Large deviation principle for empirical measures of once-reinforced random walks on finite graphs

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Abstract

Given any connected graph \( G \) and any \( \delta \in (0, \infty) \). Note edge weight function at the \( (k + 1) \)-th step of \( \delta \)-once-reinforced random walk (\( \delta \)-ORRW) \( (X_n)_{n \geq 0} \) on \( G \) is given by

\[
w_k(e) = 1 + (\delta - 1) \cdot I_{\{N(e,k) > 0\}} = I_{\{N(e,k) = 0\}} + \delta \cdot I_{\{N(e,k) > 0\}},
\]

where \( N(e,k) \) is the number of times that edge \( e \) has been traversed by the walk before time \( k \). In this paper, for \( \delta \)-ORRW on finite connected graph \( G \), we mainly verify a large deviation principle (LDP) for its empirical measures, and characterize critical exponent for exponential integrability of a class of stopping times including cover time.

To prove the LDP, through introducing a class of functionals of the empirical measure processes and solving a novel dynamic programming equation associated with these functionals by a variational representation, we prove a variational formula of the limit of the functionals, which plays a key role in completing the proof of the LDP. We also prove that rate function \( I_\delta \) of the LDP is continuous and decreasing in \( \delta \in [1, \infty) \), and is constant in \( \delta \in (0, 1] \), and is not differentiable at \( \delta = 1 \), and \( I_{\delta_1} \neq I_{\delta_2} \) for any \( 1 \leq \delta_1 < \delta_2 \). For the critical exponent, we prove that it is continuous and strictly decreasing in \( \delta \in (0, \infty) \), and there is no exponential integrability at it; and its limit as \( \delta \to 0 \) can reflect structure of the graph \( G \) to a certain extent.

Key words: Once-reinforced random walk; Cover time; Large deviation principle; Empirical measure; Critical exponent.

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1 Introduction

Given any connected locally finite undirected graph $G = (V, E)$ with vertex set $V$ and edge set $E$. If $u, v \in V$ are adjacent, then write $u \sim v$ and denote the corresponding edge by $uv = \{u, v\}$. Recall firstly that edge reinforced random walk (ERRW) $X = (X_n)_{n \geq 0}$ on $G$, which is a self-interacting non-Markovian random walk, was firstly introduced by Coppersmith and Diaconis \cite{8} in 1987 to model the idea that random walker may love to traverse edges visited before. Mathematically, let $(\mathcal{F}_n)_{n \geq 0}$ be the natural filtration generated by the history of $X$, then the ERRW $X$ is defined by that for any $n \geq 0$,

$$
\mathbb{P}(X_{n+1} = u | X_n = v, \mathcal{F}_n) = \begin{cases} 
\frac{w_n(uv)}{\sum_{u' \sim v} w_n(u'v)}, & u \sim v, \\
0, & \text{otherwise},
\end{cases}
$$

where $w_n(e)$ is random weight of edge $e$ at time $n$. In the original model, for some $\delta > 0$,

$$
w_0(e) = 1, \ w_{n+1}(e) = w_n(e) + \delta 1_{\{X_nX_{n+1}=e\}}, \ e \in E, \ n \geq 0.
$$

Note the study of the ERRW is challenging due to that its future depends on its past trajectory. Considering the absence of results on ERRWs for many years, in 1990, Davis \cite{9} introduced once-reinforced random walk (ORRW) as an \textit{a priori} simplified version of the
ERRW. For the ORRW, initial weights are all taken to be 1; and once an edge is traversed, its weight changes to a constant $\delta > 0$ and remains in $\delta$. Namely, for any $n \geq 0$ and $e \in E$,

$$w_n(e) = 1 + (\delta - 1) \cdot 1_{\{N(e, n) > 0\}} = \begin{cases} 1 & \text{if } N(e, n) = 0, \\ \delta & \text{if } N(e, n) > 0, \end{cases}$$

where $N(e, n) := |\{i < n : X_i X_{i+1} = e\}|$ is the number of times that $e$ has been traversed by the walk up to time $n$, and $|A|$ denotes the cardinality of any set $A$. Call it negatively reinforced if $\delta < 1$, and positively reinforced if $\delta > 1$; and write $\delta$-ORRW for the ORRW with parameter $\delta$. Unexpectedly, after years of researches, as a seemingly simpler model, the ORRW is more difficult to study than other ERRWs. So far the ERRWs have derived many interesting mathematical issues such as the continuous reinforced process in [24] (see also [25]), and the random Schrödinger operator in [28], and the connection to the $\mathbb{H}^2$ and $\mathbb{H}^2$ spin systems in [27] (see also [1]), and also motivated a wide-range of researchers scattered in some other scientific fields such as biology ([6, 32]) and social networks ([19, 26, 31, 34]).

This paper devotes to studying large deviation principle (LDP) for empirical measures of ORRWs and critical exponent for exponential integrability of some stopping times including cover time of ORRWs when $G$ is a finite connected graph.

To begin, let us recall some researches on ORRWs. Such walks are hard to study usually. For instance, their recurrence and transience, which is a fundamental long time behaviour, can be very difficult to obtain generally when underlying graphs are infinite. For the $\delta$-ORRW on infinite trees $G$, there is a $\delta_c \in [0, \infty]$ such that it is recurrent for $\delta < \delta_c$ and transient for $\delta > \delta_c$ (Kious and Sidoravicius (2018) [18], Collevecchio, Kious and Sidoravicius (2020) [7]), where $\delta_c$ is the branching-ruin number of $G$. See [7, Sections 2-3] for information on when $\delta_c = 0$, $\delta_c = \infty$ and $\delta_c \in (0, \infty)$. The recurrence for all $\delta > 0$ is known to hold on lines with parallel edges including $\mathbb{Z}$ (Vervoort (2002) [33]), and on $\mathbb{Z} \times \{0, 1\}$ (Vervoort (2002) [33], Sellke (2006) [30], Huang, Liu, Sidoravicius and Xiang (2021) [16]). Recall the $\delta$-ORRW on ladders $\mathbb{Z} \times \{0, 1, \ldots, d\}$ with $d \geq 2$ is recurrent for $\delta \in (1 - 1/(d + 1), 1 + 1/(d - 1))$ ([33, 30]). Generally, Kious, Schapira and Singh (2018) [17] proved that the $\delta$-ORRW on ladder graphs $\mathbb{Z} \times \Gamma$ with $\Gamma$ being a finite connected graph is recurrent for suitably large $\delta$. Additionally, there is no recurrence/transience result on $\mathbb{Z}^d$ for any $d \geq 2$ and any $\delta > 0$; and it was conjectured by Sidoravicius (see [18, p. 2122]) that any $\delta$-ORRW on $\mathbb{Z}^2$ is recurrent and the $\delta$-ORRW on $\mathbb{Z}^d$ with $d \geq 3$ undergoes a phase transition, being recurrent for large $\delta$ and transient for small $\delta$.

Now a natural interesting question arises: When connected graph $G = (V, E)$ is finite with $|E| \geq 1$, which important long time behaviour can we study for the $\delta$-ORRW? Due to $G$ is finite and connected, intuitively the $\delta$-ORRW can visit all edges of $G$ in a finite time almost surely (we will see later this is indeed the case); namely (edge) cover time

$$C_E = \inf\{t : \forall e \in E, \exists s \leq t, \ X_{s-1}X_s = e\}$$

is a.s. finite. This follows from exponential integrability of $C_E$ specified in Proposition 7.2. Note the $\delta$-ORRW becomes a simple random walk (SRW) on $G$ after all edges being traversed. These suggest (and really imply) that almost surely, empirical measure process $(L_n)_{n \geq 1}$ of the $\delta$-ORRW $X$, defined by

$$L_n(A) = \frac{1}{n} \sum_{i=0}^{n-1} \tilde{\delta}_{X_i}(A)$$

for $A \subseteq V$, where $\tilde{\delta}_x$ is the Dirac measure centered at $x \in V$,

$$L_n(A) = \frac{1}{n} \sum_{i=0}^{n-1} \tilde{\delta}_{X_i}(A)$$

for $A \subseteq V$, where $\tilde{\delta}_x$ is the Dirac measure centered at $x \in V$, (1.1)

converges weakly to the unique invariant probability of the SRW on $G$, by the ergodic theorem of the SRW. Therefore, it is natural and interesting to study LDP of the empirical measure
process \((L_n)_{n \geq 1}\). Novel aspects of the mentioned LDP are that the \(\delta\)-ORRW \(X\) is not Markovian if \(\delta \neq 1\), and how it depends on parameter \(\delta \in (0, \infty)\), and how the cover time \(C_E\) will affect the LDP. So besides the LDP, we also study critical exponent for exponential integrability of some stopping times such as \(C_E\).

Our main results are summarized as follows: (i) The empirical measure process \((L_n)_{n \geq 1}\) of the \(\delta\)-ORRW satisfies an LDP with a good rate function \(I_\delta\) (Theorem 2.4). (ii) The rate function \(I_\delta\) is continuous and decreasing in \(\delta \in [1, \infty)\), and is constant in \(\delta \in (0, 1]\), and is not differentiable at \(\delta = 1\), and \(I_{\delta_1} \neq I_{\delta_2}\) for any \(1 \leq \delta_1 < \delta_2\) (Theorem 2.6). (iii) The critical exponent \(\alpha_1^\delta(\delta)\) for exponential integrability of the (edge) cover time \(C_E\) satisfies that

\[
\alpha_1^\delta(\delta) \in (0, \infty), \quad \lim_{n \to \infty} \frac{1}{n} \log P(C_E > n) = -\alpha_1^\delta(\delta),
\]

and \(\alpha_1^\delta(\delta)\) is continuous and strictly decreasing in \(\delta \in (0, \infty)\), and \(\lim_{\delta \to +\infty} \alpha_1^\delta(\delta) = 0\), and \(\lim_{\delta \to 0} \alpha_1^\delta(\delta)\) can reflect structure of the graph \(G\) to a certain extent as shown by Figure 1. See Theorem 2.9.

Note there is an interesting clear contrast: \(I_\delta \equiv I_1\) for \(\delta \in (0, 1]\), while \(\alpha_1^\delta(\delta)\) is strictly decreasing in \(\delta \in (0, 1]\). Recall every \(\delta\)-ORRW becomes the SRW after \(C_E\). Intuitively, the \(\delta\)-ORRW with \(\delta \in (0, 1)\) prefers to leave the traversed vertices and traverses all the edges very quickly; while the \(\delta\)-ORRW with \(\delta > 1\) prefers to stay in the traversed vertices. Then the unexpected beforehand but natural phase transition for the rate function given in (ii) can be explained by (iii) to a certain degree. Finally, \(I_\delta (\delta \neq 1)\) is distinct for different starting points of the \(\delta\)-ORRW and structure of the graphs (Remark 5.10), while \(I_1\) is independent of starting points of the SRW.

![Figure 1](attachment:image.png)

Figure 1: This is a sketch map of \(\alpha_1^\delta(\cdot)\) for the ORRW on finite connected graphs. The left graph stands for the case that \(\{(\mu, E_0 \in \mathcal{J}_0) : \text{supp}(\mu |_1) \subseteq E_0, \text{supp}(\mu |_1) \cap \partial E_0 = \emptyset\} \neq \emptyset\). Here, \(\alpha_1^\delta(\delta)\) converges to \(\inf \{(\mu, E_0 \in \mathcal{J}_0) : \text{supp}(\mu |_1) \subseteq E_0, \text{supp}(\mu |_1) \cap \partial E_0 = \emptyset\} \Lambda_1(\mu) < \infty\) as \(\delta\) approaches 0, and converges to 0 as \(\delta\) approaches +\(\infty\). The right graph stands for the case that \(\{(\mu, E_0 \in \mathcal{J}_0) : \text{supp}(\mu |_1) \subseteq E_0, \text{supp}(\mu |_1) \cap \partial E_0 = \emptyset\} = \emptyset\). In this graph, \(\alpha_1^\delta(\delta)\) converges to +\(\infty\) and 0 as \(\delta\) approaches 0 and +\(\infty\) respectively.

Now we are in the position to recall some previous LDP works for Markov processes, which may be helpful to our researches. As far as we know, the first study on the LDP for the empirical measure was given by Sanov [29] in 1957. He obtained that empirical measure of an i.i.d. sequence with common distribution \(\rho\) on a Polish space \(\mathcal{X}\) satisfies an LDP with the rate function \(R(\cdot ||\rho)\). Here \(R(\cdot ||\rho)\) is the relative entropy function defined by that for any
probability $\gamma$ on $\mathcal{X}$,
\[ R(\gamma||\rho) = \left\{ \begin{array}{ll}
\int_{\mathcal{X}} \frac{d\gamma}{d\rho} \log \frac{d\gamma}{d\rho} \, d\rho & \text{if } \gamma \ll \rho, \\
\infty & \text{otherwise.}
\end{array} \right. \]

In 1975, a seminal study on LDP for empirical measures of Markov processes was done by Donsker and Varadhan [11, 12, 13]. They proved that the empirical measure of a process generated by a strongly continuous Feller Markovian semigroup on a compact metric space satisfies an LDP.

Since the classical studies in [11, 12, 29] strongly depend on homogeneous Markov property, and the $\delta$-ORRW with $\delta \neq 1$ is not Markovian, it is hard to obtain an LDP for empirical measures of the $\delta$-ORRW with $\delta \neq 1$ when following aforementioned works. In fact, for non-Markovian processes, there were some LDP results. For instance, an LDP of renewal processes was shown in [21, 22, 23]. These works depend on Markov property of the embedding chain, which is Donsker and Varadhan’s case essentially. Therefore, it is not feasible to follow them for the ORRWs.

So far as we know, there has been only one LDP-type result for the ORRW given by Zhang [35] in 2014. To describe it, given any integer $b \geq 2$, let $T$ be an infinite rooted tree such that the root $o$ has $b$ neighbours and other vertices have $b+1$ neighbours. For any vertex $x \in T$, write $h(x)$ for the distance between $x$ and $o$, namely the height of $x$ with respect to $o$, on $T$. Given any $\delta > 1$. For the $\delta$-ORRW $(X_n)_{n \geq 0}$ on $T$, there is a positive constant $s = s(\delta)$ such that $\lim_{n \to \infty} \frac{h(X_n)}{n} = s$ a.s.. Zhang proved that for any $\varepsilon > 0$, there exists $\beta = \beta(\delta, b, \varepsilon) \in (0, \infty)$ such that
\[ \lim_{n \to \infty} \frac{1}{n} \log \mathbb{P}(h(X_n) \geq (s + \varepsilon)n) = \beta; \]
and when $0 < \varepsilon < s$,
\[ 0 < \liminf_{n \to \infty} \frac{1}{n} \log \mathbb{P}(h(X_n) \leq (s - \varepsilon)n) \leq \limsup_{n \to \infty} \frac{1}{n} \log \mathbb{P}(h(X_n) \leq (s - \varepsilon)n) < \infty. \]

Recently, Budhiraja and Waterbury [4] proved a nice LDP for empirical measures of reinforced random walks on finite spaces $\{1, 2, \ldots, d\}$ by the weak convergence approach, and the rate function of this LDP is strikingly different from the Donsker-Varadhan rate function associated with the empirical measure of the related Markov chain on $\{1, 2, \ldots, d\}$. This work is similar to but independent from our results on the LDP for the empirical measures of the ORRWs. Note that the reinforced random walk they considered is essentially different from the ORRW, and our techniques are greatly different from theirs though we also use the weak convergence approach.

In this paper we use the weak convergence approach in Dupuis and Ellis’s book [14] to study the LDP. Two novel ingredients in our proofs are as follows. Firstly, we give a more general variational formula and an exponential decay (see Theorems 2.3 and 2.8), which imply both the LDP for empirical measures of the ORRWs on finite graphs and the critical exponent for exponential integrability of the cover time (see Theorems 2.6 and 2.9). Secondly, in order to overcome essential difficulties produced by non-Markov property of the ORRWs $(X_n)_{n \geq 0}$, we have to lift $(X_n)_{n \geq 0}$ to a directed graph, and modify the weak convergence approach to the LDP for empirical measures of homogeneous Markov chains. For this purpose, we introduce a class of novel functionals, i.e. restricted logarithmic Laplace functionals. We then establish a dynamic programming equation associated with these functionals, and give its solution by a variational representation (see Lemma 2.16). Next, not only do we prove the upper bound
and the lower one of the LDP, but also we verify the convergence of subsequence of the admissible control sequence (see Theorem 3.1).

This paper is organized as follows. In Section 2, we give core ideas and main results of the paper, introduce the weak convergence approach, and explain how it works in our study. In Section 3, we prove the LDP for the empirical measures of the ORRWs, and show some interesting phenomena of the rate function. In Section 4, we verify related results on critical exponent for exponential integrability of the cover time. In Section 5, we calculate the rate functions on some specific graphs.

2 Preliminaries and main results

Recall the good rate function and the LDP from [10, Section 1.2]:

Let $\mathcal{X}$ be a Polish space, and $I : \mathcal{X} \to [0, \infty)$ a function on $\mathcal{X}$, and $(\xi_n)_{n \geq 1}$ a sequence of $\mathcal{X}$-valued random variables. Say $I$ is a good rate function if (i) $I$ is lower semicontinuous, namely $\liminf_{y \to x} I(y) \geq I(x)$ for any $x \in \mathcal{X}$, and (ii) level set $\{x : I(x) \leq M\}$ is compact for any $M > 0$, and (iii) there is some $x$ with $I(x) < \infty$. Say $(\xi_n)_{n \geq 1}$ satisfies an LDP if for some good rate function $I$, the following hold:

(a) For any closed subset $C \subseteq \mathcal{X}$, $\limsup_{n \to \infty} \frac{1}{n} \log \mathbb{P}(\xi_n \in C) \leq -\inf_{x \in C} I(x)$; and

(b) for any open subset $O \subseteq \mathcal{X}$, $\liminf_{n \to \infty} \frac{1}{n} \log \mathbb{P}(\xi_n \in O) \geq -\inf_{x \in O} I(x)$.

2.1 Main results

To state our main results, we introduce some notations. Consider a $\delta$-ORRW $X = (X_n)_{n \geq 0}$ on finite connected graph $G = (V, E)$ with a fixed ordering $v_1, \ldots, v_c$ of $V$ such that $X_0 = v_1$. Let $d := |E| \geq 1$ and $\mathbb{P}_{v_1}$ be the law of the $\delta$-ORRW $X = (X_n)_{n \geq 0}$ starting at $v_1$. Construct a new directed graph $S$ with vertex set $V_S := E \times \{-1, 1\}$ as follows: For any vertex $z = (v_m v_n, \sigma) \in E \times \{-1, 1\}$ with $m < n$, define $z^+$ (resp. $z^-$), the head (resp. tail) of $z$, by that

$$z^+ = (v_m v_n, \sigma^+) := \begin{cases} v_m, & \sigma = -1, \\ v_n, & \sigma = 1, \end{cases} \quad z^- = (v_m v_n, \sigma^-) := \begin{cases} v_n, & \sigma = -1, \\ v_m, & \sigma = 1. \end{cases}$$

For any two vertices $z_1, z_2 \in S$, if $z_1^+ = z_2^-$, denoted by $z_1 \to z_2$, then we set a directed edge $\overrightarrow{z_1 z_2}$ from $z_1$ to $z_2$ (see Figure 2).

Note that for any $n \geq 1$, there is a unique $\sigma_n \in \{-1, 1\}$ satisfying $Z_n^- = X_{n-1}$ and $Z_n^+ = X_n$, where

$$Z_n = (X_{n-1} X_n, \sigma_n) := (Y_n, \sigma_n).$$

Thus we construct a stochastic process $(Z_n)_{n \geq 1} = (Y_n, \sigma_n)_{n \geq 1}$ on $S$.

Definition 2.1.

$$p_{\mu}(z_1, z_2) = p_{\mu}(z_1, z_2; \delta) := \begin{cases} \frac{1}{\sum_{z_1 \to z_2} \mu(z_1(z_2))} & z_1 \to z_2, \\ \frac{1}{\sum_{z_1 \to z_2} \mu(z_1(z_2))} + \delta \frac{1}{\sum_{z_1 \to z_2} \mu(z_1(z_2))}, & \text{otherwise}, \end{cases}$$

where $\mu$ is a measure on $E \times \{-1, 1\}$, $\mu(z_1)$ is the first marginal measure of $\mu$, and $z_1$ stands for the first coordinate of $z$. 

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Notice that $p_\mu$ is determined by the support of $\mu$, i.e., $p_\mu = p_\nu$ if $\text{supp}(\mu) = \text{supp}(\nu)$. Write $p_\mu(z_1, z_2; \delta)$ as $p_\mu(z_1, z_2)$ for fixed $\delta$. For $E' \subseteq E$, introduce a measure $p_{E'}$ on $E \times \{-1, 1\}$:

$$p_{E'}(z) = \begin{cases} 1, & z \in E' \times \{-1, 1\}, \\ 0, & z \notin E' \times \{-1, 1\}. \end{cases}$$

For convenience we replace $p_{E'}$ by $p_{E'}$.

Let

$$E_n' = \{ e \in E : \exists m \leq n, Z_m|_1(\omega) = Y_m(\omega) = e \},$$

which is the collection of edges traversed by $X$ up to time $n$. Then it is easy to check that for any $n \geq 1$, $p_{E_n'}$ is the transition probability from $Z_n$ to $Z_{n+1}$:

$$P(Z_{n+1} = z_2 | Z_n = z_1, F_n) = \begin{cases} p_{E_n'}(z_1, z_2), & z_1 \rightarrow z_2, \\ 0, & \text{otherwise}. \end{cases} \quad (2.1)$$

Actually, $(Z_n)_{n \geq 1}$ can be regarded as a vertex once-reinforced random walk on the directed graph $S$ such that the weights of vertices $(e, 1)$ and $(e, -1)$ will be reinforced together when one of them is traversed (see Figure 3 for a simple illustration).

Let

$$\mathcal{L}^n(A) = \frac{1}{n} \sum_{i=1}^{n} \delta_{Z_i}(A), \quad A \subseteq S, \quad n \geq 1, \quad (2.2)$$

be the empirical measures of $(Z_n)_{n \geq 1}$, where $\delta_z$ is the Dirac measure on $E \times \{-1, 1\}$ at $z$.

The key point of introducing $(Z_n)_{n \geq 1}$ on $S$ is that $p_{E_n} = p_{E_n'}$ is the transition probability from $Z_n$ to $Z_{n+1}$, and $(Z_n, \mathcal{L}^n)_{n \geq 1}$ is a non-homogeneous Markov process satisfying that

$$P \left( Z_{n+1} = z', \mathcal{L}^{n+1} = \mu' \mid Z_n = z, \mathcal{L}^n = \mu \right) = p_{\mu + \delta_z} \left( z, z' \right) \cdot \delta_{\mu + \delta_z} \left( \mu' \right), \quad n \geq 1,$$

where $\delta_\mu$ is the Dirac measure on $\mathcal{P}(V_S)$ at $\nu$, and $\mathcal{P}(V_S)$ is the set of all probabilities on $V_S$ equipped with the weak convergence topology.

Denote by $\delta$ the collection of all sequences of subsets $\{E_k\}_{1 \leq k \leq d}$ satisfying the following conditions (see Figure 4 for an instance):

$$\delta_{\mu} = \sum_{k=1}^{d} E_k \subseteq S, \quad \mu \cdot p_{E_k} = \mu, \quad \mu \cdot p_{E_{k+1} - E_k} = \mu_{E_k},$$

Figure 3: This figure shows the weights in Figure 2 if only edge $e_1$ has been traversed by $X$. The transition probability of $(Z_n)_{n \geq 1}$ is $p_{(e_1)}$, which is induced by weights of vertices.
Figure 4: This figure shows a case of \( \{E_k\}_k \in \mathcal{E} \) for graph with vertex set \( \{v_i : 1 \leq i \leq 4\} \), where \( v_i \sim v_{i+1} \) for \( 1 \leq i \leq 3 \) and \( v_2 \sim v_4 \).

(a) \( E_k \subseteq E_{k+1} \subseteq E, \ 1 \leq k < d \).

(b) \( |E_k| = k, \ 1 \leq k \leq d \).

(c) There exists \( v \in V \) such that \( \{v_1, v\} \in E_1 \).

(d) The unique edge in \( E_{k+1} \setminus E_k \) is adjacent to some edge in \( E_k \).

That is, \( \mathcal{E} \) collects all possible sequences of renewal subsets of \( E \) generated by \( (Z_n|_1)_{n \geq 1} \), i.e. \( E_k = \{Z_n(\omega)|_1 : n \leq \tau_k\} \) when fixing \( \omega \), where

\[
\tau_k = \inf \{j > \tau_{k-1} : Z_j|_1 \notin \{Z_i|_1 : i < j\}\} \quad \text{and} \quad \tau_0 := 0.
\] (2.3)

Let

\[
\mathcal{S} = \left\{ E' : \exists \text{connected } G' = (V', E') \subseteq G = (V, E) \text{ and } \exists v \in V \text{ with } \{v_1, v\} \in E' \right\}
\] (2.4)

be the collection of edge sets \( E' \) of all connected subgraphs \( G' \) of finite connected graph \( G \) satisfying that \( E' \) contains an edge \( \{v_1, v\} \), namely contains the starting point \( v_1 \) of the ORRW. Given any non-empty decreasing proper subset \( \mathcal{S}_0 \) of \( \mathcal{S} \). Here “decreasing” means that for any \( E_1, E_2 \in \mathcal{S} \) with \( E_1 \subset E_2 \), if \( E_2 \in \mathcal{S}_0 \), then \( E_1 \in \mathcal{S}_0 \). Define

\[
\mathcal{C}(\mathcal{S}_0) := \{ \mu \in \mathcal{P}(V_S) : \text{supp}(\mu|_1) \subseteq E' \text{ for some } E' \in \mathcal{S}_0 \}.
\] (2.5)

Then \( \mathcal{C}(\mathcal{S}_0) \) is a closed subset of \( \mathcal{P}(V_S) \) under the weak convergence topology (Lemma 3.3). For any \( \mu \in \mathcal{P}(V_S) \), set

\[
\mathcal{A}(\mu, \mathcal{S}_0) = \left\{ (\mu_k, r_k, E_k)_{1 \leq k \leq d} : \sum_{k=1}^{d} \mu_k r_k = \mu, \ \text{supp}(\mu_k) = E_k \times \{-1, 1\}, \ \sum_{k=1}^{d} r_k = 1, \ \ r_k \geq 0, \ r_l = 0 \text{ for } E_l \notin \mathcal{S}_0, \ \{E_j\}_{1 \leq j \leq d} \in \mathcal{E} \right\}.
\]

Write \( \mathcal{A} \) for the collection of all transition probabilities on \( V_S \).

**Definition 2.2.** For any \( \mu \in \mathcal{P}(V_S) \) and non-empty decreasing proper subset \( \mathcal{S}_0 \) of \( \mathcal{S} \), let

\[
\Lambda_{\delta, \mathcal{S}_0}(\mu) = \inf_{(\mu_k, r_k, E_k)_{1 \leq k \leq d} \in \mathcal{A}(\mu, \mathcal{S}_0)} \sum_{k=1}^{d} r_k \int_{V_S} R(q_k\|p_{E_k}) \ d\mu_k.
\] (2.6)

Specifically, write \( \Lambda_{\delta, \mathcal{S}}(\cdot) \) as \( \Lambda_{\delta}(\cdot) \).
Denote by
\[ E_z := \{ (E_k)_{1 \leq k \leq d} \in E : E_1 = \{ z \} \} \tag{2.7} \]
the renewal subsets of \( E \) starting from edge set \( \{ z \} \) for any \( z \in V_s \). Set
\[ \mathcal{A}_z(\mu, \mathcal{A}_0) = \{ (\mu_k, r_k, E_k)_{1 \leq k \leq d} \in \mathcal{A}(\mu, \mathcal{A}_0) : \{ E_k \}_{1 \leq k \leq d} \in E_z \}. \]

**Theorem 2.3.** For any bounded and continuous function \( h \) on \( \mathcal{P}(V_s) \), let \( V^n_{\mathcal{A}_0}(z) \) to be the minimal cost function given in (2.15). Then
\[ \lim_{n \to \infty} V^n_{\mathcal{A}_0}(z) = \inf_{\mu \in \mathcal{C}(\mathcal{A}_0)} \left\{ \Lambda^2_{\delta, \mathcal{A}_0}(\mu) + h(\mu) \right\}, \tag{2.8} \]
where
\[ \Lambda^2_{\delta, \mathcal{A}_0}(\mu) = \inf_{(\mu_k, r_k, E_k) \in \mathcal{A}_z(\mu, \mathcal{A}_0)} \sum_{k=1}^{d} r_k \int_{S} R(q_k \| p_{E_k}) \, d\mu_k. \tag{2.9} \]

Let \( \mathcal{P}(V) \) be the set of all probabilities on \( V \) equipped with the weak convergence topology, which is equivalent to the Wasserstein topology since \( V \) is finite. Then we have

**Theorem 2.4.** The empirical measure process \( (L_n)_{n \geq 1} \) defined by (1.1) of the \( \delta \)-ORRW on finite connected graph \( G = (V, E) \) satisfies an LDP on \( \mathcal{P}(V) \) with the rate function \( I_\delta \) given by that
\[ I_\delta(\nu) = \inf_{\mu \in \mathcal{P}(V_s) \cap T(\mu) = \nu} \Lambda_\delta(\mu), \quad \nu \in \mathcal{P}(V), \]
where \( T(\mu)(u) = \mu(\{ z : z^- = u \}) \) for any \( \mu \in \mathcal{P}(V_s) \) and \( u \in V \).

