DENSITY OF PERIODIC MEASURES AND LARGE DEVIATION PRINCIPLE FOR GENERALIZED \((\alpha, \beta)\)-TRANSFORMATIONS

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Abstract. We introduce generalized \((\alpha, \beta)\)-transformations, which include all \((\alpha, \beta)\) and generalized \(\beta\)-transformations, and prove that all transitive generalized \((\alpha, \beta)\)-transformations satisfy the level-2 large deviation principle with a unique measure of maximal entropy. A crucial step in our proof is to establish density of periodic measures in the set of ergodic measures.

1. Introduction

In this paper, we consider piecewise monotonic maps on the unit interval \([0, 1]\). We say that \(T: [0, 1] \to [0, 1]\) is a piecewise monotonic map if there exist integer \(k > 1\) and \(0 = c_0 < c_1 < \cdots < c_k = 1\), which we call the critical points, such that \(T|_{(c_{i-1}, c_i)}\) is strictly monotonic and continuous for each \(1 \leq i \leq k\). Throughout this paper, we further assume the following conditions for a piecewise monotonic map \(T\).

- The topological entropy \(h_{top}(T)\) of \(T\) is positive (see [1] Ch. 9) for the definition of topological entropy for piecewise monotonic maps).
- \(T\) is transitive, i.e., there exists a point \(x \in [0, 1]\) whose forward orbit \(\{T^n(x) : n \geq 0\}\) is dense in \([0, 1]\).

Under these conditions, it is proved in [10] Theorem 4 that there exists a unique measure of maximal entropy for \(T\), that is, a \(T\)-invariant measure whose metric entropy coincides with \(h_{top}(T)\). The aim of this paper is to investigate whether the large deviation principle holds for a piecewise monotonic map with the unique measure of maximal entropy as a reference.

Let \(\mathcal{M}([0, 1])\) be the set of all Borel probability measures on \([0, 1]\) endowed with the weak*-topology. We say that \(([0, 1], T)\) satisfies the (level-2) large deviation principle with the unique measure of maximal entropy \(m\) as a reference if there exists a lower semi-continuous function \(J: \mathcal{M}([0, 1]) \to [0, \infty]\), called a rate function, such that

\[
\limsup_{n \to \infty} \frac{1}{n} \log m \left( \left\{ x \in [0, 1] : \frac{1}{n} \sum_{j=0}^{n-1} \delta_{T^j(x)} \in \mathcal{K} \right\} \right) \leq - \inf_{\mathcal{K}} J
\]

2020 Mathematics Subject Classification. Primary 37A50, 37E05, 37B10; Secondary 60F10.
holds for any closed set $K \subset M([0,1])$ and
\[
\liminf_{n \to \infty} \frac{1}{n} \log m\left(\left\{ x \in [0,1] : \frac{1}{n} \sum_{j=0}^{n-1} \delta_{T^j(x)} \in U \right\}\right) \geq -\inf_U J
\]
holds for any open set $U \subset M([0,1])$. Here $\delta_y$ signifies the Dirac mass at a point $y \in [0,1]$. We refer to [6] for a general theory of large deviations and its background in statistical mechanics.

In this literature, the case that $T$ has a constant slope is important because this condition implies that the unique measure of maximal entropy is absolutely continuous to the Lebesgue measure. Hence we focus our attention to piecewise monotonic maps with a constant slope $\beta > 1$. A familiar example included in this family is a $\beta$-transformation.

- **$\beta$-transformations.**
  The $\beta$-transformation $T_\beta : [0,1] \to [0,1]$ with $\beta > 1$ was introduced by Rényi [21] and defined by
  \[
  T_\beta(x) = \begin{cases} 
  \beta x \pmod{1} & (x \neq 1), \\
  \lim_{y \to 1-0} (\beta y \pmod{1}) & (x = 1).
  \end{cases}
  \]
  This family has recently attracted attention in the setting beyond specification, since $T_\beta$ does not satisfy the specification property for Lebesgue almost parameter $\beta > 1$ [2, 22]. Inspired by $\beta$-transformations, many authors have considered various generalizations of $T_\beta$: $(\alpha, \beta)$-transformations [4, 5, 7, 9, 20], $(-\beta)$-transformations [15, 16, 17, 23], generalized $\beta$-transformations [7, 8, 24, 26],

- **$(\alpha, \beta)$-transformations.**
  The $(\alpha, \beta)$-transformation $T_{\alpha,\beta} : [0,1] \to [0,1]$ with $\beta > 1$ and $0 \leq \alpha < 1$ was introduced by Parry [18] and defined by
  \[
  T_{\alpha,\beta}(x) = \begin{cases} 
  \beta x + \alpha \pmod{1} & (x \neq 1), \\
  \lim_{y \to 1-0} (\beta y + \alpha \pmod{1}) & (x = 1).
  \end{cases}
  \]

- **Generalized $\beta$-transformations.**
  Let $\beta > 1$ and $k$ be a smallest integer not less than $\beta$, fix $E = (E_1, \ldots, E_k) \in \{+1, -1\}^k$, and consider $k$ intervals
  \[
  I_1 := [0, 1/\beta), \, I_2 := [1/\beta, 2/\beta), \ldots, I_k := [k - 1/\beta, 1].
  \]
  The generalized $\beta$-transformation $T_{\beta,E} : [0,1] \to [0,1]$ was introduced by Góra [8] and defined by
  \[
  T_{\beta,E}(x) = \begin{cases} 
  \beta x - i + 1 & (x \in I_i, \, E(i) = +1), \\
  -\beta x + i & (x \in I_i, \, E(i) = -1).
  \end{cases}
  \]
  If $E = (-1, \ldots, -1)$, then we call $T_{\beta,E}$ a $(-\beta)$-transformation.
  Pfister and Sullivan [19] established the large deviation principle for $\beta$-transformations for any $\beta > 1$, which is the first work on the large deviation principle in this family without the specification property. In [5], Chung and
the second author proved that the large deviation principle holds for \((\alpha, \beta)\) and generalized \(\beta\)-transformations in the following parameters:

- \((\alpha, \beta)\)-transformations for \(0 \leq \alpha < 1\) and \(\beta > 2\).
- Generalized \(\beta\)-transformations for \(\frac{1+\sqrt{5}}{2} < \beta < 2\) and \(E = (-1, -1)\).

