Abstract. For a class of infinite lattices of interacting anharmonic oscillators, we study the existence of the dynamics, together with Lieb-Robinson bounds, in a suitable algebra of observables.

1. Introduction. Statement of results.

Infinite lattices of nearest-neighbors interacting harmonic oscillators are a usual model in quantum statistical mechanics. Among the objects associated to this model, an important one is the dynamics describing the time evolution of some algebra of observables, related to the lattice. Such dynamics on a lattice was defined by Malyshev-Minlos [MA-MI] and by Thirring [TH], when the potential is a quadratic form.

We also note that, for bounded Hamiltonian models, Lieb and Robinson have established in [LI-R] an estimate, concerning the propagation speed for the correlation between two local observables. For these models, the existence of the dynamics is proven, for example in [NOS], in some algebra (not the same as in [MA-MI] or [TH]).

More recently, Nachtergaele, Raz, Schlein and Sims [NRSS] have derived Lieb-Robinson type inequalities for lattices of harmonic oscillators with quadratic interactions with, moreover, on each site of the lattice, a self-interaction potential in a more general class. More precisely, Lieb-Robinson type inequalities are proved ([NRSS]) for Hamiltonians associated to a finite subset Λ of the lattice, and hold uniformly in |Λ|. However, to the best of our knowledge, the existence of dynamics as |Λ| → ∞ is established when the potential is a quadratic form, but not with smaller perturbations.

The aim of this article is twofold. On the first side, we define a $C^*$–algebra $\mathcal{W}_2$ which seems to be more convenient, when the perturbation is turned on, than the Weyl algebra defined in [MA-MI] or in [TH], or than the one used in [NOS]. In particular, we prove the existence of a dynamics (defined as a limit when the number of sites goes to infinity) for local and non local observables in this algebra. On the other side, we are able to perturb the quadratic potential of interaction in a more general way than in [NRSS], with not only self-interacting terms. In this framework, we also obtain the Lieb-Robinson type inequalities, with a bound for the propagation speed of the correlations.

We make the choice here to consider a one dimensional lattice $\mathbb{Z}$ in order to simplify the notations. For each subset $\Lambda_n$ in the lattice $\mathbb{Z}$ written as $\Lambda_n = \{-n, ..., +n\}$ ($n \geq 1$), we define a Hamiltonian $H_{\Lambda_n}$ in $\mathbb{R}^{\Lambda_n}$.
by:

\[ H_{\Lambda_n} = -\frac{1}{2} \sum_{\lambda \in \Lambda_n} \frac{\partial^2}{\partial x^2_{\lambda}} + V_{\Lambda_n}, \quad V_{\Lambda_n} = V_{\Lambda_n}^{\text{quad}} + V_{\Lambda_n}^{\text{pert}}. \]

where the potential \( V_{\Lambda_n}^{\text{quad}} \) is a definite positive quadratic form on \( \mathbb{R}^{\Lambda_n} \) and where \( V_{\Lambda_n}^{\text{pert}} \) is viewed as a perturbation of \( V_{\Lambda_n}^{\text{quad}} \).

The quadratic potential is defined for all \( n \) by:

\[ V_{\Lambda_n}^{\text{quad}}(x) = \frac{a}{2} |x|^2 - b \sum_{\lambda=-n}^{n-1} x_\lambda x_{\lambda+1} \]

where \( a \) and \( b \) are two real numbers verifying \( a > 2b > 0 \).

Precise hypotheses on the perturbation potential are stated in \((H_1)\) and \((H_2)\) (see below). These assumptions imply that \( V_{\Lambda_n}^{\text{pert}} \) is a multiplication operator by a real-valued function \( v_{\Lambda_n}^{\text{pert}} \) belonging to \( C^1(\mathbb{R}^{\Lambda_n}) \), and satisfying \( v_{\Lambda_n}^{\text{pert}}(x) = o(|x|^2) \) near infinity.

Following Kato-Rellich’s theorem, the operator \( H_{\Lambda_n} \) defined in (1.1), with the hypotheses \((H_1)\) and \((H_2)\), is self-adjoint with the same domain as the harmonic oscillator on \( \mathbb{R}^{\Lambda_n} \). Hence, we can define the unitary operator \( e^{itH_{\Lambda_n}} \) (\( t \in \mathbb{R} \)).

Thus, the following operator is well-defined:

\[ \alpha_{\Lambda_n}^{(t)}(A) = e^{itH_{\Lambda_n}} A e^{-itH_{\Lambda_n}} \]

for all \( A \in \mathcal{L}(\mathcal{H}_{\Lambda_n}) \), (where \( \mathcal{H}_{\Lambda_n} = L^2(\mathbb{R}^{\Lambda_n}) \)), and for all \( t \in \mathbb{R} \). It is then natural to ask whether this sequence of operators has a limit when \( n \) tends to \( +\infty \), and for which class of operators \( A \) ? More precisely, we are looking for a Banach algebra \( \mathcal{A} \) satisfying the following conditions:

- The spaces \( \mathcal{L}(L^2(\mathbb{R}^\Lambda)) \), (where \( \Lambda \) is a finite subset of \( \mathbb{Z} \)), is isometrically immersed in the algebra (the elements of \( \mathcal{L}(\mathcal{H}_{\Lambda}) \) are under this identification called local observables supported in \( \Lambda \)).

- For all local observables \( A \), the limit as \( n \) tends to infinity of \( \alpha_{\Lambda_n}^{(t)}(A) \), denoted by \( \alpha^{(t)}(A) \), exists in this algebra \( \mathcal{A} \).

- This operator \( \alpha^{(t)} \), defined in this procedure for local observables \( A \), may be extended by density to the whole algebra \( \mathcal{A} \), and acts in a continuous way.

Several works, related to this issue, have considered the \( C^* \)-algebra \( \mathcal{A} \) of the quasi-local observables. Let us recall its definition (cf [SI]). For each finite subset \( \Lambda \) in \( \mathbb{Z} \) set \( \mathcal{H}_{\Lambda} = L^2(\mathbb{R}^{\Lambda}) \). One notes that, if \( \Lambda \subset \Lambda' \) then \( \mathcal{L}(\mathcal{H}_{\Lambda}) \) is isometrically immersed in \( \mathcal{L}(\mathcal{H}_{\Lambda'}) \). Therefore, one may define \( \mathcal{A} \) as the completion of the inductive limit of the spaces \( \mathcal{L}(\mathcal{H}_{\Lambda}) \):

\[ \mathcal{A} = \bigcup_{\Lambda \subset \mathbb{Z}} \mathcal{L}(\mathcal{H}_{\Lambda}) \]

This algebra is well-adapted in the case of bounded potentials, or when the first order derivatives are bounded (cf e.g. the work of [NOS] for the existence of a dynamics, or [ACLN] for estimates on the decay of the correlations), whereas it might not be suitable for the perturbed quadratic case studied here.
Another algebra, the Weyl algebra, is considered by Malyshev-Minlos [MA-MI] and by Thirring [TH]. This algebra fits to the non-perturbated quadratic case ($V_{\Lambda_n}^{pert} = 0$), and it is defined using the Fock space’s formalism.

The space $\mathcal{H}$ denotes the symmetrized Fock space $\mathcal{H} = F_\alpha(\ell^2(\mathbb{Z}))$, associated to the Hilbert space $\ell^2(\mathbb{Z})$. For all $\lambda \in \mathbb{Z}$, one defines the two self-adjoint operators $P_\lambda$ and $Q_\lambda$ in the Fock space, verifying the same commutation relations as the position and momentum operators in $L^2(\mathbb{R}^n)$. (Note that there is here an infinite number of these operators.) For each finite subset $\Lambda$ of $\mathbb{Z}$, the space $\mathcal{L}(\mathcal{H}_\Lambda)$ (where $\mathcal{H}_\Lambda = L^2(\mathbb{R}^\Lambda)$) is isometrically immersed in $\mathcal{L}(\mathcal{H})$. This identification extends also to non bounded operators. Thus, the multiplication operator by $x_\lambda$ and the operator $\frac{1}{i} \partial_{x_\lambda}$ ($\lambda \in \Lambda$) becomes the two operators $Q_\lambda$ and $P_\lambda$, sometimes denoted in this paper by $Q_\lambda^{(0)}$ and $Q_\lambda^{(1)}$:

$$Q_\lambda^{(0)} = Q_\lambda = x_\lambda \quad Q_\lambda^{(1)} = P_\lambda = \frac{1}{i} \partial_{x_\lambda}$$

(1.5)

The Fock spaces formalism allows us to properly define, for all real sequences $u$ and $v$ in $\ell^2(\mathbb{Z})$, the non bounded self-adjoint operator, (the Segal operator), formally defined by:

$$\Pi(u, v) = \sum_{\lambda \in \mathbb{Z}} (u_\lambda P_\lambda + v_\lambda Q_\lambda)$$

(1.6)

The two operators $P_\lambda$ and $Q_\lambda$ are generally not defined by (1.5) anymore, but, instead, $\Pi(u, v)$ is defined starting from the creation and annihilation operators associated to $\ell^2(\mathbb{Z})$ (see section 2). The corresponding unitary operator $W(u, v) = e^{i\Pi(u, v)}$ is called a Weyl operator.

The Weyl algebra introduced by Malyshev-Minlos [MA-MI] or by Thirring [TH] is the closure in $\mathcal{L}(\mathcal{H})$ of the subspace generated by the operators $W(u, v)$ ($u$ and $v$ being real sequences in $\ell^2(\mathbb{Z})$).

In the purely quadratic case ($V_{\Lambda_n}^{pert} = 0$) and for all $A$ in this Weyl algebra, an explicit analysis allows us to define properly $\alpha^{(t)}_{\Lambda_n}(A)$ (even if $A$ is not supported in $\Lambda_n$) and to define the limit operator $\alpha^{(t)}(A)$ such that, for all $f \in \mathcal{H}$:

$$\lim_{n \to \infty} \left\| [\alpha^{(t)}_{\Lambda_n}(A) - \alpha^{(t)}(A)]f \right\|_{\mathcal{H}} = 0$$

In order to derive the latter limit, uniform estimates, such as those established in [N-R-S-S], are needed.

Using the Weyl algebra defined above, it is probably difficult to also obtain these results when the potential of perturbation is turned on. The purpose of this work is then to extend the above results to the quadratic case with perturbations by involving another algebra $\mathcal{W}_2$ included in $\mathcal{L}(\mathcal{H})$. Furthermore, the Lieb-Robinson estimates in [N-R-S-S] are also extended to that framework.

Before giving the definition of $\mathcal{W}_2$, let us mention that the works of Calderon-Vaillancourt [C-V] and Beals [BE] (see also Hörmander [HO]), give an important role to a particular subalgebra of $\mathcal{L}(L^2(\mathbb{R}^n))$ or here, of $\mathcal{L}(L^2(\mathbb{R}^\Lambda))$, for all finite subset $\Lambda$ in $\mathbb{Z}$. This particular subalgebra $OPS^0(\mathbb{R}^\Lambda)$ is the set of pseudo-differential operators on $\mathbb{R}^\Lambda$, associated to symbols that are bounded, together with all of their derivatives. From Beals [BE], these operators are characterized by the following property, implying the operators $Q_\lambda^{(0)}$ and $Q_\lambda^{(1)}$ defined in (1.5) for all $\lambda \in \Lambda$. An operator $A$ in $\mathcal{L}(L^2(\mathbb{R}^\Lambda))$ in $OPS^0(\mathbb{R}^\Lambda)$ if, and only if, all the iterated commutators $(ad Q_{\lambda_1}^{(k_1)}) \cdots (ad Q_{\lambda_m}^{(k_m)}) A$, (with $\lambda_1, \ldots, \lambda_m$ are in $\Lambda$, $m \geq 0$, and $k_j \in \{0, 1\}$), are bounded operators in $L^2(\mathbb{R}^\Lambda)$. The commutators are known to a priori map from $S(\mathbb{R}^\Lambda)$ into $S'(\mathbb{R}^\Lambda)$.

Replacing $\Lambda$ by $\mathbb{Z}$, one may analogously define a decreasing sequence of subalgebras $\mathcal{W}_k$ in $\mathcal{L}(\mathcal{H})$ ($k \geq 0$). Set $\mathcal{W}_0 = \mathcal{L}(\mathcal{H})$. We denote by $\mathcal{W}_1$ the set of all $A$ in $\mathcal{W}_0$ such that, for all $\lambda \in \mathbb{Z}$, the commutators $[A, Q_\lambda^{(k)}]$ and $[A, P_\lambda]$ are bounded in $\mathcal{H}$, and such that the sum in the following norm is finite:

$$\|A\|_{\mathcal{W}_1} = \|A\|_{\mathcal{W}_0} + \sum_{\lambda \in \mathbb{Z}} \sum_{k \geq 0} \| [A, Q_\lambda^{(k)}] \|_{\mathcal{W}_0}$$

(1.7)
Note that the above commutators are properly defined in section 2. From now on, the operators \( Q^{(0)} = Q_\lambda \) and \( Q^{(1)} = P_\lambda \) are defined through the Fock space’s formalism, and not by (1.5) anymore.

Let us denote by \( \mathcal{W}_2 \) the set of all operators \( A \in \mathcal{W}_1 \) such that the commutators \([Q^{(k)}_\lambda, A]\) belongs to \( \mathcal{W}_1 \) for all \( \lambda \in \mathbb{Z} \), and such that the sum in the norm below is finite:

\[
\| A \|_{\mathcal{W}_2} = \| A \|_{\mathcal{W}_1} + \frac{1}{2} \sum_{(\lambda, \mu) \in \mathbb{Z}^2} \| [A, Q^{(j)}_\lambda, Q^{(k)}_\mu] \|_{\mathcal{L}(\mathcal{H})}
\]

(1.8)

An example. For all \( u \) and \( v \) in \( \ell^1(\mathbb{Z}) \), the Weyl operator \( W(u, v) = e^{iH(u,v)} \) is in \( \mathcal{W}_k \) \( (0 \leq k \leq 2) \).

One might define similarly a sequence of algebras \( \mathcal{W}_k \) using iterated commutations. In particular, the intersection set of these algebras could correspond to an analogous of \( OP^0 \) in infinite dimension. Other particular classes of pseudo-differential operators in infinite dimension are studied by B. Lascar (see [L1] [L2],…).

Among all of these algebras and for our point of view, it is \( \mathcal{W}_2 \) that appears to be the most suitable to our study. If \( A \) is not supposed to be an element of \( \mathcal{W}_2 \), \( A \) being only assumed to be in \( \mathcal{L}(\mathcal{H}) \) and supported on a finite subset \( E \) of \( \mathbb{Z} \), it appears to be possible to show that, for all \( f \) in \( \mathcal{H} \), the sequence \( \alpha^{(t)}_n(A)f \) weakly converges in \( \mathcal{H} \). If this limit is denoted by \( \alpha^{(t)}(A)f \), it is not clear whether the map \( t \to \alpha^{(t)} \) is continuous, neither whether \( \alpha^{(t)} \) may be extended to a suitable Banach algebra.

More precise estimates are obtained when the local observable \( A \) belongs to \( \mathcal{W}_2 \). Before that, let us describe now the perturbation potential.

Hypotheses on the perturbation potentials. The operator \( V^{pert}_{A_n} \) is written as the following sum:

\[
V^{pert}_{A_n} = \sum_{\lambda \in \Lambda_n} V_\lambda + \sum_{(\lambda, \mu) \in \Lambda^2_n} V_{\lambda\mu},
\]

where the operators \( V_\lambda \) and \( V_{\lambda\mu} \) are defined for all \( \lambda \) and \( \mu \) in \( \mathbb{Z} \), and verify the assumptions below:

Hypotheses on the perturbation potentials.

(1.10)

\[
\sum_{2 \leq j + k \leq 3} \| \xi^j_k \hat{V}_{\lambda\mu} \|_{L^1(\mathbb{R}^2)} \leq C_0 e^{-\gamma_0 |\lambda - \mu|},
\]

(1.11)

\[
| \nabla v_{\lambda\mu}(0) | \leq C_0 e^{-\gamma_0 |\lambda - \mu|}.
\]

(1.12)

\[
\sum_{2 \leq j \leq 3} \| \xi^j \hat{v}_\lambda \|_{L^1(\mathbb{R})} \leq C_0, \quad | \nabla v_\lambda(0) | \leq C_0.
\]

In particular, in the case of interactions between nearest neighbors, one has \( V_{\lambda\mu} = 0 \) if \( |\lambda - \mu| \geq 2 \). It is then sufficient that the integrals in the l.h.s. of (1.10) and (1.12) are uniformly bounded in \( \lambda \). In that
case, the hypotheses (H1) are (H2) satisfied for any \( \gamma_0 > 0 \) and in all the results below, the phrase «for all \( \gamma \in [0, \gamma_0] \)» is replaced by «for all \( \gamma > 0 \)».

For each integer \( n \), the perturbation potential \( V^{pert}_\Lambda \) and the Hamiltonian \( H_\Lambda \) are defined by (1.9) and (1.1) respectively. In [NRSS], the authors have only considered the \( V_\Lambda \)’s. We shall say that an element \( A \) of \( \mathcal{W}_2 \) has a finite support if there exists a finite subset \( E \) in \( \mathbb{Z} \), such that \( A \) is identified to an element of \( \mathcal{L}(\mathcal{H}_E) \). The smallest set having this property is called the support of \( A \) and is denoted by \( \sigma(A) \).

**Theorem 1.1.** Under the above hypotheses, for all element \( A \in \mathcal{W}_2 \) with finite support, for all \( t \in \mathbb{R} \), for all \( n \) such that \( \Lambda_n \) contains the support of \( A \), the operator \( \alpha^{(t)}_{\Lambda_n}(A) \) belongs to \( \mathcal{W}_2 \). Moreover, there exists two real positive real numbers \( C \) and \( M \) not depending on \( n \) and \( t \) such that:

\[
\|\alpha^{(t)}_{\Lambda_n}(A)\|_{\mathcal{W}_2} \leq Ce^{M|t|}\|A\|_{\mathcal{W}_2}.
\]

Furthermore, for each \( f \in \mathcal{H} \), the sequence \( \alpha^{(t)}_{\Lambda_n}(A)f \) strongly converges in \( \mathcal{H} \). Denoting this limit by \( \alpha^{(t)}(A)f \), the map \( t \to \alpha^{(t)}(A)f \) is strongly continuous, the operator \( \alpha^{(t)}(A) \) is in \( \mathcal{W}_2 \) and one has:

\[
\|\alpha^{(t)}(A)\|_{\mathcal{W}_2} \leq Ce^{M|t|}\|A\|_{\mathcal{W}_2}.
\]

In the first part of this theorem, (where \( n \) is fixed), one may think that \( \alpha^{(t)}_{\Lambda_n} \) acts in the algebra \( \mathcal{W}_k \), defined similarly as \( \mathcal{W}_1 \) and \( \mathcal{W}_2 \), but with iterated commutators of length \( k \), and for operators supported in \( \Lambda_n \). (The hypotheses (H1) and (H2) naturally need to be strengthened.) From Beals characterization, one would deduce a group action of \( \alpha^{(t)}_{\Lambda_n} \) on the operators in \( OPS^0(\mathbb{R}^A) \). An alternative approach concerning this problem may be found in the works of Bony (see [BO1] and [BO2]).

