INDEPENDENT PARAMETERS FOR SPECIAL INSTANTON BUNDLES ON $\mathbb{P}^{2n+1}$

NORBERT HOFFMANN

Abstract. Motivated by Yang-Mills theory in $4n$ dimensions, and generalizing the notion due to Atiyah, Drinfeld, Hitchin and Manin for $n = 1$, Okonek, Spindler and Trautmann introduced instanton bundles and special instanton bundles as certain algebraic vector bundles of rank $2n$ on the complex projective space $\mathbb{P}^{2n+1}$. The moduli space of special instanton bundles is shown to be rational.

1. Introduction

An instanton is a self-dual solution of the SU(2) Yang-Mills equations on the Euclidean sphere $S^4$ [8, Exp. 1]. Via the Penrose transformation, these correspond to certain algebraic vector bundles of rank 2 on the complex projective space $\mathbb{P}^3$, which are consequently called instanton bundles [2, 9].

Salamon [19] and Corrigan-Goddard-Kent [7] have independently generalized this picture to Yang-Mills theory in dimension $4n$, replacing $S^4$ and $\mathbb{P}^3$ by the quaternionic projective space $\mathbb{HP}^n$ and the twistor space over it, which is the complex projective space $\mathbb{P}^{2n+1}$. Now certain Yang-Mills connections on $\mathbb{HP}^n$ correspond to certain algebraic vector bundles on $\mathbb{P}^{2n+1}$; cf. [5, §3] and [14, Corollary to Main Theorem 2]. Motivated by this generalization of the Penrose transform, Okonek and Spindler [18] extended the notion of instanton bundles to algebraic vector bundles of rank $2n$ on $\mathbb{P}^{2n+1}$.

A fundamental tool in the study of algebraic vector bundles on projective spaces is their description in terms of monads [13, 3, 17]. It often allows to describe the vector bundles at hand in terms of some linear algebra data. In particular, the Beilinson spectral sequence [4] yields a correspondence between instanton bundles and instanton monads; we recall this in Section 2 below.

Hirschowitz and Narasimhan [10] have introduced certain special instanton bundles on $\mathbb{P}^3$, calling them special ‘t Hooft bundles. They

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have studied their moduli space, and proved in particular that it is rational. Recently Tikhomirov [23] has shown that the moduli space of all instanton bundles on $\mathbb{P}^3$ with fixed odd $c_2$ is irreducible, and together with Markushevich also that it is rational [16].

Generalizing from $\mathbb{P}^3$ to $\mathbb{P}^{2n+1}$, Spindler and Trautmann [22] have introduced the notion of special instanton bundles on $\mathbb{P}^{2n+1}$ and constructed their moduli space. They also prove that this moduli space is non-empty, irreducible, and smooth of dimension $2n(k+1)+(2n+2)^2-7$ if one fixes the quantum number $k+1$ of these instantons. The main result of the present paper, Theorem 4.9, states that the Spindler-Trautmann moduli space of special instanton bundles on $\mathbb{P}^{2n+1}$ is rational. So these instanton bundles depend on $2n(k+1)+(2n+2)^2-7$ independent complex parameters.

Like in the case of vector bundles on a curve [15, 12, 11], the proof involves the rationality of some Severi-Brauer varieties, whose Brauer classes are related to the existence of Poincaré families, or universal families, of vector bundles parameterized by the moduli space. Spindler and Trautmann determined in [22] when such Poincaré families of special instanton bundles exist. Somewhat surprisingly, Theorem 4.9 proves rationality of the moduli space even in the cases where there is no Poincaré family. A similar phenomenon has been observed for moduli spaces of vector bundles on rational surfaces [21, 8].

Another main ingredient in the proof is the no-name lemma about rationality of quotients of vector spaces modulo linear groups; cf. for example [6]. It allows us to finally reduce to a rationality problem for quotients modulo $\text{PGL}_2$, where the invariant ring is known explicitly; that’s how we prove rationality of a moduli space without at the same time constructing a Poincaré family on some open part of it. Compared to the special case $n = 1$ of bundles on $\mathbb{P}^3$, where the rationality is proved in [10], this ingredient is new. Then some results about Severi-Brauer varieties from [10] complete the proof.

The structure of the present text is as follows. In Section 2 we recall the definition of mathematical instanton bundles that we will work with, and we review the correspondence to instanton monads. In Section 3 we recall the notion of special instanton bundles due to Spindler and Trautmann, and we also review their construction of moduli spaces for these; both aspects are based on the correspondence to instanton monads. Section 4 is devoted to the proof of the main result, Theorem 4.9.
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2. Mathematical instanton bundles

In this section, we recall the notion of (mathematical) instanton bundles, and their description in terms of monads.

We work over the odd-dimensional complex projective space

\[ \mathbb{P} := \mathbb{P}^{2n+1}/\mathbb{C}, \quad n \geq 1. \]

An algebraic vector bundle \( E \) over \( \mathbb{P} \) is said to have **natural cohomology** if there is at most one \( q \) with \( H^q(\mathbb{P}, E) \neq 0 \).

**Definition 2.1.** Let \( k \geq 1 \) be an integer. A **\( k \)-instanton bundle** is an algebraic vector bundle \( E \) over \( \mathbb{P} = \mathbb{P}^{2n+1} \) with the following properties:

i) The rank of \( E \) is \( 2n \).

ii) The Chern polynomial of \( E \) is \( c_t(E) = (1 - t^2)^{-k} \).

iii) For each \( l \in \mathbb{Z} \) with \( -2n - 1 \leq l \leq 0 \), the twisted vector bundle \( E(l) := E \otimes \mathcal{O}_\mathbb{P}(1)^{\otimes l} \) has natural cohomology.

**Remark 2.2.**

i) Every \( k \)-instanton bundle is simple by [1, Theorem 2.8].

ii) Some authors moreover require that \( E \) has trivial splitting type. Note that this is an open condition. It is not used here, so we don’t include it in our definition.

