On the $q$-analog of homological algebra

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§0 Introduction.

Homological algebra can be seen as the study of the equation $d^2 = 0$ (see the epigraph to [1]). It is natural to ask why $d^2$ and not, say, $d^3$. Following this mood, we give the natural definition.

**Definition 0.1.** Let $A$ be some Abelian category (e.g. the category of modules over some ring) and $N \geq 1$ be an integer. An $N$-complex in $A$ is a sequence of objects and morphisms of $A$

$$C. = \{ ... \to C_1 \to C_0 \to C_{-1} \to ... \}$$

(0.1)

in which the composition of any $N$ consecutive morphisms equals 0.

Thus a 1-complex is just a graded object (no differential) and a 2-complex is a chain complex in the usual sense.

A source of 2-complexes is provided by simplicial sets. Recall [1] that a simplicial set $X$ is a collection of sets $X_i, i \geq 0$, maps $\partial_i : X_n \to X_{n-1}, i = 0, 1, ..., n$ (called face maps) given for any $n$ and similar degeneracy maps $s_i : X_n \to X_{n+1}, i = 0, 1, ..., n+1$. These maps are subject to certain commutation relations of which we note the following:

$$\partial_i \partial_j = \partial_{j-1} \partial_i \quad \text{for} \quad i < j$$

(0.2).

The chain complex of a simplicial set $X$, is the complex of $\mathbb{C}$-vector spaces $\mathbb{C}[X]$ whose $n$-th term is $\mathbb{C}[X_n]$, the vector space freely generated by the set $X_n$. The standard chain differential $d : \mathbb{C}[X_n] \to \mathbb{C}[X_{n-1}]$ is defined by the formula $d = \sum (-1)^i \partial_i$ and satisfies $d^2 = 0$. Let $q$ be any complex number. Define the $q$-analog of the differential $d$ to be the map

$$d_q : \mathbb{C}[X_n] \to \mathbb{C}[X_{n-1}], \quad d_q = \sum_{i=0}^{n} q^i \partial_i.$$ 

(0.3)

For $q = -1$ we obtain the usual formula.

**Proposition 0.2.** Let $q$ be a $N$-th root of unity, $q^N = 1, q \neq 1$. Then the sequence $(\mathbb{C}[X], d_q)$ is an $N$-complex.

To prove Proposition 0.2, we introduce, as it is common in the theory of $q$-analogs, basic numbers

$$[n]_q = (1 - q)^n / (1 - q) = 1 + q + ... + q^{n-1},$$

(0.4)
and basic factorials
\[ [n!]\_q = [1]_q [2]_q \cdots [n]_q = \sum_{w \in S_n} q^{l(w)}, \]
where in the last sum \( S_n \) is the symmetric group on \( n \) letters and \( l(w) \) is the length of a permutation \( w \). If \( q \) is a non-trivial \( N \)-th root of unity then \([N!]\_q = 0\). Therefore Proposition 0.2. is an immediate consequence of the following more precise statement whose proof is left to the reader.

**Lemma 0.3.** For any \( q \in \mathbb{C} \), any simplicial set \( X \), and simplex \( x \in X_m \) and any \( N \leq m \) we have
\[ d^N_q(x) = [N!]\_q \sum_{i_1 \geq \ldots \geq i_N} q^{i_1 + \ldots + i_N} \partial_{i_1} \ldots \partial_{i_N}(x). \]

The purpose of this note is to demonstrate that there is a meaningful homological algebra of \( N \)-complexes. This formalism includes \( q \)-anlogs of classical combinatorial functions taken for \( q \) an \( N \)-th root of unity. These values of \( q \) have recently attracted much attention in the study of representations of quantum groups. It seems that ”\( N \)-homological algebra might be relevant to this study.

We show that homology objects of an \( N \)-complex form naturally an \((N-1)\)-complex. The role of Euler characteristic for \( N \)-complexes is played by the value of Poincaré polynomial at \( N \)-th root of unity. In particular, for an exact \( N \)-complex this value is 0.

In \( \S 2 \) we construct the \( q \)-analog of the de Rham complex adding to commuting coordinates \( x_i \) a set of \( q \)-commuting variables \( dx_i \) and equip the resulting algebra with the differential \( d \) satisfying the \( q \)-analog of Leibniz rule. For \( q \) a primitive \( n \)-th root of 1, the differential satisfies the equation \( d^N = 0 \). This differential calculus is not covariant under usual changes of coordinates, but there is a natural quadratic Hopf algebra acting on \( x_i \) from the left and on \( dx_i \) from the right, which gives ”quantum covariance”.

For \( q^N = 1 \) and a matrix-valued 1-form \( A \) (in the \( q \)-deformed sense) we show that for the operator \((d + A)^N\) is given by the multiplication with some matrix \( N \)-form called the curvature of \( A \). In [4] G.Lusztig suggested an analogy between ”quantum geometry” at roots of unity and algebraic geometry in characteristic \( p \). Our construction of curvature can be seen in this vein as the analog of the \( p \)-curvature of connection in characteristic \( p \) (which measures non-commuting of covariant derivative with Frobenius map). In particular, for \( N = 3 \) we obtain an expression resembling Chern-Simons functional in usual gauge theory.
§1. Homology of \( N \)-complexes.

