OPERATORS INDUCED BY GRAPHS

IL WOO CHO AND PALLE E. T. JORGENSEN

ABSTRACT. In this paper, we consider the spectral-theoretic properties of certain operators induced by given graphs. Self-adjointness, unitary, hyponormality, and normality of graph-depending operators are considered. As application, we study the finitely supported operators in the free group factor L(F_N).

1. Introduction

Starting with analysis on countable directed graphs G, we introduce Hilbert spaces H_G and a family of weighted operators T on H_G. When the weights (called coefficients later in the present context) are chosen, T is called a graph operator. From its weights (or coefficients), we define the support Supp(T) of T. In full generality, it is difficult to identify analytic tools that reflect global properties of the underlying graph. We will be interested in generic properties that allow us to study spectral theory of this family of operators T. The spectral theorem will produce a spectral measure representation for T provided we can establish normality of T; self-adjointness, unitary, etc. These are the classes of operators that admit spectral analysis.

In Theorem 3.1, we give a necessary and sufficient condition on Supp(T) for T to be self-adjoint. Our analysis is of interest even in the case when G is finite. For instance, in Theorems 3.2 and 3.3, with G assumed finite, we show that there is a vertex-edge correspondence which characterizes to weighted operators T that are unitary. Also, in Section 4, hyponormality and normality of our graph operators are characterized.

1.1. Overview. A graph is a set of objects called vertices (or points or nodes) connected by links called edges (or lines). In a directed graph, the two directions are counted as being distinct directed edges (or arcs). A graph is depicted in a diagrammatic form as a set of dots (for vertices), jointed by curves or line-segments (for edges). Similarly, a directed graph is depicted in a diagrammatic form as a set of dots jointed by arrowed curves, where the arrows point the direction of the directed edges.

Recently, we studied the operator-algebraic structures induced by directed graphs. A key idea in the study of graph-depending operator algebras is that every directed graph G induces its corresponding groupoid G, called the graph groupoid of G. By considering the algebraic structure of G, we can determine the groupoid actions λ.
acting on Hilbert spaces $H$: We can obtain suitable representations $(H, \lambda)$ for $\mathbb{G}$. And this guarantees the existence of operator algebras $A_G = C_l^\infty(\lambda(\mathbb{G}))$, generated by $\mathbb{G}$ (or induced by $G$), contained in the operator algebras $B(H)$. Indeed, the operator algebras $A_G$ are the groupoid ($C^*$- or $W^*$-)subalgebras of $B(H)$.

Note that each edge $e$ of $G$ assigns a partial isometry on $H$, and every vertex $v$ of $G$ assigns a projection on $H$ (under various different types of representations of $\mathbb{G}$). We will fix a canonical representation $(H_G, L)$ of $\mathbb{G}$, and construct the corresponding von Neumann algebra

$$M_G = \overline{C_l^\infty(\lambda(\mathbb{G}))}^{w^*} \text{ in } B(H_G),$$

where $H_G$ is the graph Hilbert space $l^2(\mathbb{G})$. This von Neumann algebra $M_G$ is called the graph von Neumann algebra of $G$. (See Section 3.1 below).

In this paper, we are interested in certain elements $T$ of $M_G$. Recall that, by the definition of graph von Neumann algebras (which are groupoid von Neumann algebras), if $T \in M_G$, then

$$T = \sum_{w \in \mathbb{G}} t_w L_w \text{ with } t_w \in \mathbb{C}.$$

Define the support $\text{Supp}(T)$ of $T$ by \n
$$\text{Supp}(T) = \{w \in \mathbb{G} : t_w \neq 0\}.$$  

If the support $\text{Supp}(T)$ of $T$ is a “finite” subset of $\mathbb{G}$, then we call $T$ a graph operator. If $T$ is a graph operator, then the quantities $t_w$, for $w \in \text{Supp}(T)$, are called the coefficients of $T$.

As we see, all graph operators are (finite) linear sums of generating operators $L_w$ of $M_G$, for $w \in \mathbb{G}$, i.e., they are the operators generated by finite numbers of projections and partial isometries on $H_G$. We are interested in the operator-theoretical properties of them; in particular, self-adjointness, the unitary property, hyponormality, and normality.

In operator theory, such properties are very important in order to understand the given operators. For instance, if a given operator $T$ is normal, then $T$ satisfies the conditions in the spectral mapping theorem, and hence the $C^*$-algebra generated by $T$ is $*$-isomorphic to $C(\text{spec}(T))$, the $C^*$-algebra consisting of all continuous functions on the spectrum $\text{spec}(T)$ of $T$.

Recall that, for an operator $T$, the spectrum of $T$, defined by

$$\text{spec}(T) \overset{\text{def}}{=} \{t \in \mathbb{C} : T - t1_H \text{ is not invertible}\},$$

is a nonempty compact subset of $\mathbb{C}$.

We characterize the self-adjointness, the unitary property, hyponormality, and normality of graph operators in $M_G$. We show that such operator-theoretic properties of graph operators are characterized by the combinatorial property of given graphs and certain analytic data of coefficients of $T$. This provides another connection between operator theory, operator algebra, groupoid theory, and combinatorial graph theory.

1.2. Motivation and Applications. As application, we derive the operator-theoretic properties of finitely supported elements of the free group factors $L(F_N)$, for $N \in \mathbb{N}$. Recall that the free group factor $L(F_N)$, for $N \in \mathbb{N}$, is the group von Neumann algebra $C_l^\infty(\lambda(F_N))^{w^*}$, in $B(l^2(F_N))$, generated by the free group $F_N$ with $N$-generators, where $(l^2(F_N), \lambda)$ is the left regular unitary representation of $F_N$, consisting of the group Hilbert space $l^2(F_N)$, and the unitary representation (which is a group action)
of $F_N$ acting on $l^2(F_N)$. It is possible since the free group factors $L(F_N)$ are $*$-isomorphic to the graph von Neumann algebras $M_{O_N}$ of the one-vertex-$N$-loop-edge graphs $O_N$, for all $N \in \mathbb{N} \cup \{\infty\}$ (See Section 5 below and [5]).

Recall that a von Neumann algebra $\mathcal{M}$ in $B(H)$ is a factor, if its $W^*$-subalgebra $\mathcal{M}' \cap \mathcal{M}$ is $*$-isomorphic to $\mathbb{C}$ (or $\mathbb{C} \cdot 1_\mathcal{M}$), where

$$\mathcal{M}' \overset{\text{def}}{=} \{ x \in B(H) : xm = mx, \forall m \in \mathcal{M} \}.$$ 

It is well-known that a group $\Gamma$ is an i. c. c (or an infinite conjugacy class) group, if and only if the corresponding group von Neumann algebra $L(\Gamma)$ is a factor. Since every free group $F_N$ is i. c. c., the group von Neumann algebra $L(F_N)$ is a factor. So, we call $L(F_N)$, the free group factors.

The study of free group factors, itself, is very interesting and important in operator algebra. We are interested in the operator-theoretic properties of each element of the fixed free group factor.

We can check that the free group factors $L(F_N)$ and the graph von Neumann algebras $M_{O_N}$ of the one-vertex-$N$-loop-edge graphs $O_N$ are $*$-isomorphic. This provides a motivation for our application in Section 5. More precisely, the analysis of finitely supported operators in $L(F_N)$ is the study of graph operators in $M_{O_N}$, since there are one-to-one correspondence between finitely supported operators in $L(F_N)$, and graph operators in $M_{O_N}$.

2. Definitions and Background

Starting with a graph $G$, to understand the operator theory, we must introduce a Hilbert space $H_G$ naturally coming from $G$. Our approach is as follows: From $G$, introduce an enveloping groupoid $G$ and an associated involutive algebra $A_G$. We then introduce a conditional expectation $E$ of $A_G$ onto the subalgebra $D_G$ of diagonal elements. To get a representation of $A_G$ and an associated Hilbert space $H_G$, we then use the Stinespring construction on $E$ (e.g., see [14]). In this section, we introduce the concepts and definitions we will use.

2.1. Graph Groupoids. Let $G$ be a directed graph with its vertex set $V(G)$ and its edge set $E(G)$. Let $e \in E(G)$ be an edge connecting a vertex $v_1$ to a vertex $v_2$. Then we write $e = v_1 e v_2$, for emphasizing the initial vertex $v_1$ of $e$ and the terminal vertex $v_2$ of $e$.

For a fixed graph $G$, we can define the oppositely directed graph $G^{-1}$, with $V(G^{-1}) = V(G)$ and $E(G^{-1}) = \{ e^{-1} : e \in E(G) \}$, where each element $e^{-1}$ of $E(G^{-1})$ satisfies that

$$e = v_1 e v_2 \text{ in } E(G), \text{ with } v_1, v_2 \in V(G),$$

if and only if

$$e^{-1} = v_2 e^{-1} v_1, \text{ in } E(G^{-1}).$$

This opposite directed edge $e^{-1} \in E(G^{-1})$ of $e \in E(G)$ is called the shadow of $e$. Also, this new graph $G^{-1}$, induced by $G$, is said to be the shadow of $G$. It is clear that $(G^{-1})^{-1} = G$.

Define the shadowed graph $\hat{G}$ of $G$ by a directed graph with its vertex set

$$V(\hat{G}) = V(G) = V(G^{-1})$$

and its edge set

$$E(\hat{G}) = E(G) \cup E(G^{-1}),$$
where $G^{-1}$ is the shadow of $G$.

We say that two edges $e_1 = v_1 e_1 v'_1$ and $e_2 = v_2 e_2 v'_2$ are admissible, if $v'_1 = v_2$, equivalently, the finite path $e_1 e_2$ is well-defined on $\hat{G}$. Similarly, if $w_1$ and $w_2$ are finite paths on $G$, then we say $w_1$ and $w_2$ are admissible, if $w_1 w_2$ is a well-defined finite path on $G$, too. Similar to the edge case, if a finite path $w$ has its initial vertex $v$ and its terminal vertex $v'$, then we write $w = v_1 w v_2$. Notice that every admissible finite path is a word in $E(\hat{G})$. Denote the set of all finite path by $FP(\hat{G})$.

Then $FP(\hat{G})$ is the subset of the set $E(\hat{G})^*$, consisting of all finite words in $E(\hat{G})$.

Suppose we take a part

$$
\begin{array}{c}
\bullet \xrightarrow{e_3} \ldots \\
\uparrow \hspace{0.5cm} e_2 \\
\ldots \xrightarrow{e_1} \bullet
\end{array}
$$

in a graph $G$ or in the shadowed graph $\hat{G}$, where $e_1, e_2, e_3$ are edges of $G$, respectively of $\hat{G}$. Then the above admissibility shows that the edges $e_1$ and $e_2$ are admissible, since we can obtain a finite path $e_1 e_2$, however, the edges $e_1$ and $e_3$ are not admissible, since a finite path $e_1 e_3$ is undefined.

We can construct the free semigroupoid $F^+(\hat{G})$ of the shadowed graph $\hat{G}$, as the union of all vertices in $V(\hat{G}) = V(G) = V(G^{-1})$ and admissible words in $FP(\hat{G})$, equipped with its binary operation, the admissibility. Naturally, we assume that $F^+(\hat{G})$ contains the empty word $\emptyset$, as the representative of all undefined (or non-admissible) finite words in $E(\hat{G})$.

Remark that some free semigroupoid $F^+(\hat{G})$ of $\hat{G}$ does not contain the empty word; for instance, if a graph $G$ is a one-vertex-multi-edge graph, then the shadowed graph $\hat{G}$ of $G$ is also a one-vertex-multi-edge graph too, and hence its free semigroupoid $F^+(\hat{G})$ does not have the empty word. However, in general, if $|V(G)| > 1$, then $F^+(\hat{G})$ always contain the empty word. Thus, if there is no confusion, we always assume the empty word $\emptyset$ is contained in the free semigroupoid $F^+(\hat{G})$ of $\hat{G}$.

**Definition 2.1.** By defining the reduction (RR) on $F^+(\hat{G})$, we define the graph groupoid $G$ of a given graph $G$, by the subset of $F^+(\hat{G})$, consisting of all “reduced” finite paths on $\hat{G}$, with the inherited admissibility on $F^+(\hat{G})$ under (RR), where the reduction (RR) on $G$ is as follows:

(RR) \hspace{0.5cm} $w w^{-1} = v$ and $w^{-1} w = v'$,

for all $w = v w' \in G$, with $v, v' \in V(\hat{G})$.

Such a graph groupoid $G$ is indeed a categorial groupoid with its base $V(\hat{G})$ (See Appendix A).

**2.2. Canonical Representation of Graph Groupoids.** Let $G$ be a given countable connected directed graph with its graph groupoid $G$. Then we can define the (pure algebraic) algebra $A_G$ of $G$ by a vector space over $\mathbb{C}$, consisting of all linear combinations of elements of $G$, i.e.,

$$
A_G \overset{\text{def}}{=} \mathbb{C} \cup \left\{ \sum_{j=1}^{\infty} t_j w_j \mid w_j \in G, \ t_j \in \mathbb{C}, \ j = 1, 2, \ldots \right\},
$$

under the usual addition $+$, and the multiplication $\cdot$, dictated by the admissibility on $G$. Define now a unary operation $*$ on $A_G$ by

$$
A_G \overset{\text{def}}{=} \mathbb{C} \cup \left\{ \sum_{j=1}^{\infty} t_j w_j \mid w_j \in G, \ t_j \in \mathbb{C}, \ j = 1, 2, \ldots \right\},
$$

under the usual addition $+$, and the multiplication $\cdot$, dictated by the admissibility on $G$. Define now a unary operation $*$ on $A_G$ by
where $\overline{z}$ means the conjugate of $z$, for all $z \in \mathbb{C}$, and of course $w^{-1}$ means the shadow of $w$, for all $w \in G$. We call this unary operation $(\ast)$, the adjoint (or the shadow) on $\mathcal{A}_G$. Then the vector space $\mathcal{A}_G$, equipped with the adjoint $(\ast)$, is a well-defined (algebraic) $\ast$-algebra.

Now, define a $\ast$-subalgebra $\mathcal{D}_G$ of $\mathcal{A}_G$ by

$$\mathcal{D}_G \overset{\text{def}}{=} \mathbb{C} \cup \left( \bigcup_{k=1}^{\infty} \left\{ \sum_{j=1}^{n} t_j \cdot v_j \left| v_j \in V(\hat{G}), \ t_j \in \mathbb{C}, \ j = 1, ..., k \right. \right\} \right).$$

This $\ast$-algebra $\mathcal{D}_G$ acts like the diagonal of $\mathcal{A}_G$, so we call $\mathcal{D}_G$, the diagonal $\ast$-subalgebra of $\mathcal{A}_G$.

2.2.1. The Hilbert Space $H_G$. Below, we identify the canonical Hilbert space $H_G$. The algebra $\mathcal{A}_G$ is represented by bounded linear operators acting on $H_G$. The representation is induced by the canonical conditional expectation, via the Stinespring construction (e.g., see [14]).

We can construct a (algebraic $\ast$-)conditional expectation

$$E : \mathcal{A}_G \rightarrow \mathcal{D}_G$$

by

$$E \left( \sum_{w \in X} t_w w \right) \overset{\text{def}}{=} \sum_{v \in X \cap V(\hat{G})} t_v \cdot v,$$

for all $\sum_{w \in X} t_w w \in \mathcal{A}_G$, where $X$ means a finite subset of $G$.

