The Li-Yau-Hamilton estimate and the Yang-Mills heat equation on manifolds with boundary

Artem Pulemotov

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Department of Mathematics, Cornell University,
310 Malott Hall, Ithaca, NY 14853-4201, USA
E-mail: artem@math.cornell.edu

Abstract

The paper pursues two connected goals. Firstly, we establish the Li-Yau-Hamilton estimate for the heat equation on a manifold $M$ with nonempty boundary. Results of this kind are typically used to prove monotonicity formulas related to geometric flows. Secondly, we establish bounds for a solution $\nabla(t)$ of the Yang-Mills heat equation in a vector bundle over $M$. The Li-Yau-Hamilton estimate is utilized in the proofs. Our results imply that the curvature of $\nabla(t)$ does not blow up if the dimension of $M$ is less than 4 or if the initial energy of $\nabla(t)$ is sufficiently small.

1 Introduction

The present paper considers two related subjects. Section 2 establishes the Li-Yau-Hamilton estimate for the heat equation on a manifold with boundary. Results of this kind are known to be useful in the study of geometric flows. Sections 3 and 4 discuss estimates for the solutions of the Yang-Mills heat equation in a vector bundle over a manifold with boundary. The proofs utilize a probabilistic technique. Our results imply that the curvature of a solution does not blow up if the dimension of the manifold is less than 4 or if the initial energy is sufficiently small.

The Li-Yau-Hamilton estimate for the heat equation generalizes the well-known differential Harnack inequality of \[25\]. This estimate was originally obtained on manifolds without boundary in the paper \[16\]. It is typically used to prove monotonicity formulas related to various geometric evolution equations; see, for example, \[17\]. In their turn, such monotonicity formulas are essential for establishing the existence of solutions.
Let us mention that [11] offers a constrained version of the Li-Yau-Hamilton estimate from [10]. The paper [6] adapts the result of [16] to Kähler manifolds. We point out that an inequality similar to the Li-Yau-Hamilton estimate for the heat equation comes up in the investigation of the Ricci flow. Its precise formulation and various applications are presented in [10, Chapter 15]. Analogous results hold for the Kähler-Ricci flow. Their formulations and relevant references can be found in [9, Chapter 2] and in [29].

Suppose \( M \) is a smooth compact Riemannian manifold without boundary. Consider a positive solution \( p(t, x) \) to the heat equation on \( M \) such that the integral \( \int_M p(t, x) \, dx \) does not exceed 1 for any \( t \in (0, \infty) \). Then there exist constants \( A > 0 \) and \( B > 0 \) that depend only on the manifold \( M \) and satisfy

\[
D^2 \log p(t, x) \geq -\left( \frac{1}{2t} + A \left( 1 + \log \left( \frac{B}{\text{dim}^2 M} p(t, x) \right) \right) \right) \langle \cdot, \cdot \rangle,
\]

\( t \in (0, 1], \ x \in M. \) (1.1)

In this formula, \( D^2 \) is the second covariant derivative, and \( \langle \cdot, \cdot \rangle \) is the Riemannian metric. The inequality is to be understood in the sense of bilinear forms. If \( M \) is Ricci parallel and has nonnegative sectional curvatures, then (1.1) holds with \( A = 0 \). This is the case when \( M \) is, for example, a sphere or a flat torus. Formula (1.1) constitutes the Li-Yau-Hamilton estimate for the heat equation. It was originally obtained in [16].

Suppose now that \( M \) is a smooth compact Riemannian manifold with nonempty boundary \( \partial M \). Section 2 of the present paper establishes formula (1.1) in this case. The solution \( p(t, x) \) of the heat equation is assumed to satisfy the Neumann boundary condition. Theorem 2.1 proves (1.1) in the situation where no restrictions are imposed on the curvature of \( M \) away from \( \partial M \). But the boundary of \( M \) must be totally geodesic for this result to hold. Moreover, several derivatives of the curvature of \( M \) have to vanish at \( \partial M \). Theorem 2.6 deals with a more exclusive situation. It shows that inequality (1.1) holds with \( A = 0 \) if the manifold \( M \) is Ricci parallel and has nonnegative sectional curvatures. As before, \( \partial M \) must be totally geodesic. However, the previously mentioned derivatives of the curvature of \( M \) are no longer required to vanish at \( \partial M \). Our proofs of Theorems 2.1 and 2.6 differ considerably in their techniques.

Both incarnations of estimate (1.1) appearing in Section 2 play significant roles in establishing the results of Section 3. More precisely, they enable us to obtain a monotonicity formula related to the Yang-Mills heat equation. This formula is given by Lemma 3.9. It helps us establish an estimate for the solutions to the Yang-Mills heat equation in dimensions 5 and higher.

In order to prove Theorem 2.1, we employ the doubling method. More precisely, we consider two identical copies of \( M \) and glue them together along the boundary. This procedure produces a closed manifold \( \mathcal{M} \). The desired estimate follows by applying the results of the paper [16] on \( \mathcal{M} \). Of course, several technical questions need to be handled in order to make the doubling method work for our purpose.
The proof of Theorem 2.6 relies on the Hopf boundary point lemma for vector bundle sections appearing in [30]. The technique we use resembles those employed in [25, 16]. One may also apply the doubling method to prove Theorem 2.6. However, the approach adopted in the present paper appears to be more effective. Firstly, it enables us to avoid the assumption on the curvature of $M$ near $\partial M$ that is required to carry out the doubling procedure. Secondly, it does not rely on the previously known versions of the Li-Yau-Hamilton estimate. Last but not least, our approach seems to be more natural and to provide a better ground for further generalizations.

Section 3 of the present paper deals with the Yang-Mills heat equation in a vector bundle over a compact Riemannian manifold $M$ with nonempty boundary. In order to describe our results, we need to outline the setup. Let $E$ be a vector bundle over $M$. Suppose the time-dependent connection $\nabla(t)$ in $E$ solves the Yang-Mills heat equation

$$\frac{\partial}{\partial t} \nabla(t) = -\frac{1}{2} d^* \nabla(t) R \nabla(t), \quad t \in [0, T). \quad (1.2)$$

Here and in what follows, $d\nabla(t)$ is the exterior covariant derivative, $d^* \nabla(t)$ is its adjoint, and $R \nabla(t)$ is the curvature of $\nabla(t)$. By definition, $R \nabla(t)$ is a 2-form on $M$ with its values in the endomorphism bundle $\text{End} E$. The Yang-Mills heat equation is a potentially powerful instrument for minimizing the Yang-Mills energy functional; see, for example, [3, 31, 2]. It has a number of applications in topology and in mathematical physics. Some of these applications are comprehensively discussed in the book [13] and the dissertation [34]; see also [4]. The existence of solutions is one of the most important questions regarding the Yang-Mills heat equation.

Since $\partial M$ is assumed to be nonempty, we have to specify the boundary conditions for the time-dependent connection $\nabla(t)$. Doing so is a delicate matter. As detailed in Remark 3.11, it is more natural for us to impose the boundary conditions on the curvature $R \nabla(t)$ than on $\nabla(t)$ itself. We assume

$$\left( R \nabla(t) \right)_\text{tan} = 0, \quad \left( d^* \nabla(t) R \nabla(t) \right)_\text{tan} = 0, \quad t \in [0, T). \quad (1.3)$$

The subscript “tan” stands for the component of the corresponding $\text{End} E$-valued form that is tangent to $\partial M$. Alternatively, we may assume

$$\left( R \nabla(t) \right)_\text{norm} = 0, \quad \left( d\nabla(t) R \nabla(t) \right)_\text{norm} = 0, \quad t \in [0, T). \quad (1.4)$$

(Actually, the second equality always holds due to the Bianchi identity.) The subscript “norm” signifies the component that is normal to $\partial M$. Conditions (1.3) and (1.4) are analogous to the relative and the absolute boundary conditions for real-valued forms. The results in Section 3 prevail regardless of whether we choose (1.3) or (1.4) to hold on $\partial M$. Other ways to introduce the boundary conditions in the context of Yang-Mills theory were considered in several works including, for example, [26, 36, 38, 15, 7]. We should mention,
however, that none of these works except [7] deals with parabolic-type equations like (1.2). The relationship between the boundary conditions utilized in the present paper and the boundary conditions appearing elsewhere is discussed in Remark 3.12.

Section 3 provides estimates for the curvature $R^{\nabla(t)}$ of the solution $\nabla(t)$ to the Yang-Mills heat equation (1.2) subject to (1.3) or (1.4). Roughly speaking, we show that $R^{\nabla(t)}$ is bounded at every point of $M$ by expressions involving the initial energy of $\nabla(t)$. Theorem 3.1 considers the case where the dimension of $M$ is either 2 or 3. It yields an estimate on $R^{\nabla(t)}$ and demonstrates that $R^{\nabla(t)}$ does not blow up. Theorem 3.2 deals with the case where the dimension is equal to 4. It requires that the initial energy of $\nabla(t)$ be smaller than a constant depending on $M$. If this assumption is satisfied, the theorem produces a bound on $R^{\nabla(t)}$. It is easy to see that $R^{\nabla(t)}$ does not blow up when this bound holds. Theorem 3.3 considers the situation where the dimension of $M$ is greater than or equal to 5. It produces an estimate on $R^{\nabla(t)}$ under a rather sophisticated condition. The theorem implies that the curvature of a solution to Eq. (1.2) cannot blow up after time $\rho$ if the initial energy is smaller than a number depending on $\rho$.

When the dimension of $M$ equals 2, 3, or 4, the boundary $\partial M$ has to be convex for the results in Section 3 to hold. No other assumptions on the geometry of $M$ are required. However, if the dimension is 5 or higher, the situation is different. In this case, $\partial M$ has to be totally geodesic, and restrictions have to be imposed on the curvature of $M$. The reason for such a phenomenon lies in the fact that, when the dimension is 5 or higher, our arguments involve the Li-Yau-Hamilton estimate (1.1). Both Theorems 2.1 and 2.6 are exploited.

We thus observe a trichotomy in the behavior of the solution $\nabla(t)$ to Eq. (1.2). Theorems 3.1, 3.2, and 3.3 provide three different sets of conditions ensuring that $R^{\nabla(t)}$ does not blow up. Each of these sets corresponds to a certain range of dimensions of $M$. A similar trichotomy occurs on closed manifolds; see, for instance, [2]. However, the difference in the geometric assumptions that was discussed in the previous paragraph is not observed in this case.

Let us make a comment as to the practical importance of the results in Section 3. Proving that the curvature does not blow up is the principal ingredient in establishing the long-time existence of solutions to the Yang-Mills heat equation. The list of relevant references includes but is not limited to [13, 31, 37, 2, 7]. We should point out that all these works except [7] restrict their attention to manifolds without boundary.

The proofs of Theorems 3.1, 3.2, and 3.3 rely on the probabilistic technique developed in [2]. The origin of this technique lies in the theory of harmonic maps; see [39]. The pivotal stochastic process in our considerations is a reflecting Brownian motion on the manifold $M$. Let us mention that the probabilistic approach to Yang-Mills theory was investigated rather extensively. The paper [2] contains a series of results and a list of references on the subject.

While establishing the theorems in Section 3 we prove a noteworthy property of $\text{End} \mathcal{E}$-valued forms on $M$. The precise phrasing of this property is given by Lemma 3.5. Roughly speaking, it states that, if $\partial M$ is convex and an $\text{End} \mathcal{E}$-
valued form \( \phi \) satisfies (1.3) or (1.4), then the derivative of the squared absolute value of \( \phi \) in the direction of the outward normal to \( \partial M \) must be nonpositive. A simpler version was established in [7].

Section 4 of the present paper provides an exit time estimate for a reflecting Brownian motion on a manifold with convex boundary. This result helps us prove another inequality for the curvature of the connection \( \nabla(t) \) discussed above.

2 The Li-Yau-Hamilton estimate

Consider a smooth, compact, connected, oriented, \( n \)-dimensional Riemannian manifold \( M \) with nonempty boundary \( \partial M \). We suppose \( n \geq 2 \). This section aims to study the solutions of the heat equation on \( M \) with the Neumann boundary condition. More precisely, we will obtain two versions of the Li-Yau-Hamilton estimate for such solutions.

The Riemannian curvature tensor will be designated by \( R(X,Y,Z) \) when applied to the vectors \( X, Y, \) and \( Z \) from the tangent space \( T_xM \) at the point \( x \in M \). We use the usual notation

\[
R(X,Y,Z,W) = \langle R(X,Y)Z,W \rangle, \quad X,Y,Z,W \in T_xM.
\]

The angular brackets with no lower index refer to the scalar product in the space \( T_xM \) given by the Riemannian metric. The Ricci tensor will be written as \( \text{Ric}(X,Y) \) when applied to \( X, Y \in T_xM \). We will impose substantial assumptions on the curvature of \( M \) in Theorem 2.6 below.

The Levi-Civita connection \( D \) in the tangent bundle \( TM \) induces connections in the tensor bundles over \( M \). We preserve the notation \( D \) for all of them. Our further arguments require introducing higher-order differential operators. Let us describe the corresponding procedure. Fix a tensor field \( T \) and two or more vector fields \( Y_1, \ldots, Y_k \) on \( M \). Set \( D^1_{Y_1} T \) equal to \( D_{Y_1} T \). We define the \( k \)th covariant derivative \( D^k_{Y_1,\ldots,Y_k} T \) inductively by the formula

\[
D^k_{Y_1,\ldots,Y_k} T = D_{Y_k} \left( D^{k-1}_{Y_1,\ldots,Y_{k-1}} T \right) - \sum_{i=1}^{k-1} D^{k-1}_{Y_1,\ldots,Y_{i-1},D_{Y_k} Y_i,Y_{i+1},\ldots,Y_{k-1}} T.
\]

One can verify that the value of \( D^k_{Y_1,\ldots,Y_k} T \) at the point \( x \in M \) does not depend on the values of \( Y_1, \ldots, Y_k \) away from \( x \).

Let \( \nu \) be the outward unit normal vector field on \( \partial M \). The differentiation of real-valued functions in the direction of \( \nu \) will be denoted by \( \frac{\partial}{\partial \nu} \). If the point \( x \) lies in \( \partial M \), then the space \( T_xM \) contains the subspace \( T_x \partial M \) tangent to \( \partial M \). We write \( \Pi(X,Y) \) for the second fundamental form of \( \partial M \) applied to \( X, Y \in T_x \partial M \). By definition, \( \Pi(X,Y) = \langle D_X \nu, Y \rangle \). Some of the statements below require that \( \partial M \) be totally geodesic. In this case, \( \Pi(X,Y) = 0 \) for all \( X, Y \in T_x \partial M \) at every point \( x \in \partial M \).
Suppose the smooth positive function $p(t, x)$ defined on $(0, \infty) \times M$ solves the heat equation

$$
\left( \frac{\partial}{\partial t} - \Delta M \right) p(t, x) = 0, \quad t \in (0, \infty), \ x \in M,
$$

(2.1)

with the Neumann boundary condition

$$
\frac{\partial}{\partial \nu} p(t, x) = 0, \quad t \in (0, \infty), \ x \in \partial M.
$$

(2.2)

The notation $\Delta M$ represents the Laplace-Beltrami operator on $M$. It should be mentioned that Theorem 2.1 and Remark 2.5 below assume the inequality $\int_M p(t, x) \, dx \leq 1$ for all $t \in (0, \infty)$. Here and in what follows, the integration over a Riemannian manifold is to be carried out with respect to the Riemannian volume measure on the manifold.

