On a conjecture of H. Fang, Z. Lu and K.-I. Yoshikawa

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Abstract

In [8, Sec. 4, Conj. 4.17], Fang, Lu and Yoshikawa conjecture that a certain string-theoretic invariant of Calabi-Yau threefolds is a birational invariant. We prove a weak form of this conjecture.

1 Introduction

Let $Y$ be a smooth projective variety of dimension 3 over $\mathbb{C}$. We suppose that $Y$ is a Calabi-Yau variety (in the restricted sense). By definition, this means that $H^1(Y, \mathcal{O}_Y) = H^2(Y, \mathcal{O}_Y) = 0$ and that $\omega_Y := \det(\Omega_Y) \simeq \mathcal{O}_Y$.

In [8] (see also [22, Sec. 2]), H. Fang, Z. Lu and K.-I. Yoshikawa introduced the analytic invariant $\tau_{BCOV}(Y(\mathbb{C})) \in \mathbb{R}^*_+$. See [8, p. 177] or Definition 2.3 below for the precise definition. They conjectured the following (see [8, Sec. 4, Conj. 4.17] and [22, Sec. 2, Conj. 2.1]): if $Y$ and $Y'$ are birational Calabi-Yau varieties of dimension 3 over $\mathbb{C}$, then $\tau_{BCOV}(Y(\mathbb{C})) = \tau_{BCOV}(Y'(\mathbb{C}))$.\footnote{The conjecture made in [8, Sec. 4, Conj. 4.17] is only apparently weaker than the conjecture made in [22, Sec. 2, Conj. 2.1], because the topological types of $Y(\mathbb{C})$ and $Y'(\mathbb{C})$ coincide by a result of D. Huybrechts (see [14, middle of p. 65]).}
H. Fang, Z. Lu and K.-I. Yoshikawa explain that their definition of $\tau_{BCOV}$ is the mathematical formalisation of a definition made by the string-theorists M. Bershadsky, S. Cecotti, H. Ooguri and C. Vafa in [3] and [4] (the invariant $F_1(Y)$). Their conjecture should be viewed as a "secondary" analog of the conjecture (which is now a theorem of Batyrev and Kontsevich; see [2]) that the Hodge numbers of $Y(\mathbb{C})$ and $Y'(\mathbb{C})$ coincide. The latter conjecture was also motivated by physical considerations.

The purpose of this note is to describe the proof of the following arithmetic result, which is a step towards Yoshikawa’s conjecture.

Suppose that $X$ (resp. $X'$) is a smooth projective variety of dimension 3 over $L$. Suppose that $X$ (resp. $X'$) is a Calabi-Yau variety (in the restricted sense).

If $Z$ is a scheme, write as usual $D^b(Z) := D^b_c(Z)$ for the category derived from the homotopy classes of bounded complexes of coherent sheaves on $Z$.

If $\sigma : L \hookrightarrow \mathbb{C}$ is a subfield of $\mathbb{C}$, we shall write $X_\sigma$ for the base change $X \times_{\text{Spec } L, \sigma \text{ Spec } \mathbb{C}}$ of $X$ to $\mathbb{C}$ via the embedding $\sigma$.

Now fix an embedding $\sigma : L \hookrightarrow \mathbb{C}$.

**Theorem 1.1.** Let $T$ be a finite set of embeddings of $L$ into $\mathbb{C}$.

Let (S) be the statement : there exists $n \in \mathbb{N}^*$ and $\alpha \in L^*$ such that for all $\tau \in T$,

$$\frac{\tau_{BCOV}(X_\tau(\mathbb{C}))}{\tau_{BCOV}(X'_\tau(\mathbb{C}))} = \sqrt[n]{|\tau(\alpha)|}.$$  

(A) If $X_\sigma$ is birational to $X'_\sigma$ then (S) is verified.

(B) If $D^b(X_\sigma)$ and $D^b(X'_\sigma)$ are equivalent as triangulated $\mathbb{C}$-linear categories then (S) is verified.

In particular, if $L = \mathbb{Q}$ and $X_\sigma$ is birational to $X'_\sigma$ then there exists $n \in \mathbb{N}^*$ such that $(\tau_{BCOV}(X(\mathbb{C}))/\tau_{BCOV}(X'(\mathbb{C})))^n \in \mathbb{Q}$.

Notice that by a theorem of Bridgeland (see [5]), (B) implies (A).

We shall nevertheless give two separate proofs of (A) and (B).

Here is an outline of our proofs of (A) and (B). We first express the quantity $\tau_{BCOV}$ in terms of arithmetic Chern numbers; this is made possible by the arithmetic Riemann-Roch theorem of Bismut-Gillet-Soulé [10]. To prove (A), we use the weak factorisation conjecture
for birational maps (proved in [1]) and some lemmas describing the effect of blow-up on some global Arakelov-theoretic invariants. To prove (B), we make use of a theorem of Orlov, which asserts that if $D^b(X_\sigma)$ and $D^b(X'_\sigma)$ are equivalent as triangulated $\mathbb{C}$-linear categories then $X_\sigma$ and $X'_\sigma$ are related by a Fourier-Mukai functor. We also use a theorem of Caldararu, which describes the effect of a Fourier-Mukai functor on the singular cohomology of $X_\sigma(\mathbb{C})$ and $X'_\sigma(\mathbb{C})$.

**Remark.** It is likely that Theorem 1.1 is true without any restriction of finiteness on $T$. In particular, the quantity $\alpha$ should not depend on $T$. The reason for restricting the statement to finite $T$ is a (probably unnecessary) hypothesis of finiteness included in the definition of an arithmetic ring in Arakelov geometry.

### 2 The invariant $\tau_{BCOV}$ and arithmetic Chern numbers on Calabi-Yau threefolds

We shall apply the arithmetic Riemann-Roch theorem to certain vector bundles on $X$. In the following, we shall freely use the terminology of global Arakelov theory. For a concise summary of the necessary vocabulary, see [20, Sec. 1].

Let $f : X \to S := \text{Spec} \ L$ be the structure morphism. We may enlarge the set $T$ without changing the conclusion of Theorem 1.1, so we may assume that $T$ is conjugation-invariant. We view $L$ as an arithmetic ring, endowed with the set of embeddings $T$ into $\mathbb{C}$. We endow $X(\mathbb{C}) := \bigsqcup_{\tau \in T} X_\tau(\mathbb{C})$ with a conjugation-invariant Kähler form $\nu$. Let $\Omega := \Omega_X$ be the sheaf of differentials of $X$, endowed with the metric induced by $\nu$. We write $\omega := \omega_X$ for $\det(\Omega_X)$ and $\Omega^p := \Omega^p_X$ for $\Lambda^p(\Omega_X)$. Furthermore, we shall write $H^q(Y, \Omega^p)$ for the $L$-vector space $R^qf_*(\Omega^p)$, endowed with the $L^2$-metric induced by $\nu$.

