On equation of state for the "thermal" part of the spin current: Pauli principle contribution in the spin wave spectrum in cold fermion system

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Spin evolution opened a large field in quantum plasma research. The spin waves in plasmas were considered among new phenomena considered in spin-1/2 quantum plasmas. The spin density evolution equation found by means of the many-particle quantum hydrodynamics shows existence of the "thermal" part of the spin current, which is an analog of the thermal pressure, or the Fermi pressure for degenerate electron gas, existing in the Euler equation. However, this term has been dropped, since there has not been found any equation of state for the thermal part of the spin current (TPSC), like we have for the pressure. In this paper we derive the equation of state for the TPSC and apply it for study of spectrum of collective excitations in spin-1/2 quantum plasmas. We focus our research on the spectrum of spin waves, since this spectrum is affected by the thermal part of the spin current. We consider two kinds of plasmas: electron-ion plasma with motionless ions and degenerate electrons, and degenerate electron-positron plasmas. We also present the non-linear Pauli equation with the spinor pressure term containing described effects. The thermal part of the flux of spin current existing in the spin current evolution equation is also derived. We also consider the contribution of the TPSC in the grand generalized vorticity evolution.

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I. INTRODUCTION

Almost 15 years the spin-1/2 quantum plasmas have been the center of active consideration. Starting from the explicit derivation of the continuity, Euler, and the magnetic moment evolution (the Bloch equation) equations from the many-particle Pauli equation in 2001 [1], [2], which contains the derivation of the Coulomb and spin-spin exchange correlations, analysis of many collective phenomena has been performed. Contribution of the spin dynamics (the magnetic moment dynamics) in the plasma properties have been calculated during these research. First of all the spin contribution in plasma dynamics arises via the force of the spin-spin interaction existing in the Euler equation, requiring the account of the Bloch equation.

On this path there have been studied the shift of the electromagnetic wave frequency \( \omega \), the increase of the fast magnetosonic mode frequency and the decrease of the slow magnetosonic mode frequency \( \omega_0 \), the interaction of the magnetosonic waves in the spin-1/2 quantum plasmas \( \delta \), reduction of the energy transport in the quantum spin-1/2 plasmas due to the modification of the group velocity of the extraordinary wave at the certain range of wave numbers \( \delta \), widening of the solitary magnetosonic waves by a pressurelike term with negative sign caused by spin-spin interaction force \( \delta \), non-linear whistlers in the strongly magnetized high density plasmas forming the large-scale density fluctuations \( \delta \), investigation of the magnetic diffusivity and obtaining that the magnetic diffusivity plays a dominant role for the transition from the solitary wave to shock wave for arbitrary amplitude magnetosonic waves \( \delta \), detailed analysis of the small and arbitrary shock structures in spin 1/2 quantum plasma \( \delta \), composite nonlinear structures within the magnetosonic soliton interactions \( \delta \), the dynamics of small but finite amplitude magnetosonic waves exhibits both oscillatory and monotonic shock-like perturbations significantly affected by the spin-spin interaction \( \delta \), circularly polarized Alfven solitary waves with Gaussian form surrounded by smaller sinusoidal variations in the density envelope \( \delta \), the modification of the RayleighTaylor instability \( \delta \).

The described phenomena, in some form, exist in spinless plasmas, while there are some plasma effects requiring the spin of particles. On this path the following purely spin plasma phenomena have been found: the spin-plasma waves \( \delta \), \( \delta \), \( \delta \), \( \delta \), \( \delta \), \( \delta \), \( \delta \), the spin-electron acoustic waves \( \delta \), \( \delta \), \( \delta \), \( \delta \), \( \delta \), \( \delta \), \( \delta \), which are possibly related to the high-temperature superconductivity \( \delta \), the spin-electron acoustic soliton \( \delta \), the spin (quantum) vorticity \( \delta \), \( \delta \), \( \delta \), \( \delta \), the spin caused modulational instability of the magnetosonic waves in the dense quantum plasma \( \delta \), spin instabilities caused by specific equilibrium distribution functions \( \delta \), instability of the plasma and spin-plasma waves at the propagation of the spin polarized neutron beam through the magnetized spin-1/2 plasmas arising due to the spin-spin and spin-current interactions Ref. [19], [31]. Some effects are reviewed in Ref. [32].

Moreover, the spin gives the contribution in the plasma dynamics via the modification of the Fermi pressure \( \delta \), [33] and due to the account of the spin current evolution

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The magnetization enters the spectrum of Langmuir waves propagating parallel to the external magnetic field due to the modification of the Fermi pressure. Increase of the spin polarization of electrons increases the Fermi pressure up to 60 percent. The contribution of this phenomenon can also be found in other plasma effects.

The spin current tensor arises in the Bloch equation. Usually we present it in terms of other hydrodynamical variables, applying Takabayasi's results for single particle hydrodynamics, to truncate the set of quantum hydrodynamic (QHD) equations dropping its thermal part. However, we can find an equation for dynamical evolution of the spin current under influence of the electromagnetic field, but it does not give the thermal part of the spin current. In the general form the spin current evolution equation contains a pressure like term (the thermal part of the spin current flux), but no explicit form for this term has been found. Therefore this term has been dropped in earlier research. Let us mention now that in this paper we present the thermal part of the spin current (TPSC) and the thermal spin current flux for the degenerate electron gas, where the described quantities are caused by the Pauli blocking principle like the Fermi pressure in the Euler equation.

The spin current plays an important role in the quantum plasma physics. However, it is also in the center of attention in the condensed matter physics, especially in the application to spintronics. Spintronics is related to the spin-dependent electron transport phenomena. For the modeling of effects related to the spin currents the analytical definition of the spin current in terms of an effective single particle wave function is developed in Refs. 11, 44. Necessity of application of the full current composed of the spin and orbital currents is suggested in Ref. 42. Methodology of these research differs from the quantum hydrodynamics, where the spin current arises as a part of hydrodynamic variable set for the description of collective phenomena.

A special case of spin-1/2 quantum plasma modeling is the separate description of the spin-up and spin-down electrons. In this picture we do not consider electrons as a single fluid, but we consider the electrons as two interacting fluids. Since the Fermi pressures of the spin-up and spin-down magnetized electrons are different, the two fluid model gives extra longitudinal waves in the electron gas. These waves are related to relative motion of the spin-up and spin-down electrons and called, due to their spectrum properties, the spin-electron acoustic waves (SEAWs). If we consider the wave propagation parallel or perpendicular to the external magnetic field we find a single SEAW. While, at the oblique propagation, we find two branches of the SEAWs. These modes are affected by the Coulomb exchange interaction. The nonlinear SEAWs propagating parallel to the external magnetic field were considered, the existence of the spin electron acoustic soliton was demonstrated. It was shown that interaction of electrons via the spinons (the quantum of the SEAWs) gives a mechanism of the Cooper pairs formation. This kind of Cooper pairs gives a mechanism of the high temperature superconductivity. The thermal part of the spin current contributes in the separate spin evolution QHD either. Thus its derivation is essential for all forms of spin-1/2 QHDS.

QHD models are related to the specific form of the wave equation (Schrödinger, Pauli, or Dirac). In Ref. 50 developed a general first-principle theory of resonant nondissipative vector waves assumes no specific wave equation.

Quantum properties of plasmas are also caused in the exchange interaction, but we do not consider it here. For the description and discussion of the exchange interaction in quantum plasmas see the following recent papers, which is an analog for the Wigner function method.

Majority of effects in spin quantum plasma and spin related effects in condensed matter physics can be affected by the thermal part of the spin current. Thus, we obtain it here for the degenerate electron gas, so the thermal part of the spin current is related to the particle distribution under the Fermi step. It arises in the magnetic moment evolution equation. In its nature it does not related to the interparticle interaction. If one considers the spin-1/2 quantum plasma with spin-orbit interaction, the spin current tensor arises in the force field in the Euler equation and the spin torque in the magnetic moment evolution equation. In this regime the thermal part of the spin current contribution arises there either. Applying the thermal part of the spin current to different quantum plasma phenomena we consider the spin-plasmas waves propagating parallel and perpendicular to the external field in the electron-ion and the electron-positron plasmas, the thermal part of the spin current in the quantum vorticity, and calculate the thermal part of the spin current flux existing in the spin current evolution equation.

This paper is organized as follows. In Sec. II we present the basis of our model. We present the non-linear Pauli equation with the spinor pressure. We also describe the derivation of QHD equations from the microscopic model. In Sec. III we introduce the velocity field in the QHD equations and show the existence of the TPSC in the Bloch equation. In Sec. IV we present the explicit form of the thermal part of the spin current for the degenerate electron gas arising from the non-linear Pauli equation suggested in Sec. II. Corresponding modification of the quantum vorticity evolution equation is also described in Sec. IV. In Sec. V we present the thermal part of the spin current flux existing in the spin current evolution equation. This result is also based on the non-linear Pauli equation. In Sec. VI and below we apply...
the obtained model to the wave dispersion in plasmas. In Sec. VI we consider the contribution of the TPSC in spectrum of plasma waves, especially spin-plasma waves. In Sec. VII we show contribution of the thermal part of the spin current in the quantum hydrodynamic model of the electron-positron plasmas. In Sec. VIII we consider spectrum of the quantum electron-positron plasmas arising from presented model. In Sec. IX a summary of obtained results is presented.

