Efficiently simulating the work distribution of identical bosons with boson sampling

Wen-Qiang Liu and Zhang-qi Yin*

Center for Quantum Technology Research and Key Laboratory of Advanced Optoelectronic Quantum Architecture and Measurements (MOE), School of Physics, Beijing Institute of Technology, Beijing 100081, China
(Dated: July 6, 2023)

Boson sampling has been theoretically proposed and experimentally demonstrated to show quantum computational advantages. However, it still lacks the deep understanding of the practical applications of boson sampling. Here we propose that boson sampling can be used to simulate the work distribution of multiple identical bosons in a one-dimensional quantum piston system. We link the work distribution to boson sampling and numerically calculate the transition amplitude matrix between the multi-boson eigenstates, and then map the matrix to a linear optical network of boson sampling. The work distribution can be efficiently simulated through the output probabilities of boson sampling by using the method of the grouped probability estimation. The scheme requires at most the polynomial number of the samples and the optical elements. The work opens up a new path towards the calculation of complex quantum work distribution using only photons and linear optics.

PACS numbers: 03.67.-a, 03.67.Ac, 03.67.Lx, 42.50.Ex, 05.70.Ce
Keywords: quantum simulation, quantum work distribution, boson sampling, linear optics

I. INTRODUCTION

Work in nonequilibrium systems is a fundamental research topic and has stimulated many research interests in statistical physics [1–3]. Quantum work distribution is a key quantity in the thermodynamic analysis of any quantum system, and determines many important thermodynamic properties, such as the free energy difference [1] and nonequilibrium work relation [2]. The work distribution in a thermally isolated system can be effectively determined by the beginning-time and end-time energy measurements [4]. Previously, there are many explorations on quantum work distribution in the nonequilibrium quantum system, both theoretically [5–9] and experimentally [10–12]. However, these results mainly focused on the single-particle systems.

In recent years, multiparticle work distribution in nonequilibrium processes has received more and more attentions [13–16]. The calculation of work distribution for an identical many-particle system involves the transition probability between multiparticle eigenstates, which may be formidable difficulty due to the interference influence of these particles [17–19]. The transition probability between eigenstates of multiple identical bosons (fermions) associates with a permanent (determinant) of the corresponding transition amplitude matrix. The determinant can be efficiently calculated on a classical computer, while evaluating the permanent is a so-called ‘#P-complete’ hard [20, 21]. This seriously hinders the calculation of the work distribution with a large number of bosons using the classical computing.

Quantum boson sampling, a remarkably quantum computational supremacy candidate [22], was proposed by Aaronson and Arkhipov in 2011 [23]. Boson sampling emerges as a powerful paradigm to efficiently solve the output probability distribution of photons in a linear optical network, and provides several practical applications in graph theory [24–26], decision and function problems [27, 28], quantum chemistry [29–33], random number generation [34], and image encryption [35]. The early proof-of-principle demonstration of boson sampling has experimentally confirmed that the result of sampling is related to the matrix permanent [36–41]. Recently, boson samplings with tens [42, 43] or even more than a hundred of photons [44] have been experimentally realized, which fully showed a quantum computational advantage over the classical computer. Besides, the scalable implementations of boson sampling utilizing phononic modes of trapped ions have been proposed [45, 46].

Though the output probability of boson sampling is associated with the permanent of a matrix, it is completely different from the problem of predicting or estimating the permanent via boson sampling. In fact, it is infeasible to directly estimate the individual output probability using boson sampling, as the detected probability is exponentially small and one has to collect exponentially many samples to achieve a reasonable accuracy [23]. This is one of the main obstacles for limiting the practical applications of boson sampling. Fortunately, it is found that if these output probabilities are grouped and their sums are estimated, the polynomial samples rather than exponential ones are required for solving the certain problems [29–32, 47]. In this way, boson sampling could be a potentially effective method to solve some practical problems.

