ON $\mathcal{OL}_\infty$ STRUCTURE OF NUCLEAR $C^*$-ALGEBRAS

MARIUS JUNGE, NARUTAKA OZAWA, AND ZHONG-JIN RUAN

Abstract. We study the local operator space structure of nuclear $C^*$-algebras. It is shown that a $C^*$-algebra is nuclear if and only if it is an $\mathcal{OL}_\infty,\lambda$ space for some (and actually for every) $\lambda > 6$. The $\mathcal{OL}_\infty$ constant $\lambda$ provides an interesting invariant

$$\mathcal{OL}_\infty(A) = \inf \{ \lambda : A \text{ is an } \mathcal{OL}_\infty,\lambda \text{ space} \}$$

for nuclear $C^*$-algebras. Indeed, if $A$ is a nuclear $C^*$-algebra, then we have $1 \leq \mathcal{OL}_\infty(A) \leq 6$, and if $A$ is a unital nuclear $C^*$-algebra with $\mathcal{OL}_\infty(A) \leq (\frac{1+\sqrt{5}}{2})^2$, we show that $A$ must be stably finite.

We also investigate the connection between the rigid $\mathcal{OL}_\infty,1^+$ structure and the rigid complete order $\mathcal{OL}_\infty,1^+$ structure on $C^*$-algebras, where the latter structure has been studied by Blackadar and Kirchberg in their characterization of strong NF $C^*$-algebras. Another main result of this paper is to show that these two local structures are actually equivalent on unital nuclear $C^*$-algebras. We obtain this by showing that if a unital (nuclear) $C^*$-algebra is a rigid $\mathcal{OL}_\infty,1^+$ space, then it is inner quasi-diagonal, and thus is a strong NF algebra. It is also shown that if a unital (nuclear) $C^*$-algebra is an $\mathcal{OL}_\infty,1^+$ space, then it is quasi-diagonal, and thus is an NF algebra.

1. Introduction

The purpose of this paper is to use operator space theory to study the underlying operator space structure (more precisely, the $\mathcal{OL}_\infty$ structure) of nuclear $C^*$-algebras. Roughly speaking, an $\mathcal{OL}_\infty$ space is an operator space which can be “locally paved up” by finite-dimensional $C^*$-algebras. This notion is a natural operator space analogue of $\mathcal{L}_\infty$ spaces introduced by Lindenstrauss and Pełczyński [25]. As in Banach space theory, there are two typical ways to pave up an operator space by finite-dimensional $C^*$-algebras, i.e. by uniformly completely bounded isomorphic injections, or by completely isometric (i.e. rigid) injections. We will see that it is crucially important to distinguish these different local structures for nuclear $C^*$-algebras.

To explain our motivation, let us first recall some Banach space results from [25]. Given $1 \leq p \leq \infty$ and $\lambda > 1$, a Banach space $V$ is said to be an $\mathcal{L}_{p,\lambda}$ space if for every finite-dimensional subspace $E$ of $V$, there exists a finite-dimensional subspace $F$ of $V$ such that $E \subseteq F$ (with $\dim F = n$) and the Banach-Mazur distance

$$d(F, \ell_p(n)) = \inf \{ \|T\|\|T^{-1}\| : T : F \to \ell_p(n) \text{ a linear isomorphism} \} < \lambda.$$

A Banach space $V$ is said to be an $\mathcal{L}_{p,1^+}$ space if it is an $\mathcal{L}_{p,\lambda}$ space for every $\lambda > 1$, and is said to be a rigid $\mathcal{L}_{p,1^+}$ space if there exists a collection of finite-dimensional
subspaces $F_α$ of $V$ such that each $F_α$ is isometric to an $ℓ_p(α)$ space and the union of $F_α$ is norm dense in $V$.

It is well-known (see [23]) that for $1 ≤ p < ∞$, a Banach space $V$ is an $L_{p,1}$ space if and only if it is isometric to an $L_p(X,ℳ,μ)$ space (and thus is a rigid $L_{p,1}$ space). Therefore, $L_{p,λ}$ spaces are natural local generalization of $L_p(X,ℳ,μ)$ spaces. The situation is more subtle for $p = ∞$. It is easy to see that commutative $C^∗$-algebras $C(Ω)$ (these include $L_∞(X,ℳ,μ)$ spaces) are $L_{∞,1}$ spaces. However, there are many other $L_{∞,1}$ spaces. In general, it is known that a Banach space is an $L_{∞,1}$ space if and only if it is a predual of some $L_1(X,ℳ,μ)$ space, and it is a non-trivial result of Michael and Pelczynski [28] that a Banach space $V$ is an $L_{∞,1}$ space if and only if it is a rigid $L_{∞,1}$ space.

Operator spaces are natural non-commutative quantization of Banach spaces. An operator space can be (concretely) defined to be a norm closed linear space of operators on some Hilbert space $H$, which is equipped with a distinguished matrix norm obtained from $B(H)$. The morphisms between operator spaces are completely bounded mappings. There are many parallel results as well as many distinctions between operator spaces and Banach spaces. Nevertheless, Banach space theory always provides an important source of inspiration for the development of operator space theory. The readers are referred to the recent book of Effros and Ruan [13] and the book of Pisier [31] for details.

The operator space analogue of $L_p$ spaces was first studied by Effros and Ruan [14]. Let us first recall from Pisier [31] that if $ℬ$ is a finite-dimensional $C^∗$-algebra, then we may use complex interpolation to obtain a natural operator space structure on the non-commutative $L_p(ℬ)$ space (for $1 ≤ p ≤ ∞$). In particular, we have $L_∞(ℬ) = ℬ$ and $L_1(ℬ) = ℬ_∞$. An operator space $V$ is said to be an $𝒪L_{p,λ}$ space (for some $λ > 1$) if for every finite-dimensional subspace $E$ of $V$, there exists a finite-dimensional $C^∗$-algebra $ℬ$ and a finite-dimensional subspace $F$ of $V$ such that $E ⊆ F$ and the completely bounded Banach-Mazur distance

$$d_{cb}(F, L_p(ℬ)) < λ,$$

where (1.1) is equivalent to saying that there exists a linear isomorphism

$$φ : L_p(ℬ) → F$$

such that $∥φ∥_{cb}∥φ^{-1}∥_{cb} < λ$. An operator space $V$ is said to be an $𝒪L_{p,1}$ space if it is an $𝒪L_{p,λ}$ space for every $λ > 1$, and is said to be a rigid $𝒪L_{p,1}$ space if for every $x_1, ⋯, x_n ∈ V$ and $ε > 0$, there exists a finite-dimensional $C^∗$-algebra $ℬ$ and a complete isometry $φ$ from $L_p(ℬ)$ onto a finite-dimensional subspace $F$ of $V$ such that

$$dist(x_i, F) < ε$$

for all $i = 1, ⋯, n$. Using a standard perturbation argument, we can easily show that rigid $𝒪L_{p,1}$ spaces are $𝒪L_{p,1}$ spaces. For $p = ∞$, this has been discussed in [14].

$𝒪L_{1,λ}$ spaces have been intensively studied in [14] and [27]. It was shown in [14] that if $V = ℭ_+$ is the operator predual of a von Neumann algebra $ℛ$ on a separable Hilbert space, then $V$ is an $𝒪L_{1,λ}$ space for some $λ > 1$ (respectively, an $𝒪L_{1,1}$ space or a rigid $𝒪L_{1,1}$ space) if and only if $ℛ$ is an injective von Neumann algebra. The separability can be removed by a result of Haagerup, which was stated and proved in [16] Appendix]. This shows that various notions of $𝒪L_1$ structures are
all equivalent on the operator preduals of von Neumann algebras. Recently, Ng and Ozawa [25] have proved that a separable operator space $V$ is an $\mathcal{OL}_{1,1+}$ space if and only if $V$ is completely isometric to the operator predual of an injective von Neumann algebra. Therefore, the local structure of the operator preduals of injective von Neumann algebras has been completely understood.

Turning to the $\mathcal{OL}_{\infty,\lambda}$ space case, it was observed in [14] that every $\mathcal{OL}_{\infty,\lambda}$ space is $\lambda$-nuclear, i.e. there exist approximate diagrams of completely bounded mappings

$$
(1.2) \quad \alpha_\gamma \xrightarrow{M_n(\gamma)} V \xrightarrow{id_V} \beta_\gamma
$$

such that $\|\alpha_\gamma\|\|\beta_\gamma\|_{cb} \leq \lambda$ and $\beta_\gamma \circ \alpha_\gamma \to id_V$ in the point-norm topology. If the mappings $\alpha_\gamma$ and $\beta_\gamma$ in (1.2) are completely contractive, we say that $V$ is a nuclear operator space. The notion of nuclear operator space first appeared in Kirchberg [22]. Smith [33] showed that for $C^*$-algebras, this is equivalent to the usual definition introduced by Lance [24]. In fact, Pisier [30] proved that for $C^*$-algebras, the nuclearity is equivalent to the $\lambda$-nuclearity. Therefore, if a $C^*$-algebra $A$ is an $\mathcal{OL}_{\infty,\lambda}$ space for some $\lambda > 1$, then $A$ must be nuclear (see [14, Proposition 4.9]). Surprisingly, the local structure of nuclear $C^*$-algebras turns out to be more sophisticated. The aim of this paper is to study various notions of $\mathcal{OL}_\infty$ structures and related properties on nuclear $C^*$-algebras.

Let us first look at the rigid $\mathcal{OL}_{\infty,1+}$ structure on $C^*$-algebras. It is easy to see that if a $C^*$-algebra $A$ is a rigid $\mathcal{OL}_{\infty,1+}$ space, then each complete isometry $\varphi_{F,\varepsilon} : F \to B$ (given in the definition) extends to a complete contraction $\psi_{F,\varepsilon} : A \to B$. Then we may obtain a net of finite-rank completely contractive projections

$$
(1.3) \quad P_{F,\varepsilon} = \varphi_{F,\varepsilon} \circ \psi_{F,\varepsilon}
$$
on $A$, which converges to $id_A$ in the point-norm topology. We show in §2 (see Theorem 2.3) that the existence of such a net of finite-rank completely contractive projections on a unital $C^*$-algebra $A$ implies that $A$ is a rigid $\mathcal{OL}_{\infty,1+}$ space. We note that, in general, the range space $P(A)$ of a finite-rank completely contractive projection $P$ on $A$ is not necessarily a finite-dimensional $C^*$-algebra. It is a finite-dimensional ternary ring of operators (see Youngson [37]), and thus is a finite direct sum of rectangular matrices (see Smith [14]). This motivated us to consider rigid rectangular $\mathcal{OL}_{\infty,1+}$ spaces in §2 and to show in Theorem 2.3 that a unital $C^*$-algebra is a rigid $\mathcal{OL}_{\infty,1+}$ space if and only if it is a rigid rectangular $\mathcal{OL}_{\infty,1+}$ space. The theory of ternary ring of operators plays a key role in §2.

It is well-known that for every $C^*$-algebra $A$, there is a canonical matrix order on $A$ given by the positive cones $M_n(A)^+$ in the matrix spaces $M_n(A)$ for $n \in \mathbb{N}$. Then it is natural to consider the rigid complete order $\mathcal{OL}_{\infty,1+}$ structure on $C^*$-algebras. This, actually, has been investigated by Blackadar and Kirchberg [2] in their study of strong NF algebras. We recall (by an equivalent definition from [2]) that a $C^*$-algebra $A$ is said to be a strong NF algebra if for every $x_1, \ldots, x_n \in A$ and $\varepsilon > 0$, there exists a finite-dimensional $C^*$-algebra $B$ and a (completely) isometric and complete order isomorphism $\varphi$ from $B$ onto a finite-dimensional $*$-subspace $F$ in $A$ such that

$$
\text{dist}(x_i, F) < \varepsilon
$$
for all \( i = 1, \ldots, n \).

It is clear from the definition that every strong NF algebra is a rigid \( \mathcal{OL}_{\infty,1} \) space. One of the major results in §3 is to show that these two notions are actually equivalent for unital \( \text{C}^* \)-algebras. Indeed, we prove in Theorem 3.3 that if a unital (nuclear) \( \text{C}^* \)-algebra is a rigid \( \mathcal{OL}_{\infty,1} \) space, then it is inner quasi-diagonal, and thus is a strong NF algebra by [3]. We also prove in Theorem 3.2 that if a unital (nuclear) \( \text{C}^* \)-algebra is an \( \mathcal{OL}_{\infty,1} \) space, then it is quasi-diagonal, and thus is an NF algebra (see definition in [2]).

Summarizing our results in §2 and §3, we obtain the following equivalent conditions for unital \( \text{C}^* \)-algebras.

**Theorem 1.1.** Let \( \mathcal{A} \) be a unital \( \text{C}^* \)-algebra. Then the following are equivalent:

(i) \( \mathcal{A} \) is a strong NF algebra (equivalently, \( \mathcal{A} \) is nuclear and inner quasi-diagonal),

(ii) \( \mathcal{A} \) is a rigid \( \mathcal{OL}_{\infty,1} \) space,

(iii) \( \mathcal{A} \) is a rigid rectangular \( \mathcal{OL}_{\infty,1} \) space,

(iv) there exists a net of completely contractive projections \( P_\gamma : \mathcal{A} \to \mathcal{A} \) such that \( P_\gamma \to \text{id}_\mathcal{A} \) in the point-norm topology.

Since every rigid \( \mathcal{OL}_{\infty,1} \) space is an \( \mathcal{OL}_{\infty,1} \) space, a \( \text{C}^* \)-algebra \( \mathcal{A} \) is an \( \mathcal{OL}_{\infty,1} \) space if it satisfies any of equivalent conditions in Theorem 1.1. At this moment, we do not know whether \( \mathcal{OL}_{\infty,1} \) implies rigid \( \mathcal{OL}_{\infty,1} \), and whether nuclearity and quasi-diagonality imply \( \mathcal{OL}_{\infty,1} \) on (unital) \( \text{C}^* \)-algebras.

