Random Walk Representation of the Lattice Fermionic Propagators and the Quark Model

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Abstract

A representation of the continuum fermionic propagator as a sum of directed random walks on a lattice is presented. Also a random walk representation for the lattice fermionic propagators is developed for the case of the naive, the Wilson, and the Kogut-Susskind fermions. For the naive fermions the phenomenon of fermion doubling appears as having $2^D$ distinct spin factors being associated with a single path in $D$-dimensions. In the case of the Wilson and the Kogut-Susskind fermions, in the naive continuum limit, the path integral representation coincides with the path integral representation for the continuum fermionic propagator. Using this representation the Green’s functions of lattice QCD involving quark operators are written as a sum over the paths of valence quark, the gauge fields and the sea quarks being integrated out. Possible advantages of such a representation are illustrated by showing how one can use numerical simulations to obtain a heuristic insight into the relationship between QCD and the constituent quark model.
1 Introduction

It seems increasingly likely that in the near future the lattice simulations will provide an accurate numerical solution of Quantum Chromodynamics (QCD). Complimenting these numerical investigations are many attempts at obtaining a heuristic understanding of the dynamics of QCD by trying to isolate those fluctuations in the gauge fields that may be responsible for the confinement and chiral symmetry breaking \[1\]. A similar interplay between numerical simulations and heuristic understanding is lacking for the problems involving the quark degrees of freedom. Thus there are many striking features of strong interactions, an incomplete list would include the apparent successes of the non-relativistic quark model and the Okubo-Zweig-Iizuka rule \[2\], for which the numerical simulations provide little or no heuristic guidance.

The reasons for this are familiar, the fermionic degrees of freedom being elements of Grassmann algebra cannot be simulated numerically. These degrees of freedom have to be integrated out from the functional integral before it can be numerically evaluated. The result of the integration is expressed in terms of the fermionic determinant and the fermionic propagator \[3, 4, 5, 6, 7\]. But the numerical calculation of the determinant and the inverse of a fermionic matrix seems to offer little scope for visualizing the degrees of freedom that gave rise to them in the first place.

In this paper I suggest an alternate way of dealing with the fermionic propagator which is more amenable to heuristic investigations. The approach is based on the path integral representation, or the equivalent random walk representation\(^1\) of the fermionic propagator \[8, 9, 10, 11\]. One way of obtaining a random walk representation for the lattice fermions is to expand the propagator for the Wilson fermions in a Neumann series \[12, 13, 5\]. An alternate approach, which is more general, is based on Polyakov’s observation \[8\] that the path integral for a fermion can be regarded as a natural generalization of the path integral for a scalar particle by including an appropriate spin factor. In this paper I will take the latter approach.

With in such an approach it is natural to ask the question that how does the phenomenon of fermion doubling on a lattice \[14, 15\] manifests itself in the language of path integrals. To answer this, in Sec. 2, first I represent the

\(^1\)In this paper I will interchangeably use the words path-integral and random walk representation.
continuum fermionic propagator as a sum over directed random walks on a lattice. Then in Sec. 3 and 4, I consider the path integral representation for the lattice fermionic propagators, this includes the so called naive fermions, the Wilson fermions, and the Kogut-Susskind fermions. In the case of the naive fermions the expected phenomenon of fermion doubling appears as the existence of $2^D$ distinct spin-factors for a single path in a $D$-dimensional hypercubic lattice. Continuing the analysis in Sec. 4, it is shown that for the case of the Wilson and the Kogut-Susskind fermions, in the naive continuum limit, only one of the $2^D$ spin-factor survives and the path integral representation of the propagators coincides with that of the continuum fermionic propagator developed in Sec. 2.

Having developed a path integral representation for a free fermionic propagator it is easy to extend it to the case when a gauge field is defined on a lattice [16, 17]. Using the resulting representation of the fermionic propagator, in Sec. 5, I write the Green’s functions involving quark fields as a sum over the paths of the valence quarks. This way of writing the Green’s functions in which the integration over the gauge field is carried out before summing over the quark paths is a close transcription, which allows for numerical explorations, of the heuristic way in which Wilson [16] motivated his criterion for the quark confinement. Perhaps the advantage of this approach is that it allows us to probe the theory using a language, the paths of the valence quarks, which is easy to visualize. This is illustrated in Sec. 5 by indicating how the numerical simulation can provide a insight into the relationship between QCD and the constituent quark model. I state my conclusions in the last section.

2 Representing Continuum Fermionic Propagator as Random Walks on a Lattice

The fermionic propagator can be represented as a path integral either using an appropriate spin-factor [8], or equivalently as a sum over directed random walks [9, 10, 11]. In this section I will represent the continuum Euclidean Fermionic propagator as a sum over directed random walks on a $D$ dimensional hypercubic lattice. Following [11] I start with the Fermionic
propagator in the momentum space

\[ \Delta(p, m) = \frac{1}{m + p \mu \gamma_\mu}, \]  

(1)

where \( \gamma_\mu \) are the Euclidean Dirac matrices satisfying the anticommutation relation

\[ \{ \gamma_\mu, \gamma_\nu \} = -2 \delta_{\mu \nu}. \]  

(2)

The propagator is now written as a Laplace transform

\[ \Delta(p, m) = \frac{1}{D} \int_0^\infty \exp \left\{ -\frac{mL}{D} \right\} \tilde{G}(L, p) dL, \]  

