NEW APPROXIMATE SOLUTIONS TO THE NONLINEAR KLEIN-GORDON EQUATIONS USING PERTURBATION ITERATION TECHNIQUES

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Abstract. In this study, we present the new approximate solutions of the nonlinear Klein-Gordon equations via perturbation iteration technique and newly developed optimal perturbation iteration method. Some specific examples are given and obtained solutions are compared with other methods and analytical results to confirm the good accuracy of the proposed methods. We also discuss the convergence of the optimal perturbation iteration method for partial differential equations. The results reveal that perturbation iteration techniques, unlike many other techniques in literature, converge rapidly to exact solutions of the given problems at lower order of approximations.

1. Introduction. Many scientific phenomena can be modelled by nonlinear partial differential equations (NPDEs) and it is still complicated to deal with most of NPDEs analytically. Therefore, a broad class of semi-analytical and numerical techniques were used to solve these equations such as Adomian decomposition method [8, 22], homotopy decomposition method [6], Chebyshev spectral collocation method [25], optimal q-homotopy analysis method [35], trial equation method [12, 18, 27], homotopy analysis method [1, 34, 39], variational iteration method [19, 33, 36] and the sine-cosine method [5, 7]. In recent times, new and more effective techniques have been developed on the insufficiency of these methods such as perturbation iteration technique [3, 4], optimal homotopy asymptotic method [9, 21, 29–31] and optimal perturbation iteration method [10, 11, 13–16].

Klein-Gordon equation is one of the important NPDEs encountered in several applied physics fields. It is the relativistic wave equation version of the Schrödinger equation which represents spinless particles. Although it was named after Oskar Klein and Walter Gordon, it was purportedly first derived by Schrödinger, before he discovered the equation bearing his name. However, he denied it because he could not make it contain the spin of the electron [37, 38].

Nonlinear Klein-Gordon equation can be given in the form

\[ u_{tt} - u_{xx} + \alpha u + \beta u^2 = h(x, t), \quad (x, t) \in \Omega = [a, b] \times (0, T) \quad (1) \]

subject to initial conditions

\[ u(x, 0) = f(x), \quad u_t(x, 0) = g(x), \quad (2) \]

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where \(\alpha, \beta\) are constants; \(h(x, t), f(x)\) and \(g(x)\) are known functions and \(u(x, t)\) is unknown function. For \(\gamma = 2\) and \(\gamma = 3\) we have quadratic and cubic nonlinearity, respectively.

The nonlinear Klein–Gordon equation plays a significant role in many scientific fields such as nonlinear optics, solid state physics, fluid dynamics and quantum field theory. To examine these types of equations, scientists have used many effective techniques such as the auxiliary equation method [23], pseudo-spectral method [26], Jacobi elliptic functions [17] and the tanh–sech method [28].

In this work, we put forward new approaches to perturbation iteration method (PIM) and optimal perturbation iteration method (OPIM) to make them practicable for the nonlinear Klein-Gordon equations. We apply this technique to three specific examples. Our results reveal that the new approximate solutions obtained via OPIM are more accurate and impressive than many other techniques in literature.

2. Formulation of OPIM for the Klein-Gordon equation. Optimal perturbation iteration method (OPIM) has been recently constructed based on the perturbation iteration algorithms [3, 4]. It has been efficiently applied to strongly nonlinear equations [10, 13, 15]. In this section, we formulate OPIM for the general NPDEs. Meanwhile, we will have created perturbation iteration algorithms for the Klein-Gordon equations.

(a) Consider the following NPDE in closed form:

\[
F(u_{xx}, u_{tt}, u, \varepsilon) = 0 \tag{3}
\]

where \(u = u(x, t)\) and \(\varepsilon = 1\) is the artificial perturbation parameter which can be inserted into the Eq. (1) as:

\[
F = u_{tt} + \varepsilon (-u_{xx} + \alpha u + \beta u^\gamma - h(x, t)) = 0. \tag{4}
\]

(b) Take the approximate solution with one correction term in the perturbation expansion as

\[
u_{n+1} = u_n + \varepsilon (u_c)_n \tag{5}
\]

where \(n \in \mathbb{N} \cup \{0\}\) and \((u_c)_n\) is the \(n\)th correction term of the iteration algorithm. Upon substitution of (5) into (3) then expanding it in a Taylor series only with first derivative gives:

\[
F + F_{u_{tt}} ((u_c)_n)_{tt} \varepsilon + F_{u_{xx}} ((u_c)_n)_{xx} \varepsilon + F_u (u_c)_n \varepsilon + F_\varepsilon \varepsilon = 0 \tag{6}
\]

where

\[
F_u = \frac{\partial F}{\partial u}, F_{u_{xx}} = \frac{\partial F}{\partial u_{xx}}, F_{u_{tt}} = \frac{\partial F}{\partial u_{tt}}, F_\varepsilon = \frac{\partial F}{\partial \varepsilon}.
\]

Performing these calculations at \(\varepsilon = 0\) for the Eq. (4), we get

\[
((u_c)_n)_{tt} = (u_n)_{xx} - \alpha (u_n) - \beta (u_n)^\gamma - (u_n)_{tt} + h(x, t). \tag{7}
\]

(7) is called as the perturbation iteration algorithm (PIA) for the nonlinear Klein-Gordon equations (1). In order to initiate the iteration procedure, a first trial function \(u_0\) is selected appropriately according to the prescribed conditions. Correspondingly, the first correction term \((u_c)_0\) can be computed from the algorithm (7) by using \(u_0\) and given condition(s). Then, the first approximate solution \(u_1\) of
perturbation iteration method (PIM) can be found by knowing \( u_0 \) and \( (u_c)_0 \) and so on.

c) To improve the accuracy of the results and effectiveness of the PIM, use the equation

\[
    u_{n+1} = u_n + P_n(u_c)_n
\]

where \( P_0, P_1, P_2, \ldots \) are convergence control parameters which allow us to adjust the convergence.

