TWO-TERM SILTING COMPLEXES OVER ALGEBRAS WITH RADICAL SQUARE ZERO

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Abstract. We give an explicit description of two-term silting complexes over algebras with radical square zero in terms of tilting modules over path algebras. As an application, we prove that the number of two-term tilting complexes over Brauer algebras (respectively, Brauer cycle algebras) with $n$ edges is $\binom{2n}{n}$ (respectively, $2^{2n-1}$ if $n$ is odd and $\infty$ if $n$ is even).

Introduction

Tilting theory is essential in the representation theory of finite dimensional algebras over a field. Tilting theory originates from BGP (= Bernstein-Gelfand-Ponomarev) reflections [BGP] and tilting modules over finite dimensional algebras [BH]. This viewpoint is adopted by Rickard to develop a Morita theory for derived categories [Ric].

Recently, Adachi [Ada2] gave a characterization of of representation-finite algebras with radical square zero [Ga]. Recently, as an analog of this result, and a map $Q$, and arrow set $(i, j) \rightarrow (i', j')$ in $\mathbb{Q}_1$. This equivalence gives a characterization of representation-finite algebras with radical square zero [Ga]. Recently, as an analog of this result, Adachi [Ada2] gave a characterization of $\tau$-tilting-finite algebras with radical square zero in terms of single subquivers of $Q$ (see also [Zha]). The connection between path algebras and algebras with radical square zero also appears as quadratic duality or Koszul duality for positively graded algebras [BGS].

In this paper, we study two-term silting complexes for algebras with radical square zero over an algebraically closed field $k$, which provide one of the most fundamental classes of algebras. Let $\Lambda$ be a finite dimensional $k$-algebra with radical square zero and $Q = (Q_0, Q_1)$ the quiver of $\Lambda$, where $Q_0$ is the set of vertices and $Q_1$ is the set of arrows. Then $\Lambda$ is presented as the path algebra of $Q$ modulo the ideal generated by all paths of length 2. A crucial role is played by the stable equivalence $\mathbb{Q}_1 \cong \mathbb{Q}_0$, where the separated quiver $Q^s = (Q^s_0, Q^s_1)$ of $Q$ is defined by the vertex set $Q^s_0 := \{i^s \mid i \in Q_0\} \cup \{i^- \mid i \in Q_0\}$ and the arrow set $Q^s_1 := \{i^s \rightarrow j^- \mid i \rightarrow j \in Q_1\}$. This equivalence gives a characterization of representation-finite algebras with radical square zero [Ga]. Recently, as an analog of this result, Adachi [Ada2] gave a characterization of $\tau$-tilting-finite algebras with radical square zero in terms of single subquivers of $Q$ (see also [Zha]). The connection between path algebras and algebras with radical square zero also appears as quadratic duality or Koszul duality for positively graded algebras [BGS].

In Section 2, we prove the main result in this paper, which further strengthens the result of Ada2. We need to modify the maximal single subquiver as follows (see Remark 2.15 for detail): For a given quiver $Q$ and a map $\epsilon : Q_0 \rightarrow \{+,-\}$, let $Q_\epsilon = (Q_0, Q_1, Q_2)$ be the quiver of $Q$ with vertex set $(Q_\epsilon)_0 = Q_0$ and arrow set $(Q_\epsilon)_1 = \{i \rightarrow j \in Q_1 \mid \epsilon(i) = +, \epsilon(j) = -\}$. Then we have the following, where $\tilde{\Lambda}$ is the set of isomorphism classes of basic tilting $\Lambda$-modules, and $2$-$\text{silt} \Lambda$ is the set of isomorphism classes of basic two-term silting complexes for $\Lambda$.

Theorem 0.1 (Theorem 2.15). Let $\Lambda$ be a finite dimensional $k$-algebra with radical square zero and $Q$ the quiver of $\Lambda$. Then there is a bijection

$$\tilde{\Lambda} \cong \prod_{Q_0 \rightarrow \{+,-\}} \text{tilt}(kQ_\epsilon)^{\text{op}} \xrightarrow{1-1} 2\text{-silt} \Lambda.$$ 

Moreover, for each map $\epsilon$, the restriction of this bijection gives an isomorphism of partially ordered sets from $\text{tilt}(kQ_\epsilon)^{\text{op}}$ to its image.
On the other hand, the study of symmetric algebras with radical cube zero can be reduced to that of algebras with radical square zero. Representation theory of symmetric algebras with radical cube zero has been studied by Benson [Ben2], Erdmann and Solberg [ES], and these algebras also appear in several area such as [HK, CL, Sei]. Recently, Green and Schroll [GS1, GS2] showed that this class of algebras is precisely the Brauer configuration algebras with radical cube zero. In Section 1, we see that the same result as Theorem 0.1 holds for symmetric algebras with radical cube zero (Theorem 3.3). As an application, we calculate the number of isomorphism classes of basic two-term tilting complexes for two classes of algebras, namely Brauer line algebras and Brauer cycle algebras [Ben1].

**Theorem 0.2.**

1. Let \( \Gamma_n \) be a Brauer line algebra with \( n \) edges. Then we have
\[
\# 2\text{-tilt} \, \Gamma_n = \binom{2n}{n}.
\]

2. Let \( \Xi_n \) be a Brauer cycle algebra with \( n \) edges. Then we have
\[
\# 2\text{-tilt} \, \Xi_n = \begin{cases} 2^{2n-1} & \text{if } n \text{ is odd}, \\ \infty & \text{if } n \text{ is even}. \end{cases}
\]

**Notations.** In this paper, we use the following notations. By an algebra we mean a finite dimensional algebra over an algebraically closed field \( k \), by a module we mean a finitely generated right module unless otherwise noted. For an algebra \( \Lambda \), we denote by \( \text{mod} \, \Lambda \) (respectively, \( \text{proj} \, \Lambda \)) the category of \( \Lambda \)-modules (respectively, projective \( \Lambda \)-modules). We denote by \( \text{C}^b(\text{proj} \, \Lambda) \) the category of bounded complexes of \( \text{proj} \, \Lambda \), by \( \text{K}^b(\text{proj} \, \Lambda) \) its homotopy category. Then \( \text{K}^b(\text{proj} \, \Lambda) \) is the triangulated category whose suspension functor is given by the shift functor \([1]\). For an object \( X \), it is said to be basic if it is isomorphic to a direct sum of indecomposable objects which are mutually non-isomorphic, and \(|X|\) is the number of isomorphism classes of indecomposable direct summands of \( X \). We denote by \( \text{add} \, X \) the category of all direct summands of direct sums of copies of \( X \).

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1. **Preliminaries**

1.1 **Silting theory.** In this section, we recall basic properties of silting complexes. Let \( \Lambda \) be an algebra.

**Definition 1.1.** [AI, Definition 2.1] Let \( T \) be a complex in \( \text{K}^b(\text{proj} \, \Lambda) \).

1. We say that \( T \) is *presilting* if \( \text{Hom}_{\text{K}^b(\text{proj} \, \Lambda)}(T, T[i]) = 0 \) for all positive integers \( i > 0 \).
2. We say that \( T \) is *silting* if it is presilting and generates \( \text{K}^b(\text{proj} \, \Lambda) \) as the thick subcategory.

Moreover, it is called *tilting* if it also satisfies \( \text{Hom}_{\text{K}^b(\text{proj} \, \Lambda)}(T, T[i]) = 0 \) for \( i < 0 \).

For example, any silting complex in a symmetric algebra is tilting [AI, Example 2.8]. Let \( T = X \oplus U \) be a basic silting complex for \( \Lambda \) with \( X \) indecomposable. Take a triangle
\[
X \xrightarrow{f} U' \longrightarrow Y \longrightarrow X[1]
\]
where \( f \) is a left minimal \( (\text{add} \, U) \)-approximation of \( X \), that is, \( f \) is a left minimal morphism such that \( \text{Hom}_{\text{K}^b(\text{proj} \, \Lambda)}(f, U'') \colon \text{Hom}_{\text{K}^b(\text{proj} \, \Lambda)}(U', U'') \to \text{Hom}_{\text{K}^b(\text{proj} \, \Lambda)}(X, U'') \) is surjective for any \( U'' \in \text{add} \, U \).

Then \( \mu_X(T) \coloneqq Y \oplus U \) is again a silting complex and called a *left silting mutation* of \( T \) (with respect to \( X \)). Dually, we define a right silting mutation \( \mu_X^{-1}(T) \) of \( T \).

It is known that the set of isomorphism classes of basic silting complexes for \( \Lambda \) has a structure of partially ordered set with partial order \( \geq \) defined as \( T \geq U \) if \( \text{Hom}_{\text{K}^b(\text{proj} \, \Lambda)}(T, U[i]) = 0 \) for all \( i > 0 \). Moreover, the Hasse quiver of this partially ordered set is given as follows:

- Each vertex is an isomorphism class of a basic silting complex.
- We draw an arrow \( T \rightarrow U \) if \( U \) is a left silting mutation of \( T \).

