MIRROR MANIFOLDS
AND TOPOLOGICAL FIELD THEORY

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ABSTRACT

These notes are devoted to sketching how some of the standard facts relevant to mirror symmetry and its applications can be naturally understood in the context of topological field theory. If $X$ is a Calabi-Yau manifold, the usual nonlinear sigma model governing maps of a Riemann surface $\Sigma$ to $X$ can be twisted in two ways to give topological field theories, which I call the $A$ model and the $B$ model. Mirror symmetry relates the $A$ model (of one Calabi-Yau manifold) to the $B$ model (of its mirror). The correlation functions of the $A$ and $B$ models can be computed, respectively, by counting rational curves and by calculating periods of differential forms. This can be proved as a consequence of a reduction to weak coupling (as in §3-4 of these notes) or by a sort of fixed point theorem for the Feynman path integral (see §5). The correlation functions of the twisted models coincide, as explained in §6, with certain matrix elements of the physical, untwisted model – namely those that determine the superpotential. The conventional moduli spaces of sigma models can be thickened, in the context of topological field theory, to extended moduli spaces, indicated in §7, which are probably the natural framework for understanding the still mysterious “mirror map” between moduli spaces.

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1. Introduction

The purpose of these notes is to explain aspects of the mirror manifold problem that can be naturally understood in the context of topological field theories. (For other aspects of the problem, and detailed references, the reader should consult other articles in this volume.) Most of what I will say can be found in the existing literature, but to isolate the facts most relevant to mirror symmetry may be useful. The points that I want to explain are as follows.

First, we will consider the standard supersymmetric nonlinear sigma model in two dimensions, governing maps of a Riemann surface $\Sigma$ to a target space which for our purposes will be a Kahler manifold $X$ of $c_1 = 0$. We will see that there are two different topological field theories that can be made by twisting the standard sigma model. There are not standard names for these topological field theories; I will call them the $A$ theory and the $B$ theory, or $A(X)$ and $B(X)$ when I want to specify the choice of $X$. The $A$ theory has been studied in detail in [1], where many facts sketched below are explained; the $B$ theory has been studied less intensively. (The $A$ and $B$ theories were discussed qualitatively, in a general context of $N = 2$ superconformal field theories, by Vafa in his lecture at this meeting [2].)

Unlike the ordinary supersymmetric nonlinear sigma model, the twisted models are “soluble” in the sense that the problem of computing all the physical observables can be reduced to classical questions in geometry. This is done by a sort of fixed point theorem in field space, which gives the following results: the correlation functions of the $A$ theory are determined by counting holomorphic maps $\Sigma \to X$ obeying various conditions; the correlation functions of the $B$ theory can be computed by calculating periods of classical differential forms. (More generally, “counting” rational curves must be replaced by computing the Euler class of the vector bundle of antighost zero modes, as I have explained elsewhere [3, §3.3]; this has been implemented in the mirror manifold problem by Aspinwall and Morrison [4].)

For the most direct physical applications one is not interested in the twisted $A$ or $B$ theories, but in the original, “physical” sigma model. However, there is one very important case in which physical observables coincide with observables of the $A$ and $B$ theories. This happens for the Yukawa couplings, which are certain quantities that one wishes to compute
for Σ of genus zero. In particular, the $27^3$ and $27^3$ Yukawa couplings of superstring models coincide with certain observables of the $A$ and $B$ theories, respectively. These Yukawa couplings are therefore determined by the fixed point theorem mentioned in the last paragraph; from this we recover results that were originally found by more detailed arguments, such as the fact that the $27^3$ Yukawa couplings have no quantum corrections, and the instanton sum formula [5] for the $27^3$ Yukawa couplings.

In §2 we recall the definition of the standard sigma model. The twisted $A$ and $B$ models are described in §3 and §4, along with an explanation of their main properties, including the reduction to instanton moduli spaces and to constant maps in the $A$ and $B$ models respectively. This latter reduction is reinterpreted as a sort of fixed point theorem in §5. In §6, I explain why certain observables of the standard physical sigma model coincide with observables of the twisted models. This occurs whenever the canonical bundle of the Riemann surface becomes trivial after deleting the points at which fermion vertex operators have been inserted – in practice, mainly for amplitudes in genus zero with precisely two fermions. In §7 – the only part of these notes containing some novelty – I look a little more closely at the $A$ and $B$ models and describe the full families of topological field theories of which they are part. I strongly suspect that many properties of mirror symmetry that are not now well understood, including the structure of the mirror map between the parameter spaces, can be better understood in looking at the full topological families.

Since “elliptic genera” (see [6]) arise in the same supersymmetric nonlinear sigma models that are the basis for the study of mirror manifolds, I will also along the way make a few observations about them. In particular we will note the easy fact that if $X, Y$ are a mirror pair, they have the same elliptic genus. This is trivial in complex dimension three (since the elliptic genus of a three dimensional Calabi-Yau manifold is zero), but becomes interesting in higher dimension.

Perhaps I should emphasize that if $X$ and $Y$ are a mirror pair, then mirror symmetry relates all observables of the sigma model on $X$ to corresponding observables on $Y$ – and not just the few observables that can be related naturally to the twisted models and thus to topological field theory. The literature on mirror symmetry has focussed on these particular

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1 The restriction to genus zero appears in two ways: the superpotential that determines the Yukawa couplings has no higher genus corrections because of nonrenormalization theorems [5]; alternatively, in relating the superpotential to an observable of the twisted model, we will require in §6 that the canonical bundle of $\Sigma$ with two points deleted be trivial, which is of course true only in genus zero.
observables, which of course are also the ones we will be studying here, because of their phenomenological importance, and because, since the $B$ model is soluble classically, the relation given by mirror symmetry between observables of the $A(X)$ model and observables of the $B(Y)$ model is particularly useful.

2. Preliminaries

To begin with, we recall the standard supersymmetric nonlinear sigma model in two dimensions. It governs maps $\Phi : \Sigma \to X$, with $\Sigma$ being a Riemann surface and $X$ a Riemannian manifold of metric $g$. If we pick local coordinates $z, \bar{z}$ on $\Sigma$ and $\phi^I$ on $X$, then $\Phi$ can be described locally via functions $\phi^I(z, \bar{z})$. Let $K$ and $\overline{K}$ be the canonical and anti-canonical line bundles of $\Sigma$ (the bundles of one forms of types $(1,0)$ and $(0,1)$, respectively), and let $K^{1/2}$ and $\overline{K}^{1/2}$ be square roots of these. Let $TX$ be the complexified tangent bundle of $X$. The fermi fields of the model are $\psi^I_+$, a section of $K^{1/2} \otimes \Phi^*(TX)$, and $\psi^I_-$, a section of $\overline{K}^{1/2} \otimes \Phi^*(TX)$. The Lagrangian is

$$L = 2t \int d^2z \left( \frac{1}{2} g_{IJ}(\Phi) \partial_z \phi^I \partial_{\bar{z}} \phi^J + \frac{i}{2} g_{IJ} \psi^I_+ D_{\bar{z}} \psi^J_+ ight. + \left. \frac{i}{2} g_{IJ} \psi^I_- D_z \psi^J_- + \frac{1}{4} R_{IJKL} \psi^I_+ \psi^J_+ \psi^K_+ \psi^L_- \right).$$

(2.1)

Here $t$ is a coupling constant, and $R_{IJKL}$ is the Riemann tensor of $X$. $D_{\bar{z}}$ is the $\bar{\partial}$ operator on $K^{1/2} \otimes \Phi^*(TX)$ constructed using the pullback of the Levi-Civita connection on $TX$. In formulas (using a local holomorphic trivialization of $K^{1/2}$),

$$D_{\bar{z}} \psi^I_+ = \frac{\partial}{\partial \bar{z}} \psi^I_+ + \frac{\partial \phi^J}{\partial \bar{z}} \Gamma^I_{JK} \psi^K_+,$$

(2.2)

with $\Gamma^I_{JK}$ the affine connection of $X$. Similarly $D_z$ is the $\partial$ operator on $\overline{K}^{1/2} \otimes \Phi^*(TX)$.

The supersymmetries of the model are generated by infinitesimal transformations

$$\delta \Phi^I = i \epsilon_- \psi^I_+ + i \epsilon_+ \psi^I_-$$

$$\delta \psi^I_+ = -\epsilon_- \partial_{\bar{z}} \phi^I - i \epsilon_+ \psi^K_+ \Gamma^I_{KM} \psi^M_+$$

$$\delta \psi^I_- = -\epsilon_+ \partial_z \phi^I - i \epsilon_- \psi^K_- \Gamma^I_{KM} \psi^M_-,$$

(2.3)

This discussion will be at the classical level, and we will not worry about the anomalies that arise and spoil some assertions if the target space is not a Calabi-Yau manifold.

Here $d^2z$ is the measure $-idz \wedge d\bar{z}$. Thus if $a$ and $b$ are one forms, $\int a \wedge b = i \int d^2z (a_z b_{\bar{z}} - a_{\bar{z}} b_z)$. The Hodge $\ast$ operator is defined by $\ast dz = idz$, $\ast d\bar{z} = -id\bar{z}$.
where $\epsilon_-$ is a holomorphic section of $K^{-1/2}$, and $\epsilon_+$ is an antiholomorphic section of $\overline{K}^{-1/2}$. The formulas for the Lagrangian (2.1) and the transformation laws (2.3) have a natural interpretation upon formulating the model in superspace, that is in terms of maps of a super-Riemann surface to $X$. I will not discuss this here; for references see Rocek’s lecture in this volume.

As a small digression, let me note that usually, in discussing superconformal symmetry, one considers $\epsilon$’s that are defined locally (say in a neighborhood of a circle $C \subset \Sigma$ if one is studying the Hilbert spaces obtained by quantization on $C$). One might wonder what happens if we can find global $\epsilon$’s. On a Riemann surface of genus $g > 1$, this will not occur as there are no holomorphic sections of $K^{-1/2}$. In genus one $K$ is trivial; if we pick $K^{1/2}$ to be trivial, then there is a one dimensional space of global $\epsilon_-$’s. On the other hand, if we pick $\overline{K}^{1/2}$ to be a non-trivial line bundle (of order two) then globally $\epsilon_+$ must vanish. With the techniques that we will use below, one can readily use the symmetry generated by $\epsilon_-$ to prove that the partition function of the sigma model, on such a genus one surface, is independent of the metric of $X$ and so is a topological invariant in the target space; it is in fact the elliptic genus of $X$. (See [5] for an introduction to elliptic genera, and my lecture in that volume for an explanation of the field theoretic approach to them.) As the existence of a holomorphic section of $K^{-1/2}$ was essential here, this construction will not generalize to genus $g > 1$.

