A NECESSARY AND SUFFICIENT CONDITION FOR ISS OF IMPULSIVE SYSTEMS VIA A TIME-VARYING LYAPUNOV FUNCTION

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Abstract. We propose a time-varying ISS-Lyapunov function for impulsive systems over Banach spaces which provides a necessary and sufficient condition for ISS. Our result applies to a broad class of impulsive systems including simultaneous unstable continuous and discrete dynamics.

Key words. Impulsive system, input-to-state stability, ISS-Lyapunov function.

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1. Introduction. Impulsive systems form an important class of hybrid dynamical systems that combine continuous behavior with sudden changes of state, often related by the flow and the jump, respectively. Impulsive systems find their application in numerous fields such as biomedicine, networked control systems, and multi-agent systems; see [12], [2], and [6]. On the other hand, an essential property of dynamical systems in practice is their sensitivity to external perturbations. The notion of input-to-state stability (ISS), introduced by [13], guarantees a certain tolerance to such inputs and is therefore helpful for the classification of the system’s behavior.

ISS was initially developed for ordinary differential equations (ODEs). The efficacy of the ISS theory for ODEs and the requirement of robust stability analysis tools for partial differential equations (PDEs) led to the study of the ISS property for infinite-dimensional systems. A significant step in this direction was the introduction of converse Lyapunov theorems for ISS of semi-linear evolution equations over Banach spaces by [10]. Therefore, it seems natural to formulate a converse Lyapunov theorem for ISS of impulsive evolution equations over Banach spaces and it is the main theme of the present paper. To the authors’ best knowledge, there is no converse ISS-Lyapunov theorem in the literature for this class of systems.

A well-established tool for proving ISS is the ISS-Lyapunov function. This framework was modified to candidate ISS-Lyapunov function to also cover impulsive systems (cf. [7], [3]). However, a candidate ISS-Lyapunov function based analysis has certain restrictions. First, it only provides a sufficient condition for ISS. Second, it offers stability conclusions for a restricted class of systems: Either the flow behavior must be stable, and the jumps may be unstable, or the jumps ought to be stable, and the flow might be unstable. These ISS results are inconclusive for impulsive systems with simultaneous instability of the continuous and discrete dynamics. ISS of this significant class of impulsive systems has received little attention in the literature. It was not until recently that [4] treated ISS of such impulsive systems using a dwell-time approach based on the higher order derivatives of the Lyapunov function. However, the aforementioned work still only gives a sufficient condition of ISS and does not provide a converse ISS-Lyapunov theorem.

In this paper, we propose a time-varying formulation of ISS-Lyapunov functions in implication form that provides a necessary and sufficient condition for ISS of im-
impulsive systems over Banach spaces. Our result applies to a broad class of impulsive systems including simultaneous unstable continuous and discrete dynamics. Note that our proposed ISS-Lyapunov functions are in general time-dependent, even when the flow is time-autonomous. That is necessary because the jumps change the dynamics compared to an instant of flow without a jump. At the instants of the jumps, the Lyapunov functions will, in general, exhibit significantly different behavior than during the flow time.

The rest of the paper unfolds as follows. In Section 2, we provide some preliminaries, including necessary definitions and notations. Furthermore, we introduce the notion of time-varying ISS-Lyapunov functions for impulsive systems. In Section 3, we provide a sufficient condition of ISS for the class of impulsive systems introduced in Section 2. In Section 4, we prove the converse inclusion that for every system, which is ISS, there must exist an ISS-Lyapunov function. We conclude the paper by summarizing our findings and providing some future perspectives in Section 5.

2. Preliminaries and Input-to-State Stability. We denote the set of natural numbers by \( \mathbb{N} \), the set \( \mathbb{N}_0 = \mathbb{N} \cup \{0\} \), the set of real numbers by \( \mathbb{R} \), the set of non-negative real numbers by \( \mathbb{R}^+_0 \), the space of continuous functions from normed spaces \( X \) to \( Y \) by \( C(X,Y) \) and the ball of radius \( r > 0 \) around 0 by \( B_X(r) \). Let \( I = [t_0, \infty) \subset \mathbb{R} \). Let \( S = \{t_n\}_{n \in \mathbb{N}} \) be a set, which contains the elements of a strictly increasing sequence of impulse times \( (t_n)_{n \in \mathbb{N}} \) in \( (t_0, \infty) \) such that \( t_i \to \infty \) for \( i \to \infty \).

Let \( (X, \| \cdot \|_X) \) be a Banach space representing the state space. Let \( PC(I, X) \) be the space of piecewise continuous functions from \( I \) to \( X \), which are right-continuous and the left limit exists for all times \( t \in I \). Let the Banach space \( (U, \| \cdot \|_U) \) represent the input space. Let furthermore \( U_c \) be the space of bounded functions from \( I \) to \( U \) with norm \( \| u \|_\infty := \sup_{t \in I} \{ \| u(t) \|_U \} \). We denote the left limit of a function \( f \) at \( t \) as \( f^-(t) \). We consider the standard classes of comparison functions \( K, K_{\infty} \) and \( KL \) as defined by [13] and the positive definite function class \( P \) consisting of continuous functions \( \gamma : [0, \infty) \to [0, \infty) \), which satisfy \( \gamma(0) = 0 \) and \( \gamma(r) > 0 \) for all \( r > 0 \).

Consider an impulsive system described by the interacting continuous and discontinuous evolution maps:

\[
\begin{align*}
\dot{x}(t) &= Ax(t) + f(t, x(t), u(t)), & t \in I \setminus S, \\
x(t) &= g_i(x^-(t), u^-(t)), & t = t_i \in S, \ i \in \mathbb{N},
\end{align*}
\]

where \( u \in U_c \) and \( x : I \to X \). The closed linear operator \( A : D(A) \to X \) is the infinitesimal generator of a \( C_0 \)-semigroup \( T(t) \) on \( X \), where \( D(A) \) is a dense subset of \( X \), \( f : I \times X \times U \to X \), and \( g_i : X \times U \to X \) for all \( i \in \mathbb{N} \). We are interested in solutions in the mild sense, i.e., a function \( x \in PC(I, X) \) such that

\[
\begin{align*}
x(t) &= T(t-t_0)x_0 + \int_{t_0}^t T(t-s)f(s, x(s), u(s)) \, ds \\
&\quad + \sum_{i \in \mathbb{N} : t_i \leq t} T(t-t_i)\left(g_i(x^-(t_i), u^-(t_i)) - x^-(t_i)\right)
\end{align*}
\]

holds for all \( t \in [t_0, \infty) \) (cf. [1]). We assume that for system (2.1), a (forward-)unique global mild solution exists for every initial condition \( x(t_0) = x_0 \) and every \( u \in U_c \). Sufficient conditions for the existence and uniqueness of solutions of system (2.1) can be straightforwardly extended from [1]. For instance, the conditions in [1] apply for system (2.1) provided that the Lipschitz bounds on \( f \) and \( (g_i)_{i \in \mathbb{N}} \) are uniform with respect to control.
We denote the value of the solution trajectory at time \( t \) with the initial condition \( x(t_0) = x_0 \) and the input \( u \in U_c \) by \( x(t; t_0, x_0, u) \). We shorten the notation by \( x(t) \) if the parameters are clear from the context or can be chosen arbitrarily.