We are now in a position to formulate the first result of this section. It establishes a general version of the Li-Yau-Hamilton estimate for the function $p(t, x)$.

**Theorem 2.1.** Let the boundary $\partial M$ be totally geodesic. Suppose the following statements hold:

1. The covariant derivative $(D^k_{\nu, \ldots, \nu} R)(\nu, X, \nu, Y)$ is equal to 0 for all positive odd $k$ and all $X, Y \in T_x M$ at every point $x \in \partial M$.

2. The integral $\int_M p(t, x) \, dx$ of the solution $p(t, x)$ to the boundary value problem (2.1)–(2.2) does not exceed 1 at any $t \in (0, \infty)$.

Then there exist constants $A > 0$ and $B > 0$ independent of $p(t, x)$ such that the estimate

$$
D^2_{X, X} \log p(t, x) \geq - \left( \frac{1}{2t} + A \left( 1 + \log \left( \frac{B}{\sqrt{p(t, x)}} \right) \right) \right) \langle X, X \rangle
$$

(2.3)

holds for every $t \in (0, 1]$, $x \in M$, and $X \in T_x M$. (Recall that $n$ is the dimension of $M$.)

Conceptually, the proof consists in doubling $M$ to get a manifold without boundary and exploiting the results of [16]. A few technical aspects need to be handled. The most essential problem is to make sure the function to which we apply the theorem in [16] possesses the necessary differentiability properties.

**Proof.** Let $\mathcal{M}$ be the double of $M$. More precisely, $\mathcal{M}$ appears as the quotient $(M \times \{1, 2\})/\sim$. The equivalence relation $\sim$ is given as follows: Two distinct pairs, $(x, i)$ and $(y, j)$, satisfy $(x, i) \sim (y, j)$ if and only if $x$ coincides with $y$ and lies in $\partial M$. We preserve the notation $(x, i)$ for the equivalence class of $(x, i) \in M \times \{1, 2\}$. As described in [28], $\mathcal{M}$ carries the canonical smooth structure. One may also obtain this structure by using Theorem 5.77 in [41] and the diffeomorphism $\mu(r, x)$ defined below. We explain further in the proof how to introduce a local coordinate system around $(x, i) \in \mathcal{M}$ when $x \in \partial M$. 

6
Note that $\mathcal{M}$ is a manifold without boundary. The map $E_i(x)$ taking $x \in M$ to $(x, i) \in \mathcal{M}$ is an embedding for both $i = 1$ and $i = 2$.

The Riemannian metric on $M$ induces a Riemannian metric on $\mathcal{M}$ in a natural fashion. More precisely, the scalar product $\langle X, Y \rangle_{\mathcal{M}}$ of the vectors $X, Y \in T_{(x,i)}\mathcal{M}$ is given by the formula $\langle X, Y \rangle_{\mathcal{M}} = \langle (dE_i)^{-1}X, (dE_i)^{-1}Y \rangle$. It is not difficult to verify that $\langle \cdot, \cdot \rangle_{\mathcal{M}}$ is well-defined at every $(x, i) \in \mathcal{M}$. The proposition in [28], along with Assumption 1 of our theorem, implies that $\langle \cdot, \cdot \rangle_{\mathcal{M}}$ depends smoothly on $(x, i) \in \mathcal{M}$.

Introduce a positive function $\bar{p}(t, z)$ on $(0, \infty) \times \mathcal{M}$ by setting $\bar{p}(t, (x, i)) = \frac{1}{2}p(t, x)$. Its integral over the manifold $\mathcal{M}$ is bounded by 1. Our next goal is to demonstrate that $\bar{p}(t, z)$ solves the heat equation on $\mathcal{M}$. This would allow us to apply the results of [10] and obtain estimate (2.3) for this function. Theorem 2.1 would then follow as a direct consequence.

First and foremost, we need to prove that $\bar{p}(t, z)$ is twice continuously differentiable in the second variable. Consider the set $\mathcal{M}^\partial \subset \mathcal{M}$ equal to $E_1(\partial M)$. Of course, this set is also equal to $E_2(\partial M)$. Using the smoothness of the function $p(t, x)$ on $M$, one can easily establish the smoothness of $\bar{p}(t, z)$ outside of $\mathcal{M}^\partial$.

In consequence, it suffices to show that $\bar{p}(t, z)$ is twice continuously differentiable in a neighborhood of an arbitrarily picked point $\bar{z} \in \mathcal{M}^\partial$.

There exists a unique $\bar{x} \in \partial M$ satisfying $\bar{z} = E_1(\bar{x}) = E_2(\bar{x})$. We need to introduce local coordinates in $\mathcal{M}$ around $\bar{x}$. Suppose $\epsilon > 0$ is small enough to ensure that the mapping $\mu(r, x)$ defined on $[0, \epsilon) \times \partial M$ by the formula $\mu(r, x) = \exp_x(-rv)$ is a diffeomorphism onto its image. The existence of such an $\epsilon > 0$ is justified in [27] Chapter 11. Fix a coordinate neighborhood $U^\partial$ of $\bar{x}$ in the boundary $\partial M$ with a local coordinate system $y_1, \ldots, y_{n-1}$ in $U^\partial$ centered at $\bar{x}$. Define the set $U$ as the image of $[0, \epsilon) \times U^\partial$ under $\mu(r, x)$. Clearly, $U$ is a neighborhood of $\bar{x}$ in $M$. We extend $y_1, \ldots, y_{n-1}$ to a coordinate system $x_1, \ldots, x_n$ in $U$ by demanding that the equalities

$$x_k(\mu(r, x)) = y_k(x), \quad x_n(\mu(r, x)) = r,$$

$$r \in [0, \epsilon), \quad x \in U^\partial, \quad k = 1, \ldots, n-1,$$

hold true; cf. [28]. Importantly, $\frac{\partial}{\partial x_n}$ is tangent to the boundary on $U^\partial$ for every $i = 1, \ldots, n-1$. The vector field $\frac{\partial}{\partial x_n}$ coincides with $-\nu$ on this set.

The coordinate system $x_1, \ldots, x_n$ in $U$ gives rise to a coordinate system $z_1, \ldots, z_n$ in the neighborhood $\mathcal{U} = E_1(U) \cup E_2(U)$ of $\bar{z}$. Namely, suppose $z \in \mathcal{U}$ equals $E_i(x)$ with $x \in U$. Define $z_k(z) = x_k(x)$ when $k = 1, \ldots, n-1$ and $z_n(z) = (-1)^{i+1}x_n(x)$. We will now analyze the partial derivatives of $\bar{p}(t, z)$ with respect to the newly introduced local coordinates. By doing so, we will establish the desired differentiability properties of this function.

It is easy to understand that $\frac{\partial}{\partial z_n} \bar{p}(t, z)$ exists and coincides with $\frac{1}{2} \frac{\partial}{\partial x_n} p(t, x)$ if $z = (x, i) \in \mathcal{U}$ and $k = 1, \ldots, n-1$. Furthermore, $\frac{\partial}{\partial z_n} \bar{p}(t, z)$ is continuous on $\mathcal{U}$ for these $k$. The situation is slightly more complicated when we differentiate with respect to the last coordinate. A straightforward argument shows

$$\frac{\partial}{\partial z_n} \bar{p}(t, z) = \frac{(-1)^{i+1}}{2} \frac{\partial}{\partial x_n} p(t, x)$$
when \( z = (x, i) \in U \setminus M^0 \). The one-sided derivatives \( \partial^+_{\partial_{2n}} \tilde{p}(t, z) \) and \( \partial^-_{\partial_{2n}} \tilde{p}(t, z) \) coincide with \( \frac{1}{2} \frac{\partial}{\partial_{x_n}} p(t, x) \) and \( -\frac{1}{2} \frac{\partial}{\partial_{x_n}} p(t, x) \), respectively, if \( z = (x, i) \in M^\delta \).

The boundary condition (2.2) ensures that \( \frac{\partial}{\partial_{2n}} \tilde{p}(t, z) \) is well-defined and equal to 0 on \( M^0 \). We conclude that \( \frac{\partial}{\partial_{2n}} \tilde{p}(t, z) \) exists in \( U \). Furthermore, it is continuous on \( U \).

Let us turn our attention to the second derivatives. Analogous reasoning can be used here. The existence and the continuity of \( \frac{\partial^2}{\partial_{2n} \partial_{2k}} \tilde{p}(t, z) \) on \( U \) are clear for \( k = 1, \ldots, n - 1 \) and \( l = 1, \ldots, n \). In order to analyze \( \frac{\partial^2}{\partial_{2n} \partial_{2k}} \tilde{p}(t, z) \) with \( k = 1, \ldots, n - 1 \), observe that the formula

\[
\frac{\partial^+}{\partial_{2n}} \frac{\partial}{\partial_{2l}} \tilde{p}(t, z) = \frac{1}{2} \frac{\partial^2}{\partial_{x_n} \partial_{x_l}} p(t, x) = \frac{1}{2} \frac{\partial^2}{\partial_{x_k} \partial_{x_n}} p(t, x) = 0
\]

holds when \( z = (x, i) \in M^\delta \). A similar calculation suggests the equality \( \frac{\partial^2}{\partial_{2n} \partial_{2k}} \tilde{p}(t, z) = 0 \) on \( M^\delta \). As a consequence, \( \frac{\partial^2}{\partial_{2n} \partial_{2l}} \tilde{p}(t, z) \) is well-defined and continuous on \( U \). The same can be said about \( \frac{\partial^2}{\partial_{2k} \partial_{2n}} \tilde{p}(t, z) \). Indeed, the formula

\[
\frac{\partial^2}{\partial_{2k} \partial_{2n}} \tilde{p}(t, z) = \frac{(1)^{2i+2}}{2} \frac{\partial^2}{\partial_{x_k} \partial_{x_n}} p(t, x) = \frac{1}{2} \frac{\partial^2}{\partial_{x_i} \partial_{x_n}} p(t, x)
\]

holds when \( z = (x, i) \in U \).

Summarizing the arguments above, we arrive at the following verdict: The function \( \tilde{p}(t, z) \) is twice continuously differentiable in \( z \) on the manifold \( M \). The smoothness of \( \tilde{p}(t, z) \) in \( t \) is evident. With this in mind, one can readily verify that the heat equation

\[
\left( \frac{\partial}{\partial t} - \Delta_M \right) \tilde{p}(t, z) = 0, \quad t \in (0, \infty), \quad z \in M, \quad (2.4)
\]

is satisfied (\( \Delta_M \) denoting the Laplace-Beltrami operator on \( M \)). In addition, the integral of \( \tilde{p}(t, z) \) over \( M \) is bounded by 1. These observations enable us to apply Theorem 4.3 of [16]. As a result, we get the existence of constants \( \tilde{A} > 0 \) and \( \tilde{B} > 0 \) such that

\[
\hat{D}_{X,Y}^2 \log \tilde{p}(t, z) \geq - \left( \frac{1}{2} + \hat{A} \left( 1 + \log \left( \frac{\tilde{B}}{t^{\frac{1}{2}} \tilde{p}(t, z)} \right) \right) \right) \langle X, X \rangle
\]

for every \( t \in (0, 1], \quad z \in M, \quad \text{and} \quad X \in T_z M \). Here, \( \hat{D}_{X,Y}^2 \) refers to the second covariant derivative given by the Levi-Civita connection in \( TM \). Inequality (2.3) follows immediately with \( \hat{A} = \hat{A} \) and \( \tilde{B} = 2\hat{B} \).

\[
\square
\]

Remark 2.2. As in the proof of Theorem 2.1 let \( M \) be the double of the manifold \( M \). Given \( z \in M \), the tangent space \( T_z M \) carries a natural scalar product induced by the Riemannian metric on \( M \). This scalar product depends smoothly on \( z \in M \) if and only if the boundary \( \partial M \) is totally geodesic and Assumption 1 of Theorem 2.1 is fulfilled. The justification of this fact can be found in [28].
Remark 2.3. Since the function $\tilde{p}(t, z)$ appearing in the proof satisfies (2.4), it must be smooth on $(0, \infty) \times \mathcal{M}$. In order to verify this, one may use the uniqueness and the integral representation of solutions to the heat equation; see, e.g., [19, Proposition 4.1.2].

Remark 2.4. Estimate (2.3) means that $D^2 \cdot \log p(t, x)$ is greater than or equal to

$$-\left(\frac{1}{2t} + A\left(1 + \log \left(\frac{B}{t\pi p(t, x)}\right)\right)\right) \langle \cdot, \cdot \rangle$$

in the sense of bilinear forms for every $t \in (0, 1]$ and $x \in M$.

Remark 2.5. If Assumption 2 of Theorem 2.1 is fulfilled, then there exists a constant $C > 0$ independent of $p(t, x)$ such that

$$p(t, x) \leq Ct^{-\frac{n}{2}}, \quad t \in (0, 1], \quad x \in M. \quad (2.5)$$

Note that $\partial M$ does not have to be totally geodesic for this to hold. In the case where $p(t, x)$ tends to a delta function as $t$ tends to 0, formula (2.5) follows from the parametrix construction for the Neumann heat kernel. This observation was made in [18, Proof of Lemma 3.2]. We also refer to [40] for relevant results. In the general case, formula (2.5) can be established by using the integral representation of the solution to the heat equation; see, e.g., [19, Proposition 4.1.2]. Importantly, if all the assumptions of Theorem 2.1 are fulfilled and $C$ satisfies (2.5), then there exists a constant $A_C > 0$ such that (2.3) holds with $A = A_C$ and $B = C$.

We now state a more specific version of the Li-Yau-Hamilton estimate for the function $p(t, x)$. It shows how (2.3) simplifies when the appropriate curvature restrictions are imposed on $M$ away from the boundary. Note that the inequality $\int_M p(t, x)dx \leq 1$ is no longer required for our arguments.

**Theorem 2.6.** Let the boundary $\partial M$ be totally geodesic. Suppose the following statements hold at every point $x \in M$:

1. The covariant derivative $(D_X \text{Ric})(Y, Z)$ is equal to 0 for all $X, Y, Z \in T_xM$.

2. The sectional curvature of every plane in $T_xM$ is nonnegative. That is, $R(X, Y, Y, X) \geq 0$ for all $X, Y \in T_xM$.

Then the solution $p(t, x)$ of the boundary value problem (2.1)–(2.2) satisfies the inequality

$$D^2_{X, X} \log p(t, x) \geq -\frac{1}{2t} \langle X, X \rangle \quad (2.6)$$

for every $t \in (0, \infty)$, $x \in M$, and $X \in T_xM$.

In many situations, estimate (2.6) can be established by the same technique we used to establish Theorem 2.1. One just has to exploit Corollary 4.4 in [16] instead of Theorem 4.3 in [16]. However, we prefer to adduce a direct method of proving (2.6) here based on the Hopf lemma for vector bundle
sections; see [30]. Firstly, because this method does not require the equality 
\( (D^k_v, \ldots, R) (\nu, X, \nu, Y) = 0 \) to hold on \( \partial M \). Secondly, because it avoids using the results of [16]. Last but not least, we believe the direct method is more illuminating and gives a more fertile ground for generalizations.

