ON THE CONSTRUCTION OF EQUIVOLUME PARTITIONS OF THE
\(d\)-DIMENSIONAL UNIT CUBE

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Abstract. The aim of this note is to construct equivolume partitions of the \(d\)-dimensional unit cube with hyperplanes that are orthogonal to the main diagonal of the cube. Each such hyperplane \(H_r \subset \mathbb{R}^d\) is defined via the equation \(x_1 + \ldots + x_d + r = 0\) for a parameter \(r\) with \(-d \leq r \leq 0\). For a given \(N\), we characterise those real numbers \(r_i\) with \(1 \leq i \leq N - 1\) for which the corresponding hyperplanes \(H_{r_i}\) partition the unit cube into \(N\) sets of volume \(1/N\). As a main result we derive an algebraic and a probabilistic characterisation of the \(r_i\) for arbitrary \(d \geq 2\) and \(N \in \mathbb{N}\). Importantly, our results do not only work for equivolume partitions but also for arbitrary predefined distributions of volume among the \(N\) sets of the partition generated from hyperplanes \(H_r\).

1. Introduction

1.1. Main question. There are several straightforward ways to partition the \(d\)-dimensional unit cube \([0,1]^d\) into \(N\) sets of equal volume. We could for example partition the interval \([0,1]\) into \(N\) subintervals of equal length and use this to define \(N\) slices of equal volume. In fact, we can partition an arbitrary number of the \(d\) generating unit intervals into subintervals of equal length to generate grid-type partitions of the \(d\)-dimensional unit cube. In every such example it is straightforward to characterise the sets in the partition (since they are axis-parallel rectangles) and to prove that these sets have indeed all the same volume.

The aim of this note is to characterise another family of equivolume partitions that was recently studied in the context of jittered sampling [5] and which will be introduced in the following. Given the unit square \([0,1]^2\) and \(N - 1\) parallel lines \(H_i\) with \(i = 1, \ldots, N - 1\), which are orthogonal to the main diagonal, \(D\), of the square, we would like to arrange the lines such that we obtain an equi-volume partition of the unit square; see Figure 1.

![Figure 1. Partition of the unit cube into \(N = 6\) equivolume slices that are orthogonal to the diagonal.](image)

We denote the intersection \(H_i \cap D\) of a line with the diagonal with \(p_i\). It is straightforward to calculate all points \(p_i\) for arbitrary \(N\). In fact, note that \(H_i\) splits the unit square into two sets of volume \(i/N\) and \(1 - i/N\). If \(i \leq N/2\), we just need to look at the isosceles right triangle that \(H_i\) forms with \((0,0)\). We know that this triangle has volume \(i/N\). Denote the length of the two equal sides with \(a_i > 0\), then \(a_i^2/2 = i/N\) and \(p_i = a_i/2\). Therefore, we get that

\[
(1) \quad p_i = \sqrt{\frac{i}{2N}},
\]
for all \( i \leq N/2 \). By symmetry, we also get the points \( p_i \) with \( i > N/2 \). The main question of this note is how to generalise this simple characterisation of equi-volume partitions of the unit square to dimensions \( d > 2 \).

To state this problem formally, we introduce further notation. Let \( H_r^+ \) be the positive half space defined as the set of all \( x \in \mathbb{R}^d \) satisfying
\[
x_1 + x_2 + \ldots + x_d + r \geq 0;
\]
(2) accordingly let \( H_r^- \) be the corresponding negative half space and let \( H_r \) be the hyperplane of all points \( x \in \mathbb{R}^d \) satisfying \( x_1 + x_2 + \ldots + x_d + r = 0 \). For a given \( N \), we would like to find the points \(-d < r_1 < \ldots < r_{N-1} < 0 \) such that the corresponding hyperplanes \( H_{r_1}, \ldots, H_{r_{N-1}} \) define a partition of \([0, 1]^d\) into \( N \) equivolume sets. We call the set
\[
S(N, d) := \{ r_i : i = 1, \ldots, N-1 \}
\]
the generating set of the partition. Using [1] we see that
\[
S(N, 2) = \left\{ -\sqrt{\frac{2i}{N}} : 1 \leq i \leq N/2 \right\} \cup \left\{ -1 - \sqrt{\frac{2i}{N}} : 1 \leq i < N/2 \right\}.
\]
Note that for even \( N \) the point \(-1\) is in the set, while it is not for odd \( N \).

