Exact zeros of the Loschmidt echo and quantum speed limit time for the dynamical quantum phase transition in finite-size systems

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We study exact zeros of Loschmidt echo and quantum speed limit time for dynamical quantum phase transition in finite size systems. Our results illustrate that exact zeros of Loschmidt echo exist even in finite size quantum systems when the postquench parameter takes some discrete values in regions with the corresponding equilibrium phase different from the initial phase. As the system size increases and tends to infinity, the discrete parameters distribute continuously in the parameter regions. We further analyze the time for the appearance of the first exact zero of Loschmidt echo which is known as the quantum speed limit time $\tau_{QSL}$. We demonstrate that the maximal value of $\tau_{QSL}$ is proportional to $L$ and approaches infinity in the thermodynamical limit, when we quench the initial non-critical state to the critical phase. We also calculate the minimal value of $\tau_{QSL}$ and find that its behavior is dependent on the phase of initial state.

I. INTRODUCTION

In recent years, the development of quantum simulation platforms, such as neutral atom arrays,[16] stimulates the intensive studies on the nonequilibrium dynamics of quantum many-body systems. An interesting issue is the dynamical quantum phase transition (DQPT) which describes dynamical quantum critical phenomena presented in quench dynamics of kinds of quantum systems with initial state chosen as the ground state of a given Hamiltonian and evolving under a sudden change of a Hamiltonian parameter. The DQPT is characterized by the emergence of zero points of Loschmidt echo (LE) at a series of critical times, where the LE is defined by $\mathcal{L}(t) = |\mathcal{G}(t)|^2$ with the Loschmidt amplitude given by

$$\mathcal{G}(t) = \langle \psi_i | e^{-iH_F t} | \psi_i \rangle. \quad (1)$$

Here $|\psi_i\rangle$ is the ground state of prequench Hamiltonian, $H_F$ is the postquench Hamiltonian, and we have set $\hbar = 1$. The LE measures the overlap between initial quantum state and time-evolved state, which has wide application in various contexts ranging from the theory of quantum chaos,[17,18] and the Schwinger mechanism of particle production,[39,40] to work distribution functions in the context of nonequilibrium fluctuation theorem.[39,10] The existence of zero points of LE means the occurrence of nonanalytic behaviors of dynamical free energy, i.e., the rate function of LE given by $\lambda(t) = -\frac{1}{L} \ln \mathcal{L}(t)$, at these critical times. It has been shown that DQPT and the equilibrium quantum phase transition are closely related as the nonanalyticities in the rate function of LE occur for quenches crossing the static quantum phase transition point,[41] although a one-to-one correspondence between them does not always hold true.[42,43] The relation between the long-time average of the LE and the ground state fidelity susceptibility,[44] was also unveiled recently.[45]

In general, exact zeros of LE or nonanalyticities of dynamical free-energy only occur when the system size tends to infinity. This is very similar to Fisher zeros of the partition function in statistical physics.[46] It is well known that the Fisher zeros in a finite size system do not lie on the real temperature axis, and exact zeros only emerge in the thermodynamic limit.[47] Similarly, the exact zeros of LE in a finite size quantum system can only appear in the complex time plane. When the system size tends to infinity, the zeros approach to the real time axis for quenches crossing the quantum phase transition point. Now a question arises: can exact zeros of LE occur in real time axis for a finite size quantum system? If the answer is yes, it seems that there exists controversy with the known results and how to understand the seeming controversy?

Aiming to answer the above questions and deepen our understanding of DQPT in the finite size systems, we shall explore the exact zeros of LE by focusing on a well-known exact solvable model, i.e., the transverse field Ising model (TFIM), which is well studied and known to exhibit DQPT in the thermodynamic limit. The existence of exact solutions enables us to analytically derive the condition for the existence of exact zeros of LE in finite size systems. For a given initial state prepared as the ground state of pre-quench Hamiltonian, our results illustrate that exact zeros of LE exist even in finite size quantum systems when the post-quench parameter takes some discrete values. These discrete parameters are found to locate in regions with the corresponding equilibrium phase different from the initial phase. As the system size increases and tends to infinity, the discrete parameters distribute continuously in the parameter regions and thus are consistent with previous results.

