The mapping class group orbit of a multicurve

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Abstract

Given a set equipped with a transitive action of a group, we define the notion of an almost invariant coloring of the set. We consider the mapping class group orbit of a multicurve on a compact surface, and prove that in the case of genus at least two, no such almost invariant coloring exists. Conversely, in the case of a closed torus, one may find almost invariant colorings using arbitrarily many colors.

1 Introduction

Let $G$ be a group and $X$ an infinite set on which $G$ acts. We define a coloring (or $C$-coloring) of $X$ to be any map $c: X \to C$ into some set $C$ of “colors”. We will use the following terminology:

- A coloring $c$ is **invariant** if $c(gx) = c(x)$ for each $g \in G$ and $x \in X$.
- A coloring is **almost invariant** if, for each $g \in G$, the identity $c(x) = c(gx)$ fails for only finitely many $x \in X$.
- Two colorings are **equivalent** if they assign different colors to only finitely many elements of $X$; this is clearly an equivalence relation on the set of $C$-colorings.
- A coloring is **trivial** if it is equivalent to a monochromatic (constant) coloring.

We will only deal with the case where the action of $G$ is transitive. Then clearly the only invariant colorings are the constant ones, and hence we are only interested in studying the question of existence of almost invariant colorings. If two colorings are equivalent and one is almost invariant, so is the other, which explains the above definition of a trivial coloring. If one wants to classify all almost invariant colorings, this can clearly not be done better than up to the equivalence defined above.

A **simplification** of $c$ is a coloring obtained by post-composing $c$ with some map $i: C \to C'$ (one “identifies” some of the colors). Clearly a simplification of an almost invariant coloring is almost invariant. Now, if there exists an almost invariant, non-trivial $C$-coloring $c$, there also exists an almost invariant coloring where exactly two colors are used. To see this, partition
C into \( C_0 \sqcup C_1 \) such that \( c^{-1}(C_k), k = 0,1, \) are both infinite, and define a \( \{0,1\} \)-coloring by composing \( c \) with the map \( i: C \to \{0,1\} \) determined by \( z \in C_{i(z)} \). Hence, if one wants to prove the non-existence of almost invariant, non-trivial colorings, it suffices to consider colorings where two colors are used.

If \( S \subset G \) is a set of generators for \( G \), a coloring is almost invariant if and only if for each \( g \in S \) we have \( c(x) = c(gx) \) for all but finitely many \( x \in X \). This observation is of course particularly useful when \( G \) is finitely generated, which will be the case in this paper. Hence both \( G \) and \( X \) are countable. Also, it is easy to see that any almost invariant coloring of \( X \) can at most use finitely many colors: Assume WLOG that \( c: X \to C \) is surjective, and for \( z \in C \) let \( X_z = c^{-1}(z) \); then \( X = \bigcup_{z \in C} X_z \) is the partition of \( X \) associated to \( c \). Next, choose some finite set of generators \( g_1, \ldots, g_k \) of \( G \). The almost invariance of the coloring implies that each \( g_i \) acts as a permutation of all but finitely many \( X_z \), hence \( G \) acts as a permutation on all but finitely many of the subsets. If the partition consists of infinitely many subsets, this contradicts the assumption that \( G \) acts transitively on \( X \).

Let \( \Sigma \) be an oriented connected surface of genus \( g \) with \( r \) boundary components, where \( g \geq 2 \) and \( r \geq 0 \), or \( g = 1 \) and \( r = 0 \). Let \( \Gamma \) be the mapping class group of \( \Sigma \), the group of orientation preserving diffeomorphisms of \( \Sigma \) fixing the boundary (if any) pointwise, modulo the group of diffeomorphisms isotopic (through isotopies fixing the boundary point-wise) to the identity. Furthermore, let \( D_0 \) be a non-empty multicurve on \( \Sigma \), the isotopy class of a collection of disjoint circles in \( \Sigma \) which are not trivial nor parallel to a boundary component (components of the multicurve are allowed to be parallel to each other). Let \( X \) denote the mapping class group orbit of \( D_0 \).

The main theorem of this paper is

**Theorem 1.1.** When \( g \geq 2 \), \( r \) arbitrary, there are no non-trivial almost invariant colorings of \( X \).

The proof of this is the contents of Section 3. We are also able to prove that the “converse” is true for the closed torus:

**Theorem 1.2.** When \( g = 1 \), \( r = 0 \), there exist almost invariant colorings of \( X \) using arbitrarily many colors.

In fact, we classify all such almost invariant colorings explicitly.

## 2 Motivation

Before delving into the proofs of Theorems 1.1 and 1.2, let us explain the motivation for studying the question of existence of such almost invariant partitions. As in the introduction, let \( D_0 \) be a multicurve on \( \Sigma \), and let \( X \) be the mapping class group orbit of \( D_0 \). Let \( M = CX \) denote the complex vector
spanned by $X$ (the set of finite formal $\mathbb{C}$-linear combinations of elements of $X$), and let $\hat{M}$ denote the algebraic dual of $M$, which we may think of as the space of all formal linear combinations of elements of $X$. Both $M$ and $\hat{M}$ become modules over $\Gamma$ by extending the action $\mathbb{C}$-linearly, and there is a $\Gamma$-equivariant inclusion $\iota: M \rightarrow \hat{M}$.

