A counter-example to the central limit theorem in Hilbert spaces under a strong mixing condition

Davide Giraudo∗ Dalibor Volný†

Abstract
We show that in a separable infinite dimensional Hilbert space, uniform integrability of the square of the norm of normalized partial sums of a strictly stationary sequence, together with a strong mixing condition, does not guarantee the central limit theorem.

Keywords: Central limit theorem ; Hilbert space ; mixing conditions ; strictly stationary process.
AMS MSC 2010: 60F05 ; 60G10.
Submitted to ECP on January 9, 2014, final version accepted on August 24, 2014.

1 Introduction and notations
Let \((\Omega, F, \mu)\) be a probability space and \((S,d)\) a separable metric space. We say that the sequence of random variables \((X_n)_{n \in \mathbb{Z}}\) from \(\Omega\) to \(S\) is strictly stationary if for all integer \(d\) and all integer \(k\), the \(d\)-uple \((X_1, \ldots, X_d)\) has the same law as \((X_{k+1}, \ldots, X_{k+d})\).

Rosenblatt introduced in [18] the measure of dependence between two \(\sigma\)-algebras \(A\) and \(B\):
\[
\alpha(A,B) := \sup \{|\mu(A \cap B) - \mu(A)\mu(B)|, A \in A, B \in B\}.
\]

Another one is \(\beta\)-mixing, which is defined by
\[
\beta(A,B) := \frac{1}{2} \sup \sum_{i=1}^{I} \sum_{j=1}^{J} |\mu(A_i \cap B_j) - \mu(A_i)\mu(B_j)|,
\]
where the supremum is taken over the finite partitions \(\{A_1, \ldots, A_I\}\) and \(\{B_1, \ldots, B_J\}\) of \(\Omega\), which consist respectively of elements of \(A\) and \(B\). It was introduced by Volkonskii and Rozanov in [21].

In order to measure dependence of a sequence of random variables, say \(X := (X_j)_{j \in \mathbb{Z}}\) (assumed strictly stationary for simplicity), we define \(F^n_m\) as the \(\sigma\)-algebra generated by the \(X_j\) for \(m \leq j \leq n\), where \(-\infty \leq m \leq n \leq +\infty\).

Then mixing coefficients are defined by
\[
\alpha_X(n) := \alpha\left(F^{-\infty}_m, F^+\infty_n\right)
\]

∗Université de Rouen, France. E-mail: davide.giraudo1@univ-rouen.fr
†Université de Rouen, France.
E-mail: dalibor.volny@univ-rouen.fr
A mixing counter-example to the central limit theorem in Hilbert spaces

\[ \beta_X(n) := \beta \left( \mathcal{F}^0_{-\infty}, \mathcal{F}^+_{+\infty} \right), \] (1.2)

which will be simply written \( \alpha(n) \) (respectively \( \beta(n) \)) when there is no ambiguity.

We say that the strictly stationary sequence \( (X_j) \) is \( \alpha \)-mixing (respectively \( \beta \)-mixing) if \( \lim_{n \to \infty} \alpha(n) = 0 \) (respectively \( \lim_{n \to \infty} \beta(n) = 0 \)). Sequences which are \( \alpha \)-mixing are also called strong-mixing. Notice that the inequality \( 2\alpha(A,B) \leq \beta(A, B)\) for any two sub-\sigma-algebras \( A \) and \( B \) implies that each \( \beta \)-mixing sequence is strong mixing. We refer the reader to Bradley’s book [4] for further information about mixing conditions.

Let \( (V, \|\cdot\|) \) be a separable normed space. We can represent a strictly stationary sequence \( (X_j) \) by \( X_j = f \circ T^j \), where \( T: \Omega \to \Omega \) is measurable and measure preserving, that is, \( \mu(T^{-1}(S)) = \mu(S) \) for all \( S \in \mathcal{F} \) (see [8], p.456, second paragraph).

Given an integer \( N \), we define \( S_N(f) := \sum_{j=0}^{N-1} f \circ T^j \) and \( (\sigma_N(f))^2 := E \left[ \|S_N(f)\|^2 \right] \).

When \( V = \mathbb{R}^d \), \( d \in \mathbb{N}^* \) it is well-known that if \( (f \circ T^j)_{j \geq 0} \) satisfies the following assumptions:

1. \( \lim_{N \to +\infty} \sigma_N(f) = +\infty \);
2. \( \int f d\mu = 0 \);
3. \( \lim_{n \to +\infty} \alpha(n) = 0 \);
4. the family \( \left\{ \frac{\|S_N(f)\|^2}{(\sigma_N(f))^2}, N \geq 1 \right\} \) is uniformly integrable,

then \( \left( \frac{1}{\sigma_N(f)} S_N(f) \right)_{N \geq 1} \) converges in distribution to a Gaussian law. It was established for \( d = 1 \) by Denker [7], Mori and Yoshihara [14] using a blocking argument. Volný [22] gave a proof for \( d \) arbitrary based on approximation by an array of independent random variables.

A natural question would be: what if we replace \( \mathbb{R}^d \) by another normed space?

