THREE-DIMENSIONAL $\mathcal{N}=4$ SUPERCONFORMAL
SUPERFIELD THEORIES

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Abstract

The mirror map in the $D=3,\mathcal{N}=4$ supersymmetry connects the left and right $SU(2)$ automorphism groups and also the superfield representations of the corresponding $\mathcal{N}=4$ supermultiplets. The mirror $\mathcal{N}=4$ harmonic superspaces use the harmonics of two $SU(2)$ groups and two types of the Grassmann analyticity. The irreducible left and right $\mathcal{N}=4$ supermultiplets are defined in these harmonic superspaces. We analyze the $\mathcal{N}=4$ superconformal interactions of the gauge and matter superfields and the spontaneous breakdown of the superconformal symmetry. The most interesting superconformal action possesses the mirror symmetry and contains two nonlinear terms of the abelian left and right gauge superfields, and also the mixing $\mathcal{N}=4$ $BF$ interaction which yields the topological masses of the gauge fields and the nontrivial interaction of the scalar and pseudoscalar fields. The minimal interactions of the left and right $\mathcal{N}=4$ hypermultiplets can be included to this abelian gauge theory. We consider also the nonlinear $\mathcal{N}=4$ gauge superfield interactions.

Keywords: Harmonic superspace, extended supersymmetry, superconformal symmetry

1 Introduction

Three-dimensional superconformal field theories with $\mathcal{N}=6$ and $\mathcal{N}=8$ supersymmetries describe worldvolume of the $M2$-branes at low energies [1, 2, 3]. The Lagrangians of these theories contains matter the scalar and spinor fields interacting with the Chern-Simons gauge vector fields for a specific choice of the gauge groups. The three-dimensional $\mathcal{N}=4$ Chern-Simons models were studied in the $\mathcal{N}=1$ superspace [4]. The algebras of higher supersymmetries in these models close on the mass shell only.

The manifestly supersymmetric description of the three-dimensional Chern-Simons-matter systems is possible for $\mathcal{N} \leq 3$ supersymmetries in the framework of the superspace approach [5],[6], [7],[8]. These superfield constructions are universal for any gauge groups. The $P$-parity is preserved in the gauge group $G \times G$ if we choose the difference of two superfield Chern-Simons actions corresponding to each $G$.

The $\mathcal{N}=6$ Chern-Simons-matter model for the gauge group $U(N) \times U(N)$ ($ABJM$-model) and the $\mathcal{N}=8$ BLG-model for the gauge group $SU(2) \times SU(2)$ were investigated in the $\mathcal{N}=3$ harmonic superspace [9]. The $\mathcal{N}=3$ supersymmetry is manifest in this formalism, the higher supersymmetry transformations connect different superfields and the corresponding algebra of transformations closes on the mass shell.

The superfield description of the $\mathcal{N}=6$ Chern-Simons theory is possible in the $SO(5)/U(2)$ or $SO(6)/U(3)$ harmonic superspaces [14], [15]. In the first version, the $\mathcal{N}=6$ supersymmetry is realized on the $\mathcal{N}=5$ superfields. In the second version, the $SU(3)$ triplet of the harmonic $\mathcal{N}=6$ gauge superfields contains the Chern-Simons vector
field and the infinite number of auxiliary fields, for instance, the unusual fermion field with three spinor Lorentz indices and the $SO(6)$ indices. All auxiliary fields vanish on the mass shell in these variants of the Chern-Simons theory, however, we do not know how to include the superfield matter interaction.

The $D = 3, \mathcal{N} = 8$ Yang-Mills theory for the group $U(N)$ can be constructed in the $\mathcal{N} = 4$ harmonic superspace by the analogy with the four-dimensional case [11], however, this theory is not superconformal in three dimensions. In this paper, we analyze the possible constructions of the $\mathcal{N} = 4$ superconformal models in the three-dimensional harmonic superspace [8],[13].

The $D = 3, \mathcal{N} = 4$ superspace is covariant with respect to the Lorentz group $SO(2,1) \sim SL(2,R)$ and the automorphism group $SU_L(2) \times SU_R(2)$. The important property of the $\mathcal{N} = 4$ superspace is the discrete symmetry with respect to the mirror map

$$\mathcal{M} : \quad SU_L(2) \leftrightarrow SU_R(2)$$

which connects the representations $(r,l)$ and $(l,r)$ of the group $SU_L(2) \times SU_R(2)$. Different coordinate bases of the $\mathcal{N} = 4$ superspace are defined in appendix, for instance, the central basis (CB) is invariant under the $\mathcal{M}$-map. The irreducible $D = 3, \mathcal{N} = 4$ supermultiplets correspond to different superfield constraints, and the mirror map connects the left and right versions of these constraints. Respectively, we have the left and right hypermultiplets or the left and right vector multiplets [13].

The left and right versions of the $\mathcal{N} = 4$ harmonic superspace use the corresponding harmonics $u^\pm_k$ or $v^{(\pm)}_a$. In the appendix we define the left analytic basis (LAB) and the mirror right analytic basis (RAB) using left and right conditions of the Grassmann analyticity. The left and right analytic superfields describe the mirror irreducible supermultiplets. In the next section, we consider the superconformal transformations in different representations of the $\mathcal{N} = 4$ superspace.

Section 3 is devoted to the superconformal interactions of the $\mathcal{N} = 4$ gauge superfields. The left abelian gauge prepotential is defined as the analytic scalar superfield $V_0^{++}$ in the left harmonic superspace. The corresponding pseudoscalar superfield strength $W_{ab}$ satisfies the constraints of the right tensor multiplet. The abelian superfield $W = \sqrt{W_{ab}W^{ab}}$ plays the role of the dilaton superfield which helps us to construct the superconformal actions in the $\mathcal{N} = 4$ superspace. We use $W$ as a dynamical coupling constant in the superconformal version of the $\mathcal{N} = 4$ abelian superfield gauge action $S_W^0$ [13] and also in the superconformal interaction of the $\mathcal{N} = 4$ nonabelian gauge superfields.

It is not difficult to obtain the right analytic gauge superfield $A_0^{(++)}$ and the corresponding scalar left superfield strength $L^{kl}$ using the mirror map. The mirror abelian superconformal action $S_L^0 (A_0^{(++)})$ contains the second dilaton $L = \sqrt{L^{kl}L_{kl}}$.

The superconformal interactions of the left hypermultiplet and the improved tensor multiplet are studied in Sec. 4 by the analogy with the corresponding interactions of the $D = 4, \mathcal{N} = 2$ supermultiplets. The left $D = 3, \mathcal{N} = 4$ hypermultiplet has the minimal interaction only with the left vector multiplet. It is shown that the superconformal abelian action $S_L^0$ is equivalent to the analytic action of the improved tensor multiplet, which is dual to the free action of the left hypermultiplet.

In Sec. 5 we consider the $\mathcal{N} = 4$ superfield terms generalizing the Dirac-Born-Infeld interactions with the derivatives of the vector and scalar fields. These terms are invariant under the nonlinear transformations of the $\mathcal{N} = 8$ supersymmetry.
We analyze the $N = 4$ superconformal abelian $BF$ term $S_{BF}^0$ which connects the left $U_L(1)$ scalar gauge superfield $V_0^+$ and the right $U_R(1)$ pseudoscalar gauge superfield $A_0^{++}$ [18]. This term was considered earlier in components [16], [17]. Note that the $N = 4$ action $S_{BF}^0$ can be rewritten as the difference of two abelian Chern-Simons terms in the $N = 3$ superfield formalism [9], in this case the transformations of parity and the fourth supersymmetry connect the mirror abelian gauge superfields defined in the single $\bar{N} = 3$ analytic superspace.

