Charged condensate and helium dwarf stars

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Abstract. White dwarf stars composed of carbon, oxygen and heavier elements are expected to crystallize as they cool down below certain temperatures. Yet, simple arguments suggest that the helium white dwarf cores may not solidify, mostly because of zero-point oscillations of the helium ions that would dissolve the crystalline structure. We argue that the interior of the helium dwarfs may instead form a macroscopic quantum state in which the charged helium-4 nuclei are in a Bose–Einstein condensate, while the relativistic electrons form a neutralizing degenerate Fermi liquid. We discuss the electric charge screening, and the spectrum of this substance, showing that the bosonic long-wavelength fluctuations exhibit a mass gap. Hence, there is a suppression at low temperatures of the boson contribution to the specific heat—the latter being dominated by the specific heat of the electrons near the Fermi surface. This state of matter may have observational signatures.

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1. Introduction

Consider a system of a large number of neutral atoms, each having \( Z \) electrons, and for simplicity we focus on \( Z \leq 10 \). Suppose we increase the number density of the atoms—denoted by \( J_0/Z \), where \( J_0 \) is the electron number density—so that the average separation between the atomic nuclei

\[
d \equiv \left( \frac{3Z}{4\pi J_0} \right)^{1/3},
\]

becomes much smaller than the Bohr radius, but is still much greater than the size of the charged nuclei. This would certainly be the case when, for instance, \( J_0 \approx (0.5–5 \text{ MeV})^3 \), for which the electron Fermi momentum, \( p_F = (3\pi^2J_0)^{1/3} \), ranges from fully relativistic to ultra-relativistic.

Under these circumstances the electron Fermi energy \( E_F \) will exceed the atomic binding energy, and the neutral atoms will dissolve into the electrons and nuclei. What is the ground state of such a system, when it is held together by gravity?

The answer would depend on its temperature \( T \). At certain high temperatures a plasma of negatively charged electrons and positively charged nuclei (ions) would be the equilibrium state. However, for lower temperatures the system would undergo significant charges.

Let us consider the cooling process by ignoring the quantum mechanical effects for the nuclei (ions), while treating the electrons as a degenerate Fermi liquid\(^3\). Consider the nuclei that are bosons. The strength of interactions of a classical plasma of the charged nuclei (ions) is customarily characterized by the ratio of the average Coulomb energy of a pair of ions to the thermal energy (see, e.g., [1])

\[
\Gamma \equiv \frac{E_{\text{Coulomb}}}{2E_{\text{Thermal}}/3} = \frac{(Ze)^2/(4\pi d)}{k_B T},
\]

where \( e \) denotes the electric charge, \( k_B \) is the Boltzmann constant, and the second equality in (2) assumes the validity of the classical approximation.

\(^3\) The Coulomb energy of a pair of electrons here is smaller than the Fermi energy.
It is known that when $\Gamma \gtrsim 180$ the ion plasma is coupled strongly enough for the system to crystallize, i.e., when the temperatures become low enough and $\Gamma$ reaches the above-mentioned value, the system would undergo crystallization transition (for earlier works, see [2,3]; for later studies, see [4]–[6] and references therein). This has direct relevance to white dwarf stars (WDs): it is expected that in most of the white dwarfs, consisting of carbon, oxygen, and heavier elements, the crystallization transition takes place in the process of cooling (for a review, see, e.g., [7]).

The above discussions were classical. Although it was observed in [8] that the quantum effects of the ions may be important for cooling of the WDs, quantitative studies using a model for a quantum solid and Lindemann’s empirical rule lead to a conclusion [6] that the melting curves for most of the WDs are not changed significantly by the quantum effects.

In the present work we are interested in the system of electrons and helium-4 nuclei. The latter, being lighter than the other nuclei, and having lower charge, should be expected to reveal stronger sensitivity to the quantum effects.

