3,p}(\Omega) \) is a nontrivial solution of the problem (1.8). Multiplying the equation in (1.8) by \( u_x \) and integrating on \( \Omega \), we obtain

\[
\int_{a_1}^{b_1} \int_{a_2}^{b_2} - (|u_{xy}|^{p-2} u_{xy}) u_x \, dy \, dx = \int_{a_1}^{b_1} \int_{a_2}^{b_2} r(x,y) |u|^{p-2} u u_x \, dy \, dx. \tag{2.14}
\]

Then, applying partial integration in \( \int_{a_2}^{b_2} - (|u_{xy}|^{p-2} u_{xy}) u_x \, dy \) and using the boundary conditions in (1.8), we have

\[
\int_{a_1}^{b_1} \int_{a_2}^{b_2} |u_{xy}|^p \, dy \, dx = \int_{a_1}^{b_1} \int_{a_2}^{b_2} r(x,y) |u|^{p-2} u u_x \, dy \, dx. \tag{2.15}
\]

By taking the absolute value on right hand side of (2.15), we get

\[
\int_{a_1}^{b_1} \int_{a_2}^{b_2} |u_{xy}|^p \, dy \, dx \leq \int_{a_1}^{b_1} \int_{a_2}^{b_2} |r(x,y)| |u|^{p-1} |u_x| \, dy \, dx. \tag{2.16}
\]

Hence, applying Hölder’s inequality to the right hand side of (2.16), we find

\[
\int_{a_1}^{b_1} \int_{a_2}^{b_2} |u_{xy}|^p \, dy \, dx \leq \left( \int_{a_1}^{b_1} \int_{a_2}^{b_2} |r(x,y)|^q |u|^{p-1} |u_x| \, dy \, dx \right)^{1/q} \left( \int_{a_1}^{b_1} \int_{a_2}^{b_2} |u_x|^p \, dy \, dx \right)^{1/p}. \tag{2.17}
\]

Now, considering only the second term of right hand side in (2.17) from Fubini’s theorem, we can rewrite the inequality (2.17) as follows:

\[
\int_{a_1}^{b_1} \int_{a_2}^{b_2} |u_{xy}|^p \, dy \, dx \leq \left( \int_{a_1}^{b_1} \int_{a_2}^{b_2} |r(x,y)|^q |u|^{p-1} |u_x| \, dy \, dx \right)^{1/q} \left( \int_{a_2}^{b_2} \left[ \int_{a_1}^{b_1} |u_x|^p \, dx \right] \, dy \right)^{1/p}. \tag{2.18}
\]

Hence, using the inequality (2.2) in (2.18), we obtain

\[
\left( \int_{a_1}^{b_1} \int_{a_2}^{b_2} |u_{xy}|^p \, dy \, dx \right)^{(p-1)/p} \leq \left( \frac{(b_2 - a_2)}{2} \right) \left( \int_{a_1}^{b_1} \int_{a_2}^{b_2} |r(x,y)|^q |u|^{p-1} |u_x| \, dy \, dx \right)^{1/q}. \tag{2.19}
\]

Then, replacing the point of \( (x,y) \), which is used in Lemma 1, with the maximum point of \( |u(x,y)| \), from (2.1), we get

\[
\left( 4 \max |u(x,y)| \right)^{(p-1)/(b_1 - a_1)^{(p-1)/q}(b_2 - a_2)^{(p-1)/q}} \leq \left( \int_{a_1}^{b_1} \int_{a_2}^{b_2} |u_{xy}|^p \, dy \, dx \right)^{(p-1)/p}. \tag{2.20}
\]
Then, using the inequality (2.20) in the inequality (2.19), we have

$$\left(4 \max |u(x, y)| \right)^{(p-1)/q} / \left((b_2 - a_2)^{(p-1)/q} (b_2 - a_2)^{(p-1)/q}\right)$$

$$\leq \left((b_2 - a_2)/2\right) \left(\int_{a_1}^{b_1} \int_{a_2}^{b_2} |r(x, y)|^q |u|^q \, dy \, dx\right)^{1/q}$$

$$\leq \left((b_2 - a_2)/2\right) \left(\int_{a_1}^{b_1} \int_{a_2}^{b_2} |r(x, y)|^q \, dy \, dx\right)^{1/q} \left(\max |u(x, y)| \right)^{(p-1)}. \quad (2.21)$$

Since $u(x, y)$ is a nontrivial solution, we have $\max |u(x, y)| \neq 0$. Therefore, we obtain

$$2^{p+1} / (b_1 - a_1)^{p-1} (b_2 - a_2)^p \leq \int_{a_1}^{b_1} \int_{a_2}^{b_2} |r(x, y)|^q \, dy \, dx. \quad (2.22)$$

Thus, the proof is completed. □

**Corollary 1** Let $\lambda_1$ be the first eigenvalue of the equation that is defined on $\Omega$ as follows:

$$-(\Phi_p(u_{xy})) = \lambda_1 r(x, y) \Phi_p(u),$$

where $\Omega$ is a domain, which is defined in the beginning of the paper, and with the boundary conditions in (1.8). Then we have

$$2^{p+1} / (b_1 - a_1)^{p-1} (b_2 - a_2)^p \|r(x, y)\|^{q}_{L^q(\Omega)} \leq \lambda_1. \quad (2.23)$$

**Remark 1** If we take the Dirichlet boundary conditions, which are $u(a_1, y) = 0 = u(b_1, y)$ and $u(x, a_2) = 0 = u(x, b_2)$, instead of the Robin boundary conditions in the problem (1.8), then we obtain the identical conclusions given above.

**Remark 2** The result, which is obtained in this study, is also the necessary condition for the problem of (1.8) to have a nontrivial solution.

**Competing interests**
The authors declare that they have no competing interests.

**Authors’ contributions**
Both authors contributed equally to the writing of this paper. Both authors read and approved the final manuscript.

**Author details**
1Department of Mathematics, Faculty of Sciences, Gazi University, 06500 Teknikokullar, Ankara, Turkey. 2Department of Mathematics, Institute of Sciences, Gazi University, 06500 Teknikokullar, Ankara, Turkey.

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