It is not possible to present the action $S_{BF}^0$ as the difference of two actions in the $\bar{N} = 4$ superspace, so we consider it as the $\mathcal{M}$-symmetric analog of the Chern-Simons action for the group $U_L(1) \times U_R(1)$. The independent interactions of the mirror $\bar{N} = 4$ hypermultiplets and the corresponding gauge multiplets can be easily included to this picture. The term $S_{BF}^0$ yields the specific superconformal interaction of the left and right $\bar{N} = 4$ abelian gauge superfields in the composite $\mathcal{M}$-symmetric action $S^W + S^L_0 + S_{BF}^0$. This action describes the nontrivial interactions of the left pseudoscalar and right scalar fields with the topologically massive vector and pseudovector fields in the bosonic sector. It is not possible to formulate the non-abelian Chern-Simons models in the $D = 3, \bar{N} = 4$ superspace.

2 Superconformal $D = 3, \bar{N} = 4$ transformations

The $CB$ coordinates of the $\bar{N} = 4$ superspace $z^M = (x^m, \theta^a_k)$ are defined in appendix. The infinitesimal $\bar{N} = 4$ superconformal transformations of these coordinates have the form

\[
\delta x^{\alpha \beta} = \lambda^{\alpha \beta} = \lambda^{\alpha \beta} + a^{\alpha \beta} x^\gamma + a^{\alpha \beta} x^\gamma - i e^{\alpha \beta} \partial_{\lambda} - i e^{\alpha \beta} \partial_{\lambda} + \frac{1}{2} x^{\alpha \beta} k_{\gamma} x^\gamma + \frac{1}{8} \Theta^{2 \gamma} k_{\alpha \beta} + \frac{1}{4} x^{\alpha \beta} \eta_{\gamma} x^\gamma - \frac{1}{2} \eta_{\gamma} k_{\alpha \beta} - \frac{1}{2} \eta_{\gamma} k_{\alpha \beta} + \frac{1}{2} x^{\alpha \beta} k_{\gamma} k_{\gamma} + \frac{1}{8} \Theta^{2 \gamma} k_{\alpha \beta},
\]

\[
\delta \theta^a_k = \Theta^a_k = \Theta^a_k + a^{\alpha \beta} \eta_{\gamma} x^\gamma - \frac{1}{2} \eta_{\gamma} k_{\alpha \beta} - \frac{1}{2} \eta_{\gamma} k_{\alpha \beta} + \frac{1}{2} x^{\alpha \beta} \eta_{\gamma} x^\gamma + \frac{1}{8} \Theta^{2 \gamma} k_{\alpha \beta},
\]

where $c^m, a^m, k_m, b$ are the parameters of the conformal group $SO(3, 2), \omega_{kl}$ and $\Omega_{ab}$ are the parameters of the group $SU_L(2) \times SU_R(2), c^a_k$ and $\eta_k^a$ describe the $Q$ and $S$ supersymmetries, and $\Theta = \theta^a_k \theta^a_k$.

The standard notation of this superconformal group is $OSp(4|4)$, but we sometimes use the short notation $SC$. The superconformal transformation of the full superspace integral measure $\delta_{sc} d^{11} z = j(z) d^{11} z$ contains the superfield parameter

\[
j(z) = \partial_m \lambda^m - \partial^a_k \theta^a_k = -b - k_m x^m - i \Theta^{ka} \eta_{ka} - \frac{1}{4} \Theta^{ka} \eta_{ka},
\]

(2.2)

\[
\delta_{sc} \omega^{\alpha \beta} = - j \omega^{\alpha \beta} + \chi^a_{\rho} \omega^{\rho \beta} + \chi^a_{\rho} \omega^{\rho \beta} + \chi^a_{\rho} \omega^{\rho \beta} + \chi^a_{\rho} \omega^{\rho \beta} + \chi^a_{\rho} \omega^{\rho \beta}.
\]

(2.3)

The superconformal transformations of the flat vector differential $\omega^{\alpha \beta} = dx^{\alpha \beta} - id\theta^{\alpha \beta} \theta^{ka} - id\theta^{\alpha \beta} \theta^{ka}$ and the spinor derivative $D^{ka}_{\alpha}$ have the covariant form

\[
\delta_{sc} D^{ka}_{\alpha} = \frac{1}{2} j D^{ka}_{\alpha} - \chi^a_{\rho} D^{ka}_{\alpha} + \chi^a_{\rho} D^{ka}_{\alpha} + \chi^a_{\rho} D^{ka}_{\alpha} + \chi^a_{\rho} D^{ka}_{\alpha}.
\]

(2.3)

Three traceless $2 \times 2$ transformation matrices are constructed from the parameters of $OSp(4|4)$

\[
\chi^a_{\beta} = \frac{1}{2} \delta^a_{\beta} + \frac{1}{2} (x^{\alpha \beta} k_{\beta} - \frac{1}{2} \delta^a_{\beta} x^{\gamma \lambda} k_{\gamma} - \frac{1}{2} \delta^a_{\beta} x^{\gamma \lambda} k_{\gamma} - \frac{1}{2} \delta^a_{\beta} x^{\gamma \lambda} k_{\gamma} - \frac{1}{2} \delta^a_{\beta} x^{\gamma \lambda} k_{\gamma} - \frac{1}{2} \delta^a_{\beta} x^{\gamma \lambda} k_{\gamma},
\]

(2.3)
\[ \xi^{bc} = \Omega^{bc} - \frac{i}{2} \xi^{ab} \theta^{c\gamma} k_{b\gamma} - \frac{i}{2} (\theta^{bc} \eta^a_c + \theta^{cb} \eta^a_k), \]

\[ \lambda^{kl} = \mathcal{M} \xi^{bc} = \omega^{kl} - \frac{i}{2} \theta^{k\alpha} \theta^{\ell\beta} k_{b\gamma} - \frac{i}{2} (\theta^{k\gamma} \eta^l_{c\gamma} + \theta^{l\gamma} \eta^k_{c\gamma}) \]

and satisfy the simple relations

\[ D^{ka}_{\gamma \beta} = -\delta^{k\gamma}_{\beta} D^{ka}_{\beta \gamma} + \frac{1}{2} \delta^{k\gamma}_{\beta} D^{ka}_{\beta j} + \frac{1}{2} \epsilon^{ab} D^{ka}_{j \gamma}, \]

\[ D^{ab}_{\gamma \lambda} = \frac{1}{2} \epsilon^{nk} D^{ab}_{\lambda j} + \frac{1}{2} \epsilon^{nl} D^{ab}_{\lambda j}, \]

\[ D^{lj}_{\gamma} (z) = -i \theta^{ll\gamma} k_{\gamma} - i \eta^{l b}. \]

By analogy with [11], we define the following \( \text{OSp}(4|4) \) transformations of the left harmonics:

\[ \delta_{sc} u^+_k = \lambda^{++} u^+_k, \quad \delta_{sc} u^-_k = 0, \]

\[ \lambda^{++} = u^+_j u^+_l \lambda^{ij}(z) = \omega^{++} - \frac{i}{2} \theta^{+a+} \theta^{+b+} k_{\alpha\beta} - i \theta^{++} \eta^{a b}. \]

The \( \text{SC} \) transformations of the coordinates \( x^m_L, \theta^a_{-\alpha} \) in \( L^A \) (A.5) are manifestly analytic

\[ \lambda^m_L = \delta_{sc} x^m_L = \frac{1}{2} (\gamma^m)_{\alpha\beta} \delta_{sc} x^\alpha_L = b \cdot x^m_L + (x^m_L) x^m_L - \frac{1}{2} \eta^{Lm}_m - 2i (\gamma^m)_{\alpha\beta} \epsilon^\alpha_{b\gamma} \theta^{+b+} x_{nL} - i (\gamma^m)_{\alpha\beta} \omega^{+b+} \theta^{+b+}, \]

\[ \lambda^+_{a\alpha} = \delta_{sc} \theta^a_{-\alpha} = \epsilon^+_{a\alpha} + \frac{1}{2} b \theta^{a\alpha} + \omega^{+a} \theta^{+\alpha} - \Omega^a_{b \alpha} + \frac{1}{2} \theta^{a\gamma} \theta^{+\gamma} k_{\beta\gamma} - i \theta^{+a} \theta^{+b} \eta^{a b}. \]

