Uncertainty, entropy and decoherence of the damped harmonic oscillator in the Lindblad theory of open quantum systems

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Abstract

In the framework of the Lindblad theory for open quantum systems, expressions for the density operator, von Neumann entropy and effective temperature of the damped harmonic oscillator are obtained. The entropy for a state characterized by a Wigner distribution function which is Gaussian in form is found to depend only on the variance of the distribution function. We give a series of inequalities, relating uncertainty to von Neumann entropy and linear entropy. We analyze the conditions for purity of states and show that for a special choice of the diffusion coefficients, the correlated coherent states (squeezed coherent states) are the only states which remain pure all the time during the evolution of the considered system. These states are also the most stable under evolution in the presence of the environment and play an important role in the description of environment induced decoherence.

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1 Introduction

In the last years a large interest has arisen to study open systems, especially in quantum optics [1], quantum measurement theory [2] and in connection with decoherence and quantum to classical transition [3, 4, 5]. The consistent description of open quantum systems was investigated by various authors [8, 9, 10, 11, 12] (for a recent review see Ref. [13]). It is commonly understood [11, 14] that dissipation in an open system results from microscopic reversible interactions between the observable system and the environment. Because dissipative processes imply irreversibility and, therefore, a preferred direction in time, it is generally thought that quantum dynamical semigroups are the basic tools to introduce dissipation in quantum mechanics. The most general form of the generators of such semigroups was given by Lindblad [15], under the assumption that the evolution of the system is Markovian. This assumption is not generally valid, but it is considered a good approximation for various interesting physical models (for example, the quantum Brownian motion in a high temperature reservoir). In the Lindblad theory the density operator properties (Hermiticity, unit trace and positivity) are preserved by the master equation.

Different master equations for the evolution of the density operator have been used to describe decoherence and quantum to classical transition [3, 4, 6, 10, 17, 18, 19, 20, 21]. In particular models, the density operator becomes diagonal in some basis, indicating that interference between the states in that basis is suppressed – this is the process of decoherence of density operators. In this case the dynamical variables corresponding to the diagonalizing basis are expected to evolve approximately classically. A different theoretical approach to the problem of quantum to classical transition is the decoherent histories approach [3, 22, 23, 24].

The Lindblad formalism has been studied for the case of the damped harmonic oscillator [13, 25, 26, 27] and applied to various physical phenomena, for instance, the
damping of collective modes in deep inelastic collisions in nuclear physics [28]. In [29], a family of master equations, constructed in the form of Lindblad generators, was proposed for local Ohmic quantum dissipation. In [30] the Lindblad master equation for the harmonic oscillator was transformed into Fokker-Planck equations for quasiprobability distributions and a comparative study was made for the Glauber $P$, antinormal ordering $Q$ and Wigner $W$ representations. In [31, 32] the density matrix for the coherent state representation and the Wigner distribution function subject to different types of initial conditions were obtained for the damped harmonic oscillator. A phase space representation for open quantum systems within the Lindblad theory was given in [33].

In the present paper we are also concerned with the observable system of a harmonic oscillator which interacts with an environment. In Sec. 2 we give the description of the Lindblad model for open quantum systems and write the master equation for the density operator of the damped harmonic oscillator. Generally the master equation gains considerably in clarity if it is represented in terms of the Wigner distribution function which satisfies the Fokker-Planck equation. In Sec. 3 we transform the master equation into the Fokker-Planck equation by means of the well-known methods [11, 34, 35, 36, 37, 38] and analyze the evolution of the Wigner function of the density operator. The Wigner function is shown to have a Gaussian form. Then we derive a closed form of the density operator satisfying the master equation based on the Lindblad dynamics and describe the evolution of the considered system towards a final equilibrium state. By using the explicit form of the density operator, we calculate in Sec. 4 the von Neumann entropy and time dependent temperature of the quantum system in a state characterized by a Wigner distribution function which is Gaussian in form and analyze their temporal evolution towards the equilibrium values. Then we introduce the Wehrl entropy and present its basic properties, including the Wehrl-Lieb inequality and another inequality which gives an interesting relationship between
the Wehrl entropy and von Neumann entropy. Following Halliwell and collaborators [39, 40], we give two other simple inequalities, relating uncertainty to von Neumann entropy, and von Neumann entropy to linear entropy. The dynamical behaviour of the Wehrl entropy is compared to the von Neumann entropy and Shannon information entropy (in some cases the two last entropies are equal to each other). The concept of the classical-like Wehrl entropy is a very informative measure describing the time evolution of a quantum system. The Wehrl entropy, first introduced as a classical entropy of a quantum state, can give additional insights into the dynamics of the system, as compared to other entropies. We also discuss about the introduction of entropy production for studying the stability of stationary states. In Sec. 5 we analyze under what conditions the open system can be described by a quantum pure state and show that for a special choice of the diffusion coefficients, the correlated coherent states (squeezed coherent states), taken as initial states, remain pure for all time during the evolution of the system. In some simple models of the damped harmonic oscillator in the framework of quantum statistical theory [11, 12], it was shown that the pure Glauber coherent states remain as those during the evolution and in all other cases, the oscillator immediately evolves into mixtures. In this respect we generalize this result, the results of other authors [13], obtained by using different methods as well as our previous result of Ref. [27]. In Sec. 6 we use the linear entropy for the description of the environment induced quantum decoherence phenomenon. We state that the correlated coherent states are the most stable under evolution in the presence of the environment and make the connection with the work done in this field by other authors [17, 20, 39, 40, 44]. Finally, a summary and concluding remarks are given in Sec. 7.
2 Lindblad model for the damped quantum harmonic oscillator

The simplest dynamics for an open system which describes an irreversible process is a semigroup of transformations introducing a preferred direction in time [9, 10, 15]. In Lindblad’s axiomatic formalism of introducing dissipation in quantum mechanics, the usual von Neumann-Liouville equation ruling the time evolution of closed quantum systems is replaced by the following Markovian master equation for the density operator \( \hat{\rho}(t) \) in the Schrödinger picture [15]:

\[
\frac{d\Phi_t(\hat{\rho})}{dt} = L(\Phi_t(\hat{\rho})).
\] (1)

Here, \( \Phi_t \) denotes the dynamical semigroup describing the irreversible time evolution of the open system in the Schrödinger representation and \( L \) the infinitesimal generator of the dynamical semigroup \( \Phi_t \). The condition for the Lindblad theory is that the time scale considered for the open system (subsystem) should be very long compared to the relaxation time of the heat bath (external system), but shorter than the recurrence time of the total system assumed as a closed finite system [13, 40]. Using the structural theorem of Lindblad [15], which gives the most general form of the bounded, completely dissipative Liouville operator \( L \), we obtain the explicit form of the most general time-homogeneous quantum mechanical Markovian master equation:

\[
\frac{d\hat{\rho}(t)}{dt} = -\frac{i}{\hbar}[\hat{H}, \hat{\rho}(t)] + \frac{1}{2\hbar} \sum_j ([\hat{V}_j, \hat{V}_j^\dagger], [\hat{\rho}(t), \hat{V}_j^\dagger]) + [\hat{V}_j, \hat{\rho}(t)\hat{V}_j^\dagger]).
\] (2)

Here \( \hat{H} \) is the Hamiltonian of the system in the absence of environment. The operators \( \hat{V}_j, \hat{V}_j^\dagger \) are bounded operators on the Hilbert space \( \mathcal{H} \) of the Hamiltonian and they model the effect of the environment.

We mention that the Markovian master equations found in the literature are of this form after some rearrangement of terms, even for unbounded Liouville operators.
In this connection we assume that the general form of the master equation given by (2) is also valid for unbounded Liouville operators.

We consider an open system consisting of a particle moving in a quadratic (harmonic oscillator) potential, coupled to an environment described by non-Hermitian Lindblad operators in (2) which are a linear combination of position and momentum operators. We impose a simple condition on the operators \( \hat{V}_j, \hat{V}^\dagger_j \) that they are functions of the basic observables \( \hat{q}, \hat{p} \) of the one-dimensional quantum mechanical system (with \( [\hat{q}, \hat{p}] = i\hbar I \), where \( I \) is the identity operator on \( \mathcal{H} \)) of such kind that the obtained model is exactly solvable. A precise version of this last condition is that linear spaces spanned by first degree (respectively second degree) noncommutative polynomials in \( \hat{q} \) and \( \hat{p} \) are invariant to the action of the completely dissipative mapping \( L \). This condition implies \(^{25}\) that \( \hat{V}_j \) are at most first degree polynomials in \( \hat{q} \) and \( \hat{p} \) and \( \hat{H} \) is at most a second degree polynomial in \( \hat{q} \) and \( \hat{p} \). Because in the linear space of the first degree polynomials in \( \hat{q} \) and \( \hat{p} \) the operators \( \hat{q} \) and \( \hat{p} \) give a basis, there exist only two \( \mathbb{C} \)-linear independent operators \( \hat{V}_1, \hat{V}_2 \) which can be written in the form

\[
\hat{V}_j = a_j \hat{p} + b_j \hat{q}, \quad j = 1, 2, \tag{3}
\]

with \( a_j, b_j \) complex numbers \(^{22}\). The constant term is omitted because its contribution to the generator \( L \) is equivalent to terms in \( \hat{H} \) linear in \( \hat{q} \) and \( \hat{p} \) which for simplicity are assumed to be zero. Then the harmonic oscillator Hamiltonian \( \hat{H} \) is chosen of the general form

\[
\hat{H} = \hat{H}_0 + \mu \left( \frac{\hat{q} \hat{p} + \hat{p} \hat{q}}{2\hbar} \right), \quad \hat{H}_0 = \frac{1}{2m}\hat{p}^2 + \frac{m\omega^2}{2}\hat{q}^2. \tag{4}
\]

