Pfaffian and determinantal tau functions I

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Abstract

Adler, Shiota and van Moerbeke observed that a tau function of the Pfaff lattice is a square root of a tau function of the Toda lattice hierarchy of Ueno and Takasaki. In this paper we give a representation theoretical explanation for this phenomenon. We consider 2-BKP and two-component 2-KP tau functions. We shall show that a square of a BKP tau function is equal to a certain two-component KP tau function and a square of a 2-BKP tau function is equal to a certain two-component 2-KP tau function.

Key words: integrable systems, tau functions, BKP, DKP, two-component Toda lattice, Pfaff lattice, free fermions

1 Introduction

Sato and Sato [26] and Date, Jimbo, Kashiwara and Miwa [4] introduced and described in the beginning of the 1980’s the KP hierarchy in various setting. They introduced a tau function, which is a fundamental object in this theory. It is an element in a $GL_\infty$ group orbit and as such a solution of the KP hierarchy. Around the same time Date, Jimbo, Kashiwara and Miwa introduced in [5] also a new hierarchy of soliton equations, which they called the KP hierarchy of type B or BKP hierarchy. The corresponding tau function is an element of the $B_\infty$ group orbit and hence a solution to this new BKP hierarchy. In a straightforward calculation they show that this BKP tau function is the square root of a certain KP tau function. Their proof of this phenomenon is fundamental for the contents of this paper. Here we show that this method is also applicable for the observation of Adler, Shiota and van Moerbeke [1], [2], that a tau function of the Pfaff lattice is a square root of a certain Ueno, Takasaki [31] Toda lattice tau function. In this introduction we will recall the work of Date, Jimbo, Kashiwara and Miwa [4] and explain the relation BKP versus KP. The involutions that are used in their work, which provide the relation between the two tau functions of KP and BKP, also give the relation between the various tau functions of the Pfaff and Toda lattice. This means that the construction of the KP group element out of the BKP group element is the same. One finds in both cases that $g_{KP} = h_{BKP} \hat{h}_{BKP}$, where $\hat{h}_{BKP}$ is constructed out of $h_{BKP}$ by using one of the involutions.

The Pfaff lattice of Adler and van Moerbeke [1], [2] was discovered by Jimbo and Miwa ([10], section 7) it was rediscovered by Hirota and Ohta [13] as the coupled KP hierarchy and studied in a paper by Kac and one of the authors [15] as the charged DKP hierarchy. The Pfaff lattice studied in [16] is slightly bigger than the one studied by Adler, Shiota and van Moerbeke in [2] it is the charged BKP hierarchy of [15], called the large BKP in [22] to make difference to the small BKP of [5]. Here we study this charged or large BKP.

In addition we also consider the large 2-BKP which is two-sided evolution (via positive and negative parts of current modes, like the Toda lattice hierarchy of Ueno and Takasaki [31] in contrast to the one-sided KP). There we have two sets of higher times, $t$ and $\bar{t}$. This is the B-analogue of Takasaki’s

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2-DKP of [28] (which he called the Pfaff lattice by analogy with Pfaff lattice of Adler and van Moerbeke). Also we use two discrete variables for BKP (and also for 2-BKP): \( \tau_{m,n}(t, \bar{t}) \).

In this introduction and the first 5 sections we will only describe the one sided case, to avoid technicalities. In section 6 we will introduce the 2-sided case, which will be important for matrix models.

For matrix models we need the semi-infinite Toda and Pfaff lattices. The fermionic expressions for the related tau functions is considered in the Section 7 where a set of examples is written down. In this Section we generalize the observation of Adler, van Moerbeke and Shiota [1], [2] about the relation between Toda and Pfaff lattices.

But first we will start with the recollection of the work of Date, Jimbo, Kashiwara, and Miwa, on KP and BKP hierarchies.

**Fermions and KP hierarchy** Consider, following [11] and [10] charged fermions \( \{ f_i, f^+_i \}_{i \in \mathbb{Z}} \), satisfying

\[
[f_i, f^+_j]_+ = \delta_{ij}, \quad [f_i, f_j]_+ = 0 = [f^+_i, f^+_j]_+ , \quad i, j \in \mathbb{Z}
\]  (1)

where \([x, y]_+ = xy + yx\) is anticommutator. The elements \( f_i \) and \( f^+_i \) form the basis of a vector space, which we denote by \( V \).

Introduce the spin modules (right and left Fock spaces) with vacuum vectors, \( |0\rangle , \langle 0| \), such that

\[
f_i |0\rangle = f^+_i - 1 \quad f^+_i |0\rangle = 0 = \langle 0| f^+_i = \langle 0| f_{-1-i} , \quad i < 0
\]  (2)

Let \( g \in GL_\infty \) which may be written as

\[
g = \exp \sum_{i,j \in \mathbb{Z}} a_{ij} f_i f^+_j \quad (3)
\]

where \( f_i f^+_j \) denotes \( f_i f^+_j - \langle 0| f_i f^+_j |0\rangle \), and \( a_{ij} \) are some complex numbers. The elements \( f_i f^+_j \) together with 1 form a basis of the Lie algebra \( gl_\infty \), see [11], [10] for more details.

Then

\[
\sum_{i \in \mathbb{Z}} f_i |0\rangle \otimes f^+_i |0\rangle = 0
\]  (4)

is the KP hierarchy in the fermionic form, see [11].

One turns this equation into a hierarchy of differential equation by using the boson-fermion correspondence, see e.g. [14] for more details.

However the tau function can be be calculated in a different way, define the oscillator algebra

\[
V = \{ \alpha_n = \sum_{i \in \mathbb{Z}} : f_i f^+_i \}_{i \in \mathbb{Z}} \}
\]  (5)

We have

\[
[\alpha_n, \alpha_m] = n \delta_{n+m,0}
\]  (6)

where \([x, y]_+ = xy - yx\) is commutator. Define

\[
f(z) = \sum_{i \in \mathbb{Z}} f_i z^i, \quad f^+(z) = \sum_{i \in \mathbb{Z}} f^+_i z^{-i},
\]  (7)

then

\[
[\alpha_k, f(z)] = z^k f(z) \quad \text{and} \quad [\alpha_k, f^+(z)] = -z^k f^+(z)
\]  (8)

Now

\[
\alpha_n |0\rangle = \langle 0| \alpha_{-n} = 0, \quad n \geq 0.
\]  (9)

Then \( \tau_{KP}(t) \), defined by the following expectation value

\[
\tau_{KP}(t) = \langle 0 | \exp \left( \sum_{k=1}^{\infty} t_k \alpha_k \right) | g(0) \rangle,
\]  (10)

is a solution of the KP hierarchy.
Neutral fermions and the BKP hierarchy. Following [10] we introduce an involution $\omega$ on the Clifford algebra, defined by

$$\omega(f_i) = (-)^i f_{-i}^\dagger, \quad \omega(f_i^\dagger) = (-)^i f_{-i}$$

(11)

The fixed points of $\omega$ in the vector space $V$ with basis $f_i, f_i^\dagger$ are the elements

$$b_j = f_j + (-)^j f_j^\dagger \sqrt{2}$$

(12)

The elements

$$\hat{b}_j = i f_j - (-)^j f_j^\dagger \sqrt{2}$$

(13)

are elements in the $-1$ eigenspace of $\omega$. $b_j$ and $\hat{b}_j$ form a new basis of $V$ We have

$$[b_i, \hat{b}_j] = 0, \quad [b_i, b_j] = [b_i, \hat{b}_j] = (-)^i \delta_{i,j}$$

(14)

The fixed points in $gl_\infty$ are 1 and the elements

$$(-)^k : f_j f_k^\dagger : = (-)^k : f_k f_j^\dagger :$$

(15)

These elements form a Lie algebra of type $b_\infty$. If one considers the action of these elements on the vacua of the spin modules one obtains a level two representation of $b_\infty$ (Note that here we allow also certain infinite sums of these elements).

Note also that only the $\alpha_k$ for $k$ odd are fixed by $\omega$.

It is straightforward to check that

$$(-)^k : f_j f_k^\dagger : = (-)^k : f_k f_j^\dagger : =: b_j b_k : + : \hat{b}_j \hat{b}_k :$$

(16)

The following observation is crucial. The elements $b_j b_k :$ together with 1, or the elements $\hat{b}_j \hat{b}_k :$ together with 1, separately also form the Lie algebra $b_\infty$. If one considers the action of these elements separately on the vacua one obtains a level one representation. The corresponding module in terms of the $b_j$ or $\hat{b}_j$ is called the spin module of type B. One has

$$b_j|0\rangle = \hat{b}_j|0\rangle = \langle 0|b_{-j} = \langle 0|\hat{b}_{-j}, \quad j < 0$$

(17)

and

$$b_0|0\rangle = -ib_0|0\rangle = \frac{1}{2}\sqrt{2} f_0^\dagger f_0|0\rangle, \quad \langle 0|b_0 = i \langle 0|b_0 = \frac{1}{2}\sqrt{2} \langle 0|f_0^\dagger,$$

(18)

hence

$$(0|b_0 b_0|0\rangle = -i \langle 0|b_0 b_0|0\rangle = i \langle 0|b_0 b_0|0\rangle) = \langle 0|b_0 b_0|0\rangle = \frac{1}{2}.$$  

(19)

As in the KP case we now consider an element in the group $h \in B_\infty$, such an element may e.g. be written as

$$h = \exp \sum_{i,j \in \mathbb{Z}} a_{ij} : b_i b_j :$$

(20)

Then

$$\sum_{i \in \mathbb{Z}} (-)^i b_i h|0\rangle \otimes b_{-i} h|0\rangle = hh|0\rangle |0\rangle \otimes hh|0\rangle$$

(21)

is the BKP hierarchy in the fermionic form, see e.g. [10]. One turns this into a hierarchy of differential equations by using twisted vertex operators see [15] for more details. The corresponding BKP tau function can be obtained as follows. Define the twisted oscillator algebra

$$\gamma_n = \frac{1}{2} \sum_{i \in \mathbb{Z}} (-)^{i+1} : b_i b_{-i-n} :$$

(22)

Then $\gamma_n = 0$ if $n$ is even. We have

$$[\gamma_n, \gamma_m] = \frac{n}{2} \delta_{n+m,0}$$

(23)
\( \gamma_n|0\rangle = \langle 0|\gamma_n = 0, \quad n \geq 0. \)  

Let     
\[ b(z) = \sum_{i \in \mathbb{Z}} b_i z^i, \]
then     
\[ [\gamma_k, b(z)] = z^k b(z). \]

The BKP tau function \( \tau_{\text{BKP}}(t) \) is defined by the following expectation value     
\[ \tau_{\text{BKP}}(t) = \langle 0| \exp \left( \sum_{k=1, \text{odd}}^{\infty} t_k \gamma_k \right) \hat{h}|0\rangle \]
and is a solution of the BKP hierarchy. A crucial observation is the fact that in this construction we could have replaced the \( b_i \) by \( \hat{b}_i \) and in this way would have obtained the same result.

