(2P₂, K₄)-Free Graphs are 4-Colorable

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Abstract

In this paper, we show that every (2P₂, K₄)-free graph is 4-colorable. The bound is attained by the five-wheel and the complement of seven-cycle. This answers an open question by Wagon [6] in 1980s.

1 Introduction

All graphs in this paper are finite and simple. We say that a graph G contains a graph H if H is isomorphic to an induced subgraph of G. A graph G is H-free if it does not contain H. For a family of graphs ℱ, G is ℱ-free if G is H-free for every H ∈ ℱ. In case that ℱ consists of two graphs, we write (H₁, H₂)-free instead of {H₁, H₂}-free. As usual, let Pₜ and Cₙ denote the path on t vertices and the cycle on s vertices, respectively. The complete graph on n vertices is denoted by Kₙ. The n-wheel Wₙ is the graph obtained from Cₙ by adding a new vertex and making it adjacent to every vertex in Cₙ. For two graphs G and H, we use G + H to denote the disjoint union of G and H. For a positive integer r, we use rG to denote the disjoint union of r copies of G. The complement of G is denoted by G. A hole in a graph is an induced cycle of length at least 4. A hole is odd if it is of odd length.

A q-coloring of a graph G is a function φ : V(G) → {1, . . . , q} such that φ(u) ̸= φ(v) whenever u and v are adjacent in G. We say that G is q-colorable if G admits a q-coloring. The chromatic number of G, denoted by χ(G), is the minimum number q such that G is q-colorable. The clique number of G, denoted by ω(G), is the size of a largest clique in G. Obviously, χ(G) ≥ ω(G) for any graph G.

A family ℱ of graphs is said to be χ-bounded if there exists a function f such that for every graph G ∈ ℱ and every induced subgraph H of G it holds that χ(H) ≤ f(ω(H)). The function f is called a χ-binding function for ℱ. The class of perfect graphs (a graph G is perfect if for every induced subgraph H of G it holds that χ(H) = ω(H)), for instance, is a χ-bounded family with χ-binding function f(x) = x. Therefore, χ-boundedness is a generalization of perfection. The notion of χ-bounded families was introduced by Gyárfás [5] who make the following conjecture.

Conjecture 1 (Gyárfás [4]). For every forest T, the class of T-free graphs is χ-bounded.

Gyárfás [5] proved the conjecture for T = P₁; every P₁-free graph G has χ(G) ≤ (t−1)ω(G)−1. The result was slightly improved by Gravier, Hoang and Maffray in [3] that every P₁-free graph G has χ(G) ≤ (t−2)ω(G)−1. This implies that every P₅-free graph G has χ(G) ≤ 3ω(G)−1. Note that this χ-binding function is exponential in ω(G). For ω(G) = 3, Esperet, Lemoine, Maffray and Morel [2] obtained the optimal bound on the chromatic number: every (P₅, K₄)-free graph is 5-colorable. They also demonstrated a (P₅, K₄)-free graph whose chromatic number is 5. On

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that the other hand, a polynomial \( \chi \)-binding function for the class of 2\( P_2 \)-free graphs was shown by Wagon [6] who proved that every such graph has \( \chi(G) \leq \left( \omega(G)/2 + 1 \right) \). This implies that every (2\( P_2 \), \( K_4 \))-free graph is 6-colorable. In [6] it was asked if there exists a (2\( P_2 \), \( K_4 \))-free graph whose chromatic number is 5 or 6.

We observe that the (\( P_5 \), \( K_4 \))-free graph with chromatic number 5 given in [2] contains an induced 2\( P_2 \). This is the starting point of this research. In this paper we settle Wagon’s question by proving the following.

**Theorem 1.** Every (2\( P_2 \), \( K_4 \))-free graph \( G \) has \( \chi(G) \leq 4 \).

The bound in Theorem 1 is attained by the five-wheel \( W_5 \) and the complement of a seven-cycle \( \overline{C_7} \).

The **neighborhood** of a vertex \( v \) in a graph \( G \), denoted by \( N_G(v) \), is the set of neighbors of \( v \). We simply write \( N(v) \) if the graph \( G \) is clear from the context. Two nonadjacent vertices \( u \) and \( v \) in \( G \) are **comparable** if either \( N(v) \subseteq N(u) \) or \( N(u) \subseteq N(v) \). Observe that if \( N(u) \subseteq N(v) \), then \( \chi(G - u) = \chi(G) \). Therefore, it suffice to prove Theorem 1 for every connected (2\( P_2 \), \( K_4 \))-free graph with no pair of comparable vertices. We do so by proving a number of lemmas below.

The idea is that we assume the occurrence of some induced subgraph \( H \) in \( G \) and then argue that the theorem holds in this case. Afterwards, we can assume that \( G \) is \( H \)-free in addition to being (2\( P_2 \), \( K_4 \))-free. We then pick a different induced subgraph as \( H \) and repeat. In the end, we are able to show that the theorem holds if \( G \) contains a \( C_5 \) (see Lemma 2-Lemma 5 below). Therefore, the remaining case is that \( G \) is (odd hole, \( K_4 \))-free. In this case, the theorem follows from a known result by Chudnovsky, Robertson, Seymour and Thomas [1] that every (odd hole, \( K_4 \))-free graph is 4-colorable. This proves Theorem 1.

The proof idea stems from [2] in which it was proved that every (\( P_5 \), \( K_4 \))-free graph is 5-colorable. In particular, the graph \( H_1 \) (see Figure 1) that plays an important role in our proof was first discovered and used in [2]. However, to prove 4-colorability we need to use the argument of comparable vertices and extensively extend the structural analysis in [2]. The remainder of the paper is organized as follows. In section 2 we present some preliminary results. In section 3 and section 4 we prove Lemma 2 and Lemma 3, respectively. We then prove Lemma 4 and Lemma 5 in section 5.

2 Preliminaries

We present the structure around a five-cycle in (2\( P_2 \), \( K_4 \))-free graphs that will be used in section 4 and section 5. Let \( G \) be a (2\( P_2 \), \( K_4 \))-free graph and \( C = 12345 \) be an induced \( C_5 \).
of \( G \). All indices below are modulo 5. We partition \( V \setminus C \) into following subsets:

\[
Z = \{ v \in V \setminus C : N_C(v) = \emptyset \},
R_i = \{ v \in V \setminus C : N_C(v) = \{i - 1, i + 1\} \},
Y_i = \{ v \in V \setminus C : N_C(v) = \{i - 2, i, i + 2\} \},
F_i = \{ v \in V \setminus C : N_C(v) = C \setminus \{i\} \},
U = \{ v \in V \setminus C : N_C(v) = C \}.
\]

**Lemma 1.** Let \( G \) be a \((2P_2, K_4)\)-free graph and \( C = 12345 \) be an induced \( C_5 \) of \( G \). Then \( V(G) = C \cup Z \cup (\bigcup_{i=1}^5 R_i) \cup (\bigcup_{i=1}^5 Y_i) \cup (\bigcup_{i=1}^5 F_i) \cup U \).

**Proof.** Suppose that there is a vertex \( v \in V(G) \setminus C \) that does not belong to any of \( Z, R_i, Y_i, F_i \) and \( U \). Note that \( v \) has at least one and at most three neighbors on \( C \). Moreover, these neighbors must be consecutive on \( C \). Without loss of generality, we may assume that \( v \) is adjacent to 1 and not adjacent to 3 and 4. Now 34 and 1

It suffices to prove for \( i \in \{1, 2, 3\} \) that \( \{i, y, i - 2, i + 2\} \) induces a \( K_4 \). ■

We now prove some structural properties of these sets.

(1) \( Z \cup R_i \) is independent.

If \( Z \cup R_i \) contains an edge \( xy \), then \( xy \) and \( (i - 2)(i + 2) \) induce a \( 2P_2 \), a contradiction. ■

(2) \( U \cup Y_i \) and \( U \cup F_i \) are independent.

If either \( U \cup Y_i \) or \( U \cup F_i \) contains an edge \( xy \), then \( \{x, y, i - 2, i + 2\} \) induces a \( K_4 \). ■

(3) \( R_i \) and \( R_{i+1} \) are complete.

It suffices to prove for \( i = 1 \). If \( r_1 \in R_1 \) and \( r_2 \in R_2 \) are not adjacent, then \( 5r_1 \) and \( 3r_2 \) induce a \( 2P_2 \). ■

(4) \( Y_i \) and \( Y_{i+1} \) are complete.

It suffices to prove for \( i = 1 \). If \( y_1 \in Y_1 \) and \( y_2 \in Y_2 \) are not adjacent, then \( 5y_2 \) and \( 3y_1 \) induce a \( 2P_2 \). ■

(5) \( R_i \) and \( Y_i \) are complete.

It suffices to prove for \( i = 1 \). If \( r_1 \in R_1 \) and \( y_1 \in Y_1 \) are not adjacent, then \( 5r_1 \) and \( 3y_1 \) induce a \( 2P_2 \). ■

(6) Either \( R_i \) and \( Y_{i+1} \) are anti-complete or \( R_{i+1} \) and \( Y_i \) are anti-complete.

Suppose, by contradiction, that there exist vertices \( r_i \in R_i \), \( r_{i+1} \in R_{i+1} \), \( y_i \in Y_i \), \( y_{i+1} \in Y_{i+1} \) such that \( r_i \) and \( r_{i+1} \) are adjacent to \( y_{i+1} \) and \( y_i \), respectively. Then it follows from \( (3), (4) \) and \( (5) \) that \( \{r_i, r_{i+1}, y_i, y_{i+1}\} \) induces a \( K_4 \). ■

(7) Each vertex in \( Y_i \) is anti-complete to either \( Y_{i-2} \) or \( Y_{i+2} \).

It suffices to prove for \( i = 1 \). If \( y_1 \in Y_1 \) is adjacent to a vertex \( y_i \in Y_i \) for \( i = 3, 4 \), then \( \{1, y_1, y_3, y_4\} \) induces a \( K_4 \) by \( (4) \). ■

(8) \( F_i \) is complete to \( Y_{i-2} \cup Y_{i+2} \) and anti-complete to \( Y_{i-1} \cup Y_i \cup Y_{i+1} \).

It suffices to prove for \( i = 5 \). Let \( f \in F_5 \). Recall that \( f \) is adjacent to \( 1, 2, 3, 4 \) but not adjacent to \( 5 \) by the definition of \( F_5 \). Suppose first that \( f \) is not adjacent to a vertex \( y \in Y_2 \cup Y_3 \). Note that \( y \) is adjacent to \( 5 \) by the definition of \( Y_2 \) and \( Y_3 \). Now either \( 3f \) or \( 2f \) forms a \( 2P_2 \) with \( 5y \) depending on whether \( y \in Y_2 \) or \( y \in Y_3 \). This proves the first part of \( (8) \). Suppose now that \( f \) is adjacent to a vertex \( y \in Y_i \) for some \( i \in \{1, 4, 5\} \). Since \( i \not\in \{2, 3\} \), it follows that \( 5 \not\in \{i - 2, i + 2\} \). Therefore, \( f \) is adjacent to \( i - 2 \) and \( i + 2 \). This implies that \( \{f, y, i - 2, i + 2\} \) induces a \( K_4 \). This proves the second part of \( (8) \). ■
Let $V$ be a set of vertices. We first show that $V$ is complete to $R_i - 1 \cup R_{i+1}$.