**Remark 2.5.** One can study an LDP for the pair empirical occupation measures (introduced by [5]):
\[ L^{(2)}_n = \frac{1}{n} \sum_{k=0}^{n-1} \delta_{X_k} \delta_{X_{k+1}}, \quad n \geq 1. \]
It is easy to see that \( L^{(2)}_n \) is equivalent to the empirical measure of directed edges and hence equivalent to \( \mathcal{L}^n \) given in (2.2). The LDP for \( (\mathcal{L}^n)_{n \geq 1} \) is specified in Corollary 3.6.

**Theorem 2.6.** For any \( \delta_2 > \delta_1 \geq 1 \), \( I_{\delta_2}(\mu) \geq I_{\delta_1}(\mu) \), \( \forall \mu \in \mathcal{P}(V) \); and further there exists a \( \mu_0 \in \mathcal{P}(V) \) depending on \( \delta_1 \) and \( \delta_2 \) such that \( I_{\delta_1}(\mu_0) > I_{\delta_2}(\mu_0) \). However, \( I_{\delta} \equiv I_1 \) for any \( \delta \in (0, 1) \). In addition, \( I_{\delta} \) is uniformly continuous in \( \delta \) in the sense that
\[ \lim_{|\delta_1 - \delta_2| \to 0} \sup_{\mu \in \mathcal{P}(V); I_1(\mu) < \infty} |I_{\delta_1}(\mu) - I_{\delta_2}(\mu)| = 0, \]
and \( I_{\delta}(\mu) \) is not differentiable in \( \delta \) at \( \delta = 1 \) for some \( \mu \in \mathcal{P}(V) \).

**Remark 2.7.** Intuitively, when \( \delta \in (0, 1) \), the \( \delta \)-ORRW prefers to leave the vertices that have been traversed, and traverses all the edges very quickly, and then immediately becomes the SRW. However, when \( \delta > 1 \), the \( \delta \)-ORRW prefers to stay in the traversed vertices; and thus it makes a significant difference of long time behaviours from those of the SRW. Theorem 2.6 clearly describes this interesting phenomenon.
Recall $\mathcal{L}^n$ and $\mathcal{C}(\mathcal{I}_0)$ from (2.2) and (2.5) respectively and that $\mathbb{P}_{v_1}$ is the law of the $\delta$-ORRW starting at $v_1$.

**Theorem 2.8.** $\mathbb{P}_{v_1} (\mathcal{L}^n \in \mathcal{C}(\mathcal{I}_0))$ decays exponentially with the rate $\inf_{\mu \in \mathcal{C}(\mathcal{I}_0)} \Lambda_{\delta, \mathcal{I}_0}(\mu)$:

$$
\lim_{n \to \infty} \frac{1}{n} \log \mathbb{P}_{v_1} (\mathcal{L}^n \in \mathcal{C}(\mathcal{I}_0)) = - \inf_{\mu \in \mathcal{C}(\mathcal{I}_0)} \Lambda_{\delta, \mathcal{I}_0}(\mu).
$$

The proof of Theorem 2.8 is given in Section 4.

**Theorem 2.9.** The exponential integrability of $C_E$ has a critical exponent

$$
\alpha_c^1(\delta) = \inf_{\mu \in \mathcal{C}_1} \Lambda_{\delta, \mathcal{I}_1}(\mu)
$$

with $\mathcal{C}_1 = \{ \mu \in \mathcal{P}(V_S) : \text{supp}(\mu|_1) \neq E \}$ and $\mathcal{I}_1 = \mathcal{I} \setminus \{ E \}$ such that

$$
\mathbb{E}_{v_1} \left[ e^{\alpha C_E} \right] < \infty \text{ for } \alpha < \alpha_c^1(\delta) \text{ and } \mathbb{E}_{v_1} \left[ e^{\alpha C_E} \right] = \infty \text{ for } \alpha \geq \alpha_c^1(\delta),
$$

$$
\lim_{n \to \infty} \frac{1}{n} \log \mathbb{P}_{v_1}(C_E > n) = -\alpha_c^1(\delta).
$$

(2.10)

Furthermore, $\alpha_c^1(\delta)$ is continuous and strictly decreasing in $\delta > 0$ and uniformly continuous in $\delta \geq \delta_0$ for any $\delta_0 > 0$, and $\lim_{\delta \to 0} \alpha_c^1(\delta) = 0$.

If for any nonempty sets $E', E_0$ such that $E' \subset E_0 \in \mathcal{I}_1$, there is an invariant probability measure $\mu$ on $V_S$ with respect to some transition probability $q \in \mathcal{I}$ such that $\text{supp}(\mu|_1) = E'$, $E' \cap \partial E_0 \neq \emptyset$, then

$$
\lim_{\delta \to 0} \alpha_c^1(\delta) = \infty.
$$

Otherwise,

$$
\lim_{\delta \to 0} \alpha_c^1(\delta) = \inf_{(\mu, E_0 \in \mathcal{I}_1) : \text{supp}(\mu|_1) \subseteq E_0} \Lambda_1(\mu) < \infty.
$$

See Figure 1 for an illustration.

Theorem 2.9 is a corollary of Theorems 4.1, 4.3 and 4.4 in Section 4.

**Remark 2.10.** Note (2.10) means that for any $\epsilon > 0$, when $n$ is large enough,

$$
e^{-n(\alpha_c^1(\delta)+\epsilon)} \leq \mathbb{P}_{v_1}(C_E > n) \leq e^{-n(\alpha_c^1(\delta)-\epsilon)}.$$

In fact, by the proof of Theorem 4.1, we can obtain a stronger result:

$$
0 < \liminf_{n \to \infty} \frac{\mathbb{P}_{v_1}(C_E > n)}{e^{-\alpha_c^1(\delta)n}} \leq \limsup_{n \to \infty} \frac{\mathbb{P}_{v_1}(C_E > n)}{e^{-\alpha_c^1(\delta)n}} < \infty.
$$

Since $(Z_n, \mathcal{L}^n)_{n \geq 1}$ is non-homogeneously Markovian, we have to modify the weak convergence approach on the LDP of homogeneous Markov chains, and introduce some complicated notations and techniques in our proofs. For reader’s convenience, we provide a very brief introduction to the classical weak convergence approach in Subsection 2.2. Then we explain novel ideas of our modified methods in Subsection 2.3.
2.2 Brief introduction to weak convergence approach to homogeneous Markov chains

In this subsection we introduce weak convergence approach by showing how it works on homogeneous Markov chains with transition probability $p$. This method was given by Dupuis and Ellis [14] in 1997. The weak convergence approach considers the Laplace principle (LP) given by

**Definition 2.11.** Let $\mathcal{X}$ be a Polish space and $(\xi_n)_{n \geq 1}$ a sequence of $\mathcal{X}$-valued random variables. Call $(\xi_n)_{n \geq 1}$ satisfies the Laplace principle (LP) if for some good rate function $I$ on $\mathcal{X}$, the following condition holds for any bounded continuous function $h$ on $\mathcal{X}$:

$$
\lim_{n \to \infty} \frac{1}{n} \log \mathbb{E}[\exp \{-nh(\xi_n)\}] = - \inf_{x \in \mathcal{X}} \{I(x) + h(x)\}. \tag{2.11}
$$

It is well-known that the LP implies the LDP. Therefore, one way to demonstrate the LDP is to verify the LP.

**Theorem 2.12** ([14, Theorem 1.2.3]). The LP implies the LDP with the same rate function. Namely, if $I$ is a good rate function on Polish space $\mathcal{X}$ and $(\xi_n)_{n \geq 1}$ is a sequence of $\mathcal{X}$-valued random variables such that (2.11) holds for any bounded continuous function $h$ on $\mathcal{X}$, then $(\xi_n)_{n \geq 1}$ satisfies the LDP with the rate function $I$.

**Remark 2.13.** In fact, the LDP also implies the LP ([14, Theorem 1.2.1]). Moreover, if $(\xi_n)_{n \geq 1}$ is exponentially tight (see (1.2.17) in [10]), and the limit

$$
J(h) = \lim_{n \to \infty} \frac{1}{n} \log \mathbb{E}[\exp \{-nh(\xi_n)\}]
$$

exists for all $h \in C_b$ (the set of all bounded continuous functions on $\mathcal{X}$), then $(\xi_n)_{n \geq 1}$ satisfies an LDP with the rate function

$$
I(x) = - \inf_{h \in C_b} \{h(x) + J(h)\}, \; x \in \mathcal{X}.
$$

Refer to Bryc [3].

Write $\mathcal{P}(\mathcal{X})$ for the set of all probabilities on Polish space $\mathcal{X}$, and endow it with the weak convergence topology. From [14], we have

**Proposition 2.14.** Given Polish space $\mathcal{X}$ and $\theta \in \mathcal{P}(\mathcal{X})$. Then for any bounded continuous function $h$ on $\mathcal{X}$,

$$
- \log \int_{\mathcal{X}} e^{-h} \, d\theta = \inf_{\gamma \in \mathcal{P}(\mathcal{X})} \left\{ R(\gamma \| \theta) + \int_{\mathcal{X}} h \, d\gamma \right\}, \tag{2.12}
$$

where the infimum is uniquely attained at $\gamma_0$ with $\frac{d\gamma_0}{d\theta} = e^{-h} \cdot \frac{1}{\int_{\mathcal{X}} e^{-h} \, d\theta}$.

Comparing (2.11) with (2.12), one may consider the relative entropy $R(\cdot \| \cdot)$ in (2.12) in order to find the rate function.

Now we introduce how the weak convergence approach works when dealing with the LDP for empirical measures $(L_n)_{n \geq 1}$ of homogeneous Markov chains $(X_n)_{n \geq 0}$ on finite connected graph $G = (V, E)$ with transition probability kernel $p(x, dy)$. To begin, consider

$$
W^n(x) := - \frac{1}{n} \log \mathbb{E}_x \{\exp [-nh(L_n)]\}, \; x \in V,
$$
where \( P_x \) is the law of \((X_n)_{n \geq 0} \) with \( X_0 = x \) and \( \mathbb{E}_x \) denotes the corresponding expectation, and \( h \) is a bounded continuous function on \( \mathcal{P}(V) \). For any \( n \geq 1 \) and \( 0 \leq j \leq n \), let \( \mathcal{M}_{j/n}(G) \) be all measures \( \nu \) on \( V \) with \( \nu(V) = j/n \), and equip it with the weak convergence topology. Assume for \( j = 0, \ldots, n - 1 \), \( \nu^n_j := \nu^n_j(dy|x, \mu) \) is a transition probability from \( V \times \mathcal{M}_{j/n}(G) \) to \( V \). Call \( \{\nu^n_j, j = 0, \ldots, n - 1\} \) an admissible control sequence. Given such an admissible control sequence, define \((\overline{X}^n_j, \overline{L}^n_j)_{0 \leq j \leq n} \) on a probability space \((\Omega, \mathcal{F}, P_x)\) by induction. Set \( \overline{X}^n_0 = x, \overline{L}^n_0 = 0 \). For all \( j \in \{0, \ldots, n - 1\} \), define

\[
\overline{L}^n_{j+1} = \overline{L}^n_j + \frac{1}{n} \delta_x \nu^n_j \in \mathcal{M}_{(j+1)/n}(G),
\]

and let \( \overline{X}^n_{j+1} \) be distributed as \( \nu^n_j(\cdot | \overline{X}^n_j, \overline{L}^n_j) \) given \((\overline{X}^n_i, \overline{L}^n_i)_{0 \leq i \leq j} \). Here we apply the weak convergence approach in several steps as follows.

- **Dynamic programming**

  Denote by \( W^n(i, x, \mu) := -\frac{1}{n} \log \mathbb{E}_{i,x,\mu}(\exp[-nh(L^n)]) \), where \( \mathbb{E}_{i,x,\mu} \) is the expectation conditioned on \((X_i, L^n_i) = (x, \mu) \), where \( L^n_i = \frac{1}{n} \sum_{j=1}^{i} \delta_X \). Dupuis and Ellis \cite{D} showed that \( W^n(i, x, \mu) \) satisfies

\[
\exp[-nW^n(i, x, \mu)] = \int_V \exp \left[ -nW^n \left( i + 1, y, \mu + \frac{1}{n} \delta_x \right) \right] p(x, dy)
\]

by the Markov property. They then obtained the following dynamic programming equation by Proposition 2.14:

\[
W^n(i, x, \mu) = \inf_{\nu \in \mathcal{P}(V)} \left\{ \frac{1}{n} R(\nu(\cdot)p(x, \cdot)) + \int_V W^n \left( i + 1, y, \mu + \frac{1}{n} \delta_x \right) \nu(dy) \right\}. \tag{2.13}
\]

By \cite[Theorem 1.5.2]{D}, the unique solution to (2.13) is

\[
V^n_0(i, x, \mu) := \inf_{\nu^0_j} \mathbb{E}_{i,x,\mu} \left\{ \frac{1}{n} \sum_{j=i}^{n-1} R(\nu^n_j(\cdot | \overline{X}^n_j, \overline{L}^n_j) \| p(\overline{X}^n_j, \cdot)) + h(\overline{L}^n_j) \right\},
\]

where expectation \( \mathbb{E}_{i,x,\mu} \) is taken for \((\overline{X}^n_j, \overline{L}^n_j)_{0 \leq j \leq n} \) given \((\overline{X}^n_i, \overline{L}^n_i) = (x, \mu) \). Every \( V^n_0(i, x, \mu) \) is called a minimal cost function. Taking \( i, \mu = 0 \), one can obtain a representation for \( W^n(x) \):

\[
W^n(x) = V^n_0(x) := \inf_{\nu^n_j} \mathbb{E}_x \left\{ \frac{1}{n} \sum_{j=0}^{n-1} R(\nu^n_j(\cdot | \overline{X}^n_j, \overline{L}^n_j) \| p(\overline{X}^n_j, \cdot)) + h(\overline{L}^n_j) \right\}, \tag{2.14}
\]

which is easier to compute.

- **Convergence of the admissible control sequence**

Consider the LP by dealing with the limit of \( V^n_0(x) \) instead of that of \( W^n(x) \). To this end, the asymptotic behaviour of \( \nu^n_j \) and \( \overline{L}^n \) needs to be investigated. Define

\[
\nu^n := \frac{1}{n} \sum_{j=0}^{n-1} \delta_{\overline{X}^n_j} \times \nu^n_j(\cdot | \overline{X}^n_j, \overline{L}^n_j),
\]

12
By [14, Theorem 8.2.8], there exist some \((\nu, \mathcal{T})\) and \(\mathcal{P}\) such that \((\nu^n, \mathcal{T}^n) \Rightarrow (\nu, \mathcal{T})\), 
\(\nu = \mathcal{T} \otimes \mathcal{P}\), where \(\Rightarrow\) means the weak convergence, and \(\mathcal{P}\) is a transition probability from \(V\) to \(V\), and 
\[
\mathcal{T} \otimes \mathcal{P}(A \times B) = \int_A \mathcal{T}(dx)\mathcal{P}(x, B), \quad \forall A, B \subseteq V.
\]
In particular, \(\mathcal{T}\) is invariant for \(\mathcal{P}\), namely \(\mathcal{T}\mathcal{P} = \mathcal{T}\), where \(\mathcal{T}(\cdot) = \int_V \mathcal{T}(dx)\mathcal{P}(x, \cdot)\).

- The upper bound of the LP
  That is to verify
  \[
  \lim \inf_{n \to \infty} W^n(x) = \lim \inf_{n \to \infty} V^n_0(x) \geq \inf_{\mu} \{I(\mu) + h(\mu)\},
  \]
  where \(I(\cdot)\) is the rate function of the LDP for the empirical measures. Choose some admissible control sequence \(\{\nu^n_j\}\) such that the value of
  \[
  \mathbb{E}_x \left\{ \frac{1}{n} \sum_{j=0}^{n-1} R \left( \nu^n_j \left( \cdot, |X^n_j, L^n_j \right) \right) \left\| p \left( X^n_j, \cdot \right) \right\| + h \left( L^n_j \right) \right\}
  \]
  is close to the infimum in (2.14). By the asymptotic behaviour shown in the previous step, one can finish this proof. See [14, Section 8] for details.

- The lower bound of the LP
  That is to verify
  \[
  \lim \sup_{n \to \infty} W^n(x) = \lim \sup_{n \to \infty} V^n_0(x) \leq \inf_{\mu} \{I(\mu) + h(\mu)\}.
  \]
  Choose some measure \(\gamma\) such that \(I(\gamma) + h(\gamma)\) is close to its infimum, and try to find some ergodic measure \(\gamma^*\) close to \(\gamma\). Next, one can select some admissible control sequence with an ergodic measure \(\gamma^*\) such that
  \[
  \mathbb{E}_x \left\{ \frac{1}{n} \sum_{j=0}^{n-1} R \left( \nu^n_j \left( \cdot, |X^n_j, L^n_j \right) \right) \left\| p \left( X^n_j, \cdot \right) \right\| + h \left( L^n_j \right) \right\}
  \]
  is greater than \(V^n_0(x)\), and converges to \(I(\gamma^*) + h(\gamma^*)\) that is close to \(I(\gamma) + h(\gamma)\). Refer to [14, Section 8] for details.

Remark 2.15. The time homogeneous Markov property plays a key role in the estimates of \(\mathbb{E}_x \left\{ \frac{1}{n} \sum_{j=0}^{n-1} R \left( \nu^n_j \left( \cdot, |X^n_j, L^n_j \right) \right) \left\| p \left( X^n_j, \cdot \right) \right\| + h \left( L^n_j \right) \right\}\). Therefore, we cannot directly apply the same techniques to \((Z_n, \mathcal{L}^n)\).

2.3 Modified approach
First, we establish the dynamic programming for the \(\delta\)-ORRW \(X = (X_n)_{n \geq 0}\) on a finite connected graph \(G = (V, E)\) with \(|E| = d\). For any \(n \geq 1\) and \(0 \leq j \leq n\), write \(\mathcal{M}_{j/n}(S)\) for the set of all measures \(\mu\) on \(V_S\) with \(\mu(V_S) = j/n\), and endow it with the weak convergence topology. Note an admissible control sequence \(\{\nu^n_j, j = 0, \ldots, n-1\}\) means that
  \[
  \text{each } \nu^n_j := \nu^n_j \left( dz' \mid z, \mu \right) \text{ is a transition probability from } V_S \times \mathcal{M}_{j/n}(S) \text{ to } V_S.
  \]
Given such an admissible control sequence and any \( z \in V_\mathcal{S} \). We define \((\mathcal{Z}^n_j, \mathcal{Z}^n_j)_{0 \leq j \leq n}\) by induction on some probability space \((\Omega, \mathcal{F}, \mathbb{P}_z)\). Set \( \mathcal{Z}_0^n = z, \mathcal{Z}_0^n = 0 \). For all \( j = 0, 1, \ldots, n-1 \), define
\[
\mathcal{L}^n_{j+1} = \mathcal{L}^n_j + \frac{1}{n} \delta \mathcal{Z}^n_j \in \mathcal{M}_{(j+1)/n}(S),
\]
and let \( \mathcal{Z}^n_{j+1} \) be distributed as \( \nu^n_j (\cdot | \mathcal{Z}^n_j, \mathcal{Z}^n_j) \) given \((\mathcal{Z}^n_j, \mathcal{Z}^n_j)_{0 \leq j \leq j} \). In part, write \( \mathcal{Z}^n_j \) as \( \mathcal{Z}^n_i \).

Recall Definition 2.1 and (2.5). Define the general minimal cost function by
\[
V^n(z) := \inf_{\{\nu^n_j\}_{j \in C(S)}} \mathbb{E}_z \left\{ \frac{1}{n} \sum_{j=0}^{n-1} R \left( \nu^n_j (\cdot | \mathcal{Z}^n_j, \mathcal{Z}^n_j) \parallel p_{\mathcal{Z}^n_{j+1}} (\mathcal{Z}^n_j, \cdot) \right) + h (\mathcal{L}^n) \right\},
\]
where \( \mathbb{E}_z \) is the expectation under \( \mathbb{P}_z \), and \( h \) is a bounded continuous function on \( \mathcal{P}(V_\mathcal{S}) \).

**Lemma 2.16.** \( W^n_\mathcal{S}(z) := -\frac{1}{n} \log \mathbb{E}_z \{ \exp [-nh(\mathcal{L}^n)] \} \} \) equals the minimal cost function \( V^n_\mathcal{S}(z) \).

Particularly, write \( W^n_\mathcal{S}(z) \) as \( W^n(z) \). For proof of Lemma 2.16, see Appendix 7.3.

It is not accessible to follow the weak convergence approach in [14] since the transition probability \( p_{\mathcal{Z}^n_i} \) is not homogeneous. One of our important observations is that \( (Z_n)_{n \geq 1} \) is a homogeneous Markov process when stuck in the subgraph that all edges have been traversed. This allows us to divide the whole trajectory into some parts, so that each part of the process stays in some subgraph. We then consider each part of the process. Mathematically speaking, denote renewal times by
\[
\tau^n_k = \inf \left\{ j > \tau_{k-1} : \mathcal{Z}^n_j \notin \{ \mathcal{Z}^n_i : i < j \} \right\} \wedge n,
\]
where \( \tau^n_0 := 0 \). For convenience, let \( \tau^n_{d+1} = n \). We consider \( \mathcal{Z}^n_j \) with \( \tau^n_k \leq j < \tau^n_{k+1} \) for \( 1 \leq k \leq d \) respectively, see Figure 5. Note that \( \mathcal{Z}^n_j \) stays in \( E_k \times \{-1,1\} \) from time \( \tau^n_k \) up to time \( \tau^n_{k+1} - 1 \), and all edges in \( E_k \times \{-1,1\} \) are exactly traversed after time \( \tau^n_k \). As a result, \( \mathcal{X}^n_j \) is Markovian from \( \tau^n_k \) to \( \tau^n_{k+1} - 1 \). This allows us to apply the weak convergence approach of Markov chains in each interval \( [\tau^n_k, \tau^n_{k+1}] \) for \( 1 \leq k \leq d \). However, the essential point is to verify the convergence of the admissible control sequence in these random intervals, see Theorem 3.1 and Corollary 3.2.

For the upper and lower bound of the LP, we still need to modify the method to adapt to the situation of dividing the whole trajectory into several parts. To begin, we point out that the rate function should be some combination of those rate functions of the SRW on each subgraph \( E_k \times \{-1,1\} \) mentioned above. As a matter of fact, \( Z \) staying in \( E_k \times \{-1,1\} \) represents that \( X \) stays in \( G_k = (V_k, E_k) \), where \( V_k \) is the set consisting of those vertices adjacent to some edge in \( E_k \), see Figure 6.
Figure 6: When $X$ stays in the solid lines in (a), it is a Markov process. Then $X$ traverses the first dashed line from the left side in (a) at the renewal time, and stays in the solid lines in (b) for a period of time, during which the process is also Markovian.

For the upper bound of the LP, we select a sequence of \{\nu^n_j\}_{0 \leq j \leq n-1} for fixed $n$ such that

$$
E_\tau \left\{ \frac{1}{n} \sum_{j=0}^{n-1} R \left( \nu^n_j \left( \cdot \mid Z^n_j, Z^n_{j+1} \right) \right) \left\| p^*_{\mathcal{T}^n_{j+1}} \left( Z^n_j, \cdot \right) \right\| + h \left( Z^n \right) \right\}
$$

is close to (2.15). Meanwhile it divides the sum into several parts, i.e., $\sum_{j=0}^{n-1} \frac{1}{n} \cdot \sum_{k=1}^d r_k \int_{V_S} R(q_k \parallel p_{E_k}) \, d\gamma_k$. To complete the proof, in each part, we can apply the techniques similar to those used in the case of Markov chains in [14, Subsection 8.3].

For the lower bound of the LP, we can find a measure $\gamma$ such that $\Lambda_\delta(\gamma) + h(\gamma)$ is very close to $\inf_\mu \{ \Lambda_\delta(\mu) + h(\mu) \}$. Here, the infimum of $\Lambda_\delta(\gamma)$ (see (2.12)) is attained at $(\gamma_k, r_k, E_k, q_k)_{1 \leq k \leq d}$. However, $\gamma_k$ may not be ergodic with respect to the transition probability $q_k$. It is hard to find an admissible control sequence such that the sum of the relative entropy converges to $\sum_{k=1}^d r_k \int_{V_S} R(q_k \parallel p_{E_k}) \, d\gamma_k$. To overcome these difficulties, we introduce an auxiliary sequence $(\gamma^*_k, r_k, E_k, q^*_k)_{1 \leq k \leq d}$ such that $\gamma^*_k$ is ergodic with respect to $q^*_k$ and close to $\gamma_k$, where $q^*_k$ satisfies

$$
\int_{V_S} R(q^*_k \parallel p_{E_k}) \, d\gamma_k \leq \int_{V_S} R(q_k \parallel p_{E_k}) \, d\gamma_k.
$$

Meanwhile, we need to construct an admissible control sequence $(\tilde{\nu}^n_j)_{0 \leq j \leq n-1}$ such that

**C1** $\frac{1}{n} \sum_{j=1}^n R \left( \tilde{\nu}^n_j \left( \cdot \mid Z^n_j, Z^n_{j+1} \right) \right) \rightarrow \sum_{k=1}^d r_k \int_{V_S} R(q^*_k \parallel p_{E_k}) \, d\gamma^*_k$,

**C2** $\tau_k = s_k(n)$ with $\lim_{n \to \infty} \frac{s_k(n) - s_{k-1}(n)}{n} = r_k$, where $s_k(n)$ is deterministic and $s_{d+1}(n) = n$,

**C3** $Z_{\tau_k} \in E_k \setminus E_{k-1}$, where $E_k$ is deterministic.

In fact, **C1**, **C2** and **C3** are in a strongly restrictive relation. We have to design $(\tilde{\nu}^n_j)_{0 \leq j \leq n-1}$ delicately.
3 LDP and properties of the rate function

In this section, we prove the main results in Subsection 2.1. As pointed out in Subsections 2.2 and 2.3, we need the following four steps to verify the LDP:

(a) Solving the dynamic programming equation.
(b) Showing the convergence of the admissible control sequence.
(c) Verifying the upper bound of the LP.
(d) Verifying the lower bound of the LP.

Because we have obtained the unique solution to the dynamic programming equation of the ORRW in Lemma 2.16, we demonstrate the convergence of the admissible control sequence, the upper bound and the lower bound associated with Theorem 2.3 in Subsections 3.1, 3.2 and 3.3 respectively. We then show Theorems 2.4 and 2.8 in Subsection 3.4. In Subsection 3.5, we provide some properties of the rate function, implying Theorem 2.6.

3.1 Convergence of the admissible control sequence

Recall the first paragraph in Subsection 2.3. For any $n \geq 1$, define a measure by

$$
\nu^n := \frac{1}{n} \sum_{j=0}^{n-1} \delta_{Z^n_j} \times \nu^n_j (\cdot | Z^n_j, L^n_j).
$$

**Theorem 3.1.** Let $Z^n_0 = z^n$, $n \geq 1$ be initial conditions in finite state space $S$. For each $n$, consider some admissible control sequence $\{\nu^n_j\}_{0 \leq j \leq n-1}$ such that $\text{supp} \left( L^n_j \mid z^n_{m+1} \right) = E' \subseteq E$ for $m + 1 \leq j \leq k$ and $Z^n_{m+1} = y \in E'$, where $m = m(n) < k = k(n) \leq n$ are some random positive integers. Then there is a probability space $(\Omega, \mathcal{F}, P_{z^n})$ such that any subsequence of

$$
\left( \frac{1}{k-m} \sum_{j=m}^{k-1} \delta_{Z^n_j} \times \nu^n_j (\cdot | Z^n_j, L^n_j), \frac{1}{k-m} \sum_{j=m}^{k-1} \delta_{Z^n_j}, z^n \right)
$$

has a subsequence that converges in distribution to some $\left( \nu, \overline{\mathcal{L}}, z \right)$ with the following properties.