**A relation between KP and BKP tau functions**  We follow Jimbo and Miwa \[10\] or rather You \[33\] and define an automorphism \( \hat{*} \) on the B type Clifford algebra

\[ \hat{*}(b_h) = \hat{b}_h \]
then
\[ : b_n b_m : + : \hat{b}_n \hat{b}_m : = (-)^m : f_n f^-_m : - (-)^n : f_m f^-_n : \]
and
\[ \gamma_n + \hat{*}(\gamma_n) = \gamma_n + \hat{\gamma}_n = \alpha_n, \quad n \text{ odd}. \]

Then
\[ (\tau_{\text{BKP}}(t))^2 = \langle 0| \exp \left( \sum_{k=1, \text{odd}}^{\infty} t_k \gamma_k \right) \hat{h}|0\rangle \langle 0| \exp \left( \sum_{k=1, \text{odd}}^{\infty} t_k \hat{\gamma}_k \right) \hat{h}|0\rangle \]
and since the elements \( h \) and \( \gamma_n \) commute with \( \hat{h} \) and \( \hat{\gamma}_n \)
\[ (\tau_{\text{BKP}}(t))^2 = \langle 0| \exp \left( \sum_{k=1, \text{odd}}^{\infty} t_k (\gamma_k + \hat{\gamma}_k) \right) \hat{h}\hat{h}|0\rangle, \]

Now using \[30\], we find
\[ (\tau_{\text{BKP}}(t))^2 = \langle 0| \exp \left( \sum_{k=1, \text{odd}}^{\infty} t_k \alpha_k \right) \hat{h}\hat{h}|0\rangle = \tau_{\text{KP}}(t_1, 0, t_3, 0, t_5, 0, \ldots), \]
for \( g = \hat{h}\hat{h} \). Since \( h \) is of the form \[20\] we find, using \[20\], that
\[ g = \hat{h}\hat{h} = \exp \sum_{n,m \in \mathbb{Z}} a_{nm}(: b_n b_m : + : \hat{b}_n \hat{b}_m :) = \exp \sum_{n,m \in \mathbb{Z}} a_{nm}((-)^m : f_n f^-_m : - (-)^n : f_m f^-_n :). \]

The element \((-)^n : f_n f^-_m : - (-)^n : f_m f^-_n : \) is fixed by \( \omega \), hence \( g \in GL_\infty \) is an element in the level 2 representation of \( b_\infty \), which one gets by considering the action of the \( \omega \) invariant elements in \( gl_\infty \). In other words this \( \tau_{\text{KP}}(t_1, 0, t_3, 0, t_5, 0, \ldots) \) is an element in the \( B_\infty \) group orbit of the vacuum, where we consider \( B_\infty \) as a subgroup of \( GL_\infty \) and not as a group acting on the spin module of type B. The latter one is related to the level 1 representation of \( b_\infty \).
A relation between KP and BKP wave functions  Recall from (7) the formula for \(f(z)\), the KP wave function is defined as

\[
W_{\text{KP}}(t, z) = \frac{w_{\text{KP}}(t, z)}{\tau_{\text{KP}}(t)},
\]

with

\[
w_{\text{KP}}(t, z) = \langle 0 | f_0^t \exp \left( \sum_{k=1}^{\infty} t_k \alpha_k \right) g \langle 0 \rangle.
\]

Using the boson-fermion correspondence (see e.g. [4], [14] or (63) in the next section),

\[
f(z) = Q z e^{-\sum_{k>0} \frac{\alpha_k}{k} z^{-k}} e^{-\sum_{k>0} \frac{\alpha_k}{k} z^{-k}},
\]

which is based on (8), then gives

\[
w_{\text{KP}}(t, z) = e^{\sum_{k>0} t_k z^k} \langle 0 | \exp \left( \sum_{k=1}^{\infty} \left( t_k - \frac{z^{-k}}{k} \right) \alpha_k \right) g \langle 0 \rangle
\]

\[
= \tau_{\text{KP}}(t_1 - \frac{z^{-1}}{1}, t_2 - \frac{z^{-2}}{2}, t_3 - \frac{z^{-3}}{3}, \ldots) e^{\sum_{k>0} t_k z^k}\]

Recall \(b(z)\) from (25) and define analogously \(b(z) = \sum_{j \in \mathbb{Z}} b_j z^j\), then using (12) and (13)

\[
f(z) = \frac{1}{\sqrt{2}} \left( b(z) - i\bar{b}(z) \right) \quad \text{and} \quad f^\dagger(z) = \frac{1}{\sqrt{2}} \left( b(-z) + i\bar{b}(-z) \right)
\]

Now let as before \(g = h\bar{h}\), then

\[
w_{\text{KP}}(t_{\text{odd}}, z) = w_{\text{KP}}(t_1, 0, t_3, 0, \ldots, z) = \tau_{\text{KP}}(t_1 - \frac{z^{-1}}{1}, -\frac{z^{-2}}{2}, t_3 - \frac{z^{-3}}{3}, \ldots) e^{\sum_{k>0} t_k z^k}
\]

Using (26) and the known bosonization formula (6.5) of [10],

\[
\langle 0 | b_0 b(z) e^{\sum_{k=1, \text{odd}} \gamma_k t_k} e^{\sum_{k=1, \text{odd}} \gamma_k (t_k - \frac{z^{-k}}{k})} = \frac{1}{2} \langle 0 | e^{\sum_{k=1, \text{odd}} \gamma_k (t_k - \frac{z^{-k}}{k})}
\]

and a similar formula with “hats” ((7.4) in [10]), we obtain

\[
w_{\text{KP}}(t_{\text{odd}}, z) = \frac{1}{2} \langle 0 | \left( b_0 + i\bar{b}_0 \right) \exp \left( \sum_{k=1, \text{odd}}^{\infty} t_k (\gamma_k + \tilde{\gamma}_k) \right) (b(z) - i\bar{b}(z)) h\bar{h} \langle 0 \rangle
\]

\[
= \frac{1}{2} e^{\sum_{k>0} t_k z^k} \langle 0 | \exp \left( \sum_{k=1, \text{odd}}^{\infty} \left( t_k - \frac{2}{k} z^{-k} \right) \gamma_k \right) h\bar{h} \langle 0 \rangle
\]

\[
+ \frac{1}{2} e^{\sum_{k>0} t_k z^k} \langle 0 | \exp \left( \sum_{k=1, \text{odd}}^{\infty} t_k \gamma_k \right) h\bar{h} \langle 0 \rangle
\]

If we now divide this by \(\tau_{\text{KP}}(t_1, 0, t_3, 0, \ldots) = (\tau_{\text{BKP}}(t))^2\), we obtain an expression for \(W_{\text{KP}}(t_{\text{odd}}, z)\). Define

\[
W_{\text{BKP}}(t, z) = \frac{\langle 0 | b_0 \sum_{k=1, \text{odd}} t_k \gamma_k b(z) h\bar{h} \langle 0 \rangle}{\langle 0 | e^{\sum_{k=1, \text{odd}} t_k \gamma_k} h\bar{h} \langle 0 \rangle} = e^{\sum_{k>0, \text{odd}} t_k z^k} (1 + O(z^{-1})).
\]

Again using (25), one deduces

\[
W_{\text{BKP}}(t, z) = \tau_{\text{BKP}}(t_1 - \frac{z^{-1}}{1}, t_3 - \frac{2}{3} z^{-3}, \ldots) e^{\sum_{k>0, \text{odd}} t_k z^k} / \tau_{\text{BKP}}(t)
\]

Now comparing [10] and (42), we obtain:

\[
W_{\text{BKP}}(t, z) = W_{\text{KP}}(t_1, 0, t_3, 0, \ldots, z).
\]
2 Another realization of the KP hierarchy

In order to describe the relation between the Toda lattice hierarchy (or 2-component KP) and the Pfaff lattice (or large BKP), we need another realization of both spin modules. We start with the KP hierarchy and relabel the fermions \( f_i \) and \( f_i^\dagger \) as follows

\[
\begin{align*}
& f_0 = \psi, \quad f_0^\dagger = \psi^\dagger \quad (44) \\
& f_{2i+1} = \psi_i^{(1)}, \quad f_{2i+1}^\dagger = \psi_i^{1(1)}, \quad i \geq 0 \\
\end{align*}
\]

while

\[
\begin{align*}
& f_{2i+2} = \psi_i^{(2)}, \quad f_{2i+2}^\dagger = \psi_i^{1(2)}, \quad i \geq 0 \\
& f_{2i} = \psi_i^{(1)}, \quad f_{2i}^\dagger = \psi_i^{1(1)}, \quad i < 0 \\
& f_{2i+1} = -\psi_i^{(2)}, \quad f_{2i+1}^\dagger = -\psi_i^{1(2)}, \quad i < 0. \\
\end{align*}
\]

The minus sign in (48) will be convenient later on when we apply the involution \( \omega \) to this new realization.

Then for \( a, b = 1, 2 \)

\[
\begin{align*}
& [\psi_i^{(a)}, \psi_j^{(b)}]_+ = \delta_{a,b}\delta_{i,j}, \quad \text{and} \quad [\psi, \psi^\dagger]_+ = 1, \\
\end{align*}
\]

all other anti-commutation relations are zero. The action on the vacuum is given by

\[
\psi^\dagger |0\rangle = 0 = |0\rangle \psi \\
\psi_i^{(a)} |0\rangle = \psi_{i-1}^{(a)} |0\rangle = 0 = |0\rangle \psi_i^{(a)} = |0\rangle \psi_i^{(a-1)}, \quad i < 0
\]

Now the KP equation (4) may be rewritten as

\[
\psi g|0\rangle \otimes \psi^\dagger g|0\rangle + \sum_{n=1,2} \sum_{i \in \mathbb{Z}} \psi_i^{(a)} g|0\rangle \otimes \psi_i^{1(1)} g|0\rangle = 0
\]

where \( g \) is the same as in (4) rewritten as

\[
g = e^{\sum_{n=1,2} \sum_{m \in \mathbb{Z}} a_{nm} \psi_n^{(a)} \psi_m^{(b)} + \sum_{n=1,2} \sum_{m \in \mathbb{Z}} a_{nm} \psi_m^{1(α)} + \sum_{n=1,2} \sum_{m \in \mathbb{Z}} a_{nm} \psi_n^{1(α)} + \sum_{n=1,2} \sum_{m \in \mathbb{Z}} a_{nm} \psi_n^{1(α)} \psi_m^{1(α)} + a^{00} \psi^\dagger}
\]

Remark 1. One can identify eq. (52) with the Hirota equation for the three-component KP hierarchy [4], [10], [14] restricted on the class of \( g \) which does not depend on \( \psi_k^{(3)}, \psi_k^{1(3)} \) for all \( k \neq i \) except on \( \psi_i^{(3)} := \psi, \psi_i^{1(3)} := \psi^\dagger \) where \( i \) is arbitrary chosen.

It may also be written in the form

\[
g = g^{(0)} + g^{(1)} \psi + g^{(-1)} \psi^\dagger, \quad g^{(0)} = g^{(0,0)} + g^{(1,−1)} \psi^\dagger
\]

where \( g^{(0,0)}, \ g^{(1,−1)}, \ g^{(1)} \) and \( g^{(-1)} \) do not contain neither \( \psi \), nor \( \psi^\dagger \). Then, \( g^{(0)} \) is even and \( g^{(±1)} \) are odd in the \( \mathbb{Z}_2 \) grading of the Clifford algebra.

Define a 2-component oscillator algebra (cf [3])

\[
\alpha_n^{(a)} = \sum_{i \in \mathbb{Z}} a_{i+n}^{(a)} \psi_i^{(a)} \psi_i^{1(α)}.
\]

We have

\[
[\alpha_n^{(a)}, \alpha_m^{(b)}]_+ = n \delta_{a,b} \delta_{n+m,0}
\]

Introduce

\[
\Psi_n^{(a)} = \begin{cases} 
\psi_0^{(a)} \cdots \psi_n^{(a)} & \text{if } n > 0 \\
1 \cdots \psi_n^{1(α)} & \text{if } n = 0 \\
\psi_n^{(a)} \cdots \psi_0^{1(α)} & \text{if } n < 0
\end{cases}
\]

and introduce right and left Fock vectors

\[
|n, m, 0\rangle = \Psi_m^{(2)} \Psi_n^{(1)} |0\rangle, \quad \langle n, m, 0| = \langle 0| \Psi_n^{1(2)} \Psi_m^{1(1)}
\]

\[ |n,m,1\rangle = \psi_m^{(2)} \psi_n^{(1)} \phi(0), \quad |n,m,1\rangle = \langle 0| \psi_m^{(1)} \psi_n^{(2)} \]

which are highest weight vectors for the oscillator algebra \([55]:\)

\[ \alpha_i^{(a)}|n,m,k\rangle = 0 \quad \text{if} \quad i > 0, \quad \langle n,m,k|\alpha_i^{(a)} = 0 \quad \text{if} \quad i > 0 \]

for any \(a,n,m \) and \(k = 0,1.\)

We have the orthogonality condition \(\langle n,m,k|n',m',k'\rangle = \delta_{n,n'}\delta_{m,m'}\delta_{k,k'} \) where \(n,n',m,m' \in \mathbb{Z}, \)

\(k,k' = 0,1.\)

Now we can write

\[ g(0) = \sum_{n \in \mathbb{Z}} \left( g_n^{(0)}|n,-n,0\rangle + g_n^{(1)}|n-1,-n,1\rangle \right) \]

**Boson-fermion correspondence.** Now we set the following correspondence. The right Fock space is isomorphic to \([\theta, q_a, q_a^{-1}, i^{(a)}; a = 1, 2, i = 1, 2, \ldots].\) Let \(\sigma\) be the corresponding isomorphism. Introduce the fermionic fields

\[ \psi^{(a)}(z) = \sum_{i \in \mathbb{Z}} \psi_i^{(a)} z^i, \quad \psi^\dagger{(a)}(z) = \sum_{i \in \mathbb{Z}} \psi_i^{(a)\dagger} z^{-i-1}, \]

then

\[ \sigma(\psi) = \theta, \quad \sigma(\psi^\dagger) = \frac{\partial}{\partial \theta} \]

where \(\theta^2 = 0,\) and

\[ \sigma(\psi^{(a)}(z)) = q_\theta z^{q_\theta - \frac{2}{m_a}} E^+(a)(z), \quad \sigma(\psi^\dagger{(a)}(z)) = q_\theta^{-1} z^{-q_\theta + \frac{2}{m_a}} E^-(a)(z) \]

where

\[ E^\pm(a)(z) = e^{\pm \sum_{n \in \mathbb{Z}} \tau_n^{(a)} z^n} e^{\sum_{k=1}^{\infty} \frac{1}{k} \frac{q_\theta}{m_a^n}} \]