Further, once we know the exact zeros of LE in finite size quantum systems, it is natural to explore the minimum time of an initial state evolving to its orthogonal state which corresponds to the time for the emergence of
the first exact zero of LE. The minimum time required for arriving an orthogonal quantum state is called quantum speed limit (QSL) time, denoted as $\tau_{QSL}$. The QSL time gives fundamental limit on the scale for how fast a quantum state evolves in real-time dynamics, and the lower bound of $\tau_{QSL}$ has been discussed in closed quantum systems\textsuperscript{23,39} and open quantum systems\textsuperscript{51,63}. The discussion of QSL time can be traced back to the early time when Mandelstarn and Tamm studied time-energy uncertainty in non-relativistic quantum mechanics\textsuperscript{23}. The QSL time is also related to several interesting topics, such as quantum optimal control\textsuperscript{29} and quantum information\textsuperscript{50} and quantum geometry\textsuperscript{61,62}. In the framework of DQPT, the QSL time has been studied in a previous work\textsuperscript{23}, which however, only concerns on the dynamics of the quantum critical state. As we shall clarify in this work, if the initial system is in the quantum critical state, no exact zeros of LE can be found and thus the connection of QSL time to the DQPT is still elusive. In this work, we shall unveil how the QSL time changes with quench parameters and explore the general connection between QSL time and DQPT by considering various situations with different quench parameters. We also pay particular attention to the maximum and minimum value of $\tau_{QSL}$, as the maximum of $\tau_{QSL}$ is related to the quench dynamics close to critical point and the minimum of $\tau_{QSL}$ gives the important message for how fast the DQPT could happen. When the system size tends to infinity, we find that the maximum value of $\tau_{QSL}$ approaches infinity when the quench parameter approaches to the critical point, which is independent of the initial state. However, the behavior of the minimum value of $\tau_{QSL}$ is distinct if the initial state is chosen in different phase. We demonstrate that non-analytical behaviours appear in both the average of $\tau_{\min}(L)$ and the variance of $\tau_{\min}(L)$ when we change the quench parameter across the static quantum phase transition point.

The article is organized as follows. In Sec. II, we study the quench dynamics in finite size systems of the transverse field Ising model and give exactly the relation for the occurrence of zeros of LE. The dynamical behavior quenched to the parameter region close to the critical point is also studied. In Sec. III, we study quantum speed limit time under different quench parameters and explore its connection to DQPT. A summary and outlook is given in Sec.IV.

II. EXACT ZEROS OF LOSCHMIDT ECHO IN FINITE SIZE SYSTEMS

We consider the one-dimensional (1D) TFIM described by the following Hamiltonian\textsuperscript{23}

$$H = -J \sum_{j=1}^{L} \sigma_{j}^{z} \sigma_{j+1}^{z} - h \sum_{j=1}^{L} \sigma_{j}^{z},$$

(2)

where $J$ is the nearest-neighbor spin coupling, $h$ is the external magnetic field along the z axis and the periodic boundary condition $\sigma_{L+1}^{z} = \sigma_{1}^{z}$ is assumed. The three Pauli matrices are $\sigma_{j}^{\alpha}(\alpha = x, y, z), j = 1, \cdots, L$ with $L$ denoting total number of lattice sites. The TFIM fulfills a duality relation\textsuperscript{14,26} $U(\beta, h)U^{-1} = Jh (1/J, 1/h)$. By using the Jordan-Wigner transformation, the even-parity and odd-parity of the TFIM with periodical boundary condition can be mapped to the anti-periodical Kitaeev chain and periodical Kitaeev chain, respectively\textsuperscript{23,14,28,29}. Then we can write the Hamiltonian in the fermion representation as

$$H = -J \sum_{j=1}^{L} \left( c_{j}^{\dagger} c_{j+1} + c_{j}^{\dagger} c_{j+1}^{\dagger} + \text{H.c.} \right) - 2h \sum_{j=1}^{L} c_{j}^{\dagger} c_{j}$$

$$\pm J (c_{L}^{\dagger} c_{1} + c_{L}^{\dagger} c_{1}^{\dagger} + \text{H.c.}),$$

(3)

where the plus sign or minus sign is corresponds to the even-parity or odd-parity. For convenience, we take $J > 0$ in the following discussion so that the system is in the ferromagnetic phase when $|h/J| < 1$.