In this paper, we investigate how to use a boson sampling system to simulate the work distribution of multiple non-interaction identical bosons in a one-dimensional quantum piston. We firstly present a general theory for the work distribution of multiple bosons and establish a connection between the work distribution and boson...
sampling. Specifically, we numerically calculate the transition amplitude matrix of the multi-boson eigenstates and program it into an optical network of boson sampling. The output boson sampling probabilities can be efficiently simulated the work distribution by using the method of the grouped probability estimation (GPE) instead of the individual probability estimation (IPE). We also analyze the effect of system parameters on the work distribution and finally present a feasibility analysis of the scheme in terms of the resource cost. Our proposed way opens up a new possibility for studying the work distribution problem of quantum thermodynamics via the linear quantum optical network of boson sampling.

II. RESULTS

A. Work distribution with multiple bosons

We consider that \(N\) identical bosons in a quantum system are driven by a varied external work parameter \(\lambda\) from initial time \(t = 0\) to final time \(t = \tau\). The work parameter \(\lambda\) could be the position of the quantum piston or the spring coefficient of the harmonic oscillator. Suppose that at initial time \(t = 0\), the parameter is \(\lambda(0) = \lambda_0\) and the system is prepared in a thermal equilibrium state with a heat bath at an inverse temperature \(\beta = 1/(k_B T)\). Here \(k_B\) is the Boltzmann constant and \(T\) is the temperature of the system. Then the system is detached from the heat bath, and the work parameter is changed to \(\lambda(\tau) = \lambda_\tau\) at final time \(t = \tau\). The work distribution during this nonequilibrium process can be written as [48, 49]

\[
\rho(W) = \sum_{i_k} \sum_{f_l} P(|i_k^\lambda_0 : n_{i_k}\rangle) \times P(|i_k^\lambda_0 : n_{i_k}\rangle \rightarrow |f_l^{\lambda_\tau} : n_{f_l}\rangle) \times \delta(W - \sum_l n_{f_l} E_{f_l}^{\lambda_\tau} + \sum_k n_{i_k} E_{i_k}^{\lambda_0}).
\]

(1)

Here \(E_{i_k}^{\lambda_0}\) and \(E_{f_l}^{\lambda_\tau}\) are the \(i_k\)th eigenenergy at time \(t = 0\) and the \(f_l\)th eigenenergy at time \(t = \tau\), respectively. And the corresponding eigenstates are \(i_k\) and \(f_l\), respectively.

\[
P(|i_k^\lambda_0 : n_{i_k}\rangle) = \frac{1}{Z^\lambda_0} \exp(-\beta \sum_k n_{i_k} E_{i_k}^{\lambda_0}).
\]

(2)

Here \(Z^\lambda_0\) is a partition function, which is given by \(Z^\lambda_0 = \sum_{i_k} \exp(-\beta \sum_k n_{i_k} E_{i_k}^{\lambda_0})\). The other is the transition probability \(P(|i_k^\lambda_0 : n_{i_k}\rangle \rightarrow |f_l^{\lambda_\tau} : n_{f_l}\rangle)\) between the initial energy eigenstates and the final eigenstates. Due to the interference of multiple identical bosons, the transition probability can be expressed by [17, 18]

\[
P(|i_k^\lambda_0 : n_{i_k}\rangle \rightarrow |f_l^{\lambda_\tau} : n_{f_l}\rangle) = \prod_{i=1}^{K} \frac{1}{n_{i_k}!} \prod_{l=1}^{L} \frac{1}{n_{f_l}!} |\text{Per}(\Lambda^{(1,F)})|^2.
\]

(3)

Here the function \(\text{Per}(M)\) represents the permanent of a matrix \(M\). \(\Lambda = (|f_l^{\lambda_\tau} \langle i_k^\lambda_0 |)\) is an \(L \times K\) transition amplitude matrix. \(\hat{H}(t)\) is the Hamiltonian of the system, and \(\hat{U}\) denotes an evolutionary unitary operator of the system satisfying the time-dependent Schrödinger equation \(i\hbar \partial_t \hat{U}(t) = \hat{H}(t) \hat{U}(t)\). The vector \(I\) is an initial arrangement of the boson number in the eigenstate \(i_k\) and the vector \(F\) is the final arrangement of the boson number in the eigenstate \(f_l\) in the system. \(\Lambda^{(1,F)}\) denotes a sub-matrix of \(\Lambda\) by taking \(n_{f_l}\) copies of the \(f_l\)th row and \(n_{i_k}\) copies of the \(i_k\)th column of \(\Lambda\), which occupies a dimension of \(n_{f_l} \times n_{i_k}\). Since the total number of bosons is conserved, we have the identity relation \(\sum_{k=1}^{K} n_{i_k} = \sum_{l=1}^{L} n_{f_l} = N\).

