Solution of the Bethe Equation Through the Laplace-Adomian Decomposition Method

O. González-Gaxiola, A. León-Ramírez, G. Chacón-Acosta

Abstract The Bethe equation is a nonlinear differential equation that plays an important role in nuclear physics and a variety of applications related to it, such as the description of the behavior of an energetic particle when it penetrates into matter. Despite its importance, its unusual to find the exact solution to this nonlinear equation in literature and practically all of them are of experimental nature. In this paper, we solve this equation and present a new approach to obtain the solution through the combined use of the Adomian Decomposition Method and the Laplace Transform (LADM). In addition, we illustrate our approach solving three examples, in which initial conditions are considered within the typical numerical ranges derived from the applications. Our results indicate that LADM is highly accurate and can be considered a very useful and valuable method.

Keywords Bethe equation · Nonlinear equation · Stopping power · Adomian decomposition method · Laplace transform.

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1 Introduction

Many of the phenomena that arise in the real world can be described by means of nonlinear partial and ordinary differential equations and, in some cases, by integro-differential equations. However, most of the mathematical methods developed thus far are only capable of solving linear differential equations. In the 1980’s, George Adomian (1923-1996) introduced a powerful method to solve nonlinear differential equations, known as the Adomian decomposition method (ADM) [4,5]. The technique is based on the decomposition of a solution of nonlinear differential equations into a series of functions. Each term of the series is obtained from a polynomial generated by a power series expansion of an analytic function. The Adomian method is very simple in an abstract formulation; however, calculating the polynomials is difficult, which becomes a non-trivial task. This method has been widely

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used to solve equations that come from nonlinear models, as well as to solve fractional
differential equations [11,12,26,27]. The chaotic nature and nonlinearity of other systems,
proposed in the past, have been studied through ADM in [18]. The advantage of this method
is that it solves the problem directly without the need of linearization, perturbation, or any
other transformation, and also, requires relatively lesser computational effort compared to
most other methods.

Quantitative and accurate information on the penetration of high energy particles through
matter, in particular the systematics of energy loss, is a topic of interest in basic science
[22,25,29], medicine and technology [6,16,23,24,28,31,35]. Until the middle of the past
century, studies of charged-particle penetration were stimulated almost exclusively by the
needs of fundamental physics research, but applications in other areas gradually became
important. The first studies of charged particle penetration were stimulated by experiments
on gas discharges toward the end of the 19’th century, but experimental possibilities were
greatly enhanced after the discovery of radioactivity, in particular the pioneering work by
E. Rutherford and coworkers in the beginning of the 20’th century. Pioneering theoretical
studies by J. J. Thomson and N. Bohr date back to the same time. Subsequently, after the
development of quantum mechanics, quantum theory of particle stopping was developed by
H. Bethe, F. Bloch, W. H. Barkas, H. H. Andersen and others [21,36].

Under certain assumptions, the stopping power in a medium is given by the relativistic Bethe
equation [8]. Despite its importance in several physics models, the exact solution of this
nonlinear equation have not been obtained. In the present work, we will use the Adomian
decomposition method in combination with the Laplace transform (LADM) [33] to deter-
mine the approximate solution to the Bethe equation. We decompose the nonlinear terms of
this equation using the Adomian polynomials and then, in combination with the use of the
Laplace transform, we obtain an algorithm to solve the problem subject to initial conditions.
Finally, we illustrate our procedure and the quality of the algorithm obtained by solving sev-
eral numerical examples in which the nonlinear differential equation is solved for different
initial conditions.

Our work is divided into several sections. In the “Adomian Decomposition Method Com-
bined With Laplace Transform” section, we present, in a brief and self-contained manner,
the LADM. Several references are given to delve deeper into the subject and to study its
mathematical foundation, which is beyond the scope of the present work. In the “The Bethe
Equation” section, we present a brief introduction to the model described by the Bethe equation.
In the “Solution of the Bethe Equation Through LADM” section, we establish that
LADM can be used to solve this equation in a very simple way. In “Numerical Examples”
section, we show by means of three examples, the quality and precision of our method, com-
paring the obtained results with existing approximate solutions available in the literature and
obtained by other methods. Finally, in the “Summary and Conclusions” section, we present
the conclusions and implications of this study.

2 The Adomian Decomposition Method Combined with Laplace Transform

The ADM is a method to solve ordinary and nonlinear differential equations. Using this
method, it is possible to express analytic solutions in terms of a series [5]. In other words,
the method identifies and separates the linear and nonlinear parts of a differential equation.
By inverting and applying the highest order differential operator that is contained in the
linear part of the equation, it is possible to express the solution in terms of the rest of the
equation affected by the inverse operator. At this point, the solution is proposed through a
series of terms that will be determined and that will result in the Adomian Polynomials \[32\]. The nonlinear part can also be expressed in terms of these polynomials. The initial (or the border conditions) and the terms that contain the independent variables will be considered as the initial approximation. In this manner, and through recurrence relations, it is possible to find the terms of the series that give the approximate solution of the differential equation. In the next paragraph, we will see how to use the ADM in combination with the Laplace transform (LADM).

Let us consider the following homogeneous differential equation of first order:

$$\frac{du}{dx} + N(u) = 0$$

with the initial condition

$$u(0) = u_0$$

where \(u_0\) is a real constant and \(N\) is a nonlinear operator acting on the dependent variable \(u\) and some of its derivatives.

