The geodetic vertex covering number of a graph

V.M. Arul Flower Mary¹, J. Anne Mary Leema²*, P. Titus³ and B. Uma Devi⁴

Abstract
A subset \( S \) of vertices in a connected graph \( G \) of order at least two is called a geodetic vertex cover if \( S \) is both a geodetic set and a vertex covering set. The minimum cardinality of a geodetic vertex cover is the geodetic vertex covering number of \( G \) denoted by \( g_\alpha(G) \). Any geodetic vertex cover of cardinality \( g_\alpha(G) \) is a \( g_\alpha \)-set of \( G \). Some general properties satisfied by geodetic vertex covering number of a graph are studied. The geodetic vertex covering number of several classes of graphs are determined. Some bounds for \( g_\alpha(G) \) are obtained and the graphs attaining these bounds are characterized. A few realization results are given for the parameter \( g_\alpha(G) \).

Keywords
Geodesic, geodetic set, vertex covering set, geodetic vertex cover, geodetic vertex covering number.

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¹² Department of Mathematics, Holy Cross College (Autonomous), Affiliated College of Manonmaniam Sundaranar University, Nagercoil-629002, Tamil Nadu, India.
³ Department of Mathematics, Anna University, Tirunelveli Region, Tirunelveli-627007, Tamil Nadu, India.
⁴ Department of Mathematics, S.T. Hindu College, Affiliated to Manonmaniam Sundaranar University, Nagercoil-629002, Tamil Nadu, India.
*Corresponding author: ¹ arulflowermary@gmail.com; ²annemary88ma@gmail.com; ³ titusvino@yahoo.com; ⁴ umasub1968@gmail.com

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1. Introduction

For basic graph theoretic terminology and basic definitions not given here we refer to Harary [5]. We consider finite, undirected, connected graphs without loops and multiple edges. Denote the number of vertices and edges of a graph \( G \) as \( n = |V(G)| \) and \( m = |E(G)| \) respectively. A vertex \( v \) is a simplicial vertex or an extreme vertex of \( G \) if the subgraph induced by its neighbors is complete.

Let \( I[u,v] \) denote the set consisting of \( u, v, \) and all the vertices lying on a \( u \rightarrow v \) geodesic and for \( S \subseteq V(G) \), \( I[S] \) denote the union of all \( I[u,v] \) for \( u, v \in S \). The geodetic number \( g(G) \) of \( G \) is the minimum cardinality of its geodetic sets and any geodetic set of cardinality \( g(G) \) is a minimum geodetic set or a \( g \)-set of \( G \). The geodetic number of a graph was introduced in [1, 6] and further studied in [2–4]. A subset \( S \subseteq V(G) \) is called a vertex covering set of \( G \) if every edge has at least one end point in \( S \). A vertex covering set with minimum cardinality is a minimum vertex covering set of \( G \). The vertex covering number of \( G \) is the cardinality of any minimum vertex covering set of \( G \) denoted as \( \alpha(G) \). The vertex covering number of a graph was studied in [7].

A set of vertices (edges) in a graph \( G \) is independent if no two of the vertices (edges) are adjacent. The independence number \( \beta(G) \) of \( G \) is the maximum number of vertices in an independent set of vertices of \( G \). By a matching in a graph \( G \), we mean an independent set of edges in \( G \). A caterpillar is a tree of order 3 or more, the removal of whose end vertices produces a path called the spine of the caterpillar. A graph \( G \) is called triangle free if it does not contain cycles of length 3. A subset \( S \subseteq V(G) \) is a dominating set if every vertex in \( V \rightarrow S \) is adjacent to at least one vertex in \( S \). A geodetic dominating set of \( G \) is a subset \( S \) of vertices which is both a geodetic set and a dominating set. The minimum cardinality of a geodetic dominating set of a graph \( G \) is its geodetic domination number denoted by \( \gamma_G(G) \).