First, we restrict ourselves to separable normed spaces in order to avoid measurability issues of sums of random variables. Corollary 10.9. in [11] asserts that a separable Banach space \( B \) with norm \( \|\cdot\|_B \) is isomorphic to a Hilbert space if and only if for all random variables \( X \) with values in \( B \), the conditions \( E[X] = 0 \) and \( E[\|X\|^2] < 1 \) are necessary and sufficient for \( X \) to satisfy the central limit theorem. By “\( X \) satisfies the CLT”, we mean that if \( (X_j)_{j \geq 1} \) is a sequence of independent random variables, with the same law as \( X \), the sequence \( \left( n^{-1/2} \sum_{j=1}^{n} X_j \right)_{n \geq 1} \) weakly converges in \( B \). Hence we cannot expect a generalization in a class larger than separable Hilbert spaces. Such a space is necessarily isomorphic to \( \mathcal{H} := l^2(\mathbb{R}) \), the space of square summable sequences \( (x_n)_{n \geq 1} \) endowed with the inner product \( (x,y)_\mathcal{H} := \sum_{n=1}^{+\infty} x_n y_n \). We shall denote by \( e_n \) the sequence whose all terms are 0, except the \( n \)-th which is 1. Bold letters denote both random variables taking their values in \( \mathcal{H} \) and elements of this space.

General considerations about probability measures and central limit theorem in Banach spaces are contained in Araujo and Giné’s book [2].

**Notation** 1. If \( (a_n)_{n \geq 1}, (b_n)_{n \geq 1} \) are sequences of non-negative real numbers, \( a_n \lesssim b_n \) means that \( a_n \leq C b_n \), where \( C \) does not depend on \( n \). In an analogous way, we define \( a_n \gtrsim b_n \). When \( a_n \lesssim b_n \lesssim a_n \), we simply write \( a_n \asymp b_n \).

Our main results are
A mixing counter-example to the central limit theorem in Hilbert spaces

**Theorem A.** There exists a probability space \((\Omega, F, \mu)\) such that given \(0 < q < 1\), we can construct a strictly stationary sequence \(X = (f ◦ T^k) = (X_k)_{k \in \mathbb{N}}\) defined on \(\Omega\), taking its values in \(\mathcal{H}\), such that:

a) \(E[f] = 0, \ E[\|f\|^q_{\mathcal{H}}]\) is finite for each \(p\);

b) the limit \(\lim_{N \to \infty} \sigma_N(f)\) is infinite;

c) the process \((X_k)_{k \in \mathbb{N}}\) is \(\beta\)-mixing, more precisely, \(\beta_X(j) = O \left( \frac{1}{j^2} \right)\);

d) the family \(\left\{ \frac{\|X_n(f)\|^2}{\sigma_n^2(f)}, N \geq 1 \right\}\) is uniformly integrable;

e) if \(I \subset \mathbb{N}\) is infinite, the family \(\left\{ \frac{S_N(f)}{\sigma_n^2(f)}, N \in I \right\}\) is not tight in \(\mathcal{H}\); furthermore, given a sequence \((c_N)_{N \geq 1}\) of real numbers going to infinity, we have either

\[
\begin{align*}
\text{a) } & \lim_{N \to +\infty} \frac{\sigma_N(f)}{c_N} = 0, \text{ hence } \left( \frac{S_N(f)}{c_N} \right)_{N \geq 1} \text{ converges to } 0_{\mathcal{H}} \text{ in distribution, or} \\
\text{b) } & \limsup_{N \to +\infty} \frac{\sigma_N(f)}{c_N} > 0, \text{ and in this case the collection } \left\{ \frac{S_N(f)}{c_N}, N \geq 1 \right\} \text{ is not tight.}
\end{align*}
\]

**Theorem A’.** Let \((b_N)_{N \geq 1}\) and \((h_N)_{N \geq 1}\) be sequences of positive real numbers, with \(\lim_{N \to \infty} b_N = 0\) and \(\lim_{N \to \infty} h_N = \infty\). Then there exists a strictly stationary sequence \(X := (f ◦ T^k)_{k \in \mathbb{N}} = (X_k)_{k \in \mathbb{N}}\) of random variables with values in \(\mathcal{H}\) such that \(A, A, A\) of Theorem A and the following two properties hold:

b') we have \(\sigma^2_X(f) \leq N \cdot h_N\) and \(\frac{\sigma^2_X(f)}{h_N} \to \infty\);

c') the process \((X_k)_{k \in \mathbb{N}}\) is \(\beta\)-mixing, and there is an increasing sequence \((n_k)_{k \geq 1}\) of integers such that for each \(k\), \(\beta_X(n_k) \leq b_{n_k}\).

**Remark 2.** Theorem A shows that Denker’s result does not remain valid in its full generality in the context of Hilbert space valued random variables.

Furthermore, a careful analysis of the proof of Proposition 6 shows that for the construction given in Theorem A, we have \(\sigma^2_X(f) = N \cdot h(N)\) with \(h\) slowly varying in the strong sense. Theorem 1 of [12] does not remain valid in the Hilbert space setting. Indeed, the arguments given in pages 654-655 show that the conditions of Denker’s theorem together with the assumption that \(\sigma^2_X = N \cdot h(N)\) with \(h\) slowly varying in the strong sense imply those of Theorem 1. These arguments are still true in the Hilbert space setting.

**Remark 3.** Theorem A' gives a control of the mixing coefficients on a subsequence. When \(b_N := N^{-2}\) for example, the construction gives a better estimation for the considered subsequence than what we get by Theorem A.