During an operator space workshop organized by G. Pisier at the IHÉS in Paris in January, 2000, U. Haagerup showed the third author that if a unital \( \text{C}^* \)-algebra satisfies the condition (iv) in Theorem 1.1 then it is stably finite. The strong NF algebra case was first proved by Blackadar and Kirchberg [2]. Then it is natural to ask whether this is still true if \( \mathcal{A} \) is an \( \mathcal{OL}_{\infty,1} \) space, or an \( \mathcal{OL}_{\infty,\lambda} \) space for some \( \lambda > 1 \). Along this line, we show in Theorem 3.4 that if a unital \( \text{C}^* \)-algebra \( \mathcal{A} \) is an \( \mathcal{OL}_{\infty,\lambda} \) space with \( \lambda \leq \left( \frac{\sqrt{5} - 1}{2} \right)^{\frac{1}{2}} \), then \( \mathcal{A} \) must be stably finite.

It is well-known from Banach space theory that if \( \Omega \) is a compact topological space, then the commutative \( \text{C}^* \)-algebra \( \mathcal{C}(\Omega) \) is an \( \mathcal{L}_{\infty,1} \) space and thus a rigid \( \mathcal{L}_{\infty,1} \) space (see [28]). Since it has the minimal operator space structure, it is also a rigid \( \mathcal{OL}_{\infty,1} \) space (see [4]), and thus is a strong NF algebra by Theorem 1.1. This was also shown directly by Blackadar and Kirchberg [3]. They actually proved in [3] that a quite large class of stably finite nuclear \( \text{C}^* \)-algebras are strong NF algebras and thus are (rigid) \( \mathcal{OL}_{\infty,1} \) spaces. These include the spatial tensor product \( M_n \hat{\otimes} \mathcal{C}(\Omega) \), their finite direct sums and inductive limits (such as AF algebras and AH algebras). Moreover, they proved in [3] that subhomogeneous \( \text{C}^* \)-algebras, and thus ASH algebras are also strong NF algebras. A \( \text{C}^* \)-algebra is said to be **subhomogeneous** if all of its irreducible representations are finite-dimensional with \( \dim \leq n \) for some positive integer \( n \), and a \( \text{C}^* \)-algebra is said to be an **ASH algebra** if it is the inductive limit of subhomogeneous \( \text{C}^* \)-algebras.

In §4, we investigate the relation of the local structure of a nuclear \( \text{C}^* \)-algebra and its second dual. Using these results, we are able to show in §5 that if \( \mathcal{A} \) is a non-subhomogeneous nuclear \( \text{C}^* \)-algebra, then \( \mathcal{A} \) is an \( \mathcal{OL}_{\infty,\lambda} \) space for every \( \lambda > 6 \) (see Theorem 5). Combining the subhomogeneous and non-subhomogeneous cases, we obtain the following theorem.

**Theorem 1.2.** If \( \mathcal{A} \) is a nuclear \( \text{C}^* \)-algebra, then \( \mathcal{A} \) is an \( \mathcal{OL}_{\infty,\lambda} \) space for every \( \lambda > 6 \).
ON $\mathcal{O}_\infty$ STRUCTURE OF NUCLEAR $C^*$-ALGEBRAS

We note that Kirchberg proved in [22] that if $A$ is a separable non-type I nuclear $C^*$-algebra, then $A$ is completely isomorphic to the CAR algebra $B$ with $d_{cb}(A,B) \leq 256$, and thus is an $\mathcal{O}_\infty,\lambda$ space for every $\lambda > 256$ since the CAR algebra is a (rigid) $\mathcal{O}_\infty,1^+$ space. Our result significantly improves on the constant obtained from Kirchberg’s result.

From these results, we see that in contrast to the $p = 1$ case, the $\mathcal{O}_\infty$ constant $\lambda$ provides an interesting invariant $\mathcal{O}_\infty(A) = \{\lambda : A$ is an $\mathcal{O}_\infty,\lambda$ space $\}$ for nuclear $C^*$-algebras. We can conclude from Theorem 1.2 and Theorem 3.4 that if $A$ is a nuclear $C^*$-algebra, then we have

$$1 \leq \mathcal{O}_\infty(A) \leq 6,$$

and if $A$ is a nuclear non-stably finite unital $C^*$-algebra, we have

$$(\frac{1 + \sqrt{5}}{2})^2 < \mathcal{O}_\infty(A) \leq 6.$$ 

To end this paper, we will make some remarks and propose some open questions related to this new invariant $\mathcal{O}_\infty$ on nuclear $C^*$-algebras in §6.

Finally, we wish thank Bruce Blackadar, Uffe Haagerup, Huaxin Lin, Eberhard Kirchberg and Haskell Rosenthal for many stimulating discussions.

2. RIGID RECTANGULAR $\mathcal{O}_\infty,1^+$ SPACES

Let us first consider the definition. An operator space $V$ is said to be a rigid rectangular $\mathcal{O}_\infty,1^+$ space if for every $x_1, \ldots, x_n \in V$ and $\varepsilon > 0$, there exits a finite direct sum of rectangular matrices $B = \oplus_{k=1}^l M_{m(k),n(k)}$ and a completely isometric injection $\varphi : B \to V$ such that

$$\text{dist}(x_i, \varphi(B)) < \varepsilon$$

for all $i = 1, \ldots, n$.

It is clear that $B = \oplus_{k=1}^l M_{m(k),n(k)}$ can be identified with the $(1,2)$ off-diagonal corner of the finite-dimensional $C^*$-algebra $\tilde{B} = \oplus_{k=1}^l M_{m(k)+n(k)}$, and thus is a finite-dimensional ternary ring of operators with the canonical ternary operation obtained from $\tilde{B}$. In general, an operator space $V \subseteq B(K,H)$ is called a ternary ring of operators (or simply, TRO) if it is closed under the triple product

$$(x,y,z) \in V \times V \times V \to x y^* z \in V.$$

TORs were first introduced by Hestenes [19] (see also [18], [17], [21], [17], and [10]). A linear isomorphism between two TROs is a TRO-isomorphism if it preserves the triple products. It is known that up to completely isometric TRO-isomorphism, every TRO can be identified with the off-diagonal corner of a unital $C^*$-algebra, and every finite-dimensional TRO has the form $\oplus_{k=1}^l M_{m(k),n(k)}$ (see [19]).

If $A$ is a $C^*$-algebra and $P : A \to A$ is a finite-rank completely contractive projection, then it is known from Youngson [37] that the range space $P(A)$ is a finite-dimensional TRO, and thus has the form

$$P(A) \cong \oplus_{k=1}^l M_{m(k),n(k)}.$$

**Proposition 2.1.** Let $A$ be a $C^*$-algebra. Then $A$ is a rigid rectangular $\mathcal{O}_\infty,1^+$ space if and only if there exists a net of finite-rank completely contractive projections $P_\gamma : A \to A$ such that $P_\gamma \to \text{id}_A$ in the point-norm topology.
Proof. Let us assume that $\mathcal{A}$ is a rigid rectangular $\mathcal{OL}_{\infty,1}$ space. We let

$$I = \{ \gamma = (x_1, \cdots, x_n, \varepsilon) : x_i \in \mathcal{A}, \varepsilon > 0 \}$$

be the collection of finite subsets of $\mathcal{A}$ and $\varepsilon > 0$. Given $\gamma = (x_1, \cdots, x_n, \varepsilon) \in I$, there exists a finite-dimensional TRO $\mathcal{B}_\gamma = \oplus_{k=1}^m M_{m(k),n(k)}$ and a completely isometric injection

$$\varphi_\gamma : \mathcal{B}_\gamma \to \mathcal{A}$$

such that

$$\text{dist}(x_i, \varphi_\gamma(B_\gamma)) < \varepsilon$$

for all $i = 1, \cdots, n$. Since $\varphi_\gamma(B_\gamma)$ is an injective subspace of $\mathcal{A}$, the identity mapping on $\varphi_\gamma(B_\gamma)$ has a completely contractive extension $P_\gamma : \mathcal{A} \to \varphi_\gamma(B_\gamma)$, which is a projection from $\mathcal{A}$ onto $\varphi_\gamma(B_\gamma)$. In this case, there exist $w_1, \cdots, w_n \in \varphi_\gamma(B_\gamma)$ such that $\|v_i - w_i\| < \varepsilon$ for all $i = 1, \cdots, n$. Since $P_\gamma(w_i) = w_i$, we have

$$\|P_\gamma(v_i) - v_i\| \leq \|P_\gamma(v_i - w_i)\| + \|w_i - v_i\| \leq 2\varepsilon.$$ 

Then $\{P_\gamma\}_{\gamma \in I}$ (with the canonical partial order on the index set $I$) is a net of completely contractive projections on $\mathcal{A}$ such that $\|P_\gamma(x) - x\| \to 0$ for all $x \in \mathcal{A}$.

On the other hand, if we have a net of finite-rank completely contractive projections $P_\gamma : \mathcal{A} \to \mathcal{A}$ such that $\|P_\gamma(x) - x\| \to 0$ for all $x \in \mathcal{A}$, then each $P_\gamma(\mathcal{A})$ is a finite-dimensional TRO and thus is completely isometric to some $\oplus_{k=1}^m M_{m(k),n(k)}$. Given $x_1, \cdots, x_n \in \mathcal{A}$ and $\varepsilon > 0$, there exists a completely contractive projection $P_\gamma$ such that $\|P_\gamma(x_i) - x_i\| < \varepsilon$, and thus

$$\text{dist}(x_i, P_\gamma(\mathcal{A})) < \varepsilon$$

for all $i = 1, \cdots, n$. This shows that $\mathcal{A}$ is a rigid rectangular $\mathcal{OL}_{\infty,1}$ space.

The following proposition shows that if $\mathcal{A}$ is a unital $C^*$-algebra and $P(1)$ is sufficiently close to the unital element 1 of $\mathcal{A}$, then $P(\mathcal{A})$ must be completely isometric to a finite-dimensional $C^*$-algebra.

Proposition 2.2. Let $\mathcal{A}$ be a unital $C^*$-algebra and let $P : \mathcal{A} \to \mathcal{A}$ be a finite-rank completely contractive projection. If $\|P(1) - 1\|_{\mathcal{A}} < \frac{1}{4}$, then $P(\mathcal{A})$ is completely isometric to a finite-dimensional $C^*$-algebra.

Proof. To simplify our notation, let us use $V = P(\mathcal{A})$ to denote the range space of $P$. Since $P$ is a finite-rank completely contractive projection on $\mathcal{A}$, $V$ is a finite-dimensional TRO with triple product given by

$$x \cdot y^* \cdot z = P(xy^*z)$$

for all $x, y, z \in V$. Up to (completely isometric) TRO-isomorphism, we may write

$$V = \oplus_{k=1}^l M_{m(k),n(k)}$$

and identify $V$ with the off-diagonal corner (i.e. the $(1,2)$ corner) of the finite-dimensional $C^*$-algebra $\mathcal{B} = \oplus_{k=1}^l M_{m(k)+n(k)}$. Then

$$C = \text{span}\{x \cdot y^* : x, y \in V\}$$

and

$$D = \text{span}\{y^* \cdot z : y, z \in V\}$$
are finite-dimensional \( C^* \)-subalgebras of \( \mathcal{B} \), and \( V \) is a faithful non-degenerate \((C, D)\)-bimodule. The norms on \( C \) and \( D \) can be determined by the left module norm and right module norm on \( V \), respectively. More precisely, for every \( c \in C \) we have
\[
\|c\|_C = \sup\{\|c \cdot x\|_V : x \in V, \|x\|_V \leq 1\},
\]
and for every \( d \in D \) we have
\[
\|d\|_D = \sup\{\|x \cdot d\|_V : x \in V, \|x\|_V \leq 1\}.
\]
If we let \( a = P(1) \in V \), then \( \|a\|_V = \|a\|_A \leq 1 \), and \( a^* \cdot a \) is a positive element in \( D \) such that
\[
\|a^* \cdot a\|_D = \sup\{\|x \cdot a^* \cdot a\|_V : x \in V, \|x\|_V \leq 1\}
\]
\[
= \sup\{\|P(xa^*a)\|_V : x \in V, \|x\|_V \leq 1\}
\]
\[
\leq \|a^*a\|_A = \|a\|^2_A \leq 1.
\]
Moreover, we have
\[
\|a^* \cdot a - 1\|_D = \sup\{\|x \cdot a^* \cdot a - x\|_V : x \in V, \|x\|_V \leq 1\}
\]
\[
= \sup\{\|P(xa^*a - x)\|_V : x \in V, \|x\|_V \leq 1\}
\]
\[
\leq \sup\{\|xa^*a - 1\|_A : x \in V, \|x\|_V \leq 1\}
\]
\[
\leq \|a^*a - 1\|_A \leq \|a^* - 1\|_A + \|a - 1\|_A \leq \frac{1}{4}.
\]
This shows that \( a^* \cdot a \) is an invertible element in \( D \), and
\[
\frac{3}{4} 1_D < a^* \cdot a \leq 1_D.
\]
It follows that its square root \(|a|\) in \( D \) satisfies
\[
\frac{\sqrt{3}}{2} 1_D < |a| \leq 1_D,
\]
and thus
\[
\|1_D - |a|\|_D \leq 1 - \frac{\sqrt{3}}{2} < \frac{1}{4}.
\]
Regarding \( a \) as an element in \( \mathcal{B} \), we can consider the polar decomposition \( a = v \cdot |a| \) for some partial isometry \( v \in \mathcal{B} \). Since \(|a|\) is an invertible element in \( D \), and \( V \) is a right \( D \)-module, we can conclude that
\[
v = a \cdot |a|^{-1} \in V,
\]
and thus \( v^* \cdot v \) is a projection in \( D \). We claim that \( v^* \cdot v = 1_D \).

