GRADIENT ESTIMATES AND ERGODICITY FOR SDES DRIVEN BY MULTIPLICATIVE LÉVY NOISES VIA COUPLING

MINGJIE LIANG  JIAN WANG

Abstract. We consider SDEs driven by multiplicative pure jump Lévy noises, where Lévy processes are not necessarily comparable to $\alpha$-stable-like processes. By assuming that the SDE has a unique solution, we obtain gradient estimates of the associated semigroup when the drift term is locally Hölder continuous, and we establish the ergodicity of the process both in the $L^1$-Wasserstein distance and the total variation, when the coefficients are dissipative for large distances. The proof is based on a new explicit Markov coupling for SDEs driven by multiplicative pure jump Lévy noises, which is derived for the first time in this paper.

Keywords: stochastic differential equation, multiplicative pure jump Lévy noises, coupling, gradient estimate, ergodicity

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1. Introduction and main results

We consider the following $d$-dimensional stochastic differential equation (SDE) driven by multiplicative pure jump Lévy noises

\begin{equation}
\begin{aligned}
\quad dX_t &= b(X_t) dt + \sigma(X_t, \cdot) dZ_t, \quad X_0 = x \in \mathbb{R}^d,
\end{aligned}
\end{equation}

where $b : \mathbb{R}^d \to \mathbb{R}^d$ is measurable, $\sigma : \mathbb{R}^d \to \mathbb{R}^d \otimes \mathbb{R}^d$ is continuous, and $Z := (Z_t)_{t \geq 0}$ is a pure jump Lévy process on $\mathbb{R}^d$, i.e., the finite-dimensional distributions of the process $Z$ are uniquely characterized by the characteristic function

$$
\mathbb{E} e^{i\langle \xi, Z_t \rangle} = e^{-t\phi_Z(\xi)}, \quad \xi \in \mathbb{R}^d, t > 0
$$

with

$$
\phi_Z(\xi) = \int \left(1 - e^{i\langle \xi, z \rangle} + i\langle \xi, z \rangle 1_{\{|z| \leq 1\}}(z)\right) \nu(dz).
$$

Here, $\nu$ is the Lévy measure, i.e., a $\sigma$-finite measure on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ such that $\nu(\{0\}) = 0$ and $\int (1 \wedge |z|^2) \nu(dz) < \infty$.

Throughout this paper, we always assume that there exists a non-explosive and pathwise unique solution to SDE (1.1), see [2, 3, 7, 17, 18, 22, 29, 30] for more details. We also need the following two assumptions on the coefficient $\sigma(x)$:

- $\sigma(x)$ is uniformly non-degenerate in the sense that, there exists a constant $\Lambda \geq 1$ such that for all $\xi \in \mathbb{R}^d$,

\begin{equation}
\Lambda^{-1} |\xi| \leq \inf_{x \in \mathbb{R}^d} \{ |\sigma(x)\xi| \wedge |\sigma(x)^{-1}\xi| \} \leq \sup_{x \in \mathbb{R}^d} \{ |\sigma(x)\xi| \vee |\sigma(x)^{-1}\xi| \} \leq \Lambda |\xi|,
\end{equation}

where $|\cdot|$ denotes the Euclidean norm.

M. Liang: College of Mathematics and Informatics, Fujian Normal University, 350007 Fuzhou, P.R. China. liangmingjie@aliyun.com.

J. Wang: College of Mathematics and Informatics & Fujian Key Laboratory of Mathematical Analysis and Applications (FJKLMAA), Fujian Normal University, 350007 Fuzhou, P.R. China. jianwang@fjnu.edu.cn.
- $\sigma(x)$ is bounded and globally Lipschitz continuous, i.e., there is a constant $L_\sigma > 0$ such that for all $x, y \in \mathbb{R}^d$,

$$
\|\sigma(x) - \sigma(y)\|_{\text{H.S.}} \leq L_\sigma |x - y|,
$$

where $\| \cdot \|_{\text{H.S.}}$ denotes the Hilbert-Schmidt norm of a matrix, and $L_\sigma$ is called the Lipschitz constant.

The goal of the present paper is to establish the regularity of the semigroups and the ergodicity of the process corresponding to the SDE (1.1) driven by multiplicative pure jump Lévy noises. More explicitly, we not only extend the main results of [14] to multiplicative Lévy noises setting, but also establish the regularity of the semigroups when the drift term is locally Hölder continuous. The Lévy process in this paper can be non-symmetric and not comparable with the $\alpha$-stable-type process. The methods used in this paper rely on coupling techniques and constructing coupling operators. We emphasize that the coupling for SDEs driven by multiplicative pure jump Lévy noises, which has been open for a long time, is derived for the first time in this paper.

Coupling for SDEs driven by multiplicative Brownian motions is a well developed field, and there is a vast literature on this topic; we mention here the papers [6, 11, 20, 25] and the monographs [5, 10, 23, 24]. Notice that, in contrast with the case of coupling for SDEs driven by multiplicative Brownian motions, our case for multiplicative Lévy noises is highly non-trivial. Indeed, we cannot use the above technique based on the decomposition, and in some sense the coupling for SDEs driven by multiplicative pure jump Lévy noises, which has been open for a long time, is derived for the first time in this paper.

In the existing mathematical literature there are a few works devoted to coupling for SDEs with additive Lévy noises, i.e., the coefficient $\sigma(x)$ in (1.1) is independent of the space variable. The readers can refer to [14, 15, 28] for an essential progress.

In particular, the couplings used in [15, 28] depend heavily on the existence of the rotationally symmetric component for Lévy measure; while in the framework of [14] only the existence of absolutely continuous component of Lévy measure is required, and then the main result of [14] works for some non-symmetric and even singular Lévy measure. However, there is no result for the coupling for SDEs driven by
multivariate Lévy processes till now. The difficulty is due to the fact that in this situation an efficient coupling shall pay attention to not only the shape of Lévy measure itself but also the diffusion coefficient, both of which are usually hard to handle. An important contribution of this paper is to fill this gap.

To illustrate the contribution of our paper, we present the following statement, which is a special case of our main results in Section 4. Denote by \( X := (X_t)_{t \geq 0} \) the unique solution to the SDE (1.1). For any \( f \) which is a special case of our main results in Section 4. Denote by

\[
P_t f(x) = E^x f(X_t), \quad x \in \mathbb{R}^d, t \geq 0.
\]

We will assume that one of assumptions below holds for the Lévy measure \( \nu \):

1. \( \nu(dz) \geq 1_{\{|z| \leq \eta\}} \frac{c_0}{|z|^{d+\alpha}} \, dz \)
   for some \( \eta \in (0,1) \) and \( c_0 > 0 \).

2. When \( \sigma(x) = (\sigma_{i,j}(x))_{d \times d} \) is diagonal, i.e., \( \sigma_{i,j}(x) = 0 \) for all \( x \in \mathbb{R}^d \) and \( 1 \leq i \neq j \leq d \),
   \[
   \nu(dz) \geq 1_{\{0 < z_1 \leq \eta\}} \frac{c_0}{|z|^{d+\alpha}} \, dz
   \]
   for some \( \eta \in (0,1) \) and \( c_0 > 0 \).

**Theorem 1.1.** Assume that the diffusion coefficient \( \sigma(x) \) is bounded and Lipschitz continuous, and the drift term \( b(x) \) is locally \( \beta \)-Hölder continuous with \( \beta \in ((1 - \alpha) \vee 0,1] \) for some \( \alpha \in (0,2) \). If one of assumptions (i) and (ii) above is satisfied for the Lévy measure \( \nu \), then the following hold.

1. If \( \alpha \in (1,2) \), then for any \( \theta > 0 \), there exists a constant \( C_1 := C_1(\theta) > 0 \) such that for all \( f \in B_b(\mathbb{R}^d) \) and \( t > 0 \),
   \[
   \sup_{x \neq y} \frac{|P_t f(x) - P_t f(y)|}{|x - y|} \leq C_1 \|f\|_\infty \left( \frac{\log^{1+\theta}(1/(t \wedge 1))}{t \wedge 1} \right)^{1/\alpha}.
   \]

2. If \( \alpha \in (0,1] \), then, for any \( \theta \in (0,\alpha) \), there exists a constant \( C_2 := C_2(\theta) > 0 \) such that for all \( f \in B_b(\mathbb{R}^d) \) and \( t > 0 \),
   \[
   \sup_{x \neq y} \frac{|P_t f(x) - P_t f(y)|}{|x - y|^{\theta}} \leq C_2 \|f\|_\infty t^{-\theta/\alpha}.
   \]

Gradient estimates for the semigroup associated to SDEs driven by multiplicative subordinated Brownian motions have been obtained in [26] by using the Malliavin calculus and a finite-jump approximation argument. For SDEs with multiplicative Brownian motions and general Poisson jump processes, Takeuchi [21] obtained the derivative formula for the associated semigroups, by using stochastic diffeomorphism flows and Girsanov’s transformation. Later, based on Bismut’s approach to the Malliavin calculus with jumps, a derivative formula of Bismut-Elworthy-Li’s type was established in [27] for SDEs with multiplicative \( \alpha \)-stable-like processes and maybe including non-degenerate diffusion part. It is clear that under (1.4), the Lévy process \( (Z_t)_{t \geq 0} \) is not comparable with the \( \alpha \)-stable-like process, so that the tool based on the Malliavin calculus with jumps used in [26, 27] cannot apply.
Next, we consider the ergodicity for the SDE given by (1.1). Let $\psi$ be a strictly increasing function on $[0, \infty)$ satisfying $\psi(0) = 0$. Given two probability measures $\mu_1$ and $\mu_2$ on $\mathbb{R}^d$, define

$$ W_\psi(\mu_1, \mu_2) = \inf_{\Pi \in \mathcal{P}(\mu_1, \mu_2)} \int_{\mathbb{R}^d \times \mathbb{R}^d} \psi(|x - y|) d\Pi(x, y), $$

where $\mathcal{P}(\mu_1, \mu_2)$ is the collection of measures on $\mathbb{R}^d \times \mathbb{R}^d$ having $\mu_1$ and $\mu_2$ as marginals. When $\psi$ is concave, the above definition gives rise to a Wasserstein distance $W_\psi$ in the space of probability measures $\mu$ on $\mathbb{R}^d$ such that $\int \psi(|z|) \mu(dz) < \infty$. If $\psi(r) = r$ for all $r \geq 0$, then $W_\psi$ is the standard $L^1$-Wasserstein distance (with respect to the Euclidean norm $|\cdot|$), which will be denoted by $W_1(\mu_1, \mu_2)$ for simplicity.

Another well-known example for $W_\psi$ is given by $\psi(r) = I_{(0, \infty)}(r)$, which leads to the total variation distance $W_\psi(\mu_1, \mu_2) = \frac{1}{2} \|\mu_1 - \mu_2\|_{\text{Var}}$.

**Theorem 1.2.** Assume that the diffusion coefficient $\sigma(x)$ is Lipschitz continuous with Lipschitz constant $L_\sigma > 0$. Suppose furthermore that

(i) the Lévy measure $\nu$ satisfies that

(1.5) \[
\int_{\{|z| \geq 1\}} |z| \nu(dz) < \infty
\]

and one of assumptions (i) and (ii) before Theorem 1.1.

(ii) the drift term $b(x)$ satisfies

(1.6) \[
\frac{(b(x) - b(y), x - y)}{|x - y|} \leq \begin{cases} 
K_1|x - y|^\beta, & |x - y| < l_0, \\
-K_2|x - y|, & |x - y| \geq l_0 
\end{cases}
\]

for all $x, y \in \mathbb{R}^d$ with some constants $\beta \in ((1 - \alpha) \lor 0, 1]$, $l_0 \geq 0$ $K_1 \geq 0$ and $K_2 > 0$.

Then, there exist constants $C, \lambda > 0$ such that for all $x, y \in \mathbb{R}^d$ and $t > 0$,

$$ W_1(\delta_x P_t, \delta_y P_t) \leq Ce^{-\lambda t}|x - y| $$

and

(1.7) \[
\|\delta_x P_t - \delta_y P_t\|_{\text{Var}} \leq Ce^{-\lambda t}(1 + |x - y|).
\]

Moreover, (1.7) holds true even for $\beta = 0$ in (1.6).

As a consequence of Theorem 1.2, we have

**Corollary 1.3.** Under the setting of Theorem 1.2, there exist a unique probability measure $\mu$, some positive functions $C_1(x), C_2(x)$ and a constant $\lambda > 0$ such that

$$ W_1(\delta_x P_t, \mu) \leq C_1(x)e^{-\lambda t}, \quad x \in \mathbb{R}^d, t > 0, $$

and

$$ \|\delta_x P_t - \mu\|_{\text{Var}} \leq C_2(x)e^{-\lambda t}, \quad x \in \mathbb{R}^d, t > 0. $$

When the coefficient $\sigma(x)$ is independent of the space variable (i.e., the noise is additive) and the drift term $b$ is dissipative for large distances (i.e., satisfies (1.6)), in [28] the second author established the exponential convergence rate in the $L^p$-Wasserstein distance for any $p \geq 1$ when the Lévy noise in (1.1) has an $\alpha$-stable component. For a large class of Lévy processes whose associated Lévy measure has a rotationally invariant absolutely continuous component, Majka obtained in [15] the exponential convergence rates with respect to both the $L^1$-Wasserstein distance.
and the total variation. Recently, the results of [15, 28] are extended and improved in [14], where the associated Lévy measure of Lévy process is only assumed to have an absolutely continuous component. It is noticed that all the works above are restricted to the additive noise case. Once the coefficient $\sigma(x)$ depends on the space variable (i.e., the noise is multiplicative), the problem gets more complicated. When the coefficients are locally Lipschitz continuous and satisfy a Lyapunov type dissipative condition, it has been shown in [1, 8, 9] that there is a unique invariant probability measure associated to the SDE (1.1), which is exponentially ergodic. Recently, Xie and Zhang studied in [29] the exponential ergodicity of SDEs driven by general multiplicative Lévy noises (maybe with Brownian motions), when $b$ is locally bounded and maybe singular at infinity. To the best of our knowledge, there is no result about the exponential convergence rates with respect to the $L^1$-Wasserstein distance, when the coefficient $\sigma(x)$ is bounded and Lipschitz continuous, and $b(x)$ is dissipative for large distances.

The remainder of this paper is arranged as follows. In Section 2, we first review the refined basic coupling for Lévy processes constructed in [14], and then present a new coupling for multiplicative Lévy process, which is a key part of our paper. We also prove the existence of coupling process here. Section 3 is devoted to some explicit estimates for the coupling operator, which is a necessary ingredient of our proof. General ideas to yield the regularity of the semigroups and the ergodicity of the process via coupling are presented in Section 4. Finally, we present proofs of Theorems 1.1, 1.2 and Corollary 1.3 in the last section.

2. Coupling operator and coupling process

2.1. Coupling operator for the SDE (1.1). Denote by $X := (X_t)_{t \geq 0}$ the solution to the SDE (1.1). It is easy to see that the generator $L$ of the process $X$ acting on $C^2_b(\mathbb{R}^d)$ is given by

$$Lf(x) = \langle \nabla f(x), b(x) \rangle + \int \left( f(x + \sigma(x)z) - f(x) - \langle \nabla f(x), \sigma(x)z \rangle 1_{\{|z| \leq 1\}}(z) \right) \nu(dz). \tag{2.8}$$

The purpose of this subsection is to construct a new and efficient coupling operator for $L$, which is one of crucial ingredients in our approach. Recall that an operator $\tilde{L}$ acting on $C^2_b(\mathbb{R}^{2d})$ is a coupling of $L$, if for any $f, g \in C^2_b(\mathbb{R}^d)$,

$$\tilde{L}h(x, y) = Lf(x) + Lg(y), \tag{2.9}$$

where $h(x, y) = f(x) + g(y)$ for all $x, y \in \mathbb{R}^d$.

2.1.1. Additive Lévy noises. To illustrate clearly ideas of the construction of a proper coupling for the SDE (1.1), in this part we briefly introduce the refined basic coupling operator constructed in [14, Section 2] for SDEs driven by additive Lévy noises (that is, the case that $\sigma(x) = I_{d \times d}$ for any $x \in \mathbb{R}^d$ in the SDE (1.1)).

