Dark energy, integrated Sachs–Wolfe effect and large-scale magnetic fields

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Abstract
The impact of large-scale magnetic fields on the interplay between the ordinary and integrated Sachs–Wolfe effects is investigated in the presence of a fluctuating dark energy component. The modified initial conditions of the Einstein–Boltzmann hierarchy allow for the simultaneous inclusion of dark energy perturbations and of large-scale magnetic fields. The temperature and polarization angular power spectra are compared with the results obtained in the magnetized version of the (minimal) concordance model. Purported compensation effects arising at large scales are specifically investigated. The fluctuating dark energy component modifies, in a computable manner, the shapes of the 1- and 2-σ contours in the parameter space of the magnetized background. The allowed spectral indices and magnetic field intensities turn out to be slightly larger than those determined in the framework of the magnetized concordance model where the dark energy fluctuations are absent.

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(Some figures in this article are in colour only in the electronic version)

1. Motivations and goals
Defining \( z_{\text{rec}} = 1090.51 \pm 0.95 \) as the typical redshift of recombination according to the WMAP data alone (see [1, 2] for the 3 year data release and [3–7] for the 5 year data release), the temperature anisotropies for multipoles \( \ell < \sqrt{z_{\text{rec}}} \) are practically unaffected by the thickness of the visibility function. The resulting angular power of the temperature inhomogeneities is determined by the ordinary Sachs–Wolfe effect (SW in what follows), by the integrated Sachs–Wolfe effect (ISW effect in what follows) and by their mutual correlation. The origin of the SW and of the ISW contributions is physically distinct. When the opacity drops suddenly, the SW term is determined by the density contrast of the photons and by the curvature perturbations around \( z_{\text{rec}} \). Conversely the ISW effect depends upon an integral over the conformal time coordinate \( \tau \) extending between \( \tau_{\text{rec}} \) (corresponding to \( z_{\text{rec}} \)) and \( \tau_0 \)
(i.e. the present time when $z = 0$). The integration path determining the ISW passes through $z_A$, i.e. the redshift at which the dark energy contribution is of the same order of the total density of matter and the geometry starts accelerating. The explicit contribution of the dark energy to the SW and ISW effects (as well as to their cross-correlation) has been scrutinized, for the first time, in [8] (see also [9]) where the effects of the late dominance of a cosmological term have been taken into account in a consistent semi-analytic treatment.

The SW contribution typically peaks for comoving wavenumbers $k \simeq 0.0002\, \text{Mpc}^{-1}$ while the ISW effects contributes between $k_{\text{min}} = 0.001\, \text{Mpc}^{-1}$ and $k_{\text{max}} = 0.01\, \text{Mpc}^{-1}$. For comparison recall that the angular power spectra of the cosmic microwave background (CMB in what follows) fluctuations are customarily assigned at a pivot scale $k_p = 0.002\, \text{Mpc}^{-1}$ which corresponds, for the best-fit parameters of the WMAP 5 years alone (and in the light of the $\Lambda$CDM scenario\(^1\)) to $\ell_p \simeq 30$. Even if both contributions are reasonably separated in scales, the SW and ISW effects may partially compensate. An effective compensation would imply a suppression of the lowest multipoles (and in particular of the quadrupole) in the angular power spectrum of the temperature autocorrelations. In the context of the $\Lambda$CDM scenario with purely adiabatic fluctuations, the compensation is partial and ineffective for the suppression of the quadrupole. If non-adiabatic fluctuations in the dark energy sector are consistently included, the physical situation is different and a quadrupole suppression cannot be excluded. This possibility has been investigated especially in the context of the (observed) low quadrupole of the temperature autocorrelations [10–12].

The relative contribution of the SW and ISW terms does depend upon the evolution of the background geometry between the recombination epoch and the present time. Furthermore it does also depend upon the potential presence of non-adiabatic fluctuations as well as upon the specific way dark energy is parametrized. In the $\Lambda$CDM paradigm there are, by definition, no fluctuations in the dark energy sector. The rationale for such a statement stems directly from the value of the barotropic index of the dark energy (i.e. $w_{\text{de}} = -1$). The natural extension of the $\Lambda$CDM paradigm to the case when the dark energy fluctuations are dynamical is represented by the $w$CDM model where $w$ stands for the barotropic index of the dark energy component\(^2\). In the $w$CDM scenario the dark energy fluctuations affect virtually all CMB observables and, more specifically, the best-fit parameters will differ from the ones determined, for instance, within the standard $\Lambda$CDM scenario.

A legitimate question concerns, in this context, the role of the large-scale magnetic fields. Indeed the $\Lambda$CDM paradigm served as the simplest framework for the systematic scrutiny of magnetized CMB anisotropies (see, e.g., [13, 14] and [15–17] for more recent developments). The purpose of the present investigation is to extend the program of [13, 14] by relaxing the hypotheses of [15–17] on the dark energy component. What happens if the dominant energy density of the background is not parametrized in terms of a cosmological constant and, simultaneously, a magnetized background is present? For large angular separations the interplay between the SW and the ISW effect answers satisfactorily to the previous question, but what happens for higher multipoles? What are the changes in the parameter space of the magnetized background induced by a fluctuating dark energy component?

While there are no doubts that large-scale magnetic fields exist today in nearly all gravitationally bound systems there has been mounting evidence from diverse observations in galaxies [18, 19], clusters [20, 21], superclusters [22], high-redshift quasars [23, 24] that

\(\Lambda\) stays for the dark energy component (i.e. just a cosmological constant) and CDM stands for the cold dark matter component.

\(\text{To be more clear, since also other barotropic indices will intervene, we will denote the barotropic index of dark energy as } w_{\text{de}}.\) At the same time, since it is conventional to talk about the $w$CDM paradigm, we will adhere to this convention and refrain from writing $w_{\text{de}}$CDM.
magnetic fields could also have pretty large correlation scales (see [25] for a review on the subject). The evolution of electromagnetic fields in a plasma is subject to daily test in terrestrial laboratories and in astrophysical observations (see, e.g., [26, 27] for historic monographs on the subject). When plasma physics is used to interpret or even justify some astrophysical observations, the governing equations are exactly the ones used in terrestrial experiments. This approximation is reasonable for a sufficiently small redshift. However, as we move toward a higher redshift, the evolution of the spacetime curvature cannot be neglected anymore. The spirit of the approach summarized in [15–17] is to translate the plasma dynamics in flat spacetime to a curved background which is the one dictated, for simplicity, by the \( \Lambda \)CDM paradigm and by its neighboring extensions. Specific attention must be paid, in this respect, to the accurate treatment of the relativistic fluctuations of the geometry and to their initial conditions [13, 14].

Before passing to the layout of the paper, it is appropriate to account for the opinion of some authors who say that it is useless to consider models where dark energy perturbations are present since, after all, the \( \Lambda \)CDM scenario works well and there are no reasons either to dismiss it or to extend it. According to this perspective we should also refrain from analyzing many other effects which are customarily included in the experimental analyses [1, 2] (see also [3–7]) such as tensor modes, mildly open Universes, delayed recombination effects just to mention a few. The interest of the \( \omega \)CDM paradigm is exactly that it allows for a fluctuating dark energy component which is not present in the vanilla \( \Lambda \)CDM. In the present context the effects of large-scale magnetic fields and the ones of dark energy perturbations will mix in a way which is nontrivial as it will be explicitly shown by comparing the computed CMB observables with the measured values of the TT and TE correlations.

The content of the forthcoming sections can be summarized, in short, as follows. In section 2 the governing equations of the system will be introduced. In section 3 the ordinary and integrated SW contributions will be analyzed. Section 4 deals with the initial conditions for the magnetized Einstein–Boltzmann hierarchy in the presence of a fluctuating dark energy component. In section 5 the numerical aspects of the whole analysis have been collected. Section 6 contains our concluding considerations. Various technical results have been reported in the appendix to avoid excessive digressions from the main logical line of each section.

### 2. Governing equations

The governing equations of the pre-decoupling plasma can be written in terms of the global magnetohydrodynamical variables, i.e. the total current, the charge density and the baryon velocity (see, e.g., [28]). In most of the astrophysical applications the spacetime curvature hardly plays any role in the evolution of the plasma. Prior to decoupling, as already emphasized in section 1, the spacetime metric as well its relativistic inhomogeneities are two necessary ingredients for gauging the effects of large-scale magnetism on the CMB observables. The reduction from the two-fluid system to the one-fluid system is discussed in [15–17] (see, in particular, [28]) with special attention to the effects of the spacetime curvature. The magnetohydrodynamical reduction holds, of course under various approximations which are not that dissimilar from the ones usually employed in the case of flat spacetime. There are, however, also notable differences which can be summarized as follows.

- Since electrons and ions are non-relativistic (i.e. the plasma is cold) the system is not invariant under Weyl rescaling of the spacetime metric and this implies that the corresponding evolution equations will be qualitatively different from the case of a non-expanding spacetime.
• Even if the charge carriers are non-relativistic, the fluctuations of the geometry cannot be treated in the standard Newtonian approximation since the relevant modes of the gravitational field have wavelengths which are larger than the Hubble radius before matter–radiation equality.

With these specifications in mind, the relevant evolution equations will be written prior to decoupling when the electron–photon and electron–ion coupling is still strong.

2.1. Background variables

In a spatially flat background metric of the type \( g_{\mu\nu}(\tau) = a^2(\tau)\eta_{\mu\nu} \) (where \( \eta_{\mu\nu} \) is the Minkowski metric) the Friedmann–Lemaître equations take the form

\[
\mathcal{H}^2 = \frac{8\pi G a^2}{3} \rho_t, \quad \mathcal{H}' = 4\pi G a^2 (p_t + \rho_t), \quad \mathcal{H} = \frac{a'}{a},
\]

where the prime denotes a derivation with respect to the conformal time coordinate \( \tau \). The explicit form of the energy density and of the enthalpy density appearing in equation (2.1) is given by

\[
\rho_t = \rho_e + \rho_i + \rho_c + \rho_{\gamma} + \rho_{\nu} + \rho_{de},
\]

\[
\rho_t + p_t = \frac{4}{3}(\rho_{\nu} + \rho_{\gamma}) + \rho_c + \rho_e + \rho_i + (w_{de} + 1)\rho_{de}.
\]

In the one-fluid limit the electron and ion energy densities form a single physical entity, i.e. the baryonic matter density \( \rho_b = \rho_e + \rho_i \) where \( \rho_e = m_e n_0 \) and \( \rho_i = m_i n_0 \) are, respectively, the electron and ion matter densities. The comoving concentrations of electrons and ions (i.e. \( n_0 = a^3 \tilde{n}_0 \)) coincide because of the electric neutrality of the plasma. The remaining energy densities in equations (2.2) and (2.3) parametrize the contributions of neutrinos (with subscript \( \nu \)) of photons (with subscript \( \gamma \)) and of cold dark matter particles (with subscript c); finally \( \rho_{de} \) denotes the dark energy density while \( w_{de} \), as already mentioned, is the barotropic index of dark energy.

2.2. Fluctuations of the geometry

Diverse gauges lead to slightly different (but mathematically equivalent) physical pictures of the effects discussed in the present investigation. To avoid a pedantic presentation, the relevant equations will be swiftly introduced in the conformally Newtonian gauge with the proviso that the very same equations will be expressed, when needed, in the synchronous frame. The recipe to move between these two gauges will now be given together with the appropriate definitions of the perturbed line elements. In the conformally Newtonian gauge the perturbed entries of the metric are given by

\[
\delta_{(cn)}^{(0)} g_{00}(k, \tau) = 2a^2(\tau)\phi(k, \tau), \quad \delta_{(cn)}^{(ij)} g_{ij}(k, \tau) = 2a^2(\tau)\psi(k, \tau)\delta_{ij},
\]

while in the synchronous gauge the perturbed metric can be written as

\[
\delta_{(S)}^{(ij)} g_{ij}(k, \tau) = a^2(\tau)\left[ \hat{k}_i \hat{k}_j h(k, \tau) + 6\xi(k, \tau)\left( \hat{k}_i \hat{k}_j - \delta_{ij} \frac{\delta_{il} \delta_{lj}}{3} \right) \right],
\]

3 The numerical value of the redshift of matter–radiation equality, denoted by \( z_{eq} \) is given by \( z_{eq} + 1 = 3228.91 \left[ h_0^2 \Omega_M / 0.134 \right] \).

4 The plasma is globally neutral \( n_i = n_e = n_0 \) and the common value of the electron and ion concentrations can be expressed as \( n_0 = n_0 n_0 \), where \( n_0 \) is the comoving concentration of photons, \( n_0 = 6.219 \times 10^{-10} \left[ h_0^2 \Omega_b / (0.02773) \right]^{1/2} (2.5 \text{K})^{-3} \) and \( \Omega_b \) is, as usual, the critical fraction of baryonic matter.
where \( \hat{k}_i = k_i/|\hat{k}| \). The parametrizations of equations (2.4) and (2.5) are related by the appropriate coordinate transformations, i.e.

\[
\psi(k, \tau) = -\xi(k, \tau) + \frac{\mathcal{H}}{2k^2} \{ h(k, \tau) + 6\xi(k, \tau) \},
\]

\[
\phi(k, \tau) = -\frac{1}{2k^2} \{ h(k, \tau) + 6\xi(k, \tau) \} + \mathcal{H} \{ h(k, \tau) + 6\xi(k, \tau) \}'.
\]  

Consider, for instance, the curvature perturbations on comoving orthogonal hypersurfaces, i.e. \( \mathcal{R}(k, \tau) \), within the description provided by the conformally Newtonian gauge

\[
\mathcal{R}(k, \tau) = -\psi - \frac{\mathcal{H}(\mathcal{H} \phi + \psi')}{\mathcal{H}^2 - \mathcal{H}'} \to \xi + \frac{\mathcal{H} \xi'}{\mathcal{H}^2 - \mathcal{H}'}.
\]  

where the arrow simply labels the coordinate transformation from the conformally Newtonian gauge to the synchronous coordinate system; the last expression in equation (2.7) is obtained by shifting metric fluctuations according to equation (2.6). From the last equality in equation (2.7) it is also apparent that when \( \mathcal{R}' = 0 \), \( \xi(k) = \mathcal{R}(k) \). In more general terms, by solving equation (2.7) in terms of \( \xi \), it can be shown, after integration by part, that

\[
\xi(k, \tau) = \mathcal{R}(k, \tau) - \frac{\mathcal{H}(\tau)}{a(\tau)} \int_0^\tau a(\tau_1) \mathcal{R}'(k, \tau_1) d\tau_1,
\]  

where \( \tau_1 \) is an integration variable and where the prime denotes, as usual, a derivation with respect to \( \tau \). Both in analytical and numerical calculations the normalization of the curvature perturbations is customarily expressed in terms of \( \mathcal{R} \); for this reason, equations (2.7) and (2.8) turn out to be particularly useful in the explicit estimates. The same transformations of equation (2.6) can be used to gauge-transform the governing equations from the conformally Newtonian frame to the synchronous coordinate system. Equation (2.8) also justifies, a posteriori, the perturbed form of the line element introduced in equation (2.5): the factor of 6 in front of \( \xi \) is essential if we want \( \xi(k, \tau) \) to coincide with \( \mathcal{R}(k, \tau) \) at least in the case of the adiabatic mode when \( \mathcal{R}_c(k) \) does not depend on time prior to equality and for wavelengths larger than the Hubble radius.

