Existence and concentration of positive bound states for the 
Schrodinger-Poisson system with potential functions

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Abstract

In this article we study the existence and concentration behavior of bound states 
for a nonlinear Schrödinger-Poisson system with a parameter ε > 0. Under some 
suitable conditions on the potential functions, we prove that for ε small the system 
has a positive solution that concentrates at a point which is a global minimum of the 
minimax function associated to the related autonomous problem.

Keywords: Schrödinger-Poisson system; variational methods, concentration.

1 Introduction

In this article we will focus on the following Schrödinger-Poisson system

\[
\begin{cases}
-\varepsilon^2 \Delta v + V(x)v + K(x)\phi(x)v = |v|^{q-2}v & \text{in } \mathbb{R}^3 \\
-\Delta \phi = K(x)v^2 & \text{in } \mathbb{R}^3
\end{cases}
\] (SP_ε)

where ε > 0 is a parameter, q ∈ (4, 6) and V, K : \mathbb{R}^3 → \mathbb{R} are, respectively, an external 
potential and a charge density. The unknowns of the system are the field v associated 
with the particles and the electric potential \phi. We are interested in the existence and 
concentration behavior of solutions of (SP_ε) in the semiclassical limit ε → 0.

The first equation of (SP_ε) is a nonlinear equation in which the potential \phi satisfies a 
nonlinear Poisson equation. For this reason, (SP_ε) is called a Schrödinger-Poisson system, 
also known as Schrödinger-Maxwell system. For more informations about physical aspects, 
we refer [5, 9] and references therein.

We observe that when \phi ≡ 0, (SP_ε) reduces to the well known Schrödinger equation

\[ -\varepsilon^2 \Delta u + V(x)u = f(x, u) \quad x \in \mathbb{R}^N. \] (S)

In the last years, the nonlinear stationary Schrödinger equation has been widely inves-
tigated, mainly in the semiclassical limit as ε → 0 (see e.g. [18, 20, 21] and its references).

In [18], Rabinowitz studied problem (S) through mountain pass arguments in order to 
find least energy solutions, for ε > 0 sufficiently small. Then, Wang [20] proved that the 
solution in [18] concentrates around the global minimal of V when ε tends to 0.

In [21], Wang and Zeng considered the following Schrödinger equation

\[ -\varepsilon^2 \Delta u + V(x)u = K(x)|u|^{p-1}u + Q(x)|u|^{q-1}u, \quad x \in \mathbb{R}^N \] (WZ)

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where $1 < q < p < (n + 2)/(n - 2)$. They proved the existence of least energy solutions and their concentration around a point in the semiclassical limit. The authors used the energy function $C(s)$ defined as the minimal energy of the functional associated with $\Delta u + V(s)u = K(s)|u|^{p-1}u + Q(s)|u|^{q-1}u$, where $s \in \mathbb{R}^N$ acts as a parameter instead of an independent variable. For each $\varepsilon > 0$ sufficiently small, they proved the existence of a solution $u_\varepsilon$ for $(\varepsilon WZ)$, whose global maximum approaches to a point $y^*$ when $\varepsilon$ tends to 0. Moreover, under suitable hypothesis on the potentials $V$ in $W$, the function $s \mapsto C(s)$ assumes a minimum at $y^*$.

Motivated by those results, Alves and Soares \cite{1} investigated the same phenomenon for the following class of gradient systems

$$\begin{cases}
-\varepsilon^2 \Delta u + V(x)u = Q_u(u, v) & \text{in } \mathbb{R}^N \\
-\varepsilon^2 \Delta v + W(x)v = Q_v(u, v) & \text{in } \mathbb{R}^N \\
u(x), v(x) \to 0, \quad \text{as } |x| \to \infty \\
u, v > 0 & \text{in } \mathbb{R}^N
\end{cases}$$

In this system is natural to expect some competition between the potentials $V$ and $W$, each one trying to attract the local maximum points of the solutions to its minimum points. In fact, in \cite{1} the authors proved that functions $u_\varepsilon$ and $v_\varepsilon$ satisfies $(\text{AS})$ and concentrate around the same point which is the minimum of the respective function $C(s)$.

In \cite{23}, Yang and Han studied the following Schrödinger-Poisson system

$$\begin{cases}
\Delta v + V(x)v + K(x)\phi(x)v = |v|^{q-2}v & \text{in } \mathbb{R}^3 \\
-\Delta\phi = K(x)v^2 & \text{in } \mathbb{R}^3
\end{cases}$$

$(SP)$

Under suitable assumptions on $V$, $K$ and $f$ they proved existence and multiplicity results by using the mountain pass theorem and the fountain theorem. Later, L. Zhao, Liu and F. Zhao \cite{24}, using variational methods, proved the existence and concentration of solutions for system

$$\begin{cases}
\Delta v + \lambda V(x)v + K(x)\phi(x)v = |v|^{q-2}v & \text{in } \mathbb{R}^3 \\
-\Delta\phi = K(x)v^2 & \text{in } \mathbb{R}^3
\end{cases}$$

when $\lambda > 0$ is a parameter and $2 < p < 6$.

Several papers dealt with system $(SP)$ under variety assumptions on potentials $V$ and $K$. Most part of the literature focuses on the study of the system with $V$ or $K$ constant or radially symmetric, mainly studying existence, nonexistence and multiplicity of solutions see e.g. \cite{3} \cite{8} \cite{9} \cite{10} \cite{15} \cite{17} \cite{19}.

The double parameters’ perturbation was also considered in system $(SP_\varepsilon)$. In \cite{13}, He and Zhou studied the existence and behavior of a ground state solution which concentrates around the global minimum of the potential $V$. They considered $K \equiv 1$ and the presence of the nonlinear term $f(x,u)$.

Recently, Ianni and Vaira \cite{14} studied the Schrödinger-Poisson system $(SP_\varepsilon)$ proving that if $V$ has a non-degenerated critical point $x_0$, then there exists a solution that concentrates around this point. Moreover, they also proved that if $x_0$ is degenerated for $V$ and a local minimum for $K$, then there exist a solution concentrating around $x_0$. The proof was based in the Lyapunov-Schmidt reduction.

Using variational methods as employed by \cite{1} \cite{18} \cite{21}, we prove that there exists a solution $u_\varepsilon$ for the Schrödinger-Poisson system $(SP_\varepsilon)$ which concentrates around a point, without any additional assumption on the degenerability of such point related with the potentials $V$ and $K$, as used in \cite{14}.
More precisely, denote $C_\infty$ as the minimax value related to

$$\begin{cases}
-\Delta v + V_\infty v + K_\infty \phi v = |v|^{q-2}v & \text{in } \mathbb{R}^3 \\
-\Delta \phi = K_\infty v^2 & \text{in } \mathbb{R}^3
\end{cases}$$

where the following conditions hold

(H0) There exists $\alpha > 0$ such that $V(x), K(x) \geq \alpha > 0, \forall x \in \mathbb{R}^3$

(H1) $V_\infty$ and $K_\infty$ are defined by

$$
V_\infty = \liminf_{|x| \to \infty} V(x) > \inf_{x \in \mathbb{R}^3} V(x) \\
K_\infty = \liminf_{|x| \to \infty} K(x) > \inf_{x \in \mathbb{R}^3} K(x).
$$

We prove that if

$$C_\infty > \inf_{\xi \in \mathbb{R}^3} C(\xi)$$

then, system $(SP_\varepsilon)$ has a positive solution $v_\varepsilon$ as $\varepsilon$ tends to zero. After passing to a subsequence, $v_\varepsilon$ concentrates at a global minimum point of $C(\xi)$ for $\xi \in \mathbb{R}^3$, where the energy function $C(\xi)$ is defined to be the minimax function associated with the problem

$$\begin{cases}
-\Delta u + V(\xi)u + K(\xi)\phi(\xi)u = |u|^{q-2}u & \text{in } \mathbb{R}^3 \\
-\Delta \phi = K(\xi)u^2 & \text{in } \mathbb{R}^3
\end{cases} \quad (SP_\xi)$$

Therefore, $C(\xi)$ plays a central role in our study.

