Eliminating Odd Cycles by Removing a Matching*

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1 Introduction

Given a graph $G = (V,E)$ and a graph property $\Pi$, the $\Pi$ edge-deletion problem consists in determining the minimum number of edges required to be removed in order to obtain a graph satisfying $\Pi$ [12]. Given an integer $k \geq 0$, the $\Pi$ edge-deletion decision problem asks for a set $F \subseteq E(G)$ with $|F| \leq k$, such that the obtained graph by the removal of $F$ satisfies $\Pi$. Both versions have received widely attention on the study of their complexity, where we can cite [2, 12, 23, 26, 29, 33, 37, 38] and references therein for applications. When the obtained graph is required to be bipartite, the corresponding edge- (vertex-) deletion problem is called edge (vertex) bipartization [1, 14, 22] or edge (vertex) frustration [39]. Choi, Nakajima, and Rim [14] showed that the edge bipartization decision problem is $NP$-complete even for cubic graphs.

Furmańczyk, Kubale, and Radziszowski [22] considered vertex bipartization of cubic graphs by the removal of an independent set. In this paper we study the analogous edge deletion decision problem, that is, the problem of determining whether a finite, simple, and undirected graph $G$ admits a removal of a set of edges that is a matching in $G$ in order to obtain a bipartite graph. Formally, for a set $M$ of edges of a graph $G = (V,E)$, let $G - M$ be the graph with vertex set $V(G)$ and edge set $E(G) \setminus M$. For a matching $M \subseteq E(G)$, we say that $M$ is an odd decycling matching of $G$ if $G - M$ is bipartite. Let $\mathcal{BM}$ denote the set of all graphs admitting an odd decycling matching. We deal with the complexity of the following decision problem.

ODD DECYCLING MATCHING

**Input:** A finite, simple, and undirected graph $G$.

**Question:** Does $G \in \mathcal{BM}$?

A more restricted version of this problem is considered by Schaefer [35]. He deals with the problem of determining whether a given graph $G$ admits a 2-coloring of the vertices so that each vertex has exactly one neighbor with same color as itself. We can see that the removal of the set of edges whose endvertices have same color, which is a perfect matching of $G$, generates a bipartite graph. Schaefer proved that such a problem is $NP$-complete even for planar cubic graphs.

With respect to the minimization version, the edge-deletion decision problem in order to obtain a bipartite graph is analogous to SIMPLE MAX CUT, which was proved to be $NP$-complete by Garey, Johnson and Stockmeyer [23]. Yannakakis [37] proved its $NP$-completeness even for cubic graphs.

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Lemma 1. Graphs in every graph in and 1 showed that $G$ is obtained from a cycle $v$ of length $n$ and the central $k$-pool $C_n$ is adjacent to all vertices of $C$. Let $W_k$ be the wheel graph of order $k$, that is, the graph containing a vertex $v$, called central, and a cycle $C$ of order $k$, such that $v$ is adjacent to all vertices of $C$.

We say that a graph is a $k$-pool if it is formed by $k$ triangles edge-disjoint whose bases induce a $C_k$. Formally, a $k$-pool is obtained from a cycle $C = \{v_1, v_2, \ldots, v_k\}$ ($k \geq 3$), such that the odd-indexed vertices induce a cycle $v_1 v_2 \ldots v_k v_1$, called internal cycle of the $k$-pool, where $p_i = v_{2i-1}$, $1 \leq i \leq k$. The even-indexed vertex $b_i$ is the $i$-th-border of the $k$-pool, where $\{b_i\} = N_C(p_i) \cap N_C(p_{i+1})$ and $i + 1$ is taken modulo $2k$. Fig. 1c and Fig. 1d represent the 3-pool and 5-pool, respectively.

The claw (K$_{1,3}$) and the paw (a triangle plus an edge) graphs are the unique ones with degree sequences 1, 1, 1, 3 and 1, 2, 2, 3, respectively.

Clearly, every graph $G \in \mathcal{B}_k$ admits a proper 4-coloring. Hence every graph in $\mathcal{B}_k$ is K$_3$-free. More precisely, every graph in $\mathcal{B}_k$ is $W_k$-free, which is depicted in Fig 1a. Hence some proper 4-colorable graphs do not admit an odd decycling matching. Fig. 1 shows some other examples of forbidden subgraphs. Lemma 1 collects some properties of graphs in $\mathcal{B}_k$.

Lemma 1. For a graph $G \in \mathcal{B}_k$ and an odd decycling matching $M$ of $G$, the following assertions are true.

(i) If $G$ has a diamond $D$ as a subgraph, then $M$ contains no edge $e \notin E(D)$ incident to only one vertex of degree three of $D$.

(ii) $G[N_G(v)]$ cannot contain two disjoint $P_5$, for every $v \in V(G)$.

(iii) $G$ cannot contain a $W_k$ as a subgraph, for all $k \geq 4$.

(iv) $G$ cannot contain a $k$-pool as a subgraph, for all odd $k \geq 3$.

Proof. (i) Let $G \in \mathcal{B}_k$ be a graph that contains a diamond $D$ as a subgraph, such that $V(D) = \{u, v_1, v_2, v_3\}$ and $d_{DP}(u) = d_{DP}(v_2) = 3$. We can see that $M \cap E(D)$ equals to exactly one of the following sets: $\{uv_1, v_2v_3\}$, $\{v_1v_2, uv_3\}$, $\{uv_2\}$. For
Proof. Let \( B \) there is no vertex in Algorithm 1 returns in linear-time an odd decycling matching for subcubic graphs.

Consider a bipartition of \( G \). Let \( B \) be a largest bipartite subgraph of \( G \). Then \( d_{\delta}(B) \geq \frac{1}{2}d_{\delta}(G) \) for every \( v \in V(G) \).

Lemma 2 shows that every subcubic graph \( G \) contains an odd decycling matching, since every vertex has at most one incident edge not in a largest bipartite subgraph of \( G \). This result was also obtained by Lovász [30] with respect to 1-improper 2-coloring of graphs with maximum degree at most 3.

Consider a bipartition of \( V(G) \) into sets \( A \) and \( B \). For every vertex \( v \), we say that \( v \) is of type \((a, b)\) if \( d_{\delta}(G) \) and \( d_{\delta}(v) = b \), where \( X \) is the part (either \( A \) or \( B \)) which contains \( v \). We present a linear algorithm to find an odd decycling matching of subcubic graphs, Algorithm 1.

Theorem 3. Algorithm 1 returns in linear-time an odd decycling matching for subcubic graphs.

Proof. Let \( A \) be a maximal independent set of \( G \). Let \( B = V(G) \setminus A \). In this case, every vertex of \( A \) is of type \((k, 0)\) and there is no vertex in \( B \) of type \((0, k), k \in \{1, 2, 3\} \). Therefore, if there exists a vertex \( v \) of type \((a, b)\) with \( a < b \), then it must be in \( B \) and be of type \((1, 2)\). In order to prove the correctness of Algorithm 1, it is sufficient to show that the operations on lines 7–8 and 10–11 do not generate vertices of type \((a, b)\) with \( a < b \).

