Group theoretical foundations of the quantum theory of an interacting particle

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July 7, 2016

Abstract

1 Introduction

Group theoretical methods, due in particular to E. Wigner and G. Mackey, allow to attain a formulation of the Quantum Theory of a free particle through a purely deductive development based on symmetry principles. These approaches enforce the circumstance that Galilei’s group $G$ (or Poincaré’s group $P$, for a relativistic theory) is a group of quantum symmetry transformations for an isolated system, so that Wigner’s theorem [1],[2] on the representation of symmetries and Mackey’s imprimitivity theorem [3],[4] can be applied to deduce the explicit Quantum Theory of a free particle [5]. We outline in section 2.3 how such a deduction can be carried out. In so doing, it is avoided invoking canonical quantization, which formulates the Quantum Theory of a physical system, roughly speaking, by replacing classical magnitudes with operators in the equations of the classical theory of the system.

The extension of the group theoretical approach, so satisfactory for a free particle, to an interacting particle encounters serious problems; the main obstacle is the fact that for a non-isolated system the galileian transformations, or the transformations of Poincaré in the relativistic case, do not form a group of symmetry transformations [15], so that neither Wigner’s theorem nor Mackey’s imprimitivity theorem can directly apply.

To overcome this difficulty, the approaches that in the literature extend the cited group theoretical method to the non-relativistic interacting particle (e.g., [5],[6],[7],[15]) adopt the following sentence as a valid statement.

PROJ. Each Galilean transformation $g \in G$ is assigned a unitary or an anti-unitary operator $U_g$ which realizes the corresponding quantum transformations of observables according to $A \rightarrow U_gAU_g^*$; the correspondence $g \rightarrow U_g$ is a projective representation.
We stress that, contrary to the free particle case where PROJ is implied by the existence of symmetry conditions, for an interacting particle such an implication cannot be carried out because these symmetry conditions fail.

Now, in sect. 2.4 we prove that, if time is homogeneous, statement PROJ forces the hamiltonian operator ruling over the quantum dynamics of a spin-0 particle into the form

\[ H = -\frac{1}{2\mu} \left( \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2} \right) + \Phi(x_1, x_2, x_3). \]

Therefore a Quantum Theory where PROJ holds cannot describe all empirically known interactions; in particular, it excludes electromagnetic interactions. An empirically more adequate approach should abandon assumption PROJ.

In this work we pursue a group theoretical approach to the Quantum Theory of an interacting particle without making use of PROJ or other conditions that are not implied by physical principles. The present article accomplishes this task in the non-relativistic case; the development of the approach for a relativistic theory is in progress.

RIFARE! First we find results which hold both for a non-relativistic and for a relativistic theory, i.e. independently of which group \( \Upsilon \) of space-time transformations, \( \Upsilon = \mathcal{G} \) or \( \Upsilon = \mathcal{P} \), is taken into account.

The concept of quantum transformation corresponding to a space-time transformation \( g \in \Upsilon \), in absence of the condition of symmetry, is introduced and studied (sect. 2.2).

In particular we show how the general features of the concept of quantum transformation imply that each space-time transformations \( g \in \Upsilon \) can be assigned a unitary operator \( U_g \), also if the transformation is not a symmetry; moreover, we prove that anti-unitary operators must be excluded also in the non-zero interaction case where the usual proofs [5], which make use of the hypothesis that \( g \rightarrow U_g \) is a projective representation, do not apply.

However, the correspondence \( g \rightarrow U_g \) is not a projective representation, yet; hence, one of the conditions for applying the theorem of imprimitivity fails, and the approach stops again. To address this problem, the correspondence \( g \rightarrow U_g \) is converted into a non-physical projective representation by means of a so-called \( \sigma \)-conversion.

We show, in a non-relativistic theory, how the operators which physically represent the position of the particle can be explicitly identified in the case that the interaction admits a \( \sigma \)-conversion that leaves unaltered the covariance properties of the position with respect to the space-time transformations. Then we determine the general dynamical law for this class of interactions, where the hamiltonian operator turns out to be more general than the currently practiced hamiltonians of a particle; we show how the form of the dynamical equation is constrained by the covariance properties preserved by the \( \sigma \)-conversion admitted by the interaction; in particular, we exactly identify which covariance properties characterize electromagnetic interaction among all possible ones.
2 Space-time and quantum transformations

2.1 Mathematical tools

Let us introduce, to begin, the notation for the mathematical structures involved in
the work. The Quantum Theory of a physical system, formulated in a complex and
separable Hilbert space  \( \mathcal{H} \), needs the following mathematical structures.

- The set  \( \Omega(\mathcal{H}) \) of all self-adjoint operators of  \( \mathcal{H} \), which represent quantum observables.
- The complete, ortho-complemented lattice  \( \Pi(\mathcal{H}) \) of all projections operators of  \( \mathcal{H} \), i.e. quantum observables with possible outcomes in  \( \{0, 1\} \).
- The set  \( \Pi_1(\mathcal{H}) \) of all rank one orthogonal projections of  \( \mathcal{H} \).
- The set  \( \mathcal{S}(\mathcal{H}) \) of all density operators of  \( \mathcal{H} \), which represent quantum states.
- The set  \( \mathcal{U}(\mathcal{H}) \) of all unitary operators of the Hilbert space  \( \mathcal{H} \).

In the group theoretical approach a key role is played by the imprimitivity theorem
of Mackey, which is a representation theorem for imprimitivity systems relative to
projective representations [4]. The following definition recalls the notion of projective
representation.

**Definition 2.1.** Let  \( G \) be a separable, locally compact group with identity element \( e \). A correspondence  \( U : G \to \mathcal{U}(\mathcal{H}), g \to U_g \), with  \( U_e = \mathbb{I} \), is a projective representation of  \( G \) if the following conditions hold.

i) A complex function \( \sigma : G \times G \to \mathbb{C} \) such that  \( |\sigma(g_1, g_2)| = 1 \) for all  \( g_1, g_2 \in G \), called multiplier, exists such that  \( U_{g_1g_2} = \sigma(g_1, g_2)U_{g_1}U_{g_2} \);

ii) for all  \( \phi, \psi \in \mathcal{H} \), the mapping  \( g \to \langle U_g\phi | \psi \rangle \) is a Borel function in  \( g \).

A projective representation with multiplier  \( \sigma \) is also called  \( \sigma \)-representation.

A projective representation is said to be continuous if for any fixed  \( \psi \in \mathcal{H} \) the mapping  \( g \to U_g\psi \) from  \( G \) into  \( \mathcal{H} \) is continuous with respect to  \( g \).

Let  \( E \) be the Euclidean group, i.e. the semi-direct product  \( E = \mathbb{R}^3 \rtimes SO(3) \) between the group of spatial translations  \( \mathbb{R}^3 \) and the group of spatial proper rotations  \( SO(3) \); each transformation  \( g \in E \) bi-univocally corresponds to the pair  \( (a, R) \in \mathbb{R}^3 \times SO(3) \) such that  \( R^{-1}x - R^{-1}a \equiv g(x) \) is the result of the passive transformation of the spatial point  \( x = (x_1, x_2, x_3) \) by  \( g \). The general imprimitivity theorem is an advanced mathematical result; in this article we shall make use of this theorem relatively to the euclidean group  \( E \) only. Then we introduce the concept of imprimitivity system and the theorem for this specific case.

**Definition 2.2.** Let  \( \mathcal{H} \) be the Hilbert space of a  \( \sigma \)-representation  \( g \to U_g \) of the Euclidean group  \( E \). A projection valued (PV) measure  \( E : B(\mathbb{R}^3) \to \Pi(\mathcal{H}), \Delta \to E(\Delta) \) is an imprimitivity system for the  \( \sigma \)-representation  \( g \to U_g \) if the relation

\[
U_gE(\Delta)U_g^{-1} = E(g^{-1}(\Delta)) \equiv E(R(\Delta) + a) \tag{eq43}
\]

holds for all  \( (a, R) \in E \).
Mackey’s theorem of imprimitivity for $\mathcal{E}$. If a PV measure $E : \mathcal{B}(\mathbb{R}^3) \to \Pi(\mathcal{H})$ is an imprimitivity system for a continuous $\sigma$-representation $g \to U_g$ of the Euclidean group $\mathcal{E}$, then a $\sigma$-representation $L : SO(3) \to \mathcal{U}(\mathcal{H}_0)$ exists such that, modulo a unitary isomorphism,

$$(M.1) \quad \mathcal{H} = L_2(\mathbb{R}^3, \mathcal{H}_0),$$

$$(M.2) \quad (E(\Delta)\psi)(x) = \chi_\Delta(x)\psi(x), \quad \text{where } \chi_\Delta \text{ is the characteristic functional of } \Delta,$$

$$(M.3) \quad (U_g\psi)(x) = L_R\psi(g(x)) \equiv L_R\psi(R^{-1}x - R^{-1}a), \quad \text{for every } g = (a, R) \in \mathcal{E}.$$ 

Furthermore, the $\sigma$-representation $U$ is irreducible if and only if the “inducing” representation $L$ is irreducible.

### 2.2 Conceptual basis

In this subsection we formulate a concept of quantum transformation, which is viable also for space-time transformations that are not symmetry transformations.

For sake of synthesis, in the following by $\Upsilon$ we denote the group $G$ of galileian transformations or the group $P$ of Poincaré’s transformations, without time or space inversions; therefore, what stated for $\Upsilon$ must be understood stated for $G$ and also for $P$.

In the present work the group $\Upsilon$ is interpreted as a group of changes of reference frame in a class $\mathcal{F}$ of frames which move uniformly with respect to each other. So, given any reference frame $\Sigma$, a transformation $g \in \Upsilon$ univocally singles out the reference frame $\Sigma_g$ related to $\Sigma$ just by $g$.

Let us consider the Quantum Theory of a localizable particle, that is to say of a physical system which can be localized in a point of the physical space, so that its Quantum Theory contains three commuting self-adjoint operators $(Q_1, Q_2, Q_3) \equiv Q$ representing the three coordinates of the position. Now, the point of the space, where the particle is localized by the measurement of the position observables, is identified only if the frame is specified the values of the coordinates refer to. In fact, if $(Q_1, Q_2, Q_3) \equiv Q$ are the three self-adjoint operators which represent the three coordinates of the position with respect to $\Sigma$ and if $g \in \mathcal{E}$, then the $\alpha$-th coordinate of the position with respect to another frame $\Sigma_g$, related to $\Sigma$ by $g$, must be represented by $[g(Q)]_\alpha$, where $g(x)$ is the representation with respect to $\Sigma_g$ of the spatial point represented by $x$ with respect to $\Sigma$. In the non-relativistic case, a pure galileian boost $g \in G$ characterized by a velocity $u = (u, 0, 0)$, does not change the instantaneous position at all; hence $g(x) = x$ and $S_{g}^{\Sigma}[Q] = g(Q) = Q$, so that the operators which represent the “position with respect to $\Sigma_g$” coincide with the operators representing the position with respect to $\Sigma$. In order to transform the position quantum observables at time $t$, i.e. the operators $Q(t) = e^{iHt}Q(t)e^{-iHt}$, by a galileian boost $g$, a function $g_t$ different from $g$ must be used. Indeed, $Q(t)$ represents the position measured with a delay $t$, therefore the operators which represent the “position at time $t$ with respect to $\Sigma_g$” must be $S_{g}^{\Sigma}[Q(t)] = (Q(t) - ut, Q_2, Q_3) \equiv g_t(Q(t))$, where $g_t(x) = (x_1 - ut, x_2, x_3)$

In general, we can state that for every $g \in G$ the following covariance relations hold for all $g \in G$,

$$(i) \quad S_{g}^{\Sigma}[Q] = g(Q), \quad (ii) \quad S_{g}^{\Sigma}[Q(t)] = g_t(Q(t)), \quad (eq17)$$

where $g_t$ is a suitable function, in general different from $g$. In fact, relations (eq17) are the conditions which define the position operators of a localizable particle.
A priori we cannot exclude that also observables other than position change their representation according to the frame they are referred to; so, in order that Quantum Theory can account for such a possibility, it must appropriately extend the transformations $S^\Sigma_g$ to all quantum observables. To this aim, given two reference frames $\Sigma_1$ and $\Sigma_2$ in $F$, we introduce the following concept of relative indistinguishability between measuring procedures:

If a measuring procedure $\mathcal{M}_1$ is relatively to $\Sigma_1$ identical to what is $\mathcal{M}_2$ relatively to $\Sigma_2$, we say that $\mathcal{M}_1$ and $\mathcal{M}_2$ are indistinguishable relatively to $\Sigma_1$ and $\Sigma_2$.

