Theory of valuations on manifolds: a survey.

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Abstract

This is a non-technical survey of a recent theory of valuations on manifolds constructed in [10]-[13] and actually a guide to this series of articles. We review also some recent related results obtained by a number of people. We formulate some open questions.

0 Introduction.

In convexity there are many geometrically interesting and well known examples of valuations on convex sets: Lebesgue measure, the Euler characteristic, the surface area, mixed volumes, the affine surface area. For a description of older classical developments on this subject we refer to the surveys McMullen-Schneider [40], McMullen [39]. For the general background on convexity we refer to the book by Schneider [45].

Approximately during the last decade there was a significant progress in this classical subject which has led to new classification results of various classes of valuations, to discovery of new structures on them. This progress has shed a new light on the notion of valuation which allowed to generalize it in some cases to the more general setting of valuations on manifolds and on not necessarily convex sets (a concept which in any case has no meaning on a general manifold). On the other hand, the author’s feeling is that the notion of valuation equips smooth manifolds with a new general rich structure. Valuations on manifolds were introduced and studied in a series of four articles: [10], [11], [12] by the author, and [13] by J. Fu and the author. This theory depends heavily on and is a continuation of the classical theory of valuations on convex subsets of an affine space; it combines probably most of the results obtained on translation invariant continuous valuations on convex sets. Also the notions of normal (or characteristic) cycle and Legendrian currents and tools from geometric measure theory to study them (see e.g. [21]-[24]), turned out to be very useful in this new theory, as well as tools from representation theory.

The goal of this article is to give a non-technical overview of [10], [11], [13], [12]. We also mention a few recent closely related results by a number of people. We state a number of open questions on valuations on manifolds.

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In [11] the notion of smooth valuation on a smooth manifold was introduced. Roughly put, a smooth valuation can be thought as a finitely additive \( \mathbb{C} \)-valued measure on a class of nice subsets; this measure is required to satisfy some additional assumptions of continuity (or rather smoothness) in some sense. The basic examples of smooth valuations on a general manifold \( X \) are smooth measures on \( X \) and the Euler characteristic. Moreover, the well known intrinsic volumes of convex sets can be generalized to provide examples of smooth valuations on an arbitrary Riemannian manifold; these valuations are known as Lipschitz-Killing curvatures.

The article is organized as follows. In Section 1 we give an overview of some necessary previously known facts on translation invariant valuations on convex sets. In Section 2 we have surveyed relevant results on translation invariant valuations on convex sets due to the author but which were proved in other places than [10]-[13], and due to Bernig-Bröcker [16], and J. Fu [25]. Section 3 contains the description of [10]-[13]; this is the main section of this article. Section 4 plays a role of an appendix: this is a short guide to [10]-[13] where we indicate in what part of the series [10]-[13] one can find the proofs of the results discussed in Section 3.

This article is not a survey of the developments of valuation theory, not even during the last decade, and there are a number of interesting developments which are not discussed here. To mention just a few of them, these are: [2], [26], [32], [33], [34], [35], [36], [44]. Some of them (particularly Klain [32], Schneider [44]) were very influential on the progress discussed in this article.

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1 A brief overview of valuations on convex sets.

In this section we present a brief overview of necessary facts from the classical theory of valuations on convex sets. Let \( V \) be a finite dimensional real vector space of dimension \( n \). Let us denote by \( \mathcal{K}(V) \) the family of non-empty convex compact subsets of \( V \). Then \( \mathcal{K}(V) \) has a natural topology. To define it, let us fix a Euclidean metric on \( V \). The Hausdorff metric \( d_H \) on \( \mathcal{K}(V) \) is defined as follows:

\[
d_H(A, B) := \inf\{\varepsilon > 0 \mid A \subset (B)_\varepsilon \text{ and } B \subset (A)_\varepsilon\}
\]

where \((U)_\varepsilon\) denotes the \( \varepsilon \)-neighborhood of a set \( U \). It is well known (the Blaschke selection theorem) that \( \mathcal{K}(V) \) equipped with the Hausdorff metric \( d_H \) is a locally compact space. If we choose a different Euclidean metric on \( V \) the corresponding Hausdorff metric will define the same topology on \( \mathcal{K}(V) \).

1.1 Definition. A scalar valued functional

\[
\phi: \mathcal{K}(V) \to \mathbb{C}
\]
is called a *convex valuation* if
\[ \phi(A \cup B) = \phi(A) + \phi(B) - \phi(A \cap B) \]
whenever \( A, B, A \cup B \in \mathcal{K}(V) \).

**1.2 Remark.** In Definition 1.1 the notion we called *convex valuation* is called just *valuation* in all the classical literature. We have made this change of terminology in order to emphasize on one hand that in the sequel we will consider valuations defined on not necessarily convex sets, and on the other hand that the new notion of valuation discussed below generalizes in a sense the classical notion from Definition 1.1.

**1.3 Definition.** A convex valuation \( \phi \) is called *continuous* if \( \phi \) is continuous in the Hausdorff metric.