Let $\mathcal{E}$ be the natural exact sequence of hermitian bundles

$$0 \to f^* f_* \omega \to \omega \to 0 \to 0$$

We let $\eta := \widehat{\text{ch}}(\mathcal{E})$ be the Bott-Chern secondary class associated to $\mathcal{E}$, so that

$$f^* f_* \omega - \omega = \eta$$
in $K_0(X)$, the arithmetic Grothendieck group of $X$. We shall write $\eta^0$ for the degree 0 part of $\eta$.

We apply the arithmetic Riemann-Roch theorem to $f$ and to the formal linear combination of hermitian bundles

$$-\Omega^1 + 2 \cdot \Omega^2 - 3 \cdot \Omega^3.$$ 

We obtain the equality

$$\hat{c}_1 \left( R^* f_* \left[ -\Omega^1 + 2 \cdot \Omega^2 - 3 \cdot \Omega^3 \right] \right) - a(\tau( -\Omega^1 + 2 \cdot \Omega^2 - 3 \cdot \Omega^3))$$

$$= f_* \left( \text{Td}(\Omega^\vee) \hat{\text{ch}} \left( -\Omega^1 + 2 \cdot \Omega^2 - 3 \cdot \Omega^3 \right) \right)$$

$$- a(\int_X R(\Omega^\vee) \text{Td}(\Omega^\vee) \text{ch}(\Omega^1 + 2 \cdot \Omega^2 - 3 \cdot \Omega^3))^{(1)}$$

in $\hat{CH}^1(L)_{\mathbb{Q}}$, which is the first arithmetic Chow group of $L$, tensored with $\mathbb{Q}$.

Recall that $R(\cdot)$ is the $R$-genus of Gillet-Soule and that $\tau(\cdot)$ is the Ray-Singer analytic torsion; see [10, Introduction].

We shall first analyse the various terms appearing in this equation. Write $\zeta_{\mathbb{Q}}(s)$ for the evaluation of the Riemann zeta function at $s \in \mathbb{C}$.

**Lemma 2.1.** The equation

$$\hat{\text{c}}_1 \left( R^* f_* \left[ -\Omega^1 + 2 \cdot \Omega^2 - 3 \cdot \Omega^3 \right] \right) - a(\tau( -\Omega^1 + 2 \cdot \Omega^2 - 3 \cdot \Omega^3))$$

$$= -c^{\text{top}}(\Omega^\vee) \left[ a(\zeta_{\mathbb{Q}}(0) \text{rk}^{}(\Omega)) + \zeta_{\mathbb{Q}}(-1) \hat{c}_1(\omega) + \text{terms of degree > 1} \right]$$

$$= -c^{\text{top}}(\Omega^\vee) \left[ a(\zeta_{\mathbb{Q}}(0) \text{rk}^{}(\Omega)) + \zeta_{\mathbb{Q}}(-1) \hat{c}_1(f^* f_* \omega) - a(\zeta_{\mathbb{Q}}(-1) \eta^0) + \text{terms of degree > 1} \right]$$

holds in $\hat{CH}^1(X)_{\mathbb{Q}}$.

**Proof.** The proof is similar to the proof of [17, Lemma 3.1] so we omit it. Q.E.D.

Using the projection formula together with Lemma 2.1, we obtain that

$$[ f_* (\hat{\text{Td}}(\Omega^\vee) \hat{\text{ch}}(-\Omega^1 + 2 \cdot \Omega^2 - 3 \cdot \Omega^3)) ]^{(1)} = -a(\zeta_{\mathbb{Q}}(-1) \hat{c}_1(f_* \omega) \int_X c^{\text{top}}(\Omega^\vee)) + a(\zeta_{\mathbb{Q}}(-1) \int_X c^{\text{top}}(\Omega^\vee) \eta^0)$$

in $\hat{CH}^1(L)_{\mathbb{Q}}$ (here the superscript $^{(1)}$ refers to part of degree 1 in $\hat{CH}^1(L)_{\mathbb{Q}}$).
We have the identity of cohomology classes
\[ R(\Omega^\vee) \operatorname{Td}(\Omega^\vee) \operatorname{ch}(-\Omega^1 + 2 \cdot \Omega^2 - 3 \cdot \Omega^3) = -R(\Omega^\vee)c_{\text{top}}(\Omega^\vee) [\zeta_Q(0) \operatorname{rk}(\Omega) + \text{terms of degree } > 0] \]
and so
\[ \int_X R(\Omega^\vee) \operatorname{Td}(\Omega^\vee) \operatorname{ch}(-\Omega^1 + 2 \cdot \Omega^2 - 3 \cdot \Omega^3) \big|^{(1)} = 0 \]
since \( R^1(\Omega^\vee) = 0 \) by assumption.

As to the left-hand side of equation (\( \text{(1)} \)), we have
\[
\begin{align*}
R^*f_* \left[ -\overline{1}^1 + 2 \cdot \overline{2}^1 - 3 \cdot \overline{3}^1 \right] &= -[ H^0(X, \overline{1}^1) - H^1(X, \overline{1}^1) + H^2(X, \overline{1}^1) ] + 2[ H^0(X, \overline{2}^2) - H^1(X, \overline{2}^2) + H^2(X, \overline{2}^2) ] \\
&- 3[ H^0(X, \overline{3}^3) - H^1(X, \overline{3}^3) + H^2(X, \overline{3}^3) ] \\
&= -[ -H^1(X, \overline{1}^1) + H^2(X, \overline{1}^1) ] + 2[ -H^1(X, \overline{2}^2) + H^2(X, \overline{2}^2) ] - 3[ H^0(X, \overline{3}^3) ] \\
&= -H^1(X, \overline{1}^1) - H^1(X, \overline{2}^2) - 3 \cdot H^0(X, \overline{3}^3)
\end{align*}
\]
Here we used the fact that Serre duality is compatible with \( L^2 \)-metrics (see [11, p. 27, after eq. 9]).

