PARK–TARTER MATRIX FOR A DYON–DYON SYSTEM

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Abstract

The problem of separation of variables in a dyon–dyon system is discussed. A linear transformation is obtained between fundamental bases of this system. Comparison of the dyon–dyon system with a 4D isotropic oscillator is carried out.

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1 Introduction

In this paper, we have calculated the matrix between the spherical and parabolic bases of a dyon–dyon system [1] belonging to the same energy level. This matrix is a generalization of the Park–Tarter matrix known from the theory of hydrogen atom [2, 3] to the case when the Coulomb center carries not only the electric but also magnetic charge. Like the Park–Tarter matrix, our matrix is expressed through the Clebsch–Gordan coefficients $C^c_{a \gamma b \alpha}$, however, in our case $a \neq b$, in contrast to the case of a hydrogen atom. We have also traced the connection of the dyon–dyon problem with that of a 4-dimensional isotropic oscillator. As is known [4], these problems are related to each other by the Kustaanheimo–Stiefel transformation [5] supplemented with the 4th (angular) coordinate. We have shown that the coefficients $C^c_{a \alpha b \beta}$ coincide with the ones [6] of the expansion of the double polar basis over the Euler basis of a 4-dimensional isotropic oscillator.

2 Dyon–Dyon System

A dyon–dyon system in the space $\mathbb{R}^3$ is described by the equation

$$\left[ \left( \frac{\partial}{\partial x_j} - \frac{ie}{\hbar c} A_j \right)^2 - \frac{s^2}{r^2} \right] \psi + \frac{2M_0}{\hbar^2} \left( \epsilon^s + \frac{e^2}{r} \right) \psi = 0 \quad (1)$$

where

$$A_j = \frac{g x_3}{r(r^2 - x_3^2)}(-x_2, x_1, 0)$$

and $s = eg/\hbar c = 0, \pm 1/2, \pm 1, ...$. Each value of $s$ describes its particular dyon–dyon system. At $s = 0$, eq.(1) is reduced to the Schrödinger equation for a hydrogen atom. When $s \neq 0$, equation (1) preserves O(4)-symmetry and therefore variables in it are separated into spherical, parabolic, and prolate spheroidal coordinates [1].

The system (1) possesses a singularity on the axis $x_3$. It is also possible to consider systems with singularities either on the semiaxis $x_3 > 0$ or on $x_3 < 0$, i.e. they are described by the vector potentials

$$A_j^{(\pm)} = \frac{g}{r(r \mp x_3)}(\mp x_2, \pm x_1, 0)$$

and are connected with the system (1) by the gauge transformations

$$A_j^{(\pm)} = A_j + \frac{\partial f^{(\pm)}}{\partial x_j}, \quad \psi^{(\pm)}(\vec{x}) = \psi(\vec{x}) \exp \left( \frac{ie}{\hbar c} f^{(\pm)} \right)$$

with the gauge function $f^{(\pm)} = \pm 2g \arctan x_2/x_1$.

The variables in eq. (1) are separated in spherical and parabolic coordinates.

In the spherical coordinates

$$x_1 = r \sin \theta \cos \varphi, \quad x_2 = r \sin \theta \sin \varphi, \quad x_3 = r \cos \theta \quad (2)$$
the wave function of the dyon–dyon system is of the form [7]

\[ \psi_{nk}(r, \theta, \varphi) = R_{nk}(r)Z_k(\theta)e^{im\varphi} \]

where the functions \( Z_k(\theta) \) and \( R_{nk}(r) \) normalized by the condition

\[ \int_0^\pi \sin \theta Z_k'(\theta)Z_k(\theta)d\theta = \delta_{kk'}, \quad \int_0^\infty r^2 [R_{nk}(r)]^2 dr = 1 \]

are given by the formulae

\[ Z_k(\theta) = N_{km}(1 - \cos \theta)^{-\frac{|m-s|}{2}}(1 + \cos \theta)^{-\frac{|m+s|}{2}}P_{k}\(|m-s|,|m+s|\)(\cos \theta) \]

\[ R_{nk}(r) = C_{nk} \exp \left( -\frac{r}{r_0n} \right) \left( \frac{2r}{r_0n} \right)^k P_{k}\left(\frac{r}{r_0n}\right)^{\frac{|m-s|+|m+s|}{2}} \]

\[ F\left( -n + k + \frac{|m-s|+|m+s|}{2} + 1; 2k + |m-s| + |m+s| + 2; \frac{2r}{r_0n} \right) \]

