HOMOLOGICAL MIRROR SYMMETRY OF $\mathbb{C}P^N$ AND THEIR PRODUCTS VIA MORSE HOMOTOPY

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Abstract. We propose a way of understanding homological mirror symmetry when a complex manifold is a smooth compact toric manifold. So far, in many example, the derived category $D^b(\text{coh}(X))$ of coherent sheaves on a toric manifold $X$ is compared with the Fukaya-Seidel category of the Milnor fiber of the corresponding Landau-Ginzburg potential. We instead consider the dual torus fibration $\pi : M \to B$ of the complement of the toric divisors in $X$, where $\bar{B}$ is the dual polytope of the toric manifold $X$. A natural formulation of homological mirror symmetry in this set-up is to define $\text{Fuk}(\bar{M})$, a variant of the Fukaya category and show the equivalence $D^b(\text{coh}(X)) \simeq D^b(\text{Fuk}(\bar{M}))$. As an intermediate step, we construct the category $\text{Mo}(P)$ of weighted Morse homotopy on $P := \bar{B}$ as a natural generalization of the weighted Fukaya-Oh category proposed by Kontsevich-Soibelman [14]. We then show a full subcategory $\text{Mo}_\varepsilon(P)$ of $\text{Mo}(P)$ generates $D^b(\text{coh}(X))$ for the cases $X$ is a complex projective space and their products.

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1. Introduction

In this paper, we propose a way of understanding homological mirror symmetry for the case of smooth compact toric manifolds. So far, in many example, the derived category $D^b(coh(X))$ of coherent sheaves on a toric manifold $X$ is compared with the Fukaya-Seidel category of the Milnor fiber of the corresponding Landau-Ginzburg potential. In this paper, we consider the dual torus fibration $\pi : M \to B$, in the sense of Strominger-Yau-Zaslow construction \cite{SYZ}, of the complement of the toric divisors in $X$, where $P = \bar{B}$ is the dual polytope of the toric manifold $X$. In \cite{Fang}, Fang discusses homological mirror symmetry of $\mathbb{C}P^n$ along this line. There, he starts with considering line bundles on $\mathbb{C}P^n$ and the corresponding Lagrangians in the mirror dual side. His idea of discussing the homological mirror symmetry is to consider the category of constructible sheaves as an intermediate step. We instead apply Kontsevich-Soibelman’s approach \cite{KS} to our case and consider a category $Mo(P)$ of Morse homotopy. Namely, a natural formulation of homological mirror symmetry in our situation is to define a variant of Fukaya category $Fuk(\bar{M})$ and show the equivalence $D^b(coh(X)) \simeq D^b(Fuk(\bar{M}))$. As an intermediate step, we construct the category $Mo(P)$ of weighted Morse homotopy on $P$ as a natural generalization of the weighted Fukaya-Oh category proposed in \cite{KS}. We then show a full subcategory $Mo_F(P)$ of $Mo(P)$ generates $D^b(coh(X))$ for the case $X$ is a complex projective space and their products. For more general $X$, we may consider the Lagrangian sections discussed in \cite{Chan}, where Chan discusses the correspondence between holomorphic line bundles over projective toric manifolds and Lagrangian sections in the mirror dual. The relation of such an approach with Abouzaid’s one \cite{Abouzaid} is also mentioned there.

As a formulation of the homological mirror symmetry for non-Calabi-Yau situations based on the SYZ construction, Fukaya discusses in \cite{Fukaya} how to treat the singular fibers of the torus fibrations where the Kähler metrics go zero at singular fibers. In our set-up, the Kähler metrics go infinity at the boundaries $\partial(P)$. We believe our category $Mo(P)$ may be a correct candidate for such a situation.
This paper is organized as follows. In section 2 we recall the SYZ torus fibration set-up [18] following [16, 15]. There, a pair of dual torus fibrations $M \to B$ and $\tilde{M} \to B$ is defined. We use this set-up by identifying $\tilde{M}$ with the complement of the toric divisors in a toric manifold $X$. In section 3 we recall the correspondence of Lagrangian sections of $M \to B$ and holomorphic line bundles on $\tilde{M}$, again, following [16, 15]. In the last subsection, we demonstrate a Lagrangian section to be derived from the line bundle $\mathcal{O}(k)$ restricted on the complement of the toric divisors for $X = \mathbb{C}P^n$. by the correspondence above. In section 4, we first recall DG-categories $\mathcal{F}(M)$ and $\mathcal{V}(\tilde{M})$ associated to $M$ and $\tilde{M}$, respectively, in [11]. Kontsevich-Soibelman’s approach for the homological mirror symmetry [14] proposes an intermediate category $Mo(B)$ and the existence of an $A_\infty$-equivalence

$$Fuk(M) \simeq Mo(B) \overset{\sim}{\longrightarrow} \mathcal{F}(M).$$

This $Mo(B)$ is called the weighted Fukaya-Oh category or the category of weighted Morse homotopy on $B$. In subsection [15] we propose a modification $Mo(P)$ of $Mo(B)$ where $P = \bar{B}$ is the dual polytope of a smooth compact toric manifold $X$. In the last section, we discuss the correspondence between $D^b(\text{coh}(X))$ and $Mo(P)$ when $X$ is a complex projective space or their products. In particular, we see that we can take strongly exceptional collections $\mathcal{E}$ of $D^b(\text{coh}(X))$ consisting of line bundles and the corresponding full subcategory $Mo_\mathcal{E}(P)$ of $Mo(P)$ so that

$$Tr(Mo_\mathcal{E}(P)) \simeq D^b(\text{coh}(X))$$

where $Tr$ is the Bondal-Kapranov-Kontsevich construction of triangulated categories from $A_\infty$-categories.

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2. Toric manifolds and $T^n$-invariant manifolds

2.1. Dual torus fibrations. In this subsection, we briefly review the SYZ torus fibration set-up [18]. For more details see [16, 15]. We follow the convention of [11].

Throughout this section, we consider an $n$-dimensional tropical Hessian manifold $B$, which we will define shortly, as the base space of a torus fibration. A smooth manifold $B$ is called affine if $B$ has an open covering $\{U_\lambda\}_{\lambda \in \Lambda}$ such that the coordinate transformation is affine. This means that, for any $U_\lambda$ and $U_\mu$ such that $U_\lambda \cap U_\mu \neq \emptyset$, the coordinate systems $x_\lambda := (x_1^\lambda, \ldots, x_n^\lambda)$ and $x_\mu := (x_1^\mu, \ldots, x_n^\mu)$ are related to each other by

$$x_\mu = \varphi_\mu x_\lambda + \psi_\mu$$

with some $\varphi_\mu \in GL(n; \mathbb{R})$ and $\psi_\mu \in \mathbb{R}^n$. If in particular $\varphi_\mu \in GL(n, \mathbb{Z})$ for any $U_\lambda \cap U_\mu$, then $B$ is called tropical affine. (If in addition $\psi_\mu \in \mathbb{Z}^n$, $B$ is called integral affine.)
For simplicity, we take such an open covering \( \{ U_\lambda \}_{\lambda \in \Lambda} \) so that the open sets \( U_\lambda \) and their intersections are all contractible. It is known that \( B \) is an affine manifold iff the tangent bundle \( TB \) is equipped with a torsion free flat connection. When \( B \) is affine, then its tangent bundle \( TB \) forms a complex manifold. This fact is clear as follows. For each open set \( U = U_\lambda \), let us denote by \((x^1, \ldots, x^n; y^1, \ldots, y^n)\) the coordinates of \( U \times \mathbb{R}^n \cong TB|_U \) so that a point \( \sum_{i=1}^n y^i \frac{\partial}{\partial x^i} x \in T_x B \subset TB \) corresponds to \((x^1, \ldots, x^n; y^1, \ldots, y^n) \in U \times \mathbb{R}^n\).

We locally define the complex coordinate system by \((z^1, \ldots, z^n)\), where \( z^i := x^i + iy^i \) with \( i = 1, \ldots, n \). By the coordinate transformation \((\varphi)\), the bases are transformed by

\[
\frac{\partial}{\partial x^i} \rightarrow (\varphi^t_{\lambda \mu})^{-1} \frac{\partial}{\partial x^i} = \frac{\partial}{\partial \varphi^i_{\lambda \mu}}, \quad \frac{\partial}{\partial x^j} := (\frac{\partial}{\partial \varphi^j_{\lambda \mu}})^t,
\]

and hence the corresponding coordinates are transformed by

\[
y(\mu) := \varphi_{\lambda \mu} y(\lambda), \quad y := (y^1, \ldots, y^n)^t
\]

so that the combination \( \sum_i y^i \frac{\partial}{\partial x^i} \) is independent of the coordinate systems. This shows that the transition functions for the manifold \( TB \) are given by

\[
\begin{pmatrix}
(x(\mu)) \\
y(\mu)
\end{pmatrix} = \begin{pmatrix}
\varphi_{\lambda \mu} & 0 \\
0 & \varphi_{\lambda \mu}
\end{pmatrix} \begin{pmatrix}
x(\lambda) \\
y(\lambda)
\end{pmatrix} + \begin{pmatrix}
\psi_{\lambda \mu} \\
0
\end{pmatrix},
\]

and hence the complex coordinate systems are transformed holomorphically:

\[
z(\mu) = \varphi_{\lambda \mu} z(\lambda) + \psi_{\lambda \mu}.
\]

On the other hand, for any smooth manifold \( B \), the cotangent bundle \( T^*B \) has a (canonical) symplectic form \( \omega_{T^*B} \). For each \( U_\lambda = U \), when we denote the coordinates of \( T^*B|_U \cong U \times \mathbb{R}^n \) by \((x^1, \ldots, x^n; y_1, \ldots, y_n)\), \( \omega_{T^*B} \) is given by

\[
\omega_{T^*B} := d(\sum_{i=1}^n y_i dx^i) = \sum_{i=1}^n dx^i \wedge dy_i.
\]

This is actually defined globally since the coordinate transformations on \( T^*B \) are induced from the coordinate transformations of \( \{ U_\lambda \}_{\lambda \in \Lambda} \). Actually, one has

\[
dx(\lambda) = \varphi_{\lambda \mu} dx(\mu)
\]

and the corresponding coordinates are transformed by

\[
(2) \quad \dot{y}(\lambda) = (\varphi^t_{\lambda \mu})^{-1} \dot{y}(\mu), \quad \dot{y} := (y_1, \ldots, y_n)^t
\]

so that the combination \( \sum_{i=1}^n y_i dx^i \in T^*B \) is independent of the coordinates. From this, it follows that the symplectic form \( \omega_{T^*B} = d(\sum_{i=1}^n y_i dx^i) \) is defined globally.

By choosing a metric \( g \) on a smooth manifold \( B \), one obtains a bundle isomorphism between \( TB \) and \( T^*B \). For each \( b \in B \), this isomorphism \( TB \rightarrow T^*B \) is defined by \( \xi \mapsto g(\xi, -) \) for \( \xi \in T_b B \). This actually defines a bundle isomorphism since \( g \) is nondegenerate.
at each point \( b \in B \). This bundle isomorphism also induces a diffeomorphism from \( TB \) to \( T^*B \). In this sense, hereafter we sometimes identify \( TB \) and \( T^*B \). By this identification, \( y^i \) and \( y_i \) is related by

\[
y_i = \sum_{j=1}^{n} g_{ij} y^j,
\]

\[
g_{ij} := g\left( \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right).
\]

When an affine manifold \( B \) is equipped with a metric \( g \) which is expressed locally as

\[
g_{ij} = \frac{\partial^2 \phi}{\partial x^i \partial x^j}
\]

for some local smooth function \( \phi \), then \((B, g)\) is called a Hessian manifold. When \( B \) is a Hessian manifold, then \( TB \simeq T^*B \) is equipped with the structure of Kähler manifold as we explain below. In this sense, a Hessian manifold is also called an affine Kähler manifold.

