A FOUR-DIMENSIONAL MAPPING CLASS GROUP RELATION

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Abstract. We give a relation between (right-handed) Dehn twists on the symplectic mapping class group of a 4-dimensional Weinstein domain via holomorphic curve techniques. A key ingredient of the construction is a solution to the symplectic isotopy problem on symplectic submanifolds in Del Pezzo surfaces. In the appendix, we provide an alternative proof of a relation between Dehn twists and a fibered Dehn twist.

1. Introduction

Let \((W, \omega)\) be a symplectic manifold possibly with boundary. A compactly supported symplectic automorphism is a diffeomorphism \(\varphi: W \to W\) such that \(\varphi^* \omega = \omega\), and its support is compact and lies in the interior of \(W\). Let \(\text{Symp}_c(W, \omega)\) denote the group of compactly supported symplectic automorphisms of \((W, \omega)\) equipped with the \(C^\infty\)-topology. This paper is concerned with relations on the symplectic mapping class group \(\pi_0(\text{Symp}_c(W, \omega))\). In particular, we study the case where \((W, \omega)\) is a Weinstein domain. Here a Weinstein domain is a compact symplectic manifold \((W, \omega)\) with boundary such that a Liouville vector field is gradient-like for an exhausting Morse function \(f: W \to \mathbb{R}\) with no critical points on \(\partial W\) and \(f^{-1}(\max f) = \partial W\).

Relations in \(\pi_0(\text{Symp}_c(W, \omega))\) help us to understand the topology of Weinstein fillings of contact manifolds. Thanks to results of Akbulut and Ozbagci [AO], Loi and Pergallini [LP] and Giroux and Pardon [GP], every Weinstein domain admits a Lefschetz fibration with regular fibers symplectomorphic to a codimension 2 Weinstein domain, say \((W, \omega)\). Hence, Lefschetz fibrations relate \(\pi_0(\text{Symp}_c(W, \omega))\) to Weinstein domains via their monodromies. When \(\dim W = 2\), i.e., \(W\) is a surface, Moser’s trick tells us that \(\pi_0(\text{Symp}_c(W, \omega))\) is isomorphic to the smooth mapping class group of \(W\). Since the latter group is well studied, Ozbagci and Stipsicz [OS], for example, handle relations on smooth mapping class groups of surfaces and construct contact 3-manifolds with infinitely many Weinstein fillings up to homotopy.

When \(\dim W \geq 4\), applying the above result on Lefschetz fibrations is not so practical to construct various Weinstein domains at this moment. Although symplectic mapping class groups of some 4-dimensional Weinstein domains are understood (e.g. [Sei1], [Wu] and [Eva]), still little is known about those groups of more general Weinstein domains and especially relations on them.

Relations on symplectic mapping class groups are often derived from fibration-like structures on symplectic manifolds. Seidel [Sei2] extracts a braid relation about two
Dehn twists along Lagrangian spheres from Lefschetz fibrations. Lefschetz pencils and fibrations yield a relation between a fibered Dehn twist and (right-handed) Dehn twists: [Aur1, Sci3, Com, AA] (see also Theorem 2.7). This kind of relation involves many important ones on smooth mapping class groups of surfaces such as chain relations [AA] and the lantern relation [AS]. Note that any fibered Dehn twist defined on a surface is actually a Dehn twist. In another direction, Keating [Kea] compares relations on higher-dimensional symplectic mapping class group with 2-dimensional ones in terms of Lefschetz fibrations.

Reviewing constructions of 4-dimensional Weinstein domains, relations between different numbers of Dehn twists such as the lantern relation play a crucial role in many cases. The main theorem in this paper is to give the first example of such relations in symplectic mapping class groups of 4-dimensional Weinstein domains.

Theorem 1.1. There exists a 4-dimensional Weinstein domain \((W,\omega)\) and Lagrangian spheres \(L_{1,1},\cdots,L_{1,4},L_{2,1},\cdots,L_{2,6}\) such that the two products of (right-handed) Dehn twists along these Lagrangian spheres, \(\tau_{L_{1,1}} \circ \cdots \circ \tau_{L_{1,4}}\) and \(\tau_{L_{2,1}} \circ \cdots \circ \tau_{L_{2,6}}\), are symplectically isotopic; in other words, we have
\[
\left[\tau_{L_{1,1}} \circ \cdots \circ \tau_{L_{1,4}}\right] = \left[\tau_{L_{2,1}} \circ \cdots \circ \tau_{L_{2,6}}\right] \in \pi_0(\text{Symp}_c(W,\omega)).
\]

We will construct the Weinstein domain \((W,\omega)\) and Lagrangian spheres \(L_{i,j}\) in the theorem, employing fibration-like structures. Here is a sketch of the proof of Theorem 1.1. It is known that there are distinct two complex 3-folds \(X_1\) and \(X_2\) containing Del Pezzo surfaces of degree 6 as ample divisors. Note that such a Del Pezzo surface is diffeomorphic to \(\mathbb{CP}^2 \# 3\mathbb{CP}^2\). For \(i=1,2\), consider a Lefschetz pencil \(f_i: X_i \rightarrow \mathbb{CP}^1\) defined by a linear system containing the above ample divisor. This gives a Lefschetz fibration \(p_i: X_i \setminus \nu(f_i^{-1}(\infty)) \rightarrow D^2\), where \(\nu(f_i^{-1}(\infty))\) is a tubular neighborhood of the regular fiber \(f_i^{-1}(\infty)\) in \(X_i\). Let \((W_i,\omega_i)\) denote a regular fiber of \(p_i\) \((i=1,2)\), which is a Weinstein domain. According to the aforementioned results [Aur1] and [Com], the global monodromy of \(p_i\) is a fibered Dehn twist \(\tau_{\partial W_i}\) along \(\partial W_i\) that factorizes into a product of Dehn twists along Lagrangian spheres \(L_{i,j}\):
\[
\left[\tau_{\partial W_1}\right] = \left[\tau_{L_{1,1}} \circ \cdots \circ \tau_{L_{1,4}}\right] \quad \text{and} \quad \left[\tau_{\partial W_2}\right] = \left[\tau_{L_{2,1}} \circ \cdots \circ \tau_{L_{2,6}}\right].
\]

The key step of this construction is to show that \((W_1,\omega_1)\) and \((W_2,\omega_2)\) are symplectomorphic; this is a consequence of the symplectic isotopy problem (see Problem 3.1). We solve it for anti-canonical divisors on Del Pezzo surfaces via holomorphic curve techniques. The symplectomorphism between \((W_1,\omega_1)\) and \((W_2,\omega_2)\) allows us to identify \(\tau_{\partial W_1}\) with \(\tau_{\partial W_2}\). Setting \((W,\omega) := (W_1,\omega_1)\), we finally obtain the desired relation on \(\pi_0(\text{Symp}_c(W,\omega))\). Remark that some Lagrangian spheres in the theorem might be the same.

We would like to point out a result for the symplectic mapping class group of \(\mathbb{CP}^2 \# 3\mathbb{CP}^2\). Capping off \(\partial W\) by a disc bundle over a 2-torus, we obtain \(\mathbb{CP}^2 \# 3\mathbb{CP}^2\) with a certain symplectic structure \(\omega'\) and a relation between Dehn twists on its symplectic mapping class group canonically induced by Theorem 1.1. Li, Li and Wu [LLW] show that \(\pi_0(\text{Symp}_c(\mathbb{CP}^2 \# 3\mathbb{CP}^2,\omega'))\) is a finite group generated by Dehn twists. Moreover, it also
follows from their argument that the square of any Dehn twist is symplectically iso-
topic to the identity. Hence, one could easily find a relation between Dehn twists in
\( \pi_0(\text{Symp}(\mathbb{C}P^2 \# 3\overline{\mathbb{C}P^2}, \omega')) \). It does not seem that such a relation concludes Theorem
1.1 directly. In fact, any Dehn twist defined on a Weinstein domain, more generally a
Liouville domain has infinite order in its symplectic mapping class group \([BGZ]\).

With the help of Lefschetz fibrations and open books, the author \([Oba1]\) constructs
\((4n - 1)\)-dimensional contact manifolds admitting infinitely many Weinstein, or equiv-
alent Stein fillings up to homotopy. While the result for the remaining dimensions,
\(4n + 1\), is still open, Lazarev \([Laz]\) produces contact manifolds in every odd dimension
with arbitrarily finitely many Weinstein fillings by using an \(h\)-principle argument. As an
application of Theorem 1.1 one can partially recover his result by means of fibration-like
structures (see Proposition 4.6).

Organization of this paper. Section 2 deals with Dehn twists, fibered Dehn twists
and their relation. We first review Dehn twists in Section 2.1 and Lefschetz fibrations
in Section 2.2. After this, we present a way to obtain a Lefschetz fibration from a
given Lefschetz pencil in Section 2.3. Applying this, Section 2.4 explains a symplectic
mapping class group relation between Dehn twists and a fibered Dehn twist. Section 3
is devoted to the symplectic isotopy problem for Del Pezzo surfaces. Collecting results
on holomorphic curves in Section 3.1, we give a solution to the problem in Section 3.2.
Section 4 begins with a review of how Del Pezzo surfaces appear as ample divisors
of complex 3-folds in Section 4.1 and then proves Theorem 1.1 and its application for
Weinstein fillings of contact 5-manifolds. Finally, we conclude this paper by showing a
relation between Dehn twists and a fibered Dehn twist, restricting to the projective case
in Appendix A.

Conventions and Notations. Let \( p: E \to X \) be a vector bundle. We often call the
total space \( E \) the vector bundle. Also the zero-section of \( p \) often means the image of
the zero-section and denote it by \( X \) with slight abuse of notation. Given a symplectic
manifold \((M, \omega)\), let \( h: M \to \mathbb{R} \) be a smooth function. We define the Hamiltonian vector
field associated to \( h \) as a vector field \( X_h \) such that \( dh = -\iota_{X_h} \omega \). For a submanifold
\( Y \subset M \), we denote the pull-back of \( \omega \) to \( Y \) under the inclusion by \( \omega|_Y \).

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2. Dehn twists and fibered Dehn twists

2.1. Dehn twists. Here we briefly review the definition of Dehn twist. The reader is
referred to \([Sei4, (16c)]\) and \([Sei5, (1a)]\) for more details.

First let us recall a model Dehn twist on the cotangent bundle \( T^*S^n \) to the sphere
\( S^n \). We regard \( T^*S^n \) as the space defined by

\[ \{(q, p) \in \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \mid q \cdot q = 1, q \cdot p = 0\}, \]
where \( \cdot \) denotes the standard inner product of the Euclidean space \( \mathbb{R}^{n+1} \). Write \( \lambda_{\text{can}} \) for the canonical Liouville form on \( T^*S^n \). Consider the normalized geodesic flow \( \sigma_t \) on \( T^*S^n \setminus S^n \) given by

\[
\sigma_t(q, p) = (\cos(t)q + |p|^{-1} \sin(t)p, -|p| \sin(t)q + \cos(t)p).
\]

A model Dehn twist is a diffeomorphism of \( T^*S^n \) defined by

\[
\tau(q, p) := \begin{cases} 
\sigma_{f(|p|)}(q, p) & \text{if } p \neq 0, \\
(-q, 0) & \text{if } p = 0,
\end{cases}
\]

where \( f: [0, \infty) \to \mathbb{R} \) is a smooth function such that \( f(r) \) equals \( \pi \) near \( r = 0 \) and 0 for \( r \gg 0 \), and \( f'(r) \leq 0 \) for any \( r \in [0, \infty) \). The diffeomorphism \( \tau \) is an exact symplectomorphism of \( (T^*S^n, d\lambda_{\text{can}}) \) with compact support. Choosing a function \( f \) whose support is sufficiently small, we may assume that the support of \( \tau \) lies in a small neighborhood of the zero-section of \( T^*S^n \).

Now suppose that we have a Lagrangian sphere \( L \) in a \( 2n \)-dimensional symplectic manifold \( (W, \omega) \). Equip \( L \) with a diffeomorphism \( v: S^n \to L \), called a framing of \( L \). With the framing \( v \), the Weinstein Lagrangian tubular neighborhood theorem gives a symplectic embedding \( \iota: (D^*_\epsilon S^n, d\lambda_{\text{can}}) \to (W, \omega) \) satisfying that \( \iota|_{S^n} = v \), where \( D^*_\epsilon S^n \) is the disc cotangent bundle to \( S^n \) of radius \( \epsilon \). Define a (right-handed) Dehn twist \( \tau_L \) along \( L \) by

\[
\tau_L(x) = \begin{cases} 
\iota \circ \tau \circ \iota^{-1}(x) & \text{if } x \in \text{Im}(\iota), \\
x & \text{otherwise},
\end{cases}
\]

where we choose a model twist \( \tau \) to be supported in \( D^*_\epsilon S^n \). By definition, \( \tau_L \) is a symplectomorphism of \((W, \omega)\). In fact, it is independent of the choice not only of \( \iota \) and but also of \( v \) when \( n \leq 2 \) up to Hamiltonian isotopy [Sei4, Remark 16.1].

2.2. Lefschetz fibrations. Let \( E \) be an open manifold of even dimension and \( \Omega \) a closed 2-form on \( E \).

**Definition 2.1.** A smooth map \( \pi: (E, \Omega) \to \mathbb{C} \) is called a Lefschetz fibration if the set of critical points of \( \pi \), denoted by \( \text{Crit}(\pi) \), is a finite subset in \( E \); there exists an \( \Omega \)-compatible almost complex structure \( J \) defined in a tubular neighborhood \( U \) of \( \text{Crit}(\pi) \) in \( E \); there is a positively oriented complex structure \( j \) defined in a tubular neighborhood \( V \) of \( \text{Crit}(\pi) := \pi(\text{Crit}(\pi)) \) in \( S \), which satisfy the following conditions:

1. The map \( \pi|_U : U \to V \) is \((J, j)\)-holomorphic;
2. The 2-form \( \Omega \) is nondegenerate on \( \ker_x(D\pi) \) for any point \( x \in E \);
3. The fibers are symplectomorphic to the completion of a Liouville domain \((F, d\lambda_F)\) outside compact subsets;
4. The 2-form \( \Omega \) is Kähler near \( \text{Crit}(\pi) \);
5. The complex Hessian \( D^2_x\pi \) of \( \pi \) is nondegenerate at any critical point \( x \in \text{Crit}(\pi) \).
A fiber of a Lefschetz fibration \( \pi: (E, \Omega) \to \mathbb{C} \) is said to be \textit{singular} if it is over a critical value of \( \pi \); otherwise it is said to be \textit{regular}.

