WEIGHTED SEMIGROUP MEASURE ALGEBRA AS A WAP-ALGEBRA

H.R. Ebrahimi Vishki\(^1\), B. Khodsiani\(^2\), A. Rejali\(^3\)

A Banach algebra \(\mathcal{A}\) for which the natural embedding from \(\mathcal{A}\) into \(WAP(\mathcal{A})^*\) is bounded below is called a WAP-algebra. We study those conditions under which the weighted semigroup measure algebra \(M_b(S, \omega)\) is a WAP-algebra or a dual Banach algebra. In particular, we show that the semigroup measure algebra \(M_b(S)\) is a WAP-algebra (resp. dual Banach algebra) if and only if \(wap(S)\) separates the points of \(S\) (resp. \(S\) is compactly cancellative semigroup). Some older results, in the case where \(S\) is discrete, are also improved.

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1. Introduction and Preliminaries

The dual \(\mathcal{A}^*\) of a Banach algebra \(\mathcal{A}\) can be turned into a Banach \(\mathcal{A}\)-module equipped with the natural module operations

\[
\langle f \cdot a, b \rangle = \langle f, ab \rangle \quad \text{and} \quad \langle a \cdot f, b \rangle = \langle f, ba \rangle \quad (a, b \in \mathcal{A}, f \in \mathcal{A}^*).
\]

A dual Banach algebra is a Banach algebra \(\mathcal{A}\) enjoying a predual \(\mathcal{A}_*\) such that \(\mathcal{A}_*\), as a Banach space is a closed \(\mathcal{A}\)-submodule of \(\mathcal{A}^*\); or equivalently, the multiplication on \(\mathcal{A}\) is separately weak*-continuous. It should be remarked that the predual of a dual Banach algebra need not be unique, in general (see [5, 10]); so we usually point to the involved predual of a dual Banach algebra.

A functional \(f \in \mathcal{A}^*\) is said to be weakly almost periodic if \(\{f \cdot a : \|a\| \leq 1\}\) is relatively weakly compact in \(\mathcal{A}^*\). We denote by \(WAP(\mathcal{A})\) the set of all weakly

\(^1\)Department of Pure Mathematics and Center of Excellence in Analysis on Algebraic Structures (CEAAS), Ferdowsi University of Mashhad, IRAN, e-mail: Vishki@um.ac.ir

\(^2\)Corresponding author, Department of Mathematics, University of Isfahan, Isfahan, IRAN. e-mail: b.khodsiani@sci.ui.ac.ir

\(^3\)Department of Mathematics, University of Isfahan, Isfahan, IRAN. e-mail: rejali@sci.ui.ac.ir
almost periodic elements of $\mathfrak{A}^*$. It is easy to verify that WAP($\mathfrak{A}$) is a (norm) closed subspace of $\mathfrak{A}^*$.

It is known that the multiplication of a Banach algebra $\mathfrak{A}$ has two natural but, in general, different extensions (called Arens products) to the second dual $\mathfrak{A}^{**}$ each turning $\mathfrak{A}^{**}$ into a Banach algebra. When these products are equal, $\mathfrak{A}$ is said to be (Arens) regular. It can be verified that $\mathfrak{A}$ is Arens regular if and only if WAP($\mathfrak{A}$) = $\mathfrak{A}^*$. Further information for the Arens regularity of Banach algebras can be found in [5, 6].

WAP-algebras, as a generalization of the Arens regular algebras, have been introduced and intensively studied in [9]. A Banach algebra $\mathfrak{A}$ for which the natural embedding $x \mapsto \hat{x}$ of $\mathfrak{A}$ into WAP($\mathfrak{A}$)*, where $\hat{x}(f) = f(x)$ for $f \in$ WAP($\mathfrak{A}$), is bounded below, is called a WAP-algebra. When $\mathfrak{A}$ is either Arens regular or a dual Banach algebra, then natural embedding of $\mathfrak{A}$ into WAP($\mathfrak{A}$)* is an isometry [16, Corollary 4.6]. It has also known that $\mathfrak{A}$ is a WAP-algebra if and only if it admits an isomorphic representation on a reflexive Banach space. Convolution group algebras are the main examples of WAP-algebras; however; they are neither dual nor Arens regular in general, see [17]. For more information about WAP-algebras one may consult to the impressive paper [9].

The main aim of this paper is to investigate those conditions under which the weighted measure algebra $M_b(S, \omega)$ is either a WAP-algebra or a dual Banach algebra, where $\omega$ is a weight on a locally compact semigroup $S$.

First we recall some preliminaries about the (weighted) measure algebras. Let $S$ be a locally compact semitopological semigroup. Let $M_b(S)$ be the space of all complex regular Borel measures on $S$, which is known as a Banach algebra under the convolution product $*$ defined by the equation $(\mu * \nu, f) = \int_S \int_S f(xy)d\mu(x)d\nu(y)$ ($f \in C_0(S)$). Our mean by a weight $\omega$ on $S$ is a Borel measurable function $\omega : S \to (0, \infty)$ such that $\omega(st) \leq \omega(s)\omega(t)$, ($s, t \in S$). For $\mu \in M_b(S)$ we define $(\mu \omega)(E) = \int_E \omega d\mu$, ($E \subseteq S$ is Borel set). If $\omega \geq 1$, then $M_b(S, \omega) = \{\mu \in M_b(S) : \mu \omega \in M_b(S)\}$ is known as a Banach algebra which is called the weighted semigroup measure algebra (see [6, 12, 13, 14]). In the case where $S$ is discrete we write $\ell_1(S, \omega)$ instead of $M_b(S, \omega)$ and $c_0(S, 1/\omega)$ instead of $C_0(S, 1/\omega)$. Then the Banach algebra $\ell_1(S, \omega) = \{f : f = \sum_{s \in S} f(s)\delta_s, \ ||f||_{1, \omega} = \sum_{s \in S} |f(s)|\omega(s) < \infty\}$ (where, $\delta_s \in \ell_1(S, \omega)$ is the point mass at $s$) equipped with the convolution product is called a weighted semigroup algebra. We also suppress 1 from the notation whenever $w = 1$.