The present conventions as well as the whole approach slightly differ from the treatment of, for instance, [29] (see also [30]). In the present approach the curvature perturbations on comoving orthogonal hypersurfaces (i.e. \( \mathcal{R} \)) are simply related, in the large-scale limit, to the curvature perturbations on uniform density hypersurfaces (see, e.g., [31] and also [32, 33]). From the latter observations, most of the synchronous gauge results needed in the present analysis follow by making judicious use of equation (2.8). The need for the synchronous description is also motivated since the code used for the present numerical analysis is based, originally, on COSMICS [34] and on CMBFAST [35, 36]. The numerical code used here is an extension of what has been described and exploited in [15–17]. It is also worth mentioning, for completeness, that there exist approaches to cosmological perturbations which are fully covariant [37] and which have also been applied to the case of large-scale magnetic fields [38] without leading, in the latter case, to any explicit estimate or of the temperature and polarization angular power spectra neither in the ΛCDM paradigm nor in its neighboring extensions such as the ones analyzed in the present paper.

5 Not only the metric fluctuations will change under coordinate transformations but also the inhomogeneities of the sources. In particular, it can be easily shown that \( \delta_{\text{inh}} \rho = \delta_{\text{inh}} \rho - \rho (\mathcal{H} + 6\xi')/(2k^2) \). From the latter relation it also follows that a fluctuation of the energy density does not transform if its homogeneous background value is constant in time.
2.3. Evolution equations for the inhomogeneities

The Hamiltonian and momentum constraints stemming, respectively, from the (00) and (0i) (perturbed) Einstein equations are given, in real space, by

\[ \nabla^2 \psi - 3H(\dot{\phi} + \dot{\psi}) = 4\pi Ga^2 [\delta_t \rho_i + \delta_{\rho} \rho_{de} + \delta_B \rho_B + \delta_E \rho_E], \]  

(2.9)

\[ \tilde{\nabla}(\dot{\phi} + \dot{\psi}) = -4\pi Ga^2 \left( (p_i + \rho_i) \tilde{v}_i + \frac{\vec{E} \times \vec{B}}{4\pi} \right), \]  

(2.10)

where \( \delta_t \rho_i = \delta_{\rho} \rho_{de} + \delta_B \rho_B + \delta_E \rho_E \) is the density fluctuation of the fluid sources in the longitudinal gauge. In equation (2.10) the total velocity field \( \vec{v}_i \) obeys\(^6\)

\[ (p_i + \rho_i) \tilde{v}_i = \frac{x}{3} \rho_i \tilde{v}_i + \frac{x}{3} \rho_i (1 + R_B) \tilde{v}_{yb} + \rho_e \tilde{v}_e + (w_{de} + 1) \rho_{de} \tilde{v}_{de}; \]  

(2.11)

as already mentioned, \( w_{de} \) is the barotropic index of the dark energy component, i.e.

\[ w_{de} = \frac{\rho_{de}}{\rho_{de}} \quad c_{s de}^2 = w_{de} - \frac{u_{de}'}{3H(w_{de} + 1)} \]  

(2.12)

where the sound speed of dark energy has also been introduced. In equation (2.11), \( R_B = 3\rho_b/(4\rho_e) \) denotes the baryon-to-photon ratio. Prior to photon decoupling the baryon and photon velocities effectively coincide with \( \tilde{v}_{yb} \):

\[ \tilde{v}_{yb} \simeq \tilde{v}_y \simeq \tilde{v}_b, \quad \tilde{v}_b = \frac{m_e \tilde{v}_e + m_i \tilde{v}_i}{m_e + m_i}. \]  

(2.13)

In equations (2.9) and (2.10), the gravitational effects of the large-scale electromagnetic fields have been included in terms of the comoving electric and magnetic fields \( \vec{E}(\vec{x}, \tau) = a^2(\tau)\vec{E}(\vec{x}, \tau) \) and \( \vec{B}(\vec{x}, \tau) = a^2(\tau)\vec{B}(\vec{x}, \tau) \), i.e.

\[ \delta_t \rho_B = \frac{B^2}{8\pi a^2}, \quad \delta_t \rho_E = \frac{E^2}{8\pi a^2}, \quad \delta_B \rho_B = \frac{\delta_t \rho_B}{3}, \quad \delta_E \rho_E = \frac{\delta_t \rho_B}{3}, \]  

(2.14)

where \( B^2 = |\vec{B}|^2 \) and \( E^2 = |\vec{E}|^2 \). The evolution equations for \( \vec{E} \) and \( \vec{B} \) are

\[ \vec{\nabla} \cdot \vec{B} = 0, \quad \vec{\nabla} \cdot \vec{E} = 4\pi \rho_i, \]  

(2.15)

\[ \vec{\nabla} \times \vec{E} + \vec{B}' = 0, \quad \vec{\nabla} \times \vec{B} = 4\pi \vec{J} + \vec{E}' \]  

(2.16)

\[ \rho_i' + \vec{\nabla} \cdot \vec{J} = 0, \quad \rho_e = e(n_i - n_e), \quad \vec{J} = e(n_i \tilde{v}_i - n_e \tilde{v}_e). \]  

(2.17)

The total current \( \vec{J} \) has been expressed in terms of the two fluid variables; however, as in the case of flat-space magnetohydrodynamics, \( \vec{J} \) can be related to the electromagnetic fields by means of the generalized Ohm law [28]:

\[ \vec{J} = \sigma \left( \vec{E} + \tilde{v}_b \times \vec{B} + \frac{\vec{\nabla} \rho_e}{e n_0} - \frac{\vec{J} \times \vec{B}}{n_0 e} \right), \quad \sigma = \frac{\omega_{pe}^2}{4\pi [a \Gamma_{ee} + (4/3)[\rho_{pe}/(n_0 m_e)] \Gamma_{ee}]} \]  

(2.18)

where \( \sigma \) is the conductivity; \( \Gamma_{ee} \) and \( \Gamma_{pe} \) are, respectively, the electron–ion and electron–photon interaction rates. The three terms appearing in Ohm’s law are, besides the electric field, the drift term (i.e. \( \tilde{v}_b \times \vec{B} \)), the thermoelectric term (containing the gradient of the electron pressure\(^7\)) and the Hall term (i.e. \( \vec{J} \times \vec{B} \)). It can be shown [28] that for frequencies much

\(^6\) Following exactly the same conventions established in equations (2.2) and (2.3), the various subscripts denote the velocities of the different fluid components.

\(^7\) In equation (2.18), following [28], what appears is the comoving electron pressure given by \( p_e = n_e T_e \) where \( n_e \) and \( T_e \) are, respectively, the comoving concentration and the comoving temperature of the electrons.
smaller than the (electron) plasma frequency and for typical length-scales much larger than the Debye screening length the Hall and thermoelectric terms are subleading for the purposes of the present analysis. The electromagnetic pressure as well as the anisotropic stresses enter the perturbed \((ij)\) components of the Einstein equations:

\[
\psi'' + \mathcal{H}(\phi' + 2\psi') + (\mathcal{H}^2 + 2\mathcal{H}')\phi + \frac{1}{3} \nabla^2 (\phi - \psi) = 4\pi G a^2 [\delta_\rho p_f + \delta_\rho p_{de} + \delta_\rho p_B + \delta p_E],
\]

(2.19)

\[
\nabla^4 (\phi - \psi) = 16\pi G a^2 \left[ \rho_\nu \nabla^2 \sigma_\nu + \rho_\gamma \nabla^2 \sigma_B + \rho_\gamma \nabla^2 \sigma_E + \frac{4}{3} \rho_{de} (w_{de} + 1) \nabla^2 \sigma_{de} \right],
\]

(2.20)

where \(\delta_\rho p_f = (\delta_\rho \rho_\nu + \delta_\rho \rho_\gamma) / 3\), in analogy with \(\delta_\rho p\), denotes the fluctuation of the pressure of the fluid components. The total anisotropic stress is given, as usual, by

\[\Pi'_i = \Pi'_{i,\nu} + \Pi'_{i,B} + \Pi'_{i,E} + \Pi'_{i,de};\]

(2.21)

the various subscripts denote the respective components and, in particular, the electromagnetic contribution:

\[\Pi'_{i,B} = \frac{1}{4\pi a^4} \left[ B_i B_j - \frac{\delta_j}{3} B^2 \right], \quad \Pi'_{i,E} = \frac{1}{4\pi a^4} \left[ E_i E_j - \frac{\delta_j}{3} E^2 \right].\]

(2.22)

In equation (2.20) the notation

\[\delta_\rho \rho_\nu \Pi'_{i,\nu} = \frac{1}{2} \rho_\nu \nabla^2 \sigma_\nu + \frac{4}{3} \rho_\gamma \nabla^2 \sigma_B + \frac{4}{3} \rho_\gamma \nabla^2 \sigma_E + \rho_{de} (w_{de} + 1) \nabla^2 \sigma_{de}\]

(2.23)

has been adopted. The various species of the plasma either interact strongly with the plasma (like the electrons, the ions and the photons) or they only feel the effects of the geometry (like the CDM component and the dark energy). For large scales the baryon–photon system obeys

\[\delta_\rho' = 3 \psi' - \theta_b,\]

(2.24)

\[\theta_b' + \mathcal{H} \theta_b' = \frac{\nabla \cdot \left[ J \times \vec{B} \right]}{a^4 \rho_b} - \nabla^2 \phi + \frac{4}{3} \rho_\nu \epsilon' (\theta_\nu - \theta_b),\]

(2.25)

\[\theta_\nu' = -\frac{1}{4} \nabla^2 \delta_\nu - \nabla^2 \phi + \epsilon' (\theta_b - \theta_\nu),\]

(2.26)

\[\delta_\nu' = 4 \psi' - \frac{4}{3} \theta_\nu,\]

(2.27)

where \(\epsilon'\) is the differential optical depth; furthermore, defining as \(\vec{v}_X\) the velocity of the species \(X, \theta_X = \nabla \cdot \vec{v}_X\). The CDM and the neutrino components will evolve, respectively, as

\[\delta_\rho' = 3 \psi' - \theta_c,\]

(2.28)

\[\theta_c' + \mathcal{H} \theta_c = -\nabla^2 \phi\]

(2.29)

and as

\[\theta_\nu' = -\frac{1}{4} \nabla^2 \delta_\nu + \nabla^2 \sigma_\nu - \nabla^2 \phi,\]

(2.30)

\[\delta_\nu' = 4 \psi' - \frac{4}{3} \theta_\nu,\]

(2.31)

\[\sigma_\nu' = \frac{4}{3} \theta_\nu - \frac{3}{10} \mathcal{F}_{\nu,3},\]

(2.32)

where \(\mathcal{F}_{\nu,3}\) reminds us of the coupling of the monopole and of the dipole to the higher multipoles of the neutrino phase space distribution. The latter term will be set to zero in the class of
introducing the density contrast for the dark energy component, equations (2.33) in the standard version of the dark energy anisotropic stress has been included, so far, to have sufficiently general equations. In the future, to relax this assumption by following, for instance, the approach of [39]. By [40]) can be rather interesting in connection with dark energy models. The contribution of the invariant variable which measures the density contrast on uniform curvature hypersurfaces. also upon the gravitating effects of the electromagnetic fields and of the Ohmic current [28]:

\[ \delta' \rho \delta + 3H(\delta \rho \delta + \delta' \rho \delta) - 3 \psi' (\rho \delta + \rho \delta) + (\rho \delta + \rho \delta) \vec{\nabla} \cdot \vec{v} \delta = 0, \]

where, as already mentioned, \( \vec{v} \delta \) is the velocity of the dark energy in the conformally Newtonian frame whose evolution equation can be written as

\[ (\rho \delta + \rho \delta) \vec{v} \delta + \left[ \rho \delta + \rho \delta + 4\dot{H}(\rho \delta + \rho \delta) \right] \vec{v} \delta + \vec{\nabla} \delta \rho \delta \\
+ (\rho \delta + \rho \delta) \vec{\nabla} \phi - (\rho \delta + \rho \delta) \vec{\nabla} \sigma \delta = 0. \]  

(2.34)

In equation (2.34) \( \sigma \delta \) accounts for the possible presence of an anisotropic stress for the dark energy contribution. This contribution as argued in [39] (see also, in a related perspective, [40]) can be rather interesting in connection with dark energy models. The contribution of the dark energy anisotropic stress has been included, so far, to have sufficiently general equations. In the standard version of the \( w \)CDM scenario the dark energy has no anisotropic stress and therefore, for practical reasons, the anisotropic stress will be set to zero. It might be interesting, in the future, to relax this assumption by following, for instance, the approach of [39]. By introducing the density contrast for the dark energy component, equations (2.33) and (2.34) can also be written as

\[ \delta' \rho \delta + 3H(\delta \rho \delta + \delta' \rho \delta) - 3 \psi' (\rho \delta + \rho \delta) + (\rho \delta + \rho \delta) \vec{\nabla} \cdot \vec{v} \delta \rho \delta = 0, \]

(2.35)

\[ \delta' \rho \delta + 3H(1 - 3a^2 \delta \rho \delta + \delta^2 \rho \delta + \vec{\nabla}^2 \delta \rho \delta + (w \rho \delta + 1)) \vec{\nabla}^2 \phi - \vec{\nabla}^2 \sigma \delta = 0, \]

(2.36)

where \( \theta \delta = \vec{\nabla} \cdot \vec{v} \delta \) and \( \delta \rho \delta = \rho \delta \delta \). The governing equations can be easily translated in the synchronous coordinate system (see, in particular, equations (2.4), (2.5) and (2.6)–(2.8)). For instance the synchronous form of equation (2.9) can be obtained by using, directly, equations (2.4) and (2.5) and by appreciating that

\[ \delta \rho \delta = \delta \rho \delta + \frac{\left( h' + 6 \xi \right)}{2k^2}, \]

\[ \delta \rho \delta = \delta \rho \delta + \frac{\left( h' + 6 \xi \right)}{2k^2}. \]

(2.37)

Since \( \rho \delta = (\rho \delta + \rho \delta + \rho \delta + \rho \delta) \) and \( \rho \delta \) are separately conserved (i.e. \( \rho \delta = -3\dot{H}(\rho \delta + \rho \delta) \) and \( \rho \delta = -3\dot{H}(\rho \delta + \rho \delta) \)) the result will be as expected:

\[ 2k^2 \xi - \dot{\xi} \rho \delta = 8\pi Ga^2 \left[ \delta \rho \delta + \delta \rho \delta + \delta \rho \delta + \delta \rho \delta \right]. \]

(2.38)

In the same way all the other governing equations reported in this sections can be translated in the synchronous frame.