Let \( \text{Cpx}(\mathbb{P}) \) denote the category of complexes of coherent \( \mathcal{O}_\mathbb{P} \)-modules. By a **monad**, we mean a complex of vector bundles

\[ E^\bullet = [0 \rightarrow E^{-1} \xrightarrow{i} E^0 \xrightarrow{p} E^1 \rightarrow 0] \in \text{Cpx}(\mathbb{P}) \]

such that \( p \) is surjective, and \( i \) is an isomorphism onto a subbundle. As **morphisms** of monads, we take the morphisms in \( \text{Cpx}(\mathbb{P}) \). The **cohomology** of the monad \( E^\bullet \) is the vector bundle

\[ E := \ker(p)/\text{image}(i). \]

**Definition 2.3.** A monad \( E^\bullet \) is a **\( k \)-instanton monad** for \( k \geq 1 \), if

\[ E^{-1} \cong \mathcal{O}_\mathbb{P}(-1)^k, \quad E^0 \cong \mathcal{O}_\mathbb{P}^{2n+2k}, \quad \text{and} \quad E^1 \cong \mathcal{O}_\mathbb{P}(1)^k. \]

The following standard facts show that the categories of \( k \)-instanton bundles and of \( k \)-instanton monads are equivalent.

**Proposition 2.4.**

i) If \( E^\bullet \) is a \( k \)-instanton monad, then its cohomology \( E = \ker(p)/\text{image}(i) \) is a \( k \)-instanton bundle.
ii) If $E^*$ and $F^*$ are two $k$-instanton monads with cohomologies $E$ and $F$, then $\text{Hom}_\mathbb{P}(E,F) = \text{Hom}_{\text{Cpx}(\mathbb{P})}(E^*,F^*)$.

iii) Every $k$-instanton bundle $E$ is isomorphic to the cohomology of some $k$-instanton monad $E^*$.

Proof. i) Let $E$ be the cohomology of a $k$-instanton monad

$$0 \longrightarrow \mathcal{O}_\mathbb{P}(-1)^k \xrightarrow{i} \mathcal{O}_\mathbb{P}^{2n+2k} \xrightarrow{p} \mathcal{O}_\mathbb{P}(1)^k \longrightarrow 0.$$ 

Then the short exact sequences of vector bundles

$$0 \longrightarrow \ker(p) \longrightarrow \mathcal{O}_\mathbb{P}^{2n+2k} \xrightarrow{p} \mathcal{O}_\mathbb{P}(1)^k \longrightarrow 0,$$

$$0 \longrightarrow \mathcal{O}_\mathbb{P}(-1)^k \xrightarrow{i} \ker(p) \longrightarrow E \longrightarrow 0$$

show that $E$ has rank $2n$ and Chern polynomial $c_t(E) = (1 - t^2)^{-k}$. Now tensor these sequences with $\mathcal{O}_\mathbb{P}(l)$, and consider the resulting long exact cohomology sequences. For $l = 0$, we get an exact sequence

$$H^0(\ker(p)) \longrightarrow H^0(E) \longrightarrow H^1(\mathcal{O}_\mathbb{P}(-1)^k) = 0.$$ 

Here $H^0(\ker(p)) = 0$, since $\ker(p)$ is stable by [1, Theorem 2.8]; hence $H^0(E) = 0$. For general $l$, we see that $H^q(E(l))$ vanishes whenever

$$H^{q-1}(\mathcal{O}_\mathbb{P}(l+1)) = H^q(\mathcal{O}_\mathbb{P}(l)) = H^{q+1}(\mathcal{O}_\mathbb{P}(l-1)) = 0.$$ 

The latter holds in each of the following four cases:

- $l = -2n - 1$, and $q \neq 2n$
- $-2n \leq l \leq -2$, and $q$ arbitrary
- $l = -1$, and $q \neq 1$
- $l = 0$, and $q \geq 2$

It follows that $E(l)$ has natural cohomology for $-2n - 1 \leq l \leq 0$, the only possibly nonzero cohomology groups in this range being

$$H^{2n}(E(-2n-1)), ~ H^1(E(-1)), ~ \text{and} ~ H^1(E).$$

ii) Let $E^*$ and $F^*$ more generally be two monads, with cohomologies $E$ and $F$. Given a morphism $E \to F$, we try to lift it to a morphism of monads $E^* \to F^*$. As explained in [14, Lemma II.4.1.3], the obstructions against the existence and uniqueness of such a lift are some classes in

$$\text{Ext}^q(E^i,F^j) = H^q(\mathbb{P}, (E^i)_{\text{dual}} \otimes F^j)$$

with $i > j$ and $q \leq 2$. If $E^*$ and $F^*$ are $k$-instanton monads, then

$$\text{Ext}^q(E^i,F^j) = H^q(\mathbb{P}, \mathcal{O}_\mathbb{P}(j-i)_{\text{rank}(E_i)\cdot\text{rank}(E_j)})$$

vanishes for all $i > j$ and all $q \leq 2$.  

ii) For any vector bundle $E$ over $\mathbb{P}$, Beilinson [4] has constructed a spectral sequence with $E_1$-term
\[ E_1^{pq} = H^q(\mathbb{P}, E(p)) \otimes \Omega_{\mathbb{P}}^{-p}(-p) \]
which converges to $E$. If $E$ is a $k$-instanton bundle, then most of these terms vanish, and the claim follows; for some more details, see for example [18, Lemma 1.3 and Corollary 1.4].

\[ \square \]

Remark 2.5. Let $S$ be a scheme of finite type over $\mathbb{C}$.

A vector bundle $E$ over $\mathbb{P} \times S$ is a family of $k$-instanton bundles if its restriction to every closed point $s \in S(\mathbb{C})$ is a $k$-instanton bundle.