For any \( x \in V \) with \( \|x\|_V \leq 1 \), we have
\[
\|x \cdot v^* \cdot v - x \cdot a^* \cdot a\|_V = \|P(xv^*v - xa^*a)\|_V \leq \|v^*v - a^*a\|_A
\]
\[
\leq \|v^* - a^*\|_A \|V\|_A + \|a^*\|_A \|v - a\|_A
\]
\[
\leq 2\|v - a\|_V \leq 2\|1_D - |a|\|_D \leq \frac{1}{2},
\]
and thus
\[
\|v^* \cdot v - a^* \cdot a\|_D < \frac{1}{2}.
\]
It follows that
\[ \|v^* \cdot v - 1_D\|_D \leq \|v^* \cdot v - a^* \cdot a\|_D + \|a^* \cdot a - 1_D\|_D \leq \frac{1}{2} + \frac{1}{4} < 1. \]
Since \(v^* \cdot v\) is a projection in \(D\), we must have \(v^* \cdot v = 1_D\). Similarly, we can prove that \(v \cdot v^* = 1_C\) in \(C\).

In this case, there is a natural \(C^*\)-algebra structure on \(V\) given by
\[ x \circ y = x \cdot v^* \cdot y \quad \text{and} \quad x^\dagger = v \cdot x^* \cdot v \]
for all \(x, y \in V\) (see Zettl [38]). With this \(C^*\)-algebra structure, \(v\) is the unit element of \(V\). The mapping
\[ \theta_D : x \in V \rightarrow v^* \cdot x \in D \]
is a unital \(*\)-isomorphism from \(V\) onto \(D\). The original matrix norm on \(V\) coincides with the \(C^*\)-algebra matrix norm on \(V\) since \(\theta_D\) is clearly a complete isometry from \(V\) onto \(D\). This completes the proof.

It is clear that every rigid \(\mathcal{OL}_{\infty,1}^+\) space is a rigid rectangular \(\mathcal{OL}_{\infty,1}^+\) space. But the converse is not necessarily true since for \(n \geq 2\), \(M_{1,n}\) is clearly a rigid rectangular \(\mathcal{OL}_{\infty,1}^+\) space, but not a rigid \(\mathcal{OL}_{\infty,1}^+\) space (see [14, §4]). The following theorem shows that the two notions coincide on unital \(C^*\)-algebras.

**Theorem 2.3.** Let \(A\) be a unital \(C^*\)-algebra. Then the following are equivalent:
(i) \(A\) is a rigid \(\mathcal{OL}_{\infty,1}^+\) space;
(ii) \(A\) is a rigid rectangular \(\mathcal{OL}_{\infty,1}^+\) space;
(iii) there exists a net of finite-rank completely contractive projections \(P_\gamma : A \rightarrow A\) such that \(P_\gamma \rightarrow \text{id}_A\) in the point-norm topology.

**Proof.** It is obvious that (i) \(\Rightarrow\) (ii). The equivalence (ii) \(\Leftrightarrow\) (iii) is given by Proposition 2.1. If we have (iii), then there exists \(\gamma_0\) such that
\[ \|P_\gamma(1) - 1\| < \frac{1}{8} \]
for all \(\gamma \geq \gamma_0\). It follows from Proposition 2.2 that \(P_\gamma(A)\) are completely isometric to finite-dimensional \(C^*\)-algebras for all \(\gamma \geq \gamma_0\). Therefore, \(A\) is a rigid \(\mathcal{OL}_{\infty,1}^+\) space.

3. Quasi-diagonality and inner quasi-diagonality

Let \(\mathcal{A}\) be a unital \(C^*\)-algebra and let \(\psi : \mathcal{A} \rightarrow \mathcal{B}(H)\) be a unital completely positive mapping. It is known from the Stinespring representation theorem that there exists a Hilbert space \(K\), a unital representation (i.e. a unital \(*\)-homomorphism) \(\pi : \mathcal{A} \rightarrow \mathcal{B}(K)\), and an isometry \(V \in \mathcal{B}(H,K)\) such that
\[ \psi(x) = V^* \pi(x) V \quad (x \in \mathcal{A}). \]
If we identify \(H\) with the closed subspace \(V(H)\) in \(K\) and let \(p\) be the orthogonal projection from \(K\) onto \(H\), then we may regard \(\psi\) as the *compression* of the unital representation \(\pi\) by \(p\) and write
\[ \psi(x) = p \pi(x) p \quad (x \in \mathcal{A}). \]
From this, we may easily obtain the Schwarz inequality
\[ \psi(x)^* \psi(x) \leq \psi(x^* x) \]
or equivalently,
\[(3.3) \quad 0 \leq \psi(x^* x) - \psi(x)^* \psi(x)\]
for all \(x \in \mathcal{A}\). Choi \([3]\) proved that for any \(x \in \mathcal{A}\) if \(\psi(x^* x) = \psi(x) \cdot \psi(x)\), then
\[\psi(yx) = \psi(y) \psi(x)\]
for all \(y \in \mathcal{A}\). In this case, we say that \(x\) is in the multiplicative domain \(D_\psi\) of \(\psi\). The following is a very useful quantitative estimate of Choi’s argument. We thank the referee for pointing out this simpler argument to us.

**Lemma 3.1.** Let \(\mathcal{A}\) and \(\mathcal{B}\) be unital C*-algebras and let \(\psi : \mathcal{A} \to \mathcal{B}\) be a unital completely positive mapping. Then we have
\[(3.4) \quad \|\psi(x^* y) - \psi(x)^* \psi(y)\| \leq \|\psi(x^* x) - \psi(x)^* \psi(x)\|^\frac{1}{2}\|\psi(y^* y) - \psi(y)^* \psi(y)\|^\frac{1}{2}\]
for all \(x, y \in \mathcal{A}\).

**Proof.** We may first assume that \(\mathcal{B}\) is a unital C*-algebra on some Hilbert space, and thus as we discussed in \([3.1]\), there exists a Hilbert space \(K\), a unital representation \(\pi : \mathcal{A} \to \mathcal{B}(K)\) and an orthogonal projection \(p\) on \(K\) such that
\[\psi(x) = p \pi(x) p\]
for all \(x \in \mathcal{A}\). If we let \(t : \mathcal{A} \to \mathcal{B}(K)\) be a complete contraction defined by
\[t(x) = (1 - p) \pi(x) p,\]
then we have
\[(3.5) \quad \psi(x^* y) - \psi(x)^* \psi(y) = p \pi(x^* y) p - p \pi(x^*) p \pi(y) p = t(x)^* t(y)\]
for all \(x, y \in \mathcal{A}\). It follows from \((3.3)\) that
\[\|\psi(x^* y) - \psi(x)^* \psi(y)\|^2 = \|t(x)^* t(y)\|^2 = \|t(y)^* t(x)^* t(y)\|^2\leq \|t(x)^* t(x)\| \|t(y)^* t(y)\| = \|t(x)^* t(x)\| \|t(y)^* t(y)\|\]
This proves the inequality \((3.4)\). \(\Box\)

It is well-known that quasi-diagonality is a very important property for C*-algebras. We recall from Voiculescu [36] that a C*-algebra \(\mathcal{A}\) is quasi-diagonal if for every \(x_1, \ldots, x_n \in \mathcal{A}\) and \(\varepsilon > 0\), there is a representation \(\rho\) of \(\mathcal{A}\) on a Hilbert space \(H\) and a finite-rank projection \(p \in \mathcal{B}(H)\) such that
\[(3.6) \quad \|\rho(x_i)p\| < \varepsilon \quad \text{and} \quad \|x_i\| < \|\rho p(x_i) p\| + \varepsilon.\]
For a unital C*-algebra \(\mathcal{A}\), this is equivalent to saying that for every \(x_1, \ldots, x_n \in \mathcal{A}\) and \(0 < \varepsilon < 1\), there exists a unital completely positive mapping \(\hat{\psi} : \mathcal{A} \to \mathcal{B}\) for some finite-dimensional C*-algebra \(\mathcal{B}\) such that
\[(3.7) \quad \|\hat{\psi}(x_i x_j) - \hat{\psi}(x_i) \hat{\psi}(x_j)\| < \varepsilon \quad \text{and} \quad \|x_j\| \leq \|\hat{\psi}(x_j)\| + \varepsilon.\]

**Theorem 3.2.** If a unital (nuclear) C*-algebra \(\mathcal{A}\) is an \(\mathcal{O}_{\infty, 1}\) space, then \(\mathcal{A}\) is quasi-diagonal.
Proof. It suffices to show that for every contractive elements $x_1, \cdots, x_n \in A$ (with $x_1 = 1$) and $0 < \varepsilon < 1$, we can find a unital complete positive mapping $\hat{\psi} : A \to B$ satisfying (3.7). We first let $\delta$ be a positive number with $0 < \delta < \frac{\varepsilon}{8n}$. Then there exists a finite-dimensional subspace $F$ in $A$ such that $F$ contains all $x_j$ $(1 \leq j \leq n)$ and there exists a linear isomorphism $\varphi$ from a finite-dimensional $C^*$-algebra $B$ onto $F$ such that $\|\varphi\|_{cb}\varphi^{-1}\|_{cb} < 1 + \frac{\delta^2}{2}$. Without loss of generality, we may assume that $\|\varphi\|_{cb} < 1 + \frac{\delta^2}{2}$ and $\|\varphi^{-1}\|_{cb} \leq 1$. Since $B$ is injective, $\varphi^{-1} : F \to B$ has a completely contractive extension $\psi : A \to B$.

In the following, we first show that we may suitably choose $\varphi$ and $\psi$ such that $\psi(1)$ is a positive element in $B$. Since $1 = x_1 \in F$, $b = \varphi^{-1}(1) = \psi(1)$ is a contractive element in $B$ such that $\varphi(b) = 1$ in $A$. If we let $c = (1 - b^*b)^{\frac{1}{2}} \in B$, then $\psi(c) = c$, we can find a unital complete positive mapping $\hat{\psi} : B \to F$ and thus satisfies the quasi-diagonal condition (3.7). We first let $\delta$ be a positive number with $0 < \delta < \frac{\varepsilon}{8n}$ and $\varphi$ be the polar decomposition of $b$, then $\varphi$ and $\varphi^{-1}$ must be a unitary in $B$. Moreover, since $1 - |b|$ and $1 + |b|$ commute in $B$, we have

$$\|1 - b^*b\| = \|c\|^2 = \|\psi \circ \varphi(c)\|^2 < \|\varphi(c)\|^2 < \delta^2 + \frac{\delta^4}{4} < 2\delta^2.$$  

This shows that $b$ is an invertible element in $B$. If we let $b = v|b|$ be the polar decomposition of $b$, then $v$ must be a unitary in $B$. Moreover, since $1 - |b|$ and $1 + |b|$ commute in $B$, we have

$$\|1 - |b|| \leq \|(1 - |b|)(1 + |b|)\| = \|1 - b^*b\| = 2\delta^2.$$  

Now we can modify $\psi$ and $\varphi$ by the unitary $v$. We let $\tilde{\psi} : B \to F$ and $\tilde{\psi} : A \to B$ be mappings defined by

$$\tilde{\psi}(a) = \varphi(va) \quad \text{and} \quad \tilde{\psi}(x) = v^*\psi(x)$$

for all $a \in B$ and $x \in A$. Then it is easy to see that $\tilde{\psi}$ is a linear isomorphism from $B$ onto $F$ with $\|\tilde{\psi}\|_{cb} = \|\varphi\|_{cb} < 1 + \varepsilon$ and $\tilde{\psi}$ is a completely contractive extension of $\varphi^{-1}$ from $A$ onto $B$ such that $\tilde{\psi}(1) = v^*b = |b|$ is positive in $B$.

