Stability of a Composite Wave of Two Separate Strong Viscous Shock Waves for 1-D Isentropic Navier-Stokes System

Lin CHANG
School of Mathematics and Physics, Handan University, Handan 056006, China
(E-mail: changlin23@buaa.edu.cn)

Abstract In this paper, the large time behavior of solutions of 1-D isentropic Navier-Stokes system is investigated. It is shown that a composite wave consisting of two viscous shock waves is stable for the Cauchy problem provided that the two waves are initially far away from each other. Moreover the strengths of two waves could be arbitrarily large.

Keywords combination of two shock waves; energy estimate; asymptotic stability

2000 MR Subject Classification 35Q30; 76N10

1 Introduction

We consider the following one-dimensional isentropic Navier-Stokes system for polytropic gas in the Lagrangian coordinate,

\[
\begin{cases}
v_t - u_x = 0, & t > 0, \quad x \in \mathbb{R}, \\
u_t + p_x = \left(\mu(v) \frac{u_x}{v}\right)_x, & t > 0, \quad x \in \mathbb{R},
\end{cases}
\]

with the initial data:

\[(v, u)(x, 0) = (v_0, u_0)(x) \rightarrow (v_\pm, u_\pm), \quad \text{as} \quad x \rightarrow \pm \infty.\]

Here \(v(x, t) = \frac{1}{\rho(x, t)}\) is the specific volume, \(u(x, t)\) the fluid velocity, \(p = av^{-\gamma}\) the pressure with constant \(a > 0, \gamma > 1\) the adiabatic constant, and \(\mu(v) = \mu_0 v^{-\alpha}\) the viscosity coefficient with \(\alpha \geq 0\). Without loss of generality, we assume \(\mu_0 = 1\) in the rest of this article. When the viscosity \(\mu(v) \equiv 0\), the system (1.1) becomes the Euler system

\[
\begin{cases}
v_t - u_x = 0, \\
u_t + p_x = 0.
\end{cases}
\]

It is known that the equation (1.3) has rich wave phenomena, such as shock and rarefaction. The shock is mollified as the so-called viscous shock wave when \(\mu(v) > 0\). The time asymptotic stability of single wave pattern has been extensively studied in a large amount of literature since the pioneer works of [2, 14], see [1, 6–10, 12, 15, 19] and the reference therein, see other interesting works on the composite wave [4, 5]. However, most of above works require the strength of shock wave is small, that is, the shock is weak. The stability of large amplitude shock (strong shock) is more interesting and challenging in both mathematics and physics.

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Matsumura-Nishihara [14] showed that the viscous shock wave is stable if $|v_+ - v_-| < C(\gamma - 1)^{-1}$, that is, when $\gamma$ tends to 1, the strength of shock wave could be large. The condition is later relaxed to the condition that $|v_+ - v_-| < C(\gamma - 1)^{-2}$ in [7]. The restriction on the strength of shock was removed in [17] by a wonderful weighted energy method as $\alpha > \frac{2\gamma - 1}{\gamma}$. Vasseur-Yao [20] removed the condition $\alpha > \frac{2\gamma - 1}{\gamma}$ by introducing an elegant variable transformation. Moreover, He-Huang [3] extended the result of [20] to general pressure $p(v)$ and viscosity $\mu(v)$, where $\mu(v)$ could be any positive smooth function.

It is important to study the stability of composite wave consisting of at least two waves. From [4], it is not difficult to show the asymptotic stability of a composite wave consisting of 1-viscous shock wave and 2-viscous shock wave, provided that the strengths of the two shocks are given in (2.3) and (2.4). The intermediate state $(v_m, u_m)$ is determined by the RH condition, i.e.,

$$
\begin{cases}
-s_2(v_+ - v_m) - (u_+ - u_m) = 0, \\
-s_2(u_+ - u_m) + (p(v_+) - p(v_m)) = 0,
\end{cases}
$$

and

$$
\begin{cases}
-s_1(v_m - v_-) - (u_m - u_-) = 0, \\
-s_1(u_m - u_-) + (p(v_m) - p(v_-)) = 0.
\end{cases}
$$

We denote the composite wave consisting of the two viscous shock waves $(V_i, U_i)$, $i = 1, 2$ by $(V, U)(x, t) = (V_1 + V_2 - v_m, U_1 + U_2 - u_m)$.

We outline the strategy as follows. In order to remove the condition “small with same order”, motivated by [20] and [3], we introduce a new variable $h$, and formulate a new equation (3.3)$_2$ for $h$ in which the viscous term is moved to the mass equation (3.3)$_1$ such that the two nonlinear terms $p_x$ and $(\frac{v_{1-}}{u_{1-}})_x$ are decoupled, so the interaction between nonlinear terms is weaken, and the low order estimates are obtained. We then turn to the original system (1.1) to derive the higher order energy estimates, and finally complete the a priori estimates. On the other hand, since the strengths of 1-shock wave and 2-shock wave are arbitrarily large, the interaction between the two shocks is strong. We have to assume that 1-shock wave is initially far away from 2-shock wave so that the interaction is weak.

The rest of the paper will be arranged as follows. In section 2, the composite wave is formulated and the main result is stated. In section 3, the problem is reformulated by the anti-derivatives of the perturbations around the composite wave. In section 4, the a priori estimates are established. In section 5, the main theorem is proved.

**Notation.** The functional $\| \cdot \|_{L^p(\Omega)}$ is defined by $\| f \|_{L^p(\Omega)} = \left( \int_{\Omega} |f|^p(\xi)d\xi \right)^{\frac{1}{p}}$. The symbol $\Omega$ is often omitted when $\Omega = (-\infty, \infty)$. We denote for simplicity

$$
\| f \| = \left( \int_{-\infty}^{\infty} f^2(\xi)d\xi \right)^{\frac{1}{2}}
$$

as $p = 2$. In addition, $H^m$ denotes the $m$-th order Sobolev space of functions defined by

$$
\| f \|_m = \left( \sum_{k=0}^{m} \| \partial_x^k f \|^{2} \right)^{\frac{1}{2}}.
$$
2 Preliminaries and Main Theorem

2.1 Viscous Shock Profile

Before stating the main results, we recall the Riemann problem for the Euler equation (1.3) with the Riemann initial data

\[ (v, u)(x, 0) = \begin{cases} (v_-, u_-), & x < 0, \\ (v_+, u_+), & x > 0. \end{cases} \tag{2.1} \]