Since the conditional expectation $F$ is completely positive under a suitable topology on $\mathcal{A}_G$, we may apply the Stinespring’s construction. i.e., the diagonal subalgebra $\mathcal{D}_G$ is represented as the $l^2$-space, $l^2(V(\hat{G}))$, by the concatenation. Then we can obtain the Hilbert space $H_G$,

$$H_G \overset{\text{def}}{=} \text{the Stinespring space of } \mathcal{A}_G \text{ over } \mathcal{D}_G, \text{ by } F,$$

containing $l^2(V(\hat{G}))$. i.e., if $\pi_{(E, \mathcal{D}_G)}$ is the Stinespring representation of $\mathcal{A}_G$, acting on $l^2(V(\hat{G}))$, then

$$H_G = \pi_{(E, \mathcal{D}_G)}(\mathcal{A}_G).$$

This Stinespring space $H_G$ is the Hilbert space with its inner product $<,>_G$ satisfying that:

$$< h, \pi_{(E, \mathcal{D}_G)}(a) \cdot k >_G = < h, E(a) \cdot k >_2,$$

for all $h, k \in l^2(V(\hat{G}))$, for all $a \in \mathcal{A}_G$, where $<,>_2$ is the inner product on $l^2(V(\hat{G}))$.

i.e., The Stinespring space $H_G$ is the norm closure of $\mathcal{A}_G$, by the norm,

$$\left\| \sum_{j=1}^{n} w_i \otimes h_i \right\|_G^2 = \sum_{i=1}^{n} \sum_{k=1}^{n} < h_i, E(w_i^* w_k) \cdot h_k >_2,$$

induced by the Stinespring inner product $<,>_G$ on $\mathcal{A}_G$, for all $a_i \in \mathcal{A}_G, h_i \in l^2(V(\hat{G}))$, for all $n \in \mathbb{N}$.

**Definition 2.2.** We call this Stinespring space $H_G$, the graph Hilbert space of $G$ (or of $G$).
Denote the Hilbert space element $\pi_{(E, D_G)}(w)$ by $\xi_w$ in the graph Hilbert space $H_G$, for all $w \in \mathbb{G}$, with the identification,

$$\xi_\emptyset = 0_{H_G},$$

the zero vector in $H_G$,

where $\emptyset$ is the empty word (if exists) of $\mathbb{G}$. We can check that the subset $\{\xi_w : w \in \mathbb{G}\}$ of $H_G$ satisfies the following multiplication rule:

$$\xi_{w_1}, \xi_{w_2} = \xi_{w_1 w_2},$$

for all $w_1, w_2 \in \mathbb{G}$. Thus, we can define the canonical multiplication operators $L_w$ on $H_G$, satisfying that

$$L_w \xi_{w'} \overset{\text{def}}{=} \xi_w \xi_{w'} = \xi_{ww'},$$

for all $w, w' \in \mathbb{G}$. The existence of such multiplication operators $L_w$’s guarantees the existence of a groupoid action $L$ of $\mathbb{G}$, acting on $H_G$;

$$L : w \in \mathbb{G} \mapsto L(w) \overset{\text{def}}{=} L_w \in B(H_G).$$

This action $L$ of $\mathbb{G}$ is called the canonical groupoid action of $\mathbb{G}$ on $H_G$.

2.2.2. The Operators $L_w$. Let $w$ and $w_i$ denote reduced finite paths in $FP_i(\hat{G})$, for $i \in \mathbb{N}$, equivalently, they are the reduced words in the edge set $E(\hat{G})$, under the reduction (RR). Consider

$$(2.2.3)$$

$$L_w \left( \sum_i w_i \otimes h_i \right) = \sum_i ww_i \otimes h_i,$$

for $h_i \in l^2(\mathbb{N})$. Here, the element $\sum_i w_i \otimes h_i$ denotes a finite sum of tensors in $\mathcal{A}_G$. And $ww_i$ in (2.2.3) means concatenation of finite words. With the conditional expectation $E : \mathcal{A}_G \rightarrow D_G$ (See (2.2.1) above), we get the Stinespring representation $\hat{\pi}_{(E, D_G)}$), and the operators

$\pi_{(E, D_G)}(w) : \mathcal{H}_G \rightarrow \mathcal{H}_G$

obtained from (2.2.3) by passing to the quotient and completion as in Definition 2.2. To simplify terminology, in the sequel, we will simply write $L_w$ for the operator $\pi_{(E, D_G)}(w)$.

2.2.3. Graph von Neumann Algebras. Let $G$, $\mathbb{G}$, and $H_G$ be given as above. And let $\{L_w : w \in \mathbb{G}\}$ the multiplication operators on $H_G$, where $L$ is the canonical groupoid action of $\mathbb{G}$.

**Definition 2.3.** Let $G$ be a countable directed graph with its graph groupoid $\mathbb{G}$. The pair $(H_G, L)$ of the graph Hilbert space $H_G$ and the canonical groupoid action $L$ of $\mathbb{G}$ is called the canonical representation of $\mathbb{G}$. The corresponding groupoid von Neumann algebra

$$M_G \overset{\text{def}}{=} \overline{\text{span}}(L(\mathbb{G}))$$

generated by $\mathbb{G}$ (equivalently, by $L(\mathbb{G}) = \{L_w : w \in \mathbb{G}\}$), as a $W^*$-subalgebra of $B(H_G)$, is called the graph von Neumann algebra of $G$.

We can check that the generating operators $L_w$’s of the graph von Neumann algebra $M_G$ of $G$ satisfies that:

$$L_w^* = L_{w^{-1}}, \text{ for all } w \in \mathbb{G},$$

and

$$L_{w_1} L_{w_2} = L_{w_1 w_2}, \text{ for all } w_1, w_2 \in \mathbb{G}.$$
It is easy to check that if \( v \) is a vertex in \( G \), then the graph operator \( L_v \) is a projection, since
\[
L_v^* = L_{v^{-1}} = L_v = L_{v^2} = L_v^2.
\]
Thus, by the reduction (RR) on \( G \), we can conclude that if \( w \) is a nonempty reduced finite path in \( FP_r(\hat{G}) \), then the operator \( L_w \) is a partial isometry, since
\[
L_w^* L_w = L_{w^{-1}w},
\]
and \( w^{-1}w \) is a vertex, and hence \( L_w^* L_w \) is a projection on \( H_G \).

3. SELF-ADJOINTNESS AND UNITARY PROPERTY

In this section, we introduce our main objects of this paper: canonical representations of graph groupoids, graph von Neumann algebras, and graph operators. And we study the self-adjointness of graph operators, and the unitary property of them. We can realize that the self-adjointness and the unitary property of graph operators are characterized by the combinatorial property (admissibility) of given graphs, and certain analytic data of coefficients of the operators.

Section 3.1 introduces the graph operators, and the theorem in Section 3.2 yields the structure of the graph operators that are self-adjoint; and Section 3.3, the unitary case. The different geometries of \( G \) and the associated operators reflect different spectral representations. Section 4 below covers of normal and hyponormal graph operators. Finally, Section 5 takes up the case when the algebra is one of the free group factors (e.g., see [15], and [16]).

3.1. Graph Operators. Let \( G \) be a graph with its graph groupoid \( G \), and let \( M_G = \mathbb{C}[L(G)]^w \) be the graph von Neumann algebra of \( G \) in \( B(H_G) \), where \( (H_G, L) \) is the canonical representation of \( G \). Since \( M_G \) is a groupoid von Neumann algebra generated by \( G \), every element \( T \) of \( M_G \) satisfies the expansion,
\[
T = \sum_{w \in G} t_w L_w, \text{ with } t_w \in \mathbb{C}.
\]

For the given operator \( T \in M_G \), having the above expansion, define the subset \( \text{Supp}(T) \) of \( G \) by
\[
\text{Supp}(T) \overset{\text{def}}{=} \{ w \in G : t_w \neq 0 \}.
\]
This subset \( \text{Supp}(T) \) of \( G \) is called the support of \( T \).

Definition 3.1. Let \( T \) be an element of the graph von Neumann algebra \( M_G \) of a given graph \( G \), and let \( \text{Supp}(T) \) be the support of \( T \). If \( \text{Supp}(T) \) is a finite set, then we call \( T \) a graph operator (on \( H_G \)).

i.e., graph operators are the finitely supported operators on \( H_G \).

In the rest of this section, we will consider a very specific example, but very interesting, where a given graph \( G \) is an infinite linear graph,
\[
G = \bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \cdots.
\]

We want to investigate the matrix forms of (which is unitarily equivalent to) graph operators. Instead of determining the matrix forms of graph operators, acting on the graph Hilbert space \( H_G \), we consider the matrix forms of them, acting on the subspace \( l^2(V(\hat{G})) \), embedded in the graph Hilbert space \( H_G \).

For convenience, we let
\[
V(G) = \mathbb{N}, \text{ and } E(G) = \{ (j, j + 1) : j \in \mathbb{N} \},
\]
i.e.,

\[ G = \begin{array}{c}
(1,2) \\
(2,3) \\
(3,4) \\
\vdots
\end{array} \rightarrow
\begin{array}{c}
1 \\
2 \\
3 \\
\vdots
\end{array} \rightarrow
\begin{array}{c}
(2,3) \\
(3,4) \\
\vdots
\end{array} \rightarrow
\cdots
\]

Then, we can check that

\[ l^2 \left( \mathcal{V}(\hat{G}) \right)_{\text{Hilbert}} = l^2(\mathbb{N}) \text{ in } H_G. \]

So, we can assign the graph operator \( L_j \) to the infinite matrix

\[
\begin{pmatrix}
0 & 0 \\
0 & 0 \\
0 & 0 \\
\vdots & \vdots \\
0 & 0 \\
1 & 0 \\
0 & \ddots \\
0 & \cdots
\end{pmatrix}
\]

on \( l^2(\mathbb{N}) \), for all \( j \in \mathbb{N} = \mathcal{V}(\hat{G}) \), and we assign the graph operator \( L_{(j, j+1)} \) to the infinite matrix

\[
\begin{pmatrix}
0 & 0 \\
0 & 0 \\
0 & 0 \\
\vdots & \vdots \\
0 & 0 \\
1 & 0 \\
1 & \ddots \\
0 & \cdots
\end{pmatrix}
\]

on \( l^2(\mathbb{N}) \), for all \( j \in \mathbb{N} \). More precisely, we can assign

\[ L_j \in M_G \leftrightarrow | j > < j | \in B \left( l^2(\mathbb{N}) \right) \]

and

\[ L_{(j,j+1)} \in M_G \leftrightarrow | j > < j | + | j > < j + 1 | \in B \left( l^2(\mathbb{N}) \right), \]

where \( | j > \) means the Dirac operators, for all \( j \in \mathbb{N} \).

We use Dirac’s notation for rank-one operators, i.e.,

\[ | u > < v | x = < v, x > u, \]

defined for vectors \( u, v, x \) in a fixed Hilbert space having its inner product \(<,>\).

So, for a reduced finite path \( w = e_{i_1} e_{i_2} \ldots e_{i_k} \in \mathcal{G} \), with \( e_{i_j} = (i_j, i_j + 1) \in E(\hat{G}) \),

where

\[ i_{j+1} = i_j + 1, \text{ for } j = 1, \ldots, k - 1, \]

the graph operator \( L_w \) is determined as a matrix,

\[ A_{e_{i_1}} + A_{e_{i_2}} + \cdots + A_{e_{i_k}}, \]

where \( A_{e_{i_j}} \) are the infinite matrices (on \( l^2(\mathbb{N}) \)) of the graph operators \( L_{e_{i_j}} \).

For instance, the self-adjoint operator
\[ L_{(j,j+1)} + L_{(j,j+1)}^* = L_{(j,j+1)} + L_{(j+1,j)} \]
\[ = 2 | j >< j | + | j > j + 1 | + | j + 1 > j | \]

has its matrix form

\[ \begin{pmatrix}
0 & 0 & \ldots & \ldots \\
0 & 2 & 1 & 0 \\
\vdots & 0 & 0 & \ddots \\
0 & \ddots & \ddots & \ldots \\
& \ddots & \ddots & \ddots
\end{pmatrix}, \]

on \( l^2(\mathbb{N}) = l^2 \left( V(\hat{G}) \right) \). So, more generally, the self-adjoint operator \( L_w + L_w^* \), for \( w \in G \), becomes a certain self-adjoint Toeplitz operator on \( l^2(\mathbb{N}) \), because \( l^2(\mathbb{N}) \) is Hilbert-space isomorphic to the Hardy space \( H^2(\mathbb{T}) \), equipped with the Haar measure, where \( \mathbb{T} \) is the unit circle in \( \mathbb{C} \) (e.g., see [3], [12], and [13]).

3.2. Self-Adjoint Graph Operators. The operator-theoretic properties of (bounded linear) operators; the self-adjointness, the unitary properties, the hyponormality, and the normality are briefly introduced in Appendix B.

In this section, we will consider the self-adjointness of graph operators. Let \( G \) be a graph with its graph groupoid \( G \), and let \( M_G \) be the graph von Neumann algebra of \( G \). Take a graph operator \( T \in M_G \),

\[ T = \sum_{w \in \text{Supp}(T)} t_w L_w, \text{ with } t_w \in \mathbb{C}. \]

The following theorem characterize the self-adjointness of \( T \).

**Theorem 3.1.** Let \( T \in M_G \) be a given graph operator. Then \( T \) is self-adjoint, if and only if there exists an subset \( X \) of \( \text{Supp}(T) \) such that

\[ \text{Supp}(T) \cap FP_r(\hat{G}) = X \cup X^{-1}, \]

where \( \cup \) means the disjoint union, and

\[ t_x = \overline{t}_{x^{-1}}, \text{ for all } x \in X, \]

where \( X^{-1} \stackrel{\text{def}}{=} \{ x^{-1} : x \in X \} \), and \( \overline{z} \) means the conjugate of \( z \), for all \( z \in \mathbb{C} \), and

\[ t_v \in \mathbb{R}, \text{ for all } v \in \text{Supp}(T) \cap V(\hat{G}). \]

**Proof.** (\( \Rightarrow \)) Assume that \( T = \sum_{w \in \text{Supp}(T)} t_w L_w \) is a graph operator in \( M_G \), and suppose there exists a subset \( X \) of

\[ \text{Supp}_V(\hat{G}) \text{ denote } = \text{Supp}(T) \cap FP_r(\hat{G}) \]

such that

\[ \text{Supp}_V(T) = X \cup X^{-1}, \]

and

\[ t_x = \overline{t}_{x^{-1}}, \text{ for all } x \in X. \]
Also, assume that $t_v \in \mathbb{R}$, for all elements $v$ in

$$\text{Supp}_V(T) \overset{\text{def.}}{=} \text{Supp}(T) \cap V(\hat{G}).$$

Then the operator $T$ can be re-written by

$$T = \sum_{v \in \text{Supp}_V(T)} t_v L_v + \sum_{x \in X} t_x L_x + \sum_{x^{-1} \in X^{-1}} t_{x^{-1}} L_{x^{-1}}.$$

Moreover, we can have that

$$T^* = \left( \sum_{v \in \text{Supp}_V(T)} t_v L_v + \sum_{x \in X} t_x L_x + \sum_{x^{-1} \in X^{-1}} t_{x^{-1}} L_{x^{-1}} \right)^*$$

and $L_v$ are projections for all $v \in V(\hat{G})$

$$= \sum_{v \in \text{Supp}_V(T)} t_v L_v + \sum_{x \in X} t_x L_x + \sum_{x^{-1} \in X^{-1}} t_{x^{-1}} L_{x^{-1}}$$

since $t_v \in \mathbb{R}$, and $L_v$ are projections for all $v \in V(\hat{G})$

$$= \sum_{v \in \text{Supp}_V(T)} t_v L_v + \sum_{x \in X} t_x L_x + \sum_{x^{-1} \in X^{-1}} t_{x^{-1}} L_{x^{-1}}$$

since $\text{Supp}(T) = X \sqcup X^{-1}$

$$= T.$$  

Therefore, under hypothesis, the adjoint $T^*$ of $T$ is identical to $T$, itself, and hence the element $T$ of $M_G$ is self-adjoint.