Next we construct a unital completely positive mapping $\tilde{\psi} : A \to B$, which is "sufficiently close" to $\tilde{\psi}$ and thus satisfies the quasi-diagonal condition (3.7). For this purpose, let us assume that $B$ is a unital $C^*$-subalgebra of some matrix algebra $B(\mathbb{C}^k)$. It follows from Paulsen [29] that there exists a unital representation $\rho : A \to B(H)$ for some Hilbert space $H$ and isometries $V, W : \mathbb{C}^k \to H$ such that

$$\tilde{\psi}(x) = V^*\rho(x)W$$

for all $x \in A$. Since

$$V^*W = \tilde{\psi}(1) = |b| = \tilde{\psi}(1)^* = W^*V,$$
we obtain from (3.10) that
\[ \| V - W \|^2 = \|(V - W)^*(V - W)\| = \|V^*V + W^*W - V^*W - W^*V\| = 2\|1 - |b|\| < 4\delta^2. \]
This shows that \( \| V - W \| \leq 2\delta. \)

Since \( \mathcal{B} \) is a unital injective \( C^* \)-subalgebra in \( \mathcal{B}(\mathbb{C}^k) \), there exists a (completely positive) conditional expectation \( P : \mathcal{B}(\mathbb{C}^k) \to \mathcal{B} \). Then \( \hat{\psi} : \mathcal{A} \to \mathcal{B} \) defined by
\[ \hat{\psi}(x) = P(V^* \rho(x)V) \]
is a unital completely positive mapping from \( \mathcal{A} \) into \( \mathcal{B} \). Since the range of \( \hat{\psi} \) is contained in \( \mathcal{B} \), we have
\[ \hat{\psi} - \tilde{\psi} = P(V^* \rho V) - V^* \rho W = P(V^* \rho(V - W)), \]
and thus
\[ \| \hat{\psi} - \tilde{\psi} \|_{cb} \leq \| V - W \| < 2\delta. \] (3.11)

Given a unitary element \( u \in \mathcal{B} \), if we let \( y = \tilde{\varphi}(u) \), then \( \| y \| < 1 + \frac{\delta^2}{2} \) and \( \hat{\psi}(y) = \tilde{\psi} \circ \tilde{\varphi}(u) = u \). It follows that
\[ \| \hat{\psi}(y) - u \| = \| \hat{\psi}(y) - \tilde{\psi}(y) \| < 2\delta \| y \| < 3\delta, \]
and thus
\[ \| \hat{\psi}(y)* \hat{\psi}(y) - 1 \| \leq \| \hat{\psi}(y)* \| \| \hat{\psi}(y) - u \| + \| \hat{\psi}(y)* - u^* \| \| u \| < (2 + \frac{\delta^2}{2})3\delta < 8\delta. \]

From the Schwarz inequality (3.2), we get
\[ 1 - 8\delta \leq \hat{\psi}(y)^* \hat{\psi}(y) \leq \hat{\psi}(y^* y) < (1 + \frac{\delta^2}{2})^2, \]
and thus
\[ 0 \leq \hat{\psi}(y^* y) - \hat{\psi}(y)^* \hat{\psi}(y) \leq 8\delta + \delta^2 + \frac{\delta^4}{4} < 16\delta. \] (3.12)

On the other hand, for every \( x \in \mathcal{A} \) we get the inequality
\[ 0 \leq \hat{\psi}(x^* x) - \hat{\psi}(x)^* \hat{\psi}(x) \leq \hat{\psi}(x^* x) \] (3.13)
from (3.3). Then applying Lemma (3.1) together with (3.12) and (3.13), we obtain
\[ \| \hat{\psi}(xy) - \hat{\psi}(x)\hat{\psi}(y) \| \leq \| \hat{\psi}(x^* x) - \hat{\psi}(x)^* \hat{\psi}(x) \|^{\frac{1}{2}} \| \hat{\psi}(y^* y) - \hat{\psi}(y)^* \hat{\psi}(y) \|^{\frac{1}{2}} \leq 4\sqrt{\delta} \| x \| \leq \varepsilon \| x \| \]
for all \( x \in \mathcal{A} \).

In general, if we are given any \( y \in F \) with \( \| y \| < 1 \), then \( a = \hat{\psi}(y) \) is contained in the open unit ball of \( \mathcal{B} \), and thus can be written as a convex combination of unitary elements in \( \mathcal{B} \) by the Russo-Dye theorem, i.e. there exist unitary elements \( u_i \in \mathcal{B} \) and positive numbers \( \alpha_i \) with \( \sum \alpha_i = 1 \) such that \( a = \sum \alpha_i u_i \). In this case, we can write
\[ \tilde{\varphi}(a) = \sum \alpha_i \tilde{\varphi}(u_i) = \sum \alpha_i y_i \]
as a convex combination of \( y_i = \tilde{\varphi}(u_i) \), and thus obtain
\[ \| \hat{\psi}(xy) - \hat{\psi}(x)\hat{\psi}(y) \| \leq \sum \alpha_i \| \hat{\psi}(xy_i) - \hat{\psi}(x)\hat{\psi}(y_i) \| < \varepsilon \| x \|. \]
By the continuity of $\hat{\psi}$, we can conclude that
\[ \|\hat{\psi}(xy) - \hat{\psi}(x)\hat{\psi}(y)\| \leq \varepsilon \|x\|\|y\| \]
for all $x \in A$ and $y \in F$. This proves the first inequality in (3.7).

The second inequality follows from
\[ \|x_j\| = \|\tilde{\phi} \circ \tilde{\psi}(x_j)\| < (1 + \frac{\delta^2}{2})\|\tilde{\psi}(x_j)\| \]
\[ < \|\hat{\psi}(x_j)\| + \frac{\delta^2}{2} + 2\delta < \|\hat{\psi}(x_j)\| + \varepsilon. \]

Blackadar and Kirchberg studied NF algebras and strong NF algebras in [2]. They proved that a unital $C^*$-algebra is an NF algebra (see definition given in [2]) if and only if it is nuclear and quasi-diagonal. Moreover, they characterized strong NF algebras by nuclearity and inner quasi-diagonality in [3], where the inner condition requires that the finite-rank projection $p$ (in the definition of quasi-diagonality) is contained in $\pi(A)'$. More precisely, a $C^*$-algebra $A$ is said to be inner quasi-diagonal if for every $x_1, \cdots, x_n \in A$ and $\varepsilon > 0$, there is a representation $\rho$ of $A$ on a Hilbert space $K$ and a finite-rank projection $p \in \rho(A)' \subseteq B(K)$ such that (3.6) is satisfied. This is a stronger condition than quasi-diagonality (see examples given in [3]).

**Theorem 3.3.** If a unital (nuclear) $C^*$-algebra $A$ is a rigid $O\mathcal{L}_{\infty,1}$+ space, then it is inner quasi-diagonal, and thus is a strong NF algebra.

**Proof.** Given contractive elements $x_1, \cdots, x_n \in A$ (with $x_1 = 1$) and $0 < \varepsilon < 1$, we let $\delta$ be a positive number with $0 < \delta < \frac{\varepsilon^2}{16}$. Since $A$ is a rigid $O\mathcal{L}_{\infty,1}$+ space, there exists a complete isometry $\phi : B \to F$ from a finite-dimensional $C^*$-algebra $B$ onto a finite-dimensional subspace $F$ of $A$ such that
\[ \text{dist}(x_i, F) < \frac{\delta}{4} \]
for all $1 \leq i \leq n$. Then for each $i$, we may find an element $a_i \in B$ such that
\[ \|\phi(a_i) - x_i\| < \frac{\delta}{4} \text{ and } \|\phi(a_i)\| < 1 + \frac{\delta}{4}. \]

We note that in contrast to the argument in Theorem 3.2, $x_1 = 1$ need not be in $F$. We can only approximate it by $\phi(a_1)$. Therefore, we need a modified argument given as follows.

We let $\psi : A \to B$ be a completely contractive extension of $\phi^{-1}$, and let $b = \psi(1)$ and $c = (1 - b^*b)^{\frac{1}{2}} \in B$. Then we have
\[ \|a_1 - b\| = \|\psi \circ \phi(a_1) - \psi(1)\| \leq \|\phi(a_1) - 1\| < \frac{\delta}{4} \]
and
\[ \left\| \begin{bmatrix} b \\ c \end{bmatrix} \right\| = \|b^*b + c^*c\|^\frac{1}{2} = 1. \]
It follows that
\[
\left\| \frac{1}{c} \varphi(c) \right\|^2 < \left( \left\| \frac{\varphi(a_1)}{c} \right\| + \frac{\delta}{4} \right)^2 = \left( \left\| \frac{a_1}{c} \right\| + \frac{\delta}{4} \right)^2 < \left( \left\| \frac{b}{c} \right\| + \frac{\delta}{4} \right)^2 = (1 + \frac{\delta}{2})^2.
\]
Using the same argument given for (3.8), (3.9) and (3.10), we can prove that if \( b = v|b| \) is an invertible element in \( \mathcal{B} \) and if \( \psi \) is the polar decomposition for \( b \), then \( v \) is a unitary in \( \mathcal{B} \) and
\[
0 \leq ||1 - |b|| \leq 2\delta^2.
\]
Then we may define a complete isometry \( \tilde{\varphi} : \mathcal{B} \rightarrow F \) and a complete contraction \( \tilde{\psi} : \mathcal{A} \rightarrow \mathcal{B} \) by letting
\[
\tilde{\varphi}(a) = \varphi(va) \quad \text{and} \quad \tilde{\psi}(x) = v^*\psi(x)
\]
for all \( a \in \mathcal{B} \) and \( x \in \mathcal{A} \). It is easy to see that \( \tilde{\psi}(1) = |b| \) is a positive element in \( \mathcal{B} \) and we have \( \psi \circ \tilde{\varphi} = id_B \) and \( \varphi \circ \tilde{\psi} = id_F \).

We let
\[
\mathcal{L}_B = \left\{ \begin{bmatrix} \lambda & a \\ b^* & \mu \end{bmatrix} : \lambda, \mu \in \mathbb{C}, a, b \in \mathcal{B} \right\}
\]
denote the operator system in \( M_2(\mathcal{B}) \) induced by \( \mathcal{B} \), and let \( \Phi : \mathcal{L}_B \rightarrow M_2(\mathcal{A}) \) be the unital completely positive mapping from \( \mathcal{L}_B \) onto the operator system \( \mathcal{L}_F \) in \( M_2(\mathcal{A}) \), which is defined by
\[
\Phi \left( \begin{bmatrix} \lambda & a \\ b^* & \mu \end{bmatrix} \right) = \begin{bmatrix} \lambda & \tilde{\varphi}(b) \tilde{\varphi}(a) \\ \tilde{\varphi}(b^*) & \mu \end{bmatrix}.
\]
We let
\[
\mathcal{L}_A = \left\{ \begin{bmatrix} \lambda & x \\ y^* & \mu \end{bmatrix} : \lambda, \mu \in \mathbb{C}, x, y \in \mathcal{A} \right\}
\]
denote the operator system in \( M_2(\mathcal{A}) \) induced by \( \mathcal{A} \), and let \( \Psi : \mathcal{L}_A \rightarrow M_2(\mathcal{B}) \) be the unital completely positive mapping defined by
\[
\Psi \left( \begin{bmatrix} \lambda & x \\ y^* & \mu \end{bmatrix} \right) = \begin{bmatrix} \lambda & \tilde{\psi}(x) \tilde{\psi}(y)^* \\ \tilde{\psi}(y)^* & \mu \end{bmatrix}.
\]
Then \( \Psi \) extends to a unital completely positive mapping, which is still denoted by \( \Psi \), from \( M_2(\mathcal{A}) \) into \( M_2(\mathcal{B}) \). It is easy to see that \( \Psi \circ \Phi = id_{\mathcal{L}_F} \) and thus \( \Phi \) is a completely order isomorphism from \( \mathcal{L}_B \) onto \( \mathcal{L}_F \). We note that \( \mathcal{L}_F \) is contained in the multiplicative domain \( D_\Psi \) of \( \Psi \), i.e. for every \( \tilde{y} \in \mathcal{L}_F \), we have
\[
\Psi(\tilde{x}\tilde{y}) = \Psi(\tilde{x})\Psi(\tilde{y})
\]
for all \( \tilde{x} \in M_2(\mathcal{A}) \). To see this, given any \( \tilde{a} = \begin{bmatrix} \lambda & u \\ v & \mu \end{bmatrix} \in \mathcal{L}_B \) with \( u, v \) being unitaries in \( \mathcal{B} \), \( \tilde{a}^*\tilde{a} \) is again an element in \( \mathcal{L}_B \) and thus satisfies
\[
\tilde{a}^*\tilde{a} = \Psi(\Phi(\tilde{a}^*)\Phi(\tilde{a})) \leq \Psi(\Phi(\tilde{a})^*)\Phi(\tilde{a})^* \leq \Psi(\Phi(\tilde{a}^*\tilde{a})) = \tilde{a}^*\tilde{a}.
\]
Therefore, we have
\[
\Psi(\Phi(\tilde{a^*})\Phi(\tilde{a})) = \Psi(\Phi(\tilde{a})^*\Phi(\tilde{a}))
\]
and thus \( \tilde{y} = \Phi(\tilde{a}) \) is contained in the multiplicative domain of \( \Psi \) by Choi [9]. Since the open unit ball of \( \mathcal{B} \) is contained in convex hull of unitary elements in \( \mathcal{B} \), we may conclude that \( \mathcal{L}_F = \Phi(\mathcal{L}_B) \) is contained in the multiplicative domain \( D_\Psi \) of \( \Psi \).
We let $\mathcal{A}_\Psi$ denote the unital subalgebra of $\mathcal{A}$ generated by $L_F$. It is easy to see that $\mathcal{A}_\Psi$ is contained in the multiplicative domain $D_\Psi$. Since $L_F$ is an operator system in $\mathcal{A}$, $\mathcal{A}_\Psi$ must be self-adjoint and thus is a unital $C^*$-subalgebra of $\mathcal{A}$. Then $\Psi$ restricted to $\mathcal{A}_\Psi$ induces a unital $*$-homomorphism $\pi = \Psi_{|\mathcal{A}_\Psi}$ from $\mathcal{A}_\Psi$ into $M_2(\mathcal{B})$. Since $L_B$ is contained in the range of $\pi$, it is easy to see that $\pi$ maps $\mathcal{A}_\Psi$ onto $M_2(\mathcal{B})$. We note that this is a generalization of an argument in Choi and Effros [6, Theorem 4.1].

