ON CHEVALLEY RESTRICTION THEOREM

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§0. Introduction

Let $\mathfrak{g}$ be a complex semisimple Lie algebra with adjoint group $G$. Suppose that $\sigma \neq \text{id}$ is an involutive automorphism of $\mathfrak{g}$. Then $\sigma$ induces uniquely an involution of $G$ also denoted by $\sigma$, let $K = G^\sigma$ be a subgroup of $\sigma$-fixed points. Consider a direct decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ of $\mathfrak{g}$ into eigenspaces for $\sigma$, so

$$\mathfrak{k} = \{x \in \mathfrak{g} | \sigma(x) = x\}, \quad \mathfrak{p} = \{x \in \mathfrak{g} | \sigma(x) = -x\}.$$ 

Then, clearly, $\mathfrak{k}$ is a Lie algebra of $K$ and $\mathfrak{p}$ is a $K$-module, this representation is certainly nothing else but an isotropy representation of a symmetric space $G/K$ (see e.g. [KR]). Denote by $\mathfrak{a} \subset \mathfrak{p}$ any maximal abelian ad-diagonalizable subalgebra. Consider the “baby Weyl group” $W_0 = N_K(\mathfrak{a})/Z_K(\mathfrak{a})$. It is well-known that $W_0$ is a finite group generated by reflections as a linear group operating on $\mathfrak{a}$ (with respect to some real form of $\mathfrak{a}$). Let $\psi : \mathbb{C}[\mathfrak{p}]^K \to \mathbb{C}[\mathfrak{a}]^{W_0}$ be a restriction map of algebras of invariants. Then the famous Chevalley restriction theorem states that $\psi$ is an isomorphism (see eg. [He1], in [Vi] a more general result is obtained in the context of so-called $\theta$-groups attached to any periodic automorphism of $\mathfrak{g}$).

Now consider the following special case. The adjoint representation $G : \mathfrak{g}$ could be identified with an isotropy representation of a symmetric space induced by an involution $\sigma$ on $G \times G$ given by $\sigma(x, y) = (y, x)$. Then the cited Chevalley restriction theorem is equivalent to the “usual” Chevalley restriction theorem $\mathbb{C}[\mathfrak{g}]^G \simeq \mathbb{C}[\mathfrak{h}]^W$, where $\mathfrak{h} \subset \mathfrak{g}$ is a Cartan subalgebra and $W$ is the usual Weyl group. In the paper [J] A. Joseph obtained the remarkable “multivariable” analogue of this theorem. Namely, he proved that the restriction map

$$\mathbb{C}[\mathfrak{g} \times \mathfrak{g}]^G \to \mathbb{C}[\mathfrak{h} \times \mathfrak{h}]^W$$

is surjective. This theorem has important applications to the representation theory and the geometry of commuting varieties (cf. [Ri]), because it is essentially equivalent to the fact that the ring of $G$-invariant functions on a commuting variety is integrally closed (see [Hu, J]). The aim of this paper is to extend Joseph’s arguments in order to prove the following generalization:

**Theorem.** The restriction map $\psi : \mathbb{C}[\mathfrak{p} \times \mathfrak{p}]^K \to \mathbb{C}[\mathfrak{a} \times \mathfrak{a}]^{W_0}$ is surjective.

Actually, this (and Joseph’s) theorem also holds for any number of summands (generalization of the proof is immediate), but we will prove the Theorem in this form in order to make the notation easier.
Certainly, one could try to deduce this Theorem directly from Joseph’s result arguing as follows: take $f \in \mathbb{C}[\mathfrak{a} \times \mathfrak{a}]^{W_0}$, lift it to an element $\mathbb{C}[\mathfrak{h} \times \mathfrak{h}]^{W}$, then by Joseph’s theorem this element could be lifted to an element of $\mathbb{C}[\mathfrak{g} \times \mathfrak{g}]^{G}$ and we could restrict it to an element of $\mathbb{C}[\mathfrak{p} \times \mathfrak{p}]^{K}$. Unfortunately, the first step of this construction fails even in the case of one summand (see examples in [He2]). Moreover, I don’t know how to prove the natural analogue of the Theorem in the context of arbitrary $\theta$-groups (see [Vil]) and this seems to be an interesting problem. I should mention also that in some particular cases these results could be proved using “generalized polarizations” (see eg. [Hu]), but these results don’t cover our Theorem (as well as Joseph’s). In [Pa1] Theorem was proved in the case of an involution of maximal rank.

