Yang–Mills moduli space in the adiabatic limit

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Abstract

We consider the Yang–Mills equations for a matrix gauge group G inside the future light cone of four-dimensional Minkowski space, which can be viewed as a Lorentzian cone $C(H^3)$ over the three-dimensional hyperbolic space $H^3$. Using the conformal equivalence of $C(H^3)$ and the cylinder $\mathbb{R} \times H^3$, we show that, in the adiabatic limit when the metric on $H^3$ is scaled down, classical Yang–Mills dynamics is described by geodesic motion in the infinite-dimensional group manifold $\text{C}^\infty(S^2, G)$ of smooth maps from the boundary two-sphere $S^2 = \partial H^3$ into the gauge group $G$.

Keywords: Yang–Mills equations, adiabatic limit, moduli space

1 Yang–Mills theory with Higgs fields governs three fundamental forces of Nature. It has a number of particle-like solutions such as vortices, monopoles and instantons [1–3]. One may ask about the dynamics of vortices and monopoles which evolve according to the second-order field equations of Yang–Mills–Higgs theory. In the seminal paper [4] Manton suggested that in the ‘slow-motion limit’ monopole dynamics can be described by geodesics in the moduli space of static multi-monopole solutions2. This approach was extended both to vortices in 2 + 1-dimensions (see e.g. [6] for a review) and instantons in 4 + 1-dimensions (see e.g. [7, 8]). In contrast, almost nothing is known about time-dependent solutions of pure Yang–Mills theory in 3 + 1-dimensions. Here we aim to partially fill this gap by applying Manton’s approach to the Yang–Mills equations on Minkowski space.

2 We parametrize Minkowski space–time $\mathbb{R}^{1,3}$ with coordinates $x^\mu, \mu = 0, 1, 2, 3$, and the metric
In this article we fix an origin in $\mathbb{R}^{3,1}$ and consider the time evolution of Yang–Mills fields in the interior of its light cone. For simplicity we will restrict ourselves to the future light cone $L_+$ and its interior $T_+$ only, as the considerations for the past are similar. $L_+$ and $T_+$ are defined by

$$ \tau^2 > 0, \ x^0 > 0 $$

for $\tau^2 = -\eta_{\mu\nu}x^\mu x^\nu$. (2)

respectively.

On $T_+$ one can introduce global pseudospherical coordinates $(\tau, \chi, \theta, \varphi)$ by

$$ x^0 = \tau \cosh \chi, \quad x^1 = \tau \sinh \chi \sin \theta \cos \varphi, $$

$$ x^2 = \tau \sinh \chi \sin \theta \sin \varphi, \quad x^3 = \tau \sinh \chi \cos \theta $$

and a range of

$$ \tau \in (0, \infty), \quad \chi \in [0, \infty), \quad \theta \in [0, \pi], \quad \varphi \in [0, 2\pi) $$

with the usual identifications and a harmless coordinate singularity at $\chi = 0$. The eigentime coordinate $\tau$ foliates $T_+$ into a family of hyperbolic three-spaces $H^3(\tau)$ or ‘radius’ $\tau$, each of which is built from spheres $S^2(\tau)$ of radius $\tau$ sinh $\chi$. In these coordinates, the metric (1) acquires the form

$$ ds^2 = -d\tau^2 + \tau^2 \left\{ d\chi^2 + \sinh^2 \chi \left\{ d\theta^2 + \sin^2 \theta \, d\varphi^2 \right\} \right\}, $$

where the expression in the round brackets is the metric on $S^2$ and the expression in the curly brackets is the metric on $H^3$. For any given $\tau$, the boundary $\partial H^3(\tau)$ is reached in the limit $\tau \to \infty$ and forms a two-sphere $S^2$ ‘at infinity’.

