OLD AND NEW APPROACHES TO THE SORITES PARADOX

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Abstract. The Sorites paradox is the name of a class of paradoxes that arise when vague predicates are considered. Vague predicates lack sharp boundaries in extension and is therefore not clear exactly when such predicates apply. Several approaches to this class of paradoxes have been made since its first formulation by Eubulides of Miletus in the IV century BCE. In this paper I survey some of these approaches and point out some of the criticism that these approaches have received. A new approach that uses tools from nonstandard analysis to model the paradox is proposed.

The knowledge came upon me, not quickly, but little by little and grain by grain.
(Charles Dickens in David Copperfield)

1. TO BE OR NOT TO BE A HEAP

This paper concerns the paradoxes which arise when several orders of magnitude are considered. These paradoxes are part of the larger phenomenon known as vagueness. Indeed,

 [...] vagueness can be seen to be a feature of syntactic categories other than predicates. Names, adjectives, adverbs and so on are all susceptible to paradoxical sorites reasoning in a derivative sense.

Vague predicates share at least the following features:

(1) Admit borderline cases
(2) Lack sharp boundaries
(3) Are susceptible to sorites paradoxes

Borderline cases are the ones where it is not clear whether or not the predicate applies, independently of how much one knows about it. For instance, most basketball players are clearly tall and most jockeys are clearly not tall. But in many cases is rather unclear if the person in question is tall, even if one knows its height with great precision. Furthermore, there is no clear distinction between the set of all tall people and the set of people that are not tall. These sets lack sharp boundaries. This leads to a collection of paradoxes called Sorites paradoxes first formulated by the Ancient Greek philosopher Eubulides of Miletus in the IV century BCE. One can be stated in the following way: a single grain of wheat cannot be considered as a heap. Neither can two grains of wheat. One must admit the presence of a heap sooner or later, so where to draw the line? In fact, the name Sorites derives from

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the Greek word *sorós* which means heap. However, one can reconstruct the paradox by replacing the term 'heap' by other vague concepts such as 'tall', 'beautiful', 'bald', 'heavy', 'cold', 'rich', ...

The argument consists of a predicate $S$ (the soritical predicate) and a subject expression $a_n$ in the series regarding to which $S$ is soritical. The terms of the series are supposed to be ordered. According to Barnes [1] a predicate $S$ must satisfy three constraints in order to be considered soritical:

1. Appear to be valid for $a_1$, the first item in the series;
2. Appear to be false for $a_i$, the last item in the series;
3. Each adjacent pair in the series, $a_n$ and $a_{n+1}$ must be sufficiently similar as to appear indiscriminable in respect to $S$.

This means that the predicate $S$ needs to be sufficiently vague in order to allow small changes. Small changes do not determine the difference between a set of individual grains and a heap, between a bald man and a hairy one, between a rich person and a poor one. However, and in spite of the vagueness involved, it also needs to have a certain area on which $S$ is clearly true and an area on which $S$ is clearly false.

The difference of one grain would seem to be too small to make any difference to the application of the predicate; it is a difference so negligible as to make no apparent difference to the truth-values of the respective antecedents and consequents.

Yet the conclusion seems false. [20]

This paper surveys some approaches that have been made to deal with the phenomenon of vagueness and the Sorites type paradoxes which arise when vague predicates are used. Also, a new approach that uses tools from nonstandard analysis to model the paradox is proposed. As mentioned above, soritical arguments are tolerant to small changes but not tolerant to large changes in relevant aspects. In fact, using a special class of external sets called external numbers (see Section 4.1) it is possible to define rigorously what is meant with terms such as 'small changes' or 'large changes'. The fact that large changes come as the result of the accumulation of small changes is no surprise because it is a very well known fact from nonstandard analysis that an infinitely large sum of infinitesimals may very well become appreciable or even infinitely large. In this way, one can make a rigorous claim that a heap and a set of individual grains of wheat are indeed not of the same order of magnitude.

2. Forms of the paradox

The Sorites paradox can be stated in various ways. This implies that one cannot hope to solve the paradox by pointing out a fault particular to any one of those way. One should instead try and reveal a common fault to all possible forms that the paradox can take. I do so by considering the (standard) mathematical induction and conditional schemata. These schemata will be revisited, in Section 4.2, after a nonstandard point of view is adopted.

2.1. Induction. Mathematical induction is generally used (within standard mathematics) to prove that a mathematical statement involving a natural number $n$ holds for all possible values of $n$. This is done in two steps. On the first step (*basis*) one proves that there is a first element for which the statement holds. On
the second step (inductive step) one shows that if the statement holds for some \( n \) then it also holds for \( n + 1 \). Then, by the principle of mathematical induction, the statement is valid for all \( n \). Let \( S \) represent a soritical predicate, for example 'is not a heap' and let \( a_n \) represent the \( n \)-th element in the soritical series. In the example above it would be the sentence '\( n \) grains of wheat'.

