Uniqueness of Solutions for Certain Markovian Backward Stochastic Differential Equations

Coskun Cetin*

October 28, 2012

Abstract

This paper considers the problem of uniqueness of the solutions to a class of Markovian backward stochastic differential equations (BSDEs) which are also connected to certain nonlinear partial differential equation (PDE) through a probabilistic representation. Assuming that there is a solution to the BSDE or to the corresponding PDE, we use the probabilistic interpretation to show the uniqueness of the solutions, and provide an example of a stochastic control application.

Key Words: Markovian BSDEs, quasilinear PDEs, uniqueness of solutions

AMS Subject Classification: 60H10, 49J20, 43E20, 65C05

1. Introduction

In this paper, we study a class of decoupled forward-backward stochastic differential equations (FBSDEs) which have a Markovian structure of the following form:

\begin{align}
  dX(t) &= \mu(t, X(t))dt + \sigma(t, X(t))dW(t), \quad 0 \leq t \leq T \\
  dY(t) &= -F(t, X(t), Y(t), Z(t))dt + Z(t)dW(t), \quad 0 \leq t \leq T \\
  X(0) &= x_0; \quad Y(T) = g(X(T))
\end{align}

where the forward process $X$ has a unique solution in a probability space $(\Omega, \mathcal{F}, P)$, the random variable $Y(T) = g(X(T))$ is integrable and the driver of the backward process $Y$, $F(t, x, y, z)$, is quadratic in $z$. Due to such a growth condition on $z$, these BSDEs are called quadratic BSDEs or "BSDEs with quadratic growth" in the literature. Moreover, due to the Markovian nature of formulation, the FBSDEs of the form (1) are known to be related to certain quasilinear parabolic partial differential equations (PDEs).

After the first existence-uniqueness result for nonlinear BSDEs with Lipschitz coefficients was given by Pardoux and Peng (1990), FBSDEs and especially Markovian BSDEs have appeared in many application areas including mathematical finance, stochastic optimal control

*CSU, Department of Mathematics and Statistics, 6000 J St., Sacramento, CA 95819. Ph: (916) 278-6221. Fax: (916) 278-5586. Email: cetin@csus.edu
and analysis of nonlinear PDEs. The existence-uniqueness results for more general BSDEs were provided by Mao (1995), Lepeltier and San Martin (1997, 1998), Kobylanski (2000), Briand et al (2007), Briand and Hu (2006, 2008) and Fan and Jiang (2010), among others. Their connections with quasilinear PDE’s were first stated by Pardoux and Peng (1992), and Peng (1992) by generalising the Feynman-Kac representation of PDE’s. They also provided a uniqueness result when the coefficients involved were uniformly Lipschitz. Similar results and their connections with the stochastic control problems were also reported in El Karoui et al (1997), Ma and Yong (1999), Cetin (2005), Fuhrman et al (2006) and Richou (2011).

The existence results for the quadratic BSDEs usually assume strong growth, monotonicity, convexity/concavity or boundedness conditions on the driver or on the terminal value. The issue of uniqueness is much more complicated and usually requires stronger assumptions or some specific forms of the parameters. See Briand et al (2007), Fan and Jiang (2010) and Richou (2011) for a discussion of such special cases, and the other works in the literature. Our aim is to obtain the uniqueness results for a class of the Markovian BSDEs with quadratic growth where the solution $Y$ is bounded from below only. Such equations usually appear in the stochastic control problems, where the process $Y$ would yield the value function of a minimization problem over a suitable space of admissible controls. An application to perturbed linear-quadratic regulator (LQR) problem is provided in the last section.

The rest of the paper is organized as follows: The basic definitions and the notations of the paper are introduced in the subsection 1.1 below. A uniqueness result for solutions to a class of Markovian BSDEs is given in the section 2. The section 3 describes how such BSDEs can be used to study the properties of the solutions to some certain quasilinear PDEs which are also related to the stochastic optimal control problems where only the drift term of the state process is control-dependent.

### 1.1 Definitions and Notations

For simplicity, we consider the one-dimensional Euclidean space $\mathbb{R}$ even though most of the results hold for higher dimensions. For a given $T > 0$ and a probability space $(\Omega, F, P)$ where $F = \{F_t : 0 \leq t \leq T\}$ is the complete $\sigma-$algebra generated by a standard Brownian motion process $W$, we define the following spaces:

- $C^{p,q}([0,T])$: The space of all real-valued measurable functions $f : [0,T] \times \mathbb{R}$ such that $f(t, x)$ is $p$ (respectively, $q$) times continuously differentiable with respect to $t$ (respectively, $x$) where $p, q$ are non-negative integers.
- $L^p_F(\Omega)$: The space of $F_T$-measurable random variables $H$ such that $E[|H|^p] < \infty$.
- $L^\infty_F(\Omega)$: The space of $F_T$-measurable essentially bounded random variables.
- $L^p_f([0,T])$: The space of $F$-adapted processes $f$ such that $E[\int_0^T |f(t)|^p dt] < \infty$.
- $L^\infty_f([0,T])$: The space of $F$-adapted essentially bounded processes.
- $S^p_f([0,T])$: The space of $F$-adapted processes such that $E[\sup_{0\leq t \leq T} |f(t)|^p] < \infty$. 