**Remark 2.2.** We often require a Lefschetz fibration \( \pi: (E, \Omega) \to \mathbb{C} \) to satisfy the following condition:

- **Horizontal triviality.** There is a subset \( E_h \subset E \) such that \((E \setminus E_h) \cap \pi^{-1}(z)\) is relatively compact for all \( z \in \mathbb{C} \); the restriction \( \pi|_{E_h} \) is isomorphic to the trivial fibration

\[
\mathbb{C} \times [0, \infty) \times \partial F \to \mathbb{C},
\]

and \( \Omega \) is identified with \( \omega_C + d\lambda_F \) by the isomorphism, where \( \omega_C \) is a symplectic form on \( \mathbb{C} \).

A Lefschetz fibration satisfying this condition is referred to as one with the horizontal triviality.

**Remark 2.3.** A Lefschetz fibration \( \pi: (E, \Omega) \to \mathbb{C} \) with the horizontal triviality provides a Lefschetz fibration over \( \mathbb{C} \) with compact fibers. Indeed, removing a subset \( E_h \subset E \) as in the above remark, we have the desired Lefschetz fibration \( \pi|_{E \setminus E_h} : (E \setminus E_h, \Omega|_{E \setminus E_h}) \to \mathbb{C} \).

Set \( \text{Critv}(\pi) = \{z_1, \ldots, z_k\} \) for a Lefschetz fibration \( \pi: (E, \Omega) \to \mathbb{C} \) with horizontal triviality. Assume that \( \pi|_{\text{Crit}(\pi)} : \text{Crit}(\pi) \to \text{Critv}(\pi) \) is injective. According to the condition 2 in Definition 2.1, the symplectic connection is defined away from the singular fibers and determines the parallel transport map along a path in \( \mathbb{C} \setminus \text{Critv}(\pi) \). Fix a basepoint \( z_0 \in \mathbb{C} \setminus \text{Critv}(\pi) \) with \( |z| \) sufficiently large and set \( F_0 = \pi^{-1}(z_0) \) and \( \omega_{F_0} = \Omega|_{F_0} \). A vanishing path for \( z_i \in \text{Critv}(\pi) \) is an embedded path \( \gamma_i: [0, 1] \to \mathbb{C} \) that \( \gamma_i(0) = z_0 \), \( \gamma_i(1) = z_i \) and \( \gamma_i^{-1}(\text{Critv}(\pi)) = \{1\} \). One can associate to a vanishing path \( \gamma_i \) for \( z_i \) a framed Lagrangian sphere \( V(\gamma_i) \in (F_0, \omega_{F_0}) \), called a vanishing cycle for \( \gamma_i \). Take a small disc \( D(z_i) \) centered at \( z_i \in \text{Critv}(\pi) \) and orient \( \partial D(z_i) \) counterclockwise. Let \( \ell_i: [0, 1] \to \mathbb{C} \setminus \text{Critv}(\pi) \) be a loop obtained by welding a vanishing path \( \gamma_i \) and \( \partial D(z_i) \) together (see Figure 1). An ordered collection \( (\gamma_1, \ldots, \gamma_k) \) of vanishing paths is called a distinguished basis of vanishing paths if it satisfies the following conditions:

- Any two distinct paths \( \gamma_i \) and \( \gamma_j \) intersect only at the basepoint \( z_0 \);
- The concatenation \( \ell_1 \cdots \ell_k \) of loops \( \ell_i \) obtained by \( \gamma_i \) in the above manner is homotopic to the circle \( \ell_0(t) = z_0 e^{2\pi it} \) relatively to the basepoint \( z_0 \).
Let $\ell$ be a loop based at $z_0$ and homotopic to $\ell_0$. As is known, for a chosen distinguished basis $(\gamma_1, \cdots, \gamma_k)$, the monodromy along $\ell$ is given by
\[ \tau_{V(\gamma_k)} \circ \cdots \circ \tau_{V(\gamma_1)} \in \text{Symp}_c(F_0, \omega_{F_0}) \]
up to symplectic isotopy. We refer to this monodromy as the global monodromy basis $(\gamma_1, \cdots, \gamma_k)$. Note that the global monodromy itself does not depend on the choice of distinguished basis $(\gamma_1, \cdots, \gamma_k)$ up to Hamiltonian isotopy, whereas its factorization into Dehn twists does depend on its choice.

2.3. From Lefschetz pencils to Lefschetz fibrations. In this subsection, we will explain how to obtain a Lefschetz fibration from a given Lefschetz pencil.

2.3.1. Symplectic forms on Whitney sums. To begin with, we describe a symplectic form on Whitney sums of line bundles, which will serve as a symplectic form on a neighborhood of the base locus of a Lefschetz pencil. The following argument is inspired by the proof of [Gom] Theorem 2.3.

Let $(B, \omega_B)$ be a closed integral symplectic manifold and $p: L \to B$ a hermitian complex line bundle with $c_1(L) = [\omega_B]$. Consider the Whitney sum $\pi: L^{\oplus 2} = L \oplus L \to B$ and projectivization $f: L^{\oplus 2} \setminus B \to \mathbb{CP}^1$; that is, the map locally written as $f(b, x_1, x_2) = [x_1 : x_2]$ for $(b, x_1, x_2) \in U \times \mathbb{C} \times \mathbb{C}$, where $U$ is an open set of $B$. Let $\delta \in L^{\oplus 2}$ denote the sphere bundle of radius $\delta$ for the hermitian metric associated to $L^{\oplus 2}$. Restricting the map $(\pi, f): L^{\oplus 2} \setminus B \to B \times \mathbb{CP}^1$ to $\delta$ gives a $U(1)$-bundle structure to $\delta$. One can identify $L^{\oplus 2} \setminus B$ with the complement $(\delta \times_{U(1)} \mathbb{C}) \setminus (B \times \mathbb{CP}^1)$ of the zero-section in the associated bundle by

\[ L^{\oplus 2} \setminus B \to (\delta \times_{U(1)} \mathbb{C}) \setminus (B \times \mathbb{CP}^1), \quad z = (z_0, z_1) \mapsto [\delta z/|z|, |z|], \]

where $|z|$ denotes the norm of $z$ for the hermitian metric. Take a connection 1-form $\alpha$ on $\delta$ such that $d\alpha = -2\pi(\pi^*\omega_B - f^*\omega_{FS})$, where $\omega_{FS}$ is the Fubini–Study form on $\mathbb{CP}^1$. Now we define the 2-form $\tilde{\Omega}$ on $\delta \times \mathbb{C}$ by

\[ \tilde{\Omega} := \pi^*\omega_B - \frac{1}{2\pi}d(r^2d\theta) + \frac{1}{2\pi}d(r^2\alpha), \]

where $(r, \theta)$ are the polar coordinates on $\mathbb{C}$. Since this is $U(1)$-invariant and horizontal, it descends to a 2-form not only on $\delta \times_{U(1)} \mathbb{C}$ but also on $L^{\oplus 2}$ under the identification (2.1). We claim that the resulting 2-form $\Omega$ on $L^{\oplus 2}$ is a symplectic form in a small neighborhood of the zero-section. As $\Omega$ is closed, it suffices to show the nondegeneracy of $\Omega$. In view of the choice of $\alpha$, we find that $\tilde{\Omega}$ can be expressed as

\[ \tilde{\Omega} = \frac{1}{2\pi}d(r^2 - 1)(\alpha - d\theta) + f^*\omega_{FS} \]

\[ = \frac{1}{2\pi}d(r^2 - 1)(\alpha - d\theta) + f^*\omega_{FS} + \frac{1}{2\pi}d(r^2\alpha) + (1 - r^2)\pi^*\omega_B + r^2f^*\omega_{FS}. \]

Observe that $\Omega$ coincides with the standard symplectic form on the fibers of $\pi$ up to a constant factor. To see a horizontal direction, define the distribution $H$ of $TL^{\oplus 2}$ to be $TB$ on the zero-section and the $\alpha$-horizontal lifts of $T(B \times \{pt\}) \subset T(B \times \mathbb{CP}^1)$ to $\delta$ for each $\delta > 0$. Since $H$ is tangent to $\delta$ and the fibers of $f$, $H$ is the $\Omega$-complement of $\ker(D\pi)$. By (2.3), $\Omega|_H = (1 - r^2)\pi^*\omega_B$, and it extends smoothly to $B$. This concludes that $\Omega$
extends smoothly to the whole of $L\oplus 2$ with $\Omega|_{B} = \pi^* \omega_B$. Hence, $\Omega$ is a nondegenerate 2-form on a small neighborhood of the zero-section.

Let $L^{\oplus 2}(r_0)$ denote the associated disc bundle to $L^{\oplus 2}$ of radius $r_0 > 0$ and set

$N_0(r_0) := L^{\oplus 2}(r_0) \setminus \{(z_0, z_1) \in L^{\oplus 2}(r_0) \mid z_0 = 0\},$

$N_0(r_0, r_1) := N_0(r_1) \setminus \text{Int } N_0(r_0)$

for $0 < r_0 < r_1 < 1$. We will see below that there is a Liouville vector field on $N_0(r_0, r_1)$ pointing outward along $\partial N_0(r_0)$. First with (2.1), we regard $N_0(r_0, r_1)$ as $P_{\delta,0} \times U(1)$, where $P_{\delta,0} = P_{\delta} \setminus \{(z_0, z_1) \in P_{\delta} \mid z_0 = 0\}$ and $A(r_0, r_1) = \{z \in \mathbb{C} \mid r_0 \leq |z| \leq r_1\}$. Consider the function $h: P_{\delta,0} \times U(1) \times (0, r_1) \to \mathbb{R}$ given by

$$h((z_0, z_1, r e^{i \theta})) = r^2.$$

Its Hamiltonian vector field $X_h$ on $P_{\delta,0} \times U(1) \times (0, r_1)$ is $-2 \pi \partial_\theta$. Since the image of $P_{\delta,0} \times U(1) \times (0, r_1)$ by $f$ is $\mathbb{C} \subset \mathbb{C} \cup \{\infty\} = \mathbb{C}^1$, the pull-back $f^* \omega_{FS}$ restricts to $-f^* d\bar{c} \log(|w|^2 + 1)$ on $P_{\delta,0} \times U(1) \times A(r_0, r_1)$, where $w$ is the standard complex coordinate of $\mathbb{C}$ and $d\bar{c} := d \circ J_{\text{std}}$ for the standard complex structure $J_{\text{std}}$ on $\mathbb{C}$. By (2.20), $\Omega$ is exact on $P_{\delta,0} \times U(1) \times A(r_0, r_1)$. Hence, one can take the Liouville vector field $V$ on this region that is $\Omega$-dual to $\lambda := (r^2 - 1)(\alpha - d\theta)/2\pi - f^* d\bar{c} \log(|w|^2 + 1)$. Now we have

$$dh(V) = -\Omega(X_h, V) = \lambda(X_h) = r^2 - 1 < 0,$$

where $X_h$ is the Hamiltonian vector field associated to $h$. This shows that $V$ points inward along each level set of $h$. In other words, $V$ points outward along $\partial N_0(r_0)$ when it is considered as a part of the boundary of $N_0(r_0, r_1)$.

Moreover, $\Omega$ matches the exact symplectic form $d(r^2 - 1)(\alpha - d\theta)/2\pi$ on each fiber of $f$, for which a Liouville vector field is given by

$$(2.4) \quad \frac{r^2 - 1}{2r} \partial_r.$$

This makes the fibers of $f|_{N_0(r_0, r_1)}$ convex along the boundary component lying in $\partial N_0(r_0)$.

We summarize the above discussion into the following:

**Proposition 2.4.** Let $(B, \omega_B)$ be a closed integral symplectic manifold and $L \to B$ a hermitian line bundle with $c_1(L) = [\omega_B]$. Also let $f: L^{\oplus 2} \setminus \{0\} \to \mathbb{C}^1$ denote projectivization on each fiber. Then there exists a symplectic form $\Omega$ on a closed tubular neighborhood $\mathcal{U}$ of the zero-section $B$ of $L^{\oplus 2}$ such that it satisfies the following:

- On the zero-section, $\Omega$ agrees with $\omega_B$;
- For another closed tubular neighborhood $\mathcal{U}' \subset \mathcal{U}$ of the zero-section, a Liouville vector field for $\Omega$ defined on $U_0 := \mathcal{U} \setminus (\text{Int } \mathcal{U}' \cup f^{-1}([0 : 1]))$ points outward along its boundary component lying in $\partial \mathcal{U}'$;
- A Liouville vector field on each fiber of $f|_{U_0}$ for the restriction of $\Omega$ to the fiber points outward along its boundary component lying in $\partial \mathcal{U}'$. 
2.3.2. Lefschetz pencils and fibrations. Let \((X,\omega)\) be a closed integral symplectic manifold and \(B\) a codimension 4 symplectic submanifold of \((X,\omega)\).

**Definition 2.5.** A smooth map \(f : X \setminus B \to \mathbb{CP}^1\), also denoted by \(f : X \dashrightarrow \mathbb{CP}^1\), is called a Lefschetz pencil on \((X,\omega)\) if it satisfies the following conditions:

1. \(f\) meets the conditions of Lefschetz fibration;
2. The closure of every fiber \(F_w := f^{-1}(w)\) in \(X\) agrees with \(f^{-1}(w) \cup B\) and is a manifold in a neighborhood of \(B\);
3. The normal bundle \(N_{B/F_w}\) of \(B\) in the closure \(\overline{F_w}\) has \(c_1(N_{B/F_w}) = [\omega|_B]\);
4. There exists a tubular neighborhood \(\nu_X(B)\) of \(B\) in \(X\) such that
   - \(\nu_X(B)\) is symplectomorphic to a neighborhood of the zero-section of \(N_{B/F_w} \oplus N_{B/F_w} \to B\), where \(w\) is a regular value of \(f\) and the Whitney sum carries a symplectic form as in Section 2.3.1
   - By the above symplectomorphism, \(f|_{\nu_X(B) \setminus B}\) is identified with projectivization \((N_{B/F_w} \oplus N_{B/F_w}) \setminus B \to \mathbb{CP}^1\).

