Polish group actions and admissible sets

B. Majcher-Iwanow

Abstract. We define some coding of Borel sets in admissible sets. Using this we generalize certain results from model theory involving admissible sets to the case of continuous actions of closed permutation groups on Polish spaces. In particular we obtain counterparts of Nadel's theorems about relationships between Scott sentences and admissible sets.

2000 Mathematics Subject Classification: 03E15, 03C70

Keywords: Polish G-spaces, Canonical partitions, Admissible sets.

0 Introduction

The aim of the paper is to study actions of closed permutation groups on Polish spaces in admissible sets. Let $A$ be an admissible set. Under some natural assumptions we can define in $A$ a class of functions that can be considered as 'recipes' for Borel subsets of second countable spaces. In Section 1 we describe such a coding and establish its basic properties.

Section 2 provides another tool of our study. Let $G$ be a closed subgroup of $S_\infty$, the group of all permutations of the set of natural numbers. For every ordinal $\alpha < \omega_1$ we define $\alpha$-sets, Borel invariants that generalize on the one hand the concept of a canonical partition introduced by Becker in [2], on the other - the concept of the $\alpha$-characteristic of a sequence in a structure given by Scott (see [1], p. 298). In some other form these sets are defined and partially studied by Hjorth in [6]. Since they seem to be interesting on its own rights we examine their properties in detail. We use them for the analysis of Borel complexity of $G$-orbits (see Section 2.2).
The main results of the paper are proved in Section 3. We prove that all Borel sets naturally involved in Scott analysis can be coded in appropriate admissible sets. Then we generalize Nadel’s results concerning coding of Scott analysis of countable structures in admissible sets [12]. We also give a generalization of another model theoretical result - we characterize in admissible sets orbits that are pieces of the canonical partition with respect to some ‘finer’ topology (nice topology [2]).

A detailed description of our results is contained in Section 1.

**Notation.** A Polish space (group) is a separable, completely metrizable topological space (group). If a Polish group \( G \) continuously acts on a Polish space \( X \), then we say that \( X \) is a Polish \( G \)-space. We usually assume that \( G \) is considered under a left-invariant metric. We say that a subset of \( X \) is invariant if it is \( G \)-invariant.

We consider the group \( S_\infty \) of all permutations of the set \( \omega \) of natural numbers and all its subgroups under the usual left invariant metric \( d \) defined by

\[
d(f, g) = 2^{-\min\{k : f(k) \neq g(k)\}}, \text{ whenever } f \neq g.
\]

We shall use the letters \( a, b, c, d \) for finite sets of natural numbers. For a finite set \( d \) of natural numbers let \( id_d \) be the identity map \( d \to d \) and \( V_d \) be the group of all permutations stabilizing \( d \) pointwise, i.e., \( V_d = \{ f \in S_\infty : f(k) = k \text{ for every } k \in d \} \).

Writing \( \langle d_n \rangle \) or \( \langle V_n \rangle \) we treat \( n \) as the set of all natural numbers less than \( n \).

Let \( S_{<\infty} \) denote the set of all bijections between finite subsets of \( \omega \). We shall use small greek letters \( \delta, \sigma, \tau \) to denote elements of \( S_{<\infty} \). For any \( \sigma \in S_{<\infty} \) let \( dom[\sigma], \text{rng}[\sigma] \) denote the domain and the range of \( \sigma \) respectively.

For every \( \sigma \in S_{<\infty} \) let \( V_\sigma = \{ f \in S_\infty : f \supseteq \sigma \} \). Then for any \( f \in V_\sigma \) we have \( V_\sigma = fV_{\text{dom}[\sigma]} = V_{\text{rng}[\sigma]}f \). Thus the family \( N = \{ V_\sigma : \sigma \in S_{<\infty} \} \) consists of all left (right) cosets of all subgroups \( V_d \) as above. This is a basis of the topology of \( S_\infty \).

Given \( \sigma \in S_{<\infty} \) and \( s \subseteq \text{dom}[\sigma] \), then for any \( f \in V_\sigma \) we have \( V_{\sigma}^f = V_{\sigma[s]} \), where \( V_{\sigma}^f \) denotes the conjugate \( fvfsf^{-1} \).

In our paper we concentrate on Polish \( G \)-spaces, where \( G \) is a closed subgroup of \( S_\infty \). For such a group we shall use the relativized version of the above, i.e., \( V^G_\sigma = \{ f \in G : f \supseteq \sigma \} \), \( S_{<\infty}^G = \{ f_d : f \in G \text{ and } d \text{ is a finite set of natural numbers } \} \)

(observe that for any subgroup \( G \) and any finite set \( d \) of natural numbers we have \( id_d \in S_{<\infty}^G \)). The family \( N^G = \{ V^G_\sigma : \sigma \in S_{<\infty}^G \} \) is a basis of the standard topology of \( G \).

All basic facts concerning Polish \( G \)-spaces can be found in [4], [6] and [8].

Since we frequently use Vaught transforms, recall the corresponding definitions. The Vaught \( * \)-transform of a set \( B \subseteq X \) with respect to an open \( H \subseteq G \) is the set \( B^*H = \{ x \in X : \{ g \in H : gx \in B \} \text{ is comeagre in } H \} \), the Vaught \( \Delta \)-transform of \( B \) is the set \( B^\Delta H = \{ x \in X : \{ g \in H : gx \in B \} \text{ is not meagre in } H \} \). It is known that for any \( x \in X \) and \( g \in G \), \( gx \in B^*H \iff x \in B^*Hg \) and \( gx \in B^\Delta H \iff x \in B^\Delta Hg \).

On the other hand, if \( B \in \Sigma_0^\alpha(X) \), then \( B^\Delta H \in \Sigma_0^\alpha(X) \) and if \( B \in \Pi_\alpha^0(X) \), then \( B^*H \in \Sigma_\alpha^0(X) \).

It is worth noting that for any open \( B \subseteq X \) and any open \( K < G \) we have \( B^\Delta K = KB \). Indeed, by continuity of the action for any \( x \in KB \) and \( g \in K \) with
gx ∈ B there are open neighbourhoods $K_1 \subseteq K$ and $B_1 \subseteq KB$ of g and x respectively so that $K_1B_1 \subseteq B$; thus $x \in B^{\Delta K}$. Other basic properties of Vaught transforms can be found in [4] and [8].

It is also assumed in the paper that the reader is already acquainted with the most basic notions of admissible sets. Any necessary background can be easily provided by [1] and [5].

We only remind the reader that an admissible set $A$ is a transitive model of KPU, in the sense of [1]. Such models are considered as two-sorted structures of some language $L$ with symbols $\emptyset, \in$, where one of the sorts corresponds to urelements and usually forms a relational first-order structure with respect to the symbols of $L$ distinct from $\emptyset$ and $\in$. Here we assume that $A$ satisfies KPU with respect to all formulas of $L$ ($A$ is admissible with respect to $L$ [12]).

1 Main results, Borel multicides and Codability

In this section we introduce the main notions of the paper and formulate our main results.

To discuss Borel sets in an admissible set $A$, we shall assume that $A$ contains some countable set (possibly as a set of urelements). We will say that $\omega$ is realizable in an admissible set $A$ if the set contains a copy of the structure $\langle \omega, 0 \rangle$ as an element (observe that $\omega$ is realizable in any admissible set satisfying Infinity Axiom). If $\omega$ is realizable in an admissible set $A$, then by $\Delta$-separation $A$ contains also a copy of the set $[\omega]^{\omega}_\omega$ of all finite sets of natural numbers, a copy of $S_{<\infty}$ and, since $\subseteq$ is a $\Delta_0$-predicate, copies of the posets $\langle [\omega]^{\omega}_\omega, \subseteq \rangle$ and $\langle S_{<\infty}, \subseteq \rangle$. Since it does not cause any misunderstanding, we shall write $\omega$ and $S_{<\infty}$ even if we work not with the sets themselves but with their copies.

We start with the definition of Borel multicides, i.e. the functions that can serve as recipes for Borel sets. Borel multicides are not uniquely assigned to Borel sets, although every Borel multicode (with respect to a countable ordinal) uniquely defines some Borel set.

**Definition 1** Let $A$ be an admissible set such that $\omega$ is realizable in it. We define in $A$ two binary predicates $B_\Sigma$ and $B_\Pi$ by simultaneous induction on the ordinal $\alpha > 0$. We put

\[
B_\Sigma(1, u) \iff u \text{ is a function } \land \text{dom}[u] = \omega \land \text{rng}[u] \subseteq \{0, 1\};
\]

\[
B_\Pi(\alpha, u) \iff u = (0, u') \land B_\Sigma(\alpha, u');
\]

\[
\alpha > 1 \land B_\Sigma(\alpha, u) \iff u \text{ is a function } \land (\alpha \text{ is a limit ordinal } \Rightarrow \text{dom}[u] = \alpha) \land \\
\land (\alpha \text{ is the successor ordinal } \Rightarrow \text{dom}[u] = \omega) \land \\
\land (\forall u' \in \text{rng}[u])(\exists \beta < \alpha)(B_\Pi(\beta, u') \lor B_\Sigma(\beta, u'))
\]

If $\alpha$ is a non-zero ordinal then every $u$ such that $B_\Sigma(\alpha, u)$ is called an $\alpha$-multicode while every $u$ such that $B_\Pi(\alpha, u)$ is called a co-$\alpha$-multicode.

We use some standard tricks of the general theory of definability in admissible sets (see [1]) to show that the relations above are $\Sigma$-definable. Consider the ternary
predicate
\[ B(c, \alpha, u) \text{ iff } (c = 0 \land B_\Sigma(\alpha, u)) \lor (c = 1 \land B_\Pi(\alpha, u)). \]

We see that the predicate \( B(c, \alpha, u) \) is defined in \( A \) by a \( B \)-positive \( \Sigma \)-formula. Thus by the second recursion theorem (Section 5.2 of [1]) \( B \) is a \( \Sigma \)-relation definable in \( A \). Since \( B_\Sigma(\alpha, u) \) is equivalent to \( B(0, \alpha, u) \) and \( B_\Pi(\alpha, u) \) is equivalent to \( B(1, \alpha, u) \), the predicates \( B_\Sigma \) and \( B_\Pi \) are also \( \Sigma \)-predicates definable in \( A \).

Now let \( A \) be an admissible set such that \( \omega \) is realizable in it. Let \( X \) be an arbitrary second countable space and \( \{ A_i : i \in \omega \} \) be its basis. To every \( u \) such that for some countable ordinal \( \alpha \in A \) we have \( A \models B_\Sigma(\alpha, u) \lor B_\Pi(\alpha, u) \), we assign a Borel subset \( B_u \) of \( X \) in the following manner:
\[
\begin{align*}
\text{if } & B_\Sigma(1, u) \quad \text{then } B_u = \bigcup\{A_n : u(n) = 1\}; \\
\text{if } & B_\Pi(\alpha, u) \quad \text{then } B_u = X \setminus B_{\omega}, \text{ where } u = (0, u'); \\
\text{if } & \alpha > 1 \land B_\Sigma(\alpha, u) \quad \text{then } B_u = \bigcup\{B_{u'} : u' \in \text{rng}[u]\}.
\end{align*}
\]

The assignment sends Borel multicodes \( u \) satisfying \( B_\Sigma(\alpha, u) \) to the class \( \Sigma_0^0(X) \). It is not one-to-one, in particular \( B_u = B_v \), whenever \( B_\Sigma(\alpha, u), B_\Sigma(\alpha, v) \) and \( \text{rng}[u] = \text{rng}[v] \).

**Definition 2** Let \( A \) be an admissible set. Let \( X \) be a second countable space with a basis \( \{ A_i : i \in \omega \} \) and \( B \subseteq X \) be a Borel set. If there are \( u \in A \) and a countable ordinal \( \alpha \in \text{Ord}(A) \) such that \( A \models B_\Sigma(\alpha, u) \) (or \( A \models B_\Pi(\alpha, u) \)) and \( B = B_u \), then we say that \( B \) is constructible in \( A \) by \( u \).

Observe that the empty set, the whole space \( X \) and every basic open set \( A_i \), are constructible by 1-multicodes in any admissible set \( A \) realizing \( \omega \). The functions \( mc_\emptyset, mc_X, mc_l : \omega \to \{0, 1\} \) below are the corresponding 1-multicodes
\[
mc_\emptyset = (0, 0, 0, \ldots); \quad mc_X = (1, 1, 1, \ldots); \quad mc_l = (0, 0, \ldots, 0, 1, 0, 0, \ldots). \quad \text{\textit{(l-1)-times}}
\]

We will use this notation below.

Lemma 4 contains the most obvious properties of constructibility. In particular it states that this notion is preserved under some natural operations which we shall use below. Appropriate descriptions are given in the following definition. By the second recursion theorem the predicate \( Q_v \) defined below is a \( \Sigma \)-predicate.

**Definition 3** Let \( A \) be an admissible set such that \( \omega \) is realizable in \( A \). We define in \( A \) a ternary predicate \( Q_v \) by the following formula.
\[
Q_v(u, w, v) \iff (Q_0 \land Q_1 \land Q_2)(u, w, v) \text{ where}
\]
\[
Q_0(u, w, v) = u, w, v \text{ are functions;} \\
Q_1(u, w, v) = (\exists \alpha)(\alpha \text{ is a limit ordinal} \land \text{dom}[u] = \alpha \land \text{dom}[w] = \alpha \land \text{dom}[v] = \alpha); \\
Q_2(u, w, v) = (\forall \beta < \alpha)(\forall n \in \omega)\left((\beta = 0 \lor \beta \text{ is a limit ordinal}) \Rightarrow \right. \\
\left. \Rightarrow (v(\beta + 2n) = u(\beta + n) \land v(\beta + 2n + 1) = w(\beta + n)) \right).\]
It is easy to see that the predicate $Q_v$ defines an operation on the class of all pairs of functions with common domain a limit ordinal. We shall also use the following notation. For any of $u, w, v$ such that $Q_v(u, w, v)$ we shall write $V(u, w) = v$. If $u' = (0, u), w' = (0, w)$ then we put $\land (u', w') = (0, \land (u, w))$.

It is worth noting that if $\alpha$ is an ordinal and $u, w$ are $\alpha$-multicodes then $V(u, w)$ is also an $\alpha$-multicode. If $u, w$ are co-$\alpha$-multicodes then $\land (u, w)$ is also a co-$\alpha$-multicode.

**Lemma 4** Let $\mathbb{A}$ be an admissible set and $\alpha, \beta \in \text{Ord}(\mathbb{A})$. Let $X$ be a second countable space with a basis $\{A_i : i \in \omega\}$ and $B, C \subseteq X$ be Borel sets.

1. If $\alpha < \beta$ and $B$ is constructible in $\mathbb{A}$ by some $u \in \mathbb{A}$ such that $\mathbb{A} \models B_\Sigma(\alpha, u)$ or $\mathbb{A} \models B_\Pi(\alpha, u)$ then there are $w, w' \in \mathbb{A}$ such that $\mathbb{A} \models B_\Sigma(\beta, w)$ and $\mathbb{A} \models B_\Pi(\beta, w')$ and $B = B_w = B_{w'}$.

2. If $B$ and $C$ are constructible in $\mathbb{A}$ by some $\alpha$-multicodes $u$ and $w$ respectively then $B \cup C$ is constructible in $\mathbb{A}$ by $V(u, w)$.

3. If $B$ and $C$ are constructible in $\mathbb{A}$ by some co-$\alpha$-multicodes $u$ and $w$ respectively then $B \cap C$ is constructible in $\mathbb{A}$ by $\land (u, w)$.

**Proof.** Let $u \in \mathbb{A}$ be an $\alpha$-multicode or a co-$\alpha$-multicode. Then the function $w$ defined by $w(n) = u$, for every $n \in \omega$ ($w(\zeta) = u$, for every $\zeta < \beta$) is a $\beta$-multicode for every successor (resp. limit) ordinal $\beta > \alpha$.

For turning $u$ into co-multicodes, note that the function $u'$ defined by $u'(n) = (0, u)$ for all $n \in \omega$ ($u'(\zeta) = (0, u)$ for every $\zeta < \beta$) satisfies $B_\Sigma(\beta, z)$ and serves as a $\beta$-multicode for $B_{(0, u)}$ for every successor (resp. limit) ordinal $\beta > \alpha$. Then $w'$ can be taken as $(0, \exists w')$.

The rest of the lemma is easy. □

We now define some equivalence relation $\equiv$ on the set of multicodes (co-multicodes).

