A Priori Estimates for Solutions of Boundary Value Problems for Fractional-Order Equations

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Abstract

We consider boundary value problems of the first and third kind for the diffusion-wave equation. By using the method of energy inequalities, we find a priori estimates for the solutions of these boundary value problems.

Fractional calculus is used for the description of a large class of physical and chemical processes that occur in media with fractal geometry as well as in the mathematical modeling of economic and social-biological phenomena [1, Chap. 5]. It was proved in [1] that fractional differentiation is a positive operator; this result permits one to obtain a priori estimates for solutions of a wide class of boundary value problems for equations with fractional derivatives. The paper [2] deals with the generalization of the differentiation and integration operations from integer to fractional, real, and complex orders and with applications of fractional integration and differentiation to integral and differential equations and in function theory. In the paper [3], an a priori estimate in terms of a fractional Riemann-Liouville integral of the solution was obtained for the solution of the first initial-boundary value problem for the fractional diffusion equation. The general fractional diffusion equation \((0 < \alpha \leq 1)\) with regularized fractional derivative was considered in [4]. A more detailed bibliography on fractional partial differential equations, including the diffusion-wave equation, can be found, for example, in [5].

In the present paper, we use the method of energy inequalities to obtain a priori estimates for solutions of boundary value problems for the diffusion-wave equation with Caputo fractional derivative [6].

1 BOUNDARY VALUE PROBLEMS FOR THE FRACTIONAL DIFFUSION EQUATION

1.1. First Boundary Value Problem. In the rectangle \(\bar{Q}_T = \{(x,t) : 0 \leq x \leq l, 0 \leq t \leq T\}\), consider the first boundary value problem

\[
\partial^\alpha_0 t u = \frac{\partial}{\partial x} \left( k(x,t) \frac{\partial u}{\partial x} \right) - q(x,t) u + f(x,t), \quad 0 < x < l, \quad 0 < t \leq T,
\]
where

\[
\partial_\alpha^\alpha u(x, t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{u(x, \tau)}{(t-\tau)^\alpha} d\tau
\]

is the Caputo fractional derivative of order \(\alpha\), \(0 < \alpha < 1\).

Throughout the following, we assume that there exists a solution \(u(x, t) \in C^{2,1}(\bar{Q}_T)\) of problem (1)-(3), where \(C^{m,n}(\bar{Q}_T)\) is the class of functions that, together with their partial derivatives of order \(m\) with respect to \(x\) and order \(n\) with respect to \(t\), are continuous on \(\bar{Q}_T\).

Let us prove the following assertion.

**Lemma 1.** For any function \(v(t)\) absolutely continuous on \([0, T]\), one has the inequality

\[
v(t)\partial_\alpha^\alpha v(t) \geq \frac{1}{2} \partial_\alpha^\alpha v^2(t), \quad 0 < \alpha < 1.
\]  

**Proof.** Let us rewrite inequality (4) in the form

\[
v(t)\partial_\alpha^\alpha v(t) - \frac{1}{2} \partial_\alpha^\alpha v^2(t) = \frac{1}{\Gamma(1-\alpha)} v(t) \int_0^t \frac{v(\tau)d\tau}{(t-\tau)^\alpha} - \frac{1}{2\Gamma(1-\alpha)} \int_0^t \frac{2v(\tau)v(\tau)d\tau}{(t-\tau)^\alpha} =
\]

\[
= \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{v(\tau)(v(t) - v(\tau))d\tau}{(t-\tau)^\alpha} = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{v_\eta(\eta)d\eta}{(t-\tau)^\alpha} \int_\tau^t v(\eta)d\eta =
\]

\[
= \frac{1}{\Gamma(1-\alpha)} \int_0^t v_\eta(\eta)d\eta \int_0^\eta \frac{v(\tau)d\tau}{(t-\tau)^\alpha} \equiv I \geq 0.
\]

