On Viability of Inflation in Nonminimal Kinetic Coupling Theory

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Abstract—We consider the initial conditions for inflation in the nonminimal kinetic coupling theory. If inflation is driven solely by the kinetic term with no potential, the resulting number of e-folds depends only on the initial velocity of the scalar field, $\dot{\phi}$. We write down the expression for the number of e-folds explicitly, and show that for physically reasonable values of the coupling constant, we can get 60 e-folds only for an exponentially big initial $\phi$. When the scalar field potential is taken into account, a double inflation scenario arises, where the first inflation is kinetic term driven, while the second one is potential term driven. In this case, we need not a very large $\phi$ to start with for 60 e-folds, on the other hand, some initial conditions lead to a physically inadmissible eternal inflation. We show numerically that in the measure used in the present paper only a smaller part of initial conditions leads to eternal inflation for reasonable values of the coupling constant.

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Inflation is one of the most popular and successful scenarios explaining the physics of the early universe [1–7]. The popularity of inflation is mainly due to the fact that it allows us to explain a number of observational data, such as homogeneity, isotropy, and spatial flatness of the Universe. It also allows us to explain the formation of galaxies with the help of initial quantum oscillations, which are then amplified due to gravitational instability [8–11].

Despite a success in general, reconstruction of a particular form of inflation which can be in accordance with the observational results on cosmological perturbations is still a problem. Models popular at the beginning of the inflationary paradigm (such a massive or self-interacting minimally coupled scalar field) are already ruled out. Current observations of the amplitude of scalar perturbations as well as upper bounds on the tensor-to-scalar ratio indicate that only rather a shallow potential is compatible with the minimal coupling. Lifting the minimal coupling, it is possible to create a viable model for steeper potentials, including the Higgs field. The price for this result is rather a high value of the dimensionless constant of nonminimal coupling [12, 13]. One more viable model, the Starobinsky inflation, which uses quadratic corrections to the Einstein gravity and treats the scalar field as an effecting one, also needs a large dimensionless coupling constant [1]. It is already known that a combination of these two scenarios into a single model cannot significantly reduce the values of the coupling constants [14].

This motivates a further search for other possible scenarios. One of the recent proposals deals with a scalar field, non-minimally coupled with gravity by its kinetic term. It is interesting that only one type of coupling, the coupling with the Einstein tensor, leads to second-order equations of motion, so this theory is exceptional within the whole class of kinetic coupling theories [16, 17]. Its cosmological dynamics is richer than the dynamics in the case of a minimally coupled scalar field. In particular, it allows inflation even in the absence of the field potential. On the other hand, a nonzero potential can induce the second stage of inflation [18] which either follows the first stage directly or through some transient period [19]. The goal of the present paper is to find the initial conditions leading to an adequate inflation in this scenario.

In the current study we use only one necessary criterion for a successful inflation—the requirement that the number of e-folds is larger than 60. This problem allows for an analytical treatment for zero-potential inflation and requires a numerical study in the general case. Unlike earlier works on minimally

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coupled fields, we do not fix the initial energy of the scalar field since in the nonminimal coupling case there is some ambiguity in defining the scalar field energy due to the presence of cross-terms in the equations of motion.

We work in the units $G = c = h = 1$ and use the signature $(-, +, +, +)$.

The action of the theory has the form [16]:

$$S = \int d^4x \sqrt{-g} \left( \frac{R}{8\pi} - [g_{\mu\nu} + \kappa G_{\mu\nu}] \phi_{,\mu} \phi_{,\nu} - 2V(\phi) \right), \quad (1)$$

where $R$ is the scalar curvature, $g_{\mu\nu}$ is the metric tensor, $G_{\mu\nu}$ is the Einstein tensor, $V(\phi)$ is the scalar potential, $\kappa$ is the coupling parameter.

