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Canonical Almost Geodesic Mappings of the First Type of Spaces with Affine Connections onto Generalized $m$-Ricci-Symmetric Spaces

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Abstract: In the paper we consider almost geodesic mappings of the first type of spaces with affine connections onto generalized 2-Ricci-symmetric spaces, generalized 3-Ricci-symmetric spaces, and generalized $m$-Ricci-symmetric spaces. In either case the main equations for the mappings are obtained as a closed system of linear differential equations of Cauchy type in the covariant derivatives. The obtained results extend an amount of research produced by N.S. Sinyukov, V.E. Berezovski, J. Mikeš.

Keywords: canonical almost geodesic mappings; Cauchy-type PDEs; space with affine connection; Ricci symmetric space

MSC: 53B05, 35R01

1. Introduction

In the sixties of the preceding century, N.S. Sinyukov [1] considered almost geodesic mappings of Riemannian and affinely connected spaces, that generalize the geodesic mappings in a natural way. The main results were presented in a monograph [2] and an expository article [3]. Geodesic mappings and their generalizations are studied in detail in monographs [4–6] and researches [7–9] by J. Mikeš et al.

The theory goes back to the paper [10] by T. Levi-Civita, in which the problem on the search for Riemannian spaces with common geodesics was stated and solved in a special coordinate system. We note a remarkable fact that this problem is related to the study of equations of dynamics of mechanical systems. That direction is developing according to Petrov’s plan to build models of physical processes using mappings and transformations [11]. For example, in papers [12,13] is shown the possible physical application of almost geodesic mappings.

N.S. Sinyukov specified three types of almost geodesic mappings $\pi_1$, $\pi_2$, $\pi_3$. The problem of completeness of classification had long remained unresolved. Berezovski and Mikeš [14,15] proved that for $n > 5$ other types of almost geodesic mappings except for $\pi_1$, $\pi_2$, and $\pi_3$ do not exist. The authors have found conditions for the almost geodesic mappings $\pi_1$, $\pi_2$, $\pi_3$ intersections. It is proved that if an almost geodesic mapping $f$ is simultaneously $\pi_1$ and $\pi_2$, then $f$ is a mapping of affine connection spaces with a preserved...
linear complex of geodesic lines. If the mapping $f$ is simultaneously $\pi_1$ and $\pi_3$, then $f$ is a mapping of affine connection spaces with a preserved quadratic complex of geodesic lines. The mappings preserving above mentioned complexes studied by V.M. Chernyshenko [16] and V.A. Dobrovolski [17].

The theory of almost geodesic mappings was developed by V.A. Aminova, A.M. Mukhamedov [18], V.S. Sobchuk [19,20], N.Y. Yablonskaya [21,22], V.E. Berezovski, J. Mikeš [14,15,23–35], O. Belova, J. Mikeš, K. Strambach [36,37], M.S. Stanković, Lj. Velićnović, N. Vesić, M.Lj. Zlatanović [38–44] et al.

N.S. Sinyukov [3] proved that the main equations for canonical almost geodesic mappings of spaces with affine connections onto Ricci-symmetric spaces can be written as a closed system of partial differential equations of Cauchy type in covariant derivatives. It follows that the general solution of the system depends on a finite number of essential parameters.

The results were extended by Berezovski and Mikeš [28,34] to the cases of canonical almost geodesic mappings of the first type of spaces with affine connections onto Riemannian spaces and canonical almost geodesic mappings of the first type of spaces with affine connections onto generalized Ricci-symmetric spaces.

The paper is devoted to the study of canonical almost geodesic mappings of type $\pi_1$ of spaces with affine connections onto generalized 2-Ricci-symmetric, generalized 3-Ricci-symmetric, and generalized $m$-Ricci-symmetric spaces. The fundamental equations for the mappings are derived in the form of the closed systems of Cauchy-type PDEs. The maximum number of essential parameters on which general solutions of these systems depend was estimated. V.R. Kaigorodov [45] comprehensively studied generalized symmetric and recurrent spaces from the point of view of the General Theory of Relativity. This review contains 97 citations and is a thorough analysis of this issue.

Let us note that in works [7,8,24,46–50] were studied geodesic and holomorphically projective mappings of above mentioned spaces. For the other mappings the main equations are obtained as closed systems of PDEs of Cauchy type, see [5,6,51–54].

2. Basic Definitions of Almost Geodesic Mappings of Spaces with Affine Connections

Let us recall the basic definition, formulas and theorems of the theory presented in [2,5,6].

Consider an $n$-dimensional space $A_n$ with affine connection $\Gamma^h_{ij}$ without torsion. The space is referred to coordinates $x^1, x^2, \ldots, x^n$. We assume that all functions under consideration are sufficiently differentiable, and we suppose that $n > 2$.

A curve defined in a space with affine connection is called almost geodesic if there exists a two-dimensional (differentiable) distribution $D$ parallel along the curve (relative to the affine connection) such that for any tangent vector of the curve its parallel translation along the curve belongs to the distribution $D$ [5].

