Gradient estimates for elliptic systems from composite materials with closely spaced stiff $C^{1,\gamma}$ inclusions

Yan Li*

Abstract This paper is devoted to establishing the pointwise upper and lower bounds estimates of the gradient of the solutions to a class of general elliptic systems with Hölder continuous coefficients in a narrow region where the upper and lower boundaries is $C^{1,\gamma}$, $0 < \gamma < 1$, weaker than the previous $C^{2,\gamma}$ assumption. These estimates play a key role in the damage analysis of composite materials. From our results, the damage may initiate from the narrowest place.

Contents

1 Introduction 1
   1.1 Background and problem formulation ................................................................. 1

2 The gradient estimates for the general elliptic systems 8
   2.1 $C^{1,\gamma}$ estimates and $W^{1,p}$ estimates .......................................................... 8
   2.2 The proof of Theorem 1.1 ......................................................................................... 9

3 Proof of Corollary 1.1 23

4 Appendix: Proof of $C^{1,\gamma}$ estimates and $W^{1,p}$ estimates 27
   4.1 Proof of Theorem 2.1 ......................................................................................... 27
   4.2 Proof of Theorem 2.2 ......................................................................................... 32

1 Introduction

1.1 Background and problem formulation

Damage analysis of composite materials is of great significance in engineering, which is also an important application of gradient estimates of second-order elliptic systems in partial differential equations [16, 27, 28]. Babuška et al. [7] computationally analyzed the damage and fracture in fiber composite materials, and some numerical results of stress concentration are given.

2020 Mathematics Subject Classification. Primary: 74G70; Secondary: 35B45 35B65 74B05.
Key words and phrases. Stress concentration, Schauder estimates, gradient estimates, Hölder semi-norm, elliptic systems.
*Corresponding author, Yan Li E-mail: yanli@mail.bnu.edu.cn
which plays a key role in the problem of stress concentration in materials. At present, a lot
of progress has been made in precisely studying this field concentration phenomenon, see e.g. [2, 9, 15, 16, 24, 25, 26, 28].

From the viewpoint of mathematic, Li and Vogelius [29] described this stress concentration by
using the gradient of the solution to a specific class of elliptic systems with partially degenerated
coefficients. Before studying the gradient estimates of elliptic systems, Bonnetier et al. [14]
considered the simplified scalar equation:

$$\nabla \cdot ((1 + (a - 1)\chi_{\bigcup_i D_i})\nabla v) = 0, \quad \text{in } \Omega,$$

(1.1)

to model a problem of electric conduction, where $a \neq 1$, $\Omega \subset \mathbb{R}^n$ represents a bounded domain,
$N \in \mathbb{Z}_+$ represents the number of the inclusions, $D_i \subset \Omega$ represents the inclusions in the matrix
material and close to each other. Bonnetier and Vogelius [14] proved rigorously that the gradient
of the solution of (1.1) is indeed bounded, when $N = 2$ and $D_1, D_2$ is two touching disks with
comparable radii in $\mathbb{R}^2$. Li and Vogelius [29] extended the result to a large class of divergence
form second order elliptic equations with piecewise Hölder continuous coefficients in $\mathbb{R}^2$, when
$N \geq 2$ and the inclusions is $C^{1,\gamma}$ ($0 < \gamma < 1$) (see Definition 1.2 below). Li and Nirenberg [28]
further extended to general divergence form elliptic systems with Hölder continuous coefficients
satisfying the strong elliptic condition.

When the coefficients degenerate to infinity in $D_i$, the gradient of the solution is no longer
bounded but blows up. For the scalar case, we call it perfect conductivity problem. Let $\varepsilon$
be the distance between the two inclusions. The blow-up rate of $|\nabla u|$ is respectively $\varepsilon^{-1/2}$ in two dimensions, $(\varepsilon|\ln \varepsilon|)^{-1}$ in three dimensions and $\varepsilon^{-1}$ in four dimensions and high dimensions. See Bao, Li and Yin [11], Ammari, Kang and Lim [6], Ammari, Kang, Lee, Lee and Lim [4], and Yun [33]. There have been many papers on the problem and related ones: see e.g. [2, 3, 5, 12, 13, 24, 27, 30] and the references therein.

However, when considering the gradient estimates to the solution of linear elasticity problem,
namely Lamé system, the method of scalar equation is no longer suitable for using. Under the
assumption that the smoothness of the inclusion boundary is $C^{2,\gamma}$ ($0 < \gamma < 1$), Bao, Li and
Li [9, 10] applied an energy method and an iteration technique, which was first used in [27],
to obtained pointwise upper bound of $|\nabla u|$ in the narrow region between inclusions. Kang and Yu [25] proved that the blow up rate $\varepsilon^{-1/2}$ is optimal in some two-dimensional cases when the smoothness of inclusion boundary is of $C^{3,\gamma}$. Ju, Li and Xu [23] established the pointwise upper and lower bounds of the gradient of solutions to a class of general elliptic systems in the narrow region between two $C^{2,\gamma}$ inclusions. For more work on elliptic equations and systems related to the study of composites, see [8, 14, 16, 17, 18, 21, 22, 24, 26, 32] and the references therein.

Under a weaker smoothness assumption on the inclusion boundary, namely, $C^{1,\gamma}$, Chen and Li [15] proved that the blow up rate of gradient for the Lamé system of linear elasticity with partially
infinite coefficients is $\varepsilon^{-1/(1+\gamma)}$ in two dimensions and $\varepsilon^{-1}$ in $n \geq 3$ dimensions.

Contrary to the case where the smoothness of the inclusion boundary is $C^{2,\gamma}$ or higher, less is
known on such blow up phenomenon for the case of weaker smoothness $C^{1,\gamma}$. Based on the classical elliptic theory, a natural question is whether it is possible to obtain gradient estimates of
the solutions to a class of general elliptic systems (see Definition 1.4 below), under a weaker smooth-
ness assumption on the inclusions, namely, $C^{1,\gamma}$. In addition, we want to obtain more information
about the dependence of $|\nabla u|$, which play an important role in the study of the perfect conductivity problem and Lamé system with partially infinite coefficients.

In this paper, we investigate the gradient estimates of the solutions to a class of general elliptic systems with Hölder continuous coefficients in a general narrow region between the two $C^{1,\gamma}$ inclusions. This is a generalization of the stress concentration problem in two-phase high-contrast elastic composites with densely packed $C^{1,\gamma}$ inclusions. This estimates have a wide range of applications and play a key role in the damage analysis of composite materials. When we apply this results to the Lamé systems of linear elasticity, under the assumptions of the $C^{1,\gamma}$-regularity boundary, the results can present more dependency information about gradient. Our results show that the damage may initiate from the narrowest place.

Before state our results, we first introduce some definitions and notations, as well as fix our domain.

Let $U$ be any domain in $\mathbb{R}^n$. Denote by the symbol $C(U)$ the set of all continuous functions on $U$. For every $0 < \gamma \leq 1$, a function $u \in C(U)$ is said to be Hölder continuous with exponents $\gamma$ if $|u(x) - u(y)| \leq C|x-y|^{\gamma}$, $x, y \in U$ for some constant $C$. If $u : U \to \mathbb{R}$ is bounded and continuous, we write $\|u\|_{C(U)} := \sup_{x \in U}|u(x)|$. We use the symbol $C^k(U)$ to denote the set of all $k$-th continuously differentiable functions on $U$ for integer $k \geq 0$. For every $0 < \gamma \leq 1$, the $\gamma$-Hölder semi-norm of $u : U \to \mathbb{R}$ is

$$[u]_{C^{0,\gamma}(U)} := \sup_{x, y \in U, x \neq y} \frac{|u(x) - u(y)|}{|x-y|^{\gamma}}.$$  \hspace{1cm} (1.2)

**Definition 1.1.** Let $0 < \gamma \leq 1$, $k \in \mathbb{Z}_+$, and $\alpha = (\alpha_1, \cdots, \alpha_n) \in \mathbb{Z}^n$ be a multiindex of order $|\alpha| = \alpha_1 + \cdots + \alpha_n$. The Hölder space $C^{k,\gamma}(U)$ is defined to be the set of all $k$-th continuous differentiable real valued functions satisfying the $k$-th order derivatives are Hölder continuous with exponents $\gamma$ and

$$\|u\|_{C^{k,\gamma}(U)} := \sum_{|\alpha| \leq k}\|\partial^\alpha u\|_{C(U)} + \sum_{|\alpha| = k}[\partial^\alpha u]_{C^{0,\gamma}(U)} < \infty,$$

where $\partial^\alpha := \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}}u$. In particular, for $0 < \gamma < 1$, we often use the symbol $C^{\gamma}(U)$ to denote $C^{0,\gamma}(U)$.

**Definition 1.2.** Let $U$ be any domain in $\mathbb{R}^n$, the integer $k > 0$ and $0 < \gamma < 1$, the boundary $\partial U$ is said to be $C^k$ or $C^{k,\gamma}$ if for each point $x_0 \in \partial U$ there exist $r > 0$ and a $C^k$ or $C^{k,\gamma}$ function $T : \mathbb{R}^{n-1} \to \mathbb{R}$ such that-when relabeling and reorienting the coordinates axes if necessary-we have

$$U \cap B_r(x_0) = \{x \in B_r(x_0) \mid x_n > T(x_1, \cdots, x_{n-1})\}.$$  \hspace{1cm} (1.3)

Furthermore, the inclusion is said to be $C^k(U)$ or $C^{k,\gamma}(U)$ if its boundary is $C^k(U)$ or $C^{k,\gamma}(U)$.

**Definition 1.3.** Let $1 \leq p \leq \infty$, $k \in \mathbb{Z}_+$ and $\alpha = (\alpha_1, \cdots, \alpha_n) \in \mathbb{Z}^n$ be a multiindex of order $|\alpha| = \alpha_1 + \cdots + \alpha_n$. The Sobolev space $W^{k,p}(U)$ is defined to be the set of all locally integrable functions $u : U \to \mathbb{R}$ such that for each multiindex $\alpha \in \mathbb{Z}^n$ with $|\alpha| \leq k$, $\partial^\alpha u$ exists in the weak...
sense and belongs to the standard Lebesgue spaces $L^p(U)$ and

$$
\|u\|_{W^{k,p}(U)} := \begin{cases} 
\left( \sum_{|\alpha| \leq k} \int_U |\partial^\alpha u|^p \, dx \right)^{1/p} & (1 \leq p < \infty) \\
\sum_{|\alpha| \leq k} \text{ess sup}_U |\partial^\alpha u| & (p = \infty)
\end{cases} < \infty.
$$

Denote by the symbol $C^\infty_c(U)$ the set of all infinitely continuously differentiable functions on $U$ with compact support. We denote by $W^{k,p}_0(U)$ the closure of $C^\infty_c(U)$ in $W^{k,p}(U)$.

Next, we fix our domain. Let $D$ be a domain in $\mathbb{R}^n$, and $D_1, D_2$ be a pair of convex subdomains of $D \subset \mathbb{R}^n$. Let the distance between $D_1$ and $D_2$ be $\varepsilon > 0$ (sufficiently small positive number). Denote $P_1 := (\bar{0}_{n-1}, \varepsilon/2), P_2 := (\bar{0}_{n-1}, -\varepsilon/2)$ the nearest points between $\partial D_1$ and $\partial D_2$ such that

$$
\text{dist}(P_1, P_2) = \text{dist}(\partial D_1, \partial D_2) = \varepsilon,
$$

where for any $x' \in \mathbb{R}^{n-1}, x := (x', x_n) \in \mathbb{R}^n$. Let $B_r$ be the ball with $\bar{0}_{n-1}$ as the center and $0 < r \leq 1$ as the radius in $\mathbb{R}^{n-1}$.

Let $\varepsilon$ be as in Definition 1.2, $h_1, h_2 \in C^{1,\gamma}(B'_1), 0 < \gamma < 1$ and satisfy

$$
-\frac{\varepsilon}{2} + h_2(x') < \frac{\varepsilon}{2} + h_1(x'), \quad \text{for } |x'| \leq 1, \tag{1.4}
$$

$$
h_1(\bar{0}_{n-1}) = h_2(\bar{0}_{n-1}) = 0, \quad \nabla h_1(\bar{0}_{n-1}) = \nabla h_2(\bar{0}_{n-1}) = 0, \tag{1.5}
$$

and there exist some constants $0 < \kappa_0 < \kappa_1$ such that

$$
\kappa_0 |x'|^\gamma \leq |\nabla h_1(x')|, \quad |\nabla h_2(x')| \leq \kappa_1 |x'|^\gamma, \quad \text{for } |x'| \leq 1. \tag{1.6}
$$

To be precise, we define general narrow region in $\mathbb{R}^n$: for $r \leq 1$,

$$
\Omega_r := \left\{ (x', x_n) \in D : -\frac{\varepsilon}{2} + h_2(x') < x_n < \frac{\varepsilon}{2} + h_1(x'), |x'| \leq r \right\}. \tag{1.7}
$$

We here assume that $\partial D_1$ and $\partial D_2$ are $C^{1,\gamma}, 0 < \gamma < 1$ as in Definition 1.2 and the top and bottom boundaries of the narrow region $\Omega_1$ between $\partial D_1$ and $\partial D_2$ satisfy

$$
\left\{ (x', x_n) \in \mathbb{R}^n : x_n = -\frac{\varepsilon}{2} + h_2(x'), |x'| \leq 1 \right\} \subset \partial D_1, \tag{1.8}
$$

$$
\left\{ (x', x_n) \in \mathbb{R}^n : x_n = \frac{\varepsilon}{2} + h_1(x'), |x'| \leq 1 \right\} \subset \partial D_2,
$$

and

$$
\left\{ (x', x_n) \in \mathbb{R}^n : -\frac{\varepsilon}{2} + h_2(x') < x_n < \frac{\varepsilon}{2} + h_1(x'), |x'| \leq 1 \right\} \cap (\partial D_1 \cup \partial D_2) = \emptyset.
$$

Furthermore, we denote the top and bottom boundaries of $\Omega_r$ as

$$
\Gamma^+_r := \left\{ x \in \partial D_1 : x_n = \frac{\varepsilon}{2} + h_1(x'), |x'| \leq r \right\}, \tag{1.9}
$$

$$
\Gamma^-_r := \left\{ x \in \partial D_2 : x_n = -\frac{\varepsilon}{2} + h_2(x'), |x'| \leq r \right\}.
$$

We now introduce the definition of general elliptic system with Hölder continuous coefficients in a narrow region $\Omega_1$ in this paper.
**Definition 1.4.** Let $0 < \gamma < 1$, $m, n \in \mathbb{Z}_+$, $A_{ij}^{\alpha \beta}, B_{ij}^\alpha, C_{ij}^\beta, D_{ij} \in C^r(\Omega_1)$ for any integer $0 \leq \alpha, \beta \leq n$, $0 \leq i, j \leq m$, and the matrix of coefficients $(A_{ij}^{\alpha \beta})_{1 \leq i, j \leq m}$ satisfy the strong ellipticity condition in $\Omega_1$, namely, there exists a constant $\lambda > 0$ such that
\[
\sum_{\alpha, \beta, i, j} A_{ij}^{\alpha \beta}(x) \xi_\alpha \xi_\beta \eta_i \eta_j \geq \lambda \xi^2 |\eta|^2, \quad \forall \xi \in \mathbb{R}^n, \eta \in \mathbb{R}^m, x \in \Omega_1.
\]

(1.10)

Let
\[
\varphi := (\varphi^{(1)}, \varphi^{(2)}, \ldots, \varphi^{(m)}) \in C^{1,\gamma}(\Gamma_1^+; \mathbb{R}^m),
\]
\[
\psi := (\psi^{(1)}, \psi^{(2)}, \ldots, \psi^{(m)}) \in C^{1,\gamma}(\Gamma_1^-; \mathbb{R}^m).
\]

(1.11)

Then the systems as follows
\[
\begin{cases}
\sum_{\alpha, \beta, j} \partial_\alpha \left( A_{ij}^{\alpha \beta} \partial_\beta u^{(j)} + B_{ij}^\alpha u^{(j)} \right) + C_{ij}^\beta \partial_\beta u^{(j)} + D_{ij} u^{(j)} = 0 \text{ in } \Omega_1, \\
u = \varphi, \quad \text{on } \Gamma_1^+,
\end{cases}
\]
\[
\begin{cases}
\sum_{\alpha, \beta, j} \int_{\Omega_1} \left( A_{ij}^{\alpha \beta}(x) \partial_\beta u^{(j)}(x) + B_{ij}^\alpha(x) u^{(j)}(x) \right) \partial_\alpha \psi^{(j)}(x) \\
- C_{ij}^\beta(x) \partial_\beta u^{(j)}(x) \psi^{(j)}(x) - D_{ij} u^{(j)}(x) \psi^{(j)}(x) \, dx = 0
\end{cases}
\]

(1.12)

is called a general elliptic system, where $\Gamma_1^+$ and $\Gamma_1^-$ are defined as in (1.9).

A function $u := (u^{(1)}, u^{(2)}, \ldots, u^{(m)}) \in W^{1,2}(\Omega_1 \subset \mathbb{R}^n; \mathbb{R}^m)$ is said to be a weak solution of the general elliptic systems defined as in Definition 1.4 if, for every vector-valued function $\phi \in W_0^{1,2}(\Omega_1 \subset \mathbb{R}^n; \mathbb{R}^m),
\[
\sum_{\alpha, \beta, j} \int_{\Omega_1} \left( A_{ij}^{\alpha \beta}(x) \partial_\beta u^{(j)}(x) + B_{ij}^\alpha(x) u^{(j)}(x) \right) \partial_\alpha \psi^{(j)}(x) \\
- C_{ij}^\beta(x) \partial_\beta u^{(j)}(x) \psi^{(j)}(x) - D_{ij} u^{(j)}(x) \psi^{(j)}(x) \, dx = 0
\]

(1.13)

holds true, for any integer $i = 1, \ldots, m$.

**Remark 1.1.** It is clear that hypotheses (1.10) and (1.16) are satisfied by the Lamé system as follow, (see [37]),
\[
\lambda_1 \Delta u + (\lambda_1 + \mu_1) \nabla (\nabla \cdot u) = 0,
\]

where Lamé constant $(\lambda_1, \mu_1)$ satisfies ellipticity conditions: $\mu_1 > 0$, $\lambda_1 + \mu_1 > 0$. Therefore, the gradient estimates results in this paper include the case of Lamé systems.