In the spatially flat Friedmann–Robertson–Walker cosmological model, the action (1) yields the following field equations [20]:

$$3H^2 = 4\pi \dot{\phi}^2 (1 - 9\kappa H^2) + 8\pi V(\phi), \quad (2)$$

$$2\dot{H} + 3H^2 = -4\pi \dot{\phi}^2 [1 + \kappa (2\dot{H} + 3H^2) + 4H \ddot{\phi} \phi^{-1}] + 8\pi V(\phi), \quad (3)$$

$$(\ddot{\phi} + 3H \dot{\phi}) - 3\kappa (H^2 \ddot{\phi} + 2H \dot{H} \dot{\phi} + 3H^3 \dot{\phi}) = -V_{,\phi}. \quad (4)$$

We start with the simplest case of $V(\phi) = 0$, since in the model (1) the kinetic coupling solely can induce the inflation. The field equations in this case are as follows:

$$3H^2 = 4\pi \dot{\phi}^2 (1 - 9\kappa H^2), \quad (5)$$

$$2\dot{H} + 3H^2 = -4\pi \dot{\phi}^2 (1 + \kappa (2\dot{H} + 3H^2) + 4H \ddot{\phi} \phi^{-1}), \quad (6)$$

$$\ddot{\phi} + 3H \dot{\phi} - 3\kappa (H^2 \ddot{\phi} + 2H \dot{H} \dot{\phi} + 3H^3 \dot{\phi}) = 0. \quad (7)$$

We can rewrite this system isolating the highest derivative terms:

$$\dot{H} = \frac{-3H^2 - 4\pi \dot{\phi}^2 (1 - 9\kappa H^2)}{2 \left( 1 + \frac{4\pi \dot{\phi}^2 \kappa (1 + 9\kappa H^2)}{1 - 3\kappa H^2} \right)}, \quad (8)$$

$$\ddot{\phi} = -3H \dot{\phi} + \frac{3H \dot{\phi} \kappa [-3H^2 - 4\pi \dot{\phi}^2 (1 - 9\kappa H^2)]}{1 - 3\kappa H^2 + 4\pi \dot{\phi}^2 \kappa (1 + 9\kappa H^2)}, \quad (9)$$

$$H^2 = 4\pi \dot{\phi}^2 (1 - 9\kappa H^2), \quad (10)$$

which allows us to write down an equation containing $H$ and its derivative only:

$$\dot{H} = \frac{-3H^2 (1 - 9\kappa H^2) (1 - 3\kappa H^2)}{1 - 9\kappa H^2 + 54\kappa^2 H^4}. \quad (11)$$

This equation can be solved in a closed form, so we get:

$$\sqrt{\kappa} \ln \left( \frac{\left( 1 + 3\sqrt{\kappa} H \right)^{1/\sqrt{2}} \left( 1 + 3\sqrt{\kappa} H \right)^{-1/2}}{1 - \sqrt{3\kappa} H} \right) + \frac{1}{3H} = t + \text{const.} \quad (12)$$

The constant in this equation depends on the initial conditions. Now we introduce the number of inflationary $e$-foldings since the onset of inflation (this means that the universe expands by a factor of $e^N$ during inflation),

$$N = \ln \left( \frac{a(t_{\text{end}})}{a(t_{\text{initial}})} \right) = \int_{t_i}^{t_e} H dt. \quad (13)$$

For realistic inflationary scenarios, this parameter should satisfy the condition $N_0 > 60$ at the end of inflation. For the case of the theory with zero scalar interaction, one can obtain an analytic expression for $N$. Rewriting (11),

$$dH \frac{1 - 9\kappa H^2 + 54\kappa^2 H^4}{-3H(1 - 9\kappa H^2)(1 - 3\kappa H^2)} = H dt, \quad (14)$$

and using (13), we obtain

$$N = \int_{t_i}^{t_e} \frac{1 - 9\kappa H^2 + 54\kappa^2 H^4}{-3H(1 - 9\kappa H^2)(1 - 3\kappa H^2)} dH$$

$$= \ln \left[ H(t_e)^{-1/3} (1 - 9\kappa H(t_e)^2)^{1/6} \times (1 - 3\kappa H(t_e))^{-1/3} \right]$$

$$- \ln \left[ H(t_i)^{-1/3} (1 - 9\kappa H(t_i)^2)^{1/6} \times (1 - 3\kappa H(t_i))^{-1/3} \right]. \quad (15)$$

We also introduce the first slow-roll parameter $\epsilon$ [5]

$$\epsilon = - \frac{\ddot{H}}{H^2}, \quad (16)$$

which implies that the Universe accelerates, $\ddot{a} > 0$, when $\epsilon < 1$.