The submanifold \(B\) is called the base locus of a Lefschetz pencil \(f : X \setminus B \to \mathbb{CP}^1\).

We shall see below that a Lefschetz pencil \(f : X \setminus B \to \mathbb{CP}^1\) on \((X,\omega)\) induces a Lefschetz fibration on the complement of a regular fiber in \(X\) after perturbing the symplectic structure \(\omega\).

**Proposition 2.6.** Let \(f : X \setminus B \to \mathbb{CP}^1\) be a Lefschetz pencil on a closed integral symplectic manifold \((X,\omega)\). Suppose that \(\infty \in \mathbb{C} \cup \{\infty\} = \mathbb{CP}^1\) is a regular value of \(f\). Then, for some tubular neighborhood \(\nu_X(B)\) of \(B\), there exists a closed 2-form \(\omega'\) on \(E := X \setminus (\overline{F_\infty} \cup \nu_X(B))\) such that

1. \(f|_E : (E,\omega') \to \mathbb{C}\) is a Lefschetz fibration with horizontal triviality;
2. For every \(w \in \mathbb{C}\), \(\omega'|_{\pi^{-1}(w)} = \omega'|_{\pi^{-1}(w)}\);
3. \(\omega'\) coincides with \(\omega\) outside a collar neighborhood of the boundary \(\partial E\).

**Proof.** Set \(L := N_{B/F_\infty}\) and take a hermitian metric on it. Throughout the proof, we use the same notations as in Section 2.3.1 such as \(L^{\otimes 2}(r)\) and \(N_0(r_0, r_1)\). We also always equip a disc bundle \(L^{\otimes 2}(r)\) for \(r < 1\) with the symplectic form \(\Omega\) unless otherwise noted.

Thanks to the condition (1) in Definition 2.5 it is sufficient to perturb \(\omega\) near the boundary \(\partial E\) to complete the proof. The condition (1) in the same definition enables us to take a tubular neighborhood \(\nu_X(B)\) of \(B\) that is symplectomorphic to the interior of the disc bundle, \(\text{Int}(L^{\otimes 2}(r_1))\), of radius \(r_1\), where \(f\) agrees with projectivization \(f_P : L^{\otimes 2}(r_1) \setminus B \to \mathbb{CP}^1\). Choose \(r_0\) with \(0 < r_0 < r_1\) and let \(\nu_X(B)\) denote a tubular neighborhood of \(B\) symplectomorphic to \(\text{Int}(L^{\otimes 2}(r_0))\). With these identifications, we will deform \(\Omega\) on \(N_0(r_0, r_1)\) instead of \(\omega\) on \(\nu_X(B)\) in the rest of the proof.
The idea of the deformation is similar to that of [Ken, Lemma 9.3]. As we saw in Section 2.3.1, $\Omega$ is exact on $N_0(r_0,r_1)$, that is, $\Omega = d\lambda$; in particular, so is $\Omega$ on each fiber of $f_\nu|_{N_0(r_0,r_1)}$. Hence, one can take a fiberwise Liouville vector field $V'$ on $N_0(r_0,r_1)$. Identify a collar neighborhood $\nu_{N_0(r_0,r_1)}(\partial N_0(r_0))$ of $\partial N_0(r_0)$ with $(-\tau,0] \times (\partial N_0(r_0) \cap f_\nu^{-1}([1:0])) \times \mathbb{C}$ by the diffeomorphism $\Phi: (-\tau,0] \times (\partial N_0(r_0) \cap f_\nu^{-1}([1:0])) \times \mathbb{C} \to \nu_{N_0(r_0,r_1)}(\partial N_0(r_0))$ defined by

$$\Phi(t, (z,0), w) = \phi_t^V \left( \frac{z}{\sqrt{|w|^2 + 1}}, \frac{2w}{\sqrt{|w|^2 + 1}} \right),$$

where $\{\phi_t^V\}$ denotes the fiberwise Liouville flow of $V'$. Let $\alpha_0 := \lambda|_{\nu_{N_0(r_0)} \cap f_\nu^{-1}([1:0])}$ and $\lambda' := (e^t\alpha_0 - d^c \log(|w|^2 + 1)) - \Phi^*\lambda$.

Notice that the 1-form $\lambda'$ vanishes on every fiber of $f_\nu$. Now define the 2-form $\eta'$ on $(-\tau,0] \times (h^{-1}(\epsilon) \cap f_\nu^{-1}([1:0])) \times \mathbb{C}$ by $\eta' = d(\Phi^*\lambda) + d(\rho(t)\lambda')$, where $\rho: (-\tau,0] \to [0,1]$ is a smooth cut-off function such that $\rho(t) = 0$ near $t = -\tau$ and $\rho(t) = 1$ near $t = 0$. It follows from the choice of $\rho$ that the pushforward $\Omega' = \Phi_*(\eta')$ extends over the whole $N_0(r_0,r_1)$ in such a way that $\Omega' = \Omega$ away from $\nu_{N_0(r_0,r_1)}(\partial N_0(r_0))$. Moreover, $\partial N_0(r_0)$ is a connected component of the horizontal boundary of $N_0(r_0,r_1)$ for $f_\nu$ defined by $\Omega'$. Hence, the map $f_\nu: (N_0(r_0,r_1),\Omega') \to \mathbb{C}$ satisfies the condition for the horizontal triviality in Remark 2.2, which concludes that $\omega'$ is the desired perturbation of $\omega$ near the boundary $\partial N_0(r_0)$.

\[\square\]

2.4. Relation between a fibered Dehn twist and Dehn twists. Let $(W,d\lambda)$ be a Liouville domain and set $\alpha := \lambda|_{\partial W}$. Fix an identification of a collar neighborhood $\nu_W(\partial W)$ of $\partial W$ with $((-\epsilon,0] \times \partial W, d(e^t\alpha))$. Suppose that all Reeb orbits of $\alpha$ are $2\pi$-periodic. A fibered Dehn twist along $\partial W$ is a symplectomorphism of $(W,d\lambda)$ defined by

$$\tau_W(p) = \begin{cases} (t, \phi_{g(t)}^\alpha(x)) & \text{if } p = (t,x) \in (-\epsilon,0] \times \partial W \cong \nu_W(\partial W), \\ p & \text{otherwise,} \end{cases}$$

where $\{\phi_t^\alpha\}$ denotes the Reeb flow of $\alpha$ and $g: (-\epsilon,0] \to \mathbb{R}$ is a smooth function such that $g(t)$ equals $2\pi$ near $t = -\epsilon$ and $0$ near $t = 0$. A fibered Dehn twist has compact support by definition. Also, it is easy to see that $\tau_W$ is independent of the choice of $g$ up to Hamiltonian isotopy.

Under a certain condition, a fibered Dehn twist can be factorized into Dehn twists:

**Theorem 2.7** (Gompf [Gom, p.271] and Auroux [Aur2, p.6]). Let $(X,\omega)$ be a closed integral symplectic manifold admitting a Lefschetz pencil $f: X \setminus B \to \mathbb{CP}^1$ and $W$ the complement of a tubular neighborhood of the base locus $B$ in the closure of a regular fiber of $f$. Then, a fibered Dehn twist $\tau_W$ along the boundary $\partial W$ is symplectically isotopic to the product of Dehn twists along Lagrangian spheres in $(W,\omega|_W)$. In other words, $[\tau_W] \in \pi_0(\text{Symp}_c(W,\omega|_W))$ factorizes into the product of isotopy classes of Dehn twists.

In Appendix A we give a proof of Theorem 2.7 from the viewpoint of Lefschetz–Bott fibrations, restricting the theorem to the case where $X$ is projective.
3. Symplectic isotopy problem on Del Pezzo surfaces

To prove the main theorem (Theorem 1.1), we will symplectically identify the complements of homologous symplectic submanifolds of dimension 2 in a Kähler surface. In general, such complements are not symplectomorphic, nor even diffeomorphic each other. In some special cases, a solution to the symplectic isotopy problem gives a symplectomorphism between the complements.

**Problem 3.1** (Symplectic isotopy problem). Let \((M, \omega, J)\) be a Kähler surface and \(S\) a symplectic submanifold of dimension 2 in \((M, \omega)\). Then, is \(S\) symplectically isotopic to a complex curve in \((M, J)\)?

Suppose the problem is solved affirmatively for \((M, \omega, J)\) and also suppose any two homologous smooth complex curves in \((M, \omega, J)\) are symplectically isotopic. Then, this isotopy yields the desired symplectomorphism; see Proposition 4.3 below. The goal of this section is to solve this problem for Del Pezzo surfaces via holomorphic curve techniques.

3.1. Preliminaries of holomorphic curves.

3.1.1. Moduli spaces of holomorphic curves. Let \((\Sigma, j)\) be a smooth Riemann surface and \((M, J)\) an almost complex manifold. A map \(u: (\Sigma, j) \to (M, J)\) is called a (pseudo-)holomorphic curve, or \(J\)-holomorphic curve if \(du \circ j = J \circ du\). \(J\)-holomorphicity can be also defined in the case where \((\Sigma, j)\) is a nodal Riemann surface: For the normalization \(\varphi: \tilde{\Sigma} = \bigsqcup_i \tilde{\Sigma}_i \to \Sigma\) of \(\Sigma\), set \(\varphi_i = \varphi|_{\tilde{\Sigma}_i}\). A map \(u: (\Sigma, j) \to (M, J)\) is said to be \(J\)-holomorphic if each \(u \circ \varphi_i\) is \(J\)-holomorphic. Let us write \((\Sigma, j, x)\) for a marked nodal Riemann surface with \(m\) marked points \(x = \{x_1, \ldots, x_m\} \subset \Sigma\). A stable \(J\)-holomorphic curve \(u: (\Sigma, j, x) \to (M, J)\) is a \(J\)-holomorphic curve with the finite automorphism 

\[
\text{Aut}(u) = \{\phi \in \text{Aut}(\Sigma, j, x) \mid u \circ \phi = u\}.
\]

Fix \(A \in H_2(M; \mathbb{Z})\) and a finite subset \(z = \{z_1, \ldots, z_m\}\) of \(M\). We denote the moduli space of (unparametrized) \(J\)-holomorphic stable curves of genus \(g\) in the class \(A\) passing through \(z\) by

\[
\overline{\mathcal{M}}_{g,m}(A; J; z)
\]

and, given a symplectic structure \(\omega\) on \(M\), set

\[
\overline{\mathcal{M}}_{g,m}(A; J; z) = \bigcup_{J \in \mathcal{J}_r(M, \omega)} \overline{\mathcal{M}}_{g,m}(A; J; z) \times \{J\},
\]

where \(\mathcal{J}_r(M, \omega)\) is the set of smooth \(\omega\)-tame almost complex structures on \(M\). Here an element of \(\overline{\mathcal{M}}_{g,m}(A; J; z)\) is an isomorphism class of a stable \(J\)-holomorphic curve \(u: (\Sigma, j, x) \to (M, J)\) such that \((\Sigma, j)\) is a nodal Riemann surface of genus \(g\), \([u(\Sigma)] = A\) and \(u(x_i) = z_i\) for every \(i\). When \(m = 0\), we will suppress \(m\) and \(z\) from the notations of the moduli spaces. Also put

\[
\mathcal{M}_{g,m}(A; J; z) = \{[\Sigma, j, u, x] \in \overline{\mathcal{M}}_{g,m}(A; J; z) \mid \Sigma \text{ is smooth}\},
\]

\[
\mathcal{M}_{g,m}(A; J; z) = \bigcup_{J \in \mathcal{J}_r(M, \omega)} \mathcal{M}_{g,m}(A; J; z) \times \{J\}.
\]
We often call an element of $\mathcal{M}_{g,m}(A; j; z)$ and $\mathcal{M}_{g,m}(A; J; z)$ a smooth holomorphic curve.

As we will discuss a local structure of moduli spaces, we need to topologize these moduli spaces. We endow $\overline{\mathcal{M}}_{g,m}(A; J; z)$ with the $C^0$-topology and $\overline{\mathcal{M}}_{g,m}(A; J; z)$ with the subspace topology (see [Sie, Section 3]). Although we omit the definition of the $C^0$-topology, one of the things to keep in mind is that with this topology the moduli space $\overline{\mathcal{M}}_{g,m}(A; J; z)$ is compact and Hausdorff.

3.1.2. Holomorphic curves in dimension 4. Holomorphic curves dealt with in this paper are only over symplectic 4-manifolds. Such holomorphic curves are well studied, and here we will collect results for them.

Two complex curves in a complex surface intersect positively. The study of local structures of holomorphic curves shows an analogy of this phenomenon:

**Theorem 3.2** (Positivity of intersections [McD, MW]). Let $(M, J)$ be an almost complex 4-manifold. Suppose that $u_0 : (\Sigma_0, j_0) \to (M, J)$ and $u_1 : (\Sigma_1, j_1) \to (M, J)$ are smooth connected $J$-holomorphic curves whose images are not identical. Then, $u_0$ and $u_1$ intersect at most finitely many points, and the homological intersection number $\langle u_0(\Sigma_0) \cdot [u_1(\Sigma_1)] \rangle$ satisfies

$$\langle u_0(\Sigma_0) \cdot [u_1(\Sigma_1)] \rangle \geq \# \{ (z_0, z_1) \in \Sigma_0 \times \Sigma_1 | u_0(z_0) = u_1(z_1) \},$$

with equality if and only if all the intersections are transverse.