Therefore, to prove the lemma, it suffices to show that the integral \(I\) is nonnegative. The integral \(I\) takes nonnegative values, since

\[
I = \frac{1}{\Gamma(1-\alpha)} \int_0^t v_\eta(\eta)d\eta \int_0^\eta \frac{v(\tau)d\tau}{(t-\tau)^\alpha} = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{v_\eta(\eta)d\eta}{(t-\eta)^\alpha} \int_0^\eta \frac{v(\tau)d\tau}{(t-\tau)^\alpha} =
\]

\[
= \frac{1}{2\Gamma(1-\alpha)} \int_0^t \left( \frac{v(\tau)d\tau}{(t-\tau)^\alpha} \right)^2 d\eta =
\]
\[ \alpha \frac{\Gamma(1-\alpha)}{2\Gamma(1-\alpha)} \int_0^t (t-\eta)^{\alpha-1} \left( \int_0^\eta v_x(\tau)d\tau \right)^2 d\eta \geq 0. \]

The proof of the lemma is complete.

We use the following notation:

\[ \|u\|_0^2 = \int_0^l u^2(x,t)dx, \quad D_{0t}^{-\alpha}u(x,t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{u(x,\tau)}{(t-\tau)^{1-\alpha}}d\tau \]

is the fractional Riemann-Liouville integral of order \( \alpha \).

**Theorem 1.** If \( k(x,t) \in C^{1,0}(\overline{Q}_T), q(x,t), f(x,t) \in C(\overline{Q}_T), k(x,t) \geq c_1 > 0 \) and \( q(x,t) \geq 0 \) everywhere on \( Q_T \), then the solution \( u(x,t) \) of problem (1)-(3) satisfies the a priori estimate

\[ \|u\|_0^2 + D_{0t}^{-\alpha}\|u_x\|_0^2 \leq M \left( D_{0t}^{-\alpha}\|f\|_0^2 + \|u_0(x)\|_0^2 \right). \] (5)

**Proof.** We multiply Eq. (1) by \( u(x,t) \) and integrate the resulting relation with respect to \( x \) from 0 to 1:

\[ \int_0^l u \partial_{\alpha}^0 t u dx - \int_0^l u(ku_x)_x dx + \int_0^l qu^2 dx = \int_0^l uf dx. \] (6)

Let us transform the terms occurring in identity (6)

\[ -\int_0^l u(ku_x)_x dx = \int_0^l ku^2 dx, \quad \left| \int_0^l uf dx \right| \leq \varepsilon \|u\|_0^2 + \frac{1}{4\varepsilon} \|f\|_0^2, \quad \varepsilon > 0; \] (7)

by virtue of inequality (4) we obtain

\[ \int_0^l u(x,t)\partial_{0t}^\alpha u(x,t)dx \geq \int_0^l \frac{1}{2}\partial_{0t}^\alpha u^2(x,t)dx = \frac{1}{2}\partial_{0t}^\alpha \|u\|_0^2. \] (8)

Identity (6), with regard of the above-performed transformations, implies the inequality

\[ \frac{1}{2}\partial_{0t}^\alpha \|u\|_0^2 + c_1 \|u_x\|_0^2 \leq \varepsilon \|u\|_0^2 + \frac{1}{4\varepsilon} \|f\|_0^2. \] (9)

By virtue of the inequality \( \|u\|_0^2 \leq (l^2/2)\|u_x\|_0^2 \) for \( \varepsilon = c_1/l^2 \) from (9), we obtain the inequality

\[ \partial_{0t}^\alpha \|u\|_0^2 + c_1 \|u_x\|_0^2 \leq \frac{l^2}{2c_1} \|f\|_0^2. \] (10)
By applying the fractional differentiation operator $D^{-\alpha}_{0t}$ to both sides of inequality (10), we obtain the estimate (5) with constant $M = \max\{t^2/(2c_1), 1\}/\min\{1, c_1\}$.

It follows from the a priori estimate (5) that the solution of problem (11)-(3) is unique and continuously depends on the input data.