**Definition 5** Let $\mathbb{A}$ be an admissible set such that $\omega$ is realizable it. We define in $\mathbb{A}$ a relation $u \equiv v$ by induction on the ordinal $\alpha > 0$:

$$u \equiv v \iff \exists \alpha \left[ (B_\Sigma(\alpha, u) \land B_\Sigma(\alpha, v)) \land (\alpha = 1 \Rightarrow u = v) \land (\alpha > 1 \Rightarrow (\forall u' \in \text{rng}[u])(\exists v' \in \text{rng}[v])(u' \equiv v') \land (\forall u' \in \text{rng}[u])(\exists v' \in \text{rng}[v])(u' \equiv v')) \right]$$

Since the operations $2_{nd}$, taking the second coordinate, and $\text{rng}$, taking the range, are $\Sigma$-definable (see Section 1.5 [1]), we see that $\equiv$ is defined by a $\equiv$-positive $\Sigma$-formula. Thus by the second recursion theorem it is a $\Sigma$-relation in $\mathbb{A}$. It is clear that $u \equiv v$ implies $B_u = B_v$. The converse implication can fail. On the other hand in some situations we will be able to obtain some kind of this converse. We will use it in Section 3 in the proof of our main results.

Now we are almost ready to discuss $G$-actions in admissible sets. We only have to define some coding of information about an action in admissible sets.
Definition 6 Let $G < S_\infty$ be a closed subgroup and $(X, \tau)$ be a Polish $G$-space with a basis $\{A_l : l \in \omega\}$. Let $\mathbb{A}$ be an admissible set. We say that $x \in X$ is codable (with respect to $G$) in $\mathbb{A}$ if $\omega$ is realizable in $\mathbb{A}$ and the function

$$F_1 : S_{<\infty} \to \mathbb{A} \text{ defined by } F_1(\sigma) = \begin{cases} \emptyset & \text{if } \sigma \notin S_{G_{<\infty}}^G \\ \{l : V_\sigma^G x \cap A_l \neq \emptyset\} & \text{if } \sigma \in S_{G_{<\infty}}^G \end{cases}$$

is an element of $\mathbb{A}$.

This condition corresponds to the standard assumption of [12] that $M \in \mathbb{A}$ where $M$ is an element of the $S_{\infty}$-space of $L$-structures in the case of the logic action of $S_{\infty}$. In Section 3.2 we give a general straightforward construction which assigns an admissible set $\mathbb{A}_x$ to any element $x \in X$ such that $x$ is codable in $\mathbb{A}$.

**Remark.** It is worth noting that in the definition we can demand only that $F_1$ is $\Sigma$-definable in $\mathbb{A}$; then $F_1$ is an element of $\mathbb{A}$ by $\Sigma$-replacement (Theorem 1.4.6 from [1]). Using $\Delta$-separation (see [1], Theorems 1.4.5) we see that if $x$ is codable in $\mathbb{A}$ then the set $S_{G_{<\infty}}^G = \{\sigma : \sigma \in S_{<\infty}, F_1(\sigma) \neq \emptyset\}$ is an element of $\mathbb{A}$.

In the situation when $x$ is codable in $\mathbb{A}$ we will usually assume that the relation

$$\text{Imp}(c, l, k) \iff (c \in [\omega]<\omega \land l, k \in \omega \land A_k \subseteq V_c^G A_l)$$

is $\Sigma$-definable in $\mathbb{A}$. This assumption is not very restrictive. For example when $X_L$ is the space of all $L$-structures on $\omega$ and $G = S_{\infty}$ acts on $X_L$ by the logic action (see [4]), take any structure $M$ on $\omega$ with an appropriate coding of finite sets (for example the standard model of arithmetic). Then $\mathbb{A} = \text{Hyp}(M, \text{Imp}(c, l, k))$, the admissible set above the structure $(M, \text{Imp}(c, l, k))$ has $\text{Imp} \Delta_0$-definable (when $M = (\omega, +, \cdot)$ we do not even need to add $\text{Imp}$, because it is $\Sigma$-definable in the structure). In Section 3.2 we give some additional examples.

The following theorem is the main result of the paper.

**Theorem 7** Let $\mathbb{A}$ be an admissible set such that $\omega$ is realizable in it. Let $G < S_\infty$ be a closed group, $X$ be a Polish $G$-space with a basis $\{A_i : i > 0\}$ and $\text{Imp}$ be $\Sigma$-definable on $\mathbb{A}$.

1. Let $x \in X$ be $\Sigma$-codable in $\mathbb{A}$. Then for every $y \in X$, if $x, y$ are in the same invariant Borel subsets of $X$ which are constructible in $\mathbb{A}$ then for every $\alpha \leq \omega(\mathbb{A})$ they are in the same invariant $\Sigma_0^\alpha$-subsets of $X$.

2. If $x, y$ are $\Sigma$-codable in $\mathbb{A}$ and they belong to the same invariant Borel sets which are constructible in $\mathbb{A}$ then they are in the same $G$-orbit.

It is based on Theorem 27, which will be proved in Section 3. In fact the method is presented in Section 2, where for every ordinal $\alpha < \omega_1$ we define $\alpha$-sets $B_\alpha(x, \sigma)$, Borel invariants that generalize on the one hand the concept of a canonical partition introduced by Becker, on the other - the concept of an $\alpha$-characteristic of a structure given by Scott. $\alpha$-Sets appear in [6] in a slightly different form. Since they seem to be interesting for its own rights we examine their properties in detail. Then we use them for the analysis of Borel complexity of $G$-orbits. As a result we are able to improve
several places of Section 6.1 of [6]. We also find a simplification of some theorem from [4] on Borel orbit equivalence relations in the case of actions of closed permutation groups.

It is worth noting that our results are not so straightforward in the direction determined by Nadel. Since we do not use standard tools from logic, we even cannot formulate them in a sufficiently close form. Instead of formulas (and of structures $\phi_\alpha$ used by Hjorth in [6]) we develop coding of $\alpha$-sets $B_\alpha(x,\sigma)$ in admissible sets (see Theorem 27). As a result some fragments of Nadel’s strategy look very different in our approach. In fact we completely avoid model theory in notation and proofs.

Theorem 7 suggests that under some additional assumptions the orbit $Gx$ becomes the intersection of all $G$-invariant Borel sets containing $x$ and codable in $A$. In Section 3 we confirm this intuition in the situation as follows. Let $(\langle X, \tau \rangle, G)$ be a Polish $G$-space with a countable basis $A$ consisting of clopen sets. Along with the topology $\tau$ we shall consider another topology on $X$. The following definition comes from [3].

**Definition 8** A topology $t$ on $X$ is nice for the $G$-space $(\langle X, \tau \rangle, G)$ if the following conditions are satisfied.

(a) $t$ is a Polish topology, $t$ is finer than $\tau$ and the $G$-action remains continuous with respect to $t$.

(b) There exists a basis $B$ for $t$ such that:
   (i) $B$ is countable;
   (ii) for all $B_1, B_2 \in B$, $B_1 \cap B_2 \in B$;
   (iii) for all $B \in B$, $X \setminus B \in B$;
   (iv) for all $B \in B$ and $u \in N^G$, $B^u \in B$;
   (v) for any $B \in B$ there exists an open subgroup $H < G$ such that $B$ is invariant under the corresponding $H$-action.

A basis satisfying condition (b) is called a nice basis.

It is noticed in [3] that any nice basis also satisfies property (b)(iv) of the definition above for $\Delta$-transforms. It is also clear that any nice basis is invariant in the sense that for every $g \in G$ and $B \in B$ we have $gB \in B$ (see [10]).

In Section 3 we will prove the following theorem.

**Theorem 9** Let $G$ be a closed subgroup of $S_\infty$, $X$ be a Polish $G$-space, $t$ be a nice topology for $X$ and $B$ be its nice basis. Let $x \in X$ and let $C$ be the piece of the canonical partition with respect to $B$ containing $x$ (see [2]). Let $A$ be an admissible set such that $x$ is codable in $A$ with respect to $B$. Then the following are equivalent:

(i) $C = Gx$;
(ii) $C$ can not be partitioned into two invariant Borel sets constructible in $A$.

It is curious that this statement is related to some fact from model theory, which was found by Morozov in [11]. Our proof is based on some arguments from [10] together with the main tools of our paper.
2 Sets arising in Polish group actions

In this section we develop the generalized Scott analysis which was initiated in [6]. We suggest a slightly different approach, more suitable for the main tasks of the paper. We replace the main tool of Hjorth’s work (hereditarily countable structures \(\phi_\alpha(x,V_n)\) corresponding to Scott sentences) by some invariants \(B_\alpha(x,\sigma), x \in X, \sigma \in S^G_{<\infty}\), which are Borel subsets of the space. They may be also used as counterparts of Scott sentences.

Actually these sets already appear in [6], where they are defined in a different way\(^1\). We formulate another, more canonical definition. It seems to be more convenient for many purposes. It enables us to describe Borel complexity of the sets \(B_\alpha(x,\sigma)\) and compare it with Borel complexity of the orbit \(Gx\). Finally they are more suitable for proofs of our main results mentioned in the previous section.

On the one hand this section can be considered as an improvement, completion and systematization of the material scattered in Section 6.1 of [6]. On the other hand it contains a couple of new results (e.g. Propositions 18 and 19) and a natural example, which illustrates the introduced objects.

The section is divided into two subsections. In the first one we define sets \(B_\alpha(x,\sigma)\) and describe the main properties of them. Lemma 14 is the key lemma which we use for the main results of the paper. On the other hand we study \(\alpha\)-sets \(B_\alpha(x,\sigma)\) slightly further in order to present this material in a complete form. Propositions 15, 17 and 19 somehow summarize our study. Proposition 20 (related to some results from [4]) is a straightforward application of our approach.

In the second subsection we define a counterpart of the Scott rank and compare it with the Borel rank of the orbit.

2.1 Borel partitions

Let \(G\) be a closed subgroup of \(S_\infty\) and \(X\) be a Polish \(G\)-space with a countable basis \(\mathcal{A} = \{A_i : i \in \omega\}\). We always assume throughout the paper that every basic open set is invariant with respect to some basic clopen group \(H < G\) (it follows from the continuity of the action that such a basis exists).

By Proposition 2.C.2 of [3] there exists a unique partition of \(X, X = \bigcup\{Y_t : t \in T\}\) into invariant \(G_\delta\) sets \(Y_t\) such that every orbit of \(Y_t\) is dense in \(Y_t\). To construct this partition we define for any \(t \in 2\mathbb{N}\) the set

\[Y_t = (\bigcap\{GA_j : t(j) = 1\}) \cap (\bigcap\{X \setminus GA_j : t(j) = 0\})\]

and take \(T = \{t \in 2\mathbb{N} : Y_t \neq \emptyset\}\).

In this section we generalize this notion and define for every ordinal \(0 < \alpha < \omega_1\) some canonical partition of \(X\) approximating the original orbit partition. In fact we define such partitions not only for the whole group \(G\), but simultaneously for every basic clopen subgroup \(V^G_d\), where \(d\) is a finite subset of \(\omega\). We call the classes of the partition \(\alpha\)-sets and study their properties in detail.

\(^1\)when \(G = S_\infty\) it can be shown, that \(B_\alpha(x, id_n) = \{y : \phi_\alpha(x,V_n) = \phi_\alpha(y,V_n)\}\) for \(\alpha \geq \omega\)
Definition 10 Let $G < S_\omega$ be a closed subgroup and $(X, \tau)$ be a Polish $G$-space with a basis $\{A_l : l \in \omega\}$. For every $x \in X$ and $\sigma \in S^G_\omega$ with $\text{rng}[\sigma] = c$ and $\text{dom}[\sigma] = d$ we define a Borel set $B(x, \sigma)$ by simultaneous induction on the ordinal $\alpha$.

\[
B_1(x, \sigma) = \bigcap \{ V^c_{\sigma} A_l : V^c_{\sigma} x \cap A_l \neq \emptyset \} \cap \bigcap \{ X \setminus V^c_{\sigma} A_l : V^c_{\sigma} x \cap A_l = \emptyset \};
\]
\[
B_{\alpha+1}(x, \sigma) = \bigcap_{\beta \geq \alpha} \left( \bigcup \{ B_\beta(x, \sigma') : \sigma' \in S^G_{<\infty}, \sigma' \supseteq \sigma, \text{dom}[\sigma'] = b \} \right) \cap \bigcap_{\alpha < \lambda} \left( \bigcup \{ B_\alpha(x, \sigma') : \sigma' \in S^G_{<\infty}, \sigma' \supseteq \sigma, \text{rng}[\sigma'] = a \} \right);
\]
\[
B_\lambda(x, \sigma) = \bigcap_{\alpha < \lambda} B_\alpha(x, \sigma), \text{ for } \lambda \text{ limit } .
\]

Although the definition of a 1-set coincides with the definition of a piece of the canonical partition, it is not quite evident that the whole definition can be considered as a generalization of the definition of the canonical partition. This will be clearer when we describe some properties of the sets $B_\alpha(x, \sigma)$. These properties will be applied in the main results of the paper.

Lemma 11 Let $x, y \in X$, $\sigma \in S^G_\omega$, $\text{dom}[\sigma] = d$ and $\text{rng}[\sigma] = c$. Then for any $f \in G$, $\delta \in S^G_{<\infty}$ and ordinals $\alpha, \beta > 0$ the following statements are true.

(a) If $\beta \leq \alpha$, then $B_\beta(x, \sigma) \supseteq B_\alpha(x, \sigma)$;

(b) $B_\alpha(f x, \sigma) = B_\alpha(x, \sigma f)$, where $\sigma f$ denotes the map $\sigma f|_{f^{-1}[d]}$, in particular $B_\alpha(x, \sigma) = B_\alpha(f x, \sigma)$, for every $f \in V^c_\delta$;

(c) $f B_\alpha(x, \sigma) = B_\alpha(f x, \sigma)$, in particular $B_\alpha(x, \sigma) = B_\alpha(x, f \sigma)$, for every $f \in V^c_\sigma$;

(d) $V_c^\sigma x \subseteq B_\alpha(x, \sigma)$ and $B_\alpha(x, \sigma)$ is $V_c^G$-invariant;

(e) $B_{\alpha+1}(x, \sigma) = \bigcap_{\sigma' \geq \sigma} V^c_\sigma B_\alpha(x, \sigma') \cap \bigcap_{\alpha < \lambda} \left( \bigcup_{g \in V^c_\sigma} \{ g B_\alpha(x, \sigma') : \sigma' \in S^G_{<\infty}, \sigma' \supseteq \sigma, \text{rng}[\sigma'] = a \} \right)$;

(f) If $\delta \supseteq \sigma$ then $B_\alpha(x, \delta) \subseteq B_\alpha(x, \sigma)$;

(g) If $y \in B_\alpha(x, \sigma)$ then $B_\alpha(y, id_c) = B_\alpha(x, \sigma)$;

(h) If $\text{rng}[\delta] = c$ then either $B_\alpha(x, \sigma) = B_\alpha(y, \delta)$ or $B_\alpha(x, \sigma) \cap B_\alpha(y, \delta) = \emptyset$.

Proof. Statement of (a) follows directly from the definition.

In the proof of (b) - (h) we shall frequently use the following claim, which can be derived by easy straightforward arguments.

