Stress Analysis of Thick-Wall Hollow Functionally Graded Material Based on Plane Strain Problem

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Abstract. In this study, the mechanical behaviours of thick-walled hollow functionally graded material (FGM) cylinder subjected to uniform pressures at inner and outer surfaces are proposed. The hollow FGM cylinder has variable geometric profile, thickness and material properties like elastic modulus and mass density are all assumed to vary radially according to exponentially-varying functions in radial direction. With the help of the infinitesimal elasticity theory and plane strain assumption, analytic solutions of stress and deformation are proposed. Numerical example is given to show the performance between FEM and homogeneous cylinder. Finally, summary and the future work are presented at the end of the paper.

1. Introduction

Functionally graded materials are composite materials, the mechanical properties change continuously from one surface to the other can resist high strength, high temperature and erosion. Nowadays, FGMs have been widely applied in various fields and catch great interest for researchers.

Durodola and Attia [1] predicted displacement and stress of FGM with uniform thickness. The discussed functionally graded material can be obtained by non-uniform reinforcement of a metal-matrix with long fibres. Eraslan et al. [2] analyzed functionally graded rotating solid shaft and rotating solid disk problems under generalized plane strain and plane stress assumptions. They considered Young’s modulus of the material varies radially according to exponential and parabolic forms. Bayat et al. [3] used semi-analytical method to study the mechanical behaviour of functionally graded material with variable thickness. They divided disk into some virtual sub-domains in radial direction, the mechanical property was assumed to be constant in each sub-domain. Free-free and fixed-free boundary conditions are discussed. Rosyid et al. [4] used finite element method to find stress solutions for elastic nonhomogeneous turbine disc with variable thickness under mechanical and thermal loading. Gharibi et al. [5] studied elastic analysis of rotating hollow FGM cylindrical pressure vessel, the hollow cylindrical vessel density was constant and the vessel thickness was uniform. Nejad et al. [6] made a review of functionally graded thick cylindrical and conical shells, some critical issues and problems were presented. Thawait [7] considered rotating cylindrical pressure vessels made of functionally graded materials, the material properties of the shells vary in radial direction according to exponential distribution law. The results of stresses and deformation were found by finite element method. The related analysis on functionally graded material has been performed by You et al. [8], Mazarei et al. [9], Shahriari and Safari [10], etc.

Although the cases of hollow or solid FGM cylinder with different elastic properties are extensively studied, the analytic results for thickness, Young’s modulus and mass density all in exponentially-varying properties are not found in the literature. The purpose of this article is to find
analytical solutions of deformation and stress for functionally graded rotating cylinder with exponentially-varying thickness and material properties.

2. Theoretical Formulation and Solution

A rotating thick-walled hollow cylindrical vessel made of functionally graded material with variable thickness is considered. The inner radii and outer radii of the hollow cylinder are \( a \) and \( b \), which is subjected to internal uniform pressure \( P_i \) at inner surface and external uniform pressure \( P_o \) at outer surface, respectively. The hollow cylinder rotates about its axis at constant angular velocity \( \omega \).

In this article, Poisson’s ratio \( \nu \) of the considered functionally graded material is assumed to be constant. The thickness, Young’s modulus and mass density at inner surface of hollow FEM cylinder are \( h_i, E_i \) and \( \rho_i \). The thickness, Young’s modulus and mass density at outer surface of hollow FEM cylinder are \( h_o, E_o \) and \( \rho_o \). The cylinder thickness \( h(r) \), Young’s modulus \( E(r) \) and mass density \( \rho(r) \) of hollow FEM cylinder are assumed to vary radially according to the following exponentially-varying functions.

\[
h(r) = h_i e^{k(r-a)/b} \\
E(r) = E_i e^{m(r-a)/b} \\
\rho(r) = \rho_i e^{n(r-a)/b}
\]

where \( r \) is the radial coordinate. The index \( k \) is a geometric parameter, indexes \( m \) and \( n \) are material parameters. They can be represented as

\[
k = (b/a - 1) \ln(h_o/h_i) \\
m = (b/a - 1) \ln(E_o/E_i) \\
n = (b/a - 1) \ln(\rho_o/\rho_i)
\]