$(\Rightarrow)$ Let $T \in M_G$ be a self-adjoint graph operator, i.e., $T$ satisfies $T^* = T$. Then

$$T^* = \left( \sum_{w \in \text{Supp}(T)} t_w L_w \right)^*$$

$$= \sum_{w \in \text{Supp}(T)} (t_w L_w)^*$$

$$= \sum_{w \in \text{Supp}(T)} t_w L_w^{-1} \overset{(*)}{=} \sum_{w \in \text{Supp}(T)} t_w L_w$$

$$= T.$$  

To satisfy the above equality $(*)$, we must have

$$\text{Supp}(T^*) = \text{Supp}(T).$$

Notice that the support $\text{Supp}(T^*)$ of the adjoint $T^*$ of $T$ satisfies

$$\text{Supp}(T^*) = \text{Supp}(T)^{-1},$$

in $G$.

So, the self-adjointness of $T$ guarantees

$$\text{Supp}(T) = \text{Supp}(T)^{-1}$$

in $G$.

Therefore, since $\text{Supp}(T)$ is self-adjoint, in the sense that $\text{Supp}(T)$ is identical to $\text{Supp}(T)^{-1}$, there must exists a subset $X$ of $\text{Supp}_V(T)$ such that

$$\text{Supp}_V(T) = X \sqcup X^{-1},$$

because the following set equality always holds true;

$$\text{Supp}_V(T)^{-1} = \text{Supp}_V(T)$$
(since \( V(G)^{-1} = V(G) = V(G) = V(G^{-1}) \)).

Now, let \( X \) be a subset satisfying the above set equality,
\[
\text{Supp}_v(T) = X \cup X^{-1}, \text{ in } G.
\]

For a fixed element \( x \in X \), the coefficient \( t_x \) of \( T \) has its corresponding coefficient \( t_{x^{-1}} \) of \( T \). Assume now that there exists at least one element \( x_0 \in X \), such that
\[
t_{x_0} \neq \overline{t_{x_0^{-1}}} \text{ in } \mathbb{C}.
\]

Then the summand \( t_{x_0} L_{x_0} \) of \( T \) satisfies that
\[
(t_{x_0} L_{x_0})^* = \overline{t_{x_0}} L_{x_0^{-1}} \neq t_{x_0^{-1}} L_{x_0^{-1}},
\]
and hence \( T^* \neq T \) on \( H_G \). This contradicts our self-adjointness of \( T \).

Therefore, if \( T \) is self-adjoint, then there exists a unique subset \( X \) of the support \( \text{Supp}(T) \) of \( T \) such that
\[
\text{Supp}_v(T) = X \cup X^{-1},
\]
and
\[
t_x = \overline{t_{x^{-1}}} \text{, for all } x \in X.
\]

Similarly, assume that there exists at least one \( v_0 \in \text{Supp}_v(T) \), such that \( t_{v_0} \in \mathbb{C} \setminus \mathbb{R} \). Then the summand \( t_{v_0} L_{v_0} \) of \( T \) satisfies that
\[
(t_{v_0} L_{v_0})^* = \overline{t_{v_0}} L_{v_0^{-1}} = \overline{t_{v_0}} L_{v_0} \neq t_{v_0} L_{v_0},
\]
since \( \overline{t_{v_0}} \neq t_{v_0} \), whenever \( t_{v_0} \notin \mathbb{R} \) in \( \mathbb{C} \). This also contradicts our assumption that \( T \) is self-adjoint. \( \blacksquare \)

The above theorem characterizes the self-adjointness of graph operators \( T \) by the classification of the support \( \text{Supp}(T) \), and the coefficients of \( T \). This is interesting since the self-adjointness of graph operators are determined by the combinatorial data represented by the elements of the supports (or the admissibility of graph groupoids of given graphs), and the simple analytic data of coefficients.

**Example 3.1.** Let \( G \) be a graph,
\[
G = v_1 \xrightarrow{e_1} v_2 \xrightarrow{e_2} v_3.
\]

Let
\[
T_1 = t_{v_1} L_{v_1} + t_{e_1} L_{e_1} + t_{e_1^{-1}} L_{e_1^{-1}} + t_{e_2 e_2^{-1}} L_{e_2 e_2^{-1}} + t_{e_3 e_3^{-1}} L_{e_3 e_3^{-1}},
\]
and
\[
T_2 = t_{e_2} L_{e_2} + t_{e_3} L_{e_3} + t_{e_3^{-1}} L_{e_3^{-1}},
\]
in \( M_G \). Then we can check the self-adjointness of \( T_1 \) and \( T_2 \) immediately by the above theorem. First, consider the self-adjointness of \( T_1 \). We can see that
\[
\text{Supp}_v(T_1) = \{ v_1 \}, \text{ and } \text{Supp}_v(T_1) = \{ e_1, e_1^{-1}, e_3 e_2^{-1}, e_2 e_3^{-1} \},
\]
in \( \text{Supp}(T_1) \). So, there exists a subset \( X \) of \( \text{Supp}(T_1) \),
\[
X = \{ e_1, e_3 e_2^{-1} \}, \text{ having } X^{-1} = \{ e_1^{-1}, e_2 e_3^{-1} \},
\]
satisfying that
\[
\text{Supp}_v(T_1) = X \cup X^{-1}.
\]
(From this example, we can realize that the existence of $X$ is not uniquely determined. For instance, we may take a set $Y$, $Y = \{e_1^{-1}, e_3 e_2^{-1}\}$, having $Y^{-1} = \{e_1, e_2 e_3^{-1}\}$.

So, the graph operator $T_1$ is self-adjoint on $H_G$, if and only if $t_{e_1} \in \mathbb{R}$, and $t_{e_3 e_2^{-1}} = \overline{t_{e_2 e_3^{-1}}}$, in $\mathbb{C}$.

Also, for an operator $T_2$, we can immediately check that $T_2$ never be self-adjoint on $H_G$, because $\text{Supp}(T_2) = \text{Supp}_V(T_2) = \{e_2, e_3, e_3^{-1}\}$, and there does not exist a subset $X$, satisfying $\text{Supp}_V(T_2) = X \cup X^{-1}$.

Therefore, a graph operator $T_2$ is not self-adjoint on $H_G$.

3.3. Unitary Graph Operators. In this section, we will consider the unitary graph operators in the given graph von Neumann algebra $M_G$ of a connected directed graph $G$. To consider the unitary property of graph operators, we will restrict our interests to the case where a given connected graph $G$ is a finite graph. Recall that a graph $G$ is finite, if $|V(G)| < \infty$, and $|E(G)| < \infty$.

Assumption In this section, we assume all given graphs are “finite.”

The reason we only consider finite graphs to study the unitary property of graph operators is that: we want to determine the identity operator $i_d$ on the graph Hilbert space $H_G$, easily.

Notice that the identity operator $i_d$ in $B(H_G)$ is identified with the element $1_{M_G} = \sum_{v \in V(G)} L_v$ in $M_G$.

Remark 3.1. Remark that, even though the given graph $K$ is “infinite,” in particular, $|V(K)| = \infty$, the identity element $1_{M_K}$ of the corresponding graph von Neumann algebra $M_K$ is the operator $\sum_{v \in V(K)} L_v$, under topology. So, the identity element $1_{M_K}$ is not finitely supported. Therefore, we can verify that a finitely supported element $T$ of $M_K$ (which is our graph operator) would not be unitary, since the Cartesian product

$$\text{Supp}(T)^{r_1} \times \ldots \times \text{Supp}(T)^{r_n},$$

where $$(r_1, \ldots, r_n) \in \{\pm 1\}^n,$$

is a finite set, for all $n \in \mathbb{N}$. Thus, we restrict our interests to the case where we have “finite” graphs.

Let $1_{M_G}$ be the identity element of the graph von Neumann algebra $M_G$ of a finite graph $G$. Then, an operator $U$ on $H_G$ is unitary, if and only if

$$U^* U = 1_{M_G} = U U^*,$$
by definition, and hence, equivalently, $U^* = U^{-1}$, where $U^{-1}$ means the inverse of $U$.

Now, let’s fix a graph operator

$$T = \sum_{w \in \text{Supp}(T)} t_w L_w \text{ in } M_G.$$  

Then the adjoint $T^*$ of $T$ is

$$T^* = \sum_{w \in \text{Supp}(T)} \overline{t_w} L_{w^{-1}} \text{ in } M_G.$$  

Thus the products $T^*T$ of $TT^*$ are

$$T^*T = \sum_{(w_1, w_2) \in \text{Supp}(T)^2} \overline{t_{w_1}} t_{w_2} L_{w_1^{-1}w_2}$$  

and

$$TT^* = \sum_{(y_1, y_2) \in \text{Supp}(T)^2} t_{y_1} \overline{t_{y_2}} L_{y_1y_2^{-1}},$$  

respectively, where

$$\text{Supp}(T)^2 \overset{\text{def}}{=} \text{Supp}(T) \times \text{Supp}(T).$$

**Definition 3.2.** Let $X$ be a subset of the graph groupoid $\mathcal{G}$ of $G$. We say that this subset $X$ is alternatively disconnected, if it satisfies that:

(i) $|X| \geq 2$,

(ii) for any pair $(w_1, w_2)$ of “distinct” elements $w_1$ and $w_2$ of $X \cap \mathcal{F}_\mathcal{P}(\hat{G})$ (if it exists, or if it is nonempty), neither “$w_1^{-1}$ and $w_2$,” nor “$w_1$ and $w_2^{-1}$” is admissible in $\mathcal{G}$.

Let $G$ be a finite graph,

$$G = v_1 \bullet e_1 \overset{e_2}{\longrightarrow} v_3$$

and let

$$X_1 = \{v_1, e_1, e_2\}, \ X_2 = \{e_1^{-1}, e_2, v_3\}, \ X_3 = \{v_2, v_3\}$$

be given subsets of the graph groupoid $\mathcal{G}$ of $G$. Then, we can check that the subset $X_1$ is not alternatively disconnected, because it does not satisfy the condition (ii) of the definition. i.e., both “$e_1^{-1}$ and $e_2$,” and “$e_2^{-1}$ and $e_1$” are admissible in $\mathcal{G}$. Also, we can see the subset $X_2$ is alternatively disconnected. Indeed, neither “$e_1 = (e_1^{-1})^{-1}$ and $e_2$,” nor “$e_2^{-1}$ and $e_1^{-1}$” is admissible in $\mathcal{G}$. Clearly, the subset $X_3$ is alternatively disconnected, since it satisfies the conditions (i) and (ii) of the above theorem. Also, all vertex sets of (finite) graphs are alternatively disconnected in the above sense.

Now, let’s go back to our main interest of this section. To become a graph operator $T$ of $M_G$ to be unitary, both operators $T^*T$ and $TT^*$ must be the identity element

$$1_{M_G} = \sum_{v \in V(\hat{G})} L_v \text{ in } M_G.$$  

Thus we can obtain the following characterization.

**Theorem 3.2.** Let $G$ be a finite graph with

$$|V(G)| \geq 2,$$
and let $T \in M_G$ be a graph operator with its support $\text{Supp}(T)$. Then $T$ is unitary, if and only if

(i) $\text{Supp}(T)$ is alternatively disconnected,

(ii) the support $\text{Supp}(T)$ satisfies

$$(\text{Supp}(T))^{-1} (\text{Supp}(T)) = V(\hat{G}),$$

where $X^{-1}X \overset{\text{def}}{=} \{w_1^{-1}w_2 : w_1, w_2 \in X\}$, for all $X \subset G$, and

(iii) the coefficients of $T$ satisfy

$$\sum_{w \in \text{Supp}(T), \ w^{-1}w = v} |t_w|^2 = 1, \text{ for all } v \in V(\hat{G}),$$

in $\mathbb{C}$.

Proof. Assume that the given graph operator $T$ is unitary on $H_G$. Then, by definition,

(3.3.1)

$$T^*T = \sum_{(w_1, w_2) \in \text{Supp}(T)^2} \overline{t_{w_1}} t_{w_2} L_{w_1^{-1}w_2} = \sum_{v \in V(\hat{G})} L_v = 1_{M_G},$$

and

(3.3.2)

$$TT^* = \sum_{(y_1, y_2) \in \text{Supp}(T)^2} t_{y_1} \overline{t_{y_2}} L_{y_1^{-1}y_2} = \sum_{v \in V(\hat{G})} L_v = 1_{M_G},$$

in the graph von Neumann algebra $M_G$ of a finite connected graph $G$. Notice here that, if there exists a pair $(w_1, w_2)$ of distinct elements $w_1 \neq w_2$ in $\text{Supp}(T)$, such that $w_1^{-1}w_2 \neq \emptyset$, equivalently, $w_1^{-1}$ and $w_2$ are admissible in $G$, then there exists an nonzero summand

$$\overline{t_{w_1}} t_{w_2} L_{w_1^{-1}w_2}$$

in (3.3.1). By the distinctness of $w_1$ and $w_2$, and by the assumption $w_1^{-1}w_2 \neq \emptyset$, the element $w_1^{-1}w_2$ must be a nonempty reduced finite path in $G$. This shows that the first equality (3.3.1) does not hold, and hence it contradicts our unitary property of $T$.

Similarly, if $w_1w_2^{-1} \neq \emptyset$, then there exists an nonzero summand

$$t_{w_1} \overline{t_{w_2}} L_{w_1w_2^{-1}}$$

in (3.3.2), and hence this term breaks the unitary property of $T$, which contradicts our assumption for $T$.

Therefore, to satisfy the unitary property of $T$, the support $\text{Supp}(T)$ of $T$ is alternatively disconnected, i.e., for any pair $(w_1, w_2)$ of distinct elements in $\text{Supp}(T)$, neither “$w_1^{-1}$ and $w_2$” nor “$w_1$ and $w_2^{-1}$” is admissible in $G$. Under the alternative disconnectedness of $\text{Supp}(T)$, we can obtain the alternating form of the left-hand side of (3.3.1):

(3.3.3)
\[ T^*T = \sum_{(w_1, w_2) \in \text{Supp}(T)^2, \ w_1 = w_2} \overline{t_{w_1}} t_{w_2} L_{w_1^{-1}w_2} \]
\[ = \sum_{w \in \text{Supp}(T)} \overline{t_w} t_w L_{w^{-1}w} \]
\[ = \sum_{w \in \text{Supp}(T)} |t_w|^2 L_{w^{-1}w}. \]

Remark here that \( w^{-1}w \in V(\hat{G}) \), for all \( w \in \mathbb{G} \). By (3.3.3), we can re-write that \( T \) is unitary if and only if
\[
(3.3.4)
\]
\[ T^*T = \sum_{w \in \text{Supp}(T)} |t_w|^2 L_{w^{-1}w} = \sum_{v \in V(\hat{G})} L_v = 1_{M_G}, \]
by the finiteness of \( G \). And the second equality of (3.3.4) can be refined as follows:
\[
(3.3.5)
\]
\[ \sum_{w \in \text{Supp}(T)} |t_w|^2 L_{w^{-1}w} = \sum_{v \in V(\hat{G})} \left( \sum_{w \in \text{Supp}(T), w^{-1}w = v} |t_w|^2 \right) L_v = 1_{M_G}. \]

Therefore, by (3.3.5), the support of \( T \) must satisfy
\[
(3.3.6)
\]
\[ (\text{Supp}(T)^{-1}) (\text{Supp}(T)) = V(\hat{G}), \]
and, under the alternative disconnectedness (3.3.6) of \( T \), the coefficients of \( T \) must satisfy
\[
(3.3.7)
\]
\[ \sum_{w \in \text{Supp}(T), w^{-1}w = v} |t_w|^2 = 1, \text{ for all } v \in V(\hat{G}), \]
in \( \mathbb{C} \), where
\[ X^{-1}X \overset{\text{def}}{=} \{ w_1^{-1}w_2 : w_1, w_2 \in X \}, \text{ for all } X \subset \mathbb{G}. \]

i.e., we can obtain that \( T^*T = 1_{M_G} \), if and only if the support \( \text{Supp}(T) \) is alternatively disconnected, and it satisfies (3.3.6), and the coefficients of \( T \) satisfy (3.3.7).