As shown in Fig. 1, \(N\) identical bosons in a one-dimensional quantum piston is an interesting example to understand the work distribution. At \(t = 0\), the bosons are prepared into a thermal equilibrium state in a stretchy box of length \(\lambda_0\) (see Fig. 1a), and their population at the eigenstates is shown in Fig. 1c. Then, the box is stretched to the length \(\lambda_\tau\) (\(\lambda_\tau > \lambda_0\)) at a constant speed \(v\) (see Fig. 1b) and at this time the population of bosons is presented in Fig. 1d. In Appendix A, we provide an explicit expression for the analytical solutions of the transition amplitudes between the initial and the final energy
eigenstates [6, 50]. Note that calculating the transition probability of the bosons is a classical difficult problem, because the calculation complexity of the permanent of a general complex matrix is \#P-hard.

### B. Mapping work distribution into boson sampling

Boson sampling is considered as there are \( N \) indistinguishable bosons are scattered into a linear unitary network with \( M \) optical modes. We denote the input photon state in the Fock basis as \( |\mathbf{T}\rangle = |t_1, t_2, \ldots, t_M\rangle \). Each \( t_i \) denotes boson occupation-number in the \( i \)th optical mode and \( |\mathbf{T}\rangle \) describes \( N = \sum_{i=1}^{M} t_i \) bosons distribution in each mode. These photons are sent through the linear optical network that is characterized by a unitary transformation \( A \). According to the linear mapping relation \( a_i^\dagger \rightarrow \sum_{j=1}^{M} A_{ij} b_j^\dagger \) between the input mode creation operator \( a_i^\dagger \) and the output mode creation operator \( b_j^\dagger \) of the network, the probability of getting an output state \( |\mathbf{S}\rangle = |s_1, s_2, \ldots, s_M\rangle \) in the Fock basis is mathematically described by [36–39],

\[
P(|\mathbf{S}\rangle |\mathbf{T}\rangle) = \left| \langle \mathbf{S} | A | \mathbf{T} \rangle \right|^2 = \prod_{j=1}^{M} \frac{1}{s_j!} \prod_{i=1}^{M} \frac{1}{t_i!} \left| \text{Per}(A^{|\mathbf{S}\rangle |\mathbf{T}\rangle}) \right|^2.
\]

(4)

Remarkably, the probability \( P(|\mathbf{S}\rangle |\mathbf{T}\rangle) \) for each of input states and output states is proportional to a permanent of sub-matrix of \( A \). Combining Eq. (1), Eq. (3), and Eq. (4), we summary the corresponding relationship between the work distribution and the boson sampling in Tab. I. As shown in Tab. I, one can see that the space of the work distribution with \( N \) bosons and \( M \)-dimensional transition amplitude matrix is isomorphic to the space of boson sampling with \( N \) bosons and \( M \)-dimensional optical network. Therefore, the work distribution can be obtained by sampling from a great quantity of matrix permanents, equivalently to the boson sampling problem.

As shown in Fig. 2, we design a schematic setup for simulating the work distribution of multiple identical bosons via boson sampling. The transition amplitude matrix \( \Lambda \) in the piston system is mapped into an optical network of the boson sampling system. The transition probability between the multi-boson eigenstates is obtained by counting the photon probability distribution and then the work distribution can be simulated by using the method of GPE.
The matrix Λ has the normalization property, because of
\[ \sum_{i_k} |\langle f_i^{\lambda_0} | \tilde{U} | i_k^{\lambda_0} \rangle|^2 = \sum_{i_k} \langle f_i^{\lambda_0} | \tilde{U} | i_k^{\lambda_0} \rangle \langle i_k^{\lambda_0} | \tilde{U}^\dagger | f_i^{\lambda_0} \rangle = 1, \]
\[ \sum_{f_i} |\langle f_i^{\lambda_0} | \tilde{U} | i_k^{\lambda_0} \rangle|^2 = \sum_{f_i} \langle f_i^{\lambda_0} | \tilde{U} | i_k^{\lambda_0} \rangle \langle i_k^{\lambda_0} | \tilde{U}^\dagger | f_i^{\lambda_0} \rangle = 1. \]

