Individuality as an illusion

Adonai S. Sant’Anna*

Department of Philosophy
University of South Carolina
Columbia, SC, 29208

Abstract

Elementary particles in quantum mechanics (QM) are indistinguishable when sharing the same intrinsic properties and the same quantum state. So, we can consider quantum particles as non-individuals, although non-individuality is usually considered as a consequence of the formalism of QM, since the entanglement of states forbids any labelling process. We show how to consider non-individuality as one of the basic principles of QM, instead of a logical consequence. The advantages of our framework are discussed as well. We also show that even in classical particle mechanics it is possible to consider the existence of non-individual particles. One of our main contributions is to show how to derive the apparent individuality of classical particles from the assumption that all physical objects are non-individuals.

Key words: Non-individuality, indistinguishability, quantum mechanics, classical mechanics, quasi-set theory.

1 Introduction

The issues of non-individuality in quantum physics have motivated many research projects. See, for example, the references in [2].

*Permanent address: Department of Mathematics, Federal University of Paraná, P. O. Box 019081, Curitiba, PR, 81531-990. E-mail: adonai@ufpr.br.
Elementary particles in quantum mechanics (QM) that share the same set of state-independent (intrinsic) properties are sometimes said to be indistinguishable. It is not possible, e.g., to keep track of individual particles in order to distinguish among them when they share the same physical properties. In other words, it is not possible, in principle, to label quantum particles. This non-individuality plays a very important role in quantum mechanics [12]; it is important in the derivation of quantum statistics and in the analysis of the wave-function of atoms, for example.

On the possibility that collections of such indistinguishable entities should not be considered as sets in the usual sense, Yu. Manin [7] has proposed the search for axioms which should allow to deal with indiscernible objects. As he said,

I would like to point out that it [standard set theory] is rather an extrapolation of common-place physics, where we can distinguish things, count them, put them in some order, etc. New quantum physics has shown us models of entities with quite different behavior. Even *sets* of photons in a looking-glass box, or of electrons in a nickel piece are much less Cantorian than the *sets* of grains of sand.

We are using the philosophical jargon in saying that ‘indistinguishable’ objects are objects that share their properties, while ‘identical’ objects means ‘the very same object’.

One manner to cope with the problem of non-individuality in quantum physics is by means of quasi-set theory [4, 5, 14], which is an extension of Zermelo-Fraenkel set theory, that allows us to talk about certain indistinguishable objects that are not identical. Such indistinguishable objects are termed as non-individuals. In quasi-set theory identity does not apply to all objects. There are some situations in quasi-set theory where the sequence of symbols $x = y$ is not a well-formed formula, i.e., it is meaningless. A weaker equivalence relation called “indistinguishability” is an extension of identity in the sense that it allows the existence of *two* objects that are indistinguishable. In standard mathematics, there is no sense in saying that two objects are identical. If $x = y$, then we are talking about one single object with two labels, namely, $x$ and $y$. But it is meaningful to say that *two* objects are indistinguishable in quasi-set theory.
Quasi-set theory has found some applications in quantum physics. It has been used for an authentic proof of the quantum distributions [5]. By “authentic proof” we mean a proof where elementary quantum particles are really considered as non-individuals from the formal point of view. If the physicist says that some particles are indistinguishable (in a sense) and he/she still uses standard mathematics in order to cope with these particles, then something does not seem to be sound, since standard mathematics is based on the concept of individuality, in the sense that it is grounded on the very notion of identity. It was also proved [14] that even non-individuals may present a classical distribution like Maxwell-Boltzmann’s. That is another way to say that a Maxwell-Boltzmann distribution in an ensemble of particles does not entail any ontological character concerning such particles, as it was previously advocated by Nick Huggett [3]. Besides, in [5] the authors also introduced the quasi-set-theoretical version of the wave-function of the atom of Helium, which is a well known example where indistinguishability plays an important role.

It is worth to remark that some authors like P. Pesic [10] have advocated the idea that the non-individuality of elementary quantum particles should be considered as the starting point on the foundations of quantum theory, instead of a consequence of other fundamental principles. The usual way in QM is to consider that the non-individuality of elementary particles is a consequence of the standard formalism of QM, since the entanglement of wave-functions would not allow any labeling process to identify particles.

In this paper we offer a perspective quite different from the standard approach. We take into consideration Pesic’s ideas and try a way where non-individuality is one of the basic assumptions of quantum theory.

Our mathematical framework allows us to discuss another point. We show that a similar sort of non-individuality may happen even in classical particle mechanics. We believe that this may be useful in some realistic interpretations of quantum mechanics, like Bohmian mechanics.

2 Non-individuals in quantum mechanics

Consider two white clouds in the sky, separated by ordinary space. Consider also that after some minutes these two clouds mix together. Now, instead of two clouds, we have just one cloud.
Electrons and quantum elementary particles in general behave differently. The state of one electron is like a cloud, represented by its wave-function. But when the wave-functions of two electrons mix together (get entangled) we still have two electrons, despite the fact that we cannot distinguish the particles. When two quantum states get entangled, we have only one resulting quantum state. But the issue is that this new entangled state is somehow associated to two particles.

Let us consider a very simple example from the literature, where indistinguishability plays its role.