The superconformal parameters in \( C B \) (2.2) and \( L^A \) are connected by the simple relations

\[ j(z) = 2 \lambda - D^{-} \lambda^{++}, \quad \lambda = \frac{1}{2} j + u^+_j u^-_n \lambda^{jn}(z). \]

We consider the superconformal transformation of the non-analytic spinor coordinates

\[ \lambda^-_{a\alpha} = \delta_{sc} \theta^-_{a\alpha} = -\epsilon^-_{a\alpha} + \frac{1}{2} b \theta^{-a\alpha} + a^2 \theta^a_{-\alpha} - \Omega^a_{b \alpha} + \omega^{+a} \theta^{+\alpha} + \omega^{+a} \theta^{+\alpha} + \frac{1}{2} \theta^{a\gamma} \theta^{+\gamma} k_{\beta\gamma} + i \theta^{+a} \theta^{+b} \eta^{a b} + i \theta^{+a} \theta^{+b} \eta^{a b} - i \theta^{+a} \theta^{+b} \eta^{a b}. \]

This formula yields the \( \text{OSp}(4|4) \) transformation of the spinor derivative \( D^{+}_\beta \)

\[ \delta_{sc} D^{+\beta} = -(D^{+\beta}_\alpha \delta_{sc} \theta^{a\alpha}) D^{\alpha}_{\beta \gamma} = -\chi^\rho_{\beta} D^{+\beta}_\rho + \epsilon^\rho_{\beta} D^{+\beta}_\rho + \lambda D^{+\beta}_\rho \]

where the \( \text{LAB} \) superconformal matrices \( \chi^\rho_{\beta} \) and \( \epsilon^\rho_{\beta} \) are identical to the corresponding matrices in \( C B \) (2.3).

We obtain the \( \text{SC} \) transformations of the higher spinor derivatives from this formula

\[ \delta_{sc} D^{+\alpha \beta} = -\chi^\rho_{\alpha \beta} D^{+ \alpha \beta} - \chi^\rho_{\alpha \beta} D^{+ \alpha \beta} + 2 \lambda D^{+ \alpha \beta} + (D^{+ \beta}) D^{+ \alpha \beta} + (D^{+ \beta}) D^{+ \alpha \beta}, \]

\[ \delta_{sc} D^{+\alpha \beta} = -(D^{+ \alpha \beta}) D^{+ \alpha \beta} + (D^{+ \beta}) D^{+ \alpha \beta} + \epsilon^\alpha_{\beta} D^{+ \alpha \beta} + \epsilon^\alpha_{\beta} D^{+ \alpha \beta} + 2 \lambda D^{+ \alpha \beta}, \]

\[ \delta_{sc} (D^+) = 4 \lambda (D^+) \]

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The superconformal transformations of the left harmonic derivatives are
\[ \delta_{sc} D^{++} = -\lambda^{++} D^0, \quad \delta_{sc} D^{--} = -(D^{--} - \lambda^{++}) D^{--}. \] (2.14)

The SC transformations in the right harmonic superspace can be obtained by the mirror map from the LAB transformations, for instance,
\[ \delta_{sc} v_a^{(+)} = \xi^{(+)} v_a^{(-)}, \quad \delta_{sc} v_a^{(-)} = 0, \quad \delta_{sc} \theta_k^{(\pm)\alpha} = \mathcal{M} \delta_{sc} \theta_k^{\pm\alpha}, \quad \delta_{sc} x_R = \mathcal{M} \delta_{sc} x_L, \]
\[ \xi^{(++)} = v_a^{(+)}, \quad \xi^{(+-)} = v_a^{(+)}, \quad \Theta^{ab} = v_a^{(+)}, \quad \Omega^{ab} = -\frac{i}{2} \theta^{(+)\alpha} \theta^{(+)\beta} k_{\alpha\beta} - i \theta^{(+)\alpha} n_{\alpha}^{(+)}. \] (2.15)

3 Superconformal $\mathcal{N} = 4$ superfield gauge interactions

We consider the superfield constraints of the left hypermultiplet $q^{ka}$ and the left tensor multiplet $L^{ij} = L^{ji}$ and their SC transformations in $CB$
\[ D^a q^{lb} + D^a q^{kb} = 0, \quad \delta_{sc} q^{lb} = \frac{1}{2} j q^{lb} + \lambda^k q^{nb} + \xi_c q^{lb}, \]
\[ D^a L^{ji} + D^a L^{kj} = 0, \quad \delta_{sc} L^{kl} = j L^{kl} + \lambda^k L^{nl} + \lambda^l L^{kn}. \] (3.1)

The mirror map (1.1) allow us to obtain the superfield constraints for the right hypermultiplet $Q^{ka}$ and the right tensor multiplet $W^{ab}$
\[ D^a Q^{lb} + D^a Q^{kb} = 0, \quad \delta_{sc} Q^{lb} = 0, \]
\[ D^a W^{bc} + D^a W^{ca} + D^a W^{ab} = 0. \] (3.3)

The superfield $W^{ab}$ can be treated as the superfield strength of the left abelian spinor gauge superfield, and the mirror superfield $L^{kl}$ has a similar interpretation. $\mathcal{M}$-map can be combined with the $P$-parity, then the mirror superfields $W^{ab}$ and $L^{kl}$ have opposite parities.

In the $\mathcal{N} = 4$ harmonic superspace, the left non-abelian gauge supermultiplet is described by the matrix analytic prepotential $V^{++}(\zeta_L, u)$
\[ \delta_{\Lambda} V^{++} = -D^{++} \Lambda - [V^{++}, \Lambda], \quad (V^{++})^\dagger = -V^{++}, \quad \text{Tr} V^{++} = 0, \]
\[ \Lambda^\dagger = -\Lambda, \quad \text{Tr} \Lambda = 0 \] (3.5)

where $\Lambda(\zeta_L, u)$ is the analytic superfield matrix parameter of the gauge group $SU(N)$. We analyze the off-shell component fields $\phi_{ab}, A_m, \lambda^{ka}_\alpha$ and $X^{kl}$ in the $WZ$-gauge of the prepotential
\[ V^{++}_{WZ} = \theta^{+a\alpha} \theta^{+b}_{\alpha} \phi_{ab} + \Theta^{+m} A_m + 2 i \Theta^{+3\alpha} u_k^{-} \lambda^{ka}_\alpha + 3 i (\theta^+)^4 u_k^{-} u_l^{-} X^{kl}. \] (3.6)

where $\Theta^{+3\alpha}$ and $(\theta^+)^4$ are defined in appendix. The component fields of the $P$-even superfield $V^{++}$ include the pseudoscalar $\phi^{ab}$, spinor $\lambda^{ka}_\alpha$, vector $A_m$ and auxiliary scalar $X^{kl}$.

The non-analytic harmonic connection $V^{--}$ can be constructed in terms of the prepotential [12]
\[ V^{--}(z, u) = \sum_{n=1}^{\infty} (-1)^n \int du_1 \ldots du_n \frac{V^{++}(z, u_1) V^{++}(z, u_2) \ldots V^{++}(z, u_n)}{(u^+_1 u^+_1)(u^+_2 u^+_2) \ldots (u^+_n u^+_n)} \] (3.7)
where the harmonic distributions \((u^+_1 u^+_2)^{-1}\) are used.

The LAB representation of the nonabelian gauge superfield strength is [8, 13]
\[
W^{ab} = -\frac{1}{4} D^{+ac} D^{b} V^{--}, \quad \delta_{\Lambda} W^{ab} = [\Lambda, W^{ab}],
\]
\[
D^{++} W^{ab} + [V^{++}, W^{ab}] = 0. \quad (3.8)
\]

The abelian \(\mathcal{N} = 4\) superfield action contains the imaginary prepotential \(V_0^{++}\) and the coupling constant \(g_{3}\) of dimension \(\frac{1}{2}\)
\[
S_{QED}^L = \frac{1}{4 g_3^2} \int d^4 x \partial V^{++} V^{--}
\]
\[
= \frac{1}{4 g_3^2} \int d^4 x [ -\phi^{ab} \Box \phi_{ab} + 2 \Lambda^m \partial^m F_{nm} - 2 i \lambda_k^a \partial_{\alpha \beta} \lambda^{k a \beta} + X^{kl} X_{kl}] \quad (3.9)
\]
where \(F_{nm} = \partial_n A_m - \partial_m A_n\).

