Statistical Equivalence and Stochastic Process Limit Theorems*

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Abstract

A classical limit theorem of stochastic process theory concerns the sample cumulative distribution function (CDF) from independent random variables. If the variables are uniformly distributed then these centered CDFs converge in a suitable sense to the sample paths of a Brownian Bridge. The so-called Hungarian construction of Komlos, Major and Tusnady provides a strong form of this result. In this construction the CDFs and the Brownian Bridge sample paths are coupled through an appropriate representation of each on the same measurable space, and the convergence is uniform at a suitable rate.

Within the last decade several asymptotic statistical-equivalence theorems for nonparametric problems have been proven, beginning with Brown and Low (1996) and Nussbaum (1996). The approach here to statistical-equivalence is firmly rooted within the asymptotic statistical theory created by L. Le Cam but in some respects goes beyond earlier results.

This talk demonstrates the analogy between these results and those from the coupling method for proving stochastic process limit theorems. These two classes of theorems possess a strong inter-relationship, and technical methods from each domain can profitably be employed in the other. Results in a recent paper by Carter, Low, Zhang and myself will be described from this perspective.

1. Probability setting

1.1. Background

Let $F$ be the CDF for a probability on $[0,1]$. $F$ abs. cont., with

$$f(x) \triangleq \frac{\partial F}{\partial x} \text{ on } [0,1].$$

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Let \( X_1, \ldots, X_n \) iid from \( F \). \( \hat{F}_n \) denotes the sample CDF,
\[
\hat{F}_n(x) \triangleq \frac{1}{n} \sum_{j=1}^{n} \mathbf{1}_{[0,x]}(X_j).
\]

Let \( \hat{Z}_n \) denote the corresponding sample “bridge”,
\[
\hat{Z}_n(x) \triangleq \hat{F}_n(x) - F(x) \tag{1}
\]

Let \( W(t) \) denote the standard Wiener process on \([0,1]\) and let \( \hat{W}_n \) denote the white noise process with drift \( f \) and local variance \( f(t)/n \). Thus \( \hat{W}_n \) solves
\[
d\hat{W}_n(t) = f(t)dt + \sqrt{\frac{f(t)}{n}}dW(t).
\]

An alternate description of \( \hat{W}_n \) is that it is the Gaussian process with mean \( F(t) \) and independent increments having
\[
\text{var}\left(\hat{W}_n(t) - \hat{W}_n(s)\right) = \frac{1}{n} (F(t) - F(s)), \text{ for } 0 \leq s < t \leq 1.
\]

The analog of \( \hat{Z}_n \) is the Gaussian Bridge, defined by
\[
\hat{B}_n(t) = \frac{\hat{W}_n(t)}{W_n(1)} - F(t).
\]

There are various ways of describing the stochastic similarity between \( \hat{Z}_n \) and \( \hat{B}_n \). For example Komlos, Major, and Tusnady (1975, 1976) proved a result of the form

**Theorem (KMT):** Given any absolutely continuous \( F \) \( \{X_1, \ldots, X_n\} \) can be defined on a probability space on which \( \hat{B}_n \) can also be defined as a (randomized) function of \( \{X_1, \ldots, X_n\} \). This can be done in such a way that \( \hat{B}_n \) has the Gaussian Bridge distribution, above, and
\[
P_F \left( \sup_{t \in [0,1]} \sqrt{n} \left| \hat{Z}_n(t) - \hat{B}_n(t) \right| > a_n \right) \leq c. \tag{2}
\]

Here \( c > 0 \) and \( a_n \) are suitable positive constants with \( a_n \sim (d \log n)/\sqrt{n} \) for some \( d > 0 \). The process \( \hat{B}_n \) can be constructed as a (randomized) function of \( \hat{Z}_n \), that is, \( \hat{B}_n(t) = Q_n \left( \hat{Z}_n(t) \right) \). It should be noted that the construction depends on knowledge of \( F \).