Then, for every $g \in \Upsilon$ and every $\Sigma$ in $F$ we introduce the mapping

$$S^\Sigma_g : \Omega(\mathcal{H}) \rightarrow \Omega(\mathcal{H}), \quad A \rightarrow S^\Sigma_g[A] \quad (eq16)$$

with the following conceptually explicit interpretation.

(QT) The self-adjoint operators $A$ and $S^\Sigma_g[A]$ represent two measuring procedures $\mathcal{M}_1$ and $\mathcal{M}_2$ indistinguishable relatively to $\Sigma$ and $\Sigma_g$.

For instance, if $A$ represents a detector placed in the origin of $\Sigma$ with a given orientation relative to $\Sigma$, then $S^\Sigma_g[A]$ is the operator that represents the same detector placed in the origin of $\Sigma_g$ with that orientation relative to $\Sigma_g$. It must be noticed that (QT) presupposes that for each quantum observable $A \in \Omega(\mathcal{H})$, two measuring procedures with the required relative indistinguishability exist, at least in principle.

We call $S^\Sigma_g$ the quantum transformation corresponding to $g$.

In (eq17) the action of the transformations $S^\Sigma_g$ on the position operators $Q(t)$ is explicitly specified; for an arbitrary observable no such a kind of explicit specification can be a priori established. However, the authentic meaning (QT) of the notion of quantum transformation is sufficient to infer, at a conceptual level, the following general constraint.

(S.1) For every frame $\Sigma$ in $F$ the following statement holds.

$$S^\Sigma_{gh}[A] = S^\Sigma_h[S^\Sigma_g[A]]^\ast, \quad for \ all \ A \in \Omega(\mathcal{H}). \quad (eq40)$$

This statement stresses how in general, i.e. without further particular conditions, the quantum transformations depend on the “starting” frame.

### 2.3 Symmetry transformations

Let us now briefly outline the important implications of the existence of conditions of symmetry, in particular for a free particle. A transformation $h \in \Upsilon$ is a symmetry transformation for the physical system under investigation if a class $F$ exists such that for every frame $\Sigma$ in $F$, the frames $\Sigma$ and $\Sigma_h$ are equivalent for the formulation of the empirical theory of the system.

The symmetry property allows to apply Wigner’s theorem, and in so doing the following well known implication is obtained [1],[2],[10],[8].

SYM.1. If $g \in \Upsilon$ is a symmetry transformation then a unitary or an anti-unitary operator $U^\Sigma_g$, unique up a phase factor, exists such that

$$S^\Sigma_g[A] = U^\Sigma_g A [U^\Sigma_g]^\ast. \quad (eq41)$$
Now, according to the Principle of Relativity, for an isolated system all \( g \in \Upsilon \) are symmetry transformations. Therefore, a class \( \mathcal{F} \) exists such that the following statement holds.

**SYM.2.** In the Quantum Theory of an isolated system, for each \( g \in \Upsilon \) the quantum transformation \( S^\Sigma_g \) must be independent of \( \Sigma \), i.e. \( S^\Sigma_g = S^\Sigma_h \equiv S_g \) and \( U^\Sigma_g = e^{i\lambda} U^\Sigma_h \) (with \( \lambda \in \mathbb{R} \)), so that (eq40) and (eq41) imply

\[
S_{gh}[A] = S_g [S_h[A]] \ , \ i.e. \ U_{gh} = \sigma(g, g)U_g U_h . \tag{eq42}
\]

Thus, if \( \Upsilon \) is a group of symmetry transformations, the correspondence \( g \to U_g \) such that \( S_g[A] = U_g AU^{-1}_g \) is a projective representation \([4],[5],[12]\). As a consequence, each \( U_g \) must be unitary \([5],[10]\); in particular, \( U^*_g = U^{-1}_g \).

A free localizable particle is just a particular kind of isolated system, so that according to Sym.2 for every \( g \in \Upsilon \) a unitary operator \( U_g \) exists such that \( S_g[A] = U_g AU^{-1}_g \). The restriction of \( g \to U_g \) to the euclidean group \( \mathcal{E} \) is a projective representation of \( \mathcal{E} \) \([5]\). Then, according to to (eq17) and Sym.1, the relation \( U_g Q U^{-1}_g = g(Q) \) holds; it entails that the spectral PV measure of \( Q \) is an imprimitivity system \([5]\); therefore we can apply Mackey’s imprimitivity theorem. In so doing, to each choice of the inducing representation \( L \) in Mackey’s theorem and of \( \mu \) in (eq2), there corresponds a different theory.

The simplest choice, i.e. \( L : SO(3) \to \Psi, L_R = 1 \), identifies \( \mathcal{H} = L_2(\mathbb{R}^3) \) as the Hilbert space of the theory, and the position operators as \( (Q_\alpha \psi)(x) = x_\alpha \psi(x) \).

In a non-relativistic theory, by making use of Galilean invariance, valid for a free particle, it can be proved \([5],[14]\) that the form of the hamiltonian operator must be

\[
H = -\frac{1}{2\mu} \sum_{\alpha=1}^{3} \frac{\partial^2}{\partial x^2_\alpha} .
\]

Hence, the simplest theory deduced from symmetry principles by the group theoretical approach is the standard Quantum Theory of a free particle. By choosing \( L \) as an irreducible \( \sigma \)-representation of \( SO(3) \) of dimension \( 2s+1 \) \((s \in \mathbb{N})\), the Standard Quantum Theory of a spin-\( s \) particle is obtained.

### 2.4 The interacting particle problem

If the system under investigation is not isolated, e.g. if it is an interacting particle, then neither Sym.1 nor Sym.2 apply, so that we find an obstacle in extending the group theoretical approach to the non-relativistic interacting particle. However, in the literature several proposals can be found \([5],[6],[7],[15]\) where the group theoretical methods are extended to the interacting case. The aforesaid obstacles, in the non-relativistic case, are overcome by adopting the content of the following statement as a valid condition.

**PROJ.** Each galileian transformation \( g \) is represented in the formalism of the Quantum Theory by a unitary or by an anti-unitary operator \( U_g \) in such a way that

i) \( S^\Sigma_g[A] = U_g AU^*_g \) is the quantum transformation of the observable \( A \) corresponding to \( g \);

ii) the correspondence \( g \to U_g \) is a projective representation.

Statement PROJ is introduced as an assumption in some approaches, e.g., see \([5]\) page 201, \([6]\) page 236; Ekstein, instead, essentially derived it from another assumption,
namely from the “empirical statement that it is possible to give an operational definition of any initial state intrinsically”, i.e. independently of the presence or absence of the interaction (cfr. [15], page 1401). By making use of PROJ, some of the cited approaches [5],[6] deduce that in the non-relativistic Quantum Theory of a spin-0 particle, undergoing an interaction homogeneous in time, the hamiltonian operator $H$ must have the following form, able to describe also interactions of electromagnetic kind [5],[6].

$$H = \frac{1}{2\mu} \sum_{\alpha=1}^{3} \left( -i \frac{\partial}{\partial x_\alpha} + a_\alpha(x) \right)^2 + \Phi(x). \quad (eq8)$$

Now we shall prove, instead, the following statement.

**Stat.** Assumption PROJ implies that the hamiltonian of the Quantum Theory of a spin-0 particle undergoing an interaction homogeneous in time into the form

$$H = \frac{1}{2\mu} \sum_{\alpha=1}^{3} \left( -i \frac{\partial}{\partial x_\alpha} \right)^2 + \Phi(x).$$

To prove the sentence Stat we shall make use of the following well known results of Mathematical Physics.

**MP.1.** As an important implication of Wigner’s theorem, the general evolution law of quantum observables with respect to a homogeneous time is obtained [10]: a self-adjoint operator $H$ exists, called hamiltonian operator, such that

$$A(t) = e^{iHt} A e^{-iHt} \quad \text{and} \quad \frac{d}{dt} A(t) = i[H,A(t)]. \quad (eq3)$$

**MP.2.** Let $g \rightarrow \hat{U}_g$ be every continuous projective representation of Galilei’s group $G$, i.e. the group generated by the euclidean group $E$ and by galileian velocity boosts. Now, the nine one-parameter abelian sub-groups $T_\alpha, R_\alpha, B_\alpha$ of spatial translation, spatial rotations and galileian velocity boosts, relative to axis $x_\alpha$, are all additive; then, according to Stone’s theorem [10], there exist nine self-adjoint generators $\hat{P}_\alpha, \hat{J}_\alpha, \hat{G}_\alpha$ of the nine one-parameter unitary subgroups $\{e^{-i\hat{P}_\alpha a_\alpha}, a \in \mathbb{R}\}, \{e^{-i\hat{J}_\alpha \theta_\alpha}, \theta_\alpha \in \mathbb{R}\}, \{e^{-i\hat{G}_\alpha u_\alpha}, u_\alpha \in \mathbb{R}\}$ representing the sub-groups $T_\alpha, R_\alpha, B_\alpha$ according to the projective representation $g \rightarrow \hat{U}_g$ of the Galilei’s group $G$. The structural properties of $G$ as a Lie group imply the validity of the following commutation relations [9].

(i) $[\hat{P}_\alpha, \hat{P}_\beta] = 0$, (ii) $[\hat{J}_\alpha, \hat{P}_\beta] = i\hat{e}_{\alpha\beta\gamma} \hat{P}_\gamma$, (iii) $[\hat{J}_\alpha, \hat{J}_\beta] = i\hat{e}_{\alpha\beta\gamma} \hat{J}_\gamma$, (iv) $[\hat{J}_\alpha, \hat{G}_\beta] = i\hat{e}_{\alpha\beta\gamma} \hat{G}_\gamma$, (v) $[\hat{G}_\alpha, \hat{G}_\beta] = 0$, (vi) $[\hat{G}_\alpha, \hat{P}_\beta] = i\hat{d}_{\alpha\beta} \mu \hat{I}$, \quad (eq2)

where $\hat{e}_{\alpha\beta\gamma}$ is the Levi-Civita symbol restricted by the condition $\alpha \neq \gamma \neq \beta$, while $\mu$ is a non-zero real number which characterizes the projective representation.

**Proof of Stat.** Now we explicitly prove Stat. Since $g \rightarrow U_g$ in [PROJ] is a projective representation, according to (MP.2) the sub-groups $T_\alpha, R_\alpha, B_\alpha$ can be represented by the one-parameter unitary sub-groups $\{e^{-i\hat{P}_\alpha a_\alpha}, a \in \mathbb{R}\}, \{e^{-i\hat{J}_\alpha \theta_\alpha}, \theta_\alpha \in \mathbb{R}\}, \{e^{i\hat{G}_\alpha u_\alpha}, u_\alpha \in \mathbb{R}\}$, in such a way that the hermitean generators $\hat{P}_\alpha, \hat{J}_\alpha, \hat{G}_\alpha$ satisfy (eq2). Once defined the self-adjoint operators $F_\alpha = \frac{\hat{G}_\alpha}{\mu}$, it can be proved that relations (eq2) imply that the following relation holds for all $g \in G$.

$$U_g F U_g^{-1} = g(F). \quad (eq9)$$
Since by (eq2.v) the $F_\alpha$’s commute with each other, according to spectral theory, a unique PV measure $E : B(\mathbb{R}^3) \to \Pi(\mathcal{H})$ exists such that $F_\alpha = \int \lambda dE_\lambda^{(\alpha)}$, where

\[
E^{(1)}_\lambda = E((-\infty, \lambda] \times \mathbb{R}^2), \quad E^{(2)}_\lambda = E(\mathbb{R} \times (-\infty, \lambda] \times \mathbb{R}), \quad E^{(3)}_\lambda = E(\mathbb{R}^2 \times (-\infty, \lambda]).
\]

Then (eq9) easily implies that $\Delta \rightarrow E(\Delta)$ satisfies (eq43) and hence it is an imprimitivity system for the restriction to $E$ of $g \rightarrow U_g$; therefore Mackey’s theorem applies. In so doing, the simplest choice for $\mathcal{H}_0$, i.e. $\mathcal{H}_0 = \mathcal{F}$, leads to identify $\mathcal{H}$, $F_\alpha$, $P_\alpha$, and $U_g$ for $g \in E$ as

\[
\mathcal{H} = L_2(\mathbb{R}^3), \quad (F_\alpha \psi) (x) = x_\alpha \psi (x), \quad P_\alpha = -i \frac{\partial}{\partial x_\alpha}, \quad (U_g \psi) (x) = \psi (g(x)). \quad (eq10)
\]

Now we can easily prove that the position operators $Q$ coincide with $F = G/\mu$.

