Not all Kripke models of HA are locally PA

Erfan Khaniki*
Institute of Mathematics
Czech Academy of Sciences
Prague, Czech Republic

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To Mohammad Ardeshir

Abstract

Let $K$ be an arbitrary Kripke model of Heyting Arithmetic, HA. For every node $k$ in $K$, we can view the classical structure of $k$, $M_k$ as a model of some classical theory of arithmetic. Let $T$ be a classical theory in the language of arithmetic. We say $K$ is locally $T$, iff for every $k$ in $K$, $M_k \models T$. One of the most important problems in the model theory of HA is the following question: Is every Kripke model of HA locally PA? We answer this question negatively. We provide two new Kripke model constructions for this matter. The first one is a direct construction of a Kripke model $K \models HA + ECT_0$ ($ECT_0$ stands for Extended Church Thesis) with the root $r$ such that $M_r \not\models I\Delta_1$ and hence $K$ is not even locally $I\Delta_1$. Not only the existence of this model completely solves the problem, but this result is also almost tight in terms of the power of induction axioms that can be failed in $M_k$ because every node of a Kripke model of HA classically satisfies induction for formulas that are provably $\Delta_1$ in PA. The second Kripke model construction is an implicit way of doing the first construction which works for any reasonable consistent intuitionistic arithmetical theory $T$ with a recursively enumerable set of axioms that has Existence property. From the second construction we construct a Kripke model $K \models HA + \neg \theta + MP$ ($\theta$ is an instance of $ECT_0$ and MP is Markov principle) with the root $r$ such that $M_r \not\models I\Delta_1$. Also, we will prove that every countable Kripke model of intuitionistic first-order logic can be transformed into another Kripke model with the full infinite binary tree as the Kripke frame such that both Kripke models force the same sentences. So with the previous result, there is a binary Kripke model $K$ of HA such that $K$ is not locally PA.

1 Introduction

Heyting Arithmetic (HA) is the intuitionistic counterpart of Peano Arithmetic (PA). HA has the same non-logical axioms as PA with intuitionistic first-order logic as the underlying logic. This theory is one of the well-known and most studied theories of constructive mathematics, and it was investigated in many proof-theoretic and model-theoretic aspects in the literature (see [17] for more information). This paper aims to answer a question about the model theory of HA. Let $T$ be a classical theory in the language of arithmetic. A Kripke model of HA is called locally $T$, iff for every node $k \in K$, the classical structure associated with $k$, which we denote it by $M_k$, is a model of $T$.

One of the most important problems in the model theory of HA is the following question:

Problem 1.1 Is every Kripke model of HA locally PA?

*e.khaniki@gmail.com
This problem was first asked and investigated in the seminal paper [18] by van Dalen et al. in 1986. They proved that every finite Kripke model of HA is locally PA. Furthermore, they proved that a Kripke model of HA with the Kripke frame \((\omega, \leq)\) as the underlying frame has infinitely many locally PA nodes. This work initiated a research line around Problem [11] and also about the following general question:

**Problem 1.2** For a Kripke model \(K\) of the theory \(T\) in a language \(\sigma\), and a node \(k \in K\), what is the relationship between the sentences forced in \(k\) and the sentences satisfied in \(M_k\)?

There are several works around these problems. We will review those works in the following. Wehmeier in [19], investigated Problem [11] and extended the results of [18] to a larger class of frames. In particular, he proved that every Kripke model of HA with \((\omega, \leq)\) as the Kripke frame is indeed locally PA. Moniri, in [12], considered these problems and proved that every once-branching Kripke model of HA + MP (Markov Principle) is locally PA. Ardeshr and Hesaam in [4] generalized results of [19] to rooted narrow Kripke models of HA. Recently, Mojtahedi in [11] considered Problem [1.2] and answered this problem in the case of finite depth Kripke models. As a by-product, he generalized the result of [4] to semi-narrow Kripke models of HA.

Regarding Problem [1.1] the strongest positive result about the strength of induction axioms that are satisfied in a node of a Kripke model of HA was proved by Marković in [11]. He proved that every node of a Kripke model of HA satisfies induction for formulas that are provably \(\Delta_1\) in PA. Also, from \(\Pi^1_2\) conservativity of PA over HA, we know that every Kripke model of HA is locally \(\text{Th}_{\Pi^1_2}(PA)\).

Buss did another direction of research around these problems in [8]. For every language \(\sigma\) and every classical theory \(T\) in it, he characterized the sentences that are true in every locally \(T\) Kripke model. As a result, he proved that HA is complete with respect to the locally PA Kripke models. In a similar direction, Ardestir et al. in [7] presented a set of axiom systems for the class of end-extension Kripke models. As a by-product, they proved HA is strongly complete for its class of end-extension Kripke models.

The above results are probably all results relevant to Problem [1.1] in the literature. There are some other papers such as [1, 2] that investigated Problem [1.2] in general and partially answered this question, but their results are valid for every Kripke model; hence we cannot get much information from them for Problem [1.1].

In this paper, we will present two new model construction to answer Problems [11] and [12]. The main technical theorem of the first construction says that the theory \(HA + ECT_0 + \text{Diag}(\mathcal{M})\) for every \(\mathcal{M} \models \text{Th}_{\Pi^1_2}(PA)\) has Existence and Disjunction property (Theorem 3.8). This theorem provides the right tool for constructing rooted Kripke models of HA with control over the structure of the root (Theorem 3.8). We will construct a Kripke model of HA + ECT_0 that is not even locally \(I\Delta_1\). This answers Problem [1.2] negatively. The second construction is an implicit way of doing the first construction and it works for any reasonable consistent intuitionistic arithmetical theory with a recursively enumerable set of axioms that had Existence property (Theorem 3.15). As an application of the second construction, we will construct a Kripke model of HA + \(\neg \theta + MP\) where \(\theta\) is an instance of ECT_0 and MP is Markov principle that is not locally \(I\Delta_1\). The second construction is general and it also works for HA + ECT_0, but some Kripke models can be constructed for HA + ECT_0 with the first construction, but it is not possible with the second one. We will discuss this matter in more detail at the end of Section 3. The new model constructions imply the existence of a large class of Kripke models of reasonable intuitionistic arithmetical theories including HA, which cannot be constructed by the previous methods, so we think regardless of their application to Problem [1.1] these model constructions are also interesting in their own rights. We will also prove that every countable Kripke model of intuitionistic first-order logic can be transformed into another Kripke model with the full infinite binary tree as the Kripke frame (Lemma 3.11). Using this result, we will prove that there exists a Kripke model of HA with the full infinite binary tree as the Kripke frame that is not locally \(I\Delta_0\) (Corollary 3.13).
2 Preliminaries

2.1 Arithmetical Theories

Let \( L \) be the language of Primitive Recursive Arithmetic in which it has a function symbol for every primitive recursive function. \( \text{HA} \) is the intuitionistic theory with the following non-logical axioms:

1. Axioms of Robinson Arithmetic \( \mathbf{Q} \).
2. Axioms defining the primitive recursive functions.
3. For each formula \( \phi(x, \bar{y}) \in L \), the axiom \( \forall \bar{y} I_{\phi} \) in which
   \[
   I_{\phi} := \phi(\bar{0}) \land \forall x(\phi(x) \to \phi(Sx)) \to \forall x \phi(x).
   \]

\( \text{PA} \) is the classical theory that has the same non-logical axioms as \( \text{HA} \). \( \text{iPRA} \) (intuitionistic Primitive Recursive Arithmetic) has axioms of \( \mathbf{Q} \) and induction for every atomic formula of \( L \). The underlying logic of \( \text{iPRA} \) is intuitionistic logic. \( \text{PRA} \) is the classical counterpart of \( \text{iPRA} \).

\( T \vdash_c \phi \) means that there exists a proof of \( \phi \) from axioms of \( T \) using first-order classical logic Hilbert system. \( \vdash_i \) denotes the same thing for intuitionistic proofs. An important set of intuitionistic arithmetical theories for the purpose of this paper is defined in the following definition.