The paper is organized as follows. In §1 we will look for a proof of Chevalley restriction theorem in the case of one summand such that this proof could be generalized to the case of two summands. Another reason for doing that separately is that all the ideas distinct from the ones used by Joseph will already appear in this case. In §2 we will prove the Theorem in full generality showing that our main tool (class 1 representations) is compatible with Joseph’s idea to use the PRV Conjecture (or, better said, the Kumar-Mathieu Theorem) to “separate layers”.

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§1. Surjectivity of $\psi : \mathbb{C}[\mathfrak{p}]^{K} \to \mathbb{C}[\mathfrak{a}]^{W_0}$

Recall, that there is 1-1 correspondence between involutions of $\mathfrak{g}$ and real forms of $\mathfrak{g}$ (see [W, §1.1]). Namely, suppose that $\mathfrak{g}^{\mathbb{R}}$ is a real form of $\mathfrak{g}$ (that is, $\mathfrak{g} = \mathfrak{g}^{\mathbb{R}} + i\mathfrak{g}^{\mathbb{R}}$), let $\mathfrak{g}^{\mathbb{R}} = \mathfrak{k}^{\mathbb{R}} \oplus \mathfrak{p}^{\mathbb{R}}$ be a Cartan decomposition, let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be its complexification.

Let $B$ be a Killing form on $\mathfrak{g}$. Then the restriction of $B$ on $\mathfrak{k}^{\mathbb{R}}$ is negative-definite and on $\mathfrak{p}^{\mathbb{R}}$ is positive definite. In particular we can identify $\mathfrak{g} \simeq \mathfrak{g}^{*}$ and $\mathfrak{a} \simeq \mathfrak{a}^{*}$ by means of this scalar product. Then the restriction map $\psi : \mathbb{C}[\mathfrak{p}]^{K} \to \mathbb{C}[\mathfrak{a}]^{W_0}$ can be viewed as a homomorphism of symmetric algebras $\psi : (S\mathfrak{p})^{K} \to (S\mathfrak{a})^{W_0}$. Now, it is well-known that $K = G^{\sigma}$ is a reductive group (see eg. [Vu]), so we have a $K$-module decomposition $S\mathfrak{p} = (S\mathfrak{p})^{K} \oplus (S\mathfrak{p})_{K}$, where $(S\mathfrak{p})_{K}$ is a sum of all non-trivial $K$-submodules. Consider the Reynolds operator ($K$-equivariant projection on the first factor of this decomposition). Let $\theta : (S\mathfrak{a})^{W_0} \to (S\mathfrak{p})^{K}$ be its restriction to $(S\mathfrak{a})^{W_0}$. It is clear that we have a commutative diagram

\[
\begin{array}{ccc}
\mathfrak{a} & \xrightarrow{\theta_1} & (S\mathfrak{p})^{K} \\
\theta_2 \downarrow & & \downarrow \theta \\
(S\mathfrak{a})^{W_0} & \xrightarrow{} & (S\mathfrak{a})^{W_0}
\end{array}
\]

where $\theta_1$ is a restriction of the Reynolds operator to $S\mathfrak{a}$ and $\theta_2$ is a $W_0$-equivariant projection. Now we are ready to prove the following lemma:

Lemma 1.1. If $\theta$ is injective, then $\psi$ is surjective.
Proof. It is clear that both maps $\psi$ and $\theta$ are homogeneous (with respect to natural grading of $S\mathfrak{a}$ and $S\mathfrak{p}$). So the composition $\psi\theta$ is a homogeneous map of degree 0 from $(S\mathfrak{a})^{W_0}$ to $(S\mathfrak{a})^{W_0}$. Now it is clear that if $\theta$ is injective and $\ker\psi \cap \im\theta = 0$ then $\psi\theta$ is injective, hence surjective, so $\psi$ is surjective as well. Certainly the second assertion follows from the fact that $\psi$ is injective (and this is an easy part of Chevalley restriction theorem), but we are looking for a proof that could be applicable in the case of two summands.