Throughout this part, we consider the operator $L$ given by (2.8), where $\sigma(x) = I_{d \times d}$ for all $x \in \mathbb{R}^d$. Motivated by the (classical) basic coupling for Markov $q$-processes or Markov chains (see [5, Example 2.10] for instance), we can define the following basic coupling for the operator $L$. For any $h \in C^2_b(\mathbb{R}^{2d})$ and $x, y \in \mathbb{R}^d$, let

$$\tilde{L}h(x, y) = \langle \nabla_x h(x, y), b(x) \rangle + \langle \nabla_y h(x, y), b(y) \rangle$$
\[ + \int \left( h(x + z, y + z + (x - y)) - h(x, y) - \langle \nabla_x h(x, y), z \rangle I_{|z| \leq 1} \right) \\
- \langle \nabla_y h(x, y), z + (x - y) \rangle I_{|z + (x - y)| \leq 1} \right) \mu_{y-x} (dz) \\
+ \int \left( h(x + z, y) - h(x, y) - \langle \nabla_x h(x, y), z \rangle I_{|z| \leq 1} \right) (\nu - \mu_{y-x}) (dz) \\
+ \int \left( h(x, y + z) - h(x, y) - \langle \nabla_y h(x, y), z \rangle I_{|z| \leq 1} \right) (\nu - \mu_{x-y}) (dz). \]

Here, \( \mu_{y-x} (dz) := [\nu \land (\delta_{y-x} \ast \nu)] (dz) \), and \( \nabla_y h(x, y) \) and \( \nabla_y h(x, y) \) are defined as the gradient of \( h(x, y) \) with respect to \( x \in \mathbb{R}^d \) and \( y \in \mathbb{R}^d \) respectively. For the sake of our explanation, by the structure of the generator of Lévy process, the coupling above can be simply written as follows:

\[
(x, y) \rightarrow \begin{cases} 
(x + z, y + z + (x - y)), & \mu_{y-x} (dz); \\
(x + z, y), & (\nu - \mu_{y-x}) (dz); \\
(x, y + z), & (\nu - \mu_{x-y}) (dz).
\end{cases}
\]

(2.10)

In the first row the distance between two marginals decreases from \(|x - y|\) to \(|(x+z) - (y+z+(x-y))| = 0\), so this term plays a key role in coupling two marginals together. For this aim, its density enjoys the biggest jump rate \( \mu_{y-x} (dz) \), i.e., the maximum common part of the jump intensities from \( x \) to \( x+z \) and from \( y \) to \( y+z+(x-y) \). However, the second and the last rows are not so welcome (indeed they are not so easy to handle for Lévy jumps), because the new distance are \(|x-y+z|\) or \(|x-y-z|\), which can be much bigger than the original distance \(|x - y|\) when the jump size \( z \) is large.

To overcome this disadvantage, we figure out the following coupling:

\[
(x, y) \rightarrow \begin{cases} 
(x + z, y + z + (x - y)), & \frac{1}{2} \mu_{y-x} (dz); \\
(x + z, y + z + (y - x)), & \frac{1}{2} \mu_{x-y} (dz); \\
(x + z, y + z), & (\nu - \frac{1}{2} \mu_{y-x} - \frac{1}{2} \mu_{x-y}) (dz).
\end{cases}
\]

(2.11)

Now, in the second row the distance after the jump is 2\(|x-y|\). Though it doubles the original distance, it is better than that in (2.10) when the jump size \( z \) is large. (In some sense, it is also easier to deal with.) Besides, the distance remains unchanged in the last row.

The above coupling (2.11) has a drawback too. For example, if the original pure jump Lévy process is of finite range, then the jump intensity \( \mu_{y-x} (dz) \) is identically zero when \(|x - y|\) is large enough. Hence, two marginal processes of the coupling (2.11) will never get closer if they are initially far away. Therefore, we need further modify the coupling above. Let \( \kappa > 0 \), and for any \( x, y \in \mathbb{R}^d \), define

\[
(x - y)_\kappa = \left( 1 \land \frac{\kappa}{|x - y|} \right) (x - y).
\]

In [14, Section 2] we finally modify the coupling above into

\[
(x, y) \rightarrow \begin{cases} 
(x + z, y + z + (x - y)_\kappa), & \frac{1}{2} \mu_{y-x} (dz); \\
(x + z, y + z + (y - x)_\kappa), & \frac{1}{2} \mu_{x-y} (dz); \\
(x + z, y + z), & (\nu - \frac{1}{2} \mu_{y-x} - \frac{1}{2} \mu_{x-y}) (dz).
\end{cases}
\]

(2.12)
We see that if \(|x - y| \leq \kappa\), then the above coupling is the same as that in (2.11). If \(|x - y| > \kappa\), then according to the first two rows, the distances after the jump are \(|x - y| - \kappa\) and \(|x - y| + \kappa\), respectively. Therefore, the parameter \(\kappa\) serves as the threshold to determine whether the marginal processes jump to the same point or become slightly closer to each other. We call the coupling given by (2.12) the refined basic coupling for pure jump Lévy processes. By making full use of this coupling, we have obtained some new results for Wasserstein-type distances for SDEs with additive Lévy noises, where Lévy measure can be much singular. The reader can refer to [14] for more details.

2.1.2. Multiplicative Lévy noises. For the SDE (1.1) driven by multiplicative Lévy noises, the jump system of the generator \(L\) given by (2.8) can be simply understood as
\[
x \longrightarrow x + \sigma(x)z, \quad \nu(dz).
\]
One may follow the construction of the refined basic coupling (2.12) above, and consider the following coupling (we can prove that this indeed associates with a coupling operator)
\[
(x, y) \longrightarrow \begin{cases}
(x + \sigma(x)z, y + \sigma(y)(z + (x - y)_n)), & \frac{1}{2}\mu_{(y-x)_n}(dz); \\
(x + \sigma(x)z, y + \sigma(y)(z + (y - x)_n)), & \frac{1}{2}\mu_{(x-y)_n}(dz); \\
(x + \sigma(x)z, y + \sigma(y)z), & (\nu - \frac{1}{2}\mu_{(y-x)_n} - \frac{1}{2}\mu_{(x-y)_n})(dz).
\end{cases}
\]
However, due to the appearance of diffusion coefficient \(\sigma(x)\), we cannot compare the distance after jump and the original distance from the coupling above. Actually, for coupling of SDEs with multiplicative Lévy noises, the situation becomes more complex, and we cannot directly use the refined basic coupling. Roughly speaking, a reasonable and efficient coupling now should pay attention to the role of coefficient \(\sigma(x)\).

Before moving further, we need some notation and elementary facts. Let \(\Psi : \mathbb{R}^d \to \mathbb{R}^d\) be a continuous and bijective mapping, i.e., \(\Psi\) is invertible and satisfies that \(\Psi(\mathbb{R}^d) = \mathbb{R}^d\). We further assume that \(\Psi(0) \neq 0\). For any \(n \geq 1\), we define
\[
(2.13) \quad \mu_\Psi = \limsup_{n \to \infty} \mu_{n,\Psi} := \limsup_{n \to \infty} (\nu_n \wedge (\nu_n \Psi)),
\]
where \(\nu_n(A) = \int_{A \cap \{|z| > 1/n\}} \nu(dz)\) and \((\nu_n \Psi)(A) = \nu_n(\Psi(A))\) for all \(A \in \mathcal{B}(\mathbb{R}^d)\). The following observation is frequently used in the arguments below.

**Lemma 2.1.**

(1) For any \(A \in \mathcal{B}(\mathbb{R}^d)\),
\[
(\mu_\Psi \Psi^{-1})(A) = \mu_{\Psi^{-1}}(A).
\]

(2) Both \(\mu_\Psi\) and \(\mu_{\Psi^{-1}}\) are finite measures on \((\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))\).

**Proof.** (1) Recall that for any two finite measures \(\mu_1\) and \(\mu_2\) on \((\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))\),
\[
\mu_1 \wedge \mu_2 := \mu_1 - (\mu_1 - \mu_2)^+,
\]
where \((\mu_1 - \mu_2)^+\) and \((\mu_1 - \mu_2)^-\) refer to the Jordan-Hahn decomposition of the signed measure \(\mu_1 - \mu_2\). In detail, for any \(A \in \mathcal{B}(\mathbb{R}^d)\),
\[
(\mu_1 - \mu_2)^+(A) = \sup_{B \in \mathcal{B}(\mathbb{R}^d)} \{\mu_1(B) - \mu_2(B) : B \subset A\}.
\]
Note that \( \nu_n \) and \( \nu_n \Psi \) are finite measures on \((\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))\). By the definition of \( \mu_n, \Psi \), for any \( A \in \mathcal{B}(\mathbb{R}^d) \), we have

\[
\begin{align*}
(\mu_n, \Psi^{-1})(A) &= \mu_n, \Psi(\Psi^{-1}(A)) \\
&= [\nu_n \wedge (\nu_n \Psi)](\Psi^{-1}(A)) = [(\nu_n \Psi) \wedge \nu_n](\Psi^{-1}(A)) \\
&= (\nu_n \Psi)(\Psi^{-1}(A)) - ((\nu_n \Psi) - \nu_n)^+(\Psi^{-1}(A)) \\
&= \nu_n(A) - \sup_{B \in \mathcal{B}(\mathbb{R}^d)} \{(\nu_n \Psi)(B) - \nu_n(B) : B \subset \Psi^{-1}(A)\} \\
&= \nu_n(A) - \sup_{\tilde{B} \in \mathcal{B}(\mathbb{R}^d)} \{\nu_n(\tilde{B}) - (\nu_n \Psi^{-1})(\tilde{B}) : \tilde{B} \subset A\} \\
&= \nu_n(A) - (\nu_n - (\nu_n \Psi^{-1}))^+(A) = (\nu_n \wedge (\nu_n \Psi^{-1}))(A) \\
&= \mu_n, \Psi^{-1}(A),
\end{align*}
\]

where in equalities above we used the fact that \( \Psi \) is a bijective mapping from \( \mathbb{R}^d \) to \( \mathbb{R}^d \). Then, the first required assertion immediately follows from the equality above.

(2) Since \( \Psi(0) \neq 0 \) and \( \Psi \) is continuous, there exists a constant \( \varepsilon_0 > 0 \) such that \( c_0 := \inf\{|\Psi(z)| : |z| \leq \varepsilon_0\} > 0 \). Thus, for any \( n \geq 1 \),

\[
\begin{align*}
\int_{\mathbb{R}^d} (\mu_n, \Psi)(dz) &= \int_{\mathbb{R}^d} (\nu_n \wedge (\nu_n \Psi))(dz) \\
&\leq \int_{\{|z| > \varepsilon_0\}} \nu_n(dz) + \int_{\{|z| \leq \varepsilon_0\}} (\nu_n \Psi)(dz) \\
&\leq \nu(\{|z| > \varepsilon_0\}) + \nu(\{|z| \leq \varepsilon_0\}) \\
&\leq \nu(\{|z| \geq \varepsilon_0\}) + \nu(\{|z| \geq c_0\}) \\
&=: c(c_0, \varepsilon_0) < \infty.
\end{align*}
\]

Letting \( n \to \infty \), we get that \( \mu_\Psi(\mathbb{R}^d) \leq c(c_0, \varepsilon_0) < \infty \). By (i) and the fact that \( \Psi \) is bijective, it also holds that \( \mu_{\Psi^{-1}}(\mathbb{R}^d) < \infty \). \( \square \)

Now, we consider the jump system as follows:

\[
(x, y) \mapsto \begin{cases} 
(x + \sigma(x)z, y + \sigma(y)\Psi(z)), & \frac{1}{2} \mu_\Psi(dz); \\
(x + \sigma(x)z, y + \sigma(y)\Psi^{-1}(z)), & \frac{1}{2} \mu_{\Psi^{-1}}(dz); \\
(x + \sigma(x)z, y + \sigma(y)z), & (\nu - \frac{1}{2} \mu_\Psi - \frac{1}{2} \mu_{\Psi^{-1}})(dz).
\end{cases}
\] (2.14)
More explicitly, for any \( h \in C^2_b(\mathbb{R}^d \times \mathbb{R}^d) \) and \( x, y \in \mathbb{R}^d \), we define

\[
\tilde{L}h(x, y) = \langle \nabla_x h(x, y), b(x) \rangle + \langle \nabla_y h(x, y), b(y) \rangle + \frac{1}{2} \int \left( h(x + \sigma(x)z, y + \sigma(y)\Psi(z)) - h(x, y) \right.
\]

\[
- \langle \nabla_x h(x, y), \sigma(z) \rangle 1_{|z| \leq 1} \mu_{\Psi}(dz)
\]

\[
- \langle \nabla_y h(x, y), \sigma(z) \rangle \Psi(z) 1_{|\Psi(z)| \leq 1} \mu_{\Psi^{-1}}(dz) \right)
\]

\[
\left( h(x + \sigma(x)z, y + \sigma(y)\Psi^{-1}(z)) - h(x, y) \right.
\]

\[
- \langle \nabla_x h(x, y), \sigma(z) \rangle 1_{|z| \leq 1} \mu_{\Psi^{-1}}(dz)
\]

\[
- \langle \nabla_y h(x, y), \sigma(z) \rangle 1_{|z| \leq 1} \left( \nu - \frac{1}{2} \mu_{\Psi} - \frac{1}{2} \mu_{\Psi^{-1}} \right)(dz).
\]

(2.15)

By Lemma 2.1(1), we can prove rigorously the following statement.

**Theorem 2.2.** The operator \( \tilde{L} \) defined by (2.15) is a coupling operator of the operator \( L \) given by (2.8).

**Proof.** First, let \( h(x, y) = f(x) \) for any \( x, y \in \mathbb{R}^d \), where \( f \in C^2_b(\mathbb{R}^d) \). Obviously, it holds that

\[
\tilde{L}h(x, y) = \langle \nabla f(x), b(x) \rangle + \frac{1}{2} \int \left( f(x + \sigma(x)z) - f(x) - \langle \nabla f(x), \sigma(z) \rangle 1_{|z| \leq 1} \right) \mu_{\Psi}(dz)
\]

\[
+ \frac{1}{2} \int \left( f(x + \sigma(x)z) - f(x) - \langle \nabla f(x), \sigma(z) \rangle 1_{|z| \leq 1} \right) \mu_{\Psi^{-1}}(dz)
\]

\[
+ \int \left( f(x + \sigma(x)z) - f(x) - \langle \nabla f(x), \sigma(z) \rangle 1_{|z| \leq 1} \right)
\]

\[
\times \left( \nu - \frac{1}{2} \mu_{\Psi} - \frac{1}{2} \mu_{\Psi^{-1}} \right)(dz)
\]

\[
= Lf(x).
\]

Secondly, let \( h(x, y) = g(y) \) for any \( x, y \in \mathbb{R}^d \), where \( g \in C^2_b(\mathbb{R}^d) \). Then,

\[
\tilde{L}h(x, y) = \langle \nabla g(y), b(y) \rangle + \frac{1}{2} \int \left( g(y + \sigma(y)\Psi(z)) - g(y) \right.
\]

\[
- \langle \nabla g(y), \sigma(y)\Psi(z) \rangle 1_{|\Psi(z)| \leq 1} \mu_{\Psi}(dz)
\]

\[
+ \frac{1}{2} \int \left( g(y + \sigma(y)\Psi^{-1}(z)) - g(y) \right.
\]

\[
- \langle \nabla g(y), \sigma(y)\Psi^{-1}(z) \rangle 1_{|\Psi^{-1}(z)| \leq 1} \mu_{\Psi^{-1}}(dz)
\]
\[ + \int \left( g(y + \sigma(y)z) - g(y) - (\nabla g(y), \sigma(y)z) \mathbb{1}_{\{|z| \leq 1\}} \right) \times \left( \nu - \frac{1}{2} \mu_{\Psi} - \frac{1}{2} \mu_{\Psi^{-1}} \right)(dz) \]

\[ = \langle \nabla g(y), b(y) \rangle + \frac{1}{2} \int \left( g(y + \sigma(y)\Psi(z)) - g(y) \right. \]

\[ - \langle \nabla g(y), \sigma(y)\Psi(z) \rangle \mathbb{1}_{\{|\Psi(z)| \leq 1\}} \left( \mu_{\Psi^{-1}} \Psi(z) \right)(dz) \]

\[ + \frac{1}{2} \int \left( g(y + \sigma(y)\Psi^{-1}(z)) - g(y) \right) \]

\[ - \langle \nabla g(y), \sigma(y)\Psi^{-1}(z) \rangle \mathbb{1}_{\{|\Psi^{-1}(z)| \leq 1\}} \left( \mu_{\Psi} \Psi^{-1}(z) \right)(dz) \]

\[ + \int \left( g(y + \sigma(y)z) - g(y) - (\nabla g(y), \sigma(y)z) \mathbb{1}_{\{|z| \leq 1\}} \right) \]

\[ \times \left( \nu - \frac{1}{2} \mu_{\Psi} - \frac{1}{2} \mu_{\Psi^{-1}} \right)(dz) \]

\[ = \langle \nabla g(y), b(y) \rangle + \frac{1}{2} \int \left( g(y + \sigma(y)z) - g(y) \right) \]

\[ - \langle \nabla g(y), \sigma(y)z \mathbb{1}_{\{|z| \leq 1\}} \rangle \mu_{\Psi^{-1}}(dz) \]

\[ + \frac{1}{2} \int \left( g(y + \sigma(y)z) - g(y) \right) \]

\[ - \langle \nabla g(y), \sigma(y)z \mathbb{1}_{\{|z| \leq 1\}} \rangle \mu_{\Psi}(dz) \]

\[ + \int \left( g(y + \sigma(y)z) - g(y) - (\nabla g(y), \sigma(y)z) \mathbb{1}_{\{|z| \leq 1\}} \right) \]

\[ \times \left( \nu - \frac{1}{2} \mu_{\Psi} - \frac{1}{2} \mu_{\Psi^{-1}} \right)(dz) \]

\[ = Lg(y), \]

where the second equality follows from Lemma 2.1(1) and we used the measure transformations \( \mu_{\Psi^{-1}} \Psi \mapsto \mu_{\Psi^{-1}} \) and \( \mu_{\Psi} \Psi^{-1} \mapsto \mu_{\Psi} \) in the third equality.

