Facets of a mixed-integer bilinear covering set with bounds on variables

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Abstract

We derive a closed form description of the convex hull of mixed-integer bilinear covering set with bounds on the integer variables. This convex hull description is completely determined by considering some orthogonal disjunctive sets defined in a certain way. Our description does not introduce any new variables. We also derive a linear time separation algorithm for finding the facet defining inequalities of this convex hull. We show the effectiveness of the new inequalities using some examples.

Keywords: Bilinear programming, mixed-integer programming, convex hull, separation.

1 Introduction

Consider the following mixed-integer bilinear covering set with bounds on the integer variables.

\[
S_U = \left\{ (x, y) \in \mathbb{Z}_+^n \times \mathbb{R}_+^n : \sum_{i=1}^n x_i y_i \geq r, x \leq u \right\}, \text{ where } r > 0 \text{ and } u \in \mathbb{N}^n \text{ are given.}
\]

\(S_U\) is a nonconvex set and even its continuous relaxation is nonconvex for \(n \geq 2\). These constraints appear in the nonlinear formulation of the trim-loss problem \([3, 7]\). In a trim-loss problem, we want to determine the best way to cut large rolls of raw materials into smaller pieces (or finals) using different patterns, so that the demand of finals is met. Let \(N = \{1, \ldots, n\}\) be the index set that denotes the cutting patterns used, and \(F\) be the index set of different sizes of the finals that are to be cut. Let \(L\) be the size of each large roll and \(l_j, j \in F\) be the lengths of the finals. The demands of the finals, say \(d_j, j \in F\) are known. Let \(x_{ij}\) be the number of final \(j\) cut according in the pattern \(i, i \in N, j \in F\), and \(y_i\) be the number of rolls cut with cutting pattern \(i, i \in N\). Therefore, we have the following constraints.

\[
\sum_{i \in N} x_{ij} y_i \geq d_j, \quad j \in F, \quad (1)
\]

\[
\sum_{j \in F} l_j x_{ij} \leq L, \quad i \in N. \quad (2)
\]

Here, all the variables \(x_{ij}\) and \(y_i, i \in N, j \in F\) are non-negative integers. When the demands of the finals are high, we can consider the variables \(y_i, i \in N\) as continuous variables without...
significantly affecting the optimal value. When integrality of \( y \) can not be ignored then \( S^U \) is a relaxation. Bounds on the variables \( x_{ij}, i \in N, j \in F \) can be either given explicitly or be implicit from the knapsack constraints (2).

Let us consider a related set \( S = \{(x, y) \in \mathbb{Z}^n_+ \times \mathbb{R}^n_+ : \sum_{i=1}^n x_i y_i \geq r \} \), \( r > 0 \), i.e., the set \( S^U \) without the upper bounds on the variable \( x \). Tawarmalani et al. [15] developed a scheme to get a tighter convex relaxation using orthogonal disjunctive subsets for a class of sets including \( S \). They applied the scheme to obtain the convex hull description of \( S \) (denoted as \( conv(S) \)) using facet defining inequalities. The description consists of countably infinite number of facet defining inequalities. But these facet defining inequalities of \( conv(S) \) are not sufficient to describe \( conv(S^U) \). This is shown by the following example:

\[
\begin{align*}
\min \quad & -x_1 + 10y_1 - 2x_2 + 12y_2 \\
\text{s.t.} \quad & x_1 y_1 + x_2 y_2 \geq 20, \\
& x_1 \leq 5, x_2 \leq 6, \\
& x_i \geq 0, y_i \geq 0, x_i \in \mathbb{Z}_+, i = 1, 2.
\end{align*}
\]

Here, \( r = 20, n = 2, u = (5, 6) \). The point \((x_1, y_1, x_2, y_2) = (5, 4, 6, 0)\) is a global optimal solution with optimal value 23. But, if we solve the relaxation defined by the facet defining inequalities of \( conv(S) \) (which we describe later), along with the bound constraints on \( x \), we get the solution \( \omega = (5, 1, 6, \frac{5}{6}) \). As expected, \( \omega \) is not feasible for \( S \), and the objective value at this point is 3. The point \( \omega \in conv(S) \) because this point is the mid point of the two points \((10, 2, 0, 0), (0, 0, 12, \frac{2}{3}) \) \( \in S \). Therefore, no facet defining inequality of \( conv(S) \) can cut off the point \( \omega \) from \( conv(S^U) \). It will be shown later that the inequality \( \frac{5y_1}{20} + \frac{6y_2}{20} \geq 1 \) is a valid inequality for \( S^U \) which cuts off the point \( w \). In fact we show that this inequality is a facet defining inequality for \( conv(S^U) \).

Optimizing a linear function over \( S^U \) is a special case of nonconvex (global) optimization problems. These are known to be generally NP-Hard [14,22]. In almost all the algorithms for global optimization, we take a convex relaxation of the feasible region and solve it over successively refined partitions of the domain of the variables [1,9,17,18]. A tighter relaxation enables us to obtain tighter lower bounds on the problem and possibly converge faster in a branch and bound framework.

There are different ways to get a convex relaxation depending on the function in a constraint. A bilinear function is a particular case of quadratic functions, for which there are several ways to get convex relaxations. McCormick relaxation [4], Reformulation Linearization Technique (RLT) [5,6], Semidefinite relaxation [10,11,25], Lagrangian relaxation [24] etc. are mostly used relaxation strategies of bilinear functions. Among these, McCormick and RLT give linear relaxations. However, these relaxations are generally weak in dimensions more than two.

The above mentioned relaxation strategies were devloped for continuous variables. These strategies can still be applied to get a convex relaxation when some of the variables have integral restriction, but the relaxation is generally even weaker. Furthermore, they introduce new variables which naturally takes the problem to a higher dimensional space. In order to obtain better bounds, one needs to exploit problem specific structures, like we do here. For example, if \( f \) is a general quadratic function, and we consider the inequality \( f(x) \geq r, r \in \mathbb{R}, \) then almost all the relaxation strategies do not consider \( r \). Tawarmalani et al. [15] gave an example where considering the right hand side parameter in a suitable way gives us a tight convex relaxation.

In this article, we derive the closed form description of the convex hull of the mixed-integer bilinear covering set \( S^U \). We note that, the orthogonal disjunctive technique of Tawarmalani et al. [15] is not directly applicable for the set \( S^U \) to find \( conv(S^U) \). So, we relax the orthogonal
subsets of \( S^U \) in such a way that the result is applicable. Our work mainly addresses the following issues of the model of Tawarmalani et al. Their model has infinitely many facet defining inequalities and these inequalities along with the bound constraints gives us a weak relaxation of our set. We show that \( \text{conv} \ (S^U) \) is a polyhedron. We derive both V-Polyhedron and H-Polyhedron description of \( \text{conv} \ (S^U) \). We provide separation algorithms to find a violated facet defining inequality for both the sets \( \text{conv} \ (S^U) \) and \( \text{conv} (S) \). Our separation algorithm runs in linear time in the input size for both the cases. Lastly, we provide some computational results that show the effectiveness of our cuts.

Unless otherwise mentioned, we use the following notations throughout this article. For a given set \( A \), we use \( \text{cl}(A) \) to denote the closure of \( A \), \( \text{conv}(A) \) to denote the convex hull of \( A \) and \( 0^+(A) \) to denote the recession cone of \( A \). \( \mathbb{R}^n_+ = [0, \infty)^n = \{ x \in \mathbb{R}^n : x \geq 0 \} \). We use \( N \) to denote the set \( \{1, 2, \ldots, n\} \). For a point \( (x, y) \in \mathbb{R}^n_+ \times \mathbb{R}^m_+ \), we write \( (x, y) \) in the form \((x_1, y_1, x_2, y_2, \ldots, x_n, y_n)\). We use \( \mathcal{L}(i, x_i, y_i) \) to denote the point \((0, 0, \ldots, x_i, y_i, \ldots, 0)\), i.e., \( x_j = 0, y_j = 0, \forall j \in N, j \neq i \).

## 2 Convexification via Orthogonal Disjunction

We start by a general result derived by Tawarmalani et al. [15] for which some more notation is required. We will use the same notation as in [15] for convenience. Let \( A \) be a given set, \( \mathbb{R}^n \) be a function.

### 2.1 The Convexification Theorem

\[ A^J = \{(z,u) : t^J(z,u) \geq 1, \forall j \in J, \forall k \in K, w^J(z,u) \geq 0, \forall l \in L \}, \]

\[ C^J = \{(z,u) : t^J(z,u) \geq 0, \forall j \in J, v^J(z,u) \geq 0, \forall k \in K, w^J(z,u) \geq 0, \forall l \in L \}. \]

To describe the results, we need to additionally define positively-homogeneous functions.

**Definition 1** (Positively Homogeneous function). Let \( f : \mathbb{R}^n \to [-\infty, \infty] \) be a function. \( f \) is said to be a positively homogeneous function if \( f(\lambda x) = \lambda f(x), \forall \lambda > 0 \).

For example, any linear function is positively homogeneous, \( f(x, y) = \sqrt{xy} \) is also positively homogeneous.

### 2.1.1 The Convexification Theorem

**Theorem 1** ([15]). Let \( z = (z_1, \ldots, z_i, \ldots, z_n) \in \mathbb{R}^{\sum_{i=1}^n d_i} \), where \( z_i \in \mathbb{R}^{d_i} \) and \( Z \subseteq \mathbb{R}^{\sum_{i=1}^n d_i} \). Let \( Z_i \subseteq Z \) for \( i \in N = \{1, \ldots, n\} \). Now let us consider the following assumptions:

**A1:** \( (z_1, \ldots, z_i, \ldots, z_n) \in Z_i \Rightarrow z_j = 0 \) \( \forall j \in N, j \neq i \),

**A2:** \( \text{conv}(Z) = \text{conv}(\bigcup_{i=1}^n Z_i) \),

**A3:** \( \text{conv}(Z_i) \subseteq \text{proj}_z(A_i) \subseteq \text{cl}(\text{conv}(Z_i)) \), where,

\[ A_i = \{ \mathcal{L}(i, z_i, u_i) : (z_i, u_i) \in A \left( t_i^J, v_i^K, w_i^L \right) \} \]

such that \( t_i^J, v_i^K, w_i^L \) are positively-homogeneous functions for all \( i \in N \) and \( \mathcal{L}(i, z_i, u_i) = (0, \ldots, 0, z_i, u_i, 0, \ldots, 0) \in \mathbb{R}^{\sum_{i=1}^n d_i} \times \mathbb{R}^{\sum_{i=1}^n d_i} \),
A4: For all $i = 1,\ldots,n$, $\text{proj}_z(C_i) \subseteq \text{conv}(\bigcup_{i=1}^{n} Z_i))$, where

$$C_i = \{ \mathcal{L}(i, z_i, u_i) : (z_i, u_i) \in C \left( t_i^{i_j}, \psi_i^{k_i}, w_i^{l_i} \right) \}.$$ 

Then, $\text{conv}(Z) \subseteq \text{proj}_z(X) \subseteq \text{conv}(\text{conv}(Z))$, where,

$$X = \begin{cases}
  \sum_{i=1}^{n} t_i^{i_j}(z_i, u_i) \geq 1, & \forall (j_i)_{i \in N} \in \prod_{i=1}^{n} J_i, \\
  \sum_{i \in I} v_i^{k_i}(z_i, u_i) \geq -1, & \forall I \subseteq N, \forall (k_i)_{i \in I} \in \prod_{i \in I} K_i, \\
  t_i^{i_j}(z_i, u_i) + v_i^{k_i}(z_i, u_i) \geq 0, & \forall i \in N, \forall j_i \in J_i, \forall k_i \in K_i, \\
  t_i^{i_j}(z_i, u_i) \geq 0, & \forall j_i \in J_i, \\
  w_i^{l_i}(z_i, u_i) \geq 0, & \forall i \in N, \forall l_i \in L_i.
\end{cases}$$

Using the above theorem, we can derive the convex hull for those sets which satisfy assumptions A1 - A4. Checking whether A1, A3 and A4 are satisfied by a given set is relatively easy. Verifying A2 might be difficult in practice. To overcome this difficulty, Tawarmalani [15] have used an alternative criterion called convex extension property which is more general than the assumption A2.

2.2 The Convex Extension Property

The convex extension property, as seen in [16], plays an important role in the derivation of the convex hull description of a set $Z$ using orthogonally restricted subsets of $Z$. The property is defined as follows.

**Definition 2** (Convex Extension Property). Let $Z$ be a set in $\mathbb{R}^n$ and $Z_i \subseteq Z, i \in N$. We say that the convex extension property holds for $Z$ if it satisfies the following two properties.

(i) If $z \in Z_i$, then $z_j = 0$ for all $j \in N, j \neq i$.

