TWO-QUBIT BLOCH SPHERE

Chu-Ryang Wie
State University of New York at Buffalo, Department of Electrical Engineering, 230B Davis Hall, Buffalo, NY 14260
(Dated: March 3, 2020)

ABSTRACT
Three unit spheres can represent the two-qubit pure states. One such model previously reported [1] is refined and presented. The three spheres are named the base sphere, entanglement sphere and fiber sphere. The base sphere and the entanglement sphere represent both the reduced density matrix of one qubit (the base qubit) as well as the non-local entanglement measure, concurrence, while the fiber sphere represents the other qubit (the fiber qubit) geometrically under a local unitary operation. When the bipartite state becomes separable, the base sphere and the fiber sphere seamlessly become the single-qubit Bloch sphere of each qubit. Since either qubit may be chosen to be the base qubit, two sets of such spheres can fully represent the reduced density matrices of both qubits as well as the concurrence where the concurrence value is the same in the two sets. The concurrence in this Bloch sphere is correctly related to the reduced density matrices. In particular, the entanglement (concurrence) and the imaginary part of coherence (off-diagonal element of the reduced density matrix) are related via an angle parameter and are represented together in the entanglement sphere. We illustrate partially entangled two-qubit states on the Bloch spheres. The significance of each sphere is discussed.

1. Introduction
In 2014 we reported a complete Bloch sphere parameterization for two-qubit pure states [1], in which one qubit (called the base qubit) was assigned to the 4-sphere $S^4$ Hopf base space and the other qubit (called the fiber qubit) to the 3-sphere $S^3$ Hopf fiber space, based on the Hopf fibration [2,3] which maps a 7-sphere to 4-sphere base with a 3-sphere fiber. Since then there were reports of parameterizing the two-qubit pure state Bloch spheres where, in particular, a single angle parameter was used to represent the entanglement as per the Schmidt decomposition of bipartite pure states [4,5]. In [4], this angle was related to the von Neuman entropy which at angle $\pi/2$ was 1 (i.e., maximal entanglement). In [5], this single parameter, entanglement angle was related to the Schmidt coefficients of the bipartite pure state when the state was expressed in terms of the Schmidt bases, which are used as the antipodal points of the local Bloch spheres. In this case, a local unitary operation will rotate the Schmidt bases independently for each Bloch sphere while leaving the entanglement angle constant. Expressing the entanglement this way, by a single invariant angle appears elegant at a first glance. However, the changing Schmidt basis under the local operations, for example, means changing basis vectors for the Bloch spheres, losing their geometric and visual advantage that the Bloch spheres afford. This is also an inconvenience in quantum computing where the computational basis states are fixed, with respect to which the final measurements statistics are obtained. In addition, when the two qubits undergo a local or a coupled unitary operation, using a local Bloch sphere with an ever-changing Schmidt basis vectors doesn’t seem to be a good way to represent the two-qubit states geometrically.

In our Bloch spheres, both reduced density matrices and the entanglement measure, concurrence, are consistently represented for all range of entanglement, from separable to maximally entangled states. A reduced density matrix has three parameters (one diagonal element and the two off-

---

1 Electronic address: wie@buffalo.edu
diagonal elements – real and imaginary). Concurrence requires another parameter. The concurrence and the reduced density matrix are related by [6]

\[
\text{concurrence} = \sqrt{2(1 - \text{Tr} \rho^2)}
\]

where \( \rho \) is the reduced density matrix, leaving three parameters for the Bloch spheres to encode the non-local entanglement and the local properties per qubit. In our model, these parameters are represented on two Bloch spheres. To represent the reduced density matrices for both qubits, we use two different sets of Bloch spheres. In our Bloch sphere representation, the Bloch basis states are fixed to the computational basis states and the base and entanglement spheres have a direct physical interpretation with respect to the reduced density operators and the entanglement (concurrence). This is the strength of our model.

Consider a quantum circuit with an initial input state \( |00\rangle \). An entangled state is created by first putting one qubit, the control qubit, in a superposition state, and then a controlled unitary operation entangles the two qubits. For example, the \( x \)- or \( y \)-rotation by angle \( \eta \), \( R_x(\eta) \), creates the linear superposition on the control qubit, which is then followed by a controlled \( x \)- or \( y \)-rotation by angle \( \omega \), \( C-R_x(\omega) \), on the target qubit, to create the entanglement. In this case, the Bloch sphere of the control-qubit can have a polar angle of \( \eta \), giving an initial degree of superposition \( \sin(\eta) \) for the control qubit. After the controlled rotation, the concurrence is \( \sin(\eta)\sin(\omega/2) \). Hence the unitary operation \( R_x(\eta) \) affects both the local Bloch sphere coordinate of the control qubit and the non-local, entanglement measure, concurrence. Using more than one angle parameter to represent the concurrence in this case would be more natural.

In this paper, we start with an introduction of the two-qubit Bloch sphere model we reported in 2014 [1], present an improved version of the entanglement sphere that includes an inner sphere of a radius pertinent to the entanglement measure, provide some examples for states in a two-qubit quantum circuit, and we summarize. The significance of each sphere with a detailed discussion of the coordinates is presented in the Appendix.

