Distorted Black Hole Initial Data Using the Puncture Method

J. David Brown and Lisa L. Lowe
Department of Physics, North Carolina State University, Raleigh, NC 27695 USA

We solve for single distorted black hole initial data using the puncture method, where the Hamiltonian constraint is written as an elliptic equation in $\mathbb{R}^3$ for the nonsingular part of the metric conformal factor. With this approach we can generate isometric and non-isometric black hole data. For the isometric case, our data are directly comparable to those obtained by Bernstein et al., who impose isometry boundary conditions at the black hole throat. Our numerical simulations are performed using a parallel multigrid elliptic equation solver with adaptive mesh refinement. Mesh refinement allows us to use high resolution around the black hole while keeping the grid boundaries far away in the asymptotic region.

I. INTRODUCTION

Distorted black holes are expected to be produced by astrophysical events such as asymmetrical gravitational collapse, black hole–black hole coalescence, black hole–neutron star coalescence, and possibly neutron star–neutron star coalescence. As ground based gravitational wave detectors such as LIGO, VIRGO, TAMA, and GEO increase their sensitivities, and the planned launch date for the space based detector LISA grows near, the need for researchers to develop a theoretical understanding of these systems becomes increasingly important. In support of that effort, we need to develop the techniques and tools to evolve a dynamic, distorted black hole and extract the emitted gravitational radiation as it settles to a quiescent state.

One of the difficulties encountered in the numerical treatment of single and multiple black hole systems is the presence of nontrivial topologies. In the past, researchers have addressed this problem by using isometry conditions at inner boundaries to represent black hole throats. Another approach to black hole evolution is excision, in which a region inside each apparent horizon is removed from the computational domain. Recently it has been shown that black holes can be treated in terms of fields on $\mathbb{R}^3$ by splitting the conformal factor for the spatial metric into singular and non-singular terms. This so-called “puncture method” was applied to the Bowen–York family of black hole initial data sets by Brandt and Brügmann, and used for evolution studies by Brügmann and others (see, for example, Refs. [4, 5]).

In this paper we make a modest extension of the puncture construction for initial data to include single distorted black holes. We reproduce and extend the results of Bernstein et al. [6, 7, 8], who constructed “black hole plus Brill wave” initial data sets using isometry conditions at the black hole throat. Whereas the black holes obtained by Bernstein et al. are isometric by construction, our distorted puncture black hole data sets are isometric only when the free parameters $\mu$ and $m$, defined in Sec. III, are equal to one another. For $\mu \neq m$, we obtain non–isometric, distorted black holes.

Another difficulty encountered in the numerical modeling of black holes is the wide discrepancy in length scales involved. The computational grid needs to be large enough to capture outgoing gravitational waves and to minimize boundary effects. The grid must also have high resolution in the interior to accurately resolve the strong gravitational fields of black holes. For a finite difference code, adaptive mesh refinement (AMR) is needed to satisfy both of these requirements: high resolution and a large grid. In this paper we introduce our AMR elliptic solver that allows us to solve accurately for puncture black hole data on very large grids. Evolution studies of these data sets are underway.

In Sec. II we set up the equations defining the initial value problem for a distorted puncture black hole. We show how the puncture data can be formulated to give isometric data sets, thus enabling a comparison with the results of Bernstein et al. In Sec. III we give a brief description of our AMR elliptic solver. In Sec. IV we present sample results both for isometric and non–isometric black hole data.