It is known that the system (1.3) has two eigenvalues: \( \lambda_1 = -\sqrt{-p'(v)} < 0, \) \( \lambda_2 = -\lambda_1 > 0. \)

By the standard arguments (e.g. [18]), we define the shock curve \( S_1 \) (resp \( S_2 \))

\[ \begin{align*}
  u &= u_- - \sqrt{(v_- - v)(v^- - v^-)}, & u < u_-, \ v < v_-, \ S_1, \\
  u &= u_- - \sqrt{(v_- - v)(v^- - v^-)}, & u < u_-, \ v > v_-, \ S_2,
\end{align*} \]

and \( SS(v_-, u_-) = \{(v, u) | u \leq u_; S_1(u) < v < S_2(u)\}. \)

In this paper, we assume that \( (v_+, u_+) \in SS(v_-, u_-). \) Thus the Riemann solution of (1.3), (2.1) consists of two shock waves (and three constant states), that is, there exists an intermediate state \((v_m, u_m)\), such that \((v_m, u_m) \in S_1(v_-, u_-)\) with the shock speed \( s_1 < 0 \), and \((v_+, u_+) \in S_2(v_m, u_m)\) with the shock speed \( s_2 > 0 \). Here the shock speeds \( s_1 \) and \( s_2 \) are constants determined by the RH condition and satisfy entropy conditions

\[ \lambda_1(v_-) > s_1 > \lambda_1(v_m), \lambda_2(v_m) > s_2 > \lambda_2(v_+). \tag{2.2} \]

In what follows, we define \((\chi_1, \chi_2)\) below

\[ \chi_1 := v_- - v_m, \quad \chi_2 := v_+ - v_m. \]

We see that the 1-shock wave is a traveling wave solution of (1.1) with the formula \((V_1, U_1)(x - s_1 t)\), satisfying

\[ \begin{align*}
  -s_1 V_1' - U_1' &= 0, \\
  -s_1 U_1' + p(V_1)' &= \left( \frac{U_1'}{V_1^{\alpha+1}} \right)', \\
  (V_1, U_1)(+\infty) &= (v_m, u_m), \\
  (V_1, U_1)(-\infty) &= (v_-, u_-),
\end{align*} \tag{2.3} \]

where \( ' = d/d\xi_1, \ \xi_1 = x - s_1 t. \) Similarly, the 2-viscous shock wave \((V_2, U_2)(x - s_2 t)\) satisfies

\[ \begin{align*}
  -s_2 V_2' - U_2' &= 0, \\
  -s_2 U_2' + p(V_2)' &= \left( \frac{U_2'}{V_2^{\gamma+1}} \right)', \\
  (V_2, U_2)(+\infty) &= (v_+, u_+), \\
  (V_2, U_2)(-\infty) &= (v_m, u_m),
\end{align*} \tag{2.4} \]

where \( ' = d/d\xi_2, \ \xi_2 = x - s_2 t. \)

**Lemma 2.1** [7]. There are positive constants \( C \) and \( c_{1,2} \), such that

\[ (U_i)_x \leq 0, \quad i = 1, 2, \]

\[ |V_1 - v_m| \leq C\chi_1 e^{-c_1|x-s_1 t|}, \quad x > s_1 t, \quad t \geq 0, \]

\[ |V_2 - v_m| \leq C\chi_2 e^{-c_2|x-s_2 t|}, \quad x < s_2 t, \quad t \geq 0. \]
2.2 Location of the Shift $\beta_1$ and $\beta_2$

As mentioned before, we assume that the 2-viscous shock wave is initially far away from the 1-viscous shock profile, that is, the shock profile is $(V_2, U_2)(x - \beta)$ with some constant $\beta > 0$ as $t = 0$. The two shocks formulate a composite wave by $V(x) = V_1(x) + V_2(x - \beta) - v_m, U(x) = U_1(x) + U_2(x - \beta) - u_m$. We consider the situation where the initial data $(v_0, u_0)(x)$ is given in a neighborhood of $(V, U)(x)$. The solution is expected to tend to the composite wave

$$
\begin{align*}
V(x, t; \beta_1, \beta_2; \beta) &= V_1(x - s_1 t + \beta_1) + V_2(x - s_2 t - \beta + \beta_2) - v_m, \\
U(x, t; \beta_1, \beta_2; \beta) &= U_1(x - s_1 t + \beta_1) + U_2(x - s_2 t - \beta + \beta_2) - u_m,
\end{align*}
$$

(2.5)

where the shifts $\beta_1$ and $\beta_2$ are supposed to satisfy

$$
0 = \int_{-\infty}^{\infty} \left( v_0(x) - V(x, 0; \beta_1, \beta_2; \beta) \right) dx := \left( \begin{array}{c}
I_1(\beta_1, \beta_2; \beta) \\
I_2(\beta_1, \beta_2; \beta)
\end{array} \right).
$$

(2.6)

We shall find unique $\beta_1$ and $\beta_2$ such that $I_i(\beta_1, \beta_2; \beta) = 0, i = 1, 2$. Note that

$$
\begin{align*}
I_1(\beta_1, \beta_2; \beta) &= \int_{-\infty}^{\infty} v_0(x) - V(x, 0; \beta_1, \beta_2; \beta) dx \\
&= I_{01} + \int_{-\infty}^{\infty} V_1(x) - V_1(x + \beta_1) dx \\
&+ \int_{-\infty}^{\infty} V_2(x - \beta) - V_2(x - \beta + \beta_2) dx \\
&= I_{01} - \beta_1 (v_m - v_-) - \beta_2 (v_+ - v_m),
\end{align*}
$$

(2.7)

where

$$
I_{01} = \int_{-\infty}^{\infty} v_0(x) - V(x, 0; 0, 0; \beta) dx.
$$

Similarly one can get

$$
I_2(\beta_1, \beta_2; \beta) = I_{02} - \beta_1 (u_m - u_-) - \beta_2 (u_+ - u_m),
$$

(2.8)

where

$$
I_{02} = \int_{-\infty}^{\infty} u_0(x) - U(x, 0; 0, 0; \beta) dx.
$$

(2.9)

