Fully non-linear and exact perturbations of the Friedmann world model

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ABSTRACT
In 1988 Bardeen suggested a pragmatic formulation of cosmological perturbation theory which is powerful in practice to employ various fundamental gauge conditions easily depending on the character of the problem. The perturbation equations are presented without fixing the temporal gauge condition and are arranged so that one can easily impose fundamental gauge conditions by simply setting one of the perturbation variables in the equations equal to zero. In this way one can use the gauge degrees of freedom as an advantage in handling problems. Except for the synchronous gauge condition, all the other fundamental gauge conditions completely fix the gauge mode, and consequently, each variable in such a gauge has a unique gauge-invariant counterpart, so that we can identify the variable as the gauge-invariant one. Here, we extend Bardeen’s linear formulation to the fully non-linear order in perturbations, with the gauge advantage kept intact. The perturbation equations are exact (except for ignoring the transverse-tracefree part of the metric), and from these we can easily derive the higher order perturbation equations in a gauge-ready form. We consider scalar- and vector-type perturbations of an ideal fluid in a flat background; we also present the minimally coupled scalar field and the multiple components of ideal fluid cases. As applications, we present fully non-linear density and velocity perturbation equations in Einstein’s gravity in the zero-pressure medium, vorticity generation from pure scalar-type perturbation and fluid formulation of a minimally coupled scalar field, all in the comoving gauge. We also present the equation of gravitational waves generated from pure scalar- and vector-type perturbations.

Key words: gravitation – hydrodynamics – cosmology: theory – large-scale structure of Universe.

1 INTRODUCTION
The Friedmann world model, based on assuming spatial homogeneity and isotropy in Einstein’s gravity, is widely accepted as a successful cosmological model, enduring 90 years of observational and theoretical advances after Friedmann’s original proposition in 1922. Main observational and theoretical advances have been made in the angular anisotropies of cosmic microwave background radiation and in the position and motion of large-scale galaxy distribution. Relativistic perturbation theory is important in providing crucial testing grounds for matching the theoretical predictions with the observation, and for the theoretical explanation of the observed phenomena. The relativistic linear perturbation theory and the Newtonian exact and non-linear perturbation theory are generally accepted to be successful in the current paradigm of modern physical cosmology. This work is concerned with relativistic fully non-linear and exact perturbation formulation in the Friedmann world model. Our formulation is exact in the sense of not separating the background and perturbations, but is approximate in the sense of ignoring the transverse-tracefree (TT) part of the metric.

The cosmological linear perturbation theory in the Friedmann’s world model was pioneered by Lifshitz (1946). Lifshitz’s analysis was made in a certain gauge (coordinate) condition known as the synchronous gauge (Landau & Lifshitz 1975). A disadvantage of the synchronous gauge condition was that even after imposing the gauge condition there remains remnant gauge (coordinate) mode which needs to be traced carefully. In this way, the algebras become unnecessarily complicated. Often, removing the remnant gauge mode in the synchronous gauge merely corresponds to taking just another gauge condition; for example, setting the velocity component of the pressureless matter (being a gauge mode in the synchronous gauge) equal to zero is equivalent to simply taking the velocity component of the pressureless matter equal...
to zero (the comoving gauge of the pressureless matter); however, as will become clear in this work, even such a cure in the synchronous gauge is available only to the linear order in the presence of a zero-pressure fluid (Hwang & Noh 2006a). The remnant gauge mode in the synchronous gauge causes quite a troublesome matter to handle in the non-linear perturbation theory.

Other gauge conditions free from the remnant gauge modes were proposed by Harrison (1967) for the zero-shear gauge (often known as the longitudinal or conformal-Newtonian gauge) and Nariai (1969) for the comoving gauge; see also Hawking (1966), Sachs & Wolfe (1967) and Field & Shepley (1968). There are, in fact, infinitely many other gauge conditions which are free from such a complication, i.e. free from the remnant gauge mode, and thus equivalently gauge invariant, see later. Systematic introduction of several different gauge conditions with explicit display of gauge-invariant combination of variables was made by Bardeen (1980) with a huge success in later applications in the literature; see Kodama & Sasaki (1984) for a review.

In a less-known work, in 1988 Bardeen suggested a pragmatic way of deploying the gauge conditions depending on the problems. As in other gauge theories, the gauge choice is the degree of freedom which can be employed depending on the advantages in achieving either mathematical simplification or plausible physical interpretation. Bardeen has arranged equations so that the fundamental gauge conditions can be implemented easily. Bardeen’s formulation of linear perturbation theory was extended in Hwang (1991), Hwang & Noh (2001, 2005) and to the second-order perturbations in Noh & Hwang (2004) and Hwang & Noh (2007).

Our aim in this work is to extend the Bardeen’s formulation to the exact and fully non-linear order in perturbations keeping the gauge advantages intact. We will display some equations, but since the main point is to present the new and powerful non-linear perturbation equations, we will show detailed steps needed for the derivation in Appendices A and B. The main equations are presented in Section 3, and in Appendices E and F for multicomponent fluid case and the minimally coupled scalar field case, respectively.

In Section 2, we review Bardeen’s formulation and the gauge strategy in non-linear perturbations. In Section 3, the exact and fully non-linear equations are presented assuming scalar- and vector-type perturbations of an ideal fluid in a flat background, but without fixing the temporal gauge condition. In Section 4, we present equations valid to the third order in perturbations still without fixing the temporal gauge condition. In Section 5, we make several applications available in the comoving gauge including comparison with the Newtonian results. In Section 6, we consider the cases in other fundamental gauge conditions. In Section 7, we consider vorticity generation from pure scalar-type perturbation in the comoving gauge. In Section 8, we present the equation of gravitational waves generated from pure scalar- and vector-type perturbations which depends on the gauge choice. In Section 9, we analyse the case of a minimally coupled scalar field in the comoving gauge. We show that in the comoving gauge the ideal fluid equations remain valid for the scalar field with a particularly simple equation of state. In Section 10, we comment on the possible future extension of this work. Appendix A is a review of the Arnowitt–Deser–Misner (ADM) formulation (Arnowitt, Deser & Misner 1962). In Appendix B, we present detailed steps needed for deriving the non-linear equations in exact form. In Appendix C, we present the analysis based on the covariant formulation. In Appendix D, we introduce and clarify different definitions of the fluid three-velocities. Appendices E and F present the cases of multicomponent fluids and the minimally coupled scalar field, respectively. We set $c = 1$ except for Section 3.

## 2 CONVENTION AND GAUGE STRATEGY

Here are our metric and the energy-momentum tensor conventions. We consider scalar- and vector-type perturbations in a flat Robertson–Walker background. The metric can be written as

$$\text{d}s^2 = -\alpha^2 (1 + 2\alpha) \text{d}t^2 - 2\alpha^2 \left(\beta_{ij} + B^{(i)} \text{d}x^i \text{d}x^j\right) + \text{d}x^i \text{d}x^i + \alpha^2 \left(1 + 2\phi\right) \delta_{ij} + 2y_{ij} + C_{ij}^{(v)} + 2c_{ij}^{(e)} \right) \text{d}x^i \text{d}x^j,$$

where $\phi$ is the cosmic scale factor, and we assume $B^{(i)} = 0 \equiv C^{(v)\, ij}$ (transverse) and $C^{(e)\, i} = 0 = C^{(e)\, ij}$ (TT) with indices of $B^{(i)}$, $C^{(v)}$ and $c^{(e)}$ raised and lowered by $\delta_{ij}$ as the metric; indices $(v)$ and $(t)$ indicate the vector- and tensor-type perturbations, respectively. Indices $a, b, \ldots$ indicate the space–time indices, and $i, j, \ldots$ indicate the spatial ones; we follow the convention of Hawking & Ellis (1973).

The energy momentum tensor is given as (Ellis 1971, 1973; Ehlers 1961)

$$\tilde{T}_{ab} = \tilde{\mu} \tilde{u}_a \tilde{u}_b + \tilde{p} \left(\tilde{u}_a \tilde{u}_b + \tilde{g}_{ab}\right) + \tilde{q}_a \tilde{u}_b + \tilde{q}_b \tilde{u}_a + \tilde{\pi}_{ab},$$

where $\tilde{\mu}$ and $\tilde{p}$ are the energy density and the pressure, respectively, $\tilde{u}_a$ is a normalized fluid four-vector with $\tilde{u}^a \tilde{u}_a = 0$ and $\tilde{\pi}_{ab}$ is the anisotropic stress with $\tilde{\pi}_{ab} = \tilde{\pi}_{ba}, \tilde{\pi}_{ab}^0 = 0$ and $\tilde{\pi}_{ab} u^b = 0$; tildes indicate the covariant quantities. The fluid quantities can be read from the energy-momentum tensor as

$$\tilde{\mu} = \tilde{T}_{ab} \tilde{u}^a \tilde{u}^b, \quad \tilde{p} = \frac{1}{3} \tilde{T}_{ab} \tilde{h}^{ab},$$

$$\tilde{q}_a = -\tilde{T}_{ab} \tilde{u}^a \tilde{h}^b, \quad \tilde{\pi}_{ab} = \tilde{T}_{ab} \tilde{h}^a \tilde{h}^b - \tilde{p} \tilde{h}_{ab}.\quad (3)$$

In this work, we consider an ideal fluid with $\tilde{\pi}_{ab} = 0$, and take the energy frame by setting $\tilde{q}_a \equiv 0$. The energy-momentum tensor becomes

$$\tilde{T}_{ab} = \tilde{\mu} \tilde{u}_a \tilde{u}_b + \tilde{p} \left(\tilde{u}_a \tilde{u}_b + \tilde{g}_{ab}\right).$$

We set

$$\tilde{\mu} \equiv \mu + \delta \mu = \mu (1 + \delta), \quad \tilde{p} \equiv p + \delta p, \quad \tilde{u}_i \equiv a v_i,$$
where $\mu$ and $\rho$ are the background energy density and pressure, respectively. We set
\[ v_i \equiv -v_i + v_i^{(0)}, \]
with $v_i^{(0)}$ and $v_i^{(0)}$ are raised and lowered by $\delta_{ij}$ as the metric. More physically motivated definitions of the fluid three-velocities will be presented in Appendix D.

In this work, unless mentioned explicitly, we do not assume the perturbed metric and fluid quantities are small.

The decomposition of an arbitrary spatial vector into longitudinal and transverse parts as $B_i = \beta_i + B_i^{(v)}$ and a symmetric spatial tensor into longitudinal, trace, transverse and trace-free-transverse parts as $C_{ij} = \phi \delta_{ij} + \gamma_{ij} + 1/2 (C_{ij}^{(l)} + C_{ij}^{(t)}) + C_{ij}^{(f)}$ are possible (York 1973); here, all spatial indices are raised and lowered by $\delta_{ij}$ as the metric. We assign the transverse part as the vector-type perturbation, the trace-free-transverse part as the tensor-type perturbation and the remaining longitudinal and trace parts as the scalar-type perturbation. We can show that the decomposition is possible order by order in perturbation to non-linear order. However, only to the linear order in the spatially homogeneous-isotropic background the three types of perturbations decouple from each other. To the non-linear order we have couplings among the scalar-, vector- and tensor-types of perturbations in the equation level.

Here, we ignore the tensor-type perturbation. Restriction of our attention only to the scalar- and vector-type perturbations can be regarded as our main assumption in this work. In Section 8, however, we will consider the generation of the linear order tensor-type perturbation from the non-linear scalar- and vector-type perturbations.

In considering the linear perturbation theory, Bardeen has suggested to take the spatial gauge condition
\[ \gamma \equiv 0 \equiv C_{ij}^{(e)}, \]
but has saved the temporal gauge condition for later use. To the linear order
\[ \chi \equiv a\beta + a^2 \dot{\gamma}, \quad \Psi_{ij}^{(e)} \equiv B_{ij}^{(v)} + aC_{ij}^{(v)}, \]
are spatially gauge invariant, see below equation (21); an overdot is a time derivative based on background cosmic time $t (dt \equiv a d\eta)$. Under the spatial gauge condition in equation (7) our metric convention becomes
\[ ds^2 = -d^2 (1 + 2\alpha) d\eta^2 - 2a \chi d\eta dx^j + a^2 (1 + 2\varphi) \delta_{ij} dx^i dx^j, \]
where we set $\chi \equiv \chi_{,i} + a\Psi_{ij}^{(e)} = a (\beta_{,i} + B_i^{(v)})$.

We may also write $\chi_{ij}^{(e)} \equiv a\Psi_{ij}^{(e)}$. We will take the metric and the energy-momentum tensor conventions in equations (9) and (4) even in non-linear perturbation theory. Justification for taking the spatial gauge condition in equation (7) to the non-linear order will be made later in this section. It is essentially these spatial gauge conditions together with ignoring the tensor-type perturbation which allow our fully non-linear and exact formulation available. As will be explained, we do not lose any generality or convenience by taking these spatial gauge conditions; the only other alternative choice of spatial gauge condition leaves remnant gauge modes even from the linear order, see below equation (21).

The scalar-type perturbation equations without taking the temporal gauge condition are arranged in the following form (Bardeen 1988):
\[ \kappa \equiv 3H\alpha - 3\varphi = \frac{\Delta}{a^2} \chi, \]
\[ 4\pi G\delta\mu + H\kappa + \frac{\Delta}{a^2} \varphi = 0, \]
\[ \kappa + \frac{\Delta}{a^2} \chi - 12\pi G(\mu + p)a\dot{v} = 0, \]
\[ \dot{\kappa} + 2H\kappa - 4\pi G (\delta\mu + 3\delta p) + \left( 3H + \frac{\Delta}{a^2} \right) \alpha = 0, \]
\[ \dot{\chi} + H\chi - \varphi - \alpha = 0, \]
\[ \delta\dot{\mu} + 3H (\delta\mu + \delta p) - (\mu + p) \left( \kappa - 3H\alpha + \frac{1}{a} \Delta \varphi \right) = 0, \]
\[ \frac{[a^4(\mu + p)v]}{a^4(\mu + p)} = \frac{1}{a} \alpha - \frac{\delta p}{a(\mu + p)} = 0, \]
where $H \equiv \dot{a}/a$; here, we have assumed the flat background and an ideal fluid. These equations are arranged without taking the temporal gauge condition. One major advantage of this arrangement is that the equations are designed so that we can readily impose the various fundamental gauge condition by simply setting one of the perturbation variables equal to zero. The vector-type perturbation equations are
\[
\frac{\Delta}{2a^2} \psi^{(v)} + 8\pi G (\mu + p) \psi^{(v)} = 0,
\]
\[
\left[ a^4 (\mu + p) \psi^{(v)} \right] = 0.
\]
These vector-type equations are gauge invariant. In this work, we will present the exact and fully non-linear extension of these equations: see equations (28)–(34).

The spatial gauge (congruence) condition in equation (7) is a unique choice without remnant spatial gauge mode after taking the gauge condition; see below equation (21). Thus, we do not lose any advantage upon our choice of the spatial gauge condition. On such a spatial gauge condition, the remaining variables can be regarded as spatially gauge-invariant ones (Bardeen 1988; see equation (8)).

Concerning the temporal gauge condition, however, we have several fundamental gauge conditions available most of which remove the temporal gauge mode completely. We have the following fundamental gauge conditions

comoving gauge: \[ v = 0, \]
zero-shear gauge: \[ \chi = 0, \]
uniform-curvature gauge: \[ \varphi = 0, \]
uniform-expansion gauge: \[ \kappa = 0, \]
uniform-density gauge: \[ \delta = 0, \]
synchronous gauge: \[ \alpha = 0. \]

Also available as the gauge conditions are any non-gauge-invariant combination of these gauge conditions, thus, we could have infinite number of different temporal gauge (spatial hyperspace or slicing) conditions available. As a consequence, we can manage an arbitrary form of differential equation for any non-gauge-invariant variable using a certain, perhaps ad hoc, choice of the gauge condition (Hwang, Noh & Park 2010).

Bardeen’s arrangement of equations apparently allows the simple adaptation of any fundamental gauge conditions: we simply set a perturbation variable equal to be zero. Except for the synchronous gauge the other temporal gauge conditions together with the spatial gauge condition in equation (7) completely remove the gauge (coordinate transformation) degrees of freedom. Thus, each of the remaining perturbation variables has a unique counterpart of gauge-invariant combination involving the variable concerned and the variables used in the spatial and temporal gauge conditions; see below equation (21). Therefore, all the variables in such a gauge condition can be equivalently regarded as the corresponding gauge-invariant variables.

The gauge conditions in equations (7) and (20), and the above statements about the gauge issue remain valid to the fully non-linear order as long as we take the perturbation approach: this was shown in section VI of Noh & Hwang (2004). In the following, we explain it again.

