Spin Connections for Nonrelativistic Electrons on Curves and Surfaces

Toru Kikuchi
Yukawa Institute for Theoretical Physics, Kyoto University, Kyoto 606-8502, Japan
(Dated: May 25, 2018)

We propose a basic theory of nonrelativistic spinful electrons on curves and surfaces. In particular, we discuss the existence and effects of spin connections, which describe how spinors and vectors couple to the geometry of curves and surfaces. We derive explicit expressions of spin connections by performing simple dimensional reductions from three-dimensional flat space. The spin connections act on electrons as spin-dependent magnetic fields, which are known as “pseudomagnetic fields” in the context of, for example, graphenes and Dirac/Weyl semimetals. We propose that these spin-dependent magnetic fields are present universally on curves and surfaces, acting on electrons regardless of the nature of their spinorial degrees of freedom or their dispersion relations. We discuss that, via the spin connections, the curvature effects will cause the spin Hall effect, and induce the Dzyaloshinskii–Moriya interactions between magnetic moments on curved surfaces, without relying on relativistic spin-orbit couplings. We also note the importance of spin connections on the orbital physics of electrons in curved geometries.

Spinors are geometrical objects. Spinors rotate, together with vectors, when we rotate a physical system. They can also generate vectors through the familiar bilinear form $\psi^\dagger \sigma^i \psi$ using the Pauli matrices $\sigma^i$. Being geometrical objects, spinors can couple to the background geometry in which they live. Spin connections describe how spinors (and vectors) couple to their background geometry. Spin connections have been used to study spinors in curved spacetimes in the realm of general relativity. The same formalism can also be applied to nonrelativistic spinors on curves and surfaces embedded in our daily three-dimensional flat space.

What spin connections are, and why they are (expected to be) present on curves and surfaces, can be understood as follows. For comparison, let us first consider the case where we rigidly rotate a bulk material at an angular velocity $\Omega$. In the rotating material, the physics of an electron is dragged by the rotation of its surrounding environment (e.g., a lattice). Such dragging arises from quantum mechanical interactions owing to the overlap between the wavefunctions of the electron and the environment. As a result, the magnetic moment $M$ of an electron, for example, is forced to rotate together with its environment; its time derivative (for a laboratory observer) changes as $\partial_t M \rightarrow \partial_t M - \Omega \times M$. This is known as the Barnett effect. What happens to electrons on curves and surfaces can be regarded as the spatial counterpart of the Barnett effect. On curves and curved surfaces, the local environment of an electron gradually rotates as an electron propagates in the geometry, where the local anisotropy of the system is characterized by the tangent vectors $T$ for curves and the normal vectors $n$ for surfaces. The local physics of an electron is dragged by the rotation of its local environment as it propagates. The situation is rather analogous to the Barnett effect; the only difference is that spatial derivatives are involved rather than time derivatives. In terms of the magnetic moment, the spatial derivative $\partial_\mu$ tangential to a curve or surface changes as $\partial_\mu M \rightarrow \partial_\mu M - \Omega_\mu \times M$, with $\Omega_\mu \sim \partial_\mu T$ for curves and $\Omega_\mu \sim \partial_\mu n$ for surfaces. This $\Omega_\mu$ is to be called the spin connection on curves and surfaces. Because the magnetic moment is related to a spinor $\psi$ as $M = \psi^\dagger \sigma^i \psi$, the spinor also rotates as it propagates on a curve or surface; its spatial derivative changes as $\partial_\mu \psi \rightarrow (\partial_\mu + i \Omega_\mu \cdot \nabla) \psi$. In this way, electrons couple to the geometry in which they exist, and the spin connections $\Omega_\mu$ describe the coupling.

In the case of curves, the discussion so far can be rephrased as follows. Let us regard a curve as being deformed from a straight line. Each infinitesimal portion of the curve is related to the original portion of the straight line by a rotation. Then, using the corresponding SU(2) rotation matrix $U$, the Hamiltonian density $H_{\text{curve}}$ of the portion of the curve is related to the Hamiltonian density $H_{\text{line}}$ of the straight line by $H_{\text{curve}} = U^\dagger H_{\text{line}} U$ (cf. Ref.[19]). In particular, the derivative operator $\partial_\mu \psi$ is transformed as $U^\dagger \partial_\mu (U \psi) = (\partial_\mu + i \Omega_\mu \cdot \nabla) \psi$ with $i \Omega_\mu \cdot \nabla = U^\dagger \partial_\mu U$. Thus, the spin connection $\Omega_\mu$ appears due to the position-dependent rotation of each portion of the curve with respect to the straight line. In the case of surfaces, the discussion is more complicated due to the intrinsic curvature of surfaces, and the derivation of spin connections should be performed more systematically as will be described in this paper.

We also note the necessity of spin connections in view of the Dirac theory. Spin connections are naturally required in order for the Dirac theory on a curve or surface to be Hermitian or to yield the Schrödinger equation as its nonrelativistic limit (see Appendix A for details). A related discussion is given by Meijer et al.[20]. They discussed that the relativistic spin-orbit coupling (SOC) term on a curved geometry needs a correction; without this correction, the SOC term is not Hermitian. Interestingly, their correction coincides with the spin connection.
to be derived in this paper [Eq.(7)], Meijer et. al. added the correction only to the SOC term, not to the usual kinetic term. However, because both the kinetic term and the SOC term have the Dirac theory as the common origin, these terms should be treated in a unified way: the correction, i.e., the spin connection, should be added also to the usual kinetic term.

In most previous work in condensed matter physics that studies nonrelativistic spinful electrons on curves and surfaces, spin connections have not been taken into account. (An exception is the study of the quantum Hall effect where spin connections acting on the “orbital spin” are introduced.) There seem to be mainly two reasons for this situation.

The first reason is that the so-called “thin-layer approach”, originally used for the spinless case, has been applied too straightforwardly to the spinful case. In this approach, for the case of surfaces, one starts from a three-dimensional bulk Hamiltonian and reduces the thickness of the system, to derive a surface Hamiltonian. It is difficult to apply this approach to the spinful case. Physically, an electronic state near a surface is affected by the surface normal direction \( \mathbf{n} \); both a spinor and the normal direction \( \mathbf{n} \) are directional quantities and they are generally coupled. However, a three-dimensional bulk Hamiltonian does not contain such coupling terms, simply because it describes only a bulk and ignores the terms which are present only near the surface. It is therefore difficult to derive a surface Hamiltonian from such a bulk Hamiltonian with no information about the surface, only by narrowing the domain of the bulk Hamiltonian in the thin-layer approach.

The second reason is that the previous studies where spin connections for electrons on curves and surfaces are taken into account have mainly focused on the case of (quasi)-relativistic Dirac dispersion relation. This may be partly because the formalism of general relativistic spinors applies directly to this case; or partly because the Dirac dispersion relation appears in graphene, the most representative and mechanically flexible two-dimensional material. However, at least in principle, there is no reason for spin connections to be relevant only to a specific dispersion relation.

In this paper, we introduce the concept of spin connections and derive their expressions on curves and surfaces in as simple a way as possible. Then, we discuss the basic properties of nonrelativistic electrons constrained on the surface is measured in Cartesian coordinates, the usual differentiation, which we call here the “flat differentiation” \( \nabla^{(\text{flat})} \), acts on \( V \) simply as \( \nabla^{(\text{flat})} V^i = \partial_{\mu} V^i \). On the other hand, when a vector is measured in the curvilinear coordinates as \( V^\mu = e^i_\mu V^i \), then \( \nabla^{(\text{flat})} \) acts as \( \nabla^{(\text{flat})} V^N = \partial_{\mu} V^N + \Gamma^N_{\mu L} V^L \). The quantity \( \Gamma^N_{\mu L} \) is called a “connection” for \( \mu \), \( L \) \( k \). It is determined by the relation \( \nabla^{(\text{flat})} V^N = e^i_\mu (\nabla^{(\text{flat})} V^i) \), and is given by \( \Gamma^N_{\mu L} = e^i_\lambda \partial_{\mu} e^N_{ij} \).

Compared with \( \nabla^{(\text{flat})} \), we can define another kind of differentiation with a nontrivially curved connection, which is determined by the geometry of the surface. We call it here the “curved derivative”, and express it as \( \nabla \). The largest difference between these two types of differentiation, \( \nabla^{(\text{flat})} \) and \( \nabla \), is the way they act on the normal component \( V^\perp \) of a vector \( V^M \). A two-dimensional observer living on the surface (e.g., an electron) has the freedom to change his coordinate system \( x^\mu \) on the surface to a new one, \( x^\mu \to x'^\mu \), but he observes that \( V^\perp \) is invariant under this coordinate transformation. Then, \( V^\perp \) is a scalar for him, whereas, for us, it is just a particular component of the three-component vector \( V^M \).