It is convenient to diagonalize the Hamiltonian (3) in the momentum space by using the Fourier transform $c_{k}^{\dagger} = \frac{1}{\sqrt{L}} \sum_{j} e^{ikj} c_{j}^{\dagger}$. Here values of $k$ should be chosen in the set of $\mathcal{K}_{\text{PBC}} = \{ k = \frac{2\pi m}{L} | m = -L/2 + 1, 0, \cdots, L/2 \}$ for periodical boundary condition (PBC) and $\mathcal{K}_{\text{aPBC}} = \{ k = \frac{\pi (2m - 1)}{L} | m = 1, \cdots, L/2 \}$ for anti-periodical boundary condition (aPBC)\textsuperscript{14,26,29}. In the following discussion, we focus on the even site of lattice with even parity which is corresponding to aPBC. It should be noted that all terms of Hamiltonian come in pairs $(k, -k)$ for aPBC. Define the positive $k$ values as $\mathcal{K}^{+}_{\text{aPBC}} = \{ k = \frac{\pi (2m - 1)}{L} | m = 1, \cdots, L/2 \}$. Then the Hamiltonian in momentum space is

$$H = -2 \sum_{k \in \mathcal{K}^{+}_{\text{aPBC}}} \left[ (J \cos k + h) (c_{k}^{\dagger} c_{k} - c_{-k}^{\dagger} c_{-k}) - iJ \sin k c_{k}^{\dagger} c_{-k} + \text{H.c.} \right].$$

(4)

By using the Bogoliubov transformation

$$\beta_{k} = e^{i\frac{\beta}{2}} \cos \theta_{k} c_{k} + i \sin \theta_{k} c_{-k}^{\dagger},$$

$$\beta_{-k}^{\dagger} = e^{i\frac{\beta}{2}} \sin \theta_{k} c_{k} + i \cos \theta_{k} c_{-k}^{\dagger},$$

where $\frac{\beta}{2} = \cos 2\theta_{k}$ and $\frac{\beta}{2} = \sin 2\theta_{k}$ with $e^{i\beta} = -J \cos k - h$ and $\beta_{k} = -J \sin k$, we arrive at a Hamiltonian given by\textsuperscript{23,14,28,29}

$$H = 2 \sum_{k \in \mathcal{K}^{+}_{\text{aPBC}}} \left( E_{k} \beta_{k}^{\dagger} \beta_{k} - E_{k} \beta_{-k}^{\dagger} \beta_{-k}^{\dagger} \right),$$

(5)

where $E_{k} = \sqrt{\epsilon_{k}^{2} + \epsilon_{k}^{2}}$.

Then we consider the quench dynamics driven by the transverse field $h$ which can be described by $h(t) =$
\( h_i \Theta(-t) + h_f \Theta(t) \). The analytical formula of LE has the form

\[
    \mathcal{L}(t) = \prod_{k \in \mathbb{K}_{\text{aPBC}}} \left[ 1 - \sin^2(2\delta\theta_k) \sin^2(2E_{k_f}t) \right],
\]

where \( \delta\theta_k = \theta_{k_f} - \theta_{k_i} \).

\[
    \theta_{k_i} = \frac{1}{2} \arctan \frac{J \sin k}{J \cos k + h_i}
\]
is the Bogoliubov angle of prequench Hamiltonian,

\[
    \theta_{k_f} = \frac{1}{2} \arctan \frac{J \sin k}{J \cos k + h_f}
\]
is the Bogoliubov angle of postquench Hamiltonian, and \( E_{k_f} \) is the energy of the postquench Hamiltonian. For the 1D transverse field Ising chain, we can prove that if the postquench parameter takes these discrete values, we have \( \mathcal{L}(t) = 0 \) at