We consider gauge transformation properties under \( \tilde{\chi}^i = x^i + \tilde{\xi}^i (x^j) \) with \( \tilde{\xi}^0 = \xi^0, \tilde{\xi}^i = \xi^i \) and \( \tilde{\xi}_i = \xi_i/a + \tilde{\xi}_i^0 \) with \( \tilde{\xi}^{(v)}_{ij} = 0; \) index of \( \tilde{\xi}^i \) is raised and lowered by \( \delta_i^j \) as the metric. To the linear order we have (Bardeen 1988; Noh & Hwang 2004)

\[
\begin{align*}
\hat{\delta} &= \delta - \frac{H}{\mu} \xi^0, \quad \hat{v} = v - \xi^0, \quad \hat{a} = a - \frac{1}{a} (a\xi^0), \\
\hat{\beta} &= \beta - \xi^0 + \left( \frac{1}{3} \right) a, \quad \hat{\gamma} = \gamma - \frac{1}{a} \xi, \quad \hat{\varphi} = \varphi - a H \xi^0, \\
\hat{\chi} &= \chi - a \xi^0, \quad \hat{\kappa} = \kappa + \left( 3 H + \frac{\Delta}{a^2} \right) a \xi^0, \\
\hat{B}^{(v)}_i &= B^{(v)}_i + \xi^{(v)}_i, \quad \hat{C}^{(v)}_i = C^{(v)}_i - \xi^{(v)}_i,
\end{align*}
\]

where a prime denotes the time derivative based on \( \eta \). Apparently \( \gamma = 0 \equiv c^{(v)}_i \) in all coordinates leaves \( \xi_i = 0 \), thus fixing the spatial gauge degree of freedom completely; the only other choice taking \( \beta = 0 \equiv B^{(v)}_i \) in all coordinates gives \( \xi \neq 0 \neq \xi_i^{(v)} \), thus leaving remnant spatial gauge modes. Similarly, for the fundamental temporal gauge condition, for example, \( v = 0 \) in all coordinates leaves \( \xi^0 = 0 \), thus fixing the temporal gauge degree of freedom completely. The following combinations are gauge invariant

\[
\begin{align*}
\varphi_v &= \varphi - a H v = -a H v, \quad \varphi_{\xi} = \varphi - H \chi = -H \chi, \\
\varphi_{\beta} &= \varphi + \frac{\delta \mu}{3 (\mu + p)}, \quad \delta_v = \delta - \frac{\dot{\mu}}{\mu} v, \\
v_{\chi} &= v - \frac{1}{a} \chi, \quad e = \delta p_{(\mu)} = \delta p - \frac{\dot{\mu}}{\mu} \delta \mu, \end{align*}
\]
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etc. This shows a systematic notation of expressing the various gauge-invariant combinations. This notation is practically useful to implement the spirit of Bardeen’s formulation employing many gauge conditions which make all variables gauge invariant (Hwang 1991). The gauge-invariant combination, for example, $\varphi_\nu$ is the same as $\varphi$ in the $\nu \equiv 0$ hypersurface condition, thus $\varphi_\nu = \varphi|_{\nu = 0}$. The temporal gauge condition, for example, $\nu = 0$ fixes the temporal gauge mode completely. Thus, any perturbation variables in that gauge, for example $\varphi$, can be equivalently regarded as a temporally gauge-invariant ones, i.e. $\varphi|_{\nu = 0} = \varphi$. Similar complete gauge fixings are true for the other fundamental temporal gauge conditions. The synchronous gauge ($\alpha \equiv 0$) is an exception, leaving a remnant temporal gauge mode $\xi^{(\alpha)}(\eta, x) \propto a^{-1}$ even after fixing the gauge condition.

Now, to the non-linear order, we may set $\bar{\xi}^0 = \xi^{(0)} + \xi^{(2)} + \cdots$ and $\xi^i = \xi_i^{(1)} + \xi_i^{(2)} + \cdots$, where the number inside the parenthesis indicates the perturbation order. To the second order, the gauge transformation properties of each variable have the same form as in equations (21) with additional terms involving quadratic combinations of $\bar{\xi}^0$, $\bar{\xi}$ and perturbation variables, all to the linear order. Since each of the quadratic terms involves $\bar{\xi}^0$ or $\bar{\xi}$ to the linear order, as long as we take the spatial and temporal gauge conditions which lead to $\bar{\xi}_i^{(1)} = 0 = \xi_i^{(1)}$ (thus the synchronous gauge is excluded), we have exactly the same gauge transformation properties in equation (21) now valid for pure second-order variables: for example, we have $\bar{\delta}_\mu^{(2)} = \delta^{(2)} - \mu_\alpha \bar{\xi}_\alpha^{(2)}$, etc. Thus, by imposing the same (i.e. ones removing the gauge degrees of freedom completely) gauge conditions now to the second order, we have $\bar{\xi}_i^{(2)} = 0 = \xi_i^{(2)}$, thus leaving any variable in that gauge having a corresponding unique gauge-invariant counterpart: i.e. $\delta_\mu|_{\nu = 0} = \delta_\mu$, etc. Apparently, the same process can be continued to any higher order perturbations.

Let us elaborate the explanation in previous paragraph using an example. We consider the gauge transformation property of the energy density $\tilde{\mu} = \bar{T}_{\alpha\beta} \bar{u}^\alpha \bar{u}^\beta$ which is a scalar quantity. Under the gauge transformation introduced above equation (21), we have $\bar{\tilde{\mu}}(x) = \tilde{\mu}(x')$. Separating the background and perturbation as $\tilde{\mu} = \mu + \delta_\mu$ and comparing $\tilde{\mu}$ and $\tilde{\mu}$ at the same space–time point, say $x'$, we have

$$\delta_\mu(x') = \delta \mu(x') - \mu_\alpha \bar{\xi}_\alpha^{(1)} + \frac{1}{2} \mu_{\alpha \beta} \bar{\xi}_\alpha^{(1)} + \mu_\alpha \bar{\xi}_\alpha^{(1)}.$$

To the linear order we have

$$\delta_\mu(x') = \delta \mu(x') - \mu_\alpha \bar{\xi}_\alpha^{(0)},$$

where we used $\mu = \mu(x')$ and $\bar{\tilde{\xi}}^0 = \xi^0$ introduced above equation (21). As long as we take the spatial gauge fixing $\gamma \equiv 0 = \xi_i^{(1)}$ and take any one of the temporal gauge fixing in the pool of the fundamental temporal gauge conditions in equation (20), except for the synchronous gauge, we have $\bar{\xi}_i^{(2)} = 0 = \xi_i^{(2)}$ to the linear order.

To the second order, we have

$$\delta_\mu(x') = \delta \mu(x') - \mu_\alpha \xi^0_\alpha - \delta \mu_\beta \xi^0_\beta - \delta \mu_i \xi^0_i + \left( \frac{1}{2} \mu_{\alpha \beta} \bar{\xi}_\alpha^{(1)} + \mu_\alpha \bar{\xi}_\alpha^{(1)} \right) \xi^0_\alpha + \mu_\alpha \xi^0_\beta \kappa^\beta.$$

Now, as we have $\xi^0_\alpha = 0 = \xi^0_i$ to the linear order in our suggested gauge conditions, the above equation simply leads to

$$\delta_\mu(x') = \delta \mu(x') - \mu_\alpha \xi^0_\alpha,$$

which is the same as equation (24), now valid even to the second order. In a decomposed form, we have

$$\delta_\mu^{(2)}(x') = \delta \mu^{(2)}(x') - \mu_\alpha \xi^{(2)}_\alpha.$$

Therefore, by imposing the gauge conditions in the suggested pool, now to the second order, we again have $\bar{\xi}_i^{(2)} = 0 = \xi_i^{(2)}$. This can be continued to the higher order perturbations as well. Explicit forms of the gauge transformation properties and gauge-invariant combinations to the second-order perturbation are presented in Noh & Hwang (2004) and Hwang, Noh & Gong (2012). The gauge transformation property in the non-linear perturbation theory is also studied in Bruni et al. (1997), Matarrese, Mollerach & Bruni (1998), Sonego & Bruni (1998), Malik & Wands (2009), and Nakamura (2010).

The names of our gauge conditions can be justified to the non-linear order by examining the ADM metric, extrinsic-curvature, intrinsic-curvature and the fluid variables presented in Appendices B and C. For the conforming gauge with $\nu \equiv 0$, ignoring the vector-type perturbation, from equation (D1) we have $\bar{u}_i = 0$, thus the fluid four-vector becomes the normal four-vector. For the zero-shear gauge with $\chi \equiv 0$, ignoring the vector-type perturbation, from equation (B7) we have $\bar{K}_i = 0$, thus having vanishing shear of the normal flow vectors $\bar{n}_i$; we have $\delta n_i^{(1)} = -\bar{K}_i$, see equation (C7). For the uniform-curvature gauge with $\varphi \equiv 0$, from equation (B6) we have $\bar{R}_{\alpha \beta} = x_{\alpha \beta} = 0$, thus having vanishing curvature of the spatial hypersurface; in the presence of the background spatial curvature, we have spatially uniform curvature. For the uniform-expansion gauge with $\kappa \equiv 0$, from equation (B7) we have $\bar{K} = -3H$, thus having the trace of extrinsic curvature uniform; we have $\bar{G}^{(1)} \equiv \bar{n}^\alpha \nu^\alpha = -K$, see equation (C7). For the uniform-density gauge with $\delta \equiv 0$, from equation (5) we have $\bar{\mu} = \mu$, thus the density becomes uniform in the hypersurface.
3 Exact and Fully Non-linear Perturbation Equations Without Taking Temporal Gauge Condition

Fully non-linear extension of equations (11)–(19) will be presented below; we recover $c$ in this section. These equations are the main result of this work. Based on the ADM equations, the derivation of our fundamental equations is unexpectedly simple. In order to help the reader who will attempt the derivation we review the ADM formulation in Appendix A and present detailed steps required for the derivation in Appendix B.

Definition of $\kappa$:

$$\kappa + 3H \left( \frac{1}{N} - 1 \right) + \frac{1}{N(1 + 2\psi)} \left[ 3\psi + \frac{c}{a^2} \left( \chi^k_{,k} + \chi^k_{,j} \phi^i_{,j} \right) \right] = 0. \quad (28)$$

ADM energy constraint:

$$- \frac{3}{2} \left( H^2 - \frac{8\pi G}{3c^2} \hat{\mu} - \frac{\Lambda c^2}{3} \right) + H\kappa + \frac{c^2 \Delta \psi}{a^2(1 + 2\psi)^2} = \frac{1}{6} \kappa^2 - \frac{4\pi G}{c^2} (\hat{\mu} + \hat{p}) (\hat{\gamma}^2 - 1) + \frac{3}{2} \frac{c^2 \phi^i_{,j} \phi^j_{,i}}{a^2(1 + 2\psi)^2} - \frac{c^2}{4} \mathcal{K}_j^j. \quad (29)$$

ADM momentum constraint:

$$\frac{2}{3} \kappa_{,i} + \frac{c}{2a^2 N(1 + 2\psi)} \left( \Delta \chi_i + \frac{1}{3} \chi^k_{,ik} \right) + \frac{8\pi G}{c^2} (\hat{\mu} + \hat{p}) \partial \hat{\gamma} v_i = \frac{c}{a^2 N(1 + 2\psi)} \left\{ \left( \frac{N}{\mathcal{N}} - 1 \right) \frac{N}{1 + 2\psi} \left[ \frac{1}{2} \left( \chi^j_{,j} + \chi_{,j} \right) - \frac{1}{3} \chi_{,j} \phi^i_{,j} \right] \right\}. \quad (30)$$

Trace of ADM propagation:

$$- \frac{3}{N} H^2 - \frac{4\pi G}{c^2} (\hat{\mu} + \hat{p}) + \Lambda c^2 + \frac{1}{N} \kappa + 2H\kappa + \frac{c^2 \Delta N}{a^2 N(1 + 2\psi)} = \frac{1}{3} \kappa^2 + \frac{8\pi G}{c^2} (\hat{\mu} + \hat{p}) (\hat{\gamma}^2 - 1) - \frac{c}{a^2 N(1 + 2\psi)} \left( \chi^k_{,k} + c \phi^i_{,j} \phi^j_{,i} \right) + c^2 \mathcal{K}_j^j. \quad (31)$$

Tracefree ADM propagation:

$$\left( \frac{1}{N} \frac{\partial}{\partial t} + 3H - \kappa + \frac{c \chi^i}{a^2 N(1 + 2\psi)} \nabla \right) \left\{ \frac{c}{a^2 N(1 + 2\psi)} \left[ \frac{1}{2} \left( \chi^j_{,j} + \chi_{,j} \right) - \frac{1}{3} \delta^j_{,j} \phi^i_{,j} \right] \right\}$$

$$- \frac{c^2}{a^2(1 + 2\psi)^2} \left[ \frac{1}{1 + 2\psi} \left( \nabla^2 \chi^i_{,j} \phi^j_{,i} \right) + \frac{N}{1 + 2\psi} \left( \nabla^2 \chi^i_{,j} \phi^j_{,i} \right) - \frac{1}{3} \delta^i_{,i} \phi^j_{,j} \phi^j_{,j} \right] = \frac{8\pi G}{c^2} (\hat{\mu} + \hat{p}) \left( \phi^i_{,i} - \frac{1}{3} \delta^i_{,i} \phi^j_{,j} \phi^j_{,j} \right) + \frac{c^2}{a^2 N^2(1 + 2\psi)^2} \times \left[ \frac{1}{2} \left( \chi^i_{,i} \phi^j_{,j} - \chi^j_{,j} \phi^i_{,i} \right) + \frac{1}{3} \delta^i_{,i} \phi^j_{,j} \phi^j_{,j} \right]$$

$$- \frac{c^2}{a^2(1 + 2\psi)^2} \left[ \frac{3}{1 + 2\psi} \left( \phi^i_{,j} - \frac{1}{3} \delta^i_{,i} \phi^j_{,j} \phi^j_{,j} \right) + \frac{N}{1 + 2\psi} \left( \phi^i_{,j} \phi^j_{,i} - \frac{2}{3} \delta^i_{,i} \phi^j_{,j} \phi^j_{,j} \right) \right]. \quad (32)$$

ADM energy conservation:

$$\frac{1}{N} \left( \frac{\partial}{\partial t} + \frac{c \chi^i}{a^2(1 + 2\psi)} \nabla \right) \left[ \hat{\mu} + (\hat{\mu} + \hat{p}) (\hat{\gamma}^2 - 1) + (\hat{\mu} + \hat{p}) (3H - \kappa) \right] = \frac{1}{3} \left( 4\hat{\gamma}^2 - 1 \right) \left( \nabla^2 + \frac{3\phi^i_{,j}}{1 + 2\psi} + 2N_i^j \right) \left( \frac{\hat{\mu} + \hat{p}}{a(1 + 2\psi)} \nabla \phi^i \right)$$

$$- \left[ \frac{\hat{\mu} + \hat{p}}{a(1 + 2\psi)} \nabla \phi^i \right] - \frac{2}{1 + 2\psi} \left( \nabla \phi^i \phi^j_{,j} \phi^j_{,j} \right). \quad (33)$$

ADM momentum conservation:

$$\left( \frac{1}{N} \frac{\partial}{\partial t} + 3H - \kappa + \frac{c \chi^i}{a^2 N(1 + 2\psi)} \nabla \right) \left[ \left( \hat{\mu} + \hat{p} \right) \nabla \phi^i \right] + c^2 \phi^i_{,j} - c^2 (\hat{\mu} + \hat{p}) \frac{N_i^j}{N} = - \left[ \left( \hat{\mu} + \hat{p} \right) \nabla \phi^i \right] \frac{1}{1 + 2\psi} \phi^i_{,j}. \quad (34)$$

Equation (28) follows from the definition of $\kappa$ as $K = -3H + \kappa$; $K$ is the trace of extrinsic curvature presented in equation (B7). Equations (29)–(34) follow from the ADM equations in equations (A6)–(A11). We used the Lorentz factor

$$\hat{\gamma} = \sqrt{1 + \frac{v^2 \psi}{c^2(1 + 2\psi)}}. \quad (35)$$
introduced in equation (D11). In Appendix D, we have introduced more physically motivated fluid three-velocities, \( \vec{\mathbf{v}} \) and \( \overrightarrow{\mathbf{v}} \); \( \vec{\mathbf{v}} \) is the fluid three-velocity measured by the Eulerian observer, and \( \overrightarrow{\mathbf{v}} \) is the fluid coordinate three-velocity. The relations among the three definitions of fluid three-velocity are presented in equation (D10). The variable \( \mathcal{N} \) is related to the lapse function in equation (B3), and \( \overline{\mathcal{K}}^j \) is the tracefree part of extrinsic curvature in equation (B7). With \( \mathcal{N} \) and \( \overline{\mathcal{K}}^j \) given as

\[
\mathcal{N} = \sqrt{1 + 2\alpha + \frac{\chi^i \chi_k}{a^2(1 + 2\varphi)}} \quad \overline{\mathcal{K}}^j = \frac{1}{a^2 \mathcal{N}^2 (1 + 2\varphi)} \left\{ \frac{1}{2} \chi^i (x_{i,j} + x_{j,i}) - \frac{4}{3} \chi^j \chi^i - \frac{1}{1 + 2\varphi} \times \left[ \frac{1}{2} \chi^i \chi^j (x_{i,j} + x_{j,i}) - \frac{1}{3} \chi^j \chi^i \right] + \frac{2}{(1 + 2\varphi)^2} \left( \chi^i \chi^j \chi^k \partial^j \varphi + \chi^j \chi^k \partial^i \varphi \right) \right\},
\]

(36)
equations (28)–(34) are the complete set of fully non-linear perturbation equations valid for the scalar- and vector-type perturbations assuming ideal fluid in a flat background.

Instead of equations (33) and (34) based on the ADM equations, we can use alternative forms based on the covariant equations. Equations (C11) and (C12) give the following covariant conservation equations.