A mapping $f : A_n \rightarrow \overline{A}_n$ is called almost geodesic if any geodesic curve of $A_n$ is mapped under $f$ onto an almost geodesic curve in $\overline{A}_n$.

Suppose, that a space $A_n$ with affine connection $\Gamma^h_{ij}(x)$ admits a mapping $f$ onto a space $\overline{A}_n$ with affine connection $\overline{\Gamma}^h_{ij}(x)$, and the spaces are referred to a common coordinate system $x^1, x^2, \ldots, x^n$ with respect to the mapping.

The tensor

$$P^h_{ij}(x) = \overline{\Gamma}^h_{ij}(x) - \Gamma^h_{ij}(x)$$

(1)

is called a deformation tensor of the connections $\Gamma^h_{ij}(x)$ and $\overline{\Gamma}^h_{ij}(x)$ with respect to the mapping $f$. The symbols $\Gamma^h_{ij}(x)$ and $\overline{\Gamma}^h_{ij}(x)$ are components of affine connections of the spaces $A_n$ and $\overline{A}_n$, respectively.
It is known [2] that in order that a mapping of a space \( A_n \) onto a space \( \overline{A}_n \) to be almost geodesic, it is necessary and sufficient that the deformation tensor \( P^h_{ij}(x) \) of the mapping \( f \) in the common coordinate system \( x^1, x^2, \ldots, x^n \) has to satisfy the condition

\[
A^h_{ijk}\lambda^i\lambda^j\lambda^k = a \cdot P^h_{ijk}\lambda^i\lambda^j + b \cdot \lambda^h,
\]

where \( \lambda^h \) is an arbitrary vector, \( a \) and \( b \) are certain functions of variables \( x^1, x^2, \ldots, x^n \) and \( \lambda^1, \lambda^2, \ldots, \lambda^n \). The tensor \( A^h_{ijk} \) is defined as

\[
A^h_{ijk} \equiv P^h_{ijk} + P^h_{ijak} - P^h_{ikaj}
\]

where the comma “,” denotes the covariant derivative with respect to the connection of the space \( A_n \).

N. S. Sinyukov [2] distinguished three kinds of almost geodesic mappings, namely \( \pi_1 \), \( \pi_2 \), and \( \pi_3 \) characterized, by following conditions for the deformation tensor.

A mapping \( f : A_n \to \overline{A}_n \) is called an almost geodesic mapping of type \( \pi_1 \), if the conditions

\[
A^h_{i(jk)} = \delta^h_{(a)j(k)} + b_1 P^h_{i(kj)},
\]

are satisfied, where \( a_{ij} \) is a symmetric tensor, \( b_1 \) is a covariant vector and \( \delta^h_{ij} \) is the Kronecker delta. We denote by the round parentheses a "symmetrization" without division with respect to the mentioned indices.

If in the Equation (2) the condition \( b_1 \equiv 0 \) holds, then the almost geodesic mapping \( \pi_1 \) is called canonical. It is known [3] that any almost geodesic mapping \( \pi_1 \) may be written as the composition of a canonical almost geodesic mapping of type \( \pi_3 \) and a geodesic mapping.

A mapping \( f : A_n \to \overline{A}_n \) is called an almost geodesic mapping of type \( \pi_2 \), if the conditions

\[
P^h_{ij} = \delta^h_{i(j \varphi_j)} + \rho^h_{i(j \varphi_j)} \quad \text{and} \quad P^h_{(ij)} + P^h_{k(i \varphi_j)} = \delta^h_{(i(k \varphi_j)} + P^h_{(i(k \varphi_j)}
\]

holds, where \( \varphi_i, \varphi_j, \mu_i, \rho_i \) are covectors, \( P^h_{ij} \) is a tensor of type \((1, 1)\).

A mapping \( f : A_n \to \overline{A}_n \) is called an almost geodesic mapping of type \( \pi_3 \), if the conditions

\[
P^h_{ij} = \delta^h_{i(j \varphi_j)} + \theta^h_{i(j \varphi_j)} \quad \text{and} \quad \theta^h_{ij} = \rho \cdot \delta^h_{ij} + \theta^h_{ai}
\]

holds, where \( \theta^h \) is a torse forming vector, \( \eta_j, a_i \) are covectors, \( a_{ij} \) is a symmetric tensor and \( \rho \) is a function.

As we have noted above, the mentioned types of mappings \( \pi_1, \pi_2 \) and \( \pi_3 \) can intersect. The classification completeness for spaces with affine connections of dimension \( n > 5 \) has been proved in [14,15].

3. Ricci-Symmetric, Generalized Ricci-Symmetric, 2-Ricci-Symmetric and Generalized 2-Ricci-Symmetric Spaces

A space \( \overline{A}_n \) with an affine connection is called Ricci-symmetric if its Ricci tensor \( \overline{R}_{ij} \) satisfies the condition

\[
\overline{R}_{ijk}\lambda = 0.
\]

The symbol “;” denotes a covariant derivative with respect to the connection of the space \( \overline{A}_n \).