First, when \( B \) is affine, then \( TB \) is already equipped with the complex structure \( J_{TB} \). We fix a metric \( g \) and set a two-form \( \omega_{TB} \) on \( TB \) as

\[
\omega_{TB} := \sum_{i,j=1}^{n} g_{ij} dx^i \wedge dy^j.
\]

This \( \omega_{TB} \) is nondegenerate since \( g \) is nondegenerate. Furthermore, \( \omega_{TB} \) is closed iff \((B, g)\) is Hessian, where \( \omega_{TB} \) coincides with the pullback of \( \omega_{T^*B} \) by the diffeomorphism \( TB \to T^*B \). Thus, a Hessian manifold \((B, g)\) is equipped with the complex structure \( J_{TB} \) and the symplectic structure \( \omega_{TB} \). A metric \( g_{TB} \) on \( TB \) is then given by

\[
g_{TB}(X, Y) := \omega_{TB}(X, J_{TB}(Y))
\]

for \( X, Y \in \Gamma(T(TB)) \). This is locally expressed as

\[
g_{TB} = \sum_{i,j=1}^{n} (g_{ij} dx^i dx^j + g_{ij} dy^i dy^j).
\]

This shows that \( g_{TB} \) is positive definite. To summarize, for a Hessian manifold \((B, g)\), \((TB, J_{TB}, \omega_{TB})\) forms a Kähler manifold, where \( g_{TB} \) is the Kähler metric.

In order to define a Kähler structure on \( T^*B \), we employ the dual affine coordinates on \( B \). Since \( \sum_{j=1}^{n} g_{ij} dx^j \) is closed if \((B, g)\) is Hessian, for each \( i \), there exists a function \( x_i := \phi_i \) of \( x \) such that

\[
dx_i = \sum_{j=1}^{n} g_{ij} dx^j.
\]

We thus obtain the dual coordinate system

\[
\dot{x}^{(\lambda)} := (x_1^{(\lambda)}, \ldots, x_n^{(\lambda)})^t
\]
for each \( \lambda \). As above, we just denote \( x_i \) instead of \( \tilde{x}_i \) whenever no confusion occurs. The dual coordinates then define another affine structure on \( B \). Actually, the local description of the metric is changed by

\[
g(\lambda) = \{(g(\lambda))_{ij}\}_{i,j=1,...,n} = (\varphi^t_{\lambda\mu})^{-1} g(\mu) \varphi^{-1}_{\lambda\mu},
\]
so one has \( d\tilde{x}(\lambda) = (\varphi^t_{\lambda\mu})^{-1} dx(\mu) \) and then

\[
\tilde{x}(\lambda) = (\varphi^t_{\lambda\mu})^{-1} x(\mu) + \tilde{\psi}_\lambda,
\]
for some \( \tilde{\psi}_\lambda \in \mathbb{R}^n \). Thus, the combinations \( z_i := x_i + iy_i, i = 1, \ldots, n \), form a complex coordinate system on \( T^*B \), and \( T^*B \) forms a complex manifold. Actually, by eq. (2) and (3), one has the holomorphic coordinate transformation

\[
\tilde{z}(\mu) = (\varphi^t_{\lambda\mu})^{-1} z(\lambda) + \tilde{\psi}_\lambda, \quad \tilde{z} := (z_1, \ldots, z_n)^t.
\]

Using this dual coordinates, the symplectic form \( \omega_{T^*B} \) is expressed locally as

\[
\omega_{T^*B} = \sum_{i,j=1}^{n} g^{ij} dx_i \wedge dy_j,
\]
where \( g^{ij} \) is the \((i,j)\) element of the inverse matrix of \( \{g_{ij}\} \). Then, we set a metric on \( T^*B \) by

\[
g_{T^*B}(X,Y) := \omega_{T^*B}(X, J_{T^*B}(Y))
\]
for \( X, Y \in \Gamma(T(T^*B)) \), which is locally expressed as

\[
g_{T^*B} = \sum_{i,j=1}^{n} (g^{ij} dx_i dx_j + g^{ij} dy_i dy_j).
\]

These structures define a Kähler structure on \( T^*B \).

For a tropical Hessian manifold \( B \), we consider two \( T^n \)-fibrations over \( B \) obtained by a quotient \( M \) of \( TB \) and a quotient \( \tilde{M} \) of \( T^*B \) by fiberwise \( \mathbb{Z}^n \) action as follows.

For \( TB \), we locally consider \( TB|_U \) and define a \( \mathbb{Z}^n \)-action generated by \( y^i \mapsto y^i + 2\pi \) for each \( i = 1, \ldots, n \). For \( T^*B \), we again locally consider \( T^*B|_U \) and define a \( \mathbb{Z}^n \)-action generated by \( y_i \mapsto y_i + 2\pi \) for each \( i = 1, \ldots, n \). Both \( \mathbb{Z}^n \)-actions are well-defined globally since \( B \) is tropical affine, i.e., the transition functions of \( n \)-dimensional vector bundles \( TB \) and \( T^*B \) belong to \( GL(n; \mathbb{Z}) \).

Then, \( M := TB/\mathbb{Z}^n \) is a Kähler manifold whose symplectic structure \( \omega_M \) and complex structure \( J_M \) are those naturally induced from \( \omega_{T^*B} \) and \( J_{T^*B} \) on \( TB \). Similarly, \( \tilde{M} := T^*B/\mathbb{Z}^n \) is a Kähler manifold whose symplectic structure \( \omega_{\tilde{M}} \) and complex structure \( J_{\tilde{M}} \) are those induced from \( \omega_{T^*B} \) and \( J_{T^*B} \), respectively. The fibrations \( \pi : M \to B \) and \( \pi : \tilde{M} \to B \) are often called semi-flat torus fibrations or \( T^n \)-invariant manifolds. See [16, 15] and also [8]. Since \( M \) and \( \tilde{M} \) are dual to each other, we can construct them in the
opposite way. That is, if we consider the coordinate systems $\tilde{x}_\lambda^{(\lambda)}$ for $B$, then the tangent bundle over $B$ is $T^*B$ above, and the cotangent bundle is $TB$. Following [16, 15], we treat $M$ as a symplectic manifold and $\tilde{M}$ as a complex manifold and discuss the homological mirror symmetry.

2.2. Toric manifolds and $T^n$-invariant manifolds. The set-up in the previous subsection is originally applied to the mirror symmetry of compact Calabi-Yau manifolds $M, \tilde{M}$. We would like to extend this set-up to the case $\tilde{M}$ is the complement of the toric divisors of a smooth compact toric manifold $X$. The complement $\tilde{M}$ is actually a trivial torus fibration $\pi : \tilde{M} \to B$ where the base $B$ is identified with the interior of the dual polytope $P$ of $X$.

What may be more interesting is that $B$ is actually tropical affine in this situation. Of course, since $B$ is a contractible open set, $B = \text{Int}(P)$ has an open covering by itself, which means that $B$ is tropical affine. However, what we meant is something stronger in the following sense. The natural open covering of the smooth compact toric manifold $X$ induces an open covering $\{\tilde{U}_\lambda\}_{\lambda \in \Lambda}$ of $P$. Then we see that the coordinate transformations are tropical affine (though $\tilde{U}_\lambda \cap B = B$ for any $\lambda$.) This seems important since we need to include some information from the boundary $\partial(B)$ of $B$ when we discuss homological mirror symmetry of $X$ and its mirror dual.

Let us see the above construction explicitly for $X = \mathbb{CP}^n$. For

$$\mathbb{CP}^n = \{[t_0 : \cdots : t_n]\},$$

the natural open covering is $\{\tilde{U}_\lambda\}_{\lambda = 0, 1, \ldots, n}$ where

$$\tilde{U}_\lambda = \{[t_0 : \cdots : t_n] \mid t_\lambda \neq 0\}.$$  

The corresponding local coordinates are $(w_1^{(\lambda)}, \ldots, w_n^{(\lambda)})$ where

$$w_1^{(\lambda)} = t_0/t_\lambda, \ldots, w_\lambda^{(\lambda)} = t_{\lambda-1}/t_\lambda, w_{\lambda+1}^{(\lambda)} = t_{\lambda+1}/t_\lambda, \ldots, w_n^{(\lambda)} = t_n/t_\lambda.$$  

We identify $\tilde{M}$ with the complement of the toric divisors of $\mathbb{CP}^n$:

$$\tilde{M} = \{[t_0 : \cdots : t_n] \mid t_0 \cdot t_1 \cdots t_n \neq 0\},$$

where $\tilde{\pi} : \tilde{M} \to B$ is given by

$$\tilde{\pi}([t_0 ; \cdots ; t_n]) := ||t_0|| : \cdots : ||t_n||.$$  

So we have $U_\lambda := \tilde{U}_\lambda \cap \tilde{M} = \tilde{M}$ for any $\lambda$. We further denote $U_\lambda := \tilde{\pi}(U_\lambda)$. For each $U_\lambda$, we express

$$w_1^{(\lambda)} = e^{i\zeta^{(\lambda)}} = e^{i\lambda_i^{(\lambda)}+iw_i^{(\lambda)}},$$

Since the coordinate transformation between $z^{(\lambda)}$ and $z^{(\mu)}$ is tropical affine by (4), so is the coordinate transformation between $x^{(\lambda)}$ and $x^{(\mu)}$. 


Hereafter we consider $U := U_0$ (since $U_0 = U_1 = \cdots = U_n = B$) and drop the upper index $(0)$; for instance $w_i^{(0)} := w_i$ and $x_i^{(0)} := x_i$. The Kähler metric is then expressed in $U := \pi^{-1}(U)$ as
\[ \omega = -2i \left( \frac{\bar{w}_1 dw_1 + \cdots + \bar{w}_n dw_n}{1 + \bar{w}_1 w_1 + \cdots + \bar{w}_n w_n} \right). \]
When we express this as $\omega = \sum_{i,j} dx_i g^{ij} dy_j$, we have
\[ g^{ij} = \frac{\partial^2 \tilde{\phi}}{\partial x_i \partial x_j}, \]
\[ \tilde{\phi} = \log(1 + e^{2x_1} + \cdots + e^{2x_n}). \]
Thus, $B$ is a Hessian manifold. The dual coordinates $(x^1, \ldots, x^n)$ is obtained by
\[ dx^i = \sum_{j=1}^n \frac{\partial^2 \tilde{\phi}}{\partial x_i \partial x_j} dx^j = d \left( \frac{\partial \tilde{\phi}}{\partial x_i} \right), \]
so
\[ (5) \quad x^i = \frac{\partial \tilde{\phi}}{\partial x_i} = \frac{2e^{2x_i}}{1 + e^{2x_1} + \cdots + e^{2x_n}}. \]
By this $(x^1, \ldots, x^n)$, $B$ is expressed as
\[ B = \{(x^1, \ldots, x^n) | x^1 > 0, \ldots, x^n > 0, x^1 + \cdots + x^n < 2\}. \]
We can regard $M$, the dual torus fibration of $\tilde{M}$, as a subset of $(\mathbb{C}^\times)^n$, where $(e^{x^1+iy^1}, \ldots, e^{x^n+iy^n})$ is the coordinate system of $(\mathbb{C}^\times)^n$. Note that $\omega$ diverges at the boundary $\partial (M) = \pi^{-1}(\partial (B))$.

3. Lagrangian submanifolds and holomorphic vector bundles

In the first two subsections, we first recall the construction of line bundles on $\tilde{M}$ associated to Lagrangian sections of $M \to B$ discussed in [16, 15]. Then, in subsection 3.3, we apply this construction to the case $\tilde{M}$ is the complement of the toric divisors of $\mathbb{C}P^n$.

3.1. Lagrangian submanifolds in $M$. We fix a tropical affine open covering $\{U_\lambda\}_{\lambda \in \Lambda}$. Let $\tilde{s} : B \to M$ be a section of $M \to B$. Locally, we may regard $\tilde{s}$ as a section of $TB \simeq T^*B$. Then, $\tilde{s}$ is locally described by a collection of functions as
\[ y^i_{(\lambda)} = s^i_{(\lambda)}(x) \]
on each $U_\lambda$.
On $U_\lambda \cap U_\mu$, these local expressions are related to each other by
\[ (6) \quad s^i_{(\mu)}(x) = s^i_{(\lambda)}(x) + I_{\lambda \mu} \]
for some $I_{\lambda\mu} \in \mathbb{Z}^n$. Here, $x$ may be identified with either $x_{(\lambda)}$ or $x_{(\mu)}$. Also, $s_{(\lambda)}(x)$ and $s_{(\mu)}(x)$ are expressed by the common coordinates $y_{(\lambda)}$ or $y_{(\mu)}$. This transformation rule automatically satisfies the cocycle condition

\begin{equation}
I_{\lambda\mu} + I_{\mu\nu} + I_{\nu\lambda} = 0
\end{equation}

for $U_{\lambda} \cap U_{\mu} \cap U_{\nu} \neq \emptyset$. We denote by $s$ such a collection $\{s_{(\lambda)} : U_{\lambda} \to TB|_{U_{\lambda}}\}_{\lambda \in \Lambda}$ which is equipped with the transformation rule (6) satisfying the cocycle condition (7).

