On the self-similar motion of a gravitating Chaplygin fluid

Vladimir Dzhunushaliev\textsuperscript{a,b,c}, Vladimir Folomeev\textsuperscript{b,c,e}, Ratbay Myrzakulov\textsuperscript{d}

\textsuperscript{a} Institut für Physik, Universität Oldenburg, Postfach 2503 26111 Oldenburg, Germany
\textsuperscript{b} Institute for Basic Research, Eurasian National University, Astana 010008, Kazakhstan
\textsuperscript{c} Institute of Physicotechnical Problems and Material Science of the NAS of the Kyrgyz Republic, 265 a, Chui Street, Bishkek 720071, Kyrgyzstan
\textsuperscript{d} Department of General and Theoretical Physics, Eurasian National University, Astana 010008, Kazakhstan

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\textbf{A B S T R A C T}

A self-similar motion of the generalized Chaplygin gas in its own gravitational field is considered. The problem is studied numerically and analytically (in the limiting cases). It is shown that the model under consideration admits only expanding solutions. As the astrophysical application of the model, a description of rotating curves of spiral galaxies is suggested by using the analytical solutions obtained in the Letter.

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1. Introduction

The problem of motion of a fluid in its own gravitational field (collapse) refers to key problems of astrophysics. Until the end of the 1990s, studies of models of compact objects and a collapse process have been basically concentrated on a consideration of models in which a fluid possesses a positive (or zero) pressure. However, the situation has changed after the discovery of the accelerated expansion of the present universe\cite{1,2}. It is now believed that such acceleration is caused by the presence in the universe of some antigravitating matter with the negative pressure $p < -1/3\varepsilon$, where $\varepsilon$ is the energy density (for a review, see e.g.\cite{3,4}). It is assumed that such matter, which is called dark energy, is distributed uniformly in space and provides more than 70\% of a total mass of the universe. However, a possibility is not excluded that dark energy could affect the formation of the structure of the universe on small scales of the order of galaxies, and may be even smaller (for a more detailed discussion of this question, see Ref.\cite{5}).

A more intriguing possibility is a description of dark energy and dark matter within the framework of some unified theory by using an effective hydrodynamical description of a dark fluid. One of variants of such a theory is a model of the Chaplygin gas\cite{6}. It is assumed in this model that the dark fluid is being described by the polytropic equation of state $p_{ch} \sim -\rho_{ch}^{\gamma}$, where $p$ and $\rho$ are the pressure and the energy density, respectively, and $\gamma$ is a negative constant. In a cosmological aspect, such equation of state gives the following scenario of the evolution of the universe: in the early stages of the expansion of the universe $\rho_{ch} \sim a^{-3}$ (here $a$ is the scale factor) that corresponds to the epoch of domination of non-relativistic (dustlike) dark matter. On the other hand, at the present time and later on such fluid acts as a cosmological constant, $\rho_{ch} \sim \text{const}$. This attractive feature allows to use such model for a description of the evolution of the universe from the epoch of formation of the large scale structure until the present time.

Possessing the properties of dark matter, the Chaplygin gas can form compact astrophysical objects such as stars and galaxies. Such objects had been already considered earlier. In particular, in Ref.\cite{7} the model of the dark energy star supported by the Chaplygin gas was considered within the framework of Newtonian gravity. Choosing various values of a central density of such configuration, sizes and masses of dark stars in the early universe were estimated. In general relativity, such stars were considered in detail in Refs.\cite{8,9}.

In this Letter we consider the motion of the Chaplygin gas in its own gravitational field within the framework of Newtonian gravity. Such model can be used for a description of evolution of the dark fluid inside galaxies. Since the sizes of such objects are significantly smaller than cosmological scales then one can use the Newtonian approximation. In the paper\cite{7} the parameters of the...
Chaplygin dark star were estimated, taking into account the cosmological evolution of the Chaplygin gas. Here we will estimate parameters of a motion of the Chaplygin gas creating a compact astrophysical object. To do this, we will consider a self-similar motion of the gas with the above equation of state. Such problems for self-gravitating gaseous spheres have been already considered by a number of authors (see e.g. earlier works [10–12], and also more recent papers [13,14] and references therein). In these works the main attention was called to the models of collapsing stars with the parameter γ > 0. It will be shown below that in the case of γ < 0 (the Chaplygin gas) self-similar solutions considered here describe only expanding configurations.

