Origin of strong scarring of wavefunctions in quantum wells in a tilted magnetic field

E. E. Narimanov and A. Douglas Stone

Applied Physics, Yale University, P.O. Box 208284, New Haven CT 06520-8284
(April 21, 2018)

The anomalously strong scarring of wavefunctions found in numerical studies of quantum wells in a tilted magnetic field is shown to be due to special properties of the classical dynamics of this system. A certain subset of periodic orbits are identified which are nearly stable over a very large interval of variation of the classical dynamics; only this subset are found to exhibit strong scarring. Semiclassical arguments shed further light on why these orbits dominate the experimentally observed tunneling spectra.

The localization of certain quantum wavefunctions in real-space along unstable classical periodic orbits illustrates how quantum mechanics can violate the ergodic behavior expected from classical mechanics. Such wavefunctions are conventionally termed “scars” and their properties have been extensively studied by theorists of quantum chaos during the past decade. Recently, an experimental system has been discovered and studied, in which such scarred wavefunctions control to a large extent an observable physical property, the tunneling current through a double-barrier GaAs-AlGaAs heterostructure (“quantum well”) under high bias. When a magnetic field is applied at an angle θ with respect to the normal to the barriers (the electric field direction), the resulting dynamics makes a transition to chaos as θ is increased from zero. Calculations from Fromhold et al. on the system found many more scars than in any previously-studied quantum-chaotic Hamiltonian, and that these scarred wavefunctions carried most of the tunnel current when the system was resonant. In the initial experiments the level-broadening (due to optic phonon emission) was too large to observe the resonances due to individual levels. However in a later experiment this was done, albeit at such low quantum numbers that the concept of scarring becomes less meaningful. From extensive numerical work we know: 1) Quantum states scarred by the same periodic orbit appear over a wide range of variation of the classical dynamics, in contrast to typical systems (e.g. billiards). 2) The scars arise from only a few of the many unstable short periodic orbits in the tilted well. 3) These scars carry most of the resonant tunneling current for θ > 15°. In this Letter we will present a theory to explain why only certain orbits scar the wavefunctions and why these scars persist as the classical dynamics changes substantially. The theory also sheds further light on why these scarred states dominate the tunneling current.

First, we recall why in typical chaotic systems scarred states are relatively rare. Generally scarred states are associated with orbits which are not too unstable; a sufficient condition for strong scarring is that λT ≤ 1, where λ is the largest instability (Lyapunov) exponent associated with the orbit. Typically in chaotic Hamiltonian systems, unstable periodic orbits appear as marginally stable orbits at bifurcations and become monotonically more unstable as the classical parameter (e.g. energy) driving the system to chaos is increased. Therefore such orbits only scar over the small interval of classical parameter space when they are close to stability. In a recent detailed study of the classical dynamics of the tilted well, we have shown that in this system a subset of the short periodic orbits behave completely differently. They exist only for a finite interval as the classical parameters (energy ε, magnetic field, B and electric field E) are varied and are “pinned” near marginal stability for their entire interval of existence. Therefore these orbits are responsible for the strong scarring seen, and indeed all the scars found numerically correspond to this subset of the short orbits. We now present this argument in detail.

We model the system by two infinite potential barriers corresponding to the x−y planes at z = −d (the emitter) and z = 0 (the collector), with an electric field E = Ez, and a “tilted” magnetic field B = B cos θz + B sin θy for −d < z < 0. The periodic orbit theory for this model is rather involved and we only sketch the most salient features here. First, the classical Hamiltonian can be rescaled so that the dynamics only depends on two dimensionless parameters: β = 2Bv0/E and γ = ε/eV where V is the voltage across the well and v0 = (2ε/m*)1/2 is the velocity corresponding to the total injection energy. In the experiments γ ≈ 1.17 is constant to a good approximation and so β, the scaled magnetic field, is the single relevant variable. Our semiclassical analysis expresses the tunneling current in terms of periodic orbits in the well which connect the emitter and collector barriers (we term these “emitter” orbits, those which don’t reach the emitter “collector” orbits). We focus on the experimentally relevant situation in which γ is only slightly greater than unity, so that many of the periodic orbits are collector orbits and may be neglected. Optic phonon emission produces a temporal cut-off which allows only short periodic orbits to produce structure in the tunneling spectra.