Then it is natural to ask whether does the large deviation principle hold for the other parameters. In this paper not only we get an affirmative answer to the all parameters, but also we show that it holds for more general class of piecewise monotonic maps with constant slope, which we call generalized \((\alpha, \beta)\)-transformations.

- **Generalized \((\alpha, \beta)\)-transformations.** Let \(0 \leq \alpha < 1\), \(\beta > 1\), \(k\) be a smallest integer not less than \(\alpha + \beta\), fix \(E = (E_1, \ldots, E_k) \in \{+1, -1\}^k\), and consider \(k\) intervals
  \[ I_1 := \left[0, \frac{1 - \alpha}{\beta}\right), \quad I_2 := \left[\frac{1 - \alpha}{\beta}, \frac{2 - \alpha}{\beta}\right), \ldots, \quad I_k := \left[\frac{k - 1 - \alpha}{\beta}, 1\right]. \]

  We define the generalized \((\alpha, \beta)\)-transformation \(T_{\alpha,\beta,E} : [0, 1] \rightarrow [0, 1]\) by
  \[ T_{\alpha,\beta,E}(x) = \begin{cases} 
\alpha + \beta x - i + 1 & (x \in I_i, \ E(i) = +1), \\
-\alpha - \beta x + i & (x \in I_i, \ E(i) = -1). 
\end{cases} \]

Generalized \((\alpha, \beta)\)-transformations are clearly a generalization of both \((\alpha, \beta)\)-transformations and generalized \(\beta\)-transformations. The graphs of \(T_\beta\), \(T_{\alpha,\beta}\), \(T_{\beta,E}\) and \(T_{\alpha,\beta,E}\) are plotted in Figure 1.

Now we state our first main result of this paper.
Theorem A. Let \( T : [0, 1] \to [0, 1] \) be a transitive generalized \((\alpha, \beta)\)-transformation. Then \(([0, 1], T)\) satisfies the level-2 large deviation principle with the unique measure of maximal entropy.

A crucial step to prove Theorem A is to show density of periodic measures. For a metrizable space \( X \) and a Borel measurable map \( f : X \to X \), denote by \( \mathcal{M}(X) \) the set of all Borel probability measures on \( X \) endowed with the weak\(^*\)-topology, by \( \mathcal{M}_f(X) \subset \mathcal{M}(X) \) the set of \( f \)-invariant ones, and by \( \mathcal{M}^e_f(X) \subset \mathcal{M}_f(X) \) the set of ergodic ones. We say that \( \mu \in \mathcal{M}(X) \) is a periodic measure if there exist \( x \in X \) and \( n > 0 \) such that \( f^n(x) = x \) and \( \mu = \delta_{f^n(x)} := 1/n \sum_{j=0}^{n-1} \delta_{f^j(x)} \) hold. Then, it is clear that \( \mu \in \mathcal{M}^e_f(X) \). We denote by \( \mathcal{M}^p_f(X) \subset \mathcal{M}^e_f(X) \) the set of all periodic measures on \( X \). It is established by Chung and the second author that the level-2 large deviation principle for a piecewise monotonic map is followed by density of periodic measures in the set of ergodic measures with the irreducibility of its Markov diagram (see [5, Theorem A]). Although the irreducibility is slightly stronger than the transitivity, we show in this paper that this result remains true if the irreducibility of a Markov diagram is replaced by the transitivity of a map.

Proposition A. Let \( T : [0, 1] \to [0, 1] \) be a transitive piecewise monotonic map with \( h_{\text{top}}(T) > 0 \). Suppose that \( \mathcal{M}^p_T([0, 1]) \) is dense in \( \mathcal{M}^e_T([0, 1]) \). Then \(([0, 1], T)\) satisfies the level-2 large deviation principle with the unique measure of maximal entropy.

Hence Theorem A follows from Proposition A and the following theorem, which is the second main result of this paper:

Theorem B. Let \( T : [0, 1] \to [0, 1] \) be a transitive generalized \((\alpha, \beta)\)-transformation. Then \( \mathcal{M}^p_T([0, 1]) \) is dense in \( \mathcal{M}^e_T([0, 1]) \).

Our proof of Theorem B is based on the work by Hofbauer and Raith ([13]) where density of periodic measures was proved for a piecewise monotonic map consisting of two monotonic pieces. Since the family of generalized \((\alpha, \beta)\)-transformations includes maps with more than three monotonic pieces, we need to improve their method. One key difference between [13] and ours is Proposition 3.3 which is one of the novelty of this paper (see also Remark 3.1).

The remainder of this paper is organized as follows. In §2, we establish our definitions and prepare several facts. Subsequently, we present proofs of Proposition A and Theorem B in §3.

2. Preliminaries

2.1. Symbolic dynamics. Let \( \mathbb{N}_0 \) be the set of non-negative integers. For a finite or countable set \( A \), we denote by \( A^{\mathbb{N}_0} \) the one-sided infinite product of \( A \) equipped with the product topology of the discrete topology of \( A \). To simplify the notation, given integers \( i \leq j \) and \( x_i, x_{i+1}, \ldots, x_j \in A \), we set
\[ x_{[i,j]} := x_i x_{i+1} \cdots x_j. \] We also write \( x_{[i,j]} := x_{[i,j-1]} \) and similarly for \( x_{(i,j)} \) and \( x_{(i,j)} \). A sequence \( x \in A^{N_0} \) will be defined by all of its coordinates \( x_n \in A \) with \( n \geq 0 \). Let \( \sigma \) be the shift map on \( A^{N_0} \) (i.e., \( \sigma(x)_n = x_{n+1} \) for each \( n \geq 0 \) and \( x \in A^{N_0} \)). When a subset \( \Sigma^+ \) of \( A^{N_0} \) is \( \sigma \)-invariant and closed, we call it a subshift and call \( A \) the alphabet of \( \Sigma^+ \). For a matrix \( M = (M_{ij})_{(i,j) \in A^2} \), each entry of which is 0 or 1, we define a subshift \( \Sigma_M^+ \subset A^{N_0} \) by

\[ \Sigma_M^+ = \{ x \in A^{N_0} : M_{x_n,x_{n+1}} = 1 \text{ for all } n \geq 0 \} \]

and call \( \Sigma_M^+ \) a Markov shift with an adjacency matrix \( M \).

