Linear dilaton black holes

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Abstract

We present new solutions to Einstein-Maxwell-dilaton-axion (EMDA) gravity in four dimensions describing black holes which asymptote to the linear dilaton background. In the non-rotating case they can be obtained as the limiting geometry of dilaton black holes. The rotating solutions (possibly endowed with a NUT parameter) are constructed using a generating technique based on the $Sp(4,R)$ duality of the EMDA system. In a certain limit (with no event horizon present) our rotating solutions coincide with supersymmetric Israel-Wilson-Perjès type dilaton-axion solutions. In the presence of an event horizon supersymmetry is broken. The temperature of the static black holes is constant, and their mass does not depend on it, so the heat capacity is zero. We investigate geodesics and wave propagation in these spacetimes and find superradiance in the rotating case. Because of the non-asymptotically flat nature of the geometry, certain modes are reflected from infinity; in particular, all superradiant modes are confined. This leads to a classical instability of the rotating solutions. The non-rotating linear dilaton black holes are shown to be stable against spherical perturbations.

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1 Introduction

Non-asymptotically flat spacetimes containing event horizons recently attracted attention in connection with the conjecture of AdS/CFT correspondence and its generalizations. In most cases the asymptotic geometry considered was of the AdS type, such configurations frequently demonstrate holography, the field theory side being some conformal field theory. As is well-known, AdS geometry typically arises in the near-horizon limit of non-dilatonic BPS black holes and/or p-branes. Other examples of holographic backgrounds are provided by the linear dilaton solutions which arise as near-horizon limits of dilatonic holes/branes [1]. In such cases the dual theory is likely to be not a local field theory. In particular, the near-horizon limit of parallel NS5 branes is dual to the decoupled NS5 brane theory known as “little string theory” [2, 3]. It is therefore interesting to investigate other backgrounds which asymptote to the linear dilaton vacuum.

Here we present new four-dimensional solutions describing rotating black holes in a linear dilaton background. Actually, a non-rotating version of such a black hole was given long ago by Giddings and Strominger [4] (see also [5]) as some limit of extremal dilaton black holes [6, 7, 8, 5] (‘horizon plus throat’ geometry), but apparently was not fully appreciated as an exact solution of Einstein-Maxwell-dilaton theory describing a non-asymptotically flat black hole. We find that within the Einstein-Maxwell-dilaton-axion theory there exist rotating black hole solutions (possibly with NUT parameter) which asymptote to the linear dilaton space-time. The latter is supersymmetric (1/2 BPS), while our black holes break all supersymmetries. Presumably their higher dimensional counterparts exist as well and may serve as an arena for non-supersymmetric holography, although we do not present any evidence for this here, being concentrated on geometric and other physical properties of the new solutions in four dimensions.

In the non-rotating case our black holes can be regarded as excitations over the linear dilaton vacua, forming a two-parameter family, which can be obtained as near-horizon and near-extreme limits of the “parent” dilaton black holes. To identify the physical mass we use the Brown-York formalism [9]. These spacetimes are not asymptotically flat, so a suitable substruction procedure is needed (of the type described in [10]) to obtain finite quantities from the integrals divergent at infinity. For the non-rotating black hole the linear dilaton background is a natural (and unique) choice for such substruction background. With this prescription we are able to establish the validity of the first law of black hole thermodynamics \( dM = TdS \), in which the Hawking temperature is independent of the black hole mass, but depends on the parameter of the background. The entropy is given by the quarter of the

\[ \frac{1}{4} \times \text{area of horizon} \]
area as usual. The mass is therefore independent on the temperature, so the heat capacity is precisely zero. In other words, these black holes will be in a marginal thermodynamical equilibrium with the heat bath. We also check that the non-rotating linear dilaton black holes are classically stable against spherical perturbations.

We also present a rotating version (possibly with NUT) of linear dilaton black holes, in which case the axion is non-zero, but (as well as the dilaton) is independent of the black hole mass. These may be regarded as excitations over the rotating Israel-Perjès-Wilson (IWP) [11] solutions which correspond to a vanishing black hole mass parameter, but non-zero rotation parameter. The metric possesses an ergosphere outside the event horizon, which rotates with some angular velocity $\Omega_h$. The angular momentum $J$ computed as a surface integral at spatial infinity contains both a geometrical contribution and that coming from the Maxwell field. The Hawking temperature and the entropy depend both on the mass and the angular momentum of the black hole, these quantities are shown to satisfy the first law in the form $dM =TdS + \Omega_h dJ$.

Surprisingly, these new rotating black holes are related in a certain way to the five-dimensional Myers-Perry [12] rotating black hole with two equal angular momenta. This relation is non-local and derives from the recently found new classical duality between the $D=4$ EMDA system and six-dimensional vacuum gravity [13] (for more general non-local dualities of this kind see [14]). The symmetries of the six-dimensional solution combine in a non-trivial way the space-time symmetry of the four-dimensional EMDA solution with the internal symmetry $SO(2,3) \sim Sp(4, R)$ of the stationary EMDA system.

The Hamilton-Jacobi equation in the rotating linear dilaton black hole spacetime is shown to be separable as in the Kerr case, indicating the existence of a Stachel-Killing tensor. Timelike geodesics do not escape to infinity, while null ones do so for an infinite affine parameter. The Klein-Gordon equation is also separable, the mode behavior near the horizon exhibits the superradiance phenomenon. We show that all superradiant modes are reflected at large distances from the black hole, so there is no superradiant flux at infinity, in contrast with the case of massless fields in the Kerr spacetime. This situation is similar to the confinement of massive superradiant modes in the Kerr metric, which are reflected back to the horizon causing stimulated emission and absorption. In the classical limit this leads to an amplification effect due to the positive balance between emission and absorption. In the Kerr spacetime this phenomenon is rather small, being present only for massive modes. In our case massless modes are confined too, therefore all superradiant modes will form a cloud outside the horizon with an exponentially growing amplitude. This is in fact a
classical instability which manifests itself in the rapid transfer of angular momentum from the hole to the outside matter cloud. A similar conclusion was obtained in [15] for the case of the Kerr-AdS spacetime with reflecting boundary conditions on the AdS boundary.

For certain values of the parameters our solutions represent a naked singularity on the rotating linear dilaton background. We have found that in some cases the spectrum of the massless scalar field may be partially or entirely discrete.

The plan of the paper is as follows. In Sect. 2 we investigate the near-horizon limit of static dilaton black holes and show that a suitable limiting procedure leads to a two-parameter family of exact solutions of the EMDA theory, which includes static black holes asymptotically to the linear dilaton background. In Sect. 3 we apply the $Sp(4,R)$ generating technique to find rotating and NUT generalizations of the above solutions, forming now a five-parameter family. Then in Sect. 4 we apply a non-local uplifting procedure to find six and five-dimensional counterparts to our rotating solution without NUT parameter, and show that it corresponds to the five-dimensional vacuum Kerr solution with two equal rotation parameters. The next Sect. 5 is devoted to the determination of the physical mass, angular momentum and other parameters of our new black holes, and to the demonstration of the validity of the first black hole law. In Sect. 6 we investigate geodesics and modes of a minimally coupled scalar field showing, in particular, confinement of the superradiant modes. The stability of the static solutions against spherically symmetric perturbations is established in Sect. 7.

\section{Static linear dilaton black holes}

Consider the EMDA theory arising as a truncated version of the bosonic sector of $D = 4$, $\mathcal{N} = 4$ supergravity. It comprises the dilaton $\phi$ and (pseudoscalar) axion $\kappa$ coupled to an Abelian vector field $A_\mu$:

\begin{equation}
S = \frac{1}{16\pi} \int d^4x \sqrt{|g|} \left\{ -R + 2\partial_\mu \phi \partial^\mu \phi + \frac{1}{2} e^{4\phi} \partial_\mu \kappa \partial^\mu \kappa - e^{-2\phi} F_{\mu\nu} F^{\mu\nu} - \kappa F_{\mu\nu} \tilde{F}^{\mu\nu} \right\},
\end{equation}

where\footnote{Here $E^{\mu\nu\lambda\tau} \equiv |g|^{-1/2} \varepsilon^{\mu\nu\lambda\tau}$, with $\varepsilon^{1234} = +1$, where $x^4 = t$ is the time coordinate.} $\tilde{F}^{\mu\nu} = \frac{1}{2} E^{\mu\nu\lambda\tau} F_{\lambda\tau}$, $F = dA$. The black hole solutions to this theory were extensively studied in the recent past [16, 17, 18, 19, 20].