II. MODEL: QHD EQUATIONS

A. Non-linear Pauli equation

Equations of the spin-1/2 quantum hydrodynamics can be represented as the Non-linear Pauli equation [1], [54]. It is similar to the spinless case when the continuity and Euler equations can be represented as the single fluid effective non-linear Schrodinger equation [55]. Even spin-orbit interaction [19] and other relativistic effects [22], [58] can be included in non-linear effective equations. Nonlinearity is caused by different factors: the many-particle interaction effect and the pressure, particularly the Fermi pressure. The non-linear Schrodinger equation for two- and three-dimensional degenerate electron gases containing the Fermi pressure and the Coulomb exchange interaction for the partially polarized electron gas is obtained in Ref. [23]. However, the partial spin polarization in the Fermi pressure presented in the scalar form. If we want to include the spin separation effect revealing in two fluid model of electrons [23], [24], [19] we can construct the non-linear Pauli equation containing the pressure in the spinor form:

\[
\vec{\pi} = \begin{pmatrix} \pi_\uparrow & 0 \\ 0 & \pi_\downarrow \end{pmatrix}, \tag{1}
\]

which arises as a diagonal second rank spinor. Subindexes \(\uparrow\) and \(\downarrow\) refer to the spin-up and spin-down electrons. It can be represented in term of the Pauli matrixes \(\pi_\uparrow = (\hat{I} + \hat{\sigma}_z)/2 + \pi_\downarrow (\hat{I} - \hat{\sigma}_z)/2\), where \(\hat{I}\) is the unit second rank spinor \(\hat{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\), and \(\hat{\sigma}_z\) is one of the Pauli matrixes \(\hat{\sigma}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\).

In accordance with the Fermi pressures for the spin-up and spin-down electrons, \(\pi_s\), with \(s = \uparrow, \downarrow\), has the following explicit form \(\pi_s = (6\pi^2)^{3/2}\hbar^2n_s^{3/2}/2m\) Here we have the mass of particles \(m\), the reduced Planck constant \(\hbar = 1.06 \times 10^{27}\) erg s, and the concentrations of spin-1/2 particles with the different spin projections on a chosen direction \(n_s\).

In accordance with Refs. [1], [19], [23] we can construct the non-linear Pauli equation with the Fermi pressure in the spinor form [11]. We have

\[
\hbar \partial_t \Phi(r, t) = \left( \frac{1}{2m} \hat{D}^2 + \vec{\pi} + q\varphi - \gamma\hat{\sigma}\vec{B} \right) \Phi(r, t), \tag{2}
\]

where \(\hat{D} = \hat{D}(r, t) = \vec{p} - (q/c)\vec{A}(r, t)\), \(q\) is the charge of particles, \(c\) is the speed of light, \(\varphi\) is the scalar potential of the electromagnetic field, \(\vec{B}\) is the magnetic field, \(\vec{A}\) is the vector potential of the electromagnetic field, hence \(\vec{B} = \nabla \times \vec{A}\). \(\gamma\) is the magnetic moment of particles, \(\hat{\sigma}\) is the vector composed of the Pauli matrixes, \(\Phi\) is the macroscopic spinor wave function related to the full concentration of particles as follows \(\Phi^\dagger(r, t)\Phi(r, t) = n(r, t)\).

We should mention that equation (2) is actually in partial agreement with Ref. [23]. It corresponds to Ref. [23] in order of separate description of the spin-up and spin-down fermions and appearance of the partial Fermi pressures for each species of electrons. However equation (2) leads to a thermal part of the spin current, which was not presented in Ref. [23].

NLSE (2) can be represented as the single fluid QHD of electrons or as the two fluid model with the separate spin evolution at our choice.

In the context of the model [11], [23] we need to mention an excellent example of NLSEs applying at the study of Bose-Einstein condensates (BECs). It is well-known Gross-Pitaevskii equation. Its traditional form allows to model behavior of scalar particles (particles with zero spin or located in a single hyperfine state). However, the generalization of the Gross-Pitaevskii equation was suggested for the spinor BECs (the spin-1 and spin-2 bosons) with evolving occupations of three or five hyperfine states [59-62]. It is shown as the mentioned papers, the evolution of the occupations leads to appearance of the spin waves. These waves are similar to the spin-plasma waves found for the magnetized spin-1/2 plasmas [3, 10, 17]. We should mention that in the neutral boson the nonlinearity in the NLSE is caused by the short range interaction [60], [63], while in plasmas it is related to the Fermi pressure and the electromagnetic interaction.

B. First principles derivation of many-particle QHD equations

The particle concentration is the quantum mechanical average of the operator constructed using the classic microscopic concentration

\[
n(r, t) = \int \Psi^\dagger(R, t) \sum_{i=1}^N \delta(r - r_i)\Psi(R, t)dR, \tag{3}
\]

where we integrate over the 3N dimensional configurational space, \(dR = \prod_{i=1}^N dr_i\), and the sum of the Dirac delta functions is the operator of microscopic concentration.

Application of the microscopic concentration in classical physics allows to derive set of hydrodynamic-like equations [64], [65]. An explicit averaging of the microscopic concentration allows to obtain hydrodynamic equations for smooth functions describing macro behavior of mediums [66], [67], [68] (for more details see [69]). Similarly, an appropriate definition of the quantum many
particle concentration allows us to derive the set of quantum hydrodynamic equations with a truncation at necessary step.

To perform this derivation we need the explicit form of the Hamiltonian of the many-particle Schrödinger (Pauli) equation $\hbar \partial_t \Psi = \hat{H} \Psi$. To include the major properties of the spin-1/2 electron-ion plasmas we include the Coulomb and spin-spin interaction

\[
\hat{H} = \sum_i \left( \frac{1}{2m_i} \mathbf{D}_i^2 + q_i \varphi_i^{ext} - \gamma_i \sigma_i^\alpha \sigma_i^\beta \right) + \frac{1}{2} \sum_{i,j \neq i} (q_i q_j G_{ij} - \gamma_i \gamma_j \sigma_i^\alpha \sigma_j^\beta),
\]

where $\mathbf{D}_i = -i \hbar \nabla_i + q_i \mathbf{A}_i \varphi^{ext}/c$. Below we also consider the electron-positron plasmas. Features of spin-1/2 electron-positron plasmas will be briefly reviewed in the light of results of this paper. The complete model of spin-1/2 electron-positron plasmas will be applied to study the spectrum of the spin-plasma waves [13, 14, 15, 22].

Let us describe meaning of different terms in Hamiltonian (3) and describe notations applied there. Hamiltonian (4) is composed of the kinetic energy operator, the potential energy of charges in the external electric field, the potential energy of the magnetic moments, correspondingly. Subindexes $i$ and $j$ are the numbers of particles, $q_i$ and $m_i$ are the mass and charge of $i$-th particle, $\gamma_i$ is the magnetic moment of $i$-th particle, for electrons $\gamma_i$ reads $\gamma_i = g_e q_i \hbar / (2m_e c)$, $q_e = -|e|$, and $g_e \approx 1.00116$, difference of $g_e$ from the unit caused by the anomalous magnetic dipole moment. The Green functions of the Coulomb, and the spin-spin interactions have the following form $G_{ij} = 1/|r_i - r_j|, G_{ij}^{\alpha\beta} = 4\pi \delta^{\alpha\beta} \delta(|r_i - r_j|)/|r_i - r_j|$. We see that the Coulomb interaction is described by the scalar Green function, which is a solution of the Poisson equation $\nabla^2 \Phi = 4\pi \rho$, where $\rho$ is the charge density. The Green function of the spin-spin interaction arises as a second rank symmetric tensor. It occurs as a solution of the following equations $\nabla \times \mathbf{B} = 4\pi \nabla \times \mathbf{M}$ and $\nabla \mathbf{B} = 0$. In the Green function of the spin-spin interaction we apply the Kronecker symbol $\delta^{\alpha\beta}$ for the space components of vectors, which is the tensor representation of the unit matrix. The quantities $\varphi_i^{ext} = \varphi_i (r_i, t)$ are the scalar and the vector potentials of the external electromagnetic field: $\mathbf{B}^{\text{ext}} (r_i, t) = \nabla \times \mathbf{A}^{\text{ext}} (r_i, t)$, and $\mathbf{E}^{\text{ext}} (r_i, t) = -\nabla \varphi_i^{\text{ext}} (r_i, t) - \frac{1}{c} \partial_t \mathbf{A}^{\text{ext}} (r_i, t)$. Operator $\sigma_i^\alpha$ are the Pauli matrices, the commutation relations for them are $[\sigma_i^\alpha, \sigma_j^\beta] = 2\delta_{ij} \varepsilon^{\alpha\beta\gamma} \sigma_i^\gamma$, where we employ the Kronecker symbol on the particle numbers, which means that the commutator is equal to zero if we take the Pauli matrices describing different particles. The commutator contains the Levi-Civita symbol $\varepsilon^{\alpha\beta\gamma}$, which is a third rank antisymmetric tensor.