The original Einstein-Podolsky-Rosen (EPR) Gedanken experiment deals with measurements of position and momentum in a two-particles system. Here we use a composite 1/2-spin system introduced by D. Bohm and inspired on EPR. We refer to this kind of experimental setup as Einstein-Podolsky-Rosen-Bohm (EPRB) experiment. Our discussion on this topic is essentially based on Sakurai’s textbook [12].

It is well known that the state ket of a two-electron system in a spin-singlet state can be described by:

$$\Psi = \frac{1}{\sqrt{2}}(|z^+;z^-\rangle - |z^-;z^+\rangle),$$  \hspace{1cm} (1)

where $z$ is an arbitrary quantization direction. The physical interpretation of $|z^+;z^-\rangle$ and $|z^-;z^+\rangle$ depends on the measurement process. The component $|z^+;z^-\rangle$ means that electron 1 is in the spin up state and electron 2 is in the spin down state, while $|z^-;z^+\rangle$ means that electron 1 is in the spin down state and electron 2 is in the spin up state. Sometimes it is said that the $\Psi$ state is an entanglement of two quantum states. These two entangled states correspond to the two possible configurations after the spin measurement.

We may also rewrite equation 1 as:

$$\Psi_2 = \frac{1}{\sqrt{2}}(|z^+;z^-\rangle - |z^-;z^+\rangle),$$  \hspace{1cm} (2)

where the index corresponds to the cardinality of a collection. The $\Psi_2$ state is associated to a collection whose cardinality is 2. The main point is that the elements of the two-elements collection are all quantum particles of the same kind (since they are indistinguishable). Nevertheless, after measurement, these indistinguishable micro-atoms collapse to distinguishable macro-atoms.
a and b, i.e., they are individuals that can be identified (labelled) by their quantum states, which are no longer entangled.

We cannot associate non-individuals of the two-particles collection to different quantum states, since we are talking about indistinguishable particles. But we can associate the whole two-particle collection to the quantum state $\Psi$, as suggested in equation (2). And we certainly can associate each one-particle collection to different quantum states, since the after measurement particles are now distinguishable.

But the question is: how can we associate two indistinguishable particles to one single quantum state? The so-called entangled state has no information at all concerning the number of associated particles. The entangled state is nothing but a vector. A vector in the Hilbert space has no information about the number of particles associated to it. So, how can we get the information about the number of particles? One manner to answer this question is by means of quasi-set theory.

3 Quasi-sets

This section is strongly based on other works [4, 5, 14]. We use standard logical notation for first-order theories without identity [8].

It is important to remark that, in contrast to the notions of set and quasi-set, the term “collection” has an intuitive meaning in this paper.

Quasi-set theory $Q$ is based on Zermelo-Fraenkel-like axioms and allows the presence of two sorts of atoms (Urelemente), termed $m$-atoms (micro-atoms) and $M$-atoms (macro-atoms). Concerning the $m$-atoms, a weaker ‘relation of indistinguishability’ (denoted by the symbol $\equiv$), is used instead of identity, and it is postulated that $\equiv$ has the properties of an equivalence relation. The predicate of equality cannot be applied to the $m$-atoms, since no expression of the form $x = y$ is a formula if $x$ or $y$ denote $m$-atoms. Hence, there is a precise sense in saying that $m$-atoms can be indistinguishable without being identical.

The universe of $Q$ is composed by $m$-atoms, $M$-atoms and quasi-sets. The axiomatization is adapted from that of ZFU (Zermelo-Fraenkel with Urelemente), and when we restrict the theory to the case which does not consider $m$-atoms, quasi-set theory is essentially equivalent to ZFU, and the corresponding quasi-sets can then be termed ‘sets’ (similarly, if also the $M$-
atoms are ruled out, the theory collapses into ZFC). The $M$-atoms play the same role of the Urelemente in ZFU.

In all that follows, $\exists_Q$ and $\forall_Q$ are the quantifiers relativized to quasi-sets. That is, $Q(x)$ reads as ‘$x$ is a quasi-set’.

In order to preserve the concept of identity for the ‘well-behaved’ objects, an Extensional Equality is defined for those entities which are not $m$-atoms on the following grounds: for all $x$ and $y$, if they are not $m$-atoms, then

$$x =_E y := \forall z(z \in x \iff z \in y) \lor (M(x) \land M(y) \land x \equiv y).$$

It is possible to prove that $=_E$ has all the properties of classical identity in a first order theory and so these properties hold regarding $M$-atoms and ‘sets’. This happens because one of the axioms of quasi-set theory says that the axiom of substitutivity of standard identity holds only for extensional equality. Concerning the more general relationship of indistinguishability nothing else is said. In symbols, the first axioms of $Q$ are:

- $\forall x (x \equiv x)$,
- $\forall x \forall y (x \equiv y \Rightarrow y \equiv x)$, and
- $\forall x \forall y \forall z (x \equiv y \land y \equiv z \Rightarrow x \equiv z)$.

And the fourth axiom says that

- $\forall x \forall y (x =_E y \Rightarrow (A(x, x) \Rightarrow A(x, y)))$, with the usual syntactic restrictions on the occurrences of variables in the formula $A$.

In this text, all references to ‘$=$’ (in quasi-set theory) stand for ‘$=_E$’, and similarly ‘$\leq$’ and ‘$\geq$’ stand, respectively, for ‘$\leq_E$’ and ‘$\geq_E$’. Among the specific axioms of $Q$, few of them deserve a more detailed explanation. The other axioms are adapted from ZFU.

