HYPSERSPHERICAL EQUIVARIANT SLICES AND BASIC CLASSICAL LIE SUPeralGEBRAS

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To Hiraku Nakajima on his 60th birthday with admiration

ABSTRACT. We classify all the hyperspherical equivariant slices of reductive groups. The classification is $S$-dual to the one of basic classical Lie superalgebras.

1. Introduction

1.1. Hyperspherical varieties. The study of cotangent bundles of complex spherical varieties goes back to [13, 23], see a nice survey in [25]. It was proved that a $G$-variety $Y$ is spherical iff a typical $G$-orbit in $T^*Y$ is coisotropic; equivalently, if the algebra of invariant functions $\mathbb{C}[T^*Y]^G$ is Poisson commutative. A systematic study of symplectic varieties $X$ equipped with a Hamiltonian $G$-action satisfying the above equivalent properties (i.e. typical $G$-orbits are coisotropic; equivalently, the algebra $\mathbb{C}[X]^G$ is Poisson commutative) was undertaken in [19]. Such $G$-varieties are called coisotropic or multiplicity free. Recently, another name hyperspherical\(^1\) was coined by D. Ben-Zvi [1].

1.2. Equivariant slices. Let $G$ be a complex reductive group with the Lie algebra $\mathfrak{g}$. Let $e \in \mathfrak{g}$ be a nilpotent element in an adjoint nilpotent orbit $\mathcal{O}_e \subset \mathfrak{g}$. We include $e$ into an $\mathfrak{sl}_2$-triple $(e, h, f)$ and obtain a Slodowy slice $S_e = e + \mathfrak{z}_g(f) \subset \mathcal{O}_e$. Using a $G$-invariant nondegenerate symmetric bilinear form $(\cdot, \cdot)$ on $\mathfrak{g}$, we identify $\mathfrak{g}$ with $\mathfrak{g}^*$, and $T^*G \cong G \times \mathfrak{g}^*$ with $G \times \mathfrak{g}$. This way we obtain an embedding $G \times S_e \hookrightarrow T^*G$. According to [18], the canonical symplectic form $\omega$ on $T^*G$ restricts to a symplectic form on $G \times S_e$ (a particular case of I. Losev’s construction of model Hamiltonian varieties).

Let $Q$ be the neutral connected component of the centralizer $Z_G(e, h, f)$ ($Q$ is the maximal connected reductive subgroup of the centralizer $Z_G(e)$). Then the symplectic equivariant slice variety $G \times S_e$ is equipped with a natural Hamiltonian action of $G \times Q$. Two extreme cases are as follows. First, $e = 0$, $Q = G$. We obtain a hyperspherical equivariant slice $G \times G \curvearrowright T^*G$ (since $G \times G \curvearrowright G$ is one of the basic examples of spherical varieties). Second, $e$ is a regular nilpotent, $Q$ is trivial. We obtain a hyperspherical equivariant slice $G \curvearrowright (G \times S_{e_{\text{reg}}}) \cong T^*_e(G/U)$ (the twisted cotangent bundle of the base affine space).

1.3. Triangle parts. Let $G = \text{GL}_n$, and let $e$ be a nilpotent element of Jordan type $(n - k, 1^k)$. The Young diagram of this partition has a hook form, so such nilpotents are said to have a hook type. For $k < n - 1$, the centralizer of the corresponding $\mathfrak{sl}_2$-triple is $\text{GL}_k \times \mathbb{C}^X$ (the second factor is the center of $\text{GL}_n$). The action of $\mathbb{C}^X$ on $S_e$, being trivial, we ignore it and set $Q = \text{GL}_k$ (if $k = n - 1$, then $e = 0$, and the centralizer of $e$ is $\text{GL}_n$). Now $G \times S_e$ is a basic building block (a triangle part) of the Cherkis-Nakajima-Takayama bow varieties [7, 22]. It appeared earlier in the works of J. Hurtubise and R. Bielawski as the moduli space of solutions of certain Nahm equations. In the special case $k = n - 1$ we declare $Q := \text{GL}_{n-1}$ (embedded as the upper left block subgroup of the full centralizer $\text{GL}_n$ of $e = 0$) for uniformity. Then the equivariant slice variety $G \times S_e = T^*\text{GL}_n$ is a hyperspherical variety of $G \times Q = \text{GL}_n \times \text{GL}_{n-1}$ (since $\text{GL}_n$ is a spherical $\text{GL}_n \times \text{GL}_{n-1}$-variety: so called Gelfand-Tsetlin case). There is one more exceptional case: when $k = n$, we can extend the hyperspherical

\(^1\)Its etymology goes back to an important class of spherical varieties, namely to the toric varieties. The toric hyperkähler varieties are birational to the cotangent bundles of toric varieties, and are sometimes called hypertoric.
GL_n × GL_n-variety T^∗GL_n to the hyperspherical GL_n × GL_n-variety T^∗(GL_n × C^n) (cotangent bundle of the spherical GL_n × GL_n-variety GL_n × C^n: so called Rankin-Selberg or mirabolic case).

If G = SO_n or G = Sp_{2n} is another classical group, and e is a nilpotent element of hook type, then G × S_e is a basic building block (a triangle part) of the orthosymplectic bow varieties (this duality acts on the groups involved as well). First, this comes from the relative Langlands duality [2]; for a trailer see [1] or [3, §1.7]. For instance, in the extreme cases of §1.2, the S-dual of G × G ↷ T^∗G is G^∨ × G^∨ ↷ T^∗G^∨ (Langlands dual group), while the S-dual of G ↷ T^∗_e(G/U) is G^∨ ↷ {0}.

According to [12, 8], extended by D. Gaiotto conjectures about categorical equivalences (see e.g. [4, §2]), the S-duals of hyperspherical equivariant slices are always symplectic vector spaces equipped with hamiltonian actions of appropriate reductive groups (all the hyperspherical symplectic representations are classified in [19, 14]). It turns out that the S-duals of hyperspherical equivariant slices are exactly the symplectic representations arising from basic classical Lie superalgebras.

Recall that a basic classical Lie superalgebra g = g_0 ⊕ g_1 is a direct sum of the ones from the following list: gl(n|k), osp(m|2n), D(2, 1; α), g(3), f(4) [21] (and D(2, 1; α) is a deformation of osp(4|2)). Let G_0 be a Lie group with Lie algebra g_0. It acts naturally on g_1, and we specify the choice of G_0 by the requirement that this action is effective. For all classical basic Lie superalgebras, g_1 is equipped with a symplectic structure (coming from the invariant symmetric bilinear form on g), and the action of G_0 on g_1 is hyperspherical.