Moreover, under the hypotheses of theorem 1.1, the automorphism \( \alpha^{(t)} \), (initially defined for local observables), is extended in a unique way to the whole algebra \( \mathcal{W}_2 \) (see below). To this end, we introduce Sobolev-type spaces.

Let \( \mathcal{H}^2 \) be the subspace of the \( f \in \mathcal{H} \) such that the following norm is finite:

\[
\|f\|_{\mathcal{H}^2} = \|f\|_{\mathcal{H}} + \sup_{\lambda \in \mathbb{R}} \|Q^{(j)}_{\lambda} f\|_{\mathcal{H}} + \sup_{\lambda, \mu} \|Q^{(j)}_{\lambda} Q^{(k)}_{\mu} f\|_{\mathcal{H}}.
\]

Since a convergence in norm is needed, theorem 1.1 is now completed with the result below:

**Theorem 1.2.** There exists \( C > 0 \), \( \gamma > 0 \) and \( M > 0 \) with the following properties. For all \( A \in \mathcal{W}_2 \) with a finite support denoted by \( \sigma(A) \), for all \( n \) such that \( \Lambda_n \) contains \( \sigma(A) \) and for all \( t \in \mathbb{R} \), we have:

\[
\left\|\alpha^{(t)}_{\Lambda_n}(A) - \alpha^{(t)}(A)\right\|_{\mathcal{L}(\mathcal{H}^2, \mathcal{H})} \leq Ce^{M|t|}e^{-\gamma d(\sigma(A), \Lambda_n)}\|A\|_{\mathcal{W}_2}
\]

Moreover,

\[
\|\alpha^{(t)}(A)\|_{\mathcal{L}(\mathcal{H}^2, \mathcal{H})} \leq Ce^{M|t|}\|A\|_{\mathcal{L}(\mathcal{H}^2, \mathcal{H})}
\]

The set of all observables having a finite support is not dense in \( \mathcal{W}_2 \). In order to extend \( \alpha^{(t)} \), we shall use, instead of density, the following two results.
The propagation speed verifies, in the cyclic quadratic case:

\[ \|A_n\|_{\mathcal{W}_2} \leq \|A\|_{\mathcal{W}_2}, \quad \lim_{n \to \infty} \|A_n - A\|_{\mathcal{L}(\mathcal{H}^2, \mathcal{H})} = 0. \]

**Theorem 1.4.** Let \((A_n)\) be a sequence of operators in \(\mathcal{W}_2\). Suppose that \(\|A_n\|_{\mathcal{W}_2} \leq 1\) and assume that there exists \(A \in \mathcal{L}(\mathcal{H}^2, \mathcal{H})\) such that \(\|A_n - A\|_{\mathcal{L}(\mathcal{H}^2, \mathcal{H})}\) tends to 0. Then \(A\) may be extended to an element of \(\mathcal{L}(\mathcal{H})\) which belongs to \(\mathcal{W}_2\) and \(\|A\|_{\mathcal{W}_2} \leq 1\). Moreover, for all \(f \in \mathcal{H}\) the sequence \(A_n f\) converges to \(Af\) in \(\mathcal{H}\).

Consequently, we easily deduce from theorems 1.1 - 1.4 that \(\alpha^{(t)}\) may be extended, in a unique way, to the whole algebra \(\mathcal{W}_2\), without any conditions on the finiteness of the supports (see section 7). The map \(\alpha^{(t)}\) is not a \(\mathcal{W}_2\) norm conservative map, but it is \(\mathcal{L}(\mathcal{H})\) norm conservative. Using this point, \(\alpha^{(t)}\) is extended to the closure \(\overline{\mathcal{W}_2}\) of \(\mathcal{W}_2\) in \(\mathcal{L}(\mathcal{H})\). Thus, \(\alpha^{(t)}\) acts in \(\overline{\mathcal{W}_2}\) in a continuous way (for the simple topology) and is norm conservative.

**Lieb-Robinson’s inequalities.**

These inequalities, established in [L-R] for bounded Hamiltonians and, more recently, in [N-R-S-S] for quadratic Hamiltonians, express the propagation of the correlation between two observables with separated supports, as a function of the time and of the distance between the two supports.

For all \(h \in \mathbb{Z}\), set \(T_h\) the map in \(\ell^2(\mathbb{Z})\) defined by \((T_h u)_{\lambda} = u_{\lambda+h}\) for all \(u \in \ell^2(\mathbb{Z})\) and for all \(\lambda \in \mathbb{Z}\). With \(T_h\) we define a map in the Fock space \(\mathcal{H} = F_\alpha(\ell^2(\mathbb{Z}))\) that is still noted \(T_h\). For any \(A \in \mathcal{L}(\mathcal{H})\) we set \(\tau_h(A) = T_h^{-1}AT_h\).

In our framework, the Lieb-Robinson type inequalities have the following form:

**Theorem 1.5.** There exists a real number \(v_0\) with the following property. For any elements \(A\) and \(B\) of \(\mathcal{W}_2\) with finite supports, for any sequence \((h_n, t_n)\) tending to infinity in \(\mathbb{Z} \times \mathbb{R}\) and satisfying \(|h_n| \geq v_0 |t_n|\), for any \(f \in \mathcal{H}\), we have:

\[ \lim_{n \to \infty} \left[ \alpha^{(t_n)}(A) , \tau_{h_n}(B) \right] f = 0. \]

The infimum \(V_0\) of all the \(v_0\) satisfying the above property, defines a kind of propagation speed, which is different from the usual definitions of phase and group velocities (cf Cohen-Tannoudji [C-T]).

In the case of cyclic quadratic potentials, (that is to say, without any perturbation, but obtained by adding to \(V_{\Lambda}^{\text{quad}}\) of (1.2) an end point interaction potential \(-hz_n x_{-n}\), one finds in [N-R-S-S] an estimate of this propagation speed. (In [NRSS] this is written for a multidimensional lattice model.) We shall provide here an alternative estimate of the same type, with an elementary proof, given in section 4. The analysis of chains of harmonic oscillators with cyclic interactions usually involves the dispersion relation \(\omega(\theta) = \sqrt{a - 2b \cos \theta}\), (cf [C-T]). It is then natural to define a complex version of this relation, and to define:

\[ \Omega(z) = \sqrt{a - b(z + z^{-1})}, \quad z \in \mathbb{C} \setminus \{0\}. \]

For any \(\gamma > 0\), set:

\[ M(\gamma) = \sup_{|z| = e^{\gamma}} |\text{Im } \Omega(z)|. \]

The propagation speed verifies, in the cyclic quadratic case:

\[ V_0 \leq \inf_{\gamma > 0} \frac{M(\gamma)}{\gamma}. \]
In a more general case, this estimate is less precise. For all \( \gamma \) in \( ]0, \gamma_0[ \) (\( \gamma_0 \) being the real number appearing in the hypotheses \((H1)\) and \((H2)\)), we shall define in Proposition 3.4 a real number \( S_\gamma \) and we shall prove in section 8 that the propagation speed verifies:

\[
V_0 \leq \inf_{0 < \gamma < \gamma_0} \frac{2\sqrt{S_\gamma}}{\gamma}.
\]

The constant number \( S_\gamma \) depends only on \( a \) and \( b \), together with the norms in \( \mathcal{F}L^1(\mathbb{R}) \) or \( \mathcal{F}L^2(\mathbb{R}^2) \) of the second derivatives of the potentials of perturbation. We then note that, multiplying \( a, b \) and the potentials of perturbation by a constant \( g > 0 \), our estimates on the propagation speed is multiplied by \( \sqrt{g} \).

Section 2 is concerned with the subalgebra \( \mathcal{W}_k \). In section 3, properties on \( V_{\lambda_n} \) under the hypotheses \((H1)\) and \((H2)\) are established. Evolution operators, for finite systems on the lattice, are studied in sections 4 - 6. Sections 7 and 8 are respectively devoted to perform the limit \( n \) goes to infinity (the number of sites tends to infinity), and to derive the Lieb-Robinson’s inequalities.

We are grateful to M. Khodja for helpful discussions.

2. Algebras of operators in the Fock space.

Notations on the Fock spaces (cf \([RE-SI]\)).

For any \( E \) subset of \( \mathbb{Z} \), the symmetrized Fock space associated to the Hilbert space \( \ell^2(E) \) shall be denoted \( \mathcal{H}_E \). When \( E = \mathbb{Z} \), this space is still noted \( \mathcal{H} \). The ground state of \( \mathcal{H}_E \) is denoted by \( \Omega_E \) or \( \Omega \) when \( E = \mathbb{Z} \).

If \( E_1 \) and \( E_2 \) are two disjoint sets of \( \mathbb{Z} \) one may identify \( \mathcal{H}_{E_1 \cup E_2} \) and \( \mathcal{H}_{E_1} \otimes \mathcal{H}_{E_2} \) (the completed tensorial product). On may also identify \( \Omega_{E_1 \cup E_2} \) with \( \Omega_{E_1} \otimes \Omega_{E_2} \).

For all real sequence \( u \) in \( \ell^2(\mathbb{Z}) \) we define the two non bounded operators \( a(u) \) (annihilation operator) and \( a^*(u) \) (creation operator), being each other the formal adjoint, and verifying the following commutation relations:

\[
[a(u), a(v)] = [a^*(u), a^*(v)] = 0, \quad [a(u), a^*(v)] = (u, v),
\]

for all \( u \) and \( v \) in \( \ell^2(\mathbb{Z}) \).

We shall denote by \( (e_\lambda)_{\lambda \in \mathbb{Z}} \) the canonical basis of \( \ell^2(\mathbb{Z}) \). Starting from the ground state \( \Omega \), and applying successively the creation operators, one defines \( a^*(e_{\lambda_1})...a^*(e_{\lambda_m})\Omega \), being orthogonal elements of \( \mathcal{H} \). Let \( \mathcal{D} \) be the subspace of \( \mathcal{H} \) generated by these vectors. It is known that \( \mathcal{D} \) is dense in \( \mathcal{H} \). The space \( \mathcal{D} \) is included in the domain of all \( a(u) \) and \( a^*(u) \), \( (u \in \ell^2(\mathbb{Z})) \). For all \( f \) in \( \mathcal{D} \) there exists a finite subset \( S \subset \mathbb{Z} \) such that \( f \) is written as: \( f = g \otimes \Omega_{S^c} \) with \( g \in \mathcal{H}_S \). We then say that \( f \) is supported in \( S \).

Next we define the Segal operator \( \Pi(u, v) \) by:

\[
(2.1) \quad \Pi(u, v) = \frac{a(u) + a^*(u)}{\sqrt{2}} + \frac{a(v) - a^*(v)}{i\sqrt{2}}
\]

for all real elements \( u \) and \( v \) in \( \ell^2(\mathbb{Z}) \). An element \( f \in \mathcal{H} \) is the domain of \( \Pi(u, v) \) if there exists a sequence \( (f_n) \) in \( \mathcal{D} \) such that \( f_n \) converges to \( f \) in \( \mathcal{H} \) and such that \( \Pi(f_n, v)f_n \) has a limit in \( \mathcal{H} \). Thus, \( \Pi(u, v) \) is a self-adjoint operator. The associated Weyl operator is \( W(u, v) = e^{i\Pi(u, v)} \).

In particular, for each element \( e_\lambda \) in the canonical basis of \( \ell^2(\mathbb{Z}) \) the Segal operators are noted:

\[
(2.2) \quad Q_\lambda = Q_\lambda^{(0)} = \frac{a(e_\lambda) + a^*(e_\lambda)}{\sqrt{2}} \quad P_\lambda = Q_\lambda^{(1)} = \frac{a(e_\lambda) - a^*(e_\lambda)}{i\sqrt{2}}.
\]
Let us write down an orthonormal basis. We shall limit ourselves to the Hilbert space \( \mathcal{H}_\Lambda \) associated to a subset of \( \mathbb{Z} \) reduced to one element \( \lambda \). In this space we again used the construction of \( \mathcal{D} \) and obtain the basis \( (h_n)_{(n \geq 0)} \) being now normalized by setting:

\[
(2.3) \quad h_0 = \Omega_\{\lambda\}, \quad h_{j+1} = (j+1)^{-1/2}a^*(e_\lambda)h_j \quad (j \geq 0)
\]

The space \( \mathcal{H}_\Lambda \) may be identified with \( L^2(\mathbb{R}) \) in an isometric way. Then the basis \( (h_j) \) becomes the Hermite’s functions basis, and the operators \( Q_\lambda \) and \( P_\lambda \) respectively become the multiplication by \( x_\lambda \) and the operator \( \frac{1}{i} \frac{d}{dx}\). Effectuating the completed tensorial product, the space \( \mathcal{H}_\Lambda \) is similarly identified to \( L^2(\mathbb{R}^\Lambda) \) for each finite subset \( \Lambda \) of \( \mathbb{Z} \).

For any \( E \subset F \subseteq \mathbb{Z} \), and any operator \( T \in \mathcal{L}(E) \), we define \( i_{EF}(T) \) by the following equality:

\[
(2.4) \quad i_{EF}(T) = T \otimes I_{F \setminus E},
\]

where \( I_{F \setminus E} \) is the identity in the space \( \mathcal{H}_{F \setminus E} \). In particular, if \( F = \mathbb{Z} \) the operator \( i_{E\mathbb{Z}}(T) \) is said to be supported in \( E \).

**Sobolev spaces.** Let us denote by \( \mathcal{H}^1 \) the set of all \( f \in \mathcal{H} \) such that \( f \) belongs to the domains of the Segal operators \( Q_\lambda = Q_\lambda^{(0)} \) and \( P_\lambda = Q_\lambda^{(1)} \) for all \( \lambda \in \mathbb{Z} \), and such that the following norm is finite:

\[
(2.5) \quad \|f\|_{\mathcal{H}^1} = \|f\|_\mathcal{H} + \sup_{\lambda \in \mathbb{Z}} \|Q_\lambda^{(j)}f\|_\mathcal{H}
\]

The space \( \mathcal{H}^2 \) is the set of all \( f \in \mathcal{H}^1 \) such that \( Q_\lambda^{(0)}f \) and \( Q_\lambda^{(1)}f \) belongs to \( \mathcal{H}^1 \) for all \( \lambda \in \mathbb{Z} \), and with a finite following norm:

\[
(2.6) \quad \|f\|_{\mathcal{H}^2} = \|f\|_{\mathcal{H}^1} + \sup_{(\lambda,\mu) \in \mathbb{Z}^2} \|Q_\lambda^{(j)}Q_\mu^{(k)}f\|_\mathcal{H}
\]

These spaces are dense in \( \mathcal{H} \) since they contain \( \mathcal{D} \). If \( E \) is a subset of \( \mathbb{Z} \) then the subspace \( \mathcal{H}^k_E \) is defined analogously in its corresponding Hilbert space \( \mathcal{H}_E \).

**Commutators, and spaces with negative orders.**

For all \( A \) in \( \mathcal{L}(\mathcal{H}) \), for all \( f \in \mathcal{H}^1 \) and for any \( \lambda \in \mathbb{Z} \) the map:

\[
(2.7) \quad \mathcal{H}^1 \ni g \rightarrow \langle AQ_\lambda^{(j)}f, g \rangle - \langle Af, Q_\lambda^{(j)}g \rangle \quad 0 \leq j \leq 1
\]

is a continuous antilinear map on the space \( \mathcal{H}^1 \). We denote by \( \mathcal{H}^{-k} \) the anti-dual of \( \mathcal{H}^k \) (\( 0 \leq k \leq 2 \)). For any \( A \) in \( \mathcal{L}(\mathcal{H}) \) the map (2.7) is linear and continuous from \( \mathcal{H}^1 \) to \( \mathcal{H}^{-1} \). It is noted \( [A, Q_\lambda^{(j)}] \). One may identify \( \mathcal{H} \) with a subspace of \( \mathcal{H}^{-1} \), and the latter one is identified to a subspace of \( \mathcal{H}^{-2} \). Thus, the operators \( Q_\lambda^{(j)} \) are bounded from \( \mathcal{H}^m \) to \( \mathcal{H}^{m-1} \) (\( -1 \leq m \leq 2 \)), and this allows us to define the iterated commutators \( [Q_\lambda^{(j)}, Q_\mu^{(k)}], ([\lambda, \mu] \in \mathbb{Z}^2, 0 \leq j, k \leq 1), \) as continuous linear maps from \( \mathcal{H}^2 \) to \( \mathcal{H}^{-2} \). This map is also denoted by \( (adQ_\lambda^{(j)})(adQ_\mu^{(k)}) \) \( A \).

If there is real number \( C > 0 \) verifying :

\[
\left| \langle AQ_\lambda^{(j)}f, g \rangle - \langle Af, Q_\lambda^{(j)}g \rangle \right| \leq C \|f\|_\mathcal{H} \|g\|_\mathcal{H}
\]
for all \( f \) and \( g \) in \( \mathcal{H}^1 \) we shall say that the commutators \([A, Q^{(j)}_\lambda]\) are in \( \mathcal{L}(\mathcal{H}) \). Then, for all \( f \) in \( \mathcal{H}^1 \) there exists an element \( \mathcal{H} \) noted \([A, Q^{(j)}_\lambda]f\) such that we have:

\[
\langle APf , g \rangle - \langle Af , Q^{(j)}_\lambda g \rangle = \langle [A, Q^{(j)}_\lambda]f , g \rangle
\]

for all \( g \) in \( \mathcal{H}^1 \), and the previously defined operator \([A, Q^{(j)}_\lambda] : \mathcal{H}^1 \to \mathcal{H}\) is extended to an element of \( \mathcal{L}(\mathcal{H}) \).

Proceeding similarly, one gives a precise meaning to the commutator \([A, Q^{(j)}_\lambda, Q^{(k)}_\mu]\) is in \( \mathcal{L}(\mathcal{H}) \).

We denote by \( \mathcal{W}_1 \) the set of all \( A \) in \( \mathcal{L}(\mathcal{H}) \) having their commutators \([A, Q^{(j)}_\lambda]\) (\( 0 \leq j \leq 1 \)) in \( \mathcal{L}(\mathcal{H}) \) for all \( \lambda \) in \( \mathbb{Z} \), and having a finite following norm:

\[
\|A\|_{\mathcal{W}_1} = \|A\|_{\mathcal{L}(\mathcal{H})} + \sum_{\lambda, \in \mathbb{Z}} \|\|[A, Q^{(j)}_\lambda]\|_{\mathcal{L}(\mathcal{H})}
\]

We denote by \( \mathcal{W}_2 \) the set of elements \( A \) belonging to \( \mathcal{W}_1 \), having commutators \([A, Q^{(j)}_\lambda, Q^{(k)}_\mu]\) in \( \mathcal{L}(\mathcal{H}) \) for all \( \lambda \) and \( \mu \) in \( \mathbb{Z} \), and having a finite following norm:

\[
\|A\|_{\mathcal{W}_2} = \|A\|_{\mathcal{W}_1} + \frac{1}{2} \sum_{(\lambda, \mu) \in \mathbb{Z}^2, 0 \leq j, k \leq 1} \|[A, [Q^{(j)}_\lambda, Q^{(k)}_\mu]]\|_{\mathcal{L}(\mathcal{H})}
\]

We easily verify the next proposition.