[Various authors, such as Cs"{o}rg"{o} and Revesz (1981) and Bretagnolle and Massart (1989) have given increasingly detailed and precise values for \( a_n \) and \( c = c(a_n) \), and also uniform (in \( n \)) versions of \( \hat{B}_n \). These are not our focus.]
1.2. Extensions

1. Results like the above also extend to functional versions of the process $\hat{Z}_n$. Various authors including Dudley (1978), Massart (1989) and Koltchinskii (1994) have established results of the following form.

Let $q:[0,1] \rightarrow \mathbb{R}$ be of bounded variation. One can define

$$\hat{Z}_n(q) \triangleq \int qd(\hat{F}_n - F) = \int (F - \hat{F}_n) dq.$$  

(Thus, $\hat{Z}_n(x) = \hat{Z}_n(I_{[0,x]})$.) There is a similar definition for $\hat{B}_n(q)$ as a stochastic integral. (See, for example, Steele (2000).) Then the KMT theorem extends to a fairly broad, but not universal, class of functions, $Q$. That is, for each $F$, $\hat{B}_n$ can be defined to satisfy

$$P_F \left( \sup_{q \in Q} \sqrt{n} |\hat{Z}_n(q) - \hat{B}_n(q)| > a'_n \right) \leq c$$  

where $\{a'_n\}$ depends on $Q$. (3)

(For most classes $Q$, $a'_n \sqrt{n} \log n \rightarrow \infty$ so that $a'_n >> a_n$.)

2. Bretagnolle and Massart (1989) proved a similar result for inhomogeneous Poisson processes. Let $\{T_1, \ldots, T_N\}$ be (ordered) observations from an inhomogeneous Poisson process with cumulative intensity function $nf$ and, correspondingly, (local) intensity $nf$. Note that $N \sim \text{Poisson}(n)$ and conditionally given $N$ the values of $\{T_1, \ldots, T_N\}$ are the order statistics corresponding to an iid sample from the distribution $F$. In this context we continue to define $\hat{F}_n(t) = n^{-1} \left\{ \sum_{j=1}^{N} I_{[0,t]}(T_j) \right\}$ where the term in braces now has a Poisson distribution with mean $nF(t)$. Also, continue to define $\hat{Z}_n(t) \triangleq \hat{F}_n(t) - F(t)$ as in (1). (But, note that it is no longer true that $\hat{Z}_n(0) = 0$, w.p.1, as was the case in (1).)

Then versions of the conclusions (2) and (3) remain valid. We give an explicit statement since this result will provide a model for our later development.

**Theorem (BM):** Given any $n$ and any absolutely continuous $F$ the observations $\{T_1, \ldots, T_N\}$ of the inhomogeneous Poisson process can be defined on a probability space on which $\hat{B}_n$ can also be defined as a (randomized) function of $\{T_1, \ldots, T_N\}$. This can be done in such a way that $\hat{B}_n$ has the Gaussian Bridge distribution, above, and

$$P_F \left( \sup_{t \in [0,1]} \sqrt{n} |\hat{Z}_n(t) - \hat{B}_n(t)| > a_n \right) \leq c.$$  

(4)

Here $c > 0$ and $a_n$ are suitable constants with $a_n \sim d \log n / \sqrt{n}$.

**Remark:** Clearly there must be extensions of (3) that are valid for the Poisson case also, although we are not aware of an explicit treatment in the literature. Such a statement would conclude in this setting that

$$P_F \left( \sup_{q \in Q} \sqrt{n} |\hat{Z}_n(q) - \hat{B}_n(q)| > a'_n \right) \leq c$$  

where $\{a'_n\}$ depends on $Q$. (5)
2. Main results

The objective is a considerably modified version of (3) and (5) that is stronger in several respects and (necessarily) different in others. We will concentrate for most of the following on the statement (5) since our results are slightly stronger and more natural in this setting. The extension of (3) will be deferred to a concluding Section.