Here is the list of expected (proved in certain cases) dualities. The S-dual of GL_n × GL_n ↷ T^∗(GL_n × C^n) is G_0 = GL_n × GL_n ↷ g_1 for g = gl(n|n). From now on, to save space, we will simply write for this that the S-dual of GL_n × GL_n ↷ T^∗(GL_n × C^n) is gl(n|n). This is proved in [4], as well as the fact that the S-dual of GL_n × GL_{n−1} ↷ T^∗GL_n is gl(n|n−1). More generally, for a nilpotent e of Jordan type (n−k, 1^k) in gl_n, the S-dual of GL_n × GL_k ↷ GL_n × S_e is gl(n|k) (proved in [24]).

Furthermore, the S-dual of SO_{2m} × SO_{2m−1} ↷ T^∗SO_{2m} is osp(2m|2m−2), and the S-dual of SO_{2m+1} × SO_{2m} ↷ T^∗SO_{2m+1} is osp(2m|2m) (proved in [5]). If e ∈ so_{2m} is a nilpotent of Jordan type (2m−k, 1^k) (note that k is automatically odd), then the S-dual of SO_{2m} × SO_k ↷ SO_{2m} × S_e is expected to be osp(2m|k−1). If e ∈ so_{2m+1} is a nilpotent of Jordan type (2m+1−k, 1^k) (note that k is automatically even), then the S-dual of SO_{2m+1} × SO_k ↷ SO_{2m+1} × S_e is expected to be osp(k|2m).

Moreover, the S-dual of Sp_{2n} × Sp_{2n} ↷ (T^∗Sp_{2n}) × C^{2n} is osp(2n+1|2n) (proved in [6]). If e ∈ sp_{2n} is a nilpotent of Jordan type (2n−k, 1^k) (note that k is automatically even), then the S-dual of Sp_{2n} × Sp_{2k} ↷ Sp_{2n} × S_e is expected to be either osp(2n+1|2n−k) or osp(2n+1−k|2n) (in this case, due to a certain anomaly, there are two twisted versions of S-duality, see e.g. [6, §3.1]).
Finally, if $e \in \mathfrak{sp}_e$ is a nilpotent of Jordan type $(3,3)$, then $Q \simeq \text{SL}_2$, and the $S$-dual of $\text{Sp}_6 \times \text{SL}_2 \simeq \text{Sp}_6 \times S_e$ is expected to be $f(4)$ [6, §3.3]. If $e \in \mathfrak{g}_2$ is an element of the 8-dimensional nilpotent orbit, then $Q \simeq \text{SL}_2$, and the $S$-dual of $G_2 \times \text{SL}_2 \simeq G_2 \times S_e$ is expected to be $\mathfrak{g}(3)$ [6, §3.4].

Note that the relation of $S$-duality with supergroups was already discussed in [20].

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2. Hyperspherical equivariant slices

2.1. Generalities.

Definition 2.1.1. Let $(X, \omega)$ be a symplectic variety equipped with an action $G \curvearrowright X$ of an algebraic group respecting the symplectic form $\omega$. Then the variety $X$ is called a hyperspherical variety of $G$ if the algebra of invariant functions $\mathbb{C}[X]^G$ is Poisson commutative.

Definition 2.1.2. Let $G$ be a reductive group acting on a symplectic variety $X$, and let $\Phi_G\colon X \to \mathfrak{g}^*$ be the moment map. Then the action $G \curvearrowright X$ is called symplectically stable if semisimple elements are dense in the image $\Phi_G(X)$ (e.g. if $\Phi_G(X) = \{0\}$).

Recall that a subspace $U \subset V$ of a symplectic vector space $V$ is called coisotropic if it contains its orthogonal complement: $U \supset U^\perp$.

Proposition 2.1.3. Let an algebraic group $G$ act on a symplectic variety $(X, \omega)$.

1. [25, Chapter 2, Proposition 5] $X$ is a hyperspherical variety of the group $G$ if and only if for a general point $x \in X$ the tangent space to the orbit $G.x$ at a point $x$ is coisotropic in $T_xX$.

2. [19, Proposition 1(1)] If $X$ is a hyperspherical variety of the group $G$ then

$$\dim X \leq \dim G + \text{rk}(G) = 2\dim B,$$

where $B$ is a Borel subalgebra of $G$.

3. [19, Proposition 1(2)] Let $G \curvearrowright X$ be a symplectically stable action. Then $X$ is a hyperspherical variety of the group $G$ if and only if a general point $x \in X$ has the property

$$\dim X = m_G(X) + \text{rk}(G) - \text{rk}(G_x),$$

where $G_x \subset G$ is the stabilizer of $x$ in $G$ and $m_G(X)$ is the maximal dimension of an orbit of the action $G \curvearrowright X$.

Let $G$ be a reductive group with Lie algebra $\mathfrak{g}$ and let $e \in \mathfrak{g}$ be a nilpotent element. Choose an $\mathfrak{sl}_2$-triple $(e, f, h)$. Then $S_e = e + \mathfrak{j}_\mathfrak{g}(f)$ is a Slodowy slice to the adjoint nilpotent orbit $G.e$. Using a $G$-invariant symmetric bilinear form $(-, -)$ on $\mathfrak{g}$, we view $e$ as an element $e^* \in \mathfrak{g}^*$, and we view $S_e$ as a slice $S_e = e^* + (\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}])^* \subset \mathfrak{g}^*$. So we have an embedding $G \times S_e \subset G \times \mathfrak{g}^* \simeq T^*G$. Here we identify $T^*G$ with $G \times \mathfrak{g}^*$ by using left $G$-invariant 1-forms on $G$. Then the action $G \curvearrowright T^*G$ by left (resp. right) translations has the following form: $g.(h, \xi) = (gh, \xi)$ (resp. $g.(h, \xi) = (hg^{-1}, \text{Ad}_g^*(\xi))$). On $T^*G$ we have a canonical symplectic form $\omega$. Its restriction to $G \times S_e$ is also denoted $\omega$. At a point $(1_G, x) \in G \times S_e$ we have

$$\omega_x(\xi + u, \eta + v) = (x, [\xi, \eta]) + (u, \eta) - (v, \xi),$$

where $\xi, \eta \in \mathfrak{g}$, $u, v \in \mathfrak{j}_\mathfrak{g}(f) \subset \mathfrak{g}$. By [18, Lemma 2] the form $\omega$ on $G \times S_e$ is non-degenerate. From now on we will identify $\mathfrak{g}^*$ with $\mathfrak{g}$ (and $\text{Ad}_g^*$ with $\text{Ad}_g$, as well as $T^*G$ with $G \times \mathfrak{g}$) using $(-, -)$.

Let $Q$ be the neutral connected component of the centralizer $Z_G(e, f, h)$ and let $\mathfrak{q}$ be its Lie algebra. Then we have a symplectic action $G \times Q \curvearrowright G \times S_e$: $(g_1, q).(g_2, \xi) = (g_1 g_2 q^{-1}, \text{Ad}_q(\xi))$, where $g_1, g_2 \in G$, $q \in Q$, $\xi \in S_e \subset \mathfrak{g}$. We want to classify all hyperspherical varieties of type $G \times S_e$ with the action of $G \times Q$ as above for reductive $G$.