2. Equations and solutions

2.1. Similarity equations

We will consider the spherically symmetric motion of the Chaplygin gas in Newtonian gravity. The corresponding hydrodynamical equations can be written in the form [12]

\[
\frac{\partial M}{\partial t} + 4\pi r^2 \rho u = 0, \tag{1}
\]

\[
\frac{\partial M}{\partial t} = 4\pi r^2 \rho, \tag{2}
\]

\[
\frac{\partial u}{\partial t} + \frac{u}{\partial r} = -\frac{M}{\rho} - \frac{GM}{r^2}, \tag{3}
\]

where \( u \) is the radial velocity of the Chaplygin gas, \( M \) is the total mass of matter contained in the interval \((0, r)\), \( \rho \) and \( p \) are the density and the pressure of matter, respectively. In the case under consideration, the pressure \( p \) is small compared with the energy density \( \rho \). Eqs. (1)–(3) are invariant under the time reversal operation [10]:

\(
t \rightarrow -t, \quad u \rightarrow -u, \quad \rho \rightarrow \rho, \quad p \rightarrow p, \quad M \rightarrow M.
\)

Thus one can consider only the range of \( 0 < t < \infty \). Let us choose the polytropic equation of state of the generalized Chaplygin gas in the form

\[
p = -K \rho^\gamma, \tag{4}
\]

where \( K \) is some positive arbitrary constant and \( -1 \leq \gamma < 0 \). The case \( \gamma = -1 \) corresponds to the usual Chaplygin gas [6].

In this Letter, we will seek similarity solutions of the system of equations (1)–(3) with the equation of state (4). For this purpose, we introduce the following dimensionless similarity variables (for details, see [12]):

\[
x = \frac{r}{\sqrt{K} t^{\alpha}}, \quad u(r, t) = \sqrt{K} t^{1-n} v(x), \quad \rho(r, t) = \frac{\alpha(x)}{4\pi G t^2},
\]

\[
p(r, t) = \frac{k}{4\pi G t^{2n-4}} \left[ k \alpha(x) \right]^2,
\]

\[
M(r, t) = \frac{k^{1/2} t^{3n-2}}{(3n-2)G} m(x), \tag{5}
\]

where \( k \) is some dimensional constant. Using these similarity variables, Eqs. (1)–(3) can be rewritten in the form

\[
m(x) = \alpha^2 (nx - v),
\]

\[
(nx - v) \frac{\partial m}{\partial x} - \frac{\partial m}{\partial x} = -2x - v
\]

\[
-\gamma \alpha^\gamma - 2\frac{\partial m}{\partial x} (nx - v) \frac{\partial m}{\partial x} = -(n - 1) v - \frac{nx - v}{3n - 2} \alpha. \tag{8}
\]

The similarity variables (5) in general suppose that

\[
K = k(4\pi G)^{-1} t^{2(n+\gamma - 2)}. \tag{9}
\]

If we restrict ourselves to the case of constant \( K \) (as it is usually assumed at a consideration of the dark Chaplygin fluid), the parameter \( n \) must be restricted by the additional condition

\[
n = 2 - \gamma, \tag{10}
\]

or, taking into account (4), \( 2 < n \leq 3 \). For convenience of performing the mathematical analysis, let us rewrite Eqs. (7) and (8) in the following form

\[
\frac{dx}{dt} = \frac{\alpha}{(nx - v)^2 + \gamma \alpha^{\gamma - 1}} \times \left[ (n - 1) v + \frac{nx - v}{3n - 2} \right], \tag{11}
\]

\[
\frac{dv}{dx} = \frac{1}{(nx - v)^2 + \gamma \alpha^{\gamma - 1}} \times \left[ (n - 1)(nx - v) + \frac{(nx - v)^2}{3n - 2} \alpha + 2\gamma \frac{x - v}{x - \alpha^{\gamma - 1}} \right]. \tag{12}
\]

2.2. Limiting behavior

In general, the above equations correspond to a non-autonomous system which should be investigated numerically. However, let us estimate first the behavior of the system (11)–(12) near the origin of coordinates. We will follow Ref. [12]: we expand \( x \) and \( v \) in a Taylor series in the neighborhood of \( x = 0 \):

\[
\alpha(x) = \alpha_0 + \alpha_1 x + \alpha_2 x^2 + \cdots,
\]

\[
v(x) = v_0 + v_1 x + v_2 x^2 + \cdots. \tag{13}
\]

