Bifurcation results for semilinear elliptic problems in $\mathbb{R}^N$ *

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Abstract: In this paper we obtain, for a semilinear elliptic problem in $\mathbb{R}^N$, families of solutions bifurcating from the bottom of the spectrum of $-\Delta$. The problem is variational in nature and we apply a nonlinear reduction method which allows us to search for solutions as critical points of suitable functionals defined on finite-dimensional manifolds.

1 Introduction and Main Results

An interesting problem in bifurcation phenomena is to look for solutions bifurcating not from an eigenvalue but from a point of the continuous spectrum of the linearized operator of the involved equation. Typical examples of differential operators with continuous spectrum are the Laplace or the Schrödinger operators in all $\mathbb{R}^N$, and there are now many results on bifurcation of solutions for semilinear elliptic equations in $\mathbb{R}^N$, for example see [19], [20], [21], [18], [16]. See also [22], and the references therein, for the study of bifurcation into spectral gaps. A.Ambrosetti and the first author have studied such kind of problems in [2] and [3], obtaining several results on bifurcation of solutions for a one-dimensional differential equation. In this paper we pursue such a study, generalizing some of the results of [2] to higher dimensions, and considering also the case of a critical nonlinearity. In section 5 of this paper we also fill a gap in the proof of theorem 3.2 in [2]. We thank S. Krömer, who pointed out this gap, for his remarks and for several useful discussions.

We consider the equation

$$\begin{cases}
-\Delta \psi - \lambda \psi = a(x)|\psi|^{p-1}\psi + b(x)|\psi|^{q-1}\psi, & x \in \mathbb{R}^N, \\
\lim_{|x|\to \infty} \psi(x) = 0,
\end{cases}$$

where $N \geq 1$, $\lambda$ is a negative parameter, $1 < p < q \leq \frac{N+2}{N-2}$ if $N \geq 3$ (and $q < +\infty$ if $N = 1, 2$), $p < 1 + 4/N$ and $a, b : \mathbb{R}^N \to \mathbb{R}$ satisfy suitable hypotheses (see below). Equation (1) is an homogeneous equation, so $\psi = 0$ is a solution for all $\lambda$, the line $\{(\lambda, \psi = 0) \mid \lambda \in \mathbb{R}\}$ is a line of trivial solutions and, as $q > p > 1$, the linearized operator at $\psi = 0$ is given by $\psi \to -\Delta \psi - \lambda \psi$.

It is well known that $[0, +\infty)$ is the spectrum of $-\Delta$ on $\mathbb{R}^N$, and that it contains no eigenvalue. We will find solutions bifurcating from the bottom of the essential spectrum of $-\Delta$. To be

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precise, by “solution” we mean a couple \((\lambda, \psi_\lambda)\) such that \(\psi_\lambda \in H^1(\mathbb{R}^N)\) and \(\psi_\lambda\) is a solution of (1) in the weak sense of \(H^1(\mathbb{R}^N)\). We look for solutions bifurcating from the origin in \(H^1(\mathbb{R}^N)\), that is families \((\lambda, \psi_\lambda)\) of solutions of (1) such that \(\lambda \in (\lambda_0, 0)\) for some \(\lambda_0 < 0\) and \(\psi_\lambda \to 0\) in \(H^1(\mathbb{R}^N)\) as \(\lambda \to 0\).

Now let us state the hypotheses on the functions \(a, b\). On \(a\) we assume that there is \(A > 0\) such that either \(a - A \in L^1(\mathbb{R}^N)\) or \(a - A\) is asymptotic, at infinity, to \(1/|x|^\gamma\), for suitable \(\gamma\). To be precise, in the first case we assume the following set of hypotheses.

(a1) \(a - A\) is continuous, bounded and \(a(x) - A \in L^1(\mathbb{R}^N)\).

(a2) \(\int_{\mathbb{R}^N} (a(x) - A) dx \neq 0\).

In the second case we assume the following hypothesis:

(a3) \(a - A\) is continuous and there exist \(L \neq 0\) and \(\gamma \in ]0, N[\) such that \(|x|^{\gamma}(a(x) - A) \to L\) as \(|x| \to +\infty\).

Notice that (a1) of course implies that \(a - A \in L^p(\mathbb{R}^N)\) for all \(p \in [1, +\infty]\), while (a3) implies that \(a - A\) is bounded. For \(b\) we use some of the following assumptions.

(b1) \(b\) is continuous and bounded.

(b2) \(b \in L^{\frac{2N}{N+2}}(\mathbb{R}^N)\). If \(N \geq \frac{2q-p}{p-1}\) we also assume that there exists \(\beta \in [1, \beta^*]\) such that \(b \in L^\beta(\mathbb{R}^N)\), where

\[
\beta^* = \frac{N(p-1)}{N(p-1) - 2(q-p)} \quad \text{if} \quad N > \frac{2q-p}{p-1}, \quad \beta^* = +\infty \quad \text{if} \quad N = \frac{2q-p}{p-1}.
\]

(b3) \(b \in L^{\frac{2N}{N+2}}(\mathbb{R}^N)\). If \(\gamma \geq \frac{2q-p}{p-1}\) we also assume that there exists \(\beta \in [1, \beta^*]\) such that \(b \in L^\beta(\mathbb{R}^N)\), where

\[
\beta^* = \frac{N(p-1)}{\gamma(p-1) - 2(q-p)} \quad \text{if} \quad \gamma > \frac{2q-p}{p-1}, \quad \beta^* = +\infty \quad \text{if} \quad \gamma = \frac{2q-p}{p-1}.
\]

The value \(\gamma\) in (b3) is that given in (a3). We will assume either (b1) and (b2), or (b1) and (b3). Notice that, assuming (b1), hypotheses (b2) and (b3) are obviously satisfied when \(b \in L^1(\mathbb{R}^N)\).

We can now state our main results.

**Theorem 1.1** Assume \(1 < p \leq \frac{N+2}{N-2}\) if \(N \geq 3\), and \(q < +\infty\) if \(N = 1,2\). Suppose that (a1), (a2), (b1), (b2) hold. Then (1) has a family of solutions bifurcating from the origin in \(L^\infty(\mathbb{R}^N)\). If, besides, \(p < 1 + \frac{1}{N}\), this family of solutions bifurcates from the origin also in \(H^1(\mathbb{R}^N)\).
Theorem 1.2 Assume $1 < p < q \leq \frac{N+2}{N-2}$ if $N \geq 3$, and $q < +\infty$ if $N = 1, 2$. Suppose that $(a_3)$, $(b_1)$ and $(b_3)$ hold. Then (1) has a family of solutions bifurcating from the origin in $L^\infty(\mathbb{R}^N)$. If, besides, $p < 1 + \frac{4}{N}$, this family of solutions bifurcates from the origin also in $H^1(\mathbb{R}^N)$.

Remark 1.3 When $p \geq 1 + 4/N$, in $H^1(\mathbb{R}^N)$ the solutions can bifurcate from infinity or can be bounded away both from zero and infinity.

Remark 1.4 An interesting question is to know if the solutions that we find form a curve. We give some results in this direction in section 5.

In the proof of theorems 1.1 and 1.2 we follow the framework of [2], concerning the existence of critical points of perturbed functionals. We start by a change of variables. Let us set $u(x) = \varepsilon^{2/(1-p)}\psi(x/\varepsilon)$, $\lambda = -\varepsilon^2$, so that equation (1) becomes

$$-\Delta u + u = A|u|^{p-1}u + (a(x/\varepsilon) - A)|u|^{p-1}u + \varepsilon^{2-p}b(x)|u|^{q-1}u.$$  \hspace{1cm} (2)

It is obvious that to any family $u_\varepsilon \in H^1(\mathbb{R}^N)$ of solutions of (2), bounded as $\varepsilon \to 0$, there corresponds a family $\psi_\varepsilon(x) = \varepsilon^{2/(p-1)}u_\varepsilon(\varepsilon x)$ of solutions of (1). When $p < 1 + 4/N$ it is easy to check that $\psi_\varepsilon(x) \to 0$ in $H^1(\mathbb{R}^N)$, as $\varepsilon \to 0$. When $p \geq 1 + 4/N$ we still get solutions, and it is easy to see that they vanish, as $\varepsilon \to 0$, in $L^\infty(\mathbb{R}^N)$, but they do not vanish in $L^2(\mathbb{R}^N)$. Throughout this paper we will look for bounded families of $H^1$-solutions of (2).

The paper is organized as follows: after the introduction (section 1) we give in section 2 a brief sketch of the abstract critical point theory for perturbed functionals that we use to prove theorems 1.1 and 1.2. In section 3 we prove theorem 1.1 and in section 4 we prove theorem 1.2. In section 5 we give some results on the existence of curves of solutions bifurcating from $(0,0)$, and we fill a gap in the proof of theorem 3.2 in [2].

**Notation**

We collect below a list of the main notation used throughout the paper.

- If $E$ is a Banach space, $F : E \to E$, and $u \in E$, then $DF(u) : E \to E$, $D^2F(u) : E \times E \to E$ and $D^3F(u) : E \times E \times E \to E$ are the first, second and third differential of $F$ at $u$, which are respectively linear, bilinear and three-times linear.

- $L(E,E)$ is the space of linear continuous operators from $E$ to $E$.

- $2^* = \frac{2N}{N-2}$ is the critical exponent for the Sobolev embedding, when $N \geq 3$.