Lemma 2.1.4. The action $G \times Q \curvearrowright G \times S_e$ is symplectically stable.
Proof. Note that the restriction \((-,-)|_{\mathfrak{g}}\) to \(\mathfrak{g}\) is also nondegenerate. So \(\mathfrak{z}_g(f) = \mathfrak{u} \oplus \mathfrak{g}\) where \(\mathfrak{u}\) is the orthogonal complement to \(\mathfrak{g}\). Let \(\pi: S_e \to \mathfrak{g}\) be the corresponding projection. Then
\[
\Phi_{G \times Q}(g, \xi) = (\Phi_G(g, \xi), \Phi_Q(g, \xi)) = (\text{Ad}_g(\xi), \pi(\xi)),
\]
where \(\Phi_{G \times Q}: G \times S_e \to \mathfrak{g} \oplus \mathfrak{q}\), \(\Phi_G: G \times S_e \to \mathfrak{g}\), \(\Phi_Q: G \times S_e \to \mathfrak{q}\) are the moment maps of the actions \(G \times Q \curvearrowright G \times S_e\), \(G \curvearrowright G \times S_e\), \(Q \curvearrowright G \times S_e\) respectively. Let \(pr_{\mathfrak{g}}: \mathfrak{g} \oplus \mathfrak{q} \to \mathfrak{g}\) and \(pr_{\mathfrak{q}}: \mathfrak{g} \oplus \mathfrak{q} \to \mathfrak{q}\) be the natural projections. It is easy to see that the images \(\Phi_G(G \times S_e), \Phi_Q(G \times S_e)\) contain nonempty Zariski open subsets \(U_{\mathfrak{g}}\) and \(U_{\mathfrak{q}}\) consisting of semisimple elements respectively. Then \(pr_{\mathfrak{g}}^{-1}(U_{\mathfrak{g}}) \cap pr_{\mathfrak{q}}^{-1}(U_{\mathfrak{q}}) \cap \Phi_{G \times Q}(G \times S_e)\) is a nonempty Zariski open subset in \(\Phi_{G \times Q}(G \times S_e)\) consisting of semisimple elements.

The next Corollary follows immediately from Lemma 2.1.4 and Proposition 2.1.3(3).

Corollary 2.1.5. Consider the action \(G \times Q \curvearrowright G \times S_e\) as above. Assume that the stabilizer \(Q_p\) of a general point \(p \in S_e\) is finite. Then \(G \times S_e\) is is a hyperspherical variety of the group \(G \times Q\) if and only if
\[
\dim G \times S_e = \dim G \times Q + \text{rk}(\mathfrak{g} \oplus \mathfrak{q}).
\]

2.2. Hook nilpotents. We describe the nilpotent elements in classical Lie algebras with Jordan type given by a partition \((n - k, 1^k)\) whose Young diagram has a hook form. Let \(W = \mathbb{C}^k\), \(U = \mathbb{C}^{n-k}\), and \(V = U \oplus W\). We view \(U\) as an irreducible \(\mathfrak{sl}_2\)-module with weight vectors \(u_1, u_2, \ldots, u_{n-k}\), where \(u_1\) is the highest weight vector and \(u_{n-k}\) is the lowest one. Denote the corresponding \(\mathfrak{sl}_2\)-triple by \(e', f', h'\) In this section \(G = G(V)\) or \(G = \text{GL}(V)\), and \(e = (e', 0) \in \mathfrak{g}(U) \oplus \mathfrak{gl}(W) \subset \mathfrak{g}(V) \subset \mathfrak{gl}(V)\) is a nilpotent element of hook Jordan type \((n-k, 1^k)\). Furthermore, \(e = (e', 0), f = (f', 0), h = (h', 0)\) is the corresponding \(\mathfrak{sl}_2\)-triple. Finally, \(Q = G(W)\), or \(Q = (\mathbb{C}^\times \cdot 1_U) \times \text{GL}(W)\).

Lemma 2.2.1. (1) If \(G = G(V)\), then the stabilizer \(Q_p\) of a general point \(p \in S_e\) is finite.
(2) If \(G = \text{GL}(V)\), then the stabilizer \(\text{GL}(W)_p\) of a general point \(p \in S_e\) is finite.

Proof. Assume that \(G = G(V)\). Consider a vector space \(L\) consisting of all elements \(\xi \in \mathfrak{gl}(V) = \text{End}(V)\) with the following properties:
\[
\xi(u_1) \in W, \quad \xi(u_i) = 0 \quad \forall i \neq 1, \quad \xi(W) \subset \mathbb{C}\langle u_{n-k} \rangle, \quad (\xi(u_1), w) = -(u_1, \xi(w)) \quad \forall w \in W.
\]
It is easy to check that \(L \subset \mathfrak{g}(V)\) and \(L \subset \mathfrak{z}_g(f) \subset \mathfrak{z}_g(V)\). Note that \(L\) is isomorphic to the \(k\)-dimensional tautological \(\mathfrak{q}\)-module. In particular, the \(\mathfrak{q}\)-module \(\mathfrak{z}_g(V)\) contains \(L \oplus \mathfrak{q}\), where \(\mathfrak{q}\) is the adjoint representation. So the stabilizer of a general point of \(S_e\) is finite.

If \(G = \text{GL}(V)\), consider \(L\) consisting of all elements \(\xi \in \mathfrak{gl}(V) = \text{End}(V)\) with the following properties:
\[
\xi(u_1) \in W, \quad \xi(u_i) = 0 \quad \forall i \neq 1, \quad \xi(W) \subset \mathbb{C}\langle u_{n-k} \rangle.
\]
Then \(L \subset \mathfrak{z}_{\text{gl}(V)}(f)\) is isomorphic to the \(\text{GL}(W)\)-module \(W \oplus W^*\). So as before, the stabilizer of a general point of \(S_e\) in \(\text{GL}(W)\) is finite.

2.2.1. Hook nilpotents in \(\mathfrak{gl}_n\). Let \(G = \text{GL}_n\) and let \(e \in \mathfrak{gl}_n\) be a nilpotent element of Jordan type \((n-k, 1^k), k \neq 0\).

Proposition 2.2.2. \(\text{GL}_n \times S_e\) is a hyperspherical variety of the group \(\text{GL}_n \times \text{GL}_k\).

Proof. We have \(\dim(\text{GL}_n \times S_e) = n^2 + ((n-k) + k + k^2) = n^2 + n + k^2 + k\), see e.g. [15] for dimensions of nilpotent orbits. So \(\dim \text{GL}_n \times S_e - \dim \text{GL}_n \times \text{GL}_k = (n^2 + k^2 + n + k) - (n^2 + k^2) = n + k = \text{rk}(\mathfrak{gl}_n \oplus \mathfrak{gl}_k)\). Hence by Lemma 2.2.1 and Corollary 2.1.5, \(\text{GL}_n \times S_e\) is a hyperspherical variety of the group \(\text{GL}_n \times \text{GL}_k\). \(\square\)
2.2.2. **Hook nilpotents in \( \mathfrak{sp}_{2n} \).** Let \( G = \mathfrak{Sp}_{2n} \) and let \( e \in \mathfrak{sp}_{2n} \) be a nilpotent element of Jordan type \((2(n-k),1^{2k})\), \( k \neq 0 \).