**Definition 2.1.** For a given sequence of impulse times \( S \), we call system (2.1) input-to-state stable (ISS) if there exist functions \( \beta \in \mathcal{K} \mathcal{L} \) and \( \gamma \in \mathcal{K}_\infty \) such that for all initial values \((t_*, x_*) \in I \times X \) and every input function \( u \in U_c \), the system has a global solution, which satisfies for all \( t \in [t_*, \infty) \)

\[
\|x(t; t_*, x_*, u)\|_K \leq \beta(\|x_*\|_K, t - t_*) + \gamma(\|u\|_\infty).
\]

Now we propose the notion of time-varying ISS-Lyapunov functions for system (2.1).

**Definition 2.2.** Let \( V : I \times X \rightarrow \mathbb{R}_0^+ \) be such that \( V \in \mathcal{C}((I \setminus S) \times X, \mathbb{R}_0^+) \) and \( V \in \mathcal{PC}(I \times X, \mathbb{R}_0^+) \) hold. We call \( V \) an ISS-Lyapunov function for an impulsive system (2.1) if it fulfills all of the following conditions:

(i) There exist functions \( \alpha_1, \alpha_2 \in \mathcal{K}_\infty \) such that

\[
\alpha_1(\|x\|_X) \leq V(t, x) \leq \alpha_2(\|x\|_X)
\]

holds true for all \( t \in I \) and all \( x \in X \).

(ii) There exist functions \( \chi \in \mathcal{K}_\infty \) and \( \varphi \in \mathcal{P} \) such that for all inputs \( u \in U_c \) and all solutions \( x = x(t; t_0, x_0, u) \) of (2.1), whenever \( V(t, x) \geq \chi(\|u\|_\infty) \), the inequalities

\[
\begin{align*}
\frac{d}{dt} V(t, x) &\leq -\varphi(V(t, x)), & t &\in I \setminus S, \\
V(t_i, g_i(x, u)) &\leq V(t_i^-, x), & t_i &\in S,
\end{align*}
\]

hold true. Here, \( \frac{d}{dt} V(t, x) \) denotes the Dini-derivative

\[
\frac{d}{dt} V(t, x) = \limsup_{s \to 0} \frac{1}{s} (V(s, x(s; t, x, u)) - V(t, x)).
\]

(iii) There exists a function \( \alpha_3 \in \mathcal{K} \) such that for all \( x \in X \), all \( u \in U \), and all \( i \in \mathbb{N} \), which satisfy \( V(t_i^-, x) < \chi(\|u\|_\infty) \), the jump inequality satisfies

\[
V(t_i, g_i(x, u)) \leq \alpha_3(\|u\|_\infty).
\]

**Remark 2.3.** From inequalities (2.5) and (2.6), it follows that the Lyapunov value of a trajectory strictly falls with time when the state is outside the perturbation radius \( \chi(\|u\|_\infty) \). Note that inequality (2.5) does not necessarily imply that the flow is stable because (2.5) only holds for time intervals \([t_i, t_{i+1})\), \( i \in \mathbb{N}_0 \). In general, \( V \) is not continuous and may increase at time instants \( t = t_i \), \( i \in \mathbb{N}_0 \). Similar arguments also hold for inequality (2.6).

**Remark 2.4.** Condition (iii) is necessary for establishing ISS, and it was not taken care of in the previous relevant works such as [3]. Once the state has reached the perturbation radius \( \chi(\|u\|_\infty) \), it can escape out of it afterward if Condition (iii) is not fulfilled. Consequently, the system will not be ISS. We illustrate with the aid of an example that there are systems, which satisfy Conditions (i) and (ii), but are not ISS. Let us consider the one-dimensional system

\[
\begin{align*}
\dot{x}(t) &= -x(t), & \forall t \in \mathbb{R}_0^+ \setminus S; \\
x(t) &= \begin{cases} 
\frac{1}{2}x^{-}(t), & \text{if } |x^-| \geq |u|, \\
1, & \text{else},
\end{cases} & \forall t \in S;
\end{align*}
\]
with control input $u : \mathbb{R}_+^n \to \mathbb{R}$ and $S = (t_i)_{i \in \mathbb{N}}$, where $t_i = i$. The function $V(x) = \|x\|$ satisfies Conditions (i) and (ii) with $\chi(s) = s$, $\varphi(s) = \frac{s}{2}$, but there exists no continuous $\mathcal{K}_\infty$-function $\gamma$ such that the system is ISS as Condition (iii) is violated. For discrete-time systems, a similar issue was discussed by [5].

2.1. Example. We provide an example to illustrate how our ISS-Lyapunov functions apply to impulsive systems with simultaneous instability in both continuous and discrete dynamics. Consider the two-dimensional system in $\mathbb{R}^2$ equipped with Euclidean norm $\| \cdot \|$ and $u \in \mathbb{R}$,

\begin{align}
\dot{x}(t) &= Ax(t), & t \in \mathbb{R}_0^+ \setminus S, \\
x(t) &= g(x^-(t), u), & t \in S,
\end{align}

where $A := \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$, $g((x_1^i), u) := \begin{pmatrix} 2x_1 \\ u \tanh(x_2) \end{pmatrix}$ with $S = \{t_i\}_{i \in \mathbb{N}}$, and $t_i = \frac{\pi}{2}i$ for $i \in \mathbb{N}$.

Note that the impulsive system described by (2.8) is strictly unstable in both components of the flow and the direction of the first unit vector of the jumps. Only the jump direction of the second unit vector is stable.

It can be shown that

\begin{equation}
V(t, x) = e^{-4(t-t_n)} x^T R(t-t_n) D R^T (t-t_n) x,
\end{equation}

where $n = \max\{i \in \mathbb{N} | t_i \leq t\}$, $R$ is the rotation matrix and $D$ is the diagonal matrix defined by $R(t) = \begin{pmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{pmatrix}$ and $D = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 2^2 \end{pmatrix}$, respectively. Hence, (2.9) fulfills all the conditions given in Definition 2.2 with $\chi(s) = 8e^{6\pi}s^2$, therefore being an ISS-Lyapunov function. By the next theorem, we can conclude that system (2.8) is indeed ISS.