The method under description describes multi species plasmas, such as electron-ion, electron-positron, or more complicate mixtures of species. However, for simplicity of presentation we focus our attention on a single species. We can do it since derivations of equations for different species are almost independent from each other. More explicit description of the derivation for electron-ion plasmas is described in Ref. [22] (see formulae (28)-(45)).

At the first step, differentiating the particle concentration (5) with respect to time and applying the Schrödinger equation with the Hamiltonian (4) we find the continuity equation

\[
\partial_t n + \nabla j = 0.
\]

At derivation of the continuity equation (5) the explicit form of the particle current (the momentum density) appears as

\[
j = \int \sum_{i=1}^N \delta (\mathbf{r} - r_i) \frac{1}{2m_i} \left( \Psi^+ (R, t) \mathbf{D}_i \Psi (R, t) + h.c. \right) dR,
\]

where h.c. means the hermitian conjugation.

Next we differentiate the particle current (6) with respect to time and find the momentum balance equation

\[
\partial_t \Pi^\alpha + \frac{1}{m} \partial^\beta \Pi^{\alpha\beta} = \frac{1}{m} \mathbf{F}^\alpha,
\]

where $\Pi^{\alpha\beta}$ is the momentum flux, and $\mathbf{F}$ is the force field. Force field consists of two parts

\[
\mathbf{F} = \mathbf{F}^{ext} + \mathbf{F}^{int}.
\]

The explicit form of the momentum flux arises at the derivation of the Euler equation as follows

\[
\Pi^{\alpha\beta} = \int \sum_{i=1}^N \delta (\mathbf{r} - r_i) \frac{1}{2m_i} \left( \Psi^+ (R, t) \hat{\sigma}_i^\beta \hat{D}_i^\alpha \Psi (R, t) + (\hat{D}_i^\beta \Psi)^+ (R, t) \hat{D}_i^\alpha \Psi (R, t) + h.c. \right) dR.
\]

The first term in formula (8) is the force of the particle interaction with the external field

\[
\mathbf{F}^{ext} = qn \mathbf{E}^{ext} + \frac{q_i}{c} \nabla \mathbf{B}^{ext} + m \mathbf{M} \nabla \mathbf{B}^{ext}.
\]

It consists of three terms describing the action the electric field on charges, the action of magnetic field on the moving charges, and the action of the external magnetic field on the magnetic moments, correspondingly.

The force field (10) contains an extra function. In the last term we meet the magnetic moment density (the magnetization) $\mathbf{M}$:

\[
\mathbf{M} (r, t) = \int \sum_{i=1}^N \delta (\mathbf{r} - r_i) \gamma_i \Psi^+ (R, t) \hat{\sigma}_i \Psi (R, t) dR.
\]
The second part of the force field \([5]\) describes the Coulomb and spin-spin interparticle interactions

\[
F_{int} = -q^2 \int (\nabla G(\mathbf{r} - \mathbf{r}'))n_2(\mathbf{r}, \mathbf{r}', t) d\mathbf{r}' + \int (\nabla G^\alpha_\gamma(\mathbf{r} - \mathbf{r}'))M^\alpha_\gamma(\mathbf{r}, \mathbf{r}', t) d\mathbf{r}', \quad (12)
\]

where

\[
n_2(\mathbf{r}, \mathbf{r}', t) = \int \sum_{i,j \neq i} \delta(\mathbf{r} - \mathbf{r}_i)\delta(\mathbf{r} - \mathbf{r}_j')\Psi^*(\mathbf{R}, t)\Psi(\mathbf{R}, t) d\mathbf{R},
\]

is the two-particle concentration, and

\[
M^\alpha_\gamma(\mathbf{r}, \mathbf{r}', t) = \int \sum_{i,j \neq i} \delta(\mathbf{r} - \mathbf{r}_i)\delta(\mathbf{r} - \mathbf{r}_j') \times \gamma_i\gamma_j \Psi^*(\mathbf{R}, t)\sigma^\alpha_i\sigma^\gamma_j\Psi(\mathbf{R}, t) d\mathbf{R}, \quad (13)
\]

is the two-particle magnetization.

We have presented a general derivation of the Euler equation from the many-particle Schrodinger equation. Therefore, the force field \([12]\) arose beyond the self-consistent field approximation. Extracting the self-consistent field we can present a general two particle function in the following form

\[
f_2(\mathbf{r}, \mathbf{r}', t) = f(\mathbf{r}, t)f(\mathbf{r}', t) + g_2(\mathbf{r}, \mathbf{r}', t),
\]

where we have introduced the correlation function \(g_2(\mathbf{r}, \mathbf{r}', t)\). In this paper we restrict our analysis by the self-consistent field approximation. Hence we drop the correlations. We consider the two particle concentration as the product of the particle concentrations \(n_2(\mathbf{r}, \mathbf{r}', t) = n(\mathbf{r}, t)n(\mathbf{r}', t)\). Similarly, for the two-particle magnetization we present it as the product of the magnetization \(M^\alpha_\gamma(\mathbf{r}, \mathbf{r}', t) = M^\alpha(\mathbf{r}, t)M^\gamma(\mathbf{r}', t)\). Account of the exchange interaction leads us beyond the self-consistent field approximation. Corresponding references are presented in the introduction. Let us mention that most of the sited particles are focused on the Coulomb exchange interaction. The spin-spin exchange interaction is considered in Ref. \([2]\).

The self-consistent electric and magnetic fields appear in the following form

\[
E = -q\nabla \int G(\mathbf{r}, \mathbf{r}')n(\mathbf{r}', t) d\mathbf{r}', \quad (15)
\]

and

\[
B^\alpha = \int G^\alpha_\beta(\mathbf{r}, \mathbf{r}')M^\beta(\mathbf{r}', t) d\mathbf{r}', \quad (16)
\]

or, in more explicit, vector, form we have

\[
B = \nabla \int \left[ \left( \nabla \frac{1}{r - r'} \right) M(r', t) \right] d\mathbf{r}' + 4\pi\mathbf{M}. \quad (17)
\]

The introduced electric \([15]\) and magnetic \([10]\) fields satisfy the quasi static Maxwell equations \(\nabla \times \mathbf{E} = 4\pi\rho, \nabla \times \mathbf{B} = 0, \nabla \times \mathbf{B} = 4\pi\nabla \times \mathbf{M}, \nabla \mathbf{B} = 0\), where \(\rho\) is the charge density.

In the self-consistent field approximation the full force field can be presented as follows

\[
F = qnE + \frac{q}{c}n\mathbf{v} \times \mathbf{B} + M^\beta\nabla B^\beta. \quad (18)
\]

Considering the time evolution of the magnetic moment density defined by formula \([11]\), and applying the Schrödinger equation with Hamiltonian \([1]\), we find, in the self-consistent field approximation, the magnetic evolution equation

\[
\partial_t M^\alpha + \nabla^\beta J^\alpha_\beta = \frac{2\gamma}{\hbar} e^{\alpha\beta\gamma} M^\beta B^\gamma, \quad (19)
\]

containing the spin current tensor

\[
J^\alpha_\beta = \int \sum_{i=1}^N \delta(\mathbf{r} - \mathbf{r}_i) \frac{\gamma_i}{2m_i} \times \left( \mathbf{\Psi}^+(\mathbf{R}, t) \mathbf{D}_\beta^i \sigma^\alpha_i \mathbf{\Psi}(\mathbf{R}, t) + h.c. \right) dR. \quad (20)
\]

General form of the magnetic moment evolution equation can be found in Ref. \([1]\) (see formula 30).

III. INTRODUCTION OF THE VELOCITY FIELD

In traditional classical and quantum hydrodynamic equations we do not meet the particle current (the momentum density) \(\mathbf{j}\), the momentum flux tensor \(\Pi^{\alpha\beta}\), and the spin current \(J^{\alpha\beta}\). Instead we meet the velocity field \(\mathbf{v}\), thermal pressure tensor \(P^{\alpha\beta}\), and expect to see the thermal part of the spin current \(J^{\alpha\beta}_{th}\), which has been assumed to be equal to zero. Therefore, it is our job to separate the center mass motion and the thermal motion. This problem was considered, for instance, in Refs. \([1, 21, 71]\). In spinless case it was effectively addressed by the Madelung decomposition \([53]\) applied for the multiparticle wave function. For the spin-1/2 particles this problem requires a generalization of the Madelung decomposition, see for example \([33]\), presented by formula

\[
\psi(\mathbf{r}, t) = a(\mathbf{r}, t)e^{i\phi(\mathbf{r}, t)/\hbar} \phi(\mathbf{r}, t), \quad (21)
\]

which contains the unit spinor \(\phi(\mathbf{r}, t)\). Its explicit form is

\[
\phi(\mathbf{r}, t) = \left( \cos(\theta/2)e^{i\varphi/2} \right) e^{-i\varphi/2}. \quad (22)
\]

This spinor satisfy the next condition \(\phi^+ \phi = 1\). The generalization Madelung decomposition contains \(a\) and \(S\), which are the amplitude and phase of wave function correspondingly. Same representation can be applied to
the macroscopic wave function obeying the NLSE and microscopic many-particle wave function obeying the Schrodinger equation with Hamiltonian.