**Proposition 2.2.3.** \( \mathfrak{sp}_{2n} \times S_{e} \) is a hyperspherical variety of the group \( \mathfrak{Sp}_{2n} \times \mathfrak{Sp}_{2k} \).

**Proof.** By [16, Proposition 2.4], \( \dim S_{e} = 2k^{2} + 2k + n \) and hence \( \dim \mathfrak{sp}_{2n} \times S_{e} - \dim \mathfrak{sp}_{2n} \times \mathfrak{sp}_{2k} = \dim S_{e} - \dim \mathfrak{sp}_{2k} = (2k^{2} + 2k + n) - (2k^{2} + k) = n + k = \text{rk}(\mathfrak{sp}_{2n} \oplus \mathfrak{sp}_{2k}). \) So by Lemma 2.2.1 and Corollary 2.1.5, \( \mathfrak{sp}_{2n} \times S_{e} \) is a hyperspherical variety of the group \( \mathfrak{Sp}_{2n} \times \mathfrak{Sp}_{2k} \). \( \square \)

2.2.3. **Hook nilpotents in \( \mathfrak{so}_{2n+1} \).** Let \( G = \mathfrak{SO}_{2n+1} \) and let \( e \in \mathfrak{so}_{2n+1} \) be a nilpotent element of Jordan type \((2(n-k) + 1,1^{2k})\), \( k \neq 0 \).

**Proposition 2.2.4.** \( \mathfrak{so}_{2n+1} \times S_{e} \) is a hyperspherical variety of the group \( \mathfrak{SO}_{2n+1} \times \mathfrak{SO}_{2k} \).

**Proof.** By [16, Proposition 2.4], \( \dim S_{e} = 2k^{2} + n \) and hence \( \dim \mathfrak{so}_{2n+1} \times S_{e} - \dim \mathfrak{so}_{2n+1} \times \mathfrak{so}_{2k} = \dim S_{e} - \dim \mathfrak{so}_{2k} = (2k^{2} + n) - (2k^{2} - k) = n + k = \text{rk}(\mathfrak{so}_{2n+1} \oplus \mathfrak{so}_{2k}). \) So by Lemma 2.2.1 and Corollary 2.1.5, \( \mathfrak{so}_{2n+1} \times S_{e} \) is a hyperspherical variety of the group \( \mathfrak{SO}_{2n+1} \times \mathfrak{SO}_{2k} \). \( \square \)

2.2.4. **Hook nilpotents in \( \mathfrak{so}_{2n} \).** Let \( G = \mathfrak{SO}_{2n} \) and let \( e \in \mathfrak{so}_{2n} \) be a nilpotent element of Jordan type \((2(n-k) - 1,1^{2k+1})\), \( k \neq 0 \).

**Proposition 2.2.5.** \( \mathfrak{so}_{2n} \times S_{e} \) is a hyperspherical variety of the group \( \mathfrak{SO}_{2n} \times \mathfrak{SO}_{2k+1} \).

**Proof.** By [16, Proposition 2.4], \( \dim S_{e} = 2k^{2} + 2k + n \) and hence \( \dim \mathfrak{so}_{2n} \times S_{e} - \dim \mathfrak{so}_{2n} \times \mathfrak{so}_{2k+1} = \dim S_{e} - \dim \mathfrak{so}_{2k+1} = (2k^{2} + 2k + n) - (2k^{2} + k) = n + k = \text{rk}(\mathfrak{so}_{2n} \oplus \mathfrak{so}_{2k+1}). \) So by Lemma 2.2.1 and Corollary 2.1.5, \( \mathfrak{so}_{2n} \times S_{e} \) is a hyperspherical variety of the group \( \mathfrak{SO}_{2n} \times \mathfrak{SO}_{2k+1} \). \( \square \)

2.3. **Exceptional case in \( \mathfrak{sp}_{6} \).** Let \( G = \mathfrak{Sp}_{6} = \mathfrak{Sp}(V) \) and let \( e \) be a nilpotent of Jordan type \((3,3)\). Choose a basis in \( V \) such that the Gram matrix of the skew-symmetric bilinear form on \( V \) has the following form:

\[
M = \begin{pmatrix} 0 & I_3 \\ -I_3 & 0 \end{pmatrix}, \quad I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.
\]

In this basis the Lie algebra \( \mathfrak{sp}_{6} \) consists of matrices \( \begin{pmatrix} A & B \\ C & -A^T \end{pmatrix} \) where \( A, B, C \in \text{Mat}_{3 \times 3}(\mathbb{C}) \) such that \( B = B^T, C = C^T \). Consider the following nilpotent element \( e \in \mathfrak{sp}_{6} \) of Jordan type \((3,3)\):

\[
e = \begin{pmatrix} J & 0 \\ 0 & -J^T \end{pmatrix},
\]

where \( J \) is the Jordan block of size 3. Note that \( e' = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, f' = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, h' = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \) is an \( \mathfrak{sl}_{2} \)-triple in \( \mathfrak{sl}_{3} \). Then \( e = \begin{pmatrix} e' & 0 \\ 0 & -e' \end{pmatrix} \) and \( f = \begin{pmatrix} f' & 0 \\ 0 & -f' \end{pmatrix} \) belong to \( \mathfrak{sp}_{6} \) and form an \( \mathfrak{sl}_{2} \)-triple in \( \mathfrak{sp}_{6} \). By an easy computation \( \mathfrak{z}_{\mathfrak{sp}_{6}}(f) \) consists of matrices of the following form:

\[
\begin{pmatrix} p & 0 & 0 & b \\ 0 & -b & 0 \\ a & 0 & c \\ 0 & -c & 0 & -p^T \end{pmatrix},
\]

where \( p \in \mathfrak{z}_{\mathfrak{gl}_{1}}(f'), a, b, c, d \in \mathbb{C} \). In particular \( \dim \mathfrak{Sp}_{6} \times S_{e} = \dim \mathfrak{Sp}_{6} + \dim \mathfrak{z}_{\mathfrak{gl}_{1}}(f') + 4 = 21 + 3 + 4 = 28 \). Note that we have an embedding \( \mathfrak{sl}_{2} \hookrightarrow \mathfrak{z}_{\mathfrak{sp}_{6}}(f) \):

\[
\begin{pmatrix} a & b \\ c & -a \end{pmatrix} \mapsto \begin{pmatrix} a & 0 & 0 & 0 & 0 & b \\ 0 & a & 0 & -b & 0 \\ 0 & a & b & 0 & 0 \\ 0 & c & -a & 0 & 0 \\ 0 & c & 0 & -a & 0 \\ c & 0 & 0 & 0 & -a \end{pmatrix} \in \mathfrak{z}_{\mathfrak{sp}_{6}}(f).
\]
From now on, when we write \( \mathfrak{sl}_2 \subset \mathfrak{sp}_6 \) we will mean this embedding. Note that \( \mathfrak{sl}_2 \subset \mathfrak{z}_{\mathfrak{sp}_6}(e) \cap \mathfrak{z}_{\mathfrak{sp}_6}(f) \).

