Thermalization in 2D critical quench and UV/IR mixing

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Abstract

We consider quantum quenches in models of free scalars and fermions with a generic time dependent mass $m(t)$ that goes from $m_0$ to zero. We prove that, as anticipated in MSS [1], the post-quench dynamics can be described by a CFT with initial state of the generalized Calabrese-Cardy form $|\psi\rangle = \exp[-\kappa_2 H - \sum_{n>2}^{\infty} \kappa_n W_n]|Bd\rangle$. The $W_n$ ($n = 2, 3, ..., W_2 = H$) represent the conserved $W_\infty$ charges and $|Bd\rangle$ represents a conformal boundary state. We prove the result for pre-quench states which can be the ground state or a squeezed state, and without recourse to perturbation in the $\kappa_n$'s as in MSS. For specific quench protocols $m(t)$, we compute exact time-dependent correlators. The correlators show thermalization to a generalized Gibbs ensemble (GGE), with inverse temperature $\beta = 4\kappa_2$, and chemical potentials $\mu_n = 4\kappa_n$. By an application of inverse scattering techniques, we are able to retrieve from the $\kappa_n$'s (equivalently, from temperature and chemical potentials) the exact quench protocol $m(t)$. The other notable result, which we interpret as a UV/IR mixing, is that the long distance and long time (IR) behaviour of correlators crucially rests on taking into account all $\kappa_n$'s which, in usual RG parlance, are highly irrelevant couplings in IR. This shows a different nature of RG for non-equilibrium dynamics.

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# 1 Introduction and Summary

The dynamics of systems undergoing a quantum quench has been extensively studied in recent years [2]. In a quantum quench, some parameter of the Hamiltonian changes over a brief period of time. The initial wavefunction in the pre-quench phase, whether it is a
ground state or otherwise, typically evolves to a non-stationary state, which then evolves by
the post-quench Hamiltonian which is time-independent. An important question in such a
dynamics is whether correlators equilibrate at long times, and if so, whether the equilibrium
is described by a thermal ensemble or otherwise [2, 3, 4]. With the advent of AdS/CFT,
the issue of thermalization has assumed additional significance as it maps to the subject of
gravitational collapse to a black hole [5, 6]. This has given rise to an extensive literature on
holographic thermalization (see, e.g. [7, 8, 9], for some of the early papers on the subject).
This correspondence has a direct bearing on the issue of universality of thermalization since
a collapse to a black hole state is also typically associated with loss of most memory of the
collapsing matter. In this paper, we will find that the final equilibrium state is characterized
by an infinite number of thermodynamic parameters (chemical potentials) which retain a
partial memory of the quench protocol\(^1\); in the holographic dual, this corresponds to retention
of memory by the final black hole of the collapsing matter.

A significant step in proving thermalization in a closed 2D system was taken in a recent
paper (MSS) [1] (similar results have subsequently appeared in [10]). MSS considered 1+1
dimensional quenches\(^2\), ending with a critical post-quench Hamiltonian and made the fol-
lowing assumptions:
(a) the post-quench wavefunction is of the generalized Calabrese-Cardy (gCC) form\(^3\)
\[|\psi\rangle_{gCC} = \exp[-\kappa_2 H - \sum_{n>2} \kappa_n W_n]|Bd\rangle\]
where \(W_n\) are additional conserved charges in the system (the results are valid even without
the additional charges present in the system). It was assumed that the charges are obtained
from local currents. Below, for specificity, we will assume that the system is integrable, with
a \(W_\infty\) algebra\(^4\) and the \(W_n, n = 2, 3, \ldots (W_2 = H)\) are \(W_\infty\) charges.
(b) The spectrum of conformal dimensions in the post-quench critical theory has a gap.
(c) The dimensionless parameters \(\tilde{\kappa}_n = \kappa_n/\kappa_2^{n-1}, n > 2\) are small and can be treated pertur-
batively.
(d) The size \(l\) of the interval is small compared to \(\kappa_2\).\(^5\)

With these assumptions in place, MSS proved that the reduced density matrix of an

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\(^1\)For a quench from a ground state, the final chemical potentials retain a full memory of the quench
process. When the initial state is different, the final chemical potentials retain partial information about the
initial state and the quench protocol.

\(^2\)Unless otherwise stated, the spatial direction will be regarded as non-compact.

\(^3\)In an obvious notation, we will define the boundary state with an energy cut-off, \(\exp[-\kappa_2 H]|Bd\rangle\) as the
Calabrese-Cardy state \(|\psi\rangle_{CC}\). These states were introduced in [11] to describe 2D critical quenches.

\(^4\)This clearly holds for the theory of free scalars and fermions discussed in this paper.

\(^5\)The assumptions (c) and (d) were made for technical reasons, which can, in principle, be obviated in
other methods, e.g. if the higher spin deformations \(\kappa_{n>2}\) can be represented geometrically (like \(\kappa_2\) which
is treated as an imaginary time). Assumption (b) appears to be more essential. In case of the scalar field
model discussed in the present work, this condition implies compactifying the range of \(\phi\) on a circle.
interval of size $l$ in the state (1) asympotes to that in a GGE, defined by

$$\rho_{GGE} = \frac{e^{-\beta H - \sum_{n=3}^{\infty} \mu_n W_n}}{Z}, \quad \beta = 4\kappa_2, \quad \mu_n = 4\kappa_n, \quad n > 2$$

(2)

with a relaxation rate given by

$$\gamma = \frac{2\pi}{\beta} \left[ \Delta + \sum_{n=3}^{\infty} \tilde{\mu}_n Q_n + O(\tilde{\mu}_n^2) \right], \quad \tilde{\mu}_n = \frac{\mu_n}{\beta^{n-1}}.$$  

(3)

where $\Delta, Q_n$ are given by the conformal dimension and other $W_\infty$ charges of the most relevant operator of the CFT (by assumption (b) above, $\Delta > 0$). A consequence of this result is that the expectation value of an arbitrary string of local operators, which can be enclosed in an interval of length $l$, exponentially thermalizes to its expectation value in the GGE.

One of the motivations of the present work is to extend the proof of thermalization, without making the assumptions made in MSS, in theories of free scalars or fermions with a time-dependent mass $m(t)$ quenched to $m = 0$. We allow for nontrivial pre-quench states.

We proceed in two ways:

- We consider arbitrary quench protocols $m(t)$ and arbitrary squeezed states as pre-quench states (including the ground state) and show, by mapping the quench problem to an auxiliary one-dimensional scattering problem, that the quench leads to a wavefunction of the gCC form. This proves the main ansatz of MSS (assumption (a) above). We also show that by judiciously choosing the pre-quench states one can satisfy the perturbative assumption (c). Thus, for theories satisfying (b) and, for intervals satisfying (d), thermalization follows from first principles.

- For specific quench protocols, but with arbitrary pre-quench states as above, we compute exact time-dependent correlators, and explicitly show thermalization of one- and two-point functions, without making any of the assumptions of MSS.

One of the technical advances in this paper is the use of non-trivial pre-quench states, which we take to be squeezed states. The motivation for considering this class of states is that besides being technically accessible, these states are experimentally realizable (see, e.g. [25, 26]) and carry non-trivial quantum entanglement encoded by the squeezing function.

We list below some salient features of our analysis:

1. **Memory retention by the equilibrium ensemble:** By using inverse scattering methods applied to the above-mentioned auxiliary potential scattering, we are able to relate the post-quench wavefunction, in particular $\kappa_n$-parameters of the gCC state, to

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$^6$GGE refers to a generalized Gibbs ensemble; see, e.g. [12] for a review. Thermalization to a GGE in the context of an integrable CFT was anticipated earlier in [13, 14], and, for more general general integrable models, in [15, 16, 17, 18, 19, 20, 21, 22, 23, 24].

$^7$To be precise the overlap of the square-normalized reduced density matrix in the pure state (1) with that in the mixed state (2), behaves like $1 - (\text{const})e^{-2\gamma t}$. See MSS for more details.

$^8$Of course, as we mentioned above, the assumption (a) about the gCC form of the wavefunction is in any case true.
the quench protocol $m(t)$. In fact, if we start with the ground state of the pre-quench Hamiltonian, the $\kappa_n$ parameters completely encode $m(t)$, implying that the equilibrium ensemble specified by $\mu_n = 4\kappa_n$, carries a precise memory of the quench protocol! In case we start with a squeezed state, the equilibrium ensemble remembers a combination of the quench protocol and the knowledge of the initial state.

2. **UV/IR mixing (IR sensitivity to irrelevant operators):** As already found in MSS, the relaxation rate of various operators (3), which govern late time dynamics, depends on all the chemical potentials $\mu_n$, equivalently on the $\kappa_n$. Now from (1) it is clear that the $\kappa_n$ represent perturbing a given initial state by higher dimensional (irrelevant) operators. Indeed, our computation of the exact correlators, shows that for a large class of operators, these correlators at long times and large distances, are affected by all these chemical potentials, in apparent contradiction to IR universality (this is elaborated in Section 6). This phenomenon is actually related to the memory retention mentioned above.

3. **Holographic correspondence:** Our results show that for a given quench protocol, a GGE with a finite number of specified chemical potentials can be obtained by taking the pre-quench state to be a suitably chosen squeezed state. By using this result and the correspondence shown in MSS between thermalization to GGE and quasinormal decay to a higher spin black hole, we infer that higher spin black holes with an arbitrary set of chemical potentials get related to thermalization of squeezed states in the field theory.

**Outline:** The outline and organization of the paper is as follows:

In Section 2 we consider mass quenches in a free scalar in two dimensions. We relate the dynamics to an equivalent potential scattering problem, discussed further in Appendix A. We find that the exact time-dependent wavefunction can be related to a Bogoliubov transform of the ‘out’ vacuum (the post-quench ground state). Using this fact we write down the exact form of the scalar propagator. These results hold for a general mass quench, including quenches from a massless to a massless theory. We find that the quenched state is always describable in terms of a gCC state (using an application of the BCH formula, as described in Appendix B). In Section 2.4 we work out all this for a specific quench protocol (i.e. specific time dependence of the mass parameter). In Section 2.6 we consider cases where the pre-quench state is a squeezed state. We show that this gives us a large class of initial conditions, by tuning which we can prepare a quench state in the exact form $\exp\left[-\sum_n \kappa_n W_n\right]|D\rangle$ which has a finite number of given $\kappa_n$ coefficients.

In Section 3 we show how to generalize the above results to fermions.

In Section 4 we work out the scalar propagator for the specific quench protocol of Section 2.4. This allows us to compute various exact correlators, starting either from a ground state or from specific quench states leading to a gCC state with a finite number of $\kappa_n$ parameters. We show that these correlators thermalize exponentially to a GGE; the relaxation rate is
found non-perturbatively, which agrees with (3) in the perturbative regime.

In Section 5 real time Wightman correlators in a GGE are computed.

In Section 6 we show that the IR behaviour of exact correlators is sensitive to all the chemical potentials even though these represent perturbation by irrelevant operators. We also show that the equilibrium ensemble remembers the quench protocol.

In Section 7 we make concluding remarks and mention some open problems. In Appendices C and D we discuss some notations and general results about bosonic and fermionic theories.

2 Critical quench of a scalar field: general strategy

An important example of quantum quench is provided by free scalar field theories with time-dependent mass (our notations will closely follow [27, 28], which also contain an extensive reference to the relevant literature).

\[
S = -\frac{1}{2} \int d^2 x (\partial_\mu \phi \partial^\mu \phi - m^2(t) \phi^2)
= \frac{1}{2} \int \frac{dk dt}{2\pi} \left( (\dot{\phi}(k,t))^2 - (k^2 + m^2(t))|\phi(k,t)|^2 \right), \quad \phi(-k,t) = \phi^*(k,t)
\]

In this section we will consider a mass function \(m(t)\) (this is referred to as a ‘quench protocol’) which decreases from an asymptotic value \(m_0\) in the past to the asymptotic value \(m = 0\) in the future. This is called a critical quench since mass gap vanishes following the quench.

The equations of motion of various Fourier modes in (4) get decoupled, where each mode satisfies a Schrödinger-type equation with \(-m^2(t)\) playing the role of a potential:

\[
- \frac{d^2 \phi(k,t)}{dt^2} + V(t)\phi(k,t) = E\phi(k,t), \quad V(t) = -m^2(t), \quad E = k^2.
\]

![Figure 1: The equivalent Schrödinger problem. We have assumed a quench of the mass parameter from \(m_0\) to 0, so that \(m^2(t) \xrightarrow{t \to -\infty} m_0^2, m^2(t) \xrightarrow{t \to \infty} 0.\)]
As explained in Appendix A (see, e.g. [29], Chapter 3 for details), the solution for the field \( \phi(k,t) \) can be expressed in two distinct ways, as \((cf. (99))\)

\[
\phi(k,t) = a_{in}(k)u_{in}(k,t) + a_{in}^\dagger(-k)u_{in}^*(k,t) = a_{out}(k)u_{out}(k,t) + a_{out}^\dagger(-k)u_{out}^*(k,t),
\]

where the ‘in’ and ‘out’ wavefunctions \( u_{in, out}(k,t) \) are defined as in (100). The in- and out-oscillators are related to each other through the Bogoliubov coefficients \( \alpha(k), \beta(k) \)

\[
a_{in}(k) = \alpha(k) a_{out}(k) + \beta(k) a_{out}^\dagger(-k),
\]

\[
a_{out}(k) = \alpha^*(k) a_{in}(k) + \beta^*(k) a_{in}^\dagger(-k),
\]

which are related to the potential scattering data as explained in Appendix A. The Bogoliubov coefficients are actually functions of \(|k|\), as explained in Appendix A.2.

