Generic boundary scattering in the open XXZ chain

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Abstract

The open critical XXZ spin chain with a general right boundary and a trivial diagonal left boundary is considered. Within this framework we propose a simple computation of the exact generic boundary $S$-matrix (with diagonal and non-diagonal entries), starting from the ‘bare’ Bethe ansatz equations. Our results as anticipated coincide with the ones obtained by Ghoshal and Zamolodchikov, after assuming suitable identifications of the bulk and boundary parameters.
1 Introduction

The derivation of exact bulk and boundary $S$-matrices in the context of 2-dimensional integrable models is a fundamental problem that has attracted considerable attention during the last decades (see e.g. [1]–[6]). The main aim of the present investigation is the computation of the exact generic—with diagonal and non-diagonal entries—boundary $S$-matrix in the framework of the open XXZ spin chain. More precisely, we consider the open XXZ spin chain with the left boundary to be trivial ($K^+ \propto I$), while the right boundary is associated to a full $K$-matrix [5, 7]. The corresponding Hamiltonian reads as:

$$H = -\frac{1}{4} \sum_{i=1}^{N-1} \left( \sigma^x_i \sigma^x_{i+1} + \sigma^y_i \sigma^y_{i+1} + \cosh(i\mu) \sigma^z_i \sigma^z_{i+1} \right) - \frac{N}{4} \cosh(i\mu) - \frac{\sinh(i\mu)}{4} \sigma^z_N$$

$$+ \frac{\sinh(i\mu) \cosh(i\mu \xi)}{4 \sinh(i\mu \xi)} \sigma^z_1 - \frac{\kappa \sinh(i\mu \theta)}{2 \sinh(i\mu \xi)} \left( \cosh(i\mu \theta) \sigma^x_1 + i \sinh(i\mu \theta) \sigma^y_1 \right)$$

(1.1)

where $\sigma^{x,y,z}$ are the $2 \times 2$ Pauli matrices, and the boundary parameters $\xi, \kappa, \theta$, are the free parameters of the generic $K$-matrix and are associated to some magnetic field applied at the boundaries of the chain. Notice that we focus here on the critical regime $|e^{i\mu}| = 1$.

In general quantum spin chains are one dimensional statistical systems displaying particle-like excitations ‘holes’ which may scatter among each other [2] or with the boundary [8, 9] in the presence of non trivial boundary magnetic fields. The boundary $S$-matrix within this framework describes exactly the reflection of the the particle-like excitation with the boundary magnetic field. Note that computations concerning both bulk and boundary $S$-matrices are always valid in the thermodynamic limit of the spin chain $N \to \infty$ [2, 4, 11, 8, 9, 10]. In integrable field theories on the other hand (see e.g. [5]) the physical boundary $S$-matrix describes the reflection of a solitonic excitation with the boundary, when the system is considered on the half line [5] (see also [12, 13] for relevant studies on the sine-Gordon model on the finite interval). It is crucial to note that the critical XXZ model may be thought of as the discrete analogue of the sine-Gordon model, see for instance [14, 15, 16] for a detailed derivation of this correspondence. It was also shown that the bulk [17, 10] and the diagonal boundary $S$-matrices [18, 10, 9] of the critical XXZ chain coincide with those of the sine-Gordon model after considering suitable identifications of both bulk and boundary parameters. It is therefore naturally anticipated that the generic boundary $S$-matrices should also coincide. Such a coincidence is also natural from the algebraic point of view since both models are ruled by the same algebra defined by the reflection equation. Recall also that boundary $S$-matrices are solutions of the reflection equation up to an overall physical factor, which in the spin chain frame may be exactly computed by means of Bethe ansatz techniques.
There exist several schemes to derive physical boundary $S$-matrices, such as the bootstrap method developed in [1, 5, 12, 20], the ‘non-linear integral equation’ (NLIE) technique (see e.g. [19, 12, 20]), and the ‘physical’ Bethe ansatz formulation [17, 18]. However, the most direct means is arguably provided by the ‘bare’ Bethe ansatz approach (see for instance [4, 11, 8, 9, 10]). In this study the derivation of the generic boundary $S$-matrix is based on the ‘bare’ Bethe equations [21, 22], and this is the first time that a direct computation starting from the microscopic Bethe ansatz equations is achieved for the generic reflection matrix.

