Simplifying the algebra of first class constraints, $SO(3)$ as an example

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We discuss the problem of non abelian constrained systems and give a systematic method to convert them to an abelian system. Our method is based on solutions of the differential equations due to the algebra of first class constraints. We suggest that multiplicative constraints should be regularized at each step of calculations. A first class constrained system with $SO(3)$ algebra is studied as an example. We show that it is possible to abelianize this system locally.

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I. INTRODUCTION

The general formalism of constrained systems seems sometimes very complicated. The reason is the complicated and extensive algebra of constraints. Consider a first class system given by constraints $\phi_a$, and the canonical Hamiltonian $H_c$. The most general form of the algebra of Poisson brackets is [1]:

$$\{\phi_a, \phi_b\} = C^c_{ab} \phi_c$$  \hspace{1cm} (I.1)

$$\{\phi_a, H_c\} = V^b_a \phi_b$$

The coefficients $C^c_{ab}$ and $V^b_a$ are called structure functions. In general these coefficients may be functions of phase space variables $(q,p)$. If one wishes to keep the track of levels of consistency, additional labels showing the level should be assigned to the constraints and the algebra would be much more complicated. This complication causes so many difficulties in proving general statements in the context of constrained systems and makes this field of study difficult to follow. However, in the real models of gauge systems we never encounter a problem in which the complete set of structure functions $C^c_{ab}$ and $V^b_a$ are present and all of them depend arbitrarily on $(q,p)$. The simplest possibility is $C^c_{ab} = 0$ and $V^b_a = \text{cons.}$. Such systems are called abelian first class constrained systems. As we will see in the next section this is the case for quadratic Lagrangians (with respect to velocities and coordinates) which include most physical models. A non abelian system requires a Lagrangian or Hamiltonian with higher powers, or more complicated functional dependence on the corresponding variables.

On the other hand, it is well known [2] that the choice of constraints in not unique. In other words, different sets of constraints may describe the same constraint surface. We say that the two sets of constraints $\phi_a(q,p)$, $a = 1, \cdots m$ and $\phi'_a(q,p)$, $a' = 1, \cdots m'$ are equivalent if $\phi_a(q,p) = 0 \iff \phi'_a(q,p) = 0$. Then it is obvious that in general, one can write:

$$\phi_a(q,p) = \sum_{a'=1}^{m'} M^a_{aa'} \phi_{a'}(q,p) \hspace{1cm} a = 1, \cdots m. \hspace{1cm} (I.2)$$

A system is called reducible if it can be converted to another one with less number of constraints. Transforming from the set $\phi_a(q,p)$ to the set $\phi_{a'}(q,p)$ is called a "redefinition" of constraints.

We wish to use the redefinition process to make the algebra of the constraints as simple as possible. The best situation is one with abelian algebra. Then the system is said to be abelianized. We can also make use of the "canonical transformations" (CT’s) to abelianize a system whenever necessary. The first tool, i.e. redefinition of the constraints, changes the variables describing the constraint surface only, while the second one, i.e. CT, change the coordinates of the whole phase space canonically. It should be emphasized that the Poisson brackets are invariant under CT, while the redefinition procedure may lead to a different algebra of Poisson brackets. In fact, this is our justification to change a non abelian system to an abelian one.

Converting a system of first class constraints to an abelian system has so many advantages. In fact in any analysis based on the constraint structure of a gauge system one encounters the complicated algebra of Poisson brackets of first class constraints and one can not show the essential features of the problem in clear formulas. For example a closed formula for the generating function of gauge transformations in terms of the constraints, is not given so far. There have been proposed only some rules and instructions in this regard for the most general case of structure functions with arbitrary dependence on $(q,p)$ [1, 3].
There is a similar situation for the BRST charge which is needed to quantize a gauge system. In fact, this charge can be written as an infinite expansion in terms of recursive Poisson brackets of structure functions which in general make an open algebra which does not terminate. For an abelian system, on the other hand, besides clarity and simplicity in understanding the structure of physical as well as gauged degrees of freedom, one is able to write down the generator of gauge transformations in a closed form and the expansion of the BRST charge terminates after the first term.

As we will see through the paper, non abelian constraints may stem from the choice of coordinates for description of constraint surface. In this view, the abelianization procedure is a redefinition and purgation process for the constraints which make it possible to describe the constraint surface with a set of suitable (i.e. commuting) coordinates.

Our aim in this paper is to show that a wide number of first class constrained system can in principle be abelianized, at least locally. In fact, one can say that the origin of non abelianity stem from bad choice of the canonical coordinates of the whole phase space as well as the variables describing the constraint surface. Therefore, the abelianization means that one tries to find the suitable coordinates in which the algebra of constraints is abelian.

