Tangent cones of numerical semigroup rings

Teresa Cortadellas Benítez and Santiago Zarzuela Armengou

Abstract. In this paper we describe the structure of the tangent cone of a numerical semigroup ring \( A = k[[S]] \subseteq k[[t]] \) with multiplicity \( e \) (as a module over the Noether normalization determined by the fiber cone of the ideal generated by \( t^e \)) in terms of some classical invariants of the corresponding numerical semigroup. Explicit computations are also made by using the GAP system.

1. Introduction

Let \((A, m)\) be an one dimensional Cohen-Macaulay local ring with infinite residue field, embedding dimension \( b \), reduction number \( r \) and multiplicity \( e \). Let \((x)\) be a minimal reduction of \( m \).

In [CZ1] and [CZ2] the authors have observed that the Noether normalization

\[
F_m(x) := \bigoplus_{n \geq 0} \frac{x^n A}{x^n m} \hookrightarrow G(m) := \bigoplus_{n \geq 0} \frac{m^n}{m^{n+1}}
\]

provides a decomposition of \( G(m) \) as direct sum of graded cyclic \( F_m(x) \)-modules of the form

\[
G(m) \cong \bigoplus_{i=0}^r (F_m(x)(-i))^{\alpha_i} \bigoplus_{i=1}^{r-1} \bigoplus_{j=1}^{r-i-1} \left( \frac{F_m(x)}{(x^*)^j F_m(x)(-i)} \right)^{\alpha_{i,j}},
\]

with \( \alpha_0 = 1 \), \( \alpha_r \neq 0 \) and \( \sum_{i=1}^r \alpha_i = e - 1 \).

In the more general context of the study of fiber cone of ideals with analytic spread one, the authors analyze in [CZ1] the information provided by the set of invariants \( \{ \alpha_i, \alpha_{i,j} \} \) (that we call the invariants of the tangent cone) in order to study, for instance, the Cohen-Macaulay or Buchsbaum properties of the tangent cone; whereas in [CZ2] the connection between these invariants and the so called microinvariants introduced by Juan Elias [E] in the geometric case, and other invariants introduced by Valentina Barucci and Ralf Fröberg [BF] in the numerical semigroup case is study in detail.

1991 Mathematics Subject Classification. Primary 13A30; Secondary 13H10, 13P10.

Key words and phrases. commutative algebra, numerical semigroup ring, fiber cone.

Partially supported by MTM2007-67493.
The main purpose of this paper is to compute explicitly the values of the set of invariants \( \alpha_i, \alpha_{ij} \) for the tangent cones of numerical semigroup rings in terms of the invariants introduced by Barucci-Fröberg in [BF]. In particular, these computations can be performed numerically by using the GAP system [GAP4], as we show with several examples. We note that for the case of the microinvariants Elias himself has described the explicit computations for semigroup rings in [E] Section 4.

The content of this work arose during the talk that, reporting the results in [CZ2], the second author gave at the Exploratory Workshop on Combinatorial Commutative Algebra and Computer Algebra held in Mangalia, Romania, in May 2008. Prof. Jürgen Herzog suggested during the talk that for the numerical semigroup case one should be able to compute explicitly the invariants of the tangent cone. The authors want to thank J. Herzog for this suggestion. Also, the second author would like to thank the organizers of the workshop, Alexandru Bobe, Vi viana Ene, and Denis Ibadula from the Ovidius University in Constanta, for the invitation to participate in it and the excellent organization and warm atmosphere during the workshop.

2. Tangent cones of numerical semigroups

Let \( \mathbb{N} \) be the set of non-negative integers. Recall that a numerical semigroup \( S \) is a subset of \( \mathbb{N} \) that is closed under addition, contains the zero element and has finite complement in \( \mathbb{N} \). The least integer belonging to \( S \) is known as the multiplicity of \( S \) and it is denoted by \( e(S) \). Reciprocally, the greatest integer not belonging to \( S \) is known as the Frobenius number of \( S \) and it is denoted by \( F(S) \).

A numerical semigroup \( S \) is always finitely generated; that is, there exist integers \( n_1, \ldots, n_t \) such that \( S = \langle n_1, \ldots, n_t \rangle = \{ \alpha_1 n_1 + \cdots + \alpha_t n_t ; \alpha_i \in \mathbb{N} \} \). Moreover, every numerical semigroup has an unique minimal system of generators \( n_1, \ldots, n_{b(S)} \).

A relative ideal of \( S \) is a nonempty set \( I \) of non-negative integers such that \( I + S \subset I + d \subseteq S \) for some \( d \in S \). An ideal of \( S \) is then a relative ideal of \( S \) contained in \( S \). If \( i_1, \ldots, i_k \) is a subset of non-negative integers, then the set \( \{ i_1, \ldots, i_k \} + S = (i_1 + S) \cup \cdots \cup (i_k + S) \) is a relative ideal of \( S \) and \( i_1, \ldots, i_k \) is a system of generators of \( I \). Note that, if \( I \) is an ideal of \( S \), then \( I \cup \{ 0 \} \) is a numerical semigroup and so \( I \) is finitely generated. We denote by \( M \) the maximal ideal of \( S \), that is, \( M = S \setminus \{ 0 \} \). \( M \) is then the ideal generated by a system of generators of \( S \). If \( I \) and \( J \) are relative ideals of \( S \) then \( I + J = \{ i + j ; i \in I, j \in J \} \) is also a relative ideal of \( S \). Finally, we denote by \( \text{Ap}(I) \) the Apéry set of \( I \) with respect to \( e(S) \), defined as the set of the smallest elements in \( I \) in each residue class module \( e(S) \).