Proceeding for \( n = 0, 1, \ldots \), more approximate solutions are obtained as:

\[
    u_1 = u(x, t; P_0) = u_0 + P_0(u_c)_0
\]

\[
    u_2(x, t; P_0, P_1) = u_1 + P_1(u_c)_1
\]

\[
    \vdots
\]

\[
    u_m(x, t; P_0, \ldots, P_{m-1}) = u_{m-1} + P_{m-1}(u_c)_{m-1}
\]

d) Inserting the approximate solution \( u_m \) into the Eq. (3), the general problem will result in the following residual:

\[
    Re(x, t; P_0, \ldots, P_{m-1}) = F ((u_m)_{xx}, (u_m)_{tt}, u_m)
\]

\[
    = (u_m)_{tt} - (u_m)_{xx} + \alpha(u_m) + \beta(u_m)^\gamma - h(x, t)
\]

Apparently, when \( Re(x, t; P_0, \ldots, P_{m-1}) = 0 \) then the approximation \( u_m(x, t; P_0, \ldots, P_{m-1}) \) will be the desired exact solution. Generally such case will not arise for nonlinear equations, but optimal values of parameter \( P_0, P_1, \ldots \) can be determined using Least Squares Method. To achieve this, we need to minimize the functional

\[
    J(P_0, \ldots, P_{m-1}) = \int_a^b \int_0^T Re^2(x, t; P_0, \ldots, P_{m-1}) dx dt
\]

where \( a, b \) and \( T \) are selected from the domain of the problem. Then, \( P_0, P_1, \ldots \) can be optimally identified from the conditions

\[
    \frac{\partial J}{\partial P_0} = \frac{\partial J}{\partial P_1} = \ldots = \frac{\partial J}{\partial P_{m-1}} = 0.
\]

Since the optimal parameters are determined in this way, this technique is called optimal perturbation iteration method (OPIM). The constants \( P_0, P_1, \ldots \) can also be determined from

\[
    Re(x_0, t_0; P_i) = Re(x_1, t_1; P_i) = \cdots = Re(x_{m-1}, t_{m-1}; P_i) = 0, \quad i = 0, 1, \ldots, m - 1
\]

where \( x_i, t_i \in [a, b] \times (0, T] \). Substituting these constants into the last one of the Eqs. (9), the approximate solution of order \( m \) is obtained. For more information about computing these constants, we refer to [20, 32].

3. Convergence analysis and error estimate. In this section, we analyze the convergence of the OPIM by considering the algorithms in a different way. Let us assume that

\[
    u_0 = H_0,
    P_n(u_c)_n = H_{n+1}
\]

then we get
\[ u_0 = H_0 \]
\[ u_1 = u(x, t; P_0) = u_0 + P_0(u_0) = H_0 + H_1 \]
\[ u_2 = u(x, t; P_0, P_1) = u_1 + P_1(u_1) = H_0 + H_1 + H_2 \]
\[ \vdots \]
\[ u_n = u(x, t; P_0, \ldots, P_{n-1}) = H_0 + H_1 + \ldots + H_n \]

Accordingly, the \(n\)-th order approximate solution can also be written as:
\[ u_n(x, t; P_0, \ldots, P_{n-1}) = H_0(x, t) + \sum_{j=1}^{n} H_j(x, t; P_0, \ldots, P_{j-1}) \quad (16) \]

**Theorem 3.1.** Let \( B \) be a Banach space denoted with a suitable norm \( ||.|| \) over which the series (16) is defined and assume that the initial function \( u_0 = H_0 \) remains inside the ball of the exact solution. (16) converges if there exists \( \beta \) such that
\[ ||H_{n+1}|| \leq \beta ||H_n||. \quad (17) \]

**Proof.** We first construct a sequence as:
\[ A_0 = H_0 \]
\[ A_1 = H_0 + H_1 \]
\[ A_2 = H_0 + H_1 + H_2 \]
\[ \vdots \]
\[ A_n = H_0 + H_1 + H_2 + \cdots + H_n \]

Next, we need to show that \( \{A_n\}_{n=0}^\infty \) is a Cauchy sequence in \( B \). Consider that
\[ ||A_{n+1} - A_n|| = ||H_{n+1}|| \leq \beta ||H_n|| \leq \beta^2 ||H_{n-1}|| \leq \cdots \leq \beta^{n+1} ||H_0||. \quad (19) \]

For every \( n, k \in \mathbb{N}, n \geq k \), we have
\[ ||A_n - A_k|| = \|(A_n - A_{n-1}) + (A_{n-1} - A_{n-2}) + \cdots + (A_{k+1} - A_k)\| \]
\[ \leq ||A_n - A_{n-1}|| + ||A_{n-1} - A_{n-2}|| + \cdots + ||A_{k+1} - A_k|| \]
\[ \leq \beta^n ||H_0|| + \beta^{n-1} ||H_0|| + \cdots + \beta^{k+1} ||H_0|| = \frac{1 - \beta^{n-k}}{1 - \beta} \beta^{k+1} ||H_0|| \quad (20) \]

Since we also have \( 0 < \beta < 1 \), we can obtain from (20)
\[ \lim_{n,k \to \infty} ||A_n - A_k|| = 0. \quad (21) \]

Herewith, \( \{A_n\}_{n=0}^\infty \) is a Cauchy sequence in \( B \) and this implies that OPIM solution (16) is convergent.  

**Theorem 3.2.** If the initial function \( u_0 = H_0 \) remains inside the ball of the solution \( u(x, t) \) then \( A_n = \sum_{i=0}^{n} H_i \) also remains inside the ball of the solution.