Here

\[ q_\theta q_\theta = -q_\theta q_\theta, \quad \theta q_\theta = -q_\theta \theta \]

for \(a, b = 1, 2, a \neq b.\)

Then \([61]\) turns out to be

\[ \sigma(g(0)) = \sum_n \left( \tau_n^{(0)} q_n^{-n} + \tau_n^{(1)} q_n^{-n-1} \right) \]

Now the Hirota KP equations \([41]\) (rewritten in form \([52]\)) are

\[ 0 = -\sum_{n,m} \tau_m^{(0)} q_1^{-n} \otimes \tau_m^{(1)} q_2^{-m-1} + \]

\[ \text{res}_{z} (z^{-n} z^{-m} E^+(1)(z) \left( \tau_n q_1^{-1-n} + z^{-1} \tau_n q_2^{-n} \right) \otimes (z^{-m} E^-(1)(z) \left( \tau_m q_2^{-m-1} + z \tau_m q_2^{-m-2} \right) + \right) + \text{res}_{z} z E^+(2)(z) \left( \tau_{n+1} q_1^{-n-1} + \tau_{n+1} q_2^{-n-1} \right) \otimes z E^-(2)(z) \left( \tau_{m+1} q_2^{-m-1} + \tau_{m+1} q_2^{-m-2} \right) \]

This decomposes in a number of equations of which one is:

\[ \text{res}_{z} \left( (-z)^{-m} E^+(1)(z) \tau^{(a)} \otimes E^-(1)(z) \tau^{(a)}_m + z^{m-n-2} E^+(2)(z) \tau^{(a)}_{n+1} \otimes E^-(2)(z) \tau^{(a)}_{m-1} \right) = 0 \]

for \(a = 0, 1\) which is a version of the two-component KP equation \([10, 14],\) see also \([31]\) and

\[ \text{res}_{z} \left( z^{-2k+1} (-z)^{-m} E^+(1)(z) \tau^{(a)} \otimes E^-(1)(z) \tau^{(1-a)}_m + z^{m-n-2} E^+(2)(z) \tau^{(a)}_{n+1} \otimes E^-(2)(z) \tau^{(1-a)}_{m-1} \right) = \delta_{a1} \tau^{0}_{m} \otimes \tau^{(1)}_{m} \]

which is a version of the two-component 1st modified KP equation \([10, 15].\)
We have
\[ \tau_n^{(0)} = \langle n, -n, 0 \rangle e^{\sum_{i=1}^{\infty} (t^{(1)}_i \alpha^{(1)}_n + t^{(2)}_i \alpha^{(2)}_n)} g(0, 0, 0) \quad \text{and} \]
\[ \tau_n^{(1)} = \langle n - 1, -n, 1 \rangle e^{\sum_{i=1}^{\infty} (t^{(1)}_i \alpha^{(1)}_n + t^{(2)}_i \alpha^{(2)}_n)} g(0, 0, 0) \]
(70)
(71)
Note, using (54), we can write \[ g = g^{(0,0)} + g^{(1)} \psi + g^{(-1)} \psi^\dagger + g^{(1,-1)} \psi \psi^\dagger. \]
Then (70) and (71) can be written as follows
\[ \tau_n^{(0)} = \langle n, -n, 0 \rangle e^{\sum_{i=1}^{\infty} (t^{(1)}_i \alpha^{(1)}_n + t^{(2)}_i \alpha^{(2)}_n)} g^{(0,0)} |0, 0, 0 \rangle \quad \text{and} \]
\[ \tau_n^{(1)} = \langle n - 1, -n, 1 \rangle e^{\sum_{i=1}^{\infty} (t^{(1)}_i \alpha^{(1)}_n + t^{(2)}_i \alpha^{(2)}_n)} g^{(1)} |0, 0, 0 \rangle \]
(72)
(73)

3 B-case in this new realization

Recall the involution \( \omega \) (see (11)) on the Clifford algebra. Using the relabeling (43)-(48), this induces
\[ \omega(\psi) = \psi^\dagger, \quad \omega(\psi^\dagger) = \psi \]
(74)
\[ \omega(\psi^{(1)}_n) = \psi^{(1)}_{-1-n}, \quad \omega(\psi^{(2)}_n) = \psi^{(2)}_{-1-n}, \]
(75)
\[ \omega(\psi^{(1)}_n) = \psi^{(2)}_{-1-n}, \quad \omega(\psi^{(2)}_n) = \psi^{(1)}_{-1-n}. \]
(76)
It is straightforward to check that
\[ \omega(\alpha^{(1)}_n) = -\alpha^{(2)}_n, \quad \omega(\alpha^{(2)}_n) = -\alpha^{(1)}_n. \]
(77)

Now define a new basis in our Clifford algebra consisting of elements that are fixed by \( \omega \) and of elements \( x \) with \( \omega(x) = -x \)
\[ \varphi = \frac{\psi + \psi^\dagger}{\sqrt{2}}, \quad \varphi = i \frac{\psi - \psi^\dagger}{\sqrt{2}} \]
(78)
\[ \psi_j = \frac{\psi^{(1)}_j + \psi^{(2)}_{-1-j}}{\sqrt{2}}, \quad \psi_j = i \frac{\psi^{(1)}_j - \psi^{(2)}_{-1-j}}{\sqrt{2}} \]
(79)
\[ \psi_j^\dagger = \frac{\psi^{(1)}_j + \psi^{(2)}_{-1-j}}{\sqrt{2}}, \quad \psi_j^\dagger = -i \frac{\psi^{(1)}_j - \psi^{(2)}_{-1-j}}{\sqrt{2}} \]
(80)
Then
\[ [\psi_n, \psi^\dagger_{m}]_+ = \delta_{n,m}, \quad [\psi^\dagger_n, \psi^\dagger_m]_+ = \delta_{n,m} \]
(81)
\[ \varphi^2 = \varphi^\dagger = \frac{1}{2} \]
(82)
all other elements anticommute.

Also
\[ \psi_j |0\rangle = \psi^\dagger_j |0\rangle = \psi^\dagger_{-1-j} |0\rangle = \psi^\dagger_{-1-j} |0\rangle = 0, \quad j < 0, \]
(83)
\[ \langle 0 | \psi^\dagger_j = \langle 0 | \psi^\dagger_{-j} = \langle 0 | \psi_{-j} = 0, \quad j < 0, \]
(84)
\[ (\varphi + i \varphi^\dagger) |0\rangle = 0 = \langle 0 | (\varphi - i \varphi^\dagger) \]
(85)
Spin module for $b_{\infty}$. Here we consider the elements $\psi_1, \psi_1^\dagger, \varphi$, which are the elements invariant under $\omega$. Recall

$$[\psi_j, \psi_n^\dagger]_+ = \delta_{jn}, \quad \varphi^2 = \frac{1}{2}$$

(86)

all other elements anticommute and

$$\psi_j|0\rangle = \psi_{1-j}|0\rangle = 0 = \langle 0|\psi_j^\dagger = \langle 0|\psi_{1-j}, \quad j < 0$$

(87)

(63) induces $\omega_{1,0}$ for the fermionic fields (cf. (62))

Then all other elements anticommute and

The $b_{\infty}$ spin module splits into two parts $V_0 \oplus V_1$ where $V_0$ has the highest weight vector $|0\rangle = |0\rangle$, and $V_1$ has the highest weight vector $|1\rangle = \sqrt{2}\varphi|0\rangle$. Both $V_0$ and $V_1$ are irreducible highest weight modules for $b_{\infty}$, which is formed by the quadratic elements of the Clifford algebra together with 1.

From now we shall focus on $V_0$.

As before one can define an oscillator algebra $B$

$$\beta_k = \sum_{l \in \mathbb{Z}} :\psi_l\psi_{l+k}^\dagger:$$

(88)

In $V_0$ one has the following highest weight vectors if one restricts to the $B$

$$|n\rangle = \begin{cases} \Psi_{2l}(0) \sqrt{2}\Psi_{2l-1}\varphi|0\rangle & \text{if } n = 2l, \\ \sqrt{2}\Psi_{2l-1}|0\rangle & \text{if } n = 2l-1 \end{cases}, \quad \langle n| = \begin{cases} \langle 0|\Psi_{2l}^\dagger \varphi \Psi_{2l-1}\sqrt{2} & \text{if } n = 2l, \\ \langle 0|\varphi \Psi_{2l-1}^\dagger & \text{if } n = 2l-1 \end{cases}$$

(89)

$$\beta_k|n\rangle = 0 = \langle 0|\beta_k, \quad k > 0, \quad n \in \mathbb{Z}$$

(90)

where similar to (57)

$$\Psi_n = \begin{cases} \psi_{n-1} \cdots \psi_0 & \text{if } n > 0 \\ 1 & \text{if } n = 0 \\ \psi_n^\dagger \cdots \psi_{n-1}^\dagger & \text{if } n < 0 \end{cases}$$

$$\Psi_n = \begin{cases} \psi_0^\dagger \cdots \psi_{n-1} & \text{if } n > 0 \\ 1 & \text{if } n = 0 \\ \psi_{n-1} \cdots \psi_n & \text{if } n < 0 \end{cases}$$

(91)

Let $h$ be a group element of $B_{\infty}$. Then we can consider $h|0\rangle$, clearly such an element decomposes

$$h|0\rangle = \sum_{n \in \mathbb{Z}} h_n|n\rangle$$

(92)

As for $GL_\infty$ there is a boson-fermion correspondence given by

$$\sigma_B(\psi(z)) = pz^\frac{a}{\sqrt{2}} B^\dagger(z), \quad \sigma_B(\psi^\dagger(z)) = p^{-1}z^{-\frac{a}{\sqrt{2}}} B(z)$$

(93)

$$B^\pm(z) = e^{\pm \sum_{n > 0} s_n z^n} e^{\mp \sum_{n > 0} \frac{\omega_n}{\sqrt{2}} z^{-n}}$$

(94)

for the fermionic fields (cf. (62))

$$\psi(z) = \sum_{i \in \mathbb{Z}} \psi_i z^i, \quad \psi^\dagger(z) = \sum_{i \in \mathbb{Z}} \psi_i^\dagger z^{-i-1},$$

(95)

and (63) induces

$$\sigma_B(\varphi) = \frac{\theta + \frac{a}{\sqrt{2}}}{\sqrt{2}}$$

(96)

Then $\sigma_B(h|0\rangle) = \tau^B$. And using (62) one finds

$$\tau^B = \sum_{l \in \mathbb{Z}} (\tau_{2l}^B p^{2l} + \tau_{2l-1}^B p^{2l-1} \theta)$$

(97)

Note that $p\theta = -\theta p$.

An element $h \in B_{\infty}$ is an even element in the Clifford algebra.