Such a system should be present in cores of helium white dwarfs. These dwarf stars stayed relatively cool due to their binary companions, which kept accreting a part of the dwarf’s energy during their paired evolution. As a result, the temperature in such dwarfs was never enough to begin the helium burning process.

The electron number densities in the interval $J_0 \simeq (0.5–5 \text{ MeV})^3$ would correspond to the mass density of the helium–electron system $5 \times (10^7–10^{10}) \text{ g cm}^{-3}$. The densities closer to the lower end of this interval (which we refer to as the low densities hereafter) may well exist in cores of the helium white dwarfs. We will also discuss the densities closer to the upper end, as they may be present in some other astrophysical circumstances.

The system should be long lived, as the helium-4 nuclei are stable w.r.t. fission. Furthermore, some nuclear reactions that could contaminate the helium-4 cores by their products are suppressed. One of these is the neutronization process due to the inverse beta decay. In our case the electrons with $J_0 \simeq (0.5–5 \text{ MeV})^3$ are not energetic enough to reach the neutronization threshold of the helium-4 nucleus, which is about 20 MeV. Moreover, we would expect the rate of helium fusion via the triple alpha-particle reaction to be suppressed as the so-called pyconuclear reaction (i.e., nuclear fusion due to zero-point oscillations in a high density environment) rates are exponentially small [1]. Hence, the cores of these WDs are expected to be dominated by helium-4 for very long time (at least $10^5$ years) after their formation.

In what follows, we will focus on the case when $Z = 2$. Then, for $J_0 \simeq (0.5–5 \text{ MeV})^3$, the classical considerations of equation (2) would give the crystallization temperature of the helium ions $T_{\text{cryst}} \simeq (10^6–10^7) \text{ K}$. However, before these temperatures are reached from above, the de Broglie wavelengths of the helium nuclei would start to overlap, suggesting that the quantum effects of the ions may be important. An estimate for the critical temperature below which this will happen can be given by

$$T_c \simeq \frac{4\pi^2}{3m_Hd^2}. \quad (3)$$

For the number densities at hand we obtain $T_c \simeq (10^7–10^9) \text{ K}$. These are greater than the crystallization temperatures estimated above. For this reason, it is unjustified to
approximate the thermal energy of the ions by its classical expression \(3k_B T/2\), as was done in the denominator of \(\text{(2)}\).

Perhaps the most important quantum effect, that invalidates the predictions of \(\text{(2)}\), is related to the zero-point fluctuations: even at very low temperatures, close to the absolute zero, there will be a significant amount of energy stored in the quantum mechanical zero-point oscillations of the nuclei in a would-be crystal. This energy can be approximated as follows (see, e.g., \([1]\)):

\[
E_0 = \frac{3}{2} \omega_0^{\text{ions}} \equiv \frac{3 \omega_p^{\text{ions}}}{\sqrt{3}} = \left( \frac{3 J_0(Z e)^2}{4 m_H Z} \right)^{1/2} \simeq (3 \times 10^3 - 10^5) \text{ eV},
\]

where \(m_H \simeq 3.7 \text{ GeV}\) is the helium nucleus mass, and \(\omega_p^{\text{ions}}\) denotes the ion plasma frequency. This energy is commensurate with the classical thermal energy at temperatures \((2/3)(3 \times 10^7 - 10^9) \text{ K}\)—the values that are greater than the crystallization temperatures obtained from \(\text{(2)}\).

The above discussions suggest that a measure of coupling in such a cold plasma is the ratio of the Coulomb energy to the zero-point energy. This ratio, properly normalized to be consistent with the conventions of \([1]\) (which are different from ours), takes a small value that is not enough for the plasma to crystallize. It increases with the nuclear charge \(Z\), and the atomic number \(A\). Thus, for heavier nuclei, it will be high enough not to obstruct the crystallization. In the charged helium-4 case with the densities at hand, however, the zero-point oscillations would melt the crystal lattice\(^5\).

The above considerations lead us to conclude that the dense system of the helium-4 nuclei and electrons may not solidify, mostly because of the zero-point oscillations of the helium nuclei.