Now we discuss when the graph of $s$ forms a Lagrangian submanifold in $M$. By definition, an $n$-dimensional submanifold $L$ in a $2n$-dimensional symplectic manifold $(M,\omega_M)$ is Lagrangian iff $\omega_M|_L = 0$. This is a local condition. Thus, in order to discuss whether the graph of a section $\bar{s} : B \to M$ is Lagrangian or not, we may check the condition locally and in particular in $T^*B$.

It is known (as shown easily by taking the basis) that the graph of $\sum_{i=1}^{n} y_{i} dx^{i}$ with local functions $y_{i}$ is Lagrangian in $T^*B$ iff there exists a local function $f$ such that $\sum_{i=1}^{n} y_{i} dx^{i} = df$. Now, a section $\bar{s} : B \to M$ is locally regarded as a section of $T^*B$ by setting $y_{i} = \sum_{j=1}^{n} g_{ij} y^{j} = \sum_{j=1}^{n} g_{ij} s^{j}$, from which one has

$$\sum_{i=1}^{n} y_{i} dx^{i} = \sum_{i=1}^{n} \left( \sum_{j=1}^{n} g_{ij} s^{j} \right) dx^{i} = \sum_{j=1}^{n} s^{j} dx_{j}.$$ 

Thus, the graph of the section $\bar{s} : B \to M$ is Lagrangian iff there exists a local function $f$ such that $\sum_{j=1}^{n} s^{j} dx_{j} = df$.

Note that $y = s(x)$ defines a special Lagrangian submanifold if $s$ is affine with respect to $x^{i}$. (Thus, the zero section of $M \to B$ is a special Lagrangian submanifold.)

The gradient vector field is of the form:

\begin{equation}
\text{grad}(f) := \sum_{i,j} \frac{\partial f}{\partial x^{j}} g^{ji} \frac{\partial}{\partial x^{i}} = \sum_{i} \frac{\partial f}{\partial x^{i}} \frac{\partial}{\partial x^{i}}.
\end{equation}

3.2. Holomorphic vector bundles on $\tilde{M}$. Consider a section $\bar{s} : B \to M$ and express it as a collection $s = \{s_{(\lambda)}\}_{\lambda \in \Lambda}$ of local functions. We define a line bundle $V$ with a $U(1)$-connection on the mirror manifold $\tilde{M}$ associated to $s$. We set the covariant derivative locally as

\begin{equation}
D := d - i \sum_{i=1}^{n} s^{i}(x) dy_{i},
\end{equation}

\footnote{We switch the sign of the connection one form compared to that in \cite{11} so that the mirror correspondence of objects fits with the one in homological mirror symmetry of tori as in \cite{13} and references therein.}
whose curvature is
\[ D^2 = i \sum_{i,j=1}^{n} \frac{\partial s^i}{\partial x_j} dx_j \wedge dy_i. \]

The \((0,2)\)-part vanishes iff the matrix \( \frac{\partial s^i}{\partial x_j} \) is symmetric, which is the case when there exists a function \( f \) locally such that \( df = \sum_{i=1}^{n} s^i dx_i \). Thus, the condition that \( D \) defines a holomorphic line bundle on \( \tilde{M} \) is equivalent to that the graph of \( s \) is Lagrangian in \( M \).

This covariant derivative \( D \) is in fact defined globally. Suppose that \( D \) is given locally on each \( \tilde{M}|_{U_\lambda} \) of the \( T^n \)-fibration \( \tilde{M} \to B \) with a fixed tropical affine open covering \( \{ U_\lambda \}_{\lambda \in \Lambda} \). Namely, we continue to employ \( \{ U_\lambda \}_{\lambda \in \Lambda} \) for local trivializations of the line bundle associated to a section \( s : B \to \tilde{M} \). The transition functions for \( (V,D) \) are defined as follows. Recall that the section \( s : B \to \tilde{M} \) is expressed locally as
\[ y^i_\lambda = s^i_\lambda(x) \]
on each \( U_\lambda \), where, on \( U_\lambda \cap U_\mu \), the local expression is related to each other by
\[ s_\mu(x) = s_\lambda(x) + I_{\lambda\mu} \]
for some \( I_{\lambda\mu} \in \mathbb{Z}^n \) (see eq. (6)). Correspondingly, the transition function for the line bundle \( V \) with the connection \( D \) is given by
\[ \psi_\mu = e^{I_{\lambda\mu} \cdot \tilde{y}} \psi_\lambda \]
for local expressions \( \psi_\lambda, \psi_\mu \) of a smooth section \( \psi \) of \( V \), where \( I_{\lambda\mu} \cdot \tilde{y} := \sum_{j=1}^{n} i_j y_j \) for \( I_{\lambda\mu} = (i_1, \ldots, i_n) \). We see the compatibility
\[ (D\psi_\lambda)_\mu = D(\psi_\mu) \]
holds true since the left hand side turns out to be
\[ e^{I_{\lambda\mu} \cdot \tilde{y}}((d - i s_\lambda(x) \cdot dy)e^{-I_{\lambda\mu} \cdot \tilde{y}} \psi_\mu) \]
\[ = e^{I_{\lambda\mu} \cdot \tilde{y}}e^{-I_{\lambda\mu} \cdot \tilde{y}}((d - i (s_\lambda(x) + I_{\lambda\mu}) \cdot dy) \psi_\mu) \]
\[ = (d - i s_\mu(x) \cdot d y) \psi_\mu. \]

Since \((V,D)\) is locally-trivialized by \( \{ \tilde{M}|_{U_\lambda} \}_{\lambda \in \Lambda} \), for each \( x \in B \), \( \psi(x, \cdot) \) gives a smooth function on the fiber \( T^n \). Thus, on each \( U_\lambda \), \( \psi(x, y) \) can be Fourier-expanded as
\[ \psi(x, y)|_{U_\lambda} = \sum_{I \in \mathbb{Z}^n} \psi_{\lambda,I}(x)e^{I \cdot \tilde{y}}, \]
where $I \cdot \hat{y} := \sum_{j=1}^{n} i_j y_j$ for $I = (i_1, \ldots, i_n)$. Note that each coefficient $\psi_{\lambda,I}$ is a smooth function on $U_\lambda$. In this expression, the transition function acts to each $\psi_{\lambda,I}$ as

$$
\sum_{I \in \mathbb{Z}^n} \psi_{\mu,I} e^{I \cdot \hat{y}} = e^{I_{\lambda,\mu} \cdot \hat{y}} \sum_{I \in \mathbb{Z}^n} \psi_{\lambda,I} e^{I \cdot \hat{y}} = \sum_{I \in \mathbb{Z}^n} \psi_{\lambda,I} e^{(I + I_{\lambda,\mu}) \cdot \hat{y}} = \sum_{I \in \mathbb{Z}^n} \psi_{\lambda,I - I_{\lambda,\mu}} e^{I \cdot \hat{y}}
$$

and hence $\psi_{\mu,I} = \psi_{\lambda,I - I_{\lambda,\mu}}$.

3.3. Holomorphic line bundles on $\mathbb{C}P^n$ and the corresponding Lagrangians. In the previous subsections, we assign a line bundle on $\tilde{M}$ to each Lagrangian section in $M \to B$. In this subsection, we start from a line bundle on $\mathbb{C}P^n$. We identify $\tilde{M}$ with the complement of the toric divisors of $\mathbb{C}P^n$, and restrict the line bundle to $\tilde{M}$. We see that, by twisting it with an appropriate isomorphism, the result actually comes from a Lagrangian section in $M \to B$. In this way, we construct a Lagrangian section in $M \to B$ corresponding to $\mathcal{O}(a)$ on $\mathbb{C}P^n$ for any $a \in \mathbb{Z}$.

We continue the convention in subsection 2.2. The complement of the toric divisors of $\mathbb{C}P^n$ is

$$
\tilde{M} = \{[t_0 : \cdots : t_n] \mid t_0 \cdot t_1 \cdots t_n \neq 0\},
$$

where $e^{x_i + iy_i} = w_i = t_i/t_0$.

A connection one-form of $\mathcal{O}(a)$ is given by the one which is expressed locally on $\tilde{M}$ as

$$
A_a = -a \frac{\bar{w}_1 dw_1 + \cdots + \bar{w}_n dw_n}{1 + \bar{w}_1 w_1 + \cdots + \bar{w}_n w_n} = -a \frac{e^{2x_1} (dx_1 + i dy_1) + \cdots + e^{2x_n} (dx_n + i dy_n)}{1 + e^{2x_1} + \cdots + e^{2x_n}}.
$$

We twist this by

$$
\Psi_a := (1 + e^{2x_1} + \cdots + e^{2x_n})^{a/2},
$$

and then obtain

$$
\Psi_a^{-1} (d + A_a) \Psi_a = d - ia \frac{e^{2x_1} dy_1 + \cdots + e^{2x_n} dy_n}{1 + e^{2x_1} + \cdots + e^{2x_n}}.
$$
By the previous subsections, this is the line bundle on $\tilde{M}$ which corresponds to the Lagrangian section $L_a$ in $M \to B$ expressed as
\[
\begin{pmatrix}
y^1 \\
\vdots \\
y^n
\end{pmatrix}
= \begin{pmatrix}
s^1_a \\
\vdots \\
s^n_a
\end{pmatrix}
= a \begin{pmatrix}
\frac{e^{2x_1}}{1 + e^{x_1} + \cdots + e^{x_n}} \\
\vdots \\
\frac{e^{2x_n}}{1 + e^{x_1} + \cdots + e^{x_n}}
\end{pmatrix}
= \frac{a}{2} \begin{pmatrix}
x^1 \\
\vdots \\
x^n
\end{pmatrix}
\]
where $x^i > 0$ for $i = 1, \ldots, n$ and $x^1 + \cdots + x^n < 2$.

We see that the local function
\[
f_a = \frac{a}{2} (\log(1 + e^{2x_1} + \cdots + e^{2x_n}) - \log 2)
\]
\[
= \frac{a}{2} \log(2 - x^1 - x^2 - \cdots - x^n)
\]
satisfies $df_a = \sum_{i=1}^n s^i_a dx_i$. The corresponding gradient vector field is
\[
\text{grad}(f_a) = \sum_{i=1}^n \frac{\partial f_a}{\partial x_i} \frac{\partial}{\partial x_i} = \frac{a x^1}{2} \frac{\partial}{\partial x^1} + \cdots + \frac{a x^n}{2} \frac{\partial}{\partial x^n}
\]
by (8).

**Remark 3.1.** This Lagrangian section $L_a$ is a special Lagrangian since it is expressed locally as the graph of linear functions $y^i(x)$ of $(x^1, \ldots, x^n)$.

Furthermore, we see that $L_a$ includes a critical point of the corresponding Landau-Ginzburg potential. In fact, the Landau-Ginzburg potential $W$ is
\[
W(z^1, z^2, \ldots, z^n) := z^1 + \cdots + z^n + \frac{e^{-2}}{z^1 z^2 \cdots z^n}.
\]
The critical points are given by
\[
(z^1, \ldots, z^n) = \left(e^{-\frac{2}{n+1}} \omega^a, \ldots, e^{-\frac{2}{n+1}} \omega^a\right) =: c_a, \quad a = 0, 1, \ldots, n
\]
where $\omega = (1)^{n+1}$ is the $(n+1)$-th root of unity. Thus, we see that each critical point $c_a \in (\mathbb{C}^\times)^n$ is included in $L_a$.

**4. Homological mirror symmetry set-up**

In this section, we first recall DG-categories $\mathcal{F}(M)$ and $\mathcal{V}(\tilde{M})$ associated to $M$ and $\tilde{M}$ respectively, following [11]. Kontsevich-Soibelman’s approach for the homological mirror symmetry [14] introduces an intermediate category $\text{Mo}(B)$ and the existence of an $A_\infty$-equivalence
\[
\text{Fuk}(M) \simeq \text{Mo}(B) \xrightarrow{\sim} \mathcal{F}(M).
\]
This $Mo(B)$ is called the weighted Fukaya-Oh category or the category of weighted Morse homotopy on $B$. In subsection 4.5 we propose a modification $Mo(P)$ of $Mo(B)$ where $P = \bar{B}$ is the dual polytope of a smooth compact toric manifold $X$.