The matrix Λ is a near-unitary matrix based on the normalized property, which can determine the dimension of matrix Λ in principle. To encode the matrix Λ into a linear optical network composed of beam splitters and phase shifters, the matrix Λ should be restricted to a unitary matrix as much as possible (the product of Λ and its Hermitian conjugate is an identity matrix). We evaluate the unitary property of the matrix Λ according to the unitary fidelity, defined as [51]
\[ F = \frac{1}{d} \left| tr \sqrt{I_d^{1/2} \Lambda \Lambda^\dagger I_d^{1/2}} \right|. \]

Here, \( d \) is the dimension of Λ and \( I_d \) is a \( d \)-order identity matrix. \( \sigma = \Lambda \Lambda^\dagger \) and \( \Lambda^\dagger \) is the Hermitian conjugate of Λ. We numerically calculate the matrix elements and truncate the size of matrix Λ when the fidelity reaches 0.995 with the fixed parameters \( \lambda_0 = 1 \) and \( \lambda_r = 2 \) and a varied speed \( v \). A relationship between the matrix dimension and the expansion speed \( v \) is plotted in Fig. 3a. We also present a relationship between the matrix dimension and the final length of the box in Fig. 3b. One can see that the dimension of the matrix Λ increases almost linearly with the acceleration of the piston speed \( v \) or the final length \( \lambda_r \) of the box.

The next step is to program the transition amplitude matrix Λ into a linear optical network. There are mainly two configurations of optical network to realize arbitrary unitary matrix, one is the triangle-shaped network [52] and the other is the square-shaped network [53]. As shown in Fig. 2a, here we use the square-shaped network to realize the transition amplitude matrix, because the symmetry design of the network is more robust against the photon loss and has minimal optical depth and better stability [53]. As shown in Fig. 2b, the crossing between two optical modes \( a \) and \( a+1 \) in the interferometer consists of two 50:50 beam splitters and two phase shifters, which can be expressed mathematically by a matrix \( T_{a,a+1}(\theta, \varphi) \) [53]. \( T_{a,a+1}(\theta, \varphi) \) is called an elimination matrix and it is obtained by replacing the entries of an identity matrix with the same size as Λ at the \( a \)th and \( (a+1) \)th rows and the \( a \)th and \( (a+1) \)th columns with
\[ \begin{pmatrix} e^{i\varphi} \cos \theta & -e^{i\varphi} \sin \theta \\ e^{i\varphi} \sin \theta & e^{i\varphi} \cos \theta \end{pmatrix}, \]
and the rest of the other entries remain unchanged. Based on Gaussian elimination method, the matrix Λ can be diagonalized into a diagonal matrix \( D \) by multiplying a series of \( T_{a,a+1}^{-1} \) and its inverse matrix \( T_{a,a+1}^{-1} \). The matrix Λ is realized physically in an optical network by choosing suitable values of parameters \( \theta \) and \( \varphi \) of \( T_{a,a+1}^{-1} \) and the phase values of diagonal matrix \( D \) at each output port. The resource overhead for programming a \( d \times d \) unitary matrix into the optical network requires \( d(d-1) \) 50:50 beam splitters and \( d^2 \) phase shifters. In boson sampling system, the phase shifters to realize the diagonal matrix \( D \) can be removed as only final photon number is sampled, which will not affect the result but can reduce the resource cost. From Fig. 3, one can see that the total resource cost presents at a polynomial hierarchy with the expansion speed \( v \) of the piston or the final length \( \lambda_r \) of the box in terms of the number of required optical elements.

C. The effect of the system parameters on the work distribution

The temperature and the speed play an important role in the piston system during the work process. On one hand, from Eq. (2), one can see that the temperature of the system affects the initial distribution of the bosons. The work distribution for two bosons with different temperature is presented in Fig. 4. As shown in Fig. 4, as the temperature increases, the probability that the bosons populate higher energy levels will increase, which makes the higher energy levels need to be considered. As a result, the dimension of the transition amplitude matrix will become larger, making the calculation of the work distribution more complicated.

