An effective theory for hot non-Abelian dynamics\textsuperscript{†}

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I try to explain some recent progress in understanding the non-perturbative dynamics of hot non-Abelian gauge theories. The non-perturbative physics is due to soft spatial momenta \(|p| \sim g^2 T\) where \(g\) is the gauge coupling and \(T\) is the temperature. An effective theory for the soft field modes is obtained by integrating out the field modes with momenta of order \(T\) and of order \(g T\) in a leading logarithmic approximation. In this effective theory the time evolution of the soft fields is determined by a local Langevin-type equation. This effective theory determines the parametric form of the rate for hot electroweak baryon number violation as \(\Gamma = \kappa g^4 \log(1/g^2 T)\). The non-perturbative coefficient \(\kappa\) is independent of the gauge coupling and it can be computed by solving the effective equations of motion on a lattice.

I. INTRODUCTION

Let me start by formulating the problem I am going to discuss: How can one calculate thermal expectation values like

\[ C(t_1 - t_2) = \langle O(t_1)O(t_2) \rangle \]

in a non-Abelian gauge theory, when the leading order contribution is due to spatial momenta of order \(g^2 T\)? The operator \(O(t)\) is a gauge invariant function of the gauge fields \(A_\mu(t, x)\) at time \(t\). When I said the leading order contribution is due to momenta of order \(g^2 T\), I referred to the Fourier components of the gauge fields entering \(O(t)\). Finally,

\[ \langle \ldots \rangle = \frac{\text{tr}\{\ldots e^{-H/T}\}}{\text{tr}\{e^{-H/T}\}} \]

(2)

denotes the thermal average, where \(H\) is the Hamiltonian. As one may anticipate, it is not possible to compute such a correlation function in perturbation theory.

This problem arises in the context of electroweak baryon number violation at temperatures, well above 100\,GeV, when the electroweak symmetry is unbroken. Then the SU(2) gauge theory is very similar to hot QCD. Fortunately, it is a bit simpler because the gauge coupling is small.

In my talk I will try to explain how such correlation functions can be computed at leading order in the gauge coupling \[\text{[1]}\]. We will see that this requires the use of an effective theory which results from integrating out field modes with hard \((p \sim T)\) and semi-hard \((p \sim g T)\) spatial momenta \[\text{[1]}\]. Due to the smallness of the gauge coupling this can be done in perturbation theory. This effective theory turns out to have a relatively simple structure and it can be easily used for numerical calculations on a lattice.

Most of the discussion will be restricted to a pure gauge theory, i.e., without matter fields. I will comment on the role of matter fields when necessary.

II. HOT ELECTROWEAK BARYON NUMBER VIOLATION

One of the most intriguing aspects of the electroweak theory is its non-trivial vacuum structure which is related to baryon number violating processes. These processes play an important role in understanding the observed baryon-asymmetry of the universe \[\text{[3]}\].

Baryon number is not conserved in the electroweak theory due to the chiral anomaly. The baryon number current satisfies \[\text{[4]}\]

\[ \partial_\mu j_\mu^B = n_f \frac{g^2}{32 \pi^2} \text{tr} \left( F^2 \right) \]

(3)

where \(F\) is the SU(2) field strength tensor. The rhs of Eq. (3) is a total derivative,

\[ \text{tr} \left( F^2 \right) = \partial_\mu K^\mu. \]

(4)

\[\text{[4]}\]For spatial vectors \(k = |k|\). Four-vectors are denoted by \(K^\mu = (K^0, \mathbf{k})\) and I use the metric \(K^2 = k^2 - k^2_0\).

\[\text{[1]}\]Baryon plus lepton number is violated in the electroweak theory while baryon minus lepton number is conserved.
Thus, integrating Eq. (3) over 3-space and over time from $t_i$ to $t_f$ we can write the change of baryon number $B$ as

$$B(t_f) - B(t_i) = n_f(N_{CS}(t_f) - N_{CS}(t_i))$$

(5)

where

$$N_{CS} = \frac{g^2}{32\pi^2} \epsilon_{i j k} \int d^4x \left( F_{i j}^a A_i^a - \frac{g}{3} \epsilon^{a b c} A_i^a A_j^b A_k^c \right)$$

(6)

is the so called Chern-Simons winding number of the gauge fields. Unlike $N_{CS}$ itself, the difference $\Delta N_{CS} = N_{CS}(t_f) - N_{CS}(t_i)$ is gauge invariant since it can be written as a space-time integral of $F \tilde F$.

The Chern-Simons number is intimately related to the non-trivial topology of the gauge-Higgs field configuration space. There are paths in this space leading from the vacuum to the vacuum which cannot be continuously deformed to a single point. The minimal field energy along such a path cannot be made arbitrarily small by continuous deformation. There is a minimal energy barrier which has to be crossed. The corresponding field configuration is the so called sphaleron. Following such a non-contractible path, the Chern-Simons number changes by an integer.

The simplest mechanical analogue of this situation I can think of is a rigid pendulum in a gravitational field which can swing all the way around in a circle. The Chern-Simons number corresponds to $\phi/(2\pi)$ where $\phi$ is the rotation angle. The vacuum corresponds to the pendulum at rest and a non-contractible loop corresponds to net rotation of the pendulum by an integer times $2\pi$.