**Problem 1.** For given dimension \( d \) and number of sets \( N \), characterise the set \( S(N, d) \).

1.2. **Results.** In this note, we provide two results which yield separate characterisations for the set \( S(N, d) \). First, we use a result from [3] to derive an algebraic characterisation.

For \( d \geq 2, -d \leq r \leq 0 \) and \( 1 \leq k \leq d \), we define
\[
f(k) := \frac{1}{d!} \sum_{j=0}^{k-1} (-1)^j \binom{d}{j} (k-j)^d
\]
and set \( f(0) = 0 \).

**Theorem 1.** For arbitrary \( r \in [-d, 0] \) and corresponding positive halfspace \( H_r^+ \), define \( V_d^+: [-d, 0] \to [0, 1] \) with
\[
V_d^+(r) = \text{vol} \left( [0, 1]^d \cap H_r^+ \right).
\]
For arbitrary \( V \in [0, 1] \) let the integer \( 1 \leq k \leq d \) be such that \( f(k-1) \leq V \leq f(k) \). The parameter \( r \) of the positive half space \( H_r^+ \) such that \( V_d^+(r) = V \) is a solution of the polynomial equation:
\[
\frac{1}{d!} \sum_{j=0}^{k-1} (-1)^j \binom{d}{j} (d-j+r)^d = V_d^+(r).
\]
Solving for \( r_i \) with \( V_d^+(r_i) = 1 - i/N \) and \( 1 \leq i \leq N-1 \) gives an algebraic characterisation of the set \( S(N, d) \). However, this description is not very practical as explained in the subsequent sections. Therefore, we derive a second result which gives a probabilistic and more practical albeit approximate characterisation of \( S(N, d) \).

**Theorem 2.** For \( d \geq 2 \), let \( H_r^- = \{ x \in \mathbb{R}^d : x_1 + \ldots + x_d + r \leq 0 \} \) denote the negative half space with parameter \(-d \leq r \leq 0 \) and let \( V_d^- : [-d, 0] \to [0, 1] \) with \( V_d^-(r) = \text{vol} \left( [0, 1]^d \cap H_r^- \right) \). If \( \Phi \) is the cumulative distribution function of the standard normal distribution, then
\[
V_d^-(r) \to \Phi \left( 2\sqrt{3d} \left( -\frac{r}{d} - \frac{1}{2} \right) \right)
\]
as \( d \to \infty \) for all \( r \in [-d, 0] \), i.e., \( V_d^- \) converges pointwise to \( \Phi \).
Note that this convergence is even uniform; see \([3]\) for a uniform error bound based on the Berry-Esseen Theorem.

To get the desired characterisation of the set \(S(N, d)\) containing the values \(r_i\) such that \(V_d^+(r_i) = \text{vol}(H_r^+ \cap [0, 1]^d) = 1 - i/N\) for \(1 \leq i \leq N - 1\), we apply the well known symmetry of the quantile function,

\[
\Phi^{-1}(x) = -\Phi^{-1}(1 - x)
\]

for \(x \in [0, 1]\). Hence using the relation \(V_d^-(r) = 1 - V_d^+(r) = i/N\), we can deduce that for \(1 \leq i \leq N - 1\)

\[
r_i \approx -\frac{d}{2} + \frac{\sqrt{d}}{2\sqrt{3}} \Phi^{-1}\left(\frac{i}{N}\right)
\]

as \(d \to \infty\). Therefore with the above observations, Theorem 2 gives the following probabilistic characterisation:

\[
S(N, d) \approx \left\{-\frac{d}{2} + \frac{\sqrt{d}}{2\sqrt{3}} \Phi^{-1}\left(\frac{i}{N}\right) : i = 1, \ldots, N - 1\right\}.
\]

1.3. Outline. Section 2 recalls important results and introduces further notation. In Section 3 we illustrate the algebraic method in the special case of \(d = 3\), outlining the difficulties that occur already in very small dimensions. In Section 4 we prove Theorem 1 while Theorem 2 is proved in Section 5. The final section contains numerical results and a discussion of the accuracy of our probabilistic characterisation.