**Proposition 2.1.** If PROJ holds, then in the simplest Quantum Theory of a localizable interacting particle the equality $F = Q$ holds for the position operators satisfying the covariance properties (eq17).

**Proof.** If $g \in T_1$ and PROJ holds, so that by (MP.2) $U_g = e^{-i P_\beta a}$, then (eq17.i) implies $[Q_\alpha, P_\beta] = i \delta_{\alpha \beta} I$; since $[F_\alpha, P_\beta] = i \delta_{\alpha \beta} I$ is implied by (eq2.iv), also $[F_\alpha - Q_\alpha, P_\beta] = 0$ holds. On the other hand, making use of (eq17.i) for $U_g = e^{i G_\beta u}$ implies $[F_\alpha - Q_\gamma, F_\beta] = 0$, and hence $F_\alpha - Q_\gamma = c_\alpha \beta \gamma I$ = constant must hold for the irreducibility of $(F, P)$. Finally, taking into account (eq2.iv) and (eq17.i) for $U_g = e^{-i J_\gamma \theta}$ imply $[J_\alpha, F_\beta - Q_\beta] = i \delta_{\alpha \beta}$ imply $[J_\alpha, F_\beta - Q_\gamma] = c_\alpha \beta \gamma (F_\beta - Q_\gamma) = [J_\alpha, c_\beta \gamma I] = 0$; thus, $F_\alpha - Q_\gamma = 0$. 

Prop. 2.1 together with (eq17.ii) is sufficient to determine the form of the Hamiltonian operator $H$ consistent with PROJ. First, we determine $[G_\alpha, \dot{Q}_\beta]$. Let us start with

\[
e^{i G_\alpha u} \dot{Q}_\beta e^{-i G_\alpha u} = \dot{Q}_\beta + i [G_\alpha, \dot{Q}_\beta] u + o(u), \quad (eq11)
\]

where $o(u)$ is an infinitesimal operator of order greater than 1 with respect to $u$. By making use of $\dot{Q}_\beta = i [H, Q_\beta] = \lim_{t \to 0} \frac{Q_\beta - Q_\beta^t}{t}$, and of $e^{i G_\alpha u} Q_\beta^t e^{-i G_\alpha u} = Q_\beta - \delta_{\alpha \beta} ut I$, implied by (eq17.ii), we also find

\[
e^{i G_\alpha u} \dot{Q}_\beta e^{-i G_\alpha u} = \lim_{t \to 0} e^{i G_\alpha u} Q_\beta^t - Q_\beta e^{-i G_\alpha u} = \dot{Q}_\beta - \delta_{\alpha \beta} u I. \quad (eq12)
\]

The comparison between (eq11) and (eq12) yields

\[
[G_\alpha, \dot{Q}_\beta] = [Q_\alpha, \mu \dot{Q}_\beta] = i \delta_{\alpha \beta} I, \quad \text{which implies} \quad [F_\alpha, \mu \dot{Q}_\beta - P_\beta] = 0. \quad (eq13)
\]

This argument can be repeated with $U_g = e^{-i P_\alpha a}$ instead of $e^{i G_\alpha u}$, and also with $U_g = e^{-i J_\gamma \theta}$ instead of $e^{i G_\alpha u}$. In so doing we obtain, respectively, $[P_\alpha, \mu \dot{Q}_\beta - P_\beta] = 0$ and $[J_\alpha, \mu \dot{Q}_\beta] = i \delta_{\alpha \beta} \gamma \mu \dot{Q}_\gamma$; the first of these two equations, together with (eq13), implies $\mu \dot{Q}_\beta - P_\beta = b_\beta I$; then, by making use of the second equation, we obtain $i \delta_{\alpha \beta} \gamma (\mu \dot{Q}_\gamma - P_\gamma) = [J_\alpha, \mu \dot{Q}_\beta - P_\beta] = [J_\alpha, b_\beta I] = 0$, i.e. $\mu \dot{Q}_\beta = P_\beta$.

At this point the determination of $H$ is straightforward. From (eq2.vi) we obtain

\[
i[H, Q_\beta] = \dot{Q}_\beta = \frac{1}{\mu} P_\beta = i \left( \frac{1}{2 \mu} \sum_{\gamma} P_\gamma^2 \frac{G_\beta}{\mu \gamma} \right) = i \left( \frac{1}{2 \mu} \sum_{\gamma} P_\gamma^2 Q_\beta \right). \quad (eq14)
\]
Then the completeness of $Q$ implies that the operator $H - \frac{1}{2\mu} \sum_{\gamma} P_{\gamma}^2$ is a function of $Q$. Thus

$$H = -\frac{1}{2\mu} \left( \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2} \right) + \Phi(Q).$$

(eq15)

Thus, assumption $\text{PROJ}$ forbids the description of electro-magnetic interactions, because their physics is correctly described by the hamiltonian in (eq8).

3 Quantum Theory of an interacting particle

Coherently with the conclusion of the last section, in order to develop a Quantum Theory able to describe also electromagnetic interactions, assumption $\text{PROJ}$ must be abandoned. In this section we undertake such a development, under the hypothesis that the interaction does not destroy time homogeneity, so that the hamiltonian operator $H$ exists such that (eq3) holds.

According to the concept of quantum transformation expressed by (QT), two further constraints (S.2) and (S.3) can be established supplement (S.1).

(S.2) For every $g \in \Upsilon$, the mapping $S_{g}^{\Sigma}$ is bijective.

(S.3) For every real Borel function $f$ such that if $A$ is a self-adjoint operator, then $B = f(A)$ is a self-adjoint operator too, the following equality holds:

$$f(S_{g}^{\Sigma}[A]) = S_{g}^{\Sigma}[f(A)].$$

(eq18)

To motivate (S.3) one can argue as follows. Let $f$ be any fixed real Borel function such that if $A$ is a self-adjoint operator, then $B = f(A)$ is a self-adjoint operator too. Now, according to Quantum Theory a measurement of the quantum observable $f(A)$ can be performed by measuring $A$ and then transforming the obtained outcome $a$ by the purely mathematical function $f$ into the outcome $b = f(a)$ of $f(A)$. If a measurement procedure is relatively to $\Sigma$ identical to another measuring procedure relatively to $\Sigma_g$, then transforming the outcomes of both procedures by means of the same function $f$ should not affect the relative indistinguishability of the so modified procedures. So we should conclude that (eq18) holds. Hence, the concept (QT) seems to entail the validity of (S.2) and (S.3); for reasons we shall indicate later in remark 3.1, however, for the time being we formulate them as conditions which characterize a particular class of interactions.

3.1 General implications of quantum transformations

Conditions (S.2) and (S.3), are sufficient to show further properties of the mappings $S_{g}^{\Sigma}$.

**Proposition 3.1.** Let $S : \Omega(\mathcal{H}) \to \Omega(\mathcal{H})$ be a bijective mapping such that $S[f(A)] = f(S[A])$ for every Borel real function $f$ such that $f(A) \in \Omega(\mathcal{H})$ if $A \in \Omega(\mathcal{H})$. Then the following statements hold.

i) If $E \in \Pi(\mathcal{H})$ then $S[E] \in \Pi[\mathcal{H}]$, i.e., the mapping $S$ is an extension of a bijection of $\Pi(\mathcal{H})$. 

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ii) If \( A, B \in \Omega(\mathcal{H}) \) and \( A + B \in \Omega(\mathcal{H}) \), then \([A, B] = \Phi\) implies \( S[A + B] = S[A] + S[B]\).

This partial additivity immediately implies \( S[A] = \Phi \) if and only if \( A = \Phi \).

iii) For all \( E, F \in \Pi(\mathcal{H}) \), \( EF = \Phi \) implies \( S[E + F] = S[E] + S[F] \in \Pi(\mathcal{H}) \); as a consequence, \( E \leq F \) if and only if \( S[E] \leq S[F] \).

iv) \( S[P] \in \Pi_1(\mathcal{H}) \) if and only if \( P \in \Pi_1(\mathcal{H}) \).

**Proof.** (i) If \( E \in \Pi(\mathcal{H}) \) and \( f(\lambda) = \lambda^2 \) then \( f(E) = E \) holds; so \( S[f(E)] = f(S[E]) \) implies \( (Sg[E])^2 = f(S[E]) = S[E]^2 \equiv S[E], \) i.e. \( S^2[E] = S[E] \).

(ii) If \([A, B] = \Phi\) then a self-adjoint operator \( C \) and two functions \( f_a, f_b \) exist so that \( A = f_a(C) \) and \( B = f_b(C) \); once defined the function \( f = f_a + f_b \), we have \( S[A + B] = S[f(C)] = S[f_a(C)] + S[f_b(C)] = S[f_a(C)] + S[f_b(C)] \equiv S[A] + S[C] \).

(iii) If \( EF = \Phi \), then \([E, F] = \Phi \) and \((E + F) \in \Pi(\mathcal{H}) \) hold. Statements (i) and (ii) imply \( S[E + F] = S[E] + S[F] \in \Pi(\mathcal{H}) \).

(iv) If \( P \in \Pi_1(\mathcal{H}) \) then \( S[P] \in \Pi(\mathcal{H}) \) by (i). If \( Q \in \Pi_1(\mathcal{H}) \) and \( Q \leq S[P] \) then \( P_0 \equiv S^{-1}[Q] \leq P \) by (iii); but \( P \) is rank 1, therefore \( P_0 = P \) and \( Q = S[P] \).

**Corollary 3.1.** From Prop.3.1 immediately follows that the restriction of \( S \) to \( \Pi(\mathcal{H}) \) is a bijection that also satisfies \( S[\Phi] = \Phi, S[1] = 1, E \leq F \) iff \( S[E] \leq S[F] \), \( S[E^\perp] = (S[E])^\perp \).

In the literature different equivalent formulations of Wigner’s theorem [2],[11] have been proved. The following version shall find application for the the mapping \( S \) of Prop. 3.1.

**Wigner’s theorem.** If \( S : \Pi(\mathcal{H}) \to \Pi(\mathcal{H}) \) is an automorphism of \( \Pi(\mathcal{H}) \), i.e. if it is a bijective mapping such that

\[
E_1 \leq E_2 \iff S[E_1] \leq S[E_2] \quad \text{and} \quad S[E^\perp] = (S[E])^\perp, \quad \forall E_1, E_2, E \in \Pi(\mathcal{H}),
\]

then either a unitary operator or an anti-unitary operator \( U \) of \( \mathcal{H} \) exists such that \( S(E) = UEU^* \) for all \( E \in \Pi(\mathcal{H}) \), unique up a phase factor.

In virtue of Corollary 3.1, by Wigner’s theorem the following proposition is easily proved.

**Proposition 3.2.** If a mapping \( S \) satisfies the hypothesis of Prop. 3.1, then a unitary or an anti-unitary operator exists such that \( S[A] = UAU^* \) for every \( A \in \Omega(\mathcal{H}) \); if another unitary or anti-unitary operator \( V \) satisfies \( S[A] = VAV^* \) for every \( A \in \Omega(\mathcal{H}) \), then \( V = e^{i\theta}U \) with \( \theta \in \mathbb{R} \).

Thus, once arbitrarily fixed a reference frame \( \Sigma \) in a class \( \mathcal{F} \), since \((S.2)\) and \((S.3)\) hold for each \( g \in \Upsilon \), according to Prop. 3.2 each transformation \( g \in \Upsilon \) is assigned a unitary or an anti unitary operator \( U_g \) which realizes the corresponding quantum transformation as the automorphism \( S_g^\Sigma : \Pi(\mathcal{H}) \to \Pi(\mathcal{H}) \), \( S_g^\Sigma[A] = U_gAU_g^* \), also if \( g \) is not a symmetry transformation.