Local abelianization has been shown to be possible through solving constraint equations, \( \phi_a = 0 \) to find a number of coordinates \( \xi_a \) in terms of other variables \( \xi \) as:

\[
\xi_a = f_a(\xi) \quad (I.3)
\]

Then the new constraints \( \psi_a = \xi_a - f_a(\xi) \) have Poisson brackets which are independent of \( \xi \). Hence, the only way that the Poisson brackets may vanish on the constraint surface \( \psi_a = 0 \) is that they vanish strongly. In it is argued that if one maps each constraint to the surface of other constraints they would be abelianized. However, it is asserted in that for gauge systems such as \( SO(3) \) the maximality conditions is violated, hence the sufficient condition for \( SO(3) \) to be abelianizable is not satisfied. Another method is proposed for abelianization which is based on finding a complicated solution for the matrix \( M \) in such that the new set of constraints are abelian.

In this paper we try to study the problem of abelianization in a systematic way from the point of view of differential equations coming from the algebra of constraints. Our method is based on finding suitable coordinates to describe the constraint surface. We do this by solving differential equations due to Poisson brackets of constraints with one momentum constraint. We will find that in this way one would naturally lead to simple coordinates of the constraint surface. This method will be discussed in details in section III. Before that we will discuss in the next section some general aspects of abelian and non abelian nature of first class systems. We will also discuss the problem of regularity of the constraints. We suggest that the multiplicative constraints should be regularized before abelianization. Section IV denotes detailed calculations concerning abelianization of a first class system with \( SO(3) \) algebra. This important example shows the general features of the abelianization procedure. In section V we give our conclusions.

II. HOW NON ABELIAN CONSTRAINTS MAY HAPPEN?

Let us first consider a simple example to see in what sense a non abelian system of first class constraints may emerge. Suppose, in a system with \( q_1 \) and \( q_2 \) as coordinates, we are given two first class constraints

\[
\phi_1 = p_1 e^{\alpha q_2} \quad \phi_2 = p_2 e^{-\beta q_1} \quad (II.4)
\]

where \( \alpha \) and \( \beta \) are constants. Clearly we have

\[
\{ \phi_1, \phi_2 \} = (\alpha e^{-\beta q_1}) \phi_1 + (\beta e^{\alpha q_2}) \phi_2 \quad (II.5)
\]

which exhibits the non abelian feature of the system. It is, however, obvious that the constraints surface \( \phi_1 = 0 \) and \( \phi_2 = 0 \) is equivalent to the surface described by \( p_1 = 0 \) and \( p_2 \), since the exponentials does not vanish in the finite range of their arguments. Clearly, the constraints \( p_1 \) and \( p_2 \) are abelian. In general the situation is not so obvious, and the algebraic structure of the Poisson brackets may be so complicated that one is not able to recognize the best and simplest phase space coordinates describing the constraints surface.

It is also possible to inspect some features of non abelian systems by power counting. Suppose \( \phi_i(z^m) \) is any constraint which can be written as a polynomial of order \( m \) with respect to phase space coordinates \( z_\mu, \mu = 1, \cdots, 2N \). For a first class system the algebra of Poisson brackets reads

\[
\{ \phi_i(z^m), \phi_j(z^n) \} = a_{ij}^k(z^r) \phi_k(z^s). \quad (II.6)
\]

Since Poisson bracket requires two times of differentiation, we should have

\[
m + n - 2 = r + s. \quad (II.7)
\]

If the constraints are linear with respect to \( z_\mu \), i.e. \( m = n = 1 \), then a first class system can be achieved just for \( a_{ij}^k = 0 \). In other words linear first class constraints are essentially abelian. For quadratic Lagrangians, which is the case for most physical systems, the primary constraints which emerge due to singularity of Hessian (the matrix of second derivatives with respect to velocities), are necessarily linear. Since the Hamiltonian is also quadratic, the secondary constraints at any level would be linear, too. So, for the wide class of gauge systems with quadratic Lagrangians the system is abelian by itself. The above analysis shows that the problem of abelianization maybe converted to the problem of linearization. In other words if we are able to abelianize...
a first class system this means that there can be found suitable coordinates in which the constraint surface is described by constant values of some phase space coordinates. Specially we can choose a basis in which the constraints are some momenta.

For example a constrained system given by $\phi_1 = xp_z + y p_y$, $\phi_2 = \frac{1}{2}p_z^2$ and $\phi_3 = \frac{1}{2}p_y^2$ obeys the non abelian algebra $\{\phi_1, \phi_2\} = \phi_2$ and $\{\phi_1, \phi_3\} = \phi_3$. This system reduces to the constraints $p_z$ and $p_y$, which are linear as well as abelian.

