RESOURCES ALLOCATION: A COMMON SET OF WEIGHTS MODEL

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Abstract. Allocation problem is an important issue in management. Data envelopment analysis (DEA) is a non-parametric method for assessing a set of decision making units (DMUs). It has proven to be a useful technique to solve allocation problems. In recent years, many papers have been published in this regard and many researchers have tried to find a suitable allocation model based on DEA. Common set of weights (CSWs) is a DEA model which, in contrast with traditional DEA models, does not allow individual weights for each decision making unit. In this manner, all DMUs are assessed through choosing a same set of weights. In this article, we will use the weighted-sum method to solve the multi-objective CSW problem. Then, via introducing a set of special weights, we will connect the CSW model to a non-linear (fractional) CSW model. After linearization, the proposed model is used for allocating resources. To illustrate our model, some examples are also provided.

1. Introduction. Most of the organizations today consist of a number of individual sections. As an example, within a university, these diverse sections (units in our field of study) may be different departments. If we take a bank, the units may be different branches of that bank. Organizations might spend a surplus cost to allocate additional resource(s) in order to raise revenue of the system. While the units benefit from the system, each of them should share a part of expense(s) or resource(s). There is a tough competition among units, from economical point of view, to receive a proper share of costs or resources; thus, the important and difficult part of management fields is solving the resource allocation issue. This

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issue involves planning of how cost or resource(s) can be fairly distributed between decision making units (DMUs).

Data envelopment analysis (DEA) is a non-parametric method which was first introduced by Charnes et al. [11] to assess performance value of a set of decision making units (DMUs). After that, a number of different DEA-based models have been proposed to solve various issues in the literature. In the cost allocation field, Cook and Kress [14] used the DEA method for the first time. Following them, many researchers have attempted to allocate a fair share of resource or fixed cost to individual units by using DEA method. Yan et al. [52] considered preference cone constraints as well as using multi objective programming. They used inverse DEA to estimate inputs/outputs of a DMU when changing some inputs/outputs such that the efficiencies are preserved. Cook and Zhu [16] developed the method in [14] from CCR model into BCC and from input orientation into output orientation model. Lin [37] showed that Cook and Zhu’s method [16] is not comprehensive because when specific limitations are imposed, the obtained solution will not be feasible. He added some fixed target to their model in order to improve this condition and to obtain a feasible allocation.

Beasley [4] presented a different method. He used a non-linear model based on DEA. He utilized a common set of weights and several other models to find a unique solution. Based on [4], Kordrostami and Amirteimoori [2] presented another method to obtain the solution by efficiency invariance assumption. Jahanshahloo et al. [28] claimed Beasley’s method [4] is always feasible and efficiency invariance assumption is not a necessary constraint in [2]; they also presented two approaches for fixed allocation based on efficiency invariance and common set of weights principles. Li et al. [35] attempted to obtain a fair allocation by considering the allocated cost to each unit as the extra input. This method was also performed with efficiency invariance assumption. Bi et al. [5] used a common weight DEA method to allocate resources and set the target in the parallel production system. Li et al. [36] utilized an iterative method based on CSWs to achieve a suitable allocation. Du et al. [23] presented an iterative method to find the optimal solution to the allocation problem using Russell model and changing it to an aggregated goal programming model. Li et al. [33] used the common set of weights and efficiency invariance principle for resource allocation and target setting.

CSW is a DEA-based method to solve some problems such as ranking and resource allocation. In traditional DEA models, each DMU separately chooses a set of weights to guarantee the most favorable result for the efficiency score which may be unacceptable in reality. However, in CSW approach, this issue does not rise as all weights are chosen similarly for all DMUs using one model to maximize all DMUs efficiency scores. In view of this, it is a useful tool to solve problems in which imposing the common constraint(s) on all units are needed such as ranking or allocation ones.

Many articles have been published in the common set of weights field. Some of them have tried to find the suitable set of weights to solve a CSW problem directly while others have employed the common set of weights technique for solving issues such as ranking or allocation. Cook et al. [15] mentioned the concept of common weights in DEA for the first time and next it was developed by Roll et al. [46]. Kao and Hung [30] and Zohrehbandian et al. [54] proposed the compromised approach to generate common weights under DEA framework. Jahanshahloo et al. [27] used a multi objective model and a non-linear transformation to obtain a set of common
weights. Liu and Peng [39] proposed a method to solve CSW problem and rank DMUs by it. Saati et al. [47] presented a two phase CSW approach by using an ideal unit. Ramn et al. [45] proposed a method to obtain the common set of weights and then used it for ranking DMUs. Hosseinzadeh Lotfi et al. [40] proposed a common set of weights to solve allocation problems and used the goal programming approach to solve the CSW problem. Ramazani-Tarkhorani et al. [44] discussed difficulties in the approach of Liu and Peng [39] and suggested another approach based on lexicography method to solve multi objective problems. Carrillo and Jorge [7] simultaneously minimized the differences between weighted sum of inputs and weighted sum of outputs based on Tchebychev norm and then $L_1$ norm. Last but not least, Chiang et al. [12] used a method based on minimization of differences between weighted sum of inputs and weighted sum of outputs.

CSW problem is a multi objective linear fractional problem (MOLFP). MOLFPs constitute a notable part of multi objective programs, for whose solution to be found so much effort has been made and many literatures have been published. It goes without saying that the purpose behind solving a multi objective problem (MOP) is obtaining an efficient solution for it because this set of programming problems do not have an optimal solution in most cases. Dinkelbach [22] proposed a parametric approach to solve linear fractional problems. Almogy and Levin [1] extended the approach of Dinkelbach [22] to solve sum of ratios problems. Despite the usage of Dinkelbachs approach in many fractional problems, it had no absolutely successful extension to solve MOLFP problems. Kornbluth and Steuer [31] presented a simplex-based solution procedure for the MOLFP problems.

