On the entanglement of degenerated ground state for spin 1 and 1/2 pair

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We study the entanglement feature of the ground state of a system composed of spin 1 and 1/2 parts. The concurrence vector is shown to be consistent with the measurement of von Neumann entropy for such system. In the light of the ground state degeneracy, we suggest an *average concurrence* to measure the entanglement of Hilbert subspace. The entanglement property of both a general superposition as well as the mixture of the degenerated ground states are discussed by means of average concurrence and the negativity respectively.

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I. INTRODUCTION

Entanglement is a fundamental concept in the theory of quantum information, and the essential resource for modern applications of quantum mechanics. In recent years, much attention have been paid to the study of the entanglement of quantum systems either qualitatively or quantitatively. Entanglement possesses some resemblance to classical correlation, but it differs in some key aspects, e.g., the entangled objects can violate Bell’s inequality\[1, 2, 3\]. An useful measure of entanglement key aspects, e.g., the entangled objects can violate Bell’s inequality\[1, 2, 3\]. An useful measure of entanglement

It is known that the spin model is an important model in condensed matter physics and statistical mechanics. Since the 1980s, much attention has been paid to the study of the entanglement feature of the ground state of a physical model. However, the entanglement properties of the ground state of a physical model has not been well defined. Particularly, the ground state maybe degenerated, which request us to measure the entanglement of a Hilbert subspace. In present paper, we study the entanglement feature of the ground state for the systems composed of spin-1/2 and spin-1 parts. In Sec. II we show that the concurrence vector is a reliable measurement of entanglement for spin-1 and spin-1/2 system. In Sec. III we propose a concept, *average concurrence* to measure the entanglement of Hilbert subspace. On the basis of this concept, we discuss the entanglement property of that system in Sec. IV for the general superposition of the degenerate ground states. In Sec. V the general mixture of the degenerate ground states are discussed in terms of Negativity\[15\]. A brief summary with discussion is given in the last section.

II. CONCURRENCE VECTOR AND ITS RELIABILITY

We recently extended the concurrence originally proposed by Hill and Wootters to a concurrence vector\[14\]

\[ C = \{ (\psi | (E_\alpha - E_{-\alpha}) \otimes (E_\beta - E_{-\beta}) | \psi^* ) | \alpha, \beta \in \Delta^+ \} \]

where $\Delta^+$ denotes the set of positive roots of $A_{N-1}$ Lie algebra. The above concurrence vector can be used to
measure the entanglement of high-dimensional systems.

For a pair of qubit and qutrit which can be regarded as spin-1/2 and spin-1, the concurrence vector is a three dimensional vector given by

$$\mathbf{C} = \{(\psi) | (\sigma_x - \sigma_z) \otimes (E_a - E_{-a}) | \psi^* \} | \alpha \in \Delta^+ \} \quad (1)$$

where $$\sigma_x = (\sigma_x + i\sigma_y)/2$$, and $$\Delta^+$$ for $$A_2$$ contains three positive roots. Thus the concurrence vector here is of three dimension.

In order to show the reliability of concurrence vector, we consider the von Neumann entropy of the system consisting of spin-1/2 and spin-1 parts. As we known, any state of bipartite system can be expanded as

$$|\psi\rangle = \sum_{\mu,j} a_{\mu j} |\mu\rangle \otimes |j\rangle, \quad (2)$$

where $$a_{\mu j}$$ is complex coefficients, and in our present case, $$\mu = 1, 2$$ and $$j = 1, 2, 3$$. The reduced density matrix $$\rho_A$$ and $$\rho_B$$ can be easily obtained,

$$\rho_A = \rho_{a a^+} = \left( \begin{array}{ccc} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{array} \right).$$

It is a 2 \times 2 matrix, thus there are two eigenvalues $$\kappa_1^2$$ and $$\kappa_2^2$$, that are squares of the coefficients of Schmidt decomposition $$|\psi\rangle = \kappa_1 |x_1\rangle_A |y_1\rangle_B + \kappa_2 |x_2\rangle_A |y_2\rangle_B$$. Here the $$\kappa_1^2$$ and $$\kappa_2^2$$ are the roots of the following secular equation

