interpreting power anisotropy measurements in plasma turbulence

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ABSTRACT

A relationship between power anisotropy and wavevector anisotropy in turbulent fluctuations is derived. This can be used to interpret plasma turbulence measurements, for example in the solar wind. If fluctuations are anisotropic in shape then the ion gyroscale break point in spectra in the directions parallel and perpendicular to the magnetic field would not occur at the same frequency, and similarly for the electron gyroscale break point. This is an important consideration when interpreting solar wind observations in terms of anisotropic turbulence theories. Model magnetic field power spectra are presented assuming a cascade of critically balanced Alfvén waves in the inertial range and kinetic Alfvén waves in the dissipation range. The variation of power anisotropy with scale is compared to existing solar wind measurements and the similarities and differences are discussed.

Key words: magnetic fields — MHD — plasmas — solar wind — turbulence

1. INTRODUCTION

Plasma turbulence has been observed to be anisotropic with respect to the magnetic field direction, for example in the solar wind, the observed power and scaling of turbulent fluctuations vary depending on the angle between the local mean field and the sampling direction (Bieber et al. 1996; Horbury et al. 2008; Podesta 2009; Osman & Horbury 2009). Correlation lengths have also been observed to be anisotropic in the solar wind (Crooker et al. 1983; Mattheau et al. 1990; Osman & Horbury 2007; Weygand et al. 2009) and laboratory measurements (Robinson & Rusbridge 1971; Zweben et al. 1979). In addition, variance anisotropy has been measured in the solar wind and the fluctuations were seen to be predominantly perpendicular to the local mean magnetic field over a range of timescales (Belcher & Davis 1971; Bavassano et al. 1982; Horbury et al. 1995; Leamon & Horbury 1998). Recent theories of plasma turbulence also involve anisotropy (Goldreich & Sridhar 1993; Boldyrev 2006; Galter & Lithwick 2007; Gogoberidze 2007; Chandrakas 2008; Beresnyak & Lazarian 2008; Podesta & Bhattacharjee 2009; Schekochihin et al. 2009) and anisotropic energy transfer has been seen in simulations (Shebalin et al. 1983; Cho & Vishniac 2000; Maron & Goldreich 2001; Cho et al. 2002; Cho & Lazarian 2004, 2009). The theories usually lead naturally to a description of the anisotropy in terms of the fluctuation wavenumbers parallel and perpendicular to the mean magnetic field direction, $k_\parallel$ and $k_\perp$. For example, Goldreich & Sridhar (1995) used the “critical balance” assumption to obtain $k_\parallel \sim k_\perp^{2/3}$ for MHD (inertial range) turbulence and, more generally, theories often assume $k_\perp \gg k_\parallel$. In the solar wind, however, it is the anisotropy in power at a fixed scale that is often measured for practical reasons, rather than the spatial anisotropy of the fluctuations. In this letter the relationship between these two types of anisotropy is derived and the implications for some recent theories and solar wind measurements are discussed.

2. POWER ANISOTROPY AND WAVEVECTOR ANISOTROPY

In the solar wind, “power” can be measured as a function of frequency at different angles with respect to the mean magnetic field direction. Since the solar wind speed is much larger than the wave speed (often taken as the Alfvén speed in the inertial range), Taylor’s hypothesis (Taylor 1938) is usually well satisfied. This means that time variations measured by a spacecraft correspond to spatial fluctuations in the plasma and frequency spectra can be converted into wavenumber spectra. Power, $P$, in a turbulent field, for example the magnetic field, $B$, is calculated in various ways (Fourier transforms, wavelet transforms, structure functions etc.) but all can be related to the second order structure function trace power, $(\langle |\mathbf{B}(\mathbf{r}+\mathbf{l})-\mathbf{B}(\mathbf{r})|^2 \rangle)$, where $l$ is the scale and direction of the power measurement and the angular brackets denote an ensemble average over positions $\mathbf{r}$. A Cartesian coordinate system will be used here in which axisymmetry about the magnetic field is assumed and power is a function of scale in the directions parallel and perpendicular to the mean magnetic field, $P(l_\parallel, l_\perp)$.

