DIQUARK CONDENSATION IN TWO COLOUR QCD

R. ALOISIO AND A. GALANTE
*D*L
Dipartimento di Fisica dell’Università di L’Aquila, 67100 L’Aquila, Italy
E-mail: aloisio.galante@lngs.infn.it

V. AZCOITI
Departamento de Física Teórica, Facultad de Ciencias, Universidad de Zaragoza,
50009 Zaragoza, Spain
E-mail: azcoiti@azcoiti.unizar.es

G. DI CARLO
Istituto Nazionale di Fisica Nucleare, Laboratori Nazionali di Frascati, P.O.B. 13 -
00044 Frascati, Italy
E-mail: gdicarlo@lnf.infn.it

A. F. GRILLO
Istituto Nazionale di Fisica Nucleare, Laboratori Nazionali del Gran Sasso, 67010
Assergi (L’Aquila), Italy
E-mail: grillo@lngs.infn.it

Unquenched lattice SU(2) is studied at nonzero chemical potential in the strong
coupling limit. The topic of diquark condensation is addressed analyzing the prob-
ability distribution function of the diquark condensate. We present results at zero
external source without using any potentially dangerous extrapolation procedure.
We find strong evidences for a (high density) second order phase transition where
a diquark condensate appears, and show quantitative agreement of lattice calcula-
tions with low-energy effective Lagrangian calculations.

1 Introduction

Recently the standard scenario for the phase diagram of QCD in thechemi-
cal potential-temperature plane has changed; besides the standard hadronic
and quark-gluon plasma phases, the existence of a new state of matter has
been claimed by several groups
[1]. This new phase is characteristic of the high
density-low temperature regime; the asymptotic freedom of QCD and the
known instability of large Fermi spheres in presence of (whatever weak) at-
tractive forces results in a pairing of quarks of higher momenta (in analogy
with the Cooper pairing in solid state systems at low temperature). A con-
densation of quark pairs should be the distinctive signal of the new phase and

* TALK PRESENTED BY A. GALANTE

canton: submitted to World Scientific on November 30, 2021
has been indeed predicted using simplified phenomenological models of QCD.

Unfortunately the lattice approach, the most powerful tool to perform first principles non perturbative studies, is affected in the case of finite density QCD by the well known sign problem that has prevented until now any step towards the understanding of this new physics. This is not the case for the SU(2) theory where the fermions are in the pseudo-real representation and finite density numerical simulations are feasible.

In the following we present a detailed study of diquark condensation for the unquenched two colours model. This work follows a previous one where we considered chiral and diquark susceptibilities and found strong evidence for a phase transition separating the ordinary low density phase from a high density one where chiral symmetry is restored and the baryon number symmetry is broken. In the present work we mainly focus on an approach, based on the analysis of the probability distribution function (p.d.f.) of the order parameter, to extract the value of the order parameter directly at zero external source without using any potentially dangerous extrapolation procedure, implicit in the standard approach.

The study of diquark condensation in two colours QCD is an interesting topic by itself and, notwithstanding the relevant differences between SU(2) and SU(3) at non zero baryon density, we can hope to use some of the results to get insights about the three colours case. Even if the phase diagram of SU(3) and SU(2) are expected to share similarities, we have to be aware of the differences between the two models.

- The quark-quark condensate $\langle qq \rangle$ is coloured for the $N_c = 3$ theory and colourless for the $N_c = 2$ one. This implies that in the SU(3) case the diquark condensation has to be interpreted as an Higgs-like mechanism leading to the breaking of the local colour symmetry, while in the SU(2) case we have a standard spontaneous symmetry breaking (SSB) of the $U_V(1)$ symmetry associated with conservation of baryon number.

- The zero temperature critical chemical potential, related to the mass of the lightest baryonic state, is different in the two cases: it is $1/3$ of the nucleon mass for SU(3) and $1/2$ of the pion mass for SU(2) (i.e. zero at vanishing quark mass). For $N_c = 2$, in order to have a phase diagram similar to the SU(3) one, it is appropriate to consider a non zero bare quark mass.

- The lightest baryonic state is a fermion for the tree colours model and a boson for the two colours one.

Even if our results can not be directly extended to the SU(3) case we can
use lattice calculations to do a highly non trivial, quantitative check of the continuum predictions for the SU(2) theory. In particular we have considered results coming from low energy effective Lagrangian calculations.

In the next section we present some details of the algorithm. The last section is devoted to the presentation and discussion of numerical results.