- We will use $C$ to denote any positive constant, that can change from line to line.
2 Abstract theory for perturbed functionals

In this section we give the main ideas and results of a variational method to study critical points of perturbed functionals. The method has been developed in [4], [1], [2] and then has been applied to many different problems, see [5], [6], [7], [8], [13], [9], [10]. We deal with a family of functionals $f^\varepsilon$, defined on a Hilbert space $E$, of the form

$$f^\varepsilon(u) = \frac{1}{2}\|u\|^2 - F(u) + G(\varepsilon, u),$$

where $\|\cdot\|$ is the norm in $E$, $F : E \to \mathbb{R}$ and $G : \mathbb{R} \times E \to \mathbb{R}$. We need the following hypotheses

$$(F_0) \quad F \in C^2;$$

$$(G_0) \quad G \text{ is continuous in } (\varepsilon, u) \in \mathbb{R} \times E \text{ and } G(0, u) = 0 \text{ for all } u \in E;$$

$$(G_1) \quad G \text{ is of class } C^2 \text{ with respect to } u \in E.$$ 

We will use the notation $F'(u)$, respectively $G'(\varepsilon, u)$, to denote the functions defined by setting

$$(F'(u)|v) = DF(u)[v], \quad \forall \ v \in E,$$

and, respectively,

$$(G'(\varepsilon, u)|v) = D_\varepsilon G(\varepsilon, u)[v], \quad \forall \ v \in E,$$

where $(\cdot | \cdot)$ is the scalar product in $E$. Similarly, $F''(u)$, resp. $G''(\varepsilon, u)$, denote the maps in $L(E, E)$ defined by

$$(F''(u) v|w) = D^2 F(u)[v, w] \quad (G''(\varepsilon, u) v|w) = D^2_{\varepsilon u} G(\varepsilon, u)[v, w].$$

In section 5 we will assume that $F, G$ are $C^3$. In this case we will denote $F'''(u)$, $G'''(\varepsilon, u)$ the bilinear maps defined by

$$(F'''(u)[v_1, v_2]|v_3) = D^3 F(u)[v_1, v_2, v_3], \quad (G'''(\varepsilon, u)[v_1, v_2]|v_3) = D^3_{\varepsilon uu} G(\varepsilon, u)[v_1, v_2, v_3].$$

We also assume that $F$ satisfies

$$(F_1) \quad \text{there exists a } d\text{-dimensional } C^2 \text{ manifold } Z, \ d \geq 1, \text{ consisting of critical points of } f_0, \text{ namely such that}$$

$$z - F'(z) = 0, \quad \forall \ z \in Z.$$ 

Such a $Z$ will be called a critical manifold of $f_0$.

Let $T_z Z$ denote the tangent space to $Z$ at $z$ and $I_E$ denote the Identity map in $E$. We further suppose
\((F_2)\) \(F''(z)\) is compact \(\forall \; z \in Z;\)

\((F_3)\) \(T_z Z = \text{Ker}[I_E - F''(z)], \; \forall \; z \in Z.\)

We make the following further assumptions on \(G.\)

\((G_2)\) The maps \((\varepsilon, u) \mapsto G'(\varepsilon, u), \; (\varepsilon, u) \mapsto G''(\varepsilon, u)\) are continuous (as maps from \(\mathbb{R} \times E\) to \(E,\) resp. to \(L(E, E)\)).

\((G_3)\) there exist \(\alpha > 0\) and a continuous function \(\Gamma : Z \to \mathbb{R}\) such that, for all \(z \in Z,\)

\[
\Gamma(z) = \lim_{\varepsilon \to 0} \frac{G(\varepsilon, z)}{\varepsilon^\alpha},
\]

and

\[
G'(\varepsilon, z) = o(\varepsilon^{\alpha/2}).
\]

In [2] (see also [1], [4]) the following theorem is proved.

\textbf{Theorem 2.1} Suppose \((F_0 - F_3)\) and \((G_0 - G_3)\) hold and assume there exist \(\delta > 0\) and \(z^* \in Z\) such that

\[
\text{either} \quad \min_{\|z - z^*\| = \delta} \Gamma(z) > \Gamma(z^*), \quad \text{or} \quad \max_{\|z - z^*\| = \delta} \Gamma(z) < \Gamma(z^*). \tag{3}
\]

Then, for \(\varepsilon\) small, \(f_\varepsilon\) has a critical point \(u_\varepsilon.\)

\textbf{Proof.} We give only a sketch of the proof, divided in three steps.

\textbf{Step 1.} Using the Implicit Function Theorem one can find \(w = w(\varepsilon, z) \perp T_z Z\) such that

\[
\dot{f}_\varepsilon(z + w) \in T_z Z, \quad \|w\| = o(\varepsilon^{\alpha/2}) \quad \text{and} \quad \|D_z w(\varepsilon, z)\| \to 0 \quad \text{as} \quad \varepsilon \to 0. \tag{4}
\]

Letting \(Z_\varepsilon = \{z + w(\varepsilon, z)\},\) it turns out that \(Z_\varepsilon\) is locally diffeomorphic to \(Z\) and any critical point of \(f_\varepsilon\) restricted to \(Z_\varepsilon\) is a stationary point of \(f_\varepsilon.\)

\textbf{Step 2.} Using the Taylor expansion we obtain, for \(u = z + w(\varepsilon, z) \in Z_\varepsilon,\)

\[
f_\varepsilon(u) = c + \varepsilon^{\alpha}\Gamma(z) + o(\varepsilon^{\alpha}),
\]

where \(c\) is a constant.

\textbf{Step 3.} It readily follows that, for small \(\varepsilon\)'s, \(f_\varepsilon\) has a local constrained minimum (or maximum) on \(Z_\varepsilon\) at some \(u_\varepsilon = z_\varepsilon + w(\varepsilon, z_\varepsilon) \in Z_\varepsilon,\) with \(\|z_\varepsilon - z^*\| < \delta.\) According to step 1, such \(u_\varepsilon\) is a critical point of \(f_\varepsilon.\)

\(\blacksquare\)
In this section we prove theorem 1.1. We want to apply the abstract tools of the previous section, and we start to set

$$E = H^1(\mathbb{R}^N), \quad ||u||^2 = \int_{\mathbb{R}^N} (|\nabla u|^2 + u^2) \, dx, \quad F(u) = \frac{A}{p+1} \int_{\mathbb{R}^N} |u|^{p+1} \, dx,$$

and $G = G_1 + G_2$ where

$$G_1(\varepsilon, u) = \begin{cases} \frac{1}{p+1} \int_{\mathbb{R}^N} (a(x/\varepsilon) - A) |u|^{p+1} \, dx & \text{if } \varepsilon \neq 0, \\ 0 & \text{if } \varepsilon = 0 \end{cases}$$

and

$$G_2(\varepsilon, u) = \begin{cases} \frac{1}{p+1} \varepsilon^{2\frac{q}{p} - 1} \int_{\mathbb{R}^N} b(x/\varepsilon) |u|^{q+1} \, dx & \text{if } \varepsilon \neq 0, \\ 0 & \text{if } \varepsilon = 0 \end{cases}$$

Throughout this section we assume $N \geq 3$ and, of course, $1 < p < q \leq \frac{N+2}{N-2}$. The cases $N = 1, 2$ can be handled in the same way, and in fact are easier. We have now to verify that the hypotheses $(F_0 - F_3)$ and $(G_0 - G_3)$ are satisfied. The fact that $q > p > 1$ gives of course $(F_0)$ and $(G_1)$. It is also well known (see [11], [12], [15]) that there exists a unique positive radial solution $z_0$ of

$$-\Delta u + u = A|u|^{p-1}u, \quad x \in \mathbb{R}^N,$$

that $z_0$ is strictly radial decreasing, has an exponential decay at infinity together with its derivatives, and that $f_0$ possesses a $N-$dimensional manifold of critical points

$$Z = \{ z_0(x) = z_0(x + \theta) \mid \theta \in \mathbb{R}^N \}.$$ 

Furthermore, we know (see [17], [4] and the references therein) that $T_{z_0}Z = \ker(I_E - F''(z_0))$ for all $z_0 \in Z$. It is also easy to check that $F''(z_0)$ is compact, for all $z_0 \in Z$. In this way all the hypotheses on $F$ are satisfied, and the rest of this section is devoted to prove those on $G$. We will get this by several lemmas. Let us prove as first thing that the hypothesis $(G_0)$ is satisfied.

**Lemma 3.1** Assume $(a_1)$ and $(b_1)$. Then $G$ is continuous.

**Proof.** We prove first that $G_1$ is continuous. Assume that $(\varepsilon, u) \to (\varepsilon_0, u_0)$ in $\mathbb{R} \times H^1(\mathbb{R}^N)$, with $\varepsilon_0 \neq 0$. Then we can write

$$(p+1)|G_1(\varepsilon, u) - G_1(\varepsilon_0, u_0)| =$$

$$\left| \int_{\mathbb{R}^N} \left( a \left( \frac{x}{\varepsilon} \right) - A \right) |u|^{p+1} \, dx - \int_{\mathbb{R}^N} \left( a \left( \frac{x}{\varepsilon_0} \right) - A \right) |u_0|^{p+1} \, dx \right| \leq$$

$$\left| \int_{\mathbb{R}^N} \left( a \left( \frac{x}{\varepsilon} \right) - A \right) |u|^{p+1} \, dx - \int_{\mathbb{R}^N} \left( a \left( \frac{x}{\varepsilon_0} \right) - A \right) |u|^{p+1} \, dx \right| +$$
If \( a(x) \) is continuous and bounded, so it is easy to deduce, by dominated convergence, that the first term goes to zero, while the second one goes to zero by hypothesis. Hence we deduce \( G_1(\varepsilon, u) - G_1(\varepsilon_0, u_0) \rightarrow 0 \).

Now assume that \( (\varepsilon, u) \rightarrow (0, u_0) \). By definition \( G_1(0, u_0) = 0 \) and we have, applying Hölder inequality,

\[
(p + 1)|G_1(\varepsilon, u)| \leq \int_{\mathbb{R}^N} |a(x) - A| |u|^{p+1} dx \leq \left( \int_{\mathbb{R}^N} |a(x) - A| \frac{2}{2-p} dx \right)^{\frac{2}{2-p}} \left( \int_{\mathbb{R}^N} |u|^2 dx \right)^{\frac{p}{2}}.
\]

By the change of variables \( y = x/\varepsilon \) we get

\[
\left( \int_{\mathbb{R}^N} |a(x) - A| \frac{2}{2-p} dx \right)^{\frac{2}{2-p}} = \varepsilon^{N-2} \left( \int_{\mathbb{R}^N} |a(y) - A| \frac{2}{2-p} dy \right)^{\frac{2}{2-p}}.
\]

As \( a - A \in L^\frac{2N}{N-2} (\mathbb{R}^N) \) and \( u \in H^1(\mathbb{R}^N) \subset L^2(\mathbb{R}^N) \), we get \( G_1(\varepsilon, u) \rightarrow 0 \) as \( (\varepsilon, u) \rightarrow (0, u_0) \).

As to \( G_2 \), we argue in the same way. If \( (\varepsilon, u) \rightarrow (\varepsilon_0, u_0) \) with \( \varepsilon_0 \neq 0 \), then

\[
(q + 1)|G_2(\varepsilon, u) - G_2(\varepsilon_0, u_0)| =
\]

\[
\left| \varepsilon^{\frac{2N}{p-1}} \int_{\mathbb{R}^N} b\left( \frac{x}{\varepsilon} \right) |u|^{q+1} dx - \varepsilon_0^{\frac{2N}{p-1}} \int_{\mathbb{R}^N} b\left( \frac{x}{\varepsilon_0} \right) |u_0|^{q+1} dx \right| \leq
\]

\[
\varepsilon^{\frac{2N}{p-1}} \left| \int_{\mathbb{R}^N} b\left( \frac{x}{\varepsilon} \right) |u|^{q+1} dx - \int_{\mathbb{R}^N} b\left( \frac{x}{\varepsilon_0} \right) |u_0|^{q+1} dx \right| +
\]

\[
\left| \varepsilon^{\frac{2N}{p-1}} - \varepsilon_0^{\frac{2N}{p-1}} \right| \left| \int_{\mathbb{R}^N} b\left( \frac{x}{\varepsilon_0} \right) |u_0|^{q+1} dx \right|.
\]

The first term can be treated as above, while the second one obviously goes to zero as \( \varepsilon \rightarrow \varepsilon_0 \).