(3)

\[ \tilde{G}(L, p) = \exp \left\{ -\frac{\gamma_\mu p_\mu L}{D} \right\}, \]  

(4)

where \( D \) is the dimension of the Euclidean space which, unless stated otherwise, will be taken to be four. The motivation for expressing the propagator as a Laplace transform is that the Fourier transform of Eq. (4) can be then interpreted as the probability amplitude for the particle to reach the point \( x \) by traveling along paths of length \( L \). With this in mind, the Laplace transform is expressed as

\[ \tilde{G}(L, p) = \lim_{N \to \infty} \left( 1 - \frac{\gamma_\mu p_\mu L}{DN} \right)^N. \]  

(5)

The individual terms appearing in Eq. (5) can be written as

\[ 1 - \frac{\gamma_\mu p_\mu L}{DN} = \frac{1}{2D} \sum_{i=1}^{2D} (1 - i\gamma_\mu \hat{n}_i)(1 - i\hat{n}_i p_\mu L/N), \]  

(6)

where \( \hat{n}_i \) are the directions available to a particle on a \( D \)-dimensional hypercubic Euclidean lattice, and for \( D = 4 \) can be taken to be:

\[ \begin{align*}
\hat{n}^1 &= (1, 0, 0, 0), & \hat{n}^5 &= (-1, 0, 0, 0), \\
\hat{n}^2 &= (0, 1, 0, 0), & \hat{n}^6 &= (0, -1, 0, 0), \\
\hat{n}^3 &= (0, 0, 1, 0), & \hat{n}^7 &= (0, 0, -1, 0), \\
\hat{n}^4 &= (0, 0, 0, 1), & \hat{n}^8 &= (0, 0, 0, -1).
\end{align*} \]  

(7)

The identity, Eq. (6), can be verified by using the above choice of \( \hat{n}_i \).
This should be contrasted with the following representation

\[ 1 - \gamma^\mu p^\mu = \int d\hat{n} (1 - i\gamma \cdot \hat{n})(1 - i\hat{n} \cdot \frac{L}{N}) \tag{8} \]

where the integration is over all the directions in \( D \)-dimensional Euclidean space and results in a representation of the continuum fermionic propagator as a sum over paths in the continuum.

Substituting Eq. (6) into Eq. (5) leads to

\[ \tilde{G}(L, p) = \prod_{k=1}^{N} \left\{ \frac{1}{2D} \sum_{i_{N}=1}^{2D} \cdots \sum_{i_{1}=1}^{2D} (1 - i\gamma \cdot \hat{n}^{i_{N}}) \cdots (1 - i\gamma \cdot \hat{n}^{i_{1}}) \right\}, \tag{9} \]

which in the limit \( N \to \infty \), for a fixed \( L \), can be written as

\[ \tilde{G}(L, p) = \left( \frac{1}{2D} \right)^N \sum_{i_{N}=1}^{2D} \cdots \sum_{i_{1}=1}^{2D} \{(1 - i\gamma \cdot \hat{n}^{i_{N}}) \cdots (1 - i\gamma \cdot \hat{n}^{i_{1}}) \}
\times \exp \left( -i \frac{L}{N} \cdot \sum_{k=1}^{N} \hat{n}^{i_{k}} \right). \tag{10} \]

Taking the Fourier transform of Eq. (10) leads to

\[ \tilde{G}(L, x - y) = \lim_{N \to \infty} \left( \frac{1}{2D} \right)^N \sum_{i_{N}=1}^{2D} \cdots \sum_{i_{1}=1}^{2D} \{(1 - i\gamma \cdot \hat{n}^{i_{N}}) \cdots (1 - i\gamma \cdot \hat{n}^{i_{1}}) \}
\times \delta^{D}(x - y - \frac{L}{N} \cdot \sum_{k=1}^{N} \hat{n}^{i_{k}}). \tag{11} \]

This, as anticipated, can be interpreted as a sum over paths of length \( L \) starting at the point \( y \) and ending at the point \( x \). The paths in Eq. (11) are defined on a \( D \)-dimensional Euclidean lattice, with each step in the direction \( \hat{n}^{i} \) there is a spin-factor \((1 - i\gamma \cdot \hat{n}^{i})\) associated with it. The step size defines the lattice constant

\[ a = \frac{L}{N}. \tag{12} \]

In view of Eq. (12) the fixed length propagator can be written as

\[ \tilde{G}(L, x - y) = \tilde{G}(N; a; x - y), \]

\[ \tilde{G}(N; a; x - y) = \lim_{a \to 0} \left( \frac{1}{2D} \right)^N \sum_{i_{N}=1}^{2D} \cdots \sum_{i_{1}=1}^{2D} \{(1 - i\gamma \cdot \hat{n}^{i_{N}}) \cdots (1 - i\gamma \cdot \hat{n}^{i_{1}}) \}
\times \delta^{D}(x - y, a \sum_{k=1}^{N} \hat{n}^{i_{k}}). \tag{13} \]
In Eq. (13) the Dirac-$\delta$ function has been replaced with the Kronecker-$\delta$ function.