**Proof.** Suppose that
\[ H_0 \in B_r(u) \quad (22) \]
where
\[ B_r(u) = \{ H \in A | ||u - H|| < r \} \quad (23) \]
is the ball of the solution \( u(x, t) \). By hypothesis \( u = \lim_{n \to \infty} A_n = \sum_{i=0}^{\infty} H_i \) and from Theorem 3.1, we have
\[ ||u - A_n|| \leq \beta^{n+1} ||H_0|| < ||H_0|| < r \quad (24) \]
where \( \beta \in (0, 1) \) and \( n \in \mathbb{N} \). This completes the proof.
Note also that the selection of the trial or initial function will directly affect the convergence of the solution to be obtained. However, there is no general theorem related to the choice of this function.

**Theorem 3.3.** Assume that the obtained solution \( \sum_{i=0}^{\infty} H_i \) is convergent to the solution \( u(x,t) \). If the truncated series \( \sum_{i=0}^{k} H_i \) is used as an approximation to the exact solution of problem (3), then the maximum error is given as,

\[
E_k \leq \frac{\beta^{k+1}}{1 - \beta} \| H_0 \|
\]

**Proof.** From (20), one can get

\[
\| A_n - A_k \| \leq \frac{1 - \beta^{n-k}}{1 - \beta} \beta^{k+1} \| H_0 \|
\]

for \( n \geq k \). By knowing

\[
u(x,t) = \lim_{n \to \infty} A_n(x,t) = \sum_{i=0}^{\infty} H_i
\]
we can write

\[
\left\| u(x,t) - \sum_{i=0}^{k} H_i \right\| \leq \frac{1 - \beta^{n-k}}{1 - \beta} \beta^{k+1} \| H_0 \|
\]

and also it can be written as

\[
E_k = \left\| u(x,t) - \sum_{i=0}^{k} H_i \right\| \leq \frac{\beta^{k+1}}{1 - \beta} \| H_0 \|
\]

since \( 1 - \beta^{n-k} < 1 \). Here \( \beta \) is selected as \( \beta = \max \{ \beta_i, i = 0, 1, \ldots, n \} \) where

\[
\beta_i = \frac{\| H_{n+1} \|}{\| H_n \|}.
\]

4. **Test problems.** In this section, we give some illustrations to show the efficiency of the presented method. We will first obtain the PIM solutions to which the OPIM results depend. The accuracy of PIM and OPIM is assessed by comparison with the solutions obtained via other methods and known exact solutions.

4.1. **Example 1.** Consider the nonlinear Klein–Gordon equation (1) with \( \alpha = 0, \beta = 1, \gamma = 2, h(x,t) = -x \cos t + x^2 \cos^2 t \), i.e.,

\[
u_{tt} - \nu_{xx} + \nu^2 = -x \cos t + x^2 \cos^2 t; \ 0 \leq x,t \leq 1
\]

with the initial conditions

\[
u(x,0) = x, \nu_t(x,0) = 0.
\]

The exact solution of this problem is given as [24]:

\[
u(x,t) = x \cos t.
\]

Firstly, trial function \( u_0 \) can be taken as

\[
u_0 = x.
\]

Using the following algorithm

\[
((u_{c,n})_{tt} = (u_n)_{xx} - (u_n)_{tt} - \nu^2 - x \cos t + x^2 \cos^2 t
\]
with the Eq. (34) and initial conditions, first order problem is generated as:

\[(u_c)_0|_{tt} = x^2 \cos^2(t) - x \cos(t) - x^2; \quad u_c(x, 0) = (u_c)_1(x, 0) = 0. \quad (36)\]

which has the solution

\[(u_c)_0 = -\frac{1}{4} x (4 + t^2 x - 4 \cos(t) - x \sin^2(t)). \quad (37)\]

**PIM Solutions:**

After finding \((u_c)_0\), one can continue to iterate to find the first and second order PIM solutions as:

\[(u_1)_{PIM} = u_0 + (u_c)_0 = x - \frac{1}{4} x (4 + t^2 x - 4 \cos(t) - x \sin^2(t)) \quad (38)\]

\[(u_2)_{PIM} = u_1 + (u_c)_1 = -\frac{1}{16} + \frac{t^2}{8} - \frac{t^4}{24} - \frac{28 x^3}{9} - \frac{1}{72} t^3 \cos(3t) + \]

\[\frac{63 x^4}{2048} - \frac{3 t^2 x^4}{256} + \frac{t^6 x^4}{192} - \frac{t^8 x^4}{480} + \left( x - \frac{1}{8} \left( -25 + 4 t^2 \right) x^3 \right) \cos(t) + \]

\[\frac{1}{64} \left( 4 + \left( -2 + t^2 \right) x^4 \right) \cos(2t) + \frac{x^4 \cos(4t)}{2048} + 2 t x^3 \sin(t) - \frac{1}{32} t x^4 \sin(2t) \]

Higher orders solutions can be also obtained in a similar way.

**OPIM Solutions:**

With the aid of the Eqs. (9),(34) and (35), OPIM solutions can be computed as:

\[(u_1)_{OPIM} = u_0 + P_0 (u_c)_0 = x - \frac{P_0}{4} x (4 + t^2 x - 4 \cos(t) - x \sin^2(t)). \quad (40)\]