In [3] N.S. Sinyukov considered canonical almost geodesic mappings of the first type of spaces with affine connections onto Riemannian Ricci-symmetric spaces. Taking into account the relations between the Riemannian tensors \( \overline{R}^h_{ijk} \) and \( R^h_{ijk} \) of the spaces \( \overline{A}_n \) and \( A_n \) respectively as follows from (1), the Equations (2) for canonical almost geodesic mappings of the first type of spaces with affine connections were written in the form.
\[3(p_{ij}^h + p_{ka}^h p_a^{ij}) = R_{(ij)k}^h - \bar{R}^h_{(ij)k} + \delta^h_{(k} a_{ij)} \tag{3}\]

We give the following example. From the Equation (3) with a condition of preserving Riemannian tensor and \(a_{ij} = 0\), we obtain

\[p_{ij}^h = -p_{ka}^h p_a^{ij}.\]

In a flat space, the above equation is completely integrable. Therefore it has a solution with respect to \(P_t^h(x)\) for any initial condition \(P_t^h(x_0)\). If \(P_t^h(x_0) \neq \delta^h_{ij} \psi_{ij}(x_0)\) then the solution \(P_t^h(x_0)\) generates an almost geodesic mapping \(A_n \rightarrow \bar{A}_n\) which is not geodesic.

Considering (3) as a system of Cauchy type in \(A_n\) with respect to the functions \(P_{ij}^h\), one has found the integrability conditions of the system in the form

\[
\begin{align*}
\bar{R}_{(ij)[kl]}^h &= R_{(ij)[kl]}^h + \delta^h_{(i} a_{jk)} - \delta^h_{(i l a_{j k)}} + 3(p_{ik}^h \bar{R}_{akl}^h - p_{i k}^h R_{(a)k l}^h) \\
&\quad - p_{ak}^h (R_{(ij)l}^h - \bar{R}_{(ij)l}^h + \delta^h_{(i} a_{j l)} + p_{ij}^h (R_{(k)l}^h - \bar{R}_{(k)l}^h + \delta^h_{(k} a_{j l)})
\end{align*}
\]

where the square brackets \([\ ]\) denote an antisymmetrization (or, alternation) without division with respect to the mentioned indices.

Let us express in the left-hand side the covariant derivatives with respect to the connection of \(A_n\) in terms of the covariant derivatives with respect to the connection of \(\bar{A}_n\). Taking account of (1), we obtain

\[
\bar{R}_{(ij)[kl]}^h = \delta_{(i} a_{jk)} - \delta_{(i l a_{j k)}} + \theta_{ijkl}^h, \tag{4}\]

where

\[
\begin{align*}
\theta_{ijkl}^h &= R_{(ij)[kl]}^h + 3(p_{ij}^h \bar{R}_{akl}^h - p_{i k}^h R_{(a)k l}^h) - p_{ak}^h (R_{(ij)l}^h - \bar{R}_{(ij)l}^h + \delta_{(i} a_{j l)}) \\
&\quad + p_{al}^h (R_{(ij)k}^h - \bar{R}_{(ij)k}^h + \delta_{(i} a_{j k}) - p_{i l}^h \bar{R}_{(a)k l}^h - p_{i l}^h \bar{R}_{(a)k l}^h + \bar{R}_{(a)k l}^h + p_{k l}^h \bar{R}_{(a)k l}^h + p_{k l}^h \bar{R}_{(a)k l}^h)
\end{align*}
\]

Here the symbol \(|a|\) means that in applying symmetrization the index \(a\) is omitted. Using the Ricci identity, let us write the conditions (4) in the form

\[
\bar{R}_{iklj}^h + \bar{R}_{ijkl}^h = \delta_{(i} a_{jk)} - \delta_{(i l a_{j k)}} + \theta_{ijkl}^h. \tag{5}\]

Contracting (5) for \(h\) and \(k\), we get the relations for covariant derivatives of the Ricci tensor \(\bar{R}_{ij}\) for the space \(\bar{A}_n\)

\[
\bar{R}_{ij} = (n + 1) a_{ij, l} - a_{(ij)} + \theta_{ijkl}^h. \tag{6}\]

A space \(\bar{A}_n\) with an affine connection is called generalized Ricci-symmetric if the Ricci tensor \(\bar{R}_{ij}\) for the space satisfies the condition

\[\bar{R}_{ijk} + \bar{R}_{kji} = 0.\]

The concept of generalized Ricci-symmetric spaces was first introduced in \([28,34]\). The papers are devoted to the study of canonical almost geodesic mappings of spaces with affine connections onto the above mentioned spaces. It is proved that the main equations for the mappings can be obtained as a closed system of Cauchy-type differential equations in covariant derivatives.