Coming back to the Eq's. (II.6) and (II.7), assume a system in which the constraints are quadratic homogeneous functions of $z_\mu$ which leads to $r = 0$. In other words, for quadratic constraints we may have a non abelian closed algebra of Poisson brackets only with constant structure functions $a^f_{ij}$. For example in a system with $x$, $y$ and $z$ as coordinates, the quadratic constraints $\phi_1 = yp_x - zp_y$, $\phi_2 = zp_x - xp_z$ and $\phi_3 = xp_y - yp_x$ exhibit the $SO(3)$ algebra. Such systems require cubic terms in Lagrangian and/or Hamiltonian, assuming the primary constraints are linear. Yang-Mils models fall in this category. More complicated examples in which constraints of different powers constitute a closed algebra this category. More complicated examples in which constraints of different powers constitute a closed algebra.

It is clear that the constraints surface described by $\phi_1 \cdots \phi_6$ in (II.8) is the same as given by $\psi_1 \equiv x \approx 0$, $\psi_2 \equiv p_x \approx 0$ and $\psi_3 \equiv p_y \approx 0$ which is a mixed system composed of a pair of second class constraints $(x, p_x)$ together with the first class constraint $\psi_3$. Such systems are recognized in the literature $[10]$ as irregular constraints, which are defined as systems of constraints whose gradients vanish on the constraint surface. More precisely their Jacobian $\frac{\partial \phi^i}{\partial \xi^j}$ (where $\xi^j$ are the phase space coordinates) is not full rank on the constraint surface.

If the non abelian algebra of the first class constraints has originated from the irregular nature of the constraints, then the most direct way toward an abelian equivalent system is regularizing the system, i.e. replacing the multiplicative constraints by the equivalent linear ones. For this reason we should classify multiplicative constraints into three different categories:

1. Nonlinear constraints of the form $|f(q,p)|^k \approx 0$. Such a constraint should be linearized, i.e. replaced by linear expression $f(q,p) \approx 0$. Our evidence for this replacement is that a constraint of the form $|f(q,p)|^k \approx 0$ has no meaning other than $f(q,p) \approx 0$. This suggestion does not mean any change in the Hamiltonian or Lagrangian (as in ref. [9]).

2. Bifurcating systems of the form $f(q,p)g(q,p) \approx 0$. In this case the constraint surface obviously “bifurcates” into two different branches $f(q,p) \approx 0$ and $g(q,p) \approx 0$. These systems should be considered as the union of different branching $f(q,p) \approx 0$ and $g(q,p) \approx 0$. Each branch should be treated individually. For example, the system given by $\phi_1 = x^2$, $\phi_2 = xp_x$ and $\phi_3 = xy$ may be regularized to $\psi_1 = x$ (first class), or $\psi_1 = x, \psi_2 = p_x$ (second class) or $\psi'_1 = y, \psi_2 = p_x$ (first class). In each branch the dynamics of the system is different from the others.

3. Reducible systems of the form $f(q,p)G(q,p) \approx 0$ where $G(q,p)$ has no root. One can replace such a constraint with the simpler one $f(q,p) \approx 0$. This reduction is in fact, the essence of the abelianization procedure explained in the next section.

As the consequence of three kinds of simplifications given above the number of constraints, as well as their algebra, may change. In most cases the regularization and reduction procedures simplifies the algebra of constraints as the above examples show. However, the more important point is clarifying the first or second class nature of the system. For example the system given in (II.8) is not in reality a first class system. Besides simplifying the algebra, the important point is that the system given by (II.8) is partly second class (in the $x, p_x$ plane). Conclusively, given a set of irregular first class constraints satisfying a non abelian algebra, there is no guarantee that the system remains first class after regularization.

One problem that makes the above analysis more difficult is the possibility of combining the multiplicative constraints in the form of more complicated functions. Another problem is that the simple multiplicative functions may be hidden by changing the variables describing the phase space or the constraint surface. Such difficulties may be resolved during the process of abelianization in
which one tries to find the most suitable coordinates for describing the constraint surface.

As a prescription toward an abelian algebra we propose to regularize a system of multiplicative constraints in advance. Then the remaining algebra of first class constraints, if it is still non abelian, can be abelianized following the method of the next section. Therefore, in the following we assume that the system is already regularized. We restrict ourselves to a pure and truly first class system and give a method to abelianize the algebra of Poisson brackets. By truly first class system we mean a system that remains first class in every description of the constraint surface.