According to axis symmetry and plane strain assumption, the strain-displacement and constitutive equations are

\[
e_r = \frac{du}{dr}, \quad e_\theta = \frac{u}{r} \\
\sigma_r = \frac{E(r)}{(1+\nu)(1-2\nu)}[(1-\nu)e_r + \nu e_\theta] \\
\sigma_\theta = \frac{E(r)}{(1+\nu)(1-2\nu)}[(1-\nu)e_\theta + \nu e_r]
\]

where \( u \) is radial displacement, \( (e_r, e_\theta) \) are strains in radial and circumferential directions, and \( (\sigma_r, \sigma_\theta) \) are stresses in radial and circumferential directions, respectively. The equation of equilibrium in the radial direction is

\[
\frac{d}{dr} [h(r)r\sigma_r] - h(r)\sigma_r + h(r)\rho(r)\omega^2 r^2 = 0
\]

Substitution of Eqs. (1)-(9) into Eq. (10), let \( r = bR, \quad u = bU, \quad \bar{a} = a/b, \quad \beta = (k + m)/b \) and \( \Omega = b\sqrt{(1+\nu)(1-2\nu)}\rho_m/(1-\nu)E_m \), the governing equation of radial displacement becomes

\[
R^2 \frac{d^2 U}{dR^2} + R(1 + \beta \nu \beta R) \frac{dU}{dR} + \left(\frac{\nu}{1-\nu} \beta \nu \beta R - 1\right) U = -\Omega^2 \omega^2 R^2 e^{(n-m)(R-a)}
\]

Based on the length of the article, we only discuss \( \beta \nu \beta R > 0 \) condition. The above equation is a generalized confluent hypergeometric differential equation and the solution is

\[
U(R) = A \tilde{W}(R) + B \tilde{M}(R) + \omega^2 \tilde{F}(R) \\
\tilde{W}(R) = e^{x/2} \psi^{(-1/2)}W_{i,j}(x) \\
\tilde{M}(R) = e^{x/2} \psi^{(-1/2)}M_{i,j}(x)
\]
\[ \bar{F}(R) = e^{x/2}x^{-(1/2)} \left[ W_{i,j}(x) \int F(R) M_{i,j}(x) \, dR - M_{i,j}(x) \int F(R) W_{i,j}(x) \, dR \right] \]  

\[ F(R) = \frac{n^2 e^{(n-m)(R-a)R^2}e^{-x/2}x^{1/2}}{[(2-3v)/(1-v)]M_{i,j}(x)W_{i,j}(x)+M_{i,j}(x)W_{i,j}(x)} \]

where \( M_{i,j}(x) \) and \( W_{i,j}(x) \) are Whittaker’s functions.

\[ M_{i,j}(x) = e^{-(x/2)x^{j+1/2}}M(j - i + 1/2, 1 + 2j, x) \]  

\[ W_{i,j}(x) = e^{-(x/2)x^{j+1/2}}U(j - i + 1/2, 1 + 2j, x) \]

in which \( M(j - i + 1/2, 1 + 2j, x) \) and \( U(j - i + 1/2, 1 + 2j, x) \) are the hypergeometric and Kummer functions, respectively, with \( i = (1 - 3v)/2(1 - v), j = 1 \) and \( x = \beta b R \). The Whittaker’s functions \( M_{i,j}(x) \) and \( W_{i,j}(x) \) converge for \( |x| < 1 \).

Substituting Eq. (12) into Eqs. (7) - (9) and let \( \Sigma_r, \Sigma_\theta = (1 + v)(1 - 2v)/(1 - v)E_i \cdot (\sigma_r, \sigma_\theta) \), the radial and circumferential stresses in dimensionless forms are

\[ \Sigma_r = A \Sigma_r + B \Sigma_\theta + \Sigma_r \omega^2 \]  

\[ \Sigma_\theta = A \Sigma_\theta + B \Sigma_r + \Sigma_\theta \omega^2 \]  

