Dual generalized B-spline functions and their applications in several approximation problems

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Abstract

A new construction method of dual generalized B-spline functions is presented in this paper. Due to the importance of generalized B-splines, several interesting applications of these dual functions in Computer Aided Manufacturing are also given. The purposes of the paper are as follows: 1) As is known to all, B-spline curves, surfaces and related techniques play important roles in the fields of Computer Aided Geometric Design, Computer Aided Manufacturing and Computer Aided Design for their perfect properties. 2) Recently, on the basis of recursive formula of classical B-spline bases, Juhász and Róth (2013) put forward the generalized B-spline bases, which are generated by monotone increasing and continuous “core” functions. 3) With these flexible “core” functions, generalized B-spline curves and surfaces not only hold almost the same perfect properties which classical B-splines hold but also show more flexibility in practical applications. It is necessary to further study dual generalized B-spline bases. The whole arrangement of the paper is divided into four parts. Firstly, a review of dual bases concluding the construction method is given. Secondly, one fresh method of constructing dual generalized B-spline functions is presented in order to extend their applications. With the help of the dual bases functions, one can easily find the best approximation of any function with expressions of generalized B-spline bases; furthermore, one has no need to consider the orthogonal bases of the corresponding space. After that, several applications of dual generalized B-spline functions in Computer Aided Manufacturing such as least square approximation, curve offsetting and degree reduction are illustrated. Numerical examples show that dual functions greatly simplify these approximation problems. Furthermore, better approximation effects can be achieved by changing core functions which exhibits the effects of ”core”. The paper not only focuses on the construction of generalized B-spline bases which has never been built before, but also applies the dual bases in the practical applications of Computer Aided Manufacturing. We hope it could be a useful bridge to connect theories and practical applications.

Key words: Generalized B-spline, Dual bases, Core function, Least square, Curve offsetting, Degree reduction

1. Introduction

The classical order \( k \) B-spline functions \((k \geq 2)\) are recursively defined as a special combination of two consecutive B-spline functions of order \(k-1\). Recently, Juhász and Róth (2013) generalized the concept of the classical normalized B-spline functions by considering monotone increasing continuously differentiable nonlinear core functions instead of the classical linear one. This class of generalized B-spline functions possess many advantageous properties, such as non-negativity, local support, the partition of unity, the effect of multiple knot values, endpoint interpolation conditions. These so called core functions are not only interesting from a theoretical perspective, but they also provide a large variety of shapes. Considering the good properties of the dual bases and investigating more applications of the generalized B-spline functions in geometric approximation, the dual generalized B-spline functions in the sense of
inner product are given. Before we introduce the construction of dual generalized B-spline functions, let's give a brief review of dual function.

Dual function is a useful tool in Computer Aided Geometric Design when talking about geometric approximation. Normally, there are two kinds of dual functions, one is in the sense of differentiation, and the other is in the sense of inner product. The former can easily help us to realize bases transformations between the fixed basis and the other bases once we get the dual function of the fixed basis. The latter can effectively help us to realize least square approximation of any integrable function with a linear combination of suitable bases. Thus, dual function attracts increasing attentions in the fields of Computer Aided Geometric Design and Approximation Theory recently.

