Thermodynamics of viscoelastic solids, its Eulerian formulation, and existence of weak solutions

Tomáš Roubíček

Mathematical Institute, Charles University,
Sokolovská 83, CZ-186 75 Praha 8, Czech Republic,
and
Institute of Thermomechanics of the Czech Academy of Sciences,
Dolejškova 5, CZ-182 00 Praha 8, Czech Republic
email: tomas.roubicek@mff.cuni.cz

Abstract

The thermodynamic model of visco-elastic deformable solids at finite strains is formulated in a fully Eulerian way in rates. Also, effects of thermal expansion or buoyancy due to evolving mass density in a gravity field are covered. The Kelvin-Voigt rheology with a higher-order viscosity (exploiting the concept of multipolar materials) is used, allowing for physically relevant frame-indifferent stored energies and for local invertibility of deformation. The model complies with energy conservation and Clausius-Duhem entropy inequality. Existence and a certain regularity of weak solutions are proved by a Faedo-Galerkin semi-discretization and a suitable regularization. Subtle physical limitations of the model are illustrated on thermally expanding neo-Hookean materials or materials with phase transitions.

Mathematics Subject Classification. 35Q49, 35Q74, 35Q79, 65M60, 74A30, 74Dxx, 74J30, 80A20.

Keywords. Elastodynamics, Kelvin-Voigt viscoelasticity, thermal coupling, large strains, multipolar continua, semi-Galerkin discretization, weak solutions.

1 Introduction

Even in the isothermal cases, the visco-elastodynamics at finite strains in its basic simple-material variant and the Kelvin-Voigt viscoelastic rheology has been articulated in [4, 5] as a difficult and essentially open problem as far as the existence of weak solutions concerns. This holds, a-fortiori, in anisothermal situations.

Let us remark that the adjective “finite” is often used equivalently to “large” strains, although sometimes it wants to emphasize some bounds on strains, as used also here since the deformation gradient ranges bounded sets.

In general, there are two basic approaches: the Lagrangian and the Eulerian ones. The former formulates the equations in a certain fixed “reference” configuration which, in some cases, may have a good meaning as a configuration at which the body was manufactured and to which the deformed body is always related in some sense. This approach allows easily for deformation of the shape of the body and mass density can be fixed but, on the other hand, the description of real deforming configuration in the fixed reference configuration needs careful pull-back and push-forward operators to be involved in the equations to comply with frame indifference, which was not always well respected in literature (e.g. [10, 25, 30]), as pointed out in [2]. Also an interaction with outer spatial fields (such as gravitation or electromagnetic) is quite complicated. This approach is believed as proper for solids and,
in the isothermal situations, has been used e.g. in \cite{16,56} with some restrictions to the stored energy (admitting self-interpenetration). Due to inevitably nonlinear nonmonotone character of the problem, even the mere existence of weak solutions is very problematic and various concepts of generalized solutions have been therefore devised (cf. also discussion of difficulties in the quasistatic situations \cite{33}) or concepts of nonsimple materials with higher gradients of deformation in the conservative part of the mechanical system have been exploited \cite{30,34,49,51}, cf. also \cite{27} Chap.9. For completeness, let us still report purely elastodynamic studies under strong qualification (certain convexity) of the stored energy \cite{14,15,25} or using a certain very weak (measure-valued) concept of solutions \cite{11,17,18,41,44}. This applies also to anisothermal extension leading to thermodynamically consistent models in the Lagrangian setting in the Kelvin-Voigt rheology in \cite{13,27,34}, inviscid in \cite{9,12}, and in Maxwellian rheology in \cite{49}.

The latter mentioned approach, Eulerian, is (with some exception as e.g. \cite{40}) standardly believed to be well fitted with fluids. It is particularly suitable in situations when there is no natural reference configuration or where a reference configuration becomes less and less relevant during long-time evolution, which may however apply also to solids. A formulation of equations in the current deforming configuration needs rather velocity/strain than displacement to be involved in the momentum equation. The advantage is an easier possibility to involve interaction with outer spatial fields (here gravitational) and avoiding the pull-back and push-forward manipulation. On the other hand, there is a necessity to involve convective derivative and transport equations and also evolving the shape of the body is troublesome. In isothermal situations, such a model was formulated and analyzed as incompressible in \cite{29,31} and as compressible in \cite{24,42}. The mentioned higher gradients that would allow for reasonable analysis are now to be involved rather in the dissipative than the conservative part, so that their influence manifests only in fast evolutions. Besides, the higher gradients may lead to easier propagation of elastic waves with less dispersion and less attenuation in some frequency ranges, compared with the usual 1st-order gradients, as pointed out in \cite[Sect.3.1]{48}. In the isothermal situations, it was used in a quasistatic case in \cite{47} and in the dynamical case in \cite{49} when considering the stored energy in the actual configuration, which then gives an energy pressure in the stress tensor. In anisothermal situations, such free-energy pressure would be directly added to stress tensor in an non-integrable way and likely would cause technical difficulties.

In comparison with \cite{47,49}, the novelty of this paper is to apply the Eulerian approach to solids in anisothermal situations, using the free energy in a reference configuration (as in \cite{40} without any analysis), which does not see the energy-type pressure in the stress tensor and which is also better fitted with usually available experimental data typically obtained in the undeformed (reference) configuration. The analysis combines $L^1$-theory for the heat equation adapted to the convective time derivatives and the techniques from compressible fluid dynamics adapted for solids. The main attributes of the devised model are:

- Concept of hyperelastic materials (whose conservative-stress response comes from a potential being thermodynamically a free energy) combined with the Kelvin-Voigt viscoelastic rheology.
- The rate formulation in terms of velocity and deformation gradient is used while the deformation itself does not explicitly occur.
- Mechanical consistency in the sense that frame indifference of the free energy (which is in particular nonconvex in terms of deformation gradient) is admitted, as well as its singularity under infinite compression in relation with local non-interpenetration.
- Thermodynamic consistency of the thermally coupled system in the sense that the total energy is conserved in a closed system, the Clausius-Duhem entropy inequality holds, and temperature stays non-negative.
- The nonconservative part of the stress in the Kelvin-Voigt model containing a higher-order component reflecting the concept of nonsimple (here 2nd-grade) multipolar media is exploited.
The model allows for rigorous mathematical analysis as far as existence and certain regularity of energy-conserving weak solutions concerns.

As far as the non-negativity of temperature, below we will be able to prove only that at least some solutions enjoy this attribute, although there is an intuitive belief that all possible solutions will make it and a hope that more advanced analytical techniques would rigorously prove it.

The main notation used in this paper is summarized in the following table:

| Symbol | Definition |
|--------|------------|
| \(v\) | velocity (in m/s), |
| \(\varrho\) | mass density (in kg/m\(^3\)), |
| \(\varrho_0\) | referential mass density, |
| \(F\) | deformation gradient, |
| \(\theta\) | temperature (in K), |
| \(T\) | Cauchy stress (symmetric, in Pa), |
| \(\delta\) | hyper-stress (in Pa m), |
| \(j\) | heat flux (in W/m\(^2\)), |
| \(f\) | traction load, |
| \(\text{det}(\cdot)\) | determinant of a matrix, |
| \(\text{Cof}(\cdot)\) | cofactor matrix, |
| \(\nu_0 > 0\) | a boundary viscosity |
| \(\mathbb{R}_{\text{sym}}^{d \times d} = \{ A \in \mathbb{R}^{d \times d}; A^T = A \}\) | |
| \(\psi = \psi(F, \theta)\) | referential free energy (in J/m\(^3\)=Pa) |
| \(\varphi = \varphi(F)\) | referential stored energy (in J/m\(^3\)=Pa), |
| \(\gamma = \gamma(F, \theta)\) | referential heat part of free energy, |
| \(e(v) = \frac{1}{2} \nabla v^\top + \frac{1}{2} \nabla v\) | small strain rate (in s\(^{-1}\)), |
| \(D = D(F, \theta; e(v))\) | dissipative part of Cauchy stress, |
| \(w\) | heat part of internal energy (enthalpy, in J/m\(^3\)), |
| \((\cdot)^* = \frac{2}{3} \nu + (v \cdot \nabla)\) | convective time derivative, |
| \(\cdot\) | or : scalar products of vectors or matrices, |
| \(\kappa = \kappa(F, \theta)\) | thermal conductivity (in W/m\(^2\)K\(^{-1}\)), |
| \(c = c(F, \theta)\) | heat capacity (in Pa/K), |
| \(\nu > 0\) | a bulk hyper-viscosity coefficient, |
| \(g\) | external bulk load (gravity acceleration in m/s\(^2\)). |

Table 1. Summary of the basic notation used through the paper.

The paper is organized as follows: The formulation of the model in actual Eulerian configuration and its energetics and thermodynamics is presented in Section 2. Then, in Section 3, the rigorous analysis by a suitable regularization and a (semi) Faedo-Galerkin approximation is performed, combined with the theory of transport by regular velocity field which is briefly presented in the Appendix. Some applicable examples are then stated in Section 4.

2 The thermodynamic model and its energetics

It is important to distinguish carefully the referential and the actual time-evolving coordinates. We aim to formulate the model eventually in actual configurations, i.e. the Eulerian formulation, reflecting also the reality in many (or even most) situations that the concept of a reference configuration is only an artificial construction and, even if relevant in some situations, becomes successively more and more irrelevant during evolution at truly finite strains and large displacements. A very typical example is geophysical models for long time scales. On the other hand, some experimental material data are related to some reference configuration – typically, it concerns mass density and stored or free energies per referential volume (in J/m\(^3\)=Pa) as considered here or per mass (in J/kg).

2.1 Finite-strain kinematics and mass and momentum transport

We briefly remind the fundamental concepts and formulas which can mostly be found in the monographs, as e.g. [23, Part XI], [32, Sect. 7.2], [53, Sect. 22.1], or [55].

In finite-strain continuum mechanics, the basic geometrical concept is the time-evolving deformation \(y : \Omega \rightarrow \mathbb{R}^d\) as a mapping from a reference configuration of the body \(\Omega \subset \mathbb{R}^d\) into a physical space \(\mathbb{R}^d\). The “Lagrangian” space variable in the reference configuration will be denoted as \(X \in \Omega\) while in the “Eulerian” physical-space variable by \(x \in \mathbb{R}^d\). The basic geometrical object is the deformation gradient \(F = \nabla_X y\).
We will be interested in deformations \( \mathbf{x} = y(t, X) \) evolving in time, which are sometimes called “motions”. The difference \( \mathbf{x} - X \) represents the displacement. The important quantity is the Eulerian velocity \( \mathbf{v} = \frac{\partial}{\partial t} \mathbf{y} + (\mathbf{v} \cdot \nabla) \mathbf{y} \). Here and throughout the whole article, we use the dot-notation (\( \cdot \)) for the convective time derivative applied to scalars or, component-wise, to vectors or tensors.

Then, the velocity gradient \( \nabla \mathbf{v} = \nabla_X \mathbf{v} \nabla_x \mathbf{X} = \dot{\mathbf{F}} \mathbf{F}^{-1} \), where we used the chain-rule calculus and \( \mathbf{F}^{-1} = (\nabla_X \mathbf{x})^{-1} = \nabla_x \mathbf{X} \). This gives the transport equation-and-evolution for the deformation gradient as

\[
\dot{\mathbf{F}} = (\nabla \mathbf{v}) \mathbf{F}.
\]  

(2.1)

From this, we also obtain the evolution-and-transport equation for the determinant \( \det \mathbf{F} \) as

\[
\frac{\dot{\det \mathbf{F}}}{\det \mathbf{F}} = \text{Cof} \mathbf{F} : \dot{\mathbf{F}} = (\det \mathbf{F}) \mathbf{F}^{-\top} : \dot{\mathbf{F}} = \det \mathbf{F} \mathbf{I} : \dot{\mathbf{F}} \mathbf{F}^{-1} = (\det \mathbf{F}) \mathbf{I} : \nabla \mathbf{v} = (\det \mathbf{F}) \text{div} \mathbf{v},
\]  

(2.2)

where \( \mathbf{I} \) denotes the identity matrix, as well as the evolution-and-transport equation for \( 1/\det \mathbf{F} \) as

\[
\frac{\dot{1}}{\det \mathbf{F}} = -\frac{\text{div} \mathbf{v}}{\det \mathbf{F}}.
\]  

(2.3)

The understanding of (2.1) is a bit delicate because it mixes the Eulerian \( \mathbf{x} \) and the Lagrangian \( \mathbf{X} \); note that \( \nabla \mathbf{v} = \nabla_X \mathbf{v}(\mathbf{x}) \) while standardly \( \mathbf{F} = \nabla_X \mathbf{y} = \mathbf{F}(\mathbf{X}) \). In fact, we consider \( \mathbf{F} \circ \xi \) where \( \xi : \mathbf{x} \mapsto \mathbf{y}^{-1}(t, \mathbf{X}) \) is the so-called return (sometimes called also a reference) mapping. Thus, \( \mathbf{F} \) depends on \( \mathbf{x} \) and (2.1) and is an equality which holds for a.a. \( \mathbf{x} \). The same holds for (2.2) and (2.3).

The reference mapping \( \xi \), which is well defined through its transport equation

\[
\dot{\xi} = 0,
\]  

(2.4)

actually does not explicitly occur in the formulation of the problem (except Remark 2.2). Here, we will benefit from the boundary condition \( \mathbf{v} \cdot \mathbf{n} = 0 \) below, which causes that the actual domain \( \varOmega \) does not evolve in time. The same convention concerns temperature \( \theta \) and thus also \( \mathbf{T} \), \( \eta \), and \( \mathbf{D} \) in (2.10) and (2.11) below, which will make the problem indeed fully Eulerian. Cf. the continuum-mechanics textbooks as e.g. [23,32].

Most physical variables can be classified either as extensive variables (as mass density, or free or internal energies, or entropy) or intensive variables (as temperature or velocity). These two are transported differently in compressible media.

The mass density (in kg/m\(^3\)) is an extensive variable and its transport (expressing that the conservation of mass) writes as the continuity equation \( \frac{\partial}{\partial t} \rho + \text{div}(\rho \mathbf{v}) = 0 \), or, equivalently, the mass transport equation

\[
\dot{\rho} = -\rho \text{div} \mathbf{v}.
\]  

(2.5)

This equation also ensures the useful equality, namely transport of the momentum \( \rho \mathbf{v} \) as another extensive (vector-valued) variable:

\[
\frac{\partial}{\partial t}(\rho \mathbf{v}) + \text{div}(\rho \mathbf{v} \otimes \mathbf{v}) = \rho \frac{\partial \mathbf{v}}{\partial t} + \frac{\partial \rho}{\partial t} \mathbf{v} + \text{div}(\rho \mathbf{v}) \mathbf{v} + \rho (\mathbf{v} \cdot \nabla) \mathbf{v} = \rho \dot{\mathbf{v}}.
\]  

(2.6)

One can determine the density \( \rho \) instead of the transport equation for mass density (2.5) from the algebraic relation

\[
\rho = \frac{\rho_R}{\det \mathbf{F}},
\]  

(2.7)

where \( \rho_R \) is the mass density in the reference configuration. Indeed, relying on (2.2), one has the calculus

\[
\frac{\dot{\rho}}{\rho} = \left( \frac{\rho_R}{\det \mathbf{F}} \right) \frac{\det \mathbf{F}}{\det \mathbf{F}} = -\text{div} \mathbf{v}
\]  

(2.8)
because \( \dot{\varrho}_0 = 0 \). Later, we will consider the initial conditions \( F_0 \) for (2.11) and \( g_0 \) for (2.5). These two should be related by \( g(0) = \varrho_0 / \det F_0 \).

Alternatively to (2.5), we can also consider an evolution-and-transport equation for the “mass sparsity” as the inverse mass density \( 1/\varrho \):

\[
(2.9) \quad \frac{\dot{\varrho}}{\varrho} = \text{div} \mathbf{v}.
\]

### 2.2 Hyperelasticity and visco-elasticity, and its thermodynamics

The main ingredients of the model are the (volumetric) free energy \( \psi \) depending on deformation gradient \( F \) and temperature \( \theta \), and the temperature-dependent dissipative stress \( D \) not necessarily possessing any underlying potential. The free energy \( \psi = \psi(F, \theta) \) is considered per the referential volume, while the free energy per actual deformed volume is \( \psi(F, \theta)/\det F \). Considering the free energy by reference volume is more standard in continuum physics \([23, 32]\) than the free energy per actual evolving volume. This corresponds to experimentally available data and seems particularly more suitable for analysis of thermally coupled systems. This last benefit is related to the fact that the referential free energy does not give an energy pressure contribution to the Cauchy stress (cf. the last term in (2.21) below or \([47, \text{Rem. 2}]\)) and allows for more easy decoupling estimation strategy decoupling the mechanical part and the thermal part of the coupled system.

While \( \psi \) is referential, the free energy per actual volume is then \( \psi(F, \theta)/\det F \). From it, we can read the conservative part of the actual Cauchy stress \( T \) and the actual entropy \( \eta \) as:

\[
(2.10) \quad T = \frac{\psi'(F, \theta)F^\top}{\det F} \quad \text{and} \quad \eta = -\frac{\psi'(F, \theta)}{\det F}.
\]

In the already anticipated viscoelastic Kelvin-Voigt rheological model, we will use also a dissipative contribution to the Cauchy stress, which will make the system parabolic. In addition to the usual first-order stress \( D \), we consider a dissipative contribution to the Cauchy stress involving also a higher-order 2nd-grade hyper-stress \( \mathcal{H} \), so that the overall dissipative stress is:

\[
D - \text{div} \mathcal{H} \quad \text{with} \quad D = D(F, \theta; e(\mathbf{v})) \quad \text{and} \quad \mathcal{H} = \mathcal{H}(\nabla e(\mathbf{v}))
\]

for some \( D(F, \theta; \cdot) : \mathbb{R}^{d \times d}_{\text{sym}} \to \mathbb{R}^{d \times d}_{\text{sym}} \) and \( \mathcal{H}(E) = \nu|E|^\theta - 2E \).

More discussion about the hyper-stress contribution will be in Remarks 2.1 below.

The momentum equilibrium equation then balances the divergence of the total Cauchy stress with the inertial and gravity force:

\[
(2.12) \quad \varrho \ddot{\mathbf{v}} - \text{div}(T + D - \text{div} \mathcal{H}) = \varrho \mathbf{g}
\]

with \( T \) from (2.10) and \( D \) and \( \mathcal{H} \) from (2.11).

The second ingredient in (2.10) is subjected to the entropy equation:

\[
(2.13) \quad \frac{\partial \eta}{\partial t} + \text{div}(\mathbf{v} \eta) = \frac{\xi}{\theta} - \frac{\text{div} \mathbf{j}}{\theta} \quad \text{with} \quad \mathbf{j} = -\kappa(F, \theta) \nabla \theta
\]

with \( \xi \) the heat production rate and \( \mathbf{j} \) the heat flux. The latter equality in (2.13) is the Fourier law determining phenomenologically the heat flux \( \mathbf{j} \) proportional to the negative gradient of temperature \( \theta \) through the thermal conductivity coefficient \( \kappa = \kappa(F, \theta) \). Assuming \( \xi \geq 0 \) and \( \kappa \geq 0 \) and integrating (2.13) over the domain \( \Omega \) while imposing the non-penetrability of the boundary in the sense that the normal velocity \( \mathbf{v} \cdot \mathbf{n} \) vanishes across the boundary \( \Gamma \) of \( \Omega \), we obtain the Clausius-Duhem inequality:

\[
(2.14) \quad \frac{d}{dt} \int_\Omega \eta \, dx = \int_\Gamma \frac{\xi}{\theta} + \kappa |\nabla \theta|^2 \, dS + \int_\Gamma \left( \frac{\nabla \theta}{\theta} - \eta \mathbf{v} \right) \cdot \mathbf{n} \, dS \geq \int_\Gamma \kappa \frac{\nabla \theta \cdot \mathbf{n}}{\theta} \, dS.
\]
If the system is thermally isolated in the sense that the normal heat flux \( j \cdot n \) vanishes across the boundary \( \Gamma \), we recover the 2nd law of thermodynamics, i.e. the total entropy in isolated systems is nondecreasing in time.