Next we recall some facts for two-term silting complexes and \( g \)-vectors. We say that a complex \( T \) in \( \text{K}^b(\text{proj} \, \Lambda) \) is *two-term* if it is a complex concentrated in degree 0 and \( -1 \), i.e. of the form
\[
(\cdots \longrightarrow 0 \longrightarrow T^{-1} \xrightarrow{d_{-1}} T^0 \longrightarrow 0 \rightarrow \cdots).
\]
Up to isomorphism, we can assume the differential $d_T$ is in the radical of $\text{proj} \Lambda$. We denote by 2-silt $\Lambda$ (respectively, 2-tilt $\Lambda$) the set of isomorphism classes of basic two-term silting (respectively, tilting) complexes for $\Lambda$. We need the following observations.

**Proposition 1.7.**\[AI\] Let $T = (T^{-1} \xrightarrow{d_T} T^0)$ be a basic two-term presilting complex in $K^b(\text{proj} \Lambda)$. Then the following hold.

(i) $T$ is a direct summand of a two-term silting complex for $\Lambda$. In addition, it is silting if and only if $|T| = |\Lambda|$.
(ii) $\text{add} T^0 \cap \text{add} T^{-1} = 0$ holds.
(iii) If $T$ is silting, then $\Lambda \in \text{add}(T^0 \oplus T^{-1})$ holds.

**Lemma 1.3.** Let $T = (T^{-1} \xrightarrow{d_T} T^0)$ and $U = (U^{-1} \xrightarrow{d_U} U^0)$ be two-term complexes in $K^b(\text{proj} \Lambda)$. Then we have the following exact sequence:

$$0 \to \text{Hom}_C(T,U) \to \text{Hom}_K(T^{-1},U^{-1}) \to \text{Hom}_K(T^0,U^0) \to \text{Hom}_K(T^{-1},U^0) \to 0$$

where $\eta$ is defined as $\eta(h^{-1},h^0) := h^0d_T-h_Uh^{-1}$ for each $h^{-1} \in \text{Hom}_K(T^{-1},U^{-1})$ and $h^0 \in \text{Hom}_K(T^0,U^0)$.

**Proof.** It is obvious from the definition of $\eta$. \qed

Let $Q = (Q_0, Q_1)$ be the quiver of $\Lambda$, where $Q_0$ is the set of vertices and $Q_1$ is the set of arrows. For each vertex $i \in Q_0$, we denote by $e_i$ the primitive idempotent corresponding to $i$, and $P(i) := e_i\Lambda$. Let $K_0(\text{proj} \Lambda)$ be the Grothendieck group of the additive category $\text{proj} \Lambda$. Then the set of residue classes of $P(i)$ gives a basis of $K_0(\text{proj} \Lambda)$, and the Grothendieck group $K_0(K^b(\text{proj} \Lambda))$ is canonically isomorphic to $K_0(\text{proj} \Lambda)$.

**Definition 1.4.**\[AIR, Section 5.1\] For any two-term complex $T$ in $K^b(\text{proj} \Lambda)$, we define the $g$-vector $g^T := (g_i^T)_{i \in Q_0}$ of $T$ by the equality in $K_0(K^b(\text{proj} \Lambda))$:

$$|T| = \sum_{i \in Q_0} g_i^T \cdot [P(i)].$$

It is known that $g$-vectors uniquely determine basic two-term silting complexes.

**Proposition 1.5.**\[AIR, Theorem 5.5\] Let $T, U$ be two-term presilting complexes in $K^b(\text{proj} \Lambda)$. If they satisfy $g^T = g^U$, then $T \cong U$.

Next, let $\epsilon : Q_0 \to \{+, -\}$ be a map. We consider the following full subcategories of $\text{proj} \Lambda$:

$$\text{proj}^+ \Lambda := \text{add}(e^+\Lambda) \quad \text{and} \quad \text{proj}^- \Lambda := \text{add}(e^-\Lambda),$$

where $e^\pm := \sum_{i \in Q_0^\pm} e_i$ with $Q_0^\pm := \{i \in Q_0 \mid \epsilon(i) = \pm\}$ respectively. Moreover, let $K_2^\epsilon(\text{proj} \Lambda)$ be the full subcategory of $K^b(\text{proj} \Lambda)$ consisting of all two-term complexes $T = (T^{-1} \xrightarrow{d_T} T^0)$ such that $T^0 \in \text{proj}^\epsilon \Lambda$ and $T^{-1} \in \text{proj}^- \Lambda$.

**Remark 1.6.**

(1) The category $K_2^\epsilon(\text{proj} \Lambda)$ is closed under extensions $\cdot$, that is, if we have a triangle $U \to V \to T \to U[1]$ with $T, U \in K_2^\epsilon(\text{proj} \Lambda)$, then $V \in K_2^\epsilon(\text{proj} \Lambda))$. This is clear since $V$ is the mapping cone of $T[-1] \to U$.

(2) Any basic complex $T$ in $K_2^\epsilon(\text{proj} \Lambda)$ satisfies $\text{add} T^0 \cap \text{add} T^{-1} = 0$.

The importance of this category in silting theory is highlighted in the following result, where 2-silt $\Lambda := 2$-silt $\Lambda \cap K^b(\text{proj} \Lambda)$.

**Proposition 1.7.** Let $\Lambda$ be an algebra and $Q$ the quiver of $\Lambda$. Then we have

$$2\text{-silt} \Lambda = \coprod_{\epsilon : Q_0 \to \{+, -\}} 2\text{-silt}^\epsilon \Lambda.$$

In particular, if 2-silt $\Lambda$ is a finite set, then $\# 2\text{-silt} \Lambda = \sum_{\epsilon} \# 2\text{-silt}^\epsilon \Lambda$.

**Proof.** We will show the first equation. Clearly, the set in the right side is a disjoint union. Thus, it is enough to show that any two-term silting complex $T$ is included in 2-silt $\Lambda$ for some $\epsilon$. Let $T = (T^{-1} \xrightarrow{d_T} T^0)$ be a basic two-term silting complex for $\Lambda$. Since each indecomposable projective $\Lambda$-module appears
in precisely one of either $T^0$ or $T^{-1}$ by Proposition 1.3(ii) and (iii), we can define a map $\epsilon := \epsilon_T: Q_0 \to \{+,-\}$ as $\epsilon(i) = +$ if $P(i) \in \text{add} T^0$ and $\epsilon(i) = -$ if $P(i) \in \text{add} T^{-1}$. Then, we have $T \in 2\text{-silt}_* \Lambda$.

The latter assertion is obvious from the former one. \hfill $\square$

In Section 2 we will investigate the category $K^2_0(\text{proj} \Lambda)$ in the case when $\Lambda$ is an algebra with radical square zero.

1.2 Path algebras. Let $Q = (Q_0, Q_1)$ be a finite acyclic quiver and $kQ$ the path algebra of $Q$. Now, we recall the famous characterization due to Gabriel of representation-finite path algebras $kQ$, where tilt $kQ$ is the set of isomorphism classes of basic tilting $kQ$-modules.

**Proposition 1.8.** [ASS, VII, VIII] Let $Q$ be a connected quiver. Then the following conditions are equivalent:

1. $kQ$ is representation-finite.
2. $Q$ is one of a Dynkin quiver $A_n$, $D_n$ ($n \geq 4$), $E_6$, $E_7$ and $E_8$.
3. tilt $kQ$ is a finite set.

For Dynkin quivers, it is known that the number $\# \text{tilt} kQ$ does not depend on orientations.

**Proposition 1.9.** (see [ONFR] for example) Let $Q$ be one of a Dynkin quiver. Then the number $\# \text{tilt} kQ$ is given as follows, where $C_n$ is the $n$-th Catalan number:

\[
\begin{array}{|c|c|c|c|c|c|c|}
\hline
Q & \# \text{tilt} kQ & A_n & D_n \,(n \geq 4) & E_6 & E_7 & E_8 \\
\hline
\hline
\end{array}
\]

On the other hand, the following observation is useful for the vanishing condition of the Ext$^1$-group of modules, which comes from the viewpoint of quiver representation theory. Let $S := kQ_0$ be a semisimple algebra spanned by $Q_0$, then the vector space $V := kQ_1$ has a natural structure of $S$-$S$-bimodule.

**Proposition 1.10.** [GR, 7.2] In the above notations, we have the following exact sequence:

\[0 \to \text{Hom}_S(X, Y) \to \text{Hom}_S(XS, YS) \to \text{Hom}_S(X \otimes_S V_S, YS) \to \text{Ext}^1_S(X, Y) \to 0\]

for any $kQ$-modules $X$ and $Y$, where $\delta$ maps $f: X_S \to Y_S$ onto $\delta(f)(x \otimes \alpha) := f(x\alpha) - f(x)\alpha$ for any $x \in X$ and $\alpha \in V$. \hfill $\square$

1.3 Triangular matrix algebras. The bimodule construction of triangular matrix algebra gives a new algebra from given algebras. Let $\Lambda$ and $\Gamma$ be algebras and $M$ be a $\Lambda$-$\Gamma$-bimodule. Then we can construct the upper triangular matrix algebra $\Delta := \begin{pmatrix} \Lambda & M \\ 0 & \Gamma \end{pmatrix}$. The category $\text{mod} \Delta$ is described as follows: Let $C_\Delta$ be the category with the following data:

- **objects**: triples $(X', X'', \varphi)$ where $X' \in \text{mod} \Lambda$, $X'' \in \text{mod} \Gamma$ and $\varphi: X' \otimes \Lambda M \to X''$ is a morphism of $\Gamma$-modules.
- **morphisms**: a morphism from $(X', X'', \varphi)$ to $(Y', Y'', \varphi')$ in $C_\Delta$ is a pair $(f, g)$ where $f: X' \to Y'$ is a morphism of $\Lambda$-modules and $g: X'' \to Y''$ is a morphism of $\Gamma$-modules such that the diagram

\[
\begin{array}{ccc}
X' \otimes \Lambda M & \longrightarrow & X'' \\
\downarrow f \otimes 1_M & & \downarrow g \\
Y' \otimes \Lambda M & \longrightarrow & Y''
\end{array}
\]

commutes.

