Stress concentration analysis in rock pillars in the framework of non-local elastic model with structural parameter

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Abstract. The regularities of changes in the stress-strain state of coal in rock mass depending on the main mining geological and engineering factors are identified. Using these factors, it is possible to find the optimal support parameters in specific conditions to increase the stability of preparatory mine workings. The degree of influence of mining and engineering conditions of exploitation on displacements in marginal rocks with various types of support in excavation workings is investigated.

1. Introduction
Among the variety of underground mining techniques, room-and-pillar mining technology with the use of chain (barrier) pillars and rock pillars is one of the most common. Such technologies are applicable to coal fields, salt mines, etc. Optimization of the size of rock pillars is critical for ensuring reduction of losses of mineral products, and safety of stoping operations. Besides, loss of stability of rigid barrier and relatively more compliant rock pillars can provoke uncontrolled subsidence of the earth’s surface.

It is known that a rock massif in itself is a structurally inhomogeneous medium. The inherent internal structure of a rock formation represents one of its fundamental properties characterized by dilatancy factor, internal friction coefficient, adhesion, nonlinear mechanical behavior and anisotropy. In addition, the internal structural elements (rock-blocks, joints, faults) can act as stress concentrators in the rock massif. Alternatively, rock pillars, in themselves, are stress concentrators for the surrounding massif. When modeling deformation processes in the rock pillars, it is therefore critical to use the mathematical models that allow analyzing stress state of the medium in the high-stress gradient conditions.

Numerous studies set out to determine stress-strain state (SSS) of both rock pillars and rock massif in the vicinity of the mined-out space, and to estimate optimal room and pillar dimensions were conducted using different problem formulations [1–14]. This study utilizes the mathematical model of a medium supplemented with a structural parameter [15, 16]. This allowed the model to yield a description of local bending of the elementary volumes in the medium. The model is generally attributed to the non-local type class [17–21].

2. Mathematical model
The chain pillar configuration enables the deformation processes analysis in a 2D formulation within frames of the plane strain deformation hypothesis. We use the approach presented in [15]. Let there is
a discrete set of \( n \) elastic particles chosen at the nodes of a square lattice (Figure 1). The pore space between the particles is not filled. Normally, sliding can be introduced at the contact region between particles according to constitutive laws of plasticity, dry or viscous friction [16]. We will consider elastic version of the model under the assumption that no sliding takes place at the contacts. Let’s also assume that no concentrated moments are transmitted through the contacts, either. Then, four force vectors \( \mathbf{t} \) and four displacement vectors \( \mathbf{u} \) will be defined at the points of the particle contact with its neighbors \( A_i, B_i, C_i, D_i \) for each particle (having \( i \) number) (Figure 1). One vector is the sum of two scalar components.

![Figure 1. Internal structure of the medium: schematics of the interposition of contacts between the particles.](image)

Below we consider the types of equations. For each of the \( n \) particles, three equilibrium equations are solved (two equations for the vector of forces and one for the moment of forces). Two conditions must be set at each boundary contacts (either for forces, displacements, or combination thereof). The major problem concerns the defining equations. Four displacement vectors correspond to four points \( A_i, B_i, C_i, D_i \), implying thereby eight degrees of freedom. Given that the defining equations can only include the combinations disregarding the particle (as a rigid body) translation and rotation, three degrees of freedom must therefore be excluded. Consequently, there must be five invariant combinations of displacements. For the forces, we also have eight degrees of freedom. The equilibrium conditions exclude three degrees of freedom, which means that there also remain five. Thus, the defining equations must bind together five invariant combinations of displacements with five force characteristics. Conversely, we have only three equations of this type for a classical elastic body, suggesting that there are assumptions existing in the classical theory that are equivalent to two equations, which are equally important as the equations known as Hooke’s law.

Let us select five invariant combinations of displacements that can appear in the defining equations. The number of variants for this choice is unlimited. We will consider the one, which is the closest to linear elasticity:

\[
\begin{align*}
\frac{u_1(A_i) - u_1(C_i)}{2r} &= \frac{1}{E} \left( \frac{t_{11}(A_i) + t_{11}(C_i)}{2} \right) - \nu \left( \frac{t_{22}(B_i) + t_{22}(D_i)}{2} \right), \\
\frac{u_2(B_i) - u_2(D_i)}{2r} &= \frac{1}{E} \left( \frac{t_{22}(B_i) + t_{22}(D_i)}{2} \right) - \nu \left( \frac{t_{11}(A_i) + t_{11}(C_i)}{2} \right), \\
\frac{u_2(A_i) - u_2(C_i)}{2r} + \frac{u_1(B_i) - u_1(D_i)}{2r} &= \frac{1 + \nu}{E} \left( \frac{t_{12}(A_i) + t_{12}(C_i)}{2} + \frac{t_{21}(B_i) + t_{21}(D_i)}{2} \right),
\end{align*}
\]

