Abstract

We consider nonlinear multibody systems and present a suitable set of coordinates for the internal dynamics which allow to decouple the internal dynamics without the need to compute the Byrnes-Isidori form. Furthermore, we derive sufficient conditions on the system parameters such that the internal dynamics are bounded-input, bounded-output stable.

Keywords: multibody systems; internal dynamics; bounded-input, bounded-output stability

1. Introduction

In the last two decades it turned out that for the purpose of high-gain based output tracking the so called funnel controller developed in [1] and generalized to nonlinear systems with arbitrary relative degree in [2] is a powerful tool. However, both necessitate stability of the internal dynamics in a certain sense. Although there is progress in tracking control of systems with unstable internal dynamics, see e.g. [3][4][5][6], most controllers require the internal dynamics to be bounded-input, bounded-output stable (minimum phase property), c.f. [7][8][9][10][11][12][13] and see also the survey [14], and for the concept of minimum phase [15], and the references therein, resp. Therefore, in order to verify applicability of a certain controller it is necessary to decouple the internal dynamics, e.g. via the Byrnes-Isidori form as in [16], and investigate its stability. However, the computation of the Byrnes-Isidori form of a nonlinear multibody system is often a challenging task. In the present work we combine the idea of the Byrnes-Isidori form with a novel approach to decouple the internal dynamics without the need to compute the Byrnes-Isidori form explicitly. This results in a representation of the internal dynamics in terms of the internal variables and the system’s output.

Moreover, we present sufficient conditions on the system parameters such that the internal dynamics are bounded-input, bounded-output stable. These conditions are independent of the representation of the internal dynamics and can be verified without their explicit decoupling. Hence, applicability of a controller e.g. as in [17][18] can be determined without decomposing the system but by investigating the system’s equations only.

This paper is organized as follows. We briefly recall the concepts of Lie derivatives, relative degree and the representation of a dynamical system in Byrnes-Isidori form in Section 2. In Section 3 we derive the representation of the internal dynamics in terms of the internal variables and the system’s output. Furthermore, we give lemmata of existence and uniqueness, resp., concerning the novel structural ansatz for the internal dynamics. In Section 4 we give an abstract stability result exploiting LaSalle’s invariance principle presented in [19]. Furthermore, we consider nonlinear multibody systems without kinematic loop and provide sufficient conditions on the system parameters such that the internal dynamics are bounded-input, bounded-output stable. We finish this paper with an illustrative example in Section 5.

1.1. Nomenclature

Throughout the present paper we will use the following notation: $\mathbb{R}_{\geq 0} := [0, \infty)$; $\|x\| := \sqrt{x^T x}$ Euclidean norm of $x \in \mathbb{R}^n$; $\|A\| := \max_{\|x\|=1} \|Ax\|$ spectral norm of $A \in \mathbb{R}^{n \times n}$; $\text{Gl}_1(\mathbb{R})$ the group of invertible matrices in $\mathbb{R}^{n \times n}$; $B_r(x) := \{z \in \mathbb{R}^n \ | \ |z - x| < r \}$ open ball of radius $r > 0$, centred at $x \in \mathbb{R}^n$; $C^k(V \to W)$ the set of $k$ times continuously differentiable functions $f : V \to W$, $k \in \mathbb{N}$ and $V \subseteq \mathbb{R}^n$, $W \subseteq \mathbb{R}^m$; $L^1_{\text{loc}}(I \to \mathbb{R}^n)$ the set of locally integrable functions $f : I \to \mathbb{R}^n$, $I \subseteq \mathbb{R}$ an interval; $L^\infty(I \to \mathbb{R}^n)$ the set of essentially bounded functions $f : I \to \mathbb{R}^n$, $I \subseteq \mathbb{R}$ an interval; $\|f\|_{\infty} := \sup_{t \in I} \|f(t)\|$ the supremum norm of $f \in L^\infty(I \to \mathbb{R}^n)$.

2. System class and Byrnes-Isidori form

In this section we briefly recall some basic concepts such as relative degree and the representation of a system in Byrnes-Isidori form. We consider nonlinear multibody systems without kinematic loops which are modeled using generalized coordinates and are of the form

$$\dot{q}(t) = v(t), \quad q(0) = q_0 \in \mathbb{R}^n,$$

$$M(q(t))\dot{v}(t) = f(q(t), v(t)) + B(q(t))u(t), \quad v(0) = v^0 \in \mathbb{R}^n,$$

$$y(t) = h(q(t)), \quad (1)$$
where \( M \in \mathcal{C}(\mathbb{R}^n \to \text{GL}_n(\mathbb{R})) \) is the generalized mass matrix, \( f \in \mathcal{C}(\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n) \) are the generalized forces, \( B \in \mathcal{C}(\mathbb{R}^n \to \mathbb{R}^{n \times m}) \) is the distribution of the input and \( h \in \mathcal{C}(\mathbb{R}^n \to \mathbb{R}^m) \) is the measurement. The functions \( u : \mathbb{R}_\geq 0 \to \mathbb{R}^m \) are the inputs that exert an influence to system (1), and \( y : \mathbb{R}_\geq 0 \to \mathbb{R}^m \) are the outputs that typically represent physically meaningful measurements of system (1). Note that the dimensions of the input and output coincide but we do not assume collocation, i.e., we do not assume \( h'(q) = B(q) \). For system (1) we introduce the state variables \( x_1 := q, x_2 := v \), setting \( x := (x_1^T, x_2^T)^T \), and transform (1) into the system of first order ordinary differential equations

\[
\dot{x}(t) = \begin{pmatrix} x_2(t) \\ M(x_1(t))^{-1} f(x_1(t), x_2(t)) \end{pmatrix} + \begin{pmatrix} 0 \\ M(x_1(t))^{-1} B(x_1(t)) \end{pmatrix} u(t), \quad x(0) = x^0 \in \mathbb{R}^{2n}
\]

\[
y(t) = h(x_1(t), x_2(t)),
\]

where \( h : \mathbb{R}^{2n} \to \mathbb{R}^m \) with \( h(x_1, x_2) = h(x_1) \). In order to decouple the internal dynamics we invoke the Byrnes-Isidori form for (2). To this end, recall the definition of the Lie derivative of a function \( h \) along a vector field \( F \) at a point \( z \in U \subseteq \mathbb{R}^{2n} \), \( U \) open

\[
(L_F h)(z) := \tilde{h}'(z)F(z),
\]

where \( \tilde{h}' \) is the Jacobian of \( \tilde{h} \). We may successively define \( L_{h_{k+1}} h = L_F (L_{h_k}^{-1} h) \) with \( L_{h_k} h = \tilde{h} \). We denote with \( g_i(z) \) the columns of \( G(z) \) for \( i = 1, ..., m \) and define

\[
(L_{G_i} h)(z) := [(L_{g_1} h)(z), ..., (L_{g_m} h)(z)].
\]

In accordance with (10) we recall the concept of relative degree (10).

**Definition 2.1.** System (2) has relative degree \( r \in \mathbb{N} \) on \( U \subseteq \mathbb{R}^{2n} \) open, if for all \( z \in U \) we have

\[
\forall k \in \{0, ..., r - 2\} : (L_G L_{h_k} h)(z) = 0_{m \times m}
\]

and

\[
\Gamma(z) := (L_G L_{h_r}^{-1} h)(z) \in \text{GL}_m(\mathbb{R}),
\]

where \( \Gamma : U \to \text{GL}_m(\mathbb{R}) \) is the high-gain matrix.