Some researchers investigated the iterative algorithms to solve the MOLFP problems [18, 19, 20, 25, 29, 32, 51]. Hosseinzadeh et al. [41] presented an approach to check the strong or weak efficiency of a given feasible solution. Sadia and Gupta [48] suggested the value function and Chebyshev Goal Programming approaches to obtain an efficient solution of multi objective linear plus linear fractional programming problems. Some other researchers tried to solve multi objective linear fractional problems via fuzzy linearization strategy [6, 8, 9, 21, 26]. When several units operate under the control of one decision maker (DM), it is realistic that they lose authority on presenting decision parameters (multipliers), individually. In this case, DM has the power of determine multipliers for all units in a similar way. Furthermore, in allocation problems, the constraints connected to allocating should be simultaneously exerted on the all units; the CSW model allows the addition of such constraints. In addition, it represents a natural generalization of the traditional DEA approach. These are main reasons for selecting the CSW method to solve allocation problem. In the present research, CSW model is utilized to solve a resource allocation problem. The CSW is a MOP; thus, to solve it, the weighted-sum method should be first used. In the proposed method, we present a special set of weights to solve the weighted CSW model. We then obtain a one-objective non-linear programming problem which is convertible to a linear problem. Next, the proposed approach is utilized to solve the allocation problem. In the rest of this paper, the multi objective problems are explained and some techniques to solve them are presented in section 2. DEA approach and CSW method are presented in section 3. In section 4, the proposed method and related issues are accompanied by an example. Section 5 involves our method to solve resource allocation problems and an example is illustrated. Finally, conclusions appear in section 6.
2. Some solving methods of MOP. A noticeable part of optimization problems are in the form of multi objective problems. The DMUs assessment problem through common weights which is the foundation of our study is of this type. To this end, the multi objective problems specifically MOLFP problems are shortly introduced in this section. Then, some of the methods to solve the MOP problems are explained.

Our notation and terminology are standard.

2.1. Multi Objective Programming. The term Multi Objective Programming is used to describe simultaneous optimization of multiple objectives over one convex solution space. Without loss of generality, it may be supposed that all objectives are maximization. On this assumption, model (1) may represent the MOP in general form: (see [50])

\[
\max \{ f_1(x), \ldots, f_k(x) \} \\
s.t. \quad x \in S
\]

Each \( f_j(x) \), \( j \in \{1, \ldots, k\} \) is a convex function and \( x = (x_1, \ldots, x_n) \) is a vector of unknown variables in the convex feasible solution space \( S \). Most of the time, there may not exist a feasible solution to simultaneously optimize all objective functions of a MOP. In this regard, solving a MOP means finding an efficient solution in the feasible space \([24, 50]\).

Definition 2.1. A point \( \hat{x} \in S \) in (1) is said to be an efficient solution if and only if there does not exist another point \( \tilde{x} \in S \) such that \( f_j(\tilde{x}) \geq f_j(\hat{x}) \) for all \( j \in \{1, \ldots, k\} \) and \( f_j(\tilde{x}) > f_j(\hat{x}) \) for at least one \( j \).

MOLFP problems are the special form of MOPs which consist the optimization of several linear fractional objective functions over some linear constraints.

The CSW problem which will be discussed in section 3 is a MOLFP problem.

2.2. Some solving methods of MOP. Many different techniques are available to solve a MOP problem. Lexicography, Goal programming and Weighted-sum methods are only three methods in this category \([24, 50]\).

In the Lexicography method, functions are arranged according to priorities. To solve a MOP problem containing \( k \) objective functions using lexicography method, at most \( k \) optimization stages may be required. In the first stage, the function at the highest priority level is optimized over the feasible space \( \mathcal{S}^0 = S \). The set of optimal solutions to the first stage is named \( \mathcal{S}^1 \). If an alternative optima exists, the function in the second priority is optimized over \( \mathcal{S}^1 \). The set of optimal solutions to the second stage is named \( \mathcal{S}^2 \). We follow this until the \( k^{th} \) stage. The function in \( k^{th} \) priority is optimized over \( \mathcal{S}^{(k-1)} \). It may cease the progression through the optimization stages as soon as one optimization stages is encountered which has a unique solution. In this case, other functions in lower priority levels are ignored. This method is easy to use but the higher priority levels do not allow compromise with lower priority levels and this is a weakness. Difficulty in determining a priority for functions is another disadvantage of this method.

An efficient solution in Goal programming approach is obtained through three steps. First, the objectives are conceptualized as goals and a goal like \( \alpha_i \) is set to any objective function \( f_i(x) \). Second, deviation variables such as \( d_i^+ \) and \( d_i^- \) are presented to measure overachievement and underachievement from target level \( \alpha_i \). Finally, the weighted sum of deviation variables is minimized over solution space while a new set of \( k \) achievement constraints i.e., \( f_i(x) + d_i^- - d_i^+ = \alpha_i \), \( i = 1, \ldots, k \) are added. This approach is desirable to those set of problems which
have some objectives without monotonousness (with a maximization objective, we have monotonicity if more is always better than less. [50]). However, increasing the size of a problem in both variables and constraints aspects is the drawback of this approach.

