Elephant Random Walk with multiple extractions

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Consider a generalized Elephant Random Walk in which the step is chosen by selecting $k$ previous steps, with $k$ odd, then going in the majority direction with probability $p$ and in the opposite direction otherwise. In the $k = 1$ case, the model is the original one, and can be solved exactly by analogy with Friedman’s urn. However, the analogy cannot be extended to the $k > 2$ case already. In this paper we show how to treat the model for each $k$ by analogy with the more general urn model of Hill Lane and Sudderth. Interestingly, for $k > 2$ we find a critical dependence from the initial conditions above certain values of the memory parameter $p$, and regions of convergence with entropy that is sub-linear in the number of steps.

I. INTRODUCTION

The Elephant Random Walk (ERW) is a simple model of a random walk with long-range memory, in which each step is determined by extracting one of the previous, then going in the same direction with probability $p$, and in the opposite otherwise. [1] This simple model has received much attention since its introduction and is still a very active area.

An important advance in the study of this model was made in 2016, when Baur and Bertoin noticed [2] that the ERW can be mapped exactly into a two color urn of the Friedman’s type, [3, 4] where at each time a ball is drawn from the urn, then replaced together with a fixed numbers of new balls of both colors.

This finding allowed many quantities of interest to be computed from known results on these types of urns. [2] [5] Unluckily, the analogy with Friedman-type urns cannot be extended to more complex memory mechanisms where the step is chosen at best of multiple draws.

Let consider a generalized version of the ERW in which the step is chosen at the best of $k$ draws: define

$$X := \{X_1, X_2, \ldots, X_N\}$$

(1)

with $X_t \in \{+,-\}$ for all $1 \leq t \leq N$. The average step up to time $N$ is defined as

$$x_N := \frac{1}{N} \sum_{t \leq N} X_t.$$  

(2)
Then, the next step $X_{N+1}$ is determined as follows: first, extract $k$ previous steps from $X_N$, with $k$ odd integer. If the sum of these steps is positive, i.e., there is a majority of positive steps, then $X_{N+1}$ is positive with probability $p$ and negative otherwise. In case the sum is negative, then $X_{N+1}$ is negative with probability $p$, and positive otherwise. For $k = 1$ the original ERW is recovered.

In the present work we show how the analogy with a more general non-linear urn, known as the Hill, Lane, and Sudderth (HLS) model, [6–9] (of which Friedman’s urn is a linear sub-case, and a mathematical treatment for small and large deviations exists already [9]) allows to deal with such generalizations. Interestingly, for multiple extractions there is a value $p_c$ of the memory parameter above which the limit of the process

$$x := \lim_{N \to \infty} X_N,$$

critically depends from the initial condition $x_M$, fixed at some finite time $M < \infty$. We also study the limit entropy density for $x_N$ converging to some given $x$,

$$\psi(x) := \lim_{N \to \infty} \frac{1}{N} \log P \left( x_N = \lfloor x_N \rfloor / N \right),$$

finding that for $k > 2$ and above $p_c$ there is a region in the $p$ vs $x$ plane where $\phi(x) = 0$, i.e., where the entropy is sub-linear in the number of steps.

II. RELATION WITH HLS URNS

An HLS urn [3, 4, 6–9] is a two color urn process governed by a functional parameter $\pi(y)$ called urn function. Let consider an infinite capacity two-color urn containing black and white balls, and let

$$Y = \{Y_0, Y_1, ..., Y_N\}$$

be the process describing the type of ball $Y_t \in \{0, 1\}$ that has been extracted at each time $t \leq N$. We associate the value 1 to black balls and 0 to white balls. The process $Y$ evolves as follows: let

$$y_N := \frac{1}{N} \sum_{t \leq N} Y_t$$

be the density of black balls at time $N$, then at step $N+1$ a new ball is added, black with probability $\pi(y_N)$ and white otherwise. The limit density is

$$y := \lim_{N \to \infty} y_N.$$
Figure 1: Example of linear urn function $\pi_1(y)$ for the classic ERW with $k = 1$, the memory parameter is $p^* = 3/4$. For all $p < 1$ the point $y_0 = 1/2$ is the only point of convergence for the density of black balls, although there still is a critical $p^*$ above which the convergence of $y_N$ is slowed according to the Pemantle mechanism [5, 8, 9], see Section IV.

