The quantum bialgebra associated with the eight-vertex R-matrix

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Abstract

The quantum bialgebra related to the Baxter’s eight-vertex R-matrix is found as a quantum deformation of the Lie algebra of \( sl(2) \)-valued automorphic functions on a complex torus.
1. Recent years witnessed an extensive development of the theory of quantum groups and their quasiclassical limits - Lie bialgebras. Although since the work of Belavin and Drinfel’d [3] it is known, that solutions of the Classical Yang-Baxter equation can be classified into three categories: the rational, the trigonometric and the elliptic ones, the main development took place in the rational and the trigonometric cases. The underlying algebraic structures in these cases were found to be the affine Kac-Moody Lie algebras (classical case) and the Yangians and the quantum affine Kac-Moody algebras (quantum case) [5,6]. One of the major developments in the elliptic case was the discovery of the Sklyanin algebra [4]. This algebra, however, has no coproduct and thus it is not a bialgebra. Another important work had been done by Reyman and Semenov-Tian-Shanskii [2], who found the Lie bialgebras associated with the elliptic solutions of the Classical Yang-Baxter equation. The simplest example of their algebras is the Lie algebra of \( sl(2) \)-valued automorphic meromorphic functions on a complex torus. The Lie bialgebra structure of this algebra is given by the classical \( r \)-matrix of the Landau-Lifshitz model [2]. In the work [7] the generators and the defining relations of this Lie algebra had been found. In the present letter we address the problem of quantization of the above Lie bialgebra. The result is a quantum bialgebra related to the eight-vertex \( R \)-matrix. We use the term “quantum bialgebra” to designate a Hopf algebra without the antipode.

2. In [7] it is shown, that the Lie algebra of \( sl(2) \)-valued automorphic meromorphic functions on a complex torus which we denote by \( \mathcal{E}_{k,\nu^\pm} \) is a finitely generated (infinite-dimensional) \( \mathbb{C} \)-Lie algebra defined upon the six generators \( \{ x_k^{\pm} \}_{k=1,2,3} \) by the relations ( \( i, j, k \) below is any cyclic permutation of 1,2,3):

\[
[x_i^+, [x_j^+, x_k^\pm]] = 0,  
\]  

\[
[x_i^+, [x_j^+, x_k^\pm]] - [x_j^+, [x_i^+, x_k^\pm]] = J_{ij} x_k^\pm,  
\]  

\[
[x_i^+, x_j^-] = 0,  
\]
\[ [x_i^+, x_j^-] = \sqrt{-1}(w_i(\nu^+ - \nu^-)x_k^- - w_j(\nu^+ - \nu^-)x_k^+), \] (4)

where \( \nu^+, \nu^- \in T = \mathbb{C}/(\mathbb{Z}4K + \mathbb{Z}4iK'), \nu^+ \neq \nu^- \neq \mathbb{Z}2K + \mathbb{Z}2iK'; K, K' \) are the complete elliptic integrals of the moduli \( k \) and \( k' \) correspondingly: \( k^2 + k'^2 = 1; J_{12} = k^2, J_{23} = k'^2, J_{31} = -1; \) and \( w_1(u) = \frac{1}{\text{sn}(u,k)}; w_2(u) = \frac{\text{dn}(u,k)}{\text{sn}(u,k)}; w_3(u) = \frac{\text{cn}(u,k)}{\text{sn}(u,k)}, u \in T. \)

The structure of a Lie bialgebra upon \( E_{k,\nu^\pm} \) is defined by the classical r-matrix of the Landau-Lifshitz model \([2]\). A certain trigonometric limit \( k \to 0 \) of \( E_{k,\nu^\pm} \) coincides with the loop algebra \( A_1^{(1)}/(centre) \) \([7]\).

3. To define a quantum deformation of the Lie algebra (1-4) we introduce a deformation parameter \( \eta \in \mathbb{C} \) and recall the definition of the Baxter’s eight-vertex R-matrix \([1]\): \( R(u) = I \otimes I + \sum_{k=1}^{3} \frac{w_k(u+\eta)}{w_k(u)} \sigma_k \otimes \sigma_k \in \text{Mat}_2 \otimes \text{Mat}_2, u \in T, I \) is 2 \times 2 identity matrix \), and \( \sigma_k \) are the Pauli matrices. Let \( H = R(0)R'(0). \)