Note that the ISS result in [3] does not apply to system (2.8) because both the flow and jumps are simultaneously unstable.

3. Sufficient Condition for ISS. In this section, we provide a sufficient condition of ISS for impulsive system (2.1).

Theorem 3.1. Let there exist an ISS-Lyapunov function for impulsive system (2.1), then it is ISS.

Proof. The proof can be partitioned into two steps. For the first step, we prove that for the set

$$A_1(t) := \{x \in X | V(t, x) < \chi(\|u\|_\infty)\}$$

and every $x_0 \notin A_1(t_0)$, there exists a $\mathcal{KL}$-function $\beta$ such that the inequality

$$\|x(t; t_0, x_0, u)\|_X \leq \beta(\|x_0\|_X, t-t_0)$$

holds true if $x(t_*) \notin A_1(t_*)$ for all $t_* \in [t_0, t]$. In the second part, we show that trajectories, once they have reached the set $A_1$, will stay bounded.

Step 1: We first assume that $x(t) \notin A_1$ such that $V(t, x(t)) \geq \chi(\|u\|_\infty)$ for all $(t, x, u)$. Then, by Definition 2.2, the function $V$ fulfills the following inequalities:

\begin{align}
\dot{V}(t, x(t)) &\leq -\varphi(V(t, x(t))), & t \in I \setminus S, \\
V(t, x(t)) &\leq (V(t, x(t)))^-, & t \in S.
\end{align}
For brevity, we use \( v : I \to \mathbb{R}_+^\star \), \( v(t) := V(t, x(t)) \) and denote \( v_i := v(t_i) \) and \( v_i^- := v^-(t_i) \). If the right-hand side of (3.1) is non-zero, we can transform (3.1) to

\[
\frac{\dd v}{\dd t} \leq -\frac{\varphi(v)}{\varphi(v)} = -1. \tag{3.3}
\]

We explicitly exclude the case \( \varphi(v(t)) = 0 \), i.e., \( v(t) = 0 \) from this inequality. Note that this case is trivial because the trajectories of (3.1)–(3.2) that become equal to zero at some time \( t^\star \), will remain there for all times \( t \geq t^\star \).

Integrating (3.3) over the interval \([t_i, t_s] \), \( i \in \mathbb{N}_0 \), for some \( t_s \in [t_i, t_{i+1}) \), we obtain

\[
\int_{v_i}^{v_{(t_s)}} \frac{1}{\varphi(s)} \, ds = \int_{t_i}^{t_s} \frac{\dd v(t)}{\varphi(v(t))} \, dt \leq -(t_s - t_i), \tag{3.4}
\]

where \( s := v(t) \).

We define the function \( F : [0, \infty) \to \mathbb{R} \cup \{-\infty\} \) as

\[
F(q) := \int_{1}^{q} \frac{1}{\varphi(s)} \, ds.
\]

\( F \) is invertible, and \( F^{-1} : \text{im}(F) \to [0, \infty) \) is increasing.

Substituting \( F \) into (3.4), we get for the left limit \( t_s \not\leq t_{i+1} \),

\[
F(v^-_{i+1}) - F(v_i) \leq -(t_{i+1} - t_i). \tag{3.5}
\]

We set \( n = n(t) := \max\{n \in \mathbb{N} \mid t_n \leq t\} \) in such a way that \( t_n \) is the last jump before \( t \). By the fact that \( \varphi \) is positive together with (3.2), the inequality \( F(v_i) - F(v^-_{i}) \leq 0 \) holds for all \( i \in \mathbb{N} \). Combining this with (3.5), we obtain

\[
F(v(t)) - F(v_0) = F(v(t)) - F(v_n) + \sum_{k=1}^{n} F(v_k) - F(v^-_{k-1}) \leq -(t - t_n) - \sum_{k=1}^{n} (t_k - t_{k-1}) = -(t - t_0). \tag{3.6}
\]

This is equivalent to

\[
v(t) \leq F^{-1}(F(v_0) - (t - t_0)). \tag{3.7}
\]

This inequality fails for \( v(t_s) = 0 \) for some \( t_s \in I \), i.e., when the term \( F(v_0) - (t - t_0) \) vanishes due to the restrictions on (3.3). But when \( v(t_s) = 0 \), the inequality (3.7) for any \( t > t_s \) is trivial as \( v(t) = 0 \). However, to circumvent the issue of finding a strictly monotonous \( KL \)-function, we define

\[
\tilde{\beta}(v_0, t - t_0) := F^{-1}\left(F(v_0) - (F(v_0) - m)\left(1 - e^{-\varphi(t - t_0)}/\varphi(t)ight)\right)
\]

\[
\geq F^{-1}(F(v_0) - (t - t_0))
\]

for \( m > -\infty \), where \( m := \inf\{\text{im}(F)\} \). Otherwise, we set

\[
\tilde{\beta}(v_0, t - t_0) := F^{-1}(F(v_0) - (t - t_0)).
\]
Clearly, \( \beta \) is strictly increasing in \( v_0 \) and decreasing strictly to zero in \( t - t_0 \). We define \( \beta \in KL, \ \beta(r, s) := \alpha_1^{-1}\left(\tilde{\beta}(\alpha_2(r), s)\right) \), and obtain for \( t \in [t_0, t^*] \),

\[
\|x(t; t_0, x_0, u)\|_X \leq \beta(\|x_0\|_X, t - t_0),
\]

where \( t^* := \inf\{t \in [t_0, \infty] \mid V(t, x(t)) < \chi(\|u\|_\infty)\} \). Obviously, in case \( t^* = \infty \), \( t \) is to be chosen in \( [t_0, t^*) \).

Step 2: Next, we show that trajectories that are in \( A_1(t) \) for some \( t \in I \), stay bounded for all times. Therefore, we apply (2.7) from which we can conclude that all trajectories jumping from \( A_1^-(t) \) for some \( t = t_i \) are bounded by

\[
A_2(t) := \{ x \in X \mid V(t, x) \leq \alpha_3(\|u\|_\infty) \}.
\]

It is not possible that the trajectories leave

\[
A_3(t) := \overline{A_1(t) \cup A_2(t)}
\]

neither by jump nor by flow because the boundary of \( A_1(t) \cup A_2(t) \) is in the complement of \( A_1 \). From (3.1), it follows that \( \frac{d}{dt}V(t, x(t)) < 0 \) holds on the boundary, and jump inequality (3.2) prevents the trajectories from leaving \( A_3(t) \). We define \( \gamma \in K_\infty \),

\[
\gamma := \alpha_1^{-1} \circ \max\{\alpha_3(\cdot), \chi(\cdot)\}.
\]

Then \( \|x(t; t_0, x_0, u)\|_X \leq \gamma(\|u\|_\infty) \) holds for all \( t > t^* \).