Putting everything together, we get
\[
-\hat{c}_1(H^1(X, \overline{1}^1)) - \hat{c}_1(H^1(X, \overline{2}^2)) - 3 \hat{c}_1(H^0(X, \overline{3}^3)) - \tau\left( -\overline{1}^1 + 2 \cdot \overline{2}^1 - 3 \cdot \overline{3}^1 \right)
\]
which implies that
\[
\begin{align*}
-\hat{c}_1(-H^1(X, \overline{1}^1)) - \hat{c}_1(H^1(X, \overline{2}^2)) - \tau\left( -\overline{1}^1 + 2 \cdot \overline{2}^1 - 3 \cdot \overline{3}^1 \right) &= -\zeta_Q(-1) \hat{c}_1(f_*\varpi) \int_X c_{\text{top}}(\Omega^\vee) + \zeta_Q(-1) \int_X c_{\text{top}}(\Omega^\vee) \eta^0 \\
&= \frac{1}{12} \hat{c}_1(f_*\varpi) \int_X c_{\text{top}}(\Omega^\vee) - \frac{1}{12} \int_X c_{\text{top}}(\Omega^\vee) \eta^0 - 3 \log \operatorname{Vol}(X(\mathbb{C}), \nu)
\end{align*}
\]
where
\[
\operatorname{Vol}(X(\mathbb{C}, \nu)) := \frac{1}{3!(2\pi)^3} \int_{X(\mathbb{C})} \nu^3.
\]
Let us write
\[ \chi(X) := \int_X c_{\text{top}}(\Omega^\vee). \]
Note that \( \chi(X) = \sum_{p,q} (-1)^{p+q} \dim L(H^q(X, \Omega^p)) \) by the generalized Gauss-Bonnet theorem. Notice also that \( f_*\omega = H^0(X, \Omega^3) \) (by definition). Since \( \zeta_Q(-1) = -1/12 \), we see that
\[
-\hat{c}_1(H^1(X, \Omega^2)) - \frac{1}{12} \chi(X) \hat{c}_1(H^0(X, \Omega^3)) \\
= -\frac{1}{12} \int_X c_{\text{top}}(\Omega^\vee) \eta^0 - 3 \log \text{Vol}(X, \nu) + \hat{c}_1(H^1(X, \Omega^1)) \\
+ \tau(-\Omega^1 + 2 \cdot \Omega^2 - 3 \cdot \Omega^3)
\]
in \( \hat{CH}^1(L) \).

The \( L^2 \)-metric on \( H^2(X(\mathbb{C}), \mathbb{C}) \) is induced from a Riemannian metric on the space \( H^2(X(\mathbb{C}), \mathbb{R}) \). This is a consequence of the formula \([17, \text{before Lemma 2.7}]\). Let \( \text{Vol}_{L^2}(H^2(X(\mathbb{C}), \mathbb{Z})) \) be the volume of a fundamental domain of the lattice \( H^2(X(\mathbb{C}), \mathbb{Z})_{\text{free}} \) in \( H^2(X(\mathbb{C}), \mathbb{R}) \) for that metric. Here \( H^2(X(\mathbb{C}), \mathbb{Z})_{\text{free}} \) is the largest direct summand of \( H^2(X(\mathbb{C}), \mathbb{Z}) \), which is a free \( \mathbb{Z} \)-module.

**Lemma 2.2.** The equality
\[
\hat{c}_1(H^1(X, \Omega^1)) = a \left( -\bigoplus_{\tau \in T} \log \text{Vol}_{L^2}(H^2(X, \tau(\mathbb{C}), \mathbb{Z})) \right)
\]
holds in \( \hat{CH}^1(L)_Q \).

**Proof.** Let \( \tau \in T \) and let \( e_1, \ldots, e_r \) be a basis of \( H^2(X, \tau(\mathbb{C}), \mathbb{Z})_{\text{free}} \). By definition, we have
\[
\text{Vol}_{L^2}(H^2(X, \tau(\mathbb{C}), \mathbb{Z})) = |e_1 \wedge \cdots \wedge e_r|^2
\]
where \( |\cdot| \) refers to the natural norm on \( \Lambda^r(H^2(X, \mathbb{C}), \mathbb{C}) \). Since \( H^2(X, \tau(\mathbb{C}), \mathbb{C}) \cong H^{1,1}(X(\mathbb{C})) \) by hypothesis, we may conclude from the Lefschetz theorem on \( (1,1) \)-classes that the elements \( e_i \) are classes of algebraic cycles \( e_i \) on \( X_\tau \). Let \( \kappa_0 : K \hookrightarrow \mathbb{C} \) be a field of definition for the \( e_i \), where \( \kappa_0 \) extends \( \tau \). We may assume that \( K \) is finite over \( L \) (see \([7, \text{proof of Prop. 1.5}]\)). Write \( e_i^K \) for the model of \( e_i \) in \( X_K \) and write \( cl_{\text{dR}} \) for the cycle class map with values
in de Rham cohomology. Let \( \tau : K \to \mathbb{C} \) be another embedding of \( K \) extending \( \tau \). Since by construction

\[
H^2(X_{K,\mathbb{C}}, \mathbb{C}) \cong H_{dR}^2(X_K/K) \otimes_\mathbb{C} \mathbb{C}
\]

we see that the elements \( \text{cl}_{dR}(e_1^K) \otimes_\mathbb{C} 1 \) form a basis of \( H^2(X_{K,\mathbb{C}}, \mathbb{C}) \). Furthermore, since \( \text{cl}_{dR}(e_1^K) \otimes_\mathbb{C} 1 = \text{cl}_{dR}(e_1^K \otimes_\mathbb{C} \mathbb{C}) \), we see that the elements \( \text{cl}_{dR}(e_1^K) \otimes_\mathbb{C} 1 \) even form a basis of \( H^2(X_{K,\mathbb{C}}, \mathbb{Z}) \) free. Furthermore, there is a natural identification

\[
H^2(X_{K,\mathbb{C}}, \mathbb{Z}) \cong H^2(X_{\tau}(\mathbb{C}), \mathbb{Z})
\]

which is an isometry for the \( L^2 \)-metrics.

Now let \( f : \text{Spec } K \to \text{Spec } L \) be the natural map. We view \( \text{Spec } K \) has an arithmetic variety over \( \text{Spec } L \). By the above, we have the equalities

\[
\hat{c}_1(H^1(X_K, \Omega^1_X)) = a(-2 \bigoplus_{\tau \in T} \log |\text{cl}(e_1^K) \otimes_\mathbb{C} 1 \wedge \cdots \wedge \text{cl}(e_r^K) \otimes_\mathbb{C} 1|^2)
\]

\[
= a(- \bigoplus_{\tau \in T} \log \text{Vol}_{L^2}(X_\tau(\mathbb{C}), \mathbb{Z}))
\]

and thus

\[
[K : L] \hat{c}_1(H^1(X, \Omega^1_X)) = f_* f^* \hat{c}_1(H^1(X, \Omega^1_X)) = f_* \hat{c}_1(H^1(X_K, \Omega^1_X))
\]

\[
= [K : L] a(- \bigoplus_{\tau \in T} \log \text{Vol}_{L^2}(X_\tau(\mathbb{C}), \mathbb{Z}))
\]

and we can conclude. Q.E.D.