The Sorites paradox can now be represented in the following way:

\[
\begin{align*}
(Sa_1 \land \forall n (Sa_n \rightarrow Sa_{n+1})) \rightarrow \forall n Sa_n \\
\exists \omega (\neg Sa_\omega)
\end{align*}
\]

So, if one admits that:

1. A single grain of wheat is not a heap.
2. If a collection of \( n \) grains of wheat is not a heap then a collection of \( n + 1 \) grains of wheat is also not a heap.

One concludes (by induction) that the heap will never appear. Since at some point the heap is obviously there one might come to the conclusion that there is something wrong with induction or, at least, with applying induction to vague predicates.

2.2. **Conditional form.** The conditional form of the Sorites paradox is the most common form throughout the literature. Using the notation of the previous section it can be formalized in the following way:

If

\[
\begin{align*}
Sa_1 \\
Sa_1 \rightarrow Sa_2 \\
Sa_2 \rightarrow Sa_3 \\
\vdots \\
Sa_i \rightarrow Sa_{i+1} \\
\exists j (\neg Sa_j)
\end{align*}
\]

Assuming \( Sa_1 \), \( Sa_1 \rightarrow Sa_2 \), \( Sa_2 \rightarrow Sa_3 \), ..., \( Sa_i \rightarrow Sa_{i+1} \), by *modus ponens*, the conclusion is \( Sa_i \), where \( i \) can be arbitrarily large. This is a fairly simple reasoning where the premises are: a single grain of wheat does not make a heap; if one grain of wheat does not make a heap then two grains of wheat do not form a heap either; if two grains of wheat do not make a heap then three grains of wheat do not form a heap either... if \( i \) grains of wheat do not make a heap then \( i + 1 \) grains of wheat do not form a heap either. The conclusion is that a set of an arbitrarily large number of grains \( i \) does not make a heap. However if one observes that there is a set of \( j \) grains that form a heap it generates a paradox.\(^1\)

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\(^1\)A reasoning from classical logic was used here and one could think that the paradox may be circumvented if one would use intuitionistic logic instead. In fact, there have been attempts to use intuitionistic logic to deal with the paradox (see [31] and its defense [31]). However, according to Keefe,

[...] critics have shown that with various reasonable additional assumptions, other versions of sorites arguments still lead to paradox. [...] The bulk of the criticisms point to the conclusion that there is no sustainable account of vagueness that emerges from rejecting classical logic in favour of intuitionistic logic. [21, p. 22]

For the criticism that Keefe refers to the reader may consult [34].
3. Response attempts

There are several attempts to solve the Sorites paradox. These responses are divided into the following four types ([21], p. 19-20). A first type of response would be to deny the validity of the argument, refusing to grant that the conclusion follows from the premises. Alternatively one can question the strict truth of the inductive premise (or of one of the conditionals). A third possibility is to accept the validity of the argument and the truth of its inductive premise (or of all the conditional premises) but contest the truth of the conclusion. Finally one can grant that there are good reasons to consider both the argument form as valid, and accept the premises and deny the conclusion hence proving that the predicate is incoherent.

In this section, some of the responses to the paradox will be reviewed. For a wider account on this matter see for example [21, 38, 36, 45]. I would like to emphasize that the theories presented below correspond to a wide variety of related points of view. This means that there are many versions of the theories presented. So, when reviewing a theory I tend to give only the general lines, common to the various versions of that theory.

3.1. Ideal Languages. Natural languages such as English or Vietnamese distinguish between intension and extension of terms. The intension is the internal content of a term or concept while the extension is the range of applicability of a term by naming the particular objects that the term denotes. The two predicates 'is a creature with a heart' and 'is a creature with a kidney' (see [32]) have the same extension because the set of creatures with hearts and the set of creatures with kidneys are the same. However, having a heart and having a kidney are very different things, so one concludes that terms can name the same thing but differ in meaning. [32]

Hence, the distinction between intension and extension leads necessarily to vagueness, ambiguity and indeterminacy of meaning for words and phrases. This is in part the reason why natural languages are so powerful. In poetry, for example, beauty is achieved by taking advantage of these features. However, if one needs clarity and precision of language then she is forced to conclude that natural languages are not the way to go. According to Quine

The sorites paradox is one imperative reason for precision in science, along with more familiar reasons. [33]

An ideal language would left out all such factors in order to eliminate any vagueness.

The defenders of this response, among them Frege [16], Russell [35] and Wittgenstein [47], consider vagueness as a non-eliminable feature of natural language. The way to avoid vagueness is by creating and using ideal languages instead. This would mean that arguments of the Sorites type are not valid since they contain vague expressions.

