EXISTENCE OF TWO SOLUTIONS FOR SINGULAR Φ-LAPLACIAN PROBLEMS

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Abstract. Existence of two solutions to a parametric singular quasi-linear elliptic problem is proved. The equation is driven by the Φ-Laplacian operator and the reaction term can be non-monotone. The main tools employed are a local minimum theorem and the Mountain Pass theorem, together with the truncation technique. Global $C^{1,\tau}$ regularity of solutions is also investigated, chiefly via a priori estimates and perturbation techniques.

1. Introduction and main results

In this paper we consider the problem
\[
\begin{cases}
\Delta_\Phi u = \lambda f(x,u) & \text{in } \Omega, \\
u > 0 & \text{in } \Omega, \\
u = 0 & \text{on } \partial\Omega,
\end{cases}
\]
where $\Omega \subseteq \mathbb{R}^N$, $N \geq 2$, is a bounded domain with smooth boundary $\partial\Omega$, $\lambda > 0$ is a parameter, $f : \Omega \times (0, +\infty) \to [0, +\infty)$ is a Carathéodory function, and $\Delta_\Phi$ is the $\Phi$-Laplacian, namely,
\[
\Delta_\Phi u := \text{div} (a(|\nabla u|) \nabla u)
\]
for a suitable $C^1$ function $a : (0, +\infty) \to (0, +\infty)$. Setting $\varphi(t) = ta(t)$ for all $t > 0$, we denote by $\Phi$ the primitive of $\varphi$ satisfying $\Phi(0) = 0$. With the hypotheses below (cf. also [14 Appendix I]), $\Phi$ turns out to be the Young function generated by $\varphi$ (see [21 Definition 3.2.1]).

Definition 1.1. We say that $u \in W_0^{1,\Phi}(\Omega)$ is a (weak) solution to $(P_{\lambda,f})$ if $u > 0$ in $\Omega$ and, for any $v \in W_0^{1,\Phi}(\Omega)$, one has both $f(\cdot, u)v \in L^1(\Omega)$ and
\[
\int_\Omega a(|\nabla u(x)|) \nabla u(x) \cdot \nabla v(x) \, dx = \lambda \int_\Omega f(x,u(x))v(x) \, dx.
\]
We assume the following hypotheses (the indices $i_\Psi, s_\Psi$ are defined in (2.3) below):

H(\text{a})_1:
\[
-1 < \inf_{t>0} \frac{ta'(t)}{a(t)} \leq \sup_{t>0} \frac{ta'(t)}{a(t)} < +\infty.
\]
H(a)2: We suppose that
\[ \int_{1}^{+\infty} \Theta_{\Phi}(t) \, dt = +\infty, \quad \text{where} \quad \Theta_{\Phi}(t) := \frac{\Phi^{-1}(t)}{t^{1 + \frac{1}{\gamma}}}. \] (1.3)

Accordingly, the Sobolev-Orlicz conjugate \( \Phi_{*} \) is well defined; see Definition 2.3.

We also suppose \( s_{\Phi} < i_{\Phi_{*}} \).

H(f)1: One has
\[ \liminf_{s \to 0^{+}} f(x, s) = +\infty \quad \text{uniformly w.r.t.} \quad x \in \Omega. \] (1.4)

H(f)2: There exist \( c_{i} > 0, \ i = 1, 2, \gamma \in (0, 1) \), and a Young function \( \Upsilon \) such that
\[ f(x, s) \leq c_{1} \Upsilon^{-1}(\Upsilon(s)) + c_{2}s^{-\gamma} \] (1.5)

for almost all \( x \in \Omega \) and all \( s > 0 \), where \( \Upsilon \) is the Young conjugate of \( \Upsilon \) (see Definition 2.2).

H(f)3: There exist \( R > 0 \) and \( \mu > s_{\Phi} \) such that
\[ \mu F(x, t) \leq tf(x, t), \quad \text{being} \quad F(x, t) := \int_{R}^{t} f(x, s) \, ds, \] (1.6)

for almost all \( x \in \Omega \) and all \( t \geq R \).

Remark 1.2. Let us briefly comment the main assumptions we have done.

• Hypothesis H(a)1 is called ellipticity condition for operators with Uhlenbeck structure, namely, in the form (1.1). It implies \( 1 < i_{\Phi} \leq s_{\Phi} < +\infty \), which in turn implies \( \Phi \in \Delta_{2} \cap \nabla_{2} \) (see (2.2) below).

• The first part of H(a)2 is the Sobolev-Orlicz analogue of the Sobolev hypothesis \( p < N \). As customary, H(a)2 forces the problem in the worst regularity setting, because of the lack of the Morrey-type embedding \( W_{0}^{1,\Phi}(\Omega) \hookrightarrow C^{0,\tau}(\Omega) \) (see [21] Theorem 7.4.4 for a complete statement). Regarding the second part of H(a)2, the condition \( s_{\Phi} < i_{\Phi_{*}} \) is used only in Section 3.

• The requirement \( s_{\Upsilon} < i_{\Phi_{*}} \) in H(f)2 is made for the sake of simplicity: indeed, it implies \( \Upsilon \ll \Phi_{*} \) (see (2.3)), which in turn guarantees the compactness of the embedding \( W_{0}^{1,\Phi}(\Omega) \hookrightarrow L^{\Upsilon}(\Omega) \). Actually, in Section 3 it suffices to require \( s_{\Upsilon} \leq i_{\Phi_{*}} \). In this respect, see also Remark A.6 in the appendix.

• Condition (1.5) parallels the sub-critical growth condition used in the standard Sobolev setting. We recall that \( \Upsilon^{-1}(\Upsilon(s)) \) can be replaced with \( \frac{\Upsilon(s)}{s} \), according to the inequalities
\[ \Psi(s) \leq s\Psi^{-1}(\Psi(s)) \leq 2\Psi(s) \quad \forall s > 0, \] (1.7)
valid for any Young function \( \Psi \). For a proof of (1.7), vide [27] Proposition 2.1.1.

• Hypothesis H(f)3 is an adaptation of the Ambrosetti-Rabinowitz unilateral condition (see, e.g., [25] p.154) in the Sobolev-Orlicz setting. Following [12], H(f)3 can be weakened by requiring, instead of \( \mu > s_{\Phi} \),
\[ \mu > \limsup_{t \to +\infty} \frac{t_{\varphi}(t)}{\Phi(t)}. \]
The primary aim of the present work is to extend the results of [8, 19] to problems driven by non-homogeneous operators as the \((p, q)\)-Laplacian \(\Delta_p + \Delta_q\), being \(\Delta_r u := |\nabla u|^{r-2}\nabla u\) the classical \(r\)-Laplacian, \(r \in (1, +\infty)\). A class of operators encompassing the \((p, q)\)-Laplacian is the one described in [14, Appendix I], where ellipticity and Uhlenbeck structure are coupled with a \(p\)-growth condition that allows to work in the Sobolev setting \(W^{1,p}_0(\Omega)\). This class can be extended further, up to the \(\Phi\)-Laplacian operator, for which regularity theory and maximum principles are still available (see [23, 26]). Existence and regularity results for problems involving the \(\Phi\)-Laplacian can be found, e.g., in [12, 13, 29, 9]. Dealing with \(\Phi\)-Laplacian problems requires the usage of Sobolev-Orlicz spaces, since \(\Phi\) may have non-standard growth; this fact is discussed in the appendix, where a class of explicit examples is furnished. An introductory exposition about Orlicz and Sobolev-Orlicz spaces is provided [21, Chapters 3 and 7]; we also address to the monographs [20, 27]. The relation between Sobolev-Orlicz spaces and PDEs is the subject of [17].

Also singular \(\Phi\)-Laplacian problems have been studied during the last years. The model case \(f(x, u) = a(x) u^{-\gamma}\), with \(a \geq 0\) and \(\gamma > 0\), was investigated in [28]. A more general problem, including also convection terms (that is, \(f\) depends also on \(\nabla u\)), was studied in [10]. Due to the lack of variational setting, primarily caused by the strongly singular term (i.e., \(\gamma > 1\)), both works make use of a generalized Galerkin method to get a solution. We are not aware of other existence results pertaining singular \(\Phi\)-Laplacian problems.

In spite of the papers mentioned above, our approach is variational: first we construct a sub-solution \(u\) (Lemma 2.7) and truncate \(f\) at the level of \(u\); then we consider the truncated problem \(\text{(P}_{\lambda}\), which is equivalent to \(\text{(P}_{\lambda}\) (cf. Lemma 2.9), and find a solution by means of the local minimum theorem reported in Theorem 4.1. In order to get a second solution, we use the Mountain Pass theorem, jointly with the Ambrosetti-Rabinowitz unilateral condition (see H(f)_3 above), which implies the Palais-Smale condition (vide Lemma 4.1). We highlight that the Ambrosetti-Rabinowitz condition has been used in the context of \(\Phi\)-Laplacian problems also in [12, 29], while [13] uses the Mountain Pass theorem without the Palais-Smale condition. Here we highlight the fact we find the first solution without using the \(W^{1,\Phi}\) versus \(C^1\) local minimizer technique.

It is worth noticing that the aforementioned works [28, 10], that make no use of the Mountain Pass theorem, consider reaction terms with growth not faster than \(\Phi\) (usually called ‘linear’); on the contrary we treat, with the same technique, both linear forcing terms and super-linear ones (but ‘sub-critical’, i.e., growing slower than \(\Phi_s\)). This is remarkable in our context since, in a variational setting, linear problems possess coercive energy functionals, allowing to find a solution via the Weierstrass-Tonelli theorem instead of the Mountain Pass one. To the best of our knowledge, this is the first work treating singular \(\Phi\)-Laplacian problems with this technique, that offers a unified approach to the coercive and the non-coercive cases.

Regularity of solutions is investigated in Section 3. \(L^\infty\) estimates are provided in Lemma 3.2 by using a technique introduced by De Giorgi; see, e.g., [22, Lemma 2.5.4]. Then \(C^{1,\tau}\) regularity is obtained in Theorem 3.3 via the perturbation technique developed by Campanato [6, 7], Giaquinta and Giusti [16], combined with a result pertaining solutions to singular semi-linear elliptic problems that traces back to [15] (see also [14]).
In the appendix we discuss about the importance of using Sobolev-Orlicz spaces, providing also two examples of problems fulfilling $H(a)_1$--$H(a)_2$ and $H(f)_1$--$H(f)_3$.

2. Preliminaries

We denote by $d(x)$ the distance of $x \in \Omega$ from $\partial \Omega$, while $d_\Omega$ stands for the diameter of $\Omega$. Given any function $u : \Omega \to \mathbb{R}$ and any number $\rho \in \mathbb{R}$, $\{u < \rho\}$ stands for the set $\{x \in \Omega : u(x) < \rho\}$, and the same holds for $\{u \geq \rho\}$, $\{u = \rho\}$, etc.

To avoid unnecessary technicalities, hereafter we use ‘for all $x \in \Omega$’ instead of ‘for almost all $x \in \Omega$’ when no confusion arises.