Expression (5) involves the target function $\hat{B}_n$. In the modified version the role of target function is instead played by $\tilde{W}_n$ which is the solution to the stochastic differential equation

$$d\tilde{W}_n(t) = g(t)dt + \frac{1}{2\sqrt{n}}dW(t)$$

where $g(t) = \sqrt{f(t)}$. An alternate description of $\tilde{W}_n$ is thus

$$\tilde{W}_n = G(t) + \frac{W(t)}{2\sqrt{n}}$$

where $G(t) = \int_0^t \sqrt{f(\tau)}d\tau$. (In the special case where $f$ is the uniform density, $f=1$, then $\tilde{W}_n = W_{4n}$.)

The role of the constructed random process $\hat{Z}_n$ is now played by a differently constructed process $\tilde{Z}_n$. As before $\tilde{Z}_n$ depends only on $\{T_1,\ldots,T_N\}$, and not otherwise on their CDF, $F$. This version also involves a large set, $\mathcal{F}$, of absolutely continuous CDFs. Both $\tilde{Z}_n$ and $F$ will be described later in more detail. Here are statements of the main results.

**Theorem 1:** Let $\mathcal{F}$ be a set of densities satisfying Assumption A or A’, below. Let $\mathcal{Q}$ be the set of all functions of bounded variation. Let $\{T_1,\ldots,T_N\}$ be an inhomogeneous Poisson process with local intensity $nf$. The process $\tilde{Z}_n$ can be constructed as a (randomized) function of $\{T_1,\ldots,T_N\}$, with the construction not depending on $f$. The Gaussian process $\tilde{W}_n$ having the distribution (7) can also be defined on this same space as a (randomized) function of $\{T_1,\ldots,T_N\}$. [This construction depends on $f$ on a set of probability at most $c_n$.] This can be done in such a way that

$$\sup_{f \in \mathcal{F}} P_f \left( \sup_{q \in \mathcal{Q}} \left| \tilde{Z}_n(q) - \tilde{W}_n(q) \right| > 0 \right) \leq c_n \to 0.$$  

To be more precise, the phrase in brackets refers to the fact that there is a basic construction, independent of $f$, and that this construction must then be modified on a set of measure at most $c_n$ with this set and the modification depending on $f$.

For the situation of iid variables, as in (4), a similar result holds. In this case the matching Gaussian process is again $\tilde{W}_n$, rather than the Brownian bridge of the KMT theorem.

**Theorem 2:** Let $\mathcal{F}$ be a set of densities satisfying Assumption B, below. Let $\mathcal{Q}$ be the set of all functions of bounded variation. Given any $n$ and $f \in \mathcal{F}$, iid variables $\{X_1,\ldots,X_n\}$ with density $f$ can be defined on a probability space. A process $\tilde{Z}_n$ can
be constructed as a (randomized) function of \( \{X_1, \ldots, X_n\} \), with the construction not depending on \( f \). The Gaussian process \( \tilde{W}_n \) having the distribution (7) can also be defined on this same space as a (randomized) function of \( \{X_1, \ldots, X_n\} \). [This construction depends on \( f \), but only on a set of probability at most \( c_n \).] This can be done in such a way that

\[
\sup_{f \in \mathcal{F}} P_f \left( \sup_{q \in \mathcal{Q}} \left| \tilde{Z}_n(q) - \tilde{W}_n(q) \right| > 0 \right) \leq c_n \to 0.
\] (9)

3. Statistical background

3.1. Settings

The first purpose of the discussion here is to motivate the probabilistic results described above. A second purpose is to state the result on which to base the proof of Theorem 1. The setting involves two statistical formulations:

**Formulation 1** (nonparametric inhomogeneous Poisson process): The observations are \( T = \{T_1, \ldots, T_N\} \) from the Poisson process with local intensity \( nf, f \in \mathcal{F} \). The problem is “nonparametric” because the “parameter space”, \( \mathcal{F} \), is a very large set – too large to be smoothly parameterized by a mapping from a (subset of) a finite dimensional Euclidean space. Some possible forms for \( \mathcal{F} \) are discussed below. The statistician desires to make some sort of inference, \( \delta \), (possibly randomized) based on the observation of \( X \).

