Inequalities related to Bourin and Heinz means with a complex parameter*

T. Bottazzi, R. Elencwajg, G. Larotonda and A. Varela †

Abstract

A conjecture posed by S. Hayajneh and F. Kittaneh claims that given $A, B$ positive matrices, $0 \leq t \leq 1$, and any unitarily invariant norm it holds
\[ |||A^t B^{1-t} + B^t A^{1-t}||| \leq |||A^t B^{1-t} + A^{1-t} B^t|||.\]

Recently, R. Bhatia proved the inequality for the case of the Frobenius norm and for $t \in \left[\frac{1}{4}, \frac{3}{4}\right]$. In this paper, using complex methods we extend this result to complex values of the parameter $t = z$ in the strip $\{z \in \mathbb{C} : \text{Re}(z) \in \left[\frac{1}{4}, \frac{3}{4}\right]\}$. We give an elementary proof of the fact that equality holds for some $z$ in the strip if and only if $A$ and $B$ commute. We also show a counterexample to the general conjecture by exhibiting a pair of positive matrices such that the claim does not hold for the uniform norm. Finally, we give a counterexample for a related singular value inequality given by $s_j(A^t B^{1-t} + B^t A^{1-t}) \leq s_j(A + B)$, answering in the negative a question made by K. Audenaert and F. Kittaneh.$\dagger$

1 Introduction

We begin this paper with some notations and definitions. The context here is the algebra of $n \times n$ complex entries matrices, but the proofs adapt well to other (infinite dimensional) settings in operator theory, so let us assume that $\mathcal{A}$ stands for an operator algebra with trace, for instance $\mathcal{A} = M_n(\mathbb{C})$ with its usual trace, or $\mathcal{A} = B_2(H)$, the Hilbert-Schmidt operators acting on a separable complex Hilbert space with the infinite trace, or $\mathcal{A} = (\mathcal{A}, Tr)$ a $C^*$-algebra with a finite faithful trace.

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Definitions 1.1. Let \(|\cdot|\) denote an unitarily invariant norm on \(A\), which we assume is equivalent to a symmetric norm, that is
\[ ||XYZ||| \leq ||X||_{\infty}|||Y||| |||Z||| \]
whenever \(Y \in A\) (from now on \(\| \cdot \|_{\infty}\) will denote the norm of the operator algebra).

For convenience we will use the notation \(\tau(X) = \text{Re} \text{Tr}(X)\). Let \(|X| = \sqrt{X^*X}\) stand for the modulus of the matrix or operator \(X\), then the (right) polar decomposition of \(X\) is given by \(X = U|X|\) where \(U\) is a unitary such that \(U\) maps \(\text{Ran}|X|\) into \(\text{Ran}(X)\) and is the identity on \(\text{Ran}|X|^{\perp} = \text{Ker}(X)\). Note that \(\|X\|_2^2 = \text{Tr}(X^*X) = \text{Tr}(|X|^2)\).

Consider the inequality
\[ \tau(A^tB^zA^{1-z}B^{1-z}) \leq \tau(AB), \] (1)
for positive invertible operators \(A, B > 0\) in \(A\), and \(z \in \mathbb{C}\). We introduce some notation regarding vertical strips in the complex plane: let
\[ S_0 = \{ z \in \mathbb{C} : 0 \leq \text{Re}(z) \leq 1 \}, \quad S_{1/4} = \{ z \in \mathbb{C} : 1/4 \leq \text{Re}(z) \leq 3/4 \}; \]
we will study the validity of (1) in both \(S_0\) and \(S_{1/4}\).

Intimately related to the expression above are the inequalities
\[ |||b_t(A,B)||| \leq ||h_t(A,B)||| \] (2)
and
\[ |||b_t(A,B)||| \leq ||A + B|||, \] (3)
for positive matrices \(A, B \geq 0\) in \(A\), where
\[ b_t(A,B) = A^tB^{1-t} + B^tA^{1-t} \quad t \in [0,1]; \]
the name \(b_t\) is due to Bourin, who conjectured inequality (3) for \(n \times n\) matrices in [5], and
\[ h_t(A,B) = A^tB^{1-t} + A^{1-t}B^t \quad t \in [0,1] \]
is named after Heinz, and the well-known [7] inequality
\[ |||h_t(A,B)||| \leq |||A + B||| \]
carrying his name.