Combining both equalities above, we know that (2.9) holds true, and so the desired assertion follows. \( \square \)

According to Theorem 2.2, there exist a lot of (non-trivial) coupling operators for the generator \( L \) given by (2.8). By the refined basic coupling (2.12) (in particular the first row here), a proper choice of \( \Psi \) in (2.14) should satisfy that

\[ x + \sigma(x)z = y + \sigma(y)\Psi(z) \]

for all \( x, y \in \mathbb{R}^d \) with \( 0 < |x - y| \leq \kappa \) for some constant \( \kappa > 0 \). For this, in the remainder of this paper, we will take

\[ \Psi(z) = \Psi_{\kappa,x,y}(z) := \sigma(y)^{-1}(\sigma(x)z + (x - y)_{\kappa}), \]

where \( \kappa > 0 \) (which is a constant determinated later) and \( (x - y)_{\kappa} = \left(1 - \frac{\kappa}{|x - y|}\right)(x - y) \).

Note that, \( \Psi(z) \) depends on \( \kappa, x \) and \( y \), and for simplicity we omit \( \kappa, x, y \) in the
notation. Clearly,
\[
\Psi^{-1}(z) = \sigma(x)^{-1}(\sigma(y)z - (x - y)\kappa).
\]
In particular, with this choice, when \( \sigma(x) = I_{d\times d} \) for all \( x \in \mathbb{R}^d \), (2.14) is reduced into (2.12). Moreover, by the nondegenerate property and the continuity of \( \sigma \), we know that for any \( x, y \in \mathbb{R}^d \) with \( x \neq y \), \( \Psi : \mathbb{R}^d \to \mathbb{R}^d \) is a continuous and bijective mapping such that \( \Psi(0) \neq 0 \). In particular, Lemma 2.1 applies.

2.2. Coupling process. In this subsection, we prove the existence of the coupling process associated with the coupling operator \( \tilde{L} \) defined by (2.15). We assume that the SDE (1.1) has a unique strong solution. By the Lévy–Itô decomposition,
\[
Z_t = \int_0^t \int_{\{|z| > 1\}} z \, N(ds, dz) + \int_0^t \int_{\{|z| \leq 1\}} z \, \tilde{N}(ds, dz),
\]
where \( N(ds, dz) \) is a Poisson random measure associated with \( (Z_t)_{t \geq 0} \), i.e.,
\[
N(ds, dz) = \sum_{\{0 < s' \leq s, \Delta Z_{s'} \neq 0\}} \delta_{(s', \Delta Z_{s'})}(ds, dz),
\]
and
\[
\tilde{N}(ds, dz) = N(ds, dz) - ds \nu(dz)
\]
is the corresponding compensated Poisson measure. In order to write a coupling process explicitly, we extend the Poisson random measure \( N \) from \( \mathbb{R}_+ \times \mathbb{R}^d \) to \( \mathbb{R}_+ \times \mathbb{R}^d \times [0, 1] \) in the following way
\[
N(ds, dz, du) = \sum_{\{0 < s' \leq s, \Delta Z_{s'} \neq 0\}} \delta_{(s', \Delta Z_{s'})}(ds, dz) 1_{[0,1]}(du).
\]
and write
\[
\tilde{N}(ds, dz, du) = 1_{\{|z| > 1\} \times [0,1]} N(ds, dz, du) + 1_{\{|z| \leq 1\} \times [0,1]} \tilde{N}(ds, dz, du).
\]
Let \( Z \) be a pure jump Lévy process on \( \mathbb{R}^d \) given above. We will construct a new Lévy process \( Z^* \) on \( \mathbb{R}^d \) as follows. Suppose that a jump of \( Z \) occurs at time \( t \), and that the process \( Z \) moves from the point \( Z_{t_-} \) to \( Z_{t_-} + z \). Then, we draw a random number \( u \in [0,1] \) to determine whether the process \( Z^* \) should jump from the point \( Z^*_{t_-} \) to the points \( Z^*_{t_-} + \Psi(z) \), \( Z^*_{t_-} + \Psi^{-1}(z) \) and \( Z^*_{t_-} + z \), respectively. By taking into account the characterization (2.14) for the coupling operator \( \tilde{L} \) defined by (2.15), the random number \( u \) should be determined by the following two factors:
\[
(2.17) \quad \rho_\Psi(x, y, z) = \frac{\mu_\Psi(dz)}{\nu(dz)}, \quad \rho_{\Psi^{-1}}(x, y, z) = \frac{\mu_{\Psi^{-1}}(dz)}{\nu(dz)}, \quad x, y, z \in \mathbb{R}^d.
\]
It is clear that both \( \rho_\Psi(x, y, z) \) and \( \rho_{\Psi^{-1}}(x, y, z) \in [0, 1] \). More explicitly, we will consider the system of equations:
\[
(2.18) \quad \begin{cases}
    dX_t = b(X_t) \, dt + \sigma(X_{t_-}) \, dZ_t, & X_0 = x; \\
    dY_t = b(Y_t) \, dt + \sigma(Y_{t_-}) \, dZ^*_t, & Y_0 = y,
\end{cases}
\]
where

\[
\begin{align*}
dZ^*_t &= \int_{\mathbb{R}^d \times [0,1]} \left[ \Psi(z) \mathbb{I}_{\{u < \frac{1}{2} \varphi \rho (X_t, Y_t, z)\}} + \Psi^{-1}(z) \mathbb{I}_{\{u > \frac{1}{2} \varphi \rho (X_t, Y_t, z) + \varphi_{-1} (X_t, Y_t, z)\}} \right] N(dt, dz, du) \\
&\quad - \int_{\mathbb{R}^d \times [0,1]} \left[ \Psi(z) \left( \mathbb{I}_{\{|\Psi(z)| \leq 1\}} - \mathbb{I}_{\{|z| \leq 1\}} \right) \mathbb{I}_{\{u < \frac{1}{2} \varphi \rho (X_t, Y_t, z)\}} + \Psi^{-1}(z) \left( \mathbb{I}_{\{|\Psi^{-1}(z)| \leq 1\}} - \mathbb{I}_{\{|z| \leq 1\}} \right) \times \mathbb{I}_{\{u > \frac{1}{2} \varphi \rho (X_t, Y_t, z) + \varphi_{-1} (X_t, Y_t, z)\}} \right] \nu(dz) du dt.
\end{align*}
\]

Note that, by Lemma 2.1(2), \(\mu_\varphi\) and \(\mu_{\varphi^{-1}}\) are finite measures on \((\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))\), and so (2.19) is well defined.

**Proposition 2.3.** Suppose that the SDE (1.1) has a unique strong solution. Then, the equation (2.18) also has a unique strong solution, denoted by \((X_t, Y_t)_{t \geq 0}\), and the associated generator is just the coupling operator \(\tilde{L} \) given by (2.15). In particular, \((X_t, Y_t)_{t \geq 0}\) is a Markov coupling process for the unique strong solution to the SDE (1.1), and \(X_t = Y_t\) for all \(t \geq T\), where \(T\) is the coupling time of \((X_t)_{t \geq 0}\) and \((Y_t)_{t \geq 0}\), i.e., \(T = \inf\{t \geq 0 : X_t = Y_t\}\).

**Proof.** (1) We first simplify the formula (2.19) for \(Z^*\). We write (2.19) as

\[
\begin{align*}
dZ^*_t &= \int_{\mathbb{R}^d \times [0,1]} z \tilde{N}(dt, dz, du) \\
&\quad + \int_{\mathbb{R}^d \times [0,1]} \left[ (\Psi(z) - z) \mathbb{I}_{\{u < \frac{1}{2} \varphi \rho (X_t, Y_t, z)\}} + (\Psi^{-1}(z) - z) \mathbb{I}_{\{u > \frac{1}{2} \varphi \rho (X_t, Y_t, z) + \varphi_{-1} (X_t, Y_t, z)\}} \right] \tilde{N}(dt, dz, du) \\
&\quad - \int_{\mathbb{R}^d \times [0,1]} \left[ \Psi(z) \left( \mathbb{I}_{\{|\Psi(z)| \leq 1\}} - \mathbb{I}_{\{|z| \leq 1\}} \right) \mathbb{I}_{\{u < \frac{1}{2} \varphi \rho (X_t, Y_t, z)\}} + \Psi^{-1}(z) \left( \mathbb{I}_{\{|\Psi^{-1}(z)| \leq 1\}} - \mathbb{I}_{\{|z| \leq 1\}} \right) \times \mathbb{I}_{\{u > \frac{1}{2} \varphi \rho (X_t, Y_t, z) + \varphi_{-1} (X_t, Y_t, z)\}} \right] \nu(dz) du dt.
\end{align*}
\]

According to Lemma 2.1(1),

\[
\begin{align*}
&\int_{\mathbb{R}^d \times [0,1]} \left[ \Psi(z) \left( \mathbb{I}_{\{|\Psi(z)| \leq 1\}} - \mathbb{I}_{\{|z| \leq 1\}} \right) \mathbb{I}_{\{u < \frac{1}{2} \varphi \rho (X_t, Y_t, z)\}} + \Psi^{-1}(z) \left( \mathbb{I}_{\{|\Psi^{-1}(z)| \leq 1\}} - \mathbb{I}_{\{|z| \leq 1\}} \right) \times \mathbb{I}_{\{u > \frac{1}{2} \varphi \rho (X_t, Y_t, z) + \varphi_{-1} (X_t, Y_t, z)\}} \right] \nu(dz) du \\
&= \frac{1}{2} \int_{\mathbb{R}^d} \Psi(z) \left( \mathbb{I}_{\{|\Psi(z)| \leq 1\}} - \mathbb{I}_{\{|z| \leq 1\}} \right) \mu_\varphi(dz) \\
&\quad + \frac{1}{2} \int_{\mathbb{R}^d} \Psi^{-1}(z) \left( \mathbb{I}_{\{|\Psi^{-1}(z)| \leq 1\}} - \mathbb{I}_{\{|z| \leq 1\}} \right) \mu_{\varphi^{-1}}(dz)
\end{align*}
\]
We consider two cases: (2.20) the integrals above are well defined. In particular, we can rewrite (2.18) as follows

\[ \int_{\mathbb{R}^d} (\Psi(z) - z) \mathbb{I}_{\{|z| \leq 1\}} \mu_{\Psi^{-1}}(dz) = \frac{1}{2} \int_{\mathbb{R}^d} (z - \Psi^{-1}(z)) \mathbb{I}_{\{|\Psi^{-1}(z)| \leq 1\}} \mu_{\Psi^{-1}}(dz). \]

Combining all the equalities above together yields that

\[ dZ_t^* = \int_{\mathbb{R}^d \times [0,1]} z \tilde{N}(dt,dz,du) + \int_{\mathbb{R}^d \times [0,1]} \left[ (\Psi(z) - z) \mathbb{I}_{\{u \leq \frac{1}{2} \rho_{\Psi}(X_t, Y_{t-}, z)\}} + (\Psi^{-1}(z) - z) \mathbb{I}_{\{1 - \frac{1}{2} \rho_{\Psi}(X_t, Y_{t-}, z) < u \leq \frac{1}{2} \rho_{\Psi}(X_t, Y_{t-}, z) + \rho_{\Psi^{-1}}(X_t, Y_{t-}, z)\}} \right] \tilde{N}(dt,dz,du) =: dZ_t + dG_t^*. \]

Again, by Lemma 2.1(2), \( \mu_{\Psi} \) and \( \mu_{\Psi^{-1}} \) are finite measures on \((\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))\), and so all the integrals above are well defined. In particular, we can rewrite (2.18) as follows

\[ \begin{cases} dX_t = b(X_t) \, dt + \sigma(X_{t-}) \, dZ_t, & X_0 = x; \\ dY_t = b(Y_t) \, dt + \sigma(Y_{t-}) \, dZ_t + \sigma(Y_{t-}) \, dG_t^*, & Y_0 = y. \end{cases} \]

(2) We next follow the idea for the argument of [14, Proposition 2.2] and show that the SDE (2.20) has a unique strong solution. By assumption, the equation (1.1) (i.e., the first equation in (2.20)) has a non-explosive and pathwise unique strong solution \((X_t)_{t \geq 0}\). We show that the sample paths of \((Y_t)_{t \geq 0}\) can be obtained by repeatedly modifying those of the solution of the following equation:

\[ d\tilde{Y}_t = b(\tilde{Y}_t) \, dt + \sigma(\tilde{Y}_{t-}) \, dZ_t, \quad \tilde{Y}_0 = y. \]

Denote by \((Y_t^{(1)})_{t \geq 0}\) the solution to (2.21). Take a uniformly distributed random variable \(\zeta_1\) on \([0, 1]\), and define the stopping times \(T_1 = \inf \{ t > 0 : X_t = Y_t^{(1)} \}\) and

\[ \sigma_1 = \inf \left\{ t > 0 : \zeta_1 \leq \frac{1}{2} \left( \rho_{\Psi}(X_t, Y_t^{(1)}, \Delta Z_t) + \rho_{\Psi^{-1}}(X_t, Y_t^{(1)}, \Delta Z_t) \right) \right\}. \]

We consider two cases:

(i) On the event \(\{T_1 \leq \sigma_1\}\), we set \(Y_t = Y_t^{(1)}\) for all \(t < T_1\); moreover, by the pathwise uniqueness of the equation (1.1), we can define \(Y_t = X_t\) for \(t \geq T_1\).