We consider the passive form of the superconformal transformations of the harmonic connections [11]
\[
\delta_{\text{sc}} V^{++} = 0, \quad \delta_{\text{sc}} V^{--} = -(D^{--} \lambda^{++}) V^{--} \quad (3.10)
\]
or the active (local) form of the same transformations
\[
\delta_{\text{sc}} V^{\pm \pm}(z, u) = -(\lambda^m \partial_m + \lambda_k^a \partial_{\alpha a} + \lambda^{++} D^{--}) V^{\pm \pm}(z, u) + \delta_{\text{sc}} V^{\pm \pm}, \quad (3.11)
\]
where functions \(\lambda^M\) and \(\lambda^{++}\) are defined in the previous section. The action \(S_{QED}^L\) is not invariant under these transformations.

The abelian superfield strength has the simple form
\[
W^{ab}(V) = -\frac{1}{4} \int du D^{-ac} D^b V^{++}(z, u), \quad \delta_{\Lambda} W^{ab} = 0, \quad \overline{W}^{ab} = W_{ab}. \quad (3.12)
\]
The superfield \(W^{ab}(V)\) is \(P\)-odd if the corresponding gauge superfield \(V^{++}\) is \(P\)-even. We consider its decomposition in the left coordinates
\[
W^{ab}(x_L, \theta^{\pm \pm}, u) = \phi^{ab} - i \theta^{-a\beta} \theta^{+ \alpha} \partial_{\alpha \beta} \phi^{bc} - i \theta^{-b\beta} \theta^{+ \alpha} \partial_{\alpha \beta} \phi^{ac} + \Theta^{--ab} \Theta^{++ \cdots} \Box \phi^{cd}
\]
\[
+ \frac{i}{2} (\theta^{+a} \theta^{+ b} + \theta^{-a} \theta^{+ b}) F_{\alpha \beta} + \frac{i}{2} (\theta^{+a} u_k^+ - \theta^{-a} u_k^+) \lambda_k^a + \frac{i}{2} (\theta^{+a} u_k^+ - \theta^{-a} u_k^+) \lambda_k^a
\]
\[
+ i \theta^{-a} \theta^{-b} u_k^- u_l^- X^{kl} + i \theta^{+a} \theta^{-b} u_k^+ u_l^- X^{kl} - i \theta^{+a} \theta^{-b} u_k^+ u_l^- X^{kl} - i \theta^{+a} \theta^{-b} u_k^+ u_l^- X^{kl} + \text{higher derivative terms} \quad (3.13)
\]
where \(F_{\alpha \beta} = \partial_{\alpha \gamma} A^\gamma_{\beta} + \partial_{\beta \gamma} A^\gamma_{\alpha}\).

The superfield \(W^{ab}\) can be connected with the right analytic superfield \(W^{(++)}(\zeta_R, v) = v^{(+)i}_a v^{(+)}_b W^{ab}\). The component decomposition of this superfield representation is very short in the right analytic coordinates (A.15).

The \(SC\) transformation of the nonabelian superfield strength can be obtained using (3.10) and (2.13)
\[
\delta_{\text{sc}} W^{ab} = -\frac{1}{4} \delta_{\text{sc}} (D^{+ab} V^{--}) = j W^{ab} + \xi_{\alpha} W^{\alpha b} + \xi_{\beta} W^{\beta c}. \quad (3.14)
\]
The important $SU_R(2)$ invariant abelian dilaton superfield

$$W = \sqrt{W^{ab}W_{ab}}, \quad \delta_{sc}W = j(z)W$$

(3.15)

satisfies the superfield constraint $D^{++}_{\alpha\beta}(W^{-1}) = 0.$

The superfield $W$ is used in the superconformal abelian gauge action [13]

$$S_W^0(V) = -\frac{1}{4}\int d^{1|1}z \, du \frac{1}{W} V^{++}(z, u) V^{--}(z, u).$$

(3.16)

The $SC$ invariance of $S_W^0$ can be checked straightforwardly using transformations of the integral measure and all superfields.

The spontaneous breakdown of the superconformal symmetry arises if we redefine the pseudoscalar superfield $W^{ab}

$$W^{ab} = \varphi(C^{ab} + w^{ab}), \quad C^{ab}C_{ab} = 2,$$

$$W = \varphi\sqrt{2 + 2C^{ab}w_{ab} + w_{ab}w_{ab}}$$

(3.17)

where $w_{ab}$ is the improved superfield strength, $\varphi$ is the scale fixing parameter of dimension 1 and $C^{ab}$ are dimensionless constants describing the spontaneous breakdown of the parity and $SU_R(2)$ symmetry.

The abelian dilaton superfield $W(V^{++})$ plays the role of a dynamic coupling constant in the superconformal version of the nonabelian gauge action

$$S_N(W, V^{++}) = \int d^{11}z d\nu_1 \ldots d\nu_n \sum_{n=2}^{\infty} \frac{(-1)^{n+1} \text{Tr} \, V^{++}(z, \nu_1) \ldots V^{++}(z, \nu_n)}{(\nu_1^+ \nu_2^-) \ldots (\nu_n^+ \nu_1^-)}. $$

(3.18)

The $SC$ invariance of this action can be checked by the method of [11] using the active superconformal transformations

$$\delta_{sc}^* V^{++}(z, u) = -(\lambda^M \partial_M + \lambda^{++} D^{--}) V^{++}(z, u),$$

$$\delta_{sc}^*(W^{-1}) = (-\lambda^M \partial_M + D^{--}\lambda^{++} - 2\lambda) W^{-1}.$$  

(3.19)

The superfield constraint $W = g_3^2$ in (3.18) breaks down the conformal symmetry, in this case we obtain the superfield action of the $\mathcal{N} = 4$ Yang-Mills theory $S_N(g_3^2, V^{++})$ constructed by analogy with the four-dimensional $\mathcal{N} = 2$ gauge action [12].

The right gauge multiplets can be considered as the mirror constructions in the right harmonic superspace (A.15). The abelian right analytic prepotential $V^{++}_{R0}(\zeta_R, v) \equiv A^{++}_{0}(\zeta_R, v)$ has the gauge transformation

$$\delta_{\Lambda} A^{++}_{0} = -D^{++}\Lambda_R(\zeta_R, v), \quad D^{++}_\alpha A^{++}_{0} = 0.$$  

(3.20)

In the $WZ$-gauge, this prepotential contains the component fields of the right vector multiplet

$$A_{WZ}^{++} = \theta^\alpha_k \theta^{(+)}_\alpha \Lambda^{kl} + (\theta^{(+)}_k \gamma^m \theta^{(+)}_k) B_m + \frac{4\sqrt{2} \theta^{(+)}_k \theta^{(+)}_\beta \theta^{(+)}_\alpha v^{(-)}_a v^{(-)}_b Y^{ab}}{\rho^k_{\alpha}} + 3i(\theta^{(+)}_k \gamma^a \theta^{(+)}_\alpha v^{(-)}_a v^{(-)}_b Y^{ab}.$$  

(3.21)

Below we use the pseudoscalar right prepotential $A^{++}_{0}$, then $B_m$ is the pseudovector field, $\Lambda^{kl}$ is the scalar field and $Y^{ab}$ is the pseudoscalar auxiliary field.
We can construct the superfield strength of the right abelian prepotential
\[ L_{kl}^{++} = -\frac{1}{4} \int dv D^{(-)}_{kl} \tilde{A}_{0}^{(+)}(z, v). \tag{3.22} \]
satisfying the constraints of the left tensor multiplet (3.2). It is scalar for the $P$-odd prepotential $A_{0}^{(+)}$. The superconformal transformation of the superfield $L_{kl}^{++}$ can be obtained by the mirror map from (3.14)
\[ \delta_{sc} L_{kl}^{++} = j(z) L_{kl}^{++} + \lambda^{k}_{j}(z) L_{jl}^{+} + \lambda^{l}_{j}(z) L_{kj}^{+}. \tag{3.23} \]