In the sense of differentiation, Othman and Goldman (1997) gave an explicit formula for the dual basis functions to the odd generalized Ball basis. Xi (1997) studied the dual function of Ball basis and its corresponding Marsden Identity. Based on these, Xi (1997) realized the transformation formula between Bézier curves and Said-Ball curves. After that, Wu (2000, 2004), Jiang (2006) and Zhang et al., (2009a) presented the dual bases of different kinds of polynomial bases, such as NS Power basis, Wang-Ball bases. Thus, transformation formulas between different bases are easily obtained instead of computing the inverses of matrices. Considering Bernstein polynomials have many important properties which make them the most commonly used bases in approximation theory and CAGD, but they are not orthogonal and not appropriate to be used in the least square approximation. To solve the above mentioned problems, Jüttler (1998) gave an explicit formula for the dual basis functions of the Bernstein polynomials with respect to the usual inner product of Hilbert space. Rababah and Al-Natour (2007, 2008) generalized these results to the dual basis functions of the Bernstein polynomials with respect to the Jacobi weight function. Lewanowicz and Woźni (2006a, 2006b) discussed the dual functions for the generalized Bernstein polynomials in terms of big q-Jacobi polynomials and also investigated dual bases for "triangular" Bernstein polynomials. Zhang, Wu and Tan (2009b, 2010) gave explicit formulas for the dual basis functions of generalized Ball bases and unified their representations. Woźni (2013) presented a brand new method to construct dual bases and he further improved it and constructed B-spline dual bases in 2014 (Woźni 2009, 2010) gave explicit formulas for the dual basis functions of generalized Ball bases and unified their representations. Woźni (2013) presented a brand new method to construct dual bases and he further improved it and constructed B-spline dual bases in 2014 (Woźni 2009, 2010) gave explicit formulas for the dual basis functions of generalized Ball bases and unified their representations. Woźni (2013) presented a brand new method to construct dual bases and he further improved it and constructed B-spline dual bases in 2014 (Woźni 2009, 2010) gave explicit formulas for the dual basis functions of generalized Ball bases and unified their representations. Woźni (2013) presented a brand new method to construct dual bases and he further improved it and constructed B-spline dual bases in 2014 (Woźni 2009, 2010) gave explicit formulas for the dual basis functions of generalized Ball bases and unified their representations. Woźni (2013) presented a brand new method to construct dual bases and he further improved it and constructed B-spline dual bases in 2014 (Woźni 2009, 2010) gave explicit formulas for the dual basis functions of generalized Ball bases and unified their representations. Woźni (2013) presented a brand new method to construct dual bases and he further improved it and constructed B-spline dual bases in 2014 (Woźni 2009, 2010) gave explicit formulas for the dual basis functions of generalized Ball bases and unified their representations. Woźni (2013) presented a brand new method to construct dual bases and he further improved it and constructed B-spline dual bases in 2014 (Woźni 2009, 2010) gave explicit formulas for the dual basis functions of generalized Ball bases and unified their representations. Woźni (2013) presented a brand new method to construct dual bases and he further improved it and constructed B-spline dual bases in 2014 (Woźni 2009, 2010) gave explicit formulas for the dual basis functions of generalized Ball bases and unified their representations. Woźni (2013) presented a brand new method to construct dual bases and he further improved it and constructed B-spline dual bases in 2014 (Woźni 2009, 2010) gave explicit formulas for the dual basis functions of generalized Ball bases and unified their representations. Woźni (2013) presented a brand new method to construct dual bases and he further improved it and constructed B-spline dual bases in 2014 (Woźni 2009, 2010) gave explicit formulas for the dual basis functions of generalized Ball bases and unified their representations.

Our paper is arranged as follows. Brief introduction and the main idea of the paper are outlined in Section 1. Some notations and preliminaries are given in Section 2. The general method to construct the dual generalized B-spline bases are introduced in Section 3. Some applications of dual generalized B-spline functions are illustrated in Section 4. Conclusions are made at last.

2. Notations and preliminaries
2.1 Generalized B-spline functions

With control points \( \{d_j\}_{j=0}^n \) and knot vectors \( U = \{u_j\}_{j=0}^{n+k+1} \) \( (u_i \leq u_j \text{ when } i \leq j) \), we can define the following B-spline curve of degree \( k (k \in \mathbb{N}) \) (Juhász and Róth, 2013),

\[
p(u) = \sum_{j=0}^{n} d_j N_j^k(u), \quad u \in [u_k, u_{k+1}]
\]

Here \( N_j^k(u) (0 \leq j \leq n) \) are the normalized B-spline basis functions which are defined on the interval \([u_j, u_{j+1}]\) and fulfill the following recursive relation

\[
N_j^k(u) = \begin{cases} 
1, & u \in [u_j, u_{j+1}] \\
0, & \text{otherwise}
\end{cases}
\]
\[ N^k_j(u) = \frac{u-u_j}{u_{j+k}-u_j} N^{k-1}_{j+k-1}(u) + \frac{u_{j+k+1}-u}{u_{j+k+1}-u_{j+1}} N^{k-1}_{j+1}(u), \]
\[ = \phi_j^{k-1}(u) N^{k-1}_{j}(u) + (1-\phi_j^{k-1}(u)) N^{k-1}_{j+1}(u) \]  
\hspace{1cm} (1) 

where

\[ \phi_j^{k-1}(u) = \frac{u-u_j}{u_{j+(k-1)+1}-u_j} . \]

and the multiplicity of knot values \( u_j \) are at most \( k+1 \), and define \( \phi_j^0 = 0 \).

**Definition 1** (Juhász and Róth, 2013): Given enough smooth monotone increasing function \( \phi(u) : [0,1] \to [0,1] \) that fulfills the following two conditions

\[ \min_{u \in [0,1]} \phi(u) = \phi(0) = 0, \]
\[ \max_{u \in [0,1]} \phi(u) = \phi(1) = 1. \]  
\hspace{1cm} (2) 

Function \( \phi(u) \) is called core function.

**Definition 2** (Juhász and Róth, 2013): By means of the core function \( \phi(u) : [0,1] \to [0,1] \) and the knot vector \( U \), order \( k (k \geq 1) \) generating function can be defined as follows:

\[ \phi_j^i(u) = \begin{cases} \phi_j(u) & u \in [u_i, u_{i+1}] \\ 0 & \text{otherwise} \end{cases} \]  
\hspace{1cm} (3) 

Replace the linear function \( \phi_j^{k-1}(u) \) with arbitrary monotone increasing functions and normalized B-spline basis functions can be generalized as follows.