Let \((C, d_C), (E, d_E)\) be two \( N \)-complexes in category \( A \). A *chain morphism* \( f : (C, d_C) \to (E, d_E)\) is a collection of morphisms \( f_n : C_n \to E_n\) commuting with differentials. We shall denote by \( N-Com(A)\) the category of all \( N \)-complexes in \( A \) with morphisms just defined.

A. Homology.

**Definition 1.1.** Let \((C, d)\) be an \( N \)-complex over an Abelian category \( A \). Its homology are the objects

\[
pH_i(C) = \frac{\text{Ker}\{d^p : C_i \to C_{i-p}\}}{\text{Im}\{d^{N-p} : C_{i+N-p} \to C_i\}}
\]

where \( i \in \mathbb{Z}, p = 1, 2, \ldots, N \). An \( N \)-complex is called \( N \)-exact if all its homology objects are 0.

Clearly the homology is functorial with respect to chain morphisms of \( N \)-complexes.

**Example 1.2.** Any sequence with only \( N \) consecutive terms

\[
\ldots 0 \to C_{N-1} \xrightarrow{d_{N-1}} \ldots \xrightarrow{d_3} C_1 \xrightarrow{d_1} C_0 \to 0\ldots
\]

is an \( N \)-complex. The condition of \( N \)-exactness of this \( N \)-complex means that all the \( d_i \) are isomorphisms. This can be easily seen by induction starting from the right end of the complex.

For \( N \geq 3 \) the homology of \((C, d)\) are connected by two families of natural maps. The first family is formed by

\[
i_* : pH_i(C) \to_{p+1} H_i(C), \quad z \mod \text{Im}(d^{N-p}) \mapsto z \mod \text{Im}(d^{N-p-1})
\]

which are defined since \( \text{Ker}(d^p) \subseteq \text{Ker}(d^{p+1}) \) and \( \text{Im}(d^{N-p}) \supseteq \text{Im}(d^{N-p-1}) \). The second family consists of

\[
d_* : pH_i(C) \to_{p-1} H_{i-1}(C), \quad z \mod \text{Im}(d^{N-p}) \mapsto dz \mod \text{Im}(d^{N-p+1})
\]

The morphisms of the homology induced by a chain morphism of complexes obviously preserve these maps. We shall arrange \( pH_i \) pictorially in such a way that \( i_* \) acts horizontally.
and $d_*$- vertically. For example, the homology of a 5-complex look like

\[
\begin{array}{ccccccc}
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\
1H_{i+1} & \overset{i_*}{\to} & 2H_{i+1} & \overset{i_*}{\to} & 3H_{i+1} & \overset{i_*}{\to} & 4H_{i+1} \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\
1H_i & \overset{i_*}{\to} & 2H_i & \overset{i_*}{\to} & 3H_i & \overset{i_*}{\to} & 4H_i \\
\downarrow & \downarrow & \downarrow & \downarrow & \\
1H_{i-1} & \overset{i_*}{\to} & 2H_{i-1} & \overset{i_*}{\to} & 3H_{i-1} & \overset{i_*}{\to} & 4H_{i-1} \\
\downarrow & \downarrow & \downarrow & \\
1H_{i-2} & \overset{i_*}{\to} & 2H_{i-2} & \overset{i_*}{\to} & 3H_{i-2} & \overset{i_*}{\to} & 4H_{i-2} \\
\end{array}
\]

(1.5)

The squares in this diagram obviously commute. We form the "total object" of the homology diagram by setting

\[ H_m(C.) = \bigoplus_{2i - p = m} pH_i(C.) \]  

(1.6)

and define morphisms $D : H_m(C.) \to H_{m-1}(C.)$ to be equal $D = i_* + d_*$ (with the convention that $i_*$ and $d_*$ are set to be zero when not defined.

**Theorem 1.3.**

(a) Let $(C., d)$ be any $N$-complex in an Abelian category $\mathcal{A}$. Then the sequence $(H.(C.), D)$ is an $(N - 1)$-complex.

(b) For $N = 3$, the 2-complex $H(C.)$ is exact (in the usual sense).

This theorem can be reformulated by saying that for any Abelian category $\mathcal{A}$ we have a sequence of categories and functors

\[ \ldots \overset{H}{\to} 3 - \text{Com}(\mathcal{A}) \overset{H}{\to} 2 - \text{com}(\mathcal{A}) \overset{H}{\to} 1 - \text{Com}(\mathcal{A}). \]  

(1.7)

The part of this sequence from $3 - \text{Com}(\mathcal{A})$ to $1 - \text{Com}(\mathcal{A})$ is a complex in the usual sense i.e. $H \circ H = 0$.

Before starting to prove theorem 1.3, let us consider some examples.