Tone has established in [20] a central limit theorem for strictly stationary random fields with values in \(\mathcal{H}\) under \(\rho\)-mixing conditions. For sequences, these coefficients are defined by

\[
\rho'(n) := \sup \left\{ \frac{\|E[f]g\|_{\mathcal{H}} - \langle E[f], E[g]\rangle_{\mathcal{H}}}{\|f\|_{L^2(\mathcal{H})} \cdot \|g\|_{L^2(\mathcal{H})}} \right\},
\]

where the supremum is taken over all the non-zero functions \(f\) and \(g\) such that \(f\) and \(g\) are respectively \(\sigma(X_j, j \in S_1)\) and \(\sigma(X_j, j \in S_2)\)-measurable, where \(S_1\) and \(S_2\) are such that \(\min_{s \in S_1, t \in S_2} |s - t| \geq n\), while \(L^2(\mathcal{H})\) denote the collection of equivalence classes of random variables \(X: \Omega \to \mathcal{H}\) such that \(\|X\|_{L^2(\mathcal{H})}^2\) is integrable.

So “interlaced index sets” can be considered, which is not the case for \(\alpha\) and \(\beta\)-mixing coefficient. Taking \(f\) and \(g\) as characteristic functions of elements of \(\mathcal{F}_{-\infty}^\infty\) and
A mixing counter-example to the central limit theorem in Hilbert spaces

$\mathcal{F}^{+\infty}$ respectively, one can see that $\alpha(n) \leq \rho'(n)$, hence $\rho'$-mixing condition is more restrictive than $\alpha$-mixing condition.

A partial generalization of the finite dimensional result was proved by Politis and Romano [15], namely, the conditions $E \|X_1\|^{2+\delta}_{\mathcal{H}}$ finite for some positive $\delta$ and $\sum_j \alpha_X(j) \rightarrow \infty$ guarantees the convergence of $n^{-1/2} \sum_{j=1}^n X_j$ to a Gaussian random variable $\mathcal{N}$, whose covariance operator $S$ satisfies

$$E [(\mathcal{N}, h)^2] = \langle Sh, h \rangle_{\mathcal{H}} = \text{Var}((X_1, h)) + 2 \sum_{i=1}^{+\infty} \text{Cov} ((X_1, h), (X_{1+i}, h)).$$

Similar results were obtained by Dehling [6].

Rio’s inequality [16] asserts that given two real valued random variables $X$ and $Y$ with finite two order moments,

$$|E [XY] - E [X]E [Y]| \leq 2 \int_0^\alpha (\sigma(X), \sigma(Y)) Q_X(u)Q_Y(u)du.$$

It was extended by Merlevède et al. [13], namely, if $X$ and $Y$ are two random variables with values in $\mathcal{H}$, with respective quantile function $Q_{\|X\|_{\mathcal{H}}}$ and $Q_{\|Y\|_{\mathcal{H}}}$, then

$$|E [(X, Y)] - E [X]E [Y]| \leq 18 \int_0^\alpha Q_{\|X\|_{\mathcal{H}}}Q_{\|Y\|_{\mathcal{H}}}du,$$

where $\alpha := \alpha(\sigma(X), \sigma(Y))$.

From this inequality, they deduce a central limit theorem for a stationary sequence $(X_j)_{j \in \mathcal{Z}}$ of $\mathcal{H}$-valued zero-mean random variables satisfying

$$\int_0^1 \alpha^{-1}(u)Q_{\|X_0\|_{\mathcal{H}}}(u)du < \infty,$$  \hspace{1cm} (1.3)

where $\alpha^{-1}$ is the inverse function of $x \mapsto \alpha_X(\|x\|)$.

Discussion after Corollary 1.2 in [17] proves that the later result implies Politis’ one.

Relative optimality of condition (1.3) (cf. [9]) can give a finite-dimensional counter-example to the central limit theorem when this condition is not satisfied. Here, the condition of uniform integrability prevents such counter-examples.

Defining $\alpha_X(n) := \sup_{i,j \geq n} \alpha(\mathcal{F}^{5+\infty}, \sigma(X_i, X_j))$ and $Q_X$, the right-continuous inverse of the function $t \mapsto \mu \{\|X_0\|_{\mathcal{H}} > t\}$ that is,

$$Q_X(u) := \inf \{t \in \mathbb{R}, \mu \{\|X_0\|_{\mathcal{H}} > t\} \leq u\},$$

Dedecker and Merlevède have shown in [5] that under the assumption

$$\sum_{k=1}^{+\infty} \int_0^{\alpha_X(k)} Q_X^2(u)du < \infty,$$

we can find a sequence $(Z_i)_{i \in \mathbb{N}}$ of Gaussian random variables with values in $\mathcal{H}$ such that almost surely,

$$\left\| S_n - \sum_{i=1}^n Z_i \right\|_{\mathcal{H}} = o \left( \sqrt{n \log \log n} \right).$$

2 The proof

2.1 Construction of $f$

In order to construct a counter-example, we shall need the following lemma, which will be proved later.

We will denote $U$ the Koopman operator associated to $T$, which acts on measurable functions by $U(f)(x) := f(T(x))$. 

ECP 19 (2014), paper 62. ecp.ejpecp.org
A mixing counter-example to the central limit theorem in Hilbert spaces

**Lemma 4.** Let \((u_k)_{k \geq 1} \subset (0, 1)\) be a sequence of numbers. Then there exists a dynamical system \((\Omega, \mathcal{F}, \mu, T)\) and a sequence of random variables \((\xi_k)_{k \geq 1}\) such that

1. for each \(k \geq 1\), \(\mu(\xi_k = 1) = \mu(\xi_k = -1) = \frac{u_k}{2}\) and \(\mu(\xi_k = 0) = 1 - u_k;\)
2. the random variables \((U^i\xi_k, k \geq 1, i \in \mathbb{Z})\) are mutually independent.