Let us denote by \( \text{Val}(V) \) the space of translation invariant continuous convex valuations. Equipped with the topology of uniform convergence on compact subsets of \( \mathcal{K}(V) \) the space \( \text{Val}(V) \) becomes a Banach space. Let us give some examples:

1) A Lebesgue measure belongs to \( \text{Val}(V) \).
2) The Euler characteristic \( \chi \) belongs to \( \text{Val}(V) \) (recall that \( \chi(K) = 1 \) for any \( K \in \mathcal{K}(V) \)).
3) Fix a convex compact set \( A \in \mathcal{K}(V) \). Let \( \text{vol} \) denotes a Lebesgue measure on \( V \). Define
\[ \phi(K) := \text{vol}(K + A) \]
where \( K + A \) denotes the Minkowski sum of \( K \) and \( A \), namely \( K + A := \{ k + a \mid k \in K, a \in A \} \). Then \( \phi \in \text{Val}(V) \).
4) Let us fix an integer \( i = 0, 1, \ldots, n \). Let us fix \( A_1, \ldots, A_{n-i} \in \mathcal{K}(V) \). Then the mixed volume \( K \mapsto V(K[i], A_1, \ldots, A_{n-i}) \) belongs to \( \text{Val}(V) \), where \( K[i] \) means that \( K \) is taken \( i \) times (for the notion of mixed volume see e.g. Schneider’s book [45]).
5) There is a different construction of continuous translation invariant convex valuations based on the theory of complex and quaternionic plurisubharmonic functions; see [5], [9].

It was conjectured by P. McMullen [38] that linear combinations of mixed volumes are dense in \( \text{Val}(V) \). This conjecture was proved by the author in [4] in a much stronger form. In order to formulate this result, called Irreducibility Theorem, let us remind a few necessary facts.

Let \( \phi \) be a convex valuation. Let \( \alpha \) be a complex number. We say that \( \phi \) is \( \alpha \)-homogeneous if
\[ \phi(\lambda K) = \lambda^\alpha \phi(K) \]
for any \( \lambda > 0, K \in \mathcal{K}(V) \).

Let us denote by \( \text{Val}_\alpha(V) \) the subspace of \( \text{Val}(V) \) of \( \alpha \)-homogeneous convex valuations. For instance, the Euler characteristic is 0-homogeneous, a Lebesgue measure is \( n \)-homogeneous, and the mixed volumes from Example 4) are \( i \)-homogeneous. The following result is due to P. McMullen [37].

**1.4 Theorem ([37]).** Let \( n = \dim V \). Then
\[ \text{Val}(V) = \bigoplus_{i=0}^{n} \text{Val}_i(V). \]
1.5 Remark. 1) It is easy to see that $Val_0(V)$ is one dimensional and is spanned by the Euler characteristic $\chi$.

2) It was shown by Hadwiger \[29\] that $Val_n(V)$ is also one dimensional and is spanned by a Lebesgue measure.

We say that a convex valuation $\phi$ is even if for any $K \in \mathcal{K}(V)$ one has $\phi(-K) = \phi(K)$. Similarly $\phi$ is called odd if for any $K \in \mathcal{K}(V)$ one has $\phi(-K) = -\phi(K)$. The space of $i$-homogeneous valuations is decomposed further into a direct sum with respect to parity:

$$Val_i(V) = Val_i^{ev}(V) \oplus Val_i^{odd}(V)$$

where the notation is obvious.

The group $GL(V)$ of invertible linear transformations of $V$ acts naturally on $Val(V)$ as follows:

$$(g\phi)(K) = \phi(g^{-1}(K))$$

for $g \in GL(V), \phi \in Val(V), K \in \mathcal{K}(V)$. This action is continuous and preserves the degree of homogeneity and the parity of valuations. The Irreducibility Theorem says the following.

1.6 Theorem (\[4\]). The natural representation of $GL(V)$ in $Val_i^{ev}(V)$ and $Val_i^{odd}(V)$ is irreducible for any $i = 0, 1, \ldots, n = \dim V$ (i.e. there is no proper closed $GL(V)$-invariant subspace).

This result implies P. McMullen’s conjecture (see \[4\]). The proof used most of the known (by that time) results on translation invariant continuous convex valuations in combination with tools from representation theory \[15\].

The following definition is a special case of a more general notion of a smooth vector in a representation space of a group.

1.7 Definition. A convex valuation $\phi \in Val(V)$ is called smooth if the map $GL(V) \to Val(V)$ defined by $g \mapsto g(\phi)$ is infinitely differentiable.

Let us denote by $Val^\infty(V)$ the subset of smooth convex valuations in the sense of Definition 1.7. It is well known (for general representation theoretical reasons) that $Val^\infty(V) \subset Val(V)$ is a linear $GL(V)$-invariant subspace dense in $Val(V)$. The space $Val^\infty(V)$ has a canonical Fréchet topology. Denoting by $Val_i^\infty(V), Val_i^{ev,\infty}(V), Val_i^{odd,\infty}(V)$ the subspaces of smooth vectors in $Val_i(V), Val_i^{ev}(V), Val_i^{odd}(V)$, respectively, one easily deduces the following versions of McMullen’s decomposition:

$$Val^\infty(V) = \bigoplus_{i=0}^n Val_i^\infty(V); \quad (1)$$

$$Val_i^\infty(V) = Val_i^{ev,\infty}(V) \oplus Val_i^{odd,\infty}(V). \quad (2)$$

### 2 Some results and questions on translation invariant valuations on convex sets.

In this section we review a number of results on translation invariant (convex) valuations due to several people which are very closely related to the material discussed in Section \[\] below.
For an \( n \)-dimensional real vector space \( V \) we denote by \( \text{Val}(V) \) the space of continuous translation invariant convex valuations in the sense of Definition 1.1 and by \( \text{Val}^{\infty}(V) \) the subspace of smooth convex valuations in the sense of Definition 1.7. It turns out that valuations from \( \text{Val}^{\infty}(V) \) have the following important property (compare with Proposition 3.6 below): they can be naturally evaluated on compact sets of positive reach (or, in other terminology, semi-convex sets); this class of sets contains all the compact convex sets and all compact submanifolds with corners.