(ii) If $z \in Z$, then $z$ can be expressed as a sum of convex combination of some points $\chi_i \in \text{conv}(\text{conv}(Z_i)), i \in N$ and a conic combination of rays $\psi_i \in 0^+(\text{conv}(\text{conv}(Z_i))), i \in N$, i.e.

$$z = \sum_{i \in N} \lambda_i \chi_i + \sum_{i \in N} \mu_i \psi_i \quad \text{(CE)}$$

where $\mu_i \in \mathbb{R}_+, i \in N$ and $\lambda_i \in \mathbb{R}_+, i \in N$ with $\sum_{i \in N} \lambda_i = 1$.

A collection of sets $Z_i, i \in N$ that satisfy condition (i) in Definition 2 are known as orthogonal sets. It is clear that if a set is defined as the union of orthogonal sets, then the convex extension property must hold. There are some other sets that are not defined as such, but still satisfy the property. For example, bilinear mixed-integer and pure-integer covering sets without variable bounds satisfy this property. The convex extension property is equivalent to the following criterion given in [15].

$$\text{conv}(\text{conv}(Z)) = \text{conv} \left( \bigcup_{i=1}^{n} Z_i \right) \quad \text{(CE-P)}$$

Now, if we assume CE or CE-P instead of the assumption A2 in Theorem 1, we get $\text{cl} (\text{proj}_z X) = \text{cl} (\text{conv} (\bigcup_{i=1}^{n} Z_i)) = \text{cl} (\text{conv}(Z))$ [15]. Since in many cases we only need $\text{cl} (\text{conv}(Z))$, it is useful to consider CE or CE-P instead of the assumption A2.
3 On The Mixed-Integer Bilinear Covering Set $S$

We start by revisiting the set $S = \{(x,y) \in \mathbb{Z}_+^n \times \mathbb{R}_+^n : \sum_{i=1}^n x_i y_i \geq r\}, r > 0$, and the facet defining inequalities of its convex hull. Then we derive a property of extreme points of $\text{conv}(S)$ that we will later extend to $\text{conv}(S^U)$.

3.1 The Convex Hull Description of $S$

Tawarmalani et al. [15] showed that the set $S$ satisfies the assumptions A1, A3 and A4 of Theorem 1 and the convex extension property CE with respect to the orthogonal disjunctive subsets $S_i, i \in \mathbb{N}$, where,

$$S_i = \{L(i,x_i,y_i) \in \mathbb{Z}_+^n \times \mathbb{R}_+^n : x_i y_i \geq r\}.$$

Therefore, we can apply Theorem 1 to construct the description $\text{conv}(S)$. For this, first, we have to find the description of $\text{conv}(S_i)$. Note that the set $S_i$ is a two dimensional convex set as all variables other than $x_i$ and $y_i$ are fixed to zero. It can also be seen that the points on the curve, $x_i y_i = r$, which are of the form $L(i,k,\frac{r}{k})$, $k \in \mathbb{N}$ are the extreme points of $\text{conv}(S_i)$. It is depicted in Figure 1. Furthermore, the line passing through $L(i,k,\frac{r}{k})$ and $L(i,k+1,\frac{r}{k+1})$ gives a facet defining inequality for $\text{conv}(S_i)$ for all $k \in \mathbb{N}$, and $x_i \geq 1$ is the only remaining facet defining inequality. Therefore, the convex hull description $\text{conv}(S_i)$ can be given as,

$$\text{conv}(S_i) = \{L(i,x_i,y_i) : a_k x_i + b_k y_i \geq 1, k \in \mathbb{N}\},$$

where $a_k x_i + b_k y_i = 1$ is the line joining the two points $L(i,k,\frac{r}{k})$ and $L(i,k-1,\frac{r}{k-1})$ for $k \in \mathbb{N} \setminus \{1\}$ and $a_1 = 1, b_1 = 0$. Hence, we have $a_k = \frac{1}{2k-1}$ and $b_k = \frac{k(k-1)}{r(2k-1)}$ for all $k \in \mathbb{N}$.

We note that $\text{conv}(S_i)$ has countably infinite number of extreme points and facet defining inequalities. Consequently, $\text{conv}(S_i)$ is not a polyhedral set. We also note that the recession cone $0^+(\text{conv}(S_i))$ of $\text{conv}(S_i)$ is the set $\{(x,y) \in \mathbb{R}_+^n \times \mathbb{R}_+^n : x_j = 0, y_j = 0, j \in \mathbb{N}, j \neq i\}$.

![Figure 1: Convex Hull of $S_i$ for $r = 6$](image)

All sets $S_i, i \in \mathbb{N}$ are identical to each other except for relabeling of indices. Thus, the coefficients $a_k$ and $b_k, k \in \mathbb{N}$ are identical for each $\text{conv}(S_i), i \in \mathbb{N}$. Therefore, finding the
coefficients $a_k, b_k, k \in \mathbb{N}$ for $\text{conv}(S_1)$ is sufficient to get all the facets of $\text{conv}(S)$. The following collection of columns $(M)$ with countably infinite number of rows can be used to generate all the facets defining inequalities of $\text{conv}(S)$.

$$
\begin{bmatrix}
x_1 & x_2 & x_3 & \ldots & x_n \\
 a_1 x_1 + b_1 y_1 & a_2 x_2 + b_2 y_2 & a_3 x_3 + b_3 y_3 & \ldots & a_n x_n + b_n y_n \\
 a_1 x_1 + b_1 y_1 & a_2 x_2 + b_2 y_2 & a_3 x_3 + b_3 y_3 & \ldots & a_n x_n + b_n y_n \\
 \vdots & \vdots & \vdots & \ddots & \vdots \\
 a_k x_1 + b_k y_1 & a_k x_2 + b_k y_2 & a_k x_3 + b_k y_3 & \ldots & a_k x_n + b_k y_n \\
 \vdots & \vdots & \vdots & \ddots & \vdots 
\end{bmatrix} \quad (M)
$$

Theorem 1 states that a facet defining inequality of $\text{conv}(S)$ is constructed by adding $n$ terms from the above matrix $(M)$ taking exactly one term from each column and constraining their sum to be greater than or equal to one. All the facet defining inequalities are constructed this way. It is also clear that $\text{conv}(S)$ also has countably infinite number of facet defining inequalities, and the recession cone $0^+(\text{conv}(S))$ is the entire non-negative orthant $\mathbb{R}_+^n \times \mathbb{R}_+^n$, since $a_k, b_k \geq 0, \forall k \in \mathbb{N}$.

### 3.2 Properties of The Extreme Points of $\text{conv}(S)$

Here we will derive a few properties of the extreme points of $\text{conv}(S)$ that we use later. We first note that $\text{conv}(S)$ is a closed set. This is because, if $(x, y) \notin \text{conv}(S)$, there exists a facet defining inequality of $\text{conv}(S)$ that strongly separates the point $(x, y)$ from $\text{conv}(S)$. Therefore, the point $(x, y)$ can not be a limit point of $\text{conv}(S)$, and consequently $\text{conv}(S)$ is a closed set.

**Theorem 2.** Let $(\bar{x}, \bar{y}) \in \mathbb{R}_+^n \times \mathbb{R}_+^n$ be an extreme point of $\text{conv}(S)$, where $S$ is the mixed-integer bilinear covering set defined above. Then, there exists an index $t \in N$ such that $x_t \bar{y}_t = r$ and $x_i = 0, y_i = 0, \forall i \in N, i \neq t$, i.e., exactly one pair of the components of $(\bar{x}, \bar{y})$ is positive.

**Proof.** Since $S$ is a mixed-integer set and $(\bar{x}, \bar{y})$ is an extreme point of $\text{conv}(S)$, then $\bar{x}$ is an integral vector and $(\bar{x}, \bar{y})$ lies on the surface $\sum_{i=1}^{n} x_i y_i = r$. If possible, let there exist two pairs of components of $(\bar{x}, \bar{y})$ that are strictly greater than zero. Without loss of generality let $(\bar{x}_1, \bar{y}_1)$ and $(\bar{x}_2, \bar{y}_2)$ have all their components greater than zero. Also let $\bar{x}_1 \bar{y}_1 + \bar{x}_2 \bar{y}_2 = \alpha$.

Without loss of generality let us assume $\bar{x}_1 \bar{y}_1 \geq \frac{\alpha}{2}$. We now consider the following two points with all non negative components,

$$
\left( \frac{\bar{x}_1}{\alpha}, \frac{\bar{x}_2}{\alpha}, 0, \bar{x}_3, \bar{y}_3, \ldots, \bar{x}_n, \bar{y}_n \right) \quad \text{and} \quad \left( \frac{\bar{x}_1}{\alpha}, 2 \bar{y}_1 - \frac{\alpha}{\bar{x}_1}, \bar{x}_2, 2 \bar{y}_2, \bar{x}_3, \bar{y}_3, \ldots, \bar{x}_n, \bar{y}_n \right),
$$

and see that

$$
(\bar{x}, \bar{y}) = \frac{1}{2} \left( \frac{\bar{x}_1}{\alpha}, \frac{\bar{x}_2}{\alpha}, 0, \bar{x}_3, \bar{y}_3, \ldots, \bar{x}_n, \bar{y}_n \right) + \frac{1}{2} \left( \frac{\bar{x}_1}{\alpha}, 2 \bar{y}_1 - \frac{\alpha}{\bar{x}_1}, \bar{x}_2, 2 \bar{y}_2, \bar{x}_3, \bar{y}_3, \ldots, \bar{x}_n, \bar{y}_n \right).
$$

Also,

$$
\alpha \frac{\bar{x}_1}{\bar{x}_1} + \bar{x}_2 0 + \bar{x}_3 \bar{y}_3 + \ldots + \bar{x}_n \bar{y}_n = \alpha + \bar{x}_3 \bar{y}_3 + \ldots + \bar{x}_n \bar{y}_n
$$

$$
= \bar{x}_1 \bar{y}_1 + \bar{x}_2 \bar{y}_2 + \bar{x}_3 \bar{y}_3 + \ldots + \bar{x}_n \bar{y}_n \geq r,
$$

6
and,

$$\bar{x}_1 \left(2\bar{y}_1 - \frac{\alpha}{\bar{x}_1}\right) + \bar{x}_2 \bar{y}_2 + \bar{x}_3 \bar{y}_3 + \ldots + \bar{x}_n \bar{y}_n = 2(\bar{x}_1 \bar{y}_1 + \bar{x}_2 \bar{y}_2) - \alpha + \bar{x}_3 \bar{y}_3 + \ldots + \bar{x}_n \bar{y}_n = \bar{x}_1 \bar{y}_1 + \bar{x}_2 \bar{y}_2 + \bar{x}_3 \bar{y}_3 + \ldots + \bar{x}_n \bar{y}_n \geq r.$$

Thus, $\left(\bar{x}_1, \frac{\alpha}{\bar{x}_1}, \bar{x}_2, 0, \bar{x}_3, \bar{y}_3, \ldots, \bar{x}_n, \bar{y}_n\right)$ and $\left(\bar{x}_1, 2\bar{y}_1 - \frac{\alpha}{\bar{x}_1}, \bar{x}_2, 2\bar{y}_2, \bar{x}_3, \bar{y}_3, \ldots, \bar{x}_n, \bar{y}_n\right)$ lie in $S$ and consequently in $\text{conv}(S)$.

This shows that $(\bar{x}, \bar{y})$ cannot be an extreme point of $\text{conv}(S)$. Therefore, our assumption must be wrong which proves that $\bar{x}_i \bar{y}_i = 0$ for all $i \in N$, $i \neq t$. We still have to show that $\bar{x}_i = 0$ and $\bar{y}_i = 0$ simultaneously for all $i \in N$, $i \neq t$.

Now let $\bar{x}_i \bar{y}_i = r$. If possible, let there exist $j \in N$, $j \neq t$ such that $\bar{y}_j > 0$. Therefore, using the above arguments, $\bar{x}_j = 0$. Let $\epsilon > 0$ be such that $\bar{y}_j - \epsilon > 0$. Then $(\bar{x}, \bar{y})$ lies in the middle of two points $(\bar{x}, \bar{y})^1$ and $(\bar{x}, \bar{y})^2$ such that $(\bar{x}, \bar{y})^1$ and $(\bar{x}, \bar{y})^2$ have the same components as $(\bar{x}, \bar{y})$ except the $j^{th}$ component of the variable $\bar{y}$ and $\bar{y}_j^1 = \bar{y}_j - \epsilon$ and $\bar{y}_j^2 = \bar{y}_j + \epsilon$. Since $(\bar{x}, \bar{y})^1, (\bar{x}, \bar{y})^2 \in S, (\bar{x}, \bar{y})$ is not an extreme point of $\text{conv}(S)$, which is a contradiction.