Now, as shown in (16), if system (2) has relative degree \( r \) on an open set \( U \subseteq \mathbb{R}^{2n} \), then there exists a (local) diffeomorphism \( \Phi : U \to W \subseteq \mathbb{R}^{2n}, W \) open, such that

\[
\begin{pmatrix} \xi(t) \\ \eta(t) \end{pmatrix} = \Phi(x(t)),
\]

with \( \xi(t) \in \mathbb{R}^m, \eta(t) \in \mathbb{R}^{2n-rm} \) transforms system (2) nonlinearly into Byrnes-Isidori form

\[
y(t) = \xi(t), \\
\dot{\xi}(t) = \xi(t), \\
\vdots \\
\dot{\xi}_{r-1}(t) = \xi(t), \\
\dot{\xi}_r(t) = (L_F h)(\Phi^{-1}(\xi(t), \eta(t))) + \Gamma(\Phi^{-1}(\xi(t), \eta(t))) \eta(t), \\
\eta(t) = q(\xi(t), \eta(t)) + p(\xi(t), \eta(t))u(t).
\]

The last equation in (4) represents the internal dynamics of system (2). Note that for \( r \cdot m < 2n \) system (2) has nontrivial interal dynamics.

With the aid of Lie derivatives the diffeomorphism \( \Phi \) can be represented as

\[
\Phi(x) = \begin{pmatrix} \tilde{h}(x) \\ (L_F \tilde{h})(x) \\ \vdots \\ \tilde{\phi}_{r-1}(x) \\ \tilde{\phi}_r(x) \end{pmatrix}, \quad x \in U \subseteq \mathbb{R}^{2n}
\]

where \( \tilde{\phi}_i : U \to \mathbb{R}, i = 1, ..., 2n - rm \), are such that \( \Phi'(x) \) is invertible for all \( x \in U \). We recall \( x = (x_1^T, x_2^T)^T \in \mathbb{R}^{2n} \) and make the following assumption

(A1) Given some open set \( U_1 \subseteq \mathbb{R}^n \) and \( H(x_1) := h'(x_1) \), we have \( \Gamma(x_1) := H(x_1) M(x_1)^{-1} B(x_1) \in \text{GL}_m(\mathbb{R}) \) for all \( x_1 \in U_1 \).

**Lemma 2.2.** Consider system (2) and assume (A1). Then system (2) has relative degree \( r = 2 \) on \( U := U_1 \times \mathbb{R}^n \).

**Proof.** Let \( x = (x_1^T, x_2^T)^T \in U \). Set \( H(x_1) := h'(x_1) \) and \( \tilde{h}(x) := h'(x_1) \), then \( H(x) = [H(x_1) \ 0] \). Now, compute the Lie derivatives of (2) for \( x \in U \)

\[
(L_G \tilde{h})(x) = [H(x_1) \ 0] \begin{pmatrix} M(x_1)^{-1} B(x_1) \\
0 \\
M(x_1)^{-1} f(x_1, x_2) \\
H(x_1) x_2 \\
H(x_1) x_2 \\
M(x_1)^{-1} B(x_1) \\
\end{pmatrix} := \Gamma(x_1)
\]

where \( \Gamma(x_1) = H(x_1) M(x_1)^{-1} B(x_1) \) is invertible by assumption (A1). Therefore, according to (3) system (2) has relative degree \( r = 2 \) on \( U \).
3. Representation of the internal dynamics

In this section we introduce a novel structural ansatz for the internal dynamics and present a set of feasible coordinates to represent the internal dynamics of \( \mathcal{S} \) without the need to compute the Byrnes-Isidori form explicitly. Via \( \mathcal{S} \) with \( r = 2 \) we obtain the following representation of the diffeomorphism \( \Phi \)

\[
(\xi \eta) = \Phi(x) = \begin{pmatrix} h(x_1) \\ H(x_1)x_2 \\ \phi_1(x) \\ \phi_2(x_2) 
\end{pmatrix}, \quad x = (x_1, x_2) \in U,
\]

where \( \phi_i : U \to \mathbb{R}, i = 1, \ldots, 2n - 2m \) and, as in Lemma 2.2

\[U_1 \subseteq \mathbb{R}^n\] is an open set and \( U = U_1 \times \mathbb{R}^n\). We make the following structural ansatz for the internal state \( \eta = (\phi_1(x), \ldots, \phi_{2n-2m}(x))^T \)

\[
\eta = \begin{pmatrix} \eta_1 \\ \eta_2
\end{pmatrix} = \begin{pmatrix} \phi_1(x_1) \\ \phi_2(x_2)
\end{pmatrix}, \quad \tag{6}
\]

and \( \mathcal{F} \) have similar structure, i.e., the internal dynamics are in the form of a mechanical system as well. Now, since \( \Phi \) is a diffeomorphism we require its Jacobian to be invertible on \( U \)

\[
\forall x \in U : \Phi'(x) = \begin{pmatrix} H(x_1) & 0 \\ \ast & H(x_1)
\end{pmatrix} \in \text{GL}_{2n}(\mathbb{R}) \iff \begin{pmatrix} h(x_1) \\ H(x_1)x_2 \\ \phi_1(x_1) \\ \phi_2(x_2)
\end{pmatrix} \in \text{GL}_n(\mathbb{R}) \tag{8}
\]

where \( \ast \) of the form \( \frac{\partial}{\partial x_i} [\zeta(x_1) \cdot x_2] \), \( \zeta : U \to \mathbb{R}^q \times \mathbb{R}^n \) with \( q \in \mathbb{N} \) appropriate, resp. We aim to investigate the internal dynamics of \( \mathcal{S} \) without explicit appearance of the input \( u \). To this end, we seek for functions \( \phi_1(x), \ldots, \phi_{2n-2m}(x) \) such that \( p(\cdot) = 0 \) in equation 4, i.e., \( [L_G \phi_1](x) = 0 \) for all \( x \in U, i = 1, \ldots, 2n - 2m \). In view of \( \mathcal{S} \) this means to find functions \( \phi_1, \phi_2 \) such that

\[
\forall x \in U : \begin{pmatrix} \phi_1(x_1) \\ \phi_2(x_1) \\ 0 \\ 0
\end{pmatrix} \begin{pmatrix} 0 \\ M(x_1)^{-1}B(x_1) \\ \ast \\ \ast
\end{pmatrix} = 0 \iff \forall x_1 \in U_1 : \begin{pmatrix} \phi_2(x_1)M(x_1)^{-1}B(x_1) \\ \phi_2(x_1) \\ \ast \\ \ast
\end{pmatrix} = 0. \tag{9}
\]

In the following lemma we show the existence of functions \( \phi_1, \phi_2 \) satisfying the aforesaid.

**Lemma 3.1.** Consider the ODE \( \mathcal{S} \) and assume \( \mathcal{A}1 \). For any \( x_0^1 \in U_1 \) of \( x_0^1 \) and \( \phi_1 \in \mathcal{C}(U_0^1 \to \mathbb{R}^{n-m}), \phi_2 \in \mathcal{C}(U_0^1 \to \mathbb{R}^{(n-m)\times n}) \) such that \( \mathcal{S} \) and \( \mathcal{F} \) hold locally on \( U_0^1 \).

Proof. We fix \( x_0^1 \in U_1 \) and make use of [20, Lem. 4.1.5] which states the following. Consider \( W \in \mathcal{C}(U_1 \to \mathbb{R}^{w\times n}) \) with \( \text{rk} W(x_1) = w \) for all \( x_1 \in U_1 \). Then there exist an open neighbourhood \( V_1 \subseteq U_1 \) of \( x_0^1 \) and \( T \in \mathcal{C}(V_1 \to \text{GL}_n(\mathbb{R})) \) such that

\[
\forall x_1 \in V_1 : W(x_1)T(x_1) = [I_w \ 0].
\]

We use this to show the existence of \( \phi_1 \in \mathcal{C}(U_0^1 \to \mathbb{R}^{n-m}) \). Since by assumption \( \mathcal{A}1 \) we have \( \text{rk} H(x_1) = m \) for all \( x_1 \in U_1 \) there exist an open neighbourhood \( V_1 \subseteq U_1 \) of \( x_0^1 \) and \( T = [T_1, T_2] \in \mathcal{C}(V_1 \to \text{GL}_n(\mathbb{R})) \) such that

\[
\forall x_1 \in V_1 : H(x_1) [T_1(x_1) \ T_2(x_1)] = [I_m \ 0],
\]

i.e., \( \text{im} T_2(x_1) = \ker H(x_1) \) and \( \text{rk} T_2(x_1) = n - m \) for all \( x_1 \in V_1 \). Let \( E = [e_1^T, \ldots, e_{n-m}^T]^T \in \mathbb{R}^{(n-m)\times n} \) with \( e_i \in \mathbb{R}^{1\times n} \) a unit row-vector for \( i \in \{1, \ldots, n\} \). Then

\[
[H(x_1) E] [T_1(x_1) \ T_2(x_1)] = [I_m \ 0] [e_1 \ldots e_{n-m}] = [I_m \ 0].
\]

