The linearity problem for the unitriangular automorphism groups of free groups*

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Abstract

We prove that the unitriangular automorphism group of a free group of rank \( n \) has a faithful representation by matrices over a field, or in other words, it is a linear group, if and only if \( n \leq 3 \). Thus, we have completed a description of relatively free groups with linear the unitriangular automorphism groups. This description was initiated by Erofeev and the author in [1], where proper varieties of groups have been considered.

1 Introduction

For each positive integer \( n \), let \( F_n \) be a free group of rank \( n \) with basis (in other words, free generating set) \( \{f_1, \ldots, f_n\} \). For any \( m \leq n \) \( F_m \) is considered as subgroup \( \text{gp}(f_1, \ldots, f_m) \) of \( F_n \). For any variety of groups \( \mathcal{G} \), let \( \mathcal{G}(F_n) \) denote the verbal subgroup of \( F_n \) corresponding to \( \mathcal{G} \). Let \( G_n = F_n / \mathcal{G}(F_n) \). Then \( G_n \) is a relatively free group of rank \( n \) in the variety \( \mathcal{G} \). By basis of \( G_n \) we mean a subset \( S \) such that every map of \( S \) into \( G_n \) extends, uniquely, to an endomorphism of \( G_n \). Write \( f_i = f_i \mathcal{G}(F_n) \) for \( i = 1, \ldots, n \). Then \( f_1, \ldots, f_n \) is a basis of \( G_n \).

Let \( \mathcal{G} \) be a variety of groups. Let \( G_n \) be the relatively free group corresponding to \( \mathcal{G} \) with basis \( \{f_1, \ldots, f_n\} \). For any \( m \leq n \) \( G_m \) is considered as subgroup \( \text{gp}(f_1, \ldots, f_m) \) of \( G_n \). An automorphism \( \varphi \) of \( G_n \) is called \textit{unitriangular} (w.r.t. the given basis) if \( \varphi \) is defined by a map of the form:

\[
\varphi : f_1 \mapsto f_1, f_i \mapsto u_i f_i, \quad i = 2, \ldots, n,
\]

where \( u_i = u_i(f_1, \ldots, f_{i-1}) \) is an element of \( G_{i-1} \). Every tuple of elements \( (u_2, \ldots, u_n) \) with this condition defines, uniquely, automorphism of \( G_n \). Let \( U_n \) be subgroup consisting of all unitriangular (w.r.t. a given basis) automorphisms of \( G_n \). Then it is called \textit{the unitriangular automorphism group of} \( G_n \). As abstract group \( U_n \) does not depend of a basis.

The question of linearity of \( U_n \) for an arbitrary proper variety \( \mathcal{G} \) has been studied by Erofeev and the author in [1]. All cases of linearity of \( G_n \) have been described. The following Section 1 contains this description. Also, we observe some relative results on linearity for relatively free groups and algebras.

In this paper we study the only open after [1] case when \( \mathcal{G} \) is the variety of all groups. Our main result is given by the following theorem.

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Theorem 1.1. The group $U_n$ of unitriangular automorphisms of the free group $F_n$ of rank $n$ is linear if and only if $n \leq 3$.

Hence, we complete a description of all cases when the unitriangular automorphism group $U_n$, corresponding to an arbitrary variety of groups $\mathcal{G}$, including the variety of all groups, is linear.

2 Some results on linearity.

We observe some results concerning the linearity of the automorphism groups and their subgroups of relatively free groups and algebras. Recall that group $G$ is said to be virtually nilpotent if it has a nilpotent subgroup of finite index.

The linearity of $\text{Aut}(F_2)$ follows by [2] from the linearity of the 4-string braid group $B_4$, which is due to Krammer [3]. Bigelow [4] and also Krammer [5] determined that the braid group $B_n$ is linear for every $n$. Formanek and Procesi in [6] have demonstrated that $\text{Aut}(F_n)$ is not linear for $n \geq 3$.