Now we are going to present our main result about pointwise gradient estimates of the weak solutions to the general elliptic systems, which is:

**Theorem 1.1.** Let $\varepsilon$ be as in (1.3), $0 < \gamma < 1$, $h_1, h_2 \in C^{1,\gamma}(B^c_{1/2})$ satisfy (1.4)-(1.7), and $\Omega_\varepsilon$, $r \in \{1/2, 1\}$ be as in (1.7). Let $\varphi$ and $\psi$ be as in (1.11) and $u \in W^{1,2}(\Omega_1 \subset \mathbb{R}^n; \mathbb{R}^m)$ be a weak solution of general elliptic system as in Definition 1.4. Then there exists a positive constant $C$ independent of $\varepsilon$, such that, for any $x = (x', x_n) \in \Omega_{1/2},
\[
|\nabla u(x', x_n)| \leq \frac{C}{\varepsilon + |x'|^{1+\gamma}} |\varphi(x', \varepsilon/2 + h_1(x')) - \psi(x', -\varepsilon/2 + h_2(x'))|
\]

Moreover, if $\varphi^{(\ell)}(\vec{0}_{n-1}, \varepsilon/2) \neq \psi^{(\ell)}(\vec{0}_{n-1}, -\varepsilon/2)$ for some integer $\ell \in \{1, \ldots, m\}$, then there exists a positive constants $C$ independent of $\varepsilon$, such that, for any $x_n \in (-\varepsilon/2, \varepsilon/2)$,

$$|\nabla u(\vec{0}_{n-1}, x_n)| \geq C\left( \frac{\varphi^{(\ell)}(\vec{0}_{n-1}, \varepsilon/2) - \psi^{(\ell)}(\vec{0}_{n-1}, -\varepsilon/2)}{\varepsilon} \right).$$

For the convenience of further applications, we list the analog result about the Lamé systems in the narrow region, which is:

**Corollary 1.1.** Let $\varepsilon$ be as in (1.3), $\Omega_0, r \in \{\frac{1}{2}, 1\}$ be as in (1.7), $\Gamma^+_1, \Gamma^-_1$ be as in (1.9), and $\varphi \in C^{1,\gamma}(\Gamma^+_1; \mathbb{R}^n), \psi \in C^{1,\gamma}(\Gamma^-_1; \mathbb{R}^n)$ be as in (1.11). Let $\lambda_1, \mu_1 \in \mathbb{R}$ be the pair of Lamé constants which satisfies the strong ellipticity conditions: $\mu_1 > 0, \lambda_1 + \mu_1 > 0$. Let $u = (u^{(1)}, \cdots, u^{(m)}) \in W^{1,2}(\Omega_1)$ be the weak solution of

$$\begin{align*}
\mathcal{L}_{\lambda_1, \mu_1} u &:= \nabla \cdot (\mathbb{C} e(u)) = \lambda_1 \Delta u + (\lambda_1 + \mu_1) \nabla \cdot u = 0, \text{ in } \Omega_1, \\
u & = \varphi, \quad \text{on } \Gamma^+_1, \\
u & = \psi, \quad \text{on } \Gamma^-_1,
\end{align*}$$

where $e(u) = \frac{1}{2}(\nabla u + (\nabla u)^T)$ and the elastic tensor $\mathbb{C} = \mathbb{C}(\lambda_1, \mu_1)$ consists of elements

$$C_{ijkl}(\lambda_1, \mu_1) = \lambda_1 \delta_{ij} \delta_{kl} + \mu_1 (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}), \quad i, j, k, l = 1, 2, \cdots, n,$$

and $\delta_{ij}$ is Kronecker symbol: $\delta_{ij} = 0$ for $i \neq j, \delta_{ij} = 1$ for $i = j$. Then there exists a positive constant $C$ independent of $\varepsilon$, for any $x = (x_1, x_2) \in \Omega_{1/2}$,

$$|\nabla u(x', x_n)| \leq \frac{C}{\varepsilon + |x'|^{1+\gamma}} \left| \varphi(x', \varepsilon/2 + h_1(x')) - \psi(x', -\varepsilon/2 + h_2(x')) \right| + C\left( \left\| \varphi \right\|_{C^{1,\gamma}(\Gamma^+_1)} + \left\| \psi \right\|_{C^{1,\gamma}(\Gamma^-_1)} + \left\| u \right\|_{L^2(\Omega_1)} \right).$$

Moreover, if $\varphi^{(\ell)}(\vec{0}_{n-1}, \varepsilon/2) \neq \psi^{(\ell)}(\vec{0}_{n-1}, -\varepsilon/2)$ for some integer $\ell \in \{1, \cdots, n\}$, then there exists a positive constants $C$ independent of $\varepsilon$, such that, for any $x_n \in (-\varepsilon/2, \varepsilon/2)$,

$$|\nabla u(\vec{0}_{n-1}, x_n)| \geq C\left( \frac{\varphi^{(\ell)}(\vec{0}_{n-1}, \varepsilon/2) - \psi^{(\ell)}(\vec{0}_{n-1}, -\varepsilon/2)}{\varepsilon} \right).$$

Indeed, the proof of this result is nontrivial. Under the assumption that the smoothness of the inclusions boundaries is $C^{1,\gamma}$, we need to estimate the Hölder semi-norm of the gradient of the constructed auxiliary function when we use the iteration method to prove that the gradient of the auxiliary function is the major singular terms of the gradient of the solution to (1.12). In this paper, we consider the general elliptic system as in Definition [7,7] so the complexity of the constructed auxiliary function makes it more cumbersome to deal with some parameters in estimating the Hölder semi-norm, see Proposition [2,7] below. In addition, compared with [15], the coefficients of the elliptic systems here are no longer constant, and the right hand side term is no longer in
divergence form, which leads to the complexity of the iteration process, see Lemmas 2.1 2.2 and 2.3 below.

To be precise, this article is organized as follows.

In Section 2, we devoted to proving the Theorem 1.1. Firstly, we give the $C^{1,\gamma}$ estimates and $W^{1,p}$ estimates required in the iteration process. Then, we obtain the solutions $v_\ell$, $\ell = 1, \cdots, m$ (see (2.5) below) of $m$ elliptic systems with relatively simple boundary conditions (see (2.4) below) by decomposing the solution $u$ to general elliptic system. Next, we use a scalar auxiliary function $\tilde{u}$ (see (2.6) below) to generate a family of vector-valued auxiliary functions $\tilde{u}_i$, whose values is same as $v_\ell$ on $\Gamma_1^+$ and $\Gamma_1^-$ (see (2.9) below). In order to prove that $\nabla \tilde{u}_\ell$ is the major singular terms of $\nabla v_\ell$ by using iteration method in [9, 10, 27], it is very important to consider the estimates of the H"older semi-norm of $\nabla \tilde{u}_\ell$ in a small region (see Proposition 2.1 below). Finally, by using semi-norm estimates, $C^{1,\gamma}$ estimates and $W^{1,p}$ estimates, we complete the proof of Theorem 1.1.

In Section 3, our main task is to prove Corollary 1.1. When the general elliptic systems are simplified to the Lamé system, the pointwise estimates of the gradient is obtained under the assumptions in Definition 1.4.

In Section 4, we show the proofs of the Theorem 2.1 and Theorem 2.2 which play a key role in the proof of Theorem 1.1. Different from the Theorem 2.3 and 2.4 in [15], the elliptic systems considered in this paper are no longer simple constant coefficients, and the right hand side of the systems is no longer in divergence form. This section can be regarded as a generalization of [15, Theorem 2.3 and 2.4]. To prove Theorem 2.1, we first give the interior $C^{1,\gamma}$ estimates (see Lemma 4.2 below) of the solution to the elliptic systems with non divergence form at the right hand side, with the help of the Campanato’s approach, Schauder estimates in [19, Theorem 5.14]. Then we can obtain the boundary $C^{1,\gamma}$ estimates (see Corollary 4.7 below) on half space by using the [19, Theorem 5.21] and Lemma 4.2. Finally, we use them to obtain the estimates near the $C^{1,\gamma}$ boundary $\Gamma$ by using the technology of locally flattening the boundary. We prove the $W^{1,p}$ estimates by applying the interior $W^{1,p}$ estimates and the boundary $W^{1,p}$ estimates of the upper half space.

Finally, we make some conventions on notation. Let $\mathbb{N} := \{1, 2, \cdots\}$ and $\mathbb{Z}_+ := \mathbb{N} \cup \{0\}$. We always denote by $C$ a positive constant which is independent of the main parameters, but it may vary from line to line. If $E$ is a subset of $\mathbb{R}^n$, we denote by $\chi_E$ its characteristic function. Let $U$ and $V$ be the open subsets of $\mathbb{R}^n$, we write $V \subset U$ if $V \subset \bar{V} \subset U$ and $\bar{V}$ is compact, and say $V$ is compactly contained in $U$. The symbol $\partial D_i$ denotes the boundary of $D_i$. For any $x_0 \in \mathbb{R}^n$ and $r > 0$, the symbol $B_r(x_0)$ denotes the open ball with center $x_0$, radius $r$ in $\mathbb{R}^n$.

Throughout this article, let $\Lambda > 0$ satisfy for $\alpha, \beta = 1, \cdots, n, i, j = 1, \cdots, m$,

$$\left|A^{ij}_{ij}(x)\right| \leq \Lambda, \quad \forall x \in \Omega_i,$$

and $\kappa_2, \kappa_3 > 0$ such that

$$(1.16) \quad \|h_1\|_{C^{1,\gamma}(\Omega_i^+)} + \|h_2\|_{C^{1,\gamma}(\Omega_i^-)} \leq \kappa_2,$$

$$(1.17) \quad \|A\|_{C^{1,\gamma}(\Omega_i)} + \|B\|_{C^{1,\gamma}(\Omega_i)} + \|C\|_{C^{1,\gamma}(\Omega_i)} + \|D\|_{C^{1,\gamma}(\Omega_i)} \leq \kappa_3,$$

where

$$\|A\|_{C^{1,\gamma}(\Omega_i)} := \max_{\alpha, \beta, i, j} \|A^{ij}_{ij}(\cdot)\|_{C^{1,\gamma}(\Omega_i)}, \quad \|B\|_{C^{1,\gamma}(\Omega_i)} := \max_{\alpha, i, j} \|B_{ij}^{ij}(\cdot)\|_{C^{1,\gamma}(\Omega_i)},$$
\[ \|C\|_{C^1(\Omega_1)} := \max_{\beta, i, j} \|C_{ij}^\beta(\cdot)\|_{C^1(\Omega_1)}, \quad \|D\|_{C^1(\Omega_1)} := \max_{i, j} \|D_{ij}(\cdot)\|_{C^1(\Omega_1)}. \]

2 The gradient estimates for the general elliptic systems

In this section, our main task is to prove Theorem 1.1. By using the idea of iteration in [9, 10, 27] and the treatment of $C^{1,\gamma}$ boundary in [75], we establish the pointwise upper and lower bounds estimates of the gradient of the solution to general elliptic systems, under the assumption that the smoothness of partial boundary of the region is $C^{1,\gamma}$. Before that, we first give the $C^{1,\gamma}$ estimates and $W^{1,p}$ estimates required in the iteration process.

2.1 $C^{1,\gamma}$ estimates and $W^{1,p}$ estimates

Firstly, we give the definition of general elliptic type system in a more general region.

Definition 2.1. Let \( 0 < \gamma < 1 \) and \( Q \) be a bounded domain in \( \mathbb{R}^n, n \geq 2 \), with a $C^{1,\gamma}$ boundary portion \( \Gamma \subset \partial Q \). Let \( A_{ij}^\alpha, B_{ij}^\alpha, C_{ij}^\beta, D_{ij} \in C^\gamma(Q) \) and \( H^{(i)} \in L^\alpha(Q) \) for any \( \alpha, \beta = 1, \ldots, n \), \( i, j = 1, \ldots, m \). Let the matrix of coefficients \( (A_{ij}^\alpha) \) satisfy (1.10) and (1.16), and \( B_{ij}^\alpha, C_{ij}^\beta, D_{ij} \) satisfy (1.18). Then the systems as follows

\[
\begin{cases}
\sum_{\alpha, \beta, i} \partial_\alpha \left( A_{ij}^\alpha \partial_\beta w^{(i)} + B_{ij}^\alpha w^{(i)} \right) + C_{ij}^\beta \partial_\beta w^{(j)} + D_{ij} w^{(j)} = H^{(i)} - \partial_\alpha F_i^{(\alpha)}, & \text{in } Q, \\
w := (w^{(1)}, \ldots, w^{(m)}) = 0, & \text{on } \Gamma,
\end{cases}
\]

is called a general elliptic type system.

The $C^{1,\gamma}$ estimates for the general elliptic type system as in Definition 2.1 is:

Theorem 2.1. ($C^{1,\gamma}$ estimates) Let \( \kappa_3 \) be as in (1.18) and \( w = (w^{(1)}, \ldots, w^{(m)}) \in W^{1,2}(Q \subset \mathbb{R}^n; \mathbb{R}^m) \cap C^1(Q \cup \Gamma; \mathbb{R}^m) \) be the weak solution to the general elliptic type system as in Definition 2.1. Then, for any \( Q' \subset Q \cup \Gamma \), there exists a positive constant \( C \) depending on \( n, m, \kappa_3, \gamma, Q', Q \), such that,

\[
\|w\|_{C^{1,\gamma}(Q')} \leq C \left( \|w\|_{L^\infty(Q')} + \|F\|_{L^1(Q')} + \|H\|_{L^\infty(Q')} \right).
\]

Here and thereafter, the Hölder semi-norm of the matrix-valued function \( F = (F_i^{(\alpha)})_{i, \alpha} \) be defined as \( \|F\|_{Y, Q} := \max_{i, \alpha} \|F_i^{(\alpha)}\|_{Y, Q} \), the $L^\infty$ norm of the vector-valued function \( H := (H^{(1)}, \ldots, H^{(m)}) \) be defined as \( \|H\|_{L^\infty(Q')} := \max_i \|H^{(i)}\|_{L^\infty(Q')} \).

The $W^{1,p}$ estimates for the general elliptic type system as in Definition 2.1 is:

Theorem 2.2. ($W^{1,p}$ estimates) Let \( 0 < \gamma < 1 \) and \( w = (w^{(1)}, \ldots, w^{(m)}) \in W^{1,2}(Q \subset \mathbb{R}^n; \mathbb{R}^m) \cap C^1(Q \cup \Gamma; \mathbb{R}^m) \) be the weak solution to the general elliptic type system as in Definition 2.1. Then, for any \( 2 \leq p < \infty \) and any domain \( Q' \subset Q \cup \Gamma \), there exists a positive constant \( C \) depending on \( \lambda, \kappa_3, p, Q', Q \), such that,

\[
\|w\|_{W^{1,p}(Q')} \leq C \left( \|w\|_{L^2(Q')} + \|F\|_{Y, Q} + \|H\|_{L^\infty(Q')} \right).
\]

In particular, if \( p > n \), there exists a positive constant \( C \) depending on \( \lambda, \tau, \kappa_3, p, Q', Q \), such that, for any \( 0 < \tau \leq 1 - n/p \),

\[
\|w\|_{C^\tau(Q')} \leq C \left( \|w\|_{L^2(Q')} + \|F\|_{Y, Q} + \|H\|_{L^\infty(Q')} \right).
\]
For readers’ convenience, the proofs of Theorem 2.1 and Theorem 2.2 are given in Section 4.

2.2 The proof of Theorem 1.1

Definition 2.2. Let \( A_{ij}^\alpha, B_{ij}^\beta, C_{ij}, D_{ij}, \alpha, \beta = 1, \cdots, n, i, j = 1, \cdots, m \), be as in Definition 1.4 and \( \varphi, \psi \) be as in (1.11). Let

\[
\nu_\ell = (\nu_\ell^{(1)}, \nu_\ell^{(2)}, \cdots, \nu_\ell^{(m)}), \quad \ell = 1, 2, \cdots, m
\]

with \( \nu_\ell^{(j)} = 0 \) for \( j \neq \ell \), \( \nu_\ell^{(j)} = u_\ell \) for \( j = \ell \), and be a weak solution of the following boundary value problem:

\[
\begin{align*}
\sum_{\alpha, \beta, j} \partial_\alpha \left( A_{ij}^\alpha \partial_\beta \nu_\ell^{(j)} + B_{ij}^\beta \nu_\ell^{(j)} \right) + C_{ij} \partial_\beta \nu_\ell^{(j)} + D_{ij} \nu_\ell^{(j)} = 0, \quad \text{in } \Omega, \\
\nu_\ell = (0, \cdots, 0, \varphi(\ell), 0, \cdots, 0), \quad \text{on } \Gamma_1^+, \\
\nu_\ell = (0, \cdots, 0, \psi(\ell), 0, \cdots, 0), \quad \text{on } \Gamma_1^-.
\end{align*}
\]
(2.4)

It follows from Definition 1.4 that for the solution \( u = (u^{(1)}, \cdots, u^{(m)}) \) of general elliptic system as in Definition 1.4, one has

\[
u = v_1 + v_2 + \cdots + v_m \quad \text{and} \quad \nabla u = \sum_{\ell=1}^m \nabla \nu_\ell \quad \text{in } \Omega_1.
\]
(2.5)

In order to estimate \( |\nabla \nu_\ell| \), \( \ell = 1, \cdots, m \), we introduce a scalar function \( \tilde{u} \in C^{1,\gamma}(\mathbb{R}^n) \) such that \( \tilde{u} = 1 \) on \( \Gamma_1^+ \), \( \tilde{u} = 0 \) on \( \Gamma_1^- \) and

\[
\tilde{u}(x) := \frac{x_n - h_2(x') + \varepsilon/2}{\varepsilon + h_1(x') - h_2(x')}, \quad x \in \Omega_1,
\]
(2.6)

where \( h_1, h_2 \in C^{1,\gamma}(B_1') \) satisfy (1.4) and (1.6).

By a direct calculation, we obtain that for \( x := (x', x_n) \in \Omega_1 \),

\[
|\partial_\alpha \tilde{u}(x)| \leq \frac{C|x'|^\gamma}{\varepsilon + |x'|^1+\gamma}, \quad \alpha = 1, 2, \cdots, n-1, \quad \partial_n \tilde{u}(x) = \frac{1}{\delta(x')},
\]
(2.7)

where

\[
\delta(x') := \varepsilon + h_1(x') - h_2(x'), \quad \text{in } \Omega_1.
\]
(2.8)

Using \( \tilde{u} \) to define a family of vector-valued auxiliary functions, for \( \ell = 1, 2, \cdots, m \), we define

\[
\tilde{u}_\ell := (0, \cdots, 0, \varphi(\ell)(x', \varepsilon/2 + h_1(x'))\tilde{u}(x) + \psi(\ell)(x', -\varepsilon/2 + h_2(x'))(1 - \tilde{u}(x)), 0, \cdots, 0).
\]
(2.9)

It is obvious that \( \tilde{u}_\ell = (0, \cdots, 0, \varphi(\ell), 0, \cdots, 0) \) on \( \Gamma_1^+ \) and \( \tilde{u}_\ell = (0, \cdots, 0, \psi(\ell), 0, \cdots, 0) \) on \( \Gamma_1^- \).