In this paper, we define geodetic vertex covering number \( g_\alpha(G) \) of a graph and initiate a study of this parameter. We investigate about some general properties satisfied and some bounds attained by this parameter. Also few realization re-
The minimum cardinality of a geodetic vertex cover of \( G \) is \( C \).

**Theorem 2.3.** Every extreme vertex of a connected graph \( G \) belongs to every geodetic set of \( G \).

**Theorem 1.2.** [6] For any tree \( T \) with \( k \) end vertices, \( g(T) = k \).

Throughout this paper, \( G \) is considered as a connected graph of order at least two.

### 2. The Geodetic Vertex Cover of a Graph

**Definition 2.1.** Let \( G \) be a connected graph of order at least 2. A set \( S \) of vertices of \( G \) is a geodetic vertex cover of \( G \) if \( S \) is both a geodetic set and a vertex covering set of \( G \). The minimum cardinality of a geodetic vertex cover of \( G \) is defined as the geodetic vertex covering number of \( G \) and is denoted by \( g_\alpha(G) \). Any geodetic vertex cover of cardinality \( g_\alpha(G) \) is a \( g_\alpha \)-set of \( G \).

**Example 2.2.** Consider the graph \( G \) of Figure 2.1. Observe that \( S_1 = \{v_2, v_4, v_3\} \) is a minimum vertex covering set of \( G \) so that \( \alpha(G) = 3 \), \( S_2 = \{v_1, v_3, v_6\} \) is a minimum geodetic set of \( G \) so that \( g(G) = 3 \) and \( S_3 = \{v_1, v_2, v_3, v_4, v_6\} \) is a \( g_\alpha \)-set of \( G \) so that \( g_\alpha(G) = 5 \). Thus the geodetic vertex covering number of \( G \) is different from its vertex covering number and its geodetic number.

![Figure 2.1: G](image)

**Theorem 2.3.** Let \( G \) be any connected graph. Then \( 2 \leq \max\{\alpha(G), g(G)\} \leq g_\alpha(G) \leq n \).

_Proof._ Any geodetic set of \( G \) needs at least two vertices and so \( 2 \leq \max\{\alpha(G), g(G)\} \). From the definition of geodetic vertex covering number of \( G \), we have \( \max\{\alpha(G), g(G)\} \leq g_\alpha(G) \). Clearly \( V(G) \) is a geodetic vertex cover of \( G \). Hence \( g_\alpha(G) \leq n \). Thus \( 2 \leq \max\{\alpha(G), g(G)\} \leq g_\alpha(G) \leq n \).

**Remark 2.4.** The bounds in Theorem 2.3 are sharp. For \( C_4 \), \( \alpha(C_4) = 2 \), \( g(C_4) = 2 \) and \( g_\alpha(C_4) = 2 \). For \( K_n \) (\( n \geq 2 \)), \( g_\alpha(K_n) = n \).

**Theorem 2.6.** Every simplicial vertex of a connected graph \( G \) belongs to every geodetic vertex cover of \( G \).

_Proof._ From the definition of \( g_\alpha \)-set, every \( g_\alpha \)-set of \( G \) is a \( g \)-set of \( G \). Hence the result follows from Theorem 1.1.

**Corollary 2.7.** Let \( K_{1,n-1} \) (\( n \geq 3 \)) be a star. Then \( g_\alpha(K_{1,n-1}) = n - 1 \).

_Proof._ The result follows from Theorem 2.6.

**Corollary 2.8.** For the complete graph \( K_n \) (\( n \geq 2 \)), \( g_\alpha(K_n) = n \).

**Theorem 2.9.** If \( G \) is a connected graph of order \( n \geq 2 \), then

(i) \( g_\alpha(G) = 2 \) if and only if \( G \) is either \( K_2 \) or \( K_{2,n-2} \) (\( n \geq 3 \)).

(ii) \( g_\alpha(G) = n \) if and only if \( G = K_n \) (\( n \geq 2 \)).

