Coordinate Independence and a Physical Metric in Compact Form

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Abstract

A physical metric is constructed as one that gives a coordinate independent result for the time delay in infinite order in the perturbation expansion in the gravitational constant. A compact form for the metric is obtained. One result is that the metric functions are positive definite. Another is an exact expression for the gravitational red shift. The metric can be used to calculate general relativity predictions in higher order for any process. A relationship between the spacetimes of the physical metric and the Schwarzschild metric is discussed.

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I. INTRODUCTION

It is well known that the coordinates of the Schwartzschild metric do not correspond to observable physical coordinates. The author has introduced a method to calculate a coordinate independent result for the time delay for light propagation in a gravitational field. It has been shown also that there exists a unique metric that gives the same coordinate independent result and that the obtained result agrees with the experimental data of Shapiro et. al. with very good accuracy. For this reason, the author calls this metric a physical metric. It could also be called a coordinate independent metric or an invariant metric. In other words, predictions in the physical metric using the geodesic equation can be compared with observable data without further coordinate transformation or other adjustments. The consideration of time delay for light propagation around a mass point is extended to infinite order in the gravitational constant for the determination of the physical metric and the coefficients of the perturbation expansion are determined successively. Summation of the infinite series yields a transcendental equation that leads to a compact form for the metric functions and can be utilized for expanding the metric functions at the origin as well as at infinity. A method for getting expressions for general relativity predictions in higher order is presented. An implication of this newly constructed metric is discussed.

II. ASYMPTOTIC FORM FOR THE PHYSICAL METRIC

The physical metric is expressed as

\[ ds^2 = e^{\nu(r)} dt^2 - e^{\lambda(r)} dr^2 - e^{\mu(r)} r^2 (d\theta^2 + \sin^2\theta d\phi^2), \]

(1)

for a spherically symmetric and static mass point \( M \). From the fact that the transformation, \( r' = re^{\mu(r)/2} \), leads to the Schwartzschild metric, one can deduce the expression for the metric,

\[ e^{\nu(r)} = 1 - (r_s/r)e^{-\mu(r)/2}, \]

(2)

\[ e^{\lambda(r)} = \left(\frac{d}{dr}(re^{\mu(r)/2})\right)^2/(1 - (r_s/r)e^{-\mu(r)/2}), \]

(3)
where \( r_s = 2GM/c^2 \) is the Schwartzschild radius. An asymptotic expansion for the metric functions can be obtained from Eq. (2) and Eq. (3), yielding

\[
e^\nu(r) = \sum_{n=0}^{\infty} a_n (r_s/r)^n, \quad e^\lambda(r) = \sum_{n=0}^{\infty} b_n (r_s/r)^n, \quad \text{and} \quad e^\mu(r) = \sum_{n=0}^{\infty} c_n (r_s/r)^n, \quad (4)
\]

where

\[
a_0 = b_0 = c_0 = 1, \quad (5)
\]

\[-a_1 = b_1 = 1 \quad \text{and} \quad (6)
\]

\[a_2 = c_1/2, \quad b_2 = 1 - c_1/2 + c_1^2/4 - c_2, \quad \text{etc.} \quad (7)
\]

It is obvious that \( a_{n+1} \) and \( b_n \) can be expressed as functions of \( c_n, c_{n-1} \ldots, c_1 \).

### III. GEODESIC EQUATIONS AND TIME DELAY

The geodesic equations can be obtained from variations of the line integral over an invariant parameter \( \tau \), \( \int \left( \frac{ds}{d\tau} \right)^2 d\tau \), and their integrals are given by

\[
\frac{dt}{d\tau} = e^{-\nu(r)}, \quad (8)
\]

\[
\frac{d\phi}{d\tau} = J_\phi e^{-\mu(r)}/(r \sin \theta)^2, \quad (9)
\]

\[
\left( \frac{d\theta}{d\tau} \right)^2 = (J_\theta^2 - J_\phi^2/\sin^2 \theta)e^{-2\mu(r)/r^4}. \quad (10)
\]