The gauge-Higgs field configuration space is of course much more complicated, it is infinite dimensional rather than one dimensional as in the case of the pendulum. Fig. 1 displays a one dimensional slice of this space. It corresponds to a path on which the field configuration for a given value of $N_{CS}$ has a minimal energy.

If there were only gauge and Higgs fields, nothing spectacular would happen in a vacuum to vacuum transition which changes $N_{CS}$. However, in the presence of fermions, the change of $N_{CS}$ is accompanied by a change of baryon number due to Eq. (3).

At zero temperature, $N_{CS}$-changing transitions are possible only by tunnelling. The tunnelling probability is so small that it has no measurable physical effect.

This situation changes at high temperature. Then it becomes possible to make such transitions by thermal activation across the energy barrier. The suppression factor is then given by a Boltzmann factor rather than the tunnelling amplitude. For $T$ of the order of the electroweak phase transition or crossover temperature $T_c \sim 100$ GeV, the transition probability becomes unsuppressed.

When the temperature is sufficiently below $T_c$, most transitions are passing the energy barrier close to the sphaleron and the transition rate $\Gamma$ can be calculated in a saddle point approximation.

For $T \sim T_c$ this approximation breaks down and for $T > T_c$ there is no sphaleron solution at all. Therefore one has to employ non-perturbative methods to calculate the rate for this case. It has become popular to call the rate for $T > T_c$ the hot sphaleron rate. The idea how to compute it is the following: The Chern-Simons number is expected to perform a random walk. Then, for large $t$, thermal average of $[N_{CS}(t) - N_{CS}(0)]^2$ grows linearly with time,

$$\langle [N_{CS}(t) - N_{CS}(0)]^2 \rangle \rightarrow Vt \Gamma$$

(7)

where $V$ is the space volume. The coefficient $\Gamma$ is the probability for a $N_{CS}$-changing transition per unit time and unit volume. The rate for baryon number violation $\Gamma_{\Delta B}$ is proportional to $\Gamma$. Thus $\Gamma_{\Delta B}$ can be obtained by computing the real time correlation function [3]. At very high temperature $T \gg T_c$ the thermal mass of the Higgs field becomes large so that it decouples. Then it is sufficient to evaluate Eq. (3) in the pure gauge theory.

For the following discussion it is important to identify both the relevant length scale $R$ of the problem and the relevant size of the gauge field fluctuations $\Delta A$ [3]. First of all, we have to consider fluctuations which are large enough to make a change $\Delta N_{CS}$ of order unity. To estimate $\Delta N_{CS}$, recall that $F \tilde F = E \cdot B$. For the electric fields we have $E = -A$ in $A_0 = 0$ gauge. Thus $\int dt E \sim \Delta A$. Note that the relevant time scale drops out in this estimate. Furthermore, we estimate $B \sim \Delta A/R$ and $\int d^3x \sim R^3$. Thus we need

$$\Delta N_{CS} \sim g^2 \int d^4x E \cdot B \sim g^2 R^2 (\Delta A)^2 \sim 1.$$  

(8)

In addition, we have to require that the energy of the relevant fluctuations does not exceed the temperature. Otherwise the transition rate would be Boltzmann suppressed. Proceeding as above we require

$$\text{energy} \sim \int d^3x B^2 \sim R (\Delta A)^2 \lesssim T.$$  

(9)

Combining Eqs. (8) and (9) we find
\[ R \gtrsim (g^2T)^{-1} \quad \Delta A \sim (gR)^{-1}. \quad (10) \]

On the other hand, we know that the correlation length of magnetic fields in a hot non-Abelian plasma is of order \((g^2T)^{-1}\). Thus we can have only \( R \lesssim (g^2T)^{-1} \) which finally gives

\[ R \sim (g^2T)^{-1} \quad \Delta A \sim gT. \quad (11) \]

As I stated above, the relevant time scale \( t \), which we need to know in order to estimate the hot sphaleron rate, has dropped out in these considerations. It has to be determined from the dynamics and it cannot be obtained from purely thermodynamic considerations. Once we know \( t \) we can estimate the hot sphaleron rate as

\[ \Gamma \sim t^{-1} R^{-3}. \quad (12) \]

While Eq. (12) has been well established for a quite a while, the characteristic time scale has been understood only recently [1]. I hope that, by the end of my talk, you will be convinced that the correct answer is given by Eq. (12).

### III. NON-PERTURBATIVE PHYSICS AT HIGH TEMPERATURE

After the discussion of electroweak baryon number violation we now return to our original problem, which is to calculate correlation functions like (1).

In the previous section we have seen that the relevant length scale for electroweak baryon number violation is \((g^2T)^{-1}\) corresponding to momenta \( p \sim g^2T \). It is well known that finite temperature perturbation theory in non-Abelian gauge theories breaks down for momenta as small as \( g^2T \) [2, 3]. One can see this by considering the typical size of the transverse gauge field fluctuations which can be estimated from the equal time correlation function. For the transverse gauge fields we have

\[ A_t(x) \sim gT. \quad (13) \]

Thus the two terms in the covariant derivative \( \partial_i - igA_i \) are of the same size which makes perturbation theory impossible. Note that the thermal fluctuations are of the same size as the change \( \Delta A \) which we have estimated in Eq. (11).

When I said that perturbation theory does not work for soft momenta, one may ask: What about the plasmon at rest which has \( p = 0 \) and the celebrated calculation of its damping rate \( \Gamma_{KS} \)? The point is that the plasmon oscillation has an amplitude \( \sim g^2T \), i.e., it is a small oscillation. What we are interested in is the time evolution which makes the field change by an amount equal to its typical size \( \sim gT \).