$$\lambda^2 - \lambda + |C|^2/4 = 0, \quad (3)$$

where $$|C|$$ is precisely the norm of concurrence vector we proposed, namely, $$|C|^2 = C_1^2 + C_2^2 + C_3^2 = 4(a_{11}a_{22} - a_{12}a_{21})^2 + 4(a_{12}a_{23} - a_{13}a_{22})^2 + 4(a_{11}a_{23} - a_{13}a_{21})^2$$. From Eq. (10), we obtain

$$\kappa_{1,2}^2 = \frac{1 \pm \sqrt{1 - |C|^2}}{2}. \quad (4)$$

So the von Neumann entropy is given by

$$E_N(|\psi\rangle) = h((1 - \sqrt{1 - |C|^2})/2), \quad (5)$$

where $$h(x) = -x \log_2 x - (1 - x) \log_2 (1 - x)$$.

On the other hand, one obtain $$\rho_B$$ by tracing out the degree of freedom of part A, i.e.,

$$\rho_B = a^a a^a. \quad (6)$$

This is a 3 \times 3 matrix whose eigenvalues are denoted by $$\kappa_1^2$$, $$\kappa_2^2$$, $$\kappa_3^2$$ are roots of the algebraic equation

$$\lambda^3 - \lambda^2 + \frac{|C|^2}{4} \lambda - \det(\rho_B) = 0. \quad (7)$$

The reduced density matrix $$\rho_B$$ is of rank 2, i.e., $$\det \rho_B = 0$$, then there are only two non-zero eigenvalues. The von Neumann entropy takes the same form as in Eq. (3). Just like the case of Wootters, the von Neumann entropy here is also a monotonous function of the norm of concurrence vectors: $$|C|^2$$. Therefore, the concurrence vector is a reliable measurement of entanglement of the states of qubit-qutrit system.

### III. AVERAGE CONCURRENCE OF A HILBERT SUBSPACE

As is known that the ground state of physical systems is degenerate frequently. We need a definition of the entanglement for Hilbert subspace to evaluate the entanglement of the degenerate ground state. With the help of the concurrence vector applicable to measure the entanglement of individual state, we suggest to use average concurrence. Since a general state in a Hilbert subspace can be expanded in terms of its bases, the magnitude of concurrence vector depends on the coefficients in the state expansion. The normalization condition gives a restriction on the coefficients so that the parameter space of the Hilbert subspace manifests a compact hyper surface. It is therefore natural to define average concurrence by the following ratio,

$$\mathbf{C}_{av} = \frac{\int d\mu(p_1, p_2, \ldots) |\mathbf{C}(p_1, p_2, \ldots)|}{\int d\mu(p_1, p_2, \ldots)}. \quad (8)$$

Here $$d\mu(p_1, p_2, \ldots)$$ refers to the Haar measure with respect to the parametrization $$p_1, p_2, \ldots$$, which is invariant under unitary operations. For doubly degenerate case, a general state can be described by a superposition of two states $$|\psi_1\rangle$$ and $$|\psi_2\rangle$$, the parameter space is a three dimensional sphere $$S^3$$. The evaluation of average concurrence becomes a calculation of the integrals in Eq. (9). We will apply the average concurrence to discuss the ground state of a concrete model in next section.

### IV. THE GROUND STATE SUPERPOSITIONS

We consider a system of spin 1 and 1/2 with anisotropic Heisenberg coupling in an uniform magnetic field,

$$H = \frac{J}{2}(\sigma_x \cdot S_x + \sigma_y \cdot S_y + \Delta \sigma_z \cdot S_z) + B(\frac{1}{2}\sigma_z + S_z) \quad (9)$$

where $$\sigma$$’s refer to the Pauli matrices for spin-1/2 and $$S$$’s denote the spin operators for spin-1; $$J$$ stands for their coupling strength, and $$\Delta$$ represents the anisotropy of the coupling. Throughout this paper, the spin-1/2 states are denoted by $$|\uparrow\rangle$$ and $$|\downarrow\rangle$$, while the spin-1 states are denoted by $$|\uparrow\rangle$$, $$|\downarrow\rangle$$, and $$|\psi\rangle$$. In terms of the spin-1 matrices ($$\hbar$$ is put to unit in this paper), the Hamiltonian is written out in matrix form:

$$\left( \begin{array}{cccc} \Delta \frac{J}{2} & \frac{3}{2} & 0 & 0 \\
 -\frac{3}{2} & -\frac{J}{2} - \Delta & 0 & \sqrt{2} \\
 0 & 0 & -\frac{J}{2} - \Delta & 0 \\
 \sqrt{2} & 0 & 0 & -\Delta \frac{J}{2} - \frac{3}{2} \end{array} \right)$$