Consider \( \mathbb{SL}_2 \subset \mathbb{Z}_{\mathfrak{sp}_6}(e) \cap \mathbb{Z}_{\mathfrak{sp}_6}(f) \) corresponding to the Lie algebra \( \mathfrak{sl}_2 \subset \mathfrak{sp}_6 \). Then \( \mathbb{SL}_2 \) centralizes \( e, f \), and so it acts on the Slodowy slice \( S_e \). In this case \( Q = \mathbb{SL}_2 \) and we have the symplectic action \( \mathbb{Sp}_6 \times \mathbb{SL}_2 \subset \mathbb{Sp}_6 \times S_e \) as before.

**Proposition 2.3.1.** \( \mathbb{Sp}_6 \times S_e \) is a hyperspherical variety of the group \( \mathbb{Sp}_6 \times \mathbb{SL}_2 \).

**Proof.** Consider a Cartan subalgebra \( \mathfrak{h} \subset \mathfrak{sl}_2 \subset \mathfrak{sp}_6 \) consisting of matrices of the following form:

\[
\begin{pmatrix}
a & 0 & 0 & 0 & 0 & 0 \\
0 & a & 0 & 0 & 0 & 0 \\
0 & 0 & a & 0 & 0 & 0 \\
0 & 0 & 0 & -a & 0 & 0 \\
0 & 0 & 0 & 0 & -a & 0 \\
0 & 0 & 0 & 0 & 0 & -a
\end{pmatrix}, \quad a \in \mathbb{C}.
\]

Let \( W_k \) denote the subspace of the \( \mathfrak{sl}_2 \)-representation \( \mathfrak{z}_{\mathfrak{sp}_6}(f) \) consisting of all vectors of weight \( k \). It is easy to see that \( \mathfrak{z}_{\mathfrak{sp}_6}(f) = W_{-2} \oplus W_0 \oplus W_2 \), where \( W_{-2}, W_0, W_2 \) consist of the following matrices:

\[
\begin{pmatrix}
a & 0 & 0 & 0 & 0 & 0 \\
0 & b & 0 & 0 & 0 & 0 \\
0 & 0 & c & 0 & 0 & 0 \\
0 & 0 & 0 & d & 0 & 0 \\
0 & 0 & 0 & 0 & b & 0 \\
0 & 0 & 0 & 0 & 0 & c
\end{pmatrix} \in W_2, \quad \begin{pmatrix}
a & 0 & 0 & 0 & 0 & 0 \\
0 & b & 0 & 0 & 0 & 0 \\
0 & 0 & c & 0 & 0 & 0 \\
0 & 0 & 0 & d & 0 & 0 \\
0 & 0 & 0 & 0 & b & 0 \\
0 & 0 & 0 & 0 & 0 & c
\end{pmatrix} \in W_{-2}, \quad \begin{pmatrix}
p & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix} \in W_0,
\]

where \( p \in \mathfrak{z}_{\mathfrak{sp}_6}(f') \), \( a, b, c, d \in \mathbb{C} \). In particular, \( \dim W_2 = \dim W_{-2} = 2 \) and \( \dim W_0 = 3 \). Hence \( \mathfrak{z}_{\mathfrak{sp}_6}(f) \cong \mathfrak{sl}_2 \oplus \mathfrak{sl}_2 \oplus \mathbb{C} \) as \( \mathfrak{sl}_2 \)-representations, where \( \mathfrak{sl}_2 \) is the adjoint representation and \( \mathbb{C} \) is the trivial one. So the stabilizer of a general point of \( S_e \) in \( \mathbb{SL}_2 \) is finite since the representation \( \mathfrak{z}_{\mathfrak{sp}_6}(f) \) contains \( \mathfrak{sl}_2 \oplus \mathfrak{sl}_2 \). Note that \( \dim \mathfrak{Sp}_6 \times S_e - \dim \mathfrak{Sp}_6 \times \mathbb{SL}_2 = 28 - 24 = 4 = 3 + 1 = \text{rk}(\mathfrak{sp}_6 \oplus \mathfrak{sl}_2) \).

So by Corollary 2.1.5, \( \mathbb{Sp}_6 \times S_e \) is a hyperspherical variety of the group \( \mathbb{Sp}_6 \times \mathbb{SL}_2 \).

\[ \square \]

### 2.4. Exceptional case in \( \mathfrak{g}_2 \)

Let \( G = \mathbb{G}_2 \) and \( e \in \mathfrak{g}_2 \) be a weight vector corresponding to a short root of \( \mathfrak{g}_2 \). We will follow the notation of [9, Figure at p.340] for the roots of \( \mathfrak{g}_2 \). The positive roots will be denoted \( \alpha_i, \ i = 1, \ldots, 6 \). The negative roots will be denoted \( \beta_j = -\alpha_j \). Finally, \( \alpha_1 \) is the short simple root, and \( \alpha_2 \) is the long simple root. As always, for any root \( \gamma \), \( \mathfrak{g}_\gamma \) stands for the corresponding root subspace.

Let \( e \in \mathfrak{g}_{\alpha_1}, \ f \in \mathfrak{g}_{\beta_2} \). Then \( \mathfrak{q} = \mathfrak{g}_{\beta_2}(e, f, h) = \mathfrak{g}_{\alpha_6} \oplus \mathfrak{g}_{\beta_6} \oplus [\mathfrak{g}_{\alpha_6}, \mathfrak{g}_{\beta_6}] \cong \mathfrak{sl}_2 \). In particular \( Z_{\mathbb{G}_2}(e, f, h) \cong \mathbb{SL}_2 \). We have a symplectic action \( \mathbb{G}_2 \times \mathbb{SL}_2 \subset \mathbb{G}_2 \times S_e \).

**Proposition 2.4.1.** \( \mathbb{G}_2 \times S_e \) is a hyperspherical variety of the group \( \mathbb{G}_2 \times \mathbb{SL}_2 \).

**Proof.** Note that \( \mathfrak{z}_{\mathbb{G}_2}(f) = \mathfrak{g}_{\beta_2} \oplus (\mathfrak{g}_{\alpha_2} \oplus \mathfrak{g}_{\beta_2}) \oplus (\mathfrak{g}_{\alpha_6} \oplus \mathfrak{g}_{\beta_6} \oplus [\mathfrak{g}_{\alpha_6}, \mathfrak{g}_{\beta_6}]) \cong \mathbb{C} \oplus V \oplus \mathfrak{sl}_2 \) as an \( \mathfrak{sl}_2 \)-module, where \( \mathbb{C} \) is the trivial representation, \( V \) is the tautological 2-dimensional \( \mathfrak{sl}_2 \)-representation, and \( \mathfrak{sl}_2 \) is the adjoint representation. In particular, the stabilizer of a general point of \( S_e \) in \( \mathbb{SL}_2 \) is trivial since \( \mathfrak{z}_{\mathbb{G}_2}(f) \) contains the \( \mathbb{SL}_2 \)-submodule \( V \oplus \mathfrak{sl}_2 \) and \( \dim S_e = \dim \mathfrak{z}_{\mathbb{G}_2}(f) = 6 \).