Similarly, let there exist $i \in N$, $i \neq t$ such that $\bar{x}_i > 0$. Then from the first part of the proof, $\bar{y}_i = 0$. Therefore, $(\bar{x}, \bar{y})$ lies in the middle of two points $(\bar{x}, \bar{y})^1$ and $(\bar{x}, \bar{y})^2$ such that $(\bar{x}, \bar{y})^1$ and $(\bar{x}, \bar{y})^2$ have the same components as $(\bar{x}, \bar{y})$ except the $i^{th}$ component of the variable $x$, where $\bar{x}_i^1 = 2\bar{x}_i$ and $\bar{x}_i^2 = 0$. Since $(\bar{x}, \bar{y})^1, (\bar{x}, \bar{y})^2 \in S, (\bar{x}, \bar{y})$ is not an extreme point of $\text{conv}(S)$, which is again a contradiction.

**Theorem 3.** $(\bar{x}, \bar{y})$ is an extreme point of $\text{conv}(S)$ if and only if $(\bar{x}, \bar{y})$ is an extreme point of $\text{conv}(S_i)$ for some $i \in N$.

**Proof.** If $(\bar{x}, \bar{y})$ is an extreme point of $\text{conv}(S)$, then it is an extreme point of $\text{conv}(S_i)$ for some $i \in N$. This immediately follows from Theorem 2.

Conversely, let $(\bar{x}, \bar{y})$ be an extreme point of $\text{conv}(S_i)$ for some $i \in N$. Then, $\bar{x}_j = 0, \bar{y}_j = 0, \forall j \in N, j \neq i$. If possible, let $(\bar{x}, \bar{y})$ be expressed as a convex combination of two distinct points $(\bar{x}, \bar{y})^1$ and $(\bar{x}, \bar{y})^2$ in $S$. Since $S \subseteq \mathbb{R}_+^n \times \mathbb{R}_+^n$, then $\bar{x}_j^1 = 0, \bar{y}_j^1 = 0, \forall j \in N, j \neq i, t = 1, 2$. This implies that $(\bar{x}, \bar{y})^1$ and $(\bar{x}, \bar{y})^2$ belong to $S_i$. This is a contradiction to the fact that $(\bar{x}, \bar{y})$ is an extreme point of $\text{conv}(S_i)$. Therefore, $(\bar{x}, \bar{y})$ must be an extreme point of $\text{conv}(S)$.

It is clear from Theorem 3 that any point of the form $L(i, k, \frac{r}{k}), k \in \mathbb{N}$ is an extreme point of $\text{conv}(S)$ and vice versa, for all $i \in N$.

4 On The Mixed-Integer Bilinear Covering Set $S^U$

In this section we obtain a description of the convex hull of $S^U$ defined in Section 1. We show that unlike $\text{conv}(S), \text{conv}(S^U)$ is a polyhedron. We derive the description of $\text{conv}(S^U)$ in both the ways - the V-Polyhedron description (i.e., in terms of the extreme points and the recession cone); and, the H-Polyhedron description (i.e., a linear inequality description).

**Proposition 1.** The set $\text{conv}(S^U)$ is a polyhedron.

**Proof.** Since there is an upper bound $u$ on the integer variable $x$, we have finitely many choices for $x$ in $S^U$. For each $i \in N$, we have $u_i + 1$ different choices for $x_i$. Since $x = 0$ is not a feasible choice for $S^U$, therefore, the total number of different choices for $x$ is $\prod_{i=1}^n(u_i + 1) - 1 = \eta$ (say). Let us denote them by $x^k, k = 1, \ldots, \eta$. Now, define the following sets:

$$F_k = \left\{(x, y) \in \mathbb{Z}_+^n \times \mathbb{R}_+^n : \sum_{i=1}^n x_i y_i \geq r, x \leq u, x = x^k\right\}, k = 1, \ldots, \eta.$$
Note that the set $F_k$ is constructed by fixing $x = x_k$ to the set $S^U$. Therefore, $F_k$ is a polyhedron for each $k = 1, \ldots, \eta$ and, $S^U = \bigcup_{k=1}^{\eta} F_k$. Also note that the recession cone $0^+ (F_k)$ of $F_k$ is the set $\{ y : y \geq 0 \}$ for all $k = 1, \ldots, \eta$. Therefore, $S^U$ is a union of finite number of nonempty polyhedra with identical recession cones. So, from Corollary 4.44 in [12], we have $\text{conv} (S^U)$ is a polyhedron.

4.1 The Extreme Points Description of $\text{conv} (S^U)$

Since $\text{conv} (S^U)$ is a polyhedron, it is closed and, therefore, it contains all its extreme points. Note that all the extreme points of the polyhedral set $\{(x, y) \in \mathbb{R}_{+}^{n} : x \leq u \}$ have all components of the variable $y$ equal to zero, none of them lie in $S^U$. Therefore, all the extreme points of the set $\text{conv} (S^U)$ satisfy $\sum_{i=1}^{n} x_i y_i = r$.

Let us consider a more general set $W$ defined below:

$$W = \left\{ (x, y) \in \mathbb{Z}_{+}^{n} \times \mathbb{R}_{+}^{n} : \sum_{i=1}^{n} x_i y_i \geq r, x \leq u, Gx \leq h \right\}$$

where $G$ is a given $m \times n$ rational matrix and $h$ is a given $n \times 1$ rational vector. By the same logic as in the proof of Proposition 1, the set $\text{conv}(W)$ is closed. We describe a property of the extreme points of $\text{conv}(W)$ like that in Theorem 2.

**Theorem 4.** Consider the set $W$ defined above. Let $(\bar{x}, \bar{y})$ be an extreme point of $\text{conv}(W)$. Then there exists an index $t \in N$ such that $\bar{x}_t \bar{y}_t = r$ and $\bar{y}_i = 0$ for all $i \in N, i \neq t$.

**Proof.** The proof is similar to the steps of the proof of Theorem 2. The only difference is that, since we have additional linear constraints on the variable $x$, the values of the variables $x_i, i \in N, i \neq t$ at an extreme point may not be zero. \qed

Theorem 4 gives us an idea about the description of the extreme points of $\text{conv} (S^U)$ which we provide next.

**Theorem 5.** Let $(\bar{x}, \bar{y})$ be an extreme point of $\text{conv} (S^U)$. Then, $\bar{x}_t = p_t, \bar{y}_t = \frac{r}{p_t}$ for some $t \in N$, where $p_t \in \{1, \ldots, u_t\}$, and $\bar{x}_j \in \{0, u_j\}, \bar{y}_j = 0, \forall j \in N, j \neq t$, i.e., $(\bar{x}, \bar{y})$ has the following form,

$$\left( \bar{x}_1, 0, \bar{x}_2, 0, \ldots, \bar{x}_{t-1}, 0, p_t, \frac{r}{p_t}, \bar{x}_{t+1}, 0, \ldots, \bar{x}_n, 0 \right)$$

where $p_t \in \{1, \ldots, u_t\}$ for some $t \in N, \bar{x}_j \in \{0, u_j\}, \forall j \in N, j \neq t$.

**Proof.** It is easy to verify that any point $(x, y)$ such that $x_i = p, y_i = \frac{r}{p}$, $x_j = 0, y_j = 0$ for all $j \in N, j \neq i$, where $p \in \{1, \ldots, u_i\}$ is an extreme point of $\text{conv} (S^U)$. From Theorem 4, we see that at any extreme point of $\text{conv} (S^U)$ only one component of the variable $y$ is nonzero.

Now, suppose $(\bar{x}, \bar{y})$ is an extreme point of $\text{conv} (S^U)$. Then $(\bar{x}, \bar{y}) \in S^U$. Also let the $t^{th}$ component of the variable $y$ of $(\bar{x}, \bar{y})$ be positive. Then from Theorem 4 we have $\bar{x}_t \bar{y}_t = r, \bar{x}_i \in N, \bar{x}_j \leq u_i, \forall j \in N, j \neq i$.

We now show that $\bar{x}_j \in \{0, u_j\}$ for $\forall j \neq i$. If $\bar{x}_j \in \{0, u_j\}, j \in N, j \neq i$, then $\bar{y}_j = 0$, and therefore, $(\bar{x}, \bar{y})$ can be written as a convex combination of the two points $(\bar{x}, \bar{y})^1$ and $(\bar{x}, \bar{y})^2$ having the exact same components as $(\bar{x}, \bar{y})$, except for the $j^{th}$ components of the variable $x$, and $x_j^1 = 0, x_j^2 = u_j$. Multipliers $1 - \lambda$ and $\lambda$ respectively provide the convex combination of $(\bar{x}, \bar{y})^1$ and $(\bar{x}, \bar{y})^2$, where $\lambda = \frac{\bar{x}_j}{u_j}$. This is a contradiction to the supposition that $(\bar{x}, \bar{y})$ is an extreme point of $\text{conv} (S^U)$.
Moreover, if \( \bar{x}_j \in \{0, u_j\} \) for \( j \neq i \), then we can not write \( (\bar{x}, \bar{y}) \) as a convex combination of two different points in \( S^U \). This is because, if two such points exist, one of the points’ \( j^{th} \) component of the variable \( x \) has to be more than \( u_j \) or less than 0, which is not allowed. \( \square \)

**Corollary 1.** \( \text{conv} (S^U) \) has \( 2^{n-1} \sum_{i=1}^n u_i \) extreme points and \( n \) extreme rays.

**Proof.** We see from the proof of Theorem 5, for a single choice of \( \bar{x}_i \in \{1, \ldots, u_i\} \), we have \( 2^{n-1} \) different extreme points, and we have \( \sum_{i=1}^n u_i \) distinct such choices. Therefore, the total number of extreme points of \( \text{conv} (S^U) \) is \( 2^{n-1} \sum_{i=1}^n u_i \), which is exponentially large, but finite. Consequently, \( \text{conv} (S^U) \) is a polyhedral set.

On the other hand, we see that the recession cone \( 0^+ (\text{conv} (S^U)) \) of \( \text{conv} (S^U) \) is the set \( \{(x, y) \in \mathbb{R}_{+}^n \times \mathbb{R}_{+}^n : x = 0\} \) which has \( n \) extreme rays. \( \square \)

Note that Theorem 5 and Corollary 1 give us the V-Description of \( \text{conv} (S^U) \). We now turn our attention to the H-Description of \( \text{conv} (S^U) \).

### 4.2 A Polyhedral Relaxation of \( \text{conv} (S^U) \)

We have the following orthogonal disjunctive subsets of \( S^U \), for all \( i \in N \).

\[
S^U_i = \{ \mathcal{L}(i, x, y_i) \in \mathbb{Z}_{+}^n \times \mathbb{R}_{+}^n : x_i y_i \geq r, x_i \leq u_i \}.
\]

We note that \( S^U_i \subset S^U \), and the recession cone of \( \text{cl} (\text{conv} (S^U_i)) \) is the following set:

\[
0^+ (\text{cl} (\text{conv} (S^U_i))) = \{(x, y) \in \mathbb{R}_{+}^n \times \mathbb{R}_{+}^n : x = 0, y_j = 0, \forall j \in N, j \neq i\}.
\]

We see that the assumption A1 of Theorem 1 is satisfied by the set \( S^U \) with respect to the orthogonal disjunctive subsets \( S^U_i \) defined above. Again, we have a polyhedral description of \( \text{conv} (S^U_i) \).

\[
\text{conv} (S^U_i) = \{ \mathcal{L}(i, x, y_i) \in \mathbb{R}_{+}^n \times \mathbb{R}_{+}^n : a_k x_i + b_k y_i \geq 1, x_i \leq u_i, \forall k \in K_i \}
\]

where \( K_i = \{1, \ldots, u_i\} \), and as defined earlier, \( a_k = \frac{1}{2^{k-1}} \), \( b_k = \frac{k(k-1)}{r(2k-1)} \), \( k \in K_i \). Therefore, assumption A3 of Theorem 1 is satisfied by the set \( S^U \) with respect to the orthogonal disjunctive subsets \( S^U_i \).

But we do not know whether the assumption A2 is satisfied. Also, verifying the convex extension property CE or CE-P in this case looks difficult. So, we can not apply Theorem 1 directly to construct the description of \( \text{conv} (S^U) \).

