A realistic and deterministic description of a quantum system

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Abstract

From the analysis of the measurement process we make the hypothesis that we have to add to the quantum state \( |\psi\rangle \) a label \( z \) and a special function \( \alpha \) in order to describe completely the preparation of a (pure) quantum system. Given \((|\psi\rangle, \alpha, z)\) every observable \( A \) receives a univocal value \( \hat{A}(|\psi\rangle, \alpha, z) \). Therefore all physical quantities of the system are well defined independently from the measuring process. Making the unknown parameter \( z \) to vary in its space we get the probabilities prescribed by quantum mechanics. We show also that every evolution of quantum states can be extended to an evolution of the ternes \((|\psi\rangle, \alpha, z)\).

1 Introduction

Let’s assign, once for all, a quantum physical system (described through the Hilbert space \( \mathcal{H} \)) prepared in the (pure) state \( |\psi\rangle \) in \( \mathbb{P}(\mathcal{H}) \) and a measuring apparatus for the observable corresponding to the self-adjoint operator \( A \) on \( \mathcal{H} \).

Let’s suppose to perform infinite times the measurement of \( A \) on \( |\psi\rangle \) giving to each value a different label \( z \) (where \( z \) varies in a set \( Z \)) and to collect all the values obtained in a function \( \tilde{A} : Z \to \mathbb{R} \); for example the label space \( Z \) could be the Minkowski spacetime and each label \( z \) be the event where/when the measured value begins to be available.

Proceeding with the measurements, from the thickening and the thinning of the labels in \( Z \), let’s suppose that a probability measure \( \mu \) is emerging informing us with the number \( \mu(X) \) how probable it is to find a label in the subset \( X \) (with \( X \) in a suitable \( \sigma \)–algebra \( \mathcal{X} \) of subsets in \( Z \)).

The measure space \((Z, \mathcal{X}, \mu)\) so obtained allows us to compute through the number \( \mu \left((\tilde{A})^{-1}(B)\right) = (\tilde{A}_* \mu)(B) \) the probability that the measured values fall in a given borel subset \( B \) of \( \mathbb{R} \) and to check if in our laboratory the probabilities furnished by the Quantum Mechanics are verified: it is so if \( (\tilde{A}_* \mu)(B) = \langle E_A^B \rangle_\psi \) for every borel subset \( B \) of \( \mathbb{R} \) (where \( \{E_A^B\}_B \) is the spectral measure of the operator \( A \)).
If we introduce the cumulative monotone function $F : \mathbb{R} \to [0, 1]$ defined by $F(r) = \left\langle E_r^A \right\rangle_\psi$ and its borel measure $\nu_F(B) = \left\langle E_{B}^A \right\rangle_\psi$, this condition can be written, more synthetically, as: $\tilde{A}_\alpha \mu = \nu_F$; this will imply also: $\langle A \rangle_\psi = \int_{\mathbb{R}} \tilde{A} \cdot d\mu$, that is the attended value of QM for $A$ on $\psi$ is the natural mean value of $\tilde{A}$ with respect to $\mu$.

Given $F$ the condition $\tilde{A}_\alpha \mu = \nu_F$ is not enough to reobtain a single values function $\tilde{A}$. By experience we know that a different laboratory will provide a different function still verifying the probabilities of QM. Therefore we ask how many solutions the following mathematical problem has: assigned the cumulative function $F$ and the measure space $(Z, \mathcal{X}, \mu)$ find all borel functions $\tilde{A} : Z \to \mathbb{R}$ verifying $\tilde{A}_\alpha \mu = \nu_F$.

From a physical viewpoint to solve this problem it means to find all the values function $\tilde{A}$, relative to $A$ and $[\psi]$, in all laboratories where our scheme is applicable, starting from the function $F$ that summarizes only the attended probabilities.

If we suppose that $(Z, \mathcal{X})$ is a standard uncountable borel space and that the probability measure $\mu$ is without atoms (that is $\mu(\{z\}) = 0$ for every $z$ in $Z$) our problem has the following answer: for each borel function $\alpha : Z \to [0, 1]$ (a barrier function) such that $\alpha_\alpha \mu = \lambda_{[0,1]}$ we get the solution: $\tilde{A}_\alpha(z) = \min \{r \in \mathbb{R}; F(r) \geq \alpha(z)\}$ and every solution to the problem coincides (essentially) with one of these functions $\tilde{A}_\alpha$ (that we will call observable functions of $A$ on $[\psi]$).

We observe that, fixed the observable $A$ and the state $[\psi]$, the measured value $\tilde{A}_\alpha(z)$ depends also on the label $z$ and on the function $\alpha$. The variable $z$ plays the role of a hidden random variable since its variations give raise to the attended probabilities and (in agreement with the Kochen and Specker theorem) the measured value depends also on the function $\alpha$ that tells us how the operator $A$ representing the measurement apparatus and the physical system in the state $[\psi]$ are to be connected to give the measured value.

The value $\alpha(z)$ is not hidden, in general: in fact we will see that the function $\alpha(z)$ is worth generally: $\alpha(z) = F(\tilde{A}_\alpha(z))$. Therefore, fixed the space of labels and performing the various measurements, we can build the structure $(Z, \mathcal{X}, \mu)$ with the values function $\tilde{A}_\alpha$ as well as the basic behaviour of the function $\alpha$.

In this context it is evident why $A$ and $[\psi]$ are not enough to determine a univocal measured value: you don’t know the value of the function $\alpha$ on the label $z$. That is you don’t know how the measurement apparatus and the physical system are matematically correlated.

For example in the case of the famous Schrödinger’s cat if $[\psi]$ is the state of the system inside the box and $A$ is a test telling if your cat is awake or not, the value $\tilde{A}_\alpha(z)$ is either 1 or 0 according to $\alpha(z) > 1 - \langle A \rangle_\psi$ or $\alpha(z) \leq 1 - \langle A \rangle_\psi$.

Conversely if you know $\alpha(z)$ there is no room for uncertainty: your measurement apparatus will show on your display the outcome: $\min \{r \in \mathbb{R}; F(r) \geq \alpha(z)\}$ because the function $\alpha$ is exactly what you need to pass from $F$ to $\tilde{A}_\alpha$.

The insufficiency of $A$ and $[\psi]$ induces us to propose a revision of the mea-
surement process: when you perform the measurement of the observable $A$ on a physical system prepared in the state $|\psi\rangle$ the outcome depends also on the value $\alpha_z$ taken by the "scalar field" $\alpha$ on the "hidden" label $z$.

The condition $\alpha_\mu = \lambda_{[0,1]}$ joined with the hypothesis that the measure $\mu$ depends only on the state $|\psi\rangle$ makes it natural to see $\alpha$ as an enrichment of the quantum state and suggests to pass from considering the "incomplete" state $|\psi\rangle$ to consider the "complete" state $(|\psi\rangle, \alpha, z)$.

Called $C$ the set of all these possible "complete states" every self-adjoint operator $A$ defines a function $\hat{A} : C \rightarrow \mathbb{R}$ by $\hat{A}(|\psi\rangle, \alpha, z) = \min \{ r \in \mathbb{R} : F(r) \geq \alpha_z \}$, that is, on complete states each observable has a well defined value. The theory we are describing here is realistic: given the proper ingredients, the measurement process can only produce an outcome that you can know a priori. QM is logically compatible with a realistic description.

Moreover we show how it is possible to enrich each unitary map $U : \mathcal{H} \rightarrow \mathcal{H}$ to an automorphism $\hat{U} : C \rightarrow C$ in such a way to have a group homomorphism and moreover $\hat{U}^{-1} \hat{A} \hat{U} = \hat{A} \circ \hat{U}$. Therefore each unitary evolution in $\mathcal{H}$ can be seen as the apparent part of a deterministic hidden evolution in $C$.