**Proposition 2.1.** For all \( k \leq 2 \) the algebra \( \mathcal{W}_k \) is a Banach algebra. For all \( A \) and \( B \) in \( \mathcal{W}_k \), the following inequality holds:

\[
\|AB\|_{\mathcal{W}_k} \leq \|A\|_{\mathcal{W}_k} \|B\|_{\mathcal{W}_k}
\]

Any operator \( A \in \mathcal{W}_2 \) is bounded in the Sobolev space \( \mathcal{H}^2 \) and we have:

\[
\|A\|_{\mathcal{L}(\mathcal{H}^2, \mathcal{H}^2)} \leq 3 \|A\|_{\mathcal{W}_2}
\]

**Proof of theorem 1.4.** Let \((A_n)\) be a sequence in \( \mathcal{W}_2 \) and let \( A \) be in \( \mathcal{L}(\mathcal{H}^2, \mathcal{H}) \) satisfying:

\[
\|A_n\|_{\mathcal{W}_2} \leq 1 \quad \lim_{n \to \infty} \|A_n - A\|_{\mathcal{L}(\mathcal{H}^2, \mathcal{H})} = 0
\]

For each \( f \) in \( \mathcal{H}^2 \), one deduces that \( \|Af\| \leq \|f\| \) and \( A \) is thus extended by density to an element of \( \mathcal{L}(\mathcal{H}) \) with a norm satisfying:

\[
\|A\|_{\mathcal{L}(\mathcal{H})} \leq \liminf_{n \to \infty} \|A_n\|_{\mathcal{L}(\mathcal{H})}
\]

For all \( \lambda \) in \( \mathbb{Z} \), for all \( f \) and \( g \) in \( \mathcal{D} \) and for any \( n \geq 1 \) we see:

\[
\left| \langle AQ^{(j)}_\lambda f , g \rangle - \langle Af , Q^{(j)}_\lambda g \rangle \right| \leq \|[A_n, Q^{(j)}_\lambda]||f||_\mathcal{H} \|g||_\mathcal{H} + \varepsilon_n
\]

where the sequence \( \varepsilon_n \) tends to 0. As a consequence:

\[
\left| \langle AQ^{(j)}_\lambda f , g \rangle - \langle Af , Q^{(j)}_\lambda g \rangle \right| \leq \|f||_\mathcal{H} \|g||_\mathcal{H} \liminf_{n \to \infty} \|[A_n, Q^{(j)}_\lambda]\|
\]
Since $\mathcal{D}$ is dense in $\mathcal{H}$, this inequality is still valid for all $f$ and $g$ in $\mathcal{H}$. With the above definition the commutator $[A, Q^{(j)}_\lambda]$ is thus in $\mathcal{L}(\mathcal{H})$ and one has:

$$\|[A, Q^{(j)}_\lambda]\|_{\mathcal{L}(\mathcal{H})} \leq \liminf_{n \to \infty} \|[A_n, Q^{(j)}_\lambda]\|_{\mathcal{L}(\mathcal{H})}$$

From Fatou’s lemma one deduces:

$$\sum_{\lambda \in \mathbb{Z}} \|[A, Q^{(j)}_\lambda]\|_{\mathcal{L}(\mathcal{H})} \leq \liminf_{n \to \infty} \sum_{\lambda \in \mathbb{Z}} \|[A_n, Q^{(j)}_\lambda]\|_{\mathcal{L}(\mathcal{H})}$$

It is similarly derived that the commutator $[\,[A, Q^{(j)}_\lambda], Q^{(k)}_\mu\,]$ is in $\mathcal{L}(\mathcal{H})$ for all $\lambda$ and $\mu$ in $\mathbb{Z}$ and that:

$$\sum_{(\lambda, \mu) \in \mathbb{Z}^2} \|[\,[A, Q^{(j)}_\lambda], Q^{(k)}_\mu\,]\|_{\mathcal{L}(\mathcal{H})} \leq \liminf_{n \to \infty} \sum_{(\lambda, \mu) \in \mathbb{Z}^2} \|[\,[A_n, Q^{(j)}_\lambda], Q^{(k)}_\mu\,]\|_{\mathcal{L}(\mathcal{H})}$$

Theorem 1.4 is then an easy consequence of these points.

In order to derive theorem 1.3, we shall construct, for each subset $E$ and $F$ such that $E \subset F \subseteq \mathbb{Z}$, an almost right inverse of the operator $i_{EF}$ defined in (2.4). Set $\Omega_{F\setminus E}$ the ground state of $F \setminus E$. Let $\pi_{EF} : \mathcal{H}_E \to \mathcal{H}_F$ be the map

$$f \to \pi_{EF}(f) = f \otimes \Omega_{F\setminus E},$$

and let $\pi_{EF}^*$ be the adjoint operator $\pi_{EF}^* : \mathcal{H}_F \to \mathcal{H}_E$. Note that $\pi_{EF}^* \pi_{EF} = I$. For all $A$ in $\mathcal{L}(\mathcal{H}_F)$ one defines an operator $\rho_{F,E}(A)$ in $\mathcal{L}(\mathcal{H}_E)$ by:

$$\rho_{F,E}(A)f = \pi_{EF}^* \circ A \circ \pi_{EF}$$

Thus, an element $\rho_{F,E}(A)$ of $\mathcal{L}(\mathcal{H}_E)$ is constructed. One can easily see that, for each $A \in \mathcal{W}_2$:

$$\|\rho_{F,E}(A)\|_{\mathcal{W}_2} \leq \|A\|_{\mathcal{W}_2}$$

We have also, if $E \subset F \subset G$:

$$\rho_{GE} = \rho_{FE} \circ \rho_{GF}.$$
Proof of theorem 1.3. Let $A \in \mathcal{W}_2$. Set $A_n = i_{\Lambda_n} z \circ \rho_{z,\Lambda_n}(A)$. The $A_n$ are in $\mathcal{W}_2$ with finite supports and verify: $\|A_n\|_{\mathcal{W}_2} \leq \|A\|_{\mathcal{W}_2}$. If $m < n$ then we have from Proposition 2.5:

$$\|A_m - A_n\|_{\mathcal{L}(\mathcal{H}^2,\mathcal{H})} \leq \|\rho_{\Lambda_n,\Lambda_m}(A_n) - A_n\|_{\mathcal{L}(\mathcal{H}^2,\mathcal{H})} \leq C \sum_{\lambda \in \mathcal{W}_{\Lambda_m}} (adP_{\lambda})^{j}(adQ_{\lambda})^{k} A \|_{\mathcal{L}(\mathcal{H})}$$

The latter sequence goes to 0 when $m \to \infty$ if $A \in \mathcal{W}_2$. Consequently, the sequence $A_n$ converges, in $\mathcal{L}(\mathcal{H}^2,\mathcal{H})$, to an element $B \in \mathcal{L}(\mathcal{H}^2,\mathcal{H})$. From Theorem 1.4, $B$ is in $\mathcal{W}_2$ and $A_n f$ strongly converges to $B f$ for all $f \in \mathcal{H}$. Let us check that $B = A$. To this end, set $f$ and $g$ two elements of $\mathcal{D}$. If $\Lambda_n$ contains the support of $f$ then $A_n f = \pi_{\Lambda_n} z \pi_{\Lambda_n}^* z A f$. Therefore, if $\Lambda_n$ also contains the support of $g$:

$$\langle A_n f, g \rangle = \langle \pi_{\Lambda_n} z \pi_{\Lambda_n}^* z A f, \pi_{\Lambda_n} z \pi_{E_2 \Lambda_n} \psi \rangle = \langle \pi_{\Lambda_n}^* z A f, \pi_{E_2 \Lambda_n} \psi \rangle = \langle A f, g \rangle$$

Since $A_n f$ strongly converges to $B f$ then $\langle A f, g \rangle = \langle B f, g \rangle$ for all $f$ and $g$ in $\mathcal{D}$. Since $\mathcal{D}$ is dense in $\mathcal{H}$ the equality $B = A$ is indeed true. As a consequence $A_n$ converges to $A$ in $\mathcal{L}(\mathcal{H}^2,\mathcal{H})$ and the proof is finished. \hfill $\square$

Proposition 2.5 also implies the following result.

Corollary 2.6. For all $A$ and $B$ in $\mathcal{W}_2$ with finite supports, one has:

$$(2.17) \quad \|[A,B]\|_{\mathcal{L}(\mathcal{H}^2,\mathcal{H})} \leq C \|B\|_{\mathcal{W}_2} \sum_{\lambda \in \sigma(B)} (adP_{\lambda})^{j}(adQ_{\lambda})^{k} A \|_{\mathcal{L}(\mathcal{H})}$$

where $C$ is not depending on any of the parameters.

Proof. We make use of the operator $\rho_{F,E}$ for $F = \sigma(A) \cup \sigma(B)$ and $E = F \setminus \sigma(B)$. It is known that $\rho_{F,E}(A)$ commutes with $B$ since its support does not intersect $\sigma(B)$. It is then deduced that:

$$\|[A,B]\|_{\mathcal{L}(\mathcal{H}^2,\mathcal{H})} = \|[A - \rho_{F,E}(A), B]\|_{\mathcal{L}(\mathcal{H}^2,\mathcal{H})} \leq \left[\|B\|_{\mathcal{L}(\mathcal{H})} + \|B\|_{\mathcal{L}(\mathcal{H})}\right] \|A - \rho_{F,E}(A)\|_{\mathcal{L}(\mathcal{H}^2,\mathcal{H})}$$

From proposition 2.1,

$$\|B\|_{\mathcal{L}(\mathcal{H}^2)} + \|B\|_{\mathcal{L}(\mathcal{H})} \leq C \|B\|_{\mathcal{W}_2}$$

Using Proposition 2.5, we find a constant $C > 0$, which does not depend on any of the parameters, such that (2.17) is verified. \hfill $\square$

3. Perturbation potentials and commutators.

We have to express the perturbation potentials $V_{\lambda}$ and $V_{\lambda,\mu}$, satisfying hypotheses (H1) and (H2) in section 1, as integrals of the Weyl operators, and to verify precisely that, under our hypotheses (H1) and (H2), these integrals are convergent and define operators in Sobolev spaces. We shall do the same work for the commutators of $V_{\lambda,\mu}$ with elements of $\mathcal{W}_1$, or with Segal operators, or for iterated commutators. These norm estimates will be used in following sections.

Partial Sobolev spaces.

The Sobolev spaces defined in section 2 are not Hilbert spaces. Nevertheless, for any finite subset like $\Lambda_n$, the space $\mathcal{H}_{\Lambda_n}^k$ may be endowed with an Hilbert space norm which is equivalent, for each fixed $n$, to the norm of section 2. As an example, for $k = 1$, one may set:

$$\|f\|_{\mathcal{H}_{\Lambda_n}^1}^2 = \sum_{\lambda \in \Lambda_n \atop j \geq 0,1} \|Q_{\lambda}^{(j)} f\|_{\mathcal{H}_{\Lambda_n}}^2$$
For all \( n \), these norms and those on section 2 are equivalent but the constant involved in the inequality depends on \( n \).

Let us choose an orthonormal basis \( (\varphi_\alpha)_{\alpha \geq 0} \) in the Hilbert space \( \mathcal{H}_{\lambda^2} \). We define a map \( \Psi_\alpha \) from \( \mathcal{H}_{\lambda_n} \) into \( \mathcal{H} \) by \( \Psi_\alpha(f) = f \otimes \varphi_\alpha \). The adjoint map from \( \mathcal{H} \) to \( \mathcal{H}_{\lambda_n} \) is denoted by \( \Psi_\alpha^* \). For all \( f \in \mathcal{H} \) we have:

\[
\|f\|^2 = \sum_{\alpha \geq 0} \|\Psi_\alpha f\|^2_{\mathcal{H}_{\lambda_n}}
\]

Then, we define the space \( \mathcal{H}^k(\Lambda_n) \) as the set of all \( f \) with a finite below norm:

\[
\|f\|^2_{\mathcal{H}^k(\Lambda_n)} = \sum_{\alpha \geq 0} \|\Psi_\alpha f\|^2_{\mathcal{H}^k_{\lambda_n}}
\]

Thus, \( \mathcal{H}^k(\Lambda_n) \subset \mathcal{H} \) if \( k \geq 0 \). When \( k = 1 \), an element \( f \) of \( \mathcal{H} \) is in \( \mathcal{H}^1 \) if it belongs to \( \mathcal{H}^1(\Lambda_n) \) and if, for all \( \lambda \in \Lambda_n \), one has \( Q_\lambda^{(j)} f \in \mathcal{H} \), the sequence of these norms being bounded. This property may be used only for fixed \( n \).

**Partial Sobolev spaces with negative order.**

Set \( \mathcal{H}^{-k}(\Lambda_n) \), the anti-dual set of \( \mathcal{H}^k(\Lambda_n) \). Thus:

\[
\mathcal{H}^2(\Lambda_n) \subset \mathcal{H}^1(\Lambda_n) \subset \mathcal{H} \subset \mathcal{H}^{-1}(\Lambda_n) \subset \mathcal{H}^{-2}(\Lambda_n)
\]

If an operator \( \Phi \in \mathcal{L}(\mathcal{H}^{1}_{\lambda_n}, \mathcal{H}) \) verifies \( \langle \Phi f, g \rangle = \langle f, \Phi g \rangle \) for all \( f \) and \( g \) in \( \mathcal{H}^{1}_{\lambda_n} \), where \( \langle \cdot, \cdot \rangle \) is the scalar product in \( \mathcal{H} \), then, for all \( f \in \mathcal{H} \), the map \( g \mapsto \langle f, \Phi g \rangle \) is an element of \( \mathcal{H}^{-1}(\Lambda_n) \) denoted here by \( \Phi f \). Thus, the operator \( Q_\lambda \) is bounded from \( \mathcal{H}^k(\Lambda_n) \) into \( \mathcal{H}^{k-1}(\Lambda_n) \) \( (-1 \leq k \leq 2, \lambda \in \Lambda_n) \). We shall check that similar considerations are also valid for the operators \( i[P_\lambda, V_{\Lambda_n}] \). The commutator of these two types of operators is in \( \mathcal{L}(\mathcal{H}^1(\Lambda_n), \mathcal{H}^{-1}(\Lambda_n)) \).

**Perturbation potentials and Weyl operators.**

If \( \xi \) is real sequence in \( l^2(\mathbb{Z}) \) with a finite support then the Segal operator \( \Pi(\xi, 0) \) defined in (2.1) is also written as \( \sum \xi_\lambda Q_\lambda \). Since the hypotheses on the perturbation potentials involve only the derivatives of order 2 and 3, the following function shall be implied in the sequel:

\[
x \rightarrow F(x) = e^{ix} - 1 - ix = i^2 x^2 \int_0^1 (1 - \theta)e^{i\theta x} d\theta
\]

Set \( V_{\lambda_1, \lambda_2} (\lambda_1 \neq \lambda_2) \) the non bounded operator in \( \mathcal{H}_{\lambda_1, \lambda_2} \) being, under the identification of this space with \( L^2(\mathbb{R}^2) \), the multiplication by a function \( v_{\lambda_1, \lambda_2} \). If the latter satisfies the hypothesis (H1), one has:

\[
V_{\lambda_1, \lambda_2} = v_{\lambda_1, \lambda_2}(0) I + \sum_{1 \leq j \leq 2} (\partial_{\lambda_j} v_{\lambda_1, \lambda_2})(0) Q_{\lambda_j} + (2\pi)^{-2} \int_{\mathbb{R}^2} \overline{v_{\lambda_1, \lambda_2}(\xi)} F(\xi_1 Q_{\lambda_1} + \xi_2 Q_{\lambda_2}) d\xi
\]

Under the hypothesis (H1) the integral is convergent and it defines a bounded operator from \( \mathcal{H}^2 \) in \( \mathcal{H} \).

**Commutators.**

In order to study the commutators of \( V_{\lambda_1, \lambda_2} \) with other operators, we shall use the following relations, valid for any operators \( X \) and \( A \) in a Banach space, and for the function \( F \) in (3.2):

\[
[e^{iX}, A] = i \int_0^1 e^{i\theta X} [X, A] e^{i(1-\theta)X} d\theta
\]
Proposition 3.2. For all $V_{\lambda_1, \lambda_2}$ we obtain:

$$[P_{\lambda_j}, V_{\lambda_1, \lambda_2}] = -i\left(\partial_{\lambda_j} v_{\lambda_1, \lambda_2}\right)(0) I + \sum_{1 \leq k \leq 2} A_{\lambda_1, \lambda_2}^{jk} Q_{\lambda_k}$$

$$A_{\lambda_1, \lambda_2}^{jk} = (2\pi)^{-2} \int_{\mathbb{R}^2 \times [0,1]} v_{\lambda_1, \lambda_2}(\xi) \frac{d\xi d\theta}{\xi_{\lambda_1} \xi_{\lambda_2} e^{i\theta(\xi_{\lambda_1} Q_{\lambda_1} + \xi_{\lambda_2} Q_{\lambda_2})}}$$

Under the assumption (H1), this integral converges and defines an operator $A_{\lambda_1, \lambda_2}^{jk}$ in $\mathcal{L}(\mathcal{H})$, with a norm being $O(e^{-\gamma_0|\lambda_1 - \lambda_2|})$. Each one site operator is similarly treated. Note that the integrals are then integrals on $\mathbb{R}$. We deduce the following proposition concerning the potential $V_{\lambda_1, \lambda_2}$ defined in (1.1) and (1.9):

**Proposition 3.1.** Under the hypotheses (H1) and (H2), one may write:

$$[P_{\lambda}, V_{\lambda_n}] = -ia^{(n)}_{\lambda} + \sum_{\mu \in \mathbb{A}_n} W^{(n)}_{\lambda \mu} Q_{\mu}$$

where $a^{(n)}_{\lambda}$ is a real constant number, and where $W^{(n)}_{\lambda \mu}$ is a bounded operator in $\mathcal{H}$. Moreover, there exists $C_1 > 0$ independent of $\lambda$, $\mu$ and $n$, such that:

$$|a^{(n)}_{\lambda}| \leq C_1, \quad ||W^{(n)}_{\lambda \mu}||_{\mathcal{L}(\mathcal{H})} \leq C_1 e^{-\gamma_0|\lambda - \mu|}.$$ 

We can also apply the commutation formula (3.5), still setting $X = \xi_{\lambda_1} Q_{\lambda_1} + \xi_{\lambda_2} Q_{\lambda_2}$, but with $A \in \mathcal{W}_2$. Inserting the expression (3.3) for $V_{\lambda_1, \lambda_2}$ and using hypotheses (H1), we obtain the following proposition.

**Proposition 3.2.** For all $A$ in $\mathcal{W}_2$, for all $\lambda$ and $\mu$ in $\mathbb{Z}$, the commutator $[A, V_{\lambda \mu}]$ is in $\mathcal{L}(\mathcal{H}^1, \mathcal{H})$. There is $C > 0$, independent of all the parameters, such that:

$$||[A, V_{\lambda \mu}]||_{\mathcal{L}(\mathcal{H}^1, \mathcal{H})} \leq C e^{-\gamma_0|\lambda - \mu|} \sum_{1 \leq j + k \leq 2} ||(ad Q_{\lambda})^j (ad Q_{\mu})^k A||_{\mathcal{L}(\mathcal{H})}$$

Double commutators.