**Example 1.4.** For a 3-complex $C$, the (usual) exact sequence $H.(C)$ has the form

\[ \ldots \to H_n(C) \overset{i_*}{\to} H_{n-1}(C) \overset{d_*}{\to} H_{n-1}(C) \overset{i_*}{\to} H_{n-1}(C) \to \ldots \]  

(1.8)

In particular, consider a 3-complex having only two terms $X \overset{f}{\to} Y$. The exact sequence of homologies of this 3-complex has the form

\[ 0 \to \text{Ker}(f) \to X \to Y \to \text{Coker}(f) \to 0 \]  

(1.9)
More generally, any composable pair of morphisms in an Abelian category \( \mathcal{A} \) still gives a 3-complex \( X \xrightarrow{f} Y \xrightarrow{g} Z \). Such a 3-complex produces the six-term exact sequence
\[
0 \to \text{Ker}(f) \to \text{Ker}(gf) \to \text{Ker}(g) \to \text{Coker}(f) \to \text{Coker}(gf) \to \text{Coker}(g) \to 0. \tag{1.10}
\]

**Proof of Theorem 1.3:** Let us first prove that \( \mathbf{H}(C) \) is an \((N-1)\)-complex. Since the squares in the homology diagram (1.5) are commutative, each matrix element of \( D^{N-1} \) can be written as the composition of a string of morphisms which contains two consecutive morphisms of the form
\[
N_{-1}H_i \xrightarrow{d_1} N{-2}H_i \xrightarrow{i} N{-1}H_{i-1}
\]
(1.11)
or
\[
1H_i \xrightarrow{i} 2H_i \xrightarrow{d_0} 1H_{i-1}
\]
(1.12)
for some \( i \). It suffices to show, therefore, that such compositions vanish. Indeed, to see that (1.11) gives 0, consider a class \([x] = x \mod \text{Im}(d) \in N_{-1}H_i \) where \( d^{N-1}x = 0 \). Then \( i_\ast d_\ast [x] = dx \mod \text{Im}(d) = 0 \). Similarly for (1.12). Thus \( \mathbf{H}(C) \) is an \((N-1)\)-complex.

Let us prove the exactness of the homology sequence (1.8) for a 3-complex. First consider the term \( 1H_n \). Let \( w \in \text{Ker}(d) \) and \( i_\ast (w \mod \text{Im}(d^2)) = 0 \). This means that \( w \mod \text{Im}(d) = 0 \) i.e. \( w = dy \) for some \( y \in C_{n+1} \). But then \( y \) lies in \( \text{Ker}(d^2) \) and therefore represents an element \([y] \in 2H_{n+1} \) so that \([w] = d_\ast [y] \).

Let us show the exactness of (1.8) in the term \( 2H_n \). Suppose that \( d_\ast (z \mod \text{Im}(d^2)) = 0 \) i.e. \( dz \mod \text{Im}(d^2) = 0 \). This means that \( dz = d^2x \) for some \( x \in C_{n+1} \). Let \( c = z - dx \). Then \( dc = 0 \). Denoting by \([c]\) the class of \( c \) in \( 1H_n \), we obtain that \([z] = i_\ast [c] \) thus proving the exactness.

For the case of a 3-complex \( C \), Theorem 1.3.b) implies that if for some \( p \in \{1,2\} \) all \( \rho H_i(C) \) are zero then all the other homology groups also vanish. This fact generalizes to arbitrary \( N \)-complexes.

**Proposition 1.5.** Let \( C \) be an \( N \)-complex such that for some \( p \in \{1,...,N-1\} \) one has \( \rho H_i(C) = 0 \) for all \( i \). Then \( C \) is \( N \)-exact i.e. \( rH_i(C) = 0 \) for all \( r \) and \( i \).

**Proof:** a) Consider first the case where the given level \( p \) equals \( N-1 \). Let \( r < N-1 \). Let us show that \( rH_i = 0 \). Let \( z \) be such that \( d^r z = 0 \). We need to show that \( z = d^{N-r}u \) for some \( u \). Note that \( d^{N-1}z = 0 \) since \( r < N - 1 \). Therefore \( z = dz_1 \) for some \( z_1 \). Further, we have \( d^{N-1}z_1 = d^{N-2}z = 0 \). Hence \( z_1 = dz_2 \) for some \( z_2 \). Thus continuing, we find
\[
z_1 = dz_2, z_2 = dz_3, ..., z_{N-r-1} = dz_{N-r}
\]
which implies \( z = d^{N-r}z_{N-r} \) as required.

b) Now consider the case \( r < p < N - 1 \). Let \( d^r z = 0 \). Then \( d^p z = 0 \) and hence \( z = d^{N-p}z_1 \). Since \( p < N - 1 \), the number \( N - p - 1 \) is non-negative and we can write
\[
d^p(d^{N-p-1}z_1) = d^{N-1}z_1 = d^{p-1}z = 0.
\]
Hence $d^{N-p-1}z_1 = d^{N-p}z_2$. If $p - r > 2$ then we have

$$d^p(d^{N-p-1}z_2) = d^{p-2}z = 0 \implies d^{N-p-1}z_2 = d^{N-p}z_3$$

for some $z_3$. Thus continuing, we find $z_3, ..., z_{p-r}$ such that $d^{N-p-1}z_i = d^{N-p}z_{i+1}$. This implies

$$z = d^{N-p}z_1 = d^{N-p+1}z_2 = ... = d^{N-r}z_{p-r}$$

what proves the assertion.

c) Now consider the case $p < r < N - 1$. Write $r = kp + l$, where $k, l \in \mathbb{Z}_+, l < p$. Let $d^r z = 0$. Then $d^p(d^{r-p}z) = 0$ and so for some $z_1$ we have $d^{r-p}z = d^{N-p}z_1$. If $k > 1$ then we can continue by writing $d^p(d^{r-2p}z - d^{N-2p}z_1) = 0$ and find that $d^{r-2p}z - d^{N-2p}z_1 = d^{N-p}z_2$ for some $z_2$. Thus continuing $k$ times, find

$$d^{r-kp} = d^{N-kp}z_1 + d^{N-(k-1)p}z_2 + d^{N-(k-2)p}z_3 + ...$$

But $r - kp = l < p$. Therefore we can write $d^l(z - d^{N-r}z_1 - d^{N-r+p}z_2 - ...) = 0$. By using case b), we have

$$z - d^{N-r}z_1 - d^{N-r+p}z_2 - ... = d^{N-l}w$$

which implies that $z \in \text{Im}(d^{N-r})$. The proposition is proven.