As stated by Russell,

The fact is that all words are attributable without doubt over a certain area, but become questionable within a penumbra, outside which they are again certainly not attributable. Someone might seek to obtain precision in the use of words by saying that no word is to be applied in the penumbra, but unfortunately the penumbra
is itself not accurately definable, and all the vagueness which apply the primary use of words apply also when we try to fix a limit to their indubitable applicability. [35]

So, this response implies that it is the philosopher’s job to discover a logically ideal language. However, this doesn’t seem possible using classical logic:

All traditional logic habitually assumes that precise symbols are being employed. It is therefore not applicable to this terrestrial life, but only to an imagined celestial existence. [35]

Russell also believed that

Vagueness, clearly, is a matter of degree, depending upon the extent of the possible differences between different systems represented by the same representation. Accuracy, on the contrary, is an ideal limit. [35]

Ideal languages as a response to the sorites paradox seem to have unsatisfying features. According to Keefe,

denying the validity of the sorites argument seems to require giving up absolutely fundamental rules of inference. [21, p. 20]

So, if one chooses to go in this direction fundamental rules such as *modus ponens* or mathematical induction are to be put in question. Furthermore, by eliminating vague predicates from the language one is not solving the paradox but avoiding it by sweeping it under the rug. In fact, most philosophers nowadays believe that vagueness is an important part of natural language and cannot be separated from it.

### 3.2. The Epistemic theory.

The Epistemic theory is based on the idea that the precise boundaries to knowledge itself cannot be known. Vagueness is seen as a particular type of ignorance.

The fact that this theory is built in the classical logic framework implies that there are precise bounds for the extensions of vague predicates even if one does not know where they are located. For instance, the defenders of the epistemic theory claim that there is in fact a last grain of wheat in the series before the heap turns up, even if one is not (nor ever will be) able to identify it definitively. In fact, Williamson [45] has shown that if there is a precise boundary for penumbral cases one cannot know where it is. So, soritical predicates are indeterminate in extension but not semantically. This position has been notably defended by Williamson [45, 46] and Sorensen [39, 40].

The first and major objection to this theory is its counter-intuitive nature. The meaning of a word is (usually) determined by its use. According to Wittgenstein

For a large class of cases - though not for all - in which we employ the word ‘meaning’ it can be defined thus: the meaning of a word is its use in the language [47]

and

if we had to name anything which is the life of the sign, we should have to say that it is its use. [48]

For instance, the word ‘guitar’ means an actual guitar because one use that word to mean an actual guitar (even if one does not know how to play). Now, one does not usually use the word ‘heap’ as if a single grain of wheat could make a difference.
Neither, more generally, does one use any vague term as if it were not tolerant to small changes. One does not use vague terms as if they had precise borders. In this sense, Smith \[35\] claims that the epistemicist is forced to deny a link between meaning and use.

Another point that deserves criticism is that nothing is said about how predicates get the precise extensions that they do. It is claimed that there is in fact a last grain of wheat in the series before the heap turns up. So there should be attempts to find which one is it \[21\]. I agree that ignorance is no excuse for the lack of attempts to find the precise boundaries of vague concepts. There should be at least some reasons to believe about where these boundaries are.

3.3. Supervaluationism. According to Fine, vagueness is a semantic notion not to be confused with ambiguity nor undecidability:

Let us say, in a preliminary way, what vagueness is. I take it to be a semantic notion. Very roughly, vagueness is deficiency of meaning. As such, it is to be distinguished from generality, undecidability, and ambiguity. These latter are, if you like, lack of content, possible knowledge, and univocal meaning, respectively. \[13\]

Supervaluationism proposes to solve the problem of vagueness by modifying classical semantics, using Van Fraassen’s supervaluations. According to Van Fraassen:

A supervaluation over a model is a function that assigns T (F) exactly to those statements assigned T (F) by all the classical val-

And he concludes that

Supervaluations have truth-value gaps. \[15\]

In classical logic the connectives have truth values in a functional way. I recall that a connective of statements is truth-functional if and only if the truth value of any compound statement obtained by applying that connective is a function of the individual truth values of the constituent statements that form the compound. The classical logic connectives are all truth-functional\[3\]. Supervaluationists abandon the concept of truth-functionality.

Fine applies the distinction between extension and intension \[32\] to vagueness:

Extensional vagueness is deficiency of extension, intensional vagueness deficiency of intension. Moreover, if intension is the possibility of extension, then intensional vagueness is the possibility of extensional vagueness. \[13\]

According to this theory, a vague predicate does not need to have a unique, sharply bounded, truth function. Vague predicates have things to which they definitely apply (positive extension), things to which they definitely do not (negative extension) and a penumbra (penumbral connections). The penumbra involves cases which seem to be neither true nor false\[4\] (borderline cases). These penumbral

\[2\]This is immediately visible if one computes the logical value of a given sentence using truth tables or a proof calculus like natural deduction \[29\].

\[3\]Fine warns about the general confusion of under- and over-determinacy.

A vague sentence can be made more precise; and this operation should preserve truth-value. But a vague sentence can be made to be either true or false, and therefore the original sentence can be neither. \[13\]
connections are instances of truth-value gaps. Truth-value gaps are related with extensional vagueness. However,

Despite the connection, extensional vagueness should not be defined in terms of truth-value gaps. This is because gaps can have other sources, such as failure of reference or presupposition. 