With these choices the Markovian master equation can be written \(^{13,26}\):

\[
\frac{d\hat{\rho}}{dt} = -\frac{i}{\hbar}[\hat{H}_0, \hat{\rho}] - \frac{i}{2\hbar}(\lambda + \mu)[\hat{q}, \hat{\rho} \hat{p} + \hat{p} \hat{\rho}] + \frac{i}{2\hbar}(\lambda - \mu)[\hat{p}, \hat{\rho} \hat{q} + \hat{q} \hat{\rho}]
- \frac{D_{qp}}{\hbar^2}[\hat{q}, [\hat{q}, \hat{\rho}]] - \frac{D_{pq}}{\hbar^2}[\hat{p}, [\hat{p}, \hat{\rho}]] + \frac{D_{pq}}{\hbar^2}(\hat{[\hat{q}, \hat{p}, \hat{\rho}]]} + \hat{[\hat{p}, \hat{q}, \hat{\rho}]}). \tag{5}
\]
Here we have used the notations:

\[ D_{qq} = \frac{\hbar}{2} \sum_{j=1,2} |a_j|^2, \quad D_{pp} = \frac{\hbar}{2} \sum_{j=1,2} |b_j|^2, \]

\[ D_{pq} = D_{qp} = -\frac{\hbar}{2} \text{Re} \sum_{j=1,2} a_j^* b_j, \quad \lambda = -\text{Im} \sum_{j=1,2} a_j^* b_j, \]  

(6)

where \( D_{qq}, D_{pp} \) and \( D_{pq} \) are the diffusion coefficients and \( \lambda \) is the friction constant. They satisfy the following fundamental constraints [13, 26]:

i) \( D_{pp} > 0 \),  
ii) \( D_{qq} > 0 \),  
iii) \( D_{pp} D_{qq} - D_{pq}^2 \geq \frac{\lambda^2 \hbar^2}{4} \).  

(7)

The semigroup method is valid for the weak coupling regime, with the damping \( \lambda \) typically obeying the inequality \( \lambda \ll \omega_0 \), where \( \omega_0 \) is the lowest frequency typical of reversible motions.

In the particular case when the asymptotic state is a Gibbs state

\[ \hat{\rho}_G(\infty) = e^{-\frac{\hat{H}_0}{kT}} / \text{Tr} e^{-\frac{\hat{H}_0}{kT}}, \]

(8)

these coefficients reduce to

\[ D_{pp} = \frac{\lambda + \mu}{2} \hbar m \omega \coth \frac{\hbar \omega}{2kT}, \quad D_{qq} = \frac{\lambda - \mu}{2} \frac{\hbar}{m \omega} \coth \frac{\hbar \omega}{2kT}, \quad D_{pq} = 0, \]

(9)

where \( T \) is the temperature of the thermal bath and the fundamental constraints are satisfied only if \( \lambda > |\mu| \).

The necessary and sufficient condition for \( L \) to be translationally invariant is \( \lambda = \mu \) [13, 23, 26]. Translation invariance means that \( [p, L(\rho)] = L([p, \rho]) \). In the following general values for \( \lambda \) and \( \mu \) will be considered.

In the literature, master equations of the type (5) are encountered in concrete theoretical models for the description of different physical phenomena in quantum optics [1, 47, 48, 49], in treatments of the damping of collective modes in deep inelastic collisions of heavy ions [50] or in the quantum description of the dissipation for the one-dimensional harmonic oscillator [11, 12, 36, 37]. A classification of these equations,
whether they satisfy or not the fundamental constraints (\ref{eq:constraint}), was given in \cite{27}. Density operators satisfying the Lindblad equation give statistical predictions in full agreement with experiment in a wide variety of situations. Lindblad type equations are also frequently used in studies of quantum decoherence \cite{4,5,17,21}. For example, in the much-studied quantum Brownian model \cite{51,52}, the master equation is the Lindblad master equation with a single Lindblad operator

\[
\hat{V} = (2D)^{-1/2}(\hat{\phi} + 2\frac{i}{\hbar}\gamma D\hat{\rho}),
\]

with \( D = \hbar^2/8m\gamma kT \) (where \( \gamma \) is the dissipation and \( T \) is the temperature of the environment) and \( \hat{H} = \hat{H}_s + (\gamma/2)\{\hat{q},\hat{p}\} \), where \( \hat{H}_s \) is the distinguished subsystem Hamiltonian in the absence of the environment. The Lindblad master equation does not, in fact, completely agree with the master equations given in a number of papers on quantum Brownian motion. In particular, the master equation given by Caldeira and Leggett \cite{51} does not contain the term \([\hat{p}, [\hat{p}, \hat{\rho}]]\) and is known to violate the positivity of the density operator on short time scales \cite{53}. This difference is principal, but practically Lindblad and Caldeira-Leggett equations are similar for high temperature, when the Markovian approximation is only valid and the extra term is negligible, since its coefficient is proportional to \( T^{-1} \) \cite{27}.

In the following we denote by \( \sigma_{AA} \) the dispersion of the operator \( \hat{A} \), i.e. \( \sigma_{AA} = \langle \hat{A}^2 \rangle - \langle \hat{A} \rangle^2 \), where \( \langle \hat{A} \rangle \equiv \sigma_A = \text{Tr}(\hat{\rho}\hat{A}) \) and \( \text{Tr}\hat{\rho} = 1 \). By \( \sigma_{AB} = 1/2 \langle \hat{A}\hat{B} + \hat{B}\hat{A} \rangle - \langle \hat{A} \rangle \langle \hat{B} \rangle \) we denote the correlation of operators \( \hat{A} \) and \( \hat{B} \).

From the master equation (\ref{eq:master_equation}) we obtain the following equations of motion for the expectation values and variances of the coordinate and momentum \cite{13,20}:

\[
\frac{d\sigma_q(t)}{dt} = -(\lambda - \mu)\sigma_q(t) + \frac{1}{m}\sigma_p(t),
\]

\[
\frac{d\sigma_p(t)}{dt} = -m\omega^2\sigma_q(t) - (\lambda + \mu)\sigma_p(t)
\]
and

\[ \frac{d\sigma_{qq}(t)}{dt} = -2(\lambda - \mu)\sigma_{qq}(t) + \frac{2}{m}\sigma_{pq}(t) + 2D_{qq}, \]  
\[ (13) \]

\[ \frac{d\sigma_{pp}(t)}{dt} = -2(\lambda + \mu)\sigma_{pp}(t) - 2m\omega^2\sigma_{pq}(t) + 2D_{pp}, \]  
\[ (14) \]

\[ \frac{d\sigma_{pq}(t)}{dt} = -m\omega^2\sigma_{qq}(t) + \frac{1}{m}\sigma_{pp}(t) - 2\lambda\sigma_{pq}(t) + 2D_{pq}, \]  
\[ (15) \]

In the underdamped case \((\omega > \mu)\) considered in this paper, with the notation \(\Omega^2 \equiv \omega^2 - \mu^2\), we obtain \([13, 26]\):

\[ \sigma_q(t) = e^{-\lambda t}((\cos \Omega t + \frac{\mu}{\Omega} \sin \Omega t)\sigma_q(0) + \frac{1}{m\Omega} \sin \Omega t\sigma_p(0)), \]  
\[ (16) \]

\[ \sigma_p(t) = e^{-\lambda t}(-\frac{m\omega^2}{\Omega} \sin \Omega t\sigma_q(0) + (\cos \Omega t - \frac{\mu}{\Omega} \sin \Omega t)\sigma_p(0)) \]  
\[ (17) \]

and \(\sigma_q(\infty) = \sigma_p(\infty) = 0\). It is convenient to consider the vectors

\[ X(t) = \begin{pmatrix} m\omega\sigma_{qq}(t) \\ \sigma_{pp}(t)/m\omega \\ \sigma_{pq}(t) \end{pmatrix} \]  
\[ (18) \]

and

\[ D = \begin{pmatrix} 2m\omega D_{qq} \\ 2D_{pp}/m\omega \\ 2D_{pq} \end{pmatrix}. \]  
\[ (19) \]

With these notations the solutions for the variances can be written in the form \([13, 26]\):

\[ X(t) = (Te^{-KtT})(X(0) - X(\infty)) + X(\infty), \]  
\[ (20) \]

where the matrices \(T\) and \(K\) are given by

\[ T = \frac{1}{2\Omega} \begin{pmatrix} \mu + i\Omega & \mu - i\Omega & 2\omega \\ \mu - i\Omega & \mu + i\Omega & 2\omega \\ -\omega & -\omega & -2\mu \end{pmatrix}, \]  
\[ (21) \]

\[ K = \begin{pmatrix} 2(\lambda - i\Omega) & 0 & 0 \\ 0 & 2(\lambda + i\Omega) & 0 \\ 0 & 0 & 2\lambda \end{pmatrix}. \]  
\[ (22) \]
and

\[ X(\infty) = (TK^{-1}T)D. \]  \hspace{1cm} (23)