The indicator of the end of inflation is the condition $\epsilon_H = 1$. Thus, based on that, it is possible to calculate the value of the Hubble parameter at the end of inflation. We write down the equation for this value using (16) and (11):

$$\frac{3(1 - 9\kappa H(t_e)^2) (1 - 3\kappa H(t_e)^2)}{1 - 9\kappa H(t_e)^2 + 54\kappa^2 H(t_e)^4} = 1, \quad (17)$$

and the solution of this equation is

$$\kappa H(t_e)^2 = \frac{27 - \sqrt{513}}{54} \approx 0.081. \quad (18)$$
Now we calculate the initial values for $H$ and $\phi$ at which $N$ would be at least 60 (using Eq. (15)) at the moment when $\epsilon = 1$ (see Eqs. (16) and (16)). The initial value of the Hubble parameter is a solution of the inequality

$$\ln \left[ H(t_i)^{-1/3} (1 - 9\kappa H(t_i)^2)^{1/6} \right] \times (1 - 3\kappa H(t_i)^2)^{-1/3} < q - N, \quad (19)$$

where

$$q = \ln (H(t_e)^{-1/3} (1 - 9\kappa H(t_e)^2)^{1/6} \times (1 - 3\kappa H(t_e)^2)^{-1/3}). \quad (20)$$

Here we present some values of $q$ and $H(t_e)$ for different $\kappa$:

- $\kappa = 1$, $q \approx 0.297$, $H(t_e) \approx 0.284$;
- $\kappa = 10$, $q \approx 0.681$, $H(t_e) \approx 0.09$;
- $\kappa = 10^6$, $q \approx 2.599$, $H(t_e) \approx 0.00028$;
- $\kappa = 10^{12}$, $q \approx 4.9$, $H(t_e) \approx 2.84 \times 10^{-7}$. \quad (21)

We rewrite the inequality (19) as

$$H(t_i)^{1/3} \left[ (1 - 9\kappa H(t_i)^2)^{1/6} - e^{q-N} (1 - 3\kappa H(t_i)^2)^{1/3} H(t_i)^{1/3} \right] \times (1 - 3\kappa H(t_i)^2)^{1/3} < 0. \quad (22)$$

For obtaining a solution to the inequality, we find the boundary of the admissible interval for $\dot{H}$. To do that, we solve this equation

$$AH(t_i)^6 - BH(t_i)^4 + CH(t_i)^2 - 1 = 0, \quad (23)$$

where

- $A = 9\kappa^2 e^{6q-6N}$,
- $B = 6\kappa e^{6q-6N}$,
- $C = 9\kappa + e^{6q-6N}$. \quad (24)

Thus we obtain an interval of values of $H$ and $\phi$ at which more than 60 e-folds are accumulated during inflation:

$$H \in (H_{\text{sol}}; 1/(3\sqrt{\kappa}))$$

and $\phi \in (\phi_{\text{sol}}, \infty)$ (see the Appendix), note that $H = 1/(3\sqrt{\kappa})$ is the highest possible value of $H$ in this theory, as can be easily seen from Eq. (10).

As we see, $\dot{\phi}$ must be exponentially large ($\dot{\phi} \sim 4.97 \times 10^{77}$ for $\kappa = 1$ and $N = 60$; $\dot{\phi} \sim 4.97 \times 10^{74}$ for $\kappa = 10^{12}$ and $N = 60$), and $H$ must be very close to $1/(3\sqrt{\kappa})$ (for $N = 60$ and $\kappa = 1$, $H$ differs from $1/(3\sqrt{\kappa})$ by $1.1 \times 10^{-79}$ and for $\kappa = 10^{12}$ this difference is $1.1 \times 10^{-91}$). Note that the value of the coupling constant $\kappa$ enters into the expression for $\dot{\phi}$ only through $q$ which changes very smoothly with $\kappa$ (see Eq. (20)), so these large values of $\dot{\phi}$ cannot be easily compensated by changing $\kappa$. Thus only very specific initial conditions must be met in order to trigger an adequate inflation.

The situation changes if we introduce a scalar field potential. Now, the same as for the minimally coupled case, the potential itself can induce inflation. Moreover, the zero potential asymptotic is still a solution if the potential is less steep than the quadratic one. Thus in the model under investigation we can have two independent inflationary stages, one induced by the kinetic coupling, and the second stage induced by the potential. This makes easier to get at least 60 e-foldings during inflation. However, in this case we are faced with another problem. Namely, it is known that apart from inflation with a natural exit, the system contains other regimes, which are inappropriate for the description of our Universe. In the present paper we consider a potential which grows less rapidly than the quadratic one, leaving the quadratic (where the first inflation stage can exist but is modified) and steeper potentials (when we have only potential induced inflation) for a future work. For definiteness, we consider $V = V_0|\phi|^{1.5}$.

