SHARP CAPACITARY ESTIMATES FOR RINGS IN METRIC SPACES

NICOLA GAROFALO AND NIKO MAROLA

Abstract. We establish sharp estimates for the $p$-capacity of metric rings with unrelated radii in metric measure spaces equipped with a doubling measure and supporting a Poincaré inequality. These estimates play an essential role in the study of the local behavior of $p$-harmonic Green’s functions.

1. Introduction

In this paper we establish sharp capacitary estimates for the metric rings with unrelated radii in a locally doubling metric measure space supporting a local $(1, p)$-Poincaré inequality. A motivation for pursuing these estimates comes from the study of the asymptotic behavior of $p$-harmonic Green’s functions in this geometric setting. Similarly to the classical case (for the latter the reader should see [30], [36] and [37]), capacitary estimates play a crucial role in studying the local behavior of such singular functions. For this aspect we refer the reader to the forthcoming paper by Danielli and the authors [11].

Perhaps the most important model of a metric space with a rich non-Euclidean geometry is the Heisenberg group $\mathbb{H}^n$, whose underlying manifold is $\mathbb{C}^n \times \mathbb{R}$ with the group law $(z, t) \circ (z', t') = (z + z', t + t' - \frac{1}{2} Im(z\overline{z'})$. Korányi and Reimann [29] first computed explicitly the $Q$-capacity of a metric ring in $\mathbb{H}^n$. Here $Q = 2n + 2$ indicates the homogeneous dimension of $\mathbb{H}^n$ attached to the non-isotropic group dilations $\delta_\lambda(z, t) = (\lambda z, \lambda^2 t)$. Their method makes use of a suitable choice of “polar” coordinates in the group.

The Heisenberg group is the prototype of a general class of nilpotent stratified Lie groups, nowadays known as Carnot groups. In this more general context, Heinonen and Holopainen [18] proved sharp estimates for the $Q$-capacity of a ring. Again, here $Q$ indicates the homogeneous dimension attached to the non-isotropic dilations associated with the grading of the Lie algebra.

In the paper [6] Capogna, Danielli and the first named author established sharp $p$-capacitary estimates, for the range $1 < p < \infty$, for

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Carnot–Carathéodory rings associated with a system of vector fields of Hörmander type. In particular they proved that for a ring centered at a point \( x \) the \( p \)-capacity of the ring itself changes drastically depending on whether \( 1 < p < Q(x) \), \( p = Q(x) \) or \( p > Q(x) \). Here, \( Q(x) \) is the pointwise dimension at \( x \), and such number in general differs from the so-called local homogeneous dimension associated with a fixed compact set containing \( x \). This unsettling phenomenon is not present, for example, in the analysis of Carnot groups since in that case \( Q(x) \equiv Q \), where \( Q \) is the above mentioned homogeneous dimension of the group.

In [26, 27] Kinnunen and Martio developed a capacity theory based on the definition of Sobolev functions on metric spaces. They also provided sharp upper bounds for the capacity of a ball.

The results in the present paper encompass all previous ones and extend them. For the relevant geometric setting of this paper we refer the reader to Section 2.

Here, we confine ourselves to mention that a fundamental example of the spaces included in this paper is obtained by endowing a connected Riemannian manifold \( M \) with the Carathéodory metric \( d \) associated with a given subbundle of the tangent bundle, see [7]. If such subbundle generates the tangent space at every point, then thanks to the theorem of Chow [10] and Rashevsky [33] \((M, d)\) is a metric space. Such metric spaces are known as sub-Riemannian or Carnot-Carathéodory (CC) spaces. By the fundamental works of Rothschild and Stein [34], Nagel, Stein and Wainger [32], and of Jerison [23], every CC space is locally doubling, and it locally satisfies a \((p, p)\)-Poincaré inequality for any \( 1 \leq p < \infty \). Another basic example is provided by a Riemannian manifold \((M^n, g)\) with nonnegative Ricci tensor. In such case thanks to the Bishop comparison theorem the doubling condition holds globally, see e.g. [8], whereas the \((1, 1)\)-Poincaré inequality was proved by Buser [5]. An interesting example to which our results apply and that does not fall in any of the two previously mentioned categories is the space of two infinite closed cones \( X = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_1^2 + \ldots + x_{n-1}^2 \leq x_n^2\} \) equipped with the Euclidean metric of \( \mathbb{R}^n \) and with the Lebesgue measure. This space is Ahlfors regular, and it is shown in Hajlasz–Koskela [15, Example 4.2] that a \((1, p)\)-Poincaré inequality holds in \( X \) if and only if \( p > n \). Another example is obtained by gluing two copies of closed \( n \)-balls \( \{x \in \mathbb{R}^n : |x| \leq 1\} \), \( n \geq 3 \), along a line segment. In this way one obtains an Ahlfors regular space that supports a \((1, p)\)-Poincaré inequality for \( p > n - 1 \).