Recall that \(e_k\) is the \(k\)-th element of the canonical orthonormal system of \(\mathcal{H} = \ell^2(\mathbb{R})\). We define

\[
f_k := \sum_{i=0}^{n_k-1} U^{-i} \xi_k, \quad f := \sum_{k=1}^{+\infty} f_k e_k,
\]

where the \(\xi_i\)'s are constructed using to Lemma 4 taking \(u_k := n_k^{-2}\). Conditions on the increasing sequence of integers \((n_k)_{k \geq 1}\) will be specified latter.

Then \(X_k := f \circ T^k\) is a strictly stationary sequence. Note that \(\|f\|_{\mathcal{H}}^2\) is an integrable random variable whenever \(\sum_k \frac{1}{n_k}\) is convergent. In the sequel, the choice of \(n_k\) will guarantee this condition.

### 2.2 Preliminary results

We express \(S_N(f_k)\) as a linear combination of independent random variables. By direct computations, we get

\[
f_k = n_k \xi_k + (I - U) \sum_{i=1-n_k}^{n_k-1} (n_k + i) U^i \xi_k,
\]

hence

\[
S_N(f_k) = n_k \sum_{j=0}^{N-1} U^j \xi_k + \sum_{i=1-n_k}^{n_k-1} (n_k + i) U^i \xi_k - \sum_{i=N-n_k+1}^{N-1} (n_k + i - N) U^i \xi_k.
\]

This formula can be simplified if we distinguish the cases \(N \geq n_k\) and \(n_k < N\) (we break the third sum at the index \(i = 0\) if necessary). This gives

\[
S_N(f_k) = \sum_{j=0}^{N-1} (N - j) U^j \xi_k + \sum_{j=1-n_k}^{N-n_k} (n_k + j) U^j \xi_k + N \sum_{j=1+N-n_k}^{N-1} U^j \xi_k, \quad \text{if } N < n_k, \tag{2.3}
\]

\[
S_N(f_k) = n_k \sum_{j=0}^{N-n_k} U^j \xi_k + \sum_{j=N-n_k+1}^{N-1} (N - j) U^j \xi_k + \sum_{j=1-n_k}^{n_k-1} (n_k + j) U^j \xi_k, \quad \text{if } N \geq n_k. \tag{2.4}
\]

The computation of the expectation of the square of partial sums gives

\[
\sigma^2_N(f_k) = \begin{cases} 
\frac{1}{n_k^2} \left( 2 \sum_{j=1}^{N} j^2 + (n_k - N - 1)N^2 \right) & \text{if } N < n_k, \\
\frac{1}{n_k^2} \left( n_k^2(N - n_k + 1) + 2 \sum_{j=1}^{n_k-1} j^2 \right) & \text{if } N \geq n_k.
\end{cases} \tag{2.5}
\]
Combining (2.6) and (2.8), we get

\[ n ECP 19 \]

Proposition 7. Assume that \((n_k)_{k \geq 1}\) satisfies the condition

\[
\text{there is } p > 1 \text{ such that for each } k, \quad n_{k+1} \geq n_k^p. \tag{C}
\]

Then \(\sigma_N^2(f) \asymp N \cdot i(N)\).

Proof. Using (2.5), the fact that \(M^3 \asymp \sum_{j=1}^{M} j^2\) and \(\sigma_N^2(f) = \sum_{k \geq 1} \sigma_N^2(f_k)\), we have

\[ \sigma_N^2(f) \geq \sum_{i=1}^{i(N)} \sigma_N^2(f_k) \geq N \sum_{j=1}^{i(N)} = N \cdot i(N). \tag{2.6} \]

From (2.5) in the case \(n_k \geq N\), we deduce

\[ \sum_{k \geq i(N)+1} \sigma_N^2(f_k) \leq \sum_{k \geq i(N)+1} \frac{N^2}{n_k} \leq \frac{N^2}{n_{i(N)}+1} + \sum_{k \geq i(N)+1} \frac{N^2}{n_k n_k^{p-1}}. \tag{2.7} \]

Since \(n_{i(N)+1} \geq N\) and the series \(\sum_{k \geq 1} n_k^{1-p}\) is convergent (by the ratio test), we obtain

\[ \sum_{k \geq i(N)+1} \sigma_N^2(f_k) \leq N + N \sum_{k \geq i(N)+1} \frac{1}{n_k^{p-1}} \leq N. \tag{2.8} \]

Combining (2.6) and (2.8), we get

\[ N \cdot i(N) \leq \sigma_N^2(f) \leq \sum_{k=1}^{i(N)} \sigma_N^2(f_k) + \sum_{k \geq i(N)+1} \sigma_N^2(f_k) \leq N \cdot i(N) + N \leq N \cdot i(N). \tag{2.9} \]

\[
\Box
\]

Proposition 7. Assume that \(\sum_{k} n_k^{-a}\) is convergent for any positive real number \(a\). Then for each integer \(p\), \(\|f\|_{H^p}\) has a finite moment of order \(p\).