In order to find the description of \( \text{conv} (S^U) \), we use the following observation. There are two inequalities \( x_i y_i \geq r \) and \( x_i \leq u_i \) in the description of \( S^U_i \), and they together imply \( y_i \geq \frac{r}{u_i} \). Let \( \bar{x}_i = \frac{r}{u_i} \). Let us now define the following set:

\[
S^L_i = \{ \mathcal{L}(i, x, y_i) \in \mathbb{Z}_{+}^n \times \mathbb{R}_{+}^n : x_i y_i \geq r, y_i \geq \bar{x}_i \}.
\]

By adding the lower bound on \( y_i \) and ignoring the upper bound on \( x_i \), we have a relaxation of \( S^U_i \). The two sets \( \text{conv} (S^U_i) \) and \( \text{conv} (S^L_i) \) have exactly the same set of extreme points that are \( u_i \) in number. Figure 2 and 3 illustrate this observation.
The description of $\text{conv}(S^L_i)$ using the facet defining inequalities is quite straight forward. We have

$$\text{conv}(S^L_i) = \{ L(i, x_i, y_i) \in \mathbb{R}^n_+ \times \mathbb{R}^n_+ : a_k x_i + b_k y_i \geq 1, y_i \geq \bar{u}_i, \forall k \in K_i \} ,$$

where $K_i = \{1, \ldots, u_i\}$, and $a_k, b_k, k \in K_i$ are defined earlier. We also note that the recession cone $0^+ (\text{conv}(S^L_i))$ of $\text{conv}(S^L_i)$ is the following set

$$\{(x, y) \in \mathbb{R}^n_+ \times \mathbb{R}^n_+ : x_j = 0, y_j = 0, j \in N, j \neq i\}.$$

Let us now define the following set.

$$S^L = \bigcup_{i=1}^{n} S^L_i$$

We will later show that the description of $\text{conv}(S^L)$ is easy to obtain using orthogonal disjunctive technique and we will derive the description of $\text{conv}(S^U)$ using $\text{conv}(S^L)$.

We observe that, since $S^L = \bigcup_{i=1}^{n} S^L_i$, we have,

$$\text{cl} (\text{conv}(S^L)) = \text{conv}(S^L) = \text{cl} \left( \text{conv} \left( \bigcup_{i=1}^{n} S^L_i \right) \right),$$

i.e., the set $S^L$ satisfies the condition CE-P which is equivalent to the convex extension property with respect to the orthogonal disjunctive subsets $S^L_i, i \in N$.

**Proposition 2.** The set $S^L$ satisfies all the assumptions A1 - A4 of Theorem 1 with respect to the orthogonal disjunctive subsets $S^L_i, i \in N$.

**Proof.** We see that the assumption A1 holds from the definition of $S^L$. For the assumption A2, we have the convex extension property that is satisfied as observed above. Since we have the polyhedral description of $\text{conv}(S^L_i)$, the assumption A3 is satisfied. Lastly, we see that $0^+ (\text{cl} (\text{conv} (\bigcup_{i=1}^{n} S^L_i)))$ is the entire non-negative orthant, which implies that the assumption A4 is also satisfied. ∎
We can now use Theorem 1 to construct the convex hull description of the set $S^L$. We have $\text{conv}(S^L) = \{ L(i,x_i,y_i) \in \mathbb{R}_+^n \times \mathbb{R}_+^n : a_kx_i + b_ky_i \geq 1, y_i \geq u_i, k \in K_i \}$, where $K_i = \{1, \ldots, u_i\}$. Let us write it using a single index set as following,

$$
\text{conv}(S^L) = \left\{ L(i,x_i,y_i) \in \mathbb{R}_+^n \times \mathbb{R}_+^n : \sum_{i=1}^{n} l^k(i,x_i,y_i) \geq 1, k_i \in K_i \right\},
$$

where, $K_i = K_i \cup \{u_i + 1\}$, $l^k(i,x_i,y_i) = a_kx_i + b_ky_i$, where $a_ki = \frac{1}{2k_i - 1}, b_ki = \frac{k_i(k_i - 1)}{2k_i - 1}, k_i \in K_i$ and $l^k(u_i + 1)(i,x_i,y_i) = \frac{u_i}{k_i}$. Therefore, applying Theorem 1 we have,

$$
\text{conv}(S^L) = \left\{ (x,y) \in \mathbb{R}_+^n \times \mathbb{R}_+^n : \sum_{i=1}^{n} l^k(i,x_i,y_i) \geq 1, \forall (k_i)_{i=1}^{n} \in \prod_{i=1}^{n} K_i \right\}.
$$

The set $\text{conv}(S^L)$ is a polyhedral set as it has finite number of facet defining inequalities in its description, and the number of facets is $\prod_{i=1}^{n} |K_i| = \prod_{i=1}^{n} (u_i + 1)$, which is exponentially large. Also, it can be seen clearly that $0^+ = \text{conv}(S^L)$ is the entire non-negative orthant $\mathbb{R}_+^n \times \mathbb{R}_+^n$. Let us now analyze some properties of the set $\text{conv}(S^L)$ and its relation with the set $S^U$.

**Theorem 6.** $(\bar{x}, \bar{y})$ is an extreme point of $\text{conv}(S^L)$ if and only if $(\bar{x}, \bar{y})$ is an extreme point of $\text{conv}(S_i^L)$ for some $i \in N$.

**Proof.** Since $\text{conv}(S^L)$ is a closed set, we have $\text{conv}(S^L) = \text{cl}(\text{conv}(\bigcup_{i=1}^{n} S_i^L))$. Now, let $(\bar{x}, \bar{y})$ be an extreme point of $\text{conv}(S^L)$. Then, we have,

$$
(\bar{x}, \bar{y}) \in S^L \Rightarrow (\bar{x}, \bar{y}) \in S_i^L, \text{ for some } i \in N, \text{ since } S^L = \bigcup_{i=1}^{n} S_i^L \Rightarrow (\bar{x}, \bar{y}) \in \text{conv}(S_i^L).
$$

Now the claim is that $(\bar{x}, \bar{y})$ is an extreme point of $\text{conv}(S_i^L)$. If not, then there must exist two points in $\text{conv}(S_i^L)$ and thus in $S^L$ other than $(\bar{x}, \bar{y})$ whose convex combination is $(\bar{x}, \bar{y})$. This contradicts the fact that $(\bar{x}, \bar{y})$ is an extreme point of $\text{conv}(S^L)$. Therefore, $(\bar{x}, \bar{y})$ is an extreme point of $\text{conv}(S_i^L)$.

Conversely, let $(\bar{x}, \bar{y})$ be an extreme point of $\text{conv}(S_i^L)$ for some $i \in N$. Then, $\bar{x}_j = 0, \bar{y}_j = 0, \forall j \in N, j \neq i$. If possible, let $(\bar{x}, \bar{y})$ be expressed as a convex combination of two distinct points $(\bar{x}, \bar{y})^1$ and $(\bar{x}, \bar{y})^2$ in $S^L$. Since $S^L \subset \mathbb{R}_+^n \times \mathbb{R}_+^n$, we have $\bar{x}_j^t = 0, \bar{y}_j^t = 0, \forall j \in N, j \neq i, t = 1, 2$. This implies that $(\bar{x}, \bar{y})^1$ and $(\bar{x}, \bar{y})^2$ belong to $S_i^L$. This contradicts the fact that $(\bar{x}, \bar{y})$ is an extreme point of $\text{conv}(S_i^L)$. Therefore, $(\bar{x}, \bar{y})$ is an extreme point of $\text{conv}(S^L)$. □

**Corollary 2.** $(\bar{x}, \bar{y})$ is an extreme point of $\text{conv}(S^L)$ if and only if $(\bar{x}, \bar{y})$ is an extreme point of $\text{conv}(S_i^L)$ for some $i \in N$.

**Proof.** Since $\text{conv}(S_i^L)$ and $\text{conv}(S_i^L)$ have exactly same set of extreme points, the result follows from Theorem 6. □

It is clear from Theorem 6 and its corollary that all the extreme points of $\text{conv}(S^L)$ lie in $S^U$ and they are of the form $(0,0,\ldots,0, p_i, \frac{1}{p_i}, 0,0,\ldots,0)$ where, $p_i \in \{1, \ldots, u_i\}, i \in N$. Moreover, these are also extreme points of $\text{conv}(S^U)$ as discussed in Theorem 5. Hence, we have the following result.

**Proposition 3.** The set $\text{conv}(S^U)$ is a proper subset of $\text{conv}(S^L)$, i.e., $\text{conv}(S^L)$ is a polyhedral relaxation of $S^U$. 11
Now, for 

we construct a closed form expression of 

From the above results, we have a polyhedral relaxation 

\[ S \subseteq \text{conv}(S^L) \]

\[ \sum_{k \in T} x_k y_k = r \]. Now, let us consider the following points.

\[ \xi_i = \mathcal{L} \left( i, x_i, y_i + \frac{\sum_{k \in T, k \neq i} x_k y_k}{x_i}, 0, 0, \ldots, x_i, y_i + \frac{\sum_{k \in T, k \neq i} x_k y_k}{x_i}, \ldots, 0, 0 \right), i \in T \]

Clearly, since \( x \leq u \), for each \( i \in T \) we have \( \xi_i \in S_i^L \), and consequently \( \xi_i \in S^L \) since \( S_i^L \subseteq S^L \). Now consider the following rays in \( 0^+ (\text{conv} (S^L)) = \mathbb{R}_+^n \times \mathbb{R}_+^n \).

\[ \psi_i = \begin{cases} (x_{i-1}, 0, 0, \ldots, x_{i-1}, 0, 0, 0, 0, x_{i+1}, 0, \ldots, x_n, 0), i \in T \\ \mathcal{L}(i, x_i, y_i), i \in N \setminus T. \end{cases} \]

Now consider the following multipliers,

\[ \lambda_i = \begin{cases} \frac{x_i y_i}{\sum_{k \in T} x_k y_k}, i \in T, \\ 0, i \in N \setminus T, \end{cases} \]

and \( \mu_i = \begin{cases} \frac{x_i y_i}{\sum_{k \in T} x_k y_k}, i \in T, \\ 1, i \in N \setminus T. \end{cases} \]

It is easy to verify that \( \lambda_i \geq 0, i \in T \) and \( \sum_{i \in T} \lambda_i = 1 \), and \( \mu_i \geq 0 \) for all \( i \in N \). Now we have,

\[ (x, y) = \sum_{i \in T} \lambda_i \xi_i + \sum_{i \in N} \mu_i \psi_i \]

Since \( (x, y) \) is arbitrary, we have \( S^U \subseteq \text{conv}(S^L) \), which implies \( \text{conv}(S^U) \subset \text{conv}(S^L) \). Now, for \( n = 2 \) clearly the point \( \left( u_1 + 1, \frac{r}{u_1}, 0, 0 \right) \in S^L \) but not in \( \text{conv}(S^U) \), which implies \( \text{conv}(S^U) \) is a proper subset of \( \text{conv}(S^L) \). Thus proved.

Here we observe that \( \text{conv}(S^L) \) is a polyhedral relaxation of \( S^U \) such that each extreme point of \( \text{conv}(S^L) \) lies in \( S^U \). Moreover, both the sets \( S^U \) and \( \text{conv}(S^L) \) are subsets of the non-negative orthant. Therefore, minimizing a linear function \( c^T x + d^T y \) such that \( c \geq 0 \) over \( S^U \) and over \( \text{conv}(S^L) \) are equivalent.

### 4.3 The Convex Hull Description of \( S^U \)

From the above results, we have a polyhedral relaxation \( \text{conv}(S^L) \) of \( S^U \). We also note that all the extreme points of \( \text{conv}(S^L) \) lie in \( S^U \) even though \( \text{conv}(S^L) \) is a relaxation of \( S^U \). Here, we construct a closed form expression of \( \text{conv}(S^U) \) by adding back the bound constraints to \( \text{conv}(S^L) \).
Theorem 7. Let \( S = \text{conv} (S^L) \cap \{(x, y) \in \mathbb{R}_+^n \times \mathbb{R}_+^n : x \leq u\} = \{(x, y) \in \text{conv} (S^L) : x \leq u\}. \) Then, \( \text{conv} (S^U) = \bar{S} \).

Proof. We know from the Minkowski Resolution Theorem (Theorem 4.15 in [2]) that any polyhedral set having at least one extreme point can be completely described by its extreme points and its recession cone.

The sets \( \text{conv} (S^U) \) and \( \bar{S} \) are polyhedral and have the same recession cone \( \{(x, y) \in \mathbb{R}_+^n \times \mathbb{R}_+^n : x = 0\} \). We have to show that the sets \( \text{conv} (S^U) \) and \( \bar{S} \) have the same set of extreme points.

Theorem 5 states that any extreme point \((\bar{x}, \bar{y})\) of \( \text{conv} (S^U) \) must have the form,
\[
(\bar{x}_1, 0, \bar{x}_2, 0, \ldots, \bar{x}_{i-1}, 0, p_i, r, \bar{x}_{i+1}, 0, \ldots, \bar{x}_n, 0)
\]
where \( p_i \in \{1, \ldots, u_i\} \) for some \( i \in N, \bar{x}_j \in \{0, u_j\}, \forall j \in N, j \neq i \).