Similar to the above observation, we can get that \( TT^* = 1_{M_G} \), if and only if \( \text{Supp}(T) \) is alternatively disconnected, and it satisfies (3.3.6), and the coefficients of \( T \) satisfies
\[
(3.3.8)
\]
\[ \sum_{y \in \text{Supp}(T), yy^{-1}=x} |t_y|^2 = 1, \text{ for all } x \in V(\hat{G}). \]

However, it is easy to check that the conditions (3.3.7) and (3.3.8) are equivalent, because there exists a bijection \( g \),
\[ g : w \in \text{Supp}(T) \mapsto w^{-1} \in \text{Supp}(T)^{-1}. \]
Therefore, we can conclude that the graph operator $T$ is unitary, if and only if the support $\text{Supp}(T)$ of $T$ is alternatively disconnected, and it also satisfies the conditions (3.3.6), and the coefficients of $T$ satisfy (3.3.7) (or (3.3.8)).

Similar to the self-adjointness of graph operators, the unitary property of graph operators are also determined by the admissibility on the graph groupoids of given graphs and certain conditions on coefficients of the operators.

**Remark 3.2.** In the proof of the above theorem (the unitary characterization of graph operators), where

$$|V(G)| \geq 2,$$

the alternative disconnectedness is crucial. Since $|V(G)| > 1$, all generating operators $L_{w}$’s of the graph von Neumann algebra $M_{G}$ are partial isometries. Moreover, the products $L_{w_{1}} \ldots L_{w_{n}}$, for all $n \in \mathbb{N}$, are partial isometries, whose initial and final spaces are “not” identified with the graph Hilbert space $H_{G}$. Thus, to satisfy the unitary property, the products $L_{w_{1}w_{2}}$ either in $T^{*}T$ or in $TT^{*}$ must be the zero operator, whenever $w_{1} \neq w_{2}$ in $G$.

In the rest of this paper, we will consider following two examples.

**Example 3.2.** Let $T = \sum_{v \in V(\hat{G})} t_{v} L_{v}$ be a graph operator in $M_{G}$. Then it is unitary, if and only if (i) $t_{v} \neq 0$, and (ii) $|t_{v}|^{2} = 1$, for all $v \in V(\hat{G})$.

**Example 3.3.** Let $G$ be a connected finite graph,

$$G = v_{1} \cdot e_{1}v_{2} \cdot e_{2}v_{3} \cdot e_{3}v_{4}.$$

Let $T_{1} = t_{e_{1}} L_{v_{1}} + t_{e_{2}}^{-1} L_{e_{2}}^{-1} + t_{e_{2}e_{3}} L_{e_{2}e_{3}}$ be a given graph operator in the graph von Neumann algebra $M_{G}$ of $G$. We can check that

$$\text{Supp}(T_{1}) = \{v_{1}, e_{2}^{-1}, e_{2}e_{3}\},$$

and hence

$$\Pi_{1} = (\text{Supp}(T_{1})^{\perp})(\text{Supp}(T_{1})) = \{v_{1}, v_{2}, v_{4}\}.$$

So, $\Pi_{1} \neq V(G) = V(\hat{G})$. Therefore, this graph operator $T_{1}$ is not unitary.

Now, let $T_{2} = t_{v_{2}} L_{v_{2}} + t_{v_{3}} L_{v_{3}} + t_{e_{2}e_{3}}^{-1} L_{e_{2}e_{3}}^{-1} + t_{e_{2}e_{3}} L_{e_{2}e_{3}}$. Then the support

$$\text{Supp}(T_{2}) = \{v_{1}, v_{3}, e_{2}^{-1}, e_{2}e_{3}\}$$

satisfies that

$$\Pi_{2} = (\text{Supp}(T_{2})^{\perp})(\text{Supp}(T_{2})) = \{v_{1}, v_{2}, v_{3}, v_{4}\} = V(\hat{G}).$$

Moreover, all the pairs $(w_{1}, w_{2})$ of distinct elements $w_{1}$ and $w_{2}$ of $\text{Supp}(T_{2})$ are alternatively disconnected. For instance,

$$(e_{2}^{-1})^{-1}(e_{2}e_{3}) = e_{2}^{2}e_{3} = 0$$

and $e_{2}^{-1}v_{3}e_{2} = 0$, and $e_{2}^{-1}v_{3} = 0$,

etc. Therefore, we can obtain that the operator $T_{2}$ is unitary on the graph Hilbert space $H_{G}$, if and only if

$$\sum_{w \in \text{Supp}(T_{2}), w^{-1}w = v_{1}} |t_{w}|^{2} = |t_{v_{1}}|^{2} = 1.$$
The above unitary characterization of graph operators (induced by finite graphs) is in fact incomplete, since we did not consider the case where a given graph \( G \) satisfies \( |V(G)| = 1 \). If a finite graph \( G \) has only one vertex \( v_0 \), then it is graph-isomorphic to the one-vertex-\( |E(G)| \)-multi-loop-edge graph \( O_{|E(G)|} \). To make our unitary characterization of graph operators complete, we need the following theorem.

**Theorem 3.3.** Let \( O_n \) be the one-vertex-\( n \)-loop-edge graph with its graph groupoid \( \mathbb{O}_n \), having its unique vertex \( v_0 \), and let \( M_{O_n} \) be the graph von Neumann algebra of \( O_n \), for \( n \in \mathbb{N} \). Let

\[
T = \sum_{w \in \text{Supp}(T)} t_w L_w \in M_{O_n}
\]

be a fixed graph operator. Then \( T \) is unitary, if and only if

(i) \( (\text{Supp}(T))^{-1} \{ v_0 \} = \{ v_0 \} \),

(ii) the coefficients \( \{ t_w : w \in \text{Supp}(T) \} \) of \( T \) satisfies

\[
\sum_{(w_1, w_2) \in \text{Supp}(T)^2} t_{w_1} t_{w_2} = 1.
\]

**Proof.** \((\Rightarrow)\) Assume that a fixed graph operator \( T \) of \( M_{O_n} \) satisfies both conditions (i) and (ii). Then we can obtain that

\[
T^* T = \sum_{(w_1^{-1}, w_2) \in (\text{Supp}(T))^{-1} \times \text{Supp}(T)} (\overline{t_{w_1}} t_{w_2}) L_{w_1^{-1} w_2}
\]

by (i)

\[
= \sum_{(w_1, w_2) \in \text{Supp}(T)^2} (\overline{t_{w_1}} t_{w_2}) L_{v_0}
\]

by (ii). Notice that, by definition, \( L_{v_0} \) is the identity element of \( M_{O_n} \), i.e., \( L_{v_0} = 1_{M_{O_n}} \). Therefore, we have that

\[
(3.3.9) \quad T^* T = 1_{M_{O_n}}.
\]

Consider now \( TT^* \). Observe that, under hypothesis,

\[
TT^* = \sum_{(w_1, w_2^{-1}) \in \text{Supp}(T) \times (\text{Supp}(T))^{-1}} (\overline{t_{w_1}} t_{w_2}) L_{w_1 w_2^{-1}}
\]

by (i), and by the fact that:
Thus we can have
\[
(\text{Supp}(T))^{-1} (\text{Supp}(T)) = \{v_O\} = \{v_O^{-1}\} = (\text{Supp}(T)) (\text{Supp}(T))^{-1},
\]
\[
= \left( \sum_{(w_1, w_2) \in \text{Supp}(T)^2} (t_{w_1} \overline{t_{w_2}}) \right) L_{v_O} = \left( \sum_{(w_1, w_2) \in \text{Supp}(T)} (\overline{t_{w_1}} t_{w_2}) \right) L_{v_O}
\]
by (i). Therefore, we obtain that
\[
TT^* = 1_{M_{O_n}}.
\]
So, by (3.3.9) and (3.3.10), this graph operator \( T \) is unitary.

(\Rightarrow) Suppose a graph operator \( T \) of \( M_{O_n} \) is unitary. Assume that \( T \) does not satisfy the condition (i). Then we can pick a pair
\[
(w_1, w_2) \in (\text{Supp}(T))^2,
\]
such that \( w_1 \neq w_2 \) and \( w_1^{-1} w_2 \neq v_O \), equivalently, \( w_1^{-1} w_2 \in FP_r(\hat{O}_n) \). This means that the product \( T^*T \) of \( T^* \) and \( T \) contains a nonzero summand \( \overline{t_{w_1}} t_{w_2} \).

Thus,
\[
T^*T \neq 1_{M_{O_n}} = L_{v_O}.
\]
This contradicts our assumption that \( T \) is unitary. Assume now that \( T \) does not satisfy the condition (ii). Say
\[
\sum_{(w_1, w_2) \in \text{Supp}(T)^2} (\overline{t_{w_1}} t_{w_2}) = t_0 \neq 1, \text{ in } \mathbb{C}.
\]
For convenience, assume \( T \) satisfies the condition (i). Then the product \( T^*T \) of \( T^* \) and \( T \) is identical to
\[
T^*T = t_0 L_{v_O} \neq L_{v_O} = 1_{M_{O_n}}.
\]
This contradict the unitary property of \( T \).

The above theorem characterizes the unitary property of graph operators induced by the one-vertex-multi-loop-edge graphs.

**Conclusion (Unitary Characterization of Graph Operators)**

Let \( G \) be a finite graph and let \( M_G \) be the graph von Neumann algebra of \( G \). Let
\[
T = \sum_{w \in \text{Supp}(T)} t_w L_w \in M_G
\]
by a graph operator.

(3.3.11) Assume that \( |V(G)| = 1 \). Then \( T \) is unitary, if and only if
\[
(\text{Supp}(T))^{-1} (\text{Supp}(T)) = V(G),
\]
and
\[
\sum_{(w_1, w_2) \in \text{Supp}(T)^2} \overline{t_{w_1}} t_{w_2} = 1, \text{ in } \mathbb{C}.
\]
(3.3.12) Assume now that $|V(G)| = 2$. Then $T$ is unitary, if and only if $\text{Supp}(T)$ is alternatively disconnected, and
\[
(\text{Supp}(T))^{-1}(\text{Supp}(T)) = V(G),
\]
and
\[
\sum_{w \in \text{Supp}(T), w^{-1}w = v} |t_w|^2 = 1, \text{ for all } v \in V(G).
\]
□

4. Normality of Graph Operators

In this section, we will consider the normality of graph operators. Let $G$ be a connected directed graph with its graph groupoid $G$, and let $M_G = \mathbb{C}[L(G)]$ be the graph von Neumann algebra of $G$ in $B(H_G)$, where $(H_G, L)$ is the canonical representation of $G$, consisting of the graph Hilbert space $H_G = l^2(G)$, and the canonical groupoid action $L$ of $G$.

We are interested in the normality of graph operators in $M_G$. Recall that an operator $T$ is normal, if $T^*T = TT^*$. Before checking the normality of a graph operator $T \in M_G$, we will consider the hyponormality of $T$ in Section 4.1. Recall that an operator $T$ is hyponormal, if $T^*T - TT^*$ is positive.

The hyponormality characterization of graph operators would give the normality characterization directly. Notice here that (pure) hyponormal operators and normal operators have few common analytic properties. So, in general, we do not know how the hyponormality determines the normality. However, in our graph-operator case, the hyponormal characterization determines the normality characterization.

As we have seen in Section 3, the self-adjointness and the unitary property of graph operators are characterized by the admissibility on the graph groupoid $G$ (equivalently, the combinatorial property of $G$ or $\hat{G}$), and certain analytic data of coefficients. We hope to obtain the similar normality characterization.

4.1. Hyponormality. To consider the normality of graph operators, we first characterize the hyponormality of them. Note that the hyponormality, itself, is interesting in operator theory (e.g., see [12], and [13]). For instance, the hyponormality of Toeplitz operators have been studied widely (e.g., See [3], and cited papers of [3]).

We may understand hyponormality (or co-hyponormality) as the generalized normality. But keep in mind that hyponormal operators and normal operators do not share analytic properties much. However, in our case, we can show that hyponormality of graph operators and normality of graph operators are combinatorially related.

In this section, we characterize the hyponormality of a given graph operator $T$, in terms of the combinatorial information on a fixed graph groupoid and the analytic data on the coefficients of $T$, like in Sections 3.2, and 3.3.

Recall that an operator $T$ is positive on a Hilbert space $H$, if
\[
< T\xi, \xi > \geq 0, \text{ for all } \xi \in H.
\]

Here, $<,>$ means the inner product on $H$, and $\|\cdot\|$ means the corresponding Hilbert norm induced by $<,>$. If $T$ is a positive operator on $H$, we write $T \geq 0_H$. 
where $0_H$ is the zero operator in $B(H)$. When $T_1$ and $T_2$ are operators on $H$, we write

$$T_1 \geq T_2,$$

if the operator $T_1 - T_2$ is a positive operator on $H$. Thus, by definition, an operator $T$ is \textit{hyponormal} on $H$, if and only if $T^*T \geq TT^*$ on $H$, equivalently, the operator

$$S(T) \overset{\text{def}}{=} [T^*, T] \overset{\text{def}}{=} T^*T - TT^*$$

is a positive operator on $H$, where $[A, B]$ means the operator,

$$[A, B] \overset{\text{def}}{=} AB - BA,$$

for all $A, B \in B(H)$.

We call the operator $[A, B]$, the \textit{commutator} of $A$ and $B$. In particular, if $A = T^*$ and $B = T$, for $T \in B(H)$, the commutator $[T^*, T]$ is called the \textit{self-commutator} of $T$.

Notice here that the self-commutator $S(T) = [T^*, T]$ of every operator $T$ is self-adjoint on $H$.

Define a two maps $s, r : G \to V(\hat{G})$ by

$$r(w) \overset{\text{def}}{=} w^{-1}w, \text{ and } s(w) \overset{\text{def}}{=} ww^{-1},$$

for all $w \in G$, i.e., these maps $r$ and $s$ are the range map and the source map of the (graph) groupoid $G$, in the sense of Section 2.2.