The crucial observation is that in the special case under consideration the Bethe equations reduce to an elegant and quite familiar form after assuming suitable boundary parametrizations parallel to the ones of [5]. Specifically, the Bethe ansatz equations acquire a form similar to the one of the open XXZ spin chain with two diagonal boundaries (see also [23] for relevant comments). We are hence able to compute the reflection amplitudes in a simple fashion analogous to the diagonal case, and this is actually one of the main advantages of the approach adopted here. It is worth pointing out that in the case where two generic boundaries are implemented the entailed Bethe ansatz equations are rather involved [24, 25] without really offering any additional information as far as the boundary scattering is concerned. Note that in [25] the spectrum in the most generic case with general free bulk and boundary parameters is obtained. Given the complexity of the Bethe ansatz type equations in the most general case it is clear that our choice of simpler boundary conditions is quite practical rendering the relevant computations considerably more tractable. Moreover, in the particular case we assume here there exists a non-local boundary conserved quantity, at first order, contrary to the generic case. Comments regarding its role on the spectrum and Bethe ansatz equations, and on its relevance to $S^z$ are also presented. Our results as expected coincide with the ones obtained by Ghoshal-Zamolodchikov for the sine-Gordon model with ‘free’ boundary conditions [5]. Relevant results from the NLIE point of view were also derived in [20].

2 The open XXZ spin chain

We recall the $R$-matrix associated to the the spin $\frac{1}{2}$ XXZ model, which is a solution of the Yang-Baxter equation [4, 26],

$$R_{12}(\lambda_1 - \lambda_2) R_{13}(\lambda_1) R_{23}(\lambda_2) = R_{23}(\lambda_2) R_{13}(\lambda_1) R_{12}(\lambda_1 - \lambda_2).$$

(2.1)

The $R$-matrix has the following form:

$$R(\lambda) = \begin{pmatrix} \sinh[\mu(\lambda + \frac{i}{2} + i\sigma^z)] & \sinh(i\mu) e^{i\lambda} \sigma^- \\ \sinh(i\mu) e^{-i\lambda} \sigma^+ & \sinh[\mu(\lambda + \frac{i}{2} - i\sigma^z)] \end{pmatrix}$$

(2.2)
it is convenient for our purposes here to set $\mu = \frac{\pi}{\nu}$.

The transfer matrix of the open spin chain is defined as [27]

$$t(\lambda) = \text{tr}_0 \left\{ M \ K^+(\lambda) \ T(\lambda) \ K^-(\lambda) \ \hat{T}(\lambda) \right\}$$

$$T(\lambda) = R_{0N}(\lambda) \ldots R_{01}(\lambda), \quad \hat{T}(\lambda) = R_{10}(\lambda) \ldots R_{N0}(\lambda)$$  \hspace{1cm} (2.3)

where $M = \text{diag}(q, q^{-1})$, $q = e^{i\mu}$ and $K^\pm$ are solutions of the reflection equation [28]:

$$R_{12}(\lambda_1 - \lambda_2) \ K_1(\lambda_1) \ R_{21}(\lambda_1 + \lambda_2) \ K_2(\lambda_2) = K_2(\lambda_2) \ R_{12}(\lambda_1 + \lambda_2) \ K_1(\lambda_1) \ R_{21}(\lambda_1 - \lambda_2)$$  \hspace{1cm} (2.4)

The general solution $K(\lambda)$ is a $2 \times 2$ matrix with entries [5, 7]:

$$K_{11}(\lambda) = \sinh[\mu(-\lambda + i\xi)]e^{\mu\lambda}, \quad K_{22}(\lambda) = \sinh[\mu(\lambda + i\xi)]e^{-\mu\lambda}$$

$$K_{12}(\lambda) = \kappa q^\theta \sinh(2\mu\lambda), \quad K_{21}(\lambda) = \kappa q^{-\theta} \sinh(2\mu\lambda).$$  \hspace{1cm} (2.5)

We shall henceforth consider $K^+ = I$, and $K^- = K(\lambda)$ defined in (2.5). This particular choice of boundary conditions is compatible with certain constraints imposed upon the left and right boundary parameters [22, 21, 29, 30]. We are eventually left with three free boundary parameters associated only to the right boundary. As a matter of fact the parameter $\theta$ may be removed via a simple gauge transformation, as also happens in [5], where a $K$-matrix with $\theta = 0$ is assumed for simplicity, but without loss of generality. Using the fact that the quantity $(T \ K^- \ \hat{T})$ satisfies the reflection equation one can show that the transfer matrix (2.3) provides a family of commuting operators [27]:

$$\left[ t(\lambda), \ t(\lambda') \right] = 0.$$  \hspace{1cm} (2.6)