For instance, to form certain elementary quasi-sets, such as those containing ‘two’ objects, we cannot use something like the usual ‘pair axiom’, since its standard formulation assumes identity; we use the weak relation of indistinguishability instead:
The ‘Weak-Pair’ Axiom - For all \( x \) and \( y \), there exists a quasi-set whose elements are the indistinguishable objects from either \( x \) or \( y \). In symbols,

\[
\forall x \forall y \exists z \forall t (t \in z \iff t \equiv x \lor t \equiv y).
\]

Such a quasi-set is denoted by \([x, y]\) and, when \( x \equiv y \), we have \([x]\), by definition. We remark that this quasi-set cannot be regarded as the ‘singleton’ of \( x \), since its elements are all the objects indistinguishable from \( x \), so its ‘cardinality’ (see below) may be greater than 1. A concept of strong singleton, which plays a crucial role in the applications of quasi-set theory, may be defined.

In \( \mathcal{Q} \) we also assume a Separation Schema, which intuitively says that from a quasi-set \( x \) and a formula \( \alpha(t) \), we obtain a sub-quasi-set of \( x \) denoted by

\[
[t \in x : \alpha(t)].
\]

We use the standard notation with ‘{’ and ‘}’ instead of ‘[‘ and ‘]’ only in the case where the quasi-set is a set.

It is intuitive that the concept of function cannot also be defined in the standard way, so a weaker concept of quasi-function was introduced, which maps collections of indistinguishable objects into collections of indistinguishable objects; when there are no \( m \)-atoms involved, the concept is reduced to that of function as usually understood. Relations (or quasi-relations), however, can be defined in the usual way, although no order relation can be defined on a quasi-set of indistinguishable \( m \)-atoms, since partial and total orders require antisymmetry, which cannot be stated without identity. Asymmetry also cannot be supposed, for if \( x \equiv y \), then for every relation \( R \) such that \( \langle x, y \rangle \in R \), it follows that \( \langle x, y \rangle =_E [[x]] =_E \langle y, x \rangle \in R \), by force of the axioms of \( \mathcal{Q} \).

It is possible to define a translation from the language of ZFU into the language of \( \mathcal{Q} \) in such a way that we can obtain a ‘copy’ of ZFU in \( \mathcal{Q} \). In this copy, all the usual mathematical concepts (like those of cardinal, ordinal, etc.) can be defined; the ‘sets’ (actually, the ‘\( \mathcal{Q} \)-sets’ which are ‘copies’ of the ZFU-sets) turn out to be those quasi-sets whose transitive closure (this concept is like the usual one) does not contain \( m \)-atoms.
Although some authors like Weyl [?] sustain that (concerning cardinals and ordinals) “the concept of ordinal is the primary one”, quantum mechanics seems to present strong arguments for questioning this thesis, and the idea of presenting collections which have a cardinal but not an ordinal is one of the most basic and important assumptions of quasi-set theory.

The concept of quasi-cardinal is taken as primitive in $\mathcal{Q}$, subject to certain axioms that permit us to operate with quasi-cardinals in a similar way to that of cardinals in standard set theories. Among the axioms for quasi-cardinality, we mention those below, but first we recall that in $\mathcal{Q}$, $qc(x)$ stands for the ‘quasi-cardinal’ of the quasi-set $x$, while $Z(x)$ says that $x$ is a set (in $\mathcal{Q}$). Furthermore, $Cd(x)$ and $card(x)$ mean ‘$x$ is a cardinal’ and ‘the cardinal of $x$’, respectively, defined as usual in the ‘copy’ of ZFU.

Quasi-cardinality - Every quasi-set has an unique quasi-cardinal which is a cardinal (as defined in the ‘ZFU-part’ of the theory) and, if the quasi-set is in particular a set, then this quasi-cardinal is its cardinal stricto sensu:

$$\forall x \exists ! y (Cd(y) \land y = E \ qc(x) \land (Z(x) \Rightarrow y = E \ card(x))).$$

From the fact that $\emptyset$ is a set, it follows that its quasi-cardinality is 0 (zero).

$\mathcal{Q}$ still encompasses an axiom which says that if the quasi-cardinal of a quasi-set $x$ is $\alpha$, then for every quasi-cardinal $\beta \leq \alpha$, there is a sub-quasi-set of $x$ whose quasi-cardinal is $\beta$, where the concept of sub-quasi-set is like the usual one. In symbols,

The quasi-cardinals of sub-quasi-sets -

$$\forall x (qc(x) = E \alpha \Rightarrow \forall \beta (\beta \leq E \alpha \Rightarrow \exists y (y \subseteq x \land qc(y) = E \beta))).$$

Another axiom states that

The quasi-cardinal of the power quasi-set -

$$\forall x (qc(P(x)) = E \ 2^{qc(x)}).$$

where $2^{qc(x)}$ has its usual meaning.
These last axioms allow us to talk about the quantity of elements of a quasi-set, although we cannot count its elements in many situations.