Let $B(S)$ denote the space of all bounded Borel measurable functions on $S$. Set $B(S, 1/\omega) = \{f : S \to \mathbb{C} : f/\omega \in B(S)\}$. Let $f \in C(S, 1/\omega)$ then $f$ is called $\omega$-weakly almost periodic if the set $\{\frac{R_{s, \omega}f}{\omega(s)} : s \in S\}$ is relatively weakly compact in $C(S)$. The set of all $\omega$-weakly almost periodic functions on $S$ is denoted by
wap(S,1/ω). The space wap(S) of 1-weakly almost periodic functions on S is a C*-subalgebra of C(S) and its character space Swap, endowed with the Gelfand topology, enjoys a (Arens type) multiplication that turns it into a compact semitopological semigroup. Many other properties of wap(S) and its inclusion relations among other function algebras are completely explored in [3].

The paper is organized as follows. In section 2 we study the weighted measure algebra $M_b(S,ω)$ from the dual Banach algebra point of view. In this respect, we shall show that, $M_b(S,ω)$ is a dual Banach algebra with respect to the predual $C_0(1/ω)$ if and only if for all compact subsets $F$ and $K$ of $S$, the maps $X_{r=1}K$ and $X_{K=r=1}$ vanishes at infinity. This extends an earlire result of Abolghasemi, Rejali, and Ebrahimi Vishki [1]. We also conclude that, the measure algebra $M_b(S)$ is a dual Banach algebra with respect to $C_0(S)$ if and only if $S$ is a compactly cancellative semigroup. The later result is an extension of a known result of Dales, Lau and Strauss [7, Theorem 4.6] stating that, $ℓ_1(S)$ is a dual Banach algebra with respect to $c_0(S)$ if and only if $S$ is a weakly cancellative semigroup.

Section 3 is devoted to the study of $M_b(S,ω)$ from the WAP-algebra point of view. We shall prove that, $M_b(S,ω)$ is a WAP-algebra if and only if the evaluation map $ε:S→X$ is one to one, where $X=MM(wap(S,1/ω)).$ The main result of this section is that $M_b(S)$ is WAP-algebra if and only if $wap(S)$ separate the points of $S$. We conclude the paper with some illuminating examples.

2. Semigroup Measure Algebras as Dual Banach Algebras

It is known that the (discrete) semigroup algebra $ℓ_1(S)$ is a dual Banach algebra with respect to $c_0(S)$ if and only if $S$ is a weakly cancellative semigroup, see [7, Theorem 4.6].This result has been extended to the weighted semigroup algebras; [1, 8]. In this section we extend the aforementioned results to the non-discrete case. More precisely, we provide some necessary and sufficient conditions that the measure algebra $M_b(S,ω)$ becomes a dual Banach algebra with respect to the predual $C_0(S,1/ω)$.

Let $F$ and $K$ be nonempty subsets of a semigroup $S$ and $s∈S$. We set $s^{-1}F=\{t∈S:st∈F\}$, and $Fs^{-1}=\{t∈S:ts∈F\}$. We also write $s^{-1}t$ for $s^{-1}\{t\}$, $FK^{-1}$ for $\cup_{s∈K}Fs^{-1}$ and $K^{-1}F$ for $\cup_{s∈K}s^{-1}F$.

A semigroup $S$ is called left (respectively, right) zero semigroup if $xy=x$ (respectively, $xy=y$), for all $x,y∈S$. A semigroup $S$ is called zero semigroup if there exist $z∈S$ such that $xy=z$ for all $x,y∈S$. A semigroup $S$ is said to be left (respectively, right) weakly cancellative semigroup if $s^{-1}F$ (respectively, $Fs^{-1}$) is finite for each $s∈S$ and each finite subset $F$ of $S$. A semigroup $S$ is said to be weakly cancellative semigroup if it is both left and right weakly cancellative semigroup.
A semi-topological semigroup $S$ is said to be compactly cancellative semigroup if for every compact subsets $F$ and $K$ of $S$ the sets $F^{-1}K$ and $KF^{-1}$ are compact set.

The following lemma needs a routine argument.

**Lemma 2.1.** Let $S$ be a topological semigroup. For every compact subsets $F$ and $K$ of $S$ the sets $F^{-1}K$ and $KF^{-1}$ are closed.

In the next result we study $M_b(S,\omega)$ from the dual Banach algebra point of view.

**Theorem 2.1.** Let $S$ be a locally compact topological semigroup and $\omega$ be a continuous weight on $S$. Then the measure algebra $M_b(S,\omega)$ is a dual Banach algebra with respect to the predual $C_0(S,1/\omega)$ if and only if for all compact subsets $F$ and $K$ of $S$, the maps $\frac{KF^{-1}}{\omega}$ and $\frac{\chi_{F^{-1}K}}{\omega}$ vanishes at infinity.