\( i = 1, n, (1) \)
\[ \begin{align*}
&\frac{u_1(A) + u_1(C_i) - u_1(B_i) + u_1(D_i)}{2} = \xi \left( \frac{t_{11}(A_i) - t_{11}(C_i) - t_{21}(B_i) + t_{21}(D_i)}{2r} \right), \\
&\frac{u_2(B_i) + u_2(D_i) - u_2(A_i) + u_2(C_i)}{2} = \xi \left( \frac{t_{12}(A_i) + t_{12}(C_i) + t_{22}(B_i) - t_{22}(D_i)}{2r} \right), \\
&\quad i = 1, n,
\end{align*} \]

where \( E, \nu \) are elastic Young's modulus and Poisson's ratio; \( r \) is the linear particle size (radius); \( \xi \) is the structural parameter, its dimension is \( r^2/E \). Equations (1) are a discrete analog of Hooke's law. In the classical theory, equations (2) are formulated in the implicit form. Instead, it is assumed that the postulate about diffeomorphism [22] is fulfilled, thereby suggesting that all the functions are sufficiently smooth. This means that locally (within an elementary volume) any function can be represented as linear, for example, \( u_1 = a_1 x_1 + a_2 x_2 \), \( u_2 = a_{12} x_1 + a_{22} x_2 \), where \( a_{11}, \ldots, a_{22} \) —const.

Substituting this representation in (2) necessarily entails \( \xi = 0 \). The opposite is also true. Hence, equations (2) are "hidden" within the postulate of diffeomorphism. In the case of \( \xi \neq 0 \), we obtain a linear theory of elasticity with a structural parameter, while relations (2) describe the local bending phenomenon.

The system of equations has the form
\[ t_{11}(A_i) - t_{11}(C_i) + t_{21}(B_i) - t_{21}(D_i) + X_i(0_i) = 0, \]
\[ t_{12}(A_i) - t_{12}(C_i) + t_{22}(B_i) - t_{22}(D_i) + X_2(0_i) = 0, \quad i = 1, N, \]
\[ t_{12}(A_i) + t_{12}(C_i) - t_{22}(B_i) - t_{22}(D_i) = 0, \]
here \( X_1, X_2 \) are the mass forces acting in the center of the particle. The first two equations describe the main force vector sum that equals zero, while the last one represents the net moment of forces which is zero.

The conditions of intergrain interaction (sliding is excluded) will be written as:
\[ t_{11}(C_i) = t_{11}(A_i), \quad t_{12}(C_i) = t_{12}(A_i), \quad t_{22}(D_k) = t_{22}(B_k), \quad t_{21}(D_k) = t_{21}(B_k), \]
\[ u_i(C_i) = u_i(A_i), \quad u_i(C_i) = u_i(A_i), \quad u_i(D_k) = u_i(B_k), \quad u_i(D_k) = u_i(B_k), \]

here, \( j \) is the number of particle located to the right of particle \( i \), and \( k \) is the number of particle located above particle \( i \) in the nodes of a square lattice (Figure 1).

3. Problem formulation and calculation results

Let us consider the boundary value problem formulation with respect to the rock pillar deformation taking into account the impact from the surrounding rock mass. Let's assume that at the initial moment of time the rock massif is under gravitational and tectonic loading. Supposing, there is a linear relationship between the initial stresses and weight of the overlying layers of the rock massif
\[ t_{22}^0 = \gamma (H - x_2), \quad t_{11}^0 = \lambda t_{22}^0, \quad t_{12}^0 = 0, \]
here, the superscript indicates that the stresses are related to the initial time; \( \gamma \) is specific gravity; \( \lambda \) is the lateral earth pressure coefficient; \( H \) is a distance from the free surface (the depth of stopping operations).

Now, in the initially virgin rock mass, let’s imaginatively delineate contours of the mined-out space and outline the rock pillar. The calculation domains of the pillar and a part of the surrounding rock mass are shown in Figure 2. Here \( h, d \) are the pillar’s height and width, respectively; \( a, b, c \) are the linear dimensions of the surrounding rock mass.
Figure 2. The rock pillar deformation problem statement with account the surrounding rock mass.