All this works for a general borel standard (uncountable) space $Z$ but if we choose the phase space as space of labels there is a natural choice for the measures $\mu_{[\psi]}$ making the functions of the positions and the functions of the momentum observable functions. Note that you need different barriers functions for position and momentum observables.

In the end it is worth mentioning that in this context a phenomenon of dependence of the "state" on the way you get the outcomes appears: there are operators $A$ such that the operation of squaring the value $\hat{A}(|\psi\rangle, \alpha, z)$ is not given by $A^2$ on the "state" $(|\psi\rangle, \alpha, z)$ but on a changed "state" $(|\psi\rangle, \beta, z)$.

An observer reading the value $v = \hat{A}(|\psi\rangle, \alpha, z)$ and then computing the value $v^2$ can continue to refer it to the inaltered "state" $(|\psi\rangle, \alpha, z)$ but another observer looking at the entire process as a measurement of the observable $A^2$ is compelled to refer the outcome to the "state" $(|\psi\rangle, \beta, z)$ generally different from the previous one.

\section{Symbols}

In the real line we will consider the family $\mathcal{B}(\mathbb{R})$ of all its borel subsets; the symbol $\chi_B$ will denote the characteristic function of the subset $B$. A borel function on $\mathbb{R}$ is called here compact (bornological) if it sends bounded sets in bounded sets: all bounded real functions and all continuous real functions are compact.

Given a monotone, non-decreasing function $F : \mathbb{R} \rightarrow [0,1]$ if it is right-continuous with $\inf F = 0$ and $\sup F = 1$ it is well defined its quasi-inverse (or quantile) function $F^{-1} : [0,1] \rightarrow \mathbb{R}$ given by: $F^{-1}(s) = \min \{ r : F(r) \geq s \}$. For every monotone, non-decreasing, right-continuous function $F$ on $\mathbb{R}$ we will denote by $\nu_F$ the borel measure characterized by $\nu_F[a,b] = F(b) - F(a)$. We will denote by $\lambda_{[a,b]} = \nu_{id}$ the borel-lebesgue measure on the interval $(a,b)$. 


Given a self-adjoint operator $A$ on a Hilbert space $\mathcal{H}$ with spectral family $\{E^A_{(-\infty,r]}\}_{r \in \mathbb{R}}$ and a unitary vector $\psi$ we will denote by $F^A_\psi$ the monotone function defined by $F^A_\psi(r) = \langle E^A_{(-\infty,r]} \psi \rangle$. Note that $\nu_{F^A_\psi}(B) = \langle E^A_B \psi \rangle$ for every $B$ in $\mathcal{B}(\mathbb{R})$, moreover $F^A_\psi$ is non-decreasing, right-continuous with $\inf F^A_\psi = 0$ and $\sup F^A_\psi = 1$, so it is well defined $F^A_\psi : ]0,1[ \to \mathbb{R}$. Note that the function $F^A_\psi$ is continuous in $r$ if and only if $\langle E^A_{\{r\}} \psi \rangle = 0$.

We will denote by $\Omega$ the family of all linear operators (essentially) self-adjoint on $\mathcal{H}$ (those associated to quantum observables) and by $\Omega_b$ its subfamily of bounded operators. For every $A$ in $\Omega$ and every borel real function $b$ the composition $b \circ A$ is still in $\Omega$, for every $A$ in $\Omega_b$ and every compact borel real function $b$ the composition $b \circ A$ is still in $\Omega_b$.

The function $F : \Omega \times \mathcal{P}(\mathcal{H}) \times \mathbb{R} \to [0,1]$ given by $F(A, [\psi], r) = F^A_\psi(r)$ resumes all the probabilistic content of a quantum system represented by the Hilbert space $\mathcal{H}$.

Given two borel spaces $(X, \mathcal{A})$ and $(Y, \mathcal{B})$ a measure $\mu$ on $(X, \mathcal{A})$ and a borel map $\varphi : X \to Y$ the image measure of $\mu$ by $\varphi$ is the measure $\varphi_*\mu$ on $Y$ defined by $(\varphi_*\mu)(B) = \mu(\varphi^{-1}(B))$. Let’s remember that $(F^A_\psi)^{-1}(\lambda_{]0,1[}) = \nu_{F^A_\psi}$ (cfr. [KS]). In general, for any operator $A$ and any borel function $b$ we have: $\nu_{F^A_\psi \circ b} = b_*\nu_{F^A_\psi}$.

### Quantum mechanics on a standard space

We fix a separable Hilbert complex space $\mathcal{H}$ of infinite dimension, a borel standard (uncountable) space $(Z, \mathcal{X})$ and a probability measure without atoms $\mu_{[\psi]}$ on $Z$ for each $[\psi]$ in the projective space $\mathcal{P}(\mathcal{H})$.

What follows works for any such structure $(Z, \mathcal{X}, \mathcal{H}, \mu)$.

First of all let’s guarantee that our future definitions will not be empty.

**Theorem 1** Let $(Z, \mathcal{X})$ be a borel standard (uncountable) space

1. There exists a borel equivalence between $(Z, \mathcal{X})$ and $([0,1], \mathcal{B}([0,1]))$

2. Each borel equivalence $\sigma : Z \setminus M \to [0,1] \setminus N$, where $M \in \mathcal{X}$ and $N \in \mathcal{B}([0,1])$ with $\lambda(N) = 0$, defines a probability measure $\mu = (\sigma^{-1})_*\lambda_{[0,1]\setminus N}$ without atoms on $Z$

3. Conversely given a probability measure $\mu$ without atoms on $Z$ there exists a borel equivalence $\sigma : Z \to [0,1]$ (defined out of borel null subsets) such that $\sigma_*\mu = \lambda_{[0,1]}$.

**Proof.** 1. Cfr. [KS] Supplement to ch. 5
   2. Obvious
   3. Cfr [Ry] Thm. 9 p. 327

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Definition 2 Let’s fix a self-adjoint linear operator $A$ on $\mathcal{H}$ and a quantum state $|\psi\rangle$ in $\mathcal{P}(\mathcal{H})$. A **probability barrier associated to** $|\psi\rangle$ is a borel function $\alpha : Z \to [0,1]$ such that $\alpha,\mu_{|\psi\rangle} = \lambda_{[0,1]}$.

Definition 3 A **complete (pure) state** on $(Z, \mathcal{X}, \mathcal{H}, \mu, \lambda)$ is a triple $(|\psi\rangle, \alpha, z)$ of a pure state $|\psi\rangle$ in $\mathcal{P}(\mathcal{H})$, a barrier $\alpha_{|\psi\rangle}$ associated to $|\psi\rangle$ and a point $z$ in $Z$.

Notation 4 Let’s denote by $C$ the set of all complete states on $(Z, \mathcal{H}, \mu)$. Let’s fix a self-adjoint linear operator $A$.

Definition 5 The **values function** $v : \Omega \times C \to \mathbb{R}$ is defined by:

$$v(A; |\psi\rangle, \alpha_{|\psi\rangle}, z) = \min\left\{ r \in \mathbb{R} : \left\langle E^A_{\langle -\infty, r \rangle} \right\rangle_{\psi} \geq \alpha_{|\psi\rangle}(z) \right\} = (F^A_{\psi})^{-1} (\alpha_{|\psi\rangle}(z))$$

Notation 6 Fixed a self-adjoint linear operator $A$ on $\mathcal{H}$ its **values function** is the function $\hat{A} : C \to \mathbb{R}$ defined by: $\hat{A}(|\psi\rangle, \alpha_{|\psi\rangle}, z) = v(A; |\psi\rangle, \alpha_{|\psi\rangle}, z)$. Fixed also a pure state $|\psi\rangle$ and a barrier $\alpha_{|\psi\rangle}$ associated to $|\psi\rangle$ we will denote by $\hat{A}_{|\psi\rangle} : Z \to \mathbb{R}$ the function $\hat{A}$ with $|\psi\rangle$ and $\alpha_{|\psi\rangle}$ fixed.