\[\begin{align*}
-\frac{t^2 x^2}{4} + x \cos(t) + \frac{1}{4} x^2 \sin^2(t) + \frac{t^4 P_0}{8} - \frac{t^4 P_0}{24} + x P_0 - 2 x^2 P_0 + \frac{5}{4} t^2 x^2 P_0 - \\
\frac{1}{8} t^2 x^3 P_0 + \frac{1}{24} t^4 x^3 P_0 - x \cos(t) P_0 + 2 x^2 \cos(t) P_0 - \frac{1}{8} \sin^2(t) P_0 - \frac{1}{4} x^2 \sin^2(t) P_0 \\
+ \frac{1}{8} x^3 \sin^2(t) P_0 + \frac{16}{8} x^2 P_0^2 - \frac{3}{4} t^2 x^2 P_0^2 - \frac{457}{144} x^3 P_0^2 + \frac{1}{8} t^2 x^2 P_0^2 - \frac{1}{24} t^4 x^2 P_0^2 - \\
\frac{3}{256} t^2 x^4 P_0^2 + \frac{1}{192} t^4 x^4 P_0^2 - \frac{1}{8} x^3 P_0^2 - 2 x^2 \cos(t) P_0^2 + \frac{25}{8} x^3 \cos(t) P_0^2 + \\
\frac{1}{8} x^2 \cos(2t) P_0^2 + \frac{1}{16} x^3 \cos(2t) P_0^2 - \frac{1}{32} x^4 \cos(2t) P_0^2 + \frac{1}{64} t^2 x^4 \cos(2t) P_0^2 \\
+ \frac{x^4 \cos(4t) P_0^2}{2048} + 2 t x^3 \sin(t) P_0^2 - \frac{1}{32} t x^4 \sin(2t) P_0^2 - \frac{1}{72} x^3 \cos(3t) P_0^2 \\
+ \frac{63 x^4 P_0^2}{2048} - \frac{1}{2} t^2 x^3 \cos(t) P_0^2
\end{align*}\]

(41)

and so on. In order to find the unknown parameters \(P_0\) and \(P_1\), the information in Section 2 can be used. For the first order OPIM solution (40), the residual (10) can be computed from

\[Re(x, t; P_0) = (u_1)_{tt} - (u_1)_{xx} + (u_1)^2 + x \cos t - x^2 \cos^2 t. \quad (42)\]
Using the method of least squares given in Section 2, we get \( P_0 = 0.94300146 \). Substituting this value into the Eq. (40) yields

\[
(u_1)_{OPIM} = x - 0.23575x \left( t^2 x - x \sin^2(t) - 4 \cos(t) + 4 \right). \tag{43}
\]

Likewise, one can reach the following values

\[
P_0 = 0.99999999981 \quad P_1 = 1.0005094152
\]

for the Eq. (41). Correspondingly, the second order OPIM solution is obtained as:

\[
(u_2)_{OPIM} = \begin{bmatrix}
0.125064 t^2 - 0.0416879 t^4 - 9.59911 \times 10^{-14} x - 0.125064 x^2 +
0.000127 t^2 x^2 - 3.175x^3 - 2.356 \times 10^{-11} t^2 x^3 + 7.8554 \times 10^{-12} t^4 x^3 \\
+ 0.03077 t^4 - 0.0117247 t^2 x^4 + 0.00521099 t^4 x^4 - 0.00208439 t^6 x^4 +
+ 0.500255 t^2 x^3 \cos(t) + 0.125064 x^2 \cos(2t) + 0.0625318 x^3 \cos(2t)
- 0.031259 x^4 \cos(2t) + 0.0156333 t^2 x^4 \cos(2t) - 0.013896 x^3 \cos(3t)
+ 0.00048853 x^4 \cos(4t) + 0.00102 t^3 x^3 \sin(t) - 0.125064 \sin^2(t)
+ 0.25 x^2 \sin(t) + 0.125064 x^3 \sin^2(t) - 0.031259 x^4 \sin(2t)
\end{bmatrix}. \tag{44}
\]

This problem has been also investigated by many authors with different techniques such as DTM [24] and ADM [40]. In [24], Kanth et al. applied DTM to get the approximate solution as:

\[
u \approx x - 0.5 t^2 x + 0.0416667 t^4 x - 0.0013889 x t^6 + \cdots \tag{45}\]

which can also be obtained by applying classical ADM.

Tables 1 and 2 show comparisons the absolute errors between the approximate results of PIM, OPIM and ADM-DTM solutions with the exact values. One can realize the superiority of OPIM even for \( n = 1 \) and \( n = 2 \). It can also be said from tables 3 and 4, we obtain quite satisfactory results by applying OPIM even at lower order of solutions. Better results can be obtained by continuing the iteration. Figures 1 and 2 demonstrate the absolute errors of approximate ADM-DTM, PIM and OPIM solutions. It is clear that as the number of iterations increase, it becomes more intricate to compute the new approximate solutions. We have made use of Mathematica 9.0 to perform calculations for illustrations.

### 4.2. Example 2

Consider the nonlinear Klein–Gordon equation (1) with \( \alpha = 3/4, \beta = -3/2, \gamma = 3, h(x, t) = 0 \), i.e.,

\[
u_{tt} - u_{xx} + \frac{3}{4} u - \frac{3}{2} u^3 = 0; \quad x \geq 0, 0 \leq t \leq 1 \tag{46}\]

with the initial conditions

\[
u(x, 0) = - \text{sech}(x), \quad u_t(x, 0) = \frac{1}{2} \text{sech}(x) \tanh x. \tag{47}\]

Exact solution can be found in [21, 42] as:

\[
u(x, t) = - \text{sech} \left( x + \frac{t}{2} \right). \tag{48}\]
Figure 1. Absolute errors obtained by ADM-DTM and PIM for Example 1.

Figure 2. Absolute errors obtained by OPIM for Example 1.

