NEWTON NON-DEGENERATE FOLIATIONS AND BLOWING-UPS

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Abstract. A codimension one singular holomorphic foliation is Newton non-degenerate if it satisfies the classical conditions of Kouchnirenko and Oka, in terms of its Newton polyhedra system. In this paper we prove that a foliation is Newton non-degenerate if and only if it admits a logarithmic reduction of singularities of a combinatorial nature.

1. Introduction

The goal of this paper is to characterize the class of foliations admitting a combinatorial reduction of singularities in terms of the associated Newton polyhedra. This characterization is relative not only to a foliation $\mathcal{F}$ on a complex space $M$, but also to a normal crossings divisor $E \subset M$; hence to a foliated space $(M, E; \mathcal{F})$.

The concept of Newton non-degenerate foliated space is given by extending the classical ideas for hypersurfaces in the works [7, 10]. The definition is given as follows. We attach a polyhedron $N$ to each stratum of the natural stratification induced by $E$. Each compact face of $N$ provides a weighted initial form for a local logarithmic generator of the foliation. We ask the weighted initial forms to have no zeros in the corresponding complex torus. When these non-degeneracy conditions hold, we say that the foliated space is Newton non-degenerate.

On the other hand, a reduction of singularities for a foliated space is combinatorial when it is obtained as a finite composition of combinatorial blowing-ups with invariant centers; we say that a blowing-up is combinatorial if it is centered at an intersection of irreducible components of the divisor. In this work, we consider logarithmic reduction of singularities; that is, the objective is to obtain logarithmically regular points. These points coincide with the classical presimple points in [2, 3], when we are in the complex hyperbolic context, that is, there are no hidden saddle-nodes in the foliation.

The main result in this paper is the following one:

Theorem 1. A foliated space is Newton non-degenerate if and only if it has a combinatorial logarithmic reduction of singularities.

We show that being Newton non-degenerate is stable under combinatorial blowing-ups and blowing-downs. When each polyhedron has a single vertex, the foliated space is Newton non-degenerate if and only if it is logarithmically desingularized. We end the proof of the theorem by noting that there is a reduction of singularities for the polyhedra system, as it is shown in [8].

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2. COMBINATORIAL REDUCTION OF SINGULARITIES

Let $M$ be a $n$-dimensional nonsingular complex analytic space. A foliation $\mathcal{F}$ on $M$ is an invertible coherent $\mathcal{O}_M$-submodule $\mathcal{F} \subset \mathcal{O}_M^1$, integrable and saturated in the sense that $\mathcal{F} = \mathcal{F}^\perp$. In local terms, a foliation is generated by a Frobenius integrable germ of holomorphic one-form

$$\omega = f_1 dz_1 + f_2 dz_2 + \cdots + f_n dz_n,$$

without common factors in its coefficients.

We need to consider a normal crossings divisor $E = \cup_{i \in J} E_i$ on $M$. Besides to the usual definition of normal crossings, we ask that the adherence of the strata defined by $E$ are connected sets; that is, the sets $E_j = \cap_{i \in J} E_j$ are connected for any $J \subset I$. We refer to this conditions by saying that $E$ is a strong normal crossings divisor. We say that the pair $(M, E)$ is an ambient space. By definition, a combinatorial blowing-up between ambient spaces

$$\pi : (M’, E’) \to (M, E), \quad E’ = \pi^{-1}(E),$$

is induced by a blowing-up $M’ \to M$ with center one of the sets $E_j$.

A foliated space $(M, E; \mathcal{F})$ is the datum of an ambient space $(M, E)$ and a foliation $\mathcal{F}$ on $M$. A combinatorial blowing-up $\pi : (M’, E’) \to (M, E)$ is admissible when the center $E_j$ is invariant for $\mathcal{F}$. We write, for short

$$\pi : (M’, E’; \mathcal{F’}) \to (M, E; \mathcal{F}),$$

where $\mathcal{F’}$ is the transform of $\mathcal{F}$ by $\pi$.