Restricting the plane of motion to \( \frac{d\phi}{d\tau} = 0, \theta = \pi/2 \), the radial part of the geodesic integral is given by

\[
\left( \frac{dr}{d\tau} \right)^2 = e^{-\lambda(r)}(e^{-\nu(r)} - J^2e^{-\mu(r)/r^2} - E) \quad (11)
\]

where \( J_\phi, J_\theta \) and \( E \) are constants of integration and

\[
J^2 = J_\phi^2 = J_\theta^2. \quad (12)
\]

for the above restriction on the plane of motion. The constant \( E \) is 0 for light propagation.
From Eq. (8) and Eq. (11) with Eqs. (5) and (6), it follows that

\[
\frac{dt}{dr} = \pm e^{-\nu(r)/2} e^{-\mu(r)-\lambda(r)/r^2} \sqrt{e^{-\nu(r)-\lambda(r)/r^2}} = \pm \left( \sqrt{r^2 - r_0^2} \right)^{-1} \left( 1 + \frac{(b_1 - a_1)}{2} \right) \left( e^{-\mu(r)/2} \right) + \cdots \tag{13}
\]

for light propagation, where \( r_0 \) is the impact parameter. Integrating from \( r_0 \) to \( r \), one gets the time delay expression for light propagation,

\[
\Delta t = r_s \left( \ln \left( \frac{r + \sqrt{r^2 - r_0^2}}{r_0} \right) + \frac{(c_1 + 1)}{2} \sqrt{r - r_0} \right) + \cdots \tag{15}
\]

The second term in Eq. (15) depends on the choice of the value of \( c_1 \) and can be eliminated by a further coordinate transformation,

\[
r = r'' e^{\mu(r)/2} = 1 + c_1'/(r_s/r'') + \cdots \tag{16}
\]

Therefore, the coordinate independent prediction for time delay in general relativity should be

\[
\Delta t = r_s \ln \left( \frac{r + \sqrt{r^2 - r_0^2}}{r_0} \right) + \cdots \tag{17}
\]

in first order. This is the result also obtained by the PPN (Post Newtonian Method)[1], and agrees with the most recent observational data [2] with high accuracy (1 in 1000 accuracy). By comparing Eqs. (15) and (17), we conclude that the same coordinate independent result can be obtained by the condition,

\[
c_1 = -1. \tag{18}
\]

We note that the parameter values

\[
a_1 = -1, \text{ and } b_1 = 1 \tag{19}
\]

are coordinate independent and determined from the solution of the Einstein equation and the physical boundary condition. Thus we conclude that Eq. (18), along with Eq. (19), is the condition for the physical metric.
IV. THE PHYSICAL METRIC IN HIGHER ORDER

In order to determine the coefficients in higher order, \( c_n \), we consider time delay in the radial direction. For \( J = 0 \), one gets

\[
\frac{dt}{dr} = e^{-\nu(r)/\sqrt{e^{-\nu(r)-\lambda(r)}}} = \left( \frac{d}{dr}(re^{\mu(r)/2}) \right)/(1 - r_s/re^{\mu(r)/2}). \tag{20}
\]

Integrating this from \( r_1 \) to \( r_2 \), one gets the time difference

\[
\Delta t(r_1, r_2) = \int_{r_1}^{r_2} \frac{d}{dr}(re^{\mu(r)/2})/(1 - r_s/re^{\mu(r)/2})dr = [re^{\mu(r)/2} + r_s \ln(re^{\mu(r)/2} - r_s)]_{r_1}^{r_2} \tag{21}
\]

Expanding Eq. (21) in a power series in \( r_s/r \) (Eq. (4), one obtains

\[
\Delta t(r_1, r_2) = r_2 - r_1 + r_s \ln(r_2/r_1) + r_s^2(1/r_2 - 1/r_1)(c_2 - c_1^2/4 + c_1 - 2)/2 + r_s^3(1/r_2^2 - 1/r_1^2)(c_3 + c_2 - c_1c_2/2 + c_1^3/8 - c_1^2/2 + c_1 - 1)/2 + \cdots \tag{22}
\]

Applying further coordinate transformations, Eq. (16), to Eq. (22), one finds the coordinate independent result to be

\[
\Delta t(r_1, r_2) = r_2 - r_1 + r_s \ln(r_2/r_1) \tag{23}
\]

and the parameters for the physical metric are determined as

\[
c_2 = c_1^2/4 - c_1 + 2 = 13/4,
\]

\[
c_3 = -c_2 + c_1c_2/2 - c_1^2/8 + c_1^2/2 - c_1 + 1 = -9/4, \quad etc., \tag{24}
\]

where Eq. (18) has been used. Successive expansion yields a determination of all the parameters, \( c_n \), for the physical metric.