A monad $E^\bullet$ of vector bundles over $\mathbb{P} \times S$ is a family of $k$-instanton monads if its restriction to every closed point $s \in S(\mathbb{C})$ is a $k$-instanton monad. (This implies $E^d \cong \mathcal{O}_S(d) \boxtimes F^d$ for vector bundles $F^d$ over $S$.)

With these definitions, the above equivalence between instanton bundles and instanton monads extends to families; cf. [22, p. 585f.].

3. Moduli of special instantons

This section recalls the notion of special instantons due to Spindler-Trautmann [22], and their construction of moduli spaces for these.

Still assuming $k \geq 1$, we fix two complex vector spaces $V, W \cong \mathbb{C}^2$. Let $V \otimes r \to S^r V$ denote the $r$th symmetric power of $V$. Then
\[ E^0_{n,k} := (S^{n+k} V \otimes W)^\text{dual} \otimes \mathcal{O}_\mathbb{P} \quad \text{and} \]
\[ E^1_{n,k} := (S^k V)^\text{dual} \otimes \mathcal{O}_\mathbb{P}(1) \]
are vector bundles over $\mathbb{P} = \mathbb{P}^{2n+1}$, with ranks $2n + 2(k + 1)$ and $k + 1$.

The multiplication $\mu_r : S^r V \otimes S^n V \to S^{n+r} V$ induces a linear map $\mu_r^* : (S^{n+r} V)^\text{dual} \to (S^r V)^\text{dual} \otimes (S^n V)^\text{dual}$.

The choice of a linear isomorphism
\[ b : (S^n V \otimes W)^\text{dual} \sim H^0(\mathbb{P}, \mathcal{O}_\mathbb{P}(1)) \]
thus determines a morphism of vector bundles
\[ p_b : E^0_{n,k} \to E^1_{n,k} \]
such that $H^0(p_b)$ is the composition
\[ H^0(E^0_{n,k}) \xrightarrow{h_k^*} (S^k V)^\text{dual} \otimes (S^n V \otimes W)^\text{dual} \xrightarrow{b} H^0(E^1_{n,k}) \]
Note that this composition is injective, since $\mu_k$ is surjective.

\[ ^1 \text{In [18], all instanton bundles are assumed to be symplectic. However, this assumption is not used in the quoted Lemma 1.3 and Corollary 1.4.} \]
Define 3.1. i) A \((k + 1)\)-instanton monad \(E^\bullet \in \mathbb{C}pX(\mathbb{P})\) is called special if its stupid truncation

\[
\tau^{\geq 0}(E^\bullet) := [E^0 \xrightarrow{p} E^1]
\]

is in \(\mathbb{C}pX(\mathbb{P})\) isomorphic to a complex of the form

\[
[E^0_{n,k} \xrightarrow{pb} E^1_{n,k}]
\]

for some isomorphism \(b : (S^nV \otimes W)^{\text{dual}} \to H^0(\mathbb{P}, \mathcal{O}_\mathbb{P}(1))\).

ii) A \((k + 1)\)-instanton bundle \(E\) over \(\mathbb{P}\) is special if \(E\) is isomorphic to the cohomology of some special \((k + 1)\)-instanton monad.

Remark 3.2. This definition is equivalent to the original definition given by Spindler and Trautmann, according to [22, Proposition 4.2].

In particular, all these special instanton bundles are simple, and have trivial splitting type; cf. also [22, Proposition 4.5].

With \(k \geq 1\) still fixed, we consider all complexes of the form

\[
[E^0_{n,k} \xrightarrow{pb} E^1_{n,k}] \in \mathbb{C}pX(\mathbb{P})
\]

as above. The next step is to classify them up to isomorphy in \(\mathbb{C}pX(\mathbb{P})\).

We have an exact sequence of algebraic groups

\[
1 \rightarrow \mathbb{G}_m \xrightarrow{\iota_n} \text{GL}(V) \times \text{GL}(W) \xrightarrow{\pi_n} \text{GL}(S^nV \otimes W),
\]

defined by \(\iota_n(\lambda) := (\lambda \text{id}_V, \lambda^{-n} \text{id}_W)\) and \(\pi_n(\alpha, \beta) := S^n\alpha \otimes \beta\). In particular, \(\pi_n\) allows us to identify the 7-dimensional group

\[
G_n := \text{GL}(V) \times \text{GL}(W)/\iota_n(\mathbb{G}_m)
\]

with a closed subgroup of \(\text{GL}(S^nV \otimes W)\).

Proposition 3.3. Let \(k \geq 1\) be given. The homogeneous variety

\[
X_n := G_n \backslash \text{GL}(S^nV \otimes W)
\]

is a coarse moduli space for complexes of the form \([E^0_{n,k} \xrightarrow{pb} E^1_{n,k}]\).

Proof. Given an isomorphism \(b : (S^nV \otimes W)^{\text{dual}} \to H^0(\mathbb{P}, \mathcal{O}_\mathbb{P}(1))\) and an element \(g \in \text{GL}(S^nV \otimes W)\), we get another isomorphism

\[
g \cdot b := b \circ g^{\text{dual}} : (S^nV \otimes W)^{\text{dual}} \xrightarrow{\sim} H^0(\mathbb{P}, \mathcal{O}_\mathbb{P}(1)).
\]