Suppose that $u : (\Sigma, j) \to (M, J)$ is a simple $J$-holomorphic curve, i.e., $u$ cannot split into a non-trivial holomorphic branched covering $(\Sigma, j) \to (\Sigma', j')$ and a $J$-holomorphic curve $u' : (\Sigma', j') \to (M, J)$. Set

$$\mathcal{D}(u) = \{ \{ z, z' \} \in \mathcal{P}(\Sigma) | u(z) = u(z'), z \neq z' \} \text{ and } \mathcal{C}(u) = \{ z \in \Sigma | du(z) = 0 \},$$

where $\mathcal{P}(\Sigma)$ is the power set of $\Sigma$. Let $\delta(u; z, z')$ denote the local intersection index for $\{ z, z' \} \in \mathcal{D}(u)$ of $u$ (see [Wen1, Section 2.10]). Also let $\delta(u, z)$ denote the virtual number of double points of $u$ at $z \in \mathcal{C}(u)$ (see [MW, Theorem 7.3] and also [Mil, Section 10]).

We define the integer $\delta(u)$ by

$$\delta(u) := \sum_{\{ z, z' \} \in \mathcal{D}(u)} \delta(u; z, z') + \sum_{z \in \mathcal{C}(u)} \delta(u; z) \in \mathbb{Z}. \tag{3.1}$$

By definition, $u : (\Sigma, j) \to (M, J)$ is an embedding with the smooth domain $(\Sigma, j)$ if and only if $\delta(u) = 0$. We also note that $\delta(u) \geq 0$ for any $J$-holomorphic curve $u$.

**Theorem 3.3** (Adjunction formula [McD, MW]). Let $(M, J)$ be an almost complex 4-manifold and $u : (\Sigma, j) \to (M, J)$ a simple $J$-holomorphic curve. Also let $\tilde{\Sigma} = \bigsqcup_{j=1}^d \tilde{\Sigma}_j \to \Sigma$ be the normalization of $\Sigma$. Then we have

$$2\delta(u) = |u(\Sigma)|^2 - c_1(TM)(|u(\Sigma)|) + \sum_{j=1}^d \chi(\tilde{\Sigma}_j). \tag{3.2}$$
The moduli space $\mathcal{M}_{g,m}(A; J; z)$ is not a manifold in general. One sufficient condition for $\mathcal{M}_{g,m}(A; J; z)$ being a manifold is that every element $[\Sigma, j, u, x] \in \mathcal{M}_{g,m}(A; J; z)$ is Fredholm regular (see e.g. [Wen1] for its definition). In dimension 4, this regularity is guaranteed by a homological condition:

**Theorem 3.4** (Automatic regularity [HLS]). Let $(M, J)$ be an almost complex 4-manifold and $[\Sigma, j, u, x] \in \mathcal{M}_{g,m}(A; J; z)$ an immersed $J$-holomorphic curve. If

$$c_1(TM)(A) > m,$$

then $[\Sigma, j, u, x] \text{ is Fredholm regular.}$

Let $(\Sigma, j)$ be a nodal Riemann surface. A $J$-holomorphic curve $u: (\Sigma, j) \to (M, J)$ is said to be nodal if $u$ is an embedding, that is, for the regularization $\varphi: \tilde{\Sigma} = \bigsqcup_j \tilde{\Sigma}_j \to \Sigma$, $u \circ \varphi$ is an embedding on each $\tilde{\Sigma}_j$ and the image $(u \circ \varphi)(\tilde{\Sigma})$ has distinct tangent spaces at all the nodes. The following theorem of Sikorav tells us that the stratum consisting of nodal holomorphic curves is at least (real) codimension 2 in the compactified moduli space:

**Theorem 3.5** ([Sik, Corollary 1.4]). Let $M$ be a 4-manifold with an almost complex structure $J$ for which all elements of $\mathcal{M}_{g,m}(A; J; z)$ are Fredholm regular. Let $[\Sigma, j, u, x]$ an element of $\overline{\mathcal{M}}_{g,m}(A; J; z)$ such that $u$ is a nodal curve with $k$ nodes and $x$ does not contain nodes. Let us denote the normalization of $\Sigma$ by $\varphi: \tilde{\Sigma} = \bigsqcup_{i=1}^r \tilde{\Sigma}_i \to \Sigma$ and set $u_i = u \circ \varphi|_{\tilde{\Sigma}_i}: \tilde{\Sigma}_i \to M$. Suppose it holds that

$$c_1(TM)([u_i(\tilde{\Sigma}_i)]) > |z \cap u_i(\tilde{\Sigma}_i)|$$

for all $i = 1, \ldots, r$. Then, a neighborhood of $[\Sigma, j, u, x]$ in $\overline{\mathcal{M}}_{g,m}(A; J; z)$ is homeomorphic to an open neighborhood of the origin in $\mathbb{C}^{\dim_{\mathbb{C}} \mathcal{M}_{g,m}(A; J; z)}$ with the expected (complex) dimension

$$\dim_{\mathbb{C}} \mathcal{M}_{g,m}(A; J; z) = c_1(TM)(A) + g - 1 - m$$

of the moduli space $\mathcal{M}_{g,m}(A; J; z)$. The subset parametrizing nodal $J$-holomorphic curves is a union of complex coordinate hyperplanes $\{(\xi_1, \ldots, \xi_k) \in \mathbb{C}^k \mid \xi_i = 0\} \times \mathbb{C}^{\dim_{\mathbb{C}} \mathcal{M}_{g,m}(A; J; z) - k}$.

In addition to Sikorav’s original paper [Sik], the reader is referred to [ST, pp.998–1000] for a sketch of the proof of Theorem 3.5.

### 3.2. Symplectic isotopy problem

We shall now discuss Problem cxi for Del Pezzo surfaces. Let us begin by recalling the definition of Del Pezzo surface.

**Definition 3.6.** A Del Pezzo surface $M$ is a smooth projective surface with ample anticanonical class $-K_M$. The number $K_M^2 := K_M \cdot K_M$ is called the degree of $M$.

It is well known that a Del Pezzo surface of degree $d$ is diffeomorphic to either $\mathbb{CP}^2 \# \overline{\mathbb{CP}}^2$ or $S^2 \times S^2$ if $d = 8$; otherwise, $\mathbb{CP}^2 \# (9 - d)\overline{\mathbb{CP}}^2$ (see [Man] Theorem 24.4). Moreover, according to [Man] Remark 24.4.1, all Del Pezzo surfaces of degree $d$ for $5 \leq d \leq 7$ are biholomorphic.
The following theorem is the main result in this section.

**Theorem 3.7.** Let $M$ be a Del Pezzo surface diffeomorphic to $\mathbb{CP}^2 \# n \mathbb{CP}^2$ $(0 \leq n \leq 8)$ and $\omega$ a Kähler form on $M$. Suppose that $S$ is a 2-dimensional symplectic submanifold of $(M, \omega)$ homologous to an anti-canonical divisor. Then, $S$ is symplectically isotopic to a smooth complex curve. Furthermore, two such symplectic submanifolds on $M$ are mutually symplectically isotopic if $0 \leq n \leq 6$. 

This theorem has been partially proven by Sikorav [Sik, Theorem 1.5] and Shevchishin [She, Theorem 1] for the case $n = 0$, and Siert and Tian [ST, Theorem B] for the case $n = 0$, 1; see also [LM, Proposition 3.6] where Li and Mak give a generalization of the theorem to symplectic divisors in the latter case. 

We will prove Theorem 3.7 by combining several lemmas. Before getting into the details, let us fix the notations. Throughout the rest of this section, we set 

$$M(n) := \mathbb{CP}^2 \# n \mathbb{CP}^2$$

and write $J(n)$ for an integrable almost complex structure on $M(n)$ and $\omega_0$ for a Kähler form on $M(n)$ with respect to $J(n)$. We denote the homology classes of a complex line and $n$ disjoint exceptional spheres in $M(n)$ by 

$$H, E_1, \ldots, E_n \in H_2(M(n); \mathbb{Z}),$$

respectively. The homology class $K_n \in H_2(M(n); \mathbb{Z})$ is defined to be 

$$K_n := -3H + \sum_{j=1}^{n} E_j.$$

Given $A \in H_2(M(n); \mathbb{Z})$, we define the set $J_{\text{reg}}(M(n); A)$ of $\omega_0$-tame almost complex structures $J$ on $M(n)$ for which every simple $J$-holomorphic curve in the class $A$ is Fredholm regular. According to [MS, Section 6.2] and [Cha, Chapter 5], given $8 - n$ points $z$ on $M(n)$, there exists a residual subset $J_{\text{reg}}^*(n; z)$ of $J_{\tau}(M(n), \omega_0)$ satisfying the following properties:

- $J_{\text{reg}}^*(n; z)$ is path-connected;
- For every $J \in J_{\text{reg}}^*(n; z)$ and any decomposition $-K_n = A_1 + \cdots + A_m$ consisting of non-trivial spherical classes $A_i$, every simple $J$-holomorphic sphere representing $A_i$ is Fredholm regular;
- For any $J \in J_{\text{reg}}^*(n; z)$, all $J$-holomorphic spheres in the class $-K_n$ passing through $z$ have only nodes as singularities.

**Lemma 3.8.** Let $J$ be an element of $J_{\text{reg}}(M(n); dH - \sum_{i=1}^{n} e_i E_i)$ and $u: (\Sigma, j) \rightarrow (M(n), J)$ a non-constant smooth simple $J$-holomorphic curve of genus at most 1. Suppose that $|u(\Sigma)| = dH - \sum_{i=1}^{n} e_i E_i$ and $|u(\Sigma)| \neq H, E_i$ for every $i$. Then, $d > 0$ and $e_i \geq 0$. 
there exists $m$ \( \exists k \)  irreducible component \( \tilde{\Sigma} \)  

Proof. For notational convenience, set $J$ holomorphic spheres in the classes $H$ and $E_j$ always exist (see Wen2 Theorem 5.1 and its proof), the positivity of intersections (Theorem 3.2) shows that $d = 0$ and $e_i \geq 0$ for each $i$. Suppose $d = 0$. Then, the dimension of the moduli space of simple $J$-holomorphic curves of genus $g(\Sigma)$ in the class $-\sum_{i=1}^n e_iE_i$ is given by  

$$-\chi(\Sigma) + 2 \left( c_1(TM(n)) \left( -\sum_{i=1}^n e_iE_i \right) \right) = 2 \left( g(\Sigma) - 1 - \sum_{i=1}^n e_i \right).$$

With $g(\Sigma) = 0, 1$, we have $2 (g(\Sigma) - 1 - \sum_{i=1}^n e_i) \leq -2 \sum_{i=1}^n e_i$. Note that $u$ is non-constant and there must be a positive $e_i$. Hence, the dimension of the moduli space is negative, which contradicts the existence of $u$. Thus, $d > 0$.

The next corollary follows from the above lemma and its proof immediately.

Corollary 3.9. Let $u: (\Sigma, j) \to (M(n), J)$ be a non-constant smooth simple $J$-holomorphic curve of genus at most 1 with $J \in J_{reg}(M(n); dH - \sum_{i=1}^n e_iE_i)$. If $u(\Sigma)$ does not intersect a $J$-holomorphic sphere in the class $H$, then $[u(\Sigma)] = E_{i_0}$ for some $i_0$.

Proof. By the formula 9.2 in Theorem 3.2 we have  

$$2\delta(u) = d(d - 3) + \sum_i e_i(1 - e_i).$$

As $\delta \geq 0$, $d(d - 3) \leq 0$ and $\sum_i e_i(1 - e_i) \leq 0$, we have $d \in \{0, 3\}$ and $e_i \in \{0, 1\}$. If $d = 0$, then Corollary 3.9 implies that $[u(\Sigma)] = E_{i_0}$ for some $i_0$ and $g(\Sigma) = 0$, which is a contradiction.

Lemma 3.10. For $J \in J_{reg}(M(n); dH - \sum_{i=1}^n e_iE_i)$, let $u: (\Sigma, j) \to (M(n), J)$ be a non-constant smooth simple $J$-holomorphic curve with $[u(\Sigma)] = dH - \sum_i e_iE_i$. If $0 \leq d \leq 3$ and $g(\Sigma) = 1$, then we have $d = 3$ and $e_i \in \{0, 1\}$.

Proof. By the formula 9.2 in Theorem 3.2 we have  

$$2\delta(u) = d(d - 3) + \sum_i e_i(1 - e_i).$$

As $\delta \geq 0$, $d(d - 3) \leq 0$ and $\sum_i e_i(1 - e_i) \leq 0$, we have $d \in \{0, 3\}$ and $e_i \in \{0, 1\}$. If $d = 0$, then Corollary 3.9 implies that $[u(\Sigma)] = E_{i_0}$ for some $i_0$ and $g(\Sigma) = 0$, which is a contradiction.

Lemma 3.11. Let $(\nu_\ell)_{\ell}$ be a sequence of almost complex structures in $J_{reg}^+(n; z)$ and $(u_\nu)$ a sequence of embedded $J_\nu$-holomorphic curves of genus 1 in the class $-K_n$ passing through generic $8 - n$ points $z$ in $M(n)$. Suppose that $J_\nu \to J_\infty \in J_{reg}^+(n; z)$ in the $C^0$-topology and $u_\nu \to u_\infty$ in the $C^0$-topology. The limit stable curve $u_\infty$ is written as $(u_{\infty, 1}, \ldots, u_{\infty, N})$, where each $u_{\infty, a}$ is a $J_{\infty}$-holomorphic curve defined on a respective irreducible component $\Sigma_{\infty, a}$ of the limit nodal curve $\Sigma_\infty$. Then, all non-constant curves $u_{\infty, a}$ are simple.