Let \( \{u\} = N_{G[A]}(v) \). If \( u \) is of type \((3, 0)\), then \( v \) is moved from \( B \) to \( A \) by lines 7–8. In this case, it follows that both \( u \) and \( v \) are vertices of type \((2, 1)\) after the line 8. If \( u \) is not of type \((3, 0)\), then the lines 10–11 modify the types of \( u \) and \( v \).

- If \( u \) is of type \((1, 1)\), then \( u \) and \( v \) are modified to type \((2, 0)\) and \((3, 0)\), respectively;
- If \( u \) is of type \((1, 0)\), then \( u \) and \( v \) are modified to type \((1, 0)\) and \((3, 0)\), respectively;
- If \( u \) is of type \((2, 0)\), then \( u \) and \( v \) are modified to type \((1, 1)\) and \((3, 0)\), respectively;
- If \( u \) is of type \((2, 1)\), then \( u \) and \( v \) are modified to type \((2, 1)\) and \((3, 0)\), respectively.
2.1 Preliminaries

We can see that each neighbor $w$ of $u$ in the same part $X \in \{A, B\}$ of $u$ loses exactly one neighbor (that is $u$) in $G[X]$. Moreover, $w$ receives at most one new neighbor (that is $v$) in $G[X]$. The same occurs for every neighbor of $v$ in $V(G) \setminus X$. Therefore, in any case it is not obtained vertices of type $(a, b)$ with $a < b$, which implies that the Algorithm 1 finishes. 

Despite the simplicity of Algorithm 1, determining the size of a minimum odd decycling matching of subcubic graphs is $NP$-hard, since this problem becomes analogous to MAX CUT [24] for such a class.

2 NP-Completeness for ODD DECYCLING MATCHING

In this section we prove that ODD DECYCLING MATCHING is $NP$-complete even for planar graphs of maximum degree at most 4. We organize the proof in three parts. In the first one we show some polynomial time reductions from NOT-ALL-EQUAL 3-SAT (NAE-3SAT) [35] and POSITIVE PLANAR 1-IN-3-SAT [32]. In the second part we prove that ODD DECYCLING MATCHING is $NP$-complete for graphs with maximum degree at most 4. This proof is a more intuitive and easier to understand the gadgets and construction of the next part. The third part presents a proof that ODD DECYCLING MATCHING is $NP$-complete even for planar graphs with maximum degree at most 5. Finally, the proof finishes as a corollary from the previous results by just slightly modifying the used gadgets.

### 2.1 Preliminaries

Let $F$ be a Boolean formula in CNF with set of variables $X = \{x_1, x_2, \ldots, x_n\}$ and set of clauses $C = \{c_1, c_2, \ldots, c_m\}$. The associated graph of $F$, $G_F = (V, E)$, is the bipartite graph such that there exists a vertex for every variable and clause of $F$, where $(X, C)$ is a bipartition of $V(G_F)$ into independent sets. Furthermore, there exists an edge $x_i c_j \in E(G_F)$ if and only if $c_j$ contains either $x_i$ or $\bar{x}_i$. We say that $F$ is planar if its associated graph is planar. In order to obtain a polynomial reduction, we consider the following decision problems, which are $NP$-complete.

#### NOT-ALL-EQUAL 3-SAT (NAE-3SAT) [35]

**Input:** A Boolean formula in 3-CNF, $F$.

**Question:** Is there a truth assignment to the variables of $F$, in which each clause has one literal assigned true and one literal assigned false?

#### POSITIVE PLANAR 1-IN-3-SAT [32]

**Input:** A planar Boolean formula in 3-CNF, $F$, with no negated literals.

**Question:** Is there a truth assignment to the variables of $F$, in which each clause has exactly one literal assigned true?

In order to prove the $NP$-completeness of ODD DECYCLING MATCHING, we first present a polynomial time reduction from NAE-3SAT and POSITIVE PLANAR 1-IN-3-SAT to the following decision problems, respectively:

#### NAE-3SAT

**Input:** A Boolean formula in CNF, $F$, where each clause has either 2 or 3 literals, each variable occurs at most 3 times, and each literal occurs at most twice.

**Question:** Is there a truth assignment to the variables of $F$ in which each clause has at least one literal assigned true and at least one literal assigned false?
After that, in order to preserve the planarity, we can follow the planar embedding \( F \) exactly once in \( F \). If \( F \) is planar then we can assume that \( F \) has no negative literals. As we can see, every variable \( x \) occurs at most 3 times in the clauses of \( F' \), since every variable \( x \) with \( d_{G_F}(x) \geq 3 \) is replaced by \( c_j \) new variables that are in exactly 3 clauses of \( F' \). By the construction, each literal occurs at most twice. Moreover, if \( F \) has no negative literals, then only the new variables have a negated literal and each one occurs exactly once in \( F' \).

Now, it remains to show that if \( G_F \) is planar then we can construct \( F' \) as a planar formula. Consider a planar embedding \( \Psi \) of \( G_F \). We construct \( G_F' \) replacing each corresponding vertex \( x \) of \( F \) by a cycle of length \( 2d_{G_F}(x) \) as described above. After that, in order to preserve the planarity, we can follow the planar embedding \( \Psi \) to add a matching between vertices corresponding to variables in such a cycle and vertices corresponding to clauses \( x \notin C \) and that \( x \in c_j \). Such a matching indicates in which clause of \( C \), a given new variable will replace \( x \) in \( F' \). Thus, without loss of generality, if \( G_F \) is planar then we can assume that \( F' \) is planar as well.

Let \( F \) be an instance of NAE-3SAT (resp. \textsc{Positive Planar 1-In-3-SAT}) such that \( X = \{x_1, \ldots, x_n\} \) denotes its set of variables and \( C = \{c_1, c_2, \ldots, c_m\} \) its set of clauses. Let \( F' \) be the formula obtained from \( F \) by the above construction. As we can observe, for any truth assignment of \( F' \), all \( x \in X \) (for a given variable \( x \) of \( F \)) have the same value. Therefore, any clause of \( F' \) containing exactly two literals has true and false values. At this point, it is easy to see that \( F \) has a not-all-equal (resp. 1-in-3) truth assignment if and only if \( F' \) has a not-all-equal (resp. 1-in-3) truth assignment.

Now we show the \( NP \)-completeness of \textsc{Odd Decycling Matching}. Let us call the graph depicted in Fig 3a by \textit{head}. Vertex \( v \) is the neck of the head. Given a graph \( G \), the next lemma shows that such a structure is very useful to ensure that some edges cannot be in any odd decycling matching of \( G \). The next simple lemma is used in the correctness of our reductions.
The odd decycling matching of variable gadget that also preserves the planarity. The circles with an \( H^3 \) are all 4-colorable. The next results show that the Theorem 6.