\[ \Sigma_{rW} = e^{m(R-a)R^{-1}e^{x/2}R^{-(1/2)}x^{-3}x^{-1/2}}W_{i,j}(x) - W_{i+1,j}(x) \]  

\[ \Sigma_{rM} = e^{m(R-a)R^{-1}e^{x/2}R^{-(1/2)}x^{-3}x^{-1/2}}[x - 3 + \frac{2}{1-v}]M_{i,j}(x) + (3 - \frac{1}{1-v})M_{i+1,j}(x) \]

\[ \Sigma_{rW} = \int F(R)M_{i,j}(x) \, dR - \int F(R)W_{i,j}(x) \, dR \]

\[ \Sigma_{rM} = \int F(R)W_{i,j}(x) \, dR - \int F(R)M_{i,j}(x) \, dR \]

\[ \Sigma_{rW} = [1 - (\frac{v}{1-v})^2]e^{m(R-a)R^{-1}R^{-1}} + \frac{v}{1-v} \Sigma_{rW} \]

\[ \Sigma_{rM} = [1 - (\frac{v}{1-v})^2]e^{m(R-a)R^{-1}R^{-1}} - \frac{v}{1-v} \Sigma_{rM} \]

\[ \Sigma_{rW} = [1 - (\frac{v}{1-v})^2]e^{m(R-a)R^{-1}R^{-1}} + \frac{v}{1-v} \Sigma_{rW} \]

Coefficients \( A \) and \( B \) are arbitrary constants can to be determined from boundary conditions and mechanical loads of internal pressure \( P_i \) and external pressure \( P_0 \). In terms of \( \Sigma_r (\bar{a} = a/b) = -(1 + v)(1 - 2v)P_i/(1 - v)E_i \) and \( \Sigma_\theta (1) = -(1 + v)(1 - 2v)P_i/(1 - v)E_i \), we can find

\[ A = \frac{(1+v)(1-2v)D_{22}}{(1-v)E_i(D_{11}D_{22}-D_{12}D_{21})} P_i + \frac{(1+v)(1-2v)D_{12}}{(1-v)E_i(D_{11}D_{22}-D_{12}D_{21})} P_0 + \frac{D_{11}D_{22}+D_{12}D_{21}}{D_{11}D_{22}-D_{12}D_{21}} \omega^2 \]  

\[ B = \frac{(1+v)(1-2v)D_{21}}{(1-v)E_i(D_{11}D_{22}-D_{12}D_{21})} P_i - \frac{(1+v)(1-2v)D_{11}}{(1-v)E_i(D_{11}D_{22}-D_{12}D_{21})} P_0 - \frac{D_{11}D_{22}+D_{12}D_{21}}{D_{11}D_{22}-D_{12}D_{21}} \omega^2 \]

\[ D_{11} = \Sigma_{rW} \bigg|_{R=a} \cdot D_{12} = \Sigma_{rM} \bigg|_{R=a} \cdot D_{13} = \Sigma_{rW} \bigg|_{R=a} \]

\[ D_{21} = \Sigma_{rW} \bigg|_{R=1} \cdot D_{22} = \Sigma_{rM} \bigg|_{R=1} \cdot D_{23} = \Sigma_{rW} \bigg|_{R=1} \]

In the limiting case, when \( k = m = n = 0 \), the discussed hollow FEM cylinder is reduced to uniform and homogeneous cylinder. The governing equation of radial displacement for the rotating uniform and homogeneous cylinder in dimension forms becomes

\[ r \frac{d^2u}{dr^2} + \frac{du}{dr} - \frac{1}{r} u = -\frac{(1+v)(1-2v)}{(1-v)E_i} \rho_0 \omega^2 \frac{r^2}{r^2} \]