A space \(\bar{A}_n\) with an affine connection is called 2-Ricci-symmetric if the Ricci tensor \(\bar{R}_{ij}\) for the space satisfies the condition

\[\bar{R}_{ijkm} = 0.\]
A space $\overline{A}_n$ with an affine connection is called generalized 2-Ricci-symmetric if its Ricci tensor $\overline{R}_{ij}$ satisfies the condition

$$\overline{R}_{ijkl} + \overline{R}_{jikl} = 0. \tag{7}$$

4. Canonical Almost Geodesic Mappings of type $\pi_1$ of Spaces with Affine Connections onto Generalized 2-Ricci-Symmetric Spaces

The Equation (6) was obtained for a canonical almost geodesic mapping of type $\pi_1$ of a space with an affine connection $A_n$ onto another space with an affine connection $\overline{A}_n$.

First we differentiate covariantly the Equation (6) with respect to the connection of the space $A_n$. Then taking account of (1), let us apply the formulas

$$\overline{R}_{ijkl} = \overline{R}_{ijlk} = \overline{R}_{ikjl} - \overline{R}_{iklj} = \overline{R}_{lilk} - \overline{R}_{ilkj}, \tag{8}$$

Finally, we get

$$\overline{R}_{iljk} + \overline{R}_{jl;ik} = (n + 1)a_{ijl;k} - a_{i;lk} - a_{ij;k} - C_{ijkl} \tag{9}$$

where

$$C_{ijkl} = -\theta^n_{ij|a} + (\overline{R}_{ijl;k} + \overline{R}_{jl;k} - 2\overline{R}_{jkl} + \overline{R}_{jkl} - \overline{R}_{ijkl} + \overline{R}_{ijkl} - \overline{R}_{ijkl} - \overline{R}_{ijkl})p_{ik}^a \tag{10}$$

Moreover, let us consider canonical almost geodesic mappings of type $\pi_1$ of spaces $A_n$ with affine connections onto generalized 2-Ricci-symmetric spaces $\overline{A}_n$. Hence the Ricci tensor $\overline{R}_{ij}$ for the space $\overline{A}_n$ satisfies the conditions. Then the Equation (9) may be written in the form

$$(n + 1)a_{ijl;k} - a_{i;lk} - a_{ij;k} = -C_{ijkl}, \tag{11}$$

when $C_{ijkl}$ is defined by the formulas (10).

Let us interchange the indices $l$ and $j$ in (11) and symmetrize in the indices $i$ and $j$. Then we have

$$a_{i;lk} + a_{ij;k} = -\frac{1}{n} C_{ijkl} + \frac{2}{n} a_{ij;k}. \tag{12}$$

The Equation (11) by means of (12) can be written in the form

$$\frac{n^2 + n - 2}{n} a_{ij;k} = -C_{ijkl} - \frac{1}{n} C_{ijkl}. \tag{13}$$

In the following we have assumed that a space $A_n$ with an affine connection is given. Then taking account of the structure of the tensor $C_{ijkl}$ which was determined by formula (10), we see that the left hand side of the Equation (13) depends on the unknown functions $p_{ij}^a(x), a_{ij}(x), a_{ij,k}(x), \overline{R}_{ijkl}(x), \overline{R}_{ijkl,m}(x)$.

Differentiate covariantly the conditions of integrability (4) with respect to the connection of the space $\overline{A}_n$. Then express in the right-hand side the covariant derivatives with respect to the connection of $\overline{A}_n$ in terms of the covariant derivatives with respect to the connection of the space $A_n$. When we make use of the Ricci identity, we obtain

$$\overline{R}^h_{ijkl} = \overline{R}^h_{ijkl} + \delta^h_{ijkl}a_{km} = \delta^h_{ijkl}a_{km} + T^h_{ijkl}, \tag{14}$$
where

\[
T^h_{ijklm} = R^h_{xmk} R^a_{(ij)x} - R^x_{imk} R^h_{(ia)l} - R^h_{imk} R^x_{(ja)} - R^a_{imk} R^h_{(ja)l} - p_m^h \delta^h_{(a)kj} l - p_m^h \delta^h_{(a)jk} l - p_{ma}^h \delta^h_{(a)lj} - p_m^h \delta^h_{(a)lj} - p_{ma}^h \delta^h_{(a)jk} l \\
- p_{ma}^h \delta^h_{(a)kj} l - p_{ma}^h \delta^h_{(a)jk} l - p_m^h \delta^h_{(a)lj} - p_m^h \delta^h_{(a)lj} - p_{ma}^h \delta^h_{(a)jk} l \\
+ p_m^h \delta^h_{(a)lj} l - p_m^h \delta^h_{(a)lj} l - p_{ma}^h \delta^h_{(a)jk} l - p_{ma}^h \delta^h_{(a)jk} l \\
- \delta^h_{ijklm} + p_{am}^h \alpha^h_{ijkl} - p_{ml}^h \alpha^h_{ijkl} - p^h_{ml} \alpha^h_{ijkl} - p^h_{ml} \alpha^h_{ijkl}.
\]

Alternating the equations (14) with respect to the indices \(l\) and \(m\), we obtain

\[
R^h_{ijklm} - R^h_{ijklm} = \delta^h_{(i)jmkl} - \delta^h_{(i)jmlk} + T^h_{ijklm} + R^h_{ijkl} R^x_{ml} + R^h_{ij} R^h_{klm} \\
- R^h_{ijkl} R^h_{xml} + R^h_{ijkl} R^h_{xim} + \delta^h_{(a)jk} R^h_{iml} + \delta^h_{(a)ik} R^h_{jml} + \delta^h_{(a)ij} R^h_{klm} - \delta^h_{(a)jk} R^h_{ximl}.
\]  