The main result for system $(SP_\varepsilon)$ is the following

**Theorem 1.** Suppose (H0) – (H1) hold. If

$$C_\infty > \inf_{\xi \in \mathbb{R}^3} C(\xi), \quad (C^\infty)$$

then there exists $\varepsilon^* > 0$ such that system $(SP_\varepsilon)$ has a positive solution $v_\varepsilon$ for $\varepsilon \in (0, \varepsilon^*)$. Moreover, $v_\varepsilon$ concentrates at a local (hence global) maximum point $y^* \in \mathbb{R}^3$ such that

$$C(y^*) = \min_{\xi \in \mathbb{R}^3} C(\xi).$$

**Remark 1.** Theorem 1 complements the study made in [10, 14, 23, 24] in the following sense: we deal with the perturbation problem $(SP_\varepsilon)$ and study the concentration behavior of positive bound states.

**Remark 2.** To the best of our knowledge, it seems that the only previous paper regarding the concentration of solutions for the perturbed Schrödinger-Poisson system with potentials $V$ and $K$ is [14], where the smoothness of such potentials is considered. We only need the boundedness of $V$ and $K$. Moreover, we do not assume that the concentration point of solutions $v_\varepsilon$ for the system $(SP_\varepsilon)$ is a local minimum (or maximum) of such potentials, as in the previous paper. In our research we shall consider a different variational approach.

The outline of this paper is as follows: in Section 2 we set the variational framework. In Section 3 we study the autonomous system related to $(SP_\varepsilon)$. In section 4 we establish an existence result for system $(SP_\varepsilon)$ with $\varepsilon = 1$. In section 5, we prove Theorem 1.
2 Variational framework and preliminary results

Throughout this paper we use the following notations:

- \( H^1(\mathbb{R}^3) \) is the usual Sobolev space endowed with the standard scalar product and norm
  \[
  (u, v) = \int_{\mathbb{R}^3} (\nabla u \nabla v + uv) \, dx, \quad ||u||^2 = \int_{\mathbb{R}^3} (|\nabla u|^2 + u^2) \, dx.
  \]

- \( D^{1,2} = D^{1,2}(\mathbb{R}^3) \) represents the completion of \( C^\infty_0(\mathbb{R}^3) \) with respect to the norm
  \[
  ||u||^2_{D^{1,2}} = \int_{\mathbb{R}^3} |\nabla u|^2 \, dx.
  \]

- \( L^p(\Omega), 1 \leq p \leq \infty, \Omega \subset \mathbb{R}^3 \), denotes a Lebesgue space; the norm in \( L^p(\Omega) \) is denoted by \( ||u||_{L^p(\Omega)} \), where \( \Omega \) is a proper subset of \( \mathbb{R}^3 \);
  \( ||u||_p \) is the norm in \( L^p(\mathbb{R}^3) \).

We recall that by the Lax-Milgram theorem, for every \( v \in H^1(\mathbb{R}^3) \), the Poisson equation
\[
-\Delta \phi = v^2
\]
has a unique positive solution \( \phi = \phi_v \in D^{1,2}(\mathbb{R}^3) \) given by
\[
\phi_v(x) = \int_{\mathbb{R}^3} \frac{v^2(y)}{|x - y|} \, dy. \tag{1}
\]

The function \( \phi : H^1(\mathbb{R}^3) \to D^{1,2}(\mathbb{R}^3), \phi[v] = \phi_v \) has the following properties (see for instance Cerami and Vaira [7])

**Lemma 2.** For any \( v \in H^1(\mathbb{R}^3) \), we have

i) \( \phi \) is continuous and maps bounded sets into bounded sets;

ii) \( \phi_v \geq 0 \);

iii) there exists \( C > 0 \) such that \( ||\phi||_{D^{1,2}} \leq C||v||^2 \) and
  \[
  \int_{\mathbb{R}^3} |\nabla v|^2 \, dx = \int_{\mathbb{R}^3} \phi_v v^2 \, dx \leq C||v||^4;
  \]

iv) \( \phi_{tv} = t^2 \phi_v, \forall t > 0 \);

v) if \( v_n \to v \) in \( H^1(\mathbb{R}^3) \), then \( \phi_{v_n} \to \phi_v \) in \( D^{1,2}(\mathbb{R}^3) \).

As in [3], for every \( v \in H^1(\mathbb{R}^3) \), there exist a unique solution \( \phi = \phi_{K,v} \in D^{1,2}(\mathbb{R}^3) \) of
\[
-\Delta \phi = K(x)v^2
\]
where
\[
\phi_{K,v}(x) = \int_{\mathbb{R}^3} \frac{K(y)v^2(y)}{|x - y|} \, dy. \tag{2}
\]

and it is easy to see that \( \phi_{K,v} \) satisfies Lemma 2 if \( K \) satisfies conditions \((H_0)-(H_1)\).

Substituting (2) into the first equation of \((SP)_\varepsilon\), we obtain
\[
-\varepsilon^2 \Delta v + V(x)v + K(x)\phi_{K,v}(x)v = |v|^{q-2}v. \tag{3}
\]

Making the changing of variables \( x \mapsto \varepsilon x \) and setting \( u(x) = v(\varepsilon x) \), (3) becomes
\[
-\Delta u + V(\varepsilon x)u + K(\varepsilon x)\phi_{K,v}(\varepsilon x)u = |u|^{q-2}u. \tag{4}
\]
A simple computation shows that
\[ \phi_{K,v}(\varepsilon x) = \varepsilon^2 \phi_{\varepsilon,u}(x), \]
where
\[ \phi_{\varepsilon,u}(x) = \int_{\mathbb{R}^3} K(\varepsilon y)u^2(y) \frac{dy}{|x-y|}. \]
Substituting it into \( (\mathcal{P}_\varepsilon) \), \( (\mathcal{SP}_\varepsilon) \) can be rewritten in the following equivalent equation
\[ -\Delta u + V(\varepsilon x)u + \varepsilon^2 K(\varepsilon x)\phi_{\varepsilon,u}u = |u|^{q-2}u. \] \( (S_\varepsilon) \)
Note that if \( u_\varepsilon \) is a solution of \( (S_\varepsilon) \), then \( v_\varepsilon(x) = u_\varepsilon(\xi) \) is a solution of \( (\mathcal{M}) \).