Auslander and Baumslag [7] determined that for every finitely generated virtually nilpotent group $G$ the automorphism group $\text{Aut}(G)$ is linear. Moreover, $\text{Aut}(G)$ has a faithful matrix representation over the integers $\mathbb{Z}$. In particular, for every relatively free virtually nilpotent group $G_n$, the automorphism group $\text{Aut}(G_n)$ is linear over $\mathbb{Z}$.

Olshanskii [8] proved for any relatively free group $G_n$, which is not virtually nilpotent and is not free, that the automorphism group $\text{Aut}(G_n)$ is not linear. His approach does not give an information on the linearity of the unitriangular automorphism groups $U_n$ for such relatively free groups $G_n$.

Erofeev and the author [1] proved for every proper variety of groups $\mathcal{G}$ that the unitriangular automorphism group $U_n$ is linear if and only if the relatively free group $G_{n-1}$ is virtually nilpotent. More exactly (for $n \geq 3$): if $G_{n-1}$ is virtually nilpotent, then $U_n$ admits a faithful matrix representation over integers $\mathbb{Z}$. It was also shown in [1] that if $n \geq 3$ and $G_{n-1}$ is nilpotent then $U_n$ is nilpotent too.

Now let $C_n$ be an arbitrary relatively free algebra of rank $n$ with set of free generating elements $\{x_1, \ldots, x_n\}$. For $m \leq n$ $C_m$ can be considered as subalgebra of $C_n$ generated by $x_1, \ldots, x_m$. An automorphism $\psi$ of $C_n$ is called unitriangular w.r.t. the given set of free generating elements if it is defined by map of the form:

$$\psi : x_1 \mapsto x_1, x_i \mapsto x_i + u_i \quad i = 2, \ldots, n,$$

where $u_i = u_i(x_1, \ldots, x_{i-1})$ belongs to $C_{i-1}$. Let $U_n$ denote a subgroup of the automorphism group $\text{Aut}(C_n)$ of $C_n$, consisting of all unitriangular automorphisms. As abstract group $U_n$ does not depend from a chosen set of free generating elements of $C_n$.

The author, Chirkov and Shevelin [9] proved that, for a free Lie (free associative, absolutely free, polynomial) algebra $C_n$ of rank $n \geq 4$ over a field of zero characteristic, the unitriangular automorphism group $U_n$ is not linear. Then the following papers [10], [11] presented descriptions of the hypercentral series of groups $U_n$ corresponding to polynomial and free metabelian Lie algebras, respectively. By these results $U_n$ are not linear for $n \geq 3$. By [12], for $n \geq 3$,
the unitriangular automorphism group $U_n$ is not linear in case of polynomial algebra and in case of free associative algebra. By [3] for each relatively free algebra $C_n$ the group $U_n$ is locally nilpotent, thus it is linear.

3 The method of Formanek and Procesi.

Let $G$ be any group, and let $\mathcal{H}(G)$ denote the following HNN-extension of $G \times G$:

$$\mathcal{H}(G) = < G \times G, t : t(g,g)t^{-1} = (1,g), g \in G > . \tag{3}$$

**Theorem 3.1.** (Formanek, Procesi [6]). Let $\rho$ be a linear representation of $\mathcal{H}(G)$. Then the image of $G \times \{1\}$ has a subgroup of finite index with nilpotent derived subgroup, i.e, is nilpotent-by-abelian-by-finite.

**Theorem 3.2.** (Brendle, Hamidi-Tehrani [7]). Let $N$ be a normal subgroup of $\mathcal{H}(G)$ such that the image of $G \times \{1\}$ in $\mathcal{H}(G)/N$ is not nilpotent-by-abelian-by-finite. Then $\mathcal{H}(G)/N$ is not linear.

In [14] a group of the type described in Theorem 3.2 is called a *Formanek and Procesi group*, or *FP-group* for short.

