Choquet order and hyperrigidity for function systems

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joint work with Matthew Kennedy
\[ 1 \in S = S^* \subset C(X) \text{ is a function system.} \]

\[ K = \{ \varphi \in S^* : \varphi \geq 0, \varphi(1) = 1 \} \text{ state space, compact, convex,} \]

and \( x \in X \longrightarrow \varepsilon_x \in K \), where \( \varepsilon_x(f) = f(x) \) for \( f \in S \).
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$K = \{ \varphi \in S^* : \varphi \geq 0, \varphi(1) = 1 \}$ state space, compact, convex, and $x \in X \rightarrow \varepsilon_x \in K$, where $\varepsilon_x(f) = f(x)$ for $f \in S$.

Theorem (Kadison 1951)

$S \xrightarrow{\text{iso}} A(K) \subset C(K)$ isometric isomorphism to affine functions.
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$K = \{ \varphi \in S^* : \varphi \geq 0, \varphi(1) = 1 \}$ state space, compact, convex, and $x \in X \mapsto \varepsilon_x \in K$, where $\varepsilon_x(f) = f(x)$ for $f \in S$.

**Theorem (Kadison 1951)**

$S \xrightarrow{\text{iso}} A(K) \subset C(K)$ isometric isomorphism to affine functions.

$\partial S := \partial K$ extreme points is Choquet boundary of $S$.

$f \in S$ affine on $K$, so $S \rightarrow C(\partial K)$ completely isometric.

$\overline{\partial K}$ is the Shilov boundary of $S$. 
By Hahn-Banach and Riesz Representation Theorems, for $\varphi \in K$ there exists $\mu \in M_+(\partial K)$ representing measure $\varphi(f) = \int f \, d\mu$ for $f \in S$. 
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\varphi(f) = \int f \, d\mu \quad \text{for } f \in S.
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Choquet theory yields \( \mu \in M_+(\partial K) \).

- important in applications
- nonmetrizable case: \( \partial K \) may not be Borel; so need technical definition of support.
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$$\varphi(f) = \int f \, d\mu$$

for $f \in S$.

Choquet theory yields $\mu \in M_+(\partial K)$.

- Important in applications
- Nonmetrizable case: $\partial K$ may not be Borel; so need technical definition of support.

**Definition**

**Choquet order:** $\mu \prec_c \nu$ in $M_+(K)$ if $\int f \, d\mu \leq \int f \, d\nu$ for $f$ convex.

This implies that $\int f \, d\mu = \int f \, d\nu$ for $f \in S$, so represent same $\varphi$. 
Theorem (Choquet, Mokobodski)

$K$ metrizable.

$\mu \in M_+(K)$ is maximal in $\prec_c \iff \text{supp} \mu \subset \partial K$. 
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Theorem (Bishop-De Leuuw)

$K$ arbitrary.
$\mu$ is maximal in $\prec_c \implies \mu(A) = 0$ if $A$ Baire s.t. $A \cap \partial K = \emptyset$. 
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Mokobodski: this does not characterize maximality.

However, if $\partial K$ is closed, then $\mu$ is maximal $\iff \text{supp } \mu \subset \partial K$. 
Classical result:

**Theorem (Korovkin)**

If $\Phi_n : C[a, b] \rightarrow C[a, b]$ positive maps s.t.

$$
\lim_{n \rightarrow \infty} \Phi_n(f) = f \quad \text{for} \quad f \in \{1, x, x^2\},
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then

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Modern, significant improvement:

**Theorem (Arveson)**

If $\pi : C[a, b] \rightarrow B(\mathcal{H})$ $\ast$-repn., $\Phi_n : C[a, b] \rightarrow B(\mathcal{H})$ (completely) positive maps s.t.

$$\lim_{n \rightarrow \infty} \Phi_n(f) = \pi(f) \quad \text{for} \quad f \in \{1, x, x^2\},$$

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$$\lim_{n \rightarrow \infty} \Phi_n(f) = \pi(f) \quad \text{for all} \quad f \in C[a, b].$$
**Definition**