In general, if we consider the first-order differential operator \(L_x = \frac{d}{dx}\), then the equation \((1)\) can be written as

$$L_x u(x) + N(u(x)) = 0.$$ 

Solving for \(L_x u(x)\), we have

$$L_x u(x) = -N(u(x)).$$

The LADM consists of applying Laplace transform (denoted throughout this paper by \(\mathcal{L}\)) first on both sides of Eq. \((1)\), thereby obtaining

$$\mathcal{L}\{L_x u(x)\} = -\mathcal{L}\{N(u(x))\}.$$ 

An equivalent expression to \((5)\) is

$$su(s) - u(0) = -\mathcal{L}\{N(u(x))\},$$

using the initial condition \((2)\), we have

$$u(s) = \frac{u_0}{s} - \frac{1}{s} \mathcal{L}\{N(u(x))\}.$$ 

Now, applying the inverse Laplace transform to equation \((7)\)

$$u(x) = u_0 - \mathcal{L}^{-1}\left\{\frac{1}{s} \mathcal{L}\{N(u(x))\}\right\}.$$ 

The ADM proposes a series of solutions \(u(x)\), given by,

$$u(x) = \sum_{n=0}^{\infty} u_n(x).$$

The nonlinear term \(N(u)\) is given by

$$N(u) = \sum_{n=0}^{\infty} A_n(u_0, u_1, \ldots, u_n)$$

where \(\{A_n\}_{n=0}^\infty\) is the so-called Adomian polynomials sequence established in \[32\] and \[7\] and, in general, gives us term by term:

\(A_0 = N(u_0)\)
\[ A_1 = u_1 N'(u_0) \]
\[ A_2 = u_2 N'(u_0) + \frac{1}{2} u_2^2 N''(u_0) \]
\[ A_3 = u_3 N'(u_0) + u_1 u_2 N''(u_0) + \frac{1}{3} u_2^3 N'''(u_0) \]
\[ A_4 = u_4 N'(u_0) + \left( \frac{1}{2} u_2^2 + u_1 u_3 \right) N''(u_0) + \frac{1}{3} u_2^4 N''(u_0) + \frac{1}{5} u_2^6 N''(u_0) + \frac{1}{7} u_2^8 N''(u_0) \]
\[ \vdots \]

Other polynomials can be generated in a similar manner. Some other approaches to obtaining Adomian’s polynomials can be found in [13,15].

Using (9) and (10) in equation (8), we obtain,

\[ \sum_{n=0}^{\infty} u_n(x) = u_0 - L^{-1} \left[ \frac{1}{s} L' \left( \sum_{n=0}^{\infty} A_n(u_0, u_1, \ldots, u_n) \right) \right]. \] (11)

From equation (11), we deduce the recurrence formula, which is given as follows:

\[ \begin{align*}
  u_0(x) &= u_0, \\
  u_{n+1}(x) &= -L^{-1} \left[ \frac{1}{s} L' \left( A_n(u_0, u_1, \ldots, u_n) \right) \right], \quad n = 0, 1, 2, \ldots
\end{align*} \] (12)

Using (12), we can obtain an approximate solution of (1), (2) using

\[ u(x) \approx \sum_{n=0}^{k} u_n(x), \quad \text{where} \quad \lim_{k \to \infty} \sum_{n=0}^{k} u_n(x) = u(x). \] (13)

It is evident that, the Adomian decomposition method, combined with the Laplace transform requires less effort in comparison with the traditional Adomian decomposition method. This method considerably decreases the volume of calculations. The decomposition procedure of Adomian is easily set, without requiring the linearization of the problem. With this approach, the solution comes in the form of a convergent series with easily computed components; in many cases, the convergence of this series is very fast and only a few terms are needed to understand how the solutions behave. Convergence conditions of this series are examined by several authors, mainly in [1,2,9,10]. Additional references related to the use of the ADM, combined with the Laplace transform, can be found in [33,20,34] and references therein.

### 3 The Bethe Equation

The Bethe equation for the collision stopping power for incoming charged particles (such as an electron) when it penetrates into matter, is the nonlinear differential equation [8]:

\[ \frac{du}{dx} + \frac{\ln(u+1)}{u} = 0, \quad u(0) = u_0. \] (14)

Here \( u \) is a dimensionless measure of the kinetic energy, and \( x \) is a dimensionless measure of the distance that the particle has penetrated into the matter [3]. The value \( x = 0 \) corresponds to the surface of the material, where the particle has the high initial energy \( u(0) = u_0 \).

To the best of our knowledge, no exact solution of the nonlinear equation (14) has yet been published; therefore the research work about equation (14) has been intense. In [19] was recently obtained, using an old Chinese algorithm, an approximation to point \( R \) in which the kinetic energy \( u \) described in (14) is canceled. For technical considerations
the value $R$ in which the kinetic energy is canceled occurs when $u = 1$; the value of $R$ as a function of the initial kinetic energy $u_0$ calculated in [19] is

$$ R = \frac{u_0^2}{2(\ln u_0 - 0.55)}. $$

(15)

Whereas in [3], the value of $x_0$ as a function of the initial kinetic energy $u_0$ was given by the formula

$$ R = \frac{u_0^2}{2(\ln u_0 - 0.5)}. $$

(16)

In the following section, we will develop an algorithm using the method described in the “Adomian Decomposition Method Combined with Laplace Transform” section in order to solve the nonlinear differential equation (14) without resorting to any truncation or linearization. Then we will use that algorithm to solve three problems with initial values of kinetic energy $u_0$, included in the typical ranges [3], with which we will illustrate that the method used is efficient and highly accurate.

4 Solution of the Bethe Equation Through LADM

Comparing (14) with equation (4) we have that $L$, $x$ and $N$ becomes:

$$ L_x u = \frac{d}{dx} u_0, \quad Nu = \frac{\ln(u + 1)}{u}. $$

(17)

Now, by using equation (12) through the LADM method, we recursively obtain

$$ \begin{aligned}
    & u_0(x) = u_0, \\
    & u_{n+1}(x) = -L^{-1}[\frac{1}{5} \cdot \mathcal{L}^{-1}\{A_n(u_0, u_1, \ldots, u_n)\}], \quad n = 0, 1, 2, \ldots
\end{aligned} $$

(18)