**Proof.** Take a number \( \epsilon > 0 \). Given \( t \in [0, \infty) \), introduce the two times covariant tensor field \( L^\epsilon_t \) by the formula

\[
L^\epsilon_t(X, Y) = (t + \epsilon)D^2_{X,Y} \log p(t + \epsilon, x) + \frac{1}{2} \langle X, Y \rangle, \quad X, Y \in T_x M.
\]

Our plan is to use the Hopf boundary point lemma of [30] for showing that \( L^\epsilon_t \) is positive semidefinite at every point of \( M \). The theorem will then be proved by taking the limit as \( \epsilon \) goes to 0.

In what follows, we assume \( p(t, x) \) is defined and smooth on \([0, \infty) \times M\). This does not lead to any loss of generality. Indeed, we can always establish the desired estimate for the function \( p_\delta(t, x) = p(t + \delta, x) \), \( \delta > 0 \), and pass to the limit as \( \delta \) tends to 0.

Firstly, let us compute \( (\partial / \partial t - \Delta_{\text{tens}}) L^\epsilon_t \). The Laplacian \( \Delta_{\text{tens}} \) in this expression appears as the trace of the second covariant derivative \( D^2 \) in the bundle \( T^*_x M \otimes T^*_x M \). Recall that the connection in this bundle is induced by the Levi-Civita connection in \( TM \).

The Riemannian metric on \( M \) yields a scalar product of tensors over a point \( x \in M \). The notation \( \langle \cdot, \cdot \rangle \) is preserved for this scalar product. Set

\[
P^\epsilon(t, x) = \text{grad} \log p(t + \epsilon, x).
\]

We omit the \( (t, x) \) at \( P^\epsilon \) when this does not lead to ambiguity. Introduce the mapping \( \Phi(t, w) \) acting from \([0, \infty) \times (T^*_x M \otimes T^*_x M)\) to \( T^*_x M \otimes T^*_x M \) by the equality

\[
\Phi(t, w)(X, Y) = 2\langle R_{X, Y}, w \rangle - \langle \iota_X \text{Ric}, \iota_Y w \rangle - \langle \iota_Y \text{Ric}, \iota_X w \rangle
\]

\[+ \frac{2}{t + \epsilon} \langle \iota_X w, \iota_Y w \rangle + 2(t + \epsilon)R(X, P^\epsilon, P^\epsilon, Y)
\]

\[- \frac{1}{t + \epsilon} w(X, Y), \quad X, Y \in T_x M.
\]

Here, the tensor \( R_{X, Y} \) is defined as \( R_{X, Y}(Z, W) = R(X, Z, W, Y) \) for \( Z, W \in T_x M \), and \( \iota \) denotes the interior product. A standard calculation, together with Assumption [4] of our theorem, shows that

\[
(\partial / \partial t - \Delta_{\text{tens}}) L^\epsilon_t = D_{2P^\epsilon} L^\epsilon_t + \Phi(t, L^\epsilon_t), \quad t \in [0, \infty),
\]

at every \( x \in M \). For relevant arguments, see [16] [11] [6] and [12] Section 2.5.

Let \( W \subset T^*_x M \otimes T^*_x M \) be the set of two times covariant, symmetric, positive semidefinite tensors. Suppose \( \epsilon \) is chosen sufficiently small to ensure that \( L^\epsilon_t \) belongs to \( W \) at every point of \( M \) when \( t = 0 \). The existence of such an \( \epsilon \) follows from the smoothness of \( p(t, x) \) on \([0, \infty) \times M\). Fixing \( T > 0 \), we will apply Theorem 2.1 in [30] (the Hopf lemma) to demonstrate that \( L^\epsilon_t \) must belong to \( W \) at every point of \( M \) for all \( t \in [0, T] \).
Some more notation has to be introduced here. Given $x \in M$, define the set $W_x$ as the intersection of $W$ with $T^*_x M \otimes T^*_x M$. Evidently, $W_x$ is closed and convex in $T^*_x M \otimes T^*_x M$. Let $\omega(w)$ stand for the point in $W_x$ nearest to $w \in T^*_x M \otimes T^*_x M$. More precisely, the minimum of the scalar product $\langle w - v, w - v \rangle$ over $v \in W_x$ must be attained at $v = \omega(w)$. Denote $\lambda(w) = w - \omega(w)$.

We now verify the assumptions of Theorem 2.1 from [30]. It was already noted that $L_t^t \in W$ at every point of $M$ when $t = 0$ and that $W_x$ was closed and convex for all $x \in M$. The set $W$ is invariant under the parallel translation in $T^* M \otimes T^* M$; see [10]. The arguments preceding Corollary 10.12. The mapping $\Phi(t, w)$, obviously, satisfies inequality (2.1) in [30]. Thus, Requirement 2 of Theorem 2.1 from [30] remains the only statement to be checked. Considering Remark 2.1 of [30], it suffices to prove the inequality

$$
\langle \Phi(t, \omega(L_t^t)), \lambda(L_t^t) \rangle \leq 0, \quad t \in [0, T],
$$

over every point of $M$.

Fix $t \in [0, T]$. We omit the subscript $t$ at $L_t^t$ in order to simplify the notation. Pick an orthonormal basis $\{e_1, \ldots, e_n\}$ of the space $T_x M$ for some $x \in M$. Without loss of generality, suppose this basis diagonalizes $L^t$ at $x$. One can easily understand that

$$
\omega(L^t)(e_i, e_j) = \max \{L^t(e_i, e_j), 0\},
$$

$$
\lambda(L^t)(e_i, e_j) = \min \{L^t(e_i, e_j), 0\}, \quad i, j = 1, \ldots, n.
$$

Hence

$$
\langle \Phi(t, \omega(L^t)), \lambda(L^t) \rangle = \sum_{i=1}^{n} \Phi(t, \omega(L^t))(e_i, e_i) \min \{L^t(e_i, e_i), 0\}.
$$

If $L^t(e_i, e_i) < 0$, then $\omega(L^t)(e_i, e_j) = 0$ for all $j = 1, \ldots, n$. Using this fact along with our Assumption 2, one can readily prove that

$$
\Phi(t, \omega(L^t))(e_i, e_i) \geq 0
$$

when $L^t(e_i, e_i) < 0$. Thus, estimate (2.7) holds true.

We are now in a position to apply Theorem 2.1 of [30]. More precisely, we apply Corollary 2.3 of that theorem. Let us establish the equality

$$
\langle \lambda(L_t^t), D_{\nu}L_t^t \rangle = 0
$$

over an arbitrarily chosen point $x \in \partial M$ for all $t \in [0, T]$. This would lead us to the conclusion that $L_t^t$ is always positive semidefinite.

As before, we fix $t \in [0, T]$ and write $L^t$ instead of $L_t^t$. Pick an orthonormal basis $\{v_1, \ldots, v_{n-1}\}$ of the space $T_x \partial M$ tangent to the boundary. Suppose this basis diagonalizes the restriction of $L^t$ to $T_x \partial M \otimes T_x \partial M$. A straightforward verification shows

$$
L^t(v_i, \nu) = -(t + \epsilon) \Pi(v_i, P^t), \quad i = 1, \ldots, n - 1.
$$

(Remark that $P^t$ is tangent to $\partial M$ due to the Neumann boundary condition [2.23].) The right-hand side of the above formula is equal to 0 because
∂M is totally geodesic. Hence \( L^\epsilon(v_i, \nu) = 0 \) for \( i = 1, \ldots, n - 1 \). We conclude that the orthonormal basis \( \{v_1, \ldots, v_{n-1}, \nu\} \) diagonalizes \( L^\epsilon \) at \( x \) and

\[
\langle \lambda(L^\epsilon), D_\nu L^\epsilon \rangle = \sum_{i=1}^{n-1} \min \{L^\epsilon(v_i, v_i), 0\} \langle D_\nu L^\epsilon \rangle(v_i, v_i) \\
+ \min \{L^\epsilon(\nu, \nu), 0\} \langle D_\nu L^\epsilon \rangle(\nu, \nu).
\] (2.8)

Each of the summands on the right-hand side of (2.8) is 0. Indeed, since \( \partial M \) is totally geodesic, we can introduce the normal coordinates \( x_1, \ldots, x_n \) around \( x \) so that \( \frac{\partial}{\partial x_1} \) and \( \frac{\partial}{\partial x_n} \) coincide with \( v_1 \) and \( -\nu \), respectively, at the origin. A calculation in these coordinates yields

\[
(D_\nu L^\epsilon) (v_i, v_i) = -(t + \epsilon) [(D_\nu (\lambda P^\epsilon \Pi))(v_i)] \\
- (t + \epsilon) \Pi(v_i, D_\nu P^\epsilon) \\
+ (t + \epsilon) R(v_i, P^\epsilon, v_i, \nu), \quad i = 1, \ldots, n - 1.
\] (2.9)

(The vector \( D_\nu P^\epsilon \) is tangent to the boundary because \( \langle D_\nu P^\epsilon, \nu \rangle = \frac{\partial}{\partial x_n} L^\epsilon(v_i, \nu) = 0 \).) The second fundamental form \( \Pi \) vanishes identically. Therefore, the first two terms in (2.9) equal 0. Given \( X, Y, Z \in T_x \partial M \), it is easy to see that \( R(X, Y)Z \) coincides with the Riemannian curvature tensor of \( \partial M \) applied to these vectors. Hence \( R(v_i, P^\epsilon) v_i \) is tangent to \( \partial M \), and the third term in (2.9) equals 0, as well. As a result, \( (D_\nu L^\epsilon) (v_i, v_i) = 0 \) for \( i = 1, \ldots, n - 1 \).

Another calculation (cf. [25]) yields

\[
(D_\nu L^\epsilon)(\nu, \nu) = (t + \epsilon) \frac{\partial}{\partial \nu} \Delta_M \log p(t + \epsilon, x) - \sum_{i=1}^{n-1} (D_\nu L^\epsilon)(v_i, v_i) \\
= (t + \epsilon) \frac{\partial}{\partial \nu} \Delta_M \log p(t + \epsilon, x) = 2(t + \epsilon) \Pi(P^\epsilon, P^\epsilon).
\]

Since \( \Pi \) vanishes identically, the above implies \( (D_\nu L^\epsilon)(\nu, \nu) = 0 \). In view of (2.8), we conclude \( \langle \lambda(L^\epsilon), D_\nu L^\epsilon \rangle = 0 \) over our arbitrarily chosen \( x \in \partial M \).

Corollary 2.3 of Theorem 2.1 in [30] now suggests that \( L^\epsilon \) is positive semidefinite at every point of \( M \) for all \( t \in [0, T] \). Since no restrictions were imposed on the number \( T \), this tensor field must be positive semidefinite at every point for all \( t \in [0, \infty) \). Taking the limit as \( \epsilon \) tends to 0 proves (2.6).

3 The Yang-Mills heat equation

This section aims to study the solutions to the Yang-Mills heat equation in a vector bundle over the manifold \( M \). Roughly speaking, we show that the curvature of such a solution is bounded if the dimension of \( M \) is less than 4 or if the initial energy is sufficiently small. The proofs utilize a probabilistic method. When the dimension of \( M \) is greater than or equal to 5, our technique requires the Li-Yau-Hamilton estimate established in Section 2. Notably, this reflects on the assumptions we impose on the geometry of \( M \).
Many statements below demand that the boundary \( \partial M \) be convex. The concept of convexity is quite delicate for Riemannian manifolds. Different definitions and the relations between them are surveyed in [35]. The paper [23] is also relevant. In what follows, when saying \( \partial M \) is convex, we mean that the formula

\[
\Pi(X, X) \geq 0, \quad X \in T\partial M,
\]

must hold for the second fundamental form of \( \partial M \).

The next few paragraphs provide a description of the structure required to formulate the Yang-Mills heat equation. For a detailed exposition of the background material, see [5, 24, 21, 13, 14].

Recall that the manifold \( M \) is assumed to be compact. Let \( E \) be a vector bundle over \( M \) with the standard fiber \( \mathbb{R}^d \) and the structure group \( G \). We suppose \( G \) appears as a Lie subgroup of \( O(d) \) and acts naturally on \( \mathbb{R}^d \). The symbol \( \mathfrak{g} \) stands for the Lie algebra of \( G \). In what follows, we assume \( \mathbb{R}^d \) is equipped with the standard scalar product. Every element of \( \mathfrak{g} \) appears as a skew-symmetric endomorphism of \( \mathbb{R}^d \). Define the scalar product in this Lie algebra by the formula

\[
\langle A, B \rangle_{\mathfrak{g}} = -\text{trace} AB, \quad A, B \in \mathfrak{g}.
\]

The adjoint bundle \( \text{Ad} E \), whose standard fiber is equal to \( \mathfrak{g} \), carries the fiber metric induced by \( \langle \cdot, \cdot \rangle_{\mathfrak{g}} \).

Let \( \nabla \) be a connection in \( E \). We understand \( \nabla \) as a mapping that takes a section \( \tau \) of \( E \) to a section \( \nabla \tau \) of the bundle \( T^* M \otimes E \). It is customary to interpret \( \nabla \tau \) as an \( E \)-valued 1-form on the manifold \( M \). Consider a vector field \( X \) on \( M \). We write \( \nabla_X \tau \) to indicate the application of \( \nabla \tau \) to \( X \). Given a smooth real-valued function \( f(x) \) on \( M \), the formula

\[
\nabla_X(f\tau) = (X f)\tau + f \nabla_X \tau
\]

must be satisfied. We suppose \( \nabla \) is compatible with the structure group \( G \). The curvature of \( \nabla \) will be denoted by \( R^{\nabla} \). Let us mention that \( R^{\nabla} \) appears as a 2-form on \( M \) with its values in the bundle \( \text{Ad} E \). Our goal is to write down the Yang-Mills heat equation. In order to do this, we need to introduce the operators of covariant exterior differentiation corresponding to a connection in \( E \).

Consider the bundle \( \Lambda^p T^* M \otimes \text{Ad} E \) for a nonnegative integer \( p \). Its sections are interpreted as \( \text{Ad} E \)-valued \( p \)-forms on the manifold \( M \). The set of all these sections will be designated by \( \Omega^p(\text{Ad} E) \). The Riemannian metric on \( M \) and the fiber metric in \( \text{Ad} E \) give rise to a scalar product in the fibers of \( \Lambda^p T^* M \otimes \text{Ad} E \). We use the notation \( \langle \cdot, \cdot \rangle_E \) for this scalar product and the notation \( \cdot \cdot \cdot _E \) for the corresponding norm.

The connections \( D \) in \( TM \) and \( \nabla \) in \( E \) induce a connection in the bundle \( \Lambda^p T^* M \otimes \text{Ad} E \). It appears as a mapping from \( \Omega^p(\text{Ad} E) \) to the set of sections of \( T^* M \otimes \Lambda^p T^* M \otimes \text{Ad} E \). We preserve the notation \( \nabla \) for this connection in
Define the operator \( d \nabla \) acting from \( \Omega^p(Ad E) \) to \( \Omega^{p+1}(Ad E) \) by the formula

\[
(d \nabla \phi)(X_1, \ldots, X_{p+1}) = \sum_{i=1}^{p+1} (-1)^{i+1} (\nabla_{X_i} \phi)(X_1, \ldots, X_{i-1}, X_{i+1}, \ldots, X_{p+1}).
\]

Here, \( \phi \) belongs to \( \Omega^p(Ad E) \), and \( X_1, \ldots, X_{p+1} \) belong to \( T_xM \) for some \( x \in M \).