Theorem 7

1. $\hat{A}_{|\psi\rangle} = (F^A_{\psi})^{-1} \circ \alpha_{|\psi\rangle}$ is a borel function

2. $\hat{A}_{|\psi\rangle} \ast \mu_{|\psi\rangle} = \nu_{F^A_{\psi}}$.

Proof.

1. A quasi-inverse function is monotonic.

2. $\hat{A}_{|\psi\rangle} \ast \mu_{|\psi\rangle} = \left( (F^A_{\psi})^{-1} \circ \alpha_{|\psi\rangle} \right) \ast \mu_{|\psi\rangle} = \left( F^A_{\psi} \right)^{-1} \lambda_{[0,1]} = \nu_{F^A_{\psi}}$.

Remark 8 The property $\hat{A}_{|\psi\rangle} \ast \mu_{|\psi\rangle} = \nu_{F^A_{\psi}}$ means that for each borel subset $B$ in $\mathbb{R}$ the probability $\mu_{|\psi\rangle} \left( (\hat{A}_{|\psi\rangle})^{-1} (B) \right)$ that the value of the function $\hat{A}_{|\psi\rangle}$ falls in $B$ is the probability $\nu_{F^A_{\psi}} (B) = \left\langle E^A_B \right\rangle_{\psi}$ assigned by the Quantum Mechanics for $A$, $|\psi\rangle$ and $B$.

Exercise 9

- For each projector $E$ on $\mathcal{H}$ we have: $\langle E \rangle_{|\psi\rangle}(z) = \chi_{L_{|\psi\rangle}}(z)$ with $L_{|\psi\rangle} = \left\{ z \in Z : 1 - \langle E \rangle_{\psi} < \alpha_{|\psi\rangle}(z) < 1 \right\}$

- Let $\{E_0, ..., E_m, ..., E_\infty \}$ be a sequence of pairwise orthogonal projectors in $\mathcal{H}$ such that $I = \sum_k E_k$ and let $\lambda_0, ..., \lambda_m, ..., \lambda_\infty$ be a sequence of real numbers with $\lambda_\infty = 0$, $\{\lambda_m\}_{m \geq 0}$ strictly increasing to 0 and defining
the bounded operator \( A = \sum_k \lambda_k \cdot E_k \), its function \( \hat{A}_{\alpha[\psi]} \) is the \( \sigma \)-simple
function \( \tilde{A}_{\alpha[\psi]} = \sum_k \lambda_k \cdot \chi_{L_k[\psi]} \) where:
\[
L_k[\psi] = \left\{ z \in Z; \sum_{j=0}^{k-1} \langle E_j \rangle_\psi < \alpha[\psi](z) \leq \sum_{j=0}^{k} \langle E_j \rangle_\psi \right\}
\]
\[
L_\infty[\psi] = \left\{ z \in Z; \sum_{j=0}^{k} \langle E_j \rangle_\psi < \alpha[\psi](z) \text{ for each } k < \infty \right\}
\]

Note that the family \( \{L_k[\psi]\} \) makes a partition of \( Z \).

**Theorem 10** When \( A \) is a bounded self-adjoint operator, a function \( f : Z \to \mathbb{R} \)
borel and essentially bounded for \( \mu[\psi] \) verifies \( f_* \mu[\psi] = \nu_{\hat{A}_{\alpha[\psi]}} \) if and only if for each
borel compact function \( b \) and each unitary vector \( \psi \), we have:
\[
\int_Z (b \circ f)(z) \cdot d\mu[\psi] = \langle \psi, (b \circ A) \psi \rangle
\]

**Proof.** \( [\implies] \) Since \( f \) is essentially bounded for \( \mu[\psi] \) and \( b \) is compact the
function \( b \circ f \) is essentially bounded for \( \mu[\psi] \) and absolutely integrable, moreover:
\[
\int_Z (b \circ f)(z) \cdot d\mu[\psi] = \int_{\mathbb{R}} b \cdot d(f_* \mu[\psi]) = \int_{\mathbb{R}} b \cdot d(\nu_{\hat{A}_{\alpha[\psi]}}) = \langle b \circ A \rangle_\psi
\]
\( [\impliedby] \) For each borel subset \( B \) in \( \mathbb{R} \) we have \( (f_* \mu[\psi])(B) = \int_{\mathbb{R}} \chi_B \cdot d(f_* \mu[\psi]) = \int_Z (\chi_B \circ f) \cdot d\mu[\psi] = \langle \chi_B \circ A \rangle_\psi = \nu_{\hat{A}_{\alpha[\psi]}}(B) \) \( \blacksquare \)

**Notation 11** Introducing the symbol \( \langle \langle b \circ f \rangle \rangle_\psi = \int_Z b \circ f \cdot d\mu[\psi] \) we can write
\( \langle \langle b \circ f \rangle \rangle_\psi = \langle b \circ A \rangle_\psi \). In particular \( \langle \langle f^k \rangle \rangle_\psi = \langle A^k \rangle_\psi \) for every \( k \geq 0 \).

**Remark 12** The next property even though it appears quite often in synthetic
data generation it deserves to be full proved in our context.

**Lemma 13** Let \( F : \mathbb{R} \to [0,1] \) be a monotone, non-decreasing and right continuous
function with \( \inf F(\mathbb{R}) = 0 \) and \( \sup F(\mathbb{R}) = 1 \). Let \( f : [0,1] \to \mathbb{R} \) a borel function
such that \( f_* \lambda_{[0,1]} = \nu_F \), there exist two borel null subsets \( M \) and \( N \) in
\( [0,1] \) and a borel function \( \alpha : [0,1] \to [0,1] \) in \( N \) such that \( \alpha_* \lambda_{[0,1]} \setminus M = \lambda_{[0,1]} \setminus N \)
and \( f \upharpoonright [0,1] \setminus M = F^{-1} \circ \alpha \).

**Proof.** The function \( F \) can have a countable set \( \{x_k\}_{k \geq 1} \) of discontinuity
points; let’s write \( a_k = F^-(x_k) < F(x_k) = b_k \). Let’s denote by \( [a_k, b_k] \) the interval \( [a_k, b_k] \cap (\{a_k\} \cap F(\mathbb{R})) \).

Moreover the function \( F \) can be constant on a countable family of dis-
joint proper maximal intervals \( \{J_n\}_{n \geq 1} \). Let’s write \( F(J_n) = \{c_n\} \). We have
\( F^{-1}(F(r)) = J_n \) if \( r \in J_n \) and instead \( F^{-1}(F(r)) = \{r\} \) if \( r \notin \bigcup J_n \). If
\( a_k \in F(\mathbb{R}) \) there exists \( J_n \) such that \( \{a_k\} = F(J_n) = \{c_n\} \).

The restricted function \( F : \upharpoonright \mathbb{R} \setminus \bigcup J_n \to (0,1] \upharpoonright [\bigcup\{a_k, b_k \cup \{c_n\}\}] \) is a bi-
jective borel equivalence with \( (F^{-1})^{-1} = F^{-1} \) \( \mid \) (observe that \( ]0,1] \setminus \bigcup \{a_k, b_k \subset \)
$F(\mathbb{R})$. Taken $M_0 = f^{-1}(\bigcup J_n \cup \{x_k\})$ and $N_0 = \bigcup[a_k, b_k] \cup \{c_n\}$ we have $\lambda(M_0) = \lambda(N_0) = 0$ and the function $\alpha_0 = F \circ f \cdot [0, 1 \setminus M_0 \rightarrow [0, 1 \setminus N_0]$ is a borel function with $F^{-1} \circ \alpha_0 = f$ and $\alpha_0 \circ \lambda = \lambda$.