We may write $\mathcal{B} = \oplus_k \mathcal{B}(\mathbb{C}^{n_k})$ and $\pi = \oplus_k \pi_k$, where each $\pi_k$ is an irreducible representation from $\mathcal{A}_\Psi$ onto $M_2(\mathcal{B}(\mathbb{C}^{n_k})) = \mathcal{B}(\mathbb{C}^{n_k})$. For each $k$, we may extend $\pi_k$ to an irreducible representation $\tilde{\pi}_k : M_2(\mathcal{A}) \to \mathcal{B}(\mathcal{H}_k)$ on a larger Hilbert space $\mathcal{H}_k$. Up to unitary equivalence, we may write

$$\pi_k = id_2 \otimes \rho_k,$$

where $\rho_k : \mathcal{A} \to \mathcal{B}(\mathcal{K}_k)$ is an irreducible representation, and we can write

$$\pi_k = \begin{bmatrix} V_k^* & 0 \\ 0 & W_k^* \end{bmatrix} (id_2 \otimes \rho_k) \begin{bmatrix} V_k & 0 \\ 0 & W_k \end{bmatrix}$$

for some isometries $V_k$ and $W_k$ from $\mathbb{C}^{n_k}$ into $\mathcal{K}_k$. If we let $V = \oplus_{k=1}^n V_k$ and let $W = \oplus_{k=1}^n W_k$, then $V$ and $W$ are isometries satisfying

$$V^*W = V^*\rho(1)W = \tilde{\psi}(1) \geq 0$$

in $\mathcal{B}$. It follows (by the same argument given in Theorem 3.2) that we have

$$\|V_k - W_k\| \leq \|V - W\| < 2\delta,$$

and the orthogonal projections $p_k = V_k V_k^*$ and $q_k = W_k W_k^*$ from $K_k$ on $\mathbb{C}^{n_k}$ satisfy

$$\|p_k - q_k\| \leq \|V_k - W_k\| \|V_k^*\| + \|W_k\| \|V_k^* - W_k^*\| < 4\delta$$

for all $k$. Since the range of $\tilde{\pi}_{k(\mathcal{A}_\Psi)} = \pi_k$ is contained in $M_2(\mathcal{B}(\mathbb{C}^{n_k})) = \mathcal{B}(\mathbb{C}^{n_k} \oplus \mathbb{C}^{n_k})$, the orthogonal projection $p_k \oplus q_k = \begin{bmatrix} p_k & 0 \\ 0 & q_k \end{bmatrix}$ from $K_k \oplus K_k$ onto $\mathbb{C}^{n_k} \oplus \mathbb{C}^{n_k}$ leaves $\tilde{\pi}_k(\mathcal{A}_\Psi)$ invariant, i.e., we have

$$\tilde{\pi}_k(\tilde{x})(p_k \oplus q_k) = (p_k \oplus q_k) \tilde{\pi}_k(\tilde{x}),$$

for all $\tilde{x} \in \mathcal{A}_\Psi$. Then for any $y \in F$,

$$\tilde{\pi}_k \begin{bmatrix} 0 & y \\ 0 & 0 \end{bmatrix} \begin{bmatrix} p_k & 0 \\ 0 & q_k \end{bmatrix} = \begin{bmatrix} p_k & 0 \\ 0 & q_k \end{bmatrix} \tilde{\pi}_k \begin{bmatrix} 0 & y \\ 0 & 0 \end{bmatrix}$$

implies

$$p_k \rho_j(y) = \rho_j(y) q_k.$$

In general, $\rho_k$ are not necessarily all non-equivalent. We let $\rho_1, \cdots, \rho_r$ be non-equivalent irreducible representations, and for $k > r$, we may choose $l \leq r$ such that $\rho_k \cong \rho_l$ and identify $K_k$ with $K_l$, and $V_k$, $W_k$ with the corresponding isometries $V_l$, $W_l$ from $\mathbb{C}^{n_l}$ into $\mathcal{K}_l$. If we let $\rho = \oplus_{k=1}^r \rho_k$, $K = \oplus_{k=1}^r K_k$, then

$$p = \oplus_{k=1}^r p_k \quad \text{and} \quad q = \oplus_{k=1}^r q_k$$

are projections contained in $\oplus_{k=1}^r \mathcal{B}(\mathbb{C}^{n_k}) = \rho(\mathcal{A}^\prime)$, and we can conclude from (3.15) and (3.16) that

$$\|p - q\| = \max\{\|p_k - q_k\|\} < 4\delta$$
Theorem 3.4. Let $A$ be a unital $C^*$-algebra. If $A$ is an $\mathcal{OL}_{\infty,\lambda}$ space with $\lambda \leq \left(\frac{1+\sqrt{5}}{2}\right)^2$, then $A$ must be a stably finite $C^*$-algebra.

Proof. We first note that the $\mathcal{OL}_{\infty,\lambda}$ structure is a stable property on operator spaces, i.e., if $A$ is an $\mathcal{OL}_{\infty,\lambda}$ space, then for each $n \in \mathbb{N}$, $M_n(A)$ is again an $\mathcal{OL}_{\infty,\lambda}$ space. Therefore, it suffices to show that $A$ is a finite $C^*$-algebra. The stable finiteness follows immediately.

Let us assume that $A$ is not finite. Then there is an isometry $s \in A$ such that $p = 1 - ss^* \neq 0$. Since $A$ is an $\mathcal{OL}_{\infty,\lambda}$ space, there exists a finite-dimensional subspace $F \subseteq A$ which contains $\{s, p\}$ and there exists a linear isomorphism

$$\varphi : F \to B$$

from $F$ onto a finite-dimensional $C^*$-algebra $B$ such that

$$\|\varphi\|_{cb} < 1 \quad \text{and} \quad \|\varphi^{-1}\|_{cb} \leq \lambda.$$ 

Since $\|[s, p]\| = \|[ss^* + (1 - ss^*)]\| = 1$ and $\frac{1}{\lambda} = \frac{1}{\lambda}\|p\| < \|\varphi(p)\|$, we have

$$\|\varphi(s) \varphi(p)\| \leq \|[s, p]\| = 1$$

and thus

$$\frac{1}{\lambda^2} < \|\varphi(p)\|^2 = \|\varphi(p)\varphi(p)^*\| \leq \|1 - \varphi(s)\varphi(s)^*\|.$$ 

On the other hand, let us consider $b = (1 - \varphi(s)^*\varphi(s))^{\frac{1}{2}} \in B$. Since $s^*s = 1$ and

$$\|1 + \varphi^{-1}(b)^*\varphi^{-1}(b)\|^{\frac{1}{2}} = \left\|\begin{bmatrix} s \\ \varphi^{-1}(b) \end{bmatrix}\right\| \leq \|\varphi^{-1}\|_{cb} \left\|\begin{bmatrix} \varphi(s) \\ b \end{bmatrix}\right\| < \lambda,$$
we have
\[ \|1 - \varphi(s)^* \varphi(s)\| = \|b\|^2 \leq \|\varphi^{-1}(b)\|^2 = \|\varphi^{-1}(b)^* \varphi^{-1}(b)\| < \lambda^2 - 1 \leq \frac{\sqrt{5} - 1}{2} < 1. \]

Then \( \varphi(s)^* \varphi(s) \) and thus \( \varphi(s) \) must be invertible elements in the finite-dimensional 
\( C^* \)-algebra \( B \). Considering the polar decomposition of \( \varphi(s) \), we get
\[ \|1 - \varphi(s)\varphi(s)^*\| = \|1 - \varphi(s)^* \varphi(s)\|, \]
and thus
\[ \frac{1}{\lambda^2} \leq \|1 - \varphi(s)\varphi(s)^*\| = \|1 - \varphi(s)^* \varphi(s)\| < \lambda^2 - 1. \]

This implies \( \lambda > (\frac{1+\sqrt{5}}{2})^\frac{1}{2} \) and leads to a contradiction to the hypothesis.

4. \( \mathcal{OL}_{\infty,\lambda} \) STRUCTURE RELATED TO THE SECOND DUALS

Let us first recall that an operator space \( V \) is said to be \emph{locally reflexive} if for any finite-dimensional operator space \( \tilde{F} \), each complete contraction \( \varphi : \tilde{F} \to V^{**} \) can be approximated in the point-weak* topology by complete contractions \( \psi : \tilde{F} \to V \). Equivalently, given any finite-dimensional subspace \( G \subseteq V^* \) and \( \varepsilon > 0 \), we can find a mapping \( \psi : \tilde{F} \to V \) such that
\[ (4.1) \quad \|\psi\|_{cb} < 1 + \varepsilon \text{ and } \langle \psi(x), f \rangle = \langle \varphi(x), f \rangle \]
for all \( x \in \tilde{F} \) and \( f \in G \) (see [9] Lemma 6.4). An operator space \( V \) is said to be \emph{strongly locally reflexive} if given any finite-dimensional subspaces \( \tilde{F} \subseteq V^{**} \), \( G \subseteq V^* \) and \( \varepsilon > 0 \), there exists a linear isomorphism \( \psi \) from \( \tilde{F} \) onto a subspace \( \tilde{F}' \) of \( V \) such that
\begin{enumerate}[(a)]  
  \item \( \|\psi\|_{cb} \|\psi^{-1}\|_{cb} < 1 + \varepsilon \),
  \item \( \langle \psi(v), f \rangle = \langle v, f \rangle \) for all \( v \in \tilde{F} \) and \( f \in G \), and
  \item \( \psi(v) = v \) for all \( v \in \tilde{F} \cap V \).
\end{enumerate}
It is obvious that every strongly locally reflexive operator space is locally reflexive, and it was shown in [9] that the operator preduals of von Neumann algebras are all strongly locally reflexive.

\textbf{Lemma 4.1.} Suppose that \( V \) and \( W \) are finite-dimensional operator spaces and \( A \) is a \( C^* \)-algebra. If a linear mapping \( \varphi : V \to W \) has a completely bounded factorization
\[ \alpha \xrightarrow{A^{**}} \beta \xrightarrow{\varphi} W, \]
then for any \( \varepsilon > 0 \) we may replace \( \beta \) in above diagram by a weak* continuous completely bounded mapping \( \tilde{\beta} : A^{**} \to W \) such that \( \|\tilde{\beta}\|_{cb} \leq (1 + \varepsilon)\|\beta\|_{cb} \).

\textit{Proof.} Let \( \beta^* : W^* \to A^{***} \) be the adjoint mapping of \( \beta \). Then \( \beta^*(W^*) \) is a finite-dimensional subspace of \( A^{**} \). Since the operator dual \( A^* \) of the \( C^* \)-algebra \( A \) is locally reflexive (see [9]), for any \( \varepsilon > 0 \) and the finite-dimensional subspace \( \alpha(V) \subseteq A^* \), there exists a mapping \( \psi : \beta^*(W^*) \to A^* \) such that \( \|\psi\|_{cb} < 1 + \varepsilon \) and
\[ \langle \alpha(v), \psi \circ \beta^*(f) \rangle = \langle \alpha(v), \beta^*(f) \rangle \]
for all \( f \in W^* \) and \( v \in V \). Then the adjoint mapping
\[ \tilde{\beta} = (\psi \circ \beta^*)^* : A^{**} \to W \]
is weak* continuous such that
\[ \|\hat{\beta}\|_{cb} \leq \|\psi\|_{cb}\|\beta^*\|_{cb} < (1 + \varepsilon)\|\beta\|_{cb}, \]
and
\[ \langle \hat{\beta} \circ \alpha(v), f \rangle = \langle \alpha(v), \psi \circ \beta^*(f) \rangle = \langle \alpha(v), \beta^*(f) \rangle = \langle \beta \circ \alpha(v), f \rangle = \langle \varphi(v), f \rangle \]
for all \( f \in W^* \) and \( v \in V \). From this we can conclude that \( \varphi = \hat{\beta} \circ \alpha. \)

The following lemma is a generalization of [1, Theorem 7.6] from the case \( \mathcal{A} = K(H) \) to general locally reflexive \( C^* \)-algebras.

**Lemma 4.2.** Suppose that \( V \) and \( W \) are finite-dimensional operator spaces and \( \mathcal{A} \) is a locally reflexive \( C^* \)-algebra. If a linear mapping \( \varphi : V \to W \) has a completely bounded factorization

\[
\begin{array}{ccc}
V & \xrightarrow{\alpha} & \mathcal{A}^* \\
\downarrow{\hat{\alpha}} & & \downarrow{\hat{\beta}} \\
V & \xrightarrow{\varphi} & W
\end{array}
\]

then for any \( 0 < \varepsilon < 1 \) we may obtain a completely bounded factorization

\[
\begin{array}{ccc}
V & \xrightarrow{\alpha} & \mathcal{A}^* \\
\downarrow{\hat{\alpha}} & & \downarrow{\hat{\beta}} \\
V & \xrightarrow{\varphi} & W
\end{array}
\]

such that

\[
\|\hat{\alpha}\|_{cb} < (1 + \varepsilon)\|\alpha\|_{cb} \text{ and } \|\hat{\beta}\|_{cb} < (1 + \varepsilon)\|\beta\|_{cb}.
\]

Moreover, we can choose \( \hat{\alpha} \) such that

\[
\alpha(V) \cap \mathcal{A} \subseteq \hat{\alpha}(V).
\]

**Proof.** Given \( 0 < \varepsilon < 1 \), it follows from Lemma 4.1 that we may first replace \( \beta \) in (4.2) by a weak* continuous mapping \( \hat{\beta} : \mathcal{A}^* \to W \) such that

\[ \varphi = \hat{\beta} \circ \alpha \text{ and } \|\hat{\beta}\|_{cb} < (1 + \varepsilon)\|\beta\|_{cb}. \]

The pre-adjoint \( \hat{\beta}_*: W^* \to \mathcal{A}^* \) is a well-defined mapping from \( W^* \) into \( \mathcal{A}^* \), and \( \hat{\beta}_*(W^*) \) is a finite-dimensional subspace of \( \mathcal{A}^* \). We let \( \tilde{F} = \alpha(V) \) denote the finite-dimensional subspace in \( \mathcal{A}^* \), and let \( n = \text{dim}\tilde{F} \) and \( 0 < \delta < \frac{\varepsilon}{2n^2} \). Since \( \mathcal{A} \) is locally reflexive, there exists a mapping \( \psi : \tilde{F} \to \mathcal{A} \) such that

\[
\|\psi\|_{cb} < 1 + \delta < 1 + \varepsilon,
\]
and
\[ \langle \psi \circ \alpha(v), \hat{\beta}_*(f) \rangle = \langle \alpha(v), \hat{\beta}_*(f) \rangle \]
for all \( v \in V \) and \( f \in W^* \). This shows that

\[
\hat{\beta} \circ \psi \circ \alpha = \hat{\beta} \circ \alpha = \varphi,
\]
and we obtain (4.3) and (4.4) by letting \( \hat{\alpha} = \psi \circ \alpha \).