Let \( V = k[[t]] \) be the formal power series ring over a field \( k \). Given a numerical semigroup \( S = \langle n_1, \ldots, n_t \rangle \) minimally generated by \( 0 < e = e(S) = n_1 < \cdots < n_k = n_{b(S)} \) we consider the ring associated to \( S \) defined as \( A = k[[S]] = k[[t^{n_1}, \ldots, t^{n_k}]] \subseteq V \). Let \( m = (t^{n_1}, \ldots, t^{n_k}) \) be the maximal ideal of \( A \). Then \( A \) is a Cohen-Macaulay local ring of dimension one with multiplicity \( e \) and embedding dimension \( b \). Moreover, the conductor \( (A : V) = t^n V \) with \( C = F + 1 \) where \( F = F(S) \) is the Frobenius number of \( S \). These kind of rings are known as numerical semigroup rings. The ideals \( (t^{i_1}, \ldots, t^{i_k}) \) of \( A \) are such that for \( v \), the
$t$-adic valuation, $v((t^{i_1}, \ldots, t^{i_k})) = \{i_1, \ldots, i_k\} + S$. In particular, for the ideals $m^n$ one has $v(m^n) = nM = M + \cdots + M$. Note that $(n+1)M \subseteq nM$ for $n \geq 0$ (we will set $m^0 := A$).

Let $A = k[[S]]$ be a numerical semigroup ring of multiplicity $e$. Then, the element $t^e$ generates a minimal reduction of $m$. In terms of semigroups, $(n+1)M = e + nM$ for all $n \geq r$, the reduction number of $m$; that is, $r$ is the smallest integer $n$ such that $m^{n+1} = t^e m^n$ (in our case the reduction number does not depend on the minimal reduction).

A crucial point for our results is the use of a fact proved by Barucci-Frörberg [BF, Lemma 2.1] in the more general context of one-dimensional equicharacteristic analytically irreducible and residually rational domains. For completeness, we give an easy proof of it for the particular case we deal with in this paper. Set $k[[t^e]] = W \rightarrow A$.

**Lemma 2.1.** Let $I$ be an ideal of $S$ and $I$ the ideal of $A$ generated by $\{t^e\}_{n \in I}$. If $Ap(I) = \{\omega_0, \ldots, \omega_{e-1}\}$ is the Apery set of $I$ with respect to $e$, then $I$ is a free $W$-module generated by $t^\omega_0, \ldots, t^\omega_{e-1}$.

**Proof.** Let $n \in I$. If $n \equiv i \mod e$ then $n = \omega_i + ae$ for some $a \geq 0$. So $t^n = (t^e)\alpha t^\omega_i \in Wt^\omega_0 + \cdots + Wt^\omega_{e-1}$ and $I = Wt^\omega_0 + \cdots + Wt^\omega_{e-1}$. Observe that the sum is direct since in each summand the elements are monomials in $t$ with exponents in different residue classes mod $e$.

In particular, we may write the powers of the maximal ideal as a direct sum of cyclic $W$-modules.

**Lemma 2.2.** For each $n \geq 0$ there exist non-negative integers $\omega_{n,0}, \ldots, \omega_{n,e-1}$ such that

$$m^n = Wt^{\omega_{n,0}} + \cdots + Wt^{\omega_{n,e-1}},$$

with $\omega_{n+1,i} = \omega_{n,i} + e \cdot e$ and $e \in \{0, 1\}$.

**Proof.** Observe first that if $Ap(S) = \{0, \omega_1, \ldots, \omega_{e-1}\}$ is the Apery set of $S$ (with respect to the multiplicity $e$), then the Apery set of $M = S \setminus \{0\}$ the maximal ideal of $S$ is $Ap(M) = \{\omega, \omega_1, \ldots, \omega_{e-1}\}$.

Now, for each $n \geq 1$ let $Ap(nM) = \{\omega_{n,0}, \ldots, \omega_{n,e-1}\}$ be the Apery set of $nM$. If $\omega_{n,i} \in (n+1)M$ then $\omega_{n+1,i} = \omega_{n,i}$. Otherwise $\omega_{n,i} + e \in (n+1)M$ and belongs to the same residue class of $\omega_{n,i}$ module $e$. Since $\omega_{n+1,i} \leq \omega_{n,i} + e$ by definition, it follows that $\omega_{n+1,i} = \omega_{n,i} + e$ for some $a \geq 1$. On the other hand, if $a \geq 2$ then $\omega_{n,i} = \omega_{n+1,i} + (a - 1)e \in (n+1)M$ which contradicts the assumption, so $a = 1$. Now the proof is concluded by applying the above lemma to $nM$ for $n \geq 0$ (where $0M := S$).