Proof. Let \( M \) be an odd decycling matching of \( G \). Suppose for a contradiction that there exists an edge \( e \) incident to \( v \), such that \( e \) contains an endvertex not in \( H \). In this case, we get that \( vh_1 \) and \( vh_4 \) does not belong to \( M \), which implies that \( h_1h_4 \in M \). By the triangle \( h_1h_2h_5 \), it follows that \( h_2h_5 \) must be in \( M \). Hence the cycle \( vh_1h_2h_3h_4v \) remains in \( G-M \), a contradiction.

Now suppose that \( vh_4 \in M \). In this case, the edge \( h_1h_2 \) cannot be in \( M \), otherwise the cycle \( h_1h_2h_3h_2h_1 \) survives in \( G-M \). In the same way, the edge \( h_1h_5 \notin M \), otherwise the cycle \( h_1h_2h_3h_2h_1 \) is not destroyed by \( M \). Therefore we get that \( h_2h_5 \) must be in \( M \), which implies that \( h_3h_6 \in M \). Hence the cycle \( h_3h_4h_7h_7h_3 \) belongs to \( G-M \). Since the triangle \( h_1h_2h_5 \) has no edge in \( M \), it is not destroyed by \( M \), a contradiction.

Finally, we get that \( vh_1 \) must be in \( M \), which implies that \( h_2h_5 \in M \) as well. Therefore, it follows that \( h_3h_6 \) must be in \( M \). Hence \( h_2h_5 \) also must be in \( M \), which turns the graph bipartite. Since all choices of the edges of \( M \) are necessary, we get that there is only one possible odd decycling matching of \( H \), which is perfect. Fig. 3b shows such a matching. This concludes the proof.

\[ \square \]

2.2 NP-Completeness for Graphs with Maximum Degree at Most 4

With Lemma 5 we can establish the NP-completeness of ODD DECYCLING MATCHING. Remember that graphs in \( \mathcal{B} \mathcal{M} \) are all 4-colorable. The next results show that the NP-completeness is also obtained even for 3-colorable bounded degree graphs. First we present a more intuitive proof by a reduction from NAE-3SAT, next we present a more complex proof that also preserves the planarity. The circles with an \( H \) in the figures represent an induced subgraph isomorphic to the head graph, whose neck is the vertex touching the circle. By simplicity, this pattern will be used in the remaining figures whenever possible.

Theorem 6. ODD DECYCLING MATCHING is NP-complete even for 3-colorable graphs with maximum degree at most 4.

Proof. We prove that ODD DECYCLING MATCHING is NP-complete by a reduction from NAE-3SAT. Let \( F \) be an instance of NAE-3SAT, with \( X = \{x_1,x_2,\ldots,x_n\} \) and \( C = \{c_1,c_2,\ldots,c_m\} \) be the sets of variables and clauses of \( F \), respectively. We construct a graph \( G = (V,E) \) as follows:

- For each variable \( x_i \in X \), we construct a variable gadget \( G_{x_i} \). Such a gadget consists on a diamond \( D \) with a head, whose neck is the vertex \( u_i \), of degree two in \( D \). The vertices of degree three in \( D \) are \( \ell_1 \) and \( \ell_2 \), which are the vertices containing the literal \( x_i \), while the last one, \( \ell_3 \), of degree two represents the negative literal \( \bar{x}_i \). Fig. 4 shows the variable gadget \( G_{x_i} \).

- For each clause \( c_j \in C \), we associate a clause gadget \( G_{c_j} \). If \( c_j \) contains three literals, then \( G_{c_j} \) is a triangle with vertices \( c_{j1}, c_{j2}, \) and \( c_{j3} \). Moreover, each vertex \( \ell_k \) is adjacent to a linking vertex \( \ell_k^j \), \( k \in \{1,2,3\} \), which is a neck of a head \( H \). Such clause gadget is shown in Fig. 5b. In a similar way, if \( c_j \) has size two, then \( G_{c_j} \) is as depicted in Fig. 5a, where \( \ell_1^j \) and \( \ell_2^j \) are the vertices that connect \( G_{c_j} \) to the gadgets of the variables contained in \( c_j \).

- We link a clause gadget \( G_{c_j} \) to a variable gadget \( G_{x_i} \), such that \( x_i \in c_j \), as follows. If \( c_j \) contains the positive literal \( x_i \), then add one edge between a linking vertex \( \ell_k^j \) to either \( \ell_1^j \) or \( \ell_2^j \), otherwise we add the edge \( \ell_3^j \ell_3^j \), for some \( 1 \leq k \leq 3 \).

Since the Head graph is 3-colorable, clearly the above construction generates also a 3-colorable graph. Next we prove that \( F \) has a truth assignment if and only if the graph \( G \) obtained form the above construction has an odd decycling matching. If \( F \) has a truth assignment \( \phi \), then each clause \( c_j \) contains at least one true literal and at least one false literal. For such a clause, we associate true to \( c_{j1}^\phi \) if and only if its corresponding literal is true in \( \phi \). In the same way, for every variable gadget \( G_{x_i} \), we associate true to \( \ell_1^j \) and \( \ell_2^j \) if and only if the positive literal \( x_i \) is true in \( \phi \). Therefore, we can construct a bipartition of \( V(G) \) into sets \( T \) and \( \bar{T} \), that represent the literal assigned true and false, respectively, as follows.

- For each clause gadget \( G_{c_j} \) of 3 literals, remove the edge \( c_{j1}c_{j2}^\phi \) if \( \phi(c_j) = \phi(c_{j1}) \), \( 1 \leq z \neq w \leq 3 \);
• For each clause gadget $G_{c_j}$ of 2 literals, remove either the edge $w_j^1c_j^1$ or $w_j^2c_j^2$;

• For every variable gadget $G_{x_i}$, remove the edge $d_i^1d_i^2$;

• For each induced head $H$, remove edges as in Fig. 3b.

It is not hard to see that the obtained graph is bipartite, since each linking vertex $\ell_j^k$ is in the opposite set of $c_j^k$ and of $d_i^k$, such that $\ell_j^k d_i^k \in E(G)$. Moreover, $c_j^1$ and $c_j^2$ are in opposite sets, for every clause of length 2. Since the removed edges are clearly a matching in $G$, it follows that $G \in \mathcal{B}$.\mathcal{B}$.