### 2.1 General proof of the gCC ansatz [1] for the ground state

The two sets of oscillators define two distinct vacua \(|0, in\rangle\) and \(|0, out\rangle\), defined by \( a_{in}(k)|0, in\rangle = 0 \) and \( a_{out}(k)|0, out\rangle = 0 \). Using the first line of (7), we can express the in-vacuum in terms of the out-vacua as follows

\[
|0, in\rangle = \exp[\frac{1}{2} \sum_k \gamma(k) a_{out}^\dagger(k) a_{out}^\dagger(-k)]|0, out\rangle,
\]

where

\[
\gamma(k) = \beta^*(k)/\alpha^*(k) = r^*(k).
\]

In the last step we have used the expression for the reflection amplitude in (102). Equation (9) establishes the relation between the quantum quench problem in the QFT and the auxiliary potential scattering problem discussed in Appendix A.

In the above expression (8) we represent the states in the Heisenberg picture, as is customary in QFT in curved spacetime.

With the above ingredients in place, it’s a simple exercise, using the Baker-Campbell-Hausdorff formula (see Appendix B), to show that the in-vacuum can be written in the following form

\[
|0, in\rangle = \exp[\frac{1}{2} \sum_k \gamma(k) a_{out}^\dagger(k) a_{out}^\dagger(-k)]|0, out\rangle = \exp[- \sum_k \kappa(k) a_{out}^\dagger(k) a_{out}(k)]|D\rangle,
\]

\[
\kappa(k) = -\frac{1}{2} \log(-\gamma(k))
\]

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9This is proved by simply checking that the right hand side is annihilated by \( \alpha^*(k) a_{out}(k) - \beta^*(k) a_{out}^\dagger(-k) \). Here \( \sum_k \) is defined as the sum over discretized values of \( k \), as elaborated in Appendix C.

10This result was independently found some time ago, for the quench protocol discussed in Section 2.4, in [30]. We thank Sumit Das for sharing these results with us.
where $|D\rangle$ is a Dirichlet boundary state (114), defined in terms of the ‘out’ Fock space:

$$|D\rangle = \exp\left[-\frac{1}{2} \sum_k a_{\text{out}}^\dagger(k)a_{\text{out}}^\dagger(-k)\right]|0, \text{out}\rangle. \quad (11)$$

Using the relation with the scattering problem (as described in Appendix A), especially (9) and (105) we find that $\gamma(k)$ admits a small-momentum expansion of the form

$$\gamma(k) = -1 + \gamma_1 |k| + \gamma_2 |k|^2 + \gamma_3 |k|^3 + \ldots, \quad \gamma_n = r_n^*, \quad \text{Re}(\gamma_1) \geq 0 \quad (12)$$

Using this power series expansion, and the expression for $\kappa(k)$ in (10), we can expand $\kappa(k)$ also in a power series, as follows:

$$\kappa(k) = \kappa_2 |k| + \kappa_3 |k|^2 + \kappa_4 |k|^3 - \ldots,$$

$$\kappa_2 = \frac{\gamma_1}{2}, \quad \kappa_3 = \frac{1}{4} (\gamma_1^2 + 2\gamma_2), \quad \kappa_4 = \frac{1}{6} (\gamma_1^3 + 3\gamma_1\gamma_2 + 3\gamma_3), \ldots \quad (13)$$

Note that it follows that $\text{Re} \kappa_2 \geq 0$. Below we will find explicit examples of this power series for specific quench protocols $m(t)$ which interpolate from $m_0$ to $m = 0$. For quenches involving a single real scalar field, we will find that the above expansion (13) has only odd powers of $|k|$,\footnote{This is consistent with the fact that a real scalar field provides a representation of the $W_{\infty}$ algebra \cite{31} where the odd $W_n$’s vanish. See below.} and, explicitly $\kappa_2 > 0$.\footnote{For massless→massless quench, $\kappa_2$ turns out to be purely imaginary (see Section 2.5).} Putting everything together, we find the following expression for the ground state $|0, \text{in}\rangle$, in a gCC form (1) with the boundary state identified as a Dirichlet state (11):

$$|0, \text{in}\rangle = \exp[-\kappa_2 H - \sum_{n=2}^\infty \kappa_{2n} W_{2n}] |D\rangle \quad (14)$$

where $W_{2n}, n = 1, 2, \ldots (W_2 = H)$ are the even $W_{\infty}$ charges \cite{31} of the final massless scalar field theory, which we define here as follows:\footnote{The normalization convention here for the $W$-charges differs from that of \cite{31}.} \footnote{If the time-dependence of the Hamiltonian stops after a finite time, the post-quench Hamiltonian coincides with the $W_2$ charge, and the other $W_{2n}$ charges also represent conserved charges of the post-quench evolution.}

$$H \equiv W_2 = \sum_k |k|a_{\text{out}}^\dagger(k)a_{\text{out}}(k), \quad W_{2n} = \sum_k |k|^{2n-1}a_{\text{out}}^\dagger(k)a_{\text{out}}(k), \quad n = 2, 3, \ldots \quad (15)$$

The values of these charges are given by

$$\langle W_{2l}\rangle = \sum_k |k|^{2l-1} \langle N(k) \rangle, \quad l = 1, 2, 3, \ldots,$$

where $\langle N(k) \rangle \equiv \langle 0, \text{in} | a_{\text{out}}^\dagger(k)a_{\text{out}}(k)|0, \text{in}\rangle = |\beta(k)|^2 \quad (16)$

The last step famously follows by expressing the out-oscillators in terms of the in-oscillators using (7).
Note that (14) is a relation between Heisenberg states, that is, the LHS, evolved to any time \( t \), equals the RHS evolved to the same time \( t \). Thus, if the time-dependence of the Hamiltonian stops at time \( t = t_0 \), then we have\(^{15}\)

\[
T \left( e^{\int_{-\infty}^{t} H(t') dt'} \right) |0, in\rangle = e^{-\kappa_2 H - \sum_{n=2}^{\infty} \kappa_2 W_n} |D\rangle, \quad t \geq t_0
\]  

**Conclusion:** Thus, we find that the ground state, under a quantum quench to zero mass, is exactly represented in the generalized Calabrese-Cardy (gCC) form, as predicted in [1].

We will indeed, find below that the above conclusion holds even when we start from more general states in the initial massive theory.

### 2.2 Thermalization to GGE

As proved for general initial gCC-type initial states (1) in MSS [1], for a perturbative domain in the \( \kappa_n \) parameters, and as we will show below explicitly for a large number of specific cases, the post-quench state, which is of the form (1) shows subsystem thermalization to the GGE (2):

\[
|\psi(\kappa_2, \{\kappa_n\})\rangle_{gCC} \xrightarrow{\text{subsystem thermalization}} \rho\text{GGE}(\beta, \{\mu_n\}), \quad \beta = 4\kappa_2, \mu_n = 4\kappa_n
\]  

Note the alternative form of this equation:

\[
\exp\left[-\sum_k \kappa(k)\hat{N}(k)\right] |D\rangle \xrightarrow{\text{subsystem thermalization}} \frac{1}{Z} \exp\left[-\sum_k \mu(k)\hat{N}(k)\right], \quad \mu(k) = 4\kappa(k)
\]  

Both equations are to be interpreted as the statement that a reduced density matrix on the LHS asymptotically approaches that in the RHS. We will compute explicit correlators below which satisfy the same property.

The energy and \( W \)-charges (as well as the number operator) are conserved in the post-quench CFT dynamics, we have

\[
\langle H\rangle_{gCC} = \langle H\rangle_{GGE}, \quad \langle W_n\rangle_{gCC} = \langle W_n\rangle_{GGE}, \quad \langle N(k)\rangle_{gCC} = \langle N(k)\rangle_{GGE}
\]  

Thus, the charges (16) measured for the post-quench state also refer to those of the GGE. In particular, note that

\[
\langle N(k)\rangle = |\beta(k)|^2 = \frac{|\gamma(k)|^2}{1 - |\gamma(k)|^2} = \frac{1}{e^{4\kappa(k)} - 1}
\]  

This relation can be identified with a similar relation in [17]. To prove the above equation, we have used (9), (10) and (103).

\(^{15}\)In case the time-development continues asymptotically, but as \( e^{-t/t_0} \) as in (24), then (17) is again true for \( t \gg t_0 \), up to terms of magnitude \( e^{-t/t_0} \).
2.3 The propagator

Using the defining property of the in-vacuum $|0, in\rangle$, and the mode expansion of $\phi(x, t)$ in terms of the in-modes, it is easy to derive the following basic two-point function

$$\langle 0, in|\phi(x_1, t_1)\phi(x_2, t_2)|0, in \rangle = \int \frac{dk}{2\pi} u_{in}(k, t_1)u_{in}^*(k, t_2) e^{ik(x_1-x_2)}$$

$$= \int \frac{dk}{2\pi} \left[ |\alpha(k)|^2 u_{out}(k, t_1)u_{out}^*(k, t_2) + \alpha(k)\beta^*(k)u_{out}(k, t_1)u_{out}(-k, t_2) + \alpha^*(k)\beta(k)u_{out}^*(-k, t_1)u_{out}(k, t_2) + |\beta(k)|^2 u_{out}^*(-k, t_1)u_{out}(-k, t_2) \right] e^{ik(x_1-x_2)} \quad (22)$$

In the second step we have used the relation (104) between the ‘in’ and ‘out’ modes. Using (9) and (103), we can find

$$|\alpha(k)|^2 = \frac{1}{1 - |\gamma(k)|^2}, \quad |\beta(k)|^2 = \frac{|\gamma(k)|^2}{1 - |\gamma(k)|^2},$$

$$\alpha(k)\beta^*(k) = \frac{\gamma(k)}{1 - |\gamma(k)|^2}, \quad \alpha^*(k)\beta(k) = \frac{\gamma^*(k)}{1 - |\gamma(k)|^2} \quad (23)$$

The propagator (22) has recently appeared in [32] who used it to study the relation between smooth fast quenches and instantaneous quenches. Related expressions, in a somewhat different form, have appeared in [33].

Using the relation (9) to relate $\gamma(k)$ to the reflection coefficient $r^*(k)$ (see Appendix A), we find that the above propagator can be expressed in terms of the solution of the auxiliary potential scattering problem. In Section 4 we will determine this propagator exactly for a specific quench protocol.

2.4 A specific quench protocol

We will now work out some of the above ideas for the specific mass function

$$m^2(t) = m_0^2(1 - \tanh(\rho t))/2 \quad (24)$$

The Schrödinger problem with a tanh potential can be exactly solved (see, e.g. [29], Chapter 3, where this model appears in a simple model of cosmological particle creation). Using this fact, we can find the following explicit solutions for $u_{in}(k, t)$ and $u_{out}(k, t)$:

$$u_{in}(k, t) = \frac{e^{-i\omega_{in}t}}{\sqrt{2\omega_{in}}} \, _2F_1\left(\frac{i\omega_-}{\rho}, \frac{i\omega_+}{\rho}; 1 - \frac{i\omega_{in}}{\rho}; -e^{2\rho t}\right) \quad (25)$$

$$u_{out}(k, t) = \frac{e^{-i\omega_{out}t}}{\sqrt{2\omega_{out}}} \, _2F_1\left(\frac{i\omega_-}{\rho}, \frac{i\omega_+}{\rho}; i\omega_{out}/\rho + 1; -e^{-2\rho t}\right) \quad (26)$$

where $_2F_1$ is a hypergeometric function and

$$\omega_{in} = \sqrt{k^2 + m_0^2}, \quad \omega_{out} = |k|, \quad \omega_{\pm} = \frac{1}{2}(\omega_{out} \pm \omega_{in})$$
Using (104) (see Appendix A for details) and properties of hypergeometric functions [34] for large arguments, we find the following Bogoliubov coefficients

\[ \alpha(k) = \sqrt{\frac{\omega_{\text{out}}}{\omega_{\text{in}}}} \frac{\Gamma\left(-\frac{i\omega_{\text{out}}}{\rho}\right) \Gamma\left(1 - \frac{i\omega_{\text{in}}}{\rho}\right)}{\Gamma\left(-\frac{i\omega_{\text{out}}}{2\rho}\right) \Gamma\left(1 - \frac{i\omega_{\text{in}}}{2\rho}\right)}, \quad \beta(k) = \sqrt{\frac{\omega_{\text{out}}}{\omega_{\text{in}}}} \frac{\Gamma\left(\frac{i\omega_{\text{out}}}{\rho}\right) \Gamma\left(1 + \frac{i\omega_{\text{in}}}{\rho}\right)}{\Gamma\left(\frac{i\omega_{\text{out}}}{2\rho}\right) \Gamma\left(1 + \frac{i\omega_{\text{in}}}{2\rho}\right)} \]

Using these values, and the general method of Section 2, we find that the ground state is of the gCC form (14),

\[ |0,\text{in}\rangle = \exp[-\kappa_2 H - \sum_{n=2}^{\infty} \kappa_{2n} W_{2n}] |D\rangle \]

where the \( \kappa_n \)'s are found by using (10), as follows:

\[ \kappa_2 = \frac{i \left( \gamma + \psi^{(0)} \left( - \frac{\text{ima}}{\rho} \right) \right)}{\rho}, \quad \kappa_4 = \frac{-im_0 \psi^{(2)} \left( - \frac{\text{ima}}{\rho} \right) + 6\rho \psi^{(1)} \left( - \frac{\text{ima}}{\rho} \right) + 7im_0 \psi^{(2)} (1) + \pi^2 \rho}{24m_0 \rho^3} \]

(27)

where \( \psi^{(n)}(z) \) is the \( n \)-th derivative of the digamma function \( \psi(z) = \Gamma'(z)/\Gamma(z) \). In an expansion in \( 1/m_0, m_0/\rho \) (to be interpreted in the sense of Appendix E), these coefficients read as follows

\[ \kappa_2 = \frac{1}{m_0} \left( 1 + \frac{\pi^2 m_0^2}{12 \rho^2} - \frac{\zeta(3)}{4} m_0^3 + \ldots \right), \quad \kappa_4 = \frac{1}{m_0^3} \left( -\frac{5}{160} + \frac{\pi^2}{288} \frac{m_0^2}{\rho^2} + \ldots \right), \ldots \]

(28)

Note that these are functions of both the scales \( m_0, \rho \) characterizing the quench protocol. The coefficient of odd powers of \( (m_0/\rho) \) in this expansion turns out to be purely imaginary. Note that the \( \kappa_n \)'s (in this case the first two, \( \kappa_2 \) and \( \kappa_4 \)) encode the quench protocol (24) completely; since the \( \kappa_n \)'s are related in a one-to-one fashion to equilibrium chemical potentials \( \mu_n = 4\kappa_n \) (2), it follows that from the equilibrium state one can retrieve the quench history (see Section 6.2 for more details).