Proof. For notational convenience, set $C_{\infty} := u_\infty(\Sigma_\infty)$ and $C_{\infty, a} := u_{\infty, a}(\Sigma_{\infty, a})$. Suppose that $[C_{\infty, a}] = d_a H - \sum_{i=1}^n e_{a,i}E_i$. Let $m_a \in \mathbb{Z}_{\geq 0}$ be the multiplicity of $C_{\infty, a}$ and $k_a \in \mathbb{Z}$ the number of points of $C_{\infty, a} \cap z$. In view of the dimension of the moduli space, for each $a$, we have $2k_a \leq d_a (d_a + 3) - \sum_{i=1}^n e_{a,i}(e_{a,i} + 1)$. We will show that if there exists $m_a \geq 2$, then we have  

$$2 \sum_{a=1}^N k_a \leq \sum_{a=1}^N d_a (d_a + 3) - \sum_{i=1}^n \left( \sum_{a=1}^N e_{a,i}(e_{a,i} + 1) \right) < 2(8 - n).$$


which contradicts the fact that $C_\infty$ passes through the fixed $8 - n$ points $z$.

We first observe that $\sum_{a=1}^N e_{a,i_0}(e_{a,i_0} + 1) \geq 2$ for any fixed $i_0$. Indeed, if all $e_{a,i_0}$ are 0 or $-1$, then $\sum_{a=1}^N m_a e_{a,i_0} \leq 0 < 1$, which contradicts the fact that $\sum_{a=1}^N m_a e_{a,i_0} = 1$. Hence, there exists $e_{a_0,i_0}$ that is not 0 or $-1$, and we have

\[
\sum_{a=1}^N e_{a,i_0}(e_{a,i_0} + 1) \geq 1 \cdot (1 + 1) = 2.
\]  

(3.3)

In what follows, without loss of generality, we may assume that $m_1 \geq 2$.

**Case 1.** $d_1 = 0$.

In this case, by Corollary 3.9, we have $[C_{\infty,1}] = E_{i_0}$ for some $i_0$. Hence we obtain

\[
\sum_{2 \leq a \leq N} m_a e_{a,i_0} = 1 - m_1 e_{1,i_0} \geq 3
\]

for the fixed $i_0$, which shows that for some $a_0 \geq 2$,

\[
e_{a_0,i_0} > 0.
\]

Thus, it follows from Corollary 3.9 again that $d_{a_0}$ must be positive. This leads to the following two cases.

**Case 1-(i).** $d_{a_0} = 1, 2$.

In this case, observe that $\sum_{a=1}^N d_a (d_a + 3) \leq 14$. Thus,

\[
\sum_{a=1}^N d_a (d_a + 3) - \sum_{i=1}^N \sum_{a=1}^N e_{a,i}(e_{a,i} + 1) \leq 14 - 2n < 16 - 2n.
\]

**Case 1-(ii).** $d_{a_0} = 3$.

As $d_{a_0} = 3$, we have $m_{a_0} = 1$ and hence $d_a = 0$ for all $a \neq a_0$. Combining Corollary 3.9 with the inequality (3.3) shows that $e_{a,i_0} \in \{-1, 0\}$ for all $a \neq a_0$ and $e_{a_0,i_0} \geq 3$. Then, with the inequality (3.3) we have

\[
\sum_{a=1}^N d_a (d_a + 3) - \sum_{a=1}^N \sum_{i=1}^N e_{a,i}(e_{a,i} + 1) \\
\leq \sum_{a=1}^N d_a (d_a + 3) - \sum_{i=1,i \neq i_0}^N \sum_{a=1}^N e_{a,i}(e_{a,i} + 1) - \sum_{a=1}^N e_{a_0,i_0}(e_{a_0,i_0} + 1) \\
\leq 3 \cdot (3 + 3) - (n - 1) \cdot 2 - 3(3 + 1) = 8 - 2n < 16 - 2n,
\]

which completes Case 1.

**Case 2.** $d_1 = 1$.

Notice that $2 \leq m_1 \leq 3$ in this case. Suppose that $m_1 = 2$. As $d_a \geq 0$ for any $a$ by Lemma 3.8, there exists a unique $a_0 > 1$ satisfying $d_{a_0} = 1$, and the rest of $d_a$ must be
0. Hence, we obtain \( \sum a d_a (d_a + 3) \leq 8 \), which also holds for the case \( m_1 = 3 \). Therefore, we conclude that

\[
\sum_{a=1}^{N} d_a (d_a + 3) - \sum_{i=1}^{n} \left( \sum_{a=1}^{N} e_{a,i} (e_{a,i} + 1) \right) \leq 8 - 2n < 16 - 2n,
\]

which finishes the proof of the lemma.

Lemma 3.12. Let \( u_{\infty} = (u_{\infty,1}, \ldots, u_{\infty,N}) \): \((\Sigma_{\infty}, J_{\infty}) \to (M_n, J_{\infty})\) be the limit curve in Lemma 3.11. Then, each \( u_{\infty,a} \) is a non-constant curve that is either an embedding of a genus 1 curve or a nodal sphere with one node. In particular, \( N = 1 \).

Proof. We shall use the same notations as in the proof of Lemma 3.11 and call each \( u_{\infty,a} \) a component of \( u_{\infty} \). The following argument is inspired by the proof of [Bar, Theorem 2].

First assume that all components \( u_{\infty,a} \) are non-constant. Let \( k_a \) be the number of points of \( z \) where \( C_{\infty,a} \) passes, and put \( z_a = z \cap C_{\infty,a} \). Let \( \mathcal{M}_{g_a,k_a}^* (d_a H - \sum_{i=1}^{n} e_{a,i} E_i; J_{\infty}; z_a) \) be the subspace of \( \mathcal{M}_{g_a,k_a}^* (d_a H - \sum_{i=1}^{n} e_{a,i} E_i; J_{\infty}; z_a) \) consisting of simple \( J_{\infty} \)-holomorphic curves. The expected dimensions of the moduli spaces \( \mathcal{M}_{0,k_a}^* (d_a H - \sum_{i=1}^{n} e_{a,i} E_i; J_{\infty}; z_a) \) and \( \mathcal{M}_{1,k_a}^* (d_a H - \sum_{i=1}^{n} e_{a,i} E_i; J_{\infty}; z_a) \) are given by

\[
2 \left( \frac{\chi(\Sigma_{\infty,a})}{2} \dim \mathcal{M} + c_1(M) \left( d_a H - \sum_{i=1}^{n} e_{a,i} E_i \right) \right) - 3\chi(\Sigma_{\infty,a}) - 2k_a \\
= \begin{cases} 
2 (-1 + 3d_a - \sum_{i=1}^{n} e_{a,i} - k_a) & \text{if } g(\Sigma_{\infty,a}) = 0, \\
2 (3d_a - \sum_{i=1}^{n} e_{a,i} - k_a) & \text{if } g(\Sigma_{\infty,a}) = 1.
\end{cases}
\]

Since \( J_{\infty} \) is generic, this shows that the moduli space \( \mathcal{M}_{0,k_a}^* (d_a H - \sum_{i=1}^{n} e_{a,i} E_i; J_{\infty}; z_a) \) (resp. \( \mathcal{M}_{1,k_a}^* (d_a H - \sum_{i=1}^{n} e_{a,i} E_i; J_{\infty}; z_a) \)) is non-empty only if \( k_a \leq 3d_a - \sum_{i=1}^{n} e_{a,i} - 1 \) (resp. \( k_a \leq 3d_a - \sum_{i=1}^{n} e_{a,i} \)). For each point \( z \) of \( z \), more than one curve \( C_{\infty,a} \) may pass through \( z \). Hence if all \( C_{\infty,a} \) are spheres, we have

\[
8 - n \leq \sum_{a=1}^{N} k_a \leq \sum_{a=1}^{N} \left( 3d_a - \sum_{a=1}^{n} e_{a,i} - 1 \right) = 9 - n - N,
\]

which implies that \( N = 1 \). Suppose that one of \( C_{\infty,a} \), say \( C_{\infty,1} \), has genus 1. In this case,

\[
8 - n \leq \sum_{a=1}^{N} k_a \leq \left( 3d_1 - \sum_{i=1}^{n} e_{1,i} \right) + \sum_{a=2}^{N} \left( 3d_a - \sum_{i=1}^{n} e_{a,i} - 1 \right) = 9 - n - (N - 1),
\]

which shows \( N \leq 2 \). Note that by Lemma 3.10 \( d_1 = 3 \) and \( e_{1,i} \in \{0, 1\} \) for all \( i \). If \( N = 2 \), \( d_2 \) must be 0, and Corollary 3.3 implies \( [C_{\infty,2}] = E_{i_0} \) for some \( i_0 \). Hence, we have \( e_{1,i_0} = 2 \), which is a contradiction. Thus, \( N = 1 \). As the domain of \( u_{\infty} = u_{\infty,1} \) is either a smooth curve of genus 1 or a nodal sphere with one node, the lemma follows from the formula (3.2) and the definition of \( J_{\text{reg}}(n; z) \).
In general, some $u_{\infty,a}$ might be a constant curve, which does not happen actually: Suppose $u_{\infty}$ has a constant component. If a unique non-constant component is a genus 1 curve, then there is a constant sphere $u_{\infty,a}$ having two marked points in $\mathbf{x}$ by the stability condition. However, these points are mapped to the same point of $\mathbf{z}$, which is a contradiction. Next assume that a unique non-constant curve, say $u_{\infty,1}$, is a nodal sphere with one node. In order to prevent $u_{\infty}$ from having a constant sphere with two marked points, the domain of $u_{\infty}$ needs to be a nodal Riemann surface $\Sigma_\infty$ that consists of two spheres $\Sigma_{\infty,1}$ and $\Sigma_{\infty,2}$ glued together at two nodes; $\Sigma_{\infty,1}$ has $7-n$ marked points and $\Sigma_{\infty,2}$ has one marked point; see Figure 2. When $n=8$, there cannot be such a nodal Riemann surface with marked points in the first place; otherwise, the expected dimension of the moduli space of such $J_\infty$-holomorphic curves $u: \Sigma_\infty \to M$ is negative. This violates the existence of $u_{\infty}$. Therefore, we conclude that $u_{\infty}$ has no constant component.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure2.png}
\caption{Nodal Riemann surface $\Sigma_\infty$ with two nodes. Red points indicate marked points: $\Sigma_{\infty,1}$ contains $7-n$ marked points; $\Sigma_{\infty,2}$ contains one marked point.}
\end{figure}

\textit{Proof of Theorem 3.7.} Throughout this proof, we often denote an element $[\Sigma, j, u, x]$ of $\overline{M}_{1,8-n}(-K_n; J; z)$ just by $[u]$ for simplicity.

Take an $\omega_0$-tame almost complex structure $J$ on $M(n)$ making $S$ $J$-holomorphic and also choose $8-n$ points $z$ on $S$. Notice that in general, $J \not\in J^*_\text{reg}(n; z)$. Thanks to the automatic regularity (Theorem 3.1), the projection $\overline{M}_{1,8-n}(-K_n; J; z) \to J^*(M(n); \omega_0)$ is a submanifold at a $J$-holomorphic curve $u: (\Sigma, j) \to M(n)$ parametrizing $S$. Hence one can slightly deform $([u], J)$ in the moduli space $\overline{M}_{1,8-n}(-K_n; J; z)$ and obtain $([u'], J') \in \overline{M}_{1,8-n}(-K_n; J; z)$ with $J' \in J^*_\text{reg}(n; z)$ and $u'$ symplectically isotopic to $u$.

The complex structure $J(n)$ may not be an element of $J^*_\text{reg}(n; z)$. Choose distinct $8-n$ points $z' = \{z'_1, \ldots, z'_{8-n}\} \subset M(n)$ sufficiently close to $z$ for which $J(n) \in J^*_\text{reg}(n; z')$. A result of Boothby [Boo] yields a Hamiltonian isotopy $(\phi_t)_{t \in [0,1]}$ of $(M(n), \omega_0)$ such that $\phi_0 = \text{Id}$ and $\phi_1(z'_i) = z_i$, and we have the regular almost complex structure

$$(\phi_1)_*J(n) := d\phi_1 \circ J(n) \circ (d\phi_1)^{-1} \in J^*_\text{reg}(n; z).$$

Thus, once we have a symplectic isotopy $(S_t)$ between $J'$- and $(\phi_1)_*J(n)$-holomorphic curves, the composition of this isotopy and $\phi_t^{-1}(S_1)$ gives a symplectic isotopy between $S$ and a $J(n)$-holomorphic curve, i.e., a complex curve.
Now we may assume that $S$ is a $J$-holomorphic, and $J$ and $J(n)$ are elements of $\mathcal{J}^*_\text{reg}(n; z)$. The moduli space along a path $h(t)$ in $\mathcal{J}^*_\text{reg}(n; z)$ joining $J$ and $J(n)$,

$$\mathcal{M}_{1,8-n}(-K_n; h; z) = \bigcup_{t \in [0, 1]} \{t\} \times \mathcal{M}_{1,8-n}(-K_n; h(t); z)$$

is a 3-dimensional manifold. Moreover, the natural projection $p: \mathcal{M}_{1,8-n}(-K_n; h; z) \to [0, 1]$ is a submersion.

Let $I \subset [0, 1]$ be the set of $\tau \in I$ such that for any $t \leq \tau$, a smooth $J_t := h(t)$-holomorphic curve $u_t$ exists and is symplectically isotopic to $S$. Since $p$ is a submersion, $I$ is an open subset of $[0, 1]$. Hence all we have to show is that $I$ is closed in $[0, 1]$. Let $(t_\nu)$ be an increasing sequence of $I$ converging to $\tau$. For each $t_\nu$, choose a $J_{t_\nu}$-holomorphic curve $u_{t_\nu}$ that passes through $z$ and is symplectically isotopic to $S$. By the Gromov compactness, we may assume that $[u_{t_\nu}]$ converges to a stable $J_\tau$-holomorphic curve $[u_\tau]$. Since $J_\tau \in \mathcal{J}^*_\text{reg}(n; z)$, Lemma 3.12 implies that if $u_\tau$ is singular, i.e., an element of $\overline{\mathcal{M}}_{1,8-n}(-K_n; J_\tau; z) \setminus \mathcal{M}_{1,8-n}(-K_n; J_\tau; z)$, then it is a nodal sphere with one node. Now Theorem 3.5 proves that the moduli space of such nodal curves contributes a 1-dimensional stratum to $\overline{\mathcal{M}}_{1,8-n}(-K_n; h; z)$. Thus, its complement in $\overline{\mathcal{M}}(-K_n; h; z)$ is still connected near $[u_\tau]$. It turns out that a $J_\tau$-holomorphic smoothing of $u_\tau$ is symplectically isotopic to $u_{t_\nu}$ for $\nu \gg 0$, in particular to $S$. Therefore, $\tau \in I$, and $I$ is a closed subset of $[0, 1]$. As $[0, 1]$ is connected, we have $I = [0, 1]$. This concludes the existence of a symplectic isotopy between $S$ and a $J(n)$-holomorphic curve.