The improved left superfield $l_{kl}$ is defined by the formula
\[ L_{kl}^{++} = \gamma (c_{kl}^{+} + t_{kl}^{+}), \quad c^{2} = c_{kl}^{+} c_{kl}^{-} = 2 \tag{3.24} \]
where $\gamma$ is some constant of dimension 1 and $c_{kl}$ are dimensionless constants of the spontaneous breakdown of $SU_L(2)$. The mirror dilaton superfield is
\[ L = \sqrt{L_{kl} L_{kl}^{-}} = \gamma \sqrt{2 + 2c_{kl} l_{kl} + l_{kl} l_{kl}^{-}}, \quad \delta_{sc} L = j L. \tag{3.25} \]
The mirror superconformal abelian interaction contains this superfield $L$
\[ S_{0}^{L} = -\frac{1}{4} \int d^{11} z dv \frac{1}{L} A_{0}^{(++)} A_{0}^{((-)}), \quad D^{(++)} A_{0}^{((-)} = D^{((-)} A_{0}^{(++)}. \tag{3.26} \]
In Sec. 4 we analyze the equivalent left analytic action of the improved tensor multiplet which is dual to the free hypermultiplet action.

We use the spinor derivative of the superfield $L$ in the superconformal spinor connection
\[ \Gamma^{ka}_{\alpha} = L^{-1} D^{ka}_{\alpha} L, \quad \delta_{sc} \Gamma^{ka}_{\alpha} = D^{ka}_{\alpha} j + \frac{1}{2} j \Gamma^{ka}_{\alpha} - \chi^{ka}_{\alpha} \Gamma^{ka}_{\rho} + \lambda^{l}_{j} \Gamma^{ia}_{\alpha} + \zeta^{a}_{c} \Gamma^{kc}_{\alpha} \tag{3.27} \]
which helps to construct the SC covariant derivatives of superfields, for instance,
\[ \hat{D}^{kd}_{\alpha} W^{ab} = D^{kd}_{\alpha} W^{ab} - 2 \Gamma^{kd}_{\alpha} W^{ab} + \Gamma^{ka}_{\alpha} W^{bd} + \Gamma^{kb}_{\alpha} W^{ad} \tag{3.28} \]
or its mirror image $\hat{D}^{kd}_{\alpha} L^{ij}$. The superfield $W/L$ is the superconformal invariant.

\section{$\mathcal{N} = 4$ tensor multiplets and hypermultiplets}

The three-dimensional left analytic hypermultiplet $q^{+a}$ has the free action
\[ S_{0}^{q} = \frac{1}{2} \int d\zeta_{L}^{+} d\zeta^{+} q^{+a} D^{++} q^{+a}, \quad \tilde{q}^{+a} = q^{+a}. \tag{4.1} \]
The natural dimension of $q^{+a}$ is equal $\frac{1}{2}$, and the corresponding superconformal transformation contains the analytic parameter (2.9)
\[ \delta_{sc} q^{+a} = \lambda q^{+a}, \quad \delta_{sc} S_{0}^{q} = 0. \tag{4.2} \]
This hypermultiplet interacts with the left $U_L(1)$ gauge prepotential

$$S(q^+, V_0^{++}) = \frac{1}{2} \int d\zeta_L^{-4} du a^+[D^{++}q^+ + (\tau_3)_b V_0^{++}q^b]$$  \hspace{1cm} (4.3)$$

where $\tau_3$ is the Pauli matrix. The mirror map $M_S(q^+, V_0^{++}) = S(Q^{(+)}, A_0^{(++)})$ yields the interaction of the right hypermultiplet $Q_k^{(+)}$ with the right abelian prepotential $A_0^{(++)}$.

The dual free $\omega$-hypermultiplet can be described analogously

$$S_0^\omega = \frac{1}{2} \int d\zeta_L^{-4} du D^{++}\omega D^{++}\omega, \quad \delta_{sc}\omega = \lambda \omega.$$  \hspace{1cm} (4.4)$$

In the gauge group $SU(N)$, we can use the adjoint representation for the $\omega$ superfield, then the hypermultiplet-gauge interaction reads

$$S(\omega, V) = \frac{1}{2} \int d\zeta_L^{-4} du \text{Tr} (D^{++}\omega + [V^{++}, \omega])^2.$$  \hspace{1cm} (4.5)$$

The sum of this action and the $\mathcal{N} = 4$ Yang-Mills action $S_N(g_3^2, V^{++})$ is invariant under the $\mathcal{N} = 8$ supersymmetry transformations constructed by the analogy with the $D = 4, \mathcal{N} = 4$ case [11]

$$\delta V^{++} = g_3 \epsilon^{kaa} u_k^+ \theta_a^+ \omega, \quad \delta \omega = -\frac{1}{2g_3}(D^{+})^4(\epsilon^{kaa} u_k^- \theta_a^- V^-),$$  \hspace{1cm} (4.6)$$

where $\epsilon^{kaa}$ are spinor parameters.

The left tensor multiplet is described by the left analytic superfield $L^{++} = u_k^+ u_l^+ L^{kl}(z)$

$$L^{++} = -\frac{1}{4} u_k^+ u_l^+ \int dv \, v_a^{(-)} v_b^{(-)} D^{kaa} D^{bba} A_0^{(++)}(z, v), \quad D_a^{++} L^{++} = 0,$$  \hspace{1cm} (4.7)$$

$$D^{++} L^{++} = 0, \quad \delta_{sc} L^{++} = 2\lambda L^{++}.$$  \hspace{1cm} (4.8)$$

The component representation of this superfield is

$$L^{++} = u_k^+ u_l^+ \Lambda^{kl} - i\Theta^{+m} u_k^+ u_l^- \partial^m \Lambda^{kl} - i\theta^+ a u_k^+ \rho_a^k + i\Theta^{++} Y^{ab}$$

$$+ i(\theta^{+b} \gamma^m \theta^+_b) \varepsilon_{mpq} \partial^p B^q + \Theta^{+3a} u_k^- \partial_{a\beta} \rho_{k\alpha}^3 + (\theta^+)^4 u_k^- u_l^- \Box \Lambda^{kl}.$$  \hspace{1cm} (4.9)$$

The free action of the left tensor multiplet is equivalent to the free action of the right gauge multiplet.

Now we consider the improved form of the left analytic tensor superfield

$$L^{++} = \gamma(c^{++} + t^{++}), \quad c^\pm = c^{kl} u_k^\pm u_l^\pm, \quad c^{++} c^{--} - (c^0)^2 = 1,$$

$$\delta_{sc} L^{++} = 2\lambda (l^{++} + c^{++}) - 2\lambda L^{++} c^0.$$  \hspace{1cm} (4.10)$$

The three-dimensional superconformal interaction of $l^{++}$ is similar to the analogous four-dimensional action [11]

$$\tilde{S}_0^L = -\frac{1}{\gamma} \int d\zeta_L^{-4} du (g^{++})^2, \quad g^{++}(l) = \frac{l^{++}}{1 + \sqrt{1 + l^{++}c^-}}.$$  \hspace{1cm} (4.11)$$
We note that this action is a dual form of the action (3.26). The scalar part of the component action of the improved tensor multiplet has the form

\[
\tilde{K}_L = \int d^3x \frac{\sqrt{2}}{\Lambda} \left\{ \frac{1}{4} \partial_m \Lambda^{kl} \partial^m \Lambda_{kl} - \frac{1}{6\Lambda^2} \Lambda^{rs} \Lambda^{kl} \partial_m \Lambda_{kl} \partial^m \Lambda_{rs} + \frac{1}{4} \psi^{ab} \psi_{ab} \right\} \quad (4.12)
\]

\[
\Lambda = \sqrt{\Lambda^{kl} \Lambda_{kl}} = \sqrt{2} \phi \sqrt{1 + c^{kl} \lambda_{kl} + \frac{1}{2} \lambda^{kl} \lambda_{kl} + \frac{1}{4} \psi^{cd} \psi_{cd}}.
\]

The similar component \( D = 4, N = 2 \) action was defined in [19].