Substituting \( \eta \) from (2.10) into (2.13) written in the form \( \theta \dot{\eta} = \xi - \text{div} \; j - \theta \eta \text{div} \; v \) and using the calculus \((\psi'_F(F, \theta)/\det F)_{\Gamma} : \hat{F} + \psi'_{\partial F}(F, \theta) (\text{div} \; v)/\det F = \psi''_{\partial F}(F, \theta) \hat{F}/\det F\), we obtain the heat-transfer equation
\[
\begin{align*}
\frac{c(F, \theta)}{\det F} \dot{\theta} = D(F, \theta; e(v)) : e(v) + \nu |\nabla e(v)|^p + \frac{\psi''_{\partial F}(F, \theta)}{\det F} : \hat{F} - \text{div} \; j \\
&\text{with the heat capacity } c(F, \theta) = -\frac{\psi''_{\partial F}(F, \theta)}{\det F}.
\end{align*}
\tag{2.15}
\]

The referential internal energy is given by the Gibbs relation \( \psi - \theta \psi' \). In terms of the thermal part of the actual internal energy \( w(F, \theta) := (\psi(F, \theta) - \theta \psi'_{\partial F}(F, \theta) - \psi(0))/\det F \), the heat equation can be written in the so-called enthalpy formulation:
\[
\begin{align*}
\frac{\partial w}{\partial t} + \text{div}(vw + j) = D(F, \theta; e(v)) : e(v) + \nu |\nabla e(v)|^p + \frac{\psi'_{\partial F}(F, \theta) - \psi'_{\partial F}(F, 0)}{\det F} : \hat{F} \\
&\text{with } w = \psi(F, \theta) - \theta \psi'_{\partial F}(F, \theta) - \psi(0).
\end{align*}
\tag{2.16}
\]

Here, alternatively, we could use also \( \hat{F} = (\nabla v)F \) for writing \( \psi'_{\partial F}(F, \theta) : \hat{F} = \psi'_{\partial F}(F, \theta) F^\top : (\nabla v) = \psi_F'(F, \theta) F^\top : e(v) \); here the assumed frame indifference of \( \psi(\cdot, \theta) \) assumed in (2.5) below, leading to symmetry of \( \psi'_{\partial F}(F, \theta) F^\top \), was employed, cf. (2.18d) below. In comparison with (2.15) which needs differentiability of \( \psi' \), the form (2.16) needs differentiability of \( \psi \) only. Using again the algebra \( F^{-1} = \text{Cof} F^\top/\det F \) and the calculus \( \det'(F) = \text{Cof} F \) and (2.1), we realize that
\[
\begin{align*}
\frac{\partial w}{\partial t} + \text{div}(vw + j) &\quad = \dot{w} + w \text{div} \; v - \frac{\gamma(F, \theta) - \theta \gamma_{\partial F}(F, \theta)}{\det F} \text{div} \; v \\
&\quad = \left( \frac{\gamma_{\partial F}(F, \theta)}{\det F} - \theta \frac{\gamma_{\partial F}(F, \theta)}{\det F} \right) : \hat{F} + c(F, \theta) \dot{\theta} + \frac{\gamma(F, \theta) - \theta \gamma_{\partial F}(F, \theta)}{\det F} \text{div} \; v \\
&\quad = c(F, \theta) \dot{\theta} + \frac{\psi'_F(F, \theta)}{\det F} : \hat{F} - \theta \frac{\gamma_{\partial F}(F, \theta)}{\det F} : \hat{F} = 0,
\end{align*}
\tag{2.17}
\]
where we abbreviated \( \gamma(F, \theta) = \psi(F, \theta) - \psi(F, 0) \). Thus we can see that (2.16) is indeed equivalent with (2.15).

Let us formulate the thermo-visco-elastodynamic system for \((\rho, v, F, \theta)\), composing the equations (2.1), (2.5), (2.12), and (2.16):
\[
\begin{align*}
\frac{\partial \rho}{\partial t} &\quad = -\text{div} \; (\rho \mathbf{v}) , \tag{2.18a} \\
\frac{\partial (\rho \mathbf{v})}{\partial t} &\quad = \text{div} \left( T(F, \theta) + D(F, \theta; e(v)) - \text{div} \; \mathcal{H}(\nabla e(v)) - \rho \mathbf{v} \otimes \mathbf{v} \right) + \rho g \\
&\quad \text{with } T(F, \theta) = -\frac{\psi'_F(F, \theta) F^\top}{\det F} \text{ and } \mathcal{H}(\nabla e(v)) = \nu |\nabla e(v)|^p - 2 |\nabla e(v)|^2 \nabla e(v) , \tag{2.18b} \\
\frac{\partial F}{\partial t} &\quad = (\nabla v)F - (v \cdot \nabla) F \tag{2.18c} , \\
\frac{\partial w}{\partial t} &\quad = D(F, \theta; e(v)) : e(v) + \nu |\nabla e(v)|^p + \frac{\gamma'(F, \theta) F^\top}{\det F} : e(v) + \text{div} \; (\kappa(F, \theta) \nabla \theta - w \mathbf{v}) \\
&\quad \text{with } w = \omega(F, \theta) := \frac{\gamma(F, \theta) - \theta \gamma_{\partial F}(F, \theta)}{\det F} \text{ where } \gamma(F, \theta) = \psi(F, \theta) - \psi(F, 0) . \tag{2.18d}
\end{align*}
\]
Denoting by \( \mathbf{n} \) the unit outward normal to the (fixed) boundary \( \Gamma \) of the domain \( \Omega \), we complete this system by suitable boundary conditions:

\[
\begin{align*}
\mathbf{v} \cdot \mathbf{n} &= 0, \\
\nabla \mathbf{e}(\mathbf{v}) : (\mathbf{n} \otimes \mathbf{n}) &= 0, \quad \text{and} \quad \kappa(F, \theta) \nabla \theta \cdot \mathbf{n} &= h(\theta) + \frac{\nu}{2} |\mathbf{v}|^2
\end{align*}
\]

(2.19a)

(2.19b)

with \( \nu > 0 \) a boundary viscosity coefficient and with \( [\cdot]_T \) a tangential part of a vector. Here \( \text{div}_v = \text{tr}(\nabla_s) \) denotes the \((d-1)\)-dimensional surface divergence with \( \text{tr}(\cdot) \) being the trace of a \((d-1) \times (d-1)\)-matrix and \( \nabla_s v = \nabla v - \frac{\partial}{\partial \mathbf{n}} \mathbf{n} \) being the surface gradient of \( v \). The first condition (i.e. normal velocity zero) expresses nonpenetrability of the boundary was used already for (2.11) and is most frequently adopted in literature for Eulerian formulation. This simplifying assumption fixes the shape of \( \Omega \) in its referential configuration allows also for considering the fixed boundary even for such time-evolving Eulerian description. The latter condition in (2.19a) involving a boundary viscosity comes from the Navier boundary condition largely used in fluid dynamics and is here connected with the technique used below, which is based on the total energy balance as the departing point and which, unfortunately, does not allow to cope with \( \nu = 0 \) and simultaneously \( \mathbf{f} \neq 0 \). This boundary viscosity naturally may contribute to the heat production on the boundary as well as to the outflow of the heat energy to the outer space. For notational simplicity, we consider that it is just equally distributed, one part remaining on the boundary of \( \Omega \) and the other part leaving outside, which is related to the coefficient \( 1/2 \) in the latter condition in (2.19b).

**Remark 2.1 (Gradient theories in rates).** So-called gradient theories in continuum-mechanical viscoelastic models are nowadays very standard, referred as nonsimple materials, determining some internal length scales, allowing for modelling various dispersion and attenuation of propagation of elastic waves, and often facilitating mathematical analysis, cf. [35]. They can be applied to the conservative stress through the free energy or to the dissipation stress. Here, in contrast to the Lagrangian solid-mechanical models as in [27, 34, 49, 51], we have used the latter option in (2.11) which is better fitted to the rate formulation and which can make the velocity field enough regular, as vitally needed for the transport of \( \rho \) and \( F \) in the Eulerian models. The higher gradient hyper-stress as used below in (2.11) follows the theory by E. Fried and M. Gurtin [21], as already anticipated in the general nonlinear context of multipolar fluids by J. Neˇ cas at al. [36–38] or solids [43, 52], inspired by R.A. Toupin [54] and R.D. Mindlin [35]. Let us emphasize that, without such higher-order gradients, the (possibly) irregular velocity fields may make the treatment of the transport problem extremely nontrivial due to the possible onset of singularities, whose occurrence in solids may be debatable, cf. [1].

**Remark 2.2 (Spatially inhomogeneous media).** Since the free energy \( \psi \) is considered in the referential domain, we can naturally generalize the model for an initially inhomogeneous medium making \( \psi \) and also \( D X \)-dependent, i.e. \( \psi = \psi(X, F, \theta) \) and \( D = D(X, F, \theta; e) \). In the Eulerian formulation, we should complete the system (2.18) by the transport equation (2.4) for the return mapping \( \xi \) and compose \( \psi \circ \xi \) defined as \( [\psi \circ \xi](x, F, \theta) = \psi(\xi(x), F, \theta) \) and analogously for \( D \). It yields also inhomogeneous heat capacity. Analogous generalization concerns also an inhomogeneous heat transfer coefficient \( \kappa = \kappa(X, F, \theta) \) or the external heat flux \( h = h(X, \theta) \). Another generalization may concern an anisotropic materials with \( \kappa \) being \( \mathbb{R}^{d \times d} \)-valued.

### 2.3 Energetics behind the system (2.18)–(2.19)

The mechanical energy-dissipation balance of the visco-elastodynamic model (2.18)–c can be seen when testing the momentum equation (2.18b) by \( \mathbf{v} \) while using the continuity equation (2.18a) and the evolution-and-transport equation (2.18c) for \( F \). We will select out the temperature-independent stored energy \( \varphi \) by denoting the mere stored energy \( \varphi := \psi(\cdot, 0) \) and the resting temperature-dependent
part $\gamma(\cdot, \theta) := \psi(\cdot, \theta) - \varphi(\cdot)$ as used in (2.18d). Thus, we have the split:

$$\psi(F, \theta) = \varphi(F) + \gamma(F, \theta) \quad \text{with} \quad \gamma(F, 0) = 0. \quad (2.20)$$

As already said, $\psi$ together with $\varphi$ and $\gamma$ are considered per the referential volume.

Using the algebra $F^{-1} = \text{Cof} F^T / \text{det} F$ and the calculus $\text{det}'(F) = \text{Cof} F$, we can write the conservative part of the Cauchy stress as

$$\frac{\varphi'(F)}{\text{det} F} F^T = \frac{\varphi'(F) - \varphi(F) F^{-T} F^T + \varphi(F) \text{det} F}{\text{det} F} F^T + \frac{\varphi(F)}{\text{det} F} F^T = \frac{[\varphi(F)]'}{\text{det} F} F^T + \frac{\varphi(F)}{\text{det} F} F^T. \quad (2.21)$$

Let us recall that $[\varphi/\text{det}(F)$ in (2.21) is the stored energy per actual (not referential) volume. Using the calculus (2.21), we obtain

\[
\int_{\Omega} \text{div} T \cdot v \, dx = \int_{\Gamma} (T n) \cdot v \, dS - \int_{\Omega} T : e(v) dx
\]

\[
= \int_{\Gamma} (T n) \cdot v \, dS - \int_{\Omega} \left( \frac{\varphi'(F)}{\text{det} F} + \frac{\gamma'(F, \theta)}{\text{det} F} \right) F^T : e(v) \, dx
\]

\[
= \int_{\Gamma} (T n) \cdot v \, dS - \int_{\Omega} \left[ \frac{\varphi(F)}{\text{det} F} F^T + \frac{\varphi(F)}{\text{det} F} + \frac{\gamma'(F, \theta)}{\text{det} F} F^T \right] : e(v) \, dx
\]

\[
= \int_{\Gamma} (T n) \cdot v \, dS - \int_{\Omega} \frac{\varphi(F)}{\text{det} F} F^T : e(v) \, dx - \int_{\Omega} \frac{\gamma'(F, \theta)}{\text{det} F} F^T : e(v) \, dx. \quad (2.22)
\]

Here, we used the matrix algebra $A: (BC) = (B^T A): C = (A C^T) : B$ for any square matrices $A$, $B$, and $C$ and also we used (2.18c) together with the Green formula and the nonpenetrability boundary condition for

\[
\int_{\Omega} \left[ \frac{\varphi(F)}{\text{det} F} \right]' (\nabla v) = \int_{\Omega} \left[ \frac{\varphi(F)}{\text{det} F} \right]' (\frac{\partial F}{\partial t} + (v \cdot \nabla) F) + \frac{\varphi(F)}{\text{det} F} \text{div} v \, dx
\]

\[
= \frac{d}{dt} \int_{\Omega} \frac{\varphi(F)}{\text{det} F} \, dx + \int_{\Omega} \nabla (\frac{\varphi(F)}{\text{det} F}) \cdot v + \frac{\varphi(F)}{\text{det} F} \text{div} v \, dx
\]

\[
= \frac{d}{dt} \int_{\Omega} \frac{\varphi(F)}{\text{det} F} \, dx + \int_{\Gamma} \frac{\varphi(F)}{\text{det} F} (v \cdot n_\Gamma) \, dS. \quad (2.23)
\]

The further contribution from the dissipative part of the Cauchy stress uses Green’s formula over $\Omega$ twice and the surface Green formula over $\Gamma$. We abbreviate the hyper-stress $\mathcal{H}(\nabla e(v)) = \nu |\nabla e(v)|^{p-2} \nabla e(v)$. Then

\[
\int_{\Omega} \text{div} \left( D(F, \theta; e(v)) - \text{div} \mathcal{H}(\nabla e(v)) \right) \cdot v \, dx
\]

\[
= \int_{\Gamma} v \cdot \left( D(F, \theta; e(v)) - \text{div} \mathcal{H}(\nabla e(v)) \right) n \, dS - \int_{\Omega} (D(F, \theta; e(v)) - \text{div} \mathcal{H}(\nabla e(v))) \cdot \nabla v \, dx
\]

\[
= \int_{\Gamma} v \cdot \left( D(F, \theta; e(v)) - \text{div} \mathcal{H}(\nabla e(v)) \right) n - n \cdot \mathcal{H}(\nabla e(v)) \cdot \nabla v \, dS
\]

\[
- \int_{\Omega} D(F, \theta; e(v)) : e(v) + \mathcal{H}(\nabla e(v)) : \nabla^2 v \, dx
\]

8
\[ = \int_{\Omega} \mathcal{H}(\nabla e(v)) : (\partial_t v \otimes n \otimes n) + n \cdot \mathcal{H}(\nabla e(v)) : \nabla v + v \cdot (D(F, \theta; e(v)) - \text{div}\mathcal{H}(\nabla e(v))) n \, dS \]

\[ - \int_{\Omega} D(F, \theta; e(v)) : e(v) + \mathcal{H}(\nabla e(v)) : \nabla^2 v \, dx \]

\[ = \int_{\Omega} \left( \mathcal{H}(\nabla e(v)) : (\partial_t v \otimes n \otimes n) - \left( (D(F, \theta; e(v)) - \text{div}\mathcal{H}(\nabla e(v))) n + \text{div}_s(n \cdot \mathcal{H}(\nabla e(v))) \right) \right) \cdot v \, dS - \int_{\Omega} D(F, \theta; e(v)) : e(v) + \nu |\nabla e(v)|^p \, dx , \quad (2.24) \]

where we used the decomposition of \( \nabla v \) into its normal part \( \partial_n v \) and the tangential part, i.e. written componentwise \( \nabla v_i = (n \cdot \nabla v_i) n + \nabla_t v_i \).

Furthermore, the inertial force \( (2.6) \) in \( (2.18b) \) tested by \( v \) gives the rate of kinetic energy \( \frac{1}{2} |v|^2 / 2 \) when using again the continuity equation \( (2.5) \) tested by \( |v|^2 / 2 \) for the identity

\[ \frac{\partial}{\partial t} \left( \frac{\theta}{2} |v|^2 \right) = \rho v \frac{\partial v}{\partial t} + \frac{\partial \rho}{\partial t} \frac{|v|^2}{2} = \rho v \frac{\partial v}{\partial t} - \text{div}(\rho v) \frac{|v|^2}{2} . \quad (2.25) \]

Integrating over \( \Omega \) and using the Green formula with the boundary condition \( v \cdot n = 0 \) and relying on \( (2.6) \), we obtain

\[ \int_{\Omega} \left( \frac{\partial}{\partial t} (\rho v) + \text{div}(\rho v \otimes v) \right) \cdot v \, dx = \int_{\Omega} \rho \dot{v} \cdot v \, dx \]

\[ = \int_{\Omega} \rho v \frac{\partial v}{\partial t} + \rho v \cdot (v \cdot \nabla) v \, dx = \frac{d}{dt} \int_{\Omega} \frac{\theta}{2} |v|^2 \, dx + \int_{\Gamma} \rho \frac{\theta}{2} |v|^2 \nu \cdot n \, dS . \quad (2.26) \]

Merging the boundary integrals in \( (2.22) \) and in \( (2.21) \) and using the boundary condition \( ((T + D - \text{div}\mathcal{H}) n - \text{div}_s(\mathcal{H} n)) + \nu \delta v = f \), we thus obtain (for this moment formally) the mechanical energy dissipation balance

\[ \frac{d}{dt} \int_{\Omega} \frac{\theta}{2} |v|^2 \, dx + \frac{\varphi(F)}{\text{det} F} \, dx + \int_{\Omega} D(F, \theta; e(v)) : e(v) + \nu |\nabla e(v)|^p \, dx + \int_{\Gamma} \nu \delta |v|^2 \, dS \]

\[ = \int_{\Omega} \rho g \cdot v - \frac{\gamma(\mathcal{F}(F, \theta) F^T)}{\text{det} F} : e(v) \, dx + \int_{\Gamma} f \cdot v \, dS . \quad (2.27) \]

When we add \( (2.18d) \) tested by 1, the adiabatic and the dissipative heat sources cancel with those in \( (2.27) \). Thus we obtain (at least formally) the total energy balance

\[ \frac{d}{dt} \int_{\Omega} \frac{\theta}{2} |v|^2 \, dx + \frac{\varphi(F)}{\text{det} F} \, dx + \omega(F, \theta) \, dx + \int_{\Gamma} \nu \delta \frac{\theta}{2} |v|^2 \, dS = \int_{\Omega} \rho g \cdot v \, dx + \int_{\Gamma} f \cdot v \, dS + \frac{h(\theta)}{\text{power of adiabatic effects}} \]

which expresses the 1st law of thermodynamics. Also, the 2nd law of thermodynamics is satisfied, viz the Clausius-Duhem inequality \( (2.11) \) above. Another aspect important both thermodynamically and also for mathematical analysis is the non-negativity of temperature, related to the 3rd law of thermodynamics. This will be demonstrated later when we exploit some information about the quality of the velocity field extracted from \( (2.27) \), cf. \( (3.43) \) below.
3 The analysis – weak solutions of (2.18)

We will provide a proof of existence and certain regularity of weak solutions. To this aim, the concept of multipolar viscosity is essential but, anyhow, still quite nontrivial and carefully ordered arguments will be needed. The peculiarities are that the inertial term in the Eulerian setting involves varying mass density requiring sophisticated techniques from compressible fluid dynamics, the momentum equation is very geometrically nonlinear, and the heat equation has an $L^1$-structure with an $F$-dependent heat capacity and with the convective time derivative, besides ever-troubling adiabatic effects due to the necessarily general (nonlinear) coupling of mechanical and thermal effect in the deforming configuration. Moreover, it is well known that the so-called Laueinert phenoomenon may occur in nonlinear static elasticity where the stored energy has singularity at det $F = 0$ in the sense that minimizing energy on $W^{1,\infty}(\Omega; \mathbb{R}^d)$ may yield a strictly bigger infimum than minimizing on a “correct” $W^{1,p}$-space, as pointed out in [8,16,19], following the original observation in [28]. This brings difficulties in a Galerkin approximation and which are copied also for evolution problems, while time discretization would be even more problematic due to necessarily very nonconvex stored energy.

We will consider an initial-value problem, prescribing the initial conditions

$$\varrho|_{t=0} = \varrho_0, \quad v|_{t=0} = v_0, \quad F|_{t=0} = F_0, \quad \text{and} \quad \vartheta|_{t=0} = \vartheta_0. \quad (3.1)$$

Referring to the referential mass density $\varrho_0$, the initial conditions should satisfy $\varrho_0 = \varrho_0/\det F_0$.

3.1 Weak solutions and the main existence result

We will use the standard notation concerning the Lebesgue and the Sobolev spaces, namely $L^p(\Omega; \mathbb{R}^n)$ for Lebesgue measurable functions $\Omega \to \mathbb{R}^n$ whose Euclidean norm is integrable with $p$-power, and $W^{k,p}(\Omega; \mathbb{R}^n)$ for functions from $L^p(\Omega; \mathbb{R}^n)$ whose all derivatives up to the order $k$ have their Euclidean norm integrable with $p$-power. We also write briefly $H^k = W^{k,2}$. The notation $p^*$ will denote the exponent from the embedding $W^{1,p}(\Omega) \subset L^{p^*}(\Omega)$, i.e. $p^* = dp/(d-p)$ for $p < d$ while $p^* \geq 1$ arbitrary for $p = d$ or $p^* = +\infty$ for $p > d$. Moreover, for a Banach space $X$ and for $I = [0,T]$, we will use the notation $L^p(I; X)$ for the Bochner space of Bochner measurable functions $I \to X$ whose norm is in $L^p(I)$ while $W^{1,p}(I; X)$ denotes for functions $I \to X$ whose distributional derivative is in $L^p(I; X)$. Also, $C(\cdot)$ and $C^1(\cdot)$ will denote spaces of continuous and continuously differentiable functions.