In addition, the nonlinear term is decomposed as

$$ Nu = \frac{\ln(u + 1)}{u} = \sum_{n=0}^{\infty} A_n(u_0, u_1, \ldots, u_n) $$

(19)

where $\{A_n\}_{n=0}^{\infty}$ is the so-called Adomian polynomials sequence, the terms are calculated according to [13,14,15]. The first few polynomials are given by

$$ A_0(u_0) = \frac{\ln(u_0 + 1)}{u_0}, $$

(20)

$$ A_1(u_0, u_1) = \frac{u_1}{u_0} - \frac{u_1 \ln(u_0 + 1)}{u_0^2}, $$

(21)

$$ A_2(u_0, u_1, u_2) = -\frac{u_1^2}{u_0^2} - \frac{u_1^2}{2u_0 (u_0 + 1)^2} + \frac{u_2}{u_0 (u_0 + 1)} + \frac{u_2^2}{u_0^2} - \frac{u_2 \ln(u_0 + 1)}{u_0^3}. $$

(22)
\[
A_3(u_0, \ldots, u_3) = \frac{u_1^3}{u_0(u_0+1)} + \frac{u_1^3}{2u_0^2(u_0+1)^2} + \frac{u_1^3}{3u_0(u_0+1)^3} - \frac{2u_2u_1}{u_0(u_0+1)} - \frac{u_2u_1}{u_0(u_0+1)^2} - \frac{u_3}{u_0(u_0+1)} - \frac{u_3^2\ln(u_0+1)}{u_0^3} - \frac{2u_2u_1\ln(u_0+1)}{u_0} \tag{23}
\]

\[
A_4(u_0, \ldots, u_4) = -\frac{u_1^4}{u_0(u_0+1)} - \frac{u_1^4}{2u_0^2(u_0+1)^2} - \frac{u_1^4}{2u_0^2(u_0+1)^2} - \frac{u_1^4}{3u_0^2(u_0+1)^3} - \frac{u_1^4}{4u_0(u_0+1)^4} + \frac{3u_2u_1^2}{u_0(u_0+1)} + \frac{3u_2u_1^2}{u_0^2(u_0+1)^2} + \frac{2u_2u_1^2}{u_0^2(u_0+1)^2} - \frac{2u_3u_1}{u_0(u_0+1)^2} - \frac{u_4}{u_0(u_0+1)} + \frac{u_4\ln(u_0+1)}{u_0^3} + \frac{2u_3u_1\ln(u_0+1)}{u_0} \tag{24}
\]

\[
A_5(u_0, \ldots, u_5) = \frac{u_1^5}{u_0(u_0+1)} + \frac{u_1^5}{2u_0^2(u_0+1)^2} + \frac{u_1^5}{3u_0^3(u_0+1)^3} + \frac{u_1^5}{4u_0^4(u_0+1)^4} + \frac{u_1^5}{5u_0^5(u_0+1)^5} - \frac{4u_2u_1^4}{u_0(u_0+1)} - \frac{4u_2u_1^4}{u_0^2(u_0+1)^2} - \frac{4u_2u_1^4}{3u_0^3(u_0+1)^3} - \frac{4u_2u_1^4}{4u_0^4(u_0+1)^4} + \frac{5u_3u_1^3}{u_0(u_0+1)^2} + \frac{5u_3u_1^3}{u_0^2(u_0+1)^2} + \frac{5u_3u_1^3}{u_0^3(u_0+1)^3} + \frac{5u_3u_1^3}{u_0^4(u_0+1)^4} - \frac{3u_4u_1^2}{u_0(u_0+1)} + \frac{3u_4u_1^2}{u_0^2(u_0+1)^2} + \frac{3u_4u_1^2}{u_0^3(u_0+1)^3} - \frac{3u_4u_1^2}{u_0^4(u_0+1)^4} + \frac{u_5}{u_0} + \frac{u_5}{u_0^2(u_0+1)^2} + \frac{u_5}{u_0^3(u_0+1)^3} + \frac{u_5}{u_0^4(u_0+1)^4} \tag{25}
\]
\[ A_6(u_0, \ldots, u_6) = \frac{\ln (u_0 + 1)}{u_0} \frac{u_0^4}{u_0^4} - \frac{u_0^4}{u_0^4 (u_0 + 1)} \frac{u_0^4}{u_0^4} - 2u_0^4 (u_0 + 1) u_2^2 - \frac{u_0^4}{u_0^4 (u_0 + 1)^3} \]

\[ - \frac{u_0^4}{4u_0^4 (u_0 + 1)^4} - \frac{u_0^4}{5u_0^4 (u_0 + 1)^5} - \frac{u_0^4}{u_0^4 (u_0 + 1)^6} + \frac{5u_0^4}{u_0^4 (u_0 + 1)^7} \]

\[ + 6u_0^4 (u_0 + 1)^3 - \frac{5u_0^4}{u_0^4 (u_0 + 1)^2} + \frac{5u_0^4}{u_0^4 (u_0 + 1)^3} + 4u_0^4 (u_0 + 1)^4 + u_0 (u_0 + 1)^5 \]

\[ - 5 \ln (u_0 + 1) u_2 u_2^2 + 4 \ln (u_0 + 1) u_2 u_2^2 u_2^3 - \frac{4u_2 u_2^3}{u_0^4 (u_0 + 1)} - \frac{2u_2 u_2^3}{u_0^4 (u_0 + 1)^2} \]

\[ + \frac{4u_2 u_2^3}{3u_0^4 (u_0 + 1)^3} - \frac{u_2 u_2^3}{u_0 (u_0 + 1)^3} + 6 \ln (u_0 + 1) u_2 u_2^3 \]

\[ + \frac{3u_2 u_2^3}{u_0^4 (u_0 + 1)^3} \]

\[ + \frac{2u_2 u_2^3}{u_0 (u_0 + 1)^3} - \frac{3u_2 u_2^3}{u_0^4 (u_0 + 1)^3} + \frac{6u_2 u_2^3 u_2^3}{u_0^4 (u_0 + 1)^3} \]

\[ + \frac{3u_2 u_2^3 u_2^3}{u_0^4 (u_0 + 1)^3} + \frac{2u_2 u_2^3 u_2^3}{u_0 (u_0 + 1)^3} - \frac{2 \ln (u_0 + 1) u_2 u_2^3}{u_0^4} - \frac{6 \ln (u_0 + 1) u_2 u_2^3 u_2^3}{u_0^4} \]