Table 1. Absolute errors of the second order ADM-DTM (ADM-DTM-2nd), PIM (PIM-2nd), OPIM (OPIM-2nd) approximate solutions at \( x = 0.5 \) for Example 1.

| t   | ADM-DTM-2nd  | PIM-2nd    | OPIM-2nd   |
|-----|--------------|------------|------------|
| 0.1 | \( 2.083 \times 10^{-6} \) | \( 4.859 \times 10^{-7} \) | \( 3.802 \times 10^{-3} \) |
| 0.2 | \( 3.329 \times 10^{-5} \) | \( 3.107 \times 10^{-7} \) | \( 2.94 \times 10^{-7} \) |
| 0.3 | \( 1.682 \times 10^{-4} \) | \( 3.533 \times 10^{-6} \) | \( 3.45 \times 10^{-6} \) |
| 0.4 | \( 5.305 \times 10^{-4} \) | \( 1.98 \times 10^{-5} \) | \( 1.955 \times 10^{-5} \) |
| 0.5 | \( 1.291 \times 10^{-3} \) | \( 7.532 \times 10^{-5} \) | \( 7.472 \times 10^{-5} \) |
| 0.6 | \( 2.668 \times 10^{-3} \) | \( 2.241 \times 10^{-4} \) | \( 2.229 \times 10^{-4} \) |
| 0.7 | \( 4.921 \times 10^{-3} \) | \( 5.625 \times 10^{-4} \) | \( 5.604 \times 10^{-4} \) |
| 0.8 | \( 8.353 \times 10^{-3} \) | \( 1.247 \times 10^{-3} \) | \( 1.243 \times 10^{-3} \) |
| 0.9 | \( 1.33 \times 10^{-2} \) | \( 2.512 \times 10^{-3} \) | \( 2.507 \times 10^{-3} \) |
| 1   | \( 2.015 \times 10^{-2} \) | \( 4.696 \times 10^{-3} \) | \( 4.689 \times 10^{-3} \) |
Table 2. Absolute errors of the third order ADM-DM (ADM-DMT-3rd), PIM (PIM-3rd), OPIM (OPIM-3rd) approximate solutions at $x = 0.5$ for Example 1.

| t    | ADM-DMT-3rd  | PIM-3rd     | OPIM-3rd     |
|------|-------------|-------------|-------------|
| 0.1  | $6.943 \times 10^{-10}$ | $3.831 \times 10^{-12}$ | $2.081 \times 10^{-12}$ |
| 0.2  | $4.441 \times 10^{-8}$  | $9.748 \times 10^{-10}$ | $3.316 \times 10^{-11}$ |
| 0.3  | $5.054 \times 10^{-7}$  | $2.468 \times 10^{-8}$  | $1.667 \times 10^{-10}$ |
| 0.4  | $2.836 \times 10^{-6}$  | $2.424 \times 10^{-7}$  | $5.221 \times 10^{-10}$ |
| 0.5  | $1.08 \times 10^{-5}$   | $1.413 \times 10^{-6}$  | $1.259 \times 10^{-9}$  |
| 0.6  | $3.219 \times 10^{-5}$  | $5.914 \times 10^{-6}$  | $2.574 \times 10^{-9}$  |
| 0.7  | $8.099 \times 10^{-5}$  | $1.965 \times 10^{-5}$  | $4.686 \times 10^{-9}$  |
| 0.8  | $1.8 \times 10^{-4}$    | $5.501 \times 10^{-5}$  | $7.838 \times 10^{-9}$  |
| 0.9  | $3.638 \times 10^{-4}$  | $1.35 \times 10^{-4}$   | $1.227 \times 10^{-8}$  |
| 1.   | $6.822 \times 10^{-4}$  | $2.979 \times 10^{-4}$  | $1.825 \times 10^{-8}$  |

Table 3. The absolute errors of second order approximation by OPIM with the exact solution of Example 1.

| x    | t = 0.1     | t = 0.2     | t = 0.3     | t = 0.4     | t = 0.5     |
|------|-------------|-------------|-------------|-------------|-------------|
| 0.1  | $5.506 \times 10^{-9}$ | $3.537 \times 10^{-7}$ | $4.019 \times 10^{-6}$ | $2.248 \times 10^{-5}$ | $8.521 \times 10^{-5}$ |
| 0.2  | $5.341 \times 10^{-9}$ | $3.492 \times 10^{-7}$ | $3.981 \times 10^{-6}$ | $2.229 \times 10^{-5}$ | $8.458 \times 10^{-5}$ |
| 0.3  | $5.023 \times 10^{-9}$ | $3.392 \times 10^{-7}$ | $3.889 \times 10^{-6}$ | $2.183 \times 10^{-5}$ | $8.293 \times 10^{-5}$ |
| 0.4  | $4.522 \times 10^{-9}$ | $3.215 \times 10^{-7}$ | $3.72 \times 10^{-6}$  | $2.096 \times 10^{-5}$ | $7.981 \times 10^{-5}$ |
| 0.5  | $3.802 \times 10^{-9}$ | $2.94 \times 10^{-7}$  | $3.45 \times 10^{-6}$  | $1.955 \times 10^{-5}$ | $7.472 \times 10^{-5}$ |
| 0.6  | $2.832 \times 10^{-9}$ | $2.546 \times 10^{-7}$ | $3.055 \times 10^{-6}$ | $1.747 \times 10^{-5}$ | $6.719 \times 10^{-5}$ |
| 0.7  | $1.577 \times 10^{-9}$ | $2.012 \times 10^{-7}$ | $2.512 \times 10^{-6}$ | $1.46 \times 10^{-5}$  | $5.674 \times 10^{-5}$ |
| 0.8  | $4.911 \times 10^{-12}$ | $1.318 \times 10^{-7}$ | $1.797 \times 10^{-6}$ | $1.08 \times 10^{-5}$  | $4.29 \times 10^{-5}$  |
| 0.9  | $1.918 \times 10^{-9}$ | $4.417 \times 10^{-8}$ | $8.865 \times 10^{-7}$ | $5.939 \times 10^{-6}$ | $2.519 \times 10^{-5}$ |
| 1.   | $4.225 \times 10^{-9}$ | $6.376 \times 10^{-8}$ | $2.428 \times 10^{-7}$ | $1.01 \times 10^{-7}$  | $3.129 \times 10^{-6}$ |

Table 4. The absolute errors of third order approximation by OPIM with the exact solution of Example 1.