As remarked above, in \( \mathcal{Q} \) there may exist quasi-sets whose elements are \( m \)-atoms only, called ‘pure’ quasi-sets. Furthermore, it may be the case that the \( m \)-atoms of a pure quasi-set \( x \) are indistinguishable from one another. In this case, the axiomatization provides the grounds for saying that nothing in the theory can distinguish among the elements of \( x \). But, in this case, one could ask what it is that sustains the idea that there is more than one entity in \( x \). The answer is obtained through the above mentioned axioms (among others, of course). Since the quasi-cardinal of the power quasi-set of \( x \) has quasi-cardinal \( 2^{qc(x)} \), then if \( qc(x) = \alpha \), for every quasi-cardinal \( \beta \leq \alpha \) there exists a sub-quasi-set \( y \subseteq x \) such that \( qc(y) = \beta \), according to the axiom about the quasi-cardinality of the sub-quasi-sets. Thus, if \( qc(x) = \alpha \neq 0 \), the axiomatization does not forbid the existence of \( \alpha \) sub-quasi-sets of \( x \) which can be regarded as ‘singletons’.

Of course the theory cannot prove that these ‘unitary’ sub-quasi-sets (supposing now that \( qc(x) \geq 2 \)) are distinct, since we have no way of ‘identifying’ their elements, but quasi-set theory is compatible with this idea. In other words, it is consistent with \( \mathcal{Q} \) to advocate that \( x \) has \( \alpha \) elements, which may be regarded as absolutely indistinguishable objects. Since the elements of \( x \) may share the relation \( \equiv \), they may be further understood as belonging to the same ‘equivalence class’ but in such a way that we cannot assert either that they are identical or that they are distinct from one another.

The collections \( x \) and \( y \) are defined as similar quasi-sets (in symbols, \( \text{Sim}(x, y) \)) if the elements of one of them are indistinguishable from the elements of the other one, that is, \( \text{Sim}(x, y) \) if and only if \( \forall z \forall t (z \in x \land t \in y \Rightarrow z \equiv t) \). Furthermore, \( x \) and \( y \) are \( Q \)-Similar (\( Q\text{Sim}(x, y) \)) if and only if they are similar and have the same quasi-cardinality. Then, since the quotient quasi-set \( x/\equiv \) may be regarded as a collection of equivalence classes of indistinguishable objects, then the ‘weak’ axiom of extensionality is:

\[
\forall Qx \forall Qy (\forall z (z \in x/\equiv \Rightarrow \exists t (t \in y/\equiv \land Q\text{Sim}(z, t))) \land \forall t (t \in y/\equiv \Rightarrow \exists z (z \in x/\equiv \land Q\text{Sim}(t, z)))) \Rightarrow x \equiv y
\]

In other words, this axiom says that those quasi-sets that have the same
quantity of elements of the same sort (in the sense that they belong to the same equivalence class of indistinguishable objects) are indistinguishable.

**Definition 1** A strong singleton of $x$ is a quasi-set $x'$ which satisfies the following property:

$$x' \subseteq [x] \land qc(x') =_E 1$$

**Definition 2** A $n$-singleton of $x$ is a quasi-set $[x]_n$ which satisfies the following property:

$$[x]_n \subseteq [x] \land qc([x]_n) =_E n$$

### 4 Standard quantum mechanics

In this section we introduce an axiomatic framework for standard quantum mechanics. We have eight primitive concepts, namely, $P$, $H$, $O$, $I_P$, $M$, $T$, $f$, and $Pr$. All of them are sets (ZF set theory). $P$ is a set of particles, $T$ is a time interval, $H$ is a Hilbert space, $O$ is a set of Hermitean operators defined on $H$ and corresponding to observables, $I_P$ is a set of intrinsic properties, like rest mass, absolute value of spin, electric charge, etc., $M$ is a set of functions termed the “measurement functions”, $f$ is a function which associates to each particle at each instant of time an intrinsic property and a quantum state, and $Pr$ is a function that has a restriction which is a probability function.

**Definition 3** $QP = \langle P, T, H, O, I_P, M, f, Pr \rangle$ is a quantum system if and only if the following axioms are satisfied:

**QP1** $P$ is a non-empty and finite set.

**QP2** $T$ is a non-degenerate interval of real numbers.

**QP3** $H$ is a Hilbert space with a norm induced by its inner product. All vectors of $H$ are complex functions whose domain is $\mathbb{R}^3 \times T$ (space and time).

**QP4** $O$ is a set of hermitean operators on $H$, such that the eigenvectors of each operator from $O$ form a basis of $H$.

**QP5** $I_P$ is a set of ordered $n$-tuples of real numbers.
QP6 $M$ is a class of functions $M_{O_i} : H \to H$, where each $O_i \in O$.

QP7 $f : P \times T \to I_P \times H$ is a function.

QP8 $Pr : H \times H \to \mathbb{R}$ is a real valued function, such that $Pr(u, v) = |\langle u | v \rangle|^2$, where $\langle u | v \rangle$ is the inner product between $u$ and $v$.

Definition 4 For each operator $O_i \in O$ there is a set $S_i$ of normalized eigenvectors $s^i_r$ of $O_i$, where each eigenvector $s^i_r$ is associated to an eigenvalue $\lambda_i$, and the possible values of $r$ depend on the dimension of $H$.

QP9 If $H$ is spanned by a base $S_i$ of normalized eigenvectors of an hermitean operator $O_i \in O$, then for every vector $u \in H$, $M_{O_i}(u) \in S_i$.