Next, we define a functor $F: C_\Delta \to \text{mod} \Delta$ as follows: For an object $(X', X'', \varphi)$ in $C_\Delta$, set $F(X', X'', \varphi) := X' \oplus X''$ as abelian group, and the right $\Delta$-module structure is given by

\[(x, y) \begin{pmatrix} a & m \\ 0 & b \end{pmatrix} := (xa, \varphi(x \otimes m) + yb)\]

for $x \in X', y \in X'', a \in \Lambda, b \in \Gamma$ and $m \in M$. If we have a morphism $(f, g): (X', X'', \varphi) \to (Y', Y'', \varphi')$ in $C_\Delta$, then $F(f, g) := \begin{pmatrix} f & 0 \\ 0 & g \end{pmatrix}: X' \oplus X'' \to Y' \oplus Y''$. 
**Proposition 1.11.** [ARS, III, Proposition 2.2] In the above notations, the functor $F: C_\Delta \to \text{mod} \Delta$ is an equivalence of additive categories.

2. Main results

In this section, we study the silting theory for the algebras with radical square zero. The following quiver plays an important role in this paper.

**Definition 2.1.** Let $Q = (Q_0, Q_1)$ be a finite quiver. For a given map $\epsilon: Q_0 \to \{+, -\}$, let $Q_\epsilon = \langle (Q_\epsilon)_0, (Q_\epsilon)_1 \rangle$ be the subquiver of $Q$ with vertex set $(Q_\epsilon)_0 = Q_0$ and arrow set 

$$(Q_\epsilon)_1 = \{ i \to j \in Q_1 \mid i \in Q_0^\epsilon \text{ and } j \in Q_0\},$$

where we recall $Q_0^+ = \epsilon^{-1}(+) \text{ and } Q_0^- = \epsilon^{-1}(-)$.

Note that, for any map $\epsilon: Q_0 \to \{+, -\}$, the quiver $Q_\epsilon$ is bipartite, that is, every vertex in $Q_\epsilon$ is either a sink or a source. In particular, the path algebra $kQ_\epsilon$ is an algebra with radical square zero.

**Example 2.2.** Let $Q$ be the following quiver:

$$Q: \quad \begin{array}{ccc} 1 & \rightarrow & 2 \\ \rightarrow & & \rightarrow \\ 3 & \leftarrow & 2 \end{array}$$

Then we obtain the following example of quivers $Q_\epsilon$ for $\epsilon$, where we identify $\epsilon$ with $(\epsilon(1), \epsilon(2), \epsilon(3))$:

$$Q_{(+++)} = [1 \quad 2 \quad 3], \quad Q_{(+- +)} = [1 \oplus \quad 2 \quad 3], \quad Q_{(-++)} = [1 \quad 2 \quad 3].$$

The main result in this paper is the following. It will be proved in Section 2.2.

**Theorem 2.3.** Let $\Lambda$ be an algebra with radical square zero and $Q$ the quiver of $\Lambda$. For each map $\epsilon: Q_0 \to \{+, -\}$, we have an isomorphism of partially ordered sets

$$\text{2-silt}_\Lambda \xrightarrow{\sim} \text{tilt}(kQ_\epsilon)^{\text{op}}.$$  

Therefore, the Hasse quivers of $\text{2-silt}_\Lambda$ and $\text{tilt}(kQ_\epsilon)^{\text{op}}$ are isomorphic. Furthermore, we have a bijection

$$\text{2-silt}_\Lambda \xrightarrow{\sim} \prod \text{tilt}(kQ_\epsilon)^{\text{op}}.$$  

2.1 Reduction of the quiver.

In this subsection, we will “reduce” the problem of silting theory on the algebra $\Lambda$ with radical square zero to that on $kQ_\epsilon$. We start with the following easy lemma.

**Lemma 2.4.** For any complexes $T, U$ in $K^b_2(\text{proj} \Lambda)$, we have

$$\text{Hom}_{K^b_2(\text{proj} \Lambda)}(T, U) = \text{Hom}_{C^b(\text{proj} \Lambda)}(T, U).$$

**Proof.** We will show that any morphism between complexes which is null-homotopic is zero. Let $T = (T^{-1} \xrightarrow{d_T} T^0)$ and $U = (U^{-1} \xrightarrow{d_U} U^0)$ are complexes in $K^b_2(\text{proj} \Lambda)$. Up to isomorphism, we can assume each differential $d_T$ and $d_U$ is in the radical of $\text{proj} \Lambda$. Assume $(f^{-1}, f^0) \in \text{Hom}_{K^b_2(\text{proj} \Lambda)}(T, U)$ is null-homotopic. By definition, there exists a morphism $h: T^0 \to U^{-1}$ of $\Lambda$-modules such that $f^{-1} = h \circ d_T$ and $f^0 = d_U \circ h$. Since both of $T$ and $U$ are in $K^b_2(\text{proj} \Lambda)$, we have $\text{add} T^0 \cap \text{add} U^{-1} = 0$, and hence $h$ is in the radical. Therefore, $f^{-1} = h \circ d_T = 0$ and $f^0 = d_U \circ h = 0$ hold since $\Lambda$ is an algebra with radical square zero. Thus, the assertion holds.

Let $Q$ be the quiver of $\Lambda$. In the rest of this subsection, we fix a map $\epsilon: Q_0 \to \{+, -\}$ and let $\Lambda_\epsilon := kQ_\epsilon$.

**Proposition 2.5.**

1. $\Lambda_\epsilon = \Lambda/\langle i \to j \mid (\epsilon(i), \epsilon(j)) \neq (+, -) \rangle$.

2. There are an injection $i: \Lambda_\epsilon \hookrightarrow \Lambda$ and a surjection $\pi: \Lambda \twoheadrightarrow \Lambda_\epsilon$ of algebras such that $\pi \circ i = \text{id}_{\Lambda_\epsilon}$.

**Proof.** (1) Since $\Lambda$ is an algebra with radical square zero, we have $\Lambda = kQ_0 \oplus kQ_1$ as a $\mathbb{Z}$-graded algebra. Similarly, $\Lambda_\epsilon = kQ_0 \oplus k(Q_\epsilon)_1$. It is obvious that the ideal $I = \langle i \to j \mid (\epsilon(i), \epsilon(j)) \neq (+, -) \rangle$ in $\Lambda$ is precisely the $k$-vector space spanned by all arrows which are not contained in $(Q_\epsilon)_1$. Therefore, we have $\Lambda/I = kQ_0 \oplus k(Q_\epsilon)_1 = \Lambda_\epsilon$ as $k$-algebras.

(2) Since $\Lambda_\epsilon = kQ_0 \oplus k(Q_\epsilon)_1$ holds, $\Lambda_\epsilon$ is a subspace of $\Lambda$ and clearly $1_\Lambda \in \Lambda_\epsilon$. Therefore, it is a subalgebra of $\Lambda$. Thus, we have $i: \Lambda_\epsilon \hookrightarrow \Lambda$, and it is obvious that the composition $\pi \circ i: \Lambda_\epsilon \hookrightarrow \Lambda$ is the identity on $\Lambda_\epsilon$, where $\pi: \Lambda \twoheadrightarrow \Lambda_\epsilon$ is a surjection given by (1).
By Proposition 2.6, we have the following two functors:

\[-\otimes \Lambda : \text{proj } \Lambda \to \text{proj } \Lambda^e \quad \text{and} \quad -\otimes \Lambda : \text{proj } \Lambda \to \text{proj } \Lambda^e\]

such that \(1_{\text{proj } \Lambda_e} \simeq -\otimes \Lambda \otimes \Lambda \), and the restriction gives the full functors \(\text{proj}^+ \Lambda \to \text{proj}^+ \Lambda_e\) and \(\text{proj}^- \Lambda \to \text{proj}^- \Lambda_e\).