In this section we will study a solution, describing charged static dilaton black holes, found in Refs. [6, 7, 8]. Recall the electrically charged solution (whose magnetic dual is
obtained by the replacement \((\phi, F) \rightarrow (\hat{\phi} = -\phi, \hat{F} = e^{-2\phi}F)\):

\[
\begin{align*}
\hat{ds}^2 &= (1 - \frac{r_+}{r})dt^2 - (1 - \frac{r_+}{r})^{-1}dr^2 - r^2(1 - \frac{r_-}{r})d\Omega^2, \\
\hat{e}^{2\phi} &= e^{2\phi}(1 - \frac{r_-}{r}), \quad \kappa = 0, \\
\hat{F} &= \frac{Qe^{\phi\infty}}{r^2}dr \wedge dt.
\end{align*}
\] (2.2)

\[
\hat{e}^{2\phi} = e^{2\phi\infty}(1 - \frac{r_-}{r}) = e^{2\phi\infty} \sqrt{\frac{r_+ + r_-}{2}}, (2.3)
\]

\[
F = \frac{Qe^{\phi\infty}}{r^2}dr \wedge dt. (2.4)
\]

where \(\phi\infty\) is the asymptotic value of the dilaton field. The mass and charge of the black hole are

\[
M = \frac{r_+}{2}, \quad Q = e^{-\phi\infty}Q = e^{-\phi\infty} \sqrt{\frac{r_+ + r_-}{2}}. (2.5)
\]

The extreme black hole \((r_+ = r_-)\) is singular, the singularity being marginally trapped. In [5] and [4] (the “infinite throat with linear dilaton” case) the near-horizon limit of the extreme dilaton black hole has been considered, leading to

\[
\begin{align*}
\hat{ds}^2 &= \frac{\rho}{\sqrt{2}Q}dt^2 - \frac{\sqrt{2}Q}{\rho}(d\rho^2 + \rho^2 d\Omega^2), \\
\hat{e}^{2\phi} &= e^{2\phi\infty} \frac{\rho}{\sqrt{2}Q}, \quad \hat{F} = \frac{1}{2Q}e^{\phi\infty}d\rho \wedge dt.
\end{align*}
\] (2.6)

\[
\hat{e}^{2\phi} = e^{2\phi\infty} \frac{\rho}{\sqrt{2}Q}, F = \frac{Qe^{\phi\infty}}{2Q}e^{-\phi\infty}d\rho \wedge dt. (2.7)
\]

The string metric associated with the magnetic version of (2.6)-(2.7) is

\[
\hat{ds}^2 = e^{2\hat{\phi}}ds^2 = e^{-2\phi\infty} \left[ dt^2 - dw^2 - \frac{2Q}{\sqrt{2}Q} d\Omega^2 \right], (2.8)
\]

with \(w = \sqrt{2}Q \ln \rho\). This spacetime is a cylinder \(R^2 \times S^2\), and the associated dilaton

\[
\hat{\phi} = -\frac{w}{\sqrt{2}Q} + \text{const.} (2.9)
\]

is linear.

Returning to the Einstein metric (2.6), we see from the conformal map (2.8) that \(\rho = 0\) \((w = -\infty)\) is a null bifurcate singularity, while \(\rho = \infty\) \((w = +\infty)\) is at spacelike and null infinity. While all curvature invariants go to zero at spacelike infinity, this metric is not asymptotically flat. In fact the following coordinate transformation

\[
\xi = \alpha^{-1}\sqrt{\rho} \cosh(2\alpha^2 t), \quad \eta = \alpha^{-1}\sqrt{\rho} \sinh(2\alpha^2 t) (2.10)
\]

(with \(\alpha^{-2} = 4\sqrt{2}Q\)) transforms (2.6) to

\[
\hat{ds}^2 = d\eta^2 - d\xi^2 - \frac{\xi^2 - \eta^2}{4}d\Omega^2. (2.11)
\]

When \(\xi\) goes to infinity with \(\eta\) held fixed this last metric asymptotes to the conical spacetime

\[
\hat{ds}^2 = d\eta^2 - d\xi^2 - \frac{\xi^2}{4}d\Omega^2. (2.12)
\]
Now we shall generalize the "linear dilaton vacuum" metric (2.6) by studying the near-horizon and near-extremal limit of (2.2)-(2.4). Putting

\[ r_- = \epsilon^{-1}r_0, \quad r_+ = \epsilon^{-1}r_0 + \epsilon b, \quad r = \epsilon^{-1}r_0 + \epsilon\rho, \quad t = \epsilon^{-1}\bar{t}, \]  

(2.13)

where the dimensionless parameter \( \epsilon \) shall eventually be taken to zero, the solution (2.2)-(2.4) becomes

\[ ds^2 = \frac{\rho - b}{r_0 + \epsilon^2\rho}d\bar{t}^2 - \frac{r_0 + \epsilon^2\rho}{\rho - b}d\rho^2 - \rho(r_0 + \epsilon^2\rho)d\Omega^2, \]  

(2.14)

\[ e^{2\phi} = \frac{\rho}{r_0 + \epsilon^2\rho}, \quad F = \frac{\epsilon Q}{(r_0 + \epsilon^2\rho)^2}d\rho \wedge d\bar{t}, \]  

(2.15)

where we have chosen \( \phi_\infty = -\ln \epsilon \). Finally taking \( \epsilon \to 0 \) and relabelling \( \bar{t} \to t, \rho \to r \), we obtain

\[ ds^2 = \frac{r - b}{r_0}dt^2 - \frac{r_0}{r - b}dr^2 - r_0r d\Omega^2, \]  

(2.16)

\[ e^{2\phi} = \frac{r}{r_0}, \quad F = \frac{1}{\sqrt{2r_0}}dr \wedge dt, \]  

(2.17)

which is a generalisation of (2.6)-(2.7). This configuration, which was previously given in [4] (the "horizon plus infinite throat case") is an exact solution of EMDA.

The Penrose diagrams corresponding to the different values of \( b \) \( (b < 0, b = 0 \) and \( b > 0) \) are shown in Fig.1. For \( b > 0, (2.16) \) describes a static black hole, with a spacelike singularity \( r = 0 \) hidden behind a horizon located at \( r = b \). The corresponding Penrose diagram is identical to that of the Schwarzschild black hole. However the spacetime is not asymptotically flat. The case \( b = 0 \) corresponds to the extreme black hole (2.6), with a null singularity. Finally for \( b < 0, (2.16) \) describes a naked timelike singularity located at \( r = 0 \).

The two parameters \( r_0 \) and \( b \) have somewhat different physical meanings. The first describes the electric charge \( Q \) of the solution which is given by the flux through a 2-sphere

\[ Q = \frac{1}{4\pi} \int e^{-2\phi}F^0r\sqrt{g}d\Omega = \frac{r_0}{\sqrt{2}}. \]  

(2.18)

It is associated, not with a specific black hole, but rather with a given linear dilaton background and its black hole family. The parameter \( b \) characterizing a black hole is proportional to its mass, as suggested by the following thermodynamical argument. Carrying out the Wick rotation \( \tau = it \) and putting \( r = b + x^2 \) the two dimensional sector \( (t, r) \) of (2.16) becomes:

\[ ds^2 = \frac{x^2}{r_0}dt^2 + 4r_0dx^2, \]  

(2.19)
which is free from conical singularity provided \( \tau \) is an angular variable of period \( P = 4\pi r_0 \). Therefore the Hawking temperature of the black hole is

\[
T = \frac{1}{P} = \frac{1}{4\pi r_0} \tag{2.20}
\]

(computation of the surface gravity gives the same value). Assuming that the entropy is given as usual by the quarter of the horizon area

\[
S = \frac{A}{4} = \pi r_0 b \tag{2.21}
\]

we obtain from the thermodynamic first law

\[
dM = TdS \tag{2.22}
\]

the value

\[
M = \frac{b}{4}. \tag{2.23}
\]

This mass value will be confirmed in Sect. 5 by an independent computation using the method of Brown and York [9].

Note that here the parameter \( r_0 \) is kept fixed, being related to the linear dilaton background, not to the hole. It would be interesting, however, to check whether this is correct in the framework of the grand canonical ensemble approach [21, 22]. This is a non-trivial task because of the lack of asymptotic flatness (in particular, the value of the Maxwell potential entering the standard first law for charged black holes diverges at infinity), and remains beyond the scope of the present paper, where we rather use microcanonical considerations.