**B. Homotopies**

It is possible to define the analog of homotopies of morphisms. Usually it is done as a particular case of internal Hom- construction for complexes. We shall use the same approach. To this end we shall make the following convention.

**Convention 1.6.** In the sequel we shall consider only categories which are linear. By $\epsilon_N$ or $\epsilon$, when $N$ is clear from the context, we shall always denote the primitive root of unity $\exp(2\pi i/N)$.

For completeness sake we shall consider, along with $N$-complexes in a category $\mathcal{A}$, arbitrary sequences of the form (0.1) (which will be called just sequences). Morphisms of sequences are defined similarly to morphisms of $N$-complexes. The category of sequences in $\mathcal{A}$ will be denoted $\text{Seq}(\mathcal{A})$.

**Definition 1.7.**

(a) Let $(C_i, d_C), (E_i, d_E)$ be two sequences in an Abelian category $\mathcal{A}$. Their $q$-Hom-sequence is the sequence $\underline{\text{Hom}}(C_i, E_i)$ of vector spaces which has terms

$$\underline{\text{Hom}}(C_i, E_i)_n = \prod_i \text{Hom}_\mathcal{A}(C_i, E_{i+n}). \tag{1.13}$$

If $(f_i : C_i \to E_{i+n})$ is an element of $\underline{\text{Hom}}(C_i, E_i)_n$ then its differential in the sequence $\underline{\text{Hom}}$ equals

$$d(f_i) = (g_i : C_i \to E_{i+n-1}), \text{ where } g_i = d_E f_i - q^i f_{i+1} d_C. \tag{1.14}$$
(b) Let \((V, d_V), (W, d_W)\) be two sequences of \(C\)-vector spaces. Their \(q\)-tensor product is the sequence \(V \otimes W\) defined by

\[
(V \otimes W)_n = \bigoplus_{i+j=n} V_i \otimes W_j; \quad d(v \otimes w) = d_V(v) \otimes w + q^{\deg(v)}v \otimes d_W(w) \tag{1.15}
\]

where \(v\) and \(w\) are supposed to be homogeneous elements.

**Proposition 1.8.** Let \(q = \epsilon_N\) be a primitive \(N\)-th root of 1. Then the \(q\)-tensor product and internal \(q\)-\(\text{Hom}\) of two \(N\)-complexes are \(N\)-complexes of vector spaces.

**Proof:** Let us introduce the Gaussian binomial coefficients

\[
\left[ \begin{array}{c} N \\ k \end{array} \right]_q = \frac{(1 - q^N)(1 - q^{N-1})... (1 - q)}{(1 - q^k)...(1-q)(1- q^{N-k})...(1-q)}. \tag{1.16}
\]

They satisfy the recursion properties

\[
\left[ \begin{array}{c} N+1 \\ k \end{array} \right]_q = \left[ \begin{array}{c} N \\ k \end{array} \right]_q q^k + \left[ \begin{array}{c} n \\ k-1 \end{array} \right]_q. \tag{1.17}
\]

and are symmetric with respect to the replacement of \(k\) by \(N - k\). These properties imply the following fact.

**Lemma 1.9.** Let \(q \in C\) and \((V, d_V), (W, d_W)\) - two sequences of vector spaces, \(v \in V_a, w \in W_b\). Then for any \(n \geq 0\) we have the following equality in the \(q\)-tensor product \(V \otimes W\).

\[
d^n(v \otimes w) = \sum_{k=0}^n q^{(n-k)a} \left[ \begin{array}{c} n \\ k \end{array} \right]_q d^k u \otimes d^{n-k}v. \tag{1.18}
\]

If \(C, E\) are two sequences in an Abelian category \(A\) then for any collection \(f = (f_i : C_i \to E_{i+a})\) in \(\text{Hom}(C, E)_a\) and any \(n \geq 0\) we have the following equality in the \(q\)-\(\text{Hom}\)-sequence:

\[
d^n(f) = \sum_{k=0}^n (-1)^{n-k} q^{(n-k)a} \left[ \begin{array}{c} n \\ k \end{array} \right]_q d^k_E \circ f \circ d^{n-k}_C. \tag{1.19}
\]

Proposition 1.8 follows from Lemma 1.9 when we note that for \(q = \epsilon_N\) all the Gaussian coefficients \(\left[ \begin{array}{c} N \\ k \end{array} \right]_q\) vanish except \(\left[ \begin{array}{c} N \\ 0 \end{array} \right]_q = \left[ \begin{array}{c} N \\ N \end{array} \right]_q = 1.\)