We have

$$[h \otimes h, S_B] = 0, \quad S_B = \varphi \otimes \varphi + \sum_{i \in \mathbb{Z}} (\psi_i \otimes \psi_i^\dagger + \psi_i^\dagger \otimes \psi_i)$$

(98)
Since
\[ S_B |0\rangle \otimes |0\rangle = \varphi |0\rangle \otimes \varphi |0\rangle \] (99)
One obtains
\[ S_B (h|0\rangle \otimes h|0\rangle) = h\varphi |0\rangle \otimes h\varphi |0\rangle \] (100)
Note that \( h = a + \sqrt{2}b\varphi \) where \( a \) is an even element expressed in \( \psi_i \) and \( \psi_i^\dagger \), and \( b \) is an odd element expressed in \( \psi_i \) and \( \psi_i^\dagger \). Then
\[
S_B \left( a + \sqrt{2}b\varphi \right) |0\rangle \otimes \left( a + \sqrt{2}b\varphi \right) |0\rangle = \left( a\varphi + \frac{1}{2}\sqrt{2}b \right) |0\rangle \otimes \left( a\varphi + \frac{1}{2}\sqrt{2}b \right) |0\rangle
\] (101)
or
\[
\left( a\varphi - \frac{1}{2}\sqrt{2}b \right) |0\rangle \otimes \left( a\varphi - \frac{1}{2}\sqrt{2}b \right) |0\rangle + \\
\sum_{i \in \mathbb{Z}} \psi_i \left( a + \sqrt{2}b\varphi \right) |0\rangle \otimes \psi_i^\dagger \left( a + \sqrt{2}b\varphi \right) |0\rangle + \\
\sum_{i \in \mathbb{Z}} \psi_i^\dagger \left( a + \sqrt{2}b\varphi \right) |0\rangle \otimes \psi_i \left( a + \sqrt{2}b\varphi \right) |0\rangle
\]
\[
= \left( a\varphi + \frac{1}{2}\sqrt{2}b \right) |0\rangle \otimes \left( a\varphi + \frac{1}{2}\sqrt{2}b \right) |0\rangle
\]
Hence
\[
\sum_{i \in \mathbb{Z}} \left[ \psi_i \left( a + \sqrt{2}b\varphi \right) |0\rangle \otimes \psi_i^\dagger \left( a + \sqrt{2}b\varphi \right) |0\rangle + \psi_i^\dagger \left( a + \sqrt{2}b\varphi \right) |0\rangle \otimes \psi_i \left( a + \sqrt{2}b\varphi \right) |0\rangle \right]
\]
\[
= \sqrt{2}a\varphi |0\rangle \otimes b|0\rangle + b|0\rangle \otimes \sqrt{2}a\varphi |0\rangle
\]
Now clearly:
\[
\sigma (a|0\rangle) = \sum_{l \in \mathbb{Z}} \tau_{2l}^B p^{2l} \theta \quad \sigma \left( \sqrt{2}b\varphi |0\rangle \right) = \sum_{l \in \mathbb{Z}} \tau_{2l-1}^B p^{2l-1} \theta
\] (102)
\[
\sigma \left( \sqrt{2}a\varphi |0\rangle \right) = \sum_{l \in \mathbb{Z}} \tau_{2l}^B p^{2l} \theta \quad \sigma (b|0\rangle) = \sum_{l \in \mathbb{Z}} \tau_{2l-1}^B p^{2l-1}
\] (103)
Also
\[
\tau_l^B (s) = \langle l|s \sum_i \beta_i h|0\rangle
\] (104)
Equation (101) turns into the large BKP hierarchy or Pfaff lattice:
\[
\text{res}_z \left( z^{n-m-2} B^+ (z) \tau_{n-1}^B \otimes B^- (z) \tau_{m+1}^B + \frac{(-1)^{n+m}}{2z} \tau_n^B \otimes \tau_m^B + z^{m-n-2} B^- (z) \tau_{n+1}^B \otimes B^+ (z) \tau_{m-1}^B \right) = \frac{1}{2} \tau_n^B \otimes \tau_m^B
\] (105)

4 A relation between the tau functions

In the same way as the small BKP is related to the KP hierarchy, the large BKP (or Pfaff lattice) is related to the 2-component KP (or Toda Lattice hierarchy). In fact one can use the same involution \( \hat{\sigma} \) of (28) on the Clifford algebra, it induces
\[
\hat{\gamma} (\varphi) = \hat{\varphi}, \quad \hat{\gamma} (\bar{\varphi}) = \bar{\varphi}
\] (106)
\[
\hat{\gamma} (\psi_i) = \hat{\psi}_i, \quad \hat{\gamma} (\bar{\psi}_i) = \bar{\psi}_i, \quad \hat{\gamma} (\psi_i^\dagger) = \bar{\psi}_i^\dagger, \quad \hat{\gamma} (\bar{\psi}_i^\dagger) = \hat{\psi}_i^\dagger
\] (107)
Then
\[
\hat{\gamma} (\beta_l) = \hat{\beta}_l = \sum_{j \in \mathbb{Z}} \hat{\psi}_j \hat{\psi}_{j+l}^\dagger
\] (108)
We want to consider

$$e^{\sum s_i (\beta_i + \beta_i^*) h} \cdot (h) |0\rangle = e^{\sum s_i \beta_i^* h} \cdot e^{\sum s_i \beta_i h} |0\rangle$$  \hspace{1cm} (109)$$

Now

$$\psi_j^\dagger \psi_k^\dagger + \psi_j \psi_k = \frac{1}{2} \left( \psi_j^{(1)} + \psi_j^{(2)} \right) \left( \psi_k^{(1)} + \psi_k^{(2)} \right) + \frac{1}{2} \left( \psi_j^{(1)} - \psi_j^{(2)} \right) \left( \psi_k^{(1)} - \psi_k^{(2)} \right)$$

Thus

$$\beta + \beta^* = \sum_j \left( \psi_j \psi_j^{(1)} + \psi_j \psi_j^{(2)} \right) = \sum_j \left( \psi_j^{(1)} \psi_j^{(1)} + \psi_j^{(2)} \psi_j^{(2)} \right) = \alpha^{(1)} + \alpha^{(2)}$$

and hence

$$e^{\sum s_i (\beta_i + \beta_i^*) h} = e^{\sum s_i \alpha^{(1)} h} \cdot e^{\sum s_i \alpha^{(2)} h}$$

First note that

$$\psi = \frac{\varphi - i \hat{\varphi}}{\sqrt{2}}, \quad \psi^\dagger = \frac{\varphi + i \hat{\varphi}}{\sqrt{2}}$$  \hspace{1cm} (111)$$

$$\psi_j^{(1)} = \frac{\psi_j - i \hat{\psi}_j}{\sqrt{2}}, \quad \psi_j^{(1)} = \frac{\psi_j + i \hat{\psi}_j}{\sqrt{2}}$$  \hspace{1cm} (112)$$

Thus

$$\psi_j^{(1)} \psi_{j+1} = \frac{1}{2} \left( \psi_j - i \hat{\psi}_j \right) \left( \psi_j + i \hat{\psi}_j \right) = i \psi_j \hat{\psi}_j$$  \hspace{1cm} (114)$$

$$\psi_j^{(1)} \psi_{j+1} = \frac{1}{2} \left( \psi_j + i \hat{\psi}_j \right) \left( \psi_j - i \hat{\psi}_j \right) = -i \psi_j \hat{\psi}_j$$  \hspace{1cm} (115)$$

The following lemma will be useful:

**Lemma 1.**

$$\langle -2n, 2n, 0 \rangle = (-1)^n \langle 0 | \Psi^\dagger_{-2n} \hat{\Psi}^\dagger_{-2n}$$  \hspace{1cm} (116)$$

$$\langle 1 - 2n, 2n - 1, 0 \rangle = (-1)^{n+1} \langle 0 | \sqrt{2} \varphi \Psi^\dagger_{1-2n}$$  \hspace{1cm} (117)$$

**Proof.** Let $n > 0$. Then

$$\langle -2n, 2n, 0 \rangle = \langle 0 | \Psi^\dagger_{-2n} \Psi^\dagger_{2n} = \langle 0 | \psi_{-1}^{(1)} \psi_{1}^{(1)} \cdots \psi_{-2n}^{(1)} \psi_{2n}^{(2)} \cdots \psi_{-2n}^{(2)} \psi_{2n} \cdots \psi_{-2n}^{(1)} \cdots \psi_{2n}^{(2)} \rangle$$

$$= (-1)^n \langle 0 | \psi_{-1} \psi_{1} \cdots \psi_{-2n} \psi_{1} \cdots \psi_{-2n} \rangle = (-1)^n \langle 0 | \Psi^\dagger_{-2n} \Psi^\dagger_{2n}$$

If $n < 0$ then we have

$$\langle -2n, 2n, 0 \rangle = \langle 0 | \Psi^\dagger_{-2n} \Psi^\dagger_{2n} = \langle 0 | \psi_{0}^{(1)} \psi_{0}^{(1)} \cdots \psi_{-2n}^{(1)} \psi_{-1}^{(2)} \cdots \psi_{2n}^{(2)} \rangle$$

$$= (-1)^n \langle 0 | \psi_{0} \psi_{0} \cdots \psi_{-2n} \psi_{1} \cdots \psi_{2n} \rangle = (-1)^n \langle 0 | \Psi^\dagger_{-2n} \Psi^\dagger_{2n}$$

Next we consider the case

$$\langle 1 - 2n, 2n - 1, 0 \rangle = \langle 0 | \Psi^\dagger_{-2n+1} \Psi^\dagger_{2n-1}$$  \hspace{1cm} (118)$$

If $n > 0$ then:

$$\langle 1 - 2n, 2n - 1, 0 \rangle = \langle 0 | \psi_{-1} \cdots \psi_{-2n+1} \psi_{0}^{(1)} \cdots \psi_{2n-2}^{(2)} \rangle$$

$$= (i)^{2n-1} \langle 0 | \psi_{-1} \cdots \psi_{-2n+1} \psi_{1} \cdots \psi_{2n-1} = (i)^{2n-1} \langle 0 | \Psi^\dagger_{-2n+1} \Psi^\dagger_{2n-1}$$

Now use that

$$\langle 0 | \varphi \hat{\varphi} = \frac{i}{2} \langle 0 | (\psi + \psi^\dagger) (\psi + \psi^\dagger) = \frac{i}{2} \langle 0 | \psi^\dagger \psi = -\frac{i}{2} \langle 0 |$$  \hspace{1cm} (119)$$
Thus
\[
(1 - 2n, 2n - 1, 0) = 2(i)^{2n}(0)\varphi_0^\dagger \Psi_{2n+1}^\dagger \Psi_{2n}^\dagger = (-1)^{n+1}(0) \left( \sqrt{2}\varphi \Psi_{2n}^\dagger \right)
\]
If \( n < 0 \) then
\[
(1 - 2n, 2n - 1, 0) = \langle 0|\psi^{(1)}_0 \cdots \psi^{(1)}_{2n-1} \psi^{(2)}_{2n-1} = (1)^{1-2n}(0)\Psi_{2n+1}^\dagger \Psi_{2n}^\dagger
\]
\[
= (-1)^{n+1}(0) \left( \sqrt{2}\varphi \Psi_{2n}^\dagger \right)
\]
Which finishes the proof of the lemma

Thus
\[
\tau_{2n}^{(0)}(s, -s) = (1)^n \langle 0|\Psi_{2n+1}^\dagger \Psi_{2n}^\dagger e^{\sum_k s_k \beta_k e^{\sum_k s_k \beta_k} h^0}\phi_n^0 \rangle = (1)^n \tau_{2n}^B(s)\tau_{2n}^B(s)
\]
And
\[
\tau_{2n}^{(0)}(s, -s) = (1)^n \langle 0|\Psi_{2n+1}^\dagger \Psi_{2n}^\dagger e^{\sum_k s_k \beta_k e^{\sum_k s_k \beta_k} h^0}\phi_n^0 \rangle = (1)^n \tau_{2n}^B(s)\tau_{2n}^B(s)
\]
Thus

**Proposition 1.** A BKP tau function satisfies
\[
\left( \tau_n^B(s) \right)^2 = (-1)^{n+1}\tau_n^{(0)}(s, -s)
\]

**Remark 2.** If \( g_i = h_i \), \( i = 1, \ldots, k \), then \( g = g_1 \cdots g_k = h \) where \( h = h_1 \cdots h_k \).

**Remark 3.** It follows from Proposition 1 that the square of the BKP function
\[
V_n(t, z) := 
\sum_{m=1}^{\infty} \tau_{m+1}(t - [z^{-1}])
\]

is related to the two-component KP Green function
\[
K_n(x, y, t^{(1)}, t^{(2)}) := (n + 1, n - 1)\sum_m \tilde{\tau}_{m+1}^{(1)}(t^{(1)})(t^{(2)})(y)h^0 \]

as follows
\[
(V_n(t, z))^2 = K_n(z, z, t, -t)
\]
Let us note that two-component KP is useful to study matrix models. The Green function \( K(x, y) \) is widely used for computing various correlation functions. However, these models do not possess the property of factorization \( g = hh \).

