Effects of model inaccuracies on reaching movements with intermittent control

Igor Gindin¹, Miri Benyamini¹ Miriam Zacksenhouse¹*  
¹ Faculty of Mechanical Engineering, Technion Israel’s Institute of Technology, Haifa 32000, Israel  
*mermz@technion.ac.il

S3: Equivalent discrete-time systems with delays

Summary of relevant equations from main text

The dynamics of the LTI plant is described by the system matrix \( \bar{A} \) and control matrix \( \bar{B} \) (Eq. (1)), which may differ from the system matrix \( A \) and control matrix \( B \) of the internal model (Eq. (2)):

\[
\dot{x}(t) = \bar{A}x(t) + \bar{B}u(t) + w(t) \quad (1)
\]

\[
\dot{x}_{IM}(t) = Ax_{IM}(t) + Bu(t) \quad (2)
\]

where \( x \in \mathbb{R}^n \) is the state of the plant, \( u(t) \in \mathbb{R}^m \) is the control signal, \( w(t) \in \mathbb{R}^n \) is the process noise, and \( x_{IM} \in \mathbb{R}^n \) is the state of the internal model.

The compound effect of process and measurement delay is accounted for by introducing measurement delay \( \tau \):

\[
y(t) = Cx(t - \tau) + v(t - \tau) \quad (3)
\]

Observer combines the internal model (Eq. (2)) and delayed measurement (Eq. (3)) to generate the estimated state \( \hat{x} \) according to:

\[
\dot{\hat{x}}(t - \tau) = \bar{A}\hat{x}(t - \tau) + \bar{B}u(t - \tau) + L(t)(y(t) - C\hat{x}(t - \tau)) \quad (4)
\]

Predictor predicts the current state, \( x_p(t) \), given the estimated state, \( \hat{x}(t - \tau) \), and the control signal \( u(\sigma) \) for \( \sigma \in [t - \tau, t) \), based on the internal model (Eq. (2)):

\[
x_p(t) = e^{A\tau}\hat{x}(t - \tau) + \int_{t-\tau}^{t} e^{A(t-\sigma)}Bu(\sigma)d\sigma \quad (5)
\]

LTI systems, i.e., LTI plants with time-invariant observer and controller gains, \( L \) and \( K \), can be described by the overall state \( x_{ov}(t - \tau) = [x(t - \tau)' \quad \dot{x}(t - \tau)']' \).

Combining Eqs. (1), (3) and (4) yields:

\[
\dot{x}_{ov}(t - \tau) = A_{\sigma}x_{ov}(t - \tau) + B_{\sigma}u(t - \tau) + w_{ov}(t - \tau) \quad (6)
\]

where \( A_{\sigma} \) and \( B_{\sigma} \) are defined in the main text, and \( w_{ov}(t - \tau) = [w(t - \tau)' \quad Lv(t - \tau)']' \) is the overall process noise.

Intermitent control performs predictions at discrete times \( t_m \), which can be evenly spaced (periodic IC, \( t_m = mh \), where \( h \) is the sampling period) or event driven (not considered in this work). At \( t_m \), the predictor receives \( \hat{x}(t_m - \tau) \) from the observer and...
generates \( x_p(t_m) \) according to Eq. [5]. The latter provides the initial condition for the hold state, \( x_h(t) \) that determines the control signal:

\[
u(t) = -K(t)x_h(t)
\]  

(7)

Between samples, \( x_h(t) \) evolves continuously according to the feedback matrix 
\((A_F(t) = A - BK(t))\), defining the SMH:

\[
\begin{aligned}
\dot{x}_h(t) &= A_F(t)x_h(t), \quad t \in [t_{m-1}, t_m) \\
x_h(t_{m}^+) &= x_p(t_m), \quad \forall m \in \mathbb{Z}^+
\end{aligned}
\]  

(8)