**Definition 3** (Juhász and Róth, 2013): Generalized B-spline basis functions which are determined by the knot vector \( U \) and the relevant generating function \( \phi_j^i(u) \) are

\[ N_j^k(u) = \begin{cases} 1 & u \in [u_i, u_{i+1}] \\ 0 & \text{otherwise} \end{cases} \]  
\[ \phi_j^k(u) = \phi_j^{k-1}(u) N_j^{k-1}(u) + (1-\phi_j^{k-1}(u)) N_{j+1}^{k-1}(u), u \in R, k \geq 1 \]  
\hspace{1cm} (4) 

Now, we can obtain generalized arbitrary degree B-splines with respect to the uniform and non-uniform knot vectors by the above definitions. Fig. 1 gives the cubic generalized B-spline basis functions \( N_j^3(u) \) which are defined on interval \([u_i, u_{i+4}]\) and their corresponding core function is

\[ \phi(u) = u^p, \quad u \in [0,1], \quad p > 1. \]

Generalized B-spline basis functions with same core function but different parameter \( p \) are illustrated in Fig.1. The left figure shows the corresponding uniform generalized B-spline basis functions, and the right figure illustrates the corresponding non-uniform generalized B-spline basis functions. Let \( p=1 \), the generalized B-spline basis functions degenerate into normal B-spline basis functions.
2. Properties of dual functions in the sense of inner product

Suppose that linear space \( K_N \) is spanned by independent functions \( b_0, b_1, ..., b_N \). We call the following functions \( d_0^{(N)}, d_1^{(N)}, ..., d_N^{(N)} \) are the dual functions of the space \( K_N \) if the following conditions are satisfied

\[
\begin{align*}
\text{span}\{d_0^{(N)}, d_1^{(N)}, ..., d_N^{(N)}\} &= \text{span}\{b_0, b_1, ..., b_N\} \\
\langle b_j, d_i^{(N)} \rangle &= \delta_{ij} (0 \leq i, j \leq N)
\end{align*}
\]

Here \( \langle \cdot, \cdot \rangle \) defines the inner product, \( \delta_{ij} \) equals to 1 when \( i = j \) and 0 otherwise (the so-called Kronecker delta).

Now we introduce some significant properties of dual bases in the sense of inner product. On one hand, any \( h \in K_N \) can be represented as a linear combination of bases \( b_0, b_1, ..., b_N \):

\[
h = \sum_{k=0}^{N} c_k b_k, \quad \text{where} \quad c_k = \langle h, d_k^{(N)} \rangle \quad (k = 0, 1, ..., N). \tag{6}
\]

On the other hand, in the space \( K_N \), the element \( q^* \) is the best approximation of any function \( g \) in the following sense

\[
\| g - q^* \| = \min_{q \in K_N} \| g - q \|,
\]

where \( \| \cdot \| = \sqrt{\langle \cdot, \cdot \rangle} \) and \( q^* \) has the following forms of expression:

\[
q^* = \sum_{k=0}^{N} e_k b_k, \quad \text{where} \quad e_k = \langle g, d_k^{(N)} \rangle \quad (k = 0, 1, ..., N), \tag{7}
\]

which helps one find optimal element without considering the orthogonal bases of the space \( K_N \).

3. Dual generalized B-spline functions

In view of generalizing the applications of generalized B-spline functions, especially their applications in approximation problems, one new method of constructing dual bases of generalized B-spline functions is introduced in this section.

3.1 Previous construction method

Firstly, let’s have a brief review of one general method to construct the dual functions \( D_m = \{d_0^{(m)}, d_1^{(m)}, ..., d_m^{(m)}\} \) which was given by Woźny (2013).
In the simplest case of \( m=0 \), let \( d_0^{(0)} = \{b_0, b_1\} \), \( D_0 = \{d_0^{(0)}\} \) is obtained. Assume that for the basis \( B_m = \{b_0, b_1, \ldots, b_n\} \), the elements of the dual basis \( D_m = \{d_0^{(m)}, d_1^{(m)}, \ldots, d_n^{(m)}\} \) with respect to the inner product \( \langle \cdot, \cdot \rangle \) are already known. Woźni proposed a method to construct the dual functions

\[
D_{m+1} = \{d_0^{(m+1)}, d_1^{(m+1)}, \ldots, d_n^{(m+1)}\}
\]

for the basis

\[
B_{m+1} = \{b_0, b_1, \ldots, b_n, b_{m+1}\} = B_m \cup \{b_{m+1}\}
\]
satisfying the following conditions

\[
\begin{align*}
\text{span} D_{m+1} &= \text{span} B_{m+1} \\
\langle b_i, d_j^{(m+1)} \rangle &= \delta_{ij} \quad (0 \leq i, j \leq m+1)
\end{align*}
\]