The curved derivative \( \nabla \) takes the viewpoint of this two-dimensional observer and acts as \( \nabla_\mu V^\perp = \partial_\mu V^\perp \) since \( V^\perp \) is just a scalar. On the other hand, the flat derivative acts as \( \nabla^{(\text{flat})} V^\perp = \partial_\mu V^\perp + \Gamma^\perp_{\mu N} V^N \). This means that \( \Gamma^\perp_{\mu N} \) is truncated to be zero for the curved derivative \( \nabla \). The same reasoning for \( V^\parallel \) requires that \( \Gamma^\parallel_{\mu \nu} \) is also truncated for \( \nabla \mu \). These truncations also determine how \( \nabla_\mu \) acts on the tangential components \( V^\nu \) of a vector \( V^M \): \( \nabla_\mu V^\nu = \partial_\mu V^\nu + \Gamma^\nu_{\mu \lambda} V^\lambda \). To summarize, the flat derivative \( \nabla^{(\text{flat})}_\mu \) and the curved derivative \( \nabla_\mu \) act differently on the curvilinear indices as

\[
\begin{align*}
\nabla^{(\text{flat})}_\mu V^N &= \partial_\mu V^N + \Gamma^N_{\mu L} V^L; \\
\nabla_\mu V^\perp &= \partial_\mu V^\perp, \quad \nabla_\mu V^\parallel = \partial_\mu V^\parallel + \Gamma^\parallel_{\mu \lambda} V^\lambda. \quad (1)
\end{align*}
\]

After all, dimensional reduction of \( \Gamma^M_{NL} \) is performed for \( \nabla \), where the direction to be reduced is the normal direction at each point of the surface.

Next, let us look at the way the curved derivative \( \nabla \) acts on a vector \( V^\mu \) with Cartesian indices. It is given by \( \nabla_\mu V^i = e^j_i \partial_\mu V^j + \Omega^j_i \nabla^j \), with

\[
\Omega^j_i = e^j_\nu \Gamma^\nu_{\mu \lambda} e^\lambda_j + e^N_\nu \partial_\mu e^N_j. \quad (2)
\]

We call \( \Omega^j_i \) the “spin connection” on a surface: it is a connection represented in Cartesian indices. We can express \( \Omega^j_i \) as follows in terms of the surface normal vector \( n^i = \partial x^i / \partial x^\perp \) at each point on the surface. By subtracting from Eq.(2) the relation \( \Omega = e^N_\nu \Gamma^\nu_{\mu L} = e^N_\nu \partial_\mu e^N_j \), which follows immediately from \( \Gamma^N_{\mu L} = e^i_\nu \partial_\mu e^N_i \), we can see that the spin connection is given by the truncated components of the connection \( \Gamma^M_{NL} \) (see Appendix B for...
rotates as we move on the surface. To summarize, the flat
the spin connection expresses the way the normal vector
the Cartesian indices:
\[ \Omega_{\mu} = \partial_{\mu} \nu - \Omega_{\mu} \nu \]
\( = n^i \partial_{\mu} n^i - n^j \partial_{\mu} n^j. \) \hspace{1cm} (3)
As seen explicitly in Eq. (3), \( \Omega_{\mu} \) is antisymmetric with
respect to \( i \) and \( j \), and it can therefore be written as \( \Omega_{\mu}^i = \epsilon_{ijk} \Omega_{\mu}^k \). This means that, when a vector is transported
by \( \nabla_\mu V^i = (\partial_\mu V - \Omega_\mu \times V)^i = 0 \), the vector spins (or precesses)
at the rate \( \Omega_{\mu} \), as in Fig.1. Hence its name, a spin connection, is natural. In particular, \( \nabla_\mu n^i = 0 \):
the spin connection expresses the way the normal vector rotates
as we move on the surface. To summarize, the flat
and curved derivatives, \( \nabla_{(\text{flat})}^\mu \) and \( \nabla_{\mu} \), act differently on the
Cartesian indices:
\[ \nabla_{(\text{flat})}^\mu V^i = \partial_{\mu} V^i; \quad \nabla_\mu V^i = \partial_{\mu} V^i - \epsilon_{ijk} \Omega_{\mu}^k V^j, \] \hspace{1cm} (4)
which is equivalent to Eq. (1).
We have so far discussed vectors. Let us next consider spinors and the way the curved derivative \( \nabla \) acts
on them. We can easily see this by taking the spin magnetic
moment vector \( V^i = \psi^i \sigma^i \psi \) with \( \psi \) a spinor and \( \sigma^i \)
the Pauli matrices. For the curved derivative \( \nabla \) to act on the
magnetic moment as in Eq. (4), the derivative must
act on a spinor as
\[ \nabla_\mu \psi = \left( \partial_\mu + i \Omega_\mu \frac{\sigma}{2} \right) \psi \quad \text{with} \quad \Omega_\mu = n \times \partial_\mu n, \] \hspace{1cm} (5)
assuming the Leibniz rule for the derivative. The expression
of the spin connection in Eq. (5) agrees with those in, for example, Refs. [13,23,24]. As a spinor propagates on a surface, it receives a torque and gets rotated at the rate \( \Omega_\mu \). Physically, this torque is to be interpreted as being exerted by the environment (e.g., a lattice), which confines the spinor to the surface. The normal vector \( n \) gives the local anisotropy of the environment, and \( \Omega_\mu \sim \partial_\mu n \)
is the rotation rate of such local anisotropy.
Let us next study some of the basic properties of electrons confined on surfaces. We can see vividly the effects of the spin connection by rotating the spinor frame to what we call here the "intrinsic frame". We have so far used the laboratory spinor frame, where we take the spin-quantization axis to lie along the \( z \) direction in the laboratory (Cartesian) coordinates. For electrons on a surface, a more natural frame is the intrinsic spinor frame where the spin-quantization axes are taken to lie along the directions of the normal vectors \( n \) distributed over the surface. The transformation from the laboratory spinor frame \( \psi \) to the intrinsic spinor frame \( \tilde{\psi} \) is described by \( \tilde{\psi} = U \psi \) with an SU(2) matrix \( U \) satisfying \( U \nu \cdot n U = \sigma^3 \)
with \( \sigma^3 = \left( \begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix} \right) \). The spin connection \( \omega_\mu \) in the intrinsic spinor frame is defined as \( (\partial_\mu + i \Omega_\mu) \psi = U (\partial_\mu + i \omega_\mu) \psi \), where \( \omega_\mu \equiv \Omega_\mu \cdot \frac{\sigma}{2} \), and it is given by (see Appendix [C] for details)
\[ \omega_\mu = U \nu \cdot n \psi - iU \nu \partial_\mu U \]
\[ = (1 - \cos \theta) \partial_\mu \psi \frac{\sigma^3}{2}, \] \hspace{1cm} (6)
with \( n^i = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta)^i \). From Eq. (6), we see that the intrinsic frame diagonalizes the spin connection. In particular, the Schrödinger equation for a freely propagating electron reads \( i \hbar \partial \psi / \partial t = - (\frac{\hbar^2}{2m_e}) \nabla^2 \psi \) with \( m_e \) being the electron mass, and the Laplacian \( \nabla^2 \) is given by \( \nabla^2 \psi = g^{\mu \nu} (\partial^\mu \nabla_\nu - \Gamma^\gamma_{\mu \nu} \nabla_\gamma) \psi = \frac{1}{\sqrt{g}} \nabla_\mu (\sqrt{g} g^{\mu \nu} \nabla_\nu) \psi \). Here, \( g_{\mu \nu} \equiv (\partial x^\mu / \partial x'^\mu)(\partial x'^\nu / \partial x^\nu) \) is the induced metric on the surface and \( \nabla_\mu \) is given as in Eq. (5) with \( \nu \) replaced by \( \omega_\mu \). The spin-up and down components decouple in the Schrödinger equation; an electron with spin in the \( \pm n \) direction propagates keeping the \( n \)-up/down states.

The expression in Eq. (6) is that of the gauge field of a magnetic monopole[3]. Monopole gauge fields with magnetic charges \( \pm \frac{1}{2} \) are coupled with the spin \( n \)-up/down components. As a result, the spin \( n \)-up and \( n \)-down components are subjected to the ‘magnetic fields’ \( B_m n \) and \( -B_m n \), respectively (see Appendix [C] for details). The field strength \( B_m \) is given by \( B_m = K/2 \), where \( K \equiv \det(\nabla_{(\text{flat})}^\mu n^\nu) \) is the Gaussian curvature[13,22,24] at a point on the surface (e.g., \( K = a^{-2} \) everywhere on a sphere of radius \( a \)). When \( K \sim 1 \) m\(^{-2}\), then \( (\hbar/e) B_m \sim 3 \times 10^2 \) tesla, with \( e \) the electric charge. Owing to this field strength of the spin connection, the curvature effects cause cyclotron motions of electrons, which are in opposite directions for the spin \( n \)-up and \( n \)-down components, yielding a vortical spin current circulating around each point of the surface.