**Proposition 2.6.** For any \(X \in \text{proj}^+ \Lambda\), \(Y \in \text{proj}^- \Lambda\), we have

\[
\text{Hom}_\Lambda(Y, X) \simeq \text{Hom}_{\Lambda_e}(Y \otimes \Lambda \otimes \Lambda, X \otimes \Lambda \otimes \Lambda).
\]

Similarly, for any \(X' \in \text{proj}^+ \Lambda_e\), \(Y' \in \text{proj}^- \Lambda_e\), we have

\[
\text{Hom}_\Lambda(Y' \otimes \Lambda, X' \otimes \Lambda, \Lambda) \simeq \text{Hom}_{\Lambda_e}(Y', X').
\]

**Proof.** Since \(\Lambda\) is an algebra with radical square zero, any morphism \(f : Y \to X\) of projective \(\Lambda\)-modules is given as a linear combination of arrows in \(Q\). In particular, if \(X \in \text{proj}^+ \Lambda\) and \(Y \in \text{proj}^- \Lambda\), then \(f\) is a linear combination of arrows from vertices in \(Q^+_0\) to those in \(Q^-_0\), i.e., arrows in \((Q_e)_1\). Therefore, \(f\) can be regarded as a morphism in \(\text{Hom}_\Lambda(Y \otimes \Lambda, X \otimes \Lambda, \Lambda)\), and vice versa. Therefore, we obtain the desired isomorphism.

The latter assertion follows from the former one. \(\square\)

By Proposition 2.6, the tensor products provide the following well-defined functors:

\[
\text{K}_2(\text{proj } \Lambda) \xrightarrow{F} \text{K}_2(\text{proj } \Lambda^e),
\]

where \(F := -\otimes \Lambda\) and \(F' := -\otimes \Lambda\). It is easy to see that \(1_{\text{K}_2(\text{proj } \Lambda)} \simeq F \circ F'\) holds. Although neither \(F\) nor \(F'\) is an equivalence of categories in general, they are enough to calculate silting complexes for \(\epsilon\).

**Proposition 2.7.** Let \(T, U\) be complexes in \(\text{K}_2(\text{proj } \Lambda)\). Then we have an isomorphism of bifunctors

\[
\text{Hom}_{\text{K}_2(\text{proj } \Lambda)}(T, U[1]) \simeq \text{Hom}_{\text{K}_2(\text{proj } \Lambda)}(FT, FU[1]).
\]

**Proof.** Since \(T\) and \(U\) are two-term, we have the following diagram of exact sequences by Lemma 1.3:

\[
\begin{array}{c}
\bigoplus_{j=-1,0} \text{Hom}_\Lambda(T^j, U^j) \xrightarrow{\eta} \text{Hom}_\Lambda(T^{-1}, U^0) \to \text{Hom}_{\text{K}_2(\text{proj } \Lambda)}(T, U[1]) \to 0 \\
\bigoplus_{j=-1,0} \text{Hom}_\Lambda((FT)^j, (FU)^j) \xrightarrow{\eta'} \text{Hom}_\Lambda((FT)^{-1}, (FU)^0) \to \text{Hom}_{\text{K}_2(\text{proj } \Lambda)}(FT, FU[1]) \to 0
\end{array}
\]

where \(r\) is a surjection defined as \(r(f^{-1}, f^0) := (f^{-1} \otimes 1_{\Lambda_e}, f^0 \otimes 1_{\Lambda_e})\). Since the left square in this diagram commutes, we obtain the desired isomorphism. \(\square\)

As a consequence, we obtain the following result.

**Corollary 2.8.** The functor \(F\) gives an isomorphism of partially ordered sets

\[
\text{2-silt } \Lambda \to \text{2-silt } \Lambda_e.
\]

**Proof.** By Proposition 2.6, it is easy to see that the functor \(F\) gives a bijection of isomorphism classes of indecomposable objects in both categories. Then the assertion is immediate from Proposition 1.2(4) and Proposition 2.7. \(\square\)

### 2.2 Bipartite Quiver

Thanks to Corollary 2.8 it is enough to consider the following situation: Let \(Q = (Q_0, Q_1)\) be a bipartite quiver. Define a map \(\epsilon : Q_0 \to \{+, -\}\) as \(\epsilon(i) = +\) if \(i\) is a source and \(\epsilon(i) = -\) if \(i\) is a sink. Then \(Q = Q_e\).

Let

\[
\Lambda_e := kQ, \quad S^+ := kQ_0^+ \quad \text{and} \quad S^- := kQ_0^-.
\]

Then \(V := kQ_1\) has a natural structure of \(S^+\)-\(S^-\)-bimodule. Consider the opposite algebra \(\Lambda_e^{\text{op}}\), then there are isomorphisms of algebras:

\[
\Lambda_e \simeq \begin{pmatrix} S^+ & V \\ 0 & S^- \end{pmatrix} \quad \text{and} \quad \Lambda_e^{\text{op}} \simeq \begin{pmatrix} S^- & V^* \\ 0 & S^+ \end{pmatrix}.
\]
where \( V^* = \text{Hom}_V(V,k) \) is the \( k \)-dual of \( V \). Under this identification, each \( \Lambda^{op}_p \)-module \( X \) can be written as a triple \((X^-,X^+,\varphi_X)\) such that \( X^- \in \text{mod } S^- \), \( X^+ \in \text{mod } S^+ \) and \( \varphi_X : X^- \otimes_S V^* \to X^+ \) as in Section 1.4.

On the other hand, since \( Q \) is bipartite we have the following equivalences of categories:

\[
\text{(2.8)} \quad - \otimes_{S^-} e^+\Lambda_e : \text{mod } S^+ \to \text{proj}^+_\Lambda_e \quad \text{and} \quad - \otimes_{S^+} e^-\Lambda_e : \text{mod } S^- \to \text{proj}^-\Lambda_e.
\]

**Proposition 2.9.** For any \( M \in \text{mod } S^+ \) and \( N \in \text{mod } S^- \), we have an isomorphism of bifunctors

\[
\text{(2.9)} \quad \widehat{(-)} : \text{Hom}_{S^+}(N \otimes_{S^-} V^*, M) \cong \text{Hom}_{\Lambda_e}(N \otimes_{S^-} e^-\Lambda_e, M \otimes_{S^+} e^+\Lambda_e)
\]

**Proof.** Let \( M \in \text{mod } S^+ \) and \( N \in \text{mod } S^- \). Note that \( V = e^+\Lambda_e e^- \) since \( Q \) is bipartite. By the hom-tensor adjoint, we have

\[
\text{Hom}_{S^+}(N \otimes_{S^-} V^*, M) \cong \text{Hom}_{S^-}(N, \text{Hom}_{\Lambda_e}(e^-\Lambda_e, M \otimes_{S^+} e^+\Lambda_e)).
\]

Thus, we obtain the desired isomorphism. It is easy to check that it is a natural transformation. \( \square \)

Now, we set a two-term complex \( G(X) := ((X^- \otimes_{S^-} e^-\Lambda_e) \oplus (X^+ \otimes_{S^+} e^+\Lambda_e)) \in \mathcal{K}_2^e(\text{proj} \Lambda_e) \) for any \( X \in \text{mod } \Lambda^{op}_p \). The following observation is crucial.

**Theorem 2.10.** (1) The correspondence \( G \) gives an equivalence

\[
G : \text{mod } \Lambda^{op}_p \cong \mathcal{K}_2^e(\text{proj} \Lambda_e).
\]

(2) For any \( \Lambda^{op}_p \)-modules \( X \) and \( Y \), the following hold:

(i) \( \text{Ext}_{\Lambda^{op}_p}^1(X,Y) \cong \text{Hom}_{\text{proj} \Lambda_e}(G^0 X, G^0 Y[1]) \).

(ii) \( (\dim X)_i = |g_i(G(X))| \) for all \( i \in Q_0 \), where \( \dim X \) is the dimension vector of \( X \) and \( |g_i(G(X))| \) is the absolute value of \( g_i(G(X)) \) given in Definition 1.4.

**Proof.** (1) For any \( \Lambda^{op}_p \)-modules \( X \) and \( Y \), we obtain the following diagram of exact sequences by Lemma 1.3 and 1.10

\[
\begin{array}{ccccccccc}
0 & \to & \text{Hom}_{\Lambda^{op}_p}(X,Y) & \to & \bigoplus_{\sigma = +, -} \text{Hom}_{S^-}(X^\sigma,Y^\sigma) & \to & \text{Hom}_{S^+}(X^- \otimes_{S^-} V^+, Y^+) & \to & \text{Ext}_{\Lambda^{op}_p}^1(X,Y) & \to & 0 \\
\downarrow \scriptstyle{p} & & \downarrow \scriptstyle{\delta} & & \downarrow \scriptstyle{\phi} & & \downarrow \scriptstyle{\gamma} & & \downarrow \scriptstyle{\zeta} & & \downarrow \scriptstyle{\theta} \\
0 & \to & \text{Hom}_{\text{proj} \Lambda_e}(G X, G Y) & \to & \bigoplus_{j = 0, 1} \text{Hom}_{\Lambda_e}((G X)^j, (G Y)^j) & \to & \text{Hom}_{\Lambda_e}((G X)^-1, (G Y)^0) & \to & \text{Hom}_{\text{proj} \Lambda_e}(G T, G Y[1]) & \to & 0
\end{array}
\]

where we use Lemma 2.4 for the leftmost term in the bottom sequence. The commutativity of the middle square follows from Proposition 2.9. Thus we obtain isomorphisms \( p \) and \( \theta \) in the diagram. Moreover, it is easy to check that \( p \) is bifunctorial, and hence \( G \) gives a fully faithful functor \( \text{mod } \Lambda^{op}_p \to \mathcal{K}_2^e(\text{proj} \Lambda_e) \). Furthermore, it is dense since \( G \) gives a bijection of isomorphism classes of indecomposable objects in both categories by Proposition 2.9. Therefore, it is an equivalence.