Covariant energy conservation:

\[
\left[ \frac{\partial}{\partial t} + \frac{1}{a(1 + 2\varphi)} \left( \frac{\mathcal{N}}{\gamma} v^i + \frac{c}{a} \chi^i \right) \nabla_i \right] \tilde{\mu} + (\tilde{\mu} + \tilde{p}) \left[ (3H - \kappa) \mathcal{N} + \frac{(N v^i)_j}{a^2 (1 + 2\varphi)} + \frac{\mathcal{N} v^i_\chi}{a^2 (1 + 2\varphi)} + \frac{1}{\varphi} \left( \frac{\partial}{\partial t} + \frac{c \chi^i}{a^2 (1 + 2\varphi)} \nabla_i \right) \tilde{\gamma} \right] = 0.
\]

(37)

Covariant momentum conservation:

\[
\left[ \frac{\partial}{\partial t} + \frac{1}{a(1 + 2\varphi)} \left( \frac{\mathcal{N}}{\gamma} v^i + \frac{c}{a} \chi^i \right) \nabla_i \right] a v_i + \frac{1}{\hat{\mu} + \hat{p}} \left\{ c^2 \frac{\mathcal{N}}{\gamma} \tilde{p}_j + a v_i \left[ \frac{\partial}{\partial t} + \frac{1}{a(1 + 2\varphi)} \left( \frac{\mathcal{N}}{\gamma} v^i + \frac{c}{a} \chi^i \right) \nabla_i \right] \tilde{p} \right\} + c^2 \tilde{\gamma} \mathcal{N}_{\chi} \gamma^2 \frac{c}{a} \gamma v_i \left[ \frac{\mathcal{N}}{a^2 (1 + 2\varphi)} \right] = 0.
\]

(38)

These are alternative forms of equations (33) and (34), respectively. For comparison between the ADM and covariant conservation equations, see equation (C17). We can consider this set of equations as exact, or treat it perturbatively to the fully non-linear order. The perturbation variables are \( \delta \mu, v_i, \kappa, \chi_i, \varphi \) and \( a, \delta \rho \) should be provided by an equation of state. Notice that we have not separated the background order equations yet; we only have assumed that \( a \) is a function of time. The vector-type perturbation is contained in \( v_i \) and \( \chi_i \) as

\[
v_i = -v_i + v_i^{(\nu)}, \quad \chi_i = c \chi_i + a \Psi_i^{(\nu)}.
\]

(39)

For the vector-type perturbation, equations (30) and (34) to the linear order give equations (18) and (19), respectively. For the pure scalar-type perturbation, we set \( v_i^{(\nu)} = 0 = \Psi_i^{(\nu)} \), thus \( v_i = -v_i \) and \( \chi_i = c \chi_i \). The dimensions are

\[
[a] = [g_{ab}] = [\varphi] = [\varphi] = [\chi^i] = [\Psi^i] = [v^i/c] = [\vec{\mathbf{v}}/c] = [\overrightarrow{\mathbf{v}}/c] = [\tilde{\gamma}] = 1, \quad [v/c] = L, \quad [x^i] = [ct] = [\alpha] = [a] = 1, \quad [k] = T^{-1}, \quad \tilde{T}_{ab} = [\tilde{\mu}] = [\overline{\mathcal{K}}^2] = [\overline{\mathcal{K}}], \quad [G_{\overline{\mathcal{K}}}^2] = T^{-2}, \quad [\Lambda] = L^{-2}.
\]

(40)

In the above set of equations, we have not taken the temporal gauge condition yet. In a sense the equations are in a sort of gauge-ready form. As the temporal gauge condition, we can impose any one condition in equation (20), except for the synchronous gauge which leaves the remnant gauge mode; see explanation in the next paragraph below equation (22). Thus, as the gauge conditions we have

comoving gauge: \( v \equiv 0 \),
zero-shear gauge: \( \chi = 0 \),
uniform-curvature gauge: \( \varphi = 0 \),
uniform-expansion gauge: \( \kappa = 0 \),
uniform-density gauge: \( \delta = 0 \),

(41)

now valid to all perturbation orders. These are the fundamental gauge conditions available to the fully non-linear order. Also available ones as the gauge conditions are setting any non-gauge-invariant linear combination of these fundamental gauge conditions equal to zero. We can also take the different gauge condition for the different perturbation order. In these ways, we have infinite number of gauge conditions available, which was true even to the linear order, now to each perturbation order. Under these gauge conditions which remove the gauge mode completely, all perturbation variables have the unique gauge-invariant counterparts, thus we can identify these as the gauge-invariant variables. Therefore, the non-linear perturbation variables in any of our suggested gauge conditions mentioned above can be regarded as gauge-invariant ones.
In Appendix E, we will present the case of multiple-component fluid system. The above set of equations remains valid with the fluid quantities interpreted as the collective ones. We will present the relations of the collective fluid quantities with the individual one, see equations (E14)–(E17) and the prescription explained above equation (E14).

In Appendix F, we will present the case of a minimally coupled scalar field. The fluid equations in this section are valid with the fluid quantities expressed in terms of the scalar field in equation (F2). We additionally have the equation of motion of the field in equation (F7).

4 Third-order perturbation equations in a gauge-ready form

In the non-linear perturbation approach, we assume the perturbation variables $\delta, v, \kappa, \chi, \alpha$ and $\varphi$ are small. As one example, here we present pure scalar-type perturbation equations valid to the third order in perturbations without fixing the temporal gauge condition, thus $v = -v_j$ and $\chi = \chi_j$. Equations up to third-order perturbations are needed to get the leading non-linear contribution to the power spectrum.

Definition of $\kappa$:

$$\kappa - 3H\alpha + 3\varphi + \frac{\Delta}{a^2} \chi = \frac{1}{2a^2} \chi^k \chi_k \left( 3H + 3\varphi - 6H\varphi - 9H\alpha + \frac{\Delta}{a^2} \chi \right) - \frac{1}{a^2} \chi^k \chi_k (1 - \alpha - 4\varphi)$$

$$+ \frac{3}{2} H\alpha^2 (-3 + 5\alpha) + \left( \frac{3}{a^2} \varphi + \frac{\Delta}{a^2} \chi \right) \left( \alpha + 2\varphi - \frac{3}{2} \alpha_\varphi - 2\alpha\varphi - 4\varphi^2 \right). \quad (42)$$

ADM energy constraint:

$$-\frac{3}{2} \left( H^2 - \frac{8\pi G}{3} \mu - \frac{\Lambda}{3} \right) + 4\pi G \delta \mu + H\kappa + \frac{\Delta}{a^2} \varphi \left( 1 - 2\varphi + \frac{\delta \mu + \delta p}{\mu + p} \right) + 4 \left( \frac{\Delta}{a^2} \varphi \right) \varphi (1 - 3\varphi)$$

$$+ \frac{3}{2a^2} \varphi \varphi (1 - 6\varphi) - \frac{1}{4a^2} \left( \chi^{ij} \chi_{ij} - \frac{1}{3} (\Delta \chi)^2 \right) \right\} \left( 1 - 2\alpha - 4\varphi \right). \quad (43)$$

ADM momentum constraint:

$$\frac{2}{3} \left[ \kappa + \frac{\Delta}{a^2} \chi - 12\pi G (\mu + p) a v \right] = 8\pi G (\mu + p) a v_j \left( \delta \mu + \delta p \right) \left( \frac{\mu + p}{\mu + p} + \frac{1}{2} v^i v_i \right)$$

$$+ \frac{2}{3} \left( \frac{\Delta}{a^2} \chi \right)_j \left( \alpha + 2\varphi - \frac{3}{2} \alpha_\varphi - 2\alpha\varphi - 4\varphi^2 + \frac{1}{2a^2} \chi^k \chi_k \right)$$

$$+ \frac{1}{a^2} \left[ \chi_{ij} (1 - 3\alpha - 2\varphi) - \chi_{ij} (1 - \alpha - 4\varphi) + \frac{1}{a^2} \chi^k \chi_k \right] \left( \nabla^i \nabla_j - \frac{1}{3} \delta^i_j \Delta \right) \chi$$

$$+ \frac{1}{a^2} \left[ (\Delta \chi) \varphi_j + \chi_{ij} \Delta \varphi + \frac{1}{3} (\chi^k \chi_k) \right] (1 - \alpha - 4\varphi)$$

$$- \frac{1}{a^2} \left[ \chi_{ij} \varphi^k (\chi_{ik} + \varphi_k) + \frac{1}{3} \chi^k (\chi_{ik} \varphi_k + 3\varphi_i \chi_{ik} - 2\varphi_i \alpha \alpha) \right]. \quad (44)$$

Trace of ADM propagation:

$$-3H - 3H^2 - 4\pi G (\mu + 3p) + \Delta + \kappa + 2H\kappa - 4\pi G (\delta \mu + 3\delta p) + \left( 3H + \frac{\Delta}{a^2} \right) \alpha$$

$$= \kappa \left( \alpha - \frac{3}{2} \alpha^2 + \frac{1}{2a^2} \chi^k \chi_k \right) + \frac{1}{3} \alpha^2 + 8\pi G (\mu + p) v^i v_i \left( 1 - 2\varphi + \frac{\delta \mu + \delta p}{\mu + p} \right)$$

$$+ \frac{3}{2} H \left[ 3\alpha^2 - \frac{1}{a^2} \chi^k \chi_k (1 - 3\alpha - 2\varphi) - 5\alpha^3 \right] + \left( \alpha + 2\varphi - \frac{3}{2} \alpha_\varphi - 2\alpha\varphi - 4\varphi^2 + \frac{1}{2a^2} \chi^k \chi_k \right) \frac{\Delta}{a^2} \varphi$$

$$+ \left( 1 - \alpha - 2\varphi \right) \frac{\Delta}{2a^2} \left[ \alpha^2 - \frac{1}{a^2} \chi^k \chi_k (1 - \alpha - 2\varphi) - \alpha^2 \right] - \frac{1}{a^2} \left[ \chi^k \chi_k (1 - \alpha - 2\varphi) + \chi^i \chi_{ij} + 4 \left( \Delta \chi \right) \chi^{ij} \right]. \quad (45)$$

Tracefree ADM propagation:

$$\frac{1}{a^2} \left( \nabla^i \nabla_j - \frac{1}{3} \delta^i_j \Delta \right) \left( \chi + H\chi - \alpha - \varphi \right) = \left( \frac{3}{dt} + 3H \right) \left[ \frac{1}{a^2} \left( \alpha + 2\varphi - \frac{3}{2} \alpha_\varphi - 2\alpha\varphi - 4\varphi^2 + \frac{1}{2a^2} \chi^k \chi_k \right) \right]$$. 

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\[
\begin{align*}
&\times \left( \nabla^i \nabla_j - \frac{1}{3} \delta^i_j \Delta \right) x + \left( \chi^i \varphi_{,j} + \chi_{,j} \varphi^i - \frac{2}{3} \delta^i_j \chi^k \varphi_{,k} \right) (1 - \alpha - 4\varphi) \right) \right) \right] \\
&+ \left[ \left( \frac{3}{2} \alpha^2 + \frac{1}{2\alpha^2} \chi^k \chi_{,k} \right) \frac{\partial}{\partial t} + \kappa - \frac{1}{\alpha^2} (1 - \alpha - 2\varphi) \chi^k \nabla_k \right] \left\{ \frac{1}{a^2} \left( 1 - \alpha - 2\varphi \right) \left( \nabla^i \nabla_j - \frac{1}{3} \delta^i_j \Delta \right) x \right. \\
&\left. + \left( \chi^i \varphi_{,j} + \chi_{,j} \varphi^i - \frac{2}{3} \delta^i_j \chi^k \varphi_{,k} \right) \right\} \right] - \frac{1}{a^2} \psi^k \left( \chi^i \chi_{,j} + \chi_{,j} \chi^i - \frac{2}{3} \delta^i_j \chi_{,k} \chi^k \right) \\
&- \frac{1}{a^2} \psi^k \left( \chi^i \varphi_{,j} + \chi_{,j} \varphi^i - \frac{2}{3} \delta^i_j \chi^k \varphi_{,k} \right) \right) \right] - \frac{1}{a^2} \psi^k \left( \chi^i \chi_{,j} + \chi_{,j} \chi^i - \frac{2}{3} \delta^i_j \chi_{,k} \chi^k \right) \\
&- \frac{1}{a^2} \left( \alpha + 2\varphi - \frac{3}{2} \alpha^2 - 2\alpha \varphi - 4\varphi^2 + \frac{1}{2\alpha^2} \chi^k \chi_{,k} \right) \left( \nabla^i \nabla_j - \frac{1}{3} \delta^i_j \Delta \right) \right) \alpha \\
&+ \frac{1}{2\alpha^2} (1 - \alpha - 2\varphi) \left( \nabla^i \nabla_j - \frac{1}{3} \delta^i_j \Delta \right) \left[ -\alpha^2 + \frac{1}{a^2} \chi^k \chi_{,k} (1 - \alpha - 2\varphi) + \alpha^3 \right] \\
&- \frac{3}{a^2} \left( \psi^i \varphi_{,j} - \frac{1}{3} \delta^i_j \psi^k \chi^k \right) (1 - 6\varphi) - \frac{3}{a} \left( \psi^i \chi^i \chi_{,j} + \chi_{,j} \chi^i - \frac{2}{3} \delta^i_j \chi^k \varphi_{,k} \right) (1 - 2\alpha - 4\varphi) \\
&- \frac{4}{a^2} \psi (1 - 3\varphi) \left( \nabla^i \nabla_j - \frac{1}{3} \delta^i_j \Delta \right) \varphi + 8\pi G (\mu + p) \left( \varphi^i v^j - \frac{1}{3} \delta^i_j v^k v_k \right) \left( 1 - 2\varphi + \frac{\delta \mu + \delta p}{\mu + p} \right). \\
\end{align*}
\]

**ADM energy conservation:**

\[
\begin{align*}
\mu + 3H (\mu + p) + \delta \mu + 3H (\delta \mu + \delta p) - (\mu + p) \kappa - \mu \alpha - (\mu + p) \frac{\Delta}{\alpha} v &= (\delta \mu + \delta p) \kappa + \frac{4}{3} (\mu + p) \kappa v^i v_i \\
&+ \frac{1}{2} \mu \left[ -3\alpha^2 + \frac{1}{a^2} \chi^k \chi_{,k} (1 - 3\alpha - 2\varphi) + 5\alpha^3 \right] + \delta \mu \left( \alpha - \frac{3}{2} \alpha^2 + \frac{1}{2\alpha^2} \chi^k \chi_{,k} \right) - \frac{1}{a^2} \chi^k \delta \mu \chi_{,k} (1 - \alpha - 2\varphi) \\
&- (1 - \alpha) \frac{\partial}{\partial t} \left[ (\mu + p) v^i v_i \left( 1 - 2\varphi + \frac{\delta \mu + \delta p}{\mu + p} \right) \right] - 4H (\mu + p) v^i v_i \left( 1 - 2\varphi + \frac{\delta \mu + \delta p}{\mu + p} \right) \\
&+ \frac{1}{a} (\mu + p) \nabla_k \left\{ v^i \left[ -2\varphi + \frac{\delta \mu + \delta p}{\mu + p} (1 - 2\varphi) + 4\varphi^2 + \frac{1}{2} v^k v_k \right] \right\} \\
&+ \frac{1}{a} (\mu + p) v^i \left[ 1 + \frac{\delta \mu + \delta p}{\mu + p} \right] \left[ 3\varphi_{,i} (1 - 4\varphi) + 2\alpha \varphi (1 - 2\varphi) \right] \\
&+ \frac{1}{a^2} (\mu + p) v_i \left[ \frac{2}{a} \chi^i \chi_{,j} - 2v_{ij} \chi^j - \chi_{,ij} v^j + \frac{1}{3} (\Delta \chi) v_i \right]. \\
\end{align*}
\]

**ADM momentum conservation:**

\[
\begin{align*}
\left\{ \frac{1}{a^2} \left[ a^2 (\mu + p) v^i \right] - (\mu + p) \alpha - \delta \varphi \right\}_i &= - \left( \frac{\partial}{\partial t} + 3H \right) \left[ a (\mu + p) v^i \left( \frac{\delta \mu + \delta p}{\mu + p} + \frac{1}{2} v^k v_k \right) \right] \\
&+ \left[ \left( \alpha - \frac{3}{2} \alpha^2 + \frac{1}{2\alpha^2} \chi^k \chi_{,k} \right) \frac{\partial}{\partial t} + \kappa - \frac{1}{\alpha^2} (1 - \alpha - 2\varphi) \chi^k \nabla_k \right] \left[ a (\mu + p) v^i \left( 1 + \frac{\delta \mu + \delta p}{\mu + p} \right) \right] \\
&+ \left[ (\mu + p) v^i v_i \left( 1 - 2\varphi + \frac{\delta \mu + \delta p}{\mu + p} \right) \right] - \frac{1}{a} (\mu + p) v_k \left( 1 + \frac{\delta \mu + \delta p}{\mu + p} \right) \left[ \chi^i \chi_{,i} (1 - \alpha - 2\varphi) \right] \\
&+ (\mu + p) v^k \left[ v_{,i} (\alpha + 3\varphi) + v_{,k} (\alpha - \varphi) \right] \\
&+ (\mu + p) \alpha \left[ -2\alpha + \frac{\delta \mu + \delta p}{\mu + p} (1 - 2\alpha) + 4\alpha^2 \right] + \frac{1}{2a^2} \left[ 1 + \frac{\delta \mu + \delta p}{\mu + p} \right] \left[ \chi^i \chi_{,i} (1 - \alpha - 2\varphi) \right]. \\
\end{align*}
\]

To the background order, equations (43), (45) and (47), respectively, give

\[
H^2 = \frac{8\pi G}{3} \mu + \frac{\Lambda}{3} \frac{\dot{a}}{a} = - \frac{4\pi G}{3} (\mu + 3p) + \frac{\Lambda}{3}, \quad \mu + 3H (\mu + p) = 0.
\]

To the linear order, assuming the background equations separately, equations (42)–(48) give equations (11)–(17).