Observe that for each $n \geq 0$ and each $0 \leq i \leq e - 1$, $Wt^{\omega_{n+1,i}} \subseteq Wt^{\omega_{n,i}}$. Also, that for $n \geq r$ we have $\omega_{n+1,i} = \omega_{n,i} + e$.

Our next result gives a description of the set of invariants $\{\alpha_i, \alpha_{i,j}\}$ of the tangent cone in terms of the Apery sets of the family of ideals $nM$, for $0 \leq n \leq r$. Previously, and just for the purposes of this paper, we introduce the following notation:

Let $E = \{a_0, \ldots, a_n\}$ be a set of integers. We call it a ladder if $a_0 \leq \cdots \leq a_n$. Given a ladder, we say that a subset $L = \{a_i, \ldots, a_{i+k}\}$ with $k \geq 1$ is a landing of
Then define the following integers: $W_t$ IS $\omega$ and reduction number $r$ diagram of graded rings

length $k$ if $a_{i-1} < a_i = \cdots = a_{i+k} < a_{i+k+1}$ (where $a_{-1} = -\infty$ and $a_{n+1} = \infty$). In this case, the index $i$ is the beginning of the landing: $s(L)$ and the index $i + k$ is the end of the landing: $e(L)$. A landing $L$ is said to be a true landing if $s(L) \geq 1$.

Given two landings $L$ and $L'$, we set $L < L'$ if $s(L) < s(L')$. Let $l(E) + 1$ be the number of landings and assume that $L_0 < \cdots < L_{l(E)}$ is the set of landings. Then, we define following numbers:

- $s_j(E) = s(L_j)$, $e_j(E) = e(L_j)$, for each $0 \leq j \leq l(E)$;
- $c_j(E) = s_j - e_{j-1}$, for each $1 \leq j \leq l(E)$.

Note that the above numbers are defined under the conditions

$$
a_0 = \cdots = a_{c_0(E)} < \cdots < a_{c_0(E)+c_1(E)} = \cdots = a_{c_l(E)} < \cdots < a_n
$$

**Theorem 2.3.** Let $A = k[[S]]$ be a numerical semigroup ring of multiplicity $e$ and reduction number $r$. Let $M$ be the maximal ideal of $S$ and put

$$Ap(nM) = \{\omega_{n,0}, \ldots, \omega_{n,i}, \ldots, \omega_{n,e-1}\}$$

for $0 \leq n \leq r$.

For any $1 \leq i \leq e - 1$, consider the ladder of values $W^i = \{\omega_{n,i}\}_{0 \leq n \leq r}$ and define the following integers:

1. $l_i = l(W^i)$;
2. $d_i = e_l(W^i)$;
3. $b^i_j = e_{j-1}(W^i)$ and $c^i_j = c_j(W^i)$, for $1 \leq j \leq l_i$.

Then

$$G(m) \cong F_m(t^e) \oplus \bigoplus_{i=1}^{e-1} F_m(t^e)(-d_i) \oplus \bigoplus_{j=1}^{l} \frac{F_m(t^e)}{((t^e)^{s_j} F_m(t^e)(-b^i_j))}.$$  

**Proof.** For all $n \geq 0$, we have by Lemma 2.2 that $m^n = W t^{\omega_{n,0}} + \cdots + W t^{\omega_{n,e-1}}$, with $\omega_{0,n} = t^e n$ for all $n$, and so we have the following commutative diagram of graded rings

$$G(m) \cong \bigoplus_{n \geq 0} \left( \bigoplus_{i=0}^{e-1} \frac{W t^{\omega_{n,i}}}{W t^{\omega_{n+1,i}}} \right) = G$$

$$\uparrow$$

$$F_m(t^e) \cong \bigoplus_{n \geq 0} \left( \left( t^e \right)^n W \right) = F$$

and we can read the structure of $G(m)$ as $F_m(t^e)$-module as the structure of $G$ as $F$-module. Note that $G$ may also be written as

$$G \cong \bigoplus_{i=0}^{e-1} \left( \bigoplus_{n \geq 0} \frac{W t^{\omega_{n,i}}}{W t^{\omega_{n+1,i}}} \right)$$
Now, let us fix \(1 \leq i \leq e - 1\). Assume first that \(b_i^1 = d_i\). Then, the component of degree degree \(n\) of

\[
\bigoplus_{n \geq 0} \frac{Wt^{\omega_{n,i}}}{Wt^{\omega_{n+1,i}}} \text{ is}
\]

\[
\begin{cases}
0 & \text{if } 0 \leq n < b_i^1 \\
\frac{Wt^{\omega_{0,i} + e(n-b_i^1)}}{Wt^{\omega_{0,i} + e(n-b_i^1+1)}} & \text{if } b_i^1 \leq n < b_i^1 + c_i^1 \\
\vdots & \text{if } b_j^i + c_{j-1}^i \leq n < b_j^i \\
\frac{Wt^{\omega_{0,i} + e(n+c_i^1 + \cdots + c_{j-1}^i - b_j^i)}}{Wt^{\omega_{0,i} + e(n+1+c_i^1 + \cdots + c_{j-1}^i - b_j^i)}} & \text{if } b_j^i \leq n < b_j^i + c_j^i \\
\vdots & \text{if } b_i^j + c_j^i \leq n < d_i \\
0 & \text{if } n \geq d_i
\end{cases}
\]