Recall that we have McMullen’s decomposition (1) with respect to the degree of homogeneity:

\[
\text{Val}^{\infty}(V) = \bigoplus_{i=0}^{n} \text{Val}^{\infty}_{i}(V). 
\] (3)

The following result was proved in [6].

**2.1 Theorem.** The space \( \text{Val}^{\infty}(V) \) has a canonical continuous product \( \text{Val}^{\infty}(V) \times \text{Val}^{\infty}(V) \to \text{Val}^{\infty}(V) \). Then \( \text{Val}^{\infty}(V) \) becomes a graded algebra:

\[
\text{Val}^{\infty}_{i}(V) \cdot \text{Val}^{\infty}_{j}(V) \subset \text{Val}^{\infty}_{i+j}(V).
\]

Moreover, it satisfies the Poincaré duality, namely for any \( i = 0, 1, \ldots, n \) the product

\[
\text{Val}^{\infty}_{i}(V) \times \text{Val}^{\infty}_{n-i}(V) \to \text{Val}^{\infty}_{n}(V) (= \mathbb{C} \cdot \text{vol})
\]

is a perfect pairing, in other words, the induced map

\[
\text{Val}^{\infty}_{i}(V) \to (\text{Val}^{\infty}_{n-i}(V))^* \otimes \text{Val}^{\infty}_{n}(V)
\]

is injective and has a dense image in the weak topology.

Thus \( \text{Val}^{\infty}(V) \) is a graded algebra satisfying the Poincaré duality; such algebras are often called Frobenius algebras. The proof of this theorem, besides the construction of the product from [6], uses the full generality of the Irreducibility Theorem.

The following result is a version of the hard Lefschetz theorem for even valuations. In order to formulate it, let us fix a Euclidean metric on \( V \). Let us denote by \( V_1 \in \text{Val}^{\infty}_{1}(V) \) the first intrinsic volume on \( V \) (see e.g. [45], p. 210); by Hadwiger’s theorem [29] this is, up to a constant, the only non-zero 1-homogeneous \( SO(n) \)-invariant translation invariant continuous valuation (which is automatically smooth, see Proposition 2.7 below).

**2.2 Theorem ([7]).** Fix an integer \( i, 0 \leq i < n/2 \). Then the operator

\[
\text{Val}^{\text{ev},\infty}_{i} \to \text{Val}^{\text{ev},\infty}_{n-i}
\]

defined by \( \phi \mapsto V_1^{n-2i} \cdot \phi \) is an isomorphism.

The proof of this theorem in [7] uses, besides much of the machinery of the valuation theory, the results on the Radon transform on Grassmannians due to Gelfand, Graev, Roşu [27] and the solution of the cosine transform problem for Grassmannians by Bernstein and the author [14]. The analogous result was conjectured in [17] for odd valuations.
2.3 Conjecture ([7]). Fix an integer \( i, 0 \leq i < n/2 \). Then the operator

\[ \text{Val}^{\text{odd}, \infty}_i(V) \to \text{Val}^{\text{odd}, \infty}_{n-i} \]

defined by \( \phi \mapsto V_i^{n-2i} \cdot \phi \) is an isomorphism.

We would like to state another version of the hard Lefschetz theorem for valuations which was proved by the author in [5] in the even case, and by Berndig-Bröcker [16] in full generality very recently. Let us consider the operator

\[ L: \text{Val}^{\infty}(V) \to \text{Val}^{\infty}(V) \]

defined by

\[ (L\phi)(K) := \frac{d}{d\varepsilon}|_{\varepsilon=0}\phi(K + \varepsilon \cdot D) \tag{4} \]

where \( D \) is the unit Euclidean ball. (Note that by a result of P. McMullen [37], \( \phi(K + \varepsilon \cdot D) \) is a polynomial in \( \varepsilon \geq 0 \) of degree at most \( n \).) It is easy to see that the operator \( L \) decreases the degree of homogeneity by one. Then one has

2.4 Theorem. Fix an integer \( i, n/2 < i \leq n \). Then

\[ L^{2i-n}: \text{Val}^{\infty}_i(V) \to \text{Val}^{\infty}_{n-i}(V) \]

is an isomorphism.

Note that the author’s proof [7] in the even case used the tools from integral geometry (such as Radon and cosine transforms on Grassmannians), while Bernig and Bröcker [16] used their description of the forms on the cotangent bundle defining the zero valuation (see Remark 3.9 (2) of this article) in terms of the Rumin operator, in combination with notion of Laplacian on translation invariant valuations they introduced.

2.5 Remark. The terminology ”Poincaré duality” and ”hard Lefschetz theorem” comes from the formal analogy of these properties of the algebra of valuations with the corresponding properties of the cohomology algebra of compact Kähler manifolds. The Poincaré duality is one of the most basic properties of general compact oriented manifolds, and the hard Lefschetz theorem is one of the most basic properties of general compact Kähler manifolds (see e.g. [28]).

Now let us discuss valuations invariant under a group. Let us fix from now on a Euclidean metric on \( V \). Let \( G \) be a compact subgroup of the orthogonal group. Let us denote by \( \text{Val}^G(V) \) the subspace of \( \text{Val}(V) \) of \( G \)-invariant convex valuations. One has the following result.

2.6 Proposition. The space \( \text{Val}^G(V) \) is finite dimensional if and only if \( G \) acts transitively on the unit sphere of \( V \).
The "if" part of this proposition was proved in \[3\], Theorem 8.1. The "only if" part is announced in print for the first time here; its proof is not very difficult. Thus in the case when \(G\) acts transitively on the unit sphere one may hope to obtain an explicit finite classification list of valuations \(\text{Val}^G(V)\). Thus from now on we will assume that \(G\) acts transitively on the unit sphere.