This defines a simply transitive action of \(\text{GL}(S^nV \otimes W)\) on the set of all such isomorphisms \(b\).
Every pair $(\alpha, \beta) \in \text{GL}(V) \times \text{GL}(W)$ yields a commutative diagram

\[
\begin{array}{ccc}
(S^{n+k}V \otimes W)_{\text{dual}} & \xrightarrow{b \circ \mu^*_k} & (S^kV)_{\text{dual}} \otimes H^0(\mathbb{P}, \mathcal{O}_\mathbb{P}(1)) \\
(S^{n+k-1} \otimes \beta^{-1})_{\text{dual}} & \xrightarrow{(S^k(-1))_{\text{dual}}} & (S^kV)_{\text{dual}} \otimes H^0(\mathbb{P}, \mathcal{O}_\mathbb{P}(1))
\end{array}
\]

with $b' := \pi_n(\alpha, \beta) \cdot b$, and hence an isomorphism of complexes

\[
\begin{array}{ccc}
E^0_{n,k} & \xrightarrow{p_b} & E^1_{n,k} \\
\downarrow & & \downarrow \\
E^0_{n,k} & \xrightarrow{p_{b'}} & E^1_{n,k}
\end{array}
\]

Conversely, suppose that $[E^0_{n,k} \xrightarrow{p_b} E^1_{n,k}]$ and $[E^0_{n,k} \xrightarrow{p_{b'}} E^1_{n,k}]$ are isomorphic in $\text{Cpx}(\mathbb{P})$ for some isomorphisms $b$ and $b'$. Then there is a pair $(\alpha, \beta) \in \text{GL}(V) \times \text{GL}(W)$ with $b' = \pi_n(\alpha, \beta) \cdot b$, according to step 2) in the proof of [22, Proposition 6.1].

Pick one isomorphism $b_0 : (S^nV \otimes W)_{\text{dual}} \rightarrow H^0(\mathbb{P}, \mathcal{O}_\mathbb{P}(1))$. Assign to each complex $p_{g-b_0} : [E^0_{n,k} \rightarrow E^1_{n,k}]$ with $g \in \text{GL}(S^nV \otimes W)$ the moduli point $G_ng$ in the coset space $X_n = G_n \backslash \text{GL}(S^nV \otimes W)$. The coset $G_ng$ depends only on the isomorphism class of the complex. This turns $X_n$ into a coarse moduli space for the complexes in question.

The group $G_n$ acts via its quotient $\text{PGL}(V)$ on $\mathbb{P}V := (V \setminus \{0\})/\mathbb{G}_m$. Thus the homogeneous variety $X_n$ carries a natural conic bundle

\[
C := \mathbb{P}V \times^{G_n} \text{GL}(S^nV \times W) \twoheadrightarrow X_n
\]

with fibers $\mathbb{P}V$. The fiberwise symmetric power

\[
C^{(2n+k)} \twoheadrightarrow X_n
\]

is a projective bundle with fibers

\[
(\mathbb{P}V)^{(2n+k)} = \mathbb{P}H^0(\mathbb{P}V, \mathcal{O}(2n + k)) = \mathbb{P}(S^{2n+k}V)_{\text{dual}}.
\]

Note that every element $f \in (S^{2n+k}V)_{\text{dual}}$ induces a linear map

\[
\mu^*_{n+k}(f) : S^nV \rightarrow (S^{n+k}V)_{\text{dual}}.
\]

We form the associated Grassmannian bundle

\[
\text{Gr}_k(C^{(2n+k)}) \twoheadrightarrow X_n,
\]

which parameterizes linear subspaces $\mathbb{P}U \subseteq (\mathbb{P}V)^{(2n+k)}$ of dimension $k$, or equivalently linear subspaces $U \subseteq (S^{2n+k}V)_{\text{dual}}$ of dimension $k + 1$. 

Theorem 3.4 (Spindler-Trautmann). Given \( k \geq 1 \), let
\[
M_F(k+1) \subseteq \text{Gr}_k(O^{(2n+k)}) \longrightarrow X_n
\]
denote the open locus of all linear subspaces \( U \subseteq (S^{2n+k}V)_{\text{dual}} \) such that \( \mu_{n+k}^*(f) \) is injective for all \( 0 \neq f \in U \). Then \( M_F(k+1) \) is a coarse moduli space for special \((k+1)\)-instanton bundles over \( P = \mathbb{P}^{2n+1} \).

**Proof.** This statement is Theorem 6.3 in [22]. For the convenience of the reader, we give an outline of the proof.

The starting point is that every special \((k+1)\)-instanton bundle defines a special \((k+1)\)-instanton monad, and hence by truncation a point in \( X_n \). The remaining part of the monad will then be parameterized by the fiber of \( M_F(k+1) \) over this point in \( X_n \).

To be more specific, note that the choice of an isomorphism \( b : (S^n V \otimes W)_{\text{dual}} \sim \rightarrow H^0(\mathbb{P}, \mathcal{O}_F(1)) \) yields a commutative diagram
\[
\begin{array}{ccc}
(S^{n+k} V \otimes W \otimes S^n V \otimes W)_{\text{dual}} & \xrightarrow{\mu_k^*} & (S^k V)_{\text{dual}} \otimes S^2((S^n V \otimes W)_{\text{dual}}) \\
\downarrow b & & \downarrow S^2 b \\
\text{Hom}_F(\mathcal{O}_F(-1), E_{n,k}^0) & \xrightarrow{(pb)_*} & \text{Hom}_F(\mathcal{O}_F(-1), E_{n,k}^1).
\end{array}
\]

The kernel of the horizontal map \( \mu_k^* \) consists of all multilinear forms \( \varphi : S^{n+k} V \otimes W \otimes S^n V \otimes W \longrightarrow \mathbb{C} \) which satisfy the condition
\[
-\varphi(v_1, \ldots, v_n, v_{n+1}, \ldots, v_{n+k}, v', v', \ldots, v', w, w') = \varphi(v', \ldots, v', v_{n+1}, \ldots, v_{n+k}, v', v_1, \ldots, v_n, w)
\]
for all \( v_i, v'_j \in V \) and \( w, w' \in W \). Using that \( \varphi \) is also symmetric in \( v_1, \ldots, v_{n+k} \) and in \( v_1', \ldots, v'_n \) separately, it is easy to deduce that \( \varphi \) is symmetric in all the \( v \)'s and alternating in the \( w \)'s. This proves
\[
\ker(\mu_k^*) = (S^{2n+k} V \otimes \Lambda^2 W)_{\text{dual}}.
\]