The next step is trying to find the dual functions \( d_j^{(m+1)} \) (\( 0 \leq j \leq m+1 \)) which has the following form

\[
d_j^{(m+1)} = \sum_{k=0}^n c_{j,k}^{(m+1)} d_k^{(m)} + c_{j,m}^{(m+1)} b_{m+1}
\]

where the coefficients \( c_{j,k}^{(m+1)} \) are chosen to fulfill the duality conditions (Eq.10) which satisfy the following system of linear equations

\[
\begin{bmatrix}
1 & 0 & \cdots & 0 & \cdots & 0 & \nu_0^{(m+1)} \\
0 & 1 & \cdots & 0 & \cdots & 0 & \nu_1^{(m+1)} \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 1 & \cdots & 0 & \nu_j^{(m+1)} \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & \cdots & 1 & \nu_m^{(m+1)} \\
\end{bmatrix}
\begin{bmatrix}
1 \\
0 \\
\vdots \\
0 \\
\vdots \\
0
\end{bmatrix}
= \begin{bmatrix}
0 \\
c_{j,0}^{(m+1)} \\
\vdots \\
c_{j,m}^{(m+1)} \\
\vdots \\
c_{j,m+1}^{(m+1)}
\end{bmatrix}
\]

where

\[
u_l^{(m+1)} = \langle b_{m+1}, d_l^{(m)} \rangle \quad (0 \leq l \leq m), \quad \nu_i^{(m+1)} = \langle b_i, b_{m+1} \rangle \quad (0 \leq i \leq m+1).
\]

Let \( M_{m+1} \) represent the matrix of the linear equations (Eq.12). In the reference (Woźni, 2013), it has been verified that matrix \( M_{m+1} \) possesses a simple \( LU \) decomposition, \( M_{m+1} \) is a reversible matrix and it can be inverted in the time \( O(m^2) \).

### 3.2 New construction method and dual generalized B-spline functions

Based on the above construction method, we propose another general method of constructing dual functions. Suppose \( D_m = \{d_0^{(m)}, d_1^{(m)}, \ldots, d_n^{(m)}\} \) is known, we are going to construct dual functions \( D_{m+1} = \{d_0^{(m+1)}, d_1^{(m+1)}, \ldots, d_n^{(m+1)}\} \).

**Theorem 3.1** The elements of the dual bases \( D_{m+1} = \{d_0^{(m+1)}, d_1^{(m+1)}, \ldots, d_n^{(m+1)}\} \) can be constructed by following formulas

\[
d_j^{(m+1)} = \sum_{k=0}^n c_{j,k}^{(m+1)} d_k^{(m)} + c_{j,m+1}^{(m+1)} b_{m+1},
\]

and

\[
d_j^{(m+1)} = \sum_{k=0}^j c_{j,k}^{(m+1)} d_k^{(m)} + \sum_{k=j+1}^{n} c_{j,k}^{(m+1)} d_k^{(m)} \quad (j = m, m-1, \ldots, 0).
\]
Proof: Firstly, since \( d_{m+1}^{(m)} \in \text{span} D_{m+1} = \text{span} \{ D_m \cup \{ b_{m+1} \} \} \), it can be expressed as the linear combination of the corresponding bases
\[
d_{m+1}^{(m)} = \sum_{k=0}^{m} c_{m+1,k}^{(m)} d_k^{(m)} + c_{m+1,m}^{(m)} b_{m+1},
\]
where the coefficients \( c_{m+1,k}^{(m)} (0 \leq k \leq m+1) \) are chosen to fulfill the duality conditions (Eq.10). By solving the simple system of linear equations:
\[
\begin{bmatrix}
1 & 0 & \cdots & 0 & \langle b_0, b_{m+1} \rangle \\
0 & 1 & \cdots & 0 & \langle b_1, b_{m+1} \rangle \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 1 & \langle b_m, b_{m+1} \rangle \\
\end{bmatrix}
\begin{bmatrix}
c_{m+1,0}^{(m)} \\
c_{m+1,1}^{(m)} \\
\vdots \\
c_{m+1,m}^{(m)} \\
1
\end{bmatrix}
= \begin{bmatrix}
0 \\
0 \\
\vdots \\
0 \\
1
\end{bmatrix},
\]
the coefficients \( c_{m+1,k}^{(m)} (0 \leq k \leq m+1) \) can be obtained. Thus, we have Eq.13.