2.7 Proposition (\[5\]; \[6\], Theorem 0.9(ii)). Under the assumption that \(G\) acts transitively on the unit sphere one has

\[
\text{Val}^G(V) \subset \text{Val}^\infty(V).
\]

2.8 Remark. It is important to emphasize that Proposition 2.7 implies that any \(G\)-invariant convex valuation from \(\text{Val}^G(V)\) can be naturally evaluated on compact submanifolds with corners, and in fact on the larger class of compact sets of positive reach.

Obviously \(\text{Val}^G(V)\) is a subalgebra, and one has McMullen’s decomposition with respect to the degree of homogeneity

\[
\text{Val}^G(V) = \bigoplus_{i=0}^{n} \text{Val}^G_i(V).
\]

Define

\[
h_i := \dim \text{Val}^G_i(V).
\]

Then \(\text{Val}^G_0(V)\) is spanned by the Euler characteristic, and \(\text{Val}^G_n(V)\) is spanned by a Lebesgue measure.

2.9 Theorem. Assume that the group \(G\) acts transitively on the unit sphere.

(i) (\[6\], Theorem 0.9) \(\text{Val}^G(V)\) is a finite dimensional graded algebra (with the grading given by the degree of homogeneity) satisfying the Poincaré duality, i.e.

\[
\text{Val}^G_i(V) \times \text{Val}^G_{n-i}(V) \to \text{Val}^G_n( = \mathbb{C} \cdot \text{vol})
\]

is a perfect pairing. In particular \(h_i = h_{n-i}\).

(ii) (\[6\], Theorem 0.9) Moreover, \(h_1 = h_{n-1} = 1\). \(\text{Val}^G_1(V)\) is spanned by the first intrinsic volume \(V_1\); \(\text{Val}^G_{n-1}(V)\) is spanned by the \((n-1)\)-st intrinsic volume \(V_{n-1}\).

(iii) (\[7\]) Assume in addition that \(-\text{Id} \in G\). Let \(i\) be an integer \(0 \leq i < n/2\). Then \(\text{Val}^G(V)\) satisfies a version of the hard Lefschetz theorem: the operator

\[
\text{Val}^G_i(V) \to \text{Val}^G_{n-i}(V)
\]

defined by \(\phi \mapsto V_1^{n-2i} \cdot \phi\) is an isomorphism.

2.10 Conjecture. In Theorem 2.9(iii) the assumption \(-\text{Id} \in G\) is unnecessary. More precisely, for any compact group \(G\) acting transitively on the unit sphere the operator

\[
\text{Val}^G_i(V) \to \text{Val}^G_{n-i}(V)
\]

defined by \(\phi \mapsto V_1^{n-2i} \cdot \phi\) is an isomorphism for any integer \(i\), \(0 \leq i < n/2\).
Of course, Conjecture 2.10 is an immediate consequence of Conjecture 2.8.

Let us state another version of the hard Lefschetz theorem for $Val^G(V)$ which was proved in [5] under the assumption $-Id \in G$ and in [10] in general (and which is a consequence of Theorem 2.4).

2.11 Theorem. Let $G$ be a group acting transitively on the unit sphere. Let $L: Val^\infty(V) \to Val^\infty(V)$ be the operator defined by (4). Let $i$ be an integer, $n/2 < i \leq n$. Then
\[ L^{2i-n}: Val^G_i(V) \to Val^G_{n-i}(V) \]
is an isomorphism. In particular $h_i \geq h_{i+1}$ for $i \geq n/2$.

Let us consider now concrete examples of compact groups $G$ acting transitively on the unit sphere. The cases when $G$ is equal either to the full orthogonal group $O(n)$ or to the special orthogonal group $SO(n)$ are classical, and there is the following famous result by Hadwiger [29].

2.12 Theorem ([29]). $Val^{O(n)}(\mathbb{R}^n) = Val^{SO(n)}(\mathbb{R}^n)$, and a basis of this space is
\[ \chi, V_1, V_2, \ldots, V_{n-1}, vol \]
where $V_i$ denotes the $i$-th intrinsic volume.

Given the Hadwiger theorem 2.12, it is not hard to describe the algebra structure of $Val^{O(n)}(\mathbb{R}^n) = Val^{SO(n)}(\mathbb{R}^n)$. One has

2.13 Proposition ([6], Theorem 2.6). The morphism of algebras
\[ \mathbb{C}[x]/(x^{n+1}) \to Val^{O(n)}(\mathbb{R}^n) = Val^{SO(n)}(\mathbb{R}^n) \]
given by $x \mapsto V_1$ defines an isomorphism of graded algebras.

Now let us discuss the other compact groups $G$. It is known in topology that the condition that $G$ acts transitively on the sphere, is quite restrictive. In particular, there exists an explicit classification of connected compact Lie groups acting transitively on the sphere, due to A. Borel [17], [18] and Montgomery-Samelson [42]. They have obtained the following list:

6 infinite series: $SO(n), U(n), SU(n), Sp(n), Sp(n) \cdot Sp(1), Sp(n) \cdot U(1)$;
3 exceptions: $G_2, Spin(7), Spin(9)$.