Using Eq. (3) and replacing the integration over $L$ by a summation over $N$ leads to the following expression for the continuum fermionic propagator in the coordinate space

$$\Delta(x - y, m) = \lim_{a \to 0} a \frac{D}{N=1} \sum_{N=1}^{\infty} \exp\left(-\frac{maN}{D}\right) \tilde{G}(N; a; x - y).$$

(14)

Though the sum in the above equation starts from $N = 1$ but in the limit $a \to 0$, which is equivalent to the $\|x - y\| >> a$, only terms with very large value of $N$ contributes because of the presence the $\delta$-function in Eq. (13).

Above representation of the continuum fermionic propagator as a sum over directed random walks on a lattice can be extended to the case when a gauge field is defined on a lattice. Following [16], see also [17], the continuum fermionic propagator in the presence of a background gauge field, $U$, which is defined on a lattice can be written as:

$$\Delta(x - y; m; U) = \lim_{a \to 0} a \frac{D}{N=1} \sum_{N=1}^{\infty} \exp\left(-\frac{maN}{D}\right) \tilde{G}(N; a; x - y; U),$$

(15)

where $\tilde{G}(N; a; x - y; U)$ is given by

$$\tilde{G}(N; a; x - y; U) = \left(\frac{1}{2D}\right)^N \sum_{i_N=1}^{2D} \cdots \sum_{i_1=1}^{2D} (1 - i\gamma.\hat{n}^i_N) \cdots (1 - i\gamma.\hat{n}^i_1)$$

$$\times (U_{\hat{n}^N} \cdots U_{\hat{n}^1}) \frac{\delta^D(x - y, a \sum_{k=1}^{N} \hat{n}^{i_k})}{a^D}$$

(16)

where $U_{\hat{n}^k}$ is a $SU(N)$ matrix associated with the link of the lattice that corresponds to the $k_{th}$ step of the random walk.

It is interesting to note that in the quenched approximation, where one neglects the contribution of the fermionic determinant, using above representation of the fermion propagator one can give a non-perturbative definition of QCD that keeps the fermionic degrees of freedom in continuum while defining the gauge degrees of freedom on a lattice. This would be of significance if one could also define the fermionic determinant in a similar manner (for some attempts along this direction see [18]), for this would give a non-perturbative definition of QCD that avoids the problem of fermion doubling.
3 Random Walk Representation of the Naive Lattice Fermionic Propagator

In this and the next section a representation of the lattice fermionic propagators as a sum over paths on a lattice will be obtained. The naive lattice fermionic propagator \[3, 6, 7\] in the momentum space is given by

\[
\Delta_n(p, m) = \frac{1}{m + \frac{i}{2}\gamma_\mu \sin(p_\mu a)},
\]

where \(a\) is the lattice constant and the \(\gamma_\mu\)s are the Euclidean Dirac matrices satisfying Eq. (2). It is well known that in the continuum limit Eq. (17) describes the propagation of \(2^D\) distinct fermions \[14\], an example of the general phenomenon of fermion doubling \[15\]. Writing the naive fermionic propagator as a sum over paths will allow us to understand this phenomenon in the language of path integrals.

The development of the path integral representation for the naive lattice fermionic propagator starts, as before, with a formal Laplace transform of Eq. (17)

\[
\Delta_n(p, m) = \frac{1}{D} \int_0^\infty \exp\left\{-\frac{mL}{D}\right\} \tilde{G}_n(L, p) dL,
\]

and with the identity

\[
1 - \frac{\gamma_\mu \sin(p_\mu a)L}{DNa} = \frac{1}{2D} \sum_{i=1}^{2D} (1 - i\gamma_\mu \hat{n}_i) (1 - i\hat{n}_i \sin(p_\mu a) \frac{L}{Na}),
\]

where \(\hat{n}_i\) are the unit vectors, along both the positive and negative directions of a \(D\)-dimensional Euclidean hypercubic lattice and are defined in Eq. (7).

Using Eq. (19) and Eq. (20), as in section 2, we can write \(\tilde{G}_n(L, p)\) as

\[
\tilde{G}_n(L, p) = \lim_{N \to \infty} \left( \frac{1}{2D} \right)^N \sum_{i_1=1}^{2D} \cdots \sum_{i_N=1}^{2D} \left\{ (1 - i\gamma_\mu \hat{n}^{i_N}) \cdots (1 - i\gamma_\mu \hat{n}^{i_1}) \right\}
\times \exp \left\{ -i \frac{L}{Na} \sin(p_\mu a) \sum_{k=1}^N \hat{n}_\mu^i \right\}.
\]
The Fourier transform of Eq. (21), defined by

\[ \tilde{G}_n(L, x - y) = \int_{-\pi/a}^{\pi/a} \frac{d^D p}{(2\pi)^D} \exp(ip.(x - y)) \tilde{G}_n(L, p) \]  

(22)

with \( x \) and \( y \) belonging to the lattice, can be written as

\[ \tilde{G}_n(L, x - y) = \lim_{N \to \infty} \left( \frac{1}{2D} \right)^N \sum_{i_1=1}^{2D} \cdots \sum_{i_N=1}^{2D} (1 - i\gamma \hat{n}^{i_N}) \cdots (1 - i\gamma \hat{n}^{i_1}) \times I[x - y, \{\hat{n}^i\}], \]  

(23)

where \( I[x - y, \{\hat{n}^i\}] \) is given by

\[ I[x - y, \{\hat{n}^i\}] = \int_{-\pi/a}^{\pi/a} \frac{d^D p}{(2\pi)^D} \exp(ip.(x - y)) \exp\left\{ -i \frac{L}{Na} \sin(p_\mu a) \sum_{k=1}^{N} \hat{n}^{i_k}_\mu \right\}. \]  