This kind of spin-dependent gauge field is known as a strain-induced[24] "pseudomagnetic field" in, for example, graphene[13,22,23] and transition-metal dichalcogenides[24].
and Dirac/Weyl semimetal [47] (see also Ref. [45]). The main interest seems to have been focused on pseudomagnetic fields acting on valley or orbital degrees of freedom rather than on the true spin (exceptions are, e.g., Refs [41–43], which treat spin connections acting on electron spin). Interest has also been focused on the case of the Dirac/Weyl dispersion relations. As discussed at the beginning of this paper, spin connections effectively represent the effects of the local rotation of the environment on electronic quantum states; their presence is quite general, regardless of the kinds of spinorial degrees of freedom (whether spin or pseudospin [29]) and the electron dispersion relations.

The curvature effects are expected to induce the spin Hall effect, because the spin-up and -down components of electrons undergo $E \times (\pm B_m)$ drifts in opposite directions, where $E$ is an externally applied electric field and $\pm B_m$ are the pseudomagnetic fields. We also observe that electrons on surfaces generally have finite spin currents in their ground states: because the surface curvature $K$ is generally position dependent, the strengths of vortical spin currents caused by the pseudomagnetic fields $\pm (K/2)n$ are also position dependent, and the vortical spin currents are not canceled among nearby points; the net result is a finite spin current flowing on the surface. In a flat space, spin currents in ground states have been known to be induced by relativistic spin-orbit coupling with broken inversion symmetry (such as the Rashba effect [39]). It has recently been shown [44–46] that spin currents are a direct origin of the Dzyaloshinskii–Moriya interaction [41–43] in magnets. Therefore, curvature-induced spin currents in ground states on surfaces will lead to intricate magnetic interactions, which we can engineer by the geometry of surfaces. More details will be reported elsewhere.

Next, let us consider spin connections on curves, which seem to have been discussed rarely in the literature. For a curve, a natural orthonormal frame [27] is defined at each point by the unit vectors $T(s) \equiv \gamma'(s)$, $N(s) \equiv T'(s)/|T'(s)|$, and $B(s) \equiv T(s) \times N(s)$, where $\gamma'(s)$ specifies a point on the curve and $s$ is the arc length parametrizing the curve. We define a curvilinear coordinate system $x^M$ with $M = s, n, b$ by $x^i(x^M) = \gamma^i + x^sN^i + x^nB^i$. Following the same procedure as in the case of surfaces, we truncate all the components of $T^{M}$ except the tangential component $T^s$. Then, the spin connection for a curve is given by (see Appendix [E] for details)

$$\Omega_s = \kappa B + \tau T$$

in the laboratory frame, where $\kappa \equiv N \cdot T'$ and $\tau \equiv B \cdot N'$ are the curvature and the torsion of the curve, respectively. The quantity $\Omega_s$ in Eq. (7) satisfies $dX/ds = \Omega_s \times X$ for $X = T, N, B$ (known as the Frenet–Serret formulas [27, 45]), representing the way the orthonormal frame rotates as it moves along the curve. The nontrivially curved derivative for a spinor on a curve reads $\nabla_s \psi \equiv \left(\frac{d}{ds} + i\Omega_s \cdot \frac{s}{2}\right)\psi$. A vector or a spinor transported by $\nabla_s V = 0$ or $\nabla_s \psi = 0$, respectively, changes its direction following the rotation of the $(T, N, B)$ frame, as in Fig. 2. When we go from the laboratory frame to the intrinsic frame consisting of $(T, N, B)$ as its basis vectors, the spin connection vanishes (see Appendix [F]). This is expected, because a one-dimensional curve is geometrically trivial in the intrinsic sense: its internal metric is always $g_{ss} = 1$.

Finally, let us mention that spin connections also act on atomic orbitals ($p$-orbitals, $d$-orbitals, etc.) of electrons, where the orbital degrees of freedom are expressed as an additional index $\alpha$ on a wavefunction $\psi_{\alpha}$. The discussion is completely parallel to the spin [27] case. The nontrivial derivative acting on the orbital magnetic moment $\psi^\dagger \ell \psi$, with the orbital angular momentum matrix $\ell$, implies that it also acts on a spinor as $\nabla_{\mu} \psi = (\partial_{\mu} + i\ell_{\mu}) \psi$. The explicit forms of $\ell$ are given as $\ell^i_{\alpha\beta} = -i\varepsilon_{\alpha\beta}$ for $p$-orbitals, $\ell^i_{\alpha\beta} = -2i\varepsilon_{ijk} ([\xi^\alpha, \xi^\beta])_{jk}$ for $d$-orbitals [29], etc. The spin connections acting on the orbitals will be useful for describing the orbital physics on curved geometries made of, for example, transition-metal dichalcogenides [20, 21]. Similar to the spin [27] case, the orbital Hall effect [22, 23] and orbital angular-momentum currents in ground states, will be induced purely by the curvature effects. More generally, spin connections act on all kinds of particles (e.g., atoms, magnons and photons), which have their own angular momenta.

In summary, we have discussed the physics of spin connections on curves and surfaces, derived their explicit expressions with their geometrical meaning clarified, and studied their basic effects on electrons. Many fundamental questions of electrons on curves and surfaces are to be investigated. Derivation of spin connections from more microscopic viewpoints should be performed for their theoretical foundation. Theories on
transportation phenomena and order-parameter physics on curved geometry are necessary to be compared with experiments. The presence of spin connections on curved geometry will enrich many classic physics such as the Hubbard model, the Kondo effect, superconductivity, and the quantum Hall effect, which are usually studied on flat geometry. Photonic crystals and possibly optical lattices give a highly controllable platform for curvature physics. Spintronics using circuits is technologically important application. Spin connections arising from phonon excitations and lattice deformation are essential ingredients to fully understand mechanical effects on electrons in solids, where the backreaction on lattice will also have intriguing effects. It is of interdisciplinary interest to study biochemical objects such as DNA helices from physics viewpoints.

ACKNOWLEDGMENTS

The author thanks Y. Avishai, T. Kawakami, M. Sato, A. Shiadate, K. Taguchi, R. Takahama, K. Totsuka and P. Wiegmann for helpful comments. In particular, the author thanks H. Saarikoski for comments and reading the manuscript, and M. Matsuo for his generosity: when the author discussed with him, it turned out that he had long had a similar idea independently. The author is a Yukawa Research Fellow supported by the Yukawa Memorial Foundation.

Appendix A: The Dirac theory favors nontrivial spin connections.

Here, we show that the relativistic Dirac theory on curves and surfaces imposes the presence of nontrivial spin connections for its self-consistency, and, as the result, show that non-relativistic theories obtained as descendants of the Dirac theory should include nontrivial spin connections.

Let us first consider a surface. We employ the curvilinear coordinate system $x^M = (x^\mu, x^i)$ and the Cartesian coordinate system $x^i = (x, y, z)$ as in the main text. The full Dirac theory needs a four-component Dirac spinor $\Psi$. The Dirac action on a surface embedded in a four-dimensional flat Minkowski spacetime will read

$$S = -i \int dt d^{2}x \sqrt{g} \gamma^{0} \{ \gamma^{0} \partial_{t} + e_{i}^{\mu} \gamma^{i} (\partial_{x}^{\mu} + i \Omega_{\mu}^{i}) + m_{e} \} \Psi, \quad (A1)$$

where $e_{i}^{\mu} = \partial x^{i} / \partial x^{\mu}$ and $g_{\mu \nu} = e_{i}^{\mu} e_{i}^{\nu}$ with $e_{i}^{\mu} = \partial x^{i} / \partial x^{\mu}$ is the induced metric on the surface, and $m_{e}$ is the electron mass. We employ the natural unit, $\hbar = 1$ and $c = 1$. The $\gamma$-matrices are given by $\gamma^{0} = -i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $\gamma^{i} = -i \begin{pmatrix} 0 & 0 \\ 0 & \sigma^{i} \end{pmatrix}$ in the Dirac representation, satisfying $(\gamma^{0})^{2} = -1$, $(\gamma^{i}, \gamma^{j}) = 2 \delta^{ij}$ and $\{ \gamma^{0}, \gamma^{i} \} = 0$ [we take the Minkowski metric as diag$(-1, 1, 1, 1)$]. We assume that the spin connection $\Omega_{\mu}^{i}$ has an expansion as $\Omega_{\mu}^{i} = \frac{1}{2} \Omega_{\mu}^{ij} \Sigma^{j}$ with $\Sigma^{i} \equiv -\frac{1}{2} [\gamma^{i}, \gamma^{j}]$.