2 Overview of the numerical technique

2.1 Simulation scheme

The standard way to study SSB is to introduce first an explicit symmetry breaking term in the action. If we do that for the diquark in the SU(2) model we have to add a term \( j(\psi\tau_2\psi + \bar{\psi}\tau_2\bar{\psi}) \) and, after integrating the Grassmann field, the fermionic contribution for \( N_f = 4 \) quark flavours becomes proportional to the Pfaffian of a \( 4V \times 4V \) matrix:

\[
Z_{\text{ferm}}(j) = Pf(B + j) = \pm \sqrt{\det(B + j)} \quad (1)
\]

where \( B \) is

\[
B = \begin{pmatrix} 0 & 1/2\Delta \tau_2 \\ -1/2\Delta^* \tau_2 & 0 \end{pmatrix}. \quad (2)
\]

and \( \Delta \) is the usual lattice Dirac operator (it contains the mass and \( \mu \) dependence). Using the relation \( \tau_2\Delta \tau_2 = \Delta^* \) we can easily prove that \( B \) is antihermitian and \( \det(B + j) \geq 0 \) for any \( j \).

It can also be shown that the eigenvalues of \( B \) are doubly degenerate and \( B^2 \) is in a block diagonal form with two hermitian blocks on the diagonal having the same eigenvalues. It follows that to compute \( \det(B + j) \) for any \( j \) (i.e. to obtain all the eigenvalues of \( B \)) it is sufficient to diagonalize only one block of \( B^2 \) (reducing the problem to the diagonalization of a \( 2V \times 2V \) hermitian matrix) and then take the two pure imaginary square roots of the (real and negative) eigenvalues.

To avoid the sign ambiguity in (1) it is customary to consider a theory with \( N_f = 8 \) quark flavours where the fermionic partition function becomes \( Z_{\text{ferm}}(j) = \det(B + j) \). On the other side if we are able to work directly at zero diquark source things become much simpler. In the \( j = 0 \) limit the sign ambiguity disappears and the Pfaffian is positive definite since \( Pf(B) \equiv \det \Delta \) and the last quantity is real and positive for any value of \( \mu \). Then we can easily consider any value of \( N_f \) writing \( Z_{\text{ferm}}(j = 0) = (\det(B))^{N_f/8} \). In the next section we show how it is possible to take advantage of this feature and extract the value of the order parameter for the diquark condensation for all \( N_f \) without any extrapolation to \( j = 0 \).
We have studied the phase structure of the theory in the limit of infinite
gauge coupling ($\beta = 0$). To simulate the $\beta = 0$ limit of the theory we have
measured fermionic observables on gauge configurations generated randomly,
i.e. with only the Haar measure of the gauge group as a weight. This choice
implies a Gaussian distribution of the plaquette energy around zero which,
according to the results of Morrison and Hands\cite{4}, has a net overlap with the
importance sample of gauge configurations at the values of $\mu$ and $m$ used in our
calculations. The validity of this procedure has also been tested comparing
different physical observables (number density and chiral condensate) with
Hybrid Montecarlo results\cite{5}.

We have considered the theory in a $4^4$ and $6^4$ lattice diagonalizing 300
gauge configurations in the first lattice volume and 100 in the second one.
As we pointed out in the introduction, to stay closer to the SU(3) case the
simulations have been performed at non zero quark mass. We chose $m = 0.025, 0.05, 0.20$ and values of the chemical potential ranging from $\mu = 0$ to
$\mu = 1.0$.

All numerical simulations have been performed on a cluster of Pentium
II and PentiumPro at the INFN Gran Sasso National Laboratory.

2.2 Analysis of the probability distribution function

The use of the p.d.f. to analyze the spontaneous symmetry breaking in spin
system or Quantum Field Theories with bosonic degrees of freedom is a stan-
dard procedure. Less standard is its application to QFT with Grassmann
fields where, for obvious reasons, the fermionic degrees of freedom have to
be integrated analytically. Nevertheless this method has been developed to
extract the chiral condensate in the chiral limit from simulations of QFT with
fermions\cite{6}.

The same ideas can be used to study the vacuum structure of two colour
QCD at non zero density and specifically to extract the diquark condensate
at $j = 0$. We refer to the original paper for a full description of the p.d.f.
technique and present a brief introduction focusing on the peculiarities of the
diquark condensate case.