(a) In $(\Omega, \mathcal{F}, P_z)$, the limiting quantities $\nu$ and $\overline{\mathcal{L}}$ can be realized respectively as a random probability on $(E' \times \{-1, 1\}) \times (E' \times \{-1, 1\})$ and a random probability on $E' \times \{-1, 1\} =: S'$. And $P_z$-a.s. $\omega \in \Omega$, $\overline{\mathcal{L}}(dz_1 \mid \omega)$ is the first marginal of $\nu(dz_1 \times dz_2 \mid \omega)$:

$$
\overline{\mathcal{L}}(A \mid \omega) = \nu \left( A \times (E' \times \{-1, 1\}) \mid \omega \right), \ A \subseteq S'.
$$

(b) There exists a transition probability $\nu(dz_2 \mid z_1) = \nu(dz_2 \mid z_1, \omega)$ from $S' \times \Omega$ to $S'$ such that $P_z$-a.s. $\omega \in \Omega$,

$$
\nu(A \times B \mid \omega) = \int_A \nu(B \mid z_1, \omega) \overline{\mathcal{L}}(dz_1 \mid \omega), \ A, B \subseteq S'.
$$
\( \tau \) denotes renewal times. Set
\[
\tau_{j}^{n} = \inf\{k > n : \tau_{k}^{n} - \tau_{j-1}^{n} \leq 1\}
\]
and the display in part (b) will be summarized as
\[
\nu(dz_{1} \times dz_{2}) = \nu(dz_{1}) \otimes \nu(dz_{2} | z_{1}) = \nu(dz_{1}) \times \nu(z_{1} | dz_{2}).
\]

Theorem 3.1 is proved in Appendix 7.4. Noting (2.4) and (2.7), and that
\[
\sum_{k=1}^{d} \frac{n_{k}}{n} \cdot \frac{1}{n_{k}} \sum_{j=s_{k}}^{s_{k+1}-1} \delta_{z_{j}}^{n} = \nu^{n},
\]
for \( s_{k} = \sum_{j=1}^{k-1} n_{j} + 1 \) with \( \sum_{k=1}^{d} n_{k} = n \), we obtain the following corollary by Theorem 3.1.

**Corollary 3.2.** Let \( Z_{0}^{\omega} = z^{\omega} \), \( n \geq 1 \) be initial conditions on \( S \). Let
\[
\tau_{k} = \inf\{j > \tau_{k-1}^{n} : Z_{j}^{\omega} \notin \{Z_{i}^{\omega} : i < j\}\}, \ 1 \leq k \leq d,
\]
denote renewal times. Set \( \tau_{k}^{n} = \tau_{k} \wedge n \), where \( \tau_{0} = 0 \) and \( \tau_{d+1} = \infty \). Define
\[
n_{k}(n) = \tau_{k+1}^{n} - \tau_{k}^{n}, \ E_{k,n}^{\omega} = \{Z_{j}^{\omega} : 1 \leq j \leq \tau_{k}^{n}\}, \ 0 \leq k \leq d.
\]
Then every subsequence of
\[
\left( \frac{1}{n_{k}} \sum_{j=\tau_{k}^{n}}^{\tau_{k+1}^{n}-1} \delta_{z_{j}}^{n} \times \nu_{j}^{n}(\cdot | Z_{j}^{\omega}, L_{j}^{\omega}) \right)_{1 \leq k \leq d, n \geq 0}
\]
has a subsequence such that for any \( 1 \leq k \leq d \),
\[
\left( \frac{1}{n_{k}} \sum_{j=\tau_{k}^{n}}^{\tau_{k+1}^{n}-1} \delta_{z_{j}}^{n} \times \nu_{j}^{n}(\cdot | Z_{j}^{\omega}, L_{j}^{\omega}) \right)_{1 \leq k \leq d, n \geq 0}
\]
converges in distribution to some \( \left( \nu_{k}, Z_{k}, R_{k}, E_{k,\omega}^{\omega}, z \right) \) with the properties mentioned in Theorem 3.1 for \( \omega \in \{\omega \in \Omega : \lim_{n \to \infty} \frac{\tau_{k+1}^{n} - \tau_{k}^{n}}{n} = 0\}^{\mathcal{C}} \), where \( R_{k} \) and \( E_{k,\omega}^{\omega} \) are \([0,1] \) and \( \mathcal{S} \) valued random variables respectively. Moreover,
\[
\sum_{k=1}^{d} R_{k} L_{k} = L, \ \sum_{k=1}^{d} R_{k} = 1 \text{ and } \{E_{k,\omega}^{\omega}(\omega)\}_{k} \in \mathcal{E}_{z} \text{ for fixed } \omega,
\]
where \( L \) is the limit of \( L^{n} \).
Proof. Note \( \frac{m}{n} \in [0, 1] \) and \( \mathcal{S} \) is compact. For every subsequence of \( n \), by Theorem 3.1 and the diagonal argument, we can choose a common subsubsequence \( n' \) such that for all \( k = 1, \ldots, d \),

\[
\left( \frac{1}{n_k(n')} \sum_{j=\tau_k'}^{\tau_{k+1}'-1} \hat{\nu}_j'(\cdot | \hat{Z}_j', \hat{L}_j'), \frac{1}{n_k(n')} \sum_{j=\tau_k'}^{\tau_{k+1}'-1} \hat{\nu}_j'(n', \cdot | \hat{Z}_j', \hat{L}_j'), E_{\tau,k,n'}^\omega, z' \right)_{n' \geq 1}
\]

converges in distribution to some \( \left( \nu_k, \mathcal{L}_k, \mathcal{R}_k, E_{\omega, \infty}^\omega, z \right) \) having the properties mentioned in Theorem 3.1.

Moreover, by the same argument as Theorem 8.2.8 in [14], the limit of \( \mathcal{L}_n' \) exists, denoted by \( \mathcal{L} \). Noting that \( \sum_{k=1}^{d} \frac{n_k}{n} \sum_{j=\tau_k}^{\tau_{k+1}-1} \hat{\nu}_j = \mathcal{L} \), \( \sum_{k=1}^{d} n_k = n \), and \( \{ E_{k,n'}^\omega(\omega) \} \in \mathcal{E}_\omega \) for fixed \( \omega \), as \( n \to \infty \), we have

\[
\sum_{k=1}^{d} \mathcal{R}_k \mathcal{L}_k = \mathcal{L}, \sum_{k=1}^{d} \mathcal{R}_k = 1 \text{ and } \{ E_{k,\infty}^\omega \}_k \in \mathcal{E}_\omega \text{ for fixed } \omega.
\]

\[\blacksquare\]

3.2 The upper bound of Theorem 2.3

Here we give the proof of the upper bound, i.e.,

\[
\liminf_{n \to \infty} V_{\gamma, 0}^n(z) \geq \inf_{\mu \in \mathcal{C}(\mathcal{S})} \{ A^x_{\gamma, \mathcal{S}}(\mu) + h(\mu) \}
\]

for any \( h \in C_0(\mathcal{S}(V_S)) \) (the set of all bounded continuous functions on \( \mathcal{S}(V_S) \)).

For any \( \varepsilon > 0 \), there exists some \( \{ \nu_j^n \}_{0 \leq j \leq n-1} \) such that

\[
V_{\gamma, 0}^n(z) + \varepsilon \geq \mathbb{E}_z \left\{ \frac{1}{n} \sum_{j=0}^{n-1} R \left( \nu_j^n (\cdot | \hat{Z}_j^n, \hat{L}_j^n) \left\| p_{\hat{Z}_j^n, \hat{L}_j^n}^{\omega} \right\| \right) \right\}
\]

Consider this sequence of \( \{ \nu_j^n \}_{0 \leq j \leq n-1} \), and note

\[
R(\gamma \| \theta) = R(\mu \times \gamma \| \mu \times \theta), \gamma, \theta, \mu \in \mathcal{S}(V_S),
\]

we have that

\[
R \left( \nu_j^n (dy | \hat{Z}_j^n, \hat{L}_j^n) \right) \left\| p_{\hat{Z}_j^n, \hat{L}_j^n}^{\omega} \right\| 
\]

\[
= R \left( \hat{Z}_j^n(dx) \times \nu_j^n(dy | \hat{Z}_j^n, \hat{L}_j^n) \right) \left\| \hat{Z}_j^n(dx) \times p_{\hat{Z}_j^n, \hat{L}_j^n}^{\omega} \right\|
\]

\[
= R \left( \hat{Z}_j^n(dx) \times \nu_j^n(dy | \hat{Z}_j^n, \hat{L}_j^n) \right) \left\| \hat{Z}_j^n(dx) \times p_{\hat{Z}_j^n, \hat{L}_j^n}^{\omega} \right\|
\]

Since \( p_{\hat{Z}_j^n, \hat{L}_j^n}^{\omega} = p_{\hat{Z}_j^n, \hat{L}_j^n}^{\omega} \) during time \( \tau_k \) and \( \tau_k + 1 - 1 \), by the convexity of \( R(\cdot \| \cdot) \), we obtain that with each \( n_k = n_k(n) := \tau_k^n - \tau_k^n \),

\[
\mathbb{E}_z \left\{ \frac{1}{n} \sum_{j=0}^{n-1} R \left( \nu_j^n (\cdot | \hat{Z}_j^n, \hat{L}_j^n) \left\| p_{\hat{Z}_j^n, \hat{L}_j^n}^{\omega} \right\| \right) + h(\mathcal{L}^\omega) \right\}
\]

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where \( n_l = 0 \) if \( \omega \in \{ \omega : E_{l,n}^\omega \notin \mathcal{J}_0 \} \). Since \( S \) is finite, \( p_{E^\omega_{l,n}}(z, z') > 0 \) for \( z \rightarrow z' \), and the relative entropy on the RHS of the inequality above is bounded, by Fatou’s Lemma, Corollary 3.2 and the lower semicontinuity and boundedness of \( R(\cdot \| \cdot) \), we obtain that

\[
\liminf_{n \to \infty} V_{\mathcal{J}_0}^n(z) + \varepsilon \geq \liminf_{n \to \infty} \mathbb{E}_z \left\{ \frac{1}{n} \sum_{j=0}^{n-1} R \left( \nu_j^a \left( \cdot \big| Z_j^n, \mathcal{L}_j^n \right) \right) \left\| \hat{Z}_j^n(x) \times p_{E_{l,n}} \left( x, dy \right) \right\| + h \left( \mathcal{L}^n \right) \right\}
\]

\[
\geq \mathbb{E}_z \left\{ \sum_{k=1}^d R_k \left( \mathcal{L}_k \times \bar{q}_k \left\| \mathcal{L}_k \times p_{E_{l,n}} \right\| + h \left( \mathcal{L} \right) \right) \right\}
\]

\[
= \mathbb{E}_z \left\{ \sum_{k=1}^d R_k \int_{V_S} R \left( \bar{q}_k \left\| p_{E_{l,n}} \right\| d\mathcal{L}_k + h \left( \mathcal{L} \right) \right) \right\}. \tag{3.1}
\]

Due to \( n_l = 0 \) if \( \omega \in \{ \omega : E_{l,n}^\omega \notin \mathcal{J}_0 \} \), we see \( \mathcal{R}_i(\omega) = 0 \) for \( \omega \in \{ \omega : E_{l,n}^\omega \notin \mathcal{J}_0 \} \), which implies that \( \mathcal{L} \in C(\mathcal{J}) \) by \( \sum_{k=1}^d \mathcal{L}_k \mathcal{R}_k = \mathcal{L} \). Note for all \( \omega \),

\[
(\mathcal{L}_k(\omega), \mathcal{R}_k(\omega), E_{l,\infty}^\omega(\omega))_{k} \in \mathcal{A}_z \left( \mathcal{L}(\omega), \mathcal{J}_0 \right),
\]

and \( \mathcal{L}_k \bar{q}_k = \mathcal{L}_k \) on \( \{ \omega : \mathcal{R}_k > 0 \} \). Hence

\[
(3.1) \geq \inf_{\omega} \left\{ \sum_{k=1}^d R_k \int_{V_S} R \left( \bar{q}_k \left\| p_{E_{l,n}} \right\| d\mathcal{L}_k + h \left( \mathcal{L} \right) \right) \right\}
\]

\[
\geq \inf_{\mu \in C(\mathcal{J}_0)} \{ \Lambda_{3,\mathcal{J}_0}(\mu) + h(\mu) \},
\]

where \( \Lambda_{3,\mathcal{J}_0} \) is given in Theorem 2.3. It completes the proof of the upper bound since \( \varepsilon \) is arbitrary.

3.3 The lower bound of Theorem 2.3

Here we give the proof of the lower bound, i.e., for any \( h \in C_b(\mathcal{P}(V_S)) \),

\[
\limsup_{n \to \infty} V_{\mathcal{J}_0}^n(z) \leq \inf_{\mu \in C(\mathcal{J}_0)} \{ \Lambda_{3,\mathcal{J}_0}(\mu) + h(\mu) \}.
\]

We first prove
Lemma 3.3. \( \mathcal{C}(\mathcal{F}_0) \) is a closed subset of \( \mathcal{P}(V_S) \) under the weak convergence topology.

Proof. Assume conversely that there is a sequence of \( \mu_n \Rightarrow \mu \) in \( \mathcal{P}(V_S) \) with \( \mu_n \in \mathcal{C}(\mathcal{F}_0) \) and \( \mu \notin \mathcal{C}(\mathcal{F}_0) \). Then, for all \( E' \in \mathcal{F}_0 \), \( \text{supp}(\mu|_1 \setminus E') \neq \emptyset \).

Since \( V_S \) is finite, we can choose a subsequence \( \mu_{n_k} \) such that there exist sets \( E_0 \) and \( E_1 \in \mathcal{F}_0 \) such that \( E_0 \subset E_1 \), \( E_0 = \text{supp}(\mu_{n_k}|_1) \). Note \( \text{supp}(\mu|_1 \setminus E_1) \neq \emptyset \). This contradicts to \( \mu_{n_k} \Rightarrow \mu \) by selecting some \( e \in \text{supp}(\mu|_1) \setminus E_1 \) and noting the fact that \( \mu|_1(\{e\}) \neq 0 \) and \( \mu_{n_k}|_1(\{e\}) = 0 \).

Take arbitrarily \( h \in C_b(\mathcal{P}(V_S)) \) and \( \varepsilon > 0 \). Since \( \mathcal{C}(\mathcal{F}_0) \) is closed, there exists some \( \gamma \in \mathcal{C}(\mathcal{F}_0) \) such that

\[
\Lambda_{\delta, \mathcal{F}_0}^n(\gamma) + h(\gamma) \leq \inf_{\mu \in \mathcal{C}(\mathcal{F}_0)} \{ \Lambda_{\delta, \mathcal{F}_0}^n(\mu) + h(\mu) \} + \varepsilon.
\]

Our goal is to find some admissible control sequence \( \{\nu^n_j\}_{0 \leq j \leq n-1} \) such that

\[
\limsup_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} R \left( \nu^n_j \left( \cdot, \mathcal{Z}^n_j, \mathcal{L}^n_j \right) \right) \leq \Lambda_{\delta, \mathcal{F}_0}^n(\gamma) + h(\gamma).
\]

is close to \( \Lambda_{\delta, \mathcal{F}_0}^n(\gamma) + h(\gamma) \).

Firstly, we show that \( \Lambda_{\delta, \mathcal{F}_0}^n(\gamma) \) can be dominated by \( \sum_{k=1}^{d} \int_{S} \! \! \! \! \! r_k R(q_k^* \| p_{E_k}) \, d\gamma_k^* \) for some ergodic measure \( \gamma_k^* \) with respect to a transition probability \( q_k^* \), some non-negative \( r_k \) with \( \sum_{k=1}^{d} r_k = 1 \), and some \( \{E_k\}_{1 \leq k \leq d} \in \mathcal{E}_2 \). By the lower semicontinuity of the relative entropy and the compactness of the state space \( S \), there exists some \( \{E_k\}_{1 \leq k \leq d} \in \mathcal{E}_2 \) and \( q_k^*, \gamma_k, r_k, k = 1, \ldots, d \) such that \( \sum_{k=1}^{d} \gamma_k r_k = \gamma, \gamma_k q_k = \gamma_k, \text{supp}(\gamma_k) = E_k \times \{-1, 1\}, r_1 = 0 \) for \( E_1 \notin \mathcal{F}_0 \) and

\[
\Lambda_{\delta, \mathcal{F}_0}^n(\gamma) = \sum_{k=1}^{d} r_k \int_{V_S} \! \! \! \! \! R(q_k \| p_{E_k}) \, d\gamma_k.
\]

By [14, Lemma 8.6.3], for all \( \delta > 0 \), there is a sequence \( \{\gamma_k^*\}_{1 \leq k \leq d} \) of probability measures with the following properties (define \( \gamma^* = \sum_{k=1}^{d} \gamma_k^* r_k \)):

(a) \( \|\gamma_k^* - \gamma_k\| \leq \delta. \)

(b) \( \text{supp} \gamma_k^* = E_k \times \{-1, 1\}. \)

(c) There exists a sequence \( \{q_k^*\}_{1 \leq k \leq d} \) of transition probabilities on \( G \) such that each \( \gamma_k^* \) is the unique invariant probability of \( q_k^* \), and the associated Markov chain is ergodic. In addition,

\[
\Lambda_{\delta, \mathcal{F}_0}^n(\gamma^*) \leq \sum_{k=1}^{d} \int_{V_S} \! \! \! \! \! r_k R(q_k^* \| p_{E_k}) \, d\gamma_k^* \leq \Lambda_{\delta, \mathcal{F}_0}^n(\gamma).
\]

Secondly, we select the admissible control sequence to be \( q_k^* \) during times \( s_k \) and \( s_{k+1} - 1 \). Note \( q_k^* \) is ergodic, and \( \gamma_k^* \) is invariant for \( q_k^* \) and supported on \( E_k \times \{-1, 1\} \). The process \( \{\mathcal{Z}^n_j\}_j \) stays in \( E_k \times \{-1, 1\} \) from \( s_k \) to \( s_{k+1} - 1 \). Furthermore, if \( \{\mathcal{Z}^n_j\}_j \) reaches \( (E_{k+1} \setminus E_k) \times \{-1, 1\} \) exactly at time \( s_{k+1} \), then

\[
\frac{1}{s_{k+1} - s_k} \sum_{j=s_k}^{s_{k+1}-1} R \left( q_k^* \left( \mathcal{Z}^n_j, \cdot \right) \parallel p_{\mathcal{Z}^n_{j+1}} \left( \mathcal{Z}^n_j, \cdot \right) \right)
\]
is stated in Appendix Z on the undirected graph, we have to lift it to \( Z^\tau \) on directed graph S. We construct \( \nu_j^\tau \) such that \( Z_j^\tau \) satisfies \( \nu_j^\tau = s_k, Z_j^\tau_k \in (E_k \setminus E_{k-1}) \times \{-1, 1\} \) for \( 1 \leq k \leq d \) (define \( s_{d+1} = n \)). Considering \( X_j^a = (Z_j^\tau)^- \) on \( G_k = (V_k, E_k) \) with \( V_k = \{ v \in V, \exists e \in E_k \text{ s.t. } v \in e \} \), by Lemma 3.4, we can choose the transition probability of \( X_j^a \) to be \( p_k^a(z_1^-, z_2^+) := q_k^a(z_1^+, z_2^-) \) for \( j \in [s_k - 1, s_k + 1 - 2d - 3] \), and \( \bar{\nu}_j^a \) for \( j \in [s_k + 1 - 2d - 2, s_k + 1 - 2] \). This allows that

\[
\frac{1}{s_{k+1} - s_k} \sum_{j=s_k}^{s_{k+1}-1} R\left(q_k^a(Z_j^\tau, \cdot) \parallel p_{E_k}(Z_j^\tau, \cdot)\right) = \lim_{n \to \infty} E_z \left\{ \frac{1}{n} \sum_{k=1}^{d} \sum_{j=s_k}^{s_{k+1}-1} R\left(q_k^a(Z_j^\tau, \cdot) \parallel p_{E_k}(Z_j^\tau, \cdot)\right) + h\left(Z_j^\tau \right) \right\} \geq \limsup_{n \to \infty} V_{\gamma_0}(z).
\]

However, unfortunately, (3.2) may not hold. Because under the transition probability \( q_k^a \), the process may not be able to reach \( (E_{k+1} \setminus E_k) \times \{ -1, 1 \} \) at time \( s_{k+1} \). Our solution is to construct an admissible control sequence to be \( q_k^a \) for \( j \) in interval \([s_k, s_k + 1)\), and \( \bar{\nu}_j^a \) for \( j \) in interval \([s_k + 1, s_{k+1} - 1)\), where \( k = 1, \ldots, d \) and \( s_k \) is non-random and will be given in Lemma 3.4, see Figure 7. And \( \bar{\nu}_j^a \) for \( j \in [s_k + 1, s_{k+1}) \) should satisfy:

- The starting point \( Z_{s_k+1}^a \) is random but the terminal point \( Z_{s_{k+1}}^a \) is deterministic;
- The length of the interval, \( s_{k+1} - s_k \), is a deterministic constant independent with \( n \).

Here we give the following lemma to find this embedding path, which is deterministic when the starting point is fixed.

**Figure 7**: The process \( Z_j^a \) stays in \( E_k \) during times \( s_k^a \) and \( s_k^a + 1 - 1 \). The transition probability of \( Z_j^a \) is \( q_k^a \) and \( \bar{\nu}_j^a \) in the interval \([s_k, s_k + 1 - 1)\) and \([s_{k+1}, s_{k+1} - 1)\) respectively, where \( 1 \leq k \leq d \) and \( s_k \) will be given latter.

**Lemma 3.4**: Given a finite connected undirected graph \( G_0 = (V_0, E_0) \) with no more than \( d \) edges. Assume that for some \( s + 2d < t \) and \( u_0, v_0 \in V_0 \), there exists a path \( u_0 u_1 \ldots u_{t-s} = v_0 \). Then for every random walk \( (U_n)_{s \leq n < t-2d} \) on \( G_0 \) conditioned on \( U_s = u_0 \) with transition probabilities \( p_j := p(j, u; j + 1, v) \) for \( j \) in interval \([s, t - 2d)\) and for all \( u, v \in V_0 \), there exist some transition probabilities \( p_j^a \) independent with \( p_j \) for \( j \) in interval \([t - 2d, d)\) such that the corresponding extended walk satisfies \( U_t = v_0 \).

The proof of Lemma 3.4 is stated in Appendix 7.5. As this lemma is for the random walk on an undirected graph, we have to lift it to \( Z_j^a \) on directed graph S. We construct \( \nu_j^a \) such that \( Z_j^\tau \) satisfies \( \nu_j^a = s_k, Z_j^\tau_k \in (E_k \setminus E_{k-1}) \times \{-1, 1\} \) for \( 1 \leq k \leq d \) (define \( s_{d+1} = n \)). Considering \( X_j^a = (Z_j^\tau)^- \) on \( G_k = (V_k, E_k) \) with \( V_k = \{ v \in V, \exists e \in E_k \text{ s.t. } v \in e \} \), by Lemma 3.4, we can choose the transition probability of \( X_j^a \) to be \( p_k^a(z_1^-, z_2^+) := q_k^a(z_1^+, z_2^-) \) for \( j \in [s_k - 1, s_k + 1 - 2d - 3] \), and \( \bar{\nu}_j^a \) for \( j \in [s_k + 1 - 2d - 2, s_k + 1 - 2] \). This allows that
\(X_j^n\) stays in \(G_k\) from time \(s_{k+1} - 2d - 2\) up to \(s_{k+1} - 2\), and \(X_j^n\) visits \(G_{k+1}\) at time \(s_{k+1} - 1\) whenever \(X_{s_{k+1} - 2d - 2}^n\) is. Then we set the admissible control sequence \(\nu_j^n \) to be \(q_k^*\) for \(j \in [s_k, s_{k+1} - 2d - 2]\), and \(\bar{\nu}_j^n(z_1, z_2) := \bar{p}_{j+1}^n(z_1^-, z_2^-)\) for \(j \in [s_{k+1} - 2d - 1, s_{k+1} - 1]\). Now, this \(\nu_j^n\) is the admissible control sequence that we need, and \(\overline{\nu}_j^n \in C(\mathcal{R}_0)\).

Finally, by the ergodicity of \(q_k^*\) and the \(L^1\) ergodic theorem [14, Theorem A.4.4],

\[
\inf_{\mu \in C(\mathcal{R}_0)} \{ \Lambda^z_{\delta, \mathcal{R}_0}(\mu) + h(\mu) \} + 2\varepsilon \geq \Lambda^z_{\delta, \mathcal{R}_0}(\gamma) + h(\gamma) + \varepsilon
\]

where \(s_k = \sum_{j=1}^k n_j\), and \(\lim_{n \to \infty} n_k/n = r_k\) (for those \(k\) with \(r_k = 0\), \(\frac{1}{n} \sum_{j=s_k}^{s_{k+1}-1} \cdot \) can be ignored, which allows us to assume that \(s_{k+1} - s_k > 2d\)). By the arbitrariness of \(\varepsilon\), the proof of the lower bound is completed.

### 3.4 Proof of Theorem 2.4

Now we give the rate function of the LDP for the ORRW on finite connected graph \(G\) by Lemma 2.16. Before that, we firstly verify that \(\Lambda^z_{\delta, \mathcal{R}_0}\) is a good rate function as follows.

**Lemma 3.5.** The rate function \(\Lambda^z_{\delta, \mathcal{R}_0}\) is lower semicontinuous, and \(\{ \mu : \Lambda^z_{\delta, \mathcal{R}_0}(\mu) \leq M \}\) is compact.

The proof of this lemma will be given in Appendix 7.6.

Recall Theorem 2.12. By taking \(\mathcal{R}_0 = \mathcal{I}\) in Theorem 2.3, and noting that

\[
\lim_{n \to \infty} -\frac{1}{n} \log \mathbb{E}_{v_1} \exp\{-nh(\mathcal{L}^n)\} = \min_{z, z^- = v_1} \lim_{n \to \infty} W^n(z),
\]

we obtain the LDP for the empirical measures of \((Z_n)_{n \geq 1}\) on \(S\):

**Corollary 3.6.** The empirical measure process \((\mathcal{L}^n)_{n \geq 1}\) of \((Z_n)_{n \geq 1} = (Y_n, \sigma_n)_{n \geq 1}\) with reinforcement factor \(\delta > 0\) satisfies an LP (resp. LDP) with the rate function \(\Lambda^z_{\delta}\), namely

\[
\lim_{n \to \infty} \frac{1}{n} \log \mathbb{E}_{v_1} \exp\{-nh(\mathcal{L}^n)\} = -\inf_{\mu} \{ \Lambda^z_{\delta}(\mu) + h(\mu) \}, \quad h \in C_b(\mathcal{P}(V_S)).
\]

By this corollary and the contraction principle, we immediately obtain Theorem 2.4 as follows:

**Proof of Theorem 2.4.** Define \(T : z \in V_S \to z^- \in V\). Then

\[
T(\mathcal{L}^n)(\cdot) := \mathcal{L}^n \circ T^{-1}(\cdot) = L_n(\cdot), \quad n \geq 1.
\]
Since the empirical measure process \((\mathcal{L}^n)_{n \geq 1}\) of \((Z_n)_{n \geq 1}\) satisfies the LDP, to prove the theorem, we only need to verify the continuity of \(T\). Given any \(\mu_0 \in \mathcal{P}(V_S)\) and \(\varepsilon > 0\). For any \(\mu \in \mathcal{P}(V_S)\) satisfying that \(|\mu(z) - \mu_0(z)| < \varepsilon, z \in V_S\), we have that

\[
|T(\mu)(v) - T(\mu_0)(v)| \leq \sum_{z : z = v} |\mu(z) - \mu_0(z)| < d\varepsilon, \ v \in V,
\]

which implies the continuity.

Let \(\mathcal{P}(E)\) be the set of all probabilities on \(E\) equipped with the weak convergence topology. Then similarly to Theorem 2.4, one can prove the following

**Corollary 3.7.** The empirical measure process of edge process \((X_n X_{n+1})_{n \geq 0}\) with reinforcement factor \(\delta > 0\) satisfies an LDP with the rate function

\[
\tilde{\Lambda}_\delta(\nu) := \inf_{T(\mu) = \nu} \Lambda_\delta(\mu), \ \nu \in \mathcal{P}(E), \ \text{where} \ \tilde{T}(\mu)(e) = \mu|_1(e), \ e \in E.
\]

### 3.5 Properties of \(\Lambda_\delta\) and \(I_\delta\)

**Lemma 3.8.** For any \(\delta_2 > \delta_1 \geq 1\), \(\Lambda_{\delta_1}(\mu) \geq \Lambda_{\delta_2}(\mu), \ \mu \in \mathcal{P}(V_S)\), and \(\Lambda_{\delta_1}(\mu_0) > \Lambda_{\delta_2}(\mu_0)\) for some \(\mu_0 \in \mathcal{P}(V_S)\) depending on \(\delta_1\) and \(\delta_2\). However, \(\Lambda_{\delta} \equiv \Lambda_1\) for \(\delta \in (0, 1)\). In addition, \(\Lambda_\delta\) is uniformly continuous in \(\delta\) in the sense that

\[
\lim_{|\delta_2 - \delta_1| \to 0} \sup_{\mu \in \mathcal{P}(V_S) : \Lambda_1(\mu) < \infty} |\Lambda_{\delta_2}(\mu) - \Lambda_{\delta_1}(\mu)| = 0,
\]

and is non-differentiable at \(\delta = 1\) for some \(\mu \in \mathcal{P}(V_S)\).