Since $\lambda(f^{-1}(\{x_k\})) = b_k - a_k = \lambda([a_k, b_k[)$ for each $k \geq 1$ there exists a borel null subset $M_k$ of $f^{-1}(\{x_k\})$ and a borel null subset $N_k$ of $[a_k, b_k[$ and a borel and measure equivalence $\alpha_k : f^{-1}(\{x_k\}) \setminus M_k \rightarrow [a_k, b_k \setminus N_k$.

Let’s take $M = \bigcup_{n \geq 0} M_n$ and $N = \bigcup_{n \geq 0} N_n$, the map $\alpha : [0, 1 \setminus M \rightarrow [0, 1 \setminus N$ defined by $\alpha_0 = F \circ f$ on $[0, 1 \setminus f^{-1}(\bigcup J_n \cup \{x_k\})$ and $\alpha_k$ on $f^{-1}(\{x_k\}) \setminus M_k$ verifies the thesis of the lemma. 

**Theorem 14** A borel function $f_{[\psi]} : Z \rightarrow \mathbb{R}$ verifies $f_{[\psi]} \ast \mu_{[\psi]} = \nu_{F_{\psi}^A}$ if and only if there exists a barrier $\alpha_{[\psi]}$ associated to $[\psi]$ such that $f_{[\psi]} = (F_{\psi}^A)^{-1} \circ \alpha_{[\psi]}$ (out of a borel $\mu_{[\psi]}$–null subset).

**Proof.** $[\Longrightarrow]$ We already proved that $\left( F_{\psi}^A \right)^{-1} \circ \alpha_{[\psi]} \ast \mu_{[\psi]} = \nu_{F_{\psi}^A}$.

$[\Longleftarrow]$ Since $Z$ is an uncountable borel standard space we can find a borel $\mu_{[\psi]}$–null subspace $M$ in $Z$, a borel null subspace $N$ in $[0, 1]$ and a borel equivalence $\sigma : Z \setminus M \rightarrow [0, 1 \setminus N$ with $\sigma \ast \mu_{[\psi]} = \lambda|_{0, 1}$. Therefore the function $f \circ \sigma^{-1} : [0, 1 \setminus N \rightarrow \mathbb{R}$ verifies: $(f \circ \sigma^{-1}) \ast \lambda|_{0, 1} = \nu_{F_{\psi}^A}$.

For the previous Lemma there exists a borel function $\beta : [0, 1 \setminus N \rightarrow [0, 1] with $\beta \ast \lambda|_{0, 1} = \lambda|_{0, 1}$ and $f \circ \sigma^{-1} = (F_{\psi}^A)^{-1} \circ \beta$ out of a borel null subset, therefore $f = (F_{\psi}^A)^{-1} \circ \alpha_{[\psi]}^A$ out of a borel null subset where $\alpha_{[\psi]}^A = \beta \circ \sigma$ is a barrier associated to $[\psi]$. 

**Remark 15** From the definition of quasi-inverse function for $f_{[\psi]} = (F_{\psi}^A)^{-1} \circ \alpha_{[\psi]}$ we have: $F_{\psi}^A(f_{[\psi]}(z)^-) \leq \alpha_{[\psi]}(z) \leq F_{\psi}^A(f_{[\psi]}(z))$. Therefore if $f_{[\psi]}(z)$ is not a discontinuity point of $F_{\psi}^A$ we have: $\alpha_{[\psi]}(z) = F_{\psi}^A(f_{[\psi]}(z))$.

### 4 Representations of bounded operators by functions

**Definition 16** A complex of barriers is a family $\alpha = \{\alpha_{[\psi]}^A\}_{A \in \Omega \setminus \psi} \in \mathbb{P}(\mathcal{H})$ where each $\alpha_{[\psi]}^A$ is a barrier for $[\psi]$.

**Notation 17** Fixed a complex of barriers $\alpha$ for each self-adjoint operator $A \in \Omega$ it is defined the function $A_{\alpha} : \mathbb{P}(\mathcal{H}) \times Z \rightarrow \mathbb{R}$ by $A_{\alpha}([\psi], z) = v(A; [\psi], \alpha_{[\psi]}^A; z) = \left( (F_{\psi}^A)^{-1} \circ \alpha_{[\psi]}^A \right)(z)$. Two such functions $\hat{H}_{\alpha}$ and $\hat{K}_{\beta}$ will be considered equivalent if, for each $[\psi]$ in $\mathbb{P}(\mathcal{H})$ the functions $\hat{H}_{\alpha[\psi]}$ and $\hat{K}_{\beta[\psi]}$ are equal out of a borel subset $\mu_{[\psi]}$–null.
Notation 18 Given two self-adjoint linear operators \( A_1 \) and \( A_2 \) and a complex of barriers \( \alpha \) if \( A_{1\alpha} \equiv A_{2\alpha} \) then \( A_1 = A_2 \).

Proof. For each pure state \( |\psi\rangle \) from \( \hat{A}_{1\alpha[|\psi\rangle]} \equiv \hat{A}_{2\alpha[|\psi\rangle]} \) it follows \( \nu_{F_{\psi}^A} = \nu_{F_{\psi}^{A}} \) and therefore \( F_{\psi}^{A_1} = F_{\psi}^{A_2} \) since both are right-continuous functions with \( \inf = 0 \) and \( \sup = 1 \).

Then \( \langle E_{(\alpha, r)} \rangle_{\psi} = \langle E_{(\alpha, r)} \rangle_{\psi} \) for each \( r \) in \( \mathbb{R} \) and each unitary vector \( \psi \). That is \( A_1 = A_2 \). ■

Lemma 19 For each \( A \) in \( \Omega_b \) and each \( [\psi] \) in \( \mathbb{P}(\mathcal{H}) \): \( \mu_{[\psi]} \left[ (\hat{A}_{\alpha[|\psi\rangle]})^{-1} a, b \right] = 0 \) if and only if \( (\hat{A}_{\alpha[|\psi\rangle]})^{-1} a, b \) = \( \emptyset \).

Proof. Since \( \hat{A}_{\alpha[|\psi\rangle]} \mu_{[\psi]} = \nu_{F_{\psi}^A} \) we have \( \mu_{[\psi]} \left[ (\hat{A}_{\alpha[|\psi\rangle]})^{-1} a, b \right] = 0 \) if and only if \( F_{\psi}^A(a) = F_{\psi}^A(b) \) that is equivalent to \( (\nu_{F_{\psi}^A})^{-1} a, b \) = \( \emptyset \) that implies \( (\hat{A}_{\alpha[|\psi\rangle]} a, b \) = \( \emptyset \). ■

Theorem 20 For each \( A \) in \( \Omega_b \) we have \( \hat{A}_{\alpha}(\mathbb{P}(\mathcal{H}) \times Z) = \text{spec}(A) \).