A thorough overview of analysis on metric spaces can be found in Heinonen [17]. One should also consult Semmes [35] and David and Semmes [12].

The present note is organized as follows. In Section 2 we list our main assumptions and gather the necessary background material. In Section 3 we establish sharp capacitary estimates for spherical rings.
with unrelated radii. Section 4 closes the paper with a small remark on the existence of $p$-harmonic functions. In the setting of metric measure spaces Holopainen and Shanmugalingam [22] constructed a $p$-harmonic Green’s function, called a singular function there, having most of the characteristics of the fundamental solution of the Laplace operator. See also Holopainen [21].

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2. Preliminaries

We begin by introducing our main assumptions on the metric space $X$ and on the measure $\mu$.

2.1. General assumptions. Throughout the paper $X = (X, d, \mu)$ is a locally compact metric space endowed with a metric $d$ and a positive Borel regular measure $\mu$ such that $0 < \mu(B(x, r)) < \infty$ for all balls $B(x, r) := \{y \in X : d(y, x) < r\}$ in $X$. We assume that for every compact set $K \subset X$ there exist constants $C_K \geq 1$, $R_K > 0$ and $\tau_K \geq 1$, such that for any $x \in K$ and every $0 < 2r \leq R_K$, one has:

(i) the closed balls $\overline{B}(x, r) = \{y \in X : d(y, x) \leq r\}$ are compact;

(ii) (local doubling condition) $\mu(B(x, 2r)) \leq C_K \mu(B(x, r))$;

(iii) (local weak $(1, p_0)$-Poincaré inequality) there exists $1 < p_0 < \infty$ such that for all $u \in N^{1, p_0}(B(x, \tau_K r))$ and all weak upper gradients $g_u$ of $u$

$$\int_{B(x, r)} |u - u_{B(x, r)}| \, d\mu \leq C_K r \left( \int_{B(x, \tau_K r)} g_u^{p_0} \, d\mu \right)^{1/p_0},$$

where $u_{B(x, r)} := \frac{1}{\mu(B(x, r))} \int_{B(x, r)} u \, d\mu := \frac{1}{\mu(B(x, r))} \int_{B(x, r)} u \, d\mu / \mu(B(x, r))$. Given an open set $\Omega \subseteq X$, and $1 < p < \infty$, the notation $N^{1, p}(\Omega)$ indicates the $p$-Newtonian space on $\Omega$ defined below.

Hereafter, the constants $C_K, R_K$ and $\tau_K$ will be referred to as the local parameters of $K$. We also say that a constant $C$ depends on the local doubling constant of $K$ if $C$ depends on $C_K$.

The above assumptions encompass, e.g., all Riemannian manifolds with $\text{Ric} \geq 0$, but they also include all Carnot–Carathéodory spaces, and therefore, in particular, all Carnot groups. For a detailed discussion of these facts we refer the reader to the paper by Garofalo–Nhieu [13]. In the case of Carnot–Carathéodory spaces, recall that if the Lie algebra generating vector fields grow at infinity faster than linearly, then the compactness of metric balls of large radii may fail in general. Consider for instance in $\mathbb{R}$ the smooth vector field of Hörmander type $X_1 = (1 + x^2) \frac{d}{dx}$. Some direct calculations prove that the distance relative
to $X_1$ is given by $d(x, y) = | \arctan(x) - \arctan(y) |$, and therefore, if $r \geq \pi/2$, we have $B(0, r) = \mathbb{R}$.

2.2. Local doubling property. We note that assumption (ii) implies that for every compact set $K \subset X$ with local parameters $C_K$ and $R_K$, for any $x \in K$ and every $0 < r \leq R_K$, one has for $1 \leq \lambda \leq R_K/r$,

$$\mu(B(x, \lambda r)) \leq C \lambda^Q \mu(B(x, r)), \quad (2.1)$$

where $Q = \log_2 C_K$, and the constant $C$ depends only on the local doubling constant $C_K$. The exponent $Q$ serves as a local dimension of the doubling measure $\mu$ restricted to the compact set $K$. In addition to such local dimension, for $x \in X$ we define the pointwise dimension $Q(x)$ by

$$Q(x) = \sup\{ q > 0 : \exists C > 0 \text{ such that } \lambda^q \mu(B(x, r)) \leq C \mu(B(x, \lambda r)), \text{ for all } \lambda \geq 1, 0 < r < \infty \}.$$

The inequality (2.1) readily implies that $Q(x) \leq Q$ for every $x \in K$. Moreover, it follows that

$$\lambda^{Q(x)} \mu(B(x, r)) \leq C \mu(B(x, \lambda r)) \quad (2.2)$$

for any $x \in K$, $0 < r \leq R_K$ and $1 \leq \lambda \leq R_K/r$, and the constant $C$ depends on the local doubling constant $C_K$. Furthermore, for all $0 < r \leq R_K$ and $x \in K$

$$C_1 r^Q \leq \frac{\mu(B(x, r))}{\mu(B(x, R_K))} \leq C_2 r^{Q(x)} \quad (2.3)$$

where $C_1 = C(K, C_K)$ and $C_2 = C(x, K, C_K)$.