**Proof.** We divide our proof into three parts as follows.

**\(\Lambda_\delta \equiv \Lambda_1\) for \(\delta \leq 1\):** Denote by \(p\) the transition probability of SRW on \(S\). Note that

\[
R(q(x, \cdot) \| p_{E_l}(x, \cdot)) = R(q(x, \cdot) \| p(x, \cdot)) + \int_{V_S} \log \frac{p(x, y)}{p_{E_l}(x, y)} q(x, dy).
\]

For \(\delta \leq 1\), every sequence of \(\{E_k\}_{1 \leq k \leq d}\) and all \(l = 1, \ldots, d\), we have \(\log \frac{p}{p_{E_l}} \geq 0\) since \(p(x, y) \geq p_{E_l}(x, y)\) for \(x, y \in E_l \times \{-1, 1\}\) (attained at \(l = d\)), which implies that

\[
R(q_l(x, \cdot) \| p_{E_l}(x, \cdot)) \geq R(q_l(x, \cdot) \| p(x, \cdot)).
\]

This deduces \(\Lambda_\delta \geq \Lambda_1\). Noting \(E_d = E, p_E = p\), and that \(r_1 = \cdots = r_{d-1} = 0\) when \(r_d = 1\), we have

\[
\Lambda_\delta(\mu) \leq \inf_{(\mu_k, r_k, E_k) \in \mathcal{F}(\mu, \mathcal{F}), r_d = 1} \sum_{k=1}^{d} r_k \int_{V_S} R(q_k \| p_{E_k}) \ d\mu_k = \inf_{q \in \mathcal{F}, \mu(q) = \mu} \int_{V_S} R(q \| p) \mu(dx) = \Lambda_1(\mu).
\]

The last identity above comes from [14, Theorem 8.4.3], since when \(\delta = 1\), \((Z_n)_{n \geq 1}\) is the SRW on \(S\). Therefore, \(\Lambda_\delta = \Lambda_1\).
Monotonicity for $\delta \geq 1$: In a similar way, we can obtain the monotonicity. For all $1 \leq \delta_1 < \delta_2$,

$$R(q_l(x, \cdot)\|p_{E_1}(x, \cdot; \delta_2)) - R(q_l(x, \cdot)\|p_{E_1}(x, \cdot; \delta_1)) = \int_{V_S} \log \frac{p_{E_1}(x, y; \delta_1)}{p_{E_1}(x, y; \delta_2)} q_l(x, dy),$$

where $\log \frac{p_{E_1}(x, y; \delta_1)}{p_{E_1}(x, y; \delta_2)} \leq 0$. It implies $\Lambda_{\delta_2}(\mu) \leq \Lambda_{\delta_1}(\mu)$ for all $\mu$. Now we show the strict monotonicity for some $\mu$. Note that for $x, y \in E_l \times \{-1, 1\}$,

$$\frac{p_{E_1}(x, y; \delta_1)}{p_{E_1}(x, y; \delta_2)} = \frac{k(x)\delta_2 + d(x) - k(x)}{\delta_2} \cdot \frac{\delta_1}{k(x)\delta_1 + d(x) - k(x)}, \quad (3.3)$$

where $d(x)$ (resp. $k(x)$) is the degree of $x$ in the graph $S$ (resp. $E_l \times \{-1, 1\}$). Since $d(x) \leq d, k(x) \geq 1$ for $x \in \partial(E_l \times \{-1, 1\})$, $(3.3)$ is less than $\frac{\delta_1}{\delta_2} \cdot \frac{\delta_2 + d - 1}{\delta_1 + d - 1} < 1$.

Specifically we choose an edge $e_0$ adjacent to the starting point $v_1$ of $(X_n)_{n \geq 0}$ and $\mu \in \mathcal{P}(\mathcal{S})$ such that $\mu_0(\{e_0, 1\}) = \mu_0(\{e_0, -1\}) = \frac{1}{2}$. Now we show that the infinums in $\Lambda_{\delta_1}, \Lambda_{\delta_2}$ are both attained at $r_1 = 1, E_1 = \{e_0\}$. Note that $p_{E_1}(x, y) \leq p_{\{e_0\}}(x, y)$ for $x, y \in \{e_0\} \times \{-1, 1\}$. For all $(\mu_k, r_k, E_k)_{1 \leq k \leq d} \in \mathcal{A}_d(\mu_0, \mathcal{S})$ and $q_k \in \mathcal{S}$ with $\mu_k q_k = \mu_k$, by the convexity of the rate function $\inf_{q, \mu_k} q \int_{V_S} R(q||p) \ d\mu_k$ (see [14, Proposition 8.5.2]),

$$\sum_{k=1}^d r_k \int_{V_S} R(q_k\|p_{E_k}) \ d\mu_k \geq \sum_{k=1}^d r_k \int_{V_S} R(q_k\|p_{\{e_0\}}) \ d\mu_k$$

$$\geq \sum_{k=1}^d r_k \inf_{q_k: \mu_k q_k=\mu_k} \int_{V_S} R(q_k\|p_{\{e_0\}}) \ d\mu_k$$

$$\geq \inf_{q: \mu_0 q=q} \int_{V_S} R(q\|p_{\{e_0\}}) \ d\mu_0,$$

which implies the attainment $r_1 = 1, E_1 = \{e_0\}$. Note that $\mu_0(\partial(\{e_0\} \times \{-1, 1\})) \geq \frac{1}{2}$. This implies that (assume $\Lambda_{\delta_1}$ is attained at $q \in \mathcal{S}$)

$$\Lambda_{\delta_2}(\mu_0) - \Lambda_{\delta_1}(\mu_0) \leq \int_{V_S} \left[ R(q(x, \cdot)\|p_{\{e_0\}}(x, \cdot; \delta_2)) - R(q(x, \cdot)\|p_{\{e_0\}}(x, \cdot; \delta_1)) \right] \ d\mu_0$$

$$= \int_{V_S} \left[ \int_{V_S} \log \frac{p_{\{e_0\}}(x, y; \delta_1)}{p_{\{e_0\}}(x, y; \delta_2)} q(x, dy) \right] \ d\mu_0$$

$$\leq \log \left\{ \frac{\delta_1}{\delta_2} \cdot \frac{\delta_2 + d - 1}{\delta_1 + d - 1} \right\} d\mu_0(\partial(\{e_0\} \times \{-1, 1\}))$$

$$\leq \frac{1}{2} d \log \left\{ \frac{\delta_1}{\delta_2} \cdot \frac{\delta_2 + d - 1}{\delta_1 + d - 1} \right\} < 0.$$  

Continuity: When considering continuity, we control $\sup_{\mu} |\Lambda_{\delta_1}(\mu) - \Lambda_{\delta_2}(\mu)|$ with some continuous function of $\delta_1, \delta_2$. Noting that $\Lambda_{\delta} = \Lambda_1$ for $\delta \leq 1$, we only need to show the uniform continuity for $\delta \geq 1$. Fix some $\mu \in \mathcal{P}(\nu) \{\mu \in \mathcal{P}(V_S) : 1(\mu) < \infty\}$. For $1 \leq \delta_1 < \delta_2$,

$$R(q_l(x, \cdot)\|p_{E_1}(x, \cdot; \delta_2)) - R(q_l(x, \cdot)\|p_{E_1}(x, \cdot; \delta_1)) = \int_{V_S} \log \frac{p_{E_1}(x, y; \delta_1)}{p_{E_1}(x, y; \delta_2)} q_l(x, dy).$$

If $x \notin \partial(E_l \times \{-1, 1\})$, this value should be 0. Otherwise, we can obtain that

$$\frac{p_{E_1}(x, y; \delta_1)}{p_{E_1}(x, y; \delta_2)} = \frac{k(x)\delta_2 + d(x) - k(x)}{\delta_2} \cdot \frac{\delta_1}{k(x)\delta_1 + d(x) - k(x)},$$
where \(d(x)\) (resp. \(k(x)\)) is the degree of \(x\) in the graph \(S\) (resp. \(E_l \times \{-1, 1\}\)). This value is greater than \(\frac{d_1}{d_2}\), which implies that

\[
R(q_l(x, \cdot) \| p_{E_l}(x, \cdot; \delta_2)) - R(q_l(x, \cdot) \| p_{E_l}(x, \cdot; \delta_1)) \geq \log \frac{d_1}{d_2}.
\]

Choose some \(\mu_k, r_k, E_k, q_k\) such that

\[
\Lambda_{\delta_1}(\mu) = \sum_{k=1}^d r_k \int_{S} R(q_k(x, \cdot) \| p_{E_k}(x, \cdot; \delta_2)) \, d\mu_k.
\]

By the definition of \(\Lambda_{\delta_1}(\mu)\) and the monotonicity of \(\Lambda_{\delta}\),

\[
0 \geq \Lambda_{\delta_2}(\mu) - \Lambda_{\delta_1}(\mu) \geq \sum_{k=1}^d r_k \int_S [R(q_k(x, \cdot) \| p_{E_k}(x, \cdot; \delta_2)) - R(q_k(x, \cdot) \| p_{E_k}(x, \cdot; \delta_1))] \, d\mu_k \geq \log \frac{d_1}{d_2}.
\]

Since \(\mu\) is arbitrary, we obtain that

\[
0 \leq \sup_{\mu \in \mathcal{P}(S)} |\Lambda_{\delta_2}(\mu) - \Lambda_{\delta_1}(\mu)| \leq \log \frac{d_2}{d_1}.
\]

By the squeeze theorem, \(\lim_{1 < \delta \to 0} \sup_{\mu \in \mathcal{P}(S)} |\Lambda_{\delta_2}(\mu) - \Lambda_{\delta_1}(\mu)| = 0\), which completes the proof of the uniform continuity.

**Non-differentiability at \(\delta = 1\)**: First, by \(\Lambda_{\delta} = I_1\) for \(\delta \leq 1\), we obtain that the left derivative at \(\delta = 1\) should be 0. Now we pay attention to the right derivative. Set \(\mu_0\) to be what we choose in the proof of the monotonicity for \(\delta \geq 1\), i.e., \(\mu_0(\{e_0, 1\}) = \mu_0(\{e_0, -1\}) = \frac{1}{2}\).

From the proof of the monotonicity, we know that the infimum of \(\Lambda_{\delta}\) (\(\delta > 1\)) is attained at \(r_1 = 1, \ E_1 = \{e_0\}\). Assume \(e_0 = v_1v_2\). Set \(d_i\) to be the degree of \(v_i\) for \(i = 1, 2\). Then \(\Lambda_\delta(\mu_0) - \Lambda_1(\mu_0) = \frac{1}{2} \log \frac{d_1 - 1 + \delta}{d_1} + \frac{1}{2} \log \frac{d_2 - 1 + \delta}{d_2}\). Hence we can obtain the right derivative (fix \(\mu = \mu_0\))

\[
\lim_{\delta \to 1^+} \frac{\Lambda_\delta(\mu_0) - \Lambda_1(\mu_0)}{\delta - 1} = \frac{1}{2d_1} + \frac{1}{2d_2} - 1.
\]

Since the graph is connected, and does not equal \(\{v_1, v_2\}\), the right derivative at \(\delta = 1\) for \(\mu_0\) is less than 0, which implies the non-differentiability at \(\delta = 1\).

**Proof of Theorem 2.6.** Based on Lemma 3.8 and the similar method, we can prove Theorem 2.6 as follows.

\(I_\delta \equiv I_1\) for \(\delta \leq 1\): By Lemma 3.8 we immediately obtain that \(I_\delta(\mu) = I_1(\mu)\) for all \(\mu \in \mathcal{P}(V)\).

**Monotonicity for \(\delta \geq 1\):** For \(1 \leq \delta_1 < \delta_2\), note that in the proof of Lemma 3.8 we obtain that \(\Lambda_{\delta_2}(\mu_0) - \Lambda_{\delta_1}(\mu_0) < 0\) for \(\mu_0 \in \mathcal{P}(V_S)\) with \(\mu_0(\{e_0, 1\}) = \mu_0(\{e_0, -1\}) = \frac{1}{2}\), where \(e_0\) is some edge containing starting point \(v_1\). Assuming \(e_0 = \{v_1, u_0\}\), and setting \(\nu_0 \in \mathcal{P}(V)\) with \(\nu_0(v_1) = \nu_0(u_0) = \frac{1}{2}\), we observe that \(\mu_0\) is the unique solution to \(T(\mu) = \nu_0\). By Lemma 3.8 we immediately get \(I_{\delta_2}(\nu_0) = \Lambda_{\delta_2}(\mu_0) < \Lambda_{\delta_1}(\mu_0) = I_{\delta_1}(\nu_0)\), and \(I_{\delta_2}(\nu) \leq I_{\delta_1}(\nu)\) for all \(\nu \in \mathcal{P}(V)\).

**Continuity:** By the proof of the continuity in Lemma 3.8, we know that for \(1 \leq \delta_1 < \delta_2\),

\[
0 \leq \sup_{\mu \in \mathcal{P}(S)} |\Lambda_{\delta_2}(\mu) - \Lambda_{\delta_1}(\mu)| \leq \log \frac{d_2}{d_1}.
\]

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where $\mathcal{P}_\Lambda(S) := \{\mu \in \mathcal{P}(V_S) : \Lambda_1(\mu) < \infty\}$. It immediately shows that

$$0 \leq \sup_{\nu \in \mathcal{P}_1(G)} |I_{\delta_2}(\nu) - I_{\delta_1}(\nu)| \leq \sup_{\nu \in \mathcal{P}_1(G)} (\Lambda_{\delta_1}(\mu_\nu) - \Lambda_{\delta_2}(\mu_\nu)) \leq \log \frac{\delta_2}{\delta_1},$$

where $\mathcal{P}_1(G) := \{\nu \in \mathcal{P}(V) : I_1(\nu) < \infty\}$, $T(\mu_\nu) = \nu$, and $I_{\delta_2}(\nu)$ is attained at $\mu_\nu$. (Since $\{\mu : T(\mu) = \nu\}$ is closed, the infimum can be reached at some $\mu_\nu$ with $T(\mu_\nu) = \nu$.) Now we get the uniform continuity of $I_\delta$.

**Non-differentiability at $\delta = 1$**: Fixing $\mu = \mu_0$, we note that $I_\delta(\nu_0) = \Lambda_\delta(\mu_0)$, $I_1(\nu_0) = \Lambda_1(\mu_0)$. Hence by the same argument in the proof of the non-differentiable for $\Lambda_\delta$ at $\delta = 1$, we observe that the left derivative equals 0, and that the right derivative is less than 0, which completes the proof. ■

### 4 Critical exponent for exponential integrability of cover time

This section devotes to critical exponent for exponential integrability of cover time $C_E$ and its properties (refer to Theorem 2.9). *In fact*, we can consider a general stopping time

$$T_{\mathcal{E}_0} := \inf \{n > 0 : \text{supp}(\mathcal{L}_n^{|1|}) \notin \mathcal{E}_0\},$$

and compute its critical exponent for exponential integrability, and describe its phase diagram. Theorem 2.9 on $C_E$ is a corollary of the following results of $T_{\mathcal{E}_0}$ in Subsection 4.1.

#### 4.1 Main results on critical exponent of exponential integrability of $T_{\mathcal{E}_0}$

**Theorem 4.1.** The exponential integrability of $T_{\mathcal{E}_0}$ has a critical exponent

$$\alpha_c(\delta) := \inf_{\mu \in C(\mathcal{E}_0)} \Lambda_{\delta,\mathcal{E}_0}(\mu)$$

in the sense that

$$\left\{ \begin{array}{ll}
\mathbb{E}_{v_1}(e^{\alpha_T_{\mathcal{E}_0}}) < \infty & \text{if } \alpha < \alpha_c(\delta), \\
\mathbb{E}_{v_1}(e^{\alpha_T_{\mathcal{E}_0}}) = \infty & \text{if } \alpha \geq \alpha_c(\delta).
\end{array} \right.$$  

Meanwhile

$$\lim_{n \to \infty} \frac{1}{n} \log \mathbb{P}_{v_1}(T_{\mathcal{E}_0} > n) = -\alpha_c(\delta).$$

**Proposition 4.2.** $\alpha_c(\delta) = \inf_{\mu \in C(\mathcal{E}_0)} \Lambda_{\delta,\mathcal{E}_0}(\mu)$ can be expressed as

$$\inf_{(\mu,E_0 \in \mathcal{E}_0, \mu_1 \subseteq E_0, \mu_2 = \mu)} \int_{V_S} R(q\|p_{E_0}) \, d\mu. \quad (4.1)$$

Proof. Note that (4.1) is greater than $\alpha_c(\delta)$ since the infimum condition of this expression is a specific case of the condition $\mu \in C(\mathcal{E}_0)$. Now we show the opposite inequality. Noting

$$\alpha_c(\delta) = \inf_{(\mu,E_0 \in \mathcal{E}_0, \mu_1 \subseteq E_0, \mu_2 = \mu)} \sum_{k=1}^d r_k \int_{V_S} R(q_k\|p_{E_k}) \, d\mu_k$$

we complete the proof. ■
Theorem 4.3. For \( \mathcal{S}_0 \neq \mathcal{S} \), \( \alpha_c(\delta) \) is decreasing and continuous in \( \delta > 0 \), and uniformly continuous in \( \delta > \delta_0 \) for any \( \delta_0 > 0 \). In addition, \( \lim_{\delta \to +\infty} \alpha_c(\delta) = 0 \).

Theorem 4.4. \( \alpha_c(\delta) \) is strictly decreasing in \( \delta > 0 \). If there exist some \( E_0 \in \mathcal{S}_0 \) and some nonempty set \( E' \subseteq E_0 \) such that \( E' \) is the support of the first marginal measure of some invariant measure on \( V_S \) and \( E' \cap \partial E_0 = \emptyset \), then

\[
\lim_{\delta \to 0} \alpha_c(\delta) = \inf_{(\mu, E_0 \in \mathcal{S}_0) : \text{supp}(\mu) \subseteq E_0, \text{supp}(\mu) \cap \partial E_0 = \emptyset} \Lambda_1(\mu) < \infty.
\]

Otherwise, \( \lim_{\delta \to 0} \alpha_c(\delta) = \infty \).

Specifically, set \( \mathcal{S}_0 = \mathcal{S} \setminus \{E\} \), then \( C_0 := C(\mathcal{S}_0) = \{\mu : \text{supp}(\mu) \neq E\} \), and by Theorems 4.1, 4.3 and 4.4, Theorem 2.9 holds.

Here we provide a simple equivalent criterion only depending on structure of graphs for \( \lim_{\delta \to 0} \alpha_c(\delta) < +\infty \) in the above theorem.

Corollary 4.5. If for some edge \( e \in E \) with \( \{e\} \in \mathcal{S}_0 \), the collection of all edges adjacent to \( e \), denoted by \( E_e := \{e' \in E : e' \cap e \neq \emptyset\} \), belongs to \( \mathcal{S}_0 \), then

\[
\lim_{\delta \to 0} \alpha_c(\delta) = \inf_{(\mu, E_0 \in \mathcal{S}_0) : \text{supp}(\mu) \subseteq E_0, \text{supp}(\mu) \cap \partial E_0 = \emptyset} \Lambda_1(\mu) < \infty.
\]

Otherwise, \( \lim_{\delta \to 0} \alpha_c(\delta) = \infty \).

Proof. By Theorem 4.4, we only need to show the equivalence of the conditions given in Theorem 4.4 and Corollary 4.5.

If for some \( E_0 \in \mathcal{S}_0 \), there exists \( E' \subseteq E_0 \) such that \( E' \) can be the support of the first marginal measure of some invariant measure on \( V_S \) and \( E' \cap \partial E_0 = \emptyset \), then for some edge \( e \in E' \) we have \( e \notin \partial E_0 \), which implies \( E_e \subseteq E_0 \). Hence \( E_e \in \mathcal{S}_0 \) and, of course, \( \{e\} \in \mathcal{S}_0 \).

If there exists some \( \{e\} \in \mathcal{S}_0 \) with \( E_e \in \mathcal{S}_0 \), then choose \( E_0 = \{e\} \) and \( E' = E_e \). We can verify these \( E_0 \) and \( E' \) satisfy the condition in Theorem 4.4. \( \blacksquare \)

Here we show that \( \lim_{\delta \to 0} \alpha_c(\delta) \) diverges on some specific graphs by Corollary 4.5.

Example 4.6. For star-shaped graphs and complete graphs, the condition does not hold, and hence \( \lim_{\delta \to 0} \alpha_c(\delta) = \infty \). For the rooted tree with two or more levels, the condition holds, which implies that \( \lim_{\delta \to 0} \alpha_c(\delta) < \infty \).

Example 4.7. For \( \mathcal{S}_0 = \mathcal{S} \setminus \{E\} \neq \emptyset \), a necessary condition of \( \lim_{\delta \to 0} \alpha_c(\delta) = \infty \) is the existence of a vertex that connects all other vertices.

Proof. Here we give a short proof by contradiction. Otherwise, we can select a vertex \( u_0 \) with the maximal degree (this degree should be greater than 1, otherwise there are at most 2 vertices in the graph and \( \mathcal{S}_0 = \emptyset \)). Set \( u_1 \) to be some vertex not adjacent to \( u_0 \), and choose an edge \( e \) adjacent to \( u_1 \). Note that the degree of \( u_0 \) is at least 2, and that \( u_1 \in e \) is not the neighbour of \( u_0 \). There exists a vertex \( u \sim u_0 \) such that \( u \notin e \). Therefore, \( e \notin E_{ uu_0 } \), which deduces \( E_{ uu_0 } \neq E \). This leads to a contradiction. \( \blacksquare \)

The stopping times \( \mathcal{S}_0 \) include not only \( C_E \), but also some interesting stopping times such as hitting time and cover time of subgraphs. For \( G' = (V', E') \subseteq G = (V, E) \), set

\[
\mathcal{S}_1 = \{E'' \in \mathcal{S} : E'' \subseteq E'\}, \quad \mathcal{S}_2 = \{E'' \in \mathcal{S} : E' \setminus E'' \neq \emptyset\}.
\]
Define
\[ C_1 := \mathcal{C}(S) = \{ \mu \in \mathcal{P}(V_S) : \text{supp}(\mu |_1) \subseteq E' \}, \]
\[ C_2 := \mathcal{C}(S) = \{ \mu \in \mathcal{P}(V_S) : E' \supset \text{supp}(\mu |_1) \neq \emptyset \}. \]

Then for the $\delta$-ORRW $(X_n)_{n \geq 0}$, $T_{G', \delta} := T_{\mathcal{G}_1} = \inf \{ n \geq 1 : X_{n-1} X_n \notin E' \}$ is hitting time of $E \setminus E'$, and $C_{E', \delta} := T_{\mathcal{G}_2} = \inf \{ n \geq 1 : E' \subseteq \{ X_{m-1} X_m : m \leq n \} \}$ is edge cover time of subgraph $G'$.

We can apply Theorems 4.1, 4.3 and 4.4 to obtain the same properties of $T_{G', \delta}$ and $C_{E', \delta}$ as those of $C_E$. Similarly to Theorem 2.9, we can prove an interesting phenomenon of $T_{G', \delta}$.

**Corollary 4.8.** For any $0 < \delta_1 < \delta_2$, there exists some $N = N(\delta_1, \delta_2)$ such that
\[ \mathbb{P}_{v_1} (T_{G', \delta_1} > n) < \mathbb{P}_{v_1} (T_{G', \delta_2} > n), \ n > N. \]

**Proof of Corollary 4.8.** Denote by $\alpha_\epsilon^*(\delta) := \inf_{\mu \in C_1} \Lambda_{\delta, \mathcal{G}_1}(\mu)$. Similarly to Theorem 2.9, we can verify that $\lim_{n \to \infty} \frac{1}{n} \log \mathbb{P}_{v_1} (T_{G', \delta_1} > n) = -\alpha_\epsilon^*(\delta_1)$ for $i \in \{1, 2\}$ and $\alpha_\epsilon^*(\delta_1) > \alpha_\epsilon^*(\delta_2) > 0$.

Thus for any $\epsilon \in (0, \alpha_\epsilon^*(\delta_2))$, there is some $N = N(\epsilon, \delta_1, \delta_2)$ such that for all $n > N$,
\[ \exp (-n(\alpha_\epsilon^*(\delta_1) + \epsilon)) < \mathbb{P}_{v_1} (T_{G', \delta_1} > n) < \exp (-n(\alpha_\epsilon^*(\delta_1) - \epsilon)), \ i = 1, 2. \]

Then for small enough $\epsilon$, $\mathbb{P}_{v_1} (T_{G', \delta_1} > n) < \mathbb{P}_{v_1} (T_{G', \delta_2} > n)$ for any $n > N$. \qed

### 4.2 Proof of Theorem 2.8

**Proof of Theorem 2.8.** It suffices to show that for any $z \in V_S$ with $z^- = v_1$,
\[ \lim_{n \to \infty} \frac{1}{n} \log \mathbb{P}_{v_1} (L^n \in C(\mathcal{G}_0) | Z_1 = z) = -\inf_{\mu \in C(\mathcal{G}_0)} \Lambda_{\delta, \mathcal{G}_0}(\mu), \]

since $\inf_{\mu \in C(\mathcal{G}_0)} \Lambda_{\delta, \mathcal{G}_0}(\mu) = \min_{z : z^- = v_1} \inf_{\mu \in C(\mathcal{G}_0)} \Lambda_{\delta, \mathcal{G}_0}(\mu)$ and
\[ \lim_{n \to \infty} \frac{1}{n} \log \mathbb{P}_{v_1} (L^n \in C(\mathcal{G}_0)) = \max_{z : z^- = v_1} \lim_{n \to \infty} \frac{1}{n} \log \mathbb{P}_{v_1} (L^n \in C(\mathcal{G}_0) | Z_1 = z). \]

Let $h_0$ be some bounded continuous function vanishing on $C(\mathcal{G}_0)$. Then
\[ \lim_{n \to \infty} \frac{1}{n} \log \mathbb{E}_{v_1} \left[ e^{-nh_0(L^n)} 1_{\{ L^n \in C(\mathcal{G}_0) \}} | Z_1 = z \right] = \lim_{n \to \infty} \frac{1}{n} \log \mathbb{P}_{v_1} (L^n \in C(\mathcal{G}_0) | Z_1 = z). \] (4.2)

Note that by Lemma 2.16,
\[ -\frac{1}{n} \log \mathbb{E}_{v_1} \left[ e^{-nh_0(L^n)} 1_{\{ L^n \in C(\mathcal{G}_0) \}} | Z_1 = z \right] = \inf_{\nu_j \in C(\mathcal{G}_0)} \mathbb{E}_z \left\{ \frac{1}{n} \sum_{j=0}^{n-1} R(\nu_j^n \| \pi_{\nu_j^{n+1}}) \right\}. \] (4.3)

Actually, setting $\tilde{\nu}_j^n = \nu_j^n (\cdot) \in C(\mathcal{G}_0))$, we can get a smaller value, which implies (4.3). By Theorem 2.3, we obtain
\[ \lim_{n \to \infty} \inf_{\nu_j \in C(\mathcal{G}_0)} \mathbb{E}_z \left\{ \frac{1}{n} \sum_{j=0}^{n-1} R(\nu_j^n \| \pi_{\nu_j^{n+1}}) \right\} = -\inf_{\mu \in C(\mathcal{G}_0)} \Lambda_{\delta, \mathcal{G}_0}(\mu). \] (4.4)

By (4.2), (4.3) and (4.4), we complete the proof. \qed
4.3 Proofs of critical exponent for exponential integrability of $T_{\mathcal{S}_0}$

Proof of Theorem 4.1. Since $\mathbb{P}_{v_1} (L^n \in C(\mathcal{S}_0)) = \mathbb{P}_{v_1} (T_{\mathcal{S}_0} > n)$, by Theorem 2.8, we obtain

$$
\lim_{n \to \infty} \frac{1}{n} \log \mathbb{P}_{v_1} (T_{\mathcal{S}_0} > n) = -\alpha_c(\delta).
$$

(4.5)

Note that

$$
\mathbb{E}_{v_1} [e^{\alpha T_{\mathcal{S}_0}}] = \sum_{n=0}^{\infty} (e^{\alpha} - 1) \cdot e^{\alpha n} \mathbb{P}_{v_1} (T_{\mathcal{S}_0} > n).
$$

(4.6)

If $\alpha < \alpha_c(\delta)$, then choose some $0 < \varepsilon < \alpha_c(\delta) - \alpha$, by (4.5) there exists some $N > 0$ such that for all $n > N$, $\mathbb{P}_{v_1} (T_{\mathcal{S}_0} > n) < e^{-n(\alpha_c(\delta) - \varepsilon)}$, which implies

$$
e^{\alpha n} \mathbb{P}_{v_1} (T_{\mathcal{S}_0} > n) < e^{n(\alpha - \alpha_c(\delta))}.
$$

Since $\alpha + \varepsilon - \alpha_c(\delta) < 0$, by (4.6), we have that $\mathbb{E}_{v_1} [e^{\alpha T_{\mathcal{S}_0}}] < \infty$.