For later reference, the “out”-number operator (16) turns out to be

\[ \langle N(k) \rangle = \text{csch} \left( \frac{\pi k}{\rho} \right) \sinh^2 \left( \frac{\pi}{2\rho} \sqrt{k^2 + m_0^2} \right) \text{csch} \left( \frac{\pi \sqrt{k^2 + m_0^2}}{\rho} \right) \]

(29)

Once again, we verify, as in (17), that the time-evolved ground state can be exactly described by a gCC state, of the form (1).\(^\text{16}\)

\(^{16}\)We should note the distinction of this statement with the exact form in (17). Since for the “tanh” protocol, there is no finite time \( t_0 \) beyond which the Hamiltonian is time-dependent, one should use (17) here as an asymptotic statement, for \( t \gg 1/\rho \), with exponentially small corrections \( O(e^{-\rho t}) \).
2.4.1 Sudden limit

We will be especially interested in the sudden limit \( \rho \to \infty \) of the above quench protocol

\[
m^2(t) = m_0^2 \Theta(-t)
\]

(30)

For later use, we note that in the sudden limit \( \rho \to \infty \),

\[
\rho \to \infty,
\]

(31)

the Bogoliubov coefficients become

\[
\alpha(k) = \frac{1}{2} \frac{|k| + \omega_{in}}{\sqrt{|k|\omega_{in}}}, \quad \beta(k) = \frac{1}{2} \frac{|k| - \omega_{in}}{\sqrt{|k|\omega_{in}}}
\]

(32)

whereas the in- and out-waves become

\[
u_{in}(k, t) = e^{-i\omega_{in}t} \frac{1}{\sqrt{2\omega_{in}}} \quad \nu_{out}(k, t) = e^{-i\omega_{out}t} \frac{1}{\sqrt{2\omega_{out}}}
\]

(33)

The \( \kappa_n \) coefficients in this limit are given by taking the \( \rho \to \infty \) limit of (28):

\[
\kappa_2 = \frac{1}{m_0}, \quad \kappa_4 = \frac{1}{m_0^3} \left( -\frac{5}{160} \right), \ldots
\]

(34)

Thus,

\[
|0, in\rangle = \exp\left[ -\frac{H}{m_0} + \frac{5W_4}{160m_0^3} + \ldots \right]|D\rangle
\]

(35)

which is a gCC state.\(^\text{17}\) In the sudden limit, the number operator (29) becomes

\[
\langle N(k) \rangle = \frac{\left( \sqrt{k^2 + m_0^2} - |k| \right)^2}{4\sqrt{k^2 + m_0^2}|k|}
\]

(36)

A more precise and careful version of the sudden limit, than (31) is described in Appendix E.

2.5 Quenching from critical to critical

We will consider a quantum quench for the scalar field where both the initial and final masses vanish (i.e. a quench from a critical Hamiltonian to a critical Hamiltonian).

\(^\text{17}\)One might be alarmed by the positive sign of the \( W_4 \)-coefficient in this state. This would mean that if all the higher \( \kappa_n \) were absent, \( \kappa(k) \) would have grown as \( +k^3 \), hence implying a divergent norm of the gCC state \( e^{-\sum_k \kappa(k)N(k)}|D\rangle \). However, such catastrophies are avoided by higher \( \kappa_n \) coefficients, as they must, since the gCC state is equal, as a Heisenberg state, to the initial ground state, which has a finite norm. We will have more to say in Appendix E on other possible divergences associated with the sudden limit.
A typical mass function which follows this property is [28]:

$$m^2(t) = m_0^2 \text{sech}^2(\rho t).$$  \hspace{1cm} (37)

Using the coordinate transformation $y = e^{2\rho t}$. The equation of motion, analogous to (5), becomes

$$\phi''(k, y) + \frac{\phi'(k, y)}{y} + \left( \frac{k^2}{4\rho^2 y} + \frac{m_0^2}{\rho^2 (1 + y)} \right) \phi(k, y) = 0$$  \hspace{1cm} (38)

With $\alpha = 1/2 + \frac{1}{\rho} \sqrt{4m_0^2 + \rho^2}$, this equation can be solved to give

$$u(k, t) = e^{-ikt} (1 + e^{2\rho t})^\alpha \left[ C_1 e^{2ikt} _2F_1 \left( \alpha, \frac{ik}{\rho} + \alpha, 1 + \frac{ik}{\rho}, -e^{2\rho t} \right) + C_2 _2F_1 \left( \alpha, -\frac{ik}{\rho} + \alpha, 1 - \frac{ik}{\rho}, -e^{2\rho t} \right) \right]$$  \hspace{1cm} (39)

$C_1 = 1$ and $C_2 = 0$ gives the incoming solution $u_{in}(k)$ which satisfies the property (100). On taking the $t \to +\infty$ limit of $u_{in}(k)$ we can express $u_{in}(k)$ in the form $\alpha(k) u_{out}(k) + \beta(k) u_{out}^*(k)$ (see Appendix A for more details), where

$$\alpha(k) = \frac{\Gamma \left( \frac{ik}{\rho} + 1 \right) \Gamma \left( \frac{ik}{\rho} \right)}{\Gamma \left( \frac{ik}{\rho} - \alpha + 1 \right) \Gamma \left( \frac{ik}{\rho} + \alpha \right)}$$  \hspace{1cm} (40)

$$\beta(k) = i \sin(\pi \alpha) \text{cosech} \left( \frac{\pi k}{\rho} \right)$$  \hspace{1cm} (41)

Using (9) and (10), we can express the in-vacuum in a gCC form (1) with

$$\kappa(k) = \frac{ik \rho}{2m_0^2} - \frac{k^2 \rho^2}{4m_0^6} - \frac{ik^3 \rho^3}{6m_0^8} + \frac{k^4 \rho^4}{8m_0^{10}} + \frac{ik^5 \rho^5}{10m_0^{12}} + \cdots,$$  \hspace{1cm} (42)

which leads to

$$\kappa_2 = \frac{i \rho}{2m_0^2}, \kappa_3 = -\frac{\rho^2}{4m_0^4}.$$

Note that $\kappa_2$ is imaginary. By contrast, $\kappa_2$ in a massive quench, is real and positive (see e.g. (34)), and is identified with $\beta/4$ where $\beta$ is the inverse temperature of the associated thermal state. With imaginary $\kappa_2$, such an identification is clearly problematic. We will find in the next section that starting with an appropriate squeezed state, one can manufacture a CC state with positive $\kappa_2$. 

Figure 2: A mass-profile describe quantum quench from a critical Hamiltonian back to the critical Hamiltonian. Here $m^2(t) \xrightarrow{t \to \pm \infty} 0$. 
2.6 Quenching squeezed states

Suppose, instead of the ground state we start with a squeezed state\(^{18}\) of the pre-quench Hamiltonian:

\[
|\psi, in\rangle = |f\rangle \equiv \exp\left[\frac{1}{2} \sum_k f(k)a^\dagger_{in}(k)a^\dagger_{in}(-k)\right]|0, in\rangle \tag{43}
\]

This is clearly a Bogoliubov transformation of $|0, in\rangle$. To see this, note that $|f\rangle$ is annihilated by $a_{in}(k) - f(k)a_{in}(-k)$,

\[
0 = \left[a_{in}(k) - f(k)a^\dagger_{in}(-k)\right]|f(k), in\rangle = \left[\alpha^*(k)a_{out}(k) - \beta^*(k)a^\dagger_{out}(-k) - f(k)\left\{\alpha(k)a^\dagger_{out}(-k) - \beta(k)a_{out}(k)\right\}\right]|f(k), out\rangle = \left[\{\alpha^*(k) + f(k)\beta(k)\}a_{out}(k) - \{\beta^*(k) + f(k)\alpha(k)\}a^\dagger_{out}(-k)\right]|f(k), out\rangle \tag{44}
\]

Thus, it follows that the squeezed pre-quench state is also expressible as a generalized CC state

\[
|\psi, in\rangle = |f\rangle = \exp\left[\frac{1}{2} \sum_k \gamma_{eff}(k)a^\dagger_{out}(k)a^\dagger_{out}(-k)\right]|0, out\rangle \tag{45}
\]

where the effective $\gamma_{eff}(k)$ is

\[
\gamma_{eff}(k) = \frac{\beta^*(k) + f(k)\alpha(k)}{\alpha^*(k) + f(k)\beta(k)} \tag{46}
\]

Using the result (45) and the method leading to (10), we can again show

\[
|f\rangle = \exp\left[\sum_k -\kappa_{eff}(k)a^\dagger_{out}(k)a_{out}(-k)\right]|D\rangle,
\]

\[
\kappa_{eff}(k) \equiv -\frac{1}{2} \log \left(-\gamma_{eff}(k)\right) \tag{47}
\]

where $\kappa_{eff}(k)$ has an expansion of the form (13) as argued below.

**General arguments from scattering theory:** Using elements of scattering theory described in Appendix A, we can rewrite (46) as follows

\[
\gamma_{eff}(k) = \left(\frac{f(k) - r'(k)}{1 - r''(k)f(k)}\right)\left(\frac{\alpha^*(k)}{\alpha(k)}\right) \tag{48}
\]

---

\(^{18}\)These states have importance in diverse contexts [25, 35] including quantum entanglement [26]. Time-development of these states can address the issue of dynamical evolution of quantum entanglement, among other things.

\(^{19}\)We assume that the norm of the squeezed state is finite, which is ensured by the finiteness of the integral $\int dk/(2\pi) \log(1 - |f(k)|^2)$. 

---
Here we have used $\beta^*(k) = -r'(k)\alpha(k)$, where $r'(k)$ is the dual reflection coefficient (98), which has a small momentum expansion (106) $r'(k) = 1 + O(k)$. Assuming $f(k)$ to be regular at $k = 0$ so that it admits an expansion $f(k) = f(0) + O(k)$, we find that the first factor in the RHS has an expansion $-1 + O(k)$. Using (107), the RHS has an expansion $-1 + O(k)$, which ensures an expansion of $\kappa_{\text{eff}}$ in (47) of the form (13).\footnote{This does not ensure $\text{Re}(\kappa_2) > 0$ by itself. We have to tailor the choice of $f(k)$’s to ensure it, as done in the examples below.} We will list a number of examples below to find such an expansion of $\kappa(k)$.

**Explicit Examples:** In the first two examples, we fix the quench protocol to be given by the ‘tanh’ function (24), in the sudden limit $\rho \to \infty$. We will determine the $\kappa_{\text{eff}}$ explicitly by using (47) and the expressions for the Bogoliubov coefficients (32). In the

- **Gaussian:** For a Gaussian squeezing function with variance proportional to $m_0^2$, i.e. $f = \exp[-k^2/(a^2m_0^2)]$, we get

$$\kappa_{\text{eff}}(k) = \frac{|k|}{a^2m_0} + \frac{2|k|}{\sqrt{k^2 + m_0^2}} \frac{(6a^4 + 1)|k|^3}{12a^6m_0^3} - \frac{(30a^8 - 10a^4 - 3)|k|^5}{240a^{10}m_0^5} + O(|k|^7) \quad (49)$$

- **Preparing CC states and gCC states with specified parameters:** It is clear from (47) that given specific Bogoliubov coefficients, e.g. (32), we can obtain any desired expression for $\kappa_{\text{eff}}(k)$ by tailoring the choice of the squeezing function $f(k)$. Thus, e.g.

$$f(k) = 1 - \frac{2|k|}{\sqrt{k^2 + m_0^2} \tanh(\kappa_{2,0}|k| + \kappa_{4,0}|k|^3) + |k|} \quad (50)$$

yields a function $\kappa_{\text{eff}}(k) = \kappa_{2,0}|k| + \kappa_{4,0}|k|^3$ with specified parameters $\kappa_2 = \kappa_{2,0}$, $\kappa_4 = \kappa_{4,0}$. This identifies the squeezed state with a gCC state with these $\kappa$-parameters.\footnote{Note that we choose here $\kappa_{2,0}$, $\kappa_{4,0}$ to be positive to ensure that the gCC state is of finite norm; see footnote 19.}

$$|\psi, in\rangle = |f\rangle = \exp[-(\kappa_{2,0}H + \kappa_{4,0}W_4)|D\rangle \quad (51)$$

Specializing even more, we can manufacture $\kappa_{2,0} = 1/m_0, \kappa_4 = 0$, i.e. $\kappa_{\text{eff}} = \kappa_{2,0}k$, (cf. (34)) by choosing

$$f(k) = 1 - \frac{2|k|}{\sqrt{k^2 + m_0^2} \tanh(k|m_0|) + |k|} \quad (52)$$

which yields a CC state of the form

$$|\psi, in\rangle = |f\rangle = \exp \left[ -\frac{1}{m_0} \sum_k |k|a_k^\dagger a_k \right] |D\rangle \quad (53)$$

We note that these squeezing functions are localised functions which vanish at both $k \to 0$ and $k \to \infty$ limits and hence the resultant squeezed state is normalisable. Note that the functions $f(k)$ are even functions, and hence are actually functions of $|k|$.
Critical to critical: Applying the above method to the quench protocol discussed in Section 2.5, we find that the following choice of the squeezing function

\[ f(k) = \frac{a(-e^{2|k|}(\kappa_{2,0} + \kappa_{4,0}k^2)) + a - i|k|}{-a + (a + i|k|)e^{2|k|}(\kappa_{2,0} + \kappa_{4,0}k^2)} \]

leads to a gCC state \( e^{-\kappa_{2,0}H - \kappa_{4,0}W} |D\rangle \). Specializing to

\[ f(k) = \frac{a(-e^{2\kappa_{2,0}|k|}) + a - i|k|}{-a + (a + i|k|)e^{2\kappa_{2,0}|k|}} \]

leads to a CC state \( e^{-\kappa_{2,0}H} |D\rangle \).