The analogy between the ERW before and the HLS urns is easily unveiled by taking $X_t = 2Y_t - 1$ and then writing the density of black balls in terms of the average step

$$y = \frac{1 + x}{2}. \tag{8}$$

Start with $k = 1$, i.e., the simple ERW: the probability of extracting a black ball from the urn is $y$, then the ERW will go in the positive direction with probability

$$\pi_1(y) := py + (1 - p)(1 - y) = (1 - p) + (2p - 1)y, \tag{9}$$

that is a linear urn function. [5] In fact, it is proven in [9] that linear HLS urns of the kind $\pi(y) = a + by$ include the Friedman urn model as special case. Let now move to the case $k = 3$: the probability of a positive step is that of extracting two positive and one negative, plus that of extracting three positive:

$$P_3(y) := y^3 + 3y^2(1 - y) = 3y^2 - 2y^3, \tag{10}$$

then, the corresponding urn function is

$$\pi_3(y) := p \left(3y^2 - 2y^3\right) + (1 - p) \left(1 - (3y^2 - 2y^3)\right) = (1 - p) + (2p - 1) \left(3y^2 - 2y^3\right) \tag{11}$$
Figure 2: Three examples of the urn function $\pi_3(y)$ for the generalized ERW with $k = 3$. The figure shows urn functions for three non-trivial memory parameters, $p^* = 2/3$, $p_c = 5/6$ and $p^{**} = 11/12$. Below $p_c$ the urn function down-crosses the line $y$ at $y_0 = 1/2$, that is the only convergence point. For $p > p_c$ the point $y_0$ becomes an up-crossing (unstable equilibrium), and the urn function crosses the diagonal $y$ also in $y_-$ and $y_+$, that are both down-crossings and are the new stable attractors for the process $x_N$.

and cannot be reduced to the linear case. In general, the probability of finding a positive majority when extracting an odd number $k$ of steps is

$$P_k(y) := \sum_{h > k/2} \frac{k!}{h!(k-h)!} y^h (1 - y)^{k-h}$$  \hspace{1cm} (12)$$

where the $h$ sum runs from $(k + 1)/2$ to $k$. Then, the urn function that describes an ERW with odd number $k > 2$ of extractions is

$$\pi_k(y) := p P_k(y) + (1 - p) (1 - P_k(y)) = (1 - p) + (2p - 1) P_k(y)$$  \hspace{1cm} (13)$$

it is a $k$–th degree polynomial, and is therefore non-linear for all non-trivial values of the memory parameter $p$. 


III. STRONG CONVERGENCE

The convergence properties of HLS urns for continuous urn functions have been studied in [6–9], finding that the points of convergence of $y_N$ always belong to the set of solutions of the equation

$$\pi(y) = y,$$  \hspace{1cm} (14)

and that these solutions are stable only if the derivative of the urn function in those points is greater than one, ie if $\pi(y)$ crosses $y$ from top to bottom (down-crossing). For the classic ERW with $k = 1$, the urn function $\pi_1(y)$ crosses $y$ at 1/2 for any value of $p < 1$, and therefore 1/2 is the only possible point of convergence for the associated density of black balls $y_N$, see Figure 1. This imply that the average step $x_N$ converges to zero almost surely

$$\lim_{N \to \infty} x_N = 0, \text{ a.s.}$$  \hspace{1cm} (15)

for all values of $p < 1$ and the initial condition $x_M$, a phase diagram for the ERW is shown in Figure 3.