The formula (23) is remarkable because it gives a very simple connection between the asymptotic values \( (t \to \infty) \) of \( \sigma_{qq}(t), \sigma_{pp}(t), \sigma_{pq}(t) \) and the diffusion coefficients \( D_{qq}, D_{pp}, D_{pq} \):

\[ \sigma_{qq}(\infty) = \frac{1}{2m^2\lambda(\lambda^2 + \omega^2 - \mu^2)}(m^2(2\lambda(\lambda + \mu) + \omega^2)D_{qq} + D_{pp} + 2m(\lambda + \mu)D_{pq}), \hspace{1cm} (24) \]

\[ \sigma_{pp}(\infty) = \frac{1}{2\lambda(\lambda^2 + \omega^2 - \mu^2)}((m\omega)^2D_{qq} + (2\lambda(\lambda - \mu) + \omega^2)D_{pp} - 2m\omega^2(\lambda - \mu)D_{pq}), \hspace{1cm} (25) \]

\[ \sigma_{pq}(\infty) = \frac{1}{2m\lambda(\lambda^2 + \omega^2 - \mu^2)}(-(\lambda + \mu)(m\omega)^2D_{qq} + (\lambda - \mu)D_{pp} + 2m(\lambda^2 - \mu^2)D_{pq}). \hspace{1cm} (26) \]

These relations show that the asymptotic values \( \sigma_{qq}(\infty), \sigma_{pp}(\infty), \sigma_{pq}(\infty) \) do not depend on the initial values \( \sigma_{qq}(0), \sigma_{pp}(0), \sigma_{pq}(0) \). In the considered underdamped case we have

\[ Te^{-Kt}T = \frac{e^{-2\lambda t}}{2\Omega^2} \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}, \hspace{1cm} (27) \]

where \( b_{ij}, i, j = 1, 2, 3 \) are time-dependent oscillating functions given by (3.78) in [26].

### 3 Wigner distribution function and density operator

One useful way to study the consequences of the master equation (3) for the density operator of the one-dimensional damped harmonic oscillator is to transform it into more familiar forms, such as the equations for the \( c \)-number quasiprobability distributions Glauber \( P \), antinormal ordering \( Q \) and Wigner \( W \) associated with the density operator [30]. In this case the resulting differential equations of the Fokker-Planck type for the distribution functions can be solved by standard methods [34, 35, 36, 37, 38].
employed in quantum optics and observables directly calculated as correlations of these
distribution functions.

In [30, 31, 33] we have transformed the master equation (5) for the density operator
into the following Fokker-Planck equation satisfied by the Wigner distribution function
\( W(q, p, t) \):

\[
\frac{\partial W}{\partial t} = -\frac{p}{m} \frac{\partial W}{\partial q} + m\omega^2 q \frac{\partial W}{\partial p} + (\lambda - \mu) \frac{\partial}{\partial q} (qW) + (\lambda + \mu) \frac{\partial}{\partial p} (pW) \\
+ D_{qq} \frac{\partial^2 W}{\partial q^2} + D_{pp} \frac{\partial^2 W}{\partial p^2} + D_{pq} \frac{\partial^2 W}{\partial p \partial q}. \tag{28}
\]

Since the drift coefficients are linear in the variables \( q \) and \( p \) and the diffusion coefficients are constant with respect to \( q \) and \( p \), Eq. (28) describes an Ornstein-Uhlenbeck process [54]. Following the method developed by Wang and Uhlenbeck [54], we solved in [31] this Fokker-Planck equation, subject to either the wave packet type or the \( \delta \)-function type of initial conditions in the underdamped case \( (\omega > \mu) \) of the harmonic oscillator. One gets two-dimensional Gaussian distributions with different variances.

Wigner function allows us to compute the moments of \( q \) and \( p \) at any time \( t \) in terms of the initial moments. By computing the long time limits of these moments, the form of the long time limit of the Wigner function may be obtained, since it is completely determined by its moments.

For an initial Gaussian Wigner function the solution of Eq. (28) is

\[
W(q, p, t) = \frac{1}{\sqrt{2\pi} \sigma(t)} \exp \left\{ -\frac{1}{\sigma(t)} \left[ \sigma_{pp}(t)(q - \sigma_q(t))^2 + \sigma_{qq}(t)(p - \sigma_p(t))^2 - 2\sigma_{pq}(t)(q - \sigma_q(t))(p - \sigma_p(t)) \right] \right\}, \tag{29}
\]

where \( \sigma(t) \) is the determinant of the dispersion (correlation) matrix \( M(t) \),

\[
\sigma(t) = \text{det } M(t) = \sigma_{qq}(t)\sigma_{pp}(t) - \sigma_{pq}^2(t) \tag{30}
\]

and

\[
M(t) = \begin{pmatrix}
\sigma_{qq}(t) & \sigma_{pq}(t) \\
\sigma_{qp}(t) & \sigma_{pp}(t)
\end{pmatrix}. \tag{31}
\]
We see that the initial Wigner function remains Gaussian and therefore the property of positivity is preserved in time. When time \( t \to \infty \), \( \sigma_q(t) \) and \( \sigma_p(t) \) vanish, all dependence on \( \sigma_q(0) \) and \( \sigma_p(0) \) drops out of the exponentials in \( W \) and we obtain the steady state solution:

\[
W_\infty(q,p) = \frac{1}{2\pi \sqrt{\sigma(\infty)}} \exp\left\{ -\frac{1}{2} \left[ \sigma_{pp}(\infty)q^2 + \sigma_{qq}(\infty)p^2 - 2\sigma_{pq}(\infty)qp \right] \right\},
\]

(32)

where \( \sigma(\infty) = \sigma_{qq}(\infty)\sigma_{pp}(\infty) - \sigma_{pq}^2(\infty) \) and \( \sigma_{qq}(\infty), \sigma_{pp}(\infty), \sigma_{pq}(\infty) \) are given by Eqs. (24) – (26). All stationary solutions to the evolution equations obtained in the long time limit are possible as a result of a balance between the wave packet spreading induced by the Hamiltonian and the localizing effect of the Lindblad operators.

We obtain now the explicit form of the density operator of the damped harmonic oscillator in the Lindblad theory by using a technique analogous to those applied in the description of quantum relaxation [55, 56, 57, 58]. Namely, we apply, like in [55, 56], the relation \( \hat{\rho} = \frac{\hbar}{2\pi} \sqrt{\delta} \exp\left\{ \frac{1}{2}\ln 4\delta - \frac{1}{2} \left( 1 + \frac{2\hbar^2}{4\sigma(t) - \hbar^2} \right) \right\} \),

(33)

The normal ordering operator \( N \) can be applied upon the Wigner function \( W_s(q,p) \) in Gaussian form by using McCoy’s theorem [35, 59]. Following Jang [56], we obtained in [32] the following expression of the density operator:

\[
\hat{\rho}(t) = \frac{\hbar}{\sqrt{\delta}} \exp\left\{ \frac{1}{2} \ln \frac{4\delta}{4\sigma(t) - \hbar^2} - \frac{1}{2\hbar} \right\} \cosh^{-1}\left( 1 + \frac{2\hbar^2}{4\sigma(t) - \hbar^2} \right).
\]

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\[ \times \{ \sigma_{pp}(t)(\hat{q} - \sigma_q(t))^2 + \sigma_{qq}(t)(\hat{p} - \sigma_p(t))^2 - 2\sigma_{pq}(t)[(\hat{q} - \sigma_q(t))(\hat{p} - \sigma_p(t)) - i\frac{\hbar}{2}] \} \}, \quad (34) \]

where

\[ \delta = \sigma_{qq}(t)\sigma_{pp}(t) - (\sigma_{pq}(t) - \frac{i\hbar}{2})^2. \quad (35) \]

The density operator (34) has a Gaussian form, as expected from the initial form of the Wigner distribution function. While the Wigner distribution is expressed in terms of real variables \( q \) and \( p \), the density operator is a function of operators \( \hat{q} \) and \( \hat{p} \). When time \( t \to \infty \), the density operator tends to

\[ \hat{\rho}(\infty) = \frac{2\hbar}{\sqrt{4\sigma(\infty) - \hbar^2}} \times \exp\left\{ -\frac{1}{2\hbar}\ln \frac{2\sqrt{\sigma(\infty) + \hbar}}{2\sqrt{\sigma(\infty) - \hbar}} \{ \sigma_{pp}(\infty)\hat{q}^2 + \sigma_{qq}(\infty)\hat{p}^2 - \sigma_{pq}(\infty)(\hat{q}\hat{p} + \hat{p}\hat{q}) \} \} \]. \quad (36) \]

In the particular case (37),

\[ \sigma_{qq}(\infty) = \frac{\hbar}{2m\omega} \coth \frac{\hbar\omega}{2kT}, \quad \sigma_{pp}(\infty) = \frac{\hbar m\omega}{2} \coth \frac{\hbar\omega}{2kT}, \quad \sigma_{pq}(\infty) = 0 \]

and the asymptotic state is a Gibbs state (38):

\[ \hat{\rho}_G(\infty) = 2\sinh \frac{\hbar\omega}{2kT} \exp\left\{ -\frac{1}{kT}(\frac{1}{2m}\hat{p}^2 + \frac{m\omega^2}{2}\hat{q}^2) \right\}. \quad (38) \]

The Wigner function can be expressed as the Fourier transform of the off-diagonal matrix elements of the density operator in the coordinate representation [62]:

\[ W(q, p) = \frac{1}{\pi\hbar} \int dy < q - y | \hat{\rho} | q + y > e^{2ipy/\hbar}. \quad (39) \]