2.6.1 The propagator in a squeezed state

The propagator in a squeezed state \( |\psi, in\rangle = |f\rangle \) is obtained by replacing \( \alpha \rightarrow \alpha_{eff}, \beta \rightarrow \beta_{eff} \) in (22):

\[
\langle \psi, in | \phi(x_1, t_1) \phi(x_2, t_2) | \psi, in \rangle = \int \frac{dk}{2\pi} \left[ |\alpha_{eff}(k)|^2 u_{out}(k, t_1) u_{out}^*(k, t_2) + \alpha_{eff}(k) \beta_{eff}(k) u_{out}(k, t_1) u_{out}(-k, t_2) 
+ \alpha_{eff}^*(k) \beta_{eff}(k) u_{out}^*(-k, t_1) u_{out}(k, t_2) + |\beta_{eff}(k)|^2 u_{out}^*(-k, t_1) u_{out}(-k, t_2) \right] e^{ik(x_1 - x_2)} \tag{54}
\]

3 Fermion theories with time-dependent mass

We will now consider fermion field theories with a time-dependent mass:

\[
S = - \int d^2x (i \bar{\Psi} \gamma^\mu \partial_\mu \Psi - m(t) \bar{\Psi} \Psi)
\]

Once again, a general analysis of an auxiliary Schrödinger problem can be performed [36], to infer the emergence of the general Calabrese-Cardy (gCC) state. However, we present below the analysis for a mass quench specific quench protocol, involving a \( \tanh \) function, which describes quantum quench from a non-critical to a critical Hamiltonian.

We start with the Dirac equation with the following time-dependent mass:[36, 28]

\[
m(t) = \frac{m_0}{2} (1 - \tanh(\rho t))
\]

The Dirac equation is

\[
(i \gamma^\mu \partial_\mu - m(t)) \Psi = 0 \tag{55}
\]

The ansatz for a solution of this equation is

\[
\Psi(k; x, t) = (\gamma^0 \partial_t - \gamma^1 \partial_x - im(t)) e^{\pm ikx} \Phi(k, t) \tag{56}
\]

where \( \Phi(k, t) \) is a two-component spinor that satisfies the following equation

\[
(\partial_t^2 + k^2 + m^2(t) - i\gamma^0 \dot{m}(t)) \Phi(k, t) = 0 \tag{57}
\]
Defining $\Phi = (\phi_+, \phi_-)^T$, the equations decouple in the eigenbasis of $\gamma^0$ in Dirac basis,

$$(\partial_t^2 + k^2 + m^2(t) \mp i m(t)) \phi_\pm(k, t) = 0$$

(58)

where $\phi_+(t)$ is the solution corresponding to $\gamma^0$ eigenvalue 1 and its part with asymptotic positive energy eigenvalues appears with the spinor $u(0)$ in the mode expansion of $\Psi(x, t)$. Similarly, $\phi_-(t)$ is the solution corresponding to $\gamma^0$ eigenvalue $-1$ and its part with asymptotic negative energy eigenvalues appears with the spinor $v(0)$ in the mode expansion of $\Psi(x, t)$. The conventions and the explicit solutions are described in Appendix D. The explicit solutions lead to the following expressions of Bogoliubov coefficients $\alpha_\pm(k)$ and $\beta_\pm(k)$

$$\alpha_\pm(k) = \frac{\Gamma \left( -\frac{|k|}{\rho} \right) \Gamma \left( 1 - \frac{i \omega_{in}}{\rho} \right)}{\Gamma \left( 1 - i \frac{|k| + m_0 + \omega_{in}}{2\rho} \right) \Gamma \left( 1 - i \frac{|k| - m_0 + \omega_{in}}{2\rho} \right)}$$

(59)

$$\beta_\pm(k) = \frac{\Gamma \left( \frac{|k|}{\rho} \right) \Gamma \left( 1 - \frac{i \omega_{in}}{\rho} \right)}{\Gamma \left( 1 - i \frac{|k| - m_0 + \omega_{in}}{2\rho} \right) \Gamma \left( 1 - i \frac{|k| + m_0 + \omega_{in}}{2\rho} \right)}$$

(60)

In terms of the ‘out’ oscillators, the ‘in’ ground state is

$$|\psi\rangle = \exp \left[ \sum_{k=-\infty}^{\infty} \gamma(k) a_{k,\text{out}}^\dagger b_{-k,\text{out}}^\dagger \right] |0, \text{in}\rangle$$

where $\gamma(k) = \chi(k) \frac{\beta_\pm(k)^T}{\alpha_\pm(k)^T}$ (123). Using a similar BCH formula to (10) for fermionic creation and annihilation operators, we get

$$|\Psi\rangle = e^{-\kappa_2 H + \kappa_4 W_4 - \kappa_6 W_6 - \cdots} |D\rangle$$

(61)

where

$$\kappa_2 = \frac{1}{2m} + \frac{\pi^2 m}{12 \rho^2} + \frac{1}{m} O(m/\rho)^3, \quad \kappa_4 = \frac{1}{12 m^3} - \frac{\pi^2}{24 m \rho^2} + \frac{1}{m^3} O(m/\rho)^3, \quad \kappa_6 = \frac{3}{80 m^5} - \frac{\pi^2}{96 m^3 \rho^2} + \frac{1}{m^5} O(m/\rho)^3$$

and $|D\rangle$ is the Dirichlet state of the fermionic theory. Using the chiral mode expansion (119) and (120),

$$|D\rangle = e^{\sum_k \text{sign}(k) a_k^\dagger b_{-k}^\dagger} |0\rangle$$

(62)

In writing the $W_\infty$ charges for the fermions, we have used the currents mentioned in the Appendix D.\footnote{We choose the overall normalization of the $W_{2n}(z)$-currents so that the $W_{2n}$ charges are given by $W_{2n} = \sum_k |k|^{2n-1} [a^\dagger(k) a(k) + b^\dagger(k) b(k)]$.}
4 Exact time-dependent correlators

4.1 Ground state

In this section, we will consider the specific quench protocol discussed in Section 2.4.1. Using the general computation (22) of the propagator and the specific values (32) and (33), we find

\[ G_{q,0}(x_1, t_1; x_2, t_2) \equiv \langle 0, in | \phi(x_1, t_1) \phi(x_2, t_2) | 0, in \rangle = \]
\[ \int \frac{dk}{2\pi} G_{q,0}(k) \left[ \left( 2|k| \left( |k| + \sqrt{k^2 + m_0^2} \right) + m_0^2 \right) \left( \Theta(k)e^{-ik(x_2^0 - x_1^0)} + \Theta(-k)e^{ik(x_2^0 - x_1^0)} \right) + \left( 2|k| \left( |k| - \sqrt{k^2 + m_0^2} \right) + m_0^2 \right) \left( \Theta(k)e^{-ik(x_2^+ - x_1^+)} + \Theta(-k)e^{ik(x_2^+ - x_1^+)} \right) \right] \]
\[ -m_0^2 \left( \Theta(k)e^{-ik(x_2^- - x_1^-)} + \Theta(-k)e^{ik(x_2^- - x_1^-)} \right) - m_0^2 \left( \Theta(k)e^{-ik(x_2^+ - x_1^+)} + \Theta(-k)e^{ik(x_2^+ - x_1^+)} \right) \]  \hspace{1cm} (63)

where we have defined \( x_i^\pm = x_i \pm t_i, i = 1, 2 \). Note that the last two lines involve the combinations \( t_1 + t_2 \), which reflect the fact that time-translation invariance is lost due to the time-dependent perturbation. In the above expression

\[ G_{q,0}(k) = \frac{1}{8|k|^2 \sqrt{k^2 + m_0^2}} \]  \hspace{1cm} (64)

is the significant part of the above propagator. Songularities of this quantity in the \( k \)-plane are explained Figure 3: these are a double pole at \( k = 0 \) and two branch points on the imaginary axis, at \( k = \pm im_0 \).

After performing the Fourier transforms, the propagator is given by:

\[ G_{q,0}(x_1, t_1; x_2, t_2) = \frac{1}{8\pi} \left( -G_{1,3}^{2,1} \left( \frac{m_0^2}{4} (x_2^0 - x_1^0) \right)^2 | 0, 1, \frac{1}{2} \right) + G_{1,3}^{2,1} \left( \frac{m_0^2}{4} (x_2^0 - x_1^0) \right)^2 | 0, 1, \frac{1}{2} \right) + G_{1,3}^{2,1} \left( \frac{m_0^2}{4} (x_2^+ - x_1^0) \right)^2 | 0, 1, \frac{1}{2} \right) - G_{1,3}^{2,1} \left( \frac{m_0^2}{4} (x_2^+ - x_1^0) \right)^2 | 0, 1, \frac{1}{2} \right) + 4K_0 (m_0 | x_2^0 - x_1^0 |) + 4K_0 (m_0 | x_2^+ - x_1^0 |) + 2i\pi \text{ sgn} (x_2^0 - x_1^-) - 2i\pi \text{ sgn} (x_2^+ - x_1^+)] \] \hspace{1cm} (65)

For \( x_2 - x_1 = r \) and \( t_1 = t_2 = t \), in the asymptotic limit this becomes

\[ G_{q,0}(0; r, t) = \frac{1}{8} (m_0 (2t - r)) + \frac{1}{8\sqrt{2\pi m_0}} \left( e^{-m_0(2t-r)} + e^{-m_0 r + 2t} + 2e^{-m_0 r} \right) + ... \hspace{1cm} r < 2t \]
\[ = \frac{1}{8\sqrt{2\pi m_0}} \left( e^{-m_0(r-2t)} + e^{-m_0 r + 2t} + 2e^{-m_0 r} \right) + ... \hspace{1cm} r > 2t \]

Note that the quantities defined in Section 2.4.1 are obtained by a naive definition of the sudden limit (31). As explained in Appendix E, although for \( W_4 \) and higher charges, this definition has be refined as in (124), for correlator calculations we can continue to use the naive definition.
The linear terms are dictated by the double pole at the origin of the \( k \)-plane. These agree with the expressions obtained by \[33\] in the so-called deep quench limit (see Section 6 for more details). The ellipsis represent higher transients.

**Correlators:**

- Two-point functions of vertex operators \( O_q = e^{iq\phi} \): The dominant behaviour in the IR limit is given by exponentiaing the linear part in the above \( \langle \phi\phi \rangle \) propagator (after subtracting the coincident part). We get

\[
\langle 0,\text{in}|e^{iq\phi(0,t)}e^{-iq\phi(r,t)}|0,\text{in}\rangle = e^{-\frac{q^2}{4m_0}r}, \quad t > r/2
\]

This result agrees with that in \[33\]. The dominant exponential is, again, given by the double pole at the origin of the \( k \)-plane. As remarked in Figure 3, the thermal correlator is also dominated by this double pole at the origin. It is no surprise therefore that the above result (66) exactly agrees with the thermal result (87), with the identification \( \beta = 4\kappa_2 = 4/m_0 \).

- Two-point functions of the holomorphic operator: \( O = \partial\phi \):

\[
\langle 0,\text{in}||\partial\phi(x_1,t_1)\partial\phi(x_2,t_2)||0,\text{in}\rangle = \int \frac{dk}{2\pi} \frac{e^{ikr}}{\sqrt{k^2 + m_0^2}} \left[ \Theta(-k)(2|k|(k^2 + m_0^2)^{1/2} + 2k^2 + m_0^2) + \Theta(k)(-2|k|(k^2 + m_0^2)^{1/2} + 2k^2 + m_0^2) \right]
\]

where we have chosen \( r = x_1 - x_2, t_1 = t_2 \) (note that there is no time-dependence for equal times in this case, as we expect for holomorphic operators since these do not ‘see’ the boundary that represents the quench).

**Note that the derivatives annihilate the double pole at the origin of the k-plane, hence the two-point function is dictated solely by the distant singularity. Consequently, the rate of fall-off is NOT universal (see Section 6 for further details).**

- Two-point functions \( \langle \partial\phi \bar{\partial}\phi \rangle \):

\[
\langle 0,\text{in}||\partial\phi(x_1,t)\bar{\partial}\phi(x_2,t)||0,\text{in}\rangle = -\int \frac{dk}{2\pi} \frac{m_0^2 e^{ik(r+2t)}}{8(k^2 + m_0^2)^{1/2}} = -\frac{m_0^2}{8\pi} K_0(m_0(r + 2t))
\]

24 We define \( \partial = \frac{1}{2}(\partial_x + \partial_t), \bar{\partial} = \frac{1}{2}(\partial_x - \partial_t) \).
One-point function \( \langle \partial \phi \bar{\partial} \phi \rangle \):

\[
\langle 0, \text{in} | \partial \phi \bar{\partial} \phi(x, t) | 0, \text{in} \rangle = -\int \frac{dk}{2\pi} \frac{m_0^2 e^{2kt}}{8(k^2 + m_0^2)^{1/2}} = -\frac{m_0^2}{8\pi} K_0(2m_0t)
\]

\[
t \to \infty \quad -e^{-2m_0t} \left[ \frac{m_0^3/2 \sqrt{T}}{16\sqrt{\pi}} - \frac{\sqrt{m_0} (\frac{1}{T})^{3/2}}{256\sqrt{\pi}} + O \left( \frac{1}{T} \right)^{5/2} \right]
\]

We also present a calculation of the energy density. In the \( t \to \infty \) limit,

\[
\frac{E}{L} = \frac{m_0^2}{8\pi} \quad (70)
\]

Note that it does not agree with (88) with \( \beta = 4/m_0 \). In other words, the higher chemical potentials affect the asymptotic energy density.