Figure 1 is a schematic of power contours with elliptical shapes, elongated in the field parallel direction. Power anisotropy measurements are made at a fixed scale (dashed red line).
indicated by the red dashed line which is at a fixed radius from the origin. At different points along the red dashed line a varying power is sampled; this effect is readily seen in the solar wind (Bieber et al. [1996; Horbury et al. 2008; Podesta 2009; Osman & Horbury 2009). A relationship between power anisotropy and wavevector anisotropy will now be derived. Note that the derivation does not depend on any particular contour shape; the elliptical shapes in the figure are for illustrative purposes only.

Let us consider two contours of size \( (l_{\parallel 1}, l_{\perp 1}) \) and \( (l_{\parallel 2}, l_{\perp 2}) \) such that \( l_{\parallel 2} = l_{\perp 1} \) (Figure 1). This length, \( l_{\parallel 2} = l_{\perp 1} \), is the scale which will be used for power anisotropy evaluation. Power in the parallel and perpendicular directions is defined as \( P_{\parallel} = P(l_{\parallel 1}, 0) \) and \( P_{\perp} = P(0, l_{\perp 1}) \), and \( \alpha \) is the scaling in the perpendicular direction. \( P_{\perp} \sim l_{\perp 1}^\alpha \). Dividing the equations for \( P_{\perp} \) for each contour we get

\[
\frac{P_{\perp 1}}{P_{\perp 2}} = \left( \frac{l_{\perp 1}}{l_{\perp 2}} \right)^\alpha.
\]

By the definition of a contour, \( P_{\perp 2} = P_{\parallel 2} \) and from the above definition of the two contours \( l_{\parallel 2} = l_{\perp 1} \). Substituting these into Equation (1) gives

\[
\frac{P_{\perp 1}}{P_{\parallel 2}} = \left( \frac{l_{\perp 1}}{l_{\perp 2}} \right)^\alpha.
\]

Since \( P_{\perp 1}/P_{\parallel 2} \) is just the power anisotropy at a fixed scale (as in solar wind measurements) the numeric subscripts may be dropped. Rearranging Equation (2) and converting lengths into wavenumbers using \( k = 2\pi/l \) we get

\[
\left( \frac{k_{\perp}}{k_{\parallel}} \right)_2 = \left( \frac{P_{\perp}}{P_{\parallel}} \right)_1^{1/\alpha}.
\]

This relationship is independent of the scaling of the parallel spectrum and allows us to calculate the wavevector anisotropy of the smaller power contour from a measurement of \( P_{\perp}/P_{\parallel} \). A similar relationship can be derived for the larger power contour,

\[
\left( \frac{k_{\perp}}{k_{\parallel}} \right)_1 = \left( \frac{P_{\perp}}{P_{\parallel}} \right)_1^{1/\alpha},
\]

where \( \beta \) is the scaling of the parallel spectrum, \( P_{\parallel} \sim l_{\parallel}^\beta \).

In the solar wind, spacecraft measurements are of the “reduced power” which is an integral of the actual spatial distribution of power (Fredricks & Coroniti 1976). The reduced power prediction for some turbulence models has been calculated (Horbury et al. 2008) and it is important to consider this when comparing measurements to theory. The \( k_{\perp} \) and \( k_{\parallel} \) of turbulence theories usually describe typical length scales associated with the fluctuations. It is often assumed that second order statistics, such as power, relate to these quantities, for example Cho & Vishniac (2000) state that in their simulations, contours of second order structure functions “reflect the shapes of the eddies.” Under this assumption it is possible to equate the wavevector anisotropy of the power contours to the “typical” wavevector anisotropy of the fluctuations.