Then \( < x | \hat{\rho} | y > \) can be obtained from the inverse Fourier transform of the Wigner function:

\[ < x | \hat{\rho} | y > = \int dp \exp \left( \frac{i}{\hbar} p(x - y) \right) W(p, \frac{x + y}{2}), \quad (40) \]
namely

\[ < x | \hat{\rho}(t) | y > = \left( \frac{1}{2\pi \sigma_{qq}(t)} \right)^{\frac{1}{2}} \exp \left\{ - \frac{1}{2 \sigma_{qq}(t)} \left( \frac{x+y}{2} - \sigma_q(t) \right)^2 \right\} \]

\[- \frac{1}{2\hbar^2} \left( \sigma_{pp}(t) - \frac{\sigma_{pq}(t)^2}{\sigma_{qq}(t)} \right) (x-y)^2 + \frac{i \sigma_{pq}(t)}{\hbar \sigma_{qq}(t)} \left( \frac{x+y}{2} - \sigma_q(t) \right) (x-y) + \frac{i}{\hbar} \sigma_p(t)(x-y). \] (41)

In the long time limit \( \sigma_q(t) = 0, \sigma_p(t) = 0 \) and we have

\[ < x | \hat{\rho}(\infty) | y > = \left( \frac{1}{2\pi \sigma_{qq}(\infty)} \right)^{\frac{1}{2}} \]

\[ \times \exp \left\{ - \frac{1}{8 \sigma_{qq}(\infty)} (x+y)^2 - \frac{1}{2\hbar^2} \left( \sigma_{pp}(\infty) - \frac{\sigma_{pq}(\infty)^2}{\sigma_{qq}(\infty)} \right) (x-y)^2 + \frac{i \sigma_{pq}(\infty)}{2 \hbar \sigma_{qq}(\infty)} (x^2 - y^2) \right\}. \] (42)

## 4 Entropy and uncertainty

The most natural measure of uncertainty of the quantum mechanical state is the entropy. Physically, entropy can be interpreted as a measure of the lack of knowledge (disorder) of the system. The von Neumann entropy measures deviations from pure state behaviour. For an isolated system the entropy is time independent due to the unitarity of the evolution operator. For open systems such as the damped harmonic oscillator, the evolution is not unitary and the entropy becomes time dependent. Denoting by \( \hat{\rho}(t) \) the density operator of the damped harmonic oscillator in the Schrödinger picture, the von Neumann entropy \( S(t) \) is given by the expectation value of the logarithmic operator \( \ln \hat{\rho} \) (we put Boltzmann’s constant \( k = 1 \)):

\[ S(t) = - < \ln \hat{\rho}(t) > = - \text{Tr}(\hat{\rho}(t) \ln \hat{\rho}(t)). \] (43)

Accordingly, the calculation of the entropy reduces to the problem of finding the explicit form of the density operator. Using Eq. (34) we obtained in [32] the following expression of the von Neumann entropy:

\[ S(t) = (\nu + 1) \ln(\nu + 1) - \nu \ln \nu, \] (44)

where we denote \( \hbar \nu = \sqrt{\sigma(t)} - \hbar/2 \). It is worth noting that the entropy depends only upon the variance of the Wigner distribution. When time \( t \to \infty \), \( \nu \) tends
to \( s \equiv \sqrt{\sigma(\infty)/\hbar} - 1/2 \) and the entropy relaxes to its equilibrium value \( S(\infty) = (s + 1) \ln(s + 1) - s \ln s \). The expression \( (44) \) is analogous to those previously obtained in the theory of quantum oscillator relaxation and for the description of a system of collective RPA phonons. It should also be noted that the expression \( (44) \) has the same form as the entropy of a system of harmonic oscillators in thermal equilibrium. In the later case \( \nu \) represents the average of the number operator. Although the expression \( (44) \) for the entropy has a well-known form, the function \( \nu \) induces a specific behaviour of the entropy. From the expression of the variances, which appear in \( (30) \), we can see that the time dependence of the entropy is given by the damping factors \( \exp(-4\lambda t) \), \( \exp(-2\lambda t) \) and the oscillating function \( \exp(2i\Omega t) \). The complex oscillating factor \( \exp(2i\Omega t) \) reduces to a function of the frequency \( \omega \), namely \( \exp(2i\omega t) \), for \( \mu \to 0 \) or if \( \mu/\Omega \ll 1 \) (i.e. the frequency \( \omega \) is very large as compared to \( \mu \)).

The von Neumann entropy gives zero for all pure states \( \hat{\rho} = \hat{\rho}^2 \), so it measures the purity of quantum states, being different from zero (in fact positive) only for mixed states. This entropy does not differentiate between various pure states. We have pure states for \( \sigma = \hbar^2/4 \), which is just the case when we have equality in the generalized uncertainty relation \( \sigma \geq \hbar^2/4 \) (see Sec. 5). By calculating

\[
\frac{dS}{d\nu} = \ln(1 + \frac{1}{\nu}),
\]

we see that \( S \) is increasing when \( \nu \) increases. But in general the entropy is not a monotonic function of time, because we cannot decide a priori about the sign of \( dS/dt \). In fact, in the underdamped case considered in this paper, the entropy oscillates and only when \( t \to \infty \) it relaxes to its equilibrium value, which is independent of the initial state of the system. If we require that \( dS(t)/dt \geq 0 \), then the environment operators \( \hat{V}_j \) must be Hermitian. This assertion was proved in Ref. by a direct calculation and the use of the inequality \((x - y) \log(x/y) \geq 0\) for positive \( x, y \).
In the case of a thermal bath (8), a time dependent effective temperature $T_e$ can be defined [56, 57], by noticing that when $t \to \infty$, $\nu$ tends, according to (2.24) in Ref. [31], to the average thermal phonon number $\langle n \rangle = (\exp(\hbar\omega/kT) - 1)^{-1}$. Thus $\nu$ can be considered as giving the time evolution of the thermal phonon number, so that we can put in this case

\[
(\exp \frac{\hbar\omega}{kT_e} - 1)^{-1} = \nu. \tag{46}
\]

From (46) the effective temperature $T_e$ can be expressed as

\[
T_e(t) = \frac{\hbar\omega}{k[\ln(\nu + 1) - \ln \nu]}. \tag{47}
\]

Accordingly, we can say that at time $t$ the system is in thermal equilibrium at temperature $T_e$. In terms of the effective temperature, the von Neumann entropy takes the form [32]

\[
S(t) = \frac{\hbar\omega}{T_e(\exp \frac{\hbar\omega}{kT_e} - 1)} - k \ln[1 - \exp(-\frac{\hbar\omega}{kT_e})]. \tag{48}
\]

As $t$ increases, the effective temperature approaches thermal equilibrium with the bath, $T_e \to T$.

The von Neumann entropy of the density operator is often connected with uncertainty, decoherence and correlations of the distinguished system with its environment [2, 1, 17, 18, 20]. Zurek, Paz and Habib, for example, looked for classes of initial states which generate the least amount of entropy at time $t$. They regarded such states as the most stable under evolution in the presence of an environment. They argued that these states are coherent states, at least approximately. The coherent states are known to be as close as possible to classical states: quantum fluctuations are minimal in these states and equal to those of the vacuum. Anastopoulos and Halliwell have shown that it is really the correlated coherent states, rather than the ordinary coherent states which are the most stable [40]. One of the reasons for looking at the von Neumann
entropy is that it is constant for unitary evolution, thus for open systems such as that considered here, it is mainly a measure of environmentally induced effects (see Sec. 6).

Entropic uncertainty relations have been obtained by Bialynicki-Birula in Refs. [66, 67]. These relations express restrictions imposed by quantum theory on probability distributions of canonically conjugate variables in terms of corresponding information entropies [66]. Entropic uncertainty relations derived from phase space quasiprobability distributions and connection to the decay of quantum coherences have been discussed in Refs. [68]. Further discussion on entropic uncertainty relations associated with the phase-number uncertainties can be found in Ref. [69].

Following [39, 40], we shall give the connection between uncertainty and the von Neumann entropy, by considering the phase space quasiprobability $Q_H$ distribution

$$Q_H(p, q) = \langle \alpha | \hat{\rho} | \alpha \rangle,$$

where

$$\langle x | \alpha \rangle = \langle x | p, q \rangle = \frac{1}{(2\pi s_{qq})^{1/4}} \exp\left(-\frac{(x - q)^2}{4s_{qq}} + \frac{i}{\hbar} px\right)$$

are the standard coherent states, $s_{qq}$ and $s_{pp}$ are the variances of the coherent states and $s_{qq}s_{pp} = \hbar^2/4$. It is normalized according to

$$\int \frac{dp dq}{2\pi \hbar} Q_H(p, q) = 1.$$

Between the two quasiprobability distributions $Q_H$ and Wigner $W$ there is the following relation [40]:

$$Q_H(p, q) = 2 \int dp' dq' \exp\left(-\frac{(p - p')^2}{2s_{pp}} - \frac{(q - q')^2}{2s_{qq}}\right) W(q', p').$$

Eq. (52) is the Husimi distribution [70]. In contrast to the other two quasiprobability distributions Glauber $P$ and Wigner $W$, the $Q_H$ representation is always a positive and well-behaved function [52, 71]. The distribution $Q_H(p, q)$ is therefore a Wigner function, Gaussian smeared over an $\hbar$-sized region of phase space.
An information-theoretic measure of the uncertainty in phase space contained in the distribution (52), is given by the Shannon information of the $Q_H$ distribution:

$$I = -\int \frac{dp dq}{2\pi \hbar} Q_H(p,q) \ln Q_H(p,q).$$

(53)