If \( (\varepsilon, u) \rightarrow (0, u_0) \), we have

\[
(q + 1)|G_2(\varepsilon, u)| \leq \varepsilon^{\frac{2N}{p-1}} \int_{\mathbb{R}^N} b\left( \frac{x}{\varepsilon} \right) |u|^{q+1} dx \leq C \varepsilon^{\frac{2N}{p-1}}.
\]

So also \( G_2 \) is a continuous function, hence \( G \) is continuous and the lemma is proved.
In the next lemma we prove that \((G_2)\) is satisfied.

**Lemma 3.2** Assume \((a_1)\) and \((b_1)\). Then \(G'\) and \(G''\) are continuous.

**Proof.** Let us consider \(G'_1\), and assume \((\varepsilon, u) \to (\varepsilon_0, u_0)\) with \(\varepsilon_0 \neq 0\). We obtain

\[ ||G'_1(\varepsilon, u) - G'_1(\varepsilon_0, u_0)|| = \sup_{||v|| \leq 1} \left\{ ||(G'_1(\varepsilon, u) - G'_1(\varepsilon_0, u_0))v|| \right\} = \]

\[ \sup_{||v|| \leq 1} \left\{ \left| \int_{\mathbb{R}^N} \left( a \left( \frac{x}{\varepsilon} \right) - A \right) |u|^{p-1} u v \, dx - \int_{\mathbb{R}^N} \left( a \left( \frac{x}{\varepsilon_0} \right) - A \right) |u_0|^{p-1} u_0 v \, dx \right| \right\} \leq \]

\[ \sup_{||v|| \leq 1} \left\{ \left| \int_{\mathbb{R}^N} \left( a \left( \frac{x}{\varepsilon} \right) - A \right) |u|^{p-1} u v \, dx - \int_{\mathbb{R}^N} \left( a \left( \frac{x}{\varepsilon_0} \right) - A \right) |u_0|^{p-1} u_0 v \, dx \right| \right\} + \]

\[ \sup_{||v|| \leq 1} \left\{ \left| \int_{\mathbb{R}^N} \left( a \left( \frac{x}{\varepsilon_0} \right) - A \right) |u_0|^{p-1} u_0 v \, dx - \int_{\mathbb{R}^N} \left( a \left( \frac{x}{\varepsilon} \right) - A \right) |u|^{p-1} u v \, dx \right| \right\}. \]

For the first term we can write

\[ \left| \int_{\mathbb{R}^N} \left( a \left( \frac{x}{\varepsilon} \right) - A \right) |u|^{p-1} u v \, dx - \int_{\mathbb{R}^N} \left( a \left( \frac{x}{\varepsilon_0} \right) - A \right) |u_0|^{p-1} u_0 v \, dx \right| \leq \]

\[ \left( \int_{\mathbb{R}^N} a \left( \frac{x}{\varepsilon_0} \right) - a \left( \frac{x}{\varepsilon} \right) \right) \frac{p+1}{p} |u_0|^{p+1} \left( \int_{\mathbb{R}^N} |v|^{p+1} \, dx \right)^{\frac{1}{p+1}} \leq \]

\[ C \left( \int_{\mathbb{R}^N} a \left( \frac{x}{\varepsilon_0} \right) - a \left( \frac{x}{\varepsilon} \right) \right) \frac{p+1}{p} |u_0|^{p+1}, \]

where \(C\) is independent of \(v\), \(||v|| \leq 1\). As above, this term tends to zero, by dominated convergence. For the second term we have

\[ \left| \int_{\mathbb{R}^N} \left( a \left( \frac{x}{\varepsilon_0} \right) - A \right) |u_0|^{p-1} u_0 v \, dx - \int_{\mathbb{R}^N} \left( a \left( \frac{x}{\varepsilon} \right) - A \right) |u|^{p-1} u v \, dx \right| \leq \]

\[ C \int_{\mathbb{R}^N} |u|^{p-1} u - |u_0|^{p-1} u_0 |v| \, dx \leq C \left( \int_{\mathbb{R}^N} |u|^{p-1} u - |u_0|^{p-1} u_0 \right)^{\frac{p+1}{p}} \left( \int_{\mathbb{R}^N} |v|^{p+1} \, dx \right)^{\frac{1}{p+1}} \leq \]

\[ C \left( \int_{\mathbb{R}^N} |u|^{p-1} u - |u_0|^{p-1} u_0 \right)^{\frac{p+1}{p}} \leq \]
Let us now assume \((\varepsilon, u) \to (0, u_0)\). Hence we conclude \(|G'_1(\varepsilon, u) - G'_1(\varepsilon_0, u_0)|\to 0\) as \((\varepsilon, u) \to (\varepsilon_0, u_0), \varepsilon_0 \neq 0\). Let us now assume \((\varepsilon, u) \to (0, u_0)\). By definition, \(G'_1(0, u) = 0\) and

\[
||G'_1(\varepsilon, u)|| = \sup_{||v|| \leq 1} \left\{ \left| \int_{\mathbb{R}^N} \left( a \left( \frac{x}{\varepsilon} \right) - A \right) |u|^{p-1} v \, dx \right| \right\} \leq \\
\sup_{||v|| \leq 1} \left\{ \int_{\mathbb{R}^N} \left( a \left( \frac{x}{\varepsilon} \right) - A \right) |u|^{p^*} \, dx \right\} \left( \int_{\mathbb{R}^N} |u|^{2^*} \, dx \right) \frac{2}{2^*} \left( \int_{\mathbb{R}^N} |v|^{2^*} \, dx \right)^\frac{1}{2^*} \leq \\
C \varepsilon^{2-p} \left( \int_{\mathbb{R}^N} |a(y) - A| |u|^{2^*} \, dy \right) \left( \int_{\mathbb{R}^N} |u|^{2^*} \, dx \right) \frac{2^{2^*} - 1}{2^{2^*}} \left( \int_{\mathbb{R}^N} |v|^{2^*} \, dx \right)^\frac{1}{2^*}
\]

and this term vanishes as \(\varepsilon \to 0\).

In this way we have proved that \(G'_1\) is continuous. Similar arguments work for \(G'_2\). Indeed, if \((\varepsilon, u) \to (\varepsilon_0, u_0)\) with \(\varepsilon_0 \neq 0\), we obtain

\[
||G'_2(\varepsilon, u) - G'_2(\varepsilon_0, u_0)|| = \sup_{||v|| \leq 1} \left| \left( \int_{\mathbb{R}^N} \left( (G'_2(\varepsilon, u) - G'_2(\varepsilon_0, u_0)) \right) v \right) \right| = \\
\sup_{||v|| \leq 1} \left\{ \int_{\mathbb{R}^N} b \left( \frac{x}{\varepsilon} \right) |u|^{p-1} v \, dx - \varepsilon_0^{2-p} \int_{\mathbb{R}^N} b \left( \frac{x}{\varepsilon_0} \right) |u_0|^{p-1} v \, dx \right\} \leq \\
\sup_{||v|| \leq 1} \left\{ \int_{\mathbb{R}^N} b \left( \frac{x}{\varepsilon} \right) |u|^{p-1} v \, dx - \int_{\mathbb{R}^N} b \left( \frac{x}{\varepsilon_0} \right) |u_0|^{p-1} v \, dx \right\} + \\
\varepsilon^{2-p} \sup_{||v|| \leq 1} \left\{ \int_{\mathbb{R}^N} b \left( \frac{x}{\varepsilon} \right) |u|^{p-1} v \, dx - \int_{\mathbb{R}^N} b \left( \frac{x}{\varepsilon_0} \right) |u_0|^{p-1} v \, dx \right\} \leq \\
\varepsilon_0^{2-p} \int_{\mathbb{R}^N} b \left( \frac{x}{\varepsilon_0} \right) |u_0|^{p-1} v \, dx \, dx.
\]

The first term can be treated exactly as before, the second term obviously vanishes as \(\varepsilon \to \varepsilon_0\). Let us now assume \((\varepsilon, u) \to (0, u_0)\). We obtain

\[
||G'_2(\varepsilon, u)|| = \sup_{||v|| \leq 1} \left| \left( (G'_2(\varepsilon, u)) \right) v \right| \leq \\
\sup_{||v|| \leq 1} \varepsilon^{2-p} \int_{\mathbb{R}^N} b \left( \frac{x}{\varepsilon} \right) |u|^{p} v \, dx \leq C \varepsilon^{2-p}.
\]

Now we have proved that \(G'\) is continuous. The argument to prove the continuity of \(G''\) is almost the same and we leave it to the reader. ■

Let us now verify that \((G_3)\) is satisfied.
Lemma 3.3 Let us assume \((a_1), (b_1)\) and \((b_2)\). Let us define, for \(\theta \in \mathbb{R}^N\),
\[
\Gamma(\theta) = -\frac{1}{p+1} z_0^{p+1}(\theta) \int_{\mathbb{R}^N} (a(y) - A) dy.
\]
(5)

Then
\[
\lim_{\varepsilon \to 0} \frac{G(\varepsilon, z_0)}{\varepsilon^N} = \Gamma(\theta)
\]
and
\[
G'(\varepsilon, z_0) = O(\varepsilon^{\frac{N}{2} + 1}).
\]
(7)

PROOF. As above we will study separately \(G_1\) and \(G_2\). By the change of variables \(y = \frac{x}{\varepsilon}\) we have
\[
G_1(\varepsilon, z_0) = -\frac{1}{p+1} \int_{\mathbb{R}^N} \left( a \left( \frac{x}{\varepsilon} \right) - A \right) z_0^{p+1}(x + \theta) dx = \\
-\frac{\varepsilon^N}{p+1} \int_{\mathbb{R}^N} (a(y) - A) z_0^{p+1}(\varepsilon y + \theta) dy.
\]
Since \(a - A \in L^1(\mathbb{R}^N)\) and \(z_0\) is bounded and continuous, by dominated convergence we get
\[
\lim_{\varepsilon \to 0} \frac{G_1(\varepsilon, z_0)}{\varepsilon^N} = \Gamma(\theta)
\]
(8)