Now, fix a graph operator

$$T = \sum_{w \in \text{Supp}(T)} t_w L_w \in M_G,$$

acting on the graph Hilbert space $H_G$. The self-commutator $S(T)$ of $T$ is computed as follows:

$$S(T) = T^*T - TT^*$$

(4.1.1)

$$= \sum_{(w_1, w_2) \in \text{Supp}(T)^2} t_{w_1} t_{w_2} L_{w_1^{-1}w_2} - \sum_{(y_1, y_2) \in \text{Supp}(T)^2} t_{y_1} t_{y_2} L_{y_1^{-1}y_2},$$

$$= \sum_{(w_1, w_2) \in \text{Supp}(T)^2} t_{w_1} t_{w_2} \left( L_{w_1^{-1}w_2} - L_{w_2w_1^{-1}} \right)$$

(4.1.2)

$$= \sum_{(w_1, w_2) \in \text{Supp}(T)^2, w_1^{-1}w_2 \neq \emptyset} t_{w_1} t_{w_2} \left( L_{w_1^{-1}w_2} - L_{w_2w_1^{-1}} \right)$$

$$= \left( \sum_{w \in \text{Supp}(T)} |t_w|^2 \left( L_{r(w)} - L_{s(w)} \right) \right)$$

$$+ \sum_{(w_1, w_2) \in \text{Supp}(T), w_1^{-1}w_2 \neq \emptyset} t_{w_1} t_{w_2} \left( L_{w_1^{-1}w_2} - L_{w_2w_1^{-1}} \right)$$

where $r(w) = w^{-1}w$, and $s(w) = ww^{-1}$, for all $w \in G$

$$= \left( \sum_{w \in \text{Supp}(T)} |t_w|^2 \left( L_{r(w)} - L_{s(w)} \right) \right)$$

$$+ \sum_{(w_1, w_2) \in \text{Supp}(T), w_1^{-1}w_2 \neq \emptyset} \mathcal{S}_{(w_1, w_2)},$$

where
\[ S_{(w_1, w_2)} \overset{\text{def}}{=} \left( t_{w_1} t_{w_2} \left( L_{w_1^{-1} w_2} - L_{w_2 w_1^{-1}} \right) + t_{w_1} t_{w_2} \left( L_{w_2^{-1} w_1} - L_{w_1 w_2^{-1}} \right) \right), \]

for all \((w_1, w_2) \in \text{Supp}(T)^2\), such that \(w_1 \neq w_2^{\pm 1}\)

\[(4.1.3)\]

\[= \left( \sum_{w \in \text{Supp}(T)} |t_w|^2 (L_{r(w)} - L_{s(w)}) \right) + \left( \sum_{(w_1, w_2) \in \text{Supp}(T), w_1 \neq w_2^{\pm 1}, w_1^{-1} w_2 \neq \emptyset} \left( \left( t_{w_1} t_{w_2} L_{w_1^{-1} w_2} + t_{w_1} t_{w_2} L_{w_2^{-1} w_1} \right) - \left( t_{w_1} t_{w_2} L_{w_2 w_1^{-1}} + t_{w_1} t_{w_2} L_{w_1 w_2^{-1}} \right) \right) \right). \]

The computation (4.1.3) indeed shows that the self-commutator \(S(T)\) is self-adjoint on the graph Hilbert space \(H_G\), since each summand of (4.1.3) is self-adjoint. Recall that if two operators are self-adjoint, then the addition of these two operators is again self-adjoint.

The hyponormality of \(T\) is guaranteed by the positivity of the self-adjoint operator \(S(T)\). In general, it is not easy to check when a self-adjoint operator \(S\) is positive, because, for example, it is hard to see when the spectrum \(\text{spec}(S)\) (contained in \(\mathbb{R}\)) is contained in \(\mathbb{R}^+_0 = \{ r \in \mathbb{R} : r \geq 0 \}\).

However, in our graph-operator case, we can check the positivity of \(S(T)\) of \(T\), by (4.1.1), (4.1.2), (4.1.3), and the computations, \[< S(T) \xi_x, \xi_x >, \text{ and }< S(T) \xi_x, \xi_y >, \]

for \(x, y \in \mathbb{G} \setminus \{ \emptyset \}\) (equivalently, for \(\xi_x, \xi_y \in \mathcal{B}_{H_G}\) in \(H_G\)), where \(<, >\) means the inner product on \(H_G\). To check the positivity of \(S(T)\), we have to show that \[< S(T) \xi, \xi > \geq 0, \text{ for all } \xi \in H_G. \]

Since the collection of vectors \[\eta = \sum_{x \in \mathbb{G}} r_x \xi_x \in H_G, \text{ with } r_x \in \mathbb{C}, \]
is dense in \(H_G\), it is enough to show that \[< S(T) \eta, \eta > \geq 0, \text{ for all } \eta \in \mathcal{H}_G, \]

where

\[\mathcal{H}_G \overset{\text{def}}{=} \left\{ \eta = \sum_{x \in \mathbb{G}} r_x \xi_x | r_x \in \mathbb{C}, \xi_x \in \mathcal{B}_{H_G} \right\} \subseteq H_G. \]

**Lemma 4.1.** Let \(L_w \in M_G\) be a generating operator of \(M_G\) induced by \(w \in \mathbb{G}\). Then

\[(4.1.4)\]

\[< L_w \xi_x, \xi_y > = \delta_{r(w), s(x)} \delta_{w x, y}, \]

where \(\delta\) means the Kronecker delta.

**Proof.** Compute \[< L_w \xi_x, \xi_y > = < \xi_{wx}, \xi_y > \]

\[= \begin{cases} < \xi_{wx}, \xi_y > & \text{if } r(w) = s(x) \\ < \xi_{\emptyset}, \xi_y > = 0 & \text{otherwise} \end{cases} \]
\[= \delta_{r(w), s(x)} < \xi_{wx}, \xi_y >\]

\[= \begin{cases} \delta_{r(w), s(x)} \cdot 1 & \text{if } wx = y \\ \delta_{r(w), s(x)} \cdot 0 & \text{otherwise} \end{cases}\]

since \(\xi_{wx}, \xi_y \in B_{\mathcal{H}_G} \cup \{0_{\mathcal{H}_G}\}\)

Therefore,

\[< L_w \xi_x, \xi_y >= \delta_{r(w), s(x)} \delta_{wx, y},\]

for all \(w, x, y \in \mathcal{G}\). \(\square\)

By (4.1.4), we can obtain the following lemma.

Lemma 4.2. Let \(T = \sum_{w \in \text{Supp}(T)} t_w L_w \in M_\mathcal{G}\) be a graph operator, and let \(\xi = \sum_{x \in \mathcal{G}} r_x \xi_x \in \mathcal{H}_\mathcal{G}\) be a vector in \(\mathcal{H}_\mathcal{G}\). Then

\[< S(T) \xi, \xi > = \sum_{(x,y) \in \mathcal{G}^2} r_x r_y \left( \sum_{(w_1,w_2) \in \text{Supp}(T)^2, w_1^{-1}w_2 \neq \emptyset} \overline{t_{w_1}} t_{w_2} \delta_{r(w_1^{-1}w_2), s(x)} \delta_{w_1^{-1}w_2, y} \right. \]

\[\left. - \sum_{(y_1,y_2) \in \text{Supp}(T)^2, y_1y_2^{-1} \neq \emptyset} t_{y_1} \overline{t_{y_2}} \delta_{r(y_1y_2^{-1}), s(x)} \delta_{y_1y_2^{-1}, x} \right).\]

\(\square\)

The proof of the above theorem is straightforward, by (4.1.1) and (4.1.4). Now, we denote the summands

\[r_x r_y \left( \sum_{(w_1,w_2) \in \text{Supp}(T)^2, w_1^{-1}w_2 \neq \emptyset} \overline{t_{w_1}} t_{w_2} \delta_{r(w_1^{-1}w_2), s(x)} \delta_{w_1^{-1}w_2, y} \right. \]

\[\left. - \sum_{(y_1,y_2) \in \text{Supp}(T)^2, y_1y_2^{-1} \neq \emptyset} t_{y_1} \overline{t_{y_2}} \delta_{r(y_1y_2^{-1}), s(x)} \delta_{y_1y_2^{-1}, x} \right)\]

of (4.1.5) by \(\Delta_{xy}\). By (4.1.3), each summand \(\Delta_{xy}\) has its (kind of) pair \(\Delta_{yx}\), in the formula (4.1.5),

\[\Delta_{yx} = r_y r_x \left( \sum_{(w_1,w_2) \in \text{Supp}(T)^2, w_1^{-1}w_2 \neq \emptyset} \overline{t_{w_2}} t_{w_1} \delta_{r(w_2^{-1}w_1), s(y)} \delta_{w_2^{-1}w_1, y} \right. \]

\[\left. - \sum_{(y_1,y_2) \in \text{Supp}(T)^2, y_1y_2^{-1} \neq \emptyset} t_{y_2} \overline{t_{y_1}} \delta_{r(y_2y_1^{-1}), s(y)} \delta_{y_2y_1^{-1}, y} \right).\]

i.e.,

\[< S(T) \xi, \xi > = \sum_{(x,y) \in \mathcal{G}^2} r_x r_y \Delta_{xy} = \sum_{(x,y) \in \mathcal{G}^2} r_y r_x \Delta_{yx},\]

for all \(\xi = \sum_{x \in \mathcal{G}} r_x \xi_x \in \mathcal{H}_\mathcal{G}\). \(\subseteq \mathcal{H}_\mathcal{G}\).

The following theorem is the characterization of hyponormal graph operators.

Theorem 4.3. Let \(T = \sum_{w \in \text{Supp}(T)} t_w L_w\) be a graph operator in the graph von Neumann algebra \(M_\mathcal{G}\) of \(G\). Then \(T\) is hyponormal, if and only if
\( (4.1.7) \)
\[ \{ r(w) : w \in \Pi_{T \cdot T} \} \supseteq \{ r(w) : w \in \Pi_{TT^*} \} \text{ in } V(\hat{G}), \]
where
\[ \Pi_{T \cdot T} \overset{\text{def}}{=} \left( (\text{Supp}(T))^{-1} (\text{Supp}(T)) \right) \setminus \{ \emptyset \}, \]
and
\[ \Pi_{TT^*} \overset{\text{def}}{=} \left( (\text{Supp}(T)) (\text{Supp}(T))^{-1} \right) \setminus \{ \emptyset \}, \]
in \( G \), and the coefficients of \( T \) satisfies
\( (4.1.8) \)
\[ \left( \sum_{w_1^{-1}w_2 \in \Pi_{T \cdot T}, \, \, r(w_1^{-1}w_2) = v} t_{w_1} t_{w_2} \right) \geq \left( \sum_{y_1y_2^{-1} \in \Pi_{TT^*}, \, \, r(y_1y_2^{-1}) = v} t_{y_1} t_{y_2} \right), \]
in \( \mathbb{R} \subset C \), for all \( v \in V(\hat{G}) \).

\textbf{Proof.} Let \( T = \sum_{w \in \text{Supp}(T)} t_w L_w \) be a given graph operator in \( M_G \). Then, by \( (4.1.1) \), the self-commutator \( S(T) \) of \( T \) is
\[ S(T) = TT^* - TT^* = \sum_{(w_1, w_2)} t_{w_1} t_{w_2} L_{w_1^{-1}w_2} - \sum_{(y_1, y_2)} t_{y_1} t_{y_2} L_{y_1y_2^{-1}}, \]
identified with
\[ \sum_{(w_1, w_2) \in \text{Supp}(T)^2, \, \, w_1^{-1}w_2 \neq \emptyset} t_{w_1} t_{w_2} \left( L_{w_1^{-1}w_2} - L_{w_2w_1^{-1}} \right), \]
by \( (4.1.2) \). Then, for all \( \xi = \sum_{x \in G} r_x \xi_x \in \mathcal{H}_G \) in \( H_G \), we can obtain the formula
\( (4.1.5) \), which states:
\[ \langle S(T) \xi, \xi \rangle = \sum_{(x,y) \in G^2} \sum_{(w_1, w_2) \in \text{Supp}(T)^2, \, \, w_1^{-1}w_2 \neq \emptyset} t_{w_1} t_{w_2} \delta_{r(w_1^{-1}w_2)}(x) \delta_{w_1^{-1}w_2}(y) \]
\[ - \sum_{(y_1, y_2) \in \text{Supp}(T)^2, \, \, y_1y_2^{-1} \neq \emptyset} t_{y_1} t_{y_2} \delta_{r(y_1y_2^{-1})}(x) \delta_{y_1y_2^{-1}}(y), \]
satisfying \( (4.1.6) \).
\( (\Leftarrow) \) Consider now that the terms
\[ \Delta_{xy}^o \overset{\text{def}}{=} \frac{1}{r_x r_y} \Delta_{xy} = \left( \sum_{(w_1, w_2) \in \text{Supp}(T)^2, \, \, w_1^{-1}w_2 \neq \emptyset} t_{w_1} t_{w_2} \delta_{r(w_1^{-1}w_2)}(x) \delta_{w_1^{-1}w_2}(y) \right) \]
\[ - \sum_{(y_1, y_2) \in \text{Supp}(T)^2, \, \, y_1y_2^{-1} \neq \emptyset} t_{y_1} t_{y_2} \delta_{r(y_1y_2^{-1})}(x) \delta_{y_1y_2^{-1}}(y), \]
in \( (4.1.5) \). Each \( \Delta_{xy}^o \) can be re-formulated by
\( (4.1.9) \)
\[ \Delta_{xy}^o = \sum_{v \in V_T} \left( \sum_{(w_1, w_2) \in \text{Supp}(T)^2, \, \, w_1^{-1}w_2 \neq \emptyset, \, \, r(w_1^{-1}w_2) = v} t_{w_1} t_{w_2} \delta_{v}(x) \delta_{w_1^{-1}w_2}(y) \right) \]
\[ - \sum_{(y_1, y_2) \in \text{Supp}(T)^2, \, \, y_1y_2^{-1} \neq \emptyset, \, \, r(y_1y_2^{-1}) = v} t_{y_1} t_{y_2} \delta_{v}(x) \delta_{y_1y_2^{-1}}(y), \]
where
\[ V_T \overset{\text{def}}{=} \{ r(w) : w \in \text{Supp}(T) \} \cup \{ s(w) : w \in \text{Supp}(T) \}, \]
in \( V(\hat{G}) \). Let’s denote the summand of \( \Delta^o_{xy} \) by \( \Delta^o_{xy}(v) \), for \( v \in V_T \), i.e.,
\[ \Delta^o_{xy} = \sum_{v \in V_T} \Delta^o_{xy} \]
Thus, if
\[ \Delta^o_{xy}(v) \geq 0, \text{ for all } v \in V(\hat{G}), \]
then we can make \( < S(T) \xi, \xi > \) be positive in \( \mathbb{R} \), for an “arbitrary” \( \xi \in \mathcal{H}_G \), and hence the operator \( T \) is hyponormal, by (4.1.6).
So, since the above vector \( \xi \) is arbitrary in \( \mathcal{H}_G \), we can obtain that: if the
\[ \text{(4.1.7)} \]
holds in \( \mathbb{R}(\subset \mathbb{C}) \), for all \( v \in V_T \), then \( T \) is hyponormal, since
\[ < S(T) \eta, \eta > \geq 0, \text{ for all } \eta \in \mathcal{H}_G(\subset \mathcal{H}_G). \]
Equivalently, if both (4.1.7) and (4.1.8) hold, then \( T \) is hyponormal on \( H_G \).