For more on doubling measures, see, e.g. Heinonen [17] and the references therein.

2.3. Upper gradients. A path is a continuous mapping from a compact interval, and we say that a nonnegative Borel function $g$ on $X$ is an upper gradient of an extended real valued function $f$ on $X$ if for all rectifiable paths $\gamma$ joining points $x$ and $y$ in $X$ we have

$$|f(x) - f(y)| \leq \int_\gamma g \, ds. \quad (2.4)$$

whenever both $f(x)$ and $f(y)$ are finite, and $\int_\gamma g \, ds = \infty$ otherwise. See Cheeger [9] and Shanmugalingam [38] for a detailed discussion of upper gradients.

If $g$ is a nonnegative measurable function on $X$ and if (2.4) holds for $p$-almost every path, then $g$ is a weak upper gradient of $f$. By saying that (2.4) holds for $p$-almost every path we mean that it fails only for a path family with zero $p$-modulus (see, for example, [38]).
A function \( f : X \to \mathbb{R} \) is \emph{Lipschitz}, denoted by \( f \in \text{Lip}(X) \), if there exists a constant \( L \geq 0 \) such that \( |f(x) - f(y)| \leq Ld(x, y) \) for every \( x, y \in X \). The \emph{upper pointwise Lipschitz constant} of \( f \) at \( x \) defined by

\[
\text{Lip}_f(x) = \limsup_{r \to 0} \sup_{y \in B(x, r)} \frac{|f(y) - f(x)|}{r}
\]

is an upper gradient of \( f \). We note that for \( c \in \mathbb{R} \), \( \text{Lip}_f(x) = 0 \) for \( \mu \)-a.e. \( x \in \{ y \in X : f(y) = c \} \).

If \( f \) has an upper gradient in \( L^p(X) \), then it has a \emph{minimal weak upper gradient} \( g_f \in L^p(X) \) in the sense that for every weak upper gradient \( g \in L^p(X) \) of \( f \), \( g_f \leq g \) \( \mu \)-almost everywhere (a.e.), see Corollary 3.7 in Shanmugalingam [39], and Lemma 2.3 in J. Björn [3] for the pointwise characterization of \( g_f \).

Thanks to the results in Cheeger [9], if \( X \) satisfies assumptions (ii) and (iii), then for \( f \in \text{Lip}(X) \) one has \( g_f(x) = \text{Lip}_f(x) \) for \( \mu \)-a.e. \( x \in X \).

We recall the following version of the chain rule.

**Lemma 2.1.** Let \( u \in \text{Lip}(X) \) and \( f : \mathbb{R} \to \mathbb{R} \) be absolutely continuous and differentiable. Then

\[
g_{f \circ u}(x) \leq |(f' \circ u)(x)| \text{Lip}(u)(x)
\]

for \( \mu \)-almost every \( x \in X \).

### 2.4. Capacity.

Let \( \Omega \subset X \) be open and \( E \subset \Omega \) a Borel set. The \emph{relative \( p \)-capacity} of \( E \) with respect to \( \Omega \) is the number

\[
\text{Cap}_p(E; \Omega) = \inf \int_{\Omega} g^p d\mu,
\]

where the infimum is taken over all functions \( u \in N^{1,p}(X) \) such that \( u = 1 \) on \( E \) and \( u = 0 \) on \( X \setminus \Omega \). If such a function do not exist, we set \( \text{Cap}_p(K; \Omega) = \infty \). When \( \Omega = X \) we simply write \( \text{Cap}_p(E) \).

Observe that if \( E \subset \Omega \) is compact the infimum above could be taken over all functions \( u \in \text{Lip}_0(\Omega) = \{ f \in \text{Lip}(X) : f = 0 \text{ on } X \setminus \Omega \} \) such that \( u = 1 \) on \( E \).

Suppose that \( \Omega \subset X \) is open. A \emph{condenser} is a triple \( (E, F; \Omega) \), where \( E, F \subset \Omega \) are disjoint non-empty compact sets. For \( 1 \leq p < \infty \) the \emph{\( p \)-capacity of a condenser} is the number

\[
\text{cap}_p(E, F; \Omega) = \inf \int_{\Omega} g^p d\mu,
\]

where the infimum is taken over all \( p \)-weak upper gradients \( g \) of all functions \( u \) in \( \Omega \) such that \( u = 0 \) on \( E \), \( u = 1 \) on \( F \), and \( 0 \leq u \leq 1 \).