If $A$, $B$ and $X$ are three operators such that $[X, B]$ is the identity operator up to a multiplicative factor, and if $F$ is the function given by (3.2), then it is deduced from (3.4) and (3.5) that:

$$[[F(X), A], B] = i^2 [X, A] \int_0^1 e^{i\theta X} [X, A] e^{i(1-\theta)X} d\theta$$

This formula is applied with $X = \xi_{\lambda_1} Q_{\lambda_1} + \xi_{\lambda_2} Q_{\lambda_2}$, $B = P_{\lambda_j}$ ($j = 1, 2$) and $A \in \mathcal{L}(\mathcal{H})$ (in particular $A \in \mathcal{W}_1$). Inserting the expression (3.3) for $V_{\lambda_1, \lambda_2}$ and using the hypotheses (H1), on gets:

$$[[V_{\lambda_1, \lambda_2}, P_{\lambda_j}], A] = \sum_{1 \leq k \leq 2} S_{\lambda_1, \lambda_2}^{jk} ([A, Q_{\lambda_k}])$$

where we set, for all $\Phi$ in $\mathcal{L}(\mathcal{H}^2(\Lambda_n), \mathcal{H}^{-2}(\Lambda_n))$

$$S_{\lambda_1, \lambda_2}^{jk} (\Phi) = (2\pi)^{-2} \int_{\mathbb{R}^2 \times [0,1]} v_{\lambda_1, \lambda_2}(\xi) \xi_{\lambda_1} \xi_{\lambda_2} e^{i\theta X(\xi)} \circ \Phi \circ e^{i(1-\theta)X(\xi)} d\xi d\theta$$
with the notation $X(\xi) = \xi_{\lambda_1}Q_{\lambda_1} + \xi_{\lambda_2}Q_{\lambda_2}$.

Next we shall deduce the following proposition.

**Proposition 3.3.** For all $\lambda$ and $\mu$ in $\Lambda_n$ ($n \geq 1$), there exists a linear continuous map $K_{\lambda\mu}$ from $\mathcal{L}(\mathcal{H}^2(\Lambda_n), \mathcal{H}^{-2}(\Lambda_n))$ into itself, leaving invariant the subspaces $\mathcal{L}(\mathcal{H}^1(\Lambda_n), \mathcal{H}^{-1}(\Lambda_n))$ and $\mathcal{L}(\mathcal{H})$, such that, for all $A$ in $\mathcal{L}(\mathcal{H})$, we have:

\[
[A, [P_\lambda, V_{\Lambda_n}]] = \sum_{\mu \in \Lambda_n} K_{\lambda\mu}([A, Q_\mu])
\] (3.12)

Moreover, when restricted to $\mathcal{L}(\mathcal{H})$, $K_{\lambda\mu}$ is in $\mathcal{L}(\mathcal{L}(\mathcal{H}))$, and there exists $C_0 > 0$, independent of $n$, $\lambda$ and $\mu$, such that

\[
\|K_{\lambda\mu}\|_{\mathcal{L}(\mathcal{L}(\mathcal{H}))} \leq C_0 e^{-\gamma_0|\lambda - \mu|}.
\] (3.13)

**Proof.** Under our hypotheses, the operator $\Phi \rightarrow S_{\lambda\lambda\lambda\lambda}^{12}(\Phi)$ maps $\mathcal{L}(\mathcal{H}^2(\Lambda_n), \mathcal{H}^{-2}(\Lambda_n))$ into itself. It also maps $\mathcal{L}(\mathcal{H})$ into itself, with a norm $\leq C_0 e^{-\gamma_0|\lambda - \mu|}$. For one site potentials $V_\lambda$, we define similar operators $S_{\lambda}$ such that $[V_\lambda, P_\lambda, A] = S_{\lambda}([A, Q_\lambda])$, for all $A \in W_1$. We then set, for all $\lambda$ and $\mu$ in $\Lambda_n$ such that $\lambda \neq \mu$:

\[
K^{(n)}_{\lambda\mu}(\Phi) = \begin{cases} S^{12}_{\lambda\mu}(\Phi) + S^{21}_{\mu\lambda}(\Phi) & \text{if } |\lambda - \mu| \geq 2 \\ -\phi \Phi + S^{12}_{\lambda\mu}(\Phi) + S^{21}_{\mu\lambda}(\Phi) & \text{if } |\lambda - \mu| = 1 \end{cases}
\]

and if $\lambda = \mu$,

\[
K^{(n)}_{\lambda\lambda}(\Phi) = a\Phi + S_{\lambda}(\Phi) + \sum_{\mu \in \Lambda_n \setminus \lambda} (S^{11}_{\lambda\mu}(\Phi) + T^{22}_{\mu\lambda}(\Phi))
\]

The equality (3.12) and the estimates (3.13) follows. \qed

A consequence of proposition 3.2, (that shall be used in the sequel), is that the left product by the matrix $\|K_{\lambda\mu}(t)\|$ leaves invariant the set of matrices with exponential decay. In particular, it is precisely the function $S_\gamma$, implied in the next proposition which will determine the propagation speed in section 8.

**Proposition 3.4.** Under the hypotheses (H1) and (H2) and for all $\gamma$ in $]0, \gamma_0[$ (or in $]0, \infty[$ in the case of interaction with nearest neighbors), there exists $S_\gamma > 0$ such that, for all $n$, for all $\lambda$ and $\nu$ in $\Lambda_n$:

\[
\sum_{\mu \in \Lambda_n} \|K_{\lambda\mu}\|_{\mathcal{L}(\mathcal{L}(\mathcal{H}))} e^{-\gamma|\mu - \nu|} \leq S_\gamma e^{-\gamma|\lambda - \nu|} \sum_{\mu \in \Lambda_n} \|W_{\lambda\mu}\|_{\mathcal{L}(\mathcal{H})} e^{-\gamma|\mu - \nu|} \leq S_\gamma e^{-\gamma|\lambda - \nu|}
\]

where the $K_{\lambda\mu}$ are the operators constructed in Proposition 3.3 and where the $W_{\lambda\mu}$ are those of Proposition 3.1.

**Triple commutators.**

If $X$, $A$, $B$, $C$ are operators such that $[X, B]$ and $[X, C]$ are equal to the identity operator up to a multiplicative factor, and if $F$ is the function defined by (3.2), then we deduce from (3.9) and (3.4) that:

\[
\left[ \left[ [F(X), B], A \right], C \right] = i^2 [X, B] \int_0^1 e^{i\theta X} \left[ [X, A], C \right] e^{i(1-\theta)X} d\theta + i^3 [X, B] [X, C] \int_0^1 e^{i\theta X} [X, A] e^{i(1-\theta)X} d\theta
\]
We shall apply this formula with \( X = \xi_\lambda, Q_\lambda \) and \( \xi_\lambda, Q_\lambda \), \( B = P_{\lambda_1}(1 \leq j \leq 2) \), \( A \in W_2 \) and \( C \) being a Segal operator. Inserting the expression of \( V_{\lambda_1, \lambda_2} \) given in (3.3) and using the hypothesis (H1), we obtain:

\[
\left[ \left[ V_{\lambda_1, \lambda_2}, P_{\lambda_1} \right], A \right], C \} = \sum_{1 \leq k \leq 2} S^j_{\lambda_1, \lambda_2} \left( \left[ A, Q_{\lambda_k} \right], C \right) + T^j_{\lambda_1, \lambda_2} \left( [A, Q_{\lambda_k}], C \right)
\]

where \( S^j_{\lambda_1, \lambda_2}(\Phi) \) is the operator defined in (3.11) and \( T^j_{\lambda_1, \lambda_2}(\Phi, C) \) is defined by:

\[
T^j_{\lambda_1, \lambda_2}(\Phi, C) = (2\pi)^{-2} \int_{\mathbb{R}^2 \times [0,1]} e^{i\lambda_2(\xi)} \xi \lambda \lambda_k [X(\xi), C] e^{i\theta X(\xi)} \circ \Phi \circ e^{i(1-\theta)X(\xi)} d\xi d\theta
\]

If \( C \) is Segal operator (linear combination of \( P_\lambda \) and \( Q_\lambda \)) then \([X(\xi), C] \) is a constant and the above integral converges using the hypothesis (H1). It is at this point that the hypothesis: "\( |x|^3 e^{i\lambda}(\xi) \) belongs to \( L^1(\mathbb{R}^2) \)" is involved. We proceed similarly for all one site operators \( V_\lambda \). Summing up as in Proposition 3.3, one obtains the next result:

**Proposition 3.5.** For all \( \lambda \) and \( \mu \) in \( \Lambda_n \) \((n \geq 1)\), for all Segal operator \( \Psi \), there exists a map \( \Phi \rightarrow R_{\lambda, \mu}(\Phi, \Psi) \) from \( L(\mathcal{H}(\Lambda_n), \mathcal{H}^{-1}(\Lambda_n)) \) into itself such that, for all \( A \in L(\mathcal{H}) \) supported in \( \Lambda_n \), we have:

\[
\left[ [A, P_\lambda, V_\Lambda_n] \right., \Psi \left. \right] = \sum_{\mu \in \Lambda_n} K_{\lambda, \mu} \left([A, Q_\mu], \Psi \right) + R_{\lambda, \mu}([A, Q_\mu], \Psi)
\]

where \( \Phi \rightarrow K_{\lambda, \mu}(\Phi) \) is the map of Proposition 3.3. If \( \Phi \) is in \( L(\mathcal{H}) \) then \( R_{\lambda, \mu}(\Phi, \Psi) \) is in \( L(\mathcal{H}) \). One has \( R_{\lambda, \mu}(\Phi, P_\rho) = 0 \) for all \( \rho \). One also see that \( R_{\lambda, \mu}(\Phi, P_\rho) = 0 \) excepted when the set \( \{\lambda, \mu, \rho\} \) has only two distinct elements \((\lambda = \mu) \) or \( \lambda = \rho \) or \( \mu = \rho \). In that case, one gets:

\[
\|R_{\lambda, \mu}(\Phi, P_\rho)\|_{L(\mathcal{H})} \leq C e^{-\gamma_0(\lambda-\mu)} \|\Phi\|_{L(\mathcal{H})} \quad \text{if} \quad \lambda \neq \mu
\]

\[
\|R_{\lambda, \mu}(\Phi, P_\rho)\|_{L(\mathcal{H})} \leq C e^{-\gamma_0(\lambda-\rho)} \|\Phi\|_{L(\mathcal{H})} \quad \text{if} \quad \lambda = \mu
\]


4. Evolution of the position and impulsion operators.

Using the Fock space notations, the Hamiltonian \( \mathcal{H}_\Lambda_n \) in (1.1) is written as:

\[
\mathcal{H}_\Lambda_n = \sum_{\lambda \in \Lambda_n} \left( P^2_\lambda + \frac{a}{2} Q^2_\lambda \right) - b \sum_{\lambda=-n}^{n-1} \lambda Q_\lambda Q_{\lambda+1} + V^{pert}_{\Lambda_n}
\]

where the operator \( V^{pert}_{\Lambda_n} \) is expressed as the sum (1.9). The terms in the sum verify the hypotheses (H1) and (H2) and let us recall that these two hypotheses are analyzed in section 3. Let us first start by giving the domain of self-adjointness of \( \mathcal{H}_\Lambda_n \).

**Proposition 4.1.** In the Hilbert space \( \mathcal{H}_\Lambda_n \), the operator \( \mathcal{H}_\Lambda_n \) is self-adjoint with the domain \( \mathcal{H}^2_{\Lambda_n} \). The operator \( e^{it\mathcal{H}_\Lambda_n} \) is bounded in \( \mathcal{H}^k_{\Lambda_n} \) \((k = 0, 1, 2)\). The operator \( e^{it\mathcal{H}_\Lambda_n} \otimes I_{\Lambda_n} \) is bounded in \( \mathcal{H}^k(\Lambda_n) \) defined in section 3 \((-2 \leq k \leq 2)\).

**Proof.** We know that \( \mathcal{H}_\Lambda_n \) is naturally identified to \( L^2(\mathbb{R}^{2n}) = L^2(\mathbb{R}^{2n+1}) \) in such a way that the operators \( P_\lambda \) and \( Q_\lambda \) become:

\[
P_\lambda = \frac{1}{i} \frac{\partial}{\partial \xi_\lambda} \quad Q_\lambda = x_\lambda
\]

The spaces \( \mathcal{H}^k_{\Lambda_n} \) are then identified to the usual spaces \( B^k \) of the theory of globally elliptic operators (cf Helffer [HE]). When \( V^{pert}_{\Lambda_n} = 0 \), the operator \( \mathcal{H}_\Lambda_n \) is a Schrödinger operator, where the potential is a definite positive quadratic form (if \( a > 2b > 0 \)). In this case, it is well-known that \( \mathcal{H}_\Lambda_n \) is self-adjoint with domain
We define an operator \( W_{\lambda} \). Let us show that the addition of \( V_{\text{pert}} \) does not modify this result. With the preceding identification and under our hypotheses, \( V_{\lambda} \) and \( V_{\lambda\mu} \) are multiplications by the functions \( v_{\lambda} \) and \( v_{\lambda\mu} \), with second-order derivatives going to 0 at infinity. (These functions are Fourier transforms of functions being in \( L^1(\mathbb{R}) \) or in \( L^1(\mathbb{R}^2) \).) Consequently, these functions \( v_{\lambda}(x_{\lambda})/|x_{\lambda}|^2 \) and \( v_{\lambda\mu}(x_{\lambda}, x_{\mu})/|x_{\lambda}|^2 + |x_{\mu}|^2 \) goes to 0 at infinity. The above Proposition thus follows from Kato-Rellich’s theorem. As a consequence, the operator \( e^{itH_{\lambda\mu}} \) is a well-defined bounded operator in \( \mathcal{H} \) and in the domain of \( H_{\Lambda_n} \), that is to say in \( \mathcal{H}_{\Lambda_n}^2 \). By interpolation it is also bounded in \( \mathcal{H}_{\Lambda_n}^1 \). The latter statement comes from (3.1) if \( 0 \leq k \leq 2 \) and is deduced by duality if \( k \leq 0 \).

Consequently, if \( A \) belongs to \( \mathcal{L}(\mathcal{H}^k(\Lambda_n), \mathcal{H}^{k'}(\Lambda_n)) \) then the operator

\[
\alpha^{(t)}_{\lambda}(A) = (e^{itH_{\lambda\mu}} \otimes I) \circ A \circ (e^{-itH_{\lambda\mu}} \otimes I)
\]

in the same spaces. In particular, the operator \( \alpha^{(t)}_{\lambda}(Q^{(j)}_{\lambda}) \) \( (\lambda \in \Lambda_n) \) belongs to \( \mathcal{L}(\mathcal{H}^1(\Lambda_n), \mathcal{H}) \).

**Proposition 4.2.** For all \( \lambda \) and \( \mu \) in \( \Lambda_n \), there exists \( C^1 \) maps \( t \to A^{(n)}_{\lambda\mu}(t), t \to B^{(n)}_{\lambda\mu}(t), \) and \( t \to R^{(n)}_{\lambda}(t) \) from \( \mathbb{R} \) into \( \mathcal{L}(\mathcal{H}) \) such that (omitting the superscript \( n \) in the expressions):

\[
\alpha^{(t)}_{\lambda}(Q_{\lambda}) = \sum_{\mu \in \Lambda_n} \left[ A_{\lambda\mu}(t)Q_{\mu} + B_{\lambda\mu}(t)P_{\mu} \right] + R_{\lambda}(t)
\]

\[
\alpha^{(t)}_{\lambda}(P_{\lambda}) = \sum_{\mu \in \Lambda_n} \left[ A'_{\lambda\mu}(t)Q_{\mu} + B'_{\lambda\mu}(t)P_{\mu} \right] + R'_{\lambda}(t)
\]

Moreover, for all \( \gamma \) in \([0, \gamma_0]\), for all \( M > \sqrt{S_{\gamma}} \) (where \( S_{\gamma} \) is the constant number appearing in Proposition 3.3), there exists \( C > 0 \) such that:

\[
\|A_{\lambda\mu}(t)\| + \|B_{\lambda\mu}(t)\| + \|A'_{\lambda\mu}(t)\| + \|B'_{\lambda\mu}(t)\| \leq Ce^{M|t|} e^{-\gamma|\lambda-\mu|}
\]

\[
\|R_{\lambda}(t)\| + \|R'_{\lambda}(t)\| \leq Ce^{M|t|}
\]

First step. We shall study the differential system satisfied by:

\[
Q_{\lambda}(t) = \alpha^{(t)}_{\lambda}(Q_{\lambda}) \quad P_{\lambda}(t) = \alpha^{(t)}_{\lambda}(P_{\lambda})
\]

One observes that \( t \to Q_{\lambda}(t) \) and \( t \to P_{\lambda}(t) \) are \( C^1 \) functions from \( \mathbb{R} \) into \( \mathcal{L}(\mathcal{H}^1(\Lambda_n), \mathcal{H}) \) verifying:

\[
Q'_{\lambda}(t) = P_{\lambda}(t) \quad P'_{\lambda}(t) = -i\alpha^{(t)}_{\lambda}([P_{\lambda}, V_{\lambda\mu}])
\]

With the operators \( W^{(n)}_{\lambda\mu} \) and the constant in \( a^{(n)}_{\lambda} \) of Proposition 3.1, it follows that:

\[
P'_{\lambda}(t) = -a^{(n)}_{\lambda} - i \sum_{\mu \in \Lambda_n} \alpha^{(t)}_{\lambda}(W_{\lambda\mu}Q_{\mu})
\]

We define an operator \( \mathcal{L}(\mathcal{H}) \) by setting:

\[
\widetilde{W}_{\lambda\mu}(t) = \alpha^{(t)}_{\lambda}(W^{(n)}_{\lambda\mu})
\]
With these notations, the preceding system is written as:

\[(4.8) \quad Q_\lambda'(t) = P_\lambda(t) \quad P_\lambda'(t) = -a_\lambda^{(n)} - i \sum_{\mu \in \Lambda_n} W_{\lambda \mu}(t) \circ Q_\mu(t)\]

To conclude, \(t \to (Q_\lambda(t), P_\lambda(t))\) is the unique \(C^1\) map from \(\mathbb{R}\) into \(\mathcal{L}(\mathcal{H}^1(\Lambda_n), \mathcal{H})\) solution to (4.8) and satisfying \(Q_\lambda(0) = Q_\lambda\) and \(P_\lambda(0) = P_\lambda\).