In the generalized ERW with multiple extractions we see that a new phase appear below some critical $p_c$. For $k = 3$ already Eq. (14) is a third degree equation, and it can be solved with the well known formula. In general, we find tree solutions: $y_0 = 1/2$ and

$$y_{\pm}(p) = \frac{1}{2} \pm \frac{\sqrt{12p^2 - 16p + 5}}{2(2p - 1)},$$  \hspace{1cm} (16)

the term inside the square root is positive for $p \leq 1/2$ and $p \geq 5/6$, but notice that $y_{\pm} \in [0, 1]$ only if $p \in [1/2, 1]$, then, for $p$ below the critical value $p_c = 5/6$ there is still a unique stable solution at 1/2 that crosses $y$ from top to bottom (down crossing), see Figure 2.

Above $p_c$ the function still crosses $y$ at the point 1/2, but it now does from bottom to top, ie it is an up-crossing, and is therefore not stable. By the way, notice that for $p > p_c$ two new solutions $y_+$ and $y_-$ also appear, that are both down-crossings and can be stable attractors for $y_N$. Follows that for $p$ below $p_c$ there are two attractors

$$x_{\pm}(p) = \pm \frac{\sqrt{12p^2 - 16p + 5}}{2p - 1},$$  \hspace{1cm} (17)

separated by an unstable equilibrium at $x_0 = 0$. Then, limit of $x_N$ for $p > p_c$ is supported by

$$\lim_{N \to \infty} x_N \in \{x_-(p), x_+(p)\}, \text{ a.s}$$  \hspace{1cm} (18)

for any initial condition $x_M$. Since the urn functions that we are considering never touch 0 or 1 for any $p \in (0, 1)$, for any initial condition $x_M$ that is fixed at finite time there is a strictly positive
probability to reach the nearby of any other $x$ in finite time at the very beginning of the process, then both points $x_\pm$ carries some non-zero probability mass for any initial condition at a finite time. Anyway, it can be shown that the probability mass of the point farther from $x_M$ will be exponentially suppressed in $M$, and fixing the initial condition at some $M = o(N)$ still divergent in $N$ will suppress one of the two possibilities, and concentrate the probability mass in the attraction point $x_\pm$ that is closest to the initial $x_M$. The phases for $k = 3$ are shown in Figure 4.

Notice that in the limit of infinite $k$ the probability of finding a majority of positive steps converges to

$$P_\infty (y) := \theta (1 - 2y)$$

(19)

and the urn function converges to a step function

$$\pi_\infty (y) := (1 - p) + (2p - 1) \theta (1 - 2y)$$

(20)

that still crosses the diagonal at point $y_0 = 1/2$ (from top to bottom) for $p < p_c = 1/2$, and at $y_- = p$, $y_+ = 1 - p$ if the memory parameter is above $p_c$. Then also in the infinite $k$ limit there is a $p_c$ above which we find the same region of the phase diagram that is observed for $k = 3$. In fact, the phase diagram shows the same structure for all $k > 2$, apart from different $p_c$ and $x_\pm (p)$. For this reason, hereafter we restrict our analysis to the cases $k = 1$ and $k = 3$ only.

IV. LARGE DEVIATIONS

We can further refine the phase diagram by studying the entropy of the event that $y_N$ converges to $y$,

$$\phi (y) := \lim_{N \to \infty} \frac{1}{N} \log P \left( y_N = \lfloor yN \rfloor / N \right),$$

(21)

that is related to $\psi (x)$ by Eq. [8]:

$$\psi (x) = \phi \left( \frac{1 + x}{2} \right).$$

(22)

From Large Deviations theory of HLS urns (see Corollary 5 of [9]) we know that for any continuous and invertible urn function the limit $\phi (y)$ exists, that it is strictly convex and negative from $y = 0$ up to the first point where the urn function crosses the diagonal, is zero from that point to the last crossing, and then is convex negative again.