Evaluating the eigenvalues of the $K$-matrix will be particularly useful for both adopting an appropriate parametrization for our Bethe ansatz equations as well as for comparing effectively with the Ghoshal-Zamolodchikov result. Before writing down the $K$-matrix eigenvalues it will be useful to introduce some notation:

$$\frac{e^{-i\mu\xi}}{2\kappa} = i \cosh[i\mu(\beta^- + \gamma^-)], \quad \frac{e^{i\mu\xi}}{2\kappa} = i \cosh(i\mu\zeta), \quad p^\pm = \frac{1}{2}(\beta^- + \gamma^- \pm \zeta)$$  \hspace{1cm} (2.7)

compare also with the parametrization used e.g. in [20, 21, 22, 29, 30]. Such a parametrization is also quite natural from the point of view of boundary Temperley-Lieb algebras [31]. The latter formulas (2.7) lead to the following relations among the boundary parameters:

$$\frac{\cosh(i\mu\xi)}{2i\kappa} = \cosh(i\mu p^+) \cosh(i\mu p^-), \quad \cosh^2(i\mu p^+) + \cosh^2(i\mu p^-) = 1 - \frac{1}{4\kappa^2}$$  \hspace{1cm} (2.8)

(c.f. see similar parametrization in [5], and section 3). ‘Renormalized’ physical boundary parameters will be ultimately identified with the boundary parameters of [5], as will become
transparent in section 3. The parameter $\theta$ is hidden is the sum $\beta^- + \gamma^-$ (see e.g. [22, 29]), but in any case we shall consider it henceforth to be zero for simplicity. After diagonalizing the $K$-matrix we obtain the two eigenvalues:

$$\varepsilon_{1,2}(\lambda) = -2i\kappa \sinh[\mu(\lambda \pm ip^+)] \sinh[\mu(\lambda \pm ip^-)].$$  (2.9)

It is worth stressing that the Bethe ansatz equations provide essentially the eigenvalues of the physical boundary $S$-matrix –with ‘renormalized’ boundary parameters. Nevertheless, the structure of the boundary $S$-matrix is a priori known due to the fact that is a generic solution of the reflection equation (2.4). The objective now is to obtain the overall physical factor in front of the boundary $S$-matrix; this will be achieved in the subsequent sections by means of the Bethe ansatz approach.

### 2.1 Bethe ansatz equations

The spectrum and Bethe ansatz equations for the spin $\frac{1}{2}$ XXZ chain with general boundary conditions were derived in [22, 21], whereas in [29, 30, 32] the spectrum and Bethe ansatz equations obtained for various representations. We shall focus here on the spin $\frac{1}{2}$ case, the spectrum in this case is given by:

$$\Lambda(\lambda) = K_1^+(0|\lambda) K_1^- (p^+|\lambda) \sinh^{2N}[\mu(\lambda + i)] \prod_{j=1}^{M} \frac{\sinh[\mu(\lambda + \lambda_j)] \sinh[\mu(\lambda - \lambda_j - i)]}{\sinh[\mu(\lambda + \lambda_j + i)] \sinh[\mu(\lambda - \lambda_j)]} \sinh[\mu(\lambda + \lambda_j + 2i)] \sinh[\mu(\lambda - \lambda_j + i)] \sinh[\mu(\lambda + \lambda_j + i)] \sinh[\mu(\lambda - \lambda_j)].$$  (2.10)

where $K_{1,4}^\pm$, for the particular choice of boundary conditions are given by (see also Appendix in [29] for the explicit definitions of $K_{1,4}^\pm$)

$$K_1^-(p^+|\lambda) = -2i\kappa e^{\mu \lambda} \sinh[\mu(\lambda - ip^-)] \sinh[\mu(\lambda + ip^+)],$$

$$K_4^-(p^+|\lambda) = -2i\kappa e^{\mu \lambda} \sinh[\mu(\lambda + ip^- + i)] \sinh[\mu(\lambda + ip^+ + i)] \frac{\sinh(2\mu \lambda)}{\sinh(i\mu)}$$

$$K_1^+(0|\lambda) = e^{-\mu \lambda} \frac{\sinh[2\mu(\lambda + i)]}{\sinh[2\mu(\lambda + i)]}, \quad K_4^+(0|\lambda) = e^{-\mu \lambda} \frac{\sinh(i\mu)}{\sinh[2\mu(\lambda + i)]}. $$  (2.11)