Since
\[
d_m^{(m)} \in \text{span}\left\{ d_0^{(m)}, d_1^{(m)}, \ldots, d_{m-1}^{(m)}, b_{m+1}^{(m)} \right\},
\]
\[
d_{m+1}^{(m)} \in \text{span}\left\{ d_0^{(m)}, d_1^{(m)}, \ldots, d_{m-1}^{(m)}, b_{m+1}^{(m)} \right\},
\]
\[
d_j^{(m)} \in \text{span}\left\{ d_0^{(m)}, d_1^{(m)}, \ldots, d_{m-1}^{(m)}, d_j^{(m)} \right\},
\]
they can be expressed as
\[
d_{m+1}^{(m)} = \sum_{k=0}^{m} c_{j,k}^{(m+1)} d_k^{(m)} + \sum_{k=j+1}^{m+1} c_{j,k}^{(m+1)} d_k^{(m)} \quad (j = m, m-1, \ldots, 0).
\]

Then for any fixed \( j \) \( (j = m, m-1, \ldots, 0) \) we can calculate the \( c_{j,k}^{(m+1)} (0 \leq k \leq m+1) \) based on the duality conditions by solving a much simpler system of linear equations
\[
\begin{bmatrix}
1 & 0 & \cdots & 0 & \cdots & 0 & 0 \\
0 & 1 & \cdots & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 1 & \cdots & 0 & 0 \\
\end{bmatrix}
\begin{bmatrix}
c_{j,0}^{(m+1)} \\
c_{j,1}^{(m+1)} \\
\vdots \\
c_{j,m}^{(m+1)} \\
1
\end{bmatrix}
= \begin{bmatrix}
0 \\
0 \\
\vdots \\
0 \\
1
\end{bmatrix},
\]
where \( u_k^{(m+1)} = \langle b_{m+1}, d_k^{(m)} \rangle \) \( (k = 0, 1, \ldots, j) \). Thus we finish constructing \( D_{m+1} \). \( \square \)

Let \( A_{m+1} \) denote the matrix of the system (Eq.17),
\[
A_{m+1} = \begin{bmatrix}
E_{m+1} & O_{m+1} \\
C_{m+1} & 1
\end{bmatrix} \in \mathbb{R}^{(m+2) \times (m+2)}
\]
where \( E_{m+1} \) represents the identity matrix of dimension \( m+1 \), \( O_{m+1} \) denotes the column vector of dimension \( m+1 \) which elements are zeros. And \( C_{m+1} = [u_0^{(m+1)} \ u_1^{(m+1)} \ \cdots \ u_j^{(m+1)} \ 0 \ \cdots \ 0] \) is a row vector of dimension \( m+1 \).

The following theorem is obviously true based on (Eq.17).

Theorem 3.2 The matrix \( A_{m+1} \) is lower triangular matrix and non-singular and its inverse matrix \( A_{m+1}^{-1} \) has the
following form
\[
A_{n+1}^{-1} = \begin{bmatrix}
E_{n+1}
& O_{n+1}

-C_{n+1}
& 1
\end{bmatrix}
\]

By observing formula (Eq.17), it is obvious that the whole elements of \((j+1)\)th column of the matrix \(A_{n+1}^{-1}\) are
\[
\begin{bmatrix}
E_{j+1}^{(m+1)}
& \cdots
& E_{j+1}^{(m+1)}

-C_{j+1}^{(m+1)}
& \cdots
& -C_{j+1}^{(m+1)}
\end{bmatrix}
(j = 0, 1, \ldots, m + 1).
\]

Comparing with the matrix "\(A_{n+1}\)" of linear system in (Woźni, 2013) which has \(LU\) decomposition but is not a lower triangular, the matrix "\(A_{n+1}\)" of our method has a much simpler structure. Thus we have the following remark.

**Remark 3.1.** The matrix \(A_{n+1}\) can be inverted in the time \(O(1)\) once we obtain the values of \(u_{k}^{(m+1)} (k = 0, 1, \ldots, j)\).

Now, dual generalized B-spline basis functions can be constructed as follows.

Given a knot vector \(U = [u_0, u_1, \ldots, u_{m+1}]\) \((u_i \leq u_j \text{ if } i \leq j)\), here \(k, n \in N\), and a core function \(\phi(u): [0, 1] \rightarrow [0, 1]\) . On the basis of the definition for generalized B-spline basis functions, we can get generalized B-spline basis functions \(N_{i}^{k,n}(u) (i = 0, 1, \ldots, n)\) of degree \(k\) for knots \(u_0, u_1, \ldots, u_{m+1}\). Let \(D_{j}^{k,n}(u) (i = 0, 1, \ldots, n)\) be the dual generalized B-spline basis functions satisfying the duality conditions, two spaces generated by them are
\[
\Omega_{\Phi} = \text{span} \{N_{0}^{k,n}(u), N_{1}^{k,n}(u), \ldots, N_{n}^{k,n}(u)\}
\]
\[
\Omega_{D} = \text{span} \{D_{0}^{k,n}(u), D_{1}^{k,n}(u), \ldots, D_{n}^{k,n}(u)\}
\]
then we will have
\[
\Omega_{\Phi} = \Omega_{D}
\]
\[
\{N_{i}^{k,n}(u), D_{j}^{k,n}(u)\} = \delta_{ij} \quad (0 \leq i, j \leq n),
\]
where \(\langle \cdot, \cdot \rangle\) denotes a given inner product.