(15)

Taking account of the properties of a curvature tensor \(R^h_{ijk}\), we may write the conditions (15) in the form

\[
R^h_{ijkl} + R^h_{jikl} = \delta^h_{(i)jmkl} - \delta^h_{(i)jmlk} - N^h_{ijklm},
\]

(16)

where

\[
N^h_{ijklm} = T^h_{ijklm} + R^h_{ij} R^x_{klm} + R^x_{ijkl} R^h_{jm} + R^h_{ij} R^x_{kml} - R^h_{ij} R^x_{klm} \\
+ \delta^h_{(a)jk} R^h_{iml} + \delta^h_{(a)ik} R^h_{jml} + \delta^h_{(a)ij} R^h_{klm} - \delta^h_{(a)jk} R^h_{ximl}.
\]

Alternating the equations (16) with respect to the indices \(j\) and \(k\), we get

\[
R^h_{ijkl} - R^h_{kml;ij} = \delta^h_{(i)jmkl} - \delta^h_{(i)jmlk} + \delta^h_{(i)jk} R^h_{iml} + \delta^h_{(i)ik} R^h_{jml} - \delta^h_{(i)ij} R^h_{klm} + \delta^h_{(i)jk} R^h_{ximl}.
\]

(17)

Let us interchange \(i\) and \(k\) in (16) and subtract it from (17). Then we have

\[
2R^h_{ijkl} = \delta^h_{(i)jm} - \delta^h_{(i)a} R^h_{kmj} + \delta^h_{(i)km} - \delta^h_{(i)jm} + \delta^h_{(i)klj} - \delta^h_{(i)aml} - \Omega^h_{ijklm},
\]

(18)

where

\[
\Omega^h_{ijklm} = N^h_{ijklm} + N^h_{ijlkm} - N^h_{ijklm} - N^h_{ijlkm} + N^h_{ijlkm} + N^h_{ijklm} - N^h_{ijklm} + N^h_{ijklm} - N^h_{ijklm} + N^h_{ijklm} + N^h_{ijklm} - N^h_{ijklm} + N^h_{ijklm} + N^h_{ijklm} - N^h_{ijklm}.
\]

Let us express in the left-hand side of the Equation (18) the covariant derivatives with respect to the connection of \(A_n\) in terms of the covariant derivatives with respect to the connection of \(A_n\). We have

\[
2R^h_{ijkl} = \delta^h_{(i)jm} - \delta^h_{(i)a} R^h_{kmj} + \delta^h_{(i)km} - \delta^h_{(i)jm} + \delta^h_{(i)klj} - \delta^h_{(i)aml} + S^h_{ijklm}.
\]

(19)
where

\[ S^h_{ijklm} = \Omega^h_{ijklm} - 2\left[ \mathcal{R}^h_{ijkl} P^h_{a_k} - \mathcal{R}^h_{ijmn} P^h_{a_k} - \mathcal{R}^h_{ijam} P^h_{a_k} - \mathcal{R}^h_{jmal} P^h_{a_k} - \mathcal{R}^h_{jmaa} P^h_{a_k} - \mathcal{R}^h_{jmll} P^h_{a_k} \right] \]

+ \left( \mathcal{R}^h_{jml} P^h_{a_i} - \mathcal{R}^h_{jml} P^h_{a_i} - \mathcal{R}^h_{jml} P^h_{a_i} - \mathcal{R}^h_{jmal} P^h_{a_i} \right) P^h_{a_k}

- \left( \mathcal{R}^h_{jml} P^h_{a_i} - \mathcal{R}^h_{jml} P^h_{a_i} - \mathcal{R}^h_{jmal} P^h_{a_i} \right) P^h_{a_k}

- \left( \mathcal{R}^h_{jml} P^h_{a_i} - \mathcal{R}^h_{jml} P^h_{a_i} - \mathcal{R}^h_{jmal} P^h_{a_i} \right) P^h_{a_k}

- \left( \mathcal{R}^h_{jml} P^h_{a_i} - \mathcal{R}^h_{jml} P^h_{a_i} - \mathcal{R}^h_{jmal} P^h_{a_i} \right) P^h_{a_k}.