Since \( \text{rk} T_2(x_0^1) = n - m \) it is possible to choose \( i_1, \ldots, i_{n-m} \) such that \( \text{ET}_2(x_0^1) \in \text{GL}_{n-m}(\mathbb{R}) \). As \( T_2 \in \mathcal{C}(V_1 \to \mathbb{R}^{n\times(n-m)}) \) the mapping \( x_1 \mapsto \det(\text{ET}_2(x_1)) \) is continuous on \( V_1 \), hence there exists an open neighbourhood \( \tilde{V}_1 \subseteq V_1 \) of \( x_0^1 \) such that \( \det(\text{ET}_2(x_1)) \neq 0 \) for all \( x_1 \in \tilde{V}_1 \). Thus,

\[
\forall x_1 \in \tilde{V}_1 : \text{rk} \begin{pmatrix} H(x_1) E 
\end{pmatrix} = n.
\]

Therefore, with

\[
\phi_1 : \tilde{V}_1 \to \mathbb{R}^{n-m}, \quad x_1 \mapsto E x_1
\]

we have \( \phi_1 \in \mathcal{C}(\tilde{V}_1 \to \mathbb{R}^{n-m}) \) and the first condition in \( \mathcal{S} \) is satisfied on \( \tilde{V}_1 \) since \( \phi_1(x_1) = E \).

Now, we show the existence of \( \phi_2 \in \mathcal{C}(U_0^1 \to \mathbb{R}^{n-m}) \). Observe that by assumption \( \mathcal{A}1 \) we have \( \text{rk} B(x_1) = m \) and \( \text{rk} M(x_1) = n \) for all \( x_1 \in U_1 \). Therefore, again via [20, Lem. 4.1.5], there exist an open neighbourhood \( \hat{V}_1 \subseteq U_1 \) of \( x_0^1 \) and \( T = [T_1, T_2] \in \mathcal{C}(\hat{V}_1 \to \text{GL}_n(\mathbb{R})) \) such that

\[
\forall x_1 \in \hat{V}_1 : [M(x_1)^{-1}B(x_1) \ T_1(x_1) \ T_2(x_1)] = [I_m \ 0],
\]

i.e., \( \text{im} T_2(x_1) = \ker(M(x_1)^{-1}B(x_1))^\top \) and \( \text{rk} T_2(x_1) = n - m \) for all \( x_1 \in \hat{V}_1 \). Now, we observe for \( \phi_2(x_1) = T_2(x_1)^\top \)

\[
[H(x_1) \phi_2(x_1)] [M(x_1)^{-1}B(x_1) \ phi_2(x_1)] = \begin{pmatrix} 1 \phi_2(x_1) \ast \\ \ast \phi_2(x_1) \ast \end{pmatrix} = 0 \quad \in \mathbb{R}^{n\times n}, \quad x_1 \in \tilde{V}_1
\]

which is invertible on \( \tilde{V}_1 \) since by assumption \( \mathcal{A}1 \) the high-gain matrix \( \Gamma(x_1) \) is invertible on \( U_1 \), and \( \text{rk} T_2(x_1) = n - m \) for all \( x_1 \in \tilde{V}_1 \). Hence the second condition in \( \mathcal{S} \) is satisfied on \( \tilde{V}_1 \). Moreover, equation 9 is true on \( \tilde{V}_1 \) by construction of \( \phi_2 \). We set \( U_0^1 := \tilde{V}_1 \cap V_1 \) which completes the proof. \( \square \)
The proof of the previous lemma justifies the structural ansatz for the internal state \( \eta \) in [6]. Hereinafter let \( U_1 = U_0^0 \) with \( U_0^0 \) as in Lemma 3.1. While \( \phi_1 \) can basically be chosen freely up to [8], \( \phi_2 \) is uniquely determined up to an invertible left transformation. To find all possible representations, let \( P : U_1 \to \mathbb{R}^{n \times m} \) and \( V : U_1 \to \mathbb{R}^{n \times (n-m)} \) be such that

\[
\forall x_1 \in U_1 : \ [P(x_1), V(x_1)] \begin{bmatrix} H(x_1) \\ \phi_2(x_1) \end{bmatrix} = I_n, \tag{10}
\]

which exist by [8]. Then \( P, V \) have pointwise full column rank, by which the pseudoinverse of \( V \) is given by \( V^\dagger(x_1) = (V(x_1)^\dagger V(x_1))^{-1} V(x_1)^\dagger \), \( x_1 \in U_1 \). For \( x_1 \in U_1 \) we define

\[
\phi_2(x_1) := V^\dagger(x_1) \left( I_n - M(x_1)^{-1} B(x_1) \Gamma(x_1)^{-1} H(x_1) \right). \tag{11}
\]

**Lemma 3.2.** We use the notation and assumptions from Lemma 3.1. Then the function \( \phi_2 : U_1 \to \mathbb{R}^{(n-m) \times n} \) is uniquely determined by [8] and [9] up to an invertible left transformation. All possible functions are given by [11] for feasible choices of \( V \) satisfying [10].

**Proof.** Assume that [8] and [9] hold, and hence we have [10] for some corresponding \( P \) and \( V \). We multiply [10] from the left by \( V(x_1)^\dagger \) and subtract \( V(x_1)^\dagger P(x_1) H(x_1) \) from both sides, and obtain

\[
\phi_2(x_1) = V^\dagger(x_1) \left( I_n - P(x_1) H(x_1) \right), \quad x_1 \in U_1.
\]

Invoking [9], we further obtain from [10] that

\[
P(x_1) = M(x_1)^{-1} B(x_1) \left( H(x_1) M(x_1)^{-1} B(x_1) \right)^{-1},
\]

and hence \( P \) is uniquely determined by \( M, H, B \). Therefore, \( \phi_2 \) is given as in [11]. Furthermore, it follows from [10] that

\[
\begin{bmatrix} H(x_1) \\ \phi_2(x_1) \end{bmatrix} [P(x_1), V(x_1)] = \begin{bmatrix} I_n & 0 \\ 0 & I_{n-m} \end{bmatrix},
\]

from which we may deduce \( \phi_2(x_1) V(x_1) = I_{n-m} \) and in addition \( \text{im} V(x_1) = \ker H(x_1) \). Hence, the representation of \( \phi_2 \) in [11] only depends on the choice of the basis of \( \ker H(x_1) \). Now, let \( V(x_1) := V(x_1) R(x_1) \), \( x_1 \in U_1 \), for some \( R : U_1 \to \mathbb{G}_{n-m} \) and consider

\[
\tilde{\phi}_2(x_1) = V^\dagger(x_1) \left( I_n - M(x_1)^{-1} B(x_1) \Gamma(x_1)^{-1} H(x_1) \right).
\]

A short calculation shows \( \tilde{\phi}_2(x_1) = R(x_1)^{-1} \phi_2(x_1) \) for all \( x_1 \in U_1 \).