We denote by \( H_{\varepsilon} = \{ u \in H^1(\mathbb{R}^3) : \int_{\mathbb{R}^3} V(\varepsilon x)u^2 < \infty \} \) the Sobolev space endowed with the norm
\[ \| u \|_{\varepsilon}^2 = \int_{\mathbb{R}^3} (|\nabla u|^2 + V(\varepsilon x)u^2) \, dx. \]

At this step, we see that \( (S_\varepsilon) \) is variational and its solutions are critical points of the functional
\[ I_\varepsilon(u) = \frac{1}{2} \int_{\mathbb{R}^3} (|\nabla u|^2 + V(\varepsilon x)u^2) \, dx + \frac{\varepsilon^2}{4} \int_{\mathbb{R}^3} K(\varepsilon x)\phi_{\varepsilon,u}(x)u^2 \, dx - \frac{1}{q} \int_{\mathbb{R}^3} |u|^q \, dx. \] \( (5) \)

3 Autonomous Case

In this section we study the following autonomous system

\[ \begin{cases} 
-\Delta u + V(\xi)u + K(\xi)\phi(\xi)u = |u|^{q-2}u & \text{in } \mathbb{R}^3 \\
-\Delta \phi = K(\xi)u^2 & \text{in } \mathbb{R}^3 
\end{cases} \] \( (\mathcal{SP}_\xi) \)

where \( \xi \in \mathbb{R}^3 \).

We associate with system \( (\mathcal{SP}_\xi) \) the functional \( I_\xi : H_\xi \mapsto \mathbb{R} \)
\[ I_\xi(u) = \frac{1}{2} \int_{\mathbb{R}^3} (|\nabla u|^2 + V(\xi)u^2) \, dx + \frac{1}{4} \int_{\mathbb{R}^3} K(\xi)\phi_{\xi,u}(\xi)u^2 \, dx - \frac{1}{q} \int_{\mathbb{R}^3} |u|^q \, dx. \] \( (5) \)

Hereafter, the Sobolev space \( H_\xi = H^1(\mathbb{R}^3) \) is endowed with the norm
\[ \| u \|_\xi = \int_{\mathbb{R}^3} (|\nabla u|^2 + V(\xi)u^2) \, dx. \]

By standard arguments, the functional \( I_\xi \) verifies the Mountain-Pass Geometry, more exactly it satisfies the following lemma

**Lemma 3.** The functional \( I_\xi \) satisfies

(i) There exist positive constants \( \alpha, \rho \) such that \( I_\xi(u) \geq \alpha \) for \( \| u \|_\xi = \rho \).

(ii) There exists \( u_1 \in H^1(\mathbb{R}^3) \) with \( \| u_1 \|_\xi > \rho \) such that \( I_\xi(u_1) < 0 \).
Applying a variant of the Mountain Pass Theorem (see [22]), we obtain a sequence $\{u_n\} \subset H^1(\mathbb{R}^3)$ such that

$$I_\xi(u_n) \to C(\xi) \text{ and } I'_\xi(u_n) \to 0,$$

where

$$C(\xi) = \inf_{\gamma \in \Gamma} \max_{0 \leq t \leq 1} I_\xi(\gamma(t)), \quad C(\xi) \geq \alpha$$

and

$$\Gamma = \{\gamma \in C([0,1], H^1(\mathbb{R}^3)) | \gamma(0) = 0, \gamma(1) = u_1\}.$$

We observe that $C(\xi)$ can be also characterized as

$$C(\xi) = \inf_{u \neq 0} \max_{t > 0} I_\xi(t u).$$

**Proposition 4.** Let $\xi \in \mathbb{R}^3$. Then system (SP $\xi$) has a positive solution $u \in H^1(\mathbb{R}^3)$ such that $I'_\xi(u) = 0$ and $I_\xi(u) = C(\xi)$, for any $q \in (4,6)$.

**Proof.** The proof is an easy adaptation of Theorem 1.1 in [4] and we omit it. \qed

**Lemma 5.** The function $\xi \mapsto C(\xi)$ is continuous.

**Proof.** The proof consists in proving that there exist sequences $(\zeta_n)$ and $(\lambda_n)$ in $\mathbb{R}^3$ such that $C(\zeta_n), C(\lambda_n) \to C(\xi)$ as $n \to 0$, where

$$\zeta_n \to \xi \text{ and } C(\zeta_n) \geq C(\xi), \forall n$$

$$\lambda_n \to \xi \text{ and } C(\lambda_n) \geq C(\xi), \forall n$$

as we know by Alves and Soares [1] with slightly modifications. \qed

**Remark 3.** The function $(\mu, \nu) \mapsto c_{\mu, \nu}$ is continuous, where $c_{\mu, \nu}$ is the minimax level of

$$I_{\mu, \nu}(u) = \frac{1}{2} \int_{\mathbb{R}^3} (|\nabla u|^2 + \mu u^2) \, dx + \frac{1}{4} \int_{\mathbb{R}^3} \nu \phi_u(x) u^2 \, dx - \frac{1}{q} \int_{\mathbb{R}^3} |u|^q \, dx.$$  

(7)

**Remark 4.** We denote by $C_\infty$ the minimax value related to the functional

$$I_\infty(u) = \frac{1}{2} \int_{\mathbb{R}^3} (|\nabla u|^2 + V_\infty u^2) \, dx + \frac{1}{4} \int_{\mathbb{R}^3} K_\infty \phi_u u^2 \, dx - \frac{1}{q} \int_{\mathbb{R}^3} |u|^q \, dx$$

where $V_\infty$ and $K_\infty$, given by condition $(H_1)$, belong to $(0, \infty)$. Otherwise, define $C_\infty = \infty$. $I_\infty(u)$ is well defined for $u \in H_\infty$, where $H_\infty$ is a Sobolev space endowed with the norm

$$\|u\|_\infty = \int_{\mathbb{R}^3} (|\nabla u|^2 + V_\infty u^2) \, dx$$

equivalent to the usual Sobolev norm on $H^1(\mathbb{R}^3)$. 6
4 System \((S_1)\)

Setting \(\varepsilon = 1\), in this section we consider the following system

\[
\begin{align*}
-\Delta u + V(x)u + K(x)\phi(x)u &= |u|^{q-2}u \quad \text{in } \mathbb{R}^3 \\
-\Delta \phi &= K(x)u^2 \\
\end{align*}
\]

whose solutions are critical points of the corresponding functional

\[
I(u) = \frac{1}{2} \int_{\mathbb{R}^3} (|\nabla u|^2 + V(x)u^2) \, dx + \frac{1}{4} \int_{\mathbb{R}^3} K(x)\phi_u(x)u^2 \, dx - \frac{1}{p} \int_{\mathbb{R}^3} |u|^p \, dx
\]

which is well defined for \(u \in H_1\), where

\[
H_1 = \{u \in H^1(\mathbb{R}^3) : \int_{\mathbb{R}^3} V(x)u^2 \, dx < \infty\}
\]

with the same norm notation of the Sobolev space \(H^1(\mathbb{R}^3)\).

Similar to the autonomous case, the functional \(I\) satisfies the mountain pass geometry, then there exists a sequence \((u_n) \subset H_1\) such that

\[
I(u_n) \to c \quad \text{and} \quad I'(u_n) \to 0 \tag{8}
\]

where

\[
c = \inf_{\gamma \in \Gamma} \max_{0 \leq t \leq 1} I(\gamma(t))
\]

and

\[
\Gamma = \{\gamma \in C([0, 1], H^1(\mathbb{R}^3)) : \gamma(0) = 0, I(\gamma(1)) < 0\}.
\]

An important tool in our analysis is the following theorem:

**Theorem 6.** If \(c < C_\infty\), then \(c\) is a critical value for \(I\).

**Proof.** From ALVES, SOUTO and SOARES [2], \((u_n)\) is bounded in \(H_1\). As a consequence, passing to a subsequence if necessary, \(u_n \rightharpoonup u\) in \(H_1\). From Proposition 2 (v), \(\phi_{u_n} \rightharpoonup \phi_u\) in \(D^{1,2}(\mathbb{R}^3)\), as \(n \to \infty\). Then, \((u, \phi_u)\) is a weak solution of \((SP_1)\). Similar to the proof of Lemma 3 \(I(u) = c\). It remains to show that \(u \neq 0\).