We have also seen earlier that any extreme point of the set \( \text{conv} (S^L) \) is of the form \((0, 0, \ldots, 0, p_i, \frac{r}{p_i}, 0, 0, \ldots, 0)\), where \( p_i \in \{1, \ldots, u_i\}, i \in N \). We see that the hyperplane \( x_i = u_i \) passes through the extreme point \((0, 0, \ldots, 0, u_i, \frac{r}{u_i}, 0, 0, \ldots, 0)\) of \( \text{conv} (S^L) \), for each \( i \in N \). Note that \((0, 0, \ldots, 0, u_i, \frac{r}{u_i}, 0, \ldots, 0)\) is the only extreme point of \( \text{conv} (S^L) \) the hyperplane \( x_i = u_i \) passes through. Therefore, the constraint \( x_i \leq u_i \) does not remove any extreme points of \( \text{conv} (S^L) \) having only the \( i^{th} \) component nonzero. Also, \( x_i \leq u_i \) does not cut off any other extreme points of \( \text{conv} (S^L) \) since all of them have the \( x_i \) coordinate zero. Therefore, adding the constraints \( x \leq u \) only creates new extreme points without cutting off any from \( \text{conv} (S^L) \), and they are of the following form,
\[
(w_1, 0, w_2, 0, \ldots, w_{i-1}, 0, p_i, r, w_{i+1}, 0, \ldots, w_n, 0)
\]
where, \( w_j \in \{0, u_j\}, j \in N, j \neq i, p_i \in \{1, \ldots, u_i\}, i \in N \). One can verify that the points of the above form are extreme points of \( \bar{S} \) using the same arguments as in the proof of Theorem 5.

Therefore, two polyhedra \( \text{conv} (S^L) \) and \( \bar{S} \) have exactly same set of extreme points and the same recession cone. Thus, \( \text{conv} (S^U) = \bar{S} \).

\[\square\]

4.4 Facet Defining Inequalities of \( \text{conv} (S^U) \)

We note that all the facet defining inequalities for the set \( \text{conv}(S) \) are also valid inequalities for \( \text{conv} (S^U) \). Here, we will focus our attention on the new inequalities that are generated by our procedure and their efficiency.

We have seen from Theorem 7 that each facet defining inequality of \( \text{conv} (S^U) \) is either a bound constraint \( x_i \leq u_i \) for some \( i \in N \) or a facet defining inequality of \( \text{conv} (S^L) \). Recall from Proposition 2 that the facet defining inequalities of \( \text{conv} (S^L) \) are of the form,
\[
\sum_{i=1}^{n} t^{k_i} (x_i, y_i) \geq 1, (k_i)_{i=1}^{n} \in \prod_{i=1}^{n} K_i
\]
where, \( K_i = K_i \cup \{u_i + 1\}, K_i = \{1, \ldots, u_i\}, t^{k_i} (x_i, y_i) = a_{k_i} x_i + b_{k_i} y_i, a_{k_i} = \frac{1}{2k_i - 1}, b_{k_i} = \frac{k_i(k_i - 1)}{r(2k_i - 1)}, k_i \in K_i \) and \( t^{(u_i + 1)} (x_i, y_i) = \frac{u_i}{u_i}, u_i = \frac{r}{u_i} \). If for some \( (k_i)_{i=1}^{n} \in \prod_{i=1}^{n} K_i, k_i \in K_i, \forall i \in N \) then the inequality \( \sum_{i=1}^{n} t^{k_i} (x_i, y_i) \geq 1 \) is identical to one of the facet defining inequalities of
conv(S). Now let $Q \subseteq N$ be a non-empty index set such that $k_i = (u_i + 1)_i$ for all $i \in Q$. Then, the inequalities of the following form
\[ \sum_{i \in Q} l^k_i (x_i, y_i) + \sum_{i \in N \setminus Q} l^k_i (x_i, y_i) \geq 1, \]
or equivalently
\[ \sum_{i \in Q} \frac{y_i}{u_i} + \sum_{i \in N \setminus Q} l^k_i (x_i, y_i) \geq 1, (k_i)_{i=1}^n \in \prod_{i=1}^n \overline{K}_i \quad \text{(NF)} \]
are generated by applying our approach. They are not valid for $\text{conv}(S)$.

For a given point $(x, y) \in \mathbb{R}_+^n \times \mathbb{R}_+^n$, we now discuss how to check whether $(x, y) \in \text{conv}(S^U)$, and how to find a facet defining inequality (if any) to cut off the point.

5 The Separation Problem

We now have the closed form description of the $\text{conv}(S^U)$, and it is clear that it consists of finitely many facet defining inequalities, even though the number is exponentially large. The immediate question is, how do we solve the associated separation problem? We just saw that the facets of $\text{conv}(S^U)$ are either the bound constraints on the variable $x$ or the facets of the set $\text{conv}(S^L)$. Let $(\bar{x}, \bar{y})$ be a point in $\mathbb{R}^n \times \mathbb{R}^n$. If $\bar{x} \not\in u$, then a bound constraint is sufficient to separate $(\bar{x}, \bar{y})$. Since the bound constraints can be checked and added easily, we will define the separation problem for the facets defining inequalities of $\text{conv}(S^L)$ only.

The facet defining inequalities of $\text{conv}(S^L)$ can be listed in a different way for easier understanding.

\[ \text{conv}(S^L) = \left\{ (x, y) \in \mathbb{R}_+^n \times \mathbb{R}_+^n : \sum_{i=1}^n l^k_i (x_i, y_i) \geq 1, \forall (k_i)_{i=1}^n \in \prod_{i=1}^n \overline{K}_i \right\}, \]
where, $\overline{K}_i = K_i \cup \{u_i + 1\}, K_i = \{1, \ldots, u_i\}$. Consider the following collection of columns.

\[
\begin{bmatrix}
  l^{11}(x_1, y_1) & l^{12}(x_2, y_2) & l^{13}(x_3, y_3) & \ldots & l^{1n}(x_n, y_n) \\
  l^{21}(x_1, y_1) & l^{22}(x_2, y_2) & l^{23}(x_3, y_3) & \ldots & l^{2n}(x_n, y_n) \\
  l^{31}(x_1, y_1) & l^{32}(x_2, y_2) & l^{33}(x_3, y_3) & \ldots & l^{3n}(x_n, y_n) \\
  \vdots & \vdots & \vdots & \ddots & \vdots \\
  l^{(u_1+1)1}(x_1, y_1) & l^{(u_2+1)2}(x_2, y_2) & l^{(u_3+1)3}(x_3, y_3) & \ldots & l^{(u_n+1)n}(x_n, y_n)
\end{bmatrix}
\]

(MX)

Note that $M_X$ may have different number of elements in each column depending upon $u$. The facet defining inequalities of $\text{conv}(S^L)$ can be constructed by adding $n$ terms from $M_X$, taking exactly one term from each column and constraining the sum to be at least one.

Let us reconsider the example in Section 1.

\[
\begin{align*}
\text{min} & \quad -x_1 + 10y_1 - 2x_2 + 12y_2 \\
\text{s.t.} & \quad x_1 y_1 + x_2 y_2 \geq 20, \\
& \quad x_1 \leq 5, x_2 \leq 6, \\
& \quad x_1 \geq 0, y_1 \geq 0, i = 1, 2.
\end{align*}
\]

Note that the point $(5, 1, 6, \frac{5}{2})$ is not feasible for this problem. As discussed in Section 1, this point can not be cut off by any of the facet defining inequalities of $\text{conv}(S)$ and the objective
value at this point is 3. But we see that this point is violated by the inequality \( \frac{5y_1}{20} + \frac{6y_2}{20} \geq 1 \) which is of the form (NF). Note that the inequalities \( \frac{5y_1}{20} \geq 1 \) and \( \frac{6y_2}{20} \geq 1 \) are valid for \( S_1 \) and \( S_2 \), respectively and combining them in the way described above we obtain the inequality \( \frac{5y_1}{20} + \frac{6y_2}{20} \geq 1 \). After adding this inequality to \( \text{conv}(S) \), we get the optimal solution \((5, 4, 6, 0)\) with optimal value 23.

Let us suppose that \((\bar{x}, \bar{y}) \notin \text{conv}(S^U)\). Then, there exists an inequality of the following form:
\[
\sum_{i=1}^{n} t^{k_i}(x_i, y_i) \geq 1, \text{ for some } (k_i)_{i=1}^{n} \in \prod_{i=1}^{n} K_i
\]
which is violated by \((\bar{x}, \bar{y})\). This implies \(\sum_{i=1}^{n} t^{k_i}(\bar{x}_i, \bar{y}_i) < 1\) and consequently \(t^{k_i}(\bar{x}_i, \bar{y}_i) < 1, \forall i = 1, ..., n\), for the \((k_i)_{i=1}^{n} \in \prod_{i=1}^{n} K_i\). We define the following separation problem which is a binary linear program.

Let \( z_{ik_i} = \begin{cases} 1, & \text{if the term } t^{k_i}(x_i, y_i) \text{ is in the inequality,} \\ 0, & \text{otherwise,} \end{cases} \)

\[
\zeta = \min \sum_{i} \sum_{k_i} t^{k_i}(\bar{x}_i, \bar{y}_i) z_{ik_i} \\
\sum_{k_i} z_{ik_i} = 1, \forall i \in N, \\
z_{ik_i} \in \{0, 1\}, \forall i \in N, k_i \in K_i.
\]

For each \( i \in N \), the constraint \( \sum_{k_i} z_{ik_i} = 1 \) implies that exactly one term from the column \( i \) of \( M_X \) should be selected to construct the inequality.

Note that the value of \( \zeta \) is greater than or equal to one if and only if \((\bar{x}, \bar{y})\) lies in \( \text{conv}(S^U) \).

Now suppose \( \zeta < 1 \). Therefore, at the corresponding solution \( \bar{z} \), exactly \( n \) components will be 1 and all others be zero. So adding the corresponding \( t^{k_i}(x_i, y_i) \) for which \( z_{ik_i} = 1 \) and setting the sum to greater or equal to one will give one such most violated inequality. Mathematically we can write it as,
\[
\sum_{i} \sum_{k_i} t^{k_i}(x_i, y_i) z_{ik_i} \geq 1.
\]

### 5.1 Efficient Separation for \( \text{conv}(S^U) \)

As we have seen earlier, the separation problem for this case is a 0–1 problem with \( \sum_{i \in N} (u_i + 1) \) binary variables and \( n \) linear equality constraints. Generally, such problems are hard to solve. But, in this case we can solve this separation problem efficiently. We find the minimum element from each column of \( M_X \) at \((\bar{x}, \bar{y})\) and add them. Clearly, if the the sum is greater than or equal to 1, the point \((\bar{x}, \bar{y})\) is feasible to \( \text{conv}(S^U) \). On the other hand, if the sum is less than 1, adding the corresponding terms from each column and setting it to greater than or equal to 1, will give us a violated facet defining inequality.

Column \( i \) of \( M_X \) has \( (u_i + 1) \) terms, \( i \in N \). To solve the separation problem, we need to find the minimum value at \((\bar{x}, \bar{y})\) from each column. This step takes \( O(u_i + 1) \) time which is pseudo-polynomial in the size of input. We now present a polynomial time algorithm for the separation problem.
Proposition 4. There exists an efficient separation of the facet defining inequalities of $\text{conv} \left(S^U\right)$.

Proof. Let $(\tilde{x}, \tilde{y}) \in \mathbb{R}_+^n \times \mathbb{R}_+^n$ be a given point. For each column of $M_X$, we want to find the term which gives the minimum evaluation at the point $(\tilde{x}, \tilde{y})$, i.e., minimum value of the column $i$ at $(\tilde{x}_i, \tilde{y}_i)$. Let

$$
\xi_i = \min \left\{ \frac{\tilde{x}_i}{2w-1} + \frac{\tilde{y}_i w(w-1)}{r(2w-1)}, w = 1, \ldots, u_i \right\}, \text{ where } \frac{u_i}{r_i} = \frac{r}{u_i}.
$$

Note that $\xi_i \geq 0$. To find $\xi_i$, we consider the following cases.

Case 1: If $\tilde{y}_i = 0$, then clearly $\xi_i = 0$ at the last term, i.e., at $\frac{\tilde{y}_i}{u_i}$ since $\frac{\tilde{y}_i}{u_i} = 0$.

Case 2: If $\tilde{x}_i = 0$, then again $\xi_i = 0$ at $w = 1$.