_Proof._ (i) Let \( g_\alpha(G) = 2 \). Let \( S = \{u, v\} \) be a minimum geodetic vertex cover of \( G \). We claim that \( G = K_2 \) or \( G = K_{2,n-2} \) (\( n \geq 3 \)). Suppose that \( G = K_2 \), then there is nothing to prove. If not, then \( n \geq 3 \) and since \( S = \{u, v\} \) is a \( g_\alpha \)-set of \( G \), \( u \) and \( v \) cannot be adjacent in \( G \). Let \( W = V - S \). We claim that every vertex of \( W \) is adjacent to both \( u \) and \( v \) and no two vertices of \( W \) are adjacent.

**Claim 1.** Every vertex of \( W \) is adjacent to both \( u \) and \( v \).

Suppose there is a vertex \( w \in W \) such that \( w \) is adjacent to at most one vertex in \( S \). Then \( w \) lies on a \( u-v \) geodesic of length at least 3. Let \( P : u = v_0, v_1, \ldots, v_i = w, v_{i+1}, \ldots, v_m = v \) be a \( u-v \) geodesic. Then the edges in \( E(P) - \{v_0v_1, v_{m-1}v_m\} \) are not covered by any of the vertices \( u \) and \( v \), which is a contradiction.

![Figure 2.2: G](image)
Claim 2. No two vertices of $W$ are adjacent.
Suppose there exist vertices $w_i, w_j \in W$ such that $w_i$ and $w_j$ are adjacent. Since from Claim 1, every vertex of $W$ is adjacent to both $u$ and $v$ and $S = \{u, v\}$ is a $g$- set of $G$, $w_i$ and $w_j$ lie on the $u - v$ geodesics, respectively. Then the edge $w_i w_j$ is not covered by any of the vertices of $S$, which is a contradiction. Hence no two vertices of $W$ are adjacent in $G$.

Thus $G$ is the complete bipartite graph $K_{2,n-2}(n \geq 3)$ with partite sets $S$ and $W$.

Conversely, let $G = K_2$ or $K_{2,n-2}$ $(n \geq 3)$. If $G = K_2$, then by Corollary 2.8, $g_a(G) = 2$. If not, let $G = K_{2,n-2}$ $(n \geq 3)$. Let $U = \{u_1, u_2\}$ and $W = \{w_1, w_2, ..., w_{n-2}\}$ be the bipartition of $G$. Clearly every vertex $w_i$ $(1 \leq i \leq n-2)$ lies on the geodesic $u_1, w_i, u_2$, and the vertices $u_1$ and $u_2$ cover all the edges of $G$. Hence $U$ is a geodetic vertex cover of $G$ and so $g_a(G) = 2$.

(ii) Assume that $G = K_n$ $(n \geq 2)$. Then by Corollary 2.8, $g_a(G) = n$. Conversely, let $g_a(G) = n$. We prove that $G = K_n$ $(n \geq 2)$. For $n = 2$, the result holds from (i). Let $n \geq 3$. Contrarily assume that there exist two non-adjacent vertices $u$ and $v$ in $G$. Let a vertex $x$ be adjacent to $u$ lying on a $u - v$ geodesic. Then $V(G) - \{x\}$ is a geodetic vertex cover of $G$, giving a contradiction to $g_a(G) = n$. Thus $G = K_n$.

Theorem 2.10. Let $G$ be a connected graph with $n \geq 3$. Then $g_a(G) = 3$ if and only if either $G = K_3$ or there exists a minimum geodetic set $S$ on 3 vertices such that $V(G) - S$ is an independent set or there exists a minimum geodetic set $S$ on 2 vertices such that $\{V(G) - S\}$ is a star.

Proof. Let $g_a(G) = 3$. Let $S = \{u, v, w\}$ be a minimum geodetic vertex cover of $G$. Since $g(G) \leq g_a(G)$, we have $g(G) = 2$ or 3.