2.4. Large-scale evolution

The Hamiltonian constraint of equation (2.9) can be expressed as

\[ \xi = R + \frac{\nabla^2 \psi}{12\pi Ga^2(p \delta + \rho \delta)}, \]

(2.39)

where \( R \) has been introduced in equation (2.7) and \( \xi [17] \) (see also [32, 33]) is a further gauge-invariant variable which measures the density contrast on uniform curvature hypersurfaces. The constraint (2.39) can also be used as an explicit definition of \( \xi \) whose evolution depends also upon the gravitating effects of the electromagnetic fields and of the Ohmic current [28]:

\[ \frac{\partial \xi}{\partial \tau} = -\frac{\dot{\xi} \rho \delta}{\rho \delta + \rho \delta} + \frac{\dot{\xi} \left( \frac{c_{\text{at}}^2 - 1}{3} \right)}{\rho \delta + \rho \delta} + \frac{\dot{\xi} \vec{E} \cdot \vec{J}}{\frac{3\delta a^4}{\rho \delta + \rho \delta} + 3\delta a^4} + \frac{\dot{\xi} \rho \delta + 3\left( c_{\text{at}}^2 + 1 \right) \dot{H} \delta \rho \delta \rho \delta}{3\delta \rho \delta \rho \delta + 3\delta \rho \delta \rho \delta} - \frac{\theta \delta}{3}. \]

(2.40)
\[
\frac{\partial \vec{J}}{\partial \tau} = -\left(\mathcal{H} + a\Gamma_{\text{ia}} - \frac{4\rho_{\gamma}\Gamma_{\text{e}}}{3m_{\gamma}m_{e}}\right) \vec{J} + \frac{\omega_0^2}{4\pi} \left(\vec{E} + \vec{v}_{\text{b}} \times \vec{B} + \frac{\vec{\nabla}p_{\text{e}}}{e n_0} - \frac{\vec{J} \times \vec{B}}{en_0}\right) + \frac{4\rho_{\gamma}\Gamma_{\text{e}}}{3m_{e}} (\vec{v}_{\text{b}} - \vec{v}_{\gamma}).
\]

In equation (2.40) the contribution of non-adiabatic pressure fluctuations has been included for the sake of completeness even if \(\delta p_{\text{had}}\) will not play a specific role. One of the possible extensions of the present work could indeed be the inclusion of non-adiabatic modes in the dark energy sector (as argued, for instance, in [10] in connection with the problem of the quadrupole suppression). In connection with equations (2.39), (2.40) and (2.41), few remarks are in order:

- since both \(\zeta\) and \(\mathcal{R}\) are invariant under gauge transformations, they do coincide, in any frame, in the limit \(k \ll \mathcal{H}\);
- the right-hand side of equation (2.40) contains three qualitatively different contributions: the purely electric contribution (proportional to \(\delta_{\text{e}}\rho_{\text{E}}\) and its derivative), the purely magnetic contribution (proportional to \(\delta_{\text{m}}\rho_{\text{B}}\)) and the Ohmic contribution (containing explicitly the Ohmic current \(\vec{J}\));
- the factor \(\theta_{\text{e}}\) on the right-hand side of equation (2.40) is of order \(k^2/\mathcal{H}\) and it is subleading for typical length-scales larger than the Hubble radius;
- equation (2.42) reduces, at large scales, to equation (2.18): this statement can be directly verified by taking into account that the rate of electron–ion interactions (i.e. \(\Gamma_{\text{e}}\)) is always much larger than the rate of electron–photon interaction (i.e. \(\Gamma_{\gamma}\)); prior to decoupling both \(\Gamma_{\text{e}}\) and \(\Gamma_{\gamma}\) are larger than \(\mathcal{H}\).

The physical hierarchies between the electric, magnetic and Ohmic contributions follow from the typical length scales of the problem and also from the frequency range dictated by the magnetohydrodynamical approximation (see [28] and, in particular, the left plot of figure 1):

\[
\frac{\delta_{\text{e}}\rho_{\text{E}}}{(p_1 + p_2)} < \mathcal{H} \delta_{\text{e}}\rho_{\text{E}} \ll \frac{\vec{E} \cdot \vec{J}}{a^4 (p_1 + p_2)} < \frac{\mathcal{H} \delta_{\text{m}}\rho_{\text{B}}}{(p_1 + p_2)}.
\]

(2.42)

The magnetic contribution appearing on the right-hand side of equation (2.40) can also be written as

\[
\mathcal{H} \frac{(3c_s^2 - 1) \rho_{\text{E}}}{3(1 + w_i)} R_y \Omega_{\text{B}}, \quad R_y + R_v = 1,
\]

(2.43)

where \(R_y = 0.405\) and \(R_v = 0.595\) are, respectively, the photon and neutrino fractions present in the radiation plasma and

\[
\Omega_{\text{B}}(\vec{x}, \tau) = \frac{\delta_{\text{m}}\rho_{\text{B}}(\vec{x}, \tau)}{\rho_{\gamma}(\tau)}.
\]

(2.44)

Recalling the parametrization of the anisotropic stress \(\sigma_{\text{B}}\), the divergence of the magnetohydrodynamical Lorentz force, i.e. \(\vec{\nabla} \cdot [\vec{J} \times \vec{B}]\), can be expressed as a combination of \(\nabla^2 \sigma_{\text{B}}\) and \(\nabla^4 \Omega_{\text{B}}\). These vector identities are important in various analytic estimates (see [15–17] and discussions therein). In the case where the dark energy is simply given by the cosmological constant the total barotropic index and the total sound speed can be written as

\[
w_i = \frac{p_1}{\rho_1} = \frac{1 - 3\alpha^2}{3(1 + \alpha + \frac{\alpha^2}{\alpha^2})}, \quad \frac{\alpha^2}{\alpha^2} = \frac{p_1'}{\rho_1'} = w_i = \frac{\alpha}{3(1 + w_i)} \frac{\partial w_i}{\partial \alpha} = \frac{4}{3(3\alpha + 4)},
\]

(2.45)
where \( \alpha = a/a_{eq} \) and
\[
\alpha_\Lambda = \frac{d_\Lambda}{a_{eq}} = 2246.81 \left( \frac{h_0^2 \Omega_{M0}}{0.1326} \right)^{4/3} \left( \frac{h_0^2 \Omega_\Lambda}{0.3835} \right), \quad \alpha_0 = 3195.18 \left( \frac{h_0^2 \Omega_{M0}}{0.1326} \right). \tag{2.46}
\]
Using equations (2.45) into equation (2.43) we also have, in Fourier space,
\[
\gamma (\frac{3\ell^2 - 1}{3(1 + w)}) \rho_0 R_\delta \Omega_B(k) = \frac{\gamma}{(3\alpha + 4)^2} R_\delta \Omega_B(k). \tag{2.47}
\]

3. Large-scale compensations

The evolution of the brightness perturbations can be written as
\[
\Delta_1 + (i k\mu + \epsilon') \Delta_1 = \psi' - i k\mu \phi + \epsilon' \left[ \Delta_{10} + \mu \nu_b + \frac{(1 - 3\mu^2)}{4} S_p(k, \tau) \right], \tag{3.1}
\]
where \( S_p \) can be expressed as the sum of the quadrupole of the intensity, of the monopole of the polarization and of the quadrupole of the polarization, i.e. respectively, \( S_p(k, \tau) = (\Delta_{12} + \Delta_{20} + \Delta_{22}) \). The multipole expansion of the brightness perturbation reads, within the present conventions,
\[
\Delta_1(k, \mu, \tau_0) = \sum_{\ell} (-i)^{\ell} (2\ell + 1) \Delta_{1\ell}(k, \mu, \tau_0) P_\ell(\mu), \tag{3.2}
\]
where \( P_\ell(\mu) \) are the standard Legendre polynomials. The line of sight solution of equation (3.1) can be written as
\[
\Delta_1(k, \mu, \tau_0) = \int_0^{\tau_0} \mathcal{K}(\tau) \left[ \Delta_{10} + \phi + \mu \nu_b + \frac{(1 - 3\mu^2)}{4} S_p \right] e^{-i \mu x(\tau)} d\tau,
\]
where the term \(-i k\mu \phi \) has been integrated by parts and where \( \mathcal{K}(\tau) \) denotes the visibility function whose explicit form can be approximated by a Gaussian profile [41–44] with different methods. At large scales, the Gaussian can be considered effectively by a Dirac delta function approximately centered at recombination.

For typical multipoles \( \ell \leq \sqrt{\frac{1}{\Omega_{\Lambda 0}}} \) the finite width of the visibility function is immaterial. This means that for sufficiently small \( \ell \) everything goes as if the opacity suddenly drops at recombination. This implies that the visibility function presents a sharp (i.e. infinitely thin peak at the recombination time). Thus, \( \mathcal{K}(\tau) \) is proportional to a Dirac delta function and \( e^{-\epsilon(\tau, \tau_0)} \) is proportional to a Heaviside theta function. Under the latter approximations, equation (3.3) leads to the wanted separation between SW and ISW contributions:
\[
\Delta_1(k, \mu, \tau_0) = \Delta_1^{(SW)}(k, \mu, \tau_0) + \Delta_1^{(ISW)}(k, \mu, \tau_0), \tag{3.4}
\]
\[
\Delta_1^{(SW)}(k, \mu, \tau_0) = \frac{\delta_\nu}{4} + \phi \right|_{\tau_0} e^{-i \mu x(\tau)}, \tag{3.5}
\]
\[
\Delta_1^{(ISW)}(k, \mu, \tau_0) = \int_{\tau_0}^{\tau} (\phi' + \psi') e^{-i \mu x(\tau)} d\tau. \tag{3.6}
\]

8 As usual, \( \mu = \hat{k} \cdot \hat{n} \) denotes the projection of the Fourier mode on the direction of propagation of the CMB photon.
9 Of course one might also wish to continue with the integrations by parts and integrate all the \( \mu \)-dependent terms. Such a step will however produce various time derivatives of the \( \mathcal{K}(\tau) \) which are difficult to evaluate explicitly when the visibility function is infinitely thin. See also, in this respect, the discussion of appendix A.
where it has been used that the $\delta_\gamma$ obeying, for instance, equation (2.27) is also related to $\Delta_{10}$ as $\delta_\gamma(k, \tau) = 4\Delta_{10}(k, \tau)$. Equations (3.5) and (3.6) can be evaluated within three complementary approximation schemes:

(a) in the first approximation we can assume that $\alpha_{\text{rec}}/\alpha_{\text{eq}} \gg 1$ and that, simultaneously, the phase appearing in equation (3.6) is $\tau$-independent (i.e. $i\mu x(\tau) \simeq ik\mu(\tau_0 - \tau_{\text{rec}})$);

(b) in a more accurate perspective the assumption that $\alpha_{\text{rec}}/\alpha_{\text{eq}} \gg 1$ can be dropped; after all

$$\alpha_{\text{rec}} = \frac{\alpha_{\text{rec}}}{\alpha_{\text{eq}}} = \frac{z_{\text{eq}} + 1}{z_{\text{rec}} + 1} = 3.04\left(\frac{H_0^2\Omega_{M0}}{0.134}\right).$$

(3.7)

since $\alpha_{\text{rec}} \simeq 3$ is not extremely larger than 1, significant corrections can be expected;

(c) finally, equation (3.7) can be taken into account in conjunction with the $\tau$-dependence of the phase in the integrand of equation (3.6).

In the absence of large-scale magnetic fields, it is sometimes practical to estimate the large-scale temperature autocorrelations by only keeping the ordinary SW contribution evaluated in the limit $\alpha_{\text{rec}} \gg 1$. The same approximation (with the same level of accuracy) can also be used when large-scale magnetic fields are included in the analysis. It would be incorrect to use different levels of accuracy for the curvature contribution and for the magnetized contribution.

3.1. Simplified analysis of the compensation

In the case labeled by (a) in the above list of items $|\alpha_{\text{rec}}/\alpha_{\text{eq}}| \gg 1$ and the ordinary SW contribution turns out to be

$$\left[\frac{\delta_\gamma(k, \tau)}{4} + \psi(k, \tau)\right]_{\text{rec}} = 2\psi(k, \tau_{\text{rec}}) - \frac{3}{2}\psi(k, \tau_*) \equiv 2\psi(k, \tau_{\text{rec}}) + \mathcal{R}_+(k) - \frac{R_\gamma}{4}\Omega_B(k),$$

(3.8)

where it has been used, according to equation (2.27), that in the large-scale limit

$$\delta_\gamma(k, \tau_{\text{rec}}) = 4\psi(k, \tau_{\text{rec}}) + \delta_\gamma(k, \tau_*) - 4\psi(k, \tau_*) + \mathcal{O}(k^2\tau^2).$$

(3.9)

Equation (3.8) then follows from equation (3.9) by recalling that the initial conditions at $\tau_*$ are given, in the case of the magnetized adiabatic mode, as

$$\delta_\gamma(k, \tau_*) = -2\phi_+(k) - 2\Omega_B(k), \quad \mathcal{R}_+(k) = -\psi_+(k) - \frac{\phi_+(k)}{2}.$$ 

(3.10)

Since the phase appearing in the integrand of equation (3.6) is estimated by positing $x(\tau) \simeq \gamma_{\text{rec}}$, the cancellation between the ordinary and the integrated SW contributions is maximal, and, in this sense, such an approximation can be improved:

$$\Delta_{10}^{(\text{SW})}(k, \mu, \tau_0) + \Delta_{10}^{(\text{SW})}(k, \mu, \tau_0)$$

$$= \left\{2\psi(k, \tau_0) - 2\psi(k, \tau_{\text{rec}})\right\} + \left[2\psi(k, \tau_{\text{rec}}) + \mathcal{R}_+(k) - \frac{R_\gamma}{4}\Omega_B(k)\right]e^{-i\gamma_{\text{rec}}},$$

$$= \left[\mathcal{R}_+(k) + 2\psi(k, \tau_0) - \frac{R_\gamma}{4}\Omega_B(k)\right]e^{-i\gamma_{\text{rec}}}.$$ 

(3.11)

The second (intermediate) equality appearing in equation (3.11) has been included just to show the explicit cancellation at $\tau_{\text{rec}}$. In a pure CDM model (i.e. no dark energy, no magnetic fields and $\Omega_{M0} = 1$), the quantity in squared brackets in equation (3.11) can be written as

$$\mathcal{R}_+(k) + 2\psi(k, \tau_0) - \frac{R_\gamma}{4}\Omega_B(k) = -\frac{\mathcal{R}_+(k)}{5} + \frac{R_\gamma}{4}\Omega_B(k).$$

(3.12)

The time labeled by $\tau_*$ is such that $\tau \ll \tau_{\text{eq}}$ and, simultaneously, $k/H_* \simeq k\tau_* \ll 1$. 