Now we give the \( \sigma \)-model representation of the solutions (2.16)-(2.17). In the stationary sector of EMDA, reduction to three dimensions can be performed by using the metric ansatz

\[
g_{\mu\nu} = \begin{pmatrix}
    f & -f\omega_i \\
    -f\omega_i & -f^{-1}h_{ij} + f\omega_i\omega_j
\end{pmatrix}, \tag{2.24}
\]

and parametrizing the vector field \( A_\mu \) by an electric potential \( v \) and a magnetic potential \( u \) according to

\[
F_{i0} = \frac{1}{\sqrt{2}}\partial_i v, \quad e^{-2\phi}F^{ij} + \kappa\tilde{F}^{ij} = \frac{f}{\sqrt{2h}}\epsilon^{ijk}\partial_k u. \tag{2.25}
\]

From the mixed components of the four–dimensional Einstein equations, \( \omega_k \) may be dualized to a twist potential \( \chi \) by [17]

\[
\partial_i \chi + v\partial_i u - u\partial_i v = -\frac{f^2}{\sqrt{h}}h_{ij}\epsilon^{jkl}\partial_k \omega_l. \tag{2.26}
\]
The six potentials \( f, \chi, u, v, \phi, \kappa \) parametrize a target space isomorphic to the coset \( Sp(4, R)/U(2) \), the EMDA field equations reducing to those of the corresponding three-dimensional \( \sigma \) model. A matrix representative of this coset can be chosen to be the symmetric symplectic matrix

\[
M = \begin{pmatrix}
P^{-1} & P^{-1}Q \\
QP^{-1} & P + QP^{-1}Q
\end{pmatrix},
\]

(2.27)

where \( P \) and \( Q \) are the real symmetric \( 2 \times 2 \) matrices

\[
P = -e^{-2\phi} \begin{pmatrix} v^2 - fe^{2\phi} & v \\ v & 1 \end{pmatrix}, \quad Q = \begin{pmatrix} wv - \chi & w \\ w & -\kappa \end{pmatrix},
\]

(2.28)

with \( w = u - \kappa v \).

When all the potentials depend on a single scalar potential \( \sigma \) (such as in the case of spherical symmetry), this potential can always be chosen to be harmonic (\( \nabla^2_h \sigma = 0 \)). The remaining \( \sigma \)-model equations, which reduce to the geodesic equation on the target space, with \( \sigma \) as affine parameter, are solved by

\[
M = Ae^{B\sigma},
\]

(2.29)

where the matrices \( A \) and \( B \) are constant. In the case of asymptotically flat solutions, the vacuum matrix \( A \) is \( diag(+1, -1, +1, -1) \) in the gauge \( \sigma(\infty) = 0 \). In the present case, the matrix \( M \) associated with the linear dilaton black hole solution (2.16)-(2.17) is

\[
M_\ell = \begin{pmatrix}
\frac{ra}{r-b} & -\frac{r}{r-b} & 0 & 0 \\
-\frac{r}{r-b} & \frac{b}{r_0} & 0 & 0 \\
0 & 0 & -\frac{b}{r_0} & -1 \\
0 & 0 & 1 & -\frac{ra}{r}
\end{pmatrix},
\]

(2.30)

The corresponding harmonic potential is

\[
\sigma = -\frac{1}{b} \ln \left| \frac{r-b}{r} \right|,
\]

(2.31)

and the matrices \( A \) and \( B \) are

\[
A = \begin{pmatrix}
0 & -1 & 0 & 0 \\
-1 & \frac{b}{r_0} & 0 & 0 \\
0 & 0 & -\frac{b}{r_0} & -1 \\
0 & 0 & 1 & 0
\end{pmatrix}, \quad B = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & b & 0 & 0 \\
0 & 0 & 0 & -r_0 \\
0 & 0 & 0 & -b
\end{pmatrix}
\]

(2.32)
In the case $Tr(B^2) = 0$, (2.29) is a null geodesic in the target space $[20]$, and the corresponding solution is of the Majumdar-Papetrou type. Here this occurs for a vanishing black hole mass $b = 0$. As obvious from (2.6), the spatial metric $h_{ij}$ is flat in this case, so that one can linearly superpose harmonic potentials $\sigma_i = 1/r_i$ to construct multi-center solutions

$$ds^2 = \sum_i \frac{1}{\sigma_i} dt^2 - \sum_i \sigma_id\vec{x}_i^2.$$  \hspace{1cm} (2.33)

This is related to the fact that the linear dilaton vacuum solution (2.6)-(2.7) is BPS, in the sense that it has $N = 2$ supersymmetry $[5]$, i.e. half of the original symmetries of $N = 4$, $D = 4$ supergravity are preserved.

3 Kerr extension

In this section we wish to generalize the static solution (2.16)-(2.17) to a rotating one. To do this, we shall first find the linear transformation $U$ that transforms the matrix $M_S$ associated with the Schwarzschild solution into the matrix $M_\ell$ associated with the static black hole solution (2.30). Note that this transformation changes the matrix representative, i.e. the potentials contained in the matrices $P$ and $Q$, without changing the reduced three-space metric $h_{ij}$, which is invariant under $U$. Then we shall apply the same transformation to the matrix representative $M_K$ of the Kerr solution to generate a rotating black hole metric $M_{\ell K}$.

The Schwarzschild solution

$$ds^2 = (1 - \frac{2M}{r})dt^2 - (1 - \frac{2M}{r})^{-1}dr^2 - r^2d\Omega^2,$$

$$e^{2\phi} = 1, \quad \kappa = 0, \quad F = 0$$ \hspace{1cm} (3.2)

has the same reduced 3-metric as (2.16) provided we identify

$$b = 2M.$$ \hspace{1cm} (3.4)

The corresponding Schwarzschild matrix is

$$M_S = \begin{pmatrix}
\frac{r}{r-2M} & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & \frac{r-2M}{r} & 0 \\
0 & 0 & 0 & -1
\end{pmatrix}. \hspace{1cm} (3.5)$$
Using (3.1), (3.5) and (2.30) we derive the expression of the matrix $U$

$$
U = \begin{pmatrix}
-\sqrt{\frac{r_0}{2M}} & \sqrt{\frac{2M}{r_0}} & 0 & 0 \\
\sqrt{\frac{r_0}{2M}} & 0 & 0 & 0 \\
0 & 0 & 0 & \sqrt{\frac{r_0}{2M}} \\
0 & 0 & \sqrt{\frac{2M}{r_0}} & \sqrt{\frac{r_0}{2M}} \\
\end{pmatrix}.
$$

(3.6)

Consider now the Kerr-NUT solution

$$
ds^2 = \frac{\Delta - a^2 \sin^2 \theta}{\Sigma} (dt - \omega d\varphi)^2 - \Sigma \left( \frac{dr^2}{\Delta} + d\theta^2 + \frac{\Delta \sin^2 \theta}{\Delta - a^2 \sin^2 \theta} d\varphi^2 \right),
$$

(3.7)

$$
e^{2\phi} = 1, \quad \kappa = 0, \quad F = 0,
$$

(3.8)

with

$$
\Delta = r^2 - 2Mr + a^2 - N^2, \quad \Sigma = r^2 + (N + a \cos \theta)^2,
$$

(3.9)

$$
\omega = -2 \frac{N \Delta \cos \theta + a (Mr + N^2) \sin^2 \theta}{\Delta - a^2 \sin^2 \theta}.
$$

(3.10)

The corresponding matrix $M_K$ is

$$
M_K = \begin{pmatrix}
    f^{-1} & 0 & -\chi f^{-1} & 0 \\
0 & -1 & 0 & 0 \\
-\chi f^{-1} & 0 & f + \chi^2 f^{-1} & 0 \\
0 & 0 & 0 & -1 \\
\end{pmatrix},
$$

(3.11)

where

$$
f = \frac{\Delta - a^2 \sin^2 \theta}{\Sigma}, \quad \chi = -2 \frac{N(M - r) + aM \cos \theta}{\Sigma}.
$$

(3.12)

Applying the transformation (3.1) to the matrix $M_K$ we obtain

$$
M_{tK} = U^T M_K U = \begin{pmatrix}
    \frac{r_0(1-f)}{2Mf} & -\frac{1}{f} & 0 & \frac{r_0 \chi}{2Mf} \\
-\frac{1}{f} & \frac{2M}{r_0} & 0 & -\frac{\chi}{f} \\
0 & 0 & -\frac{2M}{r_0} & -1 \\
\frac{r_0 \chi}{2Mf} & -\frac{\chi}{f} & -1 & \frac{r_0}{2Mf}(\chi^2 + f^2 - f) \\
\end{pmatrix}.
$$

(3.13)

Comparing (3.13) and (2.27) we obtain the rotating black hole solution generalizing (2.16)-(2.17)

$$
ds^2 = \frac{\Delta - a^2 \sin^2 \theta}{\Gamma} (dt - \omega d\varphi)^2 - \Gamma \left( \frac{dr^2}{\Delta} + d\theta^2 + \frac{\Delta \sin^2 \theta}{\Delta - a^2 \sin^2 \theta} d\varphi^2 \right),
$$

(3.14)

$$
e^{2\phi} = \frac{r^2 + (N + a \cos \theta)^2}{\Gamma}, \quad \kappa = \frac{r_0 N(r - M) - aM \cos \theta}{M \left( r^2 + (N + a \cos \theta)^2 \right)},
$$

(3.15)

$$
v = \frac{r^2 + (N + a \cos \theta)^2}{\Gamma}, \quad u = \frac{r_0 N(r - M) - aM \cos \theta}{M \left( r^2 + (N + a \cos \theta)^2 \right)}.
$$

(3.16)
with
\[
\Delta = r^2 - 2Mr + a^2 - N^2, \quad \Gamma = \frac{r_0}{M}(Mr + N^2 +aN \cos \theta), \quad \omega = -\frac{r_0}{M} \frac{N\Delta \cos \theta + a(Mr + N^2) \sin^2 \theta}{\Delta - a^2 \sin^2 \theta}.
\] (3.17)

This new solution can again be slightly generalized by the action of the S-duality transformation \( SL(2, R) \)
\[
\tilde{z} = \tilde{\kappa} + i e^{-2\phi} \frac{az + \beta}{\gamma z + \delta}, \quad \tilde{v} = \delta v + \gamma u, \quad \tilde{u} = \beta v + \alpha u \quad (\alpha \delta - \beta \gamma = 1),
\] (3.19)
which generates a magnetic charge without modifying the spacetime metric.