Moreover, as usual, we will use $C$ for a generic constant which may vary from estimate to estimate. To devise a weak formulation of the initial-boundary-value problem (2.19) and (3.1) for the system (2.18), we use the by-part integration in time and the Green formula for the inertial force in the form of the left-hand side of (2.18) multiplied by a smooth test function $\tilde{v}$. The by-part integration in time and the Green formula is used also for $\dot{F}$ in the evolution rule (2.18a) tested by a smooth $\tilde{S}$ with $\tilde{S}(T) = 0$ together $\nu \cdot n = 0$, which gives

$$\int_0^T \int_{\Omega} \dot{F} : \dot{S} \, dx \, dt = \int_0^T \int_{\Omega} (\frac{\partial F}{\partial t} + (v \cdot \nabla) F) : \dot{S} \, dx \, dt = \int_0^T \int_{\Omega} (v \cdot n) (F : \dot{S}) \, dx \, dt$$

$$- \int_0^T \int_{\Omega} F : \frac{\partial \tilde{S}}{\partial t} + (\text{div} v) F \tilde{S} + F : ((v \cdot \nabla) \tilde{S}) \, dx \, dt - \int_{\Omega} F(0) \tilde{S}(0) \, dx. \quad (3.2)$$

Definition 3.1 (Weak solutions to (2.18)). For $p, q \in [1, \infty)$, a quadruple $(\varrho, v, F, \theta)$ with $\varrho \in L^\infty(I \times \Omega)$, $v \in L^q(I; W^{1,q}(\Omega; \mathbb{R}^d)) \cap L^p(I; W^{2,p}(\Omega; \mathbb{R}^d))$, $F \in L^\infty(I \times \Omega; \mathbb{R}^{d \times d})$, and $\theta \in L^1(I; W^{1,1}(\Omega))$ will be called a weak solution to the system (2.18) with the boundary conditions (2.19) and the initial condition (3.1) if det $F > 0$ a.e., $\psi'(F, \theta) F^T/\det F \in L^1(I \times \Omega; \mathbb{R}^{d \times d})$, $\gamma_F(F, \theta) F^T/\det F \in L^q(I \times \Omega; \mathbb{R}^{d \times d})$, $D(F, \theta; e(v)) \in L^q(I \times \Omega; \mathbb{R}^{d \times d})$, such that the integral identities

$$\int_0^T \int_{\Omega} \frac{\partial \varrho}{\partial t} + \varrho v \cdot \nabla \varrho \, dx \, dt + \int_{\Omega} \varrho_0(0) \, dx = 0, \quad (3.3a)$$
for any $\bar{\varrho}$ smooth with $\bar{\varrho}(T) = 0$ on $\Omega$, and

$$
\int_0^T \left( \left( \frac{\psi_F(F, \theta)}{\det F} + D(F, \theta; e(v)) - \rho v \otimes v \right) : e(\bar{v}) + \nu |\nabla e(v)|^p |\nabla e(v)|^2 \right) \omega(\bar{v}) \imath dt = \int_0^T \int_\Gamma (f - \nu_\gamma e) \cdot e(\bar{v}) \omega(\bar{v}) \imath dx + \int_\Gamma (f - \nu_\gamma e) \cdot \bar{v} \omega(\bar{v}) \imath dx + \int_\Omega \rho_0 e(\bar{v}) \omega(\bar{v}) \imath dx
$$

(3.3b)

holds for any $\bar{v}$ and smooth with $\bar{v} \cdot n = 0$ and $\bar{v}(T) = 0$, and

$$
\int_0^T \left( F \cdot \frac{\partial S}{\partial t} + \left( (\text{div} v) F + (\nabla v) F \right) \cdot \bar{S} + F : \left( (v \cdot \nabla) \bar{S} \right) \right) \omega(\bar{v}) \imath dx + \int_\Omega F_0 \bar{S}(0) \imath dx = 0
$$

(3.3c)

holds for any $\bar{S}$ smooth with $\bar{S}(T) = 0$, and also the integral identity

$$
\int_0^T \left( \frac{\partial \bar{S}}{\partial t} \right) \omega(F, \theta) \imath dt + \left( (\nabla F) F + (\nabla F) F \right) : \bar{S} + F : \left( (v \cdot \nabla) \bar{S} \right) \omega(\bar{v}) \imath dx + \int_\Omega F_0 \bar{S}(0) \imath dx = 0
$$

(3.3d)

with $\omega(\cdot, \cdot)$ from (2.18d) holds for any $\bar{\theta}$ smooth with $\bar{\theta}(T) = 0$.

Before stating the main analytical result, let us summarize the data qualification which will be, to some extent, fitted to the examples in Section II. For some $\delta > 0$, and some $1 < q < \infty$ and $d < p < \infty$, we assume:

$$
\Omega \text{ a smooth bounded domain of } \mathbb{R}^d, \quad d = 2, 3
$$

(3.4a)

$$
\varphi \in C^1(\text{GL}^+(d)), \quad \inf \varphi(\text{GL}^+(d)) > 0,
$$

(3.4b)

$$
\gamma \in C^1(\text{GL}^+(d) \times \mathbb{R}^+), \quad \forall (F, \theta, \bar{\theta}) \in \text{GL}^+(d) \times \mathbb{R}^+ \times \mathbb{R}^+:
$$

$$
\frac{\gamma_F(F, m, \theta) - \gamma_F(F, m, \bar{\theta})}{\theta - \bar{\theta}} \leq \frac{-\delta}{\det F} \text{ and } \left| \frac{\gamma_F(F, \theta)}{\det F} \right| \leq C \left( 1 + \frac{\varphi(F) + \theta}{\det F} \right),
$$

(3.4c)

$$
\forall K \subset \text{GL}^+(d) \text{ compact } \exists C_K < \infty \forall (F, \theta) \in K \times \mathbb{R}^+:
$$

$$
\omega_F'(F, \theta) + |\omega_F''(F, \theta)| \leq C_K, \quad |\omega_F'(F, \theta)| \leq C_K(1 + \theta),
$$

(3.4d)

$$
n > 0, \quad \nu > 0, \text{ and } D(F, \theta; \cdot) \text{ monotone, } D \in C(\text{GL}^+(d) \times \mathbb{R}^+ \times \mathbb{R}^{d \times d}, \mathbb{R}^{d \times d})
$$

\(\forall (F, \theta, e, \bar{e}) \in \text{GL}^+(d) \times \mathbb{R}^+ \times \mathbb{R}^{d \times d}^2:
$$

$$
D(F, \theta; 0) = 0, \quad |D(F, \theta; e)| \leq \frac{1 + |e|^{q-1}}{\delta},
$$

(3.4e)

$$
k \in C(\text{GL}^+(d) \times \mathbb{R}^+) \text{ bounded, } \inf k(\text{GL}^+(d) \times \mathbb{R}^+) > 0
$$

(3.4f)

$$
h : I \times \Gamma \times \mathbb{R}^+ \rightarrow \mathbb{R} \text{ Carathéodory function, } \theta h(t, x, \theta) \leq C(1 + \theta^2) \text{ and } h(t, x, \theta) \leq h_{\text{max}}(t, x) \text{ for some } h_{\text{max}} \in L^1(I \times \Gamma),
$$

(3.4g)

$$
g \in L^1(I; L^\infty(\Omega; \mathbb{R}^d)), \quad f \in L^2(I \times \Gamma; \mathbb{R}^d), \quad f \cdot n = 0
$$

(3.4h)

$$
F_0 \in W^{1,r}(\Omega; \mathbb{R}^{d \times d}), \quad r > d, \quad \text{ with } \text{ ess inf } \Omega \det F_0 > 0
$$

(3.4i)

$$
\theta_0 \in L^\infty(\Omega) \cap W^{1,r}(\Omega), \quad r > d, \quad \text{ with } \text{ ess inf } \Omega \theta_0 > 0
$$

(3.4j)

$$
\theta_0 \in L^1(\Omega), \quad \theta_0 \geq 0 \text{ a.e. on } \Omega,
$$

(3.4k)
where $\omega = \omega(F, \theta)$ in (3.4a) is from (2.18a) and where $\text{GL}^+(d) = \{F \in \mathbb{R}^{d \times d}; \det F > 0\}$ denotes the orientation-preserving general linear group. One should note that (3.4b) is used in (3.6) below and, although $\varphi$ occurs in (2.18) only through its derivative, actually requires $\varphi$ to be bounded from below. Independently, the blow-up of energy in compression, i.e. $\varphi(F) \to \infty$ if $\det F \to 0^+$, is allowed. The first condition in (3.4c) is just a condition on the heat capacity $c = c(F, \theta) = \omega(F, \theta)$ and, in particular, implies the coercivity $\omega(F, \theta) \geq \delta/\det F$ since $\omega(F, 0) = 0$. The condition (3.4d) is well fitted with the standard situation that the boundary flux is $h(\theta) = f(\theta_{\text{ext}}) - f(\theta)$ with an increasing function $f$ and with $\theta_{\text{ext}} \geq 0$ a prescribed external temperature, so that one can choose $h_{\text{max}} = f(\theta_{\text{ext}})$ provided we prove that $\theta \geq 0$. Also the condition $h(t, x, \theta) \leq C(1 + \theta^2)$ is well compatible with this ansatz provided $f(0) = 0$ and $f(\theta_{\text{ext}}) \in L^2(I \times \Gamma)$.

Moreover, the expected symmetry of such Cauchy stress $T$ is granted by frame indifference of $\psi(\cdot, \theta)$. This means that

$$\forall (F, \theta) \in \text{GL}^+(d) \times \mathbb{R}, \quad Q \in \text{SO}(d): \quad \psi(F, \theta) = \psi(QF, \theta),$$

where $Q \in \text{SO}(d) = \{Q \in \mathbb{R}^{d \times d}; Q^T Q = QQ^T = I\}$ is the special orthogonal group. Let us note that the frame indifference (3.5) means that $\psi(F, \theta) = \psi(F^T F, \theta)$ for some $\hat{\psi}$, which further means that $T = F^T \hat{\psi}(F^T F, \theta)^T$ as $F$. Such $T$ is obviously symmetric.

**Theorem 3.2 (Existence and regularity of weak solutions).** Let $\min(p, q) > d$ and the assumptions (3.4) and (3.5) hold. Then:

(i) there exist a weak solution $(\rho, \mathbf{v}, F, \theta)$ according Definition 3.1 with a non-negative mass density $\rho \in L^\infty(I; W^{1,s}(\Omega))$ such that $\frac{\partial}{\partial \tau} \rho \in L^s(I; L^s/(s+1)(\Omega))$ with $3 \leq s < p(pd+4p-2d)/(4p-2d)$, and a non-negative temperature $\theta \in L^\infty(I; L^1(\Omega)) \cap L^\mu(I; W^{1,\mu}(\Omega))$ with $1 \leq \mu < (d+2)/(d+1)$, and further $\frac{\partial}{\partial \tau} F \in L^{\min(p, \mu)}(I; L^p(\Omega; \mathbb{R}^{d \times d}))$ and $\nabla F \in L^\infty(I; L^\infty(\Omega; \mathbb{R}^{d \times d}))$.

(ii) Moreover, this solution complies with energetics in the sense that the energy dissipation balance (2.27) as well as the total energy balance (2.28) integrated over the time interval $[0, t]$ with the initial conditions (3.1) hold.

It should be said that the uniqueness of the weak solution is left open and, if it ever holds, would likely need more regularity than stated above.

### 3.2 Formal estimates

Formally, the assumptions (3.4) yield some a-priori bounds which can be obtained from the total energy balance (2.28) and the mechanical energy-dissipation balance (2.27) for any sufficiently regular solution $(\rho, \mathbf{v}, F, \theta)$ with $\theta \geq 0$ a.e. in $I \times \Omega$. Later in Section 3.3 we will prove the existence of such solutions, but unfortunately we are not able to claim that every weak solution has $\theta$ non-negative.

First, we use the total energy balance (2.28) which does not see any adiabatic and dissipative-heat terms that are problematic as far as estimation is concerned. Assuming that $\theta \geq 0$, we have also $\omega(F, \theta) \geq 0$ and thus we are “only” to estimate the right-hand side in (2.28). One first issue is estimation of the gravity force $gg$ when tested by the velocity $\mathbf{v}$, which can be estimated by the Hölder/Young inequality as

$$\int_\Omega g \mathbf{v} \, dx = \int_\Omega \sqrt{\frac{\partial g}{\partial F}} \sqrt{\mathbf{v} \cdot \mathbf{g}} \, dx \leq \left( \frac{\partial g}{\partial F} \right)_{L^2(\Omega)} \left( \sqrt{\mathbf{v} \cdot \mathbf{g}} \right)_{L^2(\Omega; \mathbb{R}^d)} \|g\|_{L^\infty(\Omega; \mathbb{R}^d)}$$

$$\leq \frac{1}{2} \left( \left( \frac{\partial g}{\partial F} \right)_{L^2(\Omega)}^2 + \|\sqrt{\mathbf{v} \cdot \mathbf{g}}\|_{L^2(\Omega; \mathbb{R}^d)}^2 \right) \|g\|_{L^\infty(\Omega; \mathbb{R}^d)}$$

$$= \|g\|_{L^\infty(\Omega; \mathbb{R}^d)} \int_\Omega \frac{\partial g}{2 \det F} + \frac{\partial \|\mathbf{v}\|^2}{2} \, dx$$
\[3.4b \quad \|g\|_{L^\infty(\Omega;\mathbb{R}^d)} \left( \max_{\Omega} \varphi_{00}(F) \left( \frac{\mathcal{O}}{2} \varphi(F) \right) \right) \leq \left( \int_{\Omega} \frac{\varphi(F)}{\det F} \, dx + \int_{\Omega} \frac{\theta}{2} \|v\|^2 \, dx \right). \tag{3.6}\]

The integral on the right-hand side of (3.6) can then be treated by the Gronwall lemma, for which one needs the qualification (3.4a) of \(g\).

The boundary terms in (2.28) can be estimated, at the current time instant \(t \in I\), as

\[\int_{\Gamma} f \cdot v + h(\theta) \, dS \leq \frac{1}{\nu_F} \|f\|^2_{L^2(\Gamma;\mathbb{R}^d)} + \frac{\nu_F}{4} \|v\|^2_{L^2(\Gamma;\mathbb{R}^d)} + \|h_{\max}\|_{L^1(\Gamma)} \tag{3.7}\]

with the term \(\nu_F \|v\|^2_{L^2(\Gamma;\mathbb{R}^d)} / 4\) to be absorbed in the left-hand side of (2.28); here we used the modelling assumption that the part of the heat produced by the boundary viscosity leaves the system, otherwise we would have to confine ourselves to \(f = 0\). Then, integrating (2.28) in time, one can use the qualification of \(f\) in (3.4a).

As a result, we obtain the (formal) a-priori estimates

\[\left\| \sqrt{\varphi} v \right\|_{L^\infty(I;L^2(\Omega;\mathbb{R}^d))} \leq C, \tag{3.8a}\]

\[\left\| \varphi(F) \right\|_{\det F} \bigg\|_{L^\infty(I;L^1(\Omega))} \leq C \tag{3.8b}\]

and, when recalling that we consider now only solutions with \(\theta \geq 0\) and realizing that \(\omega(F, \theta) \geq \delta \theta / \det F\) in (3.4c), also

\[\left\| \frac{\theta}{\det F} \right\|_{L^\infty(I;L^1(\Omega))} \leq C. \tag{3.8c}\]

Now we come to (2.27). The issue is now the estimation of the adiabatic term in (2.27). Here we exploit the frame indifference (3.5) so that \(\gamma'(F, \theta)F^\top\) is symmetric and thus \(\gamma'(F, \theta)F^\top \cdot \nabla v = \gamma'(F, \theta)F^\top \cdot v\). Then we estimate

\[\int_{\Omega} \left[ \frac{\gamma'(F, \theta)F^\top \cdot v}{\det F} \right] \, dx \leq \left\| \frac{\gamma'(F, \theta)F^\top \cdot v}{\det F} \right\|_{L^1(\Omega)} \|v\|_{L^\infty(\Omega;\mathbb{R}^d)} \tag{3.9}\]

\[\leq C \left[ \left| \frac{\varphi(F)}{\det F} \right|_{L^1(\omega)} \|v\|_{L^\infty(\Omega;\mathbb{R}^d)} \right] \leq C \left( \left\| \nabla v \right\|_{L^p(\Omega;\mathbb{R}^d)} + \|v\|_{L^2(\Gamma;\mathbb{R}^d)} \right) \tag{3.9}\]

The last two terms in (3.9) can be absorbed in the left-hand side of (2.27). Thus we obtain also the estimates

\[\|e(v)\|_{L^q(I \times \Omega;\mathbb{R}^d)} \leq C \quad \text{and} \quad \|\nabla e(v)\|_{L^p(I \times \Omega;\mathbb{R}^d)} \leq C. \tag{3.10}\]

When assuming \(p > d\), the estimate (3.10) is essential by preventing the evolution of singularities of the quantities transported by such a smooth velocity field. Here, due to the qualification of \(F_0\) and \(\varphi_0 = \varphi_0 / \det F_0\) in (3.4a) and (3.4b), it yields the estimates

\[\|F\|_{L^\infty(I;W^{1,r}(\Omega;\mathbb{R}^d))} \leq C_r, \quad \left\| \frac{1}{\det F} \right\|_{L^\infty(I;W^{1,r}(\Omega))} \leq C_r, \tag{3.11a}\]

\[\left\| q \right\|_{L^\infty(I;W^{1,r}(\Omega))} \leq C_r, \quad \left\| \frac{1}{q} \right\|_{L^\infty(I;W^{1,r}(\Omega))} \leq C_r \quad \text{for any} \ 1 \leq r \leq +\infty; \tag{3.11b}\]

\[\left\| \frac{\theta}{\det F} \right\|_{L^\infty(I;W^{1,r}(\Omega))} \leq C_r, \quad \left\| \frac{1}{\theta} \right\|_{L^\infty(I;W^{1,r}(\Omega))} \leq C_r. \tag{3.11c}\]
assumptions (3.4i-k):

\[
\nabla \left( \frac{1}{\text{det} F_0} \right) = -\frac{\text{det}'(F_0) \cdot \nabla F_0}{(\text{det} F_0)^2} = -\frac{\text{Cof} F_0 \cdot \nabla F_0}{(\text{det} F_0)^2} \in L^r(\Omega; \mathbb{R}^d),
\]

\[
\nabla q_0 = \frac{\nabla \theta_R}{\text{det} F_0} - \frac{\text{Cof} F_0 \cdot \nabla F_0}{\text{det} F_0^2} \in L^r(\Omega; \mathbb{R}^d), \quad \text{and}
\]

\[
\nabla \left( \frac{1}{q_0} \right) = \nabla \left( \frac{\text{det} F_0}{\theta_R} \right) = \frac{\text{Cof} F_0 \cdot \nabla F_0}{\theta_R} - \frac{\text{det} F_0 \nabla \theta_R}{\theta_R^2} \in L^r(\Omega; \mathbb{R}^d).
\]

From (3.8a,d) and (3.11a,b), we then have also

\[
\|v\|_{L^\infty(I;L^2(\Omega;\mathbb{R}^d))} \leq \|\sqrt{\theta} v\|_{L^\infty(I;L^2(\Omega;\mathbb{R}^d))} \| \frac{1}{\sqrt{\theta}} \|_{L^\infty(I \times \Omega)} \leq C \quad \text{and} \quad (3.11c)
\]

\[
\|\theta\|_{L^\infty(I;L^1(\Omega))} \leq \|\frac{\theta}{\text{det} F}\|_{L^\infty(I;L^1(\Omega))} \|\text{det} F\|_{L^\infty(I \times \Omega)} \leq C. \quad (3.11d)
\]

### 3.3 Proof of Theorem 3.2

Let us outline the main technical difficulties more in detail: The time discretization (Rothe’s method) standardly needs convexity of \( \varphi \) (which is not a realistic assumption in finite-strain mechanics) possibly weakened if there is some viscosity in \( F \) (which is not directly considered here, however). The conformal space discretization (i.e. the Faedo-Galerkin method) can not directly copy the energetics because the “nonlinear” test of (2.18c) by \( \left[ \varphi / \text{det} \right]'(F) \) needed in (2.22) is problematic in this approximation as \( \left[ \varphi / \text{det} \right]'(F) \) is not in the respective finite-dimensional space in general and similarly also the test of (2.18a) by \( |v|^2 \) is problematic.

For the approximation method used in the proof below, we assume the data \( \psi, D, \kappa, \) and \( h \) to be defined also for the negative temperature by extending them as

\[
\psi(F, \theta) := \varphi(F) + \theta (\ln(-\theta) - 1), \quad D(F, \theta; e) := D(F, -\theta; e),
\]

\[
\kappa(F, \theta) := \kappa(F, -\theta), \quad \text{and} \quad h(t, x, \theta) := h(t, x, -\theta) \quad \text{for} \quad \theta < 0 \quad (3.12)
\]

with \( \varphi \) from the split (2.20). This definition makes \( \psi : \text{GL}^+(d) \times \mathbb{R} \rightarrow \mathbb{R} \) continuous and implies \( \gamma_F'(F, \theta) = 0 \) and \( \omega(F, \theta) = \theta / \text{det} F \) so that \( \omega_F'(F, \theta) = \theta F^{-\top} / \text{det} F \) for \( \theta \) negative. In particular, both \( \omega(F, \cdot) \) and \( \gamma_F(F, \cdot) \) are continuous.

For clarity, we will divide the proof into nine steps.