**Equivalent discrete-time systems with delays**

To facilitate the stability analysis of continuous time-delayed systems, they are converted to equivalent discrete-time systems using the standard zero-order hold [1]. Given the delay \( \tau \), the discretization time \( \Delta \) is selected so \( k_r = \tau / \Delta \) is an integer number. Thus, the equivalent discrete-time system of the plant, (1) and (3), and internal model, (2), is described by the following difference equations:

\[
x(k + 1) = \tilde{A}_d x(k) + \tilde{B}_d u(k) + w_d(k)
\]  

(9)

\[
y(k) = C x(k - k_r) + v_d(k - k_r)
\]  

(10)

\[
x_{IM}(k + 1) = A_d x_{IM}(k) + B_d u(k)
\]  

(11)

where \( \tilde{A}_d = \exp(\tilde{A}\Delta) \), \( \tilde{B}_d = \tilde{A}^{-1}(\exp(\tilde{A}\Delta) - I)\tilde{B} \), \( A_d = \exp(A\Delta) \) and \( B_d = A^{-1}(\exp(A\Delta) - I)B \). The covariance matrices of the discrete process and measurement noise are \( W_d = W \Delta \) and \( V_d = V / \Delta \), respectively [1].

Stability is analyzed for time-invariant systems with constant observer and controller gain matrices, \( L_d \) and \( K_d \), respectively. These gain matrices can be computed using standard optimal estimation and control tools under the assumption of a constant model for an infinite horizon cost function. In this case, the optimal Kalman gain matrix is \( L_d = P_d C' (C P_d C' + V_d)^{-1} \) where \( P_d \) is the solution of the discrete time algebraic Riccati equation: \( P_d = A_d P_d A_d^T - (A_d P_d C' (C P_d C' + V_d)^{-1} (C P_d A_d^T + W_d)) \), while the optimal feedback gain matrix is \( K_d = (B_d S_d B_d + R)^{-1} B_d S_d A_d \) where \( S_d \) is the solution of the discrete time algebraic Riccati equation:

\[
S_d = A_d^T S_d A_d - (A_d^T S_d B_d)(B_d^T S_d B_d + R_{\infty})^{-1}(B_d^T S_d A_d + Q_{\infty}).
\]

Due to the measurement delay, the current measurement \( y(k) \) depends on the delayed state \( x(k - k_r) \). Hence the observer updates the estimated state \( \hat{x}(k - k_r) \) according to:

\[
\hat{x}(k - k_r + 1) = A_d \hat{x}(k - k_r) + B_d u(k - k_r) + L_d(y(k) - C \hat{x}(k - k_r)).
\]  

(12)

The predicted state is the solution of [11], given the estimated state \( \hat{x}(k - k_r) \):

\[
x_p(k) = A_d^{k_r} \hat{x}(k - k_r) + \sum_{i=0}^{k_r-1} A_d^{k_r-i-1} B_d u(i + k - k_r).
\]  

(13)

The control signal is proportional to the predicted state:

\[
u(k) = -K_d x_p(k).
\]  

(14)

Equations [13] and [14] imply that \( x_p(k) \) depends on \( x_p(k - k_r), ..., x_p(k - 1) \). This dependence is captured by defining the extended state \( x_{ex}(k) = [x(k - k_r)' \hat{x}(k - k_r)' x_p(k - 1)' ... x_p(k - k_r)']' \). Thus, the dynamics of the overall discrete system described by [6] - [14] can be expressed as
\( x_{ex}(k+1) = A_{ex}x_{ex}(k) + w_{ex}(k) \), where \( w_{ex}(k) = [w_d(k - k_T)' L_d v_d(k - k_T)' 0 \ldots 0] \) is the overall discrete process noise and

\[
A_{ex} = \begin{pmatrix}
\bar{A}_d & 0 & 0 & 0 & \cdots & 0 & -\bar{B}_d K_d \\
L_d C & (A_d - L_d C) & 0 & 0 & \cdots & 0 & -B_d K_d \\
0 & A_d^{k_T} & -B_d K_d & -A_d B_d K_d & \cdots & -A_d^{k_T - 2} B_d K_d & -A_d^{k_T - 1} B_d K_d \\
0 & 0 & I & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & I & 0
\end{pmatrix}
\]

References

1. Stengel R F, Optimal Control and Estimation. Courier Corporation (1994).