2. Summary of The Original Two-Qubit Bloch Sphere Model

We summarize here the two-qubit Bloch sphere model we presented in the 2014 paper [1]. The two-qubit state vector \( |\psi\rangle \) with complex amplitudes, \( \alpha \), \( \beta \), \( \gamma \) and \( \delta \)

\[
|\psi\rangle = \alpha |00\rangle + \beta |01\rangle + \gamma |10\rangle + \delta |11\rangle
\]

(1)

with a normalization condition can be represented by a point on the 7-sphere \( S^7 \), the unit sphere in Euclidean 8-space. This 7-sphere can be mapped to a 4-sphere \( S^4 \), the so-called a Hopf base, with a fiber \( S^3 \) by the Hopf fibration process [3]. The 4-sphere \( S^4 \) is the Hopf base which is a unit sphere in the Euclidean 5-space given as \( x_0^2 + x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1 \) in Cartesian coordinates. It is assumed that one qubit is assigned to the \( S^4 \) base and the other qubit to the \( S^3 \) fiber. Here we will refer to the qubit assigned to the base as the base qubit, and the other qubit as the fiber qubit.

A general single qubit state may be written as the eigenspinor of a Pauli matrix for direction specified by \( \theta \) and \( \phi \) in spherical coordinates, and given as a 2-vector in complex field. This spinor represents a point on the \( S^3 \) Bloch sphere. If the complex imaginary unit ‘i’ is replaced by a variable pure-imaginary unit quaternion ‘i’ , this spinor represents a point on a 4-sphere \( S^4 \). This 2-vector in quaternion field is given by [1]

\[
|\bar{\psi}\rangle = \begin{pmatrix} \cos \frac{\theta}{2} \\ \sin \frac{\theta}{2}e^{i\phi} \end{pmatrix}
\]

(2)
Where, the tilde on the state vector is to indicate that it is not a common quantum state. The general pure-imaginary unit quaternion \( t \) can be parameterized with two angles, \( \chi \) and \( \xi \), in terms of the quaternion imaginary units:

\[
t = i \sin \chi \cos \xi + j \sin \chi \sin \xi + k \cos \chi \tag{3}
\]

where, \( i, j \) and \( k \) are the anti-commuting imaginary units of quaternions. The spinor in quaternion field, Eq.(2), represents a point on the Hopf base 4-sphere and can be shown on two 2-spheres with angle coordinates \((\theta, \phi)\) for the base sphere and \((\chi, \xi)\) for the entanglement sphere. The entanglement sphere is limited to the northern hemisphere \((\chi \leq \pi/2)\) [1].

The \( S^4 \) Cartesian coordinates, \( x_0, x_1, x_2, x_3, \) and \( x_4 \) in the Euclidean 5-space were defined in terms of four angles \( \theta, \phi, \chi, \) and \( \xi \) in ref.[1] and are given in the Appendix. These two spheres were shown to represent the reduced density matrix of the base qubit and the entanglement measure concurrence [1]. On these two spheres, the separable states and the maximally entangled states are as shown in Figure 2. The entanglement sphere for these two extreme cases is similar in appearance to the zero superposition (classical bits, the computational basis states) and the uniform superposition (maximum superposition) states, respectively, on the single qubit Bloch sphere as shown in Figure 1.

**Figure 1** Single qubit Bloch sphere: (a) the zero superposition states, \(|0>\) and \(|1>\), and (b) the uniform superposition states. The sine value of the polar angle of the state vector is zero for (a), and unity for (b).
Figure 2 Two qubit Bloch spheres, showing the base sphere and entanglement sphere of $S^d$ Hopf base for (a) the separable states, and (b) the maximally entangled states (MES).
For (a), either sphere coordinate means that the state is separable, and for (b) both sphere conditions must be satisfied for MES.

The $S^d$ fiber can be represented by a unit quaternion $q_f$ which, together with the 2-vector in Eq.(2) of Hopf base $S^d$, completes the Bloch sphere parameterization.

$$\begin{pmatrix} \cos \frac{\theta_f}{2} \\ \sin \frac{\theta_f}{2} e^{\phi_f} \end{pmatrix} q_f = \begin{pmatrix} \alpha + \beta \gamma \\ \gamma + \delta \gamma \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} \alpha + \gamma \delta \\ \beta + \delta \gamma \end{pmatrix}$$

where the first 2-vector on the right hand side is when qubit-A is assigned to the Hopf base, and the second is when qubit-B is assigned to the Hopf base. It also should be noted that, in this work, the complex amplitudes of $\alpha, \beta, \gamma$ and $\delta$ must use a quaternion imaginary unit ‘$k$’ as their imaginary unit in order to work with quaternions. That is, a complex number $\alpha$ with the real part $\alpha_r$ and the imaginary part $\alpha_i$, Complex($\alpha_r, \alpha_i$), becomes a Quaternion($\alpha_r, 0, 0, \alpha_i$) in order for $\alpha$ to be regarded a quaternion. The unit quaternion $q_f$ for the $S^d$ fiber is parameterized by three angles: $(\theta_f, \phi_f)$ for the fiber sphere and $\zeta_f$ for the phase factor.

$$q_f = (\cos \frac{\theta_f}{2} + \sin \frac{\theta_f}{2} e^{k \phi_f}) e^{k \zeta_f} \quad (5)$$

As either of the two qubits may be assigned to the Hopf base, our model consists of two equivalent sets of three Bloch spheres, where one set is for one qubit assigned to the base, and the other set for the other qubit assigned to the base. Only the Hopf base $S^d$ (consisting of the base sphere and the entanglement sphere) has all local information about the base qubit as well as the non-local information of entanglement of the bipartite state.