(24)

Next consider \( \tilde{G}_n(L, x - y) \) in the limit \( a \to 0 \). For this it will be convenient to, using the periodicity of the lattice momentum space, shift the range of integration from \( -\pi/a \leq p_\mu < \pi/a \) to \( -\frac{\pi}{2a} \leq p_\mu < \frac{3\pi}{2a} \). Also the range of for each \( p_\mu \) is divided into two regions, in the first region \( p_\mu \) ranges from \( -\frac{\pi}{2a} \leq p_\mu < \frac{\pi}{2a} \), and in the second region it ranges from \( \frac{\pi}{2a} \leq p_\mu < \frac{3\pi}{2a} \). In this manner the lattice momentum space divides into \( 2^D \) hypercubes centered around points for which \( p_\mu \), the \( \mu \)th component, is either zero or \( \frac{\pi}{a} \). These centers will be denoted by the vector \( \vec{K}^h \). Using this division of the momentum space Eq. (24) can be written as

\[ I[x - y; \{\hat{n}^i\}] = \sum_{h=1}^{2^D} I_h[x - y; \{\hat{n}^i\}], \]  

(25)

with \( I_h[x - y; \{\hat{n}^i\}] \) given by

\[ I_h[x - y; \{\hat{n}^i\}] = \int_{h} \frac{d^D p}{(2\pi)^D} \exp(-i \frac{L}{Na} \sin(p_\mu a) \sum_{k=1}^{N} \hat{n}^{i_k}_\mu) \exp(ip.(x - y)). \]  

(26)

In Eq. (23), as noted above, the \( \mu \)th component of \( p_\mu \) lies either in the range \( -\frac{\pi}{2a} \leq p_\mu < \frac{\pi}{2a} \) or in the range \( \frac{\pi}{2a} \leq p_\mu < \frac{3\pi}{2a} \). Defining a new integration variable \( k_\mu \),

\[ p_\mu = \vec{K}^h_\mu + k_\mu, \]  

(27)
allows one to write Eq. (26) as

\[ I_h = e^{iK^h(x-y)} \int_{-\pi/a}^{\pi/a} \frac{d^Dk}{(2\pi)^D} \exp(-iL \sin(\bar{K}^h a + k\mu a)/a) \sum_{k=1}^{N} \hat{n}^i_k \exp(ik.(x-y)). \]

Taking the limit \(a \to 0\) of the above expression leads to

\[ \lim_{a \to 0} I_h[x,y;\{\hat{n}^i\}] = e^{iK^h(x-y)} \delta^D((x-y) - \frac{L}{N} \cos(\bar{K}^h a)k\mu \sum_{k=1}^{N} \hat{n}^i_k), \]

where \(\cos(\bar{K}^h a)\) is either +1 or -1, depending on whether \(\bar{K}^h\) is 0 or \(\pi/a\). Thus \(\tilde{G}_n(L,x)\) in the limit \(a \to 0\) has the following form

\[ \tilde{G}_n(L,x) = \left(\frac{1}{2D}\right)^N \sum_{i_N=1}^{2D} \cdots \sum_{i_1=1}^{2D} (1 - i\gamma \cdot \hat{n}^i_N) \cdots (1 - i\gamma \cdot \hat{n}^i_1) \times \sum_{h=1}^{2D} \{e^{iK^h(x-y)} \delta^D(x - \frac{L}{N} \cos(\bar{K}^h a)k\mu \sum_{k=1}^{N} \hat{n}^i_k)\}. \]

\(\tilde{G}_n(L,x)\) can now be interpreted as a sum over \(2^D\) distinct random walks of length \(L\) but for the factor of \(\cos(\bar{K}^h a)\). This factor can removed from the \(\delta\) function by suitable redefinition of the spin-factors. To see this consider the case when the \(\nu_{th}\) component of \(\bar{K}^h\) is \(\frac{\pi}{a}\), then substitute the set of unit vectors \(\{\hat{n}^i\}\) by a new set of vectors obtained by changing the sign of the \(\nu_{th}\) component of the vectors \(\{\hat{n}^i\}\). This transformation keeps Eq. (30) invariant as it corresponds merely to relabeling of the vectors \(\{\hat{n}^i\}\). Because of this the spin factor for a link \(1 - i(\cdots + \gamma_{\nu} \cdot \hat{n}^i_{\nu} + \cdots)\), where no summation over \(\nu\) is implied, changes to \(1 - i(\cdots - \gamma_{\nu} \cdot \hat{n}^i_{\nu} + \cdots)\). This can be brought to its original form by doing a similarity transformation on the gamma matrices

\[ \gamma_{\nu} = -\gamma_{\nu}^h = S^h \gamma_{\nu}(S^h)^{-1}. \]