Proof. For a number \( r \) in \( \mathbb{R} \) it holds \( r \notin \text{spec}(A) \) if and only if there exists \( \sigma > 0 \) with \( E_{\sigma^r} \langle r - \sigma, r + \sigma \rangle = 0 \) (cfr. [W] Thm. 7.22) that is equivalent to \( \mu_{[\psi]} \left[ \hat{A}_{\alpha[|\psi\rangle]} \langle r - \sigma, r + \sigma \rangle = 0 \right. \) and to \( \hat{A}_{\alpha[|\psi\rangle]} \langle r - \sigma, r + \sigma \rangle = \emptyset \) for each \( [\psi] \) in \( \mathbb{P}(\mathcal{H}) \) that is \( r \notin \hat{A}_{\alpha}(\mathbb{P}(\mathcal{H}) \times Z) \). ■

Definition 21 A representation of \( \Omega_b \) with functions of \( (Z, X, \mu) \) is a map \( \hat{\rho} : \Omega_b \rightarrow \mathbb{R}^{\mathbb{P}(\mathcal{H}) \times Z} \) such that for each \( A \) in \( \Omega_b \) and for each \( [\psi] \) in \( \mathbb{P}(\mathcal{H}) \):

1. the function \( \hat{A}_{[|\psi\rangle]} : Z \rightarrow \mathbb{R} \) is borel and essentially bounded for \( \mu_{[\psi]} \)

2. \( \hat{A}_{[|\psi\rangle]} \mu_{[\psi]} = \nu_{F_{\psi}^A} \)

Theorem 22 Given a representation \( \hat{\rho} : \Omega_b \rightarrow \mathbb{R}^{\mathbb{P}(\mathcal{H}) \times Z} \) of \( \Omega_b \) on \( (Z, X, \mu) \) there exists a complex of barriers \( \alpha \) such that \( \hat{A} \equiv \hat{A}_{\alpha} \).

Proof. Given a representation \( \hat{\rho} : \Omega_b \rightarrow \mathbb{R}^{\mathbb{P}(\mathcal{H}) \times Z} \) for each \( A \) and \( [\psi] \) we have: \( \hat{A}_{[|\psi\rangle]} \mu_{[\psi]} = \nu_{F_{\psi}^A} \). This implies there exists a barrier \( \alpha_{A[|\psi\rangle]} \) associated to \( [\psi] \) such that \( \hat{A}_{[|\psi\rangle]} \equiv (F_{\psi}^A)^{-1} \circ \alpha_{A[|\psi\rangle]} = \hat{A}_{\alpha[|\psi\rangle]} \). ■

Theorem 23 A map \( \hat{\rho} : \Omega_b \rightarrow \mathbb{R}^{\mathbb{P}(\mathcal{H}) \times Z} \) with each function \( \hat{A}_{[|\psi\rangle]} : Z \rightarrow \mathbb{R} \) borel and \( \mu_{[|\psi\rangle]} \)-essentially bounded for each \( [\psi] \) in \( \mathbb{P}(\mathcal{H}) \), is a representation of \( \Omega_b \) if and only if for each bounded self-adjoint operator \( A \) on \( \mathcal{H} \), for each \( [\psi] \) in \( \mathbb{P}(\mathcal{H}) \) and for each borel compact function \( b \) we have:

\[
\int_Z \left( b \circ \hat{A}_{[|\psi\rangle]} \right)(z) \cdot d\mu_{[\psi]} = \langle \psi, (b \circ A) \psi \rangle \quad \text{(when \( \|\psi\| = 1 \))}
\]
Remark 24. Given a representation \( \hat{\cdot} \) of \( \Omega_b \) for every self-adjoint bounded operator \( A \) and every borel compact function \( b : \mathbb{R} \to \mathbb{R} \) the attended value \( \langle b \circ A \rangle_\psi \) is equal to the mean value \( \int_{\mathbb{R}} (b \circ A_\psi)(z) \cdot d\mu_\psi(z) \) and also to the mean value \( \int_{\mathbb{R}} (b \circ A_\psi)(z) \cdot d\mu_\psi(z) \) suggesting that it could be \( (b \circ A)_\psi = b \hat{A}_\psi \). This is true, for example, if \( b : \mathbb{R} \to \mathbb{R} \) is a strictly increasing homeomorphism: in fact in this case we have (cfr. [W], 7.3): \( F_{b \circ A}^\psi = F^\psi_\psi \circ b^{-1} \), \( (F_{b \circ A}^\psi)^{-1} = b \circ (F^\psi_\psi)^{-1} \) and then \( (b \circ \hat{A})_\alpha = b \circ \hat{A}_\alpha \) for every self-adjoint operator \( A \) and every complex of barriers \( \alpha \). However the next theorem tells us that this is not universally true whatever representation you consider.

Theorem 25. 1. It does not exist a complex of barriers \( \alpha \) such that \( b \circ \hat{A}_\alpha = (b \circ \hat{A})_\alpha \) for every self-adjoint bounded operator \( A \) and every compact borel function \( b \).

2. For every self-adjoint bounded operator \( A \) and every compact borel function \( b \) there exists a suitable complex \( \beta \) of barriers such that \( b \circ \hat{A}_\alpha \equiv (b \circ \hat{A})_\beta \).

Proof.

1. For every projector \( E \) the function \( \hat{E}_\alpha \) is a characteristic function. Fixed \( \langle \psi_0 \rangle \) in \( \mathbb{P}(\mathcal{H}) \) with \( \| \psi_0 \| = 1 \) and \( z_0 \) in \( \mathbb{Z} \) it is well defined the function \( G : \{ \psi, \psi \text{ is unitary} \} \to \{ 0,1 \} \) by \( G(\psi) = \langle \hat{E}_{\psi} \rangle_\alpha ([\psi_0] ; z_0) = \chi \{ 1 - \alpha_{\langle \psi_0 \rangle} (z_0) = \langle \psi_0, \psi_0 \rangle \} (\psi) \).

Let’s make the extra hypothesis that such a complex of barriers \( \alpha \) would exist, it will imply that for the function \( G \) we have: \( \sum_{k \geq 1} G(\psi_k) = 1 \) for every orthonormal basis \( \{ \psi_k \}_{k \geq 1} \). We can choose a suitable sequence \( \{ c_k \}_{k \geq 1} \) of distinct real numbers in such a way that the self-adjoint operator \( \hat{A} = \sum_{k \geq 1} c_k \cdot E_{\psi_k} \) is bounded. We have for the extra hypothesis:

\[
1 = \hat{\alpha} = \left( \sum_{k \geq 1} E_{\psi_k} \right)_\alpha = \left[ \left( \sum_{k \geq 1} c_k \right) \circ A \right]_\alpha = \left( \sum_{k \geq 1} c_k \right) \circ \hat{A}_\alpha
\]

Therefore \( \sum_{k \geq 1} (E_{\psi_k})_\alpha = \sum_{k \geq 1} \left( \chi_{c_k} \circ A \right)_\alpha = \sum_{k \geq 1} \left( \chi_{c_k} \circ \hat{A}_\alpha \right) = \left( \sum_{k \geq 1} c_k \right) \circ \hat{A}_\alpha = 1 \). Then \( H \) is a Gleason frame function of weight 1 (cfr. [G]) and there exists a bounded self-adjoint operator \( D \) such that \( G(\psi) = \langle \psi, D\psi \rangle \) for every unitary \( \psi \). This implies that \( G \) is continuous and constantly 0 or constantly 1, contradicting, in both cases, \( \sum_{k \geq 1} G(\psi_k) = 1 \) for any orthonormal basis \( \{ \psi_k \}_{k \geq 1} \).

2. For the function \( (b \circ \hat{A}_\alpha)_{\psi} \) we have \( (b \circ \hat{A}_\alpha)_{\psi} \cdot \mu_{\psi} = b_\alpha \nu_{F^A_\psi} = \nu_{F_{b \circ A}^\psi} \).

In fact: \( \nu_{F_{b \circ A}^\psi} (B) = \langle E_B^A \rangle_\psi = \langle \chi_B (b \circ A) \rangle_\psi = \langle \chi_{b^{-1}(B)} (A) \rangle_\psi = \nu_{F^A_\psi} (b^{-1}(B)) \) (cfr. [W] Ch. 7.3).