V. THE PHYSICAL METRIC IN COMPACT FORM

From Eq. (21) and Eq. (23), it follows that

\[
re^{\mu(r)/2} + r_s \ln(re^{\mu(r)/2} - r_s) - r - r_s \ln r = \text{const}, \tag{25}
\]

where the constant can be determined from the asymptotic form, Eq. (4), to be

\[
\text{const} = (c_1/2)r_s = -(1/2)r_s. \tag{26}
\]
Then, Eq. (??) can be transformed into

\[ f(x) - 1 + x/2 + x \ln(f(x) - x) = 0, \quad (27) \]

where

\[ x = r_s/r \quad (28) \]

and

\[ f(x) = e^{\mu(r)/2}. \quad (29) \]

The solution of Eq. (27) is given by

\[ f(x) = x(1 + \text{LambertW}\left(\frac{e^{-1.5-x}}{x}\right)), \quad (30) \]

where the Lambert W function is the solution of the equation

\[ W(z)e^{W(z)} = z. \quad (31) \]

The compact form for the angular metric function in the physical metric, Eq. (27) or Eq. (30) can be used to calculate an asymptotic expansion or an expansion at the origin. A tedious but doable calculation yields

\[ re^{\mu(r)/2} = rf(r_s/r) \]

\[ = r(1 - \frac{1}{2}(r_s/r) + \frac{3}{2}(r_s/r)^2 - \frac{3}{8}(r_s/r)^3 + \cdots) \quad (32) \]

for the asymptotic form and

\[ re^{\mu(r)/2} = r_s(1 + \alpha_0(r/r_s) + (1 - \alpha_0)((r/r_s)^2 + \frac{1}{2}(1 - 3\alpha_0)(r/r_s)^3 + \cdots))) \]

\[ = r_s(1 + .2231301601(r/r_s) + .1733430918(r/r_s)^2 + .02865443810(r/r_s)^3 + \cdots) \quad (34) \]

for the expansion at the origin, where

\[ \alpha_0 = e^{-\frac{3}{2}} = .2231301601. \quad (36) \]

Using Eqs. (2), (3) and (33-36), one can express the metric functions in asymptotic form and expanded at the origin,

\[ e^\nu(r) = 1 - r_s/re^{\mu(r)/2} = 1 - r_s/r - \frac{1}{2}(r_s/r)^2 + \frac{5}{4}(r_s/r)^3 + \cdots \]

\[ = \alpha_0(r/r_s)(1 + (1 - 2\alpha_0)(r/r_s) + \frac{1}{2}(1 - 8\alpha_0 + 9\alpha_0^2)(r/r_s)^2 + \cdots) \quad (38) \]

\[ = .2231301601(r/r_s) + .1235560234(r/r_s)^2 - .03759270901(r/r_s)^3 + \cdots, \quad (39) \]
\[ e^{\lambda(r)} = \left( \frac{d}{dr} \left( r e^{\mu(r)/2} \right) \right)^2 \left/ \left( 1 - \frac{r_s}{r} e^{\mu(r)/2} \right) \right. = 1 + \frac{r_s}{r} - \frac{3}{2} \left( \frac{r_s}{r} \right)^2 - \frac{3}{4} \left( \frac{r_s}{r} \right)^3 + \cdots \] (40)

\[ = \alpha_0 \left( \frac{r_s}{r} \right) \left( 1 + (3 - 2\alpha_0) \left( \frac{r_s}{r} \right) + \frac{1}{2} \left( 7 - 16\alpha_0 + 9\alpha_0^2 \right) \left( \frac{r_s}{r} \right)^2 + \cdots \right) \] (41)