Proof. We shall use Rosenthal’s inequality (Theorem 3, [19]): there exists a constant \(C\) depending only on \(q\) such that if \(M\) is an integer, \(Y_1, \ldots, Y_M\) are independent real valued zero mean random variables for which \(E|Y_i|^q < \infty\) for each \(i\), then

\[ E \left( \sum_{j=1}^{M} Y_j \right)^{q} \leq C \left( \sum_{j=1}^{M} E|Y_j|^q + \left( \sum_{j=1}^{M} E[Y_j^2] \right)^{q/2} \right). \tag{2.10} \]

If \(q = 2p\) is given then we have

\[ E|f_k|^{2p} \leq n_k^{-1} + n_k^{-q} \leq n_k^{-1}. \tag{2.11} \]

\[
\Box
\]

We provide a sufficient condition for the uniform integrability of the family \(S := \left\{ \frac{\|S_N(f)\|_{H^p}}{\sigma_N^2(f)}, N \geq 1 \right\}\).

Proposition 8. If \((n_k)_{k \geq 1}\) satisfies (C), then \(S\) is uniformly integrable.
A mixing counter-example to the central limit theorem in Hilbert spaces

Proof. For \( N \geq 1 \), we have:

\[
\frac{\|S_N(f)\|_H^2}{\sigma_N^2(f)} = \sum_{j=1}^{i(N)-1} \frac{|S_N(f_j)|^2}{\sigma_N^2(f)} + \frac{|S_N(f_{i(N)})|^2}{\sigma_N^2(f)} + \frac{|S_N(f_{i(N)+1})|^2}{\sigma_N^2(f)} + \sum_{j \geq i(N)+2} \frac{|S_N(f_j)|^2}{\sigma_N^2(f)}.
\]

hence it is enough to prove that the families

\[
S_1 := \left\{ \sum_{k=1}^{i(N)-1} \frac{|S_N(f_k)|^2}{\sigma_N^2(f)}, N \geq 1 \right\},
\]

\[
S_2 := \left\{ \frac{|S_N(f_{i(N)})|^2}{\sigma_N^2(f)}, N \geq 1 \right\} =: \{ \mu_N, N \geq 1 \},
\]

\[
S_3 := \left\{ \frac{|S_N(f_{i(N)+1})|^2}{\sigma_N^2(f)}, N \geq 1 \right\} =: \{ \nu_N, N \geq 1 \}, \text{ and}
\]

\[
S_4 := \left\{ \sum_{k \geq i(N)+2} \frac{|S_N(f_k)|^2}{\sigma_N^2(f)}, N \geq 1 \right\}
\]

are uniformly integrable. For \( S_1 \) and \( S_4 \), we shall show that these families are bounded in \( \mathbb{L}^p \) for \( p \in (1, 2] \) as in (C).

- for \( S_1 \): using the expression in (2.4) and (2.10) with \( q := 2p > 2 \), we have

\[
E \left[ |S_N(f_k)|^{2p} \right] \leq C \left( 2 \sum_{j=1}^{n_k} j^{2p} + \frac{n_k^{2p} (N - n_k)}{n_k^2} \right) + C \left( 2 \sum_{j=1}^{n_k} j^2 + \frac{(N - n_k) n_k^2}{n_k^2} \right)^p
\]

\[
\leq \frac{1}{n_k^2} \left( n_k^{2p+1} + (N - n_k) n_k^{2p} \right) + \frac{1}{n_k^2} \left( n_k^2 + (N - n_k) n_k^2 \right)^p
\]

\[
= \frac{N n_k^{2p+1}}{n_k^2} + \frac{N p n_k^{2p}}{n_k^2}
\]

\[
= N n_k^{2p} + N^p
\]

hence

\[
\left\| |S_N(f_k)|^2 \right\|_p \leq N^{1/p} n_k^{2p-1} + N,
\]

which gives

\[
\left\| \sum_{k=1}^{i(N)-1} \frac{|S_N(f_k)|^2}{\sigma_N^2(f)} \right\|_p \leq \frac{i(N)-1}{\sigma_N^2(f)} \left( \sum_{k=1}^{i(N)-1} \frac{2}{n_k^{2p-1}} + N \right)
\]

\[
\leq \frac{i(N) n_k^{2p-1} + N i(N)}{\sigma_N^2(f)}.
\]

From (2.6), we get

\[
\left\| \sum_{k=1}^{i(N)-1} \frac{|S_N(f_k)|^2}{\sigma_N^2(f)} \right\|_p \leq \frac{n_k^{2p-1}}{n_{i(N)}} + 1 = \frac{2^{p-1}}{n_{i(N)}} + 1.
\]

Since \( p - 2 \leq 0 \), we obtain that \( S_1 \) is bounded in \( \mathbb{L}^p \) hence uniformly integrable.
A mixing counter-example to the central limit theorem in Hilbert spaces

• for \( S_2 \): using (2.4) in the case \( n_k \leq N \) and Proposition 6, we get
  \[
  \|u_N\|_1 \lesssim \frac{N}{\sigma_N^2(f)} \lesssim \frac{1}{\sqrt{N}},
  \]
  \[(2.12)\]
  Since \( \|u_N\|_1 \to 0 \) and \( u_N \in L^1 \) for each \( N \), the family \( S_2 \) is uniformly integrable.

• for \( S_3 \): using (2.3) in the case \( n_k > N \) and Proposition 6, we get
  \[
  \|v_N\|_1 \lesssim \frac{N^2}{\sigma_{n(N)}^2(f)} \lesssim \frac{N}{N \cdot \sqrt{i(N)}},
  \]
  \[(2.13)\]
  Since \( \|v_N\|_1 \to 0 \) and \( v_N \in L^1 \) for each \( N \), the family \( S_3 \) is uniformly integrable.