For other properties as well as equivalent definitions of the capacity we refer to Kilpeläinen et al. [25], Kinnunen–Martio [26, 27], and Kallunki–Shanmugalingam [24]. See also Gol’dshtein and Troyanov [14].
2.5. Newtonian spaces. We define Sobolev spaces on the metric space following Shanmugalingam [38]. Let $\Omega \subseteq X$ be nonempty and open. Whenever $u \in L^p(\Omega)$, let

$$
\|u\|_{N^{1,p}(\Omega)} = \left( \int_{\Omega} |u|^p d\mu + \inf g \int_{\Omega} g^p d\mu \right)^{1/p},
$$

where the infimum is taken over all weak upper gradients of $u$. The Newtonian space on $\Omega$ is the quotient space

$$
N^{1,p}(\Omega) = \{ u : \|u\|_{N^{1,p}(\Omega)} < \infty \}/\sim,
$$

where $u \sim v$ if and only if $\|u - v\|_{N^{1,p}(\Omega)} = 0$. The Newtonian space is a Banach space and a lattice, moreover, Lipschitz functions are dense; for the properties of Newtonian spaces we refer to [38] and Björn et al. [1].

To be able to compare the boundary values of Newtonian functions we need a Newtonian space with zero boundary values. Let $E$ be a measurable subset of $X$. The Newtonian space with zero boundary values is the space

$$
N^{1,p}_0(E) = \{ u|_E : u \in N^{1,p}(X) \text{ and } u = 0 \text{ on } X \setminus E \}.
$$

The space $N^{1,p}_0(E)$ equipped with the norm inherited from $N^{1,p}(X)$ is a Banach space, see Theorem 4.4 in Shanmugalingam [39].

We say that $u$ belongs to the local Newtonian space $N^{1,p}_{loc}(\Omega)$ if $u \in N^{1,p}(\Omega')$ for every open $\Omega' \Subset \Omega$ (or equivalently that $u \in N^{1,p}(E)$ for every measurable $E \Subset \Omega$).

3. Capacitary estimates

The aim of this section is to establish sharp capacity estimates for metric rings with unrelated radii. We emphasize an interesting feature of Theorems 3.2 and 3.4 that cannot be observed, for example, in the setting of Carnot groups. That is the dependence of the estimates on the center of the ring. This is a consequence of the fact that in this generality $Q(x_0) \neq Q$ where $x_0 \in X$, see Section 2. The results in this section will play an essential role in the subsequent developments, see the forthcoming paper by Danielli and the authors [11].

For now on, let $0 < r < \frac{1}{10} \text{diam}(X)$ and fix a ball $B(x_0, r) \subset X$. We have the following estimate.

**Lemma 3.1.** Let $u \in \text{Lip}(X)$ such that $u = 0$ in $X \setminus B(x_0, r)$. Then

$$
|u(x)| \leq C \left( r^{p_0 - 1} \int_{B(x_0, r)} \frac{(\text{Lip } u)^{p_0}(y)d(x, y)}{\mu(B(x, d(x, y)))} d\mu(y) \right)^{1/p_0},
$$

for all $x \in B(x_0, r)$.

For the proof see, e.g., Mäkeläinen [31], Theorem 3.2 and Remark 3.3.
We are ready to prove sharp capacitary estimates for metric rings with unrelated radii.

**Theorem 3.2.** (Estimates from below) Let $\Omega \subset X$ be a bounded open set, $x_0 \in \Omega$, and $Q(x_0)$ be the pointwise dimension at $x_0$. Then there exists $R_0 = R_0(\Omega) > 0$ such that for any $0 < r < R < R_0$ we have

$$\text{Cap}_{p_0}(\overline{B}(x_0, r), B(x_0, R)) \geq \begin{cases} C_1(1 - \frac{r}{R})^{p_0(p_0-1)} \mu(B(x_0, r))^{p_0-1}, & \text{if } 1 < p_0 < Q(x_0), \\ C_2(1 - \frac{r}{R})^{Q(x_0)(Q(x_0)-1)} \left( \log \frac{R}{r} \right)^{1-Q(x_0)}, & \text{if } p_0 = Q(x_0), \\ C_3(1 - \frac{r}{R})^{p_0(p_0-1)} \left( \frac{R}{r} \right)^{p_0(Q(x_0) - p_0 - 1)} - r^{-p_0(Q(x_0) - p_0 - 1)} \right)^{1-p_0}, & \text{if } p_0 > Q(x_0), \end{cases}$$

where

$$C_1 = C \left( 1 - \frac{1}{2^{Q(x_0) - p_0}} \right)^{p_0-1},$$

$$C_2 = C \frac{\mu(B(x_0, r))}{\mu(B(x_0, r))^{1-Q(x_0)}},$$

$$C_3 = C \frac{\mu(B(x_0, r))}{\mu(B(x_0, r))^{1-Q(x_0)}} \left( 2^{p_0 - Q(x_0) - p_0 - 1} - 1 \right)^{p_0-1},$$

with $C > 0$ depending only on $p_0$ and the doubling constant of $\Omega$.