The equations of motion of the scalar field to the background order and to the third-order perturbation are presented in equations (F9) and (F13), respectively.
5 COMOVING GAUGE

In Sections 5 and 6, we consider only the scalar-type perturbation, thus \( v_i = -v, \) and \( \chi_i = \chi_i. \) In the comoving gauge, we set \( v = 0. \)

We have \( v_i = 0, \) thus the fluid four-vector becomes a normal one with \( \tilde{u}_i = 0. \) Equation (34) gives

\[
\tilde{p}_i = -\left( \tilde{\mu} + \tilde{p} \right) \frac{N_i}{N}. 
\]  

Equations (28)–(34) give a (redundantly) complete set of equations for the variables \( \delta, \kappa, \varphi, \chi \) and \( \alpha. \) We can treat this set of equation either exactly or perturbatively to all orders. As explained in Section 2 and below equation (41), all perturbation variables in the comoving gauge are gauge invariant to the non-linear order as

\[
\delta = \delta_c, \quad \kappa = \kappa_c, \quad \varphi = \varphi_c, \quad \chi = \chi_c, \quad \alpha = \alpha_c.
\]

We have several ways of having closed form second-order differential equations: equations (33) and (31), together with equations (29), (30) and (34) to determine \( \alpha, \varphi, \) and \( \chi, \) give equations for \( \delta \) and \( \kappa; \) equations (28) and (31), together with equations (29), (30) and (34) to determine \( \delta, \alpha \) and \( \chi, \) give equations for \( \varphi \) and \( \kappa, \) etc. For example, assuming the background equations are valid separately, equation (33) gives

\[
\kappa = \frac{1}{\mu + p} \frac{1}{N} \left( \frac{\partial}{\partial t} + \frac{\chi^i}{a^2(1 + 2\varphi)} V_i \right) \tilde{\mu} - \frac{\mu}{\mu + p}. 
\]  

By removing \( \kappa \) in equation (31) we have a second-order differential equation for \( \delta \) as

\[
\frac{1}{N} \left[ \frac{1}{\mu + p} \frac{1}{N} \left( \frac{\partial}{\partial t} + \frac{\chi^i}{a^2(1 + 2\varphi)} V_i \right) \tilde{\mu} - \frac{\mu}{\mu + p} \right] + 2H \left[ \frac{1}{\mu + p} \frac{1}{N} \left( \frac{\partial}{\partial t} + \frac{\chi^i}{a^2(1 + 2\varphi)} V_i \right) \tilde{\mu} - \frac{\mu}{\mu + p} \right] 
- \frac{3}{N} H - 4\pi G (3\delta + 3\dot{\delta} + 6 \dot{\chi} \dot{\varphi} - 2p) + \frac{\Delta N}{3a^2(1 + 2\varphi)} \left( \chi^{i} \chi_{,i} + \varphi \dot{N},i \right) + \kappa \kappa'. 
\]

The similar equation can be derived from equation (C13) evaluated for \( v_i = 0. \)

If we have a solution for a given variable we can derive the rest of the variables using the above complete set of equations. From these solutions we can derive all the variables in any other gauge conditions using the gauge-ready form equations in (28)–(34) and explicit construction of gauge-invariant combinations; for solutions in the matter-dominated era to the second-order perturbations, see Hwang et al. (2012).

5.1 Zero-pressure case

For \( \tilde{p} = 0, \) equation (51) gives \( N, = 0. \) Thus, we can set \( N = 1 \) with

\[
\alpha = -\frac{1}{2} \frac{\chi^i \chi_{,i}}{a^2(1 + 2\varphi)}. 
\]  

Thus, in our spatial gauge condition with \( \gamma = 0, \) the comoving gauge \( (v = 0) \) is no longer synchronous \( (\alpha = 0) \) to the non-linear order even in the zero-pressure medium (Hwang & Noh 2006a).

Equations (31) and (33) together with equations (29) and (30) provide a complete set of equations for \( \delta \) and \( \kappa. \) We have

\[
\left( \frac{\dot{\mu}}{\mu} + 3H \right) (1 + \delta) + \delta - \kappa = \delta - \frac{\chi^i \delta_{,i}}{a^2(1 + 2\varphi)},
\]

\[
-3H - 3H^2 - 4\pi G \mu + \Lambda + \kappa + 2H\kappa - 4\pi G \delta \mu = \frac{1}{3} \kappa^2 - \frac{\chi^i \kappa_{,i}}{a^2(1 + 2\varphi)} + \kappa' \kappa',
\]

with \( \chi \) and \( \varphi \) determined by

\[
\kappa_{,i} + \frac{\Delta \chi_{,i}}{a^2(1 + 2\varphi)} = \frac{1}{a^2(1 + 2\varphi)^2} \left[ 2(\Delta \chi) \varphi_{,i} + \frac{1}{2} \chi_{,ik} \varphi^{,k} - \chi_{,ik} \varphi^{,k} + \frac{3}{2} \chi_{,i} \Delta \varphi - \frac{3}{2} \frac{\varphi^{,i}}{1 + 2\varphi} \left( \chi_{,j} \varphi_{,j} + \frac{3}{2} \chi_{,j} \varphi_{,j} \right) \varphi^{,i} \right],
\]

\[
-\frac{3}{2} \left( H^2 + \frac{8\pi G \mu}{3} - \Lambda \right) + H\kappa + 4\pi G \mu \delta + \frac{\Delta \varphi}{a^2(1 + 2\varphi)^2} = \frac{1}{6} \kappa^2 + \frac{3}{2} \frac{\varphi^{,i}}{1 + 2\varphi} - \frac{1}{4} \kappa' \kappa',
\]

where

\[
\kappa' \kappa' = \frac{1}{a^2(1 + 2\varphi)^2} \left[ \chi_{,i}^i \chi_{,i} - \frac{1}{3} (\Delta \chi)^2 + \frac{4}{1 + 2\varphi} \left[ \frac{1}{3} (\Delta \chi) \chi_{,i} \varphi_{,i} - \chi_{,i} \chi_{,i} \varphi_{,i} \right] + \frac{2}{1 + 2\varphi} \left( \frac{1}{3} \chi_{,i}^i \chi_{,i} \varphi_{,i} \right) \right].
\]
These equations are still exact. Assuming that the background equations are valid separately, from equations (56) and (57) together with equations (58) and (59), we can derive a closed form second-order differential equation for $\delta$ or $\kappa$. For $\delta$, we have

$$\ddot{\delta} + 2H\dot{\delta} - 4\pi G\mu \delta = \frac{1}{a^2} \left( a^2 \kappa \delta \right) - \frac{1}{a^2} \left( \chi \frac{\dot{\delta}}{1 + 2\varphi} \right) + \frac{1}{3} \kappa^2 - \frac{\chi^i \kappa_i}{a^2(1 + 2\varphi)} + \tilde{K}_i \tilde{K}^i. \tag{61}$$

Using equations (56), (58) and (59) to determine $\kappa$, $\varphi$ and $\chi$, we can (perturbatively) express this equation purely in terms of $\delta$.

Now, the equation for $\dot{\varphi}$ follows from equations (28) and (58) as

$$[\ln(1 + 2\varphi)]'_i = \frac{1}{a^2(1 + 2\varphi)} \left[ \chi^i \varphi,_{i} + \chi \frac{\kappa}{2} \Delta \varphi - \frac{1}{1 + 2\varphi} \left( \chi \varphi,_{i} + 3\chi \varphi,_{i} \right) \varphi,^i \right], \tag{62}$$

thus $\varphi = 0$ to the linear order. Together with equation (57), using equations (58) and (59) to determine $\delta$ and $\chi$, we have the closed form equations purely in terms of $\varphi$ and $\kappa$.

### 5.2 Equations to fifth order in the zero-pressure case

As one exercise demonstrating the power of our fully non-linear formulation, we present the perturbation equations to the fifth order. We consider a zero-pressure fluid in the comoving gauge. Obviously the derivation is quite simple requiring only Taylor expansion of $1/(1 + 2\varphi)$ terms. Assuming the background equations are valid separately, from equations (56)–(60) we have

$$\ddot{\delta} - \kappa = \delta \kappa - \frac{1}{a^2} \chi^i \delta,_{i} \left( 1 - 2\varphi + 4\varphi^2 - 8\varphi^3 \right), \tag{63}$$

$$\dot{\kappa} + 2H\kappa - 4\pi G\mu \kappa = \frac{1}{a^2} \left[ \frac{4}{3} (\Delta \chi,_{i}) \varphi (1 - 2\varphi + 4\varphi^2 - 8\varphi^3) - \left( \chi^i, - \frac{1}{3} \delta \chi \right) \varphi,_{i} (1 - 4\varphi + 12\varphi^2 - 32\varphi^3) \right. \tag{64}$$

$$+ \left. \left( \chi^i, \varphi,_{i} + \chi \varphi,^i \right) \right] \left( 1 - 4\varphi + 12\varphi^2 - 32\varphi^3 \right) - \left( \chi \varphi,_{i} + \frac{1}{3} \chi \varphi,^i \right) \varphi,^i (1 - 6\varphi + 24\varphi^3) \right), \tag{65}$$

$$H\kappa + 4\pi G\mu \kappa + \frac{\Delta}{a^2} \varphi = \frac{1}{6} \kappa^2 + 4 \left( \frac{\Delta}{a^2} \varphi \right) \varphi (1 - 3\varphi + 8\varphi^2 - 20\varphi^3) + \frac{3}{2} \varphi,^i \varphi,_{i} (1 - 6\varphi + 24\varphi^2 - 80\varphi^3) - \frac{1}{4} \tilde{K}_i \tilde{K}^i, \tag{66}$$

where

$$\tilde{K}_i \tilde{K}^i = \frac{1}{a^2} \left[ \chi^{i, j} \chi_{, i} - \frac{1}{3} (\Delta \chi)^2 \right] (1 - 4\varphi + 12\varphi^2 - 32\varphi^3) + 4 \left[ \frac{1}{3} (\Delta \chi) \chi^i \varphi,_{i} - \chi^{i, j} \chi_{, i} \varphi,^j \right] (1 - 6\varphi + 24\varphi^3) \tag{67}$$

$$+ 2 \left[ \frac{1}{3} (\chi^{i, j})^2 + \chi^i \chi^{i, j} \right] (1 - 8\varphi).$$

From equations (63) and (64) together with equation (65) to the fourth order and equation (66) to the third order, we can derive a closed form second-order differential equation for $\delta$ or $\kappa$.

The fifth-order perturbation equation is needed to have the next-to-leading-order non-linear contribution to the power spectrum. The leading non-linear order power spectrum demands the third-order perturbation and the results for the density and velocity power spectra are presented in Jeong et al. (2011). The fourth-order perturbation equations will be needed to have the next-to-leading-order non-linear contribution to the non-Gaussianity. The leading order non-Gaussianity demands the second-order perturbation.

### 5.3 Pure Einstein’s gravity corrections to fully non-linear order

We can arrange equations (56), (57) and (58) in the following forms

$$\ddot{\delta} - \kappa - \delta \kappa + \frac{1}{a^2} \chi^i \delta,_{i} = \frac{2\varphi \chi^i \delta,_{i}}{a^2(1 + 2\varphi)}, \tag{68}$$
\[
\begin{align*}
\dot{\kappa} + 2H\kappa - 4\pi G\delta \mu & - \frac{1}{3} \kappa^2 + \frac{1}{a^2} \chi^{ij} \kappa_{,ij} - \frac{1}{a^2} \left[ \chi^{ij} \chi_{,ij} - \frac{1}{3} (\Delta \chi)^2 \right] = \frac{2 \varphi \chi^{ij} \kappa_{,ij}}{a^2(1+2\varphi)} - \frac{4\varphi(1+\varphi)}{a^2(1+2\varphi)^2} \left[ \chi^{ij} \chi_{,ij} - \frac{1}{3} (\Delta \chi)^2 \right] \\
& + \frac{2}{a^2(1+2\varphi)^2} \left\{ \frac{2}{3} (\Delta \chi) \chi^{ij} \varphi_{,ij} - 2\chi^{ij} \chi_{,ij} \varphi_{,ij} + \frac{1}{1+2\varphi} \left( \frac{1}{3} (\chi^{ij} \varphi_{,ij})^2 + \chi^{ij} \chi_{,ij} \varphi_{,ij} \varphi_{,ij} \right) \right\} ,
\end{align*}
\]

(69)

Terms in the right-hand sides are pure Einstein’s gravity corrections. Notice that the pure Einstein’s gravity contributions involve \( \varphi \) which is related to the (spatial) curvature perturbation in the comoving gauge; see equation (B6).

By ignoring the \( \varphi \) terms, equations (68)–(70) happen to coincide exactly with the Newtonian hydrodynamic equations of the mass and the momentum conservation equations (removing the gravitational potential in the momentum conservation equation using Poisson’s equation), respectively. This statement is true to the fully non-linear order in perturbation in the presence of the cosmological constant in the background. That is, by ignoring \( \varphi \) terms (we have no justification to do this in general, though) equations (56)–(58) or equations (68)–(70), give

\[
\delta - \kappa = \delta \kappa - \frac{1}{a^2} \chi^{ij} \delta_{,ij} ,
\]

(71)

\[
\dot{\kappa} + 2H\kappa - 4\pi G (\delta \mu + 3\delta \rho) \\
= \frac{1}{3} \kappa^2 - \frac{1}{a^2} \chi^{ij} \kappa_{,ij} + \frac{1}{a^2} \left[ \chi^{ij} \chi_{,ij} - \frac{1}{3} (\Delta \chi)^2 \right] ,
\]

(72)

\[
\kappa + \frac{\Delta}{a^3} \chi = 0.
\]

(73)

By identifying \( \delta \) and \( u \) as the Newtonian density and velocity perturbations with

\[
\kappa \equiv -\frac{1}{a} \nabla \cdot u ,
\]

(74)

thus \( \chi = au \) with \( u \equiv \nabla u \), we have

\[
\delta + \frac{1}{a} \nabla \cdot u = -\frac{1}{a} \nabla \cdot (\delta u) ,
\]

(75)

\[
\frac{1}{a} \nabla \cdot (\dot{u} + Hu) + 4\pi G \delta = -\frac{1}{a^2} \nabla \cdot (u \cdot \nabla u) ,
\]

(76)

which coincide exactly with the continuity equation and the Euler equation with the gravitational potential removed using the Poisson’s equation (Peebles 1980; Vishniac 1983). We emphasize that the above coincidence between Newtonian theory and Einstein’s gravity in the absence of \( \varphi \) does not imply that we can ignore the \( \varphi \) terms in any sense. On the contrary, by showing this coincidence we would like to emphasize that in the non-linear perturbation theory, the pure Einstein’s gravity effect appears purely through the presence of \( \varphi \) terms in various ways as in equations (68)–(70).

### 5.4 Relativistic/Newtonian correspondence

The real relativistic/Newtonian correspondence valid in the comoving gauge is available only in a much narrow scope. In the zero-pressure case, Newtonian hydrodynamic equations are closed at the second order in non-linearity (Peebles 1980; Zel’dovich & Novikov 1983). By taking a divergence of the Euler equation and using Poisson’s equations, the continuity and Euler equations give equations (75) and (76).

Note that the \( \varphi \) terms start to appear from the third order in equations (68) and (69). Thus, to the second-order perturbation we do have exact relativistic/Newtonian correspondences of the density and (irrotational) velocity perturbations in the comoving gauge, and the pure Einstein’s gravity effects encoded in \( \varphi \) start to appear from the third-order perturbation in the comoving gauge (Noh & Hwang 2004; Hwang & Noh 2006b).

In the comoving gauge, however, we do not have the Newtonian gravitational potential. In the conventional Newtonian limit of Einstein’s gravity, \( \alpha \) usually corresponds to Newtonian gravitational potential. As in equation (55), \( \alpha \) vanishes to the linear order, and we no longer have the proper Newtonian correspondence for the gravitational potential in the comoving gauge. In this regards, the correspondence of density and velocity perturbation in the two theories to the second order can be regarded as a coincidence. The proper Newtonian limit can be achieved in the infinite-speed-of-light limit (this implies the weak-gravity, slow-motion, negligible pressure and sub-horizon scale limits) in the zero-shear gauge and the uniform-expansion gauge (Chandrasekhar 1965; Kofman & Pogosyan 1995). The proof in our present formulation is presented separately in Hwang & Noh (2013).
6 OTHER FUNDAMENTAL GAUGES

6.1 Zero-shear gauge

In the zero-shear gauge we simply set
\[ \chi \equiv 0. \]  
(77)

We have \( N_i = 0 = K_j \). The metric becomes
\[ ds^2 = -a^2 (1 + 2\alpha) d\eta^2 + a^2 (1 + 2\varphi) \delta_{ij} dx^i dx^j. \]  
(78)

In the linear theory, the zero-shear gauge is quite popular in the literature (Mukhanov, Feldman & Brandenberger 1992), despite its shortcomings in numerical treatment in the early-universe and in the Boltzmann code (Ma & Bertschinger 1995; Hwang & Noh 2001). Equations (28)–(34) give a (redundantly) complete set of equations for the non-linear perturbation variables \( \delta, v, \kappa, \varphi \) and \( \alpha \). All perturbation variables in the zero-shear gauge are gauge invariant with
\[ \delta = \delta_\chi, \quad v = v_\chi, \quad \kappa = \kappa_\chi, \quad \varphi = \varphi_\chi, \quad \alpha = \alpha_\chi. \]  
(79)

There are several ways of having the closed form second-order differential equations: equations (33) and (31), together with equations (29), (30) and (32) to determine \( \alpha, \varphi \) and \( \chi \), give equations for \( \dot{\delta} \) and \( \dot{\kappa} \); equations (33) and (34), together with equations (32), (29) and (30) to determine \( \alpha, \varphi \) and \( \kappa \), give equations for \( \dot{\delta} \) and \( \dot{v} \), etc.