1.2. Third Boundary Value Problem. In problem (11)-(3), we replace the boundary conditions (2) by the conditions

$$\begin{cases}
    k(0, t)u_x(0, t) = \beta_1(t)u(0, t) - \mu_1(t), \\
    -k(l, t)u_x(l, t) = \beta_2(t)u(l, t) - \mu_2(t),
\end{cases} \quad 0 \leq t \leq T. \quad (11)$$

In the rectangle $Q_T$, consider the third boundary value problem (11), (3), (11).

To obtain a priori estimates for solutions of various nonstationary problems, we use the well-known Gronwall-Bellman lemma [7, p. 152], whose generalization is provided by the following assertion.

**Lemma 2.** Let a nonnegative absolutely continuous function $y(t)$ satisfy the inequality

$$\partial^\alpha_{0t}y(t) \leq c_1 y(t) + c_2(t), \quad 0 < \alpha \leq 1, \quad (12)$$

for almost all $t \in [0, T]$, where $c_1 > 0$ and $c_2(t)$ is an integrable nonnegative function on $[0, T]$. Then

$$y(t) \leq y(0)E_\alpha(c_1 t^\alpha) + \Gamma(\alpha)E_{\alpha,\alpha}(c_1 t^\alpha)D^{-\alpha}_{0t}c_2(t), \quad (13)$$

where $E_\alpha(z) = \sum_{n=0}^{\infty} z^n/\Gamma(\alpha n + 1)$ and $E_{\alpha,\mu}(z) = \sum_{n=0}^{\infty} z^n/\Gamma(\alpha n + \mu)$ are the Mittag-Leffler functions.

**Proof.** Let $\partial^\alpha_{0t}y(t) - c_1 y(t) = g(t)$, then (e.g., see [5, p. 17])

$$y(t) = y(0)E_\alpha(c_1 t^\alpha) + \int_0^t (t - \tau)^{\alpha-1}E_{\alpha,\alpha}(c_1(t - \tau)^\alpha)g(\tau)d\tau \quad (14)$$

By virtue of the inequality $g(t) \leq c_2(t)$, the positivity of the Mittag-Leffler function $E_{\alpha,\alpha}(c_1(t - \tau)^\alpha)$ for given parameters, and the growth of the function $E_{\alpha,\alpha}(t)$, from (14), we obtain the inequality

$$y(t) \leq y(0)E_\alpha(c_1 t^\alpha) + \int_0^t (t - \tau)^{\alpha-1}E_{\alpha,\alpha}(c_1(t - \tau)^\alpha)c_2(\tau)d\tau \leq$$

$$\leq y(0)E_\alpha(c_1 t^\alpha) + \Gamma(\alpha)E_{\alpha,\alpha}(c_1 t^\alpha)D^{-\alpha}_{0t}c_2(t).$$

The proof of the lemma is complete.

**Theorem 2.** If, in addition to the assumptions of Theorem 1, $\beta_i(t), \mu_i(t) \in C[0, T], |\beta_i(t)| \leq \beta$, for all $t \in [0, T], i = 1, 2$, then the solution $u(x, t)$ of problem (11), (3), (11) admits the a priori estimate

$$\|u\|_0^2 + D_{0t}^{-\alpha}\|u_x\|_0^2 \leq M \left( D_{0t}^{-\alpha}\|f\|_0^2 + D_{0t}^{-\alpha}\mu_1^2(t) + D_{0t}^{-\alpha}\mu_2^2(t) + \|u_0(x)\|_0^2 \right). \quad (15)$$
Proof. Just as in the proof of Theorem 1, we multiply Eq. (1) by \( u(x, t) \) and integrate the resulting relation with respect to \( x \) from 0 to \( l \). By transforming the terms occurring in identity (7), we obtain relations (8) and (7) with \( \varepsilon = 1/2 \) and

\[
- \int_0^l u(ku_x)_x dx = \beta_1(t)u^2(0, t) + \beta_2(t)u^2(l, t) - \mu_1(t)u(0, t) - \mu_2(t)u(l, t) + \int_0^l ku_x^2 dx,
\]

