Note on star-triangle equivalence in conducting networks

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Abstract

By using the discrete Poisson equations the star-triangle (external) equivalence in conducting networks is considered and the Kennelly famous transformation formulae [Kennelly A E 1899 Electrical World and Engineer 34 413] are explicitly restated.

1 Introduction and outline of the paper

The homological representation and modeling [1] of networks (n/w) is based on their geometric elements, called also the chains – nodes, branches (edges), meshes (simple closed loops), and using the natural geometric boundary operator of the n/w which only depends on the geometry (topology) of the n/w. Then, both of the Kirchhoff laws can be presented in a compact algebraic form that may be called the homological Kirchhoff Laws.

In the present note, we compose the discrete Poisson equations and consider the star-triangle (external) equivalence transformation in conducting networks, see Fig. 1.1, and prove the Kennelly famous transformation formulae [2]. We use the geometrical representation that was explained in [3].

\[ |\beta\rangle : = |\beta_1\beta_2\beta_3\beta_4 \rangle, \quad |\beta\rangle' : = |\beta_1\beta_2\beta_3\rangle \quad \text{(boundary currents)} \]

\[ |\phi\rangle : = |\phi_1\phi_2\phi_3\phi_4 \rangle, \quad |\phi\rangle' : = |\phi_1\phi_2\phi_4\rangle \quad \text{(node potentials)} \]

The impedance matrices are

\[ Z := \begin{bmatrix} Z_1 & 0 & 0 \\ 0 & Z_2 & 0 \\ 0 & 0 & Z_3 \end{bmatrix}, \quad Z' := \begin{bmatrix} Z_3' & 0 & 0 \\ 0 & Z_1' & 0 \\ 0 & 0 & Z_2' \end{bmatrix} \]
The admittances $Y_n$ and $Y'_n$ are defined by

$$Y_n Z_n = Y'_n Z'_n, \quad n = 1, 2, 3$$

and the admittance matrices are

$$Y := Z^{-1} = \begin{bmatrix} Y_1 & 0 & 0 \\ 0 & Y_2 & 0 \\ 0 & 0 & Y_3 \end{bmatrix}, \quad Y' := Z'^{-1} = \begin{bmatrix} Y'_3 & 0 & 0 \\ 0 & Y'_1 & 0 \\ 0 & 0 & Y'_2 \end{bmatrix}$$

2 Star

Consider the star circuit represented on Fig. 1.1. Define

- **Node space** $C_0 := \langle v_1 v_2 v_3 v_4 \rangle_C$, $\dim C_0 = 4$
- **Branch space** $C_1 := \langle e_1 e_2 e_3 \rangle_C$, $\dim C_1 = 3$

First construct the boundary operator $\partial : C_1 \to C_0$. By definition,

$$\partial e_1 = \partial (v_1 v_4) := v_4 - v_1 =: | -1; 0; 0; 1 \rangle$$
$$\partial e_2 = \partial (v_2 v_4) := v_4 - v_2 =: | 0; -1; 0; 1 \rangle$$
$$\partial e_3 = \partial (v_3 v_4) := v_4 - v_3 =: | 0; 0; -1; 1 \rangle$$

and in the matrix representation we have

$$\partial = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 1 & 1 \end{bmatrix} \implies \partial^T = \begin{bmatrix} -1 & 0 & 0 & 1 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

Now it is easy to calculate the Laplacian as follows:

$$\Delta := \partial Y \partial^T$$

$$= \begin{bmatrix} -1 & 0 & 0 & \big[ Y_1 & 0 & 0 \big] \\ 0 & -1 & 0 & \big[ 0 & Y_2 & 0 \big] \\ 0 & 0 & -1 & \big[ 0 & 0 & Y_3 \big] \\ 1 & 1 & 1 & \big[ 0 & 0 & -1 \big] \end{bmatrix}$$

The Poisson equation

$$\Delta \phi = -|\beta\rangle$$

in coordinate form reads

$$\begin{cases} 
\beta_1 = \frac{-\phi_1 + \phi_4}{Z_1} \\
\beta_2 = \frac{-\phi_2 + \phi_4}{Z_2} \\
\beta_3 = \frac{-\phi_3 + \phi_4}{Z_3} \\
\beta_4 = -\left( \frac{-\phi_1 + \phi_4}{Z_1} + \frac{-\phi_2 + \phi_4}{Z_2} + \frac{-\phi_3 + \phi_4}{Z_3} \right) 
\end{cases}$$
We can easily check consistency:
\[ \beta_1 + \beta_2 + \beta_3 + \beta_4 = 0 \]  
(2.6)