It suffices to prove $i = 5$. If $f \in F_5$ is not adjacent to $r \in R_1 \cup R_4$, then either $f_3$ or $f_2$ forms a $2P_2$ with $5r$ depending on whether $r \in R_1$ or $r \in R_4$.

(10) If $U \neq \emptyset$, then $Y_i$ and $Y_{i+2}$ are anti-complete.

Let $u \in U$. If $y_i \in Y_i$ and $y_{i+2} \in Y_{i+2}$ are adjacent, then $y_i y_{i+2}$ and $u(i+1)$ induce a $2P_2$ since $u$ is adjacent to neither $y_i$ nor $y_{i+2}$ by (2), a contradiction.

(11) Either $F_i$ or $F_{i+2}$ is empty.

It suffices to prove for $i = 3$. Suppose that $F_i$ contains a vertex $f_i \in F_i$ for $i = 3, 5$. Then either $3f_5$ and $5f_3$ induce a $2P_2$ or $\{1, 2, f_3, f_5\}$ induces a $K_4$ depending on whether $f_3$ and $f_5$ are nonadjacent or not.

(12) If $G$ is $H_1$-free, then the following holds: if $F_i \neq \emptyset$, then $R_{i+1}$ is anti-complete to $Y_{i+2} \cup Y_i$ and $R_{i-1}$ is anti-complete to $Y_{i-2} \cup Y_i$.

It suffices to prove for $i = 5$. Let $f \in F_5$. Suppose, by contradiction, that there exists vertices $r \in R_1$ and $y \in Y_2 \cup Y_5$ such that $r$ and $y$ are adjacent. Note that $f$ is adjacent to $r$ by (9). If $y \in Y_2$, then $f$ is adjacent to $y$ by (8) and this implies that $\{f, y, r, 2\}$ induces a $K_4$. If $y \in Y_5$, then $f$ is not adjacent to $y$ by (8) and this implies that $C \cup \{1\} \cup \{f, y, r\}$ induces an $H_1$ (see Figure 1). This proves that $R_1$ is anti-complete to $Y_2 \cup Y_5$. The proof for the second part is symmetric.

(13) Each vertex in $R_i$ is anti-complete to either $Y_{i+1}$ or $Y_{i+2}$. By symmetry, each vertex in $R_i$ is anti-complete to either $Y_{i-1}$ or $Y_{i-2}$.

Suppose, by contradiction, that there exists a vertex $r_i \in R_i$ such that $r_i$ is adjacent to a vertex $y_{i+1} \in Y_{i+1}$ and a vertex $y_{i+2} \in Y_{i+2}$. By (4), $y_{i+1}$ and $y_{i+2}$ are adjacent. This implies that \{\{r_i, y_{i+1}, y_{i+2}, i - 1\}\} induces a $K_4$.

3 Eliminate $H_1$

Lemma 2. Let $G$ be a connected $(2P_2, K_4)$-free graph with no pair of comparable vertices. If $G$ contains an induced $H_1$, then $\chi(G) \leq 4$.

Proof. Let $H = C \cup \{w\}$ be an induced $H_1$ in $G$ where $C = \{1, 2, 3, 4, 5, 6\}$ induces a $C_6$ such that $ij$ is an edge if and only if $|i - j| = 1$, and $w$ is adjacent to $1, 2, 4$ and $5$. All the indices below are modulo 6. We partition $V(G)$ into following subsets:

$$Z = \{v \in V \setminus C : N_C(v) = \emptyset\},$$
$$D_{i,i+1} = \{v \in V \setminus C : N_C(v) = \{i, i+1\}\},$$
$$T_i = \{v \in V \setminus C : N_C(v) = \{i - 1, i, i + 1\}\},$$
$$F_{i,i+1} = \{v \in V \setminus C : N_C(v) = \{i - 1, i, i + 1, i + 2\}\},$$
$$W = \{v \in V \setminus C : N_C(v) = N_C(w) = \{1, 2, 4, 5\}\}.$$

Let $D = \bigcup_{i=1}^{6} D_{i,i+1}$, $T = \bigcup_{i=1}^{6} T_i$ and $F = \bigcup_{i=1}^{6} F_{i,i+1}$. We choose $H$ such that $|T| + |F|$ is maximum. We first show that $V(G) = C \cup Z \cup D \cup T \cup F \cup W$.

(1) There is no vertex $v \in V \setminus C$ such that $v$ is adjacent to $i$ but adjacent to neither $i - 1$ nor $i + 1$ for any $1 \leq i \leq 6$.

Suppose that such a vertex $v$ exists. Then it follows that $vi$ and $(i-1)(i+1)$ induce a $2P_2$.■
(2) If a vertex in $V \setminus C$ has at most two neighbors on $C$, then $v \in Z \cup D$.

Suppose not. Let $v \in V \setminus C$ that has at most two neighbors on $C$ and $v \notin Z \cup D$. Then either $v$ has exactly one neighbor on $C$ or has two neighbors on $C$ that are not consecutive. By symmetry, we may assume that $v$ is adjacent to 1 but not adjacent to 2 and 6. This contradicts (1). ■

(3) If a vertex $v \in V \setminus C$ that has exactly three neighbors on $C$, then $v \in T$.

Suppose not. Let $v \in V \setminus C$ that has exactly at three neighbors on $C$. By symmetry, we may assume that $v$ is adjacent to 1. It follows from (1) that $v$ is adjacent to either 2 or 6, say 2. If $v$ is not adjacent to 3 or 6, then it contradicts (1) for $i = 4$ or $i = 5$. Therefore, $v \in T_1$ or $v \in T_2$.

(4) If a vertex $v \in V \setminus C$ that has exactly four neighbors on $C$, then $v \in F \cup W$.

By (1), $v$ must have two consecutive neighbors on $C$. If $v$ has three consecutive neighbors on $C$, then all four neighbors must be consecutive by (1) and so $v \in F$. Now $N_C(v) = \{i, i + 1, i + 3, i + 4\}$ for some $i$. If $i = 1$, then $v \in W$. Suppose that $i = 2$ (and the case $i = 3$ is symmetric). Then either $w1$ and $v6$ induce a $2P_2$ or $\{w, v, 2, 5\}$ induces a $K_4$, depending on whether $w$ and $v$ are non-adjacent or not.

(5) There is no vertex in $V \setminus C$ that has more than four neighbors.

Suppose not. Let $v \in V \setminus C$ have at least five neighbors on $C$. By symmetry, we may assume that $v$ is adjacent to $i$ for each $1 \leq i \leq 5$. Then $\{1, 3, 5, v\}$ induces a $K_4$.

It follows from (2)-(5) that $V(G) = C \cup Z \cup D \cup T \cup F \cup W$. Note that each of the subset defined is a stable set since $G$ is $(2P_2, K_4)$-free. We further investigate the adjacency among those subsets.

(6) The set $W$ is anti-complete to $Z$.

If $w \in W$ and $z \in Z$ are adjacent, then $wz$ and 36 induce a $2P_2$, a contradiction.

(7) The set $W$ is complete to $D_{i,i+1}$ for $i \in \{2, 3, 5, 6\}$ and anti-complete to $D_{i,i+1}$ for $i \in \{1, 4\}$.

Suppose that $w \in W$ is not adjacent some vertex $d \in D_{i,i+1}$ for some $i \in \{2, 3, 5, 6\}$. By symmetry, we may assume that $i = 2$. Then $d3$ and $w4$ induce a $2P_2$, a contradiction. Suppose that $w \in W$ is adjacent some vertex $d \in D_{1,2} \cup D_{4,5}$. Then $dw$ and 36 induce a $2P_2$, a contradiction.

(8) The set $W$ is complete to $T_1 \cup T_2 \cup T_3 \cup T_5$ and anti-complete to $T_3 \cup T_6$.

Suppose that $w \in W$ is not adjacent some vertex $t \in T_i$ for some $i \in \{1, 2, 4, 5\}$. By symmetry, we may assume that $i = 1$. Then $t6$ and $w5$ induce a $2P_2$. Suppose that $w \in W$ is adjacent some vertex $t \in T_i$ for some $i \in \{3, 6\}$. By symmetry, we may assume that $i = 3$. Then $\{w, t, 2, 4\}$ induces a $K_4$.

(9) The set $W$ is anti-complete to $F_{i,i+1}$ for $i \in \{2, 3, 5, 6\}$ and complete to $F_{i,i+1}$ for $i \in \{1, 4\}$.

Suppose that $w \in W$ is not adjacent some vertex $f \in F_{i,i+1}$ for some $i \in \{2, 3, 5, 6\}$. By symmetry, we may assume that $i = 2$. Then $\{f, w, 1, 4\}$ induces a $K_4$. Suppose that $w \in W$ is not adjacent some vertex $f \in F_{i,i+1}$ for some $i \in \{1, 4\}$. By symmetry, we may assume that $i = 1$. Then $6f$ and $5w$ induce a $2P_2$.

(10) The set $Z$ is anti-complete to $D \cup T \cup (F \setminus (F_{1,2} \cup F_{4,5}))$.

Suppose that $z \in Z$ is adjacent to some vertex $x \in D \cup T \cup (F \setminus (F_{1,2} \cup F_{4,5}))$. If $x \in D \cup T$, then there exists a vertex $i \in C$ such that $x$ is not adjacent to $i - 1$ and $i + 1$. Then $zx$
and \((i - 1)(i + 1)\) induce a \(2P_2\). If \(x \in F_{i,i+1}\) for some \(i = 2, 3, 5, 6\), then \(xw \notin E\) by (9). Moreover, there exists a vertex \(j \in N_C(w)\) such that \(xj \notin E\). Then \(wj\) and \(zx\) induce a \(2P_2\).

It follows from and (6) and (10) that any vertex in \(Z\) has neighbors only in \(F_{1,2} \cup F_{4,5}\). On the other hand, \(w\) is complete to \(F_{1,2} \cup F_{4,5}\) by (9). Since \(G\) contains no pair of comparable vertices, it follows that \(Z = \emptyset\).

(11) For each \(i\), \(D_{i,i+1}\) is anti-complete to \(D_{i+1,i+2}\), complete to \(D_{i+2,i+3}\) and anti-complete to \(D_{i+3,i+4}\).

By symmetry, it suffices to prove the claim for \(i = 1\). Let \(d \in D_{1,2}\). If \(d\) is adjacent to \(d' \in D_{2,3}\), then \(46\) and \(dd'\) induce a \(2P_2\). If \(d\) is not adjacent to \(d' \in D_{3,4}\), then \(2d\) and \(3d'\) induce a \(2P_2\). If \(d\) is adjacent to \(d' \in D_{4,5}\), then \(36\) and \(dd'\) induce a \(2P_2\).

(12) For each \(i\), \(F_{i,i+1}\) is anti-complete to \(F_{i+1,i+2} \cup F_{i+3,i+4}\) and complete to \(F_{i+2,i+3}\).