By contradiction, consider \(u \equiv 0\).

From Alves, Souto and Soares [2], if there exist constants \(\eta, R\) such that

\[
\liminf_{n \to +\infty} \int_{B_R(0)} u_n^2 \, dx \geq \eta > 0
\]

then \(u \neq 0\).

Hence, there exists a subsequence of \((u_n)\), still denoted by \((u_n)\), such that

\[
\lim_{n \to +\infty} \int_{B_R(0)} u_n^2 \, dx = 0.
\]

Let \(\mu\) and \(\nu\) be such that

\[
\begin{align*}
\inf_{x \in \mathbb{R}^3} V(x) &< \mu < \liminf_{|x| \to \infty} V(x) = V_\infty \\
\inf_{x \in \mathbb{R}^3} K(x) &< \nu < \liminf_{|x| \to \infty} K(x) = K_\infty
\end{align*}
\]
and take \( R > 0 \) such that
\[
V(x) > \mu, \quad \forall \, x \in \mathbb{R}^3 \setminus B_R(0)
\]
\[
K(x) > \nu, \quad \forall \, x \in \mathbb{R}^3 \setminus B_R(0).
\]

For each \( n \in \mathbb{N} \), there exist \( t_n > 0 \), \( t_n \to 1 \) such that \( I(t_n u_n) = \max_{t \geq 0} I(t u_n) \). The convergence of \((t_n)\) follows from (13). In fact, since
\[
\|u_n\|^2 + \int_{\mathbb{R}^3} K(x) \phi_{u_n} u_n^2 \, dx = \int_{\mathbb{R}^3} |u_n|^q \, dx + o_n(1)
\]
we have
\[
t_n^2 \|u_n\|^2 + t_n^4 \int_{\mathbb{R}^3} K(x) \phi_{u_n} u_n^2 \, dx = t_n^q \int_{\mathbb{R}^3} |u_n|^q \, dx + o_n(1).
\]
Then,
\[
(1 - t_n^2) \|u_n\|^2 = (t_n^q - t_n^2) \int_{\mathbb{R}^3} |u_n|^q \, dx + o_n(1)
\]
Suppose \( t_n \to t_0 \). Letting \( n \to +\infty \),
\[
0 = (t_0^2 - 1)\ell_1 + t_0^2(t_0^q - 4)\ell_2
\]
where \( \ell_1, \ell_2 > 0 \). Hence, \( t_0 = 1 \).

Consequently, we have
\[
I(u_n) - I(t_n u_n) = \frac{1 - t_n^2}{2} \|u_n\|^2 + \frac{1}{4} (1 - t_n^4) \int_{\mathbb{R}^3} K(x) \phi_{u_n} u_n^2 \, dx + \frac{t_n^q - 1}{q} \int_{\mathbb{R}^3} |u_n|^q \, dx + o_n(1)
\]
which implies, for every \( t \geq 0 \),
\[
I(u_n) \geq I(t u_n) + o_n(1)
\]
\[
= \frac{t^2}{2} \int_{\mathbb{R}^3} \nabla u_n^2 + V(x) u_n^2 \, dx + \frac{t^4}{4} \int_{\mathbb{R}^3} K(x) \phi_{u_n} u_n^2 \, dx - \frac{t^q}{q} \int_{\mathbb{R}^3} |u_n|^q \, dx + I_{\mu,\nu}(t u_n) - I_{\mu,\nu}(t u_n) + o_n(1)
\]
\[
\geq \frac{t^2}{2} \int_{B_R(0)} (V(x) - \mu) u_n^2 \, dx + \frac{t^4}{4} \int_{B_R(0)} (K(x) - \nu) \phi_{u_n} u_n^2 \, dx + I_{\mu,\nu}(t u_n) + o_n(1),
\]
(9)
where \( I_{\mu,\nu}(u) \) is given by (7).

Consider \( \tau_n \) such that \( I_{\mu,\nu}(\tau_n u_n) = \max_{t \geq 0} I_{\mu,\nu}(t u_n) \). As in the above arguments, \( \tau_n \to 1 \).

Letting \( t = \tau_n \) in (9), we have
\[
I(u_n) \geq c_{\mu,\nu} + \frac{\tau_n^2}{2} \int_{B_R(0)} (V(x) - \mu) u_n^2 + \frac{\tau_n^4}{4} \int_{B_R(0)} (K(x) - \nu) \phi_{u_n} u_n^2 \, dx + o_n(1).
\]
Taking the limit \( n \to +\infty \), we have \( c \geq c_{\mu,\nu} \). Next, taking \( \mu \to V_\infty \) and \( \nu \to K_\infty \), we obtain \( c \geq C_\infty \), proving Theorem 5.
\[\Box\]
Proof of Theorem 1

This section is devoted to study the existence, regularity and the asymptotic behavior of solutions for the system \((SP_\varepsilon)\) for small \(\varepsilon\). The proof of Theorem 1 is divided into three subsections as follows:

5.1 Existence of a solution

Theorem 7. Suppose \((H_0) - (H_1)\) hold and consider

\[ C_\infty > \inf_{\xi \in \mathbb{R}^3} C(\xi) \quad (C_\infty) \]

Then, there exists \(\varepsilon^* > 0\) such that system \((S_\varepsilon)\) has a positive solution for every \(0 < \varepsilon < \varepsilon^*\).

Proof. By hypothesis \((C_\infty)\), there exists \(b \in \mathbb{R}^3\) and \(\delta > 0\) such that

\[ C(b) + \delta < C_\infty. \quad (10) \]

Define \(u_\varepsilon(x) = u(x - \frac{b}{\varepsilon})\), where, from Proposition \(4\) \(u\) is a solution of the autonomous Schrödinger-Poisson system \((SP_b)\)

\[
\begin{aligned}
-\Delta u + V(b)u + K(b)\phi(b)u &= |u|^{q-2}u \quad \text{in} \quad \mathbb{R}^3 \\
-\Delta \phi &= K(b)u^2 \quad \text{in} \quad \mathbb{R}^3
\end{aligned}
\]

with \(I_b(u) = C(b)\).