**Second step.** We shall now construct matrices \(A_{\lambda \mu}(t)\) such that the right hand-side of (4.3) is also solution to the same system (4.8) and satisfies the same initial data. First, we can find an operator-valued matrix \((A_{\lambda \mu}^0(t), A_{\lambda \mu}^1(t))\) in \(\mathcal{L}(\mathcal{H})\) solution to:

\[(4.9) \quad \frac{d}{dt} A_{\lambda \mu}^0(t) = A_{\lambda \mu}^1(t) \quad \frac{d}{dt} A_{\lambda \mu}^1(t) = -i \sum_{\nu \in \Lambda_n} W_{\lambda \nu}(t) A_{\nu \mu}^0(t)\]

\[A_{\lambda \mu}^0(0) = \delta_{\lambda \mu} I \quad A_{\lambda \mu}^1(0) = 0\]

Indeed, from Propositions 3.1 and 3.4 one see that the hypotheses in Proposition B.1 (Appendix B) are satisfied for all \(\gamma \in [0, \gamma_0]\). Then, there exists a solution of (4.9) satisfying the above initial condition, and also, if \(M > \sqrt{S_\gamma}\):

\[(4.10) \quad \|A_{\lambda \mu}^j(t)\|_{\mathcal{L}(\mathcal{H})} \leq C(M, \gamma)e^{M|t|}e^{-\gamma|\lambda - \mu|}\]

An operator-valued matrix \((B_{\lambda \mu}^0(t), B_{\lambda \mu}^1(t))\) solution to the same system (4.9) verifying the same estimates (4.10) is analogously constructed, satisfying the following initial conditions:

\[B_{\lambda \mu}^0(0) = 0 \quad A_{\lambda \mu}^1(0) = \delta_{\lambda \mu} I\]

From remark 2 in the appendix B, one may find operators \((R_{\lambda}^0(t), R_{\lambda}^1(t))\) of \(\mathcal{L}(\mathcal{H})\) solutions to

\[\frac{d}{dt} R_{\lambda}^0(t) = R_{\lambda}^1(t) \quad \frac{d}{dt} R_{\lambda}^1(t) = -i \sum_{\nu \in \Lambda_n} W_{\lambda \nu}(t) R_{\nu \lambda}^0(t) + i a_\lambda^{(n)}\]

\[R_{\lambda}^0(0) = R_{\lambda}^1(0) = 0\]

\[\|R_{\lambda}^j(t)\|_{\mathcal{L}(\mathcal{H})} \leq C(M, \gamma)e^{M|t|} \sum_{\mu \in \Lambda_n} e^{-\gamma|\lambda - \mu|}|a_\mu| \quad j = 0, 1\]

We define the operators of \(\mathcal{L}(\mathcal{H}^1(\Lambda_n), \mathcal{H})\) by

\[\tilde{Q}_\lambda(t) = \sum_{\mu \in \Lambda_n} \left[ A_{\lambda \mu}^0(t) Q_\mu(t) + B_{\lambda \mu}^1(t) P_\mu(t) \right] + R_{\lambda}(t) \quad j = 1, 2\]

These functions verify the same system (4.8) as the functions \(Q_\lambda(t)\), together with the same initial conditions \(\tilde{Q}_\lambda(t) = Q_\lambda, \tilde{Q}_\lambda(t) = Q_\lambda(t)\) and \(\tilde{Q}_\lambda(t) = P_\lambda(t)\), thus the equalities (4.3) and (4.4) are true and the matrices estimates (4.5) (4.6) are valid.

\[\square\]

**Example: The cyclic quadratic case.**

In the case of a positive definite quadratic form potential (without perturbation potentials), it is well-known that the equalities (4.3) and (4.4) are valid with \(R_{\lambda}(t) = 0\) and the operators \(A_{\lambda \mu}(t)\) and \(B_{\lambda \mu}(t)\) being real numbers. The following classical proposition may sum up this situation:
Proposition 4.3. In the case where the potentials \(V_\lambda\) and \(V_\mu\) (perturbation potentials) are vanishing, the operators \(\alpha^{(t)}_{\lambda_n}(Q_\lambda)\) and \(\alpha^{(t)}_{\lambda_n}(P_\lambda)\) satisfy equalities (4.2) and (4.3) where \(R^{(n)}(t) = 0\) and the \(A^{(n)}_{\lambda\mu}(t)\) and \(B^{(n)}_{\lambda\mu}(t)\) are real numbers. The matrices \(A^{(n)}(t)\) and \(B^{(n)}(t)\) are related to the matrix \(W_n\) of the quadratic form \(V^{quad}_{\Lambda_n}\) in the canonical basis by the equality:

\[
A^{(n)}(t) = \cos \left( t \sqrt{W_n} \right) \quad B^{(n)}(t) = - \frac{\sin \left( t \sqrt{W_n} \right)}{\sqrt{W_n}}
\]

One may estimate the matricial elements \(A_{\lambda\mu}(t)\) and \(A_{\lambda\mu}(t)\) using Proposition 4.2. However, in some cases, the inequalities of Proposition 4.2 together with the Lieb-Robinson inequalities may be strongly improved and explicitly written down. This is precisely the case if the perturbation potential vanishes, and if the quadratic potential takes the following form (with an interaction between the two ends of the linear chain):

\[
V^{quad}_{\Lambda_n}(x) = \frac{a}{2} |x|^2 - b \sum_{\lambda=-n}^{n-1} x_{\lambda}x_{\lambda+1} - bx_nx_{-n}
\]

In that case, we can make the estimates of proposition 4.2 more precise if the distance \(d(\lambda, \mu) = |\lambda - \mu|\) is replaced by the cyclic distance on \(\Lambda_n\), \(d_n(\lambda, \mu) = d(\lambda - \mu, (2n+1)\mathbb{Z})\).

These improved estimates follow on from [NRSS] in the cyclic quadratic case. Let us give here a simplified proof of a perhaps less precise type of estimates.

In the cyclic quadratic case, the analysis of chains of oscillators involves the dispersion relations \(\omega(\theta) = \sqrt{a - 2b \cos \theta}\) (cf Cohen-Tannoudji [CT]). It is the natural to give a corresponding complex expression by setting

\[
\Omega(z) = \sqrt{a - b(z + z^{-1})}
\]

This function is analytic in \(\mathbb{C} \setminus \{-\infty, z_1 \cup z_2, 0\}\) where \(z_1\) and \(z_2\) are the roots of \(bz^2 - az + b = 0\). Note however that, the function \(|\text{Im} \Omega(z)|\) is well defined on \(\mathbb{C} \setminus \{0\}\). Set, for all \(\gamma > 0\)

\[
M(\gamma) = \sup_{\gamma = e^\gamma} |\text{Im} \Omega(z)|
\]

This function is well-defined on \(\mathbb{C} \setminus \{0\}\).

Proposition 4.4. Under the above hypotheses and for all \(\gamma > 0\) there exists \(C(\gamma) > 0\), independent on \(n\) such that, the matrices \(A^{(n)}(t)\) and \(B^{(n)}(t)\) of Proposition 4.3 satisfy:

\[
|A^{(n)}_{\lambda\mu}(t)| + |B^{(n)}_{\lambda\mu}(t)| + \left| \frac{d}{dt} A^{(n)}_{\lambda\mu}(t) \right| + \left| \frac{d}{dt} B^{(n)}_{\lambda\mu}(t) \right| \leq C(\gamma) e^{[t|M(\gamma)|]e^{-\gamma d_n(\lambda, \mu)}}
\]

where \(M(\gamma)\) is defined in (4.12) and \(d_n(\lambda, \mu) = d(\lambda - \mu, (2n+1)\mathbb{Z})\).

Proof. The matrix \(W_n\) of the quadratic form \(V^{quad}_{\Lambda_n}\), and therefore all the matrices \(A^{(n)}(t)\) and \(B^{(n)}(t)\) are functions of the cyclic shift operator \(S_n\) defined in \(\mathbb{R}^{\Lambda_n}\) by

\[
S_n e_j = \begin{cases} 
  e_{j+1} & \text{if } -n \leq j < n \\
  e_{-n} & \text{if } j = n
\end{cases}
\]

More precisely, one has \(W_n = aI + bS_n + bS_n^{-1}\) and

\[
A^{(n)}(t) = f(S_n, t) \quad B^{(n)}(t) = g(S_n, t) \quad C^{(n)}(t) = h(S_n, t)
\]
where we set, using the function \( \Omega(z) \) defined in (4.11):

\[
(4.13) \quad f(z, t) = \cos(t\Omega(z)) \quad g(z, t) = \frac{\sin(t\Omega(z))}{\Omega(z)} \quad h(z, t) = -\sin(t\Omega(z))\Omega(z)
\]

These functions are analytic on \( \mathbb{C} \setminus \{0\} \). The proof uses the following elementary lemma:

**Lemma 4.5.** Let \( S \) be a unitary operator in an Hilbert space \( \mathcal{H} \). Set \( f(z, t) \) the function defined in (4.11) and (4.13) where \( a > 2|b| > 0 \). Then, one may write for all \( t \in \mathbb{R} \):

\[
f(S, t) = \sum_{k \in \mathbb{Z}} c_k(t) S^k
\]

Moreover, one has for all \( \gamma > 0 \), for all \( t \in \mathbb{R} \) and for all \( k \in \mathbb{Z} \),

\[
|c_k(t)| \leq e^{-\gamma|k|} \frac{1}{2\pi} \int_0^{2\pi} |f(e^{\gamma i\theta}, t)| d\theta
\]

The same result holds for the functions \( g \) and \( h \) defined in (4.13).

**End of the proof of Proposition 4.4.** Since \( S_n^{2n+1} = I \), the sum in Lemma 4.5 is written as a finite sum, and

\[
A^{(n)}(t) = f(S_n, t) = \sum_{k=0}^{2n} a_k(t) S_n^k \quad a_k(t) = \sum_{p \in \mathbb{Z}} c_{k+p(2n+1)}(t)
\]

where the \( c_j(t) \) are the coefficients of Lemma 4.5. Consequently, if \( -n \leq \lambda \leq \mu \leq n \) and \( \gamma > 0 \) one has:

\[
|A^{(n)}_{\lambda\mu}(t)| = \left| \left\langle f(S_n, t)e_{\lambda}, e_{\mu} \right\rangle \right| = |a_{\mu-\lambda}(t)| \leq \sum_{p \in \mathbb{Z}} |c_{\mu-\lambda+p(2n+1)}(t)|
\]

\[
\leq \left[ \sum_{p \in \mathbb{Z}} e^{-\gamma|\mu-\lambda+p(2n+1)|} \right] \frac{1}{2\pi} \int_0^{2\pi} |f(e^{\gamma i\theta}, t)| d\theta
\]

There exists \( C_1(\gamma) \) and \( C_2(\gamma) \) independent of \( n \), such that:

\[
\sum_{p \in \mathbb{Z}} e^{-\gamma|\mu-\lambda+p(2n+1)|} \leq C_1(\gamma) e^{-\gamma d_n(\lambda, \mu)}
\]

\[
\frac{1}{2\pi} \int_0^{2\pi} |f(e^{\gamma i\theta}, t)| d\theta \leq C_2(\gamma) e^{t|M(\gamma)|}
\]

where \( M(\gamma) \) is defined in (4.12). As a consequence, \( |A^{(n)}_{\lambda\mu}(t)| \leq C_1(\gamma)C_2(\gamma)e^{t|M(\gamma)|}e^{-\gamma d_n(\lambda, \mu)} \). Similar estimates for the matricial elements \( B^{(n)}_{\lambda\mu}(t) \) together with its derivatives may be obtained. The proof of Proposition 4.4 follows.

\[\square\]

5. Evolution of the commutators.

From Proposition 4.1, the commutators \([A, \alpha^{(t)}_{\lambda n}(Q_\Lambda)]\) and \([A, \alpha^{(t)}_{\lambda n}(P_\Lambda)]\) are defined as operators taking \( \mathcal{H}^1(\Lambda_n) \) into \( \mathcal{H}^{-1}(\Lambda_n) \), for all \( A \) in \( \mathcal{L}(\mathcal{H}) \) supported in \( \Lambda_n \), and for all \( t \in \mathbb{R} \).
Proposition 5.1. For all $A \in \mathcal{W}_1$ supported in $\Lambda_n$ and for all $t \in \mathbb{R}$ the commutators $[A, \alpha^{(t)}_{\Lambda_n}(Q)]$ are bounded in $\mathcal{H}$ $(\lambda \in \Lambda_n, 0 \leq j \leq 1)$. For all $\gamma$ in the interval $]0, \gamma_0[$ and for all $M > \sqrt{S\gamma}$ there exists $C(M, \gamma) > 0$, (independent of $n$) such that:

$$
(5.1) \quad \|[A, \alpha^{(t)}_{\Lambda_n}(Q)]\|_{\mathcal{L}(\mathcal{H})} \leq C(M, \gamma)e^{M|\mu|} \sum_{\mu \in \Lambda_n} e^{-\gamma d(\lambda, \mu)} \|[A, Q]\|_{\mathcal{L}(\mathcal{H})}
$$

First step. Assuming first that $A$ is only in $\mathcal{L}(\mathcal{H})$ we shall study the differential system satisfied by the functions:

$$
(5.2) \quad \Phi^{(j)}(t) = [A, \alpha^{(t)}_{\Lambda_n}(Q)] \quad 0 \leq j \leq 1
$$

The $\Phi^{(j)}$'s are $C^1$ maps from $\mathbb{R}$ into $\mathcal{L}(\mathcal{H}^1(\Lambda_n), \mathcal{H}^{-1}(\Lambda_n))$ and verify:

$$
\frac{d}{dt}\Phi^{(1)}(t) = \Phi^{(1)}(t) \quad \frac{d}{dt}\Phi^{(j)}(t) = -i[A, \alpha^{(t)}_{\Lambda_n}([P, V_{\Lambda_n}])] = -i\alpha^{(t)}_{\Lambda_n}([\alpha^{(-t)}_{\Lambda_n}(A), [P, V_{\Lambda_n}]])
$$

Using the operators $K_{\lambda\mu}$ of Proposition 3.3,

$$
[\alpha^{(-t)}_{\Lambda_n}(A), [P, V_{\Lambda_n}]] = \sum_{\mu \in \Lambda_n} K_{\lambda\mu}([\alpha^{(-t)}_{\Lambda_n}(A), Q_{\mu}])
$$

Next set $\widetilde{K}_{\lambda\mu}(t)$ the operator taking $\mathcal{L}(\mathcal{H}^1(\Lambda_n), \mathcal{H}^{-1}(\Lambda_n))$ into itself and defined by:

$$
(5.3) \quad \widetilde{K}_{\lambda\mu}(t)\Phi = \alpha^{(t)}_{\Lambda_n}(K_{\lambda\mu}(\alpha^{-t}_{\Lambda_n}\Phi)) \quad \forall \Phi \in \mathcal{L}(\mathcal{H}^1(\Lambda_n), \mathcal{H}^{-1}(\Lambda_n))
$$

With these notations the system becomes

$$
(5.4) \quad \frac{d}{dt}\Phi^0(t) = \Phi^{(1)}(t) \quad \frac{d}{dt}\Phi^1(t) = -i \sum_{\mu \in \Lambda_n} \widetilde{K}_{\lambda\mu}(t)(\Phi^0_{\mu}(t)).
$$

Summing up, for all $A$ in $\mathcal{L}(\mathcal{H})$, supported in $\Lambda_n$, the functions $\Phi^0(t)$ defined in (5.2) ($\lambda \in \Lambda_n$) are $C^1$ from $\mathbb{R}$ to $\mathcal{L}(\mathcal{H}^1(\Lambda_n), \mathcal{H}^{-1}(\Lambda_n))$. These maps are bounded independently of $t$ and verify (5.4). It is the unique solution to (5.4) having these properties together with:

$$
(5.5) \quad \Phi^0(0) = [A, Q_{\lambda}] \quad \Phi^1(0) = [A, P_{\lambda}]
$$

Second step. One may find operators-valued matrices $(A^0_{\lambda\mu}(t), A^1_{\lambda\mu}(t))$ in $\mathcal{L}(\mathcal{L}(\mathcal{H}))$ satisfying:

$$
(5.6) \quad \frac{d}{dt}A^0_{\lambda\mu}(t) = A^1_{\lambda\mu}(t) \quad \frac{d}{dt}A^1_{\lambda\mu}(t) = -i \sum_{\nu \in \Lambda_n} \widetilde{K}_{\lambda\nu}(t) \circ A^0_{\nu\mu}(t)
$$

$$
(5.7) \quad A^0_{\lambda\mu}(0) = \delta_{\lambda\mu}I \quad A^1_{\lambda\mu}(0) = 0
$$

In (5.6) the composition is now the composition in $\mathcal{L}(\mathcal{L}(\mathcal{H}))$ and in (5.7) the identity operator is the identity in $\mathcal{L}(\mathcal{L}(\mathcal{H}))$. Indeed, for all $\gamma$ in $]0, \gamma_0[$, the hypotheses in Proposition B.1 are satisfied, by Proposition 3.4. If $\gamma$ is in $]0, \gamma_0[$ and if $M > \sqrt{S\gamma}$ there exists $C(M, \gamma)$ such that

$$
(5.8) \quad \|[A_{\lambda\mu}(t)]\|_{\mathcal{L}(\mathcal{L}(\mathcal{H}))} \leq C(M, \gamma)e^{-\gamma|\lambda-\mu|}
$$
We can find, by a similar construction, operators-valued matrices \((B_{\lambda \mu}^0(t), B_{\lambda \mu}^1(t))\) of \(\mathcal{L}(\mathcal{H})\) satisfying the same differential system (5.6) together with the same estimates (5.8) and the new initial conditions:

\[
\begin{align*}
B_{\lambda \mu}^0(0) &= 0 \\
B_{\lambda \mu}^1(0) &= \delta_{\lambda \mu} I.
\end{align*}
\]

Suppose now that \(A\) belongs to \(\mathcal{W}_1\) and is supported in \(\Lambda_n\). The operators \([A, Q_\lambda]\) and \([A, P_\lambda]\) are in \(\mathcal{L}(\mathcal{H})\). We then define the operators in \(\mathcal{L}(\mathcal{H})\) by:

\[
\Psi_j^i(t) = \sum_{\mu \in \Lambda_n} A_{\lambda \mu}^j(t)([A, Q_\mu]) + B_{\lambda \mu}^j(t)([A, P_\mu]) \quad j = 0, 1
\]

These functions, taking values into \(\mathcal{L}(\mathcal{H})\), satisfy the same differential system (5.4) with the same initial conditions (5.5) as the functions \(\Phi_j^i(t)\) (being a priori in \(\mathcal{L}(\Lambda_n, \mathcal{H}^{-1}(\Lambda_n))\)). Uniqueness shows that \(\Phi_j^i(t) = \Psi_j^i(t)\). The functions \(\Phi_j^i(t)\) defined in (5.2) have therefore the stated properties.

For all \(\lambda\) and \(\mu\) in \(\Lambda_n\) the commutator \([Q_\lambda^j, \alpha_\Lambda^j(\mu)]\) \((0 \leq j, k \leq 1)\) is bounded from \(\mathcal{H}^1(\Lambda_n)\) into \(\mathcal{H}^{-1}(\Lambda_n)\). We shall obtain that it is an element of \(\mathcal{L}(\mathcal{H})\) and we shall estimate its norm.