Here \( P_n^{(\alpha, \beta)}(x) \) are Jacobi polynomials; \( r_0 = h^2/M_0e^2 \) is the Bohr radius. The normalization constants \( N_{km}^{(s)} \) and \( C_{nk}^{(s)} \) equal

\[ N_{km}^{(s)} = \left[ \frac{(2k + |m-s| + |m+s| + 1)k!(k + |m-s| + |m+s|)!}{2^{2|m-s|+|m+s|+1}k!(k + |m-s| + 1)k!(k + |m+s| + 1)} \right]^{1/2} \]

\[ C_{nk}^{(s)} = \frac{2}{n^2r_0^{3/2}} \left( \frac{1}{(2k + |m-s| + |m+s| + 1)!} \right)^{1/2} \left( \frac{(n + k + |m-s|+|m+s|)!}{(n - k - \frac{|m-s|+|m+s|}{2} - 1)!} \right) \]

Quantum numbers run over the values \( n = 1, 3/2, 2, ..., k = 0, 1, ..., k_{\text{max}} \), where

\[ k_{\text{max}} = n - \frac{|m-s| + |m+s|}{2} - 1 \]

The energy spectrum of the system is of the form

\[ \epsilon_n^s = -\frac{M_0e^4}{2h^2n^2} \quad (3) \]

In the parabolic coordinates

\[ x_1 = \sqrt{\xi\eta} \cos \varphi, \quad x_2 = \sqrt{\xi\eta} \sin \varphi, \quad x_3 = \frac{1}{2}(\xi - \eta) \quad (4) \]

upon the substitution

\[ \psi(\xi, \eta, \varphi) = f_1(\xi)f_2(\eta)e^{im\varphi} \]

\[ \sqrt{2\pi} \]
the variables in (1) are separated, which results in the system of equations
\[
\frac{d}{d\xi} \left( \xi \frac{df_1}{d\xi} \right) + \left[ \frac{M_0 c^2}{2\hbar^2} \xi - \frac{(m + s)^2}{4\xi} + \beta_1 \right] f_1 = 0
\]
\[
\frac{d}{d\eta} \left( \eta \frac{df_2}{d\eta} \right) + \left[ \frac{M_0 c^2}{2\hbar^2} \eta - \frac{(m - s)^2}{4\eta} + \beta_2 \right] f_2 = 0
\]
where
\[
\beta_1 + \beta_2 = \frac{M_0 c^2}{\hbar^2}
\] (5)

At \( s = 0 \), these equations coincide with the equations for a hydrogen atom in the parabolic coordinates [8], and consequently,
\[
\psi_{n_1 n_2 m}(\xi, \eta, \varphi) = \frac{\sqrt{2}}{n_2 r_0^{3/2}} f_{n_1, m+s}(\xi) f_{n_2, m-s}(\eta) e^{im\varphi} \sqrt{2\pi}
\]
where
\[
f_{pq}(x) = \frac{1}{\Gamma(|q| + 1)} \sqrt{\frac{\Gamma(p + |q| + 1)}{p!}} \exp \left( -\frac{x}{2r_0 n} \right) \left( \frac{x}{r_0 n} \right)^{\frac{|q|}{2}} F \left( -p; |q| + 1; \frac{x}{r_0 n} \right)
\]

Here \( n_1 \) and \( n_2 \) are non-negative integers
\[
n_1 = -\frac{|m + s| + 1}{2} + \frac{\hbar}{\sqrt{-2M_0 c^2}} \beta_1, \quad n_2 = -\frac{|m - s| + 1}{2} + \frac{\hbar}{\sqrt{-2M_0 c^2}} \beta_2
\]
from which and (3), (5) it follows that the parabolic quantum numbers \( n_1, n_2, m \) and \( s \) are connected with the principal quantum number \( n \) as follows:
\[
n = n_1 + n_2 + \frac{|m - s| + |m + s|}{2} + 1 \quad (6)
\]

3 Park–Tarter Generalized Matrix

We write the searched expansion in the form
\[
\psi_{n_1 n_2 m}^{(s)}(\xi, \eta, \varphi) = \sum_{k=0}^{k_{\text{max}}} T_{n_1 n_2 k m}^{(s)} \psi_{nk m}'^{(s)}(r, \theta, \varphi)
\] (7)

Our purpose is to calculate the coefficients \( T_{n_1 n_2 k m}^{(s)} \), i.e. the Park–Tarter generalized matrix. The usual Park–Tarter matrix is the matrix \( T_{n_1 n_2 k m}^{(s)} \) at \( s = 0 \).