### 1. a regularisation. Referring to the formal estimates (3.11a), we can choose \( \lambda > 0 \) so small that, for any possible sufficiently regular solution, it holds

\[
\text{det} F > \lambda \quad \text{and} \quad |F| < \frac{1}{\lambda} \quad \text{a.e. on} \quad I \times \Omega. \quad (3.13)
\]

We first regularize the stress \( T \) in (2.18b) by considering a smooth cut-off \( \pi_\lambda \in C^1(\mathbb{R}^{d \times d}) \) defined as

\[
\pi_\lambda(F) := \begin{cases} 
1 & \text{for det} F \geq \lambda \text{ and } |F| \leq 1/\lambda, \\
0 & \text{for det} F \leq \lambda/2 \text{ or } |F| \geq 2/\lambda, \\
\left( \frac{3}{\lambda^2} (2 \text{ det } F - \lambda)^2 - \frac{2}{\lambda^3} (2 \text{ det } F - \lambda)^3 \right) \times (3(\lambda |F| - 1)^2 - 2(\lambda |F| - 1)^3) & \text{otherwise},
\end{cases}
\]

(3.14)

Here \( | \cdot | \) stands for the Frobenius norm \( |F| = (\sum_{i,j=1}^d F_{ij}^2)^{1/2} \) for \( F = [F_{ij}] \), which guarantees that \( \pi_\lambda \) is frame indifferent.
Furthermore, we also regularize the singular nonlinearity $1/\det(\cdot)$ which is still employed in the right-hand-side force in the momentum equation, although the mass-density continuity equation is kept nonregularized for the inertial term. To this aim, we introduce the short-hand notation

$$\det_{\lambda}(F) := \min \left( \max \left( \det F, \frac{\lambda}{2} \right), \frac{2}{\lambda} \right) \quad \text{and} \quad T_{\lambda, \varepsilon}(F, \theta) := \left( [\pi_{\lambda, \varepsilon}](F) + \pi_{\lambda}(F) \frac{\gamma'_{\lambda}(F, \theta)}{1 + \varepsilon |\theta|} \right) \frac{F^\top}{\det F}.$$ \hfill (3.15a)

Note that also $\pi_{\lambda, \varepsilon} \in C^{1}(\mathbb{R}^{d \times d})$ if $\varphi \in C^{1}(\text{GL}^+(d))$ and that $[\pi_{\lambda, \varepsilon}]'$ together with the regularized Cauchy stress $T_{\lambda, \varepsilon}$ are bounded, continuous, and vanish if $F$ “substantially” violates the constraints \((3.13)\), specifically:

$$\left( \det F \leq \frac{\lambda}{2} \quad \text{or} \quad |F| \geq \frac{2}{\lambda} \right) \quad \Rightarrow \quad T_{\lambda, \varepsilon}(F, \theta) = 0.$$ \hfill (3.15b)

Also, $\pi_{\lambda}(F)\frac{\gamma'_{\lambda}(F, \theta)}{\det F} F^\top/\det_{\lambda}(F) = \pi_{\lambda}(F)\frac{\gamma'_{\lambda}(F, \theta)}{\det F} F^\top/\det F$ so that, due to \((3.14c)\), $T_{\lambda, \varepsilon} : \mathbb{R}^{d \times d} \times \mathbb{R} \to \mathbb{R}_{\text{sym}}$ is bounded. Recalling the extension \((3.12)\), let us note that it is defined also for negative temperatures.

Altogether, for the above chosen $\lambda$ and for any $\varepsilon > 0$, we consider the regularized system

$$\frac{\partial \rho}{\partial t} = -\text{div}(\rho v),$$ \hfill (3.16a)

$$\frac{\partial}{\partial t}(\rho v) + \text{div}(\rho v \otimes v) = \text{div}(T_{\lambda, \varepsilon}(F, \theta) + D(F, \theta; e(v)) - \text{div}\mathcal{K}) + \sqrt{\frac{\theta \varepsilon}{\det_{\lambda}(F)}} g$$

with $\mathcal{K} = \nu |\nabla e(v)|^{p-2} \nabla e(v)$, \hfill (3.16b)

$$\frac{\partial F}{\partial t} = (\nabla v) F - (v \cdot \nabla) F,$$ \hfill (3.16c)

$$\frac{\partial w}{\partial t} = \text{div}(\kappa(F, \theta) \nabla \theta - w v) + \frac{D(F, \theta; e(v)) : e(v) + \nu |\nabla e(v)|^{p}}{1 + \varepsilon |e(v)|^{q} + \varepsilon |\nabla e(v)|^{p}}$$

$$\quad + \frac{\pi_{\lambda}(F)\gamma'_{\lambda}(F, \theta)}{(1 + \varepsilon |\theta| |\det F|)} e(v) \quad \text{with} \quad w = \omega(F, \theta),$$ \hfill (3.16d)

where $\omega(\cdot, \cdot)$ is from \((2.18d)\). We complete this system with the correspondingly regularized boundary conditions on $I \times \Gamma$:

$$v \cdot n = 0, \quad \left[ (T_{\lambda, \varepsilon}(F, \theta) + D(F, \theta; e(v)) - \text{div}\mathcal{K}) n - \text{div}_{h}(\mathcal{H}(n)) \right]_{\nu_0} + \nu_0 v = f,$$ \hfill (3.17a)

$$\nabla e(v) : (n \otimes n) = 0, \quad \text{and} \quad \kappa(F, \theta) \nabla \theta \cdot n - \frac{\nu_0 |v|^2}{2 + \varepsilon |v|^2} = h_{\varepsilon}(\theta) := \frac{h(\theta)}{1 + \varepsilon |h(\theta)|},$$ \hfill (3.17b)

and initial conditions

$$\rho|_{t=0} = \rho_0, \quad v|_{t=0} = v_0, \quad F|_{t=0} = F_0, \quad \theta|_{t=0} = \theta_{0, \varepsilon} := \frac{\theta_0}{1 + \varepsilon \theta_0}.$$ \hfill (3.17c)

The corresponding weak formulation of \((3.16)-(3.17)\) a la Definition 3.1 is quite straightforward, and we will not explicitly write it, also because it will be obvious from its Galerkin version \((3.19)\) below. The philosophy of the regularization \((3.13)\) is that the estimation of the mechanical part \((3.16)\) and the thermal part \((3.16d)\) decouples since $T_{\lambda, \varepsilon}$ is bounded and that heat sources in the heat equation are bounded for fixed $\varepsilon > 0$. Simultaneously, the heat equation has a non-negative solution and, for $\varepsilon \to 0$, the physical a priori estimates are the same as the formal estimates \((3.8)-(3.11)\) and, when taking $\lambda > 0$ small to comply with \((3.13)\), the $\lambda$-regularization becomes eventually inactive, cf. Step 8 below.
Step 2: a semi-discretization. For \( \varepsilon > 0 \) fixed, we use a spatial semi-discretization, keeping the transport equations (3.16a) and (3.16c) continuous (i.e. non-discretised) when exploiting Lemma 5.1. More specifically, we make a conformal Galerkin approximation of (3.16b) by using a collection of nested finite-dimensional subspaces \( \{V_k\}_{k \in \mathbb{N}} \) whose union is dense in \( W^{2,p}(\Omega; \mathbb{R}^d) \) and a conformal Galerkin approximation of (3.16d) by using a collection of nested finite-dimensional subspaces \( \{Z_k\}_{k \in \mathbb{N}} \) whose union is dense in \( H^1(\Omega) \). Without loss of generality, we can assume \( v_0 \in V_1 \) and \( \theta_{0,\varepsilon} \in Z_1 \).

Let us denoted the approximate solution of the regularized system (3.16) by \( (\tilde{\phi}_{ek}, v_{ek}, F_{ek}, \tilde{\theta}_{ek}) : I \rightarrow W^{1,r}(\Omega) \times V_k \times W^{1,r}(\Omega; \mathbb{R}^{d \times d}) \times Z_k \). Specifically, such a quadruple should satisfy

\[
\frac{\partial \tilde{\phi}_{ek}}{\partial t} = -\text{div}(\tilde{\phi}_{ek} v_{ek}) \quad \text{and} \quad \frac{\partial F_{ek}}{\partial t} = (\nabla v_{ek}) F_{ek} - (v_{ek} \cdot \nabla) F_{ek}
\]  
(3.18)

in the weak form like (3.19a,c) together with the following integral identities

\[
\int_0^T \int_{\Omega} \left( T_{\lambda,\varepsilon}(F_{ek}, \theta_{ek}) + D(F_{ek}, \theta_{ek}; e(v_{ek})) - \tilde{\phi}_{ek} \tilde{\phi}_{ek} \otimes v_{ek} \right) : e(\tilde{v}) + \nu |\nabla e(v_{ek})|^p - 2 \nabla e(v_{ek}) : \nabla e(\tilde{v}) - \tilde{\phi}_{ek} v_{ek} \frac{\partial \tilde{\theta}_{ek}}{\partial t} \right) \, dx \, dt + \int_0^T \int_{\Omega} \nu \tilde{\phi}_{ek} v_{ek} \tilde{v} \, dS \, dt
\]

\[
= \int_0^T \int_{\Omega} \frac{\theta_{0,\varepsilon} \tilde{\theta}_{ek}}{\det(\lambda(F_{ek}))} g \tilde{v} \, dx \, dt + \int_0^T \int_{\Gamma} f \cdot \tilde{v} \, dS \, dt + \int_{\Omega} \theta_0 v_0 \tilde{v}(0) \, dx
\]
(3.19a)

for any \( \tilde{v} \in L^\infty(I; V_k) \) with \( \tilde{v} \cdot n = 0 \) on \( I \times \Gamma \) and \( \tilde{v}(T) = 0 \), and

\[
\int_0^T \int_{\Omega} \left( \frac{w_{ek} \partial \tilde{\theta}_{ek}}{\partial t} + (w_{ek} v_{ek} - \kappa(F_{ek}, \theta_{ek}) \nabla \theta_{ek}) \cdot \nabla \tilde{\theta}_{ek} \right) \, dx \, dt
\]

\[
+ \left( \frac{D(F_{ek}, \theta_{ek}; e(v_{ek})) : e(v_{ek})) + \nu |\nabla e(v_{ek})|^p}{1 + \varepsilon |e(v_{ek})|^q + \varepsilon |\nabla e(v_{ek})|^p} + \frac{\pi_{\lambda}(F_{ek}) \gamma_{\mu}(F_{ek}, \theta_{ek}) F_{ek}^T : e(v_{ek})) \tilde{\theta}_{ek}}{(1 + \varepsilon |\theta_{ek}| \det F_{ek})} \right) \, dx \, dt
\]

\[
+ \int_{\Omega} \omega(F_0, \theta_{0,\varepsilon}) \tilde{\theta}(0) \, dx + \int_0^T \int_{\Gamma} \left( h_{\varepsilon}(\theta_{ek}) + \frac{\nu_{\varepsilon} |v_{ek}|^2}{2 + \varepsilon |v_{ek}|^2} \right) \tilde{\theta} \, dS \, dt = 0 \quad \text{with} \quad w_{ek} = \omega(F_{ek}, \theta_{ek})
\]
(3.19b)

holding for any \( \tilde{\theta} \in C^1(I; Z_k) \) with \( \tilde{\theta}(T) = 0 \).

Existence of this solution is based on the standard theory of systems of ordinary differential equations first locally in time combined here with the abstract \( W^{1,r}(\Omega) \)-valued differential equations based on Lemma 5.1 used with \( n = 1 \) and \( n = d \times d \) for the scalar and the tensor transport equations (3.18), and then by successive prolongation on the whole time interval based on the \( L^\infty \)-estimates below.

Actually, Lemma 5.1 with \( v = v_{ek} \) and with the fixed initial conditions \( F_0 \) and \( \theta_0 \), defines the nonlinear operators \( \tilde{\mathfrak{F}} : I \times L^p(I; W^{2,p}(\Omega; \mathbb{R}^d)) \rightarrow W^{1,r}(\Omega; \mathbb{R}^{d \times d}) \) and \( \mathfrak{R} : I \times L^p(I; W^{2,p}(\Omega; \mathbb{R}^d)) \rightarrow W^{1,r}(\Omega) \) by

\[
F_{ek}(t) = \tilde{\mathfrak{F}}(t, v_{ek}) \quad \text{and} \quad \tilde{\theta}_{ek}(t) = \mathfrak{R}(t, v_{ek}).
\]
(3.20)

Step 3: first a priori estimates. In the Galerkin approximation, it is legitimate to use \( \tilde{v} = v_{ek} \) for (3.19a) and \( \tilde{\theta} = \theta_{ek} \) for (3.19b). We take the benefit from having the transport equations (3.18) non-discretized and, thus, we can test them by the nonlinearities \( |v_{ek}|^2/2 \) and \( \frac{|\pi_{\lambda \gamma}|}{|\det|} |F_{ek}| = |(\varphi_{\lambda \gamma} + \pi_{\lambda \gamma})|/|\det| |F_{ek}| \), respectively. In particular, we can use the calculus (2.22) and the calculus (2.25) also for the semi-Galerkin approximate solution. Specifically, from (3.19a) tested by \( v_{ek} \), like (2.27) we obtain the identity

\[
\frac{d}{dt} \int_{\Omega} \frac{\tilde{\phi}_{ek}}{2} |v_{ek}|^2 + \frac{\pi_{\lambda}(F_{ek}) \varphi(F_{ek})}{\det F_{ek}} \, dx + \int_{\Omega} D(F_{ek}, \theta_{ek}; e(v_{ek})) : e(v_{ek}) + \nu |\nabla e(v_{ek})|^p \, dx + \int_{\Gamma} \nu_{\varepsilon} |v_{ek}|^2 \, dS
\]
For \( \lambda \) and \( \hat{\epsilon} \), and where
\[
\text{while the last term in (3.21) can be estimated as}
\]
\[
\text{By the Gronwall inequality, we obtain the estimates}
\]
\[
\text{and, like (3.11c), we also have that}
\]
\[
\text{and, using (3.18), we will use the calculus}
\]
\[
\text{where the dot-notation refers to the convective time derivative here with respect to the velocity \( \mathbf{v}_{ek} \) and where \( \tilde{\omega}(\mathbf{F}, \theta) \) is a primitive function to \( \theta \mapsto \theta \omega_\theta(\mathbf{F}, \theta) \) depending smoothly on \( \mathbf{F} \), specifically}
\]
\[
\text{Integrating the last term in (3.24) over} \ \Omega \ \text{gives, by the Green formula,}
\]
\[
\text{Thus, the mentioned test by} \ \theta_{ek} \ \text{then gives}
\]
where $\omega \in L^p(\mathbf{R}^d)$ and $\theta_k \in L^q(\mathbf{R}^d)$. Let us discuss the difficult terms. In view of (3.25), it holds that

$$
\int \frac{|\nabla \omega(F_{ek}, \theta_k)|}{1 + \varepsilon|\nabla (v_{ek})|^2 + \varepsilon|\nabla (v)|^2} + \theta_k \frac{\theta_k}{1 + \varepsilon|\theta_k|} \det F_{ek} = 0,
$$

for $\omega \in L^p(\mathbf{R}^d)$ and $\theta_k \in L^q(\mathbf{R}^d)$. Realizing that

$$
\int \omega(F_{ek}, \theta_k) \partial v_{ek} - \omega(F_{ek}, \theta_k) \partial v_{ek} \partial(F_{ek}, \theta_k) + \hat{\omega}(F_{ek}, \theta_k) \partial v_{ek}
$$

and that $\hat{\omega}(F_{ek}, \theta_k) \partial v_{ek}$ is to be estimated from above, in particular relying on (3.25). We integrate (3.26) in time over an interval $[0, t]$ with $t \in I$. For the left-hand side, let us realize that $\hat{\omega}(F, \theta) \geq c \theta^2$ due to (3.4c) with $c > 0$ depending on the fixed $\lambda > 0$ used in (3.13). Then the integrated right-hand side of (3.26) is to be estimated from above, in particular relying on (3.4c) and on (3.4d). Let us discuss the difficult terms. In view of (3.25), it holds that

$$
\hat{\omega}(F, \theta) = \int_0^t \theta^2 \omega(F, \theta) d\tau.
$$

Recalling (3.4d), we have $|\hat{\omega}(F, \theta) - \theta \omega(F, \theta) - \theta \omega(F, \theta)| \leq C(1 + |\theta|^2)$. It allows for estimation

$$
\left| \int \omega(F_{ek}, \theta_k) \partial v_{ek} \partial(F_{ek}, \theta_k) \right| \leq C(1 + |\theta|^2)_2(i \Omega)\|\omega v_{ek}\|_{L^\infty(\Omega; \mathbf{R}^{3d})}.
$$

Using (3.4d) together with $\omega(F, 0) = 0$, it holds $|\omega(F, \theta)| \leq C\theta$, the convective terms $\omega(F_{ek}, \theta_k)(\partial v_{ek}) \theta_k$ and $\hat{\omega}(F_{ek}, \theta_k) \partial v_{ek}$ in (3.26) can be estimated as

$$
\left| \int \omega(F_{ek}, \theta_k) \partial v_{ek} \partial(F_{ek}, \theta_k) \right| \leq C\theta_k^2(i \Omega)\|\partial v_{ek}\|_{L^\infty(\Omega)}.
$$

The terms $\|\theta_k\|_{L^2(\Omega)}^2$ in (3.27) and in (3.28) are to be treated by the Gronwall inequality. The boundary term in (3.26) can be estimated by (3.4e), taking also into the account the extension (3.12), as

$$
\int \left( h_\varepsilon(\theta_k) - \frac{\nu_b|v_{ek}|^2}{2 + \varepsilon|v_{ek}|^2} \right) \theta_k dS \leq C_{\varepsilon, \nu_b, a} |\theta_k|^2(i \Gamma) \leq C_{\varepsilon, \nu_b, a} + a N^2 \left( \|\theta_k\|_{L^2(\Omega)}^2 + \|\nabla \theta_k\|_{L^2(\Omega)}^2 \right).
$$

where $C_{\varepsilon, \nu_b, a}$ depends also on $C$ from (3.4e) and where $N$ is the norm of the trace operator $H^1(\Omega) \to L^2(\Gamma)$. For $a > 0$ in (3.29) sufficiently small, the last term can be absorbed in the left-hand side of (3.23).

By the Gronwall inequality, exploiting again the bound (3.23), we obtain the estimate

$$
\|\theta_k\|_{L^\infty(I; L^2(\Omega))} \leq C.
$$

In addition, realizing that $|\omega(F_{ek}, \theta_k)|$ is bounded in $L^2(I; L^2(\Omega))$ due to (3.4d) and that $|\nabla F_{ek}|$ is bounded in $L^\infty(I; L^2(\Omega))$, from the calculus $\nabla w_{ek} = \omega(F_{ek}, \theta_k) \nabla \theta_k + \omega(F_{ek}, \theta_k) \nabla F_{ek} \in L^2(I \times \Omega; \mathbf{R}^{3d})$, we have also the bound

$$
\|w_{ek}\|_{L^\infty(I; L^2(\Omega))} \leq C.
$$

**Step 4: Limit passage for $k \to \infty$.** Using the Banach selection principle, we can extract some subsequence of $\{\theta_k, v_{ek}, F_{ek}, w_{ek}\}_{k \in \mathbb{N}}$ and its limit $(\theta, v, F, w) : I \to W^{1,r}(\Omega) \times L^2(\Omega; \mathbf{R}^d) \times W^{1,r}(\Omega; \mathbf{R}^{3d}) \times L^2(\Omega)$ such that

$$
\theta_k \to \theta \quad \text{weakly* in } L^\infty(I; W^{1,r}(\Omega)) \cap W^{1,\min(p,q)}(I; L^r(\Omega)),
$$

$$
v_{ek} \to v \quad \text{weakly* in } L^\infty(I; L^2(\Omega; \mathbf{R}^d)) \cap L^{\min(p,q)}(I; W^{2,p}(\Omega; \mathbf{R}^d)).
$$

(3.31)
\[ F_{\epsilon k} \rightarrow F_{\epsilon} \quad \text{weakly* in } L^\infty(I; W^{1,r}(\Omega; \mathbb{R}^{d \times d})) \cap W^{1,\min(p,q)}(I; L^r(\Omega; \mathbb{R}^{d \times d})), \quad (3.31c) \]
\[ w_{\epsilon k} \rightarrow w_{\epsilon} \quad \text{weakly* in } L^\infty(I; L^2(\Omega)) \cap L^2(I; H^1(\Omega)). \quad (3.31d) \]

By the Aubin-Lions lemma here relying on the assumption \( r > d \), we also have that
\[ \theta_{\epsilon k} \rightarrow \theta_{\epsilon} \quad \text{strongly in } C(I \times \overline{\Omega}) \quad \text{and} \quad F_{\epsilon k} \rightarrow F_{\epsilon} \quad \text{strongly in } C(I \times \overline{\Omega}; \mathbb{R}^{d \times d}). \quad (3.32) \]

This already allows for the limit passage in the evolution equations \((3.18)\), cf. \((5.4)\) below.