### 3.2 Continuity and unitarity of \( g \to U_g \)

Given \( g \in \Upsilon \), the unitary or anti-unitary operator \( U_g \) such that \( S_g^\Sigma[A] = U_gAU_g^* \) can be arbitrarily chosen within an equivalence class \( U_g \) of operators, all unitary or all anti-unitary, which differ from each other by a complex phase factor; this class \( U_g \) is
called operator ray [12]; due to Wigner’s theorem, there is a bijective correspondence
between operator rays and automorphisms of \( \Pi(\mathcal{H}) \). The possibility that the choice of
\( U_g \) within \( U_g \) makes the correspondence \( g \rightarrow U_g \) continuous has a decisive role in de-
veloping the Quantum Theory of a physical system; for instance, for the non-relativistic
Quantum Theory of a free particle, it makes possible Stone’s theorem to apply, and as a
consequence the one-parameter sub-groups \( T_\alpha, R_\alpha, B_\alpha \) can be represented as \( e^{-iF_\alpha a}, e^{-iJ_\alpha \theta}, e^{iG_\alpha u} \). According to results due to Bargmann [12], a choice of \( U_g \) in \( U_g \) leading to a continuous correspondence \( g \rightarrow U_g \) exists if the mapping \( g \rightarrow S_g^\Sigma \) is continuous,
where \( S_g^\Sigma : \Pi(\mathcal{H}) \rightarrow \Pi(\mathcal{H}) \) is the restriction to \( \Pi(\mathcal{H}) \) of the quantum transformation
corresponding to \( g \). However, Bargmann carried out his proof by requiring that all
operators \( U_g \) are unitary. Now we see how the implication proved by Bargmann holds
also if such a restriction is removed.

The continuity notion of Bargmann\(^1\) for \( g \rightarrow S_g^\Sigma \) is based on the following metric
of \( \Pi(\mathcal{H}) \).

**Definition 3.1.** Given two rank 1 projection operators \( D_1, D_2 \in \Pi(\mathcal{H}) \), the distance
\( d(D_1, D_2) \) is the minimal distance \( \| \psi_1 - \psi_2 \| \) between vectors \( \psi_1, \psi_2 \) such that \( P_1 =
|\psi_1\rangle\langle\psi_1| \text{ and } P_2 = |\psi_2\rangle\langle\psi_2| \), i.e.,
\( d(D_1, D_2) = [2(1 - |\langle \psi_1 | \psi_2 \rangle|)]^{1/2} \).

Then, following Bargmann, the continuity of a mapping from a topological group \( G \)
into the automorphisms of \( \Pi(\mathcal{H}) \), is defined as follows.

**Definition 3.2.** A correspondence \( g \rightarrow S_g \) from a topological group \( G \) into the set
of all automorphisms of \( \Pi(\mathcal{H}) \) is continuous if for any fixed \( D \in \Pi(\mathcal{H}) \) the mapping
from \( G \) into \( \Pi_1(\mathcal{H}) \), \( g \rightarrow S_g[D] \) is continuous in \( g \) with respect the distance \( d \) defined
on \( \Pi_1(\mathcal{H}) \) by Def. 3.1.

Before proving the main result Prop.3.3, we formulate three lemmas. The next one
was proved by Bargmann as Lemma 1.1 in [12].

**Lemma 3.1.** The real function \( \kappa : \Pi_1(\mathcal{H}) \times \Pi_1(\mathcal{H}) \rightarrow \mathbb{R}, \kappa(D_1, D_2) = Tr(D_1 D_2) \) is continuous in both variables \( D_1 \) and \( D_2 \) with respect to the metric of Def. 3.1.

**Lemma 3.2.** Given a topological group \( G \) and a mapping \( g \rightarrow S_g \) from \( G \) into the
automorphisms of \( \Pi(\mathcal{H}) \), for every \( g \in G \) let \( U_g \) denote the operator ray identified by
\( S_g \) according to Wigner theorem; for every \( \varphi \in \mathcal{H} \) with \( \| \varphi \| = 1 \) let us define
\[
\begin{align*}
   z_{h,g}(\varphi) = U_g \varphi - \langle U_h \varphi | U_g \varphi \rangle U_h \varphi,
\end{align*}
\]
where \( h, g \in G, U_h \in U_h \) and \( U_g \in U_g \). Then
\[
\| z_{h,g}(\varphi) \|^2 = 1 - |\langle U_h \varphi | U_g \varphi \rangle|^2 \leq d^2(S_h[D_\varphi], S_g[D_\varphi]);
\]
where \( D_\varphi = |\varphi\rangle\langle\varphi| \in \Pi(\mathcal{H}) \).

**Proof.** The proof is identical to the proof of statement (1.9) in Theorem 1.1 of [12];
indeed that proof can be successfully carried out independently of the unitary or anti-
unitary character of \( U_g \) or \( U_{h_0} \).

\(^1\)In fact Bargmann’s continuity refers to a correspondence \( g \rightarrow U_g \) from a topological group \( G \) into
the set of all unitary operator rays \( U_g \); but, since an operator ray can be bijectively identified with an
automorphism of \( \Pi(\mathcal{H}) \), Bargmann’s continuity can be reformulated in terms of automorphisms; this reform-
ulation immediately extends to all automorphisms, included those corresponding to anti-unitary operator
rays, through our Def. 3.2.
Lemma 3.3. Let $G$ be a topological group, let $g \to S_g$ be a continuous mapping from $G$ into the automorphisms of $\Pi(\mathcal{H})$, and let us fix an operator $U_g \in U_g$ for each $g \in G$.

If $U_g\varphi_0$ is continuous in $g$ as a function from $G$ into $\mathcal{H}$ for a vector $\varphi_0 \in \mathcal{H}$ with $\|\varphi_0\| = 1$, then $U_g\varphi_1$ is continuous in $g$ for every $\varphi_1 \in \mathcal{H}$ with $\|\varphi_1\| = 1$, such that $\varphi_1 \perp \varphi_0$.

Proof. We prove the lemma by adapting a part of the proof of Theorem 1.1 in [12]. Let us define $\varphi = \frac{1}{\sqrt{2}}(\varphi_0 + \varphi_1)$; of course we have $\langle U_g\varphi_0 \mid U_g\varphi \rangle = \frac{1}{\sqrt{2}}$ for all $g \in G$ independently of the unitary or anti-unitary character of $U_g \in U_g$. Then

$$
\langle U_h\varphi_0 \mid z_{h,g}(\varphi) \rangle = \langle U_h\varphi_0 - U_g\varphi_0 \mid U_g\varphi \rangle + \langle U_g\varphi_0 \mid U_g\varphi \rangle - \langle U_h\varphi \mid U_g\varphi \rangle (U_h\varphi_0 \mid U_h\varphi) = \langle U_h\varphi_0 - U_g\varphi_0 \mid U_g\varphi \rangle + \frac{1}{\sqrt{2}}(1 - \langle U_h\varphi \mid U_g\varphi \rangle).
$$

So

$$
(1 - \langle U_h\varphi \mid U_g\varphi \rangle) = \sqrt{2} \{\langle U_h\varphi_0 \mid z_{h,g}(\varphi) \rangle + \langle U_g\varphi_0 - U_h\varphi_0 \mid U_g\varphi \rangle\}. \tag{eq85}
$$

Now,

$$
\|U_g\varphi - U_h\varphi\|^2 = 2|\text{Re}(1 - \langle U_h\varphi \mid U_g\varphi \rangle)| \leq 2|1 - \langle U_h\varphi \mid U_g\varphi \rangle| \leq 2\sqrt{2}(|\langle U_h\varphi_0 \mid z_{h,g}(\varphi) \rangle| + 2\sqrt{2}|U_g\varphi_0 - U_h\varphi_0| |U_g\varphi\|) \leq 2\sqrt{2} |z_{h,g}(\varphi)| + 2\sqrt{2}\|U_g\varphi_0 - U_h\varphi_0\| \leq 2\sqrt{2} (d(S_h[\varphi], S_g[\varphi]) + \|U_g\varphi_0 - U_h\varphi_0\|),
$$

where we made use of (eq85) in the second inequality, in the third inequality we used Schwarz inequality, and in the fourth inequality Lemma 3.2 is applied. These inequalities imply that $U_g\varphi$ is continuous in $g$; indeed, the distance $d(S_h[\varphi], S_g[\varphi])$ vanishes as $g \to h$ because $g \to S_g$ is continuous according to Def. 3.2 by the first continuity hypothesis; but also $\|U_g\varphi_0 - U_h\varphi_0\|$ vanishes as $g \to h$, because $U_g\varphi_0$ is continuous in $g$ by the second continuity hypothesis.

Now, $\varphi_1 = \sqrt{2}\varphi - \varphi_0$, so that $U_g\varphi_1 = \sqrt{2}U_g\varphi_2 - U_g\varphi_0$ for all $g$ such that $U_g$ is unitary, but also for all $g$ such that $U_g$ is anti-unitary. Thus $U_g\varphi_1$ is continuous because $U_g\varphi$ and $U_g\varphi_1$ are continuous.

Let us arbitrarily fix a vector $\varphi_0 \in \mathcal{H}$, with $\|\varphi_0\| = 1$. Given any mapping $g \to S_g$ from a topological group $G$ into the automorphisms of $\Pi(\mathcal{H})$, we define the real function $\rho_{\varphi_0} : G \to \mathbb{R}$, $\rho_{\varphi_0}(g) = Tr^{1/2}(D_{\varphi_0}S_g[D_{\varphi_0}])$. Since $S_g[\varphi_0][\varphi_0] = \hat{U}_g[D_{\varphi_0}] \hat{U}_g^*$, where $\hat{U}_g$ is any operator in $U_g$, we have $\rho_{\varphi_0}(g) = |\langle \varphi_0 \mid \hat{U}_g\varphi_0 \rangle|$. Hence, $\langle \varphi_0 \mid \hat{U}_g\varphi_0 \rangle = e^{i\alpha(g)}\rho_{\varphi_0}(g)$, for some $\alpha(g) \in \mathbb{R}$. Then $\rho_{\varphi_0}(g) = |\langle \varphi_0 \mid \hat{U}_g\varphi_0 \rangle| = e^{-i\alpha(g)}\rho_{\varphi_0}(g)$, because $\hat{U}_g$ is an operator. Therefore, if for each $g \in G$ we choose $U_g = e^{-i\alpha(g)}U_g$ we obtain

$$
\rho_{\varphi_0}(g) = \langle \varphi_0 \mid U_g\varphi_0 \rangle; \text{ in particular, } U_e = I. \tag{eq80}
$$

Proposition 3.3. Let $G$ be a topological group, and let $\varphi_0$ be any fixed vector in $\mathcal{H}$ with $\|\varphi_0\| = 1$. Given a mapping $g \to S_g$ from $G$ into the automorphisms of $\Pi(\mathcal{H})$, if each $g \in G$ is assigned the operator $U_g \in U_g$ such that (eq80) holds, then $U_g\psi$ is continuous in $g$, whatever be the vector $\psi \in \mathcal{H}$. 

Proof. Bargmann proved that if \( g \to S_g \) is continuous according to Def. 3.2 and if \( U_g \) is the operator such that (eq80) holds, then \( U_g \varphi_0 \) is continuous\(^2\). Now, let \( \psi \) be any vector of \( \mathcal{H} \).

If \( \psi = 0 \), then the continuity of \( U_g \psi \) is obvious. Therefore it is sufficient to prove the proposition for \( \psi \neq 0 \).

If \( \psi = \lambda \varphi_0 \) for some \( \lambda \in \mathbb{C} \setminus \{0\} \), then we choose any \( \varphi_1 \perp \varphi_0 \), with \( \| \varphi_1 \| = 1 \). According to Lemma 3.3, \( U_g \varphi_1 \) is continuous. The same Lemma implies that \( U_g \| \psi \| \) is continuous because \( \frac{\psi}{\| \psi \|} \perp \varphi_1 \). But \( U_g \psi = \| \psi \| U_g \frac{\psi}{\| \psi \|} \) for all \( g \in G \). Therefore \( U_g \psi \) is continuous.