In [AV07], we gave jointly with Andersen an algorithm for computing the first cohomology group $H^1(\Gamma, \hat{M})$ for any multicurve $D_0$, and proved that for any surface $\Sigma$, there exists some multicurve (in that paper called a BFK-diagram) such that this cohomology group is non-trivial. We also gave an explicit example showing that in the case of a closed torus, the cohomology group $H^1(\Gamma, M)$ is non-trivial.

The short exact sequence

$$0 \rightarrow M \xrightarrow{\iota} \hat{M} \rightarrow \hat{M}/M \rightarrow 0 \quad (1)$$

of $\Gamma$-modules induces a long exact sequence of cohomology groups

$$0 \rightarrow H^0(\Gamma, M) \xrightarrow{\iota_*} H^0(\Gamma, \hat{M}) \rightarrow H^0(\Gamma, \hat{M}/M) \rightarrow H^1(\Gamma, M) \xrightarrow{\iota_*} H^1(\Gamma, \hat{M}) \rightarrow \cdots \quad (2)$$

Since $X$ is in general infinite (except when $D_0$ is the empty multicurve), it is easy to see that $H^0(\Gamma, M) = 0$ and $H^0(\Gamma, \hat{M}) = \mathbb{C}$, since no finite linear combination of elements of $X$ is invariant under $\Gamma$, while the constant linear combinations are invariant elements of $\hat{M}$.

Now, what is an invariant element of the quotient module $\hat{M}/M$? It is represented by an element $\hat{m}$ of $\hat{M}$ such that for each $\gamma \in \Gamma$, we have $\gamma \hat{m} = \hat{m}$ in $\hat{M}/M$, or in other words $\gamma \hat{m} - \hat{m} \in M$ for each $\gamma \in \Gamma$ (this is by the way exactly how the connecting homomorphism in (2) above is defined). Hence, thinking of an element of $\hat{M}$ as a coloring of $X$ by complex numbers, we see that an invariant element of $\hat{M}/M$ is represented by an almost invariant $\mathbb{C}$-coloring of $X$. In terms of the exact sequence (2) above, Theorem 1.1 implies that $H^0(\Gamma, \hat{M}/M) = \mathbb{C}$ and hence that the second $\iota_*$ is injective whenever $g \geq 2$. So by computing the image of this map we obtain a computation of $H^1(\Gamma, M)$, and this is done in [AV08], using the description of $H^1(\Gamma, \hat{M})$ given in [AV07] and methods similar to those applied in the present paper.

The study of the cohomology groups $H^1(\Gamma, M)$ is in turn motivated by the fact that the complex vector space spanned by the set of all (isotopy classes of) multicurves is isomorphic to the space $\mathcal{O} = \mathcal{O}(\mathcal{M}_{\text{SL}_2(\mathbb{C})})$ of algebraic functions on the moduli space of flat $\text{SL}_2(\mathbb{C})$ connections over $\Sigma$. As a $\Gamma$-module, $\mathcal{O}$ splits into a direct sum

$$\mathcal{O} = \bigoplus_D M_D \quad (3)$$
where $M_D = \mathbb{C}(\Gamma D)$ is the complex vector space spanned by the $\Gamma$-orbit through $D$ and the sum is taken over a set of representatives of the $\Gamma$-orbits. The splitting (3) then induces a splitting in cohomology

$$H^1(\Gamma, \mathcal{O}) = \bigoplus_D H^1(\Gamma, M_D).$$

(4)

Hence, by combining the results of the present paper with those of [AV07] we obtain in [AV08] a complete calculation of the left-hand side of (4).

3 Proof of Theorem 1.1

3.1 Useful facts

We are going to need a couple of facts regarding the mapping class group and its action on the set of multicurves. First of all, the mapping class group is generated by Dehn twists, and moreover there exists a finite set of curves such that the Dehn twists on these curves generate $\Gamma$. Furthermore, one may choose these curves so that any pair of them intersect in at most two points (see [Ger01]). Dehn twists on disjoint curves commute. When $g \geq 2$, a twist on a separating curve can be written as a product of twists on non-separating curves. Hence in this case the mapping class group is generated by a finite set of twists in non-separating curves (though we may not necessarily choose this set so that each pair of curves intersect in at most two points).

There is simple way to parametrize the set of all multicurves which was found by Dehn. For details, we refer to [PH92]. Essentially one cuts the surface into pairs of pants using $3g + r - 3$ simple closed curves $\gamma_k$, and for each pair of pants one chooses a set of three disjoint arcs connecting the three pairs of boundary components. Then for each pants curve $\gamma_k$ one records the geometric intersection number $m_k(D) = i(\gamma_k, D)$ (which is a non-negative integer) and a "twisting number" $t_k(D)$, which can be any integer. This defines a $6g + 2r - 6$-tuple of integers $(m_1(D), t_1(D), \ldots, m_{3g+r-3}(D), t_{3g+r-3}(D))$ (satisfying certain conditions), and, conversely, from any such tuple satisfying these conditions one may construct a multicurve.