Further, by comparison in the equation \((3.16d)\) with the boundary condition \((3.17b)\) in its Galerkin approximation, we obtain a bound on \( \frac{\partial}{\partial t} w_{\epsilon k} \) in seminorms \(| \cdot |_l\) on \( L^2(I; H^1(\Omega)^*) \) arising from this Galerkin approximation:
\[ |f|_l := \sup_{\tilde{\theta}(t) \in Z_l \text{ for } t \in I} \int_0^T \int_\Omega f \tilde{\theta} \, dx dt \]

More specifically, for any \( k \geq l \), we can estimate
\[ \left| \frac{\partial w_{\epsilon k}}{\partial t} \right|_{l} = \sup_{\tilde{\theta}(t) \in Z_l \text{ for } t \in I} \int_0^T \int_\Omega \left( \frac{|D(F_{\epsilon k}, \theta_{\epsilon k} : e(w_{\epsilon k}))|}{1 + \varepsilon |e(w_{\epsilon k})|^2} + \frac{\pi^V(F_{\epsilon k}, \theta_{\epsilon k}) F_{\epsilon k} : e(w_{\epsilon k})}{1 + \varepsilon |\theta_{\epsilon k}|^2} \right) \, dx dt \leq C \]

with some \( C \) depending on the estimates \((3.23)\) but independent on \( l \in \mathbb{N} \). Thus, by \((3.31d)\) and by a generalized Aubin-Lions theorem \([35\text{, Ch.8}]\), we obtain
\[ w_{\epsilon k} \rightarrow w_{\epsilon} \quad \text{strongly in } L^s(I \times \Omega) \text{ for } 1 \leq s < 2 + 4/d. \quad (3.34a) \]

Since \( \omega(F_{\epsilon k}, \cdot) \) is increasing, we can write \( \theta_{\epsilon k} = \omega(F_{\epsilon k},\cdot)^{-1}(w_{\epsilon k}) \). Thanks to the continuity of \((F, w) \mapsto [\omega(F,\cdot)]^{-1}(w) : \mathbb{R}^{d \times d} \times \mathbb{R} \rightarrow \mathbb{R} \) and the at-most linear growth in \( w \) uniformly with respect to \( F \) from any compact \( K \subset \text{GL}^+(d) \), cf. \((3.4c)\), we have also
\[ \theta_{\epsilon k} \rightarrow \theta_{\epsilon} = [\omega(F_{\epsilon},\cdot)]^{-1}(w_{\epsilon}) \quad \text{strongly in } L^s(I \times \Omega) \text{ for } 1 \leq s < 2 + 4/d; \quad (3.34b) \]

note that we do not have any direct information about \( \frac{\partial}{\partial t} \theta_{\epsilon k} \) so that we could not use the Aubin-Lions arguments straight for \( \{\theta_{\epsilon k}\}_{k \in \mathbb{N}} \). Thus, by the continuity of the corresponding Nemytskii (or here simply superposition) mappings, also the conservative part of the regularized Cauchy stress as well as the heat part of the internal energy, namely
\[ T_{\lambda,\epsilon}(F_{\epsilon k}, \theta_{\epsilon k}) \rightarrow T_{\lambda,\epsilon}(F_{\epsilon}, \theta_{\epsilon}) \quad \text{strongly in } L^c(I \times \Omega; \mathbb{R}^{d \times d}_{\text{sym}}), \quad 1 \leq c < \infty, \quad (3.34c) \]
\[ \frac{\pi^\lambda(F_{\epsilon k}, \gamma^0(F_{\epsilon k}, \theta_{\epsilon k}) F_{\epsilon k}^\top)}{(1 + \varepsilon |\theta_{\epsilon k}|^2)} \det F_{\epsilon k} \rightarrow \frac{\pi^\lambda(F_{\epsilon}, \gamma^0(F_{\epsilon}, \theta_{\epsilon}) F_{\epsilon}^\top)}{(1 + \varepsilon |\theta_{\epsilon}|^2)} \det F_{\epsilon} \quad \text{strongly in } L^c(I \times \Omega; \mathbb{R}^{d \times d}_{\text{sym}}), \quad 1 \leq c < \infty, \quad (3.34d) \]
\[ \omega(F_{\epsilon k}, \theta_{\epsilon k}) \rightarrow \omega(F_{\epsilon}, \theta_{\epsilon}) \quad \text{strongly in } L^c(I \times \Omega), \quad 1 \leq c < 2 + 4/d. \quad (3.34e) \]

It is important that \( \nabla(\theta_{\epsilon k} v_{\epsilon k}) = \nabla \theta_{\epsilon k} \otimes v_{\epsilon k} + \theta_{\epsilon k} \nabla v_{\epsilon k} \) is bounded in \( L^\infty(I; L^r(\Omega; \mathbb{R}^{d \times d})) \) due to the already obtained bounds \((3.23)\) \(c,d)\). Therefore, \( \theta_{\epsilon k} v_{\epsilon k} \) converges weakly* in \( L^\infty(I; W^{1,r}(\Omega; \mathbb{R}^{d \times d})) \). The limit of \( \theta_{\epsilon k} v_{\epsilon k} \) can be identified as \( \theta v \), because we already showed that \( \theta_{\epsilon k} \) converges strongly in \((3.32)\) and \( v_{\epsilon k} \) converges weakly due to \((3.31b)\).
By comparison, we also obtain $\frac{\partial}{\partial t}(\varrho_{ek}v_{ek})$. Let us note, indeed, that \((2.6)\) still holds for the semi-discretized system since the continuity equation has not been discretized. Specifically, for any $\tilde{\boldsymbol{v}} \in L^2(I; W^{2,p}(\Omega; \mathbb{R}^d))$ with $\tilde{\boldsymbol{v}}(t) \in V_k$ for a.a. $t \in I$, we have

$$
\int_0^T \int_\Omega \frac{\partial}{\partial t}(\varrho_{ek}v_{ek}) \tilde{\boldsymbol{v}} \, dx \, dt = \int_0^T \int_\Omega (f + \nu_\delta v_{ek}) \tilde{\boldsymbol{v}} \, dS \, dt + \int_0^T \int_\Omega \left( \frac{\varrho_{ek} \varepsilon_k}{\det \lambda(F_{ek})} g \tilde{\boldsymbol{v}} - (\varrho_{ek}v_{ek} \otimes v_{ek} - T_{\lambda,\varepsilon}(F_{ek}, \theta_{ek}) - D(F_{ek}, \theta_{ek}; e(v_{ek}))) \right) : e(\tilde{\boldsymbol{v}}) \, dx \, dt \leq C \|\tilde{\boldsymbol{v}}\|_{L^3(I; W^{1,q}(\Omega; \mathbb{R}^d) \cap L^p(I; W^{2,p}(\Omega; \mathbb{R}^d)))},
$$

which yields a bound for $\frac{\partial}{\partial t}(\varrho_{ek}v_{ek})$ in a seminorm on $L^p(I; W^{1,q}(\Omega; \mathbb{R}^d)) + L^p(I; W^{2,p}(\Omega; \mathbb{R}^d))$ induced by the Galerkin discretization by $V_k$ (and by any $V_l$ with $l \leq k$, too). It is important that $C$ in \((3.35)\) does not depend on $k$. Using again a generalization of the Aubin-Lions compact-embedding theorem, cf. \([45, Lemma 7.7]\), we then obtain

$$
\varrho_{ek}v_{ek} \to \varrho_e v_e \quad \text{strongly in } L^c(I \times \Omega; \mathbb{R}^d) \quad \text{for any } 1 \leq c < 4.
$$

Since obviously $v_{ek} = \varrho_{ek}v_{ek}(1/\varrho_{ek})$, thanks to \((3.32)\) and \((3.36a)\), we also have that

$$
v_{ek} \to v_e \quad \text{strongly in } L^c(I \times \Omega; \mathbb{R}^d) \quad \text{with any } 1 \leq c < 4.
$$

For the limit passage in the momentum equation, one uses the monotonicity of the dissipative stress, i.e., the monotonicity of the quasilinear operator

$$
\nu \mapsto \text{div} \left( \text{div}(|\nabla e(\cdot)|^{p-2} \nabla e(\nu)) - D(F, \theta; e(\nu)) \right)
$$

as well as of the time-derivative operator. One could use the already obtained weak convergences and the so-called Minty trick but, later, we will need a strong convergence of $e(v_{ek})$ to pass to the limit in the heat equation. Thus, we first prove this strong convergence, which then allows for the limit passage in the momentum equation directly. We will use the weak convergence of the inertial force

$$
\begin{align*}
\int_0^T \int_\Omega & \left( \frac{\partial}{\partial t}(\varrho_{ek}v_{ek}) + \text{div}(\varrho_{ek}v_{ek} \otimes v_{ek}) \right) \tilde{\boldsymbol{v}} \, dx \, dt \\
= & \int_\Omega \varrho_{ek}(T) v_{ek}(T) \cdot \tilde{\boldsymbol{v}}(T) - \varrho_0 v_0 \cdot \tilde{\boldsymbol{v}}(0) \, dx - \int_0^T \int_\Omega \varrho_{ek}v_{ek} \cdot \frac{\partial \tilde{\boldsymbol{v}}}{\partial t} + (\varrho_{ek}v_{ek} \otimes v_{ek}) : \nabla \tilde{\boldsymbol{v}} \, dx \, dt \\
\xrightarrow{k \to \infty} & \int_\Omega \varrho_e(T) v_e(T) \cdot \tilde{\boldsymbol{v}}(T) - \varrho_0 v_0 \cdot \tilde{\boldsymbol{v}}(0) \, dx - \int_0^T \int_\Omega \varrho_e v_e \cdot \frac{\partial \tilde{\boldsymbol{v}}}{\partial t} + (\varrho_e v_e \otimes v_e) : \nabla \tilde{\boldsymbol{v}} \, dx \, dt \\
= & \int_0^T \int_\Omega \left( \frac{\partial}{\partial t}(\varrho_e v_e) + \text{div}(\varrho_e v_e \otimes v_e) \right) \tilde{\boldsymbol{v}} \, dx \, dt.
\end{align*}
$$

Further, relying on the calculus \((2.26)\), we will use the identity

$$
\begin{align*}
\int_\Omega \frac{\varrho_{ek}(T)}{2} |v_{ek}(T) - v_e(T)|^2 \, dx = & \int_0^T \int_\Omega \left( \frac{\partial}{\partial t}(\varrho_{ek}v_{ek}) + \text{div}(\varrho_{ek}v_{ek} \otimes v_{ek}) \right) \cdot v_e \, dx \, dt \\
& + \int_0^T \frac{\varrho_0}{2} |v_0|^2 - \varrho_{ek}(T)v_{ek}(T) \cdot v_e(T) + \frac{\varrho_{ek}(T)}{2} |v_e(T)|^2 \, dx. \quad (3.38)
\end{align*}
$$

We further used that the $\varrho_{ek}(T)$ is also bounded in $W^{1,r}(\Omega)$ and $v_{ek}(T)$ is bounded in $L^2(\Omega; \mathbb{R}^d)$, together with some information about the time derivative $\frac{\partial}{\partial t}(\varrho_{ek}v_{ek})$, cf. \((3.35)\), so that we can identify the weak limit of $\varrho_{ek}(T)v_{ek}(T)$. Specifically, we have that

$$
\varrho_{ek}(T)v_{ek}(T) \to \varrho_e(T)v_e(T) \quad \text{weakly in } L^2(\Omega; \mathbb{R}^d).
$$

(3.39)
We now use the Galerkin approximation of the regularized momentum equation (3.19a) tested by \( \tilde{\nu} = v_{ek} - \bar{v}_k \) with \( \bar{v}_k : I \to V_k \) an approximation of \( v_\varepsilon \) in the sense that \( \bar{v}_k \to v_\varepsilon \) strongly in \( L^\infty(I; L^2(\Omega; \mathbb{R}^d)) \), \( e(\bar{v}_k) \to e(v_\varepsilon) \) strongly in \( L^p(I \times \Omega; \mathbb{R}^{d \times d}) \), and \( \nabla e(\bar{v}_k) \to \nabla e(v_\varepsilon) \) for \( k \to \infty \) strongly in \( L^p(I \times \Omega; \mathbb{R}^{d \times d}) \) for \( k \to \infty \). Using also the first inequality in (3.13) and (3.36), we can estimate

\[
\frac{\delta}{2} \| v_{ek}(T) - v_\varepsilon(T) \|^2_{L^2(\Omega; \mathbb{R}^d)} + \delta \| e(v_{ek} - v_\varepsilon) \|^2_{L^p(I \times \Omega; \mathbb{R}^{d \times d})} + \nu \| \nabla e(v_{ek} - v_\varepsilon) \|^2_{L^p(I \times \Omega; \mathbb{R}^{d \times d})}
\]

\[
\leq \int_\Omega \frac{\theta_{ek}(T)}{2} |v_{ek}(T) - v_\varepsilon(T)|^2 \, dx + \int_0^T \int_{\Gamma} \nu_v |v_{ek} - v_\varepsilon|^2 \, dS \, dt
\]

\[
+ \int_0^T \int_\Omega \left( D(F_{ek}, \theta_{ek}; e(v_{ek}))-D(F_{ek}, \theta_{ek}; e(v_\varepsilon)) : e(v_{ek} - v_\varepsilon)
\]

\[
+ \nu \left( \| \nabla e(v_{ek}) \|^2 - \| \nabla e(v_\varepsilon) \|^2 \right) \right) \, dx \, dt
\]

\[
= \int_0^T \int_\Omega \left( \sqrt{\frac{\theta_{ek}(T)}{2 \det F_{ek}}} \cdot g(e(v_{ek} - \bar{v}_k) - T_{\lambda, k}(F_{ek}, \theta_{ek}) : e(v_{ek} - \bar{v}_k)
\]

\[
- D(F_{ek}, \theta_{ek}; e(\bar{v}_k)) : e(v_{ek} - \bar{v}_k) - \nu \left( \| \nabla e(\bar{v}_k) \|^2 - \| \nabla e(v_\varepsilon) \|^2 \right) \right) \, dx \, dt
\]

\[
+ \left( \frac{\theta_{ek}(T)}{2} \cdot (v_{ek} - \bar{v}_k) \cdot (v_{ek} - \bar{v}_k) \right) \, dx \, dt + \int_0^T \int_{\Gamma} f + v_\varepsilon \bar{v}_k \cdot (v_{ek} - \bar{v}_k) \, dS \, dt
\]

\[
+ \int_0^T \int_{\Gamma} \left( \frac{\theta_{ek}(T)}{2} \cdot (v_{ek} - \bar{v}_k) \cdot (v_{ek} - \bar{v}_k) \right) \, dx \, dt
\]

\[
(3.40)
\]

with \( \delta > 0 \) from (3.40a) and with some \( c_p > 0 \) related to the inequality \( c_p |G - \bar{G}|^p \leq (|G|^{p-2}G - |\bar{G}|^{p-2}\bar{G}) : (G - \bar{G}) \) holding for \( p \geq 2 \). The term \( \theta_k \) in (3.40) is

\[
\theta_k = \int_0^T \int_\Omega \frac{\theta_{ek}(T)}{2} \cdot (v_{ek}(T) - \bar{v}_k(T)) \, dx \, dt + \int_0^T \int_{\Gamma} \nu_v v_{ek} \cdot (v_{ek} - \bar{v}_k) \, dS \, dt
\]

\[
+ \int_0^T \int_\Omega D(F_{ek}, \theta_{ek}; e(v_{ek})) : e(v_{ek} - \bar{v}_k) + \nu \| \nabla e(v_{ek}) \|^2 - \| \nabla e(v_\varepsilon) \|^2 \right) \right) \, dx \, dt
\]

and it converges to zero due to the strong approximation properties of the approximation \( \bar{v}_k \) of \( v_\varepsilon \). Here we used (3.37)–(3.38) and also the strong convergence (3.32), (3.34b), and (3.36). Knowing already (3.34c) and that \( e(v_{ek} - \bar{v}_k) \to 0 \) weakly in \( L^p(I \times \Omega; \mathbb{R}^{d \times d}) \), we have that \( \int_0^T \int_\Omega T_{\lambda, k}(F_{ek}, \theta_{ek}) : e(v_{ek} - \bar{v}_k) \, dx \, dt \to 0 \). Thus, we obtain the desired strong convergence

\[
e(v_{ek}) \to e(v_\varepsilon) \quad \text{strongly in } L^q(I \times \Omega; \mathbb{R}^{d \times d}) \quad \text{and} \quad (3.41a)
\]

\[
\nabla e(v_{ek}) \to \nabla e(v_\varepsilon) \quad \text{strongly in } L^p(I \times \Omega; \mathbb{R}^{d \times d}) \quad \text{and} \quad (3.41b)
\]

\[
\nabla e(v_{ek}) \to \nabla e(v_\varepsilon) \quad \text{strongly in } L^2(\Omega; \mathbb{R}^d) \quad \text{for any } t \in I. \quad (3.41c)
\]

It also implies, by continuity of the trace operator \( L^2(I; H^1(\Omega)) \to L^2(I \times \Gamma) \), that

\[
v_{ek} \rvert_{I \times \Gamma} \to v_\varepsilon \rvert_{I \times \Gamma} \quad \text{strongly in } L^2(I \times \Gamma; \mathbb{R}^d). \quad (3.41d)
\]

Having (3.11) at disposal, the limit passage in the Galerkin-approximation of (3.19a) to the weak solution of (3.16b) is then easy.

Strong convergence (3.41a,b,d) allows for passing to the limit also in the dissipative heat sources and the other terms in the Galerkin approximation of the heat equation (3.19b) are even easier.
For further purposes, let us mention that the energy dissipation balance (3.21) is inherited in the limit, i.e.

\[
\frac{d}{dt} \int_\Omega \frac{1}{2} |v_\varepsilon|^2 + \frac{\pi \lambda(F_\varepsilon) \varphi(F_\varepsilon)}{\det F_\varepsilon} \, dx + \int_\Omega D(F_\varepsilon, \varepsilon; e(v_\varepsilon); e(v_\varepsilon)) : e(v_\varepsilon) + \nu |\nabla e(v_\varepsilon)|^p \, dx \\
+ \int_{\Gamma} \nu \varepsilon |v_\varepsilon|^2 \, dS = \int_\Omega \frac{\varphi(F_\varepsilon)}{\det \lambda(F_\varepsilon)} - \frac{\pi \lambda(F_\varepsilon) \gamma'_F(F_\varepsilon, \varepsilon) F_\varepsilon^\top : e(v_\varepsilon) \, dx + \int_{\Gamma} f : v_\varepsilon \, dS .
\] (3.42)

**Step 5 - non-negativity of temperature**: We can now perform various nonlinear tests of the regularized but non-discretized heat equation. The first test can be by the negative part of temperature \( \theta^- \) := \( \min(0, \theta_\varepsilon) \). Let us recall the extension \( \omega(F_\varepsilon, \theta^-) = \theta^- \) and \( \omega'(F_\varepsilon, \theta^-) = 0 \) Note also that \( \theta^- \in L^2(I; H^1(\Omega)) \), so that it is indeed a legal test for (3.16d). Here we rely on the data qualification \( \nu, \nu_\varepsilon \geq 0, \kappa = \kappa(F, \theta) \geq 0, D(F, \theta; e) : e \geq 0, \theta_0 \geq 0, \) and \( h(\theta) \geq 0 \) for \( \theta \leq 0 \), cf. (3.41). Realizing that \( \nabla \theta^- = 0 \) whenever \( \theta > 0 \) so that \( \nabla \theta \cdot \nabla \theta^- = |\nabla \theta^-|^2 \) and that \( \gamma'_{F,F}(\varepsilon, \theta^-) \theta^- = 0 \) and \( h(\theta) \theta^- = h(\theta^-) \theta^- = 0 \), this test gives

\[
\frac{1}{2} \frac{d}{dt} \|\theta^-\|_{L^2(\Omega)} \leq \int_\Omega \varepsilon \cdot \nabla \theta^- + (D(F_\varepsilon, \varepsilon; e(v_\varepsilon)): e(v_\varepsilon) + \nu |\nabla e(v_\varepsilon)|^p + \frac{\pi \lambda(F_\varepsilon) \gamma'_F(F_\varepsilon, \varepsilon) F_\varepsilon^\top : e(v_\varepsilon)}{(1+\varepsilon |\theta^-|)} \det F_\varepsilon) \theta^- \, dx
\]

\[
\leq \int_\Omega w_\varepsilon \cdot \nabla \theta^- \, dx = \int_\Omega \varepsilon \cdot \nabla \theta^- \, dx = - \int_\Omega \varepsilon \cdot \nabla \theta^- + \|\nabla \theta^-\|^2 \, dx \\
= - \frac{1}{2} \int_\Omega \|\nabla \theta^-\|^2 \, dx \leq \|\theta^-\|^2_{L^2(\Omega)} \|\nabla \theta^-\|_{L^\infty(\Omega)} .
\] (3.43)

Recalling the assumption \( \theta_\varepsilon \geq 0 \) so that \( \theta^- = 0 \) and exploiting the information \( v_\varepsilon \in L^{\min(p,q)}(I; W^{1,p}(\Omega; \mathbb{R}^d)) \) with \( p > d \) inherited from (3.23), by the Gronwall inequality we obtain \( \|\theta_\varepsilon\|_{L^\infty(I; L^2(\Omega))} \leq 0 \), so that \( \theta_\varepsilon \geq 0 \) a.e. on \( I \times \Omega \).