Utilizing (2.6)–(2.9), R-H condition (1.4)–(1.5), we have

$$
\begin{pmatrix}
I_{01} \\
I_{02}
\end{pmatrix} = -\beta_1 \begin{pmatrix} v_- - v_m \\ u_- - u_m \end{pmatrix} - \beta_2 \begin{pmatrix} v_m - v_+ \\ u_m - u_+ \end{pmatrix}.
$$

Thus, one gets

$$
\begin{align*}
\beta_1 &= \frac{I_{01}s_2 + I_{02}}{\chi_1(s_1 - s_2)}, \\
\beta_2 &= \frac{I_{01}s_1 + I_{02}}{\chi_2(s_1 - s_2)}.
\end{align*}
$$

(2.10)
2.3 Main Theorem

To state the main theorem, we assume
\[ v_0(x) - V(x, 0; 0, 0; \beta) \in H^1 \cap L^1, \quad u_0(x) - U(x, 0; 0, 0; \beta) \in H^1 \cap L^1. \]  
(2.11)

Then we can define
\[ (\phi_0, \psi_0)(x) = \int_{-\infty}^{x} [v_0(y) - V(y, 0; \beta_1, \beta_2; \beta), u_0(y) - U(y, 0; \beta_1, \beta_2; \beta)] dy. \]  
(2.12)

In view of \( I_i(\beta_1, \beta_2; \beta) = 0, i = 1, 2 \), we further assume that
\[ (\phi_0, \psi_0) \in L^2. \]  
(2.13)

We abbreviate \((V(x,t;\beta_1,\beta_2;\beta), U(x,t;\beta_1,\beta_2;\beta))\) as \((V,U)\) in the rest part of this paper. We are ready to state the main result.

**Theorem 2.2.** Suppose (2.11)–(2.13) hold and \((v_+, u_+) \in SS(v_-, u_-)\). There exists a positive constant \(\delta_0\), such that if
\[ \|\phi_0\|_2 + \|\psi_0\|_2 + \beta^{-1} \leq \delta_0, \]
then the Cauchy problem (1.1), (1.2) has a unique global solution \((v, u)(x,t)\) satisfying
\[ (v - V, u - U) \in C^0([0, +\infty); H^2), v - V \in L^2(0, +\infty; H^1), \]
\[ u - U \in L^2(0, +\infty; H^2). \]  
(2.14)

Furthermore,
\[ \sup_{x \in \mathbb{R}} |v - V| \to 0, \quad \sup_{x \in \mathbb{R}} |u - U| \to 0, \quad \text{as } t \to +\infty. \]  
(2.15)

3 Reformulation of the Original Problem

Set
\[ \phi(x, t) := \int_{-\infty}^{x} (v - V)(y, t) dy, \]
\[ \psi(x, t) := \int_{-\infty}^{x} (u - U)(y, t) dy, \]  
(3.1)

which means that we look for the solution \((v, u)(x,t)\) in the form
\[ v(x, t) = \phi_x(x, t) + V(x, t; \beta_1, \beta_2; \beta), \]
\[ u(x, t) = \psi_x(x, t) + U(x, t; \beta_1, \beta_2; \beta). \]  
(3.2)

From (2.3)–(2.5), we know the shock profile \((V,U)\) satisfies
\[ \begin{cases} 
V_t - U_x = 0, \\
U_t + p(V)_x - \left( \frac{U_x}{V^{\alpha+1}} \right)_x = W_x, \\
(V, U)(\pm \infty, t; \beta_1, \beta_2; \beta) = (v_\pm, u_\pm), 
\end{cases} \]  
(3.3)

where
\[ W = \frac{U_{2x}}{V^{\alpha+1}_2} + \frac{U_{1x}}{V^{\alpha+1}_1} - \frac{U_x}{V^{\alpha+1}} + p(V) + p(v_m) - p(V_1) - p(V_2). \]
Motivated by [13], substitute (3.2) into (1.1) and integrate the resulting system with respect to $x$, we have
\[
\begin{aligned}
\phi_t - \psi_x &= 0, \\
\psi_t - f(V, U_x)\phi_x - \frac{\psi_{xx}}{V^{\alpha+1}} &= F - W,
\end{aligned}
\] (3.4)
with the initial condition:
\[
(\phi_0, \psi_0)(x) \in \mathbb{H}^2, \quad x \in \mathbb{R},
\] (3.5)
where
\[
f(V, U_x) = -p'(V) - (\alpha + 1) \frac{U_x}{V^{\alpha+1}} > 0,
\] (3.6)
\[
F = \frac{u_x}{\psi^{\alpha+1}} - \frac{U_x}{V^{\alpha+1}} + (\alpha + 1) \frac{U_x \phi_x}{V^{\alpha+2}} - [p(v) - p(V) - p'(V)\phi_x].
\] (3.7)
We will seek the solution in the functional space $X_\delta(0, T)$ for any $0 \leq T < +\infty$,
\[
X_\delta(0, T) := \left\{(\phi, \psi) \in C([0, T]; \mathbb{H}^2)|\phi_x \in L^2(0, T; \mathbb{H}^1), \psi_x \in L^2(0, T; \mathbb{H}^2)\right\},
\]
where $\delta \ll 1$ is small.

**Proposition 3.1.** *(A priori estimate)* Suppose that $(\phi, \psi) \in X_\delta(0, T)$ is the solution of (3.4), (3.5) for some time $T > 0$. There exists a positive constant $\delta_0$ independent of $T$, such that
\[
\sup_{0 \leq t \leq T} \|(\phi, \psi)(t)\|_2 \leq \delta \leq \delta_0,
\]
for $t \in [0, T]$, then
\[
\|(\phi, \psi)(t)\|_2^2 + \int_0^t (\|\phi_x(t)\|^2 + \|\psi_x(t)\|^2)dt \leq C_0(\|(\phi_0, \psi_0)\|_2^2 + e^{-C_-\beta}),
\]
where $C_0 > 1$ and $C_-$ are two positive constants independent of $T$.

As long as Proposition 3.1 is proved, the local solution $(\phi, \psi)$ can be extend to $T = +\infty$. We have the following Lemma.

