Complexity spectrum of some discrete dynamical systems

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We first study birational mappings generated by the composition of the matrix inversion and of a permutation of the entries of $3 \times 3$ matrices. We introduce a semi-numerical analysis which enables to compute the Arnold complexities for all the $9!$ possible birational transformations. These complexities correspond to a spectrum of eighteen algebraic values. We then drastically generalize these results, replacing permutations of the entries by homogeneous polynomial transformations of the entries possibly depending on many parameters. Again it is shown that the associated birational, or even rational, transformations yield algebraic values for their complexities.

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I. INTRODUCTION AND RECALLS

Birational transformations have been seen to be a powerful tool to analyze the symmetries of the parameter space of lattice models of statistical mechanics [1,2] and to seek for some possible Yang-Baxter integrability [3,4]. Beyond the lattice statistical mechanics framework, birational transformations are worthy to be studied per se, as discrete dynamical systems. Discrete dynamical systems have been intensively studied (see for example [5,6]). Among them polynomial examples, like the Henon map [7], have been precious to understand some features of chaos. Beyond, rational mappings are of special interest since they allow some analytical calculations. Furthermore, the rational transformations also allow numerical calculations which can be performed with any wanted precision: the existence of singularities in the rational transformations one iterates, and their possible “proliferation” is not in fact a numerical obstruction. We will first consider mappings generated by the composition of the matrix inverse and some arbitrary, but fixed, permutation of the entries of $q \times q$ matrices. The results, displayed in this paper, are given for $q = 3$, but are actually valid, mutatis mutandis [2], for arbitrary $q$ values.

A. Recalling a previous $3 \times 3$ analysis

Integrability of a mapping amounts to saying that all the orbits of the iteration correspond to elliptic, or rational, algebraic curves [8,9]. From the point of view of the growth of the complexity of the successive iterations [8,9], such integrability in curves always yields a polynomial growth of the calculations [10,11], instead of the exponential growth one generically expects. Conversely polynomial growth is not restricted to integrability in curves but may correspond to orbits “densifying” Abelian varieties [8,9].

1When one iterates a rational transformations the “size” of the successive rational expressions, corresponding to the $N$-th iterate, grows, in general, exponentially. In particular the degree of these successive rational expressions has, generically, an exponential growth [10,11]. Growth of the calculations related with factorizations were also introduced by Veselov for some particular Cremona transformations [10,12].
A first exhaustive analysis of all the 9! birational transformations generated by the composition of the matrix inversion and of a (fixed) permutation of the entries of $3 \times 3$ matrices has already been performed concentrating on the extraction of integrable mappings \([4]\). This analysis was exhaustive, but restricted to particular integrability criteria\(^2\). Even from this “integrability-digger” point of view some integrable mappings are missing (for example the so-called “Class III” mappings of \([7]\), as well as some polynomial growth situations). In the first part of this paper we will revisit these 9! = 362880 birational mappings without any \textit{a priori} integrability criterion and with the help of a new equivalence relation among permutations (symmetry). This analysis exactly yields all the polynomial growth situations, and, far beyond, \textit{classifies} the exponential growth situations. The classification relies on the value of the \textit{Arnold complexity} \([6]\) of the mapping. This complexity can be obtained \([17]\) from generating functions associated with factorization schemes \([3]\) detailed below.

### B. Factorization scheme and generating functions

We use the same notations as in \([3,4,6]\), that is, we introduce the following two transformations, the usual matrix inverse \(\hat{I}\) and the \textit{homogeneous} matrix inverse \(I\):

\[
\hat{I} : M_0 \rightarrow M_0^{-1}, \quad \text{and} \quad I : M_0 \rightarrow \det(M_0) \cdot M_0^{-1}
\]  
(1)

The homogeneous inverse \(I\) is a homogeneous polynomial transformation, which associates, with each entry of \(M_0\), its corresponding cofactor. Transformation \(t\) is any (fixed) permutation of the entries of the \(3 \times 3\) matrix. We also introduce the (generically infinite order) transformations:

\[
K = t \cdot I \quad \text{and} \quad \hat{K} = t \cdot \hat{I}
\]  
(2)

Transformation \(\hat{K}\) is clearly a \textit{birational transformation} \([3,6]\).

For all the various birational transformations associated with permutations of the entries of \(3 \times 3\) matrices, the following factorization relations happen to occur \textit{at each iteration step} \(\text{\footnote{Associated with particular recursions \([14]\)!}}\):

\[
f_1 = \det(M_0), \quad M_1 = K(M_0), \quad f_2 = \frac{\det(M_1)}{f_1^{n_1}}, \quad M_2 = \frac{K(M_1)}{f_1^{n_1}}, \quad f_3 = \frac{\det(M_2)}{f_1^{n_1} f_2^{n_2}}, \quad M_3 = \frac{K(M_2)}{f_1^{n_1} f_2^{n_2}}
\]

and for arbitrary \(n\):

\[
\det(M_n) = f_{n+1} \cdot f_{n}^{\rho_1} \cdot f_{n-1}^{\rho_2} \cdot f_{n-2}^{\rho_3} \cdot f_{n-3}^{\rho_4} \cdots f_1^{\rho_n} \tag{3}
\]

\[
K(M_n) = M_{n+1} \cdot f_{n+1}^{\eta_1} \cdot f_{n}^{\eta_2} \cdot f_{n-1}^{\eta_3} \cdot f_{n-2}^{\eta_4} \cdots f_1^{\eta_n} \tag{4}
\]

\[
\det(M_n) \cdot M_{n+1} = (f_{n+1}^{\eta_1} f_{n+1}^{\phi_1} f_{n}^{\phi_2} f_{n-1}^{\phi_3} f_{n-2}^{\phi_4} \cdots f_1^{\phi_n}) \cdot K(M_n) \tag{5}
\]

defining the positive integer exponents \(\eta_n, \phi_n\) and \(\rho_n\). The \(f_n\)’s are homogeneous polynomials of the entries of \(M_0\). These factorizations allow to define, at each iteration step, the successive \(S\) and \(\rho\) exponents \(\alpha_n, \beta_n, \phi_n\)

We will denote \(\alpha_n\), the degree of the determinant of matrix \(M_n\), and \(\beta_n\), the degree of polynomial \(f_n\) and \(\alpha(x), \beta(x), \eta(x), \rho(x)\) and \(\phi(x)\), the generating functions of the degrees \(\alpha_n\)’s, \(\beta_n\)’s, and of the exponents \(\eta_n\)’s, \(\rho_n\)’s and \(\phi_n\)’s in the factorization schemes:

\[
\alpha(x) = \sum_{n=0}^{\infty} \alpha_n \cdot x^n, \quad \beta(x) = \sum_{n=1}^{\infty} \beta_n \cdot x^n, \quad \eta(x) = \sum_{n=0}^{\infty} \eta_n \cdot x^n, \quad \phi(x) = \sum_{n=1}^{\infty} \phi_n \cdot x^n, \quad \rho(x) = \sum_{n=0}^{\infty} \rho_n \cdot x^n
\]