By symmetry, it suffices to prove the claim for \(i = 1\). Let \(f \in F_{1,2}\). If \(f\) is adjacent to a vertex \(f' \in F_{2,3}\), then \(\{1, 3, f, f'\}\) induces a \(K_4\). If \(f\) is not adjacent to a vertex \(f' \in F_{3,4}\), then \(5f'\) and \(6f\) induce a \(2P_2\). If \(f\) is adjacent to a vertex \(f' \in F_{4,5}\), then \(\{3, 6, f, f'\}\) induces a \(K_4\).

(13) The sets \(T_1\) and \(T_{i+1}\) are anti-complete for \(i \in \{1, 4\}\).

By symmetry, it suffices to prove this for \(i = 1\). If \(t_1 \in T_1\) and \(t_2 \in T_2\) are adjacent, then \(w\) is adjacent to both \(t_1\) and \(t_2\) by (8). But now \(\{t_1, t_2, w, 1\}\) induces a \(K_4\).

(14) The sets \(T_3\) and \(T_1 \cup T_5\) are complete. By symmetry, \(T_6\) and \(T_2 \cup T_4\) are complete.

Suppose that \(t_3 \in T_3\) is not adjacent to some vertex \(t \in T_1 \cup T_5\). By (8), \(w\) is adjacent to \(t\) but not to \(t_3\). Then \(3t_3\) and \(wt\) induce a \(2P_2\), a contradiction.

(15) The sets \(T_1\) and \(T_{i+3}\) are complete for each \(1 \leq i \leq 6\).

By symmetry, it suffices to prove this for \(i = 1\). If \(t_1 \in T_1\) and \(t_4 \in T_4\) are not adjacent, then \(2t_1\) and \(3t_4\) induce a \(2P_2\).

(16) For each \(i\), \(D_{i,i+1}\) is anti-complete to \(T_{i-1} \cup T_1 \cup T_{i-1} \cup T_{i+2}\) and complete to \(T_{i+3} \cup T_{i+4}\).

We note that \(D_{1,2}\) and \(D_{4,5}\) are symmetric, and \(D_{2,3}, D_{3,4}, D_{5,6}\) and \(D_{6,1}\) are symmetric. So, it suffices to prove the claim for \(D_{1,2}\) and \(D_{2,3}\).

Let \(d \in D_{1,2}\). Suppose that \(d\) is adjacent to some vertex \(t \in T_6 \cup T_1 \cup T_2 \cup T_3\). By symmetry, we may assume that \(i \in \{1, 3\}\). If \(i = 1\), then \(td\) and \(35\) induce a \(2P_2\). If \(i = 3\), then \(w\) is not adjacent to \(d\) and \(t\) by (7) and (8). Then \(dt\) and \(w5\) induce a \(2P_2\). Now suppose that \(d\) is not adjacent to some vertex \(t \in T_4 \cup T_5\). By symmetry, we may assume that \(t \in T_4\). Then \(d2\) and \(t3\) induce a \(2P_2\). This proves the claim for \(D_{1,2}\).

Let \(d \in D_{2,3}\). Suppose that \(d\) is adjacent to some vertex \(t \in T_2 \cup T_3\). By symmetry, we may assume that \(t \in T_2\). Then \(dt\) and \(46\) induce a \(2P_2\). Suppose that \(d\) is not adjacent to some vertex \(t \in T_5 \cup T_6\). By symmetry, we may assume that \(t \in T_5\). Then \(d3\) and \(t4\) induce a \(2P_2\).

By (7) and (8), \(\{2, w\}\) is complete to \(D_{2,3} \cup T_1\). It follows from \(K_4\)-freeness of \(G\) that \(D_{2,3}\) is anti-complete to \(T_1\). It remains to show that \(D_{2,3}\) is anti-complete to \(T_4\). Suppose that \(d\) is adjacent to some vertex \(t_4 \in T_4\). Note that \(C' = C \setminus \{1\} \cup \{t_4\}\) induces a \(C_6\) and \(H' = C' \cup \{w\}\) induces a subgraph isomorphic to \(H_1\). By (13) and (14), all vertices in \(T_1 \cup T_4 \cup T_5 \cup T_6\) remain to be \(T\)-vertices with respect to \(C'\). Moreover, all vertices in \(T_3 \cup F\) remain to be \(F\)-vertices or \(T\)-vertices. By the choice of \(C\), there exists a vertex \(t \in T_3\) that is not adjacent to \(t_4\). Then \(dt_4\) and \(1t_2\) induce a \(2P_2\), a contradiction. This proves the claim for \(D_{2,3}\).
(17) For each $i$, $F_{i,i+1}$ is anti-complete to $T_i \cup T_{i+1}$ and complete to $T_{i+3} \cup T_{i+4}$.
By symmetry of $C$, it suffices to prove this for $i = 1$. Let $f \in F_{1,2}$. If $f$ is adjacent to some vertex $t \in T_1 \cup T_2$, then either $\{6,2,f,t\}$ or $\{1,3,f,t\}$ induces a $K_4$ depending on whether $t \in T_1$ or $t \in T_2$. Suppose that $f$ is not adjacent to some vertex $t \in T_4 \cup T_5$. By symmetry, we may assume that $t \in T_4$. Then $6f$ and $5t$ induce a $2P_2$, a contradiction. ■

(18) The sets $F_{i,i+1}$ and $T_{i-1}$ are complete for $i \in \{2,5\}$, and $F_{i,i+1}$ and $T_{i+2}$ are complete for $i \in \{3,6\}$.
Let $f \in F_{i,i+1}$ and $t \in T_i$ be nonadjacent. By (9) and (8), $w$ is adjacent to $t$ but not $f$. It can be readily checked that in each of the cases $wt$ and $f3$ or $wt$ and $f6$ induce a $2P_2$. ■

(19) The set $D_{1,2}$ is anti-complete to $F_{5,1} \cup F_{2,3}$ and complete to $F_{4,5}$.
The set $D_{3,4}$ is anti-complete to $F_{3,4} \cup F_{5,6}$ and complete to $F_{3,5}$.
The set $D_{2,3}$ is anti-complete to $F_{4,5}$ and complete to $F_{5,6}$.
The set $D_{6,1}$ is anti-complete to $F_{1,2}$ and complete to $F_{2,3} \cup F_{3,4}$.
The set $D_{5,6}$ is anti-complete to $F_{4,5}$ and complete to $F_{2,3} \cup F_{3,4}$.

Note that $D_{1,2}$ and $D_{6,1}$ are symmetric, and $D_{2,3}, D_{3,4}, D_{5,6}$ and $D_{6,1}$ are symmetric. So, it suffices to prove the claim for $D_{1,2}$ and $D_{2,3}$. Let $d \in D_{1,2}$. If $d$ is adjacent to some vertex $f \in F_{6,1} \cup F_{2,3}$, then $w$ is not adjacent to $d$ and $f$ by (7) and (9). Now $df$ and $w4$ or $d4$ and $w5$ induce a $2P_2$ depending on whether $f \in F_{6,1}$ or $f \in F_{2,3}$. If $d$ is not adjacent to some vertex $f \in F_{4,5}$, then $d2$ and $f3$ induce a $2P_2$. This proves the claim for $D_{1,2}$.

Now let $d \in D_{2,3}$. By (7), it follows that $wd \in E$. If $d$ is adjacent to a vertex $f \in F_{1,2}$, then $\{d, f, 2, w\}$ induces a $K_4$ by (9). If $d$ is not adjacent to a vertex $f \in F_{5,6} \cup F_{6,1}$, then $6f$ and $wd$ induce a $2P_2$ by (9). This proves the claim for $D_{2,3}$. ■

We proceed with a few claims that help to show that certain sets are empty.

**Claim 1.** Either $D_{1,2}$ or $D_{4,5}$ is empty.

**Proof of Claim 1.** Suppose not. Let $d_{12} \in D_{1,2}$ and $d_{45} \in D_{4,5}$. By (7)-(19), $N(d_{12}) \subseteq N(w)$ unless $d_{12}$ has a neighbor $f \in F_{3,4} \cup F_{5,6}$. Similarly, $N(d_{45}) \subseteq N(w)$ unless $d_{45}$ has a neighbor $f' \in F_{3,4} \cup F_{5,6}$. By (11) and (19), $d_{12}f$ and $d_{45}f'$ induce a $2P_2$, a contradiction. ■

**Claim 2.** Each vertex in $T_1$ has a non-neighbor in $T_5$ and each vertex in $T_5$ has a non-neighbor in $T_1$.
By symmetry, each vertex in $T_2$ has a non-neighbor in $T_4$ and each vertex in $T_4$ has a non-neighbor in $T_2$.

**Proof of Claim 2.** Let $t_1 \in T_1$. Let

$$X = \{6,1,2\} \cup W \cup D_{3,4} \cup D_{4,5} \cup T_3 \cup T_4 \cup T_2 \cup F_{3,4} \cup F_{4,5}.$$  

Note that $N(4) = \{X \cup T_5 \cup F_{5,6} \cup N(t_1) \subseteq X \cup T_5 \cup F_{5,6} \cup T_6 \}$ by the properties we have proved. Since $G$ contains no pair of comparable vertices, $t_1$ has a neighbor $t_6 \in T_6$ and there exists a vertex $t \in N(4) \setminus N(t_1)$. Clearly, $t \in F_{5,6} \cup T_5$. If $t \in F_{5,6}$, then $4t$ and $t_1 t_6$ induce a $2P_2$ since $F_{5,6}$ and $T_6$ are anti-complete by (17). This shows that $t_1$ has a non-neighbor $t \in T_5$.
By symmetry, each vertex in $T_5$ has a non-neighbor in $T_1$. ■

**Claim 3.** Each vertex in $T_6$ has a neighbor in $T_1 \cup T_5$. By symmetry, each vertex in $T_3$ has a neighbor in $T_2 \cup T_4$. 

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Figure 2: The adjacency among $T_i$ and $D_{i,i+1}$. A thick line between two sets means that the two sets are complete, a thin line means the edges between the two sets can be arbitrary, and no line means that the two sets are anti-complete. For clarity, edges between two $D_{i,i+1}$ are not shown.

**Proof of Claim 3.** Let $t_6 \in T_6$. Let

$$X = \{5, 6, 1\} \cup D_{2,3} \cup D_{3,4} \cup T_2 \cup T_3 \cup T_4 \cup F_{2,3} \cup F_{3,4}.$$  

Note that $N(3) = X \cup F_{1,2} \cup F_{4,5}$ and $N(t_6) \subseteq X \cup T_1 \cup T_5 \cup F_{12} \cup F_{45}$. Since $G$ contains no pair of comparable vertices, $t_6$ has a neighbor in $T_1 \cup T_5$. \hfill $\Box$

**Claim 4.** If $D_{5,6} \cup D_{6,1} \neq \emptyset$, then $T_2$ and $T_4$ are complete. By symmetry, if $D_{2,3} \cup D_{3,4} \neq \emptyset$, then $T_1$ and $T_3$ are complete.