Taking a logarithmic point of view with respect to the divisor $E$, we consider $\mathcal{F}$ as being locally defined at a point $p \in M$ by a meromorphic one-form

$$\eta = \sum_{j \in J} a_j \frac{dx_j}{x_j} + \sum_{j \in c(J)} a_j dy_j,$$

without common factors in its coefficients, where $E = (\prod_{j \in J} x_j = 0), \quad J \cap c(J) = \emptyset$ and $\# J \cup c(J) = \dim M$. This expression allows us to define the logarithmic singular locus $\logSing(\mathcal{F}, E)$ as follows

$$\logSing(\mathcal{F}, E) = \{ p \in M ; \nu_p(\mathcal{F}, E) > 0 \},$$

where $\nu_p(\mathcal{F}, E)$ denotes to the minimum of the orders at $p$ of the coefficients of $\eta$.

We say that $E_j$ is a log-admissible center of blowing-up for $(M, E; \mathcal{F})$ when $E_j \subset \logSing(\mathcal{F}, E)$. Let us note that a log-admissible center $E_j$ is admissible.

**Definition 2.1.** A foliated space $(M, E; \mathcal{F})$ is of logarithmic toric type if there is a finite sequence of log-admissible combinatorial blowing-ups

$$(M, E; \mathcal{F}) \leftrightarrow (M_1, E_1; \mathcal{F}_1) \leftrightarrow \cdots \leftrightarrow (M’, E’; \mathcal{F’})$$

such that $\logSing(\mathcal{F’}, E’) = \emptyset$.

**Remark 2.1.** The reduction of singularities for holomorphic foliations [2, 3, 11], attempts two objectives: to obtain either presimple points or, the more restrictive, simple points. Roughly speaking, simple points are “presimple ones without resonances”. There is a context in which presimple points coincide with logarithmically non-singular points: the case of complex hyperbolic foliations (see [9]). We recall that a foliation $\mathcal{F}$ on $M$ is complex hyperbolic (see [14]) if there is no holomorphic map $\phi : (\mathbb{C}^2, 0) \to M$ such that 0 is a saddle-node for $\phi^{-1} \mathcal{F}$. In the two
dimensional case, being complex hyperbolic is equivalent to have a reduction of singularities without saddle-nodes; this is the case considered in [1].

3. Newton Non-degenerate Foliations

In this section we define Newton non-degenerate foliated spaces.

Given a foliated space \((M,E;F)\), we attach to it a Newton polyhedra system

\[ \mathcal{N}_{M,E,F} = \{N_J\}_{J \in \mathcal{H}_{M,E}}, \quad \mathcal{H}_{M,E} = \{J \subset I; \ E_j \neq \emptyset\}, \]

following the definitions in [8]. We associate to each \(J \in \mathcal{H}_{M,E}\) a positively convex polyhedron \(N_J \subset \mathbb{R}_{\geq 0}^J\) as follows. Take a point \(p\) in the stratum \(S_J = E_j \setminus \bigcup_{j \notin J} E_j\) and a local logarithmic generator \(\eta\) of \(\mathcal{F}\) at \(p\) as in Equation [1]. We decompose each coefficient \(a_j\) of \(\eta\) as

\[ a_j = \sum_{\sigma \in \mathbb{Z}^J_{\geq 0}} a_{j,\sigma}(y) x^\sigma, \quad x^\sigma = \prod_{j \in J} x_j^{\sigma_j}. \]

The polyhedron \(N_J\) is the positively convex hull of the set

\[ \{\sigma \in \mathbb{Z}^J; \text{ there is } j \in J \cup c(J) \text{ with } a_{j,\sigma} \neq 0\}. \]

The definition of \(N_J\) is independent of the choice of \(p\), the particular coordinate system (adapted to the divisor) and the local generator \(\eta\) that we consider.

Remark 3.1. The construction of the polyhedra \(N_J\) is compatible with the natural projections \(\text{pr} : \mathbb{R}^J \to \mathbb{R}^J\), when \(J \subset J'\), in the sense that \(N_J = \text{pr}(N_{J'})\). This “coherence” property is the essential condition considered in the theory of polyhedra systems in [8].

In order to give the definition of Newton non-degenerate foliated space, we attach “initial forms” to the compact faces of the polyhedra. The non-degeneracy condition concerns these initial forms. Next, we give the precise statements and definitions.