As it was shown in [11], the action of the improved tensor multiplet is dual to the action of the free hypermultiplet. The alternative form of the action (4.11) contains unconstrained superfield \( l^{++} \) (or \( g^{++} \)) and the analytic Lagrange multiplier \( \psi \)

\[
\tilde{S}_L^0 = -\frac{1}{\gamma} \int d\zeta_L^{-1} du \left\{ [g^{++}(l)]^2 - \psi D^{++} l^{++} \right\} . \quad (4.13)
\]

Using the algebraic equation \( g^{++}(\psi) = -(1 + c^{--} D^{++} \psi)^{-1} D^{++} \psi \) we obtain the action \( S(\psi) \) which is equivalent to the free hypermultiplet action. Thus the superconformal right gauge action (3.26) is dual to the free action.

The link between the \( CB \) and \( LAB \) representations of the left tensor multiplet is the formula

\[
L^{-1}(l_{kl}) = \frac{1}{\sqrt{2\gamma}} (1 + c^{kl} l_{kl} + \frac{1}{2} l^{kl} l_{kl})^{-1/2} = \frac{1}{\sqrt{2\gamma}} \int du[1 + Z(l, u)]^{-3/2},
\]

\[
Z(l, u) = l^{++}c^{--} = l^{kl}(z)c^{im}u_k^+ u_i^+ u_j^- u_n^- . \quad (4.14)
\]

We note that both parts of this relation describe the same \( SU_L(2) \)-invariant solution of the Laplace equation in variables \( l_{kl} \): \( \Delta_l L^{-1}(l_{kl}) = 0 \). This integral representation is based on the formula

\[
\int du (l^{++}c^{--})^n = \frac{1}{2n + 1} l^{(i_1k_1l_1k_2 \cdots l_{i_nk_n})}_{c(i_1k_1c_2k_2 \cdots c_{i_nk_n})} \quad (4.15)
\]

where brackets mean the total symmetrization of the \((l_{kl})\)-polynomials in \(2n\) indices.

The superconformal interaction \( S_0^W \) (3.16) is equivalent to the action of the improved right tensor multiplet \( w^{(++)} \)

\[
W^{(++)} = v_a^{(+)} v_b^{(+)} W^{ab} = \varphi(C^{(++)} + w^{(++)}) \quad (4.16)
\]

which is mirror to the left action (4.11). The corresponding scalar terms are

\[
K_R = \int d^3x \frac{\sqrt{2}}{\phi} \left\{ \frac{1}{4} \partial_m \phi^{ab} \partial^m \phi_{ab} - \frac{1}{6\phi^2} \phi^{cd} \phi^{ab} \partial_m \phi_{ab} \partial^m \phi_{cd} + \frac{1}{4} \chi^{kl} \chi_{kl} \right\} . \quad (4.17)
\]

\[
\phi = \sqrt{\phi^{ab} \phi_{ab}} = \varphi \sqrt{2 + 2C^{ab} \varphi_{ab} + \chi^{ab} \varphi_{ab}} .
\]

The action of the improved right tensor multiplet \( w^{(++)} \) is also dual to some free action.
5 Nonlinear $\mathcal{N} = 4$ gauge interactions

Now we discuss the possible $\mathcal{N} = 4$ superfield gauge interactions which correspond to the higher degrees of the derivative terms $F_{mn} = \partial_m A_n - \partial_n A_m$, $F^B_{mn} = \partial_m B_n - \partial_n B_m$, $\partial_m \phi^{ab}$ and $\partial_m \Lambda^{kl}$ in the component actions. The nonlinear supersymmetric $D = 4$ abelian gauge terms were analyzed in [20],[21],[22].

We choose the nonlinear self-interaction terms of the analytic multiplet $L^{++}$ in the following form:

$$S(L^{++}) = \frac{1}{93} \int d\zeta^{-4} d\mu ((L^{++})^2 [1 - 12c^2(D^+)^4(L^-)^2)]$$

$$-24c^2L^{++}(D^+)^4[(L^{++})^2L^{--}] + O(L^6),$$

$$L^{++} = u_k^+ u_l^+ L^{kl}(A^{(++)}_0), \quad L^{++} = \frac{1}{2}D^{--}L^{++}, \quad L^{--} = \frac{1}{2}(D^{--})^2L^{++}$$

where $c$ is some constant of the dimension $-2$. In components, these terms describe the second and fourth degrees of $F_{mn}$ and $\partial_m \Lambda^{kl}$.

By the analogy with the nonlinear realizations of the 4-dimensional supersymmetries [20] we can find the nonlinear $\delta_f$-transformation of the superfield $L^{++}$

$$\delta_f L^{++} = f^{++}[1 + c^2(D^+)^4(L^-)^2] + 2c^2L^{++}(D^+)^4(f^{--}L^{--}) - 4c^2(D^+)^4(f^{++}L^{++}L^{--})$$

$$+4c^2(D^+)^4[(f^{--}(L^{--})^2] + O(L^4),$$

$$f^{++} = a_{kl}^+ u_k^+ u_l^+ + c^{-1} \theta^a \alpha^+ u_k^+ c_{\alpha a}, \quad f^{++} = \frac{1}{2}D^{--}f^{++}, \quad f^{--} = \frac{1}{2}(D^{--})^2f^{++}$$

where $a_{kl}$ and $c_{\alpha a}$ are parameters of the bosonic and fermionic translations. This transformation satisfies the condition $D^{++}\delta_f L^{++} = 0$.

The nonlinear action (5.1) is invariant under the $\delta_f$-transformation up to the third order in superfields. To prove this invariance we take into account only two independent third-order structures in the variation $\delta_f S(L^{++})$

$$f^{++}L^{++}(D^+)^4(L^-)^2, \quad (L^{++})^2(D^+)^4(f^{--}L^{--}).$$

The transformation (5.2) describes the spontaneous breakdown of the $D = 3, \mathcal{N} = 8$ supersymmetry. The similar nonlinear action can be found for the superfield $W^{(++)}$ in the right analytic superspace.

The fourth order nonlinear superfield terms can also be studied in the full $D = 3, \mathcal{N} = 4$ superspace

$$S_4 = \int d^{11}z [A_1 W^4 + A_2 L^4 + A_3 W^2 L^2],$$

$$W^2 = \varphi^2(2 + 2C^{ab}w_{ab} + w^{ab}w_{ab}), \quad L^2 = \gamma^2(2 + 2c_{kl}l_{kl} + k_{kl}l_{kl})$$

where $A_1, A_2$ and $A_3$ are some constants.

Higher degrees of the gauge field strength arise from the superfield terms with the spinor derivatives of $W^{ab}$, for instance,

$$S_6 \sim \int d^{11}z W^4 A_{\alpha \beta} A^{\alpha \beta},$$

$$A_{\alpha \beta} = \varepsilon_{kl}D_{(\alpha}^{ka}D_{\beta)}^{lb}W_{ab} \sim iF_{\alpha \beta}(x) + O(\theta).$$

(5.5)
All these nonlinear gauge interactions break down the superconformal symmetry.

The superconformal $\mathcal{N} = 2, D = 4$ nonlinear gauge interactions were considered in [22] where the chiral dilaton superfield was used. We have two $\mathcal{N} = 4$ abelian dilaton-type superfields $W$ and $L$ so it is not difficult to construct the non-polynomial superconformal generalization of the nonlinear terms $S_4$

$$S_{NL} = \int d^{11}z \left\{ W^4L^{-5} + L^4W^{-5} \right\}. \quad (5.6)$$

We note that additional superconformal terms with the $SC$-invariant combination $W/L$ could be added to this interaction. Using the connection (3.27) we can construct the superconformal generalizations of the derivative terms (5.5).