**Proposition 1.10.** Fix a base \(q \in C\). For any three sequences \(C, D, E\) in a category \(A\) we have a natural morphism of sequences of vector spaces

\[
\text{Hom}(C, D) \otimes \text{Hom}(D, E) \to \text{Hom}(C, E)
\]

which takes an element \((f_p) \otimes (g_p)\) of \(\text{Hom}(C, D)_m \otimes \text{Hom}(D, E)_n\) to the element \((\epsilon)^{mn} g_{p+m} \circ f_p\) of \(\text{Hom}(C, E).\)

**Proof:** A straightforward checking.
Proposition 1.10. means that for the differential of the composition of morphisms (not necessarily chain morphisms) we have the \( q \)-Leibniz rule:

\[
d(fg) = (df)g + q^{dg(f)}f(dg).
\] (1.20)

It is clear from definition 1.7 that a chain morphism between \( N \)-complexes \( C \) and \( E \) is nothing but an element of degree 0 in the complex \( \text{Hom}(C, E) \) annihilated by the differential \( d \). Call a morphism \( f : C \to E \) null-homotopic if it lies in the image of the operator \( d^{N-1} \) in \( \text{Hom}(C, E) \). In other words, \( f \) is null-homotopic if there exists a collection of morphisms \( s_i : C_i \to E_{i+N-1} \) such that

\[
f_i = s_{i-N+1}d^{N-1} + \epsilon ds_{i-N+2}d^{N-2} + \ldots + \epsilon^{N-1}d^{N-1}s_i.
\]

This follows from Lemma 1.9 after suitable renormalization of \( s_i \).

**Proposition 1.11.** A null-homotopic morphism \( f : C \to E \) of \( N \)-complexes induces a zero map on all the homology objects \( pH_i \).

**Proof:** Let \( z \in C_i \) and \( d^p z = 0 \) for some \( p < N \). Then

\[
f(z) = \epsilon^{N-p}d^{N-p}s_{i-N+1}d^{N-1}z + \epsilon^{N-p+1}d^{N-p+1}s_{i-N+2}d^{N-2}z + \ldots + \epsilon^{N-1}d^{N-1}s_i z.
\]

This expression obviously lies in \( \text{Im}(d^{N-p}) \) so the class of \( f(z) \) in \( pH_i \) equal 0, QED.

**Proposition 1.12.** Null-homotopic morphisms form an ideal in the category \( N-\text{Com}(\mathcal{A}) \).

**Proof:** Let \( f, g \) be two morphisms of \( N \)-complexes such that the composition \( fg \) is defined. Suppose that \( f \) is null-homotopic, i.e. \( f = d^{N-1} s \). Since \( dg = 0 \), we have, by iterating Leibniz rule (1.9) that \( d^{N-1}(sg) = (d^{N-1}s)g = fg \) so \( fg \) is null-homotopic. Similarly we prove that if \( g \) is null-homotopic that so is \( fg \).

The quotient category of \( N-\text{Com}(\mathcal{A}) \) modulo the ideal of null-homotopic morphisms will be denoted \( N-\text{Hot}(\mathcal{A}) \) and called the homotopy category of \( N \)-complexes.

**C. \( q \)-Euler characteristic.**

We shall consider, for simplicity, finite \( N \)-complexes of finite-dimensional complex vector spaces. If \( C \) is such an \( N \)-complex, we define its Poincaré polynomial \( P_C(q) = \sum \dim(C_i)q^i \). The usual Euler characteristic for a 2-complex is the value of the Poincaré polynomial at (-1). Its main property is that it vanishes for an exact 2-complex. Let us give a generalization of this fact for \( N \)-complexes.

**Proposition 1.13.** Let \( C \) be an \( N \)-exact \( N \)-complex and \( \epsilon \) be a primitive \( N \)-th root of 1. Then \( P_C(\epsilon) = 0 \).

**Proof:** Denote individual components of the differential by \( d_i : C_i \to C_{i-1} \). Let us denote the dimension of a vector space \( V \) shortly \([V]\). Then we have

\[
[C_i] = [\text{Ker}(d_i)] + [\text{Im}(d_i)],
\] (1.21)
\[ [C_i] = [\text{Ker}(d_{i+N-1} \circ \ldots \circ d_{i+1} \circ di)] + [\text{Im}(d_{i+N-1} \circ \ldots \circ d_{i+1} \circ di)]. \] (1.22)

By $N$-exactness we have
\[ \text{Ker}(d_i) = \text{Im}(d_{i-1} \circ \ldots \circ d_{i-N+1}), \quad \text{Im}(d_i) = \text{Ker}(d_{i+N} \circ \ldots \circ d_{i+1}). \] (1.23)

Therefore by (1.21) and (1.23) we have
\[ P_C(\epsilon) = \sum \epsilon^i([\text{Ker}(d_i)] + [\text{Im}(d_i)]) = \]
\[ = \sum \epsilon^i([\text{Im}(d_{i+N-1} \circ \ldots \circ d_{i+1} \circ di)] + [\text{Ker}(d_{i+N-1} \circ \ldots \circ d_{i+1} \circ di)]) \]

Now formula (1.22) implies that $P_C(\epsilon) = \epsilon P_C(\epsilon)$ hence $P_C(\epsilon) = 0$. 
§2. Differential forms and de Rham $N$-complexes.

The theory of differential forms which we are about to develop, will be explicitly coordinate-dependent. Fix a complex number $q$.