As a result of these transformations \(\tilde{G}_n(L,x)\) can be written as

\[ \tilde{G}_n(L,x) = \sum_{h=1}^{2D} \tilde{G}_h(L,x), \]

\[ \tilde{G}_h(L,x) = \left(\frac{1}{2D}\right)^N \sum_{i_N=1}^{2D} \cdots \sum_{i_1=1}^{2D} (1 - i\gamma^h \cdot \hat{n}^i_N) \cdots (1 - i\gamma^h \cdot \hat{n}^i_1) \times \delta^D(x - \frac{L}{N} \sum_{k=1}^{N} \hat{n}^i_k), \]

\[ \text{(33)} \]
which can be interpreted as a sum over paths of length $L$ with the step-size $a$ where $a$ is the lattice constant. Finally writing the formal Laplace transform, Eq. (18), as a sum over lengths in lattice units leads to

$$\lim_{a \to 0} \Delta_n(x, m) = \frac{a}{D} \sum_{N=1}^{\infty} \exp\left(\frac{-maN}{D}\right) \sum_{h=1}^{2D} \tilde{G}_h(N, x),$$

$$\tilde{G}_h(N, x) = e^{iK_h \cdot x} \left(\frac{1}{2D}\right)^N \sum_{i=1}^{2D} \cdots \sum_{i=1}^{2D} (1 - i\gamma^h \cdot \hat{n}^i) \cdots (1 - i\gamma^h \cdot \hat{n}^i)$$

$$\times \frac{\delta^D(x, a \sum_{k=1}^{N} \hat{n}^i)}{a^D}. \quad (34)$$

The above equation shows that in the continuum limit the naive fermionic propagator represents propagation of $2^D$ distinct fermions. In the same limit the propagator of each of these fermions can be written as a sum over paths, or random walks, on the lattice. Different species differ by the spin-factor associated with them and by an overall phase factor. Since the phase factor can be absorbed in the normalization of the single particle wave function and the spin factors differ only by a similarity transformation of the $\gamma$-matrices, therefore we have in Eq. (34) a simultaneous propagation of $2^D$ Dirac particles. Further analysis along the line of Ref. [14] reveals that these doublers appear in pairs with opposite chirality.

4 Random Walk Representation for Wilson and Kogut-Susskind Fermions

Having elucidated the phenomenon of fermion doubling in the language of path integrals for fermions, it is natural to see how the Wilson and the Kogut-Susskind fermions avoid or mitigate this problem. I start with the Wilson fermions, the corresponding propagator in the momentum space is [3, 6, 7]

$$\Delta_W(p) = \frac{1}{m + \frac{1}{a} \sin(p\mu a) + \frac{1}{a} \sum_{\mu=1}^{D} (1 - \cos(p\mu a))}. \quad (35)$$

The development of a path integral representation for the Wilson fermions is similar to the previous two cases, again one writes the propagator as a
formal Laplace transform

\[ \Delta_W(p) = \frac{1}{D} \int_0^\infty dL \exp\left(-\frac{mL}{D}\right) \tilde{G}_W(L, p), \]  

(36)

\[ \tilde{G}_W = \exp\left(\frac{L}{D a} (\gamma_\mu \sin(p_\mu a) + \sum_{\mu=1}^D (1 - \cos(p_\mu a)))\right). \]  

(37)

Then using the following identity

\[ 1 - \frac{L}{D N a} (\gamma_\mu \sin(p_\mu a) + \sum_{\mu=1}^D (1 - \cos(p_\mu a))) = \]  

(38)

\[ \frac{1}{2D} \sum_{i=1}^{2D} \left\{ ((1 - i \gamma_\mu \hat{n}^i)(1 - i \frac{L}{N a} \hat{n}^i_\mu \sin(p_\mu a))) - \frac{L}{D N a} \sum_{\mu=1}^D (1 - \cos(p_\mu a)) \right\}, \]

which can be verified with \( \hat{n}^i \) as the unit vectors defined by Eq. (7), to write \( \tilde{G}_W(L, p) \) as

\[ \tilde{G}_W(L, p) = \lim_{N \to \infty} \left( \frac{1}{2D} \right)^N \sum_{i_N=1}^{2D} \cdots \sum_{i_1=1}^{2D} (1 - i \gamma_\mu \hat{n}^{i_N}_\mu \cdots \hat{n}^{i_1}_\mu) \]  

(39)

\[ \times \exp\left(-i \frac{L}{N a} \sin(p_\mu a) \cdot \sum_{k=1}^N \hat{n}^i_\mu - \frac{L}{D a} \sum_{\mu=1}^D (1 - \cos(p_\mu a)) \right). \]

Taking the Fourier transform of Eq. (39) leads to

\[ \tilde{G}_W(L, x) = \lim_{N \to \infty} \left( \frac{1}{2D} \right)^N \sum_{i_N=1}^{2D} \cdots \sum_{i_1=1}^{2D} (1 - i \gamma_\mu \hat{n}^{i_N}_\mu \cdots \hat{n}^{i_1}_\mu) \]  

(40)

\[ \times I_W[x, \{ \hat{n}^i \}], \]

\[ I_W[x, \{ \hat{n}^i \}] = \int_{a}^{+\infty} \exp(ip \cdot x) \exp\left(-i \frac{L}{N a} \sin(p_\mu a) \cdot \sum_{k=1}^N \hat{n}^i_\mu \right. \]  

\[ - \frac{L}{D a} \sum_{\mu=1}^D (1 - \cos(p_\mu a))). \]  

(41)

Eq. (41) differs from Eq. (24) only by the presence of the Wilson term therefore one can again write it as

\[ I_W[x; \{ \hat{n}^i \}] = \sum_{h=1}^{2D} I_{Wh}[x; \{ \hat{n}^i \}], \]  