Hence \( b \) induces, via the above diagram, an isomorphism
\[
(S^{2n+k} V \otimes \Lambda^2 W)_{\text{dual}} \sim \rightarrow \text{Hom}_F(\mathcal{O}_F(-1), \ker(pb));
\]

cf. also [22, Proposition 3.2]. Given a special \((k+1)\)-instanton monad
\[
E^* = [0 \rightarrow \mathcal{O}_F(-1)^{(k+1)} \xrightarrow{i} E_{n,k}^0 \xrightarrow{pb} E_{n,k}^1 \rightarrow 0] \in \text{Cpx}(\mathbb{P}),
\]
the components $O_p(-1) \to \ker(p_b)$ of $i$ thus span a linear subspace of $(S^{2n+k}V \oplus \Lambda^2 W)^\text{dual}$, which corresponds to a linear subspace $U \subseteq (S^{2n+k}V)^\text{dual}$.

Since $i$ is an isomorphism onto a subbundle, $U$ has dimension $k + 1$, and $\mu^*_{n+k}(f)$ is injective for all $0 \neq f \in U$; cf. [22, (3.7)]. In this way, the special $(k + 1)$-instanton monad $E^\bullet$ defines a point in $M_{\mathbb{P}}(k + 1)$.

Conversely, every linear subspace $U \subseteq (S^{2n+k}V)^\text{dual}$ of dimension $k + 1$ yields, by means of the above isomorphism, a linear subspace

$$(\Lambda^2 W)^\text{dual} \otimes U \subseteq \text{Hom}_P(O(-1), \ker(p_b))$$

of dimension $k + 1$, and hence a morphism of vector bundles

$$i_U : O_P(-1)^{(k+1)} \cong (\Lambda^2 W)^\text{dual} \otimes U \otimes O_P(-1) \to \ker(p_b).$$

If $\mu^*_{n+k}(f)$ is injective for all $0 \neq f \in U$, then $i_U$ is an isomorphism onto a subbundle of $\ker(p_b)$, so

$$0 \to O_P(-1)^{(k+1)} \overset{i_U}{\to} E^0_{n,k} \overset{p_b}{\to} E^1_{n,k} \to 0$$

is a special $(k + 1)$-instanton monad. This shows that $M_{\mathbb{P}}(k + 1)$ is a coarse moduli scheme for special $(k + 1)$-instanton monads, and hence also for special $(k + 1)$-instanton bundles due to Proposition 2.4.

\begin{remark}
To make this precise using the formalism of moduli functors, one would have to define what a family of special $(k+1)$-instanton bundles is, say parameterized by a scheme $S$ of finite type over $\mathbb{C}$.

A family of isomorphisms $b : (S^n V \otimes W)^\text{dual} \to H^0(\mathbb{P}, O_{\mathbb{P}}(1))$ is an isomorphism of the corresponding trivial vector bundles over $S$; it induces as before a complex of vector bundles

$$[E^0_{n,k} \boxtimes O_S \to E^1_{n,k} \boxtimes O_S]$$

over $\mathbb{P} \times S$. One could define that a family $E^\bullet$ of $(k + 1)$-instanton monads is special if its truncation $\tau_{\geq 0}(E^\bullet)$ is étale-locally in $S$ isomorphic to a complex of this form, and that a family of $(k + 1)$-instanton bundles $E$ is special if the corresponding family of $(k + 1)$-instanton monads is. Then the arguments in the above proof of Theorem 3.4 extend routinely to families of special instantons.

Note however that the moduli functor in [22, p. 585] is slightly different from this one for non-reduced $S$, since they only require that the restriction of $E$ to all closed points $s \in S(\mathbb{C})$ is special.

\begin{remark}
The moduli space $M_{\mathbb{P}}(k + 1)$ is non-empty by [22, 3.7]. It is by construction an irreducible smooth variety of dimension

$$\dim M_{\mathbb{P}}(k + 1) = 2n(k + 1) + \dim X_n = 2n(k + 1) + (2n + 2)^2 - 7.$$
4. Rationality

Let $G$ be a linear algebraic group over $\mathbb{C}$. Suppose that $G$ acts on an integral algebraic variety $X$ of finite type over $\mathbb{C}$. We denote by $\mathbb{C}(X)^G$ the field of $G$-invariant rational functions on $X$.

**Lemma 4.1.** Suppose that $G$ acts on the integral variety $X'$ over $\mathbb{C}$ as well, and that there is an open orbit $Gx' \subseteq X'$ with $x' \in X'(\mathbb{C})$. Then
\[ \mathbb{C}(X \times X')^G \cong \mathbb{C}(X)^{\text{Stab}_G(x')} . \]

**Proof.** This is a special case of the standard ‘lemma of Seshadri’, or ‘slice lemma’; cf. for example [6, Theorem 3.1]. \[ \square \]

The action of $G$ on $X$ is called *almost free* if there is a dense open subvariety $X^0 \subseteq X$ such that the stabiliser subgroup $\text{Stab}_G(x) \subseteq G$ is trivial for each closed point $x \in X^0(\mathbb{C})$.

**Lemma 4.2.** Suppose $V \cong \mathbb{C}^2$, and $n \geq 2$. The natural action of $\text{PGL}(V)$ on the Grassmannian $\text{Gr}_2(S^nV \oplus S^nV)$ is almost free.