with \(W\)-isomorphisms

\[
\frac{Wt^{\omega_{0,i} + e(n+c_i^1 + \cdots + c_{j-1}^i - b_j^i)}}{Wt^{\omega_{0,i} + e(n+c_i^1 + \cdots + c_{j-1}^i - b_j^i+1)}} \cong \frac{W(t^e)(n-b_j^i)}{W(t^e)(n-b_j^i+1)}
\]

for \(1 \leq j \leq l_i\) and \(b_j^i \leq n \leq b_j^i + c_j^i\),

\[
\frac{Wt^{\omega_{0,i} + e(n+1+c_i^1 + \cdots + c_{j-1}^i - d_i)}}{Wt^{\omega_{0,i} + e(n+1+c_i^1 + \cdots + c_{j-1}^i - d_i)}} \cong \frac{W(t^e)(n-d_i)}{W(t^e)(n-d_i+1)}
\]

for \(n \geq d_i\) and

\[
\bigoplus_{n \geq 0} \frac{Wt^{\omega_{n,i}}}{Wt^{\omega_{n+1,i}}} = \bigoplus_{j=0}^{l_i} F \cdot (t^{\omega_{0,i} + e(c_i^1 + \cdots + c_j^i)})^* \cong \bigoplus_{j=1}^{l_i} F \cdot ((t^e)^*) \cdot (-b_j^i) \oplus F(-d_i).
\]

\(\square\)
Example 2.4. Let $S = \langle 5, 6, 13 \rangle$. It is easy to prove that the maximal ideal of $S$ has reduction number 4 and also to calculate the Apery sets of the ideals $nM$ for $n \geq 0$. The following table shows these values for $n \leq 4$:

|       | $\text{Ap}(S)$ | $\text{Ap}(M)$ | $\text{Ap}(2M)$ | $\text{Ap}(3M)$ | $\text{Ap}(4M)$ |
|-------|----------------|----------------|----------------|----------------|----------------|
| $\omega_{0,0}$ | 0              | 6              | 12             | 13             | 19             |
| $\omega_{0,1}$ |               | 5              | 6              | 12             | 13             |
| $\omega_{0,i}$ |               |               | 10             | 11             | 12             |
| $\omega_{0,e-1}$ |               |               | 15             | 16             | 17             |
| $\omega_{n,0}$ |               |               |               | 20             | 21             |
| $\omega_{n,1}$ |               |               |               |               | 22             |
| $\omega_{n,i}$ |               |               |               |               | 23             |
| $\omega_{n,e-1}$ |               |               |               |               | 24             |

Then the tangent cone $G$ of $k[[t^5, t^6, t^{13}]]$ has the following structure over $F$ the fiber cone of \((t^5)\):

\[
F \oplus F \cdot (t^6)^* \oplus F \cdot (t^{12})^* \oplus F \cdot (t^{13})^* \oplus F \cdot (t^{18})^* \oplus F \cdot (t^{19})^* \oplus F \cdot (t^{24})^*,
\]

and so isomorphic to

\[
F \oplus F \oplus F \oplus (F/(t^5)^* F)(-2) \oplus F \oplus F \oplus F \\
\oplus F/(t^5)^* F)(-2) \oplus F \oplus F.
\]

Remark 2.5. Observe that the necessary information to determine the structure of $G$ as $F$-module is contained in the table that we call the Apery table of $S$.

Thus, if we analyze the increment of the values by columns then we obtain the values of $d_i$, $b_j$, $c_i$ for $1 \leq i \leq e-1$ and $1 \leq j \leq l_i$ and we may write, putting $x = t^e$,

\[
G \cong F \oplus \bigoplus_{i=1}^{e-1} \left( F(-d_i) \bigoplus_{j=1}^{l_i} F(x^j)^* F(-b_j) \right).
\]

Also, if we separate free and torsion submodules and collect the summands by the degrees of the generators we can rewrite the above expression in the form

\[
G \cong \bigoplus_{i=0}^{r} (F(-i))^{\alpha_i} \bigoplus_{i=1}^{r-1} \bigoplus_{j=1}^{r-i-1} \left( \frac{F}{(x^*)^{\alpha_i-j}} F(-i) \right).
\]
Hence setting $\beta_{0,i} = \alpha_i + \sum_{j=1}^{r-i-1} \alpha_{i,j}$ and $\beta_{1,i} = \sum_{k+l=i} \alpha_{k,l}$ we get that

$$0 \longrightarrow \bigoplus_{i=1}^{r-1} F(-i)^{\beta_{0,i}} \longrightarrow \bigoplus_{i=0}^{r-1} F(-i)^{\beta_{1,i}} \longrightarrow 0$$

gives a minimal graded free resolution, or equivalently the graded Betti numbers of $G$ as $F$-module.