For \( \beta_4 = 0 \) we have
\[ -\phi_1 + \phi_4 \frac{Z_1}{Z_1} + -\phi_2 + \phi_4 \frac{Z_2}{Z_2} + -\phi_3 + \phi_4 \frac{Z_3}{Z_3} = 0 \]  
(2.7)

from which it follows that
\[ \phi_4 = \frac{\phi_1 Y_1 + \phi_2 Y_2 + \phi_3 Y_3}{Y_1 + Y_2 + Y_3} \]  
(2.8)

### 3 Triangle

Next consider the triangle circuit on Fig. 1.1. We denote the spanning nodes and branches by the same letters. Then the linear spans are

- **Node space** \( C_0 := \langle v_1 v_2 v_3 \rangle_C \), \( \dim C_0 = 3 \)
- **Branch space** \( C_1 := \langle e_1 e_2 e_3 \rangle_C \), \( \dim C_1 = 3 \)

Construct the boundary operator \( \partial : C_1 \to C_0 \). We can see that
\[ \partial e_1 = \partial (v_1 v_2) := v_2 - v_1 =: [-1; 1; 0] \]  
(3.1)
\[ \partial e_2 = \partial (v_2 v_3) := v_3 - v_2 =: [0; -1; 1] \]  
(3.2)
\[ \partial e_3 = \partial (v_3 v_1) := v_1 - v_3 =: [1; 0; -1] \]  
(3.3)

and the matrix representation is
\[ \partial = \begin{bmatrix} -1 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix} \implies \partial^T = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & 0 & -1 \end{bmatrix} \]  
(3.4)

The Laplacian is
\[ \Delta' := \partial Y' \partial^T \]  
(3.5a)
\[ = \begin{bmatrix} -1 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} Y'_3 & 0 & 0 \\ 0 & Y'_1 & 0 \\ 0 & 0 & Y'_2 \end{bmatrix} \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & 0 & -1 \end{bmatrix} \]  
(3.5b)
\[ = \begin{bmatrix} Y'_3 + Y'_2 & -Y'_3 & -Y'_2 \\ -Y'_3 & Y'_1 + Y'_2 & -Y'_1 \\ -Y'_2 & -Y'_1 & Y'_1 + Y'_2 \end{bmatrix} \]  
(3.5c)

The Poisson equation is
\[ \Delta' |\phi'\rangle = -|\beta'\rangle \]  
(3.6)

Hence we have
\[
\begin{align*}
\beta_1 &= -\frac{-\phi_1 + \phi_2}{Z'_3} - \frac{\phi_1 - \phi_3}{Z'_2} \\
\beta_2 &= -\frac{-\phi_1 + \phi_2}{Z'_3} + \frac{-\phi_2 + \phi_3}{Z'_1} \\
\beta_3 &= -\frac{-\phi_2 + \phi_3}{Z'_1} + \frac{\phi_1 - \phi_3}{Z'_2}
\end{align*}
\]  
(3.7)

Check the consistency:
\[ \beta_1 + \beta_2 + \beta_3 = 0 \]  
(3.8)
4 Equivalence

Now consider the star-triangle equivalence as exposed on Fig. 1.1 and prove the Kennelly theorem.

**Theorem 4.1** (A. E. Kennelly [2]). If the (external/boundary) equivalence presented on Fig. 1.1 holds, then one has

\[
Z_nZ_n' = Z_1'Z_2'Z_3'/(Z_1' + Z_2' + Z_3') = Z_1Z_2 + Z_2Z_3 + Z_3Z_1, \quad n = 1, 2, 3
\]

(4.1)

**Proof.** As soon as the boundary currents on Fig. 1.1 are considered the same, then we have

\[
-\phi_1 + \phi_4 = \frac{-\phi_1 + \phi_2}{Z_1} - \frac{\phi_1 - \phi_3}{Z_2} \quad (4.2a)
\]

\[
-\phi_2 + \phi_4 = -\frac{-\phi_1 + \phi_2}{Z_3} + \frac{-\phi_2 + \phi_3}{Z_1} \quad (4.2b)
\]

\[
-\phi_3 + \phi_4 = -\frac{-\phi_2 + \phi_3}{Z_3} + \frac{\phi_1 - \phi_3}{Z_2} \quad (4.2c)
\]

where \( \phi_4 \) is given by (2.8). Hence we obtain equations for the potentials \( \phi_1, \phi_2, \phi_3, \)

\[
-\phi_1 + \phi_4 = (-\phi_1 + \phi_2)\frac{Z_1}{Z_3} - (\phi_1 - \phi_3)\frac{Z_1}{Z_2} \quad (4.3a)
\]

\[
-\phi_2 + \phi_4 = -(-\phi_1 + \phi_2)\frac{Z_2}{Z_3} + (-\phi_2 + \phi_3)\frac{Z_2}{Z_1} \quad (4.3b)
\]

\[
-\phi_3 + \phi_4 = -(-\phi_2 + \phi_3)\frac{Z_3}{Z_1} + (\phi_1 - \phi_3)\frac{Z_3}{Z_2} \quad (4.3c)
\]