The next case which has been studied in detail is the group $G = U(m)$ acting on the standard Hermitian space $\mathbb{C}^m$. Define $n := 2m = \dim_{\mathbb{R}} \mathbb{C}^m$. Let us denote by $IU(m)$ the group of isometries of the Hermitian space $\mathbb{C}^m$ preserving the complex structure (then $IU(m) = \mathbb{C}^m \rtimes U(m)$). Let $Gr_j$ denote the Grassmannian of affine complex subspaces of $\mathbb{C}^m$ of complex dimension $j$. Clearly $Gr_j$ is a homogeneous space of $IU(m)$ and it has a unique (up to a constant) $IU(m)$-invariant measure (called Haar measure). For every non-negative integers $p$ and $k$ such that $2p \leq k \leq 2m$ let us introduce the following valuations:
\[ U_{k,p}(K) = \int_{E \in Gr_{m-p}} V_{k-2p}(K \cap E) dE. \]
Then $U_{k,p} \in Val^{U(m)}(\mathbb{C}^m)$.
2.14 Theorem ([5]). The valuations $U_{k,p}$ with $0 \leq p \leq \frac{\min\{k,2m-k\}}{2}$ form a basis of the space $Val_k^{U(m)}(\mathbb{C}^m)$.

The proof of this result used, besides the even case of the Irreducibility Theorem, the even case of the hard Lefschetz theorem (Theorem 2.11), and representation theoretical computations of Howe-Lee [30]. Some applications of Theorem 2.14 to integral geometry of complex spaces can be found in [5].

The description of the algebra structure of $Val^{U(m)}(\mathbb{C}^m)$ turned out to be a more difficult problem than for the group $SO(n)$, and the answer is much more interesting. It was obtained recently by J. Fu [25] in terms of generators and relations. His result is as follows.

2.15 Theorem ([25]). The graded algebra $Val^{U(m)}(\mathbb{C}^m)$ is isomorphic to the graded algebra $\mathbb{C}[s,t]/(f_{m+1}, f_{m+2})$ where the generators $s$ and $t$ have degrees 2 and 1 respectively, and $f_j$ is the degree $j$ component of the power series $\log(1 + s + t)$.

Note that valuations invariant under the other groups from the list (6)-(7) have not been classified, with the only exception of the group $SU(2)$ acting on $\mathbb{C}^2 \cong \mathbb{R}^4$. The explicit basis of $Val^{SU(2)}(\mathbb{C}^2)$ in geometric terms was obtained by the author in [8]; the algebra structure of it has not been computed. Some non-trivial examples of convex valuations invariant under the quaternionic groups $Sp(m), Sp(m) \cdot Sp(1)$ were constructed by the author in [9] using quaternionic plurisubharmonic functions.

Another direction which has not been studied in detail is the description of special classes of valuations on non-affine manifolds (see Section 3 below). For instance one can prove the following result (compare with Proposition 2.13) which is announced here for the first time; the details will appear elsewhere. Let $X^n$ denote either the $n$-dimensional sphere or the $n$-dimensional hyperbolic space with the standard Riemannian metric.

2.16 Proposition. The algebra of isometry invariant smooth valuations on $X^n$ is isomorphic to the algebra of truncated polynomials $\mathbb{C}[x]/(x^{n+1})$ where a generator is the first Lipschitz-Killing curvature.

The results analogous to Theorems 2.14 and 2.15 for the complex projective and hyperbolic spaces seem to be of interest and are still to be obtained.

3 Main results: valuations on manifolds.

Let $X$ be a smooth manifold. Let $n = \dim X$. We assume also for simplicity that $X$ is countable at infinity, i.e. $X$ can be presented as a countable union of compact sets. Let us denote by $\mathcal{P}(X)$ the family of all compact submanifolds with corners (for the notion of manifold with corners, see e.g. the book [11], Chapter 1). We will be interested in finitely additive measures on $\mathcal{P}(X)$ which satisfy some additional conditions of continuity (even some smoothness).

3.1 Remark. The class $\mathcal{P}(X)$ is neither closed under finite unions nor under finite intersections. Thus the notion of finite additivity should be explained. Roughly, finite additivity holds whenever it makes sense (see Section 2.2 of Part II [11] for the details).
Let $P \in \mathcal{P}(X)$ be a compact submanifold with corners. Let us remind the definition of the characteristic cycle of $P$ denoted by $CC(P)$. For any point $x \in P$ let $T_xP \subset T_xX$ denote the tangent cone of $P$ at the point $x$. It is defined as follows:

$$T_xP := \{ \xi \in T_xX \mid \text{there exists a } C^1 - \text{map } \gamma : [0, 1] \to P \text{ such that } \gamma(0) = x \text{ and } \gamma'(0) = \xi \}.$$  

$T_xP$ is a convex polyhedral cone, and if $P$ has no corners then $T_xP$ is the usual tangent space at $x$. Let $(T_xP)^\circ \subset T_x^*X$ denote the dual cone. Recall that the dual cone $C^o$ of a convex cone $C$ in a linear space $W$ is defined by

$$C^o := \{ y \in W^* \mid y(x) \geq 0 \text{ for any } x \in C \}.$$  

Define the characteristic cycle

$$CC(P) := \bigcup_{x \in P} (T_xP)^\circ.$$  

Then it is well known that $CC(P)$ has the following properties:

1. $CC(P) \subset T^*X$ is an $n$-dimensional submanifold with singularities;
2. $CC(P)$ is Lagrangian, $\mathbb{R}_{>0}$-invariant;
3. if $X$ is oriented then $CC(P)$ is an $n$-cycle, i.e. $\partial(CC(P)) = 0$.

3.2 Remark. (1) If $P$ has no corners then $CC(P)$ is the usual co-normal bundle.

(2) The notion of characteristic cycle (or almost equivalent notion of normal cycle) is well known. First the notion of normal cycle was introduced by Wintgen [47], and then studied further by Zähle [48] by the tools of geometric measure theory. Characteristic cycles of subanalytic sets of real analytic manifolds were introduced by J. Fu [24] using the tools of geometric measure theory, and independently by Kashiwara (see [31], Chapter 9) using the tools of sheaf theory. J. Fu’s article [24] develops an approach to define the normal cycle for more general sets than subanalytic or convex ones (see Theorem 3.2 in [24]).