Hence, to prove (6) we have to show that
\[
\lim_{\varepsilon \to 0} \frac{G_2(\varepsilon, z_0)}{\varepsilon^N} = 0.
\]
(9)

We distinguish two cases. Assume first \(N < 2\frac{q-p}{p-1}\). In this case
\[
\varepsilon^{-N} G_2(\varepsilon, z_0) = -\frac{1}{q+1} \varepsilon^{\frac{q-p}{p-1} - N} \int_{\mathbb{R}^N} b \left( \frac{x}{\varepsilon} \right) z_0^{q+1}(x + \theta) dx,
\]
and this expression of course vanishes as \(\varepsilon \to 0\), because the integral is bounded. Hence, let us assume \(N \geq 2\frac{q-p}{p-1}\). We obtain
\[
\varepsilon^{-N} G_2(\varepsilon, z_0) = -\frac{1}{q+1} \varepsilon^{\frac{q-p}{p-1} - N} \int_{\mathbb{R}^N} b \left( \frac{x}{\varepsilon} \right) z_0^{q+1}(x + \theta) dx \leq \\
C \varepsilon^{\frac{q-p}{p-1} - N} \left( \int_{\mathbb{R}^N} \left| b \left( \frac{x}{\varepsilon} \right) \right|^\beta dx \right)^{\frac{1}{\beta}} \left( \int_{\mathbb{R}^N} z_0^{\frac{q+1}{\beta}}(x + \theta) dx \right)^{\frac{\beta-1}{\beta}} \leq \\
C \varepsilon^{\frac{q-p}{p-1} - N + \frac{N}{\beta}} \left( \int_{\mathbb{R}^N} |b(y)|^\beta dy \right)^{\frac{1}{\beta}},
\]
where \(\beta\) is given by \((b_2)\). This term goes to zero since \(2\frac{q-p}{p-1} - N + \frac{N}{\beta} > 0\). We have now proved (9), hence, by (8), (6) is also proved.
Let us go to the proof of (7). Again we will study separately $G_1'$ and $G_2'$. We have

$$||G_1'(\varepsilon, z_0)|| = \sup_{||v|| \leq 1} |(G_1'(\varepsilon, z_0)v)| = \sup_{||v|| \leq 1} \left| \int_{\mathbb{R}^N} \left( a\left(\frac{x}{\varepsilon}\right) - A \right) z_0^p v dx \right| \leq$$

$$\sup_{||v|| \leq 1} \left\{ \left( \int_{\mathbb{R}^N} \left| a\left(\frac{x}{\varepsilon}\right) - A \right| \frac{2N}{2N+4} \frac{x}{\varepsilon} z_0^p dx \right)^{\frac{N+2}{2N}} \left( \int_{\mathbb{R}^N} |v|^2 dx \right)^{\frac{N}{2N+2}} \right\} \leq$$

$$C\varepsilon^{\frac{N}{2}+1} \left( \int_{\mathbb{R}^N} |a(y) - A| \frac{2N}{2N+4} z_0^p dy \right)^{\frac{N+2}{N}},$$

hence

$$G_1'(\varepsilon, z_0) = O(\varepsilon^{\frac{N}{2}+1}).$$

(10)

As to $G_2'(\varepsilon, z_0)$ we obtain, arguing ad before,

$$||G_2'(\varepsilon, z_0)|| \leq C\varepsilon^{\frac{2N-p}{2N+4}} \left( \int_{\mathbb{R}^N} \left| b\left(\frac{x}{\varepsilon}\right) \right| \frac{2N}{2N+4} z_0^q \frac{2N}{2N+4} dx \right)^{\frac{N+2}{2N}}.$$

By the usual change of variables $y = \frac{x}{\varepsilon}$ and using (b2), we obtain

$$G_2'(\varepsilon, z_0) = o(\varepsilon^{\frac{N}{2}+1}).$$

From this and (10) we readily get (7).

Remark 3.4 Notice that in the abstract results of section 2 the function $\Gamma$ is defined on the manifold $\mathcal{Z}$ of critical point of the unperturbed functional $f_0$. In the present case this manifold is diffeomorphic to $\mathbb{R}^N$, so we consider $\Gamma$ as a function defined on $\mathbb{R}^N$. ■

We now conclude the proof of theorem 1.1. We know that $z_0$ has a strict (global) maximum in $x = 0$, so that $\Gamma$ has a (strict) global maximum or minimum (depending on the sign of $\int (a(y) - A) dy$) at $\theta = 0$. We can then apply theorem 2.1, setting $z^* = 0$ and, for example, $\delta = 1$. We obtain a family $\{(\varepsilon, u_\varepsilon)\} \subset \mathbb{R} \times H^1(\mathbb{R}^N)$ such that $u_\varepsilon$ is a critical point of $f_\varepsilon$, hence a solution of (2), and $\{u_\varepsilon\}$ is a bounded set in $H^1(\mathbb{R}^N)$. To be precise, we have

$$u_\varepsilon(x) = z_0(x + \theta_\varepsilon) + w(\varepsilon, \theta_\varepsilon)(x)$$

where $|\theta_\varepsilon| \leq 1$ and $w(\varepsilon, \theta_\varepsilon) \to 0$ as $\varepsilon \to 0$. As $p < 1 + 4/N$, we obtain a family $\{\lambda, \psi_\lambda\}$ of solutions of (1) such that $\psi_\lambda \to 0$ in $H^1(\mathbb{R}^N)$ as $\lambda \to 0$.

Remark 3.5 The hypothesis (a2) is not used to prove the properties ($G_0 - G_3$). It is used to apply theorem 2.1, and in particular to say that there are $z^*, \delta$ such that (3) holds. If $\int (a(y) - A) dy = 0$ then $\Gamma$, as defined in (5), is identically zero. It has critical points, but of course they are not stable under perturbations, so we can not conclude that they give rise to critical points of $f_\varepsilon$. ■
4 Second bifurcation result

In this section we prove theorem 1.2. As before we have to prove that $G, G'$ and $G''$ are continuous functions. Notice that in the proof of $(G_0)$ and $(G_2)$ we will consider just the function $G_1$, because the arguments of lemmas 3.1 and 3.2 for the function $G_2$ use only hypothesis $(b_1)$ which is unchanged. On the contrary in the proof of $(G_3)$ we will study both $G_1$ and $G_2$. As above we assume $N \geq 3$ and $1 < p < q \leq \frac{N+2}{N-2}$. Let us see first that $(G_0)$ is satisfied.

**Lemma 4.1** Assume $(a_3)$. Then $G_1$ is continuous.

**Proof.** In the case $(\varepsilon, u) \rightarrow (\varepsilon_0, u_0)$, $\varepsilon_0 \neq 0$ we can repeat word by word the arguments of lemma 3.1, because there we used only the fact that $a$ is continuous and bounded, which is still true in the present case. Hence, let us suppose $(\varepsilon, u) \rightarrow (0, u_0)$. Let us fix $\eta > 0$ and, by $(a_3)$, $M_\eta > 0$ such that $|a(y) - A| < \eta$ if $|y| > M_\eta$. We obtain

\[ (p+1)|G_1(\varepsilon, u)| = \left| \int_{\mathbb{R}^N} \left( a \left( \frac{x}{\varepsilon} \right) - A \right) |u|^{p+1} \, dx \right| \leq \]

\[ \int_{|x| \leq M_\eta} \left| a \left( \frac{x}{\varepsilon} \right) - A \right| |u|^{p+1} \, dx + \int_{|x| > M_\eta} \left| a \left( \frac{x}{\varepsilon} \right) - A \right| |u|^{p+1} \, dx \leq \]

\[ C \int_{|x| \leq M_\eta} |u|^{p+1} \, dx + \eta \int_{\mathbb{R}^N} |u|^{p+1} \, dx \leq \]

\[ C \int_{\mathbb{R}^N} \left( |u|^{p+1} - |u_0|^{p+1} \right) \, dx + C \int_{|x| \leq M_\eta} \left| u_0 \right|^{p+1} \, dx + \eta \int_{\mathbb{R}^N} |u|^{p+1} \, dx. \]

As $\varepsilon \rightarrow 0$ and $u \rightarrow u_0$ the first two terms vanish, so

\[ \limsup_{(\varepsilon, u) \rightarrow (0, u_0)} |G_1(\varepsilon, u)| \leq C \eta. \]

This is true for any $\eta > 0$, so we conclude $G_1(\varepsilon, u) \rightarrow 0$ when $(\varepsilon, u) \rightarrow (0, u_0)$, and the lemma is proved. ■

This proves that $(G_1)$ holds. We want now to show that also $(G_2)$ holds.

**Lemma 4.2** Assume $(a_3)$. Then $G'_1$ and $G''_1$ are continuous.

**Proof.** We will show the continuity of $G'_1$, the argument for $G''_1$ is similar. Assume $(\varepsilon, u) \rightarrow (\varepsilon_0, u_0)$ with $\varepsilon_0 \neq 0$. In this case one can argue exactly as in lemma 3.2 to obtain $||G'_1(\varepsilon, u) - G'_1(\varepsilon_0, u_0)|| \rightarrow 0$. Hence, let us now assume $(\varepsilon, u) \rightarrow (0, u_0)$. For each $\eta > 0$ let us fix $M_\eta > 0$ as in the previous lemma. We obtain

\[ ||G'_1(\varepsilon, u)|| = \sup_{||v|| \leq 1} \left| \left( a \left( \frac{x}{\varepsilon} \right) - A \right) |u|^{p-1} uv \, dx \right| \leq \]

\[ \sup_{||v|| \leq 1} \left\{ \int_{|x| \leq M_\eta} \left| a \left( \frac{x}{\varepsilon} \right) - A \right| |u|^{p-1} v \, dx \right\} + \sup_{||v|| \leq 1} \left\{ \int_{|x| > M_\eta} \left| a \left( \frac{x}{\varepsilon} \right) - A \right| |u|^{p-1} v \, dx \right\} \leq \]
Arguing as before we then obtain

\[
\limsup_{(\varepsilon,u) \to (0,u_0)} ||G_1'(\varepsilon,u)|| \leq C\eta
\]

for all \( \eta > 0 \), hence \( \lim_{(\varepsilon,u) \to (0,u_0)} ||G_1'(\varepsilon,u)|| = 0 \) and the lemma is proved. \( \Box \)

In the next lemma we prove that \((G_3)\) is satisfied.