**Proof.** Suppose that $M_b(S,\omega)$ is a dual Banach algebra with respect to $C_0(S,1/\omega)$ and let $\varepsilon > 0$. Let $K, F$ be nonempty compact subsets of $S$ with a net $(x_\alpha)$ in $\{t \in F^{-1}K : 1/\omega(t) \geq \varepsilon\}$. Let $C^+_0(S)$ denote the non-negative continuous functions with compact support on $S$ and set $C^+_0(S,1/\omega) = \{f \in C_0(S,1/\omega) : f/\omega \in C^+_0(S)\}$. Since $\omega$ is continuous we may choose $f \in C^+_0(S,1/\omega)$ with $f(K) = 1$. There exists a net $(t_\alpha) \in F$ such that $t_\alpha x_\alpha \in K$ and the compactness of $F$ guarantees the existence of a subnet $(t_\gamma)$ of $(t_\alpha)$ such that $t_\gamma \to t_0$ for some $t_0$ in $S$. Indeed, since for each $s \in S$,

$$\lim_{\gamma} (\frac{\delta_{t_\gamma} f}{\omega})(s) = \lim_{\gamma} \frac{f(t_\gamma s)}{\omega(s)} = \frac{f(t_0 s)}{\omega(s)} = \frac{\delta_{t_0} f}{\omega}(s),$$

there exists a $\gamma_0$ such that

$$\{t \in \cup_{\gamma \geq \gamma_0} t^{-1}_\gamma K : 1/\omega(t) \geq \varepsilon\} \subseteq \cup_{\gamma \geq \gamma_0} \{r \in S : (\frac{\delta_{t_\gamma} f}{\omega})(r) \geq \varepsilon\}$$

$$\subseteq \{r \in S : (\frac{\delta_{t_0} f}{\omega})(r) \geq \frac{\varepsilon}{2}\}.$$ 

Let $H = \{t_\gamma : \gamma \geq \gamma_0\} \cup \{t_0\}$. Then

$$\{t \in H^{-1}K : 1/\omega(t) \geq \varepsilon\} = \{t \in \cup_{\gamma \geq \gamma_0} t^{-1}_\gamma K \cup t^{-1}_0 K : 1/\omega(t) \geq \varepsilon\}$$

as a closed subset of $\{r \in S : (\frac{\delta_{t_0} f}{\omega})(r) \geq \frac{\varepsilon}{2}\}$ is compact. It follows that the net $(x_\gamma)$ in $\{t \in H^{-1}K : 1/\omega(t) \geq \varepsilon\}$ has a convergent subnet. Thus $\{t \in F^{-1}K : 1/\omega(t) \geq \varepsilon\}$ is compact and that $\frac{\chi_{F^{-1}K}}{\omega}$ vanishes at infinity. Similarly $\frac{\chi_{KF^{-1}}}{\omega}$ vanishes at infinity.

The proof of sufficiency can be adopted from [1, Proposition 3.1]. Let $f \in C_0(S,1/\omega)$, $\mu \in M_b(S,\omega)$ and $\varepsilon > 0$ be arbitrary. Then there exist compact subsets $F$ and $K$ of $S$ such that $|f(s)| < \varepsilon$ for all $s \not\in K$ and $|\mu\omega|(S \setminus F) < \varepsilon$. Let
Let The measure algebra (1) For a semigroup The set If for S = M \{ empty or the whole S either cases (being left zero, right zero or zero) the sets F C ω semigroup. Then there exists a weight algebra ℓ (1, Theorem 2.2) Examples 2.1.

Proof. Corollary 2.2

Applying Theorem 2.1 for a discrete semigroup, we arrive at the next result.

As immediate consequences of Theorem 2.1 we have the next corollary.

Corollary 2.1. Let S be a locally compact topological semigroup.

(1) The measure algebra M_b(S) is a dual Banach algebra with respect to C_0(S) if and only if S is compactly cancellative.

(2) If M_b(S) is a dual Banach algebra with respect to C_0(S) then M_b(S, ω) is a dual Banach algebra with respect to C_0(S, 1/ω).

Applying Theorem 2.1 for a discrete semigroup, we arrive at the next result.

Corollary 2.2 ([1, Theorem 2.2]). For a semigroup S, the weighted semigroup algebra ℓ_1(S, ω) is a dual Banach algebra with respect to the predual c_0(S, 1/ω) if and only if the maps \( \frac{X^{-1}}{ω} \) and \( \frac{X}{ω} \) are in c_0(S) for all \( s, t \in S \).

We have also the next result as an application of Theorem 2.1.

Corollary 2.3. Let S be either a left zero, a right zero or a zero locally compact semigroup. Then there exists a weight ω on S such that M_b(S, ω) is a dual Banach algebra with respect to C_0(S, 1/ω) if and only if S is σ-compact.