(2)-(i) The isomorphism \( \theta \) in the above diagram is the desired one.

(2)-(ii) It is obvious from the construction of the two-term complex \( G(X) \). \( \square \)

The following result follows from Theorem 2.10.

**Corollary 2.11.** The equivalence \( G \) gives an isomorphism of partially ordered sets

\[
\text{(2.10)} \quad 2\text{-silt}_e \Lambda_e \cong \text{tilt} \Lambda_e^{op}.
\]

Now we are ready to prove Theorem 2.12.

**Proof of Theorem 2.12.** Let \( \epsilon : Q_0 \to \{+, -\} \) be a map. We apply Corollary 2.11 for \( \Lambda_e = kQ_\epsilon \). Then the composition \( 2\text{-silt}_e \Lambda \) \( \to \text{tilt} \Lambda_e^{op} \) gives the desired isomorphism.

The latter assertion follows from Proposition 1.7. \( \square \)

Now, we consider the algebra \( \Lambda \) with radical square zero again. In silting theory, it is important to know whether a given silting complex is silting (Definition 1.4.2)). For algebras with radical square zero, we have the following result.
Proposition 2.12. Let $Q$ be the quiver of $\Lambda$ and $\epsilon: Q_0 \to \{+,-\}$ be a map. Then the following conditions are equivalent.

1. $\{j \to i \in Q_1 \mid i \in Q^+_0 \text{ and } j \in Q^-_0\} = \emptyset$.
2. Any complex in $2\text{-silt}_\Lambda$ is tilting.
3. There exists a tilting complex in $2\text{-silt}_\Lambda$.

Proof. Let $T = (T^{-1} \xrightarrow{d_T} T^0) \in 2\text{-silt}_\Lambda$. By definition, $T$ is tilting if and only if $\text{Hom}_{\text{proj}^+(\text{proj}_\Lambda)}(T, T[-1]) = 0$. However, for any $f \in \text{Hom}_\Lambda(T^0, T^{-1})$, we have $f \circ d_T = 0 = d_T \circ f$ since $d_T$ and $f$ lie in the radical of $\text{proj}_\Lambda$. Thus, we have $\text{Hom}_{\text{proj}^{op}(\text{proj}_\Lambda)}(T, T[-1]) \cong \text{Hom}_\Lambda(T^0, T^{-1})$. On the other hand, it is easy to see that the condition (1) holds if and only if $\text{Hom}_\Lambda(P' \circ P'', \Lambda) = 0$ for any $P' \in \text{proj}^+_\Lambda$ and $P'' \in \text{proj}^-_\Lambda$.

(1) $\Rightarrow$ (2) Assume (1). Then $T$ is a tilting complex since $\text{Hom}_\Lambda(T^0, T^{-1}) = 0$ holds from the previous discussion with $T^0 \in \text{proj}^+_\Lambda$ and $T^{-1} \in \text{proj}^-_\Lambda$.

(2) $\Rightarrow$ (3) It is clear.

(3) $\Rightarrow$ (1) We show the contraposition. Assume that there exists an arrow $\alpha: j \to i$ in $Q$ with $i \in Q^+$, $j \in Q^-$. Then $0 \neq \alpha \in \text{Hom}_\Lambda(P(i), P(j))$. Since $T$ is silting, $P(i)$ and $P(j)$ are direct summands of $T^0$ and $T^{-1}$ respectively by Proposition 1.2. Thus we have $0 \neq \alpha \in \text{Hom}_\Lambda(T^0, T^{-1}) \cong \text{Hom}_\Lambda(T, T[-1])$. It implies that $T$ is not tilting.

Example 2.13. Let $\Lambda$ be an algebra with radical square zero given by the following quiver:

\begin{center}
\begin{tikzpicture}
\node (1) at (0,0) {$1$};
\node (2) at (1,0) {$2$};
\draw[->] (1) to (2);
\end{tikzpicture}
\end{center}

Namely, $\Lambda = kQ/I$ where $I$ is an ideal in $kQ$ generated by all path of length 2. Then we obtain the following figure of Hasse quivers, where we denote the Hasse quiver of a partially ordered set $\mathcal{S}$ by $\mathbb{H}(\mathcal{S})$:

\begin{center}
\begin{tikzpicture}
\node (0) at (0,0) {$0 \rightarrow P(1)$};
\node (1) at (1,0) {$P(2)$};
\draw[->] (0) to (1);
\node (2) at (0,-1) {$P(1) \rightarrow 0$};
\node (3) at (1,-1) {$P(2)$};
\draw[->] (2) to (3);
\node (4) at (0,-2) {$P(1) \rightarrow P(2)$};
\node (5) at (1,-2) {$P(2) \rightarrow P(1)$};
\draw[->] (4) to (5);
\end{tikzpicture}
\end{center}

where $S(i)$ is the simple $(kQ, \Lambda)^{\text{op}}$-module corresponding to $i \in Q_0$, and each indecomposable module is described as its composition series. By Theorem 2.3, we have an isomorphism $\mathbb{H}(2\text{-silt}_\Lambda) \cong \mathbb{H}(\text{tilt}(kQ, \Lambda)^{\text{op}})$ for each $\epsilon$. Moreover, we can compute the number $\# 2\text{-silt}_\Lambda = \sum_\epsilon \# 2\text{-silt}_\Lambda = \sum_\epsilon \# \text{tilt}(kQ, \Lambda)^{\text{op}} = 6$.

2.3 \textbf{\textit{$\tau$-tilting-finite algebras with radical square zero}.} In this subsection, we deduce Adachi’s characterization of $\tau$-tilting-finite algebras with radical square zero [Ada2] from our result. Recall that, by $\tau$-tilting theory [AIR], we have $\# 2\text{-silt}_\Lambda = \# s\tau\text{-tilt}_\Lambda$ where $s\tau\text{-tilt}_\Lambda$ is the set of isomorphism classes of basic support $\tau$-tilting $\Lambda$-modules. An algebra $\Lambda$ is said to be $\tau$-tilting-finite if $\# s\tau\text{-tilt}_\Lambda < \infty$. 

\begin{center}
\begin{tikzpicture}
\node (0) at (0,0) {$S(1)$};
\node (1) at (1,0) {$S(2)$};
\node (2) at (0,-1) {$S(1) \oplus S(2)$};
\node (3) at (1,-1) {$S(2) \oplus S(1)$};
\draw[->] (0) to (1);
\draw[->] (0) to (2);
\draw[->] (1) to (3);
\end{tikzpicture}
\end{center}

\begin{center}
\begin{tikzpicture}
\node (0) at (0,0) {$S(1)$};
\node (1) at (1,0) {$S(2)$};
\node (2) at (0,-1) {$S(1) \oplus S(2)$};
\node (3) at (1,-1) {$S(2) \oplus S(1)$};
\draw[->] (0) to (1);
\draw[->] (0) to (2);
\draw[->] (1) to (3);
\end{tikzpicture}
\end{center}

\begin{center}
\begin{tikzpicture}
\node (0) at (0,0) {$S(1)$};
\node (1) at (1,0) {$S(2)$};
\node (2) at (0,-1) {$S(1) \oplus S(2)$};
\node (3) at (1,-1) {$S(2) \oplus S(1)$};
\draw[->] (0) to (1);
\draw[->] (0) to (2);
\draw[->] (1) to (3);
\end{tikzpicture}
\end{center}

\begin{center}
\begin{tikzpicture}
\node (0) at (0,0) {$S(1)$};
\node (1) at (1,0) {$S(2)$};
\node (2) at (0,-1) {$S(1) \oplus S(2)$};
\node (3) at (1,-1) {$S(2) \oplus S(1)$};
\draw[->] (0) to (1);
\draw[->] (0) to (2);
\draw[->] (1) to (3);
\end{tikzpicture}
\end{center}
Definition 2.14. Let $Q = (Q_0, Q_1)$ be a finite quiver.

1. Define a quiver $Q^* = (Q_0^*, Q_1^*)$, called separated quiver of $Q$, with vertex set $Q_0^* := \{i^+ | i \in Q_0\}$ and arrow set $Q_1^* := \{i^+ \to j^- | i \to j \text{ in } Q_1\}$.
2. A full subquiver $Q'$ of $Q^*$ is called single subquiver if $Q'$ contains at most one of $i^+$ and $i^-$ for each $i \in Q_0$.

Remark 2.15. Our quivers $Q_e$ for $e$ can be identified with the maximal single subquivers of $Q^*$. In fact, vertices $i \in Q_0^*$ correspond to $i^\pm$ for each $i \in Q_0$.