Let us fix a combinatorial stratum \(J \in \mathcal{H}_{M,E}\) and a \(J\)-weight vector \(\rho : \mathbb{R}^J \to \mathbb{R}\), that is, \(\rho\) is a linear map such that \(\rho(\sigma) > 0\), for every non-zero \(\sigma \in \mathbb{R}^J_{\geq 0}\). Note that the set of values \(\mathcal{V}_\rho = \rho(\mathbb{Z}^J_{\geq 0})\) has the same ordinal as \(\mathbb{Z}_{\geq 0}\). Given a point \(p \in S_J\), we obtain a \(J\)-weighted filtration of the local ring \(\mathcal{O}_{M,p}\). The corresponding \(J\)-graded algebra is

\[ \mathfrak{G}_p^J = \oplus_{v \in \mathcal{V}_\rho} \mathfrak{G}_p^J(v), \quad \mathfrak{G}_p^J(v) = I_v/I_v^+. \]

The ideal \(I_v\), respectively \(I_v^+\), is generated by the monomials \(x^\sigma\) with \(\rho(\sigma) \geq v\), respectively \(\rho(\sigma) > v\), where \((x,y)\) is a local coordinate system at \(p\) such that \(E = \prod_{j \in J} (x_j = 0)\). Note that \(\mathfrak{G}_p^J(0) \simeq \mathcal{O}_{S_J,p}\), hence \(\mathfrak{G}_p^J\) is an \(\mathcal{O}_{S_J,p}\)-graded algebra. Moreover, we have an isomorphism

\[ \mathfrak{G}_p^J \simeq \mathcal{O}_{S_J,p}[, T], \quad T = (T_j)_{j \in J} \]

in the category of \(\mathcal{O}_{S_J,p}\)-graded algebras, where the weight of \(T^\sigma\) is \(\rho(\sigma)\). In this way, we build the corresponding \(\mathfrak{G}_p^J\)-graded module \(\mathfrak{A}_p^J = \oplus_v \mathfrak{A}_p^J(v)\), obtained from the free \(\mathcal{O}_{M,p}\)-module \(\Omega_{M,p}^1(\log E)\) of the germs at \(p\) of logarithmic one-forms with poles along \(E\). Thanks to the isomorphisms in Equation [2] the disjoint union \(\mathfrak{G}_p = \bigcup_{p \in S_J} \{p\} \times \mathfrak{G}_p^J\) has a topology such that the natural projection \(\mathfrak{G}_p \to S_J\) is an “espace étalé”. We obtain that \(\mathfrak{G}_p\) is an \(\mathcal{O}_{S_J}\)-graded algebra locally isomorphic to \(\mathcal{O}_{S_J}[, T]\). We also get a \(\mathfrak{G}_p\)-graded module \(\mathfrak{A}_p\) with fibers \(\mathfrak{A}_p^J\). (Details of these constructions can be found in [9]).
Remark 3.2. The constructions of $\mathfrak{G}^\rho$ and $\mathfrak{A}^\rho$ do not depend on the choices of local coordinates $(x, y)$.

Given a non-negative logarithmic germ of one-form $\eta \in \Omega_{M,E}(\log E)$, there is a well-defined $\rho$-initial form $L_\rho \eta \in \mathfrak{A}_\rho^\eta(\nu)$, where $\nu = \nu_\rho(\eta)$ is the $\rho$-value of $\eta$. We define the $\rho$-initial form $L_\rho F$ of the foliation $F$ as being the $\mathfrak{G}_\rho$-submodule of $\mathfrak{A}_\rho$ locally generated by the $L_\rho \eta$, where $\eta$ is a local generator of $F$ as in Equation 1

By means of the isomorphisms $\mathfrak{A}_\rho \simeq (\mathfrak{G}_\rho)^n \simeq (\mathcal{O}_{S,J,p}[T])^n$ and taking notations as in Equation 1 and Equation 3, we associate to the initial form $L_\rho \eta$ the family $\mathcal{L}_\rho(\eta) = (A_j[T])_{j \in J \cup \{J\}}$ of homogeneous polynomials $A_j[T]$ defined by

$$A_j[T] = \sum_{\sigma \in \Delta_j \cap \mathbb{Z}_{\geq 0}} a_{j,\sigma}(y) T^\sigma,$$

where $\Delta_\rho = \{ \sigma \in \mathbb{R}^J; \rho(\sigma) = \nu_\rho(\eta) \}$. It makes sense to consider the set $\mathcal{L}_\rho(\eta) = 0$ of common zeros of the $A_j[T]$ as a subset of $(S_J,p) \times \mathbb{C}^J$, just by taking the value of the functions $a_{j,\sigma}(y)$ at the points of $S_J$ near $p$.