| x    | t = 0.1     | t = 0.2     | t = 0.3     | t = 0.4     | t = 0.5     |
|------|-------------|-------------|-------------|-------------|-------------|
| 0.1  | $8.322 \times 10^{-14}$ | $1.326 \times 10^{-12}$ | $6.67 \times 10^{-12}$ | $2.088 \times 10^{-11}$ | $5.038 \times 10^{-11}$ |
| 0.2  | $3.329 \times 10^{-13}$ | $5.305 \times 10^{-12}$ | $2.668 \times 10^{-11}$ | $8.353 \times 10^{-11}$ | $2.015 \times 10^{-10}$ |
| 0.3  | $7.49 \times 10^{-13}$ | $1.194 \times 10^{-11}$ | $6.003 \times 10^{-11}$ | $1.88 \times 10^{-10}$  | $4.534 \times 10^{-10}$ |
| 0.4  | $1.332 \times 10^{-12}$ | $2.122 \times 10^{-11}$ | $1.067 \times 10^{-10}$ | $3.341 \times 10^{-10}$ | $8.06 \times 10^{-10}$  |
| 0.5  | $2.081 \times 10^{-12}$ | $3.316 \times 10^{-11}$ | $1.667 \times 10^{-10}$ | $5.221 \times 10^{-10}$ | $1.259 \times 10^{-9}$  |
| 0.6  | $2.996 \times 10^{-12}$ | $4.774 \times 10^{-11}$ | $2.401 \times 10^{-10}$ | $7.518 \times 10^{-10}$ | $1.814 \times 10^{-9}$  |
| 0.7  | $4.078 \times 10^{-12}$ | $6.499 \times 10^{-11}$ | $3.268 \times 10^{-10}$ | $1.023 \times 10^{-9}$  | $2.469 \times 10^{-9}$  |
| 0.8  | $5.326 \times 10^{-12}$ | $8.488 \times 10^{-11}$ | $4.268 \times 10^{-10}$ | $1.337 \times 10^{-9}$  | $3.224 \times 10^{-9}$  |
| 0.9  | $6.741 \times 10^{-12}$ | $1.074 \times 10^{-10}$ | $5.402 \times 10^{-10}$ | $1.692 \times 10^{-9}$  | $4.081 \times 10^{-9}$  |
| 1.   | $8.322 \times 10^{-12}$ | $1.326 \times 10^{-10}$ | $6.67 \times 10^{-10}$ | $2.088 \times 10^{-9}$  | $5.038 \times 10^{-9}$  |
From (7), perturbation iteration algorithm can be constructed as:

\[
((u_n)_{tt})_{tt} = (u_n)_{xx} - 3\left(\frac{u_n}{4}\right) + \frac{3}{2}(u_n)^3
\]

for the Eq. (46). \(u_0 = -\text{sech}(x)\) can be taken as a starting function for both PIM and OPIM.

**PIM Solutions:**

Using the conditions (47) and proceeding as in the first example one can reach the following PIM solutions:

\[
(u_1)_{PIM} = -\text{sech}(x) + \frac{1}{16} \left[3t^2\text{sech}^3(x) - t^2 \text{cosh}(2x)\text{sech}^3(x) + 8t\text{sech}(x)\tan(h(x))\right]
\]

\[
(u_2)_{PIM} = u_1 + \left[-\frac{129t^4\text{sech}^9(x)}{16384} - \frac{57t^6\text{sech}^9(x)}{20480} + \frac{27t^8\text{sech}^9(x)}{131072}\right]
\]

\[
(u_3)_{PIM} = u_2 + \left[-\frac{243t^{10}\text{cosh}(4x)\tan^2(x)\text{sech}^9(x)}{9175040} + \frac{81t^6\text{cosh}(6x)\tan^2(x)\text{sech}^9(x)}{20480}\right]
\]

and so on.

**OPIM Solutions:**

Following the procedure mentioned in Section 2 with the Eqs. (47),(49), we get

\[
(u_1)_{OPIM} = -\text{sech}(x) + \frac{P_0}{16} \left[3t^2\text{sech}^3(x) - t^2 \text{cosh}(2x)\text{sech}^3(x) + 8t\text{sech}(x)\tan(h(x))\right]
\]
and so on. Proceeding as mentioned in Section 2, the absolute errors for different values of $x$ with the exact solution and other methods, we created tables 5 and 6 which show 

\[ u_{2}^{OPIM} = u_{1} + P_{1} \times \]

\[
\begin{bmatrix}
-\frac{3}{16} P_{0} t^{2} \text{sech}^{3}(x) - \frac{1}{4} t^{2} \text{sech}^{3}(x) + \frac{3}{8} t^{2} \text{sech}(x) - \frac{3}{256} P_{0} t^{4} \text{sech}^{3}(x) \\
-\frac{1}{128} P_{0} t^{4} \cosh(2x) \text{sech}^{5}(x) + \frac{3}{16} P_{0} t^{3} \sinh(2x) \text{sech}^{5}(x) \\
+ \frac{111 P_{0}^{3} t^{7} \sinh(2x) \text{sech}^{9}(x)}{114688} - \frac{9 P_{0}^{3} t^{7} \sinh(4x) \text{sech}^{9}(x)}{28672} \\
+ \frac{3 P_{0}^{3} t^{7} \sinh(6x) \text{sech}^{9}(x)}{114688} + \frac{3 P_{0}^{3} t^{5} \sinh(6x) \text{sech}^{9}(x)}{10240} \\
-\frac{3}{64} P_{0} t^{4} \cosh(2x) \tanh^{2}(x) \text{sech}^{3}(x) + \cdots
\end{bmatrix}
\]

\[ (u_{3})^{OPIM} = u_{2} + P_{2} \times \]