3.3 Definition. Let $\phi : \mathcal{P}(X) \to \mathbb{C}$ be a finitely additive measure. We say that $\phi$ is continuous if for any uniformly bounded sequence $\{P_N\} \subset \mathcal{P}(X)$ and $P \in \mathcal{P}(X)$ such that $CC(P_N) \xrightarrow{\text{flat}} CC(P)$ one has

$$\phi(P_N) \to \phi(P).$$  

3.4 Remark. In the above definition the convergence is understood in the sense of local flat convergence of currents on $T^*X$. For the definition of this notion we refer to the book [20]. Here we only notice that it is well known that if $\{K_N\}$ is a uniformly bounded sequence of convex compact subsets in $\mathbb{R}^n$ (or more generally, compact uniformly bounded subsets with reach at least $\delta > 0$) then $K_N \to K$ in the Hausdorff metric iff $CC(K_N) \xrightarrow{\text{flat}} CC(K)$.

The above assumption of continuity of measures is the most important condition. It turns out mostly for technical reasons that one should impose on valuations some other conditions. We will skip here the precise definitions since most of these conditions are technical and their necessity is not very clear for the moment. Let us denote by $V^\infty(X)$ the set of smooth measures which are called smooth valuations. $V^\infty(X)$ is a linear space and has a natural nuclear Fréchet topology. This space is the main object we are going to discuss. For understanding of this survey it is possible to accept the description of all smooth valuations given in Proposition 3.8 below as a definition of smooth valuations.
3.5 Examples. (1) Any smooth density on $X$ belongs to $V^\infty(X)$.
(2) The Euler characteristic belongs to $V^\infty(X)$.

Recall that we denote by $Val^\infty(\mathbb{R}^n)$ the space of smooth translation invariant convex valuations on $\mathbb{R}^n$ in the sense of Section 1. Let us denote by $(V^\infty(\mathbb{R}^n))^{tr}$ the subspace of $V^\infty(\mathbb{R}^n)$ of translation invariant valuations.

3.6 Proposition. Restriction to convex subsets of $\mathbb{R}^n$ defines the map

$$(V^\infty(\mathbb{R}^n))^{tr} \to Val^\infty(\mathbb{R}^n).$$

This map is an isomorphism.

3.7 Remark. In other words, this proposition says that the elements of $Val^\infty(\mathbb{R}^n)$ extend canonically to a class of not necessarily convex sets.

The next proposition provides a description of $V^\infty(X)$. We will assume for simplicity that the manifold $X$ is oriented though this is not strictly necessary, and the result can be appropriately generalized to any smooth manifold.

3.8 Proposition. Let $X$ be an oriented manifold. Let $\phi \in V^\infty(X)$. There exists a $C^\infty$-smooth differential $n$-form $\omega$ on $T^*X$ which has the support compact relative to the canonical projection $T^*X \to X$, and such that for any $P \in \mathcal{P}(X)$

$$\phi(P) = \int_{CC(P)} \omega.$$ (8)

And vice versa, any expression of the above form is a smooth valuation.

3.9 Remark. (1) For a given valuation $\phi$, the above form $\omega$ is highly non-unique. For instance, for the Euler characteristic the above form is well known and it was constructed by Chern [19]. The construction depends on an extra choice of a Riemannian metric on $X$.

(2) The forms defining the zero valuation were described in a recent preprint by Bernig and Bröcker [16] by an explicitly written system of differential and integral equations. In particular, they realized key role of the Rumin operator on differential forms on contact manifolds [43] for this problem. We refer to [16] for the precise statements.

(3) The fact that the expressions of the form (8) are smooth valuations heavily uses the tools from geometric measure theory (see [21]-[24]). The converse statement uses the Irreducibility Theorem (Theorem 1.6) in combination with the Casselman-Wallach theorem (see e.g. [46]) from representation theory.

3.10 Proposition (Sheaf property). The correspondence for any open subset $U \subset X$

$$U \mapsto V^\infty(U)$$

with the obvious restriction maps is a sheaf on $X$ which we denote by $V^\infty_X$.

Let us denote by $Val(TX)$ the (infinite dimensional) vector bundle over $X$ such that its fiber over a point $x \in X$ is equal to the space $Val^\infty(T_xX)$ of smooth translation invariant convex valuations on $T_xX$. By McMullen’s theorem it has a grading by the degree of homogeneity: $Val^\infty(TX) = \oplus_{i=0}^n Val^\infty_i(TX)$. 

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3.11 Theorem. There exists a canonical filtration of $V^\infty(X)$ by closed subspaces

$$V^\infty(X) = W_0 \supset W_1 \supset \cdots \supset W_n$$

such that the associated graded space $\bigoplus_{i=0}^n W_i/W_{i+1}$ is canonically isomorphic to the space of smooth sections $C^\infty(X, Val^\infty_i(TX))$.

3.12 Remark. (1) For $i = n$ the above isomorphism means that $W_n$ coincides with the space of smooth densities on $X$.

(2) For $i = 0$ the above isomorphism means that $W_0/W_1$ is canonically isomorphic to the space of smooth functions $C^\infty(X)$.

(3) Actually $U \mapsto W_i(U)$ defines a subsheaf $W_i$ of $V^\infty_X$.