Remark 26. When \( b \circ \hat{A}_\alpha \equiv (b \circ A)_\beta \neq (b \circ A)_\alpha \) the composition with the function \( b \) compels us to consider a new complex of barriers \( \beta \), different from the
previous $\alpha$ used to define $\hat{A}_\alpha$. An observer reading the value $v = \hat{A}([\psi], \alpha^A_{[\psi]}, z)$ and then computing the value $b(v)$ can continue to refer to it as the unaltered “state” $([\psi], \alpha^A_{[\psi]}, z)$ but another observer looking at the entire process as a measurement of the observable $b \circ A$ seems compelled to refer the outcome to the “state” $([\psi], \beta^A_{[\psi]}, z)$ generally different from the previous one. A possible explanation could be that the “state” of a physical system can depend on the way the outcomes emerge (cfr. [Ro]), however other explanations are possible.

**Example 27** Let’s take $Z = \{0, 1\}$ and $\mu([\psi]) = \lambda$ for every $[\psi]$ in $\mathbb{P}(\mathcal{H})$. Let’s decompose the space $\mathcal{H}$ as $\mathcal{H} = I \oplus J \oplus K$ with $I$, $J$ and $K$ orthogonal two by two and let’s denote by $E$, $F$ and $G$ the orthogonal projections, respectively, on $I$, $J$ and $K$. Let’s take a unitary vector $\psi$ such that $\langle E \psi \rangle = 1/8$, $\langle F \psi \rangle = 1/4$ and $\langle G \psi \rangle = 5/8$ and let’s consider $A = E - G$ (with $A^2 = E + G$). It is not difficult to compute $\left( \left( F\psi \right)^{-1} \right)^2 = \chi_{[0,5/8]} \chi_{[7/8,1]}(s)$ and $\left( \left( F\psi \right)^{-1} \right)^2 = \chi_{[1/4,1]}(s)$. This means $\left( \hat{A}^2 \right)_\beta = \chi_{[1/4,1]}(s)$ and $\left( \hat{A}_\alpha \right)^2 = \chi_{[0,5/8]} \chi_{[7/8,1]}(s)$ so $\left( \hat{A}_\alpha \right)^2 = \left( \hat{A}^2 \right)_\beta$ if we take $\beta(s) \equiv \sigma(\alpha(s))$ where $\sigma(u) = (u + 3/8) \mod 1$.

## 5 Observable functions

**Definition 28** A function $f : \mathbb{P}(\mathcal{H}) \times Z \rightarrow \mathbb{R}$ will be called an observable function if each $f_{[\psi]} : Z \rightarrow \mathbb{R}$ is a borel function and there exists a self-adjoint operator $A$ on $\mathcal{H}$ such that $f_{[\psi]} \mu_{[\psi]} = \nu_{F\psi}$ for every $[\psi]$ in $\mathbb{P}(\mathcal{H})$.

**Remark 29** The observable functions associated to an operator $A$ emulate on each $[\psi]$ the values taken by the concrete apparatuses measuring the observable $A$ on $[\psi]$.

**Remark 30** For every self-adjoint operator $A$ and every complex of barriers $\alpha$ the function $\hat{A}_\alpha$ is an observable function. Conversely for every observable function $f$ there exist a self-adjoint operator $A$ and a complex of barriers $\alpha$ such that $f_{[\psi]} \equiv \hat{A}_{\alpha|[\psi]}$ for every $[\psi]$ in $\mathbb{P}(\mathcal{H})$.

**Notation 31** Let’s denote by $\mathcal{O}$ the set of all the observable functions and by $\text{Op} : \mathcal{O} \rightarrow \Omega$ the well defined map sending an observable function to its associated operator; we will denote by $\mathcal{O}_b$ the subset of $\mathcal{O}$ of all observables whose operator is bounded. Fixed a complex of barriers $\alpha$ we will denote by $\mathcal{O}_\alpha$ the family of all the functions $\hat{A}_\alpha$ with $A$ a self-adjoint operator and by $\mathcal{O}_{ab}$ the family of functions $\hat{A}_\alpha$ with $A$ a bounded self-adjoint operator. We have $\mathcal{O} = \bigcup_{\alpha} \mathcal{O}_\alpha$ and $\mathcal{O}_b = \bigcup_{\alpha} \mathcal{O}_{ab}$.

**Theorem 32** For each observable function $f$ and each borel function $b$ the composition $b \circ f$ is an observable function with $\text{Op}(b \circ f) = b \circ \text{Op}(f)$. 

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Proof. If \( f \) is in \( \mathcal{O} \) and \( b \) is borel then each \( (b \circ f)\lvert_{\psi} = b \circ f\lvert_{\psi} \) is borel and:

\[
(b \circ f)\lvert_{\psi} \mu_{\psi} = b \circ f\lvert_{\psi} \nu_{\psi} \psi q_{n(f)} = \nu_{f \circ \psi q_{n(f)}}.
\]

Theorem 33 Let’s consider the Hilbert space \( \mathcal{H} = \bigoplus_{s=-\sigma,\ldots,\sigma} L^2(\mathbb{R}_s^3, \lambda) \) of a particle of spin \( \sigma \), the standard borel space \( Z = \bigcup_s \mathbb{R}^6_s \) and the probability measures:

\[
\mu_{\psi} = \sum_{\parallel \psi \parallel \neq 0} \frac{1}{\parallel \psi \parallel} \cdot \parallel \psi \parallel^2 \cdot \lambda_{\mathbb{R}^6_\psi} \quad \text{(when \( \parallel \psi \parallel = 1 \)).}
\]

1. For every borel function \( f \) the function \( f(p_1, p_2, p_3) \) is an observable function with operator \( f(P_1, P_2, P_3) \)

2. For every borel function \( g \) the function \( g(q_1, q_2, q_3) \) is an observable function with operator \( g(Q_1, Q_2, Q_3) \)

3. The function \( s \) is an observable function with operator \( S \) (defined by \( S((\psi_s)_s) = (s \cdot \psi_s)_s \))

Proof.

1. Applying the Thm. 7.17 of [W] to the operator \( g(Q_1, Q_2, Q_3) \) it is not difficult to prove that \( E_{(-\infty, r]}^g(\psi) = \chi(g(q) \leq r) \cdot \psi \) and \( F_{\psi}^g(r) = \sum_{\parallel \psi \parallel \neq 0} \int_{\mathbb{R}^6} \chi(g(q) \leq r) \cdot \parallel \psi \parallel \cdot dqdp \) Therefore \( (g(q), \mu_{\psi})_{(-\infty, r]} = \nu_{\psi q_{n(g)}}_{(-\infty, r]} \).

2. Using \( f(P) = F^{-1}f(Q) \cdot F \) we have \( F_{\psi}^{f(P)} = \sum_{\parallel \psi \parallel \neq 0} F_{\psi}^{f(P)} = \sum_{\parallel \psi \parallel \neq 0} F_{\psi}^{f(P)} \). Moreover \( [f(P)]_{\mu_{\psi}}_{(-\infty, r]} = \sum_{\parallel \psi \parallel \neq 0} \int_{\mathbb{R}^6} \chi(f(p) \leq r) \cdot \parallel \psi \parallel \cdot dqdp = \sum_{\parallel \psi \parallel \neq 0} \int_{\mathbb{R}^3} \chi(f(p) \leq r) \cdot \parallel \psi \parallel \cdot dqdp = (\sum_{\parallel \psi \parallel \neq 0} F_{\psi}^{f(P)}(r) = \nu_{\psi q_{n(f)}}_{(-\infty, r]} \).

3. Applying the Thm. 7.17 of [W] to the operator \( S \) it is not difficult to prove that \( E_{(-\infty, r]}^S(\psi, s) = \chi_{(-\infty, r]}(s) \cdot \psi(x) \) and \( F_{\psi}^S(r) = \sum_{s \leq r} \parallel \psi \parallel \). Therefore \( s, \mu_{\psi} = \nu_{\psi q_{n(S)}} \).