We substitute
\[
\xi = r(1 + \cos \theta), \quad \eta = r(1 - \cos \theta),
\]
into the left-hand side of expansion (7), let \( r \) tend to infinity, take the formula
\[
F(-n; c; x) \sim (-1)^n \frac{\Gamma(c)}{\Gamma(c + n)} x^n, \quad (x \to \infty)
\]
and the orthogonality condition for the function $Z_{n_1n_2km}^{(s)}$ into account. All this leads to the formula

$$T_{n_1n_2km}^{(s)} = (-1)^k B_{n_1n_2km}^{(s)} I_{n_1n_2km}^{(s)}$$

where

$$B_{n_1n_2km}^{(s)} = \frac{(2k + |m - s| + |m + s| + 1)k!(k + |m - s| + |m + s|)!}{2^{n_1+n_2+|m-s|+|m+s|} \Gamma(k + |m - s| + 1) \Gamma(k + |m + s| + 1)} \left[ \frac{1}{(n_1)!(n_2)!} \Gamma(n_1 + |m + s| + 1) \Gamma(n_2 + |m - s| + 1) \right]^{1/2}$$

and the second factor is equal to the integral

$$I_{n_1n_2km}^{(s)} = \int_{-1}^{1} (1 - x)^{n_2+|m-s|}(1 + x)^{n_1+|m+s|} P_k^{(|m-s|,|m+s|)}(x) dx$$

Then taking advantage of the Rodrigues formula [9]

$$P_n^{(a,\beta)}(x) = \frac{(-1)^n}{2^n n!} (1 - x)^{-a} (1 + x)^{-\beta} \frac{d^n}{dx^n} \left[ (1 - x)^{a+n} (1 + x)^{\beta+n} \right]$$

and the integral representation for the Clebsch–Gordan coefficients [10]

$$C_{a_c\gamma}^{c\gamma} = \delta_{a+c=\gamma} \left[ \frac{(2c + 1)(J + 1)! (J - 2c)! (c + \gamma)!}{(J - 2a)! (J - 2b)! (a - \alpha)! (a + \alpha)! (b - \beta)! (b + \beta)! (c - \gamma)!} \right]^{1/2} \cdot \frac{(-1)^{a-c+\beta}}{2^{J+1}} \int_{-1}^{1} (1 - x)^{a-\alpha} (1 + x)^{b-\beta} \frac{d^{c-\gamma}}{dx^{c-\gamma}} \left[ (1 - x)^{J-2a} (1 + x)^{J-2b} \right] dx$$

$$\left( J = a + b + c \right), \text{ we obtain}$$

$$T_{n_1n_2ms}^{nk} = (-1)^{n_2+k} C_{a_c\gamma}^{c\gamma}$$

where

$$a = \frac{n_1 + n_2 + |m + s|}{2}, \quad b = \frac{n_1 + n_2 + |m - s|}{2}, \quad c = k + \frac{|m - s| + |m + s|}{2}$$

$$\alpha = \frac{n_1 - n_2 + |m + s|}{2}, \quad \beta = \frac{n_2 - n_1 + |m - s|}{2}, \quad \gamma = \frac{|m - s| + |m + s|}{2}$$

At $s = 0$ formula (8) turns into the Park–Tarter formula, as would be expected.
4 Dyon–Dyon System and 4D Oscillator

Let us demonstrate that if in eq. (1) we make the changes

\[ s \to -i \frac{\partial}{\partial \gamma}, \quad \psi(\vec{x}) \to \psi(\vec{x}, \gamma) = \psi(\vec{x}) \frac{e^{i s \gamma}}{\sqrt{4\pi}} \]  

\[(\gamma \in [0, 4\pi]),\] it will transform into the Schrödinger equation for a 4D isotropic oscillator.