Definition 2.1. A function $\Psi : [0, +\infty) \to [0, +\infty)$ is said to be a Young function\footnote{Some textbooks, as [1], use the notion of N-function; here we adopt the nomenclature used in [21]. See [1] Section 8.1 and [21] Remark 3.2.7 for further details.} if it is continuous, strictly increasing, convex, and the following holds true:

$$\lim_{t \to 0^+} \frac{\Psi(t)}{t} = 0, \quad \lim_{t \to +\infty} \frac{\Psi(t)}{t} = +\infty.$$  \hfill (2.1)

Definition 2.2. Let $\Psi$ be a Young function. We denote by $\Psi^*$ the Young conjugate of $\Psi$, defined via Legendre transformation as

$$\Psi^*(t) := \max_{s \geq 0} \{st - \Psi(s)\} \quad \forall t \geq 0.$$

Definition 2.3. Let $\Psi$ be a Young function satisfying (1.3) with $\Psi$ in place of $\Phi$. Suppose also, without loss of generality (cf. [21, Exercise 7.2.2]), that

$$\int_0^1 \Theta_\Psi(s) \, ds < +\infty.$$

The Sobolev-Orlicz conjugate of $\Psi$, indicated as $\Psi_*$, is defined via its inverse as

$$\Psi_*(t) := \int_0^t \Theta_\Psi(s) \, ds.$$

Definition 2.4. Let $\Psi$ be a Young function. We write $\Psi \in \Delta_2$ if there exist $k, T > 0$ such that

$$\Psi(2t) \leq k \Psi(t) \quad \forall t \geq T.$$

We write $\Psi \in \nabla_2$ if there exist $\eta > 1$ and $T > 0$ such that

$$\Psi(t) \leq \frac{1}{2\eta} \Psi(\eta t) \quad \forall t \geq T.$$

Equivalent statements are collected in [27] Theorem 2.3.3 and Corollary 2.3.4; here we only mention

$$\Psi \in \Delta_2 \iff \Psi^* \in \nabla_2 \iff \limsup_{t \to +\infty} \frac{t\Psi'(t)}{\Psi(t)} < +\infty,$$

$$\Psi \in \nabla_2 \iff \Psi^* \in \Delta_2 \iff \liminf_{t \to +\infty} \frac{t\Psi'(t)}{\Psi(t)} > 1.$$  \hfill (2.2)

For a comparison with power-law functions, see [27] Corollary 2.3.5.
Let $\Psi \in \Delta_2$. We endow the Orlicz space
\[ L^\Psi(\Omega) := \{ u : \Omega \to \mathbb{R} \text{ measurable} : \int_\Omega \Psi(|u(x)|) \, dx < +\infty \} \]
with the Luxembourg norm
\[ \|u\|_{L^\Psi(\Omega)} := \inf \left\{ \lambda > 0 : \int_\Omega \Psi\left(\frac{|u(x)|}{\lambda}\right) \, dx \leq 1 \right\} . \]
Suppose that
\[ 1 < i_\Psi := \inf_{t>0} \frac{t\Psi'(t)}{\Psi(t)} \leq \sup_{t>0} \frac{t\Psi'(t)}{\Psi(t)} =: s_\Psi < +\infty, \tag{2.3} \]
which implies $\Psi \in \Delta_2 \cap \nabla_2$ by (2.2). We define the functions $\zeta_\Psi, \bar{\zeta}_\Psi : [0, +\infty) \to [0, +\infty)$ as
\[ \zeta_\Psi(t) := \min\{t^{i_\Psi}, t^{s_\Psi}\}, \quad \bar{\zeta}_\Psi(t) := \max\{t^{i_\Psi}, t^{s_\Psi}\}. \]
One has (cf. [13, Lemma 2.1])
\[ \zeta_\Psi(k)\Psi(t) \leq \Psi(kt) \leq \bar{\zeta}_\Psi(k)\Psi(t) \quad \forall k, t \geq 0 \tag{2.4} \]
and
\[ \zeta_\Psi\left(\|w\|_{L^\Psi(\Omega)}\right) \leq \int_\Omega \Psi(|w(x)|) \, dx \leq \bar{\zeta}_\Psi\left(\|w\|_{L^\Psi(\Omega)}\right) \tag{2.5} \]
for all $w \in L^\Psi(\Omega)$. We also recall (see [13, Lemmas 2.4-2.5]) that
\[ s_\Psi' \leq i_\Psi \leq s_\Psi \leq s_\Psi^*, \tag{2.6} \]
and, provided $s_\Psi < N,$
\[ i_\Psi^* \leq i_\Psi \leq s_\Psi \leq s_\Psi^*, \tag{2.7} \]
being $r' := \frac{r}{r-1}$ and $r^* := \frac{Nr}{N-r}$ respectively the Young and the Sobolev conjugates of $r$.

**Definition 2.5.** Let $\Psi_1, \Psi_2$ be two Young functions. We write $\Psi_1 \lessdot \Psi_2$ if there exist $c, T > 0$ such that
\[ \Psi_1(t) \leq \Psi_2(ct) \quad \forall t \geq T. \tag{2.8} \]
We write $\Psi_1 \ll \Psi_2$ if for any $c > 0$ there exists $T = T(c) > 0$ such that (2.8) holds true. Equivalently,
\[ \lim_{t \to +\infty} \frac{\Psi_1(t)}{\Psi_2(\eta t)} = 0 \quad \forall \eta > 0. \]
Further characterizations can be found in [27, Theorem 2.2.2]. It is worth recalling the following chain of (non-reversible) implications:
\[ s_{\Psi_1} < i_{\Psi_2} \quad \Rightarrow \quad \Psi_1 \ll \Psi_2 \quad \Rightarrow \quad \Psi_1 < \Psi_2. \tag{2.9} \]
We consider the Sobolev-Orlicz space
\[ W^{1,\Phi}(\Omega) := \{ u \in L^\Phi(\Omega) : |\nabla u| \in L^\Phi(\Omega) \}, \]
\[Since \Psi \in \Delta_2, we make no distinction between Orlicz space and Orlicz class; see [21, Theorem 3.7.3].
equipped with the norm \( \|u\|_{W^{1,\Phi}(\Omega)} := \|u\|_{L^\Phi(\Omega)} + \| \nabla u \|_{L^\Phi(\Omega)} \), and its subspace \( W_0^{1,\Phi}(\Omega) \), which is the closure of \( C_c^\infty(\Omega) \) under \( \| \cdot \|_{W^{1,\Phi}(\Omega)} \). According to the Poincaré inequality (see, e.g., \([10\text{, p.8}]\)), we are allowed to endow \( W_0^{1,\Phi}(\Omega) \) with the norm

\[
\|u\|_{W_0^{1,\Phi}(\Omega)} := \|\nabla u\|_{L^\Phi(\Omega)}.
\]

Since \( \Phi \in \Delta_2 \cap \nabla_2 \), the space \( W_0^{1,\Phi}(\Omega) \) is separable and reflexive (cf. \([11\text{, Theorem 8.31}]\)). Its dual space will be denoted by \( W^{−1,\Phi^*}(\Omega) \), while \( \langle \cdot, \cdot \rangle \) represent the duality brackets between \( W^{−1,\Phi^*}(\Omega) \) and \( W_0^{1,\Phi}(\Omega) \). We recall that \( W_0^{1,\Phi}(\Omega) \hookrightarrow L^{\Phi^*}(\Omega) \) continuously and \( W_0^{1,\Phi}(\Omega) \hookrightarrow L^\Phi(\Omega) \) compactly for all \( \Upsilon \ll \Phi^* \); see \([21\text{, Theorems 7.2.3 and 7.4.4}]\).

Although the next result is folklore, we briefly sketch its proof for the sake of completeness.

**Lemma 2.6.** Under \( H(a) \), the operator \( −\Delta_\Phi : W_0^{1,\Phi}(\Omega) \to W^{−1,\Phi^*}(\Omega) \) defined as

\[
\langle −\Delta_\Phi u, v \rangle := \int_\Omega a(|\nabla u|) \nabla u \cdot \nabla v \, dx \quad \forall u, v \in W_0^{1,\Phi}(\Omega)
\]

is well defined, bounded, continuous, coercive, strictly monotone, and of type \((S_+)\). Moreover, the functional \( H : W_0^{1,\Phi}(\Omega) \to \mathbb{R} \) defined as

\[
H(u) := \int_\Omega \Phi(|\nabla u|) \, dx
\]

is convex, weakly lower semi-continuous, and of class \( C^1 \), with \( H' = −\Delta_\Phi \) in \( W^{−1,\Phi^*}(\Omega) \).

**Proof.** According to the Hölder inequality (see \([17\text{, p.62}]\)) we get

\[
|\langle −\Delta_\Phi u, v \rangle| \leq \int_\Omega \varphi(|\nabla u|)|\nabla v| \, dx \leq s_\Phi \int_\Omega \frac{\Phi(|\nabla u|)}{|\nabla u|} |\nabla v| \, dx \\
\leq s_\Phi \left( \frac{\Phi(|\nabla u|)}{|\nabla u|} \right)_{L^\Phi(\Omega)} \|\nabla v\|_{L^\Phi(\Omega)} = s_\Phi \left( \frac{\Phi(|\nabla u|)}{|\nabla u|} \right)_{L^\Phi(\Omega)} \|v\|_{W_0^{1,\Phi}(\Omega)}.
\]

Exploiting \([17]\) we infer

\[
\int_\Omega \Phi \left( \frac{\Phi(|\nabla u|)}{|\nabla u|} \right) \, dx \leq \int_\Omega \Phi(|\nabla u|) \, dx < +\infty.
\]

By \([2.11]\), \([2.12]\) we deduce that \( −\Delta_\Phi \) is well defined. Boundedness and continuity follow from \([2.12]\) and \([28\text{, Lemma 7.3}]\).

In order to prove the coercivity of \( −\Delta_\Phi \), we exploit \([25]\) to obtain

\[
\int_\Omega a(|\nabla u|)|\nabla u|^2 \, dx = \int_\Omega \varphi(|\nabla u|)|\nabla u| \, dx \geq i_\Phi \int_\Omega \Phi(|\nabla u|) \, dx \\
\geq i_\Phi \zeta_\Phi(\|\nabla u\|_{L^\Phi(\Omega)})
\]

for all \( u \in W_0^{1,\Phi}(\Omega) \). Hence

\[
\frac{\langle −\Delta_\Phi u, u \rangle}{\|u\|_{W_0^{1,\Phi}(\Omega)}} \geq i_\Phi \zeta_\Phi(\|u\|_{W_0^{1,\Phi}(\Omega)}) \to +\infty \quad \text{as} \quad \|u\|_{W_0^{1,\Phi}(\Omega)} \to +\infty.
\]

The strict monotonicity and the \((S_+)\) property of \( −\Delta_\Phi \) are guaranteed by \([9\text{, Propositions A.2-A.3}]\).
Convexity of $H$ directly follows from convexity of $\Phi$, while Lebesgue’s dominated convergence theorem and [28, Lemma 7.3] ensure that $H$ is continuous. As a consequence, $H$ is weakly lower semi-continuous. The fact that $H$ is of class $C^1$ has been proved in [13, Lemma A.3]. □

Firstly we consider the problem

\[
\begin{aligned}
-\Delta \Phi u &= f(x, u) \quad \text{in } \Omega, \\
u &= 0 \quad \text{in } \Omega, \\
u &= 0 \quad \text{on } \partial \Omega.
\end{aligned}
\]

(P$_{1,f}$)

**Lemma 2.7.** Suppose $H(a)_1$ and $H(f)_1$. Then problem (P$_{1,f}$) admits a sub-solution $u \in C^{1,\tau}_0(\Omega)$, with $\tau \in (0,1]$ opportune, satisfying

\[
k_1 d(x) \leq u(x) \leq k_2 d(x) \quad \forall x \in \Omega
\]

for suitable $k_1, k_2 > 0$.