\[
    t = t^*_n = \frac{\pi}{2E_{k_f}} \left( n + \frac{1}{2} \right),
\]

with

\[
    E_{k_f}/J = \sqrt{(\cos k + h_f/J)^2 + \sin^2 k},
\]
i.e., there exist exact zeros of LE as long as Eq. (7) is fulfilled. According to Eq. (7), if \( h_i/J \in (-1, 1) \), the exact zeros of LE emerge only for \( h_f/J \in (-\infty, -1) \cup (1, \infty) \). For the 1D transverse field Ising chain, we can prove that the Loschmidt echo fulfills the following dynamical duality relation

\[
    \mathcal{L}(\gamma_i, \gamma_f, t) = \mathcal{L}(\gamma_i^{-1}, \gamma_f^{-1}, \gamma_f t),
\]

where \( \gamma_i = h_i/J \) and \( \gamma_f = h_f/J \) are the dimensionless parameters. Due to the existence of dynamical duality relation (see appendix for details), we only need to consider the case of \( h_i/J \in (-1, 1) \) as the cases of \( h_i/J \in (-\infty, -1) \) and \( h_i/J \in (1, \infty) \) can be obtained by using the dynamical duality relation.

As displayed in Fig. 1(a) for the system with lattice size \( L = 14 \), for a given \( h_i/J \), only \( L/2 \) discrete values of \( h_f/J \) satisfy Eq. (7). Continuously varying \( h_i/J \) leads to the formation of a series of curves in the parameter space spanned by \( h_i/J \) and \( h_f/J \). When we increase the lattice size, the number of curves increases linearly and the distribution of curves becomes more and more dense, as shown in Fig. 1(b) for the system with \( L = 400 \).

Figure 1. The combination of \( h_i/J \) and \( h_f/J \) which fulfill Eq. (7). (a) \( L = 14 \) and (b) \( L = 400 \).

To characterize the average distance between neighboring curves, we define the quantity \( \Delta \) as

\[
    \Delta = \frac{1}{L-1} \sum_{k=1}^{L/2} \left| \frac{1}{J} \left[ h_f(k + \frac{2\pi}{L}) - h_f(k) \right] \right|,
\]

where \( h_f(k) \) is the solution of Eq. (7) for a \( k \) mode. In the thermodynamic limit we can turn the sum into an integral and it can be found that \( \Delta \) is approximately equal to \( 4/L \) which approaches to 0 as \( L \to \infty \). This is also confirmed by the numerical result as displayed in Fig. 2(a). Therefore the discrete values of \( h_f/J \) tend to distribute continuously in the thermodynamic limit, which is consistent with the general knowledge about the DQPT that zeros of LE appear when we quench the system across the critical point.

Figure 2. (a) Green dots are the numerical results of Eq. (11) for finite size systems. Red dashed line is the result in the large size limit \( \Delta \approx 4/L \). We set prequench parameter as \( h_i/J = 1.5 \) (−1), (b) The \( \Delta_c(h_i) \) with respect to \( 1/L^2 \) for different values of \( h_i/J \). The discrete marks are the numerical results and the dashed lines are the results of Eq. (12).

If we quench the system to the critical point \( h_f/J = 1 \) (−1), from Eq. (7), we can see that no exact zeros of LE are available unless the initial state is prepared in the other critical point \( h_i/J = -1 \) (1). Interesting, if
we restrict \( h_i/J \geq 0 \) and \( h_f/J \geq 0 \), no exact zero of Loschmidt echo can be found if the value of \( h_i/J \) or \( h_f/J \) is in the interval \( -\cos \frac{L}{L-1} - \sec \frac{L}{L-1} \) for the finite size system. The interval \( -\cos \frac{L}{L-1} - \sec \frac{L}{L-1} \approx \left( 1 - \frac{1}{2} \left( \frac{L}{L-1} \right)^2, 1 + \frac{1}{2} \left( \frac{L}{L-1} \right)^2 \right) \) is around the critical point \( h_c/J = 1 \) and the boundary of the interval are reciprocal due to the existence of dynamical duality for the TFIM. Moreover, we can define a quantity \( \Delta_c(h_i) \) which represents the shortest distance between \( h_c/J = 1 \) and the solutions of Eq. (7) for \( h_i/J = -1.5, -0.2, 0.6, 2 \) represented by different marks. The approximate formula of \( \Delta_c(h_i) \) for large \( L \) can be derived from Eq. (7), which reads as