Let \(t_\varepsilon\) be such that \(I_\varepsilon(t_\varepsilon u_\varepsilon) = \max_{t \geq 0} I_\varepsilon(t u_\varepsilon)\). Similar to the proof of Theorem \(6\), we have

\[ \lim_{\varepsilon \to 0} t_\varepsilon = 1. \]

Then, since

\[ c_\varepsilon = \inf_{\gamma \in \mathcal{I}} \max_{0 \leq t \leq 1} I_\varepsilon(\gamma(t)) = \inf_{u \in H^1_{0 \neq 0}} \max_{t \geq 0} I_\varepsilon(t u) \leq \max_{t \geq 0} I_\varepsilon(t u) = I_\varepsilon(t_\varepsilon u_\varepsilon) \]

we have

\[ \limsup_{\varepsilon \to 0} c_\varepsilon \leq \limsup_{\varepsilon \to 0} I_\varepsilon(t_\varepsilon u_\varepsilon) = I_b(u) = C(b) < C(b) + \delta, \]

which implies that, from \(10\)

\[ \limsup_{\varepsilon \to 0} c_\varepsilon < C_\infty. \]

Therefore, there exists \(\varepsilon^* > 0\) such that \(c_\varepsilon < C_\infty\) for every \(0 < \varepsilon < \varepsilon^*\). In view of Theorem \(6\) system \((S_\varepsilon)\) has a positive solution for every \(0 < \varepsilon < \varepsilon^*\). \(\Box\)

5.2 Regularity of the solution

The first result is a suitable version of Brezis and Kato \(6\) and the second one is a particular version of Theorem 8.17 from Gilbarg and Trudinger \(12\).
Proposition 8. Consider $u \in H^1(\mathbb{R}^3)$ satisfying
\[-\Delta u + b(x)u = f(x,u) \quad \text{in } \mathbb{R}^3\]
where $b : \mathbb{R}^3 \to \mathbb{R}$ is a $L^\infty_{\text{loc}}(\mathbb{R}^3)$ function and $f : \mathbb{R}^3 \to \mathbb{R}$ is a Caratheodory function such that
\[0 \leq f(x,s) \leq C_f(s^r + s), \quad \forall s > 0, \, x \in \mathbb{R}^3.\]

Then, $u \in L^t(\mathbb{R}^3)$ for every $t \geq 2$. Moreover, there exists a positive constant $C = C(t, C_f)$ such that
\[
\|u\|_{L^t(\mathbb{R}^3)} \leq C \|u\|_{H^1(\mathbb{R}^3)}.
\]

Proposition 9. Consider $t > 3$ and $g \in L^\frac{t}{2}(\Omega)$, where $\Omega$ is an open subset of $\mathbb{R}^3$. Then, if $u \in H^1(\Omega)$ is a subsolution of
\[
\Delta u = g \quad \text{in } \Omega
\]
we have, for any $y \in \mathbb{R}^3$ and $B_{2R}(y) \subset \Omega$, $R > 0$
\[
\sup_{B_{R}(y)} u \leq C \left( \|u^+\|_{L^2(B_{2R}(y))} + \|g\|_{L^\frac{t}{2}(B_{2R}(y))} \right)
\]
where $C = C(t, R)$.

In view of Propositions 8 and 9, the positive solutions of $(SP_{\varepsilon})$ are in $C^2(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$ for all $\varepsilon > 0$. Similar arguments was employed by He and Zou [13].

5.3 Concentration of solutions

Lemma 10. Suppose $(H_0) - (H_1)$ hold. Then, there exists $\beta_0 > 0$ such that
\[c_{\varepsilon} \geq \beta_0,
\]
for every $\varepsilon > 0$. Moreover,
\[
\limsup_{\varepsilon \to 0} c_{\varepsilon} \leq \inf_{\xi \in \mathbb{R}^3} C(\xi).
\]

Proof. Let $w_{\varepsilon} \in H_{\varepsilon}$ be such that $c_{\varepsilon} = I_{\varepsilon}(w_{\varepsilon})$. Then, from condition $(H_0)$
\[c_{\varepsilon} = I_{\varepsilon}(w_{\varepsilon}) \geq \inf_{w \in H^1_{\varepsilon}} \sup_{t \geq 0} J(tu) = \beta_0, \quad \forall \varepsilon > 0,
\]
where
\[
J(u) = \frac{1}{2} \int_{\mathbb{R}^3} (|\nabla u|^2 + \alpha u^2) \, dx + \frac{1}{4} \int_{\mathbb{R}^3} \alpha \phi_{\varepsilon} u^2 \, dx - \frac{1}{q} \int_{\mathbb{R}^3} |u|^q \, dx.
\]

Let $\xi \in \mathbb{R}^3$ and consider $w \in H^1(\mathbb{R}^3)$ a least energy solution for system $(SP_{\xi})$, that is, $I_{\xi}(w) = C(\xi)$ and $I'_{\xi}(w) = 0$. Let $w_{\varepsilon}(x) = w(x - \frac{\xi}{\varepsilon})$ and take $t_{\varepsilon} > 0$ such that
\[c_{\varepsilon} \leq I_{\varepsilon}(t_{\varepsilon} w_{\varepsilon}) = \max_{t \geq 0} I_{\varepsilon}(tw_{\varepsilon}).
\]
Similar to the proof of Theorem 6, \( t_\varepsilon \to 1 \) as \( \varepsilon \to 0 \), then
\[
c_\varepsilon \leq I_\xi(t_\varepsilon w_\varepsilon) \to I_\xi(w) = C(\xi), \quad \text{as } \varepsilon \to 0
\]
which implies that \( \limsup_{\varepsilon \to 0} c_\varepsilon \leq C(\xi), \quad \forall \xi \in \mathbb{R}^3 \).
Therefore,
\[
\limsup_{\varepsilon \to 0} c_\varepsilon \leq \inf_{\xi \in \mathbb{R}^3} C(\xi).
\]

Lemma 11. There exist a family \((y_\varepsilon) \subset \mathbb{R}^3\) and constants \( R, \beta > 0 \) such that
\[
\liminf_{\varepsilon \to 0} \int_{B_R(y_\varepsilon)} u_\varepsilon^2 \, dx \geq \beta, \quad \text{for each } \varepsilon > 0.
\]

Proof. By contradiction, suppose that there exists a sequence \( \varepsilon_n \to 0 \) such that
\[
\limsup_{n \to \infty} \sup_{y \in \mathbb{R}^3} \int_{B_R(y)} u_n^2 \, dx = \beta, \quad \text{for all } R > 0.
\]
where, for the sake of simplicity, we denote \( u_n(x) = u_{\varepsilon_n}(x) \). Hereafter, denote \( \phi_{\varepsilon_n,u_n}(x) = \phi_{u_{\varepsilon_n},u_n}(x) \).
From Lemma I.1 in [16], we have
\[
\int_{\mathbb{R}^3} |u_n|^q \, dx \to 0, \quad \text{as } n \to \infty.
\]
But, since,
\[
\int_{\mathbb{R}^3} (|\nabla u_n|^2 + V(\varepsilon_n x)u_n^2) \, dx + \int_{\mathbb{R}^3} \varepsilon_n^2 K(\varepsilon_n x)\phi_{u_{\varepsilon_n},u_n}^2 \, dx = \int_{\mathbb{R}^3} |u_n|^q \, dx
\]
we have
\[
\int_{\mathbb{R}^3} (|\nabla u_n|^2 + V(\varepsilon_n x)u_n^2) \, dx \to 0, \quad \text{as } n \to \infty.
\]
Therefore,
\[
\lim_{n \to \infty} c_{\varepsilon_n} = \lim_{n \to \infty} I_{\varepsilon_n}(u_n) = 0
\]
which is an absurd, since for some \( \beta_0 > 0, \ c_\varepsilon \geq \beta_0 \), from Lemma 10.