### 3. Computing the invariants of the tangent cone: Examples

The GAP - Groups, Algorithms, Programming - is a system for Computational Discrete Algebra [GAP4]. On the basis of GAP, Manuel Delgado, Pedro A. García-Sánchez and José Morais have developed the NumericalSgps package [NumericalSgps]. Its aim is to make available a computational tool to deal with numerical semigroups. By Theorem 2.3 we can determine the structure of the tangent cone of a numerical ring $k[[S]] \subseteq k[[t]]$ of multiplicity $e$ as a module over the fiber cone of $(t^e)$ if we know the Apery sets of the sum ideals $nM$, where $M$ is the maximal ideal of $S$. On the other hand, from the definition we have that the Apery set of $nM$ can be calculated as

$$\text{Ap}(nM) = nM \setminus ((e+S) + nM)$$

(see also [BF] Lemma 2.1 (2)), a computation that can be performed by using the NumericalSgps package. The following examples are just a sample of these computations.

**Example 3.1.** Let $S = \langle 10, 11, 19 \rangle$. By using the NumericalSgps package of GAP we calculate the reduction number of $M$ which is 8 and also the Apery sets of the ideals $nM$ for $n \leq 8$. The following is the Apery table in this case:

| $\text{Ap}(S)$ | 0 | 11 | 22 | 33 | 44 | 55 | 66 | 57 | 38 | 19 |
|----------------|---|----|----|----|----|----|----|----|----|----|
| $\text{Ap}(M)$ | 10 | 11 | 22 | 33 | 44 | 55 | 66 | 57 | 38 | 19 |
| $\text{Ap}(2M)$ | 15 | 21 | 22 | 33 | 44 | 55 | 66 | 57 | 38 | 29 |
| $\text{Ap}(3M)$ | 20 | 31 | 32 | 33 | 44 | 55 | 66 | 57 | 48 | 39 |
| $\text{Ap}(4M)$ | 25 | 41 | 42 | 43 | 44 | 55 | 66 | 67 | 58 | 49 |
| $\text{Ap}(5M)$ | 30 | 51 | 52 | 53 | 54 | 55 | 66 | 77 | 68 | 59 |
| $\text{Ap}(6M)$ | 35 | 61 | 62 | 63 | 64 | 65 | 66 | 77 | 78 | 69 |
| $\text{Ap}(7M)$ | 40 | 71 | 72 | 73 | 74 | 75 | 76 | 77 | 88 | 79 |
| $\text{Ap}(8M)$ | 45 | 81 | 82 | 83 | 84 | 85 | 86 | 87 | 88 | 89 |

By Theorem 2.3 the tangent cone $G$ of $k[[t^{10}, t^{11}, t^{19}]]$ has the following structure over $F$ the fiber cone of $(t^{10})$:

$$F \oplus (F(-1))^2 \oplus F(-2) \oplus F(-3) \oplus F(-4) \oplus F(-5) \oplus F(-6) \oplus F(-7) \oplus F(-8) \oplus (F/x^2 F)(-3) \oplus (F/x^5 F)(-2)$$
where \( x := (t^{10})^* \). Thus we get that the minimal graded free resolution of \( G \) as \( F \)-module has the following values for its Betti numbers:
\[
\begin{align*}
\beta_{0,1} &= \beta_{0,2} = \beta_{0,3} = 2, \\
\beta_{0,0} &= \beta_{0,4} = \beta_{0,5} = \beta_{0,6} = \beta_{0,7} = \beta_{0,8} = 1, \\
\beta_{1,5} &= \beta_{1,7} = 1.
\end{align*}
\]

**Example 3.2.** Let \( S = \langle 10, 19, 47 \rangle \). By using the NumericalSgps package of GAP we calculate the reduction number of \( M \) which is 9 and also the Apery sets of the ideals \( nM \) for \( n \leq 9 \) and we get the following Apery table:

| \( Ap(S) \) | 0 | 141 | 132 | 113 | 94 | 85 | 66 | 47 | 38 | 19 |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
| \( Ap(M) \) | 10 | 141 | 132 | 113 | 94 | 85 | 66 | 47 | 38 | 19 |
| \( Ap(2M) \) | 20 | 141 | 132 | 113 | 94 | 85 | 66 | 57 | 38 | 29 |
| \( Ap(3M) \) | 30 | 141 | 132 | 113 | 104 | 85 | 76 | 57 | 48 | 39 |
| \( Ap(4M) \) | 40 | 151 | 132 | 123 | 104 | 95 | 76 | 67 | 58 | 49 |
| \( Ap(5M) \) | 50 | 151 | 142 | 123 | 114 | 95 | 86 | 77 | 68 | 59 |
| \( Ap(6M) \) | 60 | 161 | 142 | 133 | 114 | 105 | 96 | 87 | 78 | 69 |
| \( Ap(7M) \) | 70 | 161 | 152 | 133 | 124 | 115 | 106 | 97 | 88 | 79 |
| \( Ap(8M) \) | 80 | 171 | 152 | 143 | 134 | 125 | 116 | 107 | 98 | 89 |
| \( Ap(9M) \) | 90 | 171 | 162 | 153 | 144 | 135 | 126 | 117 | 108 | 99 |