Case 1. $g(G) = 3$. If $n = 3$, then by Theorem 2.9, $G = K_3$. If $n \geq 4$, then $V(G) - S \neq \Phi$. Since $S$ is a $g_a$-set of $G$, every edge of $G$ is incident with at least one vertex in $S$. Hence $V(G) - S$ is an independent set of vertices of $G$.

Case 2. $g(G) = 2$. Let $S' = \{u, v\} \subset S$ be a $g$-set of $G$. Also since $S = \{u, v, w\}$ is a $g_a$-set of $G$, the edges not covered by the vertices of $S'$ should have exactly one end in $w$. Suppose the other ends of any two of these edges, say $x$ and $y$ are adjacent, then the edge $xy$ will not be covered by any of the vertices of $S'$, which is a contradiction. Hence $\{V(G) - S\}$ must be a star.

Conversely, if $G = K_3$, by Corollary 2.8, $g_a(G) = 3$. If $G$ has a minimum geodetic set $S$ on 3 vertices such that $V(G) - S$ is independent, then every edge of $G$ has at least one end in $S$ so that $S$ is both a minimum geodetic set and a vertex cover of $G$. Hence $S$ is a $g_a$-set of $G$ and so $g_a(G) = 3$. If $G$ has a minimum geodetic set $S$ on 2 vertices such that $\{V(G) - S\}$ is a star, then $S$ is not a vertex cover of $G$. Let $w$ be the cut vertex of the star induced by $V(G) - S$. Then $S' = S \cup \{w\}$ will be a vertex cover of $G$ so that $S'$ is a geodetic vertex cover of $G$.

Theorem 2.11. Let $G$ be a connected graph with $g(G) \geq n - 1$. Then $g_a(G) = g(G)$.

Proof. Let $G$ be a connected graph with $g(G) \geq n - 1$. By Theorem 2.3, $g(G) \leq g_a(G) \leq n$. If $g(G) = n$, then $g_a(G) = n$ and so $g(G) = g_a(G)$. If $g(G) = n - 1$, then let $S = \{x_1, x_2, ..., x_{n-1}\}$ be a $g$-set of $G$. There exists a vertex $x$ not in $S$. Then $x$ lies on a geodesic $P$ joining any two vertices of $S$. Let $x_i$ be adjacent to the vertices $x_i$ and $x_j$ on $P$ for some $i \neq j$. Then all the edges of $G$ including $x_i x_j$ and $x_j x_i$ are covered by the vertices of $S$. Hence $S$ is a $g_a$-set of $G$. Thus $g(G) = g_a(G)$.

Remark 2.12. The converse of Theorem 2.11 need not be true. For the graph $G$ given in Figure 2.3, $S = \{v_1, v_2, v_3\}$ is both a minimum geodetic set and a minimum geodetic vertex cover of $G$ so that $g_a(G) = g(G) = 3$ but $g(G) < n - 1$.

Figure 2.3: $G$

Theorem 2.13. Let $G$ be a connected graph of order $n \geq 2$. Then $g_a(G) = g(G)$ if and only if either $G = K_n$ or there exists a minimum geodetic set $S$ such that $V(G) - S$ is independent.

Proof. Let $g_a(G) = g(G)$. If $G = K_n$, then clearly $g_a(G) = g(G) = n$. If not, let $S$ be a $g$-set of $G$. Since $g_a(G) = g(G)$, $S$ is a geodetic vertex cover of $G$. Hence every edge of $G$ is incident with at least one vertex in $S$ and so no edge of $G$ has two ends in $V(G) - S$. Thus no pair of vertices of $V(G) - S$ are adjacent and hence $V(G) - S$ is an independent set.

Conversely, if $G = K_n$, then clearly $g_a(G) = g(G) = n$. If $S$ is a minimum geodetic set of $G$ such that $V(G) - S$ is independent, then no edge of $G$ has two ends in $V(G) - S$. Thus every edge of $G$ has at least one end in $S$. Hence $S$ is a minimum geodetic vertex cover of $G$ so that $|S| = g(G) = g_a(G)$.