**Definition 2.1** \( I \) is the set of all intuitionistic arithmetical theories \( T \) in \( L \) such that:

1. \( T \) is consistent.
2. \( \text{iPRA} \subseteq T \).
3. The set of axioms of \( T \) is recursively enumerable.

Note that with the power of primitive recursive functions we can define finite sequences of numbers, so we can code finite objects such as formulas, proofs, and etc. as numbers. This is a standard technique and it is called Gödel numbering (see [16]). With the help of this coding we can talk about proofs of theories in arithmetical theories (see [16]). For every \( L \) sentence \( \phi \), \( \Gamma \phi \) denotes the number associated with \( \phi \). If \( \phi(x) \) is an \( L \) formula, then \( \Gamma \phi(c) \) denotes the number associated with \( \psi(x) \) when we substitute the numeral with value \( c \) for \( x \). Suppose \( T \in I \). Let \( \text{Axiom}(x, y) \) be the primitive recursive function such that for every \( L \) sentence \( \phi \), \( \phi \) is a \( T \)-axiom iff \( \exists x \text{Axiom}(x, \Gamma \phi) \) is true. Then it is possible to define the provability predicate of \( T \), \( \text{Proof}_T(x, y) \) as a primitive recursive function as follows. Let \( \langle \cdot \rangle \) be a natural primitive recursive coding function. Then \( \text{Proof}_T(x, y) \) is true iff there exist two sequences \( L \) sentences \( \{ \phi_i \}_{i \leq n} \) and numbers \( \{ w_i \}_{i \leq n} \) for some \( n \) such that:

1. \( x = \langle \langle w_1, \Gamma \phi_1 \rangle, ..., \langle w_n, \Gamma \phi_n \rangle \rangle \).
2. For every \( i \leq n \):
   
   (a) If \( w_i > 0 \), then \( \text{Axiom}(w_i - 1, \Gamma \phi_i) \) is true.
   
   (b) If \( w_i = 0 \), then \( \phi_i \) can be derived from \( \{ \phi_j \}_{j < i} \) by one of the rules of Natural deduction system for intuitionistic first-order logic.
3. \( y = \Gamma \phi_n \).

The \( \Sigma_1 \) formula \( \text{Pr}_T(y) \) is the abbreviation for \( \exists x \text{Proof}(x, y) \). So consistency of \( T \), \( \text{Con}(T) \), is \( \neg \text{Pr}_T(\Gamma \bot) \). The following theorem states the useful facts about \( \text{Pr}_T \).
Theorem 2.2 For every $T \in \mathcal{I}$ the following statements are true:

1. For every $L$ sentence $\phi$, if $T \vdash \phi$, then $\text{PRA} \vdash_c \Pr_T(\neg \phi)$.
2. $\text{PRA} \vdash_c \forall x, y(\Pr_T(x) \land \Pr_T(x \rightarrow y) \rightarrow \Pr_T(y))$.
3. $\text{PRA} \vdash_c \forall x, y(\Pr_T(x) \land \Pr_T(y) \rightarrow \Pr_T(x \land y))$.
4. For every $L$ formula $\phi(x)$ with $x$ as the only free variable, $\text{PRA} \vdash_c \Pr_T(\forall x\phi(x)) \rightarrow \forall x\Pr_T(\neg \phi(x))$.
5. For every $\Sigma_1$ formula $\phi(\vec{x})$, $\text{PRA} \vdash_c \forall \vec{x}(\phi(\vec{x}) \rightarrow \Pr_T(\neg \phi(x_1, \ldots, x_n)))$.

Proof. See [10] for a detailed discussion of these statements. \&

2.2 Realizability

For proving the first model construction theorem, we need some definitions and theorems about Kleene’s realizability.

Definition 2.3 Let $T(x, y, z)$ be the primitive recursive function called Kleene’s $T$-predicate and $U(x)$ be the primitive recursive function called result-extracting function. Note that $HA \vdash_c \forall x, y, z, z'(T(x, y, z) = 0 \land T(x, y, z') = 0 \rightarrow z = z')$.

We use $T(x, y, z)$ instead of $T(x, y, z) = 0$ for simplicity. For more information, see section 7 of the third chapter of [17].

Let $j_1(x)$ and $j_2(x)$ be the primitive recursive projections of the pairing function $j(x, y) = 2^x \cdot (2y + 1) \div 1$. Kleene’s realizability is defined as follows.

Definition 2.4 $x \mathcal{R} \phi \ (x \text{ realizes } \phi)$ is defined by induction on the complexity of $\phi$ where $x \notin FV(\phi)$.

1. $x \mathcal{R} p := p$ for atomic $p$,
2. $x \mathcal{R} (\psi \land \eta) := j_1(x) \mathcal{R} \psi \land j_2(x) \mathcal{R} \eta$,
3. $x \mathcal{R} (\psi \lor \eta) := (j_1(x) = 0 \land j_2(x) \mathcal{R} \psi) \lor (j_1(x) \neq 0 \land j_2(x) \mathcal{R} \eta)$,
4. $x \mathcal{R} (\psi \rightarrow \eta) := \forall y(y \mathcal{R} \psi \rightarrow \exists u(T(x, y, u) \land U(u) \mathcal{R} \eta), u \notin FV(\eta))$,
5. $x \mathcal{R} \exists \psi(y) := j_2(x) \mathcal{R} \psi(j_1(x))$,
6. $x \mathcal{R} \forall \psi(y) := \forall y \exists u(T(x, y, u) \land U(u) \mathcal{R} \psi(y)), u \notin FV(\psi)$.

Definition 2.5 A formula $\phi \in L$ is almost negative iff $\phi$ does not contain $\lor$, and $\exists$ only immediately in front of atomic formulas.

Definition 2.6 The extended Church’s thesis is the following schema, where $\phi$ is almost negative:

$$\text{ECT}_0 := \forall x(\phi(x) \rightarrow \exists y \psi(x, y)) \rightarrow \exists z \forall x(\phi(x) \rightarrow \exists u(T(z, x, u) \land \psi(x, U(u))))$$

Next theorem explains the relationships between $HA$, $\text{ECT}_0$ and Kleene’s realizability.

Theorem 2.7 For every formula $\phi \in L$:

1. $HA + \text{ECT}_0 \vdash_i \phi \iff \exists x(x \mathcal{R} \phi)$,
2. $HA + \text{ECT}_0 \vdash_i \phi \iff HA \vdash_i \exists x(x \mathcal{R} \phi)$.
Another important properties of HA are the Existence and Disjunction properties. We will use notation $\overline{n}$ as the syntactic term corresponds to natural number $n$.

**Theorem 2.8** The following statements are true:

1. **Disjunction property**: For every sentences $\phi, \psi \in L$, if $HA \vdash i \phi \lor \psi$, then $HA \vdash i \phi$ or $HA \vdash i \psi$,

2. **Existence property**: For every sentence $\exists x \phi(x) \in L$, if $HA \vdash i \exists x \phi(x)$, then there exists a natural number $n$ such that $HA \vdash i \phi(\overline{n})$.

**Proof.** See Theorem 5.10 of the third chapter of [17]. ⊣

Although HA is an intuitionistic theory, it can prove decidability of some restricted class of formulas. The next theorem explains this fact.

**Theorem 2.9** For every quantifier free formula $\phi \in L$, $HA \vdash i \phi \lor \neg \phi$.

**Proof.** See [17]. ⊣

### 2.3 Kripke models

A Kripke model for a language $\sigma$ is a triple $K = (K, \leq, M)$ such that:

1. $(K, \leq)$ is a nonempty partial order.

2. For every $k \in K$, $M_k \in M$ is a classical structure in the language $\sigma(M_k) = \sigma \cup \{ c \mid c \in M_k \}$.

3. For every $k, k' \in K$, if $k \leq k'$, then $\sigma(M_k) \subseteq \sigma(M_{k'})$ and also $M_{k'} \models \text{Diag}^+(M_k)$ ($M_k$ is a sub-structure of $M_{k'}$).

For every Kripke model $K$, there is a uniquely inductively defined relation $\models K \times (\bigcup_{k \in K} \sigma(M_k))$ that is called forcing.