It is easy to understand that \( d \nabla \) plays the role of the covariant exterior derivative corresponding to \( \nabla \). The operator \( d^\ast \nabla \) acting from \( \Omega^{p+1}(Ad E) \) to \( \Omega^p(Ad E) \) is defined by the equality

\[
(d^\ast \nabla \psi)(X_1, \ldots, X_p) = -\sum_{i=1}^n (\nabla_{e_i} \psi)(e_i, X_1, \ldots, X_p).
\]

Here, \( \psi \) belongs to \( \Omega^{p+1}(Ad E) \), the vectors \( X_1, \ldots, X_p \) belong to \( T_xM \) for some \( x \in M \), and \( \{e_1, \ldots, e_n\} \) is an orthonormal basis of \( T_xM \). We set \( d^\ast \nabla \) to be equal to zero on \( \Omega^0(Ad E) \). In view of Lemma 3.4 below, this operator may be understood as the formal adjoint of \( d \nabla \).

Fix a number \( T > 0 \). Consider a connection \( \nabla(t) \) in \( E \) depending on \( t \in [0, T) \). The parameter \( t \) will be interpreted as time. We require that \( \nabla(t) \) be compatible with the structure group \( G \) for all \( t \in [0, T) \). Suppose \( \nabla(t) \) satisfies the Yang-Mills heat equation

\[
\frac{\partial}{\partial t} \nabla(t) = \frac{-1}{2} d^\ast \nabla(t) R^{\nabla(t)}, \quad t \in [0, T).
\]

(3.2)

In particular, this connection must be once continuously differentiable in \( t \in [0, T) \). The factor \( \frac{1}{2} \) appears in the right-hand side because we want to achieve maximum conformity with the probabilistic results employed below. In interpreting \( \frac{\partial}{\partial t} \nabla(t) \), one should remember that \( \nabla(t) \) lies, for each \( t \in [0, T) \), in the linear space of mappings taking sections of \( E \) to sections of \( T^*M \otimes E \). Our next step is to specify the boundary conditions for \( \nabla(t) \). Doing this is quite a delicate matter. We discuss some of the nuances in Remarks 3.11 and 3.12 in the end of this section.

Every \( Ad E \)-valued \( p \)-form \( \phi \in \Omega^p(Ad E) \) can be decomposed into the sum of its the tangential component \( \phi_{\text{tan}} \) and its normal component \( \phi_{\text{norm}} \) on the boundary of \( M \). Roughly speaking, \( \phi_{\text{tan}} \) coincides with the restriction of \( \phi \) to the vectors from \( T \partial M \). If \( \phi \) lies in \( \Omega^0(Ad E) \), then \( \phi_{\text{tan}} \) equals \( \phi \) on \( \partial M \). We are now ready to impose the boundary conditions on \( \nabla(t) \). Assume the equalities

\[
\left( R^{\nabla(t)} \right)_{\text{tan}} = 0, \quad \left( d^\ast \nabla(t) R^{\nabla(t)} \right)_{\text{tan}} = 0
\]

hold on \( \partial M \) for all \( t \in [0, T) \). One should view (3.3) as a version of the relative boundary conditions on real-valued forms; see, for example, \( \text{[32]} \). Alternatively, we may assume the formulas

\[
\left( R^{\nabla(t)} \right)_{\text{norm}} = 0, \quad \left( d\nabla(t) R^{\nabla(t)} \right)_{\text{norm}} = 0
\]

(3.4)
hold on $\partial M$ for all $t \in [0, T)$. (Actually, the second one is always satisfied due to the Bianchi identity.) These should be viewed as a version of the absolute boundary conditions; again, [32] is a good reference. The arguments in the present paper will prevail regardless of whether we choose Eqs. (3.3) or Eqs. (3.4) to hold on $\partial M$. For other problems and techniques, however, only one of the choices may be appropriate.

We should make an important comment at this point. In essence, Eqs. (3.3) and (3.4) are restrictions on the curvature form $R^{\nabla(t)}$. Another possible strategy is to impose the boundary conditions directly on the connection $\nabla(t)$. We postpone a discussion of this issue until after the proofs of our results; see Remarks 3.11 and 3.12.

Introduce the function

$$YM(t) = \int_{M} |R^{\nabla(t)}|^2_E \, dx$$

for $t \in [0, T)$. In accordance with the conventions of Section 2, the integration is to be carried out with respect to the Riemannian volume measure on $M$. It is reasonable to call $YM(t)$ the energy at time $t$. A standard argument involving Lemma 3.4 below shows that $YM(t)$ is non-increasing in $t \in [0, T)$; see [7] and also, for example, [21, 31, 8].

We now state the main results of Section 3. Our first theorem concerns the lower-dimensional case. It offers a bound for $R^{\nabla(t)}$ in terms of the initial energy $YM(0)$ and demonstrates that $R^{\nabla(t)}$ does not blow up at time $T$. In what follows, the notation $R^{\nabla(t)}(x)$ refers to the curvature of $\nabla(t)$ at the point $x \in M$.

**Theorem 3.1.** Let the dimension $\dim M$ equal 2 or 3. Suppose $\partial M$ is convex in the sense of (3.1). Then the solution $\nabla(t)$ of Eq. (3.2), subject to the boundary conditions (3.3) or (3.4), satisfies the estimate

$$\sup_{x \in M} \left| R^{\nabla}(\rho)(x) \right|^2_E \leq \max \left\{ \frac{4 \, YM(0)}{\rho^2}, \theta_1 e^{\theta_2 \sqrt{YM(0)}} \right\}$$

for all $\rho \in (0, T)$. Here, $\theta_1 > 0$ and $\theta_2 > 0$ are constants depending only on the manifold $M$.

A similar result can be obtained in dimension 4 provided that the initial energy $YM(0)$ is smaller than a certain value $\xi$. We emphasize that $\xi$ depends on nothing but $M$.

**Theorem 3.2.** Let the dimension $\dim M$ equal 4. Suppose the boundary $\partial M$ is convex in the sense of (3.1). Then there exists a constant $\xi > 0$ depending only on the manifold $M$ and satisfying the following statement: The solution $\nabla(t)$ of Eq. (3.2) with the boundary conditions (3.3) or (3.4) obeys the estimate

$$\sup_{x \in M} \left| R^{\nabla}(\rho)(x) \right|^2_E \leq \max \left\{ \frac{4 \sqrt{YM(0)}}{\rho^2}, \sqrt{YM(0)} \right\}, \quad \rho \in (0, T),$$

if the initial energy $YM(0)$ is smaller than $\xi$. 

15
We turn our attention to dimensions 5 and higher. In this case, the proof of the result will require the Li-Yau-Hamilton estimate established in Section 2. This forces us to impose stronger geometric assumptions on the manifold $M$.

The following theorem yields a bound on $R^\nabla(\rho)$ provided $YM(0)$ is smaller than a certain value $\xi(\rho)$ depending on $\rho \in [0, T)$. This result implies that the curvature of a solution to Eq. (3.2) cannot blow up after time $\rho$ if the initial energy does not exceed $\xi(\rho)$. In the above setting, the connection $\nabla(t)$ is defined for each $t \in [0, T)$ and depends differentiably on $t$ on this interval. Therefore, $R^\nabla(t)$ does not blow up at time $T$ if $YM(0) < \xi(\rho)$ for some $\rho \in (0, T)$.

**Theorem 3.3.** Let the dimension $\dim M$ be greater than or equal to 5. Suppose the boundary $\partial M$ is totally geodesic. Moreover, suppose either Assumption 1 of Theorem 2.1 or Assumptions 1 and 2 of Theorem 2.6 are fulfilled for $M$. Then there exists a positive non-decreasing function $\xi(s)$ on $(0, \infty)$ that depends on nothing but $M$ and satisfies the following statement: Given $\rho \in (0, T)$, the solution $\nabla(t)$ of Eq. (3.2) with the boundary conditions (3.3) or (3.4) obeys the estimate

$$\sup_{x \in M} \left| R^\nabla(\rho)(x) \right|^2 E \leq \max \left\{ \frac{16\sqrt{YM(0)}}{\rho^2}, \sqrt{YM(0)} \right\}$$

(3.7)

if the initial energy $YM(0)$ is smaller than $\xi(\rho)$.

The assertions of Theorems 3.1, 3.2, and 3.3 may be refined. We present them here in the less general form in order to ensure that the technical details do not obscure the qualitative meaning. The possible refinements are explained in Remarks 3.6, 3.7, and 3.10.

To prove the three theorems above, we employ the probabilistic technique developed in [2]. The main stochastic process to be used for our arguments is a reflecting Brownian motion on the manifold $M$. Its transition density is the Neumann heat kernel on $M$. Before introducing the probabilistic machinery, we need to state two geometric results.

First of all, it is necessary to formulate a version of the integration by parts formula. Let us recollect some conventions and notation. The boundary of $M$ carries a natural Riemannian metric inherited from $M$. The orientation of $\partial M$ is induced by that of $M$. The integration over $\partial M$ is to be carried out with respect to the Riemannian volume measure on $\partial M$. The letter $\nu$ stands for the outward unit normal vector field on the boundary. The letter $\iota$ stands for the interior product.

We are now ready to lay down integration by parts formula. Our source for this result is the paper [7].

**Lemma 3.4.** Let $\nabla$ be a connection in $E$ compatible with the structure group $G$. Consider $\text{Ad} E$-valued forms $\phi \in \Omega^p(\text{Ad} E)$ and $\psi \in \Omega^{p+1}(\text{Ad} E)$ with $p = 0, \ldots, \dim M - 1$. The equality

$$\int_M (\langle d\nabla \phi, \psi \rangle_E - \langle \phi, d^\nabla \psi \rangle_E) \, dx = \int_{\partial M} \langle \phi, \iota_\nu \psi \rangle_E \, dx$$

holds true.
As mentioned above, an argument involving Lemma 3.4 proves that YM(t) is non-increasing in \( t \in [0, T] \); see, for instance, [31, 7]. This fact is crucial for our further considerations.

The next step is to understand what Eqs. (3.3) and (3.4) can tell us about the behavior of \( |R^{\nabla(t)}(x)|_E^2 \) near the boundary of \( M \). In order to do this, we present the following result. It may be viewed as a variant of Lemma 3.1\(^1\) in [7] for manifolds with convex boundary. The proof utilizes a computation carried out in [7]. Given \( \phi \in \Omega^p(\text{Ad}E) \) and \( x \in M \), the notation \( \phi(x) \) refers to the restriction of \( \phi \) to \( (T_xM)^p \).

**Lemma 3.5.** Let the boundary \( \partial M \) be convex in the sense of (3.1). Suppose \( \nabla \) is a connection in \( E \) compatible with the structure group \( G \). Consider an \( \text{Ad}E \)-valued \( p \)-form \( \phi \in \Omega^p(\text{Ad}E) \) with \( p = 0, \ldots, \dim M \). If either the equations

\[
\phi_{\tan} = 0, \quad (d\nabla \phi)_{\tan} = 0 \quad (3.8)
\]

or the equations

\[
\phi_{\text{norm}} = 0, \quad (d\nabla \phi)_{\text{norm}} = 0 \quad (3.9)
\]

are satisfied on \( \partial M \), then the formula

\[
\frac{\partial}{\partial \nu} |\phi(x)|_E^2 \leq 0, \quad x \in \partial M, \quad (3.10)
\]

holds true.

**Proof.** We begin by selecting a local coordinate system on \( M \) convenient for our arguments. Choose a point \( \tilde{x} \in \partial M \). Let \( \{e_1, \ldots, e_{n-1}\} \) be an orthonormal basis of the space \( T_{\tilde{x}}\partial M \) such that

\[
\Pi(e_i, e_j) = \delta^j_i \lambda_i, \quad i, j = 1, \ldots, n-1.
\]

In this formula, \( \delta^j_i \) is the Kronecker symbol, and \( \lambda_i \) are the principal curvatures at \( \tilde{x} \). Since \( \partial M \) is convex, \( \lambda_i \) must be nonnegative for all \( i = 1, \ldots, n-1 \). Take a coordinate neighborhood \( U^\partial \) of \( \tilde{x} \) in \( \partial M \) with a coordinate system \( y_1, \ldots, y_{n-1} \) in \( U^\partial \) centered at \( \tilde{x} \). We assume \( \frac{\partial}{\partial \nu} \) coincides with \( e_i \) at \( \tilde{x} \) for each \( i = 1, \ldots, n-1 \). As in the proof of Theorem 2.1 consider the mapping \( \mu(r, x) \) defined on \( [0, \epsilon) \times \partial M \) by the formula \( \mu(r, x) = \exp_x(-r\nu) \). The number \( \epsilon > 0 \) is chosen small enough for \( \mu(r, x) \) to be a diffeomorphism onto its image. The set \( U = \mu([0, \epsilon) \times U^\partial) \) is a neighborhood of \( \tilde{x} \) in the manifold \( M \). We extend \( y_1, \ldots, y_{n-1} \) to a coordinate system \( x_1, \ldots, x_n \) in \( U \) by demanding that the equalities

\[
x_k(\mu(r, x)) = y_k(x), \quad x_n(\mu(r, x)) = r, \quad r \in [0, \epsilon), \quad x \in U^\partial, \quad k = 1, \ldots, n-1,
\]

\(^1\)This statement was labeled Lemma 3.1 in a preliminary version of [7]. It may appear under a different tag in the final manuscript.
hold true; cf. [28]. The vector \( \frac{\partial}{\partial x_i} \) coincides with \( e_i \) at \( \hat{x} \) for each \( i = 1, \ldots, n-1 \). It is easy to see that \( \frac{\partial}{\partial x_n} \) is tangent to the boundary on the set \( U^\vartheta \) for \( i = 1, \ldots, n-1 \). The vector field \( \frac{\partial}{\partial x_n} \) coincides with \( -\nu \) at every point of \( U^\vartheta \).

Having fixed a suitable local coordinate system on \( M \), we now proceed to the actual proof of the lemma. Without loss of generality, suppose Eqs. (3.9) hold for \( \phi \) on \( \partial M \). If this is not the case and Eqs. (3.8) hold instead, we can replace \( \phi \) with the form \( *\phi \) satisfying (3.9). (The symbol \( * \) denotes the Hodge star operator.) Since \( |\phi(x)|_E \) equals \( *|\phi(x)|_E \) for all \( x \in M \), proving the lemma for \( *\phi \) would suffice.

From the technical point of view, it is convenient for us to assume that \( \phi \) belongs to \( \Omega^p(\text{Ad}E) \) with \( p \) between 1 and \( \dim M \). This restriction is not significant. Indeed, if \( \phi \) is an \( \text{Ad}E \)-valued 0-form on \( M \), then estimate (3.10) follows directly from the second formula in (3.9).