The twisted models that I will explain below and which will be the basis for whatever I have to say about mirror manifolds are the closest analogs of the usual supersymmetric sigma model for which global fermionic symmetries exist regardless of the genus. In particular the closest analog of the elliptic genus for $g > 1$ involves the half-twisted model introduced at the end of this section.

2.1. Kahler Manifolds And The Twisted Models

We now wish to describe the additional structure – $N = 2$ supersymmetry, to be precise – that arises if $X$ is a Kahler manifold. In this case, local complex coordinates on $X$ will be denoted as $\phi^i$; their complex conjugates are $\bar{\phi}^i = \overline{\phi^i}$. ($\phi^I$ will still denote local real coordinates, say the real and imaginary parts of the $\phi^i$.) The complexified tangent bundle $TX$ of $X$ has a decomposition as $TX = T^{1,0}X \oplus T^{0,1}X$. The projections of $\psi_+$ in $K^{1/2} \otimes \Phi^*(T^{1,0}X)$ and $K^{1/2} \otimes \Phi^*(T^{0,1}X)$, respectively, will be denoted as $\psi_+^i$ and $\psi_-^i$.
Likewise the projections of $\psi_-$ in $\overline{K}^{1/2} \otimes \Phi^*(T^{1.0} X)$ and $\overline{K}^{1/2} \otimes \Phi^*(T^{0.1} X)$ will be denoted as $\psi_1^-$ and $\psi_2^-$, respectively. The Lagrangian can be written

$$L = 2t \int d^2 z \left( \frac{1}{2} g_{ij} \partial_z \phi^i \partial_z \phi^j + i \bar{\psi}_-^i D_z \psi_-^i g_{ii} + i \bar{\psi}_+^i D_z \psi_+^i + R_{iijj} \bar{\psi}_+^i \psi_+^j \bar{\psi}_-^j \right).$$

(2.4)

As for the fermionic symmetries, these are twice as numerous as before because of the Kahler structure; this is analogous to the decomposition of the exterior derivative on a complex manifold as $d = \overline{\partial} + \partial$. I will write the following formulas out in detail because we will need various specializations of them in describing the twisted models. In terms of infinitesimal fermionic parameters $\alpha_-, \bar{\alpha}_-$ (which are holomorphic sections of $K^{-1/2}$) and $\alpha_+, \bar{\alpha}_+$ (antiholomorphic sections of $\overline{K}^{-1/2}$), the transformation laws are

\[
\begin{align*}
\delta \phi^i &= i \alpha_- \psi_+^i + i \alpha_+ \psi_-^i \\
\delta \bar{\psi}_+^i &= i \bar{\alpha}_- \psi_+^i + i \bar{\alpha}_+ \psi_-^i \\
\delta \psi_+^i &= -\bar{\alpha}_- \partial_z \phi^i - i \alpha_+ \psi_-^j \Gamma^i_{jm} \psi_+^m \\
\delta \bar{\psi}_-^i &= -\alpha_- \partial_z \phi^i - i \bar{\alpha}_+ \psi_-^j \Gamma^i_{jm} \psi_-^m \\
\delta \psi_-^i &= -\bar{\alpha}_+ \partial_z \phi^i - i \alpha_- \psi_+^j \Gamma^i_{jm} \psi_-^m \\
\delta \bar{\psi}_+^i &= -\alpha_+ \partial_z \phi^i - i \bar{\alpha}_- \psi_-^j \Gamma^i_{jm} \psi_-^m.
\end{align*}
\]

(2.5)

Now, the twisted models are constructed as follows:

(1) Instead of taking $\psi_+^i$ and $\psi_-^i$ to be sections of $K^{1/2} \otimes \Phi^*(T^{1.0} X)$ and $K^{1/2} \otimes \Phi^*(T^{0.1} X)$, respectively, we take them to be sections of $\Phi^*(T^{1.0} X)$ and $K \otimes \Phi^*(T^{0.1} X)$, respectively. I will call this a $+$ twist. The terms in the Lagrangian containing $\psi_+$ are unchanged (except that $D_z$ must now be interpreted as the $\overline{\partial}$ operator of the appropriate bundle). Alternatively, we can make what I will call a $-$ twist, taking $\psi_+^i$ and $\psi_-^i$ to be sections of $K \otimes \Phi^*(T^{1.0} X)$ and $\Phi^*(T^{0.1} X)$, respectively.

(2) Similarly, we twist $\psi_-$ by either a $+$ twist, taking $\psi_+^i$ to be a section of $\Phi^*(T^{1.0} X)$ and $\psi_-^i$ to be a section of $\overline{K} \otimes \Phi^*(T^{0.1} X)$, or a $-$ twist, taking $\psi_-^i$ to be a section of $\overline{K} \otimes \Phi^*(T^{1.0} X)$ and $\psi_-^i$ to be a section of $\Phi^*(T^{0.1} X)$. Again the Lagrangian is unchanged ($D_z$ is now interpreted as the $\partial$ operator of the appropriate bundle).

If one is twisting only $\psi_+$ or only $\psi_-$, it does not much matter if one makes a $+$ twist or a $-$ twist. The two choices differ by a reversal of the complex structure of $X$, and we are interested in considering all possible complex structures anyway. However, when we twist both $\psi_+$ and $\psi_-$, there are two essentially different theories that can be constructed.
By making a + twist of $\psi_+$ and a − twist of $\psi_−$, we make what I will call the $A$ theory. By making − twists of both $\psi_+$ and $\psi_−$ we make what I will call the $B$ theory. When I want to make the dependence on $X$ explicit, I will call these theories $A(X)$ and $B(X)$.

Locally the twisting does nothing at all, since locally $K$ and $\overline{K}$ are trivial anyway. In particular, in the twisted models, the transformation laws (2.3) are still valid, but globally the parameters $\alpha, \overline{\alpha}$, etc., must be interpreted as sections of different line bundles. For instance, in the $A$ model, $\alpha_−$ and $\overline{\alpha}_+$ are functions while $\alpha_+$ and $\overline{\alpha}_−$ are sections of $\overline{K}^{-1}$ and $K^{-1}$. One can therefore canonically pick $\alpha_−$ and $\overline{\alpha}_+$ to be constants (and the others to vanish); this gives canonical global fermionic symmetries of the $A$ model that are responsible for its simplicity. Similarly the $B$ model has two canonical global fermionic symmetries. These global fermionic symmetries are nilpotent and behave as BRST-like symmetries.

There is an obvious variant that is possible, and that is to twist only $\psi_+$ or only $\psi_−$, leaving the other untwisted. I will call this the half-twisted model. The half-twisted model has only one canonical fermionic symmetry, just the situation which for $g = 1$ usually leads to the elliptic genus. The half-twisted model would appear to be the most reasonable framework for generalizing the elliptic genus to $g > 1$. It also is of phenomenological interest in the following sense. In superstring compactifications on Calabi-Yau manifolds of $\dim \mathbb{C}X = 3$, the $A$ model is suitable for computing the $\overline{27}^3$ Yukawa couplings and the $B$ model for computing the $27^3$ Yukawa couplings. To get a full understanding of the low energy theory, one also needs the $1 \cdot 27 \cdot \overline{27}$ and $1^3$ Yukawa couplings; these are not naturally studied in either the $A$ or $B$ model but involve BRST invariant observables of the half-twisted model. The analysis below of the $A$ model can be applied to the half-twisted model to show that correlations of BRST observables reduce to a sum over tree level computations in instanton fields.

Mirror symmetry (since it reverses one of the $U(1)$ quantum numbers in the $N = 2$ superconformal algebra) can be taken to exchange the + and − twists of $\psi_+$ while leaving $\psi_−$ alone. As a result, mirror symmetry exchanges $A$ and $B$ models (which differ by the choice of twist of $\psi_+$, for fixed twist of $\psi_−$) and maps a half-twisted model to another half twisted model (since in these models one makes no twist of $\psi_−$ anyway).

Since the elliptic genus of a complex manifold is the same as the partition function of the half-twisted (or untwisted) model on a Riemann surface of genus one, mirror pairs have the same elliptic genus.

We now turn to a detailed description of the $A$ and $B$ models.
3. The $A$ Model

In the $A$ model, we regard $\psi^i_+$ and $\psi^i_-$ as sections of $\Phi^*(T^{1,0}X)$ and $\Phi^*(T^{0,1}X)$, respectively. It is convenient to combine them into a section $\chi$ of $\Phi^*(TX)$ (so henceforth $\chi^i = \psi^i_+$, and $\bar{\chi}^i = \psi^i_-$. As for $\psi^i_-$, it is in the $A$ model a $(1,0)$ form on $\Sigma$ with values in $\Phi^*(T^{0,1}X)$; we will denote it as $\psi^i_\bar{z}$. On the other hand, $\psi^i_+$ is now a $(0,1)$ form with values in $\Phi^*(T^{1,0}X)$, and will be denoted as $\psi^i_z$.

The topological transformation laws are found from (2.5) by setting $\alpha_+ = \tilde{\alpha}_- = 0$ and setting $\alpha_-$ and $\tilde{\alpha}_+$ to constants, which we will call $\alpha$ and $\tilde{\alpha}$. The result is

$$
\delta \phi^i = i\alpha \chi^i \\
\delta \phi^\bar{i} = i\tilde{\alpha} \bar{\chi}^i \\
\delta \chi^i = \delta \chi^\bar{i} = 0 \\
\delta \psi^i_z = -\alpha \partial_{\bar{z}} \phi^i - i\alpha \chi^j \bar{\Gamma}^i_{jm} \psi^m_z \\
\delta \psi^\bar{i}_z = -\tilde{\alpha} \partial_z \phi^i - i\alpha \bar{\chi}^j \Gamma^i_{jm} \psi^m_z .
$$

(3.1)

The supersymmetry algebra of the original model collapses for these topological transformation laws to $\delta^2 = 0$, which holds modulo the equations of motion. (By including auxiliary fields, as in [1], one can get $\delta^2 = 0$ off-shell.)

Henceforth, we will generally for simplicity set $\alpha = \tilde{\alpha}$. (The additional structure that we will overlook is related to the Hodge decomposition of the cohomology of the moduli space of holomorphic maps of $\Sigma$ to $X$.) In this case, the first two lines of (3.1) combine to $\delta \Phi^I = i\alpha \chi^I$. Also, we will sometimes express the transformation laws in terms of the BRST operator $Q$, such that $\delta W = -i\alpha \{Q, W\}$ for any field $W$. Of course $Q^2 = 0$.