We can also study the nonlinear superconformal interactions of the nonabelian superfield strength (3.8) with the abelian dilaton superfields, for instance,

$$\int d^{11}z L^{-5} (Tr W^{ab}W_{ab})^2. \quad (5.7)$$

### 6 $\mathcal{N} = 4$ BF interaction of the left and right gauge multiplets

We see that the left and right $\mathcal{N} = 4$ supermultiplets live in different analytic superspaces, so it is extremely difficult to construct interactions of these supermultiplets. Nevertheless, there is the simple left-right gauge $BF$ interaction which was considered in the component fields[16, 17] and also in the biharmonic $\mathcal{N} = 4$ superspace [18].

The $LAB$ form of this $BF$ interaction reads

$$S_{BF}^0 = \frac{i}{2} \beta \int d\zeta L^{-4} d\nu V_0^{++}L_0^{++}, \quad (6.1)$$

where $\beta$ is the coupling constant, $V_0^{++}$ is the left abelian prepotential, and $L^{++}$ is the scalar analytic superfield strength of the right abelian pseudoscalar prepotential $A_0^{(++)}$ (4.7). This interaction is manifestly superconformal and preserves the $P$-parity and the $\mathcal{M}$ symmetry $V_0^{++} \leftrightarrow A_0^{(++)}$.

The equations of motion for the abelian $BF$ model

$$W^{ab}(V) = 0, \quad L^{++}(B) = 0 \quad (6.2)$$

have the pure gauge solutions only.

Using the superfield decompositions (3.6) and (4.9) we obtain the component form of the abelian $\mathcal{N} = 4$ $BF$ action [16, 17]

$$S_{BF}^0 = \beta \int d^3x (2\varepsilon^{mn}A_m\partial_n B_p - \frac{1}{2} \phi^{ab}Y_{ab} - \frac{1}{2} \Lambda^{ik}X_{ik} + 2\rho^k_{\alpha}X_{ka}). \quad (6.3)$$

The mirror symmetry $S_{BF}^0 = \mathcal{M}S_{BF}^0$ is evident in this representation.

If we identify $SU_L(2)$ and $SU_R(2)$ indices of these fields, this action can be treated as the difference of two abelian Chern-Simons actions for two $\mathcal{N} = 3$ vector multiplets connected by the parity transformation and the transformation of the fourth supersymmetry.
The gauge $\mathcal{N} = 3$ prepotentials and superfield strengths live in the same analytic superspace, and the fourth supersymmetry transformation connect different gauge supermultiplets. We note that the $\mathcal{N} = 4$ superfield generalization of the Chern-Simons action does not exist for the group $U(1)$, because the corresponding gauge prepotentials and superfield strengths are defined in different superspaces.

One can add the minimal interactions of the left hypermultiplets $q^+_a$ (4.3) and the right hypermultiplets $Q_{k}^{(+)} = \mathcal{M}q^+_a$ to the $U_L(1) \times U_R(1)$ BF action

$$S(q, Q, V_0, A_0) = S_{0 \text{BF}} + S(q^+, V_0^{++}) + S(Q^{(+)}, A_0^{(++)}).$$

(6.4)

We note that this model has the manifest $\mathcal{N} = 4$ symmetry even if it be reformulated in the $\mathcal{N} = 3$ superspace. In this formalism, generators of the linear transformations of the fourth supersymmetry have the opposite sign on the left and right $\mathcal{N} = 3$ analytic superfields: $Q^+_a = \pm D^0_\alpha$. More complex nonlinear transformations of the higher supersymmetries are defined in [9] for the nonabelian $\mathcal{N} = 3$ superfields, the algebra of these transformations closes only on the equations of motion.

The abelian BF term can be treated as the nontrivial interaction in the following superconformal composite action:

$$S^\beta(V^+_0, A_0^{(++)}) = S^W_0 + S^L_0 + S_{0 \text{BF}}$$

(6.5)

where the first two terms describe the left and right gauge superfields. It is evident that this interaction possesses the discrete $\mathcal{M}$ symmetry.

The scalar part of this action contains the kinetic term for the field $\Lambda^{kl}$ (4.12), the analogous kinetic term for the field $\Phi^{ab}$ (4.17) and the mixed potential term

$$P = -\frac{\sqrt{2}\beta^2}{4} \int d^3x \phi^{ab} \sqrt{\Lambda^{kl}\Lambda_{kl}} + \Lambda^{kl}\Lambda_{kl} \sqrt{\phi^{ab}\phi_{ab}}$$

(6.6)

which arises from terms with the auxiliary fields $X^{kl}$ and $Y^{ab}$. Thus, the superconformal action $S^\beta(V^+_0, A_0^{(++)})$ describes the nontrivial interaction of the abelian left and right $\mathcal{N} = 4$ gauge multiplets.

Using the transformation of the vector and pseudovector gauge fields in (6.5)

$$A_m = \sqrt{\frac{2}{\beta}}(A^+_m + A^-_m), \quad B_m = \sqrt{\frac{2}{\beta}}(A^+_m - A^-_m)$$

(6.7)

we obtain the sum of two quadratic gauge actions

$$S(A_m^+, A_m^-) = \frac{1}{2} \int d^3x [A_m^+ K^{nm}_{+} A_n^+ + A_m^- K^{'nm}_{-} A_n^-],$$

$$K^{nm}_{\pm} = \eta^{mn} \square - \delta^m \partial^n \pm \mu \varepsilon^{mpn} \partial_p$$

(6.8)

where $\mu = \beta \sqrt{\phi}$ is the parameter of the gauge-invariant mass terms.

It is easy to build the non-abelian version of the BF theory in the full $\mathcal{N} = 4$ superspace using the superfield $\mathcal{W}^{ab}$ (3.8) and the Lagrange multiplier $B_{ab}$, however, we do not know interactions of this additional superfield $B_{ab}$. We also cannot construct interactions of the left $\mathcal{N} = 4$ non-abelian gauge superfield with the right non-abelian gauge superfield.
7 Conclusions

We analyzed the superconformal interactions of $D = 3, \mathcal{N} = 4$ superfields in different representations. The left and right $\mathcal{N} = 4$ harmonic superspace are treated as two mirror analogs of the $D = 4, \mathcal{N} = 2$ harmonic superspace. The interaction of the left gauge and left hypermultiplet superfields are natural, and the mirror picture connects right $\mathcal{N} = 4$ superfields, while it is difficult to construct interactions of the left and right supermultiplets. We consider the dilaton superfield $W$ using the abelian gauge superfield strength $W^{ab}$ and the mirror dilaton superfield $L$ constructed from the right abelian gauge superfield. $W$ and/or $L$ play the role of the dynamic coupling constants in the superconformal gauge theory. These dilatons allow us to construct the improved superconformal versions of the abelian and nonabelian $\mathcal{N} = 4$ gauge theories. The left $\mathcal{N} = 4$ superconformal abelian gauge model $S_W^0$ is dual to the free hypermultiplet theory, this property is preserved in the mirror model $S_L^0$.

The $BF$ interaction in the $\mathcal{N} = 4$ superspace is considered as the analog of the Chern-Simons action for the group $U_L(1) \times U_R(1)$. The left and right hypermultiplets interact with the corresponding $\mathcal{N} = 4$ gauge superfields in this theory. We propose the combining superconformal interaction of the abelian left and right gauge superfields including the improved left and right gauge actions $S_W^0 + S_L^0$ and the mixing $BF$ term. This term yields the nontrivial interaction of the scalar and vector fields from two mirror supermultiplets.

On the non-superconformal level, we find the nonlinear realization of the $\mathcal{N} = 8$ supersymmetry connecting different nonlinear gauge terms in the effective action. Using the superconformal covariant derivatives of the superfield strengths we obtain the higher nonlinear superconformal interactions of the left and right gauge superfields.