• for \( S_4 \): as for \( S_1 \), we shall show that this family is bounded in \( L^p \) with \( p \in (1, 2) \). We have, using (2.3) and (2.10)
  \[
  \mathbb{E} \left[ |S(f_k)|^2 \right] \lesssim \frac{1}{n_k^p} \left( N^{2p+1} + N^{2p}(n_k - N) \right) + \frac{1}{n_k^p} \left( N^3 + (n_k - N)N^2 \right)^p
  = \frac{N^{2p}}{n_k} + \frac{N^{2p}}{n_k}
  \lesssim \frac{N^{2p}}{n_k}
  \]
  as \( N \leq n_k \). We thus get that
  \[
  \left\{ \sum_{k \geq i(N)+2} |S(f_k)|^2 \right\}_p \lesssim N^2 \sum_{k \geq i(N)+2} \frac{1}{n_k^p}.
  \]
  Also, using (2.5), we have
  \[
  \sigma_N^2(f) \gtrsim N^2 \sum_{k \geq i(N)+1} \frac{1}{n_k^p} 
  \]
  The condition \( n_{k+1} \geq n_k^p \) gives boundedness in \( L^p \) of \( S_4 \).

This concludes the proof of A.

\[\square\]

**Proposition 9.** Assume that \( (n_k)_{k \geq 1} \) is such that \( S \) is uniformly integrable and \( \sum n_k^{-1} \) is convergent. Then for each \( I \subset \mathbb{N} \) infinite, the collection \( \{ S_N(f), N \in I \} \) is not tight in \( \mathcal{H} \). Its finite-dimensional distributions converge to 0 in probability. Furthermore, if \( (c_N)_{N \geq 0} \) is a sequence of positive numbers going to infinity, we have either

- \( \lim_{N \to +\infty} \frac{S_N(f)}{c_N} = 0 \), hence \( \left( \frac{S_N(f)}{c_N} \right)_{N \geq 1} \) converges to \( 0 \) in distribution, or
- \( \limsup_{N \to +\infty} \frac{\sigma_N(f)}{c_N} > 0 \), and in this case the sequence \( \left\{ \frac{S_N(f)}{c_N}, N \geq 1 \right\} \) is not tight.

**Proof.** We first prove that the finite dimensional distributions of \( \frac{S_N(f)}{\sigma_N(f)} \) converge weakly to 0.

For each \( d \in \mathbb{N} \), we have \( \frac{\langle S_N(f), e_d \rangle_{\mathcal{H}}}{\sigma_N(f)} \to 0 \) in distribution. Indeed, we have by (2.2) that
  \[
  \langle S_N(f), e_d \rangle_{\mathcal{H}} = n_d \sum_{i=0}^{N-1} U^i \xi_d + (I - U^N) \sum_{i=1}^{n_d} u_d i U^i \xi_d. \]
  We conclude noticing that \( \sigma_N(f)^{-1} (I - U^N) \sum_{i=1}^{n_d} u_d i U^i \xi_d \) goes to 0 in probability as \( N \) goes to infinity, using Proposition 6 and the estimate
  \[
  \mathbb{E} \left( n_d \sum_{i=0}^{N-1} U^i \xi_d \right)^2 = N \lesssim \frac{\sigma_N^2(f)}{i(N)}
  \]
  for each \( d \in \mathbb{N} \).
This can be extended replacing \( e_d \) by any \( v \in \mathcal{H} \) by an application of Theorem 4.2. in [3]. By Proposition 4.15 in [2], the only possible limit is the Dirac measure at \( 0_H \).

Assume that the sequence \( \left\{ \frac{S_N(f)}{\sigma_N(f)}, N \geq 1 \right\} \) is tight. The sequence \( \left( \frac{\|S_N(f)\|^2}{\sigma_N(f)^2} \right)_{N \geq 1} \) is a uniformly integrable sequence of random variables of mean 1. A weakly convergent subsequence would go to \( 0_H \). According to Theorem 5.4 in [3], we should have that the limit random variable has expectation 1. This contradiction gives the result when \( I = \mathbb{N} \setminus \{0\} \). Applying this reasoning to subsequences, one can see that for any infinite subset \( I \) of \( \mathbb{N} \setminus \{0\} \), the family \( \left\{ \frac{S_N(f)}{\sigma_N(f)}, N \in I \right\} \) is not tight.

Let \( (c_N)_{N \geq 1} \) be a sequence of positive real numbers such that \( \lim_{N \to +\infty} c_N = +\infty \).

- first case: \( \frac{\sigma_N(f)}{c_N} \) converges to 0. In this case, the sequence \( \left( \frac{\|S_N(f)\|^2}{c_N^2} \right)_{N \geq 1} \) converges to 0 in \( L^1 \), hence the sequence \( \left( \frac{S_N(f)}{c_N} \right)_{N \geq 1} \) converges in distribution to \( 0_H \).
- second case: \( \lim \sup_{N \to \infty} \frac{\sigma_N(f)}{c_N} > 0 \). Hence there is some \( r > 0 \) and a sequence of integers \( l_i \uparrow \infty \) such that for each \( i \), \( \frac{\sigma_{l_i}(f)}{c_{l_i}} \geq \frac{1}{r} \), that is, \( c_{l_i} \leq r \sigma_{l_i}(f) \).

Assume that the family \( \left\{ \frac{S_{l_i}(f)}{c_{l_i}}, i \geq 1 \right\} \) is tight. This means that given a positive \( \varepsilon \), one can find a compact set \( K = K(\varepsilon) \) such that for each \( i \), \( \mu \left( \frac{S_{l_i}(f)}{c_{l_i}} \in K \right) > 1 - \varepsilon \).