Our next step is to write down an expression for the derivative \( \frac{\partial}{\partial x_n} |\phi(x)|_E^2 \) using the coordinate system introduced above. Observe that, in the neighborhood \( U \) of the point \( \hat{x} \), one can represent \( \phi \) by the equality

\[ \phi(x) = \alpha(x) \wedge dx_n + \beta(x). \]

Here, \( \alpha \) and \( \beta \) are \( \text{Ad}E \)-valued forms defined on \( U \) and given by the formulas

\[ \alpha(x) = \sum \alpha_I(x) \, dx^I, \quad \beta(x) = \sum \beta_J(x) \, dx^J. \]

The sums are taken over all the multi-indices \( I = (i_1, \ldots, i_{p-1}) \) and \( J = (j_1, \ldots, j_p) \) with \( 1 \leq i_1 < \cdots < i_{p-1} < n \) and \( 1 \leq j_1 < \cdots < j_p < n \). The mappings \( \alpha_I(x) \) and \( \beta_J(x) \) defined on \( U \) are local sections of the bundle \( \text{Ad}E \). The notations \( dx^I \) and \( dx^J \) refer to \( dx_{i_1} \wedge \cdots \wedge dx_{i_{p-1}} \) and \( dx_{j_1} \wedge \cdots \wedge dx_{j_p} \). If \( p = 1 \), then \( \alpha \) should be interpreted as an \( \text{Ad}E \)-valued 0-form on \( U \). If \( p = n \), then \( \beta \) equals zero.

Following the computation from [7] Proof of Lemma 3.1, we arrive at the formula

\[ \frac{1}{2} \frac{\partial}{\partial \nu} |\phi(x)|_E^2 = \sum \langle \beta_J(x), \beta_K(x) \rangle_E \langle D_{\nu}dx^I, dx^K \rangle_\Lambda, \]

\[ x \in U \cap \partial M. \quad (3.11) \]

The summation is now carried out over all \( J = (j_1, \ldots, j_p) \) and \( K = (k_1, \ldots, k_p) \) with \( 1 \leq j_1 < \cdots < j_p < n \) and \( 1 \leq k_1 < \cdots < k_p < n \). The angular brackets with the lower index \( \Lambda \) stand for the scalar product in \( \Lambda T^*M \) induced by the Riemannian metric on \( M \). If \( p = n \), then the sum in (3.11) should be interpreted as 0.

We have thus laid down an expression for \( \frac{\partial}{\partial \nu} |\phi(x)|_E^2 \) in our local coordinates. The next step is to establish estimate (3.10) at the point \( \hat{x} \) using formula (3.11). The argument will rely on the properties of the coordinate system fixed in \( U \). Remark that \( \hat{x} \) was originally chosen as an arbitrary point in \( \partial M \). Therefore, establishing (3.10) at this point would suffice to prove the lemma.
Let us take a closer look at the scalar product $\langle D_\nu dx^J, dx^K \rangle_\Lambda$ in the right-hand side of (3.11). The formula

$$\langle D_\nu dx^J, dx^K \rangle_\Lambda = \sum_{l=1}^p \det \left( \begin{array}{cccc} \langle dx_{j_1}, dx_{k_1} \rangle_\Lambda & \cdots & \langle dx_{j_1}, dx_{k_p} \rangle_\Lambda \\ \vdots & & \vdots \\ \langle dx_{j_{l-1}}, dx_{k_1} \rangle_\Lambda & \cdots & \langle dx_{j_{l-1}}, dx_{k_p} \rangle_\Lambda \\ \langle D_\nu dx_{j_1}, dx_{k_1} \rangle_\Lambda & \cdots & \langle D_\nu dx_{j_1}, dx_{k_p} \rangle_\Lambda \\ \langle dx_{j_{l+1}}, dx_{k_1} \rangle_\Lambda & \cdots & \langle dx_{j_{l+1}}, dx_{k_p} \rangle_\Lambda \\ \vdots & & \vdots \\ \langle dx_{j_p}, dx_{k_1} \rangle_\Lambda & \cdots & \langle dx_{j_p}, dx_{k_p} \rangle_\Lambda \end{array} \right)$$

holds on $U \cap \partial M$. Our choice of the coordinate system provides the identities

$$\langle dx_{l_1}, dx_{m} \rangle_\Lambda = \delta^m_{l_1},$$

$$\langle D_\nu dx_{l_1}, dx_{m} \rangle_\Lambda = -\Pi \left( \frac{\partial}{\partial x_{l_1}}, \frac{\partial}{\partial x_{m}} \right) = -\delta^m_{l_1} \lambda_l, \quad l, m = 1, \ldots, n - 1,$$

at the point $\tilde{x}$. (Recall that $\delta^m_{l_1}$ is the Kronecker symbol, and $\lambda_l$ are the principal curvatures.) As a consequence,

$$\langle D_\nu dx^J, dx^K \rangle_\Lambda = -\left( \lambda_{j_1} + \cdots + \lambda_{j_p} \right)$$

at $\tilde{x}$ when $J$ coincides with $K$, and

$$\langle D_\nu dx^J, dx^K \rangle_\Lambda = 0$$

at $\tilde{x}$ when $J$ differs from $K$.

Let us substitute the obtained equalities into (3.11). We conclude that

$$\frac{1}{2} \frac{\partial}{\partial \nu} |\phi(x)|^2_E = -\sum (\beta_J(x), \beta_J(x))_E \left( \lambda_{j_1} + \cdots + \lambda_{j_p} \right).$$

at the point $\tilde{x}$. The summation is carried out over all the multi-indices $J$ as described above. The scalar product $\langle \beta_J(x), \beta_J(x) \rangle_E$ is greater than or equal to 0 for every $J$. The principal curvatures $\lambda_{j_1}, \ldots, \lambda_{j_p}$ are all nonnegative because $\partial M$ is convex. As a result, estimate (3.10) holds at the point $\tilde{x}$. This proves the lemma because $\tilde{x}$ can be chosen arbitrarily.

Our intention is to employ the technique developed in [2] for establishing Theorems 3.1, 3.2, and 3.3. We now introduce the required probabilistic machinery. Consider the bundle $O(M)$ of orthonormal frames over $M$. The letter $\pi$ denotes the projection in this bundle. Let $u^Y_t$ be a horizontal reflecting Brownian motion on $O(M)$ starting at the frame $Y \in O(M)$. We assume $u^Y_t$ is defined on the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0, \infty)}, \mathbb{P})$ satisfying the
“usual hypotheses.” The symbol $E$ will be used for the expectation. The rigorous definition of a horizontal reflecting Brownian motion on the bundle of orthonormal frames can be found in [20, Chapter V] and in [18].

Introduce the process $X^y_t = \pi(\{u^y_t\})$. Here, we denote $y = \pi(Y)$. It is well-known that $X^y_t$ is a reflecting Brownian motion on $M$ starting at the point $y$. Details can be found in [20, Chapter V].

By definition, the process $u^Y_t$ satisfies the equation

$$df(t,u^Y_t) = \sum_{i=1}^{n}(H_if)(t,u^Y_t) dB^i_t + \left(\frac{\partial}{\partial t} + \frac{1}{2} \Delta_{O(M)}\right)f(t,u^Y_t) dt - (\mathcal{N}f)(t,u^Y_t) dL_t$$

(3.12)

for every smooth real-valued function $f(t,u)$ on $[0,\infty) \times O(M)$. Let us describe the objects occurring in the right-hand side. As before, $n \geq 2$ is the dimension of $M$. The notation $H_i$ refers to the canonical horizontal vector fields on $O(M)$. The process $(B^1_t,\ldots,B^n_t)$ is an $n$-dimensional Brownian motion defined on $(\Omega,F,(F_t)_{t\in[0,\infty)},\mathbb{P})$. The operator $\Delta_{O(M)}$ is Bochner’s horizontal Laplacian. It appears as the sum of $H^2_i$ with $i = 1,\ldots,n$. The symbol $\mathcal{N}$ stands for the horizontal lift of the vector field $\nu$ on $\partial M$. The non-decreasing process $L_t$ is the boundary local time. It only increases when $\pi(u^Y_t)$ belongs to $\partial M$.

Consider a smooth real-valued function $h(t,x)$ on $[0,\infty) \times M$. Applying (3.12) with $f(t,u) = h(t,\pi(u))$, we obtain an equation for the process $h(t,X^y_t)$. This simple observation is important to the proofs of Theorems 3.1, 3.2, and 3.3. It is also used for establishing Proposition 4.1 in the next section. When $f(t,u) = h(t,\pi(u))$, the formulas

$$\Delta_{O(M)}f(t,u) = \Delta_M h(t,x)|_{x=\pi(u)},$$

$$\langle\mathcal{N}f\rangle(t,u) = \frac{\partial}{\partial \nu} h(t,x)|_{x=\pi(u)}, \quad t \in [0,\infty), \; u \in O(M),$$

(3.13)

hold true.

Let $g(t,x,y)$ denote the transition density of the reflecting Brownian motion $X^y_t$. The function $\tilde{g}_y(t,x) = g(2t,x,y)$ is a smooth positive solution to the heat equation (2.1) with the Neumann boundary condition (2.2). Note that the density $g(t,x,y)$ will be playing a significant role in our further considerations. The estimates required to establish Theorems 3.1, 3.2, and 3.3 rely on those known for $g(t,x,y)$.

All the probabilistic objects we will need are now at hand. Introduce the notation

$$q(t,x) = \left|R^{\nu(t)}(x)\right|^2_E, \quad t \in [0,T), \; x \in M.$$

Given $r \in (0,T)$, define

$$\zeta^{r,y}(t) = \int_M q(r-t,x) g(t,x,y) \, dx, \quad t \in (0,r].$$
The quantity $\zeta^-(t)$ may be interpreted as $\mathbb{E}(q(r-t,X_t^r))$. Applying Remark 2.5 to the function $g_y(t,x)$ and taking the monotonicity of $\text{YM}(t)$ into account, one concludes that

$$\zeta^-(t) \leq C_1 t^{-\frac{\dim M}{2}} \text{YM}(0), \quad t \in (0, \min\{r,1\}], \quad (3.14)$$

with $C_1 > 0$ determined by (2.5). We are now in a position to prove Theorems 3.1 and 3.2. Two more lemmas are required to consider the case where $\dim M$ is 5 or higher. We will state them afterwards.

**Proof of Theorem 3.1.** Fix $\rho \in (0,T)$. Our goal is to obtain a bound on $\sup_{x \in M} q(\rho, x)$. Choose $\alpha \in (0,1)$ and denote $\rho_0 = \max\{0, \rho - \frac{1}{\alpha}\}$. Let the number $\sigma_0 \in (0, \rho - \rho_0]$ satisfy the equality

$$\sigma_0^2 = \sup_{t \in [0,\rho_0]} \sup_{x \in M} q(t, x) = \sup_{\alpha \in [0,\rho_0]} \left( \sigma_0^2 \sup_{t \in [0,\rho_0]} \sup_{x \in M} q(t, x) \right). \quad (3.15)$$

There exist $t_* \in [\rho_0 + \sigma_0, \rho]$ and $x_* \in M$ such that

$$q(t_*, x_*) = \sup_{t \in [\rho_0 + \sigma_0, \rho]} \sup_{x \in M} q(t, x). \quad (3.16)$$

It is convenient for us to write $q_0$ instead of $q(t_*, x_*)$. Our next step is to estimate the number $q_0$. The desired bound on $\sup_{x \in M} q(\rho, x)$ will follow therefrom.

Using the heat equation (3.2) and the Bochner-Weitzenböck formula, we can prove the existence of a constant $C_2 > 0$ such that

$$\left(\frac{\partial}{\partial t} - \frac{1}{2} \Delta_M\right) q(t, x) \leq C_2 \left(1 + \sqrt{q(t, x)}\right) q(t, x) \quad (3.17)$$

for $t \in [0, T)$ and $x \in M$; see [8, Lemma 2.2]. The definition of $\sigma_0$ implies

$$\sup_{t \in [t_* - \alpha \sigma_0, t_*]} \sup_{x \in M} q(t, x) \leq \sup_{t \in [\rho_0 + (1 - \alpha) \sigma_0, \rho]} \sup_{x \in M} q(t, x) \leq \frac{\sigma_0^2}{(1 - \alpha)^2 \sigma_0^2} \sup_{t \in [\rho_0, \rho_0 + \sigma_0, \rho]} \sup_{x \in M} q(t, x) = \tilde{\alpha}^2 \sigma_0 \quad (3.18)$$

with $\tilde{\alpha} = \frac{1}{1 - \alpha}$. Inequalities (3.17) and (3.18) will play an essential role in estimating the number $q_0$. Let $u_t^Y$ be a horizontal reflecting Brownian motion in the bundle $O(M)$. We suppose $u_t^Y$ starts at a frame $Y$ satisfying $\pi(Y) = x_*$. Define $X_t^{x_*} = \pi(u_t^Y)$ and consider the process

$$Z_t = e^{C_2(1+\tilde{\alpha} \sqrt{\sigma_0})} q(t_*, t, X_t^{x_*})$$

for $t \in [0, \alpha \sigma_0)$. Formulas (3.12) and (3.13) yield

$$q_0 = Z_0 = \mathbb{E}(Z_t) = \mathbb{E} \left( \int_0^t \left( -\frac{\partial}{\partial r} + \frac{1}{2} \Delta_M \right) e^{C_2(1+\tilde{\alpha} \sqrt{\sigma_0})} q(r, X_t^{x_*}) \big|_{r=t_*-s} \right) \right) ds \right) + \mathbb{E} \left( \int_0^t e^{C_2(1+\tilde{\alpha} \sqrt{\sigma_0})} \frac{\partial}{\partial r} q(t_* - s, X_s^{x_*}) dL_s \right). \quad (3.19)$$
In view of (3.17), (3.18), and Lemma 3.5, this implies $q_0 \leq \mathbb{E}(Z_t)$ for $t \in [0, \alpha \sigma_0)$. As a consequence, the formula

$$q_0 \leq e^{C_2(1+\hat{\alpha}\sqrt{\sigma}) t} \zeta^{t,x_0}(t), \quad t \in [0, \alpha \sigma_0), \quad (3.19)$$

holds true. We will now use it to prove that

$$\sup_{x \in M} q(\rho, x) \leq \max \left\{ \frac{Y M(0)}{\alpha^2 \rho^2}, \theta_1 e^{\theta_2, \alpha \sqrt{Y M(0)}} Y M(0) \right\} \quad (3.20)$$

with $\theta_1 > 0$ and $\theta_{2,\alpha} > 0$. Estimate (3.19) will follow by looking at the case where $\alpha = \frac{1}{2}$.

Let us assume $q_0 > 0$ and $Y M(0) > 0$. This does not lead to any loss of generality. Indeed, if $q_0 = 0$, then the supremum $\sup_{x \in M} q(\rho, x)$ is equal to 0 and (3.20) holds for any $\theta_1$ and $\theta_{2,\alpha}$. When $Y M(0) = 0$, we have $Y M(t_*) = 0$ due to the fact that $Y M(t)$ is non-increasing in $t \in [0, T)$. In this case, $q_0$ equals 0, and (3.20) is again satisfied for any $\theta_1$ and $\theta_{2,\alpha}$.