2. Preliminaries

It is a computationally very hard problem to calculate the volume of a convex polytope [4] which inspired much research on approximation methods as well as exact calculations. In [3] the authors study the special case in which a polytope is obtained from a hypercube by clipping hyperplanes. The resulting formulas do not require heavy machinery but are technically complicated as much as they are intriguing.

For our note, the simple case of a hypercube clipped by only one hyperplane is of interest. Our starting point is an interesting formula whose main idea can be traced back to a paper of Polya [6] and which seems to have appeared for the first time in [1]. We refer to the introduction of [3] for more information.

**Theorem 3.** [3, Theorem 1] Let \(H_r^+\) be the halfspace defined by

\[
\{x \in \mathbb{R}^d | g(x) := a \cdot x + r = a_1 x_1 + a_2 x_2 + \ldots + a_d x_d + r \geq 0\},
\]

with \(\prod_{i=1}^d a_i \neq 0\). Then we have

\[
\text{vol}\left([0, 1]^d \cap H_r^+\right) = \sum_{v \in F^0 \cap H_r^+} (-1)^{|v_0|} g(v)^d \frac{d!}{\prod_{i=1}^d a_i},
\]

in which \(F^0\) corresponds to the set of vertices of the cube, and \(|v_0|\) counts the number of zeros in the vector \(v\).

Our problem is in a way the inverse of the main problem studied in [3]. We know the volume of the intersection of a particular half space \(H_r^+\) with the unit cube (and we know the normal vector of the hyperplane) and we are interested in the offset \(r\). Thus, we can use Theorem 3 to obtain an expression for the parameter \(r\) given the volume of the intersection of the cube with the hyperplane. To illustrate the theorem we look again at the two-dimensional case. For \(i \leq N/2\) we have that \(\text{vol}([0, 1]^2 \cap H_r^+) = 1 - i/N\). Moreover, we know that the three vertices \((1, 0), (0, 1)\) and \((1, 1)\) of the unit square are contained in \([0, 1]^2 \cap H_r^+\). Hence, we obtain

\[
1 - \frac{i}{N} = -\frac{(1 - r_i)^2}{2} - \frac{(1 - r_i)^2}{2} + \frac{(2 + r_i)^2}{2},
\]
which can be simplified to

\[ r^2_i = \frac{2i}{N}. \]

This coincides, up to the choice of the correct sign, with the result we already derived in the introduction. Note that even in the two-dimensional case we see that solving the problem algebraically requires us to pick our desired solution from a range of possibilities.

### 3. The special case \( d = 3 \)

To further illustrate the algebraic method, we explicitly solve the special case \( d = 3 \). This case is still easy to visualise and already illustrates the difficulties we are facing beyond the two-dimensional case. Let \([0, 1]^3\) be the three-dimensional unit cube with vertices labelled \(v_1 = (0, 0, 0), v_2 = (1, 0, 0), v_3 = (0, 1, 0), v_4 = (0, 0, 1), v_5 = (1, 1, 0), v_6 = (1, 0, 1), v_7 = (0, 1, 1)\) and \(v_8 = (1, 1, 1)\). We ask for a characterisation of the set \(S(N, 3)\). To be more precise, every point on a hyperplane \(H_{r_i}\) satisfies

\[ x_1 + x_2 + x_3 + r_i = 0 \]

and we are interested in the points \(r_i\) such that \(\text{vol}([0, 1]^3 \cap H^+_r) = 1 - i/N\) for \(1 \leq i \leq N - 1\).

The following lemma clarifies which vertices of the unit cube are contained in a given half space \(H^+_r\) for arbitrary \(-3 \leq r < 0\). Note that every half space generated by \(r \geq 0\) trivially contains all the vertices of the cube, while every half space generated by \(r < -3\) contains none of the vertices.

**Lemma 4.** Let \(H^+_r\) be the positive half space for \(-3 \leq r < 0\). Then

\[
\begin{align*}
\{v_8\} \subset H^+_r, & \quad \text{for } -3 \leq r < -2 \\
\{v_5, v_6, v_7, v_8\} \subset H^+_r, & \quad \text{for } -2 \leq r < -1 \\
\{v_2, \ldots, v_8\} \subset H^+_r, & \quad \text{for } -1 \leq r < 0.
\end{align*}
\]

**Proof.** This follows from direct calculation for each vertex. For example, for \(v_8 = (1, 1, 1)\) we have that

\[ x_1 + x_2 + x_3 + r = 1 + 1 + 1 + r = 3 + r \geq 0, \]

for all \(r\) with \(-3 \leq r\). Hence, \(v_8 \in H^+_r\) for \(-3 \leq r\). The other vertices can be checked in a similar fashion. \(\square\)

**Remark 5.** Note that when \(r = -1\) the volume of the intersection is \(5/6\) and when \(r = -2\), the volume of the intersection is \(1/6\). In other words, as soon as \(N > 6\) we have to consider all cases in our analysis.