In terms of these variables, the Lagrangian is simply

$$
L = 2t \int_{\Sigma} d^2z \left( \frac{1}{2} g_{i\bar{j}} \partial_{\bar{z}} \phi^i \partial_{z} \phi^\bar{j} + i \psi^i_\bar{z} D_z \chi^i g_{i\bar{z}} + i \psi^i_z D_{\bar{z}} \bar{\chi}^i \bar{g}_{\bar{z}i} - R_{i\bar{j}m} \psi^i_z \psi^\bar{j}_\bar{z} \chi^i \chi^\bar{j} \right) .
$$

(3.2)

It is now a key fact that this can be written modulo terms that vanish by the $\psi$ equation of motion as

$$
L = it \int_{\Sigma} d^2z \{ Q, V \} + t \int_{\Sigma} \Phi^*(K) 
$$

(3.3)

where

$$
V = g_{i\bar{j}} \left( \psi^i_\bar{z} \partial_{z} \phi^i + \partial_{\bar{z}} \phi^\bar{j} \psi^i_\bar{z} \right) ,
$$

(3.4)

while

$$
\int_{\Sigma} \Phi^*(K) = \int_{\Sigma} d^2z \left( \partial_{\bar{z}} \phi^i \partial_{z} \phi^\bar{j} g_{i\bar{j}} - \partial_{z} \phi^i \partial_{\bar{z}} \phi^\bar{j} g_{i\bar{j}} \right) 
$$

(3.5)
is the integral of the pullback of the Kahler form \( K = -ig_{ij}dz^i\bar{dz}^\bar{j}. \) Thus \( \int \Phi^* (K) \) depends only on the cohomology class of \( K \) and the homotopy class of the map \( \Phi. \) If, for instance, \( H^2(X, \mathbb{Z}) \cong \mathbb{Z} \), and the metric \( g \) is normalized so that the periods of \( K \) are integer multiples of \( 2\pi, \) then
\[
\int_\Sigma \Phi^* (K) = 2\pi n, \tag{3.6}
\]
where \( n \) is an integer, the instanton number or degree. We will adopt this terminology for simplicity; this involves no essential distortion.

Instead of saying that (3.3) is true modulo the \( \psi \) equations of motion, we could modify the BRST transformation law of \( \psi \) (by adding terms that vanish on shell) to make (3.3) hold exactly. We will not spell out the requisite additional terms in the transformation law, which do not affect the analysis below, since the operators \( O_a \) that we will consider are independent of \( \psi. \)

3.1. Reduction To Weak Coupling

From equation (3.3), we can give a quick explanation of one of the key properties of the model, which is the reduction to weak coupling. Suppose that we wish to calculate the path integral for fields of degree \( n. \) With insertions of some BRST invariant operators \( O_a \) (the details of which we will discuss presently), one wishes to compute
\[
\langle \prod_a O_a \rangle_n = e^{-2\pi nt} \int_{B_n} D\phi \, D\chi \, D\psi \, e^{-it\{Q, \int V\}} \cdot \prod_a O_a. \tag{3.7}
\]
Here \( B_n \) is the component of the field space for maps of degree \( n, \) and \( \langle \cdots \rangle_n \) is the degree \( n \) contribution to the expectation value. We have made use of (3.6) to pull out an explicit factor \( e^{-2\pi nt} \) which will turn out to contain the entire \( t \) dependence of \( \langle \cdots \rangle_n. \)

Standard arguments using the \( Q \) invariance and the fact that \( Q^2 = 0 \) show that \( \langle \{Q, W\} \rangle_n = 0 \) for any \( W. \) It is therefore also true that, as long as \( \{Q, O_a\} = 0 \) for all \( a, \) (3.7) is invariant under \( O_a \to O_a + \{Q, S_a\} \) for any \( S_a. \) Thus, the \( O_a \) should be considered as representatives of BRST cohomology classes.

Likewise, and this is the key point, (3.7) is independent of \( t \) (as long as \( \text{Re} \, t > 0 \) so that the path integral converges) except for the explicit factor of \( e^{-2\pi nt} \) that has been pulled out. In fact, differentiating the other \( t \) dependent factor \( \exp(-it\{Q, \int V\}) \) with respect to \( t \) just brings down irrelevant factors of the form \( \{Q, \ldots\}. \) Therefore, the path integral in (3.7) can be computed by taking the limit of large \( \text{Re} \, t. \) This is the conventional weak coupling limit (for maps of degree \( n). \)
Looking back at the original form of the Lagrangian (2.4), or for that matter at the form of $V$, one sees that for given $n$, the bosonic part of $L$ is minimized for holomorphic maps of $\Sigma$ to $X$, that is maps obeying

$$\partial \bar{z} \phi^i = \partial z \phi^\bar{i} = 0.$$  (3.8)

The weak coupling limit therefore involves a reduction to the moduli space $\mathcal{M}_n$ of holomorphic maps of degree $n$. The entire path integral, for maps of degree $n$, reduces to an integral over $\mathcal{M}_n$ weighted by one loop determinants of the non-zero modes. (A possibly more fundamental explanation of this reduction will be given in §5.) In particular, $\langle \ldots \rangle_n$ vanishes for $n < 0$, as there are no holomorphic maps of negative degree.

We can also now explain why the model is a topological field theory, in the sense that correlation functions $\langle \prod_a \mathcal{O}_a \rangle$ are independent of the complex structure of $\Sigma$ and $X$, and depend only on the cohomology class of the Kahler form $K$. This is certainly true of $\int_\Sigma \Phi^*(K)$. For the rest, all dependence of the Lagrangian on the complex structure of $\Sigma$ or $X$ is buried in the definition of $V$, which appears in the path integral only in the form $\{Q, V\}$; varying the path integral with respect to the complex structure of $\Sigma$ or $X$ will therefore bring only irrelevant factors of the form $\{Q, \ldots\}$.

### 3.2. The Ghost Number Anomaly

The Lagrangian (3.2) has at the classical level a “ghost number” conservation law, with $\chi$ having ghost number 1, $\psi$ having ghost number $-1$, and $\phi$ having ghost number 0. The BRST operator $Q$ has ghost number 1.

At the quantum level, the ghost number is not really a symmetry because of the anomaly associated with the index or Riemann-Roch theorem. Let $a_n$ be the number of $\chi$ zero modes, that is the dimension of the space of solutions of the equations $D \bar{z} \chi^i = D_z \chi^\bar{i} = 0$. Similarly, let $b_n$ be the number of $\psi$ zero modes, solutions of $D \bar{z} \psi^{\bar{i}} = D_z \psi_i = 0$.

The index theorem gives a simple formula for the difference $w_n = a_n - b_n$. In particular, $w_n$ is a topological invariant. For instance, if $X$ is a Calabi-Yau manifold of complex dimension $d$, and $\Sigma$ has genus $g$, then $w_n = 2d(1 - g)$, independent of $n$. The expression $\langle \prod \mathcal{O}_a \rangle_n$ will vanish unless the sum of the ghost numbers of the $\mathcal{O}_a$ is equal to $w_n$.

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4 Though not indicated in our notation, $a_n$ and $b_n$ may depend on the particular map $\Phi$ considered; we return to this presently.
An essential fact is that the equation for a $\chi$ zero mode is precisely the linearization of the instanton equation (3.8), and consequently the space of $\chi$ zero modes is precisely $TM_n$, the tangent space to $M_n$. In particular, if $M_n$ is a smooth manifold, then $a_n$ is its (real) dimension and in particular is a constant.

The number $w_n = a_n - b_n$ is often called the “virtual dimension” of $M_n$. The reason for this terminology is that in a sufficiently generic situation (which may be unattainable in complex geometry) one would expect that if $w_n > 0$ (the only situation that will be of interest) then $b_n = 0$ and hence $w_n = a_n$.

Somewhat more generally, as long as $M_n$ is smooth so that $a_n$ is a constant, $b_n$ will also be constant. Hence the space $V$ of $\psi$ zero modes will vary as the fibers of a vector bundle $\mathcal{V}$ over $M_n$. At singularities of $M_n$, $a_n$ and $b_n$ may jump.

3.3. Observables Of The A Model

To prepare for actual calculations, we need a preliminary discussion of the observables of the $A$ model.

The BRST cohomology of the $A$ model, in the space of local operators, can be represented by operators that are functions of $\phi$ and $\chi$ only. They have the following simple construction.

Let $W = W_{I_1I_2...I_n}(\phi)d\phi^{I_1}d\phi^{I_2}...d\phi^{I_n}$ be an $n$-form on $X$. We can define a corresponding local operator

$$O_W(P) = W_{I_1I_2...I_n}\chi^{I_1}...\chi^{I_n}(P).$$

(3.9)

The ghost number of $O_W$ is $n$. A simple calculation shows that

$$\{Q, O_W\} = -O_{dW},$$

(3.10)

with $d$ the exterior derivative on $W$. Therefore, taking $W \rightarrow O_W$ gives a natural map from the de Rham cohomology of $X$ to the BRST cohomology of the quantum field theory $A(X)$. If one restricts oneself to local operators (a more general class is considered in §7), this map is an isomorphism.

Particularly convenient are the following representatives of the cohomology. Let $H$ be a submanifold of $X$ (or more generally any homology cycle). The “Poincaré dual” of $H$ is a cohomology class that counts intersections with $H$. It can be represented by a differential form $W(H)$ that has delta function support on $H$. Hopefully it will cause no confusion if we refer to $O_{W(H)}$ as $O_H$. The ghost number of $O_H$ is the codimension of $H$.

---

5 This happy circumstance prevents complications related to the fact that (3.3) only holds on shell or after modifying the $\psi$ transformation laws.
3.4. Evaluation Of The Path Integral

Now let us carry out the evaluation of the path integral. We pick some homology cycles \( H_a, \ a = 1 \ldots s, \) of codimensions \( q_a. \) We also pick points \( P_a \in \Sigma. \) We want to compute the quantity

\[
\langle O_{H_1}(P_1) \ldots O_{H_s}(P_s) \rangle_n = e^{-2\pi n t} \int_{B_n} D\phi \ D\chi \ D\psi \ e^{it \int \{Q,V\}} \cdot \prod O_{H_a}(P_a). \tag{3.11}
\]

This quantity will vanish unless \( w_n, \) the virtual dimension of moduli space, is equal to \( \sum a q_a. \)

The path integral in (3.11) reduces, upon using the independence of \( t \) and taking \( \text{Re} \ t \to \infty, \) to an integral over the moduli space \( M_n \) of instantons. Moreover, as we have picked \( O_{H_a}(P_a) \) to have delta function support for instantons \( \Phi \) such that

\[
\Phi(P_a) \in H_a, \tag{3.12}
\]

the path integral actually reduces to an integral over the moduli space \( \tilde{M}_n \) of instantons obeying (3.12).