10 The time labeled by $\tau_*$ is such that $\tau \ll \tau_{\text{eq}}$ and, simultaneously, $k/H_* \simeq k\tau_* \ll 1$. 

11
The estimate provided by equation (3.12) can be improved by taking into account the observation of equation (3.7). In the latter case, it can be shown that the SW contribution turns out to be

$$\Delta_1^{(SW)}(k, \mu, \tau_0) = \left[-\frac{r_s(k)}{5} SW_R(\alpha_{rec}) + \frac{R_r \Omega_B(k)}{20} SW_B(\alpha_{rec})\right] e^{-i\mu \tau_{rec}}.$$  \hfill (3.13)

$$SW_R(\alpha) = 1 + \frac{4}{3\alpha} - \frac{16}{3\alpha^2} + \frac{16(\sqrt{\alpha + 1} - 1)}{3\alpha^3},$$ \hfill (3.14)

$$SW_B(\alpha) = 1 - \frac{12}{\alpha} + \frac{48}{\alpha^2} + \frac{32(1 - \sqrt{\alpha + 1})}{\alpha^3},$$ \hfill (3.15)

where $\alpha_{rec}$ has been introduced in equation (3.7). The result of equation (3.13) corresponds to the approximation scheme labeled by (b) in the above list of items. The evolution of $\psi$ is obtained by solving explicitly the corresponding evolution equations for $R$ and by recalling the connection between $R$ and $\psi$ (see equation (2.7)). The functions reported in equations (3.14) and (3.15) have been derived in appendix A by using, consistently, the synchronous gauge description. It is interesting that the synchronous description leads to an explicit derivation of the SW and ISW contributions which is technically different from the one of the conformally Newtonian gauge. By taking the limit $\alpha_{rec} \to \infty$ (for $\alpha$ fixed) in equation (2.45), the standard CDM result is recovered. In the pure CDM case the long-wavelength limit implies

$$\frac{\psi'}{H} + \psi + \frac{H^2 - H'}{H^2} \psi = -R(k, \tau),$$ \hfill (3.16)

which can be easily solved analytically:

$$\psi(k, \alpha) = -R_s(k)I_1(\alpha) + R_r \Omega_B(k)I_2(\alpha)$$ \hfill (3.17)

where

$$I_1(\alpha) = \frac{\alpha[\alpha(9\alpha + 2) - 8] + 16(\sqrt{\alpha + 1} - 1)}{15\alpha^3},$$ \hfill (3.18)

$$I_2(\alpha) = \frac{3(\alpha[\alpha(\alpha - 2) + 8] + 16[1 - \sqrt{\alpha + 1}])}{20\alpha^3}.$$ \hfill (3.18)

In the case of the $\Lambda$CDM model $|\psi(k, \tau)|$ grows linearly as a function of the conformal time coordinate as a consequence of the presence of the dark energy phase. To obtain a quantitative estimate equation (3.16) can be used but in the case when $\alpha_{rec}$ is finite, it is given as in equations (2.45) and (2.46). Similarly, by summing up equations (2.33) and (2.34) we shall have

$$\psi'' + 2(H' - H^2)\psi = 4\pi Ga^2 \left[(w_{de} + 1)\delta_r \rho_{de} + \frac{4}{3} \delta_r \rho_B\right].$$ \hfill (3.19)

In the case where $w_{de} = -1$ (which is the $\Lambda$CDM case), equation (3.19) reduces to

$$\psi'' = \frac{16\pi Ga^2}{3} R_r \rho_B \Omega_B(k).$$ \hfill (3.20)

In the limit $\Omega_B(k) \to 0$ we will have that $\psi(k, \tau) = c_1(k) + c_2(k)\tau$. The latter expression holds asymptotically. Direct numerical integration must be eventually employed and the result can be expressed as

$$\psi(k, \tau_0) = -0.4507 R_s(k) + 0.1125 R_r \Omega_B(k).$$ \hfill (3.21)
Equation (3.21) must be compared with the value of \( \psi(k, \tau) \) at \( \tau_{rec} \) (corresponding to \( z_{rec} = 1090.51 \)):

\[
\psi(k, \tau_{rec}) = -0.5882 \mathcal{R}_s(k) + 0.1467 R_f \Omega_B(k). \tag{3.22}
\]

The result of equation (3.22) is also compatible with the estimates obtainable, from equation (2.45), in the limit \( \alpha_\Lambda \to \infty \):

\[
\psi(k, \tau_{rec}) = -0.6 \mathcal{R}_s(k) + 0.15 R_f \Omega_B(k). \tag{3.23}
\]

From equations (3.21) and (3.22) we can therefore say that

\[
\Delta \psi(k) = \psi(k, \tau_0) - \psi(k, \tau_{rec}) = 0.1375 \mathcal{R}_s - 0.0342 R_f \Omega_B(k). \tag{3.24}
\]

Concerning the result of equation (3.24) few comments are in order:

- in the absence of large-scale magnetic fields \( \Delta \psi(k) \simeq 0.137 \mathcal{R}_s(k) \); the latter result depends upon the cosmological parameters and the quoted value refers to equation (2.46);
- the comparative growth of the magnetized contribution is smaller than in the curvature case, i.e. \( \Delta \psi(k) \simeq -0.0342 R_f \Omega_B(k) \); the sign difference stems directly from the form of the Hamiltonian constraint (see equations (2.9) and (2.39)) as well as from the solution of equation (2.40) in the large-scale limit.

It is relevant to point out that the value \( \Delta \psi(k) \simeq 0.137 \mathcal{R}_s(k) \) is compatible with the results obtained in [10] (i.e. 0.14 \( \mathcal{R}_s(k) \)) under the same approximations discussed here but in the absence of large-scale magnetic fields.

### 3.2. Transfer functions and semi-analytical estimates

In the \( \Lambda \)CDM case the numerical integration is unavoidable even in the absence of large-scale magnetic fields. The semi-analytical treatment can be carried on, up to some point, and the obtained results are a useful complement of the numerical discussion (see section 5). More precisely the transfer function can be computed directly in terms of the evolution of the appropriate background quantities whose specific form will have to be studied numerically. Using equation (3.2) the line of sight solution (3.3) can be written as

\[
\Delta \ell(k, \tau_0) = -2 \int_{\tau_{rec}}^{\tau_0} \left[ \mathcal{R}_s(k) \frac{dT_R}{d\tau} - R_f \Omega_B(k) \frac{dT_B}{d\tau} \right] j_i(x(\tau)) \, d\tau
\]

\[
+ \left[ -\frac{\mathcal{R}_s(k)}{5} SWR_\gamma(\tau_{rec}) + \frac{R_f \Omega_B}{20} SWV_\Lambda(\tau_{rec}) \right] j_i(x(\tau_{rec})), \tag{3.25}
\]

where we recall that \( x(\tau) = k(\tau_0 - \tau) \) and \( x(\tau_{rec}) \) coincides, by definition with \( \tau_{rec} \); \( SWR_\gamma(\tau_{rec}) \) and \( SWV_\Lambda(\tau_{rec}) \) have already been introduced in equation (3.13). The functions \( T_R(\tau) \) and \( T_B(\tau) \) are given by

\[
T_R(\tau) = \mathcal{R}_s(k) \left[ 1 - \frac{\mathcal{H}(\tau)}{a^2(\tau)} \right] \int_0^\tau a^2(x) \, dx, \tag{3.26}
\]

\[
T_B(\tau) = \int_0^\tau B(\tau, x) \left[ \frac{a^2(x)}{\mathcal{H}(x)} - \frac{a^2(\tau)}{\mathcal{H}(\tau)} \right] dx + \int_0^\tau a^2(x) \, dx \int_0^x B(\tau, y) \, dy; \tag{3.27}
\]

for two generic arguments \( x \) and \( y \), the function \( B(x, y) \) is defined as

\[
B(x, y) = \frac{\mathcal{H}(x) \mathcal{H}(y)}{a^2(x)} \left[ 3 c_s^2(x) - 1 \right] \rho_B(y) + \frac{3[\omega_B(y) + 1]}{\rho_\Lambda(y)}; \tag{3.28}
\]

\[
\alpha_\Lambda = \frac{\rho_\Lambda}{\rho_\Lambda + 1} \frac{\mathcal{H}(x) \mathcal{H}(y)}{a^2(x)} \left[ 3 c_s^2(x) - 1 \right] \rho_B(y) + \frac{3[\omega_B(y) + 1]}{\rho_\Lambda(y)}; \tag{3.29}
\]
c_s^2(y) is the sound speed of the total plasma already introduced in equation (2.45) and here appearing as a function of a generic integration variable y. Equations (3.26) and (3.27) can be derived, after some algebra, from equations (2.7), (2.39), (2.40) and (2.41). The derivation of equations (3.26)–(3.27) is specifically discussed in appendix B.

The results of equations (3.25), (3.26) and (3.27) imply that the angular power spectrum for the temperature autocorrelations consists of three distinct terms parametrizing, respectively, the SW and the ISW terms:

\[ C^{(TT)}_\ell = C^{(SW)}_\ell + C^{(ISW)}_\ell + C^{(cross)}_\ell. \]  (3.29)

The ordinary SW contribution turns out to be, within the present approximations

\[ C^{(SW)}_\ell = \frac{4\pi}{25} S \nu^2 R^2 (\alpha_{rec}) \int_0^\infty \frac{dk}{k} P_R(k) j^2_0(k \tau_0) \]

\[ + \frac{\pi}{100} S \nu^2 (\alpha_{rec}) \int_0^\infty \frac{dk}{k} P_{B_\gamma}(k) j^2_0(k \tau_0) \]

\[ + \frac{2\pi}{25} R_\gamma \cos \beta S \nu^2 R^2 (\alpha_{rec}) S \nu^2 R_\gamma (\alpha_{rec}) \int_0^\infty \frac{dk}{k} \sqrt{P_R(k) \sqrt{P_{\Omega}(k)}} j^2_0(k \tau_0), \]  (3.30)

where \( \beta \) is the correlation angle\(^{11}\). In equation (3.30) we recalled that \( \tau_{rec} \ll \tau_0 \), implying that \( k(\tau_0 - \tau_{rec}) \approx k \tau_0 \). The relevant power spectra and the spherical Bessel functions can be written as\(^{12}\)

\[ P_R = A_R \left( \frac{k}{k_p} \right)^{n-1}, \quad P_\Omega(k) = \mathcal{E}_R \left( \frac{k}{k_L} \right)^{2(n-1)}, \quad j_0(z) = \frac{\pi}{2z} J_{\ell+1/2}(z), \]  (3.31)

the integrals over the Bessel functions can be analytically performed since, as it is well known [46]

\[ \int_0^\infty J_\nu(\alpha y) J_\mu(\alpha y) y^{-\lambda} dy = \frac{\alpha^{\lambda-1} \Gamma(\mu) \Gamma\left(\frac{\nu+\mu+\lambda+1}{2}\right)}{2^{\lambda} \Gamma\left(\frac{\nu+\mu+\lambda+1}{2}\right) \Gamma\left(\frac{\nu+\lambda+1}{2}\right)}. \]  (3.32)

The ISW contribution is more complicated and, as previously remarked, it cannot be computed in fully analytic terms since each term contains two time derivatives of the transfer functions:

\[ C^{(ISW)}_\ell = 16\pi \int_0^\infty \frac{dk}{k} P_R(k) \int_0^{\tau_{rec}} dT_R \frac{dT_R}{d\tau_1} j_0(k \Delta \tau_1) d\tau_1 \int_0^{\tau_{rec}} dT_R \frac{dT_R}{d\tau_2} j_0(k \Delta \tau_2) \]

\[ + 16\pi R_\gamma^2 \int_0^\infty \frac{dk}{k} P_\Omega(k) \int_0^{\tau_{rec}} dT_B \frac{dT_B}{d\tau_1} j_0(k \Delta \tau_1) d\tau_1 \int_0^{\tau_{rec}} dT_B \frac{dT_B}{d\tau_2} j_0(k \Delta \tau_2) \]

\[ + 32\pi R_\gamma \cos \beta \int_0^\infty \frac{dk}{k} \sqrt{P_R(k) \sqrt{P_\Omega(k)}} \int_0^{\tau_{rec}} dT_B \frac{dT_B}{d\tau_1} j_0(k \Delta \tau_1) d\tau_1 \int_0^{\tau_{rec}} dT_B \frac{dT_B}{d\tau_2} j_0(k \Delta \tau_2), \]  (3.33)

\(^{11}\) The general idea behind [13, 14] (see also [15–17]) has been to include large-scale magnetic fields at all the stages of the Einstein–Boltzmann hierarchy and, in particular, at the level of the initial conditions. This means that the solutions such as the magnetized adiabatic mode and the various non-adiabatic modes are regular (in a technical sense) and well defined when the magnetized contribution is correctly supplemented by the curvature contribution. The correlation between the components is automatic already at the level of the initial conditions. At earlier times the correlation is also suggested, incidentally, by models where large-scale magnetic fields are generated during an inflationary stage [45].