The metric (5) can be rewritten as

$$ ds^2 = -d\tau^2 + \tau^2 \delta_{ab} \, e^a \otimes e^b = \tau^2 \left\{ -\left( \tau^{-1}d\tau \right)^2 + \delta_{ab} \, e^a \otimes e^b \right\}, $$

where $\{e^a\}$ is a basis of one-forms on $H^3$ easily extracted from (5). From (6) we recognize a cone over $H^3$, i.e. $T_+ = C(H^3)$, which is conformally equivalent to a cylinder $\mathbb{R} \times H^3$ with the metric

$$ ds_{cy}^2 = -du^2 + \delta_{ab} \, e^a \otimes e^b \quad \text{for} \quad u = \ln \tau $$

and $H^3 = H^3(\tau = 1)$. We redenote the cylindrical coordinates

$$ (u, \chi, \theta, \varphi) = \left( y^0, y^1, y^2, y^3 \right) = \left( y^0, y^a \right) \quad \text{with} \quad a = 1, 2, 3. $$

From this point on we will work on the cylinder (7) since Yang–Mills theory is conformally invariant.

(3) We have set the stage to consider pure Yang–Mills theory on the cylinder $\mathbb{R} \times H^3$ with an arbitrary matrix gauge group $G$. The Yang–Mills potential $A = A_\mu dy^\mu$ takes its value in the Lie algebra $\mathfrak{g} = \text{Lie } G$ carrying a scalar product defined by the matrix trace $\text{Tr}$. The field tensor $F = dA + A \wedge A$ is defined as

$$ F = \frac{1}{2} F_{\mu\nu} \, dy^\mu \wedge dy^\nu \quad \text{with} \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu], $$

and the Yang–Mills equations read

$$ D_\mu F^{\mu\nu} = \frac{1}{\sqrt{|\text{det } g|}} \partial_\mu \left( \sqrt{|\text{det } g|} \, F^{\mu\nu} \right) + [A_\mu, F^{\mu\nu}] = 0, $$

where $g = (g_{\mu\nu})$ is the metric (7) on $\mathbb{R} \times H^3$. 

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For the metric (7) we have
\[ A = A_0 dy^0 + A_{\alpha} dy^\alpha = A_0 dy^0 + A_H, \] (11)
\[ \mathcal{F} = \mathcal{F}_{0\alpha} dy^0 \wedge dy^\alpha + \frac{1}{2} \mathcal{F}_{ab} dy^a \wedge dy^b = \mathcal{F}_{0\alpha} dy^0 \wedge dy^\alpha + \mathcal{F}_H. \] (12)
Employing the adiabatic approach [4], we deform the metric (7) and introduce
\[ \epsilon^2 = -du^2 + \epsilon^2 \delta_{ab} e^a \otimes e^b, \] (13)
where \( \epsilon \) is a real positive parameter. Then \[ \det g = \epsilon^6 \det g \],
\[ \mathcal{F}_{0\alpha} = g^0_\alpha \mathcal{F}_{\alpha\beta} - \epsilon^2 \mathcal{F}_{0\beta} \quad \text{and} \quad \mathcal{F}_{ab} = \epsilon^{-4} \mathcal{F}_{ab}, \] (14)
where in \( \mathcal{F}_{0\alpha} \) and \( \mathcal{F}_{ab} \) the indices were raised by the non-deformed metric.

The adiabatic limit of scaling down the metric on \( H^3 \) is effected by the limit \( \epsilon \to 0 \). To avoid the \( \epsilon^{-1} \) divergence of the Yang–Mills action functional, one has to impose the vanishing of the curvature (12) along \( H^3 \),
\[ \mathcal{F}_H = 0, \] (15)
which renders the connection \( A_H \) flat. Substituting (14) into the Yang–Mills equations on the cylinder \( \mathbb{R} \times H^3 \) with the metric (13) and taking the adiabatic limit \( \epsilon \to 0 \) (corresponding to 'slow evolution') together with \( \mathcal{F}_H = 0 \), we obtain
\[ g^{ab} D_a F_{b0} = 0, \] (16)
\[ D_0 F_{0b} = 0, \] (17)
which are, in fact, valid for any \( \epsilon > 0 \) as well.