**Proof of Claim 4.** Let $d \in D_{5,6} \cup D_{6,1}$. Suppose that $t_2 \in T_2$ and $t_4 \in T_4$ are not adjacent. If $d \in D_{5,6}$, then $dt_2 \in E$ and $dt_4 \notin E$ by (16). Then $dt_2$ and $4t_4$ induce a $2P_2$. If $d \in D_{6,1}$, then $dt_4 \in E$ and $dt_2 \notin E$ by (16). Then $dt_4$ and $2t_2$ induce a $2P_2$. \hfill $\Box$

**Claim 5.** One of $F_{6,1}$, $F_{1,2}$ and $F_{2,3}$ is empty. By symmetry, one of $F_{3,4}$, $F_{4,5}$ and $F_{5,6}$ is empty.

**Proof of Claim 5.** Suppose that $f_{61} \in F_{6,1}$, $f_{12} \in F_{1,2}$, and $f_{23} \in F_{2,3}$. Then $f_{61}f_{23}$ and $f_{12w}$ induce a $2P_2$ by (9) and (12). \hfill $\Box$

By Claim 1, we may assume that $D_{4,5} = \emptyset$. It follows from (13), (14) and (15) that either $T_1$ and $T_5$ are complete or $T_2$ and $T_4$ are complete for otherwise $G$ would contain a $2P_2$ (see Figure 2). By symmetry, we may assume that $T_1$ and $T_5$ are complete. It then follows from Claim 2 and Claim 3 that $T_1 \cup T_5 \cup T_6 = \emptyset$.

If $D_{5,6} \cup D_{6,1} \neq \emptyset$, then $T_2 \cup T_3 \cup T_4 = \emptyset$ due to Claim 2-Claim 4. In the following we shall use this fact without explicitly mentioning it. We divide our proof into four cases depending on whether $F_{1,2}$ and $F_{4,5}$ are empty or not. One can verify that each of the partitions of $V(G)$ into 4 subsets in the following is a 4-coloring of $G$ using the properties we have proved. For convenience, we draw Figure 3 to visualize the adjacency among $D_{i,i+1}$ and $F_{i,i+1}$. From Figure 3 it can be seen that if $T_2 \cup T_3 \cup T_4 = \emptyset$, then we can use the symmetry of $H$ under its automorphism $f : V(H) \to V(H)$ with $f(1) = 2$, $f(2) = 1$, $f(3) = 6$, $f(4) = 5$, $f(5) = 4$, $f(6) = 3$ and $f(w) = w$. 

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Case 1. Both $F_{1,2}$ and $F_{4,5}$ are not empty. Let $f_{12} \in F_{1,2}$ and $f_{45} \in F_{4,5}$. We first show that $F_{1,2} \cup F_{4,5}$ is anti-complete to $D_{2,3} \cup D_{3,4} \cup D_{5,6} \cup D_{6,1}$. By symmetry, it suffices to show that $F_{1,2} \cup F_{4,5}$ is anti-complete to $D_{2,3}$. Suppose that $d \in D_{2,3}$ and $f \in F_{1,2} \cup F_{4,5}$ are adjacent. By (19), $f \in F_{4,5}$. Then $df$ and $f_{12}$ induce a $2P_2$. On the other hand, it follows from Claim 5 and (12) that at most one of $F_{2,3}, F_{3,4}, F_{5,6}$ and $F_{6,1}$ is not empty.

- If $F_{2,3} \neq \emptyset$, then $G$ has a 4-coloring:
  \[
  F_{4,5} \cup D_{2,3} \cup D_{3,4} \cup \{1\} \cup T_4, \\
  F_{2,3} \cup D_{1,2} \cup W \cup \{6\} \cup T_3, \\
  F_{1,2} \cup \{4,5\} \cup T_2, \\
  D_{5,6} \cup D_{6,1} \cup \{2,3\}.
  \]

- Suppose that $F_{6,1} \neq \emptyset$.
  If $D_{5,6} \cup D_{6,1} \neq \emptyset$, then $G$ has a 4-coloring:
  \[
  F_{4,5} \cup D_{5,6} \cup D_{6,1} \cup \{2\}, \\
  F_{6,1} \cup D_{1,2} \cup W \cup \{3\}, \\
  F_{1,2} \cup \{4,5\}, \\
  D_{2,3} \cup D_{3,4} \cup \{1,6\}.
  \]
If \( D_{5,6} \cup D_{6,1} = \emptyset \), then \( G \) has a 4-coloring:

\[
\begin{align*}
F_{4,5} &\cup \{1, 2\} \cup T_1, \\
F_{6,1} &\cup D_{1,2} \cup W \cup \{3\}, \\
F_{1,2} &\cup \{4, 5\} \cup T_2, \\
D_{2,3} &\cup D_{3,4} \cup \{6\} \cup T_3.
\end{align*}
\]

- Suppose that \( F_{3,4} \neq \emptyset \). Note first that no vertex \( d \in D_{1,2} \) can have a neighbor in both \( F_{1,2} \) and \( F_{3,4} \) for otherwise a neighbor of \( d \) in \( F_{1,2} \), a neighbor of \( d \) in \( F_{3,4} \), \( d \) and 2 induce a \( K_4 \). Let \( D'_{1,2} \) be the set of vertices in \( D_{1,2} \) that are anti-complete to \( F_{3,4} \) and \( D''_{1,2} = D_{1,2} \setminus D'_{1,2} \). Then \( G \) has a 4-coloring:

\[
\begin{align*}
F_{4,5} &\cup D_{2,3} \cup D_{3,4} \cup \{1\} \cup T_4, \\
F_{3,4} &\cup D'_{1,2} \cup W \cup \{6\} \cup T_3, \\
F_{1,2} &\cup D''_{1,2} \cup \{4, 5\} \cup T_2, \\
D_{5,6} &\cup D_{6,1} \cup \{2, 3\}.
\end{align*}
\]

- Suppose that \( F_{5,6} \neq \emptyset \). Note first that no vertex \( d \in D_{1,2} \) can have a neighbor in both \( F_{1,2} \) and \( F_{5,6} \) for otherwise a neighbor of \( d \) in \( F_{1,2} \), a neighbor of \( d \) in \( F_{5,6} \), \( d \) and 1 induce a \( K_4 \). Let \( D'_{1,2} \) be the set of vertices in \( D_{1,2} \) that are anti-complete to \( F_{5,6} \) and \( D''_{1,2} = D_{1,2} \setminus D'_{1,2} \). By (17) and (18), \( F_{5,6} \) and \( T_3 \cup T_4 \) are complete. Since \( G \) is \( K_4 \)-free, \( T_3 \) and \( T_4 \) are anti-complete. Then \( G \) has a 4-coloring:

\[
\begin{align*}
F_{4,5} &\cup D_{5,6} \cup D_{6,1} \cup \{2\}, \\
F_{5,6} &\cup D'_{1,2} \cup W \cup \{3\}, \\
F_{1,2} &\cup D''_{1,2} \cup \{4, 5\} \cup T_2, \\
D_{2,3} &\cup D_{3,4} \cup \{1, 6\} \cup T_3 \cup T_4.
\end{align*}
\]

**Case 2.** Both \( F_{1,2} \) and \( F_{4,5} \) are empty. By (12) and the fact that \( G \) is 2\( P_2 \)-free, one of \( F_{2,3} \), \( F_{3,4} \), \( F_{5,6} \) and \( F_{6,1} \) is empty. By (11), (19), (12) and \( K_4 \)-freeness of \( G \), either \( D_{5,6} \) and \( F_{5,6} \) are anti-complete or \( D_{3,4} \) and \( F_{3,4} \) are anti-complete.

- Suppose that \( F_{6,1} = \emptyset \).

  If \( D_{5,6} \) and \( F_{5,6} \) are anti-complete, then \( G \) has a 4-coloring:

\[
\begin{align*}
F_{2,3} &\cup F_{3,4} \cup W \cup \{6\} \cup T_3, \\
F_{5,6} &\cup D_{5,6} \cup \{2, 3\}, \\
D_{1,2} &\cup D_{6,1} \cup \{4, 5\} \cup T_2, \\
D_{2,3} &\cup D_{3,4} \cup \{1\} \cup T_4.
\end{align*}
\]

Now assume that \( D_{3,4} \) and \( F_{3,4} \) are anti-complete.

If \( D_{5,6} \cup D_{6,1} \neq \emptyset \), then \( G \) has a 4-coloring:

\[
\begin{align*}
F_{2,3} &\cup F_{5,6} \cup W, \\
F_{3,4} &\cup D_{3,4} \cup \{6, 1\}, \\
D_{1,2} &\cup D_{2,3} \cup \{4, 5\}, \\
D_{5,6} &\cup D_{6,1} \cup \{2, 3\}.
\end{align*}
\]
If $D_{5,6} \cup D_{6,1} = \emptyset$, then $G$ has a 4-coloring:

\[
\begin{align*}
F_{2,3} & \cup D_{1,2} \cup W \cup \{6\} \cup T_3, \\
F_{3,4} & \cup D_{3,4} \cup \{1\} \cup T_1, \\
F_{5,6} & \cup \{2, 3\}, \\
D_{2,3} & \cup \{4, 5\} \cup T_2.
\end{align*}
\]

- Suppose that $F_{2,3} = \emptyset$.
  Suppose first that $D_{3,4}$ and $F_{3,4}$ are anti-complete.
  If $D_{5,6} \cup D_{6,1} \neq \emptyset$, then $G$ has a 4-coloring:

\[
\begin{align*}
F_{6,1} & \cup F_{5,6} \cup W \cup \{3\}, \\
F_{3,4} & \cup D_{3,4} \cup \{6, 1\}, \\
D_{1,2} & \cup D_{2,3} \cup \{4, 5\}, \\
D_{6,1} & \cup D_{5,6} \cup \{2\}.
\end{align*}
\]

If $D_{5,6} \cup D_{6,1} = \emptyset$, then $G$ has a 4-coloring:

\[
\begin{align*}
F_{6,1} & \cup F_{5,6} \cup W \cup \{3\}, \\
F_{3,4} & \cup D_{3,4} \cup \{6\} \cup T_3, \\
D_{1,2} & \cup D_{2,3} \cup \{4, 5\} \cup T_2, \\
\{1, 2\} & \cup T_4.
\end{align*}
\]

Suppose now that $D_{3,4}$ and $F_{3,4}$ are not anti-complete and that $D_{5,6}$ and $F_{5,6}$ are anti-complete. By (16) and (17), $D_{3,4} \cup F_{3,4}$ are anti-complete to $T_3 \cup T_4$. Since $G$ is $2P_2$-free, it follows that $T_3$ and $T_4$ are anti-complete. Then $G$ has a 4-coloring:

\[
\begin{align*}
F_{6,1} & \cup F_{3,4} \cup W, \\
F_{5,6} & \cup D_{5,6} \cup \{2, 3\}, \\
D_{1,2} & \cup D_{6,1} \cup \{4, 5\} \cup T_2, \\
D_{2,3} & \cup D_{3,4} \cup \{6, 1\} \cup T_3 \cup T_4.
\end{align*}
\]