Supervaluationists claim, roughly speaking, that a vague sentence is true if and only if it is true for all ways of making it completely precise, called precisifications. There are then many interpretations or precisifications. Each one of these precisifications has no penumbra because it behaves according to classical bivalence. The assignment of truth value for all such precisifications is a supervaluation.

A sentence which is true in all precisifications is called supertrue and a sentence which is false in all precisifications is called superfalse. A sentence which is true for some precisifications and false on others is neither true nor false. This means in particular that tautologies from classical logic are supertrue.

According to Keefe, truth is supertruth, meaning that a sentence is true if and only if it is true on all admissible precisifications. A precisification is acceptable only if the extensions of the concepts do not overlap. The truth of a compound sentence is determined by its truth on every precisification. For a wider account on supervaluationism the reader is referred to 

Fodor and Lepore are particularly critic of the supervaluationist approach to vagueness:

\[ \text{... there is something fundamentally wrong with using supervaluation techniques either for preserving classical logic or for providing a semantics for linguistic expressions ordinarily thought to produce truth-value gaps.} \]

However the fault they point out is not of a logical nature. Indeed they say:

Right from the start, however, we want to emphasize that the objections we are raising are philosophical rather than logical. We have no argument with supervaluations considered as a piece of formal mathematics.

Fodor and Lepore point out as the main flaws of supervaluationism the violation of intuitive semantic principles concerning disjunctions and existential quantification, the abandonment of classical rules of inference and the violation of core principles concerning the concept of truth.

Let \( S \) represent the predicate 'is not a heap'. Since all tautologies are supertrue,

\[ \neg (\forall n (Sa_n \to Sa_{n+1})) \]

is equivalent to

\[ \exists n (Sa_n \land \neg Sa_{n+1}) \]

which, semantically speaking, seems to postulate the existence of a sharp boundary and looks for that matter like a step back towards the epistemic theory. Also \((S \lor \neg S)\) is supertrue. So, for all precisifications one of the statements is true. However, the statement \( S \) is borderline and therefore neither true nor false.

Keefe argues that it is possible to surpass these difficulties at the price of adding a new operator to the language: the 'definitely' operator \( D \). This operator

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In fact, not all supervaluationists accept this last sentence.
is however not closed under certain operations such as contraposition and conditional introduction. So alternatives to the classical closure principles are proposed. However, this implies that the logic used is no longer classical.

Another argument against supervaluationism is that little information is given on what makes a precisification acceptable other than saying that precisifications must respect penumbral connections and therefore their admissibility is a vague matter. Also, supervaluationism states that precisifications behave in a classical way and have no penumbra. However, each precisification may divide the positive and negative extensions in different places.

For more on objections to supervaluationism and attempts to respond to those objections the reader is referred to [5].

3.4. Many-valued logics. Many-valued logics is a general term that refers to logics which have more than two truth-values. In these logics the principle of truth-functionality is accepted and so a sentence remains unaffected when one of its components is replaced by another with the same truth value. Many-valued logics became accepted as an independent part of logic with the works of Łukasiewicz and Post in the 1920’s. Since then many many-valued logics emerged (e.g. [25, 17, 19, 28]) and it is not possible nor desirable to describe them all in these pages. However I shall discuss an application of Kleene’s three-valued logic and applications of fuzzy logics because these seem to be the most relevant in what concerns the phenomenon of vagueness. For a more complete reference concerning many-valued logics see for example [18].

3.4.1. Kleene’s three-valued logic. Perhaps one of the simplest and best-known examples of a many-valued logic is Kleene’s three-valued logic [22]. Kleene thought of the third truth value as undefined or underdetermined. So one has three truth-values: 1 (true), 0 (false) and 1/2 (undefined or unknown). One has truth-tables for which the connectives are regular, i.e. in terms of ordering, undefined is placed below both true and false. This means that the behavior of the third truth value should be compatible with any increase in information. Kleene proposed the following truth-tables for the so-called strong connectives:

|   | p | q | p ∨ q | p ∧ q | p → q | p ←→ q |
|---|---|---|-------|-------|-------|--------|
| 1 | 1 | 1 | 1     | 1     | 1     | 1      |
| 1 | 0 | 1 | 0     | 0     | 0     | 0      |
| 1/2| 1/3| 1/3| 1/3   | 1/3   | 1/3   | 1/3    |
| 0 | 1 | 1 | 1     | 1     | 1     | 1      |
| 1/2| 0 | 0 | 0     | 1     | 1     | 1      |
| 1/2| 1/2| 1/2| 1/2   | 1/2   | 1/2   | 1/2    |
| 1/2| 0 | 1/2| 0     | 1/2   | 1/2   | 1/2    |
| 1/2| 1 | 1/2| 1/2   | 1/2   | 1/2   | 1/2    |
| 1/2| 2 | 1/2| 2     | 1/2   | 2     | 1/2    |

These tables are uniquely determined as the strongest possible regular extensions of the classical two-valued tables. Quantifiers can be defined in the following way: \( \exists x : P(x) \) is true if \( P(x) \) is true for some value of \( x \) and it is false if \( P(x) \) is false.