\[
\Delta_c(h_i) \approx \frac{\alpha(h_i)}{L^2} \tag{12}
\]

where \( \alpha(h_i) = \frac{\pi^2 (h_i + h_f)}{2(1 - h_i h_f)} \). The results of Eq. (12) for \( h_i/J = -1.5, -0.2, 0.6, 2 \) are shown in Fig. (2)(b) which are denoted by black dashed lines. So, if we quench the system from arbitrary value of \( h_i/J \) except \(-1 \) to \( h_f/J \) near the critical point, then there exists a region in which no exact zeros of LE are available for a finite size system. The width of this region is dependent on \( h_i/J \) and \( \Delta_c(h_i) \rightarrow 0 \) in the thermodynamic limit for any \( h_i/J \). Together with the result of \( \Delta \rightarrow 0 \), we can see that exact zeros of LE would exist so long as we quench across the critical point for the infinite size system, in agreement with the previous work in the thermodynamic limit.

The result of the shortest distance between \( h_f/J = -1 \) and the solutions of Eq. (7) for arbitrary value of \( h_i/J \) is similar to Eq. (12).

III. QUANTUM SPEED LIMIT TIME FOR DYNAMICAL QUANTUM PHASE TRANSITION

From the previous section, we know that there exist exact zeros of LE as we quench the ground state across the static phase transition point. It is known that the QSL time is the minimal time for the evolution of an initial state to its orthogonal state, and thus the time for the emergence of the first exact zero of LE gives the QSL time, i.e.,

\[
\tau_{QSL} = \frac{\pi}{4E_{QSL}} \tag{13}
\]

According to Eq. (9), the QSL time is dependent on the values \( h_f/J \).

As displayed in Fig. 3 for the system with \( L = 22 \), we show that a series of divergence points of rate function, corresponding to exact zeros of LE, appear in the real time axis. The QSL time corresponds to the first divergence point of rate function, which is labeled by black dashed line. To see the dependence of \( \tau_{QSL} \) on \( h_f/J \), we plot rate function versus \( Jt \) for all permitted \( h_f/J > 0 \) determined by Eq. (7). It can be observed from the Fig. 3(a) that the QSL time decreases with the increase in \( h_f/J \), when we quench from the initial phase in the region of \( 0 < h_i/J < 1 \) to the region of \( h_i/J > 1 \). On the other hand, when we quench from the region of \( h_i/J > 1 \) to \( 0 < h_f/J < 1 \), the QSL time decreases with the decrease in \( h_f/J \) as shown in Fig. (3)(b). Such an observation does not rely on the system size and can be obtained from Eq. (7) and Eq. (8). It follows that the QSL time increases as \( h_f/J \) approaches the critical point \( h_f/J = 1 \).