First we could extend $B$ to the scalar product in $S\mathfrak{p}$ in the usual way. Since the restriction of $B$ on $\mathfrak{p}^R$ is positive definite, it is clear that the restriction of $B$ on $S\mathfrak{p}^R$ is also positive definite (more precisely, the restriction of $B$ on any finite-dimensional subspace of $S\mathfrak{p}^R$ is positive definite). In particular, if $V \subset S\mathfrak{p}$ is any subspace stable under complex conjugation, then $V \cap V^\perp = 0$. Take $V$ to be a minimal $\mathbb{K}$-submodule of $S\mathfrak{p}$ generated by $S\mathfrak{a}$. Since $S\mathfrak{a}$ is stable under complex conjugation, so is $V$, because $V$ is spanned by the vectors of the form $k \cdot f$, $k \in \mathbb{K}$, $f \in S\mathfrak{a}$, and since $\sigma$ commutes with complex conjugation on $\mathfrak{g}$, $\mathbb{K}$ is stable under complex conjugation (we are in the adjoint group!).

By definition of Reynolds operator, $\im\theta = V \cap (S\mathfrak{p})^\mathbb{K}$. By the construction of $B$ on $S\mathfrak{p}$, it is clear that $\ker\psi = (S\mathfrak{a})^\perp \cap (S\mathfrak{p})^\mathbb{K}$ and by $\mathbb{K}$-invariance of $B$ we have $\ker\psi = V^\perp \cap (S\mathfrak{p})^\mathbb{K}$. Since $V \cap V^\perp = 0$, we finally obtain $\ker\psi \cap \im\theta = 0$. □

So to prove the surjectivity of $\psi$ it remains to prove that $\theta$ is injective. This idea (as well as many others in this paper) belongs to Joseph, the only exception is that Joseph didn’t use Reynolds operator directly. Instead of that he defined the “Letzer map” purely in infinitesimal terms, but it is easy to see that in the adjoint situation Joseph’s “Letzer map” actually coincides with good old Reynolds operator!

Now it is time to slightly change our notations and to introduce new ones. First of all, since $S\mathfrak{p}$ is a $\mathbb{K}$-submodule of $S\mathfrak{g}$, injectivity of $\theta$ will follow from the injectivity of the restriction on $(S\mathfrak{a})^{W_0}$ of the Reynolds operator $S\mathfrak{g} \to (S\mathfrak{g})^\mathbb{K}$. Since the definition of the Reynolds operator depends only on the $\mathbb{K}$-module structure of $S\mathfrak{g}$, we can substitute for $S\mathfrak{g}$ the universal enveloping algebra $U\mathfrak{g}$, since $U\mathfrak{g}$ has the same $\mathbb{K}$-module structure by the Poincare-Birkhoff-Witt theorem (this $\mathbb{K}$-module isomorphism could be written down explicitly as a symmetrization map $S\mathfrak{g} \to U\mathfrak{g}$). Moreover, we have an isomorphism $S\mathfrak{a} \simeq U\mathfrak{a}$. Notice that the obvious embedding $U\mathfrak{a} \subset U\mathfrak{g}$ corresponds to an embedding $S\mathfrak{a} \subset S\mathfrak{g}$ via the symmetrization map. So from now on $\theta$ will denote the map $(U\mathfrak{a})^{W_0} \to (U\mathfrak{g})^\mathbb{K}$ induced by the Reynolds operator. We need to show that it is injective.