Since for the classic ERW with $k = 1$ the urn function is a line pivoting around the center of the diagram, the CGF is zero only on the critical point $y_0 = 1/2$ and strictly negative otherwise for
Figure 3: Phase diagram $x$ vs $p$ for the entropy density $\psi(x)$ of the classic ERW $k = 1$, the diagram is shown for $p > 1/2$. The line $x_0 = 0$ is always a stable attractor for $x_N$, and the entropy is convex and strictly negative in the whole region, except at the critical line $x = 0$, where is zero. According to Eq. (14), there is a critical value at $p^* = 3/4$ at which the derivative of the urn function gets above $1/2$: for $p > p^*$ there is a shape change in $\psi(x)$ in the neighborhood of $x = 1/2$.

$p < \infty$. For the ERW with $k = 3$ the picture below $p_c = 5/6$ is qualitatively the same, but above $p_c$ there is a region

$$y \in [y_-(p), y_+(p)]$$ (23)

where the entropy density is $\phi(y) = 0$, ie the entropy it sub-linear in $N$, see the dark colored region in Figure 4. In general, the main difference between the classic ERW and the other cases $k > 2$ is the appearance of this new phase above $p_c$ where the entropy density is sub-linear in $N$.

The asymptotic behavior of the entropy in this region can be inferred by looking at the optimal trajectories of the process $y_N$ to reach some given limit value $y$. From the proof of Corollary 5 of [9] we find that the entropy density of any HLS urn with analytic urn function $\pi$ is obtained through the following LD principle: let

$$\varphi := \{\varphi(\tau) \in [0, 1] : 0 \leq \tau \leq 1\}$$ (24)

and let $Q(y)$ be the set of absolutely continuous function on $[0, 1]$ with initial value zero, final value
Figure 4: Phase diagram $x$ vs $p$ for the entropy density $\psi (x)$ of the generalized ERW $k = 3$. Above the critical value $p_c = 5/6$ the point $x_0 = 0$ becomes an unstable equilibrium, and two new symmetric attractors arise according to Eq. (17). In the white colored region we still find a convex and negative $\psi (x)$, except on the critical line, but notice that a new region appeared above $p_c = 5/6$, highlighted in darker shade, where $\psi (x) = 0$, i.e. the entropy is sub linear in $N$. The shape of $\psi (x)$ near the critical line is similar to the case $k = 1$ for $p < p_c$, except that here the point at which the derivative of the urn function rise above $1/2$ is $p^* = 2/3$. There is also another non-trivial value $p^{**} = 11/12$ at which the shape of $\psi (x)$ in the right (left) neighborhood of $x_-(x_+)$ has a change, that is, when the derivative of the associated urn function in $y_{\pm}$ goes back below $1/2$. See also Figure 2.

$y$, and such that the derivative is between zero and one,

$$ Q (y) := \{ \varphi \in C ([0, 1]) : 0 \leq \partial_{\tau} \varphi (\tau) \leq 1, \varphi (0) = 0, \varphi (1) = y \}. $$

(25)

Also, define the auxiliary function

$$ L (\alpha, \beta) := \alpha \log (\beta / \alpha) + (1 - \alpha) \log ((1 - \beta) / (1 - \alpha)). $$

(26)

Then, in [9] it is shown that the entropy density $\phi (y)$ is obtained trough solving the variational problem

$$ \phi (y) = - \inf_{\varphi \in Q (y)} I_\pi [\varphi] $$

(27)

with rate function defined as follows:

$$ I_\pi [\varphi] := \int_0^1 d\tau L (\partial_{\tau} \varphi (\tau), \pi (\varphi (\tau) / \tau)), $$