The notation $E_t[.]$ will denote the conditional expectation $E[.|F_t]$. When the initial value of a process $X$ is given at time $t$, then $E^{x,t}[.]$ refers to $E[.]$ with $X_t = x$. For a deterministic function $h(t, x): [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$, the subscript notation denotes partial derivatives: $h_t(t, x) = \frac{\partial h}{\partial t}(t, x)$, $h_x(t, x) = \frac{\partial h}{\partial x}(t, x)$ and $h_{xx}(t, x) = \frac{\partial^2 h}{\partial x^2}(t, x)$. In particular, for functions or ODE’s of one variable $t$, dot (‘) designates the derivative with respect to $t$. For a function $v \in C^{1,2}([0, T] \times \mathbb{R})$, let $L$ denote the backward evolution operator associated with the forward diffusion process $X$ in (1):

$$Lv(s, x) = v_s(s, x) + \mu(s, x)v_x(s, x) + \frac{1}{2}\sigma^2 v_{xx}(s, x).$$

Then consider the PDE

$$Lv(t, x) + F(t, x, v, \sigma v_x) = 0$$

$$v(T, x) = g(x).$$

If $\exists c > 0$ such that $\sigma(t, x) \geq c$ for all $(t, x) \in [0, T] \times \mathbb{R}$, then the PDE (3) is called uniformly parabolic. Such PDEs are known to have unique classical or generalized (e.g. viscosity) solutions under certain regularity and growth conditions. When a PDE is associated with a stochastic control problem in the form of Hamilton-Jacobi-Bellman (HJB in short) PDE, a “guess” solution to the HJB PDE usually turns out to be the solution to the corresponding control problem, thanks to the availability of a relevant verification theorem. For a summary of known results and the assumptions on such verification theorems, see Fleming and Soner (2006, IV.4) or Yong and Zhou (1999). A verification theorem is often stated heuristically in applications to conclude that the solution to the control problem is also the unique solution to the corresponding PDE, in a suitable space of continuous functions. In this paper, our emphasis is on a probabilistic description and interpretation of such equations.

2. A Uniqueness Result for a Class of Markovian BSDEs

In this section, we first assume that the BSDE in (1) has a solution $(Y, Z)$ in $S^{1+}_{F_T} \times L^2_F$ in a probability space $(\Omega, \mathcal{F}, P)$. Even though an interpretation of the weak solutions of the state variable $X$ is relevant in the PDE formulation, we are going to stick to the strong existence-uniqueness in the reference space $(\Omega, \mathcal{F}, P)$, for the simplicity of the presentation. The following result which is a special case of the Bihari’s inequality will be useful in the specification of the assumptions and the proof of our main result. For a more general version, one can refer to Bihari (1956) or Mao (1995).

Lemma 1 (Bihari’s inequality) For $T > 0$, let $f(t)$ and $v(t)$ be two continuous functions on $[0, T]$. Moreover, let $\kappa : [0, \infty) \rightarrow [0, \infty)$ be a continuous and nondecreasing function such that $\kappa(x) > 0$ for $x > 0$ and $\int_0^t \frac{ds}{\kappa(s)} = \infty$. If $f(t) \leq \int_0^t v(s)\kappa(f(s))ds$ for all $t \in [0, T]$, then $f(t) = 0$ for all $t \in [0, T]$. 

3
Now, for \((t, x) \in [0, T) \times \mathbb{R}\), we consider the following form of the driver in (1):

\[
F(t, x, y, z) = f(t, x) + h(t, x)z - \lambda(t, y) - \frac{1}{2}H(t)z^2,
\]

where the real-valued continuous functions \(f, h\) and \(\lambda\) on \([0, T] \times \mathbb{R}\), and \(H : [0, T] \rightarrow \mathbb{R}\) are continuous. The motivation for the choice of such a driver comes from the stochastic optimal control applications where the control process appears only in the drift term. Here, the driver function \(F\) may neither be Lipshitz with respect to any of the variables, nor have a linear growth in any of them. Moreover, we neither impose any convexity/concavity assumption on \(f\) or \(\lambda\), nor an exponential moment condition on the terminal condition \(g\) or \(f(t, x)\). To the best of our knowledge, no existence or uniqueness result is known to cover the BSDEs with such general drivers even though some special cases were considered in Cetin (2005), Briand et. al (2007), Briand and Hu (2008) and Richou (2011).

**Condition 1**

(i) \(H\) is a positive and continuously differentiable function which is bounded away from zero.

(ii) \(f(t, x) \geq 0\) on \([0, T] \times \mathbb{R}\), and satisfies \(f(t, X) \in L^1_{\mathcal{F}}\) where \(X\) is as in (1) and .

(iii) the function \(\lambda\) is such that

\[
2|u - v| \cdot |u\lambda(t, M - \ln u/H(t)) - v\lambda(t, M - \ln v/H(t))| \leq \varphi(t) \kappa(|u - v|^2),
\]

for \(0 \leq t \leq T\) and \(0 < u, v \leq 1\), where the function \(\kappa\) satisfies the conditions given in Lemma 1 and \(\varphi\) is a continuous function.

(iv) the terminal condition \(g(x)\) is bounded from below such that \(g(X_T) \in L^1_{\mathcal{F}_T}\).

(v) \(h(t, x)\) is bounded on \([0, T] \times \mathbb{R}\)

(vi) there is a constant \(\gamma \in (0, 1)\) such that \(2H(t)f(t, x) - \frac{h^2(t, x)}{\gamma} \geq 0\), uniformly on \([0, T] \times \mathbb{R}\) and \(h(t, X) \in L^2_{\mathcal{F}}\).

We now state a technical lemma that will be needed in the proof of the main result of the paper.

**Lemma 2** Let \(0 < r \leq 1, 0 < \epsilon < e^{-r}\) and define a function \(\kappa^{s, r}(\cdot)\) as

\[
\kappa^{s, r}(x) = \begin{cases} 
    x(\ln(x^{-1}))^r, & 0 < x \leq \epsilon \\
    \kappa^{s, r}(\epsilon) + \dot{\kappa}^{s, r}(\epsilon)(x - \epsilon), & x > \epsilon
\end{cases}
\]

where \(\dot{\kappa}^{s, r}(\epsilon) = \lim_{x \to \epsilon^-} \kappa^{s, r}(x)\). Then \(\kappa^{s, r}(\cdot)\) is an increasing, non-negative and concave (differentiable) function satisfying

(i) \(\lim_{x \to 0^+} \kappa^{s, r}(x) = 0\).

