An introduction to Hybrid High-Order methods

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Outline

1. Basics of HHO methods

2. Application to the incompressible Navier–Stokes problem
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1. Basics of HHO methods

2. Application to the incompressible Navier–Stokes problem
Features

- Capability of handling general polyhedral meshes
- Construction valid for arbitrary space dimensions
- Arbitrary approximation order (including $k = 0$)
- Robustness with respect to the variations of the physical coefficients
- Reduced computational cost after static condensation

$$N_{dof,h} = \text{card}(\mathcal{F}_h) \binom{k + d - 1}{d - 1}$$
Polyhedral meshes

Figure: Admissible meshes in 2d and 3d, and HHO solution on the agglomerated 3d mesh
Model problem

- Let $\Omega \subset \mathbb{R}^d$, $d \geq 1$, denote a bounded, connected polyhedral domain.
- For $f \in L^2(\Omega)$, we consider the Poisson problem

$$
-\Delta u = f \quad \text{in } \Omega \\
u = 0 \quad \text{on } \partial \Omega
$$

- In weak form: Find $u \in H^1_0(\Omega)$ s.t.

$$
a(u, v) := (\nabla u, \nabla v) = (f, v) \quad \forall v \in H^1_0(\Omega)
$$
At the core of HHO are projectors on local polynomial spaces

With $X = T$ or $X = F$, the $L^2$-projector $\pi^0_{X} : L^1(X) \rightarrow \mathbb{P}^l(X)$ is s.t.

$$(\pi^0_{X} v - v, w)_X = 0 \text{ for all } w \in \mathbb{P}^l(X)$$

The elliptic projector $\pi^1_{T} : W^{1,1}(T) \rightarrow \mathbb{P}^l(T)$ is s.t.

$$(\nabla (\pi^1_{T} v - v), \nabla w)_T = 0 \text{ for all } w \in \mathbb{P}^l(T) \text{ and } (\pi^1_{T} v - v, 1)_T = 0$$

Both $\pi^0_{T}$ and $\pi^1_{T}$ have optimal approximation properties in $\mathbb{P}^l(T)$

See [DP and Droniou, 2017a, DP and Droniou, 2017b]
Computing $\pi_{T}^{1,k+1}$ from $L^2$-projections of degree $k$

- The following integration by parts formula is valid for all $w \in C^{\infty}(\overline{T})$:

$$
(\nabla v, \nabla w)_T = -(v, \Delta w)_T + \sum_{F \in \mathcal{F}_T} (v, \nabla w \cdot n_{TF})_F
$$

- Specializing it to $w \in \mathbb{P}^{k+1}(T)$, we can write

$$
(\nabla \pi_{T}^{1,k+1} v, \nabla w)_T = -(\pi_{T}^{0,k} v, \Delta w)_T + \sum_{F \in \mathcal{F}_T} (\pi_{F}^{0,k} v|_F, \nabla w \cdot n_{TF})_F
$$

- Moreover, it can be easily seen that

$$
(\pi_{T}^{1,k+1} v - v, 1)_T = (\pi_{T}^{1,k+1} v - \pi_{T}^{0,k} v, 1)_T = 0
$$

- Hence, $\pi_{T}^{1,k+1} v$ can be computed from $\pi_{T}^{0,k} v$ and $(\pi_{F}^{0,k} v|_F)_F \in \mathcal{F}_T$!
Let a polynomial degree $k \geq 0$ be fixed.

For all $T \in \mathcal{T}_h$, we define the local space of discrete unknowns

$$U_T^k := \{ v_T = (v_T, (v_F)_{F \in \mathcal{F}_T}) : v_T \in P^k(T) \text{ and } v_F \in P^k(F) \quad \forall F \in \mathcal{F}_T \}$$

The local interpolator $I_T^k : H^1(T) \rightarrow U_T^k$ is s.t., for all $v \in H^1(T)$,

$$I_T^k v := (\pi^0_T v, (\pi^0_F v|_F)_{F \in \mathcal{F}_T})$$
Local potential reconstruction

Let $T \in \mathcal{T}_h$. We define the local potential reconstruction operator

$$r^{k+1}_T : U^k_T \rightarrow \mathbb{P}^{k+1}(T)$$

s.t. for all $v_T \in U^k_T$, $(r^{k+1}_T v_T - v_T, 1)_T = 0$ and

$$(\nabla r^{k+1}_T v_T, \nabla w)_T = -(v_T, \nabla w)_T + \sum_{F \in \mathcal{F}_T} (v_F, \nabla w \cdot n_{TF})_F \quad \forall w \in \mathbb{P}^{k+1}(T)$$