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Appendix

1. Central basis of $D = 3, \mathcal{N} = 4$ superspace

We consider the coordinates of the $D = 3, \mathcal{N} = 4$ superspace in the central basis (CB) and the corresponding integral measure [8, 13]:

$$z = (x^m, \theta^\alpha_{ka}), \quad d^{11}z = d^3x d^8 \theta,$$

where $i$ and $a$ are the two-component indices of the automorphism groups $SU_L(2)$ and $SU_R(2)$, respectively, $\alpha$ is the two-component index of the $SL(2, R)$ group and $m = 0, 1, 2$ is the 3D vector index. The three-dimensional $\gamma$ matrices satisfy the relations

$$(\gamma_m)^\alpha_\rho (\gamma_n)^\beta_\rho = - (\gamma_m)_{\alpha \rho} (\gamma_n)^{\rho \beta} = - \eta_{mn} \delta^\beta_\alpha + \epsilon_{mnp} (\gamma^p)^\beta_\alpha,$$  

where $\eta_{mn} = \text{diag}(1, -1, -1)$ and $\epsilon_{mnp}$ is the antisymmetric symbol.

We use the $P$-parity transformation

$$P x^0, P x^1 = - x^0, \quad P \theta^\alpha_{ka} = (\gamma_1)^\alpha_\beta \theta^\beta_{ka},$$
The mirror map $\mathcal{M}$ interchanges values of the isospinor indices of the groups $SU_L(2)$ and $SU_R(2)$, for instance, $\mathcal{M}\theta^a_{12} = \theta^a_{21}$, and connects the representations $(l, r)$ and $(r, l)$.

The $N = 4$ spinor derivatives have the form

$$D^ka = \partial^ka + i\theta^{ka\beta}\partial\alpha, \quad \partial\alpha = (\gamma^m)_{\alpha\beta}\partial_m. \quad (A.4)$$

2. Left $D = 3, N = 4$ harmonic superspace

The left $D = 3, N = 4$ harmonic superspace was considered in [8, 13]. It uses the left $SU(2)_L/U(1)$ harmonics $u^+_k$ and the corresponding left analytic basis $(LAB)$

$$\zeta_L = (x^m_L, \theta^+_a), \quad \theta^-_a, \quad x^m_L = x^m + i(\gamma^m)_{\alpha\beta}\theta^+\alpha\theta^-\beta, \quad \theta^\pm\alpha = u^+_k\theta^k\alpha. \quad (A.5)$$

We define the special conjugation in this basis

$$\tilde{u}^\pm_k = u^\pm_k, \quad \tilde{x}^m_L = x^m_L, \quad \tilde{\theta}^\pm\alpha = \theta^\mp\alpha. \quad (A.6)$$

The LAB spinor and harmonic derivatives are

$$D^+_a = \partial^+_a, \quad D^-_a = -\partial^-_a + 2i\theta^-a\beta\partial^L_m, \quad (A.7)$$

$$D^{++} = \partial^{++} - i\theta^+_a\gamma^m\theta^+a\partial^L_m + \theta^+_a\partial^+\alpha, \quad D^{--} = \partial^{--} - i\theta^-a\gamma^m\theta^-a\partial^L_m + \theta^-a\partial^-\alpha. \quad (A.8)$$

The simple combinations of the spinor derivatives are

$$D^{\pm\pmab} = D^{\pm\alpha\beta}D^{\pmb\beta}, \quad D^{\pm\alpha\beta} = D^{\pma\beta}D^{\pm\beta}, \quad (A.9)$$

They satisfy the relations

$$D^{ab\alpha\beta}D^{+\alpha\beta} = 0, \quad (A.10)$$

$$D^{++}(D^{--}D^{++})^4 = 2\partial^a\partial_mD^{(+4)} = 2\Box(D^+)^4, \quad (A.11)$$

$$D^{++}(D^{--})^2(D^{++})^4 = [\frac{32}{2}D^{ab}\partial^{\alpha\beta} + 6D^{--}\Box](D^+)^4 = [\frac{32}{2}\partial_a\partial_b\partial^{\alpha\beta} + 6\partial^{--}\Box](D^+)^4,$$

$$D^{+\alpha\beta}D^{--}(D^{+\alpha\beta})^4 = [\frac{16}{2}D^{ab}\partial^{--}ab + 6D^{--}D^{--}D^{--}\partial\alpha\beta + 12(D^{--})^2\Box](D^+)^4 \quad (A.12)$$

The analytic integral measure in LAB is

$$d\zeta_L^4du = d^4x_Ldu(D^+)^4$$

where $du$ describes the integration on $SU_L(2)/U(1)$.

We use the basic combinations of the left analytic spinor coordinates $\theta^\pm\alpha$

$$\Theta^{\pm\pm}_{ab} = \theta^\pm\alpha\partial^\pm_{ab}, \quad \Theta^{\pm\alpha}_{a\beta} = (\gamma^m)_{\alpha\beta}\Theta^{\pm\pm}_m = \theta^\pm\alpha\partial^\pm_{a\beta}, \quad \Theta^{\pm\alpha}_{a\beta} = \frac{2}{3}\theta^{\pm\beta\alpha}\Theta^{\pm\pm}_{ab}, \quad (\theta^\pm)^4 = (\theta^+_1)^2(\theta^+_2)^2. \quad (A.13)$$
They satisfy the simple identities of the Grassmann algebra

\begin{align*}
\Theta_{ab}^{\pm} & \Theta_{cd}^{\pm} = \frac{1}{2} (\varepsilon_{ac}\varepsilon_{bd} + \varepsilon_{ad}\varepsilon_{bc}) (\theta^\pm)^4, \\
\Theta_{\alpha\beta}^{\pm} & \Theta_{\gamma\rho}^{\pm} = -\frac{1}{2} (\varepsilon_{\alpha\gamma}\varepsilon_{\beta\rho} + \varepsilon_{\alpha\rho}\varepsilon_{\beta\gamma}) (\theta^\pm)^4, \\
\theta^\pm_c & \Theta_{ab}^{\pm} = \frac{1}{2} \varepsilon_{ca} \Theta_b^{\pm 3\alpha} + \frac{1}{2} \varepsilon_{cb} \Theta_a^{\pm 3\alpha}. 
\end{align*}

(A.14)

3. Right \( N = 4 \) harmonic superspace

We denote the mirror \( SU_R(2)/U(1) \) harmonics as \( v^{(\pm)}_a = M u^{\pm}_k \) and the coordinates of the right analytic basis \( RAB \) as

\begin{align*}
\zeta_R = M \zeta_L &= (x^m_R, \theta^{(+)}_k^{\alpha}), \quad \theta^{(-)}_k^{\alpha}, \\
x^m_R &= x^m + i (\gamma^m)_{\alpha\beta} \theta^{(+)}_k^{\alpha} \theta^{(-)}_k^{\beta}, \quad \theta^{(+)}_k^{\alpha} = v^{(\pm)}_a \theta^{\alpha\alpha}_k. 
\end{align*}

(A.15)

The special conjugation in \( RAB \) is analogous to the corresponding conjugation in \( LAB \) (A.6).

The spinor and harmonic derivatives in the \( RA \) basis can be obtained by the mirror map from (A.8), for instance,

\begin{align*}
D^{(+)}_\alpha = \partial^{(+)}_\alpha, & \quad D^{(-)}_\alpha = -\partial^{(-)}_\alpha + 2i \theta^{(-)}_k^{\alpha\beta} \partial_R^{\alpha\beta}, \\
D^{(++)} = \partial^{(++)} - i \theta^{(+)}_k^{\alpha\beta} \gamma^m \theta^{(+)}_k^{\alpha} \partial_m^{\alpha\beta} + \theta^{(+)}_k^{\alpha\beta} \partial^{(++)}_\alpha, 
\end{align*}

(A.16)

where the partial derivatives act on the corresponding right coordinates. The right analytic superfields \( \Phi(\zeta_R, v) \) describe right \( N = 4 \) supermultiplets.

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