Let us characterize the 'static' Yang–Mills configurations, i.e. the \( u \)-independent solutions to (15), following [9, 10]. Any flat connection \( A_H \) on \( H^3 \) is formally pure gauge
\[ A_H = \epsilon^{-1} \tilde{d} g \quad \text{with} \quad \tilde{d} = dy^\alpha \frac{\partial}{\partial y^\alpha}, \] (18)
where \( \tilde{d} \) is the exterior derivative on \( H^3 \) and \( g = g(y^\alpha) \) is a smooth map from \( H^3 \) into the gauge group \( G \). Since \( \partial H^3 = S^2 \) is not empty, the group of admissible gauge transformations is
\[ G = \left\{ g \in C^\infty(H^3, G) \mid g|_{\partial H^3} = 1 \right\}. \] (19)
The boundary condition on \( g \) obstructs the removal of
\[ A_{\partial H^3} = \epsilon^{-1} \tilde{d} g \big|_{S^2} \] (20)
by a gauge transformation and renders the flat connection (18) non-trivial. Hence, the solution space of the equation \( \mathcal{F}_H = 0 \) is the infinite-dimensional group
\[ \mathcal{N} = C^\infty(H^3, G), \] (21)
and the moduli space is the quotient group
\[ \mathcal{M} = \mathcal{N}/G = C^\infty(S^2, G). \] (22)
The current groups (19), (21) and (22) as well as the corresponding moduli spaces of flat connections are well studied in the literature, see e.g. [9–11] and references therein. In fact, (19) and (21) are groups of gauge transformations of bundles with and without framing over the boundary \( S^2 \) [12], respectively, and the moduli space (22) is their quotient. In our case, the
framing \((19)\) is equivalent to imposing the Dirichlet boundary conditions, which are natural for Yang–Mills theory on manifolds with boundary \([12]\).

Notice that the group \((22)\) contains as subgroups the loop group \(C^\infty(S^1, G)\) as well as finite-dimensional submanifolds of finite-degree holomorphic maps from \(\mathbb{C}P^1 \cong S^2\) into Kähler coset spaces \(G/H \subset G\). For a physical interpretation of the moduli space \((22)\), we remark that the gauge equivalence classes of flat connections on \(H^3\) are neither solitonic nor instantonic in character, but rather describe different static Yang–Mills vacua. Here, the term ‘static’ refers to our choice of Lorentz-invariant eigentime \(\tau^3\). Since \(\tau_2(G) = 0\) for any compact connected finite-dimensional group \(G\), the moduli space \(\mathcal{M}\) has just a single component.

(5) We introduce local coordinates \(\phi^\alpha\) with \(\alpha = 1, 2, \ldots\) on the moduli space \(\mathcal{M} = C^\infty(S^2, G)\) and assume, following Manton, that \(\mathcal{A}\) on the cylinder \(\mathbb{R} \times H^3\) given by \((11)\) depends on \(u\) (and hence on \(\tau\)) only via the moduli \(\phi^\alpha(u)\). In other words, \(\mathcal{A}_u^\alpha = g^{-1}\phi^\alpha(u); y^a, g(\phi^\alpha(u); \chi \to \infty)\) is determined by \(\phi^\alpha(u)\) and \(\mathcal{A}_0(\phi^\alpha(u))\) will be fixed in a moment. This defines a map

\[
\phi: \mathbb{R} \to \mathcal{M} \quad \text{with} \quad \phi(u) = \{\phi^\alpha(u)\}. \tag{23}
\]