\[ = 0.2231301601 \left( \frac{r_s}{r} \right) + 0.5698163437 + 0.4326494982 \left( \frac{r_s}{r} \right) + \cdots \] (42)

\[ e^{\mu(r)} = \left( \frac{r e^{\mu(r)/2}}{r} \right)^2 = 1 - \frac{13}{4} \left( \frac{r_s}{r} \right)^2 - \frac{9}{4} \left( \frac{r_s}{r} \right)^3 + \cdots \] (43)

\[ = \left( \frac{r_s}{r} \right)^2 \left( 1 + \alpha_0 \left( 2r/r_s + (2 - \alpha_0) \left( (r/r_s)^2 + (1 - \alpha_0)^2 (r/r_s)^3 + \cdots \right) \right) \right) \] (44)

\[ = \left( \frac{r_s}{r} \right)^2 + 0.4462603202 \left( \frac{r_s}{r} \right) + 0.3964732520 + 0.1346650199 \left( \frac{r_s}{r} \right) + \cdots . \] (45)

VI. THE NATURE OF THE PHYSICAL METRIC

The metric functions for the physical metric and the coordinate transformation from the Schwarzschild metric, Eq. [33], are now shown to be positive definite. In order to show this property, the solution of Eq. (27) must satisfy the condition,

\[ f(x) \geq x, \] (46)

or equivalently

\[ r e^{\mu(r)/2} \geq r_s. \] (47)

The equality in Eq. (47) is valid only at \( r = 0 \), as is seen in Eq. (35). From this condition, it follows that

\[ e^{\nu(r)} = 1 - \left( \frac{r_s}{r} \right) e^{-\mu(r)/2} \geq 0, \] (48)

where equality is valid only at the origin, \( r = 0 \). As a result, the positive definiteness of \( e^{\lambda(r)} \) follows easily,

\[ e^{\lambda(r)} = \left( \frac{d}{dr} \left( r e^{\mu(r)/2} \right) \right)^2 \left/ \left( 1 - \left( r_s/r \right) e^{\mu(r)/2} \right) \right. > 0. \] (49)

A further relationship among the metric functions follows from the generation equation, Eq. (27). Rewriting it as

\[ r e^{\mu(r)/2} - r + r_s/2 + r_s \ln \left( \left( r e^{\mu(r)/2} - r_s \right)/r \right) = 0, \] (50)

and differentiating it yields the relationship,

\[ \left( r e^{\mu(r)/2} \right)^\prime = \left( 1 - \left( r_s/r \right) e^{-\mu(r)/2} \right) \left( 1 + r_s/r \right) = e^{\nu(r)} \left( 1 + r_s/r \right). \] (51)
Hence one gets
\[ e^\lambda(r) = (1 - (r_s/r)e^{-\mu(r)/2})(1 + r_s/r)^2 = e^\nu(r)(1 + r_s/r)^2. \]  
(52)

This confirms the positive definiteness of \( e^\lambda(r) \). From
\[ (e^\nu(r))^\prime = \frac{r_s}{(r e^{\mu(r)/2})^2}(r e^{\mu(r)/2})^\prime - \frac{r_s}{(r e^{\mu(r)/2})^2} e^\nu(r)(1 + r_s/r) \]
and Eq. (51), it follows that \( e^\nu(r) \) and \( r e^{\mu(r)/2} \) are monotonically increasing functions with \( 0 \leq e^\nu(r) \ll 1 \) and \( r e^{\mu(r)/2} \gg r_s \).

From Eq. (53) and Eq. (52), it is easily seen that there should exist an inflection point for \( e^\nu(r) \). By a numerical computation, it can be shown that \( e^\nu(r) \) has a single inflection point at \( r = 0.65926 r_s \).

It is now clear that the metric functions in the physical metric are positive definite. The time metric function, \( g_{00} = e^\nu(r) \), vanishes at the origin. In other words, there is no horizon at finite distance in this metric. One may say that the horizon coincides with the origin. This is not surprising, since there is no trace of a horizon in the invariants made from the Riemann curvature tensors, \( R_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta} \). In the Schwarzschild metric, the latter is \( 12 r_s^2 / r^6 \).