The next lemma characterises the points \(r_i\) as roots of polynomials of degree 3.

**Lemma 6.** For arbitrary \(r \in [-3, 0]\) and corresponding positive halfspace \(H^+_r\), define \(V^+_3 : [-3, 0] \to [0, 1]\) with

\[
V^+_3(r) := \text{vol}([0, 1]^3 \cap H^+_r).
\]

Fixing a value \(V \in [0, 1]\), the parameter value \(r\) such that \(V^+_3(r) = V\) can be obtained as a solution of the corresponding polynomial equation:

\[
0 = \begin{cases} (3 + r)^3 - 6V & \text{if } 0 \leq V < \frac{1}{6}; \\ (3 + r)^3 - 3(2 + r)^3 - 6V & \text{if } \frac{1}{6} \leq V < \frac{5}{6}; \\ (3 + r)^3 - 3(2 + r)^3 + 3(1 + r)^3 - 6V & \text{if } \frac{5}{6} \leq V < 1. \end{cases}
\]

**Proof.** We use Theorem \(3\) and consider the three cases separately. From Lemma \(4\) and Remark \(5\) and for \(0 \leq V \leq \frac{1}{6}\), only the vertex \(v_8\) is in the intersection \(F^0 \cap H^+_r\). Therefore, \(g(v_8) = (1, 1, 1) \cdot (1, 1, 1) + r = 3 + r\), and we obtain

\[
V = \frac{(-1)^0(3 + r)^3}{3! \cdot 1}
\]
which can be rewritten as

\[ 0 = (3 + r)^3 - 6V. \]

For the second case, if \( \frac{1}{6} \leq V \leq \frac{5}{6} \), then by Lemma 4 the vertices \( \mathbf{v}_5, \mathbf{v}_6, \mathbf{v}_7 \) and \( \mathbf{v}_8 \) are in the intersection. Since the vectors \( \mathbf{v}_5, \mathbf{v}_6 \) and \( \mathbf{v}_7 \) all contain one zero, we have that \( g(\mathbf{v}_i) = 2 + r \) for \( i = 5, 6, 7 \). Hence we get,

\[ V = \frac{(-1)^0(3 + r)^3}{3! \cdot 1} + 3 \frac{(-1)^1(2 + r)^3}{3! \cdot 1} \]

or more simply

\[ 0 = (3 + r)^3 - 3(2 + r)^3 - 6V. \]

Finally, for \( \frac{5}{6} \leq V < 1 \) we have all the vertices in the intersection except \( \mathbf{v}_1 \). Note that the vectors \( \mathbf{v}_2, \mathbf{v}_3 \) and \( \mathbf{v}_4 \) all contain two zero components hence \( g(\mathbf{v}_i) = 1 + r \) for \( i = 2, 3, 4 \). Therefore,

\[ V = \frac{(-1)^0(3 + r)^3}{3! \cdot 1} + 3 \frac{(-1)^1(2 + r)^3}{3! \cdot 1} + 3 \frac{(-1)^2(1 + r)^3}{3! \cdot 1} \]

which can be rewritten as,

\[ 0 = (3 + r)^3 - 3(2 + r)^3 + 3(1 + r)^3 - 6V \]

as required. \( \square \)

Lemma 4 can be used to determine which of the three solutions that we obtain for \( r \) has to be picked. We obtain the following full characterisation of \( r \) for a given \( V \):

\[
 r = \begin{cases} 
 -3 + \sqrt[3]{6V} & \text{if } 0 \leq V < \frac{1}{6}; \\
 \frac{1}{4} \sqrt[3]{5(1 + i\sqrt{3})} \sqrt{C + 4V - 2} + \frac{3^{2/3}(1-i\sqrt{3})}{4\sqrt{C+4V-2}} - \frac{3}{2} & \text{if } \frac{1}{6} \leq V < \frac{5}{6}; \\
 (-1)^{2/3} \sqrt[3]{6(1) - 1} & \text{if } \frac{5}{6} \leq V < 1,
\end{cases}
\]

in which \( C = \sqrt{16V^2 - 16V + 1} \). The resulting function is illustrated in Figure 2.