By construction, we have

$$r^{k+1}_T \circ I^k_T = \pi^{1,k+1}_T$$

$r^{k+1}_T \circ I^k_T$ has therefore optimal approximation properties in $\mathbb{P}^{k+1}(T)$.
We would be tempted to approximate

\[ a_T(u, v) \approx (\nabla r^{k+1}_T u_T, \nabla r^{k+1}_T v_T)_T \]

This choice, however, is not stable in general. We consider instead

\[
a_T(u_T, v_T) := (\nabla r^{k+1}_T u_T, \nabla r^{k+1}_T v_T)_T + s_T(u_T, v_T)
\]

The role of \( s_T \) is to ensure \( \| \cdot \|_{1,T} \)-coercivity with

\[
\| v_T \|^2_{1,T} := \| \nabla v_T \|^2_T + \sum_{F \in \mathcal{F}_T} \frac{1}{h_F} \| v_F - v_T \|^2_F \quad \forall v_T \in U^k_T
\]
The bilinear form $s_T : \underline{U}_T^k \times \underline{U}_T^k \rightarrow \mathbb{R}$ satisfies the following properties:

- **Symmetry and positivity.** $s_T$ is symmetric and positive semidefinite.
- **Stability.** It holds, with hidden constant independent of $h$ and $T$,

  \[ a_T(v_T, v_T) \frac{1}{2} \simeq \|v_T\|_{1,T} \quad \forall v_T \in \underline{U}_T^k. \]

- **Polynomial consistency.** For all $w \in \mathbb{P}^{k+1}(T)$ and all $v_T \in \underline{U}_T^k$,

  \[ s_T(I_T^k w, v_T) = 0. \]
The following stable choice violates polynomial consistency:

\[ s_T^{\text{hdg}}(u_T, v_T) := \sum_{F \in \mathcal{F}_T} h_F^{-1}(u_F - u_T, v_F - v_T)_F \]

To circumvent this problem, we penalize the high-order differences s.t.

\[ (\delta_T^k v_T, (\delta_T F^k v_T)_F)_{F \in \mathcal{F}_T} := I_T^k r_T^{k+1} v_T - v_T \]

The classical HHO stabilization bilinear form reads

\[ s_T(u_T, v_T) := \sum_{F \in \mathcal{F}_T} h_F^{-1}((\delta_T^k - \delta_T F^k) u_T, (\delta_T^k - \delta_T F^k) v_T)_F \]
Define the global space with single-valued interface unknowns

\[ U_h^k := \left\{ v_h = ((v_T)_T \in T_h, (v_F)_F \in F_h) : 
\begin{align*}
  v_T &\in P^k(T) \quad \forall T \in T_h \text{ and } v_F \in P^k(F) \quad \forall F \in F_h
\end{align*}
\right\} \]

and its subspace with strongly enforced boundary conditions

\[ U_{h,0}^k := \left\{ v_h \in U_h^k : v_F \equiv 0 \quad \forall F \in F_h \right\} \]

The discrete problem reads: Find \( u_h \in U_{h,0}^k \) s.t.

\[ a_h(u_h, v_h) := \sum_{T \in T_h} a_T(u_T, v_T) = \sum_{T \in T_h} (f, v_T)_T \quad \forall v_h \in U_{h,0}^k \]

Well-posedness follows from coercivity and discrete Poincaré
Convergence

Theorem (Energy-norm error estimate)

Assume \( u \in H^1_0(\Omega) \cap H^{k+2}(\mathcal{T}_h) \). We have the following energy error estimate:

\[
\| \nabla_h (r_h^{k+1} u_h - u) \| + |u_h|_{s,h} \lesssim h^{k+1} |u|_{H^{k+2}(\mathcal{T}_h)},
\]

with \( (r_h^{k+1} u_h)|_T := r_T^{k+1} u_T \) for all \( T \in \mathcal{T}_h \) and \( |u_h|_{s,h}^2 := \sum_{T \in \mathcal{T}_h} s_T(u_T, u_T) \).

Theorem (Superclose \( L^2 \)-norm error estimate)

Further assuming elliptic regularity and \( f \in H^1(\mathcal{T}_h) \) if \( k = 0 \),

\[
\| r_h^{k+1} u_h - u \| \lesssim h^{k+2} \mathcal{N}_k,
\]

with \( \mathcal{N}_0 := \| f \|_{H^1(\mathcal{T}_h)} \) and \( \mathcal{N}_k := |u|_{H^{k+2}(\mathcal{T}_h)} \) for \( k \geq 1 \).
Static condensation I

- Fix a basis for $U^k_{h,0}$ with functions supported by only one $T$ or $F$
- Partition the discrete unknowns into element- and interface-based:

$$U_h = \begin{bmatrix} U_{T_h} \\ U_{F_i^h} \end{bmatrix}$$

- $U_h$ solves the following linear system:

$$\begin{bmatrix} A_{T_h T_h} & A_{T_h F_i^h} \\ A_{F_i^h T_h} & A_{F_i^h F_i^h} \end{bmatrix} \begin{bmatrix} U_{T_h} \\ U_{F_i^h} \end{bmatrix} = \begin{bmatrix} F_{T_h} \\ 0 \end{bmatrix}$$