\[
\begin{bmatrix}
\frac{27 P_{0}^{3} t^{8} \sinh(9x)}{256(\sinh(2x) + \cosh(2x) + 1)^{9}} + \frac{3 P_{0}^{3} t^{7} \sinh(x) \sinh(8x) \cosh^{2}(2x)}{56(\sinh(2x) + \cosh(2x) + 1)^{8}} \\
+ \frac{9 P_{1} t^{4} e^{5x}}{4(e^{2x} + 1)^{5}} - \frac{3 P_{1} t^{4} e^{5x} \cosh(2x)}{4(e^{2x} + 1)^{5}} - \frac{3}{8} P_{1} t^{2} \text{sech}(x) \\
\frac{7 P_{0} P_{1} t^{6} \cosh(2x) \text{sech}^{7}(x)}{3840} - \frac{87 P_{0} P_{1} t^{5} e^{5x} \sinh(x)}{20(e^{2x} + 1)^{6}} \\
+ \frac{91 P_{0} P_{1} t^{6} \sinh(7x) \cosh(2x)}{40(\sinh(2x) + \cosh(2x) + 1)^{7}} - \frac{13 P_{0} P_{1} t^{6} \sinh(7x) \cosh(4x)}{160(\sinh(2x) + \cosh(2x) + 1)^{7}} \\
+ \frac{249 P_{0}^{2} P_{1} t^{8} \sinh(13x)}{2240(\sinh(2x) + \cosh(2x) + 1)^{9}} + \cdots
\end{bmatrix}
\]

and so on. Proceeding as mentioned in Section 2, $P_{0}, P_{1}$ and $P_{2}$ are determined as:

\[ P_{0} = 1.1224051, P_{1} = -1.0077802, P_{2} = 0.0928211 \]

for the third order OPIM solution. As a result, Eq. (55) becomes the third-order OPIM approximate solution, i.e., by using the auxiliary parameters given above, we will obtain the third-order approximate solution.

This problem is also studied via Adomian decomposition method (ADM) [2], the variational iteration method (VIM) [41] and optimal homotopy asymptotic method [21]. In order to demonstrate the efficiency of the proposed method in comparison with the exact solution and other methods, we created tables 5 and 6 which show the absolute errors for different values of $x$ and $t$. The third order OPIM solution yields very encouraging results after being compared with third order approximate solution by VIM [41] and fourth order approximate solution by ADM [2].

Fig 3 displays the absolute errors for third order PIM and OPIM approximate solutions. It is evident that with the increase in the order of approximation, the accuracy increases. Thus the OPIM approximate solution converges.

The graph of analytical and estimated functions at $x = 1$ and $x = 3$ are given in Fig. 4 which also shows the reliability of the proposed method for larger domain.
Table 5. Absolute errors of third order OPIM, third order VIM and fourth order ADM solutions at t=0.1 for Example 2.

| x  | (ADM)      | (VIM)      | (OPIM)     |
|----|------------|------------|------------|
| 1  | 5.201 × 10^{-11} | 4.809 × 10^{-12} | 3.334 × 10^{-19} |
| 2  | 3.362 × 10^{-11} | 2.607 × 10^{-13} | 3.108 × 10^{-19} |
| 3  | 2.379 × 10^{-11} | 4.985 × 10^{-14} | 5.274 × 10^{-20} |
| 4  | 1.509 × 10^{-11} | 2.774 × 10^{-15} | 6.118 × 10^{-21} |
| 5  | 1.496 × 10^{-11} | 1.292 × 10^{-16} | 8.936 × 10^{-20} |
| 6  | 2.671 × 10^{-12} | 2.315 × 10^{-18} | 4.014 × 10^{-21} |
| 7  | 2.250 × 10^{-12} | 1.403 × 10^{-18} | 1.124 × 10^{-21} |
| 8  | 1.613 × 10^{-13} | 6.288 × 10^{-19} | 7.052 × 10^{-21} |
| 9  | 1.541 × 10^{-13} | 2.369 × 10^{-19} | 8.017 × 10^{-20} |
| 10 | 1.108 × 10^{-14} | 8.743 × 10^{-20} | 1.055 × 10^{-20} |

Table 6. Absolute errors of third order OPIM, third order VIM and fourth order ADM solutions at t=0.3 for Example 2.

| x  | (ADM)      | (VIM)      | (OPIM)     |
|----|------------|------------|------------|
| 1  | 4.427 × 10^{-8} | 3.177 × 10^{-8} | 5.036 × 10^{-17} |
| 2  | 6.142 × 10^{-9} | 1.651 × 10^{-9} | 6.018 × 10^{-17} |
| 3  | 3.528 × 10^{-10} | 3.211 × 10^{-10} | 3.305 × 10^{-16} |
| 4  | 2.774 × 10^{-11} | 1.788 × 10^{-11} | 9.012 × 10^{-15} |
| 5  | 8.682 × 10^{-12} | 8.318 × 10^{-13} | 7.047 × 10^{-15} |
| 6  | 1.430 × 10^{-13} | 1.741 × 10^{-14} | 2.512 × 10^{-16} |
| 7  | 5.498 × 10^{-13} | 9.126 × 10^{-15} | 6.369 × 10^{-16} |
| 8  | 1.514 × 10^{-13} | 4.081 × 10^{-15} | 8.169 × 10^{-17} |
| 9  | 4.975 × 10^{-14} | 1.537 × 10^{-15} | 9.142 × 10^{-16} |
| 10 | 4.353 × 10^{-14} | 5.674 × 10^{-16} | 8.777 × 10^{-16} |

Figure 3. Absolute errors of third order PIM and OPIM solutions for Example 2.
4.3. Example 3. We finally close our analysis by studying the following Klein-Gordon equation

\[ u_{tt} - u_{xx} + u^2 = 0; \quad 0 \leq x, t \leq 1 \]  

(57)

with the initial conditions

\[ u(x, 0) = 1 + \sin x, \quad u_t(x, 0) = 0 \]  

(58)

which has no exact solution [24, 41].

PIA can be computed from (7) as:

\[ \left( (u_c)_n \right)_{tt} = (u_n)_{xx} - (u_n)_{tt} - (u_n)^2 \]  

(59)

for the Eq. (57). \( u_0 = 1 + \sin x \) can be selected as initial function for both PIM and OPIM.