Example 2.16. Let $Q$ be the quiver as in Example 2.2. Then the separated quiver $Q^*$ is given as follows:

$$Q^*: \begin{array}{ccc}
1^+ & 2^+ & 3^+ \\
\downarrow & \downarrow & \downarrow \\
1^- & 2^- & 3^-
\end{array}$$

For example, the subquiver $Q_{(+,-)}$ of $Q$ corresponds to a maximal single subquiver of $Q^*$ with vertex set $\{1^+, 2^-, 3^+\}$. Let $\Lambda$ be the algebra with radical square zero given by the quiver $Q$. Then we can find that $\Lambda$ is not $\tau$-tilting-finite by using the following Proposition 2.17.

Now, we have the following, where the equivalence of (1) and (2) was given in [Ada2, Theorem 3.1].

Proposition 2.17. Let $\Lambda$ be an algebra with radical square zero and $Q$ the quiver of $\Lambda$. Then the following conditions are equivalent.

1. $\Lambda$ is $\tau$-tilting-finite.
2. Every single subquiver of $Q^*$ is a disjoint union of Dynkin quivers.
3. For every map $e: Q_0 \to \{+, -\}$, the quiver $Q_e$ is a disjoint union of Dynkin quivers.

Proof. (2) $\iff$ (3): This is straightforward from Remark 2.15.

(1) $\iff$ (3): By definition, $\Lambda$ is $\tau$-tilting-finite if and only if 2-silt $\Lambda$ is a finite set. By Theorem 2.3, this is equivalent to $\# \text{ silt } kQ_\op < \infty$ for every map $e: Q_0 \to \{+, -\}$. By Lemma 1.3, this is equivalent to the condition that $Q_e$ is a disjoint union of Dynkin quivers for every $e$. $\square$

3. APPLICATION TO SYMMETRIC ALGEBRAS WITH RADICAL CUBE ZERO

In this section, we study two-term tilting complexes over symmetric algebras with radical cube zero. By the following result, this is equivalent to studying two-term silting complexes over algebras with radical square zero.

Proposition 3.1. [Ada1, EJR] Let $\Lambda$ be an indecomposable symmetric algebra with $\text{rad}^2 \Lambda \neq 0$, and let $\overline{\Lambda} := \Lambda/\text{soc} \Lambda$. Then we have an isomorphism of partially ordered sets

$$2\text{-silt } \Lambda \cong 2\text{-silt } \overline{\Lambda}.$$  

In Proposition 3.1, if $\Lambda$ satisfies $\text{rad}^3 \Lambda = 0$, then $\overline{\Lambda}$ is the algebra with radical square zero.

Proposition 3.2. [CS2] Let $\Lambda$ be an indecomposable symmetric algebra with radical cube zero but $\text{rad}^2 \Lambda \neq 0$, and let $\overline{\Lambda} := \Lambda/\text{soc} \Lambda$. Then the following hold.

1. The two algebras $\Lambda$ and $\overline{\Lambda}$ have the same quiver.
2. Let $Q$ be the quiver of $\Lambda$. Then $\# \{i \to j \text{ in } Q_1\} = \# \{j \to i \text{ in } Q_1\}$ holds for any $i, j \in Q_0$.

For the symmetric algebras with radical cube zero, we have the following result. Here, for a given map $e: Q_0 \to \{+, -\}$, let $-e: Q_0 \to \{+, -\}$ be a map defined as $(-e)(i) = -e(i)$ for every $i \in Q_0$.

Theorem 3.3. Let $\Lambda$ be a symmetric algebra with radical cube zero and $Q$ the quiver of $\Lambda$. Let $e: Q_0 \to \{+, -\}$ be a map. Then we have isomorphisms of partially ordered sets

$$\text{2-silt } \Lambda \cong \text{tilt}(kQ_e)^\op \cong (\text{2-silt } \overline{\Lambda})^\op.$$  

In particular, we have $\# \text{ 2-silt } \Lambda = \# \text{ tilt}(kQ_\op) = \# \text{ 2-silt } \Lambda$.

Proof. Without loss of generality, we can assume that $\Lambda$ is indecomposable.

(a) If $\Lambda$ satisfies $\text{rad}^3 \Lambda = 0$, then it is clear.
(b) Assume \( \text{rad}^2 \Lambda \neq 0 \). It is easy to see that \( Q_{\epsilon}^{op} = Q_\tau \), holds by Proposition \ref{3.2} (2). Then, we obtain the desired isomorphisms of partially ordered sets by Theorem \ref{2.3} and Proposition \ref{3.1}.

\[
2\text{-}\text{tilt}_\epsilon \Lambda \cong 2\text{-}\text{silt}_\tau k \cong \text{tilt} k Q_{\epsilon}^{op} \cong (\text{tilt} k Q_{\epsilon}^{op})^{op} \cong (2\text{-}\text{silt}_\tau k)^{op} \cong (2\text{-}\text{tilt}_\epsilon \Lambda)^{op}.
\]

The latter assertion follows from the former one. \( \square \)

In the rest of this paper, we calculate the number \( \# 2\text{-}\text{tilt}_\epsilon \Lambda \) for two classes of algebras, namely Brauer line algebras and Brauer cycle algebras.

### 3.1 Brauer line algebras.

**Definition 3.4.** A multiplicity-free Brauer line algebra \( \Gamma_n := kQ/I \) with \( n \) edges is defined by the following quiver and relations:

\[
Q: \quad 1 \overset{\alpha_1}{\longrightarrow} 2 \overset{\alpha_2}{\longrightarrow} 3 \overset{\alpha_3}{\longrightarrow} \cdots \overset{\alpha_{n-2}}{\longrightarrow} n-1 \overset{\alpha_{n-1}}{\longrightarrow} n
\]

and

\[
I = \langle \alpha_1 \beta_1 \alpha_1, \alpha_i \alpha_{i+1}, \beta_i \alpha_i - \alpha_{i+1} \beta_{i+1}, \beta_{n-1} \alpha_{n-1} \beta_{n-1} | i = 1, \ldots, n-2 \rangle.
\]

Multiplicity-free Brauer line algebras are representation-finite symmetric algebras with radical cube zero. Our claim is the following.

**Theorem 3.5.** Let \( \Gamma_n \) be a multiplicity-free Brauer line algebra with \( n \) edges. Then we have

\[
\# 2\text{-}\text{tilt}_\epsilon \Gamma_n = \binom{2n}{n}.
\]

Before the proof of Theorem \ref{3.5}, we recall a few equations about Catalan numbers \( C_n \).

**Lemma 3.6.**\( \text{[Sta]} \) For any positive integer \( m \), the following equations hold.

1. \( C_{m+1} = \sum_{k=0}^{m} C_k C_{m-k} \).
2. \( (m+2)C_{m+1} = 2(2m+1)C_m \).

**Proof of Theorem \ref{3.5}** Let \( \Gamma_n := kQ/I \) be a multiplicity-free Brauer line algebra with \( n \) edges. Then, the separated quiver of \( Q \) is given by

\[
1^+ \longrightarrow 2^- \longrightarrow 3^+ \longrightarrow \cdots \rightarrow n-1^\pm \rightarrow n^\mp
\]

and

\[
1^- \longleftarrow 2^+ \longleftarrow 3^- \longleftarrow \cdots \leftarrow n-1^\mp \leftarrow n^\pm.
\]

By Proposition \ref{2.3}, \( \Gamma_n \) is \( \tau \)-tilting-finite since every connected single subquiver is of type \( A_n \). Thus, we have \( \# 2\text{-}\text{tilt}_\epsilon \Gamma_n = \sum \# 2\text{-}\text{tilt}_\epsilon \Gamma_n \) by Proposition \ref{2.2}. To calculate this number, we only focus on maps \( \epsilon: Q_0 \to \{+, -\} \) with \( \epsilon(1) = + \) since we have \( \# 2\text{-}\text{tilt}_\epsilon \Gamma_n = \# 2\text{-}\text{tilt}_\epsilon \Gamma_n \) by Theorem \ref{3.3}.

For a map \( \epsilon: Q_0 \to \{+, -\} \) with \( \epsilon(1) = + \), define integers \( b_1, \ldots, b_r \geq 1 \) by the equality

\[
\{1 \leq i \leq n \mid \epsilon(i) = \epsilon(i+1)\} = \{b_1, b_1 + b_2, \ldots, \sum_{s=1}^{r} b_s\}.
\]

Then the quiver \( Q_\epsilon \) for \( \epsilon \) is a disjoint union of \( Q_{b_1}, \ldots, Q_{b_r} \), where \( Q_{b_r} \) is a connected bipartite subquiver with vertex set \( \{\sum_{j=1}^{r} b_j + 1, \sum_{j=1}^{r-1} b_j + 2, \ldots, \sum_{j=1}^{r} b_j\} \). In this case, we have

\[
\# 2\text{-}\text{tilt}_\epsilon \Gamma_n = \prod_{s=1}^{r} \# 2\text{-}\text{tilt}(Q_{b_s})^{op} \text{Prop. \ref{2.3}} \prod_{s=1}^{r} C_{b_s}.
\]

by Theorem \ref{3.3}. On the other hand, if we set

\[
Z_{n,r} := \{b = (b_1, \ldots, b_r) \in \mathbb{Z}_{>0}^r \mid \sum_{s=1}^{r} b_s = n\},
\]

**Theorem 3.7.** Let \( \Gamma_n \) be a multiplicity-free Brauer line algebra with \( n \) edges. Then, the number \( \# 2\text{-}\text{tilt}_\epsilon \Gamma_n \) is given by

\[
\# 2\text{-}\text{tilt}_\epsilon \Gamma_n = \sum_{(b_1, \ldots, b_r) \in Z_{n,r}} \frac{\prod_{s=1}^{r} C_{b_s}}{r!}.
\]
then it is easy to check that the correspondence \( \epsilon : \Gamma \rightarrow \{ +, - \} \) defined as (3.2) gives a bijection from the set of maps \( \epsilon : Q_0 \rightarrow \{ +, - \} \) with \( \epsilon(1) = + \) to the set \( \bigcup_{r=1}^{n} Z_{n,r} \). Under this bijection, we have

\[
\sum_{(1)=+} \text{#-2-tilts, } \Gamma_n = \sum_{r=1}^{n} \sum_{b \in Z_{n,r}} \prod_{s=1}^{r} C_{b_s} = \sum_{r=1}^{n} P_{n,r}
\]

where \( P_{n,r} := \sum_{b \in Z_{n,r}} \prod_{s=1}^{r} C_{b_s} \). Therefore, it remains to show the following claim.