Claim. Under the assumptions of the lemma we have:

1. $\left\{ \{ f \sigma' : \sigma' \in S^G_{<\infty}, \sigma' \supseteq \sigma, \text{rng}[\sigma'] = a \} : a \supseteq \emptyset \right\} = \left\{ \{ \sigma' : \sigma' \in S^G_{<\infty}, \sigma' \supseteq f \sigma, \text{rng}[\sigma'] = b \} : b \supseteq f[\emptyset] \right\}$;

2. $\left\{ \{ f \sigma' : \sigma' \in S^G_{<\infty}, \sigma' \supseteq \sigma, \text{dom}[\sigma'] = a \} : a \supseteq d \right\} = \left\{ \{ \sigma' : \sigma' \in S^G_{<\infty}, \sigma' \supseteq f \sigma, \text{dom}[\sigma'] = b \} : b \supseteq d \right\}$;

3. If $f \in V^c_\sigma$ then $\left\{ \{ f \sigma' : \sigma' \in S^G_{<\infty}, \sigma' \supseteq \sigma, \text{dom}[\sigma'] = a \} : a \supseteq d \right\} = \left\{ \{ \sigma' : \sigma' \in S^G_{<\infty}, \sigma' \supseteq \sigma, \text{dom}[\sigma'] = b \} : b \supseteq d \right\}$.
4. \( \{\sigma' : \sigma' \in S_{G_{\infty}}^G, \sigma' \supseteq \sigma, \text{rng}[\sigma'] = a \} : a \supseteq c \) = \\
= \{\sigma' : \sigma' \in S_{G_{\infty}}^G, \sigma' \supseteq \sigma f, \text{rng}[\sigma'] = b \} : b \supseteq c \}

5. \( \{\sigma' : \sigma' \in S_{G_{\infty}}^G, \sigma' \supseteq \sigma, \text{dom}[\sigma'] = a \} : a \supseteq d \) = \\
= \{\sigma' : \sigma' \in S_{G_{\infty}}^G, \sigma' \supseteq \sigma f, \text{dom}[\sigma'] = b \} : b \supseteq f^{-1}[d] \};

6. If \( f \in V_d^G \) then \( \{\alpha f : \sigma' \in S_{G_{\infty}}^G, \sigma' \supseteq \sigma, \text{dom}[\sigma'] = a \} : a \supseteq d \) = \\
= \{\sigma' : \sigma' \in S_{G_{\infty}}^G, \sigma' \supseteq \sigma, \text{dom}[\sigma'] = b \} : b \supseteq d \};

7. For each \( \sigma' \in S_{G_{\infty}}^G \) such that \( \sigma' \supseteq \sigma \) and \( \text{dom}[\sigma'] = b \) we have \( \{g\sigma' : g \in V_c^G \} = \{\delta \in S_{G_{\infty}}^G : \delta \supseteq \sigma, \text{dom}[\delta] = b \}. \)

Now we return to the proof of the lemma.

(b) We proceed by induction on \( \alpha > 0 \). By the equality \( V_{\alpha f}^G x = V_{\alpha f}^G f x \), the statement of (b) holds for \( \alpha = 1 \). Using the inductive assumption at the successor step we get

\[
B_{\alpha + 1}(f x, \sigma) = \\
\bigcap \bigcup \{B_{\alpha}(x, \sigma f) : \sigma' \in S_{G_{\infty}}^G, \sigma' \supseteq \sigma, \text{dom}[\sigma'] = b \} \cap \\
\bigcap \bigcup \{B_{\alpha}(x, \sigma f) : \sigma' \in S_{G_{\infty}}^G, \sigma' \supseteq \sigma, \text{rng}[\sigma'] = a \}.
\]

Then we apply points 4 and 5 of the claim to get the required equality \( B_{\alpha + 1}(f x, \sigma) = B_{\alpha + 1}(x, \sigma f) \). This completes the successor step. The limit step is obvious.

(c) By an obvious inductive argument we see that \( f B_{\alpha}(x, \sigma) = B_{\alpha}(f x, f \sigma f^{-1}) \). By (b) we obtain \( B_{\alpha}(f x, f \sigma f^{-1}) = B_{\alpha}(x, f \sigma) \). These equalities obviously imply the statement.

(d) To prove the first part we use induction on \( \alpha \). The inclusion trivially holds for \( \alpha = 1 \). The limit step is immediate. Then we can easily settle the successor step, since for every \( b \supseteq d \) and \( a \supseteq c \) we have

\[
V_{\sigma}^G = \bigcup \{V_{\sigma}^G : \sigma' \supseteq \sigma, \text{dom}[\sigma'] = b \} = \bigcup \{V_{\sigma}^G : \sigma' \supseteq \sigma, \text{rng}[\sigma'] = a \}.
\]

The second part of (d) follows directly from (c).

(e) By induction, using point (c) of the lemma and points 1, 3 of the claim.

(f) We proceed inductively. First we shall consider case \( \alpha = 1 \). If \( A_l \) is a basic open set such that \( A_l \cap V_{\sigma}^G x \neq \emptyset \), then \( V_{\delta}^G x \subseteq V_{\sigma}^G x \subseteq V_{c}^G A_l \). Since \( V_{\sigma}^G A_l \) is open and the action is continuous, there is a basic open set \( A_k \) such that \( V_{\delta}^G x \cap A_k \neq \emptyset \) and \( V_{\text{rng}[\delta]}^G A_k \subseteq V_{c}^G A_l \). This in particular implies that \( B_1(x, \delta) \subseteq V_{c}^G A_l \).

On the other hand suppose that \( A_l \) is a basic open set such that \( A_l \cap V_{\sigma}^G x = \emptyset \). Since \( V_{\sigma}^G x \) is \( V_{c}^G \)-invariant, we get \( V_{c}^G A_l \cap V_{\sigma}^G x = \emptyset \). We present \( V_{c}^G A_l \) as the union
\[ \bigcup \{ V_{g \in \mathcal{A}_1} \} \] and note that for every \( g \in \mathcal{A}_1 \), we have \( V_{g} \cap \mathcal{A}_1 = \emptyset \). Thus we have \( \bigcap \{ X \setminus \bigcup_{g \in \mathcal{A}_k} A_k : V_{g} \cap \mathcal{A}_k = \emptyset \} \subseteq X \setminus \mathcal{A}_1 \) and then \( B_1(x, \delta) \subseteq X \setminus \mathcal{A}_1 \).

This yields \( B_1(x, \delta) \subseteq B_1(x, \sigma) \).

For the successor step assume that the inclusion \( B_0(x, \sigma') \subseteq B_0(x, \sigma) \) holds whenever \( \sigma' \supseteq \sigma \). For any \( b \supseteq d \) we put \( \hat{b} = b \cup \text{dom}[\delta] \). Using the inductive assumption we get

\[
\bigcup \{ B_0(x, \delta') : \delta' \supseteq \delta, \text{dom}[\delta'] = \hat{b} \} \subseteq \bigcup \{ B_0(x, \sigma') : \sigma' \supseteq \sigma, \text{dom}[\sigma'] = \hat{b} \} \subseteq \bigcup \{ B_0(x, \sigma') : \sigma' \supseteq \sigma, \text{dom}[\sigma'] = b \}.
\]

Similarly, if \( a \supseteq c \) and \( \hat{a} = a \cup \text{rng}[\delta] \) then we have

\[
\bigcup \{ B_0(x, \delta') : \delta' \supseteq \delta, \text{rng}[\delta'] = \hat{a} \} \subseteq \bigcup \{ B_0(x, \sigma') : \sigma' \supseteq \sigma, \text{rng}[\sigma'] = a \}.
\]

Hence we conclude that \( B_{\alpha+1}(x, \delta) \subseteq B_{\alpha+1}(x, \sigma) \).

The limit step is immediate.

(g) We proceed by induction. For \( \alpha = 1 \) the equality follows directly from the definition. The limit step is immediate. For the successor step, assume that the equality \( B_\alpha(x, \sigma') = B_\alpha(z, \text{id}_{\alpha}) \) holds whenever \( z \in B_\alpha(x, \sigma') \) and \( \text{rng}[\sigma'] = a \). Now take an arbitrary \( y \in B_{\alpha+1}(x, \sigma) \). By (e) we get

\[
B_{\alpha+1}(x, \sigma) = \bigcap \{ V_{c} B_\alpha(x, \sigma') : \sigma' \supseteq \sigma \} \cap \bigcap_{a \supseteq c \leq y} \{ g B_\alpha(x, \sigma') : \sigma' \supseteq \sigma, \text{rng}[\sigma'] = a \}.
\]

We see that for every \( \sigma' \in S_{\infty} \) with \( \sigma' \supseteq \sigma \) there is some \( f' \in V_{\infty} \) such that \( f' y \in B_\alpha(x, \sigma') \) and thus \( B_\alpha(x, \sigma') = B_\alpha(y, \text{id}_{\text{rng}[\sigma']} f') \) (apply the inductive assumption and (c)). Since \( f' \in V_{\infty} \) and \( \sigma' \supseteq \sigma \) then \( \text{id}_{\text{rng}[\sigma']} f' \supseteq \text{id}_c \) and \( \text{rng}[\text{id}_{\text{rng}[\sigma']} f'] = \text{rng}[\sigma'] \).

Hence the following is true

\[
(\forall \sigma' \supseteq \sigma)(\exists \delta' \supseteq \text{id}_c)(B_\alpha(x, \sigma') = B(y, \delta') \land \text{rng}[\sigma'] = \text{rng}[\delta']).
\]

On the other hand take an arbitrary \( \delta' \supseteq \text{id}_c \). Put \( a = \text{rng}[\delta'] \) and take any \( g \in V_{\infty} \) such that \( g \supseteq \delta' \). Then by (f) there is some \( \sigma' \supseteq \sigma \) such that \( \text{rng}[\sigma'] = a \) and \( gy \in B_\alpha(x, \sigma') \). By the inductive assumption, the latter implies \( B_\alpha(x, \sigma') = B_\alpha(g, \text{id}_a) \).

Then by (c) we get \( B_\alpha(x, \sigma') = B_\alpha(y, \text{id}_{\text{rng}[\sigma']} g) = B_\alpha(y, \delta') \).

We have proved that if \( y \in B_{\alpha+1}(x, \sigma) \) then the equality

\[
\{ B_\alpha(x, \sigma') : \sigma' \supseteq \sigma, \text{rng}[\sigma'] = a \} = \{ B_\alpha(y, \delta') : \delta' \supseteq \text{id}_c, \text{rng}[\delta'] = a \}
\]
is true for every \( a \supseteq c \). This implies

\[
\bigcap \bigcup_{a \geq c}\{B_{a}(x, \sigma') : \sigma' \in S_{<\infty}^{G}, \sigma' \supseteq \sigma, \text{rng}[\sigma'] = a\} = \\
\bigcap \bigcup_{a \geq c}\{B_{a}(y, \delta') : \delta' \in S_{<\infty}^{G}, \delta' \supseteq \text{id}_{c}, \text{rng}[\sigma'] = a\}
\]

and

\[
\bigcap\{V_{c}^{G}B_{a}(x, \sigma') : \sigma' \in S_{<\infty}^{G}, \sigma' \supseteq \sigma\} = \bigcap\{V_{c}^{G}B_{a}(y, \delta') : \delta' \in S_{<\infty}^{G}, \delta' \supseteq \text{id}_{c}\}
\]

which by (e) gives the required equality.

(h) follows directly from (g). \( \square \)

We are now ready to prove that the partition of \( X \) into \( \alpha \)-sets can be defined by the same scheme as the canonical partition (thus can be considered as a generalization of the latter).

**Proposition 12** Let \( A \) be a basis for \( X, x \in X, \sigma \in S_{<\infty}^{G}, \text{rng}[\sigma] = c \) and \( \alpha > 0 \) be an ordinal.

(a) Let \( B_{\alpha} = \{B_{a}(y, \delta) : y \in X, \delta \in S_{<\infty}^{G}\} \). Then we have

\[
B_{\alpha+1}(x, \sigma) = \bigcap\{V_{c}^{G}B : B \in B_{\alpha}, \ V_{\sigma}^{G}x \cap B \neq \emptyset\} \cap \\
\bigcap\{X \setminus V_{c}^{G}B : B \in B_{\alpha}, \ V_{\sigma}^{G}x \cap B = \emptyset\}.
\]

(b) Let \( B_{<\alpha} = \{B_{\gamma}(y, \delta) : y \in X, \delta \in S_{<\infty}^{G}, \gamma < \alpha\} \cup A \). Then we have

\[
B_{\alpha}(x, \sigma) = \bigcap\{V_{c}^{G}B : B \in B_{<\alpha}, \ V_{\sigma}^{G}x \cap B \neq \emptyset\} \cap \\
\bigcap\{X \setminus V_{c}^{G}B : B \in B_{<\alpha}, \ V_{\sigma}^{G}x \cap B = \emptyset\}.
\]

**Proof.** (a) The inclusion \( \supseteq \) easily follows from the definition and the lemma above. We have to work a little more with its converse. Let \( B \in B_{\alpha} \) be such that \( V_{\sigma}^{G}x \cap B \neq \emptyset \). Then, by the lemma above, there is some \( \sigma' \in S_{<\infty}^{G} \) such that \( \sigma' \supseteq \sigma \) and \( B_{a}(x, \sigma') \subseteq B \). Hence we have \( V_{c}^{G}B_{a}(x, \sigma') \subseteq V_{c}^{G}B \), which yields \( B_{a+1}(x, \sigma) \subseteq V_{c}^{G}B \).

On the other hand let \( B \in B_{\alpha} \) be such that \( V_{\sigma}^{G}x \cap B = \emptyset \). Then, by the lemma above, there is some \( a \supseteq c \) such that \( B_{a}(x, \sigma') \cap B = \emptyset \), for every \( \sigma' \in S_{<\infty}^{G} \) with \( \sigma' \supseteq \sigma \) and \( \text{rng}[\sigma'] = a \). Therefore

\[
\bigcap_{a \geq c}\{B_{a}(x, \sigma') : \sigma' \in S_{<\infty}^{G}, \sigma' \supseteq \sigma, \text{rng}[\sigma'] = a\} \cap V_{c}^{G}B = \emptyset,
\]

which yields \( B_{a+1}(x, \sigma) \subseteq X \setminus V_{c}^{G}B \).

(b) follows from (a) and the properties of \( \alpha \)-sets collected in Lemma 11. \( \square \)

Proposition 12 (b) yields the following statement.
Proposition 13 Let \( x, y \in X \), \( \alpha > 1 \) be an ordinal and \( c \subseteq \omega \) be a finite set. Then for every \( \sigma, \delta \in S^G_\infty \) with common range \( c \) the following are equivalent:

(i) \( B_\alpha(x, \sigma) = B_\alpha(y, \delta) \);
(ii) For every finite \( a \supseteq c \) and every \( \zeta < \alpha \) we have
\[
\{ B_\zeta(x, \sigma') : \sigma' \supseteq \sigma, \text{rng}[\sigma'] = a \} = \{ B_\zeta(y, \delta') : \delta' \supseteq \delta, \text{rng}[\delta'] = a \}.
\]
(iii) For every natural \( n \geq c \) and every \( \zeta < \alpha \) we have
\[
\{ B_\zeta(x, \sigma') : \sigma' \supseteq \sigma, \text{rng}[\sigma'] = n \} = \{ B_\zeta(y, \delta') : \delta' \supseteq \delta, \text{rng}[\delta'] = n \}.
\]

Proof. By Proposition 12, (i) is equivalent to the equality
\[
\{ B \in B_{<\alpha} : B \cap V^G_\sigma x \neq \emptyset \} = \{ B \in B_{<\alpha} : B \cap V^G_\delta y \neq \emptyset \}.
\]

(i) \( \Rightarrow \) (ii) Fix arbitrary \( \zeta < \alpha \) and \( a \supseteq c \). Then take any \( \sigma' \supseteq \sigma \) with \( \text{rng}[\sigma'] = a \). Since \( B_\zeta(x, \sigma') \cap V^G_\sigma x \neq \emptyset \), we see that \( B_\zeta(x, \sigma') \cap V^G_\delta y \neq \emptyset \). Hence by Lemma 11 (g), there is some \( \delta' \supseteq \delta \) with \( \text{rng}[\delta'] = \alpha \) such that \( B_\zeta(x, \sigma') = B_\zeta(y, \delta') \). Therefore
\[
\{ B_\zeta(x, \sigma') : \sigma' \supseteq \sigma, \text{rng}[\sigma'] = a \} \subseteq \{ B_\zeta(y, \delta') : \delta' \supseteq \delta, \text{rng}[\delta'] = \alpha \}.
\]
In the same way we derive the converse inclusion.

(ii) \( \Rightarrow \) (i) Take any \( B \in B_{<\alpha} \) such that \( B \cap V^G_\sigma x \neq \emptyset \). There are \( \zeta < \alpha \) and \( \sigma' \subseteq \sigma \) such that \( B_\zeta(x, \sigma') \subseteq B \). Since we can find \( \delta' \supseteq \delta \) (with \( \text{rng}[\delta'] = \text{rng}[\delta] \)) such that \( B_\zeta(x, \sigma') = B_\zeta(y, \delta') \), we see that \( B \cap V^G_\delta y \neq \emptyset \). This proves \( \{ B \in B_{<\alpha} : B \cap V^G_\sigma x \neq \emptyset \} \subseteq \{ B \in B_{<\alpha} : B \cap V^G_\delta y \neq \emptyset \} \). Similarly we obtain the converse inclusion.

(i) \( \Rightarrow \) (iii) is obvious.