Proof. Let K and F be compact subsets of S. It can be readily verified that in either cases (being left zero, right zero or zero) the sets \( F^{-1}K \) and \( KF^{-1} \) are either empty or the whole S. For each \( m \in \mathbb{N} \) we set \( S_m = \{ t \in F^{-1}K : \omega(t) \leq m \} = \{ t \in S : \omega(t) \leq m \} \). Then \( S = \bigcup_{m \in \mathbb{N}} S_m \) and so S is σ-compact. For the converse let \( S = \bigcup_{n \in \mathbb{N}} S_n \) as a disjoint union of compact sets and let z be a (left or right) zero for S. Define \( \omega(z) = 1 \) and \( \omega(x) = 1 + n \) for \( x \in S_n \) then \( \omega \) is a weight on S and \( M_b(S, ω) \) is a dual Banach algebra.

Examples 2.1. (1) The set \( S = \mathbb{R}^+ \times \mathbb{R} \) equipped with the multiplication

\[
(x, y) \cdot (x', y') = (x + x', y') \quad ((x, y), (x', y') \in S)
\]
and the weight \( \omega(x,y) = e^{-x(1 + |y|)} \) is a weighted semigroup. Set \( F = [a,b] \times [c,d] \) and \( K = [e,f] \times [g,h] \). Then \( F^{-1}K = [e-b, f-a] \times [g,h] \) and \( KF^{-1} = \left\{ \begin{array}{ll} [e-b, f-a] \times \mathbb{R} & \text{if } [c,d] \cap [g,h] \neq \emptyset \\ \emptyset & \text{if } [c,d] \cap [g,h] = \emptyset \end{array} \right. \)

Thus \( M_b(S) \) is not a dual Banach algebra by Corollary 2.1 (1). However, for all compact subsets \( F \) and \( K \) of \( S \), the maps \( \frac{x_{F^{-1}K}}{\omega} \) and \( \frac{x_{KF^{-1}}}{\omega} \) vanishes at infinity. So \( M_b(S,\omega) \) is a dual Banach algebra with respect to \( C_0(S,1/\omega) \). This shows that the converse of Corollary 2.1 (2) may not be valid.

(2) For the semigroup \( S = [0,\infty) \) endowed with the zero multiplication, neither \( M_b(S) \) nor \( l_1(S) \) is a dual Banach algebra. In fact, \( S \) is neither compactly nor weakly cancellative semigroup.

3. Semigroup Measure Algebras as WAP-Algebras

In this section we study some conditions under which the weighted measure algebra \( M_b(S,\omega) \) is a WAP-algebra. First, we provide some preliminaries.

**Definition 3.1.** Let \( \tilde{\mathcal{F}} \) be a linear subspace of \( B(S,1/\omega) \), and let \( \tilde{\mathcal{F}}_r \) denote the set of all real-valued members of \( \tilde{\mathcal{F}} \). A mean on \( \tilde{\mathcal{F}} \) is a linear functional \( \tilde{\mu} \) on \( \tilde{\mathcal{F}} \) with the property that \( \inf_{s \in S} \frac{\tilde{\mu}(s)}{\omega} \leq \tilde{\mu}(f) \leq \sup_{s \in S} \frac{\tilde{\mu}(s)}{\omega} \) \( (f \in \tilde{\mathcal{F}}_r) \). The set of all means on \( \tilde{\mathcal{F}} \) is denoted by \( M(\tilde{\mathcal{F}}) \). If \( \tilde{\mathcal{F}} \) is also an algebra with the multiplication given by \( f \circ g := (f,g)/\omega \) \( (f,g \in \tilde{\mathcal{F}}) \) and if \( \tilde{\mu} \in M(\tilde{\mathcal{F}}) \) satisfies \( \tilde{\mu}(f \circ g) = \tilde{\mu}(f)\tilde{\mu}(g) \) \( (f,g \in \tilde{\mathcal{F}}) \), then \( \tilde{\mu} \) is said to be multiplicative. The set of all multiplicative means on \( \tilde{\mathcal{F}} \) will be denoted by \( MM(\tilde{\mathcal{F}}) \).

Let \( \tilde{\mathcal{F}} \) be a conjugate closed, linear subspace of \( B(S,1/\omega) \) such that \( \omega \in \tilde{\mathcal{F}} \).

(i) For each \( s \in S \) define \( \epsilon(s) \in M(\tilde{\mathcal{F}}) \) by \( \epsilon(s)(f) = (f/\omega)(s) \) \( (f \in \tilde{\mathcal{F}}) \). The mapping \( \epsilon : S \to M(\tilde{\mathcal{F}}) \) is called the evaluation mapping. If \( \tilde{\mathcal{F}} \) is also an algebra, then \( \epsilon(S) \subseteq MM(\tilde{\mathcal{F}}) \).

(ii) Let \( \tilde{X} = M(\tilde{\mathcal{F}}) \) (resp. \( \tilde{X} = MM(\tilde{\mathcal{F}}) \), if \( \tilde{\mathcal{F}} \) is a subalgebra) be endowed with the relative weak* topology. For each \( f \in \tilde{\mathcal{F}} \) the function \( \tilde{f} \in C(\tilde{X}) \) is defined by \( \tilde{f}(\tilde{\mu}) := \tilde{\mu}(f) \) \( (\tilde{\mu} \in \tilde{X}) \).

Furthermore, we define \( \tilde{\mathcal{F}} := \{ \tilde{f} : f \in \tilde{\mathcal{F}} \} \).