The important fact is that in this parametrization, the action of the twist in the curve $\gamma_k$ on a multicurve $D$ is given by

$$t_k(\tau_{\gamma_k}^\pm D) = t_k(D) \pm m_k(D),$$

(5)

all other coordinates being unchanged. The formula (5) is intuitive in the sense that it says that for each time $D$ intersects $\gamma_k$ essentially, the action of $\tau_{\gamma_k}$ on $D$ adds 1 to the twisting number of $D$ with respect to $\gamma_k$. This can be used to prove a number of important facts.
Lemma 3.1. Let \( \gamma \) be a simple closed curve and \( D \) a multicurve. Then the following are equivalent:

1. The twist \( \tau_\gamma \) acts trivially on \( D \).
2. The twist \( \tau_\gamma \) acts trivially on each component of \( D \).
3. The geometric intersection number between \( \gamma \) and \( D \) is zero.
4. One may realize \( \gamma \) and \( D \) disjointly.

Conversely, if \( \tau_\gamma \) acts non-trivially on \( D \), all the multicurves \( \tau_n^\gamma D \), \( n \in \mathbb{Z} \), are distinct.

Proof. All of the above assertions can be proved from (5) by letting \( \gamma \) be part of a pants decomposition of the surface. This is clearly possible if \( \gamma \) is non-separating, while if \( \gamma \) is separating, observe that both connected components resulting from cutting along \( \gamma \) must have negative Euler characteristic (otherwise \( \gamma \) would be trivial or parallel to a boundary component, in which case the twist on \( \gamma \) clearly acts trivially on \( D \)).

To find a twist acting non-trivially on a multicurve, we need only find a curve which has positive geometric intersection number with the multicurve. This is possible if and only if the multicurve has a component which is not parallel to a boundary component of \( \Sigma \).

On a surface with negative Euler characteristic, there exist complete hyperbolic metrics of constant negative curvature. Within each free homotopy class of simple closed curves, there is a unique geodesic representative with respect to such a metric. If \( a \) and \( b \) are the geodesic representatives of distinct homotopy classes \( \alpha \), \( \beta \), then \( a \) and \( b \) realizes the geometric intersection number between \( \alpha \) and \( \beta \), i.e. \( \#a \cap b = i(\alpha, \beta) \).

3.2 Interesting pairs

We will assume that the elements of \( X \) have been colored red and blue, and then prove that one of these colors has only been used a finite number of times. To this end, an interesting pair is a pair \( (\tau_\gamma, D) \) where \( \tau_\gamma \) is a Dehn twist in a curve \( \gamma \) and \( D \in X \) is a multicurve such that \( \tau_\gamma D \neq D \) (equivalently, \( i(\gamma, D) > 0 \)). Since \( \tau_\gamma \) changes the color of only finitely many diagrams, the diagrams \( \tau_n^\gamma D \) all have the same color for all sufficiently large values of \( n \). This color is called the future of the interesting pair \( (\tau_\gamma, D) \), denoted \( \text{fut}(\tau_\gamma, D) \). Similarly, we may consider the past \( \text{pas}(\tau_\gamma, D) \) of an interesting pair; the common color of all diagrams \( \tau_n^{-\gamma} D \) for sufficiently large \( n \). We will also need to consider pairs of the form \( (\tau_\gamma^{-1}, D) \); the same definition of future and past applies to these, and clearly \( \text{fut}(\tau_\gamma^{\pm1}, D) = \text{pas}(\tau_\gamma^{\mp1}, D) \).
Lemma 3.2. For any interesting pair \((\tau_\alpha, D)\), we have

\[
\text{pas}(\tau_\alpha^{-1}, D) = \text{fut}(\tau_\alpha, D) = \text{pas}(\tau_\alpha, D) = \text{fut}(\tau_\alpha^{-1}, D)
\] (6)

Proof. It suffices to prove the middle identity. We may find a non-separating simple closed curve \(\beta\) different and disjoint from \(\alpha\) such that \((\tau_\beta, D)\) is also interesting. To see this, let \(\delta\) be a component of \(D\) for which \(\tau_\alpha \delta \neq \delta\), and assume that \(\alpha\) and \(\delta\) are represented by geodesics with respect to some choice of hyperbolic metric. Cutting \(\Sigma\) along \(\alpha\) then yields a (possibly non-connected) surface with geodesic boundary, in which \(\delta\) is a number of properly embedded hyperbolic arcs. At least one of the connected components of the cut surface has genus at least 1, so in this component we may find a closed geodesic \(\beta\), not parallel to a boundary component, intersecting one of the \(\delta\)-arcs. In the original surface, \(\beta\) is still a geodesic intersecting the geodesic \(\delta\); hence \(\tau_\beta \delta \neq \delta\) and \((\tau_\beta, D)\) is interesting.

Next, since \(\tau_\alpha\) and \(\tau_\beta\) commute, we see that \(\tau_\alpha^n \tau_\beta^m D\) is a \(\mathbb{Z} \times \mathbb{Z}\)-indexed family of distinct multicurves. By assumption, both \(\tau_\alpha\) and \(\tau_\beta\) change the color of finitely many multicurves. Hence, outside some bounded region in \(\mathbb{Z} \times \mathbb{Z}\), moving from one diagram to a neighbour does not change the color, and since we can connect the future of \((\tau_\alpha, D)\) to its past using such moves, the claim follows.

From now on, we will only consider the future.