QP10 Each function of $M$ is a random function, in the sense that every function of $M$ is associated to a probability function; if $v \in S_i$, then the probability that $M_{O_i}(u) = v$ is given by $Pr(u, v)$.

QP11 Every vector of $H$ obeys the Schrödinger equation.

Axiom QP1 says that we are dealing with finite systems. Axiom QP2 says that time flows on a continuum interval. QP3 and QP4 are part of the standard mathematical background of QM. QP5 says that intrinsic properties are given by real numbers. QP6 is the first axiom of this axiomatic framework concerning measurements in QM. QP7 is a very strategic axiom, since it relates intrinsic properties to quantum state properties (given by the vectors of $H$) by means of the concept of particle. In other words, the notion of particle has the role of connecting quantum states to intrinsic properties. Since quantum particles can share the same intrinsic properties and the same quantum state, this is a very easy solution to the problem of representing ensembles of multiple indistinguishable quantum particles. Within this context, particles may be physically indistinguishable (by means of their physical properties), although they are individuals in the sense of belonging to a “Cantorian” set, namely, the set $P$. If we try to describe a particle by means of its intrinsic properties and quantum states only, then indistinguishability entails identity, which forbids us to talk about collections of multiple indistinguishable quantum particles. The challenge in the next section is to consider indistinguishability on a new level of formalism, namely, on a level where indistinguishability is considered at the formalism itself.
QP8 describes a function that in some cases corresponds to a probability function, as described by axiom QP10. QP9 and QP10 are the remaining axioms describing the measurement process. The last axiom is a standard assumption which describes the time evolution of undisturbed systems.

It is a theorem that \( Pr(u, M_{O_i}(u)) \) is a real number between 0 and 1 for all \( O_i \in O \). One interesting exercise would be the description of the \( \sigma \)-algebra associated to an appropriate restriction of \( Pr \). But that is not a task for this paper.

5 Non-individuals in quantum mechanics

In this section we introduce an alternative axiomatic framework for QM, inspired on the idea of considering quantum particles truly indistinguishable even on the formal language of the axiomatic framework. The first obvious advantage of this is that we have a mathematical framework more faithful to the usual interpretation of physical phenomena. Hence, we can mathematically justify quantum distributions and other physical effects where indistinguishability plays its role. Another epistemological advantage is that a quasi-set-theoretical approach to quantum mechanics can justify the expression “indistinguishable particles” without the need for an abstract concept like \( P \), whose physical interpretation is quite difficult. In other words, if we already have all physical characteristics of a particle, given by the elements of \( I_P \) and \( H \), what is the physical meaning of \( P \) in the previous axiomatic framework? We believe that our quasi-set-theoretical solution to the problem of non-individuality in quantum mechanics is more elegant from the point of view of the foundations of physics.

We have eight primitive concepts, namely, \( [x]_n \), \( T \), \( H \), \( O \), \( I_P \), \( M \), \( P \), and \( Pr \). \( [x]_n \) is a \( n \)-singleton, \( T \) is a time interval, \( H \) is a Hilbert space, \( O \) is a set of Hermitian operators defined on \( H \) and corresponding to observables, \( I_P \) is a set of intrinsic properties, \( M \) is a set of functions termed the “measurement functions”, and \( Pr \) is a function that has a restriction which is a probability function.

**Definition 5** \( QP = ([x]_n, T, H, O, I_P, M, P, Pr) \) is a quasi-quantum system if and only if the next axioms are satisfied:
[\mathcal{x}]_n is a non-empty and finite n-singleton whose elements are micro-atoms (we are using quasi-set-theoretical terminology).

\( T \) is a non-degenerate interval of real numbers.

\( H \) is a Hilbert space with a norm induced by its inner product. All vectors of \( H \) are complex functions whose domain is \( \mathbb{R}^3 \times T \).

\( O \) is a set of hermitean operators on \( H \).

\( I_P \) is a set of ordered \( n \)-tuples of real numbers.

\( M \) is a class of functions \( M_{O_i} : H \to H \), where each \( O_i \in O \).

\( P \) is a sub-quasi-set of \( [\mathcal{x}]_n \times I_P \times H \) whose quasi-cardinality is \( n \).

\( \Pr : H \times H \to \mathbb{R} \) is a real valued function, such that \( \Pr(u, v) = |\langle u|v \rangle|^2 \), where \( \langle u|v \rangle \) is the inner product between \( u \) and \( v \).

**Definition 6** For each operator \( O_i \in O \) there is a set \( S_i \) of normalized eigenvectors \( s^i_r \) of \( O_i \), where each eigenvector is associated to an eigenvalue \( \lambda_i \). The range of values of \( r \) depends on the dimension of \( H \).