To prove (iii) \( \Rightarrow \) (ii) suppose that (ii) does not hold. Then there are some finite set \( a \supseteq c \) and ordinal \( \zeta < \alpha \) such that
\[
\{ B_\zeta(x, \sigma') : \sigma' \in S^G_\infty, \sigma' \supseteq \sigma, \text{rng}[\sigma'] = a \} \neq \{ B_\zeta(y, \delta') : \delta' \in S^G_\infty, \delta' \supseteq \delta, \text{rng}[\delta'] = a \}.
\]
Take any natural \( n \geq a \). By Lemma 11 (f), (h), we have
\[
\{ B_\zeta(x, \sigma') : \sigma' \supseteq \sigma, \text{rng}[\sigma'] = n \} \neq \{ B_\zeta(y, \delta') : \delta' \supseteq \delta, \text{rng}[\delta'] = n \},
\]
hence (iii) does not hold. \( \Box \)

The lemma below shall play the key role in the proof of the main result of the paper. It states that \( \alpha \)-sets are in some sense minimal with respect to \( \alpha \).

Lemma 14 Let \( x \in X \), \( \sigma \in S^G_\infty \), \( c = \text{rng}[\sigma] \) and \( \alpha > 0 \) be an ordinal. Then for any \( V^G_c \)-invariant \( A \in \Sigma^0_\alpha \cap \Pi^0_\alpha \) we have \( V^G_\sigma x \subseteq A \) iff \( B_\alpha(x, \sigma) \subseteq A \).

Proof. To prove \( (\Rightarrow) \) we proceed inductively.

Consider case \( \alpha = 1 \). If \( U \) is a \( V^G_c \)-invariant open set containing \( V^G_\sigma x \), then there is a basic open set \( A_{l_0} \subseteq U \) intersecting \( V^G_\sigma x \). Then \( V^G_\sigma A_{l_0} \subseteq U \) and so
\[
U \supseteq \bigcap \{ V^G_{l_1} A_l : A_l \cap V^G_\sigma x \neq \emptyset \} \supseteq B_1(x, \sigma).
\]

\( \alpha = 1 \).
If $F$ is an $V_e^G$-invariant closed set containing $V_o^G x$, then

$$F \supseteq \bigcap \{ X \setminus V_e^G A_i : F \cap A_i = \emptyset \} \supseteq \bigcap \{ X \setminus V_e^G A_i : V_o^G x \cap A_i = \emptyset \} \supseteq B_1(x, \sigma).$$

The rest of the proof is based on the following statements.

**Claim** Let $\alpha$ be an ordinal, $c \in [\omega]^\omega$ and $A \subseteq X$ be a $V_e^G$-invariant set.

1. If $A \in \Sigma^0_\alpha(X)$ then $A$ can be presented as a union $A = \bigcup D_i$ such that $\{D_i : i < \omega\} \subseteq \bigcup \{\Pi^0_{\xi}(X) : \xi < \alpha\}$ and for every $i < \omega$ there is $a_i \supseteq c$ such that $D_i$ is $V_{a_i}^G$-invariant. Moreover, if $\alpha$ is limit then each $D_i$, $i < \omega$, can be taken $V_{c_i}^G$-invariant.

2. If $A \in \Pi^0_\alpha(X)$ is a $V_e^G$-invariant set then $A$ can be presented as an intersection $A = \bigcap D_i$ such that $\{D_i : i < \omega\} \subseteq \bigcup \{\Sigma^0_{\xi}(X) : \xi < \alpha\}$ and for every $i < \omega$ there is $a_i \supseteq c$ such that $D_i$ is $V_{a_i}^G$-invariant. Moreover, if $\alpha$ is limit then each $D_i$, $i < \omega$, can be taken $V_{c_i}^G$-invariant.

**Proof of Claim.** (1) There is a countable family $\{A_i : i \in \omega\} \subseteq \bigcup_{\xi < \alpha} \Pi^0_{\xi}(X)$ such that $A = \bigcup_i A_i$. Since $A$ is $V_e^G$-invariant we have

$$A = A^{V_e^G} = \bigcup \{ \bigcup_{i} (A_i^{V_e^G} : W \subseteq V_e^G \text{ is basic, open}) \} =$$

$$= \bigcup_i \bigcup_{a \supseteq c} \left( \bigcup_{i} A_i^{V_e^G} : (\delta \in S^{G}_{<\infty}) \land (\delta \supseteq id_c) \land (\text{dom}[\delta] = a) \right).$$

It follows from the properties of Vaught transforms that if $A_i \in \Pi^0_{\xi}$ and $\text{dom}[\delta] = a$ then $A_i^{V_e^G}$ is a $V_{c_i}^G$-invariant $\Pi^0_{\xi}$-set. It completes the first part.

Now, it is clear that if $A_i \in \Pi^0_{\xi}$ then the set

$$A_i^{V_e^G} = \bigcup \{ A_i^{V_e^G} : (\delta \in S^{G}_{<\infty}) \land (\delta \supseteq id_c) \land (\text{dom}[\delta] = a) \}$$

is a $V_{c_i}^G$-invariant $\Sigma_{\xi+1}$-set. Thus it is also a $V_{c_i}^G$-invariant $\Pi_{\xi+2}$-set. Then $A$ is a countable union of $V_{c_i}^G$-invariant elements of the union $\bigcup_{\xi < \alpha} \Pi^0_{\xi+2}(X)$, which proves the additional statement for limit $\alpha$.

(2) There is a countable family $\{A_i : i \in \omega\} \subseteq \bigcup_{\xi < \alpha} \Sigma^0_{\xi}(X)$ such that $A = \bigcap_i A_i$. Since $A$ is $V_e^G$-invariant, we have

$$A = A^{V_e^G} = \bigcap_i \bigcap_{a \supseteq c} \left( \bigcup_{i} A_i^{V_e^G} : (\delta \in S^{G}_{<\infty}) \land (\delta \supseteq id_c) \land (\text{dom}[\delta] = a) \right).$$
Applying standard properties of Vaught transform again, we see that if \( A_i \in \Sigma^0_{\xi} \) and \( \text{dom}[\delta] = a \). Hence \( A_i^{V^G} \) is a \( V_a^G \)-invariant \( \Sigma^0_\xi \)-set. It completes the first part.

Now, if \( A_i \in \Sigma^0_\xi \) then the set

\[
A_i^{V^G} = \bigcap_{a \geq c} \bigcap_{i} \{ A_i^{V^G} : (\delta \in S^G_{<\omega}) \land (\delta \supseteq \text{id}_c) \land (\text{dom}[\delta] = a) \}
\]

is a \( V^G_c \)-invariant \( \Pi_{\xi+1} \)-set, thus it is also a \( V^G_c \)-invariant \( \Sigma_{\xi+2} \)-set. Then \( A \) is a countable intersection of \( V^G_c \)-invariant elements of the union \( \bigcup_{\xi<\alpha} \Sigma^0_{\xi+2}(X) \), which proves the additional statement for limit \( \alpha \).

We continue the proof of Lemma.

To go through the successor step take an arbitrary \( \alpha \) and assume that the statement holds for every \( a \in [\omega]^{<\omega} \), \( \sigma' \in S^G_{<\omega} \) with \( a = \text{rng}[\sigma'] \) and every \( V^G_a \)-invariant set \( D \in \Sigma^0_\alpha \cup \Pi^0_\alpha \) containing \( V^G_{\sigma'} x \). Let \( A \in \Sigma^0_{\alpha+1}(X) \cup \Pi^0_{\alpha+1}(X) \) be an arbitrary \( V^G_c \)-invariant set containing \( V^G_{\sigma'} x \). We shall consider two cases.

1° \( A \in \Sigma^0_{\alpha+1} \).
By Claim, \( A \) can be presented as a union \( A = \bigcup_i D_i \), where for every \( i < \omega \) there is \( a_i \geq c \) such that \( D_i \) is a \( V^G_{a_i} \)-invariant \( \Pi^0_\alpha \)-set.

Fix an arbitrary \( g \in V^G_a \). Since \( V^G_a x \subseteq A \), there are \( i \in \omega \) and \( a_i \geq c \) such that \( g x \in D_i \) and \( D_i \) is \( V^G_{a_i} \)-invariant. Put \( \sigma' = \text{id}_a, g \). Then we have \( \sigma' \supseteq \sigma \), \( \text{rng}[\sigma'] = a_i \) and \( V^G_{\sigma'} x \subseteq D_i \). Using the inductive assumption we conclude that \( B_{\alpha}(x, \sigma') \subseteq D_i \subseteq A \).

Since \( A \) is \( V^G_c \)-invariant we obtain \( B_{\alpha+1}(x, \sigma') \subseteq V^G_c B_{\alpha}(x, \sigma') \subseteq A \).

2° \( A \in \Pi^0_{\alpha+1} \). By Claim, \( A \) can be presented as a union \( A = \bigcap_i D_i \), where for every \( i < \omega \) there is \( a_i \geq c \) such that \( D_i \) is a \( V^G_{a_i} \)-invariant \( \Sigma^0_\alpha \)-set.

Fix arbitrary \( i \in \omega \) and \( a_i \geq c \) such that \( D_i \) is \( V^G_{a_i} \)-invariant. We have

\[
\bigcup \{ V^G_{\sigma'} x : (\sigma' \supseteq \sigma) \land (\text{rng}[\sigma'] = a_i) \} = V^G_{\sigma'} x \subseteq D_i.
\]

Thus for every \( \sigma' \in S^G_{<\omega} \) with \( \sigma' \supseteq \sigma \) and \( \text{rng}[\sigma'] = a_i \) we have \( V^G_{\sigma'} x \subseteq D_i \). Since \( D_i \) is a \( V^G_{a_i} \)-invariant \( \Sigma^0_\alpha \)-set, by the inductive assumption we conclude that \( B_{\alpha}(x, \sigma') \subseteq D_i \).

Therefore

\[
\bigcup \{ B_{\alpha}(x, \sigma') : (\sigma' \in S^G_{<\omega}) \land (\sigma' \supseteq \sigma) \land (\text{rng}[\sigma'] = a) \} \subseteq D_i.
\]

By Definition 10 this completes the successor step.

By Claim we can also easily go through the limit step.

The backward direction is just Lemma 11 (d). □

The next result provides another necessary and sufficient condition for the equality of \( \alpha \)-sets. It improves the result from [6], where some counterpart of (i) \( \Rightarrow \) (ii) is proved.
Proposition 15 Let $x, y \in X$, $\sigma, \delta \in S^G_{<\infty}$ and $\text{rng}[\sigma] = \text{rng}[\delta] = c$. Then for every ordinal $\alpha > 0$ the following conditions are equivalent.

(i) $B_{\alpha}(x, \sigma) = B_{\alpha}(y, \delta)$;

(ii) For every $V^G_\sigma$-invariant set $A \in \Sigma^0_\alpha(X) \cup \Pi^0_\alpha(X)$ we have

$$V^G_\sigma x \subseteq A \iff V^G_\delta y \subseteq A.$$ 

Moreover for every ordinal $\alpha > 0$ we have $B_{\alpha}(x, \sigma) = \bigcap\{A \in \Sigma^0_\alpha(X) \cup \Pi^0_\alpha(X) : A \text{ is } V^G_\sigma \text{-invariant}, V^G_\sigma x \subseteq A\}$.

In particular $B_{\alpha}(x, \sigma)$ is a $\Pi^0_{\alpha+1}$-set for every successor ordinal $\alpha$, and a $\Pi^0_\alpha$-set for every limit ordinal $\alpha$.

Proof. (i) $\Rightarrow$ (ii) follows from Lemma 11(d) and Lemma 14.

To prove (ii) $\Rightarrow$ (i) we use induction on $\alpha > 0$. The case $\alpha = 1$ and the limit step are easy. To go through the successor step, take an arbitrary $\alpha$ and assume that for every $\sigma', \delta'$ with $\text{rng}[\sigma'] = \text{rng}[\delta'] = a$ if $B_{\alpha}(x, \sigma') \neq B_{\alpha}(y, \delta')$ then we can separate $V^G_\sigma x$ from $V^G_\delta y$ by some $V^G_a$-invariant set $A \in \Sigma^0_\alpha \cup \Pi^0_\alpha$. Then suppose that $B_{\alpha+1}(x, \sigma) \neq B_{\alpha+1}(y, \delta)$. By Proposition 13 there is some $a \supseteq c$ such that one of the following cases holds

1° For some $\sigma' \supseteq \sigma$ with $\text{rng}[\sigma'] = a$ and every $\delta' \supseteq \delta$ with $\text{rng}[\delta'] = a$ we have $B_{\alpha}(x, \sigma') \neq B_{\alpha}(y, \delta')$;

2° For some $\delta' \supseteq \delta$ with $\text{rng}[\delta'] = a$ and every $\sigma' \supseteq \sigma$ with $\text{rng}[\sigma'] = a$ we have $B_{\alpha}(x, \sigma') \neq B_{\alpha}(y, \delta')$.

Since the cases are symmetric we consider only the first one. By the inductive assumption, for every $\delta' \supseteq \delta$ with $\text{rng}[\delta'] = a$ there is some $V^G_a$-invariant set $A_{\theta'} \in \Sigma^0_\alpha(X) \cup \Pi^0_\alpha(X)$ such that $V^G_{\sigma'} x \subseteq A_{\theta'}$ while $V^G_{\delta'} y$ is disjoint from $A_{\theta'}$. Then for every $\sigma' \supseteq \sigma$ with $\text{rng}[\sigma'] = a$ we have

$$V^G_{\sigma'} x \subseteq \bigcap\{A_{\theta'} : (\delta' \in S^G_{<\infty}) \land (\delta' \supseteq \delta) \land (\text{rng}[\delta'] = a)\}$$

while the set $V^G_{\delta'} y = \bigcup\{V^G_{\delta'} y : \delta' \supseteq \delta \land \text{rng}[\delta'] = a\}$ is disjoint from $\bigcap\{A_{\theta'} : \delta' \supseteq \delta \land \text{rng}[\delta'] = a\}$.

Put $A = \bigcap\{A_{\theta'}^* : \delta' \supseteq \delta \land \text{rng}[\delta'] = a\}$. Then $A \in \Pi^0_{\alpha+1}(X)$, $V^G_{\sigma'} x = (V^G_{\sigma'} x)^* V^G_{\sigma'} \subseteq A$ and $V^G_{\delta'} y$ is disjoint from $A$.

The second part of the statement is a direct consequence of the previous one and Lemma 14. □

As an immediate consequence of this proposition and Lemma 11(h) we obtain the following statement.

Corollary 16 Let $\sigma \in S^G_{<\infty}$ and $\text{rng}[\sigma] = c$.

(a) For every successor ordinal $\alpha$ the family $\{B_{\alpha}(x, \sigma) : x \in X\}$ is a partition of $X$ into $V^G_c$-invariant $\Pi^0_{\alpha+1}$-sets.

(b) For every limit ordinal $\alpha$ the family $\{B_{\alpha}(x, \sigma) : x \in X\}$ is a partition of $X$ into $V^G_c$-invariant $\Pi^0_{\alpha}$-sets.
Every piece of the canonical partition as a $G_δ$-subset of $X$, is a Polish space with the topology inherited from the original Polish topology on $X$. We generalize this fact and show that every $α$-set is a Polish space with respect to some finer topology generated by ‘earlier’ $β$-sets. From now on we shall use the following notation for every ordinal $β$:

$$B_x^0 = A$$
$$B_x^β = \{B_β(x,σ) : σ ∈ S_δ^∞ \} \quad \text{for} \quad β > 0$$
$$B_x^{≤β} = \bigcup\{B_γ : γ < β\}.$$

**Proposition 17** Let $A$ be a countable basis of $X$, $x ∈ X$ and $0 < α < ω_1$ be an ordinal. The set $B_α(x,∅)$ with the (relative) topology generated by the family $B_x^{<α}$ as basic open sets is a Polish $G_δ$-space.

**Proof.** As we have already mentioned, $B_1(x,∅)$ is a $G_δ$ subset of $X$, thus is a Polish space with respect to the (relative) topology generated by $A$. Therefore we will deal below only with $α > 1$. We shall use the following result by Sami (see [13], Lemma 4.2).

Let $⟨X,t⟩$ be a topological space and $1 ≤ ζ < ω_1$. Let $F$ be a Borel family of rank $ζ$, i.e. a family of subsets of $X$ which can be decomposed into subfamilies of two types $F = \bigcup\{P_ξ : 1 ≤ ξ < ζ\} ∪ \bigcup\{S_ξ : 1 ≤ ξ < ζ\}$ satisfying the following conditions: 1. $S_1$ consists of open sets, 2. $P_ξ = \{X \setminus A : A ∈ S_ξ\}$, for $1 ≤ ξ < ζ$, 3. every element of $S_ξ$ is a union of a countable subfamily of $\bigcup\{P_η : 1 ≤ η < ξ\}$, for $1 ≤ ξ < ζ$.