**Remark 3.1.** (i) The mapping \( f \mapsto \tilde{f} : \tilde{\mathcal{F}} \to C(\tilde{X}) \) is clearly linear and multiplicative if \( \tilde{\mathcal{F}} \) is an algebra and \( \tilde{X} = MM(\tilde{\mathcal{F}}) \). Also it preserves complex
conjugation, and is an isometry, since for any $f \in F$
\[ ||\tilde{f}|| = \sup\{||f(f)/\omega|| : \mu \in X\} \leq \sup\{||\mu(f/\omega)|| : \mu \in C(X)^*, ||\mu|| \leq 1\} \]
\[ = ||f|| = \sup\{||f(s)|| : s \in S\} = \sup\{\epsilon(f(s)) : s \in S\} \]
\[ = \sup\{\tilde{f}(\epsilon(s)) : s \in S\} \leq ||\tilde{f}||, \]
where $X = M(\mathcal{F})$ and $\mathcal{F} = \{f/\omega : f \in \tilde{F}\}$. Note that $\tilde{f}(\epsilon(s)) = \epsilon(f(s)) = \langle \frac{f}{\omega}(s)_{f/\omega}, s \in S\rangle$. This identity may be written in terms of dual map $\epsilon^* : C(X) \rightarrow C(S, \omega)$ as $\epsilon^*(\tilde{f}) = f$ for $f \in \tilde{F}$.

(ii) Let $\tilde{F}$ be a conjugate closed linear subspace of $B(S, 1/\omega)$, containing $\omega$. Then $M(\tilde{F})$ is convex and weak* compact, $\text{co} (\epsilon(S))$ is weak* dense in $M(\tilde{F})$, $\tilde{F}^*$ is the weak* closed linear span of $\epsilon(S)$, $\epsilon : S \rightarrow M(\tilde{F})$ is weak* continuous, and if $\tilde{F}$ is also an algebra, then $MM(\tilde{F})$ is weak* compact and $\epsilon(S)$ is weak* dense in $MM(\tilde{F})$.

(iii) Let $\tilde{F}$ be a $C^*$-subalgebra of $B(S, 1/\omega)$, containing $\omega$. If $\tilde{X}$ denotes the space $MM(\tilde{F})$ with the relative weak* topology, and if $\tilde{F} = \tilde{F}^*$ is an isometric isomorphism with the inverse $\epsilon^* : C(\tilde{X}) \rightarrow \tilde{F}$

Let $\tilde{F} = \text{wap}(S, 1/\omega)$. Then $\tilde{F}$ is a $C^*$-subalgebra of $\text{WAP}(M_b(S, \omega))$, see [11, Theorem 1.6, Theorem 3.3]. Set $\tilde{X} = MM(\tilde{F})$. By the above remark $\text{wap}(S, 1/\omega) \cong C(\tilde{X})$ and so

\[ M_b(\tilde{X}) \cong C(\tilde{X})^* \cong \text{wap}(S, 1/\omega)^* \subseteq \text{WAP}(M_b(S, \omega))^*. \]

Let $\epsilon : S \rightarrow \tilde{X}$ be the evaluation mapping. We also define
\[ \tilde{\epsilon} : M_b(S, \omega) \rightarrow M_b(\tilde{X}) \text{ by } \langle \tilde{\epsilon}(\mu), f \rangle = \int_S f \omega d\mu \]
for $f \in \text{wap}(S, 1/\omega) \cong C(\tilde{X})$. Then for every Borel set $B$ in $\tilde{X}$, we have $\tilde{\epsilon}(\mu)(B) = (\mu_{\omega})(\epsilon^{-1}(B))$. In particular, $\tilde{\epsilon}(\frac{\delta_{\omega}(x)}{\omega(x)}) = \delta_{\epsilon(x)}$.

The next theorem is the main result of this section.

**Theorem 3.1.** For every weighted locally compact semi-topological semigroup $(S, \omega)$ the following statements are equivalent:

1. The map $\epsilon : S \rightarrow \tilde{X}$ is one to one, where $\tilde{X} = MM(\text{wap}(S, 1/\omega))$;
2. $\tilde{\epsilon} : M_b(S, \omega) \rightarrow M_b(\tilde{X})$ is an isometric isomorphism;
3. $M_b(S, \omega)$ is a WAP-algebra.

**Proof.** (1) $\Rightarrow$ (2). Take $\mu \in M_b(S, \omega)$, say $\mu = \mu_1 - \mu_2 + i(\mu_3 - \mu_4)$, where $\mu_j \in M_b(S, \omega)^+$ for each $j = 1, 2, 3, 4$. Set $\nu_j = \epsilon(\mu_j) \in M_b(\tilde{X})^+$, and $\nu = \epsilon(\mu) = \nu_1 - \nu_2 + i(\nu_3 - \nu_4)$. Take $\delta > 0$. For each $j$, there exists a Borel set $B_j$ in $\tilde{X}$ such that $\nu_j(B_j) \geq 0$ for each Borel subset $B$ of $B_j$ with $\sum_{j=1}^4 \nu_j(B_j) > ||\nu|| - \delta$. In
fact, by the Hahn decomposition theorem for the signed measures $\lambda_1 = \nu_1 - \nu_2$ and $\lambda_2 = \nu_3 - \nu_4$, there exist four Borel sets $P_1$, $P_2$, $N_1$ and $N_2$ in $\tilde{X}$ such that $P_1 \cup N_1 = \tilde{X}$, $P_1 \cap N_1 = \emptyset$, $P_2 \cap N_2 = \tilde{X}$, $P_2 \cap N_2 = \emptyset$, and $\nu_1(E) = \lambda_1(P_1 \cap E)$, $\nu_2(E) = -\lambda_1(N_1 \cap E)$, $\nu_3(E) = \lambda_2(P_2 \cap E)$, $\nu_4(E) = -\lambda_2(N_2 \cap E)$, for every Borel set $E$ of $\tilde{X}$. That is, the measures $\nu_1$, $\nu_2$, $\nu_3$, $\nu_4$ are concentrated on $P_1$, $N_1$, $P_2$, $N_2$, respectively.