On the other hand, the moving speed of the piston is related to the transition probability \( P(|i_k^{\lambda_0} : n_{i_k} \rangle \rightarrow |f_i^{\lambda_r} : n_{f_i} \rangle) \). In the low speed limit, one can get the result \( P(|i_k^{\lambda_0} : n_{i_k} \rangle \rightarrow |f_i^{\lambda_r} : n_{f_i} \rangle) \rightarrow \delta_{i_k,f_i} (v \rightarrow 0) \) based on the quantum adiabatic theorem. Figure 5a shows approximately the initial energy distribution: the highest peak represents the energy from the beginning to the end.
of the ground state, the second highest peak corresponds to the energy of the first excited state, etc. As shown in Fig. 5, with the increase of the piston speed, the work distribution for two bosons becomes more complex. This is caused by the higher energy level transition of the bosons when the speed becomes faster.

D. An example

To understand the work distribution simulated by boson sampling well, as an interesting example, we calculate the work distribution of three bosons in detail. We consider three bosons are trapped in a box with the initial length \( \lambda_0 = 1 \), the stretching speed \( v = 0.4 \), the final length \( \lambda_r = 2 \), and the temperature \( \beta = 0.1 \). We numerically calculate the matrix elements of \( \Lambda \) based on Eq. (A5) and obtain a 5 \( \times \) 5 dimensional near-unitary matrix \( \Lambda_5 \) with a unitary fidelity \( F = 0.9992 \) (see Appendix B). We decompose the matrix \( \Lambda_5 \) into the product of a diagonal matrix \( D \) and a series of elimination matrices \( T_{a,a+1} \) based on Gaussian elimination method [53]. The result of the decomposition is expressed as

\[
\Lambda_5 = DT^{(5)}_{3,4}T^{(4)}_{4,5}T^{(3)}_{1,2}T^{(2)}_{2,3}T^{(1)}_{3,4}T^{(1)}_{1,2}.
\]

As shown in Fig. 6, the matrix \( \Lambda_5 \) is programmed into an optical network of a boson sampler. The values of phase shifter angles \( \theta \) and \( \varphi \) in the network are calculated and presented in Tab. II. The diagonal matrix \( D \) can be ignored without affecting the final photon output probability.

Finally, we calculate the cumulative work distribution by the definition as follows

\[
\chi(W) = \int W \rho(W')dW'.
\]

The result of cumulative work distribution based on Eq. (10) is plotted in the curve of Fig. 7. In real experiments,
the noise and error are inevitable. Therefore, we evaluate the effect of the noise on the cumulative work distribution by adding a random noise $\mathcal{N} \in (-0.01, 0.01)$ to the angles of the beam splitters and phase shifters. Explicitly, we randomly choose 100 groups of noise terms and calculate the cumulative work distribution under the noise effects (see error bars in Fig. 7). We find that the ratio between the error bar and the cumulative work distribution curve is 1% to 2%.

### III. Feasibility Analysis

Before we discuss the feasibility of the scheme, at first we clarify the relation between boson sampling and the matrix permanent. Boson sampling is a classically difficult problem as the sampled probability involves the permanent of a matrix. However, this does not mean that boson sampling can directly simulate the matrix permanent. Scaling up the system size of boson sampling, the probability of individual output event will become exponentially small, which causes that one has to obtain an exponential number of observations of the event to maintain the accuracy. This method of individual probability estimation is obviously infeasible. Remarkably, if one groups and sums the individual output probabilities so that the sum probability is polynomially small rather than exponentially small, then the number of needed samples would become polynomial size \cite{29, 47}. The GPE can deal with the work distribution problem well.

We next give the explicit method of GPE and evaluate the feasibility of the method by analyzing the required total sample number and the reasonable accuracy. To group the output probabilities, we introduce and define an integer energy vector $E_f \in \mathbb{Z}_{\geq 0}^M$, in which each element $E_{f_l}$ ($l = 1, \ldots, M$) is at most a polynomial large number, i.e., $E_{f_l} \leq O(\text{poly}(M))$. In fact, the each final energy $E_{f_l}$ in Eq. (1) can be expressed as a floating-point number and it can transform as the integer number $E_{f_l}$ by multiplying by a sufficiently large number. Based on the delta function in Eq. (1), the output probabilities can be grouped and sum them when $W = E_f \cdot m - E_i \cdot n$, resulting in,

$$G(W) = \sum_{m} P(m|n)\delta(W - E_f \cdot m + E_i \cdot n)$$

$$= \sum_{m \in \mathcal{G}(W)} P(m|n),$$

(11)