Substituting these expressions in Eqs. (11) and (12), one can obtain the following limiting solutions for \( x \ll 1 \) parameterized by \( \alpha_0^* \):

\[
v(x) = \frac{\alpha_0^*}{2 \gamma^{\gamma - 1}} \left( \alpha_0^* - \frac{2}{3} \right) \frac{nx - v}{3n - 2} x^3 + \cdots, \tag{14}
\]

\[
\alpha(x) = \alpha_0^* + \frac{\alpha_0^{2+n}}{6 \gamma} \left( \alpha_0^* - \frac{2}{3} \right) x^2 + \cdots. \tag{15}
\]

Note that Eqs. (7) and (8) have a particular solution with the constant density \( \alpha \)

\[
v = \frac{2}{3} x, \quad \alpha = \frac{2}{3}, \quad m = \frac{2}{3} \left( \frac{n - 2}{3} \right) x^3 \tag{16}
\]

corresponding to the homogeneous distribution of matter and describing expansion in Newtonian cosmology.

Using (14), (15) as initial conditions, let us solve Eqs. (11), (12) numerically. The results are presented in Fig. 1.

The numerical analysis indicates two regions of the solutions. In the first region \( \alpha_0 < 2/3 \). In this case there exists the critical point in which the numerators and denominators of Eqs. (11), (12) vanish simultaneously. The position of this point can be defined from the conditions:

\[
(nx - v) = \sqrt{-\gamma \alpha^{\gamma - 1}/2}, \tag{17}
\]

\[
(n - 1) v + \frac{nx - v}{3n - 2} \alpha - 2 \frac{(x - v)(nx - v)}{x} = 0. \tag{18}
\]

An asymptotic form of solutions of Eqs. (11), (12) at \( x \gg 1 \) is defined from the form of the numerical solutions. One can see from the numerical calculations that \( v \) changes more slowly than \( x \), and
Because of the above, let us use the regular similarity solutions (14), (15), (19) and (20) in the region $\alpha_+ < 2/3$ for purposes of estimation of the physical quantities (5) rewritten via the variables $(r,t)$. Using the definition for $x$ from (5), one can see that the limiting case $x \gg 1$ corresponds to the initial instant $t \to 0$ (with the arbitrary finite values of $r$). Then, using the expressions (19), (20), we have from (5)
\[ u(r, t) = Bk^{1/2}t^{(n-1)/n}, \quad \rho(r, t) = \frac{A_k^{1/n}}{4\pi G} r^{-2/n}. \] (21)
I.e., at the initial instant there is the static inhomogeneous distribution of the Chaplygin gas. On the other hand, the limiting case $x \ll 1$ corresponds to $t \to \infty$ at some finite $r$. The corresponding velocity and density of matter from (5) will then be
\[ u(r, t) = \frac{2}{3} r, \quad \rho(r, t) = \frac{\alpha_c}{4\pi G t^2}. \] (22)
I.e., asymptotically, as $t \to \infty$, the model describes the homogeneous distribution of matter which does not depend on the equation of state (i.e. on $n$). If one follows the motion of some layer of matter then after transition to the Lagrangian variables, Eq. (22) for the velocity $u$ can be easily integrated giving the following expression
\[ r = r_0^{2/3}, \] where $r_0, t_0$ are integration constants describing characteristic parameters of the configuration. In cosmology language, this equation corresponds to the motion of dust matter. I.e., within the framework of the model under consideration, the Chaplygin gas, creating the compact astrophysical object, describes asymptotically, as $t \to \infty$, a cloud of the dark fluid with the homogeneous distribution of matter with the dustlike equation of state. In this case the initially inhomogeneous distribution of matter (21) smooths asymptotically as $t \to \infty$.