The Bethe ansatz equations arise as necessary constraints such that certain ‘unwanted’ terms appearing in the eigenvalue expression are vanishing. They guarantee also analyticity of the spectrum, and are written in the familiar form:

$$e_{2p-1}^{-1}(\lambda_i; \mu) e_{2p+1}^{-1}(\lambda_i; \mu) g_1(\lambda_i; \mu) e_1^{2N+1}(\lambda_i; \mu) = -\prod_{j=1}^{M} e_2(\lambda_i - \lambda_j; \mu) e_2(\lambda_i + \lambda_j; \mu),$$  (2.12)
where we define
\[ e_n(\lambda; \mu) = \frac{\sinh[\mu(\lambda + \frac{i n}{2})]}{\sinh[\mu(\lambda - \frac{i n}{2})]}, \quad g_n(\lambda; \mu) = \frac{\cosh[\mu(\lambda + \frac{i n}{2})]}{\cosh[\mu(\lambda - \frac{i n}{2})]}. \] (2.13)

The similarity of the Bethe ansatz equations (2.12) with the Bethe equations of the case with two non-trivial diagonal boundaries is indeed noticeable (see e.g. [8, 9, 27]). This is a crucial observation, which will considerably simplify the derivation of the generic boundary S-matrix.

In [22, 29, 30] the spectrum and Bethe ansatz equations were derived starting from a particular reference state (the analogue of ‘spin up’ state), however it was shown in [33] that a second reference state exists (‘spin down’), providing another set of Bethe ansatz equations similar to (2.12), but with \( p^\pm \rightarrow -p^\pm \), guaranteeing also the completeness of the spectrum. For a detailed analysis on the completeness of the spectrum using both reference states we refer the interested reader e.g. to [24, 33, 32]. It should be stressed that the existence of the two sets of Bethe ansatz equations together with the similarity of (2.12) with the Bethe equations in the purely diagonal case are the key elements in deriving the exact boundary S-matrix.

Interestingly in the case we are considering here, where the left boundary is trivial, there exist a conserved quantity, which is somehow the analogue of the spin \( S^z \) of the diagonal case [27]. The integer \( M \) appearing in the Bethe ansatz equations is actually associated to the spectrum of the conserved quantity (see e.g. [29]). Let us briefly recall the structure of the boundary conserved quantity (boundary non-local charge), which is defined as [34, 35, 16, 36]:
\[ Q^{(N)} = q^{-\frac{1}{2}+\theta} K^{(N)} E^{(N)} + q^{\frac{1}{2}-\theta} K^{(N)} F^{(N)} = \frac{e^{-i\mu\xi}}{2\kappa \sinh(i\mu)} (K^{(N)})^2. \] (2.14)

\( K^{(N)}, E^{(N)} \) and \( F^{(N)} \) are \( N \) coproducts of the quantum algebra \( U_q(sl_2) \) [37] i.e.
\[ K^{(N)} = \bigotimes_{n=1}^{N} K_n, \quad E^{(N)} = \sum_{n=1}^{N} K^{-1} \otimes \ldots K^{-1} \otimes X \otimes K \ldots \otimes K, \]
\[ X \in \{ E, F \}. \] (2.15)

For the spin \( \frac{1}{2} \) representation in particular (\( K \rightarrow q^\sigma^z, \ E \rightarrow \sigma^+, \ F \rightarrow \sigma^- \)) we have:
\[ Q^{(N)} = -\frac{i}{\sinh(i\mu)} \cosh[i\mu(\beta^- + \gamma^- - 2S)]. \] (2.16)

The one-site \( Q^{(N)} \) operator becomes a \( 2 \times 2 \) matrix and the eigenvalues of \( S \) are \( \pm\frac{1}{2} \). For \( N = 2 \), \( S \) has 0, \( \pm 1 \) as eigenvalues, with 0 being a doubly degenerate eigenvalue, and so on. It is thus quite clear that the operator \( S \) behaves similarly to \( S^z \). Details on the diagonalization of the operator \( Q^{(N)} \) for the spin \( \frac{1}{2} \) representation can be found in [36].
From the asymptotic behaviour of the open transfer matrix (2.3) and the asymptotics of the spectrum (2.10) the following fundamental formula emerges (for more details on this matter we refer the reader to [29]):

\[ Q^{(N)}_{\varepsilon} = -\frac{i}{\sinh(i\mu)} \cosh[i\mu(\beta^- + \gamma^- + 2M - N)]. \]  

(2.17)

the subscript \( \varepsilon \) in the latter expression stands for the eigenvalue. In this case \( M \) is not fixed, as opposed to the generic case, but is associated to the spectrum of the operator \( Q^{(N)} \). From equation (2.17) the upper (lower) bounds of \( M \) (integer) are identified from the spectrum of the non-local operator \( Q^{(N)} \), indeed:

\[ M = \frac{N}{2} - S_{\varepsilon}. \]

(2.18)

The similarity of the latter relation with the one appearing in the case of diagonal boundaries is noticeable; indeed in the diagonal case \( S \) is simply replaced by \( S^z \) in (2.18). In fact, the symmetry in this case is effectively \( U(1) \) resembling the case of purely diagonal boundaries (see also [38]). In general the identification of the spectrum of the boundary non-local charge \( Q^{(N)} \) is an intriguing problem and particular cases have been analyzed in [36, 38].