The Cartesian coordinates of the base sphere are given by $(x_1, b, x_0) = (\sin \theta \sin \phi, \sin \theta \sin \phi, \cos \theta)$. The entanglement sphere may be represented by $t, bt$, or both. In this paper, we improved the entanglement sphere over the one presented earlier in ref.[1] by including both $t$ (unit radius, outer sphere) and $bt$ (radius $|b|$, inner sphere). The inner entanglement sphere has the Cartesian coordinates $(x_2, x_3, x_4) = b(\sin \zeta \cos \xi, \sin \zeta \sin \xi, \cos \zeta) = bt$. As will be discussed below, the entanglement measure concurrence is given by $\sqrt{(x_2^2 + x_3^2)} = |b| \sin(\zeta)$, and the imaginary part of coherence for the base qubit is given by $x_4 = b \cos(\zeta)$. For each set of Bloch spheres, the coordinates are obtained from the four complex amplitudes of the bipartite state as follows:

If qubit-A is assigned to the Hopf base,
Qubit-A in the $S^4$ Hopf Base

\[
1 + x_0 = 2(|\alpha|^2 + |\beta|^2) \\
x_1 + \beta t = 2(\bar{\alpha}y + \bar{\beta} \delta)
\]

The angular coordinates \((\theta_f, \phi_f - 2\zeta_f)\) of the fiber sphere (qubit-B) and the phase factor \(\zeta_f\) can be obtained by equating its definition in Eq. (5) to

\[
q_f = \cos\frac{\theta}{2}(\alpha + \beta j) + \sin\frac{\theta}{2}e^{-t\phi}(\gamma + \delta j)
\]

If qubit-B is assigned to the Hopf base,

Qubit-B in the $S^4$ Hopf Base

\[
1 + x_0 = 2(|\alpha|^2 + |\gamma|^2) \\
x_1 + \beta t = 2(\bar{\alpha}y + \bar{\gamma} \delta)
\]

\[
q_f = \cos\frac{\theta}{2}(\alpha + \gamma j) + \sin\frac{\theta}{2}e^{-t\phi}(\beta + \delta j)
\]

A detailed discussion about the significance about the coordinates of each sphere is given in the Appendix.

3. The Improved Entanglement Sphere and All Three Bloch Spheres

In our original model [1], the entanglement sphere had only the \(t\)-vector represented by a unit sphere. Here we include the \(bt\)-vector also, represented by the inner sphere of radius \(|b|\) inside the (original) \(t\) sphere of unit radius. Keeping the outer sphere of unit radius helps gauge the size of the inner sphere visually and allows the coordinates of \(t\) to be displayed. This way, all the related parameters for the concurrence and the imaginary part of coherence are presented in one sphere without having to rely on the base sphere, where the radius \(|b|\) of the inner sphere is given by its \(y\)-coordinate \((y=b)\). The intercept of the \(t\)-vector with the inner sphere gives both the non-local concurrence as the radius \(c\) of the circle parallel to the equatorial plane, \(c = \sqrt{x_2^2 + x_3^2} = |b| \sin(\chi)\), and the imaginary part of the coherence of the base qubit as the \(x_4\) coordinate, \(x_4 = b \cos(\chi)\) up to the sign. The entanglement sphere is limited to the northern hemisphere and thus the \(x_4\) coordinate is always positive and represents the imaginary coherence up to the sign. This improved version of the entanglement sphere is shown in Figure 3. All three Bloch spheres are shown in Figure 4.

Figure 3 The improved entanglement sphere has two spheres: an outer sphere of unit radius (yellow) and an inner sphere of radius \(|b|\) (green). The pure-imaginary unit quaternion \(t\) with angular coordinates \((\chi, \xi)\) meets the outer sphere and the dashed horizontal circle is drawn (magenta), and intersects the inner sphere and the dashed horizontal circle (green) of radius \(c\), concurrence. The \(z\)-coordinate of the green dashed
circle (on inner sphere) is the $x_d$ of the reduced density matrix Eq.(9). The intersection of $t$ with the inner sphere gives the coordinate $(x_2, x_3, x_d)$. The sphere on the right shows an example of entanglement sphere indicating which qubit is the base qubit (qubit-A), with values for the angular coordinates and concurrence.

**Figure 4** The two-qubit Bloch sphere: the base sphere with angle coordinates $(\theta, \phi)$, the entanglement sphere $(\chi, \xi)$, and fiber sphere $(\theta_f, \phi_f - 2\zeta_f)$ and the phase factor $\zeta_f$. The entanglement sphere illustrated is for concurrence $c = 0.5$ and $t = (j+k)/\sqrt{2}$.

The reduced density matrix of the base qubit is obtained by tracing out the fiber qubit:

$$\rho_{\text{base}} = \frac{1}{2} \left( \begin{array}{cc} 1 + x_0 & x_1 - x_4k \\ x_1 + x_4k & 1 - x_0 \end{array} \right)$$

where $x_d = b \cos(\chi)$, $|x_d| < |b|$ for a mixed state and $|x_d| = |b|$ for the pure state because $\chi = 0$ is a separable bipartite state (see Fig.2). The base sphere represents the measurement statistics with $x_0$ according to the Born rule, the real part of coherence, $x_1$, and the intermediary parameter $b$ which will be related to the imaginary part $x_d$ of coherence and to the concurrence $c$ in the entanglement sphere.