Let $\alpha$ be an index which characterizes all possible (degenerate) vacuum
states and $w_\alpha$ the probability to get the vacuum state $\alpha$ when choosing ran-
domly an equilibrium state. If $c_\alpha$ is the order parameter in the $\alpha$ state we
can write

$$
c_\alpha = \langle \frac{1}{V} \sum_{x} \psi \tau_2 \psi(x) + \bar{\psi} \tau_2 \bar{\psi}(x) \rangle_\alpha
$$

(3)

where $V$ is the lattice volume and the sum is over all lattice points. $P(c)$, the

p.d.f. of the diquark order parameter \( c \), will be given by

\[
P(c) = \sum_{\alpha} w_{\alpha} \delta(c - c_{\alpha})
\]

\[
= \lim_{V \to \infty} \frac{1}{Z} \int [dU][d\bar{\psi}][d\psi] e^{-S_G(U)} + \bar{\psi}\Delta\psi \delta \left( \frac{1}{V} \sum_x \psi \tau_2 \psi(x) + \bar{\psi} \tau_2 \bar{\psi}(x) - c \right)
\]

The main point is that while \( P(c) \) is not directly accessible with a numerical simulation its Fourier transform

\[
P(q) = \int dce^{iqc} P(c)
\]

can be easily computed. Inserting in (4) the definition of \( P(c) \) and using an integral representation for the \( \delta \)-function, after some algebra we arrive at the following formula

\[
P_V(q) = \frac{1}{Z(j = 0)} \int [dU] e^{-S_G(U)} PfB(j = \frac{iq}{V}) \left[ PfB(j = 0) \right]^{Nf_4} (5)
\]

where \( P_V(q) \) is the Fourier transformed p.d.f. of the diquark order parameter at zero external source and finite volume.

To take advantage of (5) we should be able to compute correctly the Pfaffians involved. This indeed turns out to be easy. First note that, as previously stated, \( PfB(j = 0) \) is a real and positive quantity so that the only ambiguity is related to the sign of \( PfB(j = \frac{iq}{V}) = \pm \sqrt{\text{det} B(j = iq/V)} \). If we remember that the \( 4V \) eigenvalues of the antihermitian matrix \( B \) are doubly degenerate and can be written as \( \pm i\lambda_n \) \((n = 1, \cdots , V)\) with real and positive \( \lambda_n \) we arrive at the following expression

\[
PfB(j = \frac{iq}{V}) = \pm \prod_{n=1}^{V} \left( \lambda_n^2 - \frac{q^2}{V^2} \right)
\]

(6)

The sign ambiguity can now be solved noticing that (5) has to be positive at \( j = 0 \) and, increasing \( q \), changes sign each time \( j \) is equal to one of the eigenvalues. The only possible exception would be an (accidental) extra degeneracy in the eigenvalues of \( B \) resulting in the Pfaffian not crossing zero but tangent to the horizontal axis. This situation never occurred in our simulations.

Once we have the \( P_V(q) \) and hence, by simple Fourier transform, the p.d.f. of the order parameter we need to extract the correct value for the order parameter. To do that we first have to recall that diquark condensate is the order parameter of the \( U_V(1) \) symmetry associated with baryon number conservation. Indeed \( \langle \psi \tau_2 \psi + \bar{\psi} \tau_2 \bar{\psi} \rangle \) and \( i(\psi \tau_2 \psi - \bar{\psi} \tau_2 \bar{\psi}) \) are the components...
of a vector (in a plane) which rotates by an angle $2\theta$ when we do a global phase transformation of parameter $\theta$ on the fermionic fields. Therefore, if $c_0$ is the vacuum expectation value of the diquark condensate corresponding to the $\alpha$-vacuum selected when switching-on a diquark source term, $P(c)$ can be computed as

$$P(c) = \frac{1}{2\pi} \int d\alpha \delta(c - c_0 \cos(2\alpha))$$  \hspace{1cm} (7)$$

which gives $P(c) = 1/(\pi(c_0^2 - c^2)^{1/2})$ for $-c_0 \leq c \leq c_0$ and $P(c) = 0$ otherwise. In the symmetric phase $c_0 = 0$ and $P(c)$ reduces correctly to a $\delta$-function in the origin.

The above results are valid in the thermodynamic limit while, at finite volume, the non analyticities of the p.d.f. are absent. Without entering in the details of a finite size scaling analysis we expect, for the finite volume p.d.f. $P_V(c)$, a function peaked in the origin in the symmetric phase and peaked at some non zero value in the broken phase. This is indeed the behaviour we can observe in fig. 1 where the p.d.f. of the smallest volume is reported at different values of $\mu$. It is clear that, increasing the chemical potential, the vacuum starts to be degenerate signalling a spontaneous breaking of the baryon number conservation.

We can also compare the $P_V(c)$ for two lattice volumes in the symmetric and broken phase. This is done in fig. 2 where we see clearly as, increasing the volume, the peak of the p.d.f. becomes sharper. To determine the value of the diquark condensate we used the position of the peak: a definition that
clearly converges to the correct value in the thermodynamic limit.

From fig. (2) we see also that data of the larger volume are more noisy (indeed we also get negative values for the p.d.f. in the broken phase). To have an estimate of the errors we calculated several distribution functions for the $6^4$ lattice using independent subsets of our data. We saw that the position of the peak was very stable, within 1 percent in the broken phase, and we used this quantity for the errorbars.