Equation (1) in the spherical coordinates is of the form

\[
\frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial \psi}{\partial r} \right) + \frac{1}{r^2} \left[ \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial \psi}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2 \psi}{\partial \varphi^2} \right] - \frac{2is \cos \theta \frac{\partial \psi}{\partial \varphi}}{r^2 \sin^2 \theta} - \frac{s^2}{r^2 \sin^2 \theta} \psi - \frac{2M_0}{\hbar^2} \left( \epsilon^s + \frac{\epsilon^2}{r} \right) \psi = 0 \]  

(10)

From (9) and (10) we have

\[
\left[ \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial}{\partial r} \right) - \frac{\hat{J}^2}{r^2} \right] \psi + \frac{2M_0}{\hbar^2} \left( \epsilon^s + \frac{\epsilon^2}{r} \right) \psi = 0 \]  

(11)

where

\[
\hat{J}^2 = -\left[ \frac{1}{\sin \beta} \frac{\partial}{\partial \beta} \left( \sin \beta \frac{\partial}{\partial \beta} \right) + \frac{1}{\sin^2 \beta} \left( \frac{\partial^2}{\partial \alpha^2} - 2 \cos \beta \frac{\partial^2}{\partial \alpha \partial \gamma} + \frac{\partial^2}{\partial \gamma^2} \right) \right] \]

Here we change the notation: \( \beta = \theta \) and \( \alpha = \varphi \). If we now pass from the coordinates \( r, \alpha, \beta, \gamma \) to the coordinates

\[
\begin{align*}
u_0 + iu_1 &= u \cos \frac{\beta}{2} e^{-i \frac{\alpha + \gamma}{2}}, \\
u_2 + iu_3 &= u \sin \frac{\beta}{2} e^{i \frac{\alpha + \gamma}{2}}
\end{align*}
\]  

(12)

with \( u^2 = r \), take into account that

\[
\frac{\partial^2}{\partial u^2} = \frac{1}{u^3} \frac{\partial}{\partial u} \left( u^3 \frac{\partial}{\partial u} \right) - \frac{4}{u^2} \hat{J}^2
\]

and introduce the notation

\[
E = 4\epsilon^2, \quad \epsilon^s = -\frac{M_0 \omega^2}{8}
\]

then equation (11) will turn into the Schrödinger equation for a 4D isotropic oscillator

\[
\left[ \frac{\partial^2}{\partial u^2} + \frac{2M_0}{\hbar} \left( E - \frac{M_0 \omega^2 u^2}{2} \right) \right] \psi(u) = 0
\]

whose energy spectrum is given by the formula

\[
E_N = \hbar \omega (N + 2)
\]  

(13)
Introducing the double polar coordinates
\[ u_{0} + iu_{1} = \rho_{1}e^{-i\varphi_{1}}, \quad u_{2} + iu_{3} = \rho_{2}e^{i\varphi_{2}} \]
from formulae (2), (4), (12), and (14) we get the relations
\[ \xi = 2\rho_{1}^{2}, \quad \eta = 2\rho_{2}^{2}, \quad \varphi = \varphi_{1} + \varphi_{2}, \quad \gamma = \varphi_{1} - \varphi_{2} \]
which lead to the formulae
\[ \psi_{N_{1}M_{1}M_{2}}(u, \alpha, \beta, \gamma) = 4n\sqrt{2} \lambda \delta_{n_{1},N_{1}} \delta_{n_{2},N_{2}} \delta_{m_{1}+m_{2},m_{1}+M_{2}} \psi_{n_{1}n_{2}m_{1}m_{2}}(r, \theta, \varphi, \gamma) \]
\[ \psi_{N_{1}N_{2}m_{1}m_{2}}(\rho_{1}, \rho_{2}, \varphi_{1}, \varphi_{2}) = 4n\sqrt{2} \lambda \delta_{n_{1},N_{1}} \delta_{n_{2},N_{2}} \delta_{m_{1}+m_{2},m_{1}+M_{2}} \psi_{n_{1}n_{2}m_{1}m_{2}}(\xi, \eta, \varphi, \gamma) \]
generalizing the earlier results [6, 11].