**Lemma 4.3** Assume \((a_3)\), \((b_1)\) and \((b_3)\). Define

\[
\Gamma(\theta) = -\frac{L}{p+1} \int_{\mathbb{R}^N} |x|^{-\gamma} z_0^{p+1}(x + \theta) dx.
\]

Then, for all \( \theta \in \mathbb{R}^N \), we have

\[
\lim_{\varepsilon \to 0} \frac{G(\varepsilon,z_\theta)}{\varepsilon^\gamma} = \Gamma(\theta)
\]

and

\[
G'(\varepsilon,z_\theta) = o(\varepsilon^{\gamma/2}).
\]

**Proof.** As usual we will study separately \( G_1 \) and \( G_2 \). We have

\[
G_1(\varepsilon,z_\theta) = -\frac{1}{p+1} \int_{\mathbb{R}^N} \left( a \left( \frac{x}{\varepsilon} \right) - A \right) z_0^{p+1}(x + \theta) dx =
\]

\[
-\frac{\varepsilon^\gamma}{p+1} \int_{\mathbb{R}^N} \left( a \left( \frac{x}{\varepsilon} \right) - A \right) \frac{|x|^\gamma}{\varepsilon^\gamma} \frac{z_0^{p+1}(x + \theta)}{|x|^\gamma} dx.
\]

We know that, for all \( x \neq 0 \),

\[
\left( a \left( \frac{x}{\varepsilon} \right) - A \right) \frac{|x|^\gamma}{\varepsilon^\gamma} \to L \quad \text{as} \quad \varepsilon \to 0,
\]

13
while, since $\gamma < N$ and $z_0$ has an exponential decay at infinity, $|x|^{-\gamma} z_0(x) \in L^1(\mathbb{R}^N)$. Hence by dominated convergence we get

$$\lim_{\varepsilon \to 0} \frac{G_1(\varepsilon, z_0)}{\varepsilon^\gamma} = \Gamma(\theta). \quad (13)$$

To study $\frac{1}{\varepsilon} G_2(\varepsilon, z_0)$ we can repeat the argument used in Lemma 3.3, distinguishing the cases $\gamma < \frac{2N}{p-1}$ and $\gamma \geq \frac{2N}{p-1}$, and using $(b_3)$ instead of $(b_2)$. We obtain

$$\lim_{\varepsilon \to 0} \frac{G_2(\varepsilon, z_0)}{\varepsilon^\gamma} = 0, \quad (14)$$

and (11) follows from (13) and (14).

Let us now prove (12). We study first $G'_1$ then $G'_2$. With the same arguments of lemma 3.3 we get

$$||G'_1(\varepsilon, z_0)|| \leq C \left( \int_{\mathbb{R}^N} |a(\frac{x}{\varepsilon}) - A| \frac{2N}{N+2} \frac{p^{\frac{2N}{N+2}}(x)}{|x|^\gamma \frac{2N}{N+2}} \frac{\varepsilon^{N+2}}{\varepsilon^\gamma} \right)^{\frac{N+2}{N+4}}.$$ 

We have to distinguish three cases.

**First case: $\gamma < \frac{N+2}{2}$**.

We obtain

$$\left( \int_{\mathbb{R}^N} |a(\frac{x}{\varepsilon}) - A| \frac{2N}{N+2} \frac{p^{\frac{2N}{N+2}}(x)}{|x|^\gamma \frac{2N}{N+2}} \frac{\varepsilon^{N+2}}{\varepsilon^\gamma} \right)^{\frac{N+2}{N+4}} =$$

$$\varepsilon^\gamma \left( \int_{\mathbb{R}^N} |a(\frac{x}{\varepsilon}) - A| \frac{2N}{N+2} \frac{p^{\frac{2N}{N+2}}(x)}{|x|^\gamma \frac{2N}{N+2}} \frac{\varepsilon^{N+2}}{\varepsilon^\gamma} \right)^{\frac{N+2}{N+4}} \leq C \varepsilon^\gamma,$$

because $\gamma \frac{2N}{N+2} < N$, hence $\frac{p^{\frac{2N}{N+2}}(x)}{|x|^\gamma \frac{2N}{N+2}} \in L^1(\mathbb{R}^N)$. Therefore in this case

$$||G'_1(\varepsilon, z_0)|| = O(\varepsilon^\gamma).$$

**Second case: $\gamma > \frac{N+2}{2}$**.

In this case the function $|a(x) - A| \frac{p^{\frac{2N}{N+2}}}{|x|^\gamma \frac{2N}{N+2}}$ is in $L^1(\mathbb{R}^N)$, because it is bounded and at infinity it is asymptotic to $|x|^{-\gamma \frac{2N}{N+2}}$, and $\gamma \frac{2N}{N+2} > N$. Therefore, by the usual change of variables $y = \frac{x}{\varepsilon}$, we obtain

$$\left( \int_{\mathbb{R}^N} |a(\frac{x}{\varepsilon}) - A| \frac{2N}{N+2} \frac{p^{\frac{2N}{N+2}}(x)}{|x|^\gamma \frac{2N}{N+2}} \frac{\varepsilon^{N+2}}{\varepsilon^\gamma} \right)^{\frac{N+2}{N+4}} \leq$$
\[ C \varepsilon^{\frac{N+2}{2}} \left( \int_{\mathbb{R}^N} |a(y) - A|^{\frac{2N}{N+2}} \, dy \right)^{\frac{N+2}{2N}} \leq C \varepsilon^{\frac{N+2}{2}}. \]

Hence, recalling that \( \gamma < N \), we obtain
\[ ||G'_1(\varepsilon, z_0)|| = O(\varepsilon^{\frac{N+1}{2}}) = o(\varepsilon^{\gamma/2}). \]

**Third case: \( \gamma = \frac{N+2}{2} \).**

In this case we apply Hölder inequality using as conjugate exponent \( s \), instead of \( \frac{2N}{N+2} \) and \( \frac{2N}{N-2} \), any \( s, s' \) such that \( s \) is smaller than \( \frac{2N}{N+2} \) but near to it, so that \( s' \) is bigger than \( \frac{2N}{N-2} \) but near to it. In this way we obtain
\[ ||G'_1(\varepsilon, z_0)|| = \sup_{|v| \leq 1} ||( G'_1(\varepsilon, z_0) )v || = \]
\[ \sup_{|v| \leq 1} \left| \int_{\mathbb{R}^N} \left( a \left( \frac{x}{\varepsilon} \right) - A \right) z_0^s v dx \right| \leq \sup_{|v| \leq 1} \left( \int_{\mathbb{R}^N} \left| a \left( \frac{x}{\varepsilon} \right) - A \right|^{\frac{s'}{s}} z_0^s v dx \right)^{\frac{1}{s}} \left( \int_{\mathbb{R}^N} |v|^s dx \right)^{\frac{1}{s}} \leq C \left( \int_{\mathbb{R}^N} \left| a \left( \frac{x}{\varepsilon} \right) - A \right|^{\frac{s'}{s'}} dx \right)^{\frac{1}{s'}}. \]

It is \( s' \gamma > N \), so, as before, \( |a - A|^{s'} \in L^1(\mathbb{R}^N) \). We can then apply the usual change of variables to obtain
\[ ||G'_1(\varepsilon, z_0)|| \leq C \varepsilon^{N/s'}. \]

We have that \( s' \) is near \( \frac{2N}{N+2} = \frac{N}{\gamma} \), so \( N/s' \) is near \( \gamma \). Hence we obtain
\[ ||G'_1(\varepsilon, z_0)|| = O(\varepsilon^{N/s'}) = o(\varepsilon^{\gamma/2}). \]

We have concluded the study of \( G'_1(\varepsilon, z_0) \). As to \( G'_2(\varepsilon, z_0) \), the same argument of lemma 3.3 gives
\[ ||G'_2(\varepsilon, z_0)|| = o(\varepsilon^{N/s'}) = o(\varepsilon^{\gamma/2}). \]

In this way the lemma is completely proved. \( \blacksquare \)

We want now to complete the proof of Theorem 1.2. As in the previous section, we have only to prove that the function \( \Gamma \) satisfies the hypotheses of theorem 2.1. Let us prove that
\[ \text{there is } R > 0 \text{ such that either } \min_{|\theta|=R} \Gamma(\theta) > \Gamma(0) \text{ or } \max_{|\theta|=R} \Gamma(\theta) < \Gamma(0). \] (15)

To prove (15) we first notice that \( \Gamma \) is continuous and \( \Gamma(\theta) \) is either positive on all \( \mathbb{R}^N \) or negative on all \( \mathbb{R}^N \). Then we claim that
\[ \lim_{|\theta| \to +\infty} \Gamma(\theta) = 0. \quad (16) \]

To prove (16), let us write

\[ |\Gamma(\theta)| = C \int_{\mathbb{R}^N} |x|^{-\gamma} z^{p+1}_0(x+\theta)dx = C \int_{|x| \leq 1} |x|^{-\gamma} z^{p+1}_0(x+\theta)dx + C \int_{|x| > 1} |x|^{-\gamma} z^{p+1}_0(x+\theta)dx. \]

It is \( |x|^{-\gamma} \in L^1(B_1) \), while \( z^{p+1}_0(x+\theta) \to 0 \) as \( |\theta| \to +\infty \), for all \( x \), so by dominated convergence the first integral vanishes as \( |\theta| \to +\infty \). For the second integral we write

\[ \int_{|x| > 1} |x|^{-\gamma} z^{p+1}_0(x+\theta)dx = \int_{|y-\theta| > 1} |y-\theta|^{-\gamma} z^{p+1}_0(y)dy = \int_{\mathbb{R}^N} \chi_\theta(y)|y-\theta|^{-\gamma} z^{p+1}_0(y)dy, \]

where \( \chi_\theta \) is the characteristic function of the set \( \{ y \in \mathbb{R}^N \mid |y-\theta| > 1 \} \). It is trivial to see that

\[ \chi_\theta(y)|y-\theta|^{-\gamma} \leq 1 \]

for all \( y, \theta \in \mathbb{R}^N (y \neq \theta) \), and that \( \chi_\theta(y)|y-\theta|^{-\gamma} \to 0 \) as \( |\theta| \to +\infty \). Again by dominated convergence we obtain that also the second integral vanishes when \( |\theta| \to +\infty \). So (16) is proved, hence also (15). We can apply theorem 2.1 and argue as in the previous section.