(\( \Rightarrow \)) Conversely, let a given graph operator \( T \) be hyponormal on \( H_G \), equivalently, the self-commutator \( S(T) \) is a positive operator on \( H_G \). And assume that \( S(T) \) does not satisfy either (4.1.7) or (4.1.8).
Suppose first that the condition (4.1.7) does not hold, i.e., assume
\[ \text{(4.1.12)} \]
\[ \mathcal{R}_{T \cdot T} = \{ r(w) : w \in \Pi_{T \cdot T} \} \subset \{ r(w) : w \in \Pi_{TT} \} = \mathcal{R}_{TT}. \]
This means that there exists an element \( w_0 \in \text{Supp}(T) \), such that
\[ r(w_0) \in \mathcal{R}_{T \cdot T} \text{ and } s(w_0) \in \mathcal{R}_{TT}, \]
satisfying
\[ r(w_0) \neq s(w_0) \text{ in } V(\hat{G}), \]
with
\[ s(w_0) \in \mathcal{R}_{TT} \setminus \mathcal{R}_{T \cdot T} \neq \emptyset. \]
Notice here that
\[ r(w_1 w_2) = r(w_2), \text{ and } s(w_1 w_2) = s(w_1), \]
for all \( w_1, w_2 \in \mathbb{G} \). So, our condition (4.1.12) guarantees the existence of such an element \( w_0 \) in \( \text{Supp}(T) \). Then we can obtain the summand
\[ |t_{w_0}|^2 (L_{r(w_0)} - L_{s(w_0)}) \]
of \( S(T) \), by (4.1.2). Again, by (4.1.12), we have the summand
of $S(T)$. Thus, if we take a vector $\xi_{w_0} \in \mathcal{B}_{H_G} \subset \mathcal{H}_G$ in $H_G$, then

$$< S(T)\xi_{w_0}, \xi_{w_0} > = -|t_{w_0}|^2 < L_{s(w_0)}\xi_{w_0}, \xi_{w_0} >$$

by (4.1.4)

$$= -|t_{w_0}|^2 \| \xi_{w_0} \|^2 = -|t_{w_0}|^2 < 0,$$

where

$$\| \eta \| \overset{df}{=} \sqrt{< \eta, \eta >}, \text{ for all } \eta \in H_G,$$

is the Hilbert space norm on $H_G$. This shows that there exists a vector $\xi' \in H_G$, such that $< S(T)\xi', \xi' >$ becomes negative in $\mathbb{R}$. This contradicts our assumption that $T$ is hyponormal.

Therefore, if $T$ is hyponormal, then the condition (4.1.7) must hold.

Assume now that $T$ is hyponormal, and the inequality (4.1.8) does not hold. We will assume that (4.1.7) holds true for $T$. Since (4.1.8) does not hold, there exists at least one vertex $v_0$ such that

(4.1.13)

$$\Delta^{o}_{xy}(v_0) < 0,$$

where $\Delta^{o}_{xy}$ and $\Delta^{o}_{xy}(v)$’s are defined in (4.1.10) and (4.1.11), respectively.

Then we can take a vector

$$\eta = \sum_{(w_1, w_2) \in \text{Supp}(T)^2, w_1^{-1}w_2 \neq \emptyset, r(w_1^{-1}w_2) = v_0} (\xi_x + \xi_{w_1^{-1}w_2}x)$$

$$+ \sum_{(y_1, y_2) \in \text{Supp}(T)^2, y_1y_2^{-1} \neq \emptyset, r(y_1y_2^{-1}) = v_0} (\xi_y + \xi_{y_1y_2^{-1}}y)$$

in $\mathcal{H}_G$. Then, by (4.1.3)

$$< S(T)\eta, \eta > = \Delta^{o}_{xy}(v_0) < 0.$$

Therefore, it breaks the hyponormality of $T$, which contradicts our assumption that $T$ is hyponormal. Thus, the condition (4.1.8) must hold under the hyponormality of $T$.

As we have seen above, we can conclude that a graph operator $T$ is hyponormal, if and only if the both conditions (4.1.7), and (4.1.8) hold. □

The above theorem characterize the hyponormality of graph operators in terms of the admissibility on $G$, and the analytic data of coefficients, just like Sections 3.2, and 3.3.

From below, denote

$$r(\Pi_{T^*T}) \text{ and } r(\Pi_{TT^*})$$

by

$$\mathcal{R}_{T^*T} \text{ and } \mathcal{R}_{TT^*},$$

respectively.

The above theorem provides not only the characterization of hyponormal graph operators but also the very useful process for checking “non-hyponormality.”
Example 4.2. Suppose a graph $G$ contains its subgraph,

$$
\begin{array}{c}
\bullet \quad e_1 \quad \bullet \\
\bullet \quad e_2 \quad \bullet
\end{array}
$$

and let $T = e_1 \cdot L_{e_1} + e_2 \cdot L_{e_2}$. Then this graph operator $T$ is not hyponormal, since

$$
R_{T \cdot T} = \{ r(e_1^{-1}e_1), r(e_1^{-1}e_2), r(e_2^{-1}e_1), r(e_2^{-1}e_2) \} = \{ v_2 \},
$$

and

$$
R_{TT^*} = \{ r(e_1e_1^{-1}), r(e_1e_2^{-1}), r(e_2e_1^{-1}), r(e_2e_2^{-1}) \} = \{ v_1 \},
$$
in $V(G)$. So,

$$
R_{T \cdot T} \cap R_{TT^*} = \emptyset,
$$
and hence $T$ does not satisfy the condition (4.1.7), stating

$$
R_{T \cdot T} \supseteq R_{TT^*}.
$$

Therefore, this operator $T$ is not hyponormal.

Example 4.2. Suppose a graph $G$ contains its subgraph,

$$
\begin{array}{c}
\bullet \quad e_1 \quad \bullet \\
\bullet \quad e_2 \quad \bullet
\end{array}
$$

and let $T_1 = e_1 \cdot L_{e_1} + e_1^{-1} \cdot L_{e_1^{-1}} + e_2 \cdot L_{e_2}$. Then we can have that

$$
R_{T_1 \cdot T_1} = \left\{ \begin{array}{l} r(e_1^{-1}e_1), r(e_1^{-1}e_2), r(e_1^{-1}e_2), \\ r(e_1e_1), r(e_1e_2), r(e_2e_2) \end{array} \right\} = \{ v_1, v_2 \},
$$
and

$$
R_{T_1T_1^*} = \left\{ \begin{array}{l} r(e_1e_1^{-1}), r(e_1e_1), r(e_1e_2^{-1}), \\ r(e_1e_1^{-1}), r(e_1e_2), r(e_2e_2) \end{array} \right\} = \{ v_1, v_2 \},
$$
in $V(G)$. Thus, $T_1$ satisfies the condition (4.1.7); $R_{T_1 \cdot T_1^*} = V_{T_1} = R_{T_1T_1^*}$, and hence $R_{T_1 \cdot T_1^*} \supseteq R_{T_1T_1^*}$. 

\[ \square \]
So, $T_1$ is hyponormal, if and only if (4.1.8) holds. So, $T_1$ is hyponormal, if and only if, for $v_1 \in V_T$,
\begin{equation}
|t_{e_1}|^2 - \left(|t_{e_1}|^2 + \overline{t_{e_1}}e_2 + t_2\overline{e_1} + |t_{e_2}|^2\right) \geq 0
\end{equation}
\begin{equation}
\iff |t_{e_2}|^2 \leq - t_{e_1}\overline{e_2} - \overline{t_{e_2}}e_1,
\end{equation}
and, for $v_2 \in V_T$,
\begin{equation}
\left(|t_{e_1}|^2 + \overline{t_{e_1}}e_2 + \overline{t_{e_2}}e_1 + |t_{e_2}|^2\right) - |t_{e_1}|^2 \geq 0
\end{equation}
\begin{equation}
\iff |t_{e_2}|^2 \geq -\overline{t_{e_1}} e_2 - \overline{e_2} t_{e_1}.
\end{equation}

If we combine (4.1.14) and (4.1.15), we can obtain that the given operator $T_1$ is hyponormal, if and only if
\begin{equation}
|t_{e_2}|^2 = -\overline{t_{e_1}} e_2 - \overline{e_2} t_{e_1}.
\end{equation}

In fact, the readers can easily check that the hyponormality condition (4.1.16) guarantees the "normality" of $T_1$, i.e., $T_1$ is normal, if and only if (4.1.16) holds (See Section 4.2 below).

Now, let $T_2 = t_{e_1}L_{e_1} + t_{e_1}^{-1}L_{e_1}^{-1}$. Then we have
\[ \mathcal{R}_{T_1 T_2} = \{ r(e_1^{-1}e_1), r(e_1 e_1^{-1}), r(e_1 e_1^{-1}) \} = \{ v_1, v_2 \}, \]
and
\[ \mathcal{R}_{T_2 T_1} = \{ r(e_1 e_1^{-1}), r(e_1 e_1), r(e_1 e_1) \} = \{ v_1, v_2 \}, \]
and hence the operator $T_2$ satisfies (4.1.7). So, $T_2$ is hyponormal, if and only if
\begin{equation}
|t_{e_1}|^2 \geq \left| t_{e_1}^{-1} \right|^2 \quad (\text{for } v_1),
\end{equation}
and
\begin{equation}
\left| t_{e_1}^{-1} \right|^2 \geq |t_{e_1}|^2 \quad (\text{for } v_2).
\end{equation}

Therefore, by (4.1.17), we can conclude that $T_2$ is hyponormal, if and only if
\[ |t_{e_1}|^2 = \left| t_{e_1}^{-1} \right|^2 \in \mathbb{C}. \]

This example also shows that the hyponormality of $T_2$ is equivalent to the normality of $T_2$ (See Section 4.2 below).

**Example 4.3.** Let a graph $G$ contains the following subgraph,
\[ v_1 \rightarrow e_1 \rightarrow v_2 \rightarrow e_2 \rightarrow v_3, \]
and let $T = t_{1}L_{e_1} + t_{2}L_{e_2} + t_{3}L_{v_3}$. Then we can have that
\[ \mathcal{R}_{T T} = \{ r(e_1^{-1}e_1), r(e_1^{-1}e_2), r(e_1^{-1}v_3), \}
\[ \{ r(e_2^{-1}e_1), r(e_2^{-1}e_2), r(e_2^{-1}v_3), \}
\[ \{ r(v_3 e_1), r(v_3 e_2), r(v_3 v_3) \} = \{ v_1, v_3 \}, \]
and
So, the operator $T$ does not satisfy the condition (4.1.7) for the hyponormality of $T$, and hence $T$ is not hyponormal.

**Example 4.4.** Assume that a graph $G$ contains a subgraph,

$$v_1 \bullet e_1 \rightarrow e_2 \bullet v_2,$$

and let $T = t_{v_1} L_{e_1} + t_{e_1} L_{e_2} + t_{e_2}^{-1} L_{e_2}^{-1}$. Then we can have that

$$\mathcal{R}_{TT} = \left\{ r(e_1 v_1), r(v_1 e_1), r(v_1 e_2^{-1}), r(e_1 v_1), r(e_1^{-1} e_1), r(e_1^{-1} e_2), r(e_2 v_1), r(e_2 e_1), r(e_2 e_2^{-1}) \right\} = \{ v_1, v_2 \},$$

and

$$\mathcal{R}_{TT^*} = \left\{ r(v_1 v_1), r(v_1 e_1^{-1}), r(v_2 e_2), r(v_1 v_1), r(e_1 e_1^{-1}), r(e_1 e_2), r(e_2 v_1), r(e_2 e_1^{-1}), r(e_2^{-1} e_2) \right\} = \{ v_1, v_2 \}. $$

So, $\mathcal{R}_{TT} = \mathcal{R}_{TT^*}$, and hence $T$ satisfies the condition (4.1.7). So, to make $T$ be hyponormal, the coefficients of $T$ must satisfy the condition (4.1.8). Thus we can conclude that $T$ is hyponormal, if and only if

$$\frac{1}{(4.1.18)} \left( |t_{v_1}|^2 + |t_{e_1}|^2 + |t_{e_2}^{-1} t_{e_2}| \right) \quad \text{(for } v_1 \text{)}$$

and

$$\left( \frac{1}{t_{v_1} t_{e_1} + |t_{e_1}|^2 + |t_{e_2}^{-1}|^2} \right) \geq \left( \frac{1}{t_{e_1} t_{e_2}^{-1} + |t_{e_2}^{-1}|^2} \right) \quad \text{(for } v_2 \text{)}.$$  

The above condition (4.1.18) can be rewritten by

$$\frac{1}{(4.1.19)} \left( |t_{e_1}|^2 t_{v_1} - t_{e_2}^{-1} t_{e_1} \geq |t_{e_1}|^2 \right),$$

and

$$|t_{e_1}|^2 \geq t_{e_1} t_{e_2}^{-1} - t_{v_1} t_{e_1},$$

respectively. Therefore, the given graph operator $T$ is hyponormal, if and only if

$$\frac{1}{(4.1.20)} \left( |t_{e_1}|^2 t_{v_1} - t_{e_2}^{-1} t_{e_1} \geq |t_{e_1}|^2 \right) \geq t_{e_1} t_{e_2}^{-1} - t_{v_1} t_{e_1},$$

if and only if

$$|t_{e_1}|^2 \leq \left| t_{e_1} t_{v_1} - t_{e_2} t_{e_1} \right|.$$  

**Example 4.5.** Suppose a graph $G$ contains its subgraph,

$$v_1 \bullet e_2 \rightarrow e_3 \bullet v_2,$$

and let $T = t_{e_1} L_{e_1} + t_{e_2} L_{e_2} + t_{e_3} L_{e_3}$. Then we can have that
Proof. By definition, a graph operator 

\[ \mathcal{R}_{TT} = \left\{ r(e_1^{-1}e_1), r(e_1^{-1}e_2), r(e_1^{-1}e_3), r(e_2^{-1}e_1), r(e_2^{-1}e_2), r(e_2^{-1}e_3), r(e_3^{-1}e_1), r(e_3^{-1}e_2), r(e_3^{-1}e_3) \right\} = \{ v_1, v_2 \}, \]

and

\[ \mathcal{R}_{TT^*} = \left\{ r(e_1e_1^{-1}), r(e_1e_2^{-1}), r(e_1e_3^{-1}), r(e_2e_1^{-1}), r(e_2e_2^{-1}), r(e_2e_3^{-1}), r(e_3e_1^{-1}), r(e_3e_2^{-1}), r(e_3e_3^{-1}) \right\} = \{ v_1 \}. \]

So, the operator \( T \) satisfies the condition (4.1.7) i.e.,

\[ \mathcal{R}_{TT^*} \supset \mathcal{R}_{TT^*}. \]

Thus, we can obtain that \( T \) is hyponormal, if and only if

(4.1.21)

\[ \left( |t_{e_1}|^2 + \overline{t_{e_2}} t_{e_1} + t_{e_3} \overline{t_{e_1}} \right) \geq \left( |t_{e_1}|^2 + |t_{e_2}|^2 + t_{e_2} \overline{t_{e_3}} + t_{e_3} \overline{t_{e_2}} + |t_{e_3}|^2 \right) \]

(for \( v_1 \)), and

\[ \left( \overline{t_{e_1} t_{e_2}} + \overline{t_{e_1} t_{e_3}} + |t_{e_2}|^2 + t_{e_2} \overline{t_{e_3}} + t_{e_3} t_{e_2} + |t_{e_3}|^2 \right) \geq 0 \]

(for \( v_2 \)).

4.2. Normality. In this section, we will consider the normality of graph operators. In Section 4.1, we studied the hyponormality of graph operators in terms of combinatorial information of given graphs, and certain analytic data of coefficients of operators. Throughout this section, we will use the same notations used in Section 4.1.