If $X$ is a Polish space then the topology generated by a family of intersec-
tions of finite subsets of the union $t ∪ F$ is also Polish.

Consider the family

$$\hat{B}_x^{≤α} = \begin{cases} 
\{V_cB, X \setminus V_cB : B ∈ B_x^{<β}, c ∈ [ω]^{<ω}\} ∪ \{B, X \setminus B : B ∈ B_β^β\} & \text{if} \quad α = β + 1 > 1 \\
\{V_cB, X \setminus V_cB : B ∈ B_x^{<α}, c ∈ [ω]^{<ω}\} & \text{if} \quad α \text{ is a limit ordinal}.
\end{cases}$$

Our first task is to show that $\hat{B}_x^{<α}$ is a Borel family of some countable rank. To prove this we need some preliminary work.

We define for every $1 < ξ < ω_1$ the sets $S_ξ$ and $P_ξ$. First we put:

$$S_1 = \{V_cA : A ∈ A, c ∈ [ω]^{<ω}\}$$
$$P_1 = \{X \setminus D : D ∈ S_1\}$$

$$S_2 = P_1$$
$$P_2 = \{X \setminus D : D ∈ S_2\}$$

$$S_3 = \bigcup\{V_cA : A ∩ V_σ^Gx = ∅\} ∪ \bigcup\{X \setminus V_cA : A ∩ V_σ^G ≠ ∅\} :$$
$$c ∈ [ω]^{<ω}, σ ∈ S_δ^∞, rng[σ] = c$$
$$P_3 = \{X \setminus D : D ∈ S_3\}$$
Observe that $\bigcup_{i=1}^{3}(S_i \cup P_i)$ is a Borel family of rank 4.

We proceed similarly at each successor stage. Every successor ordinal has one of the following form: $\xi + 3n + 1$, $\xi + 3n + 2$ or $\xi + 3n + 3$, where $n$ is a natural number and $\xi = 0$ or $\xi$ is a limit ordinal. We define

$$
S_{\xi+3n+1} = \{V_c B : B \in \mathcal{B}_{\xi+n}^c\}
$$

$$
P_{\xi+3n+1} = \{X \setminus V_c B : B \in \mathcal{B}_{\xi+n}^c\}
$$

$$
S_{\xi+3n+2} = P_{\xi+3n+1}
$$

$$
P_{\xi+3n+2} = S_{\xi+3n+1}
$$

$$
S_{\xi+3n+3} = \{X \setminus B : B \in \mathcal{B}_{\xi+n+1}^c\}
$$

$$
P_{\xi+3n+3} = \mathcal{B}_{\xi+n+1}^c
$$

Finally, for every limit $\xi < \omega_1$ we put:

$$
S_\xi = \{X \setminus B : B \in \mathcal{B}_\xi^c\}
$$

$$
P_\xi = \mathcal{B}_\xi^c.
$$

We claim that for every $1 < \zeta < \omega_1$ the family $\bigcup_{\xi < \zeta} (S_\xi \cup P_\xi)$ is a Borel family of rank $\zeta$. It is clear that such a family satisfies conditions $1$-$2$. We have to check that it also satisfies condition $3$. We apply an inductive argument. It is obvious for $\zeta = 2, 3, 4$. The case of a limit $\zeta$ is immediate either. For the successor step take an arbitrary $\zeta > 1$ and suppose that the family $\bigcup_{0 < \xi < \zeta} (S_\xi \cup P_\xi)$ satisfies condition $3$. We write

$$
\bigcup_{0 < \xi < \zeta} (S_\xi \cup P_\xi) = \bigcup_{0 < \xi < \zeta} (S_\xi \cup P_\xi) \cup S_\zeta \cup P_\zeta.
$$

Consider two cases.

1° $\zeta$ is limit. By Definition 10 we have

$$
P_\zeta = \mathcal{B}_\zeta^c = \left\{ \bigcap_{\xi < \zeta} B_\xi(x, \sigma) : \sigma \in \mathcal{S}_\infty^G \right\}
$$

$$
S_\zeta = \{X \setminus B : B \in \mathcal{B}_\zeta^c\} = \left\{ \bigcup_{\xi < \zeta} (X \setminus B_\xi(x, \sigma)) : \sigma \in \mathcal{S}_\infty^G \right\}.
$$

By the definition of the sets $S_\xi, P_\xi$ and the assumption that $\zeta$ is limit we see that

$$
\mathcal{B}_\infty^c \subseteq \bigcup_{\xi < \zeta} P_\xi \subseteq \bigcup_{\xi < \zeta} S_\xi.
$$

Hence $\{X \setminus B : B \in \mathcal{B}_\infty^c\} \subseteq \{X \setminus B : B \in \bigcup_{\xi < \zeta} S_\xi\} \subseteq \bigcup_{\xi < \zeta} P_\xi$. Therefore every element of $S_\zeta$ is a countable union of elements of the set $\bigcup_{\xi < \zeta} P_\xi$ which completes Case 1°.

2° $\zeta$ is a successor ordinal.

There are unique ordinals $\gamma$ and $n$ such that $n$ is a natural number, $\gamma$ equals 0 or is a limit ordinal and $\zeta$ has one of the following form: $\gamma + 3n + 1$, $\gamma + 3n + 2$ or $\gamma + 3n + 3$. If $\zeta$ takes one of the first two forms, then we are done directly from the definition.
If $\zeta = \gamma + 3n + 3$ then by Proposition 12 we have

$$P_\zeta = B^x_{\gamma+n+1} = \left\{ \cap \{ V_\zeta B : B \in B^x_{\gamma+n}, B \cap V^G_\sigma x \neq \emptyset \} \right\} \cap \cap \{ X \setminus V_\zeta B : B \in B^x_{\gamma+n}, B \cap V^G_\sigma = \emptyset \} : c \in [\omega]^{<\omega}, \sigma \in S^G_{<\infty}, \text{rng}[\sigma] = c \}$$

and

$$S_\zeta = \{ X \setminus B : B \in B^x_{\gamma+n+1} \} = \left\{ \cup \{ V_\zeta B : B \in B^x_{\gamma+n}, B \cap V^G_\sigma = \emptyset \} \cup \cup \{ X \setminus V_\zeta B : B \in B^x_{\gamma+n}, B \cap V^G_\sigma \neq \emptyset \} : c \in [\omega]^{<\omega}, \sigma \in S^G_{<\infty}, \text{rng}[\sigma] = c \} \right\}.$$

Since $\{ V_\zeta B : B \in B^x_{\gamma+n+1} \} = P_{\gamma+3n+2}$ and $\{ X \setminus V_\zeta B : B \in B^x_{\gamma+n+1} \} = P_{\gamma+3n+1}$ we conclude that $S_\zeta$ consists of countable unions of elements from $P_{\zeta+3n+1} \cup P_{\zeta+3n+2}$. This completes Case 2\(^{o}\).

Now let $\gamma$ and $k$ be the unique ordinals such that $\alpha = \gamma + k$, $k$ is a natural number and $\gamma$ equals 0 or is a limit ordinal. Define

$$\hat{\alpha} = \begin{cases} 
\gamma + 3(k - 1) + 1 & \text{if } k > 0 \\
\alpha & \text{if } k = 0.
\end{cases}$$

We see that $\hat{B}^x_{\xi,\alpha} = \cup\{ P_{\xi} : 1 \leq \xi < \hat{\alpha} \} \cup \{ S_{\xi} : 1 \leq \xi < \hat{\alpha} \}$. Hence $\hat{B}^x_{\xi,\alpha}$ is a Borel family of rank $\hat{\alpha}$.

Since $\hat{B}^x_{\xi,\alpha}$ is countable, it generates a Polish topology on $X$. Since $B_\alpha(x, \emptyset)$ is a $G_\delta$-subset of $X$ with respect to this topology, it is a Polish space with the inherited topology. As we have already noted $B^x_{\xi,\alpha} \subseteq \hat{B}^x_{\xi,\alpha}$. We now show that every set of the form $B_\alpha(x, \emptyset) \cap D$, where $D \in \hat{B}^x_{\xi,\alpha}$ is a union of elements from $\{ B_\alpha(x, \emptyset) \cap B : B \in B^x_{\xi,\alpha} \}$, i.e. the latter family can be also taken as a basis of the topology. It follows from the following claim.

**Claim.** Let $\zeta < \beta < \alpha$. Then for every $\sigma \in S^G_{<\infty}$ and $c \in [\omega]^{<\omega}$ the sets $B_\alpha(x, \emptyset) \cap V_c B_\zeta(x, \sigma)$ and $B_\alpha(x, \emptyset) \setminus V_c B_\zeta(x, \sigma)$ are unions of elements from the family $\{ B_\alpha(x, \emptyset) \cap B : B \in B^x_\beta \}$.

**Proof of Claim.** Take any $y \in B_\alpha(x, \emptyset) \cap V_c B_\zeta(x, \sigma)$. By Proposition 12 we get $B_\beta(y, id_c) \subseteq V_c B_\zeta(x, \sigma)$. On the other hand Lemma 11(h) yields $B_\alpha(y, \emptyset) = B_\alpha(x, \emptyset)$. Then, by Proposition 13 we conclude that there is some $B \in B^x_\beta$ such that $B = B_\beta(y, id_c)$ which proves the first part of the claim.

Similarly, if $y \in B_\alpha(x, \emptyset) \setminus V_c B_\zeta(x, \sigma)$, then on the one hand $B_\beta(y, id_c) \subseteq X \setminus V_c B_\zeta(x, \sigma)$, on the other hand $B_\alpha(y, \emptyset) = B_\alpha(x, \emptyset)$. Using Proposition 13 again we conclude that for some $B \in B^x_\beta$ we have $y \in B \cap B_\alpha(x, \emptyset) \subseteq B_\alpha(x, \emptyset) \setminus V_c B_\zeta(x, \sigma)$, which proves the second part.
Now it suffices to notice that the action \( a : G \times B_\alpha(x, \emptyset) \to B_\alpha(x, \emptyset) \) is continuous with respect to each argument. Since every element of \( B_\alpha^{x} \) is invariant with respect to a basic open subgroup of \( G \), the action is continuous with respect to the first coordinate. On the other hand, for every \( f \in G, \gamma < \alpha \) and \( \delta \in S_{<\infty}^G \) we have \( \{ y \in X : fy \in B_\gamma(x, \delta) \} = B_\gamma(x, f^{-1}\delta) \), which proves continuity with respect to the second coordinate. \( \square \)

From now on let \( t_\alpha^x \) denote the Polish topology on \( B_\alpha(x, \emptyset) \) described above. Observe that in the case when \( \alpha \) is a successor ordinal and \( \alpha = \beta + 1 \), the topology \( t_\alpha^x \) is also (relatively) generated by a smaller basis, namely \( B_\beta^x \). It follows directly from the claim used in the proof above and Corollary 16.

Using Proposition 17 together with Effros Theorem on \( G_\delta \)-orbits we obtain the following fact.

**Proposition 18** Let \( G \) be a closed subgroup of \( S_\infty \), \( X \) be a Polish \( G \)-space and \( x \in X \).

(a) If \( Gx \in \Pi_{\alpha+1}^0(X) \), then the following statements are true:

(i) \( Gx \) is non-meager in \( B_\alpha(x, \emptyset) \) with respect to \( t_\alpha^x \).

(ii) the map \( G \to Gx \ g \to gx \) is open with respect to \( t_\alpha^x \).

(iii) \( Gx = B_{\alpha+1}(x, \emptyset) \).

(b) If \( \alpha \) is a limit ordinal and \( Gx \in \Pi_\alpha^0(X) \), then the following statements are true:

(iv) \( Gx = B_\alpha(x, \emptyset) \).

(v) the map \( G \to Gx \ g \to gx \) is open with respect to \( t_\alpha^x \).

Proof. (a) The proof is based on the following observation.

Claim. Let \( A \subseteq B_\alpha(x, \emptyset) \) be an invariant \( \Pi_{\alpha+1}^0 \)-set. Then \( A \) is a \( G_\delta \)-set with respect to \( t_\alpha^x \).

Proof of Claim. We apply the claim used in the proof of Lemma 14. We present \( A \) as countable intersection \( A = \bigcap_{i<\omega} D_i \), such that for every \( i < \omega \), \( D_i \in \Sigma_\alpha^0(X) \) and it is invariant with respect \( V^G_{a_i} \) for some finite \( a_i \subseteq \omega \). Then we apply the claim again to each \( D_i \). For every \( i < \omega \), we find a family \( \{ D_{ij} : j < \omega \} \) satisfying the following conditions:

1. \( D_{ij} \in \bigcup_{\xi<\alpha} \Pi_{\xi}^0(X) \);

2. \( D_{ij} \) is invariant with respect to \( V^G_{a_{ij}} \), for some finite \( a_{ij} \supseteq a_i \);

3. \( D_i = \bigcup_{j<\omega} D_{ij} \).

We have

\[
A = A \cap B_\alpha(x, \emptyset) = \bigcap_{i<\omega} \bigcup_{j<\omega} (D_{ij} \cap B_\alpha(x, \emptyset)).
\]

By Lemma 14 and Lemma 11(h) we see that for every \( i, j < \omega \) the set \( D_{ij} \cap B_\alpha(x, \emptyset) \) is a union of elements of the family \( \{ B \cap B_\alpha(x, \emptyset) : B \in B_\alpha^{x} \} \), thus it is open with respect to the topology \( t_\alpha^x \). Therefore \( A \) is a \( G_\delta \) set with respect to \( t_\alpha^x \).
Hence we conclude that $Gx$ is a $G_\delta$-subset of $B_\alpha(x, \emptyset)$ with respect to the topology. Then (i) and (ii) follows from Effros theorem. (iii) follows Lemma 14.

(b) Point (iv) follows from Lemma 14, then we obtain (v) from Effros theorem. □

The second statement of the following proposition looks folklore, but we have not found it in literature.

**Proposition 19** Let $G$ be a closed subgroup of $S_\infty$, $X$ be a Polish $G$-space.

(a) Let $x \in X$. If $Gx \in \Pi^0\alpha(X)$ for some ordinal $\alpha$, then for every open basic subgroup $V^G_c \subset G$ we have $V^G_c x \in \Pi^0\alpha(X)$, and for every open subgroup $H < G$ we have $H x \in \Pi^0\alpha+2(X)$.

(b) The orbit equivalence relation induced on $X$ by the $G$-action is Borel if and only if the orbit equivalence relation induced on $X$ by the action of some of open subgroup $H < G$ is Borel.

**Proof.** (a) Let $V^G_c$ be an arbitrary basic open subgroup of $G$. We have to consider two cases.

1° $\alpha = \beta + 1$ is a successor ordinal. Then by Proposition 18(ii) there is a family $C \subseteq B^x_\beta$ such that

\[
(*) \quad V^G_c x = ( \bigcup C ) \cap Gx.
\]

Thus for some $C_0 \in C$ we have $C_0 \cap V^G_c x \neq \emptyset$, which implies $B_\beta(x, id_c) \subseteq V^G_c C_0$. Next, since the set on the right side of the equality (*) must be $V^G_c$-invariant, we have also $V^G_c x = V^G_c ( \bigcup C ) \cap Gx$. The latter implies $B_\beta(x, id_c) \cap Gx \subseteq V^G_c C_0 \cap Gx \subseteq V^G_c x$, which yields $V^G_c x = B_\beta(x, id_c) \cap Gx$. Then we are done, since $B^x_\alpha \subseteq \Pi_\alpha(X)$.

2° $\alpha$ is a limit ordinal. Then by Proposition 18(v) there is a family $C \subseteq B^x_\alpha$ such that

\[
(*) \quad V^G_c x = ( \bigcup C ) \cap Gx.
\]

Exactly as in the case $\alpha = \beta + 1$ we obtain $V^G_c x = B_\alpha(x, id_c) \cap Gx$. Then we are done, since $B^x_\alpha \subseteq \Pi_\alpha(X)$ for every limit $\alpha$.

The second part of (a) is a direct consequence of the first one.

(b) is a consequence of (a), the fact that every $G$-orbit is a countable union of $H$-orbits and the following theorem of Sami on Borel orbit equivalence relations (see [13]).

Let $G$ be a Polish group and $X$ be a Polish $G$-space. The orbit equivalence relation induced on $X$ by the $G$-action is Borel if and only if there is a countable ordinal $\alpha$ such that every $G$-orbit is a $\Pi^0\alpha$-subset of $X$.