Therefore, in the physical metric one obtains
\[ R_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta} = 12 r_s^2 / (r e^{\mu(r)/2})^6. \]  
(54)

This quantity neither vanishes nor has a singularity at a finite distance. The author will come back to this issue in the last discussion section.

VII. GRAVITATIONAL RED SHIFT

In the physical metric, the formula for the gravitational red shift is obtained from the proper time expression,
\[ d\tau_p = e^{\nu(r)/2} dt \]
\[ = \sqrt{1 - r_s/r - \frac{1}{2}(r_s/r)^2 + \frac{5}{4}(r_s/r)^3 + \cdots} dt. \]  
(56)

Since \( e^{\nu(r)} \) is positive definite, Eq. (56) is valid for all values of \( r \), in contradistinction to the case of the Schwarzschild metric. Of course, near the origin, the expansion at the origin, Eq. (39), should be used. It is worthwhile to point out that Eq. (56) is the first general relativity prediction in higher order for the gravitational red shift, which can be utilized
for experimental tests in the future. How likely is it to be used in the near future? The values of the parameter, $r_s/r$, at the surface of a white dwarf and the sun are $10^{-3}$ and $10^{-5}$ respectively. It is, therefore, not wild imagination that second order effects could be detected in gravitational red shifts in atomic spectra from white dwarfs or the sun sometime in the near future.

VIII. BENDING OF LIGHT IN HIGHER ORDER

The bending of light can be obtained from

$$\pi + \Delta \phi = 2 \int \frac{e^{-\mu(r)/2}}{r^2} \sqrt{e^{-\nu(r)/J^2} - e^{-\mu(r)/2}} dr$$

$$= 2 \int_{r_0}^{\infty} \frac{d}{dr} \left( r e^{\mu(r)/2} \right) dr \left/ \left( r e^{\mu(r)/2} \right)^2 \sqrt{1/J^2 - (1 - r_s/(r e^{\mu(r)/2}))/\left( r e^{\mu(r)/2} \right)^2} \right. \right. \right. \right.$$  \hspace{1cm} (57)

Changing the integration variable to

$$u = 1/(r e^{\mu(r)/2}),$$  \hspace{1cm} (59)

one obtains

$$\pi + \Delta \phi = 2 \int_0^{u_0} du \sqrt{u_0^2(1 - r_s u_0) - u^2(1 - r_s u)},$$  \hspace{1cm} (60)

where

$$u_0 = 1/(r_0 e^{\mu(r_0)/2}).$$  \hspace{1cm} (61)

Defining

$$u = u_0 s$$  \hspace{1cm} (62)

and

$$k = r_s u_0 = r_s/(r_0 e^{\mu(r_0)/2}),$$  \hspace{1cm} (63)

one gets

$$\pi + \Delta \phi = 2 \int_0^1 ds \sqrt{(1 - s^2)(1 - \kappa(1/s + s)).}$$  \hspace{1cm} (64)

Expanding in a power series in $\kappa$, one gets

$$\Delta \phi = 2 \sum_{n=1}^{\infty} \frac{(2n - 1)!!}{2^n n!} \kappa^n \int_0^1 \frac{1}{\sqrt{1 - s^2}} \frac{1}{1 + s + s^n}. \hspace{1cm} (65)$$
Defining the integrals,

\[ I_n = \int_0^1 \frac{1}{\sqrt{1-s^2}} \left( \frac{1}{1+s} + s \right)^n ds, \]  
\[ A_n = \int_0^1 \frac{1}{\sqrt{1-s^2}} \left( \frac{1}{1+s} \right)(\frac{1}{1+s} + s)^n ds \]  

and

\[ B_n = \int_0^1 \frac{s}{\sqrt{1-s^2}} \left( \frac{1}{1+s} + s \right)^n ds, \]

one gets recursion formulas

\[ I_n = A_{n-1} + B_{n-1}, \]  
\[ A_n = \frac{1}{2n+1}(1-nI_n + 3nI_{n-1}) \]

and

\[ B_n = \frac{1}{n+1}(1-3nA_{n-1} + 3nI_{n-1}). \]