Among various methods of finding efficient point for a MOP, the weighted-sum method is chosen. In the weighted-sum approach, each of objective functions is multiplied by a non-negative weight $\lambda_i$. Then, the weighted functions are summed and the obtained composition is optimized instead of optimizing multiple objective functions over solution space. Model (2) shows the general form of weighted-sum problem corresponded to MOP (1). See [50]

$$
\max \sum_{i=1}^{k} \lambda_i f_i(x) \\
\text{s.t. } x \in S.
$$

Traditionally, these weights are normalized. There are many reasons that make this method favorable. One of them is the effect of weighted-sum method on the size of a problem. It is obvious the size of problem does not change dramatically. Another reason is that when selecting weights, manager’s viewpoints about the importance of objective functions may be exerted. Two other important reasons to interest researchers to use weighted-sum method can be stated as the two following theorems.

**Theorem 2.2.** Let $\hat{x}$ maximize the objective function of model (2). Then, $\hat{x}$ is an efficient solution of model (1).

*Proof.* See [24].

**Theorem 2.3.** Let $\hat{x}$ be efficient in model (1). Then, there is a $(\hat{\lambda}_1, ..., \hat{\lambda}_n) \geq 0$ such that considering it as a weight vector, $\hat{x}$ maximizes the objective function of model (2).

*Proof.* See [24].

It must be noted that this method only finds some efficient points (not all of them) by presenting each set of weights; it means that in order to find other efficient points, one must change the weights.

Fig. 1 describes the weighted-sum method geometrically. This figure graphs a simple two-objective linear problem as follows:

$$
\begin{align*}
\max & \quad z_1 = c^1 x \\
\max & \quad z_2 = c^2 x \\
\text{s.t.} & \quad x \in S.
\end{align*}
$$

The vectors $c^1$ and $c^2$ represent the objective vectors of the first and second objective functions, respectively. The means of vector $\lambda c$ in Fig. 1 is $\lambda_1 c^1 + \lambda_2 c^2$ for a typical vector $\lambda$. Changing $\lambda$ such that $\lambda_1 + \lambda_2 = 1$, it may obtain different points on the dashed line between $c^1$ and $c^2$. So, considering theorem 2.2, all points on two segments $v_2v_3$ and $v_3v_4$ are efficient.

3. **DEA and CSW model.** The concept of common weights is frequently used in DEA. The CSW is a DEA model to seek the multipliers that simultaneously maximize efficiency scores to all units. In this section we first review the DEA method and then briefly explain CSW model.
3.1 Data envelopment analysis. Data envelopment analysis, as an approach to evaluate the efficiency score, is first introduced by Charnes et al. [11]. Since then, it has been used as a valuable tool in various fields. Consider n units which are hereafter named DMUs. Each DMU, \( j \in \{1, ..., n\} \), is characterized by a non-negative vector \((x_j, y_j)\) in which \(x_j \in \mathbb{R}^m\) and \(y_j \in \mathbb{R}^s\) are named input and output vectors, respectively. Both input and output vectors are assumed to have at least one positive element. To assess a unit, first the production possibility set (PPS) must be determined. PPS is characterized based on accepting all or some of five postulates. These postulates are: inclusion of observations, convexity, possibility, constant returns to scale (or unbounded ray) and minimal extrapolation [3, 17, 43]. Inclusion of observations tells us all observed units belong to PPS. Convexity argues on belonging convex hull of observed DMUs to PPS. Possibility consists if an input measure \( \hat{x} \) can produce an output measure \( \hat{y} \), then each input \( x \) greater than \( \hat{x} \) may produce any output measure \( y \) less than \( \hat{y} \). If \((x, y) \in PPS\), then \((\lambda x, \lambda y) \in PPS\) for each \( \lambda \geq 0 \) sits in constant returns to scale postulate. Finally, minimal extrapolation postulate calls PPS the smallest set that confirms four explained postulates. PPS_{CCR} (or PPS_{TC}), which is used in this article, is built on accepting all postulates. This PPS can be written as form (3) (see [11]).

\[
T_c = \{(x, y) : x \geq \sum_{j=1}^{n} \lambda_j x_j, \ y \leq \sum_{j=1}^{n} \lambda_j y_j, \ \lambda_j \geq 0\} \tag{3}
\]

Fig. 2 shows a typical PPS in two dimensions for a single input and single output case. After defining PPS, a model connected with PPS gives us the score.

\[
T_c = \{(x, y) : x \geq \sum_{j=1}^{n} \lambda_j x_j, \ y \leq \sum_{j=1}^{n} \lambda_j y_j, \ \lambda_j \geq 0\} \tag{3}
\]
of efficiency of each unit under assessment. There are two main form of models to assess a unit which are named envelopment form and multiplier form [3, 11, 17].

To assess a unit, say DMUₚ, via envelopment form in CCR, one can use model (4):

$$\theta^*_p = \min \theta$$
$$s.t. \sum_{j=1}^{n} \lambda_j x_j \leq \theta x_p$$
$$\sum_{j=1}^{n} \lambda_j y_j \geq y_p, \lambda_j \geq 0, j = 1, \ldots, n$$

The above model gives us a projection of the unit under assessment (DMUₚ) on the PPS frontier i.e. $$(\theta^*_p x_p, y_p)$$. (see [17])

**Definition 3.1.** DMUₚ is named CCR-efficient, if the optimal value of model (4) is one; on the other words, the projected unit is DMUₚ itself.

**Definition 3.2.** DMUₚ is also strong efficient if $\theta^*_p = 1$ and at least in one optimal solution of (4) all constraints are binding.