\(^5\)Priest [30] gave an alternative three-valued logic conceiving the third truth-value as overdetermined, interpreting the symbol 1/2 as being both true and false.
for all values and indefinite otherwise; \(\forall x P(x)\) is true if \(P(x)\) is true for all values of \(x\) and false if \(P(x)\) is false for some value and indefinite otherwise. Tye [43] applies Kleene’s three-valued logic to the sorites paradox. However, the objections made to the bipartite division can also be used to refute a tripartite division. In fact, Tye [43] claims that

[...] vagueness cannot be reconciled with any precise dividing lines.

because

there is no determinate fact of the matter about where truth-value changes occur.

That is to say that there is no way to assign precise truth-values to vague terms. So, as a solution, Tye proposes to use a vague metalanguage. He claims that there are sets which are genuinely vague items. For instance the set of tall men has borderline members (men which are neither clearly members nor clearly non-members of the set).

There is no determinate fact of the matter about there are objects that are neither members, borderline members, nor non-members.

[43]

Kleene’s three-valued logic has the undesirable feature of having no tautologies, because the two-valued tautologies can take the value \(\frac{1}{2}\) in the three-valued case. As an example consider the law of excluded middle \(p \lor \neg p\). In Kleene’s three-valued logic the truth table is the following:

| \(p\) | \(\neg p\) | \(p \lor \neg p\) |
|-------|------------|-----------------|
| 1     | 0          | 1               |
| 0     | 1          | 1               |
| \(\frac{1}{2}\) | \(\frac{1}{2}\) | \(\frac{1}{2}\) |

Tye tries to avoid this flaw by saying that a statement is a quasi-tautology if it has no false substitution instances. So two-valued tautologies become three-valued quasi-tautologies.

Kleene’s three-valued logic is still a precise formalization and having no tautologies seems a price too high to pay in order to be able to deal in the above sense with vagueness. Also, according to Keefe,

[...] the appeal to quasi-tautologies adds nothing: if earning this title is enough for his [Tye’s] purposes, then the fact that \(p \lor \neg p\) also earns it should be of concern. Moreover, what matters for validity does not relate to quaistautologies [sic], and assertion depends on sentences being true not being either true or indefinite, so the role for the notion seems to be merely one of appeasement. [21, p. 111]

3.4.2. Fuzzy logics. Fuzzy logics propose a graded notion of inference. Truth-values range in degree between 0 and 1 in order to capture different degrees of truth. In this way, the value 0 is attributed to sentences which are completely false and the value 1 to sentences which are completely true. The remaining sentences are truer than the false sentences, but not as true as the true ones so they have intermediate logical values according to "how true" they are. According to Bogenberger

In fuzzy logic, the truth of any statement becomes a matter of degree. [3]
Fuzzy logic is related to Zadeh’s work on fuzzy sets [49]. A fuzzy set \( A \) on \( X \) is characterized by a membership function \( f_A(x) \) with values in the interval \([0,1]\). So, a fuzzy set \( A \) is a class of objects that allow a continuum of grades of membership. The membership degree is then the degree to which the sentence ‘\( x \) is a member of \( A \)’ is true. So, one can interpret the membership degrees of fuzzy sets as truth degrees of the membership predicate in a suitable many-valued logic.

Theories of vagueness which recourse to fuzzy logics are advocated most notably by Machina [26] and Smith [38].

According to these theories the notion of heap is a vague one and it may hold true of given objects only to some (truth) degree. The premises should be considered partially true to a degree which is quite near to the maximal degree 1. This inference has to involve truth degrees for the premises and has to provide a truth degree for the conclusion in a way that in each step the truth degree becomes smaller. The sentence ‘\( n \) grains of sand do not make a heap’ tends toward being false for an increasing number of grains.

The problem of saying whether the sentence ‘\( n \) grains makes a heap’ is true or not is essentially the same as to say that that sentence is true with a certain (precise) fixed degree. This false precision is perhaps the main objection to the application of many-valued logics to the sorites paradox. According to Keefe

\[
\text{[T]he degree theorist’s assignments impose precision in a form that is just as unacceptable as a classical true/false assignment. In so far as a degree theory avoids determinacy over whether a is F, the objection here is that it does so by enforcing determinacy over the degree to which a is F. All predications of “is red” will receive a unique, exact value, but it seems inappropriate to associate our vague predicate “red” with any particular exact function from objects to degrees of truth. For a start, what could determine which is the correct function, settling that my coat is red to degree 0.322 rather than 0.321? [21, p. 113]}
\]