Theorem 2.14. Let $T$ be a tree of order $n \geq 2$. Then the following statements are equivalent.

(i) $g_a(T) = g(T)$.

(ii) $T$ is a star.
Theorem 2.16. Let $T$ be a tree of order $n$. If every cut vertex of $T$ lies on a diametral path of length $2k + 1$ or $2k + 2$, then $g_a(T) = g(T) + k$.

Proof. Let $T$ be a tree of order $n$. If every cut vertex of $T$ lies on a diametral path of length $2k + 1$ or $2k + 2$, then $g_a(T) = g(T) + k$.

Remark 2.15. The results in Theorem 2.14 are not equivalent for any connected graph $G$ of order $n \geq 2$. See graph $G$ of Figure 2.4, $S = \{v_1, v_3, v_4\}$ is both a $g_a$-set and a $g_a$-set of $G$. So $g_a(G) = g(G) = 3$. But $S' = \{v_2, v_3\}$. Hence $g_a(G) = g(G) = 2$. Moreover, $G$ is not a star.

Figure 2.4 : $G$

Theorem 2.17. Let $T$ be a tree of order $n \geq 3$ with diameter $d$. Then $g_a(T) = n - 1$ if and only if $T$ is either a star or a double star.

Proof. Let $a(T) = n - 1$. Let $P = v_0, v_1, v_2, ..., v_d$ be a diametral path of $T$. Then $d = 2$. If $d = 4$, then $S = V(T) - \{v_1, v_3\}$ is a minimum geodetic vertex cover of $T$ and so $g_a(T) \leq n - 2$, giving a contradiction. Then $d = 2$ or 3 and hence $T$ is either a star or a double star. Converse is clear.

Theorem 2.18. Let $T$ be a caterpillar of order $n \geq 2$ with diameter $d$. Then $g_a(T) = \lceil \frac{d}{2} \rceil + k - 1$, where $k$ is the number of end vertices of $T$.

Proof. Let $T$ be a caterpillar. Let $P = v_0, v_1, v_2, ..., v_d$ be a diametral path and $k$ denote number of end vertices of $T$. If $d$ is even, let $S = \{v_0, v_2, v_4, ..., v_d\}$ and if $d$ is odd, let $S = \{v_0, v_2, v_4, ..., v_{d-1}, v_d\}$. Then $|S| = \lceil \frac{d}{2} \rceil + 1$ and $S$ covers all the edges of $P$. Since any vertex of $P$ lies on the $v_0 - v_d$ geodesic, $S$ is a minimum geodetic vertex cover of the diametral path $P$. Since $T$ is a caterpillar, $S' = (V(T) - V(P)) \cup \{v_0, v_d\}$ is the set of all end vertices of $T$. Then by Theorem 2.6, every geodetic vertex cover of $T$ contains $S'$. Now, it is clear that $S'' = (S - \{v_0, v_d\}) \cup S'$ is a minimum geodetic vertex cover of $T$ and so $g_a(T) = \lceil \frac{d}{2} \rceil + 1 + k = \lceil \frac{d}{2} \rceil + k - 1$.

Theorem 2.19. (i) For the cycle $C_n$ ($n \geq 4$), $g_a(C_n) = \lceil \frac{n}{2} \rceil$.

(ii) For the wheel $W_n = K_1 + C_{n-1}$ ($n \geq 5$), $g_a(W_n) = \lceil \frac{n-1}{2} \rceil + 1$.

(iii) For the graph $G = K_1 + \cup m_j K_j$, where $\sum m_j \geq 2$, $g_a(G) = n - 1$.

Proof. (i) Let $C_n : v_1, v_2, ..., v_n, v_1$ be a cycle of order $n$. It is clear that $S = \{v_1, v_3, v_5, ..., v_{[n/2] - 1}\}$ is a $g_a$-set of $C_n$ and so $g_a(C_n) = \lceil \frac{n}{2} \rceil$.