If \( \psi \neq \lambda \varphi_0 \), define \( \varphi = \frac{\psi}{\| \psi \|} \); then a vector \( \varphi_1 \in \mathcal{H} \) exists, with \( \| \varphi_1 \| = 1 \) and \( \varphi_1 \perp \varphi_0 \), such that

\[ \varphi = a \varphi_0 + b \varphi_1 \quad \text{where } a \in \mathbb{C} \text{ but } b \in \mathbb{R}. \tag{eq81} \]

Now, a real number \( r \) and a vector \( \varphi_2 \), with \( \| \varphi_2 \| = 1 \) exist such that \( a \varphi_0 = r \varphi_2 \); this implies \( \varphi_2 \perp \varphi_1 \) and \( \varphi = r \varphi_2 + b \varphi_1 \). Lemma 3.3 implies that \( U_g \varphi_1 \) is continuous because \( \varphi_1 \perp \varphi_0 \); but the same Lemma implies that also \( U_g \varphi_2 \) is continuous, because \( \varphi_2 \perp \varphi_1 \). Therefore, since \( r \) and \( b \) are real numbers, \( U_g \varphi = r U_g \varphi_2 + b U_g \varphi_1 \) is continuous in \( g \). Thus, \( U_g \psi = \| \psi \| U_g \varphi \) is continuous too. \( \bullet \)

Another condition with helpful implications is the unitary character of the operators \( U_g \) that realize the quantum transformations according to \( S^\Sigma_g[A] = U_g A U_g^{-1} \). If the correspondence \( g \to S^\Sigma_g \) satisfied \( S^\Sigma_{g_1 g_2} = S^\Sigma_{g_1} \circ S^\Sigma_{g_2} \) so that \( g \to U_g \) would be a projective representation, then it could be easily proved, according to [3],[5],[10],[12], that every \( U_g \) must be unitary. But in presence of interaction \( S^\Sigma_g \) can be different from \( S^\Sigma_{g_2} \), so that only the more general statement (S.1) holds, and hence the unitary character of \( U_g \) cannot be implied by the cited proofs. Now we prove that anti-unitary \( U_g \) can be excluded under the only hypothesis that the correspondence \( g \to S^\Sigma_g \) is continuous according to Def. 3.2.

**Proposition 3.4.** If the mapping \( g \to S^\Sigma_g \), that assigns each \( g \in \Upsilon \) the quantum transformation of (eq16), is continuous according to Def. 3.2, then for every operator \( U_g \) such that \( S^\Sigma_g[A] = U_g A U_g^* \) for all \( A \in \Omega(\mathcal{H}) \) is unitary.

**Proof.** According to Prop. 3.3, for every \( g \in \Upsilon \) a unitary or anti-unitary operator such that \( S^\Sigma_g[A] = U_g A U_g^* \) exists which makes \( U_g \psi \) continuous in \( g \) for all \( \psi \). According to (eq80) \( U_e = \mathbb{I} \) which is unitary. Hence, because of the continuity of \( g \to U_g \psi \) for all \( \psi \), a maximal neighborhood \( K_e \) of \( e \) must exist in \( \Upsilon \) such that \( U_g \) is unitary for all \( g \in K_e \); otherwise a sequence \( g_n \to e \) would exist with \( U_{g_n} \) anti-unitary, so that \( \langle \psi \mid \varphi \rangle = \langle U_{g_n} \psi \mid U_{g_n} \varphi \rangle \) for all \( \psi, \varphi \in \mathcal{H} \), and then \( \langle \psi \mid \varphi \rangle = \lim_{n \to \infty} \langle U_{g_n} \psi \mid U_{g_n} \varphi \rangle = \langle U_e \varphi \mid U_e \psi \rangle = \langle \varphi \mid \psi \rangle \). This last equality cannot hold for all \( \psi, \varphi \in \mathcal{H} \) unless \( \mathcal{H} \) is real.

Now we prove that such a neighborhood \( K_e \) has no boundary, and since \( \Upsilon \) is a connected group, \( K_e = \Upsilon \). If \( g_0 \in \partial K_e \), two sequences \( g_n \to g_0 \) and \( h_n \to g_0 \) would exist with \( U_{g_n} \) unitary and \( U_{h_n} \) anti-unitary; therefore, the continuity of \( U_g \) would imply that \( U_{g_0} \) should simultaneously be unitary and anti-unitary. \( \bullet \)

**Remark 3.1.** The work of this subsection has shown that (S.2) and (S.3) imply that \( S^\Sigma_g[A] = U_g A U_g^{-1} \), where \( U_g \) is unitary; therefore, the spectrum of any quantum

\(^2\)In fact, Bargmann proved this statement for unitary \( U_g \); but Bargmann's proof can be successfully carried out without assuming that all \( U_g \) are unitary.
observable is left unchanged by $S^z_g$. Such an invariance has important consequences; for instance, it entails that particular kinds of interactions are not compatible with the theory.

Indeed, let the interaction be able to confine a localizable particle in a bounded region of the physical space. For sake of simplicity, we assume that space is one dimensional, so that there is only one position operator $Q$ whose possible values are confined by the interaction in the interval $[0,a]$ of the only axis of the reference frame $\Sigma$; this means that $[0,a]$ contains the spectrum of $Q$, of course: $\sigma(Q) \subseteq [0,a]$. Let $g \in \Upsilon$ be the spatial translation identified by $g(x) = x - a$. According to (eq17) we have $S^z_g(Q) = Q - a$, and hence $\sigma(S^z_g(Q)) \subseteq [-a,0]$: $S^z_g$ changes the spectrum of $Q$.

Therefore, if (S.2) and (S.3) were generally valid constraints, the confinement interaction should not be an interaction compatible with Quantum Theory, i.e. no interaction could sharply confine a particle within a bounded region. Such a drastic conclusion is based on (S.2) and (S.3) which, though endowed with conceptual soundness, do not have a formal derivation. For this reason we find appropriate, for the time being, to establish (S.2) and (S.3) as conditions which characterize the class of interactions investigated in the present work. In the following we shall see that such a class is a very large one, able, in particular, to encompass electromagnetic interaction.

### 3.3 $\sigma$-conversions

In the previous section 3.1 we have established, under a continuity condition for $g \rightarrow S^z_g$, that in the Quantum Theory of a physical system, also if it is not isolated, a continuous correspondence $U : \Upsilon \rightarrow \mathcal{U}(\mathcal{H})$ exists such that $S^z_g[A] = U_gAU_g^{-1}$. According to section 2.4, to assume that such a correspondence is a projective representation leads to a theory unable to describe particles interacting with electromagnetic fields. So, we give up Proj with the scope of developing a Quantum Theory of an interacting particle empirically more adequate. But without such a “projectivity” condition Mackey’s imprimitivity theorem does not apply. Hence, the development of our group-theoretical approach encounters a further obstacle. Now we address this obstacle.

The correspondence $g \rightarrow U_g$, can be converted into a continuous $\sigma$-representation if we multiply each operator $U_g$ by a suitable unitary operator $V_g$ of $\mathcal{H}$; namely, $V_g$ is a unitary operator such that the correspondence $g \rightarrow \hat{U}_g = V_gU_g$ turns out to be a $\sigma$-representation. The transition from the correspondence $\{g \rightarrow U_g\}$ to $\{g \rightarrow \hat{U}_g = V_gU_g\}$ will be called $\sigma$-conversion; the mapping $V : \Upsilon \rightarrow \mathcal{U}(\mathcal{H})$, $g \rightarrow V_g$ that realizes the $\sigma$-conversion will be called $\sigma$-conversion mapping. If $g \rightarrow V_g$ is a $\sigma$-conversion mapping for $g \rightarrow U_g$ and $\theta : \Upsilon \rightarrow \mathbb{R}$ is a real function, then also $g \rightarrow e^{i\theta(g)}V_g$ is a $\sigma$-conversion mapping, provided that $e^{i\theta(e)} = 1$. In any case, $V_e = \mathbb{1}$ must hold.

The following proposition shows that, in fact, our correspondence $g \rightarrow U_g$, can be always converted into a $\sigma$-representation by a $\sigma$-conversion.

**Proposition 3.5.** A correspondence $V : \Upsilon \rightarrow \mathcal{U}(\mathcal{H})$ always exists such that $\hat{U} : \Upsilon \rightarrow \mathcal{U}(\hat{\mathcal{H}})$, $g \rightarrow \hat{U}_g = V_gU_g$ is a projective representation.

**Proof.** A projective representation $\hat{U} : \Upsilon \rightarrow \mathcal{U}(\hat{\mathcal{H}})$ exists, of course; e.g., if $\Upsilon = \mathcal{G}$ we can consider the projective representations “induced” by any projective representation $\hat{L} : SO(3) \rightarrow \mathcal{U}(\mathcal{H}_0)$ of $SO(3)$, where $\mathcal{H} = L_2(\mathbb{R}^3, \mathcal{H}_0)$: $(\hat{U}\psi)(x) = \hat{L}\mathbb{R}\psi(g(x))$; if $\Upsilon = \mathcal{P}$ we can consider one of the projective representations identified by Wigner [13].
Since $\mathcal{H}$ and $\tilde{\mathcal{H}}$ have the same dimension, a unitary operator $W : \mathcal{H} \to \tilde{\mathcal{H}}$ exists. Hence $\hat{U} : \mathcal{Y} \to U(\mathcal{H})$, $g \to \hat{U}_g = W^{-1}\hat{U}_g W^{-1}$ is a projective representation of $\mathcal{Y}$ in $\mathcal{H}$. Now we define $V_g = \hat{U}_g U^{-1}_g$; then $V_g U_g = \hat{U}_g$; therefore $g \to V_g$ is a $\sigma$-conversion mapping for $g \to U_g$.

The $\sigma$-conversion allows to immediately identify a mathematical formalism for the Quantum Theory of the system, also in the case that the system is not isolated. In the case of a non-relativistic theory, where $\mathcal{Y} = \mathcal{G}$, if $g \to V_g$ is a $\sigma$-conversion mapping for $U_g$ then, according to (MP.2) in sect. 2.4, the $\sigma$-representation $g \to \hat{U}_g = V_g U_g$ has nine hermitean generators $\hat{P}_\alpha$, $\hat{J}_\alpha$, $\hat{G}_\alpha$ for which (eq2) hold. Then, following the argument of the proof of STAT in section 2.4, the common spectral measure of the triple $\mathcal{F} = \mathcal{G}/\mu$ turns out to be an imprimitivity system for the restriction of $g \to \hat{U}_g$ to $\mathcal{E}$. So, by applying the imprimitivity theorem of Mackey [5], we can explicitly identify $\mathcal{H}$ as $L_2(\mathbb{R}^3, \mathcal{H}_0)$, modulo unitary isomorphisms, where the operators $\mathcal{F}$, $\hat{P}_\alpha$, $\hat{J}_\alpha$ and $\hat{G}_\alpha$ are explicitly specified according to

$$\mathcal{H} = L_2(\mathbb{R}^3, \mathcal{H}_0), \quad (F\psi)(x) = x_\alpha \psi(x), \quad \hat{P}_\alpha = -i \frac{\partial}{\partial x_\alpha}, \quad (eq44)$$

$$\hat{J}_\alpha = F_\beta \hat{P}_\gamma - F_\gamma \hat{P}_\beta + S_\alpha, \quad \hat{G}_\alpha = \mu F_\alpha.$$

Here $(\alpha, \beta, \gamma)$ is a cyclic permutation of $(1, 2, 3)$; the $S_\alpha$ are operators that act on $\mathcal{H}_0$ only, i.e. their action is $(S_\alpha \psi)(x) = \hat{s}_\alpha \psi(x)$ where the $\hat{s}_\alpha$ are self-adjoint operators of $\mathcal{H}_0$ which form a $\sigma$-representation of the commutation rules $[\hat{s}_\alpha, \hat{s}_\beta] = i\epsilon_{\alpha\beta\gamma} \hat{s}_\gamma$. If $U$ is irreducible then, according to Mackey’s theorem, also $(\hat{s}_1, \hat{s}_2, \hat{s}_3)$ must be an irreducible representation of $[\hat{s}_\alpha, \hat{s}_\beta] = i\epsilon_{\alpha\beta\gamma} \hat{s}_\gamma$; in this case $\mathcal{H}_0$ is one of the finite-dimensional Hilbert spaces $\mathcal{F}^{2s+1}$, with $s \in \frac{1}{2}\mathbb{N}$: the $\hat{s}_\alpha$ are the familiar spin operators.

For the particular case that the physical system under investigation is a localizable particle, the mathematical formalism must contain also the triple $\mathcal{Q}$ of the self-adjoint operators representing the position with respect to $\Sigma$ for which (eq17) hold. In agreement with the literature, we say that a particle is elementary if $g \to \hat{U}_g$ is an irreducible projective representation. In the following we shall be concerned with the Quantum Theory of an elementary particle. Hence, in particular, the Hilbert space $\mathcal{H}_0$ in (eq44) has a finite dimension $2s + 1$ and the $\hat{s}_\alpha$ are the spin operators with that dimension.