**Definition 3.3.** If while assessing DMUₚ, via model (4), a unique optimal solution ($\theta^*_p = 1, \lambda^*_p = 1, \lambda^*_j = 0; j \neq p$) can obtained, then DMUₚ is called extreme efficient.

Rational model (classic multiplier form) in CCR to assess DMUₚ is in the form of model (5).

$$\theta^*_p = \max uy_p$$
$$s.t. \frac{ux_p}{vx_p} = 1, \frac{uy_j}{vx_j} \leq 0, \frac{uy_j - vx_j}{vx_j} \leq 0, j = 1, \ldots, n$$
$$u \geq 0, v \geq 0.$$

Where $u = (u_1, \ldots, u_s)$ and $v = (v_1, \ldots, v_m)$ are the unknown weighting vectors (multipliers) connected with the outputs and inputs of DMUₚ, respectively. In this model, the ratio of the weighted sum of outputs to weighted sum of inputs of each unit is presented as the efficiency score; this ratio should not exceed one. DMUₚ (under assessment) can determine a set of individual multipliers to achieve a maximum efficiency score.

Model (5) is non-linear, using Charnes-Cooper transformation [10] it can be converted to the linear form (6) as below:

$$\theta^*_p = \max uy_p$$
$$s.t. \frac{ux_p}{vx_p} = 1, \frac{uy_j}{vx_j} \leq 0, \frac{uy_j - vx_j}{vx_j} \leq 0, j = 1, \ldots, n$$
$$u \geq 0, v \geq 0.$$

Model (6) is the dual form of model (4). Let $$(u^*, v^*)$$ be the optimal solution of model (6). It may easily be proved that the hyperplane $u^* y - v^* x = 0$ is tight on the point $$(\theta^*_p x_p, y_p)$$ i.e., $u^*(y_p) - v^*(\theta^*_p x_p) = 0$.

**Definition 3.4.** Let $\bar{x} = \sum_{j=1}^{n} x_j \in \mathbb{R}^m$ and $\bar{y} = \sum_{j=1}^{n} y_j \in \mathbb{R}^s$, then $$(\bar{x}, \bar{y}) \in \mathbb{R}^{m+s}$$ is named the aggregated unit.

**Theorem 3.5.** Aggregated unit belongs to the production possibility set.

**Proof.** Based on convexity and constant returns to scale postulates, the proof is obvious. □
Theorem 3.6. Aggregated unit cannot be an extreme efficient unit.

Proof. Referring to theorem 3.5, \((\bar{x}, \bar{y}) \in T_c\), so adding \((\bar{x}, \bar{y})\) to the set of observed DMUs dose not changed the PPS. We consider it as an observed DMU and then assess it using model (6). Note that by considering it as an observed DMU, the constraint \(u_0 - v_0 \leq 0\) must be added to the set of constraints of model (6). However, this constrain is redundant because it may be obtained by summation of other constraints i.e., \(u_0 - v_0 \leq 0\) is the result of \(\sum_{j=1}^{n} (u_jy_j - v_jx_j) \leq 0\). This concludes that the dual variable corresponding to this constraint is zero in the dual form. This means \((\bar{x}, \bar{y})\) cannot be extreme efficient.

Corollary 1. Considering \((u^*, v^*)\) as an optimal solution of multiplier form (6) in assessing the aggregated unit \((\bar{x}, \bar{y})\), the hyper plane \(u^*y - v^*x = 0\) is binding on at least one observed DMU.

Proof. Considering theorem 3.6, the aggregated unit cannot be an extreme efficient unit so either it is non-extreme efficient or inefficient. Based on the envelopment form, we know \((\theta^*\bar{x}, \bar{y})\) is a unit on the frontier; therefore, there exist a non-zero vector \((\lambda_1, \ldots, \lambda_n) \geq 0\) such that \((\theta^*\bar{x}, \bar{y}) = (\sum_{j=1}^{n} \lambda_jx_j, \sum_{j=1}^{n} \lambda_jy_j)\). Since \(u^*y - v^*x = 0\) is binding on \((\theta^*\bar{x}, \bar{y})\), we may conclude \(u(\sum_{j=1}^{n} \lambda_jy_j) - v(\sum_{j=1}^{n} \lambda_jx_j) = 0\). On the other hand, suppose \(u^*y_j - v^*x_j < 0\) for all \(j \in \{1, \ldots, n\}\) (abund hypothesis), then for any non-negative and non-zero vector \((\lambda_1, \ldots, \lambda_n)\), we will have \(u(\sum_{j=1}^{n} \lambda_jy_j) - v(\sum_{j=1}^{n} \lambda_jx_j) < 0\). This contradiction proofs the corollary.

3.2. Common set of weights approach. Common set of weights approach is a technique in DEA based on multi objective view to maximize the efficiency score of DMUs, simultaneously. In traditional DEA method, each DMU can present a set of individual multipliers for inputs and outputs to achieve maximum efficiency score. This may not be acceptable in real world so the concept of common weights was proposed by Roll et al. [46]. In other words, using a CSW model, we try to obtain a set of multipliers which can simultaneously assess all units in their best performance. This is the desirable method for those problems on which similar conditions must be imposed such as ranking DMUs or allocation problems. Model (7) introduces the general form of CSW approach in CCR in output orientation (see [13]).

\[
\begin{align*}
\max & \quad \left\{ \frac{u_1y_1}{x_1}, \ldots, \frac{u_ny_n}{x_n} \right\}, \\
\text{s.t.} & \quad \frac{u_0y_0}{x_0} \leq 1, \quad j = 1, \ldots, n \\
& \quad u \geq 0, v \geq 0.
\end{align*}
\]