II. FORMULATION OF THE PROBLEM

Following Bernstein et al. [6, 7, 8] we write the line element for the physical metric $g_{ij}$ as

$$ds^2 = \psi^4[e^{2q}((dr^2 + r^2 d\theta^2) + r^2 \sin^2 \theta d\phi^2)] ,$$  

where $q$ is a specified function of the spatial coordinates and $\psi$ is the conformal factor. In this paper we restrict ourselves to the expression for $q$ used in Ref. [8], namely,

$$q(r, \theta, \phi) = 2Q_0 e^{-\eta^2} \sin^n \theta (1 + c \cos^2 \phi) ,$$  

where $\eta = \ln(2r/\mu)$. In Eq. [2], $Q_0$, $n$, $c$ and $\mu$ are adjustable constants that affect the size and type of distortion. We also restrict ourselves to data sets defined at a moment of time symmetry. Thus, the extrinsic curvature vanishes and the momentum constraints are trivially satisfied. The Hamiltonian constraint reduces to

$$\nabla^2 \psi - \frac{1}{8} \hat{R} \psi = 0 ,$$

where $\nabla^2$ and $\hat{R}$ are the Laplacian and scalar curvature of the conformal metric, defined by $\hat{g}_{ij} = \psi^{-4} g_{ij}$. 
Solutions of Eq. (3) on the manifold $\mathbb{R}^3$ are Brill waves. The distorted black hole data sets of Bernstein et al. were obtained by solving Eq. (3) on the manifold $\mathbb{R} \times S^2$, with Robin conditions at the outer boundary $[\partial(r\psi - r)/\partial r = 0$ as $r \to \infty$ and isometry conditions at the inner boundary $[\partial(\sqrt{\psi}/\partial r = 0$ at $r = \mu/2$]. We will obtain solutions of Eq. (3) on $\mathbb{R} \times S^2$ by following the puncture construction of Ref. 2. Thus, we write the conformal factor as $\psi = u + m/(2r)$ and insert this into the Hamiltonian constraint of Ref. [2].

We have had no difficulty in obtaining numerical solutions to Eq. (4). However, from an analytical point of view, it is not immediately obvious to us whether or not Eq. (4) always admits solutions, and if so whether those solutions are unique. The problem of existence and uniqueness of solutions of Eq. (4) on $\mathbb{R}^3$ is similar to the problem of existence and uniqueness of solutions of Eq. (3) on $\mathbb{R}^3$. There are two notable differences. First is the appearance of a “source” term $m\bar{R}/(16\pi)$ on the right-hand side of Eq. (4). Note that with the choice for the function $q$, the scalar curvature $\bar{R}$ goes to zero rapidly, $\bar{R} \sim (\ln(r))^2r\ln(m/\mu)r^{-2}-2\ln(2)$ at $r \to 0$, so that $\bar{R}/r$ does not blow up at the puncture. The second difference between equation (4) and the familiar initial value method as well, simply by setting the parameters $\mu$ and $m$ equal to one another: $\mu = m$. To see this, we first note that the Hamiltonian constraint on $\mathbb{R} \times S^2$ can be written as (see, for example, Appendix D of Ref. 11)

$$\hat{\nabla}^2 u - \frac{1}{8} \hat{R}u = \frac{m}{16\pi} \hat{R}. \quad (4)$$

This equation is solved for a continuous solution $u$ on $\mathbb{R}^3$ with Robin boundary conditions $\partial(ru - r)/\partial r = 0$ at $r \to \infty$. Note that the “bare mass” $m$ appears as a new parameter in the construction.1

We consider only the case $q(r) \to r^4$. Thus, we write the conformal factor as $\psi = u + m/(2r)$ and insert this into the Hamiltonian constraint of Ref. [2].

1 The parameters $\mu$ and $m$ are dimensionful. Thus, our data sets can be described in terms of the dimensionless parameters $c$, $Q_0$, $n$, the dimensionless ratio $\mu/m$, and the dimensionless coordinates $x/m, y/m,$ and $z/m$.

...
III. NUMERICAL CODE

Our numerical code solves Eq. 4 using multigrid techniques with mesh refinement. We use the Paramesh package [12, 13] to implement mesh refinement and parallelization. Paramesh divides the computational grid into blocks, with each block containing $N$ zones. A block of data is refined by bisection—that is, the block is divided into eight “child” blocks (in three spatial dimensions) each containing $N$ zones.