For the case of equal time correlation functions the non-perturbative physics associated with the scale \( g^2T \) is relatively well understood. It is determined by an effective 3-dimensional pure gauge theory. This effective theory is the result of integrating out the field modes with non-zero Matsubara frequency (“dimensional reduction”) and the 0-component of the gauge fields [4, 5]. For sufficiently small coupling this can be done in perturbation theory. The 3-dimensional theory can be used for 3-dimensional Euclidean lattice simulations which allow for a much higher precision than 4-dimensional ones [6]. It should be noted that the use of dimensional reduction is more a matter of practical convenience rather than a matter of principle. The full 4-dimensional can also be evaluated directly in a 4-dimensional Euclidean lattice simulation.

The situation is different for unequal time correlation functions. Dimensional reduction is not possible because we have to consider time dependent quantities. 4-dimensional Euclidean lattice simulations are of no use either: Formally, real time correlation functions can be obtained by an analytic continuation of their imaginary time counterparts. However, it is not possible to do an analytic continuation of a function which can be evaluated only numerically.

One simplification of the problem is that the soft field modes behave classically. This is due to the fact that the momentum scale of interest is small compared to the temperature. Then the number of field quanta in one mode with wave vector \( p \), given by the Bose-Einstein distribution function

\[ n(p) = \frac{1}{e^{p/T} - 1} \sim \frac{T}{p}, \quad (14) \]

is large. In this case we are close to the classical field limit. Thus the dynamics of the soft field modes should be governed by classical equations of motion.

The first estimate of the hot sphaleron rate, due to Khlebnikov and Shaposhnikov [7], was based on the assumption that only the soft fields play a role while the high momentum modes decouple at leading order. Then the only scale in the problem would be \( g^2T \) and, on dimensional grounds, the hot sphaleron rate would have to be of the form

\[ \Gamma_{KS} = \kappa (g^2T)^4 \quad (15) \]

with a non-perturbative numerical coefficient \( \kappa \). Subsequently attempts were made to compute \( \kappa \) by solving the classical Hamiltonian equations of motion for the gauge fields and measure \( \Gamma \).

However, it turns out that in the case of unequal time correlation functions is much more complex than in the equal time case. The main difficulty is to understand the role of the different momentum scales which are present

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**It should be noted that this argument is somewhat hand-waving and it is not easy to justify quantitatively [8, 9].**
in the problem. We will see that both momenta of order $T$ and momenta of order $gT$ are relevant to the leading order dynamics of the soft fields.

Therefore one has to integrate out the high momentum modes to obtain an effective classical theory. Schematically, the equations of motion will be of the form

$$\frac{\delta S_{\text{eff}}[A]}{\delta A_{\nu}(x)} = 0$$

where the effective action $S_{\text{eff}}[A]$ consists of the tree level action and the terms which are generated by integrating out the field modes with $p \sim T$ and $p \sim gT$. Deriving this equation of motion will be the main subject of my talk.

The reason why the hard modes have to be integrated out to obtain an effective classical theory for the soft modes is pretty obvious: For $p \sim T$ the Bose-Einstein distribution is of order 1 so that these modes certainly do not behave classically. For semi-hard momenta, however, we have $n(p) \ll 1$. The main reason why they have to be integrated out as well is that it does not seem to be possible to construct a lattice formulation of the effective theory for momenta of order $gT$ and $g^2T$ which allows to take the continuum limit (cf. Ref. [13]). Another reason is that for the former the only hard thermal loop property to be gauge invariant is the 2-point function, while for the latter there are also Abelian theories.

The generating functional of the hard thermal loops has the remarkable property that it does not seem to be gauge invariant. For semi-hard momenta, however, we have $n(p) \ll 1$. The main reason why they have to be integrated out as well is that it does not seem to be possible to construct a lattice formulation of the effective theory for momenta of order $gT$ and $g^2T$ which allows to take the continuum limit (cf. Ref. [13]). Another reason is that after integrating out the scale $gT$ we obtain an effective theory for the soft field modes which contains only one length scale and one time scale. Then it will be trivial to estimate the parametric form of the hot sphaleron rate.

### IV. Integrating Out the Hard Modes

The hard modes constitute the bulk of degrees of freedom in the hot plasma. Their physics is that of almost free massless particles moving on straight lines. Nevertheless, they have a significant influence on the soft dynamics because they are so numerous.

Integrating out the hard modes means that we have to calculate loop diagrams with external momenta $\ll T$ and internal momenta of order $T$. This generates effective propagators and vertices for the field modes with $p \ll T$. To leading order, we can restrict ourselves to one loop diagrams and we can make a high energy or eikonal approximation of the integrand. The result is nothing but the so called hard thermal loops [14]. The generating functional of the hard thermal loops has the remarkable property to be gauge invariant.

The main difference between QED and non-Abelian theories is that for the former the only hard thermal loop is the 2-point function, while for the latter there are also hard thermal loop $n$-point functions for all $n$. As we will see later, this has a significant effect on the soft dynamics, it is in fact qualitatively different in Abelian and non-Abelian theories.

For the moment, however, we restrict the discussion to the hard thermal loop 2-point function. This will make clear that the hard modes have a significant influence on the soft dynamics which was pointed out by Arnold, Son, and Yaffe [13].

