A minimax argument to a stronger version of the Jacobian conjecture∗

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Abstract
The main result of this paper is to prove the strong real Jacobian conjecture under the symmetric assumption (Theorem 1.6) and reveals the link between it and the Jacobian conjecture (Proposition 1.3). Precisely, we assume that $F: \mathbb{R}^n \to \mathbb{R}^n$ is of $C^1$ map, $n \geq 2$, if for some $\varepsilon > 0$, $0 \notin \text{Spec}(F)$ and $\text{Spec}(F + F^T) \subseteq (-\infty, -\varepsilon)$ or $(\varepsilon, +\infty)$, where $\text{Spec}(F)$ denotes all eigenvalues of $JF$ and $\text{Spec}(F + F^T)$ denotes all eigenvalues of $JF + JF^T$, then we show that $F$ is injective. It is proved by using a minimax argument.

Key words: Jacobian Conjecture; Minimax method; Injective.

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1 Introduction

In 1939, Keller (see [8]) stated the following Conjecture:

Conjecture 1.1. (Jacobian Conjecture). Let \( F : k^n \to k^n \) be a polynomial map, where \( k \) is a field of characteristic 0. If the determinant for its jacobian of the polynomial map is a non-zero constant, i.e., \( \det JF(x) \equiv C \in k^*, \forall x \in k^n \), then \( F(x) \) has a polynomial inverse map.

This is a long-standing conjecture, it is still open even in the case \( n = 2 \). There are many partial results on this conjecture, see for example [1] and [6]. Chamberland and Meisters discovered a sufficient condition for Conjecture 1.1 and formulated the following conjecture:

Conjecture 1.2. ([3], Conjecture 2.1) Let \( F : \mathbb{R}^n \to \mathbb{R}^n \) be a \( C^1 \) map. Suppose there exists an \( \varepsilon > 0 \) such that \( |\lambda| \geq \varepsilon \) for all the eigenvalues \( \lambda \) of \( F'(x) \) and all \( x \in \mathbb{R}^n \). Then \( F \) is injective.

In this note, we will show that the following conclusion is true:

Proposition 1.3. Conjecture 1.2 implies Conjecture 1.1. That is, if Conjecture 1.2 is true for all \( n \), then Conjecture 1.1 is also true.

In the current paper, let \( \text{Spec}(F) \) denote the set of all (complex) eigenvalues of \( JF(x) \) and let \( \text{Spec}(F + F^T) \) denote the set of all (complex) eigenvalues of \( JF + \{JF\}^T \), for any \( x \in \mathbb{R}^n \), where \( JF \) is the Jacobian matrix of \( F \). Fernandes, Gutierrez, and Rabanal proved Conjecture 1.2 in dimension \( n = 2 \) and obtained the following strong theorem.

Theorem 1.4. ([7]) Let \( F = (f, g) : \mathbb{R}^2 \to \mathbb{R}^2 \) be a differentiable map. For some \( \varepsilon > 0 \), if

\[
\text{Spec}(F) \cap [0, \varepsilon) = \emptyset,
\]

then \( F \) is injective.

Recently, Liu and Xu directly prove the Conjecture 1.2 under an additional condition and have the following theorem.

Theorem 1.5. ([9]) Suppose that \( F : \mathbb{R}^n \to \mathbb{R}^n \) is a \( C^1 \) map, \( n \geq 2 \). If there exists \( \varepsilon > 0 \), such that

\[
\text{Spec}(F) \in \mathbb{C}\backslash(-\varepsilon, \varepsilon) \times (-i\varepsilon, i\varepsilon) \quad \text{and} \quad \text{Spec}(F+F^T) \subseteq (-\infty, -\varepsilon) \cup (\varepsilon, +\infty),
\]

then \( F \) is injective.

In the current paper, we can obtain the following stronger results.

Theorem 1.6. Let \( F : \mathbb{R}^n \to \mathbb{R}^n \) be a \( C^1 \) map, \( n \geq 2 \). For some \( \varepsilon > 0 \), if

\[
0 \notin \text{Spec}(F) \quad \text{and} \quad \text{Spec}(F+F^T) \subseteq (-\infty, -\varepsilon) \cup (\varepsilon, +\infty),
\]

then \( F \) is injective.
Considering the Pinchuk’s counterexample (see [10]) that the eigenvalues sometimes can tend to zero although they are positive, we have the following result under adding the condition of $\text{Spec}(F + F^T)$.