Fix a torus $S \subset G$ such that $\mathfrak{a} = \text{Lie} S$. Then $S$ is a maximal $\sigma$-anisotropic torus in the terminology of [Vu]. Fix a Borel subgroup $B$ and maximal torus $T \subset B$ such that $KB$ is dense in $G$ (so $K$ is spherical in $G$ and if $\mathfrak{b}$ is a Lie algebra of $B$ then $\mathfrak{t} + \mathfrak{b} = \mathfrak{g}$), $T$ contains $S$ (such $B$, $T$ exist by [Vu]). Then automatically $\sigma(T) = T$. Let $\mathfrak{h} = \text{Lie} T$ be a Cartan subalgebra of $\mathfrak{g}$, let $\mathfrak{h}_0 = \mathfrak{h} \cap \mathfrak{t}$, so $\mathfrak{h} = \mathfrak{h}_0 \oplus \mathfrak{a}$. Let $\mathfrak{X}^*(T)$ (resp. $\mathfrak{X}_*(T)$) be a character group (resp. the set of one-parameter subgroups of $T$). We use an additive notation for multiplication in $\mathfrak{X}^*(T)$ and will sometimes identify a character with its differential. Let $P \subset \mathfrak{X}^*(T)$ be the weight lattice, let $P_+ \subset P$ be the dominant weights. Fix the sublattice $Q \subset P$ such that $\lambda \in Q$ if and only if

$$\sigma(\lambda) = -\lambda, \quad \lambda|_S \in 2\mathfrak{X}^*(S).$$

It is clear that for any $\lambda \in Q$ we have $\lambda|_{\mathfrak{h}_0} = 0$, so we may view such $\lambda$ as an element $\lambda$ of $\mathfrak{a}^\ast$. Let $w \in W, \tilde{w} = w\lambda$. Then we can extend $\tilde{w}$ to an element $\mu$ of $\mathfrak{h}$. We will denote this by $w\lambda|_{\mathfrak{h}}$. If $\mu |_{\mathfrak{h}_0} = 0$, then $\mu$ is a dominant weight in $\mathfrak{h}$.
Then it is clear that actually \( \mu \in Q \). Moreover by the structure theory of a baby Weyl group (see eg. [Wa, §1.1]), there exists \( w \in W \) such that \( \mu = w\lambda \). Another useful tip is that if two elements of \( a^* \) are conjugate with respect to \( W \), then they also conjugate by an element of \( W_0 \) (this follows from the fact that any \( W_0 \)-fundamental chamber in \( a^* \) entirely lies in some \( W \)-fundamental chamber, see [loc. cit.]). Set \( Q_+ = Q \cap P_+ \).

**Lemma 1.2.** In the identifications above, \( Q = W_0Q_+ \).

**Proof.** This follows from the results of T. Vust (see [Vu]). Namely, passing to dual lattices, it is sufficient to prove that if \( R \in X_+(S) \) is any one-parameter subgroup then there exists \( w \in W_0 \) such that \( R^w = wRw^{-1} \) is dominant, that is, for any \( b \in B \) the limit \( \lim_{t \to 0} R^w(t)bR^w(t)^{-1} \) exists. Consider a subgroup

\[
P(R) = \{ g \in G \mid \lim_{t \to 0} R(t)gR(t)^{-1} \text{ exists} \}.
\]