\footnote{In fact it is shown in \([7]\)! that other slightly more general factorizations scheme can occur on some \(K\)-invariant subvarieties (yielding smaller Arnold complexity values). Such slightly more general factorizations scheme will also be detailed below (see Appendix B). However for the transformations \(\hat{K}\) associated with permutations of \(q \times q\) matrices, for a generic initial matrix, one gets factorization schemes like \([3,4]\), also depicted in \([3]\).}
where \( \alpha_0 = 3 \) and \( \beta_1 = 3 \). It is straightforward to show 2 that the existence of the stable factorization scheme 3, 4 yields the following simple linear relations between these various “degree generating functions” and “exponents generating functions”:

\[
\begin{align*}
\alpha(x) + 3 \cdot x \cdot \eta(x) \cdot \beta(x) &= 3 + 2 \cdot x \cdot \alpha(x) \\
x \cdot \alpha(x) &= \phi(x) \cdot \beta(x) \\
3 + 3 \cdot \rho(x) \cdot \beta(x) &= (1 + x) \cdot \alpha(x)
\end{align*}
\]

(6) (7) (8)

When analytically iterating an arbitrary transformation \( K \), the degree of the successive polynomial expressions one encounters, grow exponentially: \( \alpha_n \) or \( \beta_n \approx \lambda^n \). \( \lambda \) measures the grows of the calculations and identifies with the notion of Arnold complexity 16,17. From now on \( \lambda \) will be called the complexity. When the degree generating functions \( \alpha(x) \) or \( \beta(x) \) happen to be rational functions, the complexity \( \lambda \) is obviously the inverse of the pole of smallest modulus. Recalling the “determinantal” variables 3,15 \( x_n \)’s defined by:

\[
x_n(M_0) = \det(\tilde{K}^{n+1}(M_0)) \cdot \det(\tilde{K}^n(M_0))
\]

(9)

one finds out that these determinantal variables happen to decompose on a product of the homogeneous polynomials \( f_n \)'s only:

\[
x_n(M_0) = f_{n+1} \cdot f_n \cdot f_{n-1} \cdot f_{n-2} \cdot f_{n-3} \cdot f_{n-4} \cdots
\]

(10)

which defines some, at first sight, “new” exponents \( \rho \)'s and consequently a, at first sight, “new” generating function \( W(x) \):

\[
W(x) = \sum_{n=0}^{\infty} w_n \cdot x^n
\]

(11)

It is worth noticing that the determinantal variables \( x_n \)'s induce the homogeneous polynomials \( f_n \)'s emerging from the factorization schemes 3, 4 and no other homogeneous polynomials. The variables \( x_n \)'s are well-suited since they are invariant under a multiplication of \( M_0 \) by a constant: \( M_0 \rightarrow \text{Cst} \cdot M_0 \). In other words the \( x_n \)'s are homogeneous expressions of degree zero. Concentrating on the degrees of the left-hand side, and right-hand side, of (10), one gets the following “degree equation”:

\[
0 = \beta_{n+1} \cdot w_0 + \beta_n \cdot w_1 + \cdots + \beta_{n-p} \cdot w_{p+1} + \cdots + \beta_1 \cdot w_n
\]

(12)

from which one immediately deduces the simple functional equation:

\[
W(x) \cdot \beta(x) = \beta_1 \cdot w_0 = 3 \cdot x
\]

(13)

This result is immediately generalized to \( q \times q \) matrices. Relation 13 becomes \( W(x) \cdot \beta(x) = q \cdot x \). From (13) one actually sees that \( W(x) \) is not a new generating function: it is simply related to the degree generating function \( \beta(x) \). The complexity \( \lambda \) is associated to the zeroes of \( W(x) \).

From relations 3, 6, and 7, one easily gets the “degree generating functions” \( \alpha(x) \) and \( \beta(x) \) from two of the “exponent generating functions” (for instance \( \phi(x) \) and \( \eta(x) \) or \( \eta(x) \) and \( \rho(x) \)). As a matter of fact, for most of the permutations, the factorization schemes are periodic \( (\eta_n = \eta_{n+N}, \phi_n = \phi_{n+N} \text{ and } \rho_n = \rho_{n+N} \text{ for some integer } N) \). Consequently, the exponent generating functions \( \phi(x) \) and \( \eta(x) \), or \( \rho(x) \), are rational functions with \( N \)-th root of unity poles 8 (see, for instance, the exponent generating function \( \rho(x) \) in 13 or 16). In a second step one deduces, from relations 3, or 7, rational expressions for the degree generating functions \( \alpha(x) \) and \( \beta(x) \). However it will be seen below that, for some permutations, the factorization schemes are still regular, but not with periodic exponents (see the exponent generating function \( \rho(x) \) in 13 or 17): the exponents \( \eta_n, \phi_n \text{ and } \rho_n \) grow exponentially, but one remarks that the associated generating functions \( \eta(x), \phi(x), \rho(x) \) are still rational, and thus \( \alpha(x) \) and \( \beta(x) \) are also rational. The exponent generating functions can be seen as an “encoding” of the degree generating functions, and thus of the complexity \( \lambda \). Remark that all these rational expressions involve integer coefficients, yielding algebraic values for their poles and for the growth of the calculations: the degrees of the successive rational expressions, namely

\[\text{The } w_n \text{'s are relative integers and not natural integers like exponents } \eta_n, \phi_n \text{ and } \rho_n.\]
$\alpha_n$’s and $\beta_n$’s grow like $\lambda^n$, where $\lambda$ is an algebraic number, and for regular factorization schemes (like (15) or (17), see below), the exponents $n$ and $f_n$ grow like $\mu^n$, where $\mu$ is the “scheme complexity”. Note that $\mu$ is obviously such that $\mu \leq \lambda$. Exponent $\mu$ is also the inverse of the pole of smallest modulus of the exponent generating functions. Complexity $\lambda$ allows all kinds of handy, efficient, and formal, or semi-numerical, calculations. We will present below such a semi-numerical method and apply it to all the permutations of entries of $3 \times 3$ matrices.

II. COMPLEXITY SPECTRUM ANALYSIS FOR PERMUTATIONS

A. A semi-numerical method

All these considerations allow us to design a semi-numerical method to get the value of the complexity $\lambda$ for the iteration of rational transformations. The idea is to iterate, with $K$, a generic initial matrix with integer entries. After one iteration step the entries become rational and we follow the magnitude of the successive numerators and denominators. During the first few iteration steps some “accidental” simplifications may occur, but, after this transient regime, the integer denominators (for instance) grow like $\lambda^n$, and for regular factorization schemes (like (15) or (17)), one chooses the initial matrix $M_0$ with integer entries such that the determinant, and most of its cofactors, are prime numbers as large as possible. One may impose further constraints on the initial matrices, for instance, that the first homogeneous polynomials $f_2$ and $f_3$ are also, as large as possible, prime numbers. These conditions down-size the probability that all the entries of the reduced matrices $M_n$, or the polynomials $f_n$’s, could be divisible by some accidental additional $f_1$, $f_2$ or $f_3$. Such initial matrices, well-suited for the iteration of the homogeneous transformation $K$, are also well-suited for the iteration of the (bi-)rational transformation $\hat{K}$. In practice we start with a set of initial matrices and keep only the one for which the less factorizations occur (non-generic factorization can only correspond to additional factorizations).