For a subshift \( \Sigma^+ \) on an alphabet \( A \), we set \( \mathcal{L}(\Sigma^+) := \{ x_{[0,n]} : x \in \Sigma^+, n \geq 0 \} \) and \( [u] := \{ x \in \Sigma^+ : u = x_{[0,|u|]} \} \) for each \( u \in \mathcal{L}(\Sigma^+) \), where \( |u| \) denotes the length of \( u \). A word \( v \in \mathcal{L}(\Sigma^+) \) is called a subword of \( u = u_0 \cdots u_n \in \mathcal{L}(\Sigma^+) \) if \( v = u_{[i,j]} \) for some \( 0 \leq i \leq j \leq n \). For \( u, v \in \mathcal{L}(\Sigma^+) \), we use juxtaposition \( uv \) to denote the word obtained by the concatenation and \( u^\infty \) means a one-sided infinite sequence \( uu \cdots \in A^{N_0} \). Moreover, we set \( S(u) := \{ i \in A : ui \in \mathcal{L}(\Sigma^+) \} \) for \( u \in \mathcal{L}(\Sigma^+) \). Finally, we say that \( \Sigma^+ \) is transitive if for any \( u, v \in \mathcal{L}(\Sigma^+) \), we can find \( w \in \mathcal{L}(\Sigma^+) \) such that \( uvw \in \mathcal{L}(\Sigma^+) \) holds. For the rest of this paper, we denote by \( h_\sigma(\mu) \) the metric entropy of \( \mu \in \mathcal{M}_\sigma(\Sigma^+) \).

### 2.2. Markov diagram.

Let \( X = [0,1] \) and \( T : X \rightarrow X \) be a piecewise monotonic map with critical points \( 0 = c_0 < c_1 < \cdots < c_k = 1 \). Let \( X_T := \bigcap_{n=0}^\infty T^{-n}(\bigcup_{j=1}^k (c_{j-1}, c_j)) \), and define the coding map \( I : X_T \rightarrow \{1, \ldots, k\}^{N_0} \) by

\[ (I(x))_n = j \text{ if and only if } T^n(x) \in (c_{j-1}, c_j), \]

which is injective since \( T \) is transitive (see [27 Proposition 6.1]). We denote the closure of \( I(X_T) \) in \( \{1, \ldots, k\}^\mathbb{N} \) by \( \Sigma_T^+ \). Then, \( \Sigma_T^+ \) is a subshift, and \( (\Sigma_T^+, \sigma) \) is called the coding space of \( (X, T) \). We use the following notations:

- \( a^{(i)} := \lim_{x \rightarrow c_{i-1}+0} I(x), b^{(i)} := \lim_{x \rightarrow c_{i-1}-0} I(x) \) for \( 1 \leq i \leq k-1 \).
- \( \text{adj}(a^{(i)}) := b^{(i)}, \text{adj}(b^{(i)}) := a^{(i)} \) for \( 1 \leq i \leq k-1 \).
- \( a := \lim_{x \rightarrow +0} I(x), b := \lim_{x \rightarrow +1-0} I(x) \).

We also set \( \mathcal{C} \mathcal{R} := \{ a^{(i)}, b^{(i)} : 1 \leq i \leq k-1 \} \) and call it a critical set.

In what follows, we define the Markov diagram, introduced by Hofbauer ([10]), which is a countable oriented graph with subsets of \( \Sigma_T^+ \) as vertices. Let \( C \subset \Sigma_T^+ \) be a closed subset with \( C \subset [j] \) for some \( 1 \leq j \leq k \). We say that a non-empty closed subset \( D \subset \Sigma_T^+ \) is a successor of \( C \) if \( D = [l] \cap \sigma(C) \) for some \( 1 \leq l \leq k \). The expression \( C \rightarrow D \) denotes that \( D \) is a successor of \( C \). Now, we define a set \( D_T \) of vertices by induction. First, we set \( D_0 := \{[1], \ldots, [k]\} \). If \( D_n \) is defined for \( n \geq 0 \), then we set

\[ D_{n+1} := D_n \cup \{ D : D \text{ is a successor for some } C \in D_n \}. \]
We note that $D_n$ is a finite set for each $n \geq 0$ since the number of successors of any closed subset of $\Sigma_T^+$ is at most $k$ by definition. Finally, we set

$$D_T := \bigcup_{n \geq 0} D_n.$$ 

The oriented graph $(D_T, \to)$ is called the Markov diagram of $T$. For notational simplicity, we use the expression $\mathcal{D}$ instead of $D_T$ if no confusion arises.

For $x \in \Sigma_T^+$ and $n \geq 0$, we set $D^n_T := \sigma^n(x_{[0,n]})$. We define a sequence $\{R_n^x\}_{n \geq 0}$ of integers inductively as follows. First, we set $R_0^x := 0$. If $R_n^x$ is defined for $m \geq 0$, then let

$$R_{m+1}^x := \min\{n > R_m^x : \#(S(x_{[0,n-1]})) \geq 2\}.$$ 

We also set $r_m^x := R_m^x - R_{m-1}^x$. Now, we summarize properties of Hofbauer’s Markov Diagram, which are appeared in [12] (see also [13, Page 224]).

**Proposition 2.1.** Let $(\mathcal{D}, \to)$ be the Markov diagram of $T$.

1. $\mathcal{D} = \{D^n_T : x \in \mathcal{C} \cup \{a,b\}, n \geq 0\}$.
2. $(\mathcal{D}, \to)$ has the following arrows:
   - For any $x \in \mathcal{C} \cup \{a,b\}$ and any $n \geq 0$, $D^n_T \to D_{n+1}^x$.
   - For any $x \in \mathcal{C} \cup \{a,b\}$ and any $m \geq 1$, $D_{R_{m-1}^x}^x \to D_{R_{m}^x}^{f_m(x)}$. Here $f_m(x)$ is a unique point in $\mathcal{C}$ so that $f_m(x) \neq \sigma_{R_{m-1}^x}(x)$ and $f_m(x) \in D_{R_{m-1}^x}$ hold.
3. Let $x \in \mathcal{C} \cup \{a,b\}$.
   - For any $0 \leq n \leq R_{m-1}^x - 1$, we have $D_{n+R_{m-1}^x}^x \subset D^{f_m(x)}_{n+1}$. In particular, $x_{[R_{m-1}^x - 1, R_{m}^x]} = f_m(x)_{[0,R_{m}^x]}$ holds.
   - There exists an integer $q$ such that $r_{m}^x = f_{q}^{m}(x)$.
4. If $C \subseteq \mathcal{D}$ and $\#S(C) > 2$, then $S(C) \cap D_0 \neq \emptyset$.