Denote $t_0 = \sqrt{\frac{Y M(0)}{q_0}}$. If $t_0 \geq \alpha \sigma_0$, then

$$(\rho - \rho_0)^2 \sup_{x \in M} q(\rho, x) \leq \sigma_0^2 q_0 \leq \frac{Y M(0)}{\alpha^2}$$

by virtue of the definitions of $\sigma_0$ and $t_0$. In this case, the estimate

$$\sup_{x \in M} q(\rho, x) \leq \frac{Y M(0)}{\alpha^2 (\rho - \rho_0)^2} = \frac{Y M(0)}{\alpha^2 \left( \min \{ \rho, \frac{1}{\alpha} \} \right)^2} = \max \left\{ \frac{Y M(0)}{\alpha^2 \rho^2}, Y M(0) \right\} \quad (3.21)$$

holds true, which means (3.20) is satisfied for all $\theta_1 \geq 1$ and $\theta_{2,\alpha} > 0$. If $t_0 < \alpha \sigma_0$ (note that $\alpha \sigma_0 \leq \alpha (\rho - \rho_0) \leq 1$), then formulas (3.19) and (3.14) yield

$$q_0 \leq e^{C_2(1+\hat{\alpha}\sqrt{\sigma}) t_0} \zeta^{t_0,x_0}(t_0) \leq e^{C_2 \hat{\alpha} \sqrt{Y M(0)} (C \frac{\dim M}{4} + \frac{Y M(0)}{4})} Y M(0)$$

with $\hat{C} = e^{C_2} C_1$. Hence

$$\sup_{x \in M} q(\rho, x) \leq q_0 \leq \left( e^{C_2 \hat{\alpha} \sqrt{Y M(0)} (C \frac{\dim M}{4} + \frac{Y M(0)}{4})} \right)^{\frac{4}{\dim M}}$$

$$= \left( e^{C_2 \hat{\alpha} \sqrt{Y M(0)} \hat{C}} \right)^{\frac{4}{\dim M}} Y M(0).$$

Combined with (3.21), this estimate shows that (3.20) holds for

$$\theta_1 = \max \left\{ \hat{C}^{\frac{4}{\dim M}}, 1 \right\}, \quad \theta_{2,\alpha} = \frac{4}{4 - \dim M} C_2 \hat{\alpha}.$$

We now assume $\alpha = \frac{1}{2}$. The desired result follows at once. The role of the constant $\theta_2$ is to be played by $\theta_2 \frac{1}{2}$. \qed
Remark 3.6. While proving the theorem, we have actually established a stronger result. Namely, take a number $$\alpha$$ from the interval $$(0, 1)$$. Suppose the conditions of Theorem 3.1 are satisfied. Then the estimate
$$\sup_{x \in M} \left| R^{(\rho)}(x) \right|^2_E \leq \max \left\{ \frac{YM(0)}{\alpha^2 \rho^2}, \theta_1 e^{\theta_2 \sqrt{YM(0)}}, \theta_2 \right\}, \quad \rho \in (0, T),$$
holds true. In the right-hand side, $$\theta_1 > 0$$ is a constant depending only on $$M$$, whereas $$\theta_2 > 0$$ is determined by $$\alpha$$ and $$M$$. When formulating Theorem 3.1, we restricted our attention to the case where $$\alpha = \frac{1}{2}$$. This was done for the sake of simplicity and understandability.

Proof of Theorem 3.2. Fix $$\rho \in (0, T)$$, $$\alpha \in (0, 1)$$, and $$\beta \in (0, 1)$$. Denote $$\rho_0 = \max \{0, \rho - \frac{1}{\alpha}\}$$. Let $$\sigma_0 \in (0, \rho - \rho_0]$$, $$t_* \in [\rho_0 + \sigma_0, \rho]$$, and $$x_* \in M$$ satisfy Eqs. (3.15) and (3.16). We write $$q_0$$ instead of $$q(t_*, x_*)$$. Our next step is to demonstrate that
$$\sigma_0^2 q_0 \leq \frac{YM(0)^\beta}{\alpha^2}$$
provided $$YM(0)$$ is smaller than a number $$\xi_{\alpha, \beta} > 0$$ depending only on $$\alpha$$, $$\beta$$, and the manifold $$M$$. The assertion of the theorem will be deduced from this estimate.

Suppose $$YM(0) = 0$$. Then $$YM(t) = 0$$ due to the monotonicity of $$YM(t)$$ in $$t \in [0, T)$$. Ergo, $$q_0$$ is equal to 0. It becomes evident that $$\sigma_0^2 q_0 = \frac{YM(0)^\beta}{\alpha^2}$$. We have thus proved (3.22) in the case where $$YM(0) = 0$$. Let us consider the general situation. Assume (3.22) fails to hold. Then $$q_0 > 0$$, $$YM(0) > 0$$, and the number $$t' = \sqrt{\frac{YM(0)^\beta}{q_0}}$$ lies in the interval $$(0, \alpha \sigma_0) \subset (0, 1)$$. Repeating the arguments from the proof of Theorem 3.1 and using (3.14), we conclude that the inequality
$$q_0 \leq e^{C_2(1 + \hat{\alpha} \sqrt{YM(0)} t')} e^{C_2(1 + \hat{\alpha} \sqrt{YM(0)} t')} \leq e^{C_2(1 + \hat{\alpha} \sqrt{YM(0)} t') \xi_{\alpha, \beta}}$$
must be satisfied. Here, $$\hat{\alpha}$$ stands for $$\frac{1}{1 - \alpha}$$. The constant $$\hat{C}$$ appears as $$e^{C_2 C_1}$$. It is easy to see, however, that the above inequality fails when
$$YM(0) < \xi_{\alpha, \beta} = \min \left\{ e^{C_2(1 + \hat{\alpha} \sqrt{YM(0)} t')}, 1 \right\}.$$
This contradiction establishes (3.22) under the condition $$YM(0) < \xi_{\alpha, \beta}$$. In order to complete the proof of the theorem, we estimate $$\sup_{x \in M} q(\rho, x)$$. The definition of $$\sigma_0$$ suggests that
$$(\rho - \rho_0)^2 \sup_{x \in M} q(\rho, x) \leq \sigma_0^2 q_0.$$In view of (3.22), this implies
$$\sup_{x \in M} q(\rho, x) \leq \frac{YM(0)^\beta}{\alpha^2 (\rho - \rho_0)^2}$$
$$= \frac{YM(0)^\beta}{\alpha^2 \left( \min \left\{ \rho, \frac{1}{\alpha} \right\} \right)^2} = \max \left\{ \frac{YM(0)^\beta}{\alpha^2 \rho^2}, YM(0)^\beta \right\}$$
provided $YM(0) < \xi_{\alpha,\beta}$. The assertion of the theorem follows by assuming $\alpha = \beta = \frac{1}{2}$. Inequality (3.6) holds when $YM(0) < \xi_{\frac{1}{2},\frac{1}{2}}$.

**Remark 3.7.** In the course of the proof, we have actually established a result stronger than Theorem 3.2. Namely, fix $\alpha \in (0,1)$ and $\beta \in (0,1)$. Suppose the conditions of Theorem 3.2 are satisfied. If $YM(0)$ is smaller than $\xi_{\alpha,\beta}$, then the estimate

\[
\sup_{x \in M} |R^\nabla(\rho)(x)|_E^2 \leq \max \left\{ \frac{YM(0)^\beta}{\alpha^2 \rho^2}, YM(0)^\beta \right\}, \quad \rho \in (0,T),
\]

holds true. Here, $\xi_{\alpha,\beta}$ is a number depending on $\alpha, \beta, \text{and} \ M$. When formulating Theorem 3.2, we restricted our attention to $\alpha = \beta = \frac{1}{2}$. This was done in order to make the statement more understandable.

Let us concentrate on the case where $\dim M$ is 5 or higher. First of all, we need a few auxiliary identities. Their purpose is to help us obtain a monotonicity formula related to the Yang-Mills heat equation (3.2). We establish these identities in Lemma 3.8 below. The proof is quite transparent yet worthy of attention. It demonstrates vividly how the boundary conditions imposed on $R^\nabla(t)$ interact with those satisfied by $g(t,x,y)$. In a way, this interplay of boundary conditions explains why the Brownian motion used to implement the probabilistic technique in our context should be reflected at $\partial M$.

Desiring to remain at the higher level of abstraction, we state Lemma 3.8 for a generic $\text{Ad}_E$-valued form $\phi$ and a generic function $f(x)$ on $M$. In our further arguments, it will be applied with $\phi$ equal to the curvature $R^\nabla(r-t)$ and $f(x)$ equal to the density $g(t,x,y)$.

**Lemma 3.8.** Let $\nabla$ be a connection in $E$ compatible with the structure group $G$. Suppose $f(x)$ is a real-valued function on $M$ such that $\frac{\partial}{\partial t} f(x) = 0$ on $\partial M$. Consider an $\text{Ad}_E$-valued $p$-form $\phi \in \Omega^p(\text{Ad}_E)$ with $p = 1, \ldots, \dim M$. If either Eqs. (3.8) or Eqs. (3.9) are satisfied for $\phi$ on $\partial M$, then the following formulas hold true:

\[
\begin{align*}
\int_M |\phi|_E \Delta_M f \, dx &= - \int_M \langle \text{grad} \, |\phi|_E, \text{grad} \, f \rangle \, dx, \\
\int_M \langle d \nabla d^*_\nabla \phi, f \phi \rangle_E \, dx &= \int_M \langle d^*_\nabla \phi, d \nabla (f \phi) \rangle_E \, dx, \\
\int_M \langle d \nabla (\iota_{\text{grad log} \, f} \phi), f \phi \rangle_E \, dx &= \int_M \langle \iota_{\text{grad log} \, f} \phi, d^*_\nabla (f \phi) \rangle_E \, dx.
\end{align*}
\]

**Proof.** The first identity in (3.23) is a direct consequence of the Stokes theorem and the fact that $\frac{\partial}{\partial t} f(x) = 0$. The second one can be deduced from Lemma 3.4 in a straightforward fashion. Notably, the same argument has to be used when proving $YM(t)$ is non-increasing in $t \in [0,T)$; see [7]. We will now establish the third identity in (3.23).

Let us assume Eqs. (3.8) are satisfied for $\phi$. The case where Eqs. (3.9) are satisfied instead can be treated similarly. We will show that the scalar product
\( \langle \iota_{\text{grad} \log f} \phi, \iota_\nu (f \phi) \rangle_E \) vanishes on \( \partial M \). In view of Lemma 3.8, the third identity in (3.23) would follow from this fact as an immediate consequence.

Observe that the formula \( \frac{\partial}{\partial \nu} f(x) = 0 \) implies \( \frac{\partial}{\partial \nu} \log f(x) = 0 \). Accordingly, the gradient \( \text{grad} \log f \) is tangent to \( \partial M \) at every point of \( \partial M \). This allows us to assume \( \phi \) belongs to \( \Omega^p(\text{Ad} E) \) with \( p \) between 2 and \( \dim M \). Indeed, if \( \phi \) is an \( \text{Ad} E \)-valued 1-form on \( M \), then \( \iota_{\text{grad} \log f} \phi = 0 \) due to the first formula in (3.8).

Take a point \( \tilde{x} \in \partial M \). Choose an orthonormal basis \( \{e_1, \ldots, e_n\} \) of the tangent space \( T_{\tilde{x}}M \) demanding that \( e_n \) coincide with \( \nu \). The equality \( \langle \iota_{\text{grad} \log f} \phi, \iota_\nu (f \phi) \rangle_E = \sum \langle \phi (\text{grad} \log f, e_{i_1}, \ldots, e_{i_p-1}), f \phi (\nu, e_{i_1}, \ldots, e_{i_p-1}) \rangle_E \) holds at \( \tilde{x} \). The summation is to be carried out over all the arrays \( (i_1, \ldots, i_{p-1}) \) with \( 1 \leq i_1 < \cdots < i_{p-1} \leq n \). It is easy to see that \( f \phi (\nu, e_{i_1}, \ldots, e_{i_{p-1}}) \) vanishes when \( i_{p-1} = n \). At the same time, \( \phi (\text{grad} \log f, e_{i_1}, \ldots, e_{i_{p-1}}) \) vanishes when \( i_{p-1} < n \) because \( \text{grad} \log f \) is tangent to \( \partial M \) and \( \phi_{\tan} = 0 \). We conclude that the scalar product \( \langle \iota_{\text{grad} \log f} \phi, \iota_\nu (f \phi) \rangle_E \) equals 0 at \( \tilde{x} \). Hence the third identity in (3.23). \( \square \)

The following lemma states a monotonicity formula related to the Yang-Mills heat equation (3.2). It is an important step in establishing Theorem 3.3 by means of the probabilistic technique. We emphasize that the proof of the lemma requires the Li-Yau-Hamilton estimate obtained in Section 2. For relevant results, see [2] and also [17, 8].

**Lemma 3.9.** Let the boundary \( \partial M \) be totally geodesic. Suppose either Assumption [1] of Theorem 2.1 or Assumptions [1] and [2] of Theorem 2.6 are fulfilled for \( M \). Given \( r \in (0, T) \) along with \( y \in M \), the formula

\[
\zeta^{r,y}(t_1) \leq \frac{1}{\epsilon_1} \left( \epsilon_2 u(t_1) \zeta^{r,y}(t_2) + C_3 (t_2 - t_1) \text{YM}(0) \right)
\]

holds for all \( t_1, t_2 \in (0, \min\{r, 1\}) \) satisfying \( t_1 < t_2 \). Here, \( u(t) \) is a positive increasing function on \( (0, 1] \) such that \( \lim_{t \to 0} u(t) = 0 \), and \( C_3 > 0 \) is a constant. Both \( u(t) \) and \( C_3 \) are determined solely by the manifold \( M \).

**Proof.** First, suppose Assumption [1] of Theorem 2.1 is satisfied. We will consider the other case later. Proposition 3.4 and Theorem 3.7 in [2] prove the assertion of the lemma on closed manifolds. The same line of reasoning works in our situation. However, two points need to be clarified:

- The equality between expressions (3.10) and (3.11) of [2] holds in our setting due to Lemma 3.8. The same can be said about expressions (3.14) and (3.15) of that paper.

- In order to obtain estimate (3.22) of [2] for the Neumann heat kernel \( g(t, x, y) \), one should apply formula (2.3) above to the function \( \tilde{g}_y(t, x) = g(2t, x, y) \).

25
The other arguments from the proofs of Proposition 3.4 and Theorem 3.7 in [2] work in our situation without significant modifications.

We now consider the case when there are curvature restrictions imposed on \( M \) away from the boundary. More specifically, suppose Assumptions [1] and [2] of Theorem 2.6 are satisfied. Then the assertion of the lemma can be established by repeating the arguments from the proofs of Proposition 3.4 and Theorem 3.7 in [2]. The required estimate on \( g(t, x, y) \) comes from formula (2.6) in the present paper applied to the function \( \tilde{g}_0(t, x) \).

We are now ready to prove Theorem 3.3. Afterwards, three important remarks will be made.