Having proved the non-negativity of temperature, we can now execute the strategy based of the \( L^1 \)-theory for the heat equation which led to the estimates (3.10)-(3.11), i.e. here

\[
\|v_\varepsilon\|_{L^\infty(I; L^2(\Omega; \mathbb{R}^d))} \leq C, \quad \|e(v_\varepsilon)\|_{L^p(I; L^1(\Omega; \mathbb{R}^{d \times d}))} \leq C, \quad \|\nabla e(v_\varepsilon)\|_{L^p(I; L^1(\Omega; \mathbb{R}^{d \times d}))} \leq C, \quad (3.44a)
\]

\[
\|F_\varepsilon\|_{L^\infty(I; W^{1,r}(\Omega; \mathbb{R}^{d \times d}))} \leq C, \quad \|\nabla F_\varepsilon\|_{L^p(I; L^1(\Omega; \mathbb{R}^{d \times d}))} \leq C, \quad (3.44b)
\]

\[
\|q_\varepsilon\|_{L^\infty(I; W^{1,r}(\Omega; \mathbb{R}^{d \times d}))} \leq C, \quad \|\nabla q_\varepsilon\|_{L^p(I; L^1(\Omega; \mathbb{R}^{d \times d}))} \leq C \quad \text{for any } 1 \leq r < +\infty, \quad (3.44c)
\]

\[
\|w_\varepsilon\|_{L^\infty(I; L^1(\Omega; \mathbb{R}^{d \times d}))} \leq C, \quad \|\theta_\varepsilon\|_{L^\infty(I; L^1(\Omega))} \leq C . \quad (3.44d)
\]

By interpolation exploiting the Gagliardo-Nirenberg inequality between \( L^2(\Omega) \) and \( W^{2,p}(\Omega) \), we have \( \|\cdot\|_{L^\infty(\Omega)} \leq C\|\cdot\|_{L^2(\Omega)} \cdot \|\cdot\|_{W^{2,p}(\Omega)} \) with \( 0 < r < pd/(pd + 4p - 2d) \). Using also Korn's inequality, from (3.44a) we thus obtain the estimate

\[
\|v_\varepsilon\|_{L^s(I; L^\infty(\Omega; \mathbb{R}^d))} \leq C_s \quad \text{with } 1 \leq s < p \frac{pd+4p-2d}{4p-2d} . \quad (3.44e)
\]

By comparison from \( \frac{d}{dt} q_\varepsilon = (\text{div}v_\varepsilon) q_\varepsilon - v_\varepsilon \cdot \nabla q_\varepsilon \) and from \( \frac{d}{dt} F_\varepsilon = (\nabla v_\varepsilon) F_\varepsilon - (v_\varepsilon \cdot \nabla) F_\varepsilon \), we also have

\[
\|\frac{d}{dt} q_\varepsilon\|_{L^{\min(p,q)}(I; L^r(\Omega; \mathbb{R}^{d \times d}))} \leq C, \quad \|\frac{d}{dt} F_\varepsilon\|_{L^{\min(p,q)}(I; L^r(\Omega; \mathbb{R}^{d \times d}))} \leq C . \quad (3.44f)
\]

22
The estimates (3.44d) are naturally weaker than (3.30) but, importantly, are uniform with respect to \( \varepsilon > 0 \), in contrast to (3.30) which is not uniform in this sense. The total energy balance (2.25) holds for \( \varepsilon \)-solution only as an inequality because the heat sources do not exactly cancel; more in detail, while the regularized adiabatic heat again cancels, the dissipative heat terms are regularized (and smaller) in (3.16d) and in (3.17a) but the corresponding viscous stress in (3.16b) and force in (3.17a) are not regularized. This inequality still allows to execute the above-mentioned estimation.

Let us also note that the extension (3.12) becomes now inactive and we can work with the original data defined for non-negative \( \theta \).

**Step 6 – further a-priori estimates:** We are to prove an estimate of \( \nabla \theta \varepsilon \) based on the test of the heat equation (3.16d) by \( \chi_\varepsilon(\theta_\varepsilon) \) with an increasing nonlinear function \( \chi_\varepsilon : [0, +\infty) \to [0, 1] \) defined as
\[
\chi_\varepsilon(\theta) := 1 - \frac{1}{(1+\theta)^\zeta}, \quad \zeta > 0, \tag{3.45}
\]
simplifying the original idea of L. Boccardo and T. Gallouët [7,8] in the spirit of [19], expanding the estimation strategy in [27, Sect. 8.2]. Importantly, here we have \( \chi_\varepsilon(\theta(t, \cdot)) \in H^1(\Omega) \), hence it is a legal test function, because \( 0 \leq \theta_\varepsilon(t, \cdot) \in H^1(\Omega) \) has already been proved and because \( \chi_\varepsilon \) is Lipschitz continuous on \([0, +\infty)\).

We consider \( 1 \leq \mu < 2 \) and estimate the \( L^\mu \)-norm of \( \nabla \theta \varepsilon \) by Hölder’s inequality as
\[
\int_0^T \int_\Omega |\nabla \theta_\varepsilon|^\mu \, dx \, dt \leq C_1 \left( \int_0^T \|1+\theta_\varepsilon(t, \cdot)\|_{L^{(1+\zeta)\mu/(2-\mu)}(\Omega)}^{(1+\zeta)\mu/(2-\mu)} \, dt \right)^{1-\mu/2} \left( \int_0^T \int_\Omega \chi_\varepsilon(\theta_\varepsilon) |\nabla \theta_\varepsilon|^2 \, dx \, dt \right)^{\mu/2} =: I_{\mu,\chi_\varepsilon}^{(1)}(\theta_\varepsilon) \tag{3.46}
\]
with \( \chi_\varepsilon \) from (3.45) so that \( \chi_\varepsilon'(\theta) = \zeta/(1+\theta)^{1+\zeta} \) and with a constant \( C_1 \) dependent on \( \zeta, \mu, \) and \( T \). Then we interpolate the Lebesgue space \( L^{(1+\zeta)\mu/(2-\mu)}(\Omega) \) between \( W^{1,\mu}(\Omega) \) and \( L^1(\Omega) \) to exploit the already obtained \( L^\infty(I ; L^1(\Omega)) \)-estimate in (3.44d). More specifically, by the Gagliardo-Nirenberg inequality, we obtain
\[
\|1+\theta_\varepsilon(t, \cdot)\|_{L^{\mu/(1+\zeta)}(\Omega)}^{\mu/(1+\zeta)} \leq C_2 \left( 1 + \|\nabla \theta_\varepsilon(t, \cdot)\|_{L^\mu(\Omega; \mathbb{R}^d)} \right)^{1+\zeta} \tag{3.47}
\]
with \( C_2 \) depending on \( \sigma, C_1, \) and \( C \) from (3.44d), so that \( I_{\mu,\chi_\varepsilon}^{(1)}(\theta_\varepsilon) \leq C_3(1 + \int_0^T \int_\Omega |\nabla \theta_\varepsilon|^\mu \, dx \, dt) \) with \( C_3 \) depending on \( C_2 \). Combining it with (3.46), we obtain
\[
\|\nabla \theta_\varepsilon\|_{L^\mu(I \times \Omega; \mathbb{R}^d)} = C_1 C_3(1 + \|\nabla \theta_\varepsilon\|_{L^\mu(I \times \Omega)}^{\mu})^{1-\mu/2} I_{\mu,\chi_\varepsilon}^{(2)}(\theta_\varepsilon)^{\mu/2}. \tag{3.48}
\]
Furthermore, we estimate \( I_{\chi_\varepsilon}^{(2)}(\theta_\varepsilon) \) in (3.46). Let us denote by \( \chi_\varepsilon \) a primitive function to \( \theta \mapsto \chi_\varepsilon(\theta) \omega_\theta(F, \theta) \) depending smoothly on \( F \), specifically
\[
\chi_\varepsilon(F, \theta) = \int_0^\theta \omega_\zeta(r \theta) \omega_\theta(F, r \theta) \, dr. \tag{3.49}
\]
Like (3.24) but using the partial (not convective) time derivative, we have now the calculus
\[
\int_\Omega \chi_\varepsilon(\theta) \frac{\partial \omega}{\partial t} \, dx = \int_\Omega \chi_\varepsilon(\theta) \omega_\theta(F, \theta) \frac{\partial \theta}{\partial t} + \chi_\varepsilon(\theta) \omega_\theta(F, \theta) \cdot \frac{\partial F}{\partial t} \, dx = \frac{d}{dt} \int_\Omega \chi_\varepsilon(F, \theta) \, dx \]
\[- \int_\Omega [\mathcal{L}_\zeta](F, \theta) \cdot \frac{\partial F}{\partial t} \, dx \quad \text{where} \quad \mathcal{L}_\zeta(F, \theta) := \chi_\varepsilon(F, \theta) - \chi_\varepsilon(\theta) \omega_\theta(F, \theta). \tag{3.50}
\]
In view of (3.49), it holds \([\mathcal{P}_e]_F(F, \theta) = \int_0^1 \theta \chi_\zeta(\tau \theta) \omega'_\zeta(F, \tau \theta) \, d\tau \). Altogether, testing (3.16d) with (3.17b) by \(\chi(\theta)\), gives
\[
\frac{d}{dt} \int \chi_\zeta(F, \theta) \, dx + \int \chi_\zeta(F, \theta) \, dx = \int \left( h_\zeta(\theta) + \frac{\nu}{2+\epsilon|v_\zeta|^2} \right) \chi_\zeta(\theta) \, dS
\]
\[
+ \int \left( \frac{D(F, \theta; e(v))}{1+\epsilon|v_\zeta|^2} \right) \chi_\zeta(\theta) + \omega(F, \theta) \chi_\zeta(\theta) \, d\theta
\]
\[
+ [\mathcal{P}_e]_F(F, \theta) \frac{\partial F}{\partial t} + \chi_\zeta(\theta) \left( \frac{\pi_\alpha(F, \theta; \theta) \gamma(F, \theta)}{1+\epsilon|v_\zeta|^2} \right) \chi_\zeta(\theta) \, dx.
\]

We realize that \(\chi_\zeta(\theta) = \zeta/(1+\theta)^{1+\zeta}\) as used already in (3.46) and that \(\chi_\zeta(F, \theta) \geq c_K \theta\) with some \(c_K\) for \(\theta \geq 0\) due to (3.46); again \(K\) is a compact subset of GL\(^+(d)\) related here with the already proved estimates (3.44b). The convective term in (3.51) can be estimated, for any \(\delta > 0\), as
\[
\int \omega(\theta) \, dx \leq \frac{1}{\delta} \int \chi_\zeta(\theta) |v_\zeta|^2 \, dx + \delta \int \chi_\zeta(\theta) |\nabla \theta|^2 \, dx
\]
\[
= \frac{1}{\delta} \int \chi_\zeta(\theta) |v_\zeta|^2 \, dx + \delta I_\zeta(\theta).
\]

Denoting by \(0 < \kappa_0 = \inf F, \theta \kappa(F, \theta)\) and using (3.51) integrated over \(I = [0, T]\), we further estimate:
\[
I_\zeta(\theta) \leq \frac{1}{\kappa_0} \int_0^T \int \chi_\zeta(\theta) |\nabla \theta|^2 \, dx \, dt
\]
\[
= \frac{1}{\kappa_0} \left( \int_0^T \int \chi_\zeta(F, \theta) \, dx \, dt + \int \chi_\zeta(F, \theta) \, dx \right)
\]
\[
= \frac{1}{\kappa_0} \left( \frac{\pi_\alpha(F_\zeta, \theta; \theta) \gamma(F_\zeta, \theta)}{1+\epsilon|v_\zeta|^2} \right) \chi_\zeta(\theta) \, dx + \delta \int \chi_\zeta(\theta) |v_\zeta|^2 \, dx + \delta I_\zeta(\theta).
\]

Noteworthy, when choosing \(\delta < \kappa_0\), we can absorb the last term in the left-hand side. Due to the assumption (3.4e), we can estimate the adiabatic rates \(\pi_\alpha(F_\zeta, \theta; \theta) \gamma(F_\zeta, \theta) F_\zeta^T : e(v_\zeta) \mid \det F_\zeta\) in (3.53), cf. (3.9). We also use the estimate (3.4k) and the assumption (3.4r) relying on the already proved non-negativity of temperature. By the qualification (3.4d), we have \([\mathcal{P}_e]_F(F, \theta) \leq C(1+\theta).\) This allows for estimation
\[
\|[\mathcal{P}_e]_F(F, \theta) \|_{L^r(\Omega; \mathbb{R}^{d \times d})} \leq C \|\theta \|_{L^r(\Omega)} \|\nabla \theta \|_{L^r(\Omega)}^2
\]
\[
\leq C + C_4 \|\theta \|_{L^r(\Omega)}^{1+\mu'/r} \|\nabla \theta \|_{L^r(\Omega)} \|\nabla \theta \|_{L^r(\Omega)}^{\mu'/r-1},
\]
where we use the Gagliardo-Nirenberg inequality to interpolate $L^r(\Omega)$ between $L^1(\Omega)$ and $W^{1,\mu}(\Omega)$. The penultimate term in (3.53) is a-priori bounded independently of $\varepsilon$ for $\zeta > 0$ fixed because, as $\omega(F,\theta) = \Theta(\theta)$ due to (3.4d) and due to $\chi_\zeta'(\theta) = \Theta(1/\theta)$ uniformly for $\zeta > 0$, so that we have $|\chi_\zeta'(\cdot)\omega^2(F,\cdot)(\theta) = \Theta(\theta)$. Thus the estimate (3.44d) guarantees $\chi_\zeta'(\theta)\omega^2(F,\theta) \leq \Theta(\theta)$ bounded in $L^\infty(I;L^1(\Omega))$ while $|v_\varepsilon|^2$ is surely bounded in $L^1(I;L^\infty(\Omega))$, cf. (3.44e).

In view of (3.54), one can summarize (3.53) as $I^{(2)}_\zeta(\theta) \leq C(1 + \|\nabla \theta\|_{L^\mu(I \times \Omega;\mathbb{R}^d)})$ which gives a bound for it with (3.48), one obtain the inequality as $I^{(2)}_\zeta(\theta) \leq C(1 + \|\nabla \theta\|_{L^\mu(I \times \Omega;\mathbb{R}^d)})$.

Reminding the choice $\sigma := (2-\mu)/(1+\zeta)$ from (3.44) with $\zeta > 0$ arbitrarily small, one gets after some algebra, the condition $\mu < (d+2)/(d+1)$. Obviously, for $r$ big enough (in particular if $r > d$ as assumed), the exponent on the right-hand side is lower than 1, which gives a bound for $\nabla \theta_\varepsilon$ in $L^\mu(I \times \Omega;\mathbb{R}^d)$. Altogether, we proved

$$
\|\theta_\varepsilon\|_{L^\infty(I;L^1(\Omega))} \leq C \mu \quad \text{with} \quad 1 \leq \mu < \frac{d+2}{d+1}.
$$

(3.56a)

Exploiting the calculus $\nabla w_\varepsilon = \omega_F'(F,\theta)\nabla \theta_\varepsilon + \omega_F(F,\theta)\nabla F_\varepsilon$ with $\nabla F_\varepsilon$ bounded in $L^\infty(I;L^2(\Omega;\mathbb{R}^{d\times d}))$ and relying on the assumption (3.4d), we have also the bound on $\nabla w_\varepsilon$ in $L^\mu(I;L^{2/\mu}(\Omega;\mathbb{R}^d))$, so that

$$
\|w_\varepsilon\|_{L^\infty(I;L^1(\Omega))} \leq C \mu.
$$

(3.56b)

**Step 7: Limit passage for $\varepsilon \to 0$**. We use the Banach selection principle as in Step 4, also taking (3.44) and (3.56) into account instead of the estimates (3.23) and (3.30). For some subsequence and some $(q,v,F,\theta)$, we now have

$$
\varrho_\varepsilon \to \varrho \quad \text{weakly* in} \quad L^\infty(I;W^{1,r}(\Omega)) \cap W^{1,min(p,q)}(I;L^2(\Omega))
$$

(3.57a)

and strongly in $C(I \times \Omega)$,

$$
\psi_\varepsilon \to \psi \quad \text{weakly* in} \quad L^\infty(I;L^2(\Omega;\mathbb{R}^d)) \cap L^{q}(I;W^{2,p}(\Omega;\mathbb{R}^d)),
$$

(3.57b)

$$
F_\varepsilon \to F \quad \text{weakly* in} \quad L^\infty(I;W^{1,r}(\Omega;\mathbb{R}^{d\times d})) \cap W^{1,min(p,q)}(I;L^\infty(\Omega;\mathbb{R}^{d\times d}))
$$

(3.57c)

and strongly in $C(I \times \Omega;\mathbb{R}^{d\times d})$,

$$
\theta_\varepsilon \to \theta \quad \text{weakly* in} \quad L^\mu(I;W^{1,\mu}(\Omega)), \quad 1 \leq \mu < (d+2)/(d+1).
$$

(3.57d)

Like (3.34a), by the Aubin-Lions theorem, we now have

$$
w_\varepsilon \to w \quad \text{strongly in} \quad L^c(I \times \Omega), \quad 1 \leq c < 1+2/d,
$$

(3.57e)

and then, using again continuity of $(F,w) \to [\omega'(F,\cdot)]^{-1}(w)$ as in (3.33), we also have

$$
\theta_\varepsilon \to \theta = [\omega'(F,\cdot)]^{-1}(w) \quad \text{strongly in} \quad L^c(I \times \Omega), \quad 1 \leq c < 1+2/d.
$$

(3.57f)

By the continuity of $\pi_\lambda \varphi'_F, \gamma_F, \det(\cdot)$, and $\kappa$, we have also

$$
\kappa(F_\varepsilon,\theta_\varepsilon) \to \kappa(F,\theta) \quad \text{strongly in} \quad L^c(I \times \Omega) \quad \text{for any} \quad 1 \leq c < \infty,
$$

(3.57g)

$$
T_{\lambda,\varepsilon}(F_\varepsilon,\theta_\varepsilon) \to T_{\lambda} = \frac{[\pi_\lambda \varphi'_F(F)]}{\det F} F^\top + \frac{\pi_\lambda \gamma_F(F,\theta)}{\det F} F^\top \quad \text{strongly in} \quad L^c(I \times \Omega;\mathbb{R}^{d\times d}) \quad 1 \leq c < 1+2/d.
$$

(3.57h)
The momentum equation (3.16b) (still regularized by $\varepsilon$) is to be treated like in Step 4. Here we exploit the information about $\frac{\partial}{\partial t} (\varrho_0 v_0)$ in $L^q ([0, T); W^{1,q} (\Omega; \mathbb{R}^d) + L^p ([0, T); W^{2,p} (\Omega; \mathbb{R}^d)]$ obtained like in (3.35); here we used also (3.44e). By the Aubin-Lions compact-embedding theorem, we then obtain

$$
\varrho_0 v_0 \to \varrho_0 v_0 \quad \text{strongly in } L^q (I \times \Omega; \mathbb{R}^d) \quad \text{with } s \text{ from (3.44e).}
$$

(3.58)

In fact, the argumentation (3.40) is to be slightly modified by using $(\varrho_0, v_0, T_{\lambda, \varepsilon} (F_0, v_0), \theta_0)$ in place of $(\varrho_0, v_0, T_{\lambda, \varepsilon} (F_0, v_0), \theta_0)$ and with $\bar{v}_0$ replaced by $v$. Specifically, the convergence (3.57h) is also weak in $L^\infty (I; L^1 (\Omega; \mathbb{R}^{d \times d}))$ and e($v_0$) $\to$ e($v$) strongly in $L^q (I; L^\infty (\Omega; \mathbb{R}^{d \times d}))$ so that $\int_0^T \int_\Omega T_{\lambda, \varepsilon} (F_0, \theta_0); e (v_0 - v) \, dx \, dt \to 0$. Also, $\int_0^T \int_\Omega \frac{\partial}{\partial \varepsilon} (\varrho_0 v_0) \cdot \bar{v}_0 \, dx \, dt$ is to be replaced by $(\frac{\partial}{\partial t} (\varrho_0 v_0), v)$ with $\langle \cdot , \cdot \rangle$ denoting here the duality between $\varrho_0 v_0 \in W^1 (I; W^{1,q} (\Omega; \mathbb{R}^d)) \cap L^p (I; W^{2,p} (\Omega; \mathbb{R}^d)).$

Limit passage in the heat equation (3.16d) is then simple. Altogether, we proved that $(\varrho, v, F, \theta)$ solves in the weak sense the problem (3.16)–(3.17) with $\varepsilon = 0$ and with $T_\lambda$ from (3.57h) in place of $T_{\lambda, \varepsilon} (F_0, \theta_0)$.

Step 8: the original problem. Let us note that the limit $F$ lives in $L^\infty (I; W^{1,r} (\Omega; \mathbb{R}^{d \times d})) \cap W^{1,\min(p,q)} (I; L^r (\Omega; \mathbb{R}^{d \times d}))$, cf. (3.44b,f), and this space is embedded into $C(I \times \Omega; \mathbb{R}^{d \times d})$ if $r > d$. Therefore $F$ and its determinant evolve continuously in time, being valued respectively in $C(I \times \Omega; \mathbb{R}^{d \times d})$ and $C(I)$. Let us recall that the initial condition $F_0$ complies with the bounds (3.13) and we used this $F_0$ also for the $\lambda$-regularized system. Therefore $F$ satisfies these bounds not only at $t = 0$ but also at least for small times. Yet, in view of the choice (3.13) of $\lambda$, this means that the $\lambda$-regularization is nonactive and $(\varrho, v, F, \theta)$ solves, at least for a small time, the original nonregularized problem (3.16)–(3.17) for which the a priori $L^\infty$-bounds (3.11) hold. By the continuation argument, we may see that the $\lambda$-regularization remains therefore inactive within the whole evolution of $(\varrho, v, F, \theta)$ on the whole time interval $I$.

Step 9: energy balances. It is now important that the tests and then all the subsequent calculations leading to the energy balances (2.27) and (2.28) integrated over a current time interval $[0, t]$ are really legitimate.