As a consequence, the tangent cone \( G \) of \( k[[t^{10}, t^{19}, t^{47}]] \) has the following invariants as a module over \( F \): the fiber cone of \( (t^{10})^* \):

\[
\alpha_i = 1 \text{ for } 0 \leq i < 9,
\alpha_{1,1} = \alpha_{6,1} = \alpha_{7,1} = 1, \alpha_{2,1} = \alpha_{4,1} = \alpha_{5,1} = 2, \alpha_{3,1} = 3.
\]

And the graded Betti numbers of \( G \) as \( F \)-module are
\[
\begin{align*}
\beta_{0,0} &= 1, \beta_{0,1} = 2, \beta_{0,2} = 3, \beta_{0,3} = 4, \beta_{0,4} = 3, \beta_{0,5} = 3, \beta_{0,6} = 2, \beta_{0,7} = 2, \\
\beta_{1,2} &= 1, \beta_{1,3} = 2, \beta_{1,4} = 3, \beta_{1,5} = 2, \beta_{1,6} = 2, \beta_{1,7} = 1, \beta_{1,8} = 1.
\end{align*}
\]

### 4. Buchsbaum property of numerical semigroup rings

In this section we analyze the Buchsbaum property of the tangent cones of numerical semigroup rings.

Let \( A = k[[S]] \subseteq k[[t]] \) be a numerical semigroup ring of multiplicity \( c \), embedding dimension \( b \) and reduction number \( r \). Set \( x = t^c \). Recall that \( c = \mu(m^n) + \lambda(m^{n+1}/xm^n) \) for all \( n \geq 0 \) and so \( b = \mu(m) \leq c \). Also, that \( \mu(m^n) \geq n+1 \) for \( 0 \leq n \leq r \) and so \( r \leq \mu(m^n) - 1 = e - 1 \). Let \( G \) be the tangent cone of \( A \) and \( F \) the fiber cone associated to the ideal \( (x) \). Assume that
\[
G \cong F \oplus \bigoplus_{i=1}^{c-1} \left( F(-d_i) \oplus \frac{F}{(x^c)^{d_i}} F(-b^c_i) \right)
\]
where the integers \( \{ d_i, l_i, c_i, b_{ij}; 1 \leq i \leq e - 1, 0 \leq j \leq l_i \} \) are as in Theorem 2.3.

The tangent cone \( G \) is Cohen-Macaulay if and only if \( G \) is a free graded module over \( F \). In this case the structure of \( G \) as \( F \)-graded module is

\[
G \cong F \oplus \bigoplus_{i=1}^{e-1} (F(-d_i)).
\]

Observe that this is equivalent to the fact that there are no true landings in the ladders determined by the columns of the the Apery table of \( S \).

**Example 4.1.** Let \( S = \langle 10, 17, 22, 28 \rangle \). For this numerical semigroup we use GAP to calculate the reduction number (which is 4) and the Apery table of \( S \):

| \( \text{Ap}(S) \) | 0 | 51 | 22 | 73 | 34 | 45 | 56 | 17 | 28 | 39 |
| \( \text{Ap}(M) \) | 10 | 51 | 22 | 73 | 34 | 45 | 56 | 17 | 28 | 39 |
| \( \text{Ap}(2M) \) | 20 | 51 | 32 | 73 | 34 | 45 | 56 | 27 | 38 | 39 |
| \( \text{Ap}(3M) \) | 30 | 51 | 42 | 73 | 44 | 55 | 56 | 37 | 48 | 49 |
| \( \text{Ap}(4M) \) | 40 | 61 | 52 | 73 | 54 | 65 | 66 | 47 | 58 | 59 |

and so

\[
G \cong F \oplus F(-1)^3 \oplus F(-2)^3 \oplus F(-3)^2 \oplus F(-4).
\]

It is well known that the tangent cone of a ring with reduction number at most 2 is Cohen-Macaulay. This is obvious in our case from the fact that there is no room for possible landings in the associated Apery table. The following examples show the structure of these tangent cones in the case of numerical semigroup rings in terms of the values of the associated numerical semigroup by using the main theorem of section 2.

**Corollary 4.2.** Let \( S \) be a numerical semigroup with reduction number 1 and set \( S = \langle e = n_0, \ldots, n_{b(S)-1} \rangle \). Then \( b(S) = e \) and

\[
G = F \oplus F \cdot (t^{n_1})^* \oplus \cdots \oplus F \cdot (t^{n_{e-1}})^*
\]

\[
\cong F \oplus F(-1)^{e-1}
\]

**Proof.** Observe first that \( r = 1 \) if and only if \( b(S) = \mu(m) = e \) (that is, \( A \) is of minimal multiplicity). Hence the values \( n_i \), \( n_j \) must belong to different residue classes module \( e \) for \( i \neq j \) and we may assume that \( n_i \equiv i \) module \( e \). Then, \( \text{Ap}(S) = \{0, n_1, \ldots, n_{e-1}\} \) and \( \text{Ap}(M) = \{e, n_1, \ldots, n_{e-1}\} \), which implies that \( d_i = b_i^* = 1 \) for \( 1 \leq i \leq e - 1 \). \( \square \)

**Corollary 4.3.** Let \( S \) be a numerical semigroup with reduction number 2. Then

\[
G \cong F \oplus F(-1)^{b-1} \oplus F(-2)^{e-b}
\]