**Definition 2.10** For every $k \in K$, and every sentence $\phi \in \sigma(M_k)$, the relation $k \models \phi$ is defined by induction on complexity of $\phi$:

1. $k \models p$ iff $M_k \models p$, for atomic $p$,

2. $k \models \psi \land \eta$ iff $k \models \psi$ and $k \models \eta$,

3. $k \models \psi \lor \eta$ iff $k \models \psi$ or $k \models \eta$,

4. $k \models \neg \psi$ iff for no $k' \geq k$, $k' \models \psi$,

5. $k \models \psi \to \eta$ iff for every $k' \geq k$, if $k' \models \psi$, then $k' \models \eta$,

6. $k \models \exists x \phi(x)$ iff there exists $c \in \sigma(M_k)$ such that $k \models \phi(c)$,

7. $k \models \forall x \phi(x)$ iff for every $k' \geq k$ and every $c \in \sigma(M_{k'})$, $k' \models \phi(c)$.

We use the notation $K \models \phi$ ($\phi \in \bigcap_{k \in K} \sigma(M_k)$ is a sentence) as an abbreviation that for every $k \in K$, $k \models \phi$ which simply means that the Kripke model $K$ forces $\phi$. The important property of the forcing relation is its monotonicity. This means that for every $k' \geq k$ and every $\phi \in \sigma(M_k)$, if $k \models \phi$, then $k' \models \phi$. Also, note that first-order intuitionistic logic is sound and has strong completeness with respect to the Kripke models. For more details see [17].
3 Kripke model constructions for intuitionistic arithmetical theories

3.1 The first model construction

We will explain the first model construction in this subsection. This construction will be presented in a sequence of lemmas and theorems.

Lemma 3.1 For every quantifier-free formula \( \phi \in \mathcal{L} \) there exists an atomic formula \( p \in \mathcal{L} \) with the same free variables such that \( \mathsf{HA} \vdash \phi \leftrightarrow p \).

Proof. By induction on the complexity of \( \phi \) and using Theorem 3.3.

Lemma 3.2 Let \( \langle \cdot \rangle \) and \( \langle \cdot \rangle_x \) be a primitive recursive coding and decoding functions, then for every formula \( Qx_1,\ldots,x_n \phi(x,y) \in \mathcal{L} \) where \( Q \in \{ \forall, \exists \} \) and \( n > 0 \),

\[ \mathsf{HA} \vdash Qx_1,\ldots,x_n \phi(x,y) \iff Qx \phi((x)_0,\ldots,(x)_n,y). \]

Proof. Straightforward by properties of the coding and decoding functions.

Theorem 3.3 For every \( \Pi_2 \) sentence \( \phi := \forall x \exists y \psi(x,y) \), if \( \mathsf{HA} + \mathsf{ECT}_0 \vdash \phi \), then \( \mathsf{PA} \vdash_c \phi \).

Proof. Let \( \phi \) be a \( \Pi_2 \) sentence and \( \mathsf{HA} + \mathsf{ECT}_0 \vdash \phi \). By Lemmas 3.2 and 3.1 there exists an atomic formula \( p(x,y) \) such that \( \mathsf{HA} \vdash \phi \iff \forall x \exists y p(x,y) \) and therefore \( \mathsf{HA} + \mathsf{ECT}_0 \vdash \forall x \exists y p(x,y) \). By Theorem 3.3 \( \mathsf{HA} \vdash \exists n (n r \forall x \exists y p(x,y)) \). Because \( \exists n (n r \forall x \exists y p(x,y)) \) is a sentence, there exists a natural number \( n \) such that \( \mathsf{HA} \vdash \exists n r \forall x \exists y p(x,y) \). Therefore by definition of the realizability:

1. \( \Rightarrow \mathsf{HA} \vdash \forall x \exists u(T(\bar{n},x,u) \land U(u) r \exists y p(x,y)) \),
2. \( \Rightarrow \mathsf{HA} \vdash \forall x \exists u(T(\bar{n},x,u) \land j_2(U(u)) r p(x,j_1(U(u)))) \),
3. \( \Rightarrow \mathsf{HA} \vdash \forall x \exists u(T(\bar{n},x,u) \land p(x,j_1(U(u)))) \),
4. \( \Rightarrow \mathsf{HA} \vdash \forall x \exists y p(x,u) \),

hence \( \mathsf{PA} \vdash_c \phi \).

In the rest of the paper, for every \( \mathcal{L} \) structure \( \mathfrak{M} \), \( \mathbf{T}_{\mathfrak{M}} \) means \( \mathsf{HA} + \mathsf{ECT}_0 + \mathsf{Diag}(\mathfrak{M}) \).

Theorem 3.4 If \( \mathfrak{M} \models \mathsf{Th}_{\Pi_2} (\mathsf{PA}) \), then \( \mathbf{T}_{\mathfrak{M}} \) is consistent.

Proof. Suppose \( \mathbf{T}_{\mathfrak{M}} \) is inconsistent, so there exists a finite number of \( \mathcal{L}(\mathfrak{M}) \) sentences \( \{ \phi_i(\bar{c}) \}_{i \leq n} \subseteq \mathsf{Diag}(\mathfrak{M}) \) such that \( \mathsf{HA} + \mathsf{ECT}_0 + \bigwedge_{i=1}^n \phi_i(\bar{c}) \vdash \bot \), therefore \( \mathsf{HA} + \mathsf{ECT}_0 \vdash \neg \bigwedge_{i=1}^n \phi_i(\bar{c}) \). Because \( \bar{c} \) are not used in the axioms of \( \mathsf{HA} + \mathsf{ECT}_0 \), we have \( \mathsf{HA} + \mathsf{ECT}_0 \vdash \forall x_1,\ldots,x_n (\neg \bigwedge_{i=1}^n \phi_i(x_i)). \) Note that \( \forall x_1,\ldots,x_n (\neg \bigwedge_{i=1}^n \phi_i(x_i)) \) is a \( \Pi_1 \) sentence and therefore by Theorem 3.3 \( \mathsf{PA} \vdash_c \forall x_1,\ldots,x_n (\neg \bigwedge_{i=1}^n \phi_i(x_i)) \).

This implies that \( \mathfrak{M} \models \forall x_1,\ldots,x_n (\neg \bigwedge_{i=1}^n \phi_i(x_i)) \) and especially \( \mathfrak{M} \models \neg \bigwedge_{i=1}^n \phi_i(\bar{c}) \), but by definition of \( \mathsf{Diag}(\mathfrak{M}) \) we know \( \mathfrak{M} \not\models \bigwedge_{i=1}^n \phi_i(\bar{c}) \) and this leads to a contradiction, hence \( \mathbf{T}_{\mathfrak{M}} \) is consistent.

If an \( \mathcal{L} \) structure \( \mathfrak{M} \) satisfies a strong enough theory of arithmetic, then \( \mathbf{T}_{\mathfrak{M}} \) has actually the Existence and Disjunction properties.

Theorem 3.5 (Existence and Disjunction Properties). Suppose \( \mathfrak{M} \) is a model of \( \mathsf{Th}_{\Pi_2} (\mathsf{PA}) \), then the following statements are true:

\[ \neg \exists x \phi(x) \iff \forall x \neg \phi(x), \quad \exists x \phi(x) \iff \neg \forall x \neg \phi(x). \]
1. For every $L(\mathfrak{M})$ sentence $\exists z \phi(z)$ such that $T_{\mathfrak{M}} \vdash_i \exists z \phi(z)$, there exists a constant symbol $c \in L(\mathfrak{M})$ such that $T_{\mathfrak{M}} \vdash_i \phi(c)$.

2. For every $L(\mathfrak{M})$ sentence $\phi \vee \psi$ such that $T_{\mathfrak{M}} \vdash_i \phi \vee \psi$, $T_{\mathfrak{M}} \vdash_i \phi$ or $T_{\mathfrak{M}} \vdash_i \psi$.

Proof.