(ii) For all \(0 \leq r \leq 1\) and \(0 < \epsilon < e^{-r}\), \(\exists \epsilon_1 \in (0, e^{-1})\) such that \(\kappa^{s, r}(\cdot) < \kappa^{s, 1}(\cdot)\), uniformly in \(x\).

(iii) there exists a constant \(C = C(\epsilon, r) > 0\) such that \(|x - y| \kappa^{s, r}(x) - \kappa^{s, r}(y) \leq C\kappa^{s, r}(|x - y|^2)\),

---

1Exponential moment conditions are too strong for many interesting FBSDEs where the terminal condition depends on an exponential martingale process, as in the mathematical finance applications.
for all $x, y$ in $(0, 1]$. In particular, $|x - y| |\kappa^{2r}(x) - \kappa^{2r}(y)| \leq C_1 |x - y|^2 \ln(|x - y|^2)$ also holds, with $C_1 \leq C(\epsilon, r)$.

(iv) $\int_0^x \frac{1}{\kappa^{2r}(x)} \, dx = \infty$, for $0 < r \leq 1$.

Proof. For simplicity, we write $\kappa = \kappa^{2r}$. Note that $\kappa(x)$ describes a line with a positive slope $\tilde{\kappa}(\epsilon) = [\ln(\epsilon^{-1})]^r \{1 - r/ \ln(\epsilon^{-1})\}$ for $x > \epsilon$. It is straightforward to see that $\tilde{\kappa}(x) > 0$ for $x \leq \epsilon$, $\tilde{\kappa}(x) < 0$ for $x > \epsilon$. Hence $\kappa(.)$ is a (strictly) increasing concave function and the result $\lim_{x \to 0^+} \kappa^{2r}(x) = 0$ in part (i) is a straightforward application of L'Hopital’s rule. Moreover, for fixed $r$, the expression $(\ln(x^{-1}))^p$ is strictly increasing in $p$ for $0 < x \leq \epsilon < e^{-1}$ and $r \leq p \leq 1$. So, the strict inequality $\kappa^{2r}(x) < \kappa^{1,1}(x)$ in (ii) holds for all $0 < x \leq \epsilon_1 = \epsilon < e^{-1}$. To ensure this inequality is also valid for larger $x$ and $\epsilon$ values, let $e^{-1} \leq \epsilon < \epsilon^{-r}$. Since $\kappa^{2r}(\epsilon)$ is a decreasing function, we have $1 - r = \kappa^{2r}(\epsilon^{-1}) \geq \kappa^{2r}(\epsilon)$. Then we can select $\epsilon_1$ such that $\kappa^{1,1}(\epsilon_1) \geq 1 - r$. For example, $0 < \epsilon_1 \leq e^{-r_2} < e^{-1}$ will do it, proving the part (ii).

To show (iii), without loss of generality, assume that $0 < x < y \leq 1$ and let $d = y - x > 0$. For $x \geq \epsilon$, we have $\kappa(y) - \kappa(x) = d \tilde{\kappa}(\epsilon)$, where $\tilde{\kappa}(\epsilon) = [\ln(\epsilon^{-1})]^r \leq C(\ln(d^{-2}))^r$, with $C = \max\{1, (\frac{\ln(\epsilon)}{2\ln(1-\epsilon)})^r\}$, depending on whether $d^2 \geq \epsilon$ holds\footnote{Since the expression $(\frac{\ln(\epsilon)}{2\ln(1-\epsilon)})^r$ is increasing in $r$ and decreasing in $\epsilon$, by choosing $\epsilon \approx \epsilon^{-r}$ for each $r$, $C$ can be selected to be $\lim_{\epsilon \to \epsilon^{-r}} \frac{\ln(\epsilon)}{2\ln(1-\epsilon)} = 1.0901$}.

So, $d |\kappa(y) - \kappa(x)| \leq C d^2 [\ln(d^{-2})]^r = C \kappa^{2r}(d^2)$.

For $x < \epsilon$, since $d < y$, there are two other possible cases, namely, $d^2 \leq x \leq y$ or $x < d^2 \leq y$. When $d^2 \leq x \leq y$, by mean value theorem, $\kappa(y) - \kappa(x) = d \tilde{\kappa}(z)$, for some $z$ between $x$ and $\min\{y, \epsilon\}$. But since $\tilde{\kappa}(.)$ is strictly decreasing on $(0, \epsilon]$ and $d^2 \leq x \leq z$, we obtain $\kappa(y) - \kappa(x) \leq d \tilde{\kappa}(d^2) \leq d [\ln(d^{-2})]^r$. For the case $x < d^2 \leq y$, by adding and subtracting $x [\ln(y^{-1})]^r$ to $\kappa(y) - \kappa(x)$, we get $0 < \kappa(y) - \kappa(x) = d [\ln(y^{-1})]^r + x \{[\ln(y^{-1})]^r - [\ln(x^{-1})]^r\}$, where $\ln(y^{-1}) \leq \ln(d^{-2}) < \ln(x^{-1})$. Then the inequality

$$
\kappa(y) - \kappa(x) < d [\ln(y^{-1})]^r \leq d [\ln(d^{-2})]^r
$$

easily follows, and hence, when $x < \epsilon$, (iii) holds with $C = 1$. Moreover, by part (ii), $\exists \epsilon_1 \in (0, e^{-1})$ such that $\kappa^{2r}(|x - y|^2) < \kappa^{1,1}(|x - y|^2)$ and hence the result follows for all $r \in (0, 1]$. The part (iv) is simply a result of part (ii): $\exists \epsilon_1 \in (0, e^{-1})$ such that, for all $0 < r < 1, \delta > 0$ and $0 < \epsilon < \epsilon^{-r}$,

$$
\int_0^\delta \frac{dx}{\kappa^{2r}(x)} \geq \int_0^\delta \frac{dx}{\kappa^{1,1}(x)} \geq \int_0^\delta \frac{-dx}{x \ln(x)} = \infty.
$$