**Proof.** This proof is patterned after the one of [18, Lemma 3.5]. Hypothesis $H(f)_1$ provides $\delta > 0$ such that

\[
f(x, s) \geq 1 \quad \text{for all } (x, s) \in \Omega \times (0, \delta).
\]

(2.15)

For any $n \in \mathbb{N}$, let us consider the following problem:

\[
\begin{aligned}
-\Delta \Phi u &= \frac{1}{n} \quad \text{in } \Omega, \\
u &= 0 \quad \text{on } \partial \Omega.
\end{aligned}
\]

(P$_{1,\frac{1}{n}}$)

By virtue of Lemma 2.6 Minty-Browder’s theorem [5, Theorem 5.16] can be applied; thus (P$_{1,\frac{1}{n}}$) admits a unique solution $u_n \in W^{1,\Phi}_0(\Omega)$. Lieberman’s nonlinear regularity theory [23, Theorem 1.7] guarantees that $\{u_n : n \in \mathbb{N}\}$ is bounded in $C^{1,\tau}_0(\Omega)$ for some $\tau \in (0,1]$. Hence, thanks to the Ascoli-Arzelà theorem and up to subsequences, we get $u_n \to u$ in $C^{1,\tau}_0(\Omega)$ for some $u \in C^1_0(\Omega)$. Passing to the limit in the weak formulation of (P$_{1,\frac{1}{n}}$) reveals that $u \equiv 0$ in $\Omega$. Hence it is possible to choose $\hat{n} \in \mathbb{N}$ such that

\[
\|u_{\hat{n}}\|_{L^\infty(\Omega)} < \delta.
\]

(2.16)

Set $u := u_{\hat{n}}$. The strong maximum principle [26, Theorem 1.1.1] ensures $u > 0$ in $\Omega$. Thus, by (2.15)–(2.16) one has

\[-\Delta \Phi u = \frac{1}{\hat{n}} \leq 1 \leq f(x, u)
\]

in weak sense. A standard argument involving the Boundary Point lemma [26, Theorem 5.5.1] and the Hölder continuity of $\nabla u$ gives (2.14). □

**Remark 2.8.** Without loss of generality, one can choose $\delta < R$ in (2.15), where $R > 0$ comes from $H(f)_3$. Hereafter we make this assumption, which yields $\|u\|_{L^\infty(\Omega)} \leq \delta < R$, according to (2.16).
Let us consider the auxiliary problem
\[
\begin{aligned}
-\Delta \Phi u &= \hat{f}(x, u) \quad \text{in } \Omega, \\
\quad u &= 0 \quad \text{on } \partial \Omega,
\end{aligned}
\]
where \( \hat{f} : \Omega \times \mathbb{R} \to [0, +\infty) \) is defined as
\[
\hat{f}(x, s) := \begin{cases}
\hat{f}(x, u(x)) & \text{if } |s| \leq u(x), \\
f(x, |s|) & \text{if } |s| > u(x),
\end{cases}
\]
being \( u \) as in Lemma 2.7. We also define
\[
\hat{F}(x, s) := \int_0^s \hat{f}(x, t) \, dt.
\]
Exploiting (1.5) and (2.14) one has
\[
0 \leq \hat{f}(x, s) \leq c_1 \Upsilon^{-1}(\Upsilon(|s|)) + c_1 \Upsilon^{-1}(\Upsilon(u(x))) + c_2 u(x)^{-\gamma},
\]
being \( \alpha, \beta > 0 \) such that
\[
\alpha := c_2 k_1^{-\gamma}, \quad \beta := c_1 \Upsilon^{-1}(\Upsilon(k_2 d_{\Omega})).
\]

**Lemma 2.9.** Let \( H(a)_1 \) and \( H(f)_1 \) be satisfied. Then any \( u \in W_0^1(\Omega) \) weak solution to (\( P_{1,\hat{f}} \)) is a weak solution to (\( P_{1,f} \)) and vice-versa.

**Proof.** To show the equivalence of (\( P_{1,\hat{f}} \)) and (\( P_{1,f} \)), it suffices to prove that any solution to either (\( P_{1,\hat{f}} \)) or (\( P_{1,f} \)) is greater than \( \underline{u} \); then the conclusion will follow by the definition of \( \hat{f} \).

Let \( u \in W_0^1(\Omega) \) be a weak solution to (\( P_{1,\hat{f}} \)). The weak maximum principle, jointly with \( \hat{f} \geq 0 \), ensures \( u \geq \underline{u} \). Then Lemma 2.7 and the weak comparison principle (cf., e.g., [26, Theorem 3.4.1]), applied on \( u \) and \( \underline{u} \), yields \( u \geq \underline{u} \); indeed \( -\Delta \Phi \) is a strictly monotone operator (see Lemma 2.6) and, in weak sense,
\[
-\Delta \Phi u \leq f(x, u) = \hat{f}(x, u) = -\Delta \Phi u \quad \text{in } \{ x \in \Omega : u(x) \leq u(x) \}.
\]

Now let \( u \in W_0^1(\Omega) \) be a weak solution to (\( P_{1,f} \)). Recalling that \( \|u\|_{L^\infty(\Omega)} \leq \delta \) by (2.16), from (2.16) we get
\[
-\Delta \Phi u = \frac{1}{\delta} \leq 1 \leq f(x, u) = -\Delta \Phi u \quad \text{in } \{ x \in \Omega : u(x) \leq u(x) \},
\]
where \( \delta, \delta \) come from Lemma 2.7. As above, the weak comparison principle ensures \( u \geq \underline{u} \).

**Lemma 2.10.** Suppose \( H(a)_1 \rightarrow H(a)_2 \) and \( H(f)_1 \rightarrow H(f)_2 \). Then the functional \( K : W_0^1,\Phi(\Omega) \to \mathbb{R} \) defined as
\[
K(u) := \int_{\Omega} \hat{F}(x, u) \, dx
\]
is well defined, weakly sequentially continuous, and of class \( C^1 \), with
\[
\langle K'(u), v \rangle = \int_{\Omega} \hat{f}(x, u)v \, dx \quad \forall u, v \in W_0^1,\Phi(\Omega).
\]

\[
(2.18)
\]
Moreover, \( K' : W_0^{1,\Phi}(\Omega) \to W^{-1,\overline{\Phi}}(\Omega) \) is a completely continuous operator.

**Proof.** Using (\ref{1.5}) and (\ref{1.7}) we estimate \( \hat{F} \) as

\[
|\hat{F}(x,s)| \leq \int_0^{\overline{\Phi}(s)} \frac{\hat{f}(x,t)}{t} \, dt = \int_0^{\overline{\Phi}(s)} \frac{f(x,u(x))}{t} \, dt + \int_0^{\overline{\Phi}(s)} f(x,t) \, dt
\]

\[
\leq u(x) \left[ c_1 \overline{\Phi}^{-1}(\overline{\Phi}(x)) + c_2 u(x)^{-\gamma} \right] + \int_0^{\overline{\Phi}(s)} \left[ c_1 \overline{\Phi}^{-1}(\overline{\Phi}(t)) + c_2 t^{-\gamma} \right] \, dt
\]

\[
\leq c_1 u(x) \overline{\Phi}^{-1}(\overline{\Phi}(x)) + c_2 u(x)^{1-\gamma} + c_1 |s| \overline{\Phi}^{-1}(\overline{\Phi}(|s|)) + \frac{c_2}{1-\gamma} |s|^{1-\gamma}
\]

for all \((x, s) \in \Omega \times \mathbb{R}\). Exploiting (\ref{2.14}) and \( i_{\gamma} > 1 \) we obtain

\[
|\hat{F}(x,s)| \leq C_1 + C_2 \overline{\Phi}(|s|),
\]

with positive constants

\[
C_1 := 2c_1 \overline{\Phi}(k_2 d_1) + c_2 (k_2 d_1)^{1-\gamma} + \frac{c_2}{1-\gamma},
\]

\[
C_2 := 2c_1 + \frac{c_2}{1-\gamma} \frac{1}{\overline{\Phi}(1)}.
\]

Taking into account also that \( W_0^{1,\Phi}(\Omega) \hookrightarrow L^{\overline{\Phi}}(\Omega) \) because of \( \overline{\Phi} \ll \Phi_* \), we deduce that \( K \) is well defined.

Now we compute the Gâteaux derivative of \( K \). We fix \( v \in W_0^{1,\Phi}(\Omega) \) and apply Torricelli’s theorem to deduce

\[
\lim_{t \to 0^+} \frac{K(u + tv) - K(u)}{t} = \lim_{t \to 0^+} \int_{\Omega} \frac{\hat{F}(x,u + tv) - \hat{F}(x,u)}{t} \, dx
\]

\[
= \lim_{t \to 0^+} \int_{\Omega} v \left( \int_0^1 \hat{f}(x,u + tv) \, ds \right) \, dx.
\]

According to (\ref{2.17}), (\ref{1.7}), the embedding \( W_0^{1,\Phi}(\Omega) \hookrightarrow L^{\overline{\Phi}}(\Omega) \), and the Hardy inequality \cite[Corollary 13]{11} besides supposing \( t \in (0,1) \), we infer

\[
|v \hat{f}(x,u + tv)| \leq c_1 |v| \overline{\Phi}^{-1}(\overline{\Phi}(|u| + |v|)) + \alpha d^{-\gamma} |v| + \beta |v|
\]

\[
\leq c_1 (|u| + |v|) \overline{\Phi}^{-1}(\overline{\Phi}(|u| + |v|)) + \alpha d^{-\gamma} |v| + \beta |v|
\]

\[
\leq 2c_1 \overline{\Phi}(|u| + |v|) + \alpha d_1^{1-\gamma} d^{-1} |v| + \beta |v| \in L^1(\Omega).
\]

Hence, we can pass the limit under the integral sign in (\ref{2.21}) and get (\ref{2.18}). The remaining part of the proof follows exactly as in \cite[Lemma 2.3]{8}, using the compactness of the embedding \( W_0^{1,\Phi}(\Omega) \hookrightarrow L^{\overline{\Phi}}(\Omega) \), as well as (\ref{2.17}) and \cite[Lemma 7.3]{28}. \( \square \)

\textsuperscript{3}We use Hardy’s inequality in the form

\[
\|d^{-1} v\|_{L^\Phi(\Omega)} \leq c \|\nabla v\|_{L^\Phi(\Omega)} \quad \forall v \in W_0^{1,\Phi}(\Omega),
\]

being \( c > 0 \) opportune. This inequality is valid since \( \Phi \in \nabla_2 \).
Remark 2.11. Fix any \( u \in W^{1,\Phi}_0(\Omega) \). By (2.19) we deduce that
\[
\int_{\Omega \cap \{|u| \leq \rho\}} |\hat{F}(x, u)| \, dx \leq (C_1 + C_2 \Upsilon(\rho))|\Omega| =: \Pi(\rho) \quad \forall \rho \geq 0.
\] (2.22)

3. Regularity of solutions

In this section we prove \( C^{1,\alpha} \) regularity up to the boundary for solutions to \( (P_{1,f}) \). Given a measurable function \( u : \Omega \to \mathbb{R} \), for any \( k \in \mathbb{R} \) we set \( \Omega_k := \{ x \in \Omega : u(x) \geq k \} \).

Lemma 3.1. Let \( p, r > 1 \). Suppose that \( u \in L^p(\Omega) \) satisfies
\[
\left( \int_{\Omega_k} (u - k)^p \, dx \right)^{\frac{1}{p}} \leq c \left( \int_{\Omega_k} (u - k)^p \, dx + k^p|\Omega_k| \right) \quad \text{for all } k \geq K,
\] for suitable \( c, K > 0 \). Then there exists \( M > 0 \) such that \( u \leq M \) in \( \Omega \).

Proof. Let us fix \( M > 2K \) to be chosen later, and set \( k_n := M \left( 1 - \frac{1}{2^{n+1}} \right) \quad \forall n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\} \).