In view of (2.7) and (2.9), for any \( x \in \Omega_1 \) and \( \alpha = 1, 2, \cdots, n-1 \),

\[
|\partial_\alpha \tilde{u}_\ell(x)| \leq \frac{C|x'|^\gamma}{\varepsilon + |x'|^1+\gamma} \left| \varphi(\ell)(x', \varepsilon/2 + h_1(x')) - \psi(\ell)(x', -\varepsilon/2 + h_2(x')) \right|
\]
Proof.

by parts in $\Omega$

and

\[
\frac{\varphi^{(l)}(x', \epsilon/2 + h_1(x')) - \varphi^{(l)}(x', -\epsilon/2 + h_2(x'))}{C(\epsilon + |x'|^{1+\gamma})} \leq |\partial_n \bar{u}_l(x)| \leq \frac{C(\varphi^{(l)}(x', \epsilon/2 + h_1(x')) - \varphi^{(l)}(x', -\epsilon/2 + h_2(x'))}{\epsilon + |x'|^{1+\gamma}}.
\]

Next, we estimate $|\nabla w|$, $\ell = 1, \ldots, m$. Let

\[
w_\ell = v_\ell - \bar{u}_\ell, \quad \ell = 1, 2, \ldots, m,
\]

then, by $(2.4)$ and $(2.9)$ we have $w_\ell = (w_\ell^{(1)}, w_\ell^{(2)}, \ldots, w_\ell^{(m)})$ satisfies

\[
\begin{cases}
\sum_{\alpha, \beta, j} (\partial_\alpha (A_{ij}^{\alpha \beta}(x) \partial_\beta \bar{u}_\ell^{(j)}(x)) + B_{ij}^{\alpha}(x) \bar{u}_\ell^{(j)}(x)) + D_{ij}^{\alpha}(x) \bar{u}_\ell^{(j)}(x) = H^{(i)}(x) - \sum_{\alpha} \partial_\alpha F_\ell^{\alpha}, \text{ in } \Omega_1 \\
w_\ell = 0, \quad \text{ on } \Gamma_1^+,
\end{cases}
\]

where for any $1 \leq \alpha \leq n$, $1 \leq i \leq m$,

\[
F_\ell^{(\alpha)}(x) := \sum_{\beta, j} (A_{ij}^{\alpha \beta}(x) \partial_\beta \bar{u}_\ell^{(j)}(x)) + B_{ij}^{\alpha}(x) \bar{u}_\ell^{(j)}(x),
\]

\[
H^{(i)}(x) := \sum_{\beta, j} (A_{ij}^{\alpha \beta}(x) \partial_\beta \bar{u}_\ell^{(j)}(x)) + D_{ij}^{\alpha}(x) \bar{u}_\ell^{(j)}(x).
\]

Lemma 2.1. Let $v_\ell \in W^{1,2}(\Omega_1 \subset \mathbb{R}^n; \mathbb{R}^m)$, $1 \leq \ell \leq m$, be a weak solution of $(2.4)$ in the sense of $(1.13)$, thus $w_\ell$ as in $(2.12)$ satisfies $(2.13)$. Let $\varphi, \psi$ be as in $(1.11)$, $\Gamma_1^+$, $\Gamma_1^-$ be as in $(1.9)$ and $\Omega_r$, $r \in \{1/2, 1\}$ be as in $(1.7)$. Then exists a positive constant $C$ independent of $\epsilon, \ell$, such that, for any $\ell = 1, \cdots, m$,

\[
\int_{\Omega_{1/2}} |\nabla w_\ell| \, dx \leq C \left( \|w_\ell\|^2_{L^2(\Omega_1)} + \|\varphi^{(l)}\|_{C^{1,2}(\Gamma_1^+)}^2 + \|\psi^{(l)}\|_{C^{1,2}(\Gamma_1^-)}^2 \right).
\]

Proof. For simplicity, we assume that $\psi \equiv 0$. Multiply $(2.13)$ by $w_\ell$, making use of the integration by parts in $\Omega_{1/2}$,

\[
\sum_{\alpha, \beta, i, j} \int_{\Omega_{1/2}} (A_{ij}^{\alpha \beta}(x) \partial_\beta w_\ell^{(j)}) \partial_\alpha w_\ell^{(i)} \, dx = \sum_{\alpha, \beta, i, j} \left( - \int_{\Omega_{1/2}} B_{ij}^{\alpha}(x) w_\ell^{(j)} \partial_\alpha w_\ell^{(i)} \, dx + \int_{\Omega_{1/2}} C_{ij}^{\alpha}(x) \partial_\beta w_\ell^{(j)} w_\ell^{(i)} \, dx \right)
\]

\[
+ \sum_{\alpha, \beta, i, j} \left( \int_{\Omega_{1/2}} D_{ij}^{\alpha}(x) w_\ell^{(j)} \partial_\alpha w_\ell^{(i)} \, dx - \int_{\Omega_{1/2}} H^{(i)} w_\ell^{(i)} \, dx + \int_{\Omega_{1/2}} \partial_\alpha F_\ell^{\alpha} w_\ell^{(i)} \, dx \right)
\]

\[
+ \sum_{\alpha, \beta, i, j} \left( \int_{|x'|=1/2, \ -h_2(x') < x_n < h_1(x')} (A_{ij}^{\alpha \beta}(x) \partial_\beta w_\ell^{(j)} + B_{ij}^{\alpha}(x) w_\ell^{(j)}) w_\ell^{(i)} \gamma^{(\alpha)} \, ds \right),
\]
where $\vec{v} := (v^{(1)}, \ldots, v^{(n)}) \in \mathbb{R}^n$ is the unit outer normal vector of vertical boundary on both sides of $\Omega_{1/2}$.

From the strong ellipticity condition (1.10) and Cauchy’s inequality, we have

$$\lambda \int_{\Omega_{1/2}} |\nabla w_{x}^2| dx \leq \sum_{\alpha, \beta, i, j} \left( \int_{\Omega_{1/2}} (A_{ij}^{\alpha\beta}(x) \partial_{\alpha} w_{x}^{(i)} \partial_{\beta} w_{x}^{(j)}) dx \right)$$

$$\leq \frac{\lambda}{4} \int_{\Omega_{1/2}} |\nabla w_{x}^2| dx + C \int_{\Omega_{1/2}} |w_{x}^2| dx + \sum_{i} \left( \int_{\Omega_{1/2}} H_{x}^{(i)} w_{x}^{(i)} dx \right)$$

$$+ C \int_{-\frac{r}{2} + h_{2}(x') < x_{n} < \frac{r}{2} + h_{1}(x')} (|\nabla w_{x}^2| + |w_{x}^2|) dx + \sum_{\alpha, \beta, i} \left( \int_{\Omega_{1/2}} \partial_{\alpha} F_{x}^{(i)} w_{x}^{(i)} dx \right).$$

(2.15)

By using Cauchy’s inequality again, (2.9), and (2.14), we have

$$\sum_{i} \left( \int_{\Omega_{1/2}} H_{x}^{(i)} w_{x}^{(i)} dx \right) \leq C \int_{\Omega_{1/2}} (|\nabla \tilde{u}_{x}| + |\tilde{u}_{x}| |w_{x}^2|) dx$$

$$\leq C \left( \int_{\Omega_{1/2}} (|\nabla \tilde{u}_{x}| + |\tilde{u}_{x}|^2) dx + \int_{\Omega_{1/2}} |w_{x}^2| dx \right)$$

$$\leq C \left( |w_{x}^2|^{1/2} + \|\varphi^{(i)}\|_{C^{1,\gamma}(\Gamma_{1}^{i})} \right).$$

(2.16)

Note that $\Omega_{2/3} \backslash \overline{\Omega}_{1/3} \subset \subset \Omega_{1} \backslash \overline{\Omega}_{1/4}$, applying Theorem 2.4 and (2.3) in Theorem 2.2 for (2.15) with (2.14), one has

$$\|\nabla w_{x}\|_{L^{\infty}(\Omega_{2/3} \backslash \overline{\Omega}_{1/3})} \leq \|\nabla w_{x}\|_{C^{1,\gamma}(\Omega_{2/3} \backslash \overline{\Omega}_{1/3})} \leq C \left( \|\nabla w_{x}\|_{L^{\infty}(\Omega_{1} \backslash \overline{\Omega}_{1/4})} + \|\varphi^{(i)}\|_{C^{1,\gamma}(\Gamma_{1}^{i})} \right)$$

$$\leq C \left( \|\nabla w_{x}\|_{L^{2}(\Omega_{1})} + \|\varphi^{(i)}\|_{C^{1,\gamma}(\Gamma_{1}^{i})} \right).$$

This implies that for $x = (x', x_{n}) \in \Omega_{2/3} \backslash \overline{\Omega}_{1/3}$,

$$|w_{x}(x', x_{n})| \leq C \left( \|\nabla w_{x}\|_{L^{2}(\Omega_{1})} + \|\varphi^{(i)}\|_{C^{1,\gamma}(\Gamma_{1}^{i})} \right),$$

it follows that

$$\int_{-\frac{r}{2} + h_{2}(x') < x_{n} < \frac{r}{2} + h_{1}(x')} (|\nabla w_{x}^2| + |w_{x}^2|) dx \leq C \left( \|\nabla w_{x}\|_{L^{2}(\Omega_{1})} + \|\varphi^{(i)}\|_{C^{1,\gamma}(\Gamma_{1}^{i})} \right).$$

(2.17)

By (2.10), we deduce

$$\sum_{\alpha=1}^{n-1} \int_{\Omega_{1}} |\partial_{\alpha} \tilde{u}_{x}|^2 dx$$

$$\leq C \int_{|x'| < 1} \delta(x') \left( \frac{|x'|^{2\gamma}}{(E + |x'|^{1+\gamma})^{2}} \left| \varphi^{(i)}(x', \frac{E}{2} + h_{1}(x')) \right|^2 + \|\nabla \varphi^{(i)}\|_{L^{\infty}(\Gamma_{1}^{i})} \right) dx$$

$$\leq C \|\varphi^{(i)}\|_{C^{1,\gamma}(\Gamma_{1}^{i})}.$$  

(2.18)
In the view of $\partial_{\alpha n} \tilde{u}_\ell = 0$ and (2.17), (2.18), by applying the integration by parts in $\Omega_{1/2}$ we have

$$
\sum_{\alpha, i} \left| \int_{\Omega_{1/2}} \partial_{\alpha} F_i^{(a)} w_i^{(i)} \, dx \right| \leq \sum_{\alpha, i} \left| \int_{\Omega_{1/2}} F_i^{(a)} \partial_{\alpha} w_i^{(i)} \, dx \right| + \int_{|x'| = 1/2} \left| F_i^{(a)} w_i^{(i)} \right| \, dx
$$

$$
\leq C \int_{\Omega_{1/2}} \left( \sum_{\alpha=1}^{n-1} \left| \partial_{\alpha} \tilde{u}_\ell \right| |\nabla w_\ell| + |\tilde{u}_\ell| |\nabla w_\ell| \right) \, dx
$$

$$
+ C \int_{-\frac{1}{2} + h_2(x') < x_n < \frac{1}{2} + h_1(x')} \left( \sum_{\alpha=1}^{n-1} \left| \partial_{\alpha} \tilde{u}_\ell \right| |w_\ell| + |\tilde{u}_\ell| |w_\ell| \right) \, ds
$$

$$
\leq A \int_{\Omega_{1/2}} |\nabla w_\ell|^2 \, dx + C \left( \|\varphi^{(\prime)}\|_{C^{1,n}(\Gamma_1^+)}^2 + \|\psi^{(\prime)}\|_{C^{1,n}(\Gamma_1^+)}^2 \right).
$$

this, combining with (2.15), (2.16), and (2.17) we obtain

$$
\int_{\Omega_{1/2}} |\nabla w_\ell|^2 \, dx \leq C \left( \|w_\ell\|_{L^2(\Omega_1)}^2 + \|\varphi^{(\prime)}\|_{C^{1,n}(\Gamma_1^+)}^2 \right).
$$

We have completed the proof of Lemma 2.1.

---

Let $1 \leq s \leq 1/2$ and $\Omega_{1/2}$ be as in (1.7). Let $h_1, h_2 \in C^{1,n}(B^\prime_1)$ satisfy (1.4)–(1.7) and $\varepsilon$ be as in (1.5), define set as

$$
(2.19) \quad \tilde{\Omega}_s(z) := \{(x',x_n) \in \Omega_{1/2} | -\frac{\varepsilon}{2} + h_2(x') < x_n < \frac{\varepsilon}{2} + h_1(x'), |x' - z'| < s\}.
$$

In order to iteration as in [14, 15] to prove that the gradients of the auxiliary functions $\tilde{u}_\ell$ are the major singular terms of $|\nabla u_\ell|$, we need the following estimates: namely, for a fixed point

$$
(2.20) \quad z = (z', z_n) \in \Omega_{1/2},
$$

we consider the Hölder semi-norm estimates of $|\nabla \tilde{u}_\ell|$ in $\tilde{\Omega}_s(z)$. In the following, we always assume that $\varepsilon$ as in (1.5) and $|z'|$ are sufficiently small.

**Proposition 2.1.** Let $h_1, h_2$ satisfy (1.4)–(1.7), $\Gamma_1^+, \Gamma_1^-$ be as in (1.9) and $\varphi, \psi$ be as in (1.11). Let $z = (z', z_n) \in \Omega_{1/2}$, $\delta(z')$ be as in (2.8), and $\tilde{\Omega}_s(z)$ be as in (2.19) for $0 < s \leq C \delta(z')$. Then there exists a positive constant $C$ independent of $\varepsilon$, such that, for any $\ell = 1, \cdots, m,$

$$
|\nabla \tilde{u}_\ell|_{\tilde{\Omega}_s(z)} \leq C \left[ \left| \varphi^{(\prime)}(z', \frac{\varepsilon}{2} + h_1(z')) - \varphi^{(\prime)}(z', \frac{\varepsilon}{2} + h_2(z')) \right| + C \left( \|\varphi^{(\prime)}\|_{C^{1,n}(\Gamma_1^+)} + \|\psi^{(\prime)}\|_{C^{1,n}(\Gamma_1^+)} \right) \right]
$$

$$
(2.21) \quad \left( \delta(z')^{-1} + \gamma \delta(z')^{-1} \delta(z')^{-1} + \delta(z')^{-1} \delta(z')^{-1} \right).
$$

**Proof.** Since $s \leq C \delta(z')$ and $|z'| \leq C \delta(z')^{-1}$, for any $x = (x', x_n) \in \tilde{\Omega}_s(z),$

$$
|x'| \leq |x' - z'| + |z'| < s + |z'| \leq C \delta(z')^{-1},
$$

and

$$
\nabla \tilde{u}_\ell(z', x_n) \leq C \|\varphi^{(\prime)}\|_{C^{1,n}(\Gamma_1^+)} + \|\psi^{(\prime)}\|_{C^{1,n}(\Gamma_1^+)}.
$$
and combining with mean value theorem we have

\begin{equation}
|h_i(x') - h_i(x)| \leq C|x_i'| \leq C \delta(z')^{\frac{1}{1+\gamma}} |x' - \tilde{x}'|, \quad i = 1, 2,
\end{equation}

for any $x', \tilde{x}' \in \tilde{\Omega}_d(z), x' \neq \tilde{x}'$. In view of \eqref{eq:2.22}, we obtain

\begin{equation}
|\delta(x') - \delta(\tilde{x}')| \leq |h_1(x') - h_1(\tilde{x}')| + |h_2(x') - h_2(\tilde{x}')| \leq C \delta(z')^{\frac{1}{1+\gamma}} |x' - \tilde{x}'|.
\end{equation}

In particular, taking $\tilde{x}' = z'$, and recalling that $|x' - z'| < s \leq C \delta(z')$, we have

\begin{equation}
\delta(x') \leq \delta(z') + |h_1(x') - h_1(z')| + |h_2(x') - h_2(z')| \leq C \delta(z'),
\end{equation}

and for sufficiently small $\varepsilon$ and $|z'|$,

\begin{equation}
\delta(x') \geq \delta(z') - |h_1(x') - h_1(z')| - |h_2(x') - h_2(z')| \geq \frac{1}{2} \delta(z').
\end{equation}

Next we estimate $|\partial_\alpha \bar{u}(x) - \partial_\alpha \bar{u}(\tilde{x})|, x, \tilde{x} \in \tilde{\Omega}_d(z), x \neq \tilde{x}$, for $\alpha = 1, 2, \cdots, n$. For $\alpha = 1, 2, \cdots, n - 1$, recalling \eqref{eq:2.6}, we have

\begin{equation}
\partial_\alpha \bar{u}(x', x_n) = \frac{-\partial_\alpha h_2(x')}{\delta(x')} + \frac{x_n}{\delta^2(x')} + \frac{(h_2(x') - \varepsilon/2) \partial_\alpha \delta(x')}{\delta^2(x')}
\end{equation}

\begin{equation}
= \Pi_1(x) + \Pi_2(x) + \Pi_3(x).
\end{equation}

Since $h_1(x'), h_2(x') \in C^{1,\gamma}(B')$, \eqref{eq:2.23}, and \eqref{eq:2.25}, we obtain

\begin{equation}
|\Pi_1(x) - \Pi_1(\tilde{x})| \leq \left| \frac{\partial_\alpha h_2(x')}{\delta(x')} \right| \left| \frac{1}{\delta(x')} - \frac{1}{\delta(\tilde{x}')} \right|
\end{equation}

\begin{equation}
\leq C \left( \delta(z')^{-1} |x' - \tilde{x}'|^\gamma + \delta(z')^{-\frac{1}{1+\gamma}} |x' - \tilde{x}'| \right),
\end{equation}

for any $x, \tilde{x} \in \tilde{\Omega}_d(z), x \neq \tilde{x}$. By using $|x_n| \leq \delta(z')$, we can obtain

\begin{equation}
|\Pi_2(x) - \Pi_2(\tilde{x})| \leq \left| \frac{\partial_\alpha \delta(x')}{\delta^2(x')} \right| |x_n - \tilde{x}_n| + \left| \frac{\tilde{x}_n}{\delta^2(x')} \right| \left| \partial_\alpha \delta(x') - \partial_\alpha \delta(\tilde{x}') \right|
\end{equation}

\begin{equation}
\leq C \left( \delta(z')^{-\frac{1}{1+\gamma}} |x_n - \tilde{x}_n| + \delta(z')^{-1} |x' - \tilde{x}'|^\gamma \right).
\end{equation}

It follows from \eqref{eq:2.22}, \eqref{eq:2.24}, and \eqref{eq:2.25} that

\begin{equation}
|\Pi_3(x) - \Pi_3(\tilde{x})| \leq \left| \frac{\partial_\alpha \delta(x')}{\delta^2(x')} \right| |h_2(x') - h_2(\tilde{x}')| + \left| \frac{\delta(x')}{\delta^2(x')} \right| \left| \partial_\alpha \delta(x') - \partial_\alpha \delta(\tilde{x}') \right|
\end{equation}

\begin{equation}
+ \delta(x') \left| \partial_\alpha \delta(x') \right| \left| \frac{1}{\delta(x')^2} - \frac{1}{\delta(\tilde{x}')^2} \right|
\end{equation}

\begin{equation}
\leq C \left( \delta(z')^{-\frac{1}{1+\gamma}} |x' - \tilde{x}'| + \delta(z')^{-1} |x' - \tilde{x}'|^\gamma \right).
\end{equation}

This, combining with \eqref{eq:2.26}, \eqref{eq:2.27}, and \eqref{eq:2.28}, which implies that for $\alpha = 1, 2, \cdots, n - 1$, $|\partial_\alpha \bar{u}(x) - \partial_\alpha \bar{u}(\tilde{x})|$
\begin{align}
\tag{2.29}
\leq & C \left( \delta(z')^{-\frac{1}{17}} |x' - \bar{x}'| + \delta(z')^{-1} |x' - \bar{x}'|^\gamma + \delta(z')^{-1 - \frac{1}{17}} |x_n - \bar{x}_n| \right),
\end{align}

and for \( \alpha = n \), by using (2.23) and (2.25), we obtain
\begin{align}
\tag{2.30}
|\partial_n \bar{u}(x) - \partial_n \bar{u}(\bar{x})| & \leq C \left| \frac{1}{\delta(x')} - \frac{1}{\delta(\bar{x}')} \right| \leq C \delta(z')^{-1 - \frac{1}{17}} |x' - \bar{x}'|.
\end{align}

Now we prove (2.21). Take the case when \( \psi \equiv 0 \) for instance. Firstly, for \( \alpha = n \), since \( \phi(x) \in C^{1,\gamma}(\Gamma_1^+) \), (2.27), (2.25), and (2.40), we have
\begin{align*}
|\partial_n \bar{u}(x) - \partial_n \bar{u}(\bar{x})| & \leq \left| \phi^{(l)}(x', \frac{E}{2} + h_1(x')) \right| |\partial_n \bar{u}(x) - \partial_n \bar{u}(\bar{x})| \\
& + |\partial_n \bar{u}(\bar{x})| \left| \phi^{(l)}(x', \frac{E}{2} + h_1(x')) - \phi^{(l)}(\bar{x}', \frac{E}{2} + h_1(\bar{x}')) \right| \\
& \leq C \left| \phi^{(l)}(z', \frac{E}{2} + h_1(z')) \right| \delta(z')^{-1 - \frac{1}{17}} |x' - \bar{x}'| \\
& + C \|\nabla \phi^{(l)}\|_{L^{\infty}(\Gamma_1^+)} \delta(z')^{-1} |x' - \bar{x}'| \left( \delta(z')^{-\frac{1}{17}} s + 1 \right),
\end{align*}

where we used the fact that \( |x' - \bar{x}'| < s \). Similarly, by using \( |x' - \bar{x}'| \leq s \) we obtain
\begin{align}
\tag{2.31}
\left| \partial_n \bar{u}(x) - \partial_n \bar{u}(\bar{x}) \right| & \leq C \left| \phi^{(l)}(z', \frac{E}{2} + h_1(z')) \right| \delta(z')^{-1 - \frac{1}{17}} s^{1-\gamma} \\
& + C \|\nabla \phi^{(l)}\|_{L^{\infty}(\Gamma_1^+)} \delta(z')^{-1 - \frac{1}{17}} s^{2-\gamma} + \delta(z')^{-1} s^{1-\gamma}.
\end{align}