Theorem 2 For $T > 0$ and $p \geq 1$, let the SDE in (1) have a unique solution $X$ in $L_p^T[0, T]$ with a.s. continuous paths. Moreover, let the assumptions (i)-(iv), and (v) or (v)\footnote{Condition 1 hold for the BSDE (10) with driver $F(t, x, y, z)$ as in [7]. Then the BSDE (10) has at most one solution $(Y, Z)$ in $S^1_{F_T} \times L^2_F$ such that $Y$ is bounded from below.} of Condition 1 hold for the BSDE (10) with driver $F(t, x, y, z)$ as in (7). Then the BSDE (10) has at most one solution $(Y, Z)$ in $S^1_{F_T} \times L^2_F$ such that $Y$ is bounded from below.
Proof. If the pair \((Y, Z)\) is such a solution, let \(M\) be a lower bound for \(Y\) and consider
the exponential transformation \(U(t) = \exp(-H(t)(Y(t) - M))\), for \(t \in [0, T]\). Clearly, \(U(.)\)
is bounded a.s. (between 0 and 1) and \(Y(t) = M - \ln U(t)/H(t)\) can be uniquely recovered from \(U(t)\).
The same idea applies to any solution \((Y', Z')\) to the equation \((17)\), and hence the problem reduces to showing the uniqueness of the solutions to the BSDE for the transformed process \(U(.)\). For simplicity of the notation, we take \(M = 0\). By Ito’s rule and \((11)\), a pair \((U, \Lambda)\) with \(\Lambda(t) \triangleq -H(t)UZ(t)\) and \(U(t) \triangleq \exp(-H(t)Y(t))\) satisfies the nonlinear BSDE
\[
dU(t) = \left\{ \frac{\dot{H}}{H} \ln U - H\lambda(t, -\ln U) + Hf(t, X) - \frac{h(t, X)}{U} \right\} U(t)dt + \Lambda(t)dW(t) \quad (7)
\]
with the terminal condition \(U(T) = \exp(-g(X(T)))\) and \(0 < U(.) \leq 1\) a.s. on \([0, T]\).

Now, let \((U_1, \Lambda_1)\) and \((U_2, \Lambda_2)\) be two (bounded) solutions to the BSDE \((7)\). Then, by
applying the Ito’s rule to \((U_1 - U_2)^2\) and rearranging the terms, \(P\)-a.s, the expression
\[
|U_1(t) - U_2(t)|^2 + \int_t^T (\Lambda_1 - \Lambda_2)^2(s)ds + \int_t^T 2H(s)f(s, X)(U_1 - U_2)^2(s)ds \quad (8)
\]
can be written as
\[
- \int_t^T 2(U_1 - U_2)(\Lambda_1 - \Lambda_2)(s)dW(s) - \int_t^T [\frac{\dot{H}}{H}(U_1 - U_2)(U_1 \ln U_1 - U_2 \ln U_2)](s)ds \quad (9)
\]
\[
+ \int_t^T 2H(U_1 - U_2)[U_1\lambda(s, -\ln U_1) - U_2\lambda(s, -\ln U_2)](s)ds + \int_t^T 2h(s, X)(\Lambda_1 - \Lambda_2)(U_1 - U_2)(s)ds,
\]
a.s. for \(0 \leq t < T\). Note that the integral \(\int_t^T 2H(s)f(s, X(s))(U_1 - U_2)(s)^2ds\) in \((8)\) is
non-negative a.s. by the positivity assumptions on \(H\) and \(f\), implying that both \((8)\) and \((9)\) are non-negative. In \((9)\), the first integral is a martingale, and by Lemma \(2\) with \(r = 1\) and \(\epsilon\) being sufficiently close to \(e^{-r}\), the expression \((U_1 - U_2)(U_1 \ln U_1 - U_2 \ln U_2)\) in the second
integral satisfies
\[
|U_1 - U_2||U_1 \ln U_1 - U_2 \ln U_2| \leq C \kappa^{1.1}(|U_1 - U_2|^2(.)).
\]
Moreover, thanks to the assumption \((3)\) for \(\lambda\), the third integral of \((9)\) is bounded by
\[\int_t^T H(s)|\varphi(s)|\kappa(|U_1(s) - U_2(s)|^2)ds,\] for some function \(\kappa\) as in Lemma \(1\). Finally, let the assumption (v) of Condition \(1\) hold and \(K\) be an upper bound for \(|h(t, x)|\). Then, applying the inequality \(2|ab| \leq \gamma a^2 + b^2/\gamma\) to the integrand of the last term of \((9)\) with \(\gamma = 2K\), we get
\[
\int_t^T 2h(s, X)(\Lambda_1 - \Lambda_2)(U_1 - U_2)(s)ds \leq K \int_t^T 2|\Lambda_1 - \Lambda_2|(U_1 - U_2)(s)ds
\]
\[
\leq \frac{1}{2} \int_t^T |\Lambda_1 - \Lambda_2|^2 ds + 2K^2 \int_t^T |U_1 - U_2|^2 ds.
\]
Therefore, taking the expected value of both (3) and (4), and combining with the terms above, the following upper bound for $E[|U_1 - U_2|^2(t)] + \int_t^T |\lambda_1 - \lambda_2|^2(s)ds$ is obtained:

$$2CE\int_t^T \left| \frac{\dot{H}}{H} \right| \kappa^\epsilon(x)(|U_1 - U_2|^2(s))ds + E\int_t^T |\varphi| H\kappa^\epsilon(|U_1 - U_2|^2)(s)ds + 2K^2E\int_t^T |U_1 - U_2|^2(s)ds$$

which is further bounded by $E\int_t^T v\xi(|U_1 - U_2|^2)(s)ds + 2K^2\int_t^T E|U_1 - U_2|^2(s)ds$ where $\xi(x) = \kappa^\epsilon(x) + \kappa(x)$ is concave and $v(t) = \max\{H\varphi(t), 2C\left| \frac{\dot{H}}{H} \right| \}$, satisfying the assumptions of the Lemma. Now, these bounds imply, in particular, that

$$E|U_1(t) - U_2(t)|^2 \leq E\int_t^T v(s)\xi(|U_1(s) - U_2(s)|^2)ds + 2K^2\int_t^T E|U_1(s) - U_2(s)|^2,$$

and hence by an appeal to the Gronwall’s and Jensen’s inequalities, we deduce

$$E|U_1 - U_2|^2(t) \leq e^{2K^2(T-t)} E\int_t^T v\xi(|U_1 - U_2|^2)(s)ds \leq \int_t^T v\xi(E|U_1 - U_2|^2)(s)ds.$$