(28)
In the Corollary 6 of [9] explicit formulas for the optimal trajectories are given for the HLS model. From [9], we find that since \( L \) is a negative and concave function such that \( L(\alpha, \beta) = 0 \) if and only if \( \alpha = \beta \), then, when \( I_\pi [\varphi] = 0 \) the trajectory \( \varphi \) must satisfy the differential equation

\[
\partial_\tau \varphi (\tau) = \pi \left( \varphi (\tau) / \tau \right) \tag{29}
\]

with final condition \( \varphi (1) = y \). Applying the substitution

\[
u (\tau) := \varphi (\tau) / \tau,
\]

the differential equation becomes

\[
\frac{\partial_\tau u (\tau)}{\pi (u (\tau)) - u (\tau)} = \frac{1}{\tau} \tag{31}
\]

with final condition \( u (1) = y \). The above equation can be integrated, finding that the optimal strategy to achieve the event \( y_N = [yN] / N \) according to Eq. (23) emanates from the closest unstable equilibrium point. Moreover, from the Corollary 6 of [9] follows uniqueness of the solutions, and that given two trajectories \( u_1 (\tau) \) and \( u_2 (\tau) \), with \( u_1 (1) = y_1, u_2 (1) = y_2 \), if \( y_1 < y_2 \) then also \( u_1 (\tau) < u_2 (\tau) \) for all \( \tau \in [0, 1] \).

Since any finite deviation from these trajectories has an exponential cost on a time scale \( O(N) \), the probability mass current can move along these trajectories only, then the probability current passing through \( (\varphi_1 (\tau), \varphi_2 (\tau)) \) is a constant for all \( \tau \), formally

\[
P (y_N \in (y_1, y_2)) = P \left( y_{\lfloor \tau N \rfloor} \in (\varphi_1 (\tau), \varphi_2 (\tau)) \right), \tag{32}
\]

and since the optimal trajectories can emanate only from the closest unstable equilibrium, the probability of the event \( y_N \in (y_1, y_2) \) scales in \( N \) like the entropy nearby that point. If \( y_0 \) is that point, then a martingale analysis would suggest that

\[
P (y_N \in (y_1, y_2)) = O \left( N^{-\chi-1} \right) \tag{33}
\]

for large \( N \), and with \( \chi > 1 \) derivative of the urn function at point \( y_0 \). This analysis indicates that in the sub-linear region the entropy is logarithmic in \( N \).

**V. CUMULANT GENERATING FUNCTION**

We can also study the behavior of \( \phi (y) \) in the nearby of the critical line by computing the CGF

\[
\zeta (\lambda) := \lim_{N \to \infty} \log \sum_{t \leq N} e^{-\lambda t} P (y_N = t/N), \tag{34}
\]
The right (left) behavior of $\phi(y)$ near the critical line can be deduced from the left (right) limit $\lambda \to 0^\pm$ of the CGF before. Since the critical line is symmetric around $x = 0$ we only compute the limit from right, that is related to the entropy by the Legendre Transform

$$
\phi(y) = \inf_{\lambda \in [0, \infty)} \{ \lambda y + \zeta(y) \}.
$$

In [9] is proven that, in general, the CGF satisfies the following nonlinear differential equation

$$
\partial_\lambda \zeta(\lambda) = \pi - \frac{1}{2} \left( \frac{e^{\zeta(\lambda)} - 1}{e^\lambda - 1} \right),
$$

with $\pi^{-1}$ inverse urn function. The linear urn function of the classic ERW satisfies the differential equation

$$
\partial_\lambda \zeta(\lambda) = -\frac{1}{2p-1} + \frac{1}{2p-1} \left( \frac{e^{\zeta(\lambda)} - 1}{e^\lambda - 1} \right),
$$

the above equation can be integrated exactly: adapting the results from Corollary 10 of [9] we find

$$
1 - e^{-\zeta(\lambda)} = \left( \frac{1-p}{2p-1} \right) e^{-\left( \frac{1}{2p-1} \right) \lambda} \left( 1 - e^{-\lambda} \right) \frac{1}{2p-1} \int_{1-e^{-\lambda}}^{1} dt \left( 1 - t \right) - \left( \frac{3p-2}{2p-1} \right) - \left( \frac{1}{2p-1} \right),
$$