Also, Urquhart states that

\[
\text{One immediate objection which presents itself to [fuzzy logic’s] line of approach is the extremely artificial nature of the attaching of precise numerical values to sentences like ‘73 is a large number’ or ‘Picasso’s Guernica is beautiful’. In fact, it seems plausible to say that the nature of vague predicates precludes attaching precise numerical values just as much as it precludes attaching precise classical truth values. [44]}
\]

Smith [38] tries to solve this problem, suggesting several possible solutions and concluding that the best answer is to mix fuzzy logic with a theory called plurivaluationism (not to be confused with supervaluationism) [6] called \textit{fuzzy plurivaluationism}. So, Smith accepts the semantic realism implied by the Epistemic view, but denies that vague predicates have to refer to a single bivalent model.

\[\text{[6]Supervaluationism involves only one intended (non-classical) model relevant to questions concerning meaning and truth, while plurivaluationism allows that there may be multiple (classical) models.}\]
3.5. Contextualism. Contextualism\(^7\) defends that interpretations change over time or according to context. Such shifts of contexts may occur instantaneously. For instance, at the beginning of a conversation the context is empty. Then, as the conversation goes along, these notions are sharpened in such a way that borderline cases (undecided so far) get assigned to either the extension or the anti-extension of the vague predicates in question. In fact, borderline sentences can express something true in one context and something false in another, so they are context-sensitive. In this way one can disagree about the truth-values of the propositions expressed by borderline sentences, even in situations where all the relevant information is available. This view is most prominently elaborated by Shapiro \([36, 37]\) and DeRose \([7]\).

Besides context-sensitivity Shapiro defines as central the concepts of judgment dependence, open texture, and the principle of tolerance. Judgment dependence means that both the extensions and anti-extensions for the borderline cases are solely determined by the decisions of competent speakers. These decisions are put in (and can be removed from) the conversational record. Open texture means that for a vague predicate \(S\) there exists an object \(a\) such that a competent speaker can decide whether \(Sa\) holds or not without her competency being compromised. The principle of tolerance is defined as follows. Suppose that two objects \(a, b\) differ only marginally in the relevant respect on which a vague predicate \(S\) is tolerant. Then if one competently judges \(Sa\) to hold, then \(Sb\) also holds.

One reason for skepticism about contextualism is that the problems with vague expressions seem to remain whether context-sensitivity is taken into account or not. By taking context into account one can reduce vagueness but not eliminate completely. Indeed, sets with vague boundaries are invariant to some translations. Take for instance the word 'ugly'. Even if a particular context is given (and even if one knows a great deal about another one’s ugliness) there is still no reason to suppose that there is a sharp boundary between what 'ugly' applies to and what it does not.

Smith \([38]\) argues that contextualism should not be seen as a theory of vagueness in its own right. He claims that this theory is compatible with all other mentioned theories.

4. External numbers as a model

In this section a new approach to the Sorites paradox which takes advantage of notions and concepts from nonstandard analysis is presented. I will start with a brief description of the necessary concepts to reformulate the paradox and model it by means of orders of magnitude. Orders of magnitude are given in the form of the so-called neutrices \([24, 11]\)\(^9\). I want to emphasize that the present response models

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\(^7\)Contextualism is often seen as an argument against philosophical skepticism. Skepticism claims that we don’t actually know what we think we know.

\(^8\)The reader interested in the criticism to Shapiro’s ideas can consult the review of his book \([36]\) by Matti Eklund, available online at [http://ndpr.nd.edu/news/vagueness-in-context/](http://ndpr.nd.edu/news/vagueness-in-context/)

\(^9\)The term neutrix was coined by Van der Corput in \([9]\) referring to groups of functions, with the intention of creating a mathematical tool that would enable a rigorous *ars negligendi*.
only a specific type of vagueness (of the type Sorites) and therefore is not intended as a theory for vagueness in general. Also, I am not by any means claiming that other theories are without value. For instance, the fuzzy logic approach has been quite successful in solving vagueness related to traffic and transportation processes (see for example [12] [3] [50] for other examples of applications of fuzzy set theory). According to Teodorović [42]

[...], a wide range of traffic and transportation engineering parameters are characterized by uncertainty, subjectivity, imprecision and ambiguity. Human operators, dispatchers, drivers and passengers use this subjective knowledge or linguistic information on a daily basis when making decisions.

Also,

The results obtained show that fuzzy set theory and fuzzy logic present a promising mathematical approach to model complex traffic and transportation processes [...]

However the fuzzy logic approach is also not without fault as model of imprecision, because it ultimately recourses to precise intervals to model imprecise situations. Moreover, it does not work with the actual error but only with an upper bound of the error. Those faults will be corrected with the present proposal.

4.1. Tools from Nonstandard Analysis. An “economical” version of Nonstandard Analysis due to Nelson [27, Chapter 4] (see also [10]) which is enough for the current purposes is presented below. Add to the language of conventional mathematics a new predicate st. One should read ‘x is standard’ for st(x). A formula is said to be internal if it does not involve the predicate st (i.e. it is a formula of conventional mathematics) and external otherwise. Assume the following:

(1) st(0);
(2) ∀n ∈ N(st(n) → st(n + 1));
(3) ∃ω(¬st(ω)).