### III. ABELIANIZATION

Suppose we are given a set of first class constraints \( \phi_i(q,p), i = 1, \cdots, m \), satisfying the algebra

\[
\{ \phi_i, \phi_j \} = \alpha_{ijk} \phi_k \tag{III.10}
\]

where in general \( \alpha_{ijk} \) depends on \( (q,p) \). We wish to find an equivalent set of constraints \( \tilde{\phi}_r(q,p) \) where

\[
\phi_i(q,p) = 0 \iff \tilde{\phi}_r(q,p) = 0 \tag{III.11}
\]

assuming that the system remains first class under the reduction \( \phi_i \rightarrow \tilde{\phi}_r \). If we are succeeded in this regard, then it is in principle possible to find a suitable canonical transformation which transforms all the \( \tilde{\phi}_r \) to a set of momenta, i.e.

\[
\tilde{\phi}_r \rightarrow P_r. \tag{III.12}
\]

In the first step we can choose the first constraint momentum \( P_1 \) the same as \( \phi_1(q,p) \). It is justified in principle there exists a CT which does this task. Therefore, suppose under the desired CT we have

\[
P_1 \equiv \phi_1(q,p). \tag{III.13}
\]

Suppose \( Q_1 \) is the coordinate conjugate to \( P_1 \) and the other coordinates change to \( \tilde{q}_2, \tilde{p}_2, \cdots \) under the above CT. In practice \( Q_1 \) should be determined by solving the differential equation due to \( \{Q,P\} = 1 \) and \( \tilde{q}, \tilde{p} \) are derived after determining the corresponding generating function of CT, as we will show in the example of the next section. In the new coordinates all the remaining constraints may change, so as to say

\[
\phi_1(q,p) \quad \xrightarrow{P_1} \quad \tilde{\phi}_2(q,p) \rightarrow \phi_2(Q_1, P_1, \tilde{q}, \tilde{p}) \quad \vdots \quad P_1 \tag{III.14}
\]

We first show that \( P_1 \) should be absent in the remaining constraints \( \phi_i(Q_1, P_1, \tilde{q}, \tilde{p}) \). The reason is that in general we can expand \( \phi_i \) such that:

\[
\phi_i(Q_1, P_1, \tilde{q}, \tilde{p}) = \phi_i(Q_1, 0, \tilde{q}, \tilde{p}) + P_1 f_i(Q_1, P_1, \tilde{q}, \tilde{p}) \tag{III.15}
\]

Note that we should exclude irregular cases where the gradient of constraints go to infinity at the constraint surfaces; therefore the Taylor expansion always makes sense. On the constraint surface \( P_1 = 0 \). So we have the equivalency:

\[
P_1 \phi_2(Q_1, P_1, \tilde{q}, \tilde{p}) \rightarrow \phi_2(Q_1, 0, \tilde{q}, \tilde{p}) = \phi_2(Q_1, \tilde{q}, \tilde{p}) \quad \vdots \quad P_1 \tag{III.16}
\]

Now we are at the point to consider the powerful requirement that the constraints are first class. For example the Poisson brackets \( \{\phi_i(Q_1, \tilde{q}, \tilde{p}), P_1\} \) which vanishes on the constraint surface described yet in Eq. (III.16):

\[
\frac{\partial \phi_i(Q_1, \tilde{q}, \tilde{p})}{\partial Q_1} = \sum_{j=2}^{m} C_{ij}(Q_1, \tilde{q}, \tilde{p}) \phi_j(Q_1, \tilde{q}, \tilde{p}) \quad i = 2, \cdots, m. \tag{III.17}
\]

Since \( P_1 \) is absent from LHS of the above equation, no term proportional to \( P_1 \) has been written in the RHS. This system of coupled ordinary linear first order differential equations has an analytic solution in the case where \( C_{ij} \)'s are independent of \( Q_1 \) which reads in the matrix notation as

\[
\phi(Q_1, \tilde{q}, \tilde{p}) = e^{CQ_1} \phi(0, \tilde{q}, \tilde{p}) \tag{III.18}
\]

where \( C \) is a \((m-1) \times (m-1) \) matrix with elements \( C_{ij}(\tilde{q}, \tilde{p}) \). This is the case for models in which the structure functions are constant. Since the exponential part in (III.18) can not vanish for any finite value of \( Q_1 \), the necessary and sufficient condition for vanishing of \( \phi_i(Q_1, \tilde{q}, \tilde{p}) \) is vanishing of \( \phi_i(0, \tilde{q}, \tilde{p}) \) which we can rename them as \( \eta_i(\tilde{q}, \tilde{p}) \). So, the reduction shown in Eq. (III.16) goes one step further to

\[
P_1 \phi_2(Q_1, P_1, \tilde{q}, \tilde{p}) \rightarrow \phi_2(0, \tilde{q}, \tilde{p}) = \eta_2(\tilde{q}, \tilde{p}). \tag{III.19}
\]