Let $(x_1, ..., x_n)$ be coordinates in a real affine space of dimension $n$. Introduce formal differentials $dx_1, ..., dx_n$ which we will make subjects to the relations

$$f(x)dx_i = dx_if(x)$$

(2.1)

for any differentiable function $f(x)$ and

$$dx_idx_i = 0; \quad dx_idx_j = qdx_jdx_i \quad \text{for} \quad i > j$$

(2.2)

The algebra generated by functions $f(x)$ and the symbols $dx_i$ will be called the $(q$-analog of) algebra of differential forms on $\mathbb{R}^n$ and denoted $\Omega_q,\mathbb{R}^n$. It has the obvious grading where functions have degree 0 and $dx_i$ have degree 1. A homogeneous element of degree $k$ (a $k$-form) can be written uniquely in the form

$$\omega = \sum_{1 \leq i_1 < ... < i_k \leq n} f_{i_1,...,i_k}(x)dx_{i_1}...dx_{i_k}$$

(2.3)

and thus can be identified with the collection of $f_{i_1,...,i_k}(x)$. Define the exterior differential of the form (2.3) to

$$d\omega = \sum_{1 \leq j_1 < ... < j_{k+1} \leq n} g_{j_1,...,j_{k+1}}(x)dx_{j_1}...dx_{j_{k+1}},$$

(2.4)

where

$$g_{i_1,...,i_{k+1}}(x) = \sum_{p=1}^{k+1} q^{p-1} \frac{\partial f_{j_1,...,j_p,...,j_{k+1}}}{\partial x_p}$$

(2.5)

**Proposition 2.1.**

(a) The differential $d$ satisfies the $q$-analog of the Leibniz rule: for $u \in \Omega^i, v \in \Omega^j$ we have

$$d(uv) = d(u)v + q^i ud(v).$$

(2.6)

(b) Suppose that $q$ is a primitive $N$-th root of unity. Then $d^N = 0$.

**Proof:** For 0-forms the differential is given by the usual formula: $df = \sum(\partial f/\partial x_i)dx_i$. The formula (2.5) expresses exactly the $q$-Leibniz rule for $d(f(x).dx_{i_1}...dx_{i_k})$. The $q$-Leibniz rule for a general product follows from this. Part (b) of Proposition 2.1 is a consequence of the following lemma which we leave to the reader.
Lemma 2.2. Let $A = \bigoplus A_i$ be a graded $\mathbb{C}$-algebra and $d : A \to A$ be a linear map of degree +1 satisfying the $q$-Leibniz rule (2.6) for some $q \in \mathbb{C}$. Then for each $N \geq 0$, $u \in A_i, v \in A_j$ we have

$$d^N(uv) = \sum_{p=0}^{N} q^{ip} \left[ \begin{array}{c} N \\ p \end{array} \right] q^p d^p(u)d^{N-p}(v)$$

(2.7)

Remark 2.3. Our $q$-deformed de Rham differential can be seen as a particular case of the construction of §9, which produces an $N$-complex out of every simplicial vector space. Indeed, our $\Omega^k_q$ as vector spaces (and modules over functions) do not depend on $q$, though their exterior multiplication does. Define vector spaces $V_{-1} = \Omega^n$, $V_0 = \Omega^{n-1}$ etc. For any $p$ introduce linear operators $\partial_\nu : V_p \to V_{p-1}$ for $0 \leq \nu \leq p$. The vector space $V_p = \Omega^{n-p-1}$ is a direct sum of the spaces of forms of the type $f(x)dx_{j_1}...dx_{j_{n-p-1}}$ for all sequences $1 \leq j_1 < ... < j_{n-p-1} \leq n$. We define $\partial_i$ on these spaces separately. More precisely, let $i_0 < ... < i_p$ be all the elements of $\{1,...,n\}$ not lying in $\{j_1,...,j_{n-p-1}\}$, taken in the increasing order. We set

$$\partial_\nu(f(x)dx_{j_1}...dx_{j_{n-p-1}}) = \left( \frac{\partial f}{\partial x_{i_\nu}} \right) dx_{j_1}...dx_{i_\nu}...dx_{j_{n-p-1}}$$

where $dx_{i_\nu}$ is inserted to such place that the indices of differentials increase. It is immediate to verify that operators $\partial_\nu$ thus defined satisfy the equations (0.2) i.e. that $V.$ forms an (augmented, because of $V_{-1}$) simplicial vector space without degenerations. Now the standard de Rham differential equals $\sum (-1)^\nu \partial_\nu$, whereas our $q$-deformed differential equals $\sum q^\nu \partial_\nu$.

Proposition 2.4. Let $N \leq n + 1$. Then the cohomology spaces $\mu H^i(\Omega)$ of the de Rham $N$-complex of polynomial $N$-forms equal 0 for $i + p + 1 \leq N$.