Now, we continue deriving a representation of the internal dynamics. We choose \( V \in C(U_1 \to \mathbb{R}^{n-m}) \) with orthonormal columns such that \( \text{im} V(x_1) = \ker H(x_1) \) for all \( x_1 \in U_1 \). Then via [11] and [10] the inverse of \( \begin{bmatrix} H(x_1) \\ \phi_2(x_1) \end{bmatrix} \)

is given by

\[
\begin{bmatrix} H(x_1) \\ \phi_2(x_1) \end{bmatrix}^{-1} = \left[ M(x_1)^{-1} B(x_1) \Gamma(x_1)^{-1} V(x_1) \right], \quad x_1 \in U_1. \tag{12}
\]

Recall [6] and [7], and choose \( \phi_2(x_1) \) as in [11]. Then [12] yields

\[
\begin{aligned}
\xi_2 &= \begin{bmatrix} H(x_1) \\ \phi_2(x_1) \end{bmatrix} x_2 \\
\Rightarrow x_2 &= \begin{bmatrix} H(x_1) \\ \phi_2(x_1) \end{bmatrix}^{-1} \begin{bmatrix} \xi_2 \\ \eta_2 \end{bmatrix} - M(x_1)^{-1} B(x_1) \Gamma(x_1)^{-1} \xi_2 + V(x_1) \eta_2. \tag{13}
\end{aligned}
\]

Therefore, using [2], [6] and [13], and identify \( \xi_2 = \hat{y} \), the dynamics of \( \eta_1 \) are given by

\[
\hat{\eta}_1(t) = \phi_1'(x_1(t)) x_2(t) = \phi_1'(x_1(t)) M(x_1(t))^{-1} B(x_1(t)) \Gamma(x_1(t))^{-1} \hat{y}(t) + \phi_1'(x_1(t)) V(x_1(t)) \eta_2(t), \tag{14}
\]

where \( \phi_1'(x_1) \) is of full row rank for all \( x_1 \in U_1 \) by [8], but apart from that arbitrary. Again with [6] and [7], since

\[
\begin{bmatrix} \xi_1 \\ \eta_1 \end{bmatrix} = \begin{bmatrix} h(x_1) \\ \phi_1(x_1) \end{bmatrix} =: \varphi(x_1), \tag{15}
\]

is continuously differentiable and its Jacobian is regular on \( U_1 \subseteq \mathbb{R}^n \) by [8], via the inverse function theorem \( \varphi \) defines a diffeomorphism on \( U_1 \) and hence we have

\[
x_1 = \varphi^{-1} \left( \begin{bmatrix} \xi_1 \\ \eta_1 \end{bmatrix} \right).
\]

**Corollary 3.3.** Assume [AT] holds true and that there exists \( E \in \mathbb{R}^{(n-m) \times n} \) such that \( \phi_1(x_1) := Ex_1, \ x_1 \in U_1 \), satisfies [8]. Further, assume there exists \( H \in \mathbb{R}^{m \times n} \) such that \( h \) is linear with \( h(x_1) = Hx_1 \) for all \( x_1 \in U_1 \), and let \( V \in \mathbb{R}^{n \times (n-m)} \) be such that \( \text{im} V = \ker H \). Then for \( \varphi : U_1 \to \mathbb{R}^n \) defined as in [13] we find that

\[
\varphi(U_1) = \begin{bmatrix} H \\ E \end{bmatrix} U_1 =: W_1
\]

and for all \( w_1 \in W_1 \) we have

\[
\varphi^{-1}(w_1) = \begin{bmatrix} H \\ E \end{bmatrix}^{-1} w_1 = \begin{bmatrix} H^\dagger (HH^\dagger)^{-1} - V(EV)^{-1} EH^\dagger (HH^\dagger)^{-1} V(EV)^{-1} \end{bmatrix} w_1 \]

**Proof.** Clear. 

To combine [15] with [14] we define the following functions on \( W_1 : = \varphi(U_1) \) as concatenations

\[
\phi_{1,\varphi}(:) := (\phi_1 \circ \varphi^{-1})(::), \quad M_{\varphi}(:)^{-1} := (M \circ \varphi^{-1})(::)^{-1},
\]

\[
B_{\varphi}(:) := (B \circ \varphi^{-1})(::), \quad H_{\varphi}(:) := (H \circ \varphi^{-1})(::),
\]

\[
\Gamma_{\varphi}(:)^{-1} := (\Gamma \circ \varphi^{-1})(::)^{-1}, \quad V_{\varphi}(:) := (V \circ \varphi^{-1})(::).
\]
Therefore, identifying \( \xi = y \) we obtain the representation \(16 \) of \( 14 \). Now, we explore the dynamics of \( \eta_n \). Define \( \phi_2[x_1, x_2] := \frac{\partial}{\partial x_1} \phi_2(x_1) \in \mathbb{R}^{(n-m) \times m} \) for \( x = (x_1^\top, x_2^\top)^\top \in U_1 \times \mathbb{R}^n = U \). Then from \( 2 \) and \( 3 \) we obtain for \( t \in U \):

\[
\dot{\eta}_2(t) = \phi_2[x_1(t), x_2(t)]x_2(t) + \phi_2(x_1(t))M(x_1(t))^{-1}f(x_1(t), x_2(t)).
\]

Let \( \phi_{2,\varphi}(\cdot) := (\phi_2 \circ \varphi^{-1})(\cdot) \) on \( W_1 \) and for \( w \in W_1 \) and \( v \in \mathbb{R}^n \), we define

\[
\phi_{2,\varphi}(w, v) := \phi_2 \left[ \varphi^{-1}(w), \left[ H_{\varphi}(w) \right]^{-1} v \right],
\]

\[
f_{\varphi}(w, v) := f\left[ \varphi^{-1}(w), \left[ H_{\varphi}(w) \right]^{-1} v \right].
\]

Then, the internal dynamics of \( 2 \) are given in \( 17 \).

**Remark 3.4.** The set of variables presented in this section to decouple the internal dynamics of \( 2 \) offers an alternative to the Byrnes-Isidori form as in \( 1 \), whose computation often requires a lot of effort. The representation of the internal dynamics in \( 17 \) is directly determined by the system parameters and hence a computation of the Byrnes-Isidori form is not necessary.

**Remark 3.5.** We may obtain further structure for the internal dynamics \( 17 \). Recall the concept of a conservative vector field. We call a vector field \( J : U \to \mathbb{R}^n \), \( U \subseteq \mathbb{R}^n \) open, conservative, if there exists a scalar field \( j : U \to \mathbb{R} \) such that \( j'(x) = J(x)^\top \) for all \( x \in U \). Now, if there exist \( j_i \in \mathcal{C}^1(U_1 \to \mathbb{R}) \) such that \( j'_i(x_1) = \phi_2,\varphi(x_1)^\top \) for all \( x_1 \in U_1 \), \( i = 1, \ldots, n-m \), then it is possible to choose \( \lambda_i = (\lambda_1, \ldots, \lambda_{i-1} \ldots, \lambda_{n-m+1} \ldots, \lambda_m)^\top \) for some \( \lambda_i \in \mathbb{R} \setminus \{0\} \), \( i = 1, \ldots, n-m \), and thus \( \phi_{2,\varphi}(x_1) = \Lambda \phi_2(x_1) \) for \( \Lambda = \text{diag}(\lambda_1, \ldots, \lambda_m) \). Therefore, using \( 10 \) and Lemma 3.2 we reduce the dynamics of \( \eta \) in \( 16 \) to

\[
\dot{\eta}_1(t) = \Lambda \eta_2(t).
\]

Note that the entries of \( \Lambda \) can be chosen at will. We will use this later in Section 4.2.

### 4. Stability of the Internal Dynamics

In this section we derive sufficient conditions on the system parameters such that the internal dynamics are\(44 \) bounded-input, bounded-output stable.

**4.1. Abstract stability result**

Before we give conditions on \( f \) in \( 2 \) to ensure a bounded-input, bounded-output stability of the internal dynamics \( 17 \) we present an abstract stability result. To this end, we make the following definitions.

**Definition 4.1.** Consider a dynamical control system

\[
\dot{x}(t) = f(x(t), u(t)), \quad x(0) = x^0 \in \mathbb{R}^n,
\]

where \( f \in \mathcal{C}(X \times \mathbb{R}^m \to \mathbb{R}^n) \), \( X \subseteq \mathbb{R}^n \), open, and \( u \in \mathcal{L}_\infty(\mathbb{R}^m \to \mathbb{R}^m) \). A local solution of \( 18 \) is a function \( x \in \mathcal{C}^1([0, \omega) \to \mathbb{R}^n) \), \( \omega \in \mathbb{R}^+ \) such that it satisfies \( 18 \) on \([0, \omega) \) with \( x(0) = x^0 \in \mathbb{R}^n \). If \( \omega = \infty \) we call \( x \in \mathcal{C}^1(\mathbb{R}^m \to \mathbb{R}^n) \) a global solution.