The base sphere represents the base qubit, which for a separable state is precisely the pure state Bloch sphere of the qubit; and for an entangled state, it is a mixed state Bloch sphere except that the $b$-axis is expanded by a factor $1/\cos(\chi)$ from $x_d$ as can be seen from $\rho_{\text{base}} = \frac{1}{2} (I + r \cdot \mathbf{\sigma}) = \frac{1}{2} (I + x_1 \sigma_x + x_4 \sigma_y + x_0 \sigma_z)$ where $\mathbf{\sigma}$ is the Pauli matrix $\mathbf{\sigma}=(\sigma_x, \sigma_y, \sigma_z)$ and $r = (r_x, r_y, r_z) = (x_1, x_4, x_0)$ is the Bloch vector. That is, the base sphere has its $y$-axis expanded so that it is a perfect unit sphere: $x_1^2 + b^2 + x_0^2 = 1$. Hence, a mixed-state $\rho_{\text{base}}$, which will be a distorted sphere relative to a unit sphere, contracted by the factor $\cos(\chi)$ along the $y$-axis, is not fully represented on the base sphere alone for an entangled bipartite state.

The inner entanglement sphere represents the imaginary part of coherence of base qubit $x_d = b \cos(\chi)$ up to the sign and the concurrence, $c = |b| \sin(\chi)$, of the bipartite state. These two parameters, one local and one non-local, can be seen related in the representation of the base qubit state by the 2-vector in quaternion field, defined in Eq.(2). Taking the outer product we get

$$\tilde{\rho} = |\tilde{\psi}(\tilde{\psi}) = \frac{1}{2} \left( \begin{array}{cc} 1 + x_0 & x_1 - bt \\ x_1 + bt & 1 - x_0 \end{array} \right) = \frac{i}{2} \left( \begin{array}{cc} 1 + x_0 & x_1 - x_4k - ce^{\xi}i \\ x_1 + x_4k + ce^{\xi}i & 1 - x_0 \end{array} \right)$$

Note that the imaginary part, $x_d$, of coherence and the concurrence, $c$, are the different quaternion imaginary parts of the off-diagonal element. The representation of $x_d$ and $c$ on the inner entanglement sphere (i.e., $bt$-sphere) is therefore suggested in this matrix. This relation is also suggested in the definition of concurrence [6]
\[ \text{concurrence} = \sqrt{2(1 - Tr \rho_{\text{base}}^2)} = \sqrt{1 - x_0^2 - x_1^2 - x_4^2} = \sqrt{x_2^2 + x_3^2} = c \]  (11)

Therefore, the two Bloch spheres of \( S^4 \) Hopf base represent this relation naturally by encoding the imaginary part of coherence \( x_4 \) and the concurrence \( c \) in the inner entanglement sphere. For a given \( b \)-value of the base sphere, a rising concurrence \( c \) (by the increasing \( x \)) means a falling imaginary part \( x_4 \) of coherence, and vice versa.

The fiber sphere, along with the phase factor \( \zeta_f \), represents the fiber qubit geometrically in response to a local unitary operation, but otherwise it has no local information about the fiber qubit in an entangled state. In an entangled state, its main purpose seems to be to help the three Bloch spheres accurately represent the bipartite state via Eq.(4). In a separable state, the fiber sphere is precisely the Bloch sphere of the fiber qubit.

In summary, in the entangled states, the base sphere and the entanglement sphere (of the \( S^4 \) Hopf base) represent both the local information of the base qubit via its reduced density matrix and the non-local concurrence of the bipartite state. The fiber sphere represents the fiber qubit geometrically only, showing a simple rotation under a local unitary operation, otherwise it has no physical significance other than helping the three Bloch spheres accurately represent the bipartite state. In a separable state, the base sphere and the fiber sphere become exactly the single qubit Bloch sphere of each qubit.

4. Bloch Sphere Example with Two-Qubit Quantum Circuit

Here we will discuss partially entangled two-qubit states on the Bloch spheres in terms of the rotation operators on the quantum circuit. We present the two sets of Bloch spheres, one set with qubit-A (top qubit in the quantum circuit) in the Hopf base, and the other with qubit-B (the bottom qubit in the quantum circuit) in the Hopf base.

\[ |0\rangle \xrightarrow{R_x, y(\eta)} |0\rangle \]
\[ |0\rangle \xrightarrow{R_x, y(\omega)} |0\rangle \]

Figure 5 Two-qubit quantum circuit to generate partial entanglement between the two qubits with \( 0 \leq \text{concurrence} \leq 1 \). The gates are the rotation operators. Concurrence = 0 for separable states and 1 for maximally entangled states (MES). For this circuit, \( \text{concurrence} = \sin(\eta) \sin(\omega 2) \) and \( \sin(\eta) \) is the degree of initial superposition of the control qubit.