Note that in general an infinite-dimensional \( C^* \)-algebra (for example, \( K(\ell_2) \)) is not strongly locally reflexive. So we need the following argument to obtain \( \hat{\alpha} \) satisfying (4.5). We let \( C \subseteq \text{CB}(\tilde{F}, \mathcal{A}) \) be the convex set of all mappings \( \psi : \tilde{F} \to \mathcal{A} \) satisfying (4.6) and (4.7). We let \( F = \tilde{F} \cap \mathcal{A} \) and let \( \iota : F \to \mathcal{A} \) be the inclusion mapping. We let \( C_0 \subseteq \text{CB}(F, \mathcal{A}) \) denote the convex set of all mappings \( \psi \circ \iota \), where \( \psi \in C \). We claim that \( \iota \) is in the point-norm closure of \( C_0 \). This is apparent since
if we are given an arbitrary finite-dimensional subspace $G \subseteq A^*$, then our previous argument shows that there is a mapping $\psi' : \tilde{F} \to A$ satisfying
\[ \|\psi'\|_{cb} < 1 + \delta, \]
and
\[ (\psi' \circ \alpha(v), f) = (\alpha(v), f) \]
for all $v \in V$ and $f \in \tilde{\beta}_\epsilon(W^*) + G$. The latter equation implies that
\[ \tilde{\beta} \circ \psi' \circ \alpha = \tilde{\beta} \circ \alpha = \varphi. \]
This shows that we can suitably choose a net of $\psi'$ in $C$ such that $\psi' \circ \iota \in C_0$ converges to $\iota$ in the point-weak topology. Then the usual convexity argument shows that $\iota$ is in the point-norm closure of $C_0$. Since $F$ is finite-dimensional, its closed unit ball is totally bounded and thus we may choose a mapping $\psi \in C$ such that
\[ \|\iota - \psi \circ \iota\|_{cb} < \delta. \]
From this we can conclude (see [8, Lemma 2.3]) that
\[ \|\iota - \psi \circ \iota\|_{cb} < \delta n. \]
We next perturb $\psi$ in order to satisfy
\[ (4.8) \]
It follows from [13, Lemma 5.2] that there is a projection $P$ of $\tilde{F}$ onto $F$ with $1 \leq \|P\|_{cb} \leq n^2$. Then
\[ \tilde{\psi} = (\iota - \psi) \circ P + \psi : \tilde{F} \to A \]
is a completely bounded mapping satisfying (4.8). Given any $v \in V$, $P \circ \alpha(v)$ is an element in $F = \alpha(V) \cap A$. Then there exists $v_0 \in V$ such that $P \circ \alpha(v) = \alpha(v_0)$. Since $\psi$ satisfies (4.7), we have
\[ \tilde{\beta} \circ \psi \circ P \circ \alpha(v) = \tilde{\beta} \circ \psi \circ \alpha(v_0) = \tilde{\beta} \circ \alpha(v_0) = \tilde{\beta} \circ P \circ \alpha(v), \]
and thus
\[ \tilde{\beta} \circ \psi \circ \alpha = \tilde{\beta} \circ (\iota - \psi) \circ P \circ \alpha + \tilde{\beta} \circ \psi \circ \alpha = \tilde{\beta} \circ \psi \circ \alpha = \varphi. \]
This shows that $\tilde{\psi}$ also satisfies (4.7). Finally, $\tilde{\psi}$ satisfies
\[ \|\tilde{\psi}\|_{cb} \leq \|\iota - \psi \circ \iota\|_{cb} \|P\|_{cb} + (1 + \delta) \leq \delta n^3 + (1 + \delta) < 1 + \varepsilon. \]
If we let $\tilde{\alpha} = \tilde{\psi} \circ \alpha$ and let $\tilde{\beta}$ also denote its restriction to $A$, we obtain the completely bounded factorization
\[ V \xrightarrow{\tilde{\alpha}} \tilde{F} \xrightarrow{\tilde{\psi} \circ \alpha} W \]such that
\[ \|\tilde{\alpha}\|_{cb} \leq \|\tilde{\psi}\|_{cb} \|\alpha\|_{cb} < (1 + \varepsilon)\|\alpha\|_{cb} \]
and
\[ \|\tilde{\beta}\|_{cb} < (1 + \varepsilon)\|\beta\|_{cb}. \]
We have $\alpha(V) \cap A \subseteq \tilde{\alpha}(V)$ since for any $x \in F = \alpha(V) \cap A$, there exists $v \in V$ such that

$$x = \alpha(v) = \tilde{\psi} \circ \alpha(v) = \tilde{\alpha}(v) \in \tilde{\alpha}(V),$$

where we used the fact that $\tilde{\psi}|_F = id_F$.

\[ \text{Theorem 4.3.} \quad \text{Let } A \text{ be a locally reflexive } C^*\text{-algebra and let } E \text{ be a finite-dimensional subspace of } A. \text{ If } \tilde{F} \text{ is a finite-dimensional subspace of } A^{**} \text{ such that } E \subseteq \tilde{F} \text{ and } d_{cb}(\tilde{F}, B) < \lambda \text{ for some finite-dimensional } C^*\text{-algebra } B, \text{ then there exists a finite-dimensional subspace } F \text{ of } A \text{ such that } E \subseteq F \text{ and } d_{cb}(F, B) < \lambda. \]

\[ \text{Proof.} \quad \text{Let } B \text{ be a finite-dimensional } C^*\text{-algebra and let } \alpha : B \to \tilde{F} \subseteq A^{**} \]

be a linear isomorphism from $B$ onto $\tilde{F}$ such that $\|\alpha\|_{cb} \|\alpha^{-1}\|_{cb} < \lambda$. Since $B$ is an injective operator space, the inverse mapping $\alpha^{-1} : \tilde{F} \to B$ has a completely bounded extension $\beta : A^{**} \to B$ with $\|\beta\|_{cb} = \|\alpha^{-1}\|_{cb}$. Then we obtain a completely bounded factorization

$$\xymatrix{ A^{**} \ar[r]^\alpha & B \ar[r]^{id_B} \ar[ld]^\beta & B }$$

with $\|\alpha\|_{cb} \|\beta\|_{cb} < \lambda$. It follows from Lemma 4.2 that we may find a completely bounded factorization

$$\xymatrix{ A \ar[r]^\alpha & B \ar[r]^{id_B} \ar[ld]^\beta & B }$$

such that $\|\alpha\|_{cb} \|\beta\|_{cb} < \lambda$ and

$$E \subseteq \tilde{F} \cap A = \alpha(B) \cap A \subseteq \tilde{\alpha}(B).$$

Then $F = \tilde{\alpha}(B)$ is a finite-dimensional subspace of $A$ containing $E$ and $\tilde{\alpha} : B \to F$ is a linear isomorphism with $\tilde{\alpha}^{-1} = \tilde{\beta}|_F$. From this we conclude that

$$d_{cb}(F, B) \leq \|\tilde{\alpha}\|_{cb} \|\tilde{\alpha}^{-1}\|_{cb} < \lambda.\quad \square$$

The following result is already known to Blackadar and Kirchberg [3] (see our discussion in §2). For the convenience of the readers, we include the following simple proof. Here, we do not need to assume the separability for $A$.

\[ \text{Theorem 4.4.} \quad \text{If } A \text{ is a subhomogeneous } C^*\text{-algebra, then } A \text{ is an } \mathcal{O}_{L,\infty,1}^+ \text{ space.} \]

\[ \text{Proof.} \quad \text{If } A \text{ is a subhomogeneous } C^*\text{-algebra, then all irreducible representations of } A \text{ are finite-dimensional with dim } \leq n \text{ for some positive integer } n \in \mathbb{N}, \text{ and thus its second dual } A^{**} \text{ has the form } A^{**} \cong \bigoplus_{k=1}^n L_\infty(X_k, \mathcal{M}_k, \mu_k) \hat{\otimes} M_k, \]
where we simply assume that \( L_\infty(X_k, M_k, \mu_k) = \{0\} \) if \( A \) does not have any irreducible representation of dimension \( k \). Since unital commutative \( C^* \)-algebras are rigid \( OL_{\infty,1} \) spaces (see discussion in §1), it is easy to see that \( A^{**} \cong \bigoplus_{k=1}^n L_\infty(X_k, M_k, \mu_k) \otimes M_k \) is a rigid \( OL_{\infty,1} \) space (or a strong NF algebra). The \( C^* \)-algebra \( A \) is nuclear and thus is locally reflexive. Given any finite-dimensional subspace \( E \subseteq A \), which can be regarded as a finite-dimensional subspace of \( A^{**} \), and any \( \varepsilon > 0 \), there exists a finite-dimensional subspace \( F \) in \( A^{**} \) such that

\[
d_{cb}(F, B) < 1 + \varepsilon
\]

for some finite-dimensional \( C^* \)-algebra \( B \). It follows from Theorem 4.3 that we may find a finite-dimensional subspace \( \tilde{F} \) in \( A^{**} \) such that \( E \subseteq \tilde{F} \) and

\[
d_{cb}(\tilde{F}, B) < 1 + \varepsilon.
\]

This shows that \( A \) is an \( OL_{\infty,1} \) space. \( \square \)

5. Non-subhomogeneous nuclear \( C^* \)-algebras

**Lemma 5.1.** If \( A \) is a non-subhomogeneous \( C^* \)-algebra, then there is a completely isometric and completely order preserving injection

\[
\theta : B(\ell_2) \to A^{**},
\]

which is a weak* homeomorphism from \( B(\ell_2) \) onto \( \theta(B(\ell_2)) \).

**Proof.** If \( A \) is non-subhomogeneous, then either \( A \) has an infinite-dimensional irreducible representation, or all irreducible representations of \( A \) are finite-dimensional, but the dimensions are not uniformly bounded.

If \( A \) has an infinite-dimensional irreducible representation \( \pi : A \to B(H) \) with \( \dim H = \infty \), then \( \pi \) induces a unique normal (i.e. weak*-continuous) representation \( \tilde{\pi} : A^{**} \to B(H) \) from \( A^{**} \) onto \( B(H) = \pi(A)'' \), and there is a central projection \( p \in A^{**} \) such that \( \ker \tilde{\pi} = (1 - p)A^{**} \), where \( \ker \tilde{\pi} \) is the kernel of \( \tilde{\pi} \). This induces a normal \(*\)-isomorphism

\[
B(H) \cong pA^{**}
\]

(see Takesaki’s book [35]). Since \( \dim H = \infty \) (which could be uncountable), we may identify \( \ell_2 \) with a subspace of \( H \) and identify \( B(\ell_2) \) with a von Neumann subalgebra of \( B(H) \). In this case, we obtain an injective normal \(*\)-homomorphism

\[
\theta : B(\ell_2) \to A^{**}.
\]

It is obvious that \( \theta \) is a completely isometric and completely order preserving weak* homeomorphism from \( B(\ell_2) \) onto \( \theta(B(\ell_2)) \).

Now if we assume that all irreducible representations of \( A \) are finite-dimensional, but not uniformly bounded, then there exists a strictly increasing sequence of positive integers \( n(k) \in \mathbb{N} \) such that for each \( k \), \( A \) has an irreducible representation

\[
\pi^k : A \to M_{n(k)}.
\]

By the same reason as above, for each \( k \in \mathbb{N} \) there exists a central projection \( p^k \) in \( A^{**} \) such that

\[
M_{n(k)} \cong p^k A^{**}.
\]
Since \( \pi^k \) are distinct irreducible representations of \( \mathcal{A} \) and there is no non-trivial central projections in \( M_{n(k)} \), \( \{p^k\} \) must be all distinct and mutually orthogonal. Then \( p = \sum_{k=1}^{\infty} p^k \) is a central projection in \( \mathcal{A}^{**} \) and \( \prod_{k=1}^{\infty} M_{n(k)} \) can be identified with the von Neumann subalgebra \( pA^{**} \) in \( \mathcal{A}^{**} \). For each \( k \in \mathbb{N} \), we let \( P^k \) denote the natural truncation mapping

\[
P^k : x \in \mathcal{B}(\ell_2) \to x^{n(k)} \in M_{n(k)}.
\]

It is known (see [1]) that the canonical mapping

\[
\theta : x \in \mathcal{B}(\ell_2) \to \theta(x) = (P^k(x)) \in \prod_{k=1}^{\infty} M_{n(k)} \cong pA^{**}
\]

is a completely positive and completely isometric injection. The mapping \( \theta \) is a complete order isomorphism from \( \mathcal{B}(\ell_2) \) onto \( \theta(\mathcal{B}(\ell_2)) \) since for every \( x \in \mathcal{B}(\ell_2) \), \( x \geq 0 \) if and only if \( P^k(x) \geq 0 \) for all \( k \in \mathbb{N} \). It is \( \ast \)-continuous since each truncation mapping \( P^k \) is \( \ast \)-continuous. It is also easy to see that \( \theta(\mathcal{B}(\ell_2)) \) is \( \ast \)-closed in \( \prod_{k=1}^{\infty} M_{n(k)} \cong pA^{**} \). Then \( \theta \) is a \( \ast \)-homeomorphism from \( \mathcal{B}(\ell_2) \) onto \( \theta(\mathcal{B}(\ell_2)) \).

In the following, we let \( \tau(x) = \int_0^1 x(t)dt \) denote the normal (tracial) state on \( L_\infty[0,1] \) and let \( \langle x \mid y \rangle = \tau(y^*x) \) stand for the inner product on \( L_2[0,1] \). We let \( r_k(t) = \text{sgn}sin(2^k\pi t) \) denote the Rademacher functions on \( [0,1] \). Then \( \{r_k\} \) is a sequence of self-adjoint unitary elements in \( L_\infty[0,1] \) since each \( r_k \) is a real valued function on \( [0,1] \) with \( r_k^2 = 1 \). Moreover, \( \{r_k\} \) forms an orthonormal set in \( L_2[0,1] \) since \( \langle r_k \mid r_{k'} \rangle = \delta_{k,k'} \).