**Theorem 1.7.** Let $F$ be a $C^1$ map from $\mathbb{R}^n$ to $\mathbb{R}^n$. Suppose that $0 \notin \text{Spec}(F)$ and that one of the following conditions holds:

(i) $\text{Spec}(F + F^T) \subseteq (-\infty, 0)$ and $\exists M_1, M_2 > 0$ such that
\[ \sum_{\lambda \in \text{Spec}(F + F^T)} \lambda > -M_1 \text{ and } \prod_{\lambda \in \text{Spec}(F + F^T)} |\lambda| > M_2; \]

(ii) $\text{Spec}(F + F^T) \subseteq (0, +\infty)$ and $\exists M_1, M_2 > 0$ such that
\[ \sum_{\lambda \in \text{Spec}(F + F^T)} \lambda < M_1 \text{ and } \prod_{\lambda \in \text{Spec}(F + F^T)} \lambda > M_2. \]

Then $F$ is injective.

## 2 Minimax Method

We firstly recall the following Mountain Pass theorem.

**Theorem 2.1.** Let $X$ be a Banach space, and $I \in C^1(X, \mathbb{R})$. Let $\Omega \subset X$ be an open set with $u_0 \in \Omega$ and $u_1 \notin \Omega$. Set
\[ \Gamma = \{ \gamma \in C([0, 1], X) | \gamma(i) = u_i, i = 0, 1 \} \]
and
\[ c = \inf_{\gamma \in \Gamma} \sup_{t \in [0, 1]} I(\gamma(t)). \]  
(2.1)

If further

(a) $\alpha = \inf_{\partial \Omega} I(u) > \max\{I(u_0), I(u_1)\}$;

(b) $I$ satisfies $(PS)_c$ condition.

Then $c$ is a critical value of $I$.

**Proof of Proposition 2.1.** We show that Conjecture 1 implies Conjecture 1.1. Firstly, it is sufficient to consider the case $k = \mathbb{C}$ for Conjecture 1.1 by Lefschetz principle (see [5]). Next, it further reduces to prove that for $k = \mathbb{C}$, $F$ is injective in conclusion of Conjecture 1.1 (see [2]). Finally, we prove $F$ is injective for Conjecture 1.1 in $\mathbb{C}$ by Conjecture 1.2.

Let $F$ be a polynomial map from $\mathbb{C}^n$ to $\mathbb{C}^n$ defined as following:
\[ (x_1, x_2, ..., x_n) \rightarrow (F_1, F_2, ..., F_n). \]
Define a polynomial map $\mathbf{F} : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ as

$$(\text{Re} x_1, \text{Im} x_1, \ldots, \text{Re} x_n, \text{Im} x_n) \to (\text{Re} F_1, \text{Im} F_1, \ldots, \text{Re} F_n, \text{Im} F_n).$$

Then $\det J\mathbf{F} = |\det J\mathbf{F}|^2$. Consequently, $\det J\mathbf{F}$ is not zero complex constant if and only if $\det J\mathbf{F}$ is not zero real constant.

For $\mathbf{F}$, we consider $\mathbf{F}(x) = x - H(x)$, where $H$ is a cube-homogeneous polynomial map and $JH(x)$ is a nilpotent matrix, i.e., $JH(x)^{2n} = 0$ (see [1]). From linear algebra, we know that $JH(x)$ is nilpotent if and only if its characteristic polynomial $\det (\mu I - JH(x)) = \mu^{2n}$. Therefore, we let $\mu = \lambda - 1$ and compute the characteristic polynomial of $J\mathbf{F}(x)$,

$$\det (\lambda I - J\mathbf{F}(x)) = \det ((\lambda - 1)I - JH(x)) = \det (\mu I - JH(x)) = \mu^{2n} = 0.$$

Thus, $\mu = \lambda - 1 = 0$. That is, $\text{Spec}(\mathbf{F}) = \{1\}$. It implies that $\text{Spec}(\mathbf{F})$ can not tend to zero. By Conjecture [2] we obtain that $\mathbf{F}$ is injective. Obviously, the map $\mathbf{F}$ is injective if and only if $\mathbf{F}$ is injective. Thus, $\mathbf{F}$ is injective. By the known result (see [4]), we have $\mathbf{F}$ is bijective. Furthermore the inverse is also a polynomial map. □