We have to consider the transverse (or magnetic) components of the gauge fields. The transverse polarisation function is given by

$$\delta \Pi_t(P) = \frac{1}{2} m_D^2 \left[ \frac{p^2}{p^2} + p_0 \left( 1 - \frac{p_0^2}{p^2} \right) \int \frac{d\Omega_\nu}{4\pi} \frac{1}{v \cdot P} \right]$$

where $m_D^2$ is the leading order Debye mass squared. In a pure $SU(N)$ gauge theory, it is given by

$$m_D^2 = \frac{1}{3} Ng^2 T^2.$$ 

In the hot electroweak theory it receives additional contributions due to the Higgs and fermion fields. Furthermore, $v^\mu \equiv (1,v)$, and the integral $\int d\Pi_\nu$ is over the directions of the unit vector $v$, $|v| = 1$.

For the naive “natural” scale of the problem, i.e., $p_0, p \sim gT$ we have

$$\delta \Pi_t(P) \sim g^2 T^2 \quad (p_0 \sim g^2 T, \quad p \sim g^2 T),$$

i.e., $\delta \Pi_t$ is much bigger than the tree level kinetic term. Therefore $\delta \Pi_t$ must be resummed. The resulting transverse propagator

$$\ast \Delta_t(P) = \frac{1}{-P^2 + \delta \Pi_t(P)}$$

is smaller than the tree level propagator by a factor $g^2$ when $p_0 \sim g^2 T, \quad p \sim g^2 T$. That means that for the naive “natural” frequency there are only small perturbative fluctuations which are of the same amplitude as the plasmon oscillation. It also means that the estimate (15) cannot be correct.

In order to obtain large non-perturbative fluctuations we obviously need to make $\delta \Pi_t$ smaller. We know that $\delta \Pi_t$ vanishes [13] for $p_0 \to 0$. Thus we have to consider small values of $p_0$. In the limit $p_0 \ll p$ we have

$$\delta \Pi_t(P) \simeq \frac{i}{4} m_D^2 \frac{p_0}{p}$$

which is purely imaginary. The imaginary part of $\delta \Pi_t(P)$ reflects the fact that the dynamics of the soft modes is Landau-damped by the hard modes. In fact, it is overdamping because the damping term is much larger than the tree level kinetic term $p_0^2$. In the same limit the transverse propagator (19) becomes

$$\ast \Delta_t(P) \simeq \frac{1}{p^2} \frac{i\gamma_p}{p_0 + i\gamma_p}$$

††The longitudinal hard thermal loop polarisation tensor stays large even for $p_0 \to 0$. Thus only the transverse fields can produce large, non-perturbative fluctuations.
with

$$\gamma_p = \frac{4p^3}{\pi m_D^2}. \quad (22)$$

Now we see how small $p_0$ must be in order to obtain the desired large fluctuations, namely

$$p_0 \sim \gamma_p \sim g^2 T, \quad (23)$$
corresponding to the time scale

$$t \sim \gamma_p^{-1} \sim (g^4 T)^{-1}. \quad (24)$$

Based on these considerations Arnold, Son and Yaffe estimated the hot sphaleron rate as

$$\Gamma_{ASY} = \kappa g^{10} T^4. \quad (25)$$

Subsequently, attempts were made by Huet, Son [16, 17] and Arnold [18] to construct a numerical algorithm to compute the coefficient $\kappa$ on a lattice.

In this discussion it is assumed that interactions do not change these order of magnitude estimates. As we will see in the next section, this is not the case: The field modes with momenta of order $g^2 T$ lead to interactions of the hard modes which cannot be neglected relative to the hard thermal loops.

**V. BEYOND HARD THERMAL LOOPS**

I will now argue that the hard thermal loop approximation for the high momentum modes ($p \gg g^2 T$) is not sufficient to obtain the correct effective theory for the soft modes.

Consider a hard thermal loop and imagine adding a self energy insertion on an internal line. This gives the diagram in Fig. 2a, in which the hard loop momentum $Q$ is on shell, $Q^2 = 0$. Since the external momentum $P$ is soft, the momentum $Q + P$ is almost on shell. The order of magnitude of the self energy insertion is well known from the calculation of the lifetime of hard particles [3]. From loop momenta $K$ of order $g^2 T$ one gets a contribution $\sim g^2 T^2$. Compared to the hard thermal loop, diagram 2a also contains an additional propagator which can be approximated as

$$\frac{1}{(Q + P)^2} \approx \frac{1}{2q} \frac{1}{v \cdot P} \sim \frac{1}{T} \frac{1}{v \cdot P}. \quad (26)$$

Thus we can estimate the diagram in Fig. 2a as

$$\Pi^{(a)}(P) \sim \delta \Pi_1(P) \times \frac{g^2 T}{v \cdot P} \quad (27)$$

and, by power counting, this diagram is not suppressed relative to the hard thermal loop [10] when $P \sim g^2 T$.

Note that in this estimate it did not play any role that we are considering a non-Abelian theory. The same type of diagram is also present in QED and the estimate would be exactly the same. However, there is an essential difference between the Abelian and the non-Abelian case which will become clear in a moment.

Similarly, one can see that the ladder-type diagram 2b is as large as the hard thermal loop as well. Again, this estimates are the same in QED and QCD. However, in QED the large contributions from the two diagrams turn out to be the same with opposite signs and cancel.