5 A relation between the wave functions

We introduce the 2 component KP wave function \( W_n^{(a)}(t, z) \) by
\[
W_n^{(a)}(t, z) = \frac{1}{\tau_n^{(a)}(t)} \left( (-z)^{\pm n} E^{(a)}_{\pm n}(z) \tau_n^{(a)}(t) \right)_{n \pm 1}(t)
\]
then (68) turns into
\[
\text{res}_z W_n^{(a)}(s, z)W_m^{(a)}(t, z)^T = 0
\]
and (69) turns into
\[
\text{res}_z W_n^{+(a)}(s, z)W_m^{-(a)}(t, z)^T = \frac{\delta_{n1}}{\tau_n^{(1)}(s)\tau_m^{(0)}(t)} \left( \frac{\tau_n^{(0)}(s)}{\tau_n^{(0)}(s)} \right) \left( \tau_m^{(1)}(t) \right) \left( \tau_m^{(1)}(t) \right)
\]
We introduce next the BKP wave function \( V^{\pm}(z) \). To do that we first observe that (105) can be rewritten in the matrix form

\[
\text{res } R_n^+ (z) S(z) R_m(z)^T = \text{res } T_n^+ (z) T_m(z)^T
\]

where

\[
R_n^+ (z) = \begin{pmatrix}
    z^{\pm n} B^\pm(z) \tau^B_{n+1} & (-1)^{n+1} \tau^B_{n+1} & \tau^{-n-2} B^\mp(z) \tau^B_{n+2} \\
    z^{\pm n-1} B^\pm(z) \tau^B_{n+1} & (-1)^n \tau^B_{n+1} & \tau^{-n-1} B^\mp(z) \tau^B_{n+2} \\
    z^{\pm n-2} B^\pm(z) \tau^B_{n+1} & (-1)^{n-1} \tau^B_{n+1} & \tau^{-n} B^\mp(z) \tau^B_{n+2}
\end{pmatrix}, \quad T_n^+ = \begin{pmatrix}
    \tau^B_{n+1} & 0 & 0 \\
    0 & \tau^B_{n+1} & 0 \\
    0 & 0 & \tau^B_{n+1}
\end{pmatrix}
\]

and

\[
S(z) = \text{diag}(1, \frac{1}{2}, 1)
\]

Let

\[
U_n^\pm = \begin{pmatrix}
    \frac{1}{\sqrt{n}} & \frac{\tau^B_n}{\tau^B_{n+1}} & 0 \\
    0 & \frac{1}{\sqrt{n}} & 0 \\
    0 & \frac{\tau^B_n}{\tau^B_{n+1}} & \frac{1}{\sqrt{n}}
\end{pmatrix}
\]

then \( U_n^+ = (T_n^+)^{-1} \). Now introduce the BKP wave function \( V_n^\pm(z) = U_n^+ R_n^\pm (z) \) then (127) turns into

\[
\text{res } V_n^+ (z) S(z) V_m(z)^T = \begin{pmatrix}
    \frac{1}{\sqrt{n}} & 0 & 0 \\
    0 & \frac{1}{\sqrt{n}} & 0 \\
    0 & 0 & \frac{1}{\sqrt{n}}
\end{pmatrix}
\]

We will now show that two-component KP wave functions \( W_n^{\pm(0)}(t^{(1)}, t^{(2)}, z) \) evaluated at \( t^{(1)} = s_j = -t^{(2)} \) for group elements, \( g = h \hat{h} \) may be expressed in terms of the BKP wave functions \( V_n^\pm(z) \).

**Proposition 2.** A BKP wave function is related to a 2 component KP wave function, via

\[
W_n^{\pm(0)}(s, -s, z) = \begin{pmatrix}
    (-1)^n & 0 & 0 & 1 \\
    0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0
\end{pmatrix} \begin{pmatrix}
    1 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0
\end{pmatrix}
\]

To prove this we calculate

\[
w^{(1)}_n(t^{(1)}, t^{(2)}, z) = (n + 1, -n, 0) e^{\sum_{j>0} j^\beta_j \psi^{(1)}(z)} g(0)
\]

\[
= (-z)^n e^{\sum_{j>0} t^{(1)}_j z^j} (n + 1, -n, 0) e^{\sum_{j>0} (t^{(1)} - z) j^\alpha_j \psi^{(1)}(z)} g(0)
\]

\[
= (-z)^n e^{\sum_{j>0} t^{(1)}_j z^j} \tau^{(0)}_n (t^{(1)} - [z^{-1}], t^{(2)})
\]

Now set \( g = h \hat{h} \) and \( t^{(1)}_j = s_j = -t^{(2)} \). Then

\[
w^{(1)}_n(s, -s, z) = (-z)^n e^{\sum_{j>0} s_j z^j} \tau^{(0)}_n (s - [z^{-1}], -s)
\]

\[
= (n + 1, -n, 0) e^{\sum_{j>0} s_j \psi^{(1)}(z)} h \hat{h}(0)
\]

\[
= (-i)^{n+1} 0 \psi^{(1)}_{n+1} \psi^{(2)}_{n+1} \psi^{(2)}_{n+1} e^{\sum_{j>0} s_j \psi^{(1)}(z)} (\psi(z) - i \psi(z)) h \hat{h}(0)
\]

\[
= \frac{(-i)^{n+1}}{2} (0) \psi^{(1)}_{n+1} \psi^{(1)}_{n+1} (\psi(z) + i \psi(z)) e^{\sum_{j>0} s_j \psi^{(1)}(z)} (\psi(z) - i \psi(z)) h \hat{h}(0)
\]

\[
= \frac{(-i)^{n+1}}{2} (0) \psi^{(1)}_{n+1} \psi^{(1)}_{n+1} (\psi(z) + i \psi(z)) e^{\sum_{j>0} s_j \psi^{(1)}(z)} \psi(z) h \hat{h}(0)
\]
\[ + \frac{\imath n}{2} z^n e^{\sum_{j>0} s_j z^j} \langle 0 | \hat{\Psi}_n^{-1} \hat{\Phi}_n \sum_{j>0} (s_j - \tfrac{1}{2} j) \hat{\beta}_j \rangle h_0 \langle 0 | \]

\[ + \frac{\imath n}{2} z^n e^{\sum_{j>0} s_j z^j} \langle 0 | \hat{\Psi}_n^{-1} \hat{\Phi}_n \sum_{j>0} s_j \hat{\beta}_j + (s_j - \tfrac{1}{2} j) \hat{\beta}_j \rangle h_0 \langle 0 | \]

\[ + \left( -\frac{i}{2} \right)^{n+1} z^{-n-1} e^{\sum_{j>0} s_j z^j} \langle 0 | \hat{\Psi}_{n+1}^{-1} \hat{\Phi}_{n-1} e^{\sum_{j>0} s_j \hat{\beta}_j - (s_j - \tfrac{1}{2} j) \hat{\beta}_j} h_0 \langle 0 | \right) \]

\[
= \begin{cases} 
(1) z^n e^{\sum_{j>0} s_j z^j} \left( \tau_n B (s \tau_n B (s - [z^{-1}]) - z^{-1} \tau_{n+1} B (s - [z^{-1}]) \right), & \text{if } n \text{ even} \\
(1) z^n e^{\sum_{j>0} s_j z^j} \left( \tau_n B (s \tau_n B (s - [z^{-1}]) - z^{-1} \tau_{n+1} B (s - [z^{-1}]) \right), & \text{if } n \text{ odd}
\end{cases}
\]

Thus we obtain

\[ w_n^{(1)} (s, -s, z) = (1) z^n e^{\sum_{j>0} s_j z^j} \left( \tau_n B (s \tau_n B (s - [z^{-1}]) - z^{-1} \tau_{n+1} B (s - [z^{-1}]) \right) \]

which is the first formula of

\[ (-1)^{\frac{n(n+1)}{2}} r_n^{(0)} (s - [z^{-1}], -s) = \tau_n B (s \tau_n B (s - [z^{-1}]) - z^{-1} \tau_{n+1} B (s - [z^{-1}]) \] (135)

\[ (-1)^{\frac{n(n+1)}{2}} r_n^{(0)} (s + [z^{-1}], -s) = \tau_n B (s \tau_n B (s + [z^{-1}]) + z^{-1} \tau_{n+1} B (s + [z^{-1}]) \] (136)

\[ (-1)^{\frac{n(n+1)}{2}} r_n^{(0)} (s, -s - [z^{-1}]) = \tau_n B (s \tau_n B (s + [z^{-1}]) - z^{-1} \tau_{n+1} B (s + [z^{-1}]) \] (137)

\[ (-1)^{\frac{n(n+1)}{2}} r_n^{(0)} (s, -s + [z^{-1}]) = \tau_n B (s \tau_n B (s - [z^{-1}]) + z^{-1} \tau_{n+1} B (s - [z^{-1}]) \] (138)

The other formulas can be obtained in a similar way. From these formulas and Proposition 4 one easily deduces Proposition 5.

6 The two-sided BKP (2-BKP) and two-component Toda lattice

In this section we consider a two sided version of some of the previous constructions.

2-BKP and two-component 2-KP (two-component Toda lattice) tau functions. Consider also

\[ \tau_{n,m,t}^{(0)} (t^{(1)}, t^{(2)}; \bar{t}^{(1)}, \bar{t}^{(2)} | g) := \langle n, l - m, 0 | e^{\sum_{i>0} (t_i^{(1)} a_i^{(1)} + t_i^{(2)} a_i^{(2)})} g e^{\sum_{i>0} (\bar{t}_i^{(1)} a_i^{(1)} + \bar{t}_i^{(2)} a_i^{(2)})} | m, l - m, 0 \rangle \] (139)

which may be considered as the two-component 2-KP tau function, or, the same, two-component Toda lattice tau function.\(^4\) The Hirota equations for the tau function \((139)\) may also be found in the Appendix A.2

Remark 4. We have

\[ \tau_{n,m,t}^{(0)} (t^{(1)}, t^{(2)}; \bar{t}^{(1)}, \bar{t}^{(2)} | g) = \tau_{m,n,t}^{(0)} (\bar{t}^{(1)}, \bar{t}^{(2)}; t^{(1)}, t^{(2)} | g^\dagger) \] (140)

For the proof of the Remark 4 we notice that

\[ \left( \langle n, l - m, 0 | e^{\sum_{i>0} (t_i^{(1)} a_i^{(1)} + t_i^{(2)} a_i^{(2)})} \right)^\dagger = e^{\sum_{i>0} (\bar{t}_i^{(1)} a_i^{(1)} + \bar{t}_i^{(2)} a_i^{(2)})} | m, l - m, 0 \rangle \]

and the fact that the pairing of two vectors and of two corresponding dual vectors coincides.

Later we shall omit the argument \(g\) on the left hand side of \((139)\).

In what follows we shall put \(l = 0\), \(t^{(1)}_k = s_k = -t^{(2)}_k\), \(\bar{t}^{(1)}_k = \bar{s}_k = -\bar{t}^{(2)}_k\), where \(s\) and \(\bar{s}\) are two independent sets of variables. Below \(g = h\). These are restrictions which allow to compare two-component TL tau functions with BKP tau functions.

\(^1\)see a piece between relations (9.6) and (9.7) in [10]
Consider a 2-BKP tau function
\[ \tau_{n,m}^B(s, \bar{s}| h) := \langle n \mid e^{\sum_{i>0} \bar{s}_i \bar{\beta}_i} \ h e^{\sum_{i>0} s_i \beta_{i-1}} \mid m \rangle \] (141)
which depends on two discrete parameters \( n \) and \( m \) and two sets of higher times \( s = (s_1, s_2, \ldots) \) and \( \bar{s} = (\bar{s}_1, \bar{s}_2, \ldots) \). Hirota equations of the 2-BKP. The name 2-BKP (two-sided BKP) is related to the fact that 2-BKP Hirota equations (see the Appendix A.3) contains the BKP Hirota equations with respect to the variables \( s, m \) the same as Hirota BKP equations with respect the variables \( \bar{s}, n \) (102).

**Remark 5.** We have the following symmetry
\[ \tau_{n,m}^B(s, \bar{s}| h) = \tau_{m,n}^B(\bar{s}, s|h) \] (142)
which is proved in the same way as the Remark 4. Then it follows that the 2-BKP tau functions \( \tau_{n,m}^B(s, \bar{s}) \) is a BKP tau function with respect to the variables \( \bar{s} \). This explains the name 2-BKP tau function.

In the same way as Proposition 1 was obtained we get

**Proposition 3.** A BKP tau function satisfies
\[ \left( \tau_{n,m}^B(s, \bar{s}| h) \right)^2 = (-) \frac{n(m+1)}{2} \tau_{n,m,0}(s, -s; \bar{s}, -\bar{s}| g) \] (143)
where \( g = h h \).

For proof of Proposition 3 we note that from Lemma (1) it follows that
\[ | - 2n, 2n, 0 \rangle = (-1)^n \hat{\Psi}_{-2n} \Psi_{-2n} | 0 \rangle, \quad | 1 - 2n, 2n - 1, 0 \rangle = (-1)^{n+1} \hat{\Psi}_{-2n} \sqrt{2} \hat{\Psi}_{-2n} | 0 \rangle \]
which allows to repeat all the steps of the derivation of the Proposition.