For a subset $C \subseteq \mathcal{D}$, we define a matrix $M(C) = (M(C)_{C,D})_{(C,D) \in \mathcal{C}^2}$ by

$$M(C)_{C,D} = \begin{cases} 1 & (C \to D), \\ 0 & \text{otherwise}. \end{cases}$$

Then, $\Sigma^+_M(C) = \{C \in \mathcal{C}^{00} : C_n \to C_{n+1}, n \geq 0\}$ is a one-sided Markov shift with a countable alphabet $\mathcal{C}$ and an adjacency matrix $M(C)$. For notational simplicity, we denote $\Sigma^+_C$ instead of $\Sigma^+_M(C)$. We say that $C$ is irreducible if for any $C, D \in \mathcal{C}$, there are finite vertices $C_0, \ldots, C_n \in \mathcal{C}$ such that $C_i \to C_{i+1}$ for $0 \leq i \leq n - 1$ (i.e., $C_{[0,n]} \in L(\Sigma^+_C)$), $C_0 = C$ and $C_n = D$, and if every subset of $\mathcal{D}$, which contains $C$ does not have this property. It is clear that $\Sigma^+_C$ is transitive if and only if $C$ is irreducible. We define a map $\Psi : \Sigma^+_D \to \{1, \ldots, k\}^{\mathbb{N}_0}$ by

$$\Psi((C_n)_{n \in \mathbb{N}_0}) := (x_n)_{n \in \mathbb{N}_0} \text{ for } (C_n)_{n \in \mathbb{N}_0} \in \Sigma^+_D,$$
where $1 \leq x_n \leq k$ is a unique integer such that $C_n \subset [x_n]$ holds for each $n \in \mathbb{N}_0$. Then it is not difficult to see that $\Psi$ is continuous, countable to one and $\Psi(\Sigma^+_{\mathbb{C}}) = \Sigma^+_{\mathbb{T}}$. We say that $\mu \in \mathcal{M}_d(\Sigma^+_{\mathbb{T}})$ is liftable if there is $\overline{\mu} \in \mathcal{M}_d(\Sigma^+_{\mathbb{C}})$ such that $\mu = \overline{\mu} \circ \Psi^{-1}$ holds. It is known that not every $\mu \in \mathcal{M}_d(\Sigma^+_{\mathbb{T}})$ is liftable in general although $\Psi$ is surjective (see [14]). In [10], Hofbauer provided a sufficient condition for the liftable.

**Lemma 2.2.** ([10] Lemma 3) If $\mu \in \mathcal{M}_d(\Sigma^+_{\mathbb{T}})$ has positive metric entropy, then $\mu$ is liftable.

We recall two important facts for transitive piecewise monotonic maps.

**Lemma 2.3.** ([12, Theorem 11], [13, Page 224]) There exists an irreducible subset $C \subset D$ satisfying the following properties:

- $\Psi(\Sigma^+_{\mathbb{C}}) = \Sigma^+_{\mathbb{T}}$.
- $C \subset D$ and $D \rightarrow D$ implies $D \subset C$.
- There exists an integer $n_0$ such that $D^\mathbb{C}_{n_0} \subset C$ holds for any $x \in CR \cup \{a, b\}$.

**Theorem 2.4.** ([13, Theorem 1]) Suppose that there are integers $N_0$ and $N_1$ such that for any $x \in CR$, and any $j \in \mathbb{N}$ with $R_j > N_0$, there exist an integer $1 \leq m < j$, a periodic point $p$ with period $l$ and $u \in \mathcal{L}(\Sigma^+_{\mathbb{T}})$ with $|u| \geq R_j - R_{m} - N_1$ such that $u$ is a subword of both $x_{[R_m], R_j}$ and $p_{[0, l]}$. Then $\mathcal{M}_d(0, 1)$ is dense in $\mathcal{M}_d(0, 1)$.

Hereafter, let $T : [0, 1] \rightarrow [0, 1]$ be a generalized $(\alpha, \beta)$-transformation and $k$ be a smallest integer not less than $\alpha + \beta$. To simplifies the notation, we set $A_0 := \sigma^n([a_{[0, n]}])$ and $B_0 := \sigma^n([b_{[0, n]}])$ for each $n \geq 0$ and set $R_m := R_m$, $S_m := R_m - R_{m+1} - S_{m+1} := S_{m+1} - S_m$ for each $m \geq 0$. We also denote

$A_1 := \{m \in \mathbb{N} : \sigma(f_m(a)) = a\}, \quad A_2 := \{m \in \mathbb{N} : \sigma(f_m(b)) = b\},$

$B_1 := \{m \in \mathbb{N} : \sigma(f_m(a)) = a\}$ and $B_2 := \{m \in \mathbb{N} : \sigma(f_m(b)) = b\}$.

Note that $\sigma[i] = \Sigma^+_{\mathbb{T}}$ for each $2 \leq i \leq k-1$ and $\sigma(x) \in \{a, b\}$ for any $x \in CR$. These together with Proposition 2.1 and Theorem 2.4 imply the following:

**Proposition 2.5.** Let $(D, \rightarrow)$ be the Markov diagram of $T$.

1. $D = \{A_n, B_n : n \geq 0\} \cup \{[2], \ldots, [k-1]\}$.

2. (i) For any $n \geq 0$, $A_n \rightarrow A_{n+1}$ and $B_n \rightarrow B_{n+1}$.

(ii) For $m \geq 1$,

- $A_{R_m-1} \rightarrow A_{R_m-1}$ and $a_{(R_m-1, R_m]} = a_{[0, R_m]}$ if $m \in A_1$,
- $A_{R_m-1} \rightarrow B_{R_m-1}$ and $a_{(R_m-1, R_m]} = b_{[0, R_m-1]}$ if $m \in A_2$.

(iii) For $m \geq 1$,

- $B_{S_m-1} \rightarrow A_{S_m-1}$ and $b_{(S_m-1, S_m]} = a_{[0, R_m-1]}$ if $m \in B_1$, 

• $B_{s_m - 1} \rightarrow B_{s_m - 1}$ and $b_{(s_m - 1, s_m)} = b_{(0, s_m - 1)}$ if $m \in B_2$.