4 Proof of Theorem 1.1.

3.1. For $n \leq 3$, $U_n$ is linear.

**Proof:** Since $U_1$ is trivial and $U_2$ is infinite cyclic the statement is obvious for $n = 1, 2$.

Let $n = 3$. By [11] $U_3$ is generated by automorphisms $\lambda_{2,1}, \lambda_{3,1}, \lambda_{3,2}$. Recall that $\lambda_{i,j}$ maps $f_i$ to $f_j f_i$, and fixes all other basic elements. This is applicable for any group $U_n$. The automorphisms $\lambda_{3,1} \lambda_{3,2}$ generate in $U_3$ a normal free subgroup $F_2$. The automorphism $\lambda_{2,1}$ acts as follows:

$$\lambda_{2,1}^{-1} \lambda_{3,1} \lambda_{2,1} = \lambda_{3,1}, \ \lambda_{2,1}^{-1} \lambda_{3,2} \lambda_{2,1} = \lambda_{3,1} \lambda_{3,2}. \tag{4}$$

Now we’ll show that $U_3$ is isomorphic to a subgroup of $\text{Aut}(F_2)$. Let $\tau_1$ and $\tau_2$ denote inner automorphisms of $F_2$ corresponding to $f_1$ and $f_2$ respectively. This means that any element $g$ of $F_2$ maps by $\tau_i (i = 1, 2)$ to $f_i^{-1} g f_i$. Let $\sigma_{2,1} \in \text{Aut}(F_2)$ fixes $f_1$ and maps $f_2$ to $f_1 f_2$. Obviously, $F_2 = \text{gp}(\tau_1, \tau_2)$ is a free group of rank 2. It is a normal subgroup of $V_3 = \text{gp}(\tau_1, \tau_2, \sigma_{2,1})$. A quotient $V_3 / F_2$ is the infinite cyclic generated by the image of $\sigma_{2,1}$. The corresponding action is determined by:

$$\sigma_{2,1}^{-1} \tau_1 \sigma_{2,1} = \tau_1, \ \sigma_{2,1}^{-1} \tau_2 \sigma_{2,1} = \tau_1 \tau_2. \tag{5}$$

Thus, $U_3$ and $V_3$ are both infinite cyclic extensions of $F_2$. By (4) and (5) we conclude that $\alpha : U_3 \rightarrow V_3$ defined as:

$$\alpha : \lambda_{3,j} \mapsto \tau_j, \ j = 1, 2, \ \lambda_{2,1} \mapsto \sigma_{2,1}, \tag{6}$$
is isomorphism. Since $V_3$ is a subgroup of $\text{Aut}(F_2)$, which is linear by \[2\] and \[3\]. $U_3$ is also linear.

\[\square\]

3.2. For $n \geq 4$, $U_n$ is not linear.

Proof: For $n \geq m$, $U_n$ has a subgroup that is isomorphic to $U_m$. Elements of this subgroup act naturally to $f_1, \ldots, f_m$ and fix elements $f_{m+1}, \ldots, f_n$. So, we just have to prove that $U_4$ is not linear.

By Theorem 3.2 it will be enough to find a subgroup $H$ of $U_4$ that is isomorphic to a quotient $\mathcal{H}(F_2)/N$, where $\mathcal{H}(F_2)$ is given by (3), such that the image of $G \times \{1\}$ in $\mathcal{H}(G)/N$ is not nilpotent-by-abelian-by-finite.

There are two commuting elementwise subgroups of $U_4$ each of them is isomorphic to $F_2$. Namely, there are $gp(\lambda_{3,1}, \lambda_{3,2})$ and $gp(\lambda_{4,1}, \lambda_{4,2})$. Consider them as two copies of $F_2$ via isomorphism defined by map $\lambda_{3,1} \mapsto \lambda_{4,1}, \lambda_{3,2} \mapsto \lambda_{4,2}$. Thus we have a subgroup $F_2 \times F_2$ of $U_4$. Easily to check that:

$$\lambda_{1,j}^{-1}, \lambda_{3,j}, \lambda_{4,j} = \lambda_{3,j}, \quad j = 1, 2.\tag{7}$$