\(^{12}\) The scales \( k_p \) and \( k_L \) are the two pivot scales at which the amplitudes of \( P_R(k) \) and of \( P_\Omega(k) \) are assigned. The pivotal values of these quantities will be recalled in a moment but can also be found in equations (3) and (5) of [15] (see also [16, 17] for further details).
where $\Delta t_1 = (\tau_0 - \tau_1)$ and $\Delta t_2 = (\tau_0 - \tau_2)$ and $\tau_1$ and $\tau_2$ are integration variables. The last contribution listed in equation (3.29) is the cross term whose explicit expression turns out to be

$$C_{(cross)} = \frac{16\pi}{5} SW_R^2(\alpha_{rec}) \int_0^\infty \frac{dk}{k} P_R(k) \int_0^{\tau_0} \left( \frac{dT_R}{d\tau} \right) j_i(k\tau_0) j_i(k\Delta \tau) d\tau$$

$$+ \frac{4\pi}{5} R^2_{\tau} SW_B^2(\alpha_{rec}) \int_0^\infty \frac{dk}{k} P_{\Omega}(k) \int_0^{\tau_0} \left( \frac{dT_B}{d\tau} \right) j_i(k\tau_0) j_i(k\Delta \tau) d\tau$$

$$+ \frac{16\pi}{5} R_{\tau} \cos \beta SW_B(\alpha_{rec}) \int_0^\infty \frac{dk}{k} \sqrt{P_{\Omega}(k) P_R(k)} \int_0^{\tau_0} \left( \frac{dT_B}{d\tau} \right) j_i(k\tau_0) j_i(k\Delta \tau) d\tau$$

$$+ \frac{16\pi}{5} R_{\tau} \cos \beta SW_B(\alpha_{rec}) \int_0^\infty \frac{dk}{k} \sqrt{P_{\Omega}(k) P_R(k)} \int_0^{\tau_0} \left( \frac{dT_B}{d\tau} \right) j_i(k\tau_0) j_i(k\Delta \tau) d\tau,$$

(3.34)

where $k\Delta \tau = (\tau_0 - \tau)$.

Even if the time evolution of $T_R(\tau)$ and $T_B(\tau)$ must be obtained numerically, equations (3.33) and (3.34) can be further simplified by anticipating the integration over $k$. Since both $P_R(k)$ and $P_{\Omega}(k)$ have a power dependence upon $k$, the expressions appearing in equation (3.33) can be made just more explicit by taking into account that the integral (see equation (C4))

$$\int_0^\infty k^{-\lambda} J_n(ka) J_n(kb) dk$$

(3.35)

is proportional to a confluent hypergeometric function [46, 47]. Using the quadratic transformations [47], the confluent hypergeometric functions can be related to Legendre functions. This analysis is reported in appendix C (see equations (C4) and (C14)). The expressions of appendix C generalize the results of [8, 9] holding in the Harrison–Zeldovich limit.

It is appropriate to recall the conventions leading to the power spectra mentioned in equation (3.31). This topic has been thoroughly discussed (see [15–17] and references therein) and will just be swiftly mentioned here. The power spectra of the magnetic fields are assigned exactly in the same way as the spectra of curvature perturbations: it would be rather weird to have a convention for the power spectrum of curvature perturbations and a totally different one for the magnetic power spectrum. In the $\Lambda$CDM paradigm (as in the usual $\Lambda$CDM case) the temperature and polarization inhomogeneities are solely sourced by the fluctuations of the spatial curvature $R$ whose Fourier modes obey

$$\langle R(\vec{k}) R(\vec{p}) \rangle = \frac{2\pi^2}{k^3} P_R(k) \delta^{(3)}(\vec{k} + \vec{p}), \quad P_R(k) = A_R \left( \frac{k}{k_p} \right)^{n_R-1};$$

(3.36)

in the $\Lambda$CDM case and in the light of the WMAP 5 year data alone [3–5] $A_R = (2.41 \pm 0.11) \times 10^{-9}$, as already mentioned in section 1 the pivot scale $k_p$ is 0.002 Mpc$^{-1}$. In full analogy with equation (3.36) the ensemble average of the Fourier modes of the magnetic field are given by

$$\langle B(\vec{k}) B(\vec{p}) \rangle = \frac{2\pi^2}{k^3} P_{B}(k) \delta^{(3)}(\vec{k} + \vec{p}), \quad P_B(k) = A_B \left( \frac{k}{k_L} \right)^{n_B-1};$$

(3.37)

where $P_{B}(k) = (k^2 \delta_{ij} - k_i k_j) / k^2$ and $A_B$ the spectral amplitude of the magnetic field at the pivot scale $k_L = \text{Mpc}^{-1}$ [15, 16]. In the case when $n_B > 1$ (i.e. blue magnetic field spectra), $A_B = (2\pi)^{n_B-1} B_L^2 / \Gamma[(n_B - 1)/2]$; if $n_B < 1$ (i.e. red magnetic field spectra), $A_B = [(1-n_B)/2](k_A/k_L)^{(1-n_B)} B_L^2$ where $k_A$ is the infrared cut-off of the spectrum. In
the case of white spectra (i.e. \( n_B = 1 \)) the two-point function is logarithmically divergent in real space and this is fully analog to what happens in equation (3.36) when \( n_s = 1 \), i.e. the Harrison–Zeldovich (scale-invariant) spectrum. By selecting \( k_{\text{L}}^{-1} \) of the order of the Mpc scale the comoving field \( B_{\text{L}} \) represents the (frozen-in) magnetic field intensity at the onset of the gravitational collapse of the protogalaxy [17]. On top of the power spectrum of the magnetic energy density it is necessary to compute and regularize also the power spectrum of the anisotropic stress and explicit discussions can be found in [16, 17].

4. Magnetized initial conditions with dark energy

If the dark energy component is fluctuating there are, in principle, various ways in which initial conditions of the Einstein–Boltzmann hierarchy can be set. In general terms the pre-decoupling plasma contains five\(^{13}\) physically different components (see, e.g., section 2). In the ΛCDM case the dark energy does not fluctuate and, therefore, the way initial conditions are set does not differ from the CDM scenario insofar as the entropy fluctuations of the whole plasma vanish for large scales, i.e. for \( k/\mathcal{H} \ll 1 \). The latter condition can be expressed by requiring that \( S_{ij} = 0 \) where \( i \) and \( j \) denote a generic pair of fluids in the mixture and where

\[
S_{ij} = 3(\zeta_i - \zeta_j) \quad (4.1)
\]

\[
\zeta_i = -\psi + \frac{\delta_{\text{cn}}^{(i)}}{3(w_i + 1)} = \xi + \frac{\delta^{(S)}_{i}}{3(w_i + 1)}, \quad (4.2)
\]

are, by definition, the entropy fluctuations of the system. The second equality in equation (4.2) follows by transforming the fluctuation variables from the conformally Newtonian to the synchronous gauge. In what follows the condition of equation (4.1) forbids non-adiabatic fluctuations in the dark energy sector as well as in the fluid sector.

If the dark energy background is consistently included in the initial conditions together with large-scale magnetic fields the initial conditions of the Einstein–Boltzmann hierarchy must be supplemented, in the conformally Newtonian gauge, by the following pair of conditions:

\[
\delta_{\text{de}}(k, \tau) = -\frac{3}{2}(w_{\text{de}} + 1)\phi_{\text{de}}(k) - \frac{3}{4}(w_{\text{de}} + 1)R_{\text{d}}\Omega_{\text{d}}(k) + \mathcal{O}\left(\frac{k}{\mathcal{H}}\right), \quad (4.3)
\]

\[
\theta_{\text{de}}(k, \tau) = \frac{(1 - w_{\text{de}})}{(1 + w_{\text{de}})(1 - 3w_{\text{de}})}\phi_{\text{de}}(k)k^2 - \frac{w_{\text{de}}}{(1 + w_{\text{de}})(1 - 3w_{\text{de}})}R_{\text{d}}\Omega_{\text{d}}(k)k^2\mathcal{H}, \quad (4.4)
\]

where the second equation holds under the assumption that \( w_{\text{de}} \neq -1 \) and that \( w_{\text{de}} < 0 \). For the actual numerical integration it turns out to be very useful to reshuffle the dependence of the dark energy variables in terms of two generalized potentials defined, in Fourier space, as

\[
\theta_{\text{de}}(k, \tau) = \frac{k^2}{a\sqrt{\rho_{\text{de}}}}g(k, \tau) \quad (4.5)
\]

\[
\delta_{\text{de}}(k, \tau) = \sqrt{\rho_{\text{de}}} \frac{1}{a(1 + w_{\text{de}})} \left[ f(k, \tau) - \frac{3}{2}\mathcal{H}(1 - w_{\text{de}})g(k, \tau) \right]. \quad (4.6)
\]

\[
\delta_{\text{de}}(k, \tau) = \sqrt{\rho_{\text{de}}} \frac{1}{a(1 + w_{\text{de}})} \left[ f(k, \tau) + \frac{3}{2}\mathcal{H}(1 - w_{\text{de}})g(k, \tau) \right]. \quad (4.7)
\]

\(^{13}\)This statement holds under the assumption that electrons and protons are tightly coupled by Coulomb scattering. If we count separately the electrons and the ions the total number of components increases to 6.
Using equations (4.5), (4.6) and (4.7) into equations (2.33) and (2.34) we do get a system for $f$ and $g$:

$$
g' = f + a \sqrt{\rho_c} \phi, \tag{4.8}$$

$$
f' = -2\mathcal{H} f - \left\{ k^2 - \frac{3}{4} (1-w)[2\mathcal{H}' - (3w_{de} + 5)\mathcal{H}^2] \right\} g + 3a \sqrt{\rho_c} \left[ \psi' + \frac{\mathcal{H}(1 - w_{de})}{2} \phi \right]. \tag{4.9}$$

The corresponding equations in the synchronous gauge can be obtained just by replacing, in equations (4.8) and (4.9),

$$
f \rightarrow \tilde{f}, \quad g \rightarrow \tilde{g}, \quad \phi \rightarrow 0, \quad \psi' \rightarrow \frac{h'}{6}. \tag{4.10}$$

The resulting equations are even simpler to integrate numerically:

$$
\tilde{g}' = \tilde{f}, \tag{4.11}
$$

$$
\tilde{f}' = -2\mathcal{H} \tilde{f} - \left\{ k^2 - \frac{3}{4} (1-w)[2\mathcal{H}' - (3w_{de} + 5)\mathcal{H}^2] \right\} \tilde{g} + a \sqrt{\rho_c} \frac{h'}{2}. \tag{4.12}
$$

Equations (4.11) and (4.12) are the ones which are numerically integrated. The initial conditions of the remaining components of the Einstein–Boltzmann hierarchy do not change and are given as in \cite{15, 16}. To be more specific, equation (2.54) of \cite{16} contains the initial conditions used in the numerical discussion which will be reported in section 5.

5. Numerical results

The choice of parameters of the pivotal ΛCDM model in the light of the WMAP 5 year data alone is given by \cite{3–5}

$$(\Omega_b, \Omega_c, \Omega_{de}, h_0, n_s, \epsilon_{re}) \equiv (0.0441, 0.214, 0.742, 0.719, 0.963, 0.087), \tag{5.1}$$

where consistently with the established notations (see, e.g., equations (3.1) and (3.3)), $\epsilon_{re}$ denotes the optical depth to reionization. The parameters of equation \eqref{5.1} maximize the likelihood when the barotropic index of the dark energy is fixed to $-1$.

The first step of the numerical analysis is to compute $T_R(\tau)$ and $T_B(\tau)$ when the parameters are given exactly by equation \eqref{5.1} but the barotropic index of the dark energy assumes arbitrary values which can differ from the ΛCDM choice (i.e. $w_{de} = -1$). The functions $T_R(\tau)$ and $T_B(\tau)$ have been introduced in equation \eqref{3.25} and control the two relevant large-scale contributions to the ISW effect. According to equations \eqref{3.26} and \eqref{3.27}, $T_R(\tau)$ and $T_B(\tau)$ can be assessed once the (numerical) evolution of the background geometry has been obtained. Such a technique is rather useful for the analytic estimate of the asymptotic limits and it otherwise demands the full (numerical) solution of the background geometry; the results of figures 1 and 2 $T_R(\tau)$ and $T_B(\tau)$ are obtained by direct numerical integration. It has been explicitly verified that, indeed, equations \eqref{3.26} and \eqref{3.27} are in excellent agreement with the numeric result once the full numerical expression of the scale factor is known. In figure 1 (plot on the left) on the vertical axis the common logarithm of the scale factor is illustrated while on the horizontal axis the convenient time variable is given by $y = 3.33 \times 10^{-4} (\tau/\text{Mpc})$ where $\tau$, as usual, is the conformal time coordinate. Using
y as the integration variable, equation (2.1) and the related initial conditions become very simple\(^\text{14}\):

\[
\frac{da}{dy} = \sqrt{\omega_R + \omega_M a + \omega_{\text{de}} a^{1 - 3w_{\text{de}}}}. \tag{5.2}
\]

\[
a(y_i) = \frac{\omega_M}{4} y_i^2 + \sqrt{\omega_R y_i}, \tag{5.3}
\]

where \(\omega_R = h_0^2 (\Omega_\gamma + \Omega_\kappa), \omega_M = h_0^2 (\Omega_b + \Omega_\Lambda)\) and \(\omega_{\text{de}} = h_0^2 \Omega_{\text{de}}\). The physical range of the parameters and the present normalization of the scale factor imply that \(a_0 = 1\). To illustrate

\(^{14}\) To avoid potential confusions, it is appropriate to remark that the integration variable of equations (5.2) and (5.3) has nothing to do with the variables \(x\) and \(y\) introduced in section 3. In equations (5.2) and (5.3) the variable \(y\) is simply given by \(h_0 y = \tau H_0\) where \(H_0\) is the present value of the Hubble constant and \(h_0\) its (dimensionless) indetermination. Conversely, in section 3, \(x\) and \(y\) have been used as (dimensionful) integration variables coinciding exactly with the conformal time coordinate.
the trends induced by different values of $w_{de}$ it is useful to plot the quantities of figure 1 also a bit outside their physical range (i.e. for $a > a_0$). The initial condition of equation (5.3) are imposed by making use of an exact solution of equation (5.2) (as well as of the other Friedmann–Lemaître equations (2.1)) in the limit $\rho_{de} \to 0$. The form of the initial conditions of equation (5.3) implies that $y_i$ is well before matter–radiation equality.

Always in figure 1 (plot on the right), the profile of $T_R(y)$ is illustrated. As expected, around $y_i$ (i.e. deep in the radiation-dominated epoch), $T_R(y_i) \equiv 2/3$. Then, as $y$ increases, $T_R(y)$ reaches a flat plateau for intermediate times around decoupling where $T_R(y) \simeq 3/5$ (see figure 1, plot on the right). Absent any late dominance of dark energy, $T_R(y) \to 3/5$ exactly. The (late) deviation from the CDM asymptote (i.e. 3/5) depends upon the specific value of $w_{de}$ and determines the ISW contribution. In figure 2 the function $T_B(y)$ is illustrated both on a linear time scale (plot on the left) and using a logarithmic time scale (plot on the right) which better represents the early evolution before matter–radiation equality and across decoupling. The arrow appearing in the left plot of figure 2 marks, approximately, the present time, i.e. when $a(y_0) \simeq a_0 = 1$. In short the shape of $T_B(y)$ can be understood as follows:

- unlike the case of $T_R(y)$ (which reaches a constant asymptote deep in the radiation-dominated epoch, i.e. for $y \ll 1$), $T_B(y) \to 0$ in the limit $y \to 0$;
- in the CDM paradigm $T_B(y)$ would reach the asymptotic value $3R_y/20 \simeq 0.089$;
- both in the $\Lambda$CDM and in the $w$CDM cases the would be asymptotes turn into an intermediate plateau (see figure 2, in particular plot on the right);
- the intermediate plateau is accurately estimated by $3R_y/20$ while the presence of a dark energy background is responsible for the deviation of $T_B(y)$ from the CDM asymptote.