Even if we set the spin connection $\Omega_{\mu}^{ij}$ undetermined when writing down Eq. (A1), the Hermitian (reality) condition of the action, $S^\dagger = S$, imposes a condition on $\Omega_{\mu}^{ij}$ as

$$g^{\mu \nu} (D_{\mu} e_{i}^{\nu} + \Omega_{\mu}^{ij} e_{j}^{\nu}) = 0. \quad (A2)$$

Here, $D_{\mu} e_{i}^{\nu} \equiv \partial_{\mu} e_{i}^{\nu} - \Gamma_{\mu \nu \rho} e_{i}^{\rho}$ is the partial covariant derivative acting only on the two-dimensional curved indices $\mu, \nu, \ldots$, but not on the Cartesian indices $i, j, \ldots$ Eq. (A2) is an inhomogeneous equation for $\Omega_{\mu}^{ij}$, thus imposing a nonzero $\Omega_{\mu}^{ij}$ in the laboratory frame. As a particular solution, we can see that the spin connection given in Eq. (A2),

$$\Omega_{\mu}^{ij} = e_{N}^{i} D_{\mu} e_{j}^{N} = -e_{N}^{i} D_{\mu} e_{N}^{i}, \quad (A3)$$

satisfies Eq. (A2). [The second equality in Eq. (A3) follows by (i) $D_{\mu} g^{\mu \nu} = 0$ since the nonzero components of $g^{\mu \nu} \equiv \partial_{\mu} x^{\nu} / \partial x^{\nu}$ are $g^{t t} = 1$ and $g^{\mu \nu}$, and therefore (ii) $e_{N}^{i} D_{\mu} e_{j}^{N} + e_{N}^{i} D_{\mu} e_{N}^{i} = e_{N}^{i} D_{\mu} e_{N}^{i} + e_{N}^{j} D_{\mu} e_{N}^{j} = \partial_{\mu} (e_{N}^{i} e_{N}^{j}) = 0$.] See also Ref. [29] for spin connections in the Dirac theory on curved surfaces.

From the Dirac equation $(\gamma^{0} \partial_{t} + e_{i}^{\mu} \gamma^{i} (\nabla_{\mu} + m_{e}) \Psi = 0$, with $\nabla_{\mu} \equiv \partial_{\mu} + i \Omega_{\mu}^{i}$, we obtain the Klein-Gordon equation,

$$\left( \gamma^{0} \partial_{t} + e_{i}^{\mu} \gamma^{i} D_{\nu} - m_{e} \right) (\gamma^{0} \partial_{t} + e_{i}^{\mu} \gamma^{i} (\nabla_{\mu} + m_{e}) \Psi = 0, \quad (A4)$$

where $D_{\lambda}$ is the total covariant derivative acting both on the Cartesian and the curvilinear indices (e.g., $D_{\mu} V_{\nu}^{\lambda} = \partial_{\mu} V_{\nu}^{\lambda} - \Gamma_{\mu \nu \rho}^{\lambda} V_{\rho}^{\lambda} + \Omega_{\mu}^{\lambda \nu} V_{\rho}^{\lambda}$), and $D^{\nu} \equiv g^{\mu \nu} D_{\mu}$ is the Laplacian. Here, we have used that $D_{\mu} \Psi$ or $\nabla_{\mu} \Psi$ does not act on $\gamma^{i}$. (To be precise, it acts on all of the indices of $\gamma^{i} \sigma^{m}$, and results in zero, $D_{\mu} \gamma^{i} = \partial_{\mu} \gamma^{i} + \Omega_{\mu}^{\lambda \gamma^{i}} = 0$.) We have also used $D_{\mu} e_{i}^{\nu} = 0$, which is valid only for $\Omega_{\mu}^{ij}$ chosen as Eq. (A3) (if we impose also $\nabla_{\mu} \Psi = 0$). Without the condition $D_{\mu} e_{i}^{\nu} = 0$, the derivation of the Klein-Gordon equation from the Dirac equation as in Eq. (A4) would be difficult. This is another supportive fact for the necessity of the spin connection Eq. (A3).

Let us consider the lowest-order non-relativistic limit. Separating the energy $i \partial_{t}$ into the rest energy and the non-relativistic energy as $i \partial_{t} \rightarrow m_{e} + i \partial_{t}$ with an approximation $i \partial_{t} \ll m_{e}$, we have, from Eq. (A4),

$$i \partial_{t} \Psi = \left(-\frac{1}{2m_{e}} D^{2} + \frac{1}{4m_{e}c} K\right) \Psi, \quad (A5)$$

In the Dirac representation, where the upper two components of $\Psi$ correspond to a particle and the lower two components of $\Psi$ to an antiparticle, the $\Sigma^{ij}$ in
\[ \Omega_\mu = \frac{1}{2} \Omega^i_\mu \Sigma^j \text{ is diagonal as } \Sigma^{ij} = \frac{1}{2} \varepsilon^{ijk} \begin{pmatrix} \sigma^k \\ 0 \\ \sigma^i \end{pmatrix}. \] Therefore, the upper two components of Eq. (A5) are the usual non-relativistic Schrödinger equation, and it includes the nontrivial spin connection \( \Omega^{ij}_\mu \) given in Eq. (3). [The term proportional to the Gaussian curvature \( K \) in Eq. (A5), which we neglect entirely in this paper as well as the "geometric potentials" in Refs. [4,5,13]] is akin to the electromagnetic Zeeman term \( \sigma \cdot B \), in that the field strength \( K \) of the gauge field \( \Omega \) directly couples to the fermion.

Next, let us consider a curve. The discussion is completely parallel. We employ the curvilinear coordinate system \( x^M = (s, n, b) \) as in the main text. The Dirac action on a curve reads

\[ S = -i \int dt ds \Psi^\dagger \gamma^0 \left[ \gamma^0 \partial_t + e^i_\gamma \gamma^i (\partial_s + i \Omega_s) + m_c \right] \Psi. \] (A6)

In order for \( S \) to be real, \( \Omega_s \) must satisfy

\[ \partial_t e^i_\gamma + \Omega^{ij}_s e_s^j = 0. \] (A7)

Note that \( \Gamma^s_{ss} = 0 \) (see Appendix [E]). Again, this is an inhomogeneous equation for \( \Omega^{ij}_s \) and therefore the trivial connection \( \Omega^i_s = 0 \) is not allowed. As a particular solution, we can see that the spin connection given in Eq. (7) (see Appendix [E]),

\[ \Omega^i_s = e^i_M \partial_s e^M, \] (A8)

satisfies the requirement Eq. (A7). By taking the non-relativistic limit as in the case of surfaces described above, we obtain the Schrödinger equation

\[ i \partial_t \Psi = -\frac{1}{2m_c} \nabla_s \Psi \] (A9)

with \( \nabla_s = \partial_s + i \Omega_s \).

Thus, to be Hermitan and to derive the Klein-Gordon equation (and the resulting Schrödinger equation), the Dirac theory on curves and surfaces favors the nontrivial spin connections \( \Omega \) as in Eq. (A5) and Eq. (A8), and its non-relativistic limit inherits these nontrivial spin connections. In other words, non-relativistic theories on curved geometry should have nontrivial spin connections to be consistent with the Dirac theory.

We can understand the discussion in this section as follows. In purely non-relativistic terms such as \( (\partial \mu \psi)^2 \), the spinorial index of \( \psi \) can be either geometrical or non-geometrical, since the indices are contracted (summed over) only among them. Self-consistency of such terms does not require spin connections, since, if spin connections are absent, we can always regard (or pretend) that the spinorial index of \( \psi \) is non-geometrical and hence the spin connections are absent as a matter of course. On the other hand, in terms such as \( g^{\mu \nu} e^i_\gamma e^j_\gamma \partial_\nu \Psi \) in the Dirac theory, the spinorial index of \( \Psi \) must be geometrical, since it is related to the spatial index (\( i \) or \( \mu \)) via the \( \gamma \)-matrices. In this case, self-consistency of the terms requires nontrivial spin connections, reflecting the geometrical nature of the spinor.