Thanks to the hyponormality characterization ((4.1.7) and (4.1.8)) of graph operators, we can obtain the following normality characterization of graph operators.

**Theorem 4.5.** Let \( T = \sum_{w \in \text{Supp}(T)} t_w L_w \) be a graph operator in the graph von Neumann algebra \( M_G \) of a connected graph \( G \). Then \( T \) is normal, if and only if

(4.2.1)

\[ \mathcal{R}_{TT^*} = \mathcal{R}_{TT^*}, \]

and

(4.2.2)

\[ \left( \sum_{(w_1, w_2) \in \text{Supp}(T)^2, w_1^{-1} w_2 \neq \emptyset, r(w_1^{-1} w_2) = v} t_{w_1} t_{w_2} \right) \]

\[ \geq \left( \sum_{(y_1, y_2) \in \text{Supp}(T)^2, y_1 y_2^{-1} \neq \emptyset, r(y_1 y_2^{-1}) = v} t_{y_1} t_{y_2} \right). \]

**Proof.** By definition, a graph operator \( T \) is normal on the graph Hilbert space \( H_G \), if and only if \( T^* T = T T^* \) on \( H_G \). In other words, \( T \) is normal, if and only if both \( T \) and \( T^* \) are hyponormal. Thus, \( T \) is normal, if and only if the self-commutator \( S(T) \) is identical to the zero element \( 0_{M_G} \) (which is identified with the zero operator \( 0_{H_G} \) on \( H_G \)), if and only if

\[ \langle S(T) \xi, \xi \rangle = 0, \text{ for all } \xi \in H_G. \]

Therefore, by the little modification of the proof of Theorem 4.5, we can conclude that \( T \) is normal, if and only if the combinatorial condition (4.2.1) and the analytic condition (4.2.2) hold. \( \blacksquare \)
5. Operators in Free Group Factors

In this section, we consider applications of operator-theoretic properties of graph operators. We will characterize the self-adjointness, the hyponormality, the normality, and the unitary property of finitely supported operators in the free group factor $L(F_N)$, generated by the free group $F_N$ with $N$-generators, for $N \in \mathbb{N}$.

In operator algebra, the study of free group factors $L(F_N)$ is very important (e.g., See [11]). Also, the study of elements of $L(F_N)$ is interesting, since they are (possibly, the infinite or the limit of) linear combinations of unitary operators (e.g., See [3], [4], [6], and [7]). The following theorem provides the key motivation of our applications.

**Theorem 5.1.** (Also, see [4]) The free group factor $L(F_N)$ is *-isomorphic to the graph von Neumann algebra $M_{O_N}$ of the one-vertex-$N$-loop-edge graph $O_N$, for all $N \in \mathbb{N}$.

**Proof.** Let $O_N$ be the one-vertex-$N$-loop-edge graph and let $\mathcal{O}_N$ be the graph groupoid of $O_N$. Since $O_N$ has only one vertex, say $v_O$, the graph groupoid $\mathcal{O}_N$ is in fact a group (See Section 2.2). Indeed, the graph groupoid $\mathcal{O}_N$ is a (categorial) groupoid (in the sense of Section 2.2) with its base, consisting of only one element $v_O$. Thus $\mathcal{O}_N$ is a group. Moreover, this group $\mathcal{O}_N$ has $N$-generators contained in the edge set

$$E(O_N) = \{e_1, \ldots, e_N\}$$

of $O_N$. So, we can define a morphism $g : \mathcal{O}_N \to F_N$ by a map satisfying

$$g : e_j \in E(O_N) \mapsto u_j \in X_{F_N},$$

for all $j = 1, \ldots, N$ (by the possible rearrangement), where

$$X_{F_N} = \{u_1, \ldots, u_N\}$$

is the generator set of the free group $F_N = \langle X_{F_N} \rangle$.

Then this morphism satisfies that

$$g(x_{i_1} \ldots x_{i_n}) = q_{i_1} \ldots q_{i_n} \in F_N,$$

for all $x_{i_1}, \ldots, x_{i_n} \in E(O_N)$, for $n \in \mathbb{N}$, such that

$$x_{i_j} = \begin{cases} e_{i_j} & \text{if } x_{i_j} \in E(O_N) \\ e_{i_j}^1 & \text{if } x_{i_j} \in E(O_N)^{1} \end{cases},$$

where $\hat{O}_N$ is the shadowed graph of $O_N$, and where

$$q_{i_j} = \begin{cases} u_{i_j} & \text{if } q_{i_j} \in X_{F_N} \\ u_{i_j}^1 & \text{if } q_{i_j} \in X_{F_N}^{-1} \end{cases},$$

for all $j = 1, \ldots, n$. (Remark that the graph groupoid $\mathcal{O}_N$ is generated by $E(\hat{O}_N)$, as a groupoid, and hence the group $\mathcal{O}_N$ is generated by $E(\hat{O}_N)$.)

Therefore, the morphism $g$ is a group-homomorphism. Since $g$ is preserving generators, it is bijective. So, the morphism $g$ is a group-isomorphism, and hence $\mathcal{O}_N$ and $F_N$ are group-isomorphic.

Let $(H_{O_N}, L)$ be the canonical representation of $\mathcal{O}_N$, and let $(H_{F_N}, \lambda)$ be the left regular unitary representation of $F_N$, where $H_{F_N} = l^2(F_N)$ is the group Hilbert space of $F_N$. By the existence of the group-isomorphism $g$ of $\mathcal{O}_N$ and $F_N$, the
Hilbert spaces $H_{\mathbb{O}_N}$ and $H_{\mathbb{F}_N}$ are Hilbert-space isomorphic. Indeed, there exists a linear map

$$\Phi : H_{\mathbb{O}_N} \rightarrow H_{\mathbb{F}_N}$$

satisfying that

$$\Phi \left( \sum_{w \in \mathbb{O}_N} t_w \xi_w \right) \overset{\text{def}}{=} \sum_{g(w) \in g(\mathbb{O}_N) = \mathbb{F}_N} t_w \xi_{g(w)};$$

in $H_{\mathbb{F}_N}$, for all $\sum_{w \in \mathbb{O}_N} t_w \xi_w$. It is easy to check that this linear map $\Phi$ is bounded and bijective, i.e., $\Phi$ is a Hilbert-space isomorphism, and hence $H_{\mathbb{O}_N}$ and $H_{\mathbb{F}_N}$ are Hilbert-space isomorphic.

By the existence of $\Phi$ and $g$, we can obtain the commuting diagram,

$$
\begin{array}{ccc}
H_{\mathbb{O}_N} & \xrightarrow{\Phi} & H_{\mathbb{F}_N} \\
\downarrow L & & \downarrow \lambda \\
H_{\mathbb{O}_N} & \xrightarrow{\Phi} & H_{\mathbb{F}_N}.
\end{array}
$$

This shows that the group actions $L$ of $\mathbb{O}_N$ and $\lambda$ of $\mathbb{F}_N$ are equivalent. i.e., the representations $(H_{\mathbb{O}_N}, L)$ of $\mathbb{O}_N$ and $(H_{\mathbb{F}_N}, \lambda)$ of $\mathbb{F}_N$ are equivalent.

Therefore, the group von Neumann algebras $vN(L(\mathbb{O}_N))$ and $vN(\lambda(\mathbb{F}_N))$ are $\ast$-isomorphic from each other in $B(\mathcal{H})$, where

$$H_{\mathbb{O}_N} \overset{\text{Hilbert}}{=} H_{\mathbb{F}_N},$$

where $\overset{\text{Hilbert}}{=} \overset{\text{Hilbert}}{=} \text{Hilbert}$ means “being Hilbert-space isomorphic.” i.e., the graph von Neumann algebra $M_{\mathbb{O}_N}$ and the group von Neumann algebra $L(\mathbb{F}_N)$ are $\ast$-isomorphic.

The above theorem shows that the study of $L(\mathbb{F}_N)$ is to study $M_{\mathbb{O}_N}$. So, to study finitely supported operators of $L(\mathbb{F}_N)$, we will study the graph operators in $M_{\mathbb{O}_N}$.

By Section 3.2, we can obtain the following self-adjointness characterization on $M_{\mathbb{O}_N}$.

**Proposition 5.2.** Let $T = \sum_{w \in \text{Supp}(T)} t_w L_w$ be a graph operator in $M_{\mathbb{O}_N}$. Then $T$ is self-adjoint, if and only if there exists a subset $Y$ of

$$\text{Supp}(T) \cap FP_\tau(\mathbb{O}_N),$$

such that

(5.1)

$$\text{Supp}(T) = \begin{cases}
\{v_O\} \sqcup Y \sqcup Y^{-1} & \text{if } v_O \in \text{Supp}(T) \\
Y \sqcup Y^{-1} & \text{otherwise},
\end{cases}$$

and

$t_{v_O} \in \mathbb{R}$, and $t_y = \overline{t_{\overline{y}}}^{-1}$ in $\mathbb{C}$, for all $y \in Y$.

□

The proof is done by Section 3.2. By the above proposition, we obtain the self-adjointness characterization of finitely supported elements in the free group factor $L(\mathbb{F}_N)$. 
Corollary 5.3. Let \( T = t_{j_k} u_{g_k^{-1}} + \ldots + t_{j_1} u_{g_1^{-1}} + t_0 u_e + t_1 u_{g_1} + \ldots + t_n u_{g_n} \) be an element of \( L(F_N) \), where \( u_g \overset{\text{def}}{=} \lambda(g) \), for all \( g \in F_N \), and \( e \) is the group-identity of \( F_N \). Then \( T \) is self-adjoint, if and only if
\[
(5.2) \quad k = n \text{ in } \mathbb{N},
\]
and
\[
(5.3) \quad t_0 \in \mathbb{R}, \text{ and } t_p = \overline{t_{j_p}}, \text{ for all } p = 1, \ldots, n = k.
\]
□

Now, let’s consider the hyponormality.

Proposition 5.4. Let \( T = \sum_{w \in \text{Supp}(T)} t_w L_w \) be a graph operator in \( M_{O_N} \). Then \( T \) is hyponormal, if and only if
\[
(5.4) \quad \mathcal{R}_{T^*T} = r(\Pi_{T^*T}) \supseteq \mathcal{R}_{TT^*} = r(\Pi_{TT^*}),
\]
and
\[
(5.5) \quad \left( \sum_{(w_1, w_2) \in \text{Supp}(T)^2} t_{w_1} t_{w_2} - t_{w_1} \overline{t_{w_2}} \right) \geq 0,
\]
for all \( v \in V(\hat{O}_N) \), by (4.1.7), and (4.1.8). However, the fixed graph \( O_N \) has only one vertex \( v_O \), and all elements of \( O_N \) are admissible from each other via \( v_O \) (equivalently, \( O_N \) is a group). Therefore, the condition (5.4) automatically hold true, and the inequality (5.5) can be simply re-written by
\[
(5.6) \quad \sum_{(w_1, w_2) \in \text{Supp}(T)^2} t_{w_1} t_{w_2} \geq \sum_{(y_1, y_2) \in \text{Supp}(T)^2} t_{y_1} \overline{t_{y_2}}.
\]
Therefore, the operator \( T \) is hyponormal, if and only if (5.3) holds true. □

By the above proposition, we can obtain that:

Corollary 5.5. Let \( T = \sum_{g \in \text{Supp}(T)} t_g u_g \) be a finitely supported element of \( L(F_N) \).
Then \( T \) is hyponormal, if and only if
\[
(5.7) \quad \sum_{(g_1, g_2) \in \text{Supp}(T)^2} (\overline{t_{g_1}} t_{g_2} - t_{g_1} \overline{t_{g_2}}) \geq 0.
\]
□
By the hyponormality characterization (5.7) and by Section 4.2, we can obtain the following corollary, too.

Corollary 5.6. Let $T = \sum_{g \in \text{Supp}(T)} t_g u_g$ be a finitely supported element of $L(F_N)$. Then $T$ is normal, if and only if

$$\sum_{(g_1, g_2) \in \text{Supp}(T)^2} (\overline{t_{g_1}} t_{g_2} - t_{g_1} \overline{t_{g_2}}) = 0.$$ 

□

Finally, let’s consider the unitary property of finitely supported elements of $L(F_N)$. In Section 3.3, we obtain that: a graph operator $T$ of the graph von Neumann algebra $M_{O_N}$ of the one-vertex-$N$-loop-edge graph $O_N$ is unitary, if and only if

$$(\text{Supp}(T))^{-1} (\text{Supp}(T)) = \{ v_{O} \},$$

and

$$\sum_{(w_1, w_2) \in \text{Supp}(T)^2} \overline{t_{w_1}} t_{w_2} = 1, \text{ in } \mathbb{C},$$

where $v_O$ is the unique vertex of $O_N$, for $N \in \mathbb{N}$. Therefore, we can obtain that:

Proposition 5.7. Let $T = \sum_{g \in \text{Supp}(T)} t_g u_g$ be a finitely supported element of $L(F_N)$. Then $T$ is unitary, if and only if

$$\text{(Supp}(T))^{-1} \text{(Supp}(T)) = \{ e_{F_N} \},$$

and

$$\sum_{(g_1, g_2) \in \text{Supp}(T)^2} \overline{t_{g_1}} t_{g_2} = 1, \text{ in } \mathbb{C}.$$ 

□

Appendix A. Categorial Groupoids and Groupoid Actions

We say an algebraic structure $(\mathcal{X}, \mathcal{Y}, s, r)$ is a (categorial) groupoid, if it satisfies that: (i) $\mathcal{Y} \subset \mathcal{X}$, (ii) for all $x_1, x_2 \in \mathcal{X}$, there exists a partially-defined binary operation $(x_1, x_2) \mapsto x_1 x_2$, for all $x_1, x_2 \in \mathcal{X}$, depending on the source map $s$ and the range map $r$ satisfying the followings:

(ii-1) $x_1 x_2$ is well-determined, whenever $r(x_1) = s(x_2)$ and in this case, $s(x_1 x_2) = s(x_1)$ and $r(x_1 x_2) = r(x_2)$, for $x_1, x_2 \in \mathcal{X}$,

(ii-2) $(x_1 x_2) x_3 = x_1 (x_2 x_3)$, if they are well-determined in the sense of (ii-1), for $x_1, x_2, x_3 \in \mathcal{X}$,

(ii-3) if $x \in \mathcal{X}$, then there exist $y, y' \in \mathcal{Y}$ such that $s(x) = y$ and $r(x) = y'$, satisfying $x = y x y'$ (Here, the elements $y$ and $y'$ are not necessarily distinct),

(ii-4) if $x \in \mathcal{X}$, then there exists a unique element $x^{-1}$ for $x$ satisfying
well-defined function, for all 

$X$, set 

$\hat{G}$ isomorphic shadowed graphs $\hat{G}$ (See [10] and [11]).

Thus, every group $\Gamma$ is a groupoid $\Gamma = (\Gamma, \{e\Gamma\} s, r)$ (and hence $s = r$ on $\Gamma$), where $e\Gamma$ is the group-identity of $\Gamma$. Conversely, every groupoid with its base having the cardinality 1 is a group.