- $A_{T_h T_h}$ is block-diagonal and SPD, hence inexpensive to invert
This remark suggests a two-step solution strategy:

- Element unknowns are eliminated solving the local balances

\[ U_{\mathcal{T}_h} = A_{\mathcal{T}_h}^{-1} \left( F_{\mathcal{T}_h} - A_{\mathcal{T}_h} F_{\mathcal{F}_h} U_{\mathcal{F}_h} \right) \]

- Face unknowns are obtained solving the global transmission problem

\[ A_{h}^{sc} U_{\mathcal{F}_h} = -A_{\mathcal{T}_h}^T A_{\mathcal{T}_h}^{-1} F_{\mathcal{T}_h} \]

with global system matrix

\[ A_{h}^{sc} := A_{\mathcal{F}_h} F_{\mathcal{F}_h} - A_{\mathcal{T}_h}^T A_{\mathcal{T}_h}^{-1} A_{\mathcal{T}_h} F_{\mathcal{T}_h} \]

\( A_{h}^{sc} \) is SPD and its stencil involves neighbours through faces
Figure: 2d test case, trigonometric solution. Energy (left) and $L^2$-norm (right) of the error vs. $h$ for uniformly refined \textit{triangular} (top) and \textit{hexagonal} (bottom) mesh families.
Numerical examples I
3d industrial test case, adaptive refinement, cost assessment

Figure: Geometry (left), numerical solution (right, top) and final adaptive mesh (right, bottom) for the comb-drive actuator test case [DP and Specogna, 2016]
Figure: Results for the comb drive benchmark.
Numerical examples III
3d industrial test case, adaptive refinement, cost assessment

Figure: Computing wall time (s) vs. number of DOFs for the comb drive benchmark, AGMG solver.
Numerical examples I
3d test case, singular solution, adaptive coarsening

**Figure**: Fichera corner benchmark, adaptive mesh coarsening [DP and Specogna, 2016]
Numerical examples II
3d test case, singular solution, adaptive coarsening

(a) Energy-error vs. $N_{\text{dofs}}$

(b) $L^2$-error vs. $N_{\text{ndof}}$

Figure: Error vs. number of DOFs for the Fichera corner benchmark, adaptively coarsened meshes
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- Capability of handling general polyhedral meshes
- Construction valid for both $d = 2$ and $d = 3$
- Arbitrary approximation order (including $k = 0$)
- Inf-sup stability on general meshes
- Robust handling of dominant advection
- Reduced computational cost after static condensation

\[ N_{\text{dof},h} = d \text{ card}(\mathcal{F}_h^i) \binom{k + d - 1}{d - 1} + \binom{k + d}{d} \]
A few references

- MHO for Stokes [Aghili, Boyaval, DP, 2015]
- Pressure-robust HHO for Stokes [DP, Ern, Linke, Schieweck, 2016]
- HHO for Navier–Stokes [DP and Krell, 2017]
- Péclet-robust HHO for Oseen [Aghili and DP, 2017]
- Darcy-robust HHO for Brinkman [Botti, DP, Droniou, 2018]
The incompressible Navier–Stokes equations

- Let \( d \in \{2, 3\} \), \( \nu \in \mathbb{R}_+^* \), \( f \in L^2(\Omega)^d \), \( U := H^1_0(\Omega)^d \), and \( P := L^2_0(\Omega) \).
- The INS problem reads: Find \((u, p) \in U \times P \) s.t.

\[
\nu a(u, v) + t(u, u, v) + b(v, p) = \int_{\Omega} f \cdot v \quad \forall v \in U, \\
-b(u, q) = 0 \quad \forall q \in P,
\]

where

\[
a(u, v) := \int_{\Omega} \nabla u : \nabla v, \quad b(v, q) := -\int_{\Omega} (\nabla \cdot v)q, \quad t(w, u, v) := \int_{\Omega} v^T \nabla u w
\]

- Here, we have used the matrix-product notation, so that

\[
\nabla u u = \sum_{j=1}^{d} u_j \partial_j u
\]
The incompressible Navier–Stokes equations II

- Integrating by parts and using $u = 0$ on $\partial \Omega$ and $\nabla \cdot w = 0$ in $\Omega$, we get

$$t(w, u, v) = \frac{1}{2} \int_{\Omega} v^T \nabla u \, w - \frac{1}{2} \int_{\Omega} u^T \nabla v \, w$$

- This shows that $t$ is non dissipative: For all $w, v \in U$ it holds

$$t(w, v, v) = 0$$
For $k \geq 0$, we define the global space of discrete unknowns

$$
\mathbf{U}_h^k := \left\{ \mathbf{v}_h = ((\mathbf{v}_T)_{T \in \mathcal{T}_h}, (\mathbf{v}_F)_{F \in \mathcal{F}_h}) : \right.