which solves six eigenvalues and six eigenstates. Among them, the state $$|\uparrow\rangle$$ with eigenenergy $$\frac{3}{2}B + \frac{1}{2}\Delta J$$ and
another one $|\downarrow\downarrow\rangle$ with eigenenergy $-\frac{4}{3}B + \frac{1}{2}\Delta J$ are obviously non-entangled. The other four states, whose Schmidt numbers are 2, are clearly entangled. For simplicity, we put $J$ to unit from now on.

A. In the absence of magnetic field

When $B = 0$, the diagonalization of the Hamiltonian gives rise to three distinct eigenvalues: $\frac{1}{4}\Delta$, $\frac{1}{2}(-\Delta - \sqrt{8 + \Delta^2})$, and $\frac{1}{4}(-\Delta + \sqrt{8 + \Delta^2})$, of which each energy level is doubly degenerate. The ground state energy has a critical point $\Delta_c = -1$. When $\Delta < -1$, the ground state with energy $\frac{1}{2}\Delta$ is doubly degenerate and a general ground state (not restricted to be eigenstate) is given by a superposition of those two states:

$$|\Psi_{FM}\rangle = a|\downarrow\downarrow\rangle + b|\uparrow\uparrow\rangle,$$

where the coefficients fulfil $|a|^2 + |b|^2 = 1$. The norm of concurrence vector of this state is $2|ab|$.

When $\Delta > -1$, the ground states are doubly degenerate whose energy reads

$$\frac{1}{4}(-\Delta - \sqrt{8 + \Delta^2}).$$

A general state in this two-dimensional Hilbert subspace is given by

$$|\Psi_{AF}\rangle = c|\psi_1\rangle + d|\psi_2\rangle$$

with $|c|^2 + |d|^2 = 1$, where

$$|\psi_1\rangle = \frac{1}{F_+}(|\downarrow 0\rangle - \frac{\Delta + \sqrt{\Delta^2 + 8}}{2\sqrt{2}}|\uparrow \uparrow\rangle),$$

$$|\psi_2\rangle = \frac{1}{F_-}(|\uparrow \uparrow\rangle + \frac{\Delta - \sqrt{\Delta^2 + 8}}{2\sqrt{2}}|\downarrow 0\rangle),$$

and

$$F_{\pm} = \sqrt{\left(\frac{\Delta \pm \sqrt{8 + \Delta^2}}{2\sqrt{2}}\right)^2 + 1}.$$

The norm of concurrence vector for this ground state is

$$|\mathcal{C}_{\psi_{AF}}| = \sqrt{2\left(4c^2 + 4d^2 + c^2d^2(4 + \Delta^2 + 2\sqrt{8 + \Delta^2})\right)}$$

$$\frac{1}{8 + \Delta^2}$$

(12)

The point $\Delta = -1$ is a special point in the absence of magnetic field because the ground state is 4-fold degenerated then. The general state at that point reads

$$|\Psi_c\rangle = a|\downarrow\downarrow\rangle + b|\uparrow\uparrow\rangle + c|\phi_1\rangle + d|\phi_2\rangle,$$

where

$$|\phi_1\rangle = \sqrt{\frac{2}{3}}|\downarrow 0\rangle - \frac{1}{3}|\uparrow \downarrow\rangle,$$