We now focus on the case \( N = 0 \). The stationary solution (3.14)-(3.16) is then significantly simplified to
\[
ds^2 = \frac{r^2 - 2M r + a^2}{r_0 r} dt^2 - r_0 r \left[ \frac{dr^2}{r^2 - 2Mr + a^2} + d\theta^2 + \sin^2 \theta \left( d\varphi - \frac{a}{r_0 r} dt \right)^2 \right], \quad (3.20)
\]
\[
F = \frac{1}{\sqrt{2}} \left[ \frac{r^2 - a^2 \cos^2 \theta}{r_0 r^2} d\varphi \wedge d\theta + a \sin 2\theta d\theta \wedge \left( d\varphi - \frac{a}{r_0 r} dt \right) \right], \quad (3.21)
\]
\[
e^{-2\phi} = \frac{r_0 r}{r^2 + a^2 \cos^2 \theta}, \quad \kappa = -\frac{r_0 a \cos \theta}{r^2 + a^2 \cos^2 \theta}. \quad \text{(3.22)}
\]
The Maxwell two-form is derivable from the following four-potential
\[
A = \frac{1}{\sqrt{2}} \left( \frac{r^2 + a^2 \cos^2 \theta}{r_0 r} dt + a \sin^2 \theta \, d\varphi \right), \quad (3.23)
\]
where the gauge is chosen such that the vector magnetic potential be regular on the axis \( \theta = 0 \).

Although the metric (3.20) was derived from the Kerr metric, it differs in its asymptotic behavior \( (r \to \infty) \), which is the same as for the linear dilaton metric (2.6), and in its behavior near \( r = 0 \). In the case of the Kerr metric, \( r = 0 \) is the equation of a disk through which the metric can be continued to negative \( r \), while in (3.20), \( r = 0 \) is a timelike line singularity. It follows that the Penrose diagrams of (3.20) for the three cases \( M^2 > a^2 \), \( M^2 = a^2 \) and \( M^2 < a^2 \) are identical, not to those of the Kerr spacetime, but rather to those of the Reissner-Nordström spacetime, with the charge replaced by the angular momentum parameter \( a \).

The massless case \( M = 0 \) is of particular interest. The Kerr metric with \( M = 0 \) representing flat Lorentzian spacetime, it follows that the reduced 3-metric \( h_{ij} \) in (3.20), which is identical to that of the Kerr metric with \( M = 0 \), represents flat Euclidean space. This property is characteristic of the generalized Israel-Wilson-Perjès (IWP) solutions. Indeed,
following the methods of [20], we can show that for $M = 0$ the matrix $M_{\ell K}$ depends on the two harmonic potentials

$$\sigma = \frac{r}{r^2 + a^2 \cos^2 \theta}, \quad \tau = \frac{\cos \theta}{r^2 + a^2 \cos^2 \theta},$$

through

$$M_{\ell K} = A e^{B\sigma + C\tau},$$

where the matrices $A$ and $B$ are those of [23.22] with $b = 2M = 0$, and

$$C = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & a r_0 \\
ar_0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}.$$  \hspace{1cm} (3.26)

These matrices are such that $B^2 = C^2 = BC = CB = 0$, so that the matrix $M_{\ell K}$ is linear in the two harmonic potentials,

$$M_{\ell K} = A (I + B\sigma + C\tau).$$  \hspace{1cm} (3.27)

It follows that given any two harmonic potentials $\sigma, \tau$, we can construct the IWP generalization of the rotating black hole solution (3.14)-(3.16) with $M = N = 0$

$$ds^2 = \frac{1}{r_0\sigma} (dt - \omega_i dx^i)^2 - r_0\sigma d^3x,$$  \hspace{1cm} (3.28)

$$A = \frac{1}{r_0\sigma} (dt - \omega_i dx^i),$$  \hspace{1cm} (3.29)

$$e^{-2\phi} = r_0\sigma, \quad \kappa = -ar_0\tau,$$  \hspace{1cm} (3.30)

with

$$\partial_i \omega_j = \frac{ar_0}{2} \epsilon_{ijk} \partial^k \tau.$$  \hspace{1cm} (3.31)

The general IWP solution (3.28)-(3.30) was previously constructed in [11]. The multi-center solutions given in [11], [20] are asymptotically flat, and so do not explicitly include the solution (3.14)-(3.16) with $M = N = 0$. This solution can be recovered from the one-particle rotating IWP solution given in [11] by taking the infinite mass limit. It was shown in [11] that the general IWP solution is BPS, i.e. has $N = 2$ supersymmetry when embedded in $N = 4, D = 4$ supergravity, and this result applies also to our new non-asymptotically flat solutions (3.14)-(3.16) with $M = N = 0$. 

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4 Six-dimensional vacuum dual

As shown in [13], the four-dimensional EMDA theory is equivalent to a sector of sourceless six-dimensional general relativity with two Killing vectors. That is, any solution \((ds_4^2, A_\mu, \phi, \kappa)\) of EMDA may be lifted to the Ricci-flat six-dimensional metric

\[
ds_6^2 = ds_4^2 - e^{-2\phi} \xi^2 - e^{2\phi} (\zeta + \kappa \xi)^2, \tag{4.1}\]

with

\[
\xi \equiv d\chi + \sqrt{2} A_\mu dx^\mu, \quad \zeta \equiv d\eta + \sqrt{2} B_\mu dx^\mu, \tag{4.2}\]

\[
F_{\mu\nu}(B) \equiv e^{-2\phi} \tilde{F}_{\mu\nu}(A) - \kappa F_{\mu\nu}(A). \tag{4.3}\]

Note that the \(S\)-duality of the four-dimensional dilaton-axion theory is here explicitly realized as the group of linear transformations of the 2-plane \((\chi, \eta)\).

In the present case we obtain from Eqs. (3.14)-(3.16)

\[
A = \frac{1}{\sqrt{2} \Gamma} \left( [r^2 + (N + a \cos \theta)^2] dt + \frac{r_0}{M} [N \Delta \cos \theta + a (M r + N^2) \sin^2 \theta] d\varphi \right), \tag{4.4}\]

\[
B = \frac{(M^2 + N^2) r_0}{\sqrt{2} M^2 \Gamma} \left( (N + a \cos \theta) dt - r_0 r \cos \theta d\varphi \right), \tag{4.5}\]

leading for the six-dimensional metric to a long and unenlightening expression. In the case \(N = 0\), after rescaling \(\eta \rightarrow r_0 \eta\) this reduces to

\[
ds_6^2 = -\frac{2M}{r_0} dt^2 - 2 d\chi dt - \frac{r_0}{r} (d\chi + a d\varphi - a \cos \theta d\eta)^2
- \frac{r_0 r}{\Delta} dr^2 - 4r_0 r d\Omega_3^2. \tag{4.6}\]

In (4.6),

\[
d\Omega_3^2 = \frac{1}{4} (d\theta^2 + \sin^2 \theta d\varphi^2 + (d\eta - \cos \theta d\varphi)^2) \tag{4.7}\]

is the metric of the three-sphere if \(0 < \theta < \pi\), and the angles \(\eta\) and \(\varphi\) are defined modulo \(2\pi\). This can be seen from the fact that the coordinate transformation

\[
x + iy = r \cos(\theta/2) e^{i \frac{\eta - \varphi}{2}}, \quad z + iw = r \sin(\theta/2) e^{i \frac{\eta + \varphi}{2}}, \tag{4.8}\]

describes the embedding of the three-sphere \(x^2 + y^2 + z^2 + w^2 = r^2\) in four-dimensional Euclidean space