Case 3: When $\tilde{x}_i > 0$ and $\tilde{y}_i > 0$. Let us consider the following function:

$$
f(w) = \frac{\tilde{x}_i}{2w-1} + \frac{\tilde{y}_i w(w-1)}{r(2w-1)}, w \geq 1.
$$

Our goal is to find a positive integer $q$ that minimizes $f(w)$ among all the integers in $[1, u_i]$. The function $f$ is continuously differentiable in the domain $w \geq 1$ with

$$
f'(w) = -\frac{2\tilde{x}_i}{(2w-1)^2} + \frac{\tilde{y}_i}{r} \cdot \frac{2w^2 - 2w + 1}{(2w-1)^2}
$$

$$
f''(w) = \frac{2(4\tilde{x}_i r - \tilde{y}_i)}{r(2w-1)^3}.
$$

We have the following two subcases.

Case 3.1: When $4\tilde{x}_i r - \tilde{y}_i > 0$, the function $f$ is strictly convex and has unique minimizer. Let the unique minimizer be $\bar{w}$. Then $f'(\bar{w}) = 0$ occurs at

$$
\bar{w} = \frac{1}{2} + \frac{\sqrt{4\tilde{x}_i r - \tilde{y}_i}}{2}.
$$

Therefore, when $\bar{w} \leq 1$, the integral value at which $f$ is minimized is $q = 1$. When $u_i > \bar{w} > 1$, $q = \lceil \bar{w} \rceil$ or $\lfloor \bar{w} \rfloor$ whichever gives a lower $f(q)$, and when $\bar{w} > u_i$, $q = u_i$.

Case 3.2: When $4\tilde{x}_i r - \tilde{y}_i \leq 0$, the function $f$ is concave for $w \geq 1$. Therefore, the minimum value will be attained at a boundary point, i.e., either at 1 or at $u_i$. Moreover, we see that

$$
f'(w) = -\frac{2\tilde{x}_i}{(2w-1)^2} + \frac{\tilde{y}_i}{r} \cdot \frac{2w^2 - 2w + 1}{(2w-1)^2}
$$

$$
= \frac{2\tilde{y}_i w(w-1) + \tilde{y}_i - 2\tilde{x}_i r}{r(2w-1)^2} > 0.
$$

This shows that $f$ is strictly increasing function, and $f$ is minimized at $q = 1$. Now one more comparison is required to find the value of $\xi_i$. If $\frac{\tilde{x}_i}{2q} + \frac{\tilde{y}_i (q-1)}{r(2q-1)} \leq \frac{\tilde{y}_i}{u_i}$, then $\xi_i = \frac{\tilde{x}_i}{2q} + \frac{\tilde{y}_i (q-1)}{r(2q-1)}$, else $\xi_i = \frac{\tilde{y}_i}{u_i}$.

The term corresponding to any column of $M_X$ can be computed in $O(1)$ time, and since there are $n$ columns, a violated inequality can be found in $O(n)$ time.

If $\sum_{i=1}^n \xi_i \geq 1$, the point $(\tilde{x}, \tilde{y})$ is feasible to $\text{conv} \left(S^U\right)$. Algorithm 1 describes the separation algorithm.
Algorithm 1 Separation of the facet defining inequalities \( \text{conv} \left( S^U \right) \)

1: Input : A point \((\bar{x}, \bar{y}) \in \mathbb{R}^n_+ \times \mathbb{R}^n_+, x \leq u\)
2: Output : Decide whether \((\bar{x}, \bar{y}) \in \text{conv} \left( S^U \right)\), and if not then provide a facet defining inequality of \( \text{conv} \left( S^U \right) \) that cuts off \((\bar{x}, \bar{y})\)

3: for \(i = 1, \ldots, n\) do
4: if \(\bar{y}_i = 0\) then
5: \(\hat{\omega}_i = u_i + 1\)
6: else if \(\bar{x}_i = 0\) then
7: \(\hat{\omega}_i = 1\)
8: else
9: \(q = 0\)
10: if \(4\bar{x}_i r > \bar{y}_i\) then
11: if \(\frac{1}{2} + \frac{\sqrt{\frac{x_i}{y_i}} - 1}{2} > u_i\) then
12: \(p = \left\lfloor \frac{1}{2} + \frac{\sqrt{\frac{x_i}{y_i}} - 1}{2} \right\rfloor\)
13: if \(\frac{\bar{x}_i}{2p - 1} + \frac{\bar{y}_i (p-1)}{r(2p-1)} < \frac{\bar{x}_i}{2(p+1) - 1} + \frac{\bar{y}_i (p+1)}{r(2(p+1)-1)}\) then
14: \(q = p\)
15: else
16: \(q = p + 1\)
17: end if
18: else
19: \(q = u_i\)
20: end if
21: else
22: \(q = 1\)
23: end if
24: else
25: \(q = 1\)
26: end if
27: end if
28: if \(\frac{\bar{x}_i}{2q-1} + \frac{\bar{y}_i (q-1)}{r(2q-1)} \leq \frac{\bar{y}_i}{u_i}\) then
29: \(\hat{\omega}_i = q\)
30: else
31: \(\hat{\omega}_i = u_i + 1\)
32: end if
33: end if
34: end for
35: \(R = \sum_{i \in N : \hat{\omega}_i \leq u_i} \frac{\bar{x}_i}{2\hat{\omega}_i - 1} + \frac{\bar{y}_i \hat{\omega}_i (\hat{\omega}_i - 1)}{r(2\hat{\omega}_i - 1)} + \sum_{i \in N : \hat{\omega}_i = u_i + 1} \frac{\bar{y}_i}{u_i}\)
36: if \(R \geq 1\) then
37: The point \((\bar{x}, \bar{y})\) is feasible to \(\text{conv} \left( S^U \right)\).
38: else
39: The inequality \(\sum_{i \in N : \hat{\omega}_i \leq u_i} \frac{\bar{x}_i}{2\hat{\omega}_i - 1} + \frac{\bar{y}_i \hat{\omega}_i (\hat{\omega}_i - 1)}{r(2\hat{\omega}_i - 1)} + \sum_{i \in N : \hat{\omega}_i = u_i + 1} \frac{\bar{y}_i}{u_i} \geq 1\) separates \((\bar{x}, \bar{y})\).
40: end if

\[\blacksquare\]

Corollary 3. The optimization problem having a linear objective function over the set \(S^U\) (or
equivalently over $\text{conv}(S^U)$ can be solved in polynomial time in size of input.

Proof. Since there is a polynomial time separation algorithm of the facet defining inequalities of $\text{conv}(S)$, the optimization of a linear function over $S^U$ (or equivalently over $\text{conv}(S^U)$) can also be done in polynomial time [13]. We have presented in Appendix B an efficient algorithm for the optimization problem that does not use these facet defining inequalities. \qed

5.2 Efficient Separation for $\text{conv}(S)$

The separation problem is slightly different as each column of the matrix $M$ consists of countably infinite number of elements. But, we will use similar technique to solve the separation problem as $\text{conv}(S^U)$.

**Proposition 5.** There exists an efficient separation of the facet defining inequalities of $\text{conv}(S)$.

**Proof.** We have to find the minimum of each column of the matrix $M$ at $(\bar{x}, \bar{y})$, i.e., minimum value of the column $i$ at $(\bar{x}, \bar{y})$. Mathematically we can write,

$$\xi_i = \min \left\{ \frac{\bar{x}_i}{2w - 1} + \frac{\bar{y}_i w(w - 1)}{r(2w - 1)}, w \in \mathbb{N} \right\}. $$

Note that $\xi \geq 0$ for $w \geq 1$. Our goal is to find a positive integer that minimizes $f(w)$. Below, we present an efficient way to solve this problem. We consider the following cases.

**Case 1**: When $\bar{x}_i = 0$, then clearly $\xi_i = 0$ at $\bar{w}_i = 1$.

**Case 2**: When $\bar{y}_i = 0, \bar{x}_i \neq 0$, then $\inf \left\{ \frac{\bar{x}_i}{2k-1} + \frac{\bar{y}_i k(k-1)}{r(2k-1)}, k \in \mathbb{N} \right\} = 0$, since $\frac{\bar{x}_i}{2k-1} \to 0$ as $k \to \infty$. Therefore $\xi_i$ can be taken as 0 in this case.

**Case 3**: When $\bar{x}_i > 0, \bar{y}_i > 0$, by the same logic described above for the case $\text{conv}(S^U)$, here also we have the same solution. Let $\bar{w}_i$ be the desired integer value.

$$\bar{w}_i = \begin{cases} 1, & \text{when } 4\bar{x}_i r - \bar{y}_i > 0 \text{ and } \bar{w} \leq 1, \\ \lceil \bar{w} \rceil, & \text{when } 4\bar{x}_i r - \bar{y}_i > 0, \bar{w} > 1 \text{ and } f(\lceil \bar{w} \rceil) \leq f(\lceil \bar{w} \rceil), \\ \lceil \bar{w} \rceil, & \text{when } 4\bar{x}_i r - \bar{y}_i > 0, \bar{w} > 1 \text{ and } f(\lceil \bar{w} \rceil) \geq f(\lceil \bar{w} \rceil), \\ 1, & \text{when } 4\bar{x}_i r - \bar{y}_i \leq 0, \end{cases}$$

where $\bar{w}$ is derived for the case $\text{conv}(S^U)$. Now, clearly if $\sum_{i=1}^{n} \xi_i = 0$, the point $(\bar{x}, \bar{y})$ is feasible to $\text{conv}(S)$, and if $\sum_{i=1}^{n} \xi_i < 1$, the point $(\bar{x}, \bar{y})$ is infeasible. Let $(\bar{x}, \bar{y})$ be infeasible to $\text{conv}(S)$. Then we have to find a facet defining inequality that is violated by this point. For the cases when $\bar{x}_i = 0$; and $\bar{x}_i > 0, \bar{y}_i > 0$, we know the value of $\bar{w}_i$ from CASE 1 and 3. Let $t \in \mathbb{N}$ such that the following holds,

$$\sum_{i=1}^{n} \xi_i + \sum_{i \in N: \bar{x}_i > 0, \bar{y}_i = 0} \frac{\bar{x}_i}{2t - 1} < 1.$$ 

Such a $t$ can always be found by the Archimedean property. A simple calculation shows that the least value of $t$ is $\left\lceil \frac{1 + v}{2(1 - \xi)} \right\rceil + 1$, where $\xi = \sum_{i=1}^{n} \xi_i, v = \sum_{i \in N: \bar{x}_i > 0, \bar{y}_i > 0} \bar{x}_i$. Therefore, the following inequality is violated by the point $(\bar{x}, \bar{y})$:

$$\sum_{i \in N: \bar{x}_i = 0} x_i + \sum_{i \in N: \bar{x}_i > 0, \bar{y}_i > 0} \left[ \frac{x_i}{2\bar{w}_i - 1} + \frac{y_i \bar{w}_i (\bar{w}_i - 1)}{r(2\bar{w}_i - 1)} \right] + \sum_{i \in N: \bar{x}_i > 0, \bar{y}_i = 0} \left[ \frac{x_i}{2t - 1} + \frac{y_i t(t - 1)}{r(2t - 1)} \right] \geq 1.$$
The above inequality is a facet defining inequality for $\text{conv}(S)$ from the discussion in Section 3.1. The separation algorithm is given below.

**Algorithm 2** Separation of the facet defining inequalities of $\text{conv}(S)$

1: Input : A point $(\bar{x}, \bar{y}) \in \mathbb{R}^n_+ \times \mathbb{R}^n_+$
2: Output : Decide whether $(\bar{x}, \bar{y}) \in \text{conv}(S)$, and if not then provide a facet defining inequality that cuts off $(\bar{x}, \bar{y})$
3: for $i = 1, \ldots, n$ do
4:  if $\bar{x}_i = 0$ then
5:      $\hat{w}_i = 1$, $\xi_i = 0$
6:  else if $\bar{x}_i \bar{y}_i > 0$ then
7:      if $4\bar{x}_i r > \bar{y}_i$ then
8:          if $\frac{1}{2} + \frac{\sqrt{\bar{x}_i \bar{y}_i - 1}}{2} > 1$ then
9:              $p = \left\lfloor \frac{1}{2} + \frac{\sqrt{\bar{x}_i \bar{y}_i - 1}}{2} \right\rfloor$
10:             if $\frac{\bar{x}_i}{2p-1} + \tilde{y}_i p(p-1) \leq \frac{\bar{x}_i}{2(p+1)-1} + \tilde{y}_i p(p+1)$ then
11:                $\hat{w}_i = p$
12:           else
13:                $\hat{w}_i = p + 1$
14:          end if
15:      else
16:          $\hat{w}_i = 1$
17:      end if
18:  else
19:      $\hat{w}_i = 1$
20:  end if
21: $\xi_i = \frac{\bar{x}_i}{2\hat{w}_i-1} + \frac{\tilde{y}_i \hat{w}_i(\hat{w}_i-1)}{r(2\hat{w}_i-1)}$
22: else
23:     $\xi_i = 0$
24: end if
25: end for
26: $\xi = \sum_{i \in N} \xi_i$
27: if $\xi \geq 1$ then
28: The point $(\bar{x}, \bar{y})$ is feasible to $\text{conv}(S)$.
29: else
30: $v = \sum_{i \in N; \tilde{y}_i = 0} x_i$
31: $t = \left\lfloor \frac{1 - \xi + v}{2(1 - \xi)} \right\rfloor + \gamma$, where $\gamma$ can be taken as any natural number.
32: for $i = 1, \ldots, n$ do
33:  if $\bar{x}_i > 0$ and $\tilde{y}_i = 0$ then
34:      $\hat{w}_i = t$
35:  end if
36: end for
37: The inequality $\sum_{i=1}^{n} \frac{x_i}{2\hat{w}_i-1} + \frac{y_i \hat{w}_i(\hat{w}_i-1)}{r(2\hat{w}_i-1)} \geq 1$ cuts off the point $(\bar{x}, \bar{y})$.
38: end if

It is clear that for any natural number $\gamma$, Algorithm 2 gives us a violated facet defining
inequality. Depending on the objective function, suitably chosen value of $\gamma$ may improve the bound rather than one randomly chosen.