Introduce associative \( \mathbb{C} \)-algebra \( E_{k,\nu^\pm,\eta} \) generated by the elements: 1 (identity \), \( T_{ab}^\pm, T_{ab}^- \), \( a, b \in \{1, 2\} \). The defining relations of \( E_{k,\nu^\pm,\eta} \) have the following form:

\[
T_1^+(H_{12} - H_{14})T_1^+ + T_3^+(H_{34} - H_{32})T_3^+ + T_2^+(H_{23} - H_{21})T_2^+ + T_4^+(H_{41} - H_{43})T_4^+ = 0, \] (5)

\[
T_1^+T_1^+ = T_1^+T_1^+ = 1I_1, \] (6)

\[
R_{12}(\nu^+ - \nu^-)T_1^+T_2^+ = T_2^+T_1^+R_{12}(\nu^+ - \nu^-). \] (7)

We adopt the standard convention: \( T_n^\pm \) means a matrix with \( E_{k,\nu^\pm,\eta} \)-valued entries \), which acts nontrivially only in the \( n \)-th factor of \( \mathbb{C}^{\otimes m} \) \( m = 4 \) in (5) \), \( m = 2 \) in (7) \) and coincides there with \( T_{ab}^\pm. \)

**PROPOSITION** \( E_{k,\nu^\pm,\eta} \) is a bialgebra, i.e. there exist a coproduct \( \Delta : E_{k,\nu^\pm,\eta} \to E_{k,\nu^\pm,\eta} \otimes E_{k,\nu^\pm,\eta} \) and a counit \( \varepsilon : E_{k,\nu^\pm,\eta} \to \mathbb{C} \), such, that:

\[
\Delta(ab) = \Delta(a)\Delta(b), \quad a, b \in E_{k,\nu^\pm,\eta} \] (8)

\[
(\varepsilon \otimes id)\Delta(a) = (id \otimes \varepsilon)\Delta(a) = a, \quad a \in E_{k,\nu^\pm,\eta}. \] (9)
Explicit expressions for $\Delta$, and $\varepsilon$ are given by the following formulae:

$$\Delta(T_{ab}^\pm) = \sum_{c=1}^3 T_{ac}^\pm \otimes T_{cb}^\pm, \quad \Delta(T_{ab}^\pm) = \sum_{c=1}^3 T_{cb}^\pm \otimes T_{ac}^\pm, \quad \Delta(1) = 1 \otimes 1.$$  \hfill (10)

$$\varepsilon(T_{ab}^\pm) = \delta_{ab}, \quad \varepsilon(T_{ab}^\pm) = \delta_{ab}, \quad \varepsilon(1) = 1.$$  \hfill (11)

In the quasiclassical limit $\eta \to 0$, which is described as follows:

$$T^\pm = I + 2i\eta \sum_{k=1}^3 x_k\sigma_k + O(\eta^2), \quad \mathcal{T}^\pm = I + 2i\eta \sum_{k=1}^3 x_k^\pm \sigma_k + O(\eta^2),$$  \hfill (12)

one recovers from the defining relations (5-7) the defining relations (1-4) for the generators $\{x_k^\pm\}_{k=1,2,3}$ in $E_{k,\nu^\pm}$. Note, that from (6) it follows that $x_k^\pm = -x_k^\mp$.

The eight-vertex R-matrix $R(u-v)$ appears again as an intertwiner between the tensor products $\pi_u \otimes \pi_v$ and $\pi_v \otimes \pi_u$ of the simplest 2-dimensional representations $\pi_{u,v}$ of $E_{k,\nu^\pm,\eta}$ parametrized by $u, v \in \mathbb{T}$:

$$\pi_u(T^\pm) = \frac{1}{\sqrt{D(u-\nu^\pm)}} R(u-\nu^\pm), \quad \pi_u(\mathcal{T}^\pm) = \frac{1}{\sqrt{D(u-\nu^\pm)}} R(-u + \nu^\pm),$$  \hfill (13)

$$D(u) = 1 - \sum_{k=1}^3 \frac{w_k(u + i\eta)w_k(u - i\eta)}{w_k^2(i\eta)}.$$  \hfill (14)

To establish a connection between the elliptic and the trigonometric cases we need to perform the trigonometric limit $k \to 0$ of $E_{k,\nu^\pm,\eta}$. By analogy with the classical case [7], we do this limit at $\nu^+ = i\frac{3}{2}K', \nu^- = i\frac{1}{2}K'$. The author had succeeded to recover from (5-7) under this limit all the defining relations of $U_q(A_1^{(1)}/(centre))$ at $q = e^{2\eta}$ except the quantum Serre relations.

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