Particularly, we denote the maximal value of quantum speed limit time as \( \tau_{max} = \max |\tau_{QSL}| \). It is found that \( \tau_{max} \) is corresponding to the quench process with the postquench parameter closest to \( h_c/J = 1 \). From the analytical result Eq. (7) and the formula of Eq. (6), it follows that \( E_{QSL} \approx \frac{\pi}{4} \) is minima if \( k = \frac{\pi}{L} \) or \( k = \frac{L-1}{L} \pi \). According to Eq. (7), it should also be noted that the mode \( k = \frac{\pi}{L} \) and \( k = \frac{L-1}{L} \pi \) is corresponding to \( h_f/J \) closest to \(-1 \) and \( 1 \) for the finite size system, respectively. Then we have \( \tau_{max} \approx L/(4J) \) which can be regarded as a upper bound of QSL time. As demonstrated in Fig. 4(a) we display the QSL time with respect to \( L \) for \( h_i/J = 0.2, 0.4, 0.6, 2 \) and \( h_f/J \) taken to be closest to \(-1 \) (Fig. 4(a)) and \( 1 \) (Fig. 4(b)), respectively. The red dashed lines in Fig. 4 guide the value of \( \tau_{QSL} \), which increases linearly with the increase in the system size. In the thermodynamic limit, \( k \rightarrow 0 \) and \( k \rightarrow \pi \) is corresponding to \( h_f/J \rightarrow \pi \) and \( h_f/J \rightarrow 1 \), respectively. When \( L \rightarrow \infty \), we have \( \tau_{max} \rightarrow \infty \) with \(|h_f/J| \rightarrow 1 \). This means that we can not observe the DQPT in a finite time if we quench the system from a non-critical phase to the critical phase with \(|h_f/J| = 1 \), i.e., no DQPT occurs in a finite time if we quench from a non-critical phase to the critical point due to the corresponding \( \tau_{max} \) which is approaching infinity.
For any ground state of 1D TFIM, it is also interesting to ask how fast could the ground state achieve to its orthogonal state as we quench the parameter to the thermodynamic limit. For the other case with the ferromagnetic initial state \( |\psi_0\rangle = \sqrt{\frac{J}{\pi}} \langle L | \uparrow \rangle \) the MT bound \( \tau_{\text{MT}} \) is tight if the prequench parameter \( h_i/J \) is chosen to be closest to \( \frac{\pi}{4} \). Then we have \( J_{\text{QSL}} \approx \frac{\pi}{2L} \) which is illustrated in Fig. 4(b) by the green solid line and it is shown that the asymptotic behavior of \( \tau_{\text{QSL}} \) is captured by the line of \( \frac{\pi^2}{4L} \) for \( |h_i/J| < 1 \).

\[
\tau_{\text{QSL}} \geq \tau_{\text{MT}} = \frac{\pi}{2\Delta E},
\]

where \( (\Delta E)^2 = \langle \psi_i | H^2 | \psi_i \rangle - (\langle \psi_i | H_f | \psi_i \rangle)^2 \) with \( H_f \) denotes the postquench Hamiltonian. Next, we consider the two cases discussed above. For the case with paramagnetic initial state \( |\psi_i\rangle = \sum L j=1 \frac{\sqrt{J}}{2} \uparrow \rangle \) and the postquench Hamiltonian \( H_f = -J \sum L j=1 \sigma_j^x \sigma_{j+1}^x \) with \( h_f/J = 0 \), we have \( \Delta E = J \sqrt{L} \) due to \( \langle \psi_i | H^2 | \psi_i \rangle = J^2 L \) and \( \langle \psi_i | H_f | \psi_i \rangle = 0 \). So the MT bound is \( \tau_{\text{MT}} \to 0 \) in the thermodynamical limit. For the other case with the ferromagnetic initial state \( |\psi_i\rangle = \sum L j=1 \frac{\sqrt{J}}{2} \uparrow \rangle \) (or \( |\psi_i\rangle = \sum L j=1 \frac{\sqrt{J}}{2} \downarrow \rangle \) and \( H_f = -h_f \sum L j=1 \sigma_j^z \) with \( h_f/J \to \infty \), we have \( \Delta E \to \infty \) and \( \tau_{\text{MT}} \to 0 \). It can be seen that the MT bound of QSL time is equal to zero for both two cases. In comparison with our exact result of \( \tau_{\text{QSL}} \), it can be found that the MT bound \( \tau_{\text{MT}} \) is tight if the prequench Hamiltonian lies in the ferromagnetic phase.

To see clearly how \( \tau_{\text{QSL}}(L) \) changes with \( h_i/J \), we can calculate the mean value of \( \tau_{\text{QSL}}(L) \) numerically from \( L_{\text{min}} \) to \( L_{\text{max}} \) and denote it as:

\[
\bar{\tau}_{\text{QSL}} = \frac{1}{L_{\text{max}} - L_{\text{min}}} \sum_{L=L_{\text{min}}}^{L_{\text{max}}} \tau_{\text{QSL}}(L).
\]