The radial displacement, radial and circumferential stresses are

\[ u(r) = c_1 r + c_2 \frac{1}{r} - \frac{(1+v)(1-2v)}{8(1-v)E_i} \rho_0 \omega^2 \frac{r^3}{r^3} \]

\[ \sigma_r(r) = \frac{E_i}{(1+v)(1-2v)} \left[ c_1 - \frac{1-2v}{r^2} c_2 - \frac{3-2v}{8(1-v)E_i} \rho_0 \omega^2 \frac{r^2}{r^2} \right] \]
Coefficients $c_1$ and $c_2$ are arbitrary constants can to be determined from boundary conditions and mechanical loads of internal pressure $P_i$ and external pressure $P_o$, i.e., $\sigma_r(a) = -P_i$ and $\sigma_r(b) = -P_o$. Then

$$c_1 = \frac{(3-2\nu)(1+\nu)(1-2\nu)\nu}{8(1-\nu)E_i}(a^2 + b^2)\omega^2 + \frac{(1+\nu)(1-2\nu)}{E_i(b^2-a^2)}(a^2 P_i - b^2 P_o)$$

$$c_2 = \frac{(3-2\nu)(1+\nu)\nu}{8(1-\nu)E_i}a^2 b^2\omega^2 + \frac{(1+\nu)a^2 b^2}{E_i(b^2-a^2)}(P_i - P_o)$$

3. Illustrative Examples

The analytical solutions obtain in the preceding section may be checked for a hollow FEM cylinder with the following numerical parameters. The inner and outer radii of the hollow FEM cylinder are $a = 0.2$ and $b = 1$ m. At inner and outer surfaces of the hollow FEM cylinder, which is subjected to internal uniform pressure $P_i = 200 \text{ MPa}$ and external uniform pressure $P_o = 20 \text{ MPa}$, respectively. The cylinder rotates about its axis at constant angular velocity $\omega = 200 \text{ rad/sec}$. The material at inner surface of hollow FEM cylinder is zirconia ceramic pure. The thickness, Young’s modulus and mass density at inner surface are $h_i = 0.1m$, $E_i = 151 \text{ MPa}$ and $\rho_i = 5700 \text{ kg/m}^3$. The material at outer surface of hollow FEM cylinder is aluminium metal pure. The thickness, Young’s modulus and mass density at outer surface are $h_o = 0.2m$, $E_o = 70 \text{ MPa}$ and $\rho_o = 2700 \text{ kg/m}^3$. The Poisson’s ratio is $\nu = 0.3$.

Figure 1-3 show the comparisons of mechanical behaviors between FEM and homogeneous cylinder. The convergence condition of Whittaker’s functions, $|x| < 1$, is satisfied. We can observe that the displacement, radial stress and circumferential stress in FEM cylinder has better performance than homogeneous cylinder. In figure 1 and figure 3, we can find radial displacement and circumferential stress of hollow FGM cylinder have their maximums occurring at inner surfaces, and they are decreasing with the increase of $R$. In figure 2, the radial stress, from inner to outer surface, first increases and then decreases with the increase of $R$. The maximal radial stress occurs around $R = 0.57$.

![Figure 1. Comparison of radial displacements between homogeneous and FEM cylinder](image-url)
4. Conclusion
In this article, with the help of infinitesimal elasticity theory and plane strain assumption, analytic solutions of stress and deformation about rotating hollow FGM cylinder have been derived. The hollow FGM cylinder has exponentially-varying thickness in radial direction, also Young’s modulus and mass density are represented by exponentially-varying forms in radial direction. The hollow FGM cylinder is subjected to uniform pressures at inner and outer surface of the cylinder. In this article, aluminum metal material and zirconia ceramic material are used. only ceramic-metal type of hollow FGM cylinder is considered, i.e., in the inside surface is zirconia ceramic pure and in the outside surface is aluminum metal pure. In order to compare mechanical behaviors between FEM cylinder and homogeneous cylinder, diagrams of displacement and stresses are plotted. We can find FEM cylinder has better performance than homogeneous cylinder. As shown in illustrative examples, the maximum radial displacement occurs at the inner surface. The radial stress, from inner to outer surface, first increases and then decreases with the increase of radius. Its maximum occurs near middle area of the hollow cylinder. The Circumferential stress is monotonic decreasing with its maximum occurring at the inner surface. In future research, the mechanical behaviors of metal-ceramic type of hollow FGM cylinder may be investigated. We also may discuss hollow FGM cylinder subjected to loads under different boundaries condition, i.e., the edges in inner and outer surfaces are free or constraint.

5. References
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