\[ + \frac{2u_2 u_2^3 u_2^3}{u_0^4 (u_0 + 1)^3} - \frac{u_2 u_2^3 u_2^3}{u_0 (u_0 + 1)^3} + \frac{u_2 u_2^3 u_2^3}{u_0^4 (u_0 + 1)^3} - \frac{2u_2 u_2^3 u_2^3}{u_0 (u_0 + 1)^3} + \frac{u_2 u_2^3 u_2^3}{u_0^4 (u_0 + 1)^3} - \frac{2u_2 u_2^3 u_2^3}{u_0 (u_0 + 1)^3} \]

\[ + \frac{\ln (u_0 + 1) u_6}{u_0^4} - \frac{\ln (u_0 + 1) u_6}{u_0^4 (u_0 + 1)^2} + \frac{u_6}{u_0^4 (u_0 + 1)^3} \]

\[ + \frac{\ln (u_0 + 1) u_6}{u_0^4} - \frac{\ln (u_0 + 1) u_6}{u_0^4 (u_0 + 1)^2} + \frac{u_6}{u_0^4 (u_0 + 1)^3} \]

\[ + \frac{u_2 u_4}{u_0 (u_0 + 1)^2} - \frac{u_2 u_4}{u_0 (u_0 + 1)^2} + \frac{u_2 u_4}{u_0 (u_0 + 1)^2} - \frac{u_2 u_4}{u_0 (u_0 + 1)^2} + \frac{u_2 u_4}{u_0 (u_0 + 1)^2} - \frac{u_2 u_4}{u_0 (u_0 + 1)^2} \]

(26)
\[ A_7(u_0, \ldots, u_7) = -\frac{\ln(u_0+1)u_7^3}{u_0^3} + \frac{u_7^2}{u_0^2(u_0+1)} + \frac{u_7^2}{2u_0^2(u_0+1)^3} + \frac{u_7^2}{3u_0^3(u_0+1)^3} + \]
\[ + \frac{u_7^4}{4u_0^4(u_0+1)^4} + \frac{u_7^4}{5u_0^5(u_0+1)^5} + \frac{u_7^4}{6u_0^6(u_0+1)^6} + \frac{u_7^4}{7u_0^7(u_0+1)^7} + 
\[ + \frac{6 \ln(u_0+1)u_7^2 u_1^2}{u_0^3} - \frac{6u_7^2 u_1^2}{u_0^3(u_0+1)} - \frac{u_7^2 u_1^2}{u_0^5(u_0+1)^2} - \frac{2u_7 u_1^2}{u_0^7(u_0+1)^3} - \frac{3u_7 u_1^3}{u_0^8(u_0+1)^4} - 
\[ - \frac{2u_7^3 u_1^3}{u_0^7(u_0+1)^4} - \frac{5u_7^3 u_1^3}{u_0^5(u_0+1)^5} - \frac{u_7^5 u_1^3}{u_0^7(u_0+1)^6} - \frac{u_7^3 u_1^3}{u_0^5(u_0+1)^7} + 
\[ + \frac{5u_7^3 u_1^3}{u_0^5(u_0+1)^7} + \frac{2u_7^3 u_1^3}{u_0^5(u_0+1)^7} + \frac{4u_7 u_1^3}{u_0^3(u_0+1)^3} + \frac{4u_7 u_1^3}{u_0^5(u_0+1)^5} + \frac{4u_7 u_1^3}{u_0^7(u_0+1)^7} 
\[ + \frac{12 \ln(u_0+1)u_7^2 u_2^2 u_3 u_4^2}{u_0^3} + \frac{3u_7^2 u_2^2 u_3 u_4^2}{u_0^3(u_0+1)} + \frac{3u_7^2 u_2^2 u_3 u_4^2}{u_0^5(u_0+1)^2} + \frac{u_7^2 u_2^2 u_3 u_4^2}{u_0^7(u_0+1)^3} - 
\[ - \frac{3 \ln(u_0+1)u_7^2 u_2^2 u_3 u_4^2}{u_0^3} + \frac{12u_7^2 u_2^2 u_3 u_4^2}{u_0^3(u_0+1)} - \frac{6u_7^2 u_2^2 u_3 u_4^2}{u_0^5(u_0+1)^2} - \frac{4u_7^2 u_2^2 u_3 u_4^2}{u_0^7(u_0+1)^3} - 
\[ - \frac{3u_7^3 u_2^2 u_3 u_4^2}{u_0^3(u_0+1)^3} + \frac{4u_7^3 u_2^2 u_3 u_4^2}{u_0^3(u_0+1)^3} + \frac{3u_7^3 u_2^2 u_3 u_4^2}{u_0^5(u_0+1)^2} + \frac{3u_7^3 u_2^2 u_3 u_4^2}{u_0^7(u_0+1)^3} + 
\[ + \frac{u_7^3 u_2^3 u_3 u_4^2}{u_0^3(u_0+1)^4} + \frac{6u_7^3 u_2^3 u_3 u_4^2}{u_0^3(u_0+1)^4} + \frac{3u_7^3 u_2^3 u_3 u_4^2}{u_0^5(u_0+1)^2} + \frac{2u_7^3 u_2^3 u_3 u_4^2}{u_0^7(u_0+1)^3} + 
\[ + 2 \ln(u_0+1)u_7^2 u_3 u_4^2 + \frac{3 \ln(u_0+1)u_7^2 u_3 u_4^2}{u_0^3} + \frac{6 \ln(u_0+1)u_7^2 u_3 u_4^2}{u_0^3(u_0+1)} + 
\[ - \frac{2u_7^4 u_3 u_4^2}{u_0^3(u_0+1)} - \frac{4u_7^4 u_3 u_4^2}{u_0^5(u_0+1)^2} - \frac{4u_7^4 u_3 u_4^2}{u_0^7(u_0+1)^3} - 
\[ - \frac{3u_7^5 u_3 u_4^2}{u_0^6(u_0+1)^3} - \frac{6u_7^5 u_3 u_4^2}{u_0^8(u_0+1)^5} + \frac{2u_7^5 u_3 u_4^2}{u_0^6(u_0+1)^3} + \frac{3u_7^5 u_3 u_4^2}{u_0^8(u_0+1)^5} + 
\[ + \frac{u_7^6 u_3 u_4^2}{u_0^7(u_0+1)^4} + \frac{2 \ln(u_0+1)u_7^2 u_3 u_4^2}{u_0^3} + \frac{2 \ln(u_0+1)u_7^2 u_3 u_4^2}{u_0^5(u_0+1)^2} + \frac{u_7^7}{u_0^8(u_0+1)^5} + 
\[ - \frac{\ln(u_0+1)u_7^2}{u_0^3(u_0+1)^2} - \frac{3 \ln(u_0+1)u_7^2}{u_0^5(u_0+1)^2} - \frac{2u_7 u_5}{u_0^3(u_0+1)} + \frac{2u_7 u_5}{u_0^5(u_0+1)^2} - 
\[ - \frac{u_7^2 u_5}{u_0^7(u_0+1)^3} - \frac{u_7^2 u_5}{u_0^9(u_0+1)^5}. \]
Now, recursively using (18) with the Adomian polynomials given by the later sequence \
\{A_n\}_{n=0}^\infty, we obtain, for a given initial condition \(u_0\):