We can assume that this compact set is convex and contains 0 (we consider the closed convex hull of \( K \cup \{0\} \), which is compact by Theorem 5.35 in [1]). Then we have

\[
\left\{ \frac{S_{l_i}(f)}{c_{l_i}} \in K \right\} = \left\{ \frac{S_{l_i}(f)}{\sigma_{l_i}(f)} \in \frac{c_{l_i}}{\sigma_{l_i}(f)} K \right\} \subset \left\{ \frac{S_{l_i}(f)}{\sigma_{l_i}(f)} \in rK \right\},
\]

and we would deduce tightness of \( \left\{ \frac{S_{l_i}(f)}{\sigma_{l_i}(f)}, i \geq 1 \right\} \), which cannot happen.

**Remark 10.** In the second case, it may happen that the finite dimensional distributions does not converge to degenerate ones, for example with \( c_N := N \).

\[ \square \]

### 2.3 Proof of Theorem A

Notice that if \( n_{k+1} \geq 2p^k \) for some \( p > 1 \) and \( n_1 = 2 \), then \( n_k \geq 2p^k \), hence the condition of Proposition 7 is fulfilled. We get A since each \( f_k \) has expectation 0.

We denote \( [x] := \sup \{k \in \mathbb{Z}, k \leq x\} \) the integer part of the real number \( x \).

**Proposition 11.** Let \( p > 1 \). With \( n_k := \lfloor 2p^k \rfloor \) (which satisfies (C)), we have for each positive integer \( l \),

\[
\beta_X(l) \lesssim \frac{1}{l^p}.
\]

**Proof.** We define \( \beta_k(n) \) as the \( n \)-th \( \beta \)-mixing coefficient of the sequence \( (f_k \circ T^i)_{i \geq 0} \).

By Lemma 5 of [10], we have the estimate \( \beta_k(0) \leq 4n_k^{-1} \) for each \( k \). Using then Proposition 4 of this paper (cf. [4] for a proof), we get that \( \beta_X(n_k) \lesssim \sum_{j \geq k} \frac{1}{n_j} \) for each integer \( k \). Since \( p^i \geq i \) for \( i \) large enough,

\[
\sum_{j \geq k} \frac{1}{n_j} = \sum_{i=0}^{+\infty} \frac{1}{2^{p^i+p}} = \sum_{i=0}^{+\infty} \frac{1}{2^{p^i+p}} \leq \sum_{i=0}^{+\infty} \frac{1}{2^{p^i+p}} = \frac{2}{2p^i},
\]

ECP 19 (2014), paper 62. ecp.ejpecp.org
A mixing counter-example to the central limit theorem in Hilbert spaces

we get

\[ \beta_X(N) \leq \beta_X(n_i(N)) \leq \frac{1}{n_i(N)} \leq \frac{1}{N^{1/p}}. \]

\[ \square \]

This proves A. For any \( p \), the choice \( n_k := [2^k] \) satisfies the condition of Proposition 8, which proves A. We conclude the proof by Proposition 9.

Remark 12. For each of these choices, \( \sigma_N^2(f) \) behaves asymptotically like \( N \log \log N \). Theorem A' shows that we can construct a process which satisfies the same asymptotic behavior of partial sums and has a variance close to a linear one.

A question would be: can we construct a strictly stationary sequence with all the properties of Theorem A, except A which is replaced by an assumption of linear variance?

2.4 Proof of Theorem A'

Let \( (h_N)_{N \geq 1} \) be the sequence involved in Theorem A'. We define for an integer \( u \) the quantity \( h^{-1}(u) := \inf \{ j \in \mathbb{N}, h_j \geq u \} \).

If \((b_k)_{k \geq 1}\) is the given sequence (that can be assumed decreasing), we define inductively

\[ n_{k+1} := \max \left\{ n_k^2, \left\lceil \frac{2^k}{b_k} \right\rceil, h^{-1}(k) \right\}. \]  

(2.14)

Let \( N \) be an integer. We assume without loss of generality that the growth of the sequence \((h_N)_{N \geq 1}\) is slow enough in order to guarantee that there exists \( k \) such that \( N = h^{-1}(k) \). We then have \( i(N) = k + 1 = h_N + 1 \), hence using Proposition 6, we get \( b') \).

We have \( n_k \geq 2^2 \) hence by a similar argument as in the proof of Theorem A, A is satisfied.

By a similar argument as in [10], we get \( \beta_X(n_k) \leq b_{n_k} \), hence \( c' \) holds.

Remark 13. By (1.3), we cannot expect the relationship \( \beta_X(\cdot) \leq b \) for the whole sequence.

Since for each \( k \), \( n_{k+1} \geq n_k^2 \), Proposition 8 and 9 apply. This concludes the proof of Theorem A'.

Proof of Lemma 4. Let \( \Omega := [0, 1]^{\mathbb{N}^* \times \mathbb{Z}} \), where \([0,1]\) is endowed with Borel \( \sigma \)-algebra and Lebesgue measure, and \( \Omega \) with the product structure.