4.1. **DG-category $\mathcal{V}$ associated to $\bar{M}$**. We define a DG-category $\mathcal{V} = \mathcal{V}($\bar{M}$)$ of holomorphic line bundles over $\bar{M}$ as follows. The objects are holomorphic line bundles $V$ with $U(1)$-connections $D$ associated to lifts $s$ of sections as we defined in subsection 3.2. We sometimes label these objects as $s$ instead of $(V, D)$. For any two objects $s_a = (V_a, D_a), s_b = (V_b, D_b) \in \mathcal{V}$, the space $\mathcal{V}(s_a, s_b)$ of morphisms is defined by

$$\mathcal{V}(s_a, s_b) := \Gamma(V_a, V_b) \otimes_{\mathcal{C}^{\infty}(\bar{M})} \Omega^{0,*}(\bar{M}),$$

where $\Omega^{0,*}(\bar{M})$ is the space of anti-holomorphic differential forms and $\Gamma(V_a, V_b)$ is the space of homomorphisms from $V_a$ to $V_b$. The space $\mathcal{V}(s_a, s_b)$ is a $\mathbb{Z}$-graded vector space, where the grading is defined as the degree of the anti-holomorphic differential forms. The degree $r$ part is denoted by $\mathcal{V}^r(s_a, s_b)$. We define a linear map $d_{ab} : \mathcal{V}^r(s_a, s_b) \rightarrow \mathcal{V}^{r+1}(s_a, s_b)$ as follows. We decompose $D_a$ into its holomorphic part and anti-holomorphic part $D_a = D_a^{(1,0)} + D_a^{(0,1)}$, and set $2D_a^{(0,1)} =: d_a$. Then, for $\psi \in \mathcal{V}^r(s_a, s_b)$, we set

$$d_{ab}(\psi) := d_b\psi - (-1)^r \psi d_a \in \mathcal{V}^{r+1}(s_a, s_b).$$

Note that $d_{ab}^2 = 0$ since each $(V_a, D_a)$ is holomorphic, i.e., $(d_a)^2 = 0$.

The product structure $m : \mathcal{V}(s_a, s_b) \otimes \mathcal{V}(s_b, s_c) \rightarrow \mathcal{V}(s_a, s_c)$ is defined by the composition of homomorphisms of line bundles together with the wedge product for the anti-holomorphic differential forms. More precisely, for $\psi_{ab} \in \mathcal{V}^r(s_a, s_b)$ and $\psi_{bc} \in \mathcal{V}^r(s_b, s_c)$, we set

$$m(\psi_{ab}, \psi_{bc}) := (-1)^{r_{ab}r_{bc}} \psi_{bc} \wedge \psi_{ab} (= \psi_{ab} \wedge \psi_{bc}),$$

where $\wedge$ denotes the operation consisting of the composition and the wedge product. Then, we see that $\mathcal{V}$ forms a DG-category.

In order to construct another equivalent curved DG-category, we rewrite this DG-category $\mathcal{V}$ more explicitly. For an element $\psi \in \mathcal{V}^r(s_a, s_b)$, we Fourier-expand this locally as

$$\psi(\tilde{x}, \tilde{y}) = \sum_{l \in \mathbb{Z}^n} \psi_l(\tilde{x}) e^{\tilde{H}\cdot \tilde{y}},$$

\[\text{Here we again make a minor change of the formulation of the DG category compared to [11] due to the change of sign in (9).}\]

\[\text{In [11], we construct a curved DG category $DG_{\bar{M}}$ where the objects are not necessarily holomorphic. The relation is given by $\mathcal{V} = DG_{\bar{M}}(0)$.}\]
where $\psi_I$ is a smooth anti-holomorphic differential form of degree $r$. Namely, it is expressed as

$$\psi_I = \sum_{i_1,\ldots, i_r} \psi_{I; i_1\ldots i_r} d\bar{z}_{i_1} \wedge \cdots \wedge d\bar{z}_{i_r}$$

with smooth functions $\psi_{I; i_1\ldots i_r}$. Let us express the transformation rules for $s_a$ and $s_b$ as

$$(s_a)_{(\mu)} = (s_a)_{(\lambda)} + I_a, \quad (s_b)_{(\mu)} = (s_b)_{(\lambda)} + I_b$$

with $I_a = I_a; \lambda \mu \in \mathbb{Z}^n$, $I_b = I_b; \lambda \mu \in \mathbb{Z}^n$. The transition function is then given by $\psi(\mu) = e^{i(I_{b} - I_{a}) \cdot \bar{y}} \psi(\lambda)$ and hence

$$\psi(\mu), I = \psi(\lambda), I + I_a - I_b.$$

The differential $d_{ab}$ is expressed locally as follows. Since

$$D_a = d - i \sum_{j=1}^{n} s^j_a(x) dy_j$$

$$= \sum_{j=1}^{n} \left( \frac{\partial}{\partial x_j} dx_j + \left( \frac{\partial}{\partial y_j} - i s^j_a \right) dy_j \right)$$

$$= \frac{1}{2} \sum_{j=1}^{n} \left( \frac{\partial}{\partial x_j} - i \left( \frac{\partial}{\partial y_j} - i s^j_a \right) \right) dz_j + \frac{1}{2} \sum_{j=1}^{n} \left( \frac{\partial}{\partial x_j} + i \left( \frac{\partial}{\partial y_j} - i s^j_a \right) \right) d\bar{z}_j,$$

one has

$$d_a = 2D_a^{(0,1)} = \sum_{j=1}^{n} \left( \frac{\partial}{\partial x_j} + s^j_a + i \frac{\partial}{\partial y_j} \right) d\bar{z}_j$$

and then

$$d_{ab}(\psi) = 2\bar{D}(\psi) - \sum_{i=1}^{n} (s_a - s_b)^i d\bar{z}_i \wedge \psi.$$  

4.2. **DG-category $\mathcal{F}$ associated to $M$.** We define a DG-category $\mathcal{F} = \mathcal{F}(M)$ consisting of Lagrangian sections in $M$ as follows. As we shall see, we construct it so that it is canonically isomorphic to the previous DG-category $\mathcal{V}$. We fix a tropical affine open covering $\{U_{\lambda}\}_{\lambda \in \Lambda}$ of $B$.

The objects are the same as those in $\mathcal{V}$, that is, lifts $s$ of sections of $M \to B$. For any two objects $s_a, s_b \in \mathcal{F}$, we express the transformation rules for $s_a$ and $s_b$ as

$$(s_a)_{(\mu)} = (s_a)_{(\lambda)} + I_a, \quad (s_b)_{(\mu)} = (s_b)_{(\lambda)} + I_b$$

This $\mathcal{F}$ corresponds to $DG_M(0)$ in [11].
as we did in the previous subsection. For each $\lambda \in \Lambda$ and $I \in \mathbb{Z}^n$, let $\Omega_{\lambda,I}(s_a, s_b)$ be the space of complex valued smooth differential forms on $U_\lambda$. The space $\mathcal{F}(s_a, s_b)$ is then the subspace of

$$\prod_{\lambda \in \Lambda} \prod_{I \in \mathbb{Z}^n} \Omega_{\lambda,I}(s_a, s_b)$$

such that

- $\phi_{\lambda,I} \in \Omega_{\lambda,I}(s_a, s_b)$ satisfies
  $$\phi_{\mu,I}|_{U_\lambda \cap U_\mu} = \phi_{\lambda,I+I_a-I_b}|_{U_\lambda \cap U_\mu}$$
  for any $U_\lambda \cap U_\mu \neq \emptyset$ and
- the sum $\sum_{I \in \mathbb{Z}^n} \phi_{\lambda,I} e^{iH}\hat{y}$ converges as smooth differential forms on each $M|_{U_\lambda}$.

The space $\mathcal{F}(s_a, s_b)$ is a $\mathbb{Z}$-graded vector space, where the grading is defined as the degree of the differential forms. The degree $r$ part is denoted $\mathcal{F}^r(s_a, s_b)$. We define a linear map $d_{ab} : \mathcal{F}^r(s_a, s_b) \rightarrow \mathcal{F}^{r+1}(s_a, s_b)$ which is expressed locally as

$$d_{ab}(\phi_{\lambda,I}) := d(\phi_{\lambda,I}) - \sum_{j=1}^n (s^j_a - s^j_b + i_j) dx_j \wedge \phi_{\lambda,I}$$

for $\phi_{\lambda,I} \in \Omega_{\lambda,I}(s_a, s_b)$ with $I := (i_1, \ldots, i_n) \in \mathbb{Z}^n$, where $d$ is the exterior differential on $B$. We have $d_{ab}^2 = 0$.

The composition of morphisms $m : \mathcal{F}(s_a, s_b) \otimes \mathcal{F}(s_b, s_c) \rightarrow \mathcal{F}(s_a, s_c)$ is defined by

$$m(\phi_{ab;\lambda,I}, \phi_{bc;\lambda,J}) := \phi_{ab;\lambda,I} \wedge \phi_{bc;\lambda,J} \in \Omega_{\lambda,I+J}(s_a, s_c)$$

for $\phi_{ab;\lambda,I} \in \Omega_{\lambda,I}(s_a, s_b)$ and $\phi_{bc;\lambda,J} \in \Omega_{\lambda,I}(s_b, s_c)$. These structures define a DG-category $\mathcal{F}$. Note that this $\mathcal{F}$ is believed to be $A_\infty$-equivalent to the corresponding full subcategory of the Fukaya category $\mathrm{Fuk}(M)$. (Compare this $\mathcal{F}$ with what is called the deRham model for the Fukaya category in Kontsevich-Soibelman [14], in particular a construction in the Appendix (Section 9.2).)

In subsection 4.3 we shall explain the outline of how to compare $\mathcal{F}$ with the Fukaya category.

### 4.3. Equivalence between $\mathcal{F}$ and $\mathcal{V}$

The DG-category $\mathcal{F}$ is canonically isomorphic to the DG-category $\mathcal{V}$. In fact, we see that the objects in $\mathcal{F}$ are the same as those in $\mathcal{V}$. The spaces of morphisms in $\mathcal{F}$ and in $\mathcal{V}$ are also identified canonically as follows. For a morphism $\phi_{ab} = \{\phi_{ab;\lambda,I}\} \in \mathcal{F}^r(s_a, s_b)$, each $\phi_{ab;\lambda,I}$ is expressed as

$$\phi_{ab;\lambda,I} = \sum_{i_1, \ldots, i_r} \phi_{ab;\lambda,I;i_1\ldots i_r} dx_{i_1} \wedge \cdots \wedge dx_{i_r}.$$

To this, we correspond an element in $\mathcal{V}^r(s_a, s_b)$ which is locally given as

$$\sum_{i_1, \ldots, i_r} (\phi_{ab;\lambda,I;i_1\ldots i_r} e^{iH} \hat{y}) d\bar{z}_{i_1} \wedge \cdots \wedge d\bar{z}_{i_r}.$$
on $U_\lambda$. We denote this correspondence by $\mathcal{f} : \mathcal{F} \to \mathcal{V}$. It is easily seen that our construction guarantees the following fact.

**Proposition 4.1** ([11, Proposition 4.1.]). The functor $\mathcal{f} : \mathcal{F} \to \mathcal{V}$ is a DG-isomorphism.

### 4.4. The DG-category $\mathcal{F}$ and the Fukaya category $\text{Fuk}(M)$.

The DG-category $\mathcal{F}$ is expected to be $A_\infty$-equivalent to the Fukaya category $\text{Fuk}(M)$ of Lagrangian sections. The idea discussed in [14] to relate them is to apply homological perturbation theory to the DG-category $\mathcal{F}$ (as an $A_\infty$-category) in an appropriate way so that the induced $A_\infty$-category coincides with (the full subcategory of) the Fukaya category $\text{Fuk}(M)$. More precisely, what should be induced directly from $\mathcal{F}$ is the category $\text{Mo}(B)$ of weighted Morse homotopy or the Fukaya-Oh category for the torus fibration $M \to B$ introduced in section 5.2 of [14]. Here, the Fukaya-Oh category means the $A_\infty$-category of Morse homotopy on $B$ introduced in [7]. It is shown in [9] that the Fukaya-Oh category is equivalent to the Fukaya category $\text{Fuk}(T^*B)$ consisting of the corresponding objects. The Fukaya-Oh category for the torus fibration $M \to B$ is a generalization of the Fukaya-Oh category on $B$ so that it corresponds to the Fukaya category $\text{Fuk}(M)$ instead of $\text{Fuk}(T^*B)$. Thus, a natural way of obtaining the $A_\infty$-equivalence $\mathcal{F} \simeq \text{Fuk}(M)$ is to interpolate the category $\text{Mo}(B)$ so that

$$\text{Fuk}(M) = \text{Mo}(B) \xrightarrow{\sim} \mathcal{F},$$

where an $A_\infty$-equivalence $\text{Mo}(B) \to \mathcal{F}$ is expected to be obtained by the homological perturbation theory. There are some technical difficulties in proceeding this story precisely. See subsection 5.5 of [11].