□

We now show that Proposition 19 simplifies the proof of Theorem 7.1.1 of Becker and Kechris from [4] in the particular case of actions of closed subgroups of $S_\infty$ (in [4] it is assumed that $G$ is Polish). In fact S.Solecki suggested that such applications are possible.
Proposition 20 (Becker, Kechris - special case) Let $G$ be a closed subgroup of $S_\omega$ and $X$ be a Borel $G$-space. Then the following conditions are equivalent.

(i) The orbit equivalence relation is Borel.

(ii) The map $\tau: x \to G_x$ from $X$ to the Effros space of closed subsets $\mathcal{F}(G)$ is Borel.

Proof. Note that since the action is continuous, all stabilizers of $G$ are closed. For (ii)$\Rightarrow$ (i) see [4]. To prove the converse we can assume that $X$ is a Polish $G$-space.

Now we shall define for every $x \in X$ some cardinal invariant connected with $\alpha$-sets. The definition is based on the following lemma.

Lemma 21 For every $x \in X$ there is some $\gamma < \omega_1$ such that for all $\sigma, \delta \in S_{<\omega_{\text{on}}}^G$ with $\text{rng}[\sigma] = \text{rng}[\delta]$ we have

$$(\exists \alpha < \omega_1)(B_\alpha(x, \sigma) \neq B_\alpha(x, \delta)) \Rightarrow (B_\gamma(x, \sigma) \neq B_\gamma(x, \delta)).$$

Proof. Let $d \subseteq \omega$ be an arbitrary finite set. For every pair $\{\sigma, \delta\} \subseteq S_{<\omega_{\text{on}}}^G$ with the same range $d$ consider the set $\{\alpha < \omega_1 : B_\alpha(x, \sigma) \neq B_\alpha(x, \delta)\}$. Let $\gamma_{\sigma, \delta}$ be its infimum in case the set is nonempty or 0 otherwise. Let $\gamma_d = \sup\{\gamma_{\sigma, \delta} : \{\sigma, \delta\} \subseteq S_{<\omega_{\text{on}}}^G, \text{rng}[\sigma] = \text{rng}[\delta] = d\}$. It is a countable ordinal, since the set $\{\{\sigma, \delta\} : \text{rng}[\sigma] = \text{rng}[\delta] = d\}$ is countable. Finally let $\gamma = \sup\{\gamma_d : d \in [\omega]^{<\omega}\}$. It is also a countable ordinal as a supremum of a countable set of countable ordinals. Obviously the ordinal $\gamma$ has the required property. $\square$

Definition 22 For every $x \in X$ let $\gamma^G_x(x)$ be the least ordinal $\gamma$ satisfying the statement of Lemma 21.

By Lemmas 14 and 11 every orbit is an $\alpha$-set. In the theorem below we show that every $Gx$ is a $(\gamma^G_x(x) + 2)$-set.

Theorem 23 For every $x \in X$ we have $B_{\gamma^G_x(x) + 2}(x, \emptyset) = Gx$.

Proof. Let $y \in B_{\gamma^G_x(x) + 2}(x, \emptyset)$. Then by Lemma 11 (h) we have $B_{\gamma^G_x(x) + 2}(x, \emptyset) = B_{\gamma^G_x(x) + 2}(y, \emptyset)$. The rest of the proof is based on two claims.

Claim 1. Let $\zeta, \beta$ be ordinals such that $\gamma^G_x(x) < \zeta + 1 < \beta$ and $B_\beta(x, \emptyset) = B_\beta(y, \emptyset)$. Let $\sigma, \delta \in S_{<\omega_{\text{on}}}^G$ have common range $a$. Then the equality $B_\zeta(x, \sigma) = B_\zeta(y, \delta)$ implies $B_{\zeta + 1}(x, \sigma) = B_{\zeta + 1}(y, \delta)$. 


Proof of Claim 1. By Proposition 13 there is some $\sigma'$ such that $B_{\zeta+1}(x, \sigma') = B_{\zeta+1}(y, \delta)$. Then by Lemma 11(a), (i) we have also $B_{\zeta}(x, \sigma') = B_{\zeta}(y, \delta)$ and $B_{\zeta}(x, \sigma) = B_{\zeta}(x, \sigma')$. Since $\zeta \geq \gamma^G_{\star}$, we have $B_{\zeta+1}(x, \sigma) = B_{\zeta+1}(x, \sigma')$, which proves the required equality.

Claim 2. If $B_{\gamma^G_{\star}(x)+2}(x, \emptyset) = B_{\gamma^G_{\star}(x)+2}(y, \emptyset)$ then $B_{\beta}(x, \emptyset) = B_{\beta}(y, \emptyset)$ for every $\beta$. Proof of Claim 2. It is trivially true for $\beta \leq \gamma^G_{\star} + 2$. We use induction to prove it for ordinals $\beta > \gamma^G_{\star} + 2$.

The limit step is immediate. To go through the successor step suppose that $B_{\beta}(x, \emptyset) = B_{\beta}(y, \emptyset)$, for some $\beta \geq \gamma^G_{\star} + 2$. Take an arbitrary $a \subseteq \omega$. Then by Proposition 13, for every $\zeta < \beta$ we have $\{B_{\zeta}(x, \sigma) : \text{rng}[\sigma] = a\} = \{B_{\zeta}(y, \delta) : \text{rng}[\delta] = a\}$. This equality remains true for $\zeta = \beta$. Indeed, it is obvious for a limit $\beta$ and follows from Claim 1 for a successor $\beta$. By Proposition 13 again we get $B_{\beta+1}(x, \emptyset) = B_{\beta+1}(y, \emptyset)$.

We come back to the proof of the theorem. From the fact that $G$-orbits are Borel sets, we conclude by Lemma 14, that there is an ordinal $\beta$ such that $Gx = B_{\beta}(x, \emptyset)$ and $ Gy = B_{\beta}(y, \emptyset)$. Since $B_{\gamma^G_{\star}(x)+2}(x, \emptyset) = B_{\gamma^G_{\star}(x)+2}(y, \emptyset)$, we are done by Claim 2. □

As a corollary of the theorem we obtain the following statement.

**Corollary 24** For every $x \in X$ and $\sigma \in S^G_{<\infty}$ we have $B_{\gamma^G_{\star}(x)+2}(x, \sigma) = V^G_{\sigma}x$.

**Proof.** We derive it from the definition of $\gamma^G_{\star}$ and Lemma 11 (d), (f). □

The following lemma gives a characterization of $\gamma^G_{\star}$ in terms of Borel complexity. It is a direct consequence of Proposition 15 (together with the idea that appropriate Vaught transforms make a set invariant).

**Lemma 25** For every $x \in X$, $\gamma^G_{\star}(x)$ is the least ordinal $\alpha$ with the property that

for every $\sigma, \sigma_1 \in S^G_{<\infty}$ with $\text{rng}[\sigma] = \text{rng}[\sigma_1]$ if $V^G_{\sigma}x \neq V^G_{\sigma_1}x$ then there is a Borel set of rank $\alpha$ containing one of the set $V^G_{\sigma}x, V^G_{\sigma_1}x$ and disjoint from the other.

The next proposition establishes relations between the Borel rank of the $G$-orbit of $x$ and the number $\gamma^G_{\star}(x)$. The left inequality is a direct consequence of Lemma 25 and Proposition 19, the right inequality follows from Theorem 23 and Corollary 16.

**Proposition 26** Let $x \in X$ and $\lambda$ be the multiplicative Borel rank of the orbit $Gx$ (i.e. $\lambda = \min\{\zeta : Gx \in \Pi^0_{\zeta}\}$). Then

$$\gamma^G_{\star}(x) \leq \lambda \leq \gamma^G_{\star}(x) + 3.$$ In particular if $\lambda$ is a limit ordinal, then it is equal to $\gamma^G_{\star}(x)$.
The left inequality improves the analogous inequality obtained by Hjorth, who in fact proved that \( \gamma_3^G(x) \leq \lambda + 1 \).

Proposition 26 shows that the number \( \gamma_3^G(x) \) and the Borel rank of the orbit \( Gx \) can not differ very much. Nevertheless we will show below that they can be different. The corresponding example uses Lemma 25.

**Example.** Consider the conjugacy action of \( S_\infty \) on itself. It is shown in Theorem 1.8 of [9] that any conjugacy class of \( S_\infty \) belongs to \( \Pi_3 \) and there are conjugacy classes of Borel rank 3. Let us prove that \( \gamma_3(f) = 1 \) for every \( f \in S_\infty \). Accordingly to Lemma 25, it suffices to show that for any pair of conjugates \( f \) and \( g \) and any finite set \( c \) of natural numbers if \( V^G_c f \cap V^G_c g = \emptyset \) then we can separate \( V^G_c f \) from \( V^G_c g \) by an open or a closed subset of \( S_\infty \) (in terms of that theorem \( f = v^h \) and \( g = v^{h_1} \) for some \( v, h, h_1 \in S_\infty \) with \( h \in V^G_r \) and \( h_1 \in V^G_s \)). We start with the following claim.

**Claim.** Let \( f, g \in S_\infty \) be two conjugates and \( c \in [\omega]^{< \omega} \). Then \( V^G_c f \) and \( V^G_c g \) are disjoint if and only if there are \( k \in c \) and \( m \in \mathbb{Z} \) such that

\[
\left( f^m(k) \in c \lor g^m(k) \in c \right) \land f^m(k) \neq g^m(k).
\]

**Proof.** It is well-known that \( f \) and \( g \) are conjugate if and only if their cycle types are the same. We have to consider only nonempty sets \( c \). Let \( c = \{k_0, k_1, \ldots, k_s\} \).

The proof of \((\Leftarrow)\) is easy. To prove the converse, assume that for all \( k \in c \) and \( m \in \mathbb{Z} \), \( f^m(k) \in c \lor g^m(k) \in c \) implies \( f^m(k) = g^m(k) \). We are going to define some \( h \in V^G_c \) so that \( f^h = g \).

We proceed as follows. For every \( j \leq s \) and every \( m \in \mathbb{Z} \) we put

\[
h_0(f^m(k_j)) = g^m(k_j) \quad \text{(in particular } h_0(k_j) = k_j \text{)}.
\]

It follows from the assumptions that \( h_0 \) is a well-defined bijection

\[
h_0 : \{f^m(k_j) : j \leq s, \ m \in \mathbb{Z}\} \to \{g^m(k_j) : j \leq s, \ m \in \mathbb{Z}\}
\]

such that

\[
(f|\{f^m(k_j) : j \leq s, \ m \in \mathbb{Z}\})^{h_0} = g|\{g^m(k_j) : j \leq s, \ m \in \mathbb{Z}\}.
\]

Now using the fact that \( f \) and \( g \) have the same cycle types, we see that \( h_0 \) can be extended to a permutation \( h \in S_\infty \) so that \( f^h = g \). \( \square \)

We can now finish the proof of the main statement. By the claim we find \( k, l \in c \) and \( m \in \omega \) such that

\[
\left( f^m(k) = l \lor g^m(k) = l \right) \land f^m(k) \neq g^m(k).
\]

Without loss of generality we may assume that \( f^m(k) = l \land g^m(k) \neq l \). Let \( A^l_k \) be the set of all bijections

\[
\sigma = \left( \begin{array}{cccc}
    k & a_1 & a_2 & \ldots & a_{m-2} & a_{m-1} \\
    a_1 & a_2 & a_3 & \ldots & a_{m-2} & l
\end{array} \right),
\]

where \( \{a_1, a_2, \ldots, a_{m-1}\} \subseteq \omega \).

Then the set \( \hat{A}^l_k = \bigcup \{V^G_c : \sigma \in A^l_k \} \) is open, it contains \( V^G_c f \) and is disjoint from \( V^G_c g \). \( \square \)
3 α-Sets and admissible sets

Section 3.1 contains the main results of the paper. In Section 3.2 we give some straightforward construction of admissible sets which satisfy all our assumptions.

3.1 Main results

The main notions of this section (codability and constructibility in an admissible set) were defined in Section 1. The following theorem allows us to code α-sets $B_{\alpha}(x, \sigma)$ in admissible sets. Here we use terms which appear in Definitions 1 - 6 and Lemma 4.

**Theorem 27** Let $\mathbb{A}$ be an admissible set. Let $G < S_\infty$ be a closed subgroup and $X$ be a Polish $G$-space with a basis $\{A_i : i \in \omega\}$. Suppose that $x$ is co-codable in $\mathbb{A}$ and the relation

$$\text{Imp}(c, l, k) \iff (c \in [\omega]^{\leq \omega} \land l, k \in \omega \land A_k \subseteq V_c^G A_l)$$

is $\Sigma$-definable in $\mathbb{A}$. Then for every $\sigma \in S_{<\infty}^G$ and every countable $\alpha \in \mathbb{A}$ the set $B_{\alpha}(x, \sigma)$ is constructible in $\mathbb{A}$. Moreover there is a $\Sigma$-definable in $\mathbb{A}$ binary function $u_x(\alpha, \sigma)$ which finds a co-$\alpha$-multicode for $B_{\alpha}(x, \sigma)$.

**Proof.** We start with some preliminary remarks. First for every $c \in [\omega]^{<\omega}$ and $l \in \omega$ we define in $\mathbb{A}$ a function

$$\text{imp}(c, l, k) : \omega \to \{0, 1\} \quad \text{by} \quad \text{imp}(c, l, k) = 1 \iff \mathbb{A} \models \text{Imp}(c, l, k).$$

Then since $\text{Imp}$ is a $\Sigma$-relation in $\mathbb{A}$, we have (by $\Sigma$-replacement):

- $\text{Imp}(c, l) \in \mathbb{A}$, $\mathbb{A} \models B_{\Sigma}(1, \text{imp}(c, l))$ and $B_{\text{imp}(c, l)} = V^G_c A_l$;
- $(0, \text{imp}(c, l)) \in \mathbb{A}$, $\mathbb{A} \models B_{\Pi}(1, (0, \text{imp}(c, l)))$ and $B_{(0, \text{imp}(c, l))} = X \setminus V^G_c A_l$.

As we noted in Section 1, the condition that $x$ is $\Sigma$-codable in $\mathbb{A}$ implies that $([\omega]^{<\omega}, \subseteq)$, $\langle S^G_{<\infty}, \subseteq \rangle$ $\in \mathbb{A}$. We fix in $\mathbb{A}$ some bijective enumerations

$$\rho : \omega \to [\omega]^{<\omega} \quad \text{and} \quad \mu : \omega \to S^G_{<\infty}.$$

Let $\sigma$ and $\alpha$ be as in the formulation and let $d, c$ denote the domain and the range of $\sigma$ respectively. Let $F_1$ be a $\Sigma$-function which codes $x$ in $\mathbb{A}$.

By the definition of $\alpha$-sets (see Definition 10), the set $B = B_{\alpha}(x, \sigma)$ can be naturally considered as an intersection of a pair of sets $C$ and $D$ (when $\alpha$ is limit we put $C = D$) of the form $C = \bigcap_{i \in I} C_i$ and $D = \bigcap_{i \in I} D_i$. To define a multicode for $B$ we view $C$ and $D$ as

$$C = X \setminus \bigcup_{i \in I} (X \setminus C_i) \quad \text{and} \quad D = X \setminus \bigcup_{i \in I} (X \setminus D_i).$$

Assuming that some co-multicodes for all $C_i$ and $D_i$, $i \in \omega$, are already known, we will find in $\mathbb{A}$ appropriate multicodes $w$ and $v$ for $\bigcup_{i \in I} (X \setminus C_i)$ and $\bigcup_{i \in I} (X \setminus D_i)$ respectively. It is worth noting that these multicodes will correspond to ordinals appearing in Corollary 16 as levels of Borel hierarchy. Then the co-multicode $u = \bigwedge \langle (0, w), (0, v) \rangle$ will correspond to $C \cap D$. 

25
Now we are ready to go into the details. We define in $\mathbb{A}$ binary functions $w_x(\alpha, \sigma)$ and $v_x(\alpha, \sigma)$ to the set of multicodes and a function $u_x(\alpha, \sigma)$ to the set of co-multicodes by a formula depending on variables $\alpha, \sigma, u, v, w$ (realizing a simultaneous induction on ordinals $\alpha$) as follows:

$$(\ast) \ (w_x(\alpha, \sigma) = w \land v_x(\alpha, \sigma) = v \land u_x(\alpha, \sigma) = u)$$

iff

$$(w, v \text{ are functions}) \land (u = \bigcup((0, w), (0, v))) \land \left(\Theta_1 \lor \Theta_2 \lor \Theta_3\right),$$

where the formulas $\Theta_i$, for $i = 1, 2, 3$, are defined as follows.