The previous calculations motivate the following definition:

**Definition 2.3.**

\[
\tau_{BCOV}(X(\mathbb{C})) := \exp\left[ -\frac{1}{12} \int_{X(\mathbb{C})} e^{\text{top}}(\Omega^2_{X(\mathbb{C})}) \eta^0 - 3 \log \text{Vol}(X(\mathbb{C}), \nu) - \log(\text{Vol}_{L^2}(H^2(X(\mathbb{C}), \mathbb{Z}))) \right.
\]

\[
- \left. \tau(\Omega^1_{X(\mathbb{C})}) + 2 \cdot \tau(\Omega^2_{X(\mathbb{C})}) - 3 \cdot \tau(\Omega^3_{X(\mathbb{C})}) \right]
\]
It is proven in [8, Sec. 4.4] that $\tau_{\text{BCOV}}(X)$ does not depend on the choice of $\nu$. Notice that equation (2) together with the formula [17, before Lemma 2.7] already implies the weaker statement that $a(\tau_{\text{BCOV}}(X))$ does not depend on $\nu$.

The following equation summarizes the calculations made in this section:

$$\log(\tau_{\text{BCOV}}(X(\mathbb{C}))) = -\hat{c}_1(H^1(X, \Omega^2)) - \frac{1}{12} \chi(X) \hat{c}_1(H^0(X, \Omega^3)) \quad \text{in } \hat{CH}^1(L)_{\mathbb{Q}} \quad (3)$$

3 Proof of Theorem 1.1

With the equation (3) in hand, we see that Theorem 1.1 is equivalent to the equation

$$-\hat{c}_1(H^1(X_L', \Omega^2)) - \frac{1}{12} \chi(X_L') \hat{c}_1(H^0(X_L', \Omega^3)) = -\hat{c}_1(H^1(X', \Omega^2)) - \frac{1}{12} \chi(X') \hat{c}_1(H^0(X', \Omega^3)) \quad (4)$$

Lemma 3.1. Let $L'$ be a finite field extension of $L$. We view $\text{Spec } L'$ as an arithmetic variety over $L$. With this convention, the equation

$$-\hat{c}_1(H^1(X_{L'}, \Omega^2)) - \frac{1}{12} \chi(X_{L'}) \hat{c}_1(H^0(X_{L'}, \Omega^3)) = -\hat{c}_1(H^1(X_{L}', \Omega^2)) - \frac{1}{12} \chi(X_{L}') \hat{c}_1(H^0(X_{L}', \Omega^3))$$

in $\hat{CH}^1(L')_{\mathbb{Q}}$ is equivalent to the equation (4).

Proof. Let $f : \text{Spec } L' \to \text{Spec } L$ be the natural morphism. Using the projection formula, we compute

$$[L' : L] \hat{c}_1(H^1(X, \Omega^2)) = f_* f^* \hat{c}_1(H^1(X, \Omega^2)) = f_* \hat{c}_1(H^1(X_{L'}, \Omega^2))$$

and similarly

$$[L' : L] \hat{c}_1(H^0(X, \Omega^3)) = f_* f^* \hat{c}_1(H^0(X, \Omega^3)) = f_* \hat{c}_1(H^0(X_{L'}, \Omega^3))$$

If we combine these formulae with the analogous formulae for $X'$, we may conclude. Q.E.D.

Now notice that the group $\hat{CH}^1(L')_{\mathbb{Q}}$ (where $L'$ is viewed as an arithmetic variety over $L$) is naturally isomorphic to the homonymous group $\hat{CH}^1(L'_{\mathbb{Q}}) := \hat{CH}^1(L', T'_{\mathbb{Q}})$, which is the first arithmetic Grothendieck group of the arithmetic ring $L'$, endowed with the set $T' := \{ \tau' : L' \hookrightarrow \mathbb{C} | \tau' \in T \}$.
of embeddings into \( \mathbb{C} \). Thus Lemma 3.1 implies that the truth value of Theorem 1.1 remains unchanged if we replace \( L \) by a finite extension field \( L' \) and \( T \) by the set \( T' := \{ \tau' : L' \hookrightarrow \mathbb{C} \mid \tau' \in T \} \).

Before we begin with the proof, notice that by the formula [17, before Lemma 2.7], the \( L^2 \)-metric on \( H^1(X, \Omega^2) \) is given by the formula

\[
\langle \lambda, \kappa \rangle_{L^2} = \frac{i}{(2\pi)^3} \int_{X(\mathbb{C})} \lambda \wedge \overline{\kappa} \quad (5)
\]

and the \( L^2 \)-metric on \( H^0(X, \Omega^3) \) is given by the formula

\[
\langle \lambda, \kappa \rangle_{L^2} = -\frac{i}{(2\pi)^3} \int_{X(\mathbb{C})} \lambda \wedge \overline{\kappa} \quad (6)
\]

In particular, these metrics do not depend on the choice of the Kähler form \( \nu \).

### 3.1 Proof of (A)

We now assume that there is a birational transformation from \( X_\sigma \) to \( X'_\sigma \).

**Lemma 3.2.** There is a birational transformation from \( X_L \) to \( X'_{L'} \).

**Proof.** This can be proven using a “spreading out” argument. We leave the details to the reader. Q.E.D.

Notice that the birational transformation provided by the last Lemma has a model over a finite extension of \( L \). Hence, by the discussion following Lemma 3.1, we may assume without loss of generality that there is a birational transformation from \( X \) to \( X' \) defined over \( L \).

**Lemma 3.3.**

\[
\hat{c}_1(H^0(X, \Omega^3)) = \hat{c}_1(H^0(X', \Omega^3))
\]

**Proof.** Let \( \phi \) be a birational transformation from \( X \) to \( X' \). It is shown in [13, Proof of Th. 8.19, chap. II] that there is an open set \( U \subseteq X \) and a morphism \( f : X \to X' \), with the following
properties: $f$ induces $\varphi$ and codimension $(U) \leq 2$. It is also shown in [13] Proof of Th. 8.19, chap. II] that the maps

$$H^0(X', \Omega^3) \xrightarrow{f^*} H^0(U, \Omega^3) \xleftarrow{\text{restriction to } U} H^0(X, \Omega^3)$$

are bijective. Thus, using the formula (6), we compute that

$$\hat{c}_1(H^0(X', \Omega^3)) = -\log |\int_{X'(\mathbb{C})} \lambda \wedge \overline{\lambda}| = -\log |\int_{X(\mathbb{C})} f^*(\lambda) \wedge \overline{f^*(\lambda)}| = \hat{c}_1(H^0(X, \Omega^3))$$

Here $\lambda \in H^0(X, \Omega^3)$ is any non-zero element. Q.E.D.

We recall the following theorem of Manin (and others).