In a “generic” situation, the dimension \( a_n \) of \( M_n \) coincides with the virtual dimension \( w_n. \) Moreover, requiring \( \Phi(P_a) \in H_a \) involves imposing \( q_a \) conditions. Hence “generically” the dimension of \( \tilde{M}_n \) should be \( w_n - \sum a q_a = 0. \) In such a case, \( \tilde{M}_n \) will consist of a finite set of points. Let \( \# \tilde{M}_n \) be the number of such points. In determining the contribution of any of those points to the path integral, we can take \( \text{Re} \ t \to \infty. \) The computation reduces to evaluation of a ratio of boson and fermion determinants; this ratio is however simply equal to \(+1, \) because of the BRST symmetry which ensures cancellation between bose and fermi modes.  

In a generic situation, we therefore have

\[
\langle \prod_{a=1}^s O_{H_a}(P_a) \rangle_n = e^{-2\pi n t} \cdot \# \tilde{M}_n. \tag{3.13}
\]

\[\text{In general such a BRST symmetry ensures that the ratio of fermion determinants, in expanding around a BRST fixed point, is } +1 \text{ or } -1. \text{ In the present case, the ratio is } +1 \text{ since boson determinants are always positive, and the fermion determinant of the } A(X) \text{ model is also positive since the } \chi^i, \psi^i_z \text{ determinant is the complex conjugate of the } \bar{\chi}^i, \bar{\psi}^i_z \text{ determinant.}\]
Summing over $n$ we get “generically”

$$\langle \prod_{a} O_{H_{a}}(P_{a}) \rangle = \sum_{n=0}^{\infty} e^{-2\pi n t} \cdot \# \tilde{M}_{n}. \quad (3.14)$$

In complex geometry, life is not always “generic” and $\tilde{M}_{n}$ may well have components of positive dimension. Suppose $\tilde{M}_{n}$ has real dimension $s$ (or focus on a component of that dimension). If so, by virtue of the Riemann-Roch theorem, the space $V$ of $\psi$ zero modes is $s$ dimensional and varies as the fibers of a vector bundle $V$ of (real) dimension $s$ over $\tilde{M}_{n}$. It can be argued on rather general grounds that the generalization of counting the number of points in $\tilde{M}_{n}$ is the evaluation of the Euler class $\chi(V)$ of the bundle $V$:

$$\# \tilde{M}_{n} \to \int_{\tilde{M}_{n}} \chi(V). \quad (3.15)$$

I refer to §3.3 of [3] for an explanation of this key point (and a detailed field theoretic calculation showing explicitly how $\chi(V)$ arises in a representative example); also, see [5] for some of the background.

The generalization of $(3.14)$ is therefore

$$\langle \prod_{a=1}^{s} O_{H_{a}}(P_{a}) \rangle = \sum_{n=0}^{\infty} e^{-2\pi n t} \cdot \int_{\tilde{M}_{n}} \chi(V). \quad (3.16)$$

In a real life situation, involving multiple covers of an isolated rational curve in a three dimensional Calabi-Yau manifold, the Euler class of $V$ has been evaluated by Aspinwall and Morrison [4], who as a result were able to justify a formula that had been guessed empirically by Candelas et. al. [9].

Notice that in deriving $(3.14)$ and $(3.16)$ we have not assumed that $\Sigma$ has genus zero. The restriction to genus zero will arise only when (in §6) we explain the relation of these correlation functions of the twisted model to “physical” correlation functions of the untwisted model.

Although it may be impossible in complex geometry to achieve a “generic” situation (in which the actual dimension of $M$ coincides with its virtual dimension), this can always

---

7 The $\psi$ zero modes were discussed in the subsection on the ghost number anomaly; however, the equation for such zero modes must now be corrected to permit poles at $P_{a}$ tangent to $H_{a}$. A general extension of index theory to such situations, with the properties essential here, has been given by Gromov and Shubin [7].
be done by perturbing the complex structure of $X$ to a generic non-integrable almost complex structure. (This is allowed in the $A$ model [[3]].) The importance of (3.16) comes from the fact that it is generally impractical to do calculations based on generic non-integrable deformations. The generic non-integrable deformations are sometimes useful for theoretical arguments; see the end of §7. See [[7]] for the theory of almost holomorphic curves in almost complex manifolds.

4. The $B$ Model

Now we will consider the $B$ model in a similar spirit.

In the $B$ model, $\psi^i_\pm$ are sections of $\Phi^*(T^{0,1}X)$, while $\psi^i_+$ is a section of $K_0 \Phi^*(T^{1,0}X)$, and $\psi^i_-$ is a section of $\overline{K} \otimes \Phi^*(T^{1,0}X)$. It is convenient to set

$$
\eta^i = \psi^i_+ + \psi^i_-
$$

$$
\theta_i = g_{i\bar{i}} (\psi^i_+ - \psi^i_-).
$$

(4.1)

Also, we combine $\psi^i_{\pm}$ into a one form $\rho$ with values in $\Phi^*(T^{1,0}X)$; thus, the $(1,0)$ part of $\rho$ is $\rho^i_z = \psi^i_+$, and the $(0,1)$ part of $\rho$ is $\rho^i_{\bar{z}} = \psi^i_-$. As for the supersymmetry transformations, we now set $\alpha_\pm = 0$, and set $\tilde{\alpha}_\pm$ to constants; in fact, for simplicity we will just set $\tilde{\alpha}_+ = \tilde{\alpha}_- = \alpha$. The transformation laws are then

$$
\delta \phi^i = 0
$$

$$
\delta \phi^i = i\alpha \eta^i
$$

$$
\delta \eta^i = \delta \theta_i = 0
$$

$$
\delta \rho^i = -\alpha d\phi^i.
$$

(4.2)

The BRST operator is again defined by $\delta(\ldots) = -i\alpha \{Q, \ldots\}$, and obeys $Q^2 = 0$ modulo the equations of motion.

The Lagrangian is

$$
L = t \int_\Sigma d^2z \left( g_{ij} \partial_z \phi^i \partial_{\bar{z}} \phi^j + i\eta^i (D_z \rho^i_z + D_{\bar{z}} \rho^i_{\bar{z}})g_{i\bar{i}} 
+ i\theta_i (D_{\bar{z}} \rho^i_z - D_z \rho^i_{\bar{z}}) + R_{i\bar{j}j\bar{i}} \rho^i_z \rho^j_{\bar{j}} \eta^i \theta_k g^{k\bar{j}} \right).
$$

(4.3)

This can be rewritten

$$
L = it \int \{Q, V\} + tW
$$

(4.4)
where
\[ V = g_{ij} \left( \rho^i \partial \bar{z} \phi^j + \rho^i \partial z \bar{\phi}^j \right) \] (4.5)

and
\[ W = \int_{\Sigma} \left( -\theta_i D \rho^i - \frac{i}{2} R_{i \bar{j} i \bar{j}} \rho^i \wedge \rho^j \eta^k g^{k \bar{j}} \right). \] (4.6)

Here \( D \) is the exterior derivative on \( \Sigma \) (extended to act on forms with values in \( \Phi^* (T^{1,0} X) \) by using the pullback of the Levi-Civita connection of \( X \)), and \( \wedge \) is the wedge product of forms.

We can now see that the \( B \) theory is a topological field theory, in the sense that it is independent of the complex structure of \( \Sigma \) and the Kahler metric of \( X \). Under a change of complex structure of \( \Sigma \) or Kahler metric of \( X \), the Lagrangian only changes by irrelevant terms of the form \( \{Q, \ldots \} \). This is obviously true for the \( \{Q, V\} \) term on the right hand side of (4.4). As for \( W \), it is entirely independent of the complex structure of \( \Sigma \), since it is written in terms of differential forms. It is less obvious, but true, that under change of Kahler metric of \( X \), \( W \) changes by \( \{Q, \ldots \} \). These observations are “mirror” to our earlier result that the \( A \) theory is independent of the complex structure of \( \Sigma \) and \( X \) but depends on the Kahler class of the metric of \( X \).

Similarly, the \( B \) theory is independent of the coupling constant \( t \) (except for a trivial factor which will appear shortly) as long as \( \text{Re } t > 0 \) so that the path integral converges. Under a change of \( t \), the \( t \{Q, V\} \) term changes by \( \{Q, \ldots \} \). As for the \( t \) in \( tW \), this can be removed by redefining \( \theta \to \theta/t \) (since \( V \) is independent of \( \theta \) and \( W \) is homogeneous of degree one). Hence, the theory is independent of \( t \) except for factors that come from the \( \theta \) dependence of the observables. If \( O_a \) are BRST invariant operators that are homogeneous in \( t \) of degree \( k_a \), then the \( t \) dependence of \( \langle \prod_a O_a \rangle \) is a factor of \( t^{-\sum_a k_a} \), which arises from the rescaling of \( \theta \) to remove the \( t \) from \( tW \). This trivial \( t \) dependence of the \( B \) theory should be constrained with the complicated \( t \) dependence of the \( A \) theory, coming from the instanton sum.

Because the \( t \) dependence of the \( B \) theory is trivial and known, all calculations can be performed in the limit of large \( \text{Re } t \), that is, in the ordinary weak coupling limit. In this limit, one expands around minima of the bosonic part of the Lagrangian; these are just the constant maps \( \Phi : \Sigma \to X \). The space of such constant maps is a copy of \( X \), so the path

\[ \text{The theory definitely depends on the complex structure of } X \text{, which enters in the BRST transformation laws.} \]
integral reduces to an integral over $X$. We will make this more explicit after identifying the observables.

This is to be contrasted with the $A$ theory, in which one has to integrate over moduli spaces of holomorphic curves. The difference arises because in the $A$ theory, the $t$ dependence becomes standard only after removing a factor of $t \int \Phi^*(K)$, and after doing this, the rest of the bosonic part of the action is zero for arbitrary holomorphic curves, not just constant maps. In §5, I will give an alternative and perhaps more fundamental explanation of why calculations in the $B$ theory reduce to integrals over $X$ while in the $A$ theory they reduce to integrals over instanton moduli space.

4.1. Anomalies

The fermion determinant of the $A$ model is real and positive (as the $\chi^i, \psi^i_z$ determinant is the complex conjugate of the $\chi^i, \psi_z^i$ determinant). In particular, there is no problem in defining this determinant as a function, and the $A$ model, even before taking BRST cohomology, makes at least some sense as a quantum field theory (perhaps with a cutoff, and not conformally invariant) for any complex manifold $X$, not necessarily Calabi-Yau. (In fact, in [1], the $A$ model was defined for general almost complex manifolds.) The fact that the eventual recipe (3.16) for computing correlation functions does not use the Calabi-Yau condition is related to this.