The above case with constant \( C_{ij} \) gives the general feature of the problem. The result is that given \( P_1 \) as a constraint, the remaining constraints after redefinition should be independent of \( Q_1 \). In other words the \( Q_1 \)-dependent part of them has no root and can be omitted. This feature can also be shown for the general case through the following argument. We come back to the system of linear ODE’s given by Eq. (III.17). From the theory of differential equations it is well-known that the general solutions should contain \( m-1 \) constants, which can be chosen as the initial values \( \phi_i(0, \tilde{q}, \tilde{p}) \). In fact, it can be shown that the general solution \( \phi_i(Q_1, \tilde{q}, \tilde{p}) \) is linear with respect to the initial values \( \phi_i(0, \tilde{q}, \tilde{p}) \). Just to remind the reader, the proof can be achieved by dividing the interval \((0, Q_1)\) into an infinite number of segments \( \delta Q_1 \); then the linear differential equations (III.17).
show that after \( n \) steps we have
\[
\phi_i(n\delta Q_1, \tilde{q}, \tilde{p}) = (1 + C((n - 1)\delta Q_1, \tilde{q}, \tilde{p}))_{i}\phi_i(0, \tilde{q}, \tilde{p})
\]
\[
\cdots (1 + C(0, \tilde{q}, \tilde{p}))_{n}\phi_i(0, \tilde{q}, \tilde{p}).
\]
(III.20)

Although the product of parentheses above can not be exhibited by a simple exponential (as in Eq. (III.18)), the linearity of the final values of \( \phi_i(0, \tilde{q}, \tilde{p}) \) is established. In this way one can propose the general form of the solution of Eqs. (III.17) as
\[
\phi_i(Q_1, \tilde{q}, \tilde{p}) = \sum_j \phi_j^{(j)}(Q_1, \tilde{q}, \tilde{p})\eta_j(\tilde{q}, \tilde{p})
\]
(III.21)

where
\[
\eta_j(\tilde{q}, \tilde{p}) = \phi_j(0, \tilde{q}, \tilde{p}).
\]
(III.22)

The functions \( \phi_j^{(j)}(Q_1, \tilde{q}, \tilde{p}) \) are special solutions of Eqs. (III.17) with the initial conditions
\[
\phi_j^{(j)}(0, \tilde{q}, \tilde{p}) = \delta^j_i \quad i, j = 2, \cdots m - 1.
\]
(III.23)

The general solution (III.21) can be viewed as an expansion in terms of special solution \( \phi_j^{(j)}(Q_1, \tilde{q}, \tilde{p}) \) of the ODE’s (III.17). From this point of view the functions \( \eta_j(\tilde{q}, \tilde{p}) \) can be interpreted as the constant (with respect to \( Q_1 \)) coefficients of expansion. It should be noticed that we did not solve practically the differential equations (III.17) which are resulted from the algebra of the constraints. The point is that the constraints \( \phi_2, \cdots, \phi_m \) in terms of coordinates \( (Q_1, Q_2, \cdots, Q_m) \) automatically should appear in the form of solutions of Eqs. (III.17). This point will be seen clearly in the example given in the next section.

Now we claim that the constraint surface given by \( \phi_i(Q_1, \tilde{q}, \tilde{p}) \) is the same as one given by \( \eta_i(\tilde{q}, \tilde{p}) \). It is obvious from (III.21) that \( \eta_i(\tilde{q}, \tilde{p}) = 0 \) give rise to \( \phi_i(Q_1, \tilde{q}, \tilde{p}) = 0 \). What about the inverse deduction? Our assertion here is that if in some direction the constraint surface can not be described unless some definite function of \( Q_1 \) say \( f(Q_1, \tilde{q}, \tilde{p}) \) vanishes, then the equation \( f(Q_1, \tilde{q}, \tilde{p}) = 0 \) in principle be solved to give \( Q_1 \) as \( Q_1 = g(\tilde{q}, \tilde{p}) \). Then \( P_1 \) and \( Q_1 - g(\tilde{q}, \tilde{p}) \) constitute a second class constrained system, which we have excluded it from our consideration. Hence, the necessary and sufficient condition for \( \phi_i(Q_1, \tilde{q}, \tilde{p}) = 0 \) is \( \eta_i(\tilde{q}, \tilde{p}) = 0 \). So we come to a noticeable result that the redefinition procedure has brought us to the set of equivalent constraints \( P_1, \eta_2(\tilde{q}, \tilde{p}), \cdots, \eta_m(\tilde{q}, \tilde{p}) \), with the property that \( P_1 \) commutes with all other constraints. By this procedure we have decoupled the constraint \( \phi_1 = P_1 \) from others. Decoupling is done by purging other constraints from canonical conjugate pair \( (Q_1, P_1) \).

Now we can restrict our attention to constraints \( \eta_2, \cdots, \eta_m \) which are defined in a smaller phase space \( (\tilde{q}, \tilde{p}) \) where the canonical pair \( (Q_1, P_1) \) are no longer present. Any canonical transformation in the \( (\tilde{q}, \tilde{p}) \) subspace does not affect the subspace \( (Q_1, P_1) \). Therefore, one can in principle repeat the same procedure once more and in this time assumes that \( \eta_2(\tilde{q}, \tilde{p}) \) is the momentum \( P_2 \) in some suitable coordinates. In this way after several stages all the constraints would be reduced to a set of momenta.