Figure 5 shows the entangling circuit with \( x \)- or \( y \)-rotation by angle \( \eta \), \( R_{x,y}(\eta) \), producing a linear superposition of the control-qubit, qubit-A. Note that the degree of superposition of \( |0\rangle \) and \( |1\rangle \) states is \( \sin(\eta) \) after this gate. The controlled \( x \)- or \( y \)-rotation, \( C-R_{x,y}(\omega) \), entangles the two qubits. The concurrence is given by \( \sin(\eta)\sin(\omega 2) \) after these two gates. Figure 6 shows the two sets of Bloch spheres after the \( R_y(60) \otimes I \) gate to induce the superposition in the control qubit-A and the \( C-R_x(70) \) gate to entangle the two qubits. The rotation angles are given in degrees. First, note that as expected, the concurrence value is the same in both sets of the Bloch spheres. If the control qubit (qubit-A) is assigned to the base sphere, BASE(A), the initial rotation \( R_y(60) \) rotates the base sphere precisely as if it is a free Bloch sphere. This is true for an initial \( x \)-rotation of the base sphere, and the subsequent controlled-rotation doesn’t change the base sphere. But if the
initial rotation was a y-rotation, which will keep the $b$-coordinate at zero, then the subsequent entangling operation (the controlled-rotation) causes the $b$-coordinate to develop a non-zero value in the base sphere. Therefore, after an initial y-rotation and the controlled-rotation, the base sphere rotates as if it is coupled to the entanglement sphere. Hence the initial x-rotation or the y-rotation gate acts differently after the two gates. If the control qubit is assigned to the fiber sphere, FIBER(A), the $R_x(60)$ gate rotates this fiber sphere by a rotation angle which is smaller than the angle specified, while the sense of rotation is the same as expected in the rotation operator. This initial rotation of the fiber sphere is as if it is a part of some coupled spheres.

![Bloch Spheres](image)

**Figure 6** The two sets of Bloch spheres where qubit-A is the control qubit, after $R_x(60) \otimes I$ $\rightarrow C: R_y(70)$ where the rotation angles are $\eta=60$ degrees, $\omega=70$, and $\mu=\nu=0$ in the circuit of Fig.5. Top row: qubit-A is the base qubit. Bottom row: qubit-B is the base qubit.

On the entangled bipartite state of Figure 6, effect of a local unitary operation is illustrated in Figure 7. Starting with the state given in Figure 6, a $y$-rotation by 90 degrees is applied to qubit-A with a $R_y(90) \otimes I$ gate. First, the local unitary operation does not change the non-local property of entanglement as can be seen by the same concurrence value in the entanglement spheres in Fig.7 as in Fig.6. Second, when applied to the fiber qubit, the local unitary operation rotates the fiber sphere as if it is a free Bloch sphere. This can be seen in FIBER(A) of Fig.7, which is a 90-
degree rotation about the y-axis, relative to FIBER(A) of Fig.6, and leaves the base and entanglement spheres fixed (see BASE(B) and ENTANGLEMENT(B) in Fig.6 and Fig.7). Third, if this local unitary operation on the base qubit does not change its y-coordinate (the $b$-value), which is the case with any y-rotation, then the entanglement sphere remains constant (ENTANGLEMENT(A) in Fig.6 and Fig.7) and the base sphere rotates like a free Bloch sphere (BASE(A) in Fig.6 and Fig.7), and the fiber sphere adjusts (FIBER(B) in Fig.6 and Fig.7) so that the three spheres represent the bipartite state accurately. In this case the fiber sphere (FIBER(B) in Fig.7) doesn't seem to allow any simple physical interpretation, and it seems to be only a mathematical tool to make the three spheres represent the bipartite state accurately.

Figure 7. The two sets of Bloch spheres after the partial entanglement and a local rotation induced by $R_x(60)\otimes I \rightarrow C-R_y(70) \rightarrow R_y(90)\otimes I$ where the rotation angles are $\eta=60$ degrees, $\omega=70$, $\mu=90$, and $\nu=0$ in the circuit of Fig.5, for top row, qubit-A is the base qubit, and for bottom row, qubit-B is the base qubit.

If the base qubit local unitary operation changes the y-coordinate (i.e., $b$-changing operation), like the local x- and z-rotations, then all three spheres could change in response and the base sphere rotates as if it is in a coupled system of spheres. Compare Fig.8 with Fig.6. A local y-rotation may be depicted on the base sphere alone, leaving the imaginary part $x_4$ of coherence unchanged and the entanglement sphere fixed. Hence, the following identities may be useful for the $b$-
changing $x$- and $z$-rotations in order to express them in terms of the $b$-preserving $y$-rotation in performing a local operation.

$$R_z(\mu) = R_z(-90)R_y(\mu)R_z(90); \quad R_y(\mu) = R_y(90)R_y(\mu)R_y(-90)$$

The rotation angles are given in degrees.