Remark 34 So for a suitable choice of a complex of barriers the function \( g(q_1, q_2, q_3) \) emulates the values of the operator \( g(Q_1, Q_2, Q_3) \). Then taken as \( g \) the projection on the component \( q_j \) there exists a quantum observable function giving the \( j \)-th component of the position for \( j = 1, 2, 3 \). If you have the skill to produce concrete apparatuses with assigned barriers the components of the position are outcomes for the label \( (q, p) \)!

Analogously for the components of the momentum (with different barriers).

Remark 35 Therefore on the phase space each complex of barriers \( \alpha \) with the map \( \psi \mapsto \alpha : \Omega \to \mathcal{O}_\alpha \) (together with \( \mu \)) defines a QMPS in the sense that \( \psi, A\psi = \int_{\mathbb{R}^6} A_{\alpha}(\psi) \cdot dq_{\psi} \) (for each unitary vector \( \psi \) and each self-adjoint operator \( A \) such that both terms of the equality are defined). The main differences with the usual QMPS is the dependence of \( A_{\alpha}\lvert_{\psi} \) on \( \psi \) and the lack of any explicit hypothesis of linearity on \( A \) (cfr. [CZ], [Gr], [M], [P]).
6 Dynamics

Definition 36 For each \( f \) in \( \mathcal{O}_{ab} \) the quadratic function associated to \( f \) is the function
\[
Q_f : \mathcal{H} \setminus \{0\} \to \mathbb{R} \text{ defined by } Q_f(\psi) = \|\psi\|^2 \cdot \int_Z f[\psi](z) \cdot d\mu[\psi]
\]

Remark 37 The function \( Q_f \) can be considered defined and differentiable on all \( \mathcal{H} \) (even in 0) since it is equal to \( \langle \psi, Op(f)\psi \rangle \).

Theorem 38 For each \( f \) in \( \mathcal{O}_{ab} \) we have:
\[
Op(f)\psi = \frac{1}{2} \text{Grad}_\psi Q_f
\]

Proof. Let \( A = Op(f) \) for every \( \varphi \) in \( \mathcal{H} \) we have: \( RE \langle \varphi, \frac{1}{2} \text{Grad}_\psi Q_f \rangle = \frac{1}{2} RE \{[\varphi, Q_f] \} = \frac{1}{2} RE \lim_{t \to 0} \frac{1}{t} \left( (\varphi + t\varphi, A\psi + tA\varphi) - \langle \psi, A\varphi \rangle \right) \) = \( RE \langle \varphi, A\psi \rangle \) therefore \( A\psi = \frac{1}{2} \text{Grad}_\psi Q_f \). ■

Definition 39 Given \( f \) and \( g \) in \( \mathcal{O}_{ab} \) we define \( \{f, g\}_\alpha = \{(Op(f), Op(g))\}_\alpha \)
(where \( \{A, B\} = -\frac{1}{2} [A, B] \) and \( f \circ \alpha g = (Op(f) \circ Op(g))_\alpha \) (where \( A \circ B = \frac{1}{2} (AB + BA) \)).

Remark 40 Note that here: \( A \cdot B = A \circ B + i \{A, B\} \). The space \( \mathcal{O}_{ab} \) with \( \{\cdot, \cdot\}_\alpha \) is a Lie-algebra and \( \mathcal{O}_{ab} \) with \( \circ_\alpha \) is a Jordan algebra. In the complexification of \( \mathcal{O}_{ab} \) we can consider the product \( f \times_\alpha g = (Op(f) \cdot Op(g))_\alpha = f \circ_\alpha g + i \{f, g\}_\alpha \) that makes it an associative algebra. We have: \( \langle \{f, g\}_\alpha \rangle_\psi = \langle \{Op(f), Op(g)\} \rangle_\psi \), \( \langle f \circ_\alpha g \rangle_\psi = \langle Op(f) \circ Op(g) \rangle_\psi \) and \( \{f \times_\alpha g \}_\alpha \psi = \langle Op(f) \cdot Op(g) \rangle_\psi \).
Moreover: \( Q_{f \times_\alpha g}(\psi) = \frac{1}{2} \langle \text{Grad}_\psi Q_f, \text{Grad}_\psi Q_g \rangle \), \( Q_{f \circ_\alpha g}(\psi) = RE [Q_{f \times_\alpha g}(\psi)] \) and \( Q_{\{f, g\}_\alpha}(\psi) = IM [Q_{f \times_\alpha g}(\psi)] \).

Theorem 41 Let \( H \) be a bounded self-adjoint operator on \( \mathcal{H} \), let \( \alpha \) be a complex of barriers, let \( h = \hat{H}_\alpha \) and let \( \{\psi_t\}_{t \in \mathbb{R}} \) be a differentiable unitary path in \( \mathcal{H} \).

Are equivalent:
1. \( i \frac{d}{dt}|_{t=t_0} \psi_t = H\psi_{t_0} \)
2. \( \frac{d}{dt}|_{t=t_0} \int_Z f[\psi] \cdot d\mu[\psi] = 2 \cdot \int_Z \{f, h\}_\alpha[\psi_{t_0}] \cdot d\mu[\psi_{t_0}] \) for every \( f \) in \( \mathcal{O}_{ab} \)

Proof. \([1] \implies [2]: \quad \frac{d}{dt}|_{t=t_0} \int_Z f[\psi] \cdot d\mu[\psi] = \frac{d}{dt}|_{t=t_0} \langle \psi_t, Op(f)\psi_t \rangle = \langle \psi_{t_0}, -i [Op(f), H]\psi_{t_0} \rangle = \int_Z (-i [Op(f), H])_\alpha[\psi_{t_0}] \cdot d\mu[\psi_{t_0}] = 2 \int_Z \{f, h\}_\alpha[\psi_{t_0}] \cdot d\mu[\psi_{t_0}] \). \( [2] \implies [1]: \quad \) Let’s fix \( t_0 \) in \( \mathbb{R} \) and a bounded self-adjoint operator \( A \) such that \( A\psi_{t_0} = \hat{A}_\alpha \cdot 2 \| \frac{d}{dt}|_{t=t_0} \psi_t + iH\psi_{t_0} \|^2 = \frac{d}{dt}|_{t=t_0} \langle \psi_t, A\psi_t \rangle = \langle \psi_{t_0}, -i [A, H]\psi_{t_0} \rangle = \frac{d}{dt}|_{t=t_0} \int_Z f[\psi] \cdot d\mu[\psi] - 2 \int_Z \{f, h\}_\alpha[\psi_{t_0}] \cdot d\mu[\psi_{t_0}] = 0 \). ■
Definition 42. For every observable function $f$ and every $|\psi\rangle$ in $\mathcal{P}(\mathcal{H})$ the dispersion of $f$ on $|\psi\rangle$ is the non-negative number $\Delta_f^\psi = \sqrt{\langle\langle (f - \langle f \rangle_\psi)^2 \rangle\rangle_\psi} = \sqrt{\langle\langle f^2 \rangle\rangle_\psi - \langle\langle f \rangle\rangle^2_\psi}$.

Remark 43. $\Delta_f^\psi = \Delta_{Op(f)}^\psi$ in fact: $(\Delta_f^\psi)^2 = \langle\langle (f - \langle f \rangle_\psi)^2 \rangle\rangle_\psi = \langle\langle [Op(f) - \langle f \rangle_\psi, 1]_\psi^2 \rangle\rangle_\psi = \left(\Delta_{Op(f)}^\psi\right)^2$.

Theorem 44. (Heisenberg’s uncertainty principle) For every $f$ and $g$ in $\mathcal{O}_{ab}$ we have $\Delta_f^\psi \cdot \Delta_g^\psi \geq \left| \langle\langle f, g \rangle\rangle_\psi \right|$.

Proof. $\Delta_f^\psi \cdot \Delta_g^\psi = \Delta_{Op(f)}^\psi \cdot \Delta_{Op(g)}^\psi \geq \left| \langle\langle Op(f), Op(g) \rangle\rangle_\psi \right| = \left| \langle\langle f, g \rangle\rangle_\psi \right|$.