Assume also the following axiom scheme.

External induction.

(Φ(0) ∧ ∀st n(Φ(n) → Φ(n + 1))) → ∀st n Φ(n)

Were Φ is an arbitrary formula, internal or external, and ∀st n Φ(n) is an abbreviation of ∀n(st(n) → Φ(n)). The first two axioms state that the natural numbers from conventional mathematics are all standard. Nevertheless, the third axiom states that there exist nonstandard natural numbers. External induction is a form of induction that allows to conclude that some property is true for all standard natural numbers by assuming that it is valid for 0 and that if it is valid for some standard natural number then it is also valid for its successor. Of course, the usual form of induction is still valid. However, one should be aware that (internal) induction is a principle from conventional mathematics and is therefore only applicable to internal formulas. Let me illustrate this with a simple example. Let Φ(n) := st(n). If we could apply internal induction to Φ the conclusion would be that ∀n st(n), in contradiction with the third assumption. Induction is applicable to subsets of natural numbers, so we are forced to conclude that S = {n : st(n)} is not a set. These kind of classes are sometimes called external sets.
Even this rather weak version of nonstandard analysis is enough to define different orders of magnitude.

**Definition 4.1.** A real number \( x \) is said to be:

1. limited, if there exists \( \text{st} (n) \in \mathbb{N} \) (i.e., \( n \in \mathbb{N} \land \text{st} (n) \)) such that \( |x| \leq n \).
2. unlimited, or infinitely large if \( x \) is not limited.
3. infinitesimal, or infinitely small if for any \( \text{st} (n) \in \mathbb{N}_+ \) one has \( |x| \leq \frac{1}{n} \).
4. appreciable if \( x \) is limited but not infinitesimal.
5. Two real numbers whose distance is infinitesimal are said to be infinitely close.

Sometimes it is not necessary to know precisely the value of a number to know its order of magnitude. This point of view together with intuitions and calculations from nonstandard asymptotics \[1\] lead Koudjeti and Van den Berg \[23, 24\] to introduce neutrices and external numbers. A neutrix is an additive convex subgroup of the reals and an external number is the algebraic sum of a real number with a neutrix. In a nonstandard framework, due to the existence of infinitesimals, there are many neutrices such as \( \emptyset \), the external set of all infinitesimals, and \( \mathcal{E} \), the external set of all limited numbers (numbers bounded in absolute value by a standard number). One can view an external number \( \alpha = a + A \) as the sum of a real number \( a \) with an “error”, given by a neutrix \( A \). In fact, the rules of calculation for external numbers are a sort of ”mellowed” version of the common rules of calculation of real numbers. Indeed, addition and multiplication in the external numbers are defined (with some abuse of notation) as follows.

**Definition 4.2.** Let \( \alpha = a + A \) and \( \beta = b + B \) be two external numbers, the sum and product of \( \alpha \) and \( \beta \) are defined as follows

\[
\alpha + \beta = a + b + \max\{A, B\}
\]

\[
\alpha \cdot \beta = ab + \max\{aB, bA, AB\}.
\]

The operations are well-defined because neutrices, being convex subgroups of the reals, are ordered by inclusion so the maximum of two neutrices is one of them and because the product of a real number and a neutrix is also a neutrix. Typically, external numbers are bounded but have neither infimum nor supremum and are stable for some (but not all!) translations, additions and multiplications. It is not difficult to prove that if \( A \) is a neutrix, then for all standard \( n \) it holds that \( nA = A \). In fact, it holds that \( cA = A \), for every appreciable real number \( c \). However if we let \( \omega \) be nonstandard, then \( A \subset \omega A \). Also, if we let \( \epsilon \) be infinitesimal we have \( \epsilon + \emptyset = \emptyset \) and even \( \emptyset + \emptyset = \emptyset \) but \( \epsilon \emptyset \subset \emptyset \). In this way, external numbers generate a calculus of propagation of errors not unlike the calculus of real numbers, allowing for total order and even for a sort of generalized Dedekind completeness property \[24, 11, 12\]. Thus, external numbers seem suitable as models of orders of magnitude or transitions with imprecise boundaries of the Sorites type, with the advantage of being possible to work directly with imprecisions and errors without recourse to upper bounds. Moreover, the external numbers have a rich algebraic structure. The (external) set of external numbers is a commutative regular semigroup for addition and the (external) set of external numbers which are not reduced to neutrices forms a commutative regular semigroup for multiplication. Although the distributive law does not always hold, necessary and sufficient conditions for it to hold were
given. Furthermore the structure has no zero divisors and is sufficiently strong to incorporate combinatorial laws such as the binomial law \([11]\).

A set of individual grains may be modeled by a standard subset of the external set of limited numbers (positive part of a neutrix) and the set of grains that form a heap may be modeled by the external set of the infinitely large numbers.