Now we consider that $G \in \mathcal{B}$. By Lemma 5, it follows that $d_i^1d_i^2$ must be in any odd decycling matching of $G$, for every variable gadget $G_{x_i}$. Analogously, either $w_j^1c_j^1$ or $w_j^2c_j^2$ and exactly one edge $c_j^1c_j^2$ must be included in any odd decycling matching of $G$, for every clause gadget $G_{c_j}$ of 2 and 3 literals, respectively. Therefore, for an odd decycling matching of $G$, we can associate to the parts of the bipartition of $G - M$ as true and false. Thus, it follows that:

• $d_i^1$ and $d_i^2$ are in the same part, while $d_i^k$ is in the opposite one, for every variable gadget $G_{x_i}$;

• $c_j^1$ and $c_j^2$ are in different parts, for every clause gadget $G_{c_j}$ of length 2;

• All the vertices $c_j^k$ are not in the same part, for every clause gadget $G_{c_j}$ of length 3;

Hence, every clause has at least one true and one false literal, which implies that $F$ is satisfiable. \qed

Since NAE-3SAT is polynomial time solvable for planar graphs [31], the previous construction cannot be planar. Moreover planar graphs are classical 4-colorable graphs. Hence it is interesting to know what happens in such a class. The next Subsection deals with this problem.

### 2.3 NP-Completeness for Planar Graphs

Now we will show that \textsc{Odd Decycling Matching} remains \textsc{NP}-complete even for 3-colorable planar graphs with maximum degree 4. We prove the \textsc{NP}-completeness by a reduction from \textsc{Planar 1-IN-3-SAT}. In order to prove this result, next we give a useful lemma.

**Lemma 7.** Let $b$ be a border of an odd $k$-pool graph $G$, such that $c_1$ and $c_k$ are its neighbors in $G$. It follows that every odd decycling matching of $G - b$ must contain exactly one edge of the internal cycle, which is different from $c_1c_k$. Moreover, there is only one odd decycling matching for such an edge.

**Proof.** Let $C = p_1p_2\ldots p_kp_1$ be the internal cycle of $G$ and let $b_i$ be the $i$-th-border of $G$, such that $N_G(b_i) = \{p_i, p_{i+1}\}$, $1 \leq i \leq k - 1$. Since $C$ has odd length, it follows that every odd decycling matching of $G$ contains at least one edge of $C$.

Suppose for a contradiction that $G$ has an odd decycling matching $M$ containing $p_1p_k$. In this case, we get that the edges in $\{p_1p_2, p_1b_1, p_kb_{k-1}, p_kb_{k-1}\}$ cannot be in $M$. Therefore $M$ must contain the edges $b_1p_2$ and $b_{k-1}p_{k-1}$. In the

Figure 5: Clause gadget $G_{c_j}$ in Theorem 6. Fig. 5a is such that $c_j = (x_a \lor x_b)$. Fig. 5b is such that $c_j = (x_a \lor x_b \lor x_c)$. 

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same way, we can see that the edges \{p_2p_3, p_2b_2, p_{k-1}b_{k-2}, p_{k-1}b_{k-2}\} are not in \(M\). Hence, it can be seen that all edges indent to \(p_{i-1}\) are forbidden to be in \(M\), which implies that the triangles \(p_{i-1}p_{i-1}b_{i-2}\) and \(p_{i-1}p_{i-1}b_{i-1}\) have no edge in \(M\), a contradiction by the choice of \(M\).

Let \(p_i p_{i+1}\) be an edge of \(C\) contained in an odd decycling matching \(M\) of \(G\). In a same fashion, the edges in \(\{p_i p_{i+1}, p_i b_{i+1}, p_i p_{i+1}, p_i b_{i+1}\}\) cannot be in \(M\). Following this pattern, we can see that every edge \(p_j b_j\) must be in \(M\), for every \(1 \leq j \leq i - 1\). Furthermore, it follows that \(b_z p_{z+1} \in M\), for every \(i + 1 \leq z \leq k - 1\). Since \(M\) contains one edge of every triangle of \(G\), it follows that \(M\) is unique, for every edge \(p_i p_{i+1}\). Finally, such an odd decycling matching contains only one edge of \(C\).

\[\square\]

**Theorem 8.** **ODD DECYCLING MATCHING** is \(NP\)-complete even for 3-colorable planar graphs with maximum degree at most 5.

**Proof.** Let \(F\) be an instance of \(PLANAR 1-IN-3-SAT_3\), with \(X = \{x_1, x_2, \ldots, x_n\}\) and \(C = \{c_1, c_2, \ldots, c_m\}\) be the sets of variables and clauses of \(F\), respectively. We construct a planar graph \(G = (V, E)\) of maximum degree 5 as follows:

- For each clause \(c_j \in C\), we construct a gadget \(G_{c_j}\), as depicted in Fig. 6. Such gadgets are just a 5-pool and a 7-pool less a border for clauses of size 2 and 3, respectively. Moreover, for the alternate edges of the internal cycle we subdivide them twice and append a head graph to each such a new vertex. Finally, we add two vertices \(\ell_j(k, w)\) and \(\ell_j(k, b)\), such that \(b_j^{2^{k-1}} \ell_j(k, w) \in E(G)\) and \(b_j^{2k} \ell_j(k, b) \in E(G)\), for \(k \in \{1, 2, 3\}\). For such new vertices, we append a head graph to each one.

- For each variable \(x_i \in X\), we construct a gadget \(G_{x_i}\), as depicted in Fig. 7. Such a gadget is a 7-pool less a border, where we subdivide the edges \(p_2^1 p_1^1, p_1^1 p_2^1, p_4^1 p_3^1\), and \(p_6^1 p_4^1\) twice, where every such a new vertex has a pendant head. we rename each border vertex \(b_0^{2k-1}\) \((k \in \{1, 3\})\) as \(d(k, b)\) and \(b_0^{2k}\) as \(d(k, w)\), for \(k \in \{1, 2, 3\}\). Moreover we add a new vertex \(d(2, b)\) adjacent to \(p_4^1\), which has a pendant head graph.

- The connection between clause and variable gadgets are as in Fig. 6 and Fig. 7, where each pair of arrow head edges in a variable gadget \(G_{x_i}\), corresponds to a pair of such edges in a clause gadget \(G_{c_j}\), such that \(x_i \in c_j\). More precisely, if \(x_i \in c_j\), then we add the edges \(\ell_j(k, b) d(k', b)\) and \(\ell_j(k, w) d(k', w)\), for some \(k \in \{1, 2, 3\}\) and for some \(k' \in \{1, 2\}\). On the other hand, if \(x_i \in c_j\), then we add the edges \(\ell_j(k, b) d(3, b)\) and \(\ell_j(k, w) d(3, w)\), for some \(k \in \{1, 2, 3\}\).

![Figure 6: Clause gadget \(G_{c_j}\) in Theorem 8. Fig. 6a is such that \(c_j = (x_a \lor x_b)\). Fig. 6b is such that \(c_j = (x_a \lor x_b \lor x_c)\).](image1)

![Figure 7: Variable gadget \(G_{x_i}\) in Theorem 8. Each pair of edges with one no endvertex connects \(G_{x_i}\) to one clause gadget \(G_{c_a}, G_{c_b},\) or \(G_{c_c}\), where \(x_i \in (c_a \cap c_b \cap c_c)\).](image2)
Given an edge $v$ we replace each variable vertex $v$.