**Proof.** Suppose that $\alpha \in \text{GL}(V)$ represents a nontrivial element in $\text{PGL}(V)$. Up to multiplication by $\mathbb{C}^*$, and the choice of an appropriate basis for $V$, there are three cases:

1. $\alpha = \begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix}$, $\lambda \in \mathbb{C}^*$, $\lambda^r \neq 1$ for all $r \in \{1, \ldots, n\}$,
2. $\alpha = \begin{pmatrix} 1 & 0 \\ 0 & \zeta \end{pmatrix}$, $\zeta$ a primitive $r$th root of unity, $2 \leq r \leq n$,
3. $\alpha = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

In each case, let us estimate the dimension of the fixed point set
\[ \text{Gr}_2(S^nV \oplus S^nV)^\alpha . \]

In case 1.), $S^nV$ decomposes into 1-dimensional eigenspaces under the semisimple endomorphism $\alpha$, so $S^nV \oplus S^nV$ decomposes into 2-dimensional eigenspaces. Every 2-dimensional $\alpha$-invariant subspace
\[ U \subseteq S^nV \oplus S^nV \]
is either one of these eigenspaces, or a direct sum of lines in two of them. The former are parameterized by a finite set, the latter by a finite union of products $\mathbb{P}^1 \times \mathbb{P}^1$. Hence we conclude in this case
\[ \dim \text{Gr}_2(S^nV \oplus S^nV)^\alpha = 2. \]
The union of these fixed loci has dimension $\leq 2 + \dim \text{PGL}(V) = 5$. 


In case 2.), $S^nV$ decomposes into $r$ eigenspaces $(S^nV)_{\chi}$ under the semisimple endomorphism $\alpha$. Their dimension is
\[
dim(S^nV)_{\chi} \leq \lceil (n+1)/r \rceil \leq \lceil (n+1)/2 \rceil \leq (n+2)/2.
\]
So the eigenspace $(S^nV)_{\chi}^2$ of $S^nV$ has dimension $\leq n+2$, and the space of 2-dimensional subspaces $U \subseteq (S^nV)_{\chi}^2$ has dimension
\[
\dim \text{Gr}_2((S^nV)^2_{\chi}) \leq 2n.
\]
If $\chi_1 \neq \chi_2$ are two eigenvalues of $\alpha$ on $S^nV$, then
\[
\dim(S^nV)_{\chi_1} + \dim(S^nV)_{\chi_2} \leq \dim S^nV = n + 1.
\]
So the space of direct sums $U = U_1 \oplus U_2$, where $U_i$ is a line in the eigenspace $(S^nV)_{\chi_i}^2$ of $\alpha$ on $S^nV \oplus S^nV$, has dimension
\[
\dim \mathbb{P}((S^nV)_{\chi_1}^2) \times \mathbb{P}((S^nV)_{\chi_2}^2) \leq 2 \dim(S^nV) - 2 = 2n.
\]
These two arguments cover all 2-dimensional $\alpha$-invariant subspaces $U \subseteq S^nV \oplus S^nV$. Hence we can conclude in this case
\[
\dim \text{Gr}_2(S^nV \oplus S^nV)^{\alpha} \leq 2n.
\]
Such $\alpha \in \text{GL}(V)$ yield finitely many 2-dimensional conjugacy classes in $\text{PGL}(V)$. So the union of these fixed loci has dimension $\leq 2n + 2$.

In case 3.), we can choose a basis $x, y \in V$ with
\[
\alpha(x) = x \quad \text{and} \quad \alpha(y) = x + y.
\]
The endomorphism $(\alpha - \text{id})^r$ of $S^nV$ has kernel
\[
\ker(\alpha - \text{id})^r = x^{n-r+1} \cdot S^{r-1}V
\]
for $1 \leq r \leq n+1$. In particular, the kernel of $(\alpha - \text{id})^r$ on $S^nV \oplus S^nV$ has dimension $2r$. Suppose that $U \subseteq S^nV \oplus S^nV$ is 2-dimensional and $\alpha$-invariant. It follows that $\alpha - \text{id}$ is a nilpotent endomorphism of $U$.

If $\alpha - \text{id} = 0$ on $U$, then $U$ coincides with the kernel $\mathbb{C} \cdot x^n + \mathbb{C} \cdot x^n$ of $\alpha - \text{id}$ on $S^nV \oplus S^nV$; hence $U$ is unique in this situation.

Otherwise, we have $(\alpha - \text{id})^2 = 0$ on $U$, and $U = \mathbb{C} \cdot u \oplus \mathbb{C} \cdot \alpha(u)$ for any element $u \in U$ with $\alpha(u) \neq u$. Here $u$ can be any element in the 4-dimensional kernel of $(\alpha - \text{id})^2$ on $S^nV \oplus S^nV$ with $\alpha(u) \neq u$.

Hence we conclude in this case
\[
\dim \text{Gr}_2(S^nV \oplus S^nV)^{\alpha} = 4 - 2 = 2.
\]
This $\alpha \in \text{GL}(V)$ yields one 2-dimensional conjugacy class in $\text{PGL}(V)$. So the union of these fixed loci has dimension $\leq 2 + 2 = 4$.

The Grassmannian $\text{Gr}_2(S^nV \oplus S^nV)$ has dimension $4n$. Using the assumption $n \geq 2$, we see that the closure of all these fixed points has smaller dimension. On the open complement, $\text{PGL}(V)$ acts freely. \qed
We say that the field $\mathbb{C}(X)^G$ of $G$-invariant rational functions on $X$ is **rational** if it is purely transcendental over the base field $\mathbb{C}$.