**Proposition 5.2.** Under the hypotheses (H1) and (H2) of section 1, for all \(\lambda\) and \(\mu\) in \(\Lambda_n\), the commutator \([Q_\lambda^j, \alpha_\Lambda^j(\mu)]\) \((0 \leq j, k \leq 1)\) is a bounded operator in \(\mathcal{H}\). Moreover, for all \(\gamma\) in \([0, \gamma_0]\) and for all \(M > \sqrt{S_\gamma}\), there exists \(C(M, \gamma) > 0\), (independent of \(n, t, \lambda\) and \(\mu\)) such that:

\[
\left\| [Q_\lambda^j, \alpha_\Lambda^j(\mu)] \right\| \leq C(M, \gamma) e^{M|t|} e^{-\gamma|d(\lambda, \mu)|} \quad 0 \leq j, k \leq 1
\]

**Proof.** Using the matrices \(A_{\lambda \mu}^j(t)\) and \(B_{\lambda \mu}^j(t)\) \((j = 0, 1)\) defined in the second step of Proposition 5.1 one shows that:

\[
[P_\lambda, \alpha_\Lambda^j(\mu)] = A_{\lambda \mu}^j(t)(I) \quad [Q_\lambda, \alpha_\Lambda^j(\mu)] = B_{\lambda \mu}^j(t)(I) \quad 0 \leq j \leq 1
\]

The proof uses the same points as those in Proposition 5.1. Then Proposition 5.2 follows from the estimates on these matrices being analyzed in Proposition B.1.

Let us now consider commutators of length two.

**Proposition 5.3.** If \(A\) is in \(\mathcal{W}_2\), then the commutators \([\alpha_\Lambda^j(A), Q_\lambda^{j_1}], Q_\lambda^{j_2}\) are in \(\mathcal{L}(\mathcal{H})\), \((t \in \mathbb{R}, \lambda_1\) and \(\lambda_2\) in \(\Lambda_n\), \(0 \leq j_1, j_2 \leq 1)\). Moreover, if \(\gamma\) is in \([0, \gamma_0]\) and if \(M > 2\sqrt{S_\gamma}\) there exists \(C = C(M, \gamma)\) such that:

\[
\left\| [\alpha_\Lambda^j(A), Q_\lambda^{j_1}], Q_\lambda^{j_2}\right\|_{\mathcal{L}(\mathcal{H})} \leq C e^{M|t|} \left[ \sum_{(\mu_1, \mu_2) \in A_\Lambda^j \atop 0 \leq k_1, k_2 \leq 1} e^{-\gamma|\lambda_1 - \mu_1| + |\lambda_2 - \mu_2|} \left\| [A, Q_\mu^{k_1}], Q_\mu^{k_2}\right\| + \ldots \right. \\
\left. + \sum_{\mu_0 \in \Lambda_n \atop 0 \leq k_0 \leq 1} e^{-\gamma d(\lambda_1, \lambda_2)} \left\| [A, Q_\mu^{k_0}] \right\| \right]
\]

**First step.** Set \(A\) in \(\mathcal{L}(\mathcal{H})\). We show that the functions defined for all real \(t\) by:

\[
\Phi_{\lambda_1, \lambda_2}^{j_1, j_2}(t) = \left[ [A, \alpha_\Lambda^j(Q_\lambda^{j_1})], \alpha_\Lambda^j(Q_\lambda^{j_2}) \right] \quad 0 \leq j_1, j_2 \leq 1
\]
are $C^1$ functions taking values from $\mathbb{R}$ into $\mathcal{L}(\mathcal{H}^2(\Lambda_n), \mathcal{H}^{-2}(\Lambda_n))$ and verifying the following differential system where the operators $K_{\lambda\mu}(t)$ are defined in (5.3) and where the operators $R_{\lambda\mu}$ are given by Proposition 3.5:

\begin{equation}
\frac{d}{dt} \Phi_{\lambda_1 \lambda_2}^{00}(t) = \Phi_{\lambda_1 \lambda_2}^{01}(t) + \Phi_{\lambda_1 \lambda_2}^{10}(t) \tag{5.11}
\end{equation}

\begin{equation}
\frac{d}{dt} \Phi_{\lambda_1 \lambda_2}^{10}(t) = \Phi_{\lambda_1 \lambda_2}^{11}(t) - i \sum_{\mu_1 \in \Lambda_n} K_{\lambda_1 \mu_1}(t) (\Phi_{\mu_1 \lambda_2}^{00}(t)) \tag{5.12}
\end{equation}

\begin{equation}
\frac{d}{dt} \Phi_{\lambda_1 \lambda_2}^{01}(t) = \Phi_{\lambda_1 \lambda_2}^{11}(t) - i \sum_{\mu_2 \in \Lambda_n} K_{\lambda_2 \mu_2}(t) (\Phi_{\lambda_1 \mu_2}^{00}(t)) \tag{5.13}
\end{equation}

\begin{equation}
\frac{d}{dt} \Phi_{\lambda_1 \lambda_2}^{11}(t) = -i \sum_{\mu_1 \in \Lambda_n} K_{\lambda_1 \mu_1}(t) (\Phi_{\mu_1 \lambda_2}^{01}(t)) - i \sum_{\mu_2 \in \Lambda_n} K_{\lambda_2 \mu_2}(t) (\Phi_{\lambda_1 \mu_2}^{10}(t)) + F_{\lambda_1 \lambda_2}(t) \tag{5.14}
\end{equation}

\begin{equation}
F_{\lambda_1 \lambda_2}(t) = - \sum_{\mu_1 \in \Lambda_n} \alpha_{\Lambda_n}^{(t)} \left( R_{\lambda_1 \mu_1} \left( [\alpha_{\Lambda_n}^{(-t)}(A), Q_{\mu_1}], P_{\lambda_2} \right) \right) \tag{5.15}
\end{equation}

The system of functions $\Phi_{\lambda_1 \mu_2}(t)$ is the unique solution to the differential system (5.11)... (5.15) satisfying the initial conditions:

\begin{equation}
\Phi_{\lambda_1 \lambda_2}^{ij}(0) = \left[ [A, Q_{\lambda_1}^{(j_1)}], Q_{\lambda_2}^{(j_2)} \right], \quad 0 \leq j_1, j_2 \leq 1 \tag{5.16}
\end{equation}

Let us give more details, says, for the proof of (5.14). Following the differential system satisfied by $\alpha_{\Lambda_n}^{(t)}(Q_{\lambda})$ and $\alpha_{\Lambda_n}^{(t)}(Q_{\mu})$, (see the first step of Proposition 4.2) one observes that:

\begin{equation}
\frac{d}{dt} \Phi_{\lambda_1 \lambda_2}^{11}(t) = -i \alpha_{\Lambda_n}^{(t)} \left( \left[ \left( [\alpha_{\Lambda_n}^{(-t)}(A), [P_{\lambda_1}, V_{\Lambda_n}]], P_{\lambda_2} \right) + \left( [\alpha_{\Lambda_n}^{(-t)}(A), P_{\lambda_1}], [P_{\lambda_2}, V_{\Lambda_n}] \right) \right) \right) \tag{5.17}
\end{equation}

Using the operators $K_{\lambda \mu}$ of proposition 3.3, one gets:

\begin{equation}
\left( [\alpha_{\Lambda_n}^{(-t)}(A), [P_{\lambda_1}, V_{\Lambda_n}]], P_{\lambda_2} \right) = \sum_{\mu_2 \in \Lambda_n} K_{\lambda_2 \mu_2} \left( [\alpha_{\Lambda_n}^{(-t)}(A), P_{\lambda_1}], Q_{\mu_2} \right) \tag{5.18}
\end{equation}

Also using the operators $R_{\lambda \mu}$ of proposition 3.5, one sees that:

\begin{equation}
\left( [\alpha_{\Lambda_n}^{(-t)}(A), [P_{\lambda_1}, V_{\Lambda_n}]], P_{\lambda_2} \right) = \sum_{\mu_1 \in \Lambda_n} K_{\lambda_1 \mu_1} \left( [\alpha_{\Lambda_n}^{(-t)}(A), Q_{\mu_1}], P_{\lambda_2} \right) + R_{\lambda_1 \mu_1} \left( [\alpha_{\Lambda_n}^{(-t)}(A), Q_{\mu_1}], P_{\lambda_2} \right) \tag{5.19}
\end{equation}

Equalities (5.14) and (5.15) then follows.

**Second step.** Suppose now that $A$ is in $\mathcal{W}_2$. We shall show that the operators $F_{\lambda_1 \lambda_2}(t)$ defined in (5.15) are in $\mathcal{L}(\mathcal{H})$ and we shall estimate their norms. More precisely, we shall show that if $\gamma \in [0, |\gamma_0|] \text{ and } M > \sqrt{S_\gamma}$, one has:

\begin{equation}
\| F_{\lambda_1 \lambda_2}(t) \|_{\mathcal{L}(\mathcal{H})} \leq Ce^{M|t|} \sum_{\nu \in \Lambda_n} e^{-\gamma_0|\lambda_1 - \lambda_2| - \gamma \text{dist}(\nu, \{\lambda_1, \lambda_2\})} \left\| [A, Q_{\nu}^{(k)}] \right\| \tag{5.20}
\end{equation}
Indeed, from Proposition 3.5, if $\lambda_1 \neq \lambda_2$, then the sum in (5.15) is reduced to two terms: the one with $\mu_1 = \lambda_1$ together with the one with $\mu_1 = \lambda_2$. In this case, one has:

$$
\|F_{\lambda_1,\lambda_2}(t)\|_{\mathcal{L}(\mathcal{H})} \leq C e^{-\gamma_0 |\lambda_1 - \lambda_2|} \left[ \|\alpha_{\lambda_1}^{(-t)}(A), Q_{\lambda_1}\|_{\mathcal{L}(\mathcal{H})} + \|\alpha_{\lambda_2}^{(-t)}(A), Q_{\lambda_2}\|_{\mathcal{L}(\mathcal{H})} \right]
$$

If $\lambda_1 = \lambda_2$, one has, from Proposition 3.5:

$$
\|F_{\lambda_1,\lambda_1}(t)\|_{\mathcal{L}(\mathcal{H})} \leq C \sum_{\mu_1 \in \Lambda_n} e^{-\gamma_0 |\lambda_1 - \mu_1|} \|\alpha_{\lambda_1}^{(-t)}(A), Q_{\mu_1}\|_{\mathcal{L}(\mathcal{H})}
$$

Following Proposition 5.1, one sees, if $M > \sqrt{S_\gamma}$:

$$
\|\alpha_{\lambda_1}^{(-t)}(A), Q_{\mu_1}\|_{\mathcal{L}(\mathcal{H})} = \|\alpha_{\lambda_1}^{(-t)}(A), Q_{\mu_1}\|_{\mathcal{L}(\mathcal{H})} \leq C(M, \gamma) e^{M|t|} \sum_{\nu \in \Lambda_n, 0 \leq k \leq 1} e^{-\gamma |\mu_1 - \nu|} \|\alpha_{\nu}^{(-t)}(A), Q_{\nu}^{(k)}\|_{\mathcal{L}(\mathcal{H})}
$$

and the estimates (5.17) are easily deduced.

**Third step.** If $A$ is in $\mathcal{W}_2$, then the initial data (5.16) are in $\mathcal{L}(\mathcal{H})$. From the remarks below Proposition B.1, if $\gamma$ is in $[0, \gamma_0]$, the system (5.11)...(5.14) has a solution $\Psi_{\lambda_1,\lambda_2}(t)$ in $\mathcal{L}(\mathcal{H})$ satisfying (5.16). Moreover, if $M > 2\sqrt{S_\gamma}$, there exists $C(M, \gamma)$ such that:

$$
\|\Psi_{\lambda_1,\lambda_2}(t)\|_{\mathcal{L}(\mathcal{H})} \leq C(M, \gamma) e^{M|t|} \sum_{\nu \in \Lambda_n, 0 \leq k \leq 1} e^{-\gamma |\lambda_1 - \mu_1| + |\lambda_2 - \mu_2|} \|\alpha_{\nu}^{(-t)}(A), Q_{\nu}^{(k)}\|_{\mathcal{L}(\mathcal{H})} + ...$

$$
... + C(M, \gamma) \sum_{\nu \in \Lambda_n, 0 \leq k \leq 1} e^{-\gamma |\lambda_1 - \mu_1| + |\lambda_2 - \mu_2|} \int_0^t e^{M|t-s|} \|F_{\mu_1,\mu_2}(s)\|_{\mathcal{L}(\mathcal{H})} ds
$$

The proof of this proposition then follows from the estimates of $F_{\mu_1,\mu_2}(s)$ in (5.17).

6. **Evolution for a finite number of sites.**

From Proposition 4.1, the operator $e^{itH_{\lambda_n}} \otimes I$ is bounded in the $\mathcal{H}_k(\Lambda_n)$. However, when following the proof of Proposition 4.1 the norm of this operator could depend on $n$. On the contrary, the next proposition provides a bound independent on $n$.

**Proposition 6.1.** The operator $e^{itH_{\lambda_n}} \otimes I$ is bounded in $\mathcal{H}_k$, $(0 \leq k \leq 2)$ with a norm $\leq C_k e^{M_k|t|}$ where $C_k > 0$ and $M_k > 0$ are independent of all the parameters. For all $A \in \mathcal{L}(\mathcal{H}_k, \mathcal{H}_1)$, $(k = 1, 2)$ with finite support, if $\Lambda_n$ contains the support of $A$, one has:

$$
\|\alpha_{\lambda_n}^{(t)}(A)\|_{\mathcal{L}(\mathcal{H}_k, \mathcal{H}_1)} \leq C_k e^{M_k|t|} \|A\|_{\mathcal{L}(\mathcal{H}_k, \mathcal{H}_1)}
$$

**Proof.** Set $f \in \mathcal{H}_1$. From Proposition 4.2 and for all $\lambda \in \Lambda_n$ one see:

$$
\|Q_\lambda(e^{itH_{\lambda_n}} \otimes I) f\| = \|\alpha_{\lambda_n}^{(-t)}(Q_\lambda)f\| \leq \|R_{\lambda}(-t)f\| + \sum_{\mu \in \Lambda_n} \left[ \|A_{\mu,\lambda}^{(n)}(-t)Q_{\mu}f\| + \|B_{\mu,\lambda}^{(n)}(-t)P_{\mu}f\| \right]
$$

We deduce from the estimates (4.5) and (4.6) that, if $\gamma \in [0, \gamma_0]$ and if $M_1 > \sqrt{S_\gamma}$ then

$$
\|Q_\lambda(e^{itH_{\lambda_n}} \otimes I)f\| \leq C_1 e^{M_1|t|} \|f\|_{\mathcal{H}_1}
$$
with $C_1 > 1$ independent of $n$ and $t$. If $\lambda$ is not in $\Lambda_n$ then the same inequality is valid since $Q(\lambda)\sigma$ commutes with $e^{itH_{\Lambda_n}} \otimes I$. We proceed similarly with the operators $P(\lambda)$ proving that \( \|e^{itH_{\Lambda_n}} \otimes I\|_{\mathcal{L}(\mathcal{H}^t)} \leq C_1 e^{M_1|t|} \).

**Action in $\mathcal{H}^2$.** For all $\lambda_1$ and $\lambda_2$ in $\Lambda_n$ we have from the above points:

\[
\|Q^{(j_1)}_{\lambda_1} Q^{(j_2)}_{\lambda_2} (e^{itH_{\Lambda_n}} \otimes I) f\| = \|Q^{(j_1)}_{\lambda_1} (e^{itH_{\Lambda_n}} \otimes I) \alpha^{(-t)}_{\lambda_2} Q^{(j_2)}_{\lambda_2} f\| \leq C_1 e^{M_1|t|} \|\alpha^{(-t)}_{\lambda_2} Q^{(j_2)}_{\lambda_2} f\|_{\mathcal{H}^t}
\]

One sees:

\[
\|Q^{(k)}_{\mu} \alpha^{(-t)}_{\lambda_2} (Q^{(j_2)}_{\lambda_2}) f\| \leq \left\| \left[ Q^{(k)}_{\mu}, \alpha^{(-t)}_{\lambda_2} (Q^{(j_2)}_{\lambda_2}) \right] f \right\| + \|\alpha^{(-t)}_{\lambda_2} (Q^{(j_2)}_{\lambda_2}) Q^{(k)}_{\mu} f\|
\]

\[
\leq C_1 e^{M_1|t|} \left[ \|f\| + \|Q^{(k)}_{\mu} f\|_{\mathcal{H}^t} \right]
\]

for all $\mu \in \Lambda_n$.

The two above terms have been estimated using Propositions 5.2 and 5.1 respectively. One deduces (with another constant $C_2$) that, \( \|Q^{(j_1)}_{\lambda_1} Q^{(j_2)}_{\lambda_2} (e^{itH_{\Lambda_n}} \otimes I) f\| \leq C_2 e^{2M_1|t|} \|f\|_{\mathcal{H}^2} \). The proof is completed.

\[\square\]

**Theorem 6.2.** If $A$ is in $\mathcal{W}_k$ with a finite support, and if $\Lambda_n$ contains the support of $A$, then $\alpha^{(l)}_{\Lambda_n}(A)$ is in $\mathcal{W}_k$ ($0 \leq k \leq 2$). Moreover, there exists two constants $C_k$ and $M_k$ independent of $A$, $n$ and of $t$, such that:

\[ (6.2) \quad \|\alpha^{(l)}_{\Lambda_n}(A)\|_{\mathcal{W}_k} \leq C_k e^{M_k|t|} \|A\|_{\mathcal{W}_k} \]

**Proof.** The norm in $\mathcal{L}(\mathcal{H})$ is conserved by $\alpha^{(l)}_{\Lambda_n}$. By Proposition 5.1, if $A \in \mathcal{W}_1$ is supported in $\Lambda_n$ and if $\lambda \notin \Lambda_n$ then the commutators of $A$ with $\alpha^{(-t)}_{\lambda_2} (Q^{(j_2)}_{\lambda_2})$ are bounded operators. Thus, the commutators of $\alpha^{(l)}_{\Lambda_n}(A)$ with $Q^{(j_2)}_{\lambda_2}$ are bounded operators if $\lambda \notin \Lambda_n$. Since these commutators are vanishing when $\lambda \notin \Lambda_n$ then $\alpha^{(l)}_{\Lambda_n}(A)$ is in $\mathcal{W}_1$. If $\gamma > 0$ is in $[0, \gamma_0]$, and if $M_1 > \sqrt{S_\gamma}$, we see that:

\[
\sum_{\lambda \in \Lambda_n} \|\alpha^{(l)}_{\Lambda_n}(A), Q^{(j_2)}_{\lambda_2}\| \leq C(M_1, \gamma) e^{M_1|t|} \sum_{\lambda \in \Lambda_n} \sum_{\lambda \notin \Lambda_n} e^{-\gamma d(\lambda, \mu)} \|A, Q^{(j_2)}_{\lambda_2}\|
\]

\[
\leq C(M_1, \gamma) e^{M_1|t|} \|A\|_{\mathcal{W}_1} \sup_{\lambda \in \Lambda_n, \mu \notin \Lambda_n} e^{-\gamma d(\lambda, \mu)}
\]

Consequently, there are $C_1 > 0$ and $M_1 > 0$ such that (6.2) is valid for $k = 1$.