By (3.1) and (3.2) we have, for all \( n \in \mathbb{N}_0 \),
\[
\left( \int_{\Omega_{k_{n+1}}} (u - k_{n+1})^p \, dx \right)^{\frac{1}{p}} \leq c \left( \int_{\Omega_{k_{n}}} (u - k_n)^p \, dx + k_{n+1}^p|\Omega_{k_{n+1}}| \right).
\] (3.3)

Chebichev’s inequality entails
\[
(k_{n+1} - k_n)^p|\Omega_{k_{n+1}}| \leq \int_{\Omega_{k_n}} (u - k_n)^p \, dx.
\]

Thus, recalling (3.2) and \( k > 1 \),
\[
k_{n+1}^p|\Omega_{k_{n+1}}| \leq \frac{k_{n+1}^p}{(k_{n+1} - k_n)^p} \int_{\Omega_{k_n}} (u - k_n)^p \, dx \leq 2^{(n+2)p} \int_{\Omega_{k_n}} (u - k_n)^p \, dx.
\] (3.4)

Inserting (3.4) into (3.3) gives
\[
\int_{\Omega_{k_{n+1}}} (u - k_{n+1})^p \, dx \leq \left[ c(2^{(n+2)p} + 1) \int_{\Omega_{k_n}} (u - k_n)^p \, dx \right]^{\frac{1}{p}} \leq Cb^n \left( \int_{\Omega_{k_n}} (u - k_n)^p \, dx \right)^{\frac{1}{1+a}}
\] (3.5)

for all \( n \in \mathbb{N}_0 \), where \( a := r - 1 > 0 \), \( b := 2^{pr} > 1 \), and \( C = C(c, p, r) > 0 \) is a suitable constant independent of \( k \) and \( n \). Now we apply the fast geometric convergence lemma [22 Lemma 2.4.7] to the sequence \( y_n := \int_{\Omega_{k_n}} (u - k_n)^p \, dx \), which ensures \( y_n \to 0 \) provided
\[
y_0 \leq C^{1-a} b^{-\frac{1}{a}}.
\] (3.6)
We can choose \( M \), independent of \( n \), such that (3.6) holds true: indeed, by (3.2) and the dominated convergence theorem,
\[
y_0 = \int_{\Omega_0} (u - k_0)^p \, dx = \int_{\Omega} \left( u - \frac{M}{2} \right)^p \, dx \xrightarrow{M \to +\infty} 0. \tag{3.7}
\]

Keeping \( M \) fixed as in (3.6)—(3.7), from \( y_n \to 0 \) we obtain
\[
\int_{\Omega} (u - M)^p_+ \, dx \leq \int_{\Omega} (u - k_n)^p_+ \, dx = \int_{\Omega_{k_n}} (u - k_n)^p \, dx \xrightarrow{n \to \infty} 0,
\]
which implies \( \int_{\Omega} (u - M)^p_+ \, dx = 0 \), whence \( u \leq M \) in \( \Omega \).

**Lemma 3.2.** Let \( H(a)_1 \)–\( H(a)_2 \) and \( H(f)_2 \) be satisfied. Then any \( u \in W^{1,\Phi}_0(\Omega) \) weak solution to \( (P_{1,f}) \) is essentially bounded in \( \Omega \).

**Proof.** Pick any \( k > 1 \). Testing \( (P_{1,f}) \) with \( (u - k)_+ \) and using (1.5) yield
\[
\int_{\Omega_k} \Phi(|\nabla u|) \, dx \leq i_{\Phi}^{-1} \int_{\Omega_k} \varphi(|\nabla u|)|\nabla u| \, dx = i_{\Phi}^{-1} \int_{\Omega_k} f(x, u)(u - k) \, dx \\
\leq i_{\Phi}^{-1} \left[ c_1 \int_{\Omega_k} \overline{\Phi}^{-1}(\Phi(u)) \, dx + c_2 \int_{\Omega_k} u^{1-\gamma} \, dx \right]. \tag{3.8}
\]

First we estimate each term on the right-hand side of (3.8). By convexity of \( \overline{\Phi} \) and (1.7) we have, for any \( k \) sufficiently large,
\[
u^{1-\gamma} \leq \Phi(u) \leq (\Phi(u))^{1-\gamma} \quad \text{in} \quad \Omega_k.
\tag{3.9}
\]

Moreover, by (1.7) and (2.4), besides \( s_\Phi \leq 1, k > 1 \), it turns out that
\[
\int_{\Omega_k} \overline{\Phi}^{-1}(\Phi(u)) \, dx \leq 2 \int_{\Omega_k} \Phi(u) \, dx \leq 2 \Phi(1) \int_{\Omega_k} u^{s_\Phi} \, dx \leq 2 \Phi(1) \int_{\Omega_k} u^{i_{\Phi^*}} \, dx \\
\leq 2^{i_{\Phi^*}\Phi(1)} \left[ \int_{\Omega_k} (u - k)^{i_{\Phi^*}} \, dx + k^{i_{\Phi^*}} |\Omega_k| \right]. \tag{3.10}
\]

On the other hand, to estimate the left-hand side of (3.8), we observe that the Sobolev embedding theorem and \( t^{i_{\Phi^*}} < \Phi_\ast \) in the sense of (2.8) (see (2.4)) yield
\[
W^{1,\Phi}_0(\Omega) \hookrightarrow L^{\Phi_\ast}(\Omega) \hookrightarrow L^{i_{\Phi^*}}(\Omega),
\]
where the latter space is a Lebesgue space. We deduce the embedding inequality
\[
c \|w\|_{L^{i_{\Phi^*}}(\Omega)} \leq \|\nabla w\|_{L^{\Phi_\ast}(\Omega)} \quad \forall w \in W^{1,\Phi}_0(\Omega)
\]
for a suitable \( c > 0 \). Thus, choosing \( w = (u - k)_+ \), from (2.5) we get
\[
\int_{\Omega_k} \Phi(|\nabla u|) \, dx \geq \xi_{\Phi}(\|\nabla u\|_{L^{\Phi_\ast}(\Omega)}) \geq \xi_{\Phi}(\|\nabla (u - k)_+\|_{L^{\Phi_\ast}(\Omega)}) \geq \xi_{\Phi}(c \|(u - k)_+\|_{L^{i_{\Phi^*}}(\Omega)}).
\]
Reasoning as in (3.7), for any \( k \) big enough we get \( c \|(u - k)_+\|_{L^{i_{\Phi^*}}(\Omega)} \leq 1 \), so that
\[
\int_{\Omega_k} \Phi(|\nabla u|) \, dx \geq c^{i_{\Phi^*}} \left( \int_{\Omega_k} (u - k)^{i_{\Phi^*}} \, dx \right)^{i_{\Phi^*}}. \tag{3.11}
\]
Inserting (3.9)–(3.11) into (3.8) we deduce
\[
\left( \int_{\Omega_k} (u - k)^{i\phi_*} \, dx \right)^{\frac{1}{i\phi_*}} \leq C \left[ \int_{\Omega_k} (u - k)^{i\phi_*} \, dx + k^{i\phi_*} |\Omega_k| \right]
\]
for a sufficiently large \( C > 0 \). Hence applying Lemma 3.1 with \( p = i\phi_* > 1 \) and \( r = \frac{i\phi_*}{s\Phi} > 1 \) (cf. H(a)2) yields the conclusion.

**Remark 3.3.** The \( L^\infty \) estimate provided in Lemma 3.2 is valid also when \( s = i\Phi_* \) which, in the classical Sobolev setting, represents the critical case; hence \( M \) must depend on the solution \( u \). On the other hand, in the sub-critical case \( s < i\Phi_* \) this estimate can be improved, and it turns out that \( M \) depends only on \( \|u\|_{W^{1,\Phi}(\Omega)} \) instead of \( u \) itself.

**Theorem 3.4.** Let \( H(a)1 - H(a)2 \) and \( H(f)1 - H(f)2 \) be satisfied. Then any \( u \in W^{1,\Phi}_0(\Omega) \) weak solution of \((P_{1,f})\) belongs to \( C^1,\tau_0(\Omega) \) for some \( \tau \in (0,1] \).

**Proof.** Lemma 3.2 guarantees that \( u \in L^\infty(\Omega) \). In addition, Lemma 2.9 ensures that \( u \) solves also \((P_{1,f})\). Using \( u \in L^\infty(\Omega) \) and (2.17) we get
\[
0 \leq \hat{f}(x, u(x)) \leq Cd(x)^{-\gamma} \quad \forall x \in \Omega,
\]
being \( C > 0 \) sufficiently large. Let us consider the linear problem
\[
\begin{cases}
-\Delta v = \hat{f}(x, u(x)) & \text{in } \Omega, \\
v = 0 & \text{on } \partial\Omega.
\end{cases}
\]
Problem (3.12) admits a unique solution \( v \in C^{1,\tau}_0(\Omega) \), for some \( \tau \in (0,1] \), by virtue of Minty-Browder’s theorem, Hardy’s inequality, and [19, Lemma 3.1]. It turns out that the problem
\[
\begin{cases}
-\text{div} (a(|\nabla w|)\nabla w - \nabla v(x)) = 0 & \text{in } \Omega, \\
w = 0 & \text{on } \partial\Omega
\end{cases}
\]
(3.13)
admits a unique solution \( w \in C^{1,\tau}_0(\Omega) \), by means of Minty-Browder’s theorem and Lieberman’s regularity theory, jointly with \( \nabla v \in C^{0,\tau}(\Omega) \). Since \( u \) is a solution to (3.13), by uniqueness we get \( u = w \), and thus \( u \in C^{1,\tau}_0(\Omega) \). \( \square \)

4. Existence and multiplicity results

In this last section we produce some results about \((P_{\lambda,f})\). Lemma 2.7 furnishes a sub-solution (depending on \( \lambda \)) to \((P_{\lambda,f})\). Thus, taking into account Lemma 2.9 the solutions to problem
\[
\begin{cases}
-\Delta u = \lambda \hat{f}(x, u) & \text{in } \Omega, \\
u > 0 & \text{in } \Omega, \\
u = 0 & \text{on } \partial\Omega,
\end{cases}
\]
are exactly the ones of \((P_{\lambda,f})\). The energy functional associated with \((P_{\lambda,f})\) is
\[
J_\lambda := H - \lambda K,
\]
being \( H, K \) as in Lemmas 2.6 and 2.10 respectively. So the solutions to \((P_{\lambda,f})\) are the critical points of \( J_\lambda \).
Existence of a solution is guaranteed by [8, Theorem 2.1], that we report below. This result traces back to [4, 2].

**Theorem 4.1.** Let $X$ be a reflexive Banach space, $H : X \to \mathbb{R}$ and $K : X \to \mathbb{R}$ be two continuously Gâteaux differentiable functionals such that $H$ is coercive and sequentially weakly lower semi-continuous, while $K$ is sequentially weakly upper semi-continuous with $\inf_X H = H(0) = K(0)$, and $r > 0$. Then, for every

$$\lambda \in \left[0, \frac{r}{\sup_{H^{-1}([0,r])} K}\right],$$

the functional $J_\lambda := H - \lambda K$ has a critical point $u_\lambda \in H^{-1}([0, r])$ satisfying $J_\lambda(u_\lambda) \leq J_\lambda(v)$ for all $v \in H^{-1}([0, r])$.

A second solution is furnished by the Mountain Pass theorem (vide, e.g., [24, Theorem 5.40]): the applicability of this result relies, in our context, on the Palais-Smale condition.

**Definition 4.2 ((PS)).** Let $X$ be a Banach space and $J \in C^1(X)$. We say that $J$ satisfies the Palais-Smale condition if any sequence $\{u_n\} \subseteq X$ such that $\{J(u_n)\}$ is bounded and $\|J'(u_n)\|_{X'} \to 0$ admits a convergent subsequence.