By eliminating here the potential \( \phi_4 \), we get relations for the boundary potentials,

\[
-\phi_1 + \phi_2 = (-\phi_1 + \phi_2)\frac{Z_1}{Z_3} - (\phi_1 - \phi_3)\frac{Z_1}{Z_2} + (-\phi_1 + \phi_2)\frac{Z_2}{Z_3} - (-\phi_2 + \phi_3)\frac{Z_2}{Z_1} \quad (4.4a)
\]

\[
-\phi_2 + \phi_3 = -(-\phi_1 + \phi_2)\frac{Z_2}{Z_3} + (-\phi_2 + \phi_3)\frac{Z_2}{Z_1} + (-\phi_2 + \phi_3)\frac{Z_3}{Z_1} - (\phi_1 - \phi_3)\frac{Z_3}{Z_2} \quad (4.4b)
\]

\[
-\phi_3 + \phi_1 = -(-\phi_2 + \phi_3)\frac{Z_3}{Z_1} + (\phi_1 - \phi_3)\frac{Z_3}{Z_2} - (-\phi_1 + \phi_2)\frac{Z_1}{Z_3} + (-\phi_1 + \phi_3)\frac{Z_1}{Z_2} \quad (4.4c)
\]

We can easily check consistency of the last Eqs, by summing these we easily obtain \( 0 = 0 \). This means that one equation is a linear combination of others and we can variate the independent potentials \( \phi_1, \phi_2, \phi_3 \) only in two equations. We use the first two Eqs.

By varying the independent potentials \( \phi_1, \phi_1, \phi_3 \) and setting the nontrivial potential \( \phi_3 = 1 \) in the first equation we obtain

\[
0 = \frac{Z_1}{Z_2} - \frac{Z_2}{Z_1} \quad \Rightarrow \quad \frac{Z_1Z_1'}{Z_2Z_2'} = Z_2Z_2'
\]

(4.5)

Now take \( \phi_1 = 1, \)

\[
1 = \frac{Z_1}{Z_3} + \frac{Z_1}{Z_2} + \frac{Z_2}{Z_3} \quad \Rightarrow \quad 1 = \frac{Z_1Z_3' + Z_1Z_3' + Z_2Z_2'}{Z_2Z_3'}
\]

(4.6a)

\[
= \frac{Z_1Z_3' + Z_1Z_3' + Z_1Z_1'}{Z_2Z_3'}
\]

(4.6b)
\[ Z_1 = \frac{Z_2 Z'_3}{Z'_2 + Z'_3 + Z'_1} \] (4.6c)

Next take \( \phi_2 = 1 \),

\[ 1 = \frac{Z_1}{Z'_3} + \frac{Z_2}{Z'_3} + \frac{Z_3}{Z'_3} \implies 1 = \frac{Z_1 Z'_1 + Z_2 Z'_3 + Z_2 Z'_3}{Z'_3 Z'_1} \] (4.7a)

\[ = \frac{Z_2 Z'_2 + Z_2 Z'_3 + Z_2 Z'_3}{Z'_3 Z'_1} \] (4.7b)

\[ = \frac{Z_2(Z'_2 + Z'_1 + Z'_3)}{Z'_3 Z'_1} \implies \frac{Z_2}{Z'_2 + Z'_3 + Z'_1} = \frac{Z'_3 Z'_1}{Z'_3 Z'_1} \] (4.7c)

By varying the independent potentials \( \phi_1, \phi_1, \phi_3 \) in the second equation and setting there the nontrivial potential \( \phi_1 = 1 \), we obtain

\[ 0 = \frac{Z_2}{Z'_3} - \frac{Z_3}{Z'_2} \implies Z_2 Z'_2 = Z_3 Z'_3 \] (4.8)

By setting \( \phi_3 = 1 \), we obtain

\[ 1 = \frac{Z_2}{Z'_1} + \frac{Z_3}{Z'_1} + \frac{Z_3}{Z'_2} \implies 1 = \frac{Z_2 Z'_2 + Z_3 Z'_1 + Z_3 Z'_1}{Z'_1 Z'_2} \] (4.9a)

\[ = \frac{Z_3 Z'_3 + Z_3 Z'_2 + Z_3 Z'_1}{Z'_1 Z'_2} \] (4.9b)

\[ = \frac{Z_3(Z'_3 + Z'_2 + Z'_1)}{Z'_1 Z'_2} \implies \frac{Z_3}{Z'_1 + Z'_2 + Z'_1} = \frac{Z'_3 Z'_1}{Z'_1 Z'_2} \] (4.9c)

One can easily check that other variations of the potentials \( \phi_n \ (n = 1, 2, 3) \) do not produce additional constraints.

The research was in part supported by the Estonian Research Council, grant ETF-9038.

References

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[3] Paal E and Umbleja M 2014 J. Phys.: Conf. Series 532 012022

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