For \( \alpha = 1, 2, \cdots, n - 1 \),
\begin{align}
\tag{2.32}
\left| \partial_{\alpha} \bar{u}(x) - \partial_{\alpha} \bar{u}(\bar{x}) \right| & \leq \left| \partial_{\alpha} \phi^{(l)}(x', \frac{E}{2} + h_1(x')) \bar{u}(x) - \partial_{\alpha} \phi^{(l)}(\bar{x}', \frac{E}{2} + h_1(\bar{x}')) \bar{u}(\bar{x}) \right| \\
& + \left| \phi^{(l)}(x', \frac{E}{2} + h_1(x')) \partial_{\alpha} \bar{u}(x) - \phi^{(l)}(\bar{x}', \frac{E}{2} + h_1(\bar{x}')) \partial_{\alpha} \bar{u}(\bar{x}) \right|
\end{align}

Using (2.25) and (2.30), we obtain
\begin{align*}
I & \leq C \|\nabla \phi^{(l)}\|_{L^{\infty}(\Gamma_1^+)} \left( \frac{|x_n - \bar{x}_n|}{\delta(x')} + |\bar{x}_n| \left| \frac{1}{\delta(x')} - \frac{1}{\delta(\bar{x}')} \right| \right) + C \|\nabla \phi^{(l)}\|_{C^{0,\gamma}(\Gamma_1^+)} |x' - \bar{x}'|^\gamma \\
& \leq C \|\nabla \phi^{(l)}\|_{C^{0,\gamma}(\Gamma_1^+)} \left( \delta(z')^{-1} |x_n - \bar{x}_n| + \delta(z')^{-\frac{1}{17}} |x' - \bar{x}'| + |x' - \bar{x}'|^\gamma \right).
\end{align*}

In view of \( |x' - \bar{x}'| < s \) and \( |x_n - \bar{x}_n| \leq 2 \delta(z') \), we obtain
\begin{align}
\tag{2.33}
\sup_{x, \bar{x} \in \bar{\Omega}(z'), x \neq \bar{x}} \frac{1}{|x - \bar{x}|^{\gamma}} & \leq C \left\| \nabla \phi^{(l)}\|_{C^{0,\gamma}(\Gamma_1^+)} \left( \delta(z')^{-\gamma} + \delta(z')^{-\frac{1}{17}} s^{1-\gamma} \right) \right.
\end{align}

It follows from (2.7) and (2.29) that
\begin{align*}
II & \leq C \left( |\phi^{(l)}(z', \frac{E}{2} + h_1(z'))| + s \|\nabla \phi^{(l)}\|_{L^{\infty}(\Gamma_1^+)} \right)
\end{align*}
Lemma 2.2. Let $z = (z', z_n) \in \Omega_{1/2}$ as in (1.7), $\delta(z')$ be as in (2.3), and $\bar{\Omega}_a(z)$ be as in (2.19) for $0 < s \leq C \delta(z')$. Let $A_{ij}^{ab}$ be as in Definition 1.2, $U_\ell$ be as in (2.20) for $\ell = 1, \ldots, m$ and $|\Omega_a(z)|$ be the volume of region $\Omega_a(z)$. For $i = 1, 2, \ldots, m$ and $\alpha = 1, \ldots, n$, we define

$$M_i^{(\alpha)} := (a_i)_{\beta\gamma} = \left(\frac{1}{\Omega_a(z)}\right) \int_{\Omega_a(z)} \sum_{\beta, j} (A_{ij}^{ab}(y) \partial_{\beta} \tilde{u}_{\ell}^{(j)}(y)) dy.$$

Clearly, by using (2.13) we have $w_\ell, \ell = 1, \ldots, m$ as in (2.12) satisfies

$$\sum_{\alpha, \beta, i, j} \partial_\alpha (A_{ij}^{ab} \partial_{\beta} w_\ell^{(j)} + B_{ij}^{ab} w_\ell^{(j)}) + C_{ij}^{ab} \partial_\beta w_\ell^{(j)} + D_{ij}^{ab} w_\ell^{(j)}$$

$$= H^{(i)} - \sum_\alpha \partial_\alpha (F_i^{(\alpha)} - M_i^{(\alpha)}), \quad \text{in } \Omega_1.$$

Lemma 2.2. Let $z = (z', z_n) \in \Omega_{1/2}$ as in (1.7), $\delta(z')$ be as in (2.3) and $\bar{\Omega}_{a_1}(z)$ be as in (2.19) Let $h_1, h_2$ satisfy (1.4), (1.17), $\Gamma_1^+, \Gamma_1^- \quad \text{as in (1.19)}$ and $\varphi, \psi$ be as in (1.11). Let $e$ be as in (2.13) and $w_\ell$ be as in (2.35), $\ell = 1, \ldots, m$. Then there exists a positive constant $C$ independent of $e, \ell$, such that, for $0 \leq |z'| \leq e^{1/2}$,

$$\int_{\bar{\Omega}_{a_1}(z)} |\nabla w_\ell|^2 \, dx \leq C e^{n-\frac{n}{2}+2} \left(\left|\varphi^{(i)}(z', \frac{e}{2} + h_1(z')) - \psi^{(i)}(z', -\frac{e}{2} + h_2(z'))\right|^2 + C e^{n-\frac{n}{2}+2} \left(\|w_\ell\|^2_{L^2(\Omega_1)} + \|\varphi^{(i)}\|^2_{C^{1,\gamma}(\Gamma_1)} + \|\psi^{(i)}\|^2_{C^{1,\gamma}(\Gamma_1)}\right),$$

and for $e^{1/2} < |z'| \leq 1/2$,

$$\int_{\bar{\Omega}_{a_1}(z)} |\nabla w_\ell|^2 \, dx \leq C |z'|^{(1+\gamma)(n-1)\gamma} \left(\left|\varphi^{(i)}(z', \frac{e}{2} + h_1(z')) - \psi^{(i)}(z', -\frac{e}{2} + h_2(z'))\right|^2 + C |z'|^{(1+\gamma)(n-1)\gamma} \left(\|w_\ell\|^2_{L^2(\Omega_1)} + \|\varphi^{(i)}\|^2_{C^{1,\gamma}(\Gamma_1)} + \|\psi^{(i)}\|^2_{C^{1,\gamma}(\Gamma_1)}\right).$$
Proof. For simplicity, we assume that $\psi \equiv 0$. Indeed, for $0 < t < s < 1/2$, let $\eta$ be a cut-off function satisfying $0 \leq \eta(x') \leq 1$,

$$\eta(x') = \begin{cases} 1, & \text{if } |x' - z'| < t, \\ 0, & \text{if } |x' - z'| > s, \end{cases} \quad |\nabla \eta(x')| \leq \frac{2}{s - t}.$$ 

Multiplying (2.35) by $\eta^2 w_t$ and using the integration by parts, one has

$$\sum_{\alpha,\beta,i,j} \int_{\hat{\Omega}_t(z)} \eta^2 A^{\alpha\beta}_{ij}(x) \partial_\beta w_t^{(j)} \partial_\alpha w_t^{(i)} \, dx$$

$$= - \sum_{\alpha,\beta,i,j} \int_{\hat{\Omega}_t(z)} A^{\alpha\beta}_{ij}(x) \partial_\beta w_t^{(j)} w_t^{(i)} 2\eta \partial_\alpha \eta \, dx - \sum_{\alpha,\beta,i,j} \int_{\hat{\Omega}_t(z)} B^{\alpha}_{ij}(x) w_t^{(j)} \partial_\alpha (\eta^2 w_t^{(i)})$$

$$- \sum_{\beta,i,j} \int_{\hat{\Omega}_t(z)} C_{ij}^{\beta}(x) \partial_\beta w_t^{(j)} \eta^2 w_t^{(i)} \, dx - \sum_{\alpha,i,j} \int_{\hat{\Omega}_t(z)} D_{ij}(x) w_t^{(j)} (\eta^2 w_t^{(i)}) \, dx$$

$$+ \sum_{i} \int_{\hat{\Omega}_t(z)} H^{(i)} (\eta^2 w_t^{(i)}) \, dx + \sum_{\alpha,i} \int_{\hat{\Omega}_t(z)} (F_i^{(\alpha)} - M_i^{(\alpha)}) \partial_\alpha (\eta^2 w_t^{(i)}) \, dx.$$ 

Now we can bound with (1.16), Young’s inequality $2ab \leq \xi a^2 + \frac{b^2}{\xi}$, and the properties of $\eta$,

$$\sum_{\alpha,\beta,i,j} \left| \int_{\hat{\Omega}_t(z)} A^{\alpha\beta}_{ij}(x) \partial_\beta w_t^{(j)} w_t^{(i)} 2\eta \partial_\alpha \eta \, dx \right|$$

$$\leq \xi_1 \Lambda \int_{\hat{\Omega}_t(z)} \eta^2 |\nabla w_t|^2 \, dx + \frac{4\Lambda}{\xi_1(s-t)^2} \int_{\hat{\Omega}_t(z)} |w_t|^2 \, dx,$$

combining with (1.18), we deduce

$$\sum_{\alpha,\beta,i,j} \left| \int_{\hat{\Omega}_t(z)} B^{\alpha}_{ij}(x) w_t^{(j)} \partial_\alpha (\eta^2 w_t^{(i)}) \, dx \right| = \sum_{\alpha,\beta,i,j} \left| \int_{\hat{\Omega}_t(z)} B^{\alpha}_{ij}(x) w_t^{(j)} (2\eta \partial_\alpha \eta w_t^{(i)} + \eta^2 \partial_\alpha w_t^{(i)}) \, dx \right|$$

$$\leq \frac{\xi_2 k_3}{2} \int_{\hat{\Omega}_t(z)} \eta^2 |\nabla w_t|^2 \, dx + \frac{C}{\xi_2(s-t)^2} \int_{\hat{\Omega}_t(z)} |w_t|^2 \, dx.$$ 

Similarly, using Young’s inequality again, we have

$$\sum_{\beta,i,j} \left| \int_{\hat{\Omega}_t(z)} C_{ij}^{\beta}(x) \partial_\beta w_t^{(j)} \eta^2 w_t^{(i)} \, dx \right| \leq \frac{\xi_3 k_3}{2} \int_{\hat{\Omega}_t(z)} \eta^2 |\nabla w_t|^2 \, dx + \frac{C}{\xi_3(s-t)^2} \int_{\hat{\Omega}_t(z)} |w_t|^2 \, dx.$$ 

Noticing $0 < s - t < 1$, we obtain

$$\sum_{i,j} \left| \int_{\hat{\Omega}_t(z)} D_{ij}(x) w_t^{(j)} \eta^2 w_t^{(i)} \, dx \right| \leq \frac{C}{(s-t)^2} \int_{\hat{\Omega}_t(z)} |w_t|^2 \, dx.$$ 

Recalling (2.29) and (2.14), we also have

$$\sum_{i} \left| \int_{\hat{\Omega}_t(z)} H^{(i)} \eta^2 w_t^{(i)} \, dx \right| \leq \frac{C}{(s-t)} \int_{\hat{\Omega}_t(z)} |w_t|^2 \, dx + (s-t)^2 \sum_{i} \int_{\hat{\Omega}_t(z)} |H^{(i)}|^2 \, dx.$$
For the last term in the right hand side of (2.38), using the Young’s inequality again we obtain

\[
\sum_{a,i} \int_{\hat{\Omega}_i(z)} (F_i^{(a)} - M_i^{(a)}) \partial_a (\eta^2 w_i^{(j)}) \, dx \leq \frac{c_4}{2} \int_{\hat{\Omega}_i(z)} \eta^2 |\nabla w_i|^2 \, dx + \frac{C}{\zeta_4} \sum_{a,i} \int_{\hat{\Omega}_i(z)} |F_i^{(a)} - M_i^{(a)}|^2 \, dx + \frac{C}{\zeta_4 (s - t)^2} \int_{\hat{\Omega}_i(z)} |w_i|^2 \, dx.
\]

Choosing \(0 < \zeta_i < 1, i = 1, 2, 3, 4\) satisfy

\[
\Lambda \zeta_1 + \frac{K_3}{2} \zeta_2 + \frac{K_3}{2} \zeta_3 + \frac{1}{2} \zeta_4 < \frac{\lambda}{2}.
\]

In view of the strong ellipticity condition (1.10), we obtain

\[
\lambda \int_{\hat{\Omega}_i(z)} \eta^2 |\nabla w_i|^2 \, dx \leq \sum_{a,\beta, i, j} \int_{\hat{\Omega}_i(z)} \eta^2 N_{ij}^{a\beta}(x) \partial_\beta w_i^{(j)} \partial_\alpha w_i^{(i)} \, dx,
\]

this, combining with (2.38)-(2.44), yields

\[
\int_{\hat{\Omega}_i(z)} |\nabla w_i|^2 \, dx \leq \frac{C}{(s - t)^2} \int_{\hat{\Omega}_i(z)} |w_i|^2 \, dx + C(s - t)^2 \sum_i \int_{\hat{\Omega}_i(z)} |H_i^{(i)}|^2 \, dx
\]

\[
+ C \sum_{a,i} \int_{\hat{\Omega}_i(z)} |F_i^{(a)} - M_i^{(a)}|^2 \, dx.
\]

It follows from (2.10) and (2.11) that

\[
\sum_i \int_{\hat{\Omega}_i(z)} |H_i^{(i)}|^2 \, dx \leq C \left| \varphi^{(i)}(z', \varepsilon/2 + h_i(z')) \right|^2 \int_{|x' - z'| < \varepsilon} \left( \frac{1}{\varepsilon + |x'|^{1+\gamma}} \right) \, dx'
\]

\[
+ \|\nabla \varphi^{(i)}\|_{L^\infty(\Gamma_i)} \int_{|x' - z'| < \varepsilon} \left( \frac{|x' - z'|^2}{\varepsilon + |x'|^{1+\gamma}} \right) \, dx'.
\]

\[
(2.46)
\]

\textbf{Case 1.} For \(|z'| \leq \varepsilon^{-1/\gamma} \) and \(0 < s < \varepsilon^{-1/\gamma} \), we have \(\varepsilon \leq \delta(z') \leq C \varepsilon\). By a direct calculation, we have

\[
(2.47)
\]

\[
\sum_i \int_{\hat{\Omega}_i(z)} |w_i|^2 \, dx = \int_{\hat{\Omega}_i(z)} \left| \int_{\tilde{\Omega}_i(z)} \partial_a w_i(x', x_n) \, dx_n \right|^2 \, dx \leq C \varepsilon^2 \int_{\hat{\Omega}_i(z)} |\nabla w_i|^2 \, dx,
\]

and from (2.46), we have

\[
\sum_i \int_{\hat{\Omega}_i(z)} |H_i^{(i)}|^2 \, dx \leq C \left| \varphi^{(i)}(z', \varepsilon/2 + h_i(z')) \right|^2 \frac{\varepsilon^{n-1}}{\varepsilon} + C \|\nabla \varphi^{(i)}\|_{L^\infty(\Gamma_i)}^2 \frac{s^{n+1}}{\varepsilon} := G_{11}(s).
\]

By using (1.18), (2.14), and (2.33) for any \(\alpha = 1, \cdots, n, i = 1, \cdots, m\), we have

\[
|F_i^{(a)} - M_i^{(a)}|^2 \leq \frac{C K_3}{|\hat{\Omega}_i(z)|^2} \left( |\nabla \tilde{u}_i|_{\partial \hat{\Omega}_i(z)} \int_{\hat{\Omega}_i(z)} |x - y|^\gamma \, dy + \delta(z')^{-1} \int_{\hat{\Omega}_i(z)} |x - y|^\gamma \, dy \right)^2 + C K_3 \|\varphi^{(i)}\|_{L^\infty(\Gamma_i)}
\]
\[ C \left( [\nabla \bar{u}_t]^2 \right)_{\Omega_{2}(\tau)} + \delta(\tau')^{-2} \left( \delta(\tau')^{2\gamma} + s^{2\gamma} \right) \]

thus, from Proposition 2.1 we have

\[
\sum_{\alpha,\beta} \int_{\Omega_{\alpha}(\tau)} |F_{\alpha}^\beta - M_{\alpha}^\beta|^2 \, dx \leq C \left( \left\| \phi^{(0)}(\tau', \frac{e}{2} + h(\tau')) \right\|^2_{L^2(\Gamma_1')} + s^2 \left\| \phi^{(0)} \right\|^2_{C^{1,\gamma}(\Gamma_1')} \right)
\]

(2.48)

\[
\left( \frac{s^{\gamma+1}}{e^{1 + \frac{\gamma}{\gamma+1}}} + \frac{s^{\gamma-1}}{e^{1 - \frac{\gamma}{\gamma+1}}} + \frac{s^{\gamma+1-2\gamma}}{e^{1 - \frac{2\gamma}{\gamma+1}}} + \frac{s^{\gamma-1+2\gamma}}{e^{1 + \frac{\gamma}{\gamma+1}}} \right) =: G_{12}(s).
\]

Denote \( F(t) := \int_{\Omega_{2}(\tau)} |\nabla w|^2 \, dx \). It follows from (2.45), (2.47), and (2.48) that

(2.49)

\[ F(t) \leq \left( \frac{C e^\gamma}{s - \frac{e}{2}} \right)^2 F(s) + C(s - \frac{e}{2})^{2} G_{11}(s) + C G_{12}(s). \]

Similarly as in [15], let \( k = (4c_1 e^{-\frac{\gamma}{\gamma+1}})^{-1} \) and \( t_\tau = \delta(\tau') + 2c_1 \tau e, \tau = 0, 1, 2, \ldots, k \). It is easy to see from the definition of \( G_{11}(s) \) and \( G_{12}(s) \) that

\[ G_{11}(t_{\tau+1}) \leq C e^{\gamma} \left( \left\| \phi^{(0)}(\tau', \frac{e}{2} + h(\tau')) \right\|^2 + e^2 \left\| \nabla \phi^{(0)} \right\|^2_{L^2(\Gamma_1')} \right) (\tau + 1)^{n+1}, \]

and

\[ G_{12}(t_{\tau+1}) \leq C e^{\gamma} \left( \left\| \phi^{(0)}(\tau', \frac{e}{2} + h(\tau')) \right\|^2 + e^2 \left\| \phi^{(0)} \right\|^2_{C^{1,\gamma}(\Gamma_1')} \right) (\tau + 1)^{n+3}. \]

Taking \( s = t_{\tau+1} \) and \( t = t_{\tau} \) in (2.49), we have the following iteration formula

\[ F(t_{\tau}) \leq \frac{1}{4} F(t_{\tau+1}) + C e^{\gamma} \left( \left\| \phi^{(0)}(\tau', \frac{e}{2} + h(\tau')) \right\|^2 + e^2 \left\| \nabla \phi^{(0)} \right\|^2_{L^2(\Gamma_1')} \right) (\tau + 1)^{n+1}, \]

\[ + C e^{\gamma} \left( \left\| \phi^{(0)}(\tau', \frac{e}{2} + h(\tau')) \right\|^2 + e^2 \left\| \phi^{(0)} \right\|^2_{C^{1,\gamma}(\Gamma_1')} \right) (\tau + 1)^{n+3}, \]

\[ \leq C e^{\gamma} \left( \left\| \phi^{(0)}(\tau', \frac{e}{2} + h(\tau')) \right\|^2 + e^2 \left\| \phi^{(0)} \right\|^2_{C^{1,\gamma}(\Gamma_1')} \right) (\tau + 1)^{n+3}, \]

after \( k \) iterations, and by virtue of Lemma 2.1 we have

(2.50)

\[ F(t_0) \leq \sum_{i=0}^{k-1} \left( \frac{1}{4} \right)^i F(t_i) + C e^{\gamma} \left( \left\| \phi^{(0)}(\tau', \frac{e}{2} + h(\tau')) \right\|^2 + e^2 \left\| \phi^{(0)} \right\|^2_{C^{1,\gamma}(\Gamma_1')} \right) \sum_{i=0}^{k-1} \left( \frac{1}{4} \right)^i (i + 1)^{n+3}, \]

\[ \leq C e^{\gamma} \left( \left\| \phi^{(0)}(\tau', \frac{e}{2} + h(\tau')) \right\|^2 + e^2 \left\| \phi^{(0)} \right\|^2_{C^{1,\gamma}(\Gamma_1')} + \left\| \phi^{(0)} \right\|^2_{C^{1,\gamma}(\Gamma_1')} \right). \]

This implies Lemma 2.2 with \( |\tau'| \leq \frac{e}{4^{\gamma+1}} \).