I will now argue that this cancellation does not occur in a non-Abelian theory. Let me first give a formal explanation in terms of diagrams. At the end of Sect. VII I will come back to this point and I will give a more intuitive argument.

Here is the diagrammatic argument: We have seen that the two diagrams both contain a hard and a semi-hard loop momentum. Imagine now that we calculate them in two steps: First do the integral over the hard momentum with opposite signs and cancel. In the second step we integrate over momenta of order $g^2 T$. The result of such a calculation is well known: At leading order this is nothing but the hard thermal loop four point function. And we also know that there is no such vertex in QED, even though individual diagrams would give such a contribution. Summing over the permutations of external lines the large contributions cancel. In a non-Abelian theory each of these diagrams comes with a different colour factor and the sum over permutations gives a non-vanishing result. In the second step we integrate over momenta of order $g^2 T$ which can be done using the well known expression for the hard thermal loop 4-point function.

**VI. INTEGRATING OUT THE SEMI-HARD MODES**

We have seen in the previous section that, in order to obtain an effective theory for the soft momentum fields, we cannot restrict ourselves to the hard thermal loops.
are hard thermal loop vertices. Otherwise the notation is the same as in Fig. 3.

Not only the hard modes but also the semi-hard ones affect the leading order dynamics of the soft fields. At first sight this appears like an additional complication. Somewhat surprisingly, things become much simpler instead.

We have also seen that it is convenient to proceed in two steps: In the first step we integrate out momenta of order \( T \) which yields the hard thermal loop effective theory. In the second step this effective theory is used to integrate out the scale \( gT \). In this way the calculation will be simplified because we have a partial cancellation, corresponding to the QED-type contributions, already “built in”.

**A. One loop diagrams**

We start by considering the one loop diagram in the hard thermal loop effective theory depicted in Fig. 3. It corresponds to the sum of the diagrams of the original theory in Fig. 2. We neglect the external soft momentum relative to the semi-hard one whenever it’s possible. In this way we obtain an expression in which the loop integration is logarithmically infrared divergent. This should not bother us since we do not want to compute the diagram exactly. All we want to do is to integrate out momenta of order \( gT \). Therefore we need a scale \( \mu \) which separates semi-hard momenta from soft ones,

\[
g^2 T \ll \mu \ll gT.
\]

This scale appears as a lower limit in our integral so that it is no longer divergent. The integral is difficult evaluate. What it not so difficult is to compute the terms which are singular for \( \mu \to 0 \). One finds [13]

\[
\Pi_{\mu \nu}^{(4)}(P) = -\frac{i}{4\pi} \frac{m_\text{D}}{2} N g^2 T \log \left( \frac{gT}{\mu} \right) \times \mu_0 \int \frac{d\Omega_v}{4\pi} \frac{v_{\mu}v_{\nu}}{(v \cdot P)^2}.
\]

This result is gauge fixing independent in a general covariant gauge.

Obviously, Eq. (27) is not suppressed relative to the hard thermal loop selfenergy (16): The prefactor \( g^2 T \) is compensated by one additional factor \( v \cdot P \sim g^2 T \) in the denominator. Eq. (27) is even larger than (16) by a factor of \( \log(gT/\mu) \).

Note that (23) is not transverse, which would be necessary for the effective theory, which we are going to derive, to be gauge invariant. There is, however, another loop diagram in the hard thermal loop effective theory which is of the same size as Eq. (27) and which is depicted in Fig. 3b. With the same approximations as above one obtains [13]

\[
\Pi_{\mu \nu}^{(3)}(P) = \frac{i}{\pi} m_\text{D}^2 N g^2 T \log \left( \frac{gT}{\mu} \right) \times \mu_0 \int \frac{d\Omega_{v_1}}{4\pi} \frac{v_{1 \mu}}{v_1 \cdot P} \int \frac{d\Omega_{v_2}}{4\pi} \frac{v_{2 \nu}}{v_2 \cdot P} \frac{(v_1 \cdot v_2)^2}{\sqrt{1 - (v_1 \cdot v_2)^2}}.
\]

One easily verifies that the sum of Eqs. (23) and (30) is transverse,

\[
P^\mu \left( \Pi_{\mu \nu}^{(4)}(P) + \Pi_{\mu \nu}^{(3)}(P) \right) = 0.
\]

**B. Higher loops**

There are higher loop diagrams in the hard thermal loop effective theory which are not suppressed relative hard thermal loops either. Obviously these terms need to be resummed. As I already said, the use of the hard thermal loop effective theory has simplified the calculation of (29) and (30) quite a bit. Nevertheless, higher loop diagrams are still difficult to calculate because the number of terms in the hard thermal loop \( n \)-point functions grows rapidly with increasing \( n \).

It is much more efficient to use another formulation of the hard thermal loop effective theory, which is in terms of kinetic equations. The degrees of freedom in this description are the gauge fields \( A_{\mu}(x) \) and, in addition, the fields \( W(x,v) \) which transform under the adjoint representation of the gauge group. The fields \( W(x,v) \) depend both on the space-time coordinates and on the unit vector \( v \). They describe the deviation of the distribution of hard particles with 3-velocity \( v \) from thermal equilibrium.