(3) There exist two maps $P : A_2 \rightarrow \mathbb{N}_0$ and $Q : B_1 \rightarrow \mathbb{N}_0$ such that for any $m \geq 1$, $r_m - 1 = S_{p(m)}$ and $s_m - 1 = R_{Q(m)}$.

**Theorem 2.6.** Suppose that there are integers $N_0$ and $N_1$ satisfying the following conditions:

- For any $j \in \mathbb{N}$ with $R_j > N_0$, there is an integer $1 \leq m < j$, a periodic point $p$ with period $l$ and $u \in \mathcal{L}(\Sigma_T^+)$ with $|u| \geq R_j - R_m - N_1$ such that $u$ is a subword of both $a_{(R_m, R_j)}$ and $p_{[0,l]}$.
- For any $j \in \mathbb{N}$ with $S_j > N_0$, there is an integer $1 \leq m < j$, a periodic point $p$ with period $l$ and $u \in \mathcal{L}(\Sigma_T^+)$ with $|u| \geq S_j - S_m - N_1$ such that $u$ is a subword of both $b_{(R_m, R_j)}$ and $p_{[0,l]}$.

Then $M_T^\mu([0,1])$ is dense in $M_T^\mu([0,1])$.

3. Proofs

3.1. **Proof of Proposition A.** In this subsection, we give a proof of Proposition A with the assumption that $T$ is transitive. As we mentioned in §1, this theorem appears as [5, Theorem A] with the stronger assumption that $D$ is irreducible. In [5], the hypothesis of the irreducibility for $D$ is used only in [5, Proposition 3.1]. The other part of the proof of [5, Theorem A] can be shown similarly by using the transitivity of $T$ instead of the irreducibility. Hence to prove Proposition A, it is sufficient to show the following proposition, which is analogous to [5, Proposition 3.1].

**Proposition 3.1.** Let $\mathcal{C} \subset D$ be as in Lemma 2.3. For any $\epsilon > 0$, any $\mu \in M_\sigma^e(\Sigma_T^+)$, and any neighborhood $\mathcal{U} \subset M(\Sigma_T^+)$ of $\mu$, there exist a finite set $F \subset \mathcal{C}$ and $\rho \in M_\sigma^e(\Psi(\Sigma_T^+))$ such that $\rho \in \mathcal{U}$ and $h_\sigma(\rho) \geq h_\sigma(\mu) - 2\epsilon$.

Proof. Without loss of generality, we may assume that $h_\sigma(\mu) - \epsilon > 0$, otherwise the conclusion is yield by the assumption that $M_\sigma^e(\Sigma_T^+)$ is dense in $M_\sigma^e(\Sigma_T^+)$. By Ergodic Decomposition Theorem and the affinity of the entropy map, there exists a finite convex combination of ergodic measures $\nu := \sum_{i=1}^p a_i \nu_i$ such that $h_\sigma(\nu) \in \mathcal{U}$ and $h_\sigma(\nu) \geq h_\sigma(\mu) - \epsilon$. Again by density of $M_\sigma^e(\Sigma_T^+)$ in $M_\sigma^e(\Sigma_T^+)$, we may assume $\nu_i \in M_\sigma^e(\Sigma_T^+)$ whenever $h_\sigma(\nu_i) = 0$. Then we need the following lemma.

**Lemma 3.2.** For each $1 \leq i \leq p$, there exists $\tilde{\nu}_i \in M_\sigma^e(\Sigma_T^+)$ such that $\nu_i = \tilde{\nu}_i \circ \Psi^{-1}$ holds. In particular, $\nu := \sum_{i=1}^p a_i \tilde{\nu}_i$ satisfies $\nu \in M_\sigma(\Sigma_C^+)$ and $\nu = \nu \circ \Psi^{-1}$.

Proof. We divide the proof into two cases.

**Case 1** $h_\sigma(\nu_i) = 0$. In this case, $\nu_i$ is a periodic measure. Take a periodic point $\bar{x} \in \Sigma_T^+$ in the support of $\nu_i$. Then it follows from [12, Theorem 8] and $\Psi(\Sigma_C^+) = \Sigma_T^+$ that there are finite vertices $C_0, \ldots, C_{n-1} \in \mathcal{C}$ such that $(C_{[0,n)})^\infty \in \Sigma_T^+$ and $\Psi((C_{[0,n)})^\infty) = \bar{x}$. Hence if we set $\tilde{\nu}_i := \delta_{\sigma_n}((C_{[0,n)})^\infty)$, then we have $\tilde{\nu}_i \in M_\sigma^e(\Sigma_C^+)$. 

**Case 2** $h_\sigma(\nu_i) > 0$. In this case, $\nu_i$ is an ergodic measure. Take a periodic point $\bar{x} \in \Sigma_T^+$ in the support of $\nu_i$. Then it follows from [12, Theorem 8] and $\Psi(\Sigma_C^+) = \Sigma_T^+$ that there are finite vertices $C_0, \ldots, C_{n-1} \in \mathcal{C}$ such that $(C_{[0,n)})^\infty \in \Sigma_T^+$ and $\Psi((C_{[0,n)})^\infty) = \bar{x}$. Hence if we set $\tilde{\nu}_i := \delta_{\sigma_n}((C_{[0,n)})^\infty)$, then we have $\tilde{\nu}_i \in M_\sigma^e(\Sigma_C^+)$. 

**Proof.** We divide the proof into two cases.

**Case 1** $h_\sigma(\nu_i) = 0$. In this case, $\nu_i$ is a periodic measure. Take a periodic point $\bar{x} \in \Sigma_T^+$ in the support of $\nu_i$. Then it follows from [12, Theorem 8] and $\Psi(\Sigma_C^+) = \Sigma_T^+$ that there are finite vertices $C_0, \ldots, C_{n-1} \in \mathcal{C}$ such that $(C_{[0,n)})^\infty \in \Sigma_T^+$ and $\Psi((C_{[0,n)})^\infty) = \bar{x}$. Hence if we set $\tilde{\nu}_i := \delta_{\sigma_n}((C_{[0,n)})^\infty)$, then we have $\tilde{\nu}_i \in M_\sigma^e(\Sigma_C^+)$. 