The other way to prove it is just to modify \( g \) in the Proposition by a certain right factor whose action on \( |0, 0, 0 \rangle \) recreates the vector \((151)\) to the semi-infinite lattice of \( s, \bar{s}, m, \bar{m} \).

**Miwa transforms** For tau functions of the two-component TL and 2-BKP we can write (cf. (35)):

\[ (-1)^{\frac{n(n+1)}{2} + \frac{m(m+1)}{2}} \tau_{n,m,0}^B(s - [x], -s; \bar{s}, -\bar{s}) = \tau_{n,m}^B(s, \bar{s}) \tau_{n+1,m}^B(s - [x], \bar{s}) + \tau_{n-1,m}^B(s, \bar{s}) \tau_{n+1,m}^B(s + [x], \bar{s}) \] (144)

\[ (-1)^{\frac{n(n+1)}{2} + \frac{m(m+1)}{2}} \tau_{n,m,0}^B(s + [x], -s; \bar{s}, -\bar{s}) = \tau_{n+1,m}^B(s, \bar{s}) \tau_{n,m}^B(s + [x], \bar{s}) + \tau_{n-1,m}^B(s, \bar{s}) \tau_{n+1,m}^B(s + [x], \bar{s}) \] (145)

\[ (-1)^{\frac{n(n+1)}{2} + \frac{m(m+1)}{2}} \tau_{n,m,0}^B(s, -s - [x]; \bar{s}, -\bar{s}) = \tau_{n,m}^B(s, \bar{s}) \tau_{n,m}^B(s + [x], \bar{s}) + \tau_{n+1,m}^B(s, \bar{s}) \tau_{n-1,m}^B(s + [x], \bar{s}) \] (146)

\[ (-1)^{\frac{n(n+1)}{2} + \frac{m(m+1)}{2}} \tau_{n,m,0}^B(s, -s + [x]; \bar{s}, -\bar{s}) = \tau_{n,m}^B(s, \bar{s}) \tau_{n,m}^B(s - [x], \bar{s}) + \tau_{n+1,m}^B(s, \bar{s}) \tau_{n-1,m}^B(s - [x], \bar{s}) \] (147)

Let us note that it is correct for any choice of \( h \). Then after replacing \( g = h h \) by \( g^t = h^t h^t \) in the last formulae and using the Remarks 4, 5 from (144), (147) it may be obtained:

\[ (-1)^{\frac{n(n+1)}{2} + \frac{m(m+1)}{2}} \tau_{n,m,0}^B(s, -s; \bar{s} - [x], -\bar{s}) = \tau_{n,m}^B(s, \bar{s}) \tau_{n,m}^B(s, \bar{s} - [x]) + \tau_{n,m}^B(s, \bar{s}) \tau_{n,m}^B(s, \bar{s} - [x]) \] (148)

\[ (-1)^{\frac{n(n+1)}{2} + \frac{m(m+1)}{2}} \tau_{n,m,0}^B(s, -s; \bar{s} + [x], -\bar{s}) = \tau_{n,m}^B(s, \bar{s}) \tau_{n,m}^B(s, \bar{s} + [x]) + \tau_{n,m}^B(s, \bar{s}) \tau_{n,m}^B(s, \bar{s} + [x]) \] (149)

\[ (-1)^{\frac{n(n+1)}{2} + \frac{m(m+1)}{2}} \tau_{n,m,0}^B(s, -s; \bar{s}, -\bar{s} - [x]) = \tau_{n,m}^B(s, \bar{s}) \tau_{n,m}^B(s, \bar{s} + [x]) + \tau_{n,m}^B(s, \bar{s}) \tau_{n,m}^B(s, \bar{s} + [x]) \] (150)

\[ (-1)^{\frac{n(n+1)}{2} + \frac{m(m+1)}{2}} \tau_{n,m,0}^B(s, -s; \bar{s}, -\bar{s} + [x]) = \tau_{n,m}^B(s, \bar{s}) \tau_{n,m}^B(s, \bar{s} - [x]) + \tau_{n,m}^B(s, \bar{s}) \tau_{n,m}^B(s, \bar{s} - [x]) \] (151)

### 7 An example: Toda lattice and B-type Pfaff lattice (BPL)

In many applications (like random matrices or random partitions) semi-infinite Toda lattice and semi-infinite Pfaff lattice are of use. Here we relate the semi-infinite Pfaff lattice of B-type to the semi-infinite Toda lattice.
Semi-infinite Toda lattice. First let us recall that the Hirota equation for the Toda lattice tau function [10], [31] and for the two-component KP tau function [10] coincide up to a sign factor, see the relation (9.7) in [10] and the Theorem 1.12 in [31]. Here we shall consider the semi-infinite Toda lattice which may be presented as
\[ \tau_{TL}(t, \ell) = (-)^{n+1} \tau_{2KP}(t, \ell) \] (152)
\[ \tau_{2KP}(t^{(1)}, t^{(2)}) = (N, -N) e^{\sum_{i,j} \bar{a}_i a_j} e^{\sum_{i,j} M_{ij} \psi_i^{(1)} \psi_j^{(2)}} \] (0)
(153)
(see also Appendix A.1).

The tau function (153) is the tau function \( \tau_{N,0,0} \) of (139) where \( g = e^{\sum_{i,j} M_{ij} \psi_i^{(1)} \psi_j^{(2)}} \). This choice provides semi-infinity of the TL equation which is
\[ \partial^2 \tau_{TL}^{(1)} / \partial t^{(1)} \partial t^{(2)}_1 \tau_{TL}^{(1)} - \partial \tau_{TL}^{(1)} / \partial t^{(1)} \partial \tau_{TL}^{(2)} = -\tau_{TL}^{(1)} \tau_{TL}^{(1)} \] (154)
where we put \( \tau_{TL}^{(1)}(t^{(1)}, t^{(2)}) = \delta_{N,0} \) for \( N \leq 0 \). Given \( \tau_{TL}^{(1)} \) one can construct all \( \tau_{TL}^{(1)} \), \( N > 1 \) in a recurrent way via (154).

Introduce
\[ e^{\sum_{k>0} s_k t_k} := \sum_{k=0} z^k s_k(t), \quad s_{(h)}(t) := \det (s_{h_i-j_i})_{k,j=1,\ldots,N} \] where \( h = (h_1, \ldots, h_N), h_1 > \cdots > h_N \geq 0, \) and \( s_{(h)} \) is the Schur function related to the partition \( \lambda = (\lambda_1, \cdots, \lambda_N), \lambda_i = h_i + i - N, \) see [19].

Using the relations
\[ e^{\sum_{i\in\mathbb{Z}} a_i t_i} \psi_j^{(a)} e^{-\sum_{i\in\mathbb{Z}} a_i t_i} = \sum_{k \geq 0} \psi_j^{(a)} k \] and Wick’s rule (see Appendix A.3), the tau function (153) may be presented in the determinantal form:
\[ \tau_{TL}^{(1)}(t^{(1)}, t^{(2)}) = \det \left( m_{ij}(t^{(1)}, t^{(2)}) \right)_{i,j=0,\ldots,N-1} \] (155)
where
\[ m_{ij}(t^{(1)}, t^{(2)}) = \sum_{k,l \geq 0} M_{k+l, k+l} s_k(t^{(1)}) s_l(t^{(2)}) \] (156)
As we see \( \tau_{TL} = m_{00} \).

The tau function (153) may also be written in Takasaki form [27], [29], [30] as
\[ \tau_{TL}^{(1)}(t^{(1)}, t^{(2)}) = \sum_{h, \lambda} M_{\lambda,h} s_{(h)}(t^{(1)}) s_{(\lambda)}(t^{(2)}) \] (157)
where
\[ M_{\lambda,h} := \det \left[ M_{\lambda_i, \lambda_i'} \right]_{i,j=1}^N \] (158)
Series (157), where \( M \) is specified, appear in various problems of random matrices, random partitions and in counting problems.

Another way to present tau functions of the semi-infinite TL tau functions (which is of use in many models of random matrices, see [12] and Examples below) is
\[ \tau_{TL}^{(1)}(t^{(1)}, t^{(2)}, \ell^{(1)}, \ell^{(2)}) = (N, -N) e^{\sum_{i,j=1,2} \sum_{i\in\mathbb{Z}} a_i^{(a)} t_i^{(a)}} e^{\psi^{(1)}(z_1) \psi^{(2)}(z_2) d\mu(z_1, z_2)} e^{\sum_{i\in\mathbb{Z}} a_i^{(a)} t_i^{(a)}} \] (159)
where \( d\mu(z_1, z_2) \) is a bi-measure which should be specified according to a problem we are interested in. There are additional parameters here, \( \ell^{(1)}, \ell^{(2)} \) and \( m \), which are hidden parameters of Toda lattice solutions. This is a particular case of the tau function (139).
From (159) we obtain \( \tau_{N,m}^{TL}(t^{(1)},t^{(2)},\bar{t}^{(1)},\bar{t}^{(2)}) = \delta_{n,m} \) for \( n \leq m \), and

\[
\tau_{m+1,m}^{TL}(t^{(1)},t^{(2)},\bar{t}^{(1)},\bar{t}^{(2)}) = c(t) \int z_1^m z_2^m e^{\sum_{k \geq 1}(\frac{1}{k} t^{(1)} - \frac{1}{k} \bar{t}^{(1)} + z^{(1)} - \bar{z}^{(1)})} d\mu(z_1, z_2) \tag{160}
\]

\[
c(t) = e^{-\sum_{k \geq 1}(i t^{(1)} + \bar{t}^{(2)} \bar{t}^{(1)})}
\]

**Example 1.1.** The choice \( d\mu(z_1, z_2) = e^{-|z|^2} \delta^{(2)}(z_2 - z_1) d^2 z_1 d^2 z_2, z_{1,2} \in \mathbb{C} \) yields (see [12]) both the fermionic representation for the partition function of the ensemble of normal matrices (about this ensemble see [3] and one-dimensional TL see in [6] and different fermionic representation see in [7]). In this example \( \bar{z} \) is the complex conjugate of \( z \).

Here \( \delta^{(2)} \) is the two-dimensional delta function.

**Example 1.2.** The choice \( d\mu(z_1, z_2) = e^{-|z|^2} d^2 z_1 d^2 z_2, z_{1,2} \in \mathbb{R} \) yields (see [12]) the fermionic representation for the partition function of the two-matrix model introduced in [9] (the relation of the two-matrix model and TL see in [6] and different fermionic representation see in [7]). In case we take \( z_{1,2} \in S^1 \) we obtain the model of two unitary matrices [33].

**Example 1.3.** The choice \( d\mu(z_1, z_2) = \delta(z_1 - z_2) d^2 z_1 d^2 z_2, z_{1,2} \in \mathbb{R} \) yields (see [8]) the fermionic representation for the one-matrix model (the relation of the one-matrix model and one-dimensional TL see in [6] and different fermionic representation see in [7]).

**Example 1.4.** Take \( z_2 = e^{i\phi_2}, a = 1, 2 \). The choice \( d\mu(e^{i\phi_1}, e^{i\phi_2}) = \delta(\phi_1 - \phi_2) d\phi_1 d\phi_2, 0 \leq \phi_1, \phi_2, \phi \leq 2\pi \), yields the fermionic representation for the \( \beta = 2 \) circilar ensemble (about this ensemble see Ch. 10.3 in [20] and [22]).

In case the bi-measure \( d\mu \) and the matrix \( M \) are related by the moment’s transform

\[
M_{ij} = M_{ij}(\bar{t}^{(1)},\bar{t}^{(2)},m) = \int z_1^{i+m} z_2^{j+m} e^{\sum_{k \geq 0}(\bar{t}^{(1)}_{-k} - \bar{t}^{(2)}_{-k} - z^{(1)}_{-k} - z^{(2)}_{-k})} d\mu(z_1, z_2) \tag{161}
\]

then the tau functions (153) and (155) may be equated as follows

\[
\tau_{N+1,m}^{KP}(t^{(1)},t^{(2)},\bar{t}^{(1)},\bar{t}^{(2)},m) = e^{\sum_{n=1}^{m} k_n^{(a)}} \tau_{N,m}^{TL}(\bar{t}^{(1)},\bar{t}^{(2)},t^{(1)},t^{(2)}) \tag{162}
\]

Now the tau function (155) written here depends on the additional parameters \( t^{(1)},t^{(2)},m \). Expression for \( \tau_{1}^{TL} \) given by (160) allows to obtain all \( \tau_{N}^{TL}, N > 1 \), in the recurrent way from TL Hirota equation (154).