Finally, comparing the eigenvalues of (2.16) for an one particle state \((N \text{ odd})\) –the eigenvalues for \( S \) are then \( \pm \frac{1}{2} \)– with (2.17) we conclude that:

\[ M = \frac{N - 1}{2}. \]

(2.19)

This corresponds to the state with \( S \) eigenvalue \( \frac{1}{2} \), the other state with eigenvalue \( -\frac{1}{2} \)–above the equator– may be obtained via a ‘duality’ transformation on the boundary parameters, as it happens in the diagonal case [27]; this will be demonstrated however in the subsequent section.

3 The boundary \( S \)-matrix

As already mentioned in the introduction the physical boundary \( S \) matrix describes the reflection of a particle-like excitation (kink), displayed by the XXZ spin chain, with the boundary. Our main objective here is the computation of the overall physical factor in front of the boundary \( S \)-matrix, which contains significant information regarding the existence of boundary bound states.

The physical boundary \( S \)-matrices are denoted henceforth \( K^\pm \). We define the boundary \( S \)-matrices \( K^\pm \) by the quantization condition [11, 8]

\[ \left( e^{i2\rho(\bar{\lambda})N} K^+ K^- - 1 \right) |\bar{\lambda}\rangle = 0. \]

(3.1)
\( \tilde{\lambda} \) is the rapidity of the ‘hole’ – particle-like excitation, and \( p(\tilde{\lambda}) \) is the momentum of the hole, which will be formally defined shortly. Recall that both bulk and boundary scattering may be validly evaluated in the thermodynamic limit \( (N \to \infty) \) \[2, 8, 9, 10\]. In this limit as is well known the so called string hypothesis is valid \[2\]. Based on this hypothesis (see e.g. \[2\]) it was shown that the ground state for the antiferromagnetic spin chain, which we study here, consists of a ‘sea’ of real (1-string) Bethe roots, and a particle like excitation is simply a ‘hole’ in this uniformly distributed sea of real roots \[2, 4, 11\]. For the Bethe ansatz state with one hole of rapidity \( \tilde{\lambda} \) in the sea, the counting function – obtained after taking the log of Bethe ansatz equations – is

\[
    h(\lambda) = \frac{1}{2\pi} \left\{ (2N + 1)q_1(\lambda; \mu) + r_1(\lambda; \mu) - q_{2p+1}(\lambda; \mu) - q_{2p-1}(\lambda; \mu) \right. \\
    \left. - \sum_{j=1}^{M} [q_2(\lambda - \lambda_j; \mu) + q_2(\lambda + \lambda_j; \mu)] \right\},
\]

where

\[
    q_n(\lambda; \mu) = \pi + i \log[e_n(\lambda; \mu)], \quad r_n(\lambda; \mu) = i \log[g_n(\lambda; \mu)].
\]

Note the striking similarity of the latter formula (3.2) with the one obtained in the case of two diagonal boundaries \[8, 9\]. This is a key point, as already mentioned, allowing us to proceed with a simplified derivation of the boundary S-matrix. The only difference with the fully diagonal case is that now both terms depending on \( p^\pm \) are assigned to the right boundary, otherwise we proceed exactly as in the diagonal case (see e.g. \[8, 9\]).

Since we are considering the thermodynamic limit it is clear that all the sums in the expressions above turn into integrals, and also we need a density to describe the corresponding states, see e.g. \[2, 8, 9, 10\] for details. The Fourier transform of the density \( \sigma_s(\lambda) = \frac{1}{N} \frac{d\hat{h}(\lambda)}{d\lambda} \) describing the one-hole state is given by

\[
    \hat{\sigma}_s(\omega) = 2\dot{\epsilon}(\omega) + \frac{1}{N} \frac{\hat{a}_2(\omega; \mu)}{1 + \hat{a}_2(\omega; \mu)} (e^{i\omega\tilde{\lambda}} + e^{-i\omega\tilde{\lambda}}) \\
    + \frac{1}{N} \frac{1}{1 + \hat{a}_2(\omega; \mu)} \left[ \hat{a}_1(\omega; \mu) + \hat{a}_2(\omega; \mu) + \hat{b}_1(\omega; \mu) - \hat{a}_{2p+1}(\omega; \mu) - \hat{a}_{2p+1}(\omega; \mu) \right],
\]