Recently, S. Hayajneh and F. Kittanneh proposed in [6] that the stronger (2) should also be valid in \(M_n(\mathbb{C})\); however, numerical computations (see Section 3) show that, at least for the uniform norm, this is false.
If we focus on the case \(|X| = \|X\|_2 = Tr(X^*X)^{1/2}\) (the Frobenius norm in the case of \(n \times n\) matrices) and we write \(h_t = h_t(A, B), b_t = b_t(A, B)\), then

\[
Tr|h_t|^2 = \tau(b_t^*b_t) = \tau(B^{1-t}A^t + A^{1-t}B^t)(A^tB^{1-t} + B^tA^{1-t})
\]

\[
= \tau(B^{2(1-t)}A^{2t}) + \tau(A^{2(1-t)}B^{2t}) + 2\tau(A^tB^tA^{1-t}B^{1-t})
\]

where we have repeatedly used the ciclicity of \(\tau\) (i.e. \(\tau(XY) = \tau(YX)\)) and the fact that \(\tau(Z^*) = \tau(Z)\). Likewise

\[
Tr|h_t|^2 = \tau(B^{2(1-t)}A^{2t}) + \tau(A^{2(1-t)}B^{2t}) + 2\tau(AB).
\]

Thus, proving that \(\|b_t\|_2 \leq \|h_t\|_2\) amounts to prove that

\[
\tau(A^tB^tA^{1-t}B^{1-t}) \leq \tau(AB),
\]

and in fact, it is clear that both inequalities are equivalent -as remarked in [6]-.

## 2 Main results

We will divide the problem in regions of the plane (or the line), and then we will also consider the possibility of attaining the equality; we will see that this is only possible in the trivial case, i.e. when \(A, B\) commute. We recall the generalized Hölder inequality, that we will use frequently: let \(\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1\) for \(p, q, r \geq 1\) and \(X, Y, Z\) in \(A\), then

\[
\tau(XYZ) \leq \|XYZ\|_1 \leq \|X\|_p\|Y\|_q\|Z\|_r.
\]

This is just a combination of the usual Hölder inequality together with

\[
\|XY\|_s \leq \|X\|_p\|Y\|_q
\]

provided \(s \geq 1\) and \(\frac{1}{p} + \frac{1}{q} = \frac{1}{s}\) (see [8], Theorem 2.8, for more details).

### 2.1 The inequality in the strip \(S_{1/4}\)

We begin with an easy consequence of an inequality due to Araki-Lieb and Thirrning.

**Lemma 2.1.** If \(A, B \geq 0\) and \(r \geq 2\), then

\[
\|A^{1/r}B^{1/r}\|_r \leq \tau(AB)^{1/r}.
\]

**Proof.** Note that

\[
\|A^{1/r}B^{1/r}\|_r^r = \tau([A^{1/r}B^{1/r}]^r) = \tau([A^{1/r}B^{2/r}A^{1/r}]^r)
\]

which, by the inequality of Araki-Lieb and Thirrning (see [2], and note that \(r/2 \geq 1\) is less or equal than

\[
\tau(A^{r/2r}B^{2r/2r}A^{r/2r}) = \tau(A^{1/2}BA^{1/2}),
\]

which in turn equals \(\tau(AB)\).
Note that if we exchange the variables \( z \mapsto 1 - z \) and exchange the role of \( A, B \), it suffices to consider half-strips or half-intervals around \( \text{Re}(z) = 1/2 \).

**Proposition 2.2.** If \( 0 < A, B \) and \( z \in S_{1/4} \), then

\[
\tau(A^z B^z A^{1-z} B^{1-z}) \leq \tau(AB).
\]