The computations are done using an infinite precision C-library [8]. We perform as many iterations as possible during a given CPU time $T$. This number of iterations, $n$, is such that $T \approx \lambda^n$. For $\lambda$ close to 2 and $T = 60$ seconds, $n$ is of the order of twenty and a best fit of the logarithm of the numerator as a linear function of $n$, between $n = 10$ and $n = 20$, gives the value of $\lambda$ within an accuracy of 0.1%. For smaller values of $\lambda$ (typically $\lambda < 1.5$) the number of iterations is larger, but the accuracy, for a given CPU time, is smaller. In such “difficult” cases one analytically finds the factorizations up to $n = 7$ and implement the first steps of these factorization schemes in the semi-numerical method. We are then almost guaranteed that no accidental factorizations will occur for $n > 7$, and therefore we can average over many initial matrices. Even so it remains difficult to discriminate between a truly polynomial growth [16] ($\lambda = 1$) and an exponential growth with $\lambda \simeq 1$. The complexity values close to one clearly need to be revisited by other methods we present below.

B. Equivalence relations between permutations

Even if this semi-numerical algorithm is efficient it is quite time consuming to use it directly on the $9!$ permutations. To classify the complexities associated to a large set of (birational) transformations like the one associated to the $9!$ permutations of $3 \times 3$ matrices, one certainly needs to reduce this set as much as possible. For instance one can try to find symmetries such that two permutations, related by the symmetry, yield the same complexity $\lambda$. These symmetries allow to build equivalence classes and, thus, to restrict the exhaustive analysis to a only one representant in each class. Furthermore one may have the prejudice that any non trivial symmetry could enable to explain a possible integrability structure of the mappings and beyond, the structures associated with the classification of the Arnold complexity of these mappings.

There actually exist quite trivial symmetries, corresponding to relabeling of rows and columns [13], for which the complexities of the associated $K$’s are obviously equal. It is possible to go a step further and define a set of equivalence relations $R^{(n)}$ between the permutations, yielding new equivalence classes such that any two permutations in the same “new” equivalence class, $R^{(n)}$, automatically have the same complexity $\lambda$. Equivalence relation $R^{(n)}$ amounts to saying

\[ However one should keep in mind that there is nothing specific with $3 \times 3$ matrices. These results simply generalize to $q \times q$ matrices (see for instance [4]).
that two equivalent permutations are such that the $n$-th power of their associated transformations $\widehat{K}$ are conjugated (via particular permutations, product of row permutations, column permutations and possibly the transposition, see appendix A for more details). An exhaustive inspection has shown that the equivalence relations $R^{(n)}$’s “saturate” after $n = 24$: with obvious notations $R^{(\infty)} = R^{(24)}$. One finds out that the “ultimate” $R^{(\infty)}$ equivalence classes can only have 72, or 144, elements. Among the “ultimate” $R^{(\infty)}$ classes, one wants to distinguish between the classes that were already $R^{(1)}$ classes, that we will denote from now on $R_{72}^{(1)}$, or $R_{144}^{(1)}$, according to their number of elements, and the other ones we denote $R_{72}^{(\infty)}$ or $R_{144}^{(\infty)}$. Being an $R^{(\infty)}$ equivalence class which does not reduce to a $R^{(1)}$ equivalence class, means the existence of several non trivial relations between the permutations in the $R^{(\infty)}$ equivalence class (see (A3) in appendix A). This implies strong constraints on the respective orbits. One thus expects more properties, and structures, inherited from this fact. The 362880 permutations are grouped into 2880 equivalence classes (instead of 30462 “relabeling” equivalence classes in [14]). In Tab. I the number of the respective elements , and the other ones we denote $R_{72}^{(\infty)}$, or $R_{144}^{(\infty)}$. Being an $R^{(\infty)}$ equivalence class which does not reduce to a $R^{(1)}$ equivalence class, means the existence of several non trivial relations between the permutations in the $R^{(\infty)}$ equivalence class (see [A3] in appendix A). This implies strong constraints on the respective orbits. One thus expects more properties, and structures, inherited from this fact. The 362880 permutations are grouped into 2880 equivalence classes (instead of 30462 “relabeling” equivalence classes in [14]). In Tab. I the number of the respective $R_{72}^{(1)}$, $R_{144}^{(1)}$, $R_{72}^{(\infty)}$ or $R_{144}^{(\infty)}$ classes is displayed. Since the complexities do not depend on the chosen representant, we picked a representant in each $R^{(\infty)}$ class and performed, for it, the semi-numerical method previously explained.

For $3 \times 3$ matrices, the complexities are necessarily such that: $2 \geq \lambda \geq 1$. Remarkably, instead of getting a quite complicated distribution, or spectrum, of values for the complexities, we have obtained values which are always very close, up to the accuracy of the method, to a set of seventeen values given in the left column of Tab. I (see below) and, of course, the integrable value $\lambda = 1$. To test the accuracy of the method we got complexities for two representants of the same class (that should, as we know, have the same complexity value exactly). We always obtained an equality of the corresponding complexities, up to an error of $10^{-3}$. This accuracy is however not always sufficient enough to discriminate between some complexities displayed in the left column of Tab. I. In order to fix our mind it is necessary to obtain the exact expressions of these complexity values, for instance by getting the factorization scheme (3), (4), and thus the generating functions $\alpha(x)$ and $\beta(x)$.

C. Revisiting the complexity spectrum via exact factorization schemes

For most of the $R^{(\infty)}$ equivalence classes (2832 out of 2880), the complexity values, obtained with our semi-numerical method, are extremely close to the upper limit $\lambda = 2$. In fact one can figure out that these complexity values are actually exactly equal to 2. Therefore we can focus on the analysis of the remaining 48 classes, finding systematically their factorization schemes and associated generating functions. We actually found these factorization schemes and the associated generating functions, and were actually able to see that the previous numerical spectrum exactly corresponds to eighteen algebraic values listed in Tab. I. Among these eighteen algebraic values, let us take four illustrative examples. We give for each example, the permutation representing the $R^{(\infty)}$ equivalence class, the value of $\lambda$, and $\mu$, defined in section 1B, the expressions of $\beta(x)$ and $\rho(x)$, since they respectively correspond to the simplest “degree generating function” and “exponent generating function”. The other generating functions can be deduced from these two, using linear functional relations (6), (7) and (8) between the generating functions $\alpha(x)$. Furthermore relation (10) remains valid for all the factorization schemes associated with all the various permutations studied here. We first give the permutation $t$ itself, using the notation, already used in [3], where $p_{0}p_{1}p_{2}p_{3}p_{4}p_{5}p_{6}p_{7}p_{8}$ means that $(tM)_{i} = M_{p_{i}}$, the entries of the matrix being enumerated consecutively (i.e. $M_{11} = M_{0}$, $M_{12} = M_{1}$, $M_{13} = M_{2}$, $M_{21} = M_{3}$, · · · , $M_{53} = M_{8}$).