(42)
where \( I_{Wh}[x; \{ \hat{n}^i \}] \) is given by

\[
I_{Wh}[x; \{ \hat{n}^i \}] = e^{i \tilde{K}^h x} \int_{\frac{x}{2\pi}}^{\frac{x}{2\pi}} \frac{d^D k}{(2\pi)^D} \exp(i k.x) \\
\times \exp \left( -i \frac{L \sin(\tilde{K}^h a + k^i a)}{a} \sum_{k=1}^{N} \hat{n}^i_k \\
- \sum_{\mu=1}^{D} (1 - \cos(\tilde{K}^h a + k^i a)) \right). \tag{43}
\]

In the limit \( a \to 0 \) one obtains

\[
\lim_{a \to 0} I_{Wh}[x; \{ \hat{n}^i \}] = e^{i \tilde{K}^h x} \exp(- \frac{L}{D a} \sum_{\mu=1}^{D} (1 - \cos(\tilde{K}^h a))) \\
\times \delta^D(x - \frac{L}{N} \cos(\tilde{K}^h a) \sum_{k=1}^{N} \hat{n}^i_k). \tag{44}
\]

From Eq. (44) one sees that in the limit \( a \to 0 \) only \( I_{Wh} \) corresponding to \( \tilde{K}^h = 0 \) will survive. Rest of the \( I_{Wh} \) goes to zero, approximately as \( \exp(- \frac{L}{a}) \), and therefore there is no contribution from the “doublers” and one can write

\[
\lim_{a \to 0} I_W[x; \{ \hat{n}^i \}] = \delta^D(x - \frac{L}{N} \sum_{k=1}^{N} \hat{n}^i). \tag{45}
\]

Substituting Eq. (45) in Eq. (40) leads to the identification of \( L \) as the length of the path and \( N \) as the number of steps of the random walk. Finally the propagator in the continuum limit can be written as

\[
\lim_{a \to 0} \Delta_W(x, m) = \frac{a}{D} \sum_{N=1}^{\infty} \exp(- \frac{m a N}{D}) \tilde{G}_W(N; a; x), \tag{46}
\]

\[
\tilde{G}_W(N; a; x) = \left( \frac{1}{2D} \right)^N \sum_{i_N=1}^{2D} \cdots \sum_{i_1=1}^{2D} (1 - i \gamma . \hat{n}^{i_N}) \cdots (1 - i \gamma . \hat{n}^{i_1}) \\
\times \delta^D(x - y, a \sum_{k=1}^{N} \hat{n}^i_k). \tag{47}
\]

Thus, in the continuum limit one can represent the propagator of the Wilson fermions as a sum over paths on a lattice, further more the representation is identical to the random walk representation of the continuum.
fermionic propagator. A path integral representation for the Wilson fermions in the form of a hopping-parameter expansion is well known \[7\] and is similar to the path integral representation developed above. But conceptually the two differ, in obtaining Eq. (46) no expansion in hopping parameter is done and the path integral representation emerges only in the continuum limit. Conceptually the path integral representation for the Wilson fermions is a way of transcribing the spin factor, present in the continuum path integral representation of a fermion, on to a lattice [19].

Next I consider the propagator for the Kogut-Susskind fermions. Since the development is almost identical to the previous cases only the final results will be stated. The propagator for the Kogut-Susskind fermions [6, 7] can be written as

$$\Delta_{KS}(p) = \frac{1}{m + (\gamma_\mu \otimes 1)\frac{1}{b}\sin(p_\mu b) + \frac{1}{b} \sum_{\mu=1}^{D} (1 - \cos(p_\mu b))(\gamma_5 \otimes t_\mu t_5)},$$

(48)

where $t_\mu$ are $2^{[D]}$ dimensional flavor matrices, and are given by

$$t_\mu = \gamma_\mu^\dagger,$$

$$t_5 = \gamma_5^\dagger,$$

(49)

and $b = 2a$, $a$ being the lattice spacing. The propagator, Eq. (48), describes the propagation of $2^{[D]}$ degenerate flavors. Formally the Kogut-Susskind propagator differs from the Wilson propagator only by the presence of the flavor space matrices, and that the lattice spacing has been effectively doubled, so one can immediately write down the path integral representation for the Kogut-Susskind propagator as:

$$\lim_{a \to 0} \Delta_{KS}(x, m) = \frac{b}{D} \sum_{N=1}^{\infty} \exp\left(-\frac{mbN}{D}\right) \tilde{G}_{KS}(N; b; x)$$

(50)

where $\tilde{G}_{KS}$ is given by

$$\tilde{G}_{KS}(N; b; x) = \left(\frac{1}{2D}\right)^N \sum_{i_N=1}^{2D} \cdots \sum_{i_1=1}^{2D} ((1 - i\gamma_\cdot \hat{n}^{i_N}) \cdots (1 - i\gamma_\cdot \hat{n}^{i_1})) \otimes 1$$

$$\times \delta^D(x, b \sum_{k=1}^{N} \hat{n}^{i_k}) \frac{b^D}{b^D}.\quad (51)$$
In Eq. (50) one has the path integral representation for the propagator of $2^{[2]}$ degenerate fermions. The paths are defined on a blocked lattice with a step size of $b = 2a$.