The vanishing of $T_B(y)$ for $y \to 0$ is a consequence of the evolution of $R(k, y)$ and, in particular, of equations (2.40) and (2.41). The same observation implies that whenever $\mathcal{R}_a(k) \to 0, R(k, y) \to 0$ for $y \to 0$: such an occurrence is verified (and illustrated) in the left plot of figure 3 where $\mathcal{R}(k, y)$ is shown to vanish in the limit $\mathcal{R}_a(k) \to 0$.

It can be speculated that the contribution coming from $\Omega_B(k)$ and $\mathcal{R}_a(k)$ might interfere destructively leading to an overall suppression of the large-scale contribution and, in particular, of the quadrupole. In figure 3 (plot on the right) this possibility seems excluded unless
Figure 4. The angular power spectra of the CMB observables in the light of the $\omega$CDM scenario when large-scale magnetic fields are included. In all the plots, the barotropic index of the (fluctuating) dark energy background has been fixed to $w_{de} = -0.6$. 

$\Omega_B(k) \simeq R_s(k)$. But such an extreme value would jeopardize the agreement of the theory with the acoustic region.

To scrutinize this point in a model-independent perspective, consider the condition $\Omega_B(k) \simeq R_s(k)$ in loose terms and postulate that the amplitude of curvature perturbations matches approximately the amplitude of the power spectrum of the magnetic energy density. Neglecting, for simplicity, the dependence upon the spectral indices we get a typical amplitude for the comoving amplitude of the magnetic field intensity which is of the order of $B_L \simeq 22.68$ nG where it has been assumed that $A_R = 2.4 \times 10^{-9}$, $k_L = 1$/Mpc and $k_p = 0.002$ Mpc$^{-1}$. The putative value of about 23 nG is indeed quite large and it would totally disrupt the structure of the acoustic oscillations. This aspect can be appreciated from figure 4 where a magnetic field of 10 nG already jeopardizes the observed features of the temperature autocorrelations$^{15}$.

In the minimal scenario [15, 51], the parameter space of the magnetized CMB anisotropies can be described in terms of the magnetic spectral index $n_B$ and the regularized magnetic field amplitude $B_L$. These two quantities and their relations with $\Omega_B$ and $\sigma_B$ (introduced in section 2) have been thoroughly discussed in previous papers [15–17]. The basic advantages of such a parametrization have been illustrated at the end of section 3 (see, in particular,

\[ \Omega_B(k) \simeq R_s(k) \] 

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equation (3.37) and equation (3.36) for the analog conventions employed in the definition of the power spectra of curvature perturbations).

In figure 4 the parameters are fixed to the best fit of the WMAP 5 year data alone in the light of the ΛCDM paradigm (corresponding to \( w_{de} = -1 \)). Always in figure 4 the barotropic index of the dark energy is increased from \(-1\) to \(-0.6\) while the magnetized background is switched on. The magnetic field intensity has been chosen to be rather large (i.e. \( \mathcal{O}(10 \, \text{nG}) \)). Examples of blue (i.e. \( n_B = 1.5 \)) and red (i.e. \( n_B = 0.5 \)) spectral indices are illustrated. From the TT correlations\(^{16} \) in semilogarithmic coordinates (top-right plot) as well as from the TE and EE correlations, the general interplay of the dark energy and magnetized backgrounds is more clear and can be summarized as follows:

- in the large-scale domain (i.e. \( \ell < 50 \)), both the dark energy and the magnetized contribution augment the power in the TT angular power spectra;
- possible interference effects leading to an overall suppression of the lower multipoles are excluded if the acoustic oscillations are to be reproduced correctly; this statement holds in the absence of any non-adiabatic contribution (i.e. \( \delta_{p\text{nad}} = 0 \) in equation (2.40)) but it might change if \( \delta_{p\text{nad}} \neq 0 \);
- the interplay between the magnetized and the dark energy backgrounds leads to a combined distortion of the peaks in the TT, TE and EE angular power spectra; since the level of distortion increases with the multipole, the higher peaks look also shifted in comparison with the patterns exhibited by the underlying ΛCDM model with the same parameters and in the absence of any magnetic field.

The value \( w_{de} = -0.6 \) chosen in figure 4 is rather extreme insofar as it is excluded by current fits to the WMAP 5 year data (either alone or in combination with other data sets). It is however useful highlighting some general trend induced by the dark energy fluctuations.

Figures 1, 2 and 3 also contain examples where \( w_{de} < -1 \). When analyzing the cosmological datasets in the light of a fluctuating dark energy background it can happen that the central value of the barotropic index maximizing the likelihood for a specific dataset gets smaller than \(-1\). In the latter case, future singularities are expected (see for instance [48]). In what follows, the cases \( w_{de} < -1 \) will be considered for completeness. Indeed the central values of the barotropic index is still debatable and does depend upon the datasets which are combined in the analysis. For the purposes of the present paper, the situation is summarized in the next three paragraphs.

By analyzing the WMAP 5 year data alone in the light of the \( w \)CDM model we do get \( w_{de} = -1.06^{+0.41}_{-0.42} \) while the remaining parameters are determined to be

\[
(\Omega_b, \Omega_c, \Omega_{de}, h_0, n_s, \epsilon_{re}) \\
\equiv (0.046^{+0.018}_{-0.018}, 0.22^{+0.087}_{-0.082}, 0.733^{+0.10}_{-0.11}, 0.74^{+0.15}_{-0.14}, 0.962^{+0.016}_{-0.016}, 0.086^{+0.017}_{-0.016}).
\]  \ (5.4)

in the case of the parameters of equation (5.4), the corresponding value of the normalization of the curvature perturbations is still \( A_K = 2.41 \times 10^{-9} \) at the pivot scale of \( k_p = 0.002 \, \text{Mpc}^{-1} \); the value of \( A_K \) does not change in comparison with the corresponding value determined in connection with the parameters of equation (5.1).

\(^{16}\) The TT power spectra denote, as usual, the autocorrelations of the temperature. The TE power spectra are the cross-correlations between temperature and polarization. The EE power spectra denote the polarization autocorrelations. Within the conventions employed in the present paper, the definitions of the TT, TE and EE power spectra in terms of the solutions of the heat transfer equations can be found in [16] (see also [17] and references therein). The B-mode polarization induced by Faraday rotation [49, 50] will not be specifically addressed in this context. It has been actually shown in [49] (second paper of the list) that the present data on the B-mode autocorrelations only allow for upper bounds on the Faraday-induced B-mode. An exception to this statement are the QUAD data [52–54] but, at the moment, its not clear to what extent the results are contaminated by systematics [53].
By combining the WMAP 5 year data with the data stemming from the analysis of the baryon acoustic oscillations (BAO in what follows) [55–57] the preferred range of values of $w_{de}$ becomes $-1.15^{+0.21}_{-0.22}$; the remaining parameters turn out to be [3–5]

$$\begin{align*}
(\Omega_b, \, \Omega_c, \, \Omega_{de}, \, h_0, \, n_s, \, \epsilon_{re})
\equiv & \left(0.041^{+0.0055}_{-0.0056}, \, 0.213^{+0.021}_{-0.021}, \, 0.742^{+0.026}_{-0.026}, \, 0.739^{+0.047}_{-0.048}, \, 0.958^{+0.015}_{-0.015}, \, 0.083^{+0.016}_{-0.016}\right),
\end{align*}$$

(5.5)

the value of $A_R$ determined in connection with the data of equation (5.5) increases in comparison with the corresponding values holding in the case of equations (5.1) and (5.4) and it is given by $A_R = 2.48 \times 10^{-9}$.

Finally, by combining the WMAP 5 year data with the BAO and with all the supernovae [58–60], the central value of $w_{de}$ maximizing the likelihood gets above $-1$ and it is $w_{de} = -0.972^{+0.061}_{-0.060}$; the remaining parameters are given, in this case, by [3–5]

$$\begin{align*}
(\Omega_b, \, \Omega_c, \, \Omega_{de}, \, h_0, \, n_s, \, \epsilon_{re})
\equiv & \left(0.047^{+0.018}_{-0.0018}, \, 0.231^{+0.014}_{-0.014}, \, 0.722^{+0.015}_{-0.015}, \, 0.697^{+0.014}_{-0.014}, \, 0.962^{+0.014}_{-0.014}, \, 0.085^{+0.016}_{-0.016}\right),
\end{align*}$$

(5.6)

again the value of $A_R$ determined in connection with the data of equation (5.5) diminishes in comparison with the corresponding values holding in the case of equations (5.5) and it is $A_R = 2.43 \times 10^{-9}$.

The data reported in equations (5.4), (5.5) and (5.6) will be referred to as, respectively, WMAP 5 year alone, WMAP 5 year + bao and WMAP 5 year + bao + snall, where the acronyms refer, respectively, to the baryon acoustic oscillations (bao) and to the supernova all (snall) data. Bearing in mind the latter shorthand notations it is now appropriate to scrutinize how the simultaneous presence of large-scale magnetic fields and of the dark energy fluctuations affects the temperature and the polarization observables. This aspect is illustrated in three of the forthcoming figures, i.e. in figures 5, 6 and 7 where the magnetized background is studied together with the dark energy background for the reference models discussed in equations (5.4), (5.5) and (5.6). In figure 5 the parameters of the $\omega$CDM model have been chosen as in equation (5.4) with $w_{de} = -1.06$ corresponding to the best fit in the light of the WMAP 5 year data points. It is superficially clear that the string of parameters of equation (5.1) does not differ, qualitatively, from the ones of equation (5.4). As expected, the inclusion of the magnetized contribution distorts the higher peaks. Three representative values of the parameter space of the magnetized $\omega$CDM scenario are illustrated:

1. the curves with $n_B = 1.5$ and $B_i = 10$ nG are representative of the region with high magnetic field intensity and blue spectral index (i.e. $n_B > 1$);
2. the curves with $n_B = 0.5$ and $B_i = 10$ nG are representative of the region with high magnetic field intensity and red spectral index (i.e. $n_B < 1$);
3. the curves with $n_B = 1.5$ and $B_i = 5$ nG are representative of the region with intermediate magnetic field intensities and blue spectral index.

The regions of the parameter space corresponding to (i) and (ii) are roughly incompatible with the observed angular power spectra as it can be easily argued by superimposing the curves to the binned data of the TT and TE correlations [3–5]. Furthermore, the values of the parameters of regions (i) and (ii) lie in a portion of the parameter space which has been excluded by estimating the parameters of the magnetized background in the light of the magnetized extension of the standard $\Lambda$CDM scenario. Does the inclusion of a fluctuating dark energy background make the difference? The answer to this question will be the subject of the remaining considerations of this section. Note also that the choice of the parameters
corresponding to (iii) seems to be qualitatively compatible with the observations. One of the purposes of the following discussion will be to make such a statement more quantitative.

As a last remark concerning figure 5, the value of the barotropic index of dark energy (i.e. \( w_{de} = -1.06 \)) is rather close to the \( \Lambda \)CDM determination and this occurrence excludes interference effects such as the ones observed in figure 4 where \( w_{de} \) differs from \(-1\) appreciably (i.e. \( w_{de} = -0.6 \)). In figures 6 and 7 the dark energy background and the other \( \Lambda \)CDM parameters are fixed, respectively, as in equations (5.5) and (5.6). The considerations suggested by figure 4 are confirmed, in some sense, also by figures 5 and 6. Extreme values of the magnetic field intensity \( \mathcal{O}(10 \text{ nG}) \) are incompatible with the observations both in the case of red and blue spectral indices.

The joined two-dimensional marginalized contours for the various cosmological parameters identified already by the analyses of the WMAP 3 year data are ellipses with an approximate Gaussian dependence on the confidence level as they must be in the Gaussian approximation (see, e.g., [1]). The parameter space of the magnetized CMB anisotropies can then be scanned in the presence of dark energy fluctuations by using the same technique discussed in [15, 16] and applied to the case of the \( \Lambda \)CDM paradigm.

In figure 8 the filled (ellipsoidal) spots in both plots mark the minimal value of the Chi-squared (i.e. \( \chi^2_{\text{min}} \)) obtained in the context of the WMAP 5 year data alone (see, e.g., (5.4)); the closed curves
Figure 6. Large-scale magnetic fields are included for the \( \omega \)CDM best fit when the WMAP 5 year data are combined with the ones stemming from the baryon acoustic oscillations. As in figure 5 the fluctuations in the dark energy background have been taken into account; \( \omega_{de} = -1.15 \) (as it must be from equation (5.5)).

represent the likelihood contours in the two parameters \( n_B \) and \( B_L \). In both plots of figure 8 the boundaries of the two regions contain 68.3% and 95.4% of likelihood as the values for which the \( \chi^2 \) has increased, respectively, by an amount \( \Delta \chi^2 = 2.3 \) and \( \Delta \chi^2 = 6.17 \). In the plot on the left of figure 1 the data points used for the analysis correspond to the measured TT correlations contemplating \( N_{TT} = 999 \) experimental points from \( \ell = 2 \) to \( \ell = 100 \). In the plot on the right the data points include, both, the TT and TE correlations and the total number of data points increases to \( N_{TT} + N_{TE} = 1998 \). The values of \( n_B \) and \( B_L \) minimizing the \( \chi^2 \) when only the TT correlations are considered turn out to be \( n_B = 1.883 \) and \( B_L = 4.982 \) nG; in this case the reduced \( \chi^2 \) is 1.094. When also the TE correlations are included in the analysis the reduced \( \chi^2 \), i.e. \( \chi^2_{\text{red}} \), diminishes from 1.094 to 1.033 and the values of \( n_B \) and \( B_L \) minimizing the \( \chi^2 \) become \( n_B = 1.913 \) and \( B_L = 5.163 \) nG. The points of the parameter leading to \( \chi^2_{\text{min}} \) maximize the likelihood in the Gaussian approximation (see [15] and [16]).