Note that \( \dim \mathbb{G}_2 \times S_e - \dim \mathbb{G}_2 \times \mathbb{SL}_2 = (14 + 6) - (14 + 3) = 3 = 2 + 1 = \text{rk}(\mathfrak{g}_2 \oplus \mathfrak{sl}_2) \). So by Corollary 2.1.5, \( \mathbb{G}_2 \times S_e \) is a hyperspherical variety of the group \( \mathbb{G}_2 \times \mathbb{SL}_2 \).

\[ \square \]

### 3. Non-hyperspherical slice varieties

#### 3.1. Other nilpotents in \( \mathfrak{gl}_n \)

Let \( G = \mathbb{GL}_n \) and \( e \in \mathfrak{gl}_n \) be a nilpotent element of the Jordan type \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k) \),

\[ \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_k, \quad n = \sum_{i=1}^{k} \lambda_i, \]
Let \( \mu = (\mu_1, \mu_2, \ldots, \mu_s) \) be the dual partition defined by \( \mu_i = \# \{ j : \lambda_j \geq i \} \). Then
\[
Q = \prod_{i=1}^{s} \mathrm{GL}_{\mu_i-\mu_{i+1}}, \quad \text{and} \quad \dim S_e = n + 2 \sum_{i=1}^{s} \left( \frac{\mu_i}{2} \right),
\]
see e.g. [15].

**Proposition 3.1.1.** \( \mathrm{GL}_n \times S_e \) is not a hyperspherical variety of the group \( \mathrm{GL}_n \times Q \) unless \( e \) is a nilpotent of hook type or of type \((2,2)\).

**Proof.** First of all, note that the subgroup
\[
C^\times = \{ (t_1 \mathrm{GL}_n, t_1 \mathrm{GL}_{\mu_1-\mu_2}, t_1 \mathrm{GL}_{\mu_2-\mu_3}, \ldots, t_1 \mathrm{GL}_{\mu_s}) : t \in C^\times \} \subset \mathrm{GL}_n \times Q
\]
acts trivially on \( \mathrm{GL}_n \times S_e \), so it suffices to check that the action \( (\mathrm{GL}_n \times Q)/C^\times \sim \mathrm{GL}_n \times S_e \) is not hyperspherical.

Then by Proposition 2.1.3(2), it is enough to check that
\[
\dim \mathrm{GL}_n \times S_e > 2 \dim B_{\mathrm{GL}_n \times Q} - 2,
\]
where \( B_{\mathrm{GL}_n \times Q} \) is a Borel subgroup of \( \mathrm{GL}_n \times Q \). Note that
\[
\dim B_{\mathrm{GL}_n \times Q} = \left( \frac{n+1}{2} \right) + \sum_{i=1}^{s} \left( \frac{\mu_i - \mu_{i+1} + 1}{2} \right),
\]
so (3.1.1) takes the following form:
\[
(3.1.2) \quad n^2 + (n + 2 \sum_{i=1}^{s} \left( \frac{\mu_i}{2} \right)) > 2 \left( \frac{n+1}{2} \right) + 2 \sum_{i=1}^{s} \left( \frac{\mu_i - \mu_{i+1} + 1}{2} \right) - 2
\]
\[
\iff \sum_{i=1}^{s} \mu_i^2 - \sum_{i=1}^{s} \mu_i > \sum_{i=1}^{s} (\mu_i - \mu_{i+1})^2 + \sum_{i=1}^{s} (\mu_i - \mu_{i+1}) - 2
\]
\[
\iff \sum_{i=1}^{s} \mu_i^2 - \sum_{i=1}^{s} \mu_i > \sum_{i=1}^{s} (\mu_i - \mu_{i+1})^2 + \mu_1 - 2.
\]

We will prove by induction on the length of the partition \( \mu \) that (3.1.2) is true for every partition \( \mu \) except for hook partitions and \((2,2)\).

Let us check the base of induction. Let \( \mu = (\mu_1, \mu_2) \). Then (3.1.2) takes the following form:
\[
(3.1.3) \quad 2\mu_1\mu_2 + 2 > \mu_2^2 + 2\mu_1 + \mu_2.
\]

If \( \mu_2 = 1 \), then (3.1.3) is not true. Namely, it takes the form \( 2 + 2\mu_1 = 2 + 2\mu_1 \). This case corresponds to a hook nilpotent.

If \( \mu_1 \geq \mu_2 > 1 \), then
\[
(3.1.4) \quad 2\mu_1\mu_2 + 2 > \mu_2^2 + 2\mu_1 + \mu_2 \iff 2\mu_1(\mu_2 - 1) > (\mu_2 - 1)(\mu_2 + 2).
\]

Now (3.1.4) is true for every \((\mu_1, \mu_2)\) except for \( \mu_1 = \mu_2 = 2 \). This exceptional case corresponds to a nilpotent of type \((2,2)\) in \( \mathfrak{gl}(4) \).

Let us check the step of induction. Let \( \mu = (\mu_1, \mu_2, \ldots, \mu_s, \mu_{s+1}) \). Then by induction, it suffices to verify that
\[
\mu_{s+1}^2 - \mu_{s+1} \geq (\mu_s - \mu_{s+1})^2 + \mu_{s+1}^2 - \mu_s^2 \iff \mu_{s+1}(\mu_{s+1} + 1 - 2\mu_s) \leq 0.
\]
This inequality is true for every \((\mu_s, \mu_{s+1})\) since \( \mu_s \geq \mu_{s+1} \), and it is an equality if and only if \( \mu_s = \mu_{s+1} = 1 \). This completes the proof. \( \square \)

**Remark 3.1.2.** Under the classical isomorphism \( \mathfrak{sl}_4 \cong \mathfrak{so}_6 \), a nilpotent element of type \((2,2)\) goes to a nilpotent element of type \((3,1^3)\). So the “exceptional” case in Proposition 3.1.1 is of hook type in \( \mathfrak{so}_6 \).
3.2. Other nilpotents in $\mathfrak{sp}_{2n}$. Let $G = \mathfrak{sp}_{2n}$ and let $e \in \mathfrak{sp}_{2n}$ be a nilpotent element of Jordan type $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$, and let $\mu = (\mu_1, \mu_2, \ldots, \mu_s)$ be the dual partition. Then $Q = \prod_{i=1}^{s} G_i$, where $G_i$ is $\mathfrak{sp}_{\mu_i-\mu_{i+1}}$ if $i$ is odd and $\text{SO}_{\mu_i-\mu_{i+1}}$ otherwise. By [16, Proposition 2.4], we have

\[
\dim S_e = \frac{1}{2} \left( \sum_{i=1}^{s} \mu_i^2 + \{ j : 2 \mid \lambda_j \} \right) = \frac{1}{2} \left( \sum_{i=1}^{s} \mu_i^2 + \sum (-1)^{i+1} \mu_i \right).
\]

Also, $2 \dim \mathfrak{sp}_{2k} + 2r(\mathfrak{sp}_{2k}) = (2k)^2 + 4k = 2 \dim \text{SO}_{2k+1} + 2r(\text{SO}_{2k+1})$, and $2 \dim \text{SO}_{2k} + 2r(\text{SO}_{2k}) = (2k)^2$.