This map is not free—it is constrained by \((16)\) and \((17)\). Since \(\mathcal{A}_u^\alpha\) belongs to the solution space \(\mathcal{N}\) of flatness equations for any \(u \in \mathbb{R}\), its derivative \(\partial_0 \mathcal{A}_u^\alpha\) is a solution of the flatness condition linearized around \(\mathcal{A}_u^\alpha\), i.e., \(\partial_0 \mathcal{A}_u^\alpha\) belongs to the tangent space \(\mathcal{N}_{\partial_0} T_{\mathcal{A}} \mathcal{N}\). With the help of the projection \(\pi: \mathcal{N} \to \mathcal{M}\), one can decompose \(\partial_0 \mathcal{A}_u^\alpha\) into two parts

\[
T_{\mathcal{A}} \mathcal{N} = \pi^* T_{\mathcal{A}} \mathcal{M} \oplus T_{\mathcal{A}} G \quad \Leftrightarrow \quad \partial_0 \mathcal{A}_u^\alpha = (\partial_0 \phi^\alpha) \xi_{\alpha b} + D_b \epsilon_0, \tag{24}
\]

where \(\{\xi_{\alpha b} = \xi_{\alpha b} dy^a\}\) is a local basis of vector fields on \(\mathcal{M}\), and \(\epsilon_0\) is a \(g\)-valued gauge parameter which is determined by the gauge-fixing equation

\[
g^{ab} D_b \xi_{\alpha b} = 0 \quad \Leftrightarrow \quad g^{ab} D_b \partial_0 \mathcal{A}_b^\alpha = g^{ab} D_b D_b \epsilon_0. \tag{25}
\]

Let us fix the gauge on \(\mathbb{R} \times H^3\) by choosing \(A_0 = \epsilon_0\). Then \((24)\) and \((25)\) imply that

\[
\mathcal{F}_{\alpha b} = \partial_0 \mathcal{A}_b = D_b A_0 = \partial_0 \mathcal{A}_b - D_b \epsilon_0 = \phi^\alpha \xi_{\alpha b} = \pi_\alpha \partial_0 \mathcal{A}_b, \tag{26}
\]

where the dot denotes the derivative with respect to \(y^0 = u\). From \((24)\) to \((26)\) we then see that \((16)\) is satisfied. Furthermore, we obtain

\[
\partial_0 \mathcal{A}_a = \phi^\alpha \frac{\partial \mathcal{A}_a}{\partial \phi^\alpha} \quad \Rightarrow \quad \mathcal{A}_0 = \epsilon_0 = \phi^\alpha \epsilon_\alpha, \tag{27}
\]

where the gauge parameters \(\epsilon_\alpha\) can be found as solutions to

\[
g^{ab} D_b D_b \epsilon_0 = g^{ab} D_b \frac{\partial \mathcal{A}_a}{\partial \phi^\alpha}. \tag{28}
\]

(6) Substituting \((26)\) into the remaining equation \((17)\), we arrive at

\[
g^{ab} \frac{d}{du} \left( \phi^\alpha \xi_{\alpha b} \right) = g^{ab} \phi^\beta \left[ \xi_{\alpha b}, \epsilon_0 \right]. \tag{29}
\]

\(^3\) For any adiabatic approximation one must pick some temporal foliation.
Let us multiply this equation with $a^\alpha \xi_{\alpha \nu}$ apply Tr and integrate over $H^3$. This yields
\begin{equation}
\frac{d}{du} \left( G_{\alpha \beta} \phi^{\alpha} \phi^{\beta} \right) = 0,
\end{equation}
where $G_{\alpha \beta}$ are the metric components on the moduli space $\mathcal{M}$, defined as
\begin{equation}
G_{\alpha \beta} = - \int_{H^3} d\text{vol} \ g_{ab} \text{Tr} \left( \xi_{\alpha \nu} \xi_{\beta \mu} \right).
\end{equation}
This metric is the standard left-invariant metric on the Lie group. One can get it by left translations from the Killing–Cartan metric on the tangent space at the identity in $\mathcal{M} = C^\infty (S^3, G)$. However, its calculation may not be an easy task. We postpone its study to the future.