Since our analysis is focused on the spin current let us present the explicit form for the spin current for the single particle in the external field

\[ J_{sp}^{αβ} = \gamma n s^α v^β - \frac{\hbar \gamma}{2m} s^α \partial^β s^\nu \left( s^\nu \partial^\beta s^\nu \right) \]  

arising from the generalized Madelung decomposition, where \( n(r,t) = |\psi(r,t)|^2 \) is the single-particle concentration, \( s = \phi^+ \bar{\sigma} \phi \) is the single-particle spin field, \( \bar{\sigma} \) are the Pauli matrices, \( \phi = \nabla S/m - (i \hbar/m) \phi^+ \nabla \phi - \left( q/mc \right) A \) is the single particle velocity field. The first term is the kinetic (classical-like) part of the spin current and the second term is the quantum part of the spin current, which is an analog of the quantum Bohm potential being the quantum part of the pressure.

At the analysis of many particle behavior we obtained more general formula, which contains an extra term

\[ J^{αβ} = M^α v^β - \frac{\hbar}{2m\gamma} \varepsilon^{αμν} M^μ \partial^β \left( \frac{M^ν}{n} \right) + J_{th}^{αβ}. \]  

This term is related to incoherent motion of spinning particles and presents the thermal part of the spin current.

In the many particle case formula \( \mathbf{v}_i(R,t) = \nabla i S(R,t)/m_i - (i \hbar/m_i) \phi^+ (R,t) \nabla \phi(R,t) - (q_i/m_i c) A_i \) describes quantum motion of ith particle in the system. The velocity field is defined as \( \mathbf{v} = \mathbf{j}/n \). The difference between the particle velocities and the local velocities of mass velocity (the velocity field) gives us the thermal velocity of particles. Similar description can be given for the collective part and thermal part of the spin density, for details see Ref. [71].

Application of the generalized Madelung decomposition to the momentum flux tensor \( P_{αβ} \) in the many particle Euler equation gives the following result \( P_{αβ} = nv^α v^β + P^αβ + T^αβ \), where \( P^αβ \) is the thermal pressure, and \( T^αβ \) is the quantum part giving the quantum Bohm potential. Similar picture arises for the spin current \( J_{sp}^{αβ} = nμ^α v^β + J_{quant}^{αβ} + J_{th}^{αβ} \), where \( M = nμ^α, J_{sp}^{αβ} \) is the quantum part of the spin current [52], [71], and \( J_{th}^{αβ} \) is the thermal part of the spin current [11], [71]. Substituting the particle current \( \mathbf{j} = n \mathbf{v} \), the momentum flux \( P^{αβ} \), and the spin current \( J_{sp}^{αβ} \) in the continuity, Euler and magnetic moment evolution equations we obtain for them the following representation:

\[ \partial_t n + \nabla(nv) = 0, \]  

\[ mn(\partial_t + \mathbf{v} \nabla) \mathbf{v} + \nabla P - \frac{\hbar^2}{4m} n \nabla \left( \frac{\Delta n}{n} - \frac{\nabla (n)^2}{2n^2} \right) \]

\[ + \frac{\hbar^2}{4mc^2} \partial^β \left( n(\partial^β \mu^α)^+ \nabla \mu^α \right) = qn E + qnev \times B + M^β \nabla B^β, \]  

and

\[ n(\partial_t + \mathbf{v} \cdot \nabla) \mu - \frac{\hbar}{2m\gamma} \partial^β [nμ, \partial^β μ] + \mathcal{S} = \frac{2γ}{\hbar} n[μ, B]. \]  

This derivation shows that even in the self-consistent field approximation we have two unknown functions in the quantum hydrodynamic equations. These are the thermal pressure and the thermal part of the spin current. We call these parts the thermal parts since they are related to the distribution of particles on different quantum states. While the distribution in the form of Fermi step existing in fermions at the zero temperature also gives us some "thermal" parts. The Fermi pressure is one of well-known equations of state for the pressure. The Fermi pressure gives pressure of the unpolarized spin-1/2 fermions at zero temperature. In this paper we consider partially spin polarized electrons at temperatures much smaller than their Fermi temperature. In our regime the pressure reads [33]

\[ P_{sf} = \frac{3π^2}{10} \frac{\hbar^2}{m} \left[ \left( n + \frac{M_1}{\gamma} \right)^{\frac{5}{2}} + \left( n - \frac{M_2}{\gamma} \right)^{\frac{5}{2}} \right], \]

where subindex "sf" reads single fluid. Here we refer to the single fluid model of electrons, while all plasma is considered as many fluid liquids and each species is considered as a fluid. Recently a two fluid model of spin-1/2 fermions was developed [23], [24], [49], where the spin-up and spin-down electrons are considered as two different fluids. We will apply this model to derivation of the thermal part of the spin current in the next section.

To the best of our knowledge it is impossible to give a straightforward derivation of the NLSE from the single fluid QHD, as it was done for the spinless regime [53]. However we can justify the NLSE deriving the QHD equations from the NLSE as it was done in [1].

IV. EQUATION OF STATE FOR THE THERMAL PART OF THE SPIN CURRENT

In previous section we have demonstrated the existence of the thermal part of the spin current and formulated the problem of finding an equation of state for the thermal part of the spin current.

In terms of the macroscopic effective wave function \( \Phi(r,t) \) the particle concentration \( n \), particle current \( \mathbf{j} \), and the magnetization \( \mathbf{M} \) appear as follows \( n = |\Phi|^2, \)

\[ j = (\Phi^+ \mathbf{D} \Phi + (\mathbf{D} \Phi^+) \Phi)/2m, \]

\[ M = \gamma \Phi^+ \bar{\sigma} \Phi. \]

Differentiating these functions with respect to time we find equations [23], [26], [27] derived from microscopic model in Sec. [11]. Thus, we see that the NLSE [2] is in agreement with the single fluid model of electrons. It is easy to see that the spinor pressure [11] introduced in NLSE [2] leads to the agreement of equation [2] with the two fluid model of electrons (the separate spin evolution QHD) developed in Refs. [23], [24], [49]. However, in addition, we find an
explicit form for the thermal part of the spin current
\[ \mathcal{S} = \partial_\beta j^{\beta}_\theta = \gamma_e \frac{(6\pi^2)^{2/3} \hbar}{m_e} (n_\uparrow^{2/3} - n_\downarrow^{2/3}) \{S_y, -S_x, 0\} \]

\[ = \varepsilon^{\alpha\beta\gamma} S_\alpha \frac{(6\pi^2)^{2/3} \hbar}{m_e} (n_\uparrow^{2/3} - n_\downarrow^{2/3}). \quad (29) \]

Let us rewrite formula (29) in the vector form and represent it in terms of the single fluid hydrodynamic model
\[ \mathfrak{S} = (\pi_\uparrow - \pi_\downarrow) [\mathbf{M}, \mathbf{e}_z]/\hbar \]
\[ = \frac{(3\pi^2)^{2/3} \hbar}{m_e} \left[ (n - M_z) \frac{2}{3} - (n + M_z) \frac{2}{3} \right] [\mathbf{M}, \mathbf{e}_z]. \quad (30) \]

Formulae (29) and (30) show that the term caused by the pressure in the NLSE reveals in the magnetic moment evolution equation as a torque of an effective force \( \sim [\mathbf{M}, \mathbf{e}_z] \).

In the regime of the small magnetization limit formula (30) transforms to
\[ \mathfrak{S} = -4(3\pi^2)^{2/3} \hbar \frac{M_z}{3m_e} \frac{2}{n^{1/3}} [\mathbf{S}, \mathbf{e}_z], \quad (31) \]
where we have used the spin density \( \mathbf{S} = \mathbf{M}/\gamma_e \). In the small magnetization limit the spin current \( \mathfrak{S} \) is proportional to the square of the spin density. As in the general case, the spin current is proportional to \( n^{5/3} \), since \( \mathbf{S} \sim n \).

The Landau-Lifshitz-Gilbert equation does not include the quantum part of the spin current. The convective part of the spin current is some times included in the Landau-Lifshitz-Gilbert equation: It is included, for instance, in Refs. [22], [73]; as a recent example of the Landau-Lifshitz-Gilbert equation without the convective part of the spin current see [72]. Some generalizations of the Landau-Lifshitz-Gilbert equation are presented in literature. For instance, a mutation term is introduced in Ref. [72], where it is substituted in the atomicistic Landau-Lifshitz-Gilbert equation [71]. The Landau-Lifshitz-Gilbert equation successfully describes the magnetic moment precession around and its relaxation towards the effective field acting on the magnetization on time scales down to femtoseconds [73], [76].