**Lemma 3.2.** If $(\phi_0, \psi_0) \in \mathbb{H}^2$, there exists a positive constant $\delta_1 = \frac{\delta_0}{C_0}$, such that if
\[
\|(\phi_0, \psi_0)\|_2^2 + e^{-C_-\beta} \leq \delta_1^2,
\]
then the Cauchy problem (3.4), (3.5) has a unique global solution $(\phi, \psi) \in X_{\delta_0}(0, \infty)$ satisfying
\[
\sup_{t \geq 0} (\|(\phi, \psi)(t)\|_2^2 + \int_0^t (\|\phi_x(t)\|^2 + \|\psi_x(t)\|^2)dt \leq C_0(\|(\phi_0, \psi_0)\|_2^2 + e^{-C_-\beta}).
\]

### 4 A Priori Estimate

Throughout this section, we assume that the problem (3.4), (3.5) has a solution $(\phi, \psi) \in X_\delta(0, T)$, for some $T > 0$,
\[
\sup_{0 \leq t \leq T} \|(\phi, \psi)(t)\|_2 \leq \delta.
\] (4.1)
It follows from the Sobolev inequality that $\frac{1}{2}v_m \leq v \leq \frac{3}{2} \max\{v_-, v_+\}$, and
\[
\sup_{0 \leq t \leq T} \left\{\|(\phi, \psi)(t)\|_{L^\infty} + \|(\phi_x, \psi_x)(t)\|_{L^\infty}\right\} \leq \delta.
\]
4.1 Low Order Estimate.

In order to remove the condition “small with same order”, we introduce a new perturbation \((\phi, \Psi)\) instead of \((\phi, \psi)\), where \(\Psi\) will be defined below. Motivated by [20] and [3], we introduce a new effective velocity \(h = u - v^{-(\alpha+1)}v_x\). Setting \(h_0(x) := h(x, 0)\), the equations (1.1), (1.2) become

\[
\begin{align*}
  v_t - h_x &= \left( \frac{v_x}{v^{\alpha+1}} \right)_x, \\
  h_t + p_x &= 0,
\end{align*}
\]

and

\[(v_0, h_0)(x) = (v_0, u_0 - v_0^{-(\alpha+1)}v_{0x})(x) \rightarrow (v_\pm, u_\pm), \quad \text{as} \quad x \to \pm \infty.\]

Let \(H = U - V^{-(\alpha+1)}V_x\). Then (3.3) is equivalent to

\[
\begin{align*}
  V_t - H_x &= \left( \frac{V_x}{V^{\alpha+1}} \right)_x, \\
  H_t + p(V)_x &= W_x, \\
  (V, H)(\pm \infty, t) &= (v_\pm, u_\pm).
\end{align*}
\]

We define

\[
\int_{-\infty}^{x} (h - H) dx := \Psi. \tag{4.4}
\]

Substituting (4.3) from (4.2) and integrating the resulting system with respect to \(x\), we have from (4.4), (3.1) that

\[
\begin{align*}
  \phi_t - \Psi_x - \phi_{xx} \frac{V_x}{V^{\alpha+1}} + (\alpha + 1) V_x \phi_x \frac{V_x}{V^{\alpha+2}} &= G, \\
  \Psi_t + p'(V)\phi_x = -p(v|V) - W,
\end{align*}
\]

where

\[G = \frac{v_x}{v^{\alpha+1}} - \frac{V_x}{V^{\alpha+1}} + \frac{\phi_{xx}}{V^{\alpha+1}} + (\alpha + 1) \frac{V_x \phi_x}{V^{\alpha+2}},\]

\[p(v|V) = (p(v) - p(V)) - p'(V)\phi_x,\]

with the initial data

\[\phi(x, 0) \in H^2, \quad \Psi(x, 0) \in H^1.\]

**Lemma 4.1.** Under the assumption of (4.1), it holds that

\[
\begin{align*}
  |p(v|V)| &\leq C\phi_x^2, \\
  |p(v|V)_x| &\leq C(|\phi_{xx}\phi_x| + |V_x|\phi_x^2), \\
  |G| &\leq C(|\phi_{xx}\phi_x| + |V_x|\phi_x^2), \\
  |F| &\leq C(\phi_x^2 + |\phi_x \psi_{xx}|), \\
  |F_x| &\leq C(\phi_x^2 + |\phi_x \phi_{xx}| + |\psi_{xx}\phi_x| + |\psi_{xxx}\phi_x| + |\phi_x \psi_{xx}|). \tag{4.7}
\end{align*}
\]

and

\[
\begin{align*}
  \|\Psi_0\|_2^2 \leq \|\psi_0\|_2^2 + C\|\phi_0\|_2^2, \\
  \|\psi\|_2^2 \leq \|\Psi\|_2^2 + C\|\phi\|_2^2, \\
  \|\psi_x\|_2^2 \leq \|\Psi_x\|_2^2 + C\|\phi_x\|_2^2. \tag{4.8}
\end{align*}
\]

*Here C is a constant depends only on \(v_\pm\) and \(u_\pm\).*
Proof. Note that

\[
\Psi(x, t) = \int_{-\infty}^{x} [(u - U)(y, t)]dy - \int_{-\infty}^{x} (v^{\alpha} v_{y} - v^{\alpha} V_{y})(y, t)dy
\]

(4.9)

\[
= \psi(x, t) + q(x, t) \leq \psi(x, t) + C|\phi_{x}(x, t)|, \quad \psi(x, t) = \Psi(x, t) - q(x, t) \leq \Psi(x, t) + C|\phi_{x}(x, t)|.
\]

one have (4.8) from (4.9) immediately. The estimates (4.6) and (4.7) can be found in [3] and [13] respectively. Thus the proof is completed.

It is worth to point out that the initial data \((v_0, h_0)(x)\) should satisfy the following equation

\[
0 = \int_{-\infty}^{\infty} \left( v_0(x) - V(x, 0; \beta_1, \beta_2; \beta) - h_0(x) - H(x, 0; \beta_1, \beta_2; \beta) \right) dx.
\]

(4.10)

Here \(H(x, 0; \beta_1, \beta_2; \beta) = U(x, 0; \beta_1, \beta_2; \beta) - [V(x, 0; \beta_1, \beta_2; \beta)]^{-(\alpha + 1)}[V(x, 0; \beta_1, \beta_2; \beta)]_x\). By directly calculate, we know (4.10) is equivalent to \(I_i(\beta_1, \beta_2; \beta) = 0, \; i = 1, 2\).

Lemma 4.2. Under the same conditions of Proposition 3.1, it holds that

\[
||W||_2 \leq C e^{-c_{-} t} e^{-c' t},
\]

(4.11)

where \(C, c_{-}, c'\) are constants independent of \(t\).