\[ u_0(x) = u_0, \quad (28) \]

\[ u_1(x) = -\mathcal{L}^{-1} \left[ \frac{1}{s} \mathcal{L} \{ A_0(u_0) \} \right] = -\frac{x \log(u_0 + 1)}{u_0}, \quad (29) \]

\[ u_2(x) = -\mathcal{L}^{-1} \left[ \frac{1}{s} \mathcal{L} \{ A_1(u_0, u_1) \} \right] \]
\[ = -\frac{x^2 \ln(u_0 + 1)}{2u_0^2(u_0 + 1)} \left[ -u_0 + u_0 \ln(u_0 + 1) + \ln(u_0 + 1) \right], \quad (30) \]

\[ u_3(x) = -\mathcal{L}^{-1} \left[ \frac{1}{s} \mathcal{L} \{ A_2(u_0, u_1, u_2) \} \right] \]
\[ = -\frac{x^3 \ln(u_0 + 1)}{6u_0^3(u_0 + 1)^2} \left[ u_0^2 + 3u_0^2 \ln^2(u_0 + 1) + 6u_0 \ln^2(u_0 + 1) + 3\ln^2(u_0 + 1) \
- 5u_0^2 \ln(u_0 + 1) - 4u_0 \ln(u_0 + 1) \right], \quad (31) \]

\[ u_4(x) = -\mathcal{L}^{-1} \left[ \frac{1}{s} \mathcal{L} \{ A_3(u_0, u_1, u_2, u_3) \} \right] \]
\[ = -\frac{x^4 \ln(u_0 + 1)}{24u_0^4(u_0 + 1)^3} \left[ -u_0^4 + 15u_0^3 \ln^3(u_0 + 1) + 45u_0^2 \ln^3(u_0 + 1) \
+ 45u_0 \ln^3(u_0 + 1) + 15 \ln^3(u_0 + 1) - 34u_0^2 \ln^2(u_0 + 1) - 57u_0^2 \ln^2(u_0 + 1) \
- 25u_0 \ln^2(u_0 + 1) + 15u_0 \ln(u_0 + 1) + 11u_0 \ln(u_0 + 1) \right], \quad (32) \]

\[ u_5(x) = -\mathcal{L}^{-1} \left[ \frac{1}{s} \mathcal{L} \{ A_4(u_0, u_1, \ldots, u_4) \} \right] \]
\[ = -\frac{x^5 \ln(u_0 + 1)}{120u_0^5(u_0 + 1)^4} \left[ u_0^4 + 105u_0^4 \ln^4(u_0 + 1) + 420u_0^3 \ln^4(u_0 + 1) \
+ 630u_0^2 \ln^4(u_0 + 1) + 420u_0 \ln^4(u_0 + 1) + 105 \ln^4(u_0 + 1) \
- 298u_0^3 \ln^3(u_0 + 1) - 772u_0^2 \ln^3(u_0 + 1) - 690u_0 \ln^3(u_0 + 1) \
- 210u_0^2 \ln^3(u_0 + 1) - 207u_0^2 \ln^2(u_0 + 1) + 319u_0 \ln^2(u_0 + 1) \
+ 130u_0 \ln^2(u_0 + 1) - 37u_0 \ln(u_0 + 1) - 26u_0 \ln(u_0 + 1) \right], \quad (33) \]
\[ u_6(x) = -\mathcal{L}^{-1}\left[ \frac{1}{s} \mathcal{L}\{A_5(u_0, u_1, \ldots, u_5)\} \right] \]
\[ = -\frac{x^6 \ln (u_0 + 1)}{720 u_0^5} \left[ -u_0^5 + 945 u_0^5 \ln^5 (u_0 + 1) + 4725 u_0^5 \ln^5 (u_0 + 1) \\
+ 9450 u_0^5 \ln^5 (u_0 + 1) + 9450 u_0^5 \ln^5 (u_0 + 1) + 4725 u_0 \ln^5 (u_0 + 1) \\
+ 945 \ln^5 (u_0 + 1) - 3207 u_0^5 \ln^4 (u_0 + 1) - 11310 u_0^5 \ln^4 (u_0 + 1) \\
- 15372 u_0^5 \ln^4 (u_0 + 1) - 9450 u_0^5 \ln^4 (u_0 + 1) - 2205 u_0 \ln^4 (u_0 + 1) \\
+ 3055 u_0^3 \ln^3 (u_0 + 1) + 7313 u_0^3 \ln^3 (u_0 + 1) + 6104 u_0^3 \ln^3 (u_0 + 1) \\
+ 1750 u_0^3 \ln^3 (u_0 + 1) - 954 u_0^3 \ln^2 (u_0 + 1) - 1402 u_0^3 \ln^2 (u_0 + 1) \\
- 546 u_0^3 \ln^2 (u_0 + 1) + 83 u_0^3 \ln (u_0 + 1) + 57 u_0^3 \ln (u_0 + 1) \right] \]