**Example 4.3.** The group $\text{PGL}(V)$ with $V \cong \mathbb{C}^2$ acts on the vector space $\text{End}(V)^2$ over $\mathbb{C}$ by simultaneous conjugation. The action is known to be almost free, and the field of invariants

$$\mathbb{C}\left(\text{End}(V)^2\right)^{\text{PGL}(V)}$$

is rational. In fact, sending $(\alpha_1, \alpha_2) \in \text{End}(V)^2$ to the traces of the five maps $\alpha_1, \alpha_2, \alpha_1^2, \alpha_1\alpha_2, \alpha_2^2 \in \text{End}(V)$ defines an isomorphism

$$\text{End}(V)^2//\text{PGL}(V) \xrightarrow{\sim} \mathbb{A}^5.$$

**Lemma 4.4** (No-name lemma). Let $G$ act linearly on vector spaces $M$ and $M'$ of finite dimension over $\mathbb{C}$. If $G$ is reductive, and acts almost freely on $M$, then $\mathbb{C}(M \oplus M')^G$ is purely transcendental over $\mathbb{C}(M)^G$.

**Proof.** This statement is contained for example in [6, Corollary 3.8]. □

**Example 4.5.** Let the group $\text{PGL}(V)$ with $V \cong \mathbb{C}^2$ act linearly on a finite-dimensional vector space $M'$ over $\mathbb{C}$. Then the field of invariants

$$\mathbb{C}\left(\text{End}(V)^2 \oplus M'\right)^{\text{PGL}(V)}$$

is purely transcendental over the field $\mathbb{C}$, and hence rational.

We keep the notation of the previous section, so $V, W \cong \mathbb{C}^2$, and

$$G_n = \text{GL}(V) \times \text{GL}(W)/\iota_n(\mathbb{G}_m) \quad \text{with} \quad \iota_n(\lambda) := (\lambda \text{id}_V, \lambda^{-n} \text{id}_W).$$

**Proposition 4.6.** The coarse moduli space $X_n = G_n \backslash \text{GL}(S^n V \otimes W)$ constructed in Proposition 3.3 is rational.

**Proof.** The function field of $X_n$ is by construction

$$\mathbb{C}(X_n) \cong \mathbb{C}\left((S^n V \otimes W)^{2n+2}\right)^{G_n}.$$

We start with the special case $n = 1$. The action of $G_1$ on $V \otimes W$ has an open orbit, whose points correspond to linear maps $\psi : W_{\text{dual}} \to V$ that are bijective. Thus Lemma 4.4 yields

$$\mathbb{C}(X_1) \cong \mathbb{C}\left((V \otimes W)^4\right)^{G_1} \cong \mathbb{C}\left((V \otimes W)^3\right)^{\text{Stab}_{G_1}(\psi)}.$$

The canonical projection $G_1 \to \text{PGL}(V)$ restricts to an isomorphism

$$\text{Stab}_{G_1}(\psi) \xrightarrow{\sim} \text{PGL}(V).$$

Since the linear isomorphism

$$V \otimes W \xrightarrow{(\psi^{-1})_{\text{dual}}} V \otimes V_{\text{dual}} = \text{End}(V)$$
intertwines the action of \( \text{Stab}_{G_n}(\psi) \) with that of \( \text{PGL}(V) \), we conclude
\[
\mathbb{C}(X_1) \cong \mathbb{C}((\text{End}(V)^2)^{\text{PGL}(V)}).
\]
Using Example 4.5 it follows that \( X_1 \) is rational.

For the rest of the proof, we assume \( n \geq 2 \). The group \( \text{GL}(V) \) acts on the 2-dimensional vector space
\[
V(n) := \begin{cases} 
\mathbb{C}^2 \otimes \det^f V & \text{for } n \text{ even,} \\
V \otimes \det^{n-1} V & \text{for } n \text{ odd,}
\end{cases}
\]
in such a way that the center \( \mathbb{G}_m \subseteq \text{GL}(V) \) acts with weight \( n \). Thus the action of \( \text{GL}(V) \times \text{GL}(W) \) on the 4-dimensional vector space
\[
V(n) \otimes W
\]
descends to an action of \( G_n \). This action again has an open orbit, whose points correspond to bijective linear maps \( \psi : W^\text{dual} \to V(n) \).

Viewing elements of \((S^nV \otimes W)^2\) as linear maps from \( W^\text{dual} \) to the direct sum \( S^nV \oplus S^nV \), the open locus of injective linear maps is a \( \text{GL}(W) \)-torsor over the Grassmannian \( \text{Gr}_2(S^nV \oplus S^nV) \). Thus \( G_n \) acts almost freely on \((S^nV \otimes W)^2\), due to Lemma 4.2. It follows that the field \( \mathbb{C}(X_n) \) is purely transcendental over the field
\[
\mathbb{C}((S^nV \otimes W)^2 \oplus \text{End}(V)^2 \oplus (V(n) \otimes W))^{G_n},
\]
since both are purely transcendental over \( \mathbb{C}((S^nV \otimes W)^2)^{G_n} \) according to Lemma 4.4, the former of transcendence degree \( 2n(2n + 2) \geq 24 \), and the latter of transcendence degree 12.

Thus Lemma 4.1 yields that \( \mathbb{C}(X_n) \) is purely transcendental over
\[
\mathbb{C}((S^nV \otimes W)^2 \oplus \text{End}(V)^2)^{\text{Stab}_{G_n}(\psi)}.
\]
But this field is rational according to Example 4.5, since the stabiliser of \( \psi \) in \( G_n \) projects again isomorphically onto \( \text{PGL}(V) \).

Remark 4.7. The conic bundle \( C \to X_n \) with fibers \( \mathbb{P}V \) is not Zariski-locally trivial; cf. [22, Proposition 8.5]. It follows that the \( G_n \)-torsor \( \text{GL}(S^nV \otimes W) \to X_n \) is not Zariski-locally trivial either. The obstruction against both is a Brauer class, which can be described as follows:

The proof of Proposition 4.6 shows that \( \mathbb{C}(X_n) \) is actually purely transcendental over \( \mathbb{C}(\text{End}(V)^2)^{\text{PGL}(V)} \). Over the latter, one has the so-called generic quaternion algebra; cf. [20, Chapter 14]. Its image in the Brauer group of \( \mathbb{C}(X_n) \) is the obstruction class in question.