### 2. Charged condensation

What could then be an adequate description of the quantum state discussed above? It was argued in \([9]\) that a state in which the charged helium-4 nuclei form a Bose–Einstein (BE) condensate, while the degenerate electrons neutralize the net charge of the condensate, represents a local equilibrium state of the system.

The helium nuclei could have condensed to such a state when their de Broglie wavelength started to overlap, which according to our estimates above happened before the temperature dropped to the would-be crystallization value. Such a state of the helium

\(^4\) Note that for the system of carbon nuclei we would obtain the crystallization temperature \(T_{\text{cryst}} \simeq 6 \times (10^6 - 10^7) \text{ K}\), and the corresponding critical temperatures \(T_c \simeq 2 \times (10^6 - 10^8) \text{ K}\). Thus, for the low densities the crystallization temperature exceeds the critical one, and the carbon solidification would take place. For the higher densities the opposite statement would be true. One should note, however, that we are comparing these for a fixed electron number density in the helium and carbon cases, which implies mass density three times greater for the carbon case, so the comparisons should be readjusted by this factor. Since the carbon core is not a subject of the present paper, we will not deal with this procedure here.

\(^5\) The case for the crystallization is worsened further by the fact that for the relativistic electrons with the Fermi energy \(E_F \simeq (3 \pi^2 J_0)^{1/3}\), the Coulomb interaction gets screened. The screening in such a plasma becomes effective at scales of the order the Debye wavelength, \(\lambda_D \sim (E_F/e^2 J_0)^{1/2} \sim (e^2 J_0)^{-1/4}\), and can be appreciable at the scale of the average interparticle separation \(d\). Hence, not only should the denominator in \(\text{(2)}\) be modified, but also the numerator should be reduced (multiplied by a factor) to account for the screening. The question of how significantly in the (ultra-)relativistic case this screening may affect the crystallization of the carbon nuclei is left open. Note that this screening can be important in a non-relativistic set-up as well; see, e.g., \([4]\).
nuclei cannot be resolved into individual particles. It represents a macroscopic mode with a large occupation number, somewhat similar to a quantum liquid of charged particles, where the charge is screened. This state, referred to as charged condensate, has a number of distinctive signatures: a photon propagating through this medium slows down, since it acquires a Lorentz-violating mass term, and its longitudinal polarization exhibits an unusual dispersion relation. Moreover, electric charges in this medium, as shown in [9], are screened at the distance scales $1/M$, where $M$ for the helium–electron system denotes

$$M \equiv \left(2e^2 m_H J_0\right)^{1/4}.$$  \hspace{1cm} (5)

Importantly, the screening length is shorter than the average interparticle separation, $1/M < d$ (see discussions of this in section 3)!

In [9], however, the dynamics of the electrons was neglected, since the fermions considered there were heavier. One of the goals of the present work is to include the dynamics of electrons. We will show that the screening length can be approximated by (5), as long as $m_H \gg J_0^{1/3}$ (as is the case here). However, we will also see that inclusion of the electron dynamics modifies quantitatively the spectrum of small perturbations above the ground state.

Before turning to these issues, let us deal with the gravitational part of the problem. From now on we consider the system at temperatures much lower than $\sim 10^7$ K, and ignore thermal effects. Thus, we discuss the electrons in the gravitational and electrostatic fields in the interior of an astrophysical object, e.g., in the core of a helium dwarf. The Fermi momentum, $p_F$, is related to the chemical potential $\mu$ as follows:

$$\sqrt{p_F^2 + m_e^2 + V_g + eA_0} = \mu,$$  \hspace{1cm} (6)

where $m_e$ is the electron mass, $V_g$ is the (negative) gravitational potential energy per electron, and $A_0$ denotes the gauge potential. An average $A_0$ would be non-zero for a non-zero net surface charge. However, we will assume that the net charge, even if present, is small enough, so that $|eA_0| \ll |V_g|$. Then, as is well known, the overall stability of a dwarf star is due to the balance between attractive gravity and repulsive Fermi degeneracy pressure of the electrons. Thus, we briefly discuss the interplay between $V_g$ and the Fermi energy.