**Case 2.** For \( e^{\frac{\gamma}{\gamma+1}} \leq |\tau'| \leq \frac{1}{2} \) and \( 0 < s < |\tau'| \), we have \( \frac{1}{2} |\tau'|^{1+\gamma} \leq \delta(\tau') \leq C |\tau'|^{1+\gamma} \). Estimates (2.47) and (2.48) become, respectively,

\[ \int_{\Omega_{2}(\tau)} |\nabla w|^2 \, dx \leq C |\tau'|^{2(1+\gamma)} \int_{\Omega_{2}(\tau)} |\nabla w|^2 \, dx, \]

if \( 0 < s < \frac{2}{3} |\tau'| \),
\[
\sum_i \int_{\tilde{\Omega}_i(z)} |H^{(i)}|^2 \, dx \leq C \left( \frac{\varphi^{(i)}(z', \frac{E}{2} + h_1(z'))}{|z'|^{\frac{n-1}{2}}} + C \|\nabla \varphi^{(i)}\|_{L^\infty(\tilde{\Gamma}_i)}^2 \right)^{\frac{\delta^{n+1}}{|z'|^\frac{\gamma}{2}}} := G_{21}(s),
\]
and
\[
\sum_{a,d} \int_{\tilde{\Omega}_i(z)} \left| F_i^{(a)} - \mathcal{N}_i^{(a)} \right|^2 \, dx \leq C \left( \left| \varphi^{(i)}(z', \frac{E}{2} + h_1(z')) \right|^2 + s^2 \|\varphi^{(i)}\|_{C^{1,\gamma}(\Gamma_i')}^2 \right)\left( \frac{s^{\delta^{n+1}}}{|z'|^{\frac{\gamma}{2}}} + \frac{s^{\delta^{n-1}}}{|z'|^{\frac{\gamma}{2}}} + \frac{s^{\delta^{n-1-2\gamma}}}{|z'|^{\frac{\gamma}{2}}} + \frac{s^{\delta^{n-1-2\gamma}}}{|z'|^{\frac{\gamma}{2}}} \right) = : G_{22}(s).
\]

Let \( k = (4c_2|z'|)^{-1} \) and \( t_\tau = \delta(z') + 2c_2 \tau |z'|^{1+\gamma} \), \( \tau = 0, 1, 2, \cdots, k \), one has
\[
G_{21}(s) \leq C |z'|^{(1+\gamma)(n-2)} \left( \left| \varphi^{(i)}(z', \frac{E}{2} + h_1(z')) \right|^2 + C |z'|^{2(1+\gamma)} \|\nabla \varphi^{(i)}\|_{L^\infty(\tilde{\Gamma}_i')}^2 \right)(\tau + 1)^{n+1},
\]
and
\[
G_{22}(t_{\tau+1}) \leq C |z'|^{(1+\gamma)(n-2\tau)} \left( \left| \varphi^{(i)}(z', \frac{E}{2} + h_1(z')) \right|^2 + C |z'|^{2(1+\gamma)} \|\varphi^{(i)}\|_{C^{1,\gamma}(\Gamma_i')}^2 \right)(\tau + 1)^{n+3}.
\]
Then, we obtain that, for \( 0 < t < s < \frac{1}{4} |z'| \),
\[
F(t_\tau) \leq \frac{1}{4} F(t_{\tau+1}) + C |z'|^{(1+\gamma)(n-2\tau)} \left( \left| \varphi^{(i)}(z', \frac{E}{2} + h_1(z')) \right|^2 + C |z'|^{2(1+\gamma)} \|\varphi^{(i)}\|_{C^{1,\gamma}(\Gamma_i')}^2 \right)(\tau + 1)^{n+3},
\]
after \( k \) iterations, and using Lemma 2.1 again, we have
\[
F(t_0) \leq C |z'|^{(1+\gamma)(n-2\tau)} \left( \left| \varphi^{(i)}(z', \frac{E}{2} + h_1(z')) \right|^2 + C |z'|^{2(1+\gamma)} \|\varphi^{(i)}\|_{C^{1,\gamma}(\Gamma_i')}^2 + \|\nabla \zeta\|_{L^2(\tilde{\Omega}_i')}^2 \right).
\]
This implies Lemma 2.3 with \( |z'| \geq e^{\frac{1}{1+\gamma}} \).

**Lemma 2.3.** Let \( z = (z', z_n) \in \Omega_{1/2} \) as in (17), \( \delta(z') \) be as in (2.8) and \( \tilde{\Omega}_{\delta(z')} \) be as in (2.19). Let \( h_1, h_2 \) satisfy (2.11), \( \tilde{\Gamma}_i', \Gamma_i' \) be as in (1.9) and \( \varphi, \psi \) be as in (1.11). Let \( \varepsilon \) be as in (2.8) and \( \zeta_i \) be as in (2.35), \( i = 1, \cdots, m \). Then there exists a positive constant \( C \) independent of \( \varepsilon \), such that, for \( |\zeta'| \leq e^{\frac{1}{1+\gamma}} \),
\[
|\nabla \zeta_i^{(\ell)}(z', z_n)| \leq C e^{-\frac{1}{1+\gamma}} \left| \varphi^{(i)}(z', \frac{E}{2} + h_1(z')) - \psi^{(i)}(z', \frac{E}{2} + h_2(z')) \right| + C e^{-\frac{1}{1+\gamma}} \left( \|\zeta_i\|_{L^2(\tilde{\Omega}_i')} + \|\varphi^{(i)}\|_{C^{1,\gamma}(\Gamma_i')} + \|\psi^{(i)}\|_{C^{1,\gamma}(\Gamma_i')} \right),
\]
and for $\varepsilon < \varepsilon' < \frac{1}{2}$,

$$
\left| \nabla w_{\ell}(z', \zeta) \right| \leq C|z'|^{-1} \left| \varphi^{(\ell)}(z', \frac{E}{2} + h_1(z')) - \psi^{(\ell)}(z', -\frac{E}{2} + h_2(z')) \right| + C|z'|^{-\gamma} \left( \|w_{\ell}\|_{L^2(\Omega_1)} + \|\varphi^{(\ell)}\|_{C^{1,\gamma}(\Gamma_{1}')} + \|\psi^{(\ell)}\|_{C^{1,\gamma}(\Gamma_{1}')} \right).
$$

Consequently, by (2.10) and (2.11), we have for sufficiently small $\varepsilon$ and $z \in \Omega_{1/2}$,

$$
\left| \nabla v_{\ell}(z) \right| \leq \frac{C \left| \varphi^{(\ell)}(z', \frac{E}{2} + h_1(z')) - \psi^{(\ell)}(z', -\frac{E}{2} + h_2(z')) \right|}{\varepsilon + |z'|^{1+\gamma}} + C \left( \|w_{\ell}\|_{L^2(\Omega_1)} + \|\varphi^{(\ell)}\|_{C^{1,\gamma}(\Gamma_{1}')} + \|\psi^{(\ell)}\|_{C^{1,\gamma}(\Gamma_{1}')} \right). \tag{2.50}
$$

Moreover, if $\varphi^{(\ell)}(\bar{\zeta}_{n-1}, \frac{E}{2}) \neq \psi^{(\ell)}(\bar{\zeta}_{n-1}, -\frac{E}{2})$, then there exists a positive constant $C$ independent of $\varepsilon$, such that, for any $z_n \in (-\frac{E}{2}, \frac{E}{2})$,

$$
\left| \nabla v_{\ell}(\bar{\zeta}_{n-1}, z_n) \right| \geq \frac{|\varphi^{(\ell)}(\bar{\zeta}_{n-1}, z_n) - \psi^{(\ell)}(\bar{\zeta}_{n-1}, z_n)|}{C\varepsilon}.
$$

**Proof.** For simplicity, we assume that $\psi \equiv 0$. Given $z = (z', z_n) \in \Omega_{1/2}$, making the following change of variables on $\bar{\Omega}_{\delta(z')}(z)$, as in [15],

$$
\begin{align*}
\begin{cases}
    x' - z' = \delta(z')y', \\
x_n = \delta(z')y_n,
\end{cases}
\end{align*}
$$

then $\bar{\Omega}_{\delta(z')}(z)$ becomes $Q_1$ of nearly unit size, where

$$
Q_r = \left\{ y \in \mathbb{R}^n : -\frac{\varepsilon}{2\delta(z')} + \frac{1}{\delta(z')} h_2(\delta(z')y' + z') < y_n < \frac{\varepsilon}{2\delta(z')} + \frac{1}{\delta(z')} h_1(\delta(z')y' + z'), |y'| < r \right\}. \tag{2.51}
$$

for $r \leq 1$, and the top and bottom boundaries become

$$
\Gamma_+^r := \left\{ y \in \mathbb{R}^n : y_n = \frac{\varepsilon}{2\delta(z')} + \frac{1}{\delta(z')} h_1(\delta(z')y' + z'), |y'| < r \right\},
$$

and

$$
\Gamma_-^r := \left\{ y \in \mathbb{R}^n : y_n = -\frac{\varepsilon}{2\delta(z')} + \frac{1}{\delta(z')} h_2(\delta(z')y' + z'), |y'| < r \right\},
$$

respectively. Let

$$
\tilde{w}_{\ell}(y') := w_{\ell}(\delta(z')y' + z', \delta(z')y'_n), \quad \tilde{u}_{\ell}(y') := u_{\ell}(\delta(z')y' + z', \delta(z')y'_n), \quad \text{for } (y', y_n) \in Q_1.
$$

It follows from (2.13) that $\tilde{w}_{\ell}(y)$ satisfies

$$
\begin{cases}
    \partial_{\alpha} (\tilde{A}_{ij} \partial_{\beta} \tilde{w}_{\ell}^{(j)} + \tilde{B}_{ij} \tilde{w}_{\ell}^{(j)}) + \tilde{C}_{ij} \partial_{\beta} \tilde{w}_{\ell}^{(j)} + \tilde{D}_{ij} \tilde{w}_{\ell}^{(j)} = \tilde{H}^{(i)} - \partial_{\alpha} \tilde{F}_{i}, \text{ in } Q_1, \\
    \tilde{w}_{\ell} = 0, \text{ on } \Gamma_+^1,
\end{cases} \tag{2.52}
$$

in $Q_1$. 


where
\[ \tilde{A}_{ij}^{op}(y) := A_{ij}^{op}(\delta y' + z', \delta y_n), \quad \tilde{B}_{ij}^{op}(y) := \delta B_{ij}^{op}(\delta y' + z', \delta y_n), \]
\[ \tilde{C}_{ij}^{op}(y) := \delta C_{ij}^{op}(\delta y' + z', \delta y_n), \quad \tilde{D}_{ij}(y) := \delta^2 D_{ij}(\delta y' + z', \delta y_n), \]
\[ \tilde{F}_i^\ell(y) := \tilde{A}_{ij}^{op}(y) \partial_\beta u_i^{(\ell)} + \tilde{B}_{ij}^{op}(y) u_i^{(\ell)}, \quad \tilde{H}_i^\ell(y) := \tilde{C}_{ij}^{op}(y) \partial_\beta u_i^{(\ell)} - \tilde{D}_{ij}(y) u_i^{(\ell)}. \]

(2.53)

It follows from Theorem 2.2 that

(2.54) \[ \| \tilde{w}_1 \|_{L^\infty(Q_{1/2})} \leq C \left( \| \tilde{w}_1 \|_{L^2(Q_1)} + [\tilde{F}]_{y,Q_1} + \| \tilde{H} \|_{L^\infty(Q_1)} \right). \]

Applying Theorem 2.1 for (2.52) with (2.53) on \( Q_{1/2} \), we have

\[ \| \tilde{w}_\ell \|_{C^{1/\gamma}(Q_{1/4})} \leq C \left( \| \tilde{w}_\ell \|_{L^\infty(Q_{1/2})} + [\tilde{F}]_{y,Q_1} + \| \tilde{H} \|_{L^\infty(Q_1)} \right). \]

This, combining with (2.54) and using the Poincaré inequality, yields

\[ \| \nabla \tilde{w}_\ell \|_{L^\infty(Q_{1/4})} \leq C \left( \| \nabla \tilde{w}_\ell \|_{L^2(Q_{1/2})} + [\tilde{F}]_{y,Q_1} + \| \tilde{H} \|_{L^\infty(Q_1)} \right). \]

In the following proofs, we will briefly refer to \( \delta(z') \) as \( \delta \). Recalling back to the original region \( \tilde{\Omega}_\delta(z) \), we have

\[ \| \nabla \tilde{w}_\ell \|_{L^\infty(Q_{1/4})} = \delta \| \nabla w_\ell \|_{L^\infty(\tilde{\Omega}_\delta(z))}, \quad \| \nabla \tilde{w}_\ell \|_{L^2(Q_{1/2})} = \delta^{1 - \frac{\gamma}{2}} \| \nabla w_\ell \|_{L^2(\tilde{\Omega}_\delta(z))}, \]
\[ \| \nabla \tilde{u}_\ell \|_{L^\infty(Q_{1/4})} = \delta \| \nabla u_\ell \|_{L^\infty(\tilde{\Omega}_\delta(z))}, \quad \| \nabla \tilde{u}_\ell \|_{L^2(Q_{1/2})} = \delta^{1 + \gamma} \| \nabla u_\ell \|_{L^2(\tilde{\Omega}_\delta(z))}, \]

thus,

\[ [\tilde{F}]_{y,Q_1} \leq C \delta^{1 + \gamma} \left( \| \nabla \tilde{u}_\ell \|_{L^\infty(\tilde{\Omega}_\delta(z))} + [\nabla \tilde{u}_\ell]_{y,\tilde{\Omega}_\delta(z)} + \| \tilde{u}_\ell \|_{L^\infty(\tilde{\Omega}_\delta(z))} \right), \]

and

\[ \| \tilde{H} \|_{L^\infty(Q_{1/2})} \leq C \delta^2 \left( \| \nabla \tilde{u}_\ell \|_{L^\infty(\tilde{\Omega}_\delta(z))} + \| \tilde{u}_\ell \|_{L^\infty(\tilde{\Omega}_\delta(z))} \right). \]

It follows that

\[ \| \nabla w_\ell \|_{L^\infty(\tilde{\Omega}_\delta(z))} \]
\[ \leq C \delta^{1 - \frac{\gamma}{2}} \| \nabla w_\ell \|_{L^2(\tilde{\Omega}_\delta(z))} + C \delta^{\gamma} \left( \| \nabla \tilde{u}_\ell \|_{y,\tilde{\Omega}_\delta(z)} + \| \nabla \tilde{u}_\ell \|_{L^\infty(\tilde{\Omega}_\delta(z))} + \| \tilde{u}_\ell \|_{L^\infty(\tilde{\Omega}_\delta(z))} \right). \]

**Case 1.** For \( 0 \leq |z'| \leq \epsilon^\frac{1}{1 + \gamma} \).

By (2.36) and Proposition 2.1 we have

\[ \delta^{1 - \frac{\gamma}{2}} \| \nabla w_\ell \|_{L^2(\tilde{\Omega}_\delta(z'))} \leq \frac{C}{\epsilon^{1 + \gamma}} \left| \varphi^{(l)}(z', \epsilon/2 + h_1(z')) \right| + C \| \varphi^{(l)} \|_{C^{1/\gamma}(\Gamma')} + \| w_\ell \|_{L^2(\Omega_1)}, \]

and

\[ \delta^{\gamma} \| \nabla \tilde{u}_\ell \|_{y,\tilde{\Omega}_\delta(z)} \leq \frac{C}{\epsilon^{1 + \gamma}} \left| \varphi^{(l)}(z', \epsilon/2 + h_1(z')) \right| + C \| \varphi^{(l)} \|_{C^{1/\gamma}(\Gamma')} . \]
By using (2.9), (2.10), and (2.11), we can obtain
\[ \delta^\gamma \| \bar{u}_r \|_{L^\infty(\hat{\Omega}_z(z))} \leq \| \varphi^{(f)} \|_{L^\infty(\Gamma^*_1)}, \]
and
\[ \delta^\gamma \| \nabla \bar{u}_r \|_{L^\infty(\hat{\Omega}_z(z))} \leq \frac{C}{\epsilon^{1-\gamma}} \| \varphi^{(f)}(z', \epsilon/2 + h_1(z')) \| + C \| \nabla \varphi^{(f)} \|_{L^\infty(\Gamma^*_1)}. \]
Therefore,
\[ \| \nabla w_r \|_{L^\infty(\hat{\Omega}_{w_r}(\gamma))} \leq \frac{C}{\epsilon^{1-\gamma}} \| \varphi^{(f)}(z', \epsilon/2 + h_1(z')) \| + C \| \nabla \varphi^{(f)} \|_{L^2(\Omega_1)}. \]

**Case 2.** For \( \epsilon < 1/2 \) by (2.37) and Proposition 2.1, we have
\[ \delta^\gamma \| \nabla \bar{u}_r \|_{L^2(\hat{\Omega}_z(z,\gamma))} \leq \frac{C}{\epsilon^{1-\gamma}} \| \varphi^{(f)}(z', \epsilon/2 + h_1(z')) \| + C \| \nabla \varphi^{(f)} \|_{L^2(\Omega_1)}, \]
and
\[ \delta^\gamma |\nabla \bar{u}_r|_{\gamma\hat{\Omega}_z(z)} \leq \frac{C}{\epsilon^{1-\gamma}} \| \varphi^{(f)}(z', \epsilon/2 + h_1(z')) \| + C \| \nabla \varphi^{(f)} \|_{L^2(\Omega_1)}. \]

By using (2.9), (2.10), and (2.11), we can obtain
\[ \delta^\gamma \| \bar{u}_r \|_{L^\infty(\hat{\Omega}_z(z))} \leq \| \varphi^{(f)} \|_{L^\infty(\Gamma^*_1)}, \]
and
\[ \delta^\gamma \| \nabla \bar{u}_r \|_{L^\infty(\hat{\Omega}_z(z))} \leq \frac{C}{\epsilon^{1-\gamma}} \| \varphi^{(f)}(z', \epsilon/2 + h_1(z')) \| + C \| \nabla \varphi^{(f)} \|_{L^\infty(\Gamma^*_1)}. \]

It follows that
\[ \| \nabla w_r \|_{L^\infty(\hat{\Omega}_{w_r}(\gamma))} \leq \frac{C}{\epsilon^{1-\gamma}} \| \varphi^{(f)}(z', \epsilon/2 + h_1(z')) \| + C \left( \| \varphi^{(f)} \|_{L^2(\Omega_1)} + \| \nabla \varphi^{(f)} \|_{L^2(\Omega_1)} \right). \]

Noticed that \( |\nabla \bar{u}_r| \leq |\nabla w_r| + |\nabla \bar{u}_r| \). By (2.10), (2.11), (2.36), and (2.37), we obtain (2.50).