**Proof.** In this case there exist \( \omega_i \) for \( 1 \leq j \leq b-1 \) such \( S = \langle e, \omega_i, \ldots, \omega_{i_{b-1}} \rangle \) and \( M = \{e, \omega_i, \ldots, \omega_{i_{b-1}}\} + S \). Moreover, the \( \omega_i \)'s are not in \( 2M \) and they are in different residue classes mod. \( e \). Thus, the corresponding Apery table is which gives the structure of \( G \) by theorem 2.3. \( \square \)
If the multiplicity of $A$ is less or equal to 3 then the reduction number is at most 2. So the following is one of the next cases:

**Corollary 4.4.** Let $S$ be a numerical semigroup of multiplicity 4 and embedding dimension $b$.

1. If $b = 4$ then $G \cong F \oplus F(-1)^3$.
2. If $b = 3$ then $r = 2$ or $r = 3$ and
   a. $G \cong F \oplus F(-1)^2 \oplus F(-2)$ if $r = 2$,
   b. $G \cong F \oplus F(-1) \oplus F(-2) \oplus F(-3) \oplus (F/(t^4)F)(-1)$ if $r = 3$.
3. If $b = 2$ then $G \cong F \oplus F(-1) \oplus F(-2) \oplus F(-3)$.

**Proof.** We have that $1 \leq r \leq 3$ and $2 \leq b \leq 4$. Hence, it suffices to determine all the possible Apery tables in each case and then apply theorem 2.3.

Assume first that $b = 4$. Then, $r = 1$ and the result follows from lemma 4.2. Moreover, if $S = (4, \omega_1, \omega_2, \omega_3)$ the Apery table is in this case

| Ap(S) | 0 | $\omega_1$ | $\omega_2$ | $\omega_3$ |
|-------|---|-------------|-------------|
| Ap(M) | 4 | $\omega_1$ | $\omega_2$ | $\omega_3$ |

Assume now that $b = 3$ and set $S = (4, \omega_1, \omega_2)$. Then, $\lambda(m^2/xm) = 1$ and so $r \geq 2$. If $r = 2$, equivalently, if $\mu(m^2) = 4$, there exists $\omega_3 \in S$ such that the Apery table (after a possible permutation of the columns) is

| Ap(S) | 0 | $\omega_1$ | $\omega_2$ | $\omega_3$ |
|-------|---|-------------|-------------|
| Ap(M) | 4 | $\omega_1$ | $\omega_2$ | $\omega_3$ |

Otherwise $r = 3$, equivalently, $\mu(m^2) = 3$. Taking lengths in the exact sequence

$$0 \rightarrow (m^3 + xm)/xm \rightarrow m^2/xm \rightarrow m^2/(m^3 + xm) \rightarrow 0$$

we get that $m^3 \subseteq xm$. Hence, there exist $\omega_1, \omega_2, \omega_3 \in S$ such that the Apery table (after a possible permutation of the columns) is

| Ap(S) | 0 | $\omega_1$ | $\omega_2$ | $\omega_3$ |
|-------|---|-------------|-------------|
| Ap(M) | 4 | $\omega_1$ | $\omega_2$ | $\omega_3$ |
| Ap(2M) | 8 | $\omega_1 + 4$ | $\omega_2 + 4$ | $\omega_3$ |
Finally, assume that $b = 2$. Then $r = 3$, $\mu(m^2) = 3$, $\mu(m^3) = 4$ and $\lambda(m^3/xm^2) = 1$. Thus, taking lengths in the exact sequence

$$0 \rightarrow m^3/xm^2 \rightarrow xm/xm^2 \rightarrow xm/m^3 \rightarrow 0$$

we get that $\lambda(xm/m^3) = 1$ and so $m^3$ is not contained in $xm$. Now, the Apery table (after a possible permutation of the columns) is given by

$$\begin{array}{|c|c|c|c|}
\hline
\text{Ap}(S) & 0 & \omega_1 & \omega_2 & \omega_3 \\
\text{Ap}(M) & 4 & \omega_1 & \omega_2 & \omega_3 \\
\text{Ap}(2M) & 8 & \omega_1 + 4 & \omega_2 + 4 & \omega_3 \\
\text{Ap}(3M) & 12 & \omega_1 + 8 & \omega_2 + 8 & \omega_3 + 4 \\
\hline
\end{array}$$

Example 4.5. This example illustrates the above corollary. In each case we give the specific Apery table associated to the semigroup.