### 4.5. The category $\text{Mo}(P)$ of weighted Morse homotopy.

If we start with a toric manifold $X$ and set $\tilde{M}$ as the complement of the toric divisors, we obtain $M$ as a torus fibration over the interior $B$ of the dual polytope $P$. As we discuss in the next section, from the homological mirror viewpoint, what we should discuss is not $\text{Fuk}(M)$ but a kind of Fukaya category $\text{Fuk}(\tilde{M})$ of a torus fibration over $P = \tilde{B}$. As an intermediate step, we consider the category $\text{Mo}(P)$ of weighted Morse homotopy for the dual polytope $P$. This $\text{Mo}(P)$ is a generalization of the weighted Fukaya-Oh category given in [14] to the case where the base manifold has boundaries and critical points may be degenerate.

In the present paper we consider affine Lagrangians only. However we do not limit ourselves to that cases in this subsection as the framework should work in more general toric cases. The detailed discussion in full generality should be carried out elsewhere.

**Definition of $\text{Mo}(P)$**

The definition is as follows. The objects of $\text{Mo}(P)$ are Lagrangian sections of $\pi : M \to B$ satisfying certain conditions (see for instance [3]). We extend each Lagrangian section on $B$ to that on $\tilde{B}$ smoothly. We say that two objects $L, L'$ intersects
cleanly if there exists an open set $\tilde{B}$ such that $B \subset \tilde{B}$ and $L, L'$ over $B$ can be extended to graphs of smooth sections over $\tilde{B}$ so that they intersect cleanly. We assume that any two objects $L, L'$ intersect cleanly.

For each $L$, we take a “Morse” function $f_L$ on $\tilde{B}$ so that $L$ is the graph of $df_L$. For a given ordered pair $(L, L')$, we assign a grading $|V|$ for each connected component $V$ of the intersection $\pi(L \cap L')$ in $P = \tilde{B}$ as the dimension of the stable manifold $S_v \subset \tilde{B}$ of the gradient vector field $-\text{grad}(f_L - f_{L'})$ with a point $v \in V$. This does not depend on the choice of the point $v \in V$. The space $\text{Mo}(P)(L, L')$ of morphisms is then set to be the $\mathbb{Z}$-graded vector space spanned by the connected components $V$ of $\pi(L \cap L') \subset P$ such that there exists a point $v \in V$ which is an interior point of $S_v \cap P \subset S_v$.

Now let us consider an $(l + 1)$-tuple $(L_1, \ldots, L_{l+1})$, $l \geq 2$, and take a generator $V_{i(i+1)} \in \text{Mo}(P)(L_i, L_{i+1})$ for each $i$ and $V_{1(l+1)} \in \text{Mo}(P)(L_1, L_{l+1})$. We denote by

$$\mathcal{GT}(v_{12}, \ldots, v_{l(l+1)}; v_{1(l+1)})$$

the set of gradient trees starting at $v_{12}, \ldots, v_{l(l+1)}$, where $v_{i(i+1)} \in V_{i(i+1)}$, and ending at $v_{1(l+1)} \in V_{1(l+1)}$. Here, a gradient tree $\gamma \in \mathcal{GT}(v_{12}, \ldots, v_{l(l+1)}; v_{1(l+1)})$ is a continuous map $\gamma : T \to P$ with a rooted trivalent $l$-tree $T$. Regarding $T$ as a planar tree, the leaf external vertices from the left to the right are mapped to $v_{12}, \ldots, v_{l(l+1)}$, and the root external vertex is mapped to $v_{1(l+1)}$ by $\gamma$. Furthermore, for each edge $e$ of $T$, the restriction $\gamma|_e$ is a gradient trajectory of the corresponding gradient vector field. See [14]. We then denote

$$\mathcal{GT}(V_{12}, \ldots, V_{l(l+1)}; V_{1(l+1)}) := \bigcup_{(v_{12}, \ldots, v_{l(l+1)}; v_{1(l+1)}) \in V_{12} \times \cdots \times V_{l(l+1)} \times V_{1(l+1)}} \mathcal{GT}(v_{12}, \ldots, v_{l(l+1)}; v_{1(l+1)}).$$

We say that two gradient trees $\gamma, \gamma' \in \mathcal{GT}(V_{12}, \ldots, V_{l(l+1)}; V_{1(l+1)})$ is $C^\infty$-homotopic to each other if $\gamma : T \to P$ is homotopic to $\gamma' : T \to P$ so that $\gamma|_e$ is $C^\infty$-homotopic to $\gamma'|_e$ for each edge of $T$. We further denote

$$\mathcal{HGT}(V_{12}, \ldots, V_{l(l+1)}; V_{1(l+1)}) := \{[\gamma] \mid \gamma \in \mathcal{GT}(V_{12}, \ldots, V_{l(l+1)}; V_{1(l+1)})\},$$

where $[\gamma]$ is the $C^\infty$-homotopy class of $\gamma$.

We in particular consider the case where $|V_{i(l+1)}| = |V_{12}| + \cdots + |V_{l(l+1)}| + 2 - l$, then assume that $\mathcal{HGT}(V_{12}, \ldots, V_{l(l+1)}; V_{v_{1(l+1)}})$ is a finite set. For each element in $\gamma \in \mathcal{GT}(V_{12}, \ldots, V_{v_{1(l+1)}}; V_{v_{1(l+1)}})$, we can assign the weight $e^{-A(\gamma)}$ where $A(\gamma) \in [0, \infty]$ is the symplectic area of the piecewise smooth disk in $\pi^{-1}(\gamma(T))$ as is done in Kontsevich-Soibelman [14]. This weight is invariant with respect to a $C^\infty$-homotopy. Then, we define a multilinear product

$m_l : \text{Mo}(P)(L_1, L_2) \otimes \text{Mo}(P)(L_2, L_3) \otimes \cdots \otimes \text{Mo}(P)(L_l, L_{l+1}) \to \text{Mo}(P)(L_1, L_{l+1})$ \footnote{We consider the Morse cohomology degree instead of the Morse homology degree.}
of degree $2 - l$ by

$$m_l(V_{12}, \ldots, V_{l(l+1)}) = \sum_{V_{1(l+1)}} \sum_{[\gamma] \in HGT(V_{12}, \ldots, V_{l(l+1)}; V_{1(l+1)})} \pm e^{-A(\gamma)} V_{1(l+1)},$$

where $V_{1(l+1)}$ are the bases of $Mo(P)(L_1, L_{n+1})$ of degree $|V_{12}| + \cdots + |V_{l(l+1)}| + 2 - l$, and the sign $\pm$ is given by the formula of the homological perturbation lemma of a (minimal) $A_\infty$-structure.

For a given set $E$ of objects of $Mo(P)$, we denote by $Mo_E(P)$ the full subcategory of $Mo(P)$ consisting of objects in $E$. In this paper, we consider $Mo_E(P)$ so that $E$ forms a strongly exceptional collection in $\text{Tr}(Mo_E(P))$ where we need only $m_2$, with all higher $m_k$'s vanish because of the degree reasons. As we shall explain later, we take $E$ to be the set of Lagrangian sections $L_a$ corresponding to $O(a)$ for the $X = \mathbb{C}P^n$ case, and take $E$ to be their products for the $X = \mathbb{C}P^m \times \mathbb{C}P^n$ case. Therefore we postpone to show the well-definedness of the whole $Mo(P)$ for general $P$ and that $\{m_k\}_{k=2,3,\ldots}$ defines a minimal $A_\infty$-structure.

**A strong minimality assumption** As above, we defined $m_l$ for $l \geq 2$ which we expect to form a minimal $A_\infty$-structure. In order for it to be minimal naturally, for each pair $(L, L')$, the differentials of a Morse-Bott version of the Floer complex on $Mo(P)(L, L')$ should be trivial. In our construction, we rather impose a stronger assumption for the class of objects as follows. **For any pair $(L, L')$ and any two distinct elements of the basis $V, W \in Mo(P)(L, L')$, there does not exist any gradient flow starting at a point in $V$ and ending at a point in $W$.**

**The identity morphism** For each $L \in Mo(P)$, the space $Mo(P)(L, L)$ of morphisms is generated by $P$ itself which is of degree zero. When the above $Mo(P)$ is well-defined and forms a minimal $A_\infty$-category, we believe that $P$ is the strict unit. We see that $P \in Mo(P)(L, L)$ forms at least the identity morphism with respect to $m_2$ under the strong minimality assumption above. In order to show that it is the strict unit, we need to show that $Mo(P)$ is obtained by applying homological perturbation theory to a DG category. Thus, that $Mo(P)$ forms a minimal $A_\infty$-category and that $Mo(P)$ is strictly unital should be shown at the same time.

**More explicit expression** For each $L$, let us choose a local expression $s : B \to TB$ as we did for $F$ or $V$. (A different choice leads to an isomorphic object.) This enables us to assign each generator $V$ of a morphism space a $\mathbb{Z}^n$-grading (which is different from the grading $|V|$ above). For instance, for lifts $s_a : B \to TB$ and $s_b : B \to TB$ of $L_a$ and $L_b$, consider $s_{b,I} : B \to TB$ defined by

$$y^j = s_{b,I} = s_b^j - i_j,$$
where \( I = (i_1, \ldots, i_n) \in \mathbb{Z}^n \). Denote by \( Mo_I(P)(s_a, s_b) \) the space generated by the generators of \( Mo(P)(s_a, s_b) \) which are included in the image of the intersection graph \( s_a \cap \text{graph}(s_b, i) \) by \( TB \to B \). Then, we have the decomposition

\[
Mo(P)(s_a, s_b) = \bigoplus_{I \in \mathbb{Z}^n} Mo_I(P)(s_a, s_b).
\]

In this way, each generator of \( Mo(P)(s_a, s_b) \) is assigned a \( \mathbb{Z}^n \)-grading. It turns out that the multilinear product (13) preserves these gradings.

**Connection to DG(\( X \))** We end with this subsection by explaining why we expect this \( Mo(P) \) to be a candidate of the category on the mirror dual of \( X \). We start with the DG category \( DG(X) \) of holomorphic line bundles on \( X \), as is constructed explicitly in subsection 5.1, and remove the toric divisors of \( X \) to obtain \( \check{M} \). Then, \( DG(X) \) should be regarded as a subcategory of \( V(\check{M}) \). In particular, the subcategory is not full since the smoothness condition at the removed toric divisors is imposed in \( V(\check{M}) \). The cohomologies of the morphism spaces in \( DG(X) \) then differs from those in \( V(\check{M}) \). They can be larger than those in \( V(\check{M}) \) though the morphism spaces in \( DG(X) \) are smaller at the cochain level.

In the original set-up where \( M \) and \( \check{M} \) are supposed to be compact Calabi-Yau manifolds, the cohomologies of the morphism spaces in \( V(\check{M}) \) are in one-to-one correspondence with those in \( F(M) \) since \( F(M) \) and \( V(\check{M}) \) are isomorphic DG-category (subsection 4.4). Furthermore, the cohomologies of the morphism spaces \( F(\check{M})(s_a, s_b) \) are isomorphic to \( Fuk(M)(s_a, s_b) \) at least when \( s_a \) and \( s_b \) define Lagrangian sections \( L_a \) and \( L_b \) which are transversal to each other. Namely, the cohomologies \( H(V(s_a, s_b)) \) are spanned by bases which are associated with connected components of \( L_a \cap L_b \). We would like to keep this relation even when \( \check{M} \) is noncompact. Then, if the smoothness condition at the removed toric divisors produces additional generators in \( H(DG(X)(s_a, s_b)) \) from \( H(V(M)(s_a, s_b)) \), we would like to enlarge \( M \) so that there exist the corresponding additional connected components of the intersections of the Lagrangians. Our feeling is that it seems to go well if we add the boundary of \( M \) and consider a kind of Fukaya category \( Fuk(\check{M}) \) or the corresponding category \( Mo(P = \check{B}) \) of weighted Morse homotopy. In the next section, we explicitly proceed this story successfully for \( X \) the projective spaces and their products.