Then \( P(R) \) is parabolic, moreover, \( P(R) \) and \( \sigma P(R) \) are opposite parabolics, so that \( P(R) \cap \sigma P = Z_G(R) \), the centralizer of \( R \) in \( G \). Such parabolics are called \( \sigma \)-anisotropic. Since \( Z_G(R) \) is a \( \sigma \)-invariant reductive subgroup and \( S \subset Z_G(R) \), there exists minimal \( \sigma \)-anisotropic parabolic subgroup \( P' \) of \( Z_G(R) \) such that \( P' \cap \sigma P = Z_{Z_G(R)}(S) \). Then \( P'' = P'R_u(P(R)) \) is a minimal \( \sigma \)-anisotropic parabolic subgroup in \( G \) and \( P'' \cap \sigma P'' = Z_{G}(S) \). Since \( Z_{G}(R) \subset P(R) \), we have \( P'' \subset P(R) \). Now, it is known that \( K^0SR_u(P'') \) is open in \( G \), where \( K^0 \) is an identity component of \( K \) (algebraic analogue of Iwasawa decomposition), so if \( B' \) is a Borel subgroup of \( Z_G(S) \) then \( B'' = B'R_u(P'') \) is a Borel subgroup in \( G \) that contains \( S \), is opposite to \( K \), and is contained in \( P(R) \). It remains to notice that all pairs \( (B, T) \) such that \( B \) is a Borel subgroup opposite to \( K \), \( T \subset B \) is a maximal torus that contains a maximal \( \sigma \)-anisotropic torus, are conjugate with respect to \( K \) (see [loc. cit.]). In particular, there exists \( \tilde{w} \in N_K(S) \) such that \( B = \tilde{w}B'' \). Since \( B'' \subset P(R) \), \( B \subset P(\tilde{w}R\tilde{w}^{-1}) \) and hence \( R^w = \tilde{w}R\tilde{w}^{-1} \) is dominant. \( \square \)

Returning to proof of Theorem, for any spherical subgroup \( H \) of \( G \) and any irreducible representation \( V_\lambda \) of \( G \) with highest weight \( \lambda \), it is known that \( \dim V_\lambda^H \leq 1 \) and the set

\[
\Gamma_H = \{ \lambda \in P_+ \mid \dim V_\lambda^H = 1 \}
\]

is a submonoid in \( P_+ \) (see eg. [Pa2]). In our case by [Vu], it is known that \( \Gamma_K = Q_+ \). Irreducible representations \( V_\lambda \) such that \( \lambda \in Q_+ \) are known as representations of class 1. Fix \( \lambda \in Q_+ \), for any \( v \in V_\lambda \) and \( \mu \in P \), let \( (v)_\mu \) denote a \( \mu \)-weight component of \( v \) and \( \text{Supp } v = \{ \mu \in P \mid (v)_\mu \neq 0 \} \). Let \( v_\lambda \) be a highest weight vector in \( V_\lambda \). Fix a \( K \)-invariant vector \( v_K \neq 0 \) in \( V_\lambda \).

**Lemma 1.3.**

(i) \( \text{Supp } v_K \subset Q \);

(ii) \( \lambda \in \text{Supp } v_K \);

(iii) \( \text{Supp } v_K \cap W\lambda = W_0\lambda \).

**Proof.** Fix \( \mu \in \text{Supp } v_K \). Since \( h_0v_K = 0 \) we have \( \mu(h_0) = 0 \), and so \( \sigma \mu = -\mu \). Moreover, if \( \tau \in S \) has order 2, then \( \tau \in K \), so \( \tau v_K = v_K \) and \( \mu \in 2X^*(S) \). This proves (i). To prove (ii) let us notice that if \( P^0 \) is a Borel subgroup, opposite to \( P \), then \( P^0 \cap P_+ \) is a parabolic subgroup and \( v_K \) is a highest weight vector in \( V_\lambda \). But then \( \text{Supp } v_K \cap W\lambda = W_0\lambda \). \( \square \)
(so that $B \cap B^0 = T$), then $KB^0$ is dense in $G$ (this follows from the description of $B$ as in proof of Lemma 1.2). Then $g = b^0 + \mathfrak{k}$, so $Ug = Ub^0U\mathfrak{k}$. Then

$$V_{\lambda} = (Ug)v_K = (Ub^0)v_K.$$ 

But for any $x \in Ub^0$, $\text{Supp } xv_K < \text{Supp } v_K$ (with respect to a usual order relation on $P$), so $\lambda \in \text{Supp } v_K$. The last assertion follows from (i), (ii), $N_K(S)$-invariance of $v_K$ and the discussion before Lemma 1.2. \hfill $\Box$