Using

\[ A_0 = B_0 = 1, \]

and

\[ I_0 = \frac{\pi}{2}, \]

repeated application of the recursion formulas yields

\[ I_1 = 2, \quad I_2 = -\frac{4}{3} + \frac{5}{4}\pi, \quad I_3 = \frac{122}{15} - \frac{3}{2}\pi, \quad I_4 = -\frac{104}{7} + \frac{99}{16}\pi, \ldots. \]  

With the additional use of the expression

\[ \kappa = \frac{r_s}{r_0 e^{\mu(r_0)/2}} \]
\[ = (r_s/r_0)/(1 - \frac{1}{2}(r_s/r_0) + \frac{3}{2}(r_s/r_0)^2 - \frac{3}{8}(r_s/r_0)^3 - \frac{3}{4}(r_s/r_0)^4 + \frac{21}{64}(r_s/r_0)^5 + \frac{291}{320}(r_s/r_0)^6 + \cdots) \]
\[ = (r_s/r_0)(1 + \frac{1}{2}(r_s/r_0) - \frac{5}{4}(r_s/r_0)^2 - (r_s/r_0)^3 + \frac{37}{16}(r_s/r_0)^4 + \frac{143}{64}(r_s/r_0)^5 - \frac{379}{80}(r_s/r_0)^6 + \cdots), \]

one gets a formula for higher order corrections to the bending of light,

\[ \Delta \phi = 2(r_s/r_0) + \frac{15}{16}\pi(r_s/r_0)^2 + \frac{19}{12}(r_s/r_0)^3 - \left( \frac{1}{4} + \frac{135}{1024}\pi \right)(r_s/r_0)^4 + \cdots. \]
The second order correction term is consistent with that obtained in the references[5] that use the impact parameter method. It is an interesting question whether the two methods, one using the physical metric and the other using the impact parameter, give the same result for higher order corrections. If they are different, the natural question is which method gives the correct answer. Of course, it is difficult to answer such a question based on the experimental data for general relativity tests, since one requires high precision experiments to do that. It is also an interesting question whether the impact parameter method yields a result that implies the absence of a horizon, as was shown for the physical metric method.

IX. PERIOD OF LIGHT IN A CIRCULAR ORBIT

In this section, it is shown that the period of light in a circular orbit is coordinate independent.[6] From Eq. (64), it is clear that the integral diverges if the value of \( \kappa \) approaches

\[
\kappa = \frac{2}{3}. \tag{79}
\]

In other words, light makes a circular orbit at

\[
\kappa = r_s/(r e^{\mu(r)/2}) = \frac{2}{3}. \tag{80}
\]

Using the fact that light in a circular orbit satisfies the equation

\[
\frac{dt}{d\phi} = \frac{r e^{\mu(r)/2}}{e^{\nu(r)/2}} = \frac{r e^{\mu(r)/2}}{\sqrt{1 - \frac{r_s}{r e^{\mu(r)/2}}}} = \frac{3\sqrt{3}}{2} r_s, \tag{81}
\]

the period for circular motion is given by

\[
T = \int_0^{2\pi} \frac{dt}{d\phi} d\phi = 2\pi \frac{3\sqrt{3}}{2} r_s. \tag{82}
\]

This result is obviously coordinate independent.

The solution of Eq. (80) yields the radius of the circular orbit for the particular metric. For the Schwarzschild metric \( (e^{\mu(r)/2} = 1) \), one has

\[
r = \frac{3}{2} r_s, \tag{83}
\]

while for the Eddington metric \( (e^{\mu(r)/2} = (1 + r_s/4r)^2) \), one gets

\[
r = \frac{(2 + \sqrt{3})}{4} r_s. \tag{84}
\]
For the physical metric, from Eq. (80) it follows that

$$f(x) = \frac{3}{2}x. \tag{85}$$

Combining Eq. (27) and Eq. (85), one gets the equation

$$e^{2-1/x} = 2/x, \tag{86}$$

and the solution

$$r = 1.15916r_s. \tag{87}$$

The meaning of Eqs. (81) and (82) may be understood as

$$T = \frac{2\pi r}{e^{\nu(r)/2}/e^{\mu(r)/2}} = \frac{2\pi r}{(\text{speed of light})}, \tag{88}$$

where

$$(\text{speed of light}) = \frac{rd\phi}{dt} = e^{\nu(r)/2}/e^{\mu(r)/2} \tag{89}$$

is measured by a clock at infinite distance.