Set $D_1 := P_1 \cap N_2$, $D_2 := N_1 \cap P_2$, $D_3 := P_2 \cap P_1$, $D_4 := N_2 \cap N_1$. Then the family $\{D_1, D_2, D_3, D_4\}$ is a partition for $\tilde{X}$. Further, there exists a compact set $K$ for which $||\nu|| - \delta \leq \sum_{j=1}^{4} ||\nu_{j}\|_{D_{j}}|| - \delta \leq \sum_{j=1}^{4} \nu_{j}(D_{j} \cap K) = \sum_{j=1}^{4} \nu_{j}(D_{j} \cap K)$. Set $B_j = D_j \cap K$. Then the sets $B_1, B_2, B_3, B_4$ are pairwise disjoint.

For each $j$, set $C_j = (\epsilon)^{-1}(B_j)$, a Borel set in $S$. Then $(\mu_j \omega)(C_j) = \nu_j(B_j)$. Since $\epsilon$ is injection, the sets $C_1, C_2, C_3, C_4$ are pairwise disjoint, and so $||\mu||_{\omega} = \sum_{j=1}^{4} |\mu(\omega(C_j))| \geq \sum_{j=1}^{4} |(\mu_j \omega)(C_j)| = \sum_{j=1}^{4} \nu_j(B_j) > ||\nu|| - \delta$ This holds for each $\delta > 0$, so $||\mu||_{\omega} \geq ||\nu||$. A similar argument shows that $||\mu||_{\omega} \leq ||\nu||$. Thus $||\mu||_{\omega} = ||\nu||$.

$(2) \Rightarrow (1)$. Let $P(S, \omega)$ denote the subspace of all probability measures of $M_b(S, \omega)$ and $\text{ext}(P(S, \omega))$ the extreme points of unit ball of $P(S, \omega)$. Then $\text{ext}(P(S, \omega)) = \{ \frac{\delta_x}{\omega(x)} : x \in S \} \cong S$ and $\text{ext}(P(\tilde{X})) \cong \tilde{X}$, see [4, p.151]. By the injectivity of $\tilde{\epsilon}$, it maps the extreme points of the unit ball onto the extreme points of the unit ball, thus $\epsilon : S \rightarrow \tilde{X}$ is one to one.

$(2) \Rightarrow (3)$. Since $\tilde{X}$ is compact, $M_b(\tilde{X})$ is a dual Banach algebra with respect to $C(\tilde{X})$, so it has an isometric representation $\psi$ on a reflexive Banach space $E$, see [9]. In the following commutative diagram,

\[
\begin{array}{ccc}
M_b(S, \omega) & \xrightarrow{\epsilon} & M_b(\tilde{X}) \\
\phi \downarrow & & \psi \\
B(E) & & 
\end{array}
\]

If $\tilde{\epsilon}$ is isometric, then so is $\phi$. Thus $M_b(S, \omega)$ has an isometric representation on a reflexive Banach space $E$ if $\tilde{\epsilon}$ is an isometric isomorphism. So $M_b(S, \omega)$ is a WAP-algebra if $\tilde{\epsilon}$ is an isometric isomorphism.

$(3) \Rightarrow (1)$. Let $M_b(S, \omega)$ be a WAP-algebra. Since $\ell_1(S, \omega)$ is a norm closed subalgebra of $M_b(S, \omega)$, the weighted semigroup algebra $\ell_1(S, \omega)$ is a WAP-algebra. Using the double limit criterion, it is easy to check that $\text{wap}(S, 1/\omega) = \text{WAP}(\ell_1(S, \omega))$ (see also [11, Theorem 3.7]) where we treat $\ell^\infty(S, 1/\omega)$ as an $\ell_1(S, \omega)$-bimodule. Then $\tilde{\epsilon} : \ell_1(S, \omega) \rightarrow \text{wap}(S, 1/\omega)^*$ is an isometric isomorphism. Since $\text{wap}(S, 1/\omega)$ is a $C^*$-algebra, as $(2) \Rightarrow (1)$, $\epsilon : S \rightarrow \tilde{X}$ is one to one.

**Corollary 3.1.** For a locally compact semi-topological semigroup $S$, $M_b(S, \omega)$ is a WAP-algebra if and only if $\ell_1(S, \omega)$ is a WAP-algebra.

For $\omega = 1$, it is clear that $\tilde{X} = S^{\text{wap}}$, and the map $\epsilon : S \rightarrow S^{\text{wap}}$ is one to one if and only if $\text{wap}(S)$ separates the points of $S$, see [3].
Corollary 3.2. For a locally compact semi-topological semigroup $S$, the following statements are equivalent:

1. $M_b(S)$ is a WAP-algebra;
2. $\ell_1(S)$ is a WAP-algebra;
3. The evaluation map $\epsilon : S \rightarrow S^{wap}$ is one to one;
4. $wap(S)$ separates the points of $S$.