$$|\phi_2\rangle = \sqrt{\frac{1}{3}}|\downarrow \uparrow\rangle - \sqrt{\frac{2}{3}}|\uparrow 0\rangle.$$
$3B/2 + \Delta/2$ is obviously not entangled. In the region $-(3\Delta + \sqrt{8 + \Delta^2})/4 < B < 0$, the ground state whose energy takes $(2B - \Delta - \sqrt{8 + \Delta^2})/4$ becomes
\begin{equation}
\frac{1}{N_1}(\frac{\Delta - \sqrt{8 + \Delta^2}}{2\sqrt{2}} |\uparrow 0\rangle + |\downarrow \uparrow\rangle),
\end{equation}
where $N_1$ denotes the normalization factor
\begin{equation}
N_1 = \left( (\frac{\Delta - \sqrt{8 + \Delta^2}}{2\sqrt{2}})^2 + 1 \right)^{1/2}.
\end{equation}
The norm of concurrence vector of state (16) is obtained
\begin{equation}
C(\rho) = \frac{4\sqrt{2}(\sqrt{8 + \Delta^2} - \Delta)}{8 + (\sqrt{8 + \Delta^2} - \Delta)^2}.
\end{equation}
When $\Delta = 0$, the entanglement reaches maximum 1. Furthermore, when $0 < B < (3\Delta + \sqrt{8 + \Delta^2})/4$, the ground state with energy $(-2B - \Delta - \sqrt{8 + \Delta^2})/4$ is given by
\begin{equation}
\frac{1}{N_2}(\frac{-\Delta + \sqrt{8 + \Delta^2}}{2\sqrt{2}} |\uparrow \downarrow\rangle + |\downarrow 0\rangle),
\end{equation}
where $N_2$ is normalization factor. The norm of concurrence vector is the same as Eq. (17). Once $B > (3\Delta + \sqrt{8 + \Delta^2})/4$, the ground state with energy $-3B/2 + \Delta/2$ becomes $|\downarrow \downarrow\rangle$ which is no more entangled. As a result, there are three critical points $B = 0, -(3\Delta + \sqrt{8 + \Delta^2})/4$ and $+(3\Delta + \sqrt{8 + \Delta^2})/4$ in the regime $\Delta > -1$. The norm of concurrence vector as a function of the magnetic field $B$ and anisotropy parameter $\Delta$ is plotted in Fig. 3. Clearly, when $\Delta$ approaches 1, the width of the peak of the entanglement curve approaches zero.

**V. THE GROUND STATE MIXTURES**

It is interesting to discuss the entanglement feature of a mixtures of the degenerate ground states of the same model. The *negativity* introduced by G. Vidal et al is known to be a useful measurement for the entanglement of mixed states $^{15}$, namely,
\begin{equation}
N(\rho) = \|\rho^{T_A}\|_1 - 1/2,
\end{equation}
where the trace norm is defined by $\|A\|_1 \equiv tr\sqrt{A^\dagger A}$ and $T_A$ refers to the partial transposition of $A$. The negativity vanishes for unentangled states.

Hereafter we discuss the mixture of ground state in the absence of magnetic field. In the ferromagnetic regime, $\Delta < -1$, a general mixed state is given by
\begin{equation}
\rho = p|\downarrow \downarrow\rangle\langle\downarrow \downarrow| + (1 - p)|\uparrow \uparrow\rangle\langle\uparrow \uparrow|,
\end{equation}
where $\rho = p|\downarrow \downarrow\rangle\langle\downarrow \downarrow| + (1 - p)|\uparrow \uparrow\rangle\langle\uparrow \uparrow|$, (20)

Because the convexity of the norm of concurrence vector, one can obtain
\begin{equation}
C(\rho) \leq \sum_i p_i C(p_i),
\end{equation}
Clearly, both the concurrence and negativity of the state described by (20) are zero.
In the antiferromagnetic regime $\Delta > -1$. The density matrix for a general mixture of ground state is given by
\[ \rho = p | \psi_1 \rangle \langle \psi_1 | + (1 - p) | \psi_2 \rangle \langle \psi_2 |, \quad (22) \]
where $| \psi_1 \rangle$ and $| \psi_2 \rangle$ were given in Eq. (11). Its negativity is obtained after some algebra,
\[ \mathcal{N}(\rho) = \frac{\Delta}{4\sqrt{8 + \Delta^2}} - \frac{1}{4} + f(p) + f(1 - p) \quad (23) \]
where
\[ f(p) = \sqrt{\frac{16 - 32p + (20 + \Delta^2 - \Delta\sqrt{8 + \Delta^2})p^2}{8(8 + \Delta^2)}} \]

At the critical point $\Delta = -1$, the density matrix for mixture of ground states becomes
\[ \rho = p_1 | \downarrow \downarrow \rangle \langle \downarrow \downarrow | + p_2 | \uparrow \uparrow \rangle \langle \uparrow \uparrow | + p_3 | \phi_1 \rangle \langle \phi_1 | + (1 - p_1 - p_2 - p_3) | \phi_2 \rangle \langle \phi_2 |. \quad (24) \]
where $| \phi_1 \rangle$ and $| \phi_2 \rangle$ were given in Eq. (14). The negativity for the mixture of equilibrium at zero temperature ($p_1 = 1/4$) is 0.031, and its average value is 0.077.