Identity (7), with regard of the above-performed transformations, acquires the form

\[
\frac{1}{2} \partial_{\beta}^\alpha \|u\|_0^2 + c_1 \|u_x\|_0^2 \leq
\]

\[
\leq -\beta_1(t)u^2(0, t) - \beta_2(t)u^2(l, t) + \mu_1(t)u(0, t) + \mu_2(t)u(l, t) + \frac{1}{2} \|u\|_0^2 + \frac{1}{2} \|f\|_0^2. \tag{16}
\]

By virtue of the inequalities

\[
\mu_1(t)u(0, t) \leq \frac{1}{2} u^2(0, t) + \frac{1}{2} \mu_1^2(t), \quad \mu_2(t)u(l, t) \leq \frac{1}{2} u^2(l, t) + \frac{1}{2} \mu_2^2(t),
\]

\[
u^2(0, t), u^2(l, t) \leq \varepsilon \|u_x\|_0^2 + (1/\varepsilon + 1/l)\|u\|_0^2, \quad \varepsilon > 0,
\]

from (16) with \( \varepsilon = c_1/(4\beta + 2) \), we obtain the inequality

\[
\partial_{\beta}^\alpha \|u\|_0^2 + c_1 \|u_x\|_0^2 \leq M_1 \left( \|u\|_0^2 + \mu_1^2(t) + \mu_2^2(t) + \|f\|_0^2 \right). \tag{17}
\]

By applying the fractional differentiation operator \( D_{0t}^{-\alpha} \) to both sides of inequality (17), we obtain the inequality

\[
\|u\|_0^2 + D_{0t}^{-\alpha}\|u_x\|_0^2 \leq
\]

\[
\leq M_2 \left( D_{0t}^{-\alpha}\|u\|_0^2 + D_{0t}^{-\alpha}\mu_1^2(t) + D_{0t}^{-\alpha}\mu_2^2(t) + D_{0t}^{-\alpha}\|f\|_0^2 + \|u_0(x)\|_0^2 \right). \tag{18}
\]

By eliminating the second term from the left-hand side of inequality (18) and by using Lemma 2, where \( y(t) = D_{0t}^{-\alpha}\|u(x, t)\|_0^2 \), \( \partial_{0t}y(t) = \|u(x, t)\|_0^2 \) and \( y(0) = 0 \), we obtain the inequality

\[
D_{0t}^{-\alpha}\|u\|_0^2 \leq M_3 \left( D_{0t}^{-2\alpha}\mu_1^2(t) + D_{0t}^{-2\alpha}\mu_2^2(t) + D_{0t}^{-2\alpha}\|f\|_0^2 + \frac{t^\alpha}{\Gamma(\alpha + 1)} \|u_0(x)\|_0^2 \right), \tag{19}
\]

where \( M_3 = \Gamma(\alpha)E_{\alpha, \alpha}(M_2T^\alpha) \).

Since the inequality \( D_{0t}^{-2\alpha}h(t) \leq \left( t^\alpha \Gamma(\alpha)/\Gamma(2\alpha) \right) D_{0t}^{-\alpha}h(t) \) holds for every non-negative integrable function \( h(t) \) on \([0, T] \), it follows from (18) and (19) that the a priori estimate (13) is true.
2 BOUNDARY VALUE PROBLEMS FOR THE FRACTIONAL WAVE EQUATION

2.1. First Boundary Value Problem. In the rectangle \( \bar{Q}_T = \{(x,t) : 0 \leq x \leq l, 0 \leq t \leq T\} \), consider the first boundary value problem

\[
\partial_t^{1+\alpha} u = \frac{\partial}{\partial x} \left( k(x,t) \frac{\partial u}{\partial x} \right) - q(x,t) u + f(x,t), \quad 0 < x < l, \quad 0 < t \leq T, \tag{20}
\]

\[
u(0,t) = 0, \quad u(l,t) = 0, \quad 0 \leq t \leq T, \tag{21}
\]

\[
u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad 0 \leq x \leq l, \tag{22}
\]

where \( \partial_t^{1+\alpha} u(x,t) = \int_0^t u_{\tau\tau}(x,\tau)(t-\tau)^{-\alpha} d\tau/\Gamma(1-\alpha) \) is the Caputo fractional derivative of order \( 1 + \alpha, \; 0 < \alpha < 1 \).