**Definition 4.2.** We call a function \( f : \mathbb{R}^n \to \mathbb{R} \) radially unbounded, if \( f(x) \to \infty \) for \( \|x\| \to \infty \).

Now, following \( 19 \) Thm. 4, we may formulate the first main result of the present paper as the following theorem.

**Theorem 4.3.** Let \( U \subseteq \mathbb{R}^n \) be open and consider

\[
\zeta(t) = \Psi(\zeta(t), y_1(t), y_2(t)), \quad \zeta(0) = \zeta^0 \in U,
\]

where \( \Psi \in \mathcal{C}(U \times \mathbb{R}^m \to \mathbb{R}^m) \). Assume there exist \( r_1, r_2, r_3 > 0 \) and a radially unbounded \( V \in \mathcal{C}^4(U \to \mathbb{R}) \) such that for all \( y_1 \in B_{r_1}(0), y_2 \in B_{r_2}(0) \) and for all \( \zeta \in \{ \zeta \in U \mid \|\zeta\| > r_3 \} \) we have

\[
V'(\zeta) \cdot \Psi(\zeta, y_1, y_2) \leq 0.
\]

Then for all \( y_1, y_2 \in \mathcal{L}_\infty(\mathbb{R}^m \to \mathbb{R}^m) \) with \( \|y_1\|_\infty \leq r_1 \) and \( \|y_2\|_\infty \leq r_2 \) and all global solutions \( \zeta : \mathbb{R}^+ \to U \) of \( 19 \) there exists \( \varepsilon > 0 \) such that

\[
\|\zeta\|_\infty \leq \max\{\|\zeta^0\|_\infty, r_3\} + \varepsilon.
\]

**Proof.** The following proof is inspired by the proof made in \( 19 \) Thm. 4, and some ideas are adopted from the proof in \( 20 \) Lem. 5.7.8. Let \( y_1, y_2 \in \mathcal{L}_\infty(\mathbb{R}^m \to \mathbb{R}^m) \) with \( \|y_1\|_\infty \leq r_1 \) and \( \|y_2\|_\infty \leq r_2 \), and let \( \zeta : \mathbb{R}^+ \to U \) be a global solution of \( 19 \). Further, set \( \tilde{r} := \max\{\|\zeta(0)\|_\infty, r_3\} \) and \( \nu_0 := \max\{V(\zeta) \mid \|\zeta\| = \tilde{r}\} \). Since \( V \) is radially unbounded there exists \( \varepsilon > 0 \) such that \( V(\zeta) > \nu_0 \) for all \( \zeta \in \{ \zeta \in U \mid \|\zeta\| \geq \tilde{r} + \varepsilon \} \). Seeking a contradiction, we assume there exists \( t_1 \) such that \( \|\zeta(t_1)\| = \tilde{r} + \varepsilon \). Let \( t_0 = \max\{t \in [0, t_1) \mid \|\zeta(t)\| = \tilde{r}\} \). Then we have \( \|\zeta(t)\| > r_3 \) for all \( t \in (t_0, t_1) \). Since by \( 20 \) we have \( \frac{d}{dt} V(\zeta(t)) \leq 0 \) for all \( t \in (t_0, t_1) \) and upon integration we obtain \( \nu_0 < V(\zeta(t_1)) \leq V(\zeta(t_0)) \leq \nu_0 \), a contradiction. Therefore, we conclude \( \|\zeta(t)\| \leq \tilde{r} + \varepsilon \) for all \( t \geq 0 \) and hence \( \|\zeta\|_\infty \leq \tilde{r} + \varepsilon \). \( \square \)

#### 4.2. Sufficient conditions for stability

In this section we present sufficient conditions on the system parameters, i.e., on \( f \) in \( 2 \), such that the internal dynamics are bounded-input, bounded-output stable.

**Definition 4.4.** Consider a control system \( 18 \) with output \( y(t) = h(x(t)) \), where \( h \in \mathcal{C}^1(\mathbb{R}^n \to \mathbb{R}^m) \). We call a system \( 18 \) bounded-input, bounded-output stable, if for all \( x^0 \in \mathbb{R}^n \) and all \( C_1 > 0 \) there exists \( C_2 > 0 \) such that for all \( u \in \mathcal{C}(\mathbb{R}^m \to \mathbb{R}^m) \) with \( \|u\|_\infty \leq C_1 \) and all global solutions \( x : \mathbb{R}^+ \to \mathbb{R}^n \) of \( 18 \) we have \( \|h(x)\|_\infty \leq C_2 \).
\begin{equation}
\dot{\eta}_1(t) = \phi'_{1,\varphi} \left( \begin{pmatrix} y(t) \\ \eta_1(t) \end{pmatrix} \right) M_{\varphi} \left( \begin{pmatrix} y(t) \\ \eta_1(t) \end{pmatrix} \right)^{-1} B_{\varphi} \left( \begin{pmatrix} y(t) \\ \eta_1(t) \end{pmatrix} \right) \Gamma_{\varphi} \left( \begin{pmatrix} y(t) \\ \eta_1(t) \end{pmatrix} \right)^{-1} \dot{y}(t) + \phi'_{1,\varphi} \left( \begin{pmatrix} y(t) \\ \eta_1(t) \end{pmatrix} \right) V_{\varphi} \left( \begin{pmatrix} y(t) \\ \eta_1(t) \end{pmatrix} \right) \eta_2(t).
\end{equation}

\begin{equation}
\dot{\eta}_1(t) = \phi'_{1,\varphi} \left( \begin{pmatrix} y(t) \\ \eta_1(t) \end{pmatrix} \right) M_{\varphi} \left( \begin{pmatrix} y(t) \\ \eta_1(t) \end{pmatrix} \right)^{-1} B_{\varphi} \left( \begin{pmatrix} y(t) \\ \eta_1(t) \end{pmatrix} \right) \Gamma_{\varphi} \left( \begin{pmatrix} y(t) \\ \eta_1(t) \end{pmatrix} \right)^{-1} \dot{y}(t) + \phi'_{1,\varphi} \left( \begin{pmatrix} y(t) \\ \eta_1(t) \end{pmatrix} \right) V_{\varphi} \left( \begin{pmatrix} y(t) \\ \eta_1(t) \end{pmatrix} \right) \eta_2(t),
\end{equation}

\begin{equation}
\dot{\eta}_2(t) = \phi'_{2,\varphi} \left( \begin{pmatrix} y(t) \\ \eta_1(t) \end{pmatrix} \right) M_{\varphi} \left( \begin{pmatrix} y(t) \\ \eta_1(t) \end{pmatrix} \right)^{-1} f_{\varphi} \left( \begin{pmatrix} y(t) \\ \eta_1(t) \end{pmatrix} \right), \phi'_{2,\varphi} \left( \begin{pmatrix} y(t) \\ \eta_1(t) \end{pmatrix} \right) V_{\varphi} \left( \begin{pmatrix} y(t) \\ \eta_1(t) \end{pmatrix} \right) \eta_2(t).
\end{equation}

Remark 4.5. In the context of the internal dynamics (17) of system (2) we actually consider bounded-input, bounded-state stability. As we add the output $z(t) = \eta(t)$ to system (2) we may refer to it as a bounded-input, bounded-output stable, where the output of system (2), namely $y(\cdot)$, and its derivative $\dot{y}(\cdot)$ play the role of the input of system (2).