**Proof.** Let $u \in \text{Lip}(X)$ such that $u = 0$ on $X \setminus B(x_0, R)$, $u = 1$ in $B(x_0, r)$, and $0 \leq u \leq 1$. Then by Lemma 3.1

$$1 = |u(x_0)| \leq C \left( \int_{B(x_0, r)} \frac{(\text{Lip } u)^{p_0}(y) d(x_0, y)}{\mu(B(x_0, d(x_0, y)))} d\mu(y) \right)^{1/p_0} \leq C \left( \int_{B(x_0, r)} \frac{(\text{Lip } u)(y) d(x_0, y)}{\mu(B(x_0, d(x_0, y)))} d\mu(y) \right)^{1/p_0} \leq C \left( \int_{B(x_0, r)} (\text{Lip } u)^{p_0}(y) d\mu(y) \right)^{1/p_0} \cdot \left( \int_{B(x_0, r)} \frac{d(x_0, y)^{p_0'}}{\mu(B(x_0, d(x_0, y)))^{p_0'}} d\mu(y) \right)^{1/p_0'},$$

where $p_0' = p_0/(p_0 - 1)$. We choose $k_0 \in \mathbb{N}$ so that $2^{k_0} r \leq R < 2^{k_0+1} r$. Then we get

$$\int_{B(x_0, r)} \frac{d(x_0, y)}{\mu(B(x_0, d(x_0, y)))} d\mu(y) \leq C \sum_{k=0}^{k_0} \int_{B(x_0, 2^{k+1} r) \setminus B(x_0, 2^k r)} \frac{d(x_0, y)^{p_0'}}{\mu(B(x_0, d(x_0, y)))^{p_0'}} d\mu(y).$$
\[ \leq C \sum_{k=0}^{k_0} \mu(B(x_0, 2^k r))^{p_0-1} \]

\[ \leq C \frac{r^{p'_0}}{\mu(B(x_0, r))^{p'_0-1}} \sum_{k=0}^{k_0} 2^{k(p'_0 - Q(x_0)(p'_0 - 1))}. \]

If \( 1 < p_0 < Q(x_0) \), then \( p'_0 - Q(x_0)(p'_0 - 1) < 0 \), and we obtain

\[ \begin{align*}
(3.2) \quad 1 & \leq C_1^{-1} \left( \frac{R}{R - r} \right)^{p_0(p_0 - 1)} \frac{r^{p_0}}{\mu(B(x_0, r))} \int_{B(x_0, R)} (\operatorname{Lip} u)^{p_0} d\mu. \\
(3.3) \quad 1 & \leq C_2^{-1} \left( \frac{R}{R - r} \right)^{Q(x_0)(p_0 - 1)} \frac{r^{Q(x_0)}}{\mu(B(x_0, r))} \int_{B(x_0, R)} (\operatorname{Lip} u)^{p_0} d\mu. \\
(3.4) \quad 1 & \leq C_3^{-1} \left( \frac{R}{R - r} \right)^{p_0(p_0 - 1)} \frac{r^{Q(x_0)}}{\mu(B(x_0, r))} \\
& \quad \cdot \left( 2R \right)^{\frac{p_0 - Q(x_0)}{p_0 - 1} - r^{\frac{p_0 - Q(x_0)}{p_0 - 1}}} \int_{B(x_0, R)} (\operatorname{Lip} u)^{p_0} d\mu.
\end{align*} \]

Finally, if \( p_0 > Q(x_0) \), then \( p'_0 - Q(x_0)(p'_0 - 1) > 0 \), and we have

\[ \begin{align*}
(3.2) \quad 1 & \leq C_1^{-1} \left( \frac{R}{R - r} \right)^{p_0(p_0 - 1)} \frac{r^{Q(x_0)}}{\mu(B(x_0, r))} \int_{B(x_0, R)} (\operatorname{Lip} u)^{p_0} d\mu. \\
(3.3) \quad 1 & \leq C_2^{-1} \left( \frac{R}{R - r} \right)^{Q(x_0)(p_0 - 1)} \frac{r^{Q(x_0)}}{\mu(B(x_0, r))} \int_{B(x_0, R)} (\operatorname{Lip} u)^{p_0} d\mu. \\
(3.4) \quad 1 & \leq C_3^{-1} \left( \frac{R}{R - r} \right)^{p_0(p_0 - 1)} \frac{r^{Q(x_0)}}{\mu(B(x_0, r))} \int_{B(x_0, R)} (\operatorname{Lip} u)^{p_0} d\mu.
\end{align*} \]

Taking the infimum over all competing \( u \)'s in (3.2)–(3.4) we reach the desired conclusion. \( \square \)

**Remark 3.3.** Observe that if \( X \) supports the weak \((1,1)\)-Poincaré inequality, i.e. \( p_0 = 1 \), these estimates reduce to the capacitary estimates, e.g., in Capogna et al. [6, Theorem 4.1].