(ii) On the event \(\{T_1 > \sigma_1\}\), we define \(Y_t = Y_t^{(1)}\) for all \(t < \sigma_1\) and

\[ Y_{\sigma_1} = Y_{\sigma_1}^{(1)} + \sigma(Y_{\sigma_1}^{(1)}) \Delta Z_{\sigma_1} \]

\[ + \begin{cases} \sigma(Y_{\sigma_1}^{(1)})(\Psi(\Delta Z_{\sigma_1}) - \Delta Z_{\sigma_1}), & \text{if } \zeta_1 \leq \frac{1}{2} \rho_{\Psi}(X_{\sigma_1}, Y_{\sigma_1}^{(1)}, \Delta Z_{\sigma_1}); \\ \sigma(Y_{\sigma_1}^{(1)})(\Psi^{-1}(\Delta Z_{\sigma_1}) - \Delta Z_{\sigma_1}), & \text{if } \zeta_1 > \frac{1}{2} \rho_{\Psi}(X_{\sigma_1}, Y_{\sigma_1}^{(1)}, \Delta Z_{\sigma_1}). \end{cases} \]

Next, we restrict on the event \(\{T_1 > \sigma_1\}\) and consider the SDE (2.21) with \(t > \sigma_1\) and \(\tilde{Y}_{\sigma_1} = Y_{\sigma_1}\). Denote its solution by \((Y_t^{(2)})_{t \geq 0}\). Similarly, we take another uniformly distributed random variable \(\zeta_2\) on \([0, 1]\), and define \(T_2 = \inf \{ t > \sigma_1 : X_t = Y_t^{(2)} \}\) and

\[ \sigma_2 = \inf \left\{ t > \sigma_1 : \zeta_2 \leq \frac{1}{2} \left( \rho_{\Psi}(X_t, Y_t^{(2)}, \Delta Z_t) + \rho_{\Psi^{-1}}(X_t, Y_t^{(2)}, \Delta Z_t) \right) \right\}. \]
In the same way, we can define the process \((Y_t)_{t \geq 0}\) till \(t \leq \sigma_2\). We repeat this procedure and note that, thanks to the fact that \(\mu_{\Psi}\) and \(\mu_{\Psi^{-1}}\) are finite measures on \((\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))\) (by Lemma 2.1(2)), only finite many modifications have to be made in any finite interval of time. Finally, we obtain the sample paths \((Y_t)_{t \geq 0}\).

(3) Denote by \((X_t, Y_t)_{t \geq 0}\) the unique strong solution to (2.18), and by \(\tilde{L}\) the associated Markov generator. According to the Itô formula, for any \(f \in C_b^2(\mathbb{R}^d)\), \(\tilde{L}f(x, y)\) enjoys the same formula as (2.15); that is, the generator of the process \((X_t, Y_t)_{t \geq 0}\) is just the coupling operator \(\tilde{L}\) defined by (2.15). Thus, \((X_t, Y_t)_{t \geq 0}\) is a Markov coupling of the process \(X_t\) determined by the SDE (1.1).

When \(X_{t-} = Y_{t-}\), \(\Psi(z) = z\) and so \(dZ_t = d\tilde{Z}_t\). Thus, by the Markov property of the process \((X_t, Y_t)_{t \geq 0}\) and the pathwise uniqueness of the SDE (1.1), \(X_t = Y_t\) for any \(t > T\), where \(T\) is the coupling time of the process \((X_t, Y_t)_{t \geq 0}\).

\[\]

3. Preliminary estimates on coupling operator

Let \(\tilde{L}\) be the coupling operator defined by (2.15), where \(\Psi\) is given by (2.16). Let \(f \in C([0, \infty)) \cap C_b^2((0, \infty))\) such that \(f(0) = 0\), \(f' \geq 0\), \(f'' \geq 0\) and \(f'' \leq 0\) on \((0, \infty)\).

We will give some estimates on \(\tilde{L}f(|x - y|)\).

According to (2.15), we know that for any \(f \in C([0, \infty)) \cap C_b^2((0, \infty))\) and any \(x, y \in \mathbb{R}^d\) with \(x \neq y\),

\[
\begin{align*}
\tilde{L}f(|x - y|) &= f'(|x - y|) \frac{(b(x) - b(y), x - y)}{|x - y|} \\
&+ \frac{1}{2} \int \left( f\left(|(x + \sigma(x)z) - (y + \sigma(y)\Psi(z))|\right) - f(|x - y|) \\
&- \frac{f'(|x - y|)}{|x - y|} \langle x - y, \sigma(x)z \rangle \mathbb{1}_{\{|z| \leq 1\}} \\
&+ \frac{f'(|x - y|)}{|x - y|} \langle x - y, \sigma(y)\Psi(z) \rangle \mathbb{1}_{\{|\Psi(z)| \leq 1\}} \right) \mu_{\Psi}(dz) \\
&+ \frac{1}{2} \int \left( f\left(|(x + \sigma(x)z) - (y + \sigma(y)\Psi^{-1}(z))|\right) - f(|x - y|) \\
&- \frac{f'(|x - y|)}{|x - y|} \langle x - y, \sigma(x)z \rangle \mathbb{1}_{\{|z| \leq 1\}} \\
&+ \frac{f'(|x - y|)}{|x - y|} \langle x - y, \sigma(y)\Psi^{-1}(z) \rangle \mathbb{1}_{\{|\Psi^{-1}(z)| \leq 1\}} \right) \mu_{\Psi^{-1}}(dz) \\
&+ \int \left( f\left(|(x + \sigma(x)z) - (y + \sigma(y)z)|\right) - f(|x - y|) \\
&- \frac{f'(|x - y|)}{|x - y|} \langle x - y, \sigma(x)z \rangle \mathbb{1}_{\{|z| \leq 1\}} \\
&+ \frac{f'(|x - y|)}{|x - y|} \langle x - y, \sigma(y)z \rangle \mathbb{1}_{\{|z| \leq 1\}} \right) \left( \nu - \frac{1}{2} \mu_{\Psi} - \frac{1}{2} \mu_{\Psi^{-1}} \right) (dz).
\end{align*}
\]

By Lemma 2.1,

\[
\int f'(|x - y|) \langle x - y, \sigma(y)\Psi(z) \rangle \mathbb{1}_{\{|\Psi(z)| \leq 1\}} \mu_{\Psi}(dz)
\]
by (2.18) and (2.19), we have
\[
\int f'(|x-y|) \frac{x-y}{|x-y|} (x-y, \sigma(y) \Psi(z)) 1_{\{|\Psi(z)| \leq 1\}} (\mu_{\Psi-1} \Psi)(dz)
\]
\[
= \int f'(|x-y|) \frac{x-y}{|x-y|} (x-y, \sigma(y) z) 1_{\{|z| \leq 1\} \mu_{\Psi-1}} (dz),
\]
where we note that all the integrals above are well defined since both \( \mu_{\Psi} \) and \( \mu_{\Psi-1} \) are finite measures, thanks to Lemma 2.1(2). Similarly, it also holds that
\[
\int f'(|x-y|) \frac{x-y}{|x-y|} (x-y, \sigma(y) \Psi^{-1}(z)) 1_{\{|\Psi^{-1}(z)| \leq 1\}} (\mu_{\Psi}) (dz)
\]
\[
= \int f'(|x-y|) \frac{x-y}{|x-y|} (x-y, \sigma(y) z) 1_{\{|z| \leq 1\} \mu_{\Psi}} (dz).
\]
Therefore, we arrive at for any \( f \in C([0,\infty)) \cap C^2_{\theta}((0,\infty)) \) and any \( x,y \in \mathbb{R}^d \) with \( x \neq y \),
\[
\tilde{L} f(|x-y|)
\]
\[
= \frac{f'(|x-y|)}{|x-y|} (b(x) - b(y), x - y) - \frac{f'(|x-y|)}{2|x-y|} \left( (\sigma(x) - \sigma(y)) \int_{\{|z| \leq 1\}} z (\mu_{\Psi} + \mu_{\Psi-1}) (dz), x - y \right)
\]
\[
+ \frac{1}{2} \int \left( f(|(x + \sigma(x)) z - (y + \sigma(y) \Psi(z))|) - f(|x - y|) \right) \mu_{\Psi} (dz) 
\]
\[
+ \frac{1}{2} \int \left( f(|(x + \sigma(x)) z - (y + \sigma(y) \Psi^{-1}(z))|) - f(|x - y|) \right) \mu_{\Psi-1} (dz) 
\]
\[
+ \int \left( f(|(x + \sigma(x)) z - (y + \sigma(y) z)|) - f(|x - y|) \right) 
\]
\[
- \frac{f'(|x-y|)}{|x-y|} (x-y, (\sigma(x) - \sigma(y)) z) 1_{\{|z| \leq 1\}} \left( \nu - \frac{1}{2} \mu_{\Psi} - \frac{1}{2} \mu_{\Psi-1} \right) (dz)
\]
\[
=: I_1 + I_2 + I_3 + I_4 + I_5.
\]
\textit{Remark 3.1.} We note that (3.22) also can be directly deduced from (2.18). Indeed, by (2.18) and (2.19), we have
\[
d(X_t - Y_t)
\]
\[
= (b(X_t) - b(Y_t)) dt 
\]
\[
+ \int_{\mathbb{R}^d \times [0,1]} (\sigma(X_{t^-}) - \sigma(Y_{t^-})) \Psi(z)) 1_{\{u \leq \frac{1}{2} \rho_{\Psi}(X_{t^-}, Y_{t^-}), z\}} \tilde{N}(dt, dz, du)
\]
\[
+ \int_{\mathbb{R}^d \times [0,1]} (\sigma(X_{t^-}) - \sigma(Y_{t^-})) \Psi^{-1}(z)) 
\]
\[
\times 1_{\{\frac{1}{2} \rho_{\Psi}(X_{t^-}, Y_{t^-}), u \leq \frac{1}{2} \rho_{\Psi}(X_{t^-}, Y_{t^-}) + \rho_{\Psi-1}(X_{t^-}, Y_{t^-})\}} \tilde{N}(dt, dz, du)
\]
\[
+ \int_{\mathbb{R}^d \times [0,1]} (\sigma(X_{t^-}) - \sigma(Y_{t^-})) z) 1_{\{\frac{1}{2} \rho_{\Psi}(X_{t^-}, Y_{t^-}), u \leq \frac{1}{2} \rho_{\Psi}(X_{t^-}, Y_{t^-}) \}} \tilde{N}(dt, dz, du)
\]
\[
+ \int_{\mathbb{R}^d \times [0,1]} \left[ \sigma(Y_t) \Psi(z) \left( 1_{\{|\Psi(z)| \leq 1\}} - 1_{\{|z| \leq 1\}} \right) 1_{\{u \leq \frac{1}{2} \rho_{\Psi}(X_{t^-}, Y_{t^-})\}} 
\]
\[
+ \sigma(Y_t) \Psi^{-1}(z) \left( 1_{\{|\Psi^{-1}(z)| \leq 1\}} - 1_{\{|z| \leq 1\}} \right) \right] \tilde{N}(dt, dz, du)
\]
\[
\times \mathbb{1}_{\frac{1}{2}\rho_{\psi}(X_{t-},Y_{t-}) < u \leq \frac{1}{2}(\rho_{\psi}(X_{t-},Y_{t-}) + \rho_{\psi-1}(X_{t-},Y_{t-}))} \nu(\,dz\,) \, du \, dt,
\]

where \( \rho_{\psi}(x, y, z) \) and \( \rho_{\psi-1}(x, y, z) \) are given in (2.17). Then, by the Itô formula, for any \( f \in C([0, \infty)) \cap C^2_b((0, \infty)) \) and any \( x, y \in \mathbb{R}^d \) with \( x \neq y \),

\[
\tilde{L}f(|x-y|) = f\left(\frac{|x-y|}{|x-y|}\right)(b(x) - b(y), x-y) + \frac{1}{2} \int \left( f(|x-y + \sigma(x)z - \sigma(y)\Psi(z)|) - f(|x-y|) \\
- f\left(\frac{|x-y|}{|x-y|}(x-y, \sigma(x)z - \sigma(y)\Psi(z))\mathbb{1}_{\{|z| \leq 1\}}\right) \mu_{\psi}(dz) \\
+ \frac{1}{2} \int \left( f(|x-y + \sigma(x)z - \sigma(y)\Psi^{-1}(z)|) - f(|x-y|) \\
- f\left(\frac{|x-y|}{|x-y|}(x-y, \sigma(x)z - \sigma(y)\Psi^{-1}(z))\mathbb{1}_{\{|z| \leq 1\}}\right) \mu_{\psi-1}(dz) \right) \Psi(\,dz\,) + \int \Psi(\,dz\,) \mathbb{1}_{\{|z| \leq 1\}}.
\]

According to Lemma 2.1, we know that

\[
\int \langle x-y, \sigma(y)\Psi(z) \rangle \mathbb{1}_{\{|\Psi(z)| \leq 1\}} \mu_{\psi}(dz) = \int \langle x-y, \sigma(y)\Psi^{-1}(z) \rangle \mathbb{1}_{\{|\Psi^{-1}(z)| \leq 1\}} \mu_{\psi-1}(dz).
\]
and
\begin{align*}
\int \langle x - y, \sigma(y)\Psi^{-1}(z) \rangle 1_{\{|\Psi^{-1}(z)| \leq 1\}} \mu_{\Psi^{-1}}(dz) = \int \langle x - y, \sigma(y)z \rangle 1_{\{|z| \leq 1\}} \mu_{\Psi}(dz).
\end{align*}

Hence, (3.22) follows from all the equalities above.

Next, we assume that \( f \in C([0, \infty)) \cap C^2(0, \infty) \) such that \( f(0) = 0, f' \geq 0, f'' \leq 0 \) on \( (0, \infty) \), and will compute \( I_i \) \( (i = 2, \cdots, 5) \) in (3.22) respectively.

Without loss of generality, under assumptions on \( \sigma(x) \), in the following we can assume that
\begin{align*}
\Lambda^{-1} \leq \inf_{x \in \mathbb{R}^d} \{ \|\sigma(x)\|_{\text{H.S.}} \vee \|\sigma(x)^{-1}\|_{\text{H.S.}} \} \leq \sup_{x \in \mathbb{R}^d} \{ \|\sigma(x)\|_{\text{H.S.}} \vee \|\sigma(x)^{-1}\|_{\text{H.S.}} \} \leq \Lambda.
\end{align*}

(i) It is clear that
\begin{align*}
I_2 \leq \frac{1}{2} f'(|x - y|)\|\sigma(x) - \sigma(y)\|_{\text{H.S.}} \int_{\{|z| \leq 1\}} \mu_{\Psi + \mu_{\Psi^{-1}}}(dz).
\end{align*}

(ii) By the definition of \( \Psi \) in (2.16), we have
\begin{align*}
I_3 &= \frac{1}{2} \int \left( f(|x + \sigma(x)z| - (y + \sigma(y)\sigma(x)^{-1}(\sigma(x)z + (x - y)\kappa))) - f(|x - y|) \right) \mu_{\Psi}(dz) \\
&= \frac{1}{2} \int \left( f(|x - y| - |x - y| \wedge \kappa) - f(|x - y|) \right) \mu_{\Psi}(dz) \\
&= \frac{1}{2} \mu_{\Psi}(\mathbb{R}^d) \left( f(|x - y| - |x - y| \wedge \kappa) - f(|x - y|) \right),
\end{align*}

where in the last equality we used again the fact that \( \mu_{\Psi} \) is a finite measure.