**Action in $\mathcal{W}^2$.** Proposition 5.3 shows that the commutators written as $[\alpha^{(l)}_{\Lambda_n}(A), Q^{(j_2)}_{\lambda_2}]$ are bounded operators and are vanishing if $\lambda_1$ or $\lambda_2$ is not in $\Lambda_n$. Consequently, $\alpha^{(l)}_{\Lambda_n}(A)$ is in $\mathcal{W}_2$. If $\gamma$ is in $[0, \gamma_0]$ and $M_2 = 2 \sqrt{S_\gamma}$ then Proposition 5.3 implies that inequality (6.2) is verified for $k = 2$.

\[\square\]

7. Existence of dynamics in the Weyl algebra.

The number of sites shall now go to infinity. The proofs of theorem 1.1 and 1.2 on the existence of a limit rely on the description of the difference $\alpha^{(l)}_{\Lambda_m}(A) - \alpha^{(l)}_{\Lambda_n}(A)$.

**Proposition 7.1.** There exists $C > 0$, $M > 0$ and $\gamma > 0$ satisfying the following properties. For all $A \in \mathcal{W}_2$ with finite support and for all integers $m$ and $n$ verifying $0 < m < n$ and such that $\Lambda_m$ contains the support $\sigma(A)$ of $A$, for all $t \in \mathbb{R}$, one has:

\[ (7.1) \quad \|\alpha^{(l)}_{\Lambda_m}(A) - \alpha^{(l)}_{\Lambda_n}(A)\|_{\mathcal{L}(\mathcal{H}, \mathcal{H}^0)} \leq C \|A\|_{\mathcal{W}_2} e^{M|t|} e^{-\gamma d(\sigma(A), \Lambda_n)} \]

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Proof. For \( m < n \) we denote by \( V_{\text{inter}}^{mn} \) the potential of the interaction between \( \Lambda_m \) and \( \Lambda_n \backslash \Lambda_m \):

\[
V_{\text{inter}}^{mn}(x) = -bQ_m Q_{m+1} - bQ_{-m} Q_{-m-1} + \sum_{(\lambda,\mu) \in E_{mn}} V_{\lambda\mu}
\]

where \( E_{mn} \) denotes the set of pairs of sites \((\lambda, \mu)\) such that, one of the site \((\lambda \text{ or } \mu)\) is in \( \Lambda_m \) and the other site belongs to \( \Lambda_n \backslash \Lambda_m \). For all \( \theta \in [0,1] \), set:

\[
H_{mn\theta} = H_{\Lambda_n} - (1 - \theta) V_{\text{inter}}^{mn}
\]

One may define a unitary operator by \( e^{itH_{mn\theta}} \) and set:

\[
\alpha_{mn\theta}^{(t)}(A) = (e^{itH_{mn\theta}} \otimes I) A (e^{-itH_{mn\theta}} \otimes I)
\]

Thus, if \( A \) is supported in \( \Lambda_m \) and if \( m < n \):

\[
\alpha_{mn1}^{(t)}(A) = \alpha_{\Lambda_n}^{(t)}(A) \quad \alpha_{mn\theta}^{(t)}(A) = \alpha_{\Lambda_m}^{(t)}(A)
\]

The function \( \varphi(t, \theta) = \frac{\partial}{\partial \theta} \alpha_{mn\theta}^{(t)}(A) \) verifies:

\[
\frac{\partial \varphi}{\partial t} = i[H_{mn\theta}, \varphi] + i[V_{\text{inter}}^{mn}, \alpha_{mn\theta}^{(t)}(A)] \quad \varphi(0, \theta) = 0
\]

Consequently:

\[
\frac{\partial}{\partial \theta} \alpha_{mn\theta}^{(t)}(A) = i \int_0^t \int_0^1 \alpha_{mn\theta}^{(t-s)}([V_{\text{inter}}^{mn}, \alpha_{mn\theta}^{(s)}(A)]) ds d\theta
\]

One obtains the integral representation:

\[
\alpha_{\Lambda_n}^{(t)}(A) - \alpha_{\Lambda_m}^{(t)}(A) = i \int_0^t \int_0^1 \alpha_{mn\theta}^{(t-s)}([V_{\text{inter}}^{mn}, \alpha_{mn\theta}^{(s)}(A)]) ds d\theta
\]

Applying Proposition 6.1 to the operator \( H_{mn\theta} \) which verifies the same hypotheses as \( H_{\Lambda_n} \), we deduce that there exists \( C > 0 \) and \( M > 0 \) such that:

\[
\|\alpha_{\Lambda_n}^{(t)}(A) - \alpha_{\Lambda_m}^{(t)}(A)\|_{\mathcal{L}(\mathcal{H}^1, \mathcal{H}^0)} \leq C \int_0^t \int_0^1 e^{M[t-s]} \|V_{\text{inter}}^{mn}, \alpha_{mn\theta}^{(s)}(A)\|_{\mathcal{L}(\mathcal{H}^1, \mathcal{H})} ds d\theta
\]

for all \((\lambda, \mu)\) in \( E_{mn} \). Applying Proposition 3.2 to the operator \( \alpha_{mn\theta}^{(s)}(A) \) belonging in \( \mathcal{W}_2 \) we obtain:

\[
\|([V_{\lambda\mu}, \alpha_{mn\theta}^{(s)}(A)])\|_{\mathcal{L}(\mathcal{H}^1, \mathcal{H}^0)} \leq C e^{-\gamma_0 |\lambda - \mu|} \sum_{1 \leq j+k \leq 2} \|\text{(ad } Q_{\lambda})^j (\text{ad } Q_{\mu})^k \alpha_{mn\theta}^{(s)}(A)\|
\]

Similarly:

\[
\|([Q_m Q_{m+1}, \alpha_{mn\theta}^{(s)}(A)])\|_{\mathcal{L}(\mathcal{H}^1, \mathcal{H}^0)} \leq C \sum_{1 \leq j+k \leq 2} \|\text{(ad } Q_{m})^j (\text{ad } Q_{m+1})^k \alpha_{mn\theta}^{(s)}(A)\|
\]

Summing on the pairs \((\lambda, \mu)\) in \( E_{mn} \) we get:

\[
\|([V_{\text{inter}}^{mn}, \alpha_{mn\theta}^{(s)}(A)])\|_{\mathcal{L}(\mathcal{H}^1, \mathcal{H}^0)} \leq C \sum_{(\lambda, \mu) \in E_{mn}} e^{-\gamma_0 |\lambda - \mu|} \sum_{1 \leq j+k \leq 2} \|\text{(ad } Q_{\lambda})^j (\text{ad } Q_{\mu})^k \alpha_{mn\theta}^{(s)}(A)\|
\]
Consequently:

\[(7.2)\quad \|\alpha_{\Lambda_n}^{(t)}(A) - \alpha_{\Lambda_m}^{(t)}(A)\|_{\mathcal{L}(\mathcal{H}^2, \mathcal{H})} \leq \ldots \]

\[\leq C \sum_{(\lambda, \mu) \in \Lambda_{mn}, 1 \leq j + k \leq 2} e^{-\gamma_0|\lambda - \mu|} \int_0^t \int_0^1 e^{M|t-s|} \|\left(ad Q_\lambda\right)^j (ad Q_\mu)^k \alpha^{(s)}_{mn\theta}(A)\| dsd\theta \]

Proposition 7.1 then follows from the next lemma, which shall also be used in section 8.

**Lemma 7.2.** If $\gamma$ is in $[0, \gamma_0]$ and if $M > 2\sqrt{S_\gamma}$ then there exists $C(M, \gamma)$ such that, for all $n$, for all disjoint sets $E_1$ and $E_2$ included in $\Lambda_n$, for all $A \in \mathcal{W}_2$ supported in $E_1$, we have:

\[\sum_{(\lambda_1, \lambda_2) \in \Lambda_n \times E_2} e^{-\gamma_0|\lambda_1 - \lambda_2|} \|\left(ad Q_{\lambda_1}\right)^{\alpha} \left(ad Q_{\lambda_2}\right)^{\beta} \alpha^{(s)}_{mn\theta}(A)\| \leq C(M, \gamma) \|A\|_{\mathcal{W}_2} e^{M|s|} e^{-\gamma d(E_1,E_2)}\]

This lemma is deduced from propositions 5.1 and 5.3 applied to the Hamiltonian $H_{mn\theta}$. Proposition 7.1 is a consequence of (7.2) together with this lemma, setting $E_1 = \sigma(A)$ and $E_2 = \Lambda_n \setminus \Lambda_m$.

**Proof of theorem 1.1 and 1.2.** From Proposition 7.1 the sequence $\alpha_{\Lambda_n}^{(t)}(A)$ is a Cauchy sequence in $\mathcal{L}(\mathcal{H}^2, \mathcal{H})$ and thus converges in $\mathcal{L}(\mathcal{H}^2, \mathcal{H})$ towards an element which is noted $\alpha^{(t)}(A)$. By Proposition 6.2, we have $\|\alpha_{\Lambda_n}^{(t)}(A)\|_{\mathcal{W}_2} \leq C e^{M|t|} \|A\|_{\mathcal{W}_2}$. Following theorem 1.4 the operator $\alpha^{(t)}(A)$ is in $\mathcal{W}_2$ with a norm $\leq C e^{M|t|} \|A\|_{\mathcal{W}_2}$ and for all $f \in \mathcal{H}$, the sequence $\alpha_{\Lambda_n}^{(t)}(A)f$ strongly converges to $\alpha^{(t)}(A)f$. The classical continuity of the map $t \rightarrow \alpha_{\Lambda_n}^{(t)}(A)f$ for all $n$ and for all $f$ together with the above inequalities, show the continuity of the map $t \rightarrow \alpha^{(t)}(A)f$.

**Extension of $\alpha^{(t)}$ to the algebra $\mathcal{W}_2$.** Set $A$ in $\mathcal{W}_2$ with an arbitrary support. From theorem 1.3 there exists a sequence $(A_n)$ in $\mathcal{W}_2$ with finite supports such that:

\[\|A_n\|_{\mathcal{W}_2} \leq \|A\|_{\mathcal{W}_2} \quad \lim_{n \to \infty} \|A_n - A\|_{\mathcal{L}(\mathcal{H}^2, \mathcal{H})} = 0\]

The operator $\alpha^{(t)}(A_n)$ is well-defined in view of theorem 1.1 and 1.2 since the $A_n$ have finite support and one has:

\[(7.3)\quad \|\alpha^{(t)}(A_n)\|_{\mathcal{W}_2} \leq C e^{M|t|} \|A_n\|_{\mathcal{W}_2} \leq C e^{M|t|} \|A\|_{\mathcal{W}_2}\]

If $m < n$ then we also see from theorem 1.2:

\[\|\alpha^{(t)}(A_n - A_m)\|_{\mathcal{L}(\mathcal{H}^2, \mathcal{H})} \leq C e^{M|t|} \|A_n - A_m\|_{\mathcal{L}(\mathcal{H}^2, \mathcal{H})}\]

The sequence $\alpha^{(t)}(A_n)$ thus converges in $\mathcal{L}(\mathcal{H}^2, \mathcal{H})$ to an element that is denoted $\alpha^{(t)}(A)$. From (7.3) and theorem 1.4 this element is in $\mathcal{W}_2$ and it verifies:

\[\|\alpha^{(t)}(A)\|_{\mathcal{W}_2} \leq C e^{M|t|} \|A\|_{\mathcal{W}_2}\]

The group $\alpha^{(t)}$ is therefore extended to the whole algebra $\mathcal{W}_2$. 

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8. Lieb-Robinson’s inequalities.

**Proposition 8.1.** For all $\gamma$ in $[0, \gamma_0]$ and for all $M > 2\sqrt{S_\gamma}$, there exists $C(M, \gamma) > 0$ such that, for all $A$ and $B$ in $\mathcal{W}_2$ with finite supports $\sigma(A)$ and $\sigma(B)$, for all $n$ such that $\Lambda_n$ contains $\sigma(A)$ and $\sigma(B)$, for all $t \in \mathbb{R}$ we have:

\begin{equation}
\left\| \alpha^{(t)}_{\Lambda_n}(A), B \right\|_{\mathcal{L}(\mathcal{H}^2, \mathcal{H})} \leq C(M, \gamma) \left\| A \right\|_{\mathcal{W}_2} \left\| B \right\|_{\mathcal{W}_2} e^{M|t|} e^{-\gamma d(\sigma(A), \sigma(B))}
\end{equation}

The same inequality is valid when replacing $\alpha^{(t)}_{\Lambda_n}$ by $\alpha^{(t)}$.

**Proof.** From corollary 2.6 applied with the operators $B$ and $\alpha^{(t)}_{\Lambda_n}(A)$, (both having their support in $\Lambda_n$) one has:

\[ \left\| \alpha^{(t)}_{\Lambda_n}(A), B \right\|_{\mathcal{L}(\mathcal{H}^2, \mathcal{H})} \leq C \left\| B \right\|_{\mathcal{W}_2} \sum_{\Lambda_n \sigma(B)} \left\| (ad P_\lambda)^j(ad Q_\lambda)^k(\alpha^{(t)}_{\Lambda_n}(A)) \right\| \]

Inequality (8.1) then follows by applying Lemma 7.2 to the sets $E_1 = \sigma(A)$ and $E_2 = \sigma(B)$. The analogous inequality for $\alpha^{(t)}(A)$ is then deduced since $\left\| \alpha^{(t)}_{\Lambda_n}(A) - \alpha^{(t)}(A) \right\|_{\mathcal{L}(\mathcal{H}^2, \mathcal{H})}$ tends to 0.

**Propagation speed.** Set:

\begin{equation}
V_0 = \inf_{0 < \gamma < \gamma_0} \frac{2\sqrt{S_\gamma}}{\gamma}
\end{equation}

where $S_\gamma$ is the constant given in Proposition 3.4. For the case of interaction with nearest neighbors the infimum bound is taken on $[0, \infty[$.

**Proof of theorem 1.5.** Set $A$ and $B$ in $\mathcal{W}_2$ with finite supports $\sigma(A)$ and $\sigma(B)$. Set $(h_n, t_n)$ a sequence in $\mathbb{Z} \times \mathbb{R}$ with $|t_n| \to \infty$ and with $|h_n| \geq v_1|t_n|$ where $v_1 > V_0$, $V_0$ being defined above. Set $\gamma \in [0, \gamma_0]$ such that $2\sqrt{S_\gamma} < v_1\gamma$. Set $M$ such that $2\sqrt{S_\gamma} < M < v_1\gamma$. The sequence $M|t_n| - \gamma d(\sigma(A), (\sigma(\tau_{h_n}(B))))$ tends to $-\infty$. For all $f \in \mathcal{H}$ the inequality (8.1) (with $\alpha^{(t)}_{\Lambda_n}$ replaced with $\alpha^{(t)}$) shows that:

\[ \lim_{n \to -\infty} \left\| [\alpha^{(t_n)}_{\Lambda_n}(A), \tau_{h_n}(B)]f \right\|_{\mathcal{H}} = 0 \]

The result is extended by density to all $f \in \mathcal{H}$.

**Appendix A. Approximation of operators. Proof of Proposition 2.5.**

We shall first prove Proposition 2.5 for two finite subsets $E$ and $F$ de $\mathbb{Z}$ such that $E \subset F$ with their difference $F \setminus E$ being reduced to only one element $\lambda$. Operators in $\mathcal{L}(\mathcal{H}_F)$ shall be identified using the map $i_{F\mathbb{Z}}$ with the elements of $\mathcal{L}(\mathcal{H})$ supported in $F$. We denote by $\mathcal{W}_k(F)$ the set of all $A$ in $\mathcal{L}(\mathcal{H}_F)$ such that $i_{F\mathbb{Z}}(A)$ is in $\mathcal{W}_k$.

**Proposition A.1.** There exists a constant $C > 0$ such that, for all finite subset $E$ and $F$ in $\mathbb{Z}$ written as $F = E \bigcup \{\lambda\}$ where $\lambda \in \mathbb{Z} \setminus E$, for all $T \in \mathcal{W}_2(F)$,

\begin{equation}
\left\| (T - i_{EF} \circ \rho_{FE}(T))f \right\|_{\mathcal{L}(\mathcal{H}^2, \mathcal{H})} \leq C \sum_{1 \leq j + k \leq 2} \left\| (ad P_\lambda)^j(ad Q_\lambda)^kT \right\|_{\mathcal{L}(\mathcal{H})}
\end{equation}

End of the proof of proposition 2.5. If $E \subset F \subset G$ then one has $\rho_{GE} = \rho_{FE} \circ \rho_{GF}$ and $i_{E\mathbb{Z}} = i_{F\mathbb{Z}} \circ i_{EF}$. Consequently, if $F = E \bigcup \{\lambda_1, \ldots, \lambda_m\}$ then we successively apply proposition A.1 with the set $E_k = E \bigcup \{\lambda_1, \ldots, \lambda_k\}$ ($1 \leq k \leq m$) and $E_0 = E$. We obtain, for all $T \in \mathcal{W}_2(F)$:

\[ \left\| (T - i_{EF} \circ \rho_{FE}(T))f \right\|_{\mathcal{L}(\mathcal{H}^2, \mathcal{H})} \leq \sum_{k=1}^m \left\| T_k - i_{E_{k-1}E_k} \circ \rho_{E_kE_{k-1}}(T_k) \right\|_{\mathcal{L}(\mathcal{H}^2, \mathcal{H})} T_k = \rho_{FE_k}(T). \]
Proposition 2.5 thus follows from proposition A.1 applied with the operators $T_k$.

**Notations.** $\Omega_{(\lambda)}$ denotes the ground state of the space $\mathcal{H}_{(\lambda)}$ associated to the corresponding creation and annihilation operators $a_\lambda$ and $a^*_\lambda$. One knows that $\mathcal{H}_{(\lambda)}$ is associated with the orthonormal basis $(\Psi_{j})_{(j \geq 0)}$ defined by:

$$h_0 = \Omega_{(\lambda)} \quad h_{j+1} = (j + 1)^{-1/2} a^*_\lambda h_j$$

When identifying $\mathcal{H}_{(\lambda)}$ with $L^2(\mathbb{R})$, this basis is the basis of Hermite’s functions and $a_\lambda h_j = \sqrt{j} h_{j-1} (j \geq 1)$.

We shall use the following notations for the operators belonging to the tensorial product $\mathcal{H}_F = \mathcal{H}_E \otimes \mathcal{H}_{(\lambda)}$.

We set $A = I \otimes a_\lambda$, $A^* = I \otimes a^*_\lambda$ and for all $T \in \mathcal{L}(\mathcal{H}_F)$ we set $R(T) = \rho_{EF}(T) \otimes I$ where $\rho(T)$ is defined in section 2 by $\rho(T) = \pi_{EF} T \pi_{EF}$. Thus $R(T) = i_{EF} \rho_{EF}(T)$. In order to generalize the operator $\pi_{EF}$ we define for all $j \geq 0$ an $\Psi_j$ from $\mathcal{H}_E$ into $\mathcal{H}_F$ by

$$(\Psi_j f) = f \otimes h_j$$

We denote by $\Psi_j^*$ the adjoint operator of $\mathcal{H}_F$ in $\mathcal{H}_E$.