### 5. Homological mirror symmetry of \( \mathbb{C}P^n \)

In this section, we discuss a version of homological mirror symmetry of \( \mathbb{C}P^n \) as the complex side by explicitly proceeding the story described in the last subsection. In subsection 5.1, we construct the DG category \( DG(\mathbb{C}P^n) \) of holomorphic line bundles on \( \mathbb{C}P^n \), and recall the structure of its cohomologies. Then, we discuss the homological mirror symmetry for \( \mathbb{C}P^n \) in subsection 5.2. We extend the story to \( \mathbb{C}P^m \times \mathbb{C}P^n \) in subsection 5.3.
5.1. **DG category** $DG(\mathbb{C}P^n)$ of line bundles over $\mathbb{C}P^n$. We first construct the DG category $DG(\mathbb{C}P^n)$ consisting of holomorphic line bundles $\mathcal{O}(a)$, $a \in \mathbb{Z}$. The space $DG(\mathbb{C}P^n)(\mathcal{O}(a), \mathcal{O}(b))$ of morphisms is defined as the Dolbeault resolution of $\Gamma(\mathcal{O}(a), \mathcal{O}(b))$. Namely, it is the graded vector space, each graded piece of which is given by

$$DG^r(\mathbb{C}P^n)(\mathcal{O}(a), \mathcal{O}(b)) := \Gamma(\mathcal{O}(a), \mathcal{O}(b)) \otimes \Omega^{0,r}(\mathbb{C}P^n)$$

with $\Gamma(\mathcal{O}(a), \mathcal{O}(b))$ being the space of smooth bundle morphism from $\mathcal{O}(a)$ to $\mathcal{O}(b)$. The composition of morphisms is defined in a similar way as that in $\mathcal{V}(\hat{M})$ in subsection 4.1. Each $\mathcal{O}(a)$ is associated with the connection $D_a$, which is expressed locally as

$$D_a = d - a\frac{\bar{w}_1 dw_1 + \cdots + \bar{w}_n dw_n}{1 + \bar{w}_1 w_1 + \cdots + \bar{w}_n w_n}$$

on $U = U_0$ (eq.(10)), and the differential

$$d_{ab} : DG^r(\mathbb{C}P^n)(\mathcal{O}(a), \mathcal{O}(b)) \to DG^{r+1}(\mathbb{C}P^n)(\mathcal{O}(a), \mathcal{O}(b))$$

is defined by

$$d_{ab}(\tilde{\psi}) := 2 \left( D_b^{0,1} \tilde{\psi} - (-1)^r \tilde{\psi} D_a^{0,1} \right).$$

This differential satisfies the Leibniz rule with respect to the composition. Thus, $DG(\mathbb{C}P^n)$ is a DG category.

The generator of $H^0(DG(\mathbb{C}P^n))(\mathcal{O}(a), \mathcal{O}(a + 1))$ is given by

$$1, \ w_1, \ w_2, \ldots, \ w_n$$

locally on $U$. These generate $H^0(DG(\mathbb{C}P^n))(\mathcal{O}(a), \mathcal{O}(b))$ as products of functions, so $H^0(DG(\mathbb{C}P^n))(\mathcal{O}(a), \mathcal{O}(b))$ is represented by polynomials in $(w_1, \ldots, w_n)$ of degree equal to or less than $b - a$. In particular, $H^0(DG(\mathbb{C}P^n))(\mathcal{O}(a), \mathcal{O}(b)) = 0$ for $a > b$. It is known by [2] that $\mathcal{E} := (\mathcal{O}(q), \ldots, \mathcal{O}(q + n))$ forms a full strongly exceptional collection of $D^b(\text{coh}(\mathbb{C}P^n))$ for each $q \in \mathbb{Z}$. That $\mathcal{E}$ forms a strongly exceptional collection means

$$H^0(DG(\mathbb{C}P^n))(\mathcal{O}(a), \mathcal{O}(a)) \simeq \mathbb{C},$$

$$H^0(DG(\mathbb{C}P^n))(\mathcal{O}(a), \mathcal{O}(b)) = 0, \quad a > b$$

$$H^r(DG(\mathbb{C}P^n))(\mathcal{O}(a), \mathcal{O}(b)) = 0, \quad r \neq 0$$

for any $a, b = \{q, q+1, \ldots, q+n\}$. Let $DG_{\mathcal{E}}(\mathbb{C}P^n)$ be the full DG subcategory of $DG(\mathbb{C}P^n)$ consisting of $\mathcal{E}$. Then the strongly exceptional collection $\mathcal{E}$ is full means that it generates $D^b(\text{coh}(\mathbb{C}P^n))$ in the sense that

$$\text{Tr}(DG_{\mathcal{E}}(\mathbb{C}P^n)) \simeq D^b(\text{coh}(\mathbb{C}P^n)),$$

where $\text{Tr}$ is the Bondal-Kapranov construction [4].
Note that $H^n(DG(\mathbb{C}P^n))(\mathcal{O}(q+n+1),\mathcal{O}(q)) \neq 0$; it includes an element represented by
\[
\frac{d\bar{w}_1 \cdots d\bar{w}_n}{(1 + w_1 \bar{w}_1 + \cdots + w_n \bar{w}_n)^2}.
\]

5.2. Homological mirror symmetry of $\mathbb{C}P^n$. First, we identify the DG category $DG(\mathbb{C}P^n)$ with a (non-full) subcategory $\mathcal{V}'$ of the DG category $\mathcal{V} = \mathcal{V}(\tilde{M})$ consisting of the same objects $\mathcal{O}(a), a \in \mathbb{Z}$, where
\[
\tilde{M} = \mathbb{C}P^n \setminus \{[t_0 : t_1 : \cdots : t_n] \mid t_0 \cdot t_1 \cdots t_n = 0\}.
\]

For a given morphism $\bar{\psi} \in DG^0(\mathbb{C}P^n)(\mathcal{O}(a), \mathcal{O}(b))$, we express it locally on $\mathcal{U}$ (see subsection 2.2), and remove the origin (corresponding to $t_0 = 0$). We send this to $\mathcal{V}'(\mathcal{O}(a), \mathcal{O}(b))$ using (11):
\[
\bar{\psi} \mapsto \psi := \Psi^{-1} b \circ \bar{\psi} \circ \Psi_{\bar{a}}.
\]

Clearly, this map is compatible with the differentials and the compositions in both sides. In this way, we obtain a functor
\[
\mathcal{I} : DG(\mathbb{C}P^n) \to \mathcal{V}
\]
of DG-categories. We see that $\mathcal{I}$ is faithful. However, $\mathcal{I}$ is not full since $\bar{\psi}$ is smooth at the points $\{[t_0 : t_1 : \cdots : t_n] \mid t_0 \cdot t_1 \cdots t_n = 0\}$. Thus, the image
\[
\mathcal{V}' := \mathcal{I}(DG(\mathbb{C}P^n))
\]
is a non-full DG subcategory of $\mathcal{V}$.

The local expression for morphisms are transformed by $\mathcal{I}$ as follows. By $w_i = e^{x_i + iy_i} = r_i e^{iy_i}$, we Fourier-expanded $\bar{\psi}$ as
\[
\bar{\psi} = \sum_{I \in \mathbb{Z}^n} \bar{\psi}_I(r)e^{iy_I}, \quad r := (r_1, \ldots, r_n).
\]

Now, recall (5) and then
\[
x^1 + \cdots + x^n = \frac{2(e^{2x_1} + \cdots + e^{2x_n})}{1 + e^{2x_1} + \cdots + e^{2x_n}} = 2 - \frac{2}{1 + e^{2x_1} + \cdots + e^{2x_n}},
\]
so we have
\[
\Psi_{\bar{a}} = (1 + e^{2x_1} + \cdots + e^{2x_n})^{\frac{1}{2}} = \left(\frac{2}{2 - x^1 - x^2 - \cdots - x^n}\right)^{\frac{n}{2}},
\]
and
\[
r_i = e^{x_i} = \left(\frac{x^1 + e^{2x_1} + \cdots + e^{2x_n}}{2}\right)^{\frac{1}{2}} = \left(\frac{x^i}{2 - x^1 - x^2 - \cdots - x^n}\right)^{\frac{1}{2}}.
\]
Then, \( \psi = \mathcal{I}(\tilde{\psi}) \) turns out to be

\[
\sum_{I \in \mathbb{Z}^n} \tilde{\psi}_I(r) \left( \frac{2 - x^1 - x^2 - \cdots - x^n}{2} \right)^{\frac{b-a}{2}} e^{i\tilde{\psi}_I} = \sum_{I \in \mathbb{Z}^n} \psi(x)e^{i\tilde{\psi}_I},
\]

so Fourier-componentwisely we have the transformation

\[
\psi_I(x) = \tilde{\psi}_I(r(x)) \left( \frac{2 - x^1 - x^2 - \cdots - x^n}{2} \right)^{\frac{b-a}{2}}.
\]

We can bring the generators \((14)\) of \(H^0(DG(CP^n)(\mathcal{O}(a), \mathcal{O}(a+1)))\) over \(\mathbb{C}\) to those of \(H^0(V')(\mathcal{O}(a), \mathcal{O}(a+1))\), which are given by

\[
\left[ \sqrt{\frac{2 - x^1 - x^2 - \cdots - x^n}{2}} \right], \quad \left[ \sqrt{\frac{x^1}{2}e^{iy_1}} \right], \quad \left[ \sqrt{\frac{x^2}{2}e^{iy_2}} \right], \quad \ldots, \quad \left[ \sqrt{\frac{x^n}{2}e^{iy_n}} \right].
\]

The above bases \((15)\) and \((16)\) generate the whole space \(H^0(V')(\mathcal{O}(a), \mathcal{O}(b))\) as products of these functions. Explicitly, the bases \(e_{ab,I}, I = (i_1, \ldots, i_n)\), of the vector space \(H^0(V')(\mathcal{O}(a), \mathcal{O}(b))\) are

\[
e_{ab,I} = c_{ab,I} \cdot \left( \sqrt{\frac{2 - x^1 - x^2 - \cdots - x^n}{2}} \right)^{\frac{b-a}{2}|I|} \left( \sqrt{\frac{x^1}{2}e^{iy_1}} \right)^{i_1} \cdots \left( \sqrt{\frac{x^n}{2}e^{iy_n}} \right)^{i_n},
\]

where \(i_1 \geq 0, \ldots, i_n \geq 0\) and \(|I| := i_1 + \cdots + i_n \leq b - a\), and we attach \(c_{ab,I}\) so that \(\max_{x \in P} |e_{ab,I}(x)| = 1\). Note that this is valid for \(a = b\), too, where we only have \(I = (0, \ldots, 0) =: 0\) and \(e_{aa,0}\) is the identity element; \(e_{aa,0}(x) = 1\) for any \(x \in P\).

By direct calculations, we have the following lemma.

**Lemma 5.1.** For a fixed \(a < b\) and \(e_{ab,I} \in H^0(V')(\mathcal{O}(a), \mathcal{O}(b))\), the set

\[
\{ x \in P \mid |e_{ab,I}(x)| = 1 \}
\]

consists of a point

\[
v_{ab,I} := \left( \frac{2i_1}{b-a}, \ldots, \frac{2i_n}{b-a} \right),
\]

which is the intersection \(V_{ab,I} \subset \pi(L_a \cap L_b)\) with label \(I\). This correspondence then gives a quasi-isomorphism

\[
\iota : Mo(P)(L_a, L_b) \to V'(\mathcal{O}(a), \mathcal{O}(b))
\]

of cochain complexes.

\(\square\)
For each \( a < b \) and \( I \), we later employ a function \( f_{ab;I} \) on \( P \) defined uniquely by

\[
\sum_{j=1}^{n} (s^j_a - s^j_b + i_j)dx_j = df_{ab;I}, \quad f_{ab;I}(v_{ab;I}) = 0.
\]

**Remark 5.2.** In the original set-up where \( B \) is compact, this \( f_{ab;I} \) is a Morse function, where \( v_{ab;I} \) is the critical point of degree zero. However, now in our case, \( v_{ab;I} \) may be at the boundary \( \partial(P) \). Even if we extend \( P \) to \( \tilde{B} \) naturally, \( v_{ab;I} \in \partial(P) \) may not a critical point since the symplectic form on \( M \) diverges at the boundary.

