ON CREMONA TRANSFORMATIONS OF $\mathbb{P}^3$ WITH ALL POSSIBLE BIDEGREES

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ABSTRACT. For every orderer pair $(d, e)$ of integer numbers $d, e \geq 2$, such that $\sqrt{d} \leq e \leq d$, we construct a birational map $\mathbb{P}^3 \dashrightarrow \mathbb{P}^3$ defined by homogeneous polynomials of degree $d$ whose inverse map is defined by homogeneous polynomials of degree $e$.

1. INTRODUCTION

The aim of this note is to correct a mistake in the proof of Theorem [Pa2000-2, Théorème. 2.2]. The proof of that theorem depends on the example [Pa2000-2, Exemple 2.1] which is wrong.

We propose an explicit construction of Cremona transformations of $\mathbb{P}^3$ (see §2, especially Lemma [2]) which, together with their inverse maps, provide all possible bidegrees (Theorem 3 and Corollary 4).

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2. MAIN CONSTRUCTION AND RESULTS

Let $\mathbb{P}^3$ be the projective space over an algebraically closed field $k$ of characteristic zero; we fix homogeneous coordinates $w, x, y, z$ on $\mathbb{P}^3$.

We recall that a Cremona transformation of $\mathbb{P}^3$ is a birational map $F : \mathbb{P}^3 \dashrightarrow \mathbb{P}^3$. We say $F$ has bidegree $(d, e)$ when $F$ and its inverse $F^{-1}$ are defined by homogeneous polynomials, without non trivial common factors, of degrees $d$ and $e$ respectively; notice that in this case $F^{-1}$ has bidegree $(e, d)$. If $V \subset \mathbb{P}^3$ is a dense open set over which $F^{-1}$ is defined and injective and $L \subset \mathbb{P}^3$ is a line with $L \cap V \neq \emptyset$, then $e$ is the degree of the closure of $F^{-1}(L \cap V)$; one deduces that $\sqrt{d} \leq e \leq d$ (see for example [Pa2000-2, §1]).

If $X \subset \mathbb{P}^2$ is a curve and $p \in \mathbb{P}^2$ we denote by $\text{mult}_p(X)$ the multiplicity of $X$ at $p$. If $S, S' \subset \mathbb{P}^3$ are surfaces and $C \subset S \cap S'$ is an irreducible component, we denote by $\text{mult}_C(S, S')$ the intersection multiplicity of $S$ and $S'$ along $C$.

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Consider a rational map $T : \mathbb{P}^3 \dashrightarrow \mathbb{P}^3$ defined by

$$T = (g : qt_1 : qt_2 : qt_3),$$

where $t_1, t_2, t_3 \in k[x, y, z]$ are homogeneous of degree $r$, without non trivial common factors, and $g, q \in k[w, x, y, z]$ are homogeneous of degrees $d, d - 1$, with $d \geq r \geq 1$ and $g$ irreducible. We know that $T$ is birational if $\tau := (t_1 : t_2 : t_3) : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ is birational and $g, q$ vanish at $o = (1 : 0 : 0 : 0)$ with orders $d - 1$ and $\geq d - r - 1$, respectively (see [Pa2000-1 Proposition 2.2]).

On the other hand, consider $2r - 1$ points $p_0, p_1, \ldots, p_{2r-2}$ in $\mathbb{P}^2$, $r \geq 2$, satisfying the following condition:

There exist curves $X_r, Y_{r-1} \subset \mathbb{P}^2$ of degrees $r, r - 1$, respectively, with $X_r$ irreducible, such that $\text{mult}_{p_0}(X_r) = r - 1$, $\text{mult}_{p_0}(Y_{r-1}) \geq r - 2$ and $p_i \in (I)$ $X_r \cap Y_{r-1}$ for $i = 1, \ldots, 2r - 2$.