Lemma 3.3. Assume that \(\alpha\) and \(\beta\) are simple closed curves with \(i(\alpha, \beta) \leq 1\), and that \(D\) is a multicurve such that \((\tau_\alpha, D)\), \((\tau_\beta, D)\) are interesting pairs. Then \(\text{fut}(\tau_\alpha, D) = \text{fut}(\tau_\beta, D)\).

Proof. If \(i(\alpha, \beta) = 0\) the result follows from the proof of Lemma 3.2.

Now assume \(i(\alpha, \beta) = 1\). Then \(\alpha \cup \beta\) is contained in a subsurface \(\Sigma'\) of genus 1 with one boundary component \(\gamma\). If \(D\) can not be isotoped to be contained entirely in \(\Sigma'\), either some component of \(D\) intersects \(\gamma\) essentially, or some component of \(D\) lies in the complement of \(\Sigma'\). In the former case, it is clear that \((\tau_\gamma, D)\) is interesting, so the \(i = 0\) case implies \(\text{fut}(\tau_\alpha, D) = \text{fut}(\tau_\gamma, D) = \text{fut}(\tau_\beta, D)\). In the latter case, use the fact that the complement of \(\Sigma'\) has genus at least 1 to find a simple closed curve intersecting \(D\) essentially.

Otherwise, \(D\) lives entirely in \(\Sigma'\). Let \(D_\alpha\) denote any component of \(D\) on which \(\tau_\alpha\) acts non-trivially. Then \(D_\alpha\) is a simple closed curve in a torus with one boundary component. Since \(D_\alpha\) is not a parallel copy of the boundary component, it must be a non-separating curve not parallel to \(\alpha\). Hence, thinking of \(\alpha\) as a \((1,0)\)-torus knot and \(\beta\) as a \((0,1)\)-torus knot, we conclude that \(D_\alpha\) is a \((p,q)\)-torus knot with \((p,q) \neq (1,0)\). But then any other component of \(D\) is forced to be either parallel to the boundary component of \(\Sigma'\) or to \(D_\alpha\). The only way that \(\tau_\beta\) can act on some component of \(D\) is then that \(\tau_\beta\) acts on \(D_\alpha\); hence also \((p,q) \neq (0,1)\).
Consider the schematic picture of $\Sigma'$ on Figure 1, where the boundary component is the circle in the center and $\alpha$ and $\beta$ are the sides of the square.

![Diagram of a torus with one boundary component](image)

**Figure 1**: A torus with one boundary component.

We construct two disjoint simple closed curves $\gamma_1, \gamma_2$ as follows: Draw two essential, disjoint arcs in $\Sigma'$ with the endpoints on the boundary component, and use the fact that the complement of $\Sigma'$ has genus at least 1 to close them up in such a way that they are disjoint and not homotopic to a curve contained in $\Sigma'$. By the above description of $D_\alpha$, $(\tau_{\gamma_1}, D)$ are both interesting pairs. Now the $i = 0$ case implies that

$$\text{fut}(\tau_{\alpha}, D) = \text{fut}(\tau_{\gamma_1}, D) = \text{fut}(\tau_{\gamma_2}, D) = \text{fut}(\tau_{\beta}, D).$$

The next proposition extends the above lemma to $i(\alpha, \beta) \leq 2$, but its proof is rather technical. Also, as explained in the comments following the proof, it is in fact not needed when one is only interested in surfaces with at most one boundary component.

**Proposition 3.4.** Assume that $\alpha$ and $\beta$ are simple closed curves with $i(\alpha, \beta) = 2$, and that $D$ is a multicurve such that $(\tau_{\alpha}, D)$ and $(\tau_{\beta}, D)$ are interesting. Then $\text{fut}(\tau_{\alpha}, D) = \text{fut}(\tau_{\beta}, D)$.

**Proof.** Let $N$ be a regular neighbourhood of $\alpha \cup \beta$. We distinguish between four cases.

1. At least one of $\alpha$ and $\beta$ is non-separating in $N$.
2. Both $\alpha$ and $\beta$ are separating in $N$, but non-separating in $\Sigma$.
3. Both $\alpha$ and $\beta$ are separating in $N$, but one is non-separating in $\Sigma$.
4. Both $\alpha$ and $\beta$ are separating in $\Sigma$.

In case (1), assume WLOG that $\alpha$ is non-separating. This means that when cutting $N$ along $\alpha$, there is at least one arc $b$ of $\beta$ connecting the two sides of $\alpha$. Now construct two curves $\gamma_1, \gamma_2$ as follows: Make two parallel copies of $b$ and close them up using arcs going in opposite directions along
Applying small isotopies in a tubular neighbourhood of $\alpha$ we obtain a situation as depicted in Figure 2. We observe that each $\gamma_n$ intersects $\alpha$ in exactly one point, and also they intersect each other in exactly one point $p$. Furthermore, since $i(\alpha, \beta) = 2$, the arc $b$ does not start and end at the same point of $\alpha$, so we have $i(\gamma_n, \beta) = 1$ for $n = 1, 2$.