**Corollary 4.** The optimization problem having a linear objective function over the set $S$ (or equivalently over $\text{conv}(S)$) can be solved in polynomial time.

Since there is an efficient separation algorithm of the facet defining inequalities of $\text{conv}(S)$, optimization of a linear function over $S$ (or equivalently over $\text{conv}(S)$) is also efficiently solvable [13]. In Appendix A, we have presented a polynomial time algorithm for the optimization problem.

6 Computational Results

We now study the effectiveness of our cuts by doing computational experiments on cutting stock instances of the following form.

$$\min \sum_{i=1}^{n} y_i$$

$$\sum_{i \in N} x_{ij} y_i \geq d_j, \quad j \in F,$$

$$\sum_{j \in F} l_j x_{ij} \leq L, \quad i \in N,$$

$$x_{ij} \in \mathbb{Z}_+, y \in \mathbb{R}_+, \forall i, j \in N,$$

where the notation is the same as that in Section 1. These instances have stocks of one length from which $n$ different sizes of finals are to be cut. So, there are $n$ number of mixed-integer bilinear covering constraints modeling demand satisfaction. The upper bounds $x_{ij} \leq \left\lfloor \frac{L}{r_j} \right\rfloor, \forall i \in N, j \in F$ of the integral variables are implicit from the knapsack constraints present in the formulation. Here, our objective is to minimize the total number of different patterns that are used.

We have selected for our experiments few instances used in [21] taken from applications in a chemical fiber company in Japan (Fiber-xx-xxxx), few instances generated by CUTGEN [23] (CutGen-xx-xx) and some randomly generated instances (Rand-xx).

In our computational study we compare the bounds generated by our cuts for $(\text{conv}(S^U))$ to those cuts derived by Tawarmalani et al. [15] for $(\text{conv}(S))$. In both the cases we consider the facet defining inequalities of each mixed-integer bilinear covering constraint. Adding some facet defining inequalities for each mixed-integer bilinear covering constraint together gives a polyhedral relaxation for the actual problem. For each instance, in both the cases, we start our iterations with the facet defining inequalities $\sum_{i=1}^{n} x_{ij} \geq 1, \forall j \in N$, the bound constraints
and the knapsack inequalities, i.e., we start our iterations by solving the following LP relaxation.

\[
\min \sum_{i=1}^{n} y_i \\
\text{s.t.} \quad \sum_{i=1}^{n} x_{ij} \geq 1, \forall j \in N, \\
0 \leq x_{ij} \leq \left\lfloor \frac{L_i}{l_j} \right\rfloor, \forall i \in N, j \in F \\
\sum_{j \in F} l_j x_{ij} \leq L, i \in N, \\
y \geq 0.
\]

(ILP)

Then, we add violated inequalities (if any) and resolve the LP. This process is continued until we can not find any more violated inequalities, or the number of LPs solved exceeds a limit (800).

We run the above experiment in two different settings using facet defining inequalities derived (i) for \(\text{conv}(S^U)\) and (ii) for \(\text{conv}(S)\). We consider the sets \(S^U\) and \(S\) by looking at each bilinear covering constraint separately and add one most violated cut for each such constraint using Algorithm 1 and Algorithm 2 respectively. So, we add at most \(n\) cuts in every iteration (LP solve) which are not deleted in further iterations. This means, at iteration \(k\), we solve an LP relaxation of the instance with at most \(kn\) number of linear inequalities in addition to those in ILP.

We have used PuLP [20] version 1.6.2 (installed in Python 2.7.12) to model the linear programs and CLP [8] solver to solve them. The system we used to run our code has Linux (Ubuntu 16.04) operating system with 4x Intel(R) Core(TM) i5-3570 CPU@3.40 GHz processor and 8 GB of RAM. All experiments were carried out on a single core. Computational results are given in Table 1.

| Instances   | n  | Using inequalities for \(\text{conv}(S^U)\) | Using inequalities for \(\text{conv}(S)\) |
|-------------|----|------------------------------------------|------------------------------------------|
|             |    | Iterations | Termination | Lower Bound | Iterations | Termination | Lower Bound |
| Fiber-10-5180 | 10 | 91      | Yes         | 24.1669     | 238       | Yes         | 6.8779      |
| Fiber-10-9080 | 10 | 87      | Yes         | 12.3556     | 253       | Yes         | 3.8505      |
| Fiber-11-5180 | 10 | 54      | Yes         | 15.9070     | 288       | Yes         | 6.0996      |
| Fiber-11-9080 | 10 | 75      | Yes         | 8.7798      | 269       | Yes         | 3.3582      |
| Fiber-14-5180 | 10 | 185     | Yes         | 9.3495      | 414       | Yes         | 3.3392      |
| Fiber-14-9080 | 10 | 236     | Yes         | 5.2332      | 470       | Yes         | 1.9007      |
| Fiber-15-5180 | 10 | 147     | Yes         | 16.6307     | 549       | Yes         | 3.7394      |
| Fiber-15-9080 | 10 | 284     | Yes         | 9.3677      | 475       | Yes         | 2.1147      |
| Fiber-16-5180 | 10 | 221     | Yes         | 17.1229     | 633       | Yes         | 5.1710      |
| Fiber-16-9080 | 10 | 234     | Yes         | 5.1476      | 800       | No          | 2.9283      |
| CutGen-01-01  | 10 | 102     | Yes         | 1.6806      | 250       | Yes         | 1.2376      |
| CutGen-01-02  | 10 | 84      | Yes         | 1.3060      | 245       | Yes         | 0.9744      |
| CutGen-01-25  | 10 | 83      | Yes         | 1.4492      | 218       | Yes         | 0.9984      |
| CutGen-01-100 | 10 | 114     | Yes         | 2.0188      | 257       | Yes         | 1.2535      |
| CutGen-01-20  | 10 | 143     | Yes         | 13.8252     | 248       | Yes         | 10.4968     |
| CutGen-02-60  | 10 | 98      | Yes         | 14.1568     | 259       | Yes         | 10.0957     |
| Rand-10      | 10 | 86      | Yes         | 1520.50     | 185       | Yes         | 697.22      |
| Rand-15      | 15 | 167     | Yes         | 1520.50     | 440       | Yes         | 650.29      |
| Rand-16      | 16 | 126     | Yes         | 2407.35     | 800       | No          | 735.17      |
| Rand-20      | 20 | 434     | Yes         | 1682.95     | 800       | No          | 624.92      |
| Rand-25      | 25 | 793     | Yes         | 1061.49     | 800       | No          | 542.93      |

Table 1: Comparison of iterations taken to optimize over the convex hull and the lower bounds obtained
We observe that the lower bounds improve by at least 30% and up to 400% by using facet defining inequalities for $\text{conv}(S^u)$ as compared to $\text{conv}(S)$. In the above three graphs, we present stepwise bound comparisons for the instances Fiber-15-5180, CutGen-01-25 and Rand-16. We see that the cuts for $\text{conv}(S^u)$ improve bounds much more quickly than those for $\text{conv}(S)$.

7 Concluding Remarks and Future Work

When bounds on integer variables in a bilinear covering set are finite, we are able to obtain the polyhedral description of the convex hull. Even though one can not directly apply the orthogonal disjunctive procedure here, we are still able to compute the convex hull by first creating a suitable relaxation and then applying the procedure. It would be interesting to see if similar procedure can be applied to other restrictions of the set as well. Our examples and experiments show that the new facet defining inequalities improve the bounds significantly as compared to the case when bounds are not considered.

The procedure of finding facet defining inequalities to separate a given point from the convex hull is fast, and the experiments suggest that it is effective. Our results can be applied in a straightforward manner to the following set also:

$$S^\delta = \left\{(x, y) \in \mathbb{Z}_+^n \times \mathbb{R}_+^n : \sum_{i=1}^n \delta_i x_i y_i \geq r, x \leq u \right\}$$

where $r > 0, u \in \mathbb{N}$ and $\delta_i > 0$ for all $i \in N$. Using our analysis, we can show that for this case also the facet defining inequalities of $\text{conv}(S^\delta)$ can be separated in $O(n)$ time.
In this work we did not consider the knapsack constraint in our set. Including it and also considering multiple bilinear constraints together can be taken up as future work.

The cuts generated by our criterion of ‘maximum violation’ without any normalization may not be the cut that improves the lower bound the most. Consider the following example:

$$\begin{align*}
\text{min} & \quad x_1 + y_1 + x_2 + y_2 \\
\text{s.t.} & \quad x_1 y_1 + x_2 y_2 \geq 20, \\
& \quad x_i \leq 10, \quad i = 1, 2, \\
& \quad x_i \in \mathbb{Z}_+, y_i \geq 0, \quad i = 1, 2.
\end{align*}$$

The point $$(x_1, y_1, x_2, y_2) = (5, 4, 0, 0)$$ is a global optimum solution with optimal value 9. At the first iteration, the LP solution is $$(1, 0, 0, 0)$$ with objective value 1. The best cut generated by Algorithm 1 to cut this point off is $$\frac{y_1}{2} + \frac{y_2}{3} \geq 1$$. After adding this, the solution is $$(1, 2, 0, 0)$$ with objective value 3. But, if we instead add the facet defining inequality $$\frac{x_1}{5} + \frac{3y_1}{60} + \frac{x_2}{2} + \frac{3y_2}{50} \geq 1$$, we get a better solution $$(5, 0, 0, 0)$$ with objective value 5. Also, the distance of the latter from the point $$(1, 0, 0, 0)$$ is nearly 2.7 as compared to 1.41 for the former. Finding the cut that improves the bound the most or that is farthest from the given point is another interesting question.

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Bao X., Sahinidis N.V., and Tawarmalani M. Semidefinite relaxations for quadratically constrained quadratic programming: A review and comparisons. *Mathematical Programming*, B 129:129 – 157, 2011.
A Optimization over $S$

Minimizing a linear function $c^T x + d^T y$ over $S$ is equivalent to minimizing it over $\text{conv}(S)$. For the following cases, we can solve the optimization problem by inspection.

**Case 1:** If one of the components of $c$ or $d$ is negative, then the problem is unbounded.

**Case 2:** Suppose, $c \geq 0$, $d \geq 0$ and one of the component of the vector $c$ is zero, say $c_i = 0$. If $d_i = 0$, $\min_{(x,y) \in S} c^T x + d^T y = 0$ and if $d_i > 0$, $\inf_{(x,y) \in S} c^T x + d^T y = 0$. This is because, in either case we can choose $y_i$ arbitrary small such that $x_i y_i = r$ and all other components are zero, i.e., $\mathcal{L}(t, x_i, y_i)$ is an optimal solution in either case.

**Case 3:** Suppose, $c \geq 0$, $d = 0$. Let $c_t \leq c_j$, $\forall j \in N$. Then $\mathcal{L}(t, 1, r)$ is an optimal solution with optimal value $c_t$.

Now, the only remaining case is, when $c > 0$, $d \geq 0, d \neq 0$, which we consider in the following proposition.

**Proposition 6.** Let us consider the orthogonal disjunctive subset $S_i$ of the set $S$. Then we can solve the optimization problem $\min_{(x,y) \in S_i} c_i x_i + d_i y_i$ in polynomial time.

**Proof.** From the definition, each $(x, y) \in S_i$ is of the form $\mathcal{L}(i, x_i, y_i)$. If $c_i \geq 0, d_i = 0$ for some $i \in N$, then $\mathcal{L}(i, 1, r)$ is an optimal solution with optimal value $c_i$.