Note should be added that the number of constraints may be changed, in fact reduced, at any stage of the above process of abelianization. The reason is that, for example in the first stage, linear independence of the constraints \( \phi_i(Q_1, \tilde{q}, \tilde{p}) \) does not necessarily requires that \( \phi_1(0, \tilde{q}, \tilde{p}) \) are linearly independent. Hence, from Eq. (III.22) the number of independent \( \eta_i(\tilde{q}, \tilde{p}) \) may be less than \( \phi_1(Q_1, \tilde{q}, \tilde{p}) \).

As a concrete example consider two first class non abelian constraints \( \phi_1 \) and \( \phi_2 \) with the algebra
\[
\{\phi_1, \phi_2\} = \alpha\phi_1 + \beta\phi_2.
\]
(III.24)

By a canonical transformation we map the constraint \( \phi_1 \) to momentum \( P_1 \). After projection \( \phi_2 \) on the surface \( P_1 = 0 \), the algebra (III.24) turns to
\[
\{P_1, \phi_2\} = \beta\phi_2(Q_1, \tilde{q}, \tilde{p})
\]
(III.25)

The constraint \( \phi_2 \) can be found from the differential equation
\[
\frac{\partial\phi_2}{\partial Q_1} = -\beta\phi_2
\]
(III.26)
as
\[
\phi_2(Q_1, \tilde{q}, \tilde{p}) = \eta(\tilde{q}, \tilde{p}) \exp(-\int_{Q_1}^{Q_1} \beta dQ_1)
\]
(III.27)
where \( \eta(\tilde{q}, \tilde{p}) = \phi_2(0, \tilde{q}, \tilde{p}) \). This is a realization of the solutions given in Eq. (III.21) for the general case. Since the exponential function does not vanish for finite values of its argument, vanishing of \( \phi_2 \) could be only due to \( \eta(\tilde{q}, \tilde{p}) \). The constraints \( \phi_1 \) and \( \phi_2 \) are equivalent to \( P_1 \) and \( \eta(\tilde{q}, \tilde{p}) \), where
\[
\{P_1, \eta\} = 0
\]
(III.28)

Then, we can make a CT to canonical variables in which \( \eta(\tilde{q}, \tilde{p}) = P_2 \). One may wonder if \( \beta(Q_1, q, p) \) is such that \( \int \beta dQ_1 = \ln f(Q_1, q, p) \) and the equation \( f(Q_1, q, p) = 0 \) has some roots, then one may no longer exclude vanishing of the exponential part in (III.27). If this the case, the equivalent constraints are \( P_1 \) and \( f(Q_1, q, p) \) where \( f(Q_1, q, p) \) can be solved for \( Q_1 \). This leads to a second class system which has been excluded before.

IV. ABELIANIZATION OF \( SO(3) \) CONSTRAINTS

A famous non abelian algebra is the angular momentum algebra in three dimensional configuration space.
We may assume that for a rotational invariant Hamiltonian, $L_x$, $L_y$, and $L_z$ are given as primary constraints. Consistency condition of primary constraints then gives no further secondary constraints and the set of constraints terminate here. It is also possible to consider more realistic examples in which the angular momentum constraints terminate here. It is also possible to consider the most general case of constraint

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\[
L = \frac{1}{2} \dot{\mathbf{X}}^2 - V(\mathbf{X}^2) - \xi L \tag{IV.29}
\]

where $\mathbf{X} = (x, y, z)$ and $\xi = (\xi_x, \xi_y, \xi_z)$ constitute a six dimensional configuration space, in which $L_i = \epsilon_{ijk} x_j \dot{x}_k$. In phase space $\pi_x$, $\pi_y$, and $\pi_z$, the momenta conjugate to $\xi_x$, $\xi_y$, and $\xi_z$, are primary constraints and the total Hamiltonian reads

\[
H_T = \frac{1}{2} \mathbf{P}^2 + V(\mathbf{X}^2) + \xi L + \lambda \pi \tag{IV.30}
\]

where $\mathbf{P} = (P_x, P_y, P_z)$ represents the momenta conjugate to $\mathbf{X}$, $\lambda \equiv (\lambda_x, \lambda_y, \lambda_z)$ shows Lagrange multipliers and $L_i = \epsilon_{ijk} x_j \dot{x}_k$. The consistency conditions of primary constraints $\pi_i$ give secondary constraints $L_i$ and no further constraint emerges from the consistency of $L_i$. First level constraints are abelian while the second level constraints $L_i$ obey the non abelian $SO(3)$ algebra with constant structure functions $\epsilon_{ijk}$, i.e.

\[
\{L_i, L_j\} = \epsilon_{ijk} L_k \quad i, j = 1, 2, 3. \tag{IV.31}
\]

It is also possible to get the $SO(3)$ algebra from the Lagrangian

\[
L = \frac{1}{2} \dot{\mathbf{X}}^2 - V(\mathbf{X}^2) - \epsilon \omega \mathbf{L} \times \mathbf{L} - \mathbf{L} \times \mathbf{L} \tag{IV.32}
\]

where $\omega$ is a variable. Consistency of $\omega$ gives $L_x$, $L_y$ and $L_z$ respectively as the second, third and forth level constraints.