We say that $(M,E;F)$ is non-degenerate at $p$ with respect to $\rho$ when we have

$$\mathcal{L}_\rho(\eta) = 0 \cap (S_J,p) \times (\mathbb{C}^*)^J = \emptyset.$$

This property does not depend on the chosen generator nor on the chosen local coordinate system.

Remark 3.3. The compact faces of $N_J$ are precisely the sets $F_\rho = \Delta_\rho \cap N_J$. Hence, we obtain a partition $\{ W_{J,F} \}$ of the set $W_J$ of $J$-weight vectors, by the compact faces $F$ of $N_J$. We have that $\mathcal{L}_\rho(\eta) = \mathcal{L}_\rho'(\eta)$ if and only if $F_\rho = F_\rho'$. On this way, we can talk about the “initial form of $\eta$ with respect to a compact face $F$”.

Given a point $p \in M$, let us denote by $J_p \in \mathcal{H}_{M,E}$ the combinatorial stratum such that $p \in S_{J_p}$.

Definition 3.1. A foliated space $(M,E;F)$ is Newton non-degenerate at a point $p$ if it is non-degenerate at $p$ with respect to each $J_p$-weight vector. It is Newton non-degenerate if it is so at every point of $M$.

Remark 3.4. Being Newton non-degenerate at a point is an open property.

4. Stability Property

Let us see that the property of being Newton non-degenerate is stable by combinatorial blowing-ups and blowing-downs.

Proposition 4.1. Let $\pi : (M',E';F') \to (M,E;F)$ be an admissible combinatorial blowing-up between foliated spaces. We have that $(M,E;F)$ is Newton non-degenerate if and only if $(M',E';F')$ is Newton non-degenerate.

Write $\mathcal{H} = \mathcal{H}_{M,E}$ and $\mathcal{H}' = \mathcal{H}_{M',E'}$. Assume that the center of $\pi$ is $E_J$, with $J \in \mathcal{H}$. Following notations in [3], we have that

$$\mathcal{H} = \mathcal{H}_s \cup K_J, \quad \text{where} \quad K_J = \{ K \in \mathcal{H}; J \subset K \}, \quad \mathcal{H}_s = \mathcal{H} \setminus K_J;$$

$$\mathcal{H}' = \mathcal{H}'_s \cup \left( \bigcup_{K \in K_J} \mathcal{H}'_{K,\infty} \right), \quad \text{where} \quad \mathcal{H}'_{K,\infty} = \{ (K \setminus J) \cup A \cup \{ \infty \}; A \subset J \}.$$

Let as denote by $\{ \sigma_{K,J} \} \in \mathbb{K}$ the standard basis of $\mathbb{R}^K$, where $\sigma_{K,J}(j') = \delta_{jj'}$ (Kronecker symbol). Given $K \in K_J$ and $A \subset J$, we define the subset $W_{K,A} \subset W_K$
as being the set of $K$-vector weights $\rho \in W_K$ such that there is a number $r_{\rho} > 0$ with the properties:

$$\rho(\sigma_{K,j}) = r_{\rho}, \; j \in K \setminus A; \quad \rho(\sigma_{K,j}) > r_{\rho}, \; j \in A.$$ 

On this way, we obtain a partition $\{W^A_K\}_{A \subseteq J}$ of $W_K$. For each $A \subseteq J$, we consider $K' = (K \setminus J) \cup A \cup \{\infty\} \in H^\infty_{K'}$. There is a bijection $\phi^A_K : W^A_K \to W^A_{K'}$, defined by $\rho \mapsto \rho'$, where

$$\rho'(\sigma_{K',j}) = \begin{cases} 
\rho(\sigma_{K,j}), & j \in K \setminus J; \\
\rho(\sigma_{K,j}) - r_{\rho}, & j \in A; \\
r_{\rho}, & j = \infty.
\end{cases}$$

Let us consider a weight vector $\rho \in W^A_K$ and $\rho' = \phi^A_K(\rho) \in W^A_{K'}$. The following statement completes the proof of Proposition 4.1.

**Lemma 4.1.** The foliated space $(M, E; F)$ is non-degenerate at a point $p \in S_K$ with respect to $\rho$ if and only if $(M', E'; F')$ is non-degenerate at each $q \in \pi^{-1}(p) \cap S_{K'}$ with respect to $\rho'$.