\( \text{If } H \text{ is spanned by a set } S_i \text{ of normalized eigenvectors of an Hermitian operator } O_i \in O, \text{ then for every vector } u \in H, \text{ } M_{O_i}(u) \in S_i. \)

\( \text{Each } M_{O_i} \text{ of } M \text{ is a random function; if } v \in S_i, \text{ then for all } u \in H, \text{ the probability that } M_{O_i}(u) = v \text{ is given by } \Pr(u, v). \)

\( \text{Every vector } u \text{ of } H \text{ obeys the Schrödinger equation.} \)

The main difference between this system and the previous one is on the concept of quantum particle. In the quantum system a particle is an abstract object that belongs to a set \( P \). So, the formal language has all the ways to individualize any particle. In the present system a particle is an ordered triple, where the first element is an object devoid of individuality (a micro-atom), the second element is an intrinsic property, and the third element is a quantum state. The physical meaning of the first element of the ordered triple is stated in section 8.
In the meantime, we emphasize that our approach makes possible to say that a quantum state is associated to an ensemble of indistinguishable particles. This is possible due to the concept of quasi-cardinality (see section 3). In other words, in quasi-set theory it is possible the existence of a $N$-singleton

$$X = E \langle \langle x, i_p, u \rangle \rangle_N,$$

where $x$ is a micro-atom, $i_p$ is an ordered $m$-tuple of intrinsic properties, $u$ is the quantum state associated to all particles of $X$, and $N$ is the number of particles of $X$.

So, recalling equation (2),

$$\Psi = \frac{1}{\sqrt{2}}(|z+; z-\rangle - |z-; z+\rangle),$$

we can interpret the index 2 as the quasi-cardinality of an ensemble of quantum particles that share the quantum state $\Psi$.

Now, we need to illustrate some ideas concerning the elimination of individuality even in classical particle mechanics. But first, let us recall what do we mean by classical particle mechanics.

### 6 Classical particle mechanics

Consider a very simple and well known mathematical framework for classical particle mechanics, in the newtonian formalism, introduced by McKinsey, Sugar, and Suppes [6]. We call this as McKinsey-Sugar-Suppes (MSS) system for classical particle mechanics, or MSS system, for short.

MSS system has six primitive notions: $P$, $T$, $m$, $s$, $f$, and $g$. $P$ and $T$ are sets, $m$ is a real-valued unary function defined on $P$, $s$ and $g$ are vector-valued functions defined on the Cartesian product $P \times T$, and $f$ is a vector-valued function defined on the Cartesian product $P \times P \times T$. Intuitively, $P$ corresponds to the set of particles and $T$ is to be physically interpreted as a set of real numbers measuring elapsed times (in terms of some unit of time, and measured from some origin of time). $m(p)$ is to be interpreted as the numerical value of the mass of $p \in P$. $s_p(t)$, where $t \in T$, is a 3-dimensional vector which is to be physically interpreted as the position of particle $p$ at instant $t$. $f(p, q, t)$, where $p, q \in P$, corresponds to the internal force that
particle $q$ exerts over $p$, at instant $t$. And finally, the function $g(p, t)$ is to be understood as the external force acting on particle $p$ at instant $t$.

Next we present the axiomatic formulation for MSS system:

**Definition 7** $\mathcal{CP} = \langle P, T, s, m, f, g \rangle$ is an MSS system if and only if the following axioms are satisfied:

**P1** $P$ is a non-empty, finite set.

**P2** $T$ is an interval of real numbers.

**P3** If $p \in P$ and $t \in T$, then $s_p(t)$ is a 3-dimensional vector ($s_p(t) \in \mathbb{R}^3$) such that $\frac{d^2 s_p(t)}{dt^2}$ exists.

**P4** If $p \in P$, then $m(p)$ is a positive real number.

**P5** If $p, q \in P$ and $t \in T$, then $f(p, q, t) = -f(q, p, t)$.

**P6** If $p, q \in P$ and $t \in T$, then $[s_p(t), f(p, q, t)] = -[s_q(t), f(q, p, t)]$.

**P7** If $p, q \in P$ and $t \in T$, then $m(p)\frac{d^2 s_p(t)}{dt^2} = \sum_{q \in P} f(p, q, t) + g(p, t)$.

The brackets $[\cdot]$ in axiom **P6** denote external product.

Axiom **P5** corresponds to a weak version of Newton’s Third Law: to every force there is always a counter-force. Axioms **P6** and **P5**, correspond to the strong version of Newton’s Third Law. Axiom **P6** establishes that the direction of force and counter-force is the direction of the line defined by the coordinates of particles $p$ and $q$.

Axiom **P7** corresponds to Newton’s Second Law.

**Definition 8** Let $\mathcal{P} = \langle P, T, s, m, f, g \rangle$ be a MSS system, let $P'$ be a non-empty subset of $P$, let $s', g'$, and $m'$ be, respectively, the restrictions of functions $s$, $g$, and $m$ with their first arguments restricted to $P'$, and let $f'$ be the restriction of $f$ with its first two arguments restricted to $P'$. Then $\mathcal{P}' = \langle P', T, s', m', f', g' \rangle$ is a subsystem of $\mathcal{P}$ if $\forall p, q \in P'$ and $\forall t \in T$,

$$m'(p)\frac{d^2 s_p'(t)}{dt^2} = \sum_{q \in P'} f'(p, q, t) + g'(p, t).$$

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Theorem 1 Every subsystem of a MSS system is again a MSS system.

Definition 9 Two MSS systems

\[ \mathcal{P} = \langle P, T, s, m, f, g \rangle \]

and

\[ \mathcal{P}' = \langle P', T', s', m', f', g' \rangle \]

are equivalent if and only if \( P = P' \), \( T = T' \), \( s = s' \), and \( m = m' \).

Definition 10 A MSS system is isolated if and only if for every \( p \in P \) and \( t \in T \), \( g(p, t) = \langle 0, 0, 0 \rangle \).