Throughout the following, we assume that there exists a solution \( u(x,t) \in C^{2,2} (\bar{Q}) \) of problem (20)-(22).

Theorem 3. If \( k(x,t) \in C^{1,1}(\bar{Q}_T), \; q(x,t) \in C^{0,1}(\bar{Q}_T), \; f(x,t) \in C(\bar{Q}_T), \; 0 < c_1 \leq k(x,t) \leq c_2, \; 0 < m_1 \leq q(x,t) \leq m_2 \) and \( |k_t(x,t)|, |q_t(x,t)| \leq c_3 \) everywhere on \( \bar{Q}_T \), then the solution \( u(x,t) \) of problem (20)-(22) admits the a priori estimate

\[
D_0^{\alpha-1} \|u\|^2_0 + \|u\|^2_{W^1_2(0,l)} \leq M \left( \int_0^t \| f \|^2_0 d\tau + \| u_1(x) \|^2_0 + \| u_0(x) \|^2_{W^1_2(0,l)} \right), \tag{23}
\]

where \( \|u\|^2_{W^1_2(0,l)} = \|u\|^2_0 + \|u_x\|^2_0 \).

Proof. Let us multiply Eq. (20) by \( u_t(x,t) \) and integrate the resulting relation with respect to \( x \) from 0 to \( l, \)

\[
\int_0^l u_t \partial_t^{1+\alpha} u dx - \int_0^l u_t(ku_x)_x dx + \int_0^l q u u_t dx = \int_0^l u_t f dx. \tag{24}
\]

Let us transform the terms occurring in identity (24)

\[
\int_0^l u_t \partial_t^{1+\alpha} u dx = \int_0^l u_t \partial^\alpha_0 u_t dx \geq \frac{1}{2} \partial_t^\alpha_0 \|u_t\|^2_0, \tag{25}
\]

\[
- \int_0^l u_t(ku_x)_x dx = \frac{1}{2} \partial_t \int_0^l k u_x^2 dx - \frac{1}{2} \int_0^l k u_x^2 dx.
\]
\[
\int_0^t q u u_t dx = \frac{1}{2} \frac{\partial}{\partial t} \int_0^t q u^2 dx - \frac{1}{2} \int_0^t q u_t u^2 dx, \quad \left| \int_0^t u_t f dx \right| \leq \frac{1}{2} \|u_t\|_0^2 + \frac{1}{2} \|f\|_0^2. \quad (26)
\]

By taking into account the performed transformations, from identity \((24)\), we obtain the inequality

\[
\partial_0^\alpha \|u_t\|_0^2 + \frac{\partial}{\partial t} \int_0^t k u_t^2 dx + \frac{\partial}{\partial t} \int_0^t q u^2 dx \leq M_4 \left( \|u_t\|_0^2 + \|u\|_{W^2(0,t)}^2 + \|f\|_0^2 \right); \quad (27)
\]

by integrating this relation with respect to \(\tau\) from 0 to \(t\), we obtain the inequality

\[
D_0^{\alpha-1} \|u_t\|_0^2 + \|u\|_{W^2(0,t)}^2 \leq M_5 \left( \int_0^t (\|u_t\|_0^2 + \|u\|_{W^2(0,t)}^2) d\tau + \int_0^t \|f\|_0^2 d\tau + \|u_1(x)\|_0^2 + \|u_0(x)\|_{W^2(0,t)}^2 \right). \quad (28)
\]

By omitting the first term on the left-hand side in inequality \((28)\) and by using the Gronwall-Bellman lemma \([7\), p. 152\], where \(y(t) = \int_0^t \|u\|_{W^2(0,t)}^2 d\tau\), \(y'(t) = \|u\|_{W^2(0,t)}^2\) and \(y(0) = 0\), we obtain

\[
\int_0^t \|u\|_{W^2(0,t)}^2 d\tau \leq M_6 \left( \int_0^t (\|u_t\|_0^2 + \|f\|_0^2) d\tau + \|u_1(x)\|_0^2 + \|u_0(x)\|_{W^2(0,t)}^2 \right). \quad (29)
\]