This is the so-called Wehrl entropy [72], defined in full analogy to the classical entropy in phase space. Wehrl considered this entropy as a classical analogue of the von Neumann entropy and it was introduced as a classical entropy of a quantum state. It gives additional insight into the system dynamics, as compared to other entropies. $I$ is large for spread out distributions and small for very concentrated ones. The uncertainty principle implies that a genuine phase space probability distribution in quantum mechanics cannot be arbitrarily peaked about a point in phase space. The information (53) should possess a lower bound and since the coherent states are the states most concentrated in phase space, we expect the lower bound to be the value of $I$ on a coherent state [40]. The most important property of this classical-like entropy is the following inequality, which gives a lower bound on the Shannon information:

$$I \geq 1,$$

(54)

with equality if and only if the considered state $\hat{\rho}$ is a coherent state [73]. The Wehrl entropy is a good measure of the strength of the coherent component and it clearly distinguishes coherent states, in other words it measures how close a given state is to the coherent states or how much coherence a given state has. Wehrl entropy is very sensitive to the phase space dynamics (such as, e. g., spreading) of the $Q_H$ representation. It extracts from the $Q_H$ function essential information about the investigated system. The Wehrl entropy cannot be negative. This follows from the fact that $0 \leq Q_H(\alpha) \leq 1$ and from the normalization condition (51). The $Q_H$ function can never be so concentrated as to make $I$ negative. On the contrary, classical distributions can be arbitrarily concentrated in phase space and classical entropies can take on negative values. They may even tend to $-\infty$ if the distributions tend to $\delta$-functions.
The Shannon information can be used [40] to find a relation between the von Neumann entropy and the generalized uncertainty measure

$$\sigma \equiv \sigma_{qq}\sigma_{pp} - \sigma_{pq}^2. \quad (55)$$

The Shannon information satisfies the inequality

$$I \leq \ln \left( \frac{e}{\hbar} (\det \sigma^{(Q)})^{\frac{1}{2}} \right), \quad (56)$$

where $\sigma^{(Q)}$ is the $2 \times 2$ covariance matrix of the distribution $Q_H(p, q) \,[40, 74]$. Equality holds if and only if $Q_H(p, q)$ is a Gaussian. From (52) or using the results of Ref. [30], one has

$$\det \sigma^{(Q_H)} = (\sigma_{qq} + s_{qq})(\sigma_{pp} + s_{pp}) - \sigma_{pq}^2. \quad (57)$$

There is an even stronger relationship [72]

$$I \geq S, \quad (58)$$

which establishes a connection between the Wehrl entropy and the von Neumann entropy of a given state. Using the relations (56) – (58), one obtains

$$(\sigma_{qq} + s_{qq})(\sigma_{pp} + s_{pp}) - \sigma_{pq}^2 \geq \hbar^2 e^{2(S-1)}. \quad (59)$$

Since $s_{qq}$ is arbitrary and $s_{qq}s_{pp} = \hbar^2/4$, we may minimize the left-hand side over it and we obtain the following connection between the uncertainty and entropy for a general mixed state $\hat{\rho} \,[40]$:

$$\left( \sqrt{\sigma_{qq}\sigma_{pp}} + \frac{\hbar}{2} \right)^2 - \sigma_{pq}^2 \geq \hbar^2 e^{2(S-1)}. \quad (60)$$

An analogous relation was obtained by Dodonov and Man’ko in [75].

In the regime where quantum fluctuations are more significant than thermal ones, it is appropriate to use the lower bound [74] rather than (58) (since $S$ goes to zero.
if the state is pure) and this is formally achieved by setting $S = 1$ in (60). One then deduces the usual uncertainty principle from (60) [40].

In the regime where thermal (or environmentally induced) fluctuations are dominant, one would expect $\sigma_{qq}\sigma_{pp} \gg \hbar^2/4$ and $S \gg 1$ and (60) then gives

$$\frac{\sqrt{\sigma_{qq}\sigma_{pp}}}{\hbar} \geq \frac{A}{\hbar} \geq e^S.$$  (61)

Here $A = \sqrt{\sigma}$ is the Wigner function area – a measure of the phase space area in which the Gaussian density matrix is localized and $\sqrt{\sigma_{qq}\sigma_{pp}}/\hbar$ is the number of phase space cells occupied by the state [40]. The von Neumann entropy of a Gaussian is given by (44) and for large $A$, (44) gives

$$S \approx \ln \frac{A}{\hbar}$$  (62)

and hence we have equality in (61) [40].

We can conclude that $I$ is a useful measure of both quantum and thermal fluctuations. It possesses a lower bound expressing the effect of quantum fluctuations and is closely connected to entropy, which in turn is a measure of thermal fluctuations [39].

In some models the linear entropy is introduced as a measure of purity of states:

$$S_l = \text{Tr}(\hat{\rho} - \hat{\rho}^2) = 1 - \text{Tr}\hat{\rho}^2.$$  (63)

Since $\text{Tr}\hat{\rho}^2 \leq 1$, the linear entropy is positive and it becomes zero if the state is pure. When the linear entropy is increasing, the degree of purity is decreasing. In [40] the following relation is deduced between the linear and von Neumann entropies:

$$S_l = 1 - \text{Tr}\hat{\rho}^2 \leq 1 - e^{-S}.$$  (64)

Equality in Eq. (64) is reached for pure states, when $S = S_l = 0$ and for very mixed states, when $S$ is very large and $S_l \approx 1$.

For analyzing the stability of stationary states, it could also be interesting to introduce the entropy production $\sigma_S(\hat{\rho})$ [76], defined for dynamical semigroups as follows
\[ \sigma_S(\hat{\rho}) = -\frac{d}{dt} S(\Phi_t(\hat{\rho})|\hat{\rho}^0)_{t=0}, \quad \hat{\rho} \in L_+^1(\mathcal{H}), \quad (65) \]

where

\[ S(\Phi_t(\hat{\rho})|\hat{\rho}^0) = \text{Tr}(\Phi_t(\hat{\rho}) \ln \hat{\rho}^0) + S(\Phi_t(\hat{\rho})), \quad (66) \]

\( \hat{\rho}^0 \in L_+^1(\mathcal{H}) \) is a \( \Phi_t \)-invariant state, i.e. \( \Phi_t\hat{\rho}^0 = \hat{\rho}^0 \), for any \( t \geq 0 \), \( \Phi_t(\hat{\rho}) \equiv \hat{\rho}(t) \) and \( L_+^1(\mathcal{H}) \) is the Banach space of trace class operators on \( \mathcal{H} \).

5 Purity of states

Schrödinger [78] and Robertson [79] proved for any Hermitian operators \( \hat{A} \) and \( \hat{B} \) and for pure quantum states the following generalized uncertainty relation:

\[ \sigma_{AA}\sigma_{BB} - \sigma_{AB}^2 \geq \frac{1}{4} |<[\hat{A}, \hat{B}]>|^2. \quad (67) \]

For the particular case of the operators of the coordinate \( \hat{q} \) and momentum \( \hat{p} \) the uncertainty relation (67) takes the form

\[ \sigma_{qq}\sigma_{pp} - \sigma_{pq}^2 \geq \frac{\hbar^2}{4}. \quad (68) \]

This result was generalized for arbitrary operators (in general non-Hermitian) and for the most general case of mixed states in [64]. The inequality (68) can also be represented in the following form:

\[ \sigma_{qq}\sigma_{pp} \geq \frac{\hbar^2}{4(1 - r^2)}, \quad (69) \]

where

\[ r = \frac{\sigma_{pq}}{\sqrt{\sigma_{qq}\sigma_{pp}}} \quad (70) \]

is the correlation coefficient. The equality in the relation (68) is realized for a special class of pure states, called correlated coherent states [64] or squeezed coherent states,
which are represented by Gaussian wave packets in the coordinate representation. These minimizing states, which generalize the Glauber coherent states, are eigenstates of an operator of the form [64]:

\[
\hat{a}_{r,\eta} = \frac{1}{2\eta} [1 - \frac{ir}{(1 - r^2)^{1/2}}]\hat{q} + \frac{\eta}{\hbar}\hat{p},
\]

with real parameters \( r \) and \( \eta, |r| < 1, \eta = \sqrt{\sigma_{qq}} \). Their normalized eigenfunctions, the correlated coherent states, have the form [64]:

\[
\Psi(x) = \frac{1}{(2\pi\eta^2)^{1/4}} \exp\left\{-\frac{x^2}{4\eta^2} [1 - \frac{ir}{(1 - r^2)^{1/2}}] + \frac{\alpha x}{\eta} - \frac{1}{2}(\alpha^2 + |\alpha|^2)\right\},
\]

where \( \alpha \) is a complex number. If we set \( r = 0 \) and \( \eta = (\hbar/2m\omega)^{1/2} \), where \( m \) and \( \omega \) are the mass and respectively the frequency of the harmonic oscillator, the states (72) become the usual Glauber coherent states. In Wigner representation, the states (72) have the form [64]:

\[
W_{\alpha,r,\eta}(q,p) = \frac{1}{\pi\hbar} \exp\left[ -\frac{2\eta^2}{\hbar^2}(p - \sigma_p)^2 - \frac{(q - \sigma_q)^2}{2\eta^2(1 - r^2)} + \frac{2r}{\hbar(1 - r^2)^{1/2}}(q - \sigma_q)(p - \sigma_p) \right].
\]

This is the classical normal distribution to give dispersion

\[
\sigma_{qq} = \eta^2, \quad \sigma_{pp} = \frac{\hbar^2}{4\eta^2(1 - r^2)}, \quad \sigma_{pq} = \frac{\hbar r}{2(1 - r^2)^{1/2}}
\]

and the correlation coefficient \( r \). The Gaussian distribution (73) is the only positive Wigner distribution for a pure state [80]. All other Wigner functions that describe pure states necessarily take on negative values for some values of \( q,p \).