**Lemma 3.7.** The following equation holds.

\[
\sum_{r=1}^{n} P_{n,r} = \frac{1}{2} \binom{2n}{n}.
\]

**Proof.** We will show it by induction on \( n \).

i) For \( n = 1 \), we have \( P_{1,1} = C_1 = 1 \).

ii) Assume that \( n > 1 \) and \( \sum_{r=1}^{n} P_{m,r} = \frac{1}{2} \binom{2m}{m} \) holds for \( 1 \leq m < n \).

Clearly, we have \( P_{n,1} = C_n \), and for \( 1 < r \leq n \),

\[
P_{n,r} = \sum_{b \in Z_{n,r}} \prod_{s=1}^{r} C_{b_s} = C_1 P_{n-1,r-1} + C_2 P_{n-2,r-1} + \cdots + C_{n-r+1} P_{r-1,r-1}.
\]

Therefore,

\[
\sum_{r=1}^{n} P_{n,r} = C_n + \sum_{k=1}^{n-1} C_k \left( \sum_{r=1}^{n-k} P_{n-k,r} \right) \xrightarrow{\text{induction}} C_n + \frac{1}{2} \sum_{k=1}^{n-1} C_k \left( \frac{2(n-k)}{n-k} \right)
\]

\[
\overset{\text{def}}{=} C_n + \frac{1}{2} \sum_{k=1}^{n-1} (n-k+1) C_k C_{n-k} = C_n + \frac{1}{2} \sum_{k=1}^{n-1} (k+1) C_k C_{n-k}.
\]

Adding last two equations, we have

\[
2 \times \sum_{r=1}^{n} P_{n,r} = 2C_n + \frac{1}{2}(n+2) \sum_{k=1}^{n-1} C_k C_{n-k} \overset{\text{Lem 3.6 1)}{=} 2C_n + \frac{1}{2}(n+2)(C_{n+1} - 2C_n) = 2C_n + \frac{1}{2}(n+2)(C_{n+1} - 2C_n) \overset{\text{Lem 3.6 2)}{=} \frac{2n}{n}.
\]

Thus, we obtain the desired equation for \( n \). \( \square \)

### 3.2 Brauer cycle algebras.

**Definition 3.8.** A *multiplicity-free Brauer cycle algebra* \( \Xi_n := kQ/I \) with \( n \) edges is defined by the following quiver and relations:

![Diagram of a quiver with labeled vertices and edges](attachment:image.png)

and

\[
I = \langle \alpha_n \alpha_1, \alpha_1 \alpha_{n+1}, \beta_1 \beta_n, \beta_{i+1} \beta_i, \beta_n \alpha_n - \alpha_1 \beta_1, \beta_i \alpha_i - \alpha_{i+1} \beta_{i+1} \mid i = 1, \ldots, n-1 \rangle.
\]

Multiplicity-free Brauer cycle algebras are representation-tame symmetric algebra with radical cube zero. The next result follows from Proposition 2.17 also given in [Ada2, Corollary 4.5].

**Proposition 3.9.** \( \Xi_n \) is \( r \)-tilting-finite if and only if \( n \) is odd.

In fact, the separated quiver \( Q^* \) of \( Q \) is given as follows:
Lemma 3.12. Let $n$ be odd.

\[ \sum_{r: \text{odd}} P_{n,r} = nC_{n-1} \quad \text{and} \quad \sum_{r: \text{even}} P_{n,r} = (n-1)C_{n-1}. \]

Proof. Assume $n > 1$ and $O_m - E_m = C_{m-1}$ holds for $1 \leq m < n$. Then we have

\[ O_n := \sum_{r: \text{odd}} P_{n,r} = \sum_{k=1}^{n-1} C_k \left( \sum_{r: \text{odd}} P_{n-k,r-1} \right) + C_n = \sum_{k=1}^{n-1} C_k E_{n-k} + C_n. \]

Similarly, we have

\[ E_n = \sum_{r: \text{even}} P_{n,r} = \sum_{k=1}^{n-1} C_k O_{n-k} \text{ induction } \sum_{k=1}^{n-1} C_k E_{n-k} + C_n - C_{n-1}. \]

Therefore, we have the desired equation $O_n - E_n = C_{n-1}$ for $n$. \hfill \Box

Next one is from the proof of Lemma 3.7.

Lemma 3.11. Assume $n$ is odd. Then, for any map $\epsilon: Q_0 \to \{+,-\}$, the quiver $Q_\epsilon$ is a disjoint union of odd number of connected quivers of type $A_n$.

Proof. The proof is left to the reader since it is easy. \hfill \Box

If $n$ is even, $\Xi_n$ is not $\tau$-tilting-finite by Proposition 2.17 since $Q^n$ contains a single subquiver of extended Dynkin type $A_n$. On the other hand, if $n$ is odd, $\Xi_n$ is $\tau$-tilting-finite since $Q^n$ contains only connected single subquivers of type $A_n$.

For Brauer cycle algebras, our result is the following.

Theorem 3.10. Let $\Xi_n$ be a multiplicity-free Brauer cycle algebra with odd $n$ edges. Then we have

\[ \# \text{ 2-tilt} \Xi_n = 2^{2n-1}. \]

We need the following observations.

Lemma 3.12. Let $P_{n,r}$ be as in (3.3). Then we have

(3.7) \[ \sum_{r: \text{odd}} P_{n,r} = nC_{n-1} \quad \text{and} \quad \sum_{r: \text{even}} P_{n,r} = (n-1)C_{n-1}. \]

Proof. Put $O_n := \sum_{r: \text{odd}} P_{n,r}$ and $E_n := \sum_{r: \text{even}} P_{n,r}$. We claim $O_n - E_n = C_{n-1}$ and show it by induction on $n$. After that, we obtain the desired equation (3.7) since we have already known $O_n + E_n = \sum_{r=1}^{n} P_{n,r} = \frac{1}{2} \binom{2n}{n}$ by Lemma 3.7.

i) For $n = 1$, we have $E_1 = 0 = O_1 - 1$.

ii) Assume that $n > 1$ and $O_m - E_m = C_{m-1}$ holds for $1 \leq m < n$. Then we have

\[ O_n := \sum_{r: \text{odd}} P_{n,r} = \sum_{k=1}^{n-1} C_k \left( \sum_{r: \text{odd}} P_{n-k,r-1} \right) + C_n = \sum_{k=1}^{n-1} C_k E_{n-k} + C_n. \]

Similarly, we have

\[ E_n = \sum_{r: \text{even}} P_{n,r} = \sum_{k=1}^{n-1} C_k O_{n-k} \text{ induction } \sum_{k=1}^{n-1} C_k E_{n-k} + C_n - C_{n-1}. \]

Therefore, we have the desired equation $O_n - E_n = C_{n-1}$ for $n$. \hfill \Box

Proof of Theorem 3.10. By Theorem 3.3, we only focus on maps $\epsilon: Q_0 \to \{+,-\}$ with $\epsilon(1) = +$. Let $\epsilon: Q_0 \to \{+,-\}$ with $\epsilon(1) = +$. Let $Q'$ be a connected component of $Q_\epsilon$ with $l$ vertices containing the vertex $1 \in Q_0$. 

\[ \text{Diagram for Lemma 3.12} \]

\[ \text{Diagram for Lemma 3.11} \]
(a) If $l = n$, then we have $Q' = Q$, and hence $\#2$-tilt $\Xi_n = \#\text{tilt}(kQ_{\epsilon})^{op} = C_n$ by Proposition 1.10.
(b) Assume $l = n - 1$, then the remained vertex, say $i$, must satisfy $\epsilon(i - 1) = \epsilon(i) = \epsilon(i + 1)$ since $i - 1$ and $i + 1$ are endpoints of $Q'$. However, we have $\epsilon(i + 1) = (-1)^n\epsilon(i - 1) \neq \epsilon(i - 1)$ since $n$ is odd. It is a contradiction.
(c) If $1 \leq l \leq n - 2$, then the remained part $Q_\epsilon \setminus Q'$ is a disjoint union of even components $Q_{b_1}, \ldots, Q_{b_s}$ by Lemma 3.11, where $Q_{b_i}$ is a connected bipartite quiver of type $\mathbb{A}_n$ with $b_i$ vertices. By definition, $\sum_{s=1} \#Q_{b_s} = n - l$ holds. Then we have $\#2$-tilt $\Xi_n = \#\text{tilt}(kQ_{\epsilon})^{op} = C_l \times \prod_{s=1}^{r} C_{b_s}$.