The $B$ model is very different. Because the zero forms $\eta^i$, $g^{ii} \theta_i$ are sections of $T^{0,1}X$ and the one forms $\rho^i$ are sections of $T^{1,0}X$, the fermion determinant in the $B$ model is complex. The $B$ model does not make any sense as a quantum field theory, even with cutoff, without an anomaly cancellation condition that makes it possible to define the fermion determinants as functions (not just sections of some line bundle). The relevant condition is $c_1(X) = 0$, that is, $X$ should be a Calabi-Yau manifold. Thus, in the $B$ model, the Calabi-Yau condition plays an even more fundamental role than it does in the untwisted model, where it is merely necessary for conformal invariance.

9 In two dimensions, anomalies are quadratic in the coupling of fermions to gravitational and gauge fields. To get an anomaly that depends on the twisting (since the untwisted model is not anomalous) and on $X$ (since the twisted model is a non-anomalous free field theory for $X = \mathbb{C}^n$), we must consider a term linear in the gravitational field, that is the spin connection of $\Sigma$, and linear in the gauge field, that is the pull-back of the Levi-Civita connection of $X$. The only invariant linear in the latter is $c_1(X)$, and standard considerations show that $c_1(X)$ is indeed the obstruction to defining the fermion determinant in the $B$ model.
Like the $A$ model, the $B$ model has an important $\mathbb{Z}$ grading by a quantum number that we will call the ghost number. The ghost number is 1 for $\eta$ and $\theta$, $-1$ for $\rho$, and zero for $\phi$. $Q$ is of degree 1. If $X$ is a Calabi-Yau manifold of complex dimension $d$, and $O_a$ are BRST invariant operators of ghost number $w_a$, then $\langle O_a \rangle$ vanishes in genus $g$ unless

$$\sum_a w_a = 2d(1 - g). \quad (4.7)$$

(There is actually a more refined $\mathbb{Z} \times \mathbb{Z}$ grading, which we have obscured by setting $\tilde{\alpha}_+ = \tilde{\alpha}_-$ and combining $\psi_\pm$ into $\rho$.)

4.2. The Observables

Now we wish to make the simplest observations about the observables of the $B$ model, analogous to our earlier discussion of the $A$ model.

Instead of the cohomology of $X$, as in the $A$ model, we consider $(0,p)$ forms on $X$ with values in $\wedge^q T^{1,0} X$, the $q^{th}$ exterior power of the holomorphic tangent bundle of $X$.

Such an object can be written

$$V = d\bar{z}^{i_1} d\bar{z}^{i_2} \ldots d\bar{z}^{i_p} V_{i_1 \ldots i_p}^{j_1 j_2 \ldots j_q} \frac{\partial}{\partial z^{j_1}} \ldots \frac{\partial}{\partial z^{j_q}} \quad (4.8)$$

($V$ is antisymmetric in the $j$’s as well as in the $\bar{i}$’s.) The sheaf cohomology group $H^p(X, \wedge^q T^{1,0} X)$ consists of solutions of $\bar{\partial} V = 0$ modulo $V \rightarrow V + \bar{\partial} S$.

For every $V$ as in (4.8), and $P \in \Sigma$, we can form the quantum field theory operator

$$O_V = \eta^{\bar{i}_1} \ldots \eta^{\bar{i}_p} V_{\bar{i}_1 \ldots \bar{i}_p}^{j_1 \ldots j_q} \psi_{j_1} \ldots \psi_{j_q}. \quad (4.9)$$

One finds that

$$\{Q, O_V \} = -O_{\bar{\partial}V}, \quad (4.10)$$

and consequently $O_V$ is BRST invariant if $\bar{\partial}V = 0$ and BRST exact if $V = \bar{\partial}S$ for some $S$. Thus $V \rightarrow O_V$ gives a natural map from $\oplus_{p,q} H^p(X, \wedge^q T^{1,0} X)$ to the BRST cohomology of the $B$ model. This is in fact an isomorphism (as long as one considers only local operators; in §7, we will make a slight generalization).

\footnote{In complex geometry, $T^{1,0} X$ might be called simply $TX$, but we have used that name for the complexification of the real tangent bundle of $X$.}
4.3. Correlation Functions

Now picking points \( P_a \in \Sigma \) and classes \( V_a \) in \( H^{p_a}(X, \wedge^{q_a} T^{1,0} X) \), we wish to compute

\[
\langle \prod_a \mathcal{O}_{V_a}(P_a) \rangle. \tag{4.11}
\]

We will consider only the case of genus zero. It will be clear that (4.11) vanishes unless

\[
\sum_a p_a = \sum_a q_a = d. \tag{4.12}
\]

(This is related to a more precise grading of the theory, by left- and right-moving ghost numbers, that was alluded to following equation (4.7).)

Taking the large \( t \) limit, the calculation reduces as explained earlier to an integral over the space of constant maps \( \Phi : \Sigma \to X \). In addition to the bose zero modes – the displacements of the constant map \( \Phi \) – there are fermi zero modes, which are the constant modes of \( \eta \) and \( \theta \). The nonzero bose and fermi modes enter only via their one loop determinants; these determinants are independent of the particular constant map \( \Phi : \Sigma \to X \) about which one is expanding, and so just go into the definition of the string coupling constant. So one reduces to a computation involving the zero modes only; and correlation functions of the \( B \) model will reduce to classical expressions.

Once we restrict to the space of zero modes, a function of \( \phi, \eta \) and \( \theta \) which is of \( p^{th} \) order in \( \eta \) and \( q^{th} \) order in \( \theta \) can be interpreted as a \((0,p)\) form on \( X \) with values in \( \wedge^p T^{1,0} X \). This of course is where the \( \mathcal{O} \)'s came from originally. In multiplying such functions, one automatically antisymmetrizes on the appropriate indices because of fermi statistics. Thus \( \prod_a \mathcal{O}_{V_a} \) can be interpreted, using (4.12), as a \( d \) form with values in \( \wedge^d T^{1,0} X \). The map

\[
\otimes_a H^{p_a}(X, \wedge^{q_a} T^{1,0} X) \to H^d(X, \wedge^d T^{1,0} X) \tag{4.13}
\]

is the classical wedge product.

What remains is to integrate over \( X \) the element of \( H^d(X, \wedge^d T^{1,0} X) \) obtained this way. The Calabi-Yau condition is here essential; it ensures that \( H^d(X, \wedge^d T^{1,0} X) \) is non-zero and one dimensional. The space of linear forms on this space is thus likewise one dimensional; any such non-zero form gives a method of “integration,” unique up to a constant multiple. Of course, the path integral of the \( B \) model gives formally a method of evaluating (4.11) and hence of integrating an element of \( H^d(X, \wedge^d T^{1,0}) \); this procedure formally is unique up
to a multiplicative constant (a correction to the string coupling constant). We noted in our discussion of anomalies that the $B$ model is anomalous except for Calabi-Yau manifolds.

The restriction to Calabi-Yau manifolds amounts to the fact that what can be integrated naturally are top forms or elements of $H^d(X, \Omega^d X)$. ($\Omega^d X$ is the sheaf of forms of type $(d,0)$.) In general the relation of $\Omega^d X$ and $\wedge^d T^{1,0} X$ is that they are inverses, but in the Calabi-Yau case they are both trivial, and hence isomorphic. Indeed, multiplication by the square of a holomorphic $d$ form gives a map from $\wedge^d T^{1,0} X$ to $\Omega^d X$. Empirically, the choice of a holomorphic $d$ form corresponds to the choice of the string coupling constant, though this relation is still somewhat mysterious.

5. The Fixed Point Theorem

In the last section, we explained why calculations in the $A$ model reduce to integrals over moduli spaces of holomorphic curves, while calculations in the $B$ model reduce to integrals over spaces of constant maps (and ultimately to classical expressions). I will now (as in [3], §3.1) explain this in an alternative and perhaps more fundamental way, as a sort of fixed point theorem.

Consider an arbitrary quantum field theory, with some function space $\mathcal{E}$ over which one wishes to integrate. Let $F$ be a group of symmetries of the theory. Suppose $F$ acts freely on $\mathcal{E}$. Then one has a fibration $\mathcal{E} \to \mathcal{E}/F$, and by integrating first over the fibers of this fibration, one can reduce the integral over $\mathcal{E}$ to an integral over $\mathcal{E}/F$. Provided one considers only $F$ invariant observables $\mathcal{O}$, the integration over the fibers is particularly simple and just gives a factor of $\text{vol}(F)$ (the volume of the group $F$):\[
\int_{\mathcal{E}} e^{-L} \mathcal{O} = \text{vol}(F) \cdot \int_{\mathcal{E}/F} e^{-L} \mathcal{O}. \tag{5.1}
\]

We want to apply this to the case in which $F$ is the $(0|1)$ dimensional supergroup generated by the BRST operator $Q$. This case has some very special features. The volume of the group $F$ is zero, since for a fermionic variable $\theta$,

$$
\int d\theta \cdot 1 = 0. \tag{5.2}
$$

Hence (5.1) tells us that if $Q$ acts freely, the expectation value of any $Q$ invariant operator vanishes.
To express this in another way, if $F$ acts freely, then one can introduce a collective coordinate $\theta$ for the BRST symmetry. BRST invariance tells us that $L$ and $O$ are both independent of $\theta$, and since the $\theta$ integral of a $\theta$-independent function vanishes, the path integral would vanish.

In general, $F$ does not act freely, but has a fixed point locus $E_0$. If so, let $C$ be an $F$-invariant neighborhood of $E_0$ and $E'$ its complement. Then the path integral restricted to $E'$ vanishes, by the above reasoning. So the entire contribution to the path integral comes from the integral over $C$. Here $C$ can be an arbitrarily small neighborhood, so the result is really a localization formula expressing the path integral as an integral on $E_0$. The details depend on the structure of $Q$ near $E_0$. If the vanishing of $Q$ near $E_0$ is a generic, simple zero, then the fixed point contribution is simply an integral over $E_0$ weighted by the one loop determinants of the transverse degrees of freedom. This is analogous to, say, the Atiyah-Bott fixed point theorem in topology.

Now let us carry this out in the $A$ and $B$ models. In the $A$ model, the relevant BRST transformation laws read

$$
\begin{align*}
\delta \psi^i_z &= -\alpha \partial_z \phi^i - i \tilde{\alpha} \chi^j \Gamma^{i}_{j \bar{m}} \psi^\bar{m}_z \\
\delta \psi^i_{\bar{z}} &= -\tilde{\alpha} \partial_{\bar{z}} \phi^i - i \alpha \chi^j \Gamma^{i}_{j m} \psi^m_{\bar{z}} \\
\delta \phi^I &= i \alpha \chi^I.
\end{align*}
$$

(5.3)

Requiring $\delta \phi^I = 0$, we get that $\chi^I = 0$ for a BRST fixed point, and setting $\delta \psi = 0$, we see that in addition, a fixed point must have

$$
\partial_z \phi^i = \partial_{\bar{z}} \phi^i = 0.
$$

(5.4)

This is the equation for a holomorphic curve, and shows the localization of the $A$ model on the space of such curves.

