As mentioned in the main text, in the interiors of NSs, neutrons form Cooper pairs and exhibit superfluidity [21] (see Refs. [22], [S1, S2] for recent reviews). Migdal considered an $s$-wave ($^1S_0$; spin-singlet and $s$-wave with total angular momentum $J = 0$) paring [21] as conventional ($s$-wave) metallic superconductors. Although the $^1S_0$ channel is attractive dominantly in the low-density regime, it becomes repulsive in the high-density regime in the neutron star core. There, the attraction is provided by the $^3P_2$ channel with the spin-orbit ($LS$) force inducing the neutron $^3P_2$ (spin-triplet and $p$-wave with total angular momentum $J = 2$) superfluids [39-49, S3-S20]. The ground state at the weak coupling limit is in the so-called nematic phase [45, 46]. We also comment that, although the boundary between the $s$- and $p$-wave regions may not be sharp microscopically (in reality an overlap region is predicted to exist [S20]), from a macroscopic point of view one can assume the presence of such an interface.

The $p$-wave pairings are widely considered in condensed matter physics such as $p$-wave superconductors for which electrons form Cooper pairs and $^3$He superfluids for which $^3$He atoms form Cooper pairs. Recently, $p$-wave superconductors (superfluids) have attracted great interests as topological superconductors (superfluids) [S21, S22], allowing topologically protected Majorana fermions on the boundary surfaces and vortex cores. In fact, the $^3P_2$ neutron superfluids have also been shown to be topological superfluids admitting surface Majorana fermions [S16] and Majorana bound states in their vortex cores [53, 57]. Thus, neutron star cores may be the largest topological materials in our universe.

The order parameter for $p$-wave paring (for $^3P_2$ neutron superfluids) is described by a $3 \times 3$ traceless symmetric matrix $A$ with complex components [45, 46]. Then, vortices take the following forms: an IQV is represented as

$$A = e^{i\theta}A_0$$

(S1)

with a spatial angle $\theta$ ($0 \leq \theta < 2\pi$) around the vortex core [44, 46, 47, 56], where $A_0$ is a constant matrix representing the ground state configuration. On the other hand, a HQV takes the form of

$$A = e^{i\frac{\theta}{2}}O(\theta)A_0O^T(\theta), \quad O(\theta) = \begin{pmatrix} \cos \frac{\theta}{4} & \pm \sin \frac{\theta}{4} & 0 \\ \mp \sin \frac{\theta}{4} & \cos \frac{\theta}{4} & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

(S2)

with the $D_4$ biaxial nematic ground state $A_0 \sim \text{diag}(1, -1, 0)$, where the sign $\pm (\mp)$ corresponds to HQVs with different topological charges cancelling each other [52], denoted by red and blue in the main text. The exponential factors $e^{i\theta}$ and $e^{i\frac{\theta}{2}}$ for IQV and HQV are the origin of integer and half quantizations, respectively. Importantly, the HQVs were recently observed in $^3$He-A phase, as they were created with rotation or by the Kibble-Zurek mechanism and identified by the experimental technique on the nuclear magnetic resonance [S23].
2. A story of the boojum

Originally the boojum is a particular variety of the fictional animal species called \textit{snarks} created by Lewis Carroll in his nonsense poem “The Hunting of the Snark.” The similar structures found in helium superfluids were named boojums by Mermin [58]. See [S24] for the story how he made “boojum” an internationally accepted scientific term. Now boojums have been predicted to occur in $^3$He superfluids [S25] in particular at the A-B phase boundary [S26, S27], liquid crystals [S28], Bose-Einstein condensates [S29], quantum field theory [S30–S32] and high density quark matter relevant for neutron star cores [S33–S35].

3. Proposals for laboratory experiments

We further point out that our mechanism for glitches by vortex networks through boojums can be simulated by laboratory experiments. One is with ultracold atomic gases, a mixture of a scalar Bose-Einstein condensate (BEC) and a spin-2 nematic BEC [S36–S38] (see Ref. [S39] for a review of spinor BEC), where the Gross-Pitaevskii equation for the latter is the same as the Ginzburg-Landau equation for a $^3P_2$ superfluid: a scalar BEC sandwiched between two spin-2 BECs. The other is $^3$He superfluids with the A-B phase boundary on which one vortex in the A-phase is connected to two vortices in the B-phase through a boojum [S26], therefore an A-B-A phase configuration is relevant (a B-A-B phase configuration was experimentally realized [S27]).

4. Akaike Information Criterion (AIC)

The Akaike Information Criterion (AIC) is defined as

\[
AIC = 2k - 2\ln(\hat{L}),
\]

where $\hat{L}$ is the maximum of the likelihood function $L$ (often simply called the likelihood), which is formed from the joint probability distribution of a sample of data given a set of model parameter values; it is viewed and used as a function of the parameters given by the data sample. Generally, a lower AIC indicates a better fit.

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