A. Contribution of thermal part of spin current in quantum vorticity

Considering quantum vorticity caused by the spin of particles, the quantum vorticity evolution equation has been found (see formula (13) in Ref. [27] and formula (4.6) in Ref. [77]). The quantum vorticity has been combined with the classical vorticity to obtain the grand generalized vorticity [27].

In this section we consider the contribution of the thermal part of spin current \( \mathfrak{S} \) in the grand generalized vorticity dynamics.

Quantum vorticity is defined in terms of the spin density \( \mathbf{S} \) and particle concentration \( n \) [27], [77].

\[ \mathcal{Q}_\mathcal{Q} = \frac{1}{2} \varepsilon^{\alpha\beta\gamma} \varepsilon^{\mu\nu\sigma} \left( \frac{S_\mu}{n} \right) \partial_\beta \left( \frac{S_\nu}{n} \right) \partial_\gamma \left( \frac{S_\sigma}{n} \right), \quad (32) \]
see for instance formula 12 and text below in Ref. [27] and formulae (4.10) and (4.11) in Ref. [77].

Differentiating definition (32) with respect to time and applying the quantum hydrodynamic equations [24]-[30] we find the quantum vorticity evolution equation
\[ \partial_\alpha \Omega_{q\alpha} = \nabla \times (\mathbf{v} \times \Omega_{q\alpha}) + \frac{q}{mc} \nabla \left( \frac{S_\beta}{n} \right) \times \nabla \hat{B}_{mod}, \quad (33) \]
where \( \hat{B}_{mod} = \hat{B} + (\pi_\perp - \pi_\parallel) \mathbf{e}_z/2 \gamma_e \) and \( \hat{B} = \mathbf{B} + (\hbar c/2q\eta) \partial^\beta (\eta \partial^\beta S/n) \). We see that the effective magnetic field \( \hat{B} \) is composed of the magnetic field \( \mathbf{B} \) and the contribution of the quantum Bohm potential in the magnetic moment evolution equation (the quantum part of the spin current). While the classical vorticity \( \Omega_c = \nabla \times (\mathbf{A} + m \mathbf{v} \mathbf{e}/q) \) obeys the following equation
\[ \partial_\alpha \Omega_{c\alpha} = \nabla \times (\mathbf{v} \times \Omega_{c\alpha}) + \frac{q}{mc} \nabla \left( \frac{S_\beta}{n} \right) \times \nabla \hat{B}^\beta, \quad (34) \]
as it was shown in Refs. [27], [77].

Constructing the grand generalized vorticities following Ref. [27] we have
\[ \Omega_{\pm} = \Omega_c \pm \frac{\hbar c}{2q} \Omega_{q\alpha}, \quad (35) \]
and find that the thermal part of the spin current change the canonical form of vortex dynamics for \( \Omega_- \), since \( \Omega_- \) satisfy the following equation
\[ \partial_\alpha \Omega_{-\alpha} = \nabla \times (\mathbf{v} \times \Omega_{-\alpha}) + \frac{c}{2q} \nabla \left( \frac{S_\beta}{n} \right) \times \nabla (\pi_\parallel - \pi_\perp). \quad (36) \]

The first two terms in equation (35) gives the canonical form of the vorticity evolution equation leading to the helicity conservation. The last term describes the contribution of the thermal part of the spin current. It is a generalization of equation obtained in Ref. [77] (see formula (4.7)).

Takabayasi’s works [78], [79], [80], [81] provide us with the quantum vorticity research. The quantum relativistic vorticity (see formula (II) on page 17 of Ref. [82] and corresponding four-vector Clebsch potential (the last relation in formula (f) on page 26 of Ref. [82]) can be found in Ref. [82]. Recent study of the vorticity structures can be found in Ref. [28]. Quantum spirals explicitly arising from the Pauli equation are considered in Ref. [83].
V. THERMAL PART OF THE FLUX OF THE SPIN CURRENT

The quantum hydrodynamic equations, as the classic hydrodynamics, are not restricted by the continuity, Euler and magnetic moment evolution equations. They can include the energy, pressure, thermal flux evolution equations, as it is well known from the classic five and thirteen moment models. In the spin-1/2 quantum plasmas we can derive the spin current evolution equation [34]. This equation can be derived directly from the many-particle Schrodinger (Pauli) equation [34], or we can calculate moments of the distribution functions [20, 37, 38, 56]. However the past derivations have not presented the thermal part of the flux of spin current, which existing in the spin current evolution equation.

In this section we present a derivation of the thermal part of the flux of spin current applying the NLSE [2].

In terms of the many-particle effective wave function \( \Phi(\mathbf{r}, t) \) (2) the spin current arises as \( J^{\alpha \beta} = \gamma(\Phi^+ D^\alpha \sigma^\beta \Phi + (D^\beta \sigma^\alpha \Phi^+) \Phi)/2m \). Applying this definition we derive the spin current evolution equation containing the thermal part of the spin current flux

\[
\partial_t J^{\alpha \beta} + \partial_\gamma (J^{\alpha \gamma} v^\gamma) + \frac{\gamma^2}{m} n_0 \partial_\alpha B^\alpha - \frac{2\gamma}{h} \tilde{\varepsilon} J^{\alpha \beta} = \frac{q}{m} M^\alpha E^\beta
\]

using the NLSE (2) for the macroscopic wave function \( \Phi \) time evolution. The quantum part of the spin current flux was considered in Ref. [31]. The divergence of the thermal part of spin current flux is presented by the third term on the left-hand side. On the right-hand side of equation (37), the contribution of the electromagnetic interaction in the spin current evolution is presented.

The obtained term is nonzero in the linear regime for the magnetized plasmas or magnetized dielectrics. Consequently, it gives a contribution in the wave dispersion.

If we want to derive an extended set of QHD equations containing the energy evolution [1, 53], pressure evolution, and spin current evolution we do not have to use kinetic equations [20, 37, 38], we can derive the hydrodynamic equations directly from the microscopic Schrodinger equation [1, 34, 53] or some other methods [54]. Explicit contribution of the quantum Bohm potential in the energy evolution of spin-1/2 particles was derived in Ref. [53].

VI. SPIN WAVES IN ELECTRON-ION PLASMAS

Presence of the spin waves in plasmas and their dispersion was found in 2006 in Ref. [16] by the application of the hybrid kinetic-hydrodynamic method, where the particle motion was described by the Vlasov equation, and the spin dynamics was described by the hydrodynamic like magnetic moment evolution equation. Waves propagating perpendicular to the external field were found there. The necessity of the anomalous magnetic moment of the electron was demonstrated there. Similar solution in terms of purely hydrodynamic description was found in Ref. [17]. Since hydrodynamic description does not show the cyclotron resonances the anomalous magnetic moment was not include to distinguish the spin mode from the charge waves. Applying kinetics in the extended phase space, including two extra dimensions caused spin direction evolution similar spin wave was obtained in Ref. [17]. Contribution of the quantum Bohm potential, existing in the magnetic moment evolution equation [38], in the spectrum of the spin waves was later found in Ref. [21]. Features of spin-plasma waves in the electron-positron plasmas arising due to the annihilation interaction in presence of the quantum Bohm potential were considered in Ref. [22].

Spin also gives contribution in longitudinal waves. For instance, it arises in the spectrum of Langmuir waves via the spin-orbit interaction [50].

We consider the high frequency oscillations, hence we assume that ions are motionless. We consider the electrons located in an external magnetic field \( B_0 = B_0 \hat{e}_z \). We are going to calculate the dispersion of the small amplitude wave excitations. The equilibrium state is the macroscopically motionless uniform systems of electrons, described by the equilibrium constant concentration \( n_{oe} \), which is equal to the concentration of ions \( n_{oi} \), zero velocity field \( v_{oe} = 0 \), zero electric field \( E_0 = 0 \), and the equilibrium magnetization. If the magnetization is caused by the external magnetic field we can write \( \mathbf{M}_0 = n_0 \mu_0 = \chi B_0 \), where \( \chi \) is the ratio between equilibrium magnetic susceptibility and magnetic permeability. The small perturbations are described by \( \delta n_e, \delta v_e, \delta E, \delta \mu_e \), and \( \delta B \).

After Fourier transformation of the linearized set of QHD equations (25)-(30) we find the following set of algebraic equations

\[
\omega \delta n - n_0 k \delta v = 0; \quad (38)
\]

\[
-i \omega \delta v + i k v_F \delta n + i k^2 \frac{m}{4} \delta n = \frac{q n_0}{m} \delta E + \Omega_e n_0 |\delta v, e_z| + \frac{n_0 \mu_0}{m} i k \delta B; \quad (39)
\]

\[
-i \omega \delta \mu + \left( \frac{\mu_0}{m} k^2 + w \right) |e_z, \delta \mu| = \frac{2\gamma}{h} B_0 |\delta \mu, e_z| + \frac{2\gamma}{h} \mu_0 |e_z, \delta B|; \quad (40)
\]

and

\[
-k^2 \delta E - k(k E) = \frac{\omega^2}{c^2} \delta E.
\]
tude of the electric field perturbation \( E \) present this comparison in Fig. 1. We note that TSCF is the major characteristic frequency in plasmas. Hence we call it the thermal spin current frequency \( w \) with the Langmuir frequency, which is the characteristic frequency for the thermal part of the spin current. Hence we call it the thermal spin current frequency (TSCF). It is necessary to compare the extra characteristic frequency \( w \) with the Langmuir frequency, which is the major characteristic frequency in plasmas. Hence we present this comparison in Fig. 1. We note that TSCF arises in systems of neutral particles either.