Lemma 12. The family \((\varepsilon y_\varepsilon)\) is bounded. Moreover, if \( y^* \) is the limit of the sequence \((\varepsilon_n y_{\varepsilon_n})\) in the family \((\varepsilon y_\varepsilon)\), then we have
\[
C(y^*) = \inf_{\xi \in \mathbb{R}^3} C(\xi).
\]

Proof. Consider \( u_n(x) = u_{\varepsilon_n}(x + y_{\varepsilon_n}) \). Suppose by contradiction that \((\varepsilon_n y_{\varepsilon_n})\) goes to infinity.
It follows from Lemma 11 that there exists constants \( R, \beta > 0 \) such that
\[
\int_{B_R(0)} u_{\varepsilon_n}^2 \, dx \geq \beta > 0, \quad \text{for all } n \in \mathbb{N}.
\]
Since \( u_n(x) \) satisfies
\[
- \Delta u_n + V(\varepsilon_n x + \varepsilon_n y_{\varepsilon_n})u_n + \varepsilon_n^2 K(\varepsilon_n x + \varepsilon_n y_{\varepsilon_n})\phi_{\varepsilon_n,u_n} u_n = |u_n|^{q-2} u_n, \tag{12}
\]
then, \( u_n(x) \) is bounded in \( H_\varepsilon \). Hence, passing to a subsequence if necessary, \( u_n \to \hat{u} \geq 0 \) weakly in \( H_\varepsilon \), strongly in \( L^p_{\text{loc}}(\mathbb{R}^3) \) for \( p \in (2,6) \) and a.e. in \( \mathbb{R}^3 \). From (11), \( \hat{u} \neq 0 \).

Using \( \hat{u} \) as a test function in (12) and taking the limit, we get
\[
\int_{\mathbb{R}^3} (|\nabla \hat{u}|^2 + \mu \hat{u}^2) \, dx \leq \int_{\mathbb{R}^3} (|\nabla \hat{u}|^2 + \mu \hat{u}^2) \, dx + \int_{\mathbb{R}^3} \nu \phi \hat{u}^2 \, dx \leq \int_{\mathbb{R}^3} |\hat{u}|^q \, dx \tag{13}
\]
where, \( \mu \) and \( \nu \) are positive constants such that
\[
\mu < \liminf_{|x| \to \infty} V(x) \quad \text{and} \quad \nu < \liminf_{|x| \to \infty} K(x).
\]

Consider the functional \( I_{\mu,\nu} : H^1(\mathbb{R}^3) \to \mathbb{R} \) given by
\[
I_{\mu,\nu}(u) = \frac{1}{2} \int_{\mathbb{R}^3} (|\nabla u|^2 + \mu u^2) \, dx + \frac{1}{4} \int_{\mathbb{R}^3} \nu \phi u^2 \, dx - \frac{1}{q} \int_{\mathbb{R}^3} |u|^q \, dx.
\]

Let \( \sigma > 0 \) be such that \( I_{\mu,\nu}(\sigma \hat{u}) = \max_{t > 0} I_{\mu,\nu}(t \hat{u}) \).

We claim that
\[
\sigma^2 \int_{\mathbb{R}^3} (|\nabla \hat{u}|^2 + \mu \hat{u}^2) \, dx + \sigma^4 \int_{\mathbb{R}^3} \nu \phi \hat{u}^2 \, dx = \sigma^q \int_{\mathbb{R}^3} |\hat{u}|^q \, dx. \tag{14}
\]

In fact, from (13)
\[
I_{\mu,\nu}(\sigma \hat{u}) = \frac{\sigma^2}{2} \int_{\mathbb{R}^3} (|\nabla \hat{u}|^2 + \mu \hat{u}^2) \, dx + \frac{\sigma^4}{4} \int_{\mathbb{R}^3} \nu \phi \hat{u}^2 \, dx - \frac{\sigma^q}{q} \int_{\mathbb{R}^3} |\hat{u}|^q \, dx
\]
\[
\leq \frac{\sigma^2}{2} \int_{\mathbb{R}^3} |\hat{u}|^q \, dx + \frac{\sigma^4}{4} \int_{\mathbb{R}^3} \nu \phi \hat{u}^2 \, dx - \frac{\sigma^q}{q} \int_{\mathbb{R}^3} |\hat{u}|^q \, dx
\]

it follows that \( \sigma \leq 1 \), and since \( \frac{d}{dt} I_{\mu,\nu}(t \hat{u}) \bigg|_{t=\sigma} = 0 \), we obtain
\[
\frac{d}{dt} I_{\mu,\nu}(t \hat{u}) \bigg|_{t=\sigma} = \sigma \int_{\mathbb{R}^3} (|\nabla \hat{u}|^2 + \mu \hat{u}^2) \, dx + \sigma^3 \int_{\mathbb{R}^3} \nu \phi \hat{u}^2 \, dx - \sigma^{q-1} \int_{\mathbb{R}^3} |\hat{u}|^q \, dx = 0
\]
proving (14).

From Lemma 11 equation (14) and the fact that \( \sigma \leq 1 \), we have
\[
c_{\mu,\nu} = \inf_{u \neq 0} \max_{t > 0} I_{\mu,\nu}(t u) = \inf_{u \neq 0} I_{\mu,\nu}(\sigma u) \leq I_{\mu,\nu}(\sigma \hat{u})
\]
\[
= \frac{\sigma^2}{2} \int_{\mathbb{R}^3} (|\nabla \hat{u}|^2 + \mu \hat{u}^2) \, dx + \frac{\sigma^4}{4} \int_{\mathbb{R}^3} \nu \phi \hat{u}^2 \, dx - \frac{\sigma^q}{q} \int_{\mathbb{R}^3} |\hat{u}|^q \, dx
\]
\[
= \frac{\sigma^2}{4} \int_{\mathbb{R}^3} (|\nabla \hat{u}|^2 + \mu \hat{u}^2) \, dx + \frac{\sigma^q(q-4)}{4q} \int_{\mathbb{R}^3} \nu \phi \hat{u}^2 \, dx - \frac{\sigma^q}{q} \int_{\mathbb{R}^3} |\hat{u}|^q \, dx
\]
\[
\leq \frac{1}{4} \int_{\mathbb{R}^3} (|\nabla \hat{u}|^2 + \mu \hat{u}^2) \, dx + \frac{q-4}{4q} \int_{\mathbb{R}^3} |\hat{u}|^q \, dx
\]
\[
\leq \liminf_{n \to \infty} \left( I_{\varepsilon_n}(u_n) - \frac{1}{4} I'_{\varepsilon_n}(u_n) u_n \right)
\]
\[
= \liminf_{n \to \infty} c_{\varepsilon_n} \leq \limsup_{n \to \infty} c_{\varepsilon_n} \leq \inf_{\xi \in \mathbb{R}^3} C(\xi).
\]
hence, \( c_{\mu,\nu} \leq \inf_{\xi \in \mathbb{R}^3} C(\xi) \).

If we consider
\[
\begin{align*}
\mu & \to \liminf_{|x| \to \infty} V(x) = V_\infty \quad \text{and} \quad \nu \to \liminf_{|x| \to \infty} K(x) = K_\infty
\end{align*}
\]
then, by the continuity of the function \((\mu, \nu) \mapsto c_{\mu,\nu}\) we obtain
\( C_\infty \leq \inf_{\xi \in \mathbb{R}^3} C(\xi) \), which contradicts condition \((C^\infty)\). Therefore, \((\varepsilon y_\varepsilon)\) is bounded and there exists a subsequence of \((\varepsilon y_\varepsilon)\) such that \(\varepsilon_n y_{\varepsilon_n} \to y^*\).

Now we proceed to prove that \( C(y^*) = \inf_{\xi \in \mathbb{R}^3} C(\xi) \).

Recalling that \( u_n(x) = u_{\varepsilon_n}(x + y_{\varepsilon_n}) \) and from the arguments above, \( \hat{u} \) satisfies the equation
\[
-\Delta u + V(y^*) u + K(y^*) \phi_y u = |u|^{q-2} u
\]  
(15)

The Euler-Lagrange functional associated to this equation is \( I_{y^*} : H_{y^*}(\mathbb{R}^3) \), defined as in \((10)\) with \( \xi = y^* \).