3 The proof of Theorem [1.6]

Proof. Suppose by contradiction that $\mathbf{F}$ is not injective, then $\mathbf{F}(a) = \mathbf{F}(b)$ for some $a, b \in \mathbb{R}^n, a \neq b$. We define $G(X) = \mathbf{F}(X + a) - \mathbf{F}(b), \forall X \in \mathbb{R}^n$. Then $G(0) = 0$ and putting $c = b - a$, we have $c \neq 0$ and $G(c) = 0$. Let $I(X) = G(X)^T G(X), \forall X \in \mathbb{R}^n$. Thus $I'(X) = 2G(X)^T G'(X)$ and $I(c) = I(0) = 0$.

Observe $G'(X) = F'(X + a)$, so $G'(X)$ has no zero eigenvalue. Therefore,

$$\det G'(X) \neq 0, \forall X \in \mathbb{R}^n.$$

If $I'(X) = 0, \forall X \in \mathbb{R}^n$, i.e., $G(X)^T G'(X) = 0, \forall X \in \mathbb{R}^n$, then $G'(X)G(X) = 0, \forall X \in \mathbb{R}^n$. So $G(X) = 0$ and $I(X) = 0$. Next, we prove that $I(X)$ satisfies the geometric condition-(a) in Theorem 2.3. Since $I(c) = I(0) = 0$, it is sufficient to prove that there exists $r > 0$, such that

$$I(X) > 0, \forall X \in \partial B_r(0). \tag{3.1}$$

We claim: $X = 0$ is an isolated zero point of $I(X)$. In fact, for

$$G(X) = (G_1(X), G_2(X), \ldots, G_n(X))^T,$$

so $G_i(X) = G'_i(Y_i)X$, where $Y_i$ connects $0$ to $X$ ($i = 1, 2, \ldots, n$). Define a continuous function $\beta(X)$ as

$$\beta(X) = \begin{cases} (G'_1(Y_1), G'_2(Y_2), \ldots, G'_n(Y_n))^T, & X \neq 0, \\ G'(0), & X = 0. \end{cases}$$
Thus $G(X) = \beta(X)X, \forall X \in \mathbb{R}^n$. Define

$$
\gamma(X_1, X_2, \ldots, X_n) = (G'_1(X_1), G'_2(X_2), \ldots, G'_n(X_n))^T.
$$

Thus $\gamma(X, X, \ldots) = G'(X)$ and $\gamma(Y_1, Y_2, \ldots, Y_n) = \beta(X)$. Therefore,

$$
\det \gamma(0, 0, \ldots, 0) = \det I'(0) \neq 0.
$$

By the continuity of $\gamma$, there exists a positive number $r > 0$, such that

$$
\det \gamma(X_1, X_2, \ldots, X_n) \neq 0, \text{ for } (X_1, X_2, \ldots, X_n) \in B_r(0).
$$

Thus $\det \beta(X) \neq 0, \forall X \in B_{r/\sqrt{n}}(0)$. Therefore, $0$ is an isolated zero point of $I(X)$.

Let $\alpha = \inf_{\partial B_{r/\sqrt{n}}(0)} I(X)$. It is a positive number since $I(X)$ is continuous and nonnegative and is not zero on $\partial B_{r/\sqrt{n}}(0)$. Thus, $I(X)$ satisfies the condition-(a) of Theorem 2.3.

Assume that $c$ is a critical value of $I$, that is, $\exists X_c \in \mathbb{R}^n$, such that $I'(X_c) = 0$. Thus

$$
0 < \alpha \leq c = I(X_c) = 0.
$$

Obviously, it contradicts. By Theorem 2.1, the functional $I(X)$ does not satisfy the condition-(b) i.e., the $(PS)_c$ condition does not hold. Hence, there exists an unbounded sequence $\{X_k\} \subset \mathbb{R}^n$, such that

- $(i)$ $I(X_k) \to c$; $(ii)$ $I'(X_k) \to 0$.

Suppose that $\{X_k\}$ is bounded in $\mathbb{R}^n$, then $\{X_k\}$ has a weak convergence subsequence. It is also strong convergent in $\mathbb{R}^n$. This contradicts with the fact that $I(X)$ does not satisfy the $(PS)_c$ condition.