1 ∈ \( F \subset C(X) \) is a **Korovkin set** if \( \Phi_n : C(X) \rightarrow C(X) \) are positive, 
\[
\lim_{n \to \infty} \Phi_n(f) = f \quad \text{for} \quad f \in F \quad \iff \quad \lim_{n \to \infty} \Phi_n(f) = f \quad \text{for} \quad f \in C(X).
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F is a strong Korovkin set if \( \pi : C[a, b] \to B(\mathcal{H}) \) *-repn., \( \Phi_n : C(X) \to B(\mathcal{H}) \) (completely) positive, then 
\[ \lim_{n \to \infty} \Phi_n(f) = \pi(f) \text{ for } f \in F \implies \lim_{n \to \infty} \Phi_n(f) = \pi(f) \text{ for } f \in C(X). \]
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**Theorem (Šaškin)**

$X$ compact metric. $1 \in F \subset C(X)$. $S = \overline{\text{span}}\{F \cup F^*\}$.
Then $F$ is a Korovkin set $\iff \partial S = X$. 
**Definition**

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**Theorem (Šaškin)**

X compact metric. 1 ∈ F ⊂ C(X). \( S = \text{span}\{F \cup F^*\} \).

Then F is a Korovkin set \iff \( \partial S = X. \)

**Question (Arveson)**

Characterize strong Korovkin sets.
Definition

$1 \in S = S^* \subset \mathcal{A} = C^*(S)$ is **hyperrigid** if whenever 

$\pi : \mathcal{A} \to B(\mathcal{H})$ $^*$-repn, and $\Phi_n : \mathcal{A} \to B(\mathcal{H})$ c.p.

$$\lim_{n \to \infty} \Phi_n(s) = \pi(s) \text{ for } s \in S \implies \lim_{n \to \infty} \Phi_n(a) = \pi(a) \text{ for } a \in \mathcal{A}.$$
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**Definition**

$1 \in S = S^* \subset \mathcal{A} = C^*(S)$. $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ $*$-repn.

$\pi|_S$ has **unique extension property (u.e.p.)**

if $\pi$ is the unique u.c.p. extension to $\mathcal{A}$.
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**Theorem (Arveson)**

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\( \mathcal{S} \) is hyperrigid \( \iff \) \( \pi|_{\mathcal{S}} \) has u.e.p. \( \forall \pi \) *-repn.
**Definition**

\( \pi \) \( \ast \)-repn. of \( \mathcal{A} \) is a **boundary representation for \( S \)** if 
\( \pi \) is irreducible and \( \pi|_S \) has u.e.p.
**Definition**

\( \pi \) \( \ast \)-repn. of \( A \) is a boundary representation for \( S \) if \( \pi \) is irreducible and \( \pi|_S \) has u.e.p.

**Conjecture (Arveson)**

\( S \) is hyperrigid \( \iff \) every irreducible \( \ast \)-repn. is a boundary repn.
Definition

$\pi$ $*$-repn. of $\mathcal{A}$ is a boundary representation for $S$ if

$\pi$ is irreducible and $\pi|_S$ has u.e.p.

Conjecture (Arveson)

$S$ is hyperrigid $\iff$ every irreducible $*$-repn. is a boundary repn.

Remark

For $1 \in S = S^* \subset C(\mathcal{X})$, this asks if $\partial S = \mathcal{X}$,

is $S$ is a strong Korovkin set in $C(\partial S)$?
**Definition**

**Dilation order:** $\mu \prec_d \nu \in M_+(K)$ if there exist $\ast$-repns.

\[
\pi : C(K) \to B(H), \quad \xi \in H, \quad \langle \pi(f)\xi, \xi \rangle = \int f \, d\mu \quad \forall f \in C(K)
\]

\[
\sigma : C(K) \to B(K), \quad \eta \in K, \quad \langle \pi(f)\eta, \eta \rangle = \int f \, d\nu \quad \forall f \in C(K)
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and isometry $J : H \to K$ s.t. $J\xi = \eta$ and $J^*\sigma(f)J = \pi(f) \quad \forall f \in S.$
**Definition**

Dilation order: \( \mu \prec_d \nu \in M_+(K) \) if there exist \(*\)-repns.

\[ \pi : C(K) \to B(H), \quad \xi \in \mathcal{H}, \quad \langle \pi(f)\xi, \xi \rangle = \int f \, d\mu \quad \forall f \in C(K) \]

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and isometry \( J : \mathcal{H} \to \mathcal{K} \) s.t. \( J\xi = \eta \) and \( J^*\sigma(f)J = \pi(f) \forall f \in S \).