For each \( a \), the space \( \operatorname{Mo}(P)(L_a, L_a) \) is generated by \( P \). The two conditions

\[
\max_{x \in P} |e_{aa;0}(x)| = 1, \quad \{ x \in P \mid |e_{aa;0}(x)| = 1 \} = P
\]

are clearly satisfied. We define a quasi-isomorphism \( \iota : \operatorname{Mo}(P)(L_a, L_a) \to \mathcal{V}'(\mathcal{O}(a), \mathcal{O}(a)) \) by \( \iota(P) = e_{aa;0} \).

For \( a > b \), both the space \( \operatorname{Mo}(P)(L_a, L_b) \) and the cohomology \( H(\mathcal{V}'(\mathcal{O}(a), \mathcal{O}(b))) \) are trivial. Thus, the zero map \( \iota : \operatorname{Mo}(P)(L_a, L_b) \to \mathcal{V}'(\mathcal{O}(a), \mathcal{O}(b)) \) is a quasi-isomorphism.

Now, let us fix \( q \in \mathbb{Z} \) and consider \( \mathcal{E} := (\mathcal{O}(q), \mathcal{O}(q+1), \ldots, \mathcal{O}(q+n)) \). We denote the corresponding full subcategories by \( \operatorname{DG}_{\mathcal{E}}(\mathbb{C}P^n) \subseteq \operatorname{DG}(\mathbb{C}P^n), \mathcal{V}'_{\mathcal{E}} \subseteq \mathcal{V}' \) and \( \operatorname{Mo}_{\mathcal{E}}(P) \subseteq \operatorname{Mo}(P) \). It is known that \( \mathcal{E} \) (with any \( q \)) forms a full strongly exceptional collection in \( \operatorname{Tr}(\operatorname{DG}_{\mathcal{E}}(\mathbb{C}P^n)) \simeq \operatorname{D}^b(\operatorname{coh}(\mathbb{C}P^n)) \) \[3\]. Recall that an \( A_{\infty} \)-equivalence is an \( A_{\infty} \)-functor which induces a category equivalence on the corresponding cohomology categories.

**Theorem 5.3.** For each \( q \in \mathbb{Z} \), the quasi-isomorphisms

\[
\iota : \operatorname{Mo}(P)(L_a, L_b) \to \mathcal{V}'(\mathcal{O}(a), \mathcal{O}(b))
\]

with \( a, b \in \{ q, \ldots, q+n \} \) extend to a linear \( A_{\infty} \)-equivalence

\[
\iota : \operatorname{Mo}_{\mathcal{E}}(P) \simeq \mathcal{V}'_{\mathcal{E}}.
\]

Now, we have the DG isomorphism \( \operatorname{DG}_{\mathcal{E}}(\mathbb{C}P^n) \simeq \mathcal{I}(\operatorname{DG}_{\mathcal{E}}(\mathbb{C}P^n)) = \mathcal{V}'_{\mathcal{E}}. \) Since a DG functor is a linear \( A_{\infty} \)-functor, we immediately obtain the following.

**Corollary 5.4.** One has a linear \( A_{\infty} \)-equivalence

\[
\operatorname{Mo}_{\mathcal{E}}(P) \to \operatorname{DG}_{\mathcal{E}}(\mathbb{C}P^n).
\]

\( \square \)

**Corollary 5.5.** One has an equivalence of triangulated categories

\[
\operatorname{Tr}(\operatorname{Mo}_{\mathcal{E}}(P)) \simeq \operatorname{D}^b(\operatorname{coh}(\mathbb{C}P^n)).
\]

\( \square \)
In the rest of this subsection, we show Theorem 5.3 by computing the structure of $M_{\mathcal{E}}(P)$. We already see that nontrivial morphisms in $Mo(P)$ are of degree zero only. This implies, by degree counting, that the higher $A_\infty$-products of $Mo(P)$ are trivial. Thus, what remains to show the theorem is to construct the product $m_2$ and show the compatibility of the products with respect to $\iota$.

**Lemma 5.6.** For $a < b < c$ and bases $V_{ab;I_{ab}} \in Mo(P)(L_a, L_b)$, $V_{bc;I_{bc}} \in Mo(P)(L_b, L_c)$, we have

$$\iota m_2(V_{ab;I_{ab}}, V_{bc;I_{bc}}) = e_{ab;I_{ab}} \cdot e_{bc;I_{bc}}.$$  \hspace{1cm} (19)

**proof.** Recall that each base consists of a point; $V_{ab;I_{ab}} = \{ v_{ab;I_{ab}} \}$ and so on. We take the function $f_{ab;I_{ab}}$ defined by (5.2). Since its gradient vector field is of the form

$$-\text{grad}(f_{ab;I_{ab}}) = \frac{(b - a)}{2} \left( (x^1 - i_{ab;1}) \frac{\partial}{\partial x^1} + \cdots + (x^n - i_{ab;n}) \frac{\partial}{\partial x^n} \right),$$

its gradient trajectories starting from $v_{ab;I_{ab}}$ go straight. Similarly, gradient trajectories of $-\text{grad}(f_b - f_c)$ starting from $v_{bc;I_{bc}}$ go straight. On the other hand, the only gradient trajectory of $-\text{grad}(f_a - f_c)$ ending at $v_{ac;I_{ac}}$, $I_{ac} := I_{ab} + I_{bc}$ is the one staying at $v_{ac;I_{ac}}$ since it is of degree zero. This means that these three gradient trajectories should meet at $v_{ac;I_{ac}}$. Thus we obtained the gradient tree $\gamma$ defining the product $m_2(V_{ab;I_{ab}}, V_{bc;I_{bc}})$ explicitly. (The result is that $v_{ac;I_{ac}}$ sits on the straight line segment $v_{ab;I_{ab}}v_{bc;I_{bc}}$ in all cases.) Now, $A(\gamma)$ turns out to be

$$A(\gamma) = f_{ab;I_{ab}}(v_{ac;I_{ac}}) + f_{bc;I_{bc}}(v_{ac;I_{ac}}).$$

Here, $f_{ab;I_{ab}}(v_{ac;I_{ac}})$ is the symplectic area of the triangle disk whose edges belong to $s_a(\gamma(T))$, $s_b(\gamma(T))$ and $\pi^{-1}(v_{ac;I_{ac}})$. Similarly, $f_{bc;I_{bc}}(v_{ac;I_{ac}})$ is the symplectic area of the corresponding triangle disk. We thus obtain the weight $+e^{-A(\gamma)}$.

Next, we look at the product in $\mathcal{V}'$ side. We can express the bases as $e_{ab;I_{ab}} = e^{f_{ab;I_{ab}}} \cdot e^{I_{ab}x} y$ and $e_{bc;I_{bc}} = e^{f_{bc;I_{bc}}} \cdot e^{I_{bc}x} y$. We have

$$e_{ab;I_{ab}} \cdot e_{bc;I_{bc}} = e^{f_{ab;I_{ab}} + f_{bc;I_{bc}}} \cdot e^{I_{ac}x} y.$$

The right hand side is guaranteed to be proportional to $e_{ac;I_{ac}}$, whose absolute value takes the maximal value at $v_{ac;I_{ac}}$. Namely, we have

$$e_{ab;I_{ab}} \cdot e_{bc;I_{bc}} = e^{f_{ab;I_{ab}}(v_{ac;I_{ac}}) + f_{bc;I_{bc}}(v_{ac;I_{ac}})} \cdot e_{ac;I_{ac}}.$$ 

This shows that the compatibility (19) holds true. $\square$

We need to show the compatibility (19) for any $a \leq b \leq c$. If $a = b$, then $V_{ab;I_{ab}} = P$. If $b = c$, then $V_{bc;I_{bc}} = P$. Now, we see that $Mo_{\mathcal{E}}(P)$ satisfies the strong minimality assumption in subsection 4.3, which implies that $P$ forms the identity morphism in $Mo_{\mathcal{E}}(P)$.\]
Since we already know that \( \iota(P) \) is the identity morphism in \( V'_{E} \), the compatibility follows and the proof of Theorem 5.3 is competed.

\[ \square \]

**Remark 5.7.** The product \( m_{2} \) and the linear \( A_{\infty} \)-equivalence \( \iota \) can be induced by applying homological perturbation theory to \( DG(\mathbb{C}P^{n}) \) in a suitable way. As the higher \( A_{\infty} \)-products of \( Mo(P) \) are trivial, the induced \( A_{\infty} \)-equivalence turns out to be linear by degree counting since nontrivial cohomologies of morphisms are of degree zero only.

As a biproduct of the proof, we see that \( Mo_{E}(P) \) has the following properties.

**Proposition 5.8.** For any \( L_{a}, L_{b} \in Mo_{E}(P) \) such that \( L_{a} \neq L_{b}, V_{ab} = \pi(L_{a} \cap L_{b}) \) belongs to the boundary \( \partial(P) \).

For given bases \( V_{ab} \in Mo_{E}(L_{a}, L_{b}) \) and \( V_{bc} \in Mo_{E}(L_{b}, L_{c}) \), the image \( \gamma(T) \) by any gradient tree \( \gamma \in GT(V_{ab}, V_{bc}; V_{ac}) \) belongs to the boundary \( \partial(P) \) unless \( L_{a} = L_{b} = L_{c} \).  

\[ \square \]

**Remark 5.9.** If \( L_{a} = L_{b} = L_{c} \), then \( V_{ab} = P, V_{bc} = P \) and \( V_{ac} = P \). Then \( \gamma \in GT(V_{ab}, V_{bc}; V_{ac}) \) is a constant map to a point in \( P \). If \( L_{a} = L_{b} \neq L_{c} \), then \( V_{ab} = P \) and \( V_{bc} = \{ v_{bc} \} = V_{ac} \). Then \( \gamma \in GT(V_{ab}, V_{bc}; V_{ac}) \) is the constant map to the point \( v_{bc} \in \partial(P) \).

Similarly, if \( L_{a} \neq L_{b} = L_{c} \), then \( \gamma \in GT(V_{ab}, V_{bc}; V_{ac}) \) is the constant map to the point \( v_{ab} \in \partial(P) \).

We expect that for many other toric Fano manifold \( X \) and (strongly) exceptional collection \( E \), \( Mo_{E}(P) \) may satisfy these properties.

We also believe that there exists an \( A_{\infty} \)-equivalence

\[ Mo(P) \to DG(\mathbb{C}P^{n}) \]

between the whole categories. However, it is not easy to show directly that the whole category \( Mo(P) \) is well-defined as an \( A_{\infty} \)-category since there are infinitely many gradient trees for which we should check whether our assumption hold or not. In particular, if \( Mo(P) \) is well-defined, it should have nontrivial higher \( A_{\infty} \)-products. We would come back to see this problem elsewhere.

**5.3. Homological mirror symmetry of \( \mathbb{C}P^{m} \times \mathbb{C}P^{n} \).** In this subsection we shall see how the framework presented in the last subsections works for the case of the product of projective spaces. The point here is that we need not only transversal but clean intersections of Lagrangians in the symplectic side. That’s why we included clean intersections in the definition of \( Mo(P) \) in subsection 4.5. We still do not need higher products \( m_{3}, m_{4}, \ldots \) because we can pick up full strongly exceptional collections on both sides (see remark at the end of subsection 4.5).
Let $X = \mathbb{C}P^m \times \mathbb{C}P^n$, and $\tilde{M}$ be the complement of the toric divisors. For

$$\begin{array}{c}
X \\
\mathbb{C}P^m \quad \mathbb{C}P^n
\end{array} \xleftarrow{p_1} \xrightarrow{p_2}
$$

we denote $\mathcal{O}(a, b) := p_1^*\mathcal{O}(a) \otimes p_2^*\mathcal{O}(b)$. Then $\mathcal{E} := \{\mathcal{O}(a, b)\}_{a=0,1,\ldots,m, b=0,1,\ldots,n}$ with the lexicographic order forms a strongly exceptional collection. According to Orlov [17] the semi orthogonal ordered set of admissible subcategories $(\mathcal{D}_0, \ldots, \mathcal{D}_n)$ where $\mathcal{D}_0$ is the image of $D^b(coh(\mathbb{C}P^m))$ in $D^b(coh(\mathbb{C}P^m \times \mathbb{C}P^n))$ under the pull-back functor $p_1^*$ and its twists along $\mathbb{C}P^n$ generates $D^b(coh(\mathbb{C}P^m \times \mathbb{C}P^n))$, which means that the collection of $\mathcal{O}(a, b)$’s above is full. We consider the DG category $\text{DG}(X)$ of these line bundles, and the corresponding DG-category $V(\tilde{M})$. Just in a similar way as in the previous subsection, we have a DG subcategory $\text{I}(\text{DG}(X)) = V' \subset V = V(\tilde{M})$ so that $\text{DG}(X) \simeq V'$. Their full subcategories consisting of $\mathcal{E}$ are denoted $\text{DG}_\mathcal{E}(X)$ and $V'_\mathcal{E}$.