**Theorem 3.4.** Let $Y$ be a smooth projective variety over $\mathbb{C}$. Let $Z \hookrightarrow Y$ be a smooth closed subvariety of codimension $c$ of $Y$. Let $\phi: \tilde{Y} := \text{Bl}_Z(Y) \to Y$ be the blow-up of $Y$ along $Z$. Let $e : E \to \tilde{Y}$ be the immersion of the exceptional divisor and let $\pi : E \to Z$ be the natural morphism. Let $\mathcal{O}(1)$ be the tautological vector bundle on $E$. For any $k \in \mathbb{N}$, there is an isomorphism of $\mathbb{Q}$-Hodge structures

$$H^k(Y, \mathbb{Q}) \bigoplus_{l \geq 0} \mathbb{Q}^{c-2} H^{k-2l}(Z(\mathbb{C}), \mathbb{Q})(-l - 1) \xrightarrow{\sim} H^k(\tilde{Y}, \mathbb{Q})$$

given by the formula

$$(\eta, \kappa_1, \ldots, \kappa_{c-1}) \mapsto (\phi^* \eta, e_* [\pi^*(\kappa_0) + \pi^*(\kappa_1) \cdot c_1(\mathcal{O}(1)) + \pi^*(\kappa_2) \cdot c_1(\mathcal{O}(1))^2 + \cdots + \pi^*(\kappa_{c-2}) \cdot c_1(\mathcal{O}(1))^{c-2}])$$

**Proof.** See [18]. Q.E.D.

**Lemma 3.5.** Let $C$ be a non-singular curve of genus $g$ over $L$. Then

$$\hat{c}_1(H^0(\text{Jac}(C), \Omega^1)) = \hat{c}_1(H^0(C, \Omega^1)) + (g - 1) \log(2\pi)$$

in $\hat{CH}^1(L)$, for any Kähler metrics on $C(\mathbb{C})$ and $\text{Jac}(C)(\mathbb{C})$.

**Proof.** See [21], Exp. I, Lemme 3.2.1]. Q.E.D.

**Proposition 3.6.** Let $Y$ be a smooth projective threefold over $L$. Let $Z \hookrightarrow Y$ be a smooth closed subcurve of genus $g$ of $Y$. Let $\phi: \tilde{Y} := \text{Bl}_Z(Y) \to Y$ be the blow-up of $Y$ along $Z$. Then

$$\hat{c}_1(H^1(\tilde{Y}, \Omega^2)) = \hat{c}_1(H^1(Y, \Omega^2)) + \hat{c}_1(H^0(Z, \Omega^1)) + 2g \log(2\pi)$$

for any Kähler metrics on $Y, Z$ and $\tilde{Y}$. 
Proof. Let $e : E \hookrightarrow \tilde{Y}$ be the immersion of the exceptional divisor. Let $\pi : E \to Z$ be the natural morphism. By Theorem 3.4, the map

$$H^1(Y, \Omega^2) \oplus H^0(Z, \Omega) \to H^1(\tilde{Y}, \Omega^2)$$

given by the formula

$$(\eta, \kappa) \mapsto \phi^* (\eta) + e_*(\pi^*(\kappa))$$

is an isomorphism. We compute

$$\frac{i}{(2\pi)^3} \int_{\tilde{Y}(\mathbb{C})} \left( \phi^* (\eta_1) + e_*(\pi^*(\kappa_1)) \right) \wedge \left( \phi^* (\eta_2) + e_*(\pi^*(\kappa_2)) \right) =$$

$$= \frac{i}{(2\pi)^3} \int_{\tilde{Y}(\mathbb{C})} \phi^* (\eta_1) \wedge \phi^* (\eta_2) + \frac{i}{(2\pi)^3} \int_{\tilde{Y}(\mathbb{C})} e_*(\pi^* (\eta_1) \wedge \kappa_2) +$$

$$+ \frac{i}{(2\pi)^3} \int_{\tilde{Y}(\mathbb{C})} e_*(\pi^* (\eta_2) \wedge \kappa_1) + \frac{i}{(2\pi)^3} \int_{\tilde{Y}(\mathbb{C})} e_*(\pi^* (\kappa_1)) \wedge e_*(\pi^* (\kappa_2)) =$$

$$= \frac{i}{(2\pi)^3} \int_{\tilde{Y}(\mathbb{C})} \phi^* (\eta_1) \wedge \phi^* (\eta_2) + \frac{i}{(2\pi)^3} \int_{\tilde{Y}(\mathbb{C})} e_*(\pi^* (\kappa_1)) \wedge e_*(\pi^* (\kappa_2))$$

Now using the self-intersection formula (see for instance [9 VI, 1., 1.4.2]), we may compute

$$\frac{i}{(2\pi)^3} \int_{\tilde{Y}(\mathbb{C})} e_*(\pi^* (\kappa_1)) \wedge e_*(\pi^* (\kappa_2)) = \frac{i}{(2\pi)^3} \int_{E(\mathbb{C})} e_*(\pi^* (\kappa_1) \wedge e^* e_*(\pi^* (\kappa_2))) =$$

$$= \frac{i}{(2\pi)^3} \int_{E(\mathbb{C})} e_*(c_1(\mathcal{O}_E(-1))) \wedge \pi^* (\kappa_1) \wedge \pi^* (\kappa_2)) =$$

$$= - \frac{i}{(2\pi)^3} \int_{\mathbb{Z}(\mathbb{C})} \kappa_1 \wedge \kappa_2 = \frac{1}{(2\pi)^3} \cdot -i \int_{\mathbb{Z}(\mathbb{C})} \kappa_1 \wedge \kappa_2$$

These formulae imply the conclusion of the proposition. Q.E.D.

Lemma 3.7. Let $A$ and $B$ be abelian varieties over $\overline{L}$ and let $ϕ : A_σ → B_σ$ be an isogeny (over $\mathbb{C}$). Then there is an isogeny $A → B$ (over $\overline{L}$).

Proof. By spreading out. Left to the reader. Q.E.D.