From this equation and (3.8), we can conclude

\[
\|x(t; t_0, x_0, u)\|_X \leq \beta(\|x_0\|_X, t - t_0) + \gamma(\|u\|_\infty).
\]

**4. Converse ISS-Lyapunov Theorem.** In this section, we prove that for every impulsive system (2.1), which is ISS, there exists an ISS-Lyapunov function. Therefore, the problem of determining ISS of a system reduces to finding an ISS-Lyapunov function.

Until now, we were able to work on a maximal level of generality concerning the right-hand side of impulsive system (2.1). In what follows, we need more regularity conditions. We require system (2.1) to satisfy the following Lipschitz continuity assumptions.

**Assumption 1.** Let \( f : I \times X \times U \to X \) be Lipschitz continuous in the second and third variable on bounded sets, i.e., for all \( C, D > 0 \), there exist constants \( L_1^f(C, D), L_2^f(C, D) > 0 \) such that for all \( x, y \in BX(C) \) and all \( u, v \in BU(D) \) the inequalities

(i) \( \|f(t, x, u) - f(t, y, u)\|_X \leq L_1^f(C, D) \|x - y\|_X \),

(ii) \( \|f(t, x, u) - f(t, x, v)\|_X \leq L_2^f(C, D) \|u - v\|_U \),

hold uniformly in time \( t \in I \). Furthermore, let \( f \) be continuous in the first variable for all \( t \in I \setminus S \).

Let \( g_i : X \times U \to X \) for all \( i \in \mathbb{N} \) be Lipschitz continuous in the first and second variable on bounded sets, i.e., for all \( C, D > 0 \), there exist positive constants \( L_1^{g_i}(C, D), L_2^{g_i}(C, D) > 0 \) such that for all \( x, y \in BX(C) \) and all \( u, v \in BU(D) \) the inequalities

(iii) \( \|g_i(x, u) - g_i(y, u)\|_X \leq L_1^{g_i}(C, D) \|x - y\|_X \),

(iv) \( \|g_i(x, u) - g_i(x, v)\|_X \leq L_2^{g_i}(C, D) \|u - v\|_U \),

hold.
Remark 4.1. Here, we used, and will also use in what follows, the notion of Lipschitz continuity on bounded sets. Lipschitz continuity on bounded sets implies local Lipschitz continuity and on compact sets the notions are equivalent. For non-compact sets, Lipschitz continuity on bounded sets is not equivalent to local Lipschitz continuity.

Now we state the converse ISS-Lyapunov theorem.

Theorem 4.2. Let impulsive system (2.1) be ISS and satisfy Assumption 1. Then there exists an ISS-Lyapunov function for system (2.1).

The proof of Theorem 4.2 includes the steps depicted in Fig. 1. The proof will proceed via an intermediate state called weak uniform robust asymptotic stability (WURS). First, we show that from ISS of system (2.1) follows WURS, and afterward, we establish that WURS includes the existence of a Lyapunov function.

\begin{equation}
(2.1) \text{is ISS} \xrightarrow{\text{Theorem 3.1}} \exists \text{ISS-LF for (2.1)} \quad \text{Lemma 4.5} \quad \text{Lemma 4.9} \quad \text{Lemma 4.8} \quad \exists \text{LF for (4.1)}
\end{equation}

\begin{equation}
(2.1) \text{is WURS} \quad \text{Theorem 4.8} \quad \exists \text{LF for (4.1)}
\end{equation}

Fig. 1: Diagram illustrating the proof of Theorem 4.2.

In order to introduce WURS, we define a feedback system related to (2.1). Let there exist a function \( \eta : X \to \mathbb{R}^+_0 \), which is Lipschitz-continuous on bounded subsets of \( X \), and a function \( \psi \in \mathcal{K}_\infty \) such that \( \eta(x) \geq \psi(||x||_X) \) and
\[
\dot{x}(t) = Ax(t) + f(t, x(t), d(t)\eta(x(t)))
\]
\[
= Ax(t) + \overline{f}(t, x(t), d(t)), \quad t \in I \setminus S,
\]
\[
x(t) = g_i(x^-(t), d(t)\eta(x^-(t)))
\]
\[
= g_i(x^-(t), d(t)), \quad t = t_i, \ i \in \mathbb{N},
\]
where \( d \in D := \{d : I \to \mathbb{R} \mid \|d\|_\infty \leq 1\} \). We denote trajectories of (4.1) with initial condition \( x(t_0) = x_0 \) by \( \overline{x}_d(t; t_0, x_0) \) which we abbreviate by \( \overline{x}_d(t) \) if possible.

Next, we provide several definitions.

Definition 4.3. System (4.1) is called
(i) uniformly globally asymptotically stable (UGAS) if there exists a \( \mathcal{K-L} \)-function \( \beta \) such that for all \( d \in D \), the inequality \( ||\overline{x}_d(t; t_0, x_0)||_X \leq \beta(||x_0||_X, t - t_0) \) holds for all \( x_0 \in X \) and all \( t \geq t_0 \);
(ii) uniformly globally attractive if for any \( C, \varepsilon > 0 \) there is a \( \delta = \delta(C, \varepsilon) > 0 \) such that for all \( d \in D \) the inequality \( ||\overline{x}_d(t; t_0, x_0)||_X \leq \varepsilon \) holds for all \( x_0 \in X \) satisfying \( ||x_0||_X \leq C \) and all \( t \geq t_0 + \delta(C, \varepsilon) \);
(iii) uniformly stable if there exists a class \( \mathcal{K}_\infty \)-function \( \alpha \) such that for all \( d \in D \), \( x_0 \in X \) and \( t \geq t_0 \) the inequality \( ||\overline{x}_d(t; t_0, x_0)||_X \leq \alpha(||x_0||_X) \) holds;
(iv) robustly forward complete if for all \( C > 0 \) and \( \tau > t_0 \)
\[
K(C, \tau) := \sup_{t_0 \in B_X(x), d \in D, t \in [t_0, \tau]} ||\overline{x}_d(t; t_0, x_0)||_X
\]
is finite.