3. FORM OF THE CRITICAL BALANCE POWER SPECTRUM

The “critical balance” assumption states that in a turbulent Alfvénic cascade, the linear wave timescale and the nonlinear energy transfer timescale are comparable. It was introduced by Goldreich & Sridhar (1995) and anticipated in the work of Higdon (1984). When applied to the inertial range, a spectral index in the perpendicular direction of \(-5/3\) is obtained, along with the wavevector scaling \( k_{\perp} \sim k_{\parallel}^{1/3} \). A spectral index of \(-2\) in the parallel direction follows from these statements. There is evidence in the solar wind inertial range for both the Alfvénic nature of the turbulence (Belcher & Davis 1971; Bale et al. 2005) and the anisotropic scaling (Horbury et al. 2008).

At scales smaller than the ion gyroradius, commonly termed the “dissipation range,” there is evidence for kinetic Alfvén waves (Bale et al. 2003; Sahraoui et al. 2009). When the critical balance assumption is applied to a kinetic Alfvén wave cascade the wavevector scaling becomes \( k_{\parallel} \sim k_{\parallel}^{1/3} \) and the predicted spectral indices are \(-7/3\) in the perpendicular direction and \(-5\) in the parallel direction (Cho & Lazarian 2004; Schekochihin et al. 2009).

In this theoretical framework the break between the inertial range and the dissipation range is predicted to be at \( k_{\parallel}^\rho_i \sim 1 \), where \( \rho_i \) is the ion gyroradius, but if the fluctuations here are anisotropic then their parallel length should be larger, \( k_{\parallel}^\rho_i \sim 1 \). This would imply that the observed break points in solar wind measurements are at different spacecraft frequencies for the spectra in the parallel and perpendicular directions. A similar effect would be expected at the electron break scale, \( k_{\parallel}^\rho_e \sim 1 \), where the difference in break frequency between the spectra in the parallel and perpendicular directions may be even greater if the anisotropy continues to increase throughout the dissipation range.

Since the observed break points would be at different scales for spectra in the parallel and perpendicular directions, a schematic of the spectra can be divided into five ranges (Figure 2). In range 1 both spectra display Alfvén wave (AW) scaling; in range 2 the spectrum in the parallel direction has kinetic Alfvén wave (KAW) scaling and the spectrum in the perpendicular direction has AW scaling; in range 3 both spectra display KAW scaling; in range 4 the spectrum in the parallel direction is below the electron break scale and the spectrum in the perpendicular direction has KAW scaling; in range 5 both spectra are below the electron break scale. In this figure the (logarithm of the) power anisotropy can be thought of...
as the vertical distance between the spectra and the (logarithm of the) wavevector anisotropy as the horizontal distance.

We have not included predictions for fluctuations smaller than electron scales in this figure. Gyrokinetic theory does predict scalings for an electron-entropy cascade, valid for $k_{\perp}\rho_e \gg 1$ (Schekochihin et al. 2009), however it is thought that for the solar wind gyrokinetics is no longer valid above electron scales since the frequencies are expected to be larger than the cyclotron frequency, violating one of the gyrokinetic assumptions (Howes et al. 2008).

The scalings for each of the ranges in Figure 2 are given in Table 1. Also listed in this table is the scaling of $P_\perp/P_\parallel$ in each range which follows from the scaling of $P_\perp$ and $P_\parallel$. A schematic of the variation of $P_\perp/P_\parallel$ with scale is given in Figure 3. In the inertial range a gradient of 1/3 is expected which steepens to 10/3 when $P_\parallel$ reaches the ion break scale and then becomes shallower at 8/3 when $P_\perp$ reaches the ion break scale.

The width of the kinetic Alfvén wave range in the $P_\perp$ spectrum is predicted to be $\rho_i/\rho_e = \sqrt{T_m/T_e}$, and for the model spectra in Figures 2 and 3 the temperatures have been assumed equal, $T_i = T_e$. Although $T_i/T_e$ is of order unity in the solar wind, there is some variation (Bruno & Carbone 2005) so the extent of the possible kinetic Alfvén wave range may vary. The width of each of the ranges in Figures 2 and 3 also depends on the amount of anisotropy present. For example, as the anisotropy at the ion break scale increases, the size of range 2 increases but the size of range 3 decreases. It may even be the case that if the anisotropy is very strong at the ion break scale then range 3 may not be present at all, meaning it would not be possible to measure kinetic Alfvén wave scaling in $P_\perp$ and $P_\parallel$ at the same frequency.