Using \( \hat{u} \) as a test function in \((15)\) and taking the limit, we obtain
\[
\int_{\mathbb{R}^3} (|\nabla \hat{u}|^2 + V(y^*) \hat{u}^2) \, dx \leq \int_{\mathbb{R}^3} |\hat{u}|^q \, dx.
\]

Then,
\[
I_{y^*}(\sigma \hat{u}) = \max_{t > 0} I_{y^*}(t \hat{u}).
\]

Finally, from Lemma \((10)\) and since \(0 < \sigma \leq 1\) we have
\[
\begin{align*}
\inf_{\xi \in \mathbb{R}^3} C(\xi) & \leq C(y^*) \leq I_{y^*}(\sigma \hat{u}) \\
& = \frac{\sigma^2}{4} \int_{\mathbb{R}^3} (|\nabla \hat{u}|^2 + V(y^*) \hat{u}^2) \, dx + \frac{\sigma^q (q - 4)}{4q} \int_{\mathbb{R}^3} |\hat{u}|^q \, dx \\
& \leq \frac{1}{4} \int_{\mathbb{R}^3} (|\nabla \hat{u}|^2 + V(y^*) \hat{u}^2) \, dx + \frac{q - 4}{4q} \int_{\mathbb{R}^3} |\hat{u}|^q \, dx \\
& \leq \liminf_{n \to \infty} \left[ \frac{1}{4} \int_{\mathbb{R}^3} (|\nabla u_n|^2 + V(\varepsilon_n x + \varepsilon_n y_{\varepsilon_n}) u_n^2) \, dx + \frac{q - 4}{4q} \int_{\mathbb{R}^3} |u_n|^q \, dx \right] \\
& \leq \liminf_{n \to \infty} \left( I_{\varepsilon_n}(u_n) - \frac{1}{4} I'_{\varepsilon_n}(u_n) u_n \right) \\
& \leq \liminf_{n \to \infty} \varepsilon_n \leq \inf_{\xi \in \mathbb{R}^3} C(\xi)
\end{align*}
\]
which implies that \( C(y^*) = \inf_{\xi \in \mathbb{R}^3} C(\xi) \).

As a consequence of the previous lemma, there exists a subsequence of \((\varepsilon_n y_{\varepsilon_n})\) such that \(\varepsilon_n y_{\varepsilon_n} \to y^*\).

Let \( u_{\varepsilon_n}(x + y_{\varepsilon_n}) = u_n(x) \) and consider \( \tilde{u} \in H^1 \) such that \( u_n \to \tilde{u} \).

**Lemma 13.** \( u_n \to \tilde{u} \) in \( H^1(\mathbb{R}^3) \), as \( n \to \infty \). Moreover, there exists \( \varepsilon^* > 0 \) such that \( \lim_{|x| \to \infty} u_\varepsilon(x) = 0 \) uniformly on \( \varepsilon \in (0, \varepsilon^*) \).
Proof. By applying Lemmas 10 and 12, we observe that

\[
\inf_{\xi \in \mathbb{R}^3} C(\xi) = C(y^*) + \frac{1}{4} I_{y^*}'(\tilde{u}) \tilde{u}
\]

\[
= \frac{1}{4} \int_{\mathbb{R}^3} (|\nabla \tilde{u}|^2 + V(y^*) |\tilde{u}|^2) \, dx + \left( \frac{q-4}{4q} \right) \int_{\mathbb{R}^3} |\tilde{u}|^q \, dx
\]

\[
= \lim_{n \to \infty} \frac{1}{4} \int_{\mathbb{R}^3} (|\nabla u_n|^2 + V(\varepsilon_n x + \varepsilon_n y_n) u_n^2) \, dx + \left( \frac{q-4}{4q} \right) \int_{\mathbb{R}^3} |u_n|^q \, dx
\]

\[
\leq \limsup_{n \to \infty} \frac{1}{4} \int_{\mathbb{R}^3} (|\nabla u_n|^2 + V(\varepsilon_n x + \varepsilon_n y_n) u_n^2) \, dx + \left( \frac{q-4}{4q} \right) \int_{\mathbb{R}^3} |u_n|^q \, dx
\]

\[
= \limsup_{n \to \infty} \left( I_{\varepsilon_n}(u_{\varepsilon_n}) - \frac{1}{4} I_{\varepsilon_n}'(u_{\varepsilon_n}) u_{\varepsilon_n} \right)
\]

\[
= \limsup_{n \to \infty} c_{\varepsilon_n} \leq \inf_{\xi \in \mathbb{R}^3} C(\xi)
\]

then,

\[
\lim_{n \to \infty} \int_{\mathbb{R}^3} (|\nabla u_n|^2 + V(\varepsilon_n x + \varepsilon_n y_n) u_n^2) \, dx = \int_{\mathbb{R}^3} (|\nabla \tilde{u}|^2 + V(y^*) |\tilde{u}|^2) \, dx.
\]

Now observe that

\[
c_{\varepsilon_n} = I_{\varepsilon_n}(u_{\varepsilon_n}) - \frac{1}{4} I_{\varepsilon_n}'(u_{\varepsilon_n}) u_{\varepsilon_n}
\]

\[
= \frac{1}{4} \int_{\mathbb{R}^3} (|\nabla u_n|^2 + V(\varepsilon_n x) u_n^2) \, dx + \left( \frac{q-4}{4q} \right) \int_{\mathbb{R}^3} |u_n|^q \, dx
\]

\[
= \frac{1}{4} \int_{\mathbb{R}^3} (|\nabla u_n|^2 + V(\varepsilon_n x + \varepsilon_n y_n) u_n^2) \, dx + \left( \frac{q-4}{4q} \right) \int_{\mathbb{R}^3} |u_n|^q \, dx
\]

\[
:= \alpha_n
\]

hence,

\[
\limsup_{n \to \infty} \alpha_n = \limsup_{n \to \infty} c_{\varepsilon_n} \leq C(y^*).
\]

On the other hand, using Fatou’s Lemma,

\[
\liminf_{n \to \infty} \alpha_n \geq \frac{1}{4} \int_{\mathbb{R}^3} (|\nabla \tilde{u}|^2 + V(y^*) |\tilde{u}|^2) \, dx + \left( \frac{q-4}{4q} \right) \int_{\mathbb{R}^3} |\tilde{u}|^q \, dx
\]

\[
= I_{y^*}(\tilde{u}) - \frac{1}{4} I_{y^*}'(\tilde{u}) \tilde{u}
\]

\[
\geq C(y^*)
\]

then, \( \lim_{n \to \infty} \alpha_n = C(y^*) \).

Therefore, since \( \tilde{u} \) is the weak limit of \((u_n)\) in \( H^1(\mathbb{R}^3) \), we conclude that \( u_n \to \tilde{u} \) strongly in \( H^1(\mathbb{R}^3) \).

In particular, we have

\[
\lim_{R \to \infty} \int_{|x| \geq R} u_n^{2^*} \, dx = 0 \quad \text{uniformly on } n. \quad (16)
\]

Applying Proposition 8 with \( b(x) = V(\varepsilon_n x + \varepsilon_n y_n) + \varepsilon_n^2 K(\varepsilon_n x + \varepsilon_n y_n) \phi_{u_n} \), we obtain \( u_n \in L^t(\mathbb{R}^3), \ t \geq 2 \) and

\[
||u_n||_t \leq C ||u_n||
\]
where $C$ does not depend on $n$.