4.2 Correlators in Squeezed States

The expression for the \( \langle f | \phi | f \rangle \) propagator in a squeezed is given in (54). In this section we will compute these in the squeezed states (50) which are tailored to produce a given real value of \( \kappa_2 > 0 \) and \( \kappa_4 \) (with all other \( \kappa_n = 0 \)). We find

\[
\langle \phi \rangle = \int \frac{dk}{4\pi} \left[ \coth \left( \frac{2|k|}{\kappa_2 + \kappa_4 k^2} \right) - \cos(2|k|t) \coth \left( \frac{2|k|}{\kappa_2 + \kappa_4 k^2} \right) \right]
\]

\[
\langle \partial \phi \bar{\partial} \phi \rangle = \int \frac{dk}{8\pi} e^{ikr} k \left( \frac{1}{e^{4|k|k^2} - 1} - \frac{1}{2} \cos(2|k|t) \coth \left( \frac{2|k|}{\kappa_2 + \kappa_4 k^2} \right) \right) + \frac{1}{2}
\]

\[
\langle \partial \phi \bar{\partial} \phi \rangle = \int \frac{dk}{8\pi} e^{-2ikt} k \cosh \left( 2\kappa_2 k + 2k^3\kappa_4 \right)
\]

The first two equations describe two-point functions with \((x_1, t_1) = (0, t), (x_2, t_2) = (r, t)\), whereas the third equation is a one-point function at a point \((x, t)\) (which is independent of \(x\) by translational invariance). In the propagator, the last term in the second line gives the usual \(-\log(r)\) term of free scalar in 2D spacetime.

With \( \kappa_4 = 0 \), i.e., for the CC state \( e^{-\kappa_2 H} | D \rangle \), the integrals can be done exactly and energy density can also be calculated exactly,

\[
\langle \phi \rangle = \log \left( \frac{1}{2} \coth \left( \frac{\pi r}{4\kappa_2} \right) \right) \left( \cosh \left( \frac{\pi r}{2\kappa_2} \right) + \cosh \left( \frac{\pi t}{\kappa_2} \right) \right) \quad (72)
\]

\[
\langle \partial \phi \bar{\partial} \phi \rangle_{CC} = -\frac{\pi \coth^2 \left( \frac{\pi r}{4\kappa_2} \right)}{64\kappa_2^2} \quad (73)
\]

\[
\langle \partial \phi \bar{\partial} \phi \rangle_{CC} = -\frac{\pi}{64\kappa_2^2} \coth^2 \left( \frac{2\pi t}{4\kappa_2} \right) \quad (74)
\]
These results have also been obtained using BCFT in [11]. The energy density is

\[ \frac{E}{L} = \frac{\pi}{96\kappa_2^2} \]  

(75)

This agrees with the thermal energy density in (88) with \( \beta = 4\kappa_2 \).

With non-zero \( \kappa_4 \), let us first consider \( \langle \partial \phi \partial \phi \rangle \). The associated Fourier transform can be computed by contour integration. Note that the cosech function has simple poles in the \( k \)-plane at \( 2\kappa_4 k^3 + 2\kappa_2 k = i\pi n \). Thus, there are three simple poles for each \( n \) (see Figure 3), given by

\[
k_1 = \frac{-2 \sqrt[3]{6} \kappa_2 + \sqrt[3]{6} \left( \sqrt[3]{48\kappa_3^3 - 81\pi^2\kappa_4 n^2} + 9i\pi \sqrt{\kappa_4 n} \right)^{2/3}}{6 \sqrt[3]{3} \sqrt[3]{\kappa_4^3 (16\kappa_3^2 - 27\pi^2\kappa_4 n^2)} + 9i\pi \kappa_4^2 n} \\
k_2 = \frac{4\sqrt[3]{-6\kappa_2} + i \left( \sqrt[3]{3} + i \right) \left( \sqrt[3]{48\kappa_3^3 - 81\pi^2\kappa_4 n^2} + 9i\pi \sqrt{\kappa_4 n} \right)^{2/3}}{2 \sqrt[3]{6} \sqrt[3]{\kappa_4^3 (16\kappa_3^2 - 27\pi^2\kappa_4 n^2)} + 9i\pi \kappa_4^2 n} \\
k_3 = \frac{-\sqrt[3]{-1} \left( 2\sqrt[3]{-6\kappa_2} + \left( \sqrt[3]{48\kappa_3^3 - 81\pi^2\kappa_4 n^2} + 9i\pi \sqrt{\kappa_4 n} \right)^{2/3} \right)}{6 \sqrt[3]{3} \sqrt[3]{\kappa_4^3 (16\kappa_3^2 - 27\pi^2\kappa_4 n^2)} + 9i\pi \kappa_4^2 n} \]  

(76)

In an expansion in small \( \kappa_4 \), we get

\[
k_1 = \frac{i\pi n}{2\kappa_2} + \frac{i\pi^3 \kappa_4 n^3}{8\kappa_2^3} + \frac{3i\pi^5 \kappa_4^2 n^5}{32\kappa_2^5} + \frac{3i\pi^7 \kappa_4^3 n^7}{32\kappa_2^7} \]  

(77)

\[
k_2 = \frac{i\sqrt{\kappa_2}}{\sqrt{\kappa_4}} - \frac{i\pi n}{4\kappa_2} - \frac{3i\pi^2 \sqrt{\kappa_4 n^2}}{32\kappa_2^3} - \frac{i\pi^3 \kappa_4 n^3}{16\kappa_2^3} \]  

(78)

\[
k_3 = -\frac{i\sqrt{\kappa_2}}{\sqrt{\kappa_4}} + \frac{i\pi n}{4\kappa_2} + \frac{3i\pi^2 \sqrt{\kappa_4 n^2}}{32\kappa_2^3} - \frac{i\pi^3 \kappa_4 n^3}{16\kappa_2^3} \]  

(79)

Out of these poles, it is clear that in the perturbative regime \( (\kappa_4 \ll \kappa_3^2) \), only \( k_1 \) will contribute. This is because a pole at \( k = -ik_0 \) will turn up in \( e^{-2k_0 t} \) and so large values of \( k_2 \) and \( k_3 \) will contribute highly damped solutions (note that poles in the upper half plane do not contribute for \( t > 0 \)). Thus, \( k_1 \) is the pole whose residue we are interested in for comparison with perturbative results. In practice, to get non-perturbative results, we would have to take into account the residues at the other two poles as well. Note that the pole at the origin (for \( n = 0 \)) is cancelled by the \( k \) multiplying the cosech.

From the expansion of cosech \( (2\kappa_4 (k - k_1) (k - k_2) (k - k_3) + i\pi n) \) around \( k_1 \), we find the residue of cosech to be

\[
\frac{(-1)^n}{2\kappa_4 (k_1 - k_2) (k_1 - k_3)} 
\]  

(80)
Taking the leading order of the cosech residue which is given by the $n = \pm 1$ poles, we find the total residue

$$
= -\frac{\pi}{16\kappa^2} \left(1 + 4\pi^2\tilde{\kappa}_4 + 48\pi^4\tilde{\kappa}_4^2\right) \exp \left(-\frac{4\left(\pi + 4\pi^3\tilde{\kappa}_4 + 48\pi^5\tilde{\kappa}_4^2\right)t}{4\kappa^2}\right)
$$

(81)

where $\tilde{\kappa}_4 = \frac{\kappa_4}{\pi^2\kappa^2}$.

**Comparison with MSS:** Using the charge under the $\mu_4$ current $q_4 = 3$, $\beta = 4\kappa_2$ and $\tilde{\kappa}_4 = \tilde{\mu}_4$, we match the results of MSS exactly. Note that above, $\tilde{\mu}_4^2t$ also exponentiates, so this gives the behaviour expected by MSS and higher orders.

The computation of $\langle \partial \phi \partial \phi \rangle$ follows along similar lines. Here, the poles are the same. The only difference is the residue of coth at $k_1$ which is

$$
\frac{1}{2\kappa_4 (k_1 - k_2)(k_3 - k_4)}
$$

(82)

Thus the total residue is similar to the earlier case.

$$
= \frac{\pi}{16\kappa^2} \left(1 + 4\pi^2\tilde{\kappa}_4 + 48\pi^4\tilde{\kappa}_4^2\right) \exp \left(-\frac{2\left(\pi + 4\pi^3\tilde{\kappa}_4 + 48\pi^5\tilde{\kappa}_4^2\right)t}{4\kappa^2}\right)
$$

(83)

which shows twice the relaxation rate as before (as expected from MSS).

## 5 Real time propagator in a GGE

We first review the purely thermal case briefly.

**Real time propagator in a thermal ensemble** Consider the real time, thermal Wightman propagator (see, e.g. [37] for the various definitions of propagators)

$$
G_+(x_1, t_1; x_2, t_2; \beta) \equiv \frac{1}{Z} \text{Tr} \left( e^{-\beta H} \phi(x_2, t_2) \phi(x_1, t_1) \right)
$$

$$
= \frac{1}{Z} \sum_{\{N_n\}} \langle \{N_n\} | \phi(x_1) e^{-itH} \phi(x_2) e^{-itH} | \{N_n\} \rangle
$$

(84)

By using the occupation number representation of the Hamiltonian, it is easy to derive the following result ($x = x_2 - x_1$, $t = t_2 - t_1$):

$$
G_+(x_1, t_1; x_2, t_2; \beta) = \frac{1}{2} \int \frac{dk}{2\pi} \left[G_+(k; \beta)e^{ikx-ik|t|} + G_-(k; \beta)e^{-ikx+ik|t|}\right]
$$

$$
G_\pm(k; \beta) = \frac{1}{|k|(\pm e^{\beta|k|} \mp 1)}
$$

(85)

The two-point function of $\partial \phi$ is, therefore,

$$
\frac{1}{Z} \text{Tr} \left( e^{-\beta H} \partial \phi(x_2, t_2) \partial \phi(x_1, t_1) \right) = \frac{1}{2} \int \frac{dk}{2\pi} \frac{k e^{-ikx-t}}{e^{\beta|k|} - 1} = -\frac{\pi}{4\beta^2} \sinh^2\left(\frac{\pi(x + t)}{\beta}\right)
$$

(86)
which is the well-known result obtained from CFT techniques [11].

It is also easy to compute from the above the thermal two-point function of exponential vertex operators

\[ \langle \exp[iq\phi(0,t)] \exp[-iq\phi(r,t)] \rangle_\beta = \exp[-q^2 r/2\beta] \] (87)

Note that this result agrees with the expected result [11] from conformal field theory \( \exp[-2\pi \Delta r/\beta] \), using \( \Delta = q^2/4\pi \) (see Appendix C).

The energy density in a thermal ensemble is

\[ E_L = \frac{\pi}{6\beta^2} \] (88)

We will now define the Wightman function in a GGE in an analogous fashion (for simplicity we consider only one chemical potential \( \mu_4 \) here; the generalization to arbitrary number of chemical potentials is obvious):

\[ G_+(x_1, t_1; x_2, t_2; \beta, \mu_4) = \frac{1}{Z} \text{Tr} \left( e^{-\beta H - \mu_4 W_4} \phi(x_2, t_2) \phi(x_1, t_1) \right) \]
\[ = \frac{1}{Z} \sum_{\{N_n\}} \langle \{N_n\} | \phi(x_1) e^{-itH} \phi(x_2) e^{-\beta H - \mu_4 W_4} | \{N_n\} \rangle \] (89)

By a simple evaluation, this turns out to be

\[ G_+(x_1, t_1; x_2, t_2; \beta, \mu_4) = \frac{1}{2} \int \frac{dk}{2\pi} \left[ G_+(k; \beta, \mu_4) e^{i k x - i|k|t} + G_-(k; \beta, \mu_4) e^{-i k x + i|k|t} \right], \]
\[ G_\pm(k; \beta, \mu_4) = \frac{1}{|k| \left( \pm e^{\pm(\beta|k| + \mu_4|k|^3)/2} + 1 \right)} \] (90)

The holomorphic two-point function is now given by

\[ \frac{1}{Z} \text{Tr} \left( e^{-\beta H} \partial \phi(x_2, t_2) \partial \phi(x_1, t_1) \right) = \frac{1}{2} \int_0^\infty \frac{dk}{2\pi} \frac{k e^{-ik(x+t)}}{e^{\beta|k| + \mu_4|k|^3} - 1} \]
\[ = \frac{1}{4} \int_0^\infty \frac{dk}{2\pi} \frac{k e^{-ik(x+t)}}{\text{coth}(\beta|k|/2 + \mu_4|k|^3/2) - 1} \] (91)

which exactly matches (71) provided we define, in keeping with (2)

\[ \beta = 4\kappa_2, \ \mu_4 = 4\kappa_4 \] (92)

The explicit evaluation of this Fourier transform is carried out below (71).

6 Thermalization

In the previous two sections, we found that the exact correlators show thermalization at late times. Here’s a brief summary for some specific correlators
Table 1: The 2nd and 3rd columns give equal time correlators at late times for a mass quench (30); the 4th column gives the same correlator (time-independent) in a thermal state with $\beta = 4/m_0$. In the 2nd-column the initial state is the ground state $|0, in\rangle$; in the 3rd column, the initial state is a special squeezed state (53) which is of the Calabrese-Cardy form $e^{-H/m_0}|D\rangle$. In the first two rows, we list two-point functions at separated points. In the 3rd row we list the asymptotic energy density. In the 4th row, we list the late time behaviour of a one-point function; the vanishing asymptotic value agrees with the thermal state—but we compare here the exponential decay in time between the second and third columns. Note that the asymptotic values always agree between the CC state and the thermal state, but barring the case of the exponential vertex operator, the late time behaviour differs from the CC state, signifying nontrivial modification of the behaviour by the higher chemical potentials.

6.1 UV/IR mixing

In this section we will discuss the issue of large distance/time universality (or the lack thereof) in a critical quench. A useful guide in this turns out to be the pole structure of the propagator $\langle \phi(k)\phi(-k)\rangle$, which is explained in Figure 3.
Figure 3: Singularities governing the two-point function in the complex $k$-plane: (a) of the quantity $G_{q,0}(k)$ for the ground state quench propagator (63), (b) of the quantity $G_\pm(k;\beta)$ in the thermal propagator (86), (c) of the quantity $G_\pm(k;\beta,\mu_4)$ in the GGE propagator (90) with $\beta = 2, \mu_4 = 0.2$; we have shown 30 leading poles. In each case the pole at the origin is a double pole, and yields the universal linear large distance behaviour of $\langle \phi \phi \rangle$. Due to the equivalence between the quenched state and the gCC state (14), the branch cut in (a) can be seen as a limiting case of an accumulation of single poles in a generalized version of (c) with an infinite number of chemical potentials determined by (34),(92). In two-point functions such as $\langle \partial \phi \partial \phi \rangle$, the double poles disappear and the large distance behaviour is sensitive to the sub-leading singularities, which are clearly different. This shows different types of large distance behaviour which are sensitive to the presence of higher dimensional operators.