It is known that in the small-scale limit (i.e. inside the visual horizon) in the matter-dominated era, the density, velocity and gravitational potential variables (\( \delta, v \) and \( \alpha \)) in the zero-shear gauge coincide exactly with the Newtonian results (Hwang & Noh 1999); this is true even for the variables in the uniform-expansion gauge. We have shown that the correspondences continue to be valid even to the second-order perturbation (Hwang et al. 2012) and to fully non-linear and exact order in the infinite-speed-of-light limit (Hwang & Noh 2013).

6.2 Uniform-curvature gauge

In the uniform-curvature gauge we set
\[ \varphi \equiv 0. \]  
(80)

We have \( R^{hij} = 0 \), thus flat in the flat background. The metric becomes
\[ ds^2 = -a^2 (1 + 2\alpha) d\eta^2 - 2a\chi d\eta dx^i + a^2 \delta_{ij} dx^i dx^j. \]  
(81)

Equations (28)–(34) give a complete set of equations for variables \( \delta, v, \kappa, \chi \) and \( \alpha \). All perturbation variables in the uniform-curvature gauge are gauge invariant with
\[ \delta = \delta_\varphi, \quad v = v_\varphi, \quad \kappa = \kappa_\varphi, \quad \chi = \chi_\varphi, \quad \alpha = \alpha_\varphi. \]  
(82)

In the linear theory, the uniform-curvature gauge is useful to handle the scalar field perturbation (Field & Shepley 1968; Lukash 1980a,b; Sasaki 1986; Mukhanov 1988; Hwang 1994; Hwang & Noh 2005). To the linear order, with the scalar field \( \Phi \) decomposed as \( \Phi = \phi + \delta \phi \), we have \( \delta \phi = \delta \phi - \dot{\phi} \xi_\Phi \), thus
\[ \delta \phi_\varphi \equiv \delta \phi - \frac{\phi}{H} \theta \equiv -\frac{\dot{\phi}}{H} \phi \]  
(83)

is gauge invariant. For fully non-linear treatment of the scalar field perturbation in the uniform-field gauge (\( \delta \phi \equiv 0 \) to the non-linear order), see Section 9.

6.3 Uniform-expansion gauge

In the uniform-expansion gauge we set
\[ \kappa \equiv 0. \]  
(84)

We have \( \bar{\theta}^{(n)} = -K = 3H \), thus the expansion rate of the normal frame vector field is uniform in space. Equations (28)–(34) give a complete set of equations for variables \( \delta, v, \chi, \varphi \) and \( \alpha \). All perturbation variables in the uniform-expansion gauge are gauge invariant with
\[ \delta = \delta_\kappa, \quad v = v_\kappa, \quad \varphi = \varphi_\kappa, \quad \chi = \chi_\kappa, \quad \alpha = \alpha_\kappa. \]  
(85)

As in the zero-shear gauge, the uniform-expansion gauge also shows small-scale Newtonian correspondence of \( \delta, v \) and \( \alpha \) up to the second order in perturbation (Hwang et al. 2012). It also has correct Newtonian limit in the infinite-speed-of-light limit (Hwang & Noh 2013).
6.4 Uniform-density gauge

In the uniform-density gauge we set

$$\delta \equiv 0,$$  \hspace{1cm} (86)

thus the density is uniform in the hypersurface. Equations (28)–(34) give a complete set of equations for variables $v$, $\kappa$, $\chi$, $\varphi$ and $\alpha$. All perturbation variables in the uniform-density gauge are gauge invariant with

$$v = v_j, \quad \kappa = \kappa_j, \quad \varphi = \varphi_j, \quad \chi = \chi_j, \quad \alpha = \alpha_j.$$  \hspace{1cm} (87)

The curvature perturbation $\varphi$ in the comoving gauge ($\varphi_j$), in the uniform expansion gauge ($\varphi_e$) and in the uniform-density gauge ($\varphi_d$) are known to have nice conservation behaviours in the large-scale (super-sound-horizon scale) limit to the second order in perturbations (Hwang & Noh 2007), and to general non-linear order based on the spatial gradient expansion method (Lyth, Malik & Sasaki 2005). The proof based on our exact and fully non-linear perturbation equations deserves a further examination.

7 GENERATION OF VORTICITY

We consider the generation of vorticity (rotation or vector-type perturbation) from the pure scalar-type perturbation. We consider the comoving gauge. Thus, we have $v_i = 0$ in the $v_j$ terms multiplied by perturbation terms, and have $v_i = v^{(v)}_i$ in the pure $v$ term without perturbations multiplied. From equation (34) we have

$$\frac{1}{a^2} \left[ a^2 (\mu + p) v^{(v)}_j \right] = -\tilde{\rho} - (\tilde{\mu} + \tilde{p}) \frac{N_i}{N}.$$  \hspace{1cm} (88)

Thus, we have

$$\frac{1}{a^2} \left[ a^2 (\mu + p) v^{(v)}_j \right] = -\frac{\tilde{\mu} v^{(v)}_j}{a(\mu + p)},$$  \hspace{1cm} (89)

which is true to the fully non-linear order; notice that the right-hand side vanishes for an ideal fluid with $\tilde{p} = \tilde{p}(\tilde{\mu})$.

Now, we consider the general case without taking the gauge condition. In terms of the covariant equations, from equation (8) in Hawking (1966) we have

$$\tilde{h}^{a}_{ij} \tilde{h}^{b}_{ij} \left( \tilde{\omega}_{a,b} - \tilde{\omega}_{a,d} \right) = -\frac{2}{3} \tilde{\sigma}_{a,b} + 2 \tilde{\sigma}_{a} \tilde{\omega}_{a,b}.$$  \hspace{1cm} (90)

For the covariant notations see Appendix C. Using equation (38), equation (C9) becomes

$$\tilde{\omega}_{d,j} = a v^{(v)}_{[i,j]} + \frac{1}{\tilde{\mu} + \tilde{p}} a v_{[i,j]}.$$  \hspace{1cm} (91)

For the vorticity generation from the pure scalar-type perturbation in the comoving gauge, we have $\tilde{\omega}_{d,j} = a v^{(v)}_{[i,j]}$, and from equations (90), (C11) and (C12) we arrive at equation (89).

The examination of equation (91) leads to the following conclusions valid to the fully non-linear order. In the absence of pressure, we have $\tilde{\omega}_{d,j} = 0$ for $v^{(v)}_i = 0$ independently of the gauge condition (we wish to thank the referee for clarifying comments on our previously confused state). This conclusion is consistent with equation (90) which shows that in the absence of pressure equation of $\tilde{\omega}_{d,b}$ becomes homogeneous (we have $a_d = 0$ from equation C12), thus an irrotational ($\tilde{\omega}_{d,b} = 0$) fluid remains irrotational independently of the gauge condition. For works on related issues, see Christopherson, Malik & Matravers (2009) and Lu et al. (2009).

8 GENERATION OF GRAVITATIONAL WAVES FROM SCALAR- AND VECTOR-TYPE PERTURBATIONS

Fully non-linear and exact formulation including the tensor-type perturbation is supposed to be a complicated subject which is left for future investigation. Here, we consider a much simpler case with linear tensor-type perturbation. In the presence of the tensor-type perturbation a change occurs in the spatial part of the metric in equation (9) as

$$\tilde{g}_{ij} = a^2 \left[ (1 + 2\varphi) \delta_{ij} + 2h_{ij} \right],$$  \hspace{1cm} (92)

where $h_{ij}$ is the transverse ($h_{ij}^T \equiv 0$) and tracefree ($h_{ij}^T \equiv 0$) tensor-type perturbation; indices of $h_{ij}$ are raised and lowered by $\delta_{ij}$ as the metric; only in this section $h_{ij}$ indicates the tensor-type perturbation. By keeping only linear-order terms in $h_{ij}$ we can update quantities in Appendix B. In our basic perturbation equations in (28)–(34), the linear tensor-type perturbation contributes only in equation (32) by simply adding the following term

$$\tilde{h}_{ij}^T + 3H \tilde{h}_{ij}^T - \frac{\Lambda}{a^2} \tilde{h}_{ij}^T.$$  \hspace{1cm} (93)
in the left-hand side. We ignore the tensor-type anisotropic stress. Let us write equation (32) including the linear tensor-type perturbation as

\[ \ddot{h}_{ij} + 3H \dot{h}_{ij} - \frac{\Delta}{a^2} h_{ij} \equiv n_{ij}, \]

(94)

where we moved all the terms in equation (32) to the right-hand side and called it \( n' \), then lowered the index by \( \delta_{ij} \). The right-hand side of this equation includes linear parts of the scalar- and vector-type perturbations which need to be removed to get the pure tensor-type perturbation equation generated by the non-linear scalar- and vector-type perturbations. By the following operation, we can separate the linear part of the scalar- and vector-type contributions (see equation 210 in Noh & Hwang 2004)

\[ \ddot{h}_{ij} + 3H \dot{h}_{ij} - \frac{\Delta}{a^2} h_{ij} = s_{ij}, \]

(95)

\[ s_{ij} \equiv n_{ij} - 2\Delta^{-1} \nabla \nabla n_{ij} + \frac{1}{2} \nabla \nabla (\nabla \nabla + \delta_{ij} \Delta) n_{ij}. \]

(96)

This can be regarded as the equation for gravitational waves (tensor-type perturbation) generated from pure scalar- and vector-type perturbations to the fully non-linear order. The non-linear terms in \( s_{ij} \) still depend on the temporal gauge condition, and consequently the gravitational waves generated from the scalar- and vector-type perturbations do depend on the temporal gauge choice. In the perturbation approach, \( h_{ij} \) should be regarded as the same-order perturbation as the one considered in \( s_{ij} \); i.e. with an expansion \( h_{ij} = h_{ij}^{(1)} + h_{ij}^{(2)} + \cdots \), where index (1) and (2) indicating the order of perturbation, for \( s_{ij} \) quadratic-order perturbations \( h_{ij} \) is the same as \( h_{ij}^{(2)} \), etc., thus depending on the gauge choice. This subject deserves further studies.

### 9 SCALAR FIELD IN THE COMOVING GAUGE

We consider a minimally coupled scalar field \( \tilde{\phi} \) with \( \tilde{T}_{ab} = \tilde{\phi}_{,a} \tilde{\phi}_{,b} - \frac{1}{2} \tilde{\phi}^{,c} \tilde{\phi}_{,c} V(\tilde{\phi}) \tilde{g}_{ab} \)

(97)

and the equation of motion

\[ \tilde{\phi}^{,c} = \tilde{V}_{,c}. \]

(98)

The fluid quantities can be read from equation (3).

The fluid quantities and the equation of motion to the fully non-linear order can be derived in a gauge-ready form. Here, we only consider a fluid formulation of the scalar field in the comoving gauge; the gauge-ready formulation is presented in Appendix F. From equations (4) and (97) we have

\[ \tilde{u}_i = -\frac{1}{\tilde{\mu}} \tilde{T}_{la} \tilde{u}_l = -\frac{\tilde{\phi}_{,i}}{\tilde{\phi}^{,c} \tilde{u}_c}. \]

(99)

Thus, the comoving gauge (\( v \equiv 0 \)) implies the uniform-field gauge (\( \delta \phi \equiv 0 \)) and vice versa to the fully non-linear order; we set \( \tilde{\phi} = \phi + \delta \phi \), where \( \phi \) is the background order scalar field. In this gauge we can show

\[ \tilde{\mu} = \frac{1}{2N^2} \tilde{\phi}^2 + V, \quad \tilde{p} = \frac{1}{2N^2} \tilde{\phi}^2 - V, \quad \tilde{\pi}_{ab} = 0. \]

(100)

Thus, to the background order, we have

\[ \mu = \frac{1}{2} \tilde{\phi}^2 + V, \quad p = \frac{1}{2} \tilde{\phi}^2 - V, \]

(101)

and to the fully non-linear order, we have

\[ \delta p = \delta \mu = -\frac{1}{2N^2} \tilde{\phi}^2 \left( 2a + \frac{X^X X_0}{a^2 (1 + 2\varphi)} \right), \]

(102)

with vanishing anisotropic stress. Therefore, the ideal fluid equations in equations (28)–(34) under the comoving gauge remain valid with the perturbed equation of state given as \( \delta p = \delta \mu \).

In Appendix F, we present the full formulation in the case of a minimally coupled scalar field.

### 10 DISCUSSION

Extension or feasibility of similar fully non-linear formulation including the following cases deserves future investigations: (i) background spatial curvature, (ii) anisotropic stress, (iii) scalar fields (see Appendix F for a minimally coupled scalar field), (iv) multiple components

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of fluids and fields, (v) class of generalized gravity theories, (vi) the electric and magnetic fields, (vii) the covariant equations and the Weyl tensors, (viii) the tensor-type perturbation, (ix) null geodesic for Sachs-Wolfe effect, (x) gravitational lensing, (xi) Boltzmann equations for photons, and massless and massive neutrinos, (xii) gauge transformation properties, (xiii) expression of gauge-invariant combinations, (xiv) equations in mixed gauge conditions, etc. At the moment, the first seven are trivial (some could be tedious though) extensions while the remaining ones need closer examinations for their feasibilities. Implementations of all the above cases (except for x) were made to the second order in perturbations in Noh & Hwang (2004), Hwang & Noh (2007) and Hwang et al. (2012).

Neglecting the TT part of the metric assumed in equation (9) is indeed a quite serious constraint and drawback in our formulation aiming for fully non-linear and exact analysis. In this regard, our formulation can be regarded as approximate in the sense of ignoring the TT part of the three-space hypersurface. Ignoring the TT part of the perturbation is consistent with the first-order post-Newtonian (1PN) and even 2PN approximation because gravitational waves are known to show up from the 2.5PN order (Chandrasekhar & Esposito 1970). As far as we know, the presence of TT part does not allow us to get the inverse metric in exact form, thus forbidding us to proceed the exact formulation. The TT part of the metric corresponds to gravitational waves to the linear order. In general, however, the TT part corresponds to the TT-type distortions of the spatial hypersurface in a non-linear context (Matarrese & Terranova 1996). Therefore, to the non-linear order ignoring the TT part should be regarded as a physical assumption restricting the potential applications. As emphasized in Section 8, however, the TT part can always be handled perturbatively even to the non-linear order. Although to the non-linear order, the TT part is gauge dependent, as we have shown in Section 2 it does not affect the gauge issue of the scalar- and vector-type perturbations addressed in this work, see the non-linear gauge transformation issue addressed below equation (22).

We anticipate potentially wide applications of our exact and fully non-linear perturbation formulation, not only in higher order perturbation theory, but also in the averaging, fitting and back-reaction approaches in theoretical cosmology (Ellis 1984; Ellis & Stoeger 1987; Clarkson et al. 2011).