An equivalent way to state this is to say that the stack quotient of \( \text{GL}(S^nV \otimes W) \) modulo \( \text{GL}(V) \times \text{GL}(W) \) is birational to an affine
space times the stack quotient of $\text{End}(V)^2$ modulo $\text{GL}(V)$. This can be proved along the same lines as Proposition 4.6 above.

**Remark 4.8.** The variety $X_n$, its rationality, and the local nontriviality of bundles over it in Remark 4.7 did not depend on $k$. But the existence of Poincaré families over $\mathbb{P} \times X_n$ does depend on $k$, as follows.

Fix $k \geq 1$. The closed points $x \in X_n(\mathbb{C})$ correspond to certain isomorphism classes of complexes $E^\bullet = [E^0 \to E^1] \in \text{Cpx}(\mathbb{P})$ with $E^0 \cong \mathcal{O}_{\mathbb{P}}^{2n+2(k+1)}$ and $E^1 \cong \mathcal{O}_{\mathbb{P}}(1)^{k+1}$. A complex of vector bundles $\mathcal{E}^\bullet = [\mathcal{E}^0 \to \mathcal{E}^1] \in \text{Cpx}(\mathbb{P} \times X_n)$ is a Poincaré family if for every closed point $x \in X_n(\mathbb{C})$, the corresponding isomorphism class contains the restriction of $\mathcal{E}^\bullet$ to $\mathbb{P} \times \{x\}$.

There is a universal family $b^{\text{univ}}$ of isomorphisms $b$, parameterized by $\text{GL}(S^n V \otimes W)$; cf. Remark 3.5. It induces a complex of vector bundles $[E^0_{n,k} \boxtimes \mathcal{O}_{\text{GL}(S^n V \otimes W)} \overset{b^{\text{univ}}}{\to} E^1_{n,k} \boxtimes \mathcal{O}_{\text{GL}(S^n V \otimes W)}]$ over $\mathbb{P} \times \text{GL}(S^n V \otimes W)$. On this complex, $\text{GL}(V) \times \text{GL}(W)$ acts; the image of $\mathbb{G}_m$ under $\iota_n$ acts with weight $-k$.

If $k$ is even, we can tensor the complex with the 1-dimensional representation $\text{det}^{k/2}(V)$ of $\text{GL}(V) \times \text{GL}(W)$. After that, the image of $\mathbb{G}_m$ under $\iota_n$ acts trivially, so the action of $\text{GL}(V) \times \text{GL}(W)$ descends to an action of $\mathbb{G}_n$. Then the complex descends to a complex $\mathcal{E}^\bullet$ over $\mathbb{P} \times X_n$, which is indeed a Poincaré family in the above sense.

If $k$ is odd, then one can show that no Poincaré family $\mathcal{E}^\bullet$ over $\mathbb{P} \times X_n$ exists; cf. [22, Proposition 8.5].

One way to view this is to note that the moduli stack parameterizing the complexes in question does depend on $k$. It is in fact the stack quotient $X_{n,k}$ of $\text{GL}(S^n V \otimes W)$ modulo the group $\text{GL}(V) \times \text{GL}(W)/\iota_n(\mu_k)$, which can be proved along the same lines as Proposition 3.3.

The stack $X_{n,k}$ is birational to an affine space times the stack quotient of $\text{End}(V)^2$ modulo $\text{GL}(V)/\mu_k$; cf. the previous remark. Thus the obstruction against Poincaré families is $k$ times the Brauer class over $\mathbb{C}(X_n)$ coming from the generic quaternion algebra. So we see again that the obstruction vanishes if and only if $k$ is even.

**Theorem 4.9.** For $n \geq 1$ and $k \geq 1$, the coarse moduli space $M_{\mathbb{P}}(k+1)$ of special $(k+1)$-instanton bundles over $\mathbb{P} = \mathbb{P}^{2n+1}$ is rational.

**Proof.** Recall that $M_{\mathbb{P}}(k+1)$ is constructed as an open subvariety in the Grassmannian bundle $\text{Gr}_k(\mathcal{O}_{2n+k})$ over $X_n$. We have just seen that
$X_n$ is rational. Using [10, Proposition 2.2], it follows that $\text{Gr}_k(C^{(2n+k)})$ is also rational, since $k$ and $2n + k$ have the same parity.

**Remark 4.10.** In the special case $n = 1$, which means $\mathbb{P} = \mathbb{P}^3$, the rationality of $M_{\mathbb{P}}(k+1)$ has already been proved by Hirschowitz and Narasimhan [10, Théorème 4.10]. They also prove the rationality of $X_1$ [10, Théorème 3.4.II]; their proof is different from the one given here.

**Remark 4.11.** What [10, Proposition 2.2] proves is that the Grassmannian bundle $\text{Gr}_k(C^{(2n+k)}) \rightarrow X_n$ has rational generic fiber. This does not necessarily mean that it is Zariski-locally trivial. One can show that it is Zariski-locally trivial if and only if $k$ is even; cf. Remark 4.8.

**Remark 4.12.** [22, Theorem 8.2] states that there is a Poincaré family of special instanton bundles $\mathcal{E}$ over $\mathbb{P} \times M_{\mathbb{P}}(k+1)$ if and only if $k$ is even. In fact the obstruction class is the pullback of the obstruction class on $X_n$ explained in Remark 4.8.

An equivalent way to state this is to say that the moduli stack $\mathcal{M}_\mathbb{P}(k+1)$ of special instanton bundles is birational to an affine space times the stack $\mathcal{X}_{n,k}$ in Remark 4.8. Observing that $\mathcal{M}_\mathbb{P}(k+1)$ is open in a Grassmannian bundle over $\mathcal{X}_{n,k}$, this can also be proved using Lemma 5.5 and Lemma 4.10 in [11].

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