$\Theta_1 = (\alpha = 1) \land (\text{dom}[w] = \text{dom}[v] = \omega) \land (\forall l \in \omega)(w(l) \text{ and } v(l) \text{ are as follows:})$

$$(l \in F_1(\sigma) \Rightarrow w(l) = (0, \text{imp}_{c,l})) \land (l \notin F_1(\sigma) \Rightarrow w(l) = \text{mc}_0) \land$$

$$(l \notin F_1(\sigma) \Rightarrow v(l) = \text{imp}_{c,l}) \land (l \in F_1(\sigma) \Rightarrow v(l) = \text{mc}_0)\right).$$

Accordingly to this definition $w(l)$ is a co-$1$-multicode of $X \setminus V^G_{\alpha} A_l$ in the case $V^G_{\alpha} x \cap A_l \neq \emptyset$. Otherwise $w(l)$ is a $1$-multicode for $\emptyset$. Eventually, $w$ is a $2$-multicode for the set $\bigcup\{X \setminus V^G_{\alpha} A_l : V^G_{\alpha} x \cap A_l \neq \emptyset\}$. On the other hand $v$ is a $2$-multicode for the set $\bigcup\{V^G_{\alpha} A_l : V^G_{\alpha} x \cap A_l = \emptyset\}$.

The formula $\Theta_2$ tells us how to code $\alpha$-sets at the successor step. In this formula (see below) $w(l)$ is a co-multicode for the set $X \setminus \bigcup\{B_\beta(x, \sigma') : \sigma' \supseteq \sigma \land \text{dom}[\sigma'] = \rho(l)\}$ if $\rho(l) \supseteq \text{dom}[\sigma]$. Otherwise $w(l)$ is a $1$-multicode for $\emptyset$. Eventually, $w$ is a $1$-multicode for the case $\bigcup_{b \supseteq \text{dom}[\sigma]} \left(X \setminus \bigcup\{B_\beta(x, \sigma') : \sigma' \supseteq \sigma \land \text{dom}[\sigma'] = b\}\right)$.

On the other hand $v(l)$ is a $1$-multicode for $X \setminus \bigcup\{B_\beta(x, \sigma') : \sigma' \supseteq \sigma \land \text{rng}[\sigma'] = \rho(l)\}$ in the case $\rho(l) \supseteq \text{rng}[\sigma]$. Otherwise $v(l)$ is a $1$-multicode for $\emptyset$. Eventually, $v$ is a $1$-multicode for the case $\bigcup_{a \supseteq \text{rng}[\sigma]} \left(X \setminus \bigcup\{B_\beta(x, \sigma') : \sigma' \supseteq \sigma \land \text{rng}[\sigma'] = a\}\right).$

$\Theta_3 = (\exists \beta < \alpha)[(\alpha = \beta + 1) \land (\text{dom}[w] = \text{dom}[v] = \omega) \land$

$$\land (\forall l \in \omega)(w(l) \text{ and } v(l) \text{ are defined as follows:})$$

$$(\rho(l) \not\supseteq d \Rightarrow w(l) = \text{mc}_0) \land (\rho(l) \supseteq d \Rightarrow (\exists w')(w(l) = (0, w')) \land$$

$$(w'_i \text{ is a function with } \text{dom}[w'_i] = \omega) \land (\forall j \in \omega)((\mu(j) \supseteq \sigma \land \text{dom}[\mu(j)] = \rho(l) \Rightarrow w'_i(j) = u_x(\beta, \mu(j))) \land ((\mu(j) \not\supseteq \sigma \land \text{dom}[\mu(j)] \neq \rho(l)) \Rightarrow w'_i(j) = \text{mc}_0)))\land$$

$$(\rho(l) \not\supseteq c \Rightarrow v(l) = \text{mc}_0) \land (\rho(l) \supseteq c \Rightarrow (\exists v'_i)(v(l) = (0, v'_i)) \land$$

$$(v'_i \text{ is a function with } \text{dom}[v'_i] = \omega) \land (\forall j \in \omega)((\mu(j) \supseteq \sigma \land \text{rng}[\mu(j)] = \rho(l) \Rightarrow v'_i(j) = u_x(\beta, \mu(j))) \land ((\mu(j) \not\supseteq \sigma \land \text{rng}[\mu(j)] \neq \rho(l)) \Rightarrow v'_i(j) = \text{mc}_0))))\right].$$

Finally, formula $\Theta_3$ settles the coding of $\alpha$-sets for limit ordinals.

---

2which in fact appears in the formulation of the theorem
Thus $w(\beta)$, for every $\beta < \alpha$, is a co-multicode for $X \setminus B_\beta(x, \sigma)$ and $w$ is a multicode for the union $\bigcup \{X \setminus B_\beta(x, \sigma) : \beta < \alpha\}$.

Again we shall use the second recursion theorem (Section 5.4 of [1]) to see that $(\star)$ defines a $\Sigma$-relation in $A$. Using induction and $\Sigma$-collection principle we conclude that for every $\alpha \in A$ and $\sigma \in S^G_{< \infty}$ the relation uniquely defines a co-multicode $u_x(\alpha, \sigma) \in A$ such that $B_\alpha(x, \sigma) = B_{u_x(\alpha, \sigma)}$. □

We have to prove the following technical lemma. We apply the relation $\equiv$ defined in Section 1 (Definition 5).

**Lemma 28** Let $G < S_\infty$ be a closed subgroup and $X$ be a Polish $G$-space with a basis $\{A_i : i \in \omega\}$. Let $A$ be an admissible set such that $\text{Imp}$ is $\Sigma$-definable in $A$ and $x, y$ are codable in $A$. Let $u_x, u_y$ denote the $\Sigma$-functions defined in the proof of Theorem 27. Then for every $\sigma, \delta \in S^G_{< \infty}$ with $\text{rng}[\sigma] = \text{rng}[\delta]$ and every countable ordinal $\alpha \in A$ we have

$$B_\alpha(x, \sigma) = B_\alpha(y, \delta) \implies u_x(\alpha, \sigma) \equiv u_y(\alpha, \delta).$$

**Proof.** We preserve the notation of the proof of Theorem 27. The proof is by induction on $\alpha$. Assume that

$$(\star) \text{rng}[\sigma] = \text{rng}[\delta] \text{ and } B_\alpha(x, \sigma) = B_\alpha(y, \delta).$$

If $\alpha = 1$ then we apply the formula $\Theta_1$ from the proof of Theorem 27. Since $B_1(x, \sigma) = B_1(y, \delta)$, then for every $l \in \omega$, $A_i$ intersects one of the sets $V_\sigma x, V_\delta y$ if and only if it also intersects the other. Hence we have $w_x(1, \sigma) = w_y(1, \delta), v_x(1, \sigma) = v_y(1, \delta)$. Thus $u_x(1, \sigma) \equiv u_y(1, \delta)$.

For the successor step suppose that $\alpha = \beta + 1$ and the implication

$$B_\beta(x, \sigma') = B_\beta(y, \delta') \implies u_x(\beta, \sigma') \equiv u_y(\beta, \delta').$$

holds whenever $\text{rng}[\sigma'] = \text{rng}[\delta']$. We claim that $w_x(\alpha, \sigma) \equiv w_y(\alpha, \delta)$ and $v_x(\alpha, \sigma) \equiv v_y(\alpha, \delta)$.

By Proposition 13 the condition $(\star)$ implies that for every $a \supseteq c$ we have

$$\{B_\beta(x, \sigma') : \sigma' \supseteq \sigma, \text{rng}[\sigma'] = a\} = \{B_\beta(y, \delta') : \delta' \supseteq \delta, \text{rng}[\delta'] = a\}.$$

Thus by the inductive assumption for every $l \in \omega$ such that $\rho(l) \supseteq \text{rng}[\sigma]$ the sets $\{u_x(\beta, \sigma') : \sigma' \supseteq \sigma, \text{rng}[\sigma'] = \rho(l)\}$ and $\{u_y(\beta, \delta') : \delta' \supseteq \delta, \text{rng}[\delta'] = \rho(l)\}$ represent the same $\equiv$-classes. Hence $v_x(\alpha, \sigma) \equiv v_y(\alpha, \delta)$.  

27
On the other hand for every \( b \supseteq \text{dom}[\sigma] \) there are \( \sigma_1 \supseteq \sigma \) and \( \delta_1 \supseteq \delta \) with \( \text{dom}[\sigma_1] = b \) and \( B_{\alpha}(x, \sigma_1) = B_{\alpha}(y, \delta_1) \). Then applying Lemma 11(c) (and the claim from its proof) we find \( b_1 \supseteq \text{dom}[\delta] \) (as \( \text{dom}[\delta_1] \)) such that

\[
\{ B_{\beta}(x, \sigma') : \sigma' \supseteq \sigma, \text{dom}[\sigma'] = b \} = \{ B_{\beta}(y, \delta') : \delta' \supseteq \delta, \text{dom}[\delta'] = b_1 \}.
\]

By a similar argument we see that for every \( b_1 \supseteq \text{dom}[\delta] \) there is \( b \supseteq \text{dom}[\sigma] \) such that

\[
\{ B_{\beta}(x, \sigma') : \sigma' \supseteq \sigma, \text{dom}[\sigma'] = b \} = \{ B_{\beta}(y, \delta') : \delta' \supseteq \delta, \text{dom}[\delta'] = b_1 \}.
\]

Thus for every \( l \in \omega \) such that \( \rho(l) \supseteq \text{dom}[\sigma] \) we can find \( l_1 \in \omega \) (and vice-versa for every \( l_1 \in \omega \) with \( \rho(l_1) \supseteq \text{dom}[\delta] \) there is \( l \in \omega \) with \( \rho(l) \supseteq \text{dom}[\sigma] \)) such that the sets \( \{ u_x(\beta, \sigma') : \sigma' \supseteq \sigma, \text{dom}[\sigma'] = \rho(l) \} \) and \( \{ u_y(\beta, \delta') : \delta' \supseteq \delta, \text{dom}[\delta'] = \rho(l_1) \} \) represent the same \( \equiv \)-classes. Therefore \( w_x(\alpha, \sigma) \equiv w_y(\alpha, \delta) \). By the formula \( \Theta_2 \) this finally yields \( u_x(\alpha, \sigma) \equiv u_y(\alpha, \delta) \).

For the limit step suppose that \( \alpha \in \mathbb{A} \) is a limit ordinal and the implication

\[
B_{\beta}(x, \sigma) = B_{\beta}(y, \delta) \Rightarrow u_x(\beta, \sigma) \equiv u_y(\beta, \delta)
\]

holds for every \( \beta < \alpha \). Then (\( \ast \)) implies that for every \( \beta < \alpha \) we have \( B_{\beta}(x, \sigma) = B_{\beta}(y, \delta) \), which by the inductive assumption gives \( u_x(\beta, \sigma) \equiv u_y(\beta, \delta) \). Then we are done by the formula \( \Theta_3 \). \( \Box \)

We shall now prove our main results (which were formulated in Section 1).

**Proof of Theorem 7.** Let \( \mathbb{A} \) be an admissible set such that \( \omega \) is realizable in it. Let \( G < S_{\mathcal{O}} \) be a closed group, \( X \) be a Polish \( G \)-space with a basis \( \{ A_i : i > 0 \} \) and \( \text{Imp} \) be \( \Sigma \)-definable on \( \mathbb{A} \). We want to prove the following statements:

1. Let \( x \in X \) be \( \Sigma \)-codable in \( \mathbb{A} \). Then for every \( y \in X \), if \( x, y \) are in the same invariant Borel subsets of \( X \) which are constructible in \( \mathbb{A} \) then for every \( \alpha \leq o(\mathbb{A}) \) they are in the same invariant \( \Sigma^0_\alpha \)-subsets of \( X \).
2. If \( x, y \) are \( \Sigma \)-codable in \( \mathbb{A} \) and they belong to the same invariant Borel sets which are constructible in \( \mathbb{A} \) then they are in the same \( G \)-orbit.

Part (1) is a direct consequence of Theorem 27 and Lemma 14.

We shall use the back-and-forth arguments together with \( \Sigma \)-reflection in \( \mathbb{A} \). We are going to construct a set \( \Gamma \) of triples \( (n_i, \sigma_i, A_i) \) with the following properties for every \( i \in \omega \):

a. \( n_i \in \omega \), \( \sigma_i \in S_{\leq \omega}^G \), \( r^n_{\sigma_2} = n_\omega \), \( \text{dom}[\sigma_{2i+1}] = n_{2i+1} \) and \( \text{dom}[\sigma_{2i+1}] = n_{2i+1} \) (i.e. \( r_{\sigma_{2i+1}} = n_{2i+1} \));

b. \( A_{2i} \) is a \( V_{n_{2i}}^G \)-invariant basic open set containing \( y \), \( A_{2i+1} \) is a \( V_{n_{2i+1}}^G \)-invariant basic open set containing \( x \);

c. \( n_{i+1} > n_i, \sigma_{i+1} \supseteq \sigma_i \), \( \text{diam}(A_{i+1}) < 2^{-(i+1)} \).
(d) \(A_{2i+2} \subseteq A_{2i}\), and \(A_{2i+1} \subseteq A_{2i-1}\);
\(A_{2i+1} \subseteq V^G_{\sigma_{2i}} A_{2i}\) and \(A_{2(i+1)} \subseteq V^G_{\sigma_{2i+1}} A_{2i+1}\);
(e) \(B_\alpha(x, \sigma_i) = B_\alpha(y, id_{rng[\sigma_i]})\)
for every \(\alpha < o(A)\).

We put \(n_0 = 0\), \(\sigma_0 = \emptyset\) and let \(A_0\) be any \(G\)-invariant basic open set containing \(y\).

Suppose that we have already constructed all the triples \((n_k, \sigma_k, A_k)\), for \(k \leq 2i\). In particular we have \(B_\alpha(x, \sigma_i) = B_\alpha(y, id_{rng[\sigma_i]})\) for every \(\alpha < o(A)\). Then by Lemma 11 (c) we also have \(B_\alpha(x, id_{dom[\sigma_i]}) = B_\alpha(y, \sigma_i^{-1})\), for every \(\alpha < o(A)\).

Applying assumptions of the induction (in particular \(B_1(x, \sigma_i) = B_1(y, id_{rng[\sigma_i]})\))
we see that \(V^G_{\sigma_{2i}} A_{2i}\) is an \(\sigma\)-invariant set containing \(x\). Let \(A_{2i+1} \subseteq V^G_{\sigma_{2i}} A_{2i}\) be any basic neighbourhood of \(x\) such that \(diam(A_{2i+1}) < 2^{-(2i+1)}\). Then we define \(n_{2i+1}\)
to be any natural number greater than \(n_{2i}\) and covering \(dom[\sigma_{2i}]\) such that \(A_{2i+1}\) is \(V^G_{n_{2i+1}}\)-invariant.

We claim that there is some \(\sigma_{2i+1} \in S^G_{<\infty}\) with \(\sigma_{2i+1} \supseteq \sigma_{2i}\) and \(dom[\sigma_{2i+1}] = n_{2i+1}\)
such that \(B_\alpha(x, id_{n_{2i+1}}) = B_\alpha(y, \sigma_{2i+1})\) for every \(\alpha < o(A)\). In other words we are looking for some \(\delta \in S^G_{<\infty}\) such that

\(\delta \supseteq \sigma_{2i}^{-1}, \ rng[\delta] = n_{2i+1}\) and \(B_\alpha(x, id_{n_{2i+1}}) = B_\alpha(y, \delta)\), for every \(\alpha < o(A)\).

Suppose there is no \(\delta\) satisfying \((\ast)\). Then to every \(\delta \in S^G_{<\infty}\), satisfying \(\delta \supseteq \sigma_{2i}^{-1}\) and \(rng[\delta] = n_{2i+1}\), we can assign some ordinal \(\beta_\delta \in A\) so that \(B_{\beta_\delta}(x, id_{n_{2i+1}}) \neq B_{\beta_\delta}(y, \delta)\).

By Lemma 28 the latter inequality is equivalent to the relation \(u_x(\beta_\delta, id_{n_{2i+1}}) \neq u_y(\beta_\delta, \delta)\). By Definition 5 (and the discussion after this definition) and Theorem 27
this relation can be expressed in \(A\) by a \(\Sigma\)-formula. Since the set \(\{\delta \in S^G_{<\infty} : \delta \supseteq \sigma_{2i}^{-1}, \ rng[\delta] = n_{2i+1}\}\)
is an element of \(A\), then by \(\Sigma\)-reflection in \(A\) (Section 1.4 of [1]),
there is an ordinal \(\beta \in A\) such that \(u_x(\beta, id_{n_{2i+1}}) \neq u_y(\beta, \delta)\) for every \(\delta \supseteq \sigma_{2i}^{-1}\) with \(rng[\delta] = n_{2i+1}\).