Then, by Bihari’s inequality, for all $t$, $E|U_1(t) - U_2(t)|^2 = 0$ a.s., implying also that $U_1 = U_2$ a.s. and consequently $\Lambda_1 = \Lambda_2$ a.s. By transforming back to $(Y, Z)$, the result follows. The proof is similar when the assumption (v) of Condition 1 is replaced with the alternate condition (v)’. In that case, for $0 < \gamma < 1$, we again apply the inequality $2|ab| \leq \gamma a^2 + b^2/\gamma$ to $2(\Lambda_1 - \Lambda_2)h(U_1 - U_2)$ but instead with the parameters $a = (\Lambda_1 - \Lambda_2)$ and $b = h(U_1 - U_2)$; combine the resulting integrals with the terms of (3) and finally apply the Bihari’s and Jensen’s inequalities (without an appeal to the Gronwall’s inequality) to get the result.

**Remark 3** (a) Some examples for the function $\lambda$, satisfying the condition (iv) of the Theorem, are given below:

(i) Let $\lambda_1(t, u) = \alpha(t)u^r$, where $\alpha(.)$ is a (positive) continuous function and $0 < r \leq \epsilon$. The corresponding concave function $\kappa = \kappa_1$ in (3) of Lemma is actually given by (4) of Lemma. $\kappa_1(x) = \kappa^\epsilon(x)$ for some $0 < \epsilon < e^{-\gamma}$. Note that $\lambda_1(t, u)$ is also concave in $u$.

(ii) Let $\lambda_2(t, u) = e^{-\beta(t)u}$, where $\beta : [0, 1] \rightarrow [0, \infty)$ is a continuous function. Here, $\lambda_2(t, .)$ is a convex function and the corresponding concave function in (3) is $\kappa_2(x) = x$.

(iii) Consider $\lambda_3(t, u) = \lambda_1(t, u) + \lambda_2(t, u)$, as a sum of a concave and a convex function. Now, the corresponding $\kappa_3(.)$ would be taken as $\kappa_1(.) + \kappa_2(.)$ or $\max(\kappa_1, \kappa_2)$.

(iv) Yet another example where the function $\lambda$ is super-linear in $u$ is $\lambda_4(t, u) = Cu\ln(u^{-1})$ and $\kappa_4(.) = C\ln\ln(u^{-1})$. The reader is encouraged to find other interesting examples.

(b) The existence of a (global) solution under the assumptions of the Theorem (even with a bounded terminal condition and time-homogenous parameters) is not guaranteed in general.
Briand et al. (2007) provides an example where an exponential moment condition on the driver is violated. Similarly, the generalizations of the existence-uniqueness results for the BSDEs with linear growth in \( z \) (see e.g. Fan and Jiang, 2010 and the references there) are not directly applicable to the transformed BSDE (7) due to the conditions on the functions \( f(t, x) \) and \( \lambda(t, u) \).

(c) One can perhaps try a combination of the standard localization methods and the Picard iterations (which also appeared in some of the papers cited earlier) directly to the original BSDE (10) or to (7) for the existence part. However, it is not the direction we follow in this work. Instead, we will exploit their connections with PDEs of the form (3)-(4) in the next section by also providing an application to a stochastic optimal control problem.

3. The PDE and FBSDE Representations

In this section, our aim is to show the connections between the solution \((Y, Z)\) of the Markovian FBSDE system (11) and that of the quasilinear PDEs of the form (3)-(4). Note that we haven’t assumed any conditions on the drift and diffusion parameters of the forward process \( X \) so far (hence the PDE may be degenerate). Moreover, the conditions that we imposed on the driver and the terminal condition are more general than the standard regularity and growth conditions (e.g. Lipshitz condition, boundedness of the derivatives of the coefficients, linear growth etc.) for nonlinear PDEs to ensure the existence of a smooth solution to the PDE (3)-(4). So we may only expect to have a generalized solution (e.g. a viscosity solution) to such a PDE.

3.1 PDE Characterization of the Problem

By a heuristic application of the seminal result of Pardoux and Peng (1992) and the setup above, if a function \( V(t, x) \) is a smooth solution to the equation (3), then the pair \((Y_t^{s,x}, Z_t^{s,x})\) with \( Y_t = V(t, X_t) \) and \( Z_t = \sigma(t, X_t)V_x(t, X_t) \) can be shown to be a solution to the BSDE

\[
\begin{align*}
    dY_t^{s,x} & = -F(t, X_t, Y_t, Z_t)dt + Z_tdW_t \\
    Y_T^{s,x} & = g(X_T)
\end{align*}
\]

with

\[
X_t = X_t^{s,x} = x + \int_s^t \mu(r, X_r^{s,x})dr + \int_s^t \sigma(r, X_r^{s,x})dW_r,
\]

and \( F(t, x, y, z) \) as in (4).

Remark 4 (a) The representation of (10)-(11) as a FBSDE system is not unique. Another representation may be given by the following system, by eliminating the drift term of the
forward process:

\[
\begin{align*}
\hat{X}_t^{s,x} &= x + \int_s^t \sigma(r, \hat{X}_r) dW_r \\
Y_t^{s,x} &= g(\hat{X}_T) + \int_t^T \hat{F}(r, \hat{X}_r, Z_r) dr - \int_t^T Z_r dW_r
\end{align*}
\]  

where the new driver function is \( \hat{F}(t, x, y, z) = F(t, x, y, z) + \mu(t,x) \sigma(t,x) z \) (as long as the Girsanov’s theorem applies). Each representation has some advantages depending on the complexity level of the forward and backward equations in (10)-(12). In this section, the representation (10) will be used frequently based on the assumption that the forward state dynamics (11) has a unique solution.