**Proof.** Let \( z = 1/2 + iy \), \( y \in \mathbb{R} \) denote any point in vertical line of the complex plane passing through \( x = 1/2 \). Then

\[
\tau(A^z B^z A^{1-z} B^{1-z}) = \tau(A^{iy} A^{1/2} B^{1/2} B^{iy} A^{-iy} A^{1/2} B^{1/2} B^{-iy}) \\
\leq \tau|A^{iy} A^{1/2} B^{1/2} B^{iy} A^{-iy} A^{1/2} B^{1/2} B^{-iy}| \\
\leq \|A^{iy} A^{1/2} B^{1/2} B^{iy} A^{-iy} A^{1/2} B^{1/2} B^{-iy}\|_2 \|A^{1/2} B^{1/2} B^{-iy}\|_2 = \|A^{1/2} B^{1/2}\|_2^2
\]

by the Cauchy-Schwarz inequality and the fact that \( A^{iy}, B^{iy} \) are unitary operators. Then by the previous lemma,

\[
\tau(A^z B^z A^{1-z} B^{1-z}) \leq \tau(AB)^{2/2} = \tau(AB).
\]

Now consider \( z = 1/4 + iy \), \( y \in \mathbb{R} \), a generic point in the vertical line over \( x = 1/4 \), then noting that \( \frac{1}{4} + \frac{1}{4} + \frac{1}{2} = 1 \),

\[
\tau(A^z B^z A^{1-z} B^{1-z}) = \tau(B^{1/4} A^{1/4} A^{iy} B^{iy} B^{1/4} A^{1/4} A^{-iy} A^{1/2} B^{1/2} B^{-iy}) \\
\leq \|B^{1/4} A^{1/4}\|_4^2 \|B^{1/2} A^{1/2}\|_2 \leq \tau(AB)^{2/4+1/2} = \tau(AB),
\]

where we used again the previous Lemma and the generalized Hölder’s inequality,

\[
\tau(XYZ) \leq \|X\|_p \|Y\|_q \|Z\|_r
\]

whenever \( p, q, r \geq 1 \) and \( \frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1 \).

By Hadamard’s three-lines theorem, the bound \( \tau(AB) \) is valid in the vertical strip \( 1/4 \leq \text{Re}(z) \leq 1/2 \), since it holds in the frontier of the strip. Invoking the symmetry \( z \mapsto 1 - z \) and exchanging the roles of \( A, B \) gives the desired bound on the full strip \( S_{1/4} = \{ 1/4 \leq \text{Re}(z) \leq 3/4 \} \).

Regarding the inequalities conjectured by Bourin et al., note that we can assume \( A, B > 0 \): replacing \( A \) with \( A_x = A + \varepsilon \) (and likewise with \( B \)), if the inequality (1) is valid for \( A_x, B_x \) then making \( \varepsilon \to 0^+ \) gives the general result: the following result that we state as corollary was recently obtained by R. Bhatia in \([4]\) and we should also point the reader to the paper by T. Ando, F. Hiai, K. Okubo \([1]\).

**Corollary 2.3.** For any \( A, B \geq 0 \) and any \( t \in [1/4, 3/4] \),

\[
\|A^t B^{1-t} + B^t A^{1-t}\|_2 \leq \|A^t B^{1-t} + A^{1-t} B^t\|_2 \leq \|A + B\|_2.
\]
2.2 Inequality becomes equality

Let us consider the special case when the inequality above becomes an equality. We begin with a lemma that we will use in several occasions, and will be useful when we drop the assumption on nonsingularity of $A, B$.

Lemma 2.4. Let $A, B \geq 0$, and assume

$$\tau(A^{1/2}B^{1/2}A^{1/2}B^{1/2}) = \tau(AB),$$

or

$$\|A^{1/4}B^{1/4}\|_4 = \tau(AB)^{1/4}.$$

In either case, $A$ commutes with $B$.

Proof. Name $X = A^{1/2}B^{1/2}$, and considering the inner product induced by $\tau$, $(X,Y) = \tau(XY^*)$,

$$(X,X^*) = \tau(X^2) = \tau(A^{1/2}B^{1/2}A^{1/2}B^{1/2}) = \tau(AB) = \tau(X^*X) = \|X\|_2^2 = \|X\|_2\|X^*\|_2.$$

But Cauchy-Schwarz inequality becomes an equality if and only if $X = \lambda X^*$ for some $\lambda > 0$, and since both operators have equal norm ($= \|A^{1/2}B^{1/2}\|_2$), then $X = X^*$. This means

$$A^{1/2}B^{1/2} = B^{1/2}A^{1/2},$$

and this implies that $A$ commutes with $B$. On the other hand,

$$\|A^{1/4}B^{1/4}\|_4^4 = \tau((B^{1/4}A^{1/2}B^{1/4})^2) = \tau(A^{1/2}B^{1/2}A^{1/2}B^{1/2}),$$

so what we have is just another way of writing the first equality condition. \(\square\)

Proposition 2.5. Let $A, B > 0$ and assume that there is $z_0 \in S_{1/4}$ such that

$$\tau(A^{z_0}B^{z_0}A^{1-z_0}B^{1-z_0}) = \tau(AB).$$

Then $A$ commutes with $B$ and $\tau(A^zB^zA^{1-z}B^{1-z}) = \tau(AB)$ for any $z \in \mathbb{C}$.