**Lemma 5.2.** For any \( x \in L_\infty[0,1] \), we have

\[
\lim_{k \to \infty} \tau(r_k x) = 0.
\]

**Proof.** Since \( \{r_k\} \) is an orthonormal set in \( L_2[0,1] \), we have from the Bessel inequality that for any \( x \in L_\infty[0,1] \)

\[
\sum_{k=1}^{\infty} |\tau(r_k x)|^2 = \sum_{k=1}^{\infty} |\langle x \mid r_k \rangle|^2 \leq \|x\|_2^2.
\]

This implies \( \lim_{k \to \infty} \tau(r_k x) = 0 \).

**Lemma 5.3.** Let \( F \subseteq L_\infty[0,1] \otimes M_n \) and \( S \subseteq (L_\infty[0,1] \otimes M_n)^* \) be finite-dimensional operator spaces. Then for every \( \varepsilon > 0 \), there exists a complete contraction \( u : L_\infty[0,1] \otimes M_n \to M_n \) and a complete isometry \( v : M_n \to L_\infty[0,1] \otimes M_n \) such that \( u \circ v = \text{id}_{M_n} \) and

(i) \( \|u(x)\|_{M_n} < \varepsilon \|x\| \),

(ii) \( |\langle s, v(y) \rangle| < \varepsilon \|s\|_2 \|y\| \)

for all \( x \in F \), \( y \in M_n \) and \( s \in S \).

**Proof.** Using the Rademacher functions, we may define a sequence of complete contractions \( u_k : L_\infty[0,1] \otimes M_n \to M_n \) by letting

\[
u_k(x) = (\tau \otimes \text{id})(r_k \otimes 1)x
\]
for all $x \in L_\infty[0,1] \widehat{\otimes} M_n$. Let $\{e_{ij}\}$ denote the matrix unit of $M_n$. For every $x \in L_\infty[0,1] \widehat{\otimes} M_n$, we can write

$$x = \sum_{i,j} x_{ij} \otimes e_{ij}$$

with $x_{ij} \in L_\infty([0,1])$. It follows from Lemma 5.2 that

$$u_k(x) = \sum_{i,j} \tau(r_k x_{ij}) e_{ij} \rightarrow 0$$

in $M_n$. Since $F$ is a finite-dimensional subspace of $L_\infty[0,1] \widehat{\otimes} M_n$, its closed unit ball $D_F$ is norm compact and we can conclude from (5.1) that $u_k \rightarrow 0$ uniformly on $D_F$. Then for every $\varepsilon > 0$, there exists $k_1 \in \mathbb{N}$ such that for all $k \geq k_1$, $u_k$ satisfy the condition (i), i.e.

$$\|u_k(x)\|_{M_n} \leq \varepsilon \|x\|$$

for all $x \in F$.

We can consider another sequence of mappings $v_k : M_n \rightarrow L_\infty[0,1] \widehat{\otimes} M_n$ given by

$$v_k(y) = r_k \otimes y$$

for all $y \in M_n$. Since $r_k^2 = 1$, it is clear that $v_k$ are completely isometric injections, and $u_k \circ v_k = id_{M_n}$ since

$$u_k \circ v_k(y) = (\tau \otimes id)(r_k^2 \otimes y) = \tau(1)y = y$$

for all $y \in M_n$.

To see (ii), let us first recall from [11] that there is a complete isometry

$$(L_\infty[0,1] \widehat{\otimes} M_n)_{\ast} \cong L_1[0,1] \widehat{\otimes} T_n,$$

where $L_1[0,1] \widehat{\otimes} T_n$ is the operator projective tensor product of $L_1[0,1]$ and $T_n$ introduced in [4] and [12]. Given any $s \in S \subseteq (L_\infty[0,1] \widehat{\otimes} M_n)_{\ast}$, we may write

$$s = \sum_{i,j} s_{ij} \otimes \tilde{e}_{ij},$$

where $\{\tilde{e}_{ij}\}$ is the canonical dual basis of $\{e_{ij}\}$ in $T_n$ and $s_{ij}$ are integrable functions in $L_1[0,1]$ with $\|s_{ij}\|_{L_1[0,1]} \leq \|s\|$. Since $L_\infty[0,1]$ is norm dense in $L_1[0,1]$, there exist $h_{ij} \in L_\infty[0,1]$ such that

$$\|s_{ij} - h_{ij}\|_{L_1[0,1]} < \frac{\varepsilon}{4n^2} \|s\|.$$

It follows that

$$|\tau(r_k s_{ij} - r_k h_{ij})| \leq \|r_k\|_{L_\infty[0,1]} \|s_{ij} - h_{ij}\|_{L_1[0,1]} < \frac{\varepsilon}{4n^2} \|s\|,$$

since $\lim_{k \rightarrow \infty} |\tau(r_k h_{ij})| = 0$, we may choose $k_2 \geq k_1$ such that

$$|\tau(r_k h_{ij})| < \frac{\varepsilon}{4n^2} \|s\|.$$
Then given any \( y = \sum_{i,j} y_{ij} \otimes e_{ij} \in M_n \), we deduce from (5.2) and (5.3) that
\[
|\langle s, v_{k_2}(y) \rangle| = \left| \sum_{i,j} \tau(r_{k_2} s_{ij}) y_{ij} \right| \\
\leq \sum_{i,j} |y_{ij}| |\tau(r_{k_2} s_{ij} - r_{k_2} h_{ij})| + \sum_{i,j} |y_{ij}| |\tau(r_{k_2} h_{ij})| \\
< \frac{\varepsilon}{2} \|s\| \|y\|.
\]
Since the closed unit sphere \( D_S \) of \( S \) is norm compact, we can conclude that
\[
|\langle s, v_{k_2}(y) \rangle| < \varepsilon \|s\| \|y\|
\]
for all \( y \in M_n \) and \( s \in S \). Then the mappings \( u = u_{k_2} \) and \( v = v_{k_2} \) satisfy the conditions (i) and (ii).

As we discussed in §1, a \( C^* \)-algebra \( A \) is nuclear if and only if there exist approximate diagrams of complete contractions in (5.2) which approximately commute in the point-norm topology. This is equivalent to saying that for any finite-dimensional subspace \( E \) of \( A \) and \( \varepsilon > 0 \), there exists a matrix space \( M_n \) and a commutative diagram of completely bounded mappings
\[
M_n \xrightarrow{\alpha} E \xleftarrow{\epsilon} \xrightarrow{\beta} A
\]
(5.4)
such that \( \|\alpha\|_{cb} \leq 1 \) and \( \|\beta\|_{cb} < 1 + \varepsilon \).

The following Lemma is due to Oikhberg together with the observation that the modification of the conclusion is true for operator spaces with completely bounded approximation property, but fails for general operator spaces. For completeness, we include the proof.

**Lemma 5.4.** Let \( A \) be a nuclear \( C^* \)-algebra, then for every finite-dimensional subspace \( F \subseteq A \) and \( \varepsilon > 0 \), there exists a finite-codimensional subspace \( W \) of \( V \) such that the quotient mapping \( q : A \to A/W \) induces a completely contractive linear isomorphism
\[
q|_F : F \to (F + W)/W
\]
with \( \|q|_F^{-1}\|_{cb} < (1 + \varepsilon) \).

**Proof.** Since \( A \) is nuclear, it follows from (5.4) that for every finite-dimensional subspace \( F \subseteq A \), there is a finite-rank mapping
\[
T : A \to A
\]
such that \( \|T\|_{cb} < (1 + \varepsilon) \) and \( T(x) = x \) for all \( x \in F \). Then \( W = \ker T \) is a finite-codimensional subspace of \( A \), and \( T \) determines a canonical mapping \( \hat{T} : A/W \to A \) given by
\[
\hat{T}(x + W) = T(x).
\]
Since \( q \) is a complete quotient mapping from \( A \) onto \( A/W \), we have
\[
\|\hat{T}\|_{cb} = \|T\|_{cb} < (1 + \varepsilon),
\]

and
\[ \hat{T} \circ q(x) = \hat{T}(x + W) = T(x) = x \]
for all \( x \in F \). This shows that \( \hat{T} \) restricted to \( q(F) = (F + W)/W \) is a left inverse of \( q|_F \). Then \( q|_F : F \to (F + W)/W \) is a completely contractive linear isomorphism with
\[ \|q|_F^{-1}\|_{cb} \leq \|\hat{T}\|_{cb} < (1 + \varepsilon). \]

Theorem 5.5. Let \( A \) be a non-subhomogeneous nuclear \( C^* \)-algebra. Then for any finite-dimensional subspace \( E \subseteq A \) and \( 0 < \varepsilon < \frac{1}{2} \), there exists a subspace \( \tilde{F} \subseteq A^* \) containing \( E \) and a linear isomorphism
\[ \varphi : M_n \to \tilde{F} \]
such that \( \|\varphi\|_{cb} < 3 + 2\varepsilon \) and \( \|\varphi^{-1}\|_{cb} < 2 + 12\varepsilon \).

Proof. Let \( E \) be a finite-dimensional subspace of \( A \). Since \( A \) is nuclear, it follows from (5.4) that there exists a matrix space \( M_n \) and a commutative diagram of completely bounded mappings
\[
\begin{array}{ccc}
M_n & \xrightarrow{\alpha} & A \\
E & \xrightarrow{\beta} & A \\
& \downarrow{\varphi} & \\
& & (A/W)^* \\
\end{array}
\]
such that \( \|\alpha\|_{cb} \leq 1 \) and \( \|\beta\|_{cb} < 1 + \varepsilon \). For the finite-dimensional subspace \( F = \beta(M_n) \) of \( A \), there exists a finite-codimensional subspace \( W \) of \( A \) such that the quotient mapping \( q : A \to A/W \) induces a completely contractive linear isomorphism
\[ q|_F : F \to (F + W)/W \]
with \( \|q|_F^{-1}\|_{cb} < 1 + \varepsilon \). In this case,
\[ S = W^\perp \cong (A/W)^* \]
is a finite-dimensional subspace of \( A^* \).

Since we may identify \( L_\infty[0,1] \otimes M_n \) with a von Neumann subalgebra of \( B(\ell_2) \), the canonical mapping \( \theta : B(\ell_2) \to A^* \) in Lemma 5.1 induces a weak* continuous completely isometric injection from \( L_\infty[0,1] \otimes M_n \) into \( A^{**} \), which is still denoted by \( \theta \). Since \( L_\infty[0,1] \otimes M_n \) is an injective operator space, there is a complete contraction
\[ P : A^{**} \to L_\infty[0,1] \otimes M_n \]
such that \( P \circ \theta = id_{L_\infty[0,1] \otimes M_n} \). Applying Lemma 5.3 to the finite-dimensional spaces \( P(F) \) in \( L_\infty[0,1] \otimes M_n \) and \( \theta_*(S) \) in \( (L_\infty[0,1] \otimes M_n)^* \), and \( \varepsilon' = \frac{\varepsilon}{2} \), we may find a complete contraction \( u : L_\infty[0,1] \otimes M_n \to M_n \) and a complete isometry \( v : M_n \to L_\infty[0,1] \otimes M_n \) such that \( u \circ v = id_{M_n} \) and

(i) \[ \|u \circ P(x)\| \leq \varepsilon' \|P(x)\| \leq \varepsilon'|x|, \]
(ii) \[ |\langle s, \theta \circ v(y) \rangle| \leq |\langle \theta_*(s), v(y) \rangle| \leq \varepsilon'|s|\|y\| \leq \varepsilon'|s|\|y\| \]
for all \( x \in F, \ y \in M_n \) and \( s \in S \).

The mapping \( T = u \circ P : \mathcal{A}^{**} \to M_n \) is a complete contraction such that \( \|T|_F\| < \varepsilon \) by (i). Since \( \dim F \leq n^2 \), it follows from \[\text{Lemma 2.3}\] that
\[
\|T|_F\|_{cb} \leq n^2 \|T|_F\| < \varepsilon,
\]
and thus
\[
(T|_F)|_x \leq \|T|_F\|_{cb}\|x\| < \varepsilon\|x\|
\]
for all \( x \in M_n(F) \) and \( m \in \mathbb{N} \). If we let \( G = \theta \circ v(M_n) \), then \( G \) is a finite-dimensional subspace of \( \mathcal{A}^{**} \), and
\[
\theta \circ v : M_n \to G
\]
is a completely isometric isomorphism. Given any \( \tilde{y} \in M_m(G) \), there exists a unique \( y \in M_m(M_n) \) such that \( \tilde{y} = \theta_m \circ v_m(y) \), and
\[
\|T_m(\tilde{y})\| = \|u_m \circ P_m \circ \theta_m \circ v_m(y)\| = \|u_m \circ v_m(y)\| = \|y\| = \|\tilde{y}\|.
\]

On the other hand, we let
\[
R = q^{**} : \mathcal{A}^{**} \to (\mathcal{A}/W)^{**} \cong \mathcal{A}/W
\]
denote the second adjoint of \( q \). Since \( q^* : (\mathcal{A}/W)^* \to W^\perp \subseteq \mathcal{A}^* \) is a complete isometry from \((\mathcal{A}/W)^* \) onto \( W^\perp \), we have from (ii) that for any \( \tilde{y} = \theta \circ v(y) \in G \) with \( y \in M_n \),
\[
\|R(\tilde{y})\| = \sup\{|\langle R(\tilde{y}), s \rangle| : s \in (\mathcal{A}/W)^*, \|s\| \leq 1\}
\]
\[
= \sup\{|\langle \tilde{y}, q^*(s) \rangle| : s \in (\mathcal{A}/W)^*, \|s\| \leq 1\}
\]
\[
= \{\langle \theta \circ v(y), \tilde{s}\rangle : \tilde{s} = q^*(s) \in S = W^\perp, \|\tilde{s}\| \leq 1\}
\]
\[
< \varepsilon'\|y\| = \varepsilon'\|\tilde{y}\|.
\]
This shows that
\[
\|R_G\|_{cb} \leq n^2 \|R_G\| < \varepsilon
\]
and thus
\[
\|R_m(\tilde{y})\| \leq \|R_G\|_{cb}\|\tilde{y}\| < \varepsilon\|\tilde{y}\|
\]
for all \( \tilde{y} \in M_m(G) \). Given any \( x \in M_m(F) \), we also have
\[
\|x\| = \|(q|_F^{-1})_m \circ (q|_F)_m(x)\| < (1 + \varepsilon)\|(q|_F)_m(x)\| = (1 + \varepsilon)\|R_m(x)\|.
\]