![Figure 2: When $\alpha$ is non-separating in $N$, the two sides of $\alpha$ are connected by an arc of $\beta$.](image)

Now orient $\gamma_1$ and $\gamma_2$ oppositely along $b$. Then Goldman’s bracket (see [Gol86]) of $\gamma_1$ and $\gamma_2$ is plus or minus some oriented version $\vec{\alpha}$ of $\alpha$. Now let $D_\alpha$ be some component of $D$ on which $\tau_\alpha$ acts non-trivially. We claim that at least one of $\gamma_1$ and $\gamma_2$ intersects $D_\alpha$ essentially. If this were not the case, choose geodesic representatives $\gamma'_1$, $\gamma'_2$ and $D'_\alpha$ of the three curves. Then $\gamma'_1$ is disjoint from $D'_\alpha$, and necessarily $\gamma'_1$ and $\gamma'_2$ intersect transversally in a single point $p'$. But then $(\gamma'_1 \gamma'_2)^{p'} \in \pi_1(\Sigma, p')$ is a representative of the free homotopy class of $\vec{\alpha}$ which does not intersect $D'_\alpha$, implying that $i(D'_\alpha, \alpha) = 0$, which contradicts the choice of $D_\alpha$. So one of the pairs $\left(\tau_{\gamma_n}, D\right)$ is interesting, and by Lemma 3.3 we have

\[ \text{fut}(\tau_\alpha, D) = \text{fut}(\tau_{\gamma_n}, D) = \text{fut}(\tau_\beta, D). \]

This ends case (1).

In cases (2)–(4), notice that $N$ is necessarily a sphere with four holes, and $\alpha$ and $\beta$ divide $N$ into two pairs of pants in two different ways. Denote the boundary components of $N$ by $\gamma_i$, $i = 0, 1, 2, 3$, such that $\gamma_1, \gamma_2$ are on one side of $\alpha$ and $\gamma_3, \gamma_4$ on the other, and such that $\gamma_2, \gamma_3$ are on one side of $\beta$ and $\gamma_4, \gamma_1$ on the other. Schematically we have Figure 3a on the facing page.

Throughout the rest of the proof, we assume that $\alpha, \beta, \gamma_i$, $i = 0, 1, 2, 3$, denote geodesic representatives for their isotopy classes. Also, we let $\delta$ be the geodesic representative of some component of $D$ on which $\tau_\delta$ acts non-trivially. If $\delta$ does not live entirely in $N$, a twist in one of the boundary components acts non-trivially on $\delta$, and since this boundary component is disjoint from $\alpha$ and $\beta$ we are done by Lemma 3.3. Otherwise, $\delta$ is a separating curve in $N$ which is not parallel to a boundary component. Clearly $\delta$ can not be parallel to $\beta$, since in that case $D$ could not consist of any component on which $\tau_\beta$ acts non-trivially. Hence $\delta$ is different from both $\alpha$ and $\beta$. 
In case (2), it is not hard to see that at least one of the “opposite” pairs $\gamma_1, \gamma_3$ and $\gamma_2, \gamma_4$ can be connected by an arc in the complement of $N$. Take two parallel copies of this arc, and close them up by arcs intersecting each other, $\alpha$ and $\beta$ exactly once as in Figure 3b (the two connecting arcs are related by a twist in $\alpha$. We may then argue exactly as in case (1) to see that the twist in at least one of these simple closed curves acts non-trivially on the multicurve in question.

**Figure 3**: There are four different topological cases when two curves intersect in two points.

In case (3), assume WLOG that $\beta$ is separating and $\alpha$ is non-separating. This means that it is impossible to connect any of $\gamma_0$ and $\gamma_1$ to any of $\gamma_2$ and $\gamma_3$ in the complement of $N$. But then, since $\alpha$ is non-separating, one may
connect either $\gamma_0$ to $\gamma_1$ or $\gamma_2$ to $\gamma_3$ in the complement of $N$. Assume WLOG that the latter is the case, and construct a simple closed curve $\gamma$ disjoint from $\beta$ intersecting $\gamma_2$, $\alpha$ and $\gamma_3$ exactly once each by composing the arc in the complement of $N$ with an arc in $N$, as in Figure 3c on the preceding page. Observe that the geodesic representative of $\gamma$ necessarily intersects $\gamma_2$, $\alpha$ and $\gamma_3$ exactly once and is disjoint from $\beta$, so this representative contains a subarc in $N$ starting at $\gamma_2$ and ending at $\gamma_3$. We now claim that this arc intersects $\delta$ (recall that $\delta$ has been chosen to be a geodesic). Assume the contrary. Then $\delta$ is a simple closed curve in the surface obtained by cutting $N$ along this arc, which is a pair of pants. The “legs” are $\gamma_0$ and $\gamma_1$, whereas the “waist” is composed of four segments; two copies of the connecting arc and the remaining boundary components (cut open). Since $\delta$ is simple, it is parallel to one of the boundary components of the pair of pants. But $\delta$ is certainly not parallel to any of the original boundary components, nor is it parallel to the “waist”, since the latter is parallel to $\beta$. This contradiction implies that $(\tau_\gamma, D)$ is an interesting pair, and since $\gamma$ is disjoint from $\beta$ and intersects $\alpha$ in a single point, Lemma 3.3 yields the desired result,

$$\text{fut}(\tau_\alpha, D) = \text{fut}(\tau_\gamma, D) = \text{fut}(\tau_\beta, D).$$