**Theorem 4.3.** Suppose $X$ to be a Banach space, and $J \in C^1(X)$ satisfying (PS). Let $u_0, u_1 \in X$, and $\rho > 0$ such that

$$\max\{J(u_0), J(u_1)\} \leq \inf_{\partial B(u_0, \rho)} J =: \eta_\rho, \quad \|u_1 - u_0\|_X > \rho. \quad (4.3)$$

Set

$$\Gamma := \{\gamma \in C^0([0, 1]; X) : \gamma(0) = u_0, \gamma(1) = u_1\}, \quad c := \inf_{\gamma \in \Gamma} \sup_{t \in [0, 1]} J(\gamma(t)).$$

Then $c \geq \eta_\rho$ and there exists $u \in X$ such that $J(u) = c$ and $J'(u) = 0$. Moreover, if $c = \eta_\rho$, then $u$ can be taken on $\partial B(u_0, \rho)$.

The next theorem concerns the existence of a solution to \([P_{\lambda,f}]\).

**Theorem 4.4.** Suppose $\mathbb{H}(a)_1 - \mathbb{H}(a)_2$ and $\mathbb{H}(f)_1 - \mathbb{H}(f)_2$. Then there exists $\lambda^* \in (0, +\infty]$ such that, for all $\lambda \in (0, \lambda^*)$, problem \([P_{\lambda,f}]\) admits a solution $u_\lambda \in C^1_{0, \tau}(\Omega)$, being $\tau \in (0, 1]$ opportune. Moreover, there exists $r_\lambda > 0$ (depending on $\lambda \in (0, \lambda^*)$) such that $\int_{\Omega} \Phi(|\nabla u_\lambda|) \, dx < r_\lambda$.

**Proof.** We want to apply Theorem 4.1 to $J_\lambda$ (see (4.1)). As observed above, this theorem furnishes a solution $u_\lambda \in W^{1,\Phi}_0(\Omega)$ to \([P_{\lambda,f}]\). Then the regularity of $u_\lambda$ is a consequence of Theorem 3.4.

In order to bound from above the ratio

$$\frac{\sup_{H^{-1}([0,r])} K}{r},$$

appearing in (4.2), we exploit (2.19) and estimate $K$ as follows:

$$|K(u)| \leq \int_{\Omega} |\hat{F}(x, u)| \, dx \leq C_1|\Omega| + C_2 \int_{\Omega} \Upsilon(|u|) \, dx. \quad (4.4)$$
Thus, we are led to study the function $\kappa : (0, +\infty) \to (0, +\infty)$ defined as

$$\kappa(r) := \frac{C_1|\Omega|}{r} + \frac{C_2}{r} \sup\left\{ \int_{\Omega} \Upsilon(|u|) \, dx : \int_{\Omega} \Phi(|\nabla u|) \, dx \leq r \right\}.$$

Notice that $\kappa \to +\infty$ as $r \to 0^+$. Now we distinguish four cases, depending whether $\Upsilon \ll \Phi$, $\Upsilon < \Phi$, $\Upsilon > \Phi$, or $\Upsilon \gg \Phi$. Clearly, some cases overlap.

**First case:** $\Upsilon \ll \Phi$.

Fix an arbitrary $\varepsilon \in (0, 1]$. There exists $M_\varepsilon > 0$ such that

$$\Upsilon(t) \leq \Phi(\varepsilon t) \leq \varepsilon \Phi(t) \quad \forall t > M_\varepsilon.$$ 

Hence, by Poincaré’s inequality [10, p.8] and (2.4), we get

$$\int_{\Omega} \Upsilon(|u|) \, dx = \int_{\Omega \cap \{|u| \leq M_\varepsilon\}} \Upsilon(|u|) \, dx + \int_{\Omega \cap \{|u| > M_\varepsilon\}} \Upsilon(|u|) \, dx$$

$$\leq \Upsilon(M_\varepsilon)|\Omega| + \varepsilon \int_{\Omega} \Phi(|u|) \, dx$$

$$\leq \Upsilon(M_\varepsilon)|\Omega| + \varepsilon \zeta_\Phi(2d_\Omega) \int_{\Omega} \Phi(|\nabla u|) \, dx$$

$$\leq \Upsilon(M_\varepsilon)|\Omega| + \varepsilon \zeta_\Phi(2d_\Omega)r.$$ 

Thus $\kappa$ can be estimated as

$$\kappa(r) \leq \frac{(C_1 + C_2 \Upsilon(M_\varepsilon))|\Omega|}{r} + C_2 \zeta_\Phi(2d_\Omega)\varepsilon. \quad (4.5)$$

Notice that the right-hand side of (4.5) is decreasing in $r$. Moreover, $\kappa(r) \to 0$ as $r \to +\infty$: indeed, letting $r \to +\infty$ in (4.5) reveals that

$$\limsup_{r \to +\infty} \kappa(r) \leq C_2 \zeta_\Phi(2d_\Omega)\varepsilon \quad \forall \varepsilon \in (0, 1],$$

since $\varepsilon$ was arbitrary. We set $\lambda^* = +\infty$. Then, for any $\lambda > 0$, we choose $\varepsilon = \min\{1, (2C_2 \zeta_\Phi(2d_\Omega)\lambda)^{-1}\}$ and $r_\lambda^* > 2\lambda(C_1 + C_2 \Upsilon(M_\varepsilon))|\Omega|$. According to (4.5), these choices guarantee $\kappa(r_\lambda^*) < \lambda^{-1}$, which allows to apply Theorem 4.1 with $r = r_\lambda^*$.

**Second case:** $\Upsilon < \Phi$.

There exist $M, c > 0$ such that

$$\Upsilon(t) \leq \Phi(ct) \quad \forall t > M.$$ 

Reasoning as in the first case we have

$$\int_{\Omega} \Upsilon(|u|) \, dx \leq \Upsilon(M)|\Omega| + \int_{\Omega} \Phi(c|u|) \, dx \leq \Upsilon(M)|\Omega| + \zeta_\Phi(2cd_\Omega) \int_{\Omega} \Phi(|\nabla u|) \, dx$$

$$\leq \Upsilon(M)|\Omega| + \zeta_\Phi(2cd_\Omega)r.$$ 

In this case $\kappa$ can be estimated as

$$\kappa(r) \leq \frac{(C_1 + C_2 \Upsilon(M))|\Omega|}{r} + C_2 \zeta_\Phi(2cd_\Omega). \quad (4.6)$$

We observe that the right-hand side of (4.6) is decreasing in $r$ and

$$\limsup_{r \to +\infty} \kappa(r) \leq C_2 \zeta_\Phi(2cd_\Omega).$$
We set \( \lambda^* = (C_2 \varphi_T(2cd_\Omega))^{-1} \) and, for any \( \lambda \in (0, \lambda^*) \), we take \( r^*_\lambda > \frac{(C_1 + C_2 \varphi_T(M)) |\Omega|}{\lambda^{-1} - C_2 \varphi_T(2cd_\Omega)} \). Then one applies Theorem 4.1.

**Third case:** \( \Upsilon > \Phi \).

Loosing information but not generality, we can reduce to the next case, namely, \( \Upsilon \gg \Phi \).

Indeed, in place of \( \Upsilon \) in (2.19), we can consider the intermediate function \( \hat{\Upsilon} := \sqrt{\Phi} \).

Firstly, we notice that \( \Upsilon \ll \Phi \) implies

\[
\Upsilon(t) \leq \Phi(t) \quad \forall t > M,
\]

being \( M > 0 \) opportune. So (2.19) can be re-written as

\[
|\hat{F}(x, s)| \leq C_1 + C_2 \Upsilon(M) + C_2 \hat{\Upsilon}(|s|) =: \hat{C}_1 + \hat{C}_2 \hat{\Upsilon}(|s|).
\]

Secondly, it is readily seen that \( \Upsilon \ll \hat{\Upsilon} \ll \Phi \), since, for any fixed \( \eta > 0 \), by (2.4) we have

\[
\frac{\hat{\Upsilon}(\eta t)}{\Upsilon(t)} = \sqrt{\frac{\Upsilon(\eta t)}{\Upsilon(t)}} \sqrt{\frac{\Phi_{*}(\eta t)}{\Upsilon(t)}} = \sqrt{\frac{\Phi_{*}(\eta t)}{\Upsilon(t)}} \xrightarrow{t \to +\infty} +\infty
\]

and

\[
\frac{\Phi_{*}(\eta t)}{\hat{\Upsilon}(t)} = \sqrt{\frac{\Phi_{*}(\eta t)}{\Phi_{*}(t)}} \sqrt{\frac{\Phi_{*}(\eta t)}{\Upsilon(t)}} \geq \sqrt{\frac{\Phi_{*}(\eta t)}{\Phi_{*}(t)}} \xrightarrow{t \to +\infty} +\infty.
\]

In particular we get \( \Phi \ll \hat{\Upsilon} \).

Finally, we notice the \( i_{\hat{\Upsilon}}, s_{\hat{\Upsilon}} \) are well defined as in (2.3): indeed,

\[
t \frac{\hat{\Upsilon}'(t)}{\hat{\Upsilon}(t)} = \frac{t \Upsilon'(t)}{2 \Upsilon(t)} + \frac{t \Phi_{*}'(t)}{2 \Phi_{*}(t)} \quad \forall t > 0.
\]

We deduce \( \frac{i_{\hat{\Upsilon}} + s_{\hat{\Upsilon}}}{2} \leq i_{\Phi} \leq s_{\Phi} \leq \frac{s_{\Phi} + s_{\Phi}}{2} \).

**Fourth case:** \( \Upsilon \gg \Phi \).

Thanks to (2.5) and the embedding \( W_{0,1,\Phi}^{1} = L^1(\Omega) \) we obtain

\[
\int_\Omega \Upsilon(|u|) \, dx \leq \zeta_\Phi(k \| \nabla u \|_{L^1(\Omega)}) \leq \zeta_T(k \| \nabla u \|_{L^1(\Omega)}) \leq \zeta_T(k \| \nabla u \|_{L^1(\Omega)})
\]

\[
\leq \zeta_T(k \zeta_T(k \zeta_T^{-1} (\zeta_\Phi (\| \nabla u \|_{L^1(\Omega)}))) \leq \zeta_T(k \zeta_T \left( \frac{1}{\zeta_\Phi (\| \nabla u \|_{L^1(\Omega)})} \right) \leq \zeta_T(k (1 + r^{i_{\Phi}}),
\]

where \( k > 0 \) is the best constant of the embedding mentioned above. So \( \kappa \) is estimated as

\[
\kappa(r) \leq C_1 |\Omega| + C_2 \zeta_T(k) r^{\frac{i_{\Phi}}{r}} + C_2 \zeta_T(k) r^{\frac{i_{\Phi}}{r} - 1}. \tag{4.7}
\]

We observe that \( \Upsilon \gg \Phi \) implies \( s_{\Phi} > i_{\Phi} \); otherwise we have

\[
\frac{\Upsilon(s)}{\Phi(s)} \leq \frac{\Phi'(s)}{\Phi(s)} \quad \forall s \in (0, +\infty)
\]

\[4\]Our definition should not be confused with the one in [27, Definition 6.3.1].
whence, integrating in \([1, t], \ t > 1\), and passing to the exponential,
\[
\Upsilon(t) \leq \frac{\Upsilon(1)}{\Phi(1)} \Phi(t) \quad \forall t \in (1, +\infty),
\]
in contrast with \(\Upsilon \gg \Phi\). Hence the right-hand side of (4.7), which can be re-written as
\[
\hat{k}(r) := \frac{A}{r} + Br^\theta, \quad \text{with} \quad A := C_1|\Omega| + C_2\zeta_\Upsilon(k), \ B := C_2\zeta_\Upsilon(k), \ \theta := \frac{s_\Upsilon}{i_\Phi} - 1 > 0,
\]
diverges when \(r \to +\infty\). Computing the unique critical point of \(\hat{k}\) reveals that
\[
\min_{r > 0} \hat{k}(r) = \hat{k}\left(\left(\frac{A}{\theta B}\right)^{\frac{1}{\theta+1}}\right) = \left[A^\theta B(\theta + \theta^{-\theta})\right]^{\frac{1}{\theta+1}}.
\]
In this case we set \(\lambda^* := \left[A^\theta B(\theta + \theta^{-\theta})\right]^{\frac{1}{\theta+1}}, \ r^*_\lambda := \left(\frac{A}{\theta B}\right)^{\frac{1}{\theta+1}}\) and apply Theorem 4.1.