**Proposition 3.2.1.** $\mathfrak{sp}_{2n} \times S_e$ is not a hyperspherical variety of the group $\mathfrak{sp}_{2n} \times Q$ unless $e$ is a nilpotent of hook type or of types $(2, 2)$ and $(3, 3)$.

**Proof.** By Proposition 2.1.3(2) and (3.2.1), it is enough to check that

\[
\sum_{i=1}^{s} \mu_i^2 - 2 \sum_{i=1}^{s} \mu_i > \sum_{i=1}^{s} (\mu_i - \mu_{i+1})^2
\]

\[
\Rightarrow \sum_{i=1}^{s} \mu_i^2 + \sum_{i=1}^{s} (-1)^{i+1} \mu_i > \sum_{i=1}^{s} (\mu_i - \mu_{i+1})^2 + 2 \sum_{i=1}^{s} (-1)^{i+1} \mu_i + \sum_{i=1}^{s} \mu_i \geq 2 \dim Q + 2r(\mathfrak{sp}_{2n}) + 2r(Q)
\]

\[
\Rightarrow \dim \mathfrak{sp}_{2n} \times S_e > \dim \mathfrak{sp}_{2n} + \dim Q + r(\mathfrak{sp}_{2n}) + r(Q).
\]

We will check that this inequality is true for every partition $\mu$ corresponding to a nilpotent element in $\mathfrak{sp}_{2n}$ except for partitions of hook type, $(2, 2)$ and $(2, 2, 2)$, by induction on the length of the partition $\mu$.

Let us check the base of induction. The case $\mu = (\mu_1)$ corresponds to the zero nilpotent and (3.2.2) is not true.

Let $\mu = (\mu_1, \mu_2, \mu_3)$, where $\mu_3$ may be zero. In this case (3.2.2) takes the following form:

\[
2 \mu_1 \mu_2 + 2 \mu_2 \mu_3 > \mu_2^2 + \mu_3^2 + 2 \mu_1 + 2 \mu_3.
\]

Note that $\mu_1 \mu_2 \geq \mu_2^2$ and $\mu_2 \mu_3 \geq \mu_3^2$ since $\mu_1 \geq \mu_2 \geq \mu_3$, and it is enough to check that

$\mu_1 \mu_2 + \mu_2 \mu_3 \geq 2 \mu_1 + 2 \mu_3$.

This is true for $\mu_2 > 2$. If $\mu_2 = 1$, then $\mu$ corresponds to hook nilpotent. Assume that $\mu_2 = 2$. Then (3.2.3) takes the form

$2 \mu_1 + 2 \mu_3 > 4 + \mu_3^2$.

This is true if $\mu_1 > 2$. So exceptional partitions are $(2, 2, 2), (2, 2)$ and $(2, 2, 1)$ (but note that there is no nilpotent element in $\mathfrak{sp}_{2n}$ corresponding to the dual partition $(2, 2, 1)$).

Let us check the step of induction. There will be two different situations.

First, let $\mu = (\mu_1, \mu_2, \ldots, \mu_s, \mu_{s+1}, \mu_{s+2})$ be the dual partition corresponding to a nilpotent element, where $s + 2$ is even. Then the partition $(\mu_1, \mu_2, \ldots, \mu_s)$ corresponds to a nilpotent element as well. So by induction it suffices to check that

\[
\mu_s \mu_{s+1} + \mu_{s+1} \mu_{s+2} \geq 2 \mu_{s+1}
\]

\[
\Rightarrow 2 \mu_s \mu_{s+1} + 2 \mu_{s+1} \mu_{s+2} \geq \mu_{s+1}^2 + \mu_{s+2}^2 + 2 \mu_{s+1}
\]

\[
\Leftrightarrow \mu_{s+1}^2 + \mu_{s+2}^2 - 2 \mu_{s+1} \geq (\mu_s - \mu_{s+1})^2 + (\mu_{s+1} - \mu_{s+2})^2 + \mu_{s+2}^2 - \mu_s^2
\]

This inequality holds true for every $\mu_s, \mu_{s+1}, \mu_{s+2}$ and it is an equality if and only if $\mu_s = \mu_{s+1} = \mu_{s+2} = 1$.

Second, let $\mu = (\mu_1, \mu_2, \ldots, \mu_s, \mu_{s+1})$ be the dual partition corresponding to a nilpotent element, where $s + 1$ is odd. Then the partition $(\mu_1, \mu_2, \ldots, \mu_s)$ corresponds to a nilpotent element as well. So by induction it suffices to check that

\[
\mu_{s+1}^2 - 2 \mu_{s+1} \geq (\mu_s - \mu_{s+1})^2 + \mu_{s+1}^2 - \mu_s^2 \Leftrightarrow 2 \mu_{s+1}(2 \mu_s - \mu_{s+1} - 2) \geq 0.
\]
This inequality holds true for every \((\mu_s, \mu_{s+1})\) except for \((1, 1)\) (note that there is no nilpotent element corresponding to such partition with \(\mu_s = \mu_{s+1} = 1\) and it is an equality if and only if \(\mu_s = \mu_{s+1} = 2\). This completes the proof. \[\square\]

**Remark 3.2.2.** Under the classical isomorphism \(\mathfrak{sp}_4 \cong \mathfrak{so}_5\), a nilpotent element of type \((2, 2)\) goes to a nilpotent element of type \((3, 1^2)\). So the “exceptional” case \((2, 2)\) in Proposition 3.2.1 is of hook type in \(\mathfrak{so}_5\).