**Semi-infinite TL and semi-infinite B-type Pfaff Lattice (BPL).**

**Proposition 4.**

\[
(\tau_{N}^{P}(t))^{2} = \tau_{N}^{TL}(t,-t) \tag{163}
\]

where

\[
\tau_{N}^{P}(t) = (\langle N | e^{\sum_{i \in \mathbb{Z} + \beta/2} A_{i,j} \psi_i \psi_j + \sqrt{2} \sum_{i,j} a_{i,j} \psi_i \bar{\psi}_j + \sum_{i \in \mathbb{Z}} b_{-i} \psi_i | 0 \rangle \tag{164}
\]

\[
\tau_{N}^{TL}(t,-t) = (-1)^{\frac{N(N+1)}{2}} \langle N,-N | e^{\sum_{i \in \mathbb{Z} + \beta/2} (\psi_i^{(1)} - \psi_i^{(2)}) \psi_i \sum_{i,j} (A_{i,j} - a_{i,j}) \psi_i^{(1)} \psi_i^{(2)} | 0 \rangle \tag{165}
\]

and \( A_{ij} = -A_{ji} \).

It is well-known that the determinant of a skew symmetric \( N \times N \) matrix vanishes if \( N \) is odd. The square root of a skew symmetric matrix of even size is the Pfaffian of this matrix. Let us call the sum of a skew symmetric matrix and a symmetric matrix of rank 1 the quasi-skew symmetric matrix. The square root of a quasi-skew symmetric matrix may be identified with a Pfaffian of some different matrix:

**Lemma 2.** The square root of the determinant of the quasi-skew symmetric \( N \times N \) matrix with entries \( A_{ij} = -a_{i,j}, i,j = 1, \ldots, N \), where \( A_{ij} = -A_{ji} \), is the pfaffian of the \( 2(\frac{N+1}{2}) \times 2(\frac{N+1}{2}) \) matrix \( B \) which is defined as follows. For \( N \) even, \( B = A \). For \( N \) odd, \( N = 2n - 1 \),

\[
B_{ij} = -B_{ji} := \begin{cases} A_{i,j} & \text{if} \quad 1 \leq i < j \leq 2n - 1 \\ a_i & \text{if} \quad 1 \leq i < j = 2n \end{cases} \tag{166}
\]

Thus, for both for odd and for even \( N \), we have

\[
(\text{Pf } B)^2 = \det (A_{i,j} - a_{i,j})_{i,j=1,\ldots,N} \tag{167}
\]
We have the following

**Corollary 1.**

\[
\left( \sum_{h_1 > \cdots > h_N} \sqrt{\det(A_{h_i}, h_j - a_h, a_{h_j})} s_i(h_j) \right)^2 = \sum_{h_1 > \cdots > h_N} \det(A_{h_i}, h_j - a_h, a_{h_j}) s_i(h_j) s_i(h_j) \tag{168}
\]

**Proposition 5.**

\[
\left( \tau_{n,m}(t, \bar{t}) \right)^2 = \frac{T}{\tau_{n,m}(t, -t; \bar{t}, -\bar{t})}
\]

where

\[
\tau_{n,m}(t, \bar{t}) = \langle n | e^{\sum_{i \in z} \beta_{t_i} \bar{z}_i} e^{\frac{1}{2} \int \psi(z_1) \psi(z_1) d\nu(z_1, \bar{z}_1) + \sqrt{T} \int \psi(z) d\nu(z)} e^{\sum_{i \in z} \beta_{\tau_i} t_i} | m \rangle
\]

\[
= (-)^{\frac{m+n+1}{2}} \frac{m+n+1}{\sqrt{T}} \tau_{n,m}(t, -t; \bar{t}, -\bar{t}) = \langle n, -n | e^{\sum_{i \in z} \left( \alpha_i^{(2)} - \alpha_i^{(2)} \right) t_i} \int \psi(z_1) \tau^{(2)}(z_2) d\nu(z_1, \bar{z}_1, d\nu(z_2)) e^{\sum_{i \in z} \left( \alpha_i^{(2)} - \alpha_i^{(2)} \right) t_i} | m, -m \rangle
\]

and where \( d\nu(z_1, z_2) = -d\nu(z_1, z_2) \) and \( d\nu(z) \) are measures.

Examples of the partition functions of various ensembles and also of some useful multiple integrals obtained as the semi-infinite BPL tau functions:

**Example 2.1.** The choice \( d\nu(z_1, z_2) = \frac{1}{2} \frac{\text{sgn}(z_1 - z_2)}{z_1 z_2} dz_1 dz_2 \) and \( d\nu(z) = dz, z_{1,2}, z \in \mathbb{R} \), yields the fermionic representation for the beta = 1 ensemble (orthogonal ensemble, see Ch. 7 in [20].

**Example 2.2.** The choice \( d\nu(z_1, z_2) = \frac{1}{4} (\delta_{1,2} - \delta_{1,2}) \frac{z_1 z_2}{z_1 z_2} dz_1 dz_2 \) and \( d\nu(z) = 0, z_{1,2} \in \mathbb{R} \), yields the fermionic representation for the beta = 2 ensemble (symplectic ensemble, see Ch. 8 in [20].

**Example 2.3.** Take \( z_a = e^{i\phi} \). The choice \( d\nu(e^{i\phi}, e^{i\phi}) = \frac{1}{4} \text{sgn}(\phi_1 - \phi_2) d\phi_1 d\phi_2 \) and \( d\nu(e^{i\phi}) = d\phi, 0 \leq \phi_1, \phi_2, \phi \leq 2\pi \), yields (see [25]) the fermionic representation for the beta = 1 circular ensemble (about this ensemble see Ch. 10.1 in [20].

**Example 2.4.** The choice \( d\nu(e^{i\phi}, e^{i\phi}) = \frac{1}{4} (\delta_{1,2} - \delta_{1,2}) \frac{z_a}{z_a} dz_1 dz_2 \) and \( d\nu(e^{i\phi}) = d\phi, 0 \leq \phi_1, \phi_2, \phi \leq 2\pi \), yields (see [25]) the fermionic representation for the beta = 2 circular ensemble (about this ensemble see Ch. 10.2 in [20].

**Example 2.5.** The choice \( d\nu(z_1, z_2) = \delta(z_1 + z_2) dz_1 dz_2 \) and \( d\nu(z) = \delta(z) dz, z_{1,2} \in \mathbb{R} \), yields (see [25]) the fermionic representation for the ensemble of antisymmetric matrices (see Ch. 13 in [20].

**Example 2.6.** For the Pandey-Mehta interpolating ensembles, \( 0 < \alpha^2 < \infty \) being the interpolation parameter, see [20] Ch. 14.

(a) for the ensemble interpolating between GUE and GOE, see [20] Ch. 14.1, we take

\[
d\nu(z_1, z_2) = e^{-\frac{1}{2}(1+\alpha^2)(z_1^2 + z_2^2)} \text{erf} \left( \sqrt{\frac{1 - \alpha^4}{4\alpha^2}} (z_1 - z_2) \right) \frac{dz_1 dz_2}{z_2}, \quad d\nu(z) = e^{-\frac{1}{2}(1+\alpha^2)z^2} dz, \quad z_{1,2} \in \mathbb{R}
\]

Here \( \text{erf}(x) = \int_0^x e^{-t^2} dt \).

(b) for the ensemble interpolating between GUE and GSE, see [20] Ch. 14.2, we take

\[
d\nu(z_1, z_2) = (z_1 - z_2)^{-\frac{1}{2}(1+\alpha^2)(z_1^2 + z_2^2)} e^{-\frac{1}{(1-\alpha^4)(z_1^2 + z_2^2)}} \frac{dz_1 dz_2}{z_2}, \quad z_{1,2} \in \mathbb{R}
\]

The same measures \( d\omega, d\nu \) are available to get the asymmetric two-matrix model where the first matrix is Hermitian while the second is respectively (a) either symmetric \((0 < \alpha^2 < 1)\), or antisymmetric \((1 < \alpha^2 < \infty)\) (b) either self-dual \((0 < \alpha^2 < 1)\), or anti-self-dual \((1 < \alpha^2 < \infty)\), see [25].

**Example 2.7.** The choice \( d\nu(z_1, z_2) = \frac{1}{2} (\bar{z} - z) e^{-|\bar{z}|^2} \delta^{(2)}(z_2 - z_1) dz_1 dz_2, \quad z_{1,2} \in \mathbb{C} \) and \( d\nu(z) = 0 \) yields (see [25]) the fermionic representation for the ensemble of random (non-Hermitian) quaternionic matrices (the so-called quaternion-real Ginibre ensemble (see Ch. 15.2 in [20]). Here \( \delta^{(2)} \) is the two-dimensional delta function. In this example \( \bar{z} \) is the complex conjugate of \( z \).

**Example 2.8.** For the Ginibre ensemble of real matrices (see Ch. 15.3 in [20]) the choice of the measure is more complicated, and may be found in [25].
Example 2.9. The choice $d\omega(z_1, z_2) = \frac{1}{2\pi i} \frac{dz_1 dz_2}{z_1 z_2}$, $z_{1,2} \in \mathbb{R}$, yields (see [25]) the fermionic representation for the so-called Bures ensembles which appears in quantum chaos problems where random density matrix appears [14].

Example 2.10. The choice $d\omega(z_1, z_2) = \frac{1}{2} \tanh(\pi(z_1 - z_2)) dz_1 dz_2$, $z_{1,2} \in \mathbb{R}$, yields (see [25]) the fermionic representation for the Plancherel measure for the group representation for the so-called Bures ensembles which appears in quantum chaos problems where random

$\ldots$ yielding the ensemble of symplectic matrices, details see in the forthcoming paper [18].

Examples 2.11. The choice $d\omega(z_1, z_2) = \frac{1}{2\pi i} (z_2 - z_1 - 1) \delta(z_2 - z_1) dz_1 dz_2$ and $dv(z) = 0$, $z_{1,2}, z \in S^1$ yields the ensemble of symplectic matrices, details see in the forthcoming paper [18].

Examples 2.12. The choice $d\omega(z_1, z_2) = \frac{1}{2\pi i} (z_2^{-1} - z_1^{-1}) dz_1 dz_2$ and $dv(z) = \frac{1}{2} \delta(z - 1) dz$, $z_{1,2}, z \in S^1$ yields the ensemble of orthogonal matrices, details see in the forthcoming paper [18].

The relation between realizations [163] and [170] is given by

$$A_{ij} = A_{ij}(\vec{t}, m) = \int z_1^{1+m} z_2^{1+m} e^{\sum_{k \geq 0} t_k (z_1^{k} + z_2^{k})} dv(z_1, z_2), \quad a_i = a_i(\vec{t}, m) = \int z^{1+m} e^{\sum_{k \geq 0} t_k z^{k}} dv(z)$$

(173)

In $B$-case relations [165], [165] read as

$$\tau_N^{T}(t, -t; \vec{t}, -\vec{t}, m) = \det (m_{ij}(t; \vec{t}, m))_{i,j=0, \ldots, N-1}$$

(174)

$$m_{ij}(t; \vec{t}, m) = \int z_1^{1+m} z_2^{1+m} e^{\sum_{k \geq 1} (z_1^{k} + z_2^{k}) t_k (z_1^{k} + z_2^{k}) t_k} (d\omega(z_1, z_2) - dv(z_1) dv(z_1))$$

(175)

$$= \sum_{k,l \geq 0} A_{i+k,j+l}(\vec{t}, m) s_k(t) s_l(t) - \sum_{k \geq 0} a_{i+k}(\vec{t}, m) s_k(t) \sum_{l \geq 0} a_{j+l}(\vec{t}, m) s_l(t)$$

(176)

As we see the matrix $m$ is quasi-skew symmetric one.

Then for both odd and even $N$ we have from Proposition 6

$$\tau_N^B(t; \vec{t}, m) \equiv \det (m_{ij}(t; \vec{t}, m))_{i,j=0, \ldots, N-1}$$

(177)

According to Lemma 2, $\tau_N^B$ is a certain Pfaffian (this result may be obtained also from the Wick theorem applied to the fermionic expectation value (170)).

We obtain a generalization of the result presented in [1], [2] where the relation (177) was achieved for the case of even $N$ and for skew symmetric matrices $m$. This case may referred as the $D$-type Pfaff lattice (DPL), see [28].

8 Outlook

In our next paper [18] the analogue of the Proposition 1 for the multicomponent BKP tau functions [15] will be written down. Also we shall consider the relations between various matrix integrals and between series over partitions which result from Propositions 5 and 4.

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A Appendices

A.1 Hirota equation for the TL and for the 2-component KP tau functions.