Universality: Let us first discuss the naive argument for universality in the present context. Note that in case of the sudden quench we found (35)

$$|0, in\rangle = \exp \left[ -\frac{H}{m_0} - \frac{5W_4}{160m_0^3} + \ldots \right] |D\rangle$$

which would appear to imply that, in the limit when the scale of the quench is very high: $m_0 \to \infty$, the contribution of the Hamiltonian is the most dominant and those of the higher dimensional operators $W_{2n}$, $n > 1$, are subdominant. This argument, of course, is flawed, since $m_0$ is dimensionful, and we have to specify $m_0$ is large compared to what.

There are, of course, more refined arguments for universality which define an IR limit in terms of dimensionless distances and times

$$m_0r, m_0t \gg 1$$

(93)

which is called the deep quench limit in [33]. Ref. [33] argues that in this limit, the propagator in (63) is dominated by the leading expansion of the integrand in $|k|/m_0$, which is given by a double pole. From (65), we find that the leading behaviour of this propagator is indeed
given by the linear term which is solely determined by this double pole. We find that this double pole and the consequent leading behaviour exactly coincides with that of the thermal propagator (86). Indeed, all the three propagators, the quenched one (63), the thermal one (86) and the GGE one (90), coincide in the leading behaviour. Thus, the higher order chemical potentials do not modify the leading behaviour. Note, however, that the subleading behaviours are rather different in the three cases: the exponents are different, as well as in the quenched propagator there is a prefactor involving a square root.

Lack of Universality: The long-distance/time leading behaviour of the \( \langle \phi \phi \rangle \) propagator is, of course, a rather limited part of the story. Does the above universality hold for correlators involving other operators, in particular, primary fields (recall that \( \phi \) is not a primary field)?

To address this issue for one-point functions of primary fields of the kind \( O(z, \bar{z}) = \varphi(z) \varphi^*(\bar{z}) \) which has a decay rate given by (2), (3). For the sudden quench discussed in Section 2.4.1, using (34), we find that the fractional contribution of \( W_{n>2} \) to the relaxation rate (3) is determined by the dimensionless quantity

\[
\tilde{\mu}_n = \frac{\mu_n}{\beta^{n-1}} \sim \frac{1}{m_0^{n-1}/(1/m_0)^{n-1}}
\]

which is of order one! What has happened is that, since the quench is characterized by a single scale, the chemical potentials due to the higher dimensional operators are determined by the same mass scale as the temperature, thus the dimensionless contribution due to \( W_{n>2} \) is necessarily of order one. We would expect this kind of behaviour in any single-scale quench.

Indeed, we find in (68) that the leading behaviour of the one-point function of \( \partial \phi \bar{\partial} \phi \) is not given by the thermal value (nor with any finite number of chemical potentials). This is best understood by looking at the Figure 3. The derivatives \( \partial, \bar{\partial} \) kill off the double pole at the origin in all three diagrams, leaving singularities away from the origin. These in Figure (a) differ from those in Figure (b) or in Figure (c). Figure (c), if redone with infinite number of chemical potentials as given by (34), reproduce the singularities of Figure (a).

Thus, we find that ALL higher dimensional operators are equally important in determining the long time behaviour of this operator. This is what we anticipated also from the MSS expression for the relaxation rate, as explained above.

The same story holds for two-point functions \( \langle O(x_1, t_1)O(x_2, t_2) \rangle \). The exact quench computation, even in the deep quench limit (93) is not reproduced by the thermal result or any finite number of chemical potentials. This can be explicitly seen for \( O = \partial \phi \) in the previous two sections. We have also verified this lack of universality for operators which are a composite of ‘derivative’ operators and exponential vertex operators, e.g. \( O = \partial \phi e^{i\eta \phi} \). Once again, the reason is the annihilation of the double pole at the origin by these generic operators.

It is only the pure exponential vertex operators \( O = e^{i\eta \phi} \) whose two-point functions (66) respect universality in the deep quench limit, that is it is reproduced by the thermal behaviour (these operators do not annihilate the pole at the origin).

Conclusion Generically universality, as defined above, is violated. Long time/distance behaviour is affected by perturbing the initial state by higher dimensional operators.
6.2 Memory retention

In this section we will discuss the issue of non-standard thermalization in the models studied where the equilibrium chemical potentials allow a reconstruction of the quench protocol (completely or partially depending on the situation).

Let us first consider the case of quenches from a ground state. As is clear from (10) and (9), the \( \kappa_n \)-coefficients of the gCC state (14) have a one-to-one relation to the reflection amplitude \( r(k) \) of the potential scattering problem (94) discussed in Appendix A. Now, it is well-known that the potential of a one-dimensional Schrodinger problem \([38]^{25}\) can be reconstructed from the reflection amplitude \( r(k) \). This implies, through the above correspondence between the quench problem and the scattering problem, that \( m(t) \) can be reconstructed from \( \kappa(k) \). This, in turn, means that the \( \mu_n \)'s carry complete knowledge of the quench protocol \( m(t) \). Thus, the equilibrium ensemble remembers the quench protocol! As an example, the coefficients \( \kappa_n \) in (28) can be used to determine the parameters \( m_0 \) and \( \rho \) which specify the quench protocol \( m(t) \) completely.

In case we consider a squeezed pre-quench state, the GGE is characterized by the function \( \kappa_{eff}(k) \) (47) which is given by a combination of the knowledge of the squeezing function \( f(k) \) and the quench protocol \( m(t) \) (see (48)). For a given quench protocol, the initial state, characterized by \( f(k) \) can be completely determined by the \( \kappa_n \)-parameters (see, e.g. (50)).

Thus, in case the pre-quench initial state as well as the quench protocol are unknown, the equilibrium ensemble has an imperfect recollection of the history.

7 Discussion

In this paper, we explicitly verify for actual critical quenches the ansatz made in MSS for the generalized Calabrese-Cardy form (gCC) (1) of the initial state. We show that for an arbitrary mass quench in a theory of free scalars as well as in a theory of free fermions, a large choice of pre-quench initial states (ground state or squeezed states) leads to a gCC state. We find that our results hold even when the quantum quench begins and ends in a massless theory, although in this case, the putative temperature sometimes turns out to be imaginary and the issue of thermalization in these cases is subtle.

We find that while the ground state and generic squeezed states lead to gCC states with all infinite number of \( \kappa_n \) parameters present, one can choose special squeezed states to prepare gCC states with specific values of any given number of the \( \kappa_n \)-parameters; in particular we can prepare a CC state of the form \( e^{-\kappa_2 H} |D\rangle \) from special squeezed states.

We compute the exact propagator in these quenches and hence the exact time-dependence of correlators. We find that the correlators thermalize at long times and the results verify those of MSS wherever a comparison is possible. We have a simple understanding of the identification (2) of the \( \kappa_n \)'s with the chemical potentials \( \mu_n \) in terms of poles of the propagator. In specially prepared gCC states with non-zero values of \( \kappa_2 \) and \( \kappa_4 \), we show that the exponential decay given by the relaxation rate (3) persists non-perturbatively in \( \kappa_4 \).

We point out that the presence of the extra charges in the gCC state, which are higher dimensional operators, non-trivially modify the long distance and long time behaviour of

\[25\] We thank Basudeb Dasgupta for pointing out this reference to us.
correlators, in apparent contradiction to Wilsonian universality. This is an example of a UV/IR mixing; operators which are expected to be relevant in the UV by usual RG arguments are found here to affect the IR behaviour of various correlators. We present an understanding of this in terms of poles of the propagator in the complex momentum plane. We find that while exponential vertex operators do not suffer from these ‘non-universal’ corrections, all other operators (derivatives and composites of derivatives and exponentials) do show this non-universal behaviour.

We also find another atypical behaviour, related to the above: the equilibrium ensemble remembers about the quench protocol. In case we start from the ground state of the pre-quench Hamiltonian, the chemical potentials of the GGE encode a complete knowledge of the quench protocol $m(t)$. With pre-quench squeezed state, the chemical potentials encode a combination of information about the initial state and the quench protocol.

7.1 Higher spin black holes

In MSS we have established a relation between thermalization to a GGE and, in the holographic dual, quasinormal decay to a higher spin (hs) black hole. In particular, we have found that relaxation rate in the former process is equal to the imaginary part of the quasinormal frequency involved in the latter process.

The demonstration above depended on an ansatz about the initial state being given by a gCC state. In this paper (see, e.g. (51)) we have shown explicitly that by choosing to start with a squeezed state with an appropriate squeezing function, one can explicitly generate such gCC states. In Section 4.2 we have shown explicitly (see (81)) that the exact formula for relaxation rate supports the perturbative formula (3). This, therefore, explicitly proves the relation between the quench dynamics and the quasinormal decay to higher spin black holes. Note that we now have the relaxation rate non-perturbatively, including the two non-perturbative branches (76). It would be interesting to compare these two branches with the corresponding non-perturbative branches of the hs black hole quasinormal frequency [39].

Although we have not computed an explicit collapse process to a higher spin black hole, it is natural to speculate that the memory retention by the thermal state in the field theory, discussed above, would imply that the higher spin black hole obtained from such a collapse starting from a pure AdS vacuum would remember the dynamics of the collapse which is governed by the dynamics of the quench.

We note that a massive to massless quench does not have a direct holographic dual since the theory in the past is not conformal. In this paper we have included a discussion of quenches from a critical Hamiltonian to a critical Hamiltonian, starting from ground states/excited states. This can potentially describe a collapse geometry. We hope to return to this issue at a later point.

Other open problems: Some of the obviously important extensions of the above work are to the case of (i) massive to massive quenches, (ii) higher dimensions, (iii) interacting theories. In particular, it would be interesting if the phenomena of IR non-universality persists in higher dimensions. The calculation of Bogoliubov coefficients and exact propagator for the $tanh$ protocol appears to go through [28] in higher dimensions in a straightforward manner.
However, the analysis of the poles requires more care. We hope to come back to this issue shortly.

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A Potential scattering and Bogoliubov transformation

In the text (see (5)) it has been shown that the scalar mass quench is equivalent to the following Schrödinger problem:

\[- \frac{d^2 \psi(E, x)}{dx^2} + V(x) \psi(E, x) = E \psi(E, x)\]

with the mapping (for a given fixed \(k\))

\[
\begin{array}{|c|c|}
\hline
x & t \\
\hline
E & k^2 \\
V(x) & -m^2(t) \\
\psi(E, x) & \phi(k, t) \\
\psi^*(E, x) & \phi^*(k, t) = \phi(-k, t) \\
\hline
\end{array}
\]

(94)

We will focus on the potentials of the form depicted in Figure 1. The generalization to the case of Figure 2 is straightforward.

The wavefunctions in such a potential, which asymptotes to a constant at both ends (see Figure 1) are of the general form\(^{26}\)

\[
\phi(k, t) = \begin{cases} 
A_1(k)e^{i\omega_{in}t} + B_1(k)e^{-i\omega_{in}t}, & t \to -\infty \\
A_2(k)e^{i\omega_{out}t} + B_2(k)e^{-i\omega_{out}t}, & t \to \infty 
\end{cases} \\
\omega_{in} = \sqrt{k^2 + m_0^2} \\
\omega_{out} = |k|
\]

(95)

where

\[
A_2(k) = \alpha_{LL}^*(k)A_1(k) + \beta_{LL}B_1(k), \\
B_2(k) = \alpha_{LL}(k)B_1(k) + \beta_{LL}^*(k)A_1(k).
\]

(96)

The coefficients \(\alpha_{LL}(k), \beta_{LL}(k)\) are determined by the shape of the potential \(V = -m^2(t)\)\(^{27}\)

The reflection coefficient from the right is given in our conventions, by

\[
r(k) = A_2(k)/B_2(k) \big|_{A_1=0} = \beta_{LL}(k)/\alpha_{LL}(k)
\]

(97)

\(^{26}\)We will closely follow the treatment in Landau and Lifshitz [40], Section 25.