**Proof.** Denote by $N_K$ the Newton polyhedron of $(M, E; F)$ associated to $K$ and by $N'_{K'}$, the Newton polyhedron of $(M', E'; F')$ associated to $K'$. Let $F_\rho$ be the compact face of $N_K$ defined by $\rho$ and $F'_\rho$, the compact face of $N'_{K'}$ corresponding to $\rho'$. We have that $F'_\rho$ is a translation of $\lambda^A_K(F_\rho)$, where $\lambda^A_K : \mathbb{R}^K \to \mathbb{R}^{K'}$ is the affine map given by $\sigma \mapsto \sigma'$, with

$$\sigma'(j) = \sigma(j), \; j \in K' \setminus \{\infty\}; \quad \sigma'(\infty) = \sum_{j \in J} \sigma(j)$$

(see [8, Def. 2]). The new $\rho'$-weighted initial form depends only on the old one, that is transformed as a one-differential form, with the usual equations of the blowing-up. Looking at these equations, a straightforward computation completes the result (see [9, pp. 81-82]). \(\blacksquare\)

5. **Equivalence Theorem**

The objective on this section is to prove the main statement:

**Theorem 1.** A foliated space is Newton non-degenerate if and only if it is of logarithmic toric type.

In view of Remark 2.1, we obtain the following result:

**Corollary 5.1.** A complex hyperbolic foliated space is Newton non-degenerate if and only if it has a combinatorial reduction of singularities to presimple points.

Let us start the proof of Theorem 1. We consider a foliated space $(M, E; F)$ and we denote by $N = \{N_J\}_{J \in H_{M,E}}$ the Newton polyhedra system associated to it.

**Remark 5.1.** For each $J \in H_{M,E}$ we have that the polyhedron $N_J$ has a single vertex if and only if $E_J \not\subset \logSing(F, E)$.

The proof is based in the following facts:

1. The stability of being Newton non-degenerate under combinatorial blowing-ups and blowing-downs, stated in Proposition 4.1.
(2) The statement holds when the polyhedra system is desingularized in the sense that each polyhedron $N_J$ has a single vertex. That is, under this hypothesis, we have that $(M, E; F)$ is Newton non-degenerate if and only if $\log\text{Sing}(F, E) = \emptyset$. This property can be verified in a direct way by looking at each point.

(3) The existence of reduction of singularities for polyhedra systems with centers at singular combinatorial strata $J$, that is, such that $N_J$ has not a single vertex. This result is proved in [3] Theorem 2.

(4) The compatibility between blowing-ups of polyhedra systems and combinatorial log-admissible blowing-ups of foliated spaces. More precisely, if $N_J$ has not a single vertex, then $E_J$ is a log-admissible center for a combinatorial blowing-up $(M', E'; F') \to (M, E; F)$. Moreover, the polyhedra system $N'$ associated to $(M', E'; F')$ is the transform of $N$ by the polyhedra systems blowing-up with center $J$ as it is introduced in [8].

Assume first that $(M, E; F)$ is of logarithmic toric type and let us fix a log-admissible combinatorial reduction of singularities $(M', E'; F') \to (M, E; F)$ as in Equation 2. We have that $\log\text{Sing}(F', E') = \emptyset$. Hence, given $J' \in \mathcal{H}_{M', E'}$, the property $E_{J'} \notin \log\text{Sing}(F', E')$ holds and thus $N_{J'}$ has a single vertex, in view of Remark 5.1. Now, by Fact (2) above, we conclude that $(M', E'; F')$ is Newton non-degenerate. Fact (1) allows us to conclude that $(M, E; F)$ is Newton non-degenerate.

Suppose now that the foliated space $(M, E; F)$ is Newton non-degenerate. By Fact (3), there is a reduction of singularities of the polyhedra system $\mathcal{N}'$, that induces a sequence of log-admissible combinatorial blowing-ups $(M', E'; F') \to (M, E; F)$, in view of Fact (1). On the one hand, the polyhedra system $N'$ of $(M', E'; F')$ is desingularized and, on the other, we have that $(M', E'; F')$ is Newton non-degenerate, by Fact (1). Finally, by Fact (2), we conclude that $\log\text{Sing}(F', E') = \emptyset$. Then, the foliated space $(M, E; F)$ is of logarithmic toric type.

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