Emitter orbits have the following properties. An orbit which collides with the collector n times (period-n
in the collector Poincaré map) can collide with the emitter $m$ times, where $m = 1, 2, \ldots, n$. Hence we denote the orbits as $(m, n)$. All emitter orbits (except for a single $(1, 1)$ “traversing” orbit) only exist above a threshold value of $\beta = \beta_{c1}$ and cease to exist at a higher value of $\beta = \beta_{c2}$ by an inverse bifurcation (usually a tangent bifurcation (TB)). While the “death” of the orbits follow the generic rules of hamiltonian bifurcation theory [[14]], all relevant orbits are “born” in a new kind of bifurcation, which we refer to as a cusp bifurcation (CB). CB's have non-generic properties since they appear on a closed curve in the surface of section (SOS), where the Poincaré map describing the dynamics is nonanalytic. This “critical” curve separates initial conditions at the collector barrier which will reach the emitter on the next try from those which will not. Hence orbits originating just within the curve will receive a “kick” at the emitter, while those just outside will not. This leads to a discontinuity in the stability matrix $M$ of any periodic orbit corresponding to a fixed point which crosses the boundary. In particular, all CB’s occur by the simultaneous appearance of two new orbits, which infinitesimally above threshold differ by one point of contact with the emitter (this may correspond to either one or two fewer collisions). We have shown [[15]] that the orbit which reaches the emitter more times has diverging stability ($|TrM| \to \infty$) at $\beta_{c2}$ while the other orbit in the pair can be either stable ($|TrM| < 2$) or unstable ($|TrM| > 2$) at $\beta_{c1}$. The latter case, in which two unstable orbits appear simultaneously is forbidden for generic hamiltonian systems [[14]]. Continuity arguments and numerical results [[15]] imply that the partner in a cusp bifurcation with fewer emitter collisions will have $|TrM| \sim 2$, i.e. will be born near marginal stability. Hence such orbits, being born near marginal stability and being required to return to marginal stability when they disappear in an inverse tangent bifurcation, remain only slightly unstable for their entire interval of existence, unlike unstable periodic orbits of typical chaotic systems. It is this subset of the orbits which scar strongly the quantum states over a large variation of the classical parameter $\beta$.

Bifurcation and stability diagrams illustrating this behavior are shown in Fig. 1 for the case of period-two and period-three orbits. Among the period-2 orbits the orbit denoted (1, 2) fits our criteria. At $\theta = 28^\circ$ it is born in a CB with the higher connectivity orbit $(2, 2)^+$ at $\beta \approx 4.0$ and dies in an inverse TB with the orbit $(0, 2)^-$ at $\beta \approx 7.0$. This period-two orbit and another topologically similar orbit (not shown) which appears at slightly higher value of $\beta$ account very well [[14]] for the peak-doubling regions observed at $\theta = 28^\circ$ in the experiments of Muller et al. [[1]]. This orbit was found to scar many wavefunctions in the work of Fromhold et al. [[1]]. It has a complicated evolution above $\theta = 29^\circ$, involving an “exchange” bifurcation with the topologically similar orbit just mentioned, the details of which are given elsewhere [[15]]. Note however that in roughly the same $\beta$ interval there are two other period-two emitter orbits, each with rapidly varying stability. Both born in CBs paired with an orbit with fewer collisions with the emitter, hence by our above reasoning are initially enormously unstable. Therefore they do not generate strong scars in the spectrum.

A similar story holds for one of the eight period-three orbits which appear around $\beta = 3.5$ at $\theta = 38^\circ$, the orbit we denote $(1, 3)^-$ (Fig. 1). This orbit has been discussed previously [[17],[16],[19]] in connection with the observability of trifurcations in the data of ref. [3]. It is born in a CB as the partner of a $(3, 3)$ orbit, remains near marginal stability for $3.2 < \beta < 4.4$ and dies in a TB with the $(1, 3)^+$. In the same interval there are several other unstable period-three emitter orbits which do not scar strongly. Finally, by the same reasoning we have found a $(1, 5)$ orbit which scars strongly. Quantum states scarred by each of these orbits are shown in Fig. 2.

Note that by our criteria the scarred orbits must always be $(m, n)$ orbits with $m < n$; e.g. $(1, 2)$ can scar strongly whereas $(2, 2)$ should not. On the other hand, it is easily shown [[15]] that as $\theta \to 0$ the only emitter orbits are of the type $(n, n)$. Therefore the interval of existence of the scarred orbits is small for small $\theta$. Thus, for example, the period-three scarred states are unimportant for $\theta < 20^\circ$.

We have tested this argument quantitatively by analyzing the quantum states of the tilted well for scars of the three orbits shown in Fig. 2. By generating many spectra at different values of $B, E$ we can search in the experimentally appropriate intervals of $\beta$ with $1.1 < \gamma < 1.2$ and systematically detect these scars. In Fig. 3 we plot a measure of scar strength versus action of the scarring orbit. As noted before [[13],[18]], the energies $\epsilon_n$ of scarred states satisfy an approximate Bohr-Sommerfeld quantization rule, $5(\epsilon_n) = (n + \phi)2\pi\hbar$, so we expect and find a strong periodic modulation which turns on at the tangent bifurcation at which the orbit appears [[20]].

Finally, we comment on the fact that these scarred states tend to dominate the tunneling current at large tilt angles. All of the orbits studied here and elsewhere [[15],[12]] which scar strongly have only a single collision with the emitter barrier. Since both emitter and collector surfaces of section must be symmetric when $v_y \to -v_y$, such orbits must have $v_y = 0$ at the emitter barrier [[22]]. Thus these orbits have unusually low transverse velocity at the emitter barrier compared to other orbits with the same periodicity in the collector Poincaré map. Since the emitter wavefunction is primarily a superposition of the first few Landau levels, the source of tunneling current has low transverse velocity and couples very well to these scarred states. Therefore it is due to this specific feature of the emitter state that these scarred well states dominate the tunneling current; if the emitter state had large transverse momentum these states would be anti-
selected. Since scarred states are localized in real and phase space, they can lead to quasi-selection rules for tunneling, but only if they are localized in the correct regions to couple well to the input state.