Meanwhile, we can also calculate the variance of \( \tau_{\text{QSL}}(L) \) defined by:

\[
\sigma^2_{\tau_{\text{QSL}}} = \frac{1}{L_{\text{max}} - L_{\text{min}}} \sum_{L=L_{\text{min}}}^{L_{\text{max}}} [\tau_{\text{QSL}}^2(L) - \bar{\tau}_{\text{QSL}}^2].
\]
We count from $L_{\text{min}} = 10$ to $L_{\text{max}} = 10000$ and show the numerical results of $J\bar{\tau}_{\text{min}}$ and $J^2\sigma_{\text{min}}^2$ with respect to prequench parameter $h_i/J$ in Figs. 6(a) and 6(b), respectively. It can be observed that both $\bar{\tau}_{\text{min}}$ and $\sigma_{\text{min}}^2$ have an abrupt change at $h_i/J = 1$ which corresponds to the static quantum phase transition point in the thermodynamic limit. The fluctuation of energy can be evidenced in $\sigma_{\text{min}}^2$ (Fig. 6(b)) which remains a nonzero value as $|h_i/J| < 1$ and diverges as $|h_i/J|$ approaches 1. The non-analytical behaviours appearing in the change of prequench parameter across the static quantum phase transition point indicates that clearly the minima of QSL time relies on the choice of initial states.

![Figure 6](image)

Figure 6. (a) $J\bar{\tau}_{\text{min}}$ with respect to $h_i/J$; (b) $J^2\sigma_{\text{min}}^2$ with respect to $h_i/J$. Here we count the size of system from $L_{\text{min}} = 10$ to $L_{\text{max}} = 10000$.

### IV. SUMMARY AND OUTLOOK

In summary, we have analytically calculated the exact zeros of the LE for the 1D TFIM and shown that there exist exact zeros of LE even for the finite size quantum system when the post-quench parameter takes some discrete values. As the system size tends to infinity, the discrete parameters distribute continuously in the parameter regions with the corresponding equilibrium phase different from the initial phase, which is in agreement with previous work in the thermodynamic limit. We also unveil that no exact zeros of LE are available if we quench the system to (or from) the critical point. For the finite size systems of 1D TFIM, we have unveiled how the QSL time changes with quench parameters and studied the behaviors of the maximum and minimum values of the QSL time. From our analytical result in the thermodynamic limit, it is shown that no DQPT occurs in a finite time if we quench from a non-critical phase to the critical point due to that the corresponding $\tau_{\text{max}}$ is approaching infinity. We have also illustrated the existence of non-analytical behaviors in both the average of $\tau_{\text{min}}(L)$ and the variance of $\tau_{\text{min}}(L)$ when we change the parameter of prequench Hamiltonian across the underlying static critical point.

Our work provides a firm theoretical ground for understanding why and how the DQPT occurs with the increase of system sizes and the peculiar dynamical behavior near the critical point, which paves the way for experimental investigations of DQPT for small size systems. According to our theoretical finding, we can always find exact zeros of LE by tuning the quench parameter to a series of discrete fine-tuning points for the finite size system which supports DQPT in the thermodynamical limit. At these fine-tuning points, the divergence of the corresponding rate function can be observed in some critical times. The number of fine-tuning points increases linearly with the increase of lattice size. By recording the critical times for the emergence of exact zeros of LE, one can also experimentally study the behaviors of quantum speed limit time and explore its connection to the DQPT.

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### Appendix A: The dynamical duality relation of the Loschmidt echo

Consider the quench dynamics of the quantum TFIM by suddenly switching the transverse field from the prequench parameter $h_i$ to the postquench parameter $h_f$. The Loschmidt echo can be represented as

$$\mathcal{L}(\gamma_i, \gamma_f, t) = \prod_k \mathcal{L}_k(\gamma_i, \gamma_f, t),$$

where $\gamma_i = h_i/J$ and $\gamma_f = h_f/J$ are the dimensionless parameters and the $k$-component of the Loschmidt echo is

$$\mathcal{L}_k(\gamma_i, \gamma_f, t) = 1 - \sin^2 (2\delta\theta_k) \sin^2 (2E_{k_f}t).$$