Remark that we can naturally assume that there exists the empty element $\emptyset$ in a groupoid $\mathcal{X}$. The empty element $\emptyset$ means the products $x_1 x_2$ are not well-defined, for some $x_1, x_2 \in \mathcal{X}$. Notice that if $|\mathcal{Y}| = 1$ (equivalently, if $\mathcal{X}$ is a group), then the empty word $\emptyset$ is not contained in the groupoid $\mathcal{X}$. However, in general, whenever $|\mathcal{Y}| \geq 2$, a groupoid $\mathcal{X}$ always contain the empty word. So, if there is no confusion, the existence of the empty element $\emptyset$ is automatically assumed, whenever the base $\mathcal{Y}$ of $\mathcal{X}$ contains more than one element. Under this setting, the partially-defined binary operation on $\mathcal{X}$ is well-defined on $\mathcal{X}$ (more precisely, on $\mathcal{X} \cup \{\emptyset\}$, which is identified with $\mathcal{X}$, whenever $|\mathcal{Y}| \geq 2$).

It is easy to check that our graph groupoid $\mathbb{G}$ of a countable directed graph $G$ is indeed a groupoid with its base $V(\hat{G})$, i.e., the graph groupoid $\mathbb{G}$ of a graph $G$ is a groupoid

$$\mathbb{G} = (\mathbb{G}, V(\hat{G}), s, r),$$

satisfying

$$s(w) = s(v w) = v \text{ and } r(w) = r(w v') = v',$$

for all $w = v w v' \in \mathbb{G}$ with $v, v' \in V(\hat{G})$, i.e., the vertex set $V(\hat{G}) = V(G)$ is the base of $\mathbb{G}$.

Let $\mathcal{X}_k = (\mathcal{X}_k, \mathcal{Y}_k, s_k, r_k)$ be groupoids, for $k = 1, 2$. We say that a map $f : \mathcal{X}_1 \rightarrow \mathcal{X}_2$ is a groupoid-morphism, if

(i) $f$ is a function,

(ii) $f(\mathcal{Y}_1) \subseteq \mathcal{Y}_2$,

(iii) $s_2(f(x)) = f(s_1(x))$ in $\mathcal{X}_2$, for all $x \in \mathcal{X}_1$, and

(iv) $r_2(f(x)) = f(r_1(x))$ in $\mathcal{X}_2$, for all $x \in \mathcal{X}_1$.

Equivalently, $f$ is a groupoid-morphism, if and only if (i)’ $f$ is a function, (ii)’ $f$ satisfies

$$f(x_1 x_2) = f(x_1) f(x_2) \text{ in } \mathcal{X}_2,$$

for all $x_1, x_2 \in \mathcal{X}_1$.

If a groupoid-morphism $f$ is bijective, then we say that $f$ is a groupoid-isomorphism, and the groupoids $\mathcal{X}_1$ and $\mathcal{X}_2$ are said to be groupoid-isomorphic.

Notice that, if two countable directed graphs $G_1$ and $G_2$ are graph-isomorphic, via a graph-isomorphism $g : G_1 \rightarrow G_2$, in the sense that:

(i) $g$ is bijective from $V(G_1)$ onto $V(G_2)$,

(ii) $g$ is bijective from $E(G_1)$ onto $E(G_2)$,

(iii) $g(e) = g(v_1 e v_2) = g(v_1) g(e) g(v_2)$ in $E(G_2)$,

for all $e = v_1 e v_2 \in E(G_1)$, with $v_1, v_2 \in V(G_1)$, then the graph groupoids $\mathbb{G}_1$ and $\mathbb{G}_2$ are groupoid-isomorphic. More generally, if two graphs $G_1$ and $G_2$ have graph-isomorphic shadowed graphs $\hat{G}_1$ and $\hat{G}_2$, then $\mathbb{G}_1$ and $\mathbb{G}_2$ are groupoid-isomorphic (See [10] and [11]).

Let $\mathcal{X} = (\mathcal{X}, \mathcal{Y}, s, r)$ be a groupoid. We say that this groupoid $\mathcal{X}$ acts on a set $Y$, if there exists a groupoid action $\pi$ of $\mathcal{X}$ such that: (i) $\pi(x) : Y \rightarrow Y$ is a well-defined function, for all $x \in \mathcal{X}$, and (ii) $\pi$ satisfies

$$\pi(x_1 x_2) = \pi(x_1) \circ \pi(x_2) \text{ on } Y,$$
for all \( x_1, x_2 \in X \), where \((\circ)\) means the usual composition of maps. We call the
set \( Y \), a \( X \)-set.

Let \( X_1 \subset X_2 \) be a subset, where \( X_2 = (X_2, Y_2, s, r) \) is a groupoid. Assume that
\( X_1 = (X_1, Y_1, s, r) \), itself, is a groupoid, where \( Y_1 = X_1 \cap Y_2 \). Then we say that
the groupoid \( X_1 \) is a subgraph of \( X_2 \).

Recall that we say a graph \( G_1 \) is a full-subgraph of a countable directed graph
\( G_2 \), if

\[
E(G_1) \subseteq E(G_2)
\]

and

\[
V(G_1) = \{ v \in V(G_1) : e = v e \text{ or } e = e v, \forall e \in E(G_1) \}.
\]

Remark the difference between full-subgraphs and subgraphs: We say that \( G'_1 \)
is a subgraph of \( G_2 \), if

\[
V(G'_1) \subseteq V(G_2)
\]

and

\[
E(G'_1) = \{ e \in E(G_2) : e = v_1 e v_2, \text{ for } v_1, v_2 \in V(G'_1) \}.
\]

Also, if a graph \( V \) is a graph with \( V(V) \subseteq V(G_2) \), and \( E(V) = \emptyset \), then we call
\( V \), a vertex subgraph of \( G_2 \).

We will say that \( G_1 \) is a part of \( G_2 \), if \( G_1 \) is either a full-subgraph of \( G_2 \), or
a subgraph of \( G_2 \), or a vertex subgraph of \( G_2 \). It is easy to show that the graph
groupoid \( G_1 \) of \( G_1 \) is a subgroupoid of the graph groupoid \( G_2 \) of \( G_2 \), whenever \( G_1 \)
is a part of \( G_2 \).

**Appendix B. Operator-Theoretic Properties**

Let \( H \) be an arbitrary separable Hilbert space equipped with its inner product
\(<,>\). i.e., the inner product \(<,>\) on \( H \) is the sesquilinear form,

\[
<,> : H \times H \to \mathbb{C},
\]

satisfying that:

(i) \( < t_1 \xi_1 + t_2 \xi_2, \eta > = t_1 < \xi_1, \eta > + t_2 < \xi_2, \eta > \),

(ii) \( < \xi, \eta > = \overline{< \eta, \xi >} \),

(iii) \( < \xi, \xi > \geq 0 \), and equality holds, if and only if \( \xi = 0_H \),

for all \( \xi, \xi_k, \eta, \eta_k \in H \), and \( t_k \in \mathbb{C} \), where \( \overline{t} \) mean the conjugates of \( t \), and \( 0_H \)
means the zero vector in \( H \). As usual, let \( B(H) \) be the operator algebra consisting of
all (bounded linear) operators on \( H \).

For any operator \( T \in B(H) \), there exists a unique operator \( T^* \) satisfying

\[
T \xi, \eta > = < \xi, T^* \eta >,
\]

for all \( \xi, \eta \in H \). This operator \( T^* \) is called the adjoint of \( T \).

**Definition 5.1.** Let \( T \in B(H) \) be an operator.

1. We say that an operator \( T \) is self-adjoint, if the adjoint \( T^* \) of \( T \) is identical
to \( T \), i.e.,

\[
T \text{ is self-adjoint } \iff T^* = T \text{ in } B(H).
\]

2. An operator \( T \) is said to be normal, if the product \( T^* T \) of \( T^* \) and \( T \) is
identical to the product \( T T^* \), on \( H \), i.e.,

\[
T \text{ is normal } \iff T^* T = T T^* \text{ in } B(H).
\]
(3) We call \( T \) a unitary, if it is normal, and \( T^*T = TT^* \) are identical to the identity operator \( 1_H \) on \( H \), i.e.,

\[
T \text{ is unitary} \iff T^*T = 1_H = TT^* \quad \text{in } B(H)
\]

(4) An operator \( T \in B(H) \) is called a projection, if it is self-adjoint and idempotent, in the sense that \( T^2 = T \) on \( H \), i.e.,

\[
T \text{ is a projection} \iff T^* = T = T^2 \quad \text{in } B(H).
\]

(5) We say an operator \( T \) is positive on \( H \), if

\[
<T\xi, \xi> \geq 0 \quad \text{for all } \xi \in H \quad \text{with } ||\xi|| = 1,
\]

where \( ||\xi|| \defeq \sqrt{<\xi, \xi>} \) is the Hilbert norm of \( \xi \), for all \( \xi \in H \).

(6) An operator \( T \) is said to be hyponormal, if the operator \( T^*T - TT^* \) is positive on \( H \).

Such properties of operators are well-known in operator theory. Also, if an operator \( T \) has one of the above properties, then it is a “good” operator in the theory. For instance, if \( T \) is normal, then it satisfies the spectral mapping theorem, and hence \( f(T) \) is again normal, for all continuous functions on \( \mathbb{C} \), etc. By definition, we can check that:

(2.3.1) If \( T \) is self-adjoint, then \( T \) is normal.

(2.3.2) If \( T \) is unitary, then \( T \) is normal.

(2.3.3) Every projection is self-adjoint.

(2.3.4) Every normal operator is hyponormal.

(2.3.5) If \( T \) is unitary, then \( T \) is invertible, moreover, \( T^* = T^{-1} \).

Clearly, the converses of the above facts does not hold true, in general. We say that an operator \( T \) is a partial isometry, if the product \( T^*T \) of the adjoint \( T^* \) and \( T \) is a projection. The following characterization is also known: \( T \) is a partial isometry, if and only if \( TT^*T = T \) if and only if \( T^*TT^* = T^* \). In particular, the projections \( T^*T \) and \( TT^* \) are called the initial projection, and the final projection of \( T \), respectively. i.e., the projection \( T^*T \) (resp., the projection \( TT^* \)) send the elements of \( H \) into the elements of the subspace \( H_{\text{init}}^T \) (resp., \( H_{\text{fin}}^T \)) of \( H \). We call the subspaces \( H_{\text{init}}^T \) and \( H_{\text{fin}}^T \) of \( H \), induced by a partial isometry \( T \), the initial subspace and the final subspace of \( T \) in \( H \), respectively.

A partial isometry \( T \) satisfying that \( T^*T = 1_H \) is called an isometry. Keep in mind that, even though \( T^*T = 1_H \), it is possible that \( TT^* \neq 1_H \). Clearly, if \( TT^* = 1_H \), for an isometry \( T \), then this isometry \( T \) becomes a unitary. A partial isometry \( T \), satisfying \( TT^* = 1_H \), is called a co-isometry.

In many cases, the projections \( T^*T \) and \( TT^* \) of a partial isometry \( T \) are distinct from each other, whenever \( H^T_{\text{init}} \) and \( H^T_{\text{fin}} \) are different in \( H \). This shows that partial isometries are not normal, in general. For instance, let

\[
T = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \text{with its adjoint } T^* = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},
\]

on \( H = \mathbb{C}^{2 \times 2} \). Then it is a partial isometry, since \( TT^*T = T \). And, it has its initial projection and final projection as follows:

\[
T^*T = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \text{and } TT^* = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},
\]

on \( H \). We can easily see that \( T^*T \neq TT^* \), and hence \( T \) is not normal.
Let $T \in B(H)$ be an operator. Then $T$ has its spectrum $\text{spec}(T)$, defined by a subset
\[
\text{spec}(T) \overset{\text{def}}{=} \{ t \in \mathbb{C} : T - t 1_H \text{ is not invertible on } H \},
\]
in $\mathbb{C}$. It is well-known that every spectrum $\text{spec}(T)$ is nonempty and compact in $\mathbb{C}$, whenever $T$ is a (bounded linear) operator on a (complex) Hilbert space $H$.

This numerical data for $T$ is very valuable to analyze the operators. In particular, every normal operator $T$ can be understood (or regarded) as a complex-valued function, satisfying
\[
\int_{\text{spec}(T)} t \, dE,
\]
where $E$ is the suitable (operator-valued) measure on $\text{spec}(T)$, called the spectral measure. Thus, if $f$ is a $\mathbb{C}$-valued continuous map, then
\[
f(T) = \int_{\text{spec}(T)} f(t) \, dE(t).
\]
i.e., the spectral mapping theorem holds for normal operators. Thus all operators $f(T)$ are normal, too, whenever $T$ is normal.

Also, by Gelfand, the $C^*$-algebra $C^*(T)$, generated by $T$, is $*$-isomorphic to the $C^*$-algebra $C(\text{spec}(T))$, consisting of all continuous $\mathbb{C}$-valued functions on $\text{spec}(T)$.
(However, finding spectra of operators is not easy at all.)

References

[1] A. Gibbons and L. Novak, Hybrid Graph Theory and Network Analysis, ISBN: 0-521-46117-0, (1999) Cambridge Univ. Press.
[2] B. Solel, You can see the arrows in a Quiver Operator Algebras, (2000), preprint.
[3] I. Cho, Hyponormality of Toeplitz Operators with Trigonometric Polynomial Symbols, Master Degree Thesis, (1999) Sungkyunkwan Univ.
[4] I. Cho, Graph Groupoids and Partial Isometries, ISBN: 978-3-8383-1397-9, (2009) LAP Publisher.
[5] I. Cho, and P. E. T. Jorgensen, $C^*$-Subalgebras Generated by Partial Isometries, JMP, DOI: 10.1063/1.3056588, (2009).
[6] I. Cho, and P. E. T. Jorgensen, $C^*$-Subalgebras Generated by a Single Operator in $B(H)$, ACTA Appl. Math., 108, (2009) 625 - 664.
[7] I. Cho, and P. E. T. Jorgensen, Measure Framings on Graphs and Corresponding von Neumann Algebras, (2009) Preprint.
[8] I. Raeburn, Graph Algebras, CBMS no 3, AMS (2005).
[9] P. D. Mitchener, $C^*$-Categories, Groupoid Actions, Equivalent KK-Theory, and the Baum-Connes Conjecture, arXiv:math.KT/0204291v1, (2005), Preprint.
[10] R. Gliman, V. Shpilrain and A. G. Myasnikov (editors), Computational and Statistical Group Theory, Contemporary Math, 298, (2001) AMS.
[11] F. Radulescu, Random Matrices, Amalgamated Free Products and Subfactors of the $C^*$- Algebra of a Free Group, of Noninteger Index, Invent. Math., 115, (1994) 347 - 389.
[12] P. R. Halmos. Hilbert Space Problem Book (2-nd Ed), ISBN: 0-387-90685-1, (1982) Springer-Verlag.
[13] T. Yosino, Introduction to Operator Theory, ISBN: 0-582-23743-2, (1993) Longman Sci. & Tech.
[14] F. W. Stinespring, Positive Functions on $C^*$-Algebras, Proc. Amer. Math. Soc., vol 6, (1955) 211 - 216.
[15] M. B. Stefan, Indecomposability of Free Group Factors over Nonprime Subfactors and Abelian Subalgebras, Pacific J. Math., 219, no. 2, (2005) 365 - 390.
[16] N. Tanaka, Conjugacy Classes of Zero Entropy Automorphisms on Free Group Factors, Nihonkai M. J., 6, no. 2, (1995) 171 - 175.

St. Ambrose Univ., Dept. of Math., 518 W. Locust St., Davenport, Iowa, 52803, U. S. A. / Univ. of Iowa, Dept. of Math., 14 McLean Hall, Iowa City, Iowa, 52242, U. S. A.

E-mail address: chowoo@sau.edu / jorgen@math.uiowa.edu