(b) The existence-uniqueness of the solutions to a particular form of (10) was shown in Cetin (2005, section 2.2), thanks to its stochastic control interpretation as a solution to the standard LQR problems.

**Corollary 5** Consider the assumptions of the Theorem 2 and let \( F(t, x, y, z) \) be given by (4). If \( V(t, x) \in C^{1,2}([0,T] \times \mathbb{R}) \) satisfies the PDE (3), then we have \( V(t, x) = Y_t^{t,x} \) for all \( (t, x) \in [0,T] \times \mathbb{R} \), where the pair \( (Y_t^{s,x}, Z_t^{s,x}) \) given by \( Y(t) = V(t, X_t) \) and \( Z(t) = \sigma(t, X_t)V_x(t, X_t) \) solves the system (10)-(11) uniquely. Moreover, \( V \) is the unique solution of the PDE.

**Proof.** If \( V(t, x) \) is a classical solution to the PDE (3), then define \( (\hat{Y}_t^{s,x}, \hat{Z}_t^{s,x}) \) depending on \( X_t^{s,x} \) deterministically as \( \hat{Y}_t = V(t, X_t) \) and \( \hat{Z}_t = \sigma(t, X_t)V_x(t, X_t) \). Applying Ito’s rule to \( \hat{Y}_t \triangleq V(t, X_t) \), and by (2) and (3), we get

\[
\begin{align*}
d\hat{Y} &= LV(t, X) dt + \sigma(t, X) V_x(t, X) dW \\
&= -F(t, X, V(t, X), \sigma(t, X) V_x(t, X)) dt + \sigma(t, X) V_x(t, X) dW \\
&= -F(t, X, \hat{Y}, \hat{Z}) dt + \hat{Z} dW.
\end{align*}
\]

So, by the uniqueness of the solutions to (11) from Theorem 2, the result easily follows. ■

**Remark 6** The converse of the Corollary 5 is also true in the sense that if the triple \( (X_t^{s,x}, Y_t^{s,x}, Z_t^{s,x}) \) solves the system (10)-(11) and possess some stability and path regularity properties, then the deterministic function \( V(t, x) \) defined as \( V(t, x) = Y_t^{t,x} \) is a viscosity solution of the PDE (3). Such a result is given by Briand and Hu (2008). The uniqueness may require some extra monotonicity conditions on \( F(\cdot, \cdot, y, \cdot) \). We stay working with the smooth solutions in this work.

### 3.2. A Stochastic Control Application

Now consider the following controlled state dynamics \( X_t = X_t^u \) with a control-dependent drift term:

\[
\begin{align*}
dX_t &= (\mu(t, X_t) + B(t, X_t) u_t) dt + \sigma(t, X_t) dW(t), \\
X_0 &= x_0 > 0
\end{align*}
\]
where $\mu, \sigma, B : [0, T] \times \mathbb{R} \to \mathbb{R}$ are continuous and $u$ belongs to the control space $\mathcal{U}$ of square integrable real-valued adapted processes such that the equation $\text{(13)}$ also has a strong solution $X^u \in L^2_T$. Let the cost functional be given by

$$J^u(s, x) = E_{s, x} \int_s^T [(X_t - \xi(t))^2 + k_1(t)u^2_t] \, dt + k_2(X_T - \xi(T))^2$$  \hspace{1cm} (14)$$

where $k_1(.) > 0$, $k_2 \geq 0$, and $\xi(t)$ is a continuous function, describing the target for the state process $X_t = X^u_t$ to approach or stay close. Define the value function as $V(s, x) = \inf_u J^u(s, x)$ which is finite since both $k_1(t)u^2_t$ and $k_2(X_T - \xi(T))^2$ are bounded from below. This formulation resembles the stochastic LQR problems except that here the functions $\mu$ and $\sigma$ need not be linear in $x$, and $B(t, x)$ may also depend on $x$. Assuming that the SDE $\text{(13)}$ has a solution for a sufficiently rich set of the control processes in $\mathcal{U}$, and the optimization problem $\text{(14)}$ is solvable, we can identify a corresponding FBSDE system to characterize the solution and solve it numerically.

By a formal application of the dynamic programming principle (DPP) of the standard stochastic control theory (as in Fleming and Soner, 2006), the value function should satisfy the HJB equation

$$v_t(t, x) + \frac{1}{2} \sigma^2 v_{xx}(t, x) + \inf_u \{ (x - \xi(t))^2 + k_1(t)u^2 + v_x(t, x)(\mu(t, x) + B(t, x)u) \} = 0  \hspace{1cm} (15)$$

$$k_2(x - \xi(T))^2 = v(T, x)$$

where the infimum of the (Hamiltonian) expression $(x - \xi(t))^2 + k_1(t)u^2 + v_x(t, x)(\mu(t, x) + B(t, x)u)$ is obtained with $u^*(t, x) = - \frac{B(t, x)v_x(t, x)}{2k_1(t)}$. By writing this candidate optimal control in the equation $\text{(15)}$, we obtain a quasilinear PDE of the form $\text{(3)}$, given by $\text{(16)}$ below, with $F(t, x, y, z) = (x - \xi(t))^2 - \frac{1}{2} H(t, x)z^2$, where $H(t, x) = \frac{1}{2k_1(t)} \left( \frac{B(t, x)}{\sigma(t, x)} \right)^2$. Note that the function $F$ is independent of $y^3$ and $H(t, x)$ may depend on $x$. In general, a classical solution to the equation $\text{(15)}$ is not guaranteed to exist. However if $H(t, x) = H(t)$, and if a smooth solution to the corresponding HJB PDE $\text{(10)}$ exists, then the results of the previous section apply and we have the following result:

**Theorem 7** In the setting above, suppose that $H(t, x)$ is time-dependent only: $H(t, x) = H(t)$ and $F(t, x, y, z) = (x - \xi(t))^2 - \frac{1}{2} H(t)z^2$. Assume that for all $p \geq 2$, the SDE $\text{(13)}$ has a unique square integrable solution $X^u \in S^p_T$, for $u = 0$ and $u = u^* = - \frac{B(t, x)v_x(t, x)}{2k_1(t)}$ where $v(t, x) \in C^{1,2}[0, T] \times \mathbb{R}$ satisfies the quasilinear PDE

$$v_t(t, x) + \frac{1}{2} \sigma^2 v_{xx}(t, x) + F(t, x, v, \sigma v_x) = 0, \hspace{0.5cm} v(T, x) = k_2(x - \xi(T))^2.  \hspace{1cm} (16)$$

Moreover, let $\tilde{X}_t$ denote the solution to the SDE $\text{(13)}$ for $u = 0$. Then,

(i) The pair $(Y^{s,x}_t, Z^{s,x}_t)$ with $Y_t = v(t, \tilde{X}_t)$ and $Z_t = \sigma(t)v_x(t, \tilde{X}_t)$ is a (unique) continuous solution to the BSDE

$$dY^{s,x}_t = -F(t, \tilde{X}_t, Z_t) \, dt + Z_t \, dW_t, \hspace{0.5cm} Y^{s,x}_T = k_2(X(T) - \xi(T))^2  \hspace{1cm} (17)$$

It would depend on $y$ linearly, if we considered a time-discounted cost function.
in in $S_{F_t}^1 \times L^2_F$ such that $Y$ is bounded from below.

(ii) The value function is given by $v(t, x)$ which is the unique smooth solution of the PDE (16) and satisfies $v(t, x) = Y_t^x$, for $x \in \mathbb{R}$ and $t \in [0, T)$.

Proof. The part (i) directly follows from Theorem 2, representations (10)-(11) and Corollary 5. When $u^*$ is an admissible control and value function is well-defined (finite), part (ii) is a result of Corollary 5 and the arguments of the stochastic control theory for the classical solutions of the HJB equations.

Remark 8 (a) Ideally, an applicable "verification" theorem for the control problem or some a priori bounds for the processes $X,Y$ and $Z$ would be needed (since we haven't assumed any Lipschitz or growth conditions on the SDE (13) explicitly) to get part (ii) of Theorem. In most cases, the value function will be a viscosity solution to the PDE by a "formal" appeal to a version of the DPP, if available.

(b) Under the Lipschitz conditions on $\mu$ and $\sigma$ and a boundedness assumption on $\sigma$, Fuhrman et al (2006) showed that the value function is given by the maximal solution of the BSDE (17), using some localization arguments. They also provide the LQR example as a special case and consider more general applications where the control set is constrained to take values from a closed set of $\mathbb{R}$. The uniqueness to the solutions of the BSDEs (and hence the corresponding PDEs) related to the LQR problems was also reported in Cetin (2005), by exploiting the regularity properties of the explicit solution for the value function.

Example 9 Consider the following perturbed version of the LQR problem:

$$
\begin{align*}
    dX_t &= (A(t)X_t - \delta X^3_t + B(t)u_t)dt + \sigma(t)dW(t), \\
    X_0 &= x_0 > 0
\end{align*}
$$

where the time dependent functions $A, B$ and $\sigma$ are continuous, $B$ and $\sigma$ are bounded away from zero on the interval $[0, T]$, and $u$ belongs to the control space $U$ as before. The term $\delta$ is a small perturbation constant, so the system reduces to a linear one with Lipschitz coefficients when $\delta = 0$. Even though the standard (unperturbed) LQR problems have an explicit quadratic form as a solution, this perturbed version of the problem cannot be solved explicitly. Using the same arguments above, the corresponding HJB equation is given by

$$
    v_t(t, x) + \frac{1}{2}\sigma^2(t)v_{xx}(t, x) + \inf_u \{ (x - \xi(t))^2 + k_1(t)u^2 + v_x(t, x)(A(t)x - \delta x^3 + B(t)u) \} = 0
$$

with $v(T, x) = k_2(x - \xi(T))^2$. When the terminal condition is bounded (e.g. when $k_2 = 0$, as in Tsai, 1978), using the methods of the parabolic PDEs, it can be shown to have a smooth solution. For more general functions, even when the PDE is uniformly parabolic, the existence of a classical solution is not guaranteed in general. To prove that the equation (18) also has a square integrable solution $X^{u^*}$ corresponding to the (feedback) control $u^*(t) = -\frac{B}{\sigma^2}(t)v_x(t, X(t))$, we may need some a priori estimates on the (potentially viscosity) solutions of (20). However, if the solution is smooth, the uniqueness follows from Corollary 5.
Theorem 10  Consider the perturbed state dynamics \( (18) \) together with the cost function \( (14) \) and the value function \( V^\delta(s, x) \). Then

(i) The value function \( V(s, x) \) is the unique smooth solution to the HJB PDE

\[
  v_t + \frac{1}{2} \sigma^2(t) v_{xx} + (x - \xi(t))^2 - (A(t)x - \delta x^3)v_x - \frac{1}{2} C(t) v_x^2 = 0 \tag{20}
\]

\[
v(T, x) = 0
\]

where \( C(t) = (B^2/2k_1)(t) \) over \([0, T]\).