Our code carries out multigrid V–cycles using the Full Approximation Storage (FAS) algorithm [14] on non–uniform grid structures, with zone–centered data. It works with both fixed mesh refinement (FMR) and adaptive mesh refinement (AMR). Working with FMR, we specify a non–uniform grid by hand. Working with AMR, we generally start with a coarse, uniform grid. The code V–cycles until the norm of the residual is less than the tolerance is flagged for refinement. Paramesh then re-builds the grid, the data is prolonged from the old grid to the new, and the V–cycle process begins again. Working in this mode takes the place of the Full Multigrid (nested iteration) Algorithm [14], where the solution on the final grid structure is reached by a succession of V–cycles of increasing peak resolution.

We will provide details of the AMR–multigrid algorithm in a later publication, along with numerous code tests [15].

IV. RESULTS

Bernstein et al. [6, 7, 8] produced nonaxisymmetric distorted black holes using isotropy conditions at the black hole throat. We have reproduced several of the initial data sets from Ref. [8] using the puncture method, by setting $\mu = m$ as described in Sec. II. As a check for isotropy, we have confirmed that the ADM mass at the puncture, $M_0$, agrees with the ADM mass at infinity, $M_\infty$, to several significant digits. 2

Two tests were performed to confirm the consistency of our ADM mass calculations. For the first test we used a sequence of grids with fixed interior resolution but increasing outer boundary limits. We started by solving for the initial data on a grid with boundaries (in each dimension) at $-13$ and $13$, and having two additional box–in–box refinement levels ranging (in each dimension) from $-6.5$ to $6.5$ and from $-3.25$ to $3.25$. The resolution on the finest of the three levels was $\Delta x = 0.05078$. Other grids in the sequence were created by adding, one at a time, another fixed box–in–box refinement level while doubling the size of the computational domain. This test was intended primarily as a check on the sensitivity of the ADM mass at infinity to the location of the outer boundary. We found that the ADM mass $M_\infty$ is unaffected to six digits and the ADM mass $M_0$ is unaffected to seven digits. For the second test we used a sequence of grids consisting of a fixed three–level box–in–box structure but increasing resolution throughout the computational domain. This test was intended primarily to check the convergence properties of the ADM masses $M_\infty$ and $M_0$. The test showed that the ADM masses converge with second order accuracy.

As a specific example let us consider the data set discussed in Ref. [8] with $c = -2$, $Q_0 = -0.5$, $n = 4$, and $\mu = m = 2$. The ADM masses for this data at infinity and at the puncture are $M_\infty = 2.2021$ and $M_0 = 2.2027$. Figure 1 shows a contour plot of the nonsingular part $u$ of the conformal factor in the $z = 0$ plane. The contour levels crossing the $x$ axis between $-100$ and zero are, respectively, $\{1.0015, 1.005, 1.005, 1.0015, 0.96\}$. The contour levels crossing the $y$ axis between $-100$ and zero are, respectively, $\{1.0015, 1.005, 1.01, 1.02, 1.1\}$. The inset in Fig. 1 shows the range in $x$ and $y$ from $-13$ to $13$. In the inset figure, the contours crossing the $x$ axis between $-13$ to zero are $\{1.005, 1.0015, 0.96\}$ and the contours crossing the $y$ axis between $-13$ and zero are $\{1.01, 1.02, 1.1\}$. The function $u$ forms two peaks along the $y$ axis with maximum values 2.03, and two valleys along the $x$ axis with minimum values 0.296. The value of $u$ at the puncture is 1.10. The profiles of $u$ along the three axes look similar to those shown in Fig. 2.