The important part of the BRST transformation law of the $B$ model for our present purposes is

$$
\delta \rho^i = -\alpha \, d \phi^i.
$$

(5.5)

Setting $\delta \rho^i = 0$, we see that the condition for a fixed point is $d \phi^i = 0$; that is, $\Phi : \Sigma \to X$ must be a constant map. Thus we recover the localization of the $B$ model on classical, constant configurations.
6. Relation To The “Physical” Model

So far, we have concentrated exclusively on analyzing the twisted $A$ and $B$ theories and their correlation functions. Mirror symmetry is however usually applied to the correlation functions of the untwisted, physical nonlinear sigma models. The purpose of the present section is to explain why certain correlation functions of the twisted models (either $A$ or $B$; they can be treated together) are equivalent to certain correlation functions of the physical models.

In constructing the twisted models from the physical sigma model, we “twisted” various fields by $K^{1/2}$ and $\overline{K}^{1/2}$ ($K$ being the canonical bundle of a Riemann surface $\Sigma$). To state the relation between the physical and twisted models in one sentence, it is simply that the models coincide (with a suitable identification of the observables) whenever $K$ is trivial and we choose $K^{1/2}$ and $\overline{K}^{1/2}$ to be trivial, since in that case the twisting did nothing. Although there are other examples of Riemann surfaces with trivial canonical bundle that might be considered, the important example (for standard applications of mirror symmetry) is the case that $\Sigma$ is a Riemann surface of genus zero with two points deleted.

Such a surface, of course, can be thought of as a cylinder with a complete, flat metric, say $ds^2 = d\tau^2 + d\sigma^2$, $-\infty < \tau < \infty$, $0 \leq \sigma \leq 2\pi$. In computing path integrals on such a surface, we must pick initial and final quantum states, say $|w\rangle$ and $|w'\rangle$. Considering first the twisted model, we assume that these are $Q$ invariant, and so are representatives of suitable BRST cohomology classes. Picking also points $P_a \in \Sigma$, $a = 1 \ldots s$, and BRST invariant operators $O_a$, we consider the objects

$$\langle w'| \prod_{a=1}^s O_a(P_a) |w\rangle \tag{6.1}$$

which can be represented by path integrals if we wish.

Of course, (6.1) can be interpreted more symmetrically by compactifying $\Sigma$ – adding points $P$ and $P'$ and conformally rescaling the metric to bring them to a finite distance. Then the states $|w\rangle$ and $|w'\rangle$ will correspond to BRST invariant operators $O_w(P)$ and $O_{w'}(P')$. (In the $A$ or $B$ model, the BRST cohomology is spanned by operators $O_V$, where $V$ is a de Rham cohomology class or an element of some $H^p(X, \wedge^q T^{1,0} X)$, respectively; so $O_w$ and $O_{w'}$ will automatically be operators of this type for some $V$’s.) The matrix element (6.1) is then equivalent to a correlation function

$$\langle O_{w'}(P') O_w(P) \prod_{a=1}^s O_a(P_a) \rangle \tag{6.2}$$

21
on the compactified surface $\hat{\Sigma}$. This can be evaluated according to the recipes of sections three and four, for the $A$ or $B$ model as the case may be.

Now we go back to the open surface $\Sigma$, with its trivial canonical bundle and flat metric, and make the following key observation. As $K$ is trivial, we can pick $K^{1/2}$ to be trivial. If we do so, the twisting by $K^{1/2}$ does nothing. Hence, (6.1) is equivalent to some matrix element in the untwisted model. Of course, the untwisted model has a lot of matrix elements (since the physical states have the multiplicity of a Fock space); we will get only a few of them this way. The operators $O_a$ will correspond to some bosonic vertex operators of the untwisted model; in fact, as discussed in general terms in [[10]], if the $O_a$ are chosen as harmonic representatives of the appropriate BRST cohomology classes, they are standard vertex operators of massless bosons. (In the language of [[10],[2]], these particular bosonic vertex operators generate the chiral ring – in fact, the $ca$ or $cc$ chiral ring in the case of the $A$ or $B$ model.) Let us call the harmonic representatives $B_a$.

What about the initial and final states in (6.1)? Equivalence of the twisted and untwisted models depends on choosing $K^{1/2}$ (and $\overline{K}^{1/2}$) to be trivial. This means in the language of the untwisted model that we are working in the Ramond sector; and thus the initial and final states are fermions (and in fact, harmonic representatives of the cohomology classes in question would be ground state fermions of the untwisted model). Let us call these fermi states $|f\rangle$ and $|f'\rangle$. From the point of view of the untwisted model, (6.1) might be written

$$\langle f' | \prod_{a=1}^{s} B_a | f \rangle.$$ (6.3)

I stress, though, that in going from the twisted to the untwisted model on the flat cylinder $\Sigma$, all that we have changed is the notation.

Now, (6.3) is a coupling of two ground state fermions $|f\rangle$ and $|f'\rangle$ to an arbitrary number of ground state bosons $B_a$ (which are all from the same chiral ring). Of the diversity of possible observables of the untwisted model, these are of particular importance as they determine the “superpotential.” 11 We have explained how these particular matrix elements can be identified with observables of the twisted model.

11 Actually, the cubic terms in the superpotential come from the case $s = 1$ of the above. To compute higher terms in the superpotential, one must consider the integrated, two form version of the $O$'s, which we will introduce in the next section. The analysis of the relation between twisted and untwisted models on the cylinder is unchanged.
Of course, if we wish we can compactify $\Sigma$ in the context of the untwisted model, adding points $P$ and $P'$ at infinity and conformally scaling the metric to bring them to a finite distance. At this stage the difference between the twisted and untwisted model will come in, as the isomorphism between them depends on a trivialization of $K$. In the untwisted model, when one projects the points at infinity to a finite distance, the states $|f\rangle$, $|f'\rangle$ will be replaced by fermion vertex operators $V_f$, $V_{f'}$. Hence, (6.3) has the alternative interpretation as a correlation function

$$\langle V_f(P)V_{f'}(P')\prod_{a=1}^s B_a \rangle$$

on the closed surface $\hat{\Sigma}$. Note that in contrast to (6.2), which arose from the analogous compactification in the twisted model, here the “new” vertex operators are of a different type from the old ones. This is possible because the untwisted model has vastly more observables than the twisted models. The fact that the Yukawa couplings are derived from a cubic form (with symmetry between the bose and fermi lines), which is usually regarded as a consequence of space-time supersymmetry, is manifest in the representation (6.2) of the twisted model, because the “fermions” and “bosons” are represented by the same kind of vertex operators.

### 7. Closer Look At The Observables

In this section we will, finally, take a closer look at the observables of the $A$ and $B$ theories. We will describe a structure – a hierarchy of $q$ form observables for $q = 0, 1, 2$ – which must exist on general grounds. We will then analyze this hierarchy in some detail in the $A$ and $B$ models. In the $A$ model we will obtain a simple answer which moreover has a simple and standard topological description. The analogous calculation in the $B$ model turns out to be far more complicated. We will not push it through to the end, but we will go far enough to identify the relevant structure, which turns out to be somewhat novel.

In either the $A$ or $B$ model, we described a family of observables, say $O_V(P)$, where $V$ is a de Rham cohomology class or an element of some $H^p(X,\Lambda^qT^{1,0}X)$, in the $A$ or $B$ model, and $P$ is a point in a Riemann surface $\Sigma$. Correlation functions

$$\langle \prod_{a=1}^s O_{V_a}(P_a) \rangle$$

(7.1)
are independent of the $P_a$, because of the topological invariance of the theory. We will systematically exploit the consequences of this fact. In doing so, we want to think of $O_V$ as an operator-valued zero form; to emphasize this we write it as $O^{(0)}$. We fix a particular $V$ and do not always indicate it in the notation.

Topological invariance of the theory – the fact that correlation functions of $O^{(0)}(P)$ are independent of $P$ – means that $O^{(0)}$ must be a closed zero form up to BRST commutators,

$$dO^{(0)} = \{Q, O^{(1)}\}, \quad (7.2)$$

for some $O^{(1)}$. This formula, read from right to left, means that the operator-valued one form $O^{(1)}$ is BRST invariant up to an exact form. Hence, we get new observables in the theory. If $C$ is a circle in $\Sigma$ (or more generally a one dimensional homology cycle), then

$$U(C) = \oint_C O^{(1)} \quad (7.3)$$

is a BRST invariant observable.

We can repeat this procedure. Topological invariance means that correlation functions of $U(C)$ must be invariant under small displacements of $C$; this means that $O^{(1)}$ must be a closed form up to BRST commutators,

$$dO^{(1)} = \{Q, O^{(2)}\}, \quad (7.4)$$

for some $O^{(2)}$. Also, (7.4) means that the two form $O^{(2)}$ is BRST invariant up to an exact form, so

$$W = \int_\Sigma O^{(2)} \quad (7.5)$$

is a new BRST invariant observable.

In this procedure, if $O^{(0)}$ has ghost number $q$, then $O^{(i)}$ has ghost number $q - i$, for $i = 1, 2$.

Obviously, if $X, Y$ are a mirror pair of Calabi-Yau manifolds, then the mirror symmetry $A(X) \cong B(Y)$ can be applied to the new observables that we have just described. This is likely to be particularly interesting for applications of mirror symmetry with target spaces of complex dimension greater than three.

Now, (7.4) actually leads to the existence of a more general family of topological quantum field theories. If $L$ is the original Lagrangian, and $O^{(0)}_{V_a}$ are the operator valued
zero forms, of ghost number $q_a$, with which the above procedure begins, then we get a family of topological Lagrangians,

$$L \to L + \sum_a t_a \int_\Sigma \mathcal{O}_{V_a}^{(2)}.$$

Let us call this the topological family. As the ghost number of $\mathcal{O}_{V_a}^{(2)}$ is $q_a - 2$, Lagrangians in the topological family do not necessarily conserve ghost number (even at the classical level); those that do not are not twistings of standard renormalizable sigma models. In the case of a mirror pair, the whole topological family $A(X)$ is equivalent to the topological family $B(Y)$. So far mirror symmetry has been applied only to the subfamilies of theories that conserve ghost number classically. It is very likely that aspects of mirror symmetry that are now not well understood – like the nature of the mirror map between the moduli spaces – are more transparent in the context of the full topological family.