Our work has explicitly identified the mathematical formalism of the Quantum Theory of an elementary localizable particle. However, we must stress that the operators $\hat{U}_g$ concretely identified are not the unitary operators which realize the quantum transformations: given $g \in \mathcal{G}$, in general $S^\Sigma_g[A] = \hat{U}_g A \hat{U}_g^{-1}$ does not hold. Moreover, $\mathcal{F}$ is not the triple $\mathcal{Q}$ representing the position. So, our explicit realization of the mathematical formalism of the theory is, in general, devoid of physical significance.

However, the approach can go on if we restrict our investigation to those interactions which admit $\sigma$-conversions $U_g \to \hat{U}_g = V_g U_g$ which are $Q$-covariant, i.e. $\sigma$-conversions that leave unaltered the covariance properties of the position operators $\mathcal{Q}$, i.e. such that

$$\hat{U}_g \mathcal{Q} \hat{U}_g^{-1} = g(\mathcal{Q}) \forall g \in \mathcal{G}. \quad (eq19)$$

The following proposition establishes a physically meaningful characterization of the interactions admitting $Q$-covariant $\sigma$-conversions.

**Proposition 3.6.** The interaction of a localizable particle admits a $Q$-covariant $\sigma$-conversion if and only if the position operators $\mathcal{Q}$ coincide with $\mathcal{F}$.
Proof. If $Q = F = G/\mu$, then (eq2) imply $\hat{U}_gQ\hat{U}^{-1}_g = \hat{U}_gF\hat{U}^{-1}_g = g(F)$; therefore $\hat{U}_gQ\hat{U}^{-1}_g = g(Q)$.

Conversely, if $\hat{U} : G \rightarrow \mathcal{U}(\mathcal{H})$ is obtained from $U : G \rightarrow \mathcal{U}(\mathcal{H})$ through a $Q$-covariant $\sigma$-conversion, then (eq19) for $\hat{U}_g = e^{iG\theta}$ and (eq2.2) imply $[Q_\alpha - F_\alpha, \hat{F}_\beta] = [Q_\alpha, \hat{F}_\beta] - [F_\alpha, \hat{F}_\beta] = \Theta - \Theta = \Theta$; therefore $(Q_\alpha - F_\alpha)\psi(x) = (f_\alpha(Q)\psi(x) = f_\alpha(x)\psi(x)$, where $f_\alpha(x)$ is a self-adjoint operator of $\mathcal{H}_0$. However, the $Q$-covariance and (eq2.6) imply also $[Q_\alpha - F_\alpha, \hat{P}_\beta] = [Q_\alpha, \hat{P}_\beta] - [F_\alpha, \hat{P}_\beta] = i\delta_{\alpha,\beta}I - i\delta_{\alpha,\beta} I = 0$, i.e. $[f_\alpha(Q), \hat{P}_\beta] = 0$ for all $x$; this relation, since $\hat{P} = -i\frac{\partial}{\partial f_\alpha}$, implies that $\frac{\partial f_\alpha}{\partial x}(x) = 0$, for all $\alpha, \beta$; therefore $f_\alpha(x)$ is an operator $\hat{f}_\alpha$ of $\mathcal{H}_0$ which does not depend on $x$. Now, since $\hat{f}_\alpha = Q_\alpha - F_\alpha$, also $[\hat{f}_\alpha, \hat{f}_\beta] = 0$ holds; moreover, from (eq17.1) for a pure spatial rotation $g$ about $x_\alpha$ and from (eq2.4) we obtain $[\hat{J}_\alpha, \hat{Q}_\beta - F_\beta] = i\hat{e}_{\alpha,\beta}(Q_\gamma - F_\gamma) = i\hat{e}_{\alpha,\beta}\hat{f}_\gamma$; but since $\mathcal{H}_0$ is finite dimensional, this relation can hold only if $f_\alpha = 0$.

Hence, in the general mathematical formalism of the theory, established by (eq44), the multiplication operators can be identified with the position operators if and only if the interaction has the particular regularity property of admitting a $\sigma$-conversion which preserves the covariance properties of the position operators.

We have concretely identified the position operators $Q$ in the Quantum Theory of a particle admitting a $Q$-covariant $\sigma$-conversion, but the operators $\hat{U}_g$ having from such a $\sigma$-conversion continue to be not the representative of the transformations of $G$, i.e. $S_{2g}^G[A] = \hat{U}_gA\hat{U}_g^{-1}$ does not hold. The following proposition specify how each $\hat{U}_g$ is related to the unitary operator $U_g$ that realizes the quantum transformation corresponding to $g$.

**Proposition 3.7.** For every $g \in G$, the operator $V_g$ of a $Q$-covariant $\sigma$-conversion has the form $(V_g\psi)(x) = e^{i\theta(g, x)}\psi(x)$, where $\theta(g, x)$ is a self-adjoint operator of $\mathcal{H}_0$ which depends on $x$ and on $g$.

**Proof.** The relation $V_g\hat{U}_gQ\hat{U}_g^{-1}V_g^{-1} = g(Q)$, implied by (eq19) and (eq17), imply $V_g(g(Q))V_g^{-1} = g(Q)$, i.e. $[V_g, g(Q)] = 0$. Then $[V_g, f(g(Q))] = 0$ for every sufficiently regular function $f$; by taking $f = g^{-1}$ we have $[V_g, Q] = 0$. Then $(V_g\psi)(x) = h_g(x)\psi(x)$, where $h_g(x)$ is an operator of $\mathcal{H}_0$. Finally, the unitary character of $V_g$ imposes that $h_g(x)$ must be unitary as an operator of $\mathcal{H}_0$; thus a self-adjoint operator $\theta(g, x)$ of $\mathcal{H}_0$ exists such that $h_g(x) = e^{i\theta(g, x)}$.

If $g \rightarrow S_{2g}^G$ is continuous according to Def. 3.2., then $g \rightarrow V_g$ must be continuous because $g \rightarrow \hat{U}_g = V_gU_g$ is continuous.

**Remark 3.2.** In the present approach the imprimitivity system for applying Mackey’s theorem is identified within the abstract projective representation itself, namely it is the PV measure of $G/\mu$. This is a remarkable difference with respect to the past approaches, e.g. Mackey’s approach, where the imprimitivity system is identified as the PV measure of the position operators.

### 3.4 General dynamical equation

Now we exploit the results so far obtained for deriving a general dynamical equation ruling over the time evolution of a localizable particle whose interaction admits $Q$-covariant $\sigma$-conversion. In so doing we shall suppose that the $\sigma$-conversion mapping
$g \rightarrow V_g$ is differentiable with respect to the parameters $a_\alpha, \theta_\alpha, u_\alpha$ of the group $G$.

Let us consider the pure velocity boost $g \in G$ such that $\hat{U}_g = e^{iG_\alpha u}$. According to sect. 3.3, the formalism of its Quantum Theory can be identified with that established by (eq24). Since $G_\alpha = \mu F_\alpha = \mu Q_\alpha$, we can write $\hat{U}_g = e^{i\mu Q_\alpha u}$; therefore

$$\hat{U}_g \hat{Q}_\beta \hat{U}_g^{-1} = \hat{Q}_\beta + i\mu [Q_\alpha, \hat{Q}_\beta]u + o_1(u). \quad (eq20)$$

On the other hand,

$$\hat{U}_g \hat{Q}_\beta \hat{U}_g^{-1} = \lim_{t \to 0} V_g U_g \frac{(Q_\beta^t - Q_\beta)}{t} U_g^{-1} V_g^{-1}. \quad (eq21)$$

By making use of $U_g Q_\beta^t \hat{U}_g^{-1} = Q_\beta^t - \delta_{\alpha\beta} u \mathbb{I}$, implied by (eq17), and of Prop. 3.2, Prop. 3.3 in (eq21), and then comparing with (eq20) we obtain

$$\hat{U}_g \hat{Q}_\beta \hat{U}_g^{-1} = V_g \hat{Q}_\beta V_g^{-1} - \delta_{\alpha\beta} u \mathbb{I} = \hat{Q}_\beta + i\mu [Q_\alpha, \hat{Q}_\beta]u + o_1(u). \quad (eq22)$$

But Prop. 3.7 implies that $V_g = e^{i\varsigma_\alpha(u, Q)}$, where $\varsigma_\alpha(u, x)$ is a self-adjoint operator of $\mathcal{H}_0$; replacing in (eq22) we obtain

$$\hat{Q}_\beta + i[\varsigma_\alpha(u, Q), \hat{Q}_\beta] + o_2(u) - \delta_{\alpha\beta} u \mathbb{I} = \hat{Q}_\beta + i\mu [Q_\alpha, \hat{Q}_\beta]u + o_1(u). \quad (eq23)$$

Since $e^{i\varsigma_\alpha(0, Q)} = \mathbb{I}$, the expansion of $\varsigma_\alpha$ with respect to $u$ yields $\varsigma_\alpha(u, Q) = \frac{\partial \varsigma_\alpha}{\partial u}(0, Q)u + o_3(u)$; by replacing this last relation in (eq23) we obtain

$$\mu [Q_\alpha, \hat{Q}_\beta] = [\eta_\alpha(Q), \hat{Q}_\beta] + i\delta_{\alpha\beta} \mathbb{I},$$

where $\eta_\alpha(Q) = \frac{\partial \varsigma_\alpha}{\partial u}(0, Q)$. By replacing $\hat{Q}_\beta = i[H, Q_\beta]$ in this last equation we can apply Jacobi’s identity, and in so doing we obtain $[Q_\beta, \mu \hat{Q}_\alpha] = [Q_\beta, \eta_\alpha(Q)] + i\delta_{\alpha\beta} \mathbb{I}$, i.e.

$$[Q_\beta, \eta_\alpha(Q) - \mu \hat{Q}_\alpha] = -i\delta_{\alpha\beta} \mathbb{I} = [Q_\beta, -\hat{P}_\alpha]. \quad (eq24)$$

Hence $[\eta_\alpha(Q) - \mu \hat{Q}_\alpha, Q_\beta] = 0$, therefore from (eq24) we imply that for every $x$ an operator $f_\alpha(x)$ of $\mathcal{H}_0$, must exist such that the equation $\{\eta(Q) - \mu \hat{Q}_\alpha + \hat{P}_\alpha\} \psi(x) = f_\alpha(x)\psi(x)$ holds; then we can rewrite (eq24) as

$$i[H, \mu Q_\alpha - \eta_\alpha(Q)] = \hat{P}_\alpha - f_\alpha(Q). \quad (eq25)$$

This is a general dynamical equation for a localizable particle whose interaction admits $Q$-covariant $\sigma$-conversions; according to such a law, the effects of the interaction on the dynamics are encoded in the six “fields” $\eta_\alpha, f_\alpha$. 

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### 3.5 Electromagnetic interaction for spin-0 particles

Once derived the general dynamical law (eq25) for a localizable particle with Q-covariant and homogeneous in time interaction, it is worth to re-discover the wave equation currently adopted in quantum physics as a particular case of the general equation (eq25). In this subsection we do this for a spin-0 particle, for which

\[ H_0 = I_C \]

so that

\[ H = L_2(I\mathbb{R}^3). \]

The nowadays adopted Schroedinger equation for a spin-0 particle has the form

\[
\begin{align*}
    i \frac{d}{dt} \psi_t &= \left\{ \frac{1}{2m} \sum_{\alpha=1}^{3} [\hat{P}_\alpha + a_\alpha(Q)]^2 + \Phi(Q) \right\} \psi_t,
    \tag{eq70}
\end{align*}
\]

i.e. the Hamiltonian operator is

\[ H = \left( \frac{1}{2\mu} \sum_{\alpha=1}^{3} \{ \hat{P}_\alpha + a_\alpha(Q) \}^2 + \Phi(Q) \right) \]

where \( a_\alpha(Q) \) and \( \Phi(Q) \) are self-adjoint operators of \( L^2(I\mathbb{R}^3) \) functions of \( Q \). Now we show that within our approach this specific Quantum Theory bi-univocally corresponds to the case that the functions \( \eta_\alpha \) in the general law (eq25) are constant function multiples of \( I \).

**Proposition 3.8.** The Hamiltonian operator \( H \) of an interacting spin-0 particle which admits Q-covariant \( \sigma \)-conversion has the form

\[ H = \left( \frac{1}{2\mu} \sum_{\alpha=1}^{3} \{ \hat{P}_\alpha + a_\alpha(Q) \}^2 + \Phi(Q) \right) \]

if and only if the functions \( \eta_\alpha \) in (eq25) are constant functions. In this case \( a_\alpha = -f_\alpha \).