With AMR, we are able to push the boundaries out quite far while maintaining high resolution around the
puncture. For the data described above, the computational grid extends from $-100$ to $100$ and has refined itself to reach a maximum relative truncation error of 0.008. The lowest resolution region is equivalent to a 64$^3$ grid with $\Delta x = 3.125$. The highest resolution region is equivalent to a 262,144$^3$ grid with $\Delta x = 0.000763$. The puncture method allows us to define a new class of initial data: distorted black holes that are not isometric ($\mu \neq m$). Figure 2 shows such a data set with $c = -2$, $Q_0 = -0.5$, $n = 4$, $\mu = 2$, and $m = 10$. The results were computed on a grid extending from $-100$ to $100$ in each dimension. The grid spacing for the lowest resolution region, adjacent to the grid boundaries, is $\Delta x = 3.125$. The grid spacing for the highest resolution region, surrounding the puncture, is $\Delta x = 0.01221$. The ADM mass at infinity is $M_\infty = 10.7$, while the ADM mass at the puncture is $M_0 = 12.8$. Figure 4 shows the behavior of the nonsingular part $u$ of the conformal factor along the coordinate axes. The thin solid line plots $u$ along the $y$ axis, the thick solid line plots $u$ along the $x$ axis, and the dotted line plots $u$ along the $z$ axis. The two peaks in $u$ along the $y$ axis have maximum values 4.26, and the two valleys along the $x$–axis have minimum values $\sim -1.38$. The value of $u$ at the puncture is 1.28.

Note that for the non–isometric data described above, $u$ takes on negative values in two small regions of radius $\sim 1$ at locations $\sim \pm 1$ along the $x$ axis. However, for this data set, the conformal factor $\psi = u + m/(2r)$ is everywhere positive. We have explored other data sets that contain regions with $u < 0$, and in each case we find that the conformal factor $\psi$ is positive everywhere. For data with $c = -2$, $Q_0 = -0.5$, and $n = 4$, the most extreme data sets we have studied have ratios $\mu/m = 1/300$ and $\mu/m = 100$. For the case $\mu/m = 1/300$, the function $u$ reaches a minimum value of $\sim -120$ but the conformal factor remains positive. For the case $\mu/m = 100$ the function $u$, and therefore also $\psi$, is always positive.

Acknowledgments

We would like to thank the numerical relativity group at NASA Goddard Space Flight Center, and especially Dae-Il Choi, for their help and support. We would also like to thank James Isenberg, Kevin Olson, and Peter MacNeice for helpful discussions. This work was supported by NASA Space Sciences Grant ATP02-0043-0056. Computations were carried out on the North Carolina State University IBM Blade Center Linux Cluster. The PARAMESH software used in this work was developed at NASA Goddard Space Flight Center under the HPCC and ESTO/CT projects.

[1] J. M. Bowen and J. W. York, Jr., Phys. Rev. D21, 2047 (1980).
[2] S. Brandt and B. Brugmann, Phys. Rev. Lett. 78, 3606 (1997), gr-qc/9703066.
[3] B. Brugmann, Int. J. Mod. Phys. D8, 85 (1999), gr-qc/9708035.
[4] M. Alcubierre et al., Phys. Rev. D67, 084023 (2003), gr-qc/0206072.
[5] B. Imbiriba, J. Baker, D.-I. Choi, J. Centrella, D. R. Fiske, J. D. Brown, J. R. van Meter, and K. Olson, (gr-qc/0403048) (2004).
[6] D. Bernstein, D. Hobill, E. Seidel, and L. Smarr, Phys. Rev. D50, 3760 (1994).
[7] S. R. Brandt and E. Seidel, Phys. Rev. D54, 1403 (1996), gr-qc/9601010.
[8] S. Brandt, K. Camarda, E. Seidel, and R. Takahashi, Class. Quant. Grav. 20, 1 (2003), gr-qc/0206070.
[9] Work in progress.
[10] R. M. Wald, General Relativity (University of Chicago Press, Chicago, 1984).
[11] N. O’Murchadha and J. W. York, Phys. Rev. 10, 2345 (1974).
[12] P. MacNeice, K. M. Olson, C. Mobarry, R. deFainchtein, and C. Packer, Computer Physics Comm. 126, 330 (2000).
[13] http://ct.gsfc.nasa.gov/paramesh/Users_manual/amr.html.
[14] W. H. Press, S. A. Teukolsky, W. T. Vetterling, and B. P. Flannery, Numerical Recipes (Cambridge University Press, Cambridge, 1992).
[15] J. D. Brown and L. L. Lowe (2004), gr-qc/0411112.