The negativity versus the anisotropy parameter $\Delta$ is plotted in Fig. 4. One can see from the plot that there is a singularity at $\Delta = -1$ where the quantum phase transition (ferromagnetic to antiferromagnetic) occurs. Whereas, when $\Delta$ increases from $-1$, the negativity rises at first then descends after reaching a maximal value. Finally it approaches to zero when $\Delta$ goes to infinity. The state with $p = 1/2$ is particularly interesting because it can be regarded as the thermal equilibrium at zero temperature. Note that the negativity of equilibrium state at zero temperature reaches 1/3 at $\Delta = 1$ where the system recovers its largest symmetry (isotropic Heisenberg coupling). Similar features have been noticed in some other models $[12, 13]$.

VI. SUMMARY AND DISCUSSION

We have studied the entanglement feature of the ground state for system of spin 1 and 1/2. We have shown that the concurrence vector is consistent with the measurement of von Neumann entropy for such system. Because its ground state is degenerate in cases, the simple calculation of norm of concurrence for a state is no more applicable. We therefore proposed a concept, average concurrence, to measure the entanglement of Hilbert subspace. Based on this definition, we discussed the entanglement of the superposition of the degenerated ground states. We obtained the relations between the average concurrence and the anisotropy parameter. We also studied the model by taking account of external magnetic field. The relation between the norm of concurrence and the magnetic field and the anisotropy parameters are calculated. We found that the state is not entangled when anisotropy factor $\Delta < -1$. When $\Delta > -1$, the concurrence varies with respect to the anisotropy factor $\Delta$. We also studied the entanglement of a general mixture of the degenerate ground state by employing the widely used negativity.

Our results indicate that the averages of both concurrence and negativity have singularities at the quantum critical point $\Delta = -1$. The negativity for the equilibrium at zero temperature reaches the maximal value at $\Delta = 1$ where the model possesses the largest symmetry. However, both the negativity averaged over the general mixture and the norm of concurrence vector averaged over the general superposition of the degenerate ground states do not reach maximum at $\Delta = 1$. The average concurrence takes the largest value 0.785 in the ferromagnetic regime $\Delta < -1$ and in the limit of antiferromagnetic Ising dominant regime $\Delta \rightarrow \infty$.

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[1] L.S. Bell, PHYSICS(Long Island City, N.Y.) 1, 195 (1964).
[2] M.A. Nielsen and I.L. Chuang, Quantum Computation and Quantum Information, Cambridge University Press, Cambridge 2000.
[3] D. Bouwmeester, A. Ekert, and A. Zeilinger, The Physics of Quantum Information, Springer, Berlin Heidelberg, 2001.
[4] S. Hill and W.K. Wootters, Phys. Rev. Lett. 78, 5022 (1997).
[5] W.K. Wootters, Phys. Rev. Lett. 80, 2245 (1998).
[6] P. Rungta, V. Buzek, C.M. Caves, M. Hillery, and G.J. Milburn, Phys. Rev. A 64, 042315 (2000).
[7] P. Badziag, P. Deuar, J. Mod. Optic, 49(8), 1289 (2002).
[8] A. Lozinski, A. Buchleitner, K. Zyczkowski, and T. Wellens, Europhys. Lett. 62, 168 (2003).
[9] X. Wang, Phy. Rev. A 66, 034302 (2002); X. Wang and P. Zanardi, Phys. Lett. A 301, 1 (2002).
[10] Y. Sun, Y. Chen, and H. Chen, Phys. Rev. A 68, 044301 (2003).
[11] K.M. O’Connor and W.K. Wootters, Phys. Rev. A 63, 052302 (2001).
[12] S.J. Gu, H.Q. Lin, and Y.Q. Li, Phys. Rev. A 68, 042330 (2003).
[13] S.J. Gu, S.S. Deng, Y.Q. Li, and H.Q. Lin, Phys. Rev. Lett. 93, 086402 (2004).
[14] Y.Q. Li and G.Q. Zhu, preprint quant-ph/0308130 (2003).
[15] G. Vidal and R. F. Werner, Phys. Rev. Lett. 65, 32314(2002).
[16] A. Osterloh, L. Amico, G. Falci, and R. Fazio, Nature (London) 416, 608 (2002).