Reviewing this articles introduction, one can find many different approaches to find an appropriate solution to CSW problem. In line with what came in Section 2.2, we will pick the weighted-sum method. In the next section, a special set of weights is going to be proposed to obtain a single objective instead of multi objectives in a CSW problem. Next, we convert it to a linear programming problem and solve it. It is worth to note, although there exist many other weights that satisfy the conditions of weighted sum method, as we are aware, up to now, no weights are introduced that could make the objective function linear. However, some papers e.g. Beasley [4] used the average of efficiencies to optimize (i.e. weights are \(\frac{1}{n}\)) that make a non-linear objective function. In the next section, we propose weights in such a way that make the objective function linear.
Proposed method and relevant issues. In this section, we first propose an approach based on the weighted-sum method to solve the CSW problem. Next, we provide an example to illustrate the proposed method.

4.1. Weighted-sum for the CSW problem. As noted in section 2.2, the weighted-sum method is one of several methods in solving MOPs. In this manner, by assigning a non-negative weight to each objective and summing them, one objective function is derived. Noting the CSW problem as a MOP, the weighted-sum method might be used to solve it. Let’s select \( \lambda_j = \frac{x_j}{\bar{x}} \) as a weight coefficient for the \( j \)th objective function. Considering CSW model (7) and \((\bar{x}, \bar{y})\) as the aggregated unit, we obtain the composite function \( \bar{u} \bar{y} = \sum_{j=1}^{n} \left( \frac{c_j}{x_j} \right) \left( \frac{u_j}{v_j} \right) \). Thus, model (8) will be obtained using the weighted-sum method.

\[
\max \quad \bar{u} \bar{y} \\
\text{s.t.} \quad \frac{u_j}{v_j} \leq 1, \quad j = 1, \ldots, n \\
\quad \bar{u} \geq 0, \quad v \geq 0.
\]  

One may see model (8) as the multiplier form of CCR in assessing the aggregated unit. It can convert to the linear form (9) using Charnes-Cooper transformations [10] as follows:

\[
\max \quad u \bar{y} \\
\text{s.t.} \quad v \bar{x} = 1 \\
\quad uy_j - vx_j \leq 0, \quad j = 1, \ldots, n \\
\quad u \geq 0, \quad v \geq 0.
\]

Theorem 4.1. Model (9) is feasible and bounded.

Proof. Since \( \bar{x} = \sum_{j=1}^{n} x_j \) and \( x_j \neq 0, x_j \geq 0 \) for any \( j \in \{1, \ldots, n\} \), so it has at least one positive element, say \( \bar{x}_l > 0 \). Thus, \( u = (0, \ldots, 0) \) and \( v = (0, \ldots, \frac{1}{\bar{x}_l}, \ldots, 0) \) is a feasible solution. Summation of the second set of constraints in model (9) concludes \( u \bar{y} \leq v \bar{x} \). The first constrain results the optimal objective value to be equal or less than unity.

Theorem 4.2. Suppose the optimal value of model (9) is \( u^* \bar{y} = 1 \), then CCR-score \( (\theta^*_j) \) of each unit is unity.

Proof. Let \((u^*, v^*)\) as an optimal solution of (9) with the optimal objective value of one. Considering the first constraint, we have \( u^* \bar{y} - v^* \bar{x} = 0 \). Suppose there exist a DMU, say DMU\(_r\), with CCR-score less than one so \( \frac{u^* y^*_r}{v^*_r} < 1 \) (Since \( \frac{u^* y^*_r}{v^*_r} \leq \theta^*_r < 1 \)); It results \( u^* y^*_r - v^* x^*_r < 0 \). Now, with the summation of the second set of constraints in optimality, we conclude that \( u^* \bar{y} - v^* \bar{x} < 0 \) which is a contradiction.

Corollary 2. If \((\bar{x}, \bar{y})\) is efficient (i.e., \( u^* \bar{y} = 1 \)), then all observed DMUs lie on the same hyperplane \( u^* \bar{y} - v^* \bar{x} = 0 \).

Proof. Suppose there is a unit DMU\(_r\) that is not on this hyperplane; it results \( u^* y^*_r - v^* x^*_r < 0 \). It can result in \( u^* \bar{y} - v^* \bar{x} < 0 \) which is a contradiction.

Theorem 4.3. If \((u^*, v^*)\) be the optimal solution of (9), it can be considered as an efficient solution for (7).
Proof. Suppose \((u^*, v^*)\) is not the efficient solution for model (7); based on the definition of efficient solution, there exists a feasible solution, say \((\hat{u}, \hat{v})\), such that
\[
\begin{align*}
\hat{u}y_1 &> u^*y_1 \\
\hat{v}x_1 &> v^*x_1 \\
&\hspace{1cm} \vdots \\
\hat{u}y_n &> u^*y_n \\
\hat{v}x_n &> v^*x_n
\end{align*}
\]
But according to the corollary 1, there exists at least one DMU, e.g. DMU \(r\), such that
\[
\frac{\hat{u}y_r - v^*x_r}{\hat{v}x_r} = 0
\]
It means
\[
\frac{\hat{u}y_r}{\hat{v}x_r} = 1
\]
therefore
\[
\hat{u}y_r > 1 \quad \text{(on the other word } \hat{u}y_r - \hat{v}x_r > 0 \text{)}
\]
which is a contradiction.

Notation. Each optimal solution of (9) is an efficient solution of (refQuotient7).