**Theorem 3.4.** (Estimates from above) Let \( \Omega, x_0, \) and \( Q(x_0) \) be as in Theorem 3.2. Then there exists \( R_0 = R_0(\Omega) > 0 \) such that for any \( 0 < r < R < R_0 \) we have

\[ \operatorname{Cap}_{p_0}(\overline{B(x_0, r)}, B(x_0, R)) \leq \begin{cases} 
C_4 \frac{\mu(B(x_0, r))}{r^{p_0}}, & \text{if } 1 < p_0 < Q(x_0), \\
C_5 \log \frac{R}{r}, & \text{if } p_0 = Q(x_0), \\
C_6 \left( 2R \right)^{\frac{p_0 - Q(x_0)}{p_0 - 1} - r^{\frac{p_0 - Q(x_0)}{p_0 - 1}}} & \text{if } p_0 > Q(x_0),
\end{cases} \]

where \( C_4 \) is a positive constant depending only on \( p_0 \) and the local doubling constant of \( \Omega \), whereas

\[ C_5 = C \frac{\mu(B(x_0, r))}{r^{Q(x_0)}}, \]

\[ C_6 = C \frac{\mu(B(x_0, r))}{r^{Q(x_0)}}. \]
with $C > 0$ depending only on $p_0$ and the local doubling constant of $\Omega$. Finally,

$$C_0 = C \left( \frac{p_0-Q(x_0)}{p_0-1} - 1 \right)^{-1},$$

with $C > 0$ depending on $p_0$, the local parameters of $\Omega$, and $\mu(B(x_0, R_0))$.

Proof. For $i = 0, 1$ and $p \neq Q(x_0)$, we define

$$h(t) = \begin{cases} 1, & \text{if } 0 \leq t \leq r, \\ \frac{p_0-Q_i}{p_0-1} - \frac{p_0-Q_i}{r}, & \text{if } r \leq t \leq R, \\ 0, & \text{if } t \geq R, \end{cases}$$

where $Q_0 = Q$ and $Q_1 = Q(x_0)$. Note that $h \in L^\infty(\mathbb{R})$, $\text{supp}(h') \subset [r, R]$, and that $h' \in L^\infty(\mathbb{R})$, thus $h$ is a Lipschitz function. Let $u = h \circ d(x_0, y)$. By the chain rule, see Lemma 2.1, we obtain for $\mu$-a.e.

$$\begin{align*}
    g_u^{p_0} &\leq (|(h' \circ d(x_0, y))| \text{Lip } d(x_0, y))^{p_0} \\
    &= \left| \frac{p_0-Q_1}{p_0-1} \right|^{p_0} \frac{d(x_0, y)}{r} \left( \frac{(1-Q_i)p_0}{p_0-1} - \frac{p_0-Q_i}{p_0-1} \right)^{p_0}.
\end{align*}$$

Furthermore, we have that

$$\text{Cap}_{p_0}(\overline{B}(x_0, r), B(x_0, R)) \leq \int_{B(x_0, R) \setminus \overline{B}(x_0, r)} g_u^{p_0} d\mu$$

$$\leq \sum_{k=0}^{k_0} \int_{B(x_0, 2^{k+1}r) \setminus \overline{B}(x_0, 2^k r)} g_u^{p_0} d\mu$$

$$\leq C \left| \frac{p_0-Q_1}{p_0-1} - \frac{p_0-Q_i}{p_0-1} \right|^{p_0} \sum_{k=0}^{k_0} 2^{k} \mu(B(x_0, 2^k r)),$$

where $k_0 \in \mathbb{N}$ is chosen so that $2^{k_0}r \leq R < 2^{k_0+1}r$.

At this point we need to make a distinction. If $1 < p_0 < Q(x_0) \leq Q$, then we select $i = 0$, and we have by the doubling property that

$$\text{Cap}_{p_0}(\overline{B}(x_0, r), B(x_0, R)) \leq C \left| \frac{p_0-Q_1}{p_0-1} - \frac{p_0-Q_i}{p_0-1} \right|^{p_0} \mu(B(x_0, r)) r \left( \frac{1-Q_i}{p_0-1} \right) \sum_{k=0}^{k_0} 2^{k} \left( \frac{p_0-Q_1}{p_0-1} \right)^{p_0} \mu(B(x_0, 2^k r)).$$

This completes the proof in the range $1 < p_0 < Q(x_0)$.