5 Continuous branches of solutions.

In this section we prove that in some cases the families of solutions bifurcating from \((0, 0)\), that we have found in the previous sections, form a curve. We first will prove some abstract results (following the frame of section 2), then we will apply these result to problems (1) and (2). So let us come back to the abstract frame of section 2. To make easier the passage from the abstract frame to the applications, we will denote by \( z_\theta, \theta \in \mathbb{R}^d \), the elements of \( Z \). Notice that our arguments are local in nature and we will work in the neighborhood of a fixed point, so we can assume, without loss of generality, that the manifold \( Z \) is given by a unique map \( \theta \to z_\theta \). We will indicate with \( \partial_i z, \partial_{ij} z \) the derivatives of \( z_\theta \) with respect to the parameter \( \theta \), that is

\[ \partial_i z = \frac{\partial z}{\partial \theta_i}(\theta), \quad \partial_{ij} z = \frac{\partial^2 z}{\partial \theta_i \partial \theta_j}(\theta). \]

About the manifold \( Z \) of critical points we will also assume the following hypothesis, which is satisfied in our applications.

\((H)\) \( \partial_i z | \partial_j z \) = 0 if \( i \neq j \), \( ||\partial_i z|| = c \) (independent of \( i \) and \( \theta \)), \( \partial_{ij} z | \partial_l z \) = 0 for all \( i, j, l = 1, \ldots, d \).

About the functionals \( F, G \) we will assume two different types of hypotheses. Recall that \( \alpha, \Gamma \) are those given in hypothesis \((G_3)\).
\( (F_4) \) \( F \) is of class \( C^4; \)

\( (G_4) \) \( G \) is of class \( C^4 \) with respect to \( u \) and the map \( (\varepsilon, u) \to G'''(\varepsilon, u) \) is continuous;

\( (G_5) \) \( \Gamma \) is \( C^2 \) and, if \( \theta_\varepsilon \) is a family such that \( \theta_\varepsilon \to \theta \) as \( \varepsilon \to 0 \), then

\[
\lim_{\varepsilon \to 0} \varepsilon^{-\alpha/2} G'''(\varepsilon, \theta_\varepsilon) = 0, \quad \lim_{\varepsilon \to 0} \varepsilon^{-\alpha} (G'''(\varepsilon, \theta_\varepsilon) + \varepsilon^{-\alpha} (G''(\varepsilon, \theta_\varepsilon) + \varepsilon^{-\alpha} (G'(\varepsilon, \theta_\varepsilon) + \varepsilon^{-\alpha} (G(\varepsilon, \theta_\varepsilon)))) = 0.
\]

\( \lim_{\varepsilon \to 0} \varepsilon^{-\alpha/2} G'''(\varepsilon, \theta_\varepsilon) = 0. \)

\( (F_4)' \) \( F \) is of class \( C^3; \)

\( (G_4)' \) \( G \) is of class \( C^3 \) with respect to \( u \) and the map \( (\varepsilon, u) \to G'''(\varepsilon, u) \) is continuous.

\( (G_5)' \) \( G'(\varepsilon, u) = O(\varepsilon^\alpha) \) for all \( u \in E \), \( \Gamma \) is \( C^2 \) and, if \( \theta_\varepsilon \) is a family such that \( \theta_\varepsilon \to \theta \) as \( \varepsilon \to 0 \), then

\[
\lim_{\varepsilon \to 0} \varepsilon^{-\alpha/2} G'''(\varepsilon, \theta_\varepsilon) = 0, \quad \lim_{\varepsilon \to 0} \varepsilon^{-\alpha} (G'''(\varepsilon, \theta_\varepsilon) + \varepsilon^{-\alpha} (G''(\varepsilon, \theta_\varepsilon) + \varepsilon^{-\alpha} (G'(\varepsilon, \theta_\varepsilon) + \varepsilon^{-\alpha} (G(\varepsilon, \theta_\varepsilon)))) = 0.
\]

We can now prove two abstract theorems.

**Theorem 5.1** Assume (H), \((F_0 - F_4)\) and \((G_0 - G_5)\). For a given \( \theta \in \mathbb{R}^d \), and for any small \( \varepsilon \)‘s, let us suppose that there is a critical point \( u_\varepsilon \in Z_\varepsilon \) of \( f_\varepsilon \), such that \( u_\varepsilon = z_{\theta_\varepsilon} + w(\varepsilon, \theta_\varepsilon) \) and \( \theta_\varepsilon \to \theta \) as \( \varepsilon \to 0 \). Assume that \( z_{\theta_\varepsilon} = \lim_{\varepsilon \to 0} z_{\theta_\varepsilon} \) is nondegenerate for the restriction of \( f_0 \) to \( (T_{z_{\theta_\varepsilon}}Z)^{+} \), with Morse index equal to \( m_0 \), and that the hessian matrix \( D^2 \Gamma(\theta) \) is positive or negative definite.

Then \( u_\varepsilon \), for small \( \varepsilon \)‘s, is a nondegenerate critical point for \( f_\varepsilon \) with Morse index equal to \( m_0 \) if \( D^2 \Gamma(\theta) \) is positive, to \( m_0 + d \) if \( D^2 \Gamma(\theta) \) is negative. A consequence, the critical points of \( f_\varepsilon \) form a continuous curve.

**Proof.** Let us write

\[
E = E^{+} \oplus E^{0} \oplus E^{-}
\]

where \( E^{0} = T_{z_{\theta}}Z \), \( \dim(E^{-}) = m_0 \) and there exists \( \delta > 0 \) such that

\[
\begin{align*}
D^2 f_0(z_\theta) [v, v] &> \delta ||v||^2 \quad \forall v \in E^+, \\
D^2 f_0(z_\theta) [v, v] &< -\delta ||v||^2 \quad \forall v \in E^-.
\end{align*}
\]

From the hypothesis \( f_0(z_\eta) = 0 \) for all \( \eta \in \mathbb{R}^N \) it is easy to deduce
By orthogonality of the decomposition (17), we have
\[ D^2 f_0(z_0)[\partial_i z_0, \partial_j z_0] = 0. \]

Let us define
\[ \varphi^0_i = \frac{1}{||\partial_i z_0||} \partial_i z_0. \]

The set \( \{ \varphi^0_i \}_{i=1,\ldots,d} \) is an orthonormal base for \( E^0 \). Let \( \lambda_1, \ldots, \lambda_d \) be the eigenvalues of the symmetric matrix \( D^2 f_0(z_0) \) on \( E^- \). Of course \( \lambda_i < 0 \) for all \( i \), and let \( \lambda_0 = \max \lambda_i < 0 \). Let \( \{ t^0_i \}_{i=1,\ldots,m_0} \) be an orthonormal base for \( E^- \) such that \( D^2 f_0[t^0_i, t^0_j] = 0 \) if \( i \neq j \), \( D^2 f_0[t^0_i, t^0_i] = \lambda_i \).

By orthogonality of the decomposition (17), we have \( (\varphi^0_i | t^0_j) = 0 \) for all \( i, j \). Define
\[ \varphi^\varepsilon_i = \frac{1}{||\partial_i z_0\varepsilon||} \partial_i z_0. \]

The set \( \{ \varphi^\varepsilon_i \}_{i=1,\ldots,d} \) is an orthonormal base for the tangent space \( T_{z_0} Z \), space that we denote \( E^\varepsilon_0 \). Notice that \( \varphi^\varepsilon_i \to \varphi^0_i \) as \( \varepsilon \to 0 \).

For \( i = 1, \ldots, m_0 \) we want to find \( \tau^\varepsilon_i \) such that, setting \( t^\varepsilon_i = t^0_i + \tau^\varepsilon_i \), we obtain, for all \( i, j \),
\[ (t^\varepsilon_i | \varphi^\varepsilon_j) = 0 \tag{18} \]

That is, we want
\[ 0 = (t^\varepsilon_i | \varphi^\varepsilon_j) = (t^0_i + \tau^\varepsilon_i | \varphi^\varepsilon_j + (\varphi^\varepsilon_j - \varphi^0_j)) = (t^0_i | \varphi^\varepsilon_j - \varphi^0_j) + (\tau^\varepsilon_i | \varphi^\varepsilon_j), \]
hence
\[ (\tau^\varepsilon_i | \varphi^\varepsilon_j) = -(t^0_i | \varphi^\varepsilon_j - \varphi^0_j). \]

So, we define
\[ \tau^\varepsilon_i = \sum_{j=1}^d -(t^0_i | \varphi^\varepsilon_j - \varphi^0_j)\varphi^\varepsilon_j, \]
and (18) holds. Notice that \( \tau^\varepsilon_i \to 0 \) as \( \varepsilon \to 0 \), so that
\[ t^\varepsilon_i \to t^0_i \]
as \( \varepsilon \to 0 \). As \( \{ t^0_i \}_i \) is an orthonormal base, the vectors \( \{ t^\varepsilon_i \}_i \) are linearly independent, for small \( \varepsilon \)'s.

Let us define \( E^-_0 \) the \( m_0 \)-dimensional space spanned by \( \{ t^\varepsilon_i \}_i \). For \( v \in E^-_0 \), \( ||v|| = 1 \), we have \( v = \sum_{k=1}^{m_0} \beta_k t^\varepsilon_k \) and we can write
\[ D^2 f_\varepsilon(u_\varepsilon)[v, v] = \sum_{l,k=1}^{m_0} \beta_l \beta_k D^2 f_\varepsilon(u_\varepsilon)[t^\varepsilon_l, t^\varepsilon_k + \tau^\varepsilon_l, t^\varepsilon_k] = \sum_{l,k=1}^{m_0} \beta_l \beta_k D^2 f_\varepsilon(u_\varepsilon)[t^0_l, t^0_k] + o(1), \]
where \( o(1) \) vanishes as \( \varepsilon \to 0 \), uniformly in \( v \). By hypotheses (F_0), (G_1), (G_2), we obtain
\[ D^2 f_\varepsilon(u_\varepsilon)[t^\varepsilon_l, t^\varepsilon_k] \to D^2 f_0(z_0)[t^0_l, t^0_k]. \]
As \( t^\varepsilon_k \to t^0_k \) for \( \varepsilon \to 0 \), and \( \{ t^0_k \}_k \) is orthonormal, it is easy to see that, for small \( \varepsilon \)'s, \( ||v|| = 1 \) implies \( \sum_{k=1}^{m_0} \beta_k^2 \geq \frac{1}{2} \). Hence
\[ D^2 f_\varepsilon(u_\varepsilon)[v, v] = \sum_{k=1}^{m_0} \lambda_k \beta_k^2 + o(1) \leq \frac{\lambda_0}{2} + o(1), \]

where \( o(1) \to 0 \) as \( \varepsilon \to 0 \), uniformly in \( v \) if \( ||v|| = 1 \). Hence, for small \( \varepsilon \), \( D^2 f_\varepsilon(u_\varepsilon) \) is negative definite in \( E^-_\varepsilon \). We now define

\[ E^+_\varepsilon = (E^0_\varepsilon \oplus E^-_\varepsilon)^\perp, \]

so that

\[ E = E^+_\varepsilon \oplus E^0_\varepsilon \oplus E^-_\varepsilon. \]

We want now to prove that \( D^2 f_\varepsilon(u_\varepsilon) \) is positive definite on \( E^+_\varepsilon \), for small \( \varepsilon \). Let \( P^+ \) be the orthogonal projection of \( E \) to \( E^+ \).