### 4. The General Case

We proceed to the general \( d \)-dimensional case. In order to prove Theorem 1, Lemmas 4 and 6 must be generalised.

We note that one of the key ingredients of the formula in Theorem 3 is to determine which vertices lie inside the half space and subsequently counting how many of these vertices contain \( k \) zeros, with \( 1 \leq k \leq d \). In the context of our problem, i.e. when considering hyperplanes orthogonal to the main diagonal, this can be done in a simple manner that is summarised below in Lemma 7.
Lemma 7. For \( d \geq 2 \), let \( H_r^+ \) be the positive half space for \( r \) with \( -d \leq r < 0 \). Let \( F^0 = \{v_1, \ldots, v_{2^d}\} \) be the set of vertices of the \( d \)-dimensional hypercube \([0,1]^d\). For an integer \( 1 \leq k \leq d \), let \( V_k := \{v \in F^0 : |v_0| < k\} \) where \(|v_0|\) counts the number of zeros in a vector \( v \). Then, for every \( k \leq [r + d] \) we have that

\[
V_k \subset H_r^+.
\]

Moreover, \(|V_k \setminus V_{k-1}| = \binom{d}{k-1}\).

Proof. Let \( 1 \leq k \leq d \). By definition, a vertex \( v = (v_1, \ldots, v_d) \) is in \( V_k \) if and only if \( v \) contains at most \( k - 1 \) zeros. Hence this vertex contains at least \( d - (k - 1) \) components equal to one. Therefore given \( k \) such that \( k \leq [r + d] \) and a vector \( v \in V_k \),

\[
v_1 + v_2 + \cdots + v_d + r \geq d - (k - 1) + r \geq d - (k - 1) - d + k - 1 = 0
\]
since \( k \leq [r + d] \Rightarrow r \geq -d + k - 1 \). That is, \( v \in H_r^+ \) and hence \( V_k \subset H_r^+ \) as required. Finally, there are \( \binom{d}{k-1} \) vertices containing exactly \( k - 1 \) ones and hence by definition these are precisely the elements in the difference \( V_k \setminus V_{k-1} \).

Using the previous result, we now give a closed form expression for the volume of the intersection of \([0,1]^d\) and a positive halfspace \( H_r^+ \) for an integer \( r \) with \( -d \leq r \leq 0 \). This will be used to calculate the critical values of the volume of intersection, \( V_d^+(r) \), determining the form of our algebraic equation for \( r \); see Remark 5.

Lemma 8. For \( d \geq 2 \), let \( r = -d + k \) be an integer with \( 1 \leq k \leq d \) and let \( H_r^+ \) be the corresponding positive half space. Define \( V_d^+ : [-d,0] \to [0,1] \) with

\[
V_d^+(r) := \text{vol} \left( [0,1]^3 \cap H_r^+ \right).
\]

Then

\[
V_d^+(r) = \frac{1}{d!} \sum_{j=0}^{k-1} (-1)^j \binom{d}{j} (k-j)^d.
\]

Proof. By definition, every vertex \( v \) of \([0,1]^d\) has \( d - |v_0| \) components equal to one with the rest equalling zero. Hence from the formula in Theorem 3 with \( g(v) = d - |v_0| + r \)

\[
(7) \quad \text{vol} \left( [0,1]^d \cap H_r^+ \right) = \sum_{v \in F^0 \cap H_r^+} (-1)^{|v_0|(d - |v_0| + r)^d}.
\]