Next, $V^\infty(X)$ carries a very important and non-trivial multiplicative structure (extending to the product on translation invariant valuations discussed in Theorem 2.1).

3.13 Theorem. There exists a canonical product $V^\infty(X) \times V^\infty(X) \to V^\infty(X)$ which is

(1) continuous;
(2) commutative and associative;
(3) the filtration $W_\bullet$ is compatible with it:

$$W_i \cdot W_j \subset W_{i+j};$$

(4) the Euler characteristic $\chi$ is the unit in the algebra $V^\infty(X)$;

(5) this product commutes with restrictions to open and closed submanifolds; in particular $V^\infty_X$ is a sheaf of filtered algebras.

In order to formulate one of the most interesting properties of this product, let us observe that the space of smooth valuations with compact support $V^\infty_c(X)$ admits a continuous integration functional

$$\int : V^\infty_c(X) \to \mathbb{C}$$

given by $\phi \mapsto \phi(X)$. Consider the bilinear map

$$V^\infty(X) \times V^\infty_c(X) \to \mathbb{C}$$

defined by $(\phi, \psi) \mapsto \int \phi \cdot \psi$.

3.14 Theorem. This bilinear form is a perfect pairing. In other words, the induced map

$$V^\infty(X) \to (V^\infty_c(X))^*$$

is injective and has a dense image (with respect to the weak topology in $(V^\infty_c(X))^*$).

The proof of this theorem uses the full statement of the Irreducibility Theorem for translation invariant convex valuations. We call Theorem 3.14 the Selfduality Property of valuations.
3.15 Definition. Let us define $V^{-\infty}(X) := (V_c^\infty(X))^*$. Elements of this space will be called generalized valuations.

Thus $V^{-\infty}(X)$ can be considered as a completion of $V^\infty(X)$ with respect to the weak topology. Roughly speaking, one may say that the space of valuations is essentially self-dual (at least when $X$ is compact). For any open subsets $U \subset V \subset X$ one has the natural restriction map $V^{-\infty}(V) \to V^{-\infty}(U)$ dual to the imbedding $V_c^\infty(U) \hookrightarrow V_c^\infty(V)$. The assignment $U \mapsto V^{-\infty}(U)$ is a sheaf. This sheaf is denoted by $\mathcal{V}_X^{-\infty}$.

3.16 Theorem. There exists a canonical automorphism of the algebra of smooth valuations

$$\sigma: V^\infty(X) \to V^\infty(X)$$

such that

1. $\sigma^2 = \text{Id}$, i.e. $\sigma$ is involutive;
2. $\sigma$ is continuous;
3. $\sigma$ preserves the filtration $W_\cdot$, namely $\sigma(W_i) = W_i$;
4. for any smooth translation invariant valuation $\phi$ on $\mathbb{R}^n$ one has

$$(\sigma\phi)(K) = (-1)^{\deg \phi} \phi(-K)$$

where $\deg \phi$ denotes the degree of homogeneity of $\phi$.
5. The involution $\sigma$ extends (uniquely) by continuity to $V^{-\infty}(X)$ in the weak topology.
6. $\sigma$ commutes with restrictions to open subsets, thus it defines the involution of sheaves $\mathcal{V}_X^\infty$ and $\mathcal{V}_X^{-\infty}$.

3.17 Remark. We call $\sigma$ the Euler-Verdier involution.

Let us discuss valuations on real analytic manifolds. On these manifolds the space of constructible functions imbeds canonically as a dense subspace of the space of generalized valuations, and it is useful to compare the properties of the space of valuations with the more familiar properties of the space of constructible functions.

3.18 Definition ([31], §9.7). An integer valued function $f: X \to \mathbb{Z}$ is called constructible if

1. for any $m \in \mathbb{Z}$ the set $f^{-1}(m)$ is subanalytic;
2. the family of sets $\{f^{-1}(m)\}_{m \in \mathbb{Z}}$ is locally finite.

Clearly the set of constructible $\mathbb{Z}$-valued functions is a ring with pointwise multiplication. As in [31] we denote this ring by $CF(X)$. Define

$$\mathcal{F} := CF(X) \otimes_{\mathbb{Z}} \mathbb{C}.$$  \hfill (10)

Thus $\mathcal{F}$ is a subalgebra of the $\mathbb{C}$-algebra of complex valued functions on $X$. In the rest of the article the elements of $\mathcal{F}$ will be called constructible functions.

Let $F_c(X)$ denote the subspace of $\mathcal{F}(X)$ of compactly supported constructible functions. Clearly $F_c(X)$ is a subalgebra of $\mathcal{F}(X)$ (without unit if $X$ is non-compact).
For a subset \( P \subset X \) let us denote by \( 1_{P} \) the indicator function of \( P \), namely
\[
1_{P}(x) = \begin{cases} 
1 & \text{if } x \in P \\
0 & \text{if } x \notin P.
\end{cases}
\]

For any constructible function \( f \) one can define (see [31, 24]) its characteristic cycle \( CC(f) \) so that for any subanalytic subset \( P \subset X \) which is a compact subanalytic submanifold with corners one has \( CC(1_{P}) = CC(P) \).