**Formulation 1’** (nonparametric density with random sample size): The relation between Poisson processes and density problems has been mentioned above. As a consequence, Problem 1 is equivalent to a situation where the observations are \( \{X_1, \ldots, X_N\} \) with \( N \sim \text{Poisson}(n) \) and \( \{X_1, \ldots, X_N\} \) the order statistics from a sample of size \( N \) from the distribution with density \( f \). Clearly, this situation is closely related to the more familiar one in which the observations are \( \{X_1, \ldots, X_n\} \) with \( n \) specified in advance.

**Formulation 1”** (nonparametric density with fixed sample size): This formulation refers to the more conventional density setting in which the observations are \( \{X_1, \ldots, X_n\} \) iid with density \( f \).

**Formulation 2** (white noise with drift): The statistician observes a White noise process \( d\tilde{W}_n(t), t \in [0,1] \), with drift \( g \in \mathcal{G} \) and local variance \( 1/4n \). Thus

\[
d\tilde{W}_n(t) = g(t)dt + \frac{1}{2\sqrt{n}}dW(t),
\]

and \( \tilde{W}_n(t) - G(t) = W(t)/2\sqrt{n} \) where \( G(t) = \int_0^t g(\tau)d\tau \). Again \( \mathcal{G} \) is a very large – hence “nonparametric” – parameter space. Throughout, \( \mathcal{G} \subset \mathcal{L}_2 = \{g : g^2 < \infty\} \). As of now, there need be no relation between \( f \) in Formulation 1 and \( g \) in Formulation 2, but such a relation will later be assumed in connection with Theorem 1, where

\[
g = \sqrt{f} \quad \text{and} \quad \mathcal{G} = \{\sqrt{f} : f \in \mathcal{F}\}. \tag{10}
\]
This can alternatively be considered as a statistical formulation having parameter space $F$ under the identification (10). We take this point of view in the BCLZ theorem, below.

### 3.2. Constructive asymptotic statistical equivalence

Here is one definition of the strongest form of such an equivalence.

**Definition** (asymptotic equivalence): Let $P_j^{(n)} = (X_j^{(n)}, B_j^{(n)}, F_j^{(n)})$ $j = 1, 2, n = 1, 2, \ldots$ be two sequences of statistical problems on the same sequence of parameter spaces, $\Theta^{(n)}$. Hence, $F_j^{(n)} = \{F_j^{(n)} : \theta \in \Theta^{(n)}\}$. Then $\Pi_1$ and $\Pi_2$ are asymptotically equivalent if there exist (randomized) mappings $Q_j^{(n)} : X_j^{(n)} \rightarrow X_k^{(n)}$, $j, k = 1, 2, k \neq j$, such that

$$
\sup_{\theta \in \Theta^{(n)}} \left\| F_j^{(n)}(\cdot) - \int Q_k^{(n)}(\cdot|x_k) F(\cdot) dx_k \right\|_{TV} = c_n \rightarrow 0, j, k = 1, 2, k \neq j, \quad (11)
$$

where $\|\cdot\|_{TV}$ denotes the total variation norm.

This definition involves a reformulation of the general theory originated by LeCam (1953, 1964). See also Le Cam (1986), Le Cam and Yang (2000), van der Vaart (2002) and Brown and Low (1996) for background on this theory including several alternate versions of the definition and related concepts, a number of conditions that imply asymptotic equivalence, and many applications to a variety of statistical settings. Note that both Formulations 1 and 2 involve an index, $n$, and can thus be considered as sequences of statistical problems in the sense of the definition.