Now we can finish the proof of the theorem in the case of one summand. Suppose that $f \in \text{Ker } \theta$, $f \neq 0$. To obtain a contradiction, it is sufficient to prove that $f$ (viewed as an element of $Sa$) vanishes at $\bar{\lambda}$ for all $\lambda \in Q$. Since $f$ is $W_0$-invariant we need to prove that $f(\bar{\lambda}) = 0$ for all $\lambda \in Q_+$ (by Lemma 1.2). Arguing by induction on the standard order relation in $P_+$, we may assume that $f(\bar{\mu}) = 0$ for all $\mu \in Q_+$, $\mu < \lambda$. Suppose that $f(\bar{\lambda}) \neq 0$. Let $\pi$ denote the $K$-invariant projection $V_{\lambda} \rightarrow V_{\lambda}^K$. Let $F$ denote the operator on $V_{\lambda}$ of left multiplication by $f$ (viewed as an element of $Ug$). Then, clearly, $F(v_{\lambda}) = f(\bar{\mu})v_{\lambda}$. Since $\theta f = 0$ and since Reynolds operator commutes with $K$-module homomorphisms, we have $\pi F \pi = 0$ (because $\pi F \pi$ is clearly $K$-invariant, so Reynolds operator maps it onto itself, on the other hand, Reynolds operator commutes with left (and right) multiplication on $(K$-invariant!) element $\pi$). Hence $\pi Fv_K = 0$. By Lemma 1.3 and by induction we have

$$Fv_K = \rho f(\bar{\lambda}) \sum_{w \in W_0} \bar{w}v_{\lambda},$$

where $\bar{w}$ is some representative of $w$ in $N_K(S)$ and $\rho \neq 0$. Since $\pi$ is $K$-invariant, we get $\pi(v_{\lambda}) = 0$. Therefore $(U\mathfrak{k})v_{\lambda} \neq V_{\lambda}$. But

$$V_{\lambda} = (Ug)v_{\lambda} = (U(\mathfrak{k} + b))v_{\lambda} = (U\mathfrak{k})(Ub)v_{\lambda} = (U\mathfrak{k})v_{\lambda}.$$ 

Contradiction.

§2. Surjectivity of $\psi : C[p \times p]^K \rightarrow C[a \times a]^{W_0}$

First, we can identify $C[p \times p]$ with $S(p \times p)$, and $C[a \times a]$ with $S(a \times a)$ by means of Killing form. Then we can identify $S(a \times a)$ with $U(a \times a)$. Let $\theta$ denote a restriction on $U(a \times a)^{W_0}$ of a Reynolds operator $U(g \times g) \rightarrow U(g \times g)^K$ (with respect to diagonal action).

**Lemma 2.1.** If $\theta$ is injective then $\psi$ is surjective.

**Proof.** The proof is the same as of Lemma 1.1 and the discussion after it. \hfill $\square$

Now suppose that $f \in \text{Ker } \theta$. By Lemma 1.2 we know that $Q_+ \times Q_+$ is Zarisky dense in $C \otimes (Q_+ \times Q_+)$. Hence to show that $f = 0$ it is sufficient to show that $f$ (viewed as an element of $S(a \times a)$) vanishes at all $(\lambda, \mu) \in Q_+ \times Q_+$. We will proceed by a usual induction on the standard order relation. So suppose that $f$ vanishes at all $(\lambda', \mu') < (\lambda, \mu)$. Consider the $G \times G$ module $V_{\lambda} \otimes V_{\mu}$. Let $\pi$ denote a $K$-invariant projection onto $(V_{\lambda} \otimes V_{\mu})^K$ (with respect to a diagonal action). Let $F$ denote the operator of left multiplication by $f$ (viewed as an element of $U(g \times g)$). Since $\theta f = 0$, we get $\pi F \pi = 0$ as in the proof for a case of one summand. Let $v_{\lambda}^\prime$ (resp. $v_{\mu}^\prime$)
denote a non-zero $K$-invariant vector in $V_\lambda$ (resp. $V_\mu$). Then $\pi F(v_\lambda^K \otimes v_\mu^K) = 0$. Let $v_\lambda$ (resp. $v_\mu$) denote the highest weight vector of $V_\lambda$ (resp. $V_\mu$). By Lemma 1.3