\left.
\mathbf{v}_T \in \mathbb{P}^k(T)^d \quad \forall T \in \mathcal{T}_h \text{ and } \mathbf{v}_F \in \mathbb{P}^k(F)^d \quad \forall F \in \mathcal{F}_h \right\}

The restriction to $T \in \mathcal{T}_h$ is denoted by $\mathbf{U}_T^k$, and $\mathbf{v}_T = (\mathbf{v}_T, (\mathbf{v}_F)_{F \in \mathcal{F}_T})$
The global interpolator $I_h^k : H^1(\Omega)^d \to U_h^k$ is s.t. $\forall v \in H^1(\Omega)^d$

$$I_h^k v := ((\pi_T^0 v|_T)_{T \in T_h}, (\pi_F^0 v|_F)_{F \in F_h})$$

The velocity space strongly accounting for boundary conditions is

$$U_{h,0}^k := \{v_h \in U_h^k : v_F = 0 \quad \forall F \in F_h^b\}$$

The discrete pressure space is defined setting

$$P_h^k := \left\{q_h \in P^k(T_h) \mid \int_{\Omega} q_h = 0\right\}$$
Reconstructions of differential operators

- For \( l \geq 0 \), the gradient reconstruction \( G^l_T : \mathcal{U}^k_T \to \mathbb{P}^l(T)^{d \times d} \) is s.t.
  \[
  \int_T G^l_T v_T : \tau = - \int_T v_T : (\nabla \cdot \tau) + \sum_{F \in \mathcal{F}_T} \int_F v_F : (\tau \mathbf{n}_{TF}) \quad \forall \tau \in \mathbb{P}^l(T)^{d \times d}
  \]

- The velocity reconstruction \( r^{k+1}_T : \mathcal{U}^k_T \to \mathbb{P}^{k+1}(T)^d \) is s.t.
  \[
  \int_T (\nabla r^{k+1}_T v_T - G^k_T v_T) : \nabla w = 0 \quad \forall w \in \mathbb{P}^{k+1}(T)^d, \quad \int_T r^{k+1}_T v_T - v_T = 0
  \]

- Global reconstructions are defined setting for all \( T \in \mathcal{T}_h \) and \( v_h \in \mathcal{U}^k_h \)
  \[
  (G^l_h v_h)|_T := G^l_T v_T, \quad (r^{k+1}_h v_h)|_T := r^{k+1}_T v_T, \quad D^k_h v_h := \text{tr}(G^k_h v_h)
  \]
Viscous term

- The **viscous term** is discretized by means of the bilinear form $a_h$ s.t.

$$ a_h(u_h, v_h) := \int_{\Omega} G_h^k u_h : G_h^k v_h + s_h(u_h, v_h) $$

- As in the scalar case, several possible choices for $s_h$ ensure that

$$ C_a^{-1} ||v_h||_{1,h}^2 \leq a_h(v_h, v_h) \leq C_a ||v_h||_{1,h}^2 \quad \forall v_h \in \mathbf{U}_h^k $$

with real number $C_a$ independent of $h$ and of the problem data

- **Variable viscosity** can be treated following [DP and Ern, 2015]
Pressure-velocity coupling

- The pressure-velocity coupling is realized by means of the bilinear

\[ b_h(v_h, q_h) := -\int_{\Omega} D_h v_h q_h \]

- A crucial point is that \( b_h \) satisfies the following inf-sup condition

\[
\forall q_h \in P_h^k, \quad \beta \|q_h\|_{L^2(\Omega)} \leq \sup_{v_h \in U_h^k, \|v_h\|_{1,h}=1} b_h(v_h, q_h)
\]

- This stability result is valid on general meshes and for any \( k \geq 0 \)
Recall the skew-symmetric expression of $t$:

$$t(w, u, v) = \frac{1}{2} \int_{\Omega} v^T \nabla u \ w - \frac{1}{2} \int_{\Omega} u^T \nabla v \ w$$

Inspired by this reformulation, we set

$$t_h(w_h, u_h, v_h) := \frac{1}{2} \int_{\Omega} v_h^T G_h^{2k} u_h \ w_h - \frac{1}{2} \int_{\Omega} u_h^T G_h^{2k} v_h \ w_h$$

By design, $t_h$ is non dissipative: For all $w_h, v_h$,

$$t_h(w_h, v_h, v_h) = 0$$
In practice, one does not need to actually compute $G_{h}^{2k}$.