**Theorem 1**

*Dilation order is the same as Choquet order.*
**Definition**

Dilation order: \( \mu \prec_d \nu \in M_+(K) \) if there exist *-repns.

\[ \pi : C(K) \to \mathcal{B}(\mathcal{H}), \quad \xi \in \mathcal{H}, \quad \langle \pi(f)\xi, \xi \rangle = \int f \, d\mu \quad \forall f \in C(K) \]

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and isometry \( J : \mathcal{H} \to \mathcal{K} \) s.t. \( J\xi = \eta \) and \( J^*\sigma(f)J = \pi(f) \quad \forall f \in S \).

**Theorem 1**

_Dilation order is the same as Choquet order._

**Corollary**

\[ \mu \prec_c \nu \iff \exists \Phi : C(K) \to L^\infty(\mu) \text{ positive s.t.} \]

1. \( \Phi(f) = f \) for all \( f \in A(K) \), and
2. \( \int \Phi(f) \, d\mu = \int f \, d\nu \) for all \( f \in C(K) \).
\[ \pi_\mu : C(K) \to B(L^2(\mu)) \] by \( \pi(f) = M_f \).

**Theorem 2**

\[ \pi_\mu \text{ has u.e.p. } \iff \mu \text{ is maximal in } \prec_d. \]
\( \pi_{\mu} : C(K) \to B(L^2(\mu)) \) by \( \pi(f) = M_f \).

**Theorem 2**

\( \pi_{\mu} \) has u.e.p. \( \iff \) \( \mu \) is maximal in \( \prec_d \).

**Corollary (Hyperrigidity for function systems)**

*If* \( \partial S \) *is closed, then* \( S \) *is hyperrigid in* \( C(\partial S) \).*
\( \pi_\mu : C(K) \to B(L^2(\mu)) \) by \( \pi(f) = M_f \).

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**Corollary (Hyperrigidity for function systems)**

If \( \partial S \) is closed, then \( S \) is hyperrigid in \( C(\partial S) \).

**Corollary**

If \( X \) is metrizable, \( 1 = S \subset C(X) \), \( \pi : C(X) \to B(H) \) \(*\)-repn. Then \( \pi \) has u.e.p. \( \iff \pi \) is supported on \( \partial S \).
Application to approximation theory

The following does not require metrizability, so it generalizes Šaškin’s Theorem even in the classical situation.

**Corollary**

1 ∈ S = \overline{\text{span}}\{F \cup F^*\} \subset C(X).

TFAE

1. ∂S = X
2. F is a Korovkin set.
3. F is a strong Korovkin set.
Application to classical Choquet theory

**Theorem (Cartier)**

If $K$ is metrizable, $\mu \prec_c \nu$, then $\exists \lambda : K \rightarrow M_{+,1}(K)$ s.t.

1. $x \rightarrow \lambda_x(f)$ is Borel $\forall f \in C(K)$,
2. $\lambda_x(f) = f(x)$ $\forall f \in A(K)$, and
3. $\int f \, d\nu = \int \lambda_x(f) \, d\mu$ $\forall f \in C(K)$. 
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If $K$ is metrizable, $\mu \prec_c \nu$, then $\exists \lambda : K \to M_{+1}(K)$ s.t.

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3. $\int f \, d\nu = \int \lambda_x(f) \, d\mu$ $\forall f \in C(K)$.

**Theorem 3**

If $K$ is compact convex, $\mu \prec_c \nu$, then $\exists \lambda : K \to M_{+1}(K)$ s.t.

1. $x \to \lambda_x(f)$ is Borel $\forall f \in C(K)$,
2. $\lambda_x(f) = f(x) \ a.e.(\mu)$ $\forall f \in A(K)$, and
3. $\int f \, d\nu = \int \lambda_x(f) \, d\mu$ $\forall f \in C(K)$. 
Thank you.
The end.