\[
dx^2 + dy^2 + dz^2 + dw^2 = dr^2 + \frac{r^2}{4} (d\theta^2 + d\eta^2 + d\varphi^2 - 2 \cos \theta d\eta d\varphi). \tag{4.9}\]
Remarkably, the metric (4.6) is asymptotically flat. Defining new coordinates by
\[
\rho = 2\sqrt{r_0 r}, \\
\bar{\theta} = \theta/2, \quad \varphi_\pm = (\varphi \pm \eta)/2,
\]
\[
d\psi = (2M/r_0)^{1/2}(dt + (r_0/2M) d\chi), \quad (4.10)
\]
\[
d\tau = (2M/r_0)^{-1/2} d\chi,
\]
it can be put in the form of a direct product
\[
d s_6^2 = -d\psi^2 + ds_5^2,
\]
where
\[
ds_5^2 = d\tau^2 - \frac{\mu}{\rho^2} \left( d\tau + (\bar{a}/2)(d\varphi - \cos \theta d\eta) \right)^2 - \frac{d\rho^2}{1 - \mu\rho^{-2} + \mu\bar{a}^2\rho^{-4}} - \rho^2 d\Omega_3^2 - d\rho^2 \sin^2 \bar{\theta} d\varphi_+ + \bar{a} \cos^2 \bar{\theta} d\varphi_- \right)^2 \quad (4.12)
\]
\[
(\mu = 8M r_0, \bar{a} = (2r_0/M)^{1/2}a)\) is the five-dimensional Myers-Perry black hole \([12]\) with two equal angular momentum parameters. Accordingly, the six-dimensional metric (4.6) enjoys the symmetry group \(SO(3) \times U(1) \times U(1) \times U(1)\) generated by the six Killing vectors
\[
L_1 = \cos \eta \partial_\theta - \frac{\sin \eta}{\sin \theta} \partial_\varphi - \cot \theta \sin \eta \partial_\eta, \quad L_4 = \partial_\varphi,
\]
\[
L_2 = \sin \eta \partial_\theta + \frac{\cos \eta}{\sin \theta} \partial_\varphi + \cot \theta \cos \eta \partial_\eta, \quad L_5 = \partial_\tau,
\]
\[
L_3 = \partial_\eta, \quad L_6 = \partial_\psi. \quad (4.13)
\]
It is interesting to observe that the four-dimensional rotating solution did not possess the spherical symmetry, the origin of the \(SO(3)\) component in six dimensions may be traced to the \(Sp(4, R) \sim SO(3, 2)\) invariance of stationary EMDA configurations.

Several particular cases are interesting. The Myers-Perry black hole is extremal for \(4\bar{a}^2 = \mu\) \((a^2 = M^2)\). For \(\bar{a} = 0\) \((a = 0)\), the spacetime (4.12) reduces to the static black hole
\[
ds_5^2 = (1 - \frac{\mu}{\rho^2})d\tau^2 - \frac{d\rho^2}{1 - \frac{\mu}{\rho^2}} - \rho^2 d\Omega_3^2, \quad (4.14)
\]
leading for the six-dimensional metric to the higher symmetry group \(SO(4) \times U(1) \times U(1)\). \((\mu = 0)\) reduces to five-dimensional Minkowski space.
For $M = 0$ the coordinate transformation (4.10) breaks down. In this case (4.6) reduces to the six-dimensional asymptotically flat metric
\[
 ds_6^2 = -2 \, dt \, d\chi - \frac{4r_0^2}{\rho^2} \left( d\chi + a(d\varphi - \cos \theta \, d\eta) \right)^2 - \frac{d\rho^2}{1 + 16r_0^2a^2 \rho^{-4}} - \rho^2 \, d\Omega_3^2. \tag{4.15}
\]
Finally, for $M = 0$ and $a = 0$, this reduces to the six-dimensional metric
\[
 ds_6^2 = -2 \, dt \, d\chi - \frac{4r_0^2}{\rho^2} d\chi^2 - d\rho^2 - \rho^2 \, d\Omega_3. \tag{4.16}
\]
This has flat spatial four-sections, and is easily seen to be a special case of the six-dimensional multi-center metric, generalizing the five-dimensional “antigravitating” solutions given by Gibbons [6] (Eq. (18), see also [25], Eq. (40))
\[
 ds_6^2 = 2 \, dt \, d\psi - F \, d\psi^2 - d\mathbf{x}^2, \tag{4.17}
\]
with $F(\mathbf{x})$ an arbitrary harmonic function of the four spatial dimensions.

\section{Mass, spin and the first law}

Let us discuss the physical properties of the rotating dilaton black hole solution with $N = 0$ (3.20). The space-time contains an event horizon located at $r = r_+$ which is the largest root of the equation $\Delta = (r - r_-(r - r_+)) = 0$:
\[
 r_\pm = M \pm \sqrt{M^2 - a^2}. \tag{5.1}
\]
This expression is the same as for the Kerr metric, but one has to keep in mind that $M$ is no longer the physical mass of the solution.

Again as the Kerr space-time, the space-time (3.20) does not admit a globally timelike Killing vector field, and therefore contains an ergosphere whose boundary corresponds to vanishing of the norm of the Killing vector $\partial_t$ (i.e. is given by the largest root of $g_{tt} = 0$):
\[
 r_e = M + \sqrt{M^2 - a^2 \cos^2 \theta}. \tag{5.2}
\]
All physical frames inside the ergosphere are rotating. The angular velocity of the horizon is determined by vanishing of the norm of the linear combination of the timelike and azimuthal Killing vectors $\partial_t + \Omega_h \partial_\varphi$ on the horizon (5.1)
\[
 \Omega_h = \frac{a}{r_0 r_+}. \tag{5.3}
\]
Let us compute the mass and the angular momentum of the NUT-less stationary black hole (3.20) using the method developed by Brown and York [9]. Consider a spacetime region
\( M \) foliated by a family \( \Sigma \) of spacelike slices of two-boundary \( B \), which we shall assume to be 2-spheres \( r = \text{constant} \). The metric on \( M \) is written in the ADM form \([26]\) as

\[
\begin{align*}
\,ds^2 &= -N^2 dt^2 + h_{ij}(dx^i + V^i dt)(dx^j + V^j dt).
\end{align*}
\] (5.4)

From (3.20) the unit normal to \( \Sigma \) is \( u_\mu = -N^0 \), with \( N = \sqrt{\Delta/r_0r} \), and the induced metric \( h_{\mu\nu} = g_{\mu\nu} + u_\mu u_\nu \) on \( \Sigma \) is

\[
\begin{align*}
h_{ij}dx^i dx^j &= r_0 r \left( \frac{dr^2}{\Delta} + d\Omega^2 \right).
\end{align*}
\] (5.5)

The timelike component \( ^3B \) of the three-boundary of \( M \) is the product of \( B \) with segments of timelike lines orthogonal to \( \Sigma \) at \( B \). The unit normal to \( ^3B \) is \( n_\mu = \sqrt{r_0 r/\Delta} \delta^r_\mu \), and the induced metric \( \sigma_{\mu\nu} = h_{\mu\nu} - n_\mu n_\nu \) on \( B \) is \( \sigma_{ab}dx^a dx^b = r_0r d\Omega^2 \).

The surface stress-energy-momentum tensor \( \tau^{ij} \) is defined by the functional derivative of the action with respect to the three-metric on \( ^3B \). The normal and tangential projections of \( \tau^{ij} \) on \( B \) lead to the proper energy surface density and proper momentum surface density

\[
\begin{align*}
\varepsilon &\equiv u_iu_j\tau^{ij} = -\frac{1}{\sqrt{\sigma}} \frac{\delta \tilde{S}}{\delta N}, \quad (5.6)
\end{align*}
\]

\[
\begin{align*}
j_a &\equiv -\sigma_{ai}u_j\tau^{ij} = \frac{1}{\sqrt{\sigma}} \frac{\delta \tilde{S}}{\delta V^a}. \quad (5.7)
\end{align*}
\]

Here the “renormalized” action \( \tilde{S} \) is the sum

\[
\begin{align*}
\tilde{S} = S(g) + S(m) - S(0),
\end{align*}
\] (5.8)

where \( S(g) \) and \( S(m) \) are the “gravitational” and “matter” pieces of the action \([2.1]\), evaluated at a given classical solution, and \( S(0) \) is the value of the action \( S \) at a reference or background “vacuum” classical solution. Such a substraction is necessary in the case of a non-asymptotically flat metric \([9] [10]\). Here the natural choice for the vacuum in a given charge sector is the linear dilaton metric \((2.6)\), corresponding to \( M = a = 0 \) and the specified value \( Q = r_0/\sqrt{2} \) of the electric charge.