\[ u_7(x) = -\mathcal{L}^{-1}\left[ \frac{1}{s} \mathcal{L}\{A_6(u_0, u_1, \ldots, u_6)\} \right] \]
\[ = -\frac{x^7 \ln (u_0 + 1)}{5040 u_0^6} \left[ u_0^6 + 10395 u_0^6 \ln^6 (u_0 + 1) + 62370 u_0^6 \ln^6 (u_0 + 1) \\
+ 155925 u_0^6 \ln^6 (u_0 + 1) + 207900 u_0^6 \ln^6 (u_0 + 1) + 155925 u_0^6 \ln^6 (u_0 + 1) \\
+ 62370 u_0 \ln^6 (u_0 + 1) + 10395 \ln^6 (u_0 + 1) - 40947 u_0^5 \ln^5 (u_0 + 1) \\
- 183312 u_0^5 \ln^5 (u_0 + 1) - 335706 u_0^5 \ln^5 (u_0 + 1) - 311976 u_0^5 \ln^5 (u_0 + 1) \\
- 146475 u_0^5 \ln^5 (u_0 + 1) - 27720 u_0 \ln^5 (u_0 + 1) + 49640 u_0^5 \ln^4 (u_0 + 1) \\
+ 162636 u_0^5 \ln^4 (u_0 + 1) + 207403 u_0^5 \ln^4 (u_0 + 1) + 120582 u_0^5 \ln^4 (u_0 + 1) \\
+ 26775 u_0^5 \ln^4 (u_0 + 1) - 22714 u_0^5 \ln^3 (u_0 + 1) - 51800 u_0^5 \ln^3 (u_0 + 1) \\
- 41328 u_0^5 \ln^3 (u_0 + 1) - 11368 u_0^5 \ln^2 (u_0 + 1) + 3775 u_0^5 \ln^2 (u_0 + 1) \\
+ 5388 u_0^5 \ln^2 (u_0 + 1) + 2037 u_0^5 \ln (u_0 + 1) - 177 u_0^5 \ln (u_0 + 1) - 120 u_0^5 \ln (u_0 + 1) \right] \]
\[ u_8(x) = -\mathcal{L}^{-1} \left[ \frac{1}{s^6} \mathcal{L} \{ A_7(u_0, u_1, \ldots, u_7) \} \right] \]
\[ = -\frac{x^8 \ln(u_0 + 1)}{40320 u_0^5 (u_0 + 1)} \left[ -u_0^7 + 135135 u_0^7 \ln^7(u_0 + 1) + 945945 u_0^7 \ln^7(u_0 + 1) \right. \]
\[ + 2837835 u_0^7 \ln^7(u_0 + 1) + 4729725 u_0^7 \ln^7(u_0 + 1) + 4729725 u_0^7 \ln^7(u_0 + 1) \]
\[ + 2837835 u_0^7 \ln^7(u_0 + 1) + 945945 u_0^7 \ln^7(u_0 + 1) + 135135 \ln^7(u_0 + 1) \]
\[ - 605076 u_0^7 \ln^6(u_0 + 1) - 3289587 u_0^7 \ln^6(u_0 + 1) - 7593561 u_0^7 \ln^6(u_0 + 1) \]
\[ - 9468270 u_0^7 \ln^6(u_0 + 1) - 6701310 u_0^7 \ln^6(u_0 + 1) - 2546775 u_0^7 \ln^6(u_0 + 1) \]
\[ - 405405 u_0^7 \ln^6(u_0 + 1) + 891002 u_0^7 \ln^6(u_0 + 1) + 3724256 u_0^7 \ln^6(u_0 + 1) \]
\[ + 6426369 u_0^7 \ln^5(u_0 + 1) + 5667795 u_0^7 \ln^5(u_0 + 1) + 2539845 u_0^7 \ln^5(u_0 + 1) \]
\[ + 460845 u_0^7 \ln^5(u_0 + 1) - 543482 u_0^7 \ln^4(u_0 + 1) - 1697378 u_0^7 \ln^4(u_0 + 1) \]
\[ - 2071335 u_0^7 \ln^4(u_0 + 1) + 1156750 u_0^7 \ln^4(u_0 + 1) - 247555 u_0^7 \ln^4(u_0 + 1) \]
\[ + 139931 u_0^7 \ln^3(u_0 + 1) + 309057 u_0^7 \ln^3(u_0 + 1) + 238971 u_0^7 \ln^3(u_0 + 1) \]
\[ + 63805 u_0^7 \ln^3(u_0 + 1) - 13626 u_0^7 \ln^2(u_0 + 1) - 19083 u_0^7 \ln^2(u_0 + 1) \]
\[ - 7071 u_0^7 \ln^2(u_0 + 1) + 367 u_0^7 \ln(u_0 + 1) + 247 u_0^6 \ln(u_0 + 1) \right], \quad (36) \]

In view of equations (28)-(36), and considering the equation (13), the approximate solution of the Bethe equation (14) is

\[ u_{LADM}(x) = u_0(x) + u_1(x) + u_2(x) + \cdots + u_8(x). \quad (37) \]

In Eq. (37) the approximate solution to the Bethe equation (14) depends on the initial condition \( u_0 \). Numerically \( u_0 \) is typically \( 10^4 \) to \( 10^6 \) [17].

5 Numerical Examples

In the current section, using the expressions obtained above for the solution of equation (14), we illustrate, with three examples, the effectiveness of LADM to solve the nonlinear Bethe equation. Numerical examples are computed and compared with the results available in literature. All numerical computations were done with MATHEMATICA software.