Hence loc. cit. also implies there exists a plane Cremona transformation defined by polynomials of degree $r$ with a point of multiplicity $r - 1$ at $p_0$ and passing through $p_1, \ldots, p_{2r-2}$ with multiplicity 1: indeed, if we consider $p_0 = (1 : 0 : 0)$ and take polynomials $t_1$ and $f$, of degrees $r$ and $r - 1$, defining $X_r$ and $Y_{r-1}$ respectively, then $(t_1 : yf : zf) : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ is a Cremona transformation as required; such a transformation is said to be associated to the points $p_0, p_1, \ldots, p_{2r-2}$.

Remark 1. The transformations satisfying the condition (I) are general cases of the so-called de Jonquières transformations (see [dJo1864] or [Alb2000] Def. 2.6.10]). We note that the Enriques criterion [Alb2000] Thm. 5.1.1 may be used to prove that a set of $2r - 2$ points $p_0, p_1, \ldots, p_{2r-2}$ with assigned multiplicities $r - 1, 1, \ldots, 1$, and satisfying the condition (I), defines a de Jonquières transformation.

Set $r = d$ and take an irreducible homogeneous polynomial $g = wA(x, y, z) + B(x, y, z)$ of degree $d$; that is, $g \in k - \{0\}$ in the considerations above. Denote by $T_{g, \tau}$ the Cremona transformation defined by

$$T_{g, \tau} = (g : t_1 : t_2 : t_3),$$

where $\tau = (t_1 : t_2 : t_3)$ is associated to $2d - 1$ points satisfying the condition (I).

We have

Lemma 2. Let $d \geq 2$ be an integer number. Then

(a) there exist $g$ and $\tau$ such that $T_{g, \tau}$ has bidegree $(d, 2d - 1 - m)$, for $0 \leq m \leq d - 1$.

(b) there exist $g$ and $\tau$ such that $T_{g, \tau}$ has bidegree $(d, d^2 - \ell^2 - m)$, for $0 \leq \ell < d - 1$ and $0 \leq m \leq 2d - 2$. 
Proof. We identify \( \mathbb{P}^2 \) with the plane \( \{ w = 0 \} \subset \mathbb{P}^3 \) and consider a point \( p_0 \in \mathbb{P}^2 \). Without loss of generality, we may suppose \( p_0 = (0 : 1 : 0 : 0) \). We recall \( o = (1 : 0 : 0 : 0) \).

In order to prove (a) we first choose \( g \in k[w, x, y, z] \) to be a homogeneous polynomial which vanishes along the line \( op_o \) with order \( d - 1 \) and is general with respect to this condition. In other words, one has \( g = wA + B \) with

\[
A = A_{d-1}(y, z), B = xB_{d-1}(y, z) + B_d(y, z),
\]

where \( A_i, B_i \in k[y, z] \) are general homogeneous polynomials of degree \( i \). Hence \( A = 0 \) defines a union of \( d - 1 \) distinct lines in \( \mathbb{P}^2 \) passing through \( p_0 \) and \( B = 0 \) defines an irreducible curve of degree \( d \) with an ordinary singular point of multiplicity \( d - 1 \) at \( p_0 \).

Notice that, by construction, in the open set \( \mathbb{P}^2 - \{ p_0 \} \) the curves \( A = 0 \) and \( B = 0 \) intersect at \( d(d - 1) - (d - 1)^2 = d - 1 \) points; in particular, if \( m \leq d - 1 \), there exist \( m \) points \( p_1, \ldots, p_m \in \mathbb{P}^2 \) satisfying \( A(p_i) = B(p_i) = 0 \) for \( 1 \leq i \leq m \). We consider \( m \) such points and choose \( 2d - 1 - m \) points \( p_{m+1}, \ldots, p_{2d-2} \in \mathbb{P}^2 \) with \( A(p_j) \neq 0 \) and \( B(p_j) = 0 \), for all \( j = m + 1, \ldots, 2d - 2 \), such that \( p_0, p_1, \ldots, p_{2d-2} \) satisfy (I). Let \( \tau \) be a plane Cremona transformation associated to these \( 2d - 1 \) points.