(ii) Let $C_n : v_1, v_2, ..., v_{n-1}, v_1$ be the cycle of $W_n$ and $x$, the vertex of $K_1$ in $W_n$. Then $S = \{x, v_1, v_3, ..., v_{[n/2] - 1}\}$ is a $g_a$-set of $W_n$. Hence $g_a(W_n) = \lceil \frac{n-1}{2} \rceil + 1$.

(iii) Let $G = K_1 + \cup m_j K_j$, where $\sum m_j \geq 2$. Then $n \geq 3$ and $G$ has exactly one cut vertex, say $x$, and all the remaining vertices are simplicial vertices of $G$. Then by Theorem 2.6, $S = V(G) - \{x\}$ is a subset of any geodetic vertex cover of $G$ and so $g_a(G) = n - 1$.

Theorem 2.20. For any connected graph $G$, $g_a(G) \leq n - \lceil \text{diam } G \rceil$. 

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Theorem 2.24. Every geodetic vertex cover of a connected graph G is a geodetic dominating set of G.

Proof. Let \( P : v_0, v_1, v_2, \ldots, v_d \) be a diametral path of G. If d is even, then \( S = \{v_0, v_1, v_2, v_3, \ldots, v_d\} \) covers all the edges of \( P \) and each vertex of \( P \) lies on a \( v_0 - v_d \) geodesic. Hence \( (V(G) - V(P)) \cup S \) is a \( g_a \) set. So \( g_a(G) \leq n - (d + 1) + (\frac{d}{2} + 1) = n - \frac{d}{2} \). Similarly, if d is odd, then \( S' = \{v_0, v_1, v_2, v_3, \ldots, v_{d-1}, v_d\} \) covers all the edges of \( P \) and every vertex of \( P \) lies on a \( v_0 - v_d \) geodesic. Hence \( (V(G) - V(P)) \cup S' \) is a \( g_a \) set. So \( g_a(G) \leq n - (d + 1) + (\lfloor \frac{d}{2} \rfloor + 1) = n - \lfloor \frac{d}{2} \rfloor \). Thus in both cases, we have \( g_a(G) \leq n - \lfloor \frac{\text{diam } G}{2} \rfloor \). \( \square \)

Remark 2.21. The bound in Theorem 2.20 is sharp. For the path \( P_1 : v_0, v_1, v_2, v_3, v_4, v_5, v_6, S = \{v_0, v_2, v_4, v_6\} \) is the unique minimum geodetic vertex cover of \( P_1 \) and so \( g_a(P_1) = 4 \). Also since \( \text{diam } P_1 = 6 \), we have \( n - \lfloor \frac{\text{diam } P_1}{2} \rfloor = 7 - 3 = 4 \). Thus \( g_a(G) \leq n - \lfloor \frac{\text{diam } G}{2} \rfloor \).

Theorem 2.22. If G is a triangle free graph with \( \delta(G) \geq 2 \) and \( M \) is a maximal matching of G, then \( g_a(G) \leq 2|M| \).

Proof. Let S consist of all end vertices of the edges of M. Since M is a maximal matching of G, no edge of G has its two ends in \( V(G) - S \). Hence \( V(G) - S \) is independent so that \( S \) is a vertex cover of G. Thus every edge of G has at least one end in S. Since \( \delta(G) \geq 2 \), there exist at least two neighbours \( x \) and \( y \) in S for every \( v \in V(G) - S \). Since G has no triangles, the path \( x, v, y \) is an \( x - y \) geodesic. Hence S is a \( g_a \) set of G. Thus \( g_a(G) \leq 2|M| \). \( \square \)

Theorem 2.23. Let G be a triangle free graph with \( \delta(G) \geq 2 \). Then \( g_a(G) = n - \beta(G) \), where \( \beta(G) \) is the independence number of G.