For this one could use the Thomas–Fermi (TF) approximation: ignore $A_0$ in (6), and deduce from it $p_F = ((\mu - V_g)^2 - m_e^2)^{1/2}$. Relate the latter to the number density

$$J_0 = \frac{p_F^3}{3\pi^2} = \frac{((\mu - V_g)^2 - m_e^2)^{3/2}}{3\pi^2},$$  \hspace{1cm} (7)

and use this expression in the Poisson equation for $V_g$ to derive the Chandrasekhar equation.

6 It was argued in [10] that there could exist long-lived spherically symmetric objects of a (sub)atomic size with the charged condensate of helium-4 nuclei and electrons in their interior that could be held together by electrodynamic forces, as long as there is a significant amount of uncompensated charge at the surface of such objects. This interesting possibility and its applications will be studied further elsewhere. Here, the force that keeps the system together is gravity.

7 Since this equation and its consequences are well studied, we will not discuss them here. For its derivation in the TF approach, and further details, see [11], and references therein.
Our main interest is in local properties of particles at scales that are much smaller than the size of the star. These properties are determined by electromagnetic interactions, once the average effective chemical potential for the particles in a given homogeneous region of the star is set by the gravitational dynamics. For instance, consider a small region in the core of the star. Local distortions in the number density (i.e., distortions in the local Fermi momentum $p_F(x)$) will cause the respective change in the local field potential $A_0(x)$, while the change in the local gravitational field is negligible since the gravity is so much weaker.

These arguments can be formalized by the following relation:

$$\sqrt{p_F^2(x) + m_e^2 + eA_0(x)} = \langle \mu - V_g(x) \rangle \equiv \mu_l,$$

where the brackets, $\langle \cdots \rangle$, denote the averaging of the large-scale gravity effects over the region of a roughly uniform number density in the star. Hence, by $\mu_l$ we denote the effective chemical potential that governs the fermion dynamics in a local approximately homogeneous region, e.g., in the core of the star. The size of the homogeneous region should be greater than the photon Compton wavelength for our discussions below to be valid.

Having the gravity part dealt with, let us turn to the microscopic non-gravitational physics. For a large number of quanta that form a macroscopic state, such as the charged condensate, the description in terms of the effective field theory of the order parameter, and its long-wavelength fluctuations, becomes adequate. The effective field theory will be constructed on the basis of the fundamental principles of symmetries and properties of the interactions involved. This is the formalism that we will adopt for the rest of the work.

The effective Lagrangian that captures the microscopic properties of the system contains the photon field $A_\mu$, a charged scalar field $\phi$, and fermions $\Psi^+, \Psi$, with the effective chemical potential $\mu_l$ discussed above:

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu}^2 + |\tilde{D}_\mu \phi|^2 - m_\phi^2 \phi^* \phi + \bar{\Psi}(i\gamma^\mu D_\mu - m_e)\Psi + \mu_l \Psi^+ \Psi.$$  \(9\)

In a conventional set-up, upon quantization, the fluctuations of $\phi$ would describe the helium-4 nucleus of charge $+2e$, and its antiparticle. Such a description would be valid at densities that are above the atomic, but well below the nuclear scales. Here, the vacuum expectation value of the field $\phi$ will serve as the order parameter for the condensation of the helium-4 nuclei. Fluctuations of the order parameter will be discussed below.\(^8\)

The fermions in (9) describe the electrons with charge $-e$, and positrons. The covariant derivative for them is defined as $D_\mu = \partial_\mu + ieA_\mu$, while for the scalar it reads

$$\tilde{D}_\mu \equiv \partial_\mu - 2ie\tilde{A}_\mu \equiv \partial_\mu - i(2eA_\mu + \mu_s \delta_\mu 0).$$ \(10\)

The latter is equivalent to the introduction of the chemical potential $\mu_s$ for the helium-4 nuclei.