Proof: Denote the variables $dx_i$ by $\xi_i$. Introduce the $q$-derivations $\partial/\partial \xi_i$ on $\Omega$ such that

$$\partial/\partial \xi_i(\xi_j) = \delta_{ij}, \; \partial/\partial \xi_i(x_j) = 0$$

Similarly introduce usual derivations $\partial/\partial x_i$. Then the exterior differential $d$ equals $\sum \xi_i \partial/\partial x_i$. Consider the operator $S = (\sum x_i \partial/\partial \xi_i)^{N_1}$ as a homotopy in the de Rham complex. We introduce in $\Omega^r$ the double grading $\bigoplus \Omega^r(j)$ where $\Omega^r(j)$ is the subspace of forms homogeneous of degree $i$ with respect to $\xi$’s and of degree $j$ with respect to $x$’s. The map $d^i S d^{N_1-i}$ is calculated explicitly to be an isomorphism on each $\Omega^r(s)$ with $r + s \leq N - i$. The assertion follows from this.

Let us now discuss the question of covariance of the constructed exterior differential calculus. Since the construction makes an explicit appeal to coordinates, our algebra of forms and the exterior differential are not invariant under the group $GL(n)$ of linear changes of coordinates. However, it is possible to introduce a certain Hopf algebra which
preserves all the picture. This algebra is a version of quantized $GL(n)$ which we shall now describe.

Let $a_{ij}$, $i, j = 1, ..., n$ be independent non-commutative generators which we arrange into a matrix $A = ||a_{ij}||$. We multiply formally $A$ from the left to the row vector of variables $x = (x_1, ..., x_n)$, obtaining $Ax$, where $(Ax)_i = \sum a_{ij}x_j$. Similarly we multiply $A$ from the left to the column vector $dx = (dx_1, ..., dx_n)^t$, obtaining $dx.A$, where $(dx.A)_i = \sum A_{ij}dx_j$.

Let us now require that $(Ax)_i$ and $(dx.A)_i$ commute in the same way as $x_i$ and $dx_i$ do i.e.:

$$(Ax)_i(Ax)_j = (Ax)_j(Ax)_i \quad \forall i, j \quad \text{and} \quad (dx.A)_i(dx.A)_j =$$

$$= q(dx.A)_j(dx.A)_i \quad \text{for} \quad i > j, \quad (dx.A)_i^2 = 0. \quad (2.8)$$

Here we understand that $x_i$ or $dx_i$ satisfy the imposed relations. By taking coefficients of relations (2.8) by $x_i x_j$ or $dx_j dx_i$, $i \leq j$, we obtain explicit relations:

$$a_{ij}a_{ik} = a_{ik}a_{ij}; \quad a_{ij}a_{kj} = qa_{kj}a_{ij} \quad (2.9)$$

$$a_{ii}a_{jj} - a_{jj}a_{ii} = a_{ji}a_{ij} - a_{ij}a_{ji} = q^{-1}a_{ij}a_{ji} - qa_{ji}a_{ij} \quad \text{for} \quad i < j \quad (2.10)$$

Denote by $R(n)$ the associative algebra generated by $a_{ij}$, $i, j \leq n$ subject to the relations (2.9) and (2.10).

**Proposition 2.4.**

(a) The formula $\Delta(a_{ij}) = \sum_k a_{ik} \otimes a_{kj}$ defines a Hopf algebra structure $\Delta : R(n) \otimes R(n) \to R(n)$ on $R(n)$.

(b) The formulas $\alpha(x_j) = \sum_i a_{ij} \otimes x_i, \alpha(dx_j) = \sum_j a_{ij} \otimes dx_j$ defines a left coaction $\alpha : \Omega_q \to R(n) \otimes \Omega_q$ of $R(n)$ on the $q$-de Rham algebra.

(c) The coaction $\alpha$ commutes with the $q$-exterior differential $d$ i.e., $\alpha \circ d = (1 \otimes d) \circ \alpha$.

**Proof:** Our construction of the algebra $R(n)$ is a slight modification of the construction of internal $Hom$ of quadratic algebras introduced by Y.I.Manin in [2]. Let $A$ be a homogeneous quadratic algebra with generators and relations $(V, I \subset V \otimes V)$. For such $A$ Manin introduces a Hopf algebra $\mathcal{End}(A)$ of ”endomorphisms” of these relations so that $\mathcal{End}(A)$ coacts on $A$ from the right. Similarly one can consider another group of relations on the dual space $(V^*, J \subset V^* \otimes V^*)$ and require a left coaction on the quadratic algebra $D$ defined by latter relations. One can require both coactions, thus obtaining a Hopf algebra $\mathcal{End}(D, A)$. It is straightforward to see, using methods of [2] that $\mathcal{End}(D, A)$ is always a Hopf algebra. Our construction is a particular case of $\mathcal{End}(D, A)$, where $A$ is the symmetric algebra and $D$ is the $q$-exterior algebra. This establishes (a) and (b).

**Remark.** The standard $q$-analog of group $GL(n)$ is obtained as $\mathcal{End}(D, A)$ where $D$ is the $q$-exterior algebra, as in our case, whereas $A$ is a $q$-symmetric algebra $x_i x_j = qx_jx_i$, see [2].
§3. Connections and curvature.

Let \( q \in \mathbb{C} \) be a fixed complex number. Consider a trivial vector bundle \( E \) of rank \( r \) over some domain \( U \subset \mathbb{R}^n \). Let \( x_1, ..., x_n \) be the standard coordinates in \( \mathbb{R}^n \) and \( \Omega = \Omega_q \) the graded algebra of differential forms on \( U \), introduced in §2. The differential in this algebra will be denoted by \( d \). We denote by \( \Omega(E) \) the graded \( \Omega \)-module formed by forms with values in \( \mathbb{C}^r \). By \( \Omega(\text{End}(E)) \) we denote the algebra \( \Omega \otimes \text{End}(E) \). The differential \( d \) extends to this algebra and satisfies the \( q \)-Leibniz rule 2.4.