Definition 45. A complex of equivalences is a family $\sigma = \{\sigma_{|\psi\rangle}\}_{|\psi\rangle \in \mathcal{P}(\mathcal{H})}$ of borel equivalences $\sigma_{|\psi\rangle} : Z \rightarrow [0,1]$ (defined out of null borel subsets) with $\sigma_{|\psi\rangle} \cdot \mu_{|\psi\rangle} = \lambda_{|0,1\rangle}$ for each $|\psi\rangle$ in $\mathcal{P}(\mathcal{H})$.

Definition 46. A unitary operator $\mathcal{U}$ together with a complex of equivalences $\sigma$ defines on the space of complete states an automorphism $\hat{\mathcal{U}}_\sigma : C \rightarrow C$ by:

$$\hat{\mathcal{U}}_\sigma(|\psi\rangle, \alpha_{|\psi\rangle}, z) = \left(\mathcal{U}|\psi\rangle, \left(\alpha_{|\psi\rangle} \circ \sigma^{-1}_{|\psi\rangle} \circ \sigma_{|\mathcal{U}|\psi\rangle}\right)_{\mathcal{U}|\psi\rangle}, \left(\sigma^{-1}_{\mathcal{U}|\psi\rangle} \circ \sigma_{|\psi\rangle}\right)(z)\right)$$

Remark 47. Given $\mathcal{U}$ and $\sigma$ the automorphism $\hat{\mathcal{U}}_\sigma$ can be given equivalently by:

$$\hat{\mathcal{U}}_\sigma(|\psi\rangle, \alpha_{|\psi\rangle}, z) = \left(\mathcal{U}|\psi\rangle, \left(\alpha_{|\psi\rangle} \circ \tau^{-1}_{|\psi\rangle} \circ \sigma_{|\mathcal{U}|\psi\rangle}\right)_{\mathcal{U}|\psi\rangle}, \tau_{|\psi\rangle}(z)\right)$$

where $\tau_{|\psi\rangle} = \sigma^{-1}_{|\mathcal{U}|\psi\rangle} \circ \sigma_{|\psi\rangle}$ : $Z_{|\psi\rangle} \rightarrow Z_{|\mathcal{U}|\psi\rangle}$ is a borel equivalence (defined out of $\mu-$null borel subsets) with $\tau_{|\psi\rangle} \cdot \mu_{|\psi\rangle} = \mu_{|\mathcal{U}|\psi\rangle}$.

Remark 48. Written $\hat{\mathcal{U}}_\sigma(|\psi\rangle, \alpha, z) = (\mathcal{U}|\psi\rangle, \alpha', z')$) note that $\alpha'(z') = \alpha(z)$.

Remark 49. Each automorphism $\hat{\mathcal{U}}_\sigma$ induces the projective transformation $[\mathcal{U}] : \mathcal{P}(\mathcal{H}) \rightarrow \mathcal{P}(\mathcal{H})$ defined by $[\mathcal{U}] |\psi\rangle = |\mathcal{U}|\psi\rangle$. Fixed a complex of equivalences $\sigma$ we have $\hat{\mathcal{U}}_\sigma = \hat{\mathcal{V}}_\sigma$ if and only if $[\mathcal{U}] = [\mathcal{V}]$. The automorphisms with a fixed $\sigma$ make a group isomorphic to the group of projective transformations of $\mathcal{P}(\mathcal{H})$.

Theorem 50. Given a 1-parameter group $\{\mathcal{U}_t\}_{t \in \mathbb{R}}$ of unitary transformations of $\mathcal{H}$ and a complex of equivalences $\sigma$, the family $\{\hat{\mathcal{U}}_{t\sigma}\}_{t \in \mathbb{R}}$ is a 1-parameter group of automorphisms of $C$ inducing the 1-parameter group $\{[\mathcal{U}_t]\}_{t \in \mathbb{R}}$ of projective transformations of $\mathcal{P}(\mathcal{H})$.

Proof. Obvious.
Remark 51 A 1-parameter group of automorphisms of \( C \) gives a deterministic evolution for every complete initial state \((|\psi_0\rangle, \alpha_{0|\psi_0\rangle}, z_0)\). Every evolution \({\mathcal{U}} \{ \psi_0 \}_{t \in \mathbb{R}} \) in \( \mathcal{H} \) of a unitary vector \( \psi_0 \) with \({\mathcal{U}} \}_{t \in \mathbb{R}} 1\)-parameter group of unitary transformations, can be seen, in several ways, as the apparent part of an evolution of complete states.

Theorem 52 For every self-adjoint operator \( A \), every complex of equivalences \( \sigma \) and every unitary operator \( \tilde{\mathcal{U}} \) we have:

\[
\tilde{\mathcal{U}}^{-1} \tilde{\mathcal{U}} = \tilde{A} \circ \tilde{U}_\sigma
\]

\( \square \)

Proof. \( \tilde{\mathcal{U}}^{-1} \tilde{\mathcal{U}} \langle |\psi\rangle, \alpha_{|\psi\rangle}, z \rangle = \left( F_{\tilde{\mathcal{U}}^{-1}}^\psi \right)^{-1} \left( \alpha_{|\psi\rangle} \right) (z) \) and \( \tilde{A} \circ \tilde{U}_\sigma \langle |\psi\rangle, \alpha_{|\psi\rangle}, z \rangle = \tilde{A} \langle |\psi\rangle, \left( \alpha_{|\psi\rangle} \circ \sigma_{|\psi\rangle}^{-1} \circ \sigma'_{|\psi\rangle} \right) \rangle (z) \). \( \tilde{A} \) \( \tilde{\mathcal{U}}^{-1} \tilde{\mathcal{U}} \). \( \square \)

Theorem 53 Given a bounded self-adjoint operator \( A \), a 1-parameter group \({\mathcal{U}} \) of unitary transformations of \( \mathcal{H} \), a unitary vector \( \psi_0 \), a complex of barriers \( \alpha_0 \) we have:

\[
\langle \psi_t, A \psi_t \rangle = \int Z \tilde{A}_{\alpha_0} (|\psi_t\rangle, z) \cdot d\mu_{|\psi_0\rangle}(z)
\]

where \( \psi_t = \mathcal{U} \psi_0 \).

Proof. Fixed a complex of equivalences \( \sigma \) we have: \( \langle \psi_t, A \psi_t \rangle = \langle \psi_0, (\mathcal{U}_t^{-1} \mathcal{U}_t) \psi_0 \rangle = \int Z (\mathcal{U}_t^{-1} \mathcal{U}_t)_{\alpha_0|\psi_0\rangle} (z) \cdot d\mu_{|\psi_0\rangle}(z) = \int Z (\tilde{A})(\mathcal{U}_t)_{\alpha_0|\psi_0\rangle} (z) \cdot d\mu_{|\psi_0\rangle}(z) = \int Z \left( F_{\psi_t}^A \right)^{-1} \left( \alpha_{|\psi_0\rangle} \circ \sigma_{|\psi_0\rangle}^{-1} \circ \sigma'_{|\psi_0\rangle} \circ \sigma_{|\psi_0\rangle} (z) \right) \cdot d\mu_{|\psi_0\rangle}(z) = \int Z (\tilde{A})_{\alpha_0|\psi_t\rangle}(z) \cdot d\mu_{|\psi_0\rangle}(z). \( \square \)

Remark 54 The attended value of \( A \) on \( \psi_t \) (evolution of \( \psi_0 \) at the time \( t \)) is equal to the mean value of \( \tilde{A}_{\alpha_0} \) on the points \((|\psi_t\rangle, z)\) ("evolutions" of the points \((|\psi_0\rangle, z)\) at the time \( t \)).

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