**Definition 4.4.** System (2.1) is called weakly uniformly robustly asymptotically stable (WURS), if there exist a function \( \eta : X \rightarrow \mathbb{R}_0^+ \), which is Lipschitz-continuous on bounded subsets of \( X \) and \( \psi \in \mathcal{K}_\infty \) such that \( \eta(x) \geq \psi(\|x\|_X) \), and feedback system (4.1) is UGAS for all \( d \in D \).

Now we establish the first part of the proof of Theorem 4.2.

**Lemma 4.5.** If impulsive system (2.1) is ISS, then it is WURS.

**Proof.** From the definition of ISS, it follows that there exist functions \( \beta \in \mathcal{K} \mathcal{L} \) and \( \gamma \in \mathcal{K}_\infty \). We define \( \alpha(s) := \beta(s,0) \) for all \( s \in \mathbb{R}_0^+ \). From inequality (2.3), it follows that \( \alpha(s) \geq s \) for all \( s \in \mathbb{R}_0^+ \), which means that \( \alpha \in \mathcal{K}_\infty \).

We define \( \sigma \in \mathcal{K}_\infty \) as \( \sigma(s) \leq (\gamma^{-1}(\frac{1}{2}(\frac{3}{4} \alpha^{-1}(\frac{3}{4}s))) \), and choose locally Lipschitz continuous functions \( \eta : X \rightarrow \mathbb{R}_0^+ \) and \( \psi \in \mathcal{K}_\infty \) such that \( \psi(\|x\|_X) \leq \eta(x) \leq \sigma(\|x\|_X) \).

We show that with this definition of \( \eta \), the inequality

\[
\gamma(\|d(t)\|_{\mathcal{U}}) \leq \frac{1}{2} \|x_0\|_X
\]

holds for all \( t \geq t_0 \). We first compute

\[
\gamma(\|d(t)\|_{\mathcal{U}}) \leq \gamma(\|\tau(t)\|_X) \\
\leq \frac{1}{4} \alpha^{-1}(\frac{3}{4} \alpha^{-1}(\frac{3}{4}\|d(t)\|_X)) \leq \frac{1}{4} \|d(t)\|_X
\]

which is valid for every \( d \in D \), all \( x_0 \in X \), and all \( t \in I \). By definition, \( \tau \) is right-continuous, so for each \( t_* \in I \), there exists a \( \delta > 0 \) such that

\[
\gamma(\|d(t)\|_{\mathcal{U}}) \leq \frac{1}{4} \|\tau(t; t_*, x_*)\|_X \leq \frac{1}{4} \|x_*\|_X
\]

for all \( t \in [t_*, t_* + \delta] \), where the last inequality in (4.3) follows from continuity. The next step is to fix randomly chosen \( d \in D \), \( t_0 \in I \), and \( x_0 \in X \), and define \( \overline{t} = \overline{t}(t_0, x_0, d) \) by

\[
\overline{t} = \inf \{ t \geq t_0 \mid \gamma(\|d(t)\|_{\mathcal{U}}) > \frac{1}{2} \|x_0\|_X \}.
\]

Obviously, \( \overline{t} \in (t_0, \infty] \). We prove by contradiction that \( \overline{t} = \infty \). Thus, let us assume that \( \overline{t} \) is not equal to infinity. Then, inequality (4.3) holds for all \( t \in [t_0, \overline{t}] \). By substituting inequality (4.3) into (2.3), we obtain

\[
\|\tau(t)\|_X \leq \beta(\|x_0\|_X, t - t_0) + \frac{1}{2} \|x_0\|_X \\
\leq \beta(\|x_0\|_X, t - t_0) + \frac{1}{2} \alpha(\|x_0\|) \leq \frac{3}{4} \alpha(\|x_0\|).
\]

System (2.1) is bound to the principle of causality, i.e., it only depends on past states. This means that it exhibits the same behavior on the interval \([t_0, \overline{t}]\) for \( u \in U_c \) as for

\[
\hat{u} = \begin{cases} 
  u & t \in [t_0, \overline{t}], \\
  0 & \text{else.} 
\end{cases}
\]

Therefore, we can transform (2.3) into

\[
\|\tau(t; t_*, x_*)\|_X \leq \beta(\|x_*\|_X, \overline{t} - t_*) \\
+ \gamma(\sup_{t \in [t_*, \overline{t}]} \{\|d(t)\|_{\mathcal{U}}\}).
\]

(4.5)
As system (4.1) is only right-continuous, we need an estimate for limits from the left, which also considers possible jumps. This estimate is given by

\[
\begin{align*}
|\mathbf{\Phi}_d(t)|_X &= \lim_{t_s \searrow t} |\mathbf{\Phi}_d(t; t_s, \mathbf{\Phi}_d(t_s))|_X \\
&\leq \lim_{t_s \searrow t} \beta(\|\mathbf{\Phi}_d(t_s)\|_X, t - t_s) \\
&\quad + \gamma\left(\sup_{t \in [t, \tau]} \{d(t)\eta(\|\mathbf{\Phi}_d(t_s)\|_X)\}\right) \\
&\leq \alpha(\|\mathbf{\Phi}_d(t)\|_X) + \frac{1}{2} \|\mathbf{\Phi}_d(t)\|_X \leq \frac{1}{2} \alpha(\|\mathbf{\Phi}_d(t)\|_X),
\end{align*}
\]

which is obtained from the semigroup property \(\mathbf{\Phi}_d(t) = \mathbf{\Phi}_d(t; \tau, \mathbf{\Phi}_d(\tau))\) for all \(t \in [t_0, \tau]\) and (4.5). From (4.6) follows

\[
\gamma\left(\|d(t)\eta(\|\mathbf{\Phi}_d(t)\|_X)\|_U\right) \leq \gamma(\sigma(\frac{1}{2} \alpha(\|\mathbf{\Phi}_d(t)\|_X))) \\
\leq \frac{1}{2} \alpha^{-1}\left(\frac{1}{4} \alpha^{-1}\left(\frac{1}{2} \alpha(\|\mathbf{\Phi}_d(t)\|_X)\right)\right) \\
\leq \frac{1}{2} \alpha^{-1}\left(\frac{1}{2} \|\mathbf{\Phi}_d(t)\|_X\right) \leq \frac{1}{2} \|x_0\|_X,
\]

where we have used the definition of \(\sigma\) in the second, and (4.4) in the fourth inequality.