It should also be possible to capture with external sets some modalities, like the difference between a "good" approximation, allowing to obtain an adequately precise numerical result in some context, and a "bad", useless, one. The stability of orders of magnitude under some repeated additions justifies to model them by (convex) groups of real numbers.

4.2. A nonstandard point of view on paradoxical forms. I propose to replace the standard forms presented above by the following forms which involve reasoning with nonstandard methods.

If one replaces mathematical induction by external induction, the reasoning becomes:

\[
\begin{align*}
(S_a_1 \land \forall n (S_a_n \rightarrow S_a_{n+1})) &\rightarrow \forall n S_a_n \\
\exists \omega (\neg S_a_\omega)
\end{align*}
\]

So, if one admits that:

1. A single grain of wheat is not a heap.
2. If \(n\) is a standard number and if a set of \(n\) grains of wheat is not a heap then a set of \(n + 1\) grains of wheat is also not a heap.

One concludes that in the presence of a standard number of grains of wheat one does not have a heap. The heap arises when one has a nonstandard number \(\omega \simeq +\infty\) of grains of wheat.

The conditional form, using nonstandard analysis, becomes the following.

Let \(i\) be a standard natural number. If

\[
\begin{align*}
S_a_1 \\
S_a_1 \rightarrow S_a_2 \\
S_a_2 \rightarrow S_a_3 \\
\cdots \\
S_a_i \rightarrow S_a_{i+1}
\end{align*}
\]

Then, by *modus ponens*, the conclusion is \(S_a_i\), for \(i\) an arbitrarily large but (naive) standard number. In nonstandard analysis this is modeled by allowing *modus ponens* but only a standard (naive) number of times. One calls "naive" the natural numbers which can be obtained from zero by the successive addition of one. This corresponds to Reeb's famous slogan:

\[\text{Les entiers naïfs ne remplissent pas } \mathbb{N}. \quad [9]\]

I am in fact claiming that the formalization of the predicate 'is not a heap' should be an external predicate, where not being a heap means to possess a standard number of grains.

Soritical arguments share with external numbers the fact of being tolerant to small changes but not tolerant to large changes in relevant aspects. In fact, with external numbers, using the different orders of magnitude, it is possible to define rigorously what one means with terms such as 'small changes' or 'large changes'. The fact that large changes come as the result of the accumulation of small changes is now a very natural consequence of the theory.
A simple shift from the classical forms presented in Section 2.1 and in Section 2.2 to the forms using nonstandard concepts presented in Section 4.2 does not solve the problem. A million grains of wheat should form a heap and yet that is clearly a standard number of grains. However, both these forms suggest that the set of individual grains may be modeled by the external set of limited numbers (positive part of a neutrix) and the set of grains that form a heap may be modeled by the external set of the infinitely large numbers. Indeed ‘precise’ objects possess sharp bounds and can be modeled by standard sets. 'Vague' objects have no clear bounds and should be for this matter modeled by nonstandard sets which are given by external properties.

As seen in Section 3.2, epistemicists believe in the existence of sharp bounds for vague concepts, claiming that ignorance is somehow inevitable. The current proposal takes the opposite direction. Indeed, the tolerance of vague terms, such as 'heap', to small changes indicates that such terms do not have a sharp, definite bound. By using neutrices to model such terms it should be possible to avoid the paradox and explain the tolerance to small changes.

According to Keefe [21], degree theories fail to provide an acceptable account of vagueness and are forced to make an implausible commitment to a unique numerical assignment for each sentence. Smith [38] argues that an adequate account of vagueness must involve degrees of truth and that the main objections to this theory may be overcome. His fuzzy plurivaluationism theory seems overcomplicated for the present approach to the Sorites paradox. I believe that the problem with the fuzzy logic approach is the fact that precise numbers are used to model imprecise predicates. On the contrary, with external numbers it is possible to work directly with imprecisions and errors without recourse to upper bounds, for they have neither infimum nor supremum and are tolerant to appreciable (but not infinitely large) imprecisions. Moreover, since external numbers satisfy strong algebraic laws, similar to the ones of the real numbers, those calculations are quite simple to carry on.

A final remark concerns the strength of nonstandard axioms, which may introduce undesirable consequences of external modelling. As such, within a nonstandard theory, the proposed solution of the Sorites paradox

\[
\begin{align*}
&st(0) \\
&\forall n (st(n) \rightarrow st(n+1)) \\
&\exists \omega (\neg st(\omega))
\end{align*}
\]

implies, by the group property of the standard numbers, invariance by doubling, i.e.

\[\forall n (st(n) \rightarrow st(2n))\,.
\]

One easily imagines a soritical context where this is inappropriate. However,

\[
\exists n (st(n) \land \neg st(2n))
\]

is consistent with (1). In such a context \{1, 2\} might be an acceptable axiom system indeed, though of course at the price of losing some calculation properties.

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