Corollary 9. Odd Decycling Matching is NP-complete even for 3-colorable planar graphs with maximum degree 4.
Figure 9: All possible configurations given by the removal of an odd decycling matching $M$, represented by the stressed edges, from a variable gadget $G_{x_i}$. The colors in the vertices represent the bipartition of $G_{x_i} - M$.

3 Polynomial Time Results

In this section we present our positive results about \textsc{Odd Decycling Matching} for some graph classes.

3.1 Graphs with Only Triangles as Odd Cycles

We consider now a slightly general version of \textsc{Odd Decycling Matching}, where some edges are forbidden to be in any odd decycling matching.

\begin{description}
\item[Allowed Odd Decycling Matching (AODM)]
\item[Instance:] A graph $G$ and a set $F$ of edges of $G$.
\item[Task:] Decide whether $G$ has an odd decycling matching $M$ that does not intersect $F$, and determine such a matching if it exists.
\end{description}

A matching $M$ as in \textsc{AODM} is an \textit{allowed odd decycling matching} of $(G,F)$. Since $(G,F)$ has an allowed odd decycling matching if and only if $(K,E(K) \cap F)$ has an allowed odd decycling matching for every component $K$ of $G$, we may assume that $G$ is connected. Moreover, note that if $G$ has an allowed odd decycling matching, then $G \in \mathcal{BM}$, since all allowed odd decycling matching is an odd decycling matching of $G$. With this new notion, it follows the next result.

\textbf{Theorem 10.} \textsc{AODM} can be solved in polynomial time for graphs with only triangles as odd cycles.

\textit{Proof.} Let $G$ be a graph without odd cycles of size at least 5 and let $F$ be a set of edges of $G$. As described above, we can consider $G$ connected. Moreover, we can assume $G$ as a bridge-free graph, that is, a graph whose all blocks have size at least 3.

Consider a block decomposition $\Pi$ of $G$. Clearly we can assume that $G$ has at least two blocks. Let $B$ be a block of $G$ that contains exactly one cut-vertex $v$, that is, $B$ is a final block in $\Pi$. If $G[B]$ is bipartite, then clearly $(G,F)$ has an allowed odd decycling matching if and only if $(G',F')$ admits an allowed odd decycling matching, where $G' = (V(G) \setminus V(B)) \cup \{v\}$, $E(G) \setminus E(B)$ and $F' = F \setminus E(B)$. Hence we can suppose that $B$ has a triangle $v_1v_2v_3$.

If $\{v_1v_2,v_1v_3\}$ is not an edge cut, then $G - \{v_1v_2,v_1v_3\}$ has a path $P$ from $v_1$ to $\{v_2,v_3\}$. Consider $P$ such a path of length at least 2 and such that it is a longest one. Moreover, consider $v_2$ as the first vertex reached by $P$ between $v_2$ and $v_3$. In this way $P$ must have the form $v_1w_2$, otherwise either $G[V(P) \cup \{v_3\}]$ or $P \cup \{v_1v_2\}$ would contain an odd cycle of

Figure 10: The modified variable gadget.
length at least 5, when $P$ has either an even number of vertices or an odd number, respectively. It also follows that all path between $v_1$ and $v_2$, except $v_1v_2$, has length 2. Moreover, every path between $v_1$ and $v_3$, except $v_1v_3$, has length 2 and contains $v_2$.

If $u v_3 \in E(G)$, then $N_B(v_1) = \{u, v_2, v_3\}$, otherwise let $w \in N_B(v_1)$. Since $d_B(w) \ge 2$, we get that $N_B(w) = \{v_1, v_2\}$, otherwise would exist a path between $v_1$ and $v_2$ of length at least 3, a contradiction. However the cycle $v_1u v_3 v_1 u$ has length 5, a contradiction. Thus $G[\{v_1, v_2, v_3, u\}]$ is isomorphic to a $K_4$ and $\{v_1v_2, v_1v_3, v_1u\}$ is an edge cut. Hence $B$ is a block of $G$ and, by symmetry, $v = v_1$. In this case we get that $(G, F)$ has an allowed odd decycling matching if and only if $(G', F')$ has an allowed odd decycling matching, where $F$ does contain any matching of $B$ of maximum size

$$G' = ((V(G) \setminus V(B)) \cup \{v\}, E(G) \setminus E(B)) \text{ and } F' = (F \setminus E(B)) \cup \{vx : x \in N_G(v) \setminus \{v_1, v_2, v_3\}\}.$$ 

If $u v_3 \notin E(G)$, then $\{v_2, v_1\}$ is an edge cut for every vertex $z \in N_B(v_2) \cap N_B(v_1)$. Furthermore, by symmetry, $N_B(v_2) \cap N_B(v_1)$ is an independent set. Thus $\{v_1, v_2\} \cup \{N_B(v_2) \cap N_B(v_1)\}$ is a block of $G$. If $v \in \{v_1, v_2\}$ and $|N_B(v_2) \cap N_B(v_1)| \ge 3$, then $(G, F)$ has an allowed odd decycling matching and only if $(G', F')$ has an allowed odd decycling matching, where $v_1v_2 \notin F$ and

$$G' = ((V(G) \setminus V(B)) \cup \{v\}, E(G) \setminus E(B)) \text{ and } F' = (F \setminus E(B)) \cup \{vx : x \in N_G(v) \setminus V(B)\}.$$ 

If $v \in \{v_1, v_2\}$ and $|N_B(v_2) \cap N_B(v_1)| = 2$, then $(G, F)$ has an allowed odd decycling matching if and only if $(G', F')$ has an allowed odd decycling matching, where $v_1v_2 \notin F$, or $F \supseteq B \cup \{v_1u, v_2v_3\}$, or $F \supseteq \{v_1v_3, v_2u\}$, and

$$G' = ((V(G) \setminus V(B)) \cup \{v\}, E(G) \setminus E(B)) \text{ and } F' = (F \setminus E(B)) \cup \{vx : x \in N_G(v) \setminus V(B)\}.$$ 

If $v \notin \{v_1, v_2\}$, then $(G, F)$ has an allowed odd decycling matching if and only if $(G', F')$ has an allowed odd decycling matching, where $v_1v_2 \notin F$ and

$$G' = ((V(G) \setminus V(B)) \cup \{v\}, E(G) \setminus E(B)) \text{ and } F' = (F \setminus E(B)).$$ 

Finally it remains the case that $G[\{v_1, v_2, v_3\}]$ is a block and, by symmetry, suppose $v = v_1$. Thus $(G, F)$ has an allowed odd decycling matching if and only if $(G', F')$ has an allowed odd decycling matching, where $v_2v_3 \notin F$ and

$$G' = ((V(G) \setminus V(B)) \cup \{v\}, E(G) \setminus E(B)) \text{ and } F' = (F \setminus E(B)), \text{ or } v_1v_2 \notin F \text{ or } v_1v_3 \notin F,$$

and

$$G' = ((V(G) \setminus V(B)) \cup \{v\}, E(G) \setminus E(B)) \text{ and } F' = (F \setminus E(B)) \cup \{vx : x \in N_G(v) \setminus V(B)\}.$$ 

\[ \square \]

### 3.2 Graphs with Small Dominating Sets

We will show now that ODD DECYCLING MATCHING can be solved in polynomial time when the given graph $G$ has a dominating set of constant size, which is also given as input. Such a result generalizes some known graph classes, as for example $P_3$-free graphs [13], since the graphs in $\mathcal{R, M}$ do not admit $K_5$ as a subgraph.
Theorem 11. Let $k$ be a positive integer. For a graph $G$ whose domination number is at most $k$, it is possible to decide in polynomial time whether $G$ has a matching $M$ such that $G - M$ is bipartite, and to find such a matching if it exists.