Proof. First consider the case when equality is reached in an interior point of the strip $S_{1/4}$. Note that by the maximum modulus principle, this would mean that the function

$$f(z) = \tau(A^zB^zA^{1-z}B^{1-z})$$

is constant in the strip $S_{1/4}$, in particular equality holds at $z_0 = 1/2$, and by the previous Lemma, $A$ commutes with $B$. 

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Now suppose equality is attained in the frontier, for instance at $z_0 = 1/4 + iy$ for some $y \in \mathbb{R}$. Let $X = B^{1/4}A^{1/4}A^yB^{1/4}A^{1/4}$, $Y = B^{1/2}B^yA^yA^{1/2}$. Then, if we go through the proof of Proposition 2.2 again, assuming equality

$$\tau(AB) = \tau(XY^*) = \langle X, Y \rangle \leq \|X\|_2\|Y\|_2 \leq \|B^{1/4}A^{1/4}\|_2^2\|A^{1/2}B^{1/2}\|_2 \leq \tau(AB).$$ (5)

Arguing as in the previous Lemma, there exists $\lambda > 0$ such that $X = \lambda Y$,

$$B^{1/4}A^{1/4}A^yB^{1/4}A^{1/4} = \lambda B^{1/2}B^yA^yA^{1/2}.$$

Cancelling $B^{1/4}$ on the left and $A^{1/4}$ on the right we obtain

$$A^{1/4}A^yB^{1/4} = \lambda B^{1/4}B^yA^yA^{1/4},$$

but now both elements have the same norm and this shows that $\lambda = 1$; then

$$A^{1/4+y}B^{1/4+iy} = B^{1/4+iy}A^{1/4+iy},$$

and since $A, B > 0$, the existence of analytic logarithms shows that again $A$ commutes with $B$. By symmetry, the same argument applies for any $z_0 = 3/4 + iy$ in the other border of the strip.

**Corollary 2.6.** If $A$ does not commute with $B$, the inequality is strict:

$$\tau(A^zB^tA^{1-z}B^{1-t}) < \tau(AB),$$

in some open set $\Omega \subset \mathbb{C}$ containing the closed strip $S_{1/4}$.

If we allow $A, B$ to be non invertible, holomorphy is lost, but nevertheless in the same spirit we have the following result.

**Proposition 2.7.** For given $A, B \geq 0$, there exists $\delta = \delta(A, B) > 0$ such that

$$\tau(A^tB^tA^{1-t}B^{1-t}) \leq \tau(AB)$$

holds in the interval $[1/4 - \delta, 3/4 + \delta]$. If $A$ does not commute with $B$, the inequality is strict in the whole $(1/4 - \delta, 3/4 + \delta)$.

**Proof.** If $A$ commutes with $B$, then the assertion is trivial. If not, arguing as in the last part of the proof of the previous proposition, we must have strict inequality

$$\tau(A^tB^tA^{1-t}B^{1-t}) < \tau(AB)$$

for $t = 1/4$, $t = 3/4$, and then by continuity the inequality extends a bit out of the closed interval $[1/4, 3/4]$.  