5. Discussion and Summary

As expected, a local unitary operation on the fiber qubit, as shown in FIBER(A) in Fig.7, has no effect on the base qubit properties or on the non-local entanglement property (see BASE(B) and ENTANGLEMENT(B) in Fig.7). This means that whatever local operation takes place on the fiber qubit, it does not affect the base and entanglement spheres, and there is no way to find out anything about it from the base and entanglement spheres. A local unitary interaction on the base qubit may be studied by examining the base and entanglement spheres only ($x_0$ and $x_1$ on...
BASE(A) and \( x_4 \) and \textit{concurrence} on ENTANGLEMENT(A) in Fig.7. The entanglement property, \textit{concurrence}, as shown on the entanglement sphere remains unchanged by the local operations. The reduced density matrix of the local base qubit is the only entity that is affected by the local unitary interaction, while the fiber sphere (FIBER(B) in Fig.7) carries no measurable information. Furthermore, if the local unitary operation on the base qubit is such that its \( y \)-coordinate (i.e., the \( b \)-value) remains unchanged, as in a \( y \)-rotation, then the entanglement sphere, which encodes \( x_4 \) and \textit{concurrence}, remains fixed, leaving only the base sphere to be analyzed. For a general local unitary operation on the base qubit, the reduced density matrix parameters, \( x_0, x_1, \) and \( x_4 \), are encoded on two spheres of the \( S^7 \) Hopf base. This is necessary in our approach because the imaginary part \( x_4 \) of coherence and the \textit{concurrence} are related as cosine and sine of \( \chi \), respectively, and because our Bloch spheres use the fixed basis states, the computational basis states, as the antipodal points. This raises a question about how the local dynamics may be expressed so as to determine the base and entanglement sphere parameters, \( x_0, x_1, \) and \( x_4 \) which are spread over the two spheres while leaving the concurrence constant. This question is not as obvious as expressing the effect of the local operations on the remote (and unmeasurable) fiber qubit, which is a simple single-qubit rotation. This point requires further study.

In our Bloch sphere model, the local operations evolve the base and entanglement spheres while the fiber sphere has no physical significance as far as the base qubit is concerned. Therefore, the base and entanglement spheres maintain all local and non-local information for the base qubit and the bipartite state while leaving the physically inconsequential parts to the fiber sphere. Measuring the spin direction of the local, base qubit, the probability will always be \( \cos^2(\theta/2) \) for spin up or the \( |0> \) state, whether the bipartite state is entangled or separable. A potentially useful application of our Bloch spheres is the bipartite entanglement experiments on two interacting two-level atoms using double coherent ultrashort laser pulses as reported by Yu and Li [7]. The entanglement evolution, the so-called entanglement sudden death and recurrence, may be investigated with the aid of the Bloch spheres presented in this paper. The Bloch spheres will display geometrically, and may add insight to, the evolution of the reduced density matrices (for coherence) as well as the concurrence. We are examining such an application and will report in a future publication.

We must stress here that our Bloch spheres maintain fixed, computational basis states as the antipodal points, unlike the ever-changing Schmidt basis states used as the antipodal points in refs.[4,5]. In their Bloch sphere model, the bipartite state is separated into two local spinors with the Schmidt decomposition, where each spinor evolves locally when subjected to a local unitary operation while the Schmidt coefficient, which represents the entanglement angle, changes only under a two-qubit joint operation. This Bloch sphere model seems elegant and simple. However, as the Schmidt basis vectors can change under both local and joint operations, a consistent geometric presentation of the bipartite state is likely difficult. Also, an entangled bipartite state means that any measurement on one qubit will affect the state of the other qubit. This is not a concern in our model because the fiber sphere carries no measurable information for an entangled state. Also, our Bloch sphere is practical for two-qubit quantum circuits, simpler to present and interpret geometrically relative to the unit sphere rotations, and perhaps useful for interpreting experiments such as ref.[7]. In our Bloch sphere representation, the two sets of Bloch spheres can provide a complete physical and geometrical interpretation for the two individual qubits and the bipartite state. That is, the two sets of the base and entanglement spheres provide complete and straightforward information about the reduced density matrices of each qubit and the entanglement parameter, concurrence, consistently for the entire range of entanglement, from the separable states to the maximally entangled states.
In the literature, different measures of quantum coherence have been proposed in the resource theory of coherence, still lacking yet a widely accepted and physically intuitive coherence-quantifier [8, 9, 11]. The incoherent states are the diagonal terms in the reference basis (computational basis states) [8]. If we use \( d = \sqrt{2 \text{Tr}_{\text{base}} \rho_{\text{base}}^2 - 1} \) as the coherence measure [10] for the base qubit, then \( d \) is complementary to the concurrence \( c \) in that \( d^2 + c^2 = 1 \) where the coherence can be expressed as \( d = \sqrt{x_0^2 + x_1^2 + x_4^2} \) in terms of our Bloch sphere coordinates. A rising concurrence \( c \) means a falling coherence \( d \), and vice versa. Instead, if we use the off-diagonal elements of \( \rho_{\text{base}} \) in the computational basis to represent the coherence, which we did in this paper, then its imaginary part \( x_4 \) and the concurrence \( c \) are related by \( x_4^2 + c^2 = b^2 \), which is shown in the entanglement sphere. This is also suggestive of the coherence-concurrence complementarity [12]. For a given \( b \)-coordinate in base sphere, a rising \( c \) means the falling imaginary part \( x_4 \), and vice versa. However, a local unitary operation on base qubit will keep \( c \) constant, while changing \( x_4 \) and \( b \) by exactly the same amount. A joint unitary operation on both qubits will vary all three parameters while maintaining this relation. The Bloch sphere representation could add insights to the interplay of these parameters under a joint unitary operation.