7.1. The $A$ Model

Now we will work out the details of the above for the $A$ model. This is easy enough. If

$$\mathcal{O}^{(0)} = V_{I_1 I_2 \ldots I_n} \chi^{I_1} \chi^{I_2} \ldots \chi^{I_n},$$

for some $n$ form $V$, then $d\mathcal{O}^{(0)} = \{Q, \mathcal{O}^{(1)}\}$, where

$$\mathcal{O}^{(1)} = -nV_{I_1 I_2 \ldots I_n} d\phi^{I_1} \chi^{I_2} \ldots \chi^{I_n}.$$

And $d\mathcal{O}^{(1)} = \{Q, \mathcal{O}^{(2)}\}$, with

$$\mathcal{O}^{(2)} = -\frac{n(n-1)}{2} V_{I_1 I_2 \ldots I_n} d\phi^{I_1} \wedge d\phi^{I_2} \chi^{I_3} \ldots \chi^{I_n}.$$

The above field theoretic formulas correspond to the following topological construction. Let $\mathcal{M}$ be the moduli space of holomorphic maps of $\Sigma$ to $X$ of some given homotopy type. Thus we have a family of maps $\Phi : \Sigma \to X$ parameterized by $\mathcal{M}$. Alternatively, one can think of this as a single map $\Phi : \Sigma \times \mathcal{M} \to X$. Given now an $n$ dimensional cohomology class $V \in H^*(X)$, we can pull it back to $\Phi^*(V) \in H^*(\Sigma \times \mathcal{M})$. This is an $n$ dimensional cohomology class of $\Sigma \times \mathcal{M}$. To get cohomology classes of $\mathcal{M}$, let $\gamma$ be an $s$ dimensional submanifold of $\Sigma$ (for $s = 0, 1, \text{or} 2$) and let $i : \gamma \to \Sigma$ be the inclusion. Then by integration over $\gamma$, one gets an $n - s$ dimensional class in the cohomology of $\mathcal{M}$, namely

$$i_* (\Phi^*(V)) = \int_\gamma \Phi^*(V).$$

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For instance, if $\gamma$ is a point $P \in \Sigma$, then integration over $P$ just means restricting to $P$, and (7.10) corresponds in the quantum field theory description to our old friend $O^{(0)}_{V}(P)$. For $\gamma$ a one-cycle (say a circle $C$) or a two-cycle (which must be a multiple of $\Sigma$ itself), we get the topological counterparts of the objects introduced in (7.3) and (7.5) above.

It is not too hard to verify the precise correspondence between the field theoretic and topological definitions. See [11] for further discussion of some of these matters.

7.2. The B Model

To understand the analogous issues in the B model are more difficult. In fact, because the computations involved are rather elaborate, I will first make some qualitative remarks to indicate what must be expected. Then we will just make a few illustrative computations which indicate the form of the general answer.

The operator-valued zero forms $O^{(0)}$ of the B model are determined by elements of $H^{p}(X, \wedge^{q} T^{1,0})$ for various $p$ and $q$. The corresponding two forms $O^{(2)}$ are possible perturbations of the topological Lagrangian. The case of $p = q = 1$ has particular significance, since $H^{1}(X, T^{1,0})$ is the tangent space to the space of complex structures on $X$, and the corresponding $O^{(2)}$'s are just the changes in the Lagrangian required by a change of complex structure. (The explicit calculation showing this would be just analogous to the one we will do presently for perturbations determined by elements of $H^{2}(X, \wedge^{2} T^{1,0})$.) So let us discuss what happens when the complex structure of $X$ is changed.

The complex structure of $X$ is determined by the $\bar{\partial}$ operator

$$\bar{\partial} = \sum_{i} \eta^{i} \frac{\partial}{\partial \phi^{i}}. \quad (7.11)$$

(Mathematically, $\eta^{i}$ would usually be written as the $(0, 1)$ form $d\phi^{i}$.) The transformation laws of the B model (for the fields $\phi, \eta, \theta$ from which the basic observables are constructed) are just the commutators with $\bar{\partial}$. In (7.11), I have written the $\bar{\partial}$ operator acting on $(0, q)$ forms (that is, functions of $\phi^{I}$ and $\eta^{i}$), but one can introduce the analogous $\bar{\partial}$ operator for $(0, q)$ forms with values in any holomorphic bundle. In our application, the important holomorphic bundle is $\bigoplus_{q} \wedge^{q} T^{1,0} X$. $(0, q)$ forms with values in this bundle are simply functions of $\phi^{I}, \eta^{i}$, and $\theta_{j}$.

If one makes a change in complex structure of $X$, the $\bar{\partial}$ operator changes. To first order, we get

$$\bar{\partial} \rightarrow \eta^{i} \left( \frac{\partial}{\partial \phi^{i}} + h_{i}^{j} \frac{\partial}{\partial \phi^{j}} - \frac{\partial}{\partial \phi^{k}} h_{i}^{j} \cdot \theta_{j} \frac{\partial}{\partial \theta_{k}} \right), \quad (7.12)$$

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where \( h_{ij} \) is a cocycle representing an element of \( H^1(X, T^{1,0}) \). (The perturbed \( \bar{\partial} \) operator, acting on functions or \((0, q)\) forms, may be more familiar; it is given by the same expression without the \( \theta \cdot \partial/\partial \theta \) term. This term must be included to give the perturbed \( \bar{\partial} \) operator acting on \((0, q)\) forms valued in \( \oplus_q \wedge^q T \).) When the complex structure of \( X \) is changed, the transformation laws of the \( B(X) \) model therefore also change; indeed, taking the commutator with the perturbed \( \bar{\partial} \) operator, we find

\[
\begin{align*}
\delta \phi^i &= i \alpha \eta^j h_{ij}^j, \\
\delta \theta_j &= -i \alpha \eta^i \partial_j h_{ij}^s \theta_s,
\end{align*}
\]

which replace \( \delta \phi^i = \delta \theta_j = 0 \) in the unperturbed theory. The non-zero transformation law of \( \theta_j \) is not so essential in the following sense: it reflects the change in \( T^{1,0} \) (of which \( \theta \) is a section) under a change in the complex structure of \( X \), and it can be transformed away by rotating the \( \theta^i \) to a basis appropriate to the new complex structure. The non-zero transformation law of \( \phi^i \) is unavoidable, as it is a basic expression of the change in complex structure.

**A Non-Classical Case**

So even in a "classical" case, where one is just perturbing the complex structure of \( X \), deformations of the \( B \) model require a change in the transformation laws of the basic fields. One must expect this to be true also for other, less classical deformations.

As a typical example, let \( \alpha \) be a cocycle representing an element of \( H^2(X, \wedge^2 T^{1,0}) \). The corresponding BRST invariant operator-valued zero form is

\[
\mathcal{O}^{(0)} = \alpha_{i_1 i_2 j_1 j_2} \eta^{j_1} \eta^{j_2} \theta_{j_1} \theta_{j_2}.
\]

We now wish to write \( d\mathcal{O}^{(0)} = \{Q, \mathcal{O}^{(1)}\} \), for some \( \mathcal{O}^{(1)} \). In contrast to the \( A \) model, one finds immediately that (i) this is only true modulo terms that vanish by the equations of motion; (ii) the calculations involved are rather painful. The second point is almost inevitable (in the absence of a powerful computational framework) given the first; and the first point is related, as we will see, to the fact that under perturbation of the Lagrangian, the transformation laws of the fields change.

Eventually one finds that

\[
d\mathcal{O}^{(0)} = \{Q, \mathcal{O}^{(1)}\} + G,
\]

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where

\[ \mathcal{O}^{(1)} = i\rho^j D_i \alpha_{i\bar{j}2} j_{i\bar{j}2} \eta_{\bar{i}1} \eta_{\bar{i}2} \theta_{j1} \theta_{j2} + 2d\phi^{\bar{i}1} \alpha_{i\bar{i}2} j_{i\bar{j}2} \eta_{\bar{i}2} \theta_{j1} \theta_{j2} - 2\alpha_{i\bar{i}2} j_{i\bar{j}2} \eta_{\bar{i}2} \theta_{j1} g_{j2} \kappa \star dX^k. \]  

(7.16)

The \( \star \) here is the Hodge star operator, and

\[ G = 2\alpha_{i\bar{i}2} j_{i\bar{j}2} \eta_{\bar{i}1} \eta_{\bar{i}2} \theta_{j1} Z_{j2}, \]  

(7.17)

where

\[ Z_j = D\theta_j - i\rho^m \eta^j \theta_s R_{j\bar{j}m} \eta_{\bar{j}} - g_{jj} \star D\eta^j \]  

(7.18)

vanishes by the \( \rho \) equation of motion.

The next step is to solve the equations

\[ d\mathcal{O}^{(1)} = \{ Q, \mathcal{O}^{(2)} \} + \sum_A \frac{\delta L}{\delta \Phi_A} \cdot \zeta_A. \]  

(7.19)

Here \( \Phi_A \) are all the fields of the theory (\( \phi, \eta, \theta, \rho \)). Moreover, \( \delta L/\delta \Phi_A \) are the equations of motion of the theory, so any expression that vanishes by the equations of motion is of the form \( \sum_A \delta L/\delta \Phi_Z \cdot \zeta^A \) for some \( \zeta^A \). \( \{ Q, \mathcal{O}^{(2)} \} \) means that in forming the topological family, the generalized Lagrangian

\[ \tilde{L} = L + t \int_\Sigma \mathcal{O}^{(2)} \]  

(7.20)

is not invariant under the original BRST transformations, but is invariant (to this order; that is, up to terms of order \( t^2 \)) under

\[ \tilde{\delta} \Phi_A = \delta \Phi_A + t\zeta_A. \]  

(7.21)

This shows how the modifications of the transformation laws, which we anticipate from our preliminary discussion of the role of \( H^1(X, T^{1,0}) \), depend upon having non-zero \( \zeta_A \).