Here $P(m|n)$ is the transition probability from the initial photon distribution $n = (n_1, \ldots, n_M)$ to the output distribution $m = (m_1, \ldots, m_M)$ and the vector $E_i$ is the initial eigenenergy of bosons. The sets $\mathcal{G}(W) = \{m \in \mathbb{Z}_{\geq 0}^M | E_f \cdot m - E_i \cdot n = W\}$ for each of grouped outcomes $W \in \{0, \ldots, W_{\text{max}}\}$. We note that the number of different groups is $W_{\text{max}} + 1 \leq O(\text{poly}(M))$, which causes the probability $|G(W)|$ of each group is greater than or equal to $O(1/\text{poly}(M))$ rather than the exponentially small. Based on the result of grouped probability in Eq. (11) and the work distribution expression in Eq. (1), the estimated work distribution can be rewritten as

$$\rho_{\text{est}}(W) = \sum_{i_k} P(i_k | n_{i_k})G(W).$$

(12)

This expression allows us to simulate the work distribution by grouping the output probabilities and collecting at most the polynomial samples of boson sampling.

We introduce a reasonable accuracy $\epsilon$ to evaluate the performance of the estimated work distribution and the ideal one, i.e.,

$$|\rho_{\text{est}}(W) - \rho_{\text{ide}}(W)| \leq \epsilon.$$ 

(13)

Based on the central limit theorem and Chebyshev’s inequality, the total sample number $N_{\text{tot}}$ to achieve a reasonable accuracy scales as $\text{Var}(\rho_{\text{est}}(W))/\epsilon^2$. It is clear that the variance of $\rho_{\text{est}}(W)$ (denote as $\text{Var}(\rho_{\text{est}}(W))$) is bounded by 1 because the value of the work distribution $\rho_{\text{est}}(W)$ is between 0 and 1. In other words, $N_{\text{tot}} = O(1/\epsilon^2)$ is an upper bound on the total number of samples required to simulate the work distribution, and this upper bound also implies that the reasonable target accuracy becomes $O(1/\text{poly}(M))$, instead of the exponentially small for individual output probabilities. It suggests that polynomial large samples would be still reasonable.
IV. CONCLUSION

In conclusion, we have presented a connection between the work distribution and boson sampling. We found that boson sampling can be used to efficiently simulate the work distribution by sampling the output probability of photons and using the GPE. We analyzed the computational cost with this set-up, and the results showed that at most a polynomial number of observation samples and optical elements are required to achieve a reasonable accuracy. Intuitively, calculating the work distribution may be difficult because the calculation of the transition probability between the multi-boson eigenstates is a classically hard problem. The connection provides a new possible solution for studying the work distribution that is too difficult to calculate on a classical computer. The scheme we developed here is also suitable to the simulation of the work distribution of multiple identical bosons in a contraction piston system or a harmonic oscillator system.

ACKNOWLEDGMENTS

We thank valuable discussions with Zhaohui Wei, Haitao Quan, Xianmin Jin, and Yuanhao Wang. This work was supported by National Natural Science Foundation of China under Grant No. 61771278 and Beijing Institute of Technology Research Fund Program for Young Scholars.

Appendix A: Transition amplitudes in a 1D quantum piston

We show how to obtain analytical solutions to the transition amplitudes \( f^\lambda_k \), between the initial energy eigenstates and the final energy eigenstates in a 1D quantum piston [6, 50]. The piston system evolving from time \( t = 0 \) to \( t = \tau \) follows the time-dependent Schrödinger equation \( i\hbar \dot{\Psi}(t) = H(t)\Psi(t) \). A complete orthogonal solution set of this Schrödinger equation can be expressed as [50]

\[
\Phi_j(x, t, \tau) = \exp\left[\frac{i}{\hbar \lambda(t)} \left(\frac{1}{2} M v x^2 - E_j^\lambda \lambda(0)\right)\right] \phi_j(x, \lambda(t)). 
\] (A1)

The time-dependent Schrödinger equation has the general solution of the following form

\[
\Psi(x, t) = \sum_{j} c_j \Phi_j(x, t). 
\] (A2)