• First example. Permutation 407326518 yields $\lambda \simeq 1.61803\cdots$ and $\mu = 1$ and :

$$\frac{\beta(x)}{3x} = \frac{1 - x^{2}}{1 - x - x^{2}} , \quad \rho(x) = \frac{1}{(1 - x)^{2} \cdot (1 + x)}$$  \hspace{1cm} (14)

• Second example. Permutation 417063582 yields $\lambda \simeq 1.83928\cdots$ and $\mu \simeq 1.32471\cdots$ and :

$$\frac{\beta(x)}{3x} = \frac{1 - x^{2} - x^{3}}{(1 - x)^{2} \cdot (1 + x) \cdot (1 - x - x^{2} - x^{3})} , \quad \rho(x) = \frac{(1 + x) \cdot (1 - x + x^{4})}{1 - x^{2} - x^{3}}$$  \hspace{1cm} (15)

• Third example. Permutation 164273085 yields $\lambda \simeq 1.83928\cdots$ and $\mu = 1$ and :

$$\frac{\beta(x)}{3x} = \frac{1 + x + x^{2}}{1 - x - x^{2} - x^{3}} , \quad \rho(x) = \frac{1 + x^{3} + x^{4} + x^{5}}{1 - x^{6}}$$  \hspace{1cm} (16)

• Fourth example. Permutation 174528603 yields $\lambda \simeq 1.97458\cdots$ and $\mu \simeq 1.32471\cdots$ and :
\[ \beta(x) = \frac{\lambda}{3} = \frac{1 - x^2 - x^3}{(1 - x) \cdot (1 - x - 2 x^2 - x^3 + x^4 + 2 x^5 + x^6)} \], \quad \rho(x) = \frac{1 - x + x^7 + x^8}{(1 - x + x^2) \cdot (1 - x^2 - x^3)} \] (17)

The exhaustive analysis of the factorization schemes, and the associated degree, and exponent, generating functions \((\alpha(x), \beta(x), \eta(x), \phi(x) \text{ and } \rho(x))\), confirms that the complexities are actually independent of the representent in the equivalence class. On the contrary, the factorization schemes and the associated degree, and exponent, generating functions may depend on the chosen representent in the equivalence class. In other words to two permutations in the same class of equivalence correspond the same (up to \(1 - x\) or \(1 + x\), or \(N\)-th root of unity factors) denominators for the degree generating functions \(\alpha(x)\), \(\beta(x)\). By contrast the numerators, as well as the exponents generating functions are representent dependent (see the previous four examples). Most of time the stability of the factorization scheme and thus, in a second step, the occurrence of rational generating functions, corresponds to a simple periodicity of the exponents \(\eta_n, \phi_n\) or \(\rho_n\) in the factorization scheme \((\underline{5}), (\underline{4})\). This periodicity is simply associated to the fact that the exponent generating functions have \(N\)-th root of unity poles : \(1 - x^2, 1 - x^3, 1 - x^5, \cdots\) (see \(\rho(x)\) in \((\underline{16})\)). However one sees, on examples \((\underline{15})\) and \((\underline{17})\), that one may have a stability of the factorization scheme with an exponential growth of these exponents \(\eta_n\) and \(\phi_n\). These exponent generating functions, of course, have a “scheme complexity” \(\mu\) smaller that the growth complexity \(\lambda\). This “scheme complexity” \(\mu\) is the inverse of the poles of \(\rho(x), \phi(x)\) or \(\eta(x)\), that is (for \((\underline{17})\)) \(\mu \simeq 1.32471 \cdots \leq \lambda \simeq 1.83928 \cdots\). Recalling \((\underline{16})\), for which \(\mu = 1\) and \(\lambda \simeq 1.8392 \cdots\), and \((\underline{15})\), one sees that the same growth complexity \(\lambda\) can be associated to several “scheme-complexity” \(\mu\). Conversely, comparing the fourth example \((\underline{17})\) and the second example \((\underline{15})\), one sees that one “scheme-complexity” \(\mu\) can actually yield several growth complexities \(\lambda\).

D. To sum up

All these factorization scheme calculations confirm the results of the semi-numerical method and are summarized in Tab. \(\underline{1}\). Most of the 362880 birational transformations considered here do correspond to the most “chaotic complexity”, namely the upper bound \(\lambda = 2\) : one has 359568 such \(\lambda = 2\) birational transformations, that is 99.0873 \% of all the birational transformations. It is known \((\underline{14})\), that some symmetry-classes correspond to situations where the determinantal variables \(x_n\)’s, defined by \((\underline{4})\), are periodic (denoted “Period.” in the Table \(\underline{1}\)). This \(x_n = x_{n+N}\) situation may correspond to situations where mapping \(\hat{K}\) itself, is of finite order (trivial integrability), but also to polynomial growth situations, that is, \(\lambda = 1\) exactly. One remarks that \(R_{72}^{(\infty)}\) contains all the integrable, or polynomial growth, mappings and, up to one class in \(R_{72}^{(1)}\), all the mappings such that \(x_n = x_{n+N}\), including the situations where mapping \(\hat{K}\), itself, is of finite order.

III. VARIOUS GENERALIZATIONS

We now show that all these results also apply for a much larger set of rational transformations. The number of permutations of entries of \(3 \times 3\) matrices being finite it has been possible to perform an exhaustive analysis. For more general transformations, depending on continuous parameters, is not anymore possible and we will proceed just with chosen examples. These examples always combine homogeneous transformations of the entries of a matrix together with the matrix inversion. Therefore the transmutation relations detailed in appendix A still apply, yielding again non-trivial symmetries for these new set of transformations.

A. Combining different \(K\)’s

Let us first consider permutation 146237058, and its associated \(\lambda \simeq 1.97481 \cdots\) transformation \(K_1\), and permutation 471562380 and its \(\lambda \simeq 1.54258 \cdots\) transformation \(K_2\). Let us compose the two previous transformations.

\(^6\)Considering one \(R_{72}^{(\infty)}\) equivalence class, one does not get as many factorization schemes as the number of elements in the equivalence class. It seems, inspecting directly all the 9! factorization schemes (but only up to twelve iteration steps), that, most of the time, one gets, at most, two possible factorization schemes for a given \(R_{72}^{(\infty)}\) equivalence class, and that the set of all the possible factorization schemes would be twenty one (besides the polynomial growth situations which can be quite “rich”).
From these two “atoms” we build the “molecule” \( K = K_2 \cdot K_1 \). Note that \( K = K_1 \cdot K_2 \), obviously has the same complexity.

This example is an interesting one since the complexity (obtained from the previous semi-numerical calculations) of the “molecule” \( K = K_2 \cdot K_1 \) is smaller than the product of the two complexities of \( K_1 \) and \( K_2 \): \( \lambda(K) \approx 2.897 < 1.9748 \cdot 1.5426 \approx 3.0463 \). In general the combination of two complexities \( \lambda_1 \) and \( \lambda_2 \) gives a complexity for the “molecules” larger than the product \( \lambda_1 \cdot \lambda_2 \), often equal to the upper bound (here \( \lambda_{\text{upper}} = 4 \)).

The factorization scheme of \( K \) is of the same type as the one described in \[17\], namely a “parity-dependent” factorization scheme. It is detailed in appendix (B) and yields a degree generating function \( \beta(x) \):

\[
\beta(x) = \frac{1 + 2 x - x^2 - x^4 + x^6}{1 - 3 x^2 + x^4 - x^6 - 2 x^8}
\]

The complexity of the molecule \( K \) does not identify with the complexity of \( K_1 \), or the one of \( K_2 \). It is a true new algebraic number. This algebraic expression for the complexity of the molecule is in good agreement with the semi-numerical value obtained above. We have systematically studied such “molecules” for a choice of eighteen representatives of the eighteen complexities of table I combined with themselves, and beyond, with other representatives. If one barter the permutation \( t_2 \) for another representent \( t_2^{(2)} \) in the same \( R^{(\infty)} \) class, transformation \( K_2 \) being modified accordingly (\( K_2 \to K_2^{(2)} \)), the new “molecule” \( K^{(2)} = K_2^{(2)} \cdot K_1 \) yields, in general, another algebraic value for the complexity \( \lambda \): the equivalence relation \( R^{(\infty)} \) is no longer compatible with the “molecular structure”.