If $\alpha > \alpha_c(\delta)$, then choose some $0 < \varepsilon < \alpha - \alpha_c(\delta)$, and similarly we can find some $N > 0$ such that for all $n > N$,

$$
e^{\alpha n} \mathbb{P}_{v_1} (T_{\mathcal{S}_0} > n) > e^{n(\alpha - \alpha_c(\delta))}.
$$

Since $\alpha - \varepsilon - \alpha_c(\delta) > 0$, by (4.6), we see that $\mathbb{E}_{v_1} [e^{\alpha T_{\mathcal{S}_0}}] = \infty$.

Now we pay attention to the divergence of $\mathbb{E}_{v_1} [e^{\alpha c(\delta) T_{\mathcal{S}_0}}]$. To show this we may consider $\mathbb{P}_{v_1} (T_{\mathcal{S}_0} > n)$, and prove that it has the same order as $p^n$ for some $p > 0$:

$$
0 < \lim_{n \to \infty} \frac{\mathbb{P}_{v_1} (T_{\mathcal{S}_0} > n)}{p^n} \leq \lim_{n \to \infty} \frac{\mathbb{P}_{v_1} (T_{\mathcal{S}_0} > n)}{p^n} < \infty.
$$

We denote this by $\mathbb{P}_{v_1} (T_{\mathcal{S}_0} > n) = \Theta (p^n)$. Similarly to the proof of Theorem 2.3, we choose some sequence $\{E_j\}_{1 \leq j \leq d} \in \mathscr{E}$, and consider the probability that the process stays in the subgraph $E_j \times \{-1, 1\}$ for at least time $n_i$. Recall the definition of the renewal times $\tau_k$ from (2.3). Note that under the condition $Z_{\tau_k} \in E_k$, $1 \leq k \leq d$, $(Z_n)_n$ can be regarded as a Markov process on $E_{j+1} \times \{-1, 1\}$ when it stays in the subgraph $E_j \times \{-1, 1\}$. Therefore, the probability $\mathbb{P}_{v_1} (\tau_i - \tau_{i-1} > n_i | Z_{\tau_k} \in E_k, 1 \leq k \leq d)$ has the same order as $p_i^{n_i}$ for some $p_i \in (0, 1)$, which would be verified soon.

Denote by $l = l(\{E_j\}_j) := \text{inf}\{k \leq d : E_k \notin \mathcal{S}_0\}$. Noting $\mathbb{P}_{v_1} (Z_{\tau_k} \in E_k, 1 \leq k \leq d) > 0$ for all $\{E_k\}_k \in \mathscr{E}$, we obtain

$$
\mathbb{P}_{v_1} (T_{\mathcal{S}_0} > n)
= \sum_{\{E_j\}_j \in \mathscr{E}} \mathbb{P}_{v_1} (Z_{\tau_k} \in E_k, 1 \leq k \leq d) \times
\sum_{n_1 + \cdots + n_{l-1} > n} \mathbb{P}_{v_1} (\tau_i - \tau_{i-1} = n_i, 1 \leq i \leq l - 1 | Z_{\tau_k} \in E_k, 1 \leq k \leq d)
\in \left[ \sum_{\{E_j\}_j \in \mathscr{E}} \mathbb{P}_{v_1} (Z_{\tau_k} \in E_k, 1 \leq k \leq d) \cdot \max_{1 \leq i \leq l-1} \left\{ \mathbb{P}_{v_1} (\tau_i - \tau_{i-1} > n | Z_{\tau_k} \in E_k, 1 \leq k \leq d) \right\}, \sum_{\{E_j\}_j \in \mathscr{E}} \mathbb{P}_{v_1} (Z_{\tau_k} \in E_k, 1 \leq k \leq d) \right].
$$
For all \(0 < \delta_1 < \delta_2\),

\[
R(q(x, \cdot)\|p_{E^c}(x, \cdot; \delta_2)) - R(q(x, \cdot)\|p_{E^c}(x, \cdot; \delta_1)) = \int_{V_S} \log \frac{p_{E^c}(x, y; \delta_1)}{p_{E^c}(x, y; \delta_2)} q(x, dy),
\]

where \(V_S\) is some measure on \(A^c\).
where \( \log \frac{p(x,y)}{p(x,y)} \leq 0 \). By Proposition 4.2, \( \alpha_c(\delta_2) \leq \alpha_c(\delta_1) \).

**Continuity:** Similarly to the proof of the continuity in Theorem 3.8, we obtain that

\[
0 \leq \sup_{\mu \in \mathcal{P}(S)} |\Lambda_{\delta_2} - \Lambda_{\delta_1}| \leq \log \frac{\delta_2}{\delta_1}, \quad 0 < \delta_1 < \delta_2,
\]

where \( \mathcal{P}(S) := \{ \mu \in \mathcal{P}(V_S) : \Lambda_1(\mu) < \infty \} \). It immediately shows that

\[
0 \leq |\alpha_c(\delta_2) - \alpha_c(\delta_1)| \leq (\Lambda_{\delta_2} - \Lambda_{\delta_1}) \leq \log \frac{\delta_2}{\delta_1},
\]

where \( \mu_2 \in \mathcal{C}(\mathcal{S}) \), and \( \alpha_c(\delta_2) \) is attained at \( \mu_2 \) (Since \( \mathcal{C}(\mathcal{S}) \) is closed, the infimum can be reached at some \( \mu_2 \in \mathcal{C}(\mathcal{S}) \)).

\[ \lim_{\delta \to \infty} \alpha_c(\delta) = 0 : \] Take an edge \( e_0 \) containing the starting point \( v_1 \) such that \( \{e_0\} \in \mathcal{S} \). Recalling (2.3) for the definition of the renewal times, one can see that

\[
\mathbb{P}_{v_1} (\mathcal{L} \in \mathcal{C}(\mathcal{S})) \geq \mathbb{P}_{v_1} (\tau_2 > n, Z_1|1 = e_0),
\]

which deduces that (by Theorem 2.8)

\[
- \lim_{n \to \infty} \frac{1}{n} \log \mathbb{P}_{v_1} (\tau_2 > n, Z_1|1 = e_0) \geq - \lim_{n \to \infty} \frac{1}{n} \log \mathbb{P}_{v_1} (\mathcal{L} \in \mathcal{C}(\mathcal{S})) = \alpha_c(\delta).
\]

Noting

\[
\mathbb{P}_{v_1} (\tau_2 > n, Z_1|1 = e_0) \geq \mathbb{P}_{v_1} (\tau_2 = n + 1, Z_1|1 = e_0) \geq \frac{1}{d} \left( \frac{\delta}{d - 1 + \delta} \right)^n,
\]

we have that as \( \delta \to \infty \),

\[
- \lim_{n \to \infty} \frac{1}{n} \log \mathbb{P}_{v_1} (\tau_2 > n, Z_1|1 = e_0) \leq \log \frac{d - 1 + \delta}{\delta} \to 0.
\]

By the squeeze theorem, \( \lim_{\delta \to \infty} \alpha_c(\delta) = 0 \). \( \blacksquare \)

Before proving Theorem 4.4, we provide the following

**Lemma 4.9.**

\[
\alpha_c(\delta) = \inf_{(\mu,E_0 \in \mathcal{S}) : \text{supp}(\mu_1) \subseteq E_0} \left\{ \Lambda_1(\mu) + \int_{\partial E_0 \times \{-1,1\}} \log \frac{d(z) - k(z) + k(z)\delta}{d(z)} \frac{d\mu}{d\mu} \right\}, \quad (4.7)
\]

where \( d(z), k(z) \) are the degrees of \( z \) in the graphs \( S \) and \( E_0 \times \{-1,1\} \) respectively. Moreover, the infimum is attained at some \( \mu \in \mathcal{P}(V_S) \) with \( \mu_{11} (\partial E_0) > 0 \).

**Proof.** By Proposition 4.2, \( \alpha_c(\delta) = \inf_{(\mu,E_0 \in \mathcal{S}) : \text{supp}(\mu_1) \subseteq E_0, \mu_{11} = \mu} \int_{V_S} R(q\|p_{E_0}) \ d\mu \). Note that

\[
\int_{V_S} R(q\|p_{E_0}) \ d\mu = \int_{V_S} R(q\|p_0) \ d\mu + \int_{V_S} \log \frac{d(z) - k(z) + k(z)\delta}{d(z)} \frac{d\mu}{d\mu} \mu
\]

\[
= \int_{V_S} R(q\|p_0) \ d\mu + \int_{E_0 \times \{-1,1\}} \log \frac{d(z) - k(z) + k(z)\delta}{d(z)} \frac{d\mu}{d\mu}. \quad (4.8)
\]

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By [14, Theorem 8.4.3], \( \Lambda_1(\mu) = \inf_{q \in \mathcal{F}}: \mu q = \mu \int_{V_S} R(q\|p) \, d\mu \). Hence we obtain (4.7) by taking \( \inf_{(\mu, E_0) \in \mathcal{I}_0, q} \sup(\mu \mid x) \subseteq E_0, \mu q = \mu \{ \cdot \} \) on both sides of (4.8).

Set \( g(\mu) := \Lambda_1(\mu) + \int_{\partial E_0 \times \{-1,1\}} \log \frac{d(z)-k(z)+k(z)}{d(z)} \, d\mu, \mu \in \mathcal{P}(V_S) \). Fix some \( E_0 \in \mathcal{I}_0 \) for \( \mu' \in \mathcal{P}(V_S) \) with \( \sup(\mu') \subseteq E_0 \times \{-1,1\} \) and \( \mu'\{\partial E_0\} = 0 \), to prove that there is a \( \bar{\mu} \in \mathcal{P}(V_S) \) with \( \sup(\bar{\mu}) = E_0 \times \{-1,1\} \) such that \( g(\mu') > g(\bar{\mu}) \).

Indeed, note \( g(\mu') = \Lambda_1(\mu') = \inf_{q \in \mathcal{F} : \mu' q = \mu' \int_{V_S} R(q\|p) \, d\mu', \mu' \in \mathcal{P}(V_S)} \) and assume this infimum is attained at some \( q' \in \mathcal{F} \). Generate a new directed subgraph \( S' \) by \( q' \): For any \( x, y \in E_0 \times \{1,-1\} \), \( x \to y \) in \( S' \) if and only if \( q'_{xy} > 0 \). Call a vertex set \( V' \) is an \( S' \)-connected set if for any \( x, y \in V' \) with \( x \neq y \), there is a path from \( x \) to \( y \) or from \( y \) to \( x \) in \( S' \). And call an \( S' \)-connected set \( V' \) is maximum if for any \( z \notin V' \), there does not exist a path from \( z \) to \( V' \) or from \( V' \) to \( z \). By the definition of \( S' \), there is some \( e \in E_0 \) with \( \mu'|_1(e) = 0 \) such that \( (e, \pm 1) \) do not belong to any \( S' \)-connected set. Choose a maximum \( S' \)-connected set \( V_0' \) such that there exists some \( e_1 \in E_0 \) with \( \mu'|_1(e_1) = 0 \) and some \( z_0 \in V_0' \) with \( z_0 \to (e_1,1) \) or \( z_0 \to (e_1,-1) \) by some edge in \( S \). Without loss of generality, assume \( z_0 \to (e_1,1) \). Let \( k'(x) \) be the degree of \( x \) in \( S' \). Then construct a disturbed transition probability \( q^\varepsilon \in \mathcal{F} \) as follows.

(a) When \( z_0^* := (z_0|_1, -z_0|_2) \notin V_0' \), set
\[
q^\varepsilon_{xy} = \begin{cases} 
q'_{xy} - \varepsilon/k'(x), & x = z_0, x \to y \in S', \\
\varepsilon, & x = z_0, y = z_0^*, \\
1, & x = z_0^*, y = z_0, \\
q'_{xy}, & x \neq z_0, z_0^*.
\end{cases}
\]

(b) When \( z_0^* \in V_0' \), set
\[
q^\varepsilon_{xy} = \begin{cases} 
q'_{xy} - \varepsilon/k'(x), & x = z_0, x \to y \in S', \\
\varepsilon, & x = z_0, y = (e_1,1), \\
1, & x = (e_1,1), y = (e_1,-1), \text{ or } x = (e_1,-1), y = z_0^*, \\
q'_{xy}, & x \neq z_0, (e_1,\pm 1).
\end{cases}
\]

See Figure 8 for an intuition of \( q^\varepsilon \). Let
\[
c_0 := \mu'(V_0'), \quad V_1' := V_0' \cup \{z_0^*, (e_1,\pm 1)\}.
\]
We construct a new probability measure \( \mu^\varepsilon \in \mathcal{P}(V_S) \) such that
\[
\mu^\varepsilon q^\varepsilon = \mu^\varepsilon \text{ and } \mu^\varepsilon(V_1') = c_0.
\]
Note $\mu^\varepsilon(z) = \mu'(z)$ and $q^\varepsilon(z, \cdot) = q'(z, \cdot)$ for $z \in V_S \setminus V'_1$. We only need to consider those vertices in $V'_1$ when proving $g(\mu') > g(\mu^\varepsilon)$. Specifically,

$$\mu' = \mu^\varepsilon \text{ for } \varepsilon = 0, \text{ and } \int_{V_S} R(q^\varepsilon \| p) \, d\mu' = \int_{V_S} R(q^0 \| p) \, d\mu^0.$$ 

Thus, we replace $\mu'$, $q'$ by $\mu^0$, $q^0$. Note $\{\mu^\varepsilon(z)\}_{z \in V'_1}$ can be uniquely determined by $\{q^\varepsilon_{xy}\}_{x, y \in V'_1}$. Fix $q'$ and $c_0$, for any $x, y, z \in V'_1$, $\mu^\varepsilon(z)$, $q^\varepsilon_{xy}$ and $g(\mu^\varepsilon)$ can be seen as functions in $\varepsilon$.

We claim:

(a) $\frac{d\mu^\varepsilon(z)}{d\varepsilon}$ is uniformly bounded for all $z \in V'_1$ and $\varepsilon > 0$ with $q^\varepsilon_{z0y} > 0$ for $z_0 \to y$ in $S'$.

(b) $\frac{d}{d\varepsilon} \int_{V_S} R(q^\varepsilon \| p) \, d\mu^\varepsilon \to -\infty$ for $\varepsilon \to 0$.

Assume temporarily these two claims holds. By (4.8), there is an $\varepsilon > 0$ with $g(\mu^0) > g(\mu^\varepsilon)$. This implies the infimum of $g$ must be attained at some $\mu \in \mathcal{P}(V_S)$ with $\mu|_1(\partial E_0) > 0$.

To prove the two claims. Note $\sum_{x \in V'_1} \mu^\varepsilon(x) q^\varepsilon_{xy} = \mu^\varepsilon(y)$, $y \in V'_1$, and $\sum_{x \in V'_1} \mu^\varepsilon(x) = c_0$. Hence we can compute out $(\mu^\varepsilon(x))_{x \in V'_1}$ by solving a linear equation,

$$\mu^\varepsilon(x) = \frac{\det(Q_x)}{\det(Q)}, \quad x \in V'_1,$$

where $Q = (q^\varepsilon_{xy})_{x, y \in V'_1}$ with

$$q^\varepsilon_{xy} = \begin{cases} \frac{q^x_{xy}}{y \neq x, x \neq z_0}, & y \neq x, x \neq z_0, \\ -1, & y = x \neq z_0, \\ 1, & x = z_0, \end{cases}$$

and $Q_x$ is the matrix obtained from $Q$ by replacing $q^\varepsilon_{x'x}$ by 0 for all $x' \neq z_0$ and replacing $q^\varepsilon_{z0x}$ by $c_0$. Therefore,

$$\mu^\varepsilon(z_0) = c_0 \cdot \frac{b}{\sum_{y \in V'_1 \setminus \{z_0\}} a_y(\varepsilon) + b}, \quad \mu^\varepsilon(x) = c_0 \cdot \frac{a_x(\varepsilon)}{\sum_{y \in V'_1 \setminus \{z_0\}} a_y(\varepsilon) + b}, \quad x \neq z_0,$$

where each $a_x$ is linear in $\varepsilon$, and $b > 0$ is constant; and further the claim (a) holds.

Note that

$$\int_{V_S} R(q^\varepsilon \| p) \, d\mu^\varepsilon = \sum_{x \in V_S} \mu^\varepsilon(x) \sum_{y \in V_S} \log \frac{q^\varepsilon_{xy}}{p_{xy}} q^\varepsilon_{xy}$$

$$= \sum_{x \in E_0 \setminus \{-1, 1\}} \mu^\varepsilon(x) \sum_{y \in E_0 \setminus \{-1, 1\}} \left( q^\varepsilon_{xy} \log q^\varepsilon_{xy} - q^\varepsilon_{xy} \log p_{xy} \right),$$

and the following facts:

- For $x \neq z_0$, $\frac{d q^\varepsilon_{xy}}{d\varepsilon} = 0$, and hence $\frac{d(q^\varepsilon_{xy} \log q^\varepsilon_{xy} - q^\varepsilon_{xy} \log p_{xy})}{d\varepsilon} = 0$.

- For $x = z_0$, $y \in V'_0$ with $z_0 \to y$,

$$\frac{d(q^\varepsilon_{xy} \log q^\varepsilon_{xy})}{d\varepsilon} = -\frac{1}{k'(z_0)} \cdot \left( \log (q^\varepsilon_{xy} \cdot \varepsilon / k'(z_0)) + 1 \right),$$

and $\frac{d(q^\varepsilon_{xy} \log p_{xy})}{d\varepsilon} = -\frac{1}{k'(z_0)} \log p_{xy}$. Thus,

$$\frac{d(q^\varepsilon_{xy} \log q^\varepsilon_{xy} - q^\varepsilon_{xy} \log p_{xy})}{d\varepsilon} \text{ is finite as } \varepsilon \to 0.$$
• For $x = z_0, y \in V_1 \setminus V_0$ with $z_0 \to y$,
\[
\frac{d(q_{xy}^\varepsilon \log q_{xy}^\varepsilon)}{d\varepsilon} = 1 + \log \varepsilon \to -\infty \text{ as } \varepsilon \to 0.
\]

Since $\frac{d(q_{xy}^\varepsilon \log p_{xy}^\varepsilon)}{d\varepsilon} = \log p_{xy}$, we have that
\[
\frac{d(q_{xy}^\varepsilon \log q_{xy}^\varepsilon - q_{xy}^\varepsilon \log p_{xy})}{d\varepsilon} \to -\infty \text{ as } \varepsilon \to 0.
\]

Combining with $\mu^0(z_0) > 0$ and the claim (a), we obtain the claim (b) immediately.

**Proof of Theorem 4.4.** Suppose firstly that for all nonempty sets $E', E_0$ with $E' \subseteq E_0 \in \mathcal{J}_0$ and $E' \cap \partial E_0 \neq \emptyset$, there is some invariant measure $\nu$ with $\text{supp}(\nu) = E'$. In this case, for any invariant measure $\mu$ and $E_0 \in \mathcal{J}_0$ with $\text{supp}(\mu) \subseteq E_0$, we have $\mu(\partial E_0 \times \{-1, 1\}) > 0$.

**Monotonicity:** For all $0 < \delta_1 < \delta_2$,
\[
R(q(x, \cdot)\|p_{E_0}(x, \cdot; \delta_2)) - R(q(x, \cdot)\|p_{E_0}(x, \cdot; \delta_1)) = \int_{V_S} \log \frac{p_{E_0}(x, y; \delta_1)}{p_{E_0}(x, y; \delta_2)} q(x, dy).
\]

Since $\mathcal{C}(\mathcal{J}_0)$ is closed, we may assume that $\alpha_c(\delta_1)$ is reached at $\mu_1 \in \mathcal{C}(\mathcal{J}_0)$, $E_0 \in \mathcal{J}_0$, $q_1 \in \mathcal{J}$, where $\mu_1 q_1 = \mu_1$, and $\text{supp}(\mu_1) \subseteq E_0$. Noting $\mu_1(\partial(E_0 \times \{-1, 1\})) > 0$, and that for $x \in \partial E_0 \times \{1, 1\}$ and $y \in E_0 \times \{-1, 1\}$, we have that
\[
\frac{p_{E_0}(x, y; \delta_1)}{p_{E_0}(x, y; \delta_2)} \leq \frac{\delta_1}{\delta_2}, \frac{\delta_2 + d - 1}{\delta_1 + d - 1} < 1,
\]

we obtain that
\[
\alpha_c(\delta_2) - \alpha_c(\delta_1) \\
\leq \int_{V_S} \left[ R(q_1(x, \cdot)\|p_{E_0}(x, \cdot; \delta_2)) - R(q_1(x, \cdot)\|p_{E_0}(x, \cdot; \delta_1)) \right] d\mu_1 \\
= \int_{V_S} \left[ \int_{V_S} \log \frac{p_{E_0}(x, y; \delta_1)}{p_{E_0}(x, y; \delta_2)} q_1(x, dy) \right] d\mu_1 \\
\leq \log \frac{\delta_1}{\delta_2} \cdot \frac{\delta_2 + d - 1}{\delta_1 + d - 1} \mu_1(\partial(E_0 \times \{-1, 1\})) < 0. \quad (4.9)
\]

Namely $\alpha_c(\delta_2) < \alpha_c(\delta_1)$.

$\lim_{\delta \to 0} \alpha_c(\delta) = +\infty$ : Here we use the original expression $\alpha_c(\delta) = \inf_{\mu \in \mathcal{C}(\mathcal{J}_0)} \Lambda_{\delta, \mathcal{J}_0}(\mu)$. Consider $\delta < 1$. For each $\mu \in \mathcal{C}(\mathcal{J}_0)$, we assume $\Lambda_{\delta, \mathcal{J}_0}(\mu)$ is attained at $(\mu_k, r_k, E_k)_{1 \leq k \leq d}$. Note
\[
R(q_k(x, \cdot)\|p(x, \cdot)) - R(q_k(x, \cdot)\|p_{E_k}(x, \cdot)) = \int_{V_S} \log \frac{p_{E_k}(x, y)}{p(x, y)} q_k(x, dy),
\]
where $\log \frac{p_{E_k}(x, y)}{p(x, y)} \leq \log \frac{\delta^d}{\delta + d - 1}$ for $\delta < 1$ and $x \in \partial(E_k \times \{-1, 1\})$. Meanwhile, for each measurable set $A$ and all $k \leq N := \sup\{k : E_k \in \mathcal{J}_0\}$,
\[
\sum_{k=1}^{N} r_k \mu_k(A) = \mu(A), \quad \mu(\partial E_N \times \{-1, 1\}) \geq \min_{z \in \text{supp}(\mu)} \mu(z) > 0,
\]

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\[ \mu_k(\partial(E_k \times \{-1,1\})) \geq \mu_k(\partial(E_N \times \{-1,1\})) \]

The third formula above follows from that \( \partial(E_N \times \{-1,1\}) \subseteq \partial(E_k \times \{-1,1\}) \cup (E_k \times \{-1,1\})^c \), and \( \text{supp}(\mu_k) = E_k \times \{-1,1\} \). Thus

\[
\Lambda_1(\mu) - \Lambda_{\delta,\mathcal{F}_0}(\mu) \leq \sum_{k=1}^N r_k \mu_k(\partial(E_k \times \{-1,1\})) \log \frac{d\delta}{\delta + d - 1} \\
\leq \mu(\partial(E_N \times \{-1,1\})) \log \frac{d\delta}{\delta + d - 1} \leq \min_{z \in \text{supp}(\mu)} \mu(z) \log \frac{d\delta}{\delta + d - 1}.
\]

Since \( \Lambda_1(\mu) \geq 0 \) and \( \log \frac{d\delta}{\delta + d - 1} \to -\infty (\delta \to 0) \), we get \( \lim_{\delta \to 0} \Lambda_{\delta,\mathcal{F}_0}(\mu) = +\infty \).

By the monotonicity of \( \alpha_c(\delta) \), \( \lim_{\delta \to 0} \alpha_c(\delta) \) exists. Assume \( \lim_{\delta \to 0} \alpha_c(\delta) < +\infty \). Then there are some sequence \( (\delta_n, \mu_n)_{n \geq 1} \subset (0,1) \times \mathcal{P}(\mathcal{V}_S) \) and some \( M > 0 \) such that \( \Lambda_{\delta_n,\mathcal{F}_0}(\mu_n) \leq M \) with \( \delta_n \downarrow 0 \). Since \( \mathcal{P}(\mathcal{V}_S) \) is compact, and \( \mathcal{C}(\mathcal{F}_0) \) is closed, we have that \( \mathcal{C}(\mathcal{F}_0) \) is compact, and there are a subsequence \( (\mu_{n_k})_{k \geq 1} \) of \( (\mu_n)_{n \geq 1} \) and a \( \mu \in \mathcal{C}(\mathcal{F}_0) \) with \( \mu_{n_k} \Rightarrow \mu \). For all \( \varepsilon > 0 \) and \( j \geq 1 \), by the lower semicontinuity of \( \Lambda_{\delta_n,\mathcal{F}_0} \), there exists some \( K_j > j \) such that for all \( i > K_j \), \( \Lambda_{\delta_n,\mathcal{F}_0}(\mu_{n_i}) \geq \Lambda_{\delta_n,\mathcal{F}_0}(\mu) - \varepsilon \); and thus by the decreasing of \( \Lambda_{\delta,\mathcal{F}_0} \) in \( \delta \),

\[
\Lambda_{\delta_n,\mathcal{F}_0}(\mu_{n_i}) \geq \Lambda_{\delta_n,\mathcal{F}_0}(\mu_{n_1}) \geq \Lambda_{\delta,\mathcal{F}_0}(\mu) - \varepsilon.
\]

Taking \( \liminf_{i \to \infty} \) and then letting \( \varepsilon \to 0 \), we obtain that

\[
\Lambda_{\delta_n,\mathcal{F}_0}(\mu) \leq \liminf_{i \to \infty} \Lambda_{\delta_n,\mathcal{F}_0}(\mu_{n_i}) \leq M, \ j \geq 1,
\]

which contradicts to \( \lim_{\delta \to 0} \Lambda_{\delta,\mathcal{F}_0}(\mu) = +\infty \) due to \( \delta_{n_j} \downarrow 0 \) \((j \uparrow \infty)\).

Now we turn to the opposite case. Choose \( E_0 \in \mathcal{F}_0 \) and any \( \mu \in \mathcal{P}(\mathcal{V}_S) \) such that

\[
\text{supp}(\mu|_1) = E', \ E' \subseteq E_0, \ E' \cap \partial E_0 = \emptyset.
\]

Then for any transition probability \( q \) with \( \mu q = \mu \),

\[
\int_{\mathcal{V}_S} R(q\|p_{E_0}) \ d\mu = \int_{\mathcal{V}_S} R(q\|p) \ d\mu. \tag{4.10}
\]

This implies \( \alpha_c(\delta) \leq \inf_{\mu q = \mu} \int_{\mathcal{V}_S} R(q\|p) \ d\mu = \Lambda_1(\mu) < \infty. \) Therefore,

\[
\alpha_c(\delta) \leq b := \inf_{(\mu, E_0 \in \mathcal{F}_0): \text{supp}(\mu|_1) \subseteq E_0, \text{supp}(\mu|_1) \cap \partial E_0 = \emptyset} \Lambda_1(\mu) < \infty.
\]

By Lemma 4.9, the infimum is attained at some \( \mu \) with \( \mu|_1(\partial E_0) > 0 \). In this case, (4.9) still holds, namely \( \alpha_c(\delta) \) is strictly decreasing in \( \delta \). Clearly \( \lim_{\delta \to 0} \alpha_c(\delta) \leq b \). By any integer \( n \geq 2 \), assume the infimum of \( \alpha_c(1/n) \) is attained at \( \mu_n \) and \( E_n \in \mathcal{F}_0 \). Set \( f_{n}(z) = \log \frac{d(z)-k(z)+k(z)\delta}{d(z)\delta} \).