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APPENDIX A: ADM (3+1) EQUATIONS REVIEW

The ADM formulation (Arnowitt et al. 1962) is based on splitting the space–time into the spatial and the temporal parts using a normal four-vector field $\vec{n}_a$. The metric is written as

$$\tilde{g}_{00} = -N^2 + N^i N_i, \quad \tilde{g}_{i0} = N_i, \quad \tilde{g}_{ij} = h_{ij}, \quad \tilde{g}^{00} = -N^{-2}, \quad \tilde{g}^{ii} = N^{-2} N^i, \quad \tilde{g}^{ij} = h^{ij} - N^{-2} N^i N^j,$$

where the index of $N_i$ is raised and lowered by $h_{ij}$ as the metric, and $h^{ij}$ is an inverse metric of $h_{ij}$; for meanings of the ADM variables, see Smarr & York (1978). The normal four-vector $\vec{n}_a$ is introduced as

$$\vec{n}_0 = 0, \quad \vec{n}_i = 0, \quad \vec{n}^0 = N^{-1}, \quad \vec{n}^i = -N^{-1} N^i.$$

The fluid quantities are defined as

$$E \equiv \vec{n}_a \vec{n}_b T^{ab}, \quad J_i \equiv -\vec{n}_a T^a_i, \quad S_{ij} \equiv \vec{T}_{ij}, \quad S \equiv h^{ij} S_{ij}, \quad \vec{S}_{ij} \equiv S_{ij} - \frac{1}{3} h_{ij} S,$$

where the indices of $J_i$ and $S_{ij}$ are raised and lowered by $h_{ij}$. The extrinsic curvature is introduced as

$$K_{ij} \equiv \frac{1}{2N} \left( N_{ij} + N_{j,i} - h_{ij,0} \right), \quad K \equiv h^{ij} K_{ij}, \quad \vec{K}_{ij} \equiv K_{ij} - \frac{1}{3} h_{ij} K,$$

where the index of $K_{ij}$ is raised and lowered by $h_{ij}$. A colon ':' denotes a covariant derivative based on $h_{ij}$, $\Gamma^{(h)}_{j,k}^{(h)}$ is the connection based on $h_{ij}$ as the metric, $\Gamma^{(h)}_{j,k}^{(h)}$ is the connection based on $h_{ij}$ as the metric,

$$R^{(h)}_{jkl} \equiv \Gamma^{(h)}_{j,k}^{(h)} - \Gamma^{(h)}_{j,l}^{(h)} h^{i,k} - \Gamma^{(h)}_{j,l}^{(h)} h^{i,k} - \Gamma^{(h)}_{j,m}^{(h)} h^{i,k} \Gamma^{(h)}_{l,m}^{(h)}, \quad R_{ij}^{(h)} \equiv R^{(h)}_{ij}, \quad R^{(h)} \equiv h^{ij} R_{ij}^{(h)}, \quad \vec{R}_{ij}^{(h)} \equiv R_{ij}^{(h)} - \frac{1}{3} h_{ij} R^{(h)}.$$

A complete set of the ADM equations is the following (Bardeen 1980; Noh & Hwang 2004)

$$R^{(h)} = \vec{K}^i \vec{K}_i - 2\frac{K^2}{3} + 16\pi G E + 2\Lambda,$$

$$\vec{K}_{ij} = \frac{2}{3} K_{ij} = 8\pi G J_i,$$
\[ K_{,0}N^{-1} - K_{,i}N^iN^{-1} + N^j_iN^{-1} - \overline{\mathbf{K}}^j_{,i} \overline{\mathbf{K}}_{ij} - \frac{1}{3}K^2 - 4\pi G (E + S) + \Lambda = 0, \quad \tag{A8} \]

\[ \overline{\mathbf{K}}^j_{,0}N^{-1} - \overline{\mathbf{K}}^j_{,k}N^kN^{-1} + \overline{\mathbf{K}}^j_{,i}N^iN^{-1} - \overline{\mathbf{K}}^j_{,k}N^kN^{-1} = K\overline{\mathbf{K}}^j_j - \left( N^j_j - \frac{1}{3}\delta_j^jN^k_k \right) N^{-1} + \overline{\mathbf{R}}^{jk}j_{j} - 8\pi G \overline{S}^j_j, \quad \tag{A9} \]

\[ E_{,0}N^{-1} - E_{,i}N^iN^{-1} - \left( E + \frac{1}{3}S \right) - \overline{\mathbf{S}}^j_{,i} \overline{\mathbf{K}}_{ij} + N^2 (N^j_j')_j = 0, \quad \tag{A10} \]

\[ J_{,0}N^{-1} - J_{,i}N^iN^{-1} - J_{,k}N^kN^{-1} - KJ_i + EN_{,i}N^{-1} + S_{,i,j} + S_{,i}N_jN^{-1} = 0. \quad \tag{A11} \]

These are the ADM energy constraint, the ADM momentum constraint, trace of ADM propagation, tracefree ADM propagation, the ADM energy conservation and the ADM momentum conservation equations, respectively. Equations (A6)–(A11) together with definition of \( K \) in equation (A4) provide the fundamental perturbation equations presented in equations (28)–(34). In Appendix B, we present details of steps useful for the derivation.

**APPENDIX B: DERIVATION OF FULLY NON-LINEAR PERTURBATIONS**

Our metric convention is

\[
\tilde{g}_{00} = -a^2(1 + 2\alpha), \quad \tilde{g}_{0i} = -a\chi_i, \quad \tilde{g}_{ij} = a^2(1 + 2\varphi)\delta_{ij}, \quad \tag{B1}
\]

where the index of \( \chi_i \) is raised and lowered by \( \delta_{ij} \) as the metric. We assume that \( a \) is a function of conformal time \( (\chi^0 = \eta) \) only, whereas \( \alpha \), \( \chi_i \), and \( \varphi \) are general functions of space and time, but we do not assume these to be small in amplitudes. Justification of our metric convention is made in Section 2. The inverse metric is

\[
\tilde{g}^{00} = \frac{1 + 2\varphi}{a^2(1 + 2\varphi)(1 + 2\alpha) + \chi^2\chi_i/a^2}, \quad \tilde{g}^{0i} = \frac{-\chi^i/a}{a^2(1 + 2\varphi)(1 + 2\alpha) + \chi^2\chi_i/a^2},
\]

\[
\tilde{g}^{ij} = \frac{1}{a^2(1 + 2\varphi)} \left( \delta^{ij} - \frac{\chi^i\chi^j/a^2}{(1 + 2\varphi)(1 + 2\alpha) + \chi^2\chi_i/a^2} \right).
\]

From equation (A1) the ADM metric can be identified as

\[
N = a\sqrt{1 + 2\alpha + \frac{\chi^2\chi_i}{a^2(1 + 2\varphi)}}, \quad N_i = -a\chi_i, \quad N^i = -\frac{\chi^i}{a(1 + 2\varphi)},
\]

\[
h_{ij} = a^2(1 + 2\varphi)\delta_{ij}, \quad h^{ij} = \frac{1}{a^2(1 + 2\varphi)}\tilde{g}^{ij}, \quad \tag{B3}
\]

thus

\[
\tilde{g}^{00} = -\frac{1}{a^2N^2}, \quad \tilde{g}^{0i} = -\frac{\chi^i}{a^2N^2(1 + 2\varphi)}, \quad \tilde{g}^{ij} = \frac{1}{a^2(1 + 2\varphi)} \left( \delta^{ij} - \frac{\chi^i\chi^j}{a^2N^2(1 + 2\varphi)} \right),
\]

and

\[
\tilde{n}_i = 0, \quad \tilde{n}_0 = -aN, \quad \tilde{n}^i = \frac{\chi^i}{a^2N(1 + 2\varphi)}, \quad \tilde{n}^0 = \frac{1}{aN}, \quad \tag{B4}
\]

The three-space connection and curvatures are

\[
\Gamma^{(h)}_{ijk} = \frac{1}{1 + 2\varphi} \left( \varphi, \delta^i_{jk} + \varphi^i \delta^j_{ik} - \varphi^j \delta^i_{jk} \right), \quad \Gamma^{(h)}_{ijk} = \frac{3\varphi_i}{1 + 2\varphi},
\]

\[
R^{(h)}_{ij} = -\varphi_{ij} + \frac{3}{1 + 2\varphi} \left( \varphi_{ij} - \frac{\Delta\varphi}{1 + 2\varphi} - \frac{\varphi^k\varphi_k}{(1 + 2\varphi)^2} \right) \delta_{ij}, \quad R^{(h)}_{ij} = \frac{2}{a^2(1 + 2\varphi)^2} \left( -2\Delta\varphi + 3\varphi^i\varphi_k \frac{1}{1 + 2\varphi} \right),
\]

\[
\overline{R}^{(h)}_{ij} = \frac{1}{a^2(1 + 2\varphi)^2} \left[ -\varphi_{ij} + \frac{3}{1 + 2\varphi} \left( \varphi_{ij} - \frac{\Delta\varphi}{1 + 2\varphi} - \frac{\varphi^k\varphi_k}{1 + 2\varphi} \right) \right].
\]

The extrinsic curvature gives

\[
K_{ij} = -\frac{a^2}{N} \left[ (H + \psi + 2\varphi) \delta_{ij} + \frac{1}{2a^2} (\chi_i, + \chi_{i,j} - \frac{1}{2} \chi^k \chi_{k,j} \right) - \frac{1}{(1 + 2\varphi)} (\chi_i\varphi_j + \chi_j\varphi_i - \chi^k\varphi_k\delta_{ij}) \right],
\]

\[
K = -\frac{1}{N(1 + 2\varphi)} \left[ 3(H + \psi + 2\varphi) + \frac{1}{a^2} \chi^k \chi_{k,j} + \frac{\chi^k\varphi_k}{a^2(1 + 2\varphi)} \right] \equiv -3H + \kappa,
\]
\( K_j = -\frac{1}{a^2 N(1 + 2\varphi)} \left[ \frac{1}{2} \left( x'_j + x''_j \right) - \frac{\delta_j^l x^l_x} {1 + 2\varphi} \left( x'_r + x''_r \right) \right] \),

\( K_j K^j = \frac{1}{a^2 N^2(1 + 2\varphi)^2} \left\{ \frac{1}{2} \left( x'_j + x''_j \right) + \frac{4}{1 + 2\varphi} \left[ \frac{1}{2} \left( x'_r + x''_r \right) \right] \right\} \). (B7)

The fluid four-vector is

\( \tilde{u}_i = av_i, \quad \tilde{u}_0 = -aN \left( 1 + \frac{v^4 v_k}{1 + 2\varphi} \right) \),

\( \tilde{u}' = \frac{v^i}{a(1 + 2\varphi)} + \frac{v}{a^{2}N(1 + 2\varphi)} \left[ 1 + \frac{v^4 v_k}{1 + 2\varphi} \right], \quad \tilde{u}^0 = \frac{1}{aN} \left( 1 + \frac{v^4 v_k}{1 + 2\varphi} \right) \). (B8)

The ADM fluid quantities become

\( E = \tilde{\mu} + (\tilde{\mu} + \tilde{p}) \frac{v^i v_k}{1 + 2\varphi}, \quad J_i = a \left( \tilde{\mu} + \tilde{p} \right) \frac{v^i v_k}{1 + 2\varphi}, \quad J^j = \frac{\tilde{\mu} + \tilde{p}}{a(1 + 2\varphi)} \frac{v^i v_k}{1 + 2\varphi}, \quad S'_j = \tilde{\rho} \delta^j_i + (\tilde{\mu} + \tilde{p}) v v^i v_j, \quad S = 3\tilde{p} + \frac{\tilde{\mu} + \tilde{p}}{1 + 2\varphi} v^i v_k, \quad \tilde{S}'_j = \frac{\tilde{\mu} + \tilde{p}}{1 + 2\varphi} \left( v^i v_j - \frac{\delta^i_j v^k}{1 + 2\varphi} \right). \) (B10)

Notice that for \( v_i = 0 \) (ignoring the vector-type perturbation \( v^i_{(0)} = 0 \), and taking the comoving gauge \( v^i = 0 \)) the energy-momentum tensor and the ADM fluid quantities are simplified to

\[ \tilde{T}_0^0 = -\tilde{\mu}, \quad \tilde{T}_i^j = 0, \quad \tilde{\rho} \delta^j_i + (\tilde{\mu} + \tilde{p}) v v^i v_j, \quad E = \tilde{\mu}, \quad J_i = 0, \quad S = 3\tilde{p}, \quad \tilde{S}'_j = 0. \] (B11)

Using the above quantities, from equations (A4) for \( K \) and equations (A6)–(A11) we can derive equations (28)–(34), respectively.

**APPENDIX C: COVARIANT FORMULATION**

Here, we present the covariant kinematic quantities based on the normalized fluid four-vector \( \tilde{u}_a \). We have (Ehlers 1993; Ellis 1971, 1973)

\[ \tilde{h}^{a}_{b} \tilde{h}^{b}_{c} \tilde{u}_{a,c} \equiv \tilde{h}^{a}_{b} \tilde{h}^{b}_{c} \tilde{u}_{a,c} \equiv \tilde{h}^{a}_{b} \tilde{h}^{b}_{c} \tilde{u}_{a,c} \equiv \tilde{u}_{a,b} + \tilde{u}_{a,b}, \]

\[ \tilde{h}_{a,b} = \tilde{g}_{a,b} + \tilde{u}_{a,b}, \quad \tilde{h}_{a,b} \tilde{u}^{b} = 0, \quad \tilde{h}_{a,b} \tilde{u}^{b} = 0, \quad \tilde{h}_{a,b} \tilde{u}^{b} = 0, \quad \tilde{h}_{a,b} \tilde{u}^{b} = 0, \quad \tilde{h}_{a,b} \tilde{u}^{b} = 0. \] (C1)

\[ \tilde{h}_{a,b} \tilde{u}^{b} = 0, \quad \tilde{h}_{a,b} \tilde{u}^{b} = 0. \]

An overdot with tilde indicates the covariant derivative along \( \tilde{u}^a \). The quantities \( \tilde{\vartheta}, \tilde{\varphi}, \tilde{\sigma}, \tilde{\kappa} \) and \( \tilde{\alpha}_{ab} \) are the expansion scalar, the acceleration vector, the shear tensor and the rotation (vorticity) tensor, respectively, of the \( \tilde{u}_a \) flow.

Using equations (A1), (B3), (B6) and (B8) we can show (the space–time connection \( \tilde{\Gamma}_{a}^{b} \), in terms of the ADM notations is presented in equation 6 of Noh & Hwang 2004)

\[ \tilde{\mu} = \left[ \frac{1}{aN} \left( \frac{v^4 v_k}{1 + 2\varphi} + \frac{\left( N v^4 \right) v}{1 + 2\varphi} \right) \right], \quad \tilde{\vartheta} = \left[ (3H - \kappa) \left( \frac{v^4 v_k}{1 + 2\varphi} + \frac{a N \left( N v^4 \right) v}{1 + 2\varphi} \right) \right]. \] (C2)

\[ \tilde{\vartheta} = \left[ (3H - \kappa) \left( \frac{v^4 v_k}{1 + 2\varphi} + \frac{a N \left( N v^4 \right) v}{1 + 2\varphi} \right) \right]. \] (C3)
\[ \tilde{a}_i \left( \frac{1}{1 + 2\varphi} \right) = \left( \frac{\tilde{v}_j^{i} v_k}{1 + 2\varphi} \frac{N_j}{N} + \frac{\tilde{v}_j^{i} v_k}{1 + 2\varphi} \frac{v_j^{i} \varphi_{,j}}{1 + 2\varphi} + \frac{1}{aN} \sqrt{1 + \frac{\tilde{v}_j^{i} v_k}{1 + 2\varphi} \left( (av_i)_{,0} + \frac{v_j^{i} X_{,j}^k}{1 + 2\varphi} + v_k \left( \frac{\chi^k}{1 + 2\varphi} \right)_{,j} \right)} \right). \]  

(C4)

\[ \tilde{\sigma}_{ij} = -K_{ij} \sqrt{1 + \frac{\tilde{v}_j^{i} v_k}{1 + 2\varphi}} + av_{(i, j)}{}^l - \frac{a}{1 + 2\varphi} \left( v_{(i,j),j} + (v_{(i,j)})_{,i} - 2\frac{3}{2} v_{(i,j)} \delta_{ij} \right) + av_i \left( \frac{1}{1 + 2\varphi} \frac{N_j}{N} \right) \]

\[ + \frac{v_{j,k} V_{,k}}{1 + 2\varphi} - \frac{v_{j,k} v_{,j}}{1 + 2\varphi} \left( \frac{1 + 2\varphi}{1 + 2\varphi} \right)^2 + \frac{1}{aN} \sqrt{1 + \frac{\tilde{v}_j^{i} v_k}{1 + 2\varphi} \left( (av_{j,i})_{,0} + \frac{v_{j,k} X_{,k}^i}{1 + 2\varphi} + v_i \left( \frac{\chi^i}{1 + 2\varphi} \right)_{,j} \right)} \right) \]

\[ - \frac{1}{3 a^2 v_j v_j} \left( \frac{v_{j,k} v_{,k}}{1 + 2\varphi} + (3H - \dot{k}) \sqrt{1 + \frac{\tilde{v}_j^{i} v_k}{1 + 2\varphi}} \right) \]

\[ = - \frac{1}{3} \frac{a}{\sqrt{1 + 2\varphi}} \left( \delta_{ij} + v_{ij} \right) \left( \frac{N \tilde{v}^k}{1 + 2\varphi} + \left( \partial_0 + \frac{\chi^k}{a(1 + 2\varphi)} \right) \tilde{v}_i \right) \]

\[ \left( \frac{1 + 2\varphi}{1 + 2\varphi} \right) \sqrt{1 + \frac{\tilde{v}_j^{i} v_k}{1 + 2\varphi}} \right). \]  

(C5)

\[ \tilde{\omega}_{ij} = av_{(i, j)} - av_i \left( \frac{1}{1 + 2\varphi} \frac{N_j}{N} + \frac{v_{j,k} V_{,k}}{1 + 2\varphi} - \frac{v_{j,k} v_{,j}}{1 + 2\varphi} \left( \frac{1 + 2\varphi}{1 + 2\varphi} \right)^2 + \frac{1}{aN} \sqrt{1 + \frac{\tilde{v}_j^{i} v_k}{1 + 2\varphi} \left( (av_{j,i})_{,0} + \frac{v_{j,k} X_{,k}^i}{1 + 2\varphi} + v_i \left( \frac{\chi^i}{1 + 2\varphi} \right)_{,j} \right)} \right) \]

\[ \left( \frac{1 + 2\varphi}{1 + 2\varphi} \right) \sqrt{1 + \frac{\tilde{v}_j^{i} v_k}{1 + 2\varphi}} \right). \]  

(C6)

where \( \partial_0 \equiv -\frac{\partial}{\partial \tau} \).

Taking a normal frame with \( \tilde{u}_a = \tilde{n}_a \), we have \( v_i = 0 \), thus

\[ \tilde{\theta} = -K = 3H - \dot{k}, \quad \tilde{\omega}_{ij} = \tilde{n}_i / \tilde{n}_j, \quad \tilde{\sigma}_{ij} = -K_{ij}, \quad \tilde{\omega}_{ij} = 0. \]  

(C7)

In the normal frame we have to recover the energy flux term \( \tilde{q}_a \) in equation (2), see below equation (C15).