A \( q \)-connection in \( E \) is, by definition, the operator in \( \Omega(E) \) of the form
\[
\nabla = d + A,
\]
where \( A = \sum A_i(x)dx_i \) is a matrix-valued 1-form. Such an operator also satisfies the \( q \)-Leibniz rule in the form
\[
\nabla(\omega \cdot \Sigma) = (d\omega) \cdot \Sigma + q^{\deg(\omega)} \nabla(\Sigma),
\]
(3.1)
where \( \omega \) is a scalar form of some degree \( d = \deg(\omega) \), and \( \Sigma \) is a vector-valued form.

**Theorem 3.1.** Let \( q = \epsilon_n \) be a primitive \( N \)-th root of unity. Then, for any \( q \)-connection \( \nabla \) the operator \( \nabla^N \) is given by multiplication with some matrix-valued \( N \)-form \( F_\nabla \).

We call \( F_\nabla \) the \( N \)-curvature of the connection \( \nabla \).

Theorem 3.1 is a consequence of the following lemma which is similar to Lemma 1.5.

**Lemma 3.2.** Let \( q \) be any complex number, \( \nabla \) be any \( q \)-connection, \( f \)-any scalar function, \( s \)-any vector function. Then, for any \( n \geq 0 \) we have
\[
\nabla^n(f \cdot s) = \sum_{k=0}^{n} \binom{n}{k}_q (d^k f)(\nabla^{n-k}s).
\]
(3.2)

Let us prove Theorem 3.1. If \( q \) is a primitive \( N \)-root of 1 then , by (3.2), all the Gaussian coefficients \( \binom{N}{k}_q \) vanish except \( \binom{N}{0}_q = \binom{N}{N}_q = 1 \). Substituting \( n = N \) in formula (3.4), we find that
\[
\nabla^N(f \cdot s) = (d^N f)(s) + f(\nabla^N s) = f(\nabla^N s)
\]
since \( d^N = 0 \). This means that the differential operator \( \nabla^N \) has in fact order 0 and thus has the claimed form.

Denote by \( \mathcal{G} \) the group of invertible \((r \times r)\)-matrix-valued functions on \( U \) with differentiable entries. This group acts on connections by conjugation:
\[
\nabla \mapsto g^{-1} \nabla g = g^{-1}(d + A)g.
\]
(3.5)
Therefore the transformation of \( A \) under gauge transformation is given by the usual formula:
\[
A \mapsto g^{-1} dg + g^{-1} A g.
\]
(3.6)
Clearly the curvature is covariant under gauge transformations

\[ F_{g^{-1} \nabla g} = g^{-1} F_{\nabla g}. \quad (3.7) \]

Therefore the set of connections with vanishing $N$-curvature is invariant under the gauge group. Any such connection $\nabla$ produces a vector-valued de Rham $N$-complex. However, it may be impossible to transform, even locally, a connection with vanishing curvature to a trivial connection ($A = 0$) by a gauge transformation.

It is possible to prove the analog of Bianchi identity for curvatures and to develop a version of theory of Chern forms and Chern classes for $q$-connections. We shall assume for the rest of the section that $q = \epsilon = \exp(2\pi i/N).

**Proposition 3.3.** We have $\nabla(F_{\nabla}) = 0$ for any $N$-connection $\nabla$.

**Proof:** By the $q$-Leibniz rule we have

\[ \nabla(F_{\nabla}) = \nabla \circ F_{\nabla} - F_{\nabla} \circ \nabla = \nabla^{N+1} - \nabla^{N+1} = 0. \]

**Corollary 3.4.** The scalar differential form $\text{tr}(F_{\nabla}^p)$ is closed for each $p$.

Consider the particular case $N = 3$. Then the curvature of a connection $\nabla = d + A$ is the 3-form

\[ F_A = d^2(A) + d(A).A + \epsilon A.d(A) + A.A.A. \quad (3.8) \]

The last three terms in the RHS of (3.8) have a striking resemblance with the famous Chern-Simons Lagrangian [3] (for connections in the usual sense over 3-manifolds). Moreover, the term $d^2A$ will not contribute to overall integral. In other words, we have the following fact.

**Proposition 3.5.** Let $\nabla = d + A$ be a 3-connection on $\mathbb{R}^3$ such that the functions $A_i(x)$ have compact support (or are rapidly decreasing). Then we have

\[ \int_{\mathbb{R}^3} \text{tr}F_A = \int_{\mathbb{R}^3} \text{tr}(d(A).A + \epsilon A.d(A) + A.A.A) \quad (3.9) \]

where $\epsilon = \exp(2\pi i/3)$.

**Proof:** It suffices to show that $d^2A$ has zero integral. But in explicit form we have

\[ d^2A = \left( \frac{\partial^2 A_3}{\partial x_1 \partial x_2} + \epsilon \frac{\partial^2 A_2}{\partial x_1 \partial x_3} + \epsilon^2 \frac{\partial^2 A_1}{\partial x_2 \partial x_3} \right) dx_1 dx_2 dx_3 \]

which is a divergence.
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