Now we are able to write the expansion [6]
\[ \psi_{N_{1}N_{2}m_{1}m_{2}}(\rho_{1}, \rho_{2}, \varphi_{1}, \varphi_{2}) = \sum_{J=J_{\text{min}}}^{N/2} W_{N_{1}N_{2}m_{1}m_{2}}^{N_{1}M_{1}M_{2}} \psi_{N_{1}M_{1}M_{2}}(u, \alpha, \beta, \gamma) \]
where
\[ W_{N_{1}N_{2}m_{1}m_{2}}^{N_{1}M_{1}M_{2}} = e^{i\pi\Phi} C_{a_{0},b_{0},c_{0},\gamma_{0}}^{a_{0},b_{0},c_{0},\gamma_{0}} \]
\[ a_{0} = \frac{N + |m_{1}| - |m_{2}|}{4}, \quad b_{0} = \frac{N - |m_{1}| + |m_{2}|}{4}, \quad c_{0} = J, \quad \alpha_{0} = \frac{N + |m_{1}| - |m_{2}|}{4} - N_{2}, \quad \beta_{0} = \frac{N - |m_{1}| + |m_{2}|}{4} - N_{1}, \quad \gamma_{0} = \frac{|m_{1}| + |m_{2}|}{2} \]
The lower limit of summation in (15) and quantity \( \Phi \) are given by the expressions
\[ J_{\text{min}} = \frac{1}{2}(|M_{1} - M_{2}| + |M_{1} + M_{2}|) \]
\[ \Phi = N_{2} + J - \frac{|m_{1}| + |m_{2}|}{2} - \frac{m_{2} + |m_{2}|}{2} \]

We conclude with the following two comments:
(a) Using formulae (2) and (12) and considering that \( r = u^{2}, \theta = \beta, \varphi = \alpha \), one can easily show that
\[ x_{1} = 2(u_{0}u_{2} + u_{1}u_{3}) \]
\[ x_{2} = 2(u_{0}u_{3} - u_{1}u_{2}) \]
\[ x_{3} = u_{0}^{2} + u_{1}^{2} - u_{2}^{2} - u_{3}^{2} \]
\[ \gamma = \frac{i}{2} \ln \frac{(u_{0} + iu_{1})(u_{2} + iu_{3})}{(u_{0} - iu_{1})(u_{2} - iu_{3})} \]
The first three lines are the transformation $\mathbb{R}^4 \rightarrow \mathbb{R}^3$ suggested by Kustaanheimo and Stiefel for the regularization of equations of celestial mechanics [5]. Later, this transformation found other applications, as well [12, 13]. This transformation supplemented with the coordinate $\gamma$ was used for the "synthesis" of the dyon–dyon system from the 4D isotropic oscillator [4].

(b) It is known [6] that diagonal ($m_1 = m_2$) elements of the matrix $W_{N_1 N_2 m_1 m_2}^{N M_1 M_2}$ with $N$ even coincide with the Park–Tarter matrix. From formula (16) it follows that the remaining elements of the matrix $W_{N_1 N_2 m_1 m_2}^{N M_1 M_2}$ have also a physical meaning: these are elements of the generalized Park–Tarter matrix for the dyon–dyon system.

5 Degeneracy of the Energy Levels

Let us discuss the problem of multiplicity of degeneration of the energy levels (3) and (13). From formula (6) it follows that at fixed $n, m$ and $s$ the energy levels are degenerate with the multiplicity

$$g_{nm}^s = n - \frac{|m - s| + |m + s|}{2}$$

For $s \geq 0$ the multiplicity of degeneration of levels (3) at fixed $s$ and $n$ is

$$g_n^s = \sum_{|m| \geq |s|} g_{nm}^s + \sum_{|m| \leq |s| - 1} g_{nm}^s$$

where the upper limit of summation is determined from the condition $g_{nm}^s \geq 0$,

$$|m - s| + |m + s| \leq 2n - 2$$

Therefore,

$$g_n^s = \sum_{m=-|s|+1}^{|s|-1} (n - s) + 2 \sum_{m=|s|}^{n-1} (n - m) = (n - s)(n + s) \quad (17)$$

The same result follows from analogous computations also when $s < 0$.

The quantum numbers $s$ and $n$ in formula (17) assume simultaneously either integer or half-integer values, and thus, we have

$$g_n = \sum_{s=-n+1}^{n-1} g_n^s = \frac{1}{3} n(2n - 1)(2n + 1)$$

where $g_n$ stands for the multiplicity of degeneration of the energy levels (13) of the 4d oscillator. Since $N = 2n - 2$, we arrive at the known result

$$g_N = \frac{1}{6} (N + 1)(N + 2)(N + 3)$$

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