Now let us go through the abelianization procedure of the $SO(3)$ algebra of constraints. As stated in the previous section we should first find a CT that transforms for example $L_x = y p_z - z p_y$ to a momentum $P_1$. The conjugate coordinate $Q_1$ should be determined such that

\[
\{Q_1, P_i\} = 1. \tag{IV.33}
\]

A possible solution for $Q_1$ is

\[
Q_1 = \tan^{-1} \left( \frac{\dot{z}}{\dot{y}} \right). \tag{IV.34}
\]

As is apparent, $P_1$ and $Q_1$ are functions of subspace $(y, z; p_y, p_z)$. Hence, we can exclude the subspace $(x, p_x)$. Reminding the standard method [11] for extracting a CT from a generating function, the following generator can be used

\[
F(y, z, P_1, P_2) = P_1 \tan^{-1} \left( \frac{p_x}{p_y} \right) + P_2 f(y, z). \tag{IV.35}
\]

Transformation relations then gives $Q_1$ as in [IV.34] and $Q_2 = f(y, z)$. Imposing the task $y p_z - z p_y = P_1$ on the relations $p_y = \frac{\partial f}{\partial y}$ and $p_z = \frac{\partial f}{\partial z}$ also gives $y \frac{\partial f}{\partial y} - z \frac{\partial f}{\partial z} = 0$. In this way the canonical pair $(Q_2, P_2)$ can be given as

\[
Q_2 = \frac{1}{2} \ln(y^2 + z^2) \quad P_2 = y p_y + z p_z. \tag{IV.36}
\]

Renaming the variables $(Q_1, P_1; Q_2, P_2)$ as $(\phi, p_\phi; \psi, p_\psi)$ the old variables can be written in terms of the new ones as

\[
x = X \quad y = e^\psi \cos \phi \quad p_y = e^{-\psi}(p_\phi \cos \phi - p_\psi \sin \phi) \tag{IV.37}
\]

\[
z = e^\psi \sin \phi \quad p_z = e^{-\psi}(p_\phi \cos \phi + p_\psi \sin \phi). \tag{IV.38}
\]

In terms of the new variables the constraints $(L_1, L_3)$ are:

\[
L_1 = p_\phi \quad L_2 = \eta_2(x, p_x, \psi, p_\psi) \cos \phi - \eta_3(x, p_x, \psi, p_\psi) \sin \phi \quad L_3 = \eta_2(x, p_x, \psi, p_\psi) \sin \phi + \eta_3(x, p_x, \psi, p_\psi) \cos \phi \tag{IV.38}
\]

where

\[
\eta_2(x, p_x, \psi, p_\psi) \equiv -x p_\phi e^{-\psi}, \quad \eta_3(x, p_x, \psi, p_\psi) \equiv e^{-\psi} p_\phi - e^\psi p_\psi. \tag{IV.39}
\]

From the angular momentum algebra we have

\[
\frac{\partial L_2}{\partial \phi} = -L_3, \quad \frac{\partial L_3}{\partial \phi} = L_2. \tag{IV.40}
\]

These are the same differential equations as [III.19]. Eqs. [IV.35] are in fact the solutions of Eqs. [IV.40] with respect to the variable $Q_1 = \phi$. As is seen, Eqs. [IV.35] are in the form given in Eqs. [III.21]. The solution $L_2 \propto -\sin \phi$ and $L_3 \propto \cos \phi$ is the one with initial condition $L_2(\phi = 0) = 0$ and $L_3(\phi = 0) = 1$, as stated in Eq. [III.23], and the solution $L_2 \propto \cos \phi$ and $L_3 \propto \sin \phi$ satisfy $L_2(\phi = 0) = 1$ and $L_3(\phi = 0) = 0$. Since $\eta_2 \approx 0$ on the surface $p_\phi = 0$ we see that the set of constraints $(L_1, L_3)$ finally reduces to $p_\phi$ and $\eta_3$ which commute with each other.

Important notice should be added that our transformation here is not acceptable globally. In fact, at $y = z = 0$ the transformation is singular. Therefore as indicated in some references [12] the abelianization process of $SO(3)$ algebra can be done just locally. We remind the reader that far from the origin the constraint surface given by functions $L_1$, $L_2$ and $L_3$ is the same as given by two of them. In fact, since $x \mathbf{L} = p \mathbf{L} = 0$ the constraints $L_1$, $L_2$ and $L_3$ are reducible, provided that $x \neq 0$ and/or $p \neq 0$. It seems that this subtle point is the essence that the reference [8] has not given a clear statement that the $SO(3)$ gauge system is abelianizable or not. However, an expanded version of $SO(3)$ gauge system is shown to be abelianizable in [13].