We claim that there are \( \delta_0 > 0 \) and \( \varepsilon_0 > 0 \) such that for all \( \varepsilon < \varepsilon_0 \) and all \( v \in E^+_\varepsilon, ||v|| = 1 \), it holds

\[ D^2 f_\varepsilon(u_\varepsilon)[v, v] > \delta_0. \]

We argue by contradiction. If the claim is not true, then there are sequences \( \{\varepsilon_k\}, \{v_k\} \subset E^+_\varepsilon \), with \( ||v_k|| = 1 \) and \( \varepsilon_k \to 0 \) as \( k \to \infty \), such that

\[ \frac{1}{k} > D^2 f_{\varepsilon_k}(u_{\varepsilon_k})[v_k, v_k] = D^2 f_{\varepsilon_k}(u_{\varepsilon_k})[P^+ v_k, P^+ v_k] + 2D^2 f_{\varepsilon_k}(u_{\varepsilon_k})[P^+ v_k, v_k - P^+ v_k] + D^2 f_{\varepsilon_k}(u_{\varepsilon_k})[v_k - P^+ v_k, v_k - P^+ v_k]. \]

(19)

We recall that

\[ v_k - P^+ v_k = \sum_{i=1}^{m_0} (v_k, t_i^0) t_i^0 + \sum_{i=1}^{d} (v_k, \varphi_i^0) \varphi_i^0, \]

and that, since \( v_k \in E^+_{\varepsilon_k}, (v_k, t_i^0) = 0 \) and \( (v_k, \varphi_i^0) = 0 \). Hence we have

\[ (v_k, t_i^0) = (v_k, t_i^0 - t_i^{\varepsilon_k}) \to 0 \]

as \( k \to \infty \), because \( t_i^{\varepsilon_k} \to t_i^0 \) and \( \{v_k\} \) is bounded. In the same way we get

\[ (v_k, \varphi_i^0) \to 0, \]

hence

\[ v_k - P^+ v_k \to 0. \]

(20)

Now (19) becomes

\[ \frac{1}{k} > D^2 f_{\varepsilon_k}(u_{\varepsilon_k})[P^+ v_k, P^+ v_k] + o(1) = D^2 f_0(z_0)[P^+ v_k, P^+ v_k] + (D^2 f_{\varepsilon_k}(u_{\varepsilon_k}) - D^2 f_0(z_0))[P^+ v_k, P^+ v_k] + o(1). \]

(21)
Thanks to the continuity hypotheses we have
\[(D^2 f_{s_k}(u_{x_k}) - D^2 f_0(z_0))[P^+ v_k, P^+ v_k] = o(1),\]
while
\[D^2 f_0(z_0)[P^+ v_k, P^+ v_k] > \delta||P^+ v_k||^2,\]
because \(P^+ v_k \in E^+\). By (20) we also obtain
\[||P^+ v_k|| \to 1.\]
Hence (21) gives
\[\frac{1}{k} > \delta + o(1),\]
a contradiction. So the claim is proved.
Up to now we have shown that, for small \(\varepsilon\), \(D^2 f_{\varepsilon}(u_{\varepsilon})\) is negative definite on \(E^{-}_{\varepsilon}\) and positive definite on \(E^{+}_{\varepsilon}\). We want now to study the behavior of \(D^2 f_{\varepsilon}(u_{\varepsilon})\) on \(E^0_{\varepsilon}\). We will prove that \(D^2 f_{\varepsilon}(u_{\varepsilon})\) is positive or negative definite accordingly with \(D^2 \Gamma(\theta)\), and this will conclude the proof.
As first thing we recall that we have
\[D^2 f_{\varepsilon}(u_{\varepsilon})[\partial_i z_{\theta, \varepsilon}, \partial_j z_{\theta, \varepsilon}] = (\partial_i z_{\theta, \varepsilon} \mid \partial_j z_{\theta, \varepsilon}) - (F''(u_{\varepsilon})\partial_i z_{\theta, \varepsilon} \mid \partial_j z_{\theta, \varepsilon}) + (G''(\varepsilon, u_{\varepsilon})\partial_i z_{\theta, \varepsilon} \mid \partial_j z_{\theta, \varepsilon}).\]
As \(\partial_i z_{\theta, \varepsilon} \in \ker[I_E - F''(z_{\theta, \varepsilon})]\) and \(w(0, z_{\theta, \varepsilon}) = 0\), developing \(F''(u_{\varepsilon})\) and \(G''(\varepsilon, u_{\varepsilon})\) and setting \(w_{\varepsilon} = w(\varepsilon, \theta_{\varepsilon})\), we obtain
\[D^2 f_{\varepsilon}(u_{\varepsilon})[\partial_i z_{\theta, \varepsilon}, \partial_j z_{\theta, \varepsilon}] =
(\partial_i z_{\theta, \varepsilon} \mid \partial_j z_{\theta, \varepsilon}) - (F''(z_{\theta, \varepsilon})\partial_i z_{\theta, \varepsilon} \mid \partial_j z_{\theta, \varepsilon}) - (F'''(z_{\theta, \varepsilon})[\partial_i z_{\theta, \varepsilon}, \partial_j z_{\theta, \varepsilon}] \mid w_{\varepsilon}) +
(G''(\varepsilon, z_{\theta, \varepsilon})\partial_i z_{\theta, \varepsilon} \mid \partial_j z_{\theta, \varepsilon}) + (G'''(\varepsilon, z_{\theta, \varepsilon})[\partial_i z_{\theta, \varepsilon}, \partial_j z_{\theta, \varepsilon}] \mid w_{\varepsilon}) + O(||w_{\varepsilon}||^2) =
-(F'''(z_{\theta, \varepsilon})[\partial_i z_{\theta, \varepsilon}, \partial_j z_{\theta, \varepsilon}] \mid w_{\varepsilon}) + (G'''(\varepsilon, z_{\theta, \varepsilon})\partial_i z_{\theta, \varepsilon} \mid \partial_j z_{\theta, \varepsilon}) +
(G'''(\varepsilon, z_{\theta, \varepsilon})[\partial_i z_{\theta, \varepsilon}, \partial_j z_{\theta, \varepsilon}] \mid w_{\varepsilon}) + O(||w_{\varepsilon}||^2).
\]
We have \(w_{\varepsilon} = o(\varepsilon^{\alpha/2})\) (see theorem 2.1), hence from (G5) we deduce
\[(G'''(\varepsilon, z_{\theta, \varepsilon})[\partial_i z_{\theta, \varepsilon}, \partial_j z_{\theta, \varepsilon}] \mid w_{\varepsilon}) = o(\varepsilon^\alpha)\]
so that
\[D^2 f_{\varepsilon}(u_{\varepsilon})[\partial_i z_{\theta, \varepsilon}, \partial_j z_{\theta, \varepsilon}] = -(F'''(z_{\theta, \varepsilon})[\partial_i z_{\theta, \varepsilon}, \partial_j z_{\theta, \varepsilon}] \mid w_{\varepsilon}) + (G'''(\varepsilon, u_{\varepsilon})\partial_i z_{\theta, \varepsilon} \mid \partial_j z_{\theta, \varepsilon}) + o(\varepsilon^\alpha).\]
(22)
By (4) we have
\[ z_{\theta} + w_\varepsilon = F'(z_{\theta}) + G'(\varepsilon, z_{\theta}) + \sum_l a_l \partial_l z_{\theta}. \]

Developing \( F' \) and \( G' \) we obtain

\[ z_{\theta} + w_\varepsilon = F'(z_{\theta}) - F''(z_{\theta})w_\varepsilon + G'(\varepsilon, z_{\theta}) + G''(\varepsilon, z_{\theta})w_\varepsilon + O(||w_\varepsilon||^2) = \sum_l a_l \partial_l z_{\theta}. \]

By a scalar product with \( \partial_{ij} z_{\theta} \), recalling (H) and the fact that \( z_{\theta} = F'(z_{\theta}) \), we get

\[ (w_\varepsilon | \partial_{ij} z_{\theta} ) - (F''(z_{\theta})w_\varepsilon | \partial_{ij} z_{\theta} ) + (G'(\varepsilon, z_{\theta}) | \partial_{ij} z_{\theta} ) + G''(\varepsilon, z_{\theta})w_\varepsilon | \partial_{ij} z_{\theta} ) + O(||w_\varepsilon||^2) = 0. \]

By (G) it is

\[ (G''(\varepsilon, z_{\theta})w_\varepsilon | \partial_{ij} z_{\theta} ) = (G''(\varepsilon, z_{\theta}) \partial_{ij} z_{\theta} | w_\varepsilon ) = o(\varepsilon^\alpha), \]

so we obtain

\[ (w_\varepsilon | \partial_{ij} z_{\theta} ) = (F''(z_{\theta})w_\varepsilon | \partial_{ij} z_{\theta} ) + (G'(\varepsilon, z_{\theta}) | \partial_{ij} z_{\theta} ) + o(\varepsilon^\alpha) = 0. \] (23)

Deriving twice the equation \( z_\eta = F'(z_{\theta}) \) with respect to \( \eta \in \mathbb{R}^N \) and computing the result at \( \eta = \theta \) we obtain

\[ \partial_{ij} z_{\theta} = F''(z_{\theta}) \partial_{ij} z_{\theta} + F'''(z_{\theta})[\partial_i z_{\theta}, \partial_j z_{\theta}]. \]

By a scalar product with \( w_\varepsilon \) we get

\[ (\partial_{ij} z_{\theta} | w_\varepsilon ) = (F''(z_{\theta}) \partial_{ij} z_{\theta} | w_\varepsilon ) + (F'''(z_{\theta})[\partial_i z_{\theta}, \partial_j z_{\theta}] | w_\varepsilon ). \]

Substituting this last identity in (23) we obtain

\[ -(F'''(z_{\theta})[\partial_i z_{\theta}, \partial_j z_{\theta}] | w_\varepsilon ) = (G'(\varepsilon, z_{\theta}) | \partial_{ij} z_{\theta} ) + o(\varepsilon^\alpha). \] (24)

From this and (22) we have

\[ D^2 f_\varepsilon(u_\varepsilon)[\partial_i z_{\theta}, \partial_j z_{\theta}] = (G'(\varepsilon, z_{\theta}) | \partial_{ij} z_{\theta} ) + (G''(\varepsilon, u_\varepsilon) \partial_i z_{\theta} | \partial_j z_{\theta} ) + o(\varepsilon^\alpha). \]

Hence, dividing by \( \varepsilon^\alpha \), passing to the limit and using hypothesis (G) we obtain

\[ \lim_{\varepsilon \to 0} \frac{1}{\varepsilon^\alpha} D^2 f_\varepsilon(u_\varepsilon)[\partial_i z_{\theta}, \partial_j z_{\theta}] = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon^\alpha} (G'(\varepsilon, z_{\theta}) | \partial_{ij} z_{\theta} ) + \frac{1}{\varepsilon^\alpha} (G''(\varepsilon, u_\varepsilon) \partial_i z_{\theta} | \partial_j z_{\theta} ) = \partial_{ij} \Gamma(\theta). \]

As \( D^2 \Gamma(\theta) \) is definite, so is \( D^2 f_\varepsilon(u_\varepsilon)[\partial_i z_{\theta}, \partial_j z_{\theta}] \), for small \( \varepsilon \)'s.