### 3.3. Other nilpotents in \(\mathfrak{so}_n\).

Let \(G = \text{SO}_n\) and let \(e \in \mathfrak{so}_n\) be a nilpotent element of Jordan type \(\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)\), and let \(\mu = (\mu_1, \mu_2, \ldots, \mu_s)\) be the dual partition. Then \(Q = \prod_{i=1}^s G_i\), where \(G_i\) is \(\text{SO}_{\mu_i - \mu_{i+1}}\) if \(i\) is odd and \(\text{Sp}_{\mu_i - \mu_{i+1}}\) otherwise. By [16, Proposition 2.4], we have

\[
(3.3.1) \quad \dim S_e = \frac{1}{2} \left( \sum_{i=1}^s \mu_i^2 - \{j : 2 \mid \lambda_j\} \right) = \frac{1}{2} \left( \sum_{i=1}^s \mu_i^2 - \sum_{i=1}^s (-1)^{i+1} \mu_i \right).
\]

**Proposition 3.3.1.** \(\text{SO}_n \times S_e\) is not a hyperspherical variety of the group \(\text{SO}_n \times Q\) unless \(e\) is a nilpotent of hook type or of types \((2^2), (3^2), (4^2), (2^2, 1), (2^2, 1^2)\) and \((2^4)\).

**Proof.** By Proposition 2.1.3(2) and (3.3.1), it is enough to check that

\[
(3.3.2) \quad \sum_{i=1}^s \mu_i^2 - 2\mu_1 - 2 \sum_{2|\mu_i} \mu_i > \sum_{i=1}^s (\mu_i - \mu_{i+1})^2
\]

\[
\Rightarrow \sum_{i=1}^s \mu_i^2 - \sum_{i=1}^s (-1)^{i+1} \mu_i > \sum_{i=1}^s (\mu_i - \mu_{i+1})^2 - 2 \sum_{i=2}^s (-1)^{i+1} \mu_i + \sum_{i=1}^s \mu_i \geq 2 \dim Q + 2 \text{rk}(\text{Sp}_{2n}) + 2 \text{rk}(Q)
\]

\[
\Rightarrow \dim \text{Sp}_{2n} \times S_e > \dim \text{Sp}_{2n} + \dim Q + \text{rk}(\text{Sp}_{2n}) + \text{rk}(Q).
\]

We will check that this inequality holds true for every partition \(\mu\) corresponding to a nilpotent element in \(\mathfrak{so}_n\) except for partitions of hook type, \((2^2), (3^2), (4^2), (2^2, 1), (2^2, 1^2)\) and \((2^4)\), by induction on the length of the partition \(\mu\).

Let us check the base of induction. The case \(\mu = (\mu_1)\) corresponds to the zero nilpotent and (3.3.2) is not true.

Let \(\mu = (\mu_1, \mu_2, \mu_3)\), where \(\mu_3\) may be zero. In this case (3.3.2) takes the following form:

\[
(3.3.3) \quad 2\mu_1\mu_2 + 2\mu_2\mu_3 > \mu_2^2 + \mu_3^2 + 2\mu_1 + 2\mu_2.
\]

Note that \(\mu_1\mu_2 \geq \mu_2^2\) and \(\mu_2\mu_3 \geq \mu_3^2\) since \(\mu_1 \geq \mu_2 \geq \mu_3\), and it is enough to check that

\[\mu_1\mu_2 + \mu_2\mu_3 > 2\mu_1 + 2\mu_2.\]

This is true for \(\mu_3 > 2\). Assume that \(\mu_3 = 2\). Then (3.3.3) takes the form

\[2\mu_1(\mu_2 - 1) > \mu_2^2 - 2\mu_2 + 4.\]

This is true for \(\mu_2 > 2\). If \(\mu_2 = \mu_1 = 2\) then the unique exceptional case is \((2^3)\). If \(\mu_3 = 1\), then (3.3.3) takes the form

\[2\mu_1(\mu_2 - 1) > \mu_2^2 + 1.\]

This is true for \(\mu_2 > 2\). So the exceptional cases are the hook partitions and \((\mu_1, 2, 1)\) (but there are no nilpotents of such type in \(\mathfrak{so}_n\)). If \(\mu_3 = 0\), then (3.3.3) has the form

\[2\mu_1(\mu_2 - 1) > \mu_2^2 + 2\mu_2.\]

This is true for \(\mu_2 > 4\). So the exceptional cases are the hook partitions and \((4, 4), (4, 2), (3, 2), (2, 2)\).

Let us check the step of induction. Again, there will be two different situations. First, let \(\mu = (\mu_1, \mu_2, \ldots, \mu_s, \mu_{s+1}, \mu_{s+2})\) be the dual partition corresponding to a nilpotent element, where \(s + 2\) is odd. Then the partition \((\mu_1, \mu_2, \ldots, \mu_s)\) corresponds to a nilpotent element as well. So by induction it suffices to check that

\[
\mu_{s+1}^2 + \mu_{s+2}^2 - 2\mu_{s+1} \geq (\mu_s - \mu_{s+1})^2 + (\mu_{s+1} - \mu_{s+2})^2 + \mu_{s+2}^2 - \mu_s^2.
\]
As in (3.2.4), this inequality is true for every \( \mu_s, \mu_{s+1}, \mu_{s+2} \) and it is equality if and only if \( \mu_s = \mu_{s+1} = \mu_{s+2} = 1 \).

Second, let \( \mu = (\mu_1, \mu_2, ..., \mu_s, \mu_{s+1}) \) be the dual partition corresponding to a nilpotent element, where \( s+1 \) is even. Then the partition \( (\mu_1, \mu_2, ..., \mu_s) \) corresponds to a nilpotent element as well. So by induction it suffices to check that

\[
\mu_{s+1}^2 - 2\mu_{s+1} \geq (\mu_s - \mu_{s+1})^2 + \mu_{s+1}^2 - \mu_s^2.
\]

As in (3.2.5), this inequality is true for every \( (\mu_s, \mu_{s+1}) \) except for \( (1, 1) \) (but there are no nilpotents of such type with \( \mu_s = \mu_{s+1} = 1 \)) and it is an equality if and only if \( \mu_s = \mu_{s+1} = 2 \). This completes the proof.

Remark 3.3.2. Under the classical isomorphism \( so_4 \cong sl_2 \oplus sl_2 \), a nilpotent element of type \( (2^2) \) goes to a nilpotent element of type \( (2) \oplus (1^2) \). Under the classical isomorphism \( so_6 \cong sl_4 \), a nilpotent element of type \( (3^2) \) (resp. \( (2^2, 1^2) \)) goes to a nilpotent element of type \( (3, 1) \) (resp. \( (2, 1^2) \)). Under the classical isomorphism \( so_5 \cong sp_4 \), a nilpotent element of type \( (2^2, 1) \) goes to a nilpotent element of type \( (2, 1^2) \). Under a triality outer automorphism of \( so_8 \), a nilpotent element of type \( (4^2) \) (resp. \( (2^4) \)) goes to a nilpotent element of type \( (5, 1^3) \) (resp. \( (3, 1^5) \)). So all the “exceptional” cases in Proposition 3.3.1 are of hook type in the appropriate classical Lie algebras.

3.4. Other nilpotents in exceptional Lie algebras. Scanning the tables in [17, Chapter 22], we check that the inequality \( \dim(G \times S_e) > 2 \dim B_{G \times Q} \) is always satisfied for exceptional groups \( G \) except for the cases when \( e \) is zero or regular (see §1.2) and a single case considered in §2.4.

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