Proof. Let $G$ be as in the statement. A dominating set of order at most $k$ can be found in time $O(n^k)$. Let $D$ be such a dominating set of $G$ of order at most $k$. Let $\mathcal{P}_D$ be the set of all bipartitions $P_D$ of $D$ into sets $A_D$ and $B_D$, such that $D[A_D]$ and $D[B_D]$ do not have any vertex of degree 2. Note that $|\mathcal{P}_D| = O(2^k)$.

Let $P_D \in \mathcal{P}_D$ be a bipartition of $D$. We partition all of the other vertices of $G - D$ in such a way that $P_D$ defines a bipartition of $G - M_D$, if one exists, where $M_D$ is a matching that will be removed, given the choice of $D$. We do the following tests and operations for each vertex $v \in V(G) \setminus V(D)$:

- If $d_{A_D}(v) \geq 2$ and $d_{B_D}(v) \geq 2$, then $P_D$ is not a valid partition;
- If $d_{A_D}(v) \geq 2$, then $A_D \leftarrow A_D \cup \{v\}$;
- If $d_{B_D}(v) \geq 2$, then $B_D \leftarrow B_D \cup \{v\}$.

Iteratively we allocate the vertices in $V(G) \setminus V(D)$ as described above into the respective sets $A_D$ e $B_D$, or we stop if it is not possible to acquire a valid bipartition. After these operations, $V(G) \setminus V(A_D \cup B_D)$ can be partitioned into three sets as follows:

- $X = \{u \in V(G) \setminus (A_D \cup B_D) : d_{A_D}(u) = 1 \text{ and } d_{B_D}(u) = 0\}$;
- $Y = \{u \in V(G) \setminus (A_D \cup B_D) : d_{A_D}(u) = 0 \text{ and } d_{B_D}(u) = 1\}$;
- $Z = \{u \in V(G) \setminus (A_D \cup B_D) : d_{A_D}(u) = 1 \text{ and } d_{B_D}(u) = 1\}$.

Since every vertex in $V(G) \setminus V(D)$ has a neighbor in $D$, it follows that the neighborhood of all the vertices of $X \cup Y \cup Z$ in $A_D \cup B_D$ is in $D$. In this way, we can make a choice of a matching $M_D$ to be removed, such that all of the vertices of $X \cup Y \cup Z$ in $A_D \cup B_D$ are allocated in $A_D \cup B_D$ and $G - M_D$ must be bipartite. Note that there are $(n - k)^k$ possibilities of choices for $M_D$.

In this way, we can choose a dominating set $D$ of order at most $k$ and make all possible bipartitions $P_D$ of $V(D)$ into sets $A_D$ and $B_D$ as previously described. For each one, we iteratively allocate the vertices of $V(G) \setminus V(D)$ into $A_D$ and $B_D$ in a unique way. After this operations, we test all possible matchings $M_D$ with edges between vertices of $D$ and those in $X \cup Y \cup Z$. We can see that in any case we do $O(n^3)$ operations, which concludes the proof.

Corollary 12. Odd Decycling Matching can be solved in polynomial time for $P_3$-free graphs.

Proof. Follows from the fact that every connected $P_3$-free graph has a dominating clique or a dominating $P_3$ [13], and graphs in $\mathcal{F}$ do not admit $K_5$ as a subgraph.

3.3 (Claw, Paw)-free Graphs

In this subsection we consider graphs that have no induced subgraph isomorphic to the claw ($K_{1,3}$) or to the paw ($K_{1,3}$ plus an edge).

Let $G$ be a connected (claw, paw)-free graph. We first prove that if $G \in \mathcal{F}$, then the neighborhood of any vertex $v$ of $G$ has a small size.

Lemma 13. If $G \in \mathcal{F}$ is a claw-free graph, then $\Delta(G) \leq 5$.

Proof. Suppose for a contradiction that $G$ has a vertex $v$ of degree at least 6, such that $\{v_1, v_2, v_3, v_4, v_5, v_6\} \subseteq N_G(v)$. Since $G$ is claw-free, then either $G[N_G(v)]$ has exactly two connected components, which must be cliques, or itself is connected. In the first case, since $G$ is $K_5$-free, it follows that each connected component of $G[N_G(v)]$ has size at most 3.

Moreover, by Lemma 1(iii), if both have size at least 3, then $G \notin \mathcal{F}$, a contradiction.

Now suppose that $G[N_G(v)]$ is connected. If $G[N_G(v)]$ has a $P_5 = abcd\varepsilon$ as a subgraph, then it is not difficult to see that the only odd decycling matching $M$ of $G[\{v\} \cup \{a, b, c, d, \varepsilon\}]$ should have the edges $ab, ac, bc, cd$ and $\varepsilon$. Moreover $ae \notin E(G)$, otherwise $G[\{v\} \cup \{a, b, c, d, e\}]$ has a $W_5$ as a subgraph, a contradiction by Lemma 1(iii). In this way, any other vertex $f \in N(v)$ can be adjacent only to vertex $c$, since all of the vertices in $\{v, a, b, c, d, e\}$ are matched by $M$. Hence $G[\{v\} \cup \{a, e, f\}]$ induces a claw in $G$, a contradiction.

On the other hand, necessarily $G[N_G(v)]$ has a $P_5$, otherwise there exist at least three connected components in $G[N_G(v)]$.

By symmetry, suppose that $v_1v_2$ and $v_2v_3 \in E(G)$. By Lemma 1(ii) $G[\{v_1, v_3, v_5\}]$ has exactly one edge, say $v_1v_5$. Since $G[N_G(v)]$ is connected, the same has a path $P$ of length at least 4 which connects $v_5$ to $v_2$ and $v_3$ by at least one vertex between $v_1, v_2, v_3$ and $v_5$. Since $G[N_G(v)]$ does not contain $P_5$, we can rename the vertices in $\{v_1, v_2, v_3, v_4, v_5, v_6\}$ such that $P = v_1v_2v_3v_4$. We know that $v_1v_4 \notin E(G)$, otherwise $P \cup \{v\}$ is a $W_5$, a contradiction by Lemma 1(iii). Hence $v_5$ and $v_6$ must be adjacent to at least one of $v_1$ and $v_4$, which creates a $P_5$ in $G[N_G(v)]$, a contradiction.

\[\square\]
By Lemma 13, it is not hard to see that the possible neighborhoods of a vertex \( v \) of a claw-free graph \( G \in \mathcal{B}M \) are depicted in Fig. 12.