Illustrating our results, we conclude the paper with the following examples.

Examples 3.1.

(i) We examine the semigroup algebra $\ell_1(S)$ for $S = \mathbb{N}$ equipped with various multiplications. When $S$ is equipped with the min multiplication, the semigroup algebra $\ell_1(S)$ is a WAP-algebra, while, is not neither Arens regular nor a dual Banach algebra. If we furnish $S$ with the max multiplication, then $\ell_1(S)$ is a dual Banach algebra (and so a WAP-algebra) which is not Arens regular. If we change the multiplication of $S$ to the zero multiplication then the resulted semigroup algebra is Arens regular (so a WAP-algebra) which is not a dual Banach algebra. This describes the interrelation between the concepts of being Arens regular algebra, dual Banach algebra and WAP-algebra.

(ii) Let $S$ be the set of all sequences with 0,1 values. We equip $S$ with pointwise multiplication. We denote by $e_n$ the characteristic of $n$. Let $s = \{x_n\} \in S$, and let $F_w(S)$ be the set of all elements of $S$ such that $x_i = 0$ for only finitely index $i$. It is easy to see that $F_w(S)$ is countable. Let $S = \{s_1, s_2, \cdots\}$. Recall that, each element $g \in \ell^\infty(S)$ has the presentation as $g = \sum_{s \in S} g(s)\chi_s$, see [6, p.65]. Suppose $g = \sum_{s \in S \setminus F_w(S)} g(s)\chi_s$ be in $wap(S)$, we show that $g = 0$. Let $s = \{x_n\} \in S$, and $\{k \in \mathbb{N} : x_k = 0\} = \{k_1, k_2, \cdots\}$ be an infinite set. Put $a_n = s + \sum_{j=1}^{n} e_{k_j}$ and $b_m = s + \sum_{i=m}^{\infty} e_{k_i}$. Then

$$a_nb_m = \begin{cases} \sum_{j=m}^{n} e_{k_j} + s & \text{if } m \leq n \\ s & \text{if } m > n. \end{cases}$$

Thus

$$g(s) = \lim_{n} \lim_{m} g(a_nb_m) = \lim_{m} \lim_{n} g(a_nb_m) = \lim_{m} g(s + \sum_{i=m}^{\infty} e_{k_i}) = 0.$$ 

Indeed,

$$wap(S) = \{ f \in \ell^\infty(S) : f = \sum_{i=1}^{\infty} f(s_i)\chi_{s_i}, \quad s_i \in F_w(S) \} \oplus \mathbb{C}$$

It is also clear that $F_w(S)$ is the subsemigroup of $S$ with $wap(F_w(S)) = \ell^\infty(F_w(S))$. So $\ell_1(F_w(S))$ is Arens regular. Let $T$ consist of those sequences $s = \{x_n\} \in S$
such that \( x_i = 0 \) for infinitely index \( i \), then \( T \) is a subsemigroup of \( S \) and \( \text{wap}(T) = \mathbb{C} \). Since \( \varepsilon_1 : T \to S^{\text{wap}} \) is not one to one, \( \ell_1(S) \) is not a WAP-algebra. This shows that \( \ell_1(S) \) need not be a WAP-algebra.

(iii) If we equip \( S = \mathbb{R}^2 \) with the multiplication \((x,y),(x',y') = (xx',xy + y')\), then \( M_0(S) \) is not a WAP-algebra. Indeed, every non-constant function \( f \) over \( x \)-axis is not in \( \text{wap}(S) \). Let \( f(0,z_1) \neq f(0,z_2) \) and \( \{x_m\}, \{y_m\}, \{\beta_n\} \) be sequences with distinct elements satisfying the recursive equation

\[ \beta_n x_m + y_m = \frac{m z_1 + n z_2}{m + n}. \]

Then

\[ \lim \lim_{n \to \infty} f((0,\beta_n),(x_m,y_m)) = \lim \lim_{n \to \infty} f(0,\beta_n x_m + y_m) = \lim \lim_{n \to \infty} f(0,\frac{m z_1 + n z_2}{m + n}) = f(0,z_1), \]

and similarly

\[ \lim \lim_{n \to \infty} f((0,\beta_n),(x_m,y_m)) = f(0,z_2). \]

Thus the map \( \varepsilon : S \to S^{\text{wap}} \) is not one to one, so \( M_0(S) \) is not a WAP-algebra.

(iv) Let \( S \) be the interval \([\frac{1}{2},1]\) with the multiplication \( xy = \max \{ \frac{1}{2},xy \} \), where \( xy \) is the ordinary multiplication on \( \mathbb{R} \). Then for each \( s \in S \setminus \{ \frac{1}{2} \} \), \( x \in S \), the set \( x^{-1}s \) is finite. But \( x^{-1} \frac{1}{2} = [\frac{1}{2}, \frac{1}{2x}] \). Let \( B = [\frac{1}{2}, \frac{3}{2}] \). Then for every finite subset \( F \) of \( B \),

\[ \bigcap_{x \in F} x^{-1} \frac{1}{2} \cap \bigcap_{x \in B \setminus F} x^{-1} \frac{1}{2} = [\frac{2}{3}, \frac{1}{2x_F}], \]

where \( x_F = \max F \). By [15, Theorem 4], \( \chi_\frac{1}{2} \notin \text{wap}(S) \). So \( c_0(S \setminus \{ \frac{1}{2} \}) \oplus \mathbb{C} \subseteq \text{wap}(S) \). It can be readily verified that \( \varepsilon : S^Z \to S^{\text{wap}} \) is one to one, so \( \ell_1(S) \) is a WAP-algebra but \( c_0(S) \nsubseteq \text{wap}(S) \).