Now, we only have to consider $c_i > 0, d_i > 0$. Let $\mathcal{L}(i, x^*_i, y^*_i)$ be an extreme point optimal solution. Clearly, this point should lie on the surface $x_i y_i = r$. We note that the continuous relaxation of the set $S_i$ is a strictly convex set. Therefore, the optimal solution $\mathcal{L}(i, \bar{x}_i, \bar{y}_i)$ (say) over the continuous relaxation is unique, and we have,

$$\bar{x}_i = \sqrt{\frac{r d_i}{c_i}}, \text{ and } \bar{y}_i = \frac{r}{\bar{x}_i}.$$

If $\sqrt{\frac{r d_i}{c_i}}$ is an integer, then $\mathcal{L}(i, \bar{x}_i, \bar{y}_i)$ is an optimal solution. If not, then from the geometry, it is clear that at the optimal solution either $x^*_i = \left\lfloor \sqrt{\frac{r d_i}{c_i}} \right\rfloor$ or $x^*_i = \left\lceil \sqrt{\frac{r d_i}{c_i}} \right\rceil$ whichever minimizes the objective function and is feasible.

So, to determine an optimal solution, we just have to check the signs of the objective coefficient and compute the value of $\sqrt{\frac{r d_i}{c_i}}$. This can be done in constant time.

Now we consider the set $S$. Since the objective function is linear, it is equivalent to minimize over $\text{conv}(S)$. There must be an extreme point optimal solution, provided an optimal solution exists. Suppose optimal solution exists. Then by Theorem 3, there must be an optimal solution that is an extreme point of $\text{conv}(S_i)$ for some $i \in N$, because each extreme point of $\text{conv}(S)$ is an extreme point of $\text{conv}(S_i)$ for some $i \in N$. So, if we solve the $n$ problems $\min_{\mathcal{L}(i, x_i, y_i) \in S_i} c_i x_i + d_i y_i$ for $i \in N$ and pick the minimum of the $n$ objective values, we will get the optimal value and corresponding optimal solution. We have seen earlier that each subproblem takes constant time to solve. So, we can solve this problem in linear time. The algorithm is presented in Algorithm 3.
their values, we will get the optimal solution. Now our goal is to solve the following problem.

In the following discussion we will see that the problem is efficiently solvable and we will also solve the problem for the following cases.

defining inequalities in exponential. The extreme point descriptions of $\text{conv} S$ must be an extreme point optimal solution, provided an optimal solution exists. We know the extreme point descriptions of $\text{conv} E$ are also known. From the discussion in Section 4.1, all the points in $E_i$ are extreme points of $\text{conv} S^U$ and $E = \bigcup_{t \in N} E_i$ gives the complete set of extreme points of $\text{conv} S^U$, i.e., the sets $E_i, i \in N$ defines a partition of the set $E$. If we minimize $c^T x + d^T y$ over each set $E_i, i \in N$ and compare their values, we will get the optimal solution. Now our goal is to solve the following problem.

**Algorithm 3** Algorithm to solve $\min_{(x,y) \in S} c^T x + d^T y$

1: if One of the components of the vectors $c$ or $d$ is negative then
2: The problem is unbounded.
3: else if $c \geq 0, d \geq 0$ and one of the component of the vector $c$ is zero then
4: $\inf_{(x,y) \in S} c^T x + d^T y = 0.$
5: else if $c \geq 0, d = 0$. Let $c_t \leq c_j, \forall j \in N$ then
6: $L(t,1,r)$ is an optimal solution with optimal value $c_t$.
7: else (i.e., when $c > 0, d \geq 0$)
8: for $i = 1, \ldots, n$ do
9: Solve the problem $\min_{L(i,x_i,y_i) \in S} c_i x_i + d_i y_i$.
10: Let $L^i(t,x_i,y_i)$ be an optimal solution with optimal value $v_i$.
11: end for
12: Find the minimum of $v_i, i \in N$. Let $t \in N$ such that $v_t \leq v_i$ for all $i \in N$.
13: Then $L^t(t,x_t,y_t)$ is an optimal solution with optimal value $v_t$.
14: end if

**B Optimization over $S^U$**

Now we consider the following problem:

$$\min_{(x,y) \in S^U} c^T x + d^T y \quad (P)$$

This problem is equivalent to minimizing the objective function over $\text{conv} S^U$. Also, there must be an extreme point optimal solution, provided an optimal solution exists. We know the description of $\text{conv} S^U$ in terms of the facet defining inequalities, and the number of facet defining inequalities in exponential. The extreme point descriptions of $\text{conv} S^U$ is also known. In the following discussion we will see that the problem is efficiently solvable and we will also present the algorithm.

Like the problem of optimization over $S$ discussed earlier, here also by inspection we can solve the problem for the following cases.

**CASE 1** : When $d_t < 0$ for some $t \in N$, the problem is unbounded.

**CASE 2** : When $c \leq 0, d = 0$, then $\left( u_1, \frac{1}{u_1}, u_2, 0, \ldots, u_u, 0 \right)$ is an extreme point optimal solution.

Now the remaining case is $d \geq 0$. We note that $\sum_{i \in N: c_i \leq 0} c_i u_i$ is a lower bound on the objective value. To solve this problem, we will only consider the extreme points and compare their corresponding objective values to find the optimal solution. We first partition the set of extreme points of $\text{conv} S^U$ and optimize over those partitions. Let us define the following set for each $i \in N$.

$$E_i = \left\{ (x,y) \in \mathbb{Z}_+^n \times \mathbb{R}_+^n : x_i \in \{1, \ldots, u_i\}, y_i = \frac{r}{x_i}, x_j \in \{0,u_j\}, y_j = 0, \forall j \in N, j \neq i \right\}.$$  

From the discussion in Section 4.1, all the points in $E_i$ are extreme points of $\text{conv} S^U$ and $E = \bigcup_{i \in N} E_i$ gives the complete set of extreme points of $\text{conv} S^U$, i.e., the sets $E_i, i \in N$ defines a partition of the set $E$. If we minimize $c^T x + d^T y$ over each set $E_i, i \in N$ and compare their values, we will get the optimal solution. Now our goal is to solve the following problem.

$$\zeta_i = \min_{(x,y) \in E_i} c^T x + d^T y \quad (P_i)$$

26
Note that only the \(i^{th}\) component of the variable \(y\) of each point in \(E_i\) is non-zero and rest of all are zero. Therefore, the objective function of the above problem \((P_i)\) reduces to \(c_i x_i + d_i y_i + \sum_{j \in N, j \neq i} c_j x_j\). Also, we see that for any point \((x, y) \in E_i\), the choices of the components \(x_j \in \{0, u_j\}, j \in N, j \neq i\) are independent of the choice of \(x_i \in \{1, \ldots, u_i\}\). Let \((\bar{x}, \bar{y})^i \in E_i\) be an optimal solution of \((P_i)\). Then, \(\bar{y}_j^i = \frac{r}{\bar{x}_i^i}, \bar{y}_j^i = 0, \forall j \in N, j \neq i\). Let us consider the following choices of \(x\) components of \((\bar{x}, \bar{y})^i\).

\[
\bar{x}_i^i \in \{1, \ldots, u_i\} \text{ such that } (\bar{x}_i^i, \bar{y}_i^i) \text{ minimizes } c_i x_i + d_i y_i,
\]

\[
\bar{x}_j^i = \begin{cases} 0, & \text{if } c_j > 0, \\ u_j, & \text{if } c_j \leq 0, \forall j \in N, j \neq i. \end{cases}
\]

It can be seen clearly that such above choice of the components of \((\bar{x}, \bar{y})^i\) minimizes the objective function. Now to find the value of \(\bar{x}_i^i \in \{1, \ldots, u_i\}\), we consider the following cases.

**Case 1**: When \(c_i \leq 0\), then \(\bar{x}_i^i = u_i\). This is because, since \(c_i \leq 0\), the maximum value of \(x_i\) in the domain will minimize \(c_i x_i\). Moreover, for this choice of \(\bar{x}_i^i, \bar{y}_i^i = \frac{r}{u_i}\) is also minimum, and since \(d_i \geq 0\), \((u_i, \frac{r}{u_i})\) minimizes \(c_i x_i + d_i y_i\).

**Case 2**: If \(c_i > 0\) and \(d_i = 0, \bar{x}_i^i = 1, \bar{y}_i^i = r\) as \(\bar{x}_i^i \geq 1\).

**Case 3**: The remaining case is \(c_i > 0, d_i > 0\). Since the points \(L(i, p_i, \frac{r}{p_i})\), \(p_i \in \{1, \ldots, u_i\}\) are the extreme points of \(\text{conv} (S_i^U)\), it is equivalent to minimize \(c_i x_i + d_i y_i\) over \(\text{conv} (S_i^U)\). To solve this we will use the same analysis as in the proof of Proposition 6 with slight modification as there is an upper bound \(u_i\) on the variable \(x_i\). So, in this case we have the following choice of \(\bar{x}_i^i\) and consequently \(\bar{y}_i^i = \frac{r}{\bar{x}_i^i}\).

\[
\bar{x}_i^i = \begin{cases} \sqrt{\frac{r d_i}{c_i}}, & \text{if } \sqrt{\frac{r d_i}{c_i}} \in \{1, \ldots, u_i\}, \\ 1, & \text{if } \sqrt{\frac{r d_i}{c_i}} < 1, \\ \left\lfloor \sqrt{\frac{r d_i}{c_i}} \right\rfloor \text{ or } \left\lceil \sqrt{\frac{r d_i}{c_i}} \right\rceil, & \text{whichever minimizes } c_i x_i + d_i \frac{r}{\bar{x}_i^i}, \text{ if } 1 < \sqrt{\frac{r d_i}{c_i}} < u_i \text{ and } \sqrt{\frac{r d_i}{c_i}} \notin \mathbb{Z}_+, \\ u_i, & \text{if } \sqrt{\frac{r d_i}{c_i}} > u_i. \end{cases}
\]

So, from the above analysis, we can solve the problem \((P_i)\) in linear time, as we just have to check the signs of \(n - 1\) entries and have to check the value of \(\sqrt{\frac{r d_i}{c_i}}\), whenever it exists and if not then the signs of \(c_i\) and \(d_i\). Now, we have the following algorithms to solve the problem \((P)\).
Algorithm 4 Algorithm to solve $P_i$ when $d \geq 0$

1: Let $(\bar{x}, \bar{y})^t$ be an optimal solution to $P_i$.
2: for $j \in N, j \neq i$ do
3: \( \bar{y}_j^t = 0 \)
4: if \( c_j > 0 \) then
5: \( \bar{x}_j^t = 0 \)
6: else
7: \( \bar{x}_j^t = u_j \)
8: end if
9: end for
10: if \( c_i \leq 0 \) then
11: \( \bar{x}_i^t = u_i, \bar{y}_i^t = \frac{r}{u_i} \)
12: else if \( c_i > 0, d_i = 0 \) then
13: \( \bar{x}_i^t = 1, \bar{y}_i^t = r \)
14: else
15: \( \eta = \sqrt{\frac{rd_i}{c_i}} \)
16: if \( \eta \in \mathbb{N} \) and \( \eta \leq u_i \) then
17: \( \bar{x}_i^t = \eta \)
18: else if \( \eta < 1 \) then
19: \( \bar{x}_i^t = 1 \)
20: else if \( \eta > u_i \) then
21: \( \bar{x}_i^t = u_i \)
22: else
23: if \( c_i \lfloor \eta \rfloor + d_i \frac{r}{u_i} \leq c_i \lfloor \eta \rfloor + d_i \frac{r}{\lfloor \eta \rfloor} \) then
24: \( \bar{x}_i^t = \lfloor \eta \rfloor \)
25: else
26: \( \bar{x}_i^t = \lfloor \eta \rfloor \)
27: end if
28: end if
29: \( \bar{y}_i^t = \frac{r}{\bar{x}_i^t} \)
30: end if

Algorithm 5 Algorithm to solve $\min_{(x,y)\in S^c} c^T x + d^T y$

1: if One of the components of the vector $d$ is negative then
2: The problem is unbounded.
3: else if $c \leq 0, d = 0$ then
4: \( \left( u_1, \frac{r}{u_1}, u_2, 0, \ldots, u_n, 0 \right) \) is an extreme point optimal solution.
5: else ($d \geq 0$)
6: for $i = 1, \ldots, n$ do
7: Use Algorithm 4 to solve the problem $P_i$.
8: Let $(\bar{x}, \bar{y})^t$ be an optimal solution with optimal value $\zeta_i$.
9: end for
10: Find the minimum of $\zeta_i, i \in N$. Let $t \in N$ such that $\zeta_t \leq \zeta_i$ for all $i \in N$.
11: Then $(\bar{x}, \bar{y})^t$ is an optimal solution of $P$ with optimal value $\zeta_t$.
12: end if

Since the running time of the Algorithm 4 is $O(n)$ time, and finding the minimum among $\zeta_i, i \in N$ takes $O(n)$ time, therefore the Algorithm 5 runs in $O(n^2)$ time of the input size.