**Appendix B: Details of Eq. (6)**

Here, we calculate some of the components of \( \Gamma^M_{NL} \) and then derive Eq. (6). First, the Cartesian coordinates \( x^i \) and the curvilinear coordinates \( x^M = (x^\mu, x^i) \) are related by, as in Ref. [10],

\[ x^i (x^M) = r^i (x^\mu) + x^\mu n^i (x^\mu), \] (B1)

where \( r^i (x^\mu) \) specifies a point on a surface, and \( n^i (x^\mu) \) is the surface normal at each point \( r^i (x^\mu) \). Hereafter, any quantity is eventually evaluated just on the surface, i.e., \( x^\mu = 0 \).

It is very convenient to give a name to \( \partial_i n^j \). We call it the extrinsic curvature tensor or the second fundamental form [15,21,23],

\[ K_i^j \equiv \partial_i n^j, \] (B2)

where we have extended the domain of \( n^j \) infinitesimally off the surface in a way such that \( \partial_i n^j = 0 \). We can easily see that \( n^\mu K_i^j = \partial_i n^j = 0 \) and \( K_i^j n^j = 0 \). Therefore, when we express \( K \) by the curvilinear indices as \( K^N_M = \nabla^{(flat)}_M n^N \), then \( n^M K^N_M = K^N_N = 0 \) and \( K^N_M n_M = K^\perp_M = 0 \) (note that \( n^M = \delta^M_M \) and \( n_M = \delta^\perp_M \)). Therefore, only the tangential components \( K^\perp \) are nonzero. (B3)

Moreover, since \( \nabla^{(flat)}_M n^N = \partial_M n^N + \Gamma^N_M n^L = \Gamma^N_M \) and \( \nabla^{(flat)}_M n^N = \partial_M n^N - \Gamma^L_M n^L = -\Gamma^L_M n_N \), we have

\[ \Gamma^N_M = K^N_M, \quad \Gamma^L_M = -K^L_M. \] (B4)

From this and Eq. (B3), we see that the components of \( \Gamma^M_{NL} \) with more than one normal index (\( \perp \)), such as \( \Gamma^\perp_{MN} \), are zero. The nonzero components with the normal index are

\[ \Gamma^\perp_{\mu \perp} = K^\perp, \quad \Gamma^\perp_{\mu \mu} = -K_{\mu \mu}. \] (B5)

Let us next consider the contraction of \( K \) with \( e^i_M = \partial x^i / \partial x^M \) and \( e^\gamma_M = \partial x^\gamma / \partial x^M \). This is easy, because \( K \) is a tensor and multiplication of it with \( e^i_M \) or \( e^\gamma_M \) just changes its corresponding index. For example,

\[ K_{\mu N} e^N_j = K_{\mu \nu} e^\nu_j = K^j_\mu = \nabla^{(flat)}_\mu n^j = \partial_\mu n^j. \] (B6)

(Note that we do not have to make a distinction between raised and lowered Cartesian indices.) From Eq. (B3), Eq. (B6), and \( e_\perp = e^i_\perp = n^i \), the detail of Eq. (3) reads

\[ \Omega^i_\mu = -e^i_\perp \Gamma^\perp_{\mu \perp} e_j^\perp + e^\gamma_\mu \Gamma^\perp_{\mu \gamma} e_j^\gamma = n^i K_{\mu \perp} e_j^\perp - n^j K^\perp_{\mu \perp} e^i_\perp = n^i \partial_\mu n^j + n^j \partial_\mu n^i. \] (B7)

We can derive this expression more directly from Eq. (2) or Eq. (A3) as follows. A useful identity is

\[ \partial_\mu e^\nu_\gamma = \partial_\mu e_\perp^\nu - \gamma^{\nu \rho} e_\perp^i = -K_{\mu \rho} n^i, \] (B8)
which follows from (i) $e_i^j \mathcal{D}_\mu e^i_\nu = 0$ by using $\Gamma^\lambda_{\mu\nu} = e^j_i \partial^i_{\nu} e^k_j - e^j_i \partial^i_{\mu} e^k_j$ and (ii) $K_{\mu\nu} = -n^i \partial_{\nu} e^i_\mu = -n^i \mathcal{D}_\mu e^i_\nu$ by applying $\partial_{\mu}$ on $n^i e^i_\nu = 0$. Then,

$$
\Omega^{ij}_\mu = g^{\mu\nu} e^i_\nu \mathcal{D}_\mu e^j_\nu + n^i \partial_{\mu} n^j = -e^i_\nu K^\nu_{\mu} n^j + n^i \partial_{\mu} n^j = -\partial_{\mu} n^i n^j + n^i \partial_{\mu} n^j. \quad (B9)
$$

Appendix C: Details of Eq. (6)

A useful identity to show Eq. (6) is

$$
i \mathcal{D}_\mu U^\dagger = - (n \times \partial_{\mu} n) \frac{\sigma}{2} + (1 - \cos \theta) \partial_{\mu} \phi \left( n \cdot \frac{\sigma}{2} \right). \quad (C1)
$$

Here, $U$ is defined as a SU(2) matrix satisfying $U^\dagger n \cdot \sigma U = \sigma^3$, whose particular form we choose is

$$
U = i m \cdot \sigma \quad \text{where} \quad m = \begin{pmatrix} \sin \phi \cos \theta \\ \sin \phi \sin \theta \\ \cos \phi \end{pmatrix} \quad (C2)
$$

with $n = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta)$. The definition of $U$ has ambiguity up to the $U(1)$ rotation around the $\sigma^3$-axis, $U \rightarrow U \exp(i \chi \sigma^3/2)$ with $\chi$ an arbitrary function. This $U(1)$ indeterminacy corresponds to the $U(1)$ gauge transformation on the ‘monopole gauge field’ $(1 - \cos \theta) \partial_{\mu} \phi$ in the second term in Eq. (C1). When we parameterize a general SU(2) matrix $U$ by the Euler angles $(\theta, \phi, \psi)$,

$$
U \equiv e^{-i \psi \frac{\sigma^3}{2}} e^{-i \theta \frac{\sigma}{2}} e^{-i \phi \frac{\sigma^1}{2}}, \quad (C3)
$$

the matrix $U$ in Eq. (C2) corresponds to the restriction $\psi = -\phi - \pi$, that is, $U|_{\psi=-\phi=\pi} = U$.

We define a rotation matrix $R_{ia}$ by $R_{ia} \sigma^a = U^\dagger \sigma^a U$, whose explicit form is $R_{ia} = 2m^a n^a - \delta^a$ for $U$ chosen as Eq. (C2). Then, the detail of Eq. (6) is

$$
\omega^a_{\mu} = U^\dagger \Omega_{\mu} U - iU^\dagger \partial_{\mu} U = R_{ia} \left[ (n \times \partial_{\mu} n) \frac{\sigma^a}{2} - (n \times \partial_{\mu} n) i \frac{\sigma^a}{2} \right] + (1 - \cos \theta) \partial_{\mu} \phi \left( n^i \frac{\sigma^a}{2} \right) = (1 - \cos \theta) \partial_{\mu} \phi \frac{\sigma^a}{2}, \quad (C4)
$$

where we have used $R_{ia} n^i = \delta^a$.

The calculation above can be rephrased for the case of vectors. When we define $\omega^a_\mu$ as

$$
\partial_{\mu} V^i - \varepsilon_{ijk} \Omega^{j}_{\mu} V^k = R_{ia} \left( \partial_{\mu} V^a - \varepsilon_{abc} \omega^b_{\mu} V^c \right) \quad (C5)
$$

with $V^i = R_{ia} V^a$, then

$$
\omega^a_{\mu} = R_{ia} \Omega^{i}_{\mu} + \frac{1}{2} \varepsilon_{abc} (R^{-1} \partial_{\mu} R)_{bc} = \delta^a (1 - \cos \theta) \partial_{\mu} \phi. \quad (C6)
$$

Appendix D: Details of spin-dependent magnetic field

Here, we show that the “pseudomagnetic field” $\pm B_m$ calculated from the spin connection in Eq. (6) is given as $B_m = (K/2)n$.

We can roughly expect $B_m = (K/2)n$ as follows. The pseudomagnetic field is a quantity of the second-order derivative. In order to evaluate it, we may approximate the portion of the surface near a point in interest by a sphere, whose radius is $1/\sqrt{K}$ with $K$ the (Gaussian) curvature of the point. Within this approximation, the normal vectors $n$ of the surface are the radial unit vectors of the sphere, and the intrinsic spin connection $\omega_\mu = (1 - \cos \theta) \partial_{\mu} \phi \frac{\sigma^3}{2}$ can be regarded as the electromagnetic gauge fields of magnetic monopoles with charges $\pm 1$ put at the center of the sphere. Generally, the magnetic field at a point $r$, yielded by a monopole with charge $q$ put at the origin, is given by $\frac{q}{4\pi r^2}$. Therefore, the pseudomagnetic fields are given by $\pm B_m = \pm (K/2)n$.