(v) Take \( T = (\mathbb{N} \cup \{0\},...) \) with 0 as zero of \( T \) and the multiplication defined by

\[ n.m = \begin{cases} n & \text{if } n = m \\ 0 & \text{otherwise.} \end{cases} \]

Set \( S = T \times T \) equipped with the pointwise product. Now let \( X = \{(k,0) : k \in T\} \), \( Y = \{(0,k) : k \in T\} \) and \( Z = X \cup Y \). We use the Ruppert criterion [15] to show that \( \chi_z \notin \text{wap}(S) \), for each \( z \in Z \). Let \( B = \{(k,n) : k,n \in T\} \), then

\[ (k,n)^{-1}(k,0) = \{(k,m) : m \neq n\} = B \setminus \{(k,n)\}. \]

Thus for each finite subsets
Let $F$ of $B$, 
\[
\cap\{(k,n)^{-1}(k,0) : (k,n) \in F\} \setminus \cap\{(k,0)(k,n)^{-1} : (k,n) \in F\} \setminus \cap\{(k,n)^{-1}(k,0) : (k,n) \in B \setminus F\} = (B \setminus F) \setminus F = B \setminus F
\]

and the last set is infinite. This means that $\chi_{(k,0)} \notin \text{wap}(S)$. Similarly $\chi_{(0,k)} \notin \text{wap}(S)$. Let $f = \sum_{n=0}^{\infty} f(0,n)\chi_{(0,n)} + \sum_{m=1}^{\infty} f(m,0)\chi_{(m,0)}$ be in $\text{wap}(S)$. Then for each fixed $n$ and the sequence $\{(n,k)\}$ in $S$, we have $\lim_k f(n,k) = \lim_k \lim_l f(n,l,k) = \lim_l \lim_k f(n,l,k) = f(n,0)$, which implies that $f(n,0) = 0$. Similarly $f(0,n) = 0$ and $f(0,0) = 0$. Thus $f = 0$. Since $\text{wap}(S)$ can not separate the points of $S$ so $\ell_1(S)$ is not a WAP-algebra. Let $\omega(n,m) = 2^n3^m$ for $(n,m) \in S$. Then $\omega$ is a weight on $S$ such that $\omega \in \text{wap}(S,1/\omega)$. Then the evaluation mapping $\epsilon : S \longrightarrow \tilde{X}$ is one to one. This means that $\ell_1(S,\omega)$ is a WAP-algebra while $\ell_1(S)$ is not!

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**References**

[1] M. Abolghasemi, A. Rejali and H.R. Ebrahimi Vishki, Weighted semigroup algebras as dual Banach algebras, arXiv:0808.1404 [math.FA].

[2] J.W. Baker and A. Rejali, On the Arens regularity of weighted convolution algebras, J. London Math. Soc. (2) 40, 535-546, (1989).

[3] J.F. Berglund, H.D. Junghenn and P. Milnes, Analysis on Semigroups, Wiley, New York, 1989.

[4] J.B. Conway, A Course in Functional Analysis, Springer-Verlag, Berlin, 1985.

[5] H.G. Dales, Banach Algebras and Automatic Continuity, Clarendon Press, Oxford, 2000.

[6] H.G. Dales and A.T.-M. Lau, The Second Dual of Beurling Algebras, Mem. Amer. Math. Soc. 177 (2005), no. 836.
[7] H.G. Dales, A.T.-M. Lau, and D. Strauss, Banach algebras on semigroups and on their compactification, Mem. Amer. Math. Soc. 205 (2010), 1-165.

[8] M. Daws, Connes-amenability of bidual and weighted semigroup algebras, Math. Scand. 99, no. 2 (2006), 217-246.

[9] M. Daws, Dual Banach algebras: Representation and injectivity, Studia Math. 178 (2007), 231-275.

[10] M. Daws, H.L. Pham and S. White, Conditions implying the uniqueness of the weak$^*$-topology on certain group algebras, Houston J. Math. 35, no. 1 (2009), 253-270.

[11] M. Lashkarizadeh Bami, Function algebras on weighted topological semigroups, Math. Japon. 47, no. 2 (1998), 217-227.

[12] A. Rejali, The analogue of weighted group algebra for semitopological semigroups, J. Sci. I.R. Iran 6, no. 2 (1995), 113-120.

[13] A. Rejali, The Arens regularity of weighted semigroup algebras, Sci. Math. Jpn. 60, no.1 (2004), 129-137.

[14] A. Rejali and H.R.E. Vishki, Weighted convolution measure algebras characterized by convolution algebras, J. Sci. I. Iran 18, (4) (2007), 345-349.

[15] W.A.F. Ruppert, On weakly almost periodic sets, Semigroup Fourm 32, no.3 (1985), 267-281.

[16] V.Runde, Dual Banach algebras: Connes-amenability, normal, virtual diagonals, and injectivity of the predual bimodule, Math. Scand. 95, (2004), 124-144.

[17] N.J. Young, Periodicity of functionals and representations of normed algebras on reflexive spaces, Proc. Edinburgh Math. Soc. (2) 20, (1976/77), 99-120.