For all these “molecules” the parity-dependent factorization scheme, yields algebraic numbers for the complexities of these molecules in agreement with the values obtained from our semi-numerical now applied for the “molecules”. Combining among themselves all the permutations yields a large number of different algebraic complexities, much larger than the number of complexities obtained combining only representatives of the \( R^{(\infty)} \) classes among themselves.

| \( \lambda \) | Polynomial | \( R^{(1)}_{144} \) | \( R^{(1)}_{12} \) | \( R^{(\infty)}_{144} \) | \( R^{(\infty)}_{12} \) | Total |
|---|---|---|---|---|---|---|
| 2 | \( 1 - 2x \) | 2146 | 660 | 14 | 60 | 2880 |
| 1.97481871 | \( 1 - 2x + x^2 - 2x^4 + x^6 - 2x^7 + x^8 \) | 0 | 2 | 0 | 0 | 2 |
| 1.974584654 | \( 1 - 2x + x^2 - x^4 + 2x^6 \) | 0 | 1 | 0 | 0 | 1 |
| 1.94893574 | \( 1 - 2x + x^3 - x^5 \) | 0 | 2 | 0 | 0 | 2 |
| 1.946856268 | \( 1 - x - x^2 + x^6 - x^8 - x^{10} \) | 0 | 1 | 0 | 0 | 1 |
| 1.93318498 | \( 1 - 2x + x^3 - x^5 \) | 0 | 1 | 0 | 0 | 1 |
| 1.89110302 | \( 1 - 2x + x^2 - 2x^3 + 2x^5 - 2x^7 \) | 0 | 0 | 0 | 1 | 1 |
| 1.88320350 | \( 1 - 2x + x^2 - 2x^4 + x^6 \) | 0 | 2 | 0 | 0 | 8 |
| 1.866760399 | \( 1 - 2x + x^3 - x^5 \) | 0 | 1 | 0 | 0 | 1 |
| 1.85712754 | \( 1 - 13x + x^3 - x^5 - 2x^7 \) | 0 | 1 | 0 | 0 | 1 |
| 1.83926765 | \( 1 - x - x^2 - x^3 \) | 0 | 2 | 0 | 0 | 2 |
| 1.75487766 | \( 1 - 2x + x^3 - x^5 \) | 1 | 0 | 0 | 0 | 1 |
| 1.61803399 | \( 1 - x^2 \) | 0 | 3 | 0 | 0 | 3 |
| 1.57014734 | \( 1 - x - x^2 - x^3 \) | 0 | 1 | 0 | 0 | 1 |
| 1.54279599 | \( 1 - x - x^2 - x^3 - x^6 \) | 0 | 1 | 0 | 0 | 1 |
| 1.4655123 | \( 1 + x^5 \) | 0 | 0 | 0 | 0 | 2 |
| 1 (Pol.gr.) | \( -x, -1 + x^5, \cdots \) | 0 | 0 | 0 | 9 | 9 |
| 1 (Periodic) | \( 1/x, 1/x^5, \cdots \) | 0 | 1 | 0 | 0 | 10 |

**TABLE I.**

B. From permutations to linear transformations

We get algebraic results on birational mappings associated with permutation of the entries. We now address the following question: are these structures (existence of a stable factorization scheme) dependent of the fact that we are dealing with permutations? In other terms, does one lose these algebraic properties when deforming the permutations in most general transformations? The most simple, and natural, generalization amounts to replacing the permutation of the entries by linear combination on the entries.

Let us now consider a first example, namely the quite general linear transformation depending on twenty one parameters:
complexity growth crucially dependent on the $K_1$ transposition is not isolated in the twenty-one parameters set of transformations. Actually if the parameters verify twenty one parameters leads to a permutation of any of the three classes corresponding to additional factorizations, the modified factorization scheme yielding

$$
\lambda
$$

These results are actually valid for any “sufficiently generic” choice of the twenty one parameters. One thus has a first “universality” property : the complexity $\lambda$ is “generically” not dependent of the previous twenty one parameters. Furthermore relation (19) (and consequently relation (13)) remains also valid for all the factorization schemes associated with all the linear transformations studied in this section. Complexity $\lambda \simeq 1.61803 \cdots$ (corresponding to polynomial $1-x-x^2$) is a complexity value already found in table I in the sixteenth row. It is noteworthy that no choice of the twenty one parameters leads to a permutation of any of the $9!$ permutations.

We now give another eleven parameter example associated with the following linear transformation :

$$
L : \begin{bmatrix}
m_{1,1} & m_{1,2} & m_{1,3} \\
m_{2,1} & m_{2,2} & m_{2,3} \\
m_{3,1} & m_{3,2} & m_{3,3}
\end{bmatrix} \rightarrow \begin{bmatrix}
m_{1,1} & m_{1,2} + b_1 m_{2,1} + b_2 m_{2,2} + b_3 m_{2,3} & m_{1,3} \\
m_{2,1} & a_{12} m_{1,2} + a_{21} m_{2,1} + a_{22} m_{2,2} + a_{23} m_{2,3} & a_{23} m_{3,2} m_{2,3} \\
m_{3,1} & m_{32} + c_1 m_{2,1} + c_2 m_{2,2} + c_3 m_{2,3} & m_{3,3}
\end{bmatrix}
$$

This particular form singles out the rows of the $3 \times 3$ matrix (and thus can be understood as an RCT-compatible form, see appendix A). Similarly to the previous paragraphs let us introduce the homogeneous transformation $K = L \cdot I$. Factorizations again occur at each iteration step. These factorizations correspond to a stable factorization scheme giving a growth like $\lambda^N$, where $\lambda \simeq 1.61803 \cdots$. It is of the general type described in (8) and (9). This yields the following generating functions :

$$
\beta(x) = \frac{1}{3x} (1-x-x^2), \quad \rho(x) = \frac{1}{1-x}
$$

These results are actually valid for any “sufficiently generic” choice of the twenty one parameters. One thus has a first “universality” property : the complexity $\lambda$ is “generically” not dependent of the previous twenty one parameters. Furthermore relation (19) (and consequently relation (13)) remains also valid for all the factorization schemes associated with all the linear transformations studied in this section. Complexity $\lambda \simeq 1.61803 \cdots$ (corresponding to polynomial $1-x-x^2$) is a complexity value already found in table I in the sixteenth row. It is noteworthy that no choice of the twenty one parameters leads to a permutation of any of the $9!$ permutations.