By Lemma 4.9, \( \alpha_c(1/n) = \Lambda_1(\mu_n) + \int_{\partial E_n \times \{-1,1\}} f_{n} \ d\mu_n \), where \( \mu_n|_1(\partial E_0) > 0 \). Since \( S \) is finite, there is an strictly increasing subsequence \( \{n_k\}_{k \geq 1} \) of natural numbers such that \( \mu_{n_k} \Rightarrow \mu \) for some \( \mu \in \mathcal{P}(\mathcal{V}_S) \) and \( E_{n_k} \equiv E_0 \) for some \( E_0 \in \mathcal{F}_0 \), where \( \text{supp}(\mu|_1) \subseteq E_0 \). Note for all \( z \in \partial E_0 \times \{-1,1\} \), \( \lim_{j \to \infty} f_{n_j}(z) = \infty \). Then, for any \( M > 0 \), when \( k \) is large enough, \( f_{n_k}(z) \geq M, \ z \in \partial E_0 \times \{-1,1\} \); and further

\[
\lim_{\delta \to 0} \alpha_c(\delta) = \lim_{k \to \infty} \alpha_c(1/n_k) = \lim_{k \to \infty} \left\{ \Lambda_1(\mu_{n_k}) + \int_{\partial E_0 \times \{-1,1\}} f_{n_k} \ d\mu_{n_k} \right\}
\]

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\[ \geq \liminf_{k \to \infty} \{ \Lambda_1(\mu_{n_k}) + M \mu_{n_k}(\partial E_0 \times \{-1, 1\}) \} \]
\[ \geq \Lambda_1(\mu) + M \mu(\partial E_0 \times \{-1, 1\}) , \]

which forces \( \mu(\partial E_0 \times \{-1, 1\}) = 0 \), otherwise \( b \geq \lim_{\delta \to 0} \alpha_c(\delta) = \infty \). Therefore, \( \text{supp}(\mu|_1) \subseteq E_0, \text{supp}(\mu|_1) \cap \partial E_0 = \emptyset \), and \( \lim_{\delta \to 0} \alpha_c(\delta) \geq \Lambda_1(\mu) \geq b \). This completes the proof. \[ \Box \]

**Remark 4.10.** What are intrinsic causes of the decreasing of \( \alpha_c(\delta) \) in \( \delta \) and the dependence of the rate function \( \Lambda_\delta \) on \( \delta \)? Since \( p_{E_k}(x, y; \delta) \) is increasing in \( \delta \) for all \( k < d, x \in \partial(E_k \times \{-1, 1\}) \) and \( y \in E_k \times \{-1, 1\} \) with \( x \to y \), we have that \( \int_{V_S} R(q\|p_{E_k}) \, d\mu \) is strictly decreasing in \( \delta \) for a fixed sequence \( \{E_k\}_k \), all \( \mu \) with \( \mu(\partial(E_k \times \{-1, 1\})) > 0 \) and all \( k < d \). This determines the strictly decreasing of \( \alpha_c(\delta) \), the increasing of \( \P_{v_1}(C_E > n) \) and the decreasing of

\[ \sum_{k=1}^d r_k \int_{V_S} R(q_k\|p_{E_k}) \, d\mu_k \]  \hspace{1cm} (4.11)

for \( r_d \neq 1 \) in \( \delta \). From (2.6), the infimum of (4.11) is the rate function \( \Lambda_\delta \). The reasons of the dependence of \( \Lambda_\delta \) on \( \delta \) are as follows:

For \( \delta < 1 \), \( \int_{V_S} R(q\|p_{E_k}) \, d\mu > \int_{V_S} R(q\|p) \, d\mu \). Hence the less time the process staying in the subgraph results in a smaller value of (4.11). This leads the infimum to be attained at \( r_d = 1 \), and causes \( \Lambda_\delta = \Lambda_1 \). For \( \delta > 1 \), if the infimum of (4.11) is attained at some trajectory with \( r_d \neq 1 \) (staying in the subgraph for a long time) for some \( \mu \), then \( \Lambda_\delta(\mu) < \Lambda_1(\mu) \). This implies the phase transition of the rate function. In fact this assumption is true for those \( \mu \) with \( \text{supp}(\mu|_1) \neq E \). This is because \( \int_{V_S} R(q\|p_{E|\text{supp}(\mu|_1)}) \, d\mu < \int_{V_S} R(q\|p) \, d\mu \), and

\[ (4.11) = \int_{V_S} R(q\|p_{E|\text{supp}(\mu|_1)}) \, d\mu \text{ when taking } r_{|\text{supp}(\mu|_1)} = 1, E_{|\text{supp}(\mu|_1)} = \text{supp}(\mu|_1). \]

Is there any \( \mu \) such that the infimum of (4.11) is attained at some trajectory with \( r_d = 1 \)? The answer is yes. Note the invariant probability of \( p \), i.e., \( \mu_0 \), makes \( \Lambda_\delta(\mu_0) = \Lambda_1(\mu_0) = 0 \) and the infimum be attained at \( r_d = 1 \). We conjecture that there is a neighbourhood of \( \mu_0 \) with the infimum of (4.11) being attained at \( r_d = 1 \) (i.e., \( \Lambda_\delta(\mu) = \Lambda_1(\mu) \)) if and only if \( \mu \) is in this neighbourhood. See Example 5.7 and Figure 10 for an intuitive view.

## 5 LDP for ORRW on trees and some examples

In this section, we show a simpler expression of the rate function on finite trees, and derive the rate function on star-shaped graphs and the Path \( \{0, 1, \ldots, d\} \).

### 5.1 Finite trees

Let root \( g \) of finite tree \( G = (V, E) \) be the starting point \( v_1 \) of the \( \delta \)-ORRW \( X = (X_n)_{n \geq 0} \). In this setting, use \( \mathcal{T} \) to denote the set of all transition probabilities on \( V \). Write \( e^+ \) and \( e^- \) for endpoints of edge \( e \in E \) with \( e^- \) being the parent of \( e^+ \). Note that the transition probability at time \( n \) is uniquely determined by \( X_n \) and empirical measure \( L_n \). Since edge \( e \) has been traversed if and only if \( e^+ \) has been traversed, the transition probability of \( X \) can be expressed by the following definition.
Definition 5.1.

\[ \hat{p}_\mu(x, dy) = \hat{p}_\mu(x, dy; \delta) := \begin{cases} \frac{g(\delta, \mu, y)}{\sum_{z \sim x} g(\delta, \mu, z)}, & y \sim x, \\ 0, & \text{otherwise}, \end{cases} \]

where \( \mu \) is a measure on \( V \) and \( g(\delta, \mu, y) := \delta \mathbf{1}_{\{\mu(y) \neq 0\}} + \mathbf{1}_{\{\mu(y) = 0\}} \).

For convenience, replace \( \hat{p}_\rho G_r \) by \( \hat{p}_G \), where \( G' \) is a subgraph of \( G \) and \( \rho G_r \) is a measure such that \( L G_r(\{v\}) = 1 \) (resp. 0) if \( v \in G' \) (resp. \( v \notin G' \)). Clearly, \( \hat{p}_G \) is the transition probability of \( X \) at time \( n \), where \( G_n \) is the subtree induced by \( \{X_m : m \leq n\} \).

Let \( \mathcal{S} \) be the collection of all sequences of subtrees \( \{G_j\}_{0 \leq j \leq d} \) such that \( G_j \subset G_{j+1}, \) \( |G_j| = j + 1, G_0 = \{\emptyset\}, \) and \( G_{j+1} \setminus G_j \) is connected to \( \partial G_j \), where \( d = |E| \). Set

\[ \mathcal{S}(\mu) := \left\{ (\mu_k, r_k, G_k)_{1 \leq k \leq d} : \supp(\mu_k) = G_k, \sum_{k=1}^{d} r_k \mu_k = \mu, \{G_k\}_{1 \leq k \leq d} \in \mathcal{S} \right\} \]

to be a family of \( (\mu_k, r_k, G_k)_{1 \leq k \leq d} \), where each \( \mu_k \) is a probability measure. We obtain another expression of the rate function of the LDP for empirical measure process \( (L_n)_{n \geq 1} \) as follows.

**Theorem 5.2.** The empirical measure \( L_n = \frac{1}{n} \sum_{i=0}^{n-1} \delta_{X_i} \) of the \( \delta \)-ORRW \( X \) on \( G \) satisfies an LDP with the rate function

\[ \hat{I}_\delta(\mu) = \inf_{(\mu_k, r_k, G_k) \in \mathcal{S}(\mu), q_k \in \mathcal{F}, \mu_k q_k = \mu_k} \sum_{k=1}^{d} \int_{V} R(q_k \| \hat{p}_k) \, d\mu_k. \]

In addition, each \( q_k \in \mathcal{F} \) is the unique solution to \( \mu_k q_k = \mu_k \) for given \( \mu_k \) and fixed \( \{G_j\}_j \) (see Proposition 5.4).

**Remark 5.3.** The expression of the rate function in Theorem 5.2 is simpler than (2.6) since we can compute directly on \( G \) rather than the directed graph \( S \) constructed by \( G \). This is feasible only on trees since trees have no circle. Meanwhile, the uniqueness of \( q \) may not exist for some \( \mu \) on the cyclic graph \( G \). For instance, consider complete graph on \( \{1, 2, 3\} \) and \( \mu \) with \( \mu(i) = 1/3, i = 1, 2, 3 \). Then \( q(i, i+1) = x \) is the solution to \( \mu q = \mu \) for any \( x \in [0, 1] \), where \( q(3, 4) := q(3, 1) \).

As the result of the similarity of \( \hat{p}_\mu \) and \( p_\mu \) (see Definitions 5.1 and 2.1), we can directly prove the LDP for the empirical measure process \( (L_n)_{n \geq 1} \) similarly to Theorem 2.3.

**Proposition 5.4.** For any measure \( \mu_{V'} \) supported on \( V' \subseteq V \), \( \mu_{V'} q_{V'} = \mu_{V'} \) has a unique solution \( q_{V'} \in \mathcal{F} \).

**Proof.** Denote by respectively \( f(v) \) and \( c(v) \) the parent (if exists) and the child set (if exists) of any vertex \( v \). Fix \( \mu_{V'} \in \mathcal{P}(V) \) supported on \( V' \), one can determine \( q_{V'} \) by induction from leaves. For each leaf \( v \), we have \( q_{V'}(f(v)) = 1 \) and \( q_{V'}(f(v), v) \mu_{V'}(f(v)) = \mu_{V'}(v) \). The second identity can also be written as \( q_{V'}(f(v), v) = \mu_{V'}(v) \mu_{V'}(f(v)) \). For some \( v \), if \( q_{V'}(v, u) \) and \( q_{V'}(u, v) \) are determined for all \( u \in c(v) \), then it immediately follows that when \( f(v) \) exists, \( q_{V'}(v, f(v)) = 1 - \sum_{u \in c(v)} q_{V'}(v, u) \). Meanwhile, when \( f(v) \) and \( c(v) \) exist,

\[ \mu_{V'}(v) = \mu_{V'}(f(v)) q_{V'}(f(v), v) + \sum_{u \in c(v)} \mu_{V'}(u) q_{V'}(u, v), \]

which determines \( q_{V'}(f(v), v) \). By induction, \( q_{V'} \) can be determined by \( \mu_{V'} \).

From Theorem 5.2, we can simplify the expression of \( \alpha_1^l(\delta) \) as follows.
Proposition 5.5. $\alpha_1^1(\delta)$ can be expressed as

$$\inf_{(\mu, G_0 \subseteq G, q): G_0 \neq G, \supp(\mu) \subseteq G_0, \mu q = \mu} \int_V R(q \| \tilde{p}_{G_0}) \, d\mu. \quad (5.1)$$

Remark 5.6. We can also write the rate function in Theorem 5.2 as

$$I_\delta(\mu) = \inf_{(\mu_k, r_k, G_k) \in \mathcal{A}(\mu)} \left[ - \inf_{u_k \in \mathcal{U}_1} \sum_{k=1}^d r_k \int_V \log \frac{p_{E_k} u_k}{u_k}(x) \, d\mu_k \right],$$

where $\mathcal{U}_1$ is the collection of all positive functions $u$ on $V$, $p_{E_k} f(x) := \int_V f(y) p_{E_k}(x, dy)$.

Actually, by the variational representation of the relative entropy and the minimax lemma [20, Appendix 2, Lemma 3.3], we have that

$$\inf_{q \in \mathcal{F}: \mu q = \mu} \int_V R(q(x) \| p(x, \cdot)) \, d\mu = \inf_{q \in \mathcal{F}: \mu q = \mu} \int_V \sup_{\phi \in C} \{Q(\phi) - \log P(e^\phi)\} \, d\mu,$$

where $P f(x) = \int_V f(y) p(x, dy)$, $Q f(x) = \int_V f(y) q(x, dy)$, and $C$ is the set of all functions on $V$. Taking $u = e^\phi$, we obtain that

$$\inf_{q \in \mathcal{F}: \mu q = \mu} \int_V R(q(x) \| p(x, \cdot)) \, d\mu = - \inf_{u \in \mathcal{U}_1} \int_V \log \frac{P u}{u}(x) \, d\mu,$$

by which we immediately get the required expression.

Note that this expression is similar to that given in Donsker and Varadhan [12]. Can we use their method to deal with our problem? Actually the method is not comfortable since it requires that the process is Markovian on the whole trajectory, and is hard to be separated as what we do in our method.

5.2 Star-shaped graphs

To begin we take a look at the simplest star-shaped graph.

![Figure 9](image-url) Figure 9: (a) stands for the simplest star-shaped graph $\{0, 1, 2\}$ with edges $\{0, 1\}, \{0, 2\}$. (b) indicates a star-shaped graph with $d$ edges.
Example 5.7. The empirical measure process of the $\delta$-ORRW on star-shaped graph $\{0, 1, 2\}$ with edge set $\{\{0, 1\}, \{0, 2\}\}$ starting at 0 (see Figure 9) satisfy an LDP with the good rate function

$$I_\delta(\nu) = \begin{cases} R(\nu\|\mu) - R(\nu\|T(\nu)), & \nu(0) = 1/2, \\ \infty, & \text{otherwise,} \end{cases}$$

where $\mu(0) = 2\mu(1) = 2\mu(2) = 1/2$, and

$$T(\nu)(i) = \begin{cases} (\nu(i) \lor \delta - 1/4\delta) \land \delta + 1/4\delta, & i \neq 0, \\ 1/2, & i = 0. \end{cases}$$

The proof of this example is given in Appendix 7.7. Notice that $I_\delta(\nu)$ is not differentiable in $\delta$ at $\nu$ with $\nu(1) = 0$ or $\nu(2) = 0$; and $I_\delta(\nu)$ is a line below $I_1(\nu)$ for $\nu(1) \in [0, (\delta - 1)/4\delta] \cup [(\delta + 1)/4\delta, 1/2]$ and the same as $I_1(\nu)$ for $\nu(1) \in [(\delta - 1)/4\delta, (\delta + 1)/4\delta]$. Moreover, critical exponent for exponential integrability of cover time $C_E$ is $\alpha_1^c(\delta) = 1/2 \log \frac{\delta + 1/\delta}{\delta}$, see Figure 10.

![Figure 10](image)

Figure 10: In (a), the dashed line is the rate function of the SRW and the solid line is that of the ORRW for $\delta = 2$. (b) is a sketch map of $\alpha_1^c$ for the ORRW on the simplest star-shaped graph $\{0, 1, 2\}$ with edges $\{\{0, 1\}, \{0, 2\}\}$. $E^{\alpha_1^c, E} = \infty$ when the site is above $\alpha_1^c$ and $< \infty$ when the site is below $\alpha_1^c$.

Example 5.8. The empirical measure process of the $\delta$-ORRW on star-shaped graph $\{0, 1, \ldots, d\}$ with edges $\{\{0, i\}, i = 1, \ldots, d\}$ starting at 0 satisfies an LDP with the good rate function

$$I_\delta(\nu) = \inf_{(\nu_k, r_k, s_k) \in \mathcal{B}(\nu)} \sum_{k=1}^d r_k \left[ \sum_{l=1}^k \nu_k(s_l) \log 2\nu_k(s_l) + \frac{1}{2} \log \frac{k(\delta - 1) + d}{\delta} \right] \text{ for } \nu(0) = 1/2,$$

and $I_\delta(\nu) = \infty$ otherwise, where

$$\mathcal{B}(\nu) = \left\{ (\nu_k, r_k, s_k)_k : \sum_{k=1}^d \nu_k r_k = \nu, \sum_{k=1}^d r_k = 1, r_k \geq 0, \nu_k(0) = 1/2, \right. \left. \text{supp}(\nu_k) = \{0, s_1, \ldots, s_k\}, \{s_1, \ldots, s_d\} \text{ is rearrangement of } \{1, \ldots, d\} \right\}.$$
Proof. For any subgraph \( \{0, s_1, \ldots, s_k\} \), since \( \nu_kq_k = \nu_k \), and \( \text{supp}(\nu_k) = \{0, s_1, \ldots, s_k\} \), we have that
\[
\nu_k = \frac{1}{2}, \quad q_k(x) = 2\nu_k(x), \quad \hat{p}_{G_k}(0, y) = \frac{\delta}{k(\delta - 1) + d} \text{ for } y \in \{s_1, \ldots, s_k\},
\]
which implies that \( \int_{\nu_k} R(q_k \| \hat{p}_{G_k}) \nu_k(dx) = \sum_{l=1}^{k} \nu_k(s_l) \log 2\nu_k(s_l) + \frac{1}{2} \log \frac{k(\delta - 1) + d}{\delta} \). By Theorem 5.2, we complete immediately the proof.

5.3 Path \( \{0, 1, \ldots, d\} \)

Firstly we show the rate function of the ORRW on the Path \( \{0, 1, 2\} \) with edges \( \{0, 1\}, \{1, 2\} \). Note that this graph is actually the simplest star-shaped graph mentioned in the previous subsection. However, they start from different vertices, which leads to a difference between their rate functions. Here we give the rate function of this process as the following.

Example 5.9. The empirical measure process of the \( \delta \)-ORRW on Path \( \{0, 1, 2\} \) with edges \( \{0, 1\}, \{1, 2\} \) starting at 0 satisfies an LDP with the good rate function
\[
I_{\delta}(\nu) = \begin{cases} 
R(\nu \| \mu) - R(\nu \| T'(\nu)), & \nu(1) = \frac{1}{2}, \\
\infty, & \text{otherwise}, 
\end{cases}
\]
where \( \mu(1) = 2\mu(0) = 2\mu(2) = \frac{1}{2} \), and
\[
T'(\nu)(i) = \begin{cases} 
\nu(0) \wedge \frac{\delta + 1}{4\delta}, & i = 0, \\
\frac{1}{2}, & i = 1, \\
\nu(2) \vee \frac{\delta - 1}{4\delta}, & i = 2.
\end{cases}
\]

The proof of this example is given in Appendix 7.7. The rate function in Example 5.9 is different from that in Example 5.7 if \( \delta > 1 \) and \( \nu(0) < \frac{\delta - 1}{4\delta} \), while it is equal to \( I_1 \) at the same time. See Figure 11.

![Figure 11](image_url)

Figure 11: (a) is the rate function of the ORRW starting at 0, while (b) is that of starting at 2, where the dashed line is the rate function of the SRW, and the solid line is that of the ORRW for \( \delta = 2 \). We see that those two rate functions are symmetric and equal to each other when \( \nu(0) \in ([\delta - 1)/4\delta, (\delta + 1)/4\delta] \).
Remark 5.10. From the expression of the rate function of the SRW on graphs, i.e.,

\[ I_1(\nu) = \inf_{q \in \mathcal{F}, \nu q = \nu} \int_V R(q\|p) \, d\nu, \]

which is not influenced by selection of the starting point. However, comparing Example 5.7 with Example 5.9, we figure out that different starting points may make the rate function to be different if \( \delta > 1 \). Actually, in addition, we point out that for \( \delta > 1 \), the rate functions of ORRWs on the same graph starting from different vertices are all different, i.e., for each graph \( G \) and all vertices \( v_1 \neq v_2 \) there exists a measure \( \mu \) such that \( I_\delta(v_1, \mu) \neq I_\delta(v_2, \mu) \) if \( \delta > 1 \), where \( I_\delta(u, \cdot) \) is the rate function for the ORRW with the starting point \( u \).

Here we explain the reason for the claim above. For \( \delta > 1 \), fix the starting point \( v_0 \), and set it to be the root. When we choose some measure supported on a subgraph \( G_0 \), the infimum in the display of the rate function will be attained at some \( \{G_k\}_k \) with \( G_{|F_{v_0}(G_0)|-1} = F_{v_0}(G_0) \) and \( r_k = 0 \) for \( k > |F_{v_0}(G_0)| \), where \( F_{v_0}(G_0) \) is the subgraph with the vertex set \( \{v : v \in G_0 \text{ or } v \text{ is the ancestor of some } u \in G_0\} \). Specifically, for two different starting point \( v_0, u_0 \), consider \( \nu \) supported on \( \{v_0, v_1\} \), where \( v_1 \sim v_0 \), and \( v_1 \neq u_0 \). The rate function of the ORRW starting at \( v_0 \) is actually smaller than that of the ORRW starting at \( u_0 \). Actually, the infimum of the former is attained at \( G_1 = \{v_0, v_1\}, r_1 = 1 \) (the reason is stated in the proof of the monotonicity in Lemma 3.8), while that of the latter is attained at \( G_N = F_{u_0}(\{v_0, v_1\}), r_N = 1 \) with \( N = |F_{u_0}(\{v_0, v_1\}) - 1| \). The second attainment condition here is actually one of the accessible conditions of the infimum in the display of the rate function of the first case. This condition is different from the first attainment condition, see an example in Figure 12.

![Figure 12: The solid line is the graph \( F_{u_0}(\{v_0, v_1\}) \).](image)

Now we consider the rate function of the ORRW on the Path \( \{0, 1, \ldots, d\} \) with edges \( \{i, i + 1\}, i = 0, \ldots, d - 1 \). Note that \( \mathcal{G} = \{\{0, 1, \ldots, k\} : 1 \leq k \leq d\} \) in this case. For each subgraph \( G_k = \{0, 1, \ldots, k\} \) and the fixed \( \mu_k \) on \( G_k \), we can determine the corresponding \( q_k \) by induction from the leaf 0 shown in Lemma 5.4. In fact,

\[ q_k(i, i - 1) = \frac{\sum_{j=1}^{i} (-1)^{j-1} \mu_k(i - j)}{\mu_k(i)}, \quad q_k(i, i + 1) = 1 - q_k(i, i - 1). \]

Note that \( q_k(k, k - 1) = 1 \). This implies that \( \sum_{j=0}^{k} (-1)^j \mu_k(k - j) = 0 \). Hence we obtain

\[ \int_V R(q_k\|\hat{p}_{G_k}) \, d\mu_k \]
\[
= \sum_{i=1}^{k-1} \mu_k(i) \left[ \log 2 \frac{\sum_{j=1}^{i}(1)^{j-1}\mu_k(i-j)}{\mu_k(i)} + \log \left( 2 - 2 \frac{\sum_{j=1}^{i}(1)^{j-1}\mu_k(i-j)}{\mu_k(i)} \right) \right] + \mu_k(k) \log \frac{1+\delta}{\delta},
\]
where \(\sum_{j=0}^{k}(1)^{j}\mu_k(k-j) = 0\). It shows the following corollary.

**Example 5.11.** The empirical measure process of the \(\delta\)-ORRW on the Path \(\{0, 1, \ldots, d\}\) with edges \(\{i, i+1\}, i = 0, \ldots, d-1\) starting at 0 satisfies an LDP with the good rate function

\[
I_\delta(\mu) = \inf_{(\mu_k, \lambda_k) \in \mathcal{E}(\mu)} \left\{ \sum_{k=1}^{d} \lambda_k \sum_{i=1}^{k-1} \mu_k(i) \left[ \log 2 \frac{\sum_{j=1}^{i}(1)^{j-1}\mu_k(i-j)}{\mu_k(i)} + \log \left( 2 - 2 \frac{\sum_{j=1}^{i}(1)^{j-1}\mu_k(i-j)}{\mu_k(i)} \right) \right] + \mu_k(k) \log \frac{1+\delta}{\delta} I_{\{k<d\}} \right\},
\]
where

\[
\mathcal{E}(\mu) = \left\{ (\mu_k, \lambda_k)_k : \sum_{k=1}^{d} \mu_k \lambda_k = \mu, \sum_{k=1}^{d} \lambda_k = 1, \sum_{j=0}^{k}(1)^{j}\mu_k(k-j) = 0, \text{supp}(\mu_k) = \{0, \ldots, k\}, \lambda_k \geq 0 \right\}.
\]

Furthermore, we can compute \(\alpha^1_\delta(\delta)\) on a specific path \(\{0, 1, 2, 3\}\) with edges \(\{i, i+1\}, i = 0, 1, 2\). Assume \(X\) starts from 1. By Proposition 4.2, denoted by \(x := q(1, 0)\) the transition probability from 1 to 0, we have that

\[
\alpha^1_\delta(\delta) = \min \left\{ \inf_{\mu, q: \text{supp}(\mu) \subseteq \{0, 1, 2\}} \int \mathcal{V}_S R(q||\hat{p}(0,1)) \ d\mu, \inf_{\mu, q: \text{supp}(\mu) \subseteq \{0, 1\}} \int \mathcal{V}_S R(q||\hat{p}(0,1)) \ d\mu, \right\}
\]
\[
= \min \left\{ \inf_{x \in [0, 1]} \left\{ \frac{1}{2} x \log 2x + \frac{1}{2} (1-x) \log 2(1-x) + \frac{1-x}{2} \log \frac{1+\delta}{\delta} \right\}, \right\}
\]
\[
= \min \left\{ \frac{1}{2} \log \frac{2+2\delta}{1+2\delta}, \log \frac{1+\delta}{\delta}, \log \frac{1+\delta}{\delta} \right\} = \frac{1}{2} \log \frac{2+2\delta}{1+2\delta}.
\]
Similarly we can get that \(\alpha^1_\delta(\delta) = \frac{1}{2} \log \frac{2+2\delta}{1+2\delta}\) when \(X\) starting from any vertex in \(\{0, 1, 2, 3\}\). See Figure 13.

**Remark 5.12.** On Path \(\{0, 1, 2, 3\}\), critical exponent \(\alpha^1_\delta\) is independent of the starting point of the \(\delta\)-ORRW \(X = (X_n)_{n \geq 0}\), while critical exponent for stopping time

\[
T(\mathcal{L}_0) = \inf \{n : \exists m_1, m_2 \leq n, X_{m_1} = 1, X_{m_2} = 3\}
\]
when \(X\) starting from 0 and 3 are different. We think \(\alpha^1_\delta\) depends on the starting point of the \(\delta\)-ORRW on some asymmetric graphs.
6 Conclusions

We proved that the empirical measure process \((L_n)_{n \geq 1}\) of the \(\delta\)-ORRW on finite connected graphs satisfies an LDP with the good rate function \(I_\delta\) represented by a variation formula. Through the expressions of the rate functions for finite trees and general finite connected graphs, we obtained many interesting properties on the rate function \(I_\delta\), though the expressions generally are not simple. Besides the mentioned LDP for the empirical measure process, we also studied critical exponent \(\alpha_1^1(\delta)\) for exponential integrability of some stopping times such as edge cover time \(C_E\), and got a few nice properties on the critical exponent \(\alpha_1^1(\delta)\). Refer to Theorems 2.4, 2.6 and 2.9.

In Section 5, we showed the explicit expressions and some analysis properties, such as the convexity and the differentiability, of the rate function and the critical exponent for exponential integrability of the cover time on star-shaped graphs and Path graphs. We wonder that these properties of the rate function and the critical exponent hold on general graphs.

We gave Corollary 4.8, a comparison theorem of a hitting time for different \(\delta\) by some delicate exponential estimates in Theorem 2.8. We are interested in some stronger comparison theorems to answer some questions on long time behaviors of the ORRWs. For instance, on infinite connected graph \(G\), if \(\delta_1 < \delta_2\) and the \(\delta_1\)-ORRW is recurrent, is the \(\delta_2\)-ORRW recurrent?

Finally, we pointed out that the rate function \(I_\delta\) with \(\delta \neq 1\) is distinct for different starting points of the \(\delta\)-ORRW and structure of the graphs (Remark 5.10), while \(I_1\) is independent of starting points of the SRW. It is an interesting question whether we can run an ORRW on an unknown finite graph or network to reveal some information of the structure of this graph or network.