Finally, in case (4), none of the four boundary components of $N$ can be connected in the complement of $N$. This means that at least one of the connected components of $\Sigma - N$ must have positive genus. Assume WLOG that the component $\Sigma_0$ bounded by $\gamma_0$ has positive genus. Now take some non-separating, essential arc in $\Sigma_0$ with its endpoints on $\gamma_0$ and compose it with some essential arc in $N$ disjoint from $\beta$ and intersecting $\alpha$ in exactly two points (cf. Figure 3d) to obtain a non-separating curve $\gamma$ in $\Sigma$. We claim that $\tau_\gamma$ acts non-trivially on $\delta$, i.e. that the arc in $N$ intersects $\delta$ essentially. To see this, we argue as in case (3) above. Observe that $\gamma$ has geometric intersection number 2 with $\alpha$ and $\gamma_0$. Hence, the geodesic representative of $\gamma$ intersects $\alpha$ and $\gamma_0$ exactly twice, so this geodesic contains a subarc in $N$ looking as the one depicted in Figure 3d. We claim that this arc intersects $\delta$. If this were not the case, we may cut $N$ along this arc to obtain a cylinder (bounded by one of the original boundary components and a curve coming from the cut) and a pair of pants (bounded by two of the original boundary components and a curve from the cut), and $\delta$ lives completely in one of these. Since $\delta$ is not parallel to any of the boundary components of $N$, we conclude that $\delta$ is parallel to the third boundary component of the pair of pants. But this third boundary component is clearly parallel to $\beta$, which contradicts the fact that $D$ does not contain any component parallel to $\beta$. Hence $(\tau_\gamma, D)$ is interesting, and since $\gamma$ is non-separating and intersects $\alpha$ in two points, by case (3) and Lemma 3.3 we have

$$\text{fut}(\tau_\alpha, D) = \text{fut}(\tau_\gamma, D) = \text{fut}(\tau_\beta, D),$$

which finishes the last case. \qed
Now we turn to the (finite) presentation of the mapping class group given by Gervais in [Ger01], where the generators are twists in certain curves. A key property of this presentation is that any two curves involved intersect each other in at most two points. It should be pointed out, however, that if one is only interested in surfaces with at most one boundary component, a much earlier result by Wajnryb [Waj83] yields a presentation where each pair of curves intersect in at most one point. In this case, one does not need the rather technical Proposition 3.4 above in the following (simply replace all references to [Ger01] by [Waj83] and all occurrences of “at most two points” by “at most one point”).

**Proposition 3.5.** Let \( S \) denote the set of curves from [Ger01] such that \( \{ \tau_\sigma \mid \sigma \in S \} \) generate \( \Gamma \). Let \( \alpha, \beta \in S \) be two of these curves, and let \( D_1, D_2 \in X \) be multicurves such that \((\tau_\alpha, D_1)\) and \((\tau_\beta, D_2)\) are interesting. Then

\[
\text{fut}(\tau_\alpha, D_1) = \text{fut}(\tau_\beta, D_2).
\]

**Proof.** We may find a sequence of curves \( \eta_1, \eta_2, \ldots, \eta_n \in S \) and exponents \( \epsilon_i = \pm 1 \) such that, writing \( \tau = \tau_{\eta_1}^{\epsilon_1} \cdot \tau_{\eta_2}^{\epsilon_2} \cdot \tau_{\eta_n}^{\epsilon_n} D_1 = D_2 \). For each \( 1 \leq i \leq n \) we may assume that \( (\tau_i, \tau_{i-1} \cdots \tau_1 D_1) \) is interesting; otherwise we may simply omit the corresponding \( \tau_i \). Now using alternately the fact that \( \eta_i \) and \( \eta_{i+1} \) intersect in at most two points and the obvious fact that \( \text{fut}(\tau_{\gamma}, D) = \text{fut}(\tau_{\gamma}, \tau_i D) \) for any interesting pair \((\tau_{\gamma}, D)\), we obtain a sequence of identities

\[
\text{fut}(\tau_1, D_1) = \text{fut}(\tau_1, D_1) = \text{fut}(\tau_2, D_1) = \text{fut}(\tau_2, D_1)
\]

\[
\vdots
\]

\[
\text{fut}(\tau_{n-1}, D_1) = \text{fut}(\tau_{n-1}, D_1) = \text{fut}(\tau_n, D_2)
\]

which may be augmented by the identities \( \text{fut}(\tau_\alpha, D_1) = \text{fut}(\tau_\beta, D_1) \) and \( \text{fut}(\tau_n, D_2) = \text{fut}(\tau_n, D_2) \) to obtain the desired result. \( \square \)

**Lemma 3.6.** Let \( f \in \Gamma \) be any diffeomorphism, and \((\tau_\alpha, D)\) an interesting pair. Then \((\tau_{f(\alpha)}, fD)\) is also interesting and \( \text{fut}(\tau_\alpha, D) = \text{fut}(\tau_{f(\alpha)}, fD) \).