We now give another eleven parameter example associated with the following linear transformation :

$$
L : \begin{bmatrix}
m_{1,1} & m_{1,2} & m_{1,3} \\
m_{2,1} & m_{2,2} & m_{2,3} \\
m_{3,1} & m_{3,2} & m_{3,3}
\end{bmatrix} \rightarrow \begin{bmatrix}
m_{1,1} & m_{1,2} + b_1 m_{2,1} + b_2 m_{2,2} + b_3 m_{2,3} & m_{1,3} \\
m_{2,1} & a_{12} m_{1,2} + a_{21} m_{2,1} + a_{22} m_{2,2} + a_{23} m_{2,3} & a_{23} m_{3,2} m_{2,3} \\
m_{3,1} & m_{32} + c_1 m_{2,1} + c_2 m_{2,2} + c_3 m_{2,3} & m_{3,3}
\end{bmatrix}
$$

For $K = L \cdot I$ the corresponding generating functions are :

$$
\beta(x) = \frac{1}{3x} (1-x-x^2), \quad \rho(x) = \frac{1}{1-x}
$$

The numerator of $\beta(x)$ does not appear in table Tab. I: this mapping has a new value for the complexity $\lambda \simeq 1.72088 \cdots$, not previously obtained for any of the $9!$ permutations.

Family (22), depending on eleven continuous parameters, also enables to address the following problem : is the complexity growth crucially dependent on the reversible character [14] of the transformations? In fact one may lose the birational character of $K$ when, for instance, the linear transformation $L$ becomes singular. This is very easy to realize for some condition on the eleven parameters (codimension one subvariety). For instance, taking $b_{22} = 2$, $a_{22} = 87$,
\[ a_{12} = 5, \ a_{32} = 7, \ c_{22} = 11, \] all the other parameters being zero, leads to a non invertible mapping \( K = L \cdot I \).

One easily verifies that the factorization scheme, the associated generating functions and thus the complexity \( \lambda \) are unchanged in this case and, more generally, on such singular subvarieties. With this first rational, non invertible, example one sees that the rational character of the generating functions is not a consequence of a “simple” invertibility of the mapping (see also (3)).

C. From linear transformations to homogeneous polynomial transformations

There is nothing specific with linear transformations. For instance, let us consider the following quadratic transformation depending on twenty one parameters (which is reminiscent of (19)):

\[
Q : \begin{bmatrix}
m_{1,1} & m_{1,2} & m_{1,3} \\
m_{2,1} & m_{2,2} & m_{2,3} \\
m_{3,1} & m_{3,2} & m_{3,3}
\end{bmatrix} \rightarrow
\begin{bmatrix}
m_{1,1}^2 - a_{11} m_{1,1}^2 + a_{12} m_{1,2} + a_{13} m_{1,3} + a_{21} m_{2,1}^2 + a_{22} m_{2,2}^2 + a_{23} m_{2,3}^2 + a_{31} m_{3,1}^2 + a_{32} m_{3,2}^2 + a_{33} m_{3,3}^2 & m_{1,3} \\
m_{2,1}^2 & c_{21} m_{2,1}^2 + c_{22} m_{2,2}^2 + c_{23} m_{2,3}^2 & m_{2,3} \\
m_{3,1}^2 & b_{11} m_{3,1}^2 + b_{12} m_{3,2}^2 + b_{13} m_{3,3}^2 + b_{21} m_{3,2}^2 + b_{23} m_{3,3}^2 + b_{31} m_{3,1}^2 + b_{32} m_{3,2}^2 + b_{33} m_{3,3}^2 & m_{3,3}
\end{bmatrix}
\]

The homogeneous transformation \( K = Q \cdot I \) gives again a stable factorization scheme. In this case, where \( Q \) is no longer a linear transformation, but a homogeneous polynomial transformation of degree \( r \) (here \( r = 2 \)), the factorization scheme remains of the general form (3) and (4). As far as generating functions are concerned some modifications have to be done. Firstly the \( \rho_n \)'s, and associated \( \rho(x) \), should be replaced by the \( \gamma_n \)'s defined by:

\[
\hat{K}(M_n) = \frac{K(M_n)}{\det(M_n)^r} = \frac{M_{n+1}}{f_{n+1}^0 \cdot f_{n+1}^{1} \cdot f_{n+1}^{2} \cdot \ldots}
\]

and the corresponding generating function \( \gamma(x) \). The linear relations between \( \eta(x), \phi(x) \) and \( \gamma(x) \) are slightly modified (see (24), (25) in appendix C). Secondly, relation (10) is no longer valid here. A new relation has to be introduced playing the same role. Transformation \( \hat{K} = Q \cdot I \) is a homogeneous transformation of degree \(-r\). Instead of introducing the determinantal variables \( x_n \) through (3), let us introduce \( \tilde{x}_n \) by:

\[
\tilde{x}_n(M_0) = \det(\hat{K}^{n+1}(M_0)) \cdot (\det(\hat{K}^n(M_0)))^r
\]

These new determinantal variables \( \tilde{x}_n \) are well-suited ones since they are invariant under a rescaling of \( M_0 : \tilde{x}_n(Cst \cdot M_0) = \tilde{x}_n(M_0) \). Relation (10) becomes:

\[
\tilde{x}_n(M_0) = f_{n+1}^{W_0} \cdot f_{n+1}^{W_1} \cdot f_{n-1}^{W_{n-1}} \cdot f_{n-2}^{W_{n-2}} \cdot \ldots \cdot f_0^{W_{n+1}}
\]

Again one can introduce the generating function of these exponents \( W_n \) and see that relation (13) still holds. From the stable factorization scheme of \( K = Q \cdot I \) one now gets:

\[
\beta(x) = \frac{1}{3} = \frac{1}{1 - 3x - 2x^2}, \quad \gamma(x) = 2 \cdot \frac{1 - x}{1 - 2x}
\]

This gives a complexity value \( \lambda \simeq 3.5615 \ldots \). Let us consider the expression of \( \alpha(x) \):

\[
\alpha(x) = \frac{3 \cdot (1 + x - 2x^2 + 4x^3)}{(1 + 2x) \cdot (1 - 2x) \cdot (1 - 3x - 2x^2)}
\]

On this expression one sees that other poles occur. The inverse of these additional poles, namely \( \pm 2 \), are actually smaller that the complexity value 3.56155 \ldots \. The existence of “subdominant” poles already occurred with permutations of entries, or linear transformations (see \( \beta(x) \) in (13)); we often had \( 1 - x \), or \( 1 + x \), additional factors in the expressions of the degree generating functions. With expression (29) one sees the occurrence of a \( 1 - 2x \) factor instead of \( 1 - x \) factors.

There is also nothing specific with quadratic transformations. Let us introduce the simple homogeneous polynomial of degree \( r \):

\[
\tilde{x}_n(M_0) = \det(\hat{K}^{n+1}(M_0)) \cdot (\det(\hat{K}^n(M_0)))^r
\]
\[ Q_r : \begin{bmatrix} m_{1,1} & m_{1,2} & m_{1,3} \\ m_{2,1} & m_{2,2} & m_{2,3} \\ m_{3,1} & m_{3,2} & m_{3,3} \end{bmatrix} \rightarrow \begin{bmatrix} m_{1,1}^r & m_{1,2}^r & m_{1,3}^r \\ m_{3,2}^r & m_{2,2}^r & m_{3,3}^r \\ m_{2,3}^r & m_{3,1}^r & m_{3,3}^r \end{bmatrix} \]  

(30)

and its associated homogeneous transformation \( K = Q_r \cdot I \). Its factorization scheme is very simple, it reads for \( r \geq 2 \) (for \( r = 1 \) transformation \( \beta = I \), and \( K = Q_r \cdot I \), become trivial):

\[ M_n = \frac{K(M_{n-1})}{f_n^{r-1}}, \quad \det(M_n) = f_{n+1} \cdot f_n^2 \]

which yields the following linear relations on the \( \alpha_n \)'s and \( \beta_n \)'s (see also appendix C):

\[ \alpha_n = 2 \cdot r \cdot \alpha_{n-1} - 3 \cdot r \cdot \beta_{n-1}, \quad \alpha_n = \beta_{n+1} + 2 \cdot \beta_n \]

(32)

It gives the following generating functions for arbitrary \( r \geq 2 \):

\[ \frac{\beta(x)}{3x} = \frac{1}{1 + 2 \cdot (1 - r) \cdot x - r \cdot x^2}, \quad \eta(x) = r, \quad \phi(x) = 1 + 2 \cdot x, \quad \gamma(x) = r \cdot (1 + x) \]

(33)

For homogeneous polynomials of degree \( r \) one can show that subdominant poles, like \( 1 - r \cdot x \), may occur instead of the previous \( 1 - x \) and \( 1 - 2 \cdot x \) factors.