Then, the parallel statements to the case $X = \mathbb{C}P^n$ holds.

**Theorem 5.10.** There exists a linear $A_\infty$-equivalence

$$\iota : \text{Mo}_\mathcal{E}(P) \xrightarrow{\sim} V'_\mathcal{E}$$

such that for each generator $V \in \text{Mo}_\mathcal{E}(P)(L, L')$ we have $\max_{x \in P} |\iota(V)(x)| = 1$ and

$$V = \{x \in P \mid |\iota(V)(x)| = 1 \}.$$

**Corollary 5.11.** We have a linear $A_\infty$-equivalence

$$\text{Mo}_\mathcal{E}(P) \simeq \text{DG}_\mathcal{E}(\mathbb{C}P^m \times \mathbb{C}P^n).$$

\[\square\]

**Corollary 5.12.** We have an equivalence of triangulated categories

$$\text{Tr}(\text{Mo}_\mathcal{E}(P)) \simeq D^b(coh(\mathbb{C}P^m \times \mathbb{C}P^n)).$$

\[\square\]

**Proposition 5.13.** If $L \neq L'$, any generator $V \in \text{Mo}_\mathcal{E}(P)(L, L')$ belongs to the boundary $\partial(P)$.

For bases $V \in \text{Mo}_\mathcal{E}(L, L')$ and $V' \in \text{Mo}_\mathcal{E}(L', L'')$ such that $L \neq L'$ and $L' \neq L''$, any gradient tree $\gamma \in \mathcal{G}\mathcal{T}(V, V'; V'')$ with $V'' \in \text{Mo}_\mathcal{E}(P)(L, L'')$ belongs to the boundary $\partial(P)$.

**proof of Theorem 5.10** The bases of the space $H^0(V')(\mathcal{O}(a_1, a_2), \mathcal{O}(b_1, b_2))$ are

$$e_{a_1b_1} \otimes e_{a_2b_2}; J$$
where $e_{a_1b_1;I}$ and $e_{a_2b_2;J}$ are the bases of the corresponding zero-cohomology spaces of morphisms defined in (17) for $\mathbb{C}P^m$ and $\mathbb{C}P^n$, respectively, so $I = (i_1, \ldots, i_m)$ and $J = (j_1, \ldots, j_n)$ run over

$$
i_1 \geq 0, \ldots, i_m \geq 0, \quad |I| \leq b_1 - a_1,
\quad j_1 \geq 0, \ldots, j_n \geq 0, \quad |J| \leq b_2 - a_2.$$

Each base satisfies $\max_{x \in P} |(e_{a_1b_1;I} \otimes e_{a_2b_2;J})(x)| = 1$. Let us denote by $L_{(a_1, a_2)} \in Mo(P)$ the object corresponding to $O(a_1, a_2)$. The base corresponding to $e_{a_1b_1;I} \otimes e_{a_2b_2;J}$ is then

$$\{ x \in P \mid |(e_{a_1b_1;I} \otimes e_{a_2b_2;J})(x)| = 1 \} = V_{a_1b_1;I} \times V_{a_2b_2;J} \in Mo(P)(L_{(a_1, a_2)}, L_{(b_1, b_2)}).$$

It consists of the point $(v_{a_1b_1;I}, v_{a_2b_2;J})$ if $a_1 < b_1$ and $a_2 < b_2$. Otherwise, we have

$$V_{a_1b_1;I} \times V_{a_2b_2;J} = P_1 \times \{ v_{a_2b_2;J} \}$$

for $a_1 = b_1$ and $a_2 < b_2$,

$$V_{a_1b_1;I} \times V_{a_2b_2;J} = \{ v_{a_1b_1;I} \} \times P_2$$

for $a_1 < b_1$ and $a_2 = b_2$, and

$$V_{a_1b_1;I} \times V_{a_2b_2;J} = P_1 \times P_2 = P$$

for $a_1 = b_1$ and $a_2 = b_2$, where $P_1$ and $P_2$ are the dual polytopes of $\mathbb{C}P^m$ and $\mathbb{C}P^n$, respectively.

For $V_{a_1b_1;I} \times V_{a_2b_2;J} \in Mo(P)(L_{(a_1, a_2)}, L_{(b_1, b_2)})$ and $V_{b_1c_1;K} \times V_{b_2c_2;L} \in Mo(P)(L_{(b_1, b_2)}, L_{(c_1, c_2)})$, the equation

$$m_2(V_{a_1b_1;I} \times V_{a_2b_2;J}, V_{b_1c_1;K} \times V_{b_2c_2;L}) = (e_{a_1b_1;I} \otimes e_{a_2b_2;J}) \cdot (e_{b_1c_1;K} \otimes e_{b_2c_2;L})$$

follows immediately from looking at the structure of the gradient tree $\gamma$ defining the product $m_2(V_{a_1b_1;I} \times V_{a_2b_2;J}, V_{b_1c_1;K} \times V_{b_2c_2;L})$. Actually, let us denote by $\gamma_1$ the gradient tree obtained as the composition of $\gamma$ with the projection $P \to P_1$. We see that $\gamma_1$ is the gradient tree defining the product $m_2(V_{a_1b_1;I}, V_{b_1c_1;J})$. Similarly, we consider $\gamma_2$. Then, we have $A(\gamma) = A(\gamma_1) + A(\gamma_2)$, and we see that this is compatible with the product

$$(e_{a_1b_1;I} \otimes e_{a_2b_2;J}) \cdot (e_{b_1c_1;K} \otimes e_{b_2c_2;L}) = (e_{a_1b_1;I} \cdot e_{b_1c_1;K}) \otimes (e_{a_2b_2;J} \cdot e_{b_2c_2;L})
= e^{-A(\gamma_1)} e_{a_1c_1;I+K} \otimes e^{-A(\gamma_2)} e_{a_2c_2;J+L}.$$

This completes the proof of Theorem 5.10. □

Proof of Proposition 5.13 Each $\gamma$ is obtained from the pair $(\gamma_1, \gamma_2)$ in the proof of Theorem 5.10. In particular, the image $\gamma(T)$ of a trivalent tree $T$ by $\gamma$ is obtained as $(\gamma_1, \gamma_2)(T) \subset P_1 \times P_2$. By Proposition 5.8, $\gamma_1(T) \subset \partial(P_1)$ if $V_{a_1b_1} \neq P_1$ or $V_{b_1c_1} \neq P_1$, i.e., if $a_1, b_1, c_1$ do not satisfy $a_1 = b_1 = c_1$. Similarly, $\gamma_2(T) \subset \partial(P_2)$ unless $a_2 = b_2 = c_2$.

On the other hand, when at least one of the inequalities $a_1 < b_1, b_1 < c_1, a_2 < b_2, b_2 < c_2$ is satisfied, at least one of the inequalities $a_1 < b_1, b_1 < c_1, a_2 < b_2, b_2 < c_2$ is satisfied. Thus, at least
neither \( a_1 = b_2 = c_1 \) nor \( a_2 = b_2 = c_2 \) is satisfied. This means that at least either \( \gamma_1(T) \subset \partial(P_1) \) or \( \gamma_2(T) \subset \partial(P_2) \) holds, which implies that \( \gamma(T) \subset \partial(P) \). \( \square \)

Lastly, we show more explicitly the gradient trees \( \gamma \) defining the products

\[
m_2(V_{a_1 b_1; I} \times V_{a_2 b_2; J}, V_{b_1 c_1; K} \times V_{b_2 c_2; L}).
\]

A different point from the previous subsection is that \( V_{a_1 b_1; I} \times V_{a_2 b_2; J}, V_{b_1 c_1; K} \times V_{b_2 c_2; L} \) and \( V_{a_1 c_1; I} + K \times V_{a_2 c_2; J} - L \) may not be points even if \( V_{a_1 b_1; I} \times V_{a_2 b_2; J} \neq P \) and \( V_{b_1 c_1; K} \times V_{b_2 c_2; L} \neq P \). There are \( 4 \times 4 \) types of the products as follows.

1. only one of the equations \( a_1 = b_1 \), \( b_1 = c_1 \), \( a_2 = b_2 \), \( b_2 = c_2 \) is satisfied.
2. \( a_1 = b_1 \) and \( a_2 = b_2 < c_2 \), or \( a_1 < b_1 = c_1 \) and \( a_2 < b_2 = c_2 \).
3. \( a_1 = b_1 < c_1 \) and \( a_2 < b_2 = c_2 \), or \( a_1 < b_1 = c_1 \) and \( a_2 < b_2 = c_2 \).
4. \( a_1 = b_1 = c_1 \) and \( a_2 = b_2 = c_2 \).

The case (4) corresponds to \( P \cdot P = P \) in \( Mo(P) \). The cases (3) and (2) correspond to the products \( P \cdot V = V \) or \( V \cdot P = P \) for some \( V \). The case (2") corresponds to the product \( (P_1 \times V) \cdot (P_1 \times W) = P_1 \times (V \cdot W) \) or \( (V \times P_2) \cdot (W \times P_2) = (V \cdot W) \times P_2 \), where \( V \cdot W \) is a product in \( Mo(P_1) \) or \( Mo(P_2) \). Thus, the argument reduces to the one in the previous subsection. In the case (0), all \( V_{a_1 b_1; I} \times V_{a_2 b_2; J}, V_{b_1 c_1; K} \times V_{b_2 c_2; L} \) and \( V_{a_1 c_1; I} + K \times V_{a_2 c_2; J} - L \) consist of points. Thus, the situation is just the product of the ones in Lemma 5.6.

Now we discuss more carefully the cases (1) and (2'). In the case (1), if \( a_1 = b_1 \), then

\[
V_{a_1 b_1; I} \times V_{a_2 b_2; J} = P_1 \times \{v_{a_2 b_2; J}\}
\]

which is not a point. In this case, a gradient tree \( \gamma \in \mathcal{GT}(P_1 \times \{v_{a_2 b_2; J}\}, \{v_{a_1 c_1; K}\} \times \{v_{b_2 c_2; L}\}, \{v_{a_1 c_1; K}\} \times \{v_{a_2 c_2; J} + L\}) \) belongs to \( \mathcal{GT}(\{v_{a_1 c_1; K}, v_{a_2 b_2; J}\}, \{v_{a_1 c_1; K}, v_{b_2 c_2; L}\}, \{v_{a_1 c_1; K}, v_{a_2 c_2; J} + L\}) \) and the image \( \gamma(T) \) is a straight segment connecting \( (v_{a_1 c_1; K}, v_{a_2 b_2; J}) \) and \( (v_{a_1 c_1; K}, v_{b_2 c_2; L}) \) on which \( (v_{a_1 c_1; K}, v_{a_2 c_2; J} + L) \) sits. Similarly, in the case (2'), if \( a_1 = b_1 \) and \( b_2 = c_2 \), then we have

\[
V_{a_1 b_1; I} \times V_{a_2 b_2; J} = P_1 \times \{v_{a_2 b_2; J}\}, \quad V_{b_1 c_1; K} \times V_{b_2 c_2; L} = \{v_{b_1 c_1; K}\} \times P_2.
\]

A gradient tree \( \gamma \in \mathcal{GT}(P_1 \times \{v_{a_2 b_2; J}\}, \{v_{b_1 c_1; K}\} \times P_2; \{v_{b_1 c_1; K}\} \times \{v_{a_2 b_2; J}\}) \) belongs to \( \mathcal{GT}(\{v_{b_1 c_1; K}, v_{a_2 b_2; J}\}, \{v_{b_1 c_1; K}, v_{a_2 b_2; J}\}, \{v_{b_1 c_1; K}, v_{a_2 b_2; J}\}) \), and then the image \( \gamma(T) \) is just the point \( (v_{b_1 c_1; K}, v_{a_2 b_2; J}) \) which is the intersection \( P_1 \times \{v_{a_2 b_2; J}\} \cap \{v_{b_1 c_1; K}\} \times P_2 \).

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