In fact, expanding $G_{h}^{2k}$ according to its definition, we have

$$t_{h}(\underline{w}_{h}, \underline{u}_{h}, \underline{v}_{h}) = \sum_{T \in \mathcal{T}_{h}} t_{T}(\underline{w}_{T}, \underline{u}_{T}, \underline{v}_{T}),$$

where, for all $T \in \mathcal{T}_{h}$,

$$t_{T}(\underline{w}_{T}, \underline{u}_{T}, \underline{v}_{T}) := -\frac{1}{2} \int_{T} \underline{u}_{T} \nabla \underline{v}_{T} \cdot \underline{w}_{T} + \frac{1}{2} \sum_{F \in \mathcal{F}_{T}} \int_{F} (\underline{u}_{F} \cdot \underline{v}_{T})(\underline{w}_{T} \cdot \underline{n}_{TF})$$

$$+ \frac{1}{2} \int_{T} \underline{v}_{T} \nabla \underline{u}_{T} \cdot \underline{w}_{T} - \frac{1}{2} \sum_{F \in \mathcal{F}_{T}} \int_{F} (\underline{v}_{F} \cdot \underline{u}_{T})(\underline{w}_{T} \cdot \underline{n}_{TF})$$
The discrete problem reads: Find \((u_h, p_h) \in U_h^k \times P_h^k\) s.t.

\[
\begin{align*}
\nu a_h(u_h, v_h) + t_h(u_h, u_h, v_h) + b_h(v_h, p_h) &= \int_{\Omega} f \cdot v_h \quad \forall v_h \in U_h^k, \\
-b_h(u_h, q_h) &= 0 \quad \forall q_h \in P_h^k
\end{align*}
\]
Theorem (Existence and a priori bounds)

There exists a solution \((u_h, p_h) \in U_h^k \times P_h^k\) such that

\[
\|u_h\|_{1,h} \leq C_a C_s \nu^{-1} \|f\|,
\]

\[
\|p_h\| \leq C (\|f\| + \nu^{-2} \|f\|^2),
\]

with \(C_s\) discrete Poincaré constant, and \(C > 0\) independent of \(h\) and \(\nu\).

Theorem (Uniqueness of the discrete solution)

Assume that the forcing term verifies

\[
\|f\| \leq \frac{\nu^2}{2C_a^2 C_t C_s}
\]

with \(C_t\) continuity constant of \(t_h\). Then, the solution is unique.
Convergence I

**Theorem (Convergence to minimal regularity solutions)**

*It holds up to a subsequence, as $h \to 0$,*

- $u_h \to u$ strongly in $L^p(\Omega)^d$ for $p \in [1, +\infty)$ if $d = 2$, $p \in [1, 6)$ if $d = 3$;
- $G_h^k u_h \to \nabla u$ strongly in $L^2(\Omega)^{d \times d}$;
- $s_h(u_h, u_h) \to 0$;
- $p_h \to p$ strongly in $L^2(\Omega)$.

*If the exact solution is unique, the whole sequence converges.*

**Key tools:** discrete Sobolev embeddings and Rellich–Kondrachov compactness results from [DP and Droniou, 2017a]
Theorem (Convergence rates for small data)

Assume uniqueness for both \((\mathbf{u}_h, p_h)\) and \((\mathbf{u}, p)\). Assume, moreover, the additional regularity \((\mathbf{u}, p) \in H^{k+2}(\Omega)^d \times H^{k+1}(\Omega)\), as well as

\[ \|\mathbf{f}\| \leq \frac{\nu^2}{2C_I C_a C_t (1 + C_P^2)}, \]

with \(C_a\) and \(C_t\) as above, \(C_I\) boundedness constant of \(I_h^k\), and \(C_P\) continuous Poincaré constant. Then, with hidden constant independent of both \(h\) and \(\nu\),

\[ \|\mathbf{u}_h - I_h^k \mathbf{u}\|_{1,h} + \nu^{-1}\|p_h - \pi_{h}^{0,k} p\|_{L^2(\Omega)} \lesssim h^{k+1} N_{\mathbf{u},p}. \]

with \(N_{\mathbf{u},p} := (1 + \nu^{-1}\|\mathbf{u}\|_{H^2(\Omega)^d})\|\mathbf{u}\|_{H^{k+2}(\Omega)^d} + \nu^{-1}\|p\|_{H^{k+1}(\Omega)}, \)
Static condensation

- Partition the discrete velocity unknowns as before, and the pressure unknowns into **average value + oscillations** inside each element.
- At each iteration, the linear system has the form

\[
\begin{bmatrix}
A_{T_hT_h} & \tilde{B}_{T_h} & A_{T_hF^i} & \tilde{B}_{T_h} \\
A_{F^iT_h} & \tilde{B}_{F^i} & A_{F^iF^i} & \tilde{B}_{F^i} \\
\tilde{B}^T_{T_h} & 0 & \tilde{B}^T_{F^i} & 0 \\
\tilde{B}^T_{T_h} & 0 & \tilde{B}^T_{F^i} & 0 \\
\end{bmatrix}
\begin{bmatrix}
U_{T_h} \\
\tilde{P}_{T_h} \\
U_{F^i} \\
\tilde{P}_{F^i} \\
\end{bmatrix}
= 
\begin{bmatrix}
F_{T_h} \\
0 \\
0 \\
0 \\
\end{bmatrix}
\]