Therefore by the definition of the functions \(u_x\) and \(u_y\) (in the proof of
Theorem 27) we have \(u_x(\beta + 1, id_{dom[\sigma_i]}) \neq u_y(\beta + 1, \sigma_{2i}^{-1})\). This by Lemma 28 yields
\(B_{\beta+1}(x, id_{dom[\sigma_i]}) \neq B_{\beta+1}(y, \sigma_{2i}^{-1})\) which contradicts the assumptions.

Then we take any \(\delta\) satisfying \((\ast)\) and put \(\sigma_{2i+1} = \delta^{-1}\). At even steps we use the
symmetric procedure.

Using the method just described we define a sequence \(\{\sigma_i : i < \omega\}\) of elements of \(S^G_{<\infty}\).
Since \(G\) is closed, by (b) and (c) there is \(f \in G\) such that \(\bigcap_i V^G_{\sigma_i} = \{f\}\)
and \(fx \in \bigcap_i V^G_{\sigma_{2i+1}} A_{2i+1}\).

Moreover by (a)-(d) we have \(V^G_{\sigma_{2i+1}} A_{2i+1} \subseteq V^G_{\sigma_{2i}} A_{2i+1} \subseteq V^G_{n_{2i}} A_{2i} = A_{2i}\).
Therefore \(\{y\} = \bigcap_i A_{2i} = \bigcap_i V^G_{\sigma_{2i+1}} A_{2i+1} = \{fx\}\). Thus \(y = fx\). 

This theorem has a corollary in the style of Nadel’s work [12] (see also [1], Corollary 7.7.4). It connects ranks considered in Section 2.2 with \(o(A)\), the ordinal of \(A\).

**Proposition 29** Under the assumptions of this section (of Theorem 27) \(\gamma^G_\alpha(x) \leq o(A)\). Moreover if \(\gamma^G_\alpha(x) < o(A)\) then \(Gx\) is constructible in \(A\) (thus the Borel rank of \(Gx\) is \(< o(A)\)).
Proof. We apply the characterization of \( \gamma^G(x) \) from Lemma 25. Take any \( \sigma, \delta \in S^G_{<\infty} \) with \( \text{rng}[\sigma] = \text{rng}[\delta] = c \) such that \( V^G_\sigma x \cap V^G_\delta x = \emptyset \). It suffices to show that \( V^G_\sigma x \) and \( V^G_\delta x \) can be separated by a Borel set constructible in \( \mathbb{A} \) (its Borel rank is \( < o(\mathbb{A}) \)).

First observe that if \( x \) is codable in \( \mathbb{A} \) with respect to \( G \), then it is also codable in \( \mathbb{A} \) with respect to any basic open subgroup \( V^G_\sigma \). Indeed, let \( F_1 : S_{<\infty} \to \mathbb{A} \) be a function coding \( x \) in \( \mathbb{A} \). Since \( S^G_\sigma = \{ \sigma \in S^G_{<\infty} : (\forall n \in c \cap \text{dom}[\sigma])(\sigma(n) = n) \} \) is an element of \( \mathbb{A} \), we can define in \( \mathbb{A} \) a function coding \( x \) with respect to \( V^G_\sigma \) by the following formula

\[
C_1(\sigma) = \begin{cases} 
\emptyset & \text{if } \sigma \notin S^G_\sigma \\
F_1(\sigma) & \text{if } \sigma \in S^G_\sigma.
\end{cases}
\]

It is clear that \( C_1 \) is an element of \( \mathbb{A} \).

In the following claim we consider elements of \( G \) as functions \( \omega \to \omega \) with respect to the chosen realization of \( \omega \) in \( \mathbb{A} \).

Claim. Let \( x \) be codable in \( \mathbb{A} \). Then the set \( \{ g \in G : g \in \mathbb{A} \} \) is dense in \( G \). Moreover, if \( g \in G \) is an element of \( \mathbb{A} \), then \( g \cdot x \) is codable in \( \mathbb{A} \).

Proof of Claim. If \( x \) is codable in \( \mathbb{A} \) then \( S^G_{<\infty} \) is an element of \( \mathbb{A} \). We define on \( S^G_{<\infty} \) a partial ordering \( \leq \) by the following formula

\[
\sigma \leq \delta \text{ iff } \sigma = \delta \lor \left( (\text{dom}[\sigma] = \text{dom}[\delta] \land \exists n \in \text{dom}[\sigma])(\sigma(n) < \delta(n)) \land \right.
\]

\[
(\forall k \in \text{dom}[\sigma])(k < n \to \sigma(k) = \delta(k)).
\]

Since this is a \( \Delta_0 \)-formula, we see that \( \leq \) is an element of \( \mathbb{A} \). Moreover for every \( n \) the restriction of the ordering \( \leq \) to the set \( \{ \sigma \in S^G_{<\infty} : \text{dom}[\sigma] = n \} \) becomes a lexicographical well-order.

Now, take any \( \sigma \in S^G_{<\infty} \). We have to find \( g \in G \) such that \( \sigma \subseteq g \) and \( g \in \mathbb{A} \). We define two increasing sequences: \( (k_n) \) of elements of \( \omega \) and \( (\sigma_n) \) of elements of \( S^G_{<\infty} \) by the following scheme:

\[
\begin{align*}
\sigma_0 &= \sigma \\
k_0 &= \min\{l : \text{rng}[\sigma_0] \subseteq l\} \\
\sigma^{-1}_{2n+1} &= \min\{\delta \in S^G_{<\infty} : k_{2n} + 2n + 1 = \text{dom}[\delta] \land \sigma^{-1}_{2n} \subseteq \delta\} \\
k_{2n+1} &= \min\{l : \text{dom}[\sigma_{2n+1}] \subseteq l\} \\
\sigma_{2n+2} &= \min\{\delta \in S^G_{<\infty} : k_{2n+1} + 2n + 2 = \text{dom}[\delta] \land \sigma_{2n+1} \subseteq \delta\} \\
k_{2n+2} &= \min\{l : \text{rng}[\sigma_{2n+2}] \subseteq l\}.
\end{align*}
\]

We see that \( g = \bigcup \sigma_n \) is a permutation. Since \( G \) is closed, \( g \) belongs to \( G \). On the other hand the definition of the functions \( g \) and \( n \to \sigma_n, n \in \omega \), can be formalized by a \( \Sigma \)-formula (by the second recursion theorem). Since \( \omega \in \mathbb{A} \), by \( \Sigma \)-replacement we have that both the sequence \( (\sigma_n) \) and \( g \) are elements of \( \mathbb{A} \).
Finally, if \( g \in G \) and \( g \in A \), then the operation \( S^G_{<\infty} \to S^G_{<\infty} : \sigma \to \sigma g \) defined by the \( \Delta_0 \)-formula

\[
\sigma g \text{ is a finite partial function and } (\forall n \in \text{dom}[\sigma g])(g(n)) \in \text{dom}[\sigma] \land \sigma g(n) = \sigma((g(n)))
\]
is an element \( A \).

Hence the function \( F^g_1 \) coding \( gx \) can be defined as an element of \( A \) by the following formula.

\[
F^g_1(\sigma) = \begin{cases} 
\emptyset & \text{if } \sigma \notin S^G_{<\infty} \\
F_1(\sigma g) & \text{if } \sigma \in S^G_{<\infty}.
\end{cases}
\]

Now we return to the main statement. By the claim there are \( g \in V^G_c \) and \( f \in V^G_\delta \) such that \( gx \) and \( fx \) are codable in \( A \) with respect to \( V^G_c \). Then by Theorem 7(2) (applied to \( V^G_c \)) we see that \( V^G_c gx = V^G_\sigma x \) and \( V^G_c fx = V^G_\delta x \) can be separated by Borel set constructible in \( A \).

To obtain the second part of the proposition note that by Theorem 23 we see that \( Gx \) is of the form \( B_\alpha(x,\emptyset) \) for some \( \alpha \in A \). By Theorem 27 this set is constructible in \( A \). ✷

Theorem 7 suggests that in some situations we may expect that \( Gx \) is just the intersection of all \( G \)-invariant Borel sets containing \( x \) and constructible in \( A \). We now show that under some additional assumption this is really true. This is the content of Theorem 9:

**Theorem 30** Let \( G \) be a closed subgroup of \( S_\infty \), \( X \) be a Polish \( G \)-space, \( t \) be a nice topology for \( X \) and \( B \) be its nice basis. Let \( x \in X \) and let \( A \) be an admissible set such that \( x \) is codable in \( A \) with respect to \( B \). Then the piece \( C \) of the canonical partition with respect to \( B \) with \( x \in C \) coincides with the orbit \( Gx \) if and only if \( C \) is the intersection of all invariant Borel sets containing \( x \) and constructible in \( A \).

**Proof.** Since \( C \) is invariant and Borel, the necessity is obvious. To prove sufficiency we shall use the notion of a *type* introduced in [10].

Let \( H \) be an open subgroup of \( G \) and \( \hat{X}_0 \) be an invariant \( G_\delta \)-subset of \( X \) with respect to the \( t \)-topology.

(1) A family \( F \subseteq B \) is called an \( H \)-type in \( \hat{X}_0 \), if it is maximal with respect to the following conditions:

(a) \( B \) is \( H \)-invariant, for any \( B \in F \);

(b) \( \hat{X}_0 \cap \bigcap F \neq \emptyset \).

(2) An \( H \)-type \( F \) is called principal if there is \( B_F \in F \) such that \( B_F \cap \hat{X}_0 \subseteq B \cap \hat{X}_0 \), for every \( B \in F \). We will say that \( B_F \) defines \( H \).

In paper [10] we prove the following characterization of \( G \)-orbits in terms of types (Theorem 10).
Consider the canonical partition with respect to the topology \( t \). A piece \( \hat{X}_0 \) of the canonical partition is a \( G \)-orbit if and only if for any basic clopen subgroup \( H < G \) any \( H \)-type of \( \hat{X}_0 \) is principal.

Assuming (ii) we will show that every \( H \)-type of \( C \) is principal for every clopen subgroup \( H < G \). Suppose the contrary. Then there is some basic clopen \( H < G \) and a non-principal \( H \)-type of \( \hat{X}_0 \). Then by Lemma 9 from [10], the set \( D = \bigcap_{g \in G} \left( g(\bigcup_{B \in \mathcal{F}} (C \setminus B)) \right) \) is nonempty and invariant. Since \( C \cap \bigcap \mathcal{F} \neq \emptyset \), we have also \( D \neq C \). The Borel ranks of \( D \) and \( C \setminus D \) with respect to \( \mathcal{B} \) are \( \leq 4 \) and one of these sets contains \( x \). This set by Lemma 14 includes \( B_4(x, \emptyset) \). Then we get a contradiction, since \( B_4(x, \emptyset) \) is constructible in \( A \) by Theorem 27.

In [11] A.Morozov has proved the following theorem:

Let \( A \) be a locally countable admissible set (i.e. \( \omega < o(A) \) and \( A \models (\forall s \neq \emptyset) (\exists f : \omega \rightarrow s)(f(\omega) = s) \)). Let \( \phi \) be an \( L_{\omega_1 \omega} \)-sentence for some language \( L \in A \). Then \( \phi \) is \( \omega \)-categorical if and only if \( \phi \) is complete with respect to all sentences which belong to \( A \).

Note that this theorem is quite similar to our Theorem 9. Indeed, consider the (logic) \( S_\infty \)-space \( X_L \) of all \( L \)-structures. Then identifying \( L_{\omega_1 \omega} \)-sentences \( \psi \) with the corresponding \( G_\delta \)-set \( \{ x : x \models \psi \} \) we see that sentences from \( A \) correspond to \( G_\delta \)-sets constructible in \( A \). Thus the condition that the set \( C \) of models of \( \phi \) cannot be divided by such \( G_\delta \)-sets means that \( C \) is an \( S_\infty \)-orbit.

It is worth noting that our proof of Theorem 9 is based on arguments which originally arose in model theory (see [1] and [7]).

### 3.2 Example of coding in admissible sets.

Let \( G \) be a closed subgroup of \( S_\infty \), \((X, \tau)\) be a Polish \( G \)-space and \( A = \{ A_l : l \in \omega \} \) be a countable basis of \((X, \tau)\).

To each \( x \in X \) we assign an admissible set \( A_x \) such that \( x \) is codable in \( A_x \). We start with the following two-sorted structure

\[
M_x = \langle \omega \cup S_\infty; S_\infty^G, Imp(\sigma, k, l), Sat_x(\sigma, l) \rangle
\]

defined on the disjoint union of the set \( \omega \) of natural numbers and the set \( S_\infty \) of all bijections between finite sets of natural numbers with:

1. The unary relation \( S_\infty^G \) for recognizing elements of \( S_\infty^G \);
2. the ternary relation \( Imp(\sigma, k, l) \):
   \[
   (\sigma \in S_\infty^G) \land (l, k \in \omega) \land (A_k \subseteq V^G_\sigma A_l);
   \]
3. the binary relation \( Sat_x(\sigma, l) \) defined by \( Sat_x(\sigma, l) \Leftrightarrow V^G_\sigma x \cap A_l \neq \emptyset \).
Proposition 31 Let $\mathcal{A}$ be an admissible set and $x \in X$. The element $x$ is codable in $\mathcal{A}$ so that $\text{Imp}(\sigma, k, l)$ is definable for the corresponding realization of $(\omega, <)$ if and only if $\mathcal{A}$ is admissible above $M_x$ (i.e. $M_x \in \mathcal{A}$).

Proof. ($\Rightarrow$) According to Definition 6, codability of $x$ in $\mathcal{A}$ requires that $\mathcal{A}$ contains $(\omega, <)$ or its isomorphic copy. Then as we have already noted in Section 1, $\mathcal{A}$ also contains some copy of the structure $(S_{<\infty}, \subseteq)$ . Let $F_1$ be the coding function for $x$. The predicate $S_{<\infty}^G$ is defined by the formula

$$S_{<\infty}^G(\sigma) \iff F_1(\sigma) \neq \emptyset,$$

so by $\Delta_0$-separation it also becomes an element of $\mathcal{A}$. Finally, the relation $Sat_x(\sigma, l)$ is also defined by the $\Delta_0$-formula

$$Sat_x(\sigma, l) \iff l \in F_1(\sigma).$$

The converse follows by similar arguments. $\square$

Corollary 32 Every $x \in X$ is codable in $\mathcal{A}_x = \text{Hyp}(M_x)$.

References

[1] J.Barwise, Admissible Sets and Structures, Springer-Verlag, NY, 1975.
[2] H.Becker, Polish group actions: Dichotomies and generalized elementary embeddings, J. Amer. Math. Soc. 11(1998) 397 - 449.
[3] H.Becker, Topics in invariant descriptive set theory, Annals of Pure and Appl. Logic 111(2001), 145 - 184.
[4] H.Becker and A.Kechris, The Descriptive Set Theory of Polish Group Actions, Cambridge University Press, Cambridge, 1996.
[5] Yu.Ershov, Definability and Computability, Consultants Bureau, NY, 1996.
[6] G.Hjorth, Classification and Orbit Equivalence Relations. AMS 1991
[7] W.Hodges, Building Models by Games, Cambridge University Press, Cambridge, 1985.
[8] A.Kechris, Classical Descriptive Set Theory, Springer-Verlag, New York, 1995.
[9] B.Majcher-Iwanow, Complexity of conjugacy classes of $A(Q)$, Topology Appl. 128(2003), 173-188.
[10] B.Majcher-Iwanow, $G_\delta$-pieces of canonical partitions of $G$-spaces, Math. Log. Quarterly 51(2005), 450 - 461.
[11] A. Morozov, *One more time on ω-categorical sentences*, Siberian Math. J. 40(1999), 434 - 438.

[12] M. Nadel, *Scott sentences and admissible sets*, Ann. Math. Log. 7(1974), 269 - 294.

[13] R. Sami, *Polish group actions and the Vaught conjecture*, Trans. Amer. Math. Soc. 341(1994), 335 - 353.

[14] R. Vaught, *Invariant sets in topology and logic*, Fund. Math. 82(1974), 269 - 293.

Institute of Mathematics, University of Wrocław, pl. Grunwaldzki 2/4, 50-384 Wrocław, Poland
E-mail: biwanow@math.uni.wroc.pl