The computation involved in finding \( \mathcal{O}^{(2)} \) in (7.19) (which is guaranteed to exist by the general discussion at the beginning of this section) is very complicated, and would be unilluminating if done without powerful computational methods, such as a superspace formulation. It is much easier to determine the \( \zeta_A \). The \( \zeta_A \) can be determined to eliminate terms in \( d\mathcal{O}^{(1)} \) that do not appear in any expression of the general form \( \{ Q, \mathcal{O}^{(2)} \} \). I will just state the results. First of all, one finds that \( \zeta_\eta = 0 \). This is in fact inevitable, even without computation, to preserve the fact that \( Q^2 = 0 \) (when acting on \( \phi^\bar{j} \)). The important
novelty, compared to the derivation of equation (7.15), is that $\zeta_\theta$ and $\zeta_\phi$ are non-zero. In fact,

$$\zeta_{\phi^j} = -2i\alpha\bar{\eta}^j_1\bar{\eta}^j_2\theta^k$$

(7.22)

and

$$\zeta_{\theta^i} = iD_i\alpha\bar{\eta}^i_1\bar{\eta}^i_2\theta^j_1\theta^j_2.$$  

(7.23)

Including the terms necessitated by (7.22) and (7.23), the BRST transformation laws of the topological family (or rather, the one parameter subfamily determined by the particular $O(2)$ considered here) are

$$\delta\bar{\phi}^i = i\alpha\bar{\eta}^i$$

$$\delta\bar{\eta}^i = 0$$

$$\delta\phi^i = -2it\alpha\bar{\eta}^i_1\bar{\eta}^i_2\eta^j_1\eta^j_2\theta^k$$

$$\delta\theta^i = itD_i\alpha\bar{\eta}^i_1\bar{\eta}^i_2\theta^j_1\theta^j_2.$$  

(7.24)

What sort of perturbed $\bar{\partial}$ operator will generate such transformations? Evidently, we need

$$\bar{\mathcal{D}} = \eta^i\frac{\partial}{\partial \phi^i} - 2it\eta^i_1\eta^i_2\alpha\bar{\eta}_1\bar{\eta}_2\theta^j_1\theta^j_2 + it\eta^i_1\eta^i_2D_k\alpha\bar{\eta}^i_1\bar{\eta}^i_2\bar{\theta}^j_1\theta^j_2.$$  

(7.25)

**Interpretation**

From this sample computation, it is possible to guess the general structure, as we will now indicate. Like the original $\bar{\partial}$ operator, $\bar{D}$ is a first order differential operator, which acts on functions of $\phi, \eta, \theta$. As it is the BRST operator of the perturbed model, it obeys $\bar{D}^2 = 0$ (after adding terms of order $t^2$ and higher, which we have not analyzed). However, because of the terms of third and higher order in fermions, it is definitely not a classical $\bar{\partial}$ operator.

Let $\mathcal{M}$ be the moduli space of complex structures on $X$, modulo diffeomorphism. $\mathcal{M}$ parametrizes the $B(X)$ models as we constructed them originally in §4. The “topological family” of theories (with Lagrangian $L \rightarrow L + \sum a t \int O^{(2)}_a$) is a more general family of theories parametrized by a thickened moduli space $\mathcal{N}$ which contains $\mathcal{M}$ as a subspace; the tangent space to $\mathcal{N}$ at $\mathcal{M} \subset \mathcal{N}$ is $T\mathcal{N}|\mathcal{M} = \oplus_{p,q=0}^d H^p(X, \wedge^q T^{1,0}X)$.

$\bar{D}$ is a symmetry of the perturbed model, so $\bar{D}^2$ is likewise a symmetry, which moreover is bosonic and of ghost number two. The perturbed or unperturbed model has no such symmetries, so it must be that $\bar{D}^2 = 0$. 

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The moduli space $\mathcal{M}$ of complex structures on $X$ is the space of standard $\bar{\partial}$ operators (first order operators whose leading symbol is linear in $\eta^i$ and independent of $\theta$), which obey $\bar{\partial}^2 = 0$, modulo diffeomorphisms of the $\phi^I$. The enriched moduli space $\mathcal{N}$ of topological models is apparently, in view of (7.25), the space of general $\bar{D}$ operators – first order operators on the same space, but with the leading symbol allowed to have a general dependence on $\phi, \eta, \theta$ – obeying $\bar{D}^2 = 0$, modulo diffeomorphisms of $\phi^I, \eta^I, \theta^I$. (Thus, $\partial$ operators are classified up to ordinary diffeomorphisms of $\phi$ only, but $\bar{D}$ operators are classified up to diffeomorphisms of the $\phi, \eta, \theta$ supermanifold.) The replacement of classical $\partial$ operators by the more general $\bar{D}$ operators has some of the flavor of string theory, where classical geometry is generalized in a somewhat analogous, but far more drastic, way.

I will call $\mathcal{N}$ the extended moduli space. The analogous extended moduli space in the $A$ model is just $\bigoplus_{n=0}^{2d} H^n(X, \mathcal{O})$. The study of the extended moduli space is probably the proper framework for understanding the mirror map between the $A$ and $B$ models, and so is potentially rewarding.

7.3. The Mirror Map

The weakest link in existing studies of the consequences of mirror symmetry is the construction of the mirror map between the mirror moduli spaces. Given, in other words, a mirror pair $X, Y$, one would like to identify the mirror map between the $A(X)$ moduli space and the $B(Y)$ moduli space. Understanding this map is essential for extracting the consequences of mirror symmetry. The discussion in [9] involved an elegant and very successful but not fully understood ansatz.

To construct the mirror map, it suffices to identify some sufficiently rich class of observables that can be computed both in the $A(X)$ model and in the $B(Y)$ model. I do not know how to do this, but will make a few comments. Since the $B(Y)$ model reduces to classical algebraic geometry, “everything” is computable in the $B(Y)$ model. The $A(X)$ model is another story; observables in the $A(X)$ model depend on complicated instanton sums and so in general are not really calculable (except perhaps via mirror symmetry which requires already knowing the mirror map).

The association $\mathcal{O}^{(0)} \rightarrow \mathcal{O}^{(2)}$ that we have described above gives a natural identification between the tangent space to the extended moduli space (which is the space of $\mathcal{O}^{(2)}$’s) and the BRST cohomology classes of local operators (the $\mathcal{O}^{(0)}$’s). The two point function in genus zero

$$\eta_{ab} = \langle \mathcal{O}_a^{(0)} \mathcal{O}_b^{(0)} \rangle$$ (7.26)
therefore defines a metric on the extended moduli space.\footnote{I should stress that this “topological” metric in no way coincides with the Zamolodchikov metric on the moduli space – which is much more difficult to study and involves what has been called “topological anti-topological fusion”\cite{IIB}.} (It is essential here to work with the extended moduli space \( N \); the induced metric on the ordinary moduli space \( M \subset N \) is typically degenerate or even zero.) Moreover, it can be shown that this metric is flat. For topological field theories (like the \( A \) and \( B \) models for Calabi-Yau manifolds) that arise by twisting conformal field theories, this was shown in \cite{13}. The main ingredients in the argument are the \( \mathfrak{g}^* \) action on a genus zero surface with two marked points, and the fact that for the particular BRST invariant local operator \( \mathcal{O}^{(0)} = 1 \), the corresponding two form is \( \mathcal{O}^{(2)} = 0 \). For the \( A \) model, I will give another proof of flatness presently (by showing that the metric has no instanton corrections); this proof uses the \( \mathfrak{g}^* \) action in a different way, and does not require the Calabi-Yau condition.

For the \( A(X) \) model, the extended moduli space is \( \sum_{n=0}^{2d} H^2(X, \mathbb{C}) \), and the flat metric is simply the metric on this vector space given by Poincaré duality; there are no instanton corrections, as we will show shortly. For topological field theories obtained by twisting Landau-Ginzber models, the metric was computed by Vafa \cite{14} up to a conformal factor; that the metric that so arises is flat (after a proper choice of the conformal factor, which has not yet been analyzed in the quantum field theory) is part of the rather deep study of singularities by K. Saito \cite{13}, as was explained by Blok and Varchenko \cite{10}. For the \( B(Y) \) model, the extended moduli space was roughly described above, but the metric is not yet understood. It seems to be very hard to find any observables of the \( A(X) \) model except the metric (and the related “exponential map” noted below) that are effectively computable, without instanton corrections. So understanding the metric on the extended moduli space of the \( B(Y) \) model would appear to be a promising way to understand the mirror map. (If the metric were known for both \( A(X) \) and \( B(Y) \), the mirror map would be uniquely determined up to an isometry; isometries depend on finitely many constants, which one might determine by matching a few coefficients at infinity.)

\textit{Vanishing Of Instanton Corrections To The Metric}

It remains to show that instantons do not contribute to the metric of the \( A(X) \) model. The proof is essentially dimension counting, taking account of the \( \mathfrak{g}^* \) action on a genus zero surface with two marked points. Let \( X \) be a complex manifold, not necessarily Calabi-Yau, and \( \Sigma \) a Riemann surface of genus zero. Consider a component \( \mathcal{U} \) of the moduli space of...
holomorphic maps $\Phi : \Sigma \to X$ of virtual dimension $w$. Let $H$ and $H'$ be submanifolds (or homology cycles) of codimension $q$ and $q'$ with $q + q' = w$, and let $P$ and $P'$ be two points in $\Sigma$. Let $U'$ be the subspace of $U$ parameterizing $\Phi$'s with $\Phi(P) \in H$ and $\Phi(P') \in H'$. If $U'$ is empty, the contribution of this homotopy class to $\langle O_H^{(0)} O_{H'}^{(0)} \rangle$ is zero. As the virtual dimension of $U'$ is zero, it appears superficially that $U'$ need not vanish generically.

However, let us take account of the $\mathbb{C}^*$ action on $\Sigma$ fixing $P$ and $P'$. Let $\tilde{U} = U'/\mathbb{C}^*$. $U'$ must be empty if $\tilde{U}$ is. One can think of $\tilde{U}$ as the moduli space of rational curves $C \in X$ that intersect both $H$ and $H'$ (without specifying any parameterization of $C$ or map from $\Sigma$ to $C$). The virtual dimension of $\tilde{U}$ is $-2$. Hence, $\tilde{U}$ is “generically” empty, and will really be empty at worst after making a generic nonintegrable deformation of the almost complex structure of $X$. So the instanton corrections to the metric vanish.

As a very concrete example of this counting, let $X$ be a particular Calabi-Yau manifold, a quintic hypersurface in $\mathbb{P}^4$. The virtual dimension of $U$ is six (regardless of the homotopy class of the map considered). Let $H, H'$ be two homology cycles in $X$ the sum of whose codimensions is six. So one of them, say $H$, has codimension at least three. With this particular $X$, even for generic integrable complex structures, it is believed that the number of rational curves of given positive degree in $X$ is finite; if so, the union $Z$ of these curves has codimension four. Since $4 + 3 > 6$, $H$ can be perturbed slightly so as not to intersect $Z$; hence $U'$ is empty, and the instanton contribution to the metric vanishes.

The Exponential Map

The description of the topological family by a family of Lagrangians

$$\tilde{L} = L + \sum_a t_a \int_\Sigma O_a^{(2)}$$

(7.27)

shows that once we pick a base Lagrangian $L$, corresponding to a base point $P \in \mathcal{N}$, there is a natural linear structure on $\mathcal{N}$. This might be described as an exponential map from the tangent space to $\mathcal{N}$ at $P$ to $\mathcal{N}$. In the case of the $A$ model, this structure corresponds to the linear structure on $\bigoplus_n H^n(X, \mathbb{C})$. In the case of the $B$ model, it is not presently understood. Understanding the exponential map should be roughly similar to understanding the metric on the moduli space. It is puzzling, in particular, that in the $A$ model, the linear structure on the moduli space does not seem to depend on the choice of base point, while in the $B$ model such a dependence seems almost inevitable.
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