For \( r = 2 \), one remarks that one gets a degree generating function:

\[ \frac{\beta(x)}{3x} = \frac{1}{1 - 2 \cdot x - 2 \cdot x^2} \]

(34)

which is not the limit of (28). The generic complexity corresponding to (28), namely \( \lambda \approx 3.56155 \cdots \) is changed, for (31) taken for \( r = 2 \), into \( \lambda \approx 2.73205 \cdots \). There actually exist many subvarieties of the twenty-one parameter space of transformations (31) on which the generic complexity \( \lambda \approx 3.56155 \cdots \) is modified into another (smaller) algebraic value. One remarks that the subvarieties of the twenty-one parameter space of transformation (31) (for instance, \( a_{12} = c_{22} = b_{32} = 1, a_{11} \) and \( a_{22} \) arbitrary non zero, all the other ones being zero, previously mentioned as a polynomial growth subcase) also yield “non-generic” complexities for (31).

IV. CONCLUSION

In previous papers [17,23] it has been shown that the topological entropy, and the Arnold complexity, actually identify on various simple two-dimensional birational examples, and that these quantities are actually algebraic numbers. The generating functions corresponding to these two complexity measures, namely the dynamical zeta function and the various “degree” generating functions (like \( \beta(x) \)) were shown to be simple rational expressions with integer coefficients [17], the dominant poles in these two sets of generating functions being the same. When one analyzes birational transformations depending on more than two variables, it becomes very difficult to calculate even the first coefficients of the expansion of the dynamical zeta function. On the contrary the calculations on the degree generating functions (associated to the Arnold complexity) can be quite easily performed, even for birational transformations of many variables (the \( q^2 \) entries of a matrix 3).

Analyzing exhaustively a first finite set of 362880 birational transformations (associated with all the permutations of 3 × 3 matrices), we have obtained non-trivial, but still simple, “spectrum” of eighteen algebraic Arnold complexities for the corresponding dynamical systems. In a second step it has been shown that these results can be drastically generalized along three different lines preserving the algebraic character of the complexities. Firstly, one can combine these birational transformations together, and get extremely rich sets of algebraic complexities. Secondly, one can consider (generically birational) transformations, associated with linear transformations of the entries of 3 × 3 matrices, and still get sets of algebraic complexities. Remarkably one has another universality property here: these algebraic complexities do not depend on many of the continuous parameters associated with the linear transformations. Thirdly, one still gets sets of algebraic complexities with rational transformations (associated with homogeneous polynomial transformations on the entries) which again can depend on many continuous parameters. With this last generalization we have completely lost any invertible character of the transformations. On the top of that these 3 × 3
matrix calculations can be simply generalized to $q \times q$ matrices for arbitrary $q$. Combining several of these rational transformations depending on several continuous parameters together, one certainly gets again rich sets of algebraic complexities.

We end up with an extremely large set of transformations, so large that, clearly, it should be a powerful tool to study discrete dynamical systems.

**APPENDIX A: A "TRANSMUTATION" PROPERTY OF THE MATRIX INVERSION**

Let us sketch here some non trivial symmetries between the permutations.

The transformations, considered in sections (1B), (1C), are products of matrix inversion and permutations of the entries. Any such non trivial symmetry of the birational transformations $\mathcal{K}$ should correspond to a non-trivial relation between matrix inversion and permutations of the entries of the matrix. Such relations actually exist. They correspond to a “transmutation” property between the inversion and permutations $P$ and $Q$. There actually exist two permutations $P$ and $Q$ such that:

$$P \cdot \hat{t} = \hat{t} \cdot Q$$  \hspace{1cm} (A1)

Permutations, such that a “transmutation” relation $\hat{1}$ is satisfied, do exist: one can easily build examples by combining product of permutations that permutes only rows of a $q \times q$ matrix (that we will denote by “$R$”), permutations that permutes only columns of a $q \times q$ matrix (that we will denote by “$C$”) and, possibly, the matrix transposition we denote “$T$”. Examples of permutations $P$ and $Q$, such that $\hat{1}$ is satisfied, read:

$$P = R \cdot C \cdot T^\epsilon,$$

where: $\epsilon = 0, \text{ or } 1$  \hspace{1cm} (A2)

and similarly for permutation $Q$. A permutation $P$ having such a decomposition $\hat{1}$ will be called an “RCT” permutation.

Let us consider two permutations $t_1$ and $t_2$, yielding respectively the two birational transformations $\mathcal{K}_1 = t_1 \cdot \hat{t}$ and $\mathcal{K}_2 = t_2 \cdot \hat{t}$. Let us introduce the following relation of equivalence between two permutations $t_1$ and $t_2$: $t_1$ and $t_2$ will be related if they are such that there exists an “RCT” permutation, $b_0$, such that:

$$\mathcal{K}_1^n = b_0 \cdot \mathcal{K}_2^n \cdot b_0^{-1}$$  \hspace{1cm} (A3)

Relation $\hat{1}$ can easily be seen to define a relation of equivalence between $t_1$ and $t_2$, we will denote $\mathcal{R}^{(n)}$:

$$t_1 \mathcal{R}^{(n)} t_2$$  \hspace{1cm} (A4)

Note that this $\mathcal{R}^{(n)}$ equivalence relation is compatible with the inverse in the permutation group $t \rightarrow t^{-1}$. Also note that the equivalence of two permutations, up to simple rows and columns relabeling, is an $\mathcal{R}^{(1)}$ equivalence, however, conversely, the $\mathcal{R}^{(1)}$ equivalence does not reduce to the simple, and quite trivial, equivalence of two permutations up to simple rows and columns relabeling. Obviously rows and columns relabeling of the matrices do not modify their integrability properties $\hat{1}$, as well as the growth of the calculations.

It is obvious that if $t_1 \mathcal{R}^{(n)} t_2$ then $t_1 \mathcal{R}^{(n \times p)} t_2$ for any natural integer $p$. This is a consequence of the fact that:

$$\mathcal{K}_1^n = b_0 \cdot \mathcal{K}_2^n \cdot b_0^{-1} \quad \text{yields:} \quad \mathcal{K}_1^{np} = b_0 \cdot \mathcal{K}_2^{np} \cdot b_0^{-1}$$  \hspace{1cm} (A5)

If two permutations, $t_1$ and $t_2$, are in the same equivalence class with respect to $\mathcal{R}^{(m)}$, and if $t_2$ and $t_3$ are in the same equivalence class with respect to $\mathcal{R}^{(n)}$ where $n \neq m$, $t_1$ and $t_3$ are in the same equivalence class with respect to $\mathcal{R}^{(n \times m)}$, or with respect to $\mathcal{R}^{(N)}$ for some “large enough” integer $N$. In fact it can be shown, on the example of the equivalence classification of the permutations of $3 \times 3$ matrices, that this value of $N$ corresponding to the (“asymptotic”) equivalence relation is actually equal to $N = 24$.