Proof. Let S be a maximum independent set of vertices of G so that \( |S| = \beta(G) \). Then \( V(G) - S \) is a minimum vertex cover of G. Since G is triangle free and \( \delta(G) \geq 2 \), every vertex in S has at least two neighbours which are not adjacent in \( V(G) - S \). Thus, every vertex \( v \in V(G) - S \) lies on an \( x - y \) geodesic for some vertices \( x, y \in V(G) - S \). Hence \( V(G) - S \) is also a \( g \) set of G. Thus \( V(G) - S \) is a \( g_a \) set and hence \( g_a(G) = n - \beta(G) \). \( \square \)

Theorem 2.24. Every geodetic vertex cover of a connected graph G is a geodetic dominating set of G.

Proof. Let S be a geodetic vertex cover of G. Then S is both a geodetic set and a vertex cover of G. Since S is a vertex cover, every edge of G has at least one end in S and hence every vertex in \( V(G) - S \) has at least one neighbour in S so that S is a dominating set of G. Hence S is a geodetic dominating set of G. \( \square \)

Corollary 2.25. If G is any connected graph, then \( 2 \leq \gamma_\beta(G) \leq g_a(G) \leq n \).

Remark 2.26. Note that, \( \gamma_\beta(K_2) = 2 \). See graph G of Figure 2.5, \( S_1 = \{v_1, v_2, v_3\} \) is a \( g_a \) set and \( S_2 = \{v_1, v_3\} \) is a \( \gamma_\beta \) set of G. Thus \( g_a(G) = 3 \) and \( \gamma_\beta(G) = 2 \) and so \( \gamma_\beta(G) < g_a(G) \). And \( \gamma_\beta(K_n) = g_a(K_n) = n \).

3. Realization Results

By Theorem 2.3, \( 2 \leq \max\{\alpha(G), \gamma(G)\} \leq g_a(G) \leq n \). Also, we have \( g_a(G) \leq \min\{\alpha(G) + \gamma(G), n\} \). The following theorems give realization results for these parameters.

Theorem 3.1. If a and n are positive integers such that \( 2 \leq a \leq n \), then there exists a connected graph G of order n with \( g_a(G) = a \).

Proof. We prove this theorem by considering two cases.

Case (i) \( 2 \leq a = n \). Take \( G = K_n \), then from Theorem 2.9 (ii), \( g_a(G) = n = a \).

Case (ii) \( 2 \leq a < n \). Take \( H = K_a - 1 \), the complete graph on \( a - 1 \) vertices \( u_1, u_2, \ldots, u_{a-1} \). Add \( n - a + 1 \) new vertices \( v_1, v_2, \ldots, v_{n-a}, x \) to \( H \) and join the vertices \( v_1, v_2, \ldots, v_{n-a} \) to both \( u_{a-1} \) and \( x \). Thus we get the graph G of Figure 3.1.

Figure 3.1 : G
Let \( S = \{u_1, u_2, \ldots, u_{a-2}\} \) consist of all simplicial vertices of \( G \) so that by Theorem 1.1, they must belong to every geodetic set. Observe that \( S' = S \cup \{x\} \) is a \( g - \) set and the edges of \( K_{a-1} \) and the edges \( v_iv_i(1 \leq i \leq n-a) \) are covered by the vertices of \( S' \). Now, to cover the edges \( u_{a-1}v_i(1 \leq i \leq n-a) \), we must include at least the vertex \( u_{a-1} \) to \( S' \). Hence a \( ga \)-set of \( G \) is \( S' = \{u_1, u_2, \ldots, u_{a-2}, u_{a-1}, x\} \) with \(|S'| = a < n. \)

**Theorem 3.2.** Let \( a, b \geq 2 \) be any pair of integers. Then there is a connected graph \( G \) with \( \alpha(G) = a, g(G) = b \) and \( ga(G) = a + b \).

**Proof.** Consider a cycle \( C_4 : z_1, z_2, z_3, z_4, z_1 \) of order 4 and a path \( P : y_0, y_1, \ldots, y_{2(a-2)} \) of order \( 2(a-2) + 1 \). Obtain graph \( H \) from \( C_4 \) and \( P \) by joining the vertices \( z_3 \) in \( C_4 \) and \( y_0 \) in \( P \). Add \( b - 1 \) new vertices \( x_1, x_2, \ldots, x_{b-1} \) to \( H \) and join each to the vertex \( z_1 \). The resultant graph \( G \) is shown in Figure 3.2.