Considering theorems and topical subjects in this section and the previous ones, one may conclude that the proposed weights in weighted-sum model (i.e. \(\lambda_j = \frac{v^*_j}{v_j}\)) to solve CSW help us to substitute the assessing aggregated DMU instead of solving the CSW problem. Indeed, there exists a link between aggregated unit and a set of optimal common weights. Up to know, the authors are not aware of any paper which determines a set of optimal common weights through assessing a virtual unit of PPS. It is worth to note, model (9) is a linear problem with an easy interpretation and without computation complexity.

A reader that meets Centralized Resource Allocation (CRA) [42] finds a few comparison between our method and CRA; but in CRA, total input measure is collected and re-contributed among units whereas in presented method we assume that total input measure holds a unit that can produces total output measure (aggregated DMU) and then assess it.

4.2. Example. To illustrate the proposed approach, consider four DMUs each one bearing two inputs and one output as shown in Table 1. The last column in Table 1 is appropriated to the integrated DMU i.e., \((\bar{x}, \bar{y})\). Fig. 3 projects the related PPS in \(x_1 - x_2\) plane. The dashed line \(\Delta\) is indicator of the projection of the hyperplane obtained from model (9).

Suppose a feasible solution \((u, v)\) is in hand. Let \(vx\) be a virtual input and \(uy\) virtual output. We may draw a plane with the horizontal and vertical axes to show the virtual inputs and virtual outputs, respectively. Fig. 4 is graphed considering \((u^*, v^*)\) and virtual units resulted from it are shown. \((u^*_j, v^*_j)\) and \((v^*_jx_j, u^*_jy_j)\) come in Table 2. In Fig. 4, the straight line with slope 1 is called benchmark line and the slope of each line passing through \((v^*_jx_j, u^*_jy_j)\) shows the efficiency score of DMU \(j\) obtained by \((u^*, v^*)\). Concluded from aforementioned note, it is desired that the slope of line passing through \((vx_j, uy_j)\) tends to the slope of benchmark line for each DMU \(j\).

| Table 1. Information related to example |
|-----------------------------------------|
| DMU | A | B | C | D | Agregated |
|-----|---|---|---|---|-----------|
| input1 | 1 | 2 | 6 | 3 | 12        |
| input2 | 5 | 2 | 1 | 3 | 11        |
| output | 1 | 1 | 1 | 1 | 4         |

| Table 2. Optimal solution of model (5) and virtual inputs and outputs connected with it for the example. |
|-------------------------------------------------|
| \(u^*\) | \(v^*\) | \((v^*_jx_j, u^*_jy_j)\) |
|--------|--------|-----------------|
| \(\frac{8}{37}\) | \(\frac{1}{37}\) | \((\frac{8}{37}, \frac{8}{37})\) |
| \(\frac{8}{37}\) | \(\frac{8}{37}\) | \((\frac{8}{37}, \frac{8}{37})\) |
| \(\frac{8}{37}\) | \(\frac{8}{37}\) | \((\frac{10}{37}, \frac{8}{37})\) |
| \(\frac{12}{37}\) | \(\frac{8}{37}\) | \((\frac{32}{37}, \frac{47}{37})\) |
5. **Resource allocation.** In this section, we use the proposed method in the previous section to solve the allocation problem. An example is then illustrated.

5.1. **Resource allocation approach.** Consider n DMUs as described in section 3 and a new source measure F to allocate among them. Our goal is to determine a fairly share $f_j$ of F to each DMU $j$ (as $(m+1)^{th}$ input measure) such that $\sum_{j=1}^{n} f_j = F$. To solve this problem, we use the presented method in section 4 (model (8)). To this means, note at first $\sum_{j=1}^{n} f_j = F$, therefore, we will have $\bar{x} = (\bar{x}_1, \ldots, \bar{x}_m, F)$. Assuming that $v = (v_1, \ldots, v_m, \delta)$, the efficiency score of DMU $j$ will be equal to $\theta_j = \frac{u y_j}{v x_j + f_j} + \delta f_j$. We can obtain $\theta_j = \frac{w y_j}{v x_j + f_j}$ by dividing numerator and denominator of $\frac{u y_j}{v x_j + f_j}$ by $\delta$. This means that, with loss of generality, we may set $v = (v_1, \ldots, v_m, 1)$. Now, set $\lambda_j = \frac{v x_j + f_j}{v x_j + F}$ as a weight corresponded to $j^{th}$ objective i.e., $\frac{w y_j}{v x_j + F}$. Then, model (10) may be obtained as follows:

$$\begin{align*}
\text{max} & \quad \frac{w y_j}{v x_j + F} \\
\text{s.t.} & \quad \frac{w y_j}{v x_j + F} \leq 1, \quad j = 1, \ldots, n \\
& \quad \sum_{j=1}^{n} f_j = F, \\
& \quad f_j \geq 0, \quad j = 1, \ldots, n \\
& \quad u \geq 0, v \geq 0.
\end{align*}$$

(10)
The second constraint in model (10) rises from the terms of resource allocation problem. To linearize Model (10), we at first convert it to model (11) via Charnes-Cooper transformation:

\[
\begin{align*}
\text{max} & \quad u\bar{y} \\
\text{s.t.} & \quad v\bar{x} + F = \frac{1}{t}, \\
& \quad uy_j - vx_j - f_j \leq 0, \quad j = 1, \ldots, n \\
& \quad \sum_{j=1}^{n} f_j = F, \\
& \quad f_j \geq 0, \quad j = 1, \ldots, n \\
& \quad u \geq 0, \quad v \geq 0.
\end{align*}
\] (11)