$$v_\lambda^K \otimes v_\mu^K = \sum_{\omega' \in W_0/(W_0)_{\lambda}, \omega'' \in W_0/(W_0)_{\mu}} (\tilde{\omega}' v_\lambda) \otimes (\tilde{\omega}'' v_\mu) + U;$$

where $\text{Supp} \ U$ lies in $Q \times Q$ and for any $(\lambda', \mu') \in \text{Supp} \ U$ by induction we have $f(\lambda', \mu') = 0$. So we obtain that

$$\sum_{\omega' \in W_0/(W_0)_{\lambda}, \omega'' \in W_0/(W_0)_{\mu}} f(\omega', \omega')(\tilde{\omega}' v_\lambda) \otimes (\tilde{\omega}'' v_\mu)$$

lies in the kernel of $\pi$. Now for any $\omega' \in W_0/(W_0)_{\lambda}$, $\omega'' \in W_0/(W_0)_{\mu}$ we can find an element $(w_1(\omega', \omega''), w_2(\omega', \omega'')) \in W_0 \times W_0$ such that $w_1 = y \omega'$, $w_2 = y \omega''$ for some $y \in W_0$ and such that $w_1 \lambda + w_2 \mu \in Q_+$. Denote by $w_i^1$, $w_i^2$, $i = 1, \ldots, r$ the complete set of such pairs obtained for all $w_1 \in W_0/(W_0)_{\lambda}$, $w_2 \in W_0/(W_0)_{\mu}$. Now by $W_0$-invariance of $f$ and by $K$-invariance of $\pi$ we proved that an element

$$\sum_{i=1, \ldots, r} a_i f(w_i^1 \lambda, w_i^2 \mu)(\tilde{w}_1^1 v_\lambda) \otimes (\tilde{w}_2^i v_\mu)$$

lies in the kernel of $\pi$ for some non-zero integers $a_r$. It is sufficient to prove that all $f(w_i^1 \lambda, w_i^2 \mu)$ are equal to zero. So we reduced our proof to the following claim:

**Claim.** Suppose that an element

$$A = \sum_{i=1, \ldots, r} b_i (\tilde{w}_1^i v_\lambda) \otimes (\tilde{w}_2^i v_\mu)$$

lies in the kernel of $\pi$. Then all $b_i$ are equal to zero.

Suppose that some $b_i \neq 0$. By Kostant’s refinement of PRV conjecture (see [J] or [Ku1]) for any $i$ the module $(U \mathfrak{g}) \left[ \tilde{w}_1^i v_\lambda \right] \otimes (\tilde{w}_2^i v_\mu)$ contains a unique $U \mathfrak{g}$-submodule of highest weight $w_1^i \lambda + w_2^i \mu$. Denote it highest weight vector by $v_i$. By Kumar’s refinement of PRV conjecture (see [J] or [Ku2]), these $U \mathfrak{g}$ submodules form a direct sum $S = \bigoplus_{i=1}^{r} S_i$. Set

$$S' = \bigoplus_{1 \leq i \leq r} S_i.$$

By Density theorem it follows that the module $(U \mathfrak{g})A$ contains $S'$. Let $\pi_0$ denote a $G$-invariant projection $(U \mathfrak{g})A \rightarrow S$. Take $i$ such that the weight $w_1^i \lambda + w_2^i \mu$ is minimal. Let $\pi_1$ denote a $G$-invariant projection $S' \rightarrow S_i$. Then it is clear from the above and by checking the weights that

$$\pi_1 \pi_0(A) = cv^i,$$

where $c \neq 0$. Denote by $\pi_2$ a $K$-invariant projection $S_i \rightarrow S_i^K$. Then arguing as in proof for one summand we see that $\pi_2 \pi_1 \pi_0(A) \neq 0$. But then $\pi(A) \neq 0$ as well, because $\pi_2 \pi_1 \pi_0$ is $K$-invariant and hence factors through $\pi|_{(U \mathfrak{g})A}$. This completes the proof.
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