We express all hydrodynamic variables via the amplitude of the electric field perturbation \( E_A \) and obtain a set of three algebraic equations \( \Lambda^{\alpha\beta}E_A^\beta = 0 \).

Our analysis is dedicated to point-like objects: electrons and positrons. Motionless ions are also considered as point-like objects. Model of the finite radius ions in electron-ion plasmas was considered in Ref. [87].

A. Propagation parallel to the external field field

Tensor \( \Lambda^{\alpha\beta} \) can be separated on two parts

\[
\Lambda_{\alpha\beta} = \Lambda'_{\alpha\beta} + S_{\alpha\beta},
\]

(45)

The first of them related to the charge dynamics

\[
\Lambda' = \begin{pmatrix}
\frac{\omega^2}{c^2} \Xi - k_z^2 & \frac{\omega^2}{c^2} - \frac{\omega \Omega_e}{\gamma} & 0 \\
\frac{\omega^2}{c^2} \Xi - k_z^2 & \omega^2 - \Omega_e^2 & 0 \\
0 & 0 & \frac{\omega^2}{c^2} (1 - \frac{\omega^2}{\omega^2 - \Omega_e^2})
\end{pmatrix},
\]

(46)

where \( \Xi = (1 - \frac{\omega^2}{\omega^2 - \Omega_e^2}) \), \( \Omega_e = \frac{eB_0}{mc^2} \) is the cyclotron frequency for a charge in the magnetic field.

The second part of tensor \( \Lambda^{\alpha\beta} \) is caused by the spin dynamics

\[
\hat{S} = \frac{k_z^2 \omega \mu}{\omega^2 - \Omega_e^2} \begin{pmatrix}
-\tilde{\Omega}_\gamma & -\omega & 0 \\
\omega & -\Omega_e & 0 \\
0 & 0 & 0
\end{pmatrix},
\]

(47)

where \( \Omega_\gamma = \frac{2e}{\gamma} B_0 + w + \frac{k_z^2 \omega}{2m c^2} \) is the generalized cyclotron frequency for a magnetic moment in the magnetic field.

FIG. 1: (Color online) This figure describes the ratio between the TSCF and the Langmuir frequency \( \chi = w/\omega_L \) as a function of the equilibrium concentration \( \lg n_0 \), where the concentration is measured in \( \text{cm}^{-3} \), and the external magnetic field \( \lg B_0 \), where the magnetic field is measured in G.

FIG. 2: (Color online) This figure shows the ratio between the TSCF and the cyclotron frequency \( \delta = w/|\Omega_e| \). The upper figure describes \( \delta \) as a function of magnetic field at a fixed equilibrium concentration \( n_0 = 10^{23} \text{ cm}^{-3} \). The lower figure presents \( \delta \) as a function of the external magnetic field (large equilibrium field) and the equilibrium concentration (small concentration).
field, where the last term gives the quantum part of oscillation, the second term given by formula (44) describes the collective effect caused by the Pauli principal, and 

\[ \omega_j = 8\pi \gamma \mu_0 n_0 / \hbar. \]

In formulae (46) and (47) we have applied the modified Fermi velocity \( \tilde{v}_F \) arising in accordance with the modified Fermi pressure (25).

Nonzero perturbations exist if the determinant of matrix \( \Lambda^{\alpha\beta} \) is equal to zero: \( \det \Lambda = 0 \). This condition gives the dispersion dependencies of wave perturbations.

General dispersion equation splits on two equations. One of them is for the longitudinal perturbations \( k \parallel \epsilon_z \parallel \delta \mathbf{E} \), and another one is for the transverse perturbations \( k \perp \delta \mathbf{E} \).

For the longitudinal excitations we find the spectrum of the Langmuir waves

\[ \omega^2 = \omega_{Le}^2 + \frac{1}{3} \frac{\tilde{v}^2_F}{k^2} - \frac{\hbar^2 k^4}{4m_e^2}, \tag{48} \]

which contains the modified Fermi velocity. Which leads to the dependence of the Langmuir wave dispersion on the external magnetic field as it was mentioned in Ref. 33 and numerically studied in Refs. 23, 24.

For the transverse waves we find the frequency dependence of the refractive index \( n = k c / \omega \):

\[ \frac{k^2 c^2}{\omega^2} = 1 - \frac{\omega^2}{\omega(\omega + \Omega_{Le})}, \tag{49} \]

Frequency \( \omega_\mu \) can be rewritten as \( \omega_\mu = 4\pi \Omega_{Le} \chi_e \), where \( \chi_e \) is the ratio between equilibrium magnetic susceptibility and magnetic permeability, as it was presented in Ref. 3.

Formula (49) can be presented as two equations for frequency

\[ \omega^2 - k_z^2 c^2 - \omega_{Le}^2 / \omega - \Omega_e \omega_\mu = 0, \tag{50} \]

for the left circular polarized electric field \( E_y = i E_z \), and

\[ \omega^2 - k_z^2 c^2 - \omega_{Le}^2 / \omega + \Omega_e \omega_\mu = 0, \tag{51} \]

for the right circular polarized electric field \( E_y = -i E_z \).

We start the analysis of the spin contribution in the circularly polarized electromagnetic/spin waves with the consideration of equation (51). Let us note that \( \Omega_e = - \gamma_e / \omega \epsilon_e \) since \( \epsilon_e < 0 \), and \( \Omega_\gamma = - \gamma_\gamma / \omega \) since \( \gamma_\gamma < 0 \), and we have \( | \Omega_\gamma | = 2 | \gamma_\gamma | B_0 / h + (\mu_0 c / | \gamma_\gamma | ) h k^2 / 2m - w \) for small enough \( w \). Equation (51) can be rewritten as follows \( \omega^2 - k_z^2 c^2 - \omega_{Le}^2 / (\omega + | \Omega_e |) - \omega_\mu k_z^2 c^2 / (\omega + | \Omega_\gamma |) = 0 \). The last term changes the degree of this equation, which leads to extra solution. This solution is negative \( \omega < 0 \), hence it does not have physical meaning. In the small wave vector regime, in the absence of the external magnetic field, but at the presence of inner effects creating the spin polarization of the conducting electrons, or at large TSCF, we have \( \Omega_e = w > 0 \), the comparison of the TSCF with the cyclotron frequency is presented in Fig. 2. This equation reappears as \( \omega^2 - k_z^2 c^2 - \omega_{Le}^2 / (\omega + | \Omega_e |) - \omega_\mu k_z^2 c^2 / (\omega - w) = 0 \). In this regime, we find the spin plasma waves with the left circular polarization. Since \( \omega_\mu \sim M_0 \) is rather small, the last term gives noticeable at \( \omega \ll w \). We find solution as \( \omega = w + \delta \omega \), where \( \delta \omega \ll w \), so we have

\[ \omega = w + \frac{\omega_\mu k_z^2 c^2}{w^2 - k_z^2 c^2} \tag{52} \]

As it follows from Fig. 4, for the large concentrations \( n_0 \sim 10^{23} \text{ cm}^{-3} \) and average magnetic fields \( B_0 < 10^7 \), the cyclotron frequency is negligible in compare with the TSCF. Moreover the TSCF is comparable with the Langmuir frequency, but the Langmuir frequency is larger than the TSCF \( \omega_{Le} > w \). Hence, formula (52) can be simplified to \( \omega = w - \omega_\mu k_z^2 c^2 / (k_z^2 c^2 + \omega_{Le}^2 - w^2) \). If we do not make any assumptions about magnitude of the last term, which is caused by spin evolution, in the dispersion equation we need to solve it numerically. Solutions are presented in Figs. 5, 6.

At small wave vectors \( k \) the frequency of the ordinary wave is comparable with the Langmuir frequency. If the TSCF is comparable we find the hybridization of the ordinary wave and the spin-plasma wave depicted in Figs. 5, 6. In middle and lower pictures in Fig. 3 we see the crossing of the dispersion dependencies of the ordinary wave and the spin-plasma wave. To sight the hybridization we need to decrease the scale of frequencies as it is shown in Fig. 5 at \( B_0 = 10^4 \text{ G} \).