The equations of motion for these fields are the non-Abelian generalisation of the Vlasov equations for a QED plasma [20],

\[
[D_{\mu}, F^{\mu \nu}(x)] = \frac{m_\text{D}^2}{4\pi} \int d\Omega v v_{\nu} W(x,v), \tag{32}
\]

\[
[v \cdot D, W(x,v)] = v \cdot E(x), \tag{33}
\]

where \( E \) is the non-Abelian electric field, and the 4-vector \( v \) is defined as in Eq. (10). The rhs of Eq. (32) is the current due to the hard particles. The conserved Hamiltonian corresponding to Eqs. (32) and (33) is [21,22]
\[ H = \int d^3x \{ \mathbf{E}(x) \cdot \mathbf{E}(x) + \mathbf{B}(x) \cdot \mathbf{B}(x) + m^2_D \int \frac{d\Omega}{4\pi} W(x,v)W(x,v) \}. \] (34)

One advantage of this formulations that this theory is local. What is even more important, however, is that the algebraic complexity is significantly reduced.

What does it mean to integrate out the semi-hard field modes in this formulation? First we have to split the fields into two pieces. The fields \( A, \mathbf{E} \) and \( W \) are decomposed into soft and a semi-hard modes:

\begin{align*}
A &\rightarrow A + a \\
\mathbf{E} &\rightarrow \mathbf{E} + \mathbf{e} \\
W &\rightarrow W + w.
\end{align*}

The soft modes \( A, \mathbf{E} \) and \( W \) contain the spatial Fourier components with \( p < \mu \) while the semi-hard modes \( a, \mathbf{e} \) and \( w \) consist of those with \( k > \mu \).

The interpretation of \( W \) and \( w \) is the following: Both describe deviation of the distribution of the hard particles from thermal equilibrium. \( W \) \( (w) \) is the slowly (rapidly) varying piece of this distribution varying on length scale greater (less) than \( 1/\mu \).

After this split we have two sets of equations of motion, one for the soft fields and one for the semi-hard fields. Due to the non-linear terms these sets are coupled.

The equations for the soft fields can be written as

\[ [D_\mu, F^{\mu\nu}(x)] = m^2_D \int \frac{d\Omega}{4\pi} v^\nu W(x,v), \] (36)

\[ [v \cdot D, W(x,v)] = v \cdot \mathbf{E}(x) + \xi(x,v), \] (37)

where

\[ \xi^a(x,v) = -g f^{abc} (v \cdot a^b(x)) w^c(x,v) \] (soft).

The subscript “soft” indicates that only spatial Fourier components with \( p < \mu \) are included. The field strength tensor and the covariant derivatives in Eqs. (36) and (37) contain only the soft gauge fields. Note that terms which couple soft and semi-hard fields have been neglected in Eq. (36).

Since the semi-hard modes can be treated perturbatively, one can neglect their self-interaction. One can also neglect interactions in the first Vlasov equation, so that we have

\[ \partial_\mu f^{\mu\nu}(x) = m^2_D \int \frac{d\Omega}{4\pi} v^\nu w(x,v), \] (39)

\[ v \cdot \partial w(x,v) = v \cdot \mathbf{e}(x) + h(x,v), \] (40)

where \( f^{\mu\nu} = \partial^\mu a^\nu - \partial^\nu a^\mu \). The only interaction which affects the time evolution of the semi-hard fields at leading order appears in Eq. (40), it is due to the term

\[ h^a(x,v) = -g f^{abc} [v \cdot A^b(x) w^c(x,v) + v \cdot a^b(x) W^c(x,v)]. \] (41)

Integrating out the scale \( gT \) means that we solve the classical equations of motion (39) and (40) for the semi-hard fields on the soft background. Then we have to perform the thermal average over their initial conditions using the Hamiltonian (34). In this way we eliminate the semi-hard fields and we end up with equations of motion for the soft fields only. This can be done in perturbation theory, where one expands in powers of \( h(x,v) \).

We obtain an expansion

\[ a(x) = a_0(x) + a_1(x) + a_2(x) + \cdots \]

\[ w(x,v) = w_0(x,v) + w_1(x,v) + w_2(x,v) + \cdots \] (42)

in which the \( n \)-th order term contains \( n \) powers of the soft fields \( A, \mathbf{E}, W \). The terms \( a_n, w_0 \) are independent of the soft background and they only depend on the initial conditions for the semi-hard fields. Note that the solution (32) is still formal since it contains the yet unknown non-perturbative soft fields.

Inserting (42) into Eq. (38) we obtain a series

\[ \xi(x,v) = \xi_0(x,v) + \xi_1(x,v) + \xi_2(x,v) + \cdots \] (43)

Each term in (43) is bilinear in free fields \( a_0 \) and \( w_0 \). The term \( \xi_n \) is of \( n \)-th order in the fields \( A \) and \( W \). This expression has to be plugged into the second Vlasov equation (37) for the soft fields, which then becomes

\[ [v \cdot D, W(x,v)] = v \cdot \mathbf{E}(x) + \xi_0(x,v) + \xi_1(x,v) + \xi_2(x,v) + \cdots \] (44)

As it stands, Eq. (14) is not very useful yet. On the rhs we have an infinite series of terms, each depending on the yet unknown soft fields and on the initial conditions for the semi-hard fields. In general, the average over these initial conditions can be performed only after the equations for the soft fields have been solved.