**Case 2** $h_\sigma(\nu_i) > 0$. In this case, $\nu_i$ is an ergodic measure. Take a periodic point $\bar{x} \in \Sigma_T^+$ in the support of $\nu_i$. Then it follows from [12, Theorem 8] and $\Psi(\Sigma_C^+) = \Sigma_T^+$ that there are finite vertices $C_0, \ldots, C_{n-1} \in \mathcal{C}$ such that $(C_{[0,n)})^\infty \in \Sigma_T^+$ and $\Psi((C_{[0,n)})^\infty) = \bar{x}$. Hence if we set $\tilde{\nu}_i := \delta_{\sigma_n}((C_{[0,n)})^\infty)$, then we have $\tilde{\nu}_i \in M_\sigma^e(\Sigma_C^+)$. 

**Proof.** We divide the proof into two cases.
(Case 2) $h_\sigma(\nu_i) > 0$. Since $\nu_i \in M_\sigma^c(\Sigma_C^+)$ and $h_\sigma(\nu_i) > 0$, it follows from Lemma 2.2 that there exists $\nu \in M_\sigma^c(\Sigma_D^+)$ such that $\nu := \nu_i \circ \Psi^{-1}$ holds. Since $\nu$ is ergodic, there exists an irreducible subset $C' \subset D$ such that $\nu(C') = 1$. Assume that $\mathcal{C}' \neq \mathcal{C}$. Since $\Sigma_C^+ \subset \Psi^{-1}(\Psi(\Sigma_C^+))$, we have $\nu_i(\Psi(\Sigma_C^+)) = \nu \circ \Psi^{-1}(\Psi(\Sigma_C^+)) \geq \nu(\Sigma_C^+) = 1$. This implies that $\nu_i(\Psi(\Sigma_C^+) \cap \Psi(\Sigma_C^+)) = \nu_i(\Sigma_C^+ \cap \Psi(\Sigma_C^+)) = 1$. On the other hand, by Theorem 1 (ii), $\Psi(\Sigma_C^+) \cap \Psi(\Sigma_C^+)$ is either empty or finite (see also Page 385). Hence, by the ergodicity of $\nu_i$, we have $h_\sigma(\nu_i) = 0$, which is a contradiction.

Note that $\bar{\nu}$ is an invariant measure on a transitive countable Markov shift $\Sigma_C^+$. Hence by the continuity of $\Psi$ and Main Theorem [25], we can find a finite set $F \subset \mathcal{C}$ and an ergodic measure $\bar{\nu}$ on $\Sigma_C^+$ such that $h_\sigma(\bar{\nu}) \geq h_\sigma(\bar{\nu}) - \epsilon$ and $\rho := \bar{\nu} \circ \Psi^{-1} \in \mathcal{U}$. Since $\Psi : \Sigma_C^+ \rightarrow \Sigma_D^+$ is countable to one, by Proposition 2.8, we have $h_\sigma(\nu) = h_\sigma(\bar{\nu})$ and $h_\sigma(\rho) = h_\sigma(\bar{\rho})$, which prove the proposition.

3.2. Proof of Theorem [3] The aim of this subsection is to give a proof of Theorem [3]. Let $n_0$ be as in Lemma 2.3, and $m_0$ be a smallest integer such that $R_{m_0}, S_{m_0} \geq n_0$ holds. We set $N_0 := \max\{R_{m_0}, S_{m_0}\}$ and let $n_1$ be a smallest number such that for any $C \in \mathcal{C} \cap \mathcal{D}_{n_0}$ and $D \in \mathcal{C} \cap \mathcal{D}_{N_0}$, there exist finite vertices $C_0, \ldots, C_n$ with $n \leq n_1$ such that $C_{\{0, n\}} \in \mathcal{L}(\Sigma_C^+)$, $C_0 = C$ and $C_n = D$. We note that $n_1 < \infty$ since $\mathcal{C}$ is irreducible. We set $N_1 := N_0 + n_1$. By Theorem 2.6, to prove Theorem 3, it is sufficient to show the following:

(I) For any $j \in \mathbb{N}$ with $R_j > N_0$, there is an integer $1 \leq m < j$, a periodic point $p$ with period $l$ and $u \in \mathcal{L}(\Sigma_C^+)$ with $|u| \geq R_j - R_m - N_1$ such that $u$ is a subword of both $a_{\{R_m, R_j\}}$ and $p_{\{0, l\}}$.

(II) For any $j \in \mathbb{N}$ with $S_j > N_0$, there is an integer $1 \leq m < j$, a periodic point $p$ with period $l$ and $u \in \mathcal{L}(\Sigma_C^+)$ with $|u| \geq S_j - S_m - N_1$ such that $u$ is a subword of both $b_{\{R_m, R_j\}}$ and $p_{\{0, l\}}$.

We only prove the item (I) because (II) can be shown in a similar manner. Take any $j \in \mathbb{N}$ with $R_j > N_0$. In what follows we will decompose the set $\mathbb{N}$ into six sets. Let $A_1, A_2, B_1$ and $B_2$ be as in §2.3 and set $A_3 := \{m \in \mathbb{N} : \mathcal{S}(A_{R_m-1}) \cap \mathcal{D}_0 \neq \emptyset\}$ and $B_3 := \{m \in \mathbb{N} : \mathcal{S}(B_{S_m-1}) \cap \mathcal{D}_0 \neq \emptyset\}$.

First, it is clear that $\mathbb{N} = A_1 \cup A_2 = A_1 \cup (A_2 \setminus A_3) \cup A_3$. We set $A_2^{(1)} := \{m \in A_2 : m + 1 \in A_1\}$, $A_2^{(2)} := \{m \in A_2 : P(m) \in B_3\}$, $A_2^{(3)} := \{m \in A_2 : P(m) + 1 \in B_2\}$, and $A_4 := A_2 \setminus (A_3 \cup \bigcup_{j=1}^{j=3} A_2^{(j)}).$
noting that the map $P$ is defined on $A_2$ by Proposition 2.5 (3).

Then we have $A_2 \setminus A_3 \subseteq \bigcup_{j=1}^{3} A_2^{(j)} \cup A_4$, which implies that

$$N = A_1 \cup \bigcup_{j=1}^{3} A_2^{(j)} \cup A_3 \cup A_4.$$ 

Hence we can divide the proof into six cases:

(Case A) $j \in A_1$.    
(Case B) $j \in A_2^{(1)}$.    
(Case C) $j \in A_2^{(2)}$.    
(Case D) $j \in A_2^{(3)}$.    
(Case E) $j \in A_3$.    
(Case F) $j \in A_4$.

We note that the proofs of (Case A) and (Case B) are similar to those of (Case 1) and (Case 2) in [13] respectively.