We first discuss the computation of the proper energy surface density \( \varepsilon \). The gravitational contribution is

\[
\begin{align*}
\varepsilon(g) &= \frac{1}{8\pi} k, \quad (5.9)
\end{align*}
\]

where \( k \) is the trace of the extrinsic curvature of \( B \) as embedded in \( \Sigma \),

\[
\begin{align*}
k &= -\sqrt{\frac{\Delta}{r_0r}} \frac{1}{r}. \quad (5.10)
\end{align*}
\]
The matter contribution originates solely from the gauge field piece of the action (2.1) and is given by
\[ \varepsilon(m) = \frac{1}{N\sqrt{\sigma}} A_0 \Pi^r, \] (5.11)
Here \( \Pi^r = N\sqrt{h}E^r/4\pi \) is the canonical momentum conjugate to \( A_i \), the radial component \( E^r \) reads
\[ E^r = e^{-2\phi} F^{rt} + \kappa F^{rt} = -\frac{1}{\sqrt{2r}}, \] (5.12)
(it follows from (5.12) that the electric charge is given as in the static case by \( Q = -4\pi \Pi^r = r_0/\sqrt{2} \), so that we obtain from (3.23)
\[ \varepsilon(m) = -\frac{1}{8\pi r_0} \sqrt{r_0^2 r^2 + a^2 \cos^2 \theta} \frac{\Delta}{r^2}. \] (5.13)

It is worth noting that in the Brown-York formalism there are two quantities associated with energy: the proper energy density \( \varepsilon \), whose integral gives the Brown-York total quasilocal energy \( E = \int_B d^2 x \sqrt{\sigma} \varepsilon \), and the mass \( \mathcal{M} \), which is given by the following integral over a large sphere at spatial infinity
\[ \mathcal{M} = -\int_B d^2 x \sqrt{\sigma} u_0 \varepsilon = -\int_B d^2 x \sqrt{\sigma} u_0 (\varepsilon(g) + \varepsilon(m) - \varepsilon(0)). \] (5.14)
Both quantities give the same answer for asymptotically flat spacetimes, but not for non-asymptotically flat ones. This difference, in particular, manifests itself for asymptotically AdS black holes, and it was found that it is the mass \( \mathcal{M} \) which gives the correct form of the first law, while the energy gives the redshifted quantity \[27\]. So here we will calculate the Brown-York mass (5.14). For asymptotically flat black holes, such as the Kerr-Newman one, the second term is zero and is usually ignored. In our case it is non-zero, and indeed linearly divergent, but is exactly cancelled by the electromagnetic contribution of the linear dilaton background evaluated with the same boundary data for the fields \( \sigma, N \) and \( A_0 \) \[28\], and in the present case with the same value of the electric charge. The linear divergence of the first term is cancelled by substracting the gravitational contribution of the linear dilaton background
\[ \varepsilon(g0) = -\frac{1}{8\pi \sqrt{r_0^2 r^2}}, \] (5.15)
leading to the finite result
\[ \mathcal{M} = \frac{M}{2} = \frac{b}{4}, \] (5.16)
in accordance with the value \[22\] for the static case (we recall that the parameter \( M \) is the mass of the Kerr black hole which is transformed into the black hole metric \[3\] by the

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group transformation (3.1)). Here (as usual) the black hole mass is given by the finite part of the gravitational contribution, the (divergent) matter contribution which is independent of the black hole parameters $M$ and $a$ being completely compensated by the background substraction.

The angular momentum is computed along similar lines. The gravitational and matter contributions to the azimuthal proper momentum surface density are

$$j_{(g)\varphi} = -\frac{1}{8\pi} n_\ell K^{\ell}_\varphi, \quad j_{(m)\varphi} = -\frac{1}{\sqrt{\sigma}} A_\varphi \Pi^\prime,$$  

(5.17)

where $K^{\ell}_\varphi$ is the extrinsic curvature of $\Sigma$. The total angular momentum is given by the integral

$$J = \int_B d^2x \sqrt{\sigma} j_{\varphi} = \int_B d^2x \sqrt{\sigma} (j_{(g)\varphi} + j_{(m)\varphi})$$  

(5.18)

(the spherically symmetric background does not contribute to (5.18)). Again the matter contribution $j_{(m)\varphi}$, which vanishes in the Kerr-Newman case, is often ignored (see however [29] and references therein). In the case of our rotating black hole metric (3.20) we obtain $j_{(g)\varphi} = a \sin^2 \theta / 16 r$, $j_{(m)\varphi} = 2 j_{(g)\varphi}$, leading to

$$J = \frac{ar_0}{2}.$$  

(5.19)

Now we are in a position to check that our black holes obey the differential first law of black hole mechanics:

$$dM = TdS + \Omega_\ell dJ.$$  

(5.20)

The Hawking temperature of rotating dilaton black holes can be computed either via the surface gravity, or using euclidean continuation, both leading to

$$T = \frac{r_+ - M}{2\pi r_0 r_+}.$$  

(5.21)

The entropy is given by the quarter of the horizon area

$$S = \pi r_0 r_+.$$  

(5.22)

Using (5.16), (5.21), (5.22), (5.22) and (5.19) it is straightforward to verify the first law (5.20), in which differentiations are performed for a fixed value of the electric charge $r_0/\sqrt{2}$ which is characteristic of the linear dilaton background.
To get further insight into the nature of our solutions, consider first geodesics. The Hamilton-Jacobi equation is separable in the general rotating linear dilaton black hole spacetime (3.20) in exactly the same way as in the usual Kerr metric. This indicates that a Stachel-Killing tensor exists in the present case too. The variables in the equation

\[ \frac{\partial S}{\partial x^\mu} \frac{\partial S}{\partial x^\nu} g^{\mu\nu} = \mu^2 \]  

are separated by setting

\[ S = -\mathcal{E}t + L\phi + \Theta(\theta) + \mathcal{R}(r), \]  

where \( \mathcal{E} \) is the conserved energy and \( L \) is the azimuthal momentum. We find

\[ R'^2 - \left( \frac{\mathcal{E}r_0 r - aL}{\Delta} \right)^2 + \frac{\mathcal{K}^2 + \mu^2 r_0 r}{\Delta} = 0, \]  

\[ \Theta'^2(\theta) + \frac{L^2}{\sin^2 \theta} = \mathcal{K}^2. \]  

The separation constant \( \mathcal{K} \) is an analogue of Carter’s integral for the Kerr metric, it is equal to the square of the total angular momentum. Equatorial motion corresponds to \( \mathcal{K} = L \). Finally we obtain the Hamilton-Jacobi action in the form

\[ S = -\mathcal{E}t + L\phi \pm \int \sqrt{\mathcal{K}^2 - \frac{L^2}{\sin^2 \theta}} \, d\theta \pm \int \sqrt{(\mathcal{E}r_0 r - aL)^2 - \Delta(\mathcal{K}^2 + \mu^2 r_0 r)} \frac{dr}{\Delta}. \]  

From this one can derive the trajectory and the dependence of coordinates on time \( t \).

Alternatively one may describe timelike and null geodesics using the constraint equation

\[ \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} g_{\mu\nu} = \epsilon, \quad \epsilon = 1, 0 \]  

in terms of the affine parameter. Reintroducing the (renormalized) energy and azimuthal momentum as

\[ \mathcal{E} = g_{\mu\nu} \frac{dx^\mu}{d\lambda}, \quad L = -g_{\phi\nu} \frac{dx^\mu}{d\lambda}, \]  

one obtains the following equation for the equatorial motion:

\[ \left( \frac{dr}{d\lambda} \right)^2 - \left( \frac{\mathcal{E} - aL}{r_0r} \right)^2 + \frac{\Delta}{(r_0r)^2}(L^2 + \epsilon r_0r) = 0, \]  

which can be rewritten as the conservation equation

\[ \left( \frac{dr}{d\lambda} \right)^2 + U_{\text{eff}}(r) = \mathcal{E}^2, \]
with the effective potential energy
\[ U_{\text{eff}}(r) = \frac{L^2}{r_0^2} \left( 1 - \frac{2M}{r} \right) + \frac{2aLE}{r_0r} + \frac{\epsilon \Delta}{r_0r}. \] (6.10)

For \( \epsilon = 1 \) this potential grows linearly as \( r \to \infty \), so all timelike equatorial geodesics are reflected at large \( r \). For null ones the potential is stabilized at infinity, so the geodesics with \( \mathcal{E} < L/r_0 \) are confined, while those with \( \mathcal{E} > L/r_0 \) go to infinity at an infinite affine parameter.

Let us discuss the behavior of a minimally coupled test scalar field, starting with the linear dilaton background \( M = a = 0 \). The Klein-Gordon equation
\[ (\nabla^2 + \mu^2)\psi = 0 \] (6.11)
separates in terms of the spherical harmonics
\[ \psi_{\omega lm} = C_{\omega lm} Y_{lm}(\theta, \varphi) R_{\omega l}(r) e^{-i\omega t}, \] (6.12)
where we assume \( \omega > 0 \), and the radial functions obey the equation
\[ (r^2 R_{\omega l}')' - (\nu^2 - 1/4 + \mu^2 r_0) R_{\omega l} = 0, \] (6.13)
where
\[ \nu = \sqrt{(l + 1/2)^2 - \omega^2 r_0^2}. \] (6.14)

For \( \mu \neq 0 \) the general asymptotic solution of the radial equation (6.13) reads
\[ R(r) \sim C_1 e^{-2\mu \sqrt{r_0^2}} + C_2 e^{2\mu \sqrt{r_0^2}}, \] (6.15)
where one has to put \( C_2 = 0 \) for physical solutions, the exact solution vanishing at infinity being
\[ R(r) = Cr^{-1/2} K_{2\nu}(2\mu \sqrt{r_0^2}), \] (6.16)
with \( K_{2\nu} \) a modified Bessel function. These are the massive modes reflected from infinity whose existence could be guessed from the above analysis of geodesics.