Let us consider equation (51) describing the transverse waves with the right circularly polarized wave. Due to the negative charge of electrons we have \( \omega^2 - k_z^2 c^2 - \omega_{Le}^2 / (\omega - | \Omega_e |) - \omega_\mu k_z^2 c^2 / (\omega - | \Omega_\gamma |) = 0 \) for small \( w \), so \( | \Omega_\gamma | = 2 | \gamma_\gamma | B_0 / h + (\mu_0 c / | \gamma_\gamma | ) h k^2 / 2m - w \). Since \( | \Omega_\gamma | \neq | \Omega_e | \) we see that the last term gives an extra positive solution, in compare with the spinless case, for the right circularly polarized waves. This solution has frequency near \( | \Omega_\gamma | \). Hence, we obtain

\[ \omega = | \Omega_\gamma | - \frac{\omega_\mu k_z^2 c^2}{| \Omega_\gamma | - | \Omega_e |} \tag{53} \]

where difference \( | \Omega_\gamma | - | \Omega_e | \) is equal to \( (g - 1) \gamma_e B_0 / m e + \mu_0 k_z^2 c / g e - w \). If contribution of \( w \) dominates in \( 2 \Omega_e \) equation (51) does not have extra solution.

To conclude this subsection we note that if we do not account the thermal part of the spin current or assume it to be rather small we find the spin-plasma wave as a part of the right circularly polarized wave spectrum. If the thermal part of the spin current dominates over the cyclotron frequency and the quantum part of the spin current the right circularly polarized spin-plasma wave does not exist, but we find a left circularly polarized
FIG. 3: (Color online) This figure presents solutions of equation (50) giving frequency $\gamma = \omega / |\Omega_e|$ as the functions of the dimensionless wave vector $\kappa = k_z c / |\Omega_e| = mc^2 k_z / (eB_0)$. Equation (50) gives two solutions: the ordinary (electromagnetic) wave and the spin-plasma wave. At the equilibrium concentration $n_0 = 10^{23}$ cm$^{-3}$ and several values of the magnetic field we consider the frequency of these waves as a function of the wave vector ($B_0 = 10^3$ G, $B_0 = 10^4$ G, $B_0 = 10^5$ G). The upper (lower) line in the upper figure describes the ordinary (spin-plasma) wave. At the larger magnetic field these dispersion dependencies cross each other. Hence we have hybridization of these waves.

spin-plasma wave. For the large enough TPSC we find hybridization of the spin-plasma wave and the ordinary electromagnetic wave.

FIG. 4: (Color online) This figure presents solutions of equation (50) $\omega(k_z)$ at the equilibrium concentration $n_0 = 10^{23}$ cm$^{-3}$ and continuously changing small magnetic field $B_0$. Here we use the dimensionless wave vector $\kappa = k_z c / |\Omega_e| = mc^2 k_z / (eB_0)$. It shows the hybridization of the ordinary electromagnetic and spin-plasma waves.

FIG. 5: (Color online) This figure shows in more details the hybridization of the ordinary and spin-plasma waves at $B_0 = 10^4$ G and $n_0 = 10^{23}$ cm$^{-3}$ presented in middle picture in Fig.

Using representation $\omega_p = 4\pi\Omega_e\chi_e$, as it was done in Ref. 2, the numerator can be rewritten as $4\pi\Omega^2 k_\perp^2 \chi_e$.

Dispersion equation (54) gives two solutions. Their
general form can be presented analytically:

$$\omega^2 = \frac{1}{2} \left[ \omega_{Le}^2 + k_1^2 c^2 + \Omega_\gamma^2 \right] + \sqrt{\left( \omega_{Le}^2 + k_1^2 c^2 - \Omega_\gamma^2 \right)^2 + 4 \omega_\mu \Omega_\gamma k_1^2 c^2}.$$

Solving equation (55) for the frequencies far from the shifted cyclotron frequency $|\Omega_\gamma|$ and applying the iteration method, we find the contribution of the spin dynamics in the dispersion of the electromagnetic (ordinary) waves

$$\omega^2 = \omega_{Le}^2 + k_1^2 c^2 + \frac{k_1^2 c^2 \omega_\mu \tilde{\Omega}_\gamma}{\omega_{Le}^2 + k_1^2 c^2 - \Omega_\gamma^2},$$

Considering frequencies near $|\tilde{\Omega}_\gamma|$ we obtain

$$\omega = |\tilde{\Omega}_\gamma| \left(1 - \frac{\omega_\mu}{2 \Omega_\gamma \omega_{Le}^2 + k_1^2 c^2 - \Omega_\gamma^2} \right).$$

Solutions (56) and (57) generalize solutions (8) and (9) found in Ref. [22].

The TSCF can dominate in $\tilde{\Omega}_\gamma$ in the regime of the perpendicular propagation. It reveals in the spectrum similar to the spectrum found and described for the parallel propagation.

The fluid moment hierarchy was addressed in Refs. [20], [37], [38] at the application of the generalized Wigner kinetics to the spin evolution in quantum plasmas. The spinless case was considered in Refs. [29], [38]. The account of higher moments in hydrodynamics of spin-1/2 particles generalizes the dispersion equation for the transverse waves and gives more resonance terms (see equation (16) in Ref. [20]). That leads to resonances close to double the cyclotron frequency and close to the difference of the cyclotron frequency of charge and the cyclotron frequency of magnetic moment in the external magnetic field, the last one is shifted by the anomalous magnetic moment.

The TPSC modifies the cyclotron frequency arising from the magnetic moment precession. Therefore, effects described in the previous paragraph can be also affected by the TPSC.

### VII. FEATURES OF QHD MODEL FOR SPIN-1/2 ELECTRON-POSITRON PLASMAS

The model of spin-1/2 electron-positron plasmas differs from the model of spin-1/2 electron ion plasmas due to the existence of the additional interaction between electrons and positrons called the annihilation interaction. Corresponding hydrodynamic and kinetic models were recently developed in Ref. [22].

In this paper we generalize the hydrodynamic model developed in Ref. [22] including the thermal part of the spin current.

Hamiltonian of the electron-positron plasmas differs from Hamiltonian (4), as it was demonstrated in Ref. [22]. The difference arises from the existence of the annihilation interaction. The spinless part of the annihilation interaction is similar to the Darwin interaction. Therefore, we include these interactions together:

$$\Delta \tilde{H} = -\frac{1}{2} \sum_{i,j}^{N} \pi q_i q_j \hbar^2 \delta(r_i - r_j)$$

$$-\sum_{i=1}^{N_e} \sum_{j=N_e+1}^{N_e+N_p} \pi q_i q_j \hbar^2 \left(3 + \sigma_i \cdot \sigma_j \right) \delta(r_i - r_j),$$

where the first term is the Darwin interaction between all particles (the electron-electron, positron-positron, and electron-positron interactions), and the second term is the annihilation interaction between electrons and positrons.

Choosing the coefficient in the Hamiltonian of the Darwin interaction we follow the Breit Hamiltonian for two electrons or the pair of electron and positron following from the quantum electrodynamics scattering amplitude instead of the Darwin term arising from the Dirac equation for the single electron in the external field (see book [89] and discussion in Refs. [88], [90]).

The relativistic part of the kinetic energy gives a contribution similar to the Darwin interaction. Hence it should be included in the relativistic analysis of some effects [22], [86]. However, the spectrum of spin-plasma waves is not affected by this effect, so we do not consider it here.

In the accordance with the method of many-particle quantum hydrodynamics described above we find the continuity equations for the electrons

$$\partial_t n_e + \nabla \cdot (n_e \mathbf{v}_e) = 0,$$

and the positrons

$$\partial_t n_p + \nabla \cdot (n_p \mathbf{v}_p) = 0,$$

the Euler equations for the electrons

$$mn_e (\partial_t + \mathbf{v}_e \cdot \nabla) \mathbf{v}_e + \nabla P_e - \frac{\hbar^2}{4m} n_e \nabla \left( \frac{\Delta n_e}{n_e} - \frac{(\nabla n_e)^2}{2n_e^2} \right)$$

$$+ \frac{\hbar^2}{4m \gamma_e^2} \partial^\beta \left( n_e (\partial^\beta \mu_e^\gamma) \nabla \mu_e^\beta \right) = q_e n_e \mathbf{E} + \frac{q_e}{c} n_e [\mathbf{v}_e, \mathbf{B}]$$

$$+ n_e \mu_e^\beta \nabla (B^\beta + 2\pi n_p \mu_p^\beta) + \frac{\pi q_e \hbar^2}{m^2 c^2} n_e \nabla (q_e n_e + \frac{5}{2} q_p n_p),$$

(61)
and for the positrons
\[
mn_p(\partial_t + v_p \cdot \nabla)v_p + \nabla P_p - \frac{\hbar^2}{4m} \nabla \left( \frac{\Delta_n_p}{n_p} \right) + \frac{\hbar^2}{4m^2c^2} \partial^\beta \left( n_p (\partial^\beta \mu_p \gamma) \nabla \mu_p \gamma \right) = q_p n_p E + \frac{q_p}{c} n_p [v_p, B]
\]
\[+ n_p \mu_p^3 \nabla (B^3 + 2\pi n_e \mu_e^3) + \frac{\pi q_p \hbar^2}{m^2 c^2} n_p \nabla (q_p n_p + \frac{5}{2} q_e n_e), \quad (62)
\]
where we have used the reduced magnetization \( \mu_a \) defined by the following formula \( M_a(r, t) = n_a \mu_a \).