In the case of the relation (67) the equality is generally obtained only for pure states [64]. For any density matrix in the coordinate representation (normalized to unity) the following relation must be fulfilled:

\[
\gamma = \text{Tr}\hat{\rho}^2 \leq 1.
\]

The quantity \( \gamma \) characterizes the degree of purity of the state. For pure states \( \gamma = 1 \), for highly mixed states \( \gamma \ll 1 \) and for weekly mixed states \( 1 - \gamma \ll 1 \). In the language
of the Wigner function the condition (75) has the form:

$$\gamma = 2\pi \hbar \int W^2(q,p)dqdp \leq 1.$$ (76)

Let us consider the most general mixed squeezed states described by the Wigner function of the generic Gaussian form with five real parameters:

$$W(q,p) = \frac{1}{2\pi \sqrt{\sigma}} \exp\left\{ -\frac{1}{2\sigma} [\sigma_{pp}(q-\sigma_q)^2 + \sigma_{qq}(p-\sigma_p)^2 - 2\sigma_{pq}(q-\sigma_q)(p-\sigma_p)] \right\},$$ (77)

where \(\sigma\) is given by Eq. (55). The Gaussian Wigner functions of this form correspond to the so-called quasi-free states on the \(C^*\)-algebra of the canonical commutation relations, which is the most natural framework for a unified treatment of quantum and thermal fluctuations [81]. For Gaussian states of the form (77) the coefficient of purity \(\gamma\) is given by

$$\gamma = \frac{\hbar}{2\sqrt{\sigma}}.$$ (78)

Therefore, we reobtain from (75) and (78) that \(\sigma\) has to satisfy the Schrödinger-Robertson uncertainty relation (68)

$$\sigma \geq \frac{\hbar^2}{4}.$$ (79)

This inequality must be fulfilled actually for any states, not only Gaussian. Any Gaussian pure state minimizes the relation (79). For \(\sigma > \hbar^2/4\) the function (77) corresponds to mixed quantum states, while in the case of the equality \(\sigma = \hbar^2/4\) it takes the form (73) corresponding to pure correlated coherent states (squeezed coherent states).

We have seen in the preceding section that the degree of purity of a state can also be characterized by other quantities besides \(\gamma\). The most usual one is the quantum (von Neumann) entropy. For quantum pure states this entropy is identically equal to zero. We remind [32, 55] that for Gaussian states with the Wigner functions (74) the entropy can be expressed through \(\sigma\) only, in the form (14).
In the Lindblad model for the damped harmonic oscillator, the relation (7) is a necessary condition for the generalized uncertainty inequality (68) to be fulfilled [13, 26, 27]. By using the fact that the linear positive mapping defined by \( \hat{A} \to \text{Tr}(\hat{\rho}\hat{A}) \) is completely positive, the following inequality was obtained in Refs. [13, 26]:

\[
D_{pp}\sigma_{qq}(t) + D_{qq}\sigma_{pp}(t) - 2D_{pq}\sigma_{pq}(t) \geq \frac{\hbar^2}{2}\lambda.
\]  

(80)

We have found in [27] that this inequality, which must be valid for all values of \( t \), is equivalent with the generalized uncertainty inequality (68) at any time \( t \),

\[
\sigma_{qq}(t)\sigma_{pp}(t) - \sigma_{pq}^2(t) \geq \frac{\hbar^2}{4},
\]  

(81)

if the initial values \( \sigma_{qq}(0), \sigma_{pp}(0) \) and \( \sigma_{pq}(0) \) for \( t = 0 \) satisfy this inequality. If the initial state is the ground state of the harmonic oscillator, then

\[
\sigma_{qq}(0) = \frac{\hbar}{2m\omega}, \; \sigma_{pp}(0) = \frac{m\hbar\omega}{2}, \; \sigma_{pq}(0) = 0.
\]  

(82)

By using the complete positivity property of the dynamical semigroup \( \Phi_t \), it was shown in Ref. [26] that the relation

\[
\text{Tr}(\Phi_t(\hat{\rho}) \sum_j \hat{V}_j^\dagger \hat{V}_j) = \sum_j \text{Tr}(\Phi_t(\hat{\rho}) \hat{V}_j^\dagger)\text{Tr}(\Phi_t(\hat{\rho}) \hat{V}_j)
\]

(83)

represents the necessary and sufficient condition for \( \hat{\rho}(t) = \Phi_t(\hat{\rho}) \) to be a pure state for all times \( t \geq 0 \). This equality is a generalization of the pure state condition [82, 83, 84] to all Markovian master equations (2). If \( \hat{\rho}^2(t) = \hat{\rho}(t) \) for all \( t \geq 0 \), there exists a wave function \( \psi \in \mathcal{H} \) which satisfies the nonlinear Schrödinger type equation

\[
i\hbar \frac{\partial \psi(t)}{\partial t} = \hat{H}' \psi(t),
\]  

(84)

with the non-Hermitian Hamiltonian [26]

\[
\hat{H}' = \hat{H} + i \sum_j \langle \psi(t), \hat{V}_j^\dagger \psi(t) \rangle \hat{V}_j - \frac{i}{2} \langle \psi(t), \sum_j \hat{V}_j^\dagger \hat{V}_j \psi(t) \rangle - \frac{i}{2} \sum_j \hat{V}_j^\dagger \hat{V}_j.
\]  

(85)
For environment operators \( \hat{V}_j \) of the form (3), the pure state condition (83) takes the following form [26], corresponding to equality in the relation (80):

\[
D_{pp}\sigma_{qq}(t) + D_{qq}\sigma_{pp}(t) - 2D_{pq}\sigma_{pq}(t) = \frac{\hbar^2 \lambda}{2}
\]  

(86)

and the Hamiltonian (85) becomes

\[
\hat{H}' = \hat{H} + \lambda(\sigma_p(t)\hat{q} - \sigma_q(t)\hat{p}) + \frac{i}{\hbar}[\lambda\hbar^2 - D_{pp}((\hat{q} - \sigma_q(t))^2 + \sigma_{qq}(t)) - D_{qq}((\hat{p} - \sigma_p(t))^2 + \sigma_{pp}(t))

+ \sigma_{pq}(t) + D_{pq}((\hat{q} - \sigma_q(t))(\hat{p} - \sigma_p(t)) + (\hat{p} - \sigma_p(t))(\hat{q} - \sigma_q(t)) + 2\sigma_{pq}(t))].
\]  

(87)

We will find the Gaussian states which remain pure during the evolution of the system for all times \( t \). We start by considering the pure state condition (86) and the generalized uncertainty relation (68) which transforms into the following minimum uncertainty equality for pure states:

\[
\sigma_{qq}(t)\sigma_{pp}(t) - \sigma_{pq}^2(t) = \frac{\hbar^2}{4}.
\]  

(88)

By eliminating \( \sigma_{pp} \) between the equalities (86) and (88), like in Ref. [85], we obtain:

\[
(\sigma_{qq}(t) - \frac{D_{pq}\sigma_{pq}(t) + \frac{1}{4}\hbar^2 \lambda}{D_{pp}})^2 + \frac{D_{pp}D_{qq} - D_{pq}^2}{D_{pp}^2}[\sigma_{pq}(t) - \frac{\frac{1}{2}\hbar^2 \lambda D_{pq} - \frac{1}{4}\hbar^2 \lambda^2}{D_{pp}D_{qq} - D_{pq}^2}D_{pq}] = 0.
\]  

(89)

Since the opening coefficients satisfy the inequality (7), we obtain from Eq. (89) the following relations which have to be fulfilled at any moment of time:

\[
D_{pp}D_{qq} - D_{pq}^2 = \frac{\hbar^2 \lambda^2}{4},
\]  

(90)

\[
D_{pp}\sigma_{qq}(t) - D_{pq}\sigma_{pq}(t) - \frac{\hbar^2 \lambda}{4} = 0,
\]  

(91)

\[
\sigma_{pq}(t)(D_{pp}D_{qq} - D_{pq}^2) - \frac{\hbar^2 \lambda}{4}D_{pq} = 0.
\]  

(92)
From the relations (88) and (90) – (92) it follows that the pure states remain pure for all times only if their variances are constant in time and have the form:

$$\sigma_{qq}(t) = \frac{D_{qq}}{\lambda}, \quad \sigma_{pp}(t) = \frac{D_{pp}}{\lambda}, \quad \sigma_{pq}(t) = \frac{D_{pq}}{\lambda}. \quad (93)$$