In summary, we have presented the two-qubit Bloch spheres which use two sets of three spheres to present all the local qubit information and the non-local entanglement information. The base sphere and the entanglement sphere carry the physical information on the local, base qubit as well as the non-local entanglement information, concurrence. A local unitary operation on the remote fiber qubit has no effect on the base and entanglement spheres. The two sets of Bloch spheres represent the local qubit reduced density matrix and the entanglement measure, concurrence. We presented Bloch sphere examples with rotation operators on a two-qubit quantum circuit to generate a partial entanglement.

A python program is available in GitHub to plot the Bloch spheres given the complex amplitudes of a two-qubit pure state [13].

References

1. C.R.Wie, “Bloch sphere model for two-qubit pure states,” arXiv:1403.8069v2 (2014)
2. D.W.Lyons, "An Elementary Introduction to the Hopf Fibration", Math. Mag. 72(2), 87-98 (2003)
3. F.H. Croom, Basic Concepts of Algebraic Topology, Springer-Verlag New York Inc., 1978, Sec 6.5 Homotopy Groups of Spheres
4. C.W.Wang, “The Density Operators of Qubit Systems in the Multiparticle Spacetime Algebra”, arXiv:1804.08375 (2018)
5. K.B.Wharton, “Natural Parameterization of Two-Qubit States”, arXiv:1601.04067 (2016)
6. S.Hill and W.K.Wooters, “Entanglement of a Pair of Quantum Bits”, Phys.Rev.Lett. 78, 5022 (1997)
7. X.Y.Yu, J.H.Li, “Coherent and ultrafast manipulation of entanglement sudden death and recurrence,” Opt. Lett., 35(16), 2744-2746, 2010
8. C.Napoli, T.R.Bromley, M.Cianciaruso, M.Piani, N.Johnston, G.Adesso, “Robustness of Coherence: An Operational and Observable Measure of Quantum Coherence”, Phys. Rev. Lett. 116 150502 (2016)
9. E.Chitambar, G.Gour, “Comparison of incoherent operations and measures of coherence”, Phys.Rev. A 94, 052336 (2016)
10. L.Mandel, E.Wolf, Optical Coherence and Quantum Optics (Cambridge University Press, Cambridge, UK, 1995)
APPENDIX: DISCUSSION OF THE BLOCH SPHERE COORDINATES

The base sphere represents two parameters of the reduced density operator: \( x_0 = \cos \theta \) gives the measurement probability of the base qubit by the Born rule as well as the superposition of two computational basis states; and \( x_1 = \sin \theta \cos \phi \) is the real part of coherence. The last axis (y-axis of the base sphere) represents an intermediary parameter \( b = \sin \theta \sin \phi \) for the concurrence and the imaginary part of coherence, which are depicted on the entanglement sphere. The choice of \( y = b \) axis is to make the base sphere a perfect unit sphere.

In an entangled state, the fiber sphere represents the fiber qubit geometrically under a local unitary operation, but otherwise has no information about the fiber qubit. The fiber sphere completes the representation of the bipartite state. In a separable state, the fiber sphere is precisely the Bloch sphere of the fiber qubit.

Therefore, the Cartesian coordinates of the base sphere and the inner entanglement sphere have a distinct physical meaning with respect to the local base qubit properties as well as the non-local entanglement properties as listed below. The relation between the imaginary part of coherence of the base qubit and the non-local concurrence is represented in the entanglement sphere.

The Cartesian and angular coordinates are related as follows: \( x_0 = \cos \theta; \quad x_1 = \sin \theta \cos \phi; \quad b = \sin \theta \sin \phi; \quad x_2 = b \sin \chi \cos \xi; \quad x_3 = b \sin \chi \sin \xi; \quad x_4 = b \cos \chi; \quad c = |b \sin \chi| = |\sin \theta \sin \phi \sin \chi| = \text{concurrence} \); and \( b_t = x_2 i + x_3 j + x_4 k \) where, if \( x_4 < 0 \), then \( t = -b t / |b| \) & \( b = -|b| \).

**The Base Sphere:** \((x_1, b, x_0) = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta)\)

1. \( x_0 = \cos \theta \): The base sphere z-coordinate gives the diagonal elements of reduced density matrix of the base qubit where, \((1+x_0)/2 = p_0 = \text{the probability of measuring 0}\) and \((1-x_0)/2 = p_1 = \text{the probability of measuring 1}\) on the base qubit according to the Born rule. Hence, this determines the degree \( \sin \theta \) of superposition in the base qubit.

2. \( x_1 = \sin \theta \cos \phi \): The base sphere x-coordinate relates to the real part of the off-diagonal element, or coherence, of the reduced density matrix, where \( x_1/2 \) is the real part of coherence.

3. \( b = \sin \theta \sin \phi \): The base sphere y-coordinate relates to the imaginary part of the off-diagonal element, coherence, and the degree of entanglement, concurrence. The imaginary part \( x_4 \) of coherence and the concurrence are related via the angle \( \chi \) as shown on the inner entanglement sphere. Using \( b \) instead of \( x_4 \) as the y-axis makes the base sphere a perfect unit sphere pertinent only for a pure state (or a separable bipartite state).
Discussion:  The Inner Entanglement Sphere:  \((x_3, x_4, x_4) = b(sin \chi cos \xi, sin \chi sin \xi, cos \chi) = bt\)

(4)  \(x_4 = b \cos \chi\).  Half the \(z\)-coordinate \(x_4\) of the inner entanglement sphere, \(x_4/2\), is the imaginary part of coherence of the base qubit, up to a sign.  The Bloch sphere of \(bt\) (and \(t\)) is limited to the northern hemisphere \((\chi \leq \pi/2)\) and \(x_4\) is always positive in the plot.  Therefore \(x_4/2\) should be taken as the imaginary part of coherence up to a sign.  Its sign is equal to the sign of the \(b\)-coordinate on the base sphere.