By (3) and (7) we conclude that $H$ is a homomorphic image of $\mathcal{H}(F_2)$ such that the subgroup $F_2 \times F_2$ of $\mathcal{H}(F_2)$ maps isomorphically to the just constructed subgroup of the same type of $U_4$. The image of $i$ is $\lambda_{4,3}$. Hence, $H \simeq \mathcal{H}(F_2)/N$, where $N$ is the kernel of this homomorphism. By Theorem 3.2 $H$, and so $U_4$, is not linear.

\[\square\]

Remark 1. In fact we proved that subgroup $W_4 = gp(\lambda_{3,j}, \lambda_{4,k} : j = 1, 2; l = 1, 2, 3)$ of $U_4$ is not linear. In \[1\] we noted that $[\lambda_{i,j}, \lambda_{j,k}] = \lambda_{i,k}$. Here commutator $[g, f]$ means $gfg^{-1}f^{-1}$. Any group $U_n$ is generated by the elements $\lambda_{i,j}$, for $j < i \leq n$ (see \[1\]). It follows that $W_4$ is the derived subgroup $U_4^2$ of $U_4$. Hence, we proved that the derived subgroup (the second member $\gamma_2U_4$ of the low central series) of $U_4$ is not linear. This subgroup $\gamma_2U_4$ has also characterized in $U_4$ as the stabilizer of $f_1$. In general case, for $n \geq 4$, member $\gamma_{n-2}U_n$ coincides with the elementwise stabilizer of $\{f_1, \ldots, f_{n-3}\}$. Easily to see that the derived subgroup $U_4^2$ can be embedded into $\gamma_{n-2}U_n$. Hence, for every $n \geq 4$, a member $\gamma_{n-2}U_n$ of the low central series of $U_n$ is not linear. This statement is more strong than the statement of Theorem 1.1 about nonlinearity of $U_n$ for $n \geq 4$.

Remark 2. In \[13\] an explicit faithful representation of $\text{Aut}(F_2)$ in $GL_{12}(\mathbb{Z}[t^{\pm 1}, q^{\pm 1}])$ is given. Here $\mathbb{Z}[t^{\pm 1}, q^{\pm 1}]$ is a Laurent polynomial ring. Hence, $U_3$ has a faithful matrix representation over $\mathbb{Z}[t^{\pm 1}, q^{\pm 1}]$.

Also note that $U_3$ can be presented by $< a, b : [[a, b], b] = 1 >$, where $a$ corresponds to $\lambda_{3,2}$, and $b$ corresponds to $\lambda_{2,1}$. By terminology of \[10\] $U_3$ is the first non commutative member in a series of Hydra groups $H_k \leq < a, b : [[a, b], b], \ldots, b] = 1 >, k \geq 1$, where commutator has $k$ entries of $b$. In general, Hydra groups were introduced in \[17\]. It was shown in \[16\] that Hydra groups of such form are residually torsion-free nilpotent. It seems interesting to study their linearity.

Remark 3. By \[13\] a group $G$ is called locally graded if every nontrivial finitely generated subgroup of $G$ has a proper subgroup of finite index. This class contains, for example, all locally solvable and all residually finite groups. Let $G_2$ be the relatively free group of rank 2 in the variety $\text{var}(G)$ generated by $G$. Suppose
that the derived subgroup $G'_2$ is finitely generated. Then by \[18\] $G$ is virtually nilpotent.

Let $\mathcal{G}$ be a variety consisting of locally graded groups. Obviously $\mathcal{G}$ is a proper variety. Suppose that $U_3$ is linear. Then every group $U_n$ is linear. Indeed, by \[1\] $G_2$ is virtually nilpotent. It follows that $G'_2$ is finitely generated. Any group $G_{n-1}$ generates a subvariety $\text{var}(G_{n-1})$ of $\mathcal{G}$. The relatively free group of rank 2 in this subvariety is a homomorphic image of $G_2$, and so has finitely generated derived subgroup. Then by \[1\] $G_{n-1}$ is virtually nilpotent. It follows by \[1\] that $U_n$ is linear. Thus, the linearity of $U_3$ implies the linearity of $U_n$ for every $n \geq 4$. Moreover, $\mathcal{G}$ should be virtually nilpotent.