1. Suppose $\phi(z)$ is $\psi(z, \vec{d})$ such that $\psi(z, \vec{d})$ is an $L$ formula. By assumption of the theorem there exists a finite number of $L(\mathfrak{M})$ sentences $\{\phi_i(\vec{z})\}_{i \leq n} \subseteq \text{Diag}(\mathfrak{M})$ such that

$$HA + ECT_0 + \bigwedge_{i=1}^{n} \phi_i(\vec{z}) \vdash_i \exists z \psi(z, \vec{d}),$$

so $HA + ECT_0 \vdash_i \bigwedge_{i=1}^{n} \phi_i(\vec{z}) \rightarrow \exists z \psi(z, \vec{d})$. Because $L(\mathfrak{M})$ constants that appear in $\bigwedge_{i=1}^{n} \phi_i(\vec{z}) \rightarrow \exists z \psi(z, \vec{d})$ are not used in the axioms of $HA + ECT_0$, therefore

$$HA + ECT_0 \vdash_i \forall \vec{y}, \vec{x}_1, \ldots, \vec{x}_n \left(\bigwedge_{i=1}^{n} \phi_i(\vec{x}_i, \vec{y}) \rightarrow \exists z \psi(z, \vec{y})\right).$$

Note that $\bigwedge_{i=1}^{n} \phi_i(\vec{x}_i, \vec{y})$ is a quantifier free formula, hence by Lemma 3.1 there exists an atomic formula $p$ such that $HA \vdash_i p(\vec{x}_1, \ldots, \vec{x}_n, \vec{y}) \leftrightarrow \bigwedge_{i=1}^{n} \phi_i(\vec{x}_i, \vec{y})$. Also note that by Theorem 2.9

$$HA \vdash_i p \vee \neg p,$$

Hence

$$HA + ECT_0 \vdash_i \forall \vec{y}, \vec{x}_1, \ldots, \vec{x}_n \exists z(p(\vec{x}_1, \ldots, \vec{x}_n, \vec{y}) \rightarrow \psi(z, \vec{y})).$$

By Lemma 3.2

$$HA + ECT_0 \vdash_i \forall x \exists z(p([x]) \rightarrow \psi(z, [x])).$$

Note that $\forall x \exists z(p([x]) \rightarrow \psi(z, [x]))$ is an $L$ sentence and therefore by Theorems 2.7 and 2.8 there exists a natural number $n$ such that

$$HA + ECT_0 \vdash_i \exists \vec{r} \forall x \exists z(p([x]) \rightarrow \psi(z, [x])).$$

By definition of realizability we get

$$HA + ECT_0 \vdash_i \forall x \exists u(T(\vec{n}, x, u) \land U(u) \vec{r} \exists z(p([x]) \rightarrow \psi(z, [x]))).$$

Note that $HA + ECT_0 \vdash_i \forall x \exists uT(\vec{n}, x, u)$, hence by Theorem 3.3 $PA \vdash \forall x \exists uT(\vec{n}, x, u)$ and therefore $\mathfrak{M} \models \forall x \exists uT(\vec{n}, x, u)$. Let $\mathfrak{M} \models e = (\vec{c}_1, \ldots, \vec{c}_n, \vec{d})$ and $\mathfrak{M} \models T(\vec{n}, c, f) \land U(f) = g$ for some $e, f, g \in \mathfrak{M}$. This implies $T(\vec{n}, \vec{e}, f), U(f) = g \models g \in \text{Diag}(\mathfrak{M})$ and therefore we get

$$T_{\mathfrak{M}} \vdash_i T(\vec{n}, \vec{c}, f) \land g \vec{r} \exists z(p([c]) \rightarrow \psi(z, [c])).$$

By applying realizability definition we get $T_{\mathfrak{M}} \vdash_i j_2(\vec{g}) \vec{r} (p([c]) \rightarrow \psi(j_1(\vec{g}), [c]))$. Note that by Theorem 2.7

$$HA + ECT_0 \vdash_i \forall x \exists u(p([x] \rightarrow \psi(w, [x])) \rightarrow (p([x]) \rightarrow \psi(w, [x])),$$

so

$$T_{\mathfrak{M}} \vdash_i p([c]) \rightarrow \psi(j_1(\vec{g}), [c]).$$

Because $p([c]) \in \text{Diag}(\mathfrak{M})$, we get $T_{\mathfrak{M}} \vdash_i \psi(j_1(\vec{g}), [c])$ and this implies $T_{\mathfrak{M}} \vdash_i \psi(\vec{c}, [c])$ for some $\vec{c} \in L(\mathfrak{M})$ such that $\mathfrak{M} \models j_1(\vec{g}) = c$.

2. Suppose $T_{\mathfrak{M}}$ proves $\phi \vee \psi$, therefore $T_{\mathfrak{M}} \vdash_i \exists x ((x = 0 \rightarrow \phi) \land (x \neq 0 \rightarrow \psi))$. By the previous part there exists a constant symbol $\vec{c} \in L(\mathfrak{M})$ such that $T_{\mathfrak{M}} \vdash_i (\vec{c} = 0 \rightarrow \phi) \land (\vec{c} \neq 0 \rightarrow \psi)$. Note that $\vec{c} = 0$ is an atomic formula, hence $\vec{c} = 0 \in \text{Diag}(\mathfrak{M})$ or $\vec{c} \neq 0 \in \text{Diag}(\mathfrak{M})$ and this implies $T_{\mathfrak{M}} \vdash_i \phi$ or $T_{\mathfrak{M}} \vdash_i \psi$. 
Definition 3.6 Let $M$ be an $L$ structure and $T$ be an intuitionistic theory in the language $L(M)$. For every $L(M)$ sentence $\phi$ such that $T \not\vdash \phi$, fix a Kripke model $K$ such that $K \not\models \phi$.

Definition 3.7 Let $M$ be an $L$ structure and $T$ be an intuitionistic theory in the language $L(M)$. Define $S(M, T) = \{ \phi \in L(M) | T \not\vdash \phi, \phi$ is a sentence $\}$. Define the universal model $K$ as follows. Take the disjoint union $\{ K_\phi \}_{\phi \in S(M, T)}$ and then add a new root $r$ with domain $M_r = M$.

Theorem 3.8 If $M$ is a model of $Th_{H_2}(PA)$, then $K(M, T_M)$ is a well-defined Kripke model and for every $L(M)$ sentence $\phi$, $K(M, T_M) \models \phi \iff T_M \vdash \phi$.

Proof. First note that by Theorem 3.4 $T_M \not\vdash \bot$, hence $S(M, T_M)$ is not empty and therefore $K(M, T_M)$ has other nodes except $r$. To make sure that $K(M, T_M)$ is well-defined, we should check the three conditions in the definition of Kripke models. It is easy to see that the first two conditions hold for $K(M, T_M)$. For the third condition, we need to show that for every node $k \neq r$, $L(M_k) \subseteq L(M_r)$ and $M_k \models \text{Diag}^+(M_r)$. By definition of $K(M, T_M)$, $L(M_r) \subseteq L(M_k)$ holds. For the condition $M_k \models \text{Diag}^+(M_r)$, note that $T_M \vdash \text{Diag}(M)$ which implies $M_k \models \text{Diag}(M_r)$.

$(\Rightarrow)$. Let $K(M, T_M) \models \phi$. If $T_M \not\vdash \phi$, then $K_\phi$ exists and $K_\phi \subseteq K(M, T_M)$. By the assumption we get $K_\phi \models \phi$, but this leads to a contradiction by definition of $K_\phi$, hence $T_M \vdash \phi$.

$(\Leftarrow)$. We prove this part by induction on the complexity of $\phi$:

1. $\phi = p$: Note that if $T_M \vdash \neg p$, then $p \in \text{Diag}(M)$. Because if $p \notin \text{Diag}(M)$, then $p \in \text{Diag}(M)$, hence $T_M \vdash \neg \neg p$ which leads to a contradiction by Theorem 3.4. Therefore $\neg \neg p \in \text{Diag}(M)$ and by the fact that $M \models p$ we get $K(M, T_M) \models \neg \neg p$.

2. $\phi = \psi \land \eta$: By the assumption we get $T_M \vdash \psi$ and $T_M \vdash \eta$, therefore by the induction hypothesis $K(M, T_M) \models \psi$ and $K(M, T_M) \models \eta$, hence $K(M, T_M) \models \psi \land \eta$.