Let us recall what we have proved up to now. Let us assume that \( D^2 \Gamma(\theta) \) is definite positive (the other case is analogous). We have proved that there is a constant \( \delta > 0 \) such that
\[ D^2 f_\varepsilon(u_\varepsilon)[v^-, v^-] \leq -\delta ||v^-||^2 \quad \text{for all} \quad v^- \in E^-_\varepsilon, \]

\[ D^2 f_\varepsilon(u_\varepsilon)[v^0, v^0] \geq \delta \varepsilon^n ||v^0||^2 \quad \text{for all} \quad v^0 \in E^0_\varepsilon, \]

\[ D^2 f_\varepsilon(u_\varepsilon)[v^+, v^+] \geq \delta ||v^+||^2 \quad \text{for all} \quad v^+ \in E^+_\varepsilon. \]

To conclude the proof we have to show that \( D^2 f_\varepsilon(u_\varepsilon) \) is positive definite in \( E^+_\varepsilon + E^0_\varepsilon \). This does not derive directly from the previous statements because it is not true, in general, that \( D^2 f_\varepsilon(u_\varepsilon)[v^+, v^0] = 0 \). However, thanks to \((G_5)\), we have, for any \( v^+ \in E^+_\varepsilon \), \( v^0 \in E^0_\varepsilon \),

\[ |D^2 f_\varepsilon(u_\varepsilon)[v^+, v^0]| \leq o(\varepsilon^{n/2})||v^+|| ||v^0||. \]

Hence, for small \( \varepsilon \)'s and a suitable \( \delta_1 > 0 \), we obtain

\[ D^2 f_\varepsilon(u_\varepsilon)[v^+ + v^0, v^+ + v^0] \geq \delta ||v^+|| \geq \delta ||v^+ + v^0||^2. \]

The proof is now complete.

With small changes in the previous arguments one can prove the following theorem.

**Theorem 5.2** Assume \((H), (F_0 - F_3), (G_0 - G_3), (F_4)', (G_4)', (G_5)'\). For a given \( \theta \in \mathbb{R}^d \), and for any small \( \varepsilon \)'s, let us suppose that there is a critical point \( u_\varepsilon \in Z_\varepsilon \) of \( f_\varepsilon \), such that \( u_\varepsilon = z_\theta + w(\varepsilon, \theta_\varepsilon) \) and \( \theta_\varepsilon \to \theta \) as \( \varepsilon \to 0 \). Assume that \( z_\theta = \lim_\varepsilon z_\theta_\varepsilon \) is nondegenerate for the restriction of \( f_\theta \) to \((T_{z_\theta} Z_\varepsilon)^\perp\), with Morse index equal to \( m_\theta \), and that the hessian matrix \( D^2 \Gamma(\theta) \) is positive or negative definite.

Then \( u_\varepsilon \) is a nondegenerate critical point for \( f_\varepsilon \) with Morse index equal to \( m_\theta \) if \( D^2 \Gamma(\theta) \) is positive, to \( m_\theta + d \) if \( D^2 \Gamma(\theta) \) is negative. As a consequence, the critical points of \( f_\varepsilon \) form a continuous curve.

**Proof.** To study the behavior of \( D^2 f_\varepsilon(u_\varepsilon) \) on \( E^+_\varepsilon \) and \( E^-_\varepsilon \) we repeat the arguments of the previous theorem. As to \( E^0_\varepsilon \), we recall that the hypotheses imply \( w_\varepsilon = O(\varepsilon^n) \) (see lemma 2.2 in \( [2] \)), so (22) and (24) still hold and the proof goes on as in the previous theorem.

We want now to apply these abstract results to our equation (1). In the following theorem we apply theorem 5.1. To fit hypotheses \((F_4), (G_4)\), we have to assume \( p \geq 3 \). Together with the hypothesis \( p < \frac{N+2}{N-2} \), this of course implies \( N \leq 3 \). Notice that in the following theorems we will treat curves of solutions bifurcating from 0, or \( \infty \), or bounded away both from 0 and \( \infty \). Recall that we refer in our claims to the \( H^1 \)-norm, and that in any case the \( L^\infty \)-norm is vanishing.
Theorem 5.3 Let us suppose $N = 1, 2, 3$ and $3 \leq p < q < +\infty$ if $N = 1, 2$ while $3 \leq p < q \leq 5$ if $N = 3$. Assume $(a_1)$, $(a_2)$, $(b_1)$ and $(b_2)$. Then we obtain a curve $(\lambda, \psi_\lambda)$ of solutions of (1), where $\lambda \in (\lambda_0, 0)$, for a suitable $\lambda_0 < 0$. We have the following behavior of $\psi_\lambda$ as $\lambda \to 0$:

1. If $N = 1$ and $3 \leq p < 5$, then $||\psi_\lambda|| \to 0$, so we have a curve of solutions bifurcating from the origin in $H^1(\mathbb{R}^N)$;

2. If $N = 1$ and $p = 5$, or if $N = 2$ and $p = 3$, then $||\psi_\lambda|| \to c \neq 0$, so we have, in $H^1(\mathbb{R}^N)$, a curve of solutions bounded away from 0 and $\infty$;

3. If $N = 1$ and $p > 5$, or if $N = 2$ and $p > 3$, or if $N = 3$ and $3 \leq p < 5$, then $||\psi_\lambda|| \to +\infty$, so we have, in $H^1(\mathbb{R}^N)$, a curve of solutions bifurcating from infinity.

Proof. By theorem 1.1 we get for equation (2) a family of solutions $u_\varepsilon = z_\theta + w(\varepsilon, \theta)$. Using the general devices of section 2, it is easy to see that $\theta_\varepsilon$ must converge, as $\varepsilon \to 0$, to a maximum or a minimum point of $\Gamma$. But $\Gamma$, as defined in lemma 3.3, has the unique critical point $\theta = 0$, hence $\theta_\varepsilon \to 0$ as $\varepsilon \to 0$. To apply theorem 5.1 we have to show that the hypotheses $(F_4)$, $(G_4)$, $(G_5)$ are satisfied. The assumption $p \geq 3$ gives that $F, G$ are $C^3$. As to the asymptotic assumptions, they are easily checked with arguments similar to those of section 3 and we leave this to the reader (notice that here $\alpha = N$). Recalling that $z_0$ is a nondegenerate critical point for $f_0$ we apply theorem 5.1 and we find a curve $(\varepsilon, u_\varepsilon)$ of solutions of (2), such that $u_\varepsilon(x) = z_0(x + \theta_\varepsilon) + w(\varepsilon, \theta_\varepsilon)(x)$ with $\theta_\varepsilon \to 0$ and $w(\varepsilon, \theta_\varepsilon) \to 0$ as $\varepsilon \to 0$. By the change of variables $\lambda = -\varepsilon^2$ and $\psi_\lambda(x) = \varepsilon^{2/p-1}u_\varepsilon(\varepsilon x)$ we get a family of solutions of (1), which is still a curve. Noticing that $u_\varepsilon \to z_0$ in $H^1$, it is easy to verify the statements on $\lim_{\lambda \to 0} ||\psi_\lambda||$: it is just a computation involving a change of variables, and we leave it to the reader.

In the case $N = 1$ it is possible to relax some hypotheses. In the following theorem we assume $p \geq 2$ and we do not suppose a continuous and bounded. We apply theorem 5.2.

Theorem 5.4 Let us assume $N = 1$ and $2 \leq p < q < +\infty$. Assume $a - A \in L^1(\mathbb{R})$, $\int_{\mathbb{R}}(a(x) - A)dx \neq 0$, and suppose also $(b_1)$ and $(b_2)$. Then we obtain a curve $(\lambda, \psi_\lambda)$ of solutions of (1), where $\lambda \in (\lambda_0, 0)$, for a suitable $\lambda_0 < 0$. We have the following behavior of $\psi_\lambda$ as $\lambda \to 0$:

1. If $2 \leq p < 5$, then $||\psi_\lambda|| \to 0$, so we have a curve of solutions bifurcating from the origin in $H^1(\mathbb{R}^N)$.

2. If $p = 5$ then $||\psi_\lambda|| \to c \neq 0$, so we have, in $H^1(\mathbb{R}^N)$, a curve of solutions bounded away from 0 and $\infty$.

3. If $p > 5$ then $||\psi_\lambda|| \to +\infty$, so we have, in $H^1(\mathbb{R}^N)$, a curve of solutions bifurcating from infinity.

Proof. We want to apply theorem 5.2. As $p \geq 2$, $F, G$ are $C^3$. As to the asymptotic properties of $G$, in particular $(G_5)'$, notice first that here $\alpha = 1$. We use the arguments of [2] (in particular the proof of lemma 4.1, p. 1142-1143) to study $G_1'$ and $G_1''$, while the study of $G_2'$ and $G_2''$ is the same as in the previous sections. Hence we obtain a curve $(\varepsilon, u_\varepsilon)$ of solutions
of (2). Arguing as in the previous theorem, we obtain a curve \((\lambda, \psi_\lambda)\) of solutions of (1), and we get its asymptotic properties as \(\lambda \to 0\). ■

**Remark 5.5** Theorems 5.1 and 5.2 fill a gap in the proof of theorem 3.2 in [2]. In that paper theorem 3.2 was used only in theorem 1.5, to prove that a family of solutions was a curve. Now this result is a particular case of theorem 5. A similar correction to theorem 3.2 of [2] was obtained by S. Krömer in his Diplomarbeit [14]. He also obtained there a bifurcation result analogous to theorem 1.1. ■

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