**Case (1):** $\text{Spec}(F + F^T) \subseteq (\varepsilon, +\infty)$. We let $\mu_1$ denote the minimum eigenvalue of a Hermitian matrix $A$. That is,

$$
\mu_1 = \inf_{Y \neq 0} \frac{Y^TAY}{Y^TY}.
$$

Set $A = G'(X_k) + G'(X_k)^T$ and $Y = G(X_k)$. By (3.2), we obtain

$$
\mu_1(X_k) \leq \frac{G(X_k)^T(G'(X_k) + G'(X_k)^T)G(X_k)}{G(X_k)^TG(X_k)}
$$

$$
= \frac{G(X_k)^TG'(X_k)G(X_k) + G(X_k)^TG'(X_k)^TG(X_k)}{G(X_k)^TG(X_k)}
$$

$$
= \frac{2G(X_k)^TG'(X_k)G(X_k)}{G(X_k)^TG(X_k)}.
$$

By (i), one gets

$$
G(X_k)^TG(X_k) = I(X_k) \to c > 0.
$$

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By (ii) and (3.4), we obtain
\[
\left| 2G(X_k)^T G'(X_k)G(X_k) \right| \leq \left\| 2G(X_k)^T G'(X_k) \right\| \left\| G(X_k) \right\| \\
= \left\| I'(X_k) \right\| (G(X_k)^T G(X_k))^{\frac{1}{2}} \\
= \left\| I'(X_k) \right\| \sqrt{I(X_k)} \to 0.
\] (3.5)

Combining (3.3), (3.4) with (3.5), letting \( k \to +\infty \), we see that \( \mu_1(X_k) \leq 0 \). Note that \( \text{Spec}(F + F^T) \subseteq (\varepsilon, +\infty) \), we observe that \( \mu_1(X_k) \geq \varepsilon \). Letting \( k \to +\infty \), we get a contradiction.

**Case (2):** \( \text{Spec}(F + F^T) \subseteq (\varepsilon, -\varepsilon) \). For this case, we let \( \mu_2 \) be the maximum eigenvalue of a Hermitian matrix \( A \). That is,
\[
\mu_2 = \sup_{Y \neq 0} \frac{Y^T A Y}{Y^T Y}.
\]

By the same method, we obtain \( \mu_2(X_k) \geq 0 \). Since \( \text{Spec}(F + F^T) \subseteq (\varepsilon, -\varepsilon) \) and \( \mu_2(X_k) \leq -\varepsilon \), we also get a contradiction.

**4 The proof of Theorem 1.7**

**Proof.** Firstly, we consider two cases for \( \text{Spec}(F + F^T) \) in Theorem 1.7.

Case (i): Suppose \( \text{Spec}(F + F^T) \subseteq (-\infty, 0) \) and \( \exists M_1, M_2 > 0 \) such that for any \( \lambda \in \text{Spec}(F + F^T) \), we have
\[
|\lambda| < \sum_{\lambda \in \text{Spec}(F + F^T)} |\lambda| = - \sum_{\lambda \in \text{Spec}(F + F^T)} \lambda < M_1.
\]

Therefore,
\[
M_1^n |\lambda| \geq \prod_{\lambda \in \text{Spec}(F + F^T)} |\lambda| > M_2 > 0
\]

Hence \( |\lambda| > \frac{M_2}{M_1^{n-1}} \), i.e., \( \lambda < - \frac{M_2}{M_1^{n-1}} \). It implies that \( \text{Spec}(F + F^T) \) can not tend to zero. This also contradicts to \( \lambda \geq \varepsilon \) by the case (1) in Theorem 1.7. Thus the conclusion holds.

Case (ii): Suppose \( \text{Spec}(F + F^T) \subseteq (0, +\infty) \) and \( \exists M_1, M_2 > 0 \) such that for any \( \lambda \in \text{Spec}(F + F^T) \), we have
\[
\lambda < \sum_{\lambda \in \text{Spec}(F + F^T)} \lambda < M_1.
\]

Therefore,
\[
M_1^n \lambda \geq \prod_{\lambda \in \text{Spec}(F + F^T)} \lambda > M_2 > 0
\]
Hence, \( \lambda > -\frac{M_2}{M_1^{n-1}} \). It also contradicts to \( \lambda \leq -\varepsilon \) in the case (2) of Theorem 1.7.

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