- Suppose that $F_{5,6} = \emptyset$. If $F_{6,1} = \emptyset$, then $G$ has a 4-coloring as above. So, we can assume that $F_{6,1} \neq \emptyset$. Let $f_{61} \in F_{6,1}$. If $d \in D_{2,3}$ and $f \in F_{2,3}$ are adjacent, then $\{2, f_{61}, d, f\}$ induces a $K_4$ by (12) and (19). So, $D_{2,3}$ and $F_{2,3}$ are anti-complete. By (17) and (18), $F_{6,1}$ and $T_2 \cup T_3$ are complete. Since $G$ is $K_4$-free, $T_2$ and $T_3$ are anti-complete. Then $G$ has a 4-coloring:

\[
\begin{align*}
F_{3,4} & \cup F_{6,1} \cup W, \\
F_{2,3} & \cup D_{1,2} \cup D_{2,3} \cup \{5, 6\} \cup T_2 \cup T_3, \\
D_{3,4} & \cup \{1, 2\} \cup T_4, \\
D_{5,6} & \cup D_{6,1} \cup \{3, 4\}.
\end{align*}
\]

- Suppose that $F_{3,4} = \emptyset$. If $F_{2,3} = \emptyset$, then $G$ has a 4-coloring as above. So, we can assume that $F_{2,3} \neq \emptyset$. Let $f_{23} \in F_{2,3}$. If $d \in D_{6,1}$ and $f \in F_{6,1}$ are adjacent, then $\{1, f_{23}, d, f\}$ induces a $K_4$ by (12) and (19). So, $D_{6,1}$ and $F_{6,1}$ are anti-complete.
If $D_{5,6} \cup D_{6,1} \neq \emptyset$, then $G$ has a 4-coloring:

- $F_{5,6} \cup F_{2,3} \cup W$,
- $F_{6,1} \cup D_{1,2} \cup D_{6,1} \cup \{3, 4\}$,
- $D_{5,6} \cup \{1, 2\}$,
- $D_{3,4} \cup D_{2,3} \cup \{5, 6\}$.

If $D_{5,6} \cup D_{6,1} \neq \emptyset$, then $G$ has a 4-coloring:

- $F_{5,6} \cup F_{6,1} \cup W \cup \{3\}$,
- $F_{2,3} \cup D_{1,2} \cup \{6\} \cup T_3$,
- $D_{2,3} \cup \{4, 5\} \cup T_2$,
- $D_{3,4} \cup \{1, 2\} \cup T_4$.

**Case 3.** The set $F_{1,2} = \emptyset$ but the set $F_{4,5} \neq \emptyset$. By Claim 5, either $F_{3,4} = \emptyset$ or $F_{5,6} = \emptyset$. By (11), (19), (12) and $K_4$-freeness of $G$, either $D_{2,3}$ and $F_{2,3}$ are anti-complete or $D_{6,1}$ and $F_{6,1}$ are anti-complete.

- Suppose that $F_{5,6} = \emptyset$.
  If $D_{6,1}$ and $F_{6,1}$ are anti-complete, then $G$ has a 4-coloring:

- $F_{2,3} \cup F_{3,4} \cup W \cup \{6\} \cup T_3$,
- $F_{6,1} \cup D_{1,2} \cup D_{6,1} \cup \{3, 4\}$,
- $F_{4,5} \cup D_{5,6} \cup \{1, 2\} \cup T_4$,
- $D_{2,3} \cup D_{3,4} \cup \{5\} \cup T_2$.

Now assume that $D_{2,3}$ and $F_{2,3}$ are anti-complete.
If $D_{5,6} \cup D_{6,1} \neq \emptyset$, then $G$ has a 4-coloring:

- $F_{3,4} \cup F_{6,1} \cup W$,
- $F_{2,3} \cup D_{1,2} \cup D_{2,3} \cup \{5, 6\}$,
- $F_{4,5} \cup D_{3,4} \cup \{1, 2\}$,
- $D_{5,6} \cup D_{6,1} \cup \{3, 4\}$.

If $D_{5,6} \cup D_{6,1} = \emptyset$, then $G$ has a 4-coloring:

- $F_{3,4} \cup W \cup \{6\} \cup T_3$,
- $F_{2,3} \cup D_{1,2} \cup D_{2,3} \cup \{5\} \cup T_2$,
- $F_{4,5} \cup D_{3,4} \cup \{1, 2\} \cup T_4$,
- $F_{6,1} \cup \{3, 4\}$.

- Suppose that $F_{3,4} = \emptyset$. Suppose first that $D_{2,3}$ and $F_{2,3}$ are anti-complete.
  If $D_{5,6} \cup D_{6,1} \neq \emptyset$, then $G$ has a 4-coloring:

- $F_{5,6} \cup F_{6,1} \cup W \cup \{3\}$,
- $F_{2,3} \cup D_{1,2} \cup D_{2,3} \cup \{5, 6\}$,
- $F_{4,5} \cup D_{3,4} \cup \{1, 2\}$,
- $D_{5,6} \cup D_{6,1} \cup \{4\}$.
If $D_{5,6} \cup D_{6,1} = \emptyset$, then $G$ has a 4-coloring:

\[
\begin{align*}
F_{5,6} \cup F_{6,1} \cup W \cup \{3\}, \\
F_{2,3} \cup D_{1,2} \cup D_{2,3} \cup \{6\} \cup T_3, \\
F_{4,5} \cup D_{3,4} \cup \{1,2\} \cup T_4, \\
\{4,5\} \cup T_2.
\end{align*}
\]

Now suppose that $D_{2,3}$ and $F_{2,3}$ are not anti-complete and that $D_{6,1}$ and $F_{6,1}$ are anti-complete. Then $T_2$ and $T_3$ are anti-complete for otherwise an edge between $T_2$ and $T_3$ and an edge between $D_{2,3}$ and $F_{2,3}$ induce a $2P_2$ by (16) and (17). Then $G$ has a 4-coloring:

\[
\begin{align*}
F_{5,6} \cup F_{2,3} \cup W, \\
F_{6,1} \cup D_{1,2} \cup D_{6,1} \cup \{3,4\}, \\
F_{4,5} \cup D_{5,6} \cup \{1,2\} \cup T_4, \\
D_{2,3} \cup D_{3,4} \cup \{5,6\} \cup T_2 \cup T_3.
\end{align*}
\]

**Case 4.** The set $F_{4,5} = \emptyset$ but the set $F_{1,2} \neq \emptyset$. By Claim 5, either $F_{2,3} = \emptyset$ or $F_{6,1} = \emptyset$. By (19) and (12), $F_{3,4}$ is complete to $D_{5,6} \cup F_{5,6}$. So, if $F_{3,4} \neq \emptyset$, then $D_{5,6}$ and $F_{5,6}$ are anti-complete for otherwise $G$ would contain a $K_4$. By symmetry, if $F_{5,6} \neq \emptyset$, then $D_{3,4}$ and $F_{3,4}$ are anti-complete. Moreover, either $D_{3,4}$ and $F_{3,4}$ are anti-complete or $D_{5,6}$ and $F_{5,6}$ are anti-complete. Similarly, either $D_{2,3}$ and $F_{2,3}$ are anti-complete or $D_{6,1}$ and $F_{6,1}$ are anti-complete.

- Suppose that $F_{6,1} = \emptyset$. If both $F_{3,4}$ and $F_{5,6}$ are not empty, then consider the following 4-coloring of $G - (D_{2,3} \cup D_{6,1})$:

\[
\begin{align*}
I_1 &= F_{2,3} \cup D_{1,2} \cup W \cup \{6\} \cup T_3, \\
I_2 &= F_{3,4} \cup D_{3,4} \cup \{1\} \cup T_4, \\
I_3 &= F_{5,6} \cup D_{5,6} \cup \{2,3\}, \\
I_4 &= F_{1,2} \cup \{4,5\} \cup T_2.
\end{align*}
\]

If $D_{2,3}$ and $F_{3,4}$ are anti-complete, then $G$ has a 4-coloring: $I_1$, $I_2 \cup D_{2,3}$, $I_3$ and $I_4 \cup D_{6,1}$. If $D_{6,1}$ and $F_{5,6}$ are anti-complete, then $G$ has a 4-coloring: $I_1$, $I_2 \cup D_{6,1}$ and $I_4 \cup D_{2,3}$. So, one of $F_{3,4}$ and $F_{5,6}$ is empty.

Suppose that $F_{5,6} = \emptyset$. Recall that no vertex in $D_{1,2}$ can have a neighbor in both $F_{1,2}$ and $F_{3,4}$. Let $D'_{1,2}$ be the set of vertices in $D_{1,2}$ that are anti-complete to $F_{1,2}$ and $D''_{1,2} = D_{1,2} \setminus D'_{1,2}$. Then $G$ has a 4-coloring:

\[
\begin{align*}
F_{1,2} \cup D'_{1,2} \cup \{4,5\} \cup T_2, \\
F_{2,3} \cup F_{3,4} \cup D''_{1,2} \cup W \cup \{6\} \cup T_3, \\
D_{2,3} \cup D_{3,4} \cup \{1\} \cup T_4, \\
D_{5,6} \cup D_{6,1} \cup \{2,3\}.
\end{align*}
\]

Suppose now that $F_{5,6} \neq \emptyset$ and $F_{3,4} = \emptyset$. Note that no vertex in $D_{1,2}$ can have a neighbor in both $F_{1,2}$ and $F_{5,6}$. Let $D'_{1,2}$ be the set of vertices in $D_{1,2}$ that are anti-complete to $F_{1,2}$ and $D''_{1,2} = D_{1,2} \setminus D'_{1,2}$. Moreover, recall that since $F_{5,6} \neq \emptyset$, $T_3$ and $T_4$ are anti-complete. Then $G$
has a 4-coloring:

\[
F_{1,2} \cup D_{2,3} \cup D'_{1,2} \cup \{4, 5\} \cup T_2, \\
F_{2,3} \cup F_{3,6} \cup D''_{1,2} \cup W, \\
D_{3,4} \cup \{6, 1\} \cup T_3 \cup T_4, \\
D_{5,6} \cup D_{6,1} \cup \{2, 3\}. 
\]

- Suppose that \( F_{2,3} = \emptyset \). If both \( F_{3,4} \) and \( F_{5,6} \) are not empty, then consider the following 4-coloring of \( G - (D_{2,3} \cup D_{6,1}) \):

\[
I_1 = F_{6,1} \cup D_{1,2} \cup W \cup \{3\}, \\
I_2 = F_{5,6} \cup D_{5,6} \cup \{2\}, \\
I_3 = F_{3,4} \cup D_{3,4} \cup \{6, 1\} \cup T_3 \cup T_4, \\
I_4 = F_{1,2} \cup \{4, 5\} \cup T_2.
\]

If \( D_{2,3} \) and \( F_{3,4} \) are anti-complete, then \( G \) has a 4-coloring: \( I_1, I_2, I_3 \cup D_{2,3} \) and \( I_4 \cup D_{6,1} \). If \( D_{6,1} \) and \( F_{5,6} \) are anti-complete, then \( G \) has a 4-coloring: \( I_1, I_2 \cup D_{6,1}, I_3 \) and \( I_4 \cup D_{2,3} \). So, one of \( F_{3,4} \) and \( F_{5,6} \) is empty.