### 3.3. Spaces of densities (or intensities)

Suitable families of densities, $F$, can be defined via Besov norms with respect to the Haar basis. The Besov norm with index $\alpha$ and shape parameters $p = q$ can most conveniently be defined via the stepwise approximants to $f$ at resolution level $k$. These approximants are defined as

$$
\tilde{f}_k(t) = \sum_{\ell=0}^{2^{k-1}} I_{\ell/2^k, (\ell+1)/2^k} \left( t/2^k \right) \int_{\ell/2^k}^{(\ell+1)/2^k} 2^k f,
$$

and the Besov($\alpha,p$) norm is defined as

$$
\| f \|_{\alpha,p} = \left\{ |\tilde{f}_0|^p + \sum_{k=0}^{\infty} 2^{pk\alpha} \|\tilde{f}_k - \tilde{f}_{k+1}\|_p^p \right\}^{1/p}.
$$

The statement of Theorem 1 can now be completed by stating the assumption on $F$ needed for its validity.
Assumption A: \( \mathcal{F} \) satisfies
\[
\mathcal{F} \subset \left\{ f : \inf_{0 \leq x \leq 1} f(x) \geq \varepsilon_0 \right\} \text{ for some } \varepsilon_0 > 0 \tag{12}
\]
and \( \mathcal{F} \) is compact in both Besov(1/2,2) and Besov(1/2,4).

Other function spaces are also conventional for nonparametric statistical applications of this type. The most common of these are based on either the Lipschitz norm \( \|f\|_{\beta}^{(L)} \) or the Sobolev norm \( \|f\|_{\beta}^{(S)} \). These are defined for \( \beta \leq 1 \) by
\[
\|f\|_{\beta}^{(L)} = \sup_{0 \leq x < y \leq 1} \frac{|f(y) - f(x)|}{|y - x|^\beta}, \quad \|f\|_{\beta}^{(S)} = \sum_{-\infty}^{\infty} k^{2\beta} \vartheta_k^2
\]
where \( \vartheta_k = \int_0^1 f(x) e^{ik2\pi x} dx \) denote the Fourier coefficients of \( f \). (Both spaces have natural definitions for \( \beta \geq 1 \) as well, but we need consider here only the case \( \beta \leq 1 \).)

The following implies Assumption A and hence also suffices for validity of Theorem 1.

Assumption A’: \( \mathcal{F} \) satisfies (12), and is bounded in the Lipschitz norm with index \( \beta \), and is compact in the Sobolev norm with index \( \alpha \), where \( \alpha \geq \beta \) and either \( \beta > 1/2 \) or \( \alpha \geq 3/4 \) and \( \alpha + \beta \geq 1 \).

The following assumption is noticeably stronger than either A’ or A, and is used in Theorem 2.

Assumption B: \( \mathcal{F} \) satisfies (12) and is bounded in the Lipschitz norm with index \( \beta \), where \( \beta > 1/2 \).

For more information about the relation of these spaces in this context see Brown, Cai, Low and Zhang (2002) and Brown, Carter, Low and Zhang (2002) (referred to as BCLZ below).

3.4. Statistical equivalence theorems

BCLZ then extended earlier results of Nussbaum (1996) and Klemela and Nussbaum (1998) to prove the following basic result:

**Theorem a (BCLZ):** Consider the statistical Formulations 1 and 2 with the parameter space \( \mathcal{F} \) and the relation (10). Assume \( \mathcal{F} \) satisfies Assumption A (or A’). Then the sequences of statistical problems defined in these two formulations are asymptotically statistically equivalent.