The path integral representation of the Kogut-Susskind propagator is, apart from the fact that it represents the propagation of $2^{[2]}$ degenerate fermions, identical to the path integral representation of the continuum fermions. The fact that the path integral representation of both the Wilson and the Kogut-Susskind fermions, which are derived from different lattice actions and have different lattice symmetries, coincides with each other is not surprising as these representations appear only in the continuum limit. What is, perhaps, surprising is the fact that the continuum fermionic propagator can be represented as a sum over paths even when the paths are restricted to a lattice.

5 Path Integral Representation of Green’s Function in QCD and the Quark Model

As stated in the introduction one of the motivations for developing a path integral representation for the lattice fermionic propagators was to look for a formalism that can provide some heuristic insight into the role of quark degrees of freedom in QCD. To this end, in this section, I will write the Green’s functions of lattice QCD involving quark degrees of freedom as a sum over the paths of valence quarks thus making a connection with the heuristic language of quark model.

Consider a meson propagator build from $\bar{\psi}(x)\Gamma\psi(x)$ as the interpolating field for a quark-antiquark meson with $\Gamma$ determining the spin, flavor, and the parity of the meson [6, 7]. Such a propagator can be written as

$$\langle \bar{\psi}(x)\Gamma\psi(x)\bar{\psi}(y)\Gamma\psi(y) \rangle = Z^{-1} \int DU \exp(-S_{eff}(U)) \times (\text{Tr}[\Delta(x, y, U)\Gamma\Delta(y, x, U)\Gamma] - \text{Tr}[\Delta(x, x, U)\Gamma]\text{Tr}[\Delta(y, y, U)\Gamma])$$

(52)

where the quark degrees of freedom have been integrated out giving rise to an effective gauge action and fermionic propagators. The partition function,
$Z$, and the effective gauge action, $S_{\text{eff}}(U)$, being defined by

$$
Z = \int DU \exp(-S_{\text{eff}}(U)),
$$

$$
S_{\text{eff}}(U) = S_g(U) + \ln \det \Delta^{-1}(U).
$$

$S_g(U)$ represents the lattice regularized action for the gauge field $U \text{[3, 4]}$. For the sake of definiteness, the fermionic determinant is assumed to be defined for the Wilson fermions. In what follows it will be convenient to use the following notation to represent the functional integration over the gauge fields

$$
\langle f(U) \rangle_U = Z^{-1} \int DU \exp(-S_{\text{eff}}(U)) f(U).
$$

Also the path integral representation of the quark propagator, Eq. (15), will be written in the following compact notation

$$
\Delta(x, y, U) = \sum_{l_{xy}} \exp(-\mu l) \Phi(l_{xy}) U(l_{xy}),
$$

where $l_{xy}$ denotes a path on a lattice of length $l$, in lattice units, starting at $y$ and ending at $x$, $\mu$ is a measure of the bare mass and is given by

$$
\mu = \frac{ma}{D},
$$

while the spin factor $\Phi(l_{xy})$ and the gauge field factor $U(l_{xy})$ are given explicitly by Eq. (16).

In the above defined notations Eq. (52) can be written as

$$
\langle \bar{\psi}(x) \Gamma \psi(x) \bar{\psi}(y) \Gamma \psi(y) \rangle = \langle \text{Tr}[\Delta(x, y, U) \Gamma \Delta(y, x, U) \Gamma] \rangle_U - \langle \text{Tr}[\Delta(x, x, U) \Gamma] \text{Tr}[\Delta(y, y, U) \Gamma] \rangle_U.
$$

Consider the two terms appearing on the right hand side of Eq. (58) separately and replace the fermionic propagators appearing in them by their path integral representation Eq. (58). The first term can be written as

$$
\langle \text{Tr}[\Delta(x, y, U) \Gamma \Delta(y, x, U) \Gamma] \rangle_U = \sum_{l_{xy}, l_{yx}} \exp(-\mu(l_{xy} + l_{yx})) \times \text{Tr}[\Phi(l_{xy}) \Gamma \Phi(l_{yx}) \Gamma] \times \langle \text{Tr}(U(l_{xy}) U(l_{yx})) \rangle_U.
$$
Similarly the second term can be written as
\[
\langle \text{Tr}[\Delta(x, x, U)] \text{Tr}[\Delta(y, y, U)] \rangle_U = \sum_{l_{xx}, l_{yy}} \exp(-\mu(l_{xx} + l_{yy})) \times \text{Tr}[\Phi(l_{xx})] \text{Tr}[\Phi(l_{yy})] \\
\times \langle \text{Tr}U(l_{xx}) \text{Tr}U(l_{yy}) \rangle_U. \tag{60}
\]

Another quantity of interest is the order parameter for the chiral symmetry, \( \bar{\psi}\psi \), which can be written as a sum over closed paths
\[
\langle \bar{\psi}(0)\psi(0) \rangle = \langle \text{Tr}[\Delta(0, 0, U)] \rangle_U \\
= \sum_{l_{00}} \exp(-\mu l_{00}) \text{Tr}[\Phi(l_{00})] \langle \text{Tr}U(l_{00}) \rangle_U. \tag{61}
\]