Now the system of equations cannot be solved analytically, however, it is possible to obtain its numerical solutions and to find the initial conditions leading to adequate inflation. For presentation of the numerical results, we have chosen $V_0 = 10^{-5}$ and $\kappa = 100$.

The initial conditions space of the model under study is divided into three zones depending on the output of the dynamical evolution. In the first zone, the scenario of double inflation and the subsequent exit from the second inflation is realized. The blue area (III) in Fig. 1 presents this zone. This zone is divided into two subzones, shown in Fig. 2. The first subzone (orange, B) presents the initial conditions leading to less than 60 e-folds. And in the second subzone (blue zone, A), the number of e-folds exceeds 60, so these initial conditions lead to an adequate inflation.

In zones I and II in Fig. 1, the accelerated expansion never stops. These two zones correspond to two different asymptotics. In the second zone I (marked in red in Fig. 1), the scalar field $\phi$ and the Hubble parameter $H$ tend to (see [21–23])

$$\phi(t)_{t \to \infty} \approx \left[ \frac{\sqrt{V_0} t + \text{const}}{4} \right]^4;$$

$$H_{t \to \infty} \approx \frac{1}{\sqrt{3\kappa}}. \quad (25)$$
The initial data from zone II lead to the asymptotics described as \[22, 23\]:
\[
\phi(t)_{t \to \infty} \approx \left[ \frac{33}{64\pi} \sqrt{\frac{1}{24\pi V_0}} t + \text{const} \right]^{4/11};
\]
\[
H_{t \to \infty} \approx \sqrt{\frac{8\pi V_0}{3}} \phi^{3/4} \sim t^{3/11}. \tag{26}
\]

The configuration of the orange zone (B) in Fig. 2 (insufficient or no inflation) is qualitatively the same as for the massive minimally coupled scalar field (see \[24\]), being a narrow strip inside a region of sufficient inflation. However, as we can see from Fig. 1, for large enough values of the initial scalar field there emerge two new zones with eternal accelerated expansion. These zones correspond to regimes inappropriate for the inflationary scenario. As in the present paper we consider the initial conditions of an arbitrary initial energy, the phase space is not compact, and we cannot attribute a reasonable measure to it. However, by making the transformation
\[
\alpha = \frac{\phi}{\sqrt{1 + \phi^2 + \dot{\phi}^2}};
\]
\[
\beta = \frac{\dot{\phi}}{\sqrt{1 + \phi^2 + \dot{\phi}^2}},
\]
we obtain a compact phase space \((\alpha, \beta)\), and Fig. 3 shows that the larger part of the initial conditions for the physically admissible part \(|\alpha|^2 + |\beta|^2 \leq 1\) of the plane \((\alpha, \beta)\) does not lead to eternal inflation. As for the eternal inflation, the regime (25) dominates over the regime (26) from the viewpoint of initial conditions since the green zone in Figs. 1, 3 is very small.

Our results can be summarized as follows. Though in principle inflation in the model under investigation can be driven by the kinetic coupling term only, an adequate inflation (no less than 60 e-folds) requires either exponentially large initial values of \(\dot{\phi}\) or exponentially large values of the coupling constant \(\kappa\). Both possibilities are problematic from the physical point of view. In particular, if the theory in question is a low-energy approximation of some more general underlying theory, the required initial conditions may be located far beyond a zone where this theory can be considered as a reliable approximation.
On the contrary, a nonzero scalar field potential naturally leads to a successful inflation without any fine tuning. Though in this case the necessary number of e-folds is collected mostly due to the scalar field potential (thus in the same way as in the standard inflationary scenario), a novel thing specific for kinetic coupling theories is that some part of the e-folds can be collected at the stage of a kinetic term inflation. In the time interval between these two types of inflation, the inflationary parameters can change rapidly providing significant effects on the propagation of cosmological perturbations. This effect needs a special investigation, and we leave it for a future research.