Now given \( 0 \leq k \leq d \), let \( r = -d + k \) which implies \( V_k \subset H_r^+ \), from Lemma 7. Moreover, we observe that for every \( i \leq k \), \( v \in V_i \setminus V_{i-1} \) implies \( |v_0| = i - 1 \) and \( |V_i \setminus V_{i-1}| = \binom{d}{i-1} \). Therefore we obtain a closed formula for the volume of intersection for each integer \(-d \leq r \leq 0\) as follows

\[
\text{vol} \left( [0,1]^d \cap H_r^+ \right) = \sum_{v \in F^0 \cap H_r^+} (-1)^{|v_0|(d - |v_0| + r)^d} = \sum_{i=1}^{k} \frac{(i-1)^{i-1} \binom{d}{i-1} (d - (i-1) + r)^d}{d!} = \sum_{i=1}^{k} \frac{(i-1)^{i-1} \binom{d}{i-1} (d - (i-1) + (-d + k))^d}{d!} = \sum_{i=1}^{k} \frac{(i-1)^{i-1} \binom{d}{i-1} (k - (i-1))^d}{d!} = \sum_{j=0}^{k-1} \frac{(-1)^j \binom{d}{j} (k-j)^d}{d!}.
\]
With this, we can finally prove Theorem 1.

Proof of Theorem 1. For a given fixed volume $V \in \[0, 1\]$, let $1 \leq k \leq d$ be such that $f(k - 1) \leq V \leq f(k)$. Let $r$ be such that $V_d^+(r) = V$, then we know from Lemma 8 that $V_k \subset H_r^+$. Applying a similar argument as in Lemma 8, we have for each $i \leq k$ that $|v_0| = i - 1$ for $v \in \mathcal{V}_i \setminus \mathcal{V}_{i-1}$ and $|\mathcal{V}_i \setminus \mathcal{V}_{i-1}| = \binom{d}{i-1}$. Hence using the formula from Theorem 3,

$$V = \text{vol} \left( \left[0, 1\right]^d \cap H_r^+ \right) = \sum_{v \in \mathcal{F}_0 \cap H_r^+} \frac{(-1)^{|v|}(d - |v_0| + r)^d}{d!} = k \sum_{i=1}^{k-1} \binom{d}{j} (d - j + r)^d.$$

This can be rearranged to yield

$$(8) \quad \frac{1}{d!} \sum_{j=0}^{k-1} (-1)^j \binom{d}{j} (d - j + r)^d = V.$$

Remark 9. The Abel-Ruffini Theorem tells us that such equations have, in general, no closed form solutions beyond dimension 4. Of course, we can solve them numerically to arbitrary degree of accuracy using for example Newton’s method. However, this task requires a significant amount of computation, particularly for larger $N$. We have the added downside of always obtaining the same number of roots as the dimension and it is a-priori not clear which of the $d$ roots is the right one.

5. Approximation by normal distribution

The formula in Theorem 1 is a precise characterisation of $r$ and let us plot the graph of $V_d^+(r)$ for increasing $r$; i.e. knowing that $r \in [-d, 0]$ we can explicitly calculate the volume $V_d^+(r)$ of the intersection $[0, 1]^d \cap H_r^+$. As pointed out in Remark 9, the inverse calculation is, however, limited by fundamental algebraic obstacles.

This motivates the search for a different route to a neat characterisation of the sets $S(N,d)$ for arbitrary parameters $d$ and $N$. We start from the explicit solutions that we have for $d = 3$; see Figure 2. This plot looks very similar to the quantile function of a normal distribution. In fact, if we interpret the set $S(N,3)$ as a point sample in $[-3,0]$, some numerical exploration leads us to conjecture Theorem 2 which we prove in the following.

Proof of Theorem 2. Let $X = (X_1, \ldots, X_d)$ be a uniform random point in $[0, 1]^d$ and let $P$ be its projection onto the diagonal, spanned by the unit vector $v = (1/\sqrt{d})1$, where $1 = (1, \ldots, 1)^T$, i.e.

$$P = (X \cdot v)v = \left( \frac{1}{d} \sum_{i=1}^{d} X_i \right) 1.$$

Let’s consider the length $\|P\| = \sqrt{d} \cdot \frac{1}{d} \sum_{i=1}^{d} X_i$. As $\|P\|$ has support on $[0, \sqrt{d}]$ growing with $\sqrt{d}$ we rescale and consider

$$R = \frac{1}{\sqrt{d}} \|P\| = \frac{1}{d} \sum_{i=1}^{d} X_i.$$

The value $R$ is required in order to allow the implementation of the central limit theorem later, however note that we are actually interested in the value

$$R' = -\sqrt{d} \|P\|.$$
The interpretation of the cumulative distribution function $F_R$ of $R$ is as follows:

$$F_R(t) = \mathbb{P}(R \leq t) = \mathbb{P}(|P| \leq \sqrt{dt}) = \mathbb{P}(R' \geq -dt) = \mathbb{P}(X \in H_{-dt}) = V_d^-(dt),$$

where $0 \leq t \leq 1$, and $H_d = \{ x \in \mathbb{R}^d : x_1 + \cdots + x_d + r \leq 0 \}$ is the negative half space at position $r$ with $V_d^-(r) : [-d,0] \to [0,1]$ denoting the volume of $H_d \cap [0,1]^d$. In other words,

$$V_d^-(r) = F_R\left( -\frac{r}{d} \right)$$

The value $R$ is the average of the variables $X_1, \ldots, X_d$ which are i.i.d. uniform in $[0,1]$ with mean $\mu = \mathbb{E}X_1 = 1/2$ and variance $\sigma^2 = \mathbb{V}ar(X_1) = 1/12$. The central limit theorem states that the normalized variables

$$Z_d = \sqrt{d} \frac{R - \mu}{\sigma} = 2\sqrt{3d}\left( R - \frac{1}{2} \right)$$

converge in distribution to a standard normal as $d \to \infty$. One way to express this convergence is in terms of cumulative distribution functions (CDFs). If $\Phi$ is the CDF of a standard normal, we have

$$F_{Z_d}(t) \to \Phi(t),$$

as $d \to \infty$, for all $t \in \mathbb{R}$ (i.e. $F_{Z_d}$ converges to $\Phi$ pointwise). Combining (9) and (10) we obtain

$$V_d^-(r) = F_R\left( -\frac{r}{d} \right) = F_{Z_d}\left( 2\sqrt{3d}\left( -\frac{r}{d} - \frac{1}{2} \right) \right) \to \Phi\left( 2\sqrt{3d}\left( -\frac{r}{d} - \frac{1}{2} \right) \right)$$

pointwise for all $-d \leq r \leq 0$ as $d \to \infty$. □

As outlined in the introduction, we can now derive our probabilistic characterisation via the symmetry of the quantile function. The rate of the convergence described by the Central Limit Theorem is customarily measured via the Berry-Esseen Theorem [2].

**Theorem 10 (Berry-Esseen Theorem).** Suppose that $X_1, X_2, \ldots, X_n$ are i.i.d. and let $\mu = \mathbb{E}X_1$, $\sigma^2 = \mathbb{V}ar(X_1) = \mathbb{E}\left[ (X_1 - \mu)^2 \right]$ and $\rho = \mathbb{E}\left[ |X_1 - \mu|^{3} \right]$ is the $3^{rd}$ absolute central moment. Let $R = \frac{1}{n} \sum_{i=1}^{n} X_i$ and denote by

$$F_{Z_n}(t) = \mathbb{P}\left( \sqrt{n} \frac{R - \mu}{\sigma} \leq t \right)$$

the cumulative distribution function (CDF) of the normalised sample average. If $\rho < \infty$, then

$$\sup_x |F_{Z_n}(x) - \Phi(x)| \leq \frac{C\rho}{\sigma^3 \sqrt{n}}$$

where $\Phi$ is the CDF of the standard normal distribution and $C$ is an absolute constant.

The currently best bound for the constant is $C < 0.4748$; see [7]. Hence, noting that for our i.i.d random variables $X_1, \ldots, X_d$ we have $\mu = 1/2$, $\sigma = 1/(2\sqrt{3})$ and $\rho = 1/32$ we can implement the Berry-Esseen Theorem to see that the rate of convergence noted in Theorem 2 is

$$\sup_{r \in [-d,0]} |F_{Z_d}(r) - \Phi(r)| \leq \frac{0.4748 \cdot (2\sqrt{3})^3}{32\sqrt{d}} = 0.6167 \ldots \cdot \frac{1}{\sqrt{d}}.$$

6. **Numerical Results**

In this final section we illustrate the results presented earlier in this paper.
6.1. Numerical experiments. As a first numerical experiment, we investigate the convergence in Theorem \(2\). For \(d = 3, 5\) and \(10\) and \(N = 10000\), we numerically calculate parameter values \(r_i\) \((1 \leq i \leq 9999)\) from Theorem \(1\) and subsequently use these values to calculate probabilities from the cumulative distribution function in \(3\). Comparing these probability values at \(r_i\) with the desired cumulative volume of intersection, i.e. \(i/10000\) for \(1 \leq i \leq 10000\), the error is plotted in Figure 3 for increasing dimension. As expected, for increasing dimension the error becomes on average smaller, showing the desired convergence. Note that the values we obtain in our experiments are much smaller than what is guaranteed by the general bound in \(11\).