It is clear that if \( \varphi^{(f)} \neq 0 \), then
\[ |\nabla \bar{u}_r(\hat{0}_{n-1}, x_n)| \geq |\nabla \bar{u}_r(\hat{0}_{n-1}, x_n)| + |\nabla w_r(\hat{0}_{n-1}, x_n)| \geq \frac{\| \varphi^{(f)}(\hat{0}_{n-1}, x_n) \|}{C\epsilon}. \]
The Lemma 2.3 is proved with \( \psi \equiv 0 \). \( \square \)

**Proof of Theorem 7.7.** By Lemma 2.1, Lemma 2.2, and Lemma 2.3 we have for \( x \in \Omega_{1/2} \) as in (1.7),
\[ |\nabla u(x)| \leq \sum_{\ell=1}^n |\nabla \bar{u}_r(x)| \leq \frac{C |\varphi(x', \epsilon/2 + h_1(x')) - \varphi(x', -\epsilon/2 + h_2(x'))|}{\epsilon + |x'|^{1+\gamma}} + C \left( \| \varphi \|_{C^{1,\gamma}(\Gamma^*_1)} + \| \psi \|_{C^{1,\gamma}(\Gamma^*_1)} + \| \nabla \varphi \|_{L^2(\Omega_1)} \right). \]

If \( \varphi^{(f)}(\hat{0}_{n-1}, \epsilon/2) \neq \psi^{(f)}(\hat{0}_{n-1}, -\epsilon/2) \) for some integer \( \ell \), then by Lemma 2.3 we can obtain
\[ |\nabla u(\hat{0}_{n-1}, x_n)| \geq \frac{|\varphi^{(f)}(\hat{0}_{n-1}, \epsilon/2) - \psi^{(f)}(\hat{0}_{n-1}, -\epsilon/2)|}{C\epsilon} \quad \forall x_n \in \left( -\frac{\epsilon}{2}, \frac{\epsilon}{2} \right). \]
The proof of Theorem 1.1 is completed. \( \square \)
3 Proof of Corollary 1.1

Since the Lamé systems as in (1.14) has many more applications in practice, such as shear modulus in high contrast linear elastic composites. This section establishes the gradient estimates of Lamé systems under the assumptions in Definition 1.4. We give a sketched proof of Corollary 1.1 and only list its main ingredients. In order to make the proof more clear and concise, we will prove the case when \( n = 2 \).

Let \( \nu_1 = (\nu^{(1)}_1, \nu^{(2)}_1) = (\nu^{(1)}_1, 0) \) be a weak solution of

\[
\begin{aligned}
\mathcal{L}_{\lambda_1, \mu_1} \nu_1 &= \nabla \cdot (C^0 e(\nu_1)) = 0, \quad \text{in } \Omega_1, \\
\nu_1 &= (\varphi^{(1)}, 0), \quad \text{on } \Gamma^+_1, \\
\nu_1 &= (\psi^{(1)}, 0), \quad \text{on } \Gamma^-_1,
\end{aligned}
\]

and \( \nu_2 = (\nu^{(1)}_2, \nu^{(2)}_2) = (0, \nu^{(2)}) \) be a weak solution of

\[
\begin{aligned}
\mathcal{L}_{\lambda_1, \mu_1} \nu_2 &= \nabla \cdot (C^0 e(\nu_2)) = 0, \quad \text{in } \Omega_1, \\
\nu_2 &= (0, \varphi^{(2)}), \quad \text{on } \Gamma^+_1, \\
\nu_2 &= (0, \psi^{(1)}), \quad \text{on } \Gamma^-_1.
\end{aligned}
\]

It obvious that

\[
u = \nu_1 + \nu_2 \quad \text{and} \quad \nabla \nu = \nabla \nu_1 + \nabla \nu_2.
\]

Then we still construct the auxiliary function \( \tilde{\nu}_\ell \) for \( \ell = 1, 2 \) as shown in (2.9) in Section 2:

\[
\begin{aligned}
\tilde{\nu}_1 :&= (\varphi^{(1)}(x_1, \frac{\varepsilon}{2} + h_1(x_1))u(x) + \psi^{(1)}(x_1, -\frac{\varepsilon}{2} + h_2(x_1))(1 - u(x)), 0), \\
\tilde{\nu}_2 :&= (0, \varphi^{(2)}(x_1, \frac{\varepsilon}{2} + h_1(x_1))u(x) + \psi^{(2)}(x_1, -\frac{\varepsilon}{2} + h_2(x_1))(1 - u(x))).
\end{aligned}
\]

So \( |\nabla \tilde{\nu}_\ell| \) also has the gradient estimates as in (2.10)–(2.11), and the Hölder semi-norm estimates of \( \nabla \tilde{\nu}_\ell \) as in Proposition 2.1.

Denote \( \omega_\ell = \nu_\ell - \tilde{\nu}_\ell \) for any \( \ell = 1, 2 \), which satisfies the following boundary value problem:

\[
\begin{aligned}
\mathcal{L}_{\lambda_1, \mu_1} \omega_\ell &= -\mathcal{L}_{\lambda_1, \mu_1} \tilde{\nu}_\ell, \quad \text{in } \Omega_1, \\
\omega_\ell &= 0, \quad \text{on } \Gamma^+_1, \\
\omega_\ell &= 0, \quad \text{on } \Gamma^-_1.
\end{aligned}
\]

Because the result in Corollary 1.1 independent of \( \ell \), we might as well consider only the case of \( \ell = 1 \).

Lemma 3.1. Under the hypotheses of Lemma 2.7 and in addition that \( w_1 \) is the weak solution of (3.5), then there exists a positive constant \( C \) independent of \( \varepsilon \), such that,

\[
\int_{\Omega_{1/2}} |\nabla w_1|^2 \, dx \leq C \left( \|w_1\|^2_{L^2(\Omega_1)} + \|\varphi^{(1)}\|^2_{C^{1+2}(\Gamma^+_1)} + \|\psi^{(1)}\|^2_{C^{1+2}(\Gamma^-_1)} \right).
\]
Proof. Multiplying (3.5) by $w_1$ and making use of the integration by parts in $\Omega_{1/2}$, in view of $w_1 = 0$ on $\Gamma_1$, we have

$$\int_{\Omega_{1/2}} (C^0 e (w_1), e (w_1)) \, dx = \int_{\Omega_{1/2}} (\nabla \cdot (C^0 e (\bar{u}_1))) \cdot w_1 \, dx.$$  \hspace{1cm} (3.7)

For the right hand side of (3.7), noticing that $\partial_{22} \bar{u}_1^{(1)} = 0$ in $\Omega_{1/2}$, and by using integration by parts, (2.10) and (2.17), one has

$$\begin{align*}
\left| \int_{\Omega_{1/2}} (\nabla \cdot (C^0 e (\bar{u}_1))) \cdot w_1 \, dx \right| & \leq C \left| \int_{\Omega_{1/2}} \partial_1 (\partial_1 \bar{u}_1^{(1)}) w_1^{(1)} \, dx + \int_{\Omega_{1/2}} \partial_2 (\partial_1 \bar{u}_1^{(1)}) w_2^{(1)} \, dx \right| \\
& \leq C \left( \int_{\Omega_{1/2}} |\nabla w_1| \, dx + \int \left. |w_1| |\partial_1 \bar{u}_1^{(1)}| \, dx \right|_{\frac{1}{2} + h_2 (x_1) < x_2 < \frac{1}{2} + h_1 (x_1)} \right) \\
& \leq \frac{1}{2} \int_{\Omega_{1/2}} |\nabla w_1|^2 \, dx + C \left( ||\bar{u}_1||^2_{L^2 (\Omega_1)} + ||\psi^{(1)}||^2_{C^{1,y} (\Gamma_1^\ast)} + ||\psi^{(1)}||^2_{C^{1,y} (\Gamma_1^\ast)} \right). \hspace{1cm} (3.8)
\end{align*}$$

For the left hand side of (3.7), it follows from strong ellipticity condition as in (1.10) there exists a positive constant $\lambda$, such that,

$$\lambda \int_{\Omega_{1/2}} |\nabla w_1|^2 \, dx \leq \int_{\Omega_{1/2}} (C^0 e (w_1), e (w_1)) \, dx. \hspace{1cm} (3.9)$$

By (3.7) - (3.9), we obtain (3.6). \hfill \square

From the definition of $\bar{u}_1$, we can get

$$C^0 e (\bar{u}_1) = \left( \begin{array}{c} (\lambda_1 + 2 \mu_1) \partial_1 \bar{u}_1^{(1)} \\
\mu_1 \partial_2 \bar{u}_1^{(1)} \end{array} \right).$$

Let

$$M := \int_{\Omega_{1/2}} e (\bar{u}_1 (y)) \, dy := \frac{1}{|\Omega_{1/2}|} \int_{\Omega_{1/2}} C^0 e (\bar{u}_1 (y)) \, dy.$$  \hspace{1cm} (3.10)

It follows that from (3.5) that $w_1$ satisfy

$$L_{1, \mu_1} w_1 = -L_{1, \mu_1} (\bar{u}_1 - M). \hspace{1cm} (3.11)$$

Lemma 3.2. Under the hypotheses of Lemma 2.2 and in addition that $w_1$ is the weak solution of (3.5), then there exists a positive constant $C$ independent of $\varepsilon$, such that, for $0 \leq |z_1| \leq \varepsilon \frac{1}{\mu_1}$,

$$\int_{\Omega_{1/2}(z_1)} |\nabla w_1|^2 \, dx \leq C \varepsilon^{\frac{2}{\mu_1}} \left( ||\psi^{(1)} (z_1, \frac{\varepsilon}{2} + h_2 (z_1)) - \psi^{(1)} (z_1, \frac{\varepsilon}{2} + h_2 (z_1))||^2 \right).$$

\textbf{Proof.}
Proof. In the view of Proposition 2.1 and (3.10), one has

\begin{equation}
\eta
\end{equation}

Multiplying (3.11) by \(G^2\), consider the following cut-off function \(\eta(x_1)\): for \(0 < t < s < 1/2\), let \(\eta\) be a cut-off function satisfying

\[
\eta(x_1) = \begin{cases} 
1 & \text{if } |x_1 - z_1| < t, \\
0 & \text{if } |x_1 - z_1| > s,
\end{cases}
\quad \text{and } |\eta'(x_1)| \leq \frac{2}{s-t}.
\]

Multiplying (3.11) by \(\eta^2 w_1\) and using the integration by parts, one has

\begin{equation}
\int_{\Omega(z)} (C^0 e(w_1), e(\eta^2 w_1)) \, dx = -\int_{\Omega(z)} \left( C^0 e(\bar{u}_1) - \mathcal{M}, \nabla(\eta^2 w_1) \right) \, dx.
\end{equation}

By treating (3.14) in the same way as proposition 2.1 in [15], we get

\begin{equation}
\int_{\Omega(z)} |\nabla w_1|^2 \, dx \leq \frac{C}{(s-t)^2} \int_{\Omega(z)} |w_1|^2 \, dx + C \int_{\Omega(z)} |C^0 e(\bar{u}_1) - \mathcal{M}|^2 \, dx.
\end{equation}

**Case 1.** If \(|z_1| < \frac{1}{e^{1/\gamma'}}\) and \(0 < s < e^{1/\gamma'}\), we have \(\varepsilon \leq \delta(z_1) \leq C \varepsilon\). By a direct calculation, we have

\begin{equation}
\int_{\Omega(z)} |w_1|^2 \, dx = \int_{\Omega(z)} \left| \int_{-\frac{s}{1+h_2(x_1)}}^{s} \partial_2 w_1(x_1, x_2) \, dx_2 \right|^2 \, dx \leq C \varepsilon^2 \int_{\Omega(z)} |\nabla w_1|^2 \, dx.
\end{equation}

In the view of Proposition 2.1 and (3.10), one has

\[
\int_{\Omega(z)} |C^0 e(\bar{u}_1) - \mathcal{M}|^2 \, dx \leq C |\nabla \bar{u}_1|^2 \int_{\Omega(z)} (s^2 + \delta(z_1)) \, dx \\
\leq C \left( \frac{\varepsilon^2}{2} + h_1(z_1) \right)^2 + s^2 |\nu|^2_{C^{1,\gamma}(\Gamma_1)}
\]

(3.17)

**Case 2.** For \(e^{1/\gamma'} \leq |z_1| \leq \frac{1}{2}\) and \(0 < s < |z_1|\), we have \(\frac{1}{e}|z_1|^{1+\gamma} \leq \delta(z_1) \leq C |z_1|^{1+\gamma}\). Estimates (3.16) and (3.17) become, respectively,

\begin{equation}
\int_{\Omega(z)} |w_1|^2 \, dx \leq C |z_1|^{2(1+\gamma)} \int_{\Omega(z)} |\nabla w_1|^2 \, dx, \quad \text{if } 0 < s < \frac{2}{3} |z'|.
\end{equation}
and
\[ \int_{\tilde{\Omega}_{\varepsilon}(z)} |C^0 e(\tilde{u}_1) - M|^2 \, dx \leq C \left( \left| \varphi^{(1)}(z', \frac{E}{2} + h_1(z')) \right|^2 + s^2 \left\| \varphi^{(1)} \right\|_{C^{1,\gamma}(\Gamma_1')}^2 \right) \]
\[ \left( \frac{s^3}{|z_1|^3+\gamma} + \frac{s}{|z_1|^{-1-\gamma}} + \frac{s^{3-2\gamma}}{|z_1|^{1-\gamma-2\gamma}^2} + \frac{s^{3+2\gamma}}{|z_1|^{1+2\gamma}^2} \right). \]

(3.19)

Next, similar to the proof of Lemma (2.2), we complete the proof of (3.12) and (3.13). □

**Lemma 3.3.** Under the hypotheses of Lemma 2.3 and in addition that \( w_1 \) is the weak solution of (3.5), then there exists a positive constant \( C \) independent of \( \varepsilon \) such that, for \( |z_1| \leq \varepsilon^{1+\gamma} \),
\[ |\nabla w_1(z_1, z_2)| \leq C \varepsilon^{-1+\gamma} \left| \varphi^{(1)}(z, \frac{E}{2} + h_1(z)) - \psi^{(1)}(z, \frac{E}{2} + h_2(z)) \right| \]
\[ + C \varepsilon^{-1+\gamma} \left( \left\| w_1 \right\|_{L^2(\Omega_1)} + \left\| \varphi^{(1)} \right\|_{C^{1,\gamma}(\Gamma')} + \left\| \psi^{(1)} \right\|_{C^{1,\gamma}(\Gamma')} \right), \]
\[(3.20)\]

and for \( \varepsilon^{1+\gamma} < |z_1| < \frac{1}{2} \),
\[ |\nabla w_1(z_1, z_2)| \leq C |z_1|^{-1} \left| \varphi^{(1)}(z, \frac{E}{2} + h_1(z')) - \psi^{(1)}(z, \frac{E}{2} + h_2(z)) \right| \]
\[ + C |z_1|^{-\gamma} \left( \left\| w_1 \right\|_{L^2(\Omega_1)} + \left\| \varphi^{(1)} \right\|_{C^{1,\gamma}(\Gamma')} + \left\| \psi^{(1)} \right\|_{C^{1,\gamma}(\Gamma')} \right). \]

**Proof.** Let \( Q_1 \) be as in (2.51) and \( C_{i,j,k,l} \) be as in (1.15). For any \((y_1, y_2) \in Q_1\), we denote
\[ \tilde{w}_1(y_1, y_2) := w_1(\delta(z_1)y_1 + z_1, \delta(z_1)y_2), \quad \tilde{u}_1(y_1, y_2) := \tilde{u}_1(\delta(z_1)y_1 + z_1, \delta(z_1)y_2), \]
then, after the same coordinate transformation as in Lemma 2.3, we can obtain that \( \tilde{w}_1 \) satisfies
\[ \begin{cases} - \sum_{j,k,l} \partial_j (C_{j,k,l} \partial_j \tilde{w}_1) = \sum_{j,k,l} \partial_j (C_{j,k,l} \partial_j \tilde{u}_1) & \text{in } Q_1, \\ \tilde{w}_1 = 0 & \text{on } \Gamma_1'. \end{cases} \]
\[(3.22)\]

Similar to the proof in Lemma 2.3, recalling back to the original region \( \tilde{\Omega}_{\delta(z_1)}(z) \), one has
\[ \left\| \nabla w_1 \right\|_{L^\infty(\tilde{\Omega}_{\delta(z_1)}(z))} \leq C \delta(z_1)^{-1} \left\| \nabla \tilde{w}_1 \right\|_{L^2(\tilde{\Omega}_{\delta(z_1)}(z))} + C \delta(z_1)^{-\gamma} \left\| \nabla \tilde{u}_1 \right\|_{L^2(\tilde{\Omega}_{\delta(z_1)}(z))}. \]

(3.23)

Therefore, by using (3.12) and (3.13), Proposition (2.1), we proved the (3.20) and (3.23). □

**Proof of Corollary 1.1.** Consequently, by (2.10) and (2.11), we have for sufficiently small \( \varepsilon \) and \( z \in \Omega_{1/2} \),
\[ |\nabla v_1(z)| \leq \frac{C \left| \varphi^{(1)}(z_1, \frac{E}{2} + h_1(z)) - \psi^{(1)}(z_1, \frac{E}{2} + h_2(z)) \right|}{\varepsilon + |z_1|^{1+\gamma}} \]
\[ + C \left( \left\| v_1 \right\|_{L^2(\Omega_1)} + \left\| \varphi^{(1)} \right\|_{C^{1,\gamma}(\Gamma')} + \left\| \psi^{(1)} \right\|_{C^{1,\gamma}(\Gamma')} \right). \]

(3.24)

By (3.3), we have for \( x \in \Omega_{1/2} \) as in (1.7),
\[ |\nabla u(x)| \leq |\nabla v_1(x)| + |\nabla v_2(x)| \]
Lipschitz domain

Definition 4.1. Let \( Q \) be a Lipschitz domain in \( \text{Morrey space } L^p \) endowed with the norm defined by the set \( Q \cap B(x_0, \rho) \) in the proof of Theorem 1.1, with the help of the Campanato’s approach, Schauder estimates and (see [19, Chapter 5]).

4.1 Proof of Theorem 2.1

If \( C \)

Appendix: Proof of \( C^{1,\gamma} \) estimates and \( W^{1,p} \) estimates

In this section, we show the proofs of the Theorem 2.1 and Theorem 2.2 which play a key role in the proof of Theorem 1.1 with the help of the Campanato’s approach, Schauder estimates and \( L^p \) estimates for elliptic systems in [19].

4.1 Proof of Theorem 2.1

To prove Theorem 2.1 we first introduce the definition of the spaces of Morrey and Campanato (see [19] Chapter 5).

Let \( Q \subset \mathbb{R}^n \) be any domain and \( \rho > 0 \), for any \( x_0 \in Q \) we use the symbol \( Q(x_0; \rho) \) to denote the set \( Q \cap B(x_0, \rho) \) and the symbol \( \text{dim } Q \) to denote the diameter of \( Q \). The domain \( \Omega \) is said to be a Lipschitz domain if \( \partial \Omega \) is Lipschitz defined as in Definition 1.2.

Definition 4.1. Let \( Q \) be a Lipschitz domain in \( \mathbb{R}^n \). For every \( 1 \leq p \leq +\infty \), \( \lambda > 0 \) define the Morrey space \( L^{p,\lambda}(Q) \),

\[
L^{p,\lambda}(Q) := \left\{ u \in L^p(Q) : \sup_{x_0 \in Q, \rho > 0} \rho^{-\lambda} \int_{Q(x_0, \rho)} |u|^p \, dx < +\infty \right\},
\]

endowed with the norm defined by

\[
\|u\|_{L^{p,\lambda}(Q)} := \left( \sup_{x_0 \in Q, \rho > 0} \rho^{-\lambda} \int_{Q(x_0, \rho)} |u|^p \, dx \right)^{\frac{1}{p}}.
\]

Definition 4.2. Let \( Q \) be a Lipschitz domain in \( \mathbb{R}^n \). For every \( 1 \leq p \leq +\infty \), \( \lambda > 0 \) define the Campanato space \( L^{p,\lambda}(Q) \),

\[
L^{p,\lambda}(Q) := \left\{ u \in L^p(Q) : \sup_{x_0 \in Q, \rho > 0} \rho^{-\lambda} \int_{Q(x_0, \rho)} |u - u_{x_0, \rho}|^p \, dx < +\infty \right\},
\]

endowed with the norm defined by

\[
\|u\|_{L^{p,\lambda}(Q)} := [u]_{p,\lambda} + \|u\|_{L^p}
\]

(4.1)

where \( u_{x_0, \rho} := \frac{1}{|Q(x_0, \rho)|} \int_{Q(x_0, \rho)} u \, dx \).
The follows lemma is just [19] Theorem 5.5.