We introduce the tensors \( a_{ijk} \) and \( \overline{\mathcal{R}}^h_{ijkm} \) defined by

\[ a_{ijk} = a_{ijk}, \quad (20) \]

\[ \overline{\mathcal{R}}^h_{ijkm} = \overline{\mathcal{R}}^h_{ijkm}. \quad (21) \]

Taking account of (21), we may write the Equation (19) in the form

\[ 2\mathcal{R}^h_{ijkl} = \delta^h_{ijkl} a_{klm} - \delta^h_{ijkl} a_{klm} + \delta^h_{ijkl} a_{klm} + \delta^h_{ijkl} a_{klm} - \delta^h_{ijkl} a_{klm} + S^h_{ijklm}. \quad (22) \]

In the following we have assumed that in the left-hand side of the Equation (22) the second order covariant derivatives of the tensor \( a_{ij} \) are expressed according to (13). Taking account of (20), the Equation (13) may be put in the form

\[ \frac{n^2 + n - 2}{n} a_{ij,k} = -C_{ijkl} - \frac{1}{n} C_{(ij)kl}. \quad (23) \]

We have assumed that in the right-hand sides of the Equations (22) and (23) the covariant derivatives of tensors \( a_{ij} \) and \( \overline{\mathcal{R}}^h_{ijk} \) with respect to the connection of the space \( A_n \) are expressed according to (20) and (21).

Obviously, in the space \( A_n \) the Equations (3) and (20)-(23) form a closed system of PDEs of Cauchy type with respect to the functions \( P^h_{ij}(x), a_{ij}(x), \overline{\mathcal{R}}^h_{ijk}(x), a_{ijk}(x), \overline{\mathcal{R}}^h_{ijkm}(x) \).

The functions must satisfy the algebraic conditions

\[ P^h_{ij}(x) = P^h_{ij}(x), \quad a_{ij}(x) = a_{ij}(x), \]

\[ \overline{\mathcal{R}}^h_{ijk}(x) = \overline{\mathcal{R}}^h_{ijk}(x) = 0, \quad \overline{\mathcal{R}}^h_{ijkm}(x) = \overline{\mathcal{R}}^h_{ijkm}(x) = 0. \quad (24) \]

Hence we have proved the following theorem.

**Theorem 1.** In order that a space \( A_n \) with an affine connection admits a canonical almost geodesic mapping of type \( \pi_1 \) onto a generalized 2-Ricci-symmetric space \( \overline{A}_n \), it is necessary and sufficient that the mixed system of differential equations of Cauchy type in covariant derivatives (3) and (20)-(24) have a solution with respect to the unknown functions \( P^h_{ij}(x), a_{ij}(x), a_{ijk}(x), \overline{\mathcal{R}}^h_{ijk}(x) \) and \( \overline{\mathcal{R}}^h_{ijkm}(x) \).

Also we have obtained the corollary.

**Corollary 1.** The family of all generalized 2-Ricci-symmetric spaces, which are images of a given space \( A_n \) with an affine connection with respect to canonical almost geodesic mappings of type \( \pi_1 \), depends on no more than

\[ \frac{1}{6} n(2n - 1)(n^2 - 1) + \frac{1}{2} n(1 + n)^2 \]
essential parameters.

5. Canonical Almost Geodesic Mappings of Type $\pi_1$ of Spaces with Affine Connections onto Generalized 3-Ricci-Symmetric Spaces

A space $\overline{A}_n$ with an affine connection is called generalized 3-Ricci-symmetric if the Ricci tensor $\overline{R}_{ij}$ for the space satisfies the condition

$$\overline{R}_{ijklm} + \overline{R}_{kljim} = 0. \quad (25)$$

Taking account of (1) and (8), we obtain

$$\overline{R}_{ijkl} = G_{ijkl} \quad \text{and} \quad (\overline{R}_{ijkl})_m = R_{ijkl} + G_{ajik} P_{ilm} + G_{iajk} P_{ilm} + G_{idak} P_{ilm} + G_{ilja} P_{kml}, \quad (26)$$

where

$$G_{ijkl} = R_{ijkl} - R_{ia,j} P_{ik}^a - R_{ia,i} P_{jk}^a - R_{ia,k} P_{ij}^a + (R_{ij,k} - R_{ij,l} P_{ak}^a - R_{ia,k} P_{jl}^a) P_{il}^a - (R_{ia,k} - R_{ia,l} P_{jk}^a - R_{ij,k} P_{il}^a) P_{jk}^a.$$ 

Let us introduce the tensor $P_{ijkl} = \alpha_{ijkl},$ which was determined by (25). Then the Equation (27) could be written in such form as

$$(n + 1) a_{ijkl} - a_{ij,kl} - a_{il,jk} = - C_{ijkl}, \quad (28)$$

where

$$C_{ijkl} = C_{ijlk} - G_{al[ijk]} P_{il}^a - G_{(i|a|j}) P_{il}^a - G_{(i|a|k)} P_{jm}^a - G_{(i|l|j)} a_{ik} P_{km}^a. \quad (29)$$

Let us interchange the indices $l$ and $j$ in (28) and symmetrize in the indices $i$ and $j$. Then we have

$$a_{ijkl} + a_{ij,kl} = - \frac{1}{n} C_{(i|lj)km} + \frac{2}{n} a_{ij,km}. \quad (30)$$

Because of (30) the Equation (28) may be written in the form

$$\frac{n^2 - n - 2}{n} a_{ijkl} = - C_{ijkl} - \frac{1}{n} C_{(i|lj)km}. \quad (31)$$

The Equation (19) we obtain from the integrability conditions of (4), is to hold for canonical almost geodesic mappings of type $\pi_1$ of spaces with affine connections onto generalized 3-Ricci-symmetric spaces.