- Static condensation of $U_{T_h}$ and $\tilde{P}_{T_h}$ is possible.
- **Impact of static condensation on the global matrix?**
Numerical example: Kovasznay flow

- We consider the exact solution of [Kovasznay, 1948].
- Let $\Omega := (-0.5, 1.5) \times (0, 2)$ and set
  \[
  \text{Re} := (2\nu)^{-1}, \quad \lambda := \text{Re} - \left(\text{Re}^2 + 4\pi^2\right)^{\frac{1}{2}}
  \]
- The components of the velocity are given by
  \[
  u_1(x) := 1 - \exp(\lambda x_1) \cos(2\pi x_2), \quad u_2(x) := \frac{\lambda}{2\pi} \exp(\lambda x_1) \sin(2\pi x_2),
  \]
  and pressure given by
  \[
  p(x) := -\frac{1}{2} \exp(2\lambda x_1) + \frac{\lambda}{2} \left(\exp(4\lambda) - 1\right)
  \]
Numerical example: Kovasznay flow
Cartesian mesh family, $\nu = 0.1$

| $h$         | $\|u_h - \hat{u}_h\|_{1,h}$ | OCR | $\|u_h - \hat{u}_h\|$ | OCR | $\|p_h - \hat{p}_h\|$ | OCR |
|-------------|------------------------------|-----|------------------------|-----|------------------------|-----|
| $k = 0$     |                               |     |                        |     |                        |     |
| 0.13        | 1.02                         | —   | 0.33                   | —   | 1.84                   | —   |
| $6.25 \cdot 10^{-2}$ | 0.55                         | 0.89 | 0.17                   | 1.09 | 0.21                   | 3.14 |
| $3.12 \cdot 10^{-2}$ | 0.34                         | 0.68 | $4.31 \cdot 10^{-2}$   | 1.94 | $4.36 \cdot 10^{-2}$   | 2.26 |
| $1.56 \cdot 10^{-2}$ | 0.19                         | 0.86 | $1.09 \cdot 10^{-2}$   | 1.98 | $1.37 \cdot 10^{-2}$   | 1.67 |
| $7.81 \cdot 10^{-3}$ | $9.72 \cdot 10^{-2}$         | 0.96 | $2.7 \cdot 10^{-3}$    | 2.02 | $3.78 \cdot 10^{-3}$   | 1.86 |
| $k = 1$     |                               |     |                        |     |                        |     |
| 0.13        | 0.45                         | —   | 0.15                   | —   | 0.44                   | —   |
| $6.25 \cdot 10^{-2}$ | 0.24                         | 0.94 | $3.39 \cdot 10^{-2}$   | 2.11 | $3.45 \cdot 10^{-2}$   | 3.68 |
| $3.12 \cdot 10^{-2}$ | $6.46 \cdot 10^{-2}$         | 1.86 | $4.26 \cdot 10^{-3}$   | 2.99 | $8.58 \cdot 10^{-3}$   | 2   |
| $1.56 \cdot 10^{-2}$ | $1.78 \cdot 10^{-2}$         | 1.86 | $5.58 \cdot 10^{-4}$   | 2.93 | $1.23 \cdot 10^{-3}$   | 2.8 |
| $7.81 \cdot 10^{-3}$ | $4.65 \cdot 10^{-3}$         | 1.94 | $7.11 \cdot 10^{-5}$   | 2.98 | $1.87 \cdot 10^{-4}$   | 2.72 |
| $k = 2$     |                               |     |                        |     |                        |     |
| 0.13        | 0.25                         | —   | $6.41 \cdot 10^{-2}$   | —   | $9.8 \cdot 10^{-2}$    | —   |
| $6.25 \cdot 10^{-2}$ | $4.83 \cdot 10^{-2}$         | 2.34 | $5.81 \cdot 10^{-3}$   | 3.46 | $7.55 \cdot 10^{-3}$   | 3.7 |
| $3.12 \cdot 10^{-2}$ | $7.11 \cdot 10^{-3}$         | 2.76 | $3.45 \cdot 10^{-4}$   | 4.06 | $7.71 \cdot 10^{-4}$   | 3.28 |
| $1.56 \cdot 10^{-2}$ | $1.01 \cdot 10^{-3}$         | 2.82 | $2.07 \cdot 10^{-5}$   | 4.06 | $7 \cdot 10^{-5}$      | 3.46 |
| $7.81 \cdot 10^{-3}$ | $1.34 \cdot 10^{-4}$         | 2.92 | $1.25 \cdot 10^{-6}$   | 4.06 | $6.54 \cdot 10^{-6}$   | 3.43 |
| $k = 3$     |                               |     |                        |     |                        |     |
| 0.13        | $7.84 \cdot 10^{-2}$         | —   | $2.1 \cdot 10^{-2}$    | —   | $3.46 \cdot 10^{-2}$   | —   |
| $6.25 \cdot 10^{-2}$ | $7.5 \cdot 10^{-3}$          | 3.39 | $8.03 \cdot 10^{-4}$   | 4.71 | $1.39 \cdot 10^{-3}$   | 4.64 |
| $3.12 \cdot 10^{-2}$ | $5.11 \cdot 10^{-4}$         | 3.87 | $2.52 \cdot 10^{-5}$   | 4.98 | $7.31 \cdot 10^{-5}$   | 4.24 |
| $1.56 \cdot 10^{-2}$ | $3.43 \cdot 10^{-5}$         | 3.9  | $8.15 \cdot 10^{-7}$   | 4.95 | $3.87 \cdot 10^{-6}$   | 4.24 |
| $7.81 \cdot 10^{-3}$ | $2.22 \cdot 10^{-6}$         | 3.96 | $2.59 \cdot 10^{-8}$   | 4.98 | $2.17 \cdot 10^{-7}$   | 4.16 |
### Numerical example: Kovasznay flow