**Lemma 4.1.** For $n < \lambda \leq n + p$ and $\gamma = \frac{\lambda - n}{p}$ we have $L^{p, \lambda}(Q) = C^{0, \gamma}(\Omega)$. Moreover the Hölder semi-norm $[u]_{0, \gamma}$ as in (1.2) is equivalent to $[u]_{p, \lambda}$ as in (4.1). If $\lambda > n + p$ and $u \in L^{p, \lambda}(\Omega)$, then $u$ is constant.

Referring to [19] Theorem 5.14, we can obtain the following interior estimates. In what follows, for any domain $Q \subset \mathbb{R}^n$ we denote by the symbol $L^{p, \lambda}_{\text{loc}}(Q)$ the set of all functions $u$ which satisfy for any $Q' \subset \subset Q$, $\|u\|_{L^{p, \lambda}(Q')} < \infty$.

**Lemma 4.2.** Let $Q$ is a Lipschitz domain in $\mathbb{R}^n$. Let $A_{ij}^{\alpha \beta}$ be constant and satisfy (1.10) and (1.16). Let $0 < \gamma < 1$, $\mu := n + 2\gamma - 2$ and for any $\alpha = 1, \ldots, n$, $i = 1, \ldots, m$, $F_i^{(\alpha)} \in L^{2, \mu + 2}(Q)$ and $H^{(i)} \in L^{2, \mu}(Q)$. Let $w = (w^{(1)}, \ldots, w^{(m)}) \in W^{1, 2}(Q \subset \mathbb{R}^n, \mathbb{R}^m)$ be a weak solution of

$$
\sum_{a, \beta, j} \partial_a (A_{ij}^{\alpha \beta} \partial_{\beta} w^{(j)}) = H^{(i)} - \sum_{\alpha} \partial_\alpha F_i^{(\alpha)}, \quad \text{in } Q.
$$

Then $\partial_\alpha w^{(i)} \in L^{2, \mu}_{\text{loc}}(Q)$ for any $\alpha = 1, \ldots, n$ and $i = 1, \ldots, m$, and there exists a positive constant $C$ depending on $n, m, \gamma, R, \lambda, \Lambda$ such that, for $B_R(x_0) \subset Q$,

$$
[\nabla w]_{\gamma, B_R} := \max_{\alpha, j} |\partial_\alpha w^{(i)}|_{\gamma, B_R} \leq C \left( \frac{1}{R^{1+\gamma}} \|w\|_{L^{\infty}(B_R)} + |F|_{\gamma, B_R} + \|H\|_{L^{2, \mu}(B_R)} \right),
$$

where $\|H\|_{L^{2, \mu}(B_R)} := \max_i \|H^{(i)}\|_{L^{2, \mu}(B_R)}$.

**Proof.** By Proposition 4.1 we have $F_i^{(\alpha)} \in L^{2, n + 2\gamma}(Q)$. For a given ball $B_R := B_R(x_0) \subset Q$, the decomposition of $w$ is as follows

$$
w = w_1 + w_2, \quad \text{in } B_R,
$$

where $w_1$ and $w_2$ satisfy, respectively,

$$
\begin{cases}
\sum_{\alpha, \beta, j} \partial_\alpha (A_{ij}^{\alpha \beta} \partial_{\beta} w_1^{(j)}) = 0, & \text{in } B_R, \\
w_1 = w, & \text{on } \partial B_R,
\end{cases}
$$

and

$$
\begin{cases}
\sum_{\alpha, \beta, j} \partial_\alpha (A_{ij}^{\alpha \beta} \partial_{\beta} w_2^{(j)}) = H^{(i)} - \sum_{\alpha} \partial_\alpha (F_i^{(\alpha)} - (F_i^{(\alpha)})_R), & \text{in } B_R, \\
w_2 = 0, & \text{on } \partial B_R.
\end{cases}
$$

where $(F_i^{(\alpha)})_R = \frac{1}{|B_R|} \int_{B_R} F_i^{(\alpha)} \, dx$.

By [19] Proposition 5.8, for $0 < \rho < \frac{3R}{4}$ we have

$$
\int_{B_\rho} |\nabla w_1 - (\nabla w_1)_R|^2 \, dx \leq C \left( \frac{\rho}{R} \right)^{n+2} \int_{B_{3R/4}} |\nabla w_1 - (\nabla w_1)_R|^2 \, dx,
$$
and for \( w_2 \), multiplying (4.6) by \( w_2 \) and using the integration by parts, one has

\[
\int_{B_{3R/4}} |\nabla w_2|^2 \, dx \leq CR^{\mu+2} \left( [F]_{L^2;\mu+2}(B_R) + ||H||_{L^2;\mu}(B_R) \right).
\]

Consequently,

\[
\int_{B_p} |\nabla w - (\nabla w)_p|^2 \, dx \leq \int_{B_p} |\nabla w_1 - (\nabla w_1)_p + \nabla w_2 - (\nabla w_2)_p|^2 \, dx
\]

\[
\leq C_1 \left( \frac{\rho}{R} \right)^n \int_{B_{3R/4}} |\nabla w_1 - (\nabla w_1)_R|^2 \, dx + C_2 \int_{B_{3R/4}} |\nabla w_2 - (\nabla w_2)_{3R/4}|^2 \, dx
\]

\[
\leq C_1 \left( \frac{\rho}{R} \right)^n \int_{B_{3R/4}} |\nabla w - (\nabla w)_{3R/4}|^2 \, dx + C_2 \int_{B_{3R/4}} |\nabla w_2|^2 \, dx.
\]

Inserting (4.8) in (4.9) and using [19] Lemma 5.13, we obtain

\[
\int_{B_p} |\nabla w - (\nabla w)_p|^2 \, dx \leq C \left[ \left( \frac{\rho}{R} \right)^{\mu+2} \int_{B_{3R/4}} |\nabla w|^2 \, dx + \rho^{\mu+2} ([F]_{L^2;\mu+2}(B_R) + ||H||_{L^2;\mu}(B_R)) \right]
\]

We assert that the following inequality holds:

\[
\int_{B_{3R/4}} |\nabla w|^2 \, dx \leq C \left( \frac{1}{R^2} \int_{B_R} |w|^2 \, dx + R^{\mu+2} ([F]_{L^2;\mu+2}(B_R) + ||H||_{L^2;\mu}(B_R)) \right)
\]

where \( C \) depends on \( \lambda \) and boundedness of coefficients of (4.2).

Actually, define a cut-off function \( \zeta \in C_c^{\infty}(Q) \) as follows, \( 0 \leq \zeta(x) \leq 1 \),

\[
\zeta(x) = \begin{cases} 1, & \text{on } B_{3R/4}, \\ 0, & \text{on } B_R \setminus B_{3R/4}, \end{cases} \quad |\nabla \zeta(x)| \leq \frac{8}{R},
\]

and choose as test function \( \zeta w^2 \) into (2.1). From strong ellipticity condition (1.10), we obtain

\[
\lambda \int_{B_{3R/4}} \zeta^2 |\nabla w|^2 \, dx \leq \sum_{\alpha,\beta, I, J} \int_{B_{3R/4}} \zeta^2 A^{\alpha\beta}_{ij} \partial_{\alpha} w^{(j)} \partial_{\alpha} w^{(i)} \, dx
\]

\[
= - \sum_{\alpha,\beta, I, J} \int_{B_{3R/4}} 2 \zeta w^{(i)} A^{\alpha\beta}_{ij} \partial_{\beta} w^{(j)} \partial_{\alpha} \zeta \, dx - \sum_{\alpha, I, J} \int_{B_{3R/4}} B^{\alpha}_{ij} (w^{(j)} \zeta^2) \, dx
\]

\[
- \sum_{\alpha, I, J} \int_{B_{3R/4}} w^{(i)} \zeta^2 (C^{\alpha\beta}_{ij} \partial_{\beta} w^{(j)} + D_{ij} w^{(j)}) \, dx + \sum_{I} \int_{B_{3R/4}} w^{(i)} \zeta^2 H^{(i)} \, dx
\]

\[
+ \sum_{\alpha, \beta} \int_{B_{3R/4}} (F_{\beta}^{(a)} - (F_{\beta}^{(a)})_{3R/4}) \partial_{\alpha} (w^{(i)} \zeta^2) \, dx.
\]

Then by using Cauchy’s inequality and the properties of \( \zeta \), we proved (4.11).

Therefore, using (4.10) and (4.11), we have

\[
\frac{1}{\rho^{\mu+2}} \int_{B_p} |\nabla w - (\nabla w)_p|^2 \, dx \leq C \left( \frac{1}{R^2} \int_{B_R} |w|^2 \, dx + [F]_{L^2;\mu+2}(B_R) + ||H||_{L^2;\mu}(B_R) \right)
\]
\[ \leq C \frac{1}{R^{2+2\gamma}} \int_{B_R} |w|^2 \, dx + |F|_{L^{2\mu+2}(B_R)}^2 + ||H||^2_{L^{2\mu}(B_R)}. \]

where \( C \) depends on \( n, \gamma \). For any \( x = (x', x_n) \in B_{R/2} \) and \( 0 < \rho \leq R/4 \), we have

\[ \frac{1}{R^{\mu+2}} \int_{B_{\rho}(x) \cap B_{R/2}} |\nabla w - (\nabla w)_{B_{\rho}(x) \cap B_{R/2}}|^2 \, dy \leq \frac{1}{R^{\mu+2}} \int_{B_{\rho}} |\nabla w - (\nabla w)_{\rho}|^2 \, dy \]

\[ \leq C \left( \frac{1}{R^{2+2\gamma}} \int_{B_R} |w|^2 \, dx + |F|_{L^{2\mu+2}(B_R)}^2 + ||H||^2_{L^{2\mu}(B_R)} \right). \]

By the equivalence between the Hölder space and the Campanato space (see Lemma 4.1), this implies that (4.3) holds. □

Next, we give the boundary estimate on half space \( \partial \mathbb{R}^n_+ \). Consider

\[ \sum_{a,\beta,j} \partial_a (A_{ij}^a \partial_\beta w^{(j)}) = H^{(i)} - \sum_a \partial_a F_i^{(a)}, \text{ in } \mathbb{R}^n, \]

\[ w = 0, \quad \text{on } \partial \mathbb{R}^n_+. \]

**Corollary 4.1.** In the hypothesis of Lemma 4.2, let \( w \in W^{1,2}(\mathbb{R}^n_+; \mathbb{R}^m) \) be the solution of (4.12), for any \( x_0 \in \partial \mathbb{R}^n_+ \) and \( B^+_R(x_0) := B_R(x_0) \cap \partial \mathbb{R}^n_+ \), then there is a constant \( C \) only depended to \( n, \gamma, \lambda, \Lambda \) such that,

\[ |\nabla w|_{Y, B^+_R(x_0)} \leq C \left( \frac{1}{R^{1+\gamma}} ||w||_{L^\infty(B^+_R(x_0))} + |F|_{Y, B^+_R(x_0)} + ||H||_{L^{2\mu}(B^+_R(x_0))} \right). \]

**Proof.** Firstly, we decompose \( w = w_1 + w_2 \) as shown in (4.4), where \( w_1, w_2 \) satisfy (4.5) and (4.6) in \( B^+_R(x_0) \) respectively. It follows from [19, (5.38) in Theorem 5.21] that (4.7) holds for \( w_1 \). The proof of the corollary can be obtained by the method in the proof of the Lemma 4.2 and some elementary arguments. We omit the details. □

**Proof of Theorem 2.7** Since \( \Gamma \) is \( C^{1,\gamma} \), then for any \( x_0 \in \Gamma \), there exists a neighbourhood \( U \) of \( x_0 \) and a homeomorphism \( \Psi \in C^{1,\gamma}(U) \) such that

\[ \Psi(U \cap Q) = B^+_1 := \{ y \in B_1(0), y_n > 0 \}, \quad \Psi(U \cap \Gamma) = \partial B^+_1 \cap \{ y \in \mathbb{R}^n : y_n = 0 \}. \]

Under transformation \( y = \Psi(x) = (\psi^{(1)}(x), \cdots, \psi^{(n)}(x)) \), we denote

\[ \mathcal{W}(y) := w(\Psi^{-1}(y)), \quad \mathcal{J}(y) := \left. \frac{\partial \left( (\psi^{(1)}(x), \cdots, (\psi^{(n)}(x)) \right)}{\partial (y_1, \cdots, y_n)} \right|, \quad |\mathcal{J}(y)| := \det \mathcal{J}(y), \]

and

\[ \mathcal{A}_{ij}^{\alpha \beta}(y) := \sum_{a,\beta,j} \partial_a \partial_\beta \left( \psi^{-1}(y) \right) \left( \partial_a (\psi^{-1})^{(\beta)}(y) \right)^{-1} \partial_j \left( \psi^{(a)}(y) \right) \left( \psi^{-1}(y) \right), \]

\[ \mathcal{B}_{ij}^{\alpha}(y) := \sum_{a} \mathcal{J}(y) \partial_a \left( \psi^{-1}(y) \right) \partial_j \left( \psi^{(a)}(y) \right) \left( \psi^{-1}(y) \right). \]
Then, by Corollary 4.1 we have that for $0 < \hat{\alpha} < \hat{\beta} = 1, \ldots, n$. Then (2.1) becomes
\[
\sum_{\alpha, \beta, i, j} \partial_\alpha (A_{ij}^{\beta}(\partial_\beta W^{(j)}) + B_{ij}^{\alpha}(W^{(j)})) + C_{ij}^{\beta} \partial_\beta W^{(j)} + D_{ij} W^{(j)} = \mathcal{H}^{(i)} - \sum_\alpha \partial_\alpha F_i^{(\alpha)} \text{ in } B_R^+,
\]
and $W = 0$ on $\partial B_R^+ \cap \partial \mathbb{R}_+^n$. Let $y_0 = \Psi(x_0)$. Freeze the coefficients and rewrite the above formula in the form
\[
\sum_{\alpha, \beta, j} \partial_\alpha (A_{ij}^{\beta}(y_0) \partial_\beta W^{(j)}(y)) = \sum_{\alpha, \beta, j} -\partial_\alpha ((A_{ij}^{\beta}(y_0) \partial_\beta W^{(j)}(y)) + B_{ij}^{\alpha}(y) W^{(j)}(y)) + \sum_{\alpha, \beta, j} (C_{ij}^{\beta}(y) \partial_\beta W^{(j)}(y) - D_{ij}(y) W^{(j)}(y) + \mathcal{H}^{(i)}(y) - \partial_\alpha F_i^{(\alpha)}(y)).
\]
Then, by Corollary 4.1 we have that for $0 < R < 1$,
\[
[\nabla W]_{y, B_R^+} \leq C\left(\frac{1}{R^{1+\gamma}} \lVert W \rVert_{L^{\infty}(B_R^+)} + [\mathcal{F}]_{y, B_R^+} + C \sum_{\beta, j} \left(\left( (A_{ij}^{\beta}(y) - A_{ij}^{\beta}(y_0)) \partial_\beta W^{(j)} \right)_{y, B_R^+} + [B_{ij}^{\alpha}(y) W^{(j)}]_{y, B_R^+} \right) + \sum_{\beta, j} \left( C_{ij}^{\beta}(y) \partial_\beta W^{(j)} - D_{ij}(y) W^{(j)} \right)_{L^{2}(B_R^+)} + \lVert \mathcal{H} \rVert_{L^{2}(B_R^+)} \right).
\]
Since $A_{ij}^{\beta}(y)$, $B_{ij}^{\alpha}(y)$, $C_{ij}^{\beta}(y)$, $D_{ij}(y)$, $\mathcal{H}(y) \in C^\gamma(B_R^+)$, we have
\[
\sum_{\beta, j} \left( (A_{ij}^{\beta}(y) - A_{ij}^{\beta}(y_0)) \partial_\beta W^{(j)} \right)_{y, B_R^+} \leq C \left( R^{\gamma} \lVert \nabla W \rVert_{y, B_R^+} + \lVert \nabla W \rVert_{L^{\infty}(B_R^+)} \right),
\]
\[
\sum_{j} \left[ B_{ij}^{\alpha}(y) W^{(j)} \right]_{y, B_R^+} \leq CR^\gamma \lVert W \rVert_{L^{\infty}(B_R^+)},
\]
and
\[
\sum_{\beta, j} \left( C_{ij}^{\beta}(y) \partial_\beta W^{(j)} - D_{ij}(y) W^{(j)} \right)_{L^{2}(B_R^+)} \leq C \|\nabla W\|_{L^{\infty}(B_R^+)} + \|W\|_{L^{\infty}(B_R^+)}.\]
In view of the interpolation inequality ([20, Lemma 6.35]), we can obtain
\[
\lVert \nabla W \rVert_{L^{\infty}(B_R^+)} \leq R^{\gamma} \lVert \nabla W \rVert_{y, B_R^+} + C \lVert W \rVert_{L^{\infty}(B_R^+)},
\]
where $C = C(n)$. Hence,
\[
\lVert \nabla W \rVert_{y, B_{R^2/2}^+} \leq C\left( \frac{1}{R^{1+\gamma}} \lVert W \rVert_{L^{\infty}(B_R^+)} + R^{\gamma} \lVert \nabla W \rVert_{y, B_R^+} + [\mathcal{F}]_{y, B_R^+} + \lVert \mathcal{H} \rVert_{L^{2}(B_R^+)} \right).
\]
Since $\Psi$ is a homeomorphism, thus, changing back to the variable $x$, we obtain
\[
[\nabla w]_{\gamma, N'} \leq C \left( \frac{1}{R^{1+\gamma}} \|\nabla w\|_{L^{\infty}(N)} + R^\gamma [\nabla w]_{\gamma, N} + |\nabla w|_{L^{2\mu}(N)} \right) .
\]
where $N = \Psi^{-1}(B_2^{R'})$, $N' = \Psi^{-1}(B_{2R'}^{R'})$ and $C = C(n, \gamma, \Psi)$. Furthermore, there exists a constant $0 < \sigma < 1$, independent on $R$, such that $B_{\sigma R}(x_0) \cap Q \subset N'$.

Therefore, recalling that $\Gamma \subset \partial Q$ is a boundary portion, for any domain $Q' \subset \subset Q \cup \Gamma$ and for each $x_0 \in Q' \cap \Gamma$, there exist $R_0 := R_0(x_0)$ and $C_0 = C_0(n, \gamma, x_0)$ such that,

\[
(4.13) \quad [\nabla w]_{\gamma, B_{R_0}(x_0) \cap Q'} \leq C_0 \left( \frac{R_0^2}{R_0^{1+\gamma}} [ \nabla w]_{\gamma, Q'} + \frac{1}{R_0^{1+\gamma}} \|\nabla w\|_{L^{\infty}(Q')} + [F]_{\gamma, Q'} + ||H||_{L^{2\mu}(Q')} \right) .
\]

By using the boundary estimates (4.13) near $\Gamma$, the finite covering theorem, and Lemma 4.2 we can obtain
\[
[\nabla w]_{\gamma, Q} \leq C \left( \|\nabla w\|_{L^{\infty}(Q)} + [F]_{\gamma, Q} + ||H||_{L^{2\mu}(Q)} \right) ,
\]
where $C = C(n, \gamma, Q', Q)$. The proof details can be referred to the proof of [15, Theorem 2.3]. By using the interpolation inequality ([20, Lemma 6.35]), we obtain (2.2). \hfill \Box

### 4.2 Proof of Theorem 2.2

In this subsection, we give the proof of $W^{1,p}$ estimates to the weak solution of the systems as in Definition 2.1.