3. $\phi = \psi \lor \eta$: By Theorem 3.5 $T_M \vdash \psi$ or $T_M \vdash \eta$, therefore by the induction hypothesis $K(M, T_M) \models \psi$ or $K(M, T_M) \models \eta$, hence $K(M, T_M) \models \psi \lor \eta$.

4. $\phi = \psi \rightarrow \eta$: By the assumption for every $\theta \in S(M, T_M)$, $K_\theta \models \psi \rightarrow \eta$, so for proving $K(M, T_M) \models \psi \rightarrow \eta$ we only need to show that if $r \models \psi$, then $r \models \eta$. Let $r \models \psi$, therefore we have $K(M, T_M) \models \psi$, hence by the previous part, $T_M \vdash \psi$. Note that By the assumption $T_M \vdash \psi \rightarrow \eta$, hence $T_M \models \eta$ and therefore by the induction hypothesis $K(M, T_M) \models \eta$ which implies $r \models \eta$.

5. $\phi = \exists x \psi(x)$: By Theorem 3.5 there exists a constant symbol $c \in L(M)$ such that $T_M \vdash \psi(c)$, therefore by the induction hypothesis $K(M, T_M) \models \psi(c)$, hence $K(M, T_M) \models \exists x \psi(x)$.

6. $\phi = \forall x \psi(x)$: By the assumption for every $\theta \in S(M, T_M)$, $K_\theta \models \forall x \psi(x)$, so for proving $K(M, T_M) \models \forall x \psi(x)$ we only need to show that for every $c \in M$, $r \models \psi(c)$. Let $c \in M$. By the assumption $T_M \vdash \forall x \psi(x)$, therefore $T_M \vdash \psi(c)$, hence by induction hypothesis $K(M, T_M) \models \psi(c)$. This implies that $r \models \psi(c)$. Note that $c$ is interpreted by $c \in M$, hence $k \models \psi(c)$.

The last theorem gives us the right tool for constructing a counter example for Problem 1.1. In general we can get a lot of new models for every $M \models Th_{H_2}(PA)$. For our purpose, it is sufficient to know $Th_{H_2}(PA) \not\vdash PA$ to get the result. The next two theorems established the stronger fact which says $Th_{H_2}(PA) \not\vdash \Delta_1$. $\Delta_1$ is a classical theory in the language $L$ with the following non-logical axioms:
1. Axioms of Robinson Arithmetic $Q$.

2. Axioms defining the primitive recursive functions.

3. $\Delta_1$ induction:

$$\forall \vec{y} \left[ \forall x (\phi(x, \vec{y}) \leftrightarrow \neg \psi(x, \vec{y})) \rightarrow I_{\phi} \right]$$

for every $\Sigma_1$ formulas $\phi, \psi \in \mathcal{L}$

For stating the theorems we need also another arithmetical theory that is called $B\Sigma_1$ with the following non-logical axioms:

1. Axioms of Robinson Arithmetic $Q$.

2. Axioms defining the primitive recursive functions.

3. Induction for quantifier free formulas.

4. Bounded $\Sigma_1$ collection:

$$\forall \vec{y}, x \left[ \forall z (z < x \rightarrow \exists w \phi(z, w, \vec{y})) \rightarrow \exists r \forall z (z < x \rightarrow \exists w (w < r \land \phi(z, w, \vec{y}))) \right]$$

for every $\Sigma_1$ formulas $\phi, \psi \in \mathcal{L}$

It is worth mentioning that these theories usually are defined over the language of Peano Arithmetic, and not over the language of Primitive Recursive Arithmetic, hence our definitions of $I\Delta_1$ and $B\Sigma_1$ are stronger than the usual definition, but for our use this does not cause a problem. Now we know the definitions, we will state the theorems.

**Theorem 3.9** There exists a model $\mathfrak{M} \models \text{Th}_{\Pi^2}(\mathbb{N})$ such that $\mathfrak{M} \not\models B\Sigma_1$.

**Proof.** See [3]. ⊣

**Theorem 3.10** $I\Delta_1 \vdash_c B\Sigma_1$.

**Proof.** As we explained before, this version of these theories are stronger that the original ones. Therefore by the result of [14] these two theories are the same. ⊣

**Corollary 3.11** There exists a rooted Kripke model of $\text{HA} + \text{ECT}_0$ which is not locally $I\Delta_1$.

**Proof.** By Theorem 3.9 there exists a model $\mathfrak{M} \models \text{Th}_{\Pi^2}(\mathbb{N})$ such that $\mathfrak{M} \not\models B\Sigma_1$ and hence by Theorem 3.10 $\mathfrak{M} \not\models I\Delta_1$. Note that by Theorem 3.8 $K(\mathfrak{M}, T_{\mathfrak{M}}) \models \text{HA} + \text{ECT}_0$, and $K(\mathfrak{M}, T_{\mathfrak{M}})$ is not locally $I\Delta_1$. ⊣

$\text{ECT}_0$ is a very powerful non-classical axiom schema, so a natural question is that: 
*Is it the case that for every Kripke model $K \models \text{HA} + \text{ECT}_0$ and every node $k$ in $K$, $\mathfrak{M}_k \not\models \text{PA}$?* 

This question has a negative answer, because $K(\mathbb{N}, T_N) \models \text{HA} + \text{ECT}_0$, but $\mathfrak{M}_r = \text{PA}$. 

9
3.2 The second model construction

In this subsection, we will explain the generalized construction which works for any reasonable intuitionistic arithmetical theory. We will also mention an application of it at the end of this subsection.

For every $T \in \mathcal{I}$, the Existence property of $T$ is the following $\Pi_2$ sentence:

$$\text{EP}(T):= \forall x (x = \Gamma \exists y \phi(y)) \land \text{for some formula } \phi(y) \land x \text{ is a sentence } \land \text{Pr}_T(x) \rightarrow \exists y \text{Pr}_T(\Gamma \phi(y)) \Gamma).$$

For an $L$ structure $\mathcal{M}$ and a theory $T \in \mathcal{I}$, let extension of $T$ with respect to $\mathcal{M}$ be the following theory:

$$\text{EXT}(\mathcal{M}, T) := \{ \phi \in L(\mathcal{M}) | \phi \text{ is a sentence, } \mathcal{M} \models \text{Pr}_T(\Gamma \phi) \}.$$

The following lemma states that $\text{EXT}(\mathcal{M}, T)$ is closed under finite conjunctions.

Lemma 3.12 Let $\mathcal{M} \models \text{PRA}$ and $T \in \mathcal{I}$. Then for every $L(\mathcal{M})$ sentences $\phi$ and $\psi$, if $\phi, \psi \in \text{EXT}(\mathcal{M}, T)$, then $\phi \land \psi \in \text{EXT}(\mathcal{M}, T)$.

Proof. If $\phi, \psi \in \text{EXT}(\mathcal{M}, T)$, then $\mathcal{M} \models \text{Pr}_T(\Gamma \phi) \land \text{Pr}_T(\Gamma \psi)$, so by Theorem 2.2 $\mathcal{M} \models \text{Pr}_T(\Gamma \phi \land \psi)$. Hence $\phi \land \psi \in \text{EXT}(\mathcal{M}, T)$. ⊥

Define $C_{\mathcal{M}, T} := T + \text{EXT}(\mathcal{M}, T)$. The crucial property of $C_{\mathcal{M}, T}$ is the following lemma.

Lemma 3.13 Suppose $\mathcal{M} \models \text{PRA}$. Then for every $T \in \mathcal{I}$ and every $L(\mathcal{M})$ sentence $\psi$, if $C_{\mathcal{M}, T} \vdash \psi$, then $\mathcal{M} \models \text{Pr}_T(\Gamma \psi) \Gamma)$.