If these relations are fulfilled, then the inequalities (80) and (81) are both equivalent to (7), including also the corresponding equalities (86), (88) and (90). From Eq. (20) it follows that the variances remain constant and do not depend on time only if $X(0) = X(\infty)$, that means $\sigma_{qq}(0) = \sigma_{qq}(\infty), \sigma_{pp}(0) = \sigma_{pp}(\infty), \sigma_{pq}(0) = \sigma_{pq}(\infty)$. Using the asymptotic values (24) – (26) of the variances and the relations (93), we obtain the following expressions of the diffusion coefficients which assure that the initial pure states remain pure for any $t$:

$$D_{qq} = \frac{\hbar \lambda}{2m\Omega}, \quad D_{pp} = \frac{\hbar \lambda m \omega^2}{2\Omega}, \quad D_{pq} = -\frac{\hbar \lambda \mu}{2\Omega}. \quad (94)$$

Formulas (94) are generalized Einstein relations and represent examples of quantum fluctuation-dissipation relations, connecting the diffusion with both Planck’s constant and damping constant. With the coefficients (94), the variances (24) – (26) become:

$$\sigma_{qq} = \frac{\hbar}{2m\Omega}, \quad \sigma_{pp} = \frac{\hbar m \omega^2}{2\Omega}, \quad \sigma_{pq} = -\frac{\hbar \mu}{2\Omega}. \quad (95)$$

Therefore, the quantity $\sigma$ (see Eqs. (30), (55)) is equal to its minimum possible value $\hbar^2/4$, according to the generalized uncertainty relation (79). Then the corresponding state described by a Gaussian Wigner function is a pure quantum state, namely a correlated coherent state (squeezed coherent state) with the correlation coefficient (70) $r = -\mu/\omega$. Given $\sigma_{qq}, \sigma_{pp}$ and $\sigma_{pq}$, there exists one and only one such a state minimizing the uncertainty $\sigma$ (86). We remark that the minimization of the uncertainty $\sigma$ is equivalent, by virtue of the relations (44), (45) to the minimization of the entropy $S$. A particular case of our result (corresponding to $\lambda = \mu$ and $D_{pq} = 0$) was obtained by Halliwell and Zoupas by using the quantum state diffusion method [43]. We consider
here general coefficients $\mu$ and $\lambda$ and in this respect our expressions for the diffusion coefficients and variances generalize also the ones obtained by Dekker and Valsakumar [85] and Dodonov and Man’ko [87], who used models where $\lambda = \mu$ was chosen. If $\mu = 0$, we get from (94) $D_{pq} = 0$. This case, which was considered in Ref. 27 and where we have obtained a density operator describing a pure state for any $t$, is also a particular case of the present results which give the most general Gaussian pure state which remains pure for any $t$. For $\mu = 0$, the expressions (95) become

$$
\sigma_{qq} = \frac{\hbar}{2m\omega}, \quad \sigma_{pp} = \frac{\hbar m \omega}{2}, \quad \sigma_{pq} = 0,
$$

which are the values of variances (82) for the ground state of the harmonic oscillator and the correlation coefficient (70) takes the value $r = 0$, corresponding to the case of usual coherent states.

The Lindblad equation or its equivalent Fokker-Planck equation for the Wigner function with the diffusion coefficients (94) can be used only in the underdamped case, when $\omega > \mu$. Indeed, for the coefficients (94) the fundamental constraint (7) implies that $m^2(\omega^2 - \mu^2)D_{qq}^2 > \hbar^2 / 4$, which is satisfied only if $\omega > \mu$. It can be shown that there exist diffusion coefficients which satisfy the condition (90) and make sense for $\omega < \mu$, but in this overdamped case we have always $\sigma > \hbar^2 / 4$ and the state of the oscillator cannot be pure for any diffusion coefficients [87].

The fluctuation energy of the open harmonic oscillator is

$$
E(t) = \frac{1}{2m} \sigma_{pp}(t) + \frac{1}{2} m \omega^2 \sigma_{qq}(t) + \mu \sigma_{pq}(t).
$$

If the state remains pure in time, then the variances are given by (93) and the fluctuation energy is also constant in time and is given by

$$
E = \frac{1}{\lambda} (\frac{1}{2m} D_{pp} + \frac{1}{2} m \omega^2 D_{qq} + \mu D_{pq}).
$$

Minimizing this expression with the condition (94), we obtain just the diffusion coefficients (94) and $E_{\text{min}} = \hbar \Omega / 2$. Therefore, the conservation of purity of state implies
that the fluctuation energy of the system has all the time the minimum possible value \( E_{\text{min}} \). If the asymptotic state is a Gibbs state \( \text{(8)} \), we see from Eqs. \( \text{(9)} \) and \( \text{(94)} \) that the case when the diffusion coefficients satisfy the equality \( \text{(90)} \) corresponds to a zero temperature of the thermal bath and then the influence on the oscillator is minimal. In this limiting case \( \mu = 0 \) and then \( E_{\text{min}} = \hbar \omega / 2 \), the correlation coefficient \( \text{(70)} \) vanishes and therefore the correlated coherent state (squeezed coherent state) becomes the usual coherent state.

If we choose the coefficients of the form \( \text{(94)} \), then the equation for the density operator can be represented in the form \( \text{(2)} \) with only one operator \( \hat{V} \), which up to a phase factor can be written in the form:

\[
\hat{V} = \sqrt{\frac{2}{\hbar D_{qq}}} \left[ \frac{\lambda \hbar}{2} - iD_{pq} \hat{q} + iD_{qq} \hat{p} \right],
\]

(99)

with \( [\hat{V}, \hat{V}^\dagger] = 2\hbar \lambda \).

The correlated coherent states \( \text{(72)} \) with nonvanishing momentum average, can also be written in the form:

\[
\Psi(x) = \left( \frac{1}{2\pi \sigma_{qq}^2} \right)^{1/4} \exp \left[ -\frac{1}{4\sigma_{qq}} \left( 1 - \frac{2i}{\hbar} \sigma_{pq} \right) x^2 - \frac{i}{\hbar} \sigma_p x \right]
\]

(100)

and the most general form of Gaussian density matrices compatible with the generalized uncertainty relation \( \text{(68)} \) is given by Eq. \( \text{(41)} \). For these matrices we can verify that \( \text{Tr} \rho^2 = \hbar / 2\sqrt{\sigma} \) and they correspond to the correlated coherent states \( \text{(100)} \) if \( \sigma_{qq}, \sigma_{pp} \) and \( \sigma_{pq} \) in Eq. \( \text{(41)} \) satisfy the equality in Eq. \( \text{(68)} \).

Consider the harmonic oscillator initially in a correlated coherent state (squeezed coherent state) of the form \( \text{(100)} \), with the corresponding Wigner function \( \text{(73)} \). For an environment described by the diffusion coefficients \( \text{(94)} \), the solution for the Wigner function at time \( t \) is given by \( \text{(29)} \), where \( \sigma_q(t) \) and \( \sigma_p(t) \) are given respectively by \( \text{(16)} \) and \( \text{(17)} \) and the variances by \( \text{(95)} \). Using Eq. \( \text{(41)} \), we get for the density matrix the
following time evolution:

\[
<x|\hat{\rho}(t)|y> = (\frac{m\Omega}{\pi\hbar})^{\frac{1}{2}} \exp\left[-\frac{m\Omega}{\hbar}(\frac{x+y}{2} - \sigma_q(t))^2 \right. \\
- \frac{m\Omega}{4\hbar}(x-y)^2 - \frac{im\mu}{\hbar}(\frac{x+y}{2} - \sigma_q(t))(x-y) + \frac{i}{\hbar}\sigma_p(t)(x-y) \right]. (101)
\]

In the long time limit \(\sigma_q(t) = 0, \sigma_p(t) = 0\) and we have

\[
<x|\hat{\rho}(\infty)|y> = (\frac{m\Omega}{\pi\hbar})^{\frac{1}{2}} \exp\left\{-\frac{m}{2\hbar}[\Omega(x^2+y^2) + \mu(x^2-y^2)]\right\}. (102)
\]

The corresponding Wigner function (32) has the form

\[
W_\infty(q,p) = \frac{1}{\pi\hbar} \exp\left[-\frac{1}{\hbar\Omega}\left(\frac{1}{m}p^2 + m\omega^2q^2 + 2\mu qp\right)\right]. (103)
\]

We see that the time evolution of the initial correlated coherent state of the damped harmonic oscillator is given by a Gaussian density matrix with variances constant in time. According to known general results [39, 40], the initial Gaussian density matrix remains Gaussian centered around the classical path. So, the correlated coherent state (squeezed coherent state) remains a correlated coherent state and \(\sigma_q(t)\) and \(\sigma_p(t)\) give the average time dependent location of the system along its trajectory in phase space.

### 6 Decoherence and transition from quantum to classical

The environment induced quantum decoherence (loss of quantum coherence) and the transition from quantum to classical mechanics have recently been investigated in widely different contexts [3, 4, 5, 6, 16, 17, 18, 19, 20, 21, 22, 23, 24, 40, 44]. The central idea of the quantum decoherence approach is that the transition quantum–classical is a dynamical effect within quantum mechanics, keeping \(\hbar \neq 0\). The characteristic feature of the decoherence process is that an arbitrarily chosen generic initial quantum state will be affected on a characteristic decoherence time scale and only certain stable states (which turn out to be, in a sense, decay products of the other states) will survive during the time evolution of the system.
In quantum optics the quantum interference of Gaussian states in phase space was investigated by Schleich and coworkers [88, 89] and the decay of these quantum coherences was studied in Refs. [90, 91] (for a recent review on quantum interference in phase space see the work of Bužek and Knight [92]).