Here, \( j = 1, 2, \ldots \), and \( x \) is the length of the box with the time change, \( 0 \leq x \leq \lambda(t) \). The \( j \)th eigenenergy \( E_j^\lambda \) is given by \( E_j^\lambda = \left(\frac{j \pi}{2M \lambda_0}\right)^2 \), and \( M \) is the mass of the boson. The \( j \)th eigenstate of a boson in the piston system is

\[
\phi_j(x, \lambda) = \sqrt{\frac{2}{\lambda}} \sin\left(\frac{j \pi x}{\lambda}\right). 
\] (A3)

The coefficients \( c_j \) of the solution of Schrödinger equation in Eq. (A2) can be determined by the initial condition. That is, under the initial condition \( \Psi(x, 0) = \phi_{i_k}(x, \lambda_0) = \langle x | i_k^\lambda \rangle \) (taking \( M = h = 1 \)), the coefficients become

\[
c_j(i_k) = \frac{2}{\lambda_0} \int_0^{\lambda_0} e^{-i \frac{\pi k x^2}{\lambda_0}} \sin\left(\frac{j \pi x}{\lambda_0}\right) \sin\left(\frac{i_k \pi x}{\lambda_0}\right) dx. 
\] (A4)

The transition amplitudes between the eigenstates of single-particle from time \( t = 0 \) to \( t = \tau \) can be expressed as

\[
\langle f^\lambda_i | \hat{U} | i_k^\lambda \rangle = \sum_{j} c_j(i_k) \int_0^{\lambda_\tau} \Phi_j(x, \tau) \phi_{i_k}(x, \lambda_\tau) dx. 
\] (A5)

Appendix B: Transition amplitude matrix \( \Lambda_5 \)

We numerically calculate the transition amplitudes between the initial energy eigenstates and the final energy eigenstates based on Eq. (A5) and obtain a \( 5 \times 5 \) dimensional near-unitary transition amplitude matrix \( \Lambda_5 \) when the parameters \( \lambda_0 = 1 \), \( v = 0.4 \), and \( \lambda_\tau = 2 \) are taken. The calculated result of matrix \( \Lambda_5 \) is given by

\[
\Lambda_5 = \begin{pmatrix}
0.9843 + 0.1712i & 0.0300 - 0.0273i & -0.0120 - 0.0017i & 0.0041 + 0.0051i & 0.0003 - 0.0039i \\
-0.0047 - 0.0401i & 0.8639 + 0.4990i & 0.0504 - 0.0012i & -0.0108 - 0.0147i & -0.0017 + 0.0096i \\
0.0030 + 0.0119i & 0.0054 - 0.0494i & 0.4535 + 0.8874i & 0.0394 + 0.0452i & 0.0070 - 0.0216i \\
-0.0011 - 0.0069i & -0.0021 + 0.0186i & 0.0338 - 0.0475i & -0.3230 + 0.9414i & -0.0232 + 0.0659i \\
-0.0006 + 0.0044i & 0.0039 - 0.0101i & -0.0166 + 0.0171i & 0.0662 - 0.0114i & -0.9723 + 0.2054i
\end{pmatrix}. 
\] (B1)
(2012).

[46] C. Shen, Z. Zhang, and L. M. Duan, Phys. Rev. Lett. 112, 050504 (2014).

[47] C. Oh, Y. Lim, Y. Wong, B. Fefferman, and L. Jiang, arXiv preprint arXiv:2202.01861 (2022).

[48] H. Tasaki, arXiv preprint cond-mat/0009244 (2000).

[49] P. Talkner, E. Lutz, and P. Hänggi, Phys. Rev. E 75, 050102(R) (2007).

[50] S. W. Doescher and M. H. Rice, Am. J. Phys. 37, 1246 (1969).

[51] A. Gilchrist, N. K. Langford, and M. A. Nielsen, Phys. Rev. A 71, 062310 (2005).

[52] M. Reck, A. Zeilinger, H. J. Bernstein, and P. Bertani, Phys. Rev. Lett. 73, 58 (1994).

[53] W. R. Clements, P. C. Humphreys, B. J. Metcalf, W. S. Kolthammer, and I. A. Walmsley, Optica 3, 1460 (2016).