**X. DISCUSSION**

What is the meaning of the physical metric other than that it gives coordinate invariant results and that it gives a prediction consistent with observations, at least in first order of the gravitational constant? The reason for the positive definiteness of the metric functions of the physical metric is understood easily: The transformation from $r$ to the Schwarzschild coordinate,

$$r' = r e^{\mu(r)/2}, \tag{90}$$

or rather the inverse of it is a map of the outside of the Schwarzschild spacetime onto the whole space of the physical metric. This can be seen also from Eqs. (34) or (35) and

$$\lim_{r\to0}(r e^{\mu(r)/2}) = r_s. \tag{91}$$

In other words, Eq. (90) is a map from the physical metric space to the outside of the Schwarzschild radius. It is not intended, but the requirement of coordinate independence forces us to choose this mapping. One can present two possible interpretations of this result.

[A]. An observer outside the horizon (the Schwarzschild radius) can see only the outside, but cannot see inside the horizon, since the speed of light vanishes at the horizon and
nothing can come out from the horizon in the usual interpretation of black holes. To the extent that the requirement of coordinate independence leads to physically observable consequences, (which has been proved only in first order of the gravity,) the nature of the map between the physical metric space and the Schwarzschild spacetime, Eqs. (90) and (91), is comprehensible. In order to see the whole picture of the spacetime, one has to make a map that opens up the part of the spacetime inside the horizon.

[B]. The spacetime of the physical metric is a real coordinate system that describes nature, since that is all one can get by observations. The spacetime that is opened up by a coordinate transformation is an artifact of the coordinate transformation. In this view, the horizon and anything that comes out from it has no reality. That includes Hawking radiation, time tunneling and the information paradox.

A response from the viewpoint [A] against the viewpoint [B] could be that sometimes poles in the complex plane of a physical quantity has physical reality. In a S-matrix theory, poles in the unphysical region for physical parameter spaces correspond to observable particles or bound states. Then, the viewpoint [B] would say that yes, then one has to discover a horizon as a physical observable quantity. And a dispute continues.

The only way to resolve these two viewpoints is to find direct or indirect observational evidences to support its viewpoint. In order to verify the viewpoint [A], an observation of Hawking radiations is essential. Measurements of the Hawking radiations from primordial black holes are important projects that can prove viewpoint [A]. Or one may create a method to find an evidence for the existence of horizon from some other phenomena. So far, evidences for black holes in quasars or active galactic nuclei come from phenomena related to accretion disks around massive objects. The existence of compact massive objects does not differentiate the both viewpoint, [A] or [B], since such a phenomena can be claimed from the both viewpoints.
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[1] C. W. Misner, K. S. Thorne and J. A. Wheeler, Gravitation, (Freeman, San Francisco, 1973); S. Weinberg, Gravitation and Cosmology, (Wiley and Sons, New York, 1972)

[2] R. D. Resenberg and I. I. Shapiro, Apj. 234, L219 (1979)

[3] R. M. Corless et. al., Advances in Comm. Math. 5, 329 (1996)

[4] Y. Tomozawa, Speed of Light in Gravitational Fields, arXiv astro-ph/0303047 (2003)

[5] D. K. Ross and L. I. Schiff, Phys. Rev. 141, 1215 (1966); I. I. Shapiro, Phys. Rev. 145, 1005 (1966); S. Ichinose and Y. Kaminaga, Phys. Rev. D 40,3997 (1989); J. Bodenner and C. M. Will, Am. J. Phys. 71, 70 (2003)

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