(ii) Let \( H(t) = \frac{C(t)}{\sigma^2(t)} \) satisfy the assumption of Condition 1 (i). Then the triple \( (\tilde{X}_t^{s,x}, Y_t^{s,x}, Z_t^{s,x}) \) with \( Y_t = V(t, \tilde{X}_t) \) and \( Z_t = \sigma(t)V_x(t, \tilde{X}_t) \) is a (unique) solution to the FBSDE system

\[
dY_t^{s,x} = -F(t, \tilde{X}_t, Z_t)dt + Z_t dW_t, \quad Y_T = 0 \tag{21}
\]

\[
\tilde{X}_r = x - \int_s^t (A\tilde{X}_r - \delta \tilde{X}_r^3)dr + \int_s^t \sigma(r)dW_r,
\]

where \( F(t, x, z) = (x - \rho(t))^2 - \frac{H(t)x^2}{2} \). Moreover, \( V(t, x) = Y_t^{t,x}, \) for \((t, x) \in [0, T) \times \mathbb{R}\).

Proof.  For part (i), note that the derivative of the function \( \mu(x) = Ax - \delta x^3 \) is bounded above and \( x\mu(x) \) can be bounded from above by \( \alpha - \beta x^2 \), for some positive constants \( \alpha \) and \( \beta \). So using the Lyapunov conditions for locally Lipshitz parameters, the forward SDE in \( (21) \) can be shown to have a unique global solution \( \tilde{X} \in S_p^F \), for all \( p \geq 1 \). By the relevant DPP results for bounded terminal value problems, if the value function \( V(t, x) \) is sufficiently smooth, then it should satisfy the HJB PDE \( (20) \) together with the candidate optimal control \( u^* \). Since the time dependent parameters are (uniformly) continuous on \([0, T]\), by following the similar steps as in Tsai (1978), one can show the existence of a smooth solution \( v(t, x) \) to the PDE \( (20) \), too. Then by Theorem 2, representations \( (10)-(11) \) and the Corollary 5, the BSDE in \( (21) \) has a unique solution \( (Y, Z) \) such that \( v(t, x) = Y_t^{t,x} = V(t, x) \) and \( \sigma(t)V(t, X(t)) = Z(t) \). Alternatively, it can be inferred from an applicable verification theorem, e.g. as in Tsai (1978) or Fleming and Soner (2006).

Remark 11  Such FBSDE representations would be very helpful to solve these type of non-linear PDEs (and control problems) numerically, especially in higher dimensional cases. It is especially useful if the nature of the PDE solution is not known explicitly and can be inferred from the properties of the numerical solution to the corresponding FBSDE system. We leave the discussion of the numerical solutions and their stability/convergence properties to some subsequent work, including Cetin (2012).

References

[1] Bihari, I. (1956). A generalization of a lemma of Bellman and its application to uniqueness problem of differential equations. Acta Math. Acad. Sci. Hungar. 7, 71-94

[2] Briand, P., Lepeltier, J-P, and San Martin, J. (2007). One-dimensional backward stochastic differential equations whose coefficient is monotonic in y and non-Lipschitz in z, Bernoulli 13 (1), 80-91

12
[3] Briand, P., Hu, Y. (2006). BSDEs with quadratic growth and unbounded terminal value, *Prob. Theory Related Fields* **136**, 604-618

[4] Briand, P., Hu, Y. (2008). Quadratic BSDEs with convex generators and unbounded terminal value, *Prob. Theory Related Fields* **141**, 543-567

[5] Cetin, C. (2005). Backward stochastic differential equations with quadratic growth and their applications. Ph.D. Dissertation, USC.

[6] Cetin, C (2012). A forward-backward numerical scheme for the perturbed linear-quadratic regular problems, preprint.

[7] El Karoui, N., Peng, S., Quenez, M. C. (1997). Backward stochastic differential equations in finance. *Math. Finance* **7** 1-71

[8] Fan, S. and Jiang, L. (2010). Finite and infinite time interval BSDEs with non-Lipshitz coefficients. *Statist Probab. Lett.* **80** 962-968.

[9] Fleming, W.H. and Soner, H.M. (2006). Controlled Markov Processes and Viscosity Solutions, Springer-Verlag, Second Edition.

[10] Fuhrman, M., Hu, Y. and Tessitore, G. (2006). On a class of stochastic optimal control problems related to BSDEs with quadratic growth. *SIAM J. Control Optim.*, **45**, 1279-1296

[11] Kobylanski, M (2000). Backward stochastic differential equations and partial differential equations with quadratic growth. *The Annals of Prob.* **28** 558-602

[12] Lepeltier, J.P. and San Martin, J. (1997). Backward stochastic differential equations with continuous coefficients. *Statist Probab. Lett.* **32** 425-430

[13] Lepeltier, J.P. and San Martin, J. (1998). Existence for BSDE with superlinear-quadratic coefficient. *Stochastics Stochastic Rep.* **63** 227-240

[14] Ma, J., and Yong, J. (1999). Forward-Backward Stochastic Differential Equations and Their Applications, Lecture Notes in Math, 1702, Springer

[15] Mao, X. (1995). Adapted solution of backward stochastic differential equations with non-Lipshitz coefficients. *Stoch. Process. Appl.* **58** 281-292

[16] Pardoux, E. and Peng, S. (1990). Adapted solution of a backward stochastic differential equation. *Systems Control Lett.* **14** 55-61

[17] Pardoux, E. and Peng, S. (1992). Backward stochastic differential equations and quasi-linear partial differential equations, Lecture Notes in CIS, vol. **176** Springer, 200-217

[18] Pardoux, E. and Peng, S. (1994). Some backward stochastic differential equations with non-Lipshitz coefficients, Prépublication LATP, 94-03
[19] Peng, S. (1992). Stochastic Hamilton-Bellman equations, SIAM J. Control Optim., 30, 284-304

[20] Richou, A. (2011). Markovian quadratic and superquadratic BSDEs with an unbounded terminal condition. Preprint, arXiv: 1111.5135

[21] Tsai, C-P. (1978). Perturbed stochastic linear regulator problems, SIAM J. Control Optim., 16, 396-410.

[22] Yong, J. and Zhou, X.Y. (1999). Stochastic Controls: Hamiltonian Systems and HJB Equations, Springer-Verlag, New York