Example 1

For this first example, let us consider the equation of Bethe (14) with the initial condition \( u_0 = 1 \times 10^4 \). The approximate solution of (14) is obtained by (37), which in a simplified
form is:

\[
U_{\text{LADM}}(x) = 1 \times 10^4 - \frac{\ln(10001)}{1 \times 10^4} \left[ x + \frac{x^2(10001 \ln(10001) - 10000)}{20002 \times 10^8} \right. \\
+ \frac{x^3(1 \times 10^8 + 300060003 \ln^2(10001) - 5000400000 \ln(10001))}{600120006 \times 10^{16}} \\
+ \frac{x^4(2 \times 10^{11} + 300090009003 \ln^3(10001)}{48014401440048 \times 10^{23}} \\
\left. + \frac{x^5(2 \times 10^{15} + 210084012600840021 \ln^4(10001)}{240096014400960024 \times 10^{32}} \\
\left. - \frac{740052 \times 10^{11} \ln(10001)}{14407201440144007200144 \times 10^{40}} \right] + \frac{x^6(2 \times 10^{19} + 1890945189018904189 \ln^6(10001)}{64162623074589004410000 \ln^4(10001)} \\
+ \frac{x^7(2 \times 10^{23})}{10086049512201612060481008 \times 10^{55}} \\
+ \frac{x^8(2 \times 10^{27})}{8069646493722268225693464488064 \times 10^{56}} \\
+ \frac{819306911474398129555440000 \ln^4(10001)}{2704592457661603960117585919170271 \ln^4(10001)} \\
+ \frac{1210810069290157880312936310810000 \ln^6(10001)}{17827489797387160979782169 \times 10^{8} \ln^5(10001)} \\
+ \frac{108730357029013549511 \times 10^{12} \ln^4(10001)}{27992381617954761 \times 10^{16} \ln^3(10001)} \\
+ \frac{725581674142 \times 10^{19} \ln^2(10001)}{11200011200001 \times 10^{33} \ln(10001)} \right].
\]

(38)

Figure 1 shows the graph of the solution of the Bethe equation obtained through LADM for the initial condition \(u_0 = 1 \times 10^4\), in addition we can find the physical range \(R\) of the particle with initial energy \(u_0\), which is obtained when \(u_{\text{LADM}} \approx 1\) as we can see in [3].
Example 2
In this second example, we will consider the Bethe equation (14) with the initial condition $u_0 = 1 \times 10^5$. The approximate solution of (14) is obtained by (37), which is given by:
\[ u_{\text{LADM}}(x) = 1 \times 10^5 - \frac{\ln(100001)}{1 \times 10^5} \left[ x + \frac{x^2(100001 \ln(100001) - 100000)}{200002 \times 10^{10}} \right. \\
+ \left. x^3 \left(1 \times 10^{10} + 30000600003 \ln^2(100001) - 500004 \times 10^5 \ln(100001)\right) \right] \\
+ \frac{x^4}{480014400014400048 \times 10^{30}} \left( -2 \times 10^{14} + 3000090000900003 \ln^3(100001) \right. \\
- \left. 68001400005 \times 10^5 \ln^5(100001) + 3000022 \times 10^9 \ln(100001) \right) \\
+ \frac{x^5}{24000960001440009600024 \times 10^{40}} \left( 2 \times 10^{19} \right. \\
+ 2100084001260008400021 \ln^4(100001) - 59601544013800042 \times 10^5 \ln^3(100001) \right. \\
+ 414006380026 \times 10^{19} \ln^2(100001) - 7400052 \times 10^{14} \ln(100001) \right) \\
+ \frac{x^6}{14400720014400072000144 \times 10^{50}} \left( -2 \times 10^{34} \right. \\
+ 189009450189001890094500189 \ln^5(100001) \right. \\
- \left. 64142262030744189000441 \times 10^5 \ln^4(100001) \right) \\
+ 61101462612208035 \times 10^{11} \ln^3(100001) \\
- 19080280401092 \times 10^{14} \ln^2(100001) + 16600114 \times 10^{19} \ln(100001) \right] \\
+ \frac{x^7}{100806048151202016015120064801008 \times 10^{60}} \left( 2 \times 10^{29} \right. \\
+ 207912474311185415803118512474020791 \ln^6(100001) \right. \\
- 8189766307141823954929505544000001 \ln^5(100001) \right. \\
+ 992832527614808411645355 \times 10^5 \ln^4(100001) \right. \\
- 45429036008265622736 \times 10^{14} \ln^3(100001) \right. \\
+ 75501077604074 \times 10^{19} \ln^2(100001) - 3540024 \times 10^{25} \ln(100001) \right) \\
+ \frac{x^8}{80645649693468224282241693445644808064 \times 10^{70}} \left( -2 \times 10^{14} \right. \\
+ 27028891946757645954459506756889189270271 \ln^7(100001) \right. \\
- 121021779325873113667402670935581081000000 \ln^6(100001) \right. \\
+ 1782078486405285135640796992169 \times 10^{18} \ln^5(100001) \right. \\
- 1086997947942693135049511 \times 10^{15} \ln^4(100001) \right. \\
+ 279681818779432761 \times 10^{20} \ln^3(100001) - 272523816614142 \times 10^{24} \ln^2(100001) \right. \\
+ 73400494 \times 10^{29} \ln(100001) \right]. \\
(39)
Figure 2 shows the graph of the solution of the Bethe equation obtained through LADM for the initial condition $u_0 = 1 \times 10^5$, in addition we can find the physical range $R$ of the particle with initial energy $u_0$, which is obtained when $u_{LADM} \approx 1$, see [3].

Fig. 2: Plot of the 8-th approximation of $u(x)$ obtained by LADM for $u_0 = 1 \times 10^5$ (example 2).