Suppose that \( F_{5,6} \neq \emptyset \). So, \( F_{3,4} = \emptyset \). Recall that no vertex in \( D_{1,2} \) can have a neighbor in both \( F_{1,2} \) and \( F_{3,4} \). Let \( D'_{1,2} \) be the set of vertices in \( D_{1,2} \) that are anti-complete to \( F_{1,2} \) and \( D''_{1,2} = D_{1,2} \setminus D'_{1,2} \). Moreover, \( T_3 \) and \( T_4 \) are anti-complete. Then \( G \) has a 4-coloring:

\[
F_{1,2} \cup D'_{1,2} \cup \{4, 5\} \cup T_2, \\
F_{6,1} \cup F_{3,4} \cup D''_{1,2} \cup W \cup \{3\}, \\
D_{6,1} \cup D_{5,6} \cup \{2\}, \\
D_{2,3} \cup D_{3,4} \cup \{6, 1\} \cup T_3 \cup T_4.
\]

Suppose now that \( F_{5,6} = \emptyset \). Recall that no vertex in \( D_{1,2} \) can have a neighbor in both \( F_{1,2} \) and \( F_{3,4} \). Let \( D'_{1,2} \) be the set of vertices in \( D_{1,2} \) that are anti-complete to \( F_{1,2} \) and \( D''_{1,2} = D_{1,2} \setminus D'_{1,2} \). If \( D_{5,6} \cup D_{6,1} \neq \emptyset \), then \( G \) has a 4-coloring:

\[
F_{1,2} \cup D_{6,1} \cup D'_{1,2} \cup \{4, 5\}, \\
F_{6,1} \cup F_{3,4} \cup D''_{1,2} \cup W, \\
D_{5,6} \cup \{2, 3\}, \\
D_{2,3} \cup D_{3,4} \cup \{6, 1\}.
\]

If \( D_{5,6} \cup D_{6,1} = \emptyset \), then \( G \) has a 4-coloring:

\[
F_{1,2} \cup D_{2,3} \cup D'_{1,2} \cup \{4, 5\} \cup T_2, \\
F_{3,4} \cup D''_{1,2} \cup W \cup \{6\} \cup T_3, \\
F_{5,6} \cup \{3\}, \\
D_{3,4} \cup \{1, 2\} \cup T_4.
\]

In any case we have found a 4-coloring of \( G \). This completes our proof. \qed
4 Eliminate $H_2$

**Lemma 3.** Let $G$ be a connected $(2P_2, K_4, H_1)$-free graph with no pair of comparable vertices. If $G$ contains an induced $H_2$, then $\chi(G) \leq 4$.

*Proof.* Let $H = C \cup \{ f \}$ be an induced $H_2$ where $C = 12345$ induces a $C_5$ and $f$ is adjacent to 1, 2, 3 and 4. We partition $V \setminus C$ into subsets of $Z$, $R_i$, $Y_i$, $F_i$ and $U$ as in section 2. By the fact that $G$ is $H_1$-free and (11), it follows that $F_i = \emptyset$ for $i \neq 5$. Note that $f \in F_5$. We choose $H$ such that

- $|U|$ is minimum.
- $|F_5|$ is minimum subject to the above.

(a) $U$ is complete to $R_i$ for $1 \leq i \leq 5$.

Suppose not. Let $u \in U$ be nonadjacent to $r_i \in R_i$ for some $i$. Suppose first that $1 \leq i \leq 4$. Note that $C' = C \setminus \{i\} \cup \{r_i\}$ induces a $C_5$ and $H' = C' \cup \{u\}$ induces an $H_2$. Since $5 \in C'$, it follows that $F_5 \cap U' = \emptyset$ and $U' \subseteq U$. Moreover, $u \in U$ is not in $U'$ since $u$ is not adjacent to $r_i$. This implies that $|U'| < |U|$, contradicting the choice of $H$.

Now suppose that $i = 5$. Note that $C' = C \setminus \{5\} \cup \{r_5\}$ induces a $C_5$ and $H' = C' \cup \{u\}$ induces an $H_2$. Note that $U' \subseteq F_5 \cup U$ and $u \notin U'$ since $u$ is not adjacent to $r_i$. By the choice of $H$, there exists a vertex $f' \in F_5$ such that $f'$ is adjacent to $r_5$. By (2), $u$ and $f$ are not adjacent. But then $f_r 5$ and $5u$ induce a $2P_2$.

(b) If $U \neq \emptyset$, then $R_i$ and $R_{i+2}$ are anti-complete.

Let $u \in U$. If $r_i \in R_i$ and $r_{i+2} \in R_{i+2}$ are not adjacent, then $\{r_i, r_{i+2}, i + 1, u\}$ induces a $K_4$, since $u$ is adjacent to $r_i$ and $r_{i+2}$ by (a).

Suppose first that $U \neq \emptyset$. By (b), $R_i$ and $R_{i+2}$ are anti-complete. Recall that $Y_i$ and $Y_{i+2}$ are anti-complete by (10). By (12), $R_i$ is anti-complete to $Y_5 \cup Y_2$ and $R_4$ is anti-complete to $Y_5 \cup Y_3$. By (8), $F_5$ is anti-complete to $Y_1 \cup Y_4$. By (6), either $Y_3$ and $R_2$ are anti-complete or $Y_2$ and $R_3$ are anti-complete.

If $Y_3$ and $R_2$ are anti-complete, then $G$ admits the following 4-coloring:

\[
\begin{align*}
Y_1 & \cup Y_4 \cup U \cup F_5 & (10)(2)(8) \\
Y_2 & \cup Y_5 \cup R_4 \cup \{1\} & (10)(12) \\
Y_3 & \cup R_2 \cup R_4 \cup \{2, 4\} & (b)(12) \\
R_3 & \cup R_5 \cup Z \cup \{3, 5\} & (b)(2)
\end{align*}
\]

If $Y_2$ and $R_3$ are anti-complete, then $G$ admits the following 4-coloring:

\[
\begin{align*}
Y_1 & \cup Y_4 \cup U \cup F_5 & (10)(2)(8) \\
Y_3 & \cup Y_5 \cup R_4 \cup \{4\} & (10)(12) \\
Y_2 & \cup R_1 \cup R_3 \cup \{1, 3\} & (b)(12) \\
R_2 & \cup R_5 \cup Z \cup \{2, 5\} & (b)(2)
\end{align*}
\]

This shows that if $U \neq \emptyset$, then $G$ has a 4-coloring. Therefore, we can assume in the following that $U = \emptyset$.

(c) Each vertex in $R_2 \cup R_3$ is either complete or anti-complete to $F_5$.

Suppose not. Let $r \in R_2 \cup R_3$ be adjacent to $f \in F_5$ and not adjacent to $f' \in F_5$. By symmetry, we may assume that $r \in R_2$. Note that $C' = C \setminus \{2\} \cup \{r\}$ induces a $C_5$ and $H' = C' \cup \{f\}$ induces an $H_2$. Clearly, $f' \notin F_5$. By the choice of $H$, there exists a vertex
y ∈ Y such that y ∈ F′. This means that y is not adjacent to 5 but adjacent to 1, 3, 4 and r2. This implies that y ∈ Y1. By (8), f′ and y are not adjacent. But now f′′ and yr2 induce a 2P2.

By (8), (9) and (c), only vertices in R5 ∪ Z can distinguish two vertices in F5. By (1), R5 ∪ Z is stable and so (F5, R5 ∪ Z) is a 2P2-free bipartite graph. This implies that F5 = {f} since any two vertices in F are comparable. Let R′1 = N(f) ∩ R1 and R′′1 = R1 \ R′1 for i = 2, 3, 5. We now prove properties of R′ i and R′′ i.

(d) R′ 5 is anti-complete to R′ 3 ∪ R′ 3.

Suppose that r′ 5 ∈ R′ 5 and r′ 3 ∈ R′ 3 are adjacent. Then {r′ 5, r′ 3, 1, f} induces a K4.

(e) R′ 5 is anti-complete to Y2 ∪ Y3.

Suppose that r′ 5 ∈ R′ 5 and y2 ∈ Y2 are adjacent. By (8), f and y2 are adjacent. Then {r′ 5, 4, y2, f} induces a K4.

(f) R′ 5 is anti-complete to R′ 4. By symmetry, R′ 5 is anti-complete to R1.

Suppose that r′ 5 ∈ R′ 5 and r4 ∈ R4 are adjacent. By (9), f and r4 are adjacent. Then {r′ 5, r4, 3, f} induces a K4.

(g) R′′ 5 is anti-complete to R′′ 3 ∪ R′′ 3.

Suppose that r′′ 5 ∈ R′′ 5 and r′′ 3 ∈ R′′ 3 are adjacent. Then r′′ 5 and f2 induce a 2P2.

(h) Y5 is anti-complete to R′′ 2 ∪ R′′ 3.

Suppose that y5 ∈ Y5 and r′′ 3 ∈ R′′ 3 are adjacent. By (8), f and y are not adjacent. Then y5y′′ 3 and f4 induce a 2P2.

(i) R′′ 5 is anti-complete to Y1 ∪ Y4.

Suppose that r′′ 5 ∈ R′′ 5 and y4 ∈ Y4 are adjacent. By (8), f and y4 are not adjacent. Then y′′ 5y4 and f2 induce a 2P2.

(j) R′′ 5 is anti-complete to Y1. By symmetry, R′′ 5 is anti-complete to Y4.

Suppose that r′′ 5 ∈ R′′ 5 and y1 ∈ Y1 are adjacent. By (8), f and y1 are not adjacent. Then y′′ 5y1 and f2 induce a 2P2.

(k) R′′ 5 is anti-complete to Y3. By symmetry, R′′ 5 is anti-complete to Y2.

Suppose that r′′ 5 ∈ R′′ 5 and y3 ∈ Y3 are adjacent. By (9), f and y3 are adjacent. Then {r′′ 5, y3, 3, f} induces a K4.

(l) Y5 is complete to R′′ 2 ∪ R′′ 3.

Suppose that y5 ∈ Y5 and r′ 2 ∈ R′′ 2 are not adjacent. By (8), f and y5 are not adjacent. Then fyr′ 2 and 5y5 induce a 2P2.

We now prove properties of Z.

(m) Any vertex in Z is anti-complete to either Y2 or Y3.

Suppose not. Then there exists a vertex z ∈ Z that is adjacent to a vertex yi ∈ Y i for i = 2, 3. By (8), f is adjacent to y2 and y3. Moreover, y2 and y3 are adjacent by (4). This implies that f and z are not adjacent for otherwise {f, z, y1, yi+1} would induce a K4.

We now show that z is anti-complete to Y1 ∪ Y4 ∪ Y5. Suppose not. Let z be adjacent to a vertex y ∈ Y1 ∪ Y4 ∪ Y5. Note that there exists a vertex i ∈ NC(f) such that i is not adjacent to y. Moreover, f and y are not adjacent by (8). Then zy and if induce a 2P2.
This shows that $z$ is anti-complete to $Y_1 \cup Y_4 \cup Y_5$. Recall that $Z$ is anti-complete to $R_i$ for each $i$ by (1). Therefore, $N(z) \subseteq Y_2 \cup Y_3 \subseteq N(f)$, contradicting the assumption that $G$ has no pair of comparable vertices. ■

(n) If $z \in Z$ is not adjacent to $y_i \in Y_i$, then $y_i$ is complete to $N(z) \setminus Y_i$.