(iii) For any \( R \in [1, \infty) \),
\begin{align*}
I_4 &= \frac{1}{2} \int \left( f(|x + \sigma(x)z| - (y + \sigma(y)\sigma(x)^{-1}(\sigma(y)z - (x - y)\kappa))) - f(|x - y|) \right) \mu_{\Psi^{-1}}(dz) \\
&= \frac{1}{2} \int \left( f(|x - y| + |x - y| \wedge \kappa) - f(|x - y|) \right) \mu_{\Psi^{-1}}(dz) \\
&\quad + \frac{1}{2} \int \left( f(|x - y| + \sigma(y)\sigma(x)^{-1}(\sigma(y)z - (x - y)\kappa) + (\sigma(x) - \sigma(y)\sigma(x)^{-1}\sigma(y))z) \right) \\
&\quad - f(|x - y| + |x - y| \wedge \kappa)) \mu_{\Psi^{-1}}(dz) \\
&= \frac{1}{2} \mu_{\Psi}(\mathbb{R}^d) \left( f(|x - y| + |x - y| \wedge \kappa) - f(|x - y|) \right) \\
&\quad + \frac{1}{2} \int_{\{|z| \leq R\}} \left( f(|x - y| + \sigma(y)\sigma(x)^{-1}(\sigma(y)z - (x - y)\kappa) + (\sigma(x) - \sigma(y)\sigma(x)^{-1}\sigma(y))z) \right) \\
&\quad - f(|x - y| + |x - y| \wedge \kappa)) \mu_{\Psi^{-1}}(dz) \\
&\quad + \frac{1}{2} \int_{\{|z| > R\}} \left( f(|x - y| + \sigma(y)\sigma(x)^{-1}(\sigma(y)z - (x - y)\kappa) + (\sigma(x) - \sigma(y)\sigma(x)^{-1}\sigma(y))z) \right) \\
&\quad - f(|x - y| + |x - y| \wedge \kappa)) \mu_{\Psi^{-1}}(dz) \\
&=: I_{4,1} + I_{4,2,R} + I_{4,3,R}.
\end{align*}
where in the third equality we used the fact that \( \mu_\Phi(\mathbb{R}^d) = \mu_{\Phi^{-1}}(\mathbb{R}^d) \), due to Lemma 2.1(1). By the elementary inequality

\[
(3.23) \quad f(a) - f(b) \leq f'(a - b), \quad a \geq 0, b > 0,
\]

and the fact that \( f'' \leq 0 \) on \((0, \infty)\), we have

\[
I_{4.2,R} \leq \frac{1}{2} f'(|x - y| + |x - y| \wedge \kappa) \int_{|z| \leq R} (|\sigma(x) - \sigma(y)\sigma(x)^{-1}\sigma(y)|z| \\
+ |\sigma(y)\sigma(x)^{-1} - I_{d \times d}(x - y)\kappa|) \mu_{\Phi^{-1}}(dz) \\
\leq \frac{1}{2} f'(|x - y|)\|\sigma(x) - \sigma(y)\sigma(x)^{-1}\sigma(y)\|_{H.S.} \int_{|z| \leq R} |z| \mu_{\Phi^{-1}}(dz) \\
+ \frac{1}{2} f'(|x - y|)\mu_{\Phi^{-1}}(\{z \in \mathbb{R}^d : |z| \leq R\})\|\sigma(y)\sigma(x)^{-1} - I_{d \times d}\|_{H.S.}(|x - y| \wedge \kappa) \\
\leq \frac{1}{2} f'(|x - y|)(1 + \|\sigma(y)\sigma(x)^{-1}\|_{H.S.})\|\sigma(x) - \sigma(y)\|_{H.S.} \int_{|z| \leq R} |z| \mu_{\Phi^{-1}}(dz) \\
+ \frac{1}{2} f'(|x - y|)\mu_{\Phi^{-1}}(\mathbb{R}^d)\|\sigma(x) - \sigma(y)\|_{H.S.}\|\sigma(x)^{-1}\|_{H.S.}(|x - y| \wedge \kappa),
\]

where in the last inequality we have used the facts that
\[
\|\sigma(x) - \sigma(y)\sigma(x)^{-1}\sigma(y)\|_{H.S.} \leq \|\sigma(x) - \sigma(y)\|_{H.S.} \\
+ \|\sigma(y) - \sigma(y)\sigma(x)^{-1}\sigma(y)\|_{H.S.} \\
\leq \|\sigma(x) - \sigma(y)\|_{H.S.} \\
+ \|\sigma(y)\sigma(x)^{-1}\|_{H.S.}\|\sigma(x) - \sigma(y)\|_{H.S.} \\
= (1 + \|\sigma(y)\sigma(x)^{-1}\|_{H.S.})\|\sigma(x) - \sigma(y)\|_{H.S.},
\]

and
\[
(3.25) \quad \|\sigma(y)\sigma(x)^{-1} - I_{d \times d}\|_{H.S.} \leq \|\sigma(x) - \sigma(y)\|_{H.S.}\|\sigma(x)^{-1}\|_{H.S.}.
\]

On the other hand, since for all \( a, b \geq 0 \), by \( f'' \leq 0 \) on \((0, \infty)\) and \( f(0) = 0 \),

\[
(3.26) \quad f(a + b) - f(a) = \int_a^{a+b} f'(s) \, ds = \int_0^b f'(a + s) \, ds \leq \int_0^b f'(s) \, ds = f(b),
\]

we have

\[
I_{4.3,R} \leq \frac{1}{2} \int_{|z| > R} \left( f\left(|(\sigma(x) - \sigma(y)\sigma(x)^{-1}\sigma(y))z| \right) \\
+ f\left(|x - y| + |\sigma(y)\sigma(x)^{-1}(x - y)\kappa| \right) \\
- f\left(|x - y| + |x - y| \wedge \kappa \right) \mu_{\Phi^{-1}}(dz) \\
\leq \frac{1}{2} \int_{|z| > R} f\left(1 + \|\sigma(y)\sigma(x)^{-1}\|_{H.S.}\|\sigma(x) - \sigma(y)\|_{H.S.}|z| \right) \mu_{\Phi^{-1}}(dz) \\
+ \frac{1}{2} f'(|x - y| + |x - y| \wedge \kappa)|\sigma(y)\sigma(x)^{-1} - I_{d \times d}(x - y)\kappa| \\
\times \mu_{\Phi^{-1}}(\{z \in \mathbb{R}^d : |z| > R\}) \\
\leq \frac{1}{2} \int_{|z| > R} f\left(1 + \|\sigma(y)\sigma(x)^{-1}\|_{H.S.}\|\sigma(x) - \sigma(y)\|_{H.S.}|z| \right) \mu_{\Phi^{-1}}(dz)
\]
+ \frac{1}{2} f'(|x-y|) \mu_\Psi(R^d) \| \sigma(x) - \sigma(y) \|_{H.S.} \| \sigma(x)^{-1} \|_{H.S.} (|x-y| \wedge \kappa),

where in the second inequality we used (3.24), and the last one follows from (3.25) and the facts that \( f'' \leq 0 \) on \((0, \infty)\) and \( \mu_\Psi(R^d) = \mu_{\Psi^{-1}}(R^d) \).

Combining all the conclusions above, we obtain that

\[
I_4 \leq \frac{1}{2} \mu_\Psi(R^d) \left( f(|x-y| + |x-y| \wedge \kappa) - f(|x-y|) \right) \\
+ \frac{1}{2} f'(|x-y|) \| \sigma(x) - \sigma(y) \|_{H.S.} \left[ 2 \mu_{\Psi^{-1}}(R^d) \| \sigma(x)^{-1} \|_{H.S.} (|x-y| \wedge \kappa) \\
+ (1 + \| \sigma(y) \sigma(x)^{-1} \|_{H.S.}) \int_{\{|z| \leq R\}} |z| \mu_{\Psi^{-1}}(dz) \right] \\
+ \frac{1}{2} \int_{\{|z| > R\}} f\left( (1 + \| \sigma(y) \sigma(x)^{-1} \|_{H.S.}) \| \sigma(x) - \sigma(y) \|_{H.S.} |z| \right) \mu_{\Psi^{-1}}(dz).
\]

(iv) For \( I_5 \), we have

\[
I_5 = \int_{\{|z| \leq 1\}} \left( f\left( |x-y + (\sigma(x) - \sigma(y))z| \right) - f(|x-y|) \right) \\
- \frac{f'(|x-y|)}{|x-y|} \langle x-y, (\sigma(x) - \sigma(y))z \rangle \left( \nu - \frac{1}{2} \mu_\Psi - \frac{1}{2} \mu_{\Psi^{-1}} \right)(dz) \\
+ \int_{\{|z| > 1\}} \left( f\left( |x-y + (\sigma(x) - \sigma(y))z| \right) - f(|x-y|) \right) \left( \nu - \frac{1}{2} \mu_\Psi - \frac{1}{2} \mu_{\Psi^{-1}} \right)(dz) \\
=: I_{5,1} + I_{5,2}.
\]

By (3.23) again, we find

\[
I_{5,1} \leq f'(|x-y|) \int_{\{|z| \leq 1\}} \left( |x-y + (\sigma(x) - \sigma(y))z| - |x-y| \right) \\
- \frac{1}{|x-y|} \langle x-y, (\sigma(x) - \sigma(y))z \rangle \left( \nu - \frac{1}{2} \mu_\Psi - \frac{1}{2} \mu_{\Psi^{-1}} \right)(dz) \\
\leq f'(|x-y|) \int_{\{|z| \leq 1\}} |(\sigma(x) - \sigma(y))z| \left( \nu - \frac{1}{2} \mu_\Psi - \frac{1}{2} \mu_{\Psi^{-1}} \right)(dz) \\
\leq f'(|x-y|) \frac{\| \sigma(x) - \sigma(y) \|^2_{H.S.}}{|x-y|} \int_{\{|z| \leq 1\}} |z|^2 \left( \nu - \frac{1}{2} \mu_\Psi - \frac{1}{2} \mu_{\Psi^{-1}} \right)(dz) \\
\leq f'(|x-y|) \frac{\| \sigma(x) - \sigma(y) \|^2_{H.S.}}{|x-y|} \int_{\{|z| \leq 1\}} |z|^2 \nu(dz),
\]

where in the second inequality we used the fact that

\[ |x+y| - |x| - \frac{1}{|x|} \langle x, y \rangle \leq \frac{|y|^2}{2|x|}, \quad x, y \in R^d \text{ with } x \neq 0. \]

On the other hand, using (3.23) and (3.26), we obtain that for all \( R \in [1, \infty) \),

\[
I_{5,2} = \int_{\{|z| \leq R\}} \left( f\left( |x-y + (\sigma(x) - \sigma(y))z| \right) - f(|x-y|) \right) \left( \nu - \frac{1}{2} \mu_\Psi - \frac{1}{2} \mu_{\Psi^{-1}} \right)(dz) \\
+ \int_{\{|z| > R\}} \left( f\left( |x-y + (\sigma(x) - \sigma(y))z| \right) - f(|x-y|) \right) \left( \nu - \frac{1}{2} \mu_\Psi - \frac{1}{2} \mu_{\Psi^{-1}} \right)(dz)
\]

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\[
\begin{align*}
&\leq f'(|x - y|)\|\sigma(x) - \sigma(y)\|_{\text{H.S.}} \int_{\{1 < |z| \leq R\}} |z| \left( \nu - \frac{1}{2} \mu_\Psi - \frac{1}{2} \mu_{\Psi^{-1}} \right) (dz) \\
&+ \int_{\{|z| > R\}} f(\|\sigma(x) - \sigma(y)\|_{\text{H.S.}} |z|) \left( \nu - \frac{1}{2} \mu_\Psi - \frac{1}{2} \mu_{\Psi^{-1}} \right) (dz) \\
&\leq f'(|x - y|)\|\sigma(x) - \sigma(y)\|_{\text{H.S.}} \int_{\{1 < |z| \leq R\}} |z| \nu (dz) \\
&+ \int_{\{|z| > R\}} f(\|\sigma(x) - \sigma(y)\|_{\text{H.S.}} |z|) \nu (dz).
\end{align*}
\]

Combining both inequalities above, we arrive at

\[
I_5 \leq f'(|x - y|)\|\sigma(x) - \sigma(y)\|_{\text{H.S.}} \\
\times \left[ \frac{\|\sigma(x) - \sigma(y)\|_{\text{H.S.}}}{2|x - y|} \int_{\{|z| \leq 1\}} |z|^2 \nu (dz) + \int_{\{1 < |z| \leq R\}} |z| \nu (dz) \right] \\
+ \int_{\{|z| > R\}} f(\|\sigma(x) - \sigma(y)\|_{\text{H.S.}} |z|) \nu (dz).
\]

Finally, putting all the estimates in (i)–(iv) into (3.22), we can get the following statement.

**Proposition 3.2.** Assume that (1.2) holds. Then, for all \( f \in C([0, \infty)) \cap C^2_b((0, \infty)) \) such that \( f(0) = 0, f \geq 0, f' \geq 0 \) and \( f'' \leq 0 \) on \((0, \infty)\), any \( R \in [1, \infty] \) and \( x, y \in \mathbb{R}^d \) with \( x \neq y \),

\[
\tilde{L}f(|x - y|) \leq \Theta_0(f)(x, y) + \frac{f'(|x - y|)}{|x - y|} \langle b(x) - b(y), x - y \rangle \\
\quad + f'(|x - y|)\|\sigma(x) - \sigma(y)\|_{\text{H.S.}} \Theta_{\leq R}(x, y) \\
\quad + \Theta_{> R}(f)(x, y),
\]

(3.27)

where

\[
\Theta_0(f)(x, y) := \frac{1}{2} \mu_\Psi(\mathbb{R}^d) \left( f(|x - y| + |x - y| \wedge \kappa) \\
+ f(|x - y| - |x - y| \wedge \kappa) - 2f(|x - y|) \right),
\]

\[
\Theta_{\leq R}(x, y) := \Lambda \mu_\Psi(\mathbb{R}^d) (|x - y| \wedge \kappa) + \frac{\|\sigma(x) - \sigma(y)\|_{\text{H.S.}}}{2|x - y|} \int_{\{|z| \leq 1\}} |z|^2 \nu (dz) \\
\quad + (1 + \Lambda^2) \int_{\{|z| \leq R\}} |z| \left( \mu_\Psi + \mu_{\Psi^{-1}} \right) (dz) + \int_{\{1 < |z| \leq R\}} |z| \nu (dz)
\]

and

\[
\Theta_{> R}(f)(x, y) := 2 \int_{\{|z| > R\}} f \left( (1 + \Lambda^2)\|\sigma(x) - \sigma(y)\|_{\text{H.S.}} |z| \right) \nu (dz).
\]

**Remark 3.3.** We should mention that, if \( \nu \) in the definitions of \( \mu_\Psi \) and \( \mu_{\Psi^{-1}} \) is replaced by any Borel measure \( \nu_0 \) on \((\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))\) such that \( 0 < \nu_0 \leq \nu \), then all the conclusions above still hold true.
4. Regularity and ergodicity via coupling

Assume that the SDE (1.1) has a unique strong solution, which is denoted by \( X := (X_t)_{t \geq 0} \). Let \((P_t)_{t \geq 0}\) be the associated semigroup. Let \( \tilde{L} \) be the coupling operator given by (2.15).

4.1. Regularity via coupling. The statement below shows an idea to establish regularity properties of semigroups by adopting the coupling operator \( \tilde{L} \).

Proposition 4.1. Assume that there exist a constant \( \varepsilon_0 > 0 \) and a sequence of positive and increasing functions \( \{\psi_n\}_{n \geq 1} \) such that for all \( x, y \in \mathbb{R}^d \) with \( 1/n \leq |x - y| \leq \varepsilon \leq \varepsilon_0 \),

\[
\tilde{L}\psi_n(|x - y|) \leq -\lambda(\varepsilon),
\]

where \( \lambda(\varepsilon) \) is a positive constant independent of \( n \). Then, for any \( t > 0 \) and \( f \in B_b(\mathbb{R}^d) \),

\[
\sup_{x \neq y} \frac{|P_t f(x) - P_t f(y)|}{\psi_\infty(|x - y|)} \leq 2\|f\|_\infty \sup_{\varepsilon \in (0,\varepsilon_0]} \left[ \frac{1}{\psi_\infty(\varepsilon)} + \frac{1}{t\lambda(\varepsilon)} \right],
\]

where \( \psi_\infty = \lim_{n \to \infty} \psi_n \).