Fortunately, we can simplify Eq. (44) significantly since we are only interested in the leading order dynamics of the soft field modes. Recall that each term \( \xi_n \) is bilinear in the fields \( a_0 \) and \( w_0 \). Furthermore, \( a_0 \) and \( w_0 \) are linear in the initial values for their time evolution. After solving the equations of motion for the soft fields (36) and (40), one has to average over initial conditions. Due to the
scale separation the average of the bilinear terms of semi-hard fields can be approximated by disconnected parts. But this just means that we can perform the average over initial conditions for the semi-hard fields already in the equation of motion (44). The only exception is the term $\xi_0$ because the thermal average of this term vanishes due to colour conservation: Thermal averages like $\langle a^a_0 b^b_0 \rangle$ are proportional to $\delta^{a_b}$. Inserted into Eq. (44) this gives zero due to the antisymmetry of the structure constants $f^{abc}$. The highest correlation function of $\xi_0$ we have to take into account is the two point function $\langle \xi_0(x_1, v_1) \xi_0(x_2, v_2) \rangle$. Put differently, the term $\xi_0$ acts like a Gaussian random force. It is random because it is independent of the soft fields.

Since the expectation value of $\xi_0$ vanishes, it is not sufficient to keep only this term on the rhs of Eq. (44). We also have to take into account $\xi_1$ which we can approximate by

$$\xi_1(x, v) \simeq \langle \xi_1(x, v) \rangle$$  \hspace{1cm} (45)

where $\langle \cdot \cdot \rangle$ denotes the average over initial conditions for the semi-hard fields. The higher order terms $\xi_n$ in Eq. (44) can be neglected. Then we have

$$[v \cdot D, W(x, v)] \simeq v \cdot E(x) + \xi_0(x, v) + \langle \xi_1(x, v) \rangle$$  \hspace{1cm} (46)

in which the correlation functions of $\xi_0$ can be treated as Gaussian.

VII. THE LOGARITHMIC APPROXIMATION

The next step towards the effective equations of motion for the soft fields is to evaluate the terms in the second line of Eq. (44). As in the diagram calculation in Sec. VI A one encounters terms which are logarithmically sensitive to the separation scale $\mu$. Only these logarithmic terms will be kept. The reason why this is sufficient will become clear in a moment.

We have seen that all we need to know about the noise term $\xi_0$ is its 2-point function. The leading logarithmic result reads

$$\langle \xi_0^a(x_1, v_1) \xi_0^b(x_2, v_2) \rangle = -2N g^2 T^2 m_D^2 \log \left( \frac{g T}{\mu} \right) I(v_1, v_2)$$

$$\times \delta^{ab} \delta^{(S^2)}(x_1 - x_2),$$  \hspace{1cm} (47)

with

$$I(v, v_1) = -\delta(S^2)(v - v_1) + \frac{1}{\pi^2} \frac{(v \cdot v_1)^2}{\sqrt{1 - (v \cdot v_1)^2}},$$  \hspace{1cm} (48)

where $\delta(S^2)$ is the delta function on the two dimensional unit sphere:

$$\int d\Omega_{v_1} f(v_1) \delta(S^2)(v - v_1) = f(v).$$  \hspace{1cm} (49)

With the same approximations the result for the last term in Eq. (46) is

$$\langle \xi_1(x, v) \rangle = N g^2 T \log \left( \frac{g T}{\mu} \right) \times \int \frac{d\Omega_{v_1}}{4\pi} I(v, v_1) W(x, v_1).$$  \hspace{1cm} (50)

Inserting this into Eq. (46) we obtain

$$[v \cdot D, W(x, v)] = v \cdot E(x) + \xi_0(x, v) + N g^2 T \log \left( \frac{g T}{\mu} \right) \int \frac{d\Omega_{v_1}}{4\pi} I(v, v_1) W(x, v_1).$$  \hspace{1cm} (51)

I will now argue that, for the leading order dynamics of the soft fields, the lhs of Eq. (51) can be neglected. The only spatial momentum scales which are left in the problem are $\mu$ and $g^2 T$. The field modes we are ultimately interested in, are the ones which have only momenta of order $g^2 T$. The cutoff dependence on the rhs must drop out after solving the equations of motion for the fields with spatial momenta smaller than $\mu$. Thus, after the $\mu$-dependence has cancelled, the logarithm must turn into

$$\log(g T/g^2 T)) = \log(1/g).$$

We will now simplify Eq. (51) by neglecting terms which are suppressed by inverse powers of $\log(1/g)$. We introduce moments of $W(x, v)$ and $\xi_0(x, v)$,

$$W^{i_1 \cdots i_n}(x) = \int \frac{d\Omega_{v}}{4\pi} v^{i_1} \cdots v^{i_n} W(x, v),$$

$$\xi_0^{i_1 \cdots i_n}(x) = \int \frac{d\Omega_{v}}{4\pi} v^{i_1} \cdots v^{i_n} \xi_0(x, v).$$  \hspace{1cm} (52)

Taking the first moment of Eq. (51) gives

$$[D_0, W^i(x)] = [D_j, W^{ij}(x)] = \frac{1}{3} E^i(x)$$

$$+ \xi_0^i(x) - \frac{N g^2 T}{4\pi} \log(1/g) W^i(x).$$  \hspace{1cm} (53)

Assuming that (cf. Ref. 19) $W^{ij}$ is of the same size as $W^i(x)$ and $D_0 \leq g^2 T$, the lhs of Eq. (53) is logarithmically suppressed relative to the term $\propto W^i(x)$ on the rhs and can be neglected. Then we can trivially solve (53) for $W^i(x)$,

$$W^i(x) = \frac{4\pi}{N g^2 T \log(1/g)} \left( \frac{1}{3} E^i(x) + \xi_0^i(x) \right).$$  \hspace{1cm} (54)