Proof. Let $\psi(\bar{d})$ be an $L(\mathcal{M})$ sentence such that $C_{\mathcal{M}, T} \vdash \psi(\bar{d})$. So there exists a finite number of $L(\mathcal{M})$ sentence $(\{\phi_i(\bar{c})\}_{i \leq n} \subseteq \text{EXT}(\mathcal{M}, T)$ such that

$$T \vdash \bigwedge_{i=1}^n \phi_i(\bar{c}) \rightarrow \psi(\bar{d}).$$

Because $L(\mathcal{M})$ constants that appear in $\bigwedge_{i=1}^n \phi_i(\bar{c}) \rightarrow \psi(\bar{d})$ are not used in the axioms of $T$, therefore

$$T \vdash \forall \bar{y}, \bar{x}_1, ..., \bar{x}_n \left( \bigwedge_{i=1}^n \phi_i(\bar{x}_i, \bar{y}) \rightarrow \psi(\bar{y}) \right).$$

So by Theorem 2.2

$$\mathcal{M} \models \text{Pr}_T(\Gamma \forall \bar{y}, \bar{x}_1, ..., \bar{x}_n \left( \bigwedge_{i=1}^n \phi_i(\bar{x}_i, \bar{y}) \rightarrow \psi(\bar{y}) \right)).$$

Hence again by Theorem 2.2

$$\mathcal{M} \models \text{Pr}_T(\Gamma \bigwedge_{i=1}^n \phi_i(\bar{c}_i) \rightarrow \psi(\bar{d}) \Gamma).$$

On the other hand by Lemma 3.13 $\text{EXT}(\mathcal{M}, T)$ is closed under finite conjunctions, so $\bigwedge_{i=1}^n \phi_i(\bar{c}_i) \in \text{EXT}(\mathcal{M}, T)$ which means $\mathcal{M} \models \text{Pr}_T(\Gamma \bigwedge_{i=1}^n \phi_i(\bar{c}_i) \Gamma)$. So by Theorem 2.2 $\mathcal{M} \models \text{Pr}_T(\Gamma \psi(\bar{d}) \Gamma).$ ⊥

Theorem 3.14 For every $T \in \mathcal{I}$ and every $\mathcal{M} \models \text{PRA} + \text{EP}(T) + \text{Con}(T)$, the following statements are true:

1. $C_{\mathcal{M}, T}$ is consistent.
2. $C_{\mathcal{M}, T}$ has Existence and Disjunction property.
Proof.

1. Suppose $C_{\mathcal{M},T} \vdash \bot$. Then by Lemma 3.13 $\mathcal{M} \models \text{Pr}_T(\neg \bot)$, but this is not possible because we assumed $\mathcal{M} \models \text{Con}(T)$, hence $C_{\mathcal{M},T}$ is consistent.

2. We will prove the Existence property of $C_{\mathcal{M},T}$. The Disjunction property will follow from it by the same argument as in the proof of Theorem 3.5. Let $\psi(x)$ be a formula in $L(\mathcal{M})$ with only $x$ as the free variable. Suppose $C_{\mathcal{M},T} \vdash \exists x \psi(x)$. Then by Lemma 3.13 $\mathcal{M} \models \text{Pr}_T(\exists x \psi(x))$. Note that $\mathcal{M} \models \text{EP}(T)$, hence $\mathcal{M} \models \exists x \text{Pr}_T(\psi(x))$. This means there exists a $c \in \mathcal{M}$ such that $\mathcal{M} \models \text{Pr}_T(\psi(c))$. This implies $\psi(c) \in \text{EXT}(\mathcal{M}, T)$, so $C_{\mathcal{M},T} \vdash \psi(c)$.

This is the generalized version of the Theorem 3.8.

Theorem 3.15 Let $T \in \mathcal{I}$ and $\mathcal{M} \models \text{PRA + EP}(T) + \text{Con}(T)$. Then $K(\mathcal{M}, C_{\mathcal{M},T})$ is a well-defined Kripke model and for every $L(\mathcal{M})$ sentence $\phi$, $K(\mathcal{M}, C_{\mathcal{M},T}) \models \phi$ if and only if $C_{\mathcal{M},T} \vdash \phi$.

Proof. The proof of this theorem is essentially the same as the proof of Theorem 3.8 by using the Theorem 3.14. The only part that needs some extra work is the fact that $C_{\mathcal{M},T} \vdash \text{Diag}(\mathcal{M})$ and moreover if $C_{\mathcal{M},T} \vdash p$, then $p \in \text{Diag}(\mathcal{M})$.

Let $p \in \text{Diag}(\mathcal{M})$. We know by Theorem 2.2 $\mathcal{M} \models p \rightarrow \text{Pr}_T(\neg p \land p)$. This implies $\mathcal{M} \models \text{Pr}_T(\neg p \lor p)$. So $p \in \text{EXT}(\mathcal{M}, T)$ which implies $C_{\mathcal{M},T} \vdash p$.

Now if we have $C_{\mathcal{M},T} \vdash p$ for some atomic $L(\mathcal{M})$ sentence $p$, then by Lemma 3.13 $\mathcal{M} \models \text{Pr}_T(\neg p \lor p)$. Note that $\mathcal{M} \models \text{Con}(T)$, so in presence of PRA, $\mathcal{M} \models \text{P1-RFN}(T)$ which $\text{P1-RFN}(T)$ is the following sentence:
\[
\forall x (x \in \text{P1} \land \text{Pr}_T(x) \rightarrow \text{Tr}(x))
\]
where Tr is a natural $\text{P1}$ formula which works as the truth predicate for $\text{P1}$ sentence. Substituting $\neg p \lor p$ for $x$ in $\text{P1-RFN}(T)$, we get $\mathcal{M} \models \text{Tr}(\neg p \lor p)$, hence $\mathcal{M} \models p$ which means $p \in \text{Diag}(\mathcal{M})$. ~

As we already see, using the first construction, we provide a Kripke model of $\text{HA + ECT}_0$ which is not locally $\text{I} \Delta_1$. A natural conjecture would be that the existence of such a Kripke model was possible because the base theory has a very powerful non-classical schema $\text{ECT}_0$. As an application of Theorem 3.15 we will show this is not the case. Let $H(x)$ be a $\Sigma_1$ formula that is a natural formalization of the statement "The Turing machine with code $x$ halts on input $x$". Let $\theta$ be an instance of $\text{ECT}_0$ in Definition 2.10 such that $\phi(x) := \top$ and $\psi(x, y) := (y = 0 \land H(x)) \lor (y \neq 0 \land \neg H(x))$.

We also need the definition of Markov principle.

Definition 3.16 The Markov principle is the following schema:
\[
\text{MP} := \forall \bar{y} (\forall x (\phi(x, \bar{y}) \lor \neg \phi(x, \bar{y})) \land \neg \exists x \phi(x, \bar{y}) \rightarrow \exists x \phi(x, \bar{y})).
\]

Lemma 3.17 The following statements are true:

1. $\text{HA} + \neg \theta + \text{MP}$ is consistent.

2. $\text{HA} + \neg \theta + \text{MP}$ has Existence and Disjunction properties.

Proof.

1. It is easy to see that $\text{PA} \vdash_c \neg \theta$ and also $\text{PA} \vdash_c \text{MP}$. So $\text{HA} + \neg \theta + \text{MP}$ is a sub-theory of $\text{PA}$ and it is consistent.
2. We will prove the Existence property of $\text{HA} + -\theta + \text{MP}$ here. The Disjunction property will follow from it like before. This part is a standard application of Kripke models (see [15]). Let $\exists x \psi(x)$ be an $\mathcal{L}$ sentence such that $\text{HA} + -\theta + \text{MP} \vdash x \psi(x)$, but for every natural number $n$, $\text{HA} + -\theta + \text{MP} \not\vdash \psi(n)$. It is well-known that $K(N, \text{HA} + -\theta + \text{MP})$ is a well-defined Kripke model and moreover $K(N, \text{HA} + -\theta + \text{MP}) \models \text{HA}$ (see Theorem 5.2.4 in [15]). Moreover we can assume that $K_\bot$ (Note that $\bot \in S(N, \text{HA} + -\theta + \text{MP})$) is a Kripke model with just one node with the classical structure $\mathbb{N}$. Note that $r \not\models \theta$, because otherwise by the monotonicity of forcing relation for every $\phi \in S(N, \text{HA} + -\theta + \text{MP})$, $K_\phi \models -\theta$ which is not true. Moreover for every node $k \neq r$, $k \models -\theta$, so with the last argument $r \models -\theta$ which implies $K(N, \text{HA} + -\theta + \text{MP}) \models -\theta$. Note that $\text{MP}$ is forced in every node $k \neq r$. So we only need to show that $r \models \text{MP}$. For this matter suppose $r \models \exists x (\phi(x, \bar{a}) \lor -\phi(x, \bar{a})) \land -\exists x \phi(x, \bar{a})$ where $\bar{a} \in \mathbb{N}$. If for every $n \in \mathbb{N}$, $r \not\models \phi(n, \bar{a})$, then because of decidability of $\phi(x, \bar{a})$ in the point of view of $r$, for every $n \in \mathbb{N}$, $r \models -\phi(n, \bar{a})$. This implies $K_\bot \models \forall x -\phi(x, \bar{a})$. This leads to a contradiction because $K_\bot \models -\exists x -\phi(x, \bar{a})$. This means that there exists a natural number $n$ such that $r \models \phi(n, \bar{a})$.