### 4. Numerical examples

The best approximation can be obtained with the help of dual base method. In this part, we illustrate the figures of dual generalized B-spline bases, and show the applications of dual generalized B-spline basis in degree reduction, piecewise function approximation and offset approximation.

**Example 1.** Given cubic generalized B-spline functions \(N_{i}^{3}(u) (i = 0, 1, \ldots, 8)\) with the following core function
\[
\phi(u) = \lambda(2u^3 - 3u^2 + u) - 2u^3 + 3u^2 \quad u \in [0, 1], \quad \lambda \in [0, 3],
\]
and the knot vector \(U = [0, 0, 0, \frac{1}{4}, \frac{1}{2}, \frac{1}{2}, \frac{3}{4}, 1, 1, 1, 1, 1]\), which is given in Fig.2. Then we can calculate the dual generalized B-spline functions \(D_{j}^{3}(u) (i = 0, 1, \ldots, 8)\) with the method proposed in our paper. Fig.3 and Fig.4 give the dual generalized B-spline functions respectively with respect to the following forms of inner product
\[
\langle h, f \rangle_{\Phi} = \int_{0}^{1} h(x) f(x)dx \quad \text{and} \quad \langle h, f \rangle_{H} = \sum_{i=0}^{N} h\left(\frac{i}{N}\right) f\left(\frac{i}{N}\right) \quad H \in N.
\]
Fig. 2. Figures of generalized B-spline functions of degree 3 with different $\lambda$. Fig. 2 (a) illustrates the figures of bases when $\lambda = 0$, while Fig. 2 (b) illustrates the figures of bases when $\lambda = 2.5$.

Fig. 3. Figures of corresponding dual generalized B-spline bases with respect to inner product $\langle h, f \rangle_p = \int h(x) f(x) dx$. Fig. 3 (a) illustrates the figures of dual bases when $\lambda = 0$, while Fig. 3 (b) illustrates the figures of dual bases when $\lambda = 2.5$.

Fig. 4. Figures of corresponding dual generalized B-spline bases with respect to inner product $\langle h, f \rangle = \sum_{i=1}^{H} h_i \left( \frac{s}{H} \right) f_i \left( \frac{s}{H} \right), \quad H = 160$. Fig. 4 (a) illustrates the figures of dual bases when $\lambda = 0$, while Fig. 4 (b) illustrates the figures of dual bases when $\lambda = 2.5$.

Example 2. Given a quintic generalized B-spline $N^5_i(u) (0 \leq i \leq 22)$ with the core function

$$\phi(u) = 6u^5 - 15u^4 + 10u^3, \quad u \in [0,1]$$

and the knot vector

$$U = \left[ 0, 0, 0, 0, 0, 0, 1, 2, 3, 4, 5, 6, 7, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 1, 1, 1, 1, 1, 1 \right],$$

we obtain the generalized B-spline curves $p(u) = \sum_{i=0}^{12} P_i N^5_i(u)$ with respect to the control points: $(6.05, 5), (5.2, 3), (4.35, 1.65), (3.5, 0.95), (2.25, 0.5), (0.9, 0.9), (0.5, 2.15), (0.9, 3.7), (1.9, 5), (3.5, 5.05), (4.25, 3.2), (4.5, 0.1), (6.1, 0.5)$. Fig. 5 shows the application of dual generalized B-spline in degree reduction. Fig. 5 (a) illustrates this degree 5 generalized B-spline curve which looks like an "alpha". Now, we consider the problem of finding a lower degree $k$ generalized B-spline planar curve $q(u)$

$$q(u) = \sum_{i=0}^{m} Q_i N^k_i(u), \quad m = 7 + k,$$

which is the optimal degree reduction approximation of curve $p(u)$ with respect to discrete least-squares norm in
space $B_u = \text{span}\{N^0_0(u), N^1_0(u), \ldots N^k_0(u)\}$. Suppose $D^i_u(u)(i = 0, \ldots, m)$ are dual generalized B-spline functions of $N^k(u)$ with respect to the inner product $\langle \cdot, \cdot \rangle_N$ or $\langle \cdot, \cdot \rangle_H$, and $Q_i (i = 0, \ldots, m)$ are the control points of $q(u)$. The control points of the curve $q(u)$ can be easily obtained by the formula

$$Q_i = \langle p(u), D^i_u(u) \rangle \quad i = 0, 1, \ldots, 7 + k$$

In addition, $q(u)$ has the same domain with $p(u)$, that is to say, $N^k(u)$ has the following knot vector

$$\mathcal{U} = \left[ \begin{array}{cccccc}
0, & \frac{1}{8}, & \frac{2}{8}, & \frac{3}{8}, & \ldots, & \frac{7}{8}, \ldots, 1, 1, \ldots, 1 \end{array} \right].$$

Fig. 5(b) illustrates the approximation result by using a quartic generalized B-spline curve (blue), black curve is the original degree-5 generalized B-spline curve. Fig. 6 represents the error curves for different degrees.