We have a canonical linear map
\[
\Xi: \mathcal{F} \to V^{-\infty}(X)
\]
which is uniquely characterized by the property
\[
<\Xi(1_{P}), \phi> = \phi(P)
\]
for any closed subanalytic subset \( P \) and any \( \phi \in V_{c}^{\infty}(X) \) (strictly speaking, such a subset \( P \) does not belong to \( \mathcal{P}(X) \) in general, so this construction must be justified; we refer to Section 8 of [12] for the details). The map \( \Xi \) turns out to be injective and has a dense image in the weak topology. Thus we get the following imbeddings of dense subspaces of \( V^{-\infty}(X) \)
\[
\mathcal{F} \hookrightarrow V^{-\infty}(X) \hookleftarrow V^{\infty}(X).
\]  

(12)

First one can extend the filtration \( \{W_{i}\} \) on \( V^{\infty}(X) \) to \( V^{-\infty}(X) \) by taking closures in the weak topology of each \( W_{i} \). By [12] the restriction of this filtration on \( V^{-\infty}(X) \) back to \( V^{\infty}(X) \) coincides with the original filtration \( \{W_{i}\} \). Now let us restrict the filtration obtained on \( V^{-\infty}(X) \) to \( \mathcal{F} \) and denote the induced filtration by \( \{\tilde{W}_{i}\} \). Then the filtration \( \{\tilde{W}_{i}\} \) on \( \mathcal{F} \) coincides with the filtration by the codimension of the support. More precisely we have

3.19 Proposition. Let \( X \) be a real analytic manifold of dimension \( n \). For any \( i = 0, 1, \ldots, n \)
\[
\tilde{W}_{i} = \{f \in \mathcal{F} | \text{codim (supp } f) \geq i\}
\]

For compactly supported valuations and constructible functions we have imbeddings analogous to (12)
\[
\mathcal{F}_{c} \hookrightarrow V_{c}^{-\infty}(X) \hookleftarrow V_{c}^{\infty}(X).
\]

(13)

Then the integration functional \( \int: V_{c}^{\infty}(X) \to \mathbb{C} \) extends (uniquely) by continuity in the weak topology to a linear functional
\[
\int: V_{c}^{-\infty}(X) \to \mathbb{C}.
\]

3.20 Proposition. Let \( X \) be a real analytic manifold. The restriction of the integration functional from \( V_{c}^{-\infty}(X) \) to \( \mathcal{F}_{c} \) coincides with the integration with respect to the Euler characteristic.
Recall that the integration with respect to the Euler characteristic is a linear functional $\mathcal{F}_c \to \mathbb{C}$ which is uniquely characterized by the property $\mathbb{1}_P \mapsto \chi(P)$ for any compact subanalytic subset $P \subset X$.

3.21 Remark. Observe that $\mathcal{F}$ has the natural structure of a $\mathbb{C}$-algebra with the usual pointwise multiplication. In some sense, this product should correspond to the canonical product on smooth valuations $V^\infty(X)$ discussed above. It would be interesting to make this statement rigorous. Notice however that it seems to be very unlikely that the product extends to the whole space $V^{-\infty}(X)$ of generalized valuations.

3.22 Proposition. The restriction of the Euler-Verdier involution $\sigma : V^{-\infty}(X) \to V^{-\infty}(X)$ to $\mathcal{F}$ coincides with the classical Verdier involution times $(-1)^n$ where $n = \dim X$.

For the definition of the classical Verdier involution on the constructible functions we refer to the book by Kashiwara-Schapira [31], Chapter 9. Here we notice only that if $P$ is a closed subanalytic submanifold with corners then $\sigma(\mathbb{1}_P) = (-1)^{n - \dim P} \mathbb{1}_{\text{int} P}$ where $\text{int} P$ denotes the relative interior of $P$.

4 Appendix: a short guide to [10], [11], [13], [12].

Let us describe very briefly the structure of the articles [10], [11], [13], [12].

Part I [10] still works only with convex valuations on linear spaces. There one introduces a class of smooth convex valuations playing a key role in subsequent articles since it serves as a bridge between convex valuations and general ones. Another main issue of Part I [10] is a construction of a product on this class of smooth convex valuations. It generalizes the previous construction of the product [6] for smooth convex valuations which are in addition polynomial with respect to translations. This generalization was based on the author’s earlier article [1].

Part II [11] introduces smooth valuations on general smooth manifolds. A number of descriptions of this notion are presented, and the comparison with smooth convex valuations introduced in Part I [10]. In particular, the description in terms of integration with respect to the normal (characteristic) cycle is presented. This description uses a number of results on convex valuations (including the Irreducibility Theorem) in combination with results from geometric measure theory (discussed in greater detail in Part III [13]), and the Casselman-Wallach theorem from representation theory. Also the Euler-Verdier involution on smooth valuations was introduced in [11].

The main goal of Part III [13] is to extend the product on smooth convex valuations from Part I [10] to smooth valuations on general manifolds. Roughly it works as follows. Choosing a coordinate atlas for $X$, one uses the product of valuations on $\mathbb{R}^n$, defined by the construction of Part I [10], to define the product locally. Then one shows that the products obtained on each coordinate patch coincide on pairwise intersections, and that the result does not depend on the choice of atlas. This step uses geometric measure theory. One proves commutativity, associativity, and continuity with respect to the natural topology of this product. Also in Part III [13] one reviews and proves a number of relevant results on normal cycles and geometric measure theory (following mostly [21]-[24]) which were crucial.
in the description of smooth valuations in terms of integration with respect to the normal cycle discussed in Part II \[11\].

The goal of Part IV \[12\] is twofold. First one studies further the properties of the product of smooth valuations. In particular, one proves that the filtration \( \{W_i\} \) on smooth valuations is compatible with the product, namely \( W_i \cdot W_j \subset W_{i+j} \). Then one shows that the Euler-Verdier involution is an automorphism of the algebra of smooth valuations. Then one introduces the integration functional on compactly supported valuations and proves the Selfduality Property (Theorem \[3.14\] in this text). The second main point of Part IV \[12\] is introducing the notion of generalized valuations and establishing of basic properties of them.

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