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Consider $t \in (1/4, 1/2)$ and put $X = B^{1/4} A^{1/4} t^{3/4} B^{-1/4}$, $Y = B^{1/4} A^{1/4} t^{-1/4} B^{-1/4}$.
Note that $\frac{1}{4} + \frac{3}{4} = 1$ and define $1/p = t - 1/4 \in (0, 1/4)$, $1/q = 3/4 - t \in (1/4, 1/2)$, note also that $1/p + 1/4 = t$, $1/q + 1/4 = 1 - t$. By reiterated use of Hölder’s inequality compute
\[
\tau(A^t B^t A^{1-t} B^{1-t}) \leq \|XY\|_1 \leq \|X\|_{(1-t)^{-1}} \|Y\|_{(1-t)^{-1}} \leq \|B^{1/4} A^{1/4}\|_4 \|A^{1/p} B^{1/p}\|_p \|B^{1/q} A^{1/q}\|_q \|A^{1/4} B^{1/4}\|_4.
\]
Now apply Lemma 2.4 to each of the four terms (note that $p > 4$ and $q > 2$), and we have
\[
\tau(A^t B^t A^{1-t} B^{1-t}) \leq \|B^{1/4} A^{1/4}\|_4 \|A^{1/p} B^{1/p}\|_p \|B^{1/q} A^{1/q}\|_q \|A^{1/4} B^{1/4}\|_4 \leq \tau(AB).
\]
If we assume equality of the traces, then
\[
\tau(AB) = \|B^{1/4} A^{1/4}\|_4 \|A^{1/p} B^{1/p}\|_p \|B^{1/q} A^{1/q}\|_q \|A^{1/4} B^{1/4}\|_4
\]
and in particular, it must be that $\|A^{1/4} B^{1/4}\|_4 = \tau(AB)^{1/4}$, and from Lemma 2.4 we can deduce that $A$ commutes with $B$. By the symmetry ($t \mapsto 1 - t$) the argument extends to $(1/2, 3/4)$, and again by Lemma 2.4 we already know that $A$ commutes with $B$ if equality is attained at $t = 1/2$. This finishes the proof of the assertion that the inequality is strict in $[1/4, 3/4]$ unless $A$ commutes with $B$. \hfill \Box

Remark 2.8. The inequalities in the previous proof give in fact
\[
\tau(|B^{1/4} A^t B^{1-t} A^{3/4} B^{3-t}|) \leq Tr(AB)
\]
for any $t \in [\frac{1}{4}, \frac{3}{4}]$; this is a particular instance of [7] Theorem 2.10).

3 Counterexamples

In this section we exhibit specific cases of different kind. In Example 3.1 we choose $A, B$ such that $\|b_t(A, B)\|_\infty > \|h_t(A, B)\|_\infty$, while in Example 3.2 it is shown that the $j$th singular value of $A + B$ is not always greater than the $j$th singular value of $b_t(A, B)$. This provides negative answers to [6] Conjecture 1.2 and [3] Problem 4] respectively.

Example 3.1. Consider the following positive definite matrices
\[
A = \begin{pmatrix}
1141 & 0 & 0 \\
0 & 204 & 0 \\
0 & 0 & 1/8
\end{pmatrix}
\quad \text{and} \quad
B = \begin{pmatrix}
39 & 90 & 43 \\
90 & 418 & 370 \\
43 & 370 & 426
\end{pmatrix}.
\]
\[2\text{Note that this is another proof of the inequality for real } t \in [\frac{1}{4}, \frac{3}{4}].\]
The following is the graph of \( f(t) = -\|b_t(A, B)\|_\infty + \|h_t(A, B)\|_\infty \) for \( t \in [0, \frac{1}{2}] \):

For these matrices \( -\|b_t(A, B)\|_\infty + \|h_t(A, B)\|_\infty \simeq -2.3 \) at \( t = 0.15 \).

In [3, Problem 4] K. Audenaert and F. Kittaneh asked if \( s_j(b_t(A, B)) \leq s_j(A + B) \) for every \( j \) and \( 0 < t < 1 \) (where \( s_j(M) \), \( j = 1 \ldots n \) denote the singular values of the matrix \( M \) arranged in non-increasing order).

**Example 3.2.** Consider the following positive definite matrices

\[
A = \begin{pmatrix} 6317 & 0 & 0 \\ 0 & 474 & 0 \\ 0 & 0 & 6 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 2078 & 2362 & 2199 \\ 2362 & 3267 & 2585 \\ 2199 & 2585 & 2492 \end{pmatrix}.
\]

Then, for \( t = \frac{1}{2} \) we have

\[
s(b_{\frac{1}{2}}(A, B)) = (6826.57, 878.499, 591.716)
\]

and

\[
s(A + B) = (10561.4, 3629.62, 443.017).
\]

In particular, \( s_3(b_{\frac{1}{2}}(A, B)) > s_3(A + B) \).

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