Lemma 3.8. Let $A$ and $B$ be two abelian varieties over $L$ and suppose that there exists an isogeny $ϕ : A → B$ (over $L$). Suppose that $L$ contains a square root of $\deg(ϕ)$. Then

$$\hat{c}_1(H^0(A, \Omega^0)) = \hat{c}_1(H^0(B, \Omega^0))$$

in $\hat{CH}^1(L)$, for any choice of Kähler metrics on $A$ and $B$. 
Symmetrically, there are curves an isometry of hermitian vector bundles. Using the formula [17 before Lemma 2.7], we see that for any embedding \( \tau \in T \), we have

\[
\langle \alpha_1 \wedge \alpha_2 \wedge \cdots \wedge \alpha_g, \alpha_1 \wedge \alpha_2 \wedge \cdots \wedge \alpha_g \rangle_{L^2} = \int_{B(C)} ((\alpha_1 \wedge \alpha_2 \wedge \cdots \wedge \alpha_g) \otimes \tau 1) \wedge ((\alpha_1 \wedge \alpha_2 \wedge \cdots \wedge \alpha_g) \otimes \tau 1) \]

\[
= \deg(\phi)^{-1} \int_{A(C)} ((\phi^*(\alpha_1) \wedge \phi^*(\alpha_2) \wedge \cdots \wedge \phi^*(\alpha_g)) \otimes \tau 1) \wedge ((\phi^*(\alpha_1) \wedge \phi^*(\alpha_2) \wedge \cdots \wedge \phi^*(\alpha_g)) \otimes \tau 1) \]

and thus the mapping \( H^0(B, \Omega^g) \to H^0(A, \Omega^g) \) given by the formula \( \eta \mapsto (\sqrt{\deg(\phi)}) \cdot \phi^* \) is an isometry of hermitian vector bundles. \( \text{Q.E.D.} \)

Let now \( \phi : X \dasharrow X' \) be a birational transformation. Let \( X'' \) be another smooth projective variety over \( L \), together with morphisms \( f : X'' \to X \) and \( g : X'' \to X' \) such that \( \phi \circ f \) and \( g \) coincide as birational transformations. The variety \( X'' \) can be obtained as a desingularisation of the Zariski closure of the graph of \( \phi \) in \( X \times X' \).

Denote by \( \mathcal{P}HS(\mathbb{Q}) \) the category of (pure) polarisable \( \mathbb{Q} \)-Hodge structures.

Using weak factorisation of birational maps (see [11] and Proposition 3.6) and possibly replacing \( L \) by one of its finite extensions, we conclude that there are curves \( C_1, \ldots, C_{r'} \) over \( L \) and numbers \( s_i' \in \{-1, 1\} \) so that

\[
H^3(X_{\sigma}(\mathbb{C}), \mathbb{Q}) + \sum_{l=1}^{r'} (-1)^{s_i'} H^1(C_{l,\sigma}, \mathbb{Q})(-1) = H^3(X''(\mathbb{C}), \mathbb{Q})
\]

in \( K_0(\mathcal{P}HS(\mathbb{Q})) \) and so that

\[
\hat{c}_1(H^1(X, \Omega^2)) + \sum_{l=1}^{r'} (-1)^{s_i'} \hat{c}_1(H^0(C_{l}', \Omega^1)) + 2 \sum_{l=1}^{r'} (-1)^{s_i'} \text{genus}(C_{l}') \log(2\pi) = \hat{c}_1(H^1(X'', \Omega^2)).
\]

Symmetrically, there are curves \( C_{1}', \ldots, C_{r''}, \) over \( L \) and numbers \( s_i'' \in \{-1, 1\} \) so that

\[
H^3(X'_{\sigma}(\mathbb{C}), \mathbb{Q}) + \sum_{l=1}^{r''} (-1)^{s_i''} H^1(C_{l,\sigma}, \mathbb{Q})(-1) = H^3(X''(\mathbb{C}), \mathbb{Q})
\]

in \( K_0(\mathcal{P}HS(\mathbb{Q})) \) and so that

\[
\hat{c}_1(H^1(X', \Omega^2)) + \sum_{l=1}^{r''} (-1)^{s_i''} \hat{c}_1(H^0(C_{l}'', \Omega^1)) + 2 \sum_{l=1}^{r''} (-1)^{s_i''} \text{genus}(C_{l}'') \log(2\pi) = \hat{c}_1(H^1(X'', \Omega^2)).
\]
Now by a theorem of Kontsevich (proved using motivic integration; see [16]) there is an isomorphism of $\mathbb{Q}$-Hodge structures $H^3(X_\sigma(C), \mathbb{Q}) \simeq H^3(X'_\sigma(C), \mathbb{Q})$. Thus

$$2 \sum_{l=1}^{r'} (-1)^{s'_l} \text{genus}(C'_l) \log(2\pi) = 2 \sum_{l=1}^{r''} (-1)^{s''_l} \text{genus}(C''_l) \log(2\pi).$$

Furthermore, since the category of polarisable $\mathbb{Q}$-Hodge structures is semi-simple, there exists an isomorphism of $\mathbb{Q}$-Hodge structures

$$\bigoplus_{l,s'_l=1} H^1(\text{Jac}(C'_l), \mathbb{Q}) \bigoplus \bigoplus_{l,s''_l=1} H^1(\text{Jac}(C''_l), \mathbb{Q}) \rightarrow \bigoplus_{l,s'_l=1} H^1(\text{Jac}(C'_l), \mathbb{Q}) \bigoplus \bigoplus_{l,s''_l=1} H^1(\text{Jac}(C''_l), \mathbb{Q})$$

and thus an $L$-isogeny of abelian varieties

$$\prod_{l,s'_l=1} \text{Jac}(C'_l) \prod_{l,s''_l=1} \text{Jac}(C''_l) \rightarrow \prod_{l,s'_l=1} \text{Jac}(C'_l) \prod_{l,s''_l=1} \text{Jac}(C''_l)$$

Here we used Lemma 3.7. Extend $L$ further so that the latter isogeny is defined over $L$. Then, by Lemma 3.8, we have

$$\sum_{l} (-1)^{s'_l} \hat{c}_1(H^0(\text{Jac}(C'_l), \Omega^{\dim \text{Jac}(C'_l)})) = \sum_{l} (-1)^{s''_l} \hat{c}_1(H^0(\text{Jac}(C''_l), \Omega^{\dim \text{Jac}(C''_l)}))$$

in $\hat{CH}^1(L, \mathbb{Q})$. Using Lemma 3.5 we deduce that

$$\sum_{l} (-1)^{s'_l} \hat{c}_1(H^0(C'_l, \Omega^0)) = \sum_{l} (-1)^{s''_l} \hat{c}_1(H^0(C''_l, \Omega^0)).$$

so that

$$\hat{c}_1(H^1(X, \Omega^0)) = \hat{c}_1(H^1(X', \Omega^0)).$$

Furthermore, by Lemma 3.3 we have

$$\hat{c}_1(H^0(X, \Omega^3)) = \hat{c}_1(H^0(X', \Omega^3))$$

and by a theorem of Kontsevich (see [16]) we have $\chi(X) = \chi(X')$. This implies that

$$-\hat{c}_1(H^1(X, \Omega^2)) - \frac{1}{12} \chi(X) \hat{c}_1(H^0(X, \Omega^3)) = -\hat{c}_1(H^1(X', \Omega^2)) - \frac{1}{12} \chi(X') \hat{c}_1(H^0(X', \Omega^3)).$$

Thus the equation (4) is verified and the theorem is proved.
3.2 Proof of (B)

We now assume that the categories $D^b(X_\sigma)$ and $D^b(X'_\sigma)$ are equivalent as triangulated $\mathbb{C}$-linear categories.