An anti-canonical divisor of $M(n)$ is very ample for $0 \leq n \leq 6$ [Har, Theorem V.4.6]. Hence, the assertion for two symplectic surfaces in the class $-K_n$ for $0 \leq n \leq 6$ immediately follows from the above discussion and the lemma below.

**Lemma 3.13.** Let $(M, \omega, J)$ be a Kähler manifold, and let $S$ and $S'$ be smooth very ample divisors in the same linear system. Then, $S$ and $S'$ are symplectically isotopic as symplectic submanifolds with respect to $\omega$.

**Proof.** Using the complete linear system $|S|$, embed $M$ into $\mathbb{CP}^N$. Bertini’s theorem shows that the intersection of $M$ and a generic hyperplane in $\mathbb{CP}^N$ gives a smooth divisor in $|S|$. Furthermore, the space of such hyperplanes is connected; hence we obtain a one-parameter family of smooth ample divisors connecting $S$ and $S'$. This family presents the desired symplectic isotopy. □

### 4. Mapping class group relation

#### 4.1. Del Pezzo surfaces of degree 6

A crucial ingredient of the proof of Theorem 4.1 is Del Pezzo surfaces of degree 6. In this subsection, we review how they appear as ample divisors of complex 3-manifolds following [Fu] and then study the symplectic nature of those divisors.

**4.1.1. Del Pezzo surfaces as ample divisors on complex 3-folds.** Let $X_1 := \mathbb{CP}^1 \times \mathbb{CP}^1 \times \mathbb{CP}^1$ and $M_1$ a smooth hypersurface of tri-degree $(1, 1, 1)$ in $X_1$. Also, let $X_2$ be a smooth hypersurface of bi-degree $(1, 1)$ in $\mathbb{CP}^2 \times \mathbb{CP}^2$ and $M_2$ the complete intersection of $X_2$
and another smooth hypersurface of the same bi-degree in $\mathbb{CP}^2 \times \mathbb{CP}^2$. A straightforward computation with the adjunction formula shows that $(K_{M_j})^2 = 6$ for $j = 1, 2$, and hence $M_j$ is a Del Pezzo surface of degree 6. For future use, we define the hypersurface $B_j$ in $M_j$ to be the intersection of $M_j$ and another generic smooth hypersurface of the same degree in $X_j$. By definition, the normal bundle $N_{M_j/X_j}$ of $M_j$ in $X_j$ is isomorphic to $-K_{M_j}$ and has $c_1(N_{M_j/X_j}) = \text{PD}([B_j]) \in H^2(M_j; \mathbb{Z})$, which implies that $B_j$ is an anti-canonical divisor of $M_j$. Moreover, we find that the canonical bundle $K_{B_j}$ is trivial, and $B_j$ is a genus 1 curve in $M_j$ with self-intersection number 6. Therefore, $M_j$ is a Del Pezzo surface of degree 6.

### 4.1.2. Symplectic nature of $M_1$ and $M_2$

We endow $M_j$ with a symplectic structure as follows: Let $\Omega_j$ be a symplectic form on $X_j$ induced by the Fubini–Study form on the product of projective spaces. As $M_j$ is a hyperplane section of $X_j$, we have $[\Omega_j] = \text{PD}([M_j]) \in H^2(X_j; \mathbb{Z})$. Since $M_j$ is a complex submanifold of the Kähler manifold $(X_j, \Omega_j)$, the restriction $\omega_j := \Omega_j|_{M_j}$ serves as a symplectic form on $M_j$. Note that $[\omega_j] = \text{PD}([B_j])$.

**Lemma 4.1.** Let $(M_1, \omega_1)$ and $(M_2, \omega_2)$ be Del Pezzo surfaces of degree 6 with symplectic structures as above. Then, $(M_1, \omega_1)$ and $(M_2, \omega_2)$ are symplectomorphic.

**Proof.** Since a Del Pezzo surface of degree 6 is unique up to biholomorphism (see [Man], Remark 24.4.1), one can take a biholomorphism $\phi : M_1 \to M_2$. The symplectic forms $\omega_1$ and $\phi^* \omega_2$ are compatible with the complex structure on $M_1$, and so is $(1 - t)\omega_1 + t\phi^* \omega_2$ for $t \in [0, 1]$. As $[\omega_1]$ and $[\phi^* \omega_2]$ are Poincaré dual to an anti-canonical divisor, they are cohomologous. Hence, Moser’s stability gives a symplectomorphism $\psi : (M_1, \omega_1) \to (M_1, \phi^* \omega_2)$. Thus, $\phi \circ \psi : (M_1, \omega_1) \to (M_2, \omega_2)$ is the desired symplectomorphism. \[\square\]

Next we will see that the complements of some tubular neighborhoods of $B_1$ and $B_2$ are symplectomorphic with (strictly) contactomorphic boundaries. To begin with, we briefly review a symplectic model of a tubular neighborhood of a symplectic submanifold.

Let $(M, \omega)$ be a closed integral symplectic manifold and $B$ a codimension 2 symplectic submanifold. Suppose that $[\omega] = \text{PD}[B]$. Take a complex line bundle $p : N \to B$ with $c_1(M) = [\omega]|_B$, which is isomorphic to the normal bundle of $B$ in $M$. For a hermitian metric $\| \cdot \|$ on $N$, we choose a hermitian connection $\nabla$ with curvature $-2\pi \omega|_B$. Let $\alpha^\nabla$ denote the associated transgression 1-form $\alpha^\nabla$ on $N \subset B$ defined by $\alpha^\nabla|_u(iu) = 1/2\pi$ for every $u \in N \setminus B$, and $\alpha^\nabla|_{H\nabla} = 0$, where $H\nabla$ is the horizontal distribution of $\nabla$. Define a symplectic form $\omega^\nabla$ on $\{u \in N \mid \|u\| < 1\}$ to be

$$\omega^\nabla = p^* (\omega|_B) + d(r^2 \alpha^\nabla),$$

where $r$ is the radial coordinate along the fibers induced by $\| \cdot \|$. As $d\alpha^\nabla = -p^* (\omega|_B)$, we have $\omega^\nabla = d((r^2 - 1)\alpha^\nabla)$, which is exact outside the zero-section. Set

$$\lambda^\nabla = (r^2 - 1)\alpha^\nabla.$$

Weinstein’s symplectic tubular neighborhood theorem tells us that for some $\delta > 0$, there is a symplectic embedding $\rho : (N_\delta, \omega^\nabla) \to (M, \omega)$ that maps the zero-section to $B$, where $N_\delta = \{u \in N \mid \|u\| \leq \delta\}$. 
Lemma 4.4. Let \( \Phi: (M, \omega) \to (M, \omega) \) be a symplectomorphism with \( \omega = \text{PD}(B) \). Let \( \lambda_0 \) and \( \lambda_1 \) be 1-forms defined on \( M \setminus B \) satisfying \( d\lambda_1 = d\lambda_2 = \omega \). Suppose that for each \( j = 0, 1 \), the closed tubular neighborhood of \( B \) given by a symplectic embedding \( \phi_j: (N_\delta, \omega^\vee) \to (M, \omega) \) satisfies \( \phi_j^* \lambda_j = \lambda^\vee \), where \( \lambda^\vee \) is a connection on the normal bundle \( N \) of \( B \) in \( M \) with curvature \(-2\pi \omega |_B \) and \( N_\delta \subset N \) is the disc bundle of radius \( \delta \) with respect to a hermitian metric \( | \cdot |_j \). Then, there exists a symplectomorphism \( \Psi: (M, \omega) \to (M, \omega) \) such that

Proposition 4.3. For \( j = 1, 2 \), let \((M_j, \omega_j)\) be the Del Pezzo surface of degree 6 with the symplectic structure \( \omega_j \) and let \( B_j \) be the anti-canonical divisor as above. Also, let \( s_j \) be a holomorphic section of the holomorphic line bundle \( \mathcal{O}(B_j) \) with curvature \(-2\pi \omega |_{B_j} \). Then, there exists a symplectomorphism \( \Phi: (M_1, \omega_1) \to (M_2, \omega_2) \) such that

1. \( \Phi(B_1) = B_2 \);
2. For some closed tubular neighborhoods \( \nu_{M_j}(B_j) \) of \( B_j \), \( \Phi(\nu_{M_j}(B_1)) = \nu_{M_2}(B_2) \) and \( \Phi^*((d^2 \log \| s_2 \|^2) |_{\nu_{M_j}(B_j)} |_{B_j}) = (d^2 \log \| s_1 \|^2) |_{\nu_{M_j}(B_1)} |_{B_1} \). In particular,
   \[
   \Phi^*((d^2 \log \| s_2 \|^2) |_{\partial \nu_{M_j}(B_2)}) = (d^2 \log \| s_1 \|^2) |_{\partial \nu_{M_j}(B_1)},
   \]
   that is, \( \Phi |_{\partial \nu_{M_j}(B_1)} \) gives rise to a strict contactomorphism.

The proposition will be shown by combining Lemma 4.4 with the following lemma.

Lemma 4.4. Let \((M, \omega)\) be a closed integral symplectic manifold and \( B \) a symplectic submanifold with \( \omega = \text{PD}(B) \). Let \( \lambda_0 \) and \( \lambda_1 \) be 1-forms defined on \( M \setminus B \) satisfying \( d\lambda_1 = d\lambda_2 = \omega \). Suppose that for each \( j = 0, 1 \), the closed tubular neighborhood of \( B \) given by a symplectic embedding \( \phi_j: (N_\delta, \omega^\vee) \to (M, \omega) \) satisfies \( \phi_j^* \lambda_j = \lambda^\vee \), where \( \lambda^\vee \) is a connection on the normal bundle \( N \) of \( B \) in \( M \) with curvature \(-2\pi \omega |_B \) and \( N_\delta \subset N \) is the disc bundle of radius \( \delta \) with respect to a hermitian metric \( | \cdot |_j \). Then, there exists a symplectomorphism \( \Psi: (M, \omega) \to (M, \omega) \) such that
(1) $\Psi(B) = B$;

(2) $\Psi^*\lambda_2 = \lambda_1$ near $B$.

**Proof.** Shrinking $N_{\delta_1}$, we may assume that $N_{\delta_0} \subset N_{\delta_1}$ and $\text{Im}(\rho_0) \subset \text{Im}(\rho_1)$. Also assume for a while that there exists a symplectic embedding $\kappa: (N_{\delta_0}, \omega^{\delta_0}) \to (N_{\delta_1}, \omega^{\delta_1})$ that preserves the zero-section and satisfies $\kappa^*\lambda^{\delta_1} = \lambda^{\delta_0}$; we will show this existence later. Consider the isotopy of symplectic embeddings $\rho_t: (N_{\delta_0}, \omega^{\delta_0}) \to (M, \omega)$ defined by

$$\rho_t(u) := \rho_1(t \kappa(u) + (1 - t)(\rho_1^{-1} \circ \rho_0)(u))$$

for $t \in [0, 1]$. Note that $\rho_0^2 = \rho_0$, $\rho_1^2 = \rho_1 \circ \kappa$, and $\rho_1^* \lambda_1 = \lambda^{\delta_0}$.

According to [Aur1 Proposition 4], this isotopy yields a family of symplectomorphisms $\Psi_t: (M, \omega) \to (M, \omega)$ such that $\Psi_t(B) = B$ for every $t \in [0, 1]$, $\Psi_0 = \text{Id}$, and $\Psi_t|_{\text{Im}(\rho_0)} = \rho_t \circ \rho_0^{-1}$. At $t = 1$, we have $\Psi_1^* (\lambda_1) = (\rho_1^{-1})^* (\lambda^{\delta_0}) = (\rho_0^{-1})^* \lambda^{\delta_0} = \lambda_0$ on $\text{Im}(\rho_1)$. Thus, $\Psi_1$ is the symplectomorphism we want.

To complete the proof, we shall prove the existence of $\kappa$. As in Remark 4.2, we may regard $(N_{\delta_1}, \omega^{\delta_1})$ as $(P_2 \times S^1 \mathbb{D}(\delta_2), -\omega_1^* (\omega|_B) + d(r^2 \alpha_2))$, where $\omega_2: P_2 \to B$ is the unit circle bundle of $N_1$, with respect to $\| \cdot \|_j$, carrying the connection 1-form $\alpha_1 = \alpha^{\delta_1}|_{P_1}$. Notice that the unit circle bundle $P_3$ depends on the metric $\| \cdot \|_j$, and hence $P_1$ and $P_2$ do not need to coincide as subsets of $N$. A standard argument of Gray’s theorem (see e.g. [Gei Theorem 2.2.2]) shows that there exists a strict contactomorphism between $(P_1, \alpha_1)$ and $(P_2, \alpha_2)$ that covers a symplectomorphism $(B, \omega|_B) \to (B, \omega|_B)$. This leads to a symplectic embedding

$$\kappa': (P_1 \times S^1 \mathbb{D}(\delta_1), -\omega_1^* (\omega|_B) + d(r_1^2 \alpha_1)) \to (P_2 \times S^1 \mathbb{D}(\delta_2), -\omega_2^* (\omega|_B) + d(r_2^2 \alpha_2)).$$

In view of the identification of $N_{\delta_1}$ with $P_2 \times S^1 \mathbb{D}(\delta_2)$, this embedding gives the claimed map $\kappa$. 