where we define the following Fourier transforms (recall that \( |e^{i\mu}| = 1 \) and \( \mu = \frac{\pi}{\nu} \)):

\[
    \hat{a}_n(\omega; \mu) = \frac{\sinh[(\nu - n)\frac{\omega}{2}]}{\sinh(\frac{\omega}{2})} \quad 0 < n < 2\nu
\]

\[
    \hat{b}_n(\omega; \mu) = -\frac{\sinh(\frac{\omega}{2})}{\sinh(\frac{\omega}{2})} \quad 0 < n < \nu
\]
and

\[ \hat{\epsilon}(\omega) = \frac{\hat{a}_1(\omega; \mu)}{1 + \hat{a}_2(\omega; \mu)} = \frac{1}{2 \cosh \left( \frac{\omega}{2} \right)}, \]  

(3.7)

\( \epsilon(\tilde{\lambda}) \) corresponds also to the energy of the particle-like excitation.

The boundary matrix \( K^− \), of the generic form (2.5), has two eigenvalues \( k_{1,2} \). The left boundary matrix is trivial

\[ K^+(\tilde{\lambda}) = k_0(\tilde{\lambda}) \mathbb{I}. \]  

(3.8)

From the density (3.4) and the quantization condition (3.1) we explicitly derive the quantities \( k_0, k_{1,2} \). Indeed taking into account that

\[ \epsilon(\lambda) = \frac{1}{2\pi} \frac{dp(\lambda)}{d\lambda} \]  

(3.9)

and comparing (3.4) with the quantization condition (3.1) we obtain the first eigenvalue \( k_1 \) (see e.g. [8, 9] for more details). It is worth noting that the momentum is not a conserved quantity anymore, however the momentum of the particle-like excitation may be formally defined via (3.9) (see also [8, 9]).

To obtain the second eigenvalue \( k_2 \) we apply the argument of [27, 9] for the diagonal case i.e. implement a ‘duality’ transformation on the boundary parameters such that \( p^\pm \rightarrow -p^\pm \). This is actually the equivalent of deriving the Bethe ansatz equations starting from the second reference state (‘spin down’) [33, 32]. The explicit expression for the eigenvalue \( k_1 \) is given by:

\[ k_1(\tilde{\lambda}, p^+, p^-) = \frac{-2i\kappa}{\pi^2} \cosh \left[ \frac{\pi}{\nu - 1} \left( \tilde{\lambda} - i \frac{\nu - 2}{2} p^+ \right) \right] \cosh \left[ \frac{\pi}{\nu - 1} \left( \tilde{\lambda} - i \frac{\nu - 2}{2} p^- \right) \right] k(\tilde{\lambda}, p^+ p^-), \]

where we define

\[ k(\tilde{\lambda}, p^+, p^-) = k_0(\lambda) k_1(\tilde{\lambda}, p^+) k_1(\tilde{\lambda}, p^-) \]  

(3.10)

and

\[ k_0(\tilde{\lambda}) = \exp \left\{ 2 \int_0^\infty \frac{d\omega}{\omega} \sinh \left( 2i\omega \tilde{\lambda} \right) \frac{\sinh \left( \frac{3\omega}{2} \right) \sinh \left( \frac{(\nu - 2)\omega}{2} \right)}{\sinh (2\omega) \sinh \left( \frac{(\nu - 1)\omega}{2} \right)} \right\} \]

\[ k_1(\tilde{\lambda}, x) = \frac{\pi (-2i\kappa)^{-\frac{1}{2}}}{\cosh \left[ \frac{\pi}{\nu - 1} \left( \tilde{\lambda} - i \frac{\nu - 2}{2} x \right) \right]} \exp \left\{ -2 \int_0^\infty \frac{d\omega}{\omega} \sinh \left( 2i\omega \tilde{\lambda} \right) \frac{\sinh \left( \frac{(\nu - 2x - 1)\omega}{2} \right)}{2 \sinh \left( \frac{(\nu - 1)\omega}{2} \right) \cosh(\omega)} \right\}. \]  