Next, the non-linear model (11) may be rewritten to the linear model (12) by substitution of \( u' = tu \), \( v' = tv \) and \( f'_j = tf_j \), \( j \in \{1, \ldots, n\} \).

\[
\begin{align*}
\text{max} & \quad u'\bar{y} \\
\text{s.t.} & \quad v'\bar{x} + \sum_{j=1}^{n} f'_j = 1, \\
& \quad u'y_j - v'x_j - f'_j \leq 0, \quad j = 1, \ldots, n \\
& \quad f'_j \geq 0, \quad j = 1, \ldots, n \\
& \quad u' \geq 0, \quad v' \geq 0.
\end{align*}
\] (12)

**Theorem 5.1.** Model (12) is feasible and bounded.

**Proof.** Set \( u' = (0, \ldots, 0) , \ v' = (0, \ldots, \frac{1}{F} \bar{x}, \ldots, 0) \) and \( f'_j = 0, \ f'_j = 0, \ j = 1, \ldots, n \); then, this is a feasible solution for (12). Considering the first constraint, we get \( u'^*\bar{y} \leq 1 \). \( \square \)

**Theorem 5.2.** The share of each DMU from \( F \) will be equal to \( f_j = Ff'_j/\sum_{j=1}^{n} f'_j \).

**Proof.** By considering the substitution of \( f'_j = tf_j \) and third constraint of (12), we get \( f_j = Ff'_j/t \) such that \( t = \sum_{j=1}^{n} f'_j/F \). \( \square \)

In short, the proposed approach in this article was formed from four steps.

1. Presenting the share \( f_j \) of new resource \( F \) as \( x_{m+1,j} \) for DMU \( j \), \( j = 1, \ldots, n \).
2. Using the CSW model.
3. Presenting a set of particular weights in the weighted-sum approach to substitute the CSW model with a liner one-objective problem.
4. Solving the final linear model and determining \( f_j \) s.

Earlier DEA-based researches in allocation fields mostly tried to solve this problem via using the CSW model. Some of them used the iterative method to solve the CSW model while some others solved the multi objective CSW model via Lexicography method. There are papers which have used fuzzy linearization strategy. Furthermore, among published literature in this field, there are authors that linearized CSW problems by proposing slack variables and using goal programming methods. They mainly obtained a linear model for minimizing difference between numerator and denominator of ratios \( u_j/v_j \) instead of maximizing them. In the present article, we can substitute a linear problem (LP) instead of the CSW model. This LP is in the form of assessing a virtual unit in the PPS (aggregated unit). We then used the proposed model to solve an allocation problem. To make the presented approach more tangible, we are going to illustrate an example in subsection 5.2.
5.2. **Example.** In this section, we used the dataset of the example from [14]. Our example includes twelve DMUs, each DMU consumes three input to produce two outputs. Data are presented in Table 3. Suppose we add a source with the input measure of 100 that must be shared among DMUs. Both allocated share of any DMU based on the presented method in this article and the corresponding efficiency for each DMU after allocation can be observed in Table 4.

**Table 3. Data set of [14]**

| DMU | Input1 | Input2 | Input3 | Output1 | Output2 | EFF _C CR |
|-----|--------|--------|--------|---------|---------|-----------|
| 1   | 350    | 39     | 9      | 67      | 751     | 0.75663   |
| 2   | 298    | 26     | 8      | 73      | 611     | 0.92300   |
| 3   | 422    | 31     | 7      | 75      | 584     | 0.74384   |
| 4   | 281    | 16     | 9      | 70      | 665     | 1.00000   |
| 5   | 301    | 16     | 6      | 75      | 445     | 1.00000   |
| 6   | 360    | 29     | 17     | 83      | 1070    | 0.96112   |
| 7   | 540    | 18     | 10     | 72      | 457     | 0.85863   |
| 8   | 276    | 33     | 5      | 78      | 590     | 1.00000   |
| 9   | 323    | 25     | 5      | 75      | 1074    | 1.00000   |
| 10  | 444    | 64     | 6      | 74      | 1072    | 0.83102   |
| 11  | 323    | 25     | 5      | 25      | 350     | 0.33325   |
| 12  | 444    | 64     | 6      | 104     | 1199    | 1.00000   |

**Table 4. Allocated cost to DMUs obtained by different methods**

| DMU | Allocated resource | Efficiency | DMU | Allocated resource | Efficiency |
|-----|--------------------|------------|-----|--------------------|------------|
| 1   | 8.47412            | 1.00000    | 7   | 4.84712            | 1.00000    |
| 2   | 6.90521            | 1.00000    | 8   | 6.68270            | 1.00000    |
| 3   | 6.45563            | 1.00000    | 9   | 12.32408           | 1.00000    |
| 4   | 7.56372            | 1.00000    | 10  | 12.13035           | 1.00000    |
| 5   | 4.96678            | 1.00000    | 11  | 3.76675            | 1.00000    |
| 6   | 12.22981           | 1.00000    | 12  | 13.65374           | 1.00000    |

Allocated resources, as shown in Table 4, are calculated from the formula in Theorem 5.1. Comparing the CCR-efficiency of DMUs before allocation (the last column of Table 3) and after that (efficiency column of Table 4), it may be concluded that the presented model tries to allocate the additional resource in a way to increase the efficiency score of each DMUs.