**Proof of Theorem 3.3** Fix \( \rho \in (0, T) \), \( \alpha \in (0, 1) \), and \( \beta \in (0, 1) \). We denote \( \rho_0 = \max \{(1-\alpha)\rho, \rho - \frac{1}{\alpha}\} \). Let \( \sigma_0 \in (0, \rho - \rho_0] \), \( t_\ast \in [\rho_0 + \sigma_0, \rho] \), and \( x_\ast \in M \) obey Eqs. (3.15) and (3.16). Set \( q_0 = q(t_\ast, x_\ast) \). We will show that

\[
\sigma_0^2 q_0 \leq \frac{\text{YM}(0)^\beta}{\alpha^2}
\]  

provided \( \text{YM}(0) \) is smaller than a certain value \( \xi_{\alpha, \beta}(\rho) \) depending on \( \rho \) as a non-decreasing function. The assertion of the theorem will be deduced from this estimate. Note that, aside from \( \rho \), the value \( \xi_{\alpha, \beta}(\rho) \) only depends on \( \alpha \), \( \beta \), and the manifold \( M \).

Suppose \( \text{YM}(0) = 0 \). Then \( \text{YM}(t_\ast) = 0 \) due to the monotonicity of \( \text{YM}(t) \) in \( t \in [0, T) \). As a consequence, \( q_0 \) is equal to 0. We conclude that (3.24) is satisfied when \( \text{YM}(0) = 0 \).

Denote \( T_0 = \min \{\rho_0 + \alpha \sigma_0, 1\} \). Observe that \( \alpha \sigma_0 \leq T_0 < t_\ast \). This fact is essential because it will allow us to apply Lemma 3.9 further in the proof. Assume estimate (3.24) fails to hold. Then \( q_0 > 0 \), \( \text{YM}(0) > 0 \), and the number \( t' = \sqrt{\frac{\text{YM}(0)^\beta}{q_0}} \) lies in the interval \((0, \alpha \sigma_0) \subset (0, 1)\). The arguments from the proof of Theorem 2.6 yield

\[
q_0 \leq e^{C_2 \left(1 + \alpha \sqrt{q_0}\right)} \left( \zeta^{t_\ast, x_\ast} (t') \right) \leq e^{C_2 \sqrt{\text{YM}(0)^\beta}} \left( \zeta^{t_\ast, x_\ast} (t') \right).
\]

Here, the number \( \tilde{\alpha} \) equals \( \frac{1}{1-\alpha} \). Lemma 3.9 implies

\[
\zeta^{t_\ast, x_\ast} (t') \leq C' \frac{q_0^{\alpha}}{\text{YM}(0)^{\beta}} \left( T_0^2 \zeta^{t_\ast, x_\ast} (T_0) + T_0 \text{YM}(0) \right)
\]

with \( C' = \max \{e^{n(1)}, C_3\} \). (Note that Theorems 2.1 and 2.6 are being used at this point. More precisely, the proof of Lemma 3.9 relies on them.) Formula (3.14) and the definition of \( T_0 \) enable us to conclude that

\[
q_0 \leq \frac{e^{C_2 \sqrt{\text{YM}(0)^\beta} \left( C' \frac{q_0^{\alpha}}{\text{YM}(0)^{\beta}} \right)}}{\text{YM}(0)^{\beta}} \left( T_0^2 \zeta^{t_\ast, x_\ast} (T_0) + T_0 \text{YM}(0) \right)
\]

\[
\leq e^{C_2 \sqrt{\text{YM}(0)^\beta}} C' \frac{q_0}{\text{YM}(0)^{\beta}} \left( C_1 T_0^{-\frac{\dim M}{2}} \text{YM}(0)^{1-\beta} + \text{YM}(0)^{1-\beta} \right)
\]

\[
\leq e^{C_2 \sqrt{\text{YM}(0)^\beta}} C' q_0 \text{YM}(0)^{1-\beta} \left( C_1 \left( \min \{(1-\alpha)\rho, 1\} \right)^2 \dim M + 1 \right)
\]

26
with $C'' = e^{C_2}C'$. However, this is impossible when

$$\text{YM}(0) < \xi_{\alpha,\beta}(\rho) = \min \left\{ \xi^1_{\alpha,\beta}(\rho), \xi^2_{\alpha,\beta}(\rho), 1 \right\},$$

$$\xi^1_{\alpha,\beta}(\rho) = \left( 2e^{C_2}C_1 (\min \{(1 - \alpha)\rho, 1\})^{2 - \frac{\dim M}{\alpha}} \right)^{-\frac{1}{1-\beta}},$$

$$\xi^2_{\alpha,\beta}(\rho) = (2e^{C_2}C''(\rho))^{-\frac{1}{1-\beta}}.$$

The present contradiction establishes (3.24) under the condition $\text{YM}(0) < \xi_{\alpha,\beta}(\rho)$.

To complete the proof of the theorem, we need to estimate $\sup_{x \in M} q(\rho, x)$. The definition of $\sigma_0$ suggests that

$$(\rho - \rho_0)^2 \sup_{x \in M} q(\rho, x) \leq \sigma_0^2 q_0.$$ 

According to formula (3.24), this implies

$$\sup_{x \in M} q(\rho, x) \leq \frac{\text{YM}(0)^{\beta}}{\alpha^2 (\rho - \rho_0)^2} \leq \frac{\text{YM}(0)^{\beta}}{\alpha^2 \left( \min \{\alpha \rho, \frac{1}{\alpha}\} \right)^2} = \max \left\{ \frac{\text{YM}(0)^{\beta}}{\alpha^4 \rho^2}, \text{YM}(0)^{\beta} \right\}$$

provided $\text{YM}(0) < \xi_{\alpha,\beta}(\rho)$. We now assume $\alpha = \beta = \frac{1}{2}$. The assertion of the theorem follows at once. Inequality (3.7) holds when $\text{YM}(0) < \xi(\rho) = \xi_{\frac{1}{2}, \frac{1}{2}}(\rho)$.

Remark 3.10. While proving the theorem, we have really established a stronger result. That is, suppose $\alpha \in (0, 1)$ and $\beta \in (0, 1)$. Let the conditions of Theorem 3.3 be fulfilled. Given $\rho \in (0, T)$, if $\text{YM}(0)$ is smaller than $\xi_{\alpha,\beta}(\rho)$, then the estimate

$$\sup_{x \in M} \left| \nabla^{(\rho)}(x) \right|_E^2 \leq \max \left\{ \frac{\text{YM}(0)^{\beta}}{\alpha^4 \rho^2}, \text{YM}(0)^{\beta} \right\}$$

is satisfied. Here, $\xi_{\alpha,\beta}(s)$ is a positive non-decreasing function on $(0, \infty)$ entirely determined by $\alpha, \beta$, and $M$. In the formulation of Theorem 3.3, we only dealt with the case where $\alpha = \beta = \frac{1}{2}$. This specific framework was meant to make the statement more understandable.

Remark 3.11. In the beginning of Section 3, we imposed the boundary conditions (3.3) or (3.4) on the curvature form $R^{(\nabla(t))}$. Another approach is feasible. Namely, one may formulate the boundary conditions for the connection $\nabla(t)$ directly. The paper [7] takes this particular standpoint; see also [26, 15]. It may or may not be more natural to impose the boundary conditions on $\nabla(t)$ than to impose ones on $R^{(\nabla(t))}$ depending on the considered problem and the chosen perspective. However, the approach adopted in the present paper seems to be technically simpler. The reason for this lies in the fact that, unlike $\nabla(t)$, the curvature form $R^{(\nabla(t))}$ transforms as a tensor under changes of coordinates. In particular, it is meaningful to talk about the tangential and the normal components of $R^{(\nabla(t))}$. 
Remark 3.12. In several situations, imposing the boundary conditions on the connection is virtually equivalent to imposing ones on its curvature form. Let us present an example. If a time-dependent connection satisfies the heat equation (3.2) and the conductor boundary condition in the sense of [7], then formulas (3.3) can be proved for its curvature. The converse statement holds with an adjustment. Roughly speaking, the first formula in (3.3) ensures that $\nabla(t)$ can be gauge transformed locally into a connection satisfying the conductor boundary condition. We refer to [7] for further details.

4 An exit time estimate on manifolds with convex boundary

Let $u^Y_t$ be a horizontal reflecting Brownian motion on the bundle $O(M)$. It is assumed that this process starts at the frame $Y \in O(M)$. We consider $u^Y_t$ on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0, \infty)}, P)$ satisfying the “usual hypotheses.” The definition and the basic properties of a horizontal reflecting Brownian motion on $O(M)$ were discussed in Section 3.

As before, we set $X^y_t = \pi(u^Y_t)$ with $y = \pi(Y)$. The process $X^y_t$ is a reflecting Brownian motion on $M$ starting at the point $y$. This section offers an exit time estimate for $X^y_t$ under the assumption that $\partial M$ is convex. Basically, we obtain an analogue of Lemma 4.1 in [2]; cf. [19, Theorem 3.6.1]. This result enables us to prove another estimate for the curvature of the solution $\nabla(t)$ to the Yang-Mills heat equation (3.2). More precisely, we will establish an analogue of Theorem 4.2 in [2] for manifolds with boundary.

Additional notation should be introduced at this stage. Let $\text{dist}_y(x)$ stand for the distance between $y$ and $x \in M$ with respect to the Riemannian metric on $M$. Given a radius $r > 0$, consider the ball $B(y, r) = \{ x \in M \mid \text{dist}_y(x) < r \}$. Its closure will be denoted by $\overline{B}(y, r)$.

Define

$$\tau(y, r) = \inf \{ t \geq 0 \mid X^y_t \in M \setminus \overline{B}(y, r) \}.$$ 

In other words, $\tau(y, r)$ is the first exit time of the reflecting Brownian motion $X^y_t$ from $\overline{B}(y, r)$. We now lay down an estimate for $\tau(y, r)$.

**Proposition 4.1.** Let the boundary $\partial M$ be convex in the sense of (3.7). There exist constants $\kappa_0 > 0$ and $\eta > 0$ depending only on $M$ such that the estimate

$$\mathbb{P} \{ \tau(y, r) < \kappa r^2 \} \leq e^{-\eta \kappa}$$

holds for every $r \in (0, 1)$ and $\kappa \in (0, \kappa_0)$.

The proof of the proposition will be based on Lemma 4.2 below and on Bernstein’s inequality related to local martingales. We should emphasize that $\kappa_0$ and $\eta$ are fully determined by the manifold $M$. In particular, they do not depend on the starting point $y$ of the reflecting Brownian motion $X^y_t$. 

28
Consider the squared distance function \( \text{dist}_y^2(x) = (\text{dist}_y(x))^2 \) on \( M \). The following lemma discusses the analytical features of \( \text{dist}_y^2(x) \). Generally speaking, the behavior of the squared distance function on a manifold with boundary is quite complicated. The dissertation [42] offers a series of results on the subject and a detailed description of relevant literature. The paper [1] is a slightly more recent reference. In our particular situation, however, \( \text{dist}_y^2(x) \) behaves nicely because \( \partial M \) is assumed to be convex. Many properties of \( \text{dist}_y^2(x) \) resemble those of the squared distance function on a closed manifold.

**Lemma 4.2.** Let the boundary \( \partial M \) be convex in the sense of (3.1). There exists a constant \( \epsilon_M > 0 \) independent of \( y \) such that the following statements are satisfied:

1. The squared distance function \( \text{dist}_y^2(x) \) is smooth in \( x \) on the ball \( B(y, \epsilon_M) \).

2. The normal derivative \( \frac{\partial}{\partial \nu} \text{dist}_y^2(x) \) is nonnegative for all \( x \in B(y, \epsilon_M) \cap \partial M \).

3. There is a constant \( K_{\epsilon_M} > 0 \) independent of \( y \) such that \( \Delta_M \text{dist}_y^2(x) \) is less than or equal to \( K_{\epsilon_M} \epsilon_M \) for all \( x \in B(y, \epsilon_M) \).

**Proof.** Our reasoning will be based on embedding \( M \) isometrically into a smooth connected Riemannian manifold \( N \) without boundary. Let us introduce some notation and state a definition. We write \( \text{dist}_N(z', z) \) for the distance from \( z' \in N \) to \( z \in N \) with respect to the Riemannian metric on \( N \). Accordingly, \( B_N(z', r) \) denotes the ball \( \{z \in N \mid \text{dist}_N(z', z) < r\} \) in \( N \) of radius \( r > 0 \). A set \( Q \subset N \) is said to be strongly convex in \( N \) if every pair of distinct points from \( Q \) can be joined by a unique (up to parametrization) minimizing geodesic segment that lies in \( Q \).

The first step of the proof is to specify the constant \( \epsilon_M \). Afterwards, we will establish Statements 1, 2, and 3 for this constant. There exist a smooth connected Riemannian manifold \( N \) without boundary and a mapping \( e \) from \( M \) to \( N \) such that the following requirements are satisfied:

- The dimension \( \dim N \) is equal to \( \dim M \).
- The mapping \( e \) is an isometric embedding.
- For every \( z \in e(M) \), there is a number \( \epsilon_1(z) > 0 \) such that the set \( B_N(z, \epsilon_1(z)) \cap e(M) \) is strongly convex in \( N \).

The existence of \( N \) and \( e \) is a standard consequence of (3.1); see, for instance, [23] and also [35]. In identifying \( \epsilon_M \), it will be more convenient for us to work with the image \( e(M) \) than with \( M \) itself.

We need to state one basic fact about the manifold \( N \). Namely, given a point \( z' \in N \), there is a number \( \epsilon_2(z') > 0 \) such that the following requirement is fulfilled: For every \( z'' \in B_N(z', \epsilon_2(z')) \), the inverse exponential map \( \exp_{z'}^{-1} \) is a diffeomorphism from \( B_N(z', \epsilon_2(z')) \) onto \( \exp_{z'}^{-1}(B_N(z', \epsilon_2(z'))) \). It is easy to see that \( \text{dist}_{N, z'}(z) \) is smooth in \( z \) on the set \( B_N(z', \epsilon_2(z')) \setminus \{z''\} \), and so is the
function \( \text{dist}^2_{N;\epsilon}(z) = (\text{dist}_{N;\epsilon}(z))^2 \) on the ball \( B_N(z',\epsilon_2(z')) \). This concludes the preparations we had to make before identifying \( \epsilon_M \).

Set \( \epsilon_M(z) \) equal to \( \frac{1}{4} \min\{\epsilon_1(z), \epsilon_2(z)\} \) for \( z \in e(M) \). The open cover \( (B_N(z, \epsilon_M(z)))_{z \in e(M)} \) of the compact set \( e(M) \) has a finite subcover \( (B_N(z_i, \epsilon_M(z_i)))_{i=1,\ldots,m} \). Define

\[
\epsilon_M = \min \{\epsilon_M(z_1), \ldots, \epsilon_M(z_m)\}. \tag{4.1}
\]

For every point \( z \in e(M) \), there is an index \( i \) between 1 and \( m \) such that the formulas

\[
B_N(z, \epsilon_M) \subset B_N(z_i, \epsilon_1(z_i)),
\]
\[
B_N(z, \epsilon_M) \subset B_N(z_i, \epsilon_2(z_i)) \tag{4.2}
\]

hold true. This obvious fact plays an important role in our further arguments.