It suffices to prove for $i = 1$ by symmetry. Note that $N(z) \setminus Y_1 = (N(z) \cap (Y_2 \cup Y_5)) \cup (N(z) \cap (Y_3 \cup Y_4))$. By (4), $y_i$ is complete to $N(z) \cap (Y_2 \cup Y_5)$. It remains to show that $y_1$ is complete to $N(z) \cap (Y_3 \cup Y_4)$. Suppose not. Let $y \in N(z) \cap (Y_3 \cup Y_4)$ be nonadjacent to $y_1$. By symmetry, we may assume that $y \in Y_3$. Then $zy$ and $y_1y$ induce a $2P_2$. ■

(o) If $z$ is anti-complete to $Y_i$ for some $i \in \{2, 3\}$, then $Y_i = \emptyset$.

Suppose that $z$ is anti-complete to $Y_2$ and $Y_2$ contains a vertex $y_2$. It follows from (n) that $N(z) \subseteq N(y_2)$, contradicting the assumption that $G$ contains no pair of comparable vertices. ■

If $Y_5 = \emptyset$, then $N(5) = \{1, 4\} \cup R_1 \cup R_4 \cup Y_2 \cup Y_3 \subseteq N(f)$ by (8) and (9). This contradicts the assumption $G$ contains no pair of comparable vertices. So, we assume in the following that $Y_5$ contains a vertex $y_5$. We claim now that either $R''_2$ or $R''_3$ is empty. Suppose not. Let $r''_i \in R''_i$ for $i = 2, 3$. By (3), $r''_2$ and $r''_3$ are adjacent. Moreover, $y_5$ is not adjacent to $r''_2$ and $r''_3$ by (h). Then $r''_2r''_3$ and $y_5y$ induce a $2P_2$. This proves that either $R''_2$ or $R''_3$ is empty. We consider two cases depending on whether $f$ has a neighbor in $R_5$.

Case 1. $R''_5 = \emptyset$, i.e., $f$ has no neighbors in $R_5$. Therefore, $R_5 = R''_5$. Recall that either $R''_2$ or $R''_3$ is empty. By symmetry, we may assume that $R''_2 = \emptyset$. Then $R_2 = R'_2$ and so $R_2$ and $R_4$ are anti-complete by (1). Let $Y'_2 = \{y \in Y_2 : y$ is anti-complete to $Y_5\}$ and $Y''_2 = Y_2 \setminus Y'_2$. Note that each vertex in $Y''_2$ has a neighbor in $Y_5$ by the definition and so is anti-complete to $Y_4$ by (7). Then the following is a 4-coloring of $G - (R_3 \cup Z)$:

$$
\begin{align*}
I_1 &= Y''_2 \cup Y_5 \cup R_1 \cup \{1\} \\
I_2 &= Y''_2 \cup Y_4 \cup R_4 \cup \{3\} \\
I_3 &= R_2(= R'_2) \cup R_4 \cup Y_3 \cup \{2, 4\} & \text{(Definition of $Y''_2$)} \\
I_4 &= Y_1 \cup R_5(= R''_5) \cup \{f, 5\} & \text{(12) (k)}
\end{align*}
$$

We now extend $\phi$ to $R_3$ as follows. Since $R_3$ is stable by (1), it suffices to explain how to extend $\phi$ to each vertex in $R_3$ independently. Let $r_3 \in R_3$ be an arbitrary vertex. Suppose first that $r_3 \in R'_3$. By (f) and (k), $r_3$ is anti-complete to $R_1 \cup R_2$. By (13), $r_3$ is anti-complete to either $Y_4$ or $Y_5$. Therefore, we can add $r_3$ to either $I_1$ or $I_2$. Now suppose that $r_3 \in R''_3$. By (g) and (j), $r_3$ is anti-complete to $Y_4 \cup R_5$. By (13), $r_3$ is anti-complete to either $Y_1$ or $Y_2$. Therefore, we can add $r_3$ to either $I_2$ or $I_4$. This shows that $G - Z$ admits a 4-coloring $\phi' = \{(I'_1, I'_2, I'_3, I'4)\}$ with $I_i \subseteq I'_i$ for each $1 \leq i \leq 4$.

We now obtain a 4-coloring of $G$ by either extending $\phi'$ to $Z$ or by finding another 4-coloring of $G$. If $Z$ is anti-complete to $Y_3$, then we can extend $\phi'$ by adding $Z$ to $I'_3$. So, we assume that there is a vertex $z \in Z$ that is adjacent to a vertex in $Y_3$. It then follows from (m) and (o) that $Y_2 = \emptyset$. If each vertex in $Z$ is anti-complete to one of $Y_3$, $Y_4$ and $Y_5$, then we can extend $\phi'$ to $Z$ by adding each vertex in $Z$ to $I'_1$, $I'_2$ or $I'_4$ (since $Y_2 = \emptyset$). Therefore, let $z \in Z$ be adjacent to $y_i \in Y_i$ for $i \in \{3, 4, 5\}$. We prove some additional properties using the existence of $y_3$, $y_4$ and $y_5$. First of all, $R_1$ and $R_4$ are anti-complete. Suppose not. Let $r_1 \in R_1$ and $r_4 \in R_4$ be adjacent. By (12), $y_5$ is not adjacent to $r_1$ and $r_4$. Then $r_1r_4$ and $z_5y$ induce a $2P_2$. Secondly, $y_3$ and $y_5$ are not adjacent for otherwise $\{y_3, y_4, y_5, z\}$ induces a $K_4$. Thirdly, $Y_1$ and $Y_4$ are anti-complete to each other. Suppose not. Then $Y_1$ contains a vertex $y_1$ that is not anti-complete to $Y_4$. By (7), $y_1$ is anti-complete to $Y_3$. Then $fy_3$ and $y_1y_5$ induce a $2P_2$. Now
$G$ admits the following 4-coloring:

\[
\begin{align*}
Y_1 & \cup R''_3(= R_3) \cup Y_4 \cup \{f, 5\} & (i) \\
Y_3 & \cup R''_2(= R_2) \cup \{2\} & (k) \\
R_1 & \cup R_4 \cup Y_5 \cup \{1, 4\} & (12) \\
R_4 & \cup Z \cup \{3\} & (1)
\end{align*}
\]

**Case 2.** $R'_y \neq \emptyset$. Let $r'_y \in R'_y$. If $r_1 \in R_1$ and $r_4 \in R_4$ are adjacent, then $\{r_1, r_4, r'_y, f\}$ induces a $K_4$ by (3) and (9). So, $R_1$ and $R_4$ are anti-complete. We now consider two subcases.

**Case 2.1.** $R''_3$ and $Y_3$ are not anti-complete. Let $r''_3 \in R''_2$ and $y_3 \in Y_3$ be adjacent. We claim first that $Y_1$ and $Y_4$ are anti-complete. Suppose not. Let $y_1 \in Y_1$ and $y_4 \in Y_4$ be adjacent. Then $y_3$ and $y_4$ are adjacent by (4). By (7), $y_1$ is not adjacent to $y_3$. Moreover, $y_1$ is not adjacent to $r''_3$ by (j). But now $4y_1$ and $5y_3$ induce a $2P_3$. This shows that $Y_1$ and $Y_4$ are anti-complete. Moreover, $Y_2$ and $R_3$ are anti-complete by (6). Therefore, the following is a 4-coloring $\phi$ of $G - (R''_2 \cup Z)$.

\[
\begin{align*}
I_1 & = R_4 \cup Y_5 \cup R_1 \cup \{1, 4\} & (12) \\
I_2 & = Y_1 \cup R''_3 \cup Y_4 \cup \{f, 5\} & (i) \\
I_3 & = R_3 \cup Y_2 \cup \{3\} & (6) \\
I_4 & = Y_3 \cup R''_2 \cup R''_5 \cup \{2\} & (d)(e)(k)
\end{align*}
\]

We now explain how to extend $\phi$ to each vertex in $R''_2 \cup Z$. Since $R''_2 \cup Z$ is stable by (1), this will give a 4-coloring of $G$. By (m), we can add each vertex in $Z$ to either $I_3$ or $I_4$. Let $s'' \in R''_2$ be an arbitrary vertex. Then $s''$ is anti-complete to $R''_3 \cup Y_1$ by (g) and (j). If $s''$ is not anti-complete to $Y_3$, then $s''$ is anti-complete to $Y_4$ by (13) and thus we can add $s''$ to $I_2$. Now $s''$ is anti-complete to $Y_3$. We claim that $s''$ is anti-complete to $R''_3$. Suppose not. Then $s''$ is adjacent to some vertex $r' \in R''_3$. Note that $y_3$ is not adjacent to $s''$ by our assumption. Moreover, $y_3$ is not adjacent to $r'$ by (e). Then $s''r'$ and $5y_3$ induce a $2P_2$. This shows that $s''$ is anti-complete to $R''_3$ and thus we can add $s''$ to $I_4$.

**Case 2.2.** $R''_3$ and $Y_3$ are anti-complete. By symmetry, $R''_2$ and $Y_2$ are anti-complete. It follows from (k) that $R_2$ and $Y_3$ are anti-complete and $R_3$ and $Y_2$ are anti-complete. Recall that either $R''_2$ or $R''_3$ is empty. By symmetry, we may assume that $R''_2 = \emptyset$. Then $R_2 = R''_3$. We now claim that $R''_3$ is anti-complete to $Y_4$. Suppose not. Let $r'_3 \in R''_3$ be adjacent to $y_4 \in Y_4$. By (l), $r'_3$ is adjacent to $y_5$. But this contradicts (13). So, $R''_3$ is anti-complete to $Y_4$. By symmetry, $R''_2$ is anti-complete to $Y_1$. This together with (j) implies that $R_3$ and $R_2$ are anti-complete to $Y_4$ and $Y_1$, respectively. Let $Y'_4 = \{y \in Y_4 : y$ is anti-complete to $Y_1\}$ and $Y''_4 = Y_4 \setminus Y'_4$. Note that each vertex in $Y''_4$ has a neighbor in $Y_1$ and so is anti-complete to $Y_2$ by (7). Now $G - Z$ admits a 4-coloring $\phi$:

\[
\begin{align*}
I_1 & = R_4 \cup Y_5 \cup R_1 \cup \{1, 4\} & (12) \\
I_2 & = Y_1 \cup R''_3 \cup Y'_4 \cup \{f, 5\} & (i) \\
I_3 & = R_3 \cup Y_2 \cup Y''_4 \cup \{3\} & (6) \\
I_4 & = Y_3 \cup R''_2 \cup R''_5 \cup \{2\} & (d)(e)(k)
\end{align*}
\]

We now explain how to obtain a 4-coloring of $G$ based on $\phi$. If $Z$ is anti-complete to $Y_3$, then we can add $Z$ to $I_4$. So, assume that there exists a vertex in $Z$ that is adjacent to some vertex in $Y_3$. It then follows from (m) and (o) that $Y_2 = \emptyset$. If each vertex in $Z$ is anti-complete to one of $Y_3$, $Y''_4$ and $Y_5$, then we can extend $\phi'$ to $Z$ by adding each vertex in $Z$ to $I_1$, $I_3$ or $I_4$ (since $Y_2 = \emptyset$). Therefore, let $z \in Z$ be adjacent to $y_i \in Y_i$ for $i \in \{3, 5\}$ and be adjacent to $y_4 \in Y'_4$. Note that $y_3$ and $y_5$ are not adjacent for otherwise $\{y_3, y_4, y_5, z\}$ induces a $K_4$. We
claim that $Y_1$ and $Y_4$ are anti-complete to each other. Suppose not. Then $Y_1$ contains a vertex $y_1$ that is not anti-complete to $Y_4$. By (7), $y_1$ is anti-complete to $Y_3$. Then $fy_3$ and $y_1y_5$ induce a $2P_2$. Now $G$ admits the following 4-coloring:

$$
\begin{align*}
R_4 \cup Y_5 &\cup R_1 \cup \{1,4\} \\
Y_1 \cup R'_2 &\cup Y_4 \cup \{f,5\} \\
R_3 &\cup Z \cup \{3\} \\
Y_3 &\cup R'_2 \cup R'_3 \cup \{2\}
\end{align*}
\tag{12}
$$

This completes our proof. \hfill \Box

5 Eliminate $W_5$ and $C_5$

**Lemma 4.** Let $G$ be a $(2P_2, K_4, H_1, H_2)$-free graph with no pair of comparable vertices. If $G$ contains an induced $W_5$, then $\chi(G) \leq 4$.