Here we have

$$\sin^2 (2\delta\theta_k) = \left( \frac{\zeta_f \epsilon_i \epsilon_f}{E_f E_i} - \frac{\epsilon_f \zeta_i}{E_f E_i} \right)^2$$

$$= \left( \frac{\zeta_f \epsilon_i - \epsilon_f \zeta_i}{E_i} \right)^2$$

$$= \frac{(\epsilon_f^2 + \zeta_f^2)(\epsilon_i^2 + \zeta_i^2)}{(\epsilon_i^2 + \zeta_i^2)(\epsilon_f^2 + \zeta_f^2)}$$

$$= \frac{(\gamma_i - \gamma_f)^2}{(1 + 2\gamma_i \cos k + \gamma_i^2)(1 + 2\gamma_f \cos k + \gamma_f^2)},$$

and

$$\sin^2 (2E_{k_f}t) = \sin^2 \left( 2J_f t \sqrt{1 + 2\gamma_f \cos k + \gamma_f^2} \right).$$
Now we consider the case with the prequench and postquench dimensionless parameters being $1/\gamma_i$ and $1/\gamma_f$, respectively. The $k$-component of Loschmidt echo of the corresponding model can be written as

$$L_k(\gamma_i^{-1}, \gamma_f^{-1}, t) = 1 - \sin^2(2\tilde{\theta}_k) \sin^2(2\tilde{E}_kt), \quad (A3)$$

with

$$\sin^2(2\tilde{\theta}_k) = \frac{\left(\gamma_i^{-1} - \gamma_f^{-1}\right)^2 \sin^2 k}{(1 + 2\gamma_i^{-1}\cos k + \gamma_i^{-2})(1 + 2\gamma_f^{-1}\cos k + \gamma_f^{-2})} = \frac{\sin^2 k}{(\gamma_i^2 + 2\gamma_i\cos k + 1)(\gamma_f^2 + 2\gamma_f\cos k + 1)} = \sin^2(2\tilde{\theta}_k),$$

and

$$\sin^2(2\tilde{E}_kt) = \sin^2(2E_k\gamma_f^{-1}t)$$

where $\tilde{E}_k \equiv E_k(1/\gamma_f)$ and we have used the dual relation of eigenvalues $E_k(\gamma_f) = \gamma_f E_k(1/\gamma_f)$.

Then we can observe that

$$L_k(\gamma_i, \gamma_f, t) = L_k(\gamma_i^{-1}, \gamma_f^{-1}, \gamma_f t). \quad (A4)$$

which gives rise to the dynamical dual relation Eq. (10) directly.

**Appendix B: The case under periodical boundary condition**

![Figure 7. The combination of $h_i/J$ and $h_f/J$ which fulfill Eq. (7) under the PBC for odd parity space. (a) $L = 14$ and (b) $L = 400$.](image)

In the main text, we have taken the anti-periodical boundary condition. In this appendix, we consider the odd parity of the 1D TFIM which corresponds to the periodical boundary condition of Hamiltonian Eq. (3). The formulas for determining exact zeros of LE in the main text do not change, and the constraint relation of quench parameter is the same as the Eq. (7). However, the values of $k$ under PBC should be chosen in the set of $K_{PBC} = \{k = \frac{2\pi m}{L} | m = -L/2 + 1, \cdots, 0, \cdots, L/2\}$. Here the modes corresponding to $k = 0$ and $k = \pi$ should be removed due to $\sin^2(2\tilde{\theta}_k) = 0$ and thus $L_k = 1$ for $k = 0$ or $k = \pi$. Therefore, under the PBC, for a given $h_i/J$, only $L/2 - 1$ discrete values of $h_f/J$ satisfy Eq. (7) corresponding to $k = \frac{2\pi}{L}, \cdots, \frac{2\pi(L/2-1)}{L}$. We display the result of Eq. (7) in Fig. 7(a) and (b) for $L = 14$ and $L = 400$ under the PBC, respectively, in contrast to Fig. 1 for the same system under the aPBC. When the system size tends to infinity, the discrete values of $h_f/J$ tend to distribute continuously, and therefore our conclusions do not rely on boundary conditions in the thermodynamical limit.

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