As \(\mathbf{\Phi}_d\) is right-continuous, there is a neighborhood to the right of \(\tau\) such that \(\gamma\left(\|d(t)\eta(\|\mathbf{\Phi}_d(t)\|_X)\|_U\right) \leq \frac{1}{2} \|x_0\|_X\). This is a contradiction to the definition of \(\tau\) and shows the validity of (4.2) for all \(t \geq t_0\). Therefore,

\[
\|\mathbf{\Phi}_d(t)\|_X \leq \beta(\|x\|_X, t - t_0) + \frac{1}{2} \|x\|_X
\]

holds for all \(x \in X\), all \(d \in D\), and all \(t \geq t_0\), which gives us uniform global stability of system (4.1).

To show uniform global attractivity, we use the fact that there exists such a \(\tau_1 = \tau_1(\|x\|_X)\) that \(\beta(\|x\|_X, t) \leq \frac{1}{2} \|x\|_X\) for all \(t \geq \tau_1\). Substituting this term into (4.7) yields \(\|\mathbf{\Phi}_d(t)\|_X \leq \frac{3}{2} \|x_0\|_X\) for all \(x_0 \in X\), \(d \in D\), and \(t \geq \tau_1\). By using similar arguments, we can find a sequence \((\tau_k)_{k \in \mathbb{N}}\) that only depends on \(\|x_0\|_X\) such that

\[
\|\mathbf{\Phi}_d(t)\|_X \leq \left(\frac{3}{2}\right)^k \|x_0\|_X
\]

holds for all \(x_0 \in X\), \(d \in D\), and \(t \geq \tau_k\). This implies uniform global attractivity of system (4.1). From [9, Thm. 2.2], it follows that system (4.1) is UGAS. Therefore, impulsive system (2.1) is WURS by Definition 4.4.

**Proposition 4.6.** Let Assumption 1 hold, and system (2.1) be WURS. Then, for any \(\eta\) as defined in Definition 4.4, the closed-loop system (4.1) has robustly forward complete solutions, which are Lipschitz continuous with respect to initial values in bounded subsets of \(X\) and locally Lipschitz continuous on intervals \(t \in [t_i, t_{i+1})\), \(i \in N_0\), where the Lipschitz constant depends on the norm of the initial condition \(\|x_0\|_X\).

**Proof.** The solutions of system (2.1) exist for every \(t \geq t_0\). This implies that the solutions of system (4.1) also exist for every \(t \geq t_0\). Let \(\eta\) be the function constructed in Definition 4.4. According to Definition 4.4, system (4.1) is UGAS. Therefore, for all \(C > 0\) and all \(\tau > t_0\)

\[
\sup_{x_0 \in B_X(C), d \in D, t \in [t_0, \tau]} \|\mathbf{\Phi}_d(t; t_0, x_0)\|_X \leq \beta(C, 0)
\]

exists, and system (4.1) is robustly forward complete.

We apply Lemma A.1 from the Appendix to obtain that \(\overline{f}\) and \(\overline{g}_i\) are Lipschitz continuous in space on bounded subsets of \(X\) and uniformly with respect to \(t\) and
Therefore, the preconditions of Lemma A.2 in the Appendix are fulfilled, i.e., system (4.1) has a solution, which is Lipschitz continuous with respect to the initial values in bounded subsets of \(X\). From Lemma A.3 in the Appendix, it follows that the trajectories are locally Lipschitz continuous on intervals \(t \in [t_i, t_{i+1}), i \in \mathbb{N}_0\), as claimed in the Proposition.

We now define a Lyapunov function to establish UGAS of feedback system (4.1). This is not an ISS-Lyapunov function as given in Definition 2.2 but it provides global asymptotic stability, uniformly with respect to control.

**Definition 4.7.** Let \(V : I \times X \to \mathbb{R}^+_0\) be such that \(V \in \mathcal{C}((I \setminus S) \times X, \mathbb{R}^+_0)\) and \(V \in \mathcal{PC}(I \times X, \mathbb{R}^+_0)\) hold. We call \(V\) a Lyapunov function for system (4.1), if it fulfills all of the following conditions:

(i) There exist functions \(\alpha_1, \alpha_2 \in \mathcal{K}_\infty\) such that

\[
\alpha_1(\|x\|_X) \leq V(t, x) \leq \alpha_2(\|x\|_X)
\]

holds true for all \(t \in I\) and all \(x \in X\).

(ii) There exists a \(\varphi \in \mathcal{P}\) such that for all \(u \in U_c\) and all \(x = x(t; t_0, x_0, u)\) of (2.1), the inequalities

\[
\frac{d}{dt} V(t, x) \leq -\varphi(V(t, x)) , \quad t \in I \setminus S, \\
V(t, x(t; t_i, x_i)) \leq V(t_i, x_i), \quad t_i \in S,
\]

holds true. Here, \(\frac{d}{dt} V(t, x)\) is again the Dini-derivative.

With Proposition 4.6 at hand, the following converse Lyapunov theorem [9, Thm. 3.4] delivers the implication from WURS to the existence of a Lyapunov function as given in Definition 4.7 for system (4.1).

**Theorem 4.8.** Let system (1) be WURS, and its solutions be Lipschitz continuous with respect to initial values in bounded subsets of \(X\) and locally Lipschitz continuous on intervals \(t \in [t_i, t_{i+1}), i \in \mathbb{N}_0\), where the Lipschitz constant depends on the norm of the initial condition \(\|x_0\|_X\). Then, there is a Lyapunov function \(V\) for system (4.1) as given in Definition 4.7, which is Lipschitz continuous on bounded balls in space and locally Lipschitz continuous on the intervals \([t_i, t_{i+1}), i \in \mathbb{N}_0\).

From the next lemma, we obtain that the existence of a Lyapunov function for feedback system (4.1) implies the existence of an ISS-Lyapunov function for system (2.1).

**Lemma 4.9.** Let there exist a Lyapunov function \(V\) for system (4.1) as given in Definition 4.7, which is Lipschitz continuous on bounded balls in space and locally Lipschitz continuous on the intervals \([t_i, t_{i+1}), i \in \mathbb{N}_0\). Then, there exists an ISS-Lyapunov function of the form given in Definition 2.2, which is Lipschitz continuous on bounded subsets of \(X\), and locally Lipschitz continuous on \(I \setminus S\) such that it is right continuous and the left limit exists.

**Proof.** The definition of the Lyapunov function \(V\) in Definition 4.7 immediately implies the inequalities (2.5)–(2.6) hold for all \(x \in X\) and \(u \in \mathcal{B}_U(\eta(x))\). As a consequence of Definition 4.4, (2.5)–(2.6) hold for all \(x \in X\) and \(u \in \mathcal{B}_U(\psi(\|x\|_X))\). As \(\psi\) is invertible, we can set \(\chi := \alpha_1 \circ \psi^{-1}\). By this, \(V\) satisfies (2.5)–(2.6) without additional restrictions.