The Lagrangian (9) has a global $U_s(1)$ symmetry, yielding the conservation of the number of scalars, as well as another global $U_f(1)$, responsible for the fermion number

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\(^8\) Since the helium nucleus is heavy as compared to the scales at hand, one fluctuation will also end up being heavy. It could have been ‘integrated out’ from the very beginning, keeping only the light modes, but for technical convenience we retain the heavy mode in the low energy effective Lagrangian.
conservation. One linear combination of these two symmetries is gauged, and the corresponding conserved current is coupled to the photon field. We could add the quartic scalar self-interaction term to (9), but this will not change our results significantly, as long as the quartic coupling is not strong, and $m_H \gg J_0^{1/3}$.

Since the electrons are relativistic, the quantum loops would renormalize the effective Lagrangian (9) by generating additional terms proportional to the fermion chemical potential and temperature, which would be still consistent with the symmetries of the theory. These terms would come suppressed by the fine structure constant, $\frac{e^2}{4\pi}$, and we do not expect them to modify our quantitative results significantly in the regime when $m_H \gg J_0^{1/3}$. Hence, for simplicity of presentation we do not include those terms here.

For further convenience, we introduce the following notation for the scalar field, $\phi = (1/\sqrt{2})\sigma e^{i\alpha}$, and work in the unitary gauge where the phase of the scalar is set to zero, $\alpha = 0$. In this gauge, the Lagrangian density reads

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}^2 + \frac{1}{2}(\partial_\mu \sigma)^2 + \frac{(2e)^2}{2}A_\mu^2 \sigma^2 - \frac{1}{2}m_H^2 \sigma^2 + \bar{\Psi}(i\gamma^\mu D_\mu - m_e)\Psi + \mu_s \Psi^+ \Psi. \quad (11)$$

The kinetic term for the scalar in (9) gives rise to the third term in the Lagrangian (11). If the field $A_0$ acquires an expectation value, this term plays the role of a tachyonic mass for the scalars (see, e.g., [12]). In particular, when $\langle 2eA_0 \rangle = m_H$, the scalar field condenses. That is to say, the equations of motion that follow from (11) have the following static solution:

$$\langle 2eA_0 \rangle + \mu_s = m_H, \quad \langle \sigma \rangle = \sqrt{\frac{J_0}{2m_H}}. \quad (12)$$

What fixes the value of $A_0$ is the net surface charge. In particular, when the surface charge is absent, $\langle 2eA_0 \rangle = 0$, and has to be $\mu_s = m_H$, for the condensation to occur. The system is neutral in the bulk, as the charge in the condensate is exactly cancelled by the electron charge [9].

Dynamically, the condensation proceeds in a conventional manner: at temperatures higher than $T_c$ most of the states are in thermal modes, and $\mu_s(T)$ is less than $m_H$, but increases as the temperature drops$^9$. As $T \to T_c$ a significant fraction of the modes ends up in the zero-momentum ground state, while $\mu_s(T_c)$ asymptotes to $m_H$ (for a discussion of relativistic BE condensation, see, e.g., [13]).

The last comment in this section is that we are looking at the effects of the condensate that are $1/m_H$ suppressed. In particular, the order parameter of the condensation that we are discussing (given in equation (12)) is the $1/m_H$ suppressed quantity. Clearly, in order for the condensate to make sense at some fixed non-zero temperature $T$, the de Broglie wavelength of the bosons, $\sim 1/\sqrt{m_H T}$, should be larger than the interparticle separation—the condition which would break down for very large $m_H$ (in agreement with the decoupling), but is fulfilled for the values of the parameters in our case. Furthermore, the inverse of the order parameter $\langle \sigma \rangle$ should be smaller than the physical size of the system—the condition that would also be violated for some very large $m_H$ (consistent with the decoupling), but is preserved in our case.