Appendix

We rewrite Eq. (23) in the following notations: $x = H^2$, $a_1 = A$, $b_1 = -B/3$, $c_1 = C/3$, and $d_1 = -1$:

$$a_1 x^3 + 3 b_1 x^2 + 3 c_1 x + d_1 = 0. \quad (A.1)$$

This equation has three solutions that can be found analytically. Using the substitutions

$$y = x + \frac{b_1}{a_1}, \quad p_1 = \frac{c_1 a_1 - b_1^2}{a_1^2},$$

$$2 q_1 = \frac{2 b_1^3 - 3 a_1 b_1 c_1 + d_1 a_1^2}{a_1^2},$$

we get from the Eq. (A.1) the equation

$$y^3 + 3 p_1 y + 2 q_1 = 0. \quad (A.2)$$

Now, using the Cardano formula, we get the solutions of Eq. (A.1):

$$x = \left( - q_1 \pm \sqrt{q_1^2 + p_1^3} \right)^{1/3} - \frac{p_1}{\left( - q_1 \pm \sqrt{q_1^2 + p_1^3} \right)^{1/3}} - \frac{b_1}{a_1}. \quad (A.3)$$

In our case, two of them are complex, and one is real. The real solution gives us the following exact expression for the Hubble parameter $H$ through the number of e-folds $N$ and its approximation for large $N$:

$$H^2_{sol} = \frac{S^2 - 4 e^{6(q-N)} (27 \kappa - e^{6(q-N)})}{18 e^{6(q-N)} \kappa S} + \frac{2}{9 \kappa}, \quad (A.4)$$

where

$$S = \left[ \left( \frac{36 \sqrt{972 k^2 - 27 e^{6(q-N)} \kappa + 8 e^{12(q-N)}}}{e^{6(q-N)} \kappa} \right) - 324 \kappa - 8 e^{6(q-N)} \right]^{1/3} e^{12(q-N)}. \quad (A.4)$$

Now $\dot{\phi}_{sol}$ takes the form

$$\dot{\phi}_{sol} = \sqrt{\frac{3 H^2_{sol}}{4 \pi (1 - 9 \kappa H^2_{sol})}} \approx \frac{3 \sqrt{3}}{\sqrt{13} \pi} \cdot e^{3(N-q)}. \quad (A.5)$$
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REFERENCES

1. A. A. Starobinsky, Phys. Lett. B 91, 99 (1980).
2. A. H. Guth, Phys. Rev. D 23, 347 (1981).
3. A. D. Linde, Phys. Lett. B 108, 389 (1982).
4. A. Albrecht and P. J. Steinhardt, Phys. Rev. Lett. 48, 1220 (1982).
5. A. D Linde, Particle Physics and Inflationary Cosmology (Harwood Academic, 1990); arXiv: hep-th/0503203.
6. A. R. Liddle and D. H. Lyth, Cosmological Inflation and Large Scale Structure Cambridge University Press, (2000).
7. D. Baumann, TASI Lectures on Inflation, arXiv: 0907.5424.
8. V. F. Mukhanov and G. V. Chibisov, JETP Lett. 33, 532 (1981).
9. S. W. Hawking, Phys. Lett. B 115, 295 (1982).
10. A. A. Starobinsky, Phys. Lett. B 117, 175 (1982).
11. A. H. Guth and S.-Y. Pi, Phys. Rev. Lett. 49, 1110 (1982).
12. F. L. Bezrukov and M. Shaposhnikov, Phys. Lett. B 659, 703 (2008).
13. A. O. Barvinsky, A. Y. Kamenshchik, and A. A. Starobinsky, JCAP 0811, 021 (2008).
14. Minxi He, Alexei A. Starobinsky, and Jun'ichi Yokoyama, JCAP 05, 064 (2018).
15. A. D. Linde, Phys. Lett. B 129, 177 (1983).
16. S. V. Sushkov, Phys. Rev. D 80, 103505 (2009).
17. C. Germani and A. Kehagias, Phys. Rev. Lett. 105, 011302 (2010).
18. H. M. Sadjadi and P. Goodarzi, JCAP 02, 038 (2013).
19. E. N. Saridakis and S. V. Sushkov, Phys. Rev. D 81, 083510 (2010).
20. S. V. Sushkov, Phys. Rev. D 85, 123520 (2012).
21. M. A. Skugoreva, S. V. Sushkov, and A. V. Toporensky, Phys. Rev. D 88, 083539 (2013).
22. Jiro Matsumoto and S. V. Sushkov, JCAP 1511, 047 (2015).
23. Jiro Matsumoto and S. V. Sushkov, JCAP 0118, 040 (2018).
24. S. S. Mishra, D. Müller, and A. V. Toporensky, Phys. Rev. D 102, 063523 (2020).