Using the covariant momentum conservation in equation (38), equations (C4) and (C6) become

\[ \tilde{\alpha}_{ij} = \tilde{p}_{ij} + av_{ij} \frac{\sqrt{N}}{a(1 + 2\varphi)} \left( \frac{v_i}{1 + 2\varphi} + \frac{\chi^i}{aN} \right) \tilde{v}_i \]  

\[ = \frac{1}{\mu + p} \left( \tilde{p}_i + \mu \tilde{v}_i \frac{\sqrt{N}}{a(1 + 2\varphi)} \right) \left( \frac{v_i + \frac{\chi^i}{aN}}{1 + 2\varphi} \right) \tilde{v}_i \]  

\[ \tilde{\alpha}_{ij} = av_{ij} - \frac{1}{\mu + p} av_i \tilde{p}_{ij}. \]  

(C8)

Parts of the covariant equations are (Ehlers 1961; Hawking 1966; Ellis 1971, 1973)

\[ \tilde{\alpha}_{ij} = -K_{ij} = 3H - \dot{k}, \quad \tilde{\sigma}_{ij} = \frac{N_j}{N} \]  

\[ \tilde{\omega}_{ij} = \frac{N_j}{N}, \quad \tilde{\sigma}_{ij} = -K_{ij}, \quad \tilde{\omega}_{ij} = 0. \]  

(C9)

These are the Raychaudhuri equation, and the covariant energy conservation and the covariant momentum conservation equations, respectively; the latter two equations are presented in the energy-frame with vanishing energy flux \( \tilde{q}_a \equiv 0 \) (Ellis 1971, 1973; Hwang & Vishniac 1990); the fluid four vector \( \tilde{u}_a \) in the energy frame (\( \tilde{q}_a \equiv 0 \)) coincides with the normal frame four-vector \( \tilde{n}_a \) for \( v_i = 0 \) (the irrotational and the comoving gauge). By combining equations (C10)–(C12) we have (Jackson 1972, 1993; Hwang & Vishniac 1990)

\[ \frac{\tilde{\alpha}_{ij}}{\mu + p} = 4\pi G \left( \tilde{\mu} + 3\tilde{p} \right) - \Lambda + \frac{4}{3} \tilde{\theta} - \frac{\tilde{\omega}_{ab}}{\tilde{\alpha}_{ab}} - \tilde{\omega}_{ab} \tilde{\omega}_{ab} - \tilde{\theta} \tilde{\alpha}_{ab} + \tilde{\alpha}_{ab} \tilde{X}_{ab} \left( \frac{\tilde{p}_a}{\mu + p} \right)_{,b}. \]  

(C13)

Using the covariant kinematic quantities based on \( \tilde{u}_a \) presented above, we can derive the non-linear perturbation equations in gauge-ready forms. Equations (C11) and (C12) give equations (37) and (38), respectively. Equation (C10) in general leads to a complicated combination; in the normal frame it gives equation (31) using the following relations of the fluid quantities between the energy frame and the normal frame.

The fluid quantities in the normal frame are similarly introduced as in equation (3) now based on the normal four-vector as

\[ \tilde{\alpha}_{ij} = \tilde{T}_{ab} \tilde{\alpha}_{ab} \tilde{h}^{ab}, \quad \tilde{\theta}_{ij} = \frac{1}{3} \tilde{T}_{ab} \tilde{h}^{ab} \tilde{n}_{ab}, \quad \tilde{\omega}_{ij} = -\tilde{T}_{ab} \tilde{\omega}_{ab}, \quad \tilde{\omega}_{ab} = \tilde{T}_{ab} \tilde{\omega}_{ab} \tilde{h}^{ab}, \quad \tilde{\omega}_{ij} = \tilde{T}_{ij} \tilde{h}_{ij} \tilde{h}^{ij} \tilde{h}^{ab} - \tilde{\omega}_{ab} \tilde{h}_{ab} - \tilde{\alpha}_{ab} \tilde{\alpha}_{ab} \tilde{h}_{ab}, \]  

(C14)
where $\tilde{h}^{(\alpha)}_{ab} = g_{ab} + \tilde{n}_a \tilde{n}_b$. Using equations (4), (B1), (B4), (B5) and (B8), we have

$$\tilde{\mu}^{(\alpha)} = \tilde{\mu} + (\tilde{\mu} + \tilde{p}) \frac{v^i v_k}{1 + 2 \varphi}, \quad \tilde{p}^{(\alpha)} = \tilde{p} + \frac{1}{3} (\tilde{\mu} + \tilde{p}) \frac{v^i v_k}{1 + 2 \varphi}, \quad \tilde{q}_i^{(\alpha)} = a (\tilde{\mu} + \tilde{p}) \tilde{\gamma}_i, \quad \tilde{\pi}^{(\alpha)}_{ij} = a^2 (\tilde{\mu} + \tilde{p}) \left( v_i v_j - \frac{1}{3} \delta_i^j v^k \delta_k \right). \quad (C15)$$

In the normal frame, the energy flux $\tilde{q}_i^{(\alpha)}$ plays the role of the fluid velocity ($\tilde{u}_i = a v_i$) in the energy frame.

Equations (37) and (38) follow from the covariant energy and momentum conservation equations based on projecting along $\tilde{u}_a$ and $\tilde{h}_{ab}$ as

$$0 = \tilde{T}_{ab}^{(\alpha)} \tilde{a}^b = -\tilde{\mu} - (\tilde{\mu} + \tilde{p}) \tilde{\theta} \tilde{a}_b, \quad 0 = \tilde{T}_{ab}^{\gamma} \tilde{h}_a = (\tilde{\mu} + \tilde{p}) \tilde{a}_a + \tilde{h}_{ab} \tilde{p}^b. \quad (C16)$$

Whereas the ADM energy and momentum conservation in equations (33) and (34) are based on projecting along $\tilde{n}_a$ and $\tilde{h}_{ab}^{(\alpha)} = g_{ab} + \tilde{n}_a \tilde{n}_b$. We have the following relations

$$0 = \tilde{T}_{ab}^{b} \tilde{a}^b = -\frac{1}{N} \tilde{\gamma}_0^b - \frac{1}{N} \tilde{T}_{ab}^b \tilde{a}^b - \frac{1}{N} \tilde{\gamma}_0^i \tilde{T}_{ab}^b \tilde{a}^b - \frac{1}{N} \left( \tilde{u}^i - \frac{1}{a} a N N \right) \tilde{T}_{ab}^b \tilde{a}^b, \quad 0 = \tilde{T}_{ab}^{(\alpha)} \tilde{a}^b = \tilde{T}_{ab}^{(\alpha)} \tilde{a}^b - \tilde{u}_i \tilde{T}_{ab}^{(\alpha)} \tilde{a}^b. \quad (C17)$$

These lead to equations (A10) and (A11).

**APPENDIX D: FLUID VELOCITIES**

Here, we introduce different definitions of the fluid three-velocity: $v^i$, $\tilde{v}^i$ and $\tilde{V}^i$. For relations among them, see equation (D10).

(I) Following the convention in the linear perturbation theory, we have naively introduced $v_i$ as (Bardeen 1988; Hwang & Noh 2007)

$$\tilde{u}_i = a v_i, \quad \tilde{u}_0 = -a N \tilde{\gamma} - \frac{\chi^i v_k}{1 + 2 \varphi}, \quad \tilde{u}^i = \frac{v^i}{a(1 + 2 \varphi)} + \frac{\chi^i}{a^2 N (1 + 2 \varphi)} \tilde{\gamma}, \quad \tilde{u}^0 = \frac{1}{a N} \sqrt{1 + \frac{v^i v_k}{1 + 2 \varphi}} = \frac{1}{a N} \tilde{\gamma}, \quad (D1)$$

where $\tilde{\gamma}$ is the Lorentz factor. The form of the Lorentz factor shows the non-trivial nature of $v_i$ as the velocity to the non-linear order.

(II) The fluid three-velocity $\tilde{V}^i$ measured by the Eulerian observer with normal four-velocity $\tilde{n}_i$ is defined as (Banyuls et al. 1997)

$$\tilde{V}^i = \tilde{h}^{i\alpha}_{ab} \tilde{a}^b = \frac{1}{N} \left( \tilde{a}^i - \frac{\chi^i}{a(1 + 2 \varphi)} \right), \quad (D2)$$

where $\tilde{h}^{(\alpha)}_{ab} = g_{ab} + \tilde{n}_a \tilde{n}_b$ is the projection tensor normal to $\tilde{n}^i$, and the index of $\tilde{V}^i$ is raised and lowered by the ADM three-space metric $h_{ij}$.

In the ADM notation we have

$$\tilde{u}_i = N \tilde{n}_i \tilde{V}_i, \quad \tilde{u}_0 = -N \tilde{n}_0 \left( N - N \tilde{\gamma} \tilde{V}^i \right), \quad \tilde{u}^i = \tilde{u}^0 \left( N \tilde{V}^i - N \tilde{\gamma} \tilde{V}^i \right), \quad \tilde{u}^0 = \frac{1}{N} \sqrt{1 - \tilde{V}^i \tilde{V}_i} \equiv \frac{1}{N} \tilde{\gamma}. \quad (D3)$$

In order to use the perturbation notation, we introduce

$$\tilde{V}_i \equiv \tilde{a} \tilde{u}_i, \quad \tilde{V}^i = \frac{\tilde{V}^i}{a(1 + 2 \varphi)}, \quad (D4)$$

where $\tilde{V}^i$ is raised and lowered by $\delta_{ij}$. Thus, we have

$$\tilde{u}_i = a \tilde{\gamma} \tilde{V}_i, \quad \tilde{u}_0 = -a \tilde{\gamma} \left( N + \frac{\chi^i v_k}{a(1 + 2 \varphi)} \right), \quad \tilde{u}^i = \frac{\tilde{V}^i}{a(1 + 2 \varphi)} \left( \tilde{\gamma} + \frac{1}{a N} \chi^i \right), \quad \tilde{u}^0 = \frac{1}{a N} \sqrt{1 - \frac{\chi^i v_k}{a(1 + 2 \varphi)}} \equiv \frac{1}{a N} \tilde{\gamma}. \quad (D5)$$

Using $\tilde{\gamma}$ the Lorentz factor becomes a well-known form.

(III) In the literature, we often find the fluid coordinate three-velocity introduced as (Wilson 1972; Bardeen 1980; Kodama & Sasaki 1984)

$$\tilde{V}^i = \frac{\tilde{u}^i}{\tilde{u}^0} = \frac{dx^i}{dx^0} \quad (D6)$$

where the index of $\tilde{V}^i$ is raised and lowered by the ADM three-space metric $h_{ij}$. In the ADM notation we have

$$\tilde{u}_i = \tilde{u}^0 (\tilde{V}_i + N_i), \quad \tilde{u}_0 = -\tilde{u}^0 \left[ N^2 - \tilde{N}^i (\tilde{V}_i + N_i) \right], \quad \tilde{u}^i \equiv a^0 \tilde{V}^i, \quad \tilde{u}^0 = \frac{1}{\sqrt{N^2 - (\tilde{V}^i + N^i)(\tilde{V}_i + N_i)}} \equiv \frac{1}{N} \tilde{\gamma}. \quad (D7)$$
In order to use the perturbation notation, we introduce
\[ \nabla' \equiv \nabla', \quad \nabla_i = a^2(1 + 2\varphi)\nabla_i, \]  
(D8)

where \( \nabla' \) is raised and lowered by \( \delta_{ij} \). Thus, we have
\[ \tilde{u}_i = \frac{a}{N} \tilde{\gamma} \left[ (1 + 2\varphi)\tau_i - \frac{1}{a} \chi_i \right], \quad \tilde{u}_0 = -aN\tilde{\gamma} \left[ 1 + \frac{1}{aN^2} \chi \left( \tau_i - \frac{\chi}{a(1 + 2\varphi)} \right) \right]. \]
(D9)

The \( \tilde{\gamma} \) is the same as the velocity introduced in our PN approximation (Hwang et al. 2008).

The relations among the velocities \( \nu', \nu' \) and \( \nu' \) are the following
\[ \frac{1}{\tilde{\gamma}} \nu_i = \tilde{v}_i = \frac{1}{N} \left[ (1 + 2\varphi)\tau_i - \frac{1}{a} \chi_i \right]. \]
(D10)

The Lorentz factor can be written as
\[ \tilde{\gamma} = \sqrt{1 + \frac{\nu^2_0}{1 + 2\varphi}} = \frac{1}{\sqrt{1 - \frac{\nu^2_0}{a^2 N^2}}} \left( \tau_i - \frac{\chi_i}{a(1 + 2\varphi)} \right). \]
(D11)

A three-vector can always be decomposed to a longitudinal and transverse part as
\[ \nu_i = -v_{ij} + \nu^{(s)}_j, \quad \tilde{v}_i = -\tilde{v}_{ij} + \tilde{v}^{(s)}_j, \]
(D12)

with \( \nu^{(s)}_j \equiv 0 \equiv \tilde{v}^{(s)}_j \), and similarly for \( \nu_i \). However, the relation in equation (D10) shows that \( \nu^{(s)}_j = 0 \) does not imply \( \tilde{v}^{(s)}_j = 0 \), and vice versa, and similarly for \( \tilde{\gamma} \). Due to the non-linear relations among different velocities the scalar- and vector-decompositions for \( \nu_j \) and \( \tilde{v}_j \) do not coincide with each other to the non-linear order. To the non-linear order the scalar- and vector-type perturbations are coupled in the equation level.

Our comoving slicing condition was imposed on \( \nu_j \) as \( \nu \equiv 0 \). This corresponds to \( \tilde{\nu} \equiv 0 \) for vanishing vector-type perturbation; in the presence of vector-type perturbation these two conditions differ from each other from the third-order perturbation. However, the comoving slicing in the variable \( \tau_i \) is more delicate. Even to the linear order, under our congruence (spatial gauge) condition, \( \tau \) is already gauge invariant (under our congruence condition). To the non-linear order we should regard \( \nu \equiv 0 \) or \( \tilde{\nu} \equiv 0 \) as the comoving gauge condition, and the condition in terms of \( \tau \) is rather cumbersome.

To the fully non-linear order, we have
\[ \tilde{\mu} = \frac{1}{aN} \tilde{\gamma} \tilde{\delta}_0 + \frac{1}{a(1 + 2\varphi)} \left( \nu^\ell + \frac{\tilde{\gamma}}{aN} \chi^\ell \right) \nabla_\ell \tilde{\mu} = \tilde{\gamma} \frac{1}{aN} \left( \tilde{\delta}_0 + \nu^\ell \nabla_\ell \right) \tilde{\mu}, \]
\[ \tilde{h}_0 \tilde{\mu}_j = \tilde{\mu}_j + \nu_i \left[ \frac{\tilde{\gamma}}{N} \tilde{\delta}_0 + \frac{1}{1 + 2\varphi} \left( \nu^\ell + \frac{\tilde{\gamma}}{aN} \chi^\ell \right) \nabla_\ell \right] \tilde{\mu} = \tilde{\mu}_j + \tilde{\gamma}^2 \nu_i \left[ \frac{1}{aN} \tilde{\delta}_0 + \frac{1}{1 + 2\varphi} \left( \nu^\ell + \frac{\chi^\ell}{aN} \right) \nabla_\ell \right] \tilde{\mu}. \]
(D13)

For comparison, we presented the different expressions based on three definitions of the fluid velocities.