We will now prove Statement 1 of the lemma for the constant \( \epsilon_M \) specified by (11). Using the first inclusion in (12), one can show that the image \( e(B(y, \epsilon_M)) \) coincides with \( B_N(e(y), \epsilon_M) \cap e(M) \). Furthermore, the equality

\[
\text{dist}_y(x) = \text{dist}_{N;e(y)}(e(x)), \quad x \in B(y, \epsilon_M), \tag{4.3}
\]

is satisfied. The second inclusion in (12) implies that the function \( \text{dist}_{N;e(y)}(z) \) is smooth on the ball \( B_N(e(y), \epsilon_M) \). The embedding \( e \) is a diffeomorphism onto its image. As a result, \( \text{dist}^2_y(x) \) must be smooth on \( e^{-1}(B_N(e(y), \epsilon_M)) \). Since \( e(B(y, \epsilon_M)) \) coincides with \( B_N(e(y), \epsilon_M) \cap e(M) \), the preimage \( e^{-1}(B_N(e(y), \epsilon_M)) \) is equal to \( B(y, \epsilon_M) \). Hence the desired smoothness of \( \text{dist}^2_y(x) \).

Let us establish Statement 2 of the lemma. In order to prove the nonnegativity of \( \frac{\partial}{\partial \epsilon} \text{dist}^2_y(x) \), we need to compute the gradient \( \text{grad} \text{dist}^2_y(x) \). Formula (4.3) yields

\[
\text{grad} \text{dist}^2_y(x) = (de)^{-1} \left( \text{grad} \text{dist}^2_{N;e(y)}(z) \right) \bigg|_{z=e(x)} \tag{4.4}
\]

when \( x \in B(y, \epsilon_M) \). Our next step is to identify \( \text{grad} \text{dist}^2_{N;e(y)}(z) \) on the image of the ball \( B(y, \epsilon_M) \) under the embedding \( e \).

As stated above, \( e(B(y, \epsilon_M)) \) is equal to \( B_N(e(y), \epsilon_M) \cap e(M) \). By virtue of the first inclusion in (12), this fact implies the existence of an index \( i \) between 1 and \( m \) such that \( e(B(y, \epsilon_M)) \) is contained in \( B_N(z_i, \epsilon_1(z_i)) \cap e(M) \). The latter set is strongly convex in \( N \). In consequence, the following property must hold: For every \( z \in e(B(y, \epsilon_M)) \setminus e(\{y\}) \), there is a minimizing geodesic segment \( \gamma_z(s) \) that starts at \( z \), ends at \( e(y) \), and lies in \( B_N(z_i, \epsilon_1(z_i)) \cap e(M) \). This segment is unique up to parametrization.

Let \( \Gamma_z \) denote the vector \( \frac{\partial}{\partial s} \gamma_z(s)|_{s=0} \) tangent to \( \gamma_z(s) \) at the point \( z \in e(B(y, \epsilon_M)) \setminus e(\{y\}) \). Here and in what follows, we assume \( \gamma_z(s) \) is parametrized by arc length. It is well-understood that \( \text{grad} \text{dist}_{N;e(y)}(z) \) must coincide with \(-\Gamma_z \). This fact yields the formula

\[
\text{grad} \text{dist}^2_{N;e(y)}(z) = -2 \text{dist}_{N;e(y)}(z) \Gamma_z. \tag{4.5}
\]
We emphasize that \((4.5)\) holds when \(z\) lies in \(e(B(y, \epsilon_M)) \setminus e(\{y\})\).

Given \(x \in B(y, \epsilon_M) \setminus \{y\}\), define the curve segment \(\tau_x(s)\) in the manifold \(M\) by setting \(\tau_x(s) = e^{-1}(\gamma_{e(x)}(s))\). The notation \(T_x\) refers to the vector \(\frac{d}{ds}\tau_x(s)|_{s=0}\) tangent to \(\tau_x(s)\) at the point \(x\). We have the equality \(T_x = (de)^{-1}\Gamma_{e(x)}\). Together with \((4.3)\), \((4.4)\), and \((4.5)\), this implies

\[
\langle \text{grad} \, \text{dist}_y^2(x), \nu \rangle = -2 \text{dist}_y(x) T_x.
\]

If \(x\) lies in \(\partial M\), then the vector \(T_x\) satisfies \(\langle T_x, \nu \rangle \leq 0\). Consequently,

\[
\langle \text{grad} \, \text{dist}_y^2(x), \nu \rangle \geq 0.
\]

Our arguments prove this for \(x \in (B(y, \epsilon_M) \setminus \{y\}) \cap \partial M\). If \(y\) belongs to \(\partial M\), then the estimate can be extended to \(y\) by continuity. Hence the desired nonnegativity of \(\frac{d}{ds} \text{dist}_y^2(x)\).

We will now establish Statement 3 of the lemma. A calculation based on \((4.3)\) shows that

\[
\Delta_M \text{dist}_y^2(x) = \Delta_N \text{dist}_{N,e(y)}^2(z) \bigg|_{z=e(x)} \tag{4.6}
\]

when \(x \in B(y, \epsilon_M)\). Here, \(\Delta_N\) denotes the Laplace-Beltrami operator on \(N\). Our intention is to estimate the right-hand side of Eq. \((4.6)\) using Theorem (2.28) in \([22]\); cf. \([19]\), Section 3.4.

Let us lay down a few preliminary facts. Consider a point \(z \in e(B(y, \epsilon_M)) \setminus e(\{y\})\). As proved above, one can join \(z\) with \(e(y)\) by the minimizing geodesic segment \(\gamma_z(s)\). We should point out that this segment is entirely contained in \(e(M)\). It is convenient to assume \(\gamma_z(s)\) is parametrized by arc length. Choose a constant \(K > 0\) satisfying the formula

\[
-(\dim M - 1)K^2 \leq \inf \text{Ric}(X, X).
\]

The infimum is taken over all the vectors \(X \in TM\) with \(\langle X, X \rangle = 1\). It is finite because \(M\) is compact. The following assertion is easy to verify: At every point of the segment \(\gamma_z(s)\), the Ricci curvature of \(N\) in the direction \(\frac{d}{ds}\gamma_z(s)\) is greater than or equal to \(-(\dim M - 1)K^2\).

We are now ready to estimate \(\Delta_N \text{dist}_{N,e(y)}^2(z)\) by means of Theorem (2.28) in \([22]\). Note that the manifold \(N\) is not necessarily complete. Therefore, it is essential to take account of the remark following Theorem (2.31) in \([22]\). As mentioned in the previous paragraph, one can join \(z\) with \(e(y)\) by the segment \(\gamma_z(s)\). The Ricci curvature of \(N\) in certain directions is bounded below by \(-(\dim M - 1)K^2\). With these facts at hand, Theorem (2.28) from \([22]\) implies

\[
\Delta_N \text{dist}_{N,e(y)}^2(z) \\
\leq 2(\dim N - 1)K \text{dist}_{N,e(y)}^2(z) \coth \left( K \text{dist}_{N,e(y)}^2(z) \right) + 2. \tag{4.7}
\]

We emphasize that \((4.7)\) holds when \(z\) belongs to \(e(B(y, \epsilon_M)) \setminus e(\{y\})\). For a relevant inequality, see \([19]\), Corollary 3.4.4].
Only a few simple remarks are now needed to finish the proof. Note that the function \( \text{dist}_{N,e}(y)(z) \) takes its values in the interval \((0, \epsilon_M)\) when \( z \) varies through \( e(B(y, \epsilon_M)) \setminus e(\{y\}) \). Define the constant \( K_{\epsilon_M} > 0 \) by setting
\[
K_{\epsilon_M} = 2(\dim N - 1)K \sup_{r \in (0, \epsilon_M)} (r \coth(Kr)) + 2.
\]
In view of (4.6) and (4.7), we must have
\[
\Delta_M \text{dist}_y^2(x) \leq K_{\epsilon_M}, \quad x \in B(y, \epsilon_M) \setminus \{y\}.
\]
This estimate extends to the point \( y \) by continuity. Hence the desired result.

Remark 4.3. If \( M \) were a closed manifold, then Statements 1 and 3 of Lemma 4.2 would hold for every constant \( \epsilon_M \) less than or equal to the injectivity radius of \( M \); see, for example, [19, Section 3.4].

We are now ready to establish Proposition 4.1. Our line of reasoning is borrowed from [2, Proof of Lemma 4.1]. In particular, we make use of Bernstein’s inequality related to local martingales.

**Proof of Proposition 4.1.** Introduce the process \( N_y^y = \text{dist}_y^2(X_y^y) \). Fix a constant \( \epsilon_M > 0 \) satisfying Statements 1, 2, and 3 of Lemma 4.2. Denote \( \epsilon_0 = \frac{1}{2} \min\{1, \epsilon_M\} \). Given a number \( r \in (0, 1) \), consider the hitting time
\[
\tau = \inf\{t \geq 0 \mid N_y^y(t) = \epsilon_0 r^2 \}.
\]
It is easy to see that
\[
\mathbb{P}\{\tau(y, r) < \kappa r^2\} = \mathbb{P}\left\{\sup_{t \in [0, \kappa r^2]} N_y^y(t) > r^2\right\}
\leq \mathbb{P}\left\{\sup_{t \in [0, \kappa r^2]} N_{t \wedge \tau}^y \geq \epsilon_0 r^2\right\} \quad (4.8)
\]
for all \( \kappa \in (0, \infty) \). Let us estimate the rightmost probability in this formula.

By virtue of (3.12) and (3.13), the process \( N_{t \wedge \tau}^y \) satisfies
\[
N_{t \wedge \tau}^y = \Upsilon_t + \frac{1}{2} \int_0^{t \wedge \tau} \Delta_M \text{dist}_y^2(X_y^y) \, ds - \int_0^{t \wedge \tau} \frac{\partial}{\partial y} \text{dist}_y^2(X_y^y) \, dL_s. \quad (4.9)
\]
The notation \( \Upsilon_t \) refers to the local martingale
\[
\sum_{i=1}^n \int_0^{t \wedge \tau} (\mathcal{H}_i, l) \left( u_s^y \right) \, dB_s^i
\]
with \( l(u) = \text{dist}_y^2(\pi(u)) \) when \( u \in O(M) \). Lemma 4.2 implies that the second term in the right-hand side of (4.9) is bounded above by \( \frac{1}{2} K_{\epsilon_M} t \) and the third term is nonnegative. As a consequence, the estimate
\[
\mathbb{P}\left\{\sup_{t \in [0, \kappa r^2]} N_{t \wedge \tau}^y \geq \epsilon_0 r^2\right\} \leq \mathbb{P}\left\{\sup_{t \in [0, \kappa r^2]} \left( \Upsilon_t + \frac{1}{2} K_{\epsilon_M} t \right) \geq \epsilon_0 r^2\right\} \quad (4.10)
\]
holds for all $\kappa \in (0, \infty)$.

We now set $\kappa_0 = \frac{\epsilon^2}{\lambda_M}$ and assume $\kappa \in (0, \kappa_0)$. Then the probability in the right-hand side of (4.10) cannot exceed

$$P \left\{ \sup_{t \in [0, \kappa^2]} \Upsilon_t \geq \frac{1}{2}\epsilon^2 \right\}.$$ 

Define $\eta = \frac{\epsilon^2}{128}$. As a computation shows, the quadratic variation $\langle \Upsilon, \Upsilon \rangle_t$ of the local martingale $\Upsilon_t$ satisfies

$$\langle \Upsilon, \Upsilon \rangle_t = \sum_{i=1}^n \int_0^{t \wedge \upsilon_0} (H_i l)^2 (u_s^Y) \, ds \leq 4 \int_0^{t \wedge \upsilon_0} \text{dist}_y^2(X_s^y) \, ds \leq 4\epsilon^2 \theta^2 t = 128\eta r^2 t.$$ 

In accordance with Bernstein’s inequality (see, for example, Exercise (3.16) in Chapter IV), the fact that $\langle \Upsilon, \Upsilon \rangle_t$ cannot exceed $128\eta r^2 t$ implies

$$P \left\{ \sup_{t \in [0, \kappa^2]} \Upsilon_t \geq \frac{1}{2}\epsilon^2 \right\} = P \left\{ \sup_{t \in [0, \kappa^2]} \Upsilon_t \geq 16\eta r^2 \right\} \leq e^{-\frac{\kappa}{2}}.$$ 

Combining this estimate with (4.8) and (4.10) completes the proof. \(\square\)

Proposition 4.1 may be important to the further development of the probabilistic approach to the Yang-Mills heat equation on manifolds with boundary. In particular, this result helps us obtain the following estimate for the curvature of the time-dependent connection $\nabla(t)$ discussed in Section 3. As before, we deal with the reflecting Brownian motion $X^y$ starting at the point $y$. The connection $\nabla(t)$ solves Eq. (3.2) with the boundary conditions (3.3) or (3.4).

**Theorem 4.4.** Let $\partial M$ be totally geodesic. Suppose either Assumption 7 of Theorem 2.7 or Assumptions 7 and 6 of Theorem 2.9 are fulfilled for $M$. Then there exist constants $\xi_1 > 0$ and $\theta > 0$, a function $\sigma(s)$ on $(0, \infty)$, and a function $v(s)$ on $(0, 1)$ that depend on nothing but $M$ and satisfy the following statements:

1. The values of $\sigma(s)$ and $v(s)$ lie in $(0, 1)$ and $(0, \infty)$, respectively.
2. The function $\sigma(s)$ is non-increasing, while $v(s)$ is non-decreasing.
3. Given $s_0 \in (0, T)$, $a \in (0, 1]$, and $s \in \left(0, \min\left\{ \sigma\left(a^{-1} \text{YM}(0)\right), s_0\right\}\right)$, if

$$a \left\langle \nabla^{(s_0-s)}(X^y_s) \right\rangle_E^2 \leq a \xi_1,$$

then the estimate

$$\sup_{t \in [0, \kappa^2]} \sup_{x \in \bar{B}(y, v(s))} \left\| R^{\nabla(t)}(x) \right\|_E^2 \leq \frac{a\theta}{(v(s))^4}$$

holds true.
In order to establish Theorem 4.4, we need to repeat the arguments from [2]. Let us outline the changes required for these arguments to work on a manifold with boundary. The Brownian motion $X_t(x^*)$ in [2] must be replaced by a reflecting Brownian motion starting at $x^*$. The stochastic differential equation for the process $Y_t$ in [2] then comes from Eqs. (3.12) and (3.13) of the present paper; cf. the proof of Theorem 3.1. The necessary estimates are provided by Lemma 3.5, Lemma 3.9, and Proposition 4.1. We will not discuss further details here.

**Remark 4.5.** Theorem 4.2 in [2] establishes (4.11) on a closed manifold. Section 4 of [2] contains a variety of corollaries of this estimate. Many of those results would likely generalize to manifolds with boundary by means of our Theorem 4.4.

**Remark 4.6.** The paper [2] uses its version of (4.11) to prove that the curvature of the corresponding solution of the Yang-Mills heat equation does not blow up if the dimension is less than 4 or if the initial energy is small. Such a line of reasoning may be inefficient on manifolds with boundary. For example, Theorems 3.1 and 3.2 cannot be deduced from Theorem 4.4 because their assumptions are considerably weaker. The case where $\dim M$ is 5 or higher is different. It seems likely that a statement similar to Theorem 3.3 can indeed be obtained as a consequence of Theorem 4.4.

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