**Example 3**

In this example, we will consider the Bethe equation (14) with the initial condition $u_0 = 1 \times 10^6$. The approximate solution of (14) is obtained by (37), which is given by:
\[ u_{\text{ADM}}(x) = 1 \times 10^6 - \frac{\ln(1000001)}{1 \times 10^6} \left[ x + \frac{x^2(\ln(1000001) - 1000000)}{2000002 \times 10^{12}} \right. \\
\left. + \frac{x^3}{6000012000006 \times 10^{24}} \left( 1 \times 10^{12} + 3000006000003 \ln^2(1000001) - 5000004 \times 10^6 \ln(1000001) \\
+ 30000022 \times 10^{11} \ln(1000001) \right) \right] \\
+ \frac{x^4}{48000144000144000048 \times 10^{48}} \left( -2 \times 10^{17} \\
+ 3000009000009000003 \ln^3(1000001) - 6800011400005 \times 10^6 \ln^2(1000001) \\
+ 30000022 \times 10^{11} \ln(1000001) \right) \\
+ \frac{x^5}{24000096000144000096000024 \times 10^{48}} \left( 2 \times 10^{23} \\
+ 21000084000126000084000021 \ln^4(1000001) \\
- 59600154400138000042000000 \ln^3(1000001) \\
+ 41400063800026 \times 10^{12} \ln^3(1000001) - 74000052 \times 10^{17} \ln(1000001) \right) \\
+ \frac{x^6}{144000720001440001440000720000144 \times 10^{90}} \left( -2 \times 10^{20} + 1890009450189000189000945000189 \ln^5(1000001) \\
- 64140226203074401890000441 \times 10^6 \ln^5(1000001) \\
+ 61100146206122080035 \times 10^{13} \ln^3(1000001) \\
- 1908002804001092 \times 10^{17} \ln^3(1000001) + 166000114 \times 10^{23} \ln(1000001) \right) \\
+ \frac{x^7}{100800604801512002016001512006048001008 \times 10^{22}} \left( 2 \times 10^{15} \\
+ 2079012474031185041580031185012474002079 \ln^6(1000001) \\
- 8189436624671412623952929500544000000 \ln^5(1000001) \\
+ 99280325272414806242116405355 \times 10^{12} \ln^4(1000001) \\
- 4542810360082656022736 \times 10^{17} \ln^3(1000001) \\
+ 7550010776004074 \times 10^{23} \ln^2(1000001) - 35400024 \times 10^{30} \ln(1000001) \right) \\
+ \frac{x^8}{806405644816934428224282240169344056448008064 \times 10^{84}} \left( -2 \times 10^{41} + 27027189189567567945945949556756718918960 \ln^7(1000001) \\
- 12101585791891871409365534026250935081081 \times 10^6 \ln^6(1000001) \\
+ 17820114485248527493359507969092169 \times 10^{12} \ln^5(1000001) \\
- 10869739476014267231350049511 \times 10^{18} \ln^4(1000001) \\
+ 27986261811447794212761 \times 10^{24} \ln^3(1000001) \\
- 27252038166014142 \times 10^{29} \ln^2(1000001) + 734000494 \times 10^{35} \ln(1000001) \right) \right]. \]
Figure 3 shows the graph of the solution of the Bethe equation obtained through LADM for the initial condition $u_0 = 1 \times 10^6$, in addition we can find the physical range $R$ of the particle with initial energy $u_0$, which is obtained when $u_{\text{LADM}} \approx 1$.

The results obtained in the three previous examples are shown in Table 1, in which the values of the physical range $R$ obtained using the results reported in [19] and [3] are compared for different values of $u_0$. All numerical work was performed using the Mathematica software package.

| $u_0$   | $R$ by present method | $R_1$ obtained in [19] | $R_2$ obtained in [3] | $(R_1 - R_2)/R_2 \times 100$ | $(|R - R_1|)/R_2 \times 100$ |
|---------|------------------------|-------------------------|------------------------|-------------------------------|-------------------------------|
| $1 \times 10^4$ | 5741211.4              | 5773445.1               | 5740303.8              | 0.57%                         | 0.015%                         |
| $1 \times 10^5$ | 4.5401472 $\times 10^8$  | 7.5608264 $\times 10^8$ | 4.5401197 $\times 10^8$ | 0.27%                         | 0.0006%                        |
| $1 \times 10^6$ | 3.7550206 $\times 10^{10}$ | 3.7691727 $\times 10^{10}$ | 3.7550194 $\times 10^{10}$ | 0.37%                         | 0.0003%                        |

Table 1: Table of comparison of the physical range $R$ obtained by the present method and those provided by the formulas [15] and [16].

In Table 1, we can see that our approximation made in the three previous examples to the solution of the Bethe equation is very good, since by comparing the rank $R$ that is obtained for the same initial kinetic energies $u_0$ in the only known recent works, that have addressed the problem, we can see that our approach is suitable since the maximum error is 0.015%.

We remark that, as these examples demonstrate, the Adomian decomposition method in combination with the Laplace transform avoids several difficulties in the calculation includ-
ing massive computational work are required, e.g., by discretization techniques, in determining the approximate analytic solution.

6 Summary and Conclusions

Very few exact solutions of the Bethe equation were known in the literature and practically all of them are of experimental nature. In this paper, we have obtained accurate approximations for the Bethe nonlinear differential equation solution using the Adomian decomposition method in combination with the Laplace transform, illustrating, in this way, the use of LADM in the solution of nonlinear differential equations. We have chosen the Bethe equation for its importance in physics as it describes the interaction between radiation and matter.

In order to show the accuracy and efficiency of our method, we have solved three examples, comparing our results with the only approximations that are known not from the Bethe equation itself but in the calculation of the physical range reached by a particle whose kinetic energy evolves according to said equation, which can be seen in the Table 1. Our results show that LADM produces highly accurate solutions in complicated nonlinear problems. We therefore, conclude that the Laplace-Adomian decomposition method is a notable non-sophisticated powerful tool that produces high quality approximate solutions for nonlinear differential equations using simple calculations and that attains converge with only few terms. All numerical work and graphics were performed with the Mathematica software package.

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