In the following we have assumed that a space $A_n$ with an affine connection is given. Then taking account of the structure of the tensor $C_{ijkl}$ which was determined by formulas (4), (18) and (29), we see that the left hand side of Equation (31) depends on the unknown functions $P_{ijkl}^a(x), a_{ij}(x), a_{ij,k}(x), a_{ij,km}(x), \overline{R}_{ijkl}(x), \overline{R}_{ijkl,m}(x).$

Let us introduce the tensor $a_{ijkl}$ defined by

$$a_{ijkl} = a_{ijkl}. \quad (32)$$
Because of (32) we can write the Equation (31) in the form
\[
\frac{n^2 + n - 2}{n} a_{ijklm} = -C_{ijklm} - \frac{1}{n} C_{(ij|klm)}.
\] (33)

Obviously, in the space \( A_n \) the Equations (3), (20)–(22), (32) and (33) form a closed system of PDEs of Cauchy type with respect to the functions \( p^h_{ij}(x), a_{ij}(x), R^h_{ijkl}(x), R^h_{ijklm}(x), a_{ijk}(x), a_{ijkm}(x) \). Also the functions must satisfy the algebraic conditions (24).

Hence we have proved the following theorem.

**Theorem 2.** In order that a space \( A_n \) with an affine connection admits a canonical almost geodesic mapping of type \( \pi_1 \) onto a generalized 3-Ricci-symmetric space \( \overline{A}_n \), it is necessary and sufficient that the mixed system of differential equations of Cauchy type in covariant derivatives (3), (20)–(22), (24), (32) and (33) have a solution with respect to the unknown functions \( p^h_{ij}(x), a_{ij}(x), R^h_{ijkl}(x), R^h_{ijklm}(x), a_{ijk}(x) \) and \( a_{ijkm}(x) \).

**Corollary 2.** The family of all generalized 3-Ricci-symmetric spaces, which are images of a given space \( A_n \) with an affine connection with respect to canonical almost geodesic mappings of type \( \pi_1 \), depends on no more then
\[
\frac{1}{6} n(2n - 1)(n^2 - 1) + \frac{1}{2} n(1 + n)(n^2 + n + 1)
\] essential parameters.

**6. Canonical Almost Geodesic Mappings of Type \( \pi_1 \) of Spaces with Affine Connections onto Generalized \( m \)-Ricci-Symmetric Spaces**

A space \( \overline{A}_n \) with an affine connection is called **generalized \( m \)-Ricci-symmetric** if the Ricci tensor \( R_{ij} \) for the space satisfies the condition
\[
R_{ij[p_1 \ldots [p_t \ldots [p_{t-2} \ldots p_2 \ldots p_{t-1}] \ldots p_1]} = R_{ij[p_1 \ldots [p_t \ldots [p_{t-2} \ldots p_2 \ldots p_{t-1}] \ldots p_1]} + p^h_{[p_1} R^h_{a[p_2} \Gamma^a_{p_3 \ldots p_{t-1}]} + \ldots + p^h_{[p_1} R^h_{a[p_2} \Gamma^a_{p_3 \ldots p_{2t-1]}].
\] (34)

It is obvious that generalized 2-Ricci-symmetric spaces and generalized 3-Ricci-symmetric spaces are special cases of generalized \( m \)-Ricci-symmetric spaces with \( m = 2 \) and \( m = 3 \) respectively.

If we put \( m = 1 \) in (34) we have a generalized Ricci-symmetric space. Let us differentiate covariantly \((m - 3)\) times the Equation (27) with respect to the connection of the space \( A_n \). Then express in the left-hand side the covariant derivatives with respect to the connection of the space \( A_n \) in terms of the covariant derivatives with respect to the connection of the space \( \overline{A}_n \), using the formula [24]

\[
\begin{align*}
R_{ij[p_1 \ldots [p_t \ldots [p_{t-2} \ldots p_2 \ldots p_{t-1}] \ldots p_1]} &= R_{ij[p_1 \ldots [p_t \ldots [p_{t-2} \ldots p_2 \ldots p_{t-1}] \ldots p_1]} + p^h_{[p_1} R^h_{a[p_2} \Gamma^a_{p_3 \ldots p_{t-1}]} + \ldots + p^h_{[p_1} R^h_{a[p_2} \Gamma^a_{p_3 \ldots p_{2t-1]}],
\end{align*}
\]

Transforming the left-hand side of (34), we obtain
\[
R_{ij[p_1 \ldots [p_t \ldots [p_{t-2} \ldots p_2 \ldots p_{t-1}] \ldots p_1]} + \Omega_{ij[p_1 \ldots [p_{t-2} \ldots p_2 \ldots p_{t-1}]} = (n + 1) a_{p_1 \ldots p_{t-2} \ldots p_1} - a_{p_1 \ldots p_{t-2} \ldots p_1} - a_{ij[p_1 \ldots p_{t-2} \ldots p_1]} + \Omega_{ij[p_1 \ldots [p_{t-2} \ldots p_2 \ldots p_{t-1}]}.
\] (35)

where the tensor \( \Omega_{ij[p_1 \ldots [p_{t-2} \ldots p_2 \ldots p_{t-1}]} \) depends on the unknown tensors \( p^h_{ij}, R^h_{ijkl}, R^h_{ijklm}, a_{ij}, a_{ij[p_3 \ldots p_{t-1}],} \ldots, a_{ij[p_1 \ldots p_{t-2}]} \).