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{fig5.png}
\caption{Application of dual generalized B-spline in degree reduction}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{fig6.png}
\caption{Error curves for different degrees. (red, green and black for degree 2, 3, and 4, respectively)}
\end{figure}

**Example 3.** Define a function $g(u)$ as follows (see Fig. 7)

$$g(u) = \begin{cases} 8(u - \frac{1}{8})(u - \frac{1}{2}) & u \in [0, 0.5] \\
\sin\left(2\pi u - \frac{1}{2}\right) & u \in [0.5, 1] \end{cases},$$

then we can find $p(u)$ as an approximation for $g(u)$ in the sense of least square. Now, we use generalized B-spline
bases \( N^3_i(u) \) \((i = 0,1,...,8)\) with the knot vector \( U = [0,0,0,0,1/4, 2/4, 2/4, 3/4, 4, 4, 4, 4, 4, 1,1,1,1] \). This time we use two different core functions

\[
\phi_1(u) = 2u^3 - 3u^2 + 2u, \quad u \in [0,1]
\]

\[
\phi_2(u) = \frac{\sin(2\pi w u)}{2\pi w} + u, \quad u \in [0,1], 1 \leq w \in N
\]

Let cubic generalized B-spline curve

\[
h(u) = \sum_{i=0}^{8} c_i N^3_i(u), \quad c_i = \langle g(u), D^1_i(u) \rangle
\]

be the best approximation in the sense of least square approximation, Where \( D^1_i(u)(i = 0,\cdots,8) \) denote the dual bases of generalized B-spline basis functions \( N^3_i(u)(i = 0,1,...,8) \) and \( \langle \cdot, \cdot \rangle \) represent the inner product. Fig.8 illustrates two error curves by using different core functions with respect to the following error functions

\[
e(u) = g(u) - p(u), \quad u \in [0,1]
\]

Black curve shows the approximation effect by using core function \( \phi_1(u) \) and blue curve illustrates the effect by using core function \( \phi_2(u) \). It shows a better approximation effect can be obtained by changing a suitable core function.

**Example 4.** Let us define a generalized B-spline curve of degree 3

\[
[x(u), y(u)] = q(u) = \sum_{j=0}^{12} Q_j N^3_j(u)
\]

where \( Q_j \) \((i = 0,\cdots,12)\) denote the control points \((0, 7), (3.5, 7), (5, 10), (6, 7), (9.5, 7), (7, 4.5), (8, 1), (5, 3), (2, 1), (3, 4.5), (0, 7), (3.5, 7), (5, 10)\) \( N^3_i(u)(i = 0,1,2,...,12)\) denote the generalized B-spline bases with respect to the knot vector

\[
U = [0, \frac{1}{16}, \frac{2}{16}, \cdots, \frac{15}{16}, 1]
\]

and the core function is

\[
\phi(u) = \lambda(2u^3 - 3u^2 + u) - 2u^3 + 3u^2 \quad u \in [0,1], \quad \lambda \in [0,3].
\]

Let

\[
p(u) = q(u) + d \cdot N(u)
\]
denote the offset curve of \( q(u) \), where \( d \) and \( N(u) \) represent the offset distance and unit normal vector at any point of \( q(u) \), and

\[
N(u) = \pm \frac{(-y'(u),x'(u))}{\sqrt{(y'(u))^2+(x'(u))^2}}
\]

Generally, offset curves are generally non-rational except for some special curves (PH curves or OR curves). Then by means of dual generalized B-spline functions with inner product, we can find

\[
p_d(u) = \sum_{i=0}^{12} p_i N_i^d(u),
\]

as the least square approximation of offset curve \( p(u) \) with control points,

\[
P_i = \langle p(u), D_i^d(u) \rangle
\]

Where \( D_i^d(u) \) \( (i = 0, \ldots, 12) \) represent the dual bases of generalized B-spline functions \( N_i^d(u) \) \( (i = 0, 1, \ldots, 12) \). Fig.9 is the original star curve (red) and its approximation offset curve (blue) with \( d = 0.1 \). Fig.10 shows the error curve when we use cubic dual generalized B-spline bases to approximate the offset curve with respect to inner product \( \langle h, f \rangle_H \) \( (H = 100) \).