Proof.

\[
|W| = \left| \frac{U_{1x}}{V_{1}^{\alpha+1}} + \frac{U_{2x}}{V_{2}^{\alpha+1}} - \frac{U_{x}}{V_{x}^{\alpha+1}} + p(V) + p(v_{m}) - p(V_{1}) - p(V_{2}) \right|
\]

\[
= \left| \left( \frac{U_{1x}}{V_{1}^{\alpha+1}} - \frac{U_{1x}}{V_{1}^{\alpha+1}} \right) + \left( \frac{U_{2x}}{V_{2}^{\alpha+1}} - \frac{U_{2x}}{V_{2}^{\alpha+1}} \right) \right|
\]

\[
+ \|p(V) - p(V_{1})\| + \|p(v_{m}) - p(V_{2})\|
\]

\[
= \left| \frac{U_{1x}}{V_{1}^{\alpha+1}} - \frac{1}{V_{1}^{\alpha+1}} \right| + \left| \frac{U_{2x}}{V_{2}^{\alpha+1}} - \frac{1}{V_{2}^{\alpha+1}} \right|
\]

\[
+ \|p(V_{1} + V_{2} - v_{m}) - p(V_{1})\| + \|p(v_{m}) - p(V_{2})\|
\]

\[
\leq C\{ |(V_{2} - v_{m})| + |U_{2x}| \}.
\]

(4.12)

By (2.4), we get

\[
\left| \frac{\partial^{j} U_{2}}{\partial x^{j}} \right|, \left| \frac{\partial^{j} (V_{2} - v_{m})}{\partial x^{j}} \right| \leq C |V_{2} - v_{m}|, \; \forall j \in \mathbb{N}.
\]

(4.13)

On the other hand, in the same way, it is still true to replace \((V_{2}, U_{2})\) with \((V_{1}, U_{1})\) in (4.13). We get \(\left| \frac{\partial^{n} W}{\partial x^{n}} \right| \leq C |V_{i} - v_{m}|, \; i = 1, 2; \; \forall n \in \mathbb{N}\). By (2.10) and (2.11), we know \(\beta_1, \beta_2\) are bounded. Taking \(\beta\) sufficient large, for \(n = 0, 1\), it holds that:
\[
\int_{-\infty}^{\infty} \left| \frac{\partial^n W}{\partial x^n} \right|^2 dx = \int_{-\infty}^{\frac{1}{2}} \left| \frac{\partial^n W}{\partial x^n} \right|^2 dx + \int_{\frac{1}{2}}^{\infty} \left| \frac{\partial^n W}{\partial x^n} \right|^2 dx 
\]
\[
\leq C \int_{-\infty}^{\frac{1}{2}} |V_2(x - s_2 t + \beta_2 - \beta) - v_m|^2 dx + C \int_{\frac{1}{2}}^{\infty} |V_1(x - s_1 t + \beta_1) - v_m|^2 dx 
\]
\[
\leq C \chi_2^2 \int_{-\infty}^{\frac{1}{2}} \exp[2c_2(x - s_2 t + \beta_2 - \beta)] dx + C \chi_1^2 \int_{\frac{1}{2}}^{\infty} \exp[-2c_1(x - s_1 t + \beta_1)] dx 
\]
\[
= C \chi^2 e^{-2c_2 s_2 t} e^{c_2 (2\beta_2 - \beta)} + C \chi_1^2 e^{c_2 s_1 t} e^{-c_1 (\beta - 2\beta_1)} 
\]
\[
\leq C e^{-2c_2 s_2 t} e^{-\frac{2}{n} \beta} + C e^{c_2 s_1 t} e^{-\frac{n}{2} \beta}. 
\]

We have used Lemma 2.1 in the second inequality. Setting \( e' := \min(-c_1 s_1, c_2 s_2); C_- := \frac{1}{n} \min(c_1, c_2) \), we complete the proof of the Lemma.

**Lemma 4.3.** Under the same assumptions of Proposition 3.1, it holds that
\[
\| (\phi, \Psi) \|^2(t) + \int_0^t \int_{-\infty}^{\infty} \left( \frac{1}{p'(V)} \right) t \Psi^2 dx dt + \int_0^t \| \phi_x \|^2 dt 
\]
\[
\leq C \| (\phi_0, \Psi_0) \|^2 + C \delta \int_0^t \| \phi_{xx} \|^2 dt + C e^{-C_- \beta}. 
\]

**Proof.** Multiply (4.5)\(_1\) and (4.5)\(_2\) by \( \phi \) and \( \frac{\Psi}{p'(V)} \), respectively, sum them up, and integrate result with respect to \( t \) and \( x \) over \([0, t] \times \mathbb{R}\). We have
\[
\frac{1}{2} \int_{-\infty}^{\infty} \left( \phi^2 - \frac{\Psi^2}{p'(V)} \right) dx + \int_0^t \int_{-\infty}^{\infty} \left\{ \frac{1}{2} \left( \frac{1}{p'(V)} \right) t \Psi^2 + \frac{\phi_x^2}{V^{n+1}} \right\} dx dt 
\]
\[
= \int_0^t \int_{-\infty}^{\infty} G \phi dx dt + \int_0^t \int_{-\infty}^{\infty} \frac{p(V) \Psi}{p'(V)} dx dt 
\]
\[
+ \int_0^t \int_{-\infty}^{\infty} W \frac{\Psi}{p'(V)} dx dt + \frac{1}{2} \int_{-\infty}^{\infty} \left( \phi^2 - \frac{\Psi^2}{p'(V)} \right) \big|_{t=0} dx =: \sum_{i=1}^{4} A_i. \quad (4.14) 
\]

Utilize to Lemma 4.1, we can get
\[
|A_1 + A_2| \leq C \left( \int_0^t \int_{-\infty}^{\infty} |\phi_{xx}^2| + |\phi_x \phi_{xx}| + |\Psi \phi_x^2| dx dt \right) 
\]
\[
\leq C \int_0^t \| \phi \|_{L^\infty} \int_{-\infty}^{\infty} |\phi_{xx}^2| + |\phi_{xx}^2| dx dt + C \int_0^t \| \Psi \|_{L^\infty} \int_{-\infty}^{\infty} \phi_x^2 dx dt 
\]
\[
\leq C (\| \phi \|_2 + \| \psi \|_1) \int_0^t \| \phi_x \|^2 + \| \phi_{xx} \|^2 dt 
\]
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\[ \leq C\delta \int_0^t \|\phi_x\|^2 + \|\phi_{xx}\|^2 \, dt. \quad (4.15) \]

With the help of Lemma 4.2, one has

\[ |A_3| \leq C \int_0^t \int_{-\infty}^{\infty} |W\Psi| \, dx \, dt \leq C \int_0^t \|W\|\Psi\| \, dt \leq C\delta e^{-C-\beta}. \quad (4.16) \]

Taking \( \delta \) sufficient small, using (4.14)–(4.16), we get Lemma 4.3.