Near the singularity the mass term can be neglected and we obtain the following generic solution of the equation (6.13) which is of course exact in the massless case:
\[ R(r) = C_1 r^{-\nu-1/2} + C_2 r^{\nu-1/2}. \] (6.17)

For \( \omega > (l + 1/2)/r_0 \) (imaginary \( \nu \) so that \( \nu = iq \)) these two terms represent ingoing and outgoing waves
\[ R(r) = \frac{1}{r^{1/2}} \left( C_1 e^{-iqx} + C_2 e^{iqx} \right), \] (6.18)
where \( x = \ln r \). Introducing the current

\[
j^\mu = \frac{i}{2} \psi^* \partial^\mu \psi,
\]

one can see that the flux of the (spherical) mode is

\[
\oint j^r \sqrt{-g} d\Omega = 4\pi q (|C_1|^2 - |C_2|^2).
\]

It can be checked that the solution \((6.16)\) corresponds to \(|C_1| = |C_2|\), so it can be interpreted as initiating from the singularity and reflected back to it by the linear potential barrier \(\mu^2 r/2r_0\).

Now consider our general class of rotating metrics with \(N = 0\) \((3.20)\). The Klein-Gordon equation for the modes \(\psi = \psi(r, \theta) e^{i(m\phi - \omega t)}\) takes the form

\[
\frac{1}{r_0^2} \partial_r (\Delta \partial_r \psi) + \frac{1}{r_0 \sin \theta} \partial_\theta (\sin \theta \partial_\theta \psi) + \left[ \frac{r_0}{\Delta} (\omega - am)^2 - \frac{m^2}{r^2 \sin^2 \theta} - \mu^2 \right] \psi = 0
\]

\((6.21)\)

and is again separable. In fact, putting \(\psi = R(r) \Theta(\theta)\) the above equation is split into the following two ordinary differential equations

\[
\partial_r (\Delta \partial_r R) + \left( -\mu^2 r_0^2 + \frac{(r_0 \omega - am)^2}{\Delta} \right) R = \mathcal{K}^2 R,
\]

\((6.22)\)

\[
- \frac{1}{\sin \theta} \partial_\theta (\sin \theta \partial_\theta \Theta) + \frac{m^2}{\sin^2 \theta} \Theta = \mathcal{K}^2 \Theta,
\]

\((6.23)\)

where we keep the same notation for the separation constant as before. Unlike the Kerr case, now the angular equation \((6.23)\) is that for the associated Legendre functions so we have

\[
\mathcal{K}^2 = l(l + 1), \quad l \geq |m|, \quad l = 0, 1, 2, \ldots
\]

\((6.24)\)

Let us consider the radial equation \((6.22)\) in the black hole case \(M > a\) (assuming without loss of generality \(a \geq 0\)) when a regular event horizon exists. As usual, the radial wave function behaves near the horizon as a spherical wave, and the energy spectrum is continuous. One question to be answered is that of superradiance. Since the rotating dilaton black hole contains an ergosphere outside the horizon, it potentially has a superradiant instability. In an asymptotically flat spacetime such as Kerr, superradiance manifests itself as a quantum emission of certain modes transporting the angular momentum of the black hole to infinity. If the asymptotic region is non-flat, one has to analyse the set of modes more carefully paying attention to their behavior at infinity \([30, 31]\). In particular, the situation is different (from that of the Kerr spacetime) in the AdS-Kerr spacetime, which is neither asymptotically flat nor globally hyperbolic. As was argued in \([32]\) one is free
either to impose reflecting boundary conditions at infinity, or to allow for modes \( \omega > \mu \) to propagate. In the first case the superradiant modes will be reflected back to the black hole. These waves will generate stimulated superradiant emission and absorption, whose balance in the classical limit corresponds to the classical amplification phenomenon \[33\]. As a result, the superradiant modes will grow exponentially causing transport of the angular momentum to the field cloud rotating outside the horizon \[15\], until the black hole angular momentum is lost completely. From the previous analysis it follows that in our case not all modes are reflected, so we have to explore whether the superradiant ones propagate to infinity or not.

One can eliminate the first derivative in the radial equation by introducing the tortoise coordinate

\[
dr_* = \frac{r_0r_+}{\Delta} \, dr,
\]

leading to

\[
r_* = \frac{1}{2\kappa} \ln \left( \frac{r - r_+}{r - r_-} \right),
\]

where \( \kappa = (r_+ - r_-)/2r_0r_+ \) is the surface gravity of the horizon. Note that the range of the tortoise coordinate here is \((-\infty, 0)\) unlike the case of the Kerr metric, where it is the whole real axis. Rewriting the radial equation as

\[
d^2R \over dr_*^2 - VR = 0,
\]

we obtain the potential

\[
V = \frac{\mu^2r\Delta}{r_0r_+^2} + \frac{\Delta l(l + 1)}{(r_0r_+)^2} - \frac{r^2}{r_+^2} \left( \omega - \frac{am}{r_0r} \right)^2.
\]

At the horizon \( \Delta = 0 \) the potential takes the value

\[
V = -k^2, \quad k = \omega - m\Omega_h,
\]

where \( k \) is the wave frequency with respect to the observer rotating with the horizon. Therefore radial functions at the horizon are

\[
R = e^{\pm ikr_*},
\]

so the general solution is a superposition of ingoing and outgoing waves

\[
\psi \sim f(\theta, \varphi)e^{-i(\omega t + k r_*)} + g(\theta, \varphi)e^{-i(\omega t - k r_*)}.
\]

In the case of the Kerr metric one then constructs the asymptotic modes as \( \exp[i\omega(t \pm r)] \) and finds that there is a constant flux at infinity of the modes \( \omega > 0, k < 0 \), i.e. \( \omega < m\Omega_h \)
In our case one can see that the superradiant modes do not propagate to infinity. Indeed, all terms in the potential \((6.28)\) containing the black hole parameters \(M, a\) are small when \(r \to \infty\), so the asymptotic behavior is still given by Eq. \((6.15)\) for \(\mu \neq 0\) or \((6.17)\) for \(\mu = 0\). Since all the massive modes, as well as the massless modes with \(\omega < (l + 1/2)/r_0\), are confined (do not propagate to infinity), and since \(|m| < l\) and \(a < r_+\), leading to \(m \Omega_h = ma/r_0r_+ < (l + 1/2)/r_0\), all superradiant modes are reflected. Therefore there cannot be any superradiant flux at infinity. But it would be wrong to conclude that there is no superradiance at all. In fact, these modes are confined in a region outside the black hole. Their amplitude will grow up exponentially due to the amplification of waves impinging on the horizon, leading in the quantum description to a positive balance between stimulated emission and absorption. This effect is basically classical in nature \([33]\), so that the rotating dilaton black holes are unstable, similarly to the Kerr-AdS spacetime with reflecting boundary conditions \([15]\).

It is interesting also to analyse the behavior of the Klein-Gordon modes in the case of a naked singularity, \(M < a\). One can show that there are modes reflected from the singularity, so that the spectrum is partially discrete. For the discrete spectrum the Klein-Gordon norm

\[
||\psi||^2 = \int_{\Sigma} j^\mu(x) \, d\Sigma_\mu
\]

has to be finite. Choosing the spacelike hypersurface \(\Sigma\) to be a hyperplane \(t = \text{constant}\), this condition reads here

\[
||\psi||^2 = \int (\omega g^{00} - mg^{0\varphi}) |\psi|^2 \sqrt{|g|} \, d^3x
\]

\[
= 4\pi \int_0^\infty \left( \omega - \frac{am}{r_0r} \right) \frac{(r_0r)^2}{\Delta} R^2(r) \, dr < \infty.
\]

The discussion of the eigenvalue problem depends on the value of the rotation parameter \(a\).

a) For \(a = 0, M < 0\), the radial wave function behaves near \(r = 0\) as

\[
R(r) \sim C_3 + C_4 \ln r.
\]

However in this case \(\ln r\) is a singular solution of the Klein-Gordon equation \((6.21)\) (the action of the Klein-Gordon operator on it yields a delta function), so that the regularity condition demands \(C_4 = 0\). For \(\mu = 0\), the solution of the radial equation which is bounded at infinity is

\[
R(r) = C_1 (r - 2M)^{-1/2-\nu} F \left( \frac{1}{2} + \nu + i\omega r_0, \frac{1}{2} + \nu - i\omega r_0, 1 + 2\nu; \frac{-2M}{r - 2M} \right),
\]
with $F$ a hypergeometric function. This diverges logarithmically for $r \to 0$, so that there are no regular normalisable solutions. So in this case one has again the same continuous spectrum as in the linear dilaton case, now describing normal scattering by the timelike singularity $r = 0$. On the other hand, for $\mu \neq 0$, the normalisability and regularity conditions lead to the two constraints $C_2 = 0$ in (6.14) and $C_4 = 0$, so that the energy spectrum is discrete. Our numerical computations show that, at least for small values of the angular momentum $l$, the low-lying energy levels are approximately evenly spaced.

b) For $a > 0$, $M < a$, the approximate radial equation near $r = 0$ is

$$a^2 R''(r) - \lambda^2 R(r) \simeq 0,$$

with $\lambda = \sqrt{l(l+1) - m^2}$. This is solved by a series in powers of $r$. In the present case $R = r$ is a singular solution of the Klein-Gordon equation (6.21) (again, the action on it of the Klein-Gordon operator gives a delta function), so that the regularity condition now reads

$$R'(0) = 0.$$ (6.37)

For $\mu \neq 0$ we again have two boundary conditions, $C_2 = 0$ and (6.37), leading to a discrete spectrum. For $\mu = 0$, the energy spectrum contains as before the continuous component $\omega \in (l + 1/2)/r_0 , +\infty]$, plus possibly a finite number of discrete eigenvalues in the range $[0 , (l + 1/2)/r_0]$.