By the above arguments, we have

$$K(N, \text{HA} + -\theta + \text{MP}) \models \text{HA} + -\theta + \text{MP}. $$

So $K(N, \text{HA} + -\theta + \text{MP}) \models \exists x \psi(x)$. This implies that there exists a natural number $n$ such that $r \models \psi(n)$. But this leads to a contradiction because we know $K_{\psi(n)} \not\models \psi(n)$. This implies that our assumption was false and there exists a natural number $n$ such that $\text{HA} + -\theta + \text{MP} \vdash x \psi(n)$. 

$\dashv$

**Corollary 3.18** There exists a rooted Kripke model of $\text{HA} + -\theta + \text{MP}$ which is not locally $\text{I}_1$. 

**Proof.** By Theorem 3.9 there exists a model $\mathfrak{M} \models \text{Th}_{11}(\mathbb{N})$ such that $\mathfrak{M} \not\models \text{BS}_1$ and hence by Theorem 3.14 $\mathfrak{M} \not\models \text{I}_1$. Note that by Lemma 3.17 $\text{HA} + -\theta + \text{MP}$ is consistent and has Existence property. This implies that $\text{EP}(\text{HA} + -\theta + \text{MP})$ and $\text{Con}(\text{HA} + -\theta + \text{MP})$ are true in $\mathbb{N}$. Note that these sentences are $\text{I}_2$, so they are also true in $\mathfrak{M}$. This implies that $\mathfrak{M}$ satisfies the conditions needed in the Theorem 3.17, hence

$$K(\mathfrak{M}, C_{\mathfrak{M}, \text{HA} + -\theta + \text{MP}}) \models \text{HA} + -\theta + \text{MP}$$

and also it is not locally $\text{I}_1$. 

It is worth mentioning that $\text{HA} + -\theta + \text{MP}$ does not prove anything contradictory with $\text{PA}$ and in some sense, it is closed to $\text{PA}$, but still, we were able to construct a Kripke model of it which is not locally $\text{I}_1$.

As we already mentioned in the Introduction, we can get more Kripke models for $\text{HA} + \text{ECT}_0$ from the first construction than by the second construction. We will show this fact in the rest of this subsection. For this matter, we need the following theorem.

**Theorem 3.19** For any constant $k$, there is no consistent $\Pi_k$-axiomatized theory $\mathcal{T}$ such that $\mathcal{T} \vdash \text{PA}$. 

**Proof.** See [2]. 

$\dashv$

**Theorem 3.20** The following statements are true:

1. For every $\mathcal{L}$ structure $\mathfrak{M}$, if $K(\mathfrak{M}, C_{\mathfrak{M}, \text{HA} + \text{ECT}_0}) \models \text{HA} + \text{ECT}_0$, then $K(\mathfrak{M}, T_{\mathfrak{M}}) \models \text{HA} + \text{ECT}_0$.
2. There exists an $\mathcal{L}$ structure $\mathfrak{M}$ such that $K(\mathfrak{M}, T_{\mathfrak{M}}) \models \text{HA} + \text{ECT}_0$, but $K(\mathfrak{M}, C_{\mathfrak{M}, \text{HA} + \text{ECT}_0}) \not\models \text{HA}$. 

$\dashv$
Proof.

1. Suppose $K(\mathfrak{M}, C_{\text{PA+ECT}_0}) \models \text{HA + ECT}_0$. Let $\phi := \forall x \exists y \psi(x, y)$ be a $\Pi_2$ sentence such that $\text{PA} \vdash \phi$. Then by $\Pi_2$ conservativity of HA over PA, we have $\text{HA} \vdash \phi$, hence $K(\mathfrak{M}, C_{\text{PA+ECT}_0}) \models \phi$. This implies $r \models \forall x \exists y \psi(x, y)$. So for every $\bar{a} \in \mathfrak{M}$:

(a) $\Rightarrow r \models \exists y \psi(\bar{a}, \bar{y})$.

(b) $\Rightarrow$ there exist $\bar{b} \in \mathfrak{M}$ such that $r \models \psi(\bar{a}, \bar{b})$.

(c) $\Rightarrow \mathfrak{M} \models \psi(\bar{a}, \bar{b})$.

Hence $\mathfrak{M} \models \phi$. This implies that $\mathfrak{M} \models \text{Th}_{\Pi_2}(\text{PA})$, so by Theorem 3.8 $K(\mathfrak{M}, T_{\Pi_2}) \models \text{HA + ECT}_0$.

2. By Gödel second Incompleteness theorem, PA + $\neg \text{Con(PA)}$ is consistent. So this implies that $\text{Th}_{\Pi_2}(\text{PA}) + \neg \text{Con(HA)}$ is also consistent. $\text{Th}_{\Pi_2}(\text{PA}) + \neg \text{Con(HA)}$ is a $\Pi_2$-axiomatized theory, hence by Theorem 3.13 there exists a model $\mathfrak{M} \models \text{Th}_{\Pi_2}(\text{PA}) + \neg \text{Con(HA)}$ such that $\mathfrak{M} \not\models \text{PA}$. Note that by Theorem 3.8 $K(\mathfrak{M}, T_{\Pi_2}) \models \text{HA + ECT}_0$. On the other hand $\mathfrak{M} \models \neg \text{Con(HA + ECT}_0)$, so $\perp \in \text{EXT}(\mathfrak{M}, \text{HA + ECT}_0)$. This implies $C_{\text{PA+ECT}_0} \vdash \perp$. Hence $\text{S}(\mathfrak{M}, C_{\text{PA+ECT}_0}) = \emptyset$. This means that $K(\mathfrak{M}, C_{\text{PA+ECT}_0})$ has only one node $r$ such that $\mathfrak{M}_r = \mathfrak{M}$. Note that $\mathfrak{M} \not\models \text{PA}$, so $r \not\models \text{HA}$ and this completes the proof.

$\square$

4 On binary Kripke models for intuitionistic first-order logic

In this section, we will prove that every countable rooted Kripke model $K$ (there exists a node $k$ in $K$ such that for every $k$ in $K$, $k \leq k'$ can be transformed to a Kripke model $K'$ with the infinite full binary tree as Kripke frame such that $K$ and $K'$ force the same sentences. Let $\Gamma = \{0, 1\}$ and $\Gamma^*$ be the set of all finite binary strings (including empty string $\lambda$). For every $x, y \in \Gamma^*$, $x \leq y$ iff $x$ is a prefix of $y$.

Lemma 4.1 Let $K = (K, \leq, \mathfrak{M})$ be a countable rooted Kripke model in a language $\sigma$. Then there is an onto function $f : \Gamma^* \to K$, such that:

1. $K' = (\Gamma^*, \leq, \mathfrak{M}')$ is a Kripke model with $\mathfrak{M}'$ is defined as $\mathfrak{M}'_x = \mathfrak{M}_f(x)$ for every $x \in \Gamma^*$,

2. for every $k \in K$, for every $\sigma(\mathfrak{M}_k)$ sentence $\phi$, and for every $x \in \Gamma^*$ such that $f(x) = k$, $x \models \phi$ iff $k \models \phi$.