![Fig.9. The star curve (red) and its approximating offset curve (blue)](image)

![Fig.10. The error curve of the star curve](image)

5. Conclusions

Juhász and Róth (2013) put forward the generalized B-spline bases, which is generated by monotone increasing nonlinear core functions. In consideration of the flexible usage of core functions and generalizing the applications of generalized B-spline bases, a new method to construct the corresponding dual bases is presented. The advantages of our algorithm are: the matrix of the linear equations system we built is low triangular and can be solved very easily; best approximation can be obtained by using dual generalized B-spline bases. Applications of dual generalized B-splines in degree reduction, piecewise function approximation, offset approximation are given. These applications can be extended to generalized B-spline surfaces in a similarly way. Woźny's methods (Woźny, 2013, 2014) plus our method are three general construction methods which can be used for building different dual bases with inherited property. Although no concrete dual functions are obtained, but they can be constructed step by step and greatly decrease the complexity of calculation. Combining the great applications of dual bases in approximation problems and different bases functions in different system, these general construction methods will have significant effects in real applications.

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References

Chen, X.D., Ma, W.Y. and Paul, J.C., Multi-degree reduction of Bézier curves using reparameterization, Computer-Aided Design, Vol.43, No.2 (2011), pp.161-169.

Jiang, P., Wu, H.Y. and Tan, J.Q., The dual functionals for the generalized Ball basis of Wang-Said type and basis transformation formulas, Numerical Mathematics-A Journal of Chinese Universities English Series, Vol.15, No.3(2006), pp.248-256.

Juhász, I. and Róth, Á., A class of generalized B-spline curves, Computer Aided Geometric Design, Vol.30, No.1(2013), pp.85-115.

Jüttler, B., The dual basis functions for the Bernstein polynomials, Advances in Computational Mathematics, Vol.8, No.1 (1998), pp.345-352.

Lewanowicz, S. and Woźni, P., Dual generalized Bernstein basis, Journal of Approximation Theory, Vol.138, No. 2 (2006a), pp. 129–150.

Lewanowicz, S. and Woźni, P., Connections between two-variable Bernstein and Jacobi polynomials on the triangle, Journal of Computational and Applied Mathematics, Vol.197, No.2 (2006b), pp.520–533.

Othman, W.A.M. and Goldman, R.N., The dual basis functions for the generalized Ball basis of odd degree, Computer Aided Geometric Design, Vol.14, No.5 (1997), pp.571–582.

Rababah, A. and Al-Natour, M., The weighted dual functional for the univariate Bernstein basis, Applied Mathematics and Computation, Vol.186, No.2 (2007), pp.1581–1590.

Rababah, A. and Al-Natour, M. Weighted dual functions for Bernstein basis satisfying boundary constrains, Applied Mathematics and Computation, Vol.199, No.2 (2008), pp:456–463.

Rababah, A. and Stephen, M., Linear methods for G1, G2 and G3-Multi-degree reduction of Bézier curves, Computer-Aided Design, Vol.45, No.2 (2013), pp.405-414.

Woźni, P., Construction of dual bases, Journal of Computational and Applied Mathematics, Vol.245, No.1 (2013), pp. 75-85.

Woźni, P., Construction of dual B-spline functions, Journal of Computational and Applied Mathematics, Vol.260, No.1 (2014), pp.301-311.

Woźni, P. and Lewanowicz S., Multi-degree reduction of Bézier curves with constraints, using dual Bernstein basis polynomials, Computer Aided Geometric Design, Vol.26, No.5 (2009), pp. 566–579.

Wu, H.Y., Unifying representation of Bézier curve and generalized Ball curves, Applied Mathematics: A Journal of Chinese Universities (Ser. B), Vol.15, No.1 (2000), pp.109–121.

Wu, H.Y., Dual bases for a new family of generalized Ball bases, Journal of Computational Mathematics, Vol.22, No.1 (2004), pp.79–88.

Xi, M.C., Conjugate basis of Ball basis function and its application, Mathematica Numerica Sinica, Vol.19, No.2(1997), pp.147–153 (in Chinese)

Zhang, L., Tan, J.Q. and Dong, Z.Y., The Dual bases for the Bézier-Said-Wang type generalized Ball polynomial bases and their applications. Applied Mathematics and Computation, Vol.217, No.1 (2010), pp. 3088-3101.

Zhang, L., Tan, J.Q., Shi, J. and Dong, Z.Y., The applications of weighted dual functions of Bernstein basis in curve offsetting, Journal of Computer-Aided Design & Computer Graphics, Vol.23, No.12, (2011), pp.1987-1993.

Zhang, L., Tan, J.Q. and Wu, H.Y., Liu, Z., The weighted dual functions for Wang-Bézier type generalized ball bases and their application, Applied Mathematics and Computation, Vol.215, No.1 (2009), pp.22-36.

Zhang, L., Wu, H.Y. and Tan, J.Q., Dual Bases for Wang- Bézier Basis and Their Applications, Applied Mathematics and Computation, Vol.214, No. 1(2009a), pp.218-227.

Zhang, L., Wu, H.Y. and Tan, J.Q., Dual basis functions for the NS power and their applications, Applied Mathematics and Computation, Vol.207, No. 2(2009b), pp.434–441.