**Proof.** Let $W = C \cup \{u\}$ be an induced $W_5$ such that $C = 12345$ induces a $C_5$ in this order and $u$ is complete to $C$. We partition $V \setminus C$ into subsets of $Z$, $R_i$, $Y_i$, $F_i$ and $U$ as in section 2. Note that $u \in U$. Since $G$ is $H_2$-free, it follows that $F_i = \emptyset$ for each $i$. We prove the following properties.

(a) $U$ is complete to $R$.

If $u' \in U$ is not adjacent to $r_i \in R_i$, then $C \setminus \{i\} \cup \{r_i, u\}$ induces an $H_2$. This contradicts our assumption that $G$ is $H_2$-free. \hfill \Box

(b) $R_i$ and $R_{i+2}$ are anti-complete.

Suppose that $r_i \in R_i$ and $r_{i+2} \in R_{i+2}$ are not adjacent. By (a), $u$ is adjacent to both $r_i$ and $r_{i+2}$. This implies that $\{r_i, r_{i+2}, i+1, u\}$ induces a $K_4$. \hfill \Box

(c) $R_i$ and $Y_{i+1}$ are anti-complete.

It suffices to prove for $i = 1$. If $r_1 \in R_1$ and $y_2 \in Y_2$ are adjacent, then $C \setminus \{1\} \cup \{r_1, y_2\}$ induces an $H_2$, a contradiction. \hfill \Box

(d) $Y_i$ and $Y_{i+2}$ are anti-complete.

Since $U \neq \emptyset$, (d) follows directly from (10). \hfill \Box

It follows from (b)-(d) and (1)-(2) that $G$ admits the following 4-coloring:

$$
\begin{align*}
R_1 &\cup R_3 \cup Z \cup \{1,3\} &\text{(b)} \\
R_2 &\cup Y_3 \cup R_4 \cup \{2,4\} &\text{(b)} \\
Y_1 &\cup R_5 \cup Y_4 \cup \{5\} &\text{(c)} \\
Y_2 &\cup Y_5 \cup U &\text{(d)}
\end{align*}
\tag{12}
$$

This completes our proof. \hfill \Box

**Lemma 5.** Let $G$ be a connected $(2P_2, K_4, H_1, H_2, W_5)$-free graph with no pair of comparable vertices. If $G$ contains an induced $C_5$, then $\chi(G) \leq 4$.

**Proof.** Let $C = 12345$ be an induced $C_5$ in this order. We partition $V \setminus C$ into subsets of $Z$, $R_i$, $Y_i$, $F_i$ and $U$ as in section 2. Since $G$ is $(H_2, W_5)$-free, both $U$ and $F_i$ are empty. It then follows from Lemma 1 that $V(G) = C \cup Z \cup (\bigcup_{i=1}^{5} R_i) \cup (\bigcup_{i=1}^{5} Y_i)$. We first prove the following properties of $R_i$ and $Z$. 

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(a) Each vertex in \( R_i \) is anti-complete to either \( R_{i-2} \) or \( R_{i+2} \).

It suffices to prove for \( i = 4 \). Suppose that \( r_1 \in R_4 \) is adjacent to a vertex \( r_i \in R_i \) for \( i = 1, 2 \). By (3), \( r_1 \) and \( r_2 \) are adjacent. This implies that \( \{r_1, r_2, 3, 4, 5, r_1\} \) induces a subgraph isomorphic to \( H_2 \). This contradicts the assumption that \( G \) is \( H_2 \)-free. ■

(b) \( R_i \) and \( Y_{i+1} \) are anti-complete.

It suffices to prove for \( i = 1 \). If \( r_1 \in R_1 \) and \( y_2 \in Y_2 \) are adjacent, then \( C \setminus \{1\} \cup \{r_1, y_2\} \) induces an \( H_2 \). ■

(c) Each vertex in \( Z \) cannot have a neighbor in each of \( Y_i \) for \( 1 \leq i \leq 5 \).

Suppose that \( z \in Z \) has a neighbor \( y_i \in Y_i \) for each \( 1 \leq i \leq 5 \). By (4), \( y_i \) and \( y_{i+1} \) are adjacent. This implies that \( y_i \) and \( y_{i+2} \) are not adjacent, for otherwise \( \{y_i, y_{i+1}, y_{i+2}, z\} \) induces a \( K_4 \). But now \( \{y_1, y_2, y_3, y_4, y_5, z\} \) induces a \( W_5 \). ■

(d) If \( z \in Z \) has a neighbor in each of \( Y_i, Y_{i+1}, Y_{i+2} \) and \( Y_{i+3} \), then \( Y_{i+4} \) is anti-complete to \( N(z) \).

It suffices to prove for \( i = 1 \). Let \( y_i \in Y_i \) be a neighbor of \( z \) for \( 1 \leq i \leq 4 \). By (c), \( z \) is anti-complete to \( Y_5 \) and so \( N(z) \subseteq Y_1 \cup Y_2 \cup Y_3 \cup Y_4 \) by (1). Let \( y_5 \) be an arbitrary vertex in \( Y_5 \). By (4), \( y_5 \) is complete to \( Y_1 \cup Y_4 \). Therefore it remains to show that \( y_5 \) is complete to \( N(z) \cap (Y_2 \cup Y_3) \). If \( y_5 \) is not adjacent to a vertex \( y \in N(z) \cap (Y_2 \cup Y_3) \), then either \( 3y_5 \) or \( 2y_5 \) forms a \( 2P_2 \) with \( zy \) depending on whether \( y \in Y_2 \) or \( y \in Y_3 \). ■

(e) If \( Z \) contains a vertex that has a neighbor in \( Y_i, Y_{i+1}, Y_{i+2} \) and \( Y_{i+3} \), then \( Y_{i+4} = \emptyset \).

Let \( z \in Z \) have neighbor in \( Y_i \) for \( 1 \leq i \leq 4 \). By (c), \( z \) is anti-complete to \( Y_5 \). If \( Y_5 \) contains a vertex \( y \), then \( N(z) \subseteq N(y) \) by (d). This contradicts the assumption that \( G \) contains no pair of comparable vertices. ■

Let \( Y'_4 = \{y \in Y_4 : y \text{ is anti-complete to } Y_1\} \) and \( Y''_4 = Y_4 \setminus Y'_4 \). Note that each vertex in \( Y''_4 \) has a neighbor in \( Y_1 \) by the definition and so is anti-complete to \( Y_2 \) by (7). Similarly, let \( R'_4 = \{r \in R_4 : r \text{ is anti-complete to } R_1\} \) and \( R''_4 = R_4 \setminus R'_4 \). By (a), \( R''_4 \) is anti-complete to \( R_2 \). We now consider the following two cases.

**Case 1.** \( Z \) contains a vertex that has a neighbor in four \( Y_i \). It then follows from (e) that \( Y_j = \emptyset \) for some \( j \). We may assume by symmetry that \( j = 5 \). These facts and (b) imply that \( G \) admits the following 4-coloring:

\[
\begin{align*}
Y_1 &\cup R_5 \cup Y'_4 \cup \{5\}, \\
Y_2 &\cup R_3 \cup Y''_4 \cup \{3\}, \\
R_1 &\cup Z \cup R'_4 \cup \{1\}, \\
R_2 &\cup Y_3 \cup R''_4 \cup \{2, 4\}.
\end{align*}
\]

**Case 2.** Each vertex in \( Z \) has a neighbor in at most three \( Y_i \). Note that \( G - Z \) admits the following 4-coloring \( \phi \) by (b):

\[
\begin{align*}
I_1 &= Y_1 \cup R_5 \cup Y'_4 \cup \{5\}, \\
I_2 &= Y_2 \cup R_3 \cup Y''_4 \cup \{3\}, \\
I_3 &= R_1 \cup Y_5 \cup R'_4 \cup \{1\}, \\
I_4 &= R_2 \cup Y_3 \cup R''_4 \cup \{2, 4\}.
\end{align*}
\]
We now explain how to extend $\phi$ to $Z$. For this purpose we partition $Z$ into the following two subsets:

$$Z_1 = \{ z \in Z : z \text{ is anti-complete to either } Y_3 \text{ or } Y_5 \},$$

$$Z_2 = Z \setminus Z_1.$$

We first claim that each vertex in $Z_2$ has a neighbor in $Y_4$. Suppose not. Let $z \in Z_2$ be a vertex such that $z$ is anti-complete to $Y_4$. Since $z$ has a neighbor in both $Y_3$ and $Y_5$, $z$ is anti-complete to either $Y_1$ or $Y_2$ by the assumption that each vertex in $z$ has a neighbor in at most three $Y_i$. If $z$ is anti-complete to $Y_1$, then $N(z) \subseteq Y_2 \cup Y_3 \cup Y_5 \subseteq N(5)$. If $z$ is anti-complete to $Y_2$, then $N(z) \subseteq Y_1 \cup Y_3 \cup Y_5 \subseteq N(3)$. In either case, it contradicts the assumption that $G$ contains no pair of comparable vertices. This proves the claim. Consequently, $Z_2$ is anti-complete to $Y_1 \cup Y_2$. We now claim that each vertex in $Z_2$ is anti-complete to either $Y'_4$ or $Y''_4$. Suppose not. Let $z \in Z_2$ have a neighbor $y'_4 \in Y_4$ and a neighbor $y''_4 \in Y''_4$. By the definition of $Y''_4$, it follows that $y''_4$ has a neighbor $y_1 \in Y_1$. Then $3y_1$ and $y'_4z$ induce a $2P_2$ since $y'_4$ is not adjacent to $y_1$. Now we can extend $\phi$ to $Z$ by adding each vertex in $Z_1$ to $I_3$ or $I_4$ and by adding each vertex in $Z_2$ to $I_1$ or $I_2$.

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