When $p_0 > Q(x_0)$, from the second inequality in (2.3) it follows

$$\mu(B(x_0, 2^k r)) \leq C 2^{kQ(x_0)} r^{Q(x_0)},$$

for some constant $C > 0$. Therefore, we have

$$\text{Cap}_{p_0}(\overline{B}(x_0, r), B(x_0, R)) \leq C \left| \frac{p_0-Q_1}{p_0-1} - \frac{p_0-Q_i}{p_0-1} \right|^{p_0} \mu(B(x_0, r)) r \left( \frac{1-Q_i}{p_0-1} \right) \sum_{k=0}^{k_0} 2^{k} \left( \frac{p_0-Q_1}{p_0-1} \right)^{p_0} \mu(B(x_0, 2^k r)),$$
where the constant $C$ depends on $p_0$, the local doubling constant of $\Omega$, $\Omega$, and $\mu(B(x_0, R_0))$. Then we set $i = 1$, and obtain
\[
\operatorname{Cap}_{p_0}(B(x_0, r), B(x_0, R)) \\
\leq C \left| r^{\frac{p_0 - Q(x_0)}{p_0 - 1}} - R^{\frac{p_0 - Q(x_0)}{p_0 - 1}} \right|^{-p_0} \mu(B(x_0, r)) r^{Q(x_0)} \left(1 - \frac{Q(x_0)}{p_0} \right) \sum_{k = 0}^{k_0} 2^{k (p_0 - Q(x_0))} r_0^{-p_0 - 1} \\
\leq C (2^{\frac{p_0 - Q(x_0)}{p_0 - 1}} - 1)^{-1} \left| (2R)^{\frac{p_0 - Q(x_0)}{p_0 - 1}} - r^{\frac{p_0 - Q(x_0)}{p_0 - 1}} \right|^{-p_0}.
\]
This ends the proof in the range $p_0 > Q(x_0)$.

When $p_0 = Q(x_0)$ we set
\[
h(t) = \begin{cases} 
1, & \text{if } 0 \leq t \leq r, \\
\left(\log \frac{R}{r}\right)^{-1} \log \frac{R}{t}, & \text{if } r \leq t \leq R, \\
0, & \text{if } t \geq R,
\end{cases}
\]
As above, let $u = h \circ d(x_0, y)$, and Lemma 2.1 implies for $\mu$-a.e.
\[
g_{u}^{p_0} \leq \left(\log \frac{R}{r}\right)^{-p_0} \frac{1}{d(x_0, y)^{p_0}}.
\]
We have
\[
\operatorname{Cap}_{p_0}(B(x_0, r), B(x_0, R)) \leq \int_{B(x_0, R) \setminus B(x_0, r)} g_{u}^{p_0} d\mu \\
\leq \sum_{k = 0}^{k_0} \int_{B(x_0, 2^{k+1}r) \setminus B(x_0, 2^k r)} g_{u}^{p_0} d\mu \\
\leq C \left(\log \frac{R}{r}\right)^{-Q(x_0)} \sum_{k = 0}^{k_0} 2^{k (r)} r \left(1 - \frac{Q(x_0)}{p_0} \right) \mu(B(x_0, 2^k r)) \\
\leq C \mu(B(x_0, r)) \left(\log \frac{R}{r}\right)^{-Q(x_0)},
\]
where the inequality (2.1) was used and $k_0 \in \mathbb{N}$ was chosen so that $2^{k_0} r \leq R < 2^{k_0 + 1} r$. This completes the proof. $\square$

We have the following immediate corollary.

**Corollary 3.5.** If $1 < p_0 \leq Q(x_0)$, then we have
\[
\operatorname{Cap}_{p_0}(\{x_0\}, \Omega) = 0.
\]

We close this section by stating for completeness the following well-known estimate for the conformal capacity. In the setting of Carnot groups it was first proved by Heinonen in [16]. For a discussion in metric spaces, see Heinonen–Koskela [20, Theorem 3.6] and Heinonen [17, Theorem 9.19]. In [17] a weak $(1, 1)$-Poincaré inequality is assumed. By obvious modifications, however, the proof carries out in our setting as well. We hence omit the proof.
Theorem 3.6. Suppose that $E$ and $F$ are connected closed subsets of $X$ such that $F$ is unbounded and $F \cap \partial B(z, r) \neq \emptyset$, and $E$ joins $z$ to $\partial B(z, r)$. Then there is a uniform constant $C > 0$, depending only on $X$, such that
\[
\text{cap}_Q(E, F; X) \geq C > 0.
\]

4. A remark on the existence of singular functions

In this section we give a remark on the existence of singular functions or $p_0$-harmonic Green’s functions on relatively compact domains $\Omega \subset X$. The existence of singular functions in metric space setting was proved in Holopainen–Shanmugalingam [22] in $Q$-regular metric spaces (see below) supporting a local Poincaré inequality.