(Case C) $j \in A_2^{(2)}$. Take $D \in S(B_{S_{(j)}-1}) \cap D_0$. By the definition of $n_1$, we can find an integer $n \leq n_1$ and finite vertices $C_0, \ldots, C_n \in C$ such that $C_{[0,n]} \in \mathcal{L}(\Sigma^n)$. Take $p := \Psi((B_{[s_{m_0},s_{(j)}]} C_{[0,n]})^\infty)$. Then $p$ is a periodic point with period $l := S_{(j)} - S_{m_0} + n$. Since $A_{R_j-1} \rightarrow B_{S_{(j)}}$, we have $a(R_{j-1,R_j}) = b([0,S_{(j)})$, which implies (I).

(Case D) $j \in A_2^{(3)}$. Since $P(j) + 1 \in B_2$, we can find $u \leq P(j)$ such that $B_{S_{(j)}+1} \rightarrow B_u$. We set $p := \Psi((B_{(S_{(j)},S_{(j)}+1)} B_{[u,S_{(j)}]})^\infty)$. Then $p$ is a periodic point with period $l := S_{(j)} + u - u$. Since $A_{R_j-1} \rightarrow B_{S_{(j)}}$ and $B_{S_{(j)}+1} \rightarrow B_u$, we have $a(R_{j-1,R_j}) = b([0,S_{(j)}) = b([0,u] b([u,S_{(j)}]) = b([S_{(j)},S_{(j)}+1] b([u,S_{(j)}])$. This implies (I).

(Case E) $j \in A_3$. Take $D \in S(A_{R_j-1}) \cap D_0$. By the definition of $n_1$, we can find an integer $n \leq n_1$ and finite vertices $C_0, \ldots, C_n \in C$ such that $C_{[0,n]} \in \mathcal{L}(\Sigma^n)$. Take $p := \Psi((A_{[R_{m_0},R_j]} C_{[0,n]})^\infty)$, then (I) holds.

(Case F) $j \in A_4$. In this case, $j, j + 1 \in A_2$, $P(j) + 1 \in B_1$, $j \notin A_3$ and $P(j) \notin B_3$ hold by the definition of $A_4$. We prove the following proposition, which plays a fundamental role to prove this case.

**Proposition 3.3.** We have

$$S_{(j)}^\infty \triangleright \{r_j+i-1\}_{i=1}^\infty \quad \text{or} \quad R_j^\infty \triangleright \{s_{P(j)+i-1}\}_{i=1}^\infty.$$ 

Here $\triangleright$ denotes the lexicographical order.

Proof. For the notational simplicity, set $r^{(m)} := \{r_{m+i}-1\}_{i=1}^\infty$ and $s^{(m)} := \{s_{m+i}-1\}_{i=1}^\infty$ for $m \geq 1$. Showing by contradiction, we assume

$$s_{(j)}^\infty \preceq r^{(j)} \quad \text{and} \quad R_j^\infty \preceq s^{(j)}. \quad (3.1)$$
Figure 2. Sketch of the Markov diagram in Case 1.

For $q \geq R_j + 1$ set $\mathcal{E}_{q,1} := \{m : R_j \leq R_m < q\}$ and $\mathcal{E}_{q,2} := \{m : S_P(j) \leq R_m < q\}$. Denote also $x := f_{j+1}(a)$ and $y := \text{adj}(x) = f_{P(j)+1}(b)$. Note that $f_m(a) = x$ implies $m \in A_2$. Similarly $f_m(b) = y$ implies $m \in B_1$. To prove the proposition, it is sufficient to show the following for all $q \geq R_j + 1$:

- For any $m \in \mathcal{E}_{q,1}$, we have
  \[ m \notin A_3, \quad f_{m+1}(a) = x \quad \text{and} \quad S_P^\infty(j) \preceq r(m). \]  \hspace{1cm} (3.2)

- For any $m \in \mathcal{E}_{q,2}$, we have
  \[ m \notin B_3, \quad f_{m+1}(b) = y \quad \text{and} \quad R^\infty_j \preceq s(m). \]  \hspace{1cm} (3.3)

Indeed, letting $C' := \{A_m : m \geq R_j\} \cup \{B_m : m \geq S_P(j)\}$, we have $\bigcup_{C \subseteq C'} S(C) \cap (C \setminus C') = \emptyset$ by (3.2) and (3.3). However, this is a contradiction since $C$ is irreducible and $A_{R_0} \in C \setminus C'$.

We prove (3.2) and (3.3) by induction with regard to $q \geq R_j + 1$. Let $q = R_j + 1$. Since $R_{j+1} \geq R_j + 1$, $j + 1 \notin \mathcal{E}_{R_{j+1},1}$. Similarly, $P(j) + 1 \notin \mathcal{E}_{R_{j+1},2}$ because $S_{P(j)+1} > s_{P(j)+1} \geq R_j$. Hence we have $\mathcal{E}_{R_{j+1},1} = \{j\}$ and $\mathcal{E}_{R_{j+1},2} = \{P(j)\}$. Hence (3.2) and (3.3) immediately follows by (3.1).

Let $q > R_j + 1$ and assume the inductive hypothesis for $q - 1$. Take $m \in \mathcal{E}_{q,1} \setminus \mathcal{E}_{q-1,1}$ (if no such $m$ exists, (3.2) automatically holds for all $m \in \mathcal{E}_{q,1}$). We will show that (3.2) holds for $m$. Since $m \in \mathcal{E}_{q,1}$, we have $R_m < q$ and hence $m - 1 \in \mathcal{E}_{q-1,1}$. Therefore, by the inductive hypothesis, we have $m - 1 \notin A_3$, $f_{m}(a) = x$ and $S_P^\infty(j) \preceq r(m-1)$.

**Case 1** $S_{P(j)} = r_m - 1$ (for the situation of the Markov diagram, see Figure 2). In this case, we have $S_{P(j)} = r_m - 1 = S_{P(m)}$. This implies $A_{R_m-1} \subset B_{S_{P(j)-1}}$ and $\#S(A_{R_m-1}) \leq \#S(B_{S_{P(j)-1}}) = 2$. Hence we have $m \notin A_3$. Moreover, $f_{m+1}(a) = x$ is followed by $S(A_{R_m-1}) = \{A_{R_m}, B_{S_{P(j)}}\}$
and \( f_{S_{P(j)} + 1}(b) = y \). The last condition \( S_{P(j)}^\infty \leq r^{(m)} \) follows from \( S_{P(j)} = r_m - 1 \) and \( S_{P(j)}^\infty \leq r^{(m-1)} \).