**Lemma 4.4.** Under the same assumptions of Proposition 3.1, it holds that

\[ \|(\phi, \Psi)(t)\|_1^2 + \int_0^t \|\phi_x\|_1^2 \, dt \leq C\|(\phi_0, \Psi_0)\|_1^2 + Ce^{-C-\beta}. \]

**Proof.** Multiply (4.5)_1 and (4.5)_2 by \(-\phi_{xx}\) and \(\frac{\Psi_{xx}}{p'(V)}\), respectively and sum over the result, intergrate the result with respect to \( t \) and \( x \) over \([0, t] \times \mathbb{R}\). We have

\[
\begin{align*}
\frac{1}{2} & \int_{-\infty}^{\infty} \left( \phi_x^2 - \frac{\Psi_x^2}{p'(V)} \right) dx + \int_0^t \int_{-\infty}^{\infty} \left( \frac{1}{2} \left( \frac{1}{p'(V)} \right) t \Psi_x + \frac{\phi_{xx}^2}{V_{\alpha+1}} \right) dx \, dt \\
= & \frac{1}{2} \int_{-\infty}^{\infty} \left( \phi_x^2 - \frac{\Psi_x^2}{p'(V)} \right) \bigg|_{t=0} dx \\
& - \int_0^t \int_{-\infty}^{\infty} \left[ G - (\alpha + 1) \frac{V_x}{V_{\alpha+2}} \phi_x \right] \phi_{xx} \, dx \, dt \\
& - \int_0^t \int_{-\infty}^{\infty} \left( \frac{1}{p'(V)} \right) \Psi_x \phi_x \, dx \, dt + \int_0^t \int_{-\infty}^{\infty} \Psi_x \frac{W_x}{p'(V)} \, dx \, dt \\
& + \int_0^t \int_{-\infty}^{\infty} \frac{1}{p'(V)} p(v|V) \Psi_x \, dx \, dt \\
=: & \frac{1}{2} \int_{-\infty}^{\infty} \left( \phi_x^2 - \frac{\Psi_x^2}{p'(V)} \right) \bigg|_{t=0} dx + \sum_{i=1}^4 B_i. \quad (4.17)
\end{align*}
\]

Now we estimate \( B_i \) term by term. The Cauchy inequality indicates that

\[ |B_1| \leq C \int_0^t \int_{-\infty}^{\infty} (|\phi_{xx}\phi_x| + |\phi_{xx}^2|)|\phi_{xx}| + |\phi_x\phi_{xx}| \, dx \, dt \]

\[ \leq (C\delta + \varepsilon) \int_0^t \|\phi_{xx}\|^2 \, dt + C\varepsilon \int_0^t \|\phi_x\|^2 \, dt, \quad (4.18) \]

and

\[
B_2 \leq \min(-s_1; s_2) \int_0^t \int_{-\infty}^{\infty} \left| \left( \frac{1}{4p'(V)} \right) x \right| |\Psi_x^2| \, dx \, dt \\
+ C \int_0^t \int_{-\infty}^{\infty} \left| \left( \frac{1}{p'(V)} \right) x \right| |\Psi_x^2| \, dx \, dt \\
\leq \int_0^t \int_{-\infty}^{\infty} \left( \frac{1}{4p'(V)} \right) \Psi_x^2 \, dx \, dt + C \int_0^t \|\phi_x\|^2 \, dt. \quad (4.19)
\]
The last inequality is based on the following inequality
\[
\left( \frac{1}{p'(V)} \right)_t = (p'(V))^{-1} p''(V) (V_1'(-s_1) + V_2'(s_2)) \\
\geq (p'(V))^{-1} p''(V) |V_x| \min(-s_1; s_2) \\
= \min(-s_1; s_2) \left| \left( \frac{1}{p'(V)} \right)_x \right| .
\]

Making use of Lemma 4.2, it follows that
\[
|B_4| \leq \int_0^t \int_0^\infty \left| \Psi_x W_x \right| \frac{dx dt}{p'(V)} \leq C \int_0^t \|W_x\| \|\Psi_x\| dt \leq C\delta e^{-C-\beta} . 
\]  

By (4.9) and the Sobolev inequality, we obtain
\[
|B_4| \leq C \int_0^t \int_0^\infty \left| (\phi_x \phi_{xx} + V_x \phi_x^2) \right| \frac{dx dt}{p'(V)} \leq C \int_0^t \int_0^\infty \left| (\phi_x \phi_{xx} + V_x \phi_x^2) \right| \frac{dx dt}{p'(V)} \\
+ C \int_0^t \int_0^\infty \left| (\phi_{xx} \phi_{xx} + V_x \phi_x \phi_{xx}) \phi_x \right| \frac{dx dt}{p'(V)} \\
\leq C (\|\phi\|_2 + \|\psi\|_2) \int_0^t \|\phi_x\|^2 + \|\phi_x\|^2 dt \\
\leq C \delta \int_0^t (\|\phi_{xx}\|^2 + \|\phi_x\|^2) dt . 
\]

From (4.17)–(4.21), we get
\[
\frac{1}{2} \int_0^\infty \left( \phi_x^2 - \frac{\Psi_x^2}{p'(V)} \right) dx + \frac{1}{4} \int_0^t \int_\infty^- \left( \frac{1}{p'(V)} \right)_t \Psi_x^2 dx dt + \int_0^t \int_\infty^- \frac{\phi_x^2}{V^{\alpha+1}} dx dt \\
\leq (C + C\delta + C\varepsilon) \int_0^t \|\phi_x\|^2 dt + (C\delta + \varepsilon) \int_0^t \|\phi_{xx}\|^2 dt \\
+ C e^{-C-\beta} + C (\|\phi_{xx}\|^2 + \|\Psi_x\|^2) .
\]