when $p > 1/2$ and for $\lambda > 0$. Interestingly, in the region $p > 1/2$ the CGF is never analytic at $\lambda = 0$. Expanding the expression before for small $\lambda$ we find a non vanishing term $O(\lambda^{1/(2p-1)} \log \lambda)$ for $1/(2p - 1) \in \mathbb{N}$ and $O(\lambda^{1/b})$ when $1/(2p - 1) \in \mathbb{R} \setminus \mathbb{N}$, ie derivatives $\lceil 1/(2p - 1) \rceil$ and higher are singular at zero.

Notice that when $p > p^* = 3/4$, ie when the derivative of the urn function at the point of convergence $y_0$ climbs above $1/2$, even the second order cumulant is divergent, then the shape $\phi(y)$ in the nearby of $y_0 = 1/2$ is not Gaussian anymore for $p \in (p^*, 1)$, see also Figure 3. This indicates a phase change in the convergence mechanism of $y_N$: below $p_c$, when the urn function derivative at point $y_0$ is less than $1/2$, we expect for $y_N$ to cross the critical value infinitely many times in its evolution.

But above $p_c$ the convergence of $y_N$ has a slow down, according to an interesting mechanism first described by Pemantle in [8], where $y_N$ approaches $y_0$ so slowly that it never crosses this point (almost surely), accumulating in its right neighborhood. The convergence of the classic ERW in both $p$ regimes has been further studied by Jack and Harris in [5].

Concerning the ERW with $k = 3$, the equation for the CGF cannot be integrated with analytic methods (at best of our knowledge), but looking at the behavior for small $\lambda$ we expect that below $p_c$ the same picture of the case with $k = 1$ will arise, with $p^* = 5/6$, while above $p_c$ a new special value $p^{**} = 11/12$ can be identified, where the derivative of the urn function at the convergence
point \( y_+ (p) \) goes again below \( \frac{1}{2} \). We expect that in this last region the convergence mechanism below \( p^* \) is restored, with different convergence point, see Figure 2.

VI. RELATIONS WITH OTHER MODELS

Apart from the proposed generalization, analogy with HLS urns allows to put the ERW in relation with other models of physical interest that can be embedded in this urn model.

For example, in [10] Jack has identified the urn function that describes an interesting irreversible growth model introduced by Klymko, Garrahan and Whitelam in [11] [12], the urn function is

\[
\pi_{KGW} (y) = \frac{1 + \tanh \left( J \left( 2y - 1 \right) \right)}{2},
\]

controlled by the real parameter \( J \). This urn function has a sub-linear entropy region that is similar to the \( k > 2 \) case of our generalized ERW model.

Also, the HLS framework allows to connect the ERW with the classical Random Walk Range problem, [13] [16] that studies the number of different sites visited by a random walk on the lattice \( \mathbb{Z}^d \). In [15] is shown that the Range problem can be exactly embedded into an HLS model for some non-linear urn function at any \( d > 1 \). Although for \( d = 2, 3 \) a strongly non-linear urn function is found, for \( d \geq 4 \) the urn function gets surprisingly close to a linear function in the self-avoiding walk-like region of large range values, that would be analogous to the classic ERW with \( k = 1 \).

Other than the physical models, one broad application of HLS urn is as a simplified mechanism to explain the Lock-In phenomenon in industrial and consumer behavior. [3] An influential model for Market Share between competing products that can be modeled by the urn function \( P_k (y) \) has been introduced by Arthur in [17], and further developed by other authors, see Dosi [18] [19]. In this model two competing products gain customers according to a majority mechanism, that correspond to the limit \( p \to 1 \) of our generalized ERW with \( k > 2 \).

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