Now we consider a Cremona transformation \( T_{g, \tau} : \mathbb{P}^3 \dashrightarrow \mathbb{P}^3 \) as in (I). A general member in the linear system defining \( T_{g, \tau} \) is an irreducible surface of degree \( d \), \( S \) say, with equation of the form

\[
a g + a_1 t_1 + a_2 t_2 + a_3 t_3 = 0,
\]

where \( a, a_1, a_2, a_3 \in k \) are general. Therefore \( S \) admits an ordinary singularity of multiplicity \( d - 1 \) at the generic point of \( (\text{the line}) \, op_0 \) and is smooth at the generic point of \( op_i \) for \( 1 \leq i \leq m \). If \( S' \) is another general member of that linear system, then there exists an irreducible rational curve \( \Gamma \) of degree \( e = \deg(T_{g, \tau}^{-1}) \) such that the intersection scheme \( S \cap S' \) is supported on

\[
\Gamma \cup (\cup_{i=0}^m op_i).
\]

We have

\[
\text{mult}_\Gamma(S, S') = 1, \text{mult}_{op_0}(S, S') = (d - 1)^2, \text{mult}_{op_i}(S, S') = 1, i = 1, \ldots, m,
\]

hence \( e = d^2 - (d - 1)^2 - m = 2d - 1 - m \), which proves the assertion (a).

To prove (b) we proceed analogously. This time we choose \( g = wA + B \) with

\[
A = \sum_{i=\ell}^{d-1} x^{d-1-i} A_i(y, z), B = \sum_{j=\ell}^{d} x^{d-j} B_j(y, z),
\]

where \( A_i, B_i \in k[y, z] \) are general homogeneous polynomials of degree \( i \). Since \( \ell \leq d - 2 \) there exist points \( p_1, \ldots, p_{2d-2} \in \mathbb{P}^2 \) such that \( A(p_i) = B(p_i) = 0 \) for \( 1 \leq i \leq m \) and \( A(p_j) \neq 0 \) \( B(p_j) = 0 \) for \( j = m + 1, \ldots, 2d - 2 \); indeed, in the open set \( \mathbb{P}^2 - \{ p_0 \} \), the curves \( A = 0 \) and \( B = 0 \) intersect at \( d(d - 1) - \ell^2 \geq d(d - 1) - (d - 2)^2 = 3d - 4 \) points. Thus we can define \( \tau \) as before and obtain the assertion (b).
Theorem 3. There exist Cremona transformations of bidegree \((d, e)\) for \(d \leq e \leq d^2\).

Proof. From the part (a) of Lemma \([2]\) we deduce that there exist Cremona transformations of bidegrees \((d, e)\) for \(d \leq e \leq 2d - 1\).

Now we use the part (b) of Lemma \([2]\). Suppose \(\ell < d - 1\) and think of \(e = d^2 - \ell^2 - m\) as a function \(e(\ell, m)\) depending on \(\ell, m\); to complete the proof it suffices to show that the image of that function contains \(\{2d, 2d + 1, \ldots, d^2\}\).

We note that \(e(d - 2, 2d - 2) = 2d - 2\) and \(e(0, 0) = d^2\); in other words, the part (b) of Lemma \([2]\) implies that there exist Cremona transformations of bidegrees \((d, 2d - 2)\) and \((d, d^2)\). On the other hand \(e(\ell, 0) - e(\ell - 1, 2d - 2) = 2(d - \ell) - 1 > 0\). Since \(e(\ell, m)\) decreases with respect to \(m\), we easily obtain the result. \(\square\)

For \(d = 2\) the theorem above asserts that there exist Cremona transformations of bidegrees \((2, 2), (2, 3), (2, 4)\); analogously for \(d = 3\) and bidegrees \((3, 3), (3, 4), \ldots, (3, 9)\), and so on. By symmetry we deduce

Corollary 4. There exist Cremona transformations of bidegrees \((d, e)\) with \(\sqrt{d} \leq e \leq d^2\).

Remark 5. The inequality \(\sqrt{d} \leq e \leq d^2\) is the unique obstruction to the degree for the inverse of a Cremona transformation of degree \(d\) in \(\mathbb{P}^3\).

References

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