V. CONCLUDING REMARKS

Our main objective in this paper is that it is not reasonable to consider the most general case of constraint
systems, in which a complicated open algebra of structure functions is taken into account. In fact, a complicated algebra may be escaped during the procedure of determining the constraints from the very beginning. This possibility has not been discussed here. The other possibility is simplifying the given constraints (regardless how they are produced) by using legal algebraic methods, where abelianization stands at the top of this demand. Therefore, we discussed the problem of abelianization of a system of first class constraints in details.

Our main tools for this aim is "canonical transformations" which changes the coordinates of phase space, and "redefinition" of constraints describing the constraint surface. The latter alters the Poisson brackets, while the former may make the multiplicative nature of the constraints more clear. We suggest that multiplicative constraints should be regularized at each step of calculations. This means that they should be replaced by simple roots of the corresponding expressions.

The main reason behind these calculations is to find the most suitable coordinates in which the constraint surface is described by vanishing some phase space coordinates. In general, there is no guarantee that the system remains first class after such simplification of the constraints. This point makes us to take the necessity of regularization of the constraints more serious, since otherwise it is possible to have a first class algebra of constraints out of a number of second class ones.

We observed that ordinary quadratic Lagrangians lead to abelian constraints which are linear with respect to phase space coordinates. Non abelian algebras require at least cubic or more complicated functions of coordinates and velocities in the Lagrangian.

Although we were not able to prove a wide-standing theorem, we observed that most of the time the non abelian constraints (in suitable coordinates) contain factors, such as exponentials, that have no root in the finite region of the range of variables. Therefore, one may redefine the constraints by omitting such factors. This procedure is formulated in a systematic way by solving the differential equations due to the algebra of first class constraints. This is in fact the essence of our method of abelianization of the constraints.

The important point is that most of the above mathematical manipulations (i.e. regularization, canonical transformation and redefinition of constraints) which we use to find a simple description of constraint surface are valid locally, and may fail for some singular points or finite regions of the constraint surface. For example, in the case of $SO(3)$ algebra we showed that the system can be abelianized everywhere except the origin of the phase space. In this way we can divide first class systems into "globally abelianizable" and "locally abelianizable".

The question may arise, however, that is there any advantage in employing for instance the constraints $p_\phi$ and $\eta$ in the case of $SO(3)$, instead $L_1$, $L_2$ and $L_3$, at the price of leaving the globality? The answer depends on different applications of the first class constraints in physical problems such as gauge symmetry, quantization procedure, counting the physical degrees of freedom and so on. We postpone such analysis to future works. We reserve this possibility that maybe in some cases it is better to keep a non abelian and reducible algebra of constraints instead of change it into a non reducible and abelian one. The reason may be the better possibility of tracking the physical symmetries such as rotation.

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[1] M. Henneaux and C. Teitelboim, _Quantization of Gauge Systems_, Princeton Univ. Press, 1992.
[2] J. M. Pons, J. Antonio Garcia, Int. J. Mod. Phys. A15 (2000) 4681-4721, [arXiv: hep-th/9908151].
[3] M. Henneaux, C. Teitelboim, J. D. Vergara, Nucl. Phys. B387 (1992) 391-418, [arXiv: hep-th/9205092].
[4] A. Shirzad and M. Shabani Moghadam, J. Phys. A32 (1999) 8185-8195.
[5] M. Henneaux, Phys. Rep. 126 (1985) 1.
[6] S. A. Gogilidze, A. M. Khvedelidze and V. N. Perushin, J. Math. Phys. 37 (1996) 1760-1771, [arXiv: hep-th/9504153].
[7] F. Loran, Phys. Lett. B547 (2002) 63-68, [arXiv: hep-th/0209180].
[8] F. Loran, Commun. Math. Phys. 254 (2005) 167-178, [arXiv: hep-th/0303014].
[9] O. Miskovic, J. Zanelli, J. Math. Phys. 44 (2003) 3876-3887, [arXiv: hep-th/0302033].
[10] R. Courant and D. Hilbert, _Methods of Mathematical Physics_, Interscience Publishers John Wiley and Sons, 1980.
[11] H. Goldstein, C. Poole and J. Safko, _Classical Mechanics (3rd ed)_., Addison wesley, 2002.
[12] J. Gomis, M. Henneaux, J. M. Pons, Class. Quantum. Grav. 7 (1990) 1089-1096.
[13] F. Loran, Int. J. Mod. Phys. A23 (2008) 4051-4062, [arXiv:0808.3654].