The initial stress state (5) satisfies the equilibrium equations. Upon completion of the stoping operations and formation of the rock pillar, its side surfaces, as well as the goaf roof and floor are interpreted as unloaded (i.e. the normal and tangential components of the stresses \( \sigma_n, \sigma_t \) become zero on the specified boundaries):

\[
\sigma_n\big|_{\Gamma_2} = 0, \quad \sigma_t\big|_{\Gamma_2} = 0,
\]
which translates to redistribution of the stresses in the rock pillar and surrounding rock mass.

We seek the problem solution as the total initial stress state (5), taking into account the overburden weight, incremented by additional stresses obtained as the problem (1) – (4) solution in the absence of self-weight stress with boundary conditions of type

\[
\Delta u_n\big|_{\Gamma_1} = 0, \quad \Delta u_t\big|_{\Gamma_1} = 0, \\
\Delta \sigma_n\big|_{\Gamma_2} = -\sigma_n^0, \quad \Delta \sigma_t\big|_{\Gamma_2} = -\sigma_t^0.
\]

The first condition (7) means that the boundary \( \Gamma_1 \) is essentially outlying from the pillar influence zone and is not subject to deformation (i.e. the normal and tangential components \( \Delta u_n, \Delta u_t \) of the displacement vector have no increments). The second condition on the pillar side surfaces in the roof and floor rocks means that here we have the normal tensile and shear stresses which are equal in absolute value to those corresponding to the initial stresses (5). After solving the problem with boundary conditions (7) and finding the increments \( \Delta t_{ij} \) in all points of the calculation domain, the final solution of the problem is presented as the sum

\[
t_{ij} = t_{ij}^0 + \Delta t_{ij}.
\]

Solution (8) ensures fulfillment of the boundary conditions (6) on the sidewall surfaces of the rock pillar, and in the goaf roof and floor rocks.

Equations (1)–(4) represent a closed algebraic system of equations. We solve it numerically using the Gauss method. Let’s select the following dimensionless problem parameters (all length values are attributed to the value \( h \), whereas all stress magnitudes — to the value \( \gamma H \))

\[
E = 2 \cdot 10^3, \quad \nu = 0.25, \quad h = d = 1, \quad a = b = c = 2, \quad r = 0.05, \quad \lambda = 0.4, \quad \gamma H = 1
\]
and perform calculations. First, let’s assume that \( \xi = 0 \), i.e. consider the case of the classical linear theory of elasticity. We will perform calculations with the resulting stresses (in accordance with (8)) shown as isolines for the values of a vertical component of the stress tensor \( t_{22} \) and for the maximum
shear stress $\tau_{\text{max}} = 0.5\sqrt{(t_{11} - t_{22})^2 + 4t_{12}^2}$ (Figure 3). Given that the medium structure is invisible here, local bending is not manifest, either. The highest concentration of stresses both of the vertical component (Figure 3a), and the maximum shear stress (Figure 3b) is observed in angular points of the rock pillar, acting as the natural stress concentrators.

![Figure 3. Isolines for stresses at $\xi = 0$: (a) stress $t_{22}$; (b) stress $\tau_{\text{max}}$.](image)

Now we will consistently increase the value of the structural parameter. This results in appreciable stresses redistribution in the rock pillar and in the zone nearest to the surrounding rock mass. As the structural parameter $\xi$ increases, the stresses $t_{22}$ and $\tau_{\text{max}}$ concentrating in the progressively increasing zones of the rock pillar and of the surrounding rock mass, propagate mostly vertically. At this, the maximum stress concentration tends to decrease. Figure 4 shows the stresses isolines if $\xi = 10^{-2}$, while Table 1 illustrates consistently decreasing $\tau_{\text{max}}$ as the structural parameter increases. Further increase in $\xi$ has no longer any appreciable impact on the maximum stress concentration in the study domain.

![Figure 4. Contour lines of stress at $\xi = 10^{-2}$: (a) stress $t_{22}$; (b) stress $\tau_{\text{max}}$.](image)

**Table 1.** A relationship between the maximum concentration $\tau_{\text{max}}$ and the structural parameter value

| $\xi$  | $10^6$ | $10^5$ | $10^4$ | $10^3$ | $10^2$ | $10^1$ |
|--------|--------|--------|--------|--------|--------|--------|
| $\tau_{\text{max}}$ | 1.84   | 1.81   | 1.76   | 1.59   | 1.35   | 1.04   | 0.95   |
4. Conclusions
The local bending phenomenon significantly changes stresses distribution in the rock mass surrounding the rock pillar, as compared to the classical elastic distribution. The increasing role of local inhomogeneity of the medium (local bends) largely prompts the expansion of the high stress concentration zone and its influence, accordingly, while the maximum concentration of stresses in the deformed medium decreases.

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