As a matter of notation, if $X_1 \times X_2 \times \cdots \times X_t$ is a cartesian product of varieties, we shall write

$$
\pi_{X_1X_2\cdots X_t} : X_1 \times X_2 \times \cdots \times X_t \to X_{i_1} \times X_{i_2} \times \cdots \times X_{i_j}
$$

for the natural projection.

If $M$ (resp. $M'$) is an object in $D^b(X_\sigma)$ (resp. in $D^b(X'_\sigma)$), let $F_M$ (resp. $F_{M'}$) be the functor $D^b(X_\sigma) \to D^b(X'_\sigma)$ (resp. $D^b(X'_\sigma) \to D^b(X_\sigma)$) defined by the formula

$$
F_M(\cdot) = R^\bullet \pi_{X_\sigma X'_\sigma}^* (\pi_{X_\sigma X'_\sigma}^* \pi_{X_\sigma X'_\sigma} \pi_{X_\sigma X'_\sigma} \pi_{X_\sigma X'_\sigma} \pi_{X_\sigma X'_\sigma} (\cdot))
$$

(resp.

$$
F_{M'}(\cdot) = R^\bullet \pi_{X_\sigma X'_\sigma}^* (\pi_{X_\sigma X'_\sigma}^* \pi_{X_\sigma X'_\sigma} \pi_{X_\sigma X'_\sigma} \pi_{X_\sigma X'_\sigma} \pi_{X_\sigma X'_\sigma} (\cdot))
$$

). The symbol $\otimes$ refers to the derived tensor product and $R^\bullet f_*$ refers to the functor derived from the direct image functor.

We shall make use of the following theorems.

**Theorem 3.9 (Orlov).** There exists an object $M$ (resp. $M'$) in $D^b(X_\sigma \times X'_\sigma)$ with the following properties.

(a) The object

$$
R^\bullet \pi_{X_\sigma X'_\sigma}^* (\pi_{X_\sigma X'_\sigma}^* \pi_{X_\sigma X'_\sigma} \pi_{X_\sigma X'_\sigma} \pi_{X_\sigma X'_\sigma} \pi_{X_\sigma X'_\sigma} (M)) \otimes \pi_{X_\sigma X'_\sigma} \pi_{X_\sigma X'_\sigma} (M')
$$

is isomorphic in $D^b(X_\sigma \times X_\sigma)$ to the image of the diagonal morphism in $X_\sigma \times X_\sigma$.

(b) The object

$$
R^\bullet \pi_{X_\sigma X'_\sigma}^* (\pi_{X_\sigma X'_\sigma}^* \pi_{X_\sigma X'_\sigma} \pi_{X_\sigma X'_\sigma} \pi_{X_\sigma X'_\sigma} \pi_{X_\sigma X'_\sigma} (M)) \otimes \pi_{X_\sigma X'_\sigma} \pi_{X_\sigma X'_\sigma} (M')
$$

is isomorphic in $D^b(X'_\sigma \times X'_\sigma)$ to the image of the diagonal morphism in $X'_\sigma \times X'_\sigma$. 
Proof. See [19]. Q.E.D.

The last theorem is actually valid more generally if $X_\sigma$ (resp. $X'_\sigma$) is replaced by any smooth quasi-projective scheme over $\mathbb{C}$ and if one assumes that $D^b(X_\sigma)$ and $D^b(X'_\sigma)$ are equivalent as triangulated $\mathbb{C}$-linear categories.

Write $\pi : X_\sigma \times X'_\sigma \to X_\sigma$ for the first projection and $\pi' : X_\sigma \times X'_\sigma \to X'_\sigma$ for the second projection.

**Theorem 3.10** (Caldararu). Let $M$ and $M'$ be objects satisfying the conditions (a) and (b) in Theorem 3.9 then the map

$$\Phi^H_M : H^\bullet(X_\sigma(\mathbb{C}), \mathbb{Q}) \to H^\bullet(X'_\sigma(\mathbb{C}), \mathbb{Q})$$

given by the formula

$$\Phi^H_M(\beta) := \pi'_*(\pi^*(\beta) \cdot \sqrt{\text{Td}(X_\sigma \times X'_\sigma) \cdot \text{ch}(M)})$$

is an isomorphism and for any $k \in \mathbb{N}$ we have

$$\Phi^H_M(\bigoplus_{p-q=k} H^{p,q}(X_\sigma(\mathbb{C}))) = \bigoplus_{p-q=k} H^{p,q}(X'_\sigma(\mathbb{C}))$$

and furthermore, for any $\beta, \lambda \in H^3(X_\sigma(\mathbb{C}), \mathbb{Q})$, we have

$$\int_{X(\mathbb{C})} \beta \wedge \lambda = \int_{X'(\mathbb{C})} \Phi^H_M(\beta) \wedge \Phi^H_M(\lambda)$$

Proof. See [6] or [15, 5.2]. Q.E.D.

The last theorem is actually valid more generally if $X_\sigma$ (resp. $X'_\sigma$) is replaced by any smooth projective scheme of dimension 3 over $\mathbb{C}$.

Notice that Theorem 3.10 implies that if its hypotheses are satisfied, then

$$\Phi^H_M(H^{2,1}(X_\sigma(\mathbb{C}))) = H^{2,1}(X'_\sigma(\mathbb{C}))$$

and

$$\Phi^H_M(H^{3,0}(X_\sigma(\mathbb{C}))) = H^{3,0}(X'_\sigma(\mathbb{C})).$$

Here we have used the fact that $X$ and $X'$ are Calabi-Yau varieties in the restricted sense.
Lemma 3.11. There exists a finite field extension $K$ of $L$ and an object $M_0$ (resp. $M'_0$) of $D^b(X_K \times X'_K)$ (resp. $D^b(X'_K \times X_K)$) such that

\begin{itemize}
  \item [(a)] The object
  \[
  R^\bullet \pi^*_{X_K \times X_K} (\pi_{X_K \times X_K}^* \otimes \pi_{X_K \times X_K}^* (M))
  \]
  is isomorphic in $D^b(X_K \times X_K)$ to the image of the diagonal morphism in $X_K \times X_K$.