Proof. Without loss of generality, we can assume $(K, \leq)$ is a tree (see Theorem 6.8 in the second chapter of [12]) with the root $r$. Also, we can assume that for every $k \in K$, there is a $k' \in K$ different from $k$ such that $k \leq k'$. This is true because for every $k \in K$ that does not have relation with any other nodes, we can put an infinite countable path above $k$ such that the classical structure of every node in this path is $\mathfrak{M}_k$. This transformation does not change the sentences that were forced in the original model. For every $k \in K$, define neighbor of $k$ as $N_k = \{k' \in K | k \leq k' \land k \neq k' \land \forall k'' \in K(k \leq k'' \land k'' \leq k' \rightarrow k = k'' \lor k' = k''')\}$. For every $k \in K$, fix an onto function $g_k : \mathbb{N} \to N_k$ such that for every $k' \in N_k$, $\{n \in \mathbb{N} | g_k(n) = k'\}$ is infinite. Now we define $f$ inductively with a sequence of partial function $f_0 \subseteq f_1 \subseteq ...$ and then we put $f = \bigcup_{n \in \mathbb{N}} f_n$. Put $f_0(\lambda) = r$. For a function $h$, let $\text{Dom}(h)$ be domain of $h$. Let $A_n = \{x \in \Gamma^* | x \in \text{Dom}(f_n), x_0 \notin \text{Dom}(f_n), x_1 \notin \text{Dom}(f_n)\}$. 

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Now $f_{n+1}$ is defined inductively from $f_n$ as follows:

$$
\begin{align*}
  f_{n+1}(x) &= \begin{cases} 
    f_n(x) & x \in \text{Dom}(f_n) \\
    f_n(y) & x = y0^m, \text{ for some } y \in A_n, m \in \mathbb{N} \\
    g_{f_n}(y)(m) & x = y0^m1, \text{ for some } y \in A_n, m \in \mathbb{N}.
  \end{cases}
\end{align*}
$$

It is easy to see that $\text{Dom}(f) = \Gamma^*$.

**Claim 4.2** For every $k \in \mathbb{K}$, for every $x \in \Gamma^*$ if $f(x) = k$, then

$$
\{k' \in \mathbb{K} | k \leq k'\} = \{f(y) \in \mathbb{K} | y \in \Gamma^*, x \preceq y\}.
$$

This claim is easy to prove considering the definition of $f$ and the fact that $g_k$ functions enumerate neighbors infinitely many times.

Using this claim, we can finish the proof. The proof goes by induction on the complexity of $\phi$. We will only mention a nontrivial case in the induction steps. All other cases can be treated similarly. Let $\phi := \psi \rightarrow \eta$ and $\models \psi \rightarrow \eta$. Let $x \in \Gamma^*$ be such that $f(x) = k$. Suppose for some $y \geq x$, we know $y \models \psi$. So by the induction hypothesis, $f(y) \models \psi$ and by Claim 4.2 we know $f(y) \geq k$, hence $f(y) \models \eta$, therefore by the induction hypothesis we get $y \models \eta$, so $x \models \phi$. \textbf{1}

**Corollary 4.3** There exists a Kripke model of HA with $(\Gamma^*, \preceq)$ as the Kripke frame that is not locally $\text{I} \Delta_1$.

**Proof.** Let $\mathbb{K}$ be a rooted Kripke model with the root $r$ in a language $\sigma$. Let $\mathcal{U}$ be a countable set of sentences of $\sigma$. It is easy to see that $\mathbb{K}$ can be represented by a suitable two-sorted classical structure $\mathcal{M}_\mathbb{K}$ such that:

1. For every $\phi \in \mathcal{U}$, "$r \models \phi"$ is first-order definable in $\mathcal{M}_\mathbb{K}$ by the sentence $\phi_F$.
2. For every $\phi \in \mathcal{U}$, "$\mathcal{M}_r \models \phi"$ is first-order definable in $\mathcal{M}_\mathbb{K}$ by the sentence $\phi_M$.

By applying the downward Löwenheim-Skolem theorem on $\mathcal{M}_\mathbb{K}$ we get a countable substructure of $\mathcal{M}_\mathbb{K}$ like $\mathcal{M}'_{\mathbb{K}}$ such that:

1. $\mathcal{M}'_{\mathbb{K}}$ is a representation of a countable rooted Kripke model in the language $\sigma$.
2. For every $\phi \in \mathcal{U}$, $\mathcal{M}_\mathbb{K} \models \psi$ iff $\mathcal{M}'_{\mathbb{K}} \models \psi$, for $\psi \in \{\phi_F, \phi_M\}$.

Let $\mathbb{K}(\mathcal{M}, \mathcal{T}_2\mathcal{M})$ be the rooted Kripke model from Corollary 4.1. Let $\mathcal{U} = \text{HA} \cup \{\varphi\}$ where $\varphi$ is an instance of $\Delta_1$ induction that fails in the classical structure of the root of $\mathbb{K}(\mathcal{M}, \mathcal{T}_2\mathcal{M})$. Following the same argument on $\mathbb{K}(\mathcal{M}, \mathcal{T}_2\mathcal{M})$ and $\mathcal{U}$, we get a countable rooted Kripke model $\mathbb{K}'$ of HA that is not locally $\text{I} \Delta_1$. Hence applying Lemma 4.1 on $\mathbb{K}'$ finishes the proof. \textbf{1}

5 Concluding remarks and Open problems

Problem 4.1 can be asked about other theories than HA. One can ask the same question about arithmetic over sub-intuitionist logic too. One of these logics is Visser’s Basic logic, and its extension Extended Basic logic. The model theory of arithmetic over these logics were investigated in [13] [5] [6]. From the point of view of Problem 4.1, it is proved in [4] that every irreflexive node in a Kripke model of BA (Basic Arithmetic) is locally $\text{I} \Sigma_1^+$. So in general, every irreflexive node in a Kripke model of the natural extension of BA such as EBA (Extended Basic Arithmetic) is locally $\text{I} \Sigma_1$ (see Corollary 3.33 in [6]). Also it is proved in [6] that every Kripke model of EBA is locally $\text{Th}_{\text{II}_2}(\text{I} \Sigma_1) + \text{Th}_{\text{II}_1}(\text{PA})$. Note that every Kripke model of HA is also a Kripke model of BA and
EBA. So Corollary 3.11 applies to these theories too, and this solves Problem 1.1 for these theories. Furthermore, this shows that the known positive results are the best we can get for BA and EBA.

Focusing on the proof of Theorem 2.8, we essentially use ECT\textsubscript{0} for proving the Existence and Disjunction property of T\textsubscript{\text{\neg \text{B}}}. We do not know whether ECT\textsubscript{0} is essential for such a model construction, so we have the following question:

**Problem 5.1** Does theory HA + Diag(\text{\neg \text{B}}) has the Existence property for every \( \text{\neg \text{B}} \models \text{Th}_{\Pi_{\text{\neg \text{B}}}}(\text{PA}) \)?

An important problem which we could not answer is the following:

**Problem 5.2** Is there any Kripke model K \models HA such that for every node k in K, M\text{\neg \text{B}}_{k} \not\models PA?

Another unsolved question in the direction of completeness with respect to locally PA Kripke models is the following:

**Problem 5.3** Does HA have completeness with respect to its class of locally PA Kripke models?

By the result of [8], for every sentence \( \phi \) such that HA \not\models \phi, there exists a locally PA Kripke model K such that K \not\models \phi, but this result does not say anything about whether K is a Kripke model of HA or not.

We call a rooted tree Kripke frame \((K, \leq)\), a PA-frame iff for every Kripke model K \models HA with frame \((K, \leq)\), K is locally PA. Let \( \mathcal{F}_{\text{PA}} \) be the set of all PA-frames. We know that semi narrow rooted tree Kripke frames are in \( \mathcal{F}_{\text{PA}} \). On the other hand, by Corollary 4.2 infinite full binary tree is not in \( \mathcal{F}_{\text{PA}} \). So we have the following question:

**Problem 5.4** Is there a nice characterization of \( \mathcal{F}_{\text{PA}} \)?

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