We calculate the pseudomagnetic fields more rigorously below. First, the two-dimensional Riemann tensor $R_{\nu\rho\sigma\tau}$ on a surface is defined by

$$
R_{\nu\rho\sigma\tau} \equiv \partial_{\rho} \Gamma^\lambda_{\nu\sigma} + \Gamma^\mu_{\rho\lambda} \Gamma^\lambda_{\nu\sigma} - (\rho \leftrightarrow \sigma). \quad (D1)
$$

Comparing this with an obvious identity

$$
0 = \partial_{\rho} \Gamma^\mu_{\nu\sigma} + \Gamma^\mu_{\rho\lambda} \Gamma^\lambda_{\nu\sigma} - (\rho \leftrightarrow \sigma) \quad (D2)
$$

(giving the tangential components of the Riemann tensor in the flat three-dimensional space, which are of course zero), we have

$$
R_{\nu\rho\sigma\tau} = - \Gamma^\mu_{\rho\lambda} \Gamma^\lambda_{\nu\sigma} - (\rho \leftrightarrow \sigma) = K_{\rho \sigma} K_{\nu \tau} - K_{\nu \tau} K_{\rho \sigma} \quad (D3)
$$

where $K_{\nu \tau}$ is the extrinsic curvature tensor (see Appendix B). Then, the Ricci scalar $R$ is given by

$$
R \equiv R_{\mu\nu} \frac{\sigma^\mu \sigma^\nu}{2} = (K_{\nu \tau})^2 - K_{\nu \tau} K_{\mu \sigma} = 2K, \quad (D4)
$$

where $K \equiv \det(K_{\nu \tau})$ is the Gaussian curvature, and we have used an identity det$A = \frac{1}{6}[(trA)^2 - tr(A^2)]$ valid for any $2 \times 2$ matrix $A$.

The Riemann tensor can be defined in another way as

$$
R_{\mu\nu\psi\lambda} \equiv -i [\mathcal{D}_\mu, \mathcal{D}_\nu] \psi
= (\partial_{\mu} \Omega_{\nu} - \partial_{\nu} \Omega_{\mu} + i[\Omega_{\mu}, \Omega_{\nu}]) \psi
= [n \cdot (\partial_{\mu} n \times \partial_{\nu} n)] \frac{\sigma^3}{2} \psi, \quad (D5)
$$

where $\mathcal{D}_\mu$ is the total covariant derivative acting both on the curvilinear and the Cartesian indices (e.g., $\mathcal{D}_\mu V^a = \partial_{\mu} V^a - \Gamma^a_{\mu\nu} V^\nu + \Omega^a_{\mu} V^\nu$), and $\Omega^a_{\mu} = \Omega^a_{\mu} \cdot \frac{\sigma^3}{2}$ is the spin.
connection Eq. [5] in the laboratory frame. We have used
$\partial_\mu \mathbf{n} \times \partial_\nu \mathbf{n} = [\mathbf{n} \cdot (\partial_\mu \mathbf{n} \times \partial_\nu \mathbf{n})] \mathbf{n}$. Defining $R^i_{\mu \nu} \equiv R_{\mu \nu}^i$, and $R^i_{\mu \nu} \equiv \varepsilon^{ijk} R_{\mu \nu k}$, the two ways of the definition of the
torsion tensor, Eq. (D1) and Eq. (D5), are related by
\begin{equation}
R_{\mu \nu \rho \sigma} = e^i_\mu e^j_\nu R^i_{\mu \rho}, \quad R^i_{\mu \nu} = e^j_\rho e^k_\sigma R^i_{\mu \nu \rho \sigma}, \quad (D6)
\end{equation}
where $e^j_\mu \equiv \partial x^j / \partial x^\mu$ and we have implicitly used
$n^i R^i_\mu = 0$. The relation Eq. (D6) follows from
$[\mathcal{D}_\mu, \mathcal{D}_\nu] e^j_\rho \equiv R^j_{\mu \nu} e^j_\rho - R^j_{\mu \rho} e^k_\nu e^k_\sigma$, which vanishes due to $\mathcal{D}_\mu e^j_\nu = 0$. Then, from Eq. (D5) and Eq. (D6), we have
\begin{equation}
R = e^j_\mu e^j_\nu R_{\mu \nu} \equiv [\mathbf{n} \cdot (e^j_\mu \times e^j_\nu)] [\mathbf{n} \cdot (\partial_\mu \mathbf{n} \times \partial_\nu \mathbf{n})] / g \equiv 2 \mathbf{n} \cdot (\partial_1 \mathbf{n} \times \partial_2 \mathbf{n}) \quad (D7)
\end{equation}
where we have used $e^1 \times e^2 = \frac{1}{\sqrt{g}} \mathbf{n}$ (or $e_1 \times e_2 = \sqrt{g} \mathbf{n}$). Eq. (D7) means, from Eq. (D4), that the Gaussian curvature $K = \partial x^1 \partial x^2$ with
\begin{equation}
K = \frac{1}{\sqrt{g}} \mathbf{n} \cdot (\partial_1 \mathbf{n} \times \partial_2 \mathbf{n}) \quad (D8)
\end{equation}
at a point $x = (x^1, x^2)$ is given by the solid angle spanned by the normal vectors at $x$, $(x^1 + \delta x^1, x^2)$ and $(x^1, x^2 + \delta x^2)$.

The Riemann tensor $\widetilde{R}_{\mu \nu}$ in the intrinsic frame is given by
\begin{equation}
\widetilde{R}_{\mu \nu} = \partial_\mu \omega_\nu - \partial_\nu \omega_\mu
\end{equation}
with $\omega_\mu = (1 - \cos \theta) \partial_\mu \phi (\sigma^3 / 2)$ as in Eq. [6]. Then, the pseudomagnetic field $B^i_m$ is defined by $B^i_m \equiv \frac{1}{2} \varepsilon^{ijk} F_{jk}$ with $F_{jk} \equiv e^j_\rho e^k_\sigma F_{\rho \sigma}$ and $F_{\mu \nu} \sigma^3 \equiv \widetilde{R}_{\mu \nu}$. Then, from Eq. (D8) and Eq. (D9), we have
\begin{equation}
B^i_m = \frac{1}{4} (\mathbf{n} \cdot (\partial_\mu \mathbf{n} \times \partial_\nu \mathbf{n})) (e^j_\mu \times e^j_\nu)
\end{equation}
\begin{equation}
= \frac{K}{2} \mathbf{n}. \quad (D10)
\end{equation}

Appendix E: Details of Eq. [1]

Let $\gamma(s)$ be the position of a point on a curve parametrized by its arclength $s$. The curve determines by itself a natural orthonormal frame consisting of unit vectors $(\mathbf{T}, \mathbf{N}, \mathbf{B})$, where $\mathbf{T}(s) \equiv \gamma'(s)$, $\mathbf{N}(s) \equiv \mathbf{T}'(s) / |\mathbf{T}'(s)|$ and $\mathbf{B}(s) \equiv \mathbf{T}(s) \times \mathbf{N}(s)$. With these orthonormal vectors, we define a curvilinear coordinate system $x^M$ with $M = s, n, b$ by
\begin{equation}
x^i(x^M) = \gamma^i(x^s) + x^n N^i(x^s) + x^b B^i(x^s). \quad (E1)
\end{equation}

Hereafter, all quantities are eventually evaluated just on the curve, i.e., at $x^s = x^b = 0$. The nonzero components of the geometric connection $\Gamma^M_{\cal NL}$ in this curvilinear coordinate system are
\begin{equation}
-\Gamma^n_{ss} = \Gamma^n_{ss} = \kappa, \quad -\Gamma^n_{sb} = \Gamma^n_{sn} = \tau, \quad (E2)
\end{equation}
where $\kappa \equiv \mathbf{N} \cdot T'$ and $\tau \equiv -\mathbf{N} \cdot B'$ are the curvature and the torsion of the curve, respectively. These components can be calculated directly by an expression
\begin{equation}
\Gamma^M_{\cal NL} = \frac{1}{2} G_{MPN} \left( \frac{\partial G_{PN}}{\partial x^L} + \frac{\partial G_{PL}}{\partial x^N} - \frac{\partial G_{NL}}{\partial x^P} \right), \quad (E3)
\end{equation}
where $G_{MN} = (\partial x^i / \partial x^M)(\partial x^i / \partial x^N)$ is the metric defined in the three-dimensional space. We can also calculate $\Gamma^M_{\cal NL}$ more quickly by deriving the Euler-Lagrange equation for $L = \frac{1}{2} G_{MN}(X) \dot{X}^M \dot{X}^N$ for the position $X^M(t)$ of a point particle at time $t$, as explained in Chap. 2 in Ref.[3].