It remains to show that the third property of Definition 2.2 is fulfilled. We claim that \(g_i(0, 0) = 0\) for all \(i \in \mathbb{N}\). Let us assume \(g_i(0, 0) \neq 0\). Then, for the trajectories
of system (4.1) satisfying $\mathbf{g}^-(t_i) = 0$, it follows that

$$
\mathbf{g}(t_i) = \mathbf{g}_i(\mathbf{g}^-(t_i), d(t_i)) = g_i(\mathbf{g}^-(t_i), d(t_i) \eta(\mathbf{g}^-(t_i)))
= g_i(0, 0) \neq 0.
$$

This means that $\mathbf{g} = 0$ is not a fixed point, and system (4.1) cannot be UGAS. Therefore, the claim $g_i(0, 0) \neq 0$ holds true.

Let $V(t_i, x) < \chi(\|u\|_U)$ hold, it follows from (2.4) that

$$
(4.8) \quad \|x\|_X \leq (\alpha_1^{-1} \circ \chi)(\|u\|_U) = \kappa(\|u\|_U),
$$

where we define $\kappa \in \mathcal{K}_\infty$, $\kappa(s) = (\alpha_1^{-1} \circ \chi)(s)$. For $\|x\|_X \leq C$ and $\|u\|_U \leq D$, we have

$$
\begin{align*}
\|g_i(x, u)\|_X & \leq \|g_i(x, u) - g_i(0, u)\|_X + \|g_i(0, u) - g_i(0, 0)\|_X \\
& \leq L^1_{g_i}(C, D) \|x\|_X + L^2_{g_i}(C, D) \|u\|_U \\
& \leq L^1_{g_i}(C, D) \kappa(\|u\|_U) + L^2_{g_i}(C, D) \|u\|_U = \tilde{\alpha}_3(\|u\|_U),
\end{align*}
$$

(4.9)

where the second inequality applies Assumption 1 and the last equality is a consequence of the claim $g_i(0, 0) \neq 0$. Note that $D$ is linear in $\|u\|_U$ and $C$ is linear in $\|x\|_X$. Therefore, by (4.8), $L^1_{g_i}$ and $L^2_{g_i}$ are weakly growing functions in $\|u\|_U$. It can be seen that $\tilde{\alpha}_3(s) := L^1_{g_i}(\kappa^{-1}(s), s) \kappa(s) + L^2_{g_i}(\kappa(s), s) s$ belongs to class $\mathcal{K}_\infty$. By defining the $\mathcal{K}_\infty$-function $\alpha_3 := \alpha_2 \circ \tilde{\alpha}_3$ we conclude

$$
V(t_i, g_i(x, u)) \leq \alpha_2(\|g_i(x, u)\|_X) \leq \alpha_3(\|u\|_U).
$$

We used (2.4) in the first inequality and (4.9) in the second one. So, $V$ is an ISS-Lyapunov function for system (2.1).

We now state the following theorem, which summarizes the main results. Note, that the proof of the theorem also provides a formal proof of Theorem 4.2.

**Theorem 4.10.** Let Assumption 1 hold. Then the following statements are equivalent:

(i) System (2.1) is ISS.

(ii) System (2.1) is WURS.

(iii) There exists an ISS-Lyapunov function for system (2.1), which is Lipschitz continuous on bounded subsets of $X$ and locally Lipschitz continuous on $I \setminus S$ such that it is right continuous, and the left limit exists.

**Proof.** By Theorem 3.1, Statement (iii) implies Statement (i). Lemma 4.5 implies that for every system that fulfills Statement (i), Statement (ii) also holds true.

It remains to show the implication from Statement (ii) to Statement (iii): We assume system (2.1) to be WURS. Then feedback system (4.1) is UGAS for all $d \in D$ with $\eta$ and $\psi$ as given in Definition 4.4. Proposition 4.6 then gives us that every solution of system (4.1) is Lipschitz continuous with respect to initial values in bounded subsets of $X$, and locally Lipschitz continuous on the intervals $[t_i, t_{i+1})$, $i \in \mathbb{N}_0$. Theorem 4.8 ensures that there exists a Lyapunov function $V : I \times X \to \mathbb{R}_0^+$ for feedback system (4.1), which is Lipschitz continuous on bounded balls in space and locally Lipschitz continuous on the intervals $[t_i, t_{i+1})$, $i \in \mathbb{N}_0$. From Lemma 4.9 follows the existence of an ISS-Lyapunov function for system (2.1) as stated in the theorem. □
5. Conclusion. We proposed a time-varying formulation of ISS-Lyapunov function in implication form for impulsive systems over Banach space and prove that the existence of such an ISS-Lyapunov function is a necessary and sufficient condition of ISS for these systems. Our result applies to a broad class of impulsive systems including simultaneous instability in the continuous and discrete dynamics. A possible extension is considering non-coercive Lyapunov functions, proposed in [8], to establish ISS of impulsive systems over Banach spaces with boundary inputs.

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Appendix A. Technical Lemmas.

Lemma A.1. Let Assumption 1 hold. Then $\mathcal{F}$ is Lipschitz continuous in the second variable on bounded subsets of $X$ uniformly with respect to the third argument and for all $t \in I$, i.e., for all $C > 0$, there exists a constant $L_{\mathcal{F}}(C) > 0$ such that for all $x,y \in \mathcal{B}_X(C)$ and all $d \in D$, the inequality

$$|\mathcal{F}(t,x,d) - \mathcal{F}(t,y,d)|_X \leq L_{\mathcal{F}}(C) \|x - y\|_X$$

is satisfied. Moreover, if Assumption 1 holds, $\overline{\mathcal{F}}_i$ is Lipschitz in the first variable on bounded subsets of $X$ for every $i \in \mathbb{N}$ uniformly with respect to the second argument, i.e., for all $C > 0$, there exists an $L_{\overline{\mathcal{F}}_i}(C) > 0$ such that for all $x,y \in \mathcal{B}_X(C)$ and all $d \in D$, follows

$$|\overline{\mathcal{F}}_i(x,d) - \overline{\mathcal{F}}_i(y,d)|_X \leq L_{\overline{\mathcal{F}}_i}(C) \|x - y\|_X.$$