Proof. The proof was almost known before, e.g., see that of [14, Theorem 5.1] or [13, Theorem 1.2]. For the sake of completeness, we present it here. Let \((X_t, Y_t)_{t \geq 0}\) be the coupling process constructed in Subsection 2.2, and denote by \( \tilde{E}^{(x,y)} \) and \( E^{(x,y)} \) the distribution and the expectation of \((X_t, Y_t)_{t \geq 0}\) starting from \((x, y)\), respectively. For any \( \varepsilon \in (0, \varepsilon_0] \) and \( n \geq 1 \), we set

\[
S_\varepsilon := \inf\{t \geq 0 : |X_t - Y_t| > \varepsilon\},
\]

\[
T_n := \inf\{t \geq 0 : |X_t - Y_t| \leq 1/n\}.
\]

Note that \( T_n \uparrow T \) as \( n \uparrow \infty \), where \( T \) is the coupling time, i.e.,

\[
T := \inf\{t \geq 0 : X_t = Y_t\}.
\]

For any \( x, y \in \mathbb{R}^d \) with \( 0 < |x - y| < \varepsilon \leq \varepsilon_0 \), we take \( n \) large enough such that \( |x - y| > 1/n \). Then, by (4.28), for any \( t > 0 \),

\[
0 \leq \tilde{E}^{(x,y)}\psi_n\left(|X_{t \wedge T_n \wedge S_\varepsilon} - Y_{t \wedge T_n \wedge S_\varepsilon}|\right)
\]

\[
= \psi_n(|x - y|) + \tilde{E}^{(x,y)}\left(\int_0^{t \wedge T_n \wedge S_\varepsilon} \tilde{L}\psi_n(|X_s - Y_s|) \, ds\right)
\]

\[
\leq \psi_n(|x - y|) - \lambda(\varepsilon)\tilde{E}^{(x,y)}(t \wedge T_n \wedge S_\varepsilon).
\]

Hence,

\[
\tilde{E}^{(x,y)}(t \wedge T_n \wedge S_\varepsilon) \leq \frac{\psi_n(|x - y|)}{\lambda(\varepsilon)}.
\]

Letting \( t \to \infty \),

\[
\tilde{E}^{(x,y)}(T_n \wedge S_\varepsilon) \leq \frac{\psi_n(|x - y|)}{\lambda(\varepsilon)}.
\]
On the other hand, again by (4.28), for any \( x, y \in \mathbb{R}^d \) with \( 1/n \leq |x - y| < \varepsilon \leq \varepsilon_0 \),

\[
\mathbb{E}^{(x,y)}_n(|X_t \wedge T_n \wedge S_k - Y_t \wedge T_n \wedge S_k|) = \psi_n(|x - y|) + \int_0^{t \wedge T_n \wedge S_k} \tilde{L}\psi_n(|X_u - Y_u|) \, du
\]

\[
\leq \psi_n(|x - y|).
\]

This along with the increasing property of \( \psi_n \) yields that

\[
\mathbb{P}^{(x,y)}(S_\varepsilon < T_n \wedge t) \leq \frac{\psi_n(|x - y|)}{\psi_n(\varepsilon)}.
\]

Letting \( t \to \infty \),

\[
\mathbb{P}^{(x,y)}(S_\varepsilon < T_n) \leq \frac{\psi_n(|x - y|)}{\psi_n(\varepsilon)}.
\]

Therefore, combining both estimates above, we obtain that

\[
\mathbb{P}^{(x,y)}(T_n > t) \leq \mathbb{P}^{(x,y)}(T_n \wedge S_\varepsilon > t) + \mathbb{P}^{(x,y)}(T_n > S_\varepsilon) \leq \frac{\mathbb{E}^{(x,y)}(T_n \wedge S_\varepsilon)}{t} + \frac{\psi_n(|x - y|)}{\psi_n(\varepsilon)} \leq \psi_n(|x - y|) \left[ \frac{1}{\psi_n(\varepsilon)} + \frac{1}{t\lambda(\varepsilon)} \right].
\]

It follows that

\[
\mathbb{P}^{(x,y)}(T > t) = \lim_{n \to \infty} \mathbb{P}^{(x,y)}(T_n > t) \leq \liminf_{n \to \infty} \left\{ \psi_n(|x - y|) \left[ \frac{1}{\psi_n(\varepsilon)} + \frac{1}{t\lambda(\varepsilon)} \right] \right\} \leq \psi_\infty(|x - y|) \left[ \frac{1}{\psi_\infty(\varepsilon)} + \frac{1}{t\lambda(\varepsilon)} \right].
\]

Thus, for any \( f \in B_b(\mathbb{R}^d), \ t > 0 \) and any \( x, y \in \mathbb{R}^d \) with \( 0 < |x - y| < \varepsilon \leq \varepsilon_0 \),

\[
|P_t f(x) - P_t f(y)| = |E^x f(X_t) - E^y f(Y_t)| = |E^{(x,y)}(f(X_t) - f(Y_t))| = |E^{(x,y)}(f(X_t) - f(Y_t))\mathbf{1}_{\{T > t\}}| \leq 2\|f\|_\infty \mathbb{P}^{(x,y)}(T > t) \leq 2\|f\|_\infty \psi_\infty(|x - y|) \left[ \frac{1}{\psi_\infty(\varepsilon)} + \frac{1}{t\lambda(\varepsilon)} \right].
\]

Consequently,

\[
\sup_{|x - y| \leq \varepsilon} \left| \frac{P_t f(x) - P_t f(y)}{\psi_\infty(|x - y|)} \right| \leq 2\|f\|_\infty \left[ \frac{1}{\psi_\infty(\varepsilon)} + \frac{1}{t\lambda(\varepsilon)} \right].
\]

Since \( \psi_\infty \) is increasing on \((0, \infty)\), and

\[
\sup_{|x - y| \geq \varepsilon} \left| \frac{P_t f(x) - P_t f(y)}{\psi_\infty(|x - y|)} \right| \leq 2\|f\|_\infty \psi_\infty(\varepsilon),
\]
we further obtain that for all \( \varepsilon \in (0, \varepsilon_0] \),

\[
\sup_{x \neq y} \frac{|P_t f(x) - P_t f(y)|}{\psi_\infty(|x - y|)} \leq 2 \|f\|_\infty \left[ \frac{1}{\psi_\infty(\varepsilon)} + \frac{1}{t \lambda(\varepsilon)} \right].
\]

Taking infimum with respect to \( \varepsilon \in (0, \varepsilon_0] \) in the right hand side of the inequality above proves the desired assertion. \( \square \)

**Theorem 4.2.** Suppose that the diffusion coefficient \( \sigma(x) \) is Lipschitz continuous and satisfies (1.2), and the drift term \( b(x) \) is locally \( \beta \)-Hölder continuous with \( \beta \in (0, 1] \). Assume also that there is a nonnegative and \( C^\beta((0, \infty)) \cap C^3([0, \infty)) \)-function \( \psi \) such that

(i) \( \psi(0) = 0, \psi' \geq 0, \psi'' \leq 0 \) and \( \psi''' \geq 0 \) on \((0, 2] \);

(ii) For any constants \( c_1, c_2 > 0 \),

\[
\limsup_{r \to 0} \left[ J(r)r^2 \psi''(2r) + c_1 K(r) \psi'(r) + c_2 \psi'(r)r^\beta \right] < 0,
\]

where

\[
J(r) = \inf_{x, y \in \mathbb{R}^d : |x - y| \leq r} \mu_\psi(\mathbb{R}^d)
\]

and

\[
K(r) = \sup_{x, y \in \mathbb{R}^d : |x - y| = r} \left( \mu_\psi(\mathbb{R}^d)|x - y| + \int_{\{|z| \leq 2\}} |z| (\mu_\psi + \mu_{\psi - 1})(dz) \right).
\]

Then, there are constants \( C, \varepsilon_0 > 0 \) such that for all \( f \in B_b(\mathbb{R}^d) \) and \( t > 0 \),

\[
\sup_{x \neq y} \frac{|P_t f(x) - P_t f(y)|}{|x - y|} \leq C \|f\|_\infty \inf_{\varepsilon \in (0, \varepsilon_0]} \left[ \frac{1}{\psi_\infty(\varepsilon)} + \frac{1}{t \lambda_\psi(\varepsilon)} \right],
\]

where

\[
\lambda_\psi(\varepsilon) = \sup_{0 < r \leq \varepsilon} J(r)r^2 \psi''(2r).
\]

**Proof.** It is clear that (4.29) implies

\[
\limsup_{r \to 0} J(r)r^2 \psi''(2r) < 0.
\]

Let \( \varepsilon \in (0, \kappa \wedge 1) \). For any \( x, y \in \mathbb{R}^d \) with \( 0 < |x - y| \leq \varepsilon \), by (3.27) (with \( R = 2 \)) and the assumptions that \( \sigma(x) \) is Lipschitz continuous and satisfies (1.2), and \( b(x) \) is locally \( \beta \)-Hölder continuous with \( \beta \in (0, 1] \), we find that

\[
\hat{L}\psi(|x - y|) \leq \frac{1}{2} \mu_\psi(\mathbb{R}^d)(\psi(2|x - y|) - 2\psi(|x - y|))
\]

\[
+ c_1 \psi'(|x - y|)|x - y| \left( \mu_\psi(\mathbb{R}^d)|x - y| + \int_{\{|z| \leq 2\}} |z| (\mu_\psi + \mu_{\psi - 1})(dz) \right)
\]

\[
+ c_2 \psi'(|x - y|)|x - y|^{\beta} + c_3 \int_{\{|z| \geq 2\}} \psi(c_4|x - y||z|) \nu(dz)
\]

for some constants \( c_1, c_2, c_3, c_4 > 0 \). Note that, by \( \psi''' \geq 0 \) on \((0, 2] \) and \( \psi(0) = 0 \), we have for \( r > 0 \) small enough,

\[
\psi(2r) - 2\psi(r) = \int_0^r \int_s^{r+s} \psi''(u) du \, ds \leq \psi''(2r)r^2.
\]
Then, using (4.31) and (4.29), we can choose \( \varepsilon_0 \in (0, \kappa \wedge 1) \) such that for all \( x, y \in \mathbb{R}^d \) with \( 0 < |x - y| \leq \varepsilon \leq \varepsilon_0 \),
\[
\tilde{L} \psi(|x - y|) \leq \frac{1}{2} J(|x - y|) \psi''(2|x - y|)|x - y|^2 + c_1 K(|x - y|) \psi'(|x - y|)|x - y| + c_2 \psi'(|x - y|)|x - y|^\beta + c_3 \int_{|z| \geq 2} \psi(c_4 |x - y||z|) \nu(dz)
\]
\[
\leq \frac{1}{4} \left[ J(|x - y|) \psi''(2|x - y|)|x - y|^2 + c_1 K(|x - y|) \psi'(|x - y|)|x - y| + c_2 \psi'(|x - y|)|x - y|^\beta \right] \leq \frac{1}{4} \sup_{|x - y| \leq \varepsilon} \left[ J(|x - y|) \psi''(2|x - y|)|x - y|^2 + c_1 K(|x - y|) \psi'(|x - y|)|x - y| + c_2 \psi'(|x - y|)|x - y|^\beta \right].
\]
where in the first inequality the constant \( c_2 \) may depend on \( \varepsilon_0 \) but can be chosen to be independent of \( \varepsilon \), and in the second inequality we also used the fact that
\[
\lim_{|x - y| \to 0} \int_{|z| \geq 2} \psi(c_4 |x - y||z|) \nu(dz) = 0.
\]
By (4.31) and (4.29) again, we furthermore get (by possibly choosing \( \varepsilon_0 \) small enough) that
\[
\tilde{L} \psi(|x - y|) \leq \frac{1}{8} \sup_{|x - y| \leq \varepsilon} J(|x - y|) \psi''(2|x - y|)|x - y|^2 = -\frac{1}{8} \lambda \psi(\varepsilon) < 0,
\]
Having the inequality above at hand, we can obtain the desired assertion by applying Proposition 4.1.

4.2. Ergodicity via coupling. The following proposition is essentially taken from [14, Theorem 3.1].

**Proposition 4.3.** Assume that there exist a constant \( \lambda > 0 \) and a sequence of positive functions \( \{\psi_n\}_{n \geq 1} \) such that for all \( x, y \in \mathbb{R}^d \) with \( 1/n \leq |x - y| \leq n \),
\[
\tilde{L} \psi_n(|x - y|) \leq -\lambda \psi_n(|x - y|).
\]
Then, for any \( t > 0 \) and \( x, y \in \mathbb{R}^d \),
\[
W_{\psi_{\infty}}(\delta_x P_t, \delta_y P_t) \leq \psi_{\infty}(|x - y|) e^{-\lambda t},
\]
where \( \psi_{\infty} = \lim \inf_{n \to \infty} \psi_n \).

**Proof.** The proof is inspired by that of [12, Theorem 1.3] or [28, Theorem 1.2]. We can refer to step 2 in the proof of [14, Theorem 3.1] for the details. \( \square \)

Next, we assume that (1.5) holds. This implies that we can take \( R = \infty \) in the estimate (3.27). Motivated by [14, Theorems 4.2 and 4.4], we have the following statements.

**Theorem 4.4.** Assume that the diffusion coefficient \( \sigma(x) \) is Lipschitz continuous with Lipschitz constant \( L_\sigma > 0 \) and satisfies (1.2) with some constant \( \Lambda > 0 \), and that the following conditions hold:
Assume that all assumptions but Theorem 4.5 hold, and that the following two conditions (replacing (ii) and (iii) respectively) are satisfied:

\begin{equation}
\inf_{x,y \in \mathbb{R}^d: |x-y| \leq \kappa_0} \mu_\Psi(\mathbb{R}^d) > 0
\end{equation}

and

\begin{equation}
A_1 := \sup_{x,y \in \mathbb{R}^d} \left( \Lambda \mu_\Psi(\mathbb{R}^d) (|x-y| \wedge \kappa_0) + (1 + \Lambda^2/2) \int_{\mathbb{R}^d} |z| (\mu_\Psi + \mu_{\Psi^{-1}}) (dz) \right) < \infty
\end{equation}

for some \( \kappa_0 > 0 \);

(ii) The diffusion coefficient \( \sigma(x) \) and the drift term \( b(x) \) satisfy

\[
\frac{\langle b(x) - b(y), x - y \rangle}{|x - y|} + \|\sigma(x) - \sigma(y)\|_{\text{H.S.}} (A_1 + A_2)
\]

\begin{equation}
\leq \begin{cases} 
\Phi_1(|x - y|), & |x - y| < l_0, \\
-K_2|x - y|, & |x - y| > l_0
\end{cases}
\end{equation}

for some constants \( K_2 > 0, l_0 \geq 0 \), and a nonnegative concave function \( \Phi_1 \in C([0, 2l_0]) \cap C^2((0, 2l_0]) \) such that \( \Phi_1(0) = 0 \) and \( \Phi_1'' \) is nondecreasing, where

\[
A_2 = \int_{\{|z| > 1\}} |z|^1 \nu(dz) + \frac{L_\sigma}{2} \int_{\{|z| \leq 1\}} |z|^2 \nu(dz);
\]

(iii) There is a nondecreasing and concave function \( \sigma \in C([0, 2l_0]) \cap C^2((0, 2l_0]) \) such that for some \( \kappa \in (0, \kappa_0] \), one has

\[
\sigma(r) \leq \frac{1}{2r} J(\kappa \wedge r)(\kappa \wedge r)^2, \quad r \in (0, 2l_0);
\]

and the integrals \( g_1(r) = \int_0^r \frac{1}{\sigma(s)} ds \) and \( g_2(r) = \int_0^r \frac{\Phi_1(s)}{\sigma^2(s)} ds \) are well defined for all \( r \in [0, 2l_0] \), where \( J(r) \) is defined by (4.30).