One of the assumptions used when constructing these model spectra was that they contain no additional energy injection or loss. As noted in (Schekochihin et al. 2009), at the ion break scale some energy may be transferred from the Alfvénic cascade channel to a purely electrostatic “entropy cascade.” The amount of energy transferred, if any, is not predicted and cannot be included in our model spectra here. It is also possible that power may be injected into the cascades from other sources such as plasma instabilities, for example the firehose and mirror instabilities, which evidence suggests may be important in the solar wind (Bale et al. 2009). Including effects such as these is beyond the scope of this letter.

### Table 1

| Range | $P_\perp$ Scaling | $P_\parallel$ Scaling | $P_\perp/P_\parallel$ Scaling |
|-------|-------------------|-----------------------|-----------------------------|
| 1     | -5/3              | 1/3                   |                             |
| 2     | -5/3              | -5                    | 10/3                        |
| 3     | -7/3              | -5                    | 8/3                         |
| 4     | -7/3              | -5                    |                             |
| 5     | -5/3              | -5                    |                             |

Figure 3. Schematic of power anisotropy as a function of scale for critically balanced Alfvén waves and kinetic Alfvén waves.

do not hallucinate.
exercised when drawing conclusions from this range.

The solar wind spectrum has recently been observed at frequencies higher than \( k_p \rho_e = 1 \) where a new break was seen. Sahraoui et al. (2009) suggest that above the break frequency there may be a further power law and [Alexandrova et al. (2009)] suggest that there may be an exponential falloff. Both of these studies involved solar wind intervals where the magnetic field was not aligned with the solar wind direction so that one would expect to see the spectrum in the perpendicular direction. Sahraoui et al. (2009) also plot the spectrum of the parallel component of the magnetic field which it should be pointed out for clarity, is not the same as the spectrum in the parallel direction.

5. SUMMARY AND CONCLUSIONS

In Section 2 a relationship was derived that allows the turbulent wavevector anisotropy, \( k_\perp/k_\parallel \), to be inferred from power anisotropy measurements, \( P_\perp/P_\parallel \). This is independent of any particular turbulence theory and only assumes arbitrary scaling in the parallel and perpendicular directions. Using this relation and existing solar wind measurements (Podesta 2009) the wavevector anisotropy at the ion break scale was estimated to be \( k_\perp/k_\parallel \approx 3 \) although this may be an underestimate due to the finite angular resolution of the measurements.

Model spectra of critically balanced Alfvén waves (for the inertial range) and kinetic Alfvén waves (for the dissipation range) were presented to illustrate that break points do not occur at the same scale in the observed spectra in the parallel and perpendicular directions if the turbulence is anisotropic. The variation of \( P_\perp/P_\parallel \) with scale was calculated from these model spectra resulting in five ranges, three of which have predictions of how \( P_\perp/P_\parallel \) scales: a 1/3 range, a 10/3 range and an 8/3 range. If the wavevector anisotropy is significant then some of these ranges may be small and may not even be present. Some of these features can be seen in the corresponding measurements of Podesta (2009). The main difference is extra parallel power at the ion break scale seen in the measurements but not predicted by the theory.

Although critically balanced Alfvén waves and kinetic Alfvén waves were used in Section 3 the ideas also apply to anisotropic theories of plasma turbulence in general. Some of these, for example, are a nonlocal cascade model (Gogoberidze 2007), turbulence with dynamic alignment (Boldyrev 2006; Podesta & Bhattacharjee 2009), wave turbulence in Hall MHD (Galtier 2006) and imbalanced turbulence (Lithwick et al. 2007; Beresnyak & Lazarian 2008). Chandran (2008). The anisotropy relationship derived in Section 2, and the observational considerations discussed in Sections 3 and 4 are also applicable to these theories and any possible extensions of them into the dissipation range.

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