**Proof.** If \( \eta_\alpha \) is a constant function, then (eq25) transforms into

\[
    i[H, \mu \dot{Q}_\alpha] = \hat{P}_\alpha - f_\alpha(Q)
\]

which holds if \( H_0 = \frac{1}{2\mu} \sum_{\alpha=1}^{3} \{ \hat{P}_\alpha - f_\alpha(Q) \}^2 \) replaces \( H \). Hence the operator \( H - H_0 \) must be a function \( \Phi \) of \( Q \) because of the completeness of \( Q \). Then \( \eta_\alpha(Q) = c_\alpha I \) implies

\[ H = \frac{1}{2\mu} \sum_{\alpha=1}^{3} \{ \hat{P}_\alpha - f_\alpha(Q) \}^2 + \Phi(Q) \]

Now we prove the converse. Let us suppose that \( H = \frac{1}{2\mu} \sum_{\alpha=1}^{3} \{ \hat{P}_\alpha + a_\alpha(Q) \}^2 + \Phi(Q) \); by replacing this \( H \) in (eq25) we obtain

\[
    i[H, \mu \dot{Q}_\alpha - \eta_\alpha(Q)] = \hat{P}_\alpha - f_\alpha(Q) = \\
    = \frac{i}{2\mu} \sum_{\beta} \{ \hat{P}_{\beta}^2, \mu \dot{Q}_\alpha \} + \frac{i}{2\mu} \sum_{\beta} \{ \alpha_{\beta} \hat{P}_{\beta}, \mu \dot{Q}_\alpha \} + \frac{i}{2\mu} \sum_{\beta} \{ \hat{P}_{\beta} \alpha_{\beta}, \mu \dot{Q}_\alpha \} + \\
    - \frac{i}{2\mu} \sum_{\beta} \{ \hat{P}_{\beta}^2, \eta_\alpha \} - \frac{i}{2\mu} \sum_{\beta} \{ \alpha_{\beta} \hat{P}_{\beta}, \eta_\alpha \} - \frac{i}{2\mu} \sum_{\beta} \{ \hat{P}_{\beta} \alpha_{\beta}, \eta_\alpha \} + \\
    + i[\Phi(Q), \mu \dot{Q}_\alpha - \eta_\alpha].
\]

In the last member of these equalities, the fourth, the eighth and the last term are zero. Then we have
\[ i[H, \mu Q_\alpha - \eta_\alpha(Q)] = \hat{P}_\alpha - f_\alpha(Q) \quad (eq 26) \]

\[ = \hat{P}_\alpha + \frac{i}{2} \sum_\beta (a_\beta \hat{P}_\beta Q_\alpha - Q_\alpha a_\beta \hat{P}_\beta + \hat{P}_\beta a_\beta Q_\alpha - Q_\alpha \hat{P}_\beta a_\beta) + \]

\[ - \frac{i}{2 \mu} \sum_\beta [\hat{P}^2_\beta, \eta_\alpha] - \frac{i}{2 \mu} \sum_\beta (a_\beta \hat{P}_\beta \eta_\alpha - \eta_\alpha a_\beta \hat{P}_\beta + \hat{P}_\beta a_\beta \eta_\alpha - \eta_\alpha \hat{P}_\beta a_\beta) \]

\[ = \hat{P}_\alpha + \frac{i}{2} \sum_\beta (a_\beta [\hat{P}_\beta, Q_\alpha] + [\hat{P}_\beta, Q_\alpha] a_\beta) - \frac{i}{2 \mu} \sum_\beta [\hat{P}^2_\beta, \eta_\alpha] \]

\[ - \frac{i}{2 \mu} \sum_\beta (a_\beta [\hat{P}_\beta, \eta_\alpha] + [\hat{P}_\beta, \eta_\alpha] a_\beta) \]

\[ = \hat{P}_\alpha + \frac{i}{2} (-2i a_\alpha) - \frac{i}{2 \mu} \sum_\beta [\hat{P}^2_\beta, \eta_\alpha] - \frac{i}{2 \mu} \sum_\beta \left( -2i a_\beta \frac{\partial \eta_\alpha}{\partial q_\beta} \right) \]

\[ = \hat{P}_\alpha + a_\alpha - \frac{1}{\mu} \sum_\beta a_\beta \frac{\partial \eta_\alpha}{\partial q_\beta} - \frac{i}{2 \mu} \sum_\beta [\hat{P}^2_\beta, \eta_\alpha]. \]

From the second and last members of this equations’ chain we obtain

\[-f_\alpha(Q) = a_\alpha - \frac{1}{\mu} \sum_\beta a_\beta \frac{\partial \eta_\alpha}{\partial q_\beta} - \frac{i}{2 \mu} \sum_\beta [\hat{P}^2_\beta, \eta_\alpha], \]

which implies that \( \sum_\beta [\hat{P}^2_\beta, \eta_\alpha] \) is a function of \( Q \). Therefore we have

\[ \sum_\beta [\hat{P}^2_\beta, \eta_\alpha] = \phi_\alpha(Q) = \sum_\beta (\hat{P}_\beta [\hat{P}_\beta, \eta_\alpha] + [\hat{P}_\beta, \eta_\alpha] \hat{P}_\beta) = (-i) \sum_\beta \left( \hat{P}_\beta \frac{\partial \eta_\alpha}{\partial q_\beta} + \frac{\partial \eta_\alpha}{\partial q_\beta} \hat{P}_\beta \right) \]

\[ = (-i) \sum_\beta \left( \hat{P}_\beta \frac{\partial \eta_\alpha}{\partial q_\beta} + 2 \frac{\partial \eta_\alpha}{\partial q_\beta} \hat{P}_\beta \right) \]

\[ = (-i) \sum_\beta \left( -i \frac{\partial^2 \eta_\alpha}{\partial q_\beta^2} + 2 \frac{\partial \eta_\alpha}{\partial q_\beta} \hat{P}_\beta \right). \]

As a consequence \( \sum_\beta \frac{\partial \eta_\alpha}{\partial q_\beta} \hat{P}_\beta \) must be a function of \( Q \), so that for every \( \gamma \sum_\beta \left[ Q_\gamma, \frac{\partial \eta_\alpha}{\partial q_\beta} \hat{P}_\beta \right] = 0 = \frac{\partial \eta_\alpha}{\partial q_\beta}; \) therefore \( \frac{\partial \eta_\alpha}{\partial q_\beta} = 0; \) thus \( \eta_\alpha \) is a constant function. By using this result in the equality between the second and the last members of (eq26) we obtain

\[ a_\alpha = f_\alpha. \]

### 4 Imposing wave equations

According to section 3.5, for a spin-0 particle the interaction described by (eq70), which encompasses the electromagnetic interaction, is determined by the fact that each operator \( \eta_\alpha(Q) \) appearing in the general dynamical law (eq25) is a real multiple of the identity operator: \( \eta_\alpha(Q) = \lambda_\alpha \mathbb{I} \), with \( \lambda_\alpha \in \mathbb{R} \). Hence, according to Prop. 3.7, \( e^{iG_u u} U_g = e^{i\eta_\alpha(u, Q) U_g} = e^{i(\eta_\alpha(u, Q) + a_\alpha(u, Q)) U_g} = e^{i\lambda_\alpha u} e^{i\eta_\alpha(u, Q) U_g} U_g \), where \( a_\alpha(u, Q) \) is an operator infinitesimal of order grater than 1 in \( u \) with respect to the topology of \( \mathcal{H} \), so that \( e^{iG_u u} Q^{(t)} e^{-iG_u u} = e^{i\eta_\alpha(u, Q) U_g Q^{(t)} U_g^{-1}} e^{-i\eta_\alpha(u, Q)} = e^{i\eta_\alpha(u, Q) S_y(Q^{(t)})} e^{-i\eta_\alpha(u, Q)} = \{ 1 + \omega_1(u, Q) \} S_y(Q^{(t)}) \{ 1 + \omega_2(u, Q) \} \), where \( \omega_k(u, Q) \) is an operator infinitesimal of order grater than 1 in \( u \). Therefore, the \( \sigma \)-conversion leaves invariant the transformation properties of \( Q^{(t)} \) with respect to galileian boosts at the first order in \( u \).
Finally, since \( \hat{S} \) in u then a self-adjoint operator holds. The comparison with (eq29) show that such a condition holds if and only if (eq30)

\[ \text{Let } \hat{\varphi} \in U, \text{ then } \text{if (eq30)} \text{ holds, the following relations hold.} \]

If \( \alpha \) properties of \( Q^{(t)} \) with respect to subgroups of \( G \) are left invariant at the first order by the \( \sigma \)-conversion admitted by the interaction. In section 4.1 we address the case that such a subgroup is the subgroup of boosts, for every value of the spin. In section 4.2 we address the task for the subgroup of spatial translations.

4.1 Invariance under galileian boosts

The covariance properties of \( Q^{(t)} \) with respect galileian boosts \( g \) are expressed by \( S_g[Q^{(t)}_{\beta}] = U_g Q^{(t)}_{\beta} U_g^{-1} = Q^{(t)}_{\beta} - \delta_{\alpha\beta} u t \mathbb{I} \); therefore the equality

\[ e^{i\hat{G}_{\alpha}^{(t)} u} e^{-i\hat{G}_{\alpha}^{(t)}} = Q^{(t)}_{\beta} - \delta_{\alpha\beta} u t \mathbb{I} + o^{(t)}_{\alpha}(u), \quad (eq29) \]

where \( o^{(t)}_{\alpha}(u) \) is an operator infinitesimal of order greater than 1 with respect to \( u \), is the necessary and sufficient condition in order that the \( \sigma \)-conversion leave unaltered the covariance properties of \( Q^{(t)} \) with respect to the Galileian boosts, at the first order in \( u \).

**Proposition 4.1.** A \( Q \)-covariant \( \sigma \)-conversion leaves unaltered the covariance properties of \( Q^{(t)} \) under galileian boosts at the first order in the boosts’ velocity if an only if

\[ [\eta_{\alpha}(Q), Q^{(t)}_{\beta}] = \Phi. \quad (eq30) \]

In such a case, i.e. if (eq30) holds, the following relations hold.

\[ (i) \quad [\hat{G}_{\alpha}, Q^{(t)}_{\beta}] = i\delta_{\alpha\beta} t, \quad (ii) \quad [\hat{G}_{\alpha}, \hat{Q}_{\beta}] = i\delta_{\alpha\beta}; \quad (eq53) \]

\[ (i) \quad \mu Q^{(t)}_{\beta} - \hat{P}_{\beta} t = \varphi^{(t)}(Q), \quad (ii) \quad \hat{Q}_{\beta} = \frac{1}{\mu} \left( \hat{P}_{\beta} + a_{\beta}(Q) \right), \quad (eq54) \]

where \( \varphi^{(t)}(x) \) and \( a_{\beta}(x) = \frac{d}{dt} \varphi^{(t)}(x) \) \( |_{t=0} \) are self-adjoint operators of \( \mathcal{H} \).

**Proof.** Let \( \hat{U}_g = e^{i\hat{G}_{\alpha}^{(t)} u} = V_g U_g \) be the \( \sigma \)-converted unitary operator associated with the galileian boost \( g \), where \( V_g = e^{i\alpha_\omega(u,Q)} \) according to Prop. 3.7. By starting from (eq29) and by expanding \( e^{\pm i\hat{G}_{\alpha}^{(t)} u} \) with respect to \( u \) we obtain

\[ e^{i\hat{G}_{\alpha}^{(t)} u} e^{-i\hat{G}_{\alpha}^{(t)}} = V_g U_g Q^{(t)}_{\beta} U^{-1}_g V^{-1}_g = Q^{(t)}_{\beta} + i[\eta_{\alpha}(Q), Q^{(t)}_{\beta}] u - \delta_{\alpha\beta} u t \mathbb{I} + o^{(t)}_{\alpha}(u). \quad (eq28) \]

The comparison with (eq29) show that such a condition holds if and only if (eq30) holds.