**Figure 3.** Illustration of convergence to the cumulative distribution function as proved in Theorem \(2\). The \(x\) axis represents the index, \(i\), of the set in the partition with \(1 \leq i \leq 10000\), while the \(y\)-axis contains the absolute difference \(\left| V_d^{-}\left(r_i\right) - \Phi\left(2\sqrt{3d\left(-\frac{d}{2} - \frac{1}{2}\right)}\right)\right|\). Black: \(d = 3\), Gray: \(d = 5\), Red: \(d = 10\).

Next, we would like to know how close the partitions generated by the probabilistic characterisation of \(S(N, d)\) in \(5\) are to being equivolume. We fix \(d = 5\) and set \(N = 100, 1000\) and \(10000\). Using parameter values from the normal approximation \(5\), the volume of each partition set is calculated using \(8\) and compared with the desired volume of \(1/N\), see Figure 4. We observe that the error is bounded inside the main body of the cube but much larger in the extreme ends which indicates that our deterministic distribution is not well approximated by the tails of the normal distribution. However, the number of sets in the partition that are affected from this large deviation is relatively small. Our numerical results indicate that the relative error is at most about 7% for most of the sets; i.e. if a set is expected to have volume \(10^{-k}\), the volume of the approximated set is in the interval \(10^{-k} \pm 7 \cdot 10^{k-2}\). To be more precise, note for example that in the case \(d = 5\) and \(N = 10000\) we leave the intial segment of the cube, i.e. for parameter values \(-1 \leq r \leq 0\), at partition set number \(84 = \left\lceil \frac{10000}{5!} \right\rceil\) and symmetrically return to the final segment at set number \(10000 - 84 = 9916\) for parameter ranging \(-d \leq r \leq -(d - 1)\). These values are denoted in Figure 4 by two red dashes. Hence, about 9832 of all 10000 sets, or about 98% of the sets, are well approximated.

**Figure 4.** Deviation from \(1/N\) of the volume of each partition set, for \(N = 100, 1000, 10000\) and \(d = 5\). The \(x\)-axis represents the index, \(i\), of the set in the partition, while the \(y\)-axis shows the deviation of the volume of the \(i\)-th set in the approximate partition from the desired volume of \(1/N\).
6.2. Conclusions. For a given $N$, if one is required to partition the $d$–dimensional unit cube into $N$ equivolume slices with partitioning lines orthogonal to the main diagonal, the authors suggest to proceed as follows:

- If $N \leq d!$: In this case, all partitioning planes $H_r$ belong in the range $-(d-1) \leq r \leq -1$. As our numerical experiments above confirm, in the main body of the hypercube the normal approximation \[5\] generates a quasi-equivolume partition. Therefore the recommendation would be to generate the quasi-equivolume partition using Theorem $2$. We note that the small error values we obtain in this region should play a very small role in the overall picture, particularly with respect to applications in uniform distribution theory.

- If $N > d!$: The partitioning hyperplanes now sit in every segment of the cube and we are required to consider the full range of the parameter $-d \leq r \leq 0$. As in the first case, the normal approximation yields an accurate equivolume partition in the main body of the cube. Our numerical experiments above inform that in the extreme segments, i.e. when $-d \leq r \leq -(d-1)$ and $-1 \leq r \leq 0$ the normal approximation deviates significantly from the desired volumes of the sets. To deal with this inaccuracy, we recall the algebraic formula in Theorem $1$. In the case $k = d$ (i.e. in the initial segment $-1 \leq r \leq 0$), the formula reduces to

$$r^d = d! \left(1 - V_d^+(r) \right),$$

for even $d$ and

$$r^d = d! \left(V_d^+(r) - 1 \right),$$

for odd $d$. We suggest to solve this simplified equation for $r$ to generate the hyperplanes contained in the initial segment. By symmetry, we can also easily obtain the values for $r$ in the upper most segment.

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