7 Appendices

7.1 Notations

- \(G = (V, E)\): finite connected graph with vertex set \(V\) and edge set \(E\);
- \(u \sim v\): \(u, v \in V\) are adjacent;
• $S$: a directed graph with vertex set $V_S = E \times \{-1, 1\}$ constructed from $G$;
• $z \to z'$: there is a directed edge from $z$ to $z'$ for any $z, z' \in E \times \{-1, 1\}$;
• $\mathcal{P}(T)$: the set of all probability measures on a topology space $T$, equipped with the weak convergence topology;
• $\Rightarrow$: the weak convergence for probabilities;
• $\hat{\delta}_x$: the Dirac measure centered at $x$ on a topology space;
• $X = (X_n)_{n \geq 0}$: $\delta$-ORRW on $G$; $(L_n)_{n \geq 1}$: empirical measure process of $X$;
• $Z = (Z_n)_{n \geq 1}$: $\delta$-ORRW on $S$; $(L^n)_{n \geq 1}$: empirical measure process of $Z$;
• $I_\delta(\mu)$: the rate function of LDP for empirical measure process $(L_n)_{n \geq 1}$ on $\mathcal{P}(V)$;
• $\mathcal{E}$: the collection of subset sequences $\{E_k\}_{1 \leq k \leq d}$ with $E_k \subseteq E$, where $d = |E|$;
• $\mathcal{S}$: the collection of edge sets of all connected subgraphs of $G$ that contain the starting point of the $\delta$-ORRW $X$;
• $\mathcal{S}_0$: a non-empty decreasing subset of $\mathcal{S}$ (here decreasing means for all $E_1, E_2 \in \mathcal{S}$ with $E_1 \subseteq E_2$, if $E_2 \in \mathcal{S}_0$, then $E_1 \in \mathcal{S}_0$);
• $\Lambda_{\delta, \mathcal{S}_0}$: the general rate function on $\mathcal{P}(V_S)$ conditioned on the starting point $z \in V_S$;
• $\mathcal{A}_z(\mu, \mathcal{S}_0)$: the subset of $\mathcal{A}(\mu, \mathcal{S}_0)$ satisfying $\{E_k\}_{1 \leq k \leq d} \in \mathcal{E}_z$ and $z \in V_S$;
• $\Xi_{\delta, \mathcal{S}_0}$: the collection of probability measures, whose first marginal measures are supported on some edge set in $\mathcal{S}_0$;
• $\{\nu^n_j\}_{0 \leq j \leq n-1}$: the admissible control sequence;
• $\mathbb{P}_z$: the probability generated by $\{\nu^n_j\}_{0 \leq j \leq n-1}$ conditioned on the starting point $z \in V_S$;
• $\mathbb{E}_z$: expectation of $\mathbb{P}_z$;
• $\overline{(Z^n_j, \mathcal{L}^n_j)}_{0 \leq j \leq n}$: the process generated by $\{\nu^n_j\}_{0 \leq j \leq n-1}$ and its empirical measure on $S$;
• $V^n_{\mathcal{S}_0}(z)$: minimal cost function;
• $W^n_{\mathcal{S}_0}(z)$: $-\frac{1}{n} \log \mathbb{E}_z \{\exp[-nh(\mathcal{L}^n)]\mathbf{1}_{\{\mathcal{L}^n \in \mathcal{C}(\mathcal{S}_0)\}}\}$;
• $C_E$: edge cover time of the $\delta$-ORRW $X$;
• $\alpha_{\mathcal{E}}^1(\delta)$: critical exponent for exponential integrability of $C_E$;
• $T_{\mathcal{S}_0}$: general stopping time.
7.2 Exponential integrability of cover time

Definition 7.1. For two random variables $X, Y$, call $X \lesssim Y$ if $\mathbb{P}(X \geq t) \leq \mathbb{P}(Y \geq t)$, $t \in \mathbb{R}$.

Note that $\mathbb{E}f(X) \leq \mathbb{E}f(Y)$ if $X \lesssim Y$ and $f$ is differentiable and increasing since

$$\mathbb{E}f(X) = \int_{-\infty}^{\infty} f'(t)\mathbb{P}(X > t) \, dt \leq \int_{-\infty}^{\infty} f'(t)\mathbb{P}(Y > t) \, dt = \mathbb{E}f(Y).$$

Proposition 7.2. For the $\delta$-ORRW on finite connected graph $G = (V, E)$ with $|E| = d \geq 1$, there exists some $\alpha > 0$ such that $\mathbb{E}e^{\alpha C_E} < \infty$.

Proof. For graph $G = (V, E)$, we give an order of directed edges $\vec{e}_1, \ldots, \vec{e}_{2d}$, where $\vec{e}_i = e_i^- e_i^+$, $e_i^-$ and $e_i^+$ are starting point and terminal point of $\vec{e}_i$ respectively. Denote by

$$\tilde{\tau}_k = \inf \left\{ t > \tilde{\tau}_{k-1} : \vec{X}_{t-1, t} = \vec{e}_k \right\}, \quad 1 \leq k \leq 2d,$$

a sequence of renewal times, where $\tilde{\tau}_0 = 0$, and $\tilde{\tau}_{2d+1} = \infty$. Then $C_E \leq \tilde{\tau}_{2d}$. Since it is hard to estimate $\tilde{\tau}_{2d}$ directly, we need to find some random variables to dominate $\tilde{\tau}_{2d}$. In fact, there exist $p_0 \in (0, 1)$, integer $n_0 \geq 1$ and an i.i.d. sequence $\{N_k\}_{1 \leq k \leq 2d}$ satisfying geometric distribution with parameter $p_0$ such that

$$\tilde{\tau}_k \lesssim n_0 \sum_{j=1}^{k} N_j. \quad (7.1)$$

Since $f(x) = e^{\alpha x}$ is non-negative, differentiable and increasing,

$$\mathbb{E}e^{\alpha C_E} \leq \mathbb{E}e^{\alpha n_0 \sum_{k=1}^{2d} N_k} = \left( \sum_{k=0}^{\infty} [e^{\alpha n_0 (1 - p_0)]^k p_0} \right)^{2d},$$

which is finite if and only if $e^{\alpha n_0 (1 - p_0)} < 1$. That is, for all $\alpha < \frac{1}{n_0} \log \frac{1}{1 - p_0}$, $\mathbb{E}e^{\alpha C_E} < \infty$.

Now we prove (7.1). First we determine $n_0$ and $p_0$. Noting that there exist some $n_0, \varepsilon_0 > 0$ such that for every $v, u \in V$ and stopping time $T$, there is some $n \leq n_0$ with

$$\mathbb{P}(X_{n+T} = u| X_T = v, \mathcal{F}_T) \geq \varepsilon_0.$$  

Actually, we can choose $n_0 = |V|$ and $\varepsilon_0 = \left( \frac{\delta \lambda_1}{(|V|-1)(\delta v^1)} \right)^{n_0}$. Then for the shortest path $v v_1 \ldots v_{n-1} u$ from $v$ to $u$, we have $n \leq n_0$, and

$$\mathbb{P}(X_{n+T} = u| X_T = v, \mathcal{F}_T) \geq \mathbb{P}(X_{n+T} = u, X_{n+T-1} = v_{n-1}, \ldots, X_{T+1} = v_1| X_T = v, \mathcal{F}_T) \geq \varepsilon_0.$$  

Set $p_0 = \varepsilon_0 \cdot \frac{\delta \lambda_1}{(|V|-1)(\delta v^1)}$, then for $u_1 \sim u_2$ and some $n \leq n_0$,

$$\mathbb{P}(X_{n+T+1} = u_2, X_{n+T} = u_1| X_T = v, \mathcal{F}_T) > p_0. \quad (7.2)$$

Next we construct a process $(X'_n)_{n \geq 0}$. Denote by $\tilde{\tau}'_k = \inf \left\{ t \geq \tilde{\tau}'_{k-1} : \vec{X}_{t-1, t} = \vec{e}_k \right\}$ with $\tilde{\tau}'_0 = 0$, $\tilde{\tau}'_{2d+1} = \infty$. To construct $(X'_n)_{n \geq 0}$ such that

1. $\tilde{\tau}_k \leq \tilde{\tau}'_k, \quad 1 \leq k \leq 2d,$
2. $\tilde{\tau}'_k \lesssim n_0 \sum_{j=1}^{k} N_j, \quad 1 \leq k \leq 2d,$
which can imply (7.1). Concretely, coupling a new stochastic process $X'_n := X_{K_n}$, $n \geq 0$; where $K_0 = 0$, and if $\tilde{\tau}_{k-1} \leq K_n < \tilde{\tau}_k$, then

$$K_{n+1} = \begin{cases} K_n, & X_{K_n} = \epsilon_k^{-} \text{ and } n_0 \not\mid n, \\ K_n + 1, & \text{otherwise.} \end{cases}$$

Clearly the property (1) holds. Note that $(\tilde{\tau}'_k - \tilde{\tau}'_{k-1})/n_0$ is a positive integer. By induction of $m$ and (7.2), we obtain that

$$\mathbb{P}\left(\frac{\tilde{\tau}'_k - \tilde{\tau}'_{k-1}}{n_0} \geq m | \mathcal{F}_{\tilde{\tau}'_{k-1}}\right) \leq (1 - p_0)^m \text{ for all } k = 1, \ldots, 2d.$$ 

Thus,

$$\mathbb{P}\left(\frac{\tilde{\tau}'_k - \tilde{\tau}'_{k-1}}{n_0} \geq m_k, k = 1, 2, \ldots, 2d\right) \leq (1 - p_0)^{m_1 + \cdots + m_k} = \mathbb{P}(N_k \geq m_k, k = 1, \ldots, 2d).$$

Namely $\tilde{\tau}'_k \leq n_0 \sum_{j=1}^k N_j$, the property (2) holds. $\blacksquare$

### 7.3 Proof of Lemma 2.16

Set $L^n_0 = 0$ and $L^n_{j+1} = L^n_j + \frac{1}{n} \hat{\delta} Z_j$ for $j < n$, specifically $L^n := L^n_0$. Since $Z_n$ has the transition probability $p_{\nu \gamma}$ with $\mathcal{E}_n = \{e \in E : \exists m \leq n, Z_m = \nu\}$ (see (2.1)), $(Z_j, L^n_j)_{j=0,1,\ldots,n}$ is a homogeneous Markov process with the transition probability

$$\mathbb{P}\left( (Z_{i+1}, L^n_{i+1}) \in dz \times d\nu \mid (Z_i, L^n_i) = (z_1, \mu) \right) = p_{\mu + \hat{\delta} z_1}(z_1, d\nu) \cdot \hat{\delta}^{-1} (d\nu),$$

Denote by

$$W^n_{\gamma_0}(i, z, \mu) := \frac{1}{n} \log \mathbb{E}_{i, z, \mu} \left\{ \exp\left[ -n h(L^n) \right] 1_{\{L^n \in \mathcal{C}(\mathcal{A}_0)\}} \right\},$$

where $\mathbb{E}_{i, z, \mu}$ is the expectation conditioned on $(Z_i, L^n_i) = (z, \mu)$. We simply write $W^n_{\gamma}(i, z, \mu)$ as $W^n(i, z, \mu)$.

Our proof combines the ideas of Theorems 1.5.2 and 8.2.1 in [14]. Similarly, we need to verify the terminal condition and the attainment condition of the dynamic programming equation firstly. However, the terminal condition in our case is not a bounded measurable function instead, which makes it difficult to prove. We do not know whether this result holds or not for all measurable $h$ in the terminal condition. In our proof, we let the function be $h + \infty 1_{\mu \in \mathcal{C}(\mathcal{A}_0)^c}$ for some bounded measurable function $h$. In order to meet the terminal condition

$$W^n_{\gamma_0}(n, z, \mu) = h + \infty 1_{\mu \in \mathcal{C}(\mathcal{A}_0)^c},$$

we have to give a different attainment condition in Lemma 7.4. To begin we verify that $W^n_{\gamma_0}$ is the solution of the dynamic programming equation.

#### Lemma 7.3. If $(z, \mu) \in V_S \times \mathcal{P}(V_S)$ belongs to the set

$$S := \left\{ (z, \mu) : \mu + \frac{1}{n} \hat{\delta} z \in \mathcal{C}(\mathcal{A}_0) \right\},$$

(7.4)
then \( W^n_{\mathcal{Z}_0}(i, z, \mu) \) satisfies the following equation on \( S' := \{ z' \in V_S : (z', \mu + \frac{1}{n} \delta_z) \in \mathbb{S} \} \):

\[
W^n_{\mathcal{Z}_0}(i, z, \mu) = \inf_{\nu \in \mathcal{P}(V_S) : \text{supp}(\nu) \subseteq S'} \left\{ \frac{1}{n} R(\nu(\cdot)p_{\mu+\delta_z}(z, \cdot)) + \int_{V_S} W^n_{\mathcal{Z}_0}\left(i + 1, z', \mu + \frac{1}{n} \delta_z\right) \nu(dz') \right\}, \tag{7.5}
\]

where the infimum is attained at the unique \( \nu^0_i \) denoted by

\[
\nu^0_i(A|z, \mu) := \frac{\int_A \exp[-nW^n_{\mathcal{Z}_0}(i + 1, z', \mu + \frac{1}{n} \delta_z)p_{\mu+\delta_z}(z, dz')]}{\int_{V_S} \exp[-nW^n_{\mathcal{Z}_0}(i + 1, z', \mu + \frac{1}{n} \delta_z)p_{\mu+\delta_z}(z, dz')]} \tag{7.6}
\]

In addition, \( \text{supp}(\nu^0_i(\cdot|z, \mu)) \subseteq S' \).

Proof. Note that

\[
\exp[-nW^n_{\mathcal{Z}_0}(i, z, \mu)] = \mathbb{E}_{i, z, \mu} \{ \exp[-n(h(\mathcal{L}^n))1_{\mathcal{L}^n \in \mathcal{C}(\mathcal{Z}_0)}] \}
= \mathbb{E}_{i, z, \mu} \{ \mathbb{E}_{i+1, z_{i+1}, \mathcal{L}^n_{i+1}} \exp[-n(h(\mathcal{L}^n))1_{\mathcal{L}^n \in \mathcal{C}(\mathcal{Z}_0)}] \}
= \mathbb{E}_{i, z, \mu} \{ \exp [-nW^n_{\mathcal{Z}_0}(i + 1, Z_{i+1}, \mathcal{L}^n_{i+1})] \}
= \int_{V_S} \exp \left[-nW^n_{\mathcal{Z}_0}(i + 1, z', \mu + \frac{1}{n} \delta_z)\right] p_{\mu+\delta_z}(z, dz').
\]

For fixed \( i, W^n_{\mathcal{Z}_0}(i + 1, \cdot, \mu + \frac{1}{n} \delta_z) \) is not identically infinite if \( (z, \mu) \in \mathbb{S} \). Actually,

\[
W^n_{\mathcal{Z}_0}(i + 1, \cdot, \mu + \frac{1}{n} \delta_z) \neq \infty \text{ if and only if } z' \in S'.
\]

Since \( S' \) is finite, we observe that \( W^n_{\mathcal{Z}_0}(i + 1, \cdot, \mu + \frac{1}{n} \delta_z) \) is bounded on \( S' \). Then by the variational representation in Proposition 2.14, we derive the dynamic programming equation (7.5), where the infimum is attained at the unique \( \nu^0_i \) defined in (7.6). This attainment is well-defined since \( W^n_{\mathcal{Z}_0}(i + 1, \cdot, \mu + \frac{1}{n} \delta_z) \) is not identically infinity, which implies that the denominator of (7.6) is not equal to 0. Note that

\[
\exp \left[-nW^n_{\mathcal{Z}_0}(i + 1, z', \mu + \frac{1}{n} \delta_z)\right] = 0 \text{ if } z' \notin S'.
\]

By (7.6) we immediately find that \( \nu^0_i(\cdot|z, \mu) \) is supported on \( S' \).

Now we show the attainment condition of \( W^n_{\mathcal{Z}_0}(i, z, \mu) \) as follows.

**Lemma 7.4.** For some fixed \( i \), if \( (\mathcal{Z}^n_i, \mathcal{Z}^n_i) = (z, \mu) \in \mathbb{S} \) (\( \mathbb{S} \) is defined in (7.4)), then there exists a unique sequence \( \{\nu^0_j\}_{j \geq i} \) such that the following holds: \( \nu^0_j(\cdot, \mathcal{Z}^n_j, \mathcal{Z}^n_j) \) are well-defined, and \( \mathcal{Z}^n_j \in \mathcal{C}(\mathcal{Z}_0) \) for all \( j \geq i \) and all \( \omega \). Here, the infimum of the dynamic programming equation (7.5) is attained at \( \{\nu^0_j\}_{j \geq i} \).

Proof. We prove that \( (\mathcal{Z}^n_j, \mathcal{Z}^n_j) \in \mathbb{S} \) for all \( j \geq i \) by induction: assume this property holds for some \( j \geq i \). Then by Lemma 7.3 we know that

\[
\text{supp}(\nu^0_j(\cdot|\mathcal{Z}^n_j, \mathcal{Z}^n_j)) \subseteq \{ z' : (z', \mathcal{Z}^n_{j+1}) \in \mathbb{S} \},
\]

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which implies \((\overline{Z}_{j+1}^n, \overline{Z}_j^n) \in \mathcal{S}\). Since \((\overline{Z}_j^n, \overline{Z}_j^n) \in \mathcal{S}\), we immediately complete the proof by Lemma 7.3. 

Proof of Lemma 2.16. Denote the following minimal cost function by

\[
V_{\mathcal{J}_0}^n(i, z, \mu) := \inf_{\{\nu_j^n\}} \mathbb{E}_{i, z, \mu} \left\{ \frac{1}{n} \sum_{j=1}^{n-1} R(\nu_j^n(\cdot|\overline{Z}_j^n, \overline{Z}_j^n), p_{\overline{Z}_{j+1}^n}(\overline{Z}_j^n, \cdot)) + h(\mathcal{L}^n) + \infty 1_{\mathcal{E}_n \in C(\mathcal{S})} \right\}
\]

\[
= \inf_{\{\nu_j^n\}} \mathbb{E}_{i, z, \mu} \left\{ \frac{1}{n} \sum_{j=1}^{n-1} R(\nu_j^n(\cdot|\overline{Z}_j^n, \overline{Z}_j^n), p_{\overline{Z}_{j+1}^n}(\overline{Z}_j^n, \cdot)) + h(\mathcal{L}^n) \right\},
\]

where \(\mathbb{E}_{i, z, \mu}\) is the expectation under \(\nu_j^n\) conditioned on \((\overline{Z}_i^n, \overline{Z}_i^n) = (z, \mu)\).

Now we give a stronger result as follows:

\[
W_{\mathcal{J}_0}^n(i, z, \mu) = V_{\mathcal{J}_0}^n(i, z, \mu) \text{ for } i = 0, 1, \ldots, n.
\]

We can complete the proof of this lemma by setting \(i = 0, \mu = 0\). In fact \(W_{\mathcal{J}_0}^n\) and \(V_{\mathcal{J}_0}^n\) have the same terminal condition, i.e., \(V_{\mathcal{J}_0}^n(n, z, \mu) = W_{\mathcal{J}_0}^n(n, z, \mu) = h(\mu) + \infty 1_{\mu \in C(\mathcal{S})}\). For \(i < n\), \(V_{\mathcal{J}_0}^n(i, z, \mu) = W_{\mathcal{J}_0}^n(i, z, \mu) = \infty\) if \((z, \mu) \notin \mathcal{S}\). Otherwise, we show both \(W_{\mathcal{J}_0}^n(i, z, \mu) \leq V_{\mathcal{J}_0}^n(i, z, \mu)\) and \(W_{\mathcal{J}_0}^n(i, z, \mu) \geq V_{\mathcal{J}_0}^n(i, z, \mu)\) hold.

Note that if \((\overline{Z}_i^n, \overline{Z}_i^n) \in \mathcal{S}\) then \(\nu_j^n(\cdot|\overline{Z}_j^n, \overline{Z}_j^n) \subset \mathcal{S}\) for almost surely \(\omega\). Here we verify

\[
\supp(\nu_j^n(\cdot|\overline{Z}_j^n, \overline{Z}_j^n)) \subseteq \{z': (z', \overline{Z}_{j+1}^n) \in \mathcal{S}\}
\]

by contradiction. If (7.7) does not hold, then \(\overline{Z}_{j+1}^n \notin \mathcal{C}(\mathcal{S})\), which makes a contradiction to \(\supp(\overline{Z}_i^n) \subseteq \supp(\overline{Z}_i^n)\) in \(\mathcal{C}(\mathcal{S})\) by the decreasing of \(\mathcal{S}\).

Hence by (7.7),

\[
\int_{V_S} W_{\mathcal{J}_0}^n(j + 1, z', \overline{Z}_{j+1}^n)) \nu_j^n(dz'|\overline{Z}_j^n, \overline{Z}_j^n)
\]

is bounded for all \(j \geq i\). Meanwhile, by (7.7), \((\overline{Z}_j^n, \overline{Z}_j^n) \in \mathcal{S}\) for all \(j \geq i\), which implies that \(W_{\mathcal{J}_0}^n(j, \overline{Z}_j^n, \overline{Z}_j^n)\) is bounded. By (7.5), we have

\[
\frac{1}{n} R(\nu_j^n(\cdot|\overline{Z}_j^n, \overline{Z}_j^n)) p_{\overline{Z}_{j+1}^n(\overline{Z}_j^n, \cdot))} \geq W_{\mathcal{J}_0}^n(j, \overline{Z}_j^n, \overline{Z}_j^n) - \int_{V_S} W_{\mathcal{J}_0}^n(j + 1, z', \overline{Z}_{j+1}^n)) \nu_j^n(dz'|\overline{Z}_j^n, \overline{Z}_j^n)).
\]

Hence by the Markov property and the terminal condition (7.3),

\[
\mathbb{E}_{i, z, \mu} \left\{ \frac{1}{n} \sum_{j=1}^{n-1} R(\nu_j^n(\cdot|\overline{Z}_j^n, \overline{Z}_j^n), p_{\overline{Z}_{j+1}^n}(\overline{Z}_j^n, \cdot)) + h(\mathcal{L}^n) \right\}
\]

\[
\geq \mathbb{E}_{i, z, \mu} \left\{ \sum_{j=1}^{n-1} \left[ W_{\mathcal{J}_0}^n(j, \overline{Z}_j^n, \overline{Z}_j^n) - \int_{V_S} W_{\mathcal{J}_0}^n(j + 1, z', \overline{Z}_{j+1}^n)) \nu_j^n(dz'|\overline{Z}_j^n, \overline{Z}_j^n)) \right] + W_{\mathcal{J}_0}^n(n, \overline{Z}_n^n, \overline{Z}_n^n) \right\}
\]

\[
= \mathbb{E}_{i, z, \mu} \left\{ \sum_{j=1}^{n-1} \left[ W_{\mathcal{J}_0}^n(j, \overline{Z}_j^n, \overline{Z}_j^n) - \mathbb{E}_{i, z, \mu} \{ W_{\mathcal{J}_0}^n(j + 1, \overline{Z}_{j+1}^n)) \overline{Z}_j^n, \overline{Z}_j^n) \right] + W_{\mathcal{J}_0}^n(n, \overline{Z}_n^n, \overline{Z}_n^n) \right\}
\]
Lemma 7.5. \[ \lim_{n \to \infty} k, m, \nu \] bounded measurable function \[ n W_n(j, \mathcal{Z}_j, \mathcal{L}_j) \rightarrow W_n(j + 1, \mathcal{Z}_{j+1}, \mathcal{L}_{j+1}) + W_n(n, \mathcal{Z}_n, \mathcal{L}_n) \]

= \[ W_n(i, z, \mu). \] (7.9)

Since this inequality holds for all arbitrary \( \nu_n \) such that \( \mathcal{L}_n \in \mathcal{C}(\mathcal{S}_0) \), we have \( V_n(i, z, \mu) \geq W_n(i, z, \mu) \).

We obtain that \( W_n(i, z, \mu) \geq V_n(i, z, \mu) \) for \( (z, \mu) \in \mathcal{S} \). Assuming that \( \mathcal{F}_j \) is the unique attainment in Lemma 7.4, we know that the inequality (7.8) becomes to an equality, which implies

\[ \mathbb{E}_{i,z,\mu} \left\{ \frac{1}{n} \sum_{j=i}^{n-1} R(\mathcal{V}_j^n(\cdot|\mathcal{Z}_j^n, \mathcal{L}_j^n)\|P_{\mathcal{Z}_{j+1}}(\cdot|\mathcal{Z}_j^n, \cdot)+h(\mathcal{L}_n)) \right\} = W_n(i, z, \mu) \]

by the same steps shown in (7.9). Note that \( \mathcal{L}_n \in \mathcal{C}(\mathcal{S}_0) \) by Lemma 7.4. We obtain that \( V_n(i, z, \mu) \leq W_n(i, z, \mu) \). Hence we have \( V_n(i, z, \mu) = W_n(i, z, \mu) \).

7.4 Proof of Theorem 3.1

To prove Theorem 3.1, we need the following lemma. For convenience, we define

\[ \nu_{m,k} := \frac{1}{k-m} \sum_{j=m}^{k-1} \delta_{Z_j} \times \nu_j^n(\cdot|\mathcal{Z}_j^n, \mathcal{L}_j^n), \quad \mathcal{L}_m^n := \frac{1}{k-m} \sum_{j=m}^{k-1} \delta_{Z_j}, \]

where for the non-negative integers \( m, k \) with \( m = m(n) < k = k(n) \leq n \), \( \lim_{n \to \infty} k/n \), \( \lim_{n \to \infty} m/n \) exist, and \( \lim_{n \to \infty} (k-m)/n > 0 \).

Lemma 7.5. Let \( \phi \) be a bounded measurable function mapping \( V_S \) into \( \mathbb{R} \). Then for each \( \varepsilon > 0 \), each sequence \( \{z^n\} \) in \( V_S \) and each \( n \geq 4\|g\|_{\infty}/\varepsilon \),

\[ \mathbb{P}_{z^n} \left[ \frac{k-m}{n} \int_{V_S} \phi(y) \mathcal{L}_m^n(dy) - \int_{V_S \times V_S} \phi(y) \nu_{m,k}^n(dx \times dy) \right] \geq \varepsilon \]

= \[ \mathbb{P}_{z^n} \left[ \frac{k-m}{n} \int_{V_S} \phi(y) \mathcal{L}_m^n(dy) - \int_{V_S} \phi(y) (\nu_{m,k}^n)(dy) \right] \geq \varepsilon \]

\[ \leq 16\|\phi\|_{\infty}^2 \frac{1}{n\varepsilon^2}, \]

where \( k, m, \nu_j^n \) satisfy conditions in Theorem 3.1, and for all Borel sets \( A \), \( (\nu_{k,m}^n)(A) = \nu_{k,m}^n(G \times A) \).

Proof. Define \( \mathcal{F}_j^n \) to be the \( \sigma \)-field generated by \( \{\mathcal{Z}_i^n, \mathcal{L}_i^n, i = 0, 1, \ldots, j\} \). Then for each bounded measurable function \( \phi \), since \( \mathcal{Z}_{j+1}^n \) is generated by \( \nu_j^n(\cdot|\mathcal{Z}_j^n, \mathcal{L}_j^n) \), we have

\[ \mathbb{E}_{z^n} \left[ \phi(\mathcal{Z}_{j+1}^n) - \int_{V_S} \phi(y) \nu_j^n(dy|\mathcal{Z}_j^n, \mathcal{L}_j^n) \right] = 0, \quad \mathbb{P}_{z^n} \text{ a.s.} \]

This implies that the covariance is equal to 0, i.e., for \( j < j' \),

\[ \mathbb{E}_{z^n} \left[ \phi(\mathcal{Z}_{j+1}^n) - \int_{V_S} \phi(y) \nu_j^n(dy|\mathcal{Z}_j^n, \mathcal{L}_j^n) \right] \cdot \mathbb{E}_{z^n} \left[ \phi(\mathcal{Z}_{j'+1}^n) - \int_{V_S} \phi(y) \nu_j^n(dy|\mathcal{Z}_{j'}^n, \mathcal{L}_{j'}^n) \right] \]

\[ = \mathbb{E}_{z^n} \left[ \phi(\mathcal{Z}_{j+1}^n) - \int_{V_S} \phi(y) \nu_j^n(dy|\mathcal{Z}_j^n, \mathcal{L}_j^n) \right] \cdot \mathbb{E}_{z^n} \left[ \phi(\mathcal{Z}_{j'+1}^n) - \int_{V_S} \phi(y) \nu_j^n(dy|\mathcal{Z}_{j'}^n, \mathcal{L}_{j'}^n) \right] \cdot \mathcal{F}_{j'}^n \]
Note that

$$\frac{k - m}{n} \left( \int_{V_S} \phi(y) \mathbb{Z}_{m,k}^n(dy) - \int_{V_S \times V_S} \phi(y) \nu_{m,k}^n(dx \times dy) \right)$$

$$= \frac{1}{n} \sum_{j=m}^{k-1} \left( \phi(\mathbb{Z}_{j+1}^n) - \int_{V_S} \phi(y) \nu_j^n(dy | \mathbb{Z}_{j+1}^n) \right) + \frac{1}{n} (\phi(\mathbb{Z}_m^n) - \phi(\mathbb{Z}_k^n)),$$

and $$|\phi(\mathbb{Z}_m^n) - \phi(\mathbb{Z}_k^n)| \leq 2\|\phi\|_\infty.$$ By Chebyshev’s inequality, we deduce that for any $$\varepsilon > 0$$ and $$n \geq 4\|\phi\|_\infty / \varepsilon,$$

$$\mathbb{P}_{x^n} \left[ \frac{k - m}{n} \left| \int_{V_S} \phi(y) \mathbb{Z}_{m,k}^n(dy) - \int_{V_S \times V_S} \phi(y) \nu_{m,k}^n(dx \times dy) \right| \geq \varepsilon \right]$$

$$\leq \mathbb{P}_{x^n} \left[ \frac{1}{n} \sum_{j=m}^{k-1} \left( \phi(\mathbb{Z}_{j+1}^n) - \int_{V_S} \phi(y) \nu_j^n(dy | \mathbb{Z}_{j+1}^n) \right) \geq \varepsilon / 2 \right]$$

$$\leq \frac{4}{\varepsilon^2} \mathbb{P}_{x^n} \left[ \frac{1}{n^2} \left( \sum_{j=m}^{k-1} \left( \phi(\mathbb{Z}_{j+1}^n) - \int_{V_S} \phi(y) \nu_j^n(dy | \mathbb{Z}_{j+1}^n) \right) \right)^2 \right]$$

$$= \frac{4}{\varepsilon^2} \mathbb{P}_{x^n} \left[ \frac{1}{n^2} \sum_{j=m}^{k-1} \left( \phi(\mathbb{Z}_{j+1}^n) - \int_{V_S} \phi(y) \nu_j^n(dy | \mathbb{Z}_{j+1}^n) \right)^2 \right]$$

$$\leq \frac{16\|\phi\|^2_\infty}{n \varepsilon^2},$$

which completes the proof.