Now consider

$$-\Delta u_n \leq -\Delta u_n + V(\varepsilon_n x + \varepsilon_n y_n)u_n + \varepsilon_n^2 K(\varepsilon_n x + \varepsilon_n y_n)\phi_{u_n} u_n$$

$$= |u_n|^q - u_n := g_n(x).$$

For some $t > 3$, $\|g_n\|_t \leq C$, for all $n$. Using Proposition 9, we have

$$\sup_{B_R(y)} u_n \leq C\left(\|u_n\|_{L^2(B_{2R}(y))} + \|g_n\|_{L^2_t(B_{2R}(y))}\right)$$

for every $y \in \mathbb{R}^3$, which implies that $\|u_n\|_{L^\infty(\mathbb{R}^3)}$ is uniformly bounded. Then, from (16),

$$\lim_{|x| \to \infty} u_n(x) = 0 \quad \text{uniformly on } n \in \mathbb{N}.$$ 

Consequently, there exists $\varepsilon^* > 0$ such that

$$\lim_{|x| \to \infty} u_\varepsilon(x) = 0 \quad \text{uniformly on } \varepsilon \in (0, \varepsilon^*).$$

To finish the proof of Theorem 1, it remains to show that the solutions of $(SP_\varepsilon)$ have at most one local (hence global) maximum point $y^*$ such that $C(y^*) = \min_{\xi \in \mathbb{R}^3} C(\xi)$.

From the previous Lemma, we can focus our attention only in a fixed ball $B_R(0) \subset \mathbb{R}^3$. If $w \in L^\infty(\mathbb{R}^3)$ is the limit in $C^2_{loc}(\mathbb{R}^3)$ of

$$w_n(x) = u_n(x + y_n)$$

then, from Gidas, Ni and Nirenberg [13], $w$ is radially symmetric and has a unique local maximum at zero which is a non-degenerate global maximum. Therefore, there exists $n_0 \in \mathbb{N}$ such that $w_n$ does not have two critical points in $B_R(0)$ for all $n \geq n_0$. Consider $p_\varepsilon \in \mathbb{R}^3$ this local (hence global) maximum of $w_\varepsilon$.

Recall that if $u_\varepsilon$ is a solution of $(S_\varepsilon)$, then

$$v_\varepsilon(x) = u_\varepsilon\left(\frac{x}{\varepsilon}\right)$$

is a solution of $(SP_\varepsilon)$.

Since $p_\varepsilon$ is the unique maximum of $w_\varepsilon$ then, $\hat{y}_\varepsilon = p_\varepsilon + \varepsilon y_\varepsilon$ is the unique maximum of $u_\varepsilon$. Hence, $\hat{y}_\varepsilon = \varepsilon p_\varepsilon + \varepsilon y_\varepsilon$ is the unique maximum of $v_\varepsilon$.

Once $p_\varepsilon \in B_R(0)$, that is, it is bounded, and $\varepsilon y_\varepsilon \to y^*$, we have

$$\hat{y}_\varepsilon \to y^*.$$ 

where $C(y^*) = \inf_{\xi \in \mathbb{R}^3} C(\xi)$. Consequently, the concentration of functions $v_\varepsilon$ approach to $y^*$.

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References

[1] C. O. Alves, S. H. M. Soares, Existence and concentration of positive solutions for a class of gradient systems, Nonlinear Differ. Equ. Appl., 12, 437-457, 2005.

[2] C.O. Alves, S.H.M. Soares, M.A.S. Souto, Schrödinger-Poisson equations without Ambrosetti-Rabinowitz condition, J. Math. Anal. Appl., 377 (2011), 584-592.

[3] A. Ambrosetti, On Schrödinger-Poisson systems, Milan J. Math., 76 (2008), 257-274.

[4] A. Azzollini, A. Pomponio, Ground state solutions for the nonlinear Schrödinger-Maxwell equations, J. Math. Anal. Appl., 345 (2008), 90-108.

[5] V. Benci, D. Fortunato, An eigenvalue problem for the Schrödinger-Maxwell equations, Top. Meth. Nonlinear Anal, 11, (1998) 283-293.

[6] H. Brezis, T. Kato, Remarks on the Schrödinger operator with singular complex potentials, J. Math. pures et appl., 58, (1979) 137-151.

[7] G. Cerami, G. Vaira, Positive solutions for some non-autonomous Schrödinger-Poisson systems, J. Differential Equations, 248 (2010), 521-543.

[8] G. M. Coclite, A multiplicity result for the nonlinear Schrödinger-Maxwell equations, Commun. Appl. Anal., 7 (2003), 417-423.

[9] T. D’Aprile, D. Mugnai, Solitary waves for nonlinear Klein-Gordon-Maxwell and Schrodinger-Maxwell equations, Proc. R. Soc. Edinb., Sect. A 134 (2004), 1-14.

[10] Y. Fang, J. Zhang, Multiplicity of solutions for the nonlinear Schrödinger-Maxwell system, Commun. Pure App. Anal., 10 (2011), 1267-1279.

[11] B. Gidas, Wei-Ming Ni, L. Nirenberg, Symmetry and related Properties via the Maximum Principle, Commun. Math. Phys., 68 (1979), 209-243.

[12] D. Gilbarg, N. S. Trudinger, Elliptic Partial Differential Equations of Second Order, Second edition, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], (224). Springer-Verlag, Berlin, 1983.

[13] X. He, W. Zou, Existence and concentration of ground states for Schrödinger-Poisson equations with critical growth, J. Math. Phys., 53 (2012), 023702.

[14] I. Ianni, G. Vaira, On concentration of positive bound states for the Schrödinger-Poisson problem with potentials, Adv. Nonlinear Studies, 8, 573-595, 2008.

[15] H. Kikuchi, On the existence of a solution for a elliptic system related to the Maxwell-Schrödinger equations, Nonlinear Anal., 67 (2007), 1445-1456.

[16] P. Lions, The concentration-compactness principle in the calculus of variations. The locally compact case. II, Ann. Inst. H. Poincaré Anal. Non Linéaire, 1 (1984), 223-283.

[17] C. Mercuri, Positive solutions of nonlinear Schrödinger-Poisson systems with radial potentials vanishing at infinity, Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei Mat. Appl., 19 (2008), 211-227.

[18] P.H. Rabinowitz, On a class of nonlinear Schrödinger equations, Z. Angew. Math. Phys., 43, 270-291, 1992.
[19] D. Ruiz, *The Schrödinger-Poisson equation under the effect of a nonlinear local term*, J. Funct. Anal., 237 (2006), 665-674.

[20] X. Wang, *On concentration of positive solutions bounded states of nonlinear Schrödinger equations*, Comm. Math. Phys., 153, 229-244, 1993.

[21] X. Wang, B. Zeng, *On concentration of positive bound states of nonlinear Schrödinger equations with competing potential functions*, SIAM J. Math. Anal., 28, 633-655, 1997.

[22] M. Willem, "Minimax theorems", Progress in Nonlinear Differential Equations and their Applications, 24, Birkhäuser, Boston, 1996.

[23] M-H Yang, Z-Q Han, *Existence and multiplicity results for the nonlinear Schrödinger-Poisson systems*, Nonlinear Anal., 13 (2012), 1093-1101.

[24] L. Zhao, H. Liu, F. Zhao, *Existence and concentration of solutions for the Schrödinger-Poisson equations with steep potential*, J. Differential Equations, 255 (2013), 1-23.