Observe that the set \( S = \{z_1, z_3, y_1, y_3, \ldots, y_{2(a-2) - 1}\} \) is a minimum vertex cover for \( G \) with \(|S| = a \).

![Figure 3.2: G](image)

Let \( S' = \{x_1, x_2, \ldots, x_{b-1}, y_{2(a-2)}\} \) be the set of all simplicial vertices of \( G \) so that they must belong to every geodetic set of \( G \). Moreover, \( S' \) itself is a geodetic set of \( G \) and hence \( S' \) is a minimum geodetic set of \( G \) with \(|S'| = b\). Thus \( \alpha(G) = a \) and \( g(G) = b \). Clearly \( S \cup S' \) is a \( ga \) - set and so \( ga(G) = a + b \).

**Theorem 3.3.** Every pair of integers \( a, b \) with \( 1 \leq a < b \) can be realized as the vertex covering number and geodetic vertex covering number, respectively, of some connected graph \( G \).

**Proof.** We prove this theorem by considering two cases.

**Case(i) \( 1 = a < b \).** Let \( G = K_{b+1} \) be a star. Then by Theorem 2.14 and Corollary 2.7, \( \alpha(G) = 1 \) and \( g_a(G) = b \).

**Case(ii) \( 2 \leq a < b \).** Take \( H = K_{a+1} \), the complete graph on \( a + 1 \) vertices \( u_1, u_2, \ldots, u_{a+1} \). Add \( b - a \) new vertices \( v_1, v_2, \ldots, v_{b-a} \) to \( H \) by joining each of them to \( u_{a+1} \) and get graph \( G \) of Figure 3.3.

![Figure 3.3: G](image)

Clearly the set \( S = \{u_2, u_3, \ldots, u_{a+1}\} \) is a minimum vertex cover of \( G \) with \(|S| = a \). Let \( S' = \{u_1, u_2, \ldots, u_{a+1}, v_1, v_2, \ldots, v_{b-a}\} \) be the set of all simplicial vertices of \( G \) so that by Theorem 1.1, they must belong to every geodetic set of \( G \). Moreover, \( S' \) itself is a geodetic set of \( G \) and hence \( S' \) is a minimum geodetic set of \( G \). Also, \( S' \) covers all the edges of \( G \). Hence \( S' \) is a minimum geodetic vertex cover of \( G \) with \(|S'| = b\).

**Theorem 3.4.** For any two positive integers \( a \) and \( b \) with \( 2 \leq a \leq b \), there exists a connected graph \( G \) with \( \alpha(G) = a \) and \( ga(G) = b \).

**Proof.** Let \( P : u_1, u_2, \ldots, u_{2(b-a+1)} \) be a path of order \( 2(b-a) + 1 \). Add \( a - 1 \) new vertices \( v_1, v_2, \ldots, v_{a-1} \) to \( P \) and join these to \( u_{2(b-a+1)} \). The resultant tree \( G \) is in Figure 3.4.

![Figure 3.4: G](image)

Clearly, \( S = \{u_1, v_1, v_2, \ldots, v_{a-1}\} \) is the set of all simplicial vertices of \( G \). Since \( G \) is a tree, by Theorem 1.2, \( g(G) = a \). It is clear that the set \( S \) covers the edges \( u_1u_2, v_1u_2(u_{2(b-a+1)}(1 \leq i \leq a - 1) \} \) and the remaining edges are covered by the vertices \( u_3, u_4, \ldots, u_{(b-a)+1} \). Hence the minimum geodetic vertex cover is \( S' = \{u_3, u_4, \ldots, u_{(b-a)+1}, u_1, v_1, v_2, \ldots, v_{a-1}\} \) with \(|S'| = b\).
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