**PIM Solutions:**

Performing necessary calculations, we get

\[ (u_1)_{PIM} = 1 - \frac{t^2}{2} + \sin(x) - \frac{3}{2} t^2 \sin(x) - \frac{1}{2} t^2 \sin^2(x) \]  

(60)

\[ (u_2)_{PIM} = u_1 + \begin{bmatrix} \frac{t^4}{12} - \frac{t^6}{120} - \frac{1}{12} t^4 \cos^2(x) + \frac{11}{24} t^4 \sin(x) - \frac{1}{20} t^6 \sin(x) + \frac{5}{12} t^4 \sin^2(x) \\ - \frac{11}{120} t^6 \sin^2(x) + \frac{1}{12} t^4 \sin^3(x) - \frac{1}{20} t^6 \sin^3(x) - \frac{1}{120} t^6 \sin^4(x) \end{bmatrix} \]  

(61)

\[ (u_3)_{PIM} = u_2 + \begin{bmatrix} - \frac{t^2}{4} + \frac{t^4}{6} - \frac{239}{2880} + \frac{41t^8}{13440} + \frac{5291t^{10}}{1036800} + \frac{167t^{12}}{253440} - \frac{1037t^{14}}{22364160} \\ - \frac{11t^8 \cos^2(x)}{3360} + \frac{1}{4} t^2 \cos(2x) - \frac{1}{6} t^4 \cos(2x) + \frac{61}{720} t^6 \cos(2x) \\ + \frac{11t^{10} \sin(3x)}{3840} - \frac{877t^{12} \sin(3x)}{2027520} + \frac{151t^{14} \sin(3x)}{4659200} + \cdots \end{bmatrix} \]  

(62)

and so on. These results are equivalent to those obtained by VIM in [41].
Table 7. ADM, VIM-PIM, DTM and OPIM solutions at t=0.1 for Example 3.

| x    | (ADM)   | (VIM-PIM) | (DTM)   | (OPIM-\(u_1\)) | (OPIM-\(u_2\)) |
|------|---------|-----------|---------|----------------|----------------|
| 0.0  | 0.994999| 0.995000  | 0.995000| 1.00235        | 1.00436        |
| 0.1  | 1.093291| 1.093291  | 1.093336| 1.10291        | 1.10548        |
| 0.2  | 1.190502| 1.190503  | 1.190602| 1.20252        | 1.20566        |
| 0.3  | 1.285668| 1.285668  | 1.285289| 1.30016        | 1.3039         |
| 0.4  | 1.377844| 1.377844  | 1.378073| 1.39487        | 1.39921        |
| 0.5  | 1.466118| 1.466119  | 1.466420| 1.4857         | 1.49062        |
| 0.6  | 1.549620| 1.549621  | 1.550000| 1.57173        | 1.57723        |
| 0.7  | 1.627529| 1.627531  | 1.627994| 1.65209        | 1.65815        |
| 0.8  | 1.699081| 1.699084  | 1.699640| 1.72598        | 1.73257        |
| 0.9  | 1.763575| 1.763579  | 1.764245| 1.79265        | 1.79972        |
| 1.0  | 1.820382| 1.820387  | 1.821201| 1.85142        | 1.85893        |

**OPIM Solutions:**

Using the Eqs. (9), (58), (59), approximate OPIM solutions are computed as:

\[(u_1)_{\text{OPIM}} = 1 + \sin(x) + \left( -\frac{t^2}{2} - \frac{3}{2} t^2 \sin(x) - \frac{1}{2} t^2 \sin^2(x) \right) P_0 \quad (63)\]

\[(u_2)_{\text{OPIM}} = u_1 + P_1 \times \]

\[
\begin{bmatrix}
-\frac{1}{4} (3t^2) + \frac{1}{4} t^2 \cos(2x) - \frac{3}{2} t^2 \sin(x) + \frac{t^2 P_0}{2} + \frac{t^4 P_0}{4} - \frac{1}{4} t^4 \cos(2x) P_0 \\
+ \frac{3}{2} t^2 \sin(x) P_0 + \frac{25}{48} t^4 \sin(x) P_0 + \frac{1}{2} t^2 \sin^2(x) P_0 + \cdots
\end{bmatrix} \quad (64)
\]

Using the Least Squares Method, \(P_0\) and \(P_1\) can be obtained as:

\[P_0 = -0.470347 \quad (65)\]

\[P_0 = 0.8771737 \quad P_1 = -14.2379351 \quad (66)\]

for the Eqs. (63) and (64) respectively. Therefore, Eqs. (63) and (64) become the second and third-order OPIM approximate solutions of Example 3. To illustrate the OPIM results, we present a numerical experiment to compare these solutions and results gained by using VIM [41], ADM [2] and DTM [24]. Table 7 shows a comparison between the value \(u\) obtained by first and second order OPIM, fifth order ADM, fourth order VIM-PIM and DTM at various values of \((x, t)\).

5. Results and discussion. In this study, we introduce a new efficient technique, namely optimal perturbation iteration method as an alternative to existing methods in solving NPDEs. It has been successfully applied to obtain the solution of the nonlinear Klein-Gordon equation. Examples show that the proposed technique is effective and powerful mathematical tool to cope with these types of nonlinear problems. In OPIM, it is essentially important to find the parameters \(P_0, P_1, \ldots\) and this makes it time consuming, especially for large \(n\). After 3-rd and 4-th iterations, we confront with high cost of calculations that occupy a vast space of computer’s memory. However, we have seen that this method converges rapidly at lower order of approximations for the problems discussed in applications. On the other hand, since
the method is often tedious to use by hand, one has to use a symbolic computer program to obtain approximate solutions. In this study, Mathematica 9.0 has been used to perform the complex calculations in applications. It is worth noting that one can get better results with more powerful computers or processors. Finally, we can say that the proposed method is very powerful, efficient, reliable and accurate in comparison with many competitive numerical and analytical techniques.

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