More in detail, in the calculus (2.22), we rely on that $[\phi (F)/\det F] \in L^\infty (I \times \Omega; \mathbb{R}^{d \times d})$ is surely in duality with $\frac{\partial}{\partial t} F \in L^{\min(p,q)} (I; L^r (\Omega; \mathbb{R}^{d \times d}))$ and $v \cdot \nabla F \in L^s (I; L^{r} (\Omega; \mathbb{R}^{d \times d}))$ with $s$ from (3.44c). Moreover, $\frac{\partial}{\partial \theta} \theta \in L^q (I; W^{1,q} (\Omega; \mathbb{R}^d))$ and div($v$) $\in L^p (I; W^{2,p} (\Omega; \mathbb{R}^d))$ in duality with $v \in L^q (I; W^{1,q} (\Omega; \mathbb{R}^d)) \cap L^p (I; W^{2,p} (\Omega; \mathbb{R}^d))$, as used in (2.26). Further, the calculus (2.25) relies on that $\frac{\partial}{\partial \theta} \partial \theta$ and div($v$) $\in L^r (I; L^{r/((r+s)/2)}(\Omega)))$ and thus are surely in duality with $|v|^2 \in L^s (I; L^2 (\Omega; \mathbb{R}^d))$ with $3 \leq s < r(pd+4p-2d)/(4p-2d)$, cf. (3.44c). Eventually, since $\nabla e (v) \in L^p (I \times \Omega; \mathbb{R}^{d \times d})$, we have div$^2 (\nu \nabla e (v))$ in $L^p (I; W^{2,p} (\Omega; \mathbb{R}^d))$ in duality with $v$. Also div$^2 (F, \theta; e (v)) \in L^q (I; W^{1,q} (\Omega; \mathbb{R}^d))$ is in duality with $v$ due to the growth condition (3.43). Altogether, the calculations (2.22)–(2.26) are legitimate.

Remark 3.3 (“Physical” versus “mathematical” estimates). The above results hold for arbitrarily large time horizon $T$. Anyhow, mathematical arguments vitally relied on the regularity of the initial conditions, which intuitively is an information gradually forgetting when $T \to \infty$. This is reflected by the fact that the $W^{1,r} (\Omega)$-regularity of $\varrho$ and $F$ blows up when $T \to \infty$. Anyhow, some physically relevant estimates for the autonomous thermodynamic system are indeed uniform in time, specifically those which come from the energy balances (2.27) and (2.28), i.e. (3.8) and (3.10). In contrast, the estimates (3.11) depend on the assumed regularity of the initial conditions $\varrho_0$ and $F_0$ through the hyper-viscosity $\nu$, which is surely an analytically important “mathematical quality” of the solutions but does not have a direct and sustainable physical relevance. The same qualitative difference is relevant for hyper-viscosity, which is sometimes considered as a controversial modelling aspect, and therefore the asymptotics for $\nu \to 0$ is relevant even if the limit for $\nu = 0$ is analytically open.
More specifically, the former estimates in (3.8) and (3.10) are uniform with respect to \( \nu \to 0 \), in contrast to the latter estimates in (3.8) and (3.11). A good message is that also the total-energy equality (2.28) is uniform with respect to \( \nu \), being not explicitly dependent on \( \nu \), in contrast to the energy-dissipation balance (2.27). Integrated in time, this balance involves the dissipated-energy term \( \int_0^T \int_\Omega \nu |\nabla e(v)|^p \, dx \, dt \) and it rises an interesting question whether this hyperviscosity dissipated energy (which is a-priori solely bounded) is indeed small if \( \nu > 0 \) is small. This is a very nontrivial problem and the positive answer would need more regularity of solutions than in Theorem 3.2(ii); in a linear small-strain model see \[48, \text{Remark 2}\].

4 Some examples

The assumptions (3.4) are not easy to satisfy. It is thus worth illustrating them with some examples motivated by several specific phenomena.

Example 4.1 (Neo-Hookean thermally expanding materials). For the elastic bulk modulus \( K_e \) and the elastic shear modulus \( G_e \) and a volumetric thermal expansion function \( \alpha : [0, +\infty) \to (0, +\infty) \), and example of the free energy is

\[
\psi(F, \theta) = \frac{1}{2} K_e (\det F - 1)^2 + \frac{1}{2} G_e \left( \frac{\text{tr}(FF^\top)}{(\det F)^2} - d \right) + c_0 (1-\ln \theta) - K_e \alpha(\theta) \det F
\]

for \( \det F > 0 \) otherwise \( \psi(F, \theta) = +\infty \). Here \( c_0 > 0 \) is the (referential) heat capacity. To comply with the ansatz (2.20), it should hold \( \alpha(0) = 0 \). The stored energy \( \varphi \) in (4.1), considered per referential volume, is the standard isothermal neo-Hookean model. The minimum of \( \psi(\cdot, \theta) \) with respect to \( \det F \) is at \( \det F = 1 + \alpha(\theta) \), which shows the role of the function \( \alpha \) as the volumetric thermal expansion.

The “coupling stress” is

\[
\gamma'(F, \theta) F^\top \det F = -K_e \alpha(\theta) \text{Cof} F F^\top = -K_e \alpha(\theta) \mathbb{I},
\]

where we again used the formulas \( (\det F)' = \text{Cof} F \) and \( F^{-1} = \text{Cof} F^\top / \det F \), so that the power of the adiabatic effects caused by thermal expansion in (2.27) is

\[
\gamma'(F, \theta) F^\top \det F : e(v) = -K_e \alpha(\theta) \text{div} v.
\]

The thermal part of the actual internal energy \( w = \omega(F, \theta) \) is

\[
\omega(F, \theta) = \frac{c_0 \theta}{\det F} - K_e (\alpha(\theta) - \theta \alpha'(\theta))
\]

and the (actual) heat capacity \( c(F, \theta) = \omega_0'(F, \theta) \) is

\[
c(F, \theta) = \frac{c_0}{\det F} + \theta K_e \alpha''(\theta).
\]

Always, \( \alpha \) should be convex to ensure positivity of the heat capacity, cf. the first condition in (3.4c). Yet, (4.2) fulfills the second condition in (3.4c) only if \( \alpha \) is bounded on \( \mathbb{R}^+ \), which could be fulfilled if \( \alpha \) is decreasing, but it is not much physical in real materials. More realistically, for an increasing bounded \( \alpha \), the positivity of \( c(F, \theta) \) can be “locally” achieved if \( \det F \) is bounded from above, i.e. not too much stretch of the body. This last attribute can be expected only for “reasonable” loading regimes and will intuitively be violated for extremely big loading regimes causing drastic volumetric stretching.
Example 4.2 (Thermal expansion: an alternative ansatz). Another standard concept is a multiplicative decomposition of the deformation gradient \( \mathbf{F} = \mathbf{F}_0 \mathbf{F}_T \) with the elastic strain \( \mathbf{F}_e \) and the thermal-expansion strain \( \mathbf{F}_T = \mathbf{F}_T(\theta) = (1 + \alpha(\theta))\mathbb{I} \). Here, \( \alpha(\cdot) \) means a length thermal expansion. The neo-Hookean stored energy should then be a function of the elastic strain \( \mathbf{F}_e = \mathbf{F}/(1+\alpha(\theta)) \). Realizing that \( \mathbf{F}_T \) influences only the spherical but not deviatoric part of \( \mathbf{F}_e \) and that \( \det \mathbf{F}_T(\theta) = (1+\alpha(\theta))^d \), we arrive at

\[
\psi(\mathbf{F}, \theta) = \frac{1}{2} K_e \left( \frac{\det \mathbf{F}}{(1+\alpha(\theta))^d} - 1 \right)^2 + \frac{1}{2} G_e \left( \frac{\text{tr}(\mathbf{F} \mathbf{F}^T)}{(\det \mathbf{F})^{2/d}} - d \right) + c_0 \theta (1 - \ln \theta)
\]

\[
= \frac{1}{2} K_e \left( \frac{\det \mathbf{F} - 1}{1 + \alpha(\theta)^d - 1} \right)^2 + \frac{1}{2} G_e \left( \frac{\text{tr}(\mathbf{F} \mathbf{F}^T)}{(\det \mathbf{F})^{2/d}} - d \right)
\]

\[
= \varphi(\mathbf{F})
\]

\[
+ c_0 \theta (1 - \ln \theta) - K_e \frac{(1 + \alpha(\theta))^{2d} - 1}{2(1 + \alpha(\theta))^{2d}} \det \mathbf{F}^2 + K_e \frac{(1 + \alpha(\theta))^{d} - 1}{2(1 + \alpha(\theta))^{d}} \det \mathbf{F}.
\]

One analytical advantage is that the coupling stress

\[
\frac{\gamma'_F(\mathbf{F}, \theta) \mathbf{F}^T}{\det \mathbf{F}} = K_e \left( \frac{(1 + \alpha(\theta))^{2d} - 1}{(1 + \alpha(\theta))^{2d}} \det \mathbf{F} + \frac{(1 + \alpha(\theta))^{d} - 1}{2(1 + \alpha(\theta))^{d}} \right) \mathbb{I}
\]

complies with the growth condition \((3.4c)\) since \( \varphi \) has a sufficiently fast-growing response on stretching, specifically here \( \det \mathbf{F}/\varphi(\mathbf{F}) = 0(\det \mathbf{F}) \) for \( \det \mathbf{F} \to +\infty \). On the other hand, like in Example 4.1 the heat capacity \(-\theta \gamma''_{\theta \theta}(\mathbf{F}, \theta)\) can be positive for “reasonable” loading regimes which do not lead to too drastic volumetric stretch.

Example 4.3 (Phase transitions). An interesting example of the free energy \( \psi \) occurs in (somewhat simplified) modelling of austenite-martensite transition in so-called shape-memory allows (SMA):

\[
\psi(\mathbf{F}, \theta) = (1 - \lambda(\theta)) \varphi_\lambda(\mathbf{F}) + \lambda(\theta) \varphi_M(\mathbf{F}) + c_0 \theta (1 - \ln \theta) = \varphi(\mathbf{F}) + \lambda(\theta) \varphi_{MA}(\mathbf{F}) + c_0 \theta (1 - \ln \theta)
\]

with \( \varphi_{MA} := \varphi_M - \varphi_\lambda \); cf. [31] Example 2.5. Here \( \lambda : \mathbb{R} \to [0, 1] \) denotes the volume fraction of the austenite versus martensite which is supposed to depend only on temperature. In this case, the coupling stress \( \gamma'_F(\mathbf{F}, \theta) \mathbf{F}^T/\det \mathbf{F} \) equals to \( \lambda(\theta) \varphi'_{MA}(\mathbf{F}) \mathbf{F}^T/\det \mathbf{F} \) but need not be bounded unless \( \mathbf{F} \) ranges compact sets in \( GL^+(d) \) and \( \lambda(\cdot) \) is bounded. The heat capacity then reads as

\[
c(\mathbf{F}, \theta) = \frac{c_0 + \theta \lambda''(\theta) \varphi_{MA}(\mathbf{F})}{\det \mathbf{F}}.
\]

Again, like in Examples 4.1 and 4.2 the positivity of the heat capacity is a subtle issue since \( \lambda(\cdot) \) cannot be convex and a sufficiently big \( c_0 \) to dominate \( \theta \lambda''(\theta) \varphi_{MA}(\mathbf{F}) \) is needed.

Example 4.4 (Volumetric phase transitions). The phase transition in SMA in Remark 4.3 is primarily isochoric. In contrast, other phase transitions can be volumetric, which occurs specifically in rocks which “compactify” under compression by big pressures; e.g. in the Earth’s silicate mantle there are important phase transitions at pressure 14 GPa and 24 GPa in the depths 440 and 660 km, respectively. The simplest neo-Hookean ansatz is

\[
\psi(\mathbf{F}, \theta) = v(\det \mathbf{F}) + \frac{1}{2} G_e \left( \frac{\text{tr}(\mathbf{F} \mathbf{F}^T)}{(\det \mathbf{F})^{2/d}} - d \right) + \phi(\theta).
\]
This gives the pressure depending on $F$ as $p = \partial F'(F)/\partial \det F = v'(\det F)$. This function $v$ has two constant parts - two plateaus here with the values 14 GPa and 24 GPa. Then $\varrho$ is a function of pressure $\varrho = \varrho_0/\det F = \varrho_0/[|v'|^{-1}(p)]$ and this $\varrho(\cdot)$ is discontinuous with two jumps at 14 GPa and 24 GPa. Making $v$ dependent also on temperature, one could then model the complete state equation of the type $\varrho = \varrho(p, \theta) = \varrho_0/[|v'(\cdot, \theta)|^{-1}(p)]$ with phase transitions at specific pressures which also depend on temperature, as used in material science and (geo)physics. Such $v(\det F, \theta)$ would then contribute to the heat capacity by a term $-\theta v''_{\theta\theta}(\det F, \theta) = -\theta v''_{\theta\theta}(\varrho_0/\rho(p, \theta), \theta)$ depending discontinuously also on pressure. Additional adiabatic effects due to volume change may make this phase transition either exothermic or endothermic. Since $\gamma(F, \theta) = 0$, the ansatz (4.7) easily complies with (5.3).

### 5 Appendix: transport by Lipschitz velocity fields

For $n \in \mathbb{N}$ and the vector field $z$ valued in $\mathbb{R}^n$, it is useful to state results for an initial-value problem with a general linear transport and evolution equation:

$$
\dot{z} = b(\nabla v, z) \quad \text{with} \quad z|_{t=0} = z_0,
$$

where $b : \mathbb{R}^{d \times d} \times \mathbb{R}^n \to \mathbb{R}^n$ is a bilinear mapping. We will primarily be interested in weak solutions to (5.1) defined by using the integral identity like (3.2), i.e. here

$$
\int_0^T \int_\Omega z \cdot \frac{\partial \tilde{z}}{\partial t} + ((\text{div } v)z + b(\nabla v, z)) \tilde{z} + z \cdot ((v \cdot \nabla) \tilde{z}) \, dx \, dt = -\int_\Omega z_0 \cdot \tilde{z}(0) \, dx
$$

(5.2)

to be valid for any $\tilde{z}$ smooth with $\tilde{z}(T) = 0$. The following assertion has been proved essentially in [46] or, in a special case in [50]. We thus present a bit more general situation here only with sketched proof.

**Lemma 5.1** (Evolution-and-transport equation (5.1)). Let $n \in \mathbb{N}$, $p > d$, $r > 2$, and $b : \mathbb{R}^{d \times d} \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous and bilinear. Then, for any $v \in L^1(I; W^{2,p}(\Omega; \mathbb{R}^d))$ with $v \cdot n = 0$ and any $z_0 \in W^{1,r}(\Omega; \mathbb{R}^n)$, there exists exactly one weak solution $z \in C_w(I; W^{1,r}(\Omega; \mathbb{R}^n)) \cap W^{1,1}(I; L^1(\Omega; \mathbb{R}^n))$ to (5.1) and the estimate

$$
\|z\|_{L^\infty(I; W^{1,r}(\Omega; \mathbb{R}^n)) \cap W^{1,1}(I; L^1(\Omega; \mathbb{R}^n))} \leq \mathcal{C}\left(\|\nabla v\|_{L^1(I; W^{2,p}(\Omega; \mathbb{R}^d))}, \|z_0\|_{W^{1,r}(\Omega; \mathbb{R}^n)}\right)
$$

holds with some $\mathcal{C} \in C(\mathbb{R}^2)$. The equation (5.1) actually holds a.e. on $I \times \Omega$ and $z \in C(I \times \Omega; \mathbb{R}^n)$. Moreover, the mapping

$$
v \mapsto z : L^1(I; W^{2,p}(\Omega; \mathbb{R}^d)) \to L^\infty(I; W^{1,r}(\Omega; \mathbb{R}^n))
$$

is (weak, weak*)-continuous.

**Proof.** First, for analytical reasons, let us mollify $v$ in time in order to obtain $v \in L^2(I; W^{2,p}(\Omega; \mathbb{R}^d))$. This will facilitate the (anyhow not uniform) estimate (5.8) below and eventually this mollification can be forgotten.

Let us make a parabolic regularization of (5.1) by considering

$$
\dot{z} = b(\nabla v, z) + \varepsilon \text{div}(|\nabla z|^{r-2}\nabla z),
$$

(5.5)

with an additional boundary condition $(\nabla z)n = 0$. Then we make a Faedo-Galerkin approximation of (5.5) by using a collection of nested finite-dimensional subspaces $\{V_k\}_{k \in \mathbb{N}}$ whose union is dense in $W^{1,p}(\Omega; \mathbb{R}^n)$. Without loss of generality, we can assume that $z_0 \in V_1$. Existence of this solution, let
us denote it by \( z_k \), is based on the theory of systems of ordinary differential equations first locally in time, and then by successive prolongation on the whole time interval based on the \( L^\infty \)-estimates below.

Testing the Galerkin approximation of (5.5) by \( z_k \), we can estimate

\[
\frac{d}{dt} \int_\Omega \frac{1}{2} |z_k|^2 \, dx + \varepsilon \int_\Omega |\nabla z_k|^r \, dx = \int_\Omega \left( b(\nabla v, z_k) - (v \cdot \nabla) z_k \right) \cdot z_k \, dx
\]

\[
= \int_\Omega b(\nabla v, z_k) \cdot z_k + \frac{\text{div} \, v}{2} |z_k|^2 \, dx \leq \left( B + \frac{1}{2} \right) \| \nabla v \|_{L^\infty(\Omega; \mathbb{R}^d)} \| z_k \|_{L^2(\Omega; \mathbb{R}^n)}
\]  

(5.6)

with \( B := \sup_{V \in \mathbb{R}^d, \varepsilon \in \mathbb{R}^n, |V| = 1, |z| = 1} b(V, z) \); here we used also the calculus

\[
\int_\Omega (v \cdot \nabla) z_k \cdot z_k \, dx = \int_\Omega |z_k|^2 (v \cdot n) \, dS - \int_\Omega z_k (v \cdot \nabla) z_k + (\text{div} \, v) |z_k|^2 \, dx = -\frac{1}{2} \int_\Omega (\text{div} \, v) |z_k|^2 \, dx
\]

together with the boundary condition \( v \cdot n = 0 \). Note that, beside regularity of \( \nabla v \), we needed an integrability \( v \in L^1(I \times \Omega; \mathbb{R}^d) \) to ensure legitimacy of the integrals in (5.6). By the Gronwall inequality exploiting the first left-hand-side term which does not contain the factor \( \varepsilon \), we will obtain the estimate

\[
\| z_k \|_{L^\infty(I; L^2(\Omega; \mathbb{R}^n))} \leq C \quad \text{with} \quad \| \nabla z_k \|_{L^r(I \times \Omega; \mathbb{R}^d)} \leq C \varepsilon^{-1/r} .
\]  

(5.7)

On the Galerkin-discretization level, another legitimate test of (5.5) is by \( \frac{\partial}{\partial t} z_k \). This allows to estimate

\[
\int_\Omega \left| \frac{\partial z_k}{\partial t} \right|^2 \, dx + \frac{\varepsilon}{r} \int_\Omega |\nabla z_k|^r \, dx = \int_\Omega \left( b(\nabla v, z_k) - (v \cdot \nabla) z_k \right) \frac{\partial z_k}{\partial t} \, dx
\]

\[
\leq B^2 \| \nabla v \|_{L^\infty(\Omega; \mathbb{R}^d)}^2 \| z_k \|_{L^2(\Omega; \mathbb{R}^n)}
\]

\[
+ C_r \| v \|_{L^\infty(\Omega; \mathbb{R}^d)} \left( 1 + \| \nabla z_k \|_{L^r(\Omega; \mathbb{R}^n)} \right) + \frac{1}{2} \left\| \frac{\partial z_k}{\partial t} \right\|_{L^2(\Omega; \mathbb{R}^n)}^2
\]  

(5.8)

with some \( C_r \in \mathbb{R} \); here we used that \( r > 2 \) is assumed. Using the assumed regularity of \( v \), in particular also \( v \in L^2(I \times \Omega; \mathbb{R}^d) \) for which we needed to mollify temporarily the original \( v \in L^1(I; W^{2,p}(\Omega; \mathbb{R}^d)) \), and the already obtained estimate (5.7) and the Gronwall inequality, we obtain

\[
\left\| \frac{\partial z_k}{\partial t} \right\|_{L^2(I \times \Omega; \mathbb{R}^n)} \leq \varepsilon C \varepsilon^{1/(r \varepsilon)} \quad \text{and} \quad \left\| \nabla z_k \right\|_{L^\infty(I; L^r(\Omega; \mathbb{R}^d))} \leq C \varepsilon^{1/(r \varepsilon)} .
\]  

(5.9)

Note that here the Gronwall inequality uses not the first but the second left-hand-side term which contains the factor \( \varepsilon \) so that both estimates in (5.9) are \( \varepsilon \)-dependent.

Considering \( \varepsilon > 0 \) fixed, these estimates allow for the limit passage with \( k \to \infty \) by standard arguments for quasilinear parabolic equations. The limit is a weak solution to the initial-boundary value problem for (5.5), let us denote it by \( z_\varepsilon \in L^\infty(I; L^2(\Omega; \mathbb{R}^n)) \cap L^r(I; W^{1,r}(\Omega; \mathbb{R}^n)) \). Actually, as it is determined uniquely, even the whole sequence \( \{ z_k \}_{k \in \mathbb{N}} \) converges to it.