The predictability sieve was recently proposed [20] and implemented for a harmonic oscillator with the resulting evolution of the reduced density matrix [16, 19]. For a weakly damped harmonic oscillator, pure states selected by the predictability sieve, called preferred states, turn out to be the coherent states [20, 21]. Decoherence is caused by the loss of phase coherence between the preferred quantum states in the Hilbert space of the system due to the interaction with the environment [2, 4]. Preferred states are singled out by their stability (measured, for example, by the rate of predictability loss – the rate of entropy increase) under the joint influence of the environment and the self-Hamiltonian [21]. Thus, the strength and nature of the coupling with the environment play a crucial role in selecting preferred states, which – given the distance-dependent nature of typical interactions – explains the special function of the position observable [4, 21]. Coupling with the environment also sets the decoherence time scale – the time on which quantum interference between preferred states disappears [2, 4, 17, 20, 21, 93]. Classicality is then an emergent property of an open quantum system.

The interaction with the environment also induces fluctuations in the evolution of the system. There is therefore a certain degree of tension between the demands of decoherence and approximate classical predictability: decoherence requires interaction with an environment, which inevitably produces fluctuations, but classical predictability requires that these fluctuations be small. The von Neumann entropy is an ideal parameter with which to characterize the decoherence, i.e. the rapid decay of off-diagonal coherences. In [40] an information-theoretic measure of the size of these fluctuations was proposed – the Shannon information of the $Q_H$ distribution of the
density matrix $\hat{\rho}$ for the distinguished system (see Sec. 4). This measure of uncertainty is in fact bounded from below by the von Neumann entropy of $\hat{\rho}$. This suggests that the von Neumann entropy is the key to understanding the connection between decoherence and fluctuations: it limits the amount of decoherence from above but bounds the size of the fluctuations from below. Large entropy, which is a signal of destruction of interference, therefore permits good decoherence, but leads to large fluctuations; on the other hand, small entropy allows small fluctuations, but the amount of decoherence is also small. Environmentally induced fluctuations are inescapable if one is to have decoherence. Although there is some tension, there is a broad compromise regime in which decoherence and classical predictability can each hold extremely well [40].

As we have already mentioned, besides the von Neumann entropy $S$ (43), there is another quantity which can measure the degree of mixing or purity of quantum states. It is the linear entropy $S_l$ (63). For pure states $S_l = 0$ and for a statistical mixture $S_l > 0$. The increasing of the linear entropy $S_l$ (as well as of the von Neumann entropy $S$) due to the interaction with the environment is associated with the decoherence phenomenon, given by the diffusion process [17, 20]. Due to the decoherence, the macroscopic systems obey essentially classical equations of motion, despite the quantum mechanical nature of the underlying microscopic dynamics. Dissipation increases the entropy and the pure states are converted into mixed states. The rate of entropy production is given by

$$\dot{S}_l(\hat{\rho}) = -2\text{Tr}(\dot{\hat{\rho}} \hat{\rho}) = -2\text{Tr}(\hat{\rho} L(\hat{\rho})),$$

where $L$ is the evolution operator. According to Zurek’s theory, the maximally predictive states are the pure states which minimize the entropy production in time. These states remain least affected by the openness of the system and form the preferred set of states in the Hilbert space of the system, known as the pointer basis. Their evolution is predictable with the principle of least possible entropy production.
Using Eq. (5), in our model the rate of entropy production \( (104) \) is given by:

\[
\dot{S}_l(t) = \frac{4}{\hbar^2} [D_{pp} \text{Tr}(\hat{\rho}^2 \hat{q}^2 - \hat{\rho} \hat{q} \hat{\rho} \hat{q})]
\]

\[
+ D_{qq} \text{Tr}(\hat{\rho}^2 \hat{\rho}^2 - \hat{\rho} \hat{\rho} \hat{\rho} \hat{\rho}) - D_{pq} \text{Tr}(\hat{\rho}^2 \hat{q} \hat{p} + \hat{p} \hat{q} \hat{\rho} \hat{\rho} - 2 \hat{\rho} \hat{q} \hat{\rho} \hat{p}) - \frac{\hbar^2 \lambda}{2} \text{Tr}(\hat{\rho}^2)].
\]  

(105)

When the state remains approximately pure \((\hat{\rho}^2 \approx \hat{\rho})\), we obtain:

\[
\dot{S}_l(t) = \frac{4}{\hbar^2} (D_{pp} \sigma_{qq}(t) + D_{qq} \sigma_{pp}(t) - 2 D_{pq} \sigma_{pq}(t) - \frac{\hbar^2 \lambda}{2}) \geq 0,
\]  

(106)

according to \((80)\). We see that \(\dot{S}_l(t) = 0\) when the condition \((86)\) of purity for any time \(t\) is fulfilled. The entropy production \(S_l\) is also equal to 0 at \(t = 0\) if the initial state is a pure state. We have shown in the preceding section that the only initial states which remain pure for any \(t\) are the correlated coherent states (squeezed coherent states) and therefore, we can state that in the Lindblad theory for the open quantum harmonic oscillator the correlated coherent states, which are generalized coherent states, are the maximally predictive states. Our result generalizes the previous results which assert that for many models of quantum Brownian motion in the high temperature limit the usual coherent states correspond to minimal entropy production and, therefore, they are the maximally predictive states. In our model the coherent states can be obtained as a particular case of the correlated coherent states by taking \(\mu = 0\), so that the correlation coefficient \(r = 0\) (see Eq. (70)).

Paz, Habib and Zurek \([17, 20]\) considered the harmonic oscillator undergoing quantum Brownian motion in the Caldeira-Leggett model and concluded that the minimizing states which are the initial states generating the least amount of von Neumann or linear entropy and, therefore, the most predictable or stable ones under evolution in the presence of an environment are the ordinary coherent states. Using an information-theoretic measure of uncertainty for quantum systems, Anderson and Halliwell showed in \([39]\) that the minimizing states are more general Gaussian states. Anastopoulos and Halliwell \([40]\) offered an alternative characterization of these states by noting that these
states minimize the generalized uncertainty relation. According to this assertion, we can say that in the Lindblad model the correlated coherent states (squeezed coherent states) are the most stable states which minimize the generalized uncertainty relation \((68)\). Our result confirms that one of \([40]\), but the model used in \([40]\) is different, namely the open quantum system is a particle moving in a harmonic oscillator potential and linearly coupled to an environment consisting of a bath of harmonic oscillators in a thermal state. At the same time we remind that the Caldeira-Leggett model considered in \([17, 20]\) violates the positivity of the density operator at short time scales \([53, 94]\), whereas in the Lindblad model considered here the property of positivity is automatically fulfilled.

The rate of predictability loss, measured by the rate of linear entropy increase, for a damped harmonic oscillator is also calculated in the framework of Lindblad theory in Ref. \([95]\). The initial states which minimize the predictability loss are identified as quasi-free states with a symmetry dictated by the environment diffusion coefficients. For an isotropic diffusion in phase space, the coherent states or mixtures of coherent states are selected as the most stable ones.

In order to generalize the results of Zurek and collaborators, the entropy production was considered by Gallis \([44]\) within the Lindblad theory of open quantum systems, treating environment effects perturbatively. Gallis considered the particular case with \(D_{pq} = 0\) and found out that the squeezed states emerge as the most stable states for intermediate times compared to the dynamical time scales. The amount of squeezing decreases with time, so that the coherent states are most stable for large time scales. For \(D_{pq} \neq 0\) our results generalize the ones of Gallis and establish that the correlated coherent states are the most stable under the evolution in the presence of an environment.
7 Summary and concluding remarks

Recently there is a revival of interest in quantum Brownian motion as a paradigm of quantum open systems. The possibility of preparing systems in macroscopic quantum states led to the problems of dissipation in tunneling and of loss of quantum coherence (decoherence). These problems are intimately related to the issue of quantum to classical transition and all of them point the necessity of a better understanding of open quantum systems. The Lindblad theory provides a selfconsistent treatment of damping as a general extension of quantum mechanics to open systems and gives the possibility to extend the model of quantum Brownian motion. In the present paper we have studied the one-dimensional harmonic oscillator with dissipation within the framework of this theory. From the master equation of the damped quantum oscillator we have derived the corresponding Fokker-Planck equation in the Wigner $W$ representation. The obtained equation describes an Ornstein-Uhlenbeck process and the Wigner function is a two-dimensional Gaussian. We have then obtained the explicit form of the density operator from the master and Fokker-Planck equations. The density operator in a Gaussian form is a function of the position and momentum operators in addition to several time dependent factors. In the long time limit the density operator approaches a thermal state. Then the density operator has been used to calculate the von Neumann entropy and the effective temperature. The temporal behaviour of these quantities shows how they approach their equilibrium values. Following Anderson, Anastopoulos and Halliwell [39, 40], we have put the von Neumann entropy in association with uncertainty and linear entropy. The Wehrl entropy generates an informatical-theoretical measure of the size of the intrinsic state fluctuations. As coherent states are most robust in dissipative environment [17, 20, 21, 40], this suggests the utility of the Wehrl entropy in characterizing the decoherence process in quantum mechanics. We have also shown that the only states which stay pure during
the evolution in time of the system are the correlated coherent states (squeezed coherent states) under the condition of a special choice of the environment coefficients. These states are also connected with the decoherence phenomenon and they are the most stable under the evolution in the presence of the environment. The obtained results in the framework of the Lindblad theory can be used for the description in more details of the connection between uncertainty, decoherence and correlations of open quantum systems with their environment.

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