Theorem 2 If

\[ \mathcal{P} = \langle P, T, s, m, f, g \rangle \]

and

\[ \mathcal{P}' = \langle P', T', s', m', f', g' \rangle \]

are two equivalent systems of particle mechanics, then for every \( p \in P \) and \( t \in T \)

\[ \sum_{q \in P} f(p, q, t) + g(p, t) = \sum_{q \in P'} f'(p, q, t) + g'(p, t). \]

The embedding theorem is the following:

Theorem 3 Every MSS system is equivalent to a subsystem of an isolated MSS system.

The next theorem can easily be proved by Padoa’s method:

Theorem 4 Mass and internal force are each independent of the remaining primitive notions of MSS system, i.e., they cannot be defined by means of either \( P, T, s \) or \( g \).

The next theorem is proved in [1].

Theorem 5 Time is definable from the remaining primitive concepts of MSS system.
There are some analysis in the literature concerning MSS system. See, for example, [13, 15, 16]. According to S. Obradovic [9], MSS system was “the first successful axiomatization in classical mechanics of a material point”. Besides, this system seems to be very reasonable at first sight.

Now it is important to settle some terminology for this paper. MSS system and classical particle mechanics are not the same concept. MSS system is a formal mathematical framework described in the definition given above, which is inspired on the common sense of what physicists understand as classical particle mechanics. Classical particle mechanics is not a formal theory from the logical point of view. Classical particle mechanics is a paradigm in theoretical physics which is grounded on the newtonian view of mechanics.

One of the main advantages of MSS system, compared to classical particle mechanics, is that particles are not considered as points. Particles in MSS system are abstract objects that are associated to mass, position, speed, and forces. As it happens in classical particle mechanics, in MSS system the size and the shape of particles (physical objects) are irrelevant. This is very important for the understanding of the next section.

7 Non-individuals in classical particle mechanics

Particles in MSS system are the elements of \( P \). The only intrinsic (state independent) property of each particle in MSS system is mass. All other physical properties, like position, velocity, acceleration, and forces, are state dependent. The state of a particle is the ordered pair

\[
\left( s_p(t), \frac{ds_p(t)}{dt} \right),
\]

for all instant of time \( t \in T \).

Forces, for example, are dependent on \( \frac{ds_p(t)}{dt} \), according to axiom \( P7 \).

This means that in MSS system there may exist two or more particles with the same intrinsic and state dependent physical properties. MSS system does nor forbid this kind of situation. It is legitimate to consider two particles 1 and 2 such that \( m_1 = m_2 \) and such that for a period of time \( T' \) their trajectories and velocities are identical. But in classical particle mechanics
and particularly in MSS system all particles are distinguishable. So, this kind of situation may create some confusion. In many classical books on quantum mechanics it is said that elementary particles are indistinguishable when sharing the same intrinsic properties and the same quantum state. But in MSS system and even in classical particle mechanics there may exist particles that share the same intrinsic properties and the same state. So, what makes elementary quantum particles really indistinguishable? It seems that the non-individuality of particles in quantum mechanics has a deeper origin. It does not seem to be reasonable to explain relations of indistinguishability by means of coincidence of intrinsic properties and physical state.

A physicist could argue that the notion of physical state in classical particle mechanics and the notion of quantum state in quantum mechanics are not equivalent. So, any comparison between quantum and classical behavior is not straightforward. Another way to understand the issues of non-individuality in quantum mechanics is by means of the concept of wave-function. Since wave-functions can be entangled, indistinguishability would be just a consequence of the standard formalism of quantum mechanics. Nevertheless, we show in this paper that it is possible to consider some sort of non-individuality postulate right at the start of an axiomatic framework for quantum mechanics. So, non-individuality does not need to be understood as a consequence from the postulates of quantum theory. The problem of two physical objects occupying the same position at the same time is discussed later, since it demands some further assumptions.

In the next definition we propose an axiomatic framework for a system based on classical particle mechanics, but with non-individual particles. We hope that our axiomatic framework may be useful for a better understanding of the meaning of non-individuality even in quantum theory. The intuitive meaning of the primitive concepts and postulates is given afterwards.

**Definition 11** \(\langle [x]_n, T, S, M, P, f, g \rangle\) is a quasi-MSS system if the following axioms are satisfied:

**QP1** \([x]_n\) is a \(n\)-singleton whose elements are micro-atoms. The elements of \([x]_n\) are denoted by \(x_\alpha, x_\beta, x_\gamma, x_\delta, \) and so on.

**QP2** \(T\) is a non-degenerate interval of real numbers.
QP3 \( S \) is the family of all functions \( s_a : T \to \mathbb{R}^3 \) that are twice differentiable. The images of these functions are denoted by \( s_a(t), s_b(t), s_c(t), s_d(t), \) and so on.

QP4 \( M \) is the set of all positive real numbers. The elements of \( M \) are denoted by \( m_r, m_s, m_a, m_v, \) and so on.

QP5 \( P \) is a sub-quasi-set of \([x]_n \times M \times S\) whose quasi-cardinality is \( n \).

QP6 \( f \) is a quasi-function \( f : P \times P \times T \to \mathbb{R}^3 \).

QP7 \( g \) is a quasi-function \( g : P \times T \to \mathbb{R}^3 \).