It follows from Lemma 13 and of Fig. 12 that the only possibilities to the neighborhood of a vertex \( v \) in a (claw, paw)-free graph are as in Fig. 12a, Fig. 12f, Fig. 12g, Fig. 12h and Fig. 12i. In this way we can directly conclude the following lemma.

**Lemma 14.** If \( G \in \mathcal{B}M \) is a (claw, paw)-free graph then \( \Delta(G) \leq 3 \).

Hence we can just apply Algorithm 1 to obtain a linear-time algorithm for (claw, paw)-free graphs. Moreover, by Lemma 14 we can characterize the connected (claw, paw)-free graphs that admit an odd decycling matching.

**Theorem 15.** If \( G \) is a connected (claw, paw)-free graph, then \( G \in \mathcal{B}M \) if and only if \( G \) is isomorphic to a path, a cycle, a diamond or to a \( K_4 \).

**Proof.** Let \( G \) be a connected (claw, paw)-free graph. Clearly, if \( G \) is isomorphic to a path, a cycle, a diamond or to a \( K_4 \), then \( G \in \mathcal{B}M \).

Now consider \( G \in \mathcal{B}M \). By Lemma 14 we know that does not exist any vertex of degree 4 in \( G \).

If all of the vertices of \( G \) have degree 2, then \( G \) is isomorphic to a cycle.

If \( G \) is the trivial graph, then the theorem follows. Let \( v \) be a vertex of degree 1 in \( G \). It follows that either \( G \) is a path of length at least one, or there is a vertex \( u \) of degree three. Consider \( u \) such that it is such a vertex closest to \( v \) in \( G \).

Let \( w \in N_G(u) \) be in the path \( P \) from \( v \) to \( u \) and let \( u_1 \) and \( u_2 \) be its two neighbors except for \( w \) in \( G \). Since \( d_G(w) \leq 2 \), we get that \( w \) is not adjacent neither to \( u_1 \), nor to \( u_2 \). Thus \( u_1u_2 \in E(G) \), since \( G \) is claw-free. In this way \( G[wu_1u_2] \) is isomorphic to a paw, a contradiction. Hence \( G \) must be a path.

Finally suppose that \( G \) has a vertex \( v \) of degree 3 and that \( G \) is not isomorphic to a \( K_4 \). In this way it follows that \( G[v \cup N_G(v)] \) is isomorphic to a diamond. Let \( N_G(v) = \{v_1, v_2, v_3\} \), such that \( d_G(v_2) = 3 \). Since \( v \) and \( v_2 \) have degree 3, they cannot be adjacent to any other vertex. Suppose that \( v_1 \) has a neighbor \( u \notin \{v, v_2\} \). Since \( uv, vu \notin E(G) \), then \( G[u, v, v_1, v_2] \) is isomorphic to a paw, a contradiction. It follows in a similar way that \( N_G(v_3) = \{v, v_2\} \). Hence \( G \) is isomorphic to a diamond. \( \square \)

### 4 Fixed-Parameter Tractability

In this section, we consider the parameterized complexity of **ODD DECYCLING MATCHING**, and present an analysis of its complexity when parameterized by some classical parameters.

**Definition 1.** The clique-width of a graph \( G \), denoted by \( \text{cwd}(G) \), is defined as the minimum number of labels needed to construct \( G \), using the following four operations [8]:

1. Create a single vertex \( v \) with an integer label \( \ell \) (denoted by \( \ell(v) \));
2. Disjoint union of two graphs (i.e. co-join) (denoted by \( \oplus \));
3. Join by an edge every vertex labeled \( i \) to every vertex labeled \( j \) for \( i \neq j \) (denoted by \( \eta(i, j) \));
4. Relabeling all vertices with label \( i \) by label \( j \) (denoted by \( \rho(i, j) \)).
Some graph classes with bounded clique-width include cographs [8], distance-hereditary graphs [25], graphs with bounded neighborhood diversity [28], and graphs with bounded tree-width such as forests, pseudoforests, cactus graphs, series-parallel graphs, outerplanar graphs, Halin graphs, Apollonian networks, and control flow graphs [5, 8, 11, 25, 36]. The same follows for \((P_6, \text{claw})\)-free and \((\text{claw, co-claw})\)-free graphs [9, 10].

Courcelle, Makowsky and Rotics [18] stated that for any graph \(G\) with clique-width bounded by a constant \(k\), and for each graph property \(\Pi\) that can be formulated in a \textit{monadic second order logic} (MSOL\(_1\)), there is an \(O(f(cwd(G)) \cdot n)\) algorithm that decides if \(G\) satisfies \(\Pi\) [15–19]. In this monadic second-order graph logic known as MSOL\(_1\), the graph is described by a set of vertices \(V\) and a binary adjacency relation \(edge(\ldots)\), and the graph property in question may be defined in terms of sets of vertices of the given graph, but not in terms of sets of edges.

Using Courcelle, Makowsky and Rotics’s meta-theorem based on monadic second order logic [18], in order to show the fixed-parameter tractability of ODD DECYCLING MATCHING when parameterized by clique-width, it remains to show that the related property is MSOL\(_1\)-expressible.

**Theorem 16.** ODD DECYCLING MATCHING is in FPT when parameterized by the clique-width.

**Proof.** Remind that the problem of determining whether \(G\) has an odd decycling matching is equivalent to determine whether \(G\) admits an \((1,1)\)-coloring, which is a 2-coloring of \(V(G)\) in which each color class induces a graph of maximum degree at most 1 [30]. Thus, it is enough to show that the property “\(G\) has an \((1,1)\)-coloring” is MSOL\(_1\)-expressible.

We construct a formula \(\phi(G)\) such that \(G \in \mathcal{R}\mathcal{M} \iff \phi(G)\) as follows:

\[
\exists S_1, S_2 \subseteq V(G) : (S_1 \cap S_2 = \emptyset) \land (S_1 \cup S_2 = V(G)) \land
\forall v_1 \in S_1 \left[ \exists u_1, w_1 \in S_1 : (u_1 \neq w_1) \land edge(u_1, v_1) \land edge(w_1, v_1) \right] \land
\forall v_2 \in S_2 \left[ \exists u_2, w_2 \in S_2 : (u_2 \neq w_2) \land edge(u_2, v_2) \land edge(w_2, v_2) \right].
\]

Since clique-width generalizes several graph parameters [28], we obtain the following corollary.