Hexagonal mesh family, $\nu = 0.1$

| $h$       | $\| \mathbf{u}_h - \tilde{\mathbf{u}}_h \|_{1,h}$ | OCR | $\| \mathbf{u}_h - \tilde{\mathbf{u}}_h \|$ | OCR | $\| p_h - \tilde{p}_h \|$ | OCR |
|-----------|----------------------------------|-----|-----------------|-----|-----------------|-----|
| $k = 0$   |                                  |     |                 |     |                 |     |
| 0.14      | 1.64                             | —   | 0.62            | —   | 2.1             | —   |
| $7.33 \cdot 10^{-2}$ | 0.64                         | 1.44| 0.19            | 1.81| 0.24            | 3.31|
| $3.69 \cdot 10^{-2}$ | 0.44                         | 0.56| $7.12 \cdot 10^{-2}$ | 1.42| $9.99 \cdot 10^{-2}$ | 1.28|
| $1.85 \cdot 10^{-2}$ | 0.25                         | 0.79| $2.32 \cdot 10^{-2}$ | 1.62| $3.94 \cdot 10^{-2}$ | 1.35|
| $9.27 \cdot 10^{-3}$ | 0.13                         | 0.91| $6.7 \cdot 10^{-3}$ | 1.8 | $1.32 \cdot 10^{-2}$ | 1.58|
| $k = 1$   |                                  |     |                 |     |                 |     |
| 0.14      | 0.53                             | —   | 0.22            | —   | 0.28            | —   |
| $7.33 \cdot 10^{-2}$ | 0.22                         | 1.32| $3.95 \cdot 10^{-2}$ | 2.64| $5.25 \cdot 10^{-2}$ | 2.58|
| $3.69 \cdot 10^{-2}$ | 7.26·$10^{-2}$                 | 1.63| $4.81 \cdot 10^{-2}$ | 3.07| $1.26 \cdot 10^{-2}$ | 2.08|
| $1.85 \cdot 10^{-2}$ | 1.96·$10^{-2}$                 | 1.9 | $5.81 \cdot 10^{-4}$ | 3.06| $2.37 \cdot 10^{-3}$ | 2.42|
| $9.27 \cdot 10^{-3}$ | 5.07·$10^{-3}$                 | 1.96| $6.75 \cdot 10^{-5}$ | 3.12| $4.07 \cdot 10^{-4}$ | 2.55|
| $k = 2$   |                                  |     |                 |     |                 |     |
| 0.14      | 0.28                             | —   | $7.84 \cdot 10^{-2}$ | —   | 0.11            | —   |
| $7.33 \cdot 10^{-2}$ | 5.23·$10^{-2}$                 | 2.56| $6.37 \cdot 10^{-3}$ | 3.84| $1.19 \cdot 10^{-2}$ | 3.39|
| $3.69 \cdot 10^{-2}$ | 8.32·$10^{-3}$                 | 2.68| $5.32 \cdot 10^{-4}$ | 3.62| $1.71 \cdot 10^{-3}$ | 2.84|
| $1.85 \cdot 10^{-2}$ | 1.16·$10^{-3}$                 | 2.85| $3.74 \cdot 10^{-5}$ | 3.85| $2.04 \cdot 10^{-4}$ | 3.07|
| $9.27 \cdot 10^{-3}$ | 1.52·$10^{-4}$                 | 2.94| $2.44 \cdot 10^{-6}$ | 3.95| $2.61 \cdot 10^{-5}$ | 2.98|
| $k = 3$   |                                  |     |                 |     |                 |     |
| 0.14      | $7.1 \cdot 10^{-2}$            | —   | $1.56 \cdot 10^{-2}$ | —   | $2.23 \cdot 10^{-2}$ | —   |
| $7.33 \cdot 10^{-2}$ | $9.66 \cdot 10^{-3}$        | 3.05| $1.1 \cdot 10^{-3}$ | 4.05| $2.31 \cdot 10^{-3}$ | 3.47|
| $3.69 \cdot 10^{-2}$ | $8.97 \cdot 10^{-4}$        | 3.46| $5.36 \cdot 10^{-5}$ | 4.4 | $1.7 \cdot 10^{-4}$ | 3.8 |
| $1.85 \cdot 10^{-2}$ | $6.8 \cdot 10^{-5}$         | 3.74| $2.13 \cdot 10^{-6}$ | 4.67| $1.08 \cdot 10^{-5}$ | 3.99|
| $9.27 \cdot 10^{-3}$ | $4.68 \cdot 10^{-6}$        | 3.87| $7.6 \cdot 10^{-8}$ | 4.82| $6.69 \cdot 10^{-7}$ | 4.03|
## Numerical example: Kovasznay flow