**Remark 4.5.** According to (2.5) and Theorem 4.4 we infer
\[
\zeta_\Phi(\|u_\lambda\|_{W^{1,\Phi}_0(\Omega)}) \leq \int_{\Omega} \Phi(|\nabla u_\lambda|) \, dx < r^*_\lambda.
\]
We define the ball
\[
B_\lambda := \{u \in W^{1,\Phi}_0(\Omega) : \|u\|_{W^{1,\Phi}_0(\Omega)} < \frac{1}{\zeta_\Phi^{-1}(r^*_\lambda)}\}.
\]
Taking into account Theorem 4.4 again, we deduce that \(u_\lambda\) is a minimizer for the restriction of \(J_\lambda\) to \(B_\lambda\); in particular, \(u_\lambda\) is a local minimizer for \(J_\lambda\). Incidentally, we stress the fact that this local minimizer has been provided without using any \(W^{1,\Phi}\) versus \(C^1\) local minimizer argument.

**Lemma 4.6.** Under \(H(a)_1-H(a)_2\) and \(H(f)_1-H(f)_3\), the functional \(J_\lambda\) in (4.1) satisfies the Palais-Smale condition and is unbounded from below.

**Proof.** Let \(\{u_n\} \subseteq W^{1,\Phi}_0(\Omega)\) be such that \(\{J_\lambda(u_n)\}\) is bounded and \(\|J'_\lambda(u_n)\|_{W^{-1,\Phi}_*(\Omega)} \to 0\) as \(n \to \infty\). Hence, for a suitable \(c > 0\), up to subsequences we have
\[
\int_{\Omega} \Phi(|\nabla u_n|) \, dx - \lambda \int_{\Omega} F(x, u_n) \, dx \leq c \quad (4.8)
\]
and
\[
\left|\int_{\Omega} a(|\nabla u_n|)\nabla u_n \cdot \nabla v \, dx - \lambda \int_{\Omega} \hat{F}(x, u_n)v \, dx \right| \leq \|\nabla v\|_{L^\Phi(\Omega)} \quad (4.9)
\]
for all \(n \in \mathbb{N}\) and \(v \in W^{1,\Phi}_0(\Omega)\). We prove that \(\{u_n\}\) is bounded in \(W^{1,\Phi}_0(\Omega)\) by showing the boundedness of \(\{u^-_n\}\) and \(\{u^+_n\}\).

Choosing \(v = -u^-_n\) in (4.9) and using (2.13) yield
\[
i_\Phi \zeta_\Phi(\|\nabla u^-_n\|_{L^\Phi(\Omega)}) \leq \int_{\Omega} a(|\nabla u^-_n|)|\nabla u^-_n|^2 \, dx
\]
\[
\leq \int_{\Omega} a(|\nabla u^-_n|)|\nabla u^-_n|^2 \, dx + \lambda \int_{\Omega} \hat{F}(x, u_n)u^-_n \, dx \leq \|\nabla u^-_n\|_{L^\Phi(\Omega)}^2,
\]
whence \(\{u^-_n\}\) is bounded in \(W^{1,\Phi}_0(\Omega)\).
Exploiting (2.22) and $H(f)_3$, besides Remark 2.8, we have

\[
\int_\Omega \hat{F}(x, u_n^+) \, dx = \int_{\Omega \cap \{ u_n^+ \leq R\}} \hat{F}(x, u_n^+) \, dx + \int_{\Omega \cap \{ u_n^+ > R\}} \left( \hat{F}(x, R) + F(x, u_n^+) \right) \, dx \\
\leq 2\Pi(R) + \int_{\Omega \cap \{ u_n^+ > R\}} F(x, u_n^+) \, dx \\
\leq 2\Pi(R) + \frac{1}{\mu} \int_{\Omega \cap \{ u_n^+ > R\}} f(x, u_n^+)u_n^+ \, dx \\
\leq 2\Pi(R) + \frac{1}{\mu} \int_{\Omega} \hat{f}(x, u_n^+)u_n^+ \, dx. 
\]  

(4.10)

By (4.8) and (4.10) we deduce

\[
\int_\Omega \Phi(|\nabla u_n^+|) \, dx \leq \int_\Omega \Phi(|\nabla u_n|) \, dx \leq c + \lambda \int_\Omega \hat{F}(x, u_n) \, dx \leq c + \lambda \int_\Omega \hat{F}(x, u_n^+) \, dx \\
\leq c + 2\lambda\Pi(R) + \frac{\lambda}{\mu} \int_{\Omega} \hat{f}(x, u_n^+)u_n^+ \, dx. 
\]  

(4.11)

On the other hand, choosing $v = u_n^+$ in (4.9) produces

\[
\lambda \int_{\Omega} \hat{f}(x, u_n^+)u_n^+ \, dx \leq ||\nabla u_n^+||_{L^\Phi(\Omega)} + \int_\Omega \varphi(|\nabla u_n^+|)|\nabla u_n^+| \, dx \\
\leq ||\nabla u_n^+||_{L^\Phi(\Omega)} + s_\Phi \int_{\Omega} \Phi(|\nabla u_n^+|) \, dx. 
\]  

(4.12)

Combining (4.11)–(4.12) and rearranging the terms we obtain

\[
\left(1 - \frac{s_\Phi}{\mu}\right) \int_\Omega \Phi(|\nabla u_n^+|) \, dx \leq c + 2\lambda\Pi(R) + \frac{1}{\mu} ||\nabla u_n^+||_{L^\Phi(\Omega)}. 
\]

According to $H(f)_3$ and (2.5), it turns out that $\{u_n^+\}$ is bounded in $W^{1,\Phi}_s(\Omega)$. The $(S_\Phi)$ property of $H'$ (see Lemma 2.6) and the compactness of $K'$ (see Lemma 2.10) ensure the Palais-Smale condition for $J_\lambda$; see [8, Lemma 3.1] for details.

Now we prove that $J_\lambda$ is unbounded from below. Firstly, fix any $\overline{R} > R$. Integrating (1.6) in $(\Omega, t), t > \overline{R}$, and passing to the exponential yield

\[
F(x, t) \geq \frac{F(x, \overline{R})}{\overline{R}^\mu} t^\mu =: c_\overline{R} t^\mu \quad \forall (x, t) \in \Omega \times (\overline{R}, +\infty). 
\]

(4.13)

Take any test function $u_0 \in C_c^\infty(\Omega)$ such that $u_0 \geq 0$ in $\Omega$ and $u_0 \not\equiv 0$. Then there exists a compact $K \subseteq \Omega$ such that

\[
\min_K u_0 > 0 \quad \text{and} \quad |K| > 0. 
\]

(4.14)

For any $M > 0$, set $K_M := \{ x \in \Omega : Mu_0 > \overline{R} \}$. Observe that $\{K_M\}_{M>0}$ is increasing and $K \subseteq K_M$ for large values of $M$. Using (4.13) and (2.14), besides recalling Remark
for $M$ large we get
\[
J_\lambda(Mu_0) \leq \int_\Omega \Phi(M|\nabla u_0|) \, dx - \lambda \int_{K_M} F(x, Mu_0) \, dx \\
\leq \int_\Omega \Phi(M|\nabla u_0|) \, dx - \lambda c_R M^\mu \int_{K_M} u_0^\mu \, dx \\
\leq M^{s_\Phi} \int_\Omega \Phi(|\nabla u_0|) \, dx - \lambda c_R M^\mu \int_K u_0^\mu \, dx.
\] (4.15)

By $H(f)_3$ we have $\mu > s_\Phi$, while (4.14) ensures that $\int_K u_0^\mu \, dx > 0$. Hence $J_\lambda(Mu_0) \to -\infty$ when $M \to +\infty$, as desired. \qed

**Remark 4.7.** Incidentally, we notice that (4.13) implies that $J_\lambda$ is $\Phi$-super-linear, since $\Phi < t^\mu$ in the sense of (2.8).

**Theorem 4.8.** Suppose $H(a)_1 - H(a)_2$ and $H(f)_1 - H(f)_3$. Then problem $\{P_{\lambda,f}\}$ admits two distinct solutions in $C^{1,\tau}_0(\Omega)$.

**Proof.** Let $\lambda^*, r^*$ be given by Theorem 4.4. Fix any $\lambda \in (0, \lambda^*)$. Existence of a solution $u_\lambda \in C^{1,\tau}_0(\Omega)$ to $\{P_{\lambda,f}\}$ is guaranteed by Theorem 4.4. We want to get a second solution $v_\lambda \in C^{1,\tau}_0(\Omega)$ by applying Theorem 4.3 to the functional $J_\lambda$ defined in (4.1). As in the proof of Theorem 4.4, regularity of $v_\lambda$ is a consequence of Theorem 3.4.

First we notice that $J_\lambda$ is bounded on bounded sets: indeed, by (2.5), (4.4), and the embedding inequality for $W^{1,\Phi}_0(\Omega) \hookrightarrow L^\tau(\Omega)$ we have, for an opportune $C_3 > 0$ independent of $u$,

\[
|J_\lambda(u)| \leq \zeta_\Phi(\|u\|_{W^{1,\Phi}_0(\Omega)}) + C_1|\Omega| + C_2\zeta_\tau(\|u\|_{L^\tau(\Omega)}) \\
\leq \zeta_\Phi(\|u\|_{W^{1,\Phi}_0(\Omega)}) + C_1|\Omega| + C_3\zeta_\tau(\|u\|_{W^{1,\Phi}_0(\Omega)}).
\]

Taking into account Remark 4.5 we have that $u_\lambda$ is a local minimizer for $J_\lambda$. Since Theorem 4.6 ensures that $J_\lambda$ is unbounded from below, then $u_\lambda$ is not a global minimizer. Reasoning as in the first part of the proof of [3, Theorem 2.1] guarantees (4.3) with $u_0 := u_\lambda$. Hence Theorem 4.3 furnishes $v_\lambda \in W^{1,\Phi}_0(\Omega)$ critical point to $J_\lambda$, and thus solution to both $\{P_{\lambda,f}\}$ and $\{P_{\lambda,f}\}$. Moreover $v_\lambda$ fulfills $J_\lambda(v_\lambda) \geq J_\lambda(u_\lambda)$. If $J_\lambda(v_\lambda) > J_\lambda(u_\lambda)$, then $v_\lambda \neq u_\lambda$; else, Theorem 4.3 ensures that $v_\lambda$ can be taken on $\partial B_\lambda$. In any case we have $v_\lambda \neq u_\lambda$. \qed

**Appendix A. Examples**

In this appendix we want to show the importance of working in Sobolev-Orlicz spaces instead of classical Sobolev ones. In this sight, we furnish a class of Young functions $\Phi$ and reaction terms $f$ whose corresponding problem $\{P_{\lambda,f}\}$ cannot be set in a Sobolev framework, but it fulfills $H(a)_1 - H(a)_2$ and $H(f)_1 - H(f)_3$; see Example A.4 below. Inspiring examples can be found, e.g., in [12], where existence of at least one positive solution is obtained by the Mountain Pass theorem provided $\lambda = 1$ and the reaction term is not affected by singular terms. Following [12 Example 1], at the end of this appendix we will furnish a more concrete example (vide Example A.5 below) satisfying our hypotheses.
First of all, we construct a class of ‘pathological’ Young functions $\Psi$ (with $1 < i_\Psi < s_\Psi < +\infty$), possessing distinct indices at infinity, that is,

$$\liminf_{t \to +\infty} \frac{t \Psi'(t)}{\Psi(t)} \neq \limsup_{t \to +\infty} \frac{t \Psi'(t)}{\Psi(t)}, \quad \liminf_{t \to +\infty} \frac{t \Psi''(t)}{\Psi'(t)} \neq \limsup_{t \to +\infty} \frac{t \Psi''(t)}{\Psi'(t)}.$$  

This could be hopefully useful also in other contexts to construct counterexamples in Orlicz spaces.