For \((k,j)\in \mathbb{N}^* \times \mathbb{Z} \) and \( S \subset [0,1] \), let \( P_{k,j}(S) := \prod_{(i_1,i_2)\in \mathbb{N}^* \times \mathbb{Z}} S_{i_1,i_2} \), where \( S_{i_1,i_2} = S \) if \((i_1,i_2) = (k,j) \) and \([0,1]\) otherwise. Then we define

\[ A_{k,j}^+ := P_{k,j}([0, 2^{-1}(u_k)^{-1}]), \]

\[ A_{k,j}^- := P_{k,j}([2^{-1}(u_k)^{-1}, (u_k)^{-1}]), \]

\[ A_{k,j}^{(0)} := P_{k,j}([u_k^{-1}, 1]), \]

the map \( T \) by \( T \left( (x_{k,j})_{(k,j)\in \mathbb{N}^* \times \mathbb{Z}} \right) := (x_{k,j+1})_{(k,j)\in \mathbb{N}^* \times \mathbb{Z}} \), and

\[ \xi_k := \chi A_{k,0}^+ - \chi A_{k,0}^- \].

\[ \square \]
A mixing counter-example to the central limit theorem in Hilbert spaces

References

[1] C. D. Aliprantis and K. C. Border, Infinite dimensional analysis, third ed., Springer, Berlin, 2006, A hitchiker’s guide. MR-2378491

[2] A. Araujo and E. Giné, The central limit theorem for real and Banach valued random variables, John Wiley & Sons, New York-Chichester-Brisbane, 1980, Wiley Series in Probability and Mathematical Statistics. MR-576407

[3] P. Billingsley, Convergence of probability measures, John Wiley & Sons Inc., New York, 1968. MR-0233396

[4] R. C. Bradley, Introduction to strong mixing conditions. Vol. 1, Kendrick Press, Heber City, UT, 2007. MR-2325294

[5] J. Dedecker and F. Merlevède, On the almost sure invariance principle for stationary sequences of Hilbert-valued random variables, Dependence in probability, analysis and number theory, Kendrick Press, Heber City, UT, 2010, pp. 157–175. MR-2731073

[6] H. Dehling, Limit theorems for sums of weakly dependent Banach space valued random variables, Z. Wahrsch. Verw. Gebiete 63 (1983), no. 3, 393–432. MR-705631

[7] M. Denker, Uniform integrability and the central limit theorem for strongly mixing processes, Dependence in probability and statistics (Oberwolfach, 1985), Progr. Probab. Statist., vol. 11, Birkhäuser Boston, Boston, MA, 1986, pp. 269–289. MR-899993

[8] J. L. Doob, Stochastic processes, John Wiley & Sons, Inc., New York; Chapman & Hall, Limited, London, 1953. MR-0058896

[9] P. Doukhan, P. Massart, and E. Rio, The functional central limit theorem for strongly mixing processes, Ann. Inst. H. Poincaré Probab. Statist. 30 (1994), no. 1, 63–82. MR-1262892

[10] D. Giraudo and D. Volný, A strictly stationary -mixing process satisfying the central limit theorem but not the weak invariance principle, Stochastic Processes and their Applications 124 (2014), no. 11, 3769 – 3781.

[11] M. Ledoux and M. Talagrand, Probability in Banach spaces, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], vol. 23, Springer-Verlag, Berlin, 1991, Isoperimetry and processes. MR-1102015

[12] F. Merlevède and M. Peligrad, On the weak invariance principle for stationary sequences under projective criteria, J. Theoret. Probab. Statist. 19 (2006), no. 3, 647–689. MR-2280514

[13] F. Merlevède, M. Peligrad, and S. Utev, Sharp conditions for the CLT of linear processes in a Hilbert space, J. Theoret. Probab. 10 (1997), no. 3, 681–693. MR-1468399

[14] T. Mori and K. Yoshihara, A note on the central limit theorem for stationary strong-mixing sequences, Yokohama Math. J. 34 (1986), no. 1-2, 143-146. MR-886062

[15] D. N. Politis and J. P. Romano, Limit theorems for weakly dependent Hilbert space valued random variables with application to the stationary bootstrap, Statist. Sinica 4 (1994), no. 2, 461–476. MR-1309424

[16] E. Rio, Covariance inequalities for strongly mixing processes, Ann. Inst. H. Poincaré Probab. Statist. 29 (1993), no. 4, 587–597. MR-1251142

[17] E. Rio, Théorie asymptotique des processus aléatoires faiblement dépendants, Mathématiques & Applications (Berlin) [Mathematics & Applications], vol. 31, Springer-Verlag, Berlin, 2000. MR-2117923

[18] M. Rosenblatt, A central limit theorem and a strong mixing condition, Proc. Nat. Acad. Sci. U. S. A. 42 (1956), 43–47. MR-0074711

[19] H. P. Rosenthal, On the subspaces of $L^p$ (p > 2) spanned by sequences of independent random variables, Israel J. Math. 8 (1970), 273–303. MR-0271721

[20] C. Tone, Central limit theorems for Hilbert-space valued random fields satisfying a strong mixing condition, ALEA Lat. Am. J. Probab. Math. Stat. 8 (2011), 77–94. MR-2754401

[21] V. A. Volkonskiü and Y. A. Rozanov, Some limit theorems for random functions. I, Teor. Veroyatnost. i Primenen 4 (1959), 186–207. MR-0105741
A mixing counter-example to the central limit theorem in Hilbert spaces

[22] D. Volný, *Approximation of stationary processes and the central limit problem*, Probability theory and mathematical statistics (Kyoto, 1986), Lecture Notes in Math., vol. 1299, Springer, Berlin, 1988, pp. 532–540. MR-936028

**Acknowledgments.** The authors would like to thank both referees for helpful comments, and for suggesting Remark 2.