**Proof of Theorem 2.2** First, we give the $W^{1,p}$ interior estimates. For any ball $B_{3R/4} := B_{3R/4}(x_0) \subset Q$ with $R \leq 1$, since $w \neq 0$ on $\partial B_{3R/4}$, we choose a cut-off function $\eta \in C_0^\infty(B_{3R/4})$ such that for $0 < \rho < 3R/4$,

\[
0 \leq \eta \leq 1, \quad \eta = 1 \quad \text{in} \ B_\rho, \quad |\nabla \eta| \leq \frac{C}{R - \rho} .
\]

We have $\eta w$ satisfies
\[
\sum_{\alpha, \beta, i, j} \int_{B_{3R/4}} A_{ij}^{\alpha\beta}(x_0) \partial_{\beta}(\eta w^{(j)}) \partial_{\alpha} \phi^{(i)} \, dx
\]
\[
= \sum_{\alpha, \beta, i, j} \int_{B_{3R/4}} \left( A_{ij}^{\alpha\beta}(x_0) - A_{ij}^{\alpha\beta}(x) \right) \partial_{\beta}(\eta w^{(j)}) \partial_{\alpha} \phi^{(i)} \, dx
\]
\[
+ \sum_{\alpha, i} \left( \int_{B_{3R/4}} T^{(i)} \phi^{(i)} \, dx + \int_{B_{3R/4}} K_i^{(\alpha)} \partial_{\alpha} \phi^{(i)} \right), \quad \forall \phi \in C_0^\infty(B_{3R/4}; \mathbb{R}^m) ,
\]
where
\[
T^{(i)} = - \sum_{\alpha, \beta, j} \left( (A_{ij}^{\alpha\beta}(x) \partial_{\beta} w^{(j)} + B_{ij}^{\alpha}(x) w^{(j)} \partial_{\alpha} \eta - C_{ij}^{\alpha\beta}(x)(2 w^{(j)} \partial_{\beta} \eta + \eta \partial_{\beta} w^{(j)})) \right)
\]
\[
- \sum_j D_{ij}(x)(\eta w^{(j)}) + H^{(i)} \eta + \sum_{\alpha} (F_i^{(\alpha)} - (F_i^{(\alpha)}))_{3R/4} \partial_{\alpha} \eta .
\]
By using (4.14), assume that \( w \in C^1(\mathbb{R}^n) \) satisfies \( -\Delta w = T \). Let \( v = (v^1, \ldots, v^m) \in H_0^1(B_{3R/4}; \mathbb{R}^m) \) be the weak solution of

\[
-\Delta v^{(i)} = T^{(i)}.
\]

Thus, we can obtain that \( \eta w \) satisfies

\[
\sum_{\alpha,\beta,i,j} \int_{B_{3R/4}} A^{\alpha\beta}_{ij}(x_0) \partial_{\beta}(\eta w^{(j)}) \partial_{\alpha} \phi^{(i)} \, dx
\]

\[
\quad = \sum_{\alpha,\beta,i,j} \int_{B_{3R/4}} (A^{\alpha\beta}_{ij}(x_0) - A^{\alpha\beta}_{ij}(x)) \partial_{\beta}(\eta w^{(j)}) + K^{(i)} \partial_{\alpha} \phi^{(i)} \, dx,
\]

where \( K^{(i)} := K^{(i)} + \partial_{\alpha} v^{(i)} \).

Since \( T^{(i)} \in C^0(B_{3R/4}) \), then \( (F^{(i)} - (F^{(i)})) \in L^p(B_{3R/4}) \) for any \( n \leq p < \infty \). We firstly assume that \( w \in W^{1,q}(B_{3R/4}; \mathbb{R}^n) \), \( q \geq 2 \). Then, combining with Sobolev embedding theorem and the boundedness of coefficients, \( H^{(i)} \in L^p(B_{3R/4}) \), we can get

\[
K^{(i)} \in L^{\min(p,q^*)}(B_{3R/4}) \quad \text{and} \quad T^{(i)} \in L^{\min(p,q^*)}(B_{3R/4}),
\]

where \( q^* := \frac{qa}{a-q} \) is the Sobolev conjugate of \( q \). To write simply, we use \( a \wedge b \) to represent \( \min(a, b) \). By the (4.14),

\[
-\Delta(\partial_{\alpha} v^{(i)}) = \partial_{\alpha} T^{(i)}, \quad \text{for} \quad \alpha = 1, 2, \ldots, n.
\]

The [19] Theorem 7.1] guarantees that \( \nabla(\partial_{\alpha} v^{(i)}) \in L^{p,q}(B_{3R/4}) \) and

\[
\|\nabla(\partial_{\alpha} v^{(i)})\|_{L^{p,q}(B_{3R/4})} \leq C \|T^{(i)}\|_{L^{p,q}(B_{3R/4})},
\]

where \( C \) depends on \( n, \lambda, p, q \). Combining with the Sobolev embedding theorem that

\[
\partial_{\alpha} v^{(i)} \in L^{p,q^*}(B_{3R/4}).
\]

It follows from \( p \wedge q^* \leq (p \wedge q)^* \) that \( K^{(i)} \in L^{p,q^*}(B_{3R/4}) \). Let \( s := p \wedge q^* \) and define \( T : W^{1,s}_{0}(B_{3R/4}) \to W^{1,s}_{0}(B_{3R/4}) \) as follows,

\[
T(V) = v, \quad \text{for any} \ V \in W^{1,s}_{0}(B_{3R/4}),
\]

where \( v \in W^{1,s}_{0}(B_{3R/4}) \) is the solution of the following elliptic system:

\[
\sum_{\alpha,\beta,i,j} \int_{B_{3R/4}} A^{\alpha\beta}_{ij}(x_0) \partial_{\beta}(V^{(j)}) \partial_{\alpha} \phi^{(i)} \, dx = \sum_{\alpha,\beta,i,j} \int_{B_{3R/4}} (A^{\alpha\beta}_{ij}(x_0) - A^{\alpha\beta}_{ij}(x)) \partial_{\beta}(V^{(j)}) + K^{(i)} \partial_{\alpha} \phi^{(i)} \, dx.
\]

By using [19] Theorem 7.1] again, we have

\[
\|\nabla v\|_{L^s(B_{3R/4})} \leq C \|A(x_0) - A(x)\|_{L^s(B_{3R/4})} + C \|K\|_{L^s(B_{3R/4})},
\]
where $A(x)\nabla V$ represent matrix $(A_{ij}^q(x)\partial_{x_j}V^{(i)})$, and $C$ depends on $n, \lambda, ||A||_{C^7(Q)}, p, q$.

When $R$ is sufficiently small, it is proved by Poincaré inequality and (4.15) that $T$ is a contractive mapping. $\eta w$ is the only fixed point. See [19 Theorem 7.2] for details. By (4.15), we have

$$\|\nabla(\eta w)\|_{L^1(B_R)} \leq C||A(x_0) - A(x)||\nabla(\eta w)\|_{L^1(B_R)} + C||K||_{L^1(B_R)}.$$

Therefore, for sufficiently small $R$, we can obtain

$$\|\nabla(\eta w)\|_{L^{p, q^*}(B_{3R/4})} \leq C||K||_{L^{p, q^*}(B_{3R/4})} \leq ||T||_{L^{p, q^*}(B_{3R/4})} + ||K||_{L^{p, q^*}(B_{3R/4})} \leq \frac{C}{R - \rho} \left( |F|_{L^q(B_{3R/4})} + ||H||_{L^{\infty}(B_{3R/4})} + ||w||_{W^{1, 2}(B_{3R/4})} \right).$$

where $C$ depends on $n, \lambda, ||A||_{C^7(Q)}, p, q$. Thus

(4.16) $$\|\nabla u\|_{L^{p, q^*}(B_{R_1})} \leq \frac{C}{R - R_1} \left( |F|_{L^q(B_{3R/4})} + ||H||_{L^{\infty}(B_{3R/4})} + ||w||_{W^{1, 2}(B_{3R/4})} \right).$$

Next, we prove that $\nabla w \in L^p(B_{R/2})$. Similar to the proof of [19 Theorem 2.4], choose a series of balls with radii

$$\frac{R}{2} < \cdots < R_k < \cdots < R_2 < R_1 < \frac{3R}{4}.$$

First, let $\rho = R_1, q = 2$ in (4.16), then

$$\|\nabla u\|_{L^{p, q^*}(B_{R_1})} \leq \frac{C}{R - R_1} \left( |F|_{L^q(B_{3R/4})} + ||H||_{L^{\infty}(B_{3R/4})} + ||w||_{W^{1, 2}(B_{3R/4})} \right).$$

If $p \leq 2^*$, it can be obtained by interpolation inequality (see [1 Theorem 5.8]) that

$$||w||_{L^p(B_{R_1})} \leq C||w||_{W^{1, 2}(B_{R_1})}^{\theta} ||w||_{L^2(B_{R_1})}^{1-\theta} \leq C||w||_{W^{1, 2}(B_{R_1})},$$

where $\theta = n/2 - n/p$ with $2 \leq p \leq 2^*$. Combining with (4.16), the proof is completed. If $p > 2^*$, then $\nabla w \in L^{2^*}(B_{R_1})$ and

(4.17) $$\|\nabla w\|_{L^{2^*}(B_{R_1})} \leq \frac{C}{R - R_1} \left( |F|_{L^q(B_{3R/4})} + ||H||_{L^{\infty}(B_{3R/4})} + ||w||_{W^{1, 2}(B_{3R/4})} \right).$$

By taking $R = R_1, \rho = R_2$ and $q = 2^*$ in (4.16) and combining with (4.17), one has

$$\|\nabla w\|_{L^{p, q^*}(B_{R_2})} \leq \frac{C}{(R - R_1)(R_1 - R_2)} \left( |F|_{L^q(B_{3R/4})} + ||H||_{L^{\infty}(B_{3R/4})} + ||w||_{W^{1, 2}(B_{3R/4})} \right).$$

Similarly, if $p \leq 2^{**}$, using above formula and interpolation inequality (see [1 Theorem 5.8]), we have completed the proof of the theorem.

If $p > 2^{**}$, continuing the above argument within finite steps, with the help of interpolation inequality (see [1 Theorem 5.8]), we obtain

(4.18) $$||w||_{W^{1, p}(B_{R/2})} \leq C \left( |F|_{L^q(B_{3R/4})} + ||H||_{L^{\infty}(B_{3R/4})} + ||w||_{W^{1, 2}(B_{3R/4})} \right),$$
where $C$ depends on $n, \lambda, p, \|A\|_{C^1(Q)}, \text{dist}(B_R, \partial Q))$. Similar to the proof of (4.11), we can obtain
\[ \int_{B_{3R/4}} |\nabla w|^2 \, dx \leq C \left( \|w\|_{L^2(B_R)}^2 + \|F\|_{L^2(B_R)}^2 + \|H\|_{L^2(B_R)}^2 \right). \]

This, combining with (4.18), we can obtain
\[ |w|_{W^{1,p}(B_{R/2})} \leq C \left( |w|_{L^2(B_R)} + |F|_{L^2(B_R)} + |H|_{L^2(B_R)} \right). \]

Now, we prove the $W^{1,p}$ estimates near boundary $\Gamma$ by using the technology of locally flattening the boundary, which is the same to the proof in Theorem 2.1. For simplicity, we use the same notation. Hence, we have that $W(y) := w(\Psi^{-1}(y)) \in W^{1,2}(B_{K}^+ \subset \mathbb{R}^n, \mathbb{R}^m)$ satisfies
\[ \sum_{\alpha, \beta, i, j} \int_{B_R^+} \left( A_{ij}^{\alpha \beta} \partial_{\beta} W^{(i)} + B_{ij}^{\alpha} W^{(j)} \right) \partial_{\alpha} \phi^{(i)} + C_{ij}^{\alpha} \partial_{\beta} W^{(j)} \phi^{(i)} + D_{ij} W^{(j)} \phi^{(i)} \, dy \]
\[ = \sum_{\alpha, i} \int_{B_R^+} \mathcal{H}^{(i)} \phi^{(i)} + \mathcal{F}^{(\alpha)} \partial_{\alpha} \phi^{(i)} \, dy, \]

for any $\phi \in W^{1,2}(B_{K}^+, \mathbb{R}^m)$. In this special case, we can obtain the boundary estimate of the upper half space by using the above method of proving the interior estimate (4.19), thus, for $n \leq p < \infty$ we have
\[ |w|_{W^{1,p}(B_{R/2}^+)} \leq C \left( |w|_{L^2(B_R^+)} + |F|_{L^2(B_R^+)} + |H|_{L^2(B_R^+)} \right), \]

where $C$ depends on $\lambda, \Lambda, \lambda, p, R, \Psi$. Then, changing back to the original variable $x$, we obtain
\[ |w|_{W^{1,p}(B_{R/2})} \leq C \left( |w|_{L^2(B_R)} + |F|_{L^2(B_R)} + |H|_{L^2(B_R)} \right), \]

where $N' = \Psi^{-1}(B_{K}^+), N = \Psi^{-1}(B_{K}^+)$ and $C = C(\lambda, \mu, p, R, \Psi)$. Furthermore, there exists a constant $0 < \sigma < 1$, independent on $R$, such that $B_{\sigma R}(x_0) \cap Q \subset N'$. Therefore, for any $x_0 \in Q' \cap \Gamma$, there exists $R_0 := R_0(x_0) > 0$ such that,
\[ |\nabla w|_{W^{1,p}(B_{R_0}(x_0) \cap Q')} \leq C(|w|_{L^2(Q)} + |F|_{Y,Q} + |H|_{L^2(Q)}), \]

where $C$ depends on $\lambda, \Lambda, \lambda, p, x_0, R$. Combining (4.19) and (4.20) and making use of the finite covering theorem, We have completed the proof of the Theorem 2.2. Refer to the proof of [15] Theorem 2.4 for more details. \( \square \)

References

[1] R.A. Adams; J.J.F. Fournier, Sobolev spaces. Second edition. Pure and Applied Mathematics (Amsterdam), 140. Elsevier/Academic Press, Amsterdam, 2003. xiv+305 pp.
[2] H. Ammari; G. Ciraolo; H. Kang; H. Lee; K. Yun, Spectral analysis of the Neumann-Poincaré operator and characterization of the stress concentration in anti-plane elasticity. Arch. Ration. Mech. Anal. 208 (2013) 275–304.
[3] H. Ammari; G. Dassios; H. Kang; M. Lim, Estimates for the electric field in the presence of adjacent perfectly conducting spheres. Quat. Appl. Math. 65 (2007) 339–355.

[4] H. Ammari; H. Kang; H. Lee; J. Lee; M. Lim, Optimal estimates for the electrical field in two dimensions. J. Math. Pures Appl. 88 (2007) 307–324.

[5] H. Ammari; H. Kang; H. Lee; M. Lim; H. Zribi, Decomposition theorems and fine estimates for electrical fields in the presence of closely located circular inclusions. J. Differential Equations 247 (2009) 2897–2912.

[6] H. Ammari; H. Kang; M. Lim, Gradient estimates to the conductivity problem. Math. Ann. 332 (2005) 277–286.

[7] I. Babuška; B. Andersson; P. Smith; K. Levin, Damage analysis of fiber composites. I. Statistical analysis on fiber scale. Comput. Methods Appl. Mech. Engrg. 172 (1999) 27–77.

[8] J.G. Bao; H.J. Ju; H.G. Li, Optimal boundary gradient estimates for Lamé systems with partially infinite coefficients. Adv. Math. 314 (2017) 583–629.

[9] J.G. Bao; H.G. Li; Y.Y. Li, Gradient estimates for solutions of the Lamé system with partially infinite coefficients. Arch. Ration. Mech. Anal. 215 (2015) 307–351.

[10] J.G. Bao; H.G. Li; Y.Y. Li, Gradient estimates for solutions of the Lamé system with partially infinite coefficients in dimensions greater than two. Adv. Math. 305 (2017) 298–338.

[11] E. Bao; Y.Y. Li; B. Yin, Gradient estimates for the perfect conductivity problem. Arch. Ration. Mech. Anal. 193 (2009) 195–226.

[12] E. Bao; Y.Y. Li; B. Yin, Gradient estimates for the perfect and insulated conductivity problems with multiple inclusions. Comm. Partial Differential Equations 35 (2010) 1982-2006.

[13] E. Bonnetier; F. Triki, On the spectrum of the Poincaré variational problem for two close-to-touching inclusions in 2D. Arch. Ration. Mech. Anal. 209 (2013) 541–567.

[14] E. Bonnetier; M. Vogelius, An elliptic regularity result for a composite medium with “touching” fibers of circular cross-section. SIAM J. Math. Anal. 31 (2000) 651–677.

[15] Y. Chen; H.G. Li, Estimates and asymptotics for the stress concentration between closely spaced stiff C^{1,γ} inclusions in linear elasticity. J. Funct. Anal. 281 (2021).

[16] H.J. Dong, Gradient estimates for parabolic and elliptic systems from linear laminates. Arch. Ration. Mech. Anal. 205 (2012) 119–149.

[17] H.J. Dong; H. Zhang, On an elliptic equation arising from composite materials. Arch. Rational Mech. Anal. 222 (2016) 47–89.

[18] J.E. Flaherty and J.B. Keller, Elastic behavior of composite media. Comm. Pure Appl. Math. 26 (1973) 565–580.

[19] M. Giaquinta; L. Martinazzi, An introduction to the regularity theory for elliptic systems, harmonic maps and minimal graphs. Springer Science Business Media, 2013.

[20] D. Gilbarg; N. S. Trudinger, Elliptic partial differential equations of second order. Springer 1998.

[21] Y. Gorb; L. Berlyand, Asymptotics of the effective conductivity of composites with closely spaced inclusions of optimal shape. Quart. J. Mech. Appl. Math. 58 (2005) 84–106.

[22] Y. Gorb; L. Berlyand, The effective conductivity of densely packed high contrast composites with inclusions of optimal shape. Continuum models and discrete systems, 63–74, NATO Sci. Ser. II Math. Phys. Chem., 158, Kluwer Acad. Publ., Dordrecht, 2004.

[23] H.J. Ju; H.G. Li; L.J. Xu, Estimates for elliptic systems in a narrow region arising from composite materials. Quart. Appl. Math. 77 (2019) 177–199.
[24] H. Kang; H. Lee; K. Yun, Optimal estimates and asymptotics for the stress concentration between closely located stiff inclusions. Math. Ann. 363 (2015) 1281–1306.

[25] H. Kang; S. Yu, Quantitative characterization of stress concentration in the presence of closely spaced hard inclusions in two-dimensional linear elasticity. Arch. Ration. Mech. Anal. 232 (2019) 121–196.

[26] H.G. Li; Y. Li, An extended Flaherty-Keller formula for an elastic composite with densely packed convex inclusions. [arXiv:1912.13261]

[27] H.G. Li; Y.Y. Li; E.S. Bao; B. Yin, Derivative estimates of solutions of elliptic systems in narrow regions. Quart. Appl. Math. 72 (2014) 589–596.

[28] Y.Y. Li; L. Nirenberg, Estimates for elliptic systems from composite material. Comm. Pure Appl. Math. 56 (2003) 892–925.

[29] Y.Y. Li; M. Vogelius, Gradient estimates for solutions to divergence form elliptic equations with discontinuous coefficients. Arch. Ration. Mech. Anal. 153 (2000) 91–151.

[30] M. Lim; K. Yun, Blow-up of electric fields between closely spaced spherical perfect conductors. Comm. Partial Differential Equations. 34 (2009) 1287–1315.

[31] O.A. Oleinik; A.S. Shamaev; G.A. Yosifian, Mathematical problems in elasticity and homogenization, Studies in Mathematics and its Applications, 26. North-Holland Publishing Co., Amsterdam, 1992. xiv+398 pp.

[32] S.B. Vigdergauz, On a case of the inverse problem of the two-dimensional theory of elasticity. Prikl. Mat. Mekh. 41 (1977) 902–908

[33] K. Yun, Estimates for electric fields blown up between closely adjacent conductors with arbitrary shape. SIAM J. Appl. Math. 67 (2007) 714–730.

Yan Li (Corresponding author)
School of Mathematical Sciences, Beijing Normal University, Beijing 100875, China.
E-mail: yanli@mail.bnu.edu.cn (Y. Li)