Defining $e^i = \partial x^i / \partial x^M$ and $e^i_M \equiv \partial x^M / \partial x^i$, we can easily see that $e^i_s = e^i = T^i$, $e^i_b = e^i_n = N^i$ and $e^i_b = e^i = B^i$. With all of this setup, a procedure similar to the case of surfaces gives an expression of the spin connection $\Omega^j_s$ on curves. Let us start from an obvious identity
\begin{equation}
0 = e^j_N \Gamma^n_{sL} e^L_j + e^j_s \partial_s e^N_j, \quad (E4)
\end{equation}
which simply states that the spin connection for the flat derivative is zero, $\nabla^\flat T^j_i = \partial_\mu T^j_i$. Then, the spin connection for the curved derivative is given by truncating all of the components of $\Gamma^M_{\cal NL}$ other than $\Gamma^n_{ss}$,
\begin{equation}
\Omega^j_s = e^j_N \partial_s e^N_j \quad (E5)
\end{equation}
(note that $\Gamma^n_{ss} = 0$). By comparing these equations, we have, by Eq. (E2),
\begin{equation}
\Omega^j_s = e^j_N \Gamma^n_{sn} e^n_j + e^j_s \Gamma^n_{ns} e^n_j + e^n_n \Gamma^n_{sn} e^s_j + e^n_b \Gamma^n_{sb} e^n_j = \kappa (T^i N^j - N^i T^j) + \tau (N^i B^j - B^i N^j), \quad (E6)
\end{equation}
which gives Eq. (E7) by $\Omega^j_s = \frac{1}{2} \varepsilon^{ijk} \Omega^j_{ik}$.

We can derive this expression more directly from Eq. (E5),
\begin{equation}
\Omega^j_s = T^i T^j_i + N^i N^j_i + B^i B^j_i \quad (E7)
\end{equation}
where we have used the Frenet–Serret formula.

Appendix F: Spin connections on curves vanish in the intrinsic frame.

As described in the main text, the spin connection for a curve is $\Omega_s = \kappa B + \tau T$ in the laboratory frame. Here, we show that this spin connection vanishes when we go to the intrinsic frame.
The relevant SO(3) transformation matrix is
\[ R_{ia} \equiv (N, B, T)_{ia}. \] (F1)

It is obvious that
\[ R_{ia} N^i = \delta^a_1, \quad R_{ia} B^i = \delta^a_2, \quad R_{ia} T^i = \delta^a_3, \] (F2)
which means that \( R_{ia} \) is the rotation matrix from the laboratory (Cartesian) frame to the intrinsic frame, in the latter of which \( N, B \) and \( T \) are the basis vectors.

The spin connection \( \Omega_s \) on a curve is simply the ‘angular velocity’ of \( R \):
\[ \Omega_s^i = \frac{1}{2} \varepsilon^{ijk} \left( R \frac{d}{ds} R^{-1} \right)_{jk}, \] (F3)
which is equivalent to \( dR_{ia}/ds = \varepsilon^{ijk} \Omega_s^j R_{ka} \), or the Frenet–Serret formulas \( dX/\sigma = \Omega_s \times X \) for \( X = T, N, B \). Eq. (F3) means that \( R_{ia} \) transforms \( \Omega_s^i \) to zero, as
\[ R_{ia} \Omega_s^i + \frac{1}{2} \varepsilon^{abc} \left( R^{-1} \frac{d}{ds} R \right)_{bc} = 0. \] (F4)

These can be rephrased by using the SU(2) matrix \( \mathcal{U} \) defined as \( \mathcal{U} \sigma^i \mathcal{U}^\dagger = R_{ia} \sigma^a \). From an identity
\[ -i \mathcal{U} \frac{d}{ds} \mathcal{U}^\dagger = \frac{1}{2} \varepsilon^{ijk} \left( R \frac{d}{ds} R^{-1} \right)_{jk} \sigma^i, \] (F5)
we have
\[ \Omega_s \cdot \frac{\sigma}{2} = -i \mathcal{U} \frac{d}{ds} \mathcal{U}^\dagger, \] (F6)
which is transformed to zero in the intrinsic frame as
\[ \mathcal{U}^\dagger \left( \Omega_s \cdot \frac{\sigma}{2} \right) \mathcal{U} - i \mathcal{U}^\dagger \frac{d}{ds} \mathcal{U} = 0. \] (F7)

---

1. J. J. Sakurai, *Modern Quantum Mechanics* (Addison-Wesley, 1994).
2. S. Carroll, *Spacetime and Geometry* (Addison-Wesley, 2003).
3. M. Nakahara, *Geometry, Topology and Physics*, 2nd ed. (CRC Press, 2003).
4. R. D. Kamien, *Rev. Mod. Phys.* 74, 953 (2002).
5. F. David, *Statistical Mechanics of Membranes and Surfaces*, 2nd ed., edited by D. Nelson, T. Piran, and S. Weinberg (World Scientific, 2004).
6. M. B. Green, J. H. Schwarz, and E. Witten, *Superstring Theory* (Cambridge University Press, 1987).
7. L. Parker and D. Toms, *Quantum Field Theory in Curved Spacetime: Quantized Fields and Gravity* (Cambridge University Press, 2009).
8. S. J. Barnett, *Phys. Rev.* 6, 239 (1915).
9. S. J. Barnett, *Rev. Mod. Phys.* 7, 129 (1935).
10. J. Stockhofe and P. Schmelcher, *Phys. Rev. A* 89, 033630 (2014).
11. F. E. Meijer, A. F. Morpurgo, and T. M. Klapwijk, *Phys. Rev. B* 66, 033107 (2002).
12. X. G. Wen and A. Zee, *Phys. Rev. Lett.* 69, 953 (1992).
13. J. Fröhlich and U. M. Studer, *Rev. Mod. Phys.* 65, 753 (1993).
14. H. Jensen and H. Koppe, *Ann. Phys.* 63, 586 (1971).
15. R. C. T. da Costa, *Phys. Rev. A* 23, 1982 (1981).
16. J. González, F. Guinea, and M. A. H. Vozmediano, *Phys. Rev. Lett.* 69, 172 (1992).
17. C. L. Kane and E. J. Mele, *Phys. Rev. Lett.* 78, 1932 (1997).
18. D. H. Lee, *Phys. Rev. Lett.* 103, 196804 (2009).
19. Y. Zhang and A. Vishwanath, *Phys. Rev. Lett.* 105, 206601 (2010).
20. J. Foster and J. D. Nightingale, *A Short Course in General Relativity*, 3rd ed. (Springer, 2006).
21. E. Poisson, *A Relativist’s toolkit* (Cambridge University Press, 2004).
22. K. Sato, *The Theory of Relativity* (Iwanami Shoten, in Japanese, 1996).
23. The Cartesian indices are raised and lowered by the Kronecker delta, while the curvilinear indices by the curvilinear metric tensor. We do not have to make a distinction between raised or lowered Cartesian indices.
24. This definition of spin connections is very specific to the case of curves and surfaces embedded in a flat space. See Refs. 32, 33, 34 for a more general definition.
25. A. R. Kavalov, I. K. Kostov, and A. G. Sedrakyan, *Phys. Lett. B* 175, 331 (1986).
26. See Eqs. (2.44), (2.45) and (7.3) in Ref. 31. Our results agree with these equations when we restrict vector fields, on which spin connections act, to be tangential to the surface.
27. M. P. do Carmo, *Differential Geometry of Curves and Surfaces*, 2nd ed. (Dover Publications, 2016).
28. Lattice strains can be induced either by externally applied mechanical forces or by the curvature of surfaces. Our discussion in this paper corresponds to the latter case, and can be extended to the former.
29. A. H. C. Neto, F. Guinea, N. M. R. Peres, K. S. Novoselov, and A. K. Geim, *Rev. Mod. Phys.* 81, 109 (2009).
30. N. Levy, S. A. Burke, K. L. Meaker, M. Paulasigui, A. Zettl, F. Guinea, A. H. C. Neto, and M. F. Crommie, *Science* 329, 544 (2010).
31. M. A. Vozmediano, M. I. Katsnelson, and F. Guinea, *Phys. Rep.* 496, 109 (2010).
32. M. I. Katsnelson, *Graphene: Carbon in Two Dimensions* (Cambridge University Press, 2012).
33. B. Amorim, A. Cortijo, F. De Juan, A. G. Grushin, F. Guinea, A. Gutiérrez-Rubio, H. Ochoa, V. Parente, R. Roldán, P. San-Jose, J. Schiefele, M. Sturla, and M. A.
E. I. Rashba, Phys. Rev. B 40 (1989).