**Proof of Proposition 4.3.** It follows from Lemma 4.1 that there is a symplectomorphism $\varphi': (M_1, \omega_1) \to (M_2, \omega_2)$. By construction, the symplectic submanifolds $\varphi'(B_1)$ of $(M_2, \omega_2)$ is homologous to $B_2$. Hence, from Theorem 3.7, they are symplectically isotopic, and the isotopy yields a symplectomorphism, say $\varphi''$, between the pairs $(M_2, \varphi'(B_1))$ and $(M_2, B_2)$ by [Aur1 Proposition 4]. The composition $\varphi'' \circ \varphi': (M_2, \omega_1) \to (M_2, \omega_2)$ is a symplectomorphism satisfying $\varphi'' \circ \varphi'(B_1) = B_2$. Recall that $s_j: M_j \to \mathcal{O}(B_j)$ is a holomorphic section with $s_j^{-1}(0) = B_j$ for $j = 1, 2$ and let

$$\lambda_j := \ddc \log \| s_j \|^2,$$

defined on $M_j \setminus B_j$. It turns out that Lemma 4.3 is applicable to $\lambda_1$ and $(\varphi'' \circ \varphi')^* \lambda_2$ near $B_1$; we obtain a symplectomorphism $\Psi: (M_1, \omega_1) \to (M_1, \omega_1)$ satisfying $\Psi(B_1) = B_1$ and $\Psi^*((\varphi'' \circ \varphi')^* \lambda_2) = \lambda_1$ on a neighborhood of $B_1$. Therefore, $\varphi'' \circ \varphi' \circ \Psi$ meets the requirements of the proposition. 

□
4.2. **Proof of Theorem 1.1** Let $f : X \setminus B \to \mathbb{C}P^1$ be a Lefschetz pencil on a 2$n$-dimensional closed symplectic manifold $X$ such that the closure of a regular fiber is symplectomorphic to $M$. Recall that the global monodromy of $f$ is isotopic to a fibered Dehn twist $\tau_{\partial W}$ along the boundary of $W = M \setminus \nu_M(B)$, where $\nu_M(B)$ is a tubular neighborhood of $B$ in $M$. Moreover, $\tau_{\partial W}$ factorizes into a product of Dehn twists.

**Lemma 4.5.** The fibered Dehn twist $\tau_{\partial W}$ is Hamiltonian isotopic to a product of $k$ Dehn twists along Lagrangian spheres in $M \setminus B$, where

$$k = (-1)^n(\chi(X) - 2\chi(M) + \chi(B)).$$

**Proof.** The given Lefschetz pencil $f$ induces a Lefschetz fibration $X \setminus \nu_X(M) \to \mathbb{C}$ with fibers diffeomorphic to $W = M \setminus \nu_M(B)$, where $\nu_M(B)$ denotes a tubular neighborhood of $M$ in $X$. This fibration decomposes its total space $X \setminus \nu_X(M)$ into $W \times \mathbb{C}$ and $k$ $n$-handles smoothly [Kas]. Hence, we have

$$\chi(X) - \chi(M) = \chi(X \setminus \nu(M)) = \chi(W \times \mathbb{C}) + (-1)^n \chi(M) - \chi(B) + (-1)^nk,$$

which concludes the lemma. \qed

**Proof of Theorem 1.1.** Let $X_j, M_j$ and $B_j$, for $j = 1, 2$, be the manifolds as in Section 4.1. Write $W_j$ for the complement of a tubular neighborhood of $B_j$ in $M_j$ satisfying $(W_1, \omega_1|_{W_1})$ and $(W_2, \omega_2|_{W_2})$ are symplectomorphic as in Proposition 1.3. Equip $W_1$ with a Weinstein structure by the function $-\log ||s_1||^2$ and its gradient vector field with respect to the metric $\omega_1(\cdot, J_1 \cdot)$. Note that although $W_2$ admits a Weinstein structure as well, to prove the theorem, we have no need of it.

A 1-dimensional linear system containing $M_j$ defines a Lefschetz pencil $X_j \setminus B_j \to \mathbb{C}P^1$, which induces a Lefschetz fibration $\pi_j : X_j \setminus M_j \to \mathbb{C}$. As we discussed in Section 2.3, one can truncate the total space of $\pi_j$ ($j = 1, 2$) to obtain a new Lefschetz fibration $\pi_j'$ with regular fibers symplectomorphic to $W_j$. The global monodromy of $\pi_j'$ is a product of Dehn twists along $L_{1,3}, \ldots, L_{j,k_j}$, where $k_1 = 4$ and $k_2 = 6$. Indeed, according to Lemma 4.5 we have

$$k_1 = (-1)^3((1 + 3 + 3 + 1) - 2(1 + 4 + 1) + (1 - 2 + 1)) = 4,$$

$$k_2 = (-1)^3((1 + 2 + 2 + 1) - 2(1 + 4 + 1) + (1 - 2 + 1)) = 6.$$

Theorem 2.7 states that this product is isotopic to a fibered Dehn twist along $\partial W_j$. The complements $W_1$ and $W_2$ are symplectomorphic, and their boundaries $\partial W_1$ and $\partial W_2$ carry contact structures which are strictly contactomorphic from Proposition 1.3. Thus, under the identification of $W_1$ with $W_2$, the two fibered Dehn twists $\tau_{\partial W_1}$ and $\tau_{\partial W_2}$ are symplectically isotopic in $\text{Symp}_c(W_1, \omega_1|_{W_1})$. Therefore, we have

$$[\tau_{L_{1,1}} \circ \cdots \circ \tau_{L_{1,6}}] = [\tau_{\partial W_1}] = [\tau_{\partial W_2}] = [\tau_{L_{2,1}} \circ \cdots \circ \tau_{L_{2,4}}]$$

in $\pi_0(\text{Symp}_c(W_1, \omega_1|_{W_1}))$. This is the desired relation between Dehn twists. \qed
4.3. Application of Theorem 1.1. We will construct closed contact 5-manifolds having arbitrarily finitely many Weinstein fillings up to homotopy. The construction uses open books and Lefschetz fibrations. The reader is referred to [Gel] Section 7.3 and [vK] for open books and relation of them to contact structures and also referred to [GP] for relation of Lefschetz fibrations to Weinstein domains.

Proposition 4.6. Given a positive integer n, there exists a closed contact 5-manifold $(Y_n, \xi_n)$ admitting at least n Weinstein fillings up to homotopy.

Proof. Let $W_1$ and $W_2$ be the Weinstein domains obtained above, and let $L_{1,1}, \ldots, L_{1,6}$ and $L_{2,1}, \ldots, L_{2,4}$ be Lagrangian spheres in $W_1$ and $W_2$, respectively, as in the proof of Theorem 1.1. Since $W_1$ and $W_2$ are symplectomorphic, we identify them and regards $L_{2,1}, \ldots, L_{2,4}$ as Lagrangian spheres in $W_1$. Consider the abstract open book whose page is $W_1$ and whose monodromy is $\tau_{W_1}^{n-1}$. Let $(Y_n, \xi_n)$ denote the contact manifold associated to this open book. As $[\tau_{W_1}] = [\tau_{L_{1,1}}] \circ \cdots \circ [\tau_{L_{1,6}}] = [\tau_{L_{2,1}}] \circ \cdots \circ [\tau_{L_{2,4}}] \in \pi_0(\text{Symp}_c(W_1, \omega_1|W_1))$ by Theorem 1.1 one can factorize $[\tau_{W_1}^{n-1}]$ into a product of Dehn twists in n ways:

$$[\tau_{W_1}^{n-1}] = ([\tau_{L_{1,1}}] \circ \cdots \circ [\tau_{L_{1,6}}])^k \circ ([\tau_{L_{2,1}}] \circ \cdots \circ [\tau_{L_{2,4}}])^{n-1-k}$$

for $k = 0, \ldots, n-1$. Each factorization gives a Lefschetz fibration whose vanishing cycles coincides with Lagrangian spheres appearing in the factorization, and the total space of the fibration, say $E_k$, serves as a Weinstein filling of $(Y_n, \xi_n)$. The Euler characteristic of the filling $E_k$ is given by

$$\chi(E_k) = \chi(W_1) - (6k + 4(n - 1 - k)) = \chi(W_1) - 2(k + 2n - 2).$$

It turns out that $E_k$ are mutually homotopy inequivalent. \hfill \Box

Appendix A. Alternative proof of Theorem 2.7

In this appendix, we give an alternative proof of Theorem 2.7 for projective manifolds. The precise statement to show is the following:

Theorem A.1 (Theorem 2.7 in the projective setting). Let $X \subset \mathbb{CP}^N$ be a closed projective manifold with the Kähler form $\omega$ induced by $\omega_{FS}$ on $\mathbb{CP}^N$. Also let $f: X \setminus B \to \mathbb{CP}^1$ be a Lefschetz pencil determined by hyperplane sections and $W$ the complement of a tubular neighborhood of $B$ in the closure of a regular fiber of $f$. Then, a fibered Dehn twist $\tau_{W}$ along $\partial W$ is symplectically isotopic to a product of Dehn twists along Lagrangian spheres in $W$.

A.1. Lefschetz–Bott fibrations. First we briefly review Lefschetz–Bott fibrations (see [Per] for a thorough treatment).

Definition A.2. A symplectic form $\omega$ on an almost complex manifold $(M, J)$ is said to be normally Kähler near an almost complex submanifold $N \subset M$ if there exists a tubular neighborhood $U$ of $N$ in $M$, foliated by $J$-holomorphic normal discs $\{N_x \subset U\}_{x \in N}$, such that $J$ is compatible with $\omega$ on $U$; on each normal slice $N_x$, $J$ is integrable and $\omega$ is Kähler for $J$. 
Let $E$ be an even dimensional open manifold equipped with a closed 2-form $\omega$.

**Definition A.3.** A smooth map $\pi : (E, \Omega) \to \mathbb{C}$ is called a Lefschetz–Bott fibration if the set of critical points of $\pi$, denoted by $\text{Crit}(\pi)$, is a submanifold of $E$ with finitely many connected components; there exists a compatible almost complex structure $J$ defined in a tubular neighborhood $U$ of $\text{Crit}(\pi)$ in $E$ and a positively oriented complex structure $j$ defined in a tubular neighborhood $V$ of $\text{Crit}_v(\pi)$ in $\mathbb{C}$, which satisfy the following conditions:

1. The map $\pi|_U : U \to V$ is $(J, j)$-holomorphic;
2. $\Omega$ is nondegenerate on $\ker(D_x\pi)$ for any point $x \in E$;
3. $\Omega$ is normally Kähler along $\text{Crit}(\pi)$;
4. At any critical point $x \in \text{Crit}(\pi)$, the normal complex Hessian $D^2_x\pi|_{N_x \otimes^2}$ of $\pi$ is nondegenerate.

Note that a Lefschetz–Bott fibration $\pi : (E, \Omega) \to \mathbb{C}$ with $\dim \text{Crit}(\pi) = 0$ is a Lefschetz fibration.

**Remark A.4.** We may impose the horizontal triviality on a Lefschetz–Bott fibration. If a Lefschetz–Bott fibration satisfies this condition, one can cut off cylindrical parts from the total space to obtain a map with compact fibers as in Remark 2.3.

Let $\pi : (E, \Omega) \to \mathbb{C}$ be a Lefschetz–Bott fibration on a $2n$-dimensional manifold $E$. For simplicity, we assume that each fiber of $\pi$ contains at most one connected component of $\text{Crit}(\pi)$. Fix a critical value $z_1 \in \text{Crit}_v(\pi)$ and write $C$ for the component of $\text{Crit}(\pi)$ corresponding to $z_1$. Although the dimension of $C$ can be between 0 and $2n - 4$ in general, this paper only deals with $C$ of dimension 0 and $2n - 4$. Moreover, we have already discussed Lefschetz fibrations, and it suffices to examine the case $\dim C = 2n - 4$. Fix a point $z_0 \in \mathbb{C} \setminus \text{Crit}_v(\pi)$ and choose a vanishing path $\gamma : [0, 1] \to \mathbb{C} \setminus \text{Crit}_v(\pi)$ for $z_1$ based at $z_0$. As in the case of Lefschetz fibrations, one can associate a coisotropic submanifold $V_\gamma$ in $\pi^{-1}(z_0)$ to $\gamma$, which is diffeomorphic to a circle bundle over $C$. We call $V_\gamma$ the vanishing cycle for $\gamma$. Throughout this paper, $V_\gamma$ may be assumed to be the boundary of a Liouville (or Weinstein) domain with periodic Reeb orbits. Let $\ell$ be a loop based at $z_0$ obtained from $\gamma$ as in Figure 1. The monodromy of $\pi$ along $\ell$ is given by a fibered Dehn twist along $V_\gamma$.

**A.2. Lefschetz–Bott fibrations on holomorphic line bundles.** Let $(M, J_M, \omega_M)$ be a compact Kähler manifold of (complex) dimension $n$ with $\omega_M$ integral. Suppose that $H$ is a smooth and reduced complex hypersurface in $M$ with $[\omega_M] = \text{PD}[H]$. We write $p : \mathcal{O}(-H) \to M$ for the bundle dual of a holomorphic line bundle $\mathcal{O}(H) \to M$ corresponding to $H$. Let $J$ be an integrable almost complex structure on $\mathcal{O}(-H)$ for which $p$ is $(J, J_M)$-holomorphic, and fix a hermitian metric on $\mathcal{O}(-H)$. Choose a holomorphic section $s : M \to \mathcal{O}(H)$ with $s^{-1}(0) = H$. Note that $s$ is transverse to the zero-section since $H$ is reduced. We define the map $\pi : \mathcal{O}(-H) \to \mathbb{C}$ by

$$\pi(x) := \langle s(p(x)), x \rangle,$$
where \( \langle \cdot, \cdot \rangle \) denotes the canonical coupling. We equip \( \mathcal{O}(-H) \) with a canonical symplectic structure \( \Omega = \Omega^\nabla \) determined by a hermitian connection \( \nabla \) with curvature \( 2\pi i\omega_M \) as in Section 4.1.2. By construction, \( J \) is \( \Omega \)-compatible.