  \item [(b)] The object
  \[
  R^\bullet \pi^*_{X'_K \times X'_K} (\pi_{X'_K \times X'_K}^* \otimes \pi_{X'_K \times X'_K}^* (M))
  \]
  is isomorphic in $D^b(X'_K \times X'_K)$ to the image of the diagonal morphism in $X'_K \times X'_K$.
\end{itemize}

Proof. Let $\Delta : X_\sigma \hookrightarrow X_\sigma \times X_\sigma$ (resp. $\Delta' : X_\sigma \hookrightarrow X'_\sigma \times X'_\sigma$) be the diagonal morphism. Let $U$ be a bounded complex of locally free sheaves on $X_\sigma \times X'_\sigma$ representing $M$ and let $U'$ be a bounded complex of locally sheaves on $X'_\sigma \times X_\sigma$ representing $M'$. Let $L_1$ be a finitely generated extension of $L$ (as a field), such that $U$ (resp. $U'$) has a model over $X_{L_1} \times X_{L_1}$ (resp. $X'_{L_1} \times X'_{L_1}$). Let $S$ be an affine variety over $L$, which is smooth and irreducible and whose function field is isomorphic to $L_1$ as an $L$-algebra. After possibly replacing $S$ by one of its open affine subsets, we may find bounded complexes of locally free sheaves $\tilde{U}$ (resp. $\tilde{U}'$) on $X_S \times X_S$ (resp. $X'_S \times X'_S$), which are models of $U$ and $U'$.

The conditions (a) and (b) in Theorem 3.9 are equivalent to the conditions:

\begin{itemize}
  \item There are isomorphisms of coherent sheaves
    \[
    R^0 \pi^*_{X_\sigma \times X'_\sigma} (\pi_{X_\sigma \times X'_\sigma}^* (M)) \simeq \Delta_* O_{X_\sigma}
    \]
    and
    \[
    R^i \pi^*_{X_\sigma \times X'_\sigma} (\pi_{X_\sigma \times X'_\sigma}^* (M)) \otimes \pi_{X_\sigma \times X'_\sigma}^* (M')) \simeq 0
    \]
    for all $i \neq 0$;
  \item there are isomorphisms of coherent sheaves
    \[
    R^0 \pi^*_{X'_\sigma \times X'_\sigma} (\pi_{X'_\sigma \times X'_\sigma}^* (M')) \simeq \Delta_* O_{X'_\sigma}
    \]
\end{itemize}
and
\[ \pi_* X'_s X_s^* \left( \pi_* X'_s X_s^* (M') \otimes \pi_* X'_s X_s^* (M) \right) \simeq 0 \]
for all \( i \neq 0 \).

Thus, after possibly a further reduction of the size of \( S \), we may assume that

- there are isomorphisms of coherent sheaves
  \[ R^0 \pi_* X'_s X_s^* \left( \pi_* X'_s X_s^* \left( \tilde{U} \right) \otimes X'_s X_s^* \left( \tilde{U}' \right) \right) \simeq \Delta_* O_X S \]
  and
  \[ R^i \pi_* X'_s X_s^* \left( \pi_* X'_s X_s^* \left( \tilde{U} \right) \otimes X'_s X_s^* \left( \tilde{U}' \right) \right) \simeq 0 \]
  for all \( i \neq 0 \); and
- there are isomorphisms of coherent sheaves
  \[ R^0 \pi_* X'_s X_s^* \left( \pi_* X'_s X_s^* \left( \tilde{U} \right) \otimes X'_s X_s^* \left( \tilde{U}' \right) \right) \simeq \Delta_* O_X' S \]
  and
  \[ R^i \pi_* X'_s X_s^* \left( \pi_* X'_s X_s^* \left( \tilde{U} \right) \otimes X'_s X_s^* \left( \tilde{U}' \right) \right) \simeq 0 \]
  for all \( i \neq 0 \).

To see this, use the fact that the elements of the complexes \( \tilde{U} \) and \( \tilde{U}' \) are locally free and apply the theorem on cohomology and base-change (see [12, chap. III, 7.7.4]).

Now pick a closed point \( s \in S \). The field \( K := \kappa(s) \) has all the properties we are looking for. Q.E.D.

Now replace \( L \) by a finite extension \( K \) satisfying the conclusion of Lemma 3.11. Replace \( T \) by the set \( T_K \) of embeddings of \( K \) into \( C \) lying above embeddings in \( T \). Recall that by Lemma 3.1 this does not restrict generality.

**Proposition 3.12.** There are isometries of hermitian vector bundles
\[ H^1 \left( X, \bar{\Omega}'^2 \right) \simeq H^1 \left( X', \bar{\Omega}'^2 \right) \]
and
\[ H^0(X, \Omega^3) \simeq H^0(X', \Omega^3) \]

**Proof.** Let \( M_0, M'_0 \) be as provided by Lemma 3.11. Set \( M_\sigma := M_0 \otimes_\sigma \mathbb{C} \) and \( M'_\sigma := M'_0 \otimes_\sigma \mathbb{C} \). Since \( \mathbb{C} \) is flat as an \( L \)-algebra via \( \sigma \), we see that \( M_\sigma \) and \( M'_\sigma \) satisfy properties (a) and (b) in Theorem 3.9.

Furthermore, there are comparison isomorphisms
\[ H^3(X_\sigma(\mathbb{C}), \mathbb{C}) \simeq \bigoplus_{p+q=3} H^q(X, \Omega^p) \otimes_\sigma \mathbb{C} \]
and
\[ H^3(X'_\sigma(\mathbb{C}), \mathbb{C}) \simeq \bigoplus_{p+q=3} H^q(X, \Omega^p) \otimes_\sigma \mathbb{C} \]
compatible with pull-backs, push-forwards and formation of Chern classes. We may thus conclude from Theorem 3.10 that the morphism \( \bigoplus_{p+q=3} H^q(X, \Omega^p) \to \bigoplus_{p+q=3} H^q(X, \Omega^p) \)
given by the formula in Hodge cohomology
\[ \Phi_{Hdg}^M(\beta) := \pi'_\tau^* (\pi^* (\beta) \cdot \sqrt{\text{Td}(X \times X')} \cdot \text{ch}(M_0))) \]
is an isomorphism. Therefore, again by Theorem 3.10 (see remark after the theorem), the maps
\[ \Phi_{Hdg}^M \otimes_\tau \mathbb{C}|_{H^1(X, \Omega^2)} : H^1(X_\tau, \Omega^2) \to H^1(X'_\tau, \Omega^2) \]
and
\[ \Phi_{Hdg}^M \otimes_\tau \mathbb{C}|_{H^0(X, \Omega^3)} : H^0(X_\tau, \Omega^3) \to H^0(X'_\tau, \Omega^3) \]
are isometries for any \( \tau \in T \). This implies the result. Q.E.D.

We can now conclude the proof the Theorem 1.1. Indeed, by Proposition 3.12 we have
\[ \hat{c}_1(H^1(X, \Omega^2)) = \hat{c}_1(H^1(X', \Omega^2)) \]
and
\[ \hat{c}_1(H^0(X, \Omega^3)) = \hat{c}_1(H^0(X', \Omega^3)) \]
in \( \hat{CH}^1(L) \). We conclude using equation (2).
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