$^9$ Here we ignore the temperature dependence of $m_H$ which would be present due to the quantum loop effects.
3. Charge screening and small perturbations

As a next step we would like to discuss screening of an electric charge in the condensate. To determine the screening length we consider a small, spherically symmetric object with a non-zero charge placed in the condensate. Outside of the charge, in the condensate, the equations of motion for a static $A_0$ and $\sigma$, as derived from (9), are

$$-\nabla^2 \tilde{A}_0 + 4e^2 \sigma^2 \tilde{A}_0 = eJ_0, \quad -\nabla^2 \sigma = (4e^2 \tilde{A}_0^2 - m_{H}^2)\sigma.$$  \hfill(13)

Following [9] we parametrize the perturbations as follows:

$$\tilde{A}_0(r) = \frac{m_{H}}{2e} + \delta A_0(r), \quad \sigma(r) = \sqrt{\frac{J_0}{2m_{H}}} + \delta \sigma(r).$$  \hfill(14)

To include the effects of the fermion fluctuations we substitute $J_0 = (p_F^2 / 3\pi^2)$ on the rhs of the first equation in (13), and express $p_F$ in terms of the gauge potential $A_0$ and the effective chemical potential $\mu_f$ via (8). The result is that $\delta J_0(r)$ gets related to $\delta A_0$. Thus, the coefficient in front of $\delta A_0$ in the final equation for the perturbations gets modified as compared to [9]:

$$-\nabla^2 \delta A_0 + e^2 \left[ \frac{2J_0}{m_{H}} + \frac{(3\pi^2 J_0)^{2/3}}{\pi^2} \right] \delta A_0 = -2M^2 \delta \sigma,$$  \hfill(15)

$$-\nabla^2 \delta \sigma = 2M^2 \delta A_0.$$  \hfill(16)

The above approximations is valid as long as we focus on solutions that satisfy $\delta \sigma \ll \sqrt{J_0 / 2m_{H}}$ and $\delta A_0 \ll m_{H}/2e$.

In the regime when $m_{H} \gg J_0^{1/3}$, we find that $M^2 \gg J_0^{2/3}$, and then we can neglect the second term on the rhs in (15). For large $r$ we require that $\delta A_0, \delta \sigma \rightarrow 0$. The solutions are similar to those of [9], in which the boundary conditions select the decaying functions:

$$\delta A_0(r) = \frac{e^{-Mr}}{r} \left[ c_1 \sin(Mr) + c_2 \cos(Mr) \right],$$  \hfill(17)

$$\delta \sigma(r) = \frac{e^{-Mr}}{r} \left[ -c_1 \cos(Mr) + c_2 \sin(Mr) \right].$$  \hfill(18)

The constants $c_1$ and $c_2$ are to be determined by matching these solutions to those in the interior of the small, charged object. Thus, for a particle, the screening occurs at the scale $1/M$. For the helium–electron system at hand, this scale is larger than the size of the helium nuclei, but is shorter than the average interparticle separation.

The screening length $M^{-1}$ being much smaller than the average interparticle separation $d$ is unexpected. An explanation of this screening length in terms of cancellation between the Coulomb and ‘phonon’ exchange will be given elsewhere.

Note that the above screening was derived for the electrostatic interactions. This is relevant for the heavy charged bosons sitting in the zero-momentum state in the condensate. However, the magnetic interactions for the fermions filling the Fermi sphere are not screened at scale $M^{-1}$. Instead, as will be clear from the Lagrangian (19) derived below, the magnetic interactions are screened at scale of the Compton wavelength of the massive photon. This scale will end up being much greater that the average interparticle separation $d$. 