Hexagonal mesh family, HDG trilinear form, \( \nu = 0.1 \)

| \( h \)     | \( \| \mathbf{u}_h - \mathbf{\hat{u}}_h \|_{1,h} \) | OCR   | \( \| \mathbf{u}_h - \mathbf{\hat{u}}_h \| \) | OCR   | \( \| p_h - \hat{p}_h \| \) | OCR   |
|------------|---------------------------------|-------|---------------------------------|-------|----------------------------|-------|
| **\( k = 1 \)** |                                 |       |                                 |       |                           |       |
| 0.14       | 7.33 \cdot 10^{-2}               | —     | 3.99 \cdot 10^{-2}               | —     | 4.83 \cdot 10^{-2}        | —     |
| 3.69 \cdot 10^{-2} | 7.01 \cdot 10^{-2}               | 1.65  | 4.94 \cdot 10^{-3}               | 3.04  | 9.91 \cdot 10^{-3}        | 2.31  |
| 1.85 \cdot 10^{-2} | 1.94 \cdot 10^{-2}               | 1.86  | 5.87 \cdot 10^{-4}               | 3.09  | 1.94 \cdot 10^{-3}        | 2.36  |
| 9.27 \cdot 10^{-3} | 5.04 \cdot 10^{-3}               | 1.95  | 6.64 \cdot 10^{-5}               | 3.15  | 3.5 \cdot 10^{-4}        | 2.48  |
| **\( k = 2 \)** |                                 |       |                                 |       |                           |       |
| 0.14       | 7.33 \cdot 10^{-2}               | —     | 6.36 \cdot 10^{-3}               | —     | 9.52 \cdot 10^{-3}        | —     |
| 3.69 \cdot 10^{-2} | 8.38 \cdot 10^{-3}               | 2.59  | 5.52 \cdot 10^{-4}               | 3.56  | 1.38 \cdot 10^{-3}        | 2.81  |
| 1.85 \cdot 10^{-2} | 1.18 \cdot 10^{-3}               | 2.84  | 3.92 \cdot 10^{-5}               | 3.83  | 1.73 \cdot 10^{-4}        | 3.01  |
| 9.27 \cdot 10^{-3} | 1.55 \cdot 10^{-4}               | 2.94  | 2.58 \cdot 10^{-6}               | 3.94  | 2.25 \cdot 10^{-5}        | 2.95  |
| **\( k = 3 \)** |                                 |       |                                 |       |                           |       |
| 0.14       | 6.69 \cdot 10^{-2}               | —     | 1.52 \cdot 10^{-2}               | —     | 1.65 \cdot 10^{-2}        | —     |
| 7.33 \cdot 10^{-2} | 9.61 \cdot 10^{-3}               | 2.97  | 1.1 \cdot 10^{-3}                | 4.01  | 1.91 \cdot 10^{-3}        | 3.3   |
| 3.69 \cdot 10^{-2} | 9.14 \cdot 10^{-4}               | 3.43  | 5.56 \cdot 10^{-5}               | 4.35  | 1.5 \cdot 10^{-4}        | 3.71  |
| 1.85 \cdot 10^{-2} | 6.99 \cdot 10^{-5}               | 3.72  | 2.24 \cdot 10^{-6}               | 4.65  | 9.86 \cdot 10^{-6}        | 3.94  |
| 9.27 \cdot 10^{-3} | 4.83 \cdot 10^{-6}               | 3.87  | 8.01 \cdot 10^{-8}               | 4.82  | 6.17 \cdot 10^{-7}        | 4.01  |
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