Running over all maps $\epsilon$ with $\epsilon(1) = +$, we have

$$\sum_{\epsilon(1) = +} \#2\text{-tilt}\;\Xi_n = nC_n + \sum_{l=1}^{n-2} lC_l \left\{ \sum_{r: \text{even}} P_{n-l,r} \right\} \leq 3.12$$

Therefore, it remains to show the following claim (2).

**Lemma 3.13.**

(1) [Sta] A generating function of Catalan numbers $C_n$ is given by

$$f(x) = \frac{1 - \sqrt{1 - 4x}}{2x}.$$  

(2) The following equation holds.

$$nC_n + \sum_{l=1}^{n-2} lC_l(n - l - 1)C_{n-l-1} = 4^{n-1}.$$  

Proof. (2) Let $T_n := \sum_{l=1}^{n-2} lC_l(n - l - 1)C_{n-l-1}$. We will show that the generating function of the sequence \{nC_n + T_n\} is given by

$$x = \frac{x}{1 - 4x} = x + 4x^2 + 16x^3 + \cdots.$$  

After that, we obtain the desired equation by comparing the coefficients on both sides.

Let $f(x)$ be the generating function of Catalan numbers in (1). Consider the differentiated function $f'(x) = \sum_{t=0}^{\infty} tC_t x^{t-1}$ by $x$, then we have

$$f'(x) = \frac{(1 - 2x)\sqrt{1 - 4x} - (1 - 4x)}{2x(1 - 4x)} \quad \text{and} \quad f'(x)^2 = \sum_{t=0}^{\infty} T_{t+3} x^t.$$  

Thus, the generating function of \{nC_n + T_n\} is given by

$$\sum_{t=0}^{\infty} (tC_t + T_t) x^t = f'(x)x + f'(x)^2 x^3.$$  

Consequently, we have

$$f'(x)x + f'(x)^2 x^3 = \frac{(1 - 2x)\sqrt{1 - 4x} - (1 - 4x)}{2x(1 - 4x)} + \frac{(2x^2 - 4x + 1) - (1 - 4x)\sqrt{1 - 4x}}{2x(1 - 4x)} = \frac{x}{1 - 4x}.$$  

Therefore, the generating function of \{nC_n + T_n\} is given by $x \frac{x}{1 - 4x}$. The proof is completed.  

Proof of Theorem 0.2. It was shown in [AAC, EJR] that the number of isomorphism classes of basic two-term tilting complexes over Brauer graph algebras is independent of multiplicity. Thus the assertions follow from Theorem 3.5 and 3.10.  

References

[Ada1] T. Adachi, The classification of $\tau$-tilting modules over Nakayama algebras, J. Algebra 452 (2016), 227-262.
[Ada2] T. Adachi, Characterizing $\tau$-tilting finite algebras with radical square zero, Proc. Amer. Math. Soc. 144 (2016), no. 11, 4673-4685.
[AAC] T. Adachi, T. Aihara, A. Chan, Classification of two-term tilting complexes over Brauer graph algebras, to appear in Math. Z.
[AIR] T. Adachi, O. Iyama, I. Reiten, $\tau$-tilting theory, Compos. Math. 150 (2014), no. 3, 415-452.
[Al] T. Aihara, Tilting-connected symmetric algebras, Algebr. Represent. Theory 16 (2013), no. 3, 873-894.
[AI] T. Aihara and O. Iyama, Silting mutation in triangulated categories, J. Lond. Math. Soc. 85(2) 85(2012), no. 3 633-668.

[AM] T. Aihara, Y. Mizuno, Classifying tilting complexes over preprojective algebras of Dynkin type, Algebra and Number theory 11 (2017), no. 6, 1287-1315.

[ARS] M. Auslander, I. Reiten, S. O. Smalø, Representation Theory of Artin Algebras, Cambridge studies in Advanced Mathematics 36, Cambridge University Press, Cambridge, 1995.

[ASS] I. Assem, D.Simson, A.Skowronski, Elements of the representation theory of associative algebras, London Math. Soc. Students Text 65. Cambridge University Press, Cambridge, 2006.

[Ben1] D. J. Benson, Representations and Cohomology, I: Basic Representation Theory of Finite Groups and Associative Algebras, Cambridge Studies in Advanced Mathematics 11, Cambridge University Press, New York, 1991.

[Ben2] D. J. Benson, Resolutions over symmetric algebras with radical cube zero, J. Algebra 320 (2008), no. 1, 48-56.

[BGP] I. N. Bernstein, I. M. Gelfand, V. A. Ponomarev, Cozeter functors, and Gabriel’s theorem, Uspehi Mat. Nauk 28 (1973), no. 2(170), 19-33.

[BGS] A. Beilinson, V. Ginzburg, W. Soergel, Koszul duality patterns in representation theory, J. Amer. Math. Soc. 9 (1996), no. 2, 473-527.

[BB] S. Brenner and M. C. R. Butler, Generalization of the Bernstein-Gelfand-Ponomarev reflection functors, Lecture Notes in Math. 832, Springer-Verlag, Berlin, Heidelberg, New York, 1980, pp. 103-169.

[BMRRT] A. B. Buan, R. Marsh, M. Reineke, I. Reiten, G. Todaro, Tilting theory and cluster combinatorics, Adv. Math. 204 (2006), no. 2, 572-618.

[CL] S. Cautis, A. Licata, Hessenberg categorification and Hilbert schemes, Duke Math. J. 161 (2012), no. 13, 2469-2547.

[EJR] F. Eisele, G. Janssens, T. Raedschelders, A reduction theorem for \( \tau \)-rigid modules, arXiv:1603.04293v1.

[ES] K. Erdmann, Øyvind. Solberg, Radical cube zero weakly symmetric algebras and support varieties, J. Pure Appl. Algebra 215 (2011), no. 2, 185-200.

[Ga] P. Gabriel, Unzerlegbare Darstellungen. I, Manuscripta Math. 6 (1972), 71-103; correction, ibid. 6 (1972), 309.

[GR] P. Gabriel, A.V. Roiter, Representations of finite-dimensional algebras, Encyclopedia of Math. Sci., vol. 73, Algebra 8, Springer-Verlag (1992).

[GS1] E. Green, S. Schroll, Multiserial and special multiserial algebras and their representations, Adv. Math. 302 (2016), 1111-1136.

[GS2] E. Green, S. Schroll, Brauer configuration algebras: A generalization of Brauer graph algebras, Bull. Sci. Math. 141 (2017), no. 6, 539-572.

[HK] R. S. Huerfano, M. Khovanov, A category for the adjoint representation, J. Algebra 246 (2001), no. 2, 514-542.

[HKM] M. Hoshino, Y. Kato, J. Miyachi, On \( \tau \)-structures and torsion theories induced by compact objects, J. Pure Appl. Algebra 167 (2002), no. 1, 15-35.

[HU] D. Happel and L. Unger, On a partial order of tilting modules, Algebr. Represent. Theory 8 (2005), no. 2, 147-156.

[IY] O. Iyama, Jørgensen, D. Yang, Intermediate co-\( \tau \)-structures, two-term silting objects, \( \tau \)-tilting modules, and torsion classes, Algebra Number Theory 8 (2014), no. 10, 2413-2431.

[KV] B. Keller, D. Vossieck, Aisles in derived categories, Bull. Soc. Math. Belg. Sér. A 40 (1988), no. 2, 239-253.

[Oku] T. Okuyama, Some examples of derived equivalent blocks of finite groups, preprint 1998.

[Ric1] J. Rickard, Morita theory for derived categories, J. London Math. Soc. 39 (1989), no. 3, 436-456.

[Ric2] J. Rickard, Derived categories and stable equivalence, J. Pure Appl. Algebra 61 (1989), no. 3, 303-317.

[RS] C. Riedtmann, A. Schofield, On a simplicial complex associated with tilting modules, Comment. Math. Helv. 66 (1991), no. 1, 70-78.

[ONFR] A. A. Obaid, S. K. Nauman, W. M. Fakieh, C. M. Ringel, The number of support-tilting modules for a Dynkin algebra, J. Integer Seq. 18 (2015), no. 1, Article 15.10.6, 24 pp.

[Sei] P. Seidel, Fukaya categories and Picard-Lefschetz theory, Duke Math. J. 108 (2001), no. 1, 37-108.

[Sta] R. P. Stanley, Enumerative Combinatorics, Vol. 2, Cambridge Studies in Advanced Mathematics, 62, Cambridge University Press, Cambridge, 1999.

[Yo] T. Yoshii, On algebras of bounded representation type, Osaka Math. J. 8 (1956), 51-105.

[Zha] X. Zhang, \( \tau \)-rigid modules for algebras with radical square zero, arXiv:1211.5622v5.