In a similar manner all the Green’s functions involving quark fields can be written as a sum over one or more closed (valence) quark paths, each path being weighted by its length, its spin factor, and the expectation value of the path ordered product of the gauge fields along the quark paths. The effect of the dynamical or the sea quarks being included in the expectation value of the gauge field factor. The advantage of writing the quark Green’s function using the path integral representation of the quark propagator is that it separates the contribution coming from the motion of the valence quark, in a sense the kinematics, from the contribution coming from the fluctuations in the gauge fields and the quark degrees of freedom. As a possible application of the random walk representation of the fermionic propagators, consider the relationship between QCD and the constituent quark model \[21\]. In the chiral limit the constituent quark picture for pion is not useful, pion being the Nambu-Goldstone boson, while it is a good approximation for the rho meson. One would like to understand, starting from QCD, the emergence of the constituent quark picture for the rho and its absence for the pion. For this purpose consider the connected part of the meson propagator Eq. (59), \textit{a priori} all possible paths contribute to this propagator but one expects that because of the confinement the paths in which quark and anti-quark are widely separated are strongly suppressed. This motivates us to restrict the sum in Eq. (59) to a class of paths in which the quark and the anti-quark are separated at most by, say \( r_{\text{max}} \), where \( r_{\text{max}} \) is some measure of the meson size. One expects that the constituent quark model should emerge at a coarser level of description \[21\] and such a coarser description should appear as a
result of a renormalization group transformation. In the present context one could try and see if the constituent quark model emerges after the paths have been blocked, where blocking of paths is in analogy with the blocking of spins under the real space renormalization group transformation \cite{22}. One possible way of blocking the paths is by summing over the paths in the neighborhood of a given path belonging to the above defined class. In particular consider a path $l_{xyx}$ (where $l_{xyx}$ being a compact notation for the closed path $l_{xy}l_{yx}$), imagine that that this path is enclosed in a tube of radius $r_b \approx 2r_{max}$, and then sum over all the paths which lie inside this tube to create a blocked path. Denoting the paths lying inside the tube by $\tilde{l}$, one can conjecture that in the case of the rho meson the sum over these paths,

$$\sum_{l_{xyx}} \exp(-\mu \tilde{l}) \text{Tr}[\Phi(l_{xy})\Gamma_\rho \Phi(l_{yx})\Gamma_\rho] \langle \text{Tr}U(l_{xyx}) \rangle_U, \quad (62)$$

can be approximated by a single path with the following weight factors

$$\exp(-m_{eff}(l_{xy} + l_{yx}) \text{Tr}[\Phi(l_{xy})\Gamma_\rho \Phi(l_{yx})\Gamma_\rho] \exp(-V_{eff}(l_{xy}, l_{yx})) \quad (63)$$

where $m_{eff}$ is a measure of the constituent quark mass and $\exp(-V_{eff})$ is the weight factor coming from the confining potential of the constituent quark model. The simplest possible guess for $V_{eff}$ would be $\sigma A_{xyz}$ where $\sigma$ is the string tension and $A_{xyz}$ is the minimal area enclosed by the closed path $l_{xyz}$.

On the other hand for pions one would conjecture that the blocked paths should be approximated by a single effective path with the following weight factors $\exp(-m_\pi l_{xy})$, where $m_\pi$ being the measure of the pion mass on the lattice which in turn should be related to the expectation value of the order parameter for the chiral symmetry breaking, $\langle \bar{\psi}\psi \rangle$. Such heuristic conjectures can be numerically tested, for the sum in Eq. (62) is well defined on a lattice and is a small subset of all possible paths connecting point $x$ and $y$. Perhaps the biggest obstacle in such a numerical investigation is the requirement of large lattice sizes, for the random walk representation appears only as one approaches the continuum limit. For finite lattices one will have to understand how to separate the lattice artifacts from the continuum physics. A first step in such a direction would be to use the above formalism for exploring the dynamic of lattice QCD in two dimensions.
6 Conclusions

Path integral or the random walk representation for the fermionic propagator was developed with two motivations in mind. One to try and understand the phenomena of fermion doubling on a lattice in the language of sum over paths, and other to look for a formalism for the fermionic degrees of freedom that might be more susceptible to heuristic investigations.

In the first respect, it was shown that a naive lattice fermionic propagator can be written as a sum over paths and with each path there are associated not one but $2^D$ spin factors in $D$ dimensions. As a result the naive propagator represents the propagation of $2^D$ fermions. This is in contrast to the case of the continuum fermionic propagator which can be represented as a sum over paths on a lattice with a unique spin factor associated with each path. In the case of the Wilson and the Kogut-Susskind fermions the path integral representation emerges in the continuum limit and coincides with the path integral representation of the continuum fermionic propagator. In this manner the path integral representation provides an additional insight into the relationship between various lattice fermionic propagators and their continuum counterpart, this may be of some use in exploring new ways of representing fermionic degrees of freedom on a lattice.

As to the second motivation, it was shown that using path integral representation for the quark propagators, the Green’s functions of QCD can be written as a sum over the paths of the valence quarks. Such an representation of the Green’s functions allows for the possibility of delineating the important paths that contribute to a Green’s function. Also it allows for the possibility of implementing the ideas of real space renormalization group on variables which are easy to visualize and are close to the heuristic description of QCD, namely the paths of the valence quarks. This was illustrated by showing how one can use numerical simulations to understand the relationship between QCD and the constituent quark model.

The path integral representation of the lattice fermionic propagator is unlikely to be useful for an accurate numerical solution of lattice QCD but it can be a useful tool for testing heuristic insights and for developing new intuitions.
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