Choosing \( \varepsilon \) appropriately small and \( \delta \) sufficient small, together with Lemma 4.3, we get the proof of Lemma 4.4. 

\begin{lemma}
Under the same assumptions of Proposition 3.1, it holds that
\[
\int_0^t \|\Psi_x(t)\|^2 dt \leq C \|(\phi_0, \Psi_0)\|^2 + Ce^{-C-\beta} .
\]
\end{lemma}

\begin{proof}
Multiplying (4.5.1) by \( \Psi_x \) and make use of (4.5.2), we get
\[
\Psi_x^2 = (\phi \Psi_x)_t + \{ \phi[p(v) - p(V) + W] \}_x - \phi_x (p(v) - p(V)) \\
- \frac{\Psi_x \phi_{xx}}{V^{\alpha+1}} - \phi_x W - \Psi_x G + (\alpha + 1) \frac{V_x}{V^{\alpha+2}} \phi_x \Psi_x .
\]

Integrating (4.22) with respect to \( t \) and \( x \) over \( [0, t] \times \mathbb{R} \), we obtain that
\[
\int_0^t \int_{-\infty}^\infty \Psi_x^2 dx dt
\]

\end{proof}
\[= -\int_{-\infty}^{\infty} \phi \psi \big|_{t=0} dx - \int_0^t \int_{-\infty}^{\infty} \Psi G dx dt \\
+ \int_0^t \int_{-\infty}^{\infty} (\alpha + 1) \frac{V_x}{V_{\alpha+2}} \Psi_x \phi_x dx dt + \int_{-\infty}^{\infty} \phi \psi dx \\
- \int_0^t \int_{-\infty}^{\infty} \frac{\Psi_x \phi_{xx}}{V_{\alpha+2}} dx dt - \int_0^t \int_{-\infty}^{\infty} \phi_x (p(v) - p(V)) dx dt \\
- \int_0^t \int_{-\infty}^{\infty} \phi_x W dx dt \]

\[= -\int_{-\infty}^{\infty} \phi \psi \big|_{t=0} dx + \sum_{i=1}^{6} H_i. \tag{4.23} \]

We estimate \( H_i \) term by term. By the Cauchy inequality, it follows that

\[ H_1 \leq C \int_0^t \| \phi_x \|_{L^\infty} \int_{-\infty}^{\infty} \Psi_x (|\phi_{xx}| + |\phi_x|) dx dt \\
\leq \varepsilon \int_0^t \| \Psi_x \|^2 dt + C \varepsilon \int_0^t (|\phi_{xx}|^2 + |\phi_x|^2) dt, \tag{4.24} \]

In addition, it is straightforward to imply that

\[ H_2 + H_3 + H_4 + H_5 \leq \| (\phi, \Psi_x) \|^2 + \varepsilon \int_0^t \| \Psi_x \|^2 dt + C \varepsilon \int_0^t (|\phi_{xx}|^2 + |\phi_x|^2) dt. \tag{4.25} \]

Making use of Lemma 4.2, we have

\[ H_6 = \int_0^t \int_{-\infty}^{\infty} \phi_x W dx dt \leq \int_0^t \| W \| \| \phi_x \| dt \leq C \delta e^{-C \beta}. \tag{4.26} \]

Collecting (4.23)–(4.26) and using Lemma 4.4, we get the Lemma 4.5.

Combining Lemma 4.3–Lemma 4.5, we obtain the following low order estimates

\[ \| (\phi, \Psi) \|_1^2(t) + \int_0^t \| \Psi_x \|^2 dt + \int_0^t \| \phi_x \|^2 dt \leq C \| (\phi_0, \Psi_0) \|_1^2 + C e^{-C \beta}, \]

with the help of (4.8), which can be rewritten by the variables \( \phi \) and \( \psi \) as

Lemma 4.6. Under the same assumptions of Proposition 3.1, it holds that

\[ (\| \phi \|_1^2 + \| \psi \|_1^2)(t) + \int_0^t \| \psi_x \|^2 dt + \int_0^t \| \phi_x \|^2 dt \leq C \| \phi_0 \|^2_1 + C \| \psi_0 \|^2_1 + C e^{-C \beta}. \]

4.2 High Order Estimate.

We turn to the original equation (3.4) to study the higher order estimates.

Lemma 4.7. Under the same assumptions of Proposition 3.1, it holds that

\[ \| \psi_x \|^2(t) + \int_0^t \| \psi_{xx} \|^2 dt \leq C \| \phi_0 \|^2_2 + C \| \psi_0 \|^2_2 + C e^{-C \beta}. \tag{4.27} \]
Proof. Multiplying (3.4)_2 by \(-\psi_{xx}\), integrating the result with respect to \(t\) and \(x\) over \([0, t] \times \mathbb{R}\) gives

\[
\frac{1}{2} \|\psi_x\|^2(t) + \int_0^t \int_{-\infty}^{\infty} \frac{\psi_x^2}{V^{\alpha+1}} dx dt = \frac{1}{2} \|\psi_{0x}\|^2 + \int_0^t \int_{-\infty}^{\infty} W \psi_{xx} dx dt \]

\[
- \int_0^t \int_{-\infty}^{\infty} f(V, U_x) \psi_{xx} dx dt - \int_0^t \int_{-\infty}^{\infty} F \psi_{xx} dx dt \]

\[
= \frac{1}{2} \|\psi_{0x}\|^2 + \sum_{i=1}^{3} M_i. \tag{4.28}
\]

Making use of Lemma 4.2, we have

\[
M_1 \leq \varepsilon \int_0^t \|\psi_{xx}\|^2 dt + C_\varepsilon \int_0^t \|W\|^2 dt \leq \varepsilon \int_0^t \|\psi_{xx}\|^2 dt + C e^{-C^{-\beta}}. \tag{4.29}
\]

The Cauchy inequality implies that

\[
M_2 \leq \varepsilon \int_0^t \|\psi_{xx}\|^2 dt + C_\varepsilon \int_0^t \|\phi_x\|^2 dt. \tag{4.30}
\]

By (4.7)_1 and the Sobolev inequality, yields

\[
M_3 \leq C \int_0^t \int_{-\infty}^{\infty} (|\phi_x|^2 + |\phi_x||\psi_{xx}|)|\psi_{xx}| dx dt \leq C \int_0^t \int_{-\infty}^{\infty} |\phi_x| (|\phi_x|^2 + |\psi_{xx}|^2) dx dt \leq C \int_0^t (\|\phi_x\|^2 + \|\psi_{xx}\|^2) dt. \tag{4.31}
\]

Substituting (4.29)–(4.31) into (4.28) and using Lemma 4.6, we obtain (4.27).