Now we are ready to define the mapping
\[
\varphi = \beta + \theta \circ v \circ (id_{M_n} - \alpha \circ \beta) : M_n \to \mathcal{A}^{**}.
\]
It is clear that \( \varphi \) is completely bounded with
\[
\|\varphi\|_{cb} \leq \|\beta\|_{cb} + \|\theta \circ v \circ (id_{M_n} - \alpha \circ \beta)\|_{cb}
\]
\[
= \|\beta\|_{cb} + \|id_{M_n} - \alpha \circ \beta\|_{cb}
\]
\[
< (1 + \varepsilon) + 1 = 3 + 2\varepsilon.
\]
We let \( \tilde{F} = \varphi(M_n) \). If \( x \in E \), then \( \alpha(x) \in M_n \) and
\[
\varphi(\alpha(x)) = \beta(\alpha(x)) + \theta \circ v \circ (id_{M_n} - \alpha \circ \beta)(\alpha(x)) = x.
\]
This shows that \( E = \varphi(\alpha(E)) \subseteq \tilde{F} \).
The mapping $\varphi$ is a linear isomorphism from $M_n$ onto $\tilde{F}$. To see this, we first claim that $F \cap G = \{0\}$. Given any $x \in F \cap G$, we have from (5.7) and (5.8) that

$$\|x\| < (1 + \varepsilon)\|R(x)\| < \varepsilon(1 + \varepsilon)\|x\| < 2\varepsilon\|x\|.$$ 

Since $\varepsilon < \frac{1}{2}$, we must have $\|x\| = 0$ and thus $F \cap G = \{0\}$. If $y \in M_n$ such that $\varphi(y) = 0$, then

$$\theta \circ \nu \circ (id_{M_n} - \alpha \circ \beta)(y) = -\beta(y) \in F \cap G = \{0\}.$$ 

This implies that $y = \alpha \circ \beta(y) = 0$. Therefore, $\varphi$ is an injection and thus a linear isomorphism from $M_n$ onto $\tilde{F}$.

Finally, we want to show that $\|\varphi^{-1}\|_{cb} < 2 + 12\varepsilon$. Given any $x \in M_m(F)$ and $y \in M_m(G)$, we have from (5.7) and (5.8) that

$$\frac{1}{1 + \varepsilon}\|x\| < \|R_m(x)\| \leq \|R_m(x + y)\| + \|R_m(y)\| \leq \|x + y\| + \varepsilon\|y\|.$$ 

Similarly, we have

$$\|y\| = \|T_m(y)\| \leq \|T_m(x + y)\| + \|T_m(x)\| \leq \|x + y\| + \varepsilon\|x\|.$$ 

From this we can conclude that

$$\|x\| < (1 + \varepsilon)\|x + y\| + \varepsilon(1 + \varepsilon)\|x + y\| + 2\varepsilon^2(1 + \varepsilon)\|x\|,$$

and thus

$$\|x\| < \frac{(1 + \varepsilon)^2}{1 - \varepsilon^2(1 + \varepsilon)}\|x + y\|.$$ 

Similarly, we have

$$\|y\| < \|x + y\| + \varepsilon(1 + \varepsilon)\|x + y\| + \varepsilon^2(1 + \varepsilon)\|y\|,$$

and thus,

$$\|y\| < \frac{(1 + \varepsilon)^2}{1 - \varepsilon^2(1 + \varepsilon)}\|x + y\|.$$ 

Since $\varepsilon < \frac{1}{2}$, we have

$$\frac{(1 + \varepsilon)^2}{1 - \varepsilon^2(1 + \varepsilon)} = 1 + \frac{(1 + \varepsilon)^2 - 1 + \varepsilon^2(1 + \varepsilon)}{1 - \varepsilon^2(1 + \varepsilon)} = 1 + \frac{2\varepsilon + 2\varepsilon^2 + \varepsilon^3}{1 - \varepsilon^2(1 + \varepsilon)} < 1 + 6\varepsilon,$$

and thus

$$\max\{\|x\|, \|y\|\} < (1 + 6\varepsilon)\|x + y\|.$$ 

Since $\varphi(M_n) \subset F + G$, for any $y \in M_m(M_n)$ we have

$$\|y\| = \|\alpha_m \circ \beta_m(y) + y - \alpha_m \circ \beta_m(y)\|
\leq \|\alpha_m \circ \beta_m(y)\| + \|u_m \circ v_m(y - \alpha_m \circ \beta_m(y))\|
\leq \|\beta_m(y)\| + \|v_m(y - \alpha_m \circ \beta_m(y))\|
\leq \|\beta_m(y)\| + \|\theta_m \circ v_m(y - \alpha_m \circ \beta_m(y))\|
\leq 2(1 + 6\varepsilon)\|\beta_m(y)\| + \theta_m \circ v_m(y - \alpha_m \circ \beta_m(y))
= 2(1 + 6\varepsilon)\|\varphi_m(y)\|.$$
This shows that
\[ \|\varphi^{-1}\|_{cb} \leq 2 + 12\varepsilon. \]

**Theorem 5.6.** If \( A \) is a non-subhomogeneous nuclear \( C^* \)-algebra, then \( A \) is an \( \mathcal{OL}_{\infty,\lambda} \) space for every \( \lambda > 6 \).

**Proof.** Given any \( \lambda > 6 \), we may find a sufficiently small \( \varepsilon \) such that \( \varepsilon < \frac{1}{2} \) and 
\[ (3 + 2\varepsilon)(2 + 12\varepsilon) < \lambda. \]
For any finite-dimensional subspace \( E \subseteq A \), it follows from Theorem 5.5 that there exists a finite-dimensional subspace \( \tilde{F} \subseteq A^{**} \) containing \( E \) and a linear isomorphism \( \varphi : M_n \to \tilde{F} \) with 
\[ \|\varphi\|_{cb} < 3 + 2\varepsilon \text{ and } \|\varphi^{-1}\|_{cb} < 2 + 12\varepsilon. \]
Then we have
\[ d_{cb}(\tilde{F}, M_n) \leq \|\varphi\|_{cb}\|\varphi^{-1}\|_{cb} < (3 + 2\varepsilon)(2 + 12\varepsilon) < \lambda. \]
It follows from Theorem 4.3 that there exists a finite-dimensional subspace \( F \) of \( A \) such that \( E \subseteq F \) and 
\[ d_{cb}(F, M_n) < \lambda. \]
This shows that \( A \) is an \( \mathcal{OL}_{\infty,\lambda} \) space. \( \square \)

Combining Theorem 4.4 and Theorem 5.6, we can conclude that every \( C^* \)-algebra \( A \) (either subhomogeneous or non-subhomogeneous) is an \( \mathcal{OL}_{\infty,\lambda} \) space for every \( \lambda > 6 \). This completes the proof for Theorem 1.2.

### 6. Remarks and Questions

As we discussed in §1, we can define an invariant
\[ \mathcal{OL}_{\infty}(A) = \inf\{\lambda > 1 : A \text{ is an } \mathcal{OL}_{\infty,\lambda} \text{ space}\} \]
for every nuclear \( C^* \)-algebra \( A \). In general, we have \( 1 \leq \mathcal{OL}_{\infty}(A) \leq 6 \), and \( \mathcal{OL}_{\infty}(A) = 1 \) if and only if \( A \) is an \( \mathcal{OL}_{\infty,1} \) space. This includes the case when \( A \) is a rigid \( \mathcal{OL}_{\infty,1} \) space or a strong NF algebra.

**Question 6.1.** It would be interesting to know whether \( \mathcal{OL}_{\infty,1} \) implies rigid \( \mathcal{OL}_{\infty,1} \) on unital nuclear \( C^* \)-algebras.

It is known from Theorem 3.2 that if \( A \) is a unital \( C^* \)-algebra with \( \mathcal{OL}_{\infty}(A) = 1 \), then \( A \) is nuclear and quasi-diagonal.

**Question 6.2.** Does a nuclear and quasi-diagonal unital \( C^* \)-algebra must be an \( \mathcal{OL}_{\infty,1} \) space?

To investigate Questions 6.1 and 6.2, one could look at the \( C^* \)-algebra \( B_n \) introduced by L. Brown [5], which is an essential extension
\[ 0 \to K_{\infty} \to B_n \to C(\mathbb{R}P^2) \to 0 \]
of the continuous functions on the real projective plane \( \mathbb{R}P^2 \) by \( K_{\infty} = K(\ell_2) \). One could also look at the \( C^* \)-algebra \( C^*(s \oplus s^*) \) generated by the direct sum of the unilateral shift \( s \) (on \( \ell_2 \)) and its adjoint \( s^* \), for which we have the extension
\[ 0 \to K_{\infty} \oplus K_{\infty} \to C^*(s \oplus s^*) \to C(S^1) \to 0. \]
It is known (see Blackadar and Kirchberg [3, §2]) that \( B_n \) and \( C^*(s \oplus s^*) \) are nuclear quasi-diagonal unital \( C^* \)-algebras, but they are not inner quasi-diagonal and thus
are not rigid $\mathcal{O}\mathcal{L}_{\infty,1}^+$ spaces. It would be interesting to calculate $\mathcal{O}\mathcal{L}_{\infty}(Bn)$ and $\mathcal{O}\mathcal{L}_{\infty}(C^*(s \oplus s^*))$. If $\mathcal{O}\mathcal{L}_{\infty}(Bn) = 1$ (or $\mathcal{O}\mathcal{L}_{\infty}(C^*(s \oplus s^*)) = 1$), we would get an example of a unital $C^*$-algebra, which is an $\mathcal{O}\mathcal{L}_{\infty,1}^+$ space, but is not a rigid $\mathcal{O}\mathcal{L}_{\infty,1}^+$ space. On the other hand, if $\mathcal{O}\mathcal{L}_{\infty}(Bn) > 1$ (or $\mathcal{O}\mathcal{L}_{\infty}(C^*(s \oplus s^*)) > 1$), then we would obtain an example of a unital $C^*$-algebra, which is nuclear and quasi-diagonal, but is not $\mathcal{O}\mathcal{L}_{\infty,1}^+$. This investigation would either give us a negative answer to Question 6.1 or give us a negative answer to Question 6.2.

On the other hand, it is known from Theorem 3.4 that if $A$ is a non-stably finite nuclear unital $C^*$-algebra, then we have

$$(\frac{1 + \sqrt{5}}{2})^{\frac{1}{2}} < \mathcal{O}\mathcal{L}_{\infty}(A) \leq 6.$$ 

For example, we may consider the Cuntz algebra $\mathcal{O}_n$ (with $2 \leq n < \infty$), or the Toeplitz algebra $T(S^1)$ on the unit ball of $\mathbb{C}$. Since $T(S^1)$ is the $C^*$-algebra generated by the unilateral shift $s$ (on $\ell_2$) and has an extension

$$0 \rightarrow K_{\infty} \rightarrow T(S^1) \rightarrow C(S^1) \rightarrow 0,$$

it is an infinite nuclear unital $C^*$-algebra, and thus we have

$$(\frac{1 + \sqrt{5}}{2})^{\frac{1}{2}} < \mathcal{O}\mathcal{L}_{\infty}(T(S^1)) \leq 6.$$ 

This example shows that the (rigid) $\mathcal{O}\mathcal{L}_{\infty,1}^+$ structure is not preserved by $C^*$-algebra extensions. We may also consider the Toeplitz algebra $T(S^3)$ on the unit ball of $\mathbb{C}^2$, which is a finite, but not 2-finite nuclear unital $C^*$-algebra (see [1, §6.10]).

**Question 6.3.** It would be interesting to know if there is any stably finite unital $C^*$-algebra for which $\mathcal{O}\mathcal{L}_{\infty}(A) > 1$, or $\mathcal{O}\mathcal{L}_{\infty}(A) > (\frac{1 + \sqrt{5}}{2})^{\frac{1}{2}}$.

Finally we remark that in a recent work of Junge, Nielsen, Ruan and Xu [20], we obtained the constant $\mathcal{O}\mathcal{L}_{\infty}(A) \leq 3$ for every nuclear $C^*$-algebra with completely different methods.

**Conjecture 6.4.** We conjecture that one could obtain $\mathcal{O}\mathcal{L}_{\infty}(A) \leq 2$ for every nuclear $C^*$-algebra $A$.

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ON $\mathcal{O}_p^\infty$ STRUCTURE OF NUCLEAR $C^*$-ALGEBRAS 29

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ILLINOIS, URBANA, IL 61801 USA
E-mail address, Marius Junge: junge@math.uiuc.edu

DEPARTMENT OF MATHEMATICS, TEXAS A&M UNIVERSITY, COLLEGE STATION, TX 77843
USA, AND DEPARTMENT OF MATHEMATICAL SCIENCES, UNIVERSITY OF TOKYO, KOMABA, 153-8914, JAPAN
E-mail address: ozawa@math.tamu.edu

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ILLINOIS, URBANA, IL 61801 USA
E-mail address, Zhong-Jin Ruan: ruan@math.uiuc.edu