**Proof.** Recall that \( f \circ \tau_\alpha \circ f^{-1} = \tau_{f(\alpha)} \). Hence \( \tau_{f(\alpha)}(fD) = f(\tau_\alpha D) \neq fD \), so \((\tau_{f(\alpha)}, fD)\) is interesting. Also we have \( \tau_{f(\alpha)} = f \circ \tau_\alpha \circ f^{-1} \), so \( \tau_{f(\alpha)}(fD) = f(\tau_\alpha D) \). Since the different multicurves \( \tau_\alpha^n D \) have the same color for all sufficiently large \( n \), and since \( f \) changes the color of only finitely many multicurves, the result follows. \( \square \)

**Proposition 3.7.** All interesting pairs \((\tau_\gamma, D)\) where \( \gamma \) is a non-separating curve have the same future.
Proof. Let $\tau_\alpha$ be a twist on a non-separating curve which is part of the generating set for $\Gamma$ from [Ger01]. Then Proposition 3.5, with $\alpha = \beta$, implies that the future is a property of $\tau_\alpha$ alone, and not of the particular multicurve on which $\tau_\alpha$ acts. If $\gamma$ is any non-separating curve, choose a diffeomorphism of $\Sigma$ carrying $\gamma$ to $\alpha$ and apply Lemma 3.6. □

Now we are ready to prove the main theorem.

Proof (of Theorem 1.1). Choose a finite set $\alpha_1, \ldots, \alpha_N$ of non-separating curves such that the twists in these curves generate $\Gamma$ (we do not require that these intersect pair-wise in at most two points). To be concrete, assume that the common future (cf. Proposition 3.7) of all interesting pairs $(\tau_\gamma, D)$ with $\gamma$ non-separating is red. We must then prove that only finitely many multicurves are blue. Let $B \subset X$ be the set of blue multicurves. For each blue multicurve $D \in B$, choose a generator $\tau_{\alpha_k}$ such that $(\tau_{\alpha_k}, D)$ is interesting (this must be possible since the action is transitive and the $\tau_{\alpha_k}$ generate $\Gamma$). This defines a map $f : B \to \{1, 2, \ldots, N\}$. We claim that for each $k \in \{1, \ldots, N\}$, the pre-image $f^{-1}(k)$ is finite.

To see this, for each $D \in f^{-1}(k)$ consider the “$\tau_{\alpha_k}$-string through $D$”, ie. the set $s_k(D) = \{\tau_n^{\alpha_k} D \mid n \in \mathbb{Z}\}$. Let $B_k$ be the union of the blue multicurves occurring in these strings, ie.

$$B_k = \bigcup_{D \in f^{-1}(k)} (s_k(D) \cap B),$$

so that $f^{-1}(k) \subseteq B_k$. There are only finitely many blue multicurves in each string by Proposition 3.7 and Lemma 3.2, and since $\tau_{\alpha_k}$ changes the color of at least one diagram in each string (since the strings contain both blue and red multicurves), there can be only finitely many strings by the almost invariance of the coloring. Hence, there are only finitely many blue multicurves. □

4 The genus one case

When $\Sigma$ is a closed torus, it is well-known that $\Gamma \cong \text{SL}_2(\mathbb{Z})$. A multicurve necessarily consists of some number of parallel copies of the same non-separating simple closed curve, and since $\Gamma$ acts identically on parallel curves, we may simply assume that $D_0$ is a single non-separating simple closed curve, and $X$ is the set of isotopy classes of such curves. Hence we may identify $X$ with the set of unoriented torus knots, ie. the set $P$ of pairs $(p, q)$, $p, q \in \mathbb{Z}$ and $\gcd(p, q) = 1$, where we identify the pairs $(p, q)$ and $(-p, -q)$ (since the curves are not oriented). The action of the mapping class group is then simply given by the usual action of $\text{SL}_2(\mathbb{Z})$ on pairs of
relatively prime integers, and the central element \(-I\) acts trivially, so we are really dealing with an action of \(\text{PSL}_2(\mathbb{Z})\).

As generators for \(\text{SL}_2(\mathbb{Z})\) we choose \(S = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\) and \(R = \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}\). Then \(S^2 = R^3 = I\) in \(\text{PSL}_2(\mathbb{Z})\). Letting

\[
X_1 = \{(p,q) \mid p \geq 1, q \geq 0\}
\]
\[
X_2 = \{(p,q) \mid q > -p \geq 0\}
\]
\[
X_3 = \{(p,q) \mid -p \geq q > 0\}
\]
it is easy to see that \(X_1 \cup X_2 \cup X_3 = X\), and one also verifies that \(SX_1 = X_2 \cup X_3, RX_1 = X_2, RX_2 = X_3\).

**Proposition 4.1.** Any point \((p,q) \in X\) with \(p,q > 0\), can be reached from \((1,1)\) by applying a unique sequence of elements of \(\text{SL}_2(\mathbb{Z})\) of the form \(S^{-1}R^k\), where \(k\) is 1 or 2.

**Proof.** For existence, we will use induction on \(\max(p,q)\). For \(\max(p,q) = 1\) we have \(p = q = 1\), in which case the claim is obvious (choose the empty sequence). If \(\max(p,q) > 1\), \(p\) and \(q\) are different since \(\gcd(p,q) = 1\). If \(p > q\), put

\[
(p', q') = R^{-1}S(p,q) = R^{-1}(-q, p) = (p - q, q)
\]
while if \(q > p\), put

\[
(p', q') = R^{-2}S(p,q) = R^{-2}(-q, p) = (p, q - p).
\]

In both cases, clearly \(1 \leq p', q'\) and \(\max(p', q') < \max(p,q)\), so there exists \(\gamma' = S^{-1}R^{k_{n-1}} \ldots S^{-1}R^{k_1}\) with \(\gamma'(1,1) = (p', q')\). Then \(\gamma = S^{-1}R^{k_n} \gamma'\) where \(k_n = 1\) if \(p > q\) and \(k_n = 2\) if \(p < q\) is an element of \(\text{PSL}_2(\mathbb{Z})\) of the desired form.