(5)  \(x_3 = b \sin \chi \cos \xi\) and \(x_3 = b \sin \chi \sin \xi\).  They give the concurrence by \(c = (x_3^2 + x_4^2)^{1/2} = |b| \sin \chi = |sin \theta \sin \phi_{4}|\).  A local gate may change the local property \(x_4\), but the non-local property concurrence stays constant.  This is despite the fact that they are related via the entanglement sphere angle \(\chi\) and the \(y\)-coordinate \((b\)-axis) of the base sphere via Eq.(11).  The significance of angle \(\xi\) is not clear although it seems to represent a ‘phase’ of concurrence [1].

(6)  The pure imaginary unit quaternion \(t\), which is represented in the entanglement sphere, plays a central role for concurrence and for the imaginary part of coherence, where its polar angle \(\chi\) decides the concurrence \(|b| \sin \chi\) and the imaginary part \(x_4 = |b| \cos \chi\) of coherence.  If we assume \(t\) a unit vector in the 3D space with unit vectors \(i, j, k\), then \(\chi\) is the angle between the two vectors \(t\) and \(k\).  The imaginary part of coherence of the base qubit is given by the parallel component of the vector \(|bt\) to \(k\), as \(x_4 = |bt \cdot k| = |b| \cos \chi\), and the concurrence is given by its perpendicular component as \(c = |bt \times k| = |b| \sin \chi = |sin \theta \sin \phi_{4}|\).  This also implies that, projecting the quaternion imaginary unit \(t\) onto the complex imaginary axis \(k\) (note that \(k\) is taken as the ordinary complex imaginary unit in this paper), then the base qubit density matrix \(\bar{\rho}\) on \(S^2\) is mapped to the base-qubit reduced density matrix \(\rho_{base}\) on the \(S^2\) Bloch sphere by tracing out the other qubit (fiber qubit).  That is, if we project \(t\) to \(k\), then

\[
\bar{\rho} = |\langle \bar{\psi}|\bar{\phi} \rangle| = \frac{1}{2} \left( \frac{1 + x_0}{x_1 + bt} \right) \rightarrow \rho_{base} = Tr(|\Psi \rangle \langle \Psi|) = \frac{1}{2} \left( \frac{1 + x_0}{x_1 + x_4k} \right)
\]

The Fiber Sphere:  \((x_3, y_3, z_3) = (sin \theta \cos (\phi - 2 \xi), sin \theta \sin (\phi - 2 \xi), cos \theta)\)

(7)  In a separable state, this is the Bloch sphere of the fiber qubit.  In an entangled state, this represents the fiber qubit geometrically under any local unitary operation.  This sphere rotates under the local unitary operator freely, independent of the other two spheres.

(8)  However, in an entangled state, this sphere has no information about the reduced density matrix of the fiber qubit.  In order to obtain any local information about the fiber qubit, this qubit must first become the base qubit and the Bloch spheres must be redrawn.

Discussion:

(9)  The Bloch vector \(r\) of the reduced density matrix has two components, \(r_z\) and \(r_x\), in the base sphere and one component, \(r_y\) (the imaginary part of off-diagonal element) in the entanglement sphere:  \(r = (r_o, r_y, r_z) = (x_1, x_4, x_0)\).  The other three coordinates, \(b, x_2\) and \(x_3\), give the entanglement measure, concurrence and the imaginary part \(x_4\) of the off-diagonal element via the \(b\)-coordinate of base sphere and the angle \(\chi\) of the entanglement sphere.

Therefore, the base sphere and the inner entanglement sphere (of radius \(|b|\)) represent the base qubit local information such as the measurement statistics (diagonal elements) and coherence (off-diagonal elements of the reduced density operator), as well as the non-local entanglement information, concurrence.

(10)  For \(\chi = 0\) (separable states where \(t\) is at the north pole of the entanglement sphere), the base qubit density matrix becomes a pure state.  The imaginary part \(x_4\) of coherence is at its maximum value, \(|x_4| = |b|\).

\[
\rho_{base} = \frac{1}{2} \left( \frac{1 + x_0}{x_1 + x_4k} \right) \rightarrow \frac{1}{2} \left( \frac{1 + x_0}{x_1 + bk} \right), \text{ pure state}
\]
Here, \(x_1^2 + b^2 + x_0^2 = 1\). For \(\chi = \pi/2\), the imaginary part of coherence, \(x_4\), is zero and the concurrence \(c\) is at its maximum value at \(|b|\).

Therefore, each qubit’s local information, i.e., the reduced density matrix, can be represented by assigning the qubit to the \(S^4\) Hopf base. The concurrence value in the entanglement sphere is the same regardless which qubit is in the base. The coherence is different for each qubit. Therefore, the two sets of the Bloch spheres can represent the bipartite pure states of any arbitrary entanglement.