In this case ($M < a, \mu = 0$), the radial wave function can be obtained in closed form. After the change of variables

$$r = M + cx \quad \text{with} \quad c = \sqrt{a^2 - M^2},$$ (6.38)

the radial equation (6.22) becomes

$$\partial_x \left( (1 + x^2) \partial_x R \right) + \left( -l(l+1) + \frac{(\omega' + m' x)^2}{1 + x^2} \right) R = 0$$ (6.39)

where

$$\omega' = \frac{Mr_0 \omega - am}{c}, \quad m' = r_0 \omega.$$ (6.40)

Remarkably (6.39) is identical to the massless radial equation in rotating Bertotti-Robinson spacetime (Ref.[35], Eq.(3.13) with $\mu = 0$), though the boundary conditions are quite different here. Putting

$$x \equiv \frac{\xi + 1}{\xi - 1},$$ (6.41)

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\(6.39\) reduces to the hypergeometric type equation

\[ \xi^2 (\xi - 1)^2 \partial^2_{\xi} R + \xi (\xi - 1)^2 \partial_{\xi} R - l(l+1)\xi R - \frac{1}{4}(\omega_+ - \omega_-)^2 R = 0, \]  

(6.42)

with \(\omega_\pm = \omega' \pm im'\). The solution of this equation which is bounded at infinity is

\[ R = C_1 \xi^{(\omega' - im')/2}(1 - \xi)^{1/2 + \nu} F \left( \frac{1}{2} + \nu + \omega', \frac{1}{2} + \nu - im', 1 + 2\nu; 1 - \xi \right). \]  

(6.43)

The regularity condition \((6.37)\) may then be solved numerically for given values of the quantum numbers \((l, m)\). We find that the number of discrete energy levels is equal to the azimuthal quantum number \(m\), the sign of the energy \(\omega\) being opposite to that of \(m\).

7 Stability of the static solution

Having established the superradiant classical instability of the rotating linear dilaton black holes, we now investigate linearization stability of the static solution \((2.16)-(2.17)\). A potentially dangerous perturbation is the \(s\)-mode. An electric time-dependent spherically-symmetric solution of the EMDA field equations may be written as

\[ ds^2 = e^{2\alpha} dt^2 - e^{2\alpha} dr^2 - e^{2\beta} d\Omega^2, \]

\[ F = \frac{r_0}{\sqrt{2}} e^{\alpha - 2\beta + \gamma + 2\phi} dr \wedge dt, \]

(7.1)

(7.2)

where the metric functions \(\alpha, \beta, \gamma\) and the dilaton \(\phi\) depend on \(r\) and \(t\). We assume that these fields are small perturbations of the static background fields of \((2.16)-(2.17)\),

\[ \gamma(r, t) = \gamma_0(r) + \epsilon \gamma_1(r, t), \quad \alpha(r, t) = \alpha_0(r) + \epsilon \alpha_1(r, t) \]

\[ \beta(r, t) = \beta_0(r) + \epsilon \beta_1(r, t), \quad \phi(r, t) = \phi_0(r) + \epsilon \phi_1(r, t). \]  

(7.3)

Choosing the gauge \(\beta_1(r, t) = 0\) (that is, \(e^{2\beta} = r_0 r\)), we obtain for the linearized Klein-Gordon equation and the linearized Einstein equations for the components \(R_2^2\) and \(R_{01}\):

\[ r_0^2 \frac{\ddot{\phi}_1}{(r - 2M)^2} - \phi'' - \frac{2(r - M)}{r(r - 2M)} \phi' + \frac{\alpha_1 - \gamma_1}{2r} + \frac{\phi_1 + \alpha_1}{r(r - 2M)} = 0, \]  

(7.4)

\[ \frac{\alpha_1 - \gamma_1}{2r} + \frac{\alpha_1 - \phi_1}{r(r - 2M)} = 0, \]  

(7.5)

\[ \dot{\alpha}_1 - \dot{\phi}_1 = 0, \]  

(7.6)

where \(\dot{\cdot} = \partial_t\) and \(\cdot = \partial_r\). Integrating the constraint equation \((7.6)\)

\[ \alpha_1 = \phi_1, \]  

(7.7)
and eliminating the perturbations $\alpha_1$ and $\gamma_1$ between the equations (7.4), (7.7) and (7.5) we arrive at the effective Klein-Gordon equation

$$\frac{r_0^2 \phi''_1}{(r - 2M)^2} - \phi''_1 - \frac{2(r - M)}{r(r - 2M)} \phi'_1 + \frac{2}{r(r - 2M)} \phi_1 = 0. \quad (7.8)$$

The resulting equation for growing modes $\phi_1(r, t) = e^{kt} \phi_1(r)$

$$-\phi''_1 - \frac{2(r - M)}{r(r - 2M)} \phi'_1 + \left( \frac{2}{r(r - 2M)} + \frac{k^2 r_0^2}{(r - 2M)^2} \right) \phi_1 = 0, \quad (7.9)$$

is seen to be identical to the radial Klein-Gordon equation (6.22) (with $a = 0$) for a massless scalar field with the particular value of the orbital quantum number $l = 1$ and imaginary frequency $\omega = ik$.

Introducing the new radial variable

$$r_* = \frac{1}{2M} \ln \frac{r - 2M}{r}, \quad (7.10)$$

we obtain the equation

$$-\phi''_* + (2r(r - 2M) + k^2 r_0^2 r^2) \phi = 0 \quad (7.11)$$

For real $k$ the effective potential in (7.11) is positive definite for all $r > 2M$ (in the black hole case $M > 0$), or for all $r$ (in the singular case $M \leq 0$), so that there are no bounded solutions. Therefore, the static linear dilaton black holes are linearly stable in the $s$-sector (and presumably stable with respect to other modes as well).

## 8 Conclusion

We have constructed new exact solutions to the Einstein-Maxwell-dilaton-axion system in four dimensions describing black holes which asymptote to the linear dilaton background. Although these solutions are not asymptotically flat and break all supersymmetries, they are interesting since their asymptotic region is closely related to an exact solution of string theory. Non-rotating solutions may be obtained in different ways, the simplest being to pass to the near-horizon limit of a “parent” BPS dilaton black hole. Using the $Sp(4, R)$ duality of the stationary EMDA system we were also able to find the rotating (and NUT) generalizations of the above solutions. The rotating linear dilaton black hole has some similarity with the Kerr one, but is very different asymptotically. In the limit of a vanishing mass parameter the rotating holes tend to the supersymmetric dilaton-axion generalization of the Israel-Wilson-Perjès solutions.
Rather unexpectedly, our new black holes have a very simple counterpart in six dimensions, where they can be uplifted using a non-local procedure involving dualization. Actually it is the smeared vacuum five-dimensional black hole with two equal rotation parameters. The spacetime symmetry of the six-dimensional solutions has a component inherited from the internal symmetry of the stationary EMDA system.

The physical mass and angular momentum were computed using the Brown-York definition of quasilocal gravitational energy-momentum tensor, and shown to satisfy the first law of thermodynamics consistent with the definition of the Hawking temperature via the surface gravity and of the geometric entropy as the quarter of the horizon area. A remarkable feature of the static linear dilaton black hole is that the temperature is constant (depending on a parameter of the background) and the mass is independent of it, so the heat capacity is zero. The rotating black hole possesses an ergosphere with an associated superradiance phenomenon. Analysing the asymptotic behavior of the Klein-Gordon modes, one finds that the superradiant modes do not propagate to infinity, but are confined in some region outside the black hole. This leads to an exponential growth of the superradiant modes, equivalent to classical instability of the rotating solution. The remaining static black hole is shown to be stable with respect to spherical perturbations. We have also found that in the case of a naked singularity the Klein-Gordon operator has regions of discrete spectrum. It is expected that ten-dimensional brane generalizations of our solutions exist as well, and presumably have holographic interpretation as thermal states of the little string theory.

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Figure 1: Penrose diagrams of the static spacetimes (2.16) with $b < 0$, $b = 0$ and $b > 0$. 