The TL tau function was introduced in [10] and may be defined by

\[ \tau_{n}^{\text{TL}}(t, \bar{t}) = \langle n | e^{\sum_{i>0} t_i \alpha_i} g^{\text{TL}} e^{-\sum_{i>0} \bar{t}_i \alpha_i} | n \rangle \]  

(178)

This tau function solves Hirota equation, [10], [31]

\[ \oint \frac{dz}{2\pi i} z^{n' - n} e^{V(t' - t, z)} \tau_{n'}^{\text{TL}} (t' = [z^{-1}], \bar{t}') \tau_n^{\text{TL}} (t + [z], \bar{t}) = \]  

\[ \oint \frac{dz}{2\pi i} z^{n' - n} e^{V(t' - \bar{t}, z^{-1})} \tau_{n'+1}^{\text{TL}} (t', \bar{t}' = [z]) \tau_{n-1}^{\text{TL}} (t, \bar{t} + [z]) \]  

(179)

(see [10], [31]) which includes

\[ \frac{\partial^2 \tau_n^{\text{TL}}}{\partial t_1 \partial \bar{t}_1} \tau_n^{\text{TL}} = \frac{\partial \tau_n^{\text{TL}}}{\partial t_1} \frac{\partial \tau_n^{\text{TL}}}{\partial \bar{t}_1} = -\tau_{n+1}^{\text{TL}} \tau_{n-1}^{\text{TL}} \]  

(180)

The two-component KP tau function

\[ \tau_n^{\text{2KP}}(t, \bar{t}) = \langle n, -n | e^{\sum_{i>0} (t_i \alpha_i^{(1)} + t_i \alpha_i^{(2)})} g^{\text{2KP}} | 0 \rangle \]  

(181)
solves Hirota equation
\[
\int \frac{dz}{2\pi i} (-)^{-n'-n}e^V(t'-t,z)\tau_{n'}^{2\text{KP}}(t'=[z^{-1}],\bar{\tau})\tau_n^{2\text{KP}}(t+[z^{-1}],\bar{\tau}) =
\int \frac{dz}{2\pi i} (-)^{-n'-n}e^V(t'-t,z)\tau_{n'}^{2\text{KP}}(t',[z^{-1}])\tau_n^{2\text{KP}}(t,[z^{-1}])
\] (182)
which up to the sign factor \((-)^{n+n'}\) in the first integral is [179] if we change \(z \rightarrow z^{-1}\) in the second
integral in [182]. This brings us to the relation [152].

A.2 Hirota equation for the two-sided two-component KP.

The two-sided two-component KP tau function [139]
\[
\tau_N \left( (L^{(1)}), (L^{(2)}); t^{(1)}, t^{(2)}, \bar{t}^{(1)}, \bar{t}^{(2)} \right) =
\langle N + L^{(1)}, -N + L^{(2)} | e^{\sum_{k=1}^\infty \left( \sum_{a_k} (\bar{t}^{(1)}_{k, a_k} + t^{(2)}_{k, a_k}) \right)} g e^{\sum_{k=1}^\infty \left( \sum_{a_k} (\bar{t}^{(1)}_{-k, a_k} + t^{(2)}_{-k, a_k}) \right)} | L^{(1)}, L^{(2)} \rangle
\] (183)
solves the following Hirota equations
\[
\int \frac{dz}{2\pi i} \text{Bil} \left( N', L'; \bar{t}', \bar{t}; \text{N, L; t; } \bar{t}; z \right) = 0,
\] (184)
where \(N' = (L'^{(1)}), (L'^{(2)})\) and \(L' = (L^{(1)}, L^{(2)})\) are pairs of integers, and \(t' = (t^{(1)}', t^{(2)}')\), \(\bar{t}' = (\bar{t}^{(1)}', \bar{t}^{(2)}')\), \(t^{(a)'} = (t_1^{(a)'}, t_2^{(a)'}, \ldots), \bar{t}^{(a)'} = (\bar{t}_1^{(a)'}, \bar{t}_2^{(a)'}, \ldots)\) are semi-infinite sets of higher times. Then
\[
t^{(a)'}(z) = t^{(a)'} + [z^{-1}], \quad \bar{t}^{(a)'}(z) = \bar{t}^{(a)'} + [z], \quad t^{(a)'}(z) = t^{(a)} + [z], \quad \bar{t}^{(a)'}(z) = \bar{t}^{(a)} + [z],
\]
and
\[
L^{(a)}_\pm = L^{(a)} + 1, \quad L^{(a)}_\pm = L^{(a)} + 1
\]
where \(a = 1, 2\).

Remark 6. By replacing \(z \rightarrow z^{-1}\) in the last two members of (185) we obtain Hirota equation for the 4-component KP (see [19] on multicomponent KP), where \(t^{(1)}\) and \(t^{(4)}\) may be identified respectively with \(\bar{t}^{(1)}\) and \(\bar{t}^{(1)}\).

The two-sided two-component KP is a particular case \((p = 4)\) of the \(p\)-component KP tau function, introduced in [10],
\[
\tau(N; s) := \langle N^{(1)}, \ldots, N^{(p)} | \sum_{a=1}^p \sum_{\Sigma_i > a} a^{a-1} g^{(1, \ldots, p)} | 0, 0 \rangle
\] (186)
where \(g^{(1, \ldots, p)}\) solves
\[
\left[ g^{(1, \ldots, p)} \otimes g^{(1, \ldots, p)} \sum_{a=1}^p \sum_{i \in \mathbb{Z}} \psi_i^{(a)} \otimes \psi_i^{(a)} \right] = 0
\] (187)
From (187) the multicomponent KP Hirota equations are obtained [10]:
\[
\sum_{a=1}^p \int \frac{dz}{2\pi i} (-)^{p-n} e^{V(t^{(a)}', -t^{(a)}; z)} \tau(N^{(a)}; t^{(a)}_\pm(z)) \tau(N^{(a)}; \bar{t}^{(a)}_\pm(z)) = 0
\] (188)
where
\[ N^{[a]}_\pm := \left( N^{(1)}, \ldots, N^{(a-1)}, N^{(a)} \pm 1, N^{(a+1)}, \ldots, N^{(p)} \right) \]
\[ t^{[a]}_\pm(z) := \left( t^{(1)}, \ldots, t^{(a-1)}, t^{(a)} \pm \frac{z^{-1}}{2}, t^{(a+1)}, \ldots, t^{(p)} \right) \]
\[ \kappa_a = N_p + \cdots + N_{a+1} + N_p' + \cdots + N_{a+1}' \quad (189) \]

In (188), \( N = \left( N^{(1)}, \ldots, N^{(p)} \right) \) and \( N' = \left( N^{(1)}, \ldots, N^{(p)} \right) \) are two independent sets of vacuum charges, while \( t^{(a)} = \left( t_1^{(a)}, t_2^{(a)}, \ldots, t_3^{(a)} \right) \) and \( t^{(a)'} = \left( t_1^{(a)'}, t_2^{(a)'}, \ldots, t_3^{(a)'} \right) \), \( a = 1, \ldots, p \), are two independent sets of the multicomponent KP higher times.

### A.3 Hirota equations for the two-sided BKP.

Hirota equations for the large BKP hierarchy were written in \[15\]. For the two-sided BKP hierarchy (2-BKP hierarchy) \[144\], Hirota equations are as follows \[24\]
\[ \begin{align*}
&\oint \frac{dz}{2\pi i} z^{N_1+L'-N_1-L-2} e^{V(s'-s,z)} \tau_{N_1'+1}(L', s' - [z^{-1}], \bar{s}) \tau_{N_1+1}(L, s + [z^{-1}], \bar{s}) \\
&+ \oint \frac{dz}{2\pi i} z^{N_2+L'-N_2-L-2} e^{V(s'-s,z)} \tau_{N_2'+1}(L', s' + [z^{-1}], \bar{s}) \tau_{N_2+1}(L, s - [z^{-1}], \bar{s}) \\
&= \oint \frac{dz}{2\pi i} z^{L'+L-2} e^{V(s'-s,z)} \tau_{N_2'+1}(L' + 1, s', \bar{s}' + [z]) \tau_{N_2'+1}(L - 1, s, \bar{s} - [z]) \\
&+ \oint \frac{dz}{2\pi i} z^{L'-L} e^{V(s'-s,z)} \tau_{N_2'+1}(L' - 1, s', \bar{s}' + [z]) \tau_{N_2'+1}(L + 1, s, \bar{s} + [z]) \\
&+ \frac{(-1)^{L'+L}}{2} \left( 1 - (-1)^{N_1'+N_2'} \right) \tau_{N_1'}(L', s', \bar{s}') \tau_{N_2}(L, s, \bar{s}) \quad (190)
\end{align*} \]

The difference BKP Hirota equation may be obtained from the previous one, see \[24\]
\[ \begin{align*}
&- \frac{\beta}{\alpha - \beta} \tau_N(l, t + [\beta^{-1}]) \tau_{N+1}(l, t + [\alpha^{-1}]) - \frac{\alpha}{\beta - \alpha} \tau_N(l, t + [\alpha^{-1}]) \tau_{N+1}(l, t + [\beta^{-1}]) \\
&+ \frac{1}{\alpha \beta} \tau_{N+2}(l, t + [\alpha^{-1}] + [\beta^{-1}]) \tau_{N-1}(l, t) = \tau_{N+1}(l, t + [\alpha^{-1}] + [\beta^{-1}]) \tau_N(l, t). \quad (191)
\end{align*} \]

By replacing \( z \to z^{-1} \) in the last two members of (190) it may be written as follows
\[ \begin{align*}
&\oint \frac{dz}{2\pi i} z^{N_1+L'-N_1-L-2} e^{V(s'-s,z)} \tau_{N_1'+1}(L', s' - [z^{-1}], \bar{s}) \tau_{N_1+1}(L, s + [z^{-1}], \bar{s}) \\
&+ \oint \frac{dz}{2\pi i} z^{N_2+L'-N_2-L-2} e^{V(s'-s,z)} \tau_{N_2'+1}(L' + 1, s', \bar{s}' + [z]) \tau_{N_2'+1}(L - 1, s, \bar{s} - [z]) \\
&- \oint \frac{dz}{2\pi i} z^{L'-L} e^{V(s'-s,z)} \tau_{N_2'+1}(L' - 1, s', \bar{s}' + [z]) \tau_{N_2'+1}(L + 1, s, \bar{s} + [z]) \\
&= \frac{(-1)^{L'+L}}{2} \left( 1 - (-1)^{N_1'+N_2'} \right) \tau_{N_2'}(L', s', \bar{s}') \tau_{N_2}(L, s, \bar{s}) \quad (192)
\end{align*} \]

which is up to the sign factor the Hirota equation for the 2-component BKP \[15\] tau function:
\[ \tau(N^{(1)}, N^{(2)}; s^{(1)}, s^{(2)}) := (-)^{\frac{\lambda(N^{(2)}+1)}{2}} \langle N^{(1)}, N^{(2)} \rangle e^{\sum_{a=1}^{2} \sum_{s} \beta^{a} s^{(a)} h^{(1,2)}(0,0)} \quad (193) \]

### A.4 Pfaffians, the Wick’s rule

**Pfaffian.** If \( A \) an anti-symmetric matrix of an odd order its determinant vanishes. For even order, say \( k \), the following multilinear form in \( A_{ij}, i < j \leq k \)
\[ \text{Pf}[A] := \sum_{\sigma} \text{sgn}(\sigma) A_{\sigma(1),\sigma(2)} A_{\sigma(3),\sigma(4)} \cdots A_{\sigma(k-1),\sigma(k)} \quad (194) \]
where sum runs over all permutation restricted by

\[ \sigma : \sigma(2i - 1) < \sigma(2i), \quad \sigma(1) < \sigma(3) < \cdots < \sigma(k - 1), \] (195)

coincides with the square root of \( \det A \) and is called the Pfaffian of \( A \), see, for instance [20].

**Wick’s relations.** Let each of \( w_i \) be a linear combination of Fermi operators:

\[ \hat{w}_i = \sum_{m \in \mathbb{Z}} v_{im} \psi_m + \sum_{m \in \mathbb{Z}} u_{im} \psi_m^\dagger, \quad i = 1, \ldots, n \]

Then the Wick formula is

\[
\langle l | \hat{w}_1 \cdots \hat{w}_n | l \rangle = \begin{cases} 
\text{Pf} [A]_{i,j=1,\ldots,n} & \text{if } n \text{ is even} \\
0 & \text{otherwise}
\end{cases} \tag{196}
\]

where \( A \) is \( n \) by \( n \) antisymmetric matrix with entries \( A_{ij} = \langle l | \hat{w}_i \hat{w}_j | l \rangle, \quad i < j \).