Proof of Theorem 3.1. By Lemma 7.5, we can finish the proof of Theorem 3.1 similarly to [14, Theorem 8.2.8]. In fact, since $$S$$ is finite, we may deduce that every subsequence of $$\{\nu_{m,k}^n\}_{n \geq 0}$$ is tight; and by Prohorov’s theorem, there is a subsequence that converges to some random variable $$\nu$$ in distribution. Because the map taking $$\nu_{m,k}^n$$ into $$(\nu_{m,k}^n, \mathbb{Z}_{m,k}^n) = (\nu_{m,k}^n, \mathbb{Z}_{m,k}^n) \Rightarrow (\nu, \mathbb{Z})$$ [14, Theorem A.3.6], and $$\mathbb{Z}$$ is equal to the first marginal of $$\nu$$ with probability 1. Moreover, since $$\{z^n\}$$ takes value in finite space $$V_S$$, there exists some $$z \in V_S$$ such that there exists some subsequence $$(\nu_{m,k}^n, \mathbb{Z}_{m,k}^n, z^n) \Rightarrow (\nu, \mathbb{Z}, z)$$. Noting that $$\mathbb{Z}_j^n$$ stays in $$E' \times \{-1, 1\}$$ for $$m \leq j < k$$ under $$\nu_j^n$$ (for convenience, denote by $$S' = E' \times \{-1, 1\})$$, $$\nu$$ is supported on $$S' \times S'$$, and $$\mathbb{Z}$$ is supported on $$S'$$. Let $$(\Omega, \mathbb{F}, \mathbb{P})$$ be a probability space such that $$\nu$$ is a stochastic kernel on $$S' \times S'$$ given $$\Omega$$. Since $$\mathbb{Z}$$ is the first marginal of $$\nu$$, by [14, Theorem A.5.6], there exists a stochastic kernel $$\nu(dz'|z) = \nu(dz'|z, \omega)$$ on $$S'$$ given $$S' \times \Omega$$ for which the equation in part (b) of the theorem holds $$\mathbb{P}_{z'}$$-a.s. for $$\omega \in \Omega$$.

Now we begin to verify that $$\mathbb{Z}(dz)$$ and $$\nu(dz'|z)$$ are related as in part (c). For convenience, we fix a convergent subsequence of $$(\nu_{m,k}^n, \mathbb{Z}_{m,k}^n)$$, and simply set $$n$$ to be the index. By [14, Theorem A.6.1], $$S'$$ admits an equivalent metric $$m$$ such that the bounded, uniformly continuous function space, $$U_b(S', m)$$, is separable. Set $$\Xi$$ to be a countable dense subset of $$U_b(S', m)$$.

Since $$\mathbb{Z}_{m,k}^n(dy)$$ (resp. $$\nu_{m,k}^n(dx \times dy)$$) is supported on $$S'$$ (resp. $$S' \times S'$$), Lemma 7.5 implies that for all $$\phi \in \Xi$$ and $$\varepsilon > 0$$,

$$\mathbb{P}_{x^n} \left( \frac{k - m}{n} \left( \int_{S'} \phi(y) \mathbb{Z}_{m,k}^n(dy) - \int_{S' \times S'} \phi(y) \nu_{m,k}^n(dx \times dy) \right) \geq \varepsilon \right) \to 0.$$
Thus by [14, Theorem A.3.7],
\[
\frac{k - m}{n} \left( \int_{S'} \phi(y) \overline{\mathcal{L}}_{m,k}^n(dy) - \int_{S' \times S'} \phi(y) \nu_{m,k}^n(dx \times dy) \right) \to 0.
\]
Since \((\nu_{m,k}^n, \overline{\mathcal{L}}_{m,k}^n) \Rightarrow (\nu, \overline{\mathcal{L}})\), by the Skorohod representation theorem, we can assume that with probability 1, the following limits hold for all \(\phi \in \Xi\):
\[
\lim_{n \to \infty} \frac{k - m}{n} \left( \int_{S'} \phi(y) \overline{\mathcal{L}}_{m,k}^n(dy) - \int_{S' \times S'} \phi(y) \nu_{m,k}^n(dx \times dy) \right) = 0,
\]
\[
\lim_{n \to \infty} \int_{S'} \phi(y) \overline{\mathcal{L}}_{m,k}^n(dy) = \int_{S'} \phi(y) \overline{\mathcal{L}}(dy),
\]
\[
\lim_{n \to \infty} \int_{S' \times S'} \phi(y) \nu_{m,k}^n(dx \times dy) = \int_{S' \times S'} \phi(y) \nu(dx \times dy).
\]
Combining these limits and using the decomposition in part (b) of the theorem, we conclude that for almost all \(\omega \in \{\omega \in \Omega : \lim_{n \to \infty} \frac{k - m}{n} = 0\}\), all \(\phi \in \Xi\) and hence for all \(\phi \in U_0(S', m)\),
\[
\int_{S'} \phi(y) \overline{\mathcal{L}}(dy) = \int_{S' \times S'} \phi(y) \nu(dx \times dy) = \int_{S' \times S'} \phi(y) \nu(dy|x) \overline{\mathcal{L}}(dx).
\]
By [14, Theorem A.2.2(b)], \(\overline{\mathcal{L}}(\cdot) = \int_{S'} \nu(dx \times \cdot) = \int_{S'} \nu(\cdot|x) \overline{\mathcal{L}}(dx)\), i.e. for almost all \(\omega \in \{\omega \in \Omega : \lim_{n \to \infty} \frac{k - m}{n} = 0\}\),
\[
\overline{\mathcal{L}}(B) = \int_{S'} \nu(B|x) \overline{\mathcal{L}}(dx) = \int_{S'} \pi(x, B) \overline{\mathcal{L}}(dx) = \nu|_2(B)
\]
for all Borel set \(B\).

### 7.5 Proof of Lemma 3.4

**Case 1.** If \(G_0\) is a tree with the root \(\varrho\), we denote by \(l : V_0 \to \mathbb{Z}\) the level of the vertex on \(G_0\). That is to say, \(l(\varrho) = 0\) and \(l(v) = k\) if \(v\) is the \(k\)-th generation of \(\varrho\). Then we can obtain that \(l(U_n) - l(U_m)\) has the same parity as that of \(n - m\). Since there exists a path \(u_0 u_1 \ldots u_{t-s}\) with \(u_t = v_0\), \((l(v_0) - l(u_0)) - (t - s)\) should be even. Denote by \(V_f\) the collection of all vertices with the offspring \(v_0\) (Specifically, let \(V_f = \{v_0\}\) if \(v_0 = \varrho\)). Then define a time homogeneous transition probability
\[
\tilde{p}(u, f(u)) = 1 \text{ for } u \notin V_f, \quad \tilde{p}(u, v_u) = 1 \text{ for } u \in V_f,
\]
where \(v_u\) is the unique vertex in \(c(u) \cap V_f\) (specifically, select \(v_{v_0}\) to be some deterministic vertex in \(c(v_0)\) if \(v_0 = \varrho\)), and \(f(u), c(u)\) are the parent and the child set of \(u\) respectively. Now we prove \(U_t = v_0\) under \(\tilde{p}\) for the arbitrary \(v_1\).

With law \(\tilde{p}\), the process will traversed \(v_0\) in the shortest time, and will be stuck in edge \(v_0v_t\), see Figure 14. Set \(t' = \inf\{t' > t - 2d : U_{t'} = v_0\}\). \(t'\) is non-random under law \(p\). Since \(|E_0| \leq d, t' - (t - 2d) \leq d\), which implies \(t' < t\). Note that \(t' - s\) has the same parity of \(l(v_0) - l(u_0)\). The fact that \((l(v_0) - l(u_0)) - (t - s)\) is an even number implies that \(t' - t\) is even. Since \(U_n\) is stuck in \(v_0v_t\) from time \(t'\) to time \(t\), we figure out \(U_t = v_0\).

**Case 2.** When \(G_0\) is a bipartite graph, we divide its vertices into two parts \(A, B\) such that \(u \sim v\) if and only if \(u \in A, v \in B\) or \(u \in B, v \in A\). Define a function \(l'\) on \(G_0\) such that
is a connected graph, each $w \in A$, and that $l'(v) = 1$ for $v \in B$. Then for every random walk $U$ moving on $G_0$, we observe that $l'(U_n) - l'(U_m)$ has the same parity as that of $n - m$. We consider the spanning tree $T$ of $G_0$. Now we claim that there exists a path $u_0u_1' \ldots u_{t-s}'$ on $T$ such that $u_0 = u_0', v_0 = u_{t-s}'$. By the results on tree in Case 1, we can find transition probability $\tilde{p}$.

Here we construct this path by choosing a shortest path $u_0' \ldots u_{t}'$ on $T$ from $u_0$ to $v_0$. Note that $|E_0| \leq d$, which implies the number of edges of $T$ is smaller than $d$. By $t - s > 2d$, we get that $t' < t - s$. Note that $t'$ has the same parity as $l'(v_0) - l'(u_0)$ because $u_0' \ldots u_{t}'$ is also a path on $G_0$. By the fact that $l'(v_0) - l'(u_0) = l'(u_{t-s}) - l'(u_0)$ has the same parity as $t - s$, we deduce that $t - s - t'$ is an even number. Assume $v_1$ is some vertex adjacent to $v_0$, then we choose

$$u_{s'+t'} = \begin{cases} v_1, & \text{if } t' \text{ is odd,} \\ v_0, & \text{if } t' \text{ is even.} \end{cases}$$

Then we can find that $u_{t-s}' = v_0$ since $t - s - t'$ is even.

**Case 3.** If $G_0$ is not bipartite, there exists a circle on $G_0$ with odd vertices, denoted by $C = w_1w_2 \ldots w_{2m+1}w_1$ ($w_i \neq w_j$ if $i \neq j$). For each connected component $G_t (1 \leq t \leq L)$ of $G_0 \setminus C$, choose one of its spanning tree, denoted by $T_t = (V(T_t), E(T_t))$. Set $A_0 = \{1, 2, \ldots, L\}$. Now we define $\{\hat{T}_k : 1 \leq k \leq 2m + 1\}$ by induction:

- Denote by $A_1 = \{l \in A_0 : T_l \text{ cannot be connected to } u_1 \text{ by edges in } E_0\}$. For those $r \in A_0 \setminus A_1$, choose a vertex $v_r \in T_r$ that is connected to $u_1$. Let $\hat{T}_1$ be a tree with the vertex set $\{w_1\} \cup (\bigcup_{r \in A_0 \setminus A_1} V(T_r))$ and the edge set $\{w_1v_r : r \in A_0 \setminus A_1\} \cup (\bigcup_{r \in A_0 \setminus A_1} E(T_r))$.

- For $k \leq 2m$, if $A_k, \hat{T}_k$ is determined, choose $A_{k+1} = \{r \in A_k : T_r \text{ cannot be connected to } u_{k+1} \text{ by edges in } E_0\}$.

For those $r \in A_k \setminus A_{k+1}$, choose a vertex $v_r \in T_r$ that is connected to $u_{k+1}$. Let $\hat{T}_{k+1}$ be a tree with the vertex set $\{w_{k+1}\} \cup (\bigcup_{r \in A_k \setminus A_{k+1}} V(T_r))$ and the edge set $\{w_{k+1}v_r : r \in A_k \setminus A_{k+1}\} \cup (\bigcup_{r \in A_k \setminus A_{k+1}} E(T_r))$.

We let $w_k$ be the root of $\hat{T}_k$. Without loss of generality, we assume $v_0 \neq w_{2m+1}, w_1$. Since $G_0$ is a connected graph, each $T_j$ can be connected to some $w_k$ for $k = 1, \ldots, 2m+1$, which implies
that $A_{2m+1} = \emptyset$, i.e., the vertex set of $\cup_{k=1}^{2m+1} \tilde{T}_k$ is equal to $V_0$. Intuitively, $(\cup_{k=1}^{2m+1} \tilde{T}_k) \cup C$ is similar to a key ring if we see each $\tilde{T}_k$ as a key on the ring $C$. We choose the same $p$ on each $\tilde{T}_k$ such that the process moves in tree with law $p$. If the process is on the circle $C$, we let it move counterclockwise on $C$. If this process meets the root of the tree $\tilde{T}_{k_0}$ containing $v_0$ at time $t'$, and there exists a path on $\tilde{T}_{k_0}$ such that $U_{t'} = w_{k_0}$, $U_t = v_0$, the process stops moving on $C$, and then moves on $\tilde{T}_{k_0}$. In detail, set $f(\cdot), V_f, v, (\cdot)$ on each tree $\tilde{T}_k$ to be the same notations defined in the tree case. Choose a non-homogeneous transition probability for all time $n$ and all vertex $u$ as follows: Assume $v_0 \in \tilde{T}_{k_0}$, and regard $w_{2m+2}$ (resp. $w_0$) as $w_1$ (resp. $w_{2m+1}$).

- If $u \in \tilde{T}_k$, and $n - t + l(u) + l(v_0) + k - (2m + 1)1_{\{k_0 < k\}} \in 2\mathbb{Z}$,
  \begin{align*}
  \tilde{p}_n(u, f(u)) &= 1 \text{ for } u \notin V_f \cup C, \\
  \tilde{p}_n(u, v_u) &= 1 \text{ for } u \in V_f, \\
  \tilde{p}_n(u, u_{r+1}) &= 1 \text{ for } u = u_r \in C \setminus V_f.
  \end{align*}

- If $u \in \tilde{T}_k$, $n - t + l(u) + l(v_0) + k - (2m + 1)1_{\{k_0 < k\}} \in 2\mathbb{Z} + 1$, and $|\tilde{T}_{k_0}| \neq 1$,
  \begin{align*}
  \tilde{p}_n(u, f(u)) &= 1 \text{ for } u \in G_0 \setminus C, \\
  \tilde{p}_n(u, u_{r+1}) &= 1 \text{ for } u = u_r \in C.
  \end{align*}

- If $u \in \tilde{T}_k$, $n - t + l(u) + l(v_0) + k - (2m + 1)1_{\{k_0 < k\}} \in 2\mathbb{Z} + 1$, and $|\tilde{T}_{k_0}| = 1$,
  \begin{align*}
  \tilde{p}_n(u, f(u)) &= 1 \text{ for } u \in G_0 \setminus C, \\
  \tilde{p}_n(u, u_{r-1}) &= 1 \text{ for } u = u_r \in C.
  \end{align*}

The last one is for the specific case that $v_0$ is the unique vertex in $\tilde{T}_{k_0}$. This implies that $v_0 = w_{k_0}$. In this case we let the process move on the circle $C$ reversely after traversing the edge $w_{k_0}w_{k_0+1}$.

### 7.6 Proof of Lemma 3.5

Here we prove the convexity, the lower semicontinuity and the compactness of the level set of the rate function. To this end, write $\Lambda_{\mathcal{S}, \mathcal{G}_0}$ as $\Lambda$ and let $\Lambda_{E_r}(\mu) := \inf_{q \in \mathcal{S}, \mu q = \mu} \int_{V_0} R(q|p_{E'}) \, d\mu$. Then by the definition of $\Lambda$ in Theorem 2.4, we have that

$$\Lambda(\mu) = \inf_{(\mu_k, r_k, E_k) \in \mathcal{A}(\mu, \mathcal{G}_0)} \sum_{k=1}^{d} r_k \Lambda_{E_k}(\mu_k).$$

For any $\{E_j\} \in \mathcal{E}$, define

$$\mathcal{A}|_{\{E_j\}}(\mu, \mathcal{G}_0) = \left\{ (\mu_k, r_k)_k : \sum_{k=1}^{d} \mu_k r_k = \mu, \text{ supp}(\mu_k) = E_k \times \{-1, 1\}, \right.$$ \[r_k \geq 0, \sum_{k=1}^{d} r_k = 1, r_l = 0 \text{ for } E_l \notin \mathcal{G}_0 \}.$$
\[
\Lambda_{\{E_j\}}(\mu) = \inf_{(\mu_k, r_k) \in \mathcal{P}(E_j)} \sum_{k=1}^{d} r_k \Lambda_{E_k}^0 (\mu_k).
\]

Then \( \Lambda = \inf_{\{E_j\} \in \mathcal{P}} \Lambda_{\{E_j\}} \). Noting the lower semicontinuity and the compactness of \( \Lambda \) are equivalent to those of \( \Lambda_{\{E_j\}} \), it suffices to prove those properties of \( \Lambda_{\{E_j\}} \) for fixed \( \{E_j\} \).

For all \( \mu \), there are \( \tau, \mu_k \) such that \( \Lambda_{\{E_j\}}(\mu) = \sum_{k=1}^{d} r_k \Lambda_{E_k}^0 (\mu_k) \), where \( \sum_{k=1}^{d} r_k \mu_k = \mu \), \( \sum_{k=1}^{d} r_k = 1 \), and \( \text{supp}(\mu_k) = E_k \times \{-1, 1\} \). Actually, we can choose a sequence \( \mu_k^n, r_k^n \) such that \( \sum_{k=1}^{d} r_k^n \Lambda_{E_k}^0 (\mu_k^n) < \Lambda_{\{E_j\}}(\mu) + \frac{1}{n} \), then by the diagonal argument and the lower-semicontinuity of \( \Lambda_{E_k}^0 (k = 1, \ldots, d) \), there is a subsequence such that \( (\mu_k^n, r_k^n)_{1 \leq k \leq d} \to (\mu_k, r_k)_{1 \leq k \leq d} \) and \( \sum_{k=1}^{d} r_k \Lambda_{E_k}^0 (\mu_k) \leq \Lambda_{\{E_j\}}(\mu) \). The limit satisfies the condition mentioned above.

(a) Lower semicontinuity: For all \( \mu^n \Rightarrow \mu \), choose a sequence of \( \mu_k^n, r_k^n \) such that \( \Lambda_{\{E_k\}}(\mu^n) = \sum_{k=1}^{d} r_k^n \Lambda_{E_k}^0 (\mu_k^n) \). Since \( V_S \) is finite, \( \mu_k^n \) is tight. Hence by the diagonal argument, there exists a subsequence \( n_j \) such that

- \( \mu_k^{n_j} \Rightarrow \mu_k \);
- \( \lim_{j \to \infty} \Lambda_{\{E_k\}}(\mu^{n_j}) = \lim_{n \to \infty} \Lambda_{\{E_k\}}(\mu^n) \);
- \( r_k^{n_j} \to r_k \);
- \( \mu = \sum_{k=1}^{d} r_k \mu_k \).

By the lower semicontinuity of \( \Lambda_{E_k}^0 \),

\[
\lim_{n \to \infty} \Lambda_{\{E_k\}}(\mu^n) = \sum_{k=1}^{d} r_k \Lambda_{E_k}^0 (\mu_k) \geq \Lambda_{\{E_k\}}(\mu).
\]

(b) Compactness: Note that \( \mathcal{P}(V_S) \) is compact under the weak convergence topology due to \( V_S \) is finite; and \( \{\mu : \Lambda_{\{E_k\}}(\mu) \geq M\} \) is compact because of the lower semicontinuity of \( \Lambda_{\{E_k\}} \). Therefore, the level set \( \{\mu : \Lambda_{\{E_k\}}(\mu) \leq M\} \) is compact.

Turn to the convexity of \( \Lambda_{\{E_j\}} \) for fixed \( \{E_j\} \). For all \( \lambda \in [0, 1] \), \( \mu^1, \mu^2 \in \mathcal{P}(V_S) \), by the convexity of \( \Lambda_{E_k}^0 \),

\[
\lambda \Lambda_{\{E_k\}}(\mu^1) + (1 - \lambda) \Lambda_{\{E_k\}}(\mu^2) = \sum_{k=1}^{d} \lambda r_k^1 \Lambda_{E_k}^0 (\mu_k^1) + (1 - \lambda) r_k^2 \Lambda_{E_k}^0 (\mu_k^2) \\
\geq \sum_{k=1}^{d} (\lambda r_k^1 + (1 - \lambda) r_k^2) \Lambda_{E_k}^0 \left( \frac{\lambda r_k^1 \mu_k^1 + (1 - \lambda) r_k^2 \mu_k^2}{\lambda r_k^1 + (1 - \lambda) r_k^2} \right) \\
= \Lambda_{\{E_k\}}(\lambda \mu^1 + (1 - \lambda) \mu^2).
\]

### 7.7 Proofs of Examples 5.7 and 5.9

**Proof of Example 5.7.** Note \( \mathcal{G} = \{\{0\}, G_1, G\}, \{\{0\}, G, G_2\} \), where \( G_i = \{0, i\} \) (i = 1, 2). Assume the infimum is attained at \( \{\{0\}, G_i, G\} \). Since \( \text{supp}(\mu_1) = G_i \), and \( \mu_1 q_1 = \mu_1 \), we...
have \( q_1(0, i) = 1 \). This implies that \( \mu_1(i) = \mu_1(0) = \frac{1}{2} \). Since \( q_2(j, 0) = 1 \) for all \( j = 1, 2 \) and \( \mu_2 q_2 = \mu_2, \mu_2(0) = \frac{1}{2} \). Let \( \nu(1) = x, q_2(0, 1) = y, r_1 = a, \) and \( r_2 = b, \) then

\[
\begin{align*}
&x = \frac{1}{2}(y \cdot b + a), \\
a + b = 1, \\
&\mu_2(1) = \frac{1}{2} y, \mu_2(2) = \frac{1}{2}(1 - y).
\end{align*}
\]

If \( i = 1 \), note that

\[
I_\delta(\nu) = \inf \{ (1 - b) \cdot \frac{1}{2} \log \frac{1 + \delta}{\delta} + b \cdot \frac{1}{2} (y \log 2y + (1 - y) \log 2(1 - y)) \}
\]

\[
= \inf_b \left\{ (1 - b) \cdot \frac{1}{2} \log \frac{1 + \delta}{\delta} + \frac{1}{2} \left( 2x - 1 + b \right) \log \frac{4x - 2 + 2b}{b} + (1 - 2x) \log \frac{2 - 4x}{b} \right\},
\]

where \( \frac{2x - 1 + b}{b} \in [0, 1] \). Denote by \( g(b) \) the formula in the infimum of \( I_\delta(\nu) \). Then

\[
2 \frac{\partial g(b)}{\partial b} = \log \frac{2x - 1 + b}{b} - \log \frac{1 + \delta}{\delta}.
\]

Thus, \( \frac{\partial g(b)}{\partial b} > 0 \iff \frac{1 - 2x}{b} < \frac{\delta - 1}{\delta} \). If \( \delta \leq 1 \), this inequality does not hold for all \( b > 0 \), which implies that the infimum is attained at \( b = 1 \). Hence \( I_\delta(\nu) = I_1(\nu) = R(\nu\|\mu) \).

Assume \( \delta > 1 \). Then \( b > \frac{2\delta}{\delta - 1}(1 - 2x) > 1 - 2x \). When \( \frac{2\delta}{\delta - 1}(1 - 2x) < 1 \), i.e. \( x > \frac{\delta + 1}{4\delta} \), the infimum is attained at \( b = \frac{2\delta}{\delta - 1}(1 - 2x) \),

\[
I_\delta(\nu) = x \log \frac{1 + \delta}{\delta} + \left( \frac{1}{2} - x \right) \log \frac{\delta - 1}{\delta}.
\]

When \( \frac{2\delta}{\delta - 1}(1 - 2x) \geq 1 \), i.e. \( x \leq \frac{\delta + 1}{4\delta} \), the infimum is attained at \( b = 1 \).

Another case, if \( i = 2 \), one can get the infimum by the same way. For \( \delta \leq 1 \), the infimum is attained at \( b = 1 \). For \( \delta > 1 \), if \( x < \frac{\delta - 1}{4\delta} \), the infimum is attained at \( b = \frac{2\delta}{\delta - 1} \cdot 2x \),

\[
I_\delta(\nu) = \left( \frac{1}{2} - x \right) \log \frac{1 + \delta}{\delta} + x \log \frac{\delta - 1}{\delta}.
\]

If \( x \geq \frac{\delta - 1}{4\delta} \), the infimum is attained at \( b = 1 \).

Therefore, for \( \delta \leq 1 \), \( I_\delta(\nu) = R(\nu\|\mu) \). For \( \delta > 1 \), since

\[
x \log \frac{1 + \delta}{\delta} + \left( \frac{1}{2} - x \right) \log \frac{\delta - 1}{\delta} 
\leq x \log 4x + \left( \frac{1}{2} - x \right) \log 2(1 - 2x) \quad \text{if} \quad x > \frac{\delta + 1}{4\delta},
\]

\[
\left( \frac{1}{2} - x \right) \log \frac{1 + \delta}{\delta} + x \log \frac{\delta - 1}{\delta} 
\leq x \log 4x + \left( \frac{1}{2} - x \right) \log 2(1 - 2x) \quad \text{if} \quad x < \frac{\delta - 1}{4\delta},
\]

the rate function is

\[
I_\delta(\nu) = \begin{cases} 
\frac{x \log \frac{\delta - 1}{\delta} + \left( \frac{1}{2} - x \right) \log \frac{\delta + 1}{\delta}}{\delta}, & 0 \leq x \leq \frac{\delta - 1}{4\delta}, \\
x \log 4x + \left( \frac{1}{2} - x \right) \log 2(1 - 2x), & \frac{\delta - 1}{4\delta} \leq x \leq \frac{\delta + 1}{4\delta}, \\
\left( \frac{1}{2} - x \right) \log \frac{\delta - 1}{\delta} + x \log \frac{\delta + 1}{\delta}, & \frac{\delta + 1}{4\delta} \leq x \leq \frac{1}{2}.
\end{cases}
\]
It can also be written as that for any $\delta > 0$,

$$I_\delta(\nu) = \begin{cases} R(\nu\|\mu) - R(\nu\|T(\nu)), & \nu(0) = \frac{1}{2}, \\ \infty, & \text{otherwise} \end{cases}$$

where $T$ is an operator satisfying

$$T(\nu)(i) = \begin{cases} (\nu(i) \vee \frac{\delta - 1}{4\delta}) \wedge \frac{\delta + 1}{4\delta}, & i \neq 0, \\ \frac{1}{2}, & i = 0. \end{cases}$$

Proof of Example 5.9. Note $\mathcal{G} = \{\{0\}, G_1, G\}$, which is actually the same as the first case $i = 1$ discussed in Example 5.7. Hence, when $\delta > 1$, $x > \frac{\delta + 1}{4\delta}$,

$$I_\delta(\nu) = x \log \frac{1 + \delta}{\delta} + \left(\frac{1}{2} - x\right) \log \frac{\delta - 1}{\delta}.$$ 

Otherwise $I_\delta(\nu) = I_1(\nu) = R(\nu\|\mu)$, i.e.,

$$I_\delta(\nu) = \begin{cases} x \log 4x + \left(\frac{1}{2} - x\right) \log 2(1 - 2x), & 0 \leq x \leq \frac{\delta + 1}{4\delta}, \\ \left(\frac{1}{2} - x\right) \log \frac{\delta - 1}{\delta} + x \log \frac{\delta + 1}{\delta}, & \frac{\delta + 1}{4\delta} \leq x \leq \frac{1}{2}. \end{cases}$$

It can also be written as that for any $\delta > 0$,

$$I_\delta(\nu) = \begin{cases} R(\nu\|\mu) - R(\nu\|T'(\nu)), & \nu(1) = \frac{1}{2}, \\ \infty, & \text{otherwise} \end{cases}$$

where $T$ is an operator satisfying

$$T'(\nu)(i) = \begin{cases} \nu(0) \wedge \frac{\delta + 1}{4\delta}, & i = 0, \\ \frac{1}{2}, & i = 1, \\ \nu(2) \vee \frac{\delta - 1}{4\delta}, & i = 2. \end{cases}$$

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