Set \( c_2 = (2K_2) \wedge g_1(2l_0)^{-1} \) and \( c_1 = e^{-c_2g(2l_0)} \), where the function \( g \) is defined by

\[
g(r) = g_1(r) + \frac{2}{c_2} g_2(r), \quad r \in (0, 2l_0].
\]

Then for any \( x, y \in \mathbb{R}^d \) and \( t > 0 \),

\[
W_\psi(\delta_x P_t, \delta_y P_t) \leq e^{-\lambda t \psi(|x - y|)},
\]

and so

\[
W_1(\delta_x P_t, \delta_y P_t) \leq C e^{-\lambda t |x - y|},
\]

where

\[
\psi(r) = \begin{cases} 
c_1 r + \int_0^r e^{-c_2 g(s)} ds, & r \in [0, 2l_0), \\
\psi(2l_0) + \psi'(2l_0)(r - 2l_0), & r \in (2l_0, \infty),
\end{cases}
\]

\[
\lambda = \frac{c_2}{1 + e^{c_2 g(2l_0)}} = \frac{(2K_2) \wedge g_1(2l_0)^{-1}}{1 + \exp \left( g(2l_0) [ (2K_2) \wedge g_1(2l_0)^{-1} ] \right)}
\]

and

\[
C = \frac{1 + c_1}{2c_1} = \frac{1}{2} \left( 1 + \exp \left( g(2l_0) [ (2K_2) \wedge g_1(2l_0)^{-1} ] \right) \right).
\]

**Theorem 4.5.** Assume that all assumptions but (ii) and (iii) in Theorem 4.4 hold, and that the following two conditions (replacing (ii) and (iii) respectively) are satisfied.
(ii') The diffusion coefficient $\sigma(x)$ and the drift term $b(x)$ satisfy
\[
\frac{\langle b(x) - b(y), x - y \rangle}{|x - y|} + \|\sigma(x) - \sigma(y)\|_{\text{H.S.}}(A_1 + A_2)
\]
(4.34)
\[
\leq \begin{cases} 
K_1, & |x - y| < l_0, \\
-K_2|x - y|, & |x - y| \geq l_0
\end{cases}
\]
for some constants $K_1 \geq 0$, $K_2 > 0$ and $l_0 \geq 0$, where
\[
A_2 = \int_{\{|z| > 1\}} |z| \nu(dz) + \frac{L}{2} \int_{\{|z| \leq 1\}} |z|^2 \nu(dz).
\]

(iii') There is a nondecreasing and concave function $\sigma \in C([0, 2l_0]) \cap C^2((0, 2l_0])$ such that for some $\kappa \in (0, \kappa_0 \wedge l_0]$, one has
\[
\sigma(r) \leq \frac{1}{2r} J(\kappa \wedge r)(\kappa \wedge r)^2, \quad r \in (0, 2l_0];
\]
and the integral $g(r) = \int_0^r \frac{1}{\sigma(s)} ds$ is well defined for all $r \in [0, 2l_0]$. Then there exist constants $\lambda, c > 0$ such that for any $x, y \in \mathbb{R}^d$ and $t > 0$,
\[
\|\delta_x P_t - \delta_y P_t\|_{\text{Var}} \leq ce^{-\lambda t}(1 + |x - y|).
\]

Sketch of the proofs of Theorems 4.4 and 4.5. As we mentioned before, since (1.5) holds, one can take $R = \infty$ in the estimate (3.27). Under assumptions on the diffusion coefficient $\sigma(x)$ and condition (i), (3.27) is further reduced into
\[
\tilde{L}f(|x - y|) \leq \Theta_0(f)(x, y) + f'(|x - y|) \left( \frac{\langle b(x) - b(y), x - y \rangle}{|x - y|} + \|\sigma(x) - \sigma(y)\|_{\text{H.S.}}(A_1 + A_2) \right).
\]
Then, one can use the estimate above and follow arguments of [14, Theorem 4.2 and 4.4] to prove required conclusions. \hfill \qed

5. PROOFS AND EXAMPLES

Recall from (2.16) that, for any $\kappa > 0$ and $x, y \in \mathbb{R}^d$,
\[
\Psi(z) = \Psi_{\kappa, x, y}(z) = \sigma(y)^{-1}(\sigma(x)z + (x - y)_\kappa),
\]
where $(x - y)_\kappa = \left(1 \wedge \frac{\kappa}{|x - y|}\right)(x - y)$. Hence,
\[
\Psi^{-1}(z) = \sigma(x)^{-1}(\sigma(y)z - (x - y)_\kappa).
\]

5.1. Estimates related to Lévy measures. To prove Theorems 1.1 and 1.2, we need both lower bound and upper bound for
\[
J(r) = \inf_{x, y \in \mathbb{R}^d, |x - y| \leq r} \mu_\Psi(\mathbb{R}^d) = \inf_{x, y \in \mathbb{R}^d, |x - y| \leq r} \mu_{\Psi^{-1}}(\mathbb{R}^d),
\]
where $\mu_\Psi(\mathbb{R}^d) = (\nu \wedge (\nu \Psi))(\mathbb{R}^d) = (\nu \wedge (\nu \Psi^{-1}))(\mathbb{R}^d) = \mu_{\Psi^{-1}}(\mathbb{R}^d)$.

Proposition 5.1. Suppose that there are $0 < \varepsilon_1, \varepsilon_2 \leq 1$ and $c_0 > 0$ such that
\[
\nu(dz) \geq \mathbf{1}_{\{-\varepsilon_1 < z < \varepsilon_2\}} \frac{c_0}{|z|^{d+\alpha}} dz.
\]
Then, there exists a constant $c_1 > 0$ such that for all $0 < r < \kappa$ small enough,
\[
J(r) \geq c_1 r^{-\alpha}.
\]
Proof. Let \( \nu_0(dz) = q(z) \, dz \), where
\[
q(z) = 1_{\{-\varepsilon_1 < z_1 < \varepsilon_2\}} \frac{c_0}{|z|^{d+\alpha}}.
\]
Then, \( (\nu_0 \Psi^{-1})(dz) = q_{\Psi}(z) \, dz \), where
\[
q_{\Psi}(z) = 1_{\{-\varepsilon_1 < \Psi^{-1}(z) < \varepsilon_2\}} \frac{c_0 |C_{\Psi}(x, y)|}{|\Psi^{-1}(z)|^{d+\alpha}}
\]
and \( C_{\Psi}(x, y) \) is the determinant of \( \sigma(x)^{-1} \sigma(y) \), i.e., the Jacobian matrix corresponding to the transformation \( \Psi^{-1}(z) \mapsto z \). Hence,
\[
q(z) \wedge q_{\Psi}(z) \geq c_0^\lambda 1_{\{-\varepsilon_1 < z_1 < \varepsilon_2, \sigma^{-1}(x)z_1 < \varepsilon_2 \}} \left( \frac{1}{|z|^{d+\alpha}} \wedge \frac{1}{|\Psi^{-1}(z)|^{d+\alpha}} \right)
\]
where \( c_0^\lambda := c_0 (1 \wedge \inf_{x,y \in \mathbb{R}^d} |C_{\Psi}(x, y)|) > 0 \).

In the following, we consider \( x, y \in \mathbb{R}^d \) such that \( |x - y| \leq \kappa \wedge (\varepsilon_2/(4\Lambda^2)) \wedge (\varepsilon_1/(4\Lambda)) \), where \( \kappa > 0 \) is small enough such that

(i) for all \( |z| \leq \varepsilon_1 + \varepsilon_2 \) with \( -\varepsilon_1/2 < z_1 < \varepsilon_2/2 \), it holds \( -\varepsilon_1 < (\sigma(x)^{-1} \sigma(y))z_1 < \varepsilon_2 \);

(ii) for all \( |z| \leq \Lambda^2(\varepsilon_1 + \varepsilon_2) \) with \( -\varepsilon_1/2 < (\sigma(x)^{-1} \sigma(y))z_1 < 3\varepsilon_2/4 \), it holds true that \( -\varepsilon_1 < z_1 < \varepsilon_2 \).

These two properties above are ensured by the boundedness and the globally Lipschitz continuity of \( \sigma(x) \).

If \( (\sigma(x)^{-1}(x - y))_1 \leq 0 \), then
\[
q(z) \wedge q_{\Psi}(z) \geq c_0^\lambda 1_{\{-\varepsilon_1 < z_1 < \varepsilon_2, \sigma^{-1}(x)z_1 < \varepsilon_2 \}} 1_{\{-\varepsilon_1/2 < z_1 < \varepsilon_2/2, (\sigma(x)^{-1} \sigma(y))z_1 < \varepsilon_2 \}} 1_{\{\sigma(x)^{-1}(x - y)_1 < \sigma(x)^{-1} \sigma(y)_1 \}} \frac{1}{|z|^{d+\alpha}} \wedge \frac{1}{|\Psi^{-1}(z)|^{d+\alpha}}
\]
\[
\geq \frac{c_0^\lambda}{(2\Lambda^2)^{d+\alpha}} 1_{\{0 < z_1 < \varepsilon_2, |\sigma(x)^{-1}(x - y)| < |\sigma(x)^{-1} \sigma(y)_1|, |\sigma(x)^{-1}(x - y)_1| \}} \frac{1}{|z|^{d+\alpha}}
\]
Thus, denoting by \( S_+^{d-1} = \{ \theta \in \mathbb{R}^d : |\theta| = 1 \text{ and } \theta_1 > 0 \} \) the half sphere and \( \sigma(d\theta) \) the spherical measure, we have
\[
\int_{\mathbb{R}^d} q(z) \wedge q_{\Psi}(z) \, dz \geq \frac{c_0^\lambda}{(2\Lambda^2)^{d+\alpha}} \int_{\{z_1 > 0, \Lambda^3|x - y| < \varepsilon_2 \}} 1_{\{0 < z_1 < \varepsilon_2, |\sigma(x)^{-1}(x - y)| < |\sigma(x)^{-1} \sigma(y)_1|, |\sigma(x)^{-1}(x - y)_1| \}} \frac{1}{|z|^{d+\alpha}} \, dz
\]
\[
= \frac{c_0^\lambda}{(2\Lambda^2)^{d+\alpha}} \int_{\Lambda^3|x - y| < \varepsilon_2 \}} r^{d-1} \, dr \int_{S_+^{d-1}} \frac{\sigma(d\theta)}{|r \theta|^{d+\alpha}}
\]
\[
= \frac{c_0^\lambda \omega_d}{2(2\Lambda^2)^{d+\alpha}} \left( \frac{1}{(\Lambda^3|x - y|)^\alpha} - \frac{1}{(\varepsilon_2/(2\Lambda^2))^{\alpha}} \right),
\]
where \( \omega_d = \sigma(S_+^{d-1}) \) is the area of the sphere. Noticing that \( |x - y| \leq \varepsilon_2/(4\Lambda^2) \), we further get that
\[
\int_{\mathbb{R}^d} q(z) \wedge q_{\Psi}(z) \, dz \geq \frac{c_0^\lambda \omega_d}{2^{1+d+\alpha} \Lambda^{2d+5\alpha}} \left( 1 - \frac{1}{2\alpha} \right) \frac{1}{|x - y|^{\alpha}}.
\]
Next, we follow the argument above to consider the case that \((\sigma(x)^{-1}(x-y))_1 > 0\). Note that
\[
z = \sigma(y)^{-1}\sigma(x)(\Psi^{-1}(z) + \sigma(x)^{-1}(x-y)_1).
\]
In this case, it holds
\[
q(z) \wedge q\Psi(z)
\]
\[
\geq \frac{c^*_0}{(2\Lambda^2)^{d+\alpha}} \mathbb{I}_{\{0<\Psi^{-1}(z)\leq |z|<\varepsilon/2\}} \frac{1}{|\Psi^{-1}(z)|^{d+\alpha}}
\]
\[
\geq \frac{c^*_0}{(2\Lambda^2)^{d+\alpha}} \mathbb{I}_{\{0<\Psi^{-1}(z)\leq |z|<\varepsilon/2\}} \frac{1}{|\Psi^{-1}(z)|^{d+\alpha}}
\]
\[
\geq \frac{c^*_0}{(2\Lambda^2)^{d+\alpha}} \mathbb{I}_{\{0<\Psi^{-1}(z)\leq |z|<\varepsilon/2\}} \frac{1}{|\Psi^{-1}(z)|^{d+\alpha}}
\]
\[
\geq \frac{c^*_0}{(2\Lambda^2)^{d+\alpha}} \mathbb{I}_{\{0<\Psi^{-1}(z)\leq |z|<\varepsilon/2\}} \frac{1}{|\Psi^{-1}(z)|^{d+\alpha}}
\]
Hence, we arrive at
\[
\int_{\mathbb{R}^d} q(z) \wedge q\Psi(z) \, dz \geq \frac{c^*_1}{(2\Lambda^2)^{d+\alpha}} \int_{\{z_1, 0, |x-y| \leq |z| \leq \varepsilon/2\}} \frac{1}{z^{d+\alpha}} \, dz
\]
\[
= \frac{c^*_1}{(2\Lambda^2)^{d+\alpha}} \int_{|x-y|}^{\varepsilon_1/2} r^{d-1} \, d\theta \int_{S_{\varepsilon_1}} \frac{1}{r^{d+\alpha}} \, d\theta
\]
\[
= \frac{c^*_1 \omega_d}{(2\Lambda^2)^{d+\alpha}} \left( \int_{|x-y|}^{\varepsilon_1/2} \left( 1 - \frac{1}{2\varepsilon_1^{2d+\alpha}} \right) \frac{1}{|x-y|^{d+\alpha}} \right),
\]
where \(c^*_1 := c_0(1 \wedge \inf_{x,y \in \mathbb{R}^d} |C\Psi(x, y)|) / (\sup_{x,y \in \mathbb{R}^d} |C\Psi(x, y)|) > 0\). Since \(|x-y| \leq \varepsilon_1/(4\Lambda)\), we further get that
\[
\int_{\mathbb{R}^d} q(z) \wedge q\Psi(z) \, dz \geq \frac{c^*_1 \omega_d}{(2\Lambda^2)^{d+\alpha}} \left( 1 - \frac{1}{2\varepsilon_1^{2d+\alpha}} \right) \frac{1}{|x-y|^{d+\alpha}}
\]
Therefore, we conclude that for all \(0 < r \leq \kappa \wedge (\varepsilon_2/(4\Lambda^5)) \wedge (\varepsilon_1/(4\Lambda))\),
\[
J(r) \geq \inf_{x,y \in \mathbb{R}^d: |x-y| \leq r} \int_{\mathbb{R}^d} q(z) \wedge q\Psi(z) \, dz \geq c_2 r^{-\alpha},
\]
which finishes the proof. \(\Box\)

A close inspection of the proof above shows the following statement.

**Corollary 5.2.** (1) If there are \(0 < \varepsilon_1 \leq 1\) and \(c_0 > 0\) such that
\[
\nu(dz) \geq 1_{\{|z| < \varepsilon_1\}} \frac{c_0}{|z|^{d+\alpha}} \, dz,
\]
then there exists a constant \(c_1 > 0\) such that for all \(0 < r < \kappa\) small enough,
\[
J(r) \geq c_1 r^{-\alpha}.
\]

(2) Suppose that \(\sigma(x) = (\sigma_{i,j}(x))_{d \times d}\) is diagonal, i.e., \(\sigma_{i,j}(x) = 0\) for all \(x \in \mathbb{R}^d\) and \(1 \leq i \neq j \leq d\). If there are \(0 < \varepsilon \leq 1\) and \(c_0 > 0\) such that
\[
\nu(dz) \geq 1_{\{0 < |z| < \varepsilon\}} \frac{c_0}{|z|^{d+\alpha}} \, dz,
\]
then there exists a constant $c_1 > 0$ such that for all $0 < r < \kappa$ small enough,

\[ J(r) \geq c_1 r^{-\alpha}. \]

**Proof.** One can easily obtain (1) from the proof of Proposition 5.1. For (2), we note that, when $\sigma(x)$ is diagonal, $z_1 > 0$ if and only if $(\sigma(x)^{-1}\sigma(y)z)_1 > 0$ for all $x, y \in \mathbb{R}^d$. Then, we also can follow the argument of Proposition 5.1 to get the desired assertion. \qed

Next, we consider some upper bounds related to $\mu_\Psi$.

**Proposition 5.3.** (1) Let

\[ \nu(dz) \leq \frac{c_0}{|z|^{d+\alpha}} dz \]

for some $c_0 > 0$. Then, there exists a constant $c_1 > 0$ (independent of $\kappa$) such that

\[ \mu_\Psi(\mathbb{R}^d) \leq c_1(|x - y| \wedge \kappa)^{-\alpha}. \]

(2) Let

\[ \nu(dz) \leq \frac{c_0}{|z|^{d+\alpha}} \mathbb{1}_{\{|z| \leq \eta\}} dz \]

for some $\eta, c_0 > 0$. Then, there exists a constant $c_2 > 0$ (independent of $\kappa$ and $\eta$) such that

\[ \int_{\mathbb{R}^d} |z| (\mu_\Psi + \mu_{\Psi^{-1}})(dz) \leq c_2 \begin{cases} (|x - y| \wedge \kappa)^{1-\alpha} + \eta^{1-\alpha}, & \alpha \in (0, 1), \\ \log \left( \frac{\eta}{|x - y| / \kappa} \right), & \alpha = 1, \\ (|x - y| \wedge \kappa)^{1-\alpha}, & \alpha \in (1, 2). \end{cases} \]

In particular, the estimate above holds if

\[ \nu(dz) \leq \frac{c_0}{|z|^{d+\alpha}} \mathbb{1}_{\{|z| \leq \eta\}} dz \]

for some $\eta, c_0 > 0$.