![Figure 2. P.d.f. for $4^4$ (dashed line) and $6^4$ (continuous line) in the symmetric (left) and broken (right) phase.](image)

3 Results

Here we present the results for the diquark condensate at $j = 0$ as a function of the chemical potential. Fig. 3 contains our data points for the $6^4$ volume for several values of $N_f$ at two different masses: $m = 0.025$ and $m = 0.2$.

First we can derive a peculiar phase diagram: two symmetric phases separated by a broken one and two possibly continuous transition points. This result is not new and has already been predicted by Mean Field calculations as well as numerical calculations. The high density symmetric phase has no physical relevance, it is simply the consequence of the saturation of all lattice sites with quarks. This saturation phenomena has nothing to do with continuum physics and is a pure lattice artifact. The physically interesting phase transition, i.e. the transition that has a continuum counterpart, is the first one.

To test our numerical results we have considered the analytical predictions of low energy effective Lagrangian. The $j = 0$ case is described by a simple formula where the only two parameters are the chiral condensate and one half the pion mass calculated at $j = 0, \mu = 0$. We did independent simulations
Figure 3. Diquark condensate as a function of $\mu$ for $6^4$ lattice (points) and the continuum predictions (line) for $m = 0.025$ (left) and $m = 0.2$ (right).

Table 1. Parameters for the low energy effective Lagrangian predictions.

| $m$    | $\langle \bar{\psi}\psi \rangle$ | $m_\pi/2$ |
|--------|-----------------------------------|-----------|
| 0.025  | 1.31(2)                           | 0.1696(11) |
| 0.2    | 1.23(2)                           | 0.4841(7)  |

To calculate these quantities at the same values of $m$ used in the previous analysis, obtaining the values in Table 1. Another prediction of the continuum calculation is that the value of the diquark condensate is independent of the flavour number.

The continuous line in fig. 3 is the analytical prediction for this model. We see a remarkable agreement with our strong coupling results up to values of the chemical potential where the saturation effects start to be relevant. We see also that the $N_f$ dependence in the simulation points is small.

We checked explicitly the low energy effective Lagrangian prediction at finite $\beta$ too. In that case we used standard techniques to compute chiral and diquark condensate as well as the number density in a $N_f = 8$ simulation (fig. 4). Also in that case (where no phase transition is present and the predictions are still independent of $N_f$) the agreement is very good up to values of $\mu$ where the number density becomes a consistent fraction of its maximum lattice value (i.e. $\mu/m_\pi \simeq 0.6$).

It is surprising that a $\beta = 0$ calculation has a so good agreement with a continuum prediction. Since the analytical predictions are, at the values of $\mu$ presented, well inside the validity region of the low energy approximation ($\mu <<$ the mass of the first non goldstone excitation) we can conclude that this gives an indication of a very small dependence of lattice results on $\beta$.
Figure 4. Low energy Lagrangian predictions (lines) and our $N_f = 8$ data (symbols) for chiral condensate, diquark condensate and number density at $m = 0.2$ and $j = 0.1m$ vs. $\mu/m_\pi$. $6^4$ data are squares and circles, $4^4$ data are diamonds.

Acknowledgments

This work has been partially supported by CICYT (Proyecto AEN97-1680) and by a INFN-CICYT collaboration. The Consorzio Ricerca Gran Sasso has provided part of the computer resources needed for this work.

References

1. M. Alford, K. Rajagopal and F. Wilczek, Phys. Lett. B 422, 247 (1998); Nucl. Phys. B 537, 443 (1999); R. Rapp, T. Schaefer, E.V. Shuryak and M. Velkovsky, Phys. Rev. Lett. 81, 53 (1998); T. Schafer and F. Wilczek, Phys. Rev. Lett. 82, 3956 (1999).
2. R. Aloisio, V. Azcoiti, G. Di Carlo, A. Galante and A.F. Grillo, preprint DFTUZ 2000/03.
3. J.B. Kogut, M.A. Stephanov, D. Toublan, J.J.M. Verbaarschot and A. Zhitnitsky, hep-ph/0001171.
4. S. Morrison and S. Hands, in Strong and Electroweak Matter 98, Copenhagen, Dec. 1998, hep-lat/9902012.
5. R. Aloisio, V. Azcoiti, G. Di Carlo, A. Galante and A.F. Grillo, Nucl. Phys. B 564, 489 (2000).
6. V. Azcoiti, V. Lalena and X.Q. Luo, Phys. Lett. B 354, 111 (1995).
7. E. Dagotto, A. Moreo and U. Wolff, Phys. Lett. B 186, 395 (1987).