Lemma 4.8. Under the same assumptions of Proposition 3.1, it holds that

\[
\|\phi_{xx}\|^2 + \int_0^t \|\phi_{xx}\|^2 dt \leq C \|\phi_0\|^2_2 + C \|\psi_0\|^2_1 + C e^{-C^{-\beta}} + C \delta \int_0^t \|\psi_{xxx}\|^2 dt. \tag{4.32}
\]

Proof. Differentiating (3.4)_1 with respect to \(x\), using (3.4)_2, we have

\[
\frac{\phi_{xx}}{V^{\alpha+1}} + f(V, U_x) \phi_x = \psi_t - F + W. \tag{4.33}
\]

Differentiating (4.33) in respect of \(x\) and multiplying the derivative by \(\phi_{xx}\), integrating the result in respect of \(t\) and \(x\) over \([0, t] \times \mathbb{R}\), using (2.4), one has

\[
\frac{1}{2} \int_{-\infty}^{\infty} \frac{\phi_{xx}^2}{V^{\alpha+1}} dx + \int_0^t \int_{-\infty}^{\infty} \left( f(V, U_x) + \frac{\alpha + 1}{2} U_x \right) \phi_{xx}^2 dx dt
\]
By $U_x < 0$ and (3.6), one has

$$f(V, U_x) + \frac{\alpha + 1}{2} \frac{U_x}{V^{\alpha + 2}} = -p'(V) - \frac{\alpha + 1}{2} \frac{U_x}{V^{\alpha + 2}} \geq -\max\{p'(v_-), p'(v_+)^\} > 0.$$  \hfill (4.35)

The Cauchy inequality yields

$$N_1 \leq \varepsilon \|\phi_{xx}\|^2 + C_\varepsilon \|\psi_x\|^2.$$  \hfill (4.36)

Similar to (4.29), we get

$$N_2 \leq \varepsilon \int_0^t \|\phi_{xx}\|^2 dt + C_\varepsilon e^{-Ct}. $$  \hfill (4.37)

$N_3$ can be controlled by (4.27). Using (4.7), and Cauchy inequality, we have

$$|N_4| \leq \varepsilon \int_0^t \|\phi_{xx}\|^2 dt + C_\varepsilon \int_0^t \|F_x\|^2 dt.$$  

\leq \varepsilon \int_0^t \|\phi_{xx}\|^2 dt + C_\varepsilon \int_0^t \left( \|\phi_x\|^2 + \|\psi_x\|^2 \right) dt.

The Cauchy inequality yields

$$|N_5| \leq C \int_0^t \int_{-\infty}^\infty \frac{V_x}{V^{\alpha + 2}} \phi_{xx} \phi_{xx} \psi_x \phi_{xx} dx dt \leq \varepsilon \int_0^t \|\phi_{xx}\|^2 dt + C_\varepsilon \int_0^t \|\psi_{xx}\|^2 dt.$$  \hfill (4.38)

With the help of

$$f(V, U_x) = -p''(V)V_x - (\alpha + 1) \frac{U_{xx}}{V^{\alpha + 2}} + (\alpha + 1)(\alpha + 2) \frac{U_x}{V^{\alpha + 3}} V_x < C,$$

one gets

$$|N_6| \leq \varepsilon \int_0^t \|\phi_{xx}\|^2 dt + C_\varepsilon \int_0^t \|\phi_x\|^2 dt.$$  \hfill (4.39)

Choosing $\varepsilon$ small, substituting (4.35)–(4.39) into (4.34) and using Lemma 4.6, Lemma 4.7, we have (4.32).  \hfill \square

On the other hand, differentiating the second equation of (3.4) with respect to $x$, multiplying the derivative by $-\psi_{xxx}$, integrating the resulting equality over $[0, t] \times \mathbb{R}$, using Lemma 4.6–Lemma 4.8, we can get the highest order estimate in the same way, which is listed as follows and the proof is omitted.
Lemma 4.9. Under the same assumptions of Proposition 3.1, it holds that
\begin{equation}
\|\psi_{xx}(t)\|^2 + \int_0^t \|\psi_{xxx}\|^2 dt \leq C\|\phi_0, \psi_0\|^2 + C e^{-C \beta}.
\end{equation}

Finally, Proposition 3.1 is obtained by Lemma 4.5–Lemma 4.9.

5 Proof of Theorem 2.2

It is straightforward to imply (2.14) from Lemma 3.2. It remains to show (2.15). The following useful lemma will be used.

Lemma 5.1[14]. Assume that the function \( f(t) \geq 0 \in L^1(0, +\infty) \cap BV(0, +\infty) \), then it holds that \( f(t) \to 0 \) as \( t \to \infty \).

Let us turn to the system (3.4). Differentiating (3.4) with respect to \( x \), multiplying the resulting equation by \( \phi_x \) and integrating it with respect to \( x \) on \((-\infty, \infty)\), we have
\begin{equation}
\frac{d}{dt}(\|\phi_x\|^2) \leq C(\|\phi_x\|^2 + \|\psi_{xx}\|^2).
\end{equation}

With the aid of Lemma 3.2, we have
\begin{equation}
\int_{-\infty}^{\infty} \frac{d}{dt}(\|\phi_{xx}\|^2) dt \leq C,
\end{equation}

which implies \( \|\phi_x\|^2 \in L^1(0, +\infty) \cap BV(0, +\infty) \). By Lemma 5.1, we have
\begin{equation}
\|\phi_x\| \to 0 \quad \text{as} \quad t \to +\infty.
\end{equation}

Since \( \|\phi_{xx}\| \) is bounded, the Sobolev inequality implies that
\begin{equation}
\|v - V\|^2_\infty = \|\phi_x\|^2_\infty \leq 2\|\phi_x(t)\|\|\phi_{xx}(t)\| \to 0.
\end{equation}

Similarly, we have
\begin{equation}
\|u - U\|^2_\infty = \|\psi_x\|^2_\infty \leq 2\|\psi_x(t)\|\|\psi_{xx}(t)\| \to 0.
\end{equation}

Therefore, the proof of Theorem 2.2 is completed. \( \square \)

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