Inserting the current $j^i(x) = m_D^2 W^i(x)$ into the rhs of Eq. (53) gives

$$[D_\mu, F^{\mu i}(x)] = \gamma E^i(x) + \zeta^i(x)$$  \hspace{1cm} (55)

where we have introduced

$$\zeta^i(x) \equiv \frac{4\pi m_D^2}{N g^2 T \log(1/g)} \xi_0^i(x).$$  \hspace{1cm} (56)
and

\[ \gamma = \frac{4\pi m_1^2}{3Ng^2T \log(1/g)}. \tag{57} \]

The term \( \zeta \) is a Gaussian white noise. Its correlator is easily obtained from Eq. [17].

\[ \langle \zeta^i(x_1)\zeta^j(x_2) \rangle = 2T\gamma \delta^{ij}\delta^{ab}\delta^{(4)}(x_1 - x_2). \tag{58} \]

Eq. (58) is the main result of our calculation. It is remarkable that, at the order we are considering, the effect of the high momentum modes can simply be described by a local damping term, together with a Gaussian white noise which keeps the soft modes in thermal equilibrium. This surprising result is somewhat counter-intuitive since Landau damping is generally known as a non-local effect. Qualitatively, it can be understood as follows: Recall that the hard particles have a mean free path of order \((g^2T \log(1/g))^{-1}\). To understand the physical picture behind the logarithmic approximation, we have to imagine that \(\log(1/g)\) is a large number. Then the mean free path of the hard particles is small compared to the length scale \((g^2T)^{-1}\) on which the soft non-perturbative dynamics occurs. That is, the soft field modes are Landau damped only over a small length scale.

There is an essential difference compared to Abelian gauge theories. There the mean free path is of order \((g^2T \log(1/g))^{-1}\) as well. However, the mean free path is determined by the total cross section of the hard particles which is dominated by small angle scattering. Thus, in a typical scattering event, the momentum of a charged particle in a QED plasma is hardly affected. The Landau damping can continue as a coherent process after such a scattering occurs. In a non-Abelian theory even a scattering under arbitrarily small angle can change the colour charge of a hard particle. It is precisely this colour exchange which leads to a loss of coherence in the process of Landau damping.

**VIII. THE HOT SPHALERON RATE**

With Eq. (58) we are now able to estimate the characteristic time scale \(t\) of non-perturbative gauge field fluctuations and thus of the rate for hot electroweak baryon number violation. Neglecting the term \([D_0, F^{0i}(x)]\) (see below) the lhs can be estimated as

\[ [D_j, F^{ji}(x)] \sim R^{-2} \Delta A \sim g^4T^2 \Delta A. \tag{59} \]

On the rhs we have in \(A_0 = 0\) gauge

\[ \gamma E \sim \frac{T}{\log(1/g)} \frac{\Delta A}{t}. \tag{60} \]

Putting (59) and (60) together we find

\[ t \sim \left(g^4 \log(1/g)T\right)^{-1}. \tag{61} \]

Therefore the hot sphaleron rate, at leading order, has the form

\[ \Gamma = \kappa g^{10} \log(1/g)T^4 \tag{62} \]

where \(\kappa\) is a non-perturbative coefficient which does not depend on the gauge coupling.

**IX. AN EFFECTIVE THEORY FOR THE SOFT MODES**

I will now discuss how the non-perturbative coefficient \(\kappa\) in Eq. (62), and more generally, correlation functions like (58) can be calculated on a lattice for which it is convenient to work in the \(A_0 = 0\) gauge.

The time scale for non-perturbative dynamics is much larger than the corresponding length scale. Therefore time derivatives on the lhs of Eq. (58) are negligible and the non-perturbative dynamics of the soft gauge fields at leading order is correctly described by the equation

\[ [D_j, F^{ji}(x)] = -\gamma \dot{A}(x) + \zeta(x), \tag{63} \]

which should be easy to implement in a lattice calculation.

After solving Eq. (63), the result has to be plugged into the operator \(O(t)\) of interest. The corresponding correlation function is then given by the average over the random force \(\zeta\) using Eq. (58).

**X. SUMMARY AND DISCUSSION**

We have obtained an effective theory for the non-perturbative dynamics of the soft field modes by integrating out the hard \((p \sim T)\) and semi-hard modes \((p \sim gT)\) in perturbation theory. This effective theory is described by the Langevin equation (63).

Furthermore, we have determined the parametric form of the hot electroweak baryon number violation rate at leading order. It contains a non-perturbative numerical coefficient which can be evaluated using Eq. (63).

One would expect corrections to Eq. (2) to be suppressed by a factor \((\log(1/g))^{-1}\), which, in the electroweak theory, is not small. It would therefore be interesting, both from the theoretical and from the practical point of view, to see how sub-leading terms can be computed.

***We have \(g \approx 0.66\) for \(T \sim 100\text{GeV}\) which gives \(\log(1/g) \approx 0.4\).***
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Note added: While this paper was being completed, I received a preprint by Arnold, Son and Yaffe [23], in which they derive Eq. (63) using the concept of colour conductivity. They also show that the effective theory described by this equation is insensitive to the ultraviolet which means that its lattice implementation does not require any renormalization.

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