Let us consider canonical almost geodesic mappings of type \( \pi_1 \) of spaces with affine connections \( A_n \) onto generalized \( m \)-Ricci-symmetric spaces \( \overline{A}_n \). Hence the Ricci tensor \( R_{ij} \) for the space \( \overline{A}_n \) satisfies conditions (34). Then Equation (35) could be written in such form as
\[
(n + 1) a_{p_1 \ldots p_{t-2} \ldots p_1} - a_{p_1 \ldots p_{t-2} \ldots p_1} - a_{ij[p_1 \ldots p_{t-2} \ldots p_1]} = -\Omega_{ij[p_1 \ldots [p_{t-2} \ldots p_2 \ldots p_{t-1}]}.
\] (36)
Let us interchange the indices \( j \) and \( \rho_1 \) in (36) and symmetrize in the indices \( i \) and \( \rho_1 \). Then we have

\[
a_{ij_1j_2\ldots-j_m} + a_{ij_1j_2\ldots-j_m} = -\frac{1}{n} \Omega_{ij_1j_2\ldots-j_m} + \frac{2}{n} a_{ij_1j_2\ldots-j_m}.
\]

(37)

Taking account of (37), Equation (36) may be written in the form

\[
\frac{n^2 + n - 2}{n} a_{ij_1j_2\ldots-j_m} = -\Omega_{ij_1j_2\ldots-j_m} - \frac{1}{n} \Omega_{ij_1j_2\ldots-j_m}.
\]

(38)

Let us introduce the tensors \( a_{ijkm\rho_1}, \ldots, a_{ijkm\rho_m-4\rho_m-3} \) defined by

\[
a_{ijkm\rho_1} = a_{ijkm\rho_1},
\]

\[
\ldots \ldots \ldots 
\]

\[
a_{ijkm\rho_m-4\rho_m-3} = a_{ijkm\rho_m-4\rho_m-3}.
\]

(39)

Because of (39) we can write Equation (38) in the form

\[
\frac{n^2 + n - 2}{n} a_{ij_1j_2\ldots-j_m,1} = -\Omega_{ij_1j_2\ldots-j_m} - \frac{1}{n} \Omega_{ij_1j_2\ldots-j_m}.
\]

(40)

Obviously, the right-hand side of Equation (40) depends on the unknown tensors \( P_{ij}^h, \)

\( \mathcal{K}_{ijk}^h, \mathcal{K}_{ijkm}^h, a_{ij}, a_{ijp_1}, \ldots, a_{ij\rho_m-1}. \)

Hence, in the space \( \mathcal{A}_n \) Equations (3), (20)–(22), (32), (39) and (40) form a closed system of PDEs of Cauchy type with respect to the functions \( P_{ij}^h(x), \mathcal{K}_{ijk}^h(x), \mathcal{K}_{ijkm}^h(x), a_{ij}(x), a_{ijp_1}(x), \ldots, a_{ij\rho_m-1}(x) \). Also the functions must satisfy the algebraic conditions (24).

This proves the following theorem.

**Theorem 3.** In order that a space \( \mathcal{A}_n \) with an affine connection admits a canonical almost geodesic mapping of type \( \pi_1 \) onto a generalized m-Ricci-symmetric space \( \mathcal{A}_n \), it is necessary and sufficient that the mixed system of differential equations of Cauchy type in covariant derivatives (3), (20)–(22), (24), (32), (39) and (40) have a solution with respect to the unknown functions \( P_{ij}^h(x), \mathcal{K}_{ijk}^h(x), \mathcal{K}_{ijkm}^h(x), a_{ij}(x), a_{ijp_1}(x), \ldots, a_{ij\rho_m-1}(x) \).

**Corollary 3.** The family of all generalized m-Ricci-symmetric spaces, which are images of a given space \( \mathcal{A}_n \) with an affine connection with respect to canonical almost geodesic mappings of type \( \pi_1 \), depends on no more than

\[
\frac{1}{6} n(2n-1)(n^2-1) + \frac{1}{2(n-1)} n(1+n) (n^m + n^2 - n - 1)
\]

essential parameters.

7. Conclusions

In this paper, we study canonical almost geodesic mappings of the first type of spaces with an affine connection onto generalized m-Ricci-symmetric spaces. These spaces are the essential generalization of the spaces considered in [24]. Ricci tensor is well known from the Einstein equation that describes gravitational fields. The added condition of generalized symmetric spaces is connected with the study of gravitation waves, which makes the topic of the study relevant.

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