(1) Let $S = \langle 4, 10, 11, 17 \rangle$. Then,

$$\begin{array}{|c|c|c|c|}
\hline
\text{Ap}(S) & 0 & 17 & 10 & 11 \\
\text{Ap}(M) & 4 & 17 & 10 & 11 \\
\hline
\end{array}$$

and

$$G = F \oplus F \cdot (t^{10})^* \oplus F \cdot (t^{11})^* F \cdot (t^{17})^* \cong F \oplus F(-1)^3.$$

(2) Let $S = \langle 4, 10, 11 \rangle$. Then,

$$\begin{array}{|c|c|c|c|}
\hline
\text{Ap}(S) & 0 & 21 & 10 & 11 \\
\text{Ap}(M) & 4 & 21 & 10 & 11 \\
\text{Ap}(2M) & 8 & 21 & 14 & 15 \\
\hline
\end{array}$$

and

$$G = F \oplus F \cdot (t^{10})^* \oplus F \cdot (t^{11})^* F \cdot (t^{21})^* \cong F \oplus F(-1)^2 \oplus F(-2).$$
(3) Let $S = \langle 4, 11, 29 \rangle$. Then, and
\[
G = F \oplus F \cdot (t^{11})^* \oplus F \cdot (t^{22})^* F \cdot (t^{33})^* \oplus F \cdot (t^{29})^* \\
\cong F \oplus F(-1) \oplus F(-2) \oplus F(-3) \oplus (F/(t^4)^* F)(-1).
\]

(4) Let $S = \langle 4, 11 \rangle$. Then,
\[
G = F \oplus F \cdot (t^{11})^* \oplus F \cdot (t^{22})^* F \cdot (t^{33})^* \cong F \oplus F(-1) \oplus F(-2) \oplus F(-3).
\]

The tangent cone $G$ is Buchsbaum if and only if $G \cdot H^0_0(G) = 0$. Moreover, as observed in [CZ1], $H^0_0(G)$ coincides with $T(G)$, the $F$-torsion submodule of $G$. As a consequence, if $G$ is Buchsbaum there cannot exist elements of order $> 1$ in $T(G)$ and then $e_i^j = 1$ for all $i$ and $j$. That is, if $G$ is Buchsbaum then
\[
G \cong F \oplus \bigoplus_{i=1}^{e-1} \left( F(-d_i) \bigoplus_{j=1}^{t_i} \frac{F}{(x^*)F(-b_i^j)} \right).
\]
However, this condition is not sufficient to assure the Buchsbaum property for $G$ as the following examples show.

**Example 4.6.** Consider the numerical semigroup of Example 2.4 and its Apery table:

| Ap(S) | 0  | 29 | 22 | 11 |
|-------|----|----|----|----|
| Ap(M) | 4  | 29 | 22 | 11 |
| Ap(2M) | 8  | 33 | 22 | 15 |
| Ap(3M) | 12 | 33 | 26 | 19 |

and
\[
G = F \oplus F \cdot (t^{11})^* \oplus F \cdot (t^{22})^* F \cdot (t^{33})^* \cong F \oplus F(-1) \oplus F(-2) \oplus F(-3).
\]

Consider the numerical semigroup of Example 2.4 and its Apery table:

| Ap(S) | 0  | 6  | 12 | 13 | 19 |
|-------|----|----|----|----|----|
| Ap(M) | 5  | 6  | 12 | 13 | 19 |
| Ap(2M) | 10 | 11 | 12 | 18 | 19 |
| Ap(3M) | 15 | 16 | 17 | 18 | 24 |
| Ap(4M) | 20 | 21 | 22 | 23 | 24 |
Then, we have that $0 \neq (t^6)^*, (t^{13})^* \in \frac{m}{m^2} \subseteq G$, $(t^{13})^* \in T(G)$, and $0 \neq (t^6)^* \cdot (t^{13})^* = t^{19} \in \frac{m^2}{m^3}$ and so $G$ is not Buchsbaum.

**Example 4.7.** Let $S = \langle 9, 10, 11, 23 \rangle$. The Apery table is

|      | Ap(S) | Ap(m) | Ap(2M) | Ap(3M) | Ap(4M) |
|------|-------|-------|--------|--------|--------|
| Ap(S)| 0     | 10    | 11     | 21     | 22     |
| Ap(M)| 9     | 10    | 11     | 21     | 22     |
| Ap(2M)| 18  | 19    | 20     | 21     | 22     |
| Ap(3M)| 27  | 28    | 29     | 30     | 31     |
| Ap(4M)| 36  | 37    | 38     | 39     | 40     |

Then, $0 \neq (t^{11})^*, (t^{23})^* \in \frac{m}{m^2} \subseteq G$, $(t^{23})^* \in T(G)$ and $0 \neq (t^{11})^* \cdot (t^{23})^* \in \frac{m^2}{m^3}$ and so $G$ is not Buchsbaum.

**Lemma 4.8.** If $G \cong F \oplus \bigoplus_{i=1}^{e-1} F(-d_i) \oplus \frac{F}{(x^*)F}(-b)$, then $G$ is Buchsbaum.

**Proof.** The statement is clear since, in this case, the torsion submodule $T(G)$ coincides with the socle of $G$. \qed

**Corollary 4.9.** Let $A$ be a numerical semigroup ring of multiplicity 4. Then, its tangent is always Buchsbaum.

In other terms, the above lemma says that the tangent cone $G$ of a numerical semigroup ring that verifies $\lambda(H^G_+(G)) \leq 1$ is Buchsbaum. Victoria A. Sapko has conjectured in [S] that the converse is true for the case of a 3-generated semigroup ring. Recently, Yi Huang Shen [Sh] has given a positive answer to this conjecture, but on the basis of some of our computations one could ask for a similar question for any numerical semigroup ring.

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