**Proof.** (1) Let $\nu_0(dz) = q(z) dz$, where

\[ q(z) = \frac{c_0}{|z|^{d+\alpha}}. \]

Note that if $|z| \leq (2\Lambda^3)^{-1}(|x - y| \wedge \kappa)$, then

\[ |\Psi(z)| = |\sigma(y)^{-1}\sigma(z) + \sigma(y)^{-1}(x - y)_\kappa| \geq |\sigma(y)^{-1}(x - y)_\kappa| - |\sigma(y)^{-1}\sigma(z)| \geq (2\Lambda)^{-1}(|x - y| \wedge \kappa). \]

Thus,

\[ (\nu_0 \wedge (\nu_0 \Psi))(\mathbb{R}^d) \leq \int_{\{|z| \geq (2\Lambda^3)^{-1}(|x - y| \wedge \kappa)\}} \nu_0(dz) + \int_{\{|z| \leq (2\Lambda^3)^{-1}(|x - y| \wedge \kappa)\}} (\nu_0 \Psi)(dz) \]

\[ \leq \int_{\{|z| \geq (2\Lambda^3)^{-1}(|x - y| \wedge \kappa)\}} \frac{c_0}{|z|^{d+\alpha}} dz + \int_{\{|z| \leq (2\Lambda^3)^{-1}(|x - y| \wedge \kappa)\}} \nu_0(dz) \]

\[ \leq \int_{\{|z| \geq (2\Lambda^3)^{-1}(|x - y| \wedge \kappa)\}} \frac{c_0}{|z|^{d+\alpha}} dz + \int_{\{|z| \leq (2\Lambda^3)^{-1}(|x - y| \wedge \kappa)\}} \frac{c_0}{|z|^{d+\alpha}} dz \]

\[ \leq c_1(|x - y| \wedge \kappa)^{-\alpha}. \]
This along with the fact that
\[ \mu_\Psi = \nu \wedge (\nu \Psi) \leq \nu_0 \wedge (\nu_0 \Psi) \]
proves the assertion (1).

(2) Let \( \nu_0(dz) = q(z) \, dz \), where
\[ q(z) = \frac{c_0}{|z|^{d+\alpha}} \mathbb{1}_{\{|z| \leq \eta\}}. \]
Then,
\[
\int_{\mathbb{R}^d} |z| (\nu_0 \wedge (\nu_0 \Psi))(dz) \\
\leq \int_{\{|z| \leq (2\Lambda^3)^{-1}(|x-y| \wedge \kappa) \leq |z|\}} |z| \nu_0(dz) + \int_{\{|z| \leq (2\Lambda^3)^{-1}(|x-y| \wedge \kappa)\}} (\nu_0 \Psi)(dz) \\
\leq \int_{\{|z| \leq (2\Lambda^3)^{-1}(|x-y| \wedge \kappa)\}} |z| \nu_0(dz) + (2\Lambda^3)^{-1} (|x-y| \wedge \kappa) \int_{\{|z| \leq (2\Lambda^3)^{-1}(|x-y| \wedge \kappa)\}} (\nu_0 \Psi)(dz) \\
=: I_1 + I_2.
\]
Following the argument in (1), we know that there is a constant \( c_3 > 0 \) such that for all \( x, y \in \mathbb{R}^d \),
\[ I_2 \leq c_3 (|x-y| \wedge \kappa)^{1-\alpha}. \]
For \( I_1 \), we denote by \( \tilde{x}_{d-1} = (x_2, \ldots, x_d) \) for any \( x = (x_1, x_2, \ldots, x_d) \). Then,
\[
I_1 \leq c_4 \int_{-\eta}^{\eta} d\eta_1 \int_{\{|z_1|+|\tilde{x}_{d-1}| \geq c_5(|x-y| \wedge \kappa)\}} \frac{1}{(|z_1|+|\tilde{x}_{d-1}|)^{d+\alpha-1}} d\tilde{x}_{d-1} \\
\leq c_6 \int_{0}^{\eta} ds \int_{\{|s+r| \geq c_5(|x-y| \wedge \kappa)\}} \frac{r^{d-2}}{(s+r)^{d+\alpha-1}} dr \\
= c_6 \int_{c_5(|x-y| \wedge \kappa)}^{\eta} ds \int_{0}^{\infty} \frac{r^{d-2}}{(s+r)^{d+\alpha-1}} dr \\
+ c_6 \int_{0}^{c_5(|x-y| \wedge \kappa)} ds \int_{c_5(|x-y| \wedge \kappa)}^{\infty} \frac{r^{d-2}}{(s+r)^{d+\alpha-1}} dr \\
\leq c_7 \int_{c_5(|x-y| \wedge \kappa)}^{\eta} s^{-\alpha} ds + c_7 (|x-y| \wedge \kappa)^{1-\alpha}.
\]
From the estimate above, we can get
\[ I_1 \leq c_8 \begin{cases} 
(|x-y| \wedge \kappa)^{1-\alpha} + \eta^{1-\alpha}, & \alpha \in (0, 1), \\
\log\left(\frac{\eta}{|x-y| \wedge \kappa}\right), & \alpha = 1, \\
(|x-y| \wedge \kappa)^{1-\alpha}, & \alpha \in (1, 2).
\end{cases} \]
All the estimates hold if we reply \( \Psi \) with \( \Psi^{-1} \). Therefore, according to all inequalities above and the facts that
\[ \mu_\Psi = \nu \wedge (\nu \Psi) \leq \nu_0 \wedge (\nu_0 \Psi) \]
and
\[ \frac{c_0}{|z|^{d+\alpha}} \mathbb{1}_{\{|z| \leq \eta\}} \leq \frac{c_0}{|z|^{d+\alpha}} \mathbb{1}_{\{|z| \leq \eta\}}, \]
we can prove the second desired assertion. □

5.2. Proofs of Theorems 1.1, 1.2 and Corollary 1.3. We now are in a position to present proofs of Theorems 1.1 and 1.2.

Proof of Theorem 1.1. Define $\mu_\psi = \nu_0 \wedge (\nu_0 \Psi)$, where
\[
\nu_0(dz) = 1_{\{z < 0\}} \frac{c_0}{|z|^{d+\alpha}} dz
\]
for general case, and
\[
\nu_0(dz) = 1_{\{0 < z_1 < \eta\}} \frac{c_0}{|z|^{d+\alpha}} dz
\]
when $\sigma(x)$ is diagonal. Without loss of generality, in the following we can assume that $\eta > 0$ is small enough.

(1) Suppose that $\alpha \in (1,2)$. Then, by Corollary 5.2 and Proposition 5.3, there exist constants $c_1, c_2 > 0$ such that for all $r > 0$ small enough, $J(r) \geq c_1 r^{-\alpha}$ and $K(r) \leq c_2 r^{1-\alpha}$. Letting $\theta > 0$, we take $\psi \in C_b([0, \infty)) \cap C^3((0, \infty))$ such that
\[
\psi(r) = r(1 - \log^\theta(1/r))
\]
for all $r > 0$ small enough, which can be extended on $(0,2]$ such that $\psi' \geq 0$, $\psi'' \leq 0$ and $\psi''' \geq 0$ on $(0,2]$. With this function $\psi$ and estimates for $J(r)$ and $K(r)$ above, we have for any $c_1', c_2' > 0$ and for $r > 0$ small enough,
\[
J(r) r^2 \psi''(2r) + c_1' K(r) \psi'(r) r + c_2' \psi'(r) r^\beta \\
\leq c_3 r^{1-\alpha} (\log^\theta(1/r) + c_4 r + c_5 r^{\beta-1+\alpha}) \\
\leq -c_6 r^{1-\alpha} \log^\theta(1/r),
\]
where we used the fact that $\beta > 1 - \alpha$ in the last inequality. Hence, the first required assertion follows from Theorem 4.2.

(2) When $\alpha \in (0,1]$, we take $\psi \in C_b([0, \infty)) \cap C^3((0, \infty))$ such that $\psi(r) = r^\beta$ on $[0,2]$ with $\theta \in (0, \alpha)$. Then, following the argument above and using Theorem 4.2, we can obtain the second assertion. □

Proof of Theorem 1.2. Note that $\beta \in (0,1]$, and $\sigma(x)$ is bounded and Lipschitz continuous. Under (1.6), we can take $K_1^* \geq K_1$ and $l_0^* \geq l_0$ large enough such that
\[
(5.35) \quad \frac{\langle b(x) - b(y), x - y \rangle}{|x - y|} + A \|\sigma(x) - \sigma(y)\|_{H.S.} \leq \begin{cases} K_1^* |x - y|^\beta, & |x - y| < l_0^*, \\ K_2 |x - y|, & |x - y| \geq l_0^* \end{cases}
\]
holds for all $x, y \in \mathbb{R}^d$, where
\[
A := \int_{\{|z| > 1\}} |z| \nu(dz) + \frac{L_\sigma}{2} \int_{\{|z| \leq 1\}} |z|^2 \nu(dz).
\]

Similar to the proof of Theorem 1.1, we define $\mu_\psi = \nu_0 \wedge (\nu_0 \Psi)$, where
\[
\nu_0(dz) = 1_{\{z < 0\}} \frac{c_0}{|z|^{d+\alpha}} dz
\]
for general case, and
\[
\nu_0(dz) = 1_{\{0 < z_1 < \eta\}} \frac{c_0}{|z|^{d+\alpha}} dz
\]
when $\sigma(x)$ is diagonal. We first consider the case that $\alpha \in (0,1)$. According to Corollary 5.2 and Proposition 5.3, by choosing $\eta$ and $\kappa$ small enough, we have that for $r > 0$ small enough,

$$J(r) \geq c_1 r^{-\alpha}$$

and

$$A_1 := \sup_{x,y \in \mathbb{R}^d} \left( \Lambda \mu_\psi(\mathbb{R}^d)(|x-y| \wedge \kappa_0) + (1 + \Lambda^2/2) \int_{\mathbb{R}^d} |z| (\mu_\psi + \mu_{\psi-1})(dz) \right) \leq \frac{K_2}{2L_\sigma}.$$

Hence, it follows from (5.35) that for all $x, y \in \mathbb{R}^d$,

$$\frac{\langle b(x) - b(y), x - y \rangle}{|x - y|} + (A + A_1)\|\sigma(x) - \sigma(y)\|_{H.S.} \leq K_1^* |x - y|^\beta + K_2 |x - y|^2 - [K_1^* |x - y|^\beta + K_2 |x - y|^2] 1_{\{|x-y| \geq \eta \}}.$$

So, assumption (ii) in Theorem 4.4 holds with $\Phi_1(r) = K_1^* r^\beta + K_2 r^2/2$. It is clear that, in this setting assumptions (i) and (iii) in Theorem 4.4 are satisfied too. In particular, (iii) holds with $\sigma(r) = c_2 r^{1-\alpha}$ for some $c_2 > 0$, thanks to the assumption that $\beta > 1 - \alpha$. Therefore, the required assertion with respect to the Wasserstein distance between $\delta_x P_t$ and $\delta_y P_t$ follows from Theorem 4.4. Similarly, the assertion about the total variation between $\delta_x P_t$ and $\delta_y P_t$ is a consequence of Theorem 4.5.

When $\alpha \in [1,2)$, we choose $\varepsilon \in (1 - \beta, 1)$ and define $\mu_\psi = \nu_0 \wedge (\nu_0 \Psi)$, where

$$\nu_0(dz) = 1_{\{|z| < \eta \}} \frac{c_0}{|z|^{d+\varepsilon}} dz$$

for general case, and

$$\nu_0(dz) = 1_{\{0 < |z| < \eta \}} \frac{c_0}{|z|^{d+\varepsilon}} dz$$

when $\sigma(x)$ is diagonal. Then, following the argument above, we know that (iii) in Theorem 4.4 holds with $\sigma(r) = c_3 r^{1-\varepsilon}$ for some $c_3 > 0$, and so we can obtain the desired conclusions.

Finally, we give the

**Proof of Corollary 1.3.** It follows from (1.6) that for all $x \in \mathbb{R}^d$ with $|x|$ large enough,

$$\frac{\langle b(x), x \rangle}{|x|} \leq -K_2 |x| + \frac{\langle b(0), x \rangle}{|x|} \leq -\frac{K_2}{2} |x|.$$

Let $f \in C^3(\mathbb{R}^d)$ such that $f(x) = |x|$ for all $|x| \geq 1$. Then, by (1.5) and the assumption that $\sigma(x)$ is bounded, for any $x \in \mathbb{R}^d$,

$$L f(x) = \int \left( f(x + \sigma(x)z) - f(x) - \langle \nabla f(x), \sigma(x)z \rangle 1_{\{|z| \leq 1 \}} \right) \nu(dz)$$

$$+ \langle \nabla f(x), b(x) \rangle \leq \frac{\Lambda^2}{2} \|\nabla^2 f\|_\infty \int_{\{|z| \leq 1 \}} |z|^2 \mu(dz) + \Lambda \|\nabla f\|_\infty \int_{\{|z| \geq 1 \}} |z| \nu(dz) + \langle \nabla f(x), b(x) \rangle$$

$$\leq -c_2 |x| + c_3 \leq -c_4 f(x) + c_5,$$
where $L$ is the generator of the process $X$, and $c_i$ ($i = 2, \cdots, 5$) are positive constants. On the other hand, by Theorem 1.2, there exist constants $\lambda, c > 0$ such that for any $x, y \in \mathbb{R}^d$ and $t > 0$,

$$W_1(\delta_x P_t, \delta_y P_t) \leq ce^{-\lambda t}|x - y|,$$

which yields that (e.g., see [4, Theorem 5.10])

$$\|P_t f\|_{\text{Lip}} \leq ce^{-\lambda t}\|f\|_{\text{Lip}}$$

holds for any $t > 0$ and any Lipschitz continuous function $f$, where $\|f\|_{\text{Lip}}$ denotes the Lipschitz semi-norm with respect to the Euclidean norm $|\cdot|$. By the standard approximation, we know that the semigroup $(P_t)_{t \geq 0}$ is Feller, i.e., for every $t > 0$, $P_t$ maps $C_b(\mathbb{R}^d)$ into $C_b(\mathbb{R}^d)$. This along with the Foster–Lyapunov type condition (5.36) and [16, Theorems 4.5] yields that the process $(X_t)_{t \geq 0}$ has an invariant probability measure such that whose first moment is finite. Next, we claim that the process $X$ has a unique invariant probability measure. Indeed, let $\mu_1$ and $\mu_2$ be invariant probability measures of the process $X$ such that both of them have finite moment. Then, by Theorem 1.2,

$$\|\mu_1 - \mu_2\|_{\text{Var}} = \sup_{\|f\|_{\infty} \leq 1} |\mu_1(f) - \mu_2(f)|$$

$$\leq \sup_{\|f\|_{\infty} \leq 1} \int \int |P_t f(x) - P_t f(y)| \mu_1(dx) \mu_2(dy)$$

$$\leq \int \int \|\delta_x P_t - \delta_y P_t\|_{\text{Var}} \mu_1(dy) \mu_2(dx) \leq ce^{-\lambda t}.$$

Letting $t \to \infty$, we find that $\mu_1 = \mu_2$.

According to Theorem 1.2, we can get that for any probability measures $m_1$ and $m_2$ and any $t > 0$,

$$W_1(m_1 P_t, m_2 P_t) \leq ce^{-\lambda t}W_1(m_1, m_2),$$

e.g., see [15, Section 3]. Thus, for any $t > 0$ and $x \in \mathbb{R}^d$,

$$W_1(\delta_x P_t, \mu P_t) \leq ce^{-\lambda t}W_1(\delta_x, \mu) \leq c_1(x)e^{-\lambda t}.$$

Also by Theorem 1.2,

$$\|\delta_x P_t - \mu\|_{\text{Var}} \leq \int \|\delta_x P_t - \delta_y P_t\|_{\text{Var}} \mu(dy) \leq ce^{-\lambda t}(1 + |x - y|) \mu(dy)$$

$$\leq c_2(x)e^{-\lambda t}.$$

The proof is complete. 

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