BCLZ describes in detail a construction of \( \tilde{Z}_n \) as a (randomized) function of \( \{T_1, \ldots, T_n\} \). (More precisely, BCLZ describes the construction of the Haar basis representation of \( \tilde{Z}_n \), from which \( Z_n \) can directly be recovered.) This construction is invertible, in that \( \{T_1, \ldots, T_n\} \) can be recovered as a function of \( \tilde{Z}_n \). Further, BCLZ shows that both \( \tilde{Z}_n \) and \( \tilde{W}_n \) can be represented on the same probability space so that their distributions, \( P_{\tilde{Z}_n} \) and \( P_{\tilde{W}_n} \), say, satisfy
\[
\|P_{\tilde{Z}_n} - P_{\tilde{W}_n}\|_{TV} \to 0.
\]
The mappings \( \{ Q^{(n)}_j: j=1,2, n=1,2,... \} \) that yield the equivalence of the above theorem can then be directly inferred from this construction. To save space here we refer the reader to that paper or Brown (2002) for details of the construction and proof. It can be remarked that these bear considerable similarity to parts of the construction and proof in Bretagnolle and Massart (1989) and other proofs of KMT type theorems. But there are also some basic differences, especially those related to the appearance of the square-root in the fundamental relation (10) and the total variation norm in the definition of equivalence. In addition, the fact that \( \Theta \) is uniform in \( Q \) and \( F \) entails the need for various refinements in the proof.

Theorem 1 is now an immediate logical consequence of this result from BCLZ and the following lemma.

**Lemma:** Suppose \( P_j^{(n)} = (X_j^{(n)}, B_j^{(n)}) \) are asymptotically equivalent sequences of statistical problems on the same sequence of parameter spaces, \( \Theta^{(n)} \). Let \( \{ Q_j^{(n)}: j=1,2, n=1,2,... \} \) denote a sequence of mappings that define this equivalence, as in (11). Then there are non-randomized mappings \( \{ \tilde{Q}_j^{(n)}: j=1,2, n=1,2,... \} \) such that

\[
P_f(\tilde{Q}_j^{(n)} \leq Q_j^{(n)}) \geq 1 - c_n \text{ for every } f \in F_j^{(n)}, \quad j = 1,2, n = 1,2, ...
\]

and for every \( \theta \in \Theta^{(n)} \)

\[
P_{f,\theta}(\tilde{Q}_j^{(n)}(X_j^{(n)}) \in A) = P_{f,\theta}(X_k^{(n)} \in A), \quad \theta \in \Theta^{(n)}
\]

for every measurable \( A \subset X_k^{(n)} \), \( j, k = 1,2, j \neq k, n = 1,2, \ldots \).

**Proof of Lemma:** Fix \( n, j, k \neq j, \theta \in \Theta^{(n)} \). Let \( F_k \) denote the distribution under \( \theta \) of \( X_k^{(n)} \) and let \( F'_k \) denote the distribution under \( \theta \) of \( Q_j^{(n)}(X_j^{(n)}) \). Let \( H = \min(F_k, F'_k) \). Let \( \infty \geq f'_k = \frac{dF'_k}{dH} \geq 1 \). Then define \( \tilde{Q}_j^{(n)} \) as a version of the randomized map satisfying

\[
\tilde{Q}_j(B|x) = \frac{1}{f'_k(x)}Q_j(B|x) + \frac{f'_k - 1}{f'_k}(F'_k(B) - H(B)).
\]

This completes the proof of the lemma, and consequently also that of Theorem 1. \( \Box \)

Theorem 2 requires a slightly different fundamental result. The following result is the foundation for the proof of Theorem 2. It is adapted from Theorem 2 of BCLZ. This result closely resembles Theorem a, above, but as noted in BCLZ it appears to require a modified construction for its proof. The argument there is based heavily on results in Carter (2001).

**Theorem b** (BCLZ): Consider the statistical Formulations 1” and 2 with the parameter space \( F \) and the relation (1). Assume \( F \) satisfies Assumption B. Then the sequences of statistical problems defined in these two formulations are asymptotically statistically equivalent.
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