**Lemma A.1.** Let $1 < q < p < +\infty$. Set $\alpha := \frac{p+q}{2}$ and $\beta := \frac{p-q}{2}$. Then, for any $\varepsilon < \min\{4, \frac{q-1}{\beta}\}$, the function

$$\Psi(t) := t^\alpha e^{\eta(t)} \quad \forall t \geq 0,$$  

(A.1)

with

$$\eta(t) = \begin{cases} 
\frac{\beta \varepsilon}{2e^2} (e - t)^2 - \frac{\beta \varepsilon}{1 + \varepsilon^2} & \text{for } 0 \leq t \leq e, \\
\beta \frac{\log t}{1 + \varepsilon^2} [\sin(\varepsilon \log(\log t)) - \varepsilon \cos(\varepsilon \log(\log t))] & \text{for } t \geq e,
\end{cases}$$  

(A.2)

is a Young function satisfying the following properties:

$$\inf_{t > 0} \frac{t \Psi'(t)}{\Psi(t)} = \liminf_{t \to +\infty} \frac{t \Psi'(t)}{\Psi(t)} = q < p = \limsup_{t \to +\infty} \frac{t \Psi'(t)}{\Psi(t)} = \sup_{t > 0} \frac{t \Psi'(t)}{\Psi(t)},$$  

(A.3)

$$q - 1 - \beta \varepsilon < \inf_{t > 0} \frac{t \Psi''(t)}{\Psi'(t)} \leq \liminf_{t \to +\infty} \frac{t \Psi''(t)}{\Psi'(t)} = q - 1$$

(A.4)

$$< p - 1 = \limsup_{t \to +\infty} \frac{t \Psi''(t)}{\Psi'(t)} \leq \sup_{t > 0} \frac{t \Psi''(t)}{\Psi'(t)} < p - 1 + \beta \varepsilon,$$

$$\liminf_{t \to +\infty} \frac{\Psi(t)}{t^r} = 0 \quad \text{or} \quad \limsup_{t \to +\infty} \frac{\Psi(t)}{t^r} = +\infty \quad \forall r > 1.$$  

(A.5)

If $p < N$ then $\Psi$ satisfies (1.3) with $\Psi$ in place of $\Phi$. If, in addition, $p < q^*$, then $s_\Psi < i_\Psi$.

**Proof.** Starting from (A.1), let us compute $\Psi'$, $\Psi''$ in terms of the lower order derivatives:

$$\Psi'(t) = \Psi(t) \left( \frac{\alpha}{t} + \eta'(t) \right) = \frac{\Psi(t)}{t} (\alpha + t \eta'(t)),$$  

(A.6)

$$\Psi''(t) = \Psi'(t) \left( \frac{\Psi'(t)}{\Psi(t)} + \frac{\eta'(t) - \frac{\alpha}{t}}{\alpha + \eta'(t)} \right) = \frac{\Psi'(t)}{t} \left( \alpha + t \eta'(t) + \frac{t^2 \eta''(t) - \alpha}{\alpha + t \eta'(t)} \right),$$  

(A.7)

Firstly we study $\Psi$ in the interval $[0, e]$. We have

$$\eta'(t) = \frac{\beta \varepsilon}{e^2} (t - e) \quad \text{and} \quad \eta''(t) = \frac{\beta \varepsilon}{e^2} \quad \text{for all } t \in (0, e].$$  

(A.8)

We observe that (A.6), (A.2), and $\varepsilon < 4$ entail

$$q < \alpha - \frac{\beta \varepsilon}{4} = \alpha + \min_{s \in (0, e]} s \eta'(s) \leq \frac{t \Psi'(t)}{\Psi(t)} \leq \alpha + \max_{s \in (0, e]} s \eta'(s) = \alpha < p$$  

(A.9)
for all $t \in (0, e]$. Exploiting (A.7) and (A.9), $\varepsilon < 4$, the monotonicity of $r \mapsto r + \frac{r}{\alpha + r}$, and $\eta' < 0 < \eta''$ in $(0, e]$, we obtain, for all $t \in (0, e]$,

$$
\alpha - 1 - \beta \varepsilon < \alpha - 1 - \frac{\beta \varepsilon}{4} \left(1 + \frac{1}{\alpha - \frac{\beta \varepsilon}{4}}\right)
$$

$$
= \alpha - 1 + \min_{s \in (0, e]} \left(s\eta'(s) + \frac{s\eta'(s)}{\alpha + s\eta'(s)}\right)
$$

$$
\leq t\Psi''(t) \leq \alpha - 1 + t^2 \eta''(t)
$$

$$
\leq \alpha - 1 + \frac{\beta \varepsilon}{\alpha - \frac{\beta \varepsilon}{4}} < \alpha - 1 + \beta \varepsilon.
$$

Now we analyze $\Psi$ in $[e, +\infty)$. We posit $\zeta(t) := \varepsilon(\log(\log t))$ for all $t \geq e$. Integrating by parts twice reveals that

$$
\int \sin(\varepsilon \log s) \, ds = s \sin(\varepsilon \log s) - \varepsilon \int \cos(\varepsilon \log s) \, ds
$$

$$
= s[\sin(\varepsilon \log s) - \varepsilon \cos(\varepsilon \log s)] - \varepsilon^2 \int \sin(\varepsilon \log s) \, ds,
$$

whence, performing the change of variable $s = \log t$ and recalling (A.2),

$$
\beta \int \frac{\sin(\zeta(t))}{t} \, dt = \beta \int \sin(\varepsilon \log s) \, ds = \frac{\beta s}{1 + \varepsilon^2} [\sin(\varepsilon \log s) - \varepsilon \cos(\varepsilon \log s)]
$$

$$
= \frac{\log t}{1 + \varepsilon^2} [\sin(\varepsilon \log(\log t)) - \varepsilon \cos(\varepsilon \log(\log t))] = \eta(t)
$$

for all $t \in [e, +\infty)$. Accordingly, we have

$$
\eta'(t) = \frac{\beta \sin(\zeta(t))}{t} \quad \text{and} \quad \eta''(t) = \frac{\beta}{t^2} [t\zeta'(t) \cos(\zeta(t)) - \sin(\zeta(t))]
$$

for all $t \geq e$. (A.12)

Hence we rewrite (A.6)–(A.7) as

$$
\Psi'(t) = \frac{\Psi(t)}{t} (\alpha + \beta \sin(\zeta(t))),
$$

(A.13)

$$
\Psi''(t) = \frac{\Psi'(t)}{t} \left(\alpha - 1 + \beta \sin(\zeta(t)) + \frac{\beta t\zeta'(t) \cos(\zeta(t))}{\alpha + \beta \sin(\zeta(t))}\right),
$$

(A.14)

valid for all $t \in [e, +\infty)$.

We observe that $\zeta(t) \to +\infty$ as $t \to +\infty$, so (A.9) and (A.13) guarantee (A.3). Moreover, $t\zeta'(t) \to 0$ as $t \to +\infty$ and

$$
0 \leq \frac{\beta |\cos(\zeta(t))|}{\alpha + \beta \sin(\zeta(t))} \leq \frac{\beta}{\alpha - \beta} \quad \forall t \in [e, +\infty).
$$

(A.15)

Thus, (A.14) provides the equalities in (A.4). More precisely, we notice that

$$
0 < t\zeta'(t) = \frac{\varepsilon}{\log t} \leq \varepsilon \quad \forall t \in [e, +\infty).
$$

(A.16)
Exploiting (A.14)–(A.16) we get, for all $t \geq e$,
\[
q - 1 - \beta \varepsilon < \alpha - 1 - \beta - \frac{\beta \varepsilon}{\alpha - \beta} \leq \frac{t \Psi''(t)}{\Psi'(t)} \leq \alpha - 1 + \beta + \frac{\beta \varepsilon}{\alpha - \beta} < p - 1 + \beta \varepsilon.
\] (A.17)

Because of (A.10) and (A.17), the inequalities in (A.4) hold true.

A direct computation, based on (A.8) and (A.12), shows that $\eta \in C^2(0, +\infty)$; thus, $\Psi$ enjoys the same property. Moreover, (A.3)–(A.4) and $\beta \varepsilon < q - 1$ yield $\Psi''(t), \Psi''(t) > 0$ for all $t > 0$. Thus $\Psi$ is strictly increasing and convex. Using again (A.3), together with (2.4), we deduce
\[
(\Psi(1) \min\{t^p, t^q\} = \Psi(1) \zeta_{\Psi}(t) \leq \Psi(t) \leq \Psi(1) \zeta_{\Psi}(t) = \Psi(1) \max\{t^p, t^q\} \forall t > 0,
\]
which entails (2.1). Hence $\Psi$ is a Young function.

In order to prove (A.5), let us consider two sequences $h_n, k_n \to +\infty$ such that
\[
\sin(\zeta(h_n)) = 1 \quad \text{and} \quad \cos(\zeta(k_n)) = 1 \quad \text{for all} \ n \in \mathbb{N}.
\]

Then (A.1)–(A.2) give, for any $n$ large enough,
\[
\Psi(h_n) = h_n^{\alpha + \frac{\beta}{1 + \varepsilon^2}} \quad \text{and} \quad \Psi(k_n) = k_n^{\alpha - \frac{\beta}{1 + \varepsilon^2}},
\]
ensuring (A.5).

Now suppose that $p < N$. Then, setting $\Lambda := t^p$, by (2.4) we infer $\Psi < \Lambda$ in the sense of (2.8). In particular, $\Psi^{-1}(t) \geq c\Lambda^{-1}(t)$ for all $t > 1$, being $c > 0$ small enough. Thus we get
\[
\int_1^{+\infty} \Theta_\Phi(t) \, dt \geq c \int_1^{+\infty} \Theta_\Lambda(t) \, dt = c \int_1^{+\infty} \frac{1}{t^{\sigma - 1}} \, dt = +\infty.
\]
The last statement is a direct consequence of (A.3) and (2.7).