**Corollary 17.** ODD DECYCLING MATCHING is in FPT when parameterized by the following parameters:

- neighborhood diversity;
- treewidth;
- pathwidth;
- feedback vertex set;
- vertex cover.

**Corollary 18.** ODD DECYCLING MATCHING is solvable in polynomial-time for chordal graphs.

**Proof.** Since graphs containing \(K_5\) does not have an odd decycling matching, chordal graphs in \(\mathcal{R}\mathcal{M}\) have bounded treewidth [34], and thus, ODD DECYCLING MATCHING is solvable in polynomial-time for such a class.

Courcelle’s theorem is a good classification tool, however it does not provide a precise FPT-running time bound. Next result shows the exact upper bound for ODD DECYCLING MATCHING parameterized by \(vc(G)\), the vertex cover number of \(G\).

**Theorem 19.** ODD DECYCLING MATCHING admits a \(2^{O(vc(G))} \cdot n\) algorithm.

**Proof.** Let \(S\) be a vertex cover of \(G\) such that \(|S| = vc(G)\). The algorithm follows in a similar way to Algorithm 11. Let \(\mathcal{P}\) be the set of all bipartitions \(P_5\) of \(S\) into sets \(A_S\) and \(B_S\), such that \(S[A_S]\) and \(S[B_S]\) do not have any vertex of degree 2. Note that \(|\mathcal{P}\)\(|= O(2^k)\).

For each \(P_5 \in \mathcal{P}\), we will check if an odd decycling matching of \(G\) can be obtained from \(P_5\) by applying the following operations:

For each vertex \(v \in V(G) \setminus V(S)\) do
- If \(d_{A_S}(v) \geq 2\) and \(d_{B_S}(v) \geq 2\), then \(P_5\) is not a valid partial partition;
- If \(d_{A_S}(v) \geq 2\), then \(A_S \leftarrow A_S \cup \{v\}\);
- If \(d_{B_S}(v) \geq 2\), then \(B_S \leftarrow B_S \cup \{v\}\).

After that, if for all vertices the first condition is not true, then \(V(G) \setminus V(A_S \cup B_S)\) can be partitioned into three sets:
Definition 2. A graph \( G(V, E) \) has neighborhood diversity \( nd(G) = t \) if we can partition \( V \) into \( t \) sets \( V_1, \ldots, V_t \) such that, for every \( v \in V \) and all \( i \in 1, \ldots, t \), either \( v \) is adjacent to every vertex in \( V_i \) or it is adjacent to none of them. Note that each part \( V_i \) of \( G \) is either a clique or an independent set.

The neighborhood diversity parameter is a natural generalization of the vertex cover number. In 2012, Lampis [28] showed that for every graph \( G \) we have \( nd(G) \leq 2vc(G) + vc(G) \). The optimal neighborhood diversity decomposition of a graph \( G \) can be computed in \( O(n^3) \) time [28].

Theorem 20. ALLOWED ODD DECYCLING MATCHING admits a kernel with at most \( 2 \cdot nd(G) \) vertices when parameterized by neighborhood diversity number.

Proof. Given an instance \( (G, F) \) of ALLOWED ODD DECYCLING MATCHING such that \( G \) is a graph and \( F \subseteq E(G) \) a set of forbidden edges. The kernelization algorithm consists on applying the following reduction rules:

1. If \( G \) contains a \( K_5 \), then \( G \) has no allowed odd decycling matching; otherwise

2. If a part \( V_i \) induces a \( K_5 \) and exist two vertices in \( V(G) \setminus V_i \) adjacent to \( V_i \), then \( G \) has no allowed odd decycling matching; otherwise

3. If a subgraph of \( G \) induces either a \( K_5 \) or a \( K_3 \) and does not admit an allowed odd decycling matching, then \( G \) has no allowed odd decycling matching; otherwise

4. Remove all parts isomorphic to a \( K_3 \);

5. Remove all isolated parts isomorphic to a \( K_3 \);

6. If \( V_i \) is a part that induces a \( K_5 \) and \( v \in V(G) \setminus V_i \) is adjacent to \( V_i \) (note that \( \{v\} \) is a part), then remove \( V_i \) and \( F \leftarrow F \cup \{uv : u \in N_G(v) \setminus V_i\} \);

7. If a part \( V_i \) induces an independent set of size at least 3, then contract it into a single vertex \( v_i \) (without parallel edges) and forbids all of its incident edges;

It is easy to see that all reduction rules can be applied in polynomial time, and after applying them any remaining part has size at most two. As the resulting graph \( G' \) has \( nd(G') \leq nd(G) \) then \( |V(G')| \leq 2 \cdot nd(G) \). Thus, it remains to prove that the application of each reduction rule is correct. As \( K_5 \) and \( K_3 - e \) are forbidden subgraphs, and any odd decycling matching of a \( K_3 \) is a perfect matching then rules 1, 2, 3, 4, 5 and 6 can be applied in such an order. Finally, the correctness of rule 7 follows from the following facts: (i) if \( G' \) has an allowed odd decycling matching, then \( G \) has also an allowed odd decycling matching, because bipartite graph class is closed under the operation of replacing vertices by a set of false twins, which have the same neighborhood as the replaced vertex; (ii) if \( G' \) does not admit an allowed odd decycling matching then \( G \) also does not admit an allowed odd decycling matching, because if a contracted single vertex \( v_i \) is in an odd cycle in \( G' \), then even replacing \( v_i \) by \( V_i \) (|\( V_i \| \geq 3 \)) and removing some incident edges of \( V_i \) which form a matching, some vertex of \( V_i \) remains to an odd cycle. \( \square \)
5 Conclusion

The polynomial time results obtained here can clearly be extended to other graph classes, as well as characterizations of such graph classes that have an odd decycling matching. Such polynomial time algorithms must be applicable to sparse graphs, since they are even subgraph of \( K_5 \) free, such as the \( W_4 \).

We found a good dichotomy related to the \( NP \)-completeness of \( \text{ODD DECYCLING MATCHING} \), where it is know that subcubic graphs have a linear-time algorithm and it is \( NP \)-complete even for planar graphs with degree at most 4.

As we saw, the minimization version have received attention in the literature, where we can see that \( \text{ODD DECYCLING MATCHING} \) coincides to \( \text{MAX-CUT} \) when restricted to subcubic graphs. The ideas presented in this work can be used to obtain new combinatorial approximative algorithms with a better approximation factor than the best one know, which is based on semi-defined programming [4].

Interesting properties regarding the chromatic number of graphs in \( \mathcal{H} \) can be proposed. For example, which graphs \( G \in \mathcal{H} \) are such that \( \chi(G-M) \leq \chi(G) \), for an odd decycling matching \( M \) of \( G \)? Moreover, what is the maximum size of the gap between \( \chi(G-M) \) and \( \chi(G) \)?

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