For the purpose of comparison, the allocated costs determined by Beasley [4], Du et al. [23], Li et al. [34], Hosseinzadeh Lotfi et al. [40], Si et al. [49], Cook and Kress [14], Lin and Chen [38], Yang and Zhang [53] in addition to our approach are listed in Table 5. We list them in two categories: they whose models appeared in output orientation and they who presented their models in input orientation. The words yes/no in the Efficiency invariance row show which one of methods used efficiency invariance principle. With respect to limitation in space, all results are presented with only two digits after decimal point.
Table 5. Different allocation in selected methods

| DMU | Eff. invariance | Output orientation | Input orientation |
|-----|----------------|--------------------|------------------|
|     |                | no | no | no | no | no | no | yes | no | yes |
| 1   | yes            | 8.47 | 6.78 | 5.79 | 5.54 | 8.20 | 7.65 | 14.52 | 9.83 | 7.54 |
| 2   | yes            | 6.91 | 7.21 | 7.95 | 7.53 | 7.46 | 8.41 | 6.74 | 7.53 | 8.65 |
| 3   | yes            | 6.46 | 6.83 | 6.54 | 7.35 | 4.28 | 8.62 | 9.32 | 9.93 | 7.52 |
| 4   | yes            | 7.56 | 8.47 | 11.10 | 7.87 | 9.30 | 8.11 | 5.6 | 5.20 | 9.05 |
| 5   | yes            | 4.97 | 7.08 | 8.69 | 6.38 | 4.81 | 8.69 | 5.79 | 5.20 | 9.07 |
| 6   | yes            | 12.23 | 10.06 | 13.49 | 11.50 | 15.37 | 9.57 | 8.15 | 9.10 | 8.84 |
| 7   | yes            | 4.85 | 5.09 | 7.30 | 5.90 | 0 | 8.33 | 8.36 | 5.85 | 8.17 |
| 8   | yes            | 6.68 | 7.74 | 6.83 | 7.77 | 7.34 | 9.96 | 6.26 | 8.96 | 9.06 |
| 9   | yes            | 12.32 | 15.11 | 16.68 | 11.90 | 16.33 | 8.65 | 7.31 | 8.07 | 10.46 |
| 10  | yes            | 12.13 | 10.08 | 5.42 | 11.38 | 11.60 | 8.35 | 10.08 | 9.96 | 8.01 |
| 11  | yes            | 3.77 | 1.58 | 0 | 2.74 | 0 | 2.80 | 7.31 | 8.07 | 4.55 |
| 12  | yes            | 13.65 | 13.97 | 10.41 | 14.14 | 15.31 | 11.85 | 10.08 | 12.56 | 9.12 |

With focus on units 9 and 11 in Table 5, one may find an unacceptable result. Those approaches with efficiency invariance assumption ([14, 38]) allocated the identical costs to DMU_9 and DMU_11. As shown in Table 3, DMU_9 and DMU_11 consume the same input measures while DMU_9 generates almost three times outputs than DMU_11. However, our method that does not impose the efficiency invariance assumption, allocates costs to this pair of units differently.

Another finding in Table 5 is that some approaches e.g. [23, 40] allocated a zero share to some units. This phenomenon may not be acceptable by other units. From this viewpoint, the results of our approach are more reasonable.

Beasley’s method [4] is another approach appeared in Table 5. He obtained the allocated costs after solving a noticeable number of models (some are non-linear). Also, in mentioned approach, the difference between minimum and maximum shared costs is 13.53; which is greater than 9.88 obtained by our model.

Finally, in comparison with three other methods: Li et al. [34], Si et al. [49] and Yang and Zhang [53], we consider the deference between minimal and maximal allocated costs. From this perspective, the resulted gap from our allocation is less than one obtained by Li et al. [34]. Among them, Si et al. [49] used a set of common weights with the assumption on minimizing gap on the allocated costs among all units. While, Si et al. [49] obtained a gap equal to 9.05 (=11.85-2.80) with the objective of minimizing the gap on the allocated costs among DMUs, we can obtain a gap equal to 9.88 without any assumption on the resulted gap. Finally, Yang and Zhang [53] determined an equitable share to each of units with the aim of minimizing the gap between minimum and maximum allocated costs, but they sacrificed efficiency; since in their approach some units obtained less relative efficiency compared to when they were not allocated.

Considering all above discussions, one may conclude that purposed method in this study is more preferable and acceptable by all units.

6. Conclusions. Resource allocation is an important issue in economics and management; DEA is known as a popular tool to achieve this. DEA is a method to assess a set of decision making units (DMUs). In traditional DEA, any unit can determine a set of individual multipliers corresponding to inputs and outputs to obtain maximum efficiency score. Although multipliers are unknown to the manager, real world may not widely deviate between units under assessment. CSW is a DEA-based method which assesses all units by the same multipliers. Because of
having a common base for evaluating and comparing DMUs, this method is a suitable tool to solve allocation problems. To do this, we applied the CSW approach in this article. Then, based on the weighted-sum approach, we proposed a set of appropriate weights to obtain a linear problem. The obtained problem is in the multiplier form of a virtual unit (aggregated unit) in the PPS. Next, the proposed approach was employed to solve allocation problems. We proved that each optimal solution of the proposed linear model gives an efficient point in the CSW problem. In section 4, it was proved that if the optimal value of the proposed model (9) is one, then all units are efficient. In comparison with the published literatures in this field, the proposed model is easily applicable and computationally economical.

As for the future research, since we proposed an approach to solve a CSW model over PPS\textsubscript{CCR}, one may develop the proposed method for the BCC production possibility set. Furthermore, as this article discussed the case of allocating a new resource (or cost) among units, researchers may survey the cases in which some inputs under allocation problem constraints are increased. Last but not least, this paper did not investigate the target setting issue. Since the target setting concept is a noticeable subject for some managers, this might be another topic of investigation.

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