Now, we consider a system (2) with constant mass matrix $M \in \mathbf{GL}_n(\mathbb{R})$ and constant input distribution $B \in \mathbb{R}^{n \times m}$. Since in many applications the output function $y(\cdot)$ is linear, we assume $h$ to be linear, i.e., $h(x_1) = H \cdot x_1$ for all $x_1 \in U_1$ with $H \in \mathbb{R}^{m \times n}$. Under these assumptions we have $U_1 = \mathbb{R}^n$ and $\tilde{A}1$ becomes $H M^{-1} B \in \mathbf{GL}_m(\mathbb{R})$, i.e., $\Gamma \in \mathbf{GL}_m(\mathbb{R})$. We revisit system (2) to obtain a simpler representation of the internal dynamics. First, we observe that since $H, M, B$ are constant matrices we have that

$$\phi_2 = V^T \left( I_n - M^{-1} B \Gamma^{-1} H \right) \in \mathbb{R}^{(n-m) \times n}$$

is a constant matrix, where $V \in \mathbb{R}^{n \times (n-m)}$ such that $\text{im} V = \ker H$ and $\Gamma = H M^{-1} B \in \mathbf{GL}_m(\mathbb{R})$. Hence, $\phi_2[x_1, x_2] = 0$ and $U = \mathbb{R}^m$. Furthermore, $\phi_2$ defines a conservative vector field and thus via Remark 3.3 we may choose $\phi_1(x_1) = \lambda \phi_2 \cdot x_1, x_1 \in \mathbb{R}^n$ and $\lambda \in \mathbb{R} \setminus \{0\}$. Moreover, since $h$ is linear we have $V \in \mathbb{R}^{n \times (n-m)}$ such that $\text{im} V = \ker H$. Therefore, via (13) and (15) we obtain

$$x_1 = \begin{bmatrix} H \\ \lambda \phi_2 \end{bmatrix}^{-1} \begin{pmatrix} \xi_1 \\ \eta_1 \end{pmatrix} = M^{-1} B \Gamma^{-1} \xi_1 + \lambda^{-1} V \eta_1,$$

(22)

$$x_2 = \begin{bmatrix} H \\ \phi_2 \end{bmatrix}^{-1} \begin{pmatrix} \xi_2 \\ \eta_2 \end{pmatrix} = M^{-1} B \Gamma^{-1} \xi_2 + V \eta_2.$$ 

(23)

We define $\mathcal{M} := M^{-1} B \Gamma^{-1} \in \mathbb{R}^{n \times m}$ and $\Theta := \phi_2 M^{-1}$. Then, combining the aforementioned observations, equation (17) for the internal dynamics simplify to

$$\dot{\eta}_1(t) = \lambda \eta_2(t),$$

$$\dot{\eta}_2(t) = \Theta f(\mathcal{M} \eta(t) + \lambda^{-1} V \eta_1(t), \mathcal{M} \dot{y}(t) + V \eta_2(t)).$$

where we identified $\xi_1 = y, \xi_2 = \dot{y}$, and the arguments of $f(x_1, x_2)$ have been substituted via (22) and (23), resp. Now, we assume $f$ has the structure

$$f(x_1, x_2) = -K(x_1) - D(x_2),$$

where $K \in \mathcal{C}(\mathbb{R}^n \to \mathbb{R}^n)$ may be considered as a nonlinear restoring force, and $D \in \mathcal{C}(\mathbb{R}^n \to \mathbb{R}^n)$ for example mimics a nonlinear damping. With this we revisit system (2) and obtain the following control system

$$\dot{x}(t) = \left( M^{-1} \left( -K(x_1(t)) - D(x_2(t)) \right) \right) + \left[ \begin{array}{cccc} 0 \\ M^{-1}B \end{array} \right] u(t),$$

$$y(t) = H x_1(t).$$

(24)

We define the vector field $F$ in (25). Then the internal dynamics of (24) are given via

$$\dot{y}(t) = F(\eta_1(t), \eta_2(t), y(t), \dot{y}(t)).$$

(26)

Henceforth let $V$ have orthonormal columns. For $i = 1, 2$ we assume, that there exist $z_i^T > 0$ such that for all $z_i \in Z_i := \{ z \in \mathbb{R}^{-n-m} \mid \|z\| > z_i^T \}$ and all $w \in \mathbb{R}^n$ the functions $K$ and $D$ satisfy the following conditions

There ex. a radially unbounded $V_K \in C^1(Z_1 \to \mathbb{R})$

such that $V_K(z_1) = K(V z_1)^T \Theta^T,$

$$\|K(V z_1) - K(V z_1 + w)\| \leq g_1(w),$$

(27)

$$\|D(V z_2) - D(V z_2 + w)\| \leq g_2(w),$$

(28)

$$\|D(V z_2)\| \leq \delta \|z_2\|^2, \delta > 0, \quad \|D(V z_2)\| \leq \delta \|z_2\|^2, \delta > 0.$$ 

(29)

(30)

(31)

for suitable functions $g_1, g_2 \in \mathcal{C}(\mathbb{R}^m \to \mathbb{R})$, $i = 1, 2$. From this it is clear that conditions (27)-(32) mean that the acting forces are assumed to be basically linear in a certain region, far away from the origin. Hence these are merely
\[ \mathcal{F} : \mathbb{R}^{2(n-m)} \times \mathbb{R}^{2m} \to \mathbb{R}^{2(n-m)} \\
(\zeta_1, \zeta_2, \nu_1, \nu_2) \mapsto \left( \Theta \left( -K(\mathcal{M}\nu_1 + \lambda^{-1}V\zeta_1) - D(\mathcal{M}\nu_2 + V\zeta_2) \right) \right) \] (25)

weak assumptions.

We set \( \tau := \| \Theta \| \) and for some \( r_1, r_2 \geq 0, \ c > 0 \) we define the following constants: \( \bar{K} := \max_{z \in B_{r_1}(0)} g_1(z), \ \bar{D} := \max_{z \in B_{r_2}(0)} g_2(\mathcal{M}z), \ e_1 := c(\lambda^{-2} - \frac{2}{\tau^2}), \ e_2 := \delta - c(d^2 + \lambda), \ E_1 := ct(\bar{K} + d\|M\|r_2), \ E_2 := \tau(\bar{K} + \bar{D}), \) which are all nonnegative making the feasible choice \( 0 < \lambda < 2\epsilon/\tau^2 \) and \( 0 < c < 2\delta/(d^2 + 2\lambda) \). Further, we define

\[ \tilde{Z}_i := \left\{ z \in \mathbb{R}^{n-m} \mid \| z \| > \frac{E_i}{\epsilon_i} \right\}, \ i = 1, 2. \]

As the second main result we present an explicit Lyapunov function for system (26) in the following theorem.

**Theorem 4.6.** Consider system (26) and fix \( r_1, r_2 \geq 0, \ 0 < \lambda < 2\epsilon/\tau^2 \) and \( 0 < c < 2\delta/(d^2 + 2\lambda) \). Assume conditions (27), (28) hold true for all \( z_i \in Z_i, \ i = 1, 2. \) Then for \( \mathcal{V} : \mathbb{R}^{n-m} \times \mathbb{R}^{n-m} \to \mathbb{R} \) defined by

\[ \mathcal{V}(\eta_1, \eta_2) := \frac{1}{2}\| \eta_2 \|^2 + c_\eta^\top \eta_2 + V\kappa(\lambda^{-1}\eta_1) \]

the Lie derivative along the vector field \( \mathcal{F} \) from (26) is nonincreasing for all \( \eta_1 \in B_{r_1}(0), \ \eta_2 \in B_{r_2}(0) \) and all \( \eta_i \in Z_i \cap \tilde{Z}_i, \ i = 1, 2, \) i.e.,

\[ \mathcal{V}'(\eta_1, \eta_2) \cdot \mathcal{F}(\eta_1, \eta_2, y_1, y_2) \leq 0. \] (33)

**Proof.** For \( i = 1, 2 \) let \( \eta_i \in Z_i \) and \( y_i \in B_{r_i}(0), \) and take the Lie derivative of \( \mathcal{V} \) along the vector field \( \mathcal{F} \) from (26)

\[ \mathcal{V}'(\eta_1, \eta_2) \cdot \mathcal{F}(\eta_1, \eta_2, y_1, y_2) \\
= K(\lambda^{-1}V\eta_1)\top \eta_2 + cM_2\eta_2 \\
+ \eta_2^\top \Theta( -K(M\eta_1 + \lambda^{-1}V\eta_1) - D(M_{y_2} + V\eta_2)) \\
+ \epsilon_1^\top \Theta( -K(M\eta_1 + \lambda^{-1}V\eta_1) - D(M_{y_2} + V\eta_2)) \\
\leq \eta_2^\top \Theta( K(\lambda^{-1}V\eta_1) - K(M\eta_1 + \lambda^{-1}V\eta_1)) \\
+ c\lambda\|\eta_2\|^2 \\
- \eta_2^\top \Theta D(M_{y_2} + V\eta_2) \\
- \epsilon_1^\top \Theta K(M\eta_1 + \lambda^{-1}V\eta_1) \\
- \eta_2^\top \Theta D(M_{y_2} + V\eta_2). \] (34)