Since now

\[
\left\| \frac{\partial z_\varepsilon}{\partial t} + (v \cdot \nabla) z_\varepsilon - b(\nabla v, z_\varepsilon) \right\|_{L^2(I \times \Omega; \mathbb{R}^n)} \leq \varepsilon C \varepsilon^{1/(r \varepsilon)} ,
\]  

(5.10)

by comparison we also obtain

\[
\| \text{div} (|\nabla z_\varepsilon|^{r-2} \nabla z_\varepsilon) \|_{L^2(I \times \Omega; \mathbb{R}^n)} \leq C \varepsilon^{1/(r \varepsilon)} .
\]  

(5.11)
Although this estimate blows up when $\varepsilon \to 0$, we have now at least the information that \(\text{div}(|\nabla z_\varepsilon|^{-2}\nabla z_\varepsilon) \in L^2(I \times \Omega; \mathbb{R}^n)\) and we have (5.5) continuous. Therefore, we can legitimately use \(\text{div}(|\nabla z_\varepsilon|^{-2}\nabla z_\varepsilon)\) as a test. Let us denote the Sobolev exponent to $r$ by $r^*$, i.e. $r^* = dr/(d-r)$ if $r < d$ while $r^* = +\infty$ if $r > d$ or arbitrary in $[1, +\infty)$ if $r = d$. Since $p > d$, we have $p^{-1} + (r^*)^{-1} + (r^*)^{-1} \leq 1$, and thus by the Hölder and Young inequalities, we can estimate

$$
\frac{d}{dt} \int_\Omega \frac{1}{r} |\nabla z_\varepsilon|^r \, dx \leq \frac{d}{dt} \int_\Omega \frac{1}{r} |\nabla z_\varepsilon|^r \, dx + \varepsilon \int_\Omega |\text{div}(|\nabla z_\varepsilon|^{-2}\nabla z_\varepsilon)|^2 \, dx
$$

$$
= \int_\Omega \nabla((v \cdot \nabla)z_\varepsilon - b(\nabla v, z_\varepsilon)) : (|\nabla z_\varepsilon|^{-2}\nabla z_\varepsilon) \, dx
$$

$$
= \int_\Omega \left( |\nabla z_\varepsilon|^{-2}(\nabla z_\varepsilon \cdot \nabla z_\varepsilon) : e(v) - \frac{1}{r} |\nabla z_\varepsilon|^r \text{div} v 
- (b_v v(z_\varepsilon) \nabla^2 v + b_z(\nabla v) \nabla z_\varepsilon) : (|\nabla z_\varepsilon|^{-2}\nabla z_\varepsilon) \right) \, dx
$$

$$
\leq C_r \|\nabla v\|_{L^\infty(\Omega; \mathbb{R}^d \times d)} \|\nabla z_\varepsilon\|_{L^r(\Omega; \mathbb{R}^d \times d)} + C_r \|\nabla^2 v\|_{L^p(\Omega; \mathbb{R}^d \times d \times d)} \|z_\varepsilon\|_{L^r(\Omega; \mathbb{R}^n)} \|\nabla z_\varepsilon\|_{L^r(\Omega; \mathbb{R}^d \times d)}^{-1}
$$

$$
\leq C_r \|\nabla v\|_{L^\infty(\Omega; \mathbb{R}^d \times d)} \|\nabla z_\varepsilon\|_{L^r(\Omega; \mathbb{R}^d \times d)} + C_r N \|\nabla^2 v\|_{L^p(\Omega; \mathbb{R}^d \times d \times d)} \|\nabla z_\varepsilon\|_{L^r(\Omega; \mathbb{R}^d \times d)}
$$

$$
+ C_r N \|\nabla^2 v\|_{L^p(\Omega; \mathbb{R}^d \times d \times d)} \|z_\varepsilon\|_{L^2(\Omega; \mathbb{R}^n)} (1 + \|\nabla z_\varepsilon\|_{L^r(\Omega; \mathbb{R}^d \times d)}),
$$

(5.12)

where we used $p > d$ also for the embedding of $\nabla v$ into $L^\infty(\Omega; \mathbb{R}^d \times d)$ and where we further used the calculus

$$
\int_\Omega \nabla((v \cdot \nabla)z_\varepsilon) : |\nabla z_\varepsilon|^{-2}\nabla z_\varepsilon \, dx
$$

$$
= \int_\Omega |\nabla z_\varepsilon|^{-2}(\nabla z_\varepsilon \cdot \nabla z_\varepsilon) : e(v) + (v \cdot \nabla)\nabla z_\varepsilon : |\nabla z_\varepsilon|^{-2}\nabla z_\varepsilon \, dx
$$

$$
= \int_\Omega |\nabla z_\varepsilon|^{-2}(\nabla z_\varepsilon \cdot \nabla z_\varepsilon) : e(v)
- (\text{div} v) |\nabla z_\varepsilon|^{-2} - (r-1) |\nabla z_\varepsilon|^{-2} \nabla z_\varepsilon : (v \cdot \nabla) \nabla z_\varepsilon \right) \, dx
$$

$$
= \int_\Omega |\nabla z_\varepsilon|^{-1} \nabla v \cdot n \, dS + \int_\Omega \left| |\nabla z_\varepsilon|^{-2}(\nabla z_\varepsilon \cdot \nabla z_\varepsilon) : e(v)
- \left( (\text{div} v) |\nabla z_\varepsilon|^{-2} - (r-1) |\nabla z_\varepsilon|^{-2} \nabla z_\varepsilon : (v \cdot \nabla) \nabla z_\varepsilon \right) \right| \, dx
$$

Again, the boundary integral vanishes in (5.12) if $v \cdot n = 0$. For the last inequality in (5.12), we will use $\|z_\varepsilon\|_{L^{r^*}(\Omega; \mathbb{R}^n)} \leq N(\|z_\varepsilon\|_{L^2(\Omega; \mathbb{R}^n)}) + \|\nabla z_\varepsilon\|_{L^r(\Omega; \mathbb{R}^d)}$ where $N$ is the norm of the embedding $W^{1,r}(\Omega) \subset L^{r^*}(\Omega)$ if $W^{1,r}(\Omega)$ is endowed with the norm $\|\cdot\|_{L^2(\Omega)} + \|\nabla \cdot \|_{L^r(\Omega; \mathbb{R}^d)}$.

Thus one can apply the Gronwall inequality to (5.12). The estimates (5.7) and (5.9) can thus be strengthened. Specifically, using the former estimate in (5.7) and having assumed $z_0 \in W^{1,r}(\Omega; \mathbb{R}^n)$, one obtains the estimates

$$
\|\nabla z_\varepsilon\|_{L^\infty(I; L^r(\Omega; \mathbb{R}^d \times d))} \leq C
$$

and

$$
\|\text{div}(|\nabla z_\varepsilon|^{-2}\nabla z_\varepsilon)\|_{L^2(I \times \Omega; \mathbb{R}^n)} \leq C\varepsilon^{-1/2}
$$

(5.13a)

(5.13b)

The limit passage for $\varepsilon \to 0$ in linear terms is then easy and, due to (5.13b), the quasilinear regularizing term in (5.5) vanishes as $O(\varepsilon^{1/2})$ for $\varepsilon \to 0$. Alternatively, when tested it by $\tilde{z}$ and using (5.13a), it vanishes even faster as

$$
\left| \int_0^T \int_\Omega \varepsilon |\nabla z_\varepsilon|^{-2} \nabla z_\varepsilon : \nabla \tilde{z} \, dx \, dt \right| \leq \varepsilon \|\nabla z_\varepsilon\|_{L^r(I \times \Omega; \mathbb{R}^d \times d)} \|\nabla \tilde{z}\|_{L^r(I \times \Omega; \mathbb{R}^d \times d)} = O(1)
$$

In any case, the limit for $\varepsilon \to 0$ solves the original initial-boundary value problem (5.5). As this equation is linear, this solution is unique.
The former estimate in (5.9) on $\frac{\partial}{\partial t}z_\varepsilon$ is not inherited by the limit, but we can obtain by comparison

$$\frac{\partial}{\partial t}z = b(\nabla v, z) - (v \cdot \nabla)z\text{ at least the estimate}$$

$$\left\| \frac{\partial z}{\partial t} \right\|_{L^1(I; L^r(\Omega; \mathbb{R}^n))} \leq C.$$  \hfill (5.14)

In particular, the equation in (5.5) holds a.e. on $I \times \Omega$. By the embedding $L^\infty(I; W^{1,r}(\Omega)) \cap W^{1,1}(I; L^r(\Omega)) \subset C(I \times \Omega)$, we have also $z \in C(I \times \Omega; \mathbb{R}^n)$.

(weak,weak*)-continuity of the mapping $v \mapsto z$ as (5.4) is simple if we realize also the bound of $z$ in $W^{1,1}(I; L^r(\Omega; \mathbb{R}^n))$ and use the Aubin-Lions theorem, which then gives strong convergence of $z$’s in $L^{1/\zeta}(I; L^{r-\zeta}(\Omega; \mathbb{R}^n))$ for any $0 < \zeta < 1$. Then, if varying $v$ in the weak topology of $L^1(I; W^{2,p}(\Omega; \mathbb{R}^n))$, we can pass to the limit in the bi-linear nonlinearities $(v \cdot \nabla)z$ and $b(\nabla v, z)$ in (5.1) in its weak formulation (5.2).

In Section 3, we used (5.1) with $n = d \times d$ for

$$b(\nabla v, z) = (\nabla v)z\text{ with } z = F,$$  \hfill (5.15a)

which yielded (2.18c) and, in the weak formulation, (3.3c). Simultaneously, we used it in the scalar case $n = 1$ for

$$b(\nabla v, z) = -z \text{ div } v\text{ with } z = \begin{cases} \varrho, & \text{or} \\ 1/\det F, & \end{cases}$$  \hfill (5.15b)

which was applied to (2.5) or (2.3), respectively. In the case of the continuity equation (2.5), the weak formulation (5.2) yields the weak formulation (3.3a); note that the term $(\text{div } v)z + b(\nabla v, z)$ in (5.2) vanishes in this last case. Similarly, this scalar case was applied to (2.9) for

$$b(\nabla v, z) = z \text{ div } v\text{ with } z = 1/\varrho.$$  \hfill (5.15c)

Another tensorial application with $n = d \times d$ could be on distortion $F^{-1}$, relying on:

$$b(\nabla v, z) = -z(\nabla v)\text{ with } z = F^{-1}$$

together with the qualification (3.10), which guarantees $\nabla F_0^{-1} = \nabla (\text{Cof } F_0/\det F_0) \in L^r(\Omega; \mathbb{R}^{d \times d \times d})$. A trivial vectorial application with $n = d$ and $b \equiv 0$ is for the return mapping $z = \xi$, cf. (2.4), which can then lead to a formulation in terms of the distortion $F^{-1} = \nabla \xi$, cf. in particular [22, 39]. For another nontrivial affine and even time-dependent $b$, which would allow here for a combination with Maxwellian rheology leading to a more general Jeffreys’ visco-elastic rheology, we refer to [46].

References

[1] G. Alberti, G. Crippa, and A.L. Mazzucato. Loss of regularity for the continuity equation with non-Lipschitz velocity field. Annals of PDE, 5:Art.no.9, 2019.
[2] S.S. Antman. Physically unacceptable viscous stresses. Zeitschrift angew. Math. Physik, 49:980–988, 1998.
[3] J.M. Ball. Singular minimizers and their significance in elasticity. In M.G. Crandall, P.H. Rabinowitz, and R.E.L. Turner, editors, Directions in partial differential equations, pages 1–15. Acad. Press, 1987.
[4] J.M. Ball. Some open problems in elasticity. In P. Newton, P. Holmes, and A. Weinstein, editors, Geometry, Mechanics, and Dynamics, pages 3–59. Springer, New York, 2002.
[5] J.M. Ball. Progress and puzzles in nonlinear elasticity. In J. Schröder and P. Neff, editors, *Poly-, Quasi- and Rank-One Convexity in Applied Mechanics*, CISM Intl. Centre for Mech. Sci. 516, pages 1–15. Springer, Wien, 2010.

[6] J.M. Ball and V.J. Mizel. One-dimensional variational problems whose minimizers do not satisfy the Euler-Lagrange equation. *Arch. Rational Mech. Anal.*, 90:325–388, 1985.

[7] L. Boccardo, A. Dall’aglio, T. Gallouët, and L. Orsina. Nonlinear parabolic equations with measure data. *J. Funct. Anal.*, 147:237–258, 1997.

[8] L. Boccardo and T. Gallouët. Non-linear elliptic and parabolic equations involving measure data. *J. Funct. Anal.*, 87:149–169, 1989.

[9] J. Bonet, C.H. Lee, A.J. Gil, and A. Ghavamian. A first order hyperbolic framework for large strain computational solid dynamics. Part III: Thermo-elasticity. *Comput. Methods Appl. Mech. Engrg.*, 373:Art.no. 113505, 2021.

[10] C. Carstensen and G. Dolzmann. Time-space discretization of the nonlinear hyperbolic system $u_{tt} = \text{div}(\sigma(du) + du_t)$. *SIAM J. Numer. Anal.*, 42:75–89, 2004.

[11] C. Carstensen and M.O. Rieger. Young-measure approximations for elastodynamics with non-monotone stress-strain relations. *ESAIM: Mathematical Modelling and Numerical Analysis*, 38:397–418, 2004.

[12] C. Christoforou, M. Galanopoulou, and A.E. Tzavaras. A discrete variational scheme for isentropic processes in polyconvex thermoelasticity. *Calc. Var.*, 59:Art.no.122, 2020.

[13] C. Christoforou and A.E. Tzavaras. Relative entropy for hyperbolic-parabolic systems and application to the constitutive theory of thermoviscoelasticity. *Arch. Rational Mech. Anal.*, 229:1–52, 2018.

[14] C.M. Dafermos. Quasilinear hyperbolic systems with involutions. *Arch. Rational Mech. Anal.*, 94:373–389, 1986.

[15] C.M. Dafermos and W.J. Hrusa. Energy methods for quasilinear hyperbolic initial-boundary value problems, applications to elastodynamics. In C.M. Dafermos, D.D. Joseph, and F.M. Leslie, editors, *The Breadth and Depth of Continuum Mechanics*, pages 609–634. Springer, Berlin, 1986.

[16] S. Demoulini. Weak solutions for a class of nonlinear systems of viscoelasticity. *Archive Rational Mech. Anal.*, 155:299–334, 2000.

[17] S. Demoulini, D. Stuart, and A. Tzavaras. A variational approximation scheme for three dimensional elastodynamics with polyconvex energy. *Arch. Ration. Mech. Anal.*, 157:325–344, 2001.

[18] S. Demoulini, D.M.A. Stuart, and A.E. Tzavaras. Weak-strong uniqueness of dissipative measure-valued solutions for polyconvex elastodynamics. *Archive Rational Mech. Anal.*, 205:927–961, 2012.

[19] E. Feireisl and J. Málek. On the Navier-Stokes equations with temperature-dependent transport coefficients. *Diff. Equations Nonlin. Mech.*, 14pp.(electronic), Art.ID 90616, 2006.

[20] M. Foss, W. J. Hrusa, and V. J. Mizel. The Lavrentiev gap phenomenon in nonlinear elasticity. *Archive Rat. Mech. Anal.*, 167:337–365, 2003.

[21] E. Fried and M.E. Gurtin. Tractions, balances, and boundary conditions for nonsimple materials with application to liquid flow at small-length scales. *Arch. Ration. Mech. Anal.*, 182:513–554, 2006.

[22] S.K. Godunov and I.M. Peshkov. Thermodynamically consistent nonlinear model of elastoplastic Maxwell medium. *Comput. Math. & Math. Physics*, 50:1409–1426, 2010.

[23] M.E. Gurtin, E. Fried, and L. Anand. *The Mechanics and Thermodynamics of Continua*. Cambridge Univ. Press, New York, 2010.

[24] X. Hu and N. Masmoudi. Global solutions to repulsive Hookean elastodynamics. *Arch. Ration. Mech. Anal.*, 223, 2016.

[25] T.J.R. Hughes, T. Kato, and J. E. Marsden. Well-posed quasi-linear second-order hyperbolic systems with applications to nonlinear elastodynamics and general relativity. *Arch. Ration. Mech. Anal.*, 63:273–294, 1977.

[26] K. Koumatos, C. Lattanzio, S. Spirito, and A.E. Tzavaras. Existence and uniqueness for a viscoelastic Kelvin-Voigt model with nonconvex stored energy. *Preprint arXiv:2012.10344*, 2020.

[27] M. Kružík and T. Roubíček. *Mathematical Methods in Continuum Mechanics of Solids*. Springer, Switzerland, 2019.

[28] A. Lavrentiev. Sur quelques problèmes du calcul des variations. *Ann. Mat. Pura Appl.*, 41:107–124, 1926.
[29] Z. Lei, C. Liu, and P. Zhou. Global solutions for incompressible viscoelastic fluids. *Archive Rational Mech. Anal.*, 188:371–398, 2008.

[30] W. Lian at al. Global well-posedness for a class of fourth-order nonlinear strongly damped wave equations. *Advances Calc. Var.*, 14:589–611, 2021.

[31] C. Liu and N.J. Walkington. An Eulerian description of fluids containing visco-elastic particles. *Archive Rational Mech. Anal.*, 159:229–252, 2001.

[32] Z. Martinec. *Principles of Continuum Mechanics*. Birkhäuser/Springer, Switzerland, 2019.

[33] A. Mielke, C. Ortner, and Y. Sengül. An approach to nonlinear viscoelasticity via metric gradient flows. *SIAM J. Math. Anal.*, 46:1317–1347, 2014.

[34] A. Mielke and T. Roubíček. Thermoviscoelasticity in Kelvin-Voigt rheology at large strains. *Archive Ration. Mech. Anal.*, 238:1–45, 2020.

[35] R.D. Mindlin. Micro-structure in linear elasticity. *Archive Ration. Mech. Anal.*, 16:51–78, 1964.

[36] J. Nečas. Theory of multipolar fluids. In L. Jentsch and F. Tröltzsch, editors, *Problems and Methods in Mathematical Physics*, pages 111–119, Wiesbaden, 1994. Vieweg+Teubner.

[37] J. Nečas, A. Novotný, and M. Šilhavý. Global solution to the compressible isothermal multipolar fluid. *J. Math. Anal. Appl.*, 162:223–241, 1991.

[38] J. Nečas and M. Růžička. Global solution to the incompressible viscous-multipolar material problem. *J. Elasticity*, 29:175–202, 1992.

[39] M. Pavelka, I. Peshkov, and V. Klika. On Hamiltonian continuum mechanics. *Physica D*, 408:Art.no.132510, 2020.

[40] V. Průša and K. Tůma. Temperature field and heat generation at the tip of a cutout in a viscoelastic solid body undergoing loading. *Applications Engr. Sci.*, 6:Art.no.100054, 2021.

[41] A. Prohl. Convergence of a finite element-based space-time discretization in elastodynamics. *SIAM J Numerical Anal.*, 46:2469–2483, 2008.

[42] J. Qian and Z. Zhang. Global well-posedness for compressible viscoelastic fluids near equilibrium. *Archive Rational Mech. Anal.*, 198:835–868, 2010.

[43] M. Růžička. Mathematical and physical theory of multipolar viscoelasticity. Bonner Mathematische Schriften 233, Bonn, 1992.

[44] M.O. Rieger. Young measure solutions for nonconvex elastodynamics. *SIAM J. Math. Anal.*, 34:1380–1398, 2003.

[45] T. Roubíček. *Nonlinear Partial Differential Equations with Applications*. Birkhäuser, Basel, 2nd edition, 2013.

[46] T. Roubíček. Quasistatic hypoplasticity at large strains Eulerian. *J. Nonlin. Sci.*, 32:Art.no.45, 2022.

[47] T. Roubíček. Visco-elastodynamics at large strains Eulerian. *Zeitschrift f. angew. Math. Phys.*, 73:Art.no.80, 2022.

[48] T. Roubíček. Some gradient theories in linear visco-elastodynamics towards dispersion and attenuation of waves in relation to large-strain models. 2023. (Preprint arXiv no. 2309.05089).

[49] T. Roubíček and U. Stefanelli. Finite thermoelastoplasticity and creep under small elastic strain. *Math. Mech. of Solids*, 24:1161–1181, 2019.

[50] T. Roubíček and U. Stefanelli. Visco-elastodynamics of solids undergoing swelling at large strains by an Eulerian approach. *SIAM J. Math. Anal.*, 55:2475–2876, 2023.

[51] T. Roubíček and G. Tomassetti. Dynamics of charged elastic bodies under diffusion at large strains. *Disc. Cont. Dynam. Syst. B*, 25:1415–1437, 2020.

[52] M. Šilhavý. Multipolar viscoelastic materials and the symmetry of the coefficient of viscosity. *Appl. Math.*, 37:383–400, 1992.

[53] M. Šilhavý. *The Mechanics and Thermodynamics of Continuous Media*. Springer, Berlin, 1997.

[54] R.A. Toupin. Elastic materials with couple stresses. *Arch. Ration. Mech. Anal.*, 11:385–414, 1962.

[55] C. Truesdell. *Rational Thermodynamics*. McGraw-Hill, New York, 1969.

[56] B. Tvedt. Quasilinear equations for viscoelasticity of strain-rate type. *Archive Rational Mech. Anal.*, 189:237–281, 2008.