We start off by recalling the definition of $p_0$-harmonic function on metric spaces. Let $\Omega \subset X$ be a domain. A function $u \in N^{1, p_0}_\text{loc}(\Omega) \cap C(\Omega)$ is $p_0$-harmonic in $\Omega$ if for all relatively compact $\Omega' \subset \Omega$ and for all $v$ such that $u - v \in N^{1, p_0}_\text{loc}(\Omega')$
\[
\int_{\Omega'} g_{u}^{p_0} d\mu \leq \int_{\Omega'} g_{v}^{p_0} d\mu.
\]

It is known that nonnegative $p_0$-harmonic functions satisfy Harnack’s inequality and the strong maximum principle, there are no non-constant nonnegative $p_0$-harmonic functions on all of $X$, and $p_0$-harmonic functions have locally Hölder continuous representatives. See Kinnunen–Shanmugalingam [28] (see also [2]).

In this section we also assume that $X$ is linearly locally connected: there exists a constant $C \geq 1$ such that each point $x \in X$ has a neighborhood $U_x$ such that for every ball $B(x, r) \subset U_x$ and for every pair of points $y, z \in B(x, 2r) \setminus \overline{B}(x, r)$, there exists a path in $B(x, Cr) \setminus \overline{B}(x, r/C)$ joining the points $y$ and $z$.

The following definition was given by Holopainen and Shanmugalingam in [22].

Definition 4.1. Let $\Omega$ be a relatively compact domain in $X$ and $x_0 \in \Omega$. An extended real-valued function $G = G(\cdot, x_0)$ on $\Omega$ is said to be a singular function with singularity at $x_0$ if

1. $G$ is $p_0$-harmonic and positive in $\Omega \setminus \{x_0\}$,
2. $G|_{\Omega \setminus \{x_0\}} = 0$ and $G \in N^{1, p_0}(\Omega \setminus B(x_0, r))$ for all $r > 0$,
3. $x_0$ is a singularity, i.e.,
\[
\lim_{x \to x_0} G(x) = \text{Cap}_{p_0}(\{x_0\}, \Omega)^{1/(1 - p_0)},
\]
and $\lim_{x \to x_0} G(x) = \infty$ if $\text{Cap}_{p_0}(\{x_0\}, \Omega)^{1/(1 - p_0)} = 0$,
4. whenever $0 \leq \alpha < \beta < \sup_{x \in \Omega} G(x)$,
\[
C_1(\beta - \alpha)^{-p_0} \leq \text{Cap}_{p_0}(\Omega^\beta, \Omega_\alpha) \leq C_2(\beta - \alpha)^{-p_0},
\]
where \( \Omega^\beta = \{ x \in \Omega : G(x) \geq \beta \} \), \( \Omega_\alpha = \{ x \in \Omega : G(x) > \alpha \} \), and \( 0 < C_1, C_2 < \infty \) are constants depending only on \( p_0 \).

Note that the singular function is necessarily non-constant, and continuous on \( \Omega \setminus \{ x_0 \} \).

We have the following theorem on the existence.

**Theorem 4.2.** Let \( \Omega \) be a relatively compact domain in \( X \), \( x_0 \in \Omega \), and \( Q(x_0) \) the pointwise dimension at \( x_0 \). Then there exists a singular function on \( \Omega \) with singularity at \( x_0 \). Moreover, if \( p_0 \leq Q(x_0) \), then every singular function \( G \) with singularity at \( x_0 \) satisfies the condition
\[
\lim_{x \to x_0} G(x) = \infty.
\]

Essentially, the proof follows from the Harnack inequality on spheres and Corollary 3.5. In particular, it is in the Harnack inequality on spheres that \( X \) is needed to be linearly locally connected. See, e.g., Björn et al. [4, Lemma 5.3]. We omit the proof.

**Remark 4.3.** The theorem was first proved by Holopainen and Shanmugalingam in [22, Theorem 3.4] under the additional assumption that the measure on \( X \) is \( Q \)-regular, i.e., for all balls \( B(x, r) \) a double inequality
\[
C^{-1}r^{Q} \leq \mu(B(x, r)) \leq Cr^{Q}
\]
holds. (If \( \mu \) is \( Q \)-regular then \( X \) is called an Ahlfors regular space.) There are, however, many instances where this is not satisfied. For example, the weights modifying the Lebesgue measure in \( \mathbb{R}^n \), see [19], or systems of vector fields of Hörmander type, see e.g. Capogna et al. [6], are, in general, not \( Q \)-regular for any \( Q > 0 \). In this sense our observation seems to generalize slightly the results obtained in [22].

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