3. Chaplygin gas in galaxies

Since the solutions (21) are self-similar, i.e. they are similar to each other at different times, then one can use these solutions for a description of the present distribution of the dark fluid in the universe. Since the Chaplygin gas may mimic dark matter, we want to apply the solution obtained above as a dynamical model of dark matter inside a spiral galaxy. We assume that dark matter inside a galaxy is a dynamical object evolving in time. We will apply the asymptotic solution (21) for a description of the dark matter distribution in a galaxy for $x \gg 1$ (that corresponds to $t \cong \text{const.}, r \gg \sqrt{k}r_0$). In our model a dark matter halo evolves in time, and we try to describe such evolution at some instant. As an example, let us consider a possibility in the use of the solutions (21) for modeling dark matter in spiral galaxies. For such galaxies there is the so-called Universal Rotation Curve defining the velocities of the motion of stars in a galaxy at any radius. The form of such curve can be represented by the following expression [15]
\[ V_{URC}(x) = V(R_{opt}) \left[ \left( 0.72 + 0.44\log \frac{L}{L_\odot} \right) \left( \frac{0.172 + 0.785}{x^2 + 1.52(L/L_\odot)^{0.04}} \right)^{1/2} \right] \] km s$^{-1}$, (23)
where $x = r/r_{opt}$, $R_{opt}$ is the radius encompassing 83% of the total integrated light, $L$ is the galaxy luminosity, and $\log L_\odot/L_\odot \approx 10.4$. As was pointed out in Ref. [15], Eq. (23) predicts rotation velocities at any radius with a typical accuracy of 4%. The first term in

Fig. 1. The graphs of velocities $v(x)$ (the solid lines) and densities $\alpha(x)$ (the dashed lines) at different initial values: $\alpha_0 = 0.6, 0.6, 0.5, 0.1$, from top to bottom, for both sets of the curves.
this formula gives the contribution from a stellar disc, and the second one – the contribution from a dark halo. For our purposes, let us use the second term. Then we have from (23) the following distribution of velocities which is defined by the dark halo

\[ V_{\text{DM}}^{\text{obs}}(r/R_{\text{opt}}) = V(R_{\text{opt}}) \sqrt{\frac{1.6e^{-0.4(L/L_*)}}{r^2 + 2.25(L/L_*)^{1/4}}} \]  

(24)

Let us now estimate velocities of test particles moving in the field of gravity of the dark halo created by the Chaplygin gas. The corresponding profile of the velocity distribution will then be given by the formula

\[ V_{\text{Chapl}}^{\text{obs}}(r) = \sqrt{\frac{GM(r)}{r}}. \]

The expression for the mass distribution of the dark fluid follows from Eq. (6), taking into account Eqs. (5) and (21), the expression for the mass profile at \( x \gg 1 \) will be

\[ M(r) = \frac{A_{\text{K}}^{1/n}}{(3n - 2)/G), \]

and the corresponding velocity distribution is

\[ V_{\text{Chapl}}^{\text{obs}}(r) = D r^{(n-1)/n}, \quad D \equiv \sqrt{\frac{A_{\text{K}}^{1/n}}{3n - 2}}. \]

For convenience of comparison of this expression with formula (24), let us rewrite it via the variables \( \tilde{x} \) and \( R_{\text{opt}} \) as

\[ V_{\text{DM}}^{\text{obs}}(\tilde{x}) = V(R_{\text{opt}}) \tilde{D} x^m, \]

where

\[ \tilde{D} = \frac{D R_{\text{opt}}^{(n-1)/n}}{V(R_{\text{opt}})}, \quad m = \frac{n - 1}{n}. \]

(25)

Further, choosing the parameters \( \tilde{D} \) and \( m \), one can compare Eq. (25) with formula (24). As an example, let us take \( L/L_* = 0.75 \). The corresponding parameters will then be: \( \tilde{D} \approx 0.59 \) and \( m \approx 0.67 \) (see Fig. 2). Since the parameter \( \tilde{D} \) contains the integration constant \( A \) then its value can be chosen arbitrarily. The value \( m \approx 0.67 \) corresponds to \( n \approx 3 \), i.e. \( \gamma \approx -1 \) (see Eq. (10)) that corresponds to the usual Chaplygin gas.

Summarizing the obtained results, we have considered the self-similar motion of the Chaplygin gas with the equation of state (4). For values of the parameters used in creating the cosmological models with the Chaplygin gas \((-1 < \gamma < 0)\), it was shown that only expanding solutions do exist. This is caused by the nature of the Chaplygin gas having a negative pressure. Within the framework of the model under consideration, the numerical analysis was performed, and also the analytical estimations of the behavior of the solutions were made for the limiting cases when the similarity variable \( x \gg 1 \) and \( x \ll 1 \) (see Eqs. (21) and (22), respectively). The case of \( x \gg 1 \), corresponding to the static distribution of the Chaplygin gas, has been applied for the evaluation of the velocity profile of stars in galaxies. Comparing the observational curve for a dark halo (24) and the curve for the Chaplygin gas (25), it was shown that they are in good agreement with each other at the corresponding choice of the fitting parameters \( \tilde{D} \) and \( m \) (see Fig. 2).

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