\(^{27}\)The suffix \(LL\) indicates the Landau-Lifshitz convention ([40], Section 25). Our \(\alpha, \beta\)’s (102) will differ from these by a normalization factor.
For later reference we note the reflection coefficient from the left is
\[ r'(k) = \frac{B_1(k)}{A_1(k) B_2 = -\frac{\beta_{LL}^*}{\alpha_{LL}(k)} } \tag{98} \]
To make connections with QFT later, let us write
\[ \phi(k, t) = a_{in}(k) u_{in}(k, t) + a_{in}^*(-k) u_{in}^*(-k, t) = a_{out}(k) u_{out}(k, t) + a_{out}^*(-k) u_{out}^*(-k, t) \tag{99} \]
in terms of two separate sets of linearly independent solutions: (see, e.g. [29], Chapter 3) with the defining properties:
\[ t \to -\infty : u_{in} \to e^{-i\omega_{in} t} \sqrt{\frac{\omega_{in}}{2\omega_{in}}}, t \to \infty : u_{out} \to e^{-i\omega_{out} t} \sqrt{\frac{\omega_{out}}{2\omega_{out}}} \tag{100} \]
Thus, \( u_{in} \) does not have a negative energy wave component \( \propto e^{i\omega_{in} t} \) in the past), similarly \( u_{out} \) does not have a negative energy wave component \( \propto e^{i\omega_{out} t} \) in the future.
These expressions for \( \phi \) agree with the earlier ones (95), if we identify
\[ A_1(k) = \frac{1}{\sqrt{2\omega_{in}}} a_{in}(k), B_1(k) = \frac{1}{\sqrt{2\omega_{in}}} a_{in}(-k), A_2(k) = \frac{1}{\sqrt{2\omega_{out}}} a_{out}^*(k), B_2(k) = \frac{1}{\sqrt{2\omega_{out}}} a_{out}(-k) \]
This implies
\[ a_{in}(k) = \alpha^*(k) a_{out}(k) - \beta^*(k) a_{out}^*(-k) \tag{101} \]
where the new scattering data \( \{\alpha, \beta\} \), to be identified with Bogoliubov coefficients in the quantum theory, are related to old one (96) by some normalization factors
\[ \alpha(k) = \sqrt{\frac{\omega_{out}}{\omega_{in}}} \alpha_{LL}(k), \beta(k) = \sqrt{\frac{\omega_{out}}{\omega_{in}}} \beta_{LL}(k), \tag{102} \]
\[ r(k) = \frac{\beta_{LL}(k)}{\alpha_{LL}(k)} = \frac{\beta(k)}{\alpha(k)} \]
Note that the reflection amplitudes \( r(k) \) remain unaltered. The new scattering data satisfy the normalization conditions
\[ |\alpha(k)|^2 - |\beta(k)|^2 = 1 \tag{103} \]
which follows from probability conservation in the scattering problem. Upon quantization, the coefficients \( a_{in,out}(k) \) are treated as operators in the Fock space (with \( a_{in,out}(k) \) rewritten as \( a_{in,out}^+(k) \)), as in the text (see (6)).
Note that the in- and out- wavefunctions are related to each other as follows:
\[ u_{in}(k) = \alpha(k) u_{out}(k) + \beta(k) u_{out}^*(-k) \tag{104} \]

\[ ^{28} \text{Note that our conventions ensure } \phi(-k, t) = \phi^*(k, t) \text{ which is the reality condition for } \phi(x, t) = \int (dk/2\pi) \phi(k, t) \exp[ikx]. \]
\[ ^{29} \text{We consider } \exp(\mp i\omega t) \text{ to be future/past directed, with energy defined by } i\partial/\partial t. \text{ This is to be contrasted with } p = -i\partial/\partial x \text{ with } \exp[\pm ikx] \text{ identified as right/left directed.} \]
One of the important points of this analysis is that under some broad conditions on the potential (see [40], Section 25, also [41]), the reflection amplitude has a Taylor expansion\(^{30}\)

\[
r(k) = -1 + r_1 |k| + r_2 |k|^2 + r_3 |k|^3 + ... \quad \text{Re}(r_1) \geq 0
\]  

(105)

It is also of interest to note that the other reflection amplitude \(r'(k)\) has an expansion

\[
r'(k) = 1 + r_1' |k| + r_2' |k|^2 + r_3' |k|^3 + ...
\]  

(106)

Note that

\[ -r^*(k)/r'(k) = \alpha(k)/\alpha^*(k) = 1 + o(|k|). \]  

(107)

\textbf{A.1 Examples of potentials}

Below we describe a few examples of potential scattering (see [40], Section 25) to test the validity of the power series expansions (105) and (106).

1. Consider a step potential \(U_0 \Theta(x)\). Let us choose a wavefunction in the past to come from the left, with energy slightly above the barrier \(U_0\). We will denote the transmitted wave as \(\sim e^{ikx}\); this plays the role of the ‘out’ wave. It is easy to find the left\(^{31}\) reflection coefficient \(r(k) = (|k| - \sqrt{k^2 + U_0^2})/(|k| + \sqrt{k^2 + U_0^2})\). This admits the following expansion in the right momentum \(|k|\):

\[
r(k) = -1 + \frac{2|k|}{U_0} - \frac{2k^2}{U_0^2} + \frac{|k|^3}{U_0^3} - \cdots
\]

which is consistent with the nature of the power series expansion (105) which was inferred from general arguments.

It is easy to show that the right reflection coefficient is \(r'(k) = -r(k)\). This clearly satisfies an expansion of the form (106).

2. For a rectangular barrier potential \(U_0 \Theta(x) - U_0 \Theta(x - a)\) with width \(a\), for \(E > U_0\) the left reflection coefficient \(r(k) = -\sqrt{(U_0^4 \sin^2 \left( a \sqrt{k^2 + U_0^2} \right))/(U_0^4 \sin^2 \left( a \sqrt{k^2 + U_0^2} \right) + 4|k|^2(k+U_0^2))}\) admits the following expansion in the left momentum \(|k|\):

\[
r(k) = -1 + \frac{2k^2 \cosec^2 (aU_0)}{U_0^2} + \cdots
\]

3. For a smooth barrier potential \(U_0 (1 + e^{-ax})^{-1}\), the left reflection coefficient \(r(k) = \sinh \left( \frac{\pi|k| - \sqrt{k^2 + U_0^2}}{a} \right) \cosech \left( \frac{\pi|k| + \sqrt{k^2 + U_0^2}}{a} \right)\) admits the following expansion in the left momentum \(|k|\):

\[
r(k) = -1 + \frac{2\pi|k| \coth \left( \frac{\pi U_0}{a} \right)}{a} - \frac{2\pi^2 k^2 \coth^2 \left( \frac{\pi U_0}{a} \right)}{a^2} + \cdots
\]

\(^{30}\)Roughly speaking, \(r(0) = -1\) is due to a hard-wall reflection, and \(\text{Re}(r_1) \geq 0\) follows from \(1 - |r(k)|^2 \geq 0\) which, in turn, follows from (103).

\(^{31}\)Note the left-right flip due to the mapping \(-x \rightarrow t\), as explained in footnote 29.
A.2 Even parity of the Bogoliubov coefficients

It is clear from the correspondence (94) between the QFT problem and the potential scattering problem that the $A_i, B_i$ are actually functions of the energy $E$, implying that $\alpha(k), \beta(k)$ are all actually functions of $k^2$. In particular, for real $k$, the Bogoliubov coefficients have even parity

$$\alpha(k) = \alpha(-k) = \alpha(|k|), \beta(k) = \beta(-k) = \beta(|k|), r(k) = r(-k) = r(|k|), r'(k) = r'(-k) = r'(|k|)$$

(108)

B Baker-Campbell-Hausdorff Calculation

We will show that

$$|\psi\rangle \equiv \exp\left(\frac{1}{2} \sum_k \gamma(k) a^\dagger(k) a^\dagger(-k)\right) |0\rangle = \exp\left(-\sum_k \kappa(k) a^\dagger(k) a(k)\right) |Bd\rangle$$

(109)

where

$$\kappa(k) = -\frac{1}{2} \log(\gamma(k)/\gamma_0)$$

(110)

and

$$|Bd\rangle \equiv \exp\left(\frac{1}{2} \sum_k \gamma_0 a^\dagger(k) a^\dagger(-k)\right) |0\rangle,$$

(111)

The choice $\gamma_0 = -1$ corresponds to the Dirichlet state (114) (similarly, $\gamma_0 = 1$ corresponds to Neumann boundary condition). To derive (109), we note that the right hand side can be written as

$$\exp\left[\sum_k B(k)\right] \exp\left[\sum_k A(k)\right] |0\rangle = \exp\left[\sum_k B(k)\right] \exp\left[\sum_k A(k)\right] \exp\left[-\sum_k B(k)\right] |0\rangle$$

where we have defined $B(k) = -\kappa(k) a^\dagger(k) a(k)$ and $A(k) = \gamma_0 a^\dagger(k) a^\dagger(-k)$. The identity (109) follows by noting that $[B(l), A(k)] = -\kappa(k) A(k) (\delta_{k,l} + \delta_{k,-l})$, and by using the following form of the Baker-Campbell-Hausdorff (BCH) formula

$$e^X e^Y e^{-X} = e^{exp(s)Y}$$

(112)

where $[X, Y] = sY$.

In the context of this paper, we will be interested in evaluating $\kappa(k)$ from (110) in a power series in $k$, using (12). Since the leading term in $\gamma(k)$ is $-1$, with the choice of the Dirichlet boundary state $\gamma_0 = -1$, we get the equation (10) in the text.

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32We thank Samir Mathur for drawing our attention to [42] where a relation of the form (110) was derived earlier in a somewhat different context for a single oscillator.
C Bosons

The action for a free massless scalar is

\[ S = \frac{1}{2} \int dxdt \left[ (\partial_t \phi)^2 - (\partial_x \phi)^2 \right] = -\frac{1}{2} \int dxdt \partial_\mu \phi \partial^\mu \phi \]

The normal mode expansion is (we use “box normalization” \( k = 2\pi n/L, \int \frac{dk}{2\pi} = \frac{1}{L} \sum_n \) )

\[ \phi(x,t) = \int \frac{dk}{2\pi} \left[ a(k) \frac{1}{\sqrt{2|k|}} \exp(ikx - i|k|t) + a^\dagger(k) \right. \exp(-ikx + i|k|t)] = \sum_{k \neq 0} \frac{1}{4\pi L|n|} \alpha_n \exp \left( \frac{2\pi}{L} (i n x - i|n|t) \right) + h.c \]

\[ \equiv \sum_{k \neq 0} \left[ \frac{a(k)}{\sqrt{2|k|}} \exp(ikx - i|k|t) + \frac{a^\dagger(k)}{\sqrt{2|k|}} \exp(-ikx + i|k|t) \right] \] (113)

We will often use \( a_n \equiv a(k) \), with a slight abuse of notation. The commutation relations are \([a(k), a^\dagger(l)] = \delta_{kl} \).

Boundary states In terms of standard CFT oscillators \( \alpha_n, \tilde{\alpha}_n \), the Dirichlet boundary state is given by (see, e.g. \([44]\) Eq. 4.1.13)

\[ |D\rangle = \exp[\sum_{n=1}^{\infty} \frac{1}{n} \alpha_n \tilde{\alpha}_n] |0\rangle \]

In terms of our oscillators \( a_n \equiv a_k \)

\[ \alpha_{-n} = i\sqrt{n} a_{-n}, \quad \tilde{\alpha}_{-n} = i\sqrt{n} a_{-n}^\dagger \]

\[ |D\rangle = \exp[-\sum_{n>0} a_n^\dagger a_{-n}^\dagger] |0\rangle = \exp[-\frac{1}{2} \sum_{n>0} a_n^\dagger a_{-n}^\dagger] |0\rangle = \exp[-\frac{1}{2} \sum_{k \neq 0} a^\dagger(k) a^\dagger(-k)] |0\rangle \] (114)

In the first step we used the relation between our oscillators here and the standard CFT conventions (see [43], Chap. 6).

Euclidean CFT We define \( w = x + i\tau, \bar{w} = x - i\tau, \tau = it \). The Euclidean Propagator is

\[ \langle \phi(0,0)\phi(x,\tau) \rangle = \langle \phi(0,0)\phi(w,\bar{w}) \rangle = -\frac{1}{4\pi} (\ln w + \ln \bar{w}) \]

Vertex operators Consider the exponential vertex operator \( O(w, \bar{w}) = \exp[iq\phi(w, \bar{w})] \).

\[ \langle \exp[iq\phi(0,0)] \exp[-iq\phi(w, \bar{w})] \rangle = w^{-q^2/4\pi} \bar{w}^{-q^2/4\pi} \]

Hence \( h = \bar{h} = q^2/8\pi, \Delta = q^2/4\pi \).

\(^{33}\text{We use the conventions of [43].}\)
**Boson W-currents**  We have used the following definitions of the \( W_{\infty} \) currents [31] (normal ordering is implicit),

\[
T(z) = \partial \phi(z) \partial \phi(z) \tag{115}
\]
\[
W_4(z) = 2 \partial^3 \phi \partial \phi - 3 \partial^2 \phi \partial^2 \phi \tag{116}
\]

**D  Fermions**

We have used the following conventions in the text.

\[
\eta_{\mu \nu} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \partial_{\mu} = (\partial_t, \partial_x), \quad \gamma^\mu \partial_{\mu} = \gamma^0 \partial_t - \gamma^1 \partial_x,
\]
\[
\gamma_0^d = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \gamma_1^d = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad \text{in Dirac basis.}
\]
\[
S = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}, \quad \gamma_0^c = S \gamma_0^d S^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \gamma_1^c = S \gamma_1^d S^{-1} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad \text{in chiral basis.}
\]
\[
u(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \bar{\psi}(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \text{are the spinors in the rest frame.}
\]

The spinors in a general frame are

\[
u(k, m) = \frac{1}{\sqrt{(\omega + m)}} \begin{bmatrix} (\omega + m) \\ -k \end{bmatrix}, \quad \bar{\psi}(k, m) = \frac{1}{\sqrt{(\omega + m)}} \begin{bmatrix} k \\ - (\omega + m) \end{bmatrix}
\]
\[
\bar{u}(k, m) = \frac{1}{\sqrt{(\omega + m)}} \begin{bmatrix} (\omega + m) \\ k \end{bmatrix}, \quad \bar{\bar{v}}(k, m) = \frac{1}{\sqrt{(\omega + m)}} \begin{bmatrix} k \\ (\omega + m) \end{bmatrix}
\]  \hspace{1cm} (117)

where we have used the normalization \( \bar{u}(k, m) u(k, m) = - \bar{\bar{v}}(k, m) v(k, m) = 2m \). In the chiral basis, the mode expansion in the massless limit is

\[
\Psi_c(x, t) = S \cdot \Psi(x, t) = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \end{bmatrix} \cdot \int \frac{dk}{2\pi} \frac{1}{\sqrt{2}} \left[ a_k e^{-ik \cdot x} + \text{sgn}(k) b_k^\dagger e^{ik \cdot x} \right] = \int_{-\infty}^{\infty} \frac{dk}{2\pi} \frac{1}{\sqrt{2}} \left[ (1 + \text{sgn}(k))(a_k e^{-ik \cdot x} + b_k^\dagger e^{ik \cdot x}) \right]
\]
\[
= \int_{-\infty}^{\infty} \frac{dk}{2\pi} \frac{1}{\sqrt{2}} \left[ (1 - \text{sgn}(k))(a_k e^{-ik \cdot x} - b_k^\dagger e^{ik \cdot x}) \right]
\]
\hspace{1cm} (118)