In a frequentist approach, the boundaries of the confidence regions represent exclusion plots at 68.3% and 95.4% confidence level. In figure 8 (plot on the right) the inner dashed curve corresponds to the 68.3% boundary while the outer dashed line corresponds to the 95.4% boundary as determined in the context of the magnetized \( \Lambda \)CDM scenario and with the very same set of data [15]. By comparing the dashed and full contour plots in figure 8 (right plot) the shapes of the excluded regions are compatible in the two cases. The overlap
Figure 7. Large-scale magnetic fields are included for the \( \Lambda \)CDM best fit when the WMAP 5 year data are combined with the baryon acoustic oscillations and with all supernova data. As in figures 5 and 6 the fluctuations in the dark energy background have been taken into account; \( w_{de} = -0.972 \) (in agreement with equation (5.6)).

seems even to increase by increasing the confidence range. At the same time the addition of a fluctuating dark energy background pins down systematically larger values of the magnetic field parameters. More specifically, by looking at the parameters corresponding to \( \chi^2_{\text{min}} \), we can say that

\[
(n_B, B_L)_{\Lambda \text{CDM}} = (1.598, 3.156\text{nG}) \rightarrow (n_B, B_L)_{\text{wCDM}} = (1.883, 4.982\text{nG}),
\]

\[
(n_B, B_L)_{\Lambda \text{CDM}} = (1.616, 3.218\text{nG}) \rightarrow (n_B, B_L)_{\text{wCDM}} = (1.913, 5.163\text{nG}),
\]

where equation (5.7) corresponds to \( N_{TT} = 999 \) (i.e. only the unbinned points of the TT correlations are used) while equation (5.8) does correspond to \( N_{TT} + N_{TE} = 1998 \) (i.e. the unbinned points of the TT and TE correlations are used).

Equations (5.7) and (5.8) show that the magnetic field parameters minimizing the \( \chi^2 \) (and maximizing the likelihood) do (slightly) increase when the dark energy component is allowed to fluctuate. Of course the slight numerical difference does not change the physical conclusions obtained in the context of the magnetized \( \Lambda \)CDM class of models: the preferred region is for blue spectral indices and moderate magnetic field intensities. In figure 9 the likelihood contours are obtained when the best-fit parameters are determined from the \( \Lambda \)CDM parameters arising when the WMAP 5 year data are combined with the data stemming from the baryon acoustic
Figure 8. Likelihood contours in the plane \((n_B, B_L)\). In both plots the inner curve corresponds to \(\Delta \chi^2 = 2.3\) (i.e. 68.3% or 1 \(\sigma\) of the likelihood content) while the outer curve corresponds to \(\Delta \chi^2 = 6.17\) (i.e. 95.4% or 2 \(\sigma\) of the likelihood content).

Figure 9. Same quantities as in figure 8 but in the case when \(\omega\)CDM scenario is analyzed in the light of the WMAP 5 years +bao data (see equation (5.5)).

oscillations (see equation (5.5)). In this case, again, the shapes of the likelihood contours are qualitatively a bit different but quantitatively compatible with the cases of figure 8 and of equation 9. As expected a decrease of \(k_L\) lowers the value of the regularized magnetic field intensity \(B_L\). This effect is illustrated in figure 10. In the right plot, the likelihood contours are computed when \(k_L = 0.1\) Mpc\(^{-1}\); the parameters of the \(\omega\)CDM paradigm are determined from equation (5.4). In the plot on the left the same analysis is performed for the \(\Lambda\)CDM case when, again, the other parameters are determined from the WMAP 5 year data alone. The exclusion plots on the left side of figures 8, 9 and 10 explain, \textit{a posteriori}, why in the plots of
Figure 10. The effect of change in the magnetic pivot scale $k_L$ is illustrated in the case of the WMAP 5 year data alone in the $\Lambda$CDM case (left plot) and in the $\omega$CDM case (plot on the right). We stress that while figures 8 and 9 assume a magnetic pivot scale of $1\,\text{Mpc}^{-1}$, the present figure assumes $k_L = 0.1\,\text{Mpc}^{-1}$.

figures 5, 6 and 7 the minimal amount of distortions was provided by the dot-dashed curves. The dot-dashed curves corresponded, in those figures, to the region (iii), i.e. the region where the moderate magnetic field intensities corresponded to blue magnetic spectral indices.

Before concluding this section it is appropriate to comment on two complementary themes: the first one deals with models of non-standard recombination (such as the ones discussed in [61–64]); the second point is related to the role of the magnetic pivot scale. It is one of the crucial hopes of modern cosmology that the number of ascertainable parameters increases with the quality of the observational data. In spite of this hope one must be aware that there could be degeneracies between the parameters and the models: this means, in loose terms, that the same effect can be accounted for by a variety of phenomena. An example along this direction are non-standard models of recombination. Some of the latter models contemplate more exotic effects (such as the variation of the fine structure constant) while some other recombination models simply postulate a different recombination history. Examples along this direction are given in [61] where sources of Ly$\alpha$ resonance radiation are considered around a redshift of, say, 1000. The effects discussed in [61] included possible suppressions of secondary peaks and plausible shifts of the position of the acoustic peaks. The shift in the acoustic peaks resembles a bit the one produced in the present context even if it seems quantitatively more pronounced. In spite of the fact that the authors of [61] acknowledged that their sources of Ly$\alpha$ resonance radiation were not at all natural (i.e. stars or even galactic nuclei at $z \sim 1000$) various reprises of this theme are now available in the literature. For instance in [62] not only sources of Ly$\alpha$ resonance radiation but also possible clustering in the baryonic component have been considered. Other studies along the latter lines can be found in [63, 64]. The effect of the shift in the peaks observed in figures 4 and 5 is just one of the aspects of the problem since large-scale magnetic fields have several other effects which cannot be reduced to a modification of the standard recombination scenario. Examples along this direction are the linear [49, 50] and circular polarization signatures [65, 66]. It is not implausible that, by
adding a sufficient number of supplementary parameters, one could imagine to contrive, by diverse means, all the signatures of magnetized CMB anisotropies. This type of scenario has not yet been considered in the literature and it is not clear how it could be tailored. Instead of speculating on the way one could fake the typical signatures of magnetized CMB anisotropies, let us give a procedure which could be used to disentangle the effects of delayed recombination and the ones of the magnetized CMB anisotropies. The idea is, in short, to consider the relative heights and positions of the different acoustic peaks for the TT, EE and TE correlations. The latter comparison is beyond the scopes of the present investigation but, still, it is useful to give an explicit numerical example. To illustrate this point let us consider, for instance, the heights and positions of the different acoustic peaks for the TT, EE and TE correlations. The idea is, in short, to consider the relative heights and positions of the magnetized CMB anisotropies. This type of scenario has not yet been considered in the literature and it is not clear how it could be tailored. Instead of adding a sufficient number of supplementary parameters, one could imagine to contrive, by diverse means, all the signatures of magnetized CMB anisotropies. This type of scenario has not yet been considered in the literature and it is not clear how it could be tailored. Instead of

where the results of \([15, 16]\) strictly apply). The best-fit parameters are the ones reported in equations (5.7) and (5.8) (as \((n_B, B_L)_{\Lambda CDM}\)). In particular we shall focus hereunder on the determination of equation (5.8) leading to \((\overline{H}_1, \overline{H}_2, \overline{H}_3) = (6.942, 0.447, 0.981)\) where, in the notations and conventions of \([15, 16]\), \((\overline{H}_1, \overline{H}_2, \overline{H}_3)\) are the relative heights of the first, second and third peaks\(^\text{17}\). By moving more than \(2\sigma\) away from the minimum of the \(\chi^2\) (i.e. \(n_B = 1.616\) and \(B_L = 6\ nG\)) we shall have that \((\overline{H}_1, \overline{H}_2, \overline{H}_3) = (7.142, 0.444, 1.012)\). The same analysis valid for the TT correlations can obviously be extended to the TE power spectra by monitoring the relative locations and heights of the correlations and anticorrelation peaks. In \([61]\) the authors gave an approximate formula for the quantity which we call here \(\overline{H}_2\) and which turns out to be (see equation (8) of \([61]\))

\[
\overline{H}_2 \simeq 0.7(1 + 3\epsilon_\alpha)^{-0.042} \left[1 + \left(\frac{h_0^2\Omega_b}{0.016}\right)^{4\gamma - 1/4}(2.4)^{n_t - 1}\right]. \tag{5.9}
\]

By fixing \(h_0^2\Omega_b = 0.02273\) and \(n_t = 0.963\) (i.e. the same putative values used also in our determination) we get, for \(\overline{H}_2, 0.451 \times (1 + 3\epsilon_\alpha)^{-0.042}\) where \(\epsilon_\alpha\) is the free parameter measuring the rate of production of Ly\(\alpha\) resonance photons. It is suggestive that the differences in the two scenarios can be appreciated already by looking at the second peak. Unfortunately we did not find in the literature explicit formulas parametrizing the contributions of \(\epsilon_\alpha\) to the higher acoustic peaks. At the same time the provided example shows that the two effects can be explicitly disentangled, at least in principle if not in practice. Future (i.e. higher quality) data might be essential in this respect.

There is a second point we wish to touch upon before the end of the present section and it is related with the role of the magnetic pivot scale (see, for instance, equation (3.36) of the paper under consideration). There is a degeneracy associated with the scaling of the magnetized contribution with the pivot scale \(k_L\). This degeneracy is exemplified in figure 10. Let us consider, to begin with, the case when dark energy fluctuations are absent. In this case, by comparing the contour plots of figures 9 and 10 (plots on the left in both figures) it is apparent that a decrease in \(k_L\) (from 1 to 0.1 Mpc\(^{-1}\)) reduces, as already discussed, the putative amplitude of the regularized magnetic field (say from 3.3 to 1.52 nG) but keeps almost the best-fit value of the spectral index (changing only from 1.67 to 1.61). The interesting thing happens when fluctuation dark energy is included in the analysis: comparatively larger values of the magnetic field and of the spectral index seem to be allowed. The result of the paper, in this respect, is that if we keep the spectral index and the regularized magnetic field intensity within the 2-sigma contours dramatic changes in the ISW effect are not plausible if we demand that the induced distortions do not have to jeopardize the TT and TE correlations.

\(^\text{17}\) More specifically \(\overline{H}_1 = \tilde{g}\tilde{c}_\ell /g_{\ell \rightarrow 100}, \overline{H}_2 = \tilde{g}\tilde{c}_{\ell 22}/g_{\ell \rightarrow 100}, \overline{H}_3 = \tilde{g}\tilde{c}_{\ell 33}/g_{\ell \rightarrow 100}\) where \(\tilde{g}_\ell = (\ell + 1)C_\ell^{TT} / (2\pi)\) and where \(\ell_1, \ell_2\) and \(\ell_3\) are, respectively, the locations of the first, second and third acoustic peaks.
6. Concluding remarks

The analysis of magnetized CMB anisotropies has been extended, for the first time, to the case when the dark energy background fluctuates. This class of models, conventionally labeled as magnetized $w$CDM, generalizes the magnetized $\Lambda$CDM scenario. The ordinary and integrated Sachs–Wolfe effects have been analyzed in the light of the $w$CDM paradigm. In the minimal realization of the scenario (i.e. in the absence of non-adiabatic components in the initial conditions), the contribution of the magnetized background is not sufficient to compensate the effect induced on the adiabatic mode by the late-time dynamics of the dark energy background. Possible compensating effects arise for excessively large values of the magnetic field intensity; these values are already constrained by the structure of the acoustic oscillations.

Since the modified initial conditions of the Einstein–Boltzmann hierarchy allow for the simultaneous presence of dark energy perturbations and of large-scale magnetic fields, the morphology of the distortions arising in the magnetized $w$CDM scenario has been compared with what happens when the dark energy background is not dynamical. The modifications induced by the fluctuating dark energy component on the shapes of the 1- and 2-$\sigma$ contours in the parameter space of the magnetized background have been computed. At 95% confidence level, the allowed spectral indices and magnetic field intensities turn out to be systematically larger than those determined in the framework of the magnetized $\Lambda$CDM paradigm.

The latter conclusions suggest a methodological observation. There are some, in the community, who say that the $\Lambda$CDM model together with a sufficiently structured single-field inflationary potential is the final holy grail of cosmology. Along the latter perspective the only thing we can hope for is a determination of the parameters of the inflaton potential. While this perspective is certainly legitimate, the point of view of the present paper is that it might well be that nature is more inventive than that. In other words, the success of the $\Lambda$CDM paradigm does not forbid the analysis of different scenarios, such as the one discussed in this paper, which are physically motivated and mathematically well defined.

Appendix A. Synchronous frame

It is understood that all the quantities appearing in this section of the appendix are evaluated in the synchronous frame. The evolution equation of the intensity brightness perturbations are given by

$$\Delta_I' + (ik\mu + \epsilon')\Delta_I = -\xi' - \frac{\mu^2}{2}(h + 6\xi)' + \epsilon' \left[ \Delta_{I0} + \mu v_b + \frac{(1 - 3\mu^2)S(k, \tau)}{4} \right], \quad (A.1)$$

$$v'_b + \frac{\epsilon'}{R_b} (3i\Delta_{11} + v_b) + i\mu \Omega_b - 4\sigma_b \frac{4}{4R_b}. \quad (A.2)$$

The formal solution of equation (A.1) can be written, after integrating once by parts, as

$$\Delta_I(k, \mu, \tau_0) = \int_0^{\tau_0} e^{-\epsilon(\tau, \tau_0)} e^{-i\mu x(\tau)} \ d\tau + \int_0^{\tau_0} K_\tau e^{-i\mu x(\tau)} \left[ \Delta_{I0} - \frac{(h + 6\xi)''}{2k} + \mu v_b + \frac{i\mu}{2k} (h + 6\xi)' + \frac{(1 - 3\mu^2)S(k, \tau)}{4} \right], \quad (A.3)$$

where, as in the bulk of the paper, $x(\tau) = k(\tau_0 - \tau)$. The visibility function is defined in terms of the optical depth as in equation (3.3). The integration by parts can be further used in
Recalling that, by definition, \( \alpha \) has been used explicitly. In equation (A.10) the various anisotropic stresses have been neglected; equation (A.10) is the synchronous version of equation (2.20), as it can be easily verified by using the procedure swiftly illustrated in section 1. Recalling that \( \Delta_0 = 4\delta_r \), the solution of a synchronous gauge version of equation (2.27) implies

\[
\Delta_0 = -\frac{R_y}{6}\Omega_B + \frac{h}{6} + O(k^2 \tau^2).
\]

To obtain the explicit form of equation (A.9) we need the evolution of \( \xi(k, \tau) \) and \( h(k, \tau) \) across equality. By using equations (2.8) and (2.38), we obtain

\[
\xi(k, \tau) = R_\alpha(k) - \frac{R_y}{8}\Omega_B(k)G_1(\alpha),
\]

\[
h(k, \tau) = \frac{3}{4}R_y\Omega_B(k)G_1(\alpha) + k^2\tau_1^2R_\alpha(k)G_2(\alpha) - \frac{3}{4}R_y\Omega_B(k)k^2\tau_1^2G_3(\alpha),
\]

\[
\Delta_0 = -\frac{R_y}{15}\Omega_B G_4(\alpha) + \frac{R_y}{20}\Omega_B G_5(\alpha),
\]

where \( \alpha = a/a_{eq} = [(\tau/\tau_1)^2 + 2(\tau/\tau_1)] \) solves equation (2.1) across equality and where the five functions \( G_i(\alpha) \) are given by

\[
G_1(\alpha) = \frac{2}{\alpha^2}[\alpha(\alpha - 4) + 8(\sqrt{\alpha + 1} - 1)],
\]

\[
G_2(\alpha) = \frac{[\alpha^3 - 2\alpha^2 + 8\alpha + 16(1 - \sqrt{\alpha + 1})]}{5\alpha^2},
\]
\[ G_1(\alpha) = \frac{32(\sqrt{\alpha + 1} - 1) + \alpha[3\alpha(3\alpha - 26) - 16] + 60\sqrt{\alpha + 1}[\alpha - \ln(\alpha + 1)]]}{45\alpha^3}, \quad (A.17) \]
\[ G_2(\alpha) = \frac{\sqrt{1 + \alpha}[3\alpha^2 - 4\alpha + 8] - 8}{\alpha^2}, \quad (A.18) \]
\[ G_3(\alpha) = \frac{\sqrt{\alpha + 1}[24 - \alpha^2 + 8\alpha] - 4(5\alpha + 6)}{\alpha^2}. \quad (A.19) \]

Using equations (A.15)–(A.19), the ordinary SW contribution can be written as
\[ \Delta_1^{(SW)}(k, \mu, \tau_0) = \left[ -\frac{R_\gamma}{5} - \frac{R_\gamma}{20} \right], \quad (A.20) \]
where \( SW_R(\alpha_{rec}) \) and \( SW_B(\alpha_{rec}) \) are the two functions appearing, respectively, in equations (3.14) and (3.15).