**Remark A.2.** Two motivations suggest to work in Sobolev-Orlicz spaces instead of in the classical Sobolev framework.

The first motivation is structural: if we set the problem in a reflexive Sobolev-Orlicz space $W^{1,\Psi}_0(\Omega)$ (which may be a Sobolev space), the weak formulation of problem $[P_{\lambda, f}]$ requires
\[
\int_{\Omega} a(|\nabla u|) \nabla u \nabla v \, dx < +\infty \quad \forall u, v \in W^{1,\Psi}_0(\Omega).
\]
This is a *duality* property, which fails whenever $W^{1,\Psi}_0(\Omega) \setminus W^{1,\Phi}_0(\Omega) \neq \emptyset$: indeed, taking $u \in W^{1,\Psi}_0(\Omega) \setminus W^{1,\Phi}_0(\Omega)$ and $v = u$, by (2.3) we get
\[
\int_{\Omega} a(|\nabla u|) \nabla u \nabla v \, dx = \int_{\Omega} \varphi(|\nabla u|) \, dx \geq i_{\Phi} \int_{\Omega} \Phi(|\nabla u|) \, dx = +\infty.
\]
Hence, in order to properly define the concept of ‘weak solution’, we have to require $W^{1,\Psi}_0(\Omega) \subseteq W^{1,\Phi}_0(\Omega)$, which means $\Phi < \Psi$ (in the sense of (2.8)).

Here comes the second motivation, which is technical: if we suppose $W^{1,\Psi}_0(\Omega) \subseteq W^{1,\Phi}_0(\Omega)$, then we lose the *coercivity* of $-\Delta_\Psi$. To show this, we pick $u \in W^{1,\Phi}_0(\Omega) \setminus W^{1,\Psi}_0(\Omega)$ and a sequence $\{u_n\} \subseteq W^{1,\Psi}_0(\Omega)$ such that $u_n \rightharpoonup u$ in $W^{1,\Phi}_0(\Omega)$. It turns out that $\|u_n\|_{W^{1,\Psi}_0(\Omega)} \to +\infty$; otherwise, by reflexivity of $W^{1,\Phi}_0(\Omega)$ and up to subsequences, we would have $u_n \to u^*$ in $W^{1,\Phi}_0(\Omega)$ for some $u^* \in W^{1,\Psi}_0(\Omega)$ and, by uniqueness of the
weak limit, we would conclude \( u = u^* \in W^{1,\Psi}_0(\Omega) \), in contrast with the choice of \( u \). On
the other hand, by \( (2.3) \),
\[
\sup_{n \in \mathbb{N}} \int_{\Omega} a(|\nabla u_n|)|\nabla u_n|^2 \, dx = \sup_{n \in \mathbb{N}} \int_{\Omega} \varphi(|\nabla u_n|)|\nabla u_n| \, dx \\
\leq s_\Phi \sup_{n \in \mathbb{N}} \int_{\Omega} \Phi(|\nabla u_n|) \, dx \leq s_\Phi \sup_{n \in \mathbb{N}} \zeta_\Phi(\|u_n\|_{W^{1,\Phi}_0(\Omega)}) < +\infty,
\]
which proves that \(-\Delta \Phi\) is not coercive on \( W^{1,\Psi}_0(\Omega) \). Since coercivity of the principal part is an essential ingredient for existence results and, in particular, for our approach, which relies on Theorem 4.1, we adopted the framework \( W^{1,\Phi}_0(\Omega) = W^{1,\Psi}_0(\Omega) \).

It remains to prove that \( W^{1,\Phi}_0(\Omega) \) is not a Sobolev space in general. To this end, we observe that any Young function given by Lemma A.1, say \( \Phi \), furnishes a counterexample. Indeed, suppose by contradiction that \( W^{1,\Phi}_0(\Omega) = W^{1,r}_0(\Omega) \) for some \( r > 1 \). Then we have \( \Phi < t^r \) and \( t^r < \Phi \) (in the sense of \( (2.8) \)), whence
\[
\Phi < t^r \Rightarrow \limsup_{t \to +\infty} \frac{\Phi(t)}{t^r} \leq c'_1 < +\infty,
\]
\[
t^r < \Phi \Rightarrow \liminf_{t \to +\infty} \frac{\Phi(t)}{t^r} \geq c''_2 > 0,
\]
for a suitable \( c_1, c_2 > 0 \) given by \( (2.8) \). Since \( (A.18) \) contradicts \( (A.5) \), we deduce that \( W^{1,\Phi}_0(\Omega) \) is not a Sobolev space.

**Remark A.3.** Another important aspect related to the choice of the Sobolev-Orlicz framework is represented by the reaction term: we address the reader to [12, Section 6] for a discussion about this setting and the Ambrosetti-Rabinowitz condition. Here we limit ourselves to provide an example of nonlinearity \( f = f(u) \) fulfilling \( H(f)_1-H(f)_3 \).

Suppose \( s_\Phi < i_\Phi \). By virtue of Lemma A.1 we can construct a Young function \( \Upsilon \) satisfying \( s_\Phi < i_\Upsilon < s_\Psi < i_\Phi \). Then, fixed \( \gamma \in (0,1) \), we consider
\[
f(t) = \frac{\Upsilon(t)}{t} + t^{-\gamma}.
\]
Obviously, \( f \) fulfills \( H(f)_1 \). Observe that \( (1.7) \) implies
\[
\frac{\Upsilon(t)}{t} \leq \Upsilon^{-1}(\Upsilon(t)) \quad \forall t > 0,
\]
so that \( H(f)_2 \) is satisfied with \( c_1 = c_2 = 1 \). To prove \( H(f)_3 \), choose any \( \mu \in (s_\Phi, i_\Upsilon) \). For all \( t > 0 \) we have
\[
tf(t) = \Upsilon(t) + t^{1-\gamma}
\]
and, given any \( R > 0 \),
\[
F(t) = \int_R^t \left( \frac{\Upsilon(s)}{s} + s^{-\gamma} \right) \, ds \leq \int_R^t (i_\Upsilon^{-1} \Upsilon'(s) + s^{-\gamma}) \, ds \leq \frac{1}{i_\Upsilon} \Upsilon(t) + \frac{t^{1-\gamma}}{1-\gamma}.
\]
Convexity of \( \Upsilon \) and \( \mu < i_\Upsilon \) guarantee that there exists \( R > 0 \) such that
\[
\frac{\mu}{1-\gamma} t^{1-\gamma} \leq \left( 1 - \frac{\mu}{i_\Upsilon} \right) \Upsilon(t) \quad \forall t \geq R.
\]
From (A.20)–(A.22) we get
\[
\mu F(t) \leq \frac{\mu}{i_{\Upsilon}} \Upsilon(t) + \frac{\mu}{1-\gamma} t^{1-\gamma} \leq \Upsilon(t) \leq tf(t) \quad \forall t \geq R,
\]
which entails \( H(f) \).

As announced, we conclude with two examples of problems fulfilling the hypotheses of Theorem 4.8 according to Remark A.2 we stress that it is necessary to set them in the appropriate Sobolev-Orlicz setting. Existence of two solutions for these problems is a consequence of Theorem 4.8.

**Example A.4.** Take any \( r > s > p > q > 1 \) such that \( p < N \) and \( r < q^* \). Let \( \Phi \) and \( \Upsilon \) be given by Lemma A.1 (applied with any sufficiently small \( \varepsilon > 0 \)), such that \( i_{\Phi} = q, \ s_{\Phi} = p, \ i_{\Upsilon} = s, \ s_{\Upsilon} = r \). Let \( f \) be defined as in (A.19). Then problem (P\( \lambda,f \)) admits at least two distinct weak solutions \( u, v \in C^{1,\gamma}_{0,\tau}(\Omega) \) for all \( \lambda \in (0,\lambda^*) \). Here \( \tau \in (0,1] \) and \( \lambda^* > 0 \) are given by Theorems 3.4 and 4.4 respectively.

The hypotheses of Theorem 4.8 are fulfilled: (A.4) implies \( H(a)_1 \) and the final part of Lemma A.1 gives \( H(a)_2 \), while Remark A.3 ensures \( H(f)_1 \)–\( H(f)_3 \).

**Example A.5.** The same result stated in Example A.4 holds true for the problem
\[
\begin{cases}
-\text{div} (\log(1 + |\nabla u|) |\nabla u|^{p-2} \nabla u) = \lambda(u^r + u^{-\gamma}) & \text{in } \Omega, \\
u > 0 & \text{in } \Omega, \\
u = 0 & \text{on } \partial\Omega,
\end{cases}
\]
where \( 1 < p < N - 1, \ N < p + p^2, \ r \in (p, p^* - 1), \) and \( \gamma \in (0,1) \).

Problem (A.23) comes from (P\( \lambda,f \)) by choosing, for all \( (x,t) \in \Omega \times (0, +\infty) \),
\[
a(t) := t^{p-2} \log(1+t), \quad \Phi(t) := \int_0^t s^{p-1} \log(1+s) \, ds, \quad f(x,t) := t^r + t^{-\gamma}.
\]

In order to verify the assumptions of Theorem 4.8 we explicitly observe that

1. \( H_a(t) := \frac{ta(t)}{a(t)} = \frac{t}{a(t)} \left( t^{p-2} \log(1+t) \right) \) is a decreasing function in \( (0, +\infty) \) with \( \lim_{t \to 0^+} H_a(t) = p - 1 \) and \( \lim_{t \to +\infty} H_a(t) = p - 2. \) Then we have
\[
i_a := \inf_{t > 0} H_a(t) = p - 2 < p - 1 = \sup_{t > 0} H_a(t) =: s_{a}.
\]

2. According to De L’Hôpital’s rule, \( H_\phi(t) := \frac{t \Phi(t)}{\Phi(t)} \) fulfills
\[
\lim_{t \to 0^+} H_\phi(t) = \lim_{t \to 0^+} \frac{ta(t) + t(ta(t))'}{ta(t)} = 2 + \lim_{t \to 0^+} H_a(t) = p + 1
\]
and
\[
\lim_{t \to +\infty} H_\phi(t) = 2 + \lim_{t \to +\infty} H_a(t) = p.
\]

3. One has
\[
i_a + 2 \leq i_\phi \leq s_\phi \leq s_a + 2.
\]
Indeed, for all \( s > 0 \), we have \( i_a \leq \frac{s a(s)}{a(s)} \leq s_a. \) Multiplying by \( sa(s) \), an integration by parts in \( 0, t \) gives \( i_a \Phi(t) \leq t^2 a(t) - 2\Phi(t) \leq s_a \Phi(t) \) and our claim follows.
From (A.24)–(A.27), it is readily seen that
\[ p = i_a + 2 \leq i_\Phi \leq p \quad \text{and} \quad p + 1 \leq s_\Phi \leq s_a + 2 = p + 1, \]
that is,
\[ i_\Phi = p < p + 1 = s_\Phi. \] (A.28)
Therefore, from (A.24), it is clear that \( H(a)_1 \) holds if and only if \( p > 1 \). Bearing in mind (2.7), since we have that \( s_\Phi = p + 1 < p^* = i_\Phi^* \leq i^*_\Phi \), also \( H(a)_2 \) is verified. On the other hand, \( H(f)_1 - H(f)_3 \) follow from Remark A.3 by taking \( \Upsilon(t) = t^r + 1 \), being \( i_\Upsilon = s_\Upsilon = r + 1 \) with \( p < r < p^* - 1 \).

**Remark A.6.** Regarding Example A.5, if we drop the condition \( N < p^2 + p^2 \) and replace \( r \in (p, p^* - 1) \) with jointly \( p < r \) and \( t^r \ll \Phi^* \), we can ensure only that problem (A.23) admits at least two distinct weak solutions in \( W^{1,\Phi}_0(\Omega) \). In particular, two solutions are obtained in the case \( p < r \leq p^* - 1 \), since \( t^p \ll \Phi \) forces \( t^{r+1} < t^{p^*} \ll \Phi^* \) (with an argument similar to the one in the last part of the proof of Lemma A.1). A similar conclusion holds true for Example A.4, replacing \( r < q^* \) with \( t^r \ll \Phi^* \).

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