(3.11)
The latter expressions may be written as infinite products of $\Gamma$-functions, so the comparison with the results in [5] is easier

\[ k_0(\lambda) = \prod_{n=0}^{\infty} \frac{\Gamma\left(\frac{1}{\nu-1}(-2i\tilde{\lambda} + 4n + 3) + 1\right) \Gamma\left(\frac{1}{\nu-1}(-2i\tilde{\lambda} + 4n + 1)\right)}{\Gamma\left(\frac{1}{\nu-1}(2i\tilde{\lambda} + 4n + 3) + 1\right) \Gamma\left(\frac{1}{\nu-1}(2i\tilde{\lambda} + 4n + 1)\right)} \times \frac{\Gamma\left(\frac{1}{\nu-1}(2i\tilde{\lambda} + 4n + 1)\right) \Gamma\left(\frac{1}{\nu-1}(2i\tilde{\lambda} + 4n + 4)\right)}{\Gamma\left(\frac{1}{\nu-1}(-2i\tilde{\lambda} + 4n + 1)\right) \Gamma\left(\frac{1}{\nu-1}(-2i\tilde{\lambda} + 4n + 4)\right)} \right] \]

\[ k_1(\lambda, x) = \sqrt{\frac{2\pi}{\kappa}} \prod_{n=0}^{\infty} \frac{\Gamma\left(\frac{1}{\nu-1}(-i\tilde{\lambda} + 2n - \frac{1}{2}(\nu - 2x)\right) + \frac{1}{2}) \Gamma\left(\frac{1}{\nu-1}(-i\tilde{\lambda} + 2n + \frac{1}{2}(\nu - 2x)\right) + \frac{1}{2})}{\Gamma\left(\frac{1}{\nu-1}(i\tilde{\lambda} + 2n + 2 - \frac{1}{2}(\nu - 2x)\right) + \frac{1}{2}) \Gamma\left(\frac{1}{\nu-1}(i\tilde{\lambda} + 2n + 2 + \frac{1}{2}(\nu - 2x)\right) + \frac{1}{2})} \times \frac{\Gamma\left(\frac{1}{\nu-1}(i\tilde{\lambda} + 2n + 1 - \frac{1}{2}(\nu - 2x)\right) + \frac{1}{2}) \Gamma\left(\frac{1}{\nu-1}(-i\tilde{\lambda} + 2n + 1 + \frac{1}{2}(\nu - 2x)\right) + \frac{1}{2})}{\Gamma\left(\frac{1}{\nu-1}(-i\tilde{\lambda} + 2n + 1 - \frac{1}{2}(\nu - 2x)\right) + \frac{1}{2}) \Gamma\left(\frac{1}{\nu-1}(-i\tilde{\lambda} + 2n + 1 + \frac{1}{2}(\nu - 2x)\right) + \frac{1}{2})} \right]. \]

After implementing the ‘duality’ transformation on the boundary parameters $p^\pm \to -p^\pm$ in Bethe ansatz equations (2.12) we obtain the second eigenvalue $k_2$ of the reflection matrix with:

\[ k_2(\tilde{\lambda}, p^+, p^-) = \frac{\cosh\left[\frac{\pi}{\nu-1}\left(\tilde{\lambda} + \frac{i}{2}(\nu - 2p^+)\right)\right]}{\cosh\left[\frac{\pi}{\nu-1}\left(\tilde{\lambda} - \frac{i}{2}(\nu - 2p^+)\right)\right]} \]

\[ k_1(\tilde{\lambda}, p^+, p^-) = \frac{\cosh\left[\frac{\pi}{\nu-1}\left(\tilde{\lambda} + \frac{i}{2}(\nu - 2p^-)\right)\right]}{\cosh\left[\frac{\pi}{\nu-1}\left(\tilde{\lambda} - \frac{i}{2}(\nu - 2p^-)\right)\right]} \]

compare also the latter relation with (2.9); the appearance of renormalized parameters becomes now apparent. Notice that the $p^\pm$ dependent term, expressed as a product of two $k_1$ functions (3.13), is ‘double’ compared to the diagonal case. In the diagonal limit only one of the functions $k_1(\lambda, p^\pm)$ survives, while the other one becomes unit (see also relevant discussion in [5]). The diagonal case corresponds to what is called ‘fixed’ boundary conditions in [5].

Comparison between our findings and the results of [5] for ‘free’ boundary conditions, gives rise to the following identifications of bulk and boundary parameters. The Ghoshal-Zamolodchikov bulk coupling constant $\lambda$ is related to our coupling constant $\nu$ by $\lambda = \frac{1}{\nu - 1}$; the spectral parameter $u$ in [5] is related to our variable $\tilde{\lambda}$ by $u = -i\pi \tilde{\lambda}$. Note that the original parameter $\mu = \frac{\pi}{\beta}$ appearing in the $R$-matrix (2.2) is now renormalized to $\tilde{\mu} = \frac{\pi}{\nu - 1}$.