To prove uniqueness, choose \((p,q)\) with \(\max(p,q)\) minimal such that there are two different strings

\[
\gamma_1 = S^{-1}R^{k_n}S^{-1}R^{k_{n-1}} \ldots S^{-1}R^{k_1}
\]
\[
\gamma_2 = S^{-1}R^{\ell_m}S^{-1}R^{\ell_{m-1}} \ldots S^{-1}R^{\ell_1}
\]
satisfying \(\gamma_1(1,1) = (p,q)\). Then \(S(p,q)\) is a point in \(X_2 \cup X_3\) which is obtained by applying \(R^{k_n}\) to some point of \(X_1\) and also by applying \(R^{\ell_m}\) to some (possibly other) point of \(X_1\). But since \(S(p,q)\) is an element of exactly one of \(X_2 = RX_1\) and \(X_3 = R^2X_1\), this implies that \(k_n = \ell_m\). Continuing this way, we only need to show that there is no non-trivial string

\[
\gamma = S^{-1}R^{k_n}S^{-1}R^{k_{n-1}} \ldots S^{-1}R^{k_1}
\]
such that \(\gamma(1,1) = (1,1)\). But this is trivial by observing that each element of the form \(S^{-1}R^k\) strictly increases the max-norm of any point \((p,q)\) with \(p,q \geq 1\) (since \(S^{-1}R(p,q) = (p+q,q)\) and \(S^{-1}R^2(p,q) = (p,p+q)\)). \(\square\)
This proposition allows us to label the vertices of an infinite binary tree $T$ as follows. The root is labelled by $(1, 1)$, and all remaining vertices are labelled according to the rule: If a vertex $v$ is reached by going “left” from the immediate predecessor, the label of $v$ is obtained by applying $S^{-1}R$ to the label of its predecessor; otherwise the label is obtained by applying $S^{-1}R^2$ (see Figure 4).

![Diagram of an infinite binary tree labelled by the points of $X_1$.](image-url)

Now add a single vertex below the root and label this by $(1, 0)$. This gives, by Proposition 4.1, a 1–1-correspondence between the vertices of $T$ and the points in $X_1$, and from now on we shall refer to a vertex and its label interchangeably.

By the level of a vertex of $T$ we mean its distance from $(1, 1)$ (the level of $(1, 0)$ may be taken to be $-1$); there are $2^k$ vertices at level $k$ for each $k \geq 0$, and also exactly $2^k$ vertices at level $< k$.

**Proposition 4.2.** For each $k \geq 0$, there is an almost invariant coloring of $X$ using $2^k$ different colors.

*Proof.* We start by coloring the subset $X_1$ by coloring the vertices of the tree. Assign different colors to the $2^k$ vertices at level $k$, and for each of these vertices assign the same color to all descendants. The remaining $2^k$ points of $X_1$ may be colored arbitrarily.

To obtain a coloring of all of $X$, we insist that the coloring is completely invariant under $R$. This gives a well-defined coloring, since $X_1$ is a complete set of representatives of the $R$-orbits of $X$. In order to see that this coloring is almost invariant under $\text{PSL}_2(\mathbb{Z})$, it suffices to check that the other generator $S$ changes the color of only finitely many points of $X$. Since $S$ has order two in $\text{PSL}_2(\mathbb{Z})$, $S$ changes the color of $p$ if and only if it changes the color of $Sp$. Hence we need only check that $S$ changes the color of finitely many elements of $X_1$. But for any vertex $v$ of $T$ of level $k + 1$ or higher, applying $S$ to the label of $v$ yields by construction a point of $X_2$ or $X_3$ which has the
same color as the predecessor of \( v \); hence \( S \) does not change the color of labels placed at level \( k + 1 \) or higher, and thus \( S \) changes the color of at most \( 2 \cdot 2^{k+1} \) points of \( X \).

This finishes the proof of Theorem 1.2, but we also promised a classification of all almost invariant colorings in this case.

**Proposition 4.3.** Any almost invariant coloring of \( X \) is equivalent to (a coloring which is a simplification of) a coloring of the form constructed in Proposition 4.2.

**Proof.** Let \( c : X \to C \) be some almost invariant coloring of \( X \). Since \( R \) changes the color of only finitely many points of \( X \), it changes the color of only finitely many points \( x_1, \ldots, x_N \) of \( X \). Now we change \( c \) into an equivalent coloring \( c' \) by putting \( c'(Rx_i) = c'(R^2x_i) = c(x_i) \), and \( c' = c \) otherwise. Then \( c' \) is by construction completely invariant under \( R \). Now since \( c' \) is almost invariant, there are only finitely many points of \( X \) whose \( c' \)-color changes under \( S \). Choose \( K \) such that the color of any label placed at level \( k > K \) is unchanged under \( S \). This, together with the \( R \)-invariance of \( c' \), implies that each label at level \( K \) has the same color as any of its descendants, and hence \( c' \) is (a simplification of) a coloring using \( 2^K \) different colors. \( \square \)

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