For purpose of better legibility we set \( \mu := \| M \|, \) and estimate the addends in (34) separately for \( \eta_i \in Z_i, \) resp. Note that since \( V \) from Lemma 3.1 has orthonormal columns we have \( \| Vz \| = \| z \| \) for \( z \in \mathbb{R}^{n-m}. \)

**Step i)**

\[ -\eta_2^\top \Theta D(M_{y_2} + V\eta_2) \\
\leq \epsilon_1\|\eta_2\|^2 + E_1\|\eta_2\|^2 + E_2\|\eta_2\|^2 \]

where \( \epsilon_1, \epsilon_2 > 0 \) and \( E_1, E_2 \geq 0 \) via the choice of \( c \) and \( \lambda. \) Now, we consider the function

\[ W : \mathbb{R}^{2(n-m)} \to \mathbb{R} \\
(w_1, w_2) \mapsto -\epsilon_1\|w_1\|^2 + E_1\|w_1\|^2 - \epsilon_2\|w_2\|^2 + E_2\|w_2\|^2, \] (36)

\[ \mathcal{V}'(\eta_1, \eta_2) \cdot \mathcal{F}(\eta_1, \eta_2, y_1, y_2) \\
\leq -\epsilon_1\|\eta_2\|^2 + E_1\|\eta_2\|^2 - \epsilon_2\|\eta_2\|^2 + E_2\|\eta_2\|^2, \] (35)

where \( \epsilon_1, \epsilon_2 > 0 \) and \( E_1, E_2 \geq 0 \) via the choice of \( c \) and \( \lambda. \) Now, we consider the function

\[ W : \mathbb{R}^{2(n-m)} \to \mathbb{R} \\
(w_1, w_2) \mapsto -\epsilon_1\|w_1\|^2 + E_1\|w_1\|^2 - \epsilon_2\|w_2\|^2 + E_2\|w_2\|^2, \] (36)
with $\varepsilon_i > 0$, $E_i \geq 0$ for $i = 1, 2$ as above. A short calculation yields that for $w_i \in \mathcal{Z}_i$ we have $W(w_1, w_2) \leq 0$. Comparing (35) and (36) yields assertion (33) \[ Y'(\eta_1, \eta_2) \cdot \mathcal{F}(\eta_1, \eta_2, y_1, y_2) \leq W(\eta_1, \eta_2) \leq 0, \]
for all $\eta_i \in Z_i \cap \tilde{Z}_i, i = 1, 2$, and $y_1 \in B_{r_1}(0), y_2 \in B_{r_2}(0)$. \[ \square \]

**Remark 4.7.** The sets $Z_i$ in Theorem 4.6 are determined by the system parameters only and hence conditions (27)–(32) can be verified without decoupling the internal dynamics.

Finally, we combine Theorem 4.3 and Theorem 4.6 to obtain a stability result for the internal dynamics (26) of system (24).

**Theorem 4.8.** Consider system (24) and use the assumptions from Theorem 4.6. Then the internal dynamics (26) are bounded-input, bounded-output stable.

**Proof.** Clear. \[ \square \]

**Remark 4.9.** Since $\eta_i \notin Z_i \cap \tilde{Z}_i$, means $\|\eta_i\| \leq \max(z^+, E_i/\varepsilon_i)$, it is clear that the choice of $\lambda$ and $\varepsilon$ in Theorem 4.6 does not affect the stability statement in Theorem 4.8 but only determines the region where (33) is true.

5. Example

We give an example for the results presented in this paper, in particular to illustrate conditions (27)–(32). To this end, we consider the mass on a car system investigated in [11] and [13]. Consider a car with mass $m_1$ on which a ramp with constant angle $0 < \alpha < \pi/2$ is mounted. A second mass $m_2$ lying on the ramp is coupled to the car via a spring with characteristic $K_2(s)$, and a damping with characteristic $D_2(s)$. A horizontal force can be applied via the input $u = F$. The situation is depicted in Figure 1. For convenience we assume the constant force on $m_2$ due to gravity, namely $m_2 g \sin(\alpha)$, where $g$ is the gravitational constant, to be compensated via a linear coordinate transformation, such that $K_2(0) = 0$. Then, the equations of motion for that system are given by

\[
\begin{pmatrix}
m_1 + m_2 & m_2 \cos(\alpha) \\
m_2 \cos(\alpha) & m_2
\end{pmatrix}
\begin{pmatrix}
\ddot{z}(t) \\
\ddot{s}(t)
\end{pmatrix}
= M \begin{pmatrix} 0 \\ -K_2(s(t)) - D_2(s(t)) \end{pmatrix}
+ \begin{pmatrix} 1 \\ 0 \end{pmatrix} u(t)
\]

\[ y(t) = z(t) + \cos(\alpha)s(t), \]

where the system output $y$ describes the horizontal position of mass $m_2$. In this particular example we have $n = 2$ and $m = 1$, and set $B := [1, 0]^T$, $H := [1, \cos(\alpha)]$. Note that the input and the output are not collocated, i.e., $H \notin B^T$. Let $K_2$ and $D_2$ have the following characteristics, where $\sigma$ denotes the sign function

\[
K_2 : \mathbb{R}^2 \to \mathbb{R}
\]

\[
g \mapsto \begin{cases}
\sigma(q_2)\sqrt{|q_2|} & |q_2| \leq 1 \\
2q_2 - \sigma(q_2) & |q_2| > 1
\end{cases}
\]

and

\[
D_2 : \mathbb{R}^2 \to \mathbb{R}
\]

\[
v \mapsto \begin{cases}
\sigma(v_2)v_2^2 & |v_2| \leq 1 \\
2v_2 - \sigma(v_2) & |v_2| > 1
\end{cases}
\]

Note that $K_2 \in \mathcal{C}(\mathbb{R}^2 \to \mathbb{R})$ and $D_2 \in \mathcal{C}(\mathbb{R}^2 \to \mathbb{R})$. The schematic shapes of $K_2(\cdot)$ and $D_2(\cdot)$ are depicted in Figure 2.

![Figure 1: Mass on a car system. The figure is taken from from [13].](image1.png)

![Figure 2: Schematic shape of $K_2(\cdot)$ and $D_2(\cdot)$, resp. Solid lines on $Z_i$, dashed lines on $R \setminus Z_i$, $i = 1, 2$.](image2.png)
and \( (38) \) is of the form \( (24) \). We set
\[
\gamma := \frac{1}{m_1 + m_2 \sin(\alpha)^2}, \quad m := \frac{m_1 + m_2}{m_2}
\]
and calculate
\[
M^{-1} = \gamma \begin{bmatrix} -\cos(\alpha) & 0 \\ -\cos(\alpha) & m \end{bmatrix}, \quad M^{-1} B = \gamma \begin{bmatrix} -\cos(\alpha) \\ 0 \end{bmatrix}
\]
and thus \( \Gamma = HM^{-1}B = \gamma \sin(\alpha)^2 > 0 \) for \( 0 < \alpha < \pi/2 \).

Therefore, assumption \( (A1) \) is satisfied and thus, using Lemma 2.2, system \( (38) \) has relative degree \( r = 2 \) on \( \mathbb{R}^4 \).