Most of the terms in Euler equations \( (61) \) and \( (62) \) have same meaning as similar terms in the Euler equation \( (26) \) described after equation \( (27) \). However, equations \( (61) \) and \( (62) \) contain some new terms specific for electron-positron interaction. Let us describe them here. These terms follows from additional part to the Hamiltonian \( (53) \). We have four groups of terms on the right-hand side of each Euler equation. The account of Hamiltonian \( (53) \) leads to the second part of the third group of terms and to the existence of the fourth group. The contribution to the third group caused by the spin dependent part of the annihilation interaction. The first part of the fourth term is the Darwin interaction of the particles of the same species. Its second part is the combination of the interspecies Darwin interaction (1 of 5/2) and the spinless part of the annihilation interaction. Its second part is the combination of the same species. Its second part is the combination of the interspecies Darwin interaction (3 of 5/2). The pressures \( P_e \) and \( P_p \) are given by formula \( (25) \).

We also find the magnetic moment evolution equations for the electrons
\[
n_e(\partial_t + v_e \cdot \nabla)\mu_e - \frac{\hbar}{2m \gamma_e} \partial^\beta [n_e \mu_e, \partial^\beta \mu_e] + \mathcal{S}_e
\]
\[= \frac{2\gamma_e}{\hbar} n_e [\mu_e, B] + \frac{4\pi \gamma_e}{\hbar} n_e n_p [\mu_e, \mu_p], \quad (63)
\]
and for the positrons
\[
n_p(\partial_t + v_p \cdot \nabla)\mu_p - \frac{\hbar}{2m \gamma_p} \partial^\beta [n_p \mu_p, \partial^\beta \mu_p] + \mathcal{S}_p
\]
\[= \frac{2\gamma_p}{\hbar} n_p [\mu_p, B] + \frac{4\pi \gamma_p}{\hbar} n_e n_p [\mu_p, \mu_e]. \quad (64)
\]
Equations \( (63) \) and \( (64) \) differ from equation \( (27) \) by the last term, which is caused by the spin part of the annihilation interaction (see Hamiltonian \( (53) \)). Vectors \( \mathcal{S}_e \) and \( \mathcal{S}_p \) are given by formula \( (30) \).

**VIII. SPIN WAVES IN ELECTRON-POSITRON PLASMAS**

Spin-plasma waves in electron-positron plasmas show some similarity to the electron-ion spin-plasma waves. Principal difference is in the existence of the annihilation interaction.

**A. Propagation parallel to the external field**

Looking on matrices \( \left( 40 \right) \) and \( \left( 41 \right) \) we see that at consideration of the electron-positron plasmas the non-diagonal elements disappear since the contributions of electrons and positrons cancel each other. Therefore the circularly polarized, in the electron-ion plasmas, wave are the linearly polarized in the electron-positron plasmas and the account of the spin dynamics does not change it. Nevertheless, the spin gives a contribution in the dispersion of the electromagnetic linearly polarized waves. Moreover, the spin leads to the spin-plasma waves similarly to the electron-ion plasmas described above.

The dispersion equation for the transverse waves propagating perpendicular to the external magnetic field with the account of the spin evolution together with the contribution of the thermal part of the spin current arises as follows
\[
k_z^2 c^2 - \omega^2 + 2\omega_L c \frac{\omega^2}{\omega^2 - \Omega_e^2} + 2 \gamma | 8\pi (k_z e)^2 n_0 \mu_0 (\Theta + \Lambda) = 0, \quad (65)
\]
where
\[
\Theta = \frac{2\gamma}{h} B_0 + \frac{h \mu_0}{2m \gamma} k_z^2 + \frac{4\pi \gamma}{h} n_0 \mu_0 + w, \quad (66)
\]
and
\[
\Lambda = 4\pi \gamma | n_0 \mu_0 / h. \quad (67)
\]
Frequency \( \Theta \), given by formula \( (65) \), is the shifted cyclotron frequency for a magnetic moment in the external magnetic field, it is shifted, from the traditional cyclotron frequency \( eB_0 / mc \), due to the anomalous magnetic moment of the electrons and positrons included in \( \gamma \), the quantum Bohm potential contribution in the magnetic moment evolution equation (the second term in \( (65) \)), the spin dependent part of the annihilation interaction (the third term), and, found in this paper, contribution of the thermal part of the spin current \( w \) given by formula \( (66) \). Frequency \( \Theta \) is the characteristic frequency of the spin dependent part of the annihilation interaction.

The contribution of the thermal part of the spin current in the resonance frequency for electron-ion plasmas is described above. The electron-positron plasmas differs by the presence of the annihilation interaction, which increases frequency \( \Theta \) on a constant \( \Lambda \) \( (67) \).

The frequency of the spin-plasma wave is close to the resonance frequency \( \omega_R = \sqrt{\Theta^2 - \Lambda^2} \). Let us consider this frequency in more details. The annihilation interaction arises in the resonance frequency \( \sqrt{\Theta^2 - \Lambda^2} \) twice. These contributions partially cancel each other. Considering square of \( \Theta \) explicitly we have \( \sqrt{\Theta^2 - \Lambda^2} = \sqrt{\frac{2\gamma}{h} B_0 + \frac{h \mu_0}{2m \gamma} k_z^2 + w} \left( \frac{2\gamma}{h} B_0 + \frac{h \mu_0}{2m \gamma} k_z^2 + w + 2\Lambda \right) \). It
shows that the annihilation interaction increases the resonance frequency, while the thermal part of the spin current decreases it, as it has been demonstrated in more details for the electron-ion plasmas.

B. Propagation perpendicular to the external field

In this regime the dispersion equation for the transverse waves arises as follows

\[
\omega^2 - k_x^2 c^2 - 2 \omega_{Le}^2 - 8 \pi n_0 \mu_0 \frac{2 \gamma k_x^2 (\Theta + \Lambda)}{\hbar \omega^2 - \Theta^2 + \Lambda^2} = 0.
\]

(68)

The last term in equation (68) arises due to the spin of the electrons and positrons. It increases the degree of the dispersion equation and leads to the spin-plasma wave appearance, as it was also demonstrated in Sect VI B for the electron-ion plasmas.

If contribution of the magnetization in equation (68) is small the spin-plasma wave arises at frequencies close to the resonance frequency of the last term in formula (68) \( \omega \approx \sqrt{\Theta^2 - \Lambda^2} \). Explicitly including the small shift from the resonance frequency we find the dispersion of the spin-plasma wave propagating perpendicular to the external field:

\[
\omega = \sqrt{\Theta^2 - \Lambda^2} \left( 1 - \frac{1}{\Theta - \Lambda} \frac{8 \pi |\gamma | \mu_0 k_x^2 c^2 / \hbar}{2 \omega_{Le}^2 + k_x^2 c^2 + \Lambda^2 - \Theta^2} \right).
\]

(69)

It is an analog of solution (57) found for the electron-ion plasmas.

At the parallel and perpendicular propagation of the spin-plasma waves we have the same resonance frequency \( \sqrt{\Theta^2 - \Lambda^2} \).

Formula (69) gives an approximate solution of equation (68) for the spin-plasma wave. Besides, we can present general analytical solution of equation (68) describing the electromagnetic and spin-plasma waves together:

\[
\omega^2 = \frac{1}{2} \left( 2 \omega_{Le}^2 + k_x^2 c^2 + \Theta^2 - \Lambda^2 \right)
\]

\[
\pm \sqrt{ \left( 2 \omega_{Le}^2 + k_x^2 c^2 - \Theta^2 + \Lambda^2 \right)^2 + 8 \omega_{Le} (\Theta + \Lambda) k_x^2 c^2 }.
\]

(70)

This solution includes the hybridization of two branches studied numerically for waves propagating parallel to the external field in the electron-ion plasmas and presented in Figs. 4 and 5.

IX. CONCLUSION

The thermal part of the spin current has not been considered in the condensed matter physics and physics of quantum plasmas. In this paper we have improved the magnetic moment evolution equation for degenerate electrons and applied it for the spin-1/2 quantum plasma phenomena.

The explicit forms of the thermal spin current and the spin current flux have been derived for the degenerate electrons. Corresponding modification of the quantum vorticity evolution equation has been found. The thermal part of the spin current has been applied to find the spectrums of the transverse waves, focusing on the spin-plasma waves, in the electron-ion and electron-positron plasmas. It was demonstrated that the thermal spin current decreases the frequencies of the spin-plasma waves with right circular polarization propagating parallel and perpendicular to the external magnetic field. We have found that for large enough TPSC the spin-plasma waves with right circular polarization propagating parallel to the external magnetic field disappears, but the right circularly polarized spin-plasma wave arises at frequency near the TSCF. If the TSCF and the Langmuir frequency are approximately equal to each other we have obtained hybridization of the spin-plasma wave and the ordinary electromagnetic wave and their spectrums.

All of it have been found as applications of the generalized non-linear Pauli equation with the spinor pressure term suggested in this paper.

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