Writing as \( \psi(x, t) \) and \( \bar{\psi}(x, t) \),

\[
\psi(x, t) = \int_{0}^{\infty} \frac{dk}{2\pi} (a_k e^{-ik \cdot x} + b_k^\dagger e^{ik \cdot x}) \tag{119}
\]
\[
\bar{\psi}(x, t) = \int_{-\infty}^{0} \frac{dk}{2\pi} (a_k e^{-ik \cdot x} - b_k^\dagger e^{ik \cdot x}) \tag{120}
\]
Solution of Dirac equation and Bogoliubov coefficients

Using the coordinate transformation \( y = e^{-\rho t} \) and the ansatz, we get the following equation:

\[
\phi''(y) + \frac{\phi'(y)}{y} + \phi(y) \left( \frac{k^2}{\rho^2 y^2} + \frac{m_0^2 y^2 + 2i m_0 \rho}{\rho^2 (y^2 + 1)^2} \right) = 0
\]

(121)

The ‘in’ solutions are solutions which become plane waves in far past and the ‘out’ solutions are solutions which become plane waves in far future. Due to the explicit \( i \) in the equation of \( \phi_{\pm} \), the positive energy solutions \( \phi_{\pm, in/out,p}(k, t) \) and the negative energy solutions \( e^{-i\omega_{in/out}t} \) are related as

\[
\phi_{+, in/out,m}(k, t) = \phi_{-, in/out,p}(k, t)^*, \quad \phi_{-, in/out,m}(k, t) = \phi_{+, in/out,p}(k, t)^*
\]

So, the solutions can be written as

\[
\phi_{+, in/out}(k, t) = \phi_{+, in/out,p}(k, t) + \phi_{-, in/out,p}(k, t)^*
\]

\[
\phi_{-, in/out}(k, t) = \phi_{-, in/out,p}(k, t) + \phi_{+, in/out,p}(k, t)^*
\]

The explicit solutions are

\[
\phi_{+, in}(k, t) = (e^{-2\rho t} + 1) \frac{-i m_0}{2\rho} e^{i\nu_{in} + m_0/2} \left[ F_1 \left( \frac{i (k - m_0 - \omega_{in})}{2\rho}, \frac{i (-k + m_0 - \omega_{in})}{2\rho}; 1 - \frac{i \omega_{in}}{\rho}; e^{2\rho t} \right) \right]
\]

\[
\phi_{-, in}(k, t) = (e^{-2\rho t} + 1) \frac{i m_0}{2\rho} e^{-i\nu_{in} - m_0/2} \left[ F_1 \left( \frac{i (k + m_0 - \omega_{in})}{2\rho}, \frac{i (-k + m_0 - \omega_{in})}{2\rho}; 1 - \frac{i \omega_{in}}{\rho}; e^{2\rho t} \right) \right]
\]

\[
\phi_{+, out}(k, t) = e^{-ikt} (e^{-2\rho t} + 1) \frac{-i m_0}{2\rho} e^{i\nu_{in} + m_0/2} \left[ F_1 \left( \frac{i (k - m_0 + \omega_{in})}{2\rho}, \frac{i (k + m_0 + \omega_{in})}{2\rho}; 1 + \frac{i k}{\rho}; e^{-2\rho t} \right) \right]
\]

\[
\phi_{-, out}(k, t) = e^{-ikt} (e^{-2\rho t} + 1) \frac{i m_0}{2\rho} e^{-i\nu_{in} - m_0/2} \left[ F_1 \left( \frac{i (k + m_0 - \omega_{in})}{2\rho}, \frac{i (k + m_0 + \omega_{in})}{2\rho}; 1 + \frac{i k}{\rho}; e^{-2\rho t} \right) \right]
\]

(122)

Defining the Dirac spinors as

\[
U_{in/out}(k, x, t) = K_{in/out} \left( \gamma^0 \partial_t - i k \gamma^1 - im(t) \right) e^{ikx} \phi_{+, in/out,p}(k, t) u(0)
\]

\[
V_{in/out}(k, x, t) = -K_{in/out} \left( \gamma^0 \partial_t + i k \gamma^1 - im(t) \right) e^{-ikx} \phi_{+, in/out,p}(k, t)^* v(0)
\]

where \( K_{in/out} = i \left( \frac{1}{\sqrt{\omega_{in/out} + m_{in/out}}} \right)^{1/2} \) . For constant mass, \( U(k, x, t) = u(k, m)e^{-ikx} \) and \( V(k, x, t) = v(k, m)e^{ikx} \) where \( u(k, m) \) and \( v(k, m) \) have been defined in (117). The mode expansion of \( \Psi(x, t) \) in terms of in/out modes are

\[
\Psi(x, t) = \int_{-\infty}^{\infty} \frac{dk}{\sqrt{2\omega_{in/out}}} \left[ a_{k, in/out} U_{in/out}(k, x, t) + b_{k, in/out}^* V_{in/out}(k, x, t) \right]
\]

Using properties of hypergeometric functions [34], the Bogoliubov transformations between ‘in’ and ‘out’ solutions are

\[
\phi_{+, in,p}(k, t) = \alpha_+(k) \phi_{+, out,p}(k, t) + \beta_+(k) \phi_{-, out,p}(k, t)^*
\]

\[
\phi_{-, in,p}(k, t) = \alpha_-(k) \phi_{-, out,p}(k, t) + \beta_+(k) \phi_{+, out,p}(k, t)^*
\]

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Hence, the Bogoliubov transformations between the ‘in’ and ‘out’ operators are

\[ a_{k,\text{in}} = \left( \frac{\omega_{\text{in}}}{\omega_{\text{out}}} \right)^{1/2} \frac{K_{\text{out}}}{K_{\text{in}}} \left( \alpha_+(k)^* a_{k,\text{out}} - \chi(k) \beta_+(k)^* b_{-k,\text{out}}^\dagger \right) \]

\[ b_{k,\text{in}} = \left( \frac{\omega_{\text{in}}}{\omega_{\text{out}}} \right)^{1/2} \frac{K_{\text{out}}}{K_{\text{in}}} \left( \alpha_+(k)^* b_{k,\text{out}} + \tilde{\chi}(k) \beta_-(k)^* a_{-k,\text{out}}^\dagger \right) \]

where \( \chi(k) = \tilde{\chi}(k) = \text{sgn}(k) \). It is straightforward now to find the expressions for the Bogoliubov coefficients which are reproduced in the text (60).

**Fermion W-currents** We have used the following definitions of the super-\( W_\infty \) currents [45] (normal ordering is implicit),

\[ T(z) = -i \left( \psi^* \partial \psi(z) - \partial \psi^* \psi(z) \right) \]

\[ W_4(z) = \frac{4}{5} q^2 \left( \partial^3 \psi^* \psi(z) - 9 \partial^2 \psi^* \partial \psi(z) + 9 \partial \psi^* \partial^2 \psi(z) - \psi^* \partial^3 \psi(z) \right) 
+ 25 \partial \psi^* \partial^4 \psi - \psi^* \partial^5 \psi(z) \]

**E Subtleties of the sudden limit**

In Section 2.4 we analyzed the behaviour of the quench under the “tanh” protocol for large \( \rho \) in a power series in \( m_0/\rho \). In particular, in Section 2.4.1, we defined the sudden limit as the limit (31). In this section we will give a more precise definition of this limit. In certain quantities, like the number operator (29) in Section 2.4 and the propagator in Section 4.1 etc. the distinction is not essential, but in general the naive limit entails UV divergences. E.g. all \( W \)-charges, including the energy density, under a naive \( m_0/\rho \) expansion introduced in Section 2.4 appear to have progressively higher UV divergences as one goes down the order. To treat these divergences properly, let us first analyze these. Later on, we will find that terms in this expansion can be resummed to yield finite expressions, provided we define the sudden limit by the equation (124).

Energy density

\[ E/L = \frac{1}{2\pi} \int_{-\Lambda}^{\Lambda} dk |k| N_k = m_0^2 \left( \frac{1}{8\pi} - \frac{m_0^2}{32\pi \Lambda^2} + O \left( \frac{m_0}{\Lambda} \right)^4 - \frac{m_0^2}{\rho^2} \left[ \frac{1}{48} \pi \log \left( \frac{\Lambda}{m_0} \right) \right. \right. \]
\[ + \frac{1}{96} \pi \log(4) + \frac{\pi m_0^2}{192 \Lambda^2} + O \left( \frac{m_0}{\Lambda} \right)^4 \right] + O \left( \frac{m_0}{\rho} \right)^4 \]

where we have used the asymptotic number density (29), in an \( m_0/\rho \) expansion:

\[ N_k = \frac{(k - \sqrt{k^2 + m_0^2})^2}{4k \sqrt{k^2 + m_0^2}} - \left( \frac{m_0}{\rho} \right)^2 \frac{\pi^2 m_0^2}{48 (k \sqrt{k^2 + m^2})} + O \left( \frac{m_0}{\rho} \right)^4 \]
W4 density

$$W_4/L = \int_{-\Lambda}^{\Lambda} \frac{dk}{2\pi} |k|^3 N_k = m_0^4 \left[ \frac{4 \log(\Lambda/m_0) - 3 + \log(16)}{64\pi} + \frac{m_0^2}{32\pi^2} + O \left( \frac{m_0}{\Lambda} \right)^4 \right] + \left( \frac{m_0}{\rho} \right)^2 \left( -\frac{\pi\Lambda^2}{96m_0} + \frac{1}{192} \pi (2 \log(\Lambda/m_0) - 1 + \log(4)) + \frac{\pi m_0^2}{256\Lambda^2} + O \left( \frac{m_0}{\Lambda} \right)^4 \right) + O \left( \frac{m_0}{\rho} \right)^4$$

W6 density

$$W_6/L = \int_{-\Lambda}^{\Lambda} \frac{dk}{2\pi} |k|^5 N_k = m_0^6 \left[ \left( \frac{\Lambda^2}{32\pi m_0^2} + \left( \frac{\log(m_0/\Lambda)}{16\pi} + \frac{1}{24\pi} - \frac{\log(4)}{32\pi} \right) - \frac{15m_0^2}{512\pi^2} + O \left( \frac{m_0}{\Lambda} \right)^4 \right) + \frac{m_0^2}{\rho^2} \left( -\frac{\pi\Lambda^4}{192m_0^2} + \frac{\pi\Lambda^2}{192m_0^2} + \frac{1}{128} \pi \log \left( \frac{m_0}{\Lambda} \right) \right) - \frac{1}{256} \pi \log(4) + \frac{7\pi}{1536} - \frac{5\pi m_0^2}{1536\Lambda^2} + O \left( \frac{m_0}{\Lambda} \right)^4 \right) + O \left( \frac{m_0}{\rho} \right)^4 \right]$$

E.1 Resumming the divergences

It turns out that the terms with growing UV-divergences with growing powers of $m_0/\rho$ can be resummed to the following form.

Introduce the scaling functions

$$E/L = m_0^2 \mathcal{E}(x, y), \quad W_4/L = m_0^4 F(x, y), \quad W_6/L = m_0^6 G(x, y), \quad x = m_0^2/\rho^2, \quad y = m_0^2/\Lambda^2$$

The leading singularities in the above expressions for the charges are captured by

$$\mathcal{E}(x, y) = \frac{1}{8\pi} + \frac{\left( \frac{2\pi^2 x}{3} + y \right) \log (\pi^2 x + y)}{60\pi} + \cdots = \frac{1}{8\pi} + \cdots$$

$$F(x, y) = \frac{\log \left( \frac{2\pi^4 x^2}{5} + y^2 \right) + \log (\pi^2 x + y) (40 (5y + 3) + \pi^4 x^2 + 20\pi^2 x)}{11520\pi} + \cdots$$

$$G(x, y) = \frac{1}{1536\pi} \left( \frac{\pi^4 x^2}{120} + \frac{1}{32} \pi^2 x (9y + 4) + y^2 + y + 1 \right) \left[ 8 \left( \frac{25}{\sqrt{26x^2 + 25y^2}} + \frac{1}{\pi^2 x + y} \right) + 19 \log \left( \frac{74\pi^4 x^2}{285} + y^2 \right) + 10 \log (\pi^2 x + y) \right] + \cdots$$

The correct version of the “sudden” limit, therefore, is to take the limit $\Lambda \to \infty$ first, for finite, large $\rho/m_0$ (see Figure 4). , i.e.

$$y = \frac{m_0^2}{\Lambda^2} \to 0, \quad x = \frac{m_0^2}{\rho^2} \text{ small, fixed} \quad (124)$$
In this limit, as we can see from the above expressions:

\[ E(x, 0) = \frac{1}{8\pi} + \frac{\pi}{96} x \log(x) + \cdots = \frac{1}{8\pi} + \cdots, \quad F(x, 0) \propto \log(x) + \cdots, \quad G(x, 0) \propto \log(x)/x + \cdots, \]

which implies

\[ \frac{E}{L} = m_0^2 \left( \frac{1}{8\pi} - \frac{\pi}{48} \frac{m_0^2}{\rho^2} \log(\frac{\rho}{m_0}) \right) + \cdots = m_0^2 \frac{1}{8\pi} + \cdots \]

\[ \frac{W_4}{L} \propto m_0^4 \log(\frac{\rho}{m_0}) + \cdots \]

\[ \frac{W_6}{L} \propto m_0^6 \frac{\rho^2}{m_0^2} \log(\frac{\rho}{m_0}) + \cdots \]

(125)

![Figure 4: The sudden limit.](image)

**E.2 Summary**

In those quantities, which are UV-convergent in the limit \( \Lambda/m_0 \to \infty \) (irrespective of the value of \( m_0/\rho \)), e.g. the energy density and the correlators discussed in the text, it is okay to use the naive definition of the sudden limit (31). However, for \( W_4 \) and the higher charges which have \( \log(\Lambda/m_0) \) and higher UV divergences, the only uniformly sensible way to define this limit is (124), as in figure 4.

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