**Proposition A.5.** The map \( \pi : (\mathcal{O}(-H), \Omega) \to \mathbb{C} \) is a Lefschetz–Bott fibration.

**Proof.** As \( \pi \) is a \( (J, j) \)-holomorphic map for the standard complex structure \( j \) on \( \mathbb{C} \), the kernel of \( \text{D}\pi \) at each point \( x \) is a \( J \)-complex vector space. Thanks to the \( \Omega \)-compatibility of \( J, \Omega \) is nondegenerate on \( \ker(\text{D}\pi) \). Now all we have to do is examine the behavior of \( \pi \) near the critical point set \( \text{Crit}(\pi) \).

Let us specify \( \text{Crit}(\pi) \). Take an open cover \( \{ U_\alpha \} \) of \( M \) such that \( p|_{p^{-1}(U_\alpha)} \) can be trivialized. For convenience, identifying \( \pi \) of \( \Omega \) of \( \pi \) of \( \pi \) of \( \pi \) projection to the second factor. On \( \pi \) near the critical point set \( \text{Crit}(\pi) \). By construction, \( J \) is given by

\[
\left( \frac{\partial s_\alpha(b)}{\partial b_1}(z), \ldots, \frac{\partial s_\alpha(b)}{\partial b_n}(z), s_\alpha(b) \right),
\]

where \( (b_1, \ldots, b_n) \) are coordinates on \( U_\alpha \). Hence, \( s_\alpha(b) \) must be 0 for \( (b, x) \in \text{Crit}(\pi) \). Furthermore, we obtain \( z = 0 \) by the transversality of the section \( s \). Thus,

\[
\text{Crit}(\pi) = s(M) \cap \{ \text{the image of the zero-section} \} =: H_0
\]

which is a complex submanifold of \( \mathcal{O}(-H) \) biholomorphic to \( H \). The complex Hessian matrix \( \text{D}^2\pi \) at any critical point \( (b, 0) \in H_0 \) is given by

\[
\begin{pmatrix}
\frac{\partial^2 s_\alpha(b)}{\partial b_1 \partial b_1}(b)z & \cdots & \frac{\partial^2 s_\alpha(b)}{\partial b_1 \partial b_n}(b)z & \frac{\partial s_\alpha(b)}{\partial b_1}(b) \\
\vdots & \ddots & \vdots & \vdots \\
\frac{\partial^2 s_\alpha(b)}{\partial b_n \partial b_1}(b)z & \cdots & \frac{\partial^2 s_\alpha(b)}{\partial b_n \partial b_n}(b)z & \frac{\partial s_\alpha(b)}{\partial b_n}(b) \\
\frac{\partial s_\alpha(b)}{\partial b_1}(b) & \cdots & \frac{\partial s_\alpha(b)}{\partial b_n}(b) & 0
\end{pmatrix}
= \begin{pmatrix}
0 & \cdots & 0 & \frac{\partial s_\alpha(b)}{\partial b_1}(b) \\
\vdots & \ddots & \vdots & \vdots \\
0 & \cdots & 0 & \frac{\partial s_\alpha(b)}{\partial b_n}(b) \\
\frac{\partial s_\alpha(b)}{\partial b_1}(b) & \cdots & \frac{\partial s_\alpha(b)}{\partial b_n}(b) & 0
\end{pmatrix},
\]

This shows that the matrix has rank 2 and \( \ker(\text{D}^2\pi) = \text{TH}_0^{\otimes 2} \). Consider the function \( ||s||^2 : M \to \mathbb{R}, x \mapsto ||s(x)||^2 \) and its gradient vector field \( \nabla ||s||^2 \) with respect to the Riemannian metric \( \omega(\cdot, J_M \cdot) \). We observe that \( H \) is a Morse–Bott submanifold of this function, and the unstable manifold of \( \nabla ||s||^2 \) at \( x \in H \) is a holomorphic disc near \( H \) since \( \omega_M(\nabla ||s||^2, J_M \nabla ||s||^2) > 0 \) outside \( H \) and a finite energy punctured holomorphic disc extends to a holomorphic disc. Hence, one can take a tubular neighborhood \( U \) of \( H \) in \( M \) whose normal slices are holomorphic. Restricting \( \mathcal{O}(-H) \) to \( U \), we obtain a neighborhood \( \mathcal{U} \) of \( H_0 \) in \( \mathcal{O}(-H) \) satisfying that the conditions for \( \Omega \) being normally Kähler. Indeed, \( H_0 \) is a complex submanifold of \( (\mathcal{U}, J) \); by definition, \( \Omega \) is nondegenerate and Kähler on \( \mathcal{U} \); the fact that \( \ker(\text{D}^2\pi) = \text{TH}_0^{\otimes 2} \) proves that the normal Hessian of \( \pi \) is nondegenerate at any point of \( H_0 \). This finishes the proof. \( \square \)
Remark A.6. A similar statement to the above theorem holds for closed integral symplectic manifolds generally. In fact, the author shows that a prequantization line bundle over a closed integral symplectic manifold admits a Lefschetz–Bott fibration; see [Oba2].

A.3. Proof of Theorem A.1. We are now in a position to prove Theorem A.1. The idea of the proof is connect a Lefschetz fibration obtained from a Lefschetz pencil to a Lefschetz–Bott fibration on certain space via one-parameter family of fibrations. In the following proof, given a projective manifold $M \subset \mathbb{CP}^N$, we denote the restriction of the hyperplane line bundle $O_{\mathbb{CP}^N}(1) \to \mathbb{CP}^N$ to $M$ by $O_M(1) \to M$.

Proof of Theorem A.1. Since a projective manifold is algebraic by Chow’s theorem, we may assume that the projective manifold $X$ is the zero set of homogeneous polynomials $f_1, \ldots, f_k$. Take generic hyperplane sections $s_0$ and $s_1$ that provides the given Lefschetz pencil $f: X \setminus B \to \mathbb{CP}^1$, i.e.,

$$f: X \setminus B \to \mathbb{CP}^1 = \mathbb{C} \cup \{\infty\}, \quad f(z) = \frac{s_1(z)}{s_0(z)},$$

where $B = s_0^{-1}(0) \cap s_1^{-1}(0) \cap X$. Letting $M := s_0^{-1}(0) \cap X$, the complement $X \setminus M$ can be identified with the affine manifold $Y_t$ given by

$$Y_t = \{x \in \mathbb{C}^{N+1} \mid s_0(x) = t, f_1(x) = \cdots = f_k(x) = 0\}$$

for any $t \in \mathbb{C} \setminus \{0\}$. Indeed, the projection $\mathbb{C}^{N+1} \setminus \{0\} \to \mathbb{CP}^N$ yields a diffeomorphism between $Y_t$ and $X \setminus M$. By definition of $\omega_{\text{FS}}$, $(Y_t, \rho_{\text{FS}}|Y_t)$ and $(X \setminus M, \omega_{\text{FS}}|X \setminus M)$ are symplectomorphic with $\rho_{\text{FS}} = -\frac{1}{4\pi}dd^c \log \|x\|^2$. We find that the map $\pi_t: (Y_t, \rho_{\text{FS}}|Y_t) \to \mathbb{C} = \mathbb{CP}^1 \setminus \{\infty\}, x \mapsto s_1(x)/t$ determined by $f$ is a Lefschetz fibration.

Next, we construct a Lefschetz–Bott fibration on the holomorphic line bundle $O_M(-1) := O_M(1)^*$ by employing Proposition A.5 for $p: O_M(-1) \to M$. Observe that the bundle $O_M(-1)$ is described as

$$O_M(-1) = \{(z, x) \in \mathbb{CP}^N \times \mathbb{C}^{N+1} \mid s_0(z) = f_1(z) = \cdots = f_k(z) = 0, x \in \ell_z\},$$

where $\ell_z$ denotes the complex line in $\mathbb{C}^{N+1}$ spanned by $z$. We may think of $s_1$ as a holomorphic section of $O_X(1) \to X$ that restricts to one of $O_M(1) \to M$ with $(s_1|M)^{-1}(0) = B$. Hence, Proposition A.5 concludes that the map $\pi_0: (O_M(-1), \Omega) \to \mathbb{C}$ defined by

$$\pi_0(z, x) := (s_1(z), (z, x)) = s_1(x)$$

is a Lefschetz–Bott fibration. Here $\Omega$ is a canonical symplectic form on the line bundle $O_M(-1)$ as in Section 4.1.2.

Now we relate this fibration $\pi_0$ to Lefschetz fibrations $\pi_t$ ($t \neq 0$). To do this, we shall first see $O_M(-1)$ from a different viewpoint: Let $\tau: O_M(-1) \to \mathbb{C}^{N+1}$ be the restriction of the projection $\mathbb{CP}^N \times \mathbb{C}^{N+1} \to \mathbb{C}^{N+1}$ and write $Y_0$ for the image of $O_M(-1)$ under $\tau$. As

$$Y_0 = \{x \in \mathbb{C}^{N+1} \mid s_0(x) = f_1(x) = \cdots = f_k(x) = 0\},$$
it has possibly a unique singular point at the origin. Hence, the bundle $O_M(-1)$ can be regarded as a resolution of this singularity. Next, let $\tilde{Y}_t$ denote the strict transform of $Y_t$ under blowing up $C^{N+1}$ at the origin. Set

$$\tilde{Y} := \bigcup_{t \in [0,1]} \{t\} \times \tilde{Y}_t \subset [0,1] \times O_{CP^N}(-1).$$

By definition, $\tilde{Y}_0$ coincides with $O_M(-1)$, and the others $\tilde{Y}_t$ are isomorphic to $Y_t$ since they do not contain $0 \in C^{N+1}$. Define the map $\tilde{\Pi}: \tilde{Y} \to [0,1] \times C$ by

$$\tilde{\Pi}(t,z,x) := (t, (s_1(z), (z,x))) = (t,s_1(x)),$$

which agrees with $\pi_t$ on every slice $\{t\} \times \tilde{Y}_t$ up to a constant multiple.

Take a Kähler form $\omega_0$ on the line bundle $O_{CP^N}(-1)$ such that it is a canonical symplectic form associated to a hermitian connection near the zero-section, and far from there it equals the pull-back of $\rho_{FS}$ on $C^{N+1} \setminus \{0\}$ by the blowing-up map. We define a closed 2-form on $[0,1] \times O_{CP^N}(-1)$ by $pr_2^*\omega_0$, where $pr_2: [0,1] \times O_{CP^N}(-1) \to O_{CP^N}(-1)$ is the projection, and put

$$\Omega_0 = pr_2^*\omega_0|_{\tilde{Y}}.$$

Then, each regular fiber of $\tilde{\Pi}$ is a symplectic manifold with respect to $\Omega_0$, which shows that

$$\tilde{\Pi}|_{\tilde{\Pi}^{-1}(([0,1] \times C) \setminus \text{Critv}(\tilde{\Pi}))}: \tilde{\Pi}^{-1}(([0,1] \times C \setminus \text{Critv}(\tilde{\Pi})) \to ([0,1] \times C \setminus \text{Critv}(\tilde{\Pi}))$$

is a symplectic fibration. The symplectic connection defines parallel transport along a path in $([0,1] \times C \setminus \text{Critv}(\tilde{\Pi}))$, especially the monodromy along a loop in the same space. Perturbing the 2-form $\Omega_0$ as in [Kea, Lemma 9.3], we may assume that each fibration $\tilde{\Pi}|_{\tilde{Y}_t}$ is a Lefschetz–Bott fibration with horizontal triviality. As a result, in view of Remark 2.3 and A.4, removing cylindrical parts from the fibers of $\tilde{Y}$, one can obtain a subspace $\mathcal{Y} \subset \tilde{Y}$ such that the restriction $\Pi := \Pi|_{\mathcal{Y}}: (\mathcal{Y}, \Omega_0|_{\mathcal{Y}}) \to [0,1] \times C$ has compact fibers symplectomorphic to the complement $(W, \omega_{FS}|_{W})$ of a tubular neighborhood of $B$ in $M$. The critical value set $\text{Critv}(\Pi)$ of $\Pi$ consists of the family of finite sets $\text{Critv}(\Pi|_{\tilde{\Pi}^{-1}(\{t\} \times C)}) = \text{Critv}(\Pi)|_{\tilde{Y}_t}$ with $t \in [0,1]$. Thus, $([0,1] \times C) \setminus \text{Critv}(\Pi)$ is path-connected.

Choose $x_0 \in C$ with $|x_0|$ sufficiently large and put $b_0 = (0,x_0) \in [0,1] \times C$. Identify $\Pi|_{\Pi^{-1}(b_0)}$ with $(W, \omega_{FS}|_{W})$. Take a loop $\ell_0: [0,1] \to (\{0\} \times C \setminus \text{Critv}(\Pi))$ based at $b_0$ that encloses $((\{0\} \times C) \setminus \text{Critv}(\Pi))$. As $\Pi|_{\Pi^{-1}(\{0\} \times C)}$ is a Lefschetz–Bott fibration, the monodromy along $\ell_0$ is a fibered Dehn twist along $\partial W$. Also, take a loop $\ell_1: [0,1] \to (\{1\} \times C \setminus \text{Critv}(\Pi))$ based at $(1,x_0)$ surrounding all the points of $(\{1\} \times C) \setminus \text{Critv}(\Pi)$. Since $\Pi|_{\Pi^{-1}(\{1\} \times C)}$ is a Lefschetz fibration, the monodromy of $\Pi$ along $\ell_1$ is a product of Dehn twists. We connect $\ell_1$ with the path $\gamma: [0,1] \to [0,1] \times C$, $t \mapsto (t,x_0)$ and denotes the resulting loop $\gamma \cdot \ell_1 \cdot \gamma^{-1}$ by $\ell'_1$ (see Figure 3). It turns out that the monodromy along $\ell'_1$ is isotopic to a product of Dehn twists, and $\ell'_1$ is homotopic to $\ell_0$ in $([0,1] \times C) \setminus \text{Critv}(\Pi)$. Therefore, the monodromies along $\ell_0$ and $\ell'_1$ are symplectically isotopic, which implies the desired mapping class group relation. □
Figure 3. Loops $\ell_0, \ell_1$ and path $\gamma$ in $I \times \mathbb{C}$. The red segments indicate the critical value set of $\Pi$.

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