We see by Theorem 1.1 that just presented statement, that the linearity of $U_3$ implies the linearity of $U_n$ for all $n \geq 4$, is not true for the variety of all groups. Likely, it is also non-true for some proper varieties of groups. As candidates to such varieties we can consider the varieties of groups generated by the famous Golod groups. We conjecture that for every $m \geq 3$ there is a variety $\mathcal{G}_m$ such that the groups $U_n$ are linear if and only if $n \leq m$.

References

[1] S.Yu. Erofeev, V.A. Roman’kov, On the groups of unitriangular automorphisms of relatively free groups, Sibirskii Mat. Zh., 53(2012), no. 5, 991-1000 (Russian); English translation: Siberian Math. J., 53(2012), no. 5, 792-799.

[2] J.L. Dyer, E. Formanek, E.K. Grossman, On the linearity of automorphism groups of free groups, Arch. Math. (Basel), 38(1982), 404-409.

[3] D. Krammer, The braid group $B_4$ is linear, Invent. Math., 142(2000), 451-486.

[4] S. Bigelow, Braid groups are linear, J. Amer. Math. Soc., 14(2001), no. 2, 471-486.

[5] D. Krammer, Braid groups are linear, Ann. Math., 155(2002), 131-156.

[6] E. Formanek, C. Procesi, The automorphism group of a free group is not linear, J. Algebra, 149(1992), 494-499.

[7] L. Auslender, G. Baumslag, Automorphism groups of finitely generated nilpotent groups, Bull. Amer. Math. Soc., 73(1967), 716-717.

[8] A.Yu. Olshanskii, Linear automorphism groups of relatively free groups, Turk. J. Math., 31(2007), no. 1, 105-111.

[9] V.A. Roman’kov, I.V. Chirikov, M.A. Shevelin, Nonlinearity of the automorphism groups of some free algebras, Sibirskii Mat. Zh., 45(2004), no. 5, 1184-1188 (Russian); English translation: Siberian Math. J., 45(2004), no. 5, 974-977.

[10] Yu.V. Sosnovskii, The hypercentral structure of the group of unitriangular automorphisms of a polynomial algebra, Sibirskii Mat. Zh., 48(2007), no. 3, 689-693 (Russian); English translation: Siberian Math. J., 48(2007), no. 3, 555-558.
[11] A.N. Kabanov, The hypercentral structure of the group of unitriangular automorphisms of a free metabelian Lie algebra, *Sibirskii Mat. Zh.*, 50(2001), no. 2, 329-333 (Russian); English translation: *Siberian Math. J.*, 50(2001), no. 2, 261-264.

[12] V.G. Bardakov, M.V. Neshadim, Yu.V. Sosnovskii, Groups of triangular automorphisms of a free associative algebra and a polynomial algebra, *J. Algebra*, 362(2012), 201-220.

[13] V.A. Roman’kov, The local structure of groups of triangular automorphisms of relatively free algebras, *Algebra i Logika*, 51(2012), no. 5, 638-651 (Russian); English translation: *Algebra and Logic*, 51(2012), no. 5, 425-434.

[14] T.E. Brendle, H. Hamidi-Tehrani, On the linearity problem for mapping class groups, *Algebraic and Geometric Topology*, 1(2001), 445-468.

[15] V.G. Bardakov, Linear representations of the group of conjugating automorphisms and the braid groups of some manifolds, *Sibirskii Mat. Zh.*, 46(2005), 17-31 (Russian); English translation: *Siberian Math. J.*, 46(2005), no. 1, 13-23.

[16] G. Baumslag, R. Mikhailov, Residual properties of groups defined by basic commutators, *arXiv math.*: 1301.4629v2. [math. GR] 28 Aug. 2013.

[17] W. Dison, T. Riley, Hydra groups, *arXiv math.*: 1002.1945v2 [math. GR] 14 May 2010.

[18] B. Bajorska, O. Macedonska, W. Tomaszewski, A defining property of virtually nilpotent groups, *Publ. Math. Debrecen*, 81(2012), 415-420.