By expanding \( e^{\pm i\hat{G}_{\alpha}^{(t)} u} \) with respect to \( u \) we find \( e^{i\hat{G}_{\alpha}^{(t)} u} e^{-i\hat{G}_{\alpha}^{(t)}} = Q^{(t)}_{\beta} + i[\hat{G}_{\alpha}, Q^{(t)}_{\beta}] u + o^{(t)}_{\alpha}(u) \), so that (eq29) holds if and only if \( i[\hat{G}_{\alpha}, Q^{(t)}_{\beta}] = -\delta_{\alpha\beta} u t \mathbb{I} \); therefore (eq53) hold. Finally, since \( \hat{G}_{\alpha} = \mu Q_{\alpha} \), (eq53.i) implies \( [\mu Q_{\alpha}, Q^{(t)}_{\beta}] = [\hat{G}_{\alpha}, Q^{(t)}_{\beta}] = [Q_{\alpha}, \hat{P}_{\beta}], \) and then a self-adjoint operator \( \varphi^{(t)}_{\beta} \) of \( \mathcal{H} \) must exists for every \( x \) such that (eq54) hold.
If we put \( H_0 = \frac{1}{2\mu} \sum_\gamma \left( \hat{P}_\gamma + a_\gamma(Q) \right)^2 \), then a simple calculation yields
\[
i[H_0, Q_\beta] = \frac{1}{\mu} \left( \hat{P}_\beta + a_\beta(Q) \right).
\]
Whenever (eq30) holds, Prop. 4.1 implies \( i[H_0, Q_\beta] = \hat{Q}_\beta \), i.e. \([H, Q_\beta] = [H_0, Q_\beta] \); therefore
\[
H = H_0 + \Phi(Q) = \frac{1}{2\mu} \sum_\gamma \left( \hat{P}_\gamma + a_\gamma(Q) \right)^2 + \Phi(Q),
\]
(eq55)
where \( \Phi(x) \) is a self-adjoint operator of \( \mathcal{H}_0 \), and the wave equation is
\[
i \frac{\partial}{\partial t} \psi_t = \left\{ \frac{1}{2} \sum_\gamma \left( \hat{P}_\gamma + a_\gamma(Q) \right)^2 \right\} \psi_t.
\]

According to (eq55), the dynamics of the particle is determined by the four vector valued functions \( a_\alpha, \Phi \). We can call them the “fields” which describe the effects of the interaction; in so doing, however we have not confuse them with other notions of field involved in Quantum Physics. Now we shall see how these fields are related to the fields \( \eta_\alpha, f_\alpha \) entering the general dynamical law (eq25).

From (eq54.ii) we imply \([\eta_\alpha(Q), \hat{Q}_\beta] = \frac{1}{\mu}[\eta(Q), \hat{P}_\beta] + \frac{1}{\mu}[\eta_\alpha(Q), a_\beta(Q)] \). By making use of (eq30) we obtain
\[
\frac{\partial \eta_\alpha}{\partial x_\beta}(Q) = \frac{i}{2} [\eta_\alpha(Q), a_\beta(Q)].
\]
(eq58.ii)
Now, by replacing the form (eq55) of \( H \) in (eq25) we obtain
\[
\hat{P}_\alpha - f_\alpha(Q) = i[H, \mu Q_\alpha - \eta_\alpha(Q)]
\]
(eq56)
\[
= i \left[ \frac{1}{2\mu} \sum_\beta \mu^2 \hat{Q}_{\beta}^2 + \Phi(Q), \mu Q_\alpha \right] - i \left[ \frac{1}{2\mu} \sum_\beta \mu^2 \hat{Q}_{\beta}^2 + \Phi(Q), \eta_\alpha(Q) \right]
\]
\[
= i \left\{ \frac{1}{2} \mu \sum_\beta [\hat{Q}_{\beta}^2, \mu Q_\alpha] + [\Phi(Q), \mu Q_\alpha] \right\} +
\]
\[
- i \left\{ \frac{1}{2} \mu \sum_\beta [\hat{Q}_{\beta}^2, \eta_\alpha(Q)] + [\Phi(Q), \eta_\alpha(Q)] \right\}.
\]
By making use of (eq53.ii), which implies \([\mu Q_\alpha, \hat{Q}_{\beta}] = 2i\delta_{\alpha\beta} \hat{Q}_{\beta} \), of (eq30) and of (eq54.ii), we find
\[
\hat{P}_\alpha - f_\alpha(Q) = \frac{1}{2\mu} \sum_\beta (-2i \delta_{\alpha\beta} \hat{Q}_{\beta}) + \Phi - i \frac{1}{2\mu} \Phi - i [\Phi(Q), \eta_\alpha(Q)]
\]
\[
= \mu Q_\alpha - i [\Phi(Q), \eta_\alpha(Q)] = \hat{P}_\alpha + a_\alpha(Q) - i [\Phi(Q), \eta_\alpha(Q)] = \hat{P}_\alpha - f_\alpha(Q).
\]
Therefore we have proved that
\[
f_\alpha(Q) = i [\Phi(Q), \eta_\alpha(Q)] - a_\alpha(Q).
\]
(eq58.i)
Hence, whenever (eq30) holds, the fields \( \eta_\alpha \) and \( f_\alpha \) in the general law (eq25) are determined, according to (eq58), by the fields \( a_\alpha, \Phi \).
For the particular case of a spin-0 particle we can show the following further characterization.

**Proposition 4.2.** In the simplest quantum Theory of an interacting particle, corresponding to the case $\mathcal{H}_0 = \mathfrak{G}$ in (eq44), the $Q$-covariant $\sigma$-conversions for which $\eta_\alpha(Q) =$ constant are those and only those which leave unaltered the covariant properties of $Q^{(t)}$ with respect to the Galileian boosts $g \in G$, at the first order in the boost's velocity.

**Proof.** If $\eta_\alpha =$constant then (eq30) holds, of course. Therefore, in order to prove the proposition, it is sufficient to prove the inverse implication. Hence we suppose that (eq30) holds. It implies the condition $[\eta_\alpha(Q), Q_\alpha] = 0$. On the other hand, (eq54.i) implies $Q^{(t)}_\beta = \frac{\lambda}{\mu} (\varphi^{(t)}_\beta(Q) + \dot{P}_\beta)$, which replaced in (eq30) yields $[\eta_\alpha(Q), \dot{P}_\beta] = 0$; therefore $\eta_\alpha(Q)$ is a constant operator $\lambda_\alpha I$.

### 4.2 Invariance under spatial translations

Let us now suppose that the interaction admits a $\sigma$-conversion such that if $\hat{U}_g = e^{-i\hat{P}_a \alpha}$ then

$$e^{-i\hat{P}_a \alpha} Q^{(t)}(\alpha) e^{i\hat{P}_a \alpha} = Q^{(t)}(\alpha) - \delta_{\alpha \beta} + o^{(t)}(\alpha),$$

**(eq60)**

where $o^{(t)}(\alpha)$ is an infinitesimal operator of order greater than 1 in $\alpha$. In fact, we are supposing that the interaction leaves unaltered the covariance properties of $Q^{(t)}$ with respect to spatial translations at the first order in the translation parameter $\alpha$. Now, by expanding $e^{-i\hat{P}_a \alpha}$ with respect to the translation parameter $\alpha$, (eq60) yields

$$i) \ [Q^{(t)}_\beta, \hat{P}_\alpha] = i\delta_{\alpha \beta} \text{ which implies } \ (ii) \ [\hat{Q}_\beta, \hat{P}_\alpha] = 0.$$  

**(eq50)**

Therefore we can state that

$$\dot{Q}_\beta = v_\beta(\hat{P}),$$

**(eq51)**

where $v_\beta(p)$ is a self-adjoint operator of $\mathcal{H}_0$. Since $[Q_\alpha, v_\beta(\hat{P})] = i\frac{\partial v_\beta}{\partial p_\alpha}(\hat{P})$, by making use of the Jacobi identity for $[Q_\alpha, [H, Q_\beta]]$ we obtain

$$i\frac{\partial v_\beta}{\partial p_\alpha}(\hat{P}) = [Q_\alpha, Q_\beta] = i [Q_\alpha, [H, Q_\beta]] = [Q_\beta, \dot{Q}_\alpha] = i\frac{\partial v_\beta}{\partial p_\beta}(\hat{P}).$$

This equality shows that $v(p) = (v_1(p), v_2(p), v_3(p))$ is an irrotational field; hence a function $F$ of $p$ exists such that $v_\alpha(p) = \frac{\partial F}{\partial p_\alpha}(p)$, where $F(p)$ is a self-adjoint operator of $\mathcal{H}_0$. Therefore we can establish the following equalities.

$$\dot{Q}_\alpha = v_\alpha(\hat{P}) = \frac{\partial F}{\partial p_\alpha}(\hat{P}) = i[F(\hat{P}), Q_\alpha] = i[H, Q_\alpha].$$

**(61)**

The last equation implies that a function $\Psi$ of $x$ exists such that $H - F(\hat{P}) = \Psi(\hat{Q})$, i.e.

$$H = F(\hat{P}) + \Psi(\hat{Q}),$$

**(eq52)**

where $\Psi(x)$ is a self-adjoint operator of $\mathcal{H}_0$, and the wave equation is $i\frac{\partial}{\partial \tau} \psi_t = \{F(\hat{P}) + \Psi(\hat{Q})\} \psi_t$.  

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4.3 Invariance under both

Let us suppose that the interaction admits a $\sigma$-conversion that leaves unaltered the covariance properties of $Q(t)$ under both subgroups of boosts and of spatial translations. Accordingly, the following equality holds

$$H = F(\hat{P}) + \Psi(Q) = \frac{1}{2\mu} \sum_\gamma \left( \hat{P}_\gamma + a_\gamma(Q) \right)^2 + \Phi(Q)$$

$$= \frac{1}{2\mu} \sum_\gamma \left( \hat{P}_\gamma^2 + a_\gamma(Q) \hat{P}_\gamma + \hat{P}_\gamma a_\gamma(Q) + a_\gamma^2(Q) \right) + \Phi(Q).$$

Since $a_\gamma(Q) \hat{P}_\gamma + \hat{P}_\gamma a_\gamma(Q) = [a_\gamma(Q), \hat{P}_\gamma] + 2a_\gamma \hat{P}_\gamma a_\gamma(Q) = i \frac{\partial a_\gamma}{\partial x_\gamma}(Q) + 2a_\gamma \hat{P}_\gamma a_\gamma(Q)$ the equality above implies

$$\frac{1}{2\mu} \sum_\beta \hat{P}_\beta a_\beta(Q) = \left( F(\hat{P}) - \frac{1}{2\mu} \sum_\beta \hat{P}_\beta^2 \right) + \Psi(Q) - i \frac{\partial a_\gamma}{\partial x_\gamma}(Q) - \Phi(Q) - \sum_\beta a_\beta^2(Q).$$

Then

$$\frac{1}{2\mu} \sum_\beta \hat{P}_\beta a_\beta(Q) = F_1(\hat{P}) + F_2(Q),$$

where $F_1(\hat{P}) = \left( F(\hat{P}) - \frac{1}{2\mu} \sum_\beta \hat{P}_\beta^2 \right)$ and $F_2(Q) = \Psi(Q) - i \frac{\partial a_\gamma}{\partial x_\gamma}(Q) - \Phi(Q) - \sum_\beta a_\beta^2(Q)$. Therefore

$$\left[ Q_\gamma, \frac{1}{2\mu} \sum_\beta \hat{P}_\beta a_\beta(Q) \right] = i \frac{\partial a_\gamma}{\partial x_\gamma}(Q) = \frac{\partial f_1}{\partial x_\gamma}(\hat{P}).$$

Then

$$[\hat{P}_\alpha, a_\gamma(Q)] = \frac{\partial a_\alpha}{\partial x_\alpha}(Q) = -2i\mu \left[ \hat{P}_\alpha, \frac{\partial f_1}{\partial x_\gamma}(\hat{P}) \right] = 0.$$ 

Therefore, $a_\gamma(Q)$ is an operator that acts as follows

$$[a_\gamma(Q)\psi](x) = \hat{a}_\gamma \psi(x),$$

where $\hat{a}_\gamma$ is an operator of $\mathcal{H}_0$ which does not depend on $x$.

Thus, if (eq30) and (eq60) hold, then $H = \frac{1}{2\mu} \sum_\gamma (\hat{P}_\gamma + \hat{a}_\gamma)^2 + \Phi(Q)$, and the wave equation is

$$i \frac{\partial}{\partial t} \psi_t = \left\{ \frac{1}{2\mu} \sum_\gamma (\hat{P}_\gamma + \hat{a}_\gamma)^2 + \Phi(Q) \right\} \psi_t.$$

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