\[ \text{The coupling constant $\lambda$ is defined in [5] as $\lambda = \frac{8\pi}{\beta^2} - 1$, where $\beta$ is the familiar sine-Gordon bulk coupling constant.} \]
Recall also that the boundary parameters in \([5]\) \((\eta, \vartheta)\) and \((k, \xi')\) (we denote \(\xi'\) the Ghoshal-Zamolodchikov \(\xi\)-parameter to distinguish it from the bare parameter of \((2.5)\)) satisfy \([5]\):

\[
\cos(\eta) \cosh(\vartheta) = -\frac{1}{k} \cos(\xi'), \quad \cos^2(\eta) + \cosh^2(\vartheta) = 1 + \frac{1}{k^2}, \tag{3.15}
\]

which are similar to the constraints \((2.8)\) among the ‘bare’ boundary parameters. Finally, the following identifications among the boundary parameters are valid:

\[
\vartheta = \frac{i\pi (\nu - 2p^+)}{2(\nu - 1)}, \quad \eta = \frac{\pi (\nu - 2p^-)}{2(\nu - 1)}, \quad \xi' = \frac{\pi (\nu - 2\xi)}{2(\nu - 1)}, \quad k = -2i\kappa \tag{3.16}
\]

see also relevant formulas in \([20]\). With this we conclude our derivation of the generic physical boundary \(S\)-matrix, which naturally coincides with the general reflection matrix found in \([5]\) associated to ‘free’ boundary conditions.

4 Discussion

Let us summarize the main findings of this investigation. Our main objective was the derivation of the exact generic boundary \(S\)-matrix for the open XXZ chain. This was achieved by means of the ‘bare’ Bethe ansatz approach. More precisely, we considered a particular case of the open spin chain, with a generic right boundary and a trivial left one. Then assuming appropriate boundary parametrizations we were able to write down a simple and familiar form of the Bethe ansatz equations, similar to the Bethe equations of the XXZ chain with two diagonal boundaries.

The simple form of the Bethe ansatz equations has been a crucial point in our analysis. This together with the existence of a second set of Bethe ansatz equations, obtained via a duality transformation on the boundary parameters, facilitated the derivation of the two eigenvalues of the generic physical boundary \(S\)-matrix. Moreover, we were able to associate the spectrum of the boundary conserved quantity \(Q^{(N)}\) with the spectrum of the transfer matrix, and we showed that \(Q^{(N)}\) plays essentially a role analogous to \(S_z\) in the fully diagonal case. We identified the overall physical factor in front of the boundary \(S\)-matrix of the form \((2.5)\), but with renormalized boundary parameters \((3.16)\), which are basically the parameters used in \([5]\). Thus the reflection matrix derived in \([5]\) was fully recovered.

Similar results may be deduced in the attractive regime where breathers (bound states) are also present \([10]\). In this case the computation of the generic breather boundary \(S\)-matrix goes along the same lines as in \([10]\). The \(n^\text{th}\) breather boundary \(S\)-matrix is then given by:

\[
S_b^{(n)}(\tilde{\lambda}, p^+, p^-) = S_0^{(n)}(\tilde{\lambda}) S_1^{(n)}(\tilde{\lambda}, p^+) S_1^{(n)}(\tilde{\lambda}, p^-) \tag{4.1}
\]
where explicit formulas for $S_0^{(n)}(\tilde{\lambda})$, $S_1^{(n)}(\tilde{\lambda}, x)$ are presented in [10]. Expression (4.1) is analogous to the $n^{th}$ breather reflection matrix in the sine-Gordon model obtained in [39] for ‘free’ boundary conditions. In the diagonal limit one of the $S_1^{(n)}$ terms in (4.1) becomes unit, exactly as in the solitonic reflection matrix, and the result of [10] is recovered corresponding to the ‘fixed’ boundary conditions in [39]. It is clear that such a computation immediately provides the reflection matrix for the fundamental particle in sinh-Gordon. We recall that the lightest breather ($n = 1$) in sine-Gordon corresponds essentially to the fundamental particle of the sinh-Gordon model, provided that the coupling constant $\beta \rightarrow i\beta$.

Another problem which may be treated in the same spirit is the derivation of the generic reflection matrix for the XXZ chain in the non-critical regime. In this case the boundary $S$-matrix will be expressed in terms of $\Gamma_q$-functions, see e.g [9, 40] for diagonal boundaries only. Finally an interesting direction to pursue is the computation of generic boundary $S$-matrices in the context of higher spin open XXZ chains. Specifically, the spin-1 case [17, 41] is of particular significance given its relation to the super-symmetric sine-Gordon model [42]. We hope to address these issues in forthcoming publications.

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