Identifying $\gamma^0 = u$ with the length parameter on $\mathcal{M}$, i.e. choosing the metric as
\begin{equation}
(30)
\end{equation}
becomes the geodesic equation on $\mathcal{M}$ with affine parameter $u$. To see them in more standard form, consider the action
\begin{equation}
\int_{\mathcal{M}} d\text{vol} \ G_{\alpha \beta} \phi^{\alpha} \phi^{\beta}.
\end{equation}
whose Euler–Lagrange equations are
\begin{equation}
\ddot{\phi}^{\alpha} + \Gamma^{\alpha}_{\beta \gamma} \dot{\phi}^{\beta} \dot{\phi}^{\gamma} - \phi^{\alpha} \frac{d}{du} \ln (G_{\beta \gamma} \phi^{\beta} \phi^{\gamma}) = 0 \quad \Rightarrow \quad \ddot{\phi}^{\alpha} + \Gamma^{\alpha}_{\beta \gamma} \dot{\phi}^{\beta} \dot{\phi}^{\gamma} = 0,
\end{equation}
where the Christoffel symbols are
\begin{equation}
\Gamma^{\alpha}_{\beta \gamma} = \frac{1}{2} G^{\alpha \lambda} \left( \frac{\partial}{\partial \phi^{\gamma}} G_{\beta \lambda} + \frac{\partial}{\partial \phi^{\beta}} G_{\gamma \lambda} - \frac{\partial}{\partial \phi^{\lambda}} G_{\beta \gamma} \right).
\end{equation}
This derivation reflects the equivalence of the action (33) and the functional
\begin{equation}
S = \int_{\mathcal{M}} G_{\alpha \beta} \dot{\phi}^{\alpha} \dot{\phi}^{\beta}.
\end{equation}
The latter is the effective Yang–Mills action in the adiabatic limit $\varepsilon \to 0$ and stems from the term
\begin{equation}
\int_{\mathbb{R} \times H^3} d\text{vol} \ Tr(\mathcal{F}_{ab} \mathcal{F}^{ab})
\end{equation}
in the original Yang–Mills action functional. Since $\mathcal{M}$ is a Lie group, one can construct geodesics as one-parameter subgroups, and we intend to do this in a separate publication. The physical meaning of the moduli parameters $\phi^{\alpha}$ will become clear from the properties of such solutions.

If we assume that $\mathcal{F}^{ab} = 0$ for any $\tau = e^\tau$, then (16) and (17) form all Yang–Mills equations on $\mathbb{R} \times H^3$ for any $\varepsilon \geq 0$ including $\varepsilon = 1$. Their solutions
\begin{equation}
(38)
\end{equation}
carry electrical but no magnetic charge since $\mathcal{F}_{ab} = 0$ while $\mathcal{F}_{ab} = 0$. From the implicit function theorem it follows that for any solution $A^{\tau = 0}_{\alpha \nu}$ defined by $\phi$ satisfying (34) there exist nearby solutions $A^{\varepsilon = 0}_{\alpha \nu}$ of the Yang–Mills equations for $\varepsilon$ sufficiently small, and we conjecture

4 The right-hand side of (29) disappears since $g^{ab} \phi^a \phi^b \text{Tr} (\xi_{\alpha \nu} \xi_{\beta \mu} | e_0) \equiv 0$.
5 In general $\mathcal{F}^{ab} = 0$ is mandatory unless $\varepsilon \to 0$. 

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that the moduli space of all geodesics \((34)\) in \(C^∞(S^2_\infty, G)\) is bijective to the moduli space of solutions to the Yang–Mills equations.

(7) In conclusion, we reduced Yang–Mills theory on Minkowski space in a certain adiabatic limit to a one-dimensional sigma model with the target space \(\mathcal{M} = C^∞(S^2_\infty, G)\), which should capture the low-energy dynamics of the gauge theory. We note that the group \(C^∞(\Sigma, G)\) of smooth maps from a Riemannian surface \(\Sigma\) (including the case of \(S^2\)) into a Lie group \(G\) has been considered by mathematicians (see e.g. [13, 14]) but did not yet find a true application in physics. This short article indicates relations of such groups with Yang–Mills theory in four-dimensions.

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