If two permutations $t_1$ and $t_2$ are in the same equivalence class, with respect to $\mathcal{R}^{(m)}$, the complexities (which are real positive numbers), associated with their respective birational transformations $\mathcal{K}_1$ and $\mathcal{K}_2$, we denote $\lambda_1$ and $\lambda_2$ are, as a straight consequence of $\hat{1}$, related by:

---

8The “spectrum” of values of the complexity $\lambda$ depends on $q$, see for instance $\hat{2}$. 

11
Therefore one sees that their complexities are equal: \( \lambda_1 = \lambda_2 \). In particular if one considers the (largest) equivalence classes corresponding, for \( 3 \times 3 \) matrices, to \( \mathcal{R}^{(24)} \), all the representants in one of these \( \mathcal{R}^{(24)} \) equivalence classes will have the same complexity growth \( \lambda \).

**APPENDIX B: A “MOLECULAR” FACTORIZATION SCHEME**

The factorization scheme of \( K = t_1 \cdot I \cdot t_2 \cdot I \), corresponding to permutation 146237058 and permutation 471562380 (see section IIIA), is of the same type as the one described in [7], namely a parity-dependent factorization scheme (which is a straight consequence of the fact that one acts with \( K_1 \), and then with \( K_2 \), and again ...) :

\[
\begin{align*}
 f_1 &= \det(M_0), \quad M_1 = K_1(M_0), \quad f_2 = \det(M_1), \quad M_2 = K_2(M_1), \quad f_3 = \frac{\det(M_2)}{f_2}, \quad M_3 = K_1(M_2), \\
 f_4 &= \det(M_3), \quad M_4 = K_2(M_3), \quad f_5 = \frac{\det(M_4)}{f_2}, \quad M_5 = \frac{K_1(M_4)}{f_2}, \quad f_6 = \frac{\det(M_5)}{f_2 \cdot f_4}, \quad \cdots \quad (B1)
\end{align*}
\]

and for arbitrary \( n \geq 3 \) :

\[
\begin{align*}
 \det(M_n) &= f_{n+1} \cdot f_n \cdot f_{n-2} \cdot f_{n-6} \cdot f_{n-8} \cdot f_{n-10} \cdot f_{n-12} \cdot f_{n-14} \cdots \\
 K_1(M_n) &= M_{n+1} \cdot f_{n-2} 
\end{align*}
\]

for \( n \) even and :

\[
\begin{align*}
 \det(M_n) &= f_{n+1} \cdot f_{n-1} \cdot f_{n-3} \cdot f_{n-5} \cdot f_{n-7} \cdot f_{n-9} \cdot f_{n-11} \cdot f_{n-13} \cdots \\
 K_2(M_n) &= M_{n+1} \cdot f_{n-3} \cdot f_{n-7} \cdot f_{n-9} \cdot f_{n-11} \cdot f_{n-13} \cdots 
\end{align*}
\]

for \( n \) odd. This yields the following expressions for the odd and even parts of \( \alpha(x) \) and \( \beta(x) \) (“2” for even and “1” for odd):

\[
\begin{align*}
 \beta_2(x) &= \frac{6 \cdot x^2}{1 - 3 \cdot x^2 + x^4 - x^6 - 2 \cdot x^8}, \\
 \alpha_2(x) &= \frac{3 \cdot (1 + 4 \cdot x^4 - 4 \cdot x^6 + x^8)}{(1 - x^2) \cdot (1 - 3 \cdot x^2 + x^4 - x^6 - 2 \cdot x^8)}
\end{align*}
\]

\[
\begin{align*}
 \beta_1(x) &= \frac{3 \cdot x \cdot (1 + x^2) \cdot (-1 + x)^2 (x + 1)^2}{1 - 3 \cdot x^2 + x^4 - x^6 - 2 \cdot x^8}, \\
 \alpha_1(x) &= \frac{6 \cdot x \cdot (1 + x^4 - x^6 + x^8)}{(1 - x^2) \cdot (1 - 3 \cdot x^2 + x^4 - x^6 - 2 \cdot x^8)}
\end{align*}
\]

These generating functions yield a “molecular complexity” : \( \lambda \simeq 2.8581 \cdots \). These generating functions verify a parity dependent system of functional relations which generalizes the ones described in [3] :

\[
\begin{align*}
 x \cdot \alpha_1(x) - \beta_2(x) &= F_{2p}(x) \cdot \beta_2(x), \\
 x \cdot \alpha_2(x) - \beta_1(x) &= F_{1m}(x) \cdot \beta_2(x), \\
 \alpha_2(x) - 3 - 2 \cdot x \cdot \alpha_1(x) + 3 \cdot G_{2p} \cdot \beta_2(x) &= 0, \\
 \alpha_1(x) - 2 \cdot x \cdot \alpha_2(x) + 3 \cdot G_{1m} \cdot \beta_2(x) &= 0
\end{align*}
\]

where :

\[
\begin{align*}
 F_{2p}(x) &= x^2 + 2 \cdot x^4 + x^6 + \frac{2 \cdot x^8}{1 - x^2}, \\
 F_{1m}(x) &= 2 \cdot x^3 - x^5 + \frac{x}{1 - x^2}, \\
 G_{1m}(x) &= x^3, \\
 G_{2p}(x) &= x^4 + \frac{x^8}{1 - x^2}
\end{align*}
\]

**APPENDIX C: EXPONENT GENERATING FUNCTIONS FOR HOMOGENEOUS POLYNOMIAL TRANSFORMATIONS OF DEGREE \( R \)**

Let us consider a homogeneous transformation \( Q_r \) of degree \( r \) (like [3], or like [4] for \( r = 2 \)) and its associated homogeneous transformation \( K = Q_r \cdot I \). Relations [4], [4] are still valid but yield a slight modification of the linear functional relations [1] and [4], namely :

\[
\begin{align*}
 ((q - 1) \cdot r \cdot x - 1) \cdot \alpha(x) + q - q \cdot x \cdot \eta(x) \cdot \beta(x) &= 0, \quad (C1) \\
 x \cdot \alpha(x) &= \phi(x) \cdot \beta(x) \quad (C2)
\end{align*}
\]
Let us recall that, for homogeneous transformations of degree \( r \), one must introduce, instead of \( \rho(x) \), the generating function \( \gamma(x) \) (see section (III C)) defined by:

\[
\hat{K}(M_n) = \frac{K(M_n)}{\det(M_n)^r} = \frac{M_{n+1}}{f_{n+1} \cdot f_{n} \cdot f_{n-1} \cdot \cdots}
\]

(C3)

This last relation yields a new relation:

\[
q + q \cdot \gamma(x) \cdot \beta(x) = (1 + r \cdot x) \cdot \alpha(x)
\]

(C4)

which has to be compatible with the previous two (C1), (C2):

\[
r \cdot \phi(x) = \gamma(x) + x \cdot \eta(x)
\]

(C5)