M. A. Cazalilla, H. Ochoa, and F. Guinea, Phys. Rev. Lett. 113, 077201 (2014).

H. Shapourian, T. L. Hughes, and S. Ryu, Phys. Rev. B 92, 165131 (2015).

We note that in some cases indices attached to spinors are non-geometrical. For example, the so-called isospin degrees of freedom in nuclear physics express a proton as the up-state and a neutron as the down-state. Spin connections do not act on such a kind of indices.

E. I. Rashba, Phys. Rev. B 68, 241315(R) (2003).

M. I. Katsnelson, Y. O. Kvashnin, V. V. Mazurenko, and A. I. Lichtenstein, Phys. Rev. B 82, 100403(R) (2010).

T. Kikuchi, T. Koretsune, R. Arita, and G. Tataru, Phys. Rev. Lett. 116, 247201 (2016).

T. Koretsune, T. Kikuchi, and R. Arita, J. Phys. Soc. Jpn. 87, 041018 (2018).

K. Yasuda, Theory of Magnetism (Springer, 1998).

R. Skomski, Simple Models of Magnetism (Oxford University Press, 2008).

J. Kamonari, Magnetism (Baifukan, in Japanese, 1991).

K. Yasuda, Magnetism (Iwanami Shoten, in Japanese, 1991).

S. Matsuda, Mechanics, edited by J. Maki, Y. Nagaoka, and Y. Ohtsuki (Maruzen, in Japanese, 1993).

M. E. Coury, S. L. Dudarev, W. M. Foulkes, A. P. Horsfield, P. W. Ma, and J. S. Spencer, Phys. Rev. B 93, 075101 (2016).

Q. H. Wang, K. Kalantar-Zadeh, A. Kis, J. N. Coleman, and M. S. Strano, Nat. Nanotech. 7, 699 (2012).

W. Choi, N. Choudhary, G. H. Han, J. Park, D. Akinwande, and Y. H. Lee, Mater. Today 20, 116 (2017).

F. Qin, W. Shi, T. Ideue, M. Yoshida, A. Zak, R. Tennie, T. Kikitsu, D. Inoue, D. Hashizume, and Y. Iwasa, Nat. Commun. 8, 14465 (2017).

S. Manzeli, D. Ovchinnikov, D. Pasquier, O. V. Yazeyev, and A. Kis, Nat. Rev. Mater. 2, 17033 (2017).

R. Dong and I. Kuljanishvili, J. Vac. Sci. Technol. B: Nanoelectron. Microelectron. Microstruct. 35, 030803 (2017).

B. A. Bernevig, T. L. Hughes, and S. C. Zhang, Phys. Rev. Lett. 95, 066601 (2005).

H. Kontani, T. Tanaka, D. S. Hirashima, K. Yamada, and J. Inoue, Phys. Rev. Lett. 100, 096601 (2008).

T. Tanaka, H. Kontani, M. Naito, T. Naito, D. S. Hirashima, K. Yamada, and J. Inoue, Phys. Rev. B 77, 165117 (2008).

H. Kontani, T. Tanaka, D. S. Hirashima, K. Yamada, and J. Inoue, Phys. Rev. Lett. 102, 016601 (2009).

B. A. Bernevig and S.-C. Zhang, Phys. Rev. Lett. 96, 106802 (2006).

M. Matsuo, J. Ieda, E. Saitoh, and S. Maekawa, Phys. Rev. Lett. 106, 076601 (2011).

M. Matsuo, J. Ieda, K. Harii, E. Saitoh, and S. Maekawa, Phys. Rev. B 87, 180402(R) (2013).

X. Qian, J. Lui, L. Fu, and J. Li, Science 346, 1344 (2014).

M. Matsuo, E. Saitoh, and S. Maekawa, J. Phys. Soc. Jpn. 86, 011011 (2017).

D. Kobayashi, T. Yoshikawa, M. Matsuo, R. Iguchi, S. Maekawa, E. Saitoh, and Y. Nozaki, Phys. Rev. Lett. 119, 077202 (2017).

Y. Gaididei, V. P. Kravchuk, and D. D. Sheka, Phys. Rev. Lett. 112, 257203 (2014).

O. V. Pylypowksyi, V. P. Kravchuk, D. D. Sheka, D. Makarov, O. G. Schmidt, and Y. Gaididei, Phys. Rev. Lett. 114, 197204 (2015).

D. D. Sheka, V. P. Kravchuk, and Y. Gaididei, J. Phys.: Math. Theor. 48, 125202 (2015).

R. Streubel, P. Fischer, F. Kronast, V. V. Kravchuk, and D. Makarov, J. Phys. D: Appl. Phys. 49, 363001 (2016).

J. A. Otalora, M. Yan, H. Schultheiss, R. Hertel, and A. Káky, Phys. Rev. Lett. 117, 227203 (2016).

O. A. Tretiak, M. Morini, S. Vasilyevych, and V. Slastikov, Phys. Rev. Lett. 119, 077202 (2017).

M. Charilaou and J. F. Löffler, Phys. Rev. B 95, 024409 (2017).

R. Moreno, V. L. Carvalho-Santos, A. P. Espejo, D. Laroze, O. Chubykalo-Fesenko, and D. Altbir, Phys. Rev. B 96, 184401 (2017).

D. K. Bälß, S. Günther, M. Fritzschke, K. Lenz, G. Varvaro, S. Laureti, D. Makarov, A. Mücklich, S. Faesco, M. Albrecht, and J. Fassbender, J. Phys. D 50, 115004 (2017).

S. Vojkovic, V. L. Carvalho-Santos, J. M. Fonseca, and A. S. Nunez, J. Appl. Phys. 121, 113906 (2017).

A. A. Zvyagin, Phys. Rev. B 95, 165141 (2017).

J. Kondo, J. Phys. Soc. Jpn. 74, 1 (2005).

Y. Maeno, S. Kittaka, T. Nomura, S. Yonezawa, and K. Ishida, J. Phys. Soc. Jpn. 81, 011009 (2012).

R. I. Shekhter, M. Jonson, and A. Aharony, Phys. Rev. Lett. 116, 217001 (2016).

T. Can, M. Laskin, and P. Wiegmann, Phys. Rev. Lett. 113, 046803 (2014).

T. Can, Y. H. Chiu, M. Laskin, and P. Wiegmann, Phys. Rev. Lett. 117, 266803 (2016).

A. Szameit, F. Dreisow, M. Heinrich, R. Keil, S. Nolte, A. Tünnemann, and S. Longhi, Phys. Rev. Lett. 104, 150410 (2010).

V. H. Schultheiss, S. Batz, A. Szameit, F. Dreisow, S. Nolte, A. Tünnemann, S. Longhi, and U. Peschel, Phys. Rev. Lett. 105, 143901 (2010).

F. Nagasawa, J. Takagi, Y. Kunitsuki, M. Koda, and J. Nitta, Phys. Rev. Lett. 108, 086801 (2012).

F. Nagasawa, D. Frustaglia, H. Saarikoski, K. Richter, and J. Nitta, Nat. Commun. 4, 2526 (2013).

H. Saarikoski, J. E. Vázquez-Lozano, J. P. Baltanas, F. Nagasawa, J. Nitta, and D. Frustaglia, Phys. Rev. B 91, 241406(R) (2015).

H. Saarikoski, J. P. Baltanás, J. E. Vázquez-Lozano, J. Nitta, and D. Frustaglia, J. Phys.: Condens. Matter 28, 166002 (2016).

Y. Avishai and Y. B. Band, Phys. Rev. B 95, 104429 (2017).

M. Hamada, T. Yokoyama, and S. Murakami, Phys. Rev. B 92, 060409(R) (2015).

L. Dong and Q. Niu, arXiv:1802.02887.

C. Brendel, V. Peano, O. Painter, and F. Marquardt, Proc. Natl. Acad. Sci. U.S.A. 114, E3390 (2017).

B. Göehler, V. Hamelbeck, T. Z. Markus, M. Kettner, G. F. Hanne, Z. Vager, R. Naaman, and H. Zacharias, Science 331, 894 (2011).

A. M. Guo and Q. F. Sun, Phys. Rev. Lett. 108, 218102 (2012).
93 R. Gutierrez, E. Díaz, C. Gaul, T. Brumme, F. Domínguez-Adame, and G. Cuniberti, J. Phys. Chem. C 117, 22276 (2013).

94 P. Strange, Relativistic Quantum Mechanics: With Applications in Condensed Matter and Atomic Physics (Cambridge University Press, 1998).

95 K. Nishijima, Relativistic Quantum Mechanics (Baifukan, in Japanese, 1973).