QP8 For all elements of \( P \) and for all \( t \in T \),

\[
f(x_\alpha, m_r, s_a(t); x_\beta, m_s, s_b(t); t) = -f(x_\gamma, m_s, s_b(t); x_\delta, m_r, s_a(t); t).
\]

QP9 For all elements of \( P \) and for all \( t \in T \),

\[
[s_a(t), f(x_\alpha, m_r, s_a(t); x_\beta, m_s, s_b(t); t)] = -[s_b(t), f(x_\gamma, m_s, s_b(t); x_\delta, m_r, s_a(t); t)].
\]

QP10 For all elements of \( P \) and for all \( t \) of \( T \) we have

\[
m_r \frac{d^2 s_a(t)}{dt^2} = \sum_P f(x_\alpha, m_r, s_a(t); x_\beta, m_s, s_b(t); t) + g(x_\beta, m_r, s_a(t)),
\]

where the summation is performed over \( N \) terms if there are \( N \) indistinguishable elements on \( P \), plus the remaining distinguishable elements.

Particles are ordered triples. The first element of these triples is a micro-atom from quasi-set theory. It is this first element that is responsible for the lack of individuality of particles. In other words, we consider that individuality is unnecessary for particles, although indispensable for describing physical states and intrinsic properties. The second element of the ordered
triple corresponds to the intrinsic property and the third element has all necessary information to describe the physical state of each particle, since the third element is not just position but a position function with respect to time $t$. $T$ is the set of instants of time and $M$ is the set of all possible values for mass. Axioms QP8 to QP10 are very similar to axioms P5 to P7 and have a similar physical meaning.

It is easy to see that two particles with the same mass and the same state (position and speed) are indistinguishable, i.e.,

$$(x_\alpha, m_r, s_a(t)) \equiv (x_\beta, m_r, s_a(t)),$$

since $x_\alpha \equiv x_\beta$.

MSS and quasi-MSS systems are not equivalent. In MSS system it is possible the existence of two particles that share the same mass and the same state, but being associated to different internal and external forces. This happen because the coincidence of physical properties does not entail any true indistinguishability between particles. In MSS system, particles do have individuality. In quasi-MSS system this kind of situation is not possible. Since internal and external forces are quasi-functions, indistinguishable particles are supposed to be associated to indistinguishable forces. But forces are real three-dimensional vectors which may be described by standard mathematics. So, indistinguishable forces are identical forces.

8 The physical meaning of $[x]_n$

In [2] we find an interesting statement:

[T]hat a permutation of the particles is counted as giving a different arrangement in classical statistical mechanics implies that, although they are indistinguishable, such particles can be regarded as individuals (indeed, Boltzmann himself made this explicit in the first axiom of his Lectures on Mechanics). Since this individuality resides in something over and above the intrinsic properties of the particles in terms of which they can be regarded as indistinguishable, it has been called Transcendental Individuality by Post [see [11]].
In this paper we consider a very different idea which goes to an opposite direction. We consider that the notion of individuality of physical objects may be considered as an illusion, caused by some physical laws and our interpretation of them. In other words, all physical objects in nature may be devoid of individuality, no matter if they are subject to the laws of the macroscopic world (classical mechanics) or the microscopic world (quantum mechanics). In this sense, all physical objects would have some sort of ‘Transcendental Non-individuality’, where the illusion of individuality would be caused by other physical components and our interaction with the world.

Consider, as an example, the quasi-MSS system. If we add a new axiom that says that the only internal force between particles is Newtonian gravitation

$$f(x_\alpha, m_r, s_a(t); x_\beta, m_s, s_b(t); t) = \gamma \frac{m_r m_s}{||s_a(t) - s_b(t)||^3} (s_a(t) - s_b(t)), \quad (6)$$

where $\gamma$ is the universal gravitational constant, then we will never have any two particles sharing the same state, since any coincidence of position would entail an inconsistency, namely, a singularity on the gravitational potential. In this case, any particle, despite its lack of individuality, can be distinguished by its physical state. This means that in a classical world we do not need any kind of transcendental individuality to justify the apparent individuality of physical objects. In the case of continuum mechanics another intrinsic property plays a very important role in the process of individuation: the size of the objects. So, we do not need the assumption of individuality for physical objects (either classical or quantum), although we need individual measures of state and intrinsic properties. Hence, the physical meaning of $[x]_n$ is that it drops the unnecessary individuality of a physical object which is associated to individual measures of state and state-independent properties. So, let us not confuse physical properties with physical objects like particles. Physical properties do need individuality. But physical objects don’t.

Since QM does not forbid the existence of particles sharing all their physical properties, we have the impression that non-individuality is an exclusivity of the quantum world. But that does not need to be true.

We are not proving that the whole world is made of non-individuals. But we are pointing out to the possibility that this may be a fact.
Another interesting issue concerns some realistic interpretations of QM. Bohmian mechanics, for example, is a realistic interpretation for quantum mechanics, where particles have definite trajectories and velocities. So, some sort of classical behavior among particles is unavoidable. The non-individuality of elementary particles in Bohmian mechanics has more to do with experimental facts than with the mathematical formalism. Nevertheless we may consider the non-individuality in Bohmian mechanics as an intrinsic characteristic of its own formalism. That would be a manner to explain quantum statistics into the scope of Bohmian mechanics.

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