**Case 2** \( S_{P(j)} < r_m - 1 \) (For the situation of the Markov diagram, see Figure 3.3). In this case, we have \( S_{P(j)} < r_m - 1 = S_{P(m)} < R_m = q - 1 \), which implies that \( P(m) - 1, P(m) \in E_{q-1,2} \). Hence by the inductive hypothesis (3.3) for \( q - 1 \), we have \( P(m) - 1, P(m) \notin B_3, P(m), P(m) + 1 \in B_1 \) and \( R_j \leq s(P(m) - 1), s(P(m)) \). Since \( A_{R_m - 1} \rightarrow B_{S_{P(m)}} \), we have \#\( S(A_{R_m - 1}) \leq \#S(B_{S_{P(m) - 1}}) = 2 \), which implies \( m \notin \mathcal{A}_3 \). Moreover, \( S(A_{R_m - 1}) = \{ A_{R_m}, B_{S_{P(m)}} \} \) implies

\[
f_{m+1}(a) = \text{adj}(f_{S_{P(m)}+1}(b)) = \text{adj}(y) = x.
\]

Now we consider the last condition \( S_{P(j)}^\infty \leq r^{(m)} \). Since \( Q(P(m)) = s_{P(m)} - 1 \geq R_j \) and \( Q(P(m)) < s_{P(m)} < S_{P(m)} < R_m = q - 1 \), (3.2) holds for \( Q(P(m)) \). If \( r(Q(P(m))) = r^{(m)} \), (3.2) yields \( S_{P(j)}^\infty \leq r(Q(P(m))) = r^{(m)} \). Otherwise let \( \ell = \inf \{ i \geq 1 : r(Q_{P(m)})+i \neq r_{m+i} \} \). It is easy to see \( S_{P(j)}^\infty \leq r^{(m)} \) is yield by the following: For \( i = 0, 1, \ldots, \ell \), we have

\[
A_{R_m+i} \subset A_{R_{Q(P(m))}+i} \quad \text{and} \quad \#\tau(A_{R_m+i-1}) = \#\tau(A_{R_{Q(P(m))}+i-1}) = 2 \quad (3.4)
\]

In particular, (3.4) implies \( r_{Q(P(m))}+\ell+1 < r_{m+\ell+1} \) and \( r(Q(P(m)) + 1) \leq r^{(m)} \). We close our proof of (3.2) by showing (3.4). Since \( A_{R_m - 1} \rightarrow B_{S_{P(m)}} \) and \( B_{S_{P(m)} - 1} \rightarrow A_{R_{Q(P(m))}} \), we have

\[
A_{R_m - 1} \subset B_{S_{P(m)} - 1} \subset A_{R_{Q(P(m))}}
\]

Moreover, \( f_{m+1}(a) = f_{Q(P(m)) + 1} = x \) implies \( A_{R_m}, A_{R_{Q(P(m))}} \subset [x_0] \) and \( A_{R_m} \subset A_{R_{Q(P(m))}} \). Since (3.2) holds for \( Q(P(m)) \), we have \#\( S(A_{R_m - 1}) = \#S(A_{R_{Q(P(m))} - 1}) = 2 \).
Let $0 < i \leq \ell$ and assume \([3.4]\) holds for all $0 \leq i' < i$. Since $A_{R_{m+i-1}} \subset A_{R_Q(P(m)) +i-1}$ and $r_Q(P(m)) +i-1 = r_{m+i-1}$, we have $A_{R_{m+i-1}} \subset A_{R_Q(P(m)) +i-1}$. Since $A_{R_Q(P(m))} < S_P(m) < R_m$ yields

$$R_{Q(P(m)) +i} = R_{Q(P(m))} + \sum_{i'=1}^{i} r_{Q(P(m)) +i'}$$

$$= R_{Q(P(m))} + \sum_{i'=1}^{i} r_{m+i'}$$

$$< R_m + R_{m+i} - R_m = R_{m+i},$$

we have $\#S(A_{R_{m+i-1}}) = \#S(A_{R_Q(P(m)) +i-1}) = 2$. Combining the definition of $\ell$ and the $\#S$ for $\ell$, we have $r_Q(P(m)) +i+1 < r_{m+\ell+1}$. We can prove \([3.3]\) for all $l \in E_{q,2}$ in a similar manner, which proves the proposition.

We continue to the proof of Theorem \([B]\). By Proposition \([3.3]\) we can divide (Case F) into the following three cases:

(Case F1) $R_{J}^{\infty} \succ \{s_{P(j)+i} - 1\}_{i=1}^{\infty}$.
(Case F2) $S_{P(j)}^{\infty} \succ \{r_{j+i} - 1\}_{i=1}^{\infty}$ and $P(j) - 1 \in B_1$.
(Case F3) $S_{P(j)}^{\infty} \succ \{r_{j+i} - 1\}_{i=1}^{\infty}$ and $P(j) - 1 \in B_2$.

The proofs of (Case F1), (Case F2) and (Case F3) are similar to those of (Case 3), (Case 4) and (Case 5) in \([13]\), respectively. Theorem \([B]\) is proved.

Remark 3.1. In our proof of Theorem \([B]\) we improve the method in Hofbauer and Raith’s work \([13]\) to apply generalized $(\alpha, \beta)$-transformations. For a piecewise monotonic map with two monotonic pieces, its critical set $CR$ consists of two points which are clearly adjacent. Moreover, its Markov diagram has no vertices at which more than three edges start and a word represented by a path without branching vertices coincides with the word of either of the points in $CR$. On the other hand, for a generalized $(\alpha, \beta)$-transformation, there may exist a vertex at which more than three edges start in its Markov diagram. Moreover it is difficult to check which critical point represent a word defined on a path without branching vertices. Therefore, in the proof of Proposition \([3.3]\) we need to be careful to check that every vertex has two or less edges and a pair of adjacent critical points, $x = f_j(a)$ and $y = adj(x) = f_{P(j+1)}(b)$, appears in each inductive step.

Acknowledgement. The authors would like to thank Hajime Kaneko for suggesting this problem. The first author was partially supported by JSPS KAKENHI Grant Number 21K13816 and the second author was partially supported by JSPS KAKENHI Grant Number 21K03321.
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