**Appendix B. Explicit derivation of \( T_R(\tau) \) and \( T_B(\tau) \)**

Consider equation (2.7) in terms of the conformally Newtonian variables. The first equality can be written, for immediate convenience, as
\[ \mathcal{R} = -\psi + \frac{H^2}{H} \left[ \psi + \frac{H}{H} \right], \quad \frac{\partial}{\partial t} \left( \frac{a\psi}{H} \right) = a\mathcal{R} \frac{\dot{H}}{H^2}, \quad (B.1) \]
where we only passed from the conformal time coordinate \( \tau \) to the cosmic time coordinate \( t \) and we recalled that \( dt = a(t) d\tau \); it is also practical to use the well-known identities \( H = (\dot{H} - H^3) / a^2 \) and \( H = \dot{\tau} / a \). From equation (B.1) by integrating once with respect to \( t \) the solution for \( \psi(k, t) \) then becomes
\[ \psi(k, t) = -\mathcal{R}(k, t) + \frac{H}{a} \int_0^t a\mathcal{R} \frac{\dot{H}}{H} dt' + \frac{H}{a} \int_0^t \mathcal{R}(k, t') a(t') dt', \quad (B.2) \]
where we integrated by parts and used that
\[ \frac{\partial}{\partial t} \left( \frac{a\mathcal{R}}{H} \right) = -\frac{\dot{H}}{H^2} a\mathcal{R} + a\mathcal{R} \frac{\dot{H}}{H}. \quad (B.3) \]

From equations (2.39), (2.40) and (2.41), neglecting the electric and Ohmic contributions, \( \mathcal{R}(k, t) \) is given by
\[ \mathcal{R} = -\frac{H}{p_1 + p_5} \left( \frac{c_u^2}{3} - 1 \right) \frac{H\delta_{\mathrm{nad}}}{p_1 + p_5}, \quad (B.4) \]
whose formal solution can be written as
\[ \mathcal{R}(k, t') = \mathcal{R}_*(k) - \int_0^t \frac{H(t') \delta_{\mathrm{nad}}(k, t')}{p_1 + p_5} dt'' + \int_0^t \frac{H(t'')\delta_{\mu}B(k, t'')}{p_1 + p_5} \left( \frac{c_u^2}{3} - 1 \right) dt''. \quad (B.5) \]

If we now insert equation (B.5) into equation (B.2) and go back to the conformal time coordinate we can write
\[ \psi(k, \tau) = -\mathcal{R}_*(k) T_R(\tau) + R_y \Omega_B(k) T_B(\tau), \quad (B.6) \]
where \( T_R(\tau) \) and \( T_B(\tau) \) are given by equations (3.26) and (3.27). For the sake of simplicity and for comparison with other treatments consider, for instance, the case of the \( \Lambda \)CDM scenario where
\[ a(t) = a_1 \left[ \sinh \left( \frac{1}{2} H_0 t \right) \right]^{2/3}. \quad (B.7) \]
In the cosmic time coordinate, $T_R(t)$ can be written as

$$T_R(t) = 1 - \frac{H(t)}{a(t)} \int_0^t a(t') \, dt'.$$

It is now easy to show that, for $t \to 0$, $a(t) \simeq t^{2/3}$ and $T_R(t) \simeq -(3/5)$. For $t \to \infty$ $T_R(t) \simeq e^{-H_0 t}$. Unfortunately, in the most general situation of the $w$CDM scenario, the semi-analytic estimates are not that simple and the numerical evaluation of $T_R(\tau)$ and $T_B(\tau)$ is often mandatory (see, for instance, the initial part of section 5).

**Appendix C. Generalization of certain integral representations**

In the present study there is often the need of generalizing certain integral representations of the power spectrum of the ISW and of its correlation with the SW effect. For instance, in the simplest case of adiabatic fluctuations, we shall have that the autocorrelation of the ISW and its cross correlation with the SW term boils down to the following pair of expressions:

$$C_{\ell}^{\text{(ISW)}} = \int_{\tau_0}^{\tau} dT_R \int_{\tau_0}^{\tau} dT_R f_\ell(\tau_1, \tau_2, n_s),$$

$$(C.1)$$

$$C_{\ell}^{\text{(cross)}} = \frac{1}{5} \int_{\tau_0}^{\tau} dT_R \int_{\tau_0}^{\tau} f_\ell(\tau, \tau_{\text{rec}}, n_s),$$

$$(C.2)$$

where

$$f_\ell(a, b, n_s) = \frac{8\pi^2 A_R k_1^{-n_s}}{\sqrt{ab}} \int_0^\infty k^{n_s-3} J_{\ell+1/2}(ka) J_{\ell+1/2}(kb).$$

$$(C.3)$$

The expressions of equations (C.1) and (C.2) are just illustrative since the same problem arises in the case of the transfer functions of the magnetic component. To evaluate expressions like the ones of equations (C.1) and (C.2) one might choose to perform first the (numerical) integrations over the conformal time coordinate and then the integrations over the modulus of the wavevector $k$. The idea is to reverse this order and integrate first over $k$. In doing so we are led, in general terms, to integrals like

$$\int_0^\infty k^{-\lambda} J_\nu(ka) J_\nu(kb) \, dk$$

$$= \frac{a^{\lambda} \Gamma\left(\nu + \frac{1-\lambda}{2}\right)}{2^{\nu-\lambda+1} \Gamma\left(\frac{\lambda+1}{2}\right) \Gamma(\nu+1)} F\left[\nu + 1 - \lambda, 1 - \lambda, 2; -\nu + 1, a^2/b^2\right].$$

$$(C.4)$$

In terms of the illustrative examples of equations (C.1) and (C.2), $v = (\ell+1/2)$ and $\lambda = 3-n_s$.

Equation (C.4) can be found, for instance, in [46] (see formula 6.574, page 675).

It is more practical, for the purpose of approximate evaluations, to relate the hypergeometric functions to the associated Legendre functions. This was the strategy followed in the first attempt to evaluate analytically the ISW contribution in the $\Lambda$CDM model [8]. The latter analysis has been performed in the case of (exactly) scale-invariant contribution. In spite of the fact that the explicit dependence upon the spectral index is unimportant for orders of magnitude estimates, the habit of computing analytically the ISW contribution for the exactly scale-invariant case persists [11]. In what follows the integral representation of the two-point function for the ISW will be directly related to the associated Legendre functions but for a generic spectral index.

Consider a hypergeometric function parametrized as $F(\alpha, \beta; \gamma; z)$. It can be shown that if two of the numbers $(\gamma - 1), (\alpha - \beta)$ and $(\alpha + \beta - \gamma)$ are equal (or if one of them is equal to 1/2)
The hypergeometric function appearing in equation (C.4) to the general relation (see [47], p 560) and the associated Legendre functions according to the general relation (C.5) implies, in the case of equation (C.4),

\[
F \left( \nu + \frac{1 - \lambda}{2}, \frac{1 - \lambda}{2}; \nu + 1; \frac{a^2}{b^2} \right) = \left( \frac{b}{b + a} \right)^{2\nu + 1 - \lambda} F \left( \nu + \frac{1 - \lambda}{2}, \nu + \frac{1}{2}; 2\nu + 1; \frac{4ab}{(a + b)^2} \right).
\]

It also follows that\(^{18}\)

\[
\int_0^\infty k^{-\lambda} J_\nu(ka) J_\nu(kb) \, dk = \frac{a^{\nu} b^{\nu}}{2^{\nu} (a + b)^{2\nu + 1}} \frac{\Gamma(v + \frac{1 - \lambda}{2})}{\Gamma\left(\frac{1 + \nu}{2}\right)} \Gamma(v + 1) \times F \left[ \nu + \frac{1 - \lambda}{2}, \nu + \frac{1}{2}; 2\nu + 1; \frac{4ab}{(a + b)^2} \right].
\]

The hypergeometric function appearing in equation (C.7) can be further transformed according to the general relation (see [47], p 560)

\[
F(\alpha, \beta; 2\beta; z) = \left( 1 - \frac{z}{2} \right)^{-\alpha} F \left[ \alpha + \frac{1}{2}; \beta + \frac{1}{2}; \frac{z^2}{(2 - z)^2} \right].
\]

In our case, equation (C.8) implies

\[
F \left[ \nu + \frac{1 - \lambda}{2}, \nu + \frac{1}{2}; 2\nu + 1; \frac{4ab}{(a + b)^2} \right] = \left[ \frac{a^2 + b^2}{(a + b)^2} \right]^{\nu + \frac{1}{2} \nu} \frac{\Gamma(v + \frac{1 - \lambda}{2})}{\Gamma\left(\frac{1 + \nu}{2}\right)} \Gamma(v + 1) \times F \left[ \nu + \frac{1 - \lambda}{2}, \nu + \frac{1}{2}; 2\nu + 1; \frac{4a^2b^2}{(a^2 + b^2)} \right].
\]

Using equation (C.9) we also have that

\[
\int_0^\infty k^{-\lambda} J_\nu(ka) J_\nu(kb) \, dk = \frac{a^{\nu} b^{\nu}}{2^{\nu} (a^2 + b^2)^{\nu + (1 - \lambda)/2}} \frac{\Gamma(v + \frac{1 - \lambda}{2})}{\Gamma\left(\frac{1 + \nu}{2}\right)} \Gamma(v + 1) \times F \left[ \nu + \frac{1 - \lambda}{2}, \nu + \frac{1}{2}; 2\nu + 1; \frac{4a^2b^2}{(a^2 + b^2)} \right];
\]

the hypergeometric function appearing in equation (C.10) can be finally related to the associated Legendre functions according to the general relation

\[
F \left[ \frac{1 + n + m}{2}, \frac{1}{2} + \frac{m + n}{2}; \frac{n + 3}{2}, \frac{1}{2}; \frac{z}{z^2} \right] = e^{-i\pi\nu} \frac{z^{m+1}}{(z^2 - 1)^{m/2}} \sqrt{\pi} \Gamma(m + n + 1) Q_n^m(z).
\]

\(^{18}\) We correct here a typo appearing in the formula 6.576 (number 2) page 576 of [46]. In the numerator of the formula we read \(2^\nu\) which should be instead \(b^\nu\). Equation (C.7) represents the correct version of the quoted formula where \(b^\nu\) (and not \(2^\nu\)) correctly appears in the numerator.
where $Q_n^m(z)$ are the associated Legendre functions of the second order. Matching equation (C.11) with the expression appearing in equation (C.10) we shall have

\[
F \left[ \frac{v}{2} - \frac{\lambda}{4} + \frac{3}{4} \frac{v}{2} + 1 - \frac{\lambda}{4}; v + 1; \frac{4a^2b^2}{(a^2 + b^2)^2} \right] = \left( \frac{a^2 + b^2}{2ab} \right)^{v+1/2} \left( \frac{a^2 - b^2}{a^2 + b^2} \right)^{\lambda/2} \frac{\Gamma(v + 1)}{\Gamma(v + \frac{1}{2} + \lambda)} \frac{2^{i\pi\lambda/2}}{\sqrt{\pi}} e^{-i\pi\lambda/2} Q_{n-1}^m \left( \frac{a^2 + b^2}{2ab} \right).
\]

(C.12)

where we used the fact that

\[
Q_n^{-m}(y) = e^{-\pi m} \frac{\Gamma(n - m + 1)}{\Gamma(n + m + 1)} Q_n^m(y).
\]

(C.13)

Thus, in conclusion, we shall have that

\[
\int_0^\infty k^{-\lambda} J_n(ka) J_n(kb) \, dk = \left( \frac{a^2 - b^2}{2\sqrt{ab}} \right)^{\lambda/2} \frac{\Gamma(v + \frac{1}{2} + \lambda)}{\Gamma(\frac{1}{2} + \lambda)\Gamma(v + \frac{1}{2} + \lambda)} \frac{2^{-i\pi\lambda/2}}{\sqrt{\pi}} Q_{n-1}^m \left( \frac{a^2 + b^2}{2ab} \right).
\]

(C.14)

This expression generalizes, to the best of our knowledge, the analog expressions derived in [8] in the case of scale-invariant curvature perturbations (corresponding, in the present parametrization, to $\lambda = 3 - n_s$, i.e. $\lambda = 2$ for $n_s = 1$ and $v = \ell + 1/2$). The very same expression can be used to integrate the magnetized contributions to the ISW. In the latter case, unlike the expressions of equations (C.1) and (C.2), the various terms will contain (at least) one $T_{B}(\tau)$. For instance, in the case of the magnetized contribution to the ISW, there will be two derivatives of $T_{B}(\tau)$ and the new $\lambda$ will be given as $\lambda = 4 - 2n_{B}$. All the different terms arising in equations (3.33) and (3.34) can be explicitly integrated over $k$ by using the techniques reported in this part of the appendix.

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