Proof. Let us choose $C > 0$ arbitrarily, and fix $x,y \in \mathcal{B}_X(C)$ and $d \in D$. By definition, $\eta$ is locally Lipschitz, i.e., there exists a bound $R = R(C) > 0$ such that $\eta(x) \leq R$ for all $\|x\|_X \leq C$. Then follows

$$\|\mathcal{F}(t,x,d(t)) - \mathcal{F}(t,y,d(t))\|_X$$

$$= \|f(t,x,d(t)\eta(x)) - f(t,y,d(t)\eta(y))\|_X$$

$$\leq \|f(t,x,d(t)\eta(x)) - f(t,y,d(t)\eta(x))\|_X$$

$$+ \|f(t,y,d(t)\eta(x)) - f(t,y,d(t)\eta(y))\|_X$$

$$\leq L_{f}^1(C,R)\|x - y\|_X + L_{\eta}^2(C,R) |\eta(x) - \eta(y)|$$

$$\leq L_{\mathcal{F}}(C) \|x - y\|_X,$$

where we have used Assumptions 1 (i) and (ii) in the third step. In the fourth step we have used the Lipschitz continuity of $\eta$. The Lipschitz constant $L_{\mathcal{F}}$ we define in the last step depends on $C$, $R$, and the Lipschitz constant of $\eta$. Note that $R$ and the Lipschitz constant of $\eta$ also depend on $C$. Therefore, $L_{\mathcal{F}}$ can be expressed in terms of $C$ only.

The proof for $\overline{\mathcal{F}}_i$ follows analogously. 

Lemma A.2. Assume that system (4.1) is robustly forward complete. Let $\mathcal{F}$ be Lipschitz continuous in the second variable on bounded subsets of $X$ with constant $L_{\mathcal{F}}$ uniformly with respect to the third argument and for all $t \in I$. Let for every $i \in \mathbb{N}$, the function $\overline{\mathcal{F}}_i$ be Lipschitz continuous in the first variable on bounded subsets of $X$ uniformly with respect to the second argument. Then, on compact time intervals, the solutions of (4.1) are Lipschitz continuous with respect to the initial values in bounded subsets of $X$. 
Proof. Let $C > 0$, $\tau > t_0$, $d \in D$ and $x_0, y_0 \in B_X(C)$. By Hille-Yosida theorem, there exist constants $M > 0$ and $\lambda \in \mathbb{R}$ such that $\|T(t - t_0)\|_{\mathcal{L}(X)} \leq Me^{\lambda(t-t_0)}$ for all $t \geq t_0$, where $\|\cdot\|_{\mathcal{L}(X)}$ is the norm for linear operators over $X$.

Let $t \in [t_0, t_1)$ such that $t \leq \tau$. The definition of the mild solution (2.2) gives us that

$$\mathfrak{p}(t) = T(t - t_0)x_0 + \int_{t_0}^{t} T(t - s)\mathfrak{f}(s, \mathfrak{p}(s), d(s)) \, ds$$

holds. Let $x_i^- := \mathfrak{f}(t_i^-)$, $x_i := x(t_i) = \mathfrak{g}_i(x_i^-)$ such that

$$\mathfrak{p}(t) = T(t - t_i)x_i + \int_{t_i}^{t} T(t - s)\mathfrak{f}(s, \mathfrak{p}(s), d(s)) \, ds$$

holds true for all $t \in [t_i, t_{i+1})$, $t \leq \tau$. From this follows that for any two solutions $\mathfrak{p}$ and $\mathfrak{g}$ of (4.1) with initial values $x_0$ and $y_0$, respectively, we have

$$\|\mathfrak{p}(t) - \mathfrak{g}(t)\|_X \leq Me^{\lambda(t-t_i)} \|x_i - y_i\|_X + \int_{t_i}^{t} Me^{\lambda(s-t)}L_\mathfrak{p}(K(C, \tau)) \|\mathfrak{p}(s) - \mathfrak{g}(s)\|_X \, ds,$$

where $t \in [t_i, t_{i+1})$, $t \leq \tau$, $y_i^- := \mathfrak{g}(t_i^-)$, $y_i := \mathfrak{g}(t_i) = \mathfrak{g}_i(y_i^-)$, and $L_\mathfrak{p}$ is the constant from Lemma A.1 in the Appendix and $K = K(C, \tau)$ as introduced in Definition 4.3 (iv). Using Gronwall’s inequality, this means

$$\|\mathfrak{p}(t) - \mathfrak{g}(t)\|_X \leq M \|x_i - y_i\|_X e^{(ML_\mathfrak{p}K(C,\tau)+\lambda)(t-t_i)}$$

for all $t \in [t_i, t_{i+1})$, $t \leq \tau$. The value for $t_{i+1}$ is bounded by

$$\|\mathfrak{p}(t_{i+1}) - \mathfrak{g}(t_{i+1})\|_X \leq L_\mathfrak{p}_{i+1}(K(C, \tau)) \|\mathfrak{p}^-(t_{i+1}) - \mathfrak{g}^-(t_{i+1})\|_X \leq L_\mathfrak{p}_{i+1}(K(C, \tau))M \|x_i - y_i\|_X e^{(ML_\mathfrak{p}K(C,\tau)+\lambda)(t_{i+1}-t_i)},$$

where $L_\mathfrak{p}_{i+1}$ is defined as in Lemma A.1 in the Appendix. By induction, we obtain

$$\|\mathfrak{p}(t) - \mathfrak{g}(t)\|_X \leq M \prod_{j=1}^{n} \left( L_\mathfrak{p}_{j}(K(C, \tau)) M \right) \|x_0 - y_0\|_X \times e^{(ML_\mathfrak{p}K(C,\tau)+\lambda)(t-t_0)}$$

for $t \in [t_n, t_{n+1})$, $t \leq \tau$ and $n \in \mathbb{N}_0$, where we define the product $\prod_{j=1}^{0} a_j := 1$. From this, we can conclude by extreme value theorem that

$$\|\mathfrak{p}(t) - \mathfrak{g}(t)\|_X \leq \|x_0 - y_0\|_X \max_{t \in [t_0, \tau]} \left\{ M \prod_{j=1}^{n} \left( L_\mathfrak{p}_{j}(K(C, \tau)) M \right) \times e^{(ML_\mathfrak{p}K(C,\tau)+\lambda)(t-t_0)} \mid n \in \mathbb{N}_0 : t \in [t_n, t_{n+1}) \right\}.$$ 

This gives us a Lipschitz constant for the solutions of (4.1) with respect to the initial conditions.
Lemma A.3. Let system (4.1) be robustly forward complete. Let $f$ be Lipschitz continuous in the second variable on bounded subsets of $X$ uniformly with respect to the third argument and for all $t \in I$. Then its solutions are locally Lipschitz continuous on intervals $t \in [t_i, t_{i+1})$, $i \in \mathbb{N}_0$, where the Lipschitz constant depends on $\|x_0\|_X$.

The proof follows from [11, Lemma 4.6].

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