With these notations, we can sum up some of the usual properties on Hermite’s functions with the next lemma:

**Lemma A.2.** With these notations one has:

\begin{equation}
\sum_{j=0}^{\infty} \Psi_j \Psi_j^* = I \quad \sum_{j=0}^{\infty} \|\Psi_j^* f\|^2_{\mathcal{H}_E} = \|f\|^2_{\mathcal{H}_F} \quad \forall f \in \mathcal{H}_F
\end{equation}

If we denote by $\mathcal{H}^m(E,F)$ ($m \geq 0$) the partial Sobolev space consisting of the $f \in \mathcal{H}_F$ such that:

$$\|f\|^2_{\mathcal{H}^m(E,F)} = \sum_{j=0}^{\infty} (1 + j)^m \|\Psi_j^* f\|^2_{\mathcal{H}_E} < \infty,$$

then the operator $A A^*$ with the domain $\mathcal{H}^2(E,F)$ is self-adjoint and verifies $A A^* \geq I$. One has for all $\alpha \in \mathbb{R}$ and for all $j \geq 0$:

\begin{equation}
(A A^*)^\alpha \Psi_j = (j + 1)^\alpha \Psi_j \quad \Psi_j^* (A A^*)^\alpha = (j + 1)^\alpha \Psi_j^*
\end{equation}

We have for all $j \geq 1$:

\begin{equation}
A \Psi_j = \sqrt{j} \Psi_{j-1} \quad \Psi_j^* A^* = \sqrt{j} \Psi_{j-1}^*
\end{equation}

(if $j = 0$ then the right hand-sides are replaced by 0.) For all $j \geq 0$, we have:

\begin{equation}
A^* \Psi_j = \sqrt{j+1} \Psi_{j+1} \quad \Psi_j^* A = \sqrt{j+1} \Psi_{j+1}^*
\end{equation}

For each operator $T$ in $\mathcal{L}(\mathcal{H}_F)$ we define an operators-valued matrix $a_{jk}(T)$ in $\mathcal{L}(\mathcal{H}_E)$ by:

\begin{equation}
a_{jk}(T) = \Psi_j^* T \Psi_k
\end{equation}

Thus $\pi_{EF} = \Psi_0$ and $\rho_{EF}(T) = A_{90}(T)$. The norm of an operator $T$ in $\mathcal{L}(\mathcal{H}_F)$ may be estimated starting from those of the $a_{jk}(T)$ using the following proposition which is a variant of Schur’s Lemma.

**Proposition A.3.** Set $T$ an element of $\mathcal{L}(\mathcal{H}_F)$. Suppose that there exists $M > 0$ such that, for all $k \geq 0$ and for all $\varphi$ in $\mathcal{H}_E$:

\begin{equation}
\sum_{j \geq 0} \|a_{jk}(T)\varphi\|_{\mathcal{H}_E} \leq M \|\varphi\|_{\mathcal{H}_E} \quad \sum_{j \geq 0} \|a_{jk}(T^*)\varphi\|_{\mathcal{H}_E} \leq M \|\varphi\|_{\mathcal{H}_E}
\end{equation}
Then $\|T\|_{L(H_F)} \leq M$. 

Proof. From lemma A.2, for all $f$ and $g$ in $H_F$ one gets:

$$\langle Tf, g \rangle = \sum_{jk} \langle a_{jk}(T)\Psi_k^*f, \Psi_j^*g \rangle$$

One has:

$$|\langle a_{jk}(T)\Psi_k^*f, \Psi_j^*g \rangle| \leq \|a_{jk}(T)\Psi_k^*f\| \|\Psi_j^*g\|$$

This scalar product may be bounded by:

$$|\langle a_{jk}(T)\Psi_k^*f, \Psi_j^*g \rangle| \leq \|\Psi_k^*f\| \|a_{jk}(T)^*\Psi_j^*g\|$$

Consequently:

$$|\langle a_{jk}(T)\Psi_k^*f, \Psi_j^*g \rangle| \leq \left(\|a_{jk}(T)\Psi_k^*f\| \|\Psi_k^*f\|\right)^{1/2} \left(\|a_{jk}(T)^*\Psi_j^*g\| \|\Psi_j^*g\|\right)^{1/2}$$

From Cauchy-Schwarz:

$$\left|\langle Tf, g \rangle\right|^2 \leq \left[\sum_{jk} \|a_{jk}(T)\Psi_k^*f\| \|\Psi_k^*f\|\right] \left[\sum_{jk} \|a_{jk}(T)^*\Psi_j^*g\| \|\Psi_j^*g\|\right]$$

Noticing that $(a_{jk}(T))^* = a_{kj}(T^*)$ we obtain:

$$\left|\langle Tf, g \rangle\right|^2 \leq M^2 \left[\sum_{k \geq 0} \|\Psi_k^*f\|^2\right] \left[\sum_{j \geq 0} \|\Psi_j^*g\|^2\right] \leq M^2 \|f\|_{H_F}^2 \|g\|_{H_F}^2$$

The proof of proposition A.3 is completed. 

\[\square\]

**Proposition A.4.** Set $T$ an element in $W_k(F)$. Assume that there is $M > 0$ satisfying for all $\varphi$ in $H_E$:

(A.8) \[\sup_{k \geq 0} \sum_{j \geq 0} \|a_{jk}(T)\varphi\|_{H_E} \leq M \|\varphi\|_{H_E}\]

(A.9) \[\sup_{k \geq 0} \sum_{j \geq 0} \|a_{jk}(T^*)\varphi\|_{H_E} \leq M \|\varphi\|_{H_E}\]

Then

(A.10) \[\|Tf\| \leq M \sqrt{2} \|(A^*_k)^2f\| + \sqrt{2} \|[A^*_k, T]\| \|f\|\]

for all $f$ in $H^2(E, F)$.

Proof. Set $S$ the operator $S = (AA^*)^{-1/2}T(AA^*)^{-1/2}$. By lemma A.2 we have:

$$a_{jk}(S) = \frac{a_{jk}(T)}{\sqrt{(j+1)(k+1)}}$$
Under the hypotheses of the Proposition the operator $S$ is then bounded in $\mathcal{H}_E$ with a norm $\leq M$. From lemma A.2 and for all $g$ in $\mathcal{H}_E$, we get
\[
\|(AA^*)^{-1/2}A^*g\|^2 = \sum_{j\geq 1} \frac{j}{j+1} \|\Psi^*_j g\|^2 \geq \frac{1}{2} \sum_{j\geq 0} \|\Psi^*_j g\|^2 = \frac{1}{2}\|g\|^2
\]
Consequently, for all $f$ in $\mathcal{H}^2(E,F)$:
\[
\|Tf\| \leq \sqrt{2}\|(AA^*)^{-1/2}A^*Tf\| \leq \sqrt{2}\|[A^*,T]\|\|f\| + \sqrt{2}\|(AA^*)^{-1/2}TA^*f\|
\]
Indeed the operator $(AA^*)^{-1/2}$ has a norm $\leq 1$. We have:
\[
\|(AA^*)^{-1/2}TA^*f\| \leq \|S(AA^*)^{+1/2}A^*f\| \leq M\|(AA^*)^{+1/2}A^*f\| = M\|(A^*)^2f\|
\]
Consequently, inequality (A.10) thus follows.

We shall apply proposition A.4 to the operator $T - R(T)$ noticing that $R(T)$ commutes with $A$ and $A^*$. The operator $R(T)$ is chosen such that $a_{00}(T - R(T)) = 0$. Using commutators, we shall estimate all the others elements $a_{jk}(T - R(T))$. This is the purpose of the next proposition.

**Proposition A.5.** Under the hypotheses of Proposition A.1, for all $k \geq 0$ and for all $\varphi$ in $\mathcal{H}_E$ we have:

\[
(A.11) \quad S_k(T, \varphi) := \sum_{j \geq 0} \frac{\|a_{jk}(T - R(T))\varphi\|}{\sqrt{j+1}(j+1)} \leq C\|\varphi\| \sum_{1 \leq \alpha + \beta \leq 2} \|(adP_\lambda)^\alpha (adQ_\lambda)^\beta T\| \zeta(H)
\]

and a similar expression holds when replacing $T$ by $T^*$.

Estimations of $S_0(T, \varphi)$. We shall prove that:

\[
(A.12) \quad S_0(T, \varphi) \leq \|[A,T]\| \|\varphi\|
\]
From lemma A.2 (point A.5), one sees for all $j \geq 1$ that:
\[
\sqrt{j}a_{j0}(T - R(T)) = \Psi^*_j - 1[A,T]\Psi_0
\]
Since $a_{00}(T - R(T)) = 0$, it is deduced using (A.2) that:
\[
S_0(T, \varphi) \leq \sum_{j=1}^{\infty} \frac{1}{\sqrt{j+1}} \|\Psi^*_j - 1[A,T]\Psi_0\| \leq \left[ \sum_{j=1}^{\infty} \frac{1}{j(j+1)} \right]^{1/2} \left[ \sum_{j=1}^{\infty} \|\Psi^*_j - 1[A,T]\Psi_0\varphi\|^2 \right]^{1/2} \leq \|[A,T]\| \|\Psi_0\varphi\| \leq \|\varphi\|\|\Psi_0\varphi\| \leq \|[A,T]\| \|\varphi\|\]

Inequality (A.12) is therefore true.

Recursion between the $S_k(T, \varphi)$. If $k \geq 1$ we shall prove that:

\[
(A.13) \quad S_k(T, \varphi) \leq \frac{k}{k+1} S_{k-1}(T, \varphi) + \frac{C\|\varphi\|}{k+1} \left[ \|[A,T]\| + \|[A^*,T]\| + \|[A^*,[A^*,T]]\| \right]
\]
To this end, we use the fact that, if $1 \leq j \leq k$ the we have from (A.4) (A.5):
\[
\sqrt{k}a_{jk}(T - R(T)) = \sqrt{j}a_{j-1,k-1}(T - R(T)) + \Psi^*_j[T,A^*]\Psi_{k-1}
\]
If \( j = 0 \) then the first term above has to be replaced by \( 0 \). If \( j > k \) then we use:

\[
\sqrt{j} a_j (T - R(T)) = \sqrt{k} a_{j-1,k-1} (T - R(T)) + \Psi_{j-1}^* [A,T] \Psi_k
\]

Then we are able to write \( S_k (T, \varphi) \leq S_k' (T, \varphi) + S_k'' (T, \varphi) + S_k''' (T, \varphi) \) where:

\[
S_k' (T, \varphi) = \sum_{j=1}^{\infty} \inf \left( \sqrt{\frac{j}{k}} \left| \frac{k}{j} \right| \frac{\| a_{j-1,k-1} (T - R(T)) \varphi \|}{\sqrt{(j+1)(k+1)}} \right) \leq \frac{k}{k+1} S_{k-1} (T, \varphi)
\]

\[
S_k'' (T, \varphi) = \sum_{j=k+1}^{\infty} \frac{\| \Psi_{j-1}^* [T,A] \Psi_k \varphi \|}{\sqrt{(j+1)(k+1)}}
\]

\[
S_k''' (T, \varphi) = \sum_{j=0}^{k} \frac{\| \Psi_j^* [T,A^*] \Psi_k \varphi \|}{\sqrt{(j+1)k(k+1)}}
\]

From (A.2) and since \( \| \Psi_k \varphi \| = \| \varphi \| \):

\[
S_k''' (T, \varphi) \leq \frac{1}{\sqrt{k+1}} \left[ \sum_{j=k+1}^{\infty} \frac{1}{j(j+1)} \right]^{1/2} \left[ \sum_{j=1}^{\infty} \frac{\| \Psi_{j-1}^* [T,A] \Psi_k \varphi \|}{\sqrt{(j+1)(k+1)}} \right]^{1/2} \leq \frac{1}{k+1} \|[T,A]\| \|\varphi\|
\]

If \( k = 1 \) then we see that \( S_k''' (T, \varphi) \leq \|[A^*,T]\| \|\varphi\| \). If \( k \geq 2 \) then the estimation of \( S_k''' (T, \varphi) \) involves commutators with length 2. We still have, if \( j \leq k \):

\[
\sqrt{k-1} \Psi_j^* [T,A] \Psi_{k-1} = \sqrt{j} \Psi_{j-1}^* [T,A^*] \Psi_{k-2} + \Psi_j^* [T,A^*] \Psi_{k-2}
\]

Consequently, if \( k \geq 2 \):

\[
S_k''' (T, \varphi) \leq \sum_{j=1}^{k} \sqrt{\frac{j}{k-1}} \frac{\| \Psi_{j-1}^* [T,A^*] \Psi_{k-2} \varphi \|}{\sqrt{(j+1)k(k+1)}} + ... + \sum_{j=0}^{k} \frac{\| \Psi_j^* [T,A^*] \Psi_{k-2} \varphi \|}{\sqrt{(j+1)(k+1)(k-1)}}
\]

Using again Cauchy-Schwarz and lemma A.2, we obtain if \( k \geq 2 \):

\[
S_k''' (T, \varphi) \leq \frac{1}{\sqrt{k(k-1)}} \left[ \|[A^*,T]\| \|\varphi\| + \|[A^*,[A^*,T]]\| \|\varphi\| \right]
\]

We then deduce the validity of (A.13). Inequality (A.11) follows by iteration on (A.12) and (A.13). The proposition A.1 is a consequence of Propositions A.4 and A.5 and the proof of Proposition 2.5 is finished.

Appendix B. Differential systems.

**Proposition B.1.** Suppose that we are given for all \( \lambda \) and \( \mu \) in \( \Lambda_n \) a continuous map \( t \to \Omega_{\lambda \mu} (t) \) from \( \mathbb{R} \) into \( \mathcal{L}(\mathcal{L}(\mathcal{H})) \). Assume that there are \( \gamma > 0 \) and \( S_{\gamma} > 0 \) such that, for all \( \lambda \) and \( \mu \) in \( \Lambda_n \), for all \( t \in \mathbb{R} \):

\[
(B.1) \quad \sum_{\mu \in \Lambda_n} \| \Omega_{\lambda \mu} (t) \|_{\mathcal{L}(\mathcal{L}(\mathcal{H}))} e^{-\gamma |\mu - \nu|} \leq S_{\gamma} e^{-\gamma |\lambda - \nu|}
\]

Then, for all \( s \in \mathbb{R} \), there exists functions \( t \to A_{\lambda \mu}^{(0)} (t,s) \) and \( t \to A_{\lambda \mu}^{(1)} (t,s) \) \( ((\lambda, \mu) \in \Lambda_n^2) \) being \( C^1 \) from \( \mathbb{R} \) into \( \mathcal{L}(\mathcal{L}(\mathcal{H})) \) such that:

\[
(B.2) \quad \frac{d}{dt} A_{\lambda \mu}^{(0)} (t,s) = A_{\lambda \mu}^{(1)} (t,s) \quad \frac{d}{dt} A_{\lambda \mu}^{(1)} (t,s) = \sum_{\nu \in \Lambda_n} \Omega_{\lambda \mu} (t) \circ A_{\nu \mu}^{(0)} (t,s)
\]
(B.3) \[ A^0_{\lambda\mu}(s, s) = \delta_{\lambda\mu} I \quad A^1_{\lambda\mu}(s, s) = 0 \]

(in (B.1), the composition is the one of \( \mathcal{L}(\mathcal{L}(\mathcal{H})) \) and in (B.3) the identity operator \( I \) is the one of \( \mathcal{L}(\mathcal{L}(\mathcal{H})) \).)

Moreover, if \( M > \sqrt{S_\gamma} \), there exists \( C(M, \gamma) > 0 \) independent of \( n \) such that:

(B.4) \[ \|A^{(j)}_{\lambda\mu}(t, s)\|_{\mathcal{L}(\mathcal{L}(\mathcal{H}))} \leq C(M, \gamma) e^{M|t-s|} e^{-\gamma|\lambda-\mu|} \quad \forall(\lambda, \mu) \in \Lambda_n^2 \]

There are also operator-valued matrices \( t \to B^0_{\lambda\mu}(t, s) \) and \( t \to B^1_{\lambda\mu}(t, s) \) satisfying the same system together with the same estimations and the same initial conditions:

(B.5) \[ B^0_{\lambda\mu}(s, s) = 0 \quad B^1_{\lambda\mu}(s, s) = \delta_{\lambda\mu} I \]

Proof. Let \( E_{n\gamma} \) be the set of all matrices \( A = (A_{\lambda\mu})_{((\lambda, \mu) \in \Lambda_n^2)} \) where each \( A_{\lambda\mu} \) is a map in \( \mathcal{L}(\mathcal{L}(\mathcal{H})) \) which is associated with the norm:

\[ \|A\|_{n, \gamma} = \sup_{(\lambda, \mu) \in \Lambda_n^2} e^{\gamma|\lambda-\mu|} \|A_{\lambda\mu}\|_{\mathcal{L}(\mathcal{L}(\mathcal{H}))} \]

The left composition by the operators-valued matrix \( \Omega_{\lambda\mu}(t) \) defines a map \( \Omega(t) \) in \( \mathcal{L}(E_{n\gamma}) \) with a norm \( \leq S_\gamma \). For all \( \varepsilon > 0 \) we can associate to \( E^2_{n\gamma} \) a norm such that

\[ U(t) = \begin{pmatrix} 0 & I \\ \Omega(t) & 0 \end{pmatrix} \]

is \( \leq \sqrt{S_\gamma}(1 + \varepsilon) \). The stated result is then valid.

Remark 1. In the tensorial product \( (E^2_{n\gamma}) \otimes (E^2_{n\gamma}) \) let \( V(t) \) be the map defined by \( V(t) = U(t) \otimes I + I \otimes U(t) \).

For all \( \varepsilon > 0 \) one may associate \( (E^2_{n\gamma}) \otimes (E^2_{n\gamma}) \) with a norm such that the map \( V(t) \) is \( \leq 2\sqrt{S_\gamma}(1 + \varepsilon) \).

Consequently, if \( M > 2\sqrt{S_\gamma} \) and if \( A_0 \) is in \( (E^2_{n\gamma}) \otimes (E^2_{n\gamma}) \) then the differential system:

\[ A'(t) = U(t)A(t) \quad A(0) = A_0 \]

has a solution taking values into \( (E^2_{n\gamma}) \otimes (E^2_{n\gamma}) \) and with an time exponential growth \( e^{M|t|} \).

Remark 2. If we are also given the continuous functions \( t \to F_\lambda(t) \) from \( \mathbb{R} \) and taking values into \( \mathcal{L}(\mathcal{H}) \) then the family of functions \( t \to X^{(j)}_\lambda(t) \) defined by:

\[ X^{(j)}_\lambda(t) = \sum_{\mu \in \Lambda_n} \int_0^t B_{\lambda\mu}^j(t, s)\left(F_\mu(s)\right)ds \]

satisfies the differential system:

\[ \frac{d}{dt}X^{(0)}_\lambda(t) = X^{(1)}_\lambda(t) \quad \frac{d}{dt}X^{(1)}_\lambda(t) = \sum_{\mu \in \Lambda_n} \Omega_{\lambda\mu}(t)\left(X^{(0)}_\mu(t)\right) + F_\lambda(t) \]

together with the initial conditions \( X^{(j)}_\lambda(0) = 0 \) and the following estimates (for example if \( t > 0 \):

\[ \|X^{(j)}_\lambda(t)\|_{\mathcal{L}(\mathcal{H})} \leq C(M, \gamma) \sum_{\mu \in \Lambda_n} e^{-\gamma|\lambda-\mu|} \int_0^t e^{M|t-s|}\|F_\mu(s)\|_{\mathcal{L}(\mathcal{H})} \, ds \]
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