We acknowledge helpful conversations with Greg Boebinger, Mark Fromhold and Tania Monteiro and the support of NSF grant DMR-9215065.

[1] E. J. Heller, Phys. Rev. Lett 53, 1515 - 1518 (1984).
[2] E. B. Bogomolny, Physica D 31, 169 - 189 (1988).
[3] M. V. Berry, Proc. roy. Soc. Lond. A 423, 219 - 231 (1989).
[4] S. Tomsovic and E. J. Heler, Phys. Rev. E 47, 282 (1993).
[5] T. M. Fromhold et al., Phys. Rev. Lett. 72, 2608 (1994).
[6] G. Muller et al., Phys. Rev. Lett. 75, 2875 (1995); G. S. Boebinger et al., Proc. EP2DS-IX, Surf. Sci.
[7] D. L. Shepelyansky and A. D. Stone, Phys. Rev. Lett. 74, 2098 (1995)
[8] T. M. Fromhold et al, Phys. Rev. Lett., 75, 1142 (1995).
[9] P. B. Wilkinson et al. Nature, 380, 608 (1996); Surface-Science, 361 - 362, 696 (1996);
[10] T. S. Monteiro and P. Dando, Phys. Rev. E 53, 3369 (1996)
[11] T. S. Monteiro et al, to be published in Phys. Rev. E.
[12] T. S. Monteiro, D. Delande and J. - P. Connerade, Comment, to be published in Nature.
[13] M.C. Gutzwiller Chaos in Classical and Quantum Mechanics (Springer-Verlag, New York, 1990)
[14] K. R. Meyer, Trans. Am. Math. Soc. 149, 95 (170).
[15] E. E. Narimanov and A. D. Stone, cond. matt./9704083; and unpublished.
[16] T. M. Fromhold et al, Phys. Rev. Lett 78, 2865(C) (1997); G. S. Boebinger, E. E. Narimanov and A. D. Stone, Phys. Rev. Lett. 78, 2866(C) (1997).
[17] T. M. Fromhold et al, Phys. Rev. B, 51, 18029 (1995).
[18] T. M. Fromhold et al, to be published in Chaos.
[19] O. Agam and S. Fishman, Phys. Rev. Lett. 73, 806 (1994).
[20] Actually the scar strength turns on before the tangent bifurcation signalling the birth of the orbit due to the existence of “ghost” orbits [21,10,11]; note that when increasing action (or energy) the tangent bifurcation at higher $\beta$ occurs at the lower action side of Fig. 3.
[21] A. M. Ozorio de Almeida and J. H. Hannay, J. Phys. A 20, 5873 (1987).
[22] A subtle exception to this argument are orbits which have only one collision with the emitter, but generate two fixed points in the emitter SOS, corresponding to the two different senses of traversing the orbit. Such a (1,2) orbit, considered in ref. [3], is created by a pitchfork bifurcation of the (1,2) orbit considered here, but it still has very small transverse velocity when it is near stability.
FIG. 1. Bifurcation diagrams for the relevant period-two (a) and period-three (c) orbits. The horizontal axis is the scaled magnetic field, $\beta$, the vertical axis is the $v_x$ coordinate of the fixed point(s) in the collector SOS indicated schematically in the insets. (b) and (d) are plots of the trace of the monodromy (stability) matrix for the corresponding orbits; Shaded area ($|\text{Tr}[M]| < 2$) denotes stability region. Orbits denoted $(1,2)_2$, $(1,3)^-$ (see Fig. 2) remain slightly unstable over a large variation of $\beta$, leading to strong scarring.

FIG. 2. Examples of wavefunctions scarred by the unstable $(1,2)_2$ orbit (a), $(1,3)^-$ orbit (b) and a $(1,5)$ orbit (c); $y$-$z$ projections of orbits are superimposed.
“Scar strength” (Husimi function at the location of the fixed point of the periodic orbit at the emitter barrier, calculated for the normal derivative of the wavefunction) vs. the scaled action $S(\varepsilon_n)/\hbar$ of the corresponding orbit ($\varepsilon_n$ is the energy of the corresponding eigenstate) for: (a) $(1,2)_2$ orbit, (b) $(1,3)^-\text{orbit}$, (c) $(1,5)$ orbit. The arrows indicate the values of $\beta$ for the tangent bifurcation, which give birth to the periodic orbits, the peaks of the scar strength below these values are due to the “ghost effect” [21,22]. Scaled actions below the bifurcation points were obtained by linear extrapolation of the (approximately linear) function $S(\varepsilon)/\hbar$. 

FIG. 3.