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Let us now turn to the spectrum of small perturbations. The electrons fill the Fermi surface with the Fermi energy $E_F$. As to the bosons, there are three components of the massive photon, and the heavy scalar mode. We ‘integrate out’ the fermions by taking into account their effect via the TF approximation, like we did it above for the static case. The resulting Lagrangian density for the bosonic perturbations takes the form

$$\mathcal{L}_{\text{bos}} = -\frac{1}{4} f_{\mu\nu} f^{\mu\nu} + \frac{1}{2} m_0^2 (\delta A_0)^2 - \frac{1}{2} m_\gamma^2 (\delta A_j)^2 + \frac{1}{2} (\partial_\mu \delta \sigma)^2 + 2 M^2 \delta A_0 \delta \sigma, \tag{19}$$

where $f_{\mu\nu}$ denotes the field strength for $\delta A_\mu$, $j = 1, 2, 3$ is the spatial index and

$$m_0^2 \equiv m_\gamma^2 + \frac{e^2 (3\pi^2 J_0)^{2/3}}{\pi^2}, \quad m_\gamma \equiv 2e \sqrt{\frac{J_0}{2m_H}}. \tag{20}$$

The calculation of the spectrum is a bit lengthy but straightforward. The result is as follows: two (out of three) helicities of the massive photon are transverse and propagate according to the massive dispersion relation

$$\omega^2 = \mathbf{k}^2 + m_\gamma^2, \tag{21}$$

where $\mathbf{k}$ denotes the spatial 3-momentum. Furthermore, there are two modes with the dispersion relations

$$\omega^2_\pm = \mathbf{k}^2 \left( \frac{m_0^2 + m_\gamma^2}{2m_0^2} \right) + \frac{2M^4}{m_0^2} + \frac{m_\gamma^2}{2} \pm \sqrt{4k^2 \frac{M^4 m_0^2}{m_\gamma^2} + \left[ \frac{2M^4}{m_0^2} - \frac{m_\gamma^2}{2} + \mathbf{k}^2 \left( \frac{m_0^2 - m_\gamma^2}{2m_0^2} \right) \right]^2}. \tag{22}$$

The solution with the plus subscript corresponds to the scalar mode and its mass squared in this frame is $\omega^2_+(\mathbf{k} = 0) = (4M^4/m_0^2)$, while the solution with the minus subscript corresponds to the longitudinal component of the massive vector field, with $\omega^2_-(\mathbf{k} = 0) = m_\gamma^2$. Here it is useful to note that the expression for $M$ (5) can be rewritten as $M = (m_H m_\gamma)^{1/2}$.

Furthermore, in the limit when the fermions are non-dynamical (i.e., are frozen ‘by hand’, or by some other dynamics), then $m_0 \rightarrow m_\gamma$, and the solutions reduce to the ones obtained in [9]; however, in the present set-up the difference between $m_0$ and $m_\gamma$ is greater than $m_\gamma$. Therefore, the fermion dynamics contributes an additional screening of the electrostatic interactions.

The solutions (21) and (22) are positive for arbitrary $\mathbf{k}$. From this, we conclude that the charged condensate background is stable w.r.t. small perturbations. All the group velocities obtained from (21) and (22) are subluminal.

4. Brief discussions and outlook

The spectrum of bosons exhibits a mass gap. As a result, contributions of the bosons to the specific heat of the substance at low temperatures would be suppressed as $\exp(-m_\gamma/T)$, or more strongly. Since for the densities at hand $m_\gamma \simeq (30–90)$ keV (corresponding to $\sim (3–9) \times 10^8$ K), the exponential suppression will be strong at lower temperatures.

As the star cools further, the specific heat will be dominated by the contributions of the relativistic electrons near the Fermi surface, and, hence, would vanish with the
temperature linearly, $\sim T$. This resembles the low temperature scaling in metals, but differs from that in insulating solids, or in the BE condensate of the neutral helium-4 atoms, where the low temperature specific heat is dominated by massless phonons, and scales as $\sim T^3$.

If charged condensate is realized in the helium dwarf stars, as we have argued in this work, there may be important consequences for the equation of state, cooling, and rotation, as well as the asteroseismology of the helium dwarfs. These and related issues will be studied elsewhere.

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