A covariant spatial gradient variable
\[ \tilde{\Delta}_\ell = \frac{1}{\tilde{\gamma}} \tilde{h}_0 \tilde{\mu}_j, \]
(D14)

was introduced as a covariant and gauge-invariant variable in the linear perturbation theory (Ellis & Bruni 1989; Woszczyna & Kulak 1989), see also Hawking (1966). Using the gauge transformation properties to the second order in Hwang et al. (2012), we can show that \( \tilde{\Delta}_\ell \) is gauge invariant only to the linear order. To the linear order, we have
\[ \tilde{\Delta}_0 = \nabla_0 \tilde{\delta}_0 + a \frac{\tilde{\mu}_j}{\mu} \nu^{(s)}_j, \]
(D15)

which is a sum of the spatial gradient of \( \delta \) in the comoving gauge and the vector-type perturbation, and is gauge invariant; for \( \delta_j \), see equation (22). In the comoving gauge without vector-type perturbation (thus \( \nu_j = 0 \)), we have \( \tilde{\Delta}_j = \delta_j / (1 + \delta_0) \) which is related to the density perturbation in the comoving gauge to the fully non-linear order.
APPENDIX E: MULTIPLE COMPONENTS OF IDEAL FLUID

Here, we consider the case of multiple components of ideal fluid. Even in the presence of many fluids (with vanishing fluxes and anisotropic stresses) all our equations are valid with the fluid quantities considered as the collective ones. In the presence of \( N \) fluids we have

\[
\bar{T}_{ab} = \sum_j \bar{T}_{(J)ab},
\]

with the fluid quantities of collective and individual components introduced as

\[
\bar{T}_{ab} = \mu\bar{u}_a\bar{u}_b + \bar{p}\left(\bar{g}_{ab} + \bar{u}_a\bar{u}_b\right), \quad \bar{T}_{(J)ab} = \mu(J)\bar{u}_{(J)a}\bar{u}_{(J)b} + \bar{p}(J)\left(\bar{g}_{ab} + \bar{u}_{(J)a}\bar{u}_{(J)b}\right),
\]

where indices \( I, J, \ldots = 1, 2, \ldots N \) identify the fluid component. From equation (E1) we have

\[
\bar{\mu} = \sum_j \left(\tilde{\mu}(J) + (\bar{\mu}(J) + \bar{p}(J)) \left[ (\bar{u}_J)_{(J)}^a \bar{u}_J \right]^2 - 1 \right), \quad \bar{p} = \sum_j \left(\tilde{p}(J) + \frac{1}{3} \left(\tilde{\mu}(J) + \bar{p}(J)\right) \left[ (\bar{u}_J)_{(J)}^a \bar{u}_J \right]^2 - 1 \right),
\]

\[
\bar{\mu} - 3\bar{p} = \sum_j \left(\tilde{p}(J) - 3\tilde{p}(J)\right), \quad \bar{u}_a = -\frac{1}{\bar{\mu} + \sum_k \bar{p}(k)} \sum_j \left(\tilde{\mu}(J) + \tilde{p}(J)\right) (\bar{u}_J)_{(J)}^a \bar{u}_J \bar{u}_{(J)j}.
\]

Now we introduce the normalized (\( \tilde{u}_{(J)a} \equiv -1 \)) fluid four-vector of individual component as

\[
\tilde{u}_{(J)a} = a\bar{v}_{(J)a}.
\]

The rest of \( \tilde{u}_{(J)a} \) are the same as in equation (B8) with \( v_{ij} \) replaced by \( \bar{v}_{ij} \). Similarly, for the energy-momentum tensor, the ADM fluid quantities and the covariant kinematic quantities of the individual fluid component, we can replace \( \tilde{\mu}, \bar{p} \) and \( \tilde{v}_i \) to \( \tilde{\mu}(J), \bar{p}(J) \) and \( \tilde{v}_{ij} \), respectively, in equations (B9)–(C5) with \( \bar{T}_{ab}, \bar{E}, \bar{\theta}, \) etc., replaced by \( \tilde{T}_{ab}, \tilde{E}, \tilde{\theta}, \) etc., respectively. For the ADM fluid quantities we have

\[
E = \sum_j E_{(j)}, \quad J_i = \sum_j J_{i(j)}, \quad S = \sum_j S_{(j)}, \quad \bar{S}_j = \sum_j \bar{S}^j_{(j)},
\]

but these simple relations do not hold for the covariant kinematic quantities \( \tilde{\theta}, \tilde{\omega}_i \) and \( \tilde{\theta}_i \). From equation (B8) for the collective component and for the \( I \)-component, we have

\[
\tilde{\bar{v}}_{(I)a} = \frac{\bar{v}^{(I)a}}{1 + 2\bar{p}} - \sqrt{1 + \frac{v^{(I)a} v^{(I)b}}{1 + 2\bar{p}}} \left( 1 + \frac{\bar{v}^{(I)b}}{1 + 2\bar{p}} \right).
\]

Equation (E4) gives

\[
v_i = -\frac{1}{\bar{\mu} + \sum_k \bar{p}(k)} \sum_j \left(\tilde{\mu}(J) + \tilde{p}(J)\right) (\bar{u}_{(j)})^a \bar{u}_J \bar{v}_{(j)j}.
\]

Equations (E3) and (E8) with equation (E7) provide the relations between the collective and individual fluid quantities. Using these relations our fully non-linear and exact perturbation equations remain valid even in the multiple component fluid case.

Now, we have to provide the equations followed by the individual fluid. The energy and the momentum conservation equations for individual component are the ones we need. The energy and momentum conservation equations follow from \( \bar{T}_{ab} = 0 \), thus

\[
\bar{T}_{(J)a} = \sum_j \bar{T}_{(J)a} = 0,
\]

where \( \bar{T}_{(J)a} \) is the interaction terms among fluids. In the ADM and the covariant formulations, the energy and the momentum conservation equations are presented in equations 47 and 48 of Noh & Hwang (2004), and equations 8 and 9 of Hwang & Noh (2007), respectively. In ideal fluids, these are

\[
E_{(J),0} N^{-1} - E_{(J),j} N^{-1} - K \left( E_{(J),j} + \frac{1}{3} S_{(j)} \right) = \frac{N^2}{N} \left( I_{(j)0} - I_{(j)N} \right),
\]

\[
J_{(j),0} N^{-1} - J_{(j),j} N^{-1} - J_{(j),j} N^{-1} = \frac{N^2}{N} \left( S_{(j)0} + S_{(j)j} + S_{(j)N} \right) = \bar{I}_{(j)},
\]

\[
\tilde{\bar{\mu}}(J) + (\tilde{\bar{\mu}}(J) + \tilde{\bar{p}}(J)) \tilde{\bar{\theta}}(I) = -\tilde{\bar{v}}_{(I)a} \tilde{\bar{I}}_{(I)a},
\]

\[
\tilde{\bar{\mu}}(J) + \tilde{\bar{p}}(J) \tilde{\bar{\theta}}(I) = \frac{\tilde{\bar{I}}_{(J)a}}{\tilde{\bar{\mu}}(J) + \tilde{\bar{p}}(J)}.
\]

Compare these with equations (A10), (A11), (C11) and (C12), respectively, for collective ones. From equations (E10)–(E13) we can derive the energy and the momentum conservation equations in alternative forms. The results are the same as equations (33), (34), (37) and (38) with all fluid quantities \( \tilde{\mu}, \tilde{p} \) and \( v_i \) replaced by \( \tilde{\mu}(J), \tilde{p}(J) \) and \( v_{ij} \), respectively, and add the following contributions from interactions among fluids to the right-hand sides of the equations

\[
-\frac{1}{aN} \left( I_{(j)0} + \frac{\chi}{a(1 + 2\bar{p})} I_{(j)j} \right).
\]
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\[ + I_{ij}, \quad (E15) \]

\[ - \frac{1}{a(1 + 2x)} v_{i(j)} I_{(j)} - \frac{1}{aN} \sqrt{1 + \frac{v_{i(j)} v_{i(k)}}{1 + 2x}} \left( \dot{I}_{(j)} + \frac{X^j}{a(1 + 2x)} I_{(j)} \right), \quad (E16) \]

\[ + \frac{1}{\mu_{i(j)}} + \hat{p}_{i(j)} \left\{ I_{(j)} + v_{i(j)} \left[ \frac{1}{1 + 2x} v_{i(j)} I_{(j)} + \frac{1}{1 + 2x} \sqrt{1 + \frac{v_{i(j)} v_{i(k)}}{1 + 2x}} \left( \dot{I}_{(j)} + \frac{X^j}{a(1 + 2x)} I_{(j)} \right) \right] \right\}, \quad (E17) \]

respectively; we have introduced \( I_{ij} = \tilde{I}_{ij} \), where the spatial index of \( I_{ij} \) is raised and lowered by \( \delta_{ij} \); \( I_{ij} \) is the perturbed-order quantity whereas \( \tilde{I}_{ij} \) can include the background order quantity as \( \tilde{I}_{ij} = I_{ij} + \delta I_{ij} \). These equations together with equations (E3) and (E8) complete the additional equations we need in the multiple fluid system. The vector variables \( v_{i(j)} \) and \( I_{ij} \) can be decomposed to the scalar- and vector-type perturbations as

\[ v_{i(j)} = -v_{i(j)}^{(0)} + v_{i(j)}^{(v)}, \quad I_{ij} = \delta I_{ij} + \delta I_{ij}^{(0)}, \quad (E18) \]

with \( v_{i(j)}^{(v)} = 0 \) and \( v_{i(j)}^{(0)} = 0 \).

As in the single component case, to the linear order the scalar-type perturbation \( \delta I_{ij} \), \( \delta p_{ij} \), \( v_{ij} \), \( \delta I_{ij} \) and \( \delta I_{ij} \) depend on the temporal gauge transformation whereas the vector-type perturbations are gauge invariant (Hwang 1991). In addition to the fundamental gauge conditions in equation (20), in the multicomponent case, we have the following gauge conditions available

I-component-comoving gauge:

\[ v_{ij} = 0, \quad \text{(E19)} \]

uniform-I-component-density gauge:

\[ \delta_{ij} = 0, \quad \text{(E19)} \]

to the fully non-linear order.

\textbf{APPENDIX F: MINIMALLY COUPLED SCALAR FIELD}

Here, we present the fully non-linear and exact formulation in the case of a minimally coupled scalar field. In Section 9, we analysed the case in the comoving gauge.

As the minimally coupled scalar field has no anisotropic stress, the same equations in the fluid formulation in this work are valid even for the minimally coupled scalar field case with the fluid quantities replaced by the ones representing the scalar field. In addition, we need the equation of motion in equation (98).

The fluid quantities and the equation of motion to the fully non-linear order can be derived in a gauge-ready form. From equations (4) and (97) we have

\[ \dot{\bar{\phi}} = -\frac{1}{\bar{\mu}} \bar{\phi}, \quad \bar{\mu} = \frac{1}{2} \bar{\phi}, \quad \bar{p} = \frac{1}{2} \bar{\phi} - \bar{\phi}, \quad a\bar{v}_{\phi} = -\bar{\phi}, \quad \bar{\pi}_{ab} = 0. \quad (F1) \]

We set \( \tilde{\phi} = \phi + \delta \phi \), where \( \phi \) is the background order scalar field. Thus, the comoving gauge \( (v = 0) \) together with irrotational condition (thus, \( v_{ij} = 0 \)) implies the uniform-field gauge \( (\delta \phi = 0) \) and vice versa to the fully non-linear order; i.e., the uniform-field gauge implies \( v_{ij} = 0 \), thus the comoving gauge condition as well as the \( v_{ij}^{(0)} = 0 \) condition; the case in the comoving gauge is analysed in Section 9.

From equations (97) and (3) we have

\[ v_{ij} = \frac{1}{a} \Delta^{-1} \nabla^k \left( \tilde{\phi} \tilde{\phi} / \phi \right), \quad \text{F2} \]

From equations (F6) and (F2) we have

\[ v = \frac{1}{a} \Delta^{-1} \nabla^k \left( \tilde{\phi} / \phi \right), \quad \text{F3} \]

It is convenient to have

\[ \tilde{\phi} = \tilde{\phi} / \phi = -\frac{\tilde{\phi} / \phi}{a(1 + 2x)} + \frac{\tilde{\phi} / \phi}{\gamma} \frac{\dot{\tilde{\phi}} / \phi}{\gamma}, \quad \text{F4} \]

where we introduced a new derivative

\[ \frac{D}{D} = \frac{1}{N} \left( \frac{\partial}{\partial t} + \frac{\chi^j}{a(1 + 2x)} \nabla_j \right) = \frac{1}{N} \left( \partial_0 - N^i \nabla_i \right). \quad (F5) \]
The equation of motion in equation (98) gives
\[
\dot{\phi} = -\frac{D^2\phi}{Dt^2} + (3H - \kappa) \frac{D\phi}{Dt} - \frac{(N\sqrt{1+2\dot{\phi}^2})}{a^2N(1+2\dot{\phi})^{3/2}} = -\vec{V}_\phi(\phi). \tag{F6}
\]

In order to derive equation (F6), it is convenient to use the space–time connection $\vec{\Gamma}^\mu_{\nu\rho}$ presented in terms of the ADM notations in equation (6) of Noh & Hwang (2004). We can show that equation (33) leads to equation (F6), and equation (34) is identically satisfied. Equation (F6) can be written as
\[
\dot{\phi} + \left(3HN - N\kappa - \frac{\dot{N}}{N} - \frac{\kappa}{a^2N(1+2\dot{\phi})}ight) \dot{\phi} + \frac{2\mu}{\alpha^2(1+2\dot{\phi})} \dot{\phi} + \frac{1}{\alpha^2(1+2\dot{\phi})} \left(\frac{N\ddot{\phi}}{\alpha^2(1+2\dot{\phi})} - \frac{\kappa}{\alpha^2(1+2\dot{\phi})} \dot{\phi} \right) = -N^2 \vec{V}_\phi(\phi). \tag{F7}
\]

To the background order, we have
\[
\mu = \frac{1}{2} \dot{\phi}^2 + V, \quad p = \frac{1}{2} \dot{\phi}^2 - V \tag{F8}
\]
and
\[
\dot{\phi} + 3H\dot{\phi} + V = 0. \tag{F9}
\]

The entropic perturbation $e$ becomes
\[
e \equiv \delta p - \frac{\bar{p}}{\mu} \delta \mu = \left(1 + \frac{\dot{\phi}}{3H\phi}\right) \left(\frac{\dot{\phi}^2}{\phi} - \dot{\phi}^2\right) + \frac{2\dot{\phi}}{3H\phi}(\vec{V} - V). \tag{F10}
\]

To the third order in perturbations, the complete set of scalar-type perturbation equations of a fluid without anisotropic stress was presented in Section 4 in a gauge-ready form. The same equations are valid even for the minimally coupled scalar field case with the fluid quantities replaced by the ones representing the scalar field. The equation of motion is also contained in the complete set of fluid equation, but it is convenient to present it separately. In the following, we provide the fluid quantities and the equation of motion to the third-order perturbation in a gauge-ready form. We consider both the scalar- and vector-type perturbations.

To the third order, fluid quantities in equation (F2) give
\[
v_i = \frac{1}{a} \frac{\delta \phi}{\phi} \left(1 - \frac{\dot{\phi}}{\phi} + \alpha + \frac{\delta \phi^2}{\phi^2} - \frac{\delta \phi}{\phi} - \frac{1}{a^2\phi^2} \alpha x^i \delta \phi_j - \frac{1}{2} \frac{\delta \phi}{\phi} \delta \phi_j - \frac{1}{2} \alpha^2 + \frac{1}{2a^2} \chi^i \chi_j\right), \tag{F11}
\]
and for $\bar{\mu} = \mu + \delta \mu$ and $\bar{p} = p + \delta p$ in equation (F2), we need
\[
\frac{1}{2} \frac{\dot{\phi}^2}{\phi} = \frac{1}{2} \frac{\dot{\phi}^2}{\phi} \left(1 - 2\alpha + 4a^2 - \left(v^i v_i + \frac{1}{a^2} \chi_i \right) \left(1 - 2\alpha - 2\phi\right) \right) + \frac{2}{a^2} \chi^i \chi_j (\alpha - 8a^2)
\]
\[
+ \alpha \frac{\dot{\phi}}{\phi} \left(1 - 2\alpha + 4a^2 - v^i v_i - \frac{1}{a^2} \chi_i \right) + \frac{1}{2} \delta \phi^2 \left(1 - 2\alpha\right) + \frac{1}{a^2} \chi^i \chi_j \left(1 - 2\alpha - 2\phi\right) + \frac{\delta \phi}{\phi},
\]

\[
\vec{V}(\phi) = V + V_{\phi} \delta \phi + \frac{1}{2} V_{\phi\phi} \delta \phi^2 + \frac{1}{6} V_{\phi\phi\phi} \delta \phi^3. \tag{F12}
\]

The equation of motion in equation (F7) gives
\[
\dot{\phi} + 3H\dot{\phi} + V_{\phi} + \delta \phi + 3H\delta \phi + \left(V_{\phi\phi} + \frac{\Delta}{a^2}\right) \delta \phi + 3H(\alpha - \kappa) \dot{\phi} + 2V_{\phi} \alpha = \frac{3}{2} H \left[ \alpha^2 - \frac{1}{a^2} \chi_i \chi_j \left(1 - \alpha - 2\phi\right) \right] + \left(\alpha - \frac{1}{2} \alpha^2 + \frac{1}{2a^2} \chi^i \chi_j \right) \kappa - 2\alpha \alpha + \frac{1}{a^2} \chi^i \chi_j \chi_i \chi_j (1 - 2\alpha - 2\phi)
\]
\[
- \frac{1}{a^2} \chi^i \chi_j (\alpha + \phi) + 4a^2 \alpha + \frac{1}{a^2} \chi^i \chi_j \left(1 - 2\alpha - 2\phi\right) + \frac{1}{a^2} \chi^i \chi_j \chi_i \chi_j \left(1 - 2\alpha - 2\phi\right)
\]
\[
+ \left[-3H \left(\alpha - \frac{1}{2} \alpha^2 + \frac{1}{2a^2} \chi^i \chi_j\right) + \kappa (1 + \alpha) + \alpha (1 - 2\alpha) + \frac{1}{a^2} \chi^i \chi_j \chi_i \chi_j \right] \delta \phi
\]
- \frac{2}{a^2} \chi^i \delta \phi_j (1 - 2\varphi) - \frac{1}{a^2} \chi^i \delta \phi_j (1 - 2\varphi) + \frac{1}{a^2} \left[ \left( 2\alpha - 2\varphi - 4\alpha \varphi + 4\varphi^2 + \frac{1}{a^2} \chi^i \chi_i \right) \Delta \delta \phi - \frac{1}{a^2} \chi^i \delta \phi_{ij} \right]

+ \frac{1}{a^2} \left[ \alpha^i \delta \phi_j + \psi^i \left( 1 + 2\alpha - 4\varphi \right) \delta \phi_j \right] - \frac{1}{a^2} \left[ H \left( 1 + 3\alpha - 2\varphi \right) - \dot{\alpha} - 2\dot{\varphi} - \kappa \right] \chi^i \delta \phi_j

- \left[ \frac{1}{2} V_{\psi \psi \psi} \delta \phi^3 + 2 V_{\psi \psi} \delta \phi \alpha + \frac{1}{a^2} V_{\psi \psi} \chi_i (1 - 2\varphi) + \frac{1}{6} V_{\psi \psi \psi} \delta \phi^3 + 4 \alpha^i \delta \phi^2 \alpha + \frac{1}{a^2} V_{\psi \psi} \chi_i \delta \phi \right]. \quad (F13)

Equations in Section 3 together with equation (F13) and the fluid quantities in equations (F2), (F11) and (F12) provide a complete set of third-order perturbation equations in a gauge-ready form.

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