We calculate \( V \) such that \( \text{im} V = \ker H \) and \( V^\top V = I_{2-1} \)
\[
V = \frac{1}{\sqrt{1+\cos(\alpha)^2}} \begin{bmatrix} -\cos(\alpha) \\ 1 \end{bmatrix}.
\]

Therefore, we obtain \( Z_i = \{ z \in \mathbb{R} \mid |z| > \nu \}, \ i = 1, 2 \) for \( \nu := \sqrt{1+\cos(\alpha)^2} > 0 \). Then, \( \phi_2 \) is given via
\[
\phi_2 = \frac{1}{\nu} [-\cos(\alpha), 1] (I_2 - \gamma \begin{bmatrix} -\cos(\alpha) \\ 1, \cos(\alpha) \end{bmatrix} \gamma^{-1}[1, \cos(\alpha)])
\]
and thus
\[
\Theta := \phi_2 M^{-1} = \frac{\nu}{\sin(\alpha)^2} \begin{bmatrix} 0, 1 \end{bmatrix}.
\]

Now, we validate conditions \( (27) \)–\( (32) \) step by step. Consider \( V_K : Z_1 \to \mathbb{R} \) defined by
\[
V_K : Z_1 \to \mathbb{R}, \ z_1 \mapsto \frac{1}{m_2 \sin(\alpha)^2} (z_1^2 - \nu |z_1|)
\]
which is radially unbounded. Note that \( Z_1 = \mathbb{R} \setminus [-\nu, \nu] \) and hence \( V_K \in C^1(Z_1 \to \mathbb{R}) \). Then for \( z_1 \in Z_1 \) the derivative of \( V_K \) is given by
\[
V'_K(z_1) = \frac{1}{m_2 \sin(\alpha)^2} (2z_1 - \nu \sigma(z_1))
\]
\[
= \frac{\nu}{m_2 \sin(\alpha)^2} [0, 2 z_1/\nu - \sigma(z_1)] [0 \ 1]
\]
\[
= K(V z_1)^\top \Theta^T
\]
for \( z_1 \in Z_1 \) and thus \( (27) \) is satisfied. Furthermore,
\[
\|K(V z_1) - K(V z_1 + w)\|
\]
\[
= \|\begin{bmatrix} 2 \z_3/\nu - \sigma(z_1) \\ 0 \end{bmatrix} - \begin{bmatrix} 2 \z_3/\nu - \sigma(z_1) + w_1 \\ w_2 \end{bmatrix}\| = \|w\|
\]
which proves \( (28) \) and
\[
z_1 \Theta K(V z_1) = z_1 \frac{\nu}{m_2 \sin(\alpha)^2} [0, 1] \begin{bmatrix} 0 \\ 2 \z_3/\nu - \sigma(z_1) \end{bmatrix}
\]
\[
= \frac{\nu}{m_2 \sin(\alpha)^2} [2 \z_3/\nu - |z_1|] = \frac{1}{m_2 \sin(\alpha)^2} z_1^2
\]

for \( z_1 \in Z_1 \) shows \( (29) \). Conditions \( (30) \) and \( (31) \) for \( D \) follow analogously for \( z_2 \in Z_2 = Z_1 \). For \( (32) \) consider
\[
\|D(V z_2)\|^2 = \|\begin{bmatrix} 2 \z_3/\nu - \sigma(z_2) \\ 0 \nu, \nu \end{bmatrix} \| = |2 \z_3/\nu - \sigma(z_2)|^2 
\]
\[
= \frac{2 \z_3/\nu - \sigma(z_2)}{\nu} \leq 2 |z_2| + 1 \leq \frac{3}{\nu} |z_2|, \ \ z_2 \in Z_2,
\]
which shows \( (32) \). Therefore, via Theorem 4.8 we may deduce bounded-input, bounded-output stability of the internal dynamics of the mass on a car system \( (37) \).

6. Conclusion

In the present article we elaborated two main results. First, we presented a suitable set of coordinates to represent the internal dynamics of a multibody system. This representation is completely determined by the system parameters and therefore it is not necessary to compute the Byrnes-Isidori form explicitly. Second, we gave an abstract stability result for control systems and derived sufficient conditions on the system parameters of a multibody system such that its internal dynamics are bounded-input, bounded-output stable. We highlight that the conditions can be verified beforehand and hence an explicit decoupling of the internal dynamics is not necessary.

Now, further research aims to achieve similar results for multibody systems with a more complex function \( f \), a state dependent mass matrix, such as e.g. a robotic manipulator arm with a passive joint, and systems with algebraic constraints such as e.g. systems with a kinematic loop.

Acknowledgements. I thank Thomas Berger (University of Paderborn) for many helpful discussions, suggestions and corrections.

References

[1] A. Ilchmann, E. P. Ryan, C. J. Sangwin, Tracking with prescribed transient behaviour, ESAIM: Control, Optimisation and Calculus of Variations 7 (2002) 471–493.
[2] T. Berger, H. H. Lê, T. Reis, Funnel control for nonlinear systems with known strict relative degree, Automatica 87 (2018) 345–357. doi:10.1016/j.automatica.2017.10.017
[3] T. Berger, Tracking with prescribed performance for linear nonminimum phase systems, Automatica 115 (2020) 108909. doi:10.1016/j.automatica.2020.108909
[4] T. Berger, L. Lanza, Output tracking for a non-minimum phase robotic manipulator, submitted for publication. Available at arXiv:https://arxiv.org/abs/2001.07535 (2020).
[5] R. Seifried, Integrated mechanical and control design of underactuated multibody systems, Nonlinear Dynamics 67 (2) (2012) 1539–1557.
[6] R. Seifried, Dynamics of Underactuated Multibody Systems. Modeling, Control and Optimal Design. no. 205 in Solid Mechanics and Its Applications, Springer-Verlag, 2014.
[7] C. I. Byrnes, J. C. Willems, Adaptive stabilization of multivariable linear systems, in: Proc. 23rd IEEE Conf. Decis. Control, 1984, pp. 1574–1577.
[8] C. I. Byrnes, A. Isidori, A frequency domain philosophy for nonlinear systems, with application to stabilization and to adaptive control, in: Proc. 23rd IEEE Conf. Decis. Control, Vol. 1, 1984, pp. 1569–1573.

[9] H. K. Khalil, A. Saberi, Adaptive stabilization of a class of nonlinear systems using high-gain feedback, IEEE Trans. Autom. Control 32 (1987) 1031–1035.

[10] D. E. Miller, E. J. Davison, An adaptive controller which provides an arbitrarily good transient and steady-state response, IEEE Trans. Autom. Control 36 (1) (1991) 68–81.

[11] R. Seifried, W. Blajer, Analysis of servo-constraint problems for underactuated multibody systems, Mech. Sci. 4 (2013) 113–129.

[12] T. Berger, T. Reis, Funnel control via funnel pre-compensator for minimum phase systems with relative degree two, IEEE Trans. Autom. Control 63 (7) (2018) 2264–2271. doi:10.1109/TAC.2017.2761020.

[13] T. Berger, S. Otto, T. Reis, R. Seifried, Combined open-loop and funnel control for underactuated multibody systems. Nonlinear Dynamics 95 (2019) 1977–1998. doi:10.1007/s11071-018-4672-5.

[14] A. Ilchmann, E. P. Ryan, High-gain control without identification: a survey, GAMM Mitt. 31 (1) (2008) 115–125.

[15] A. Ilchmann, F. Wirth, On minimum phase, Automatisierungstechnik 12 (2013) 805–817.

[16] A. Isidori, Nonlinear Control Systems, 3rd Edition, Communications and Control Engineering Series, Springer-Verlag, Berlin, 1995.

[17] T. Berger, T. Reis, The Funnel Pre-Compensator, Int. J. Robust Nonlinear Control 28 (16) (2018) 4747–4771. doi:10.1002/rnc.4281.

[18] D. Chowdhury, H. K. Khalil, Funnel control for nonlinear systems with arbitrary relative degree using high-gain observers, Automatica 105 (2019) 107–116. doi:10.1016/j.automatica.2019.03.012.

[19] J. LaSalle, Some extensions of Liapunov’s second method, IRE Transactions on Circuit Theory 7 (4) (1960) 520–527. doi:10.1109/tct.1960.1086720.

[20] E. D. Sontag, Mathematical Control Theory: Deterministic Finite Dimensional Systems, 2nd Edition, Springer-Verlag, New York, 1998.