Then, by omitting the second term on the left-hand side in inequality \((28)\) and by using inequality \((29)\), we obtain the inequality

\[
D_0^{\alpha-1} \|u_t\|_0^2 \leq M_7 \left( \int_0^t \|u_t\|_0^2 d\tau + \int_0^t \|f\|_0^2 d\tau + \|u_1(x)\|_0^2 + \|u_0(x)\|_{W^2(0,t)}^2 \right). \quad (30)
\]

By Lemma 2, where \(y(t) = \int_0^t \|u_t(x, \tau)\|_{L^2}^2 d\tau\), \(\partial_0^\alpha y(t) = D_0^{\alpha-1} \|u_t(x, t)\|_0^2\) and \(y(0) = 0\), from \((30)\) we obtain the inequality

\[
\int_0^t \|u_t\|_0^2 d\tau \leq M_8 \left( D_0^{\alpha-1} \|f\|_0^2 + \|u_1(x)\|_0^2 + \|u_0(x)\|_{W^2(0,t)}^2 \right). \quad (31)
\]
By virtue of the inequality $D_{0t}^{1-\alpha} \|f\|_0^2 \leq \left( t^\alpha / \Gamma(1 + \alpha) \right) \int_0^t \|f\|_0^2 d\tau$, it follows from inequalities (28), (29) and (31) that the a priori estimate (23) holds. The a priori estimate (23) implies that the solution of problem (20)-(22) exists and continuously depends on the input data.

2.2. Third Boundary Value Problem. In the rectangle $\bar{Q}_T$, consider the third boundary value problem (20), (22), (11).

Theorem 4. If, in addition to the assumptions of Theorem 3, $\beta_i(t), \mu_i(t) \in C^1[0, T], \beta_i(t) \geq \beta > 0$ and $|\beta_{it}(t)| \leq c_4$ for all $t \in [0, T], i = 1, 2$, then the solution $u(x, t)$ of problem (20), (22), (11) admits the a priori estimate

$$D_{0t}^{\alpha-1} \|u_t\|_0^2 + \|u\|_{W_2^2(0, l)}^2 \leq M \left( \int_0^t \left( \|f\|_0^2 + \mu_{1x}(\tau) + \mu_{2x}(\tau) \right) d\tau \right) +$$

$$+ M \left( \|\mu_1(t)\|_{C^2[0,T]} + \|\mu_2(t)\|_{C^2[0,T]} + \|u_1(x)\|_0^2 + \|u_0(x)\|_{W_2^2(0, l)}^2 \right). \quad (32)$$

Proof. Just as in Theorem 3, by multiplying Eq. (20) by $u_t(x, t)$ and by integrating the resulting relation with respect to $x$ from 0 to $l$, we obtain identity (24). By transforming the terms occurring in identity (24), we obtain relations (25) and (26) and

$$- \int_0^l u_t(ku_x)_x dx =$$

$$= \frac{1}{2} \frac{\partial}{\partial t} \left( \beta_1(t)u^2(0, t) + \beta_2(t)u(l, t) - 2\mu_1(t)u(0, t) - 2\mu_2(t)u(l, t) + \int_0^l ku_x^2 dx \right) -$$

$$- \frac{1}{2} \beta_{1x}(t)u^2(0, t) - \frac{1}{2} \beta_{2x}(t)u^2(l, t) + \mu_{1x}(t)u(0, t) + \mu_{2x}(t)u(l, t) - \frac{1}{2} \int_0^l k_{1x}u_x^2 dx.$$
\[ + \frac{\partial}{\partial t} \left( \beta_1(t)u^2(0, t) + \beta_2(t)u(l, t) - 2\mu_1(t)u(0, t) - 2\mu_2(t)u(l, t) \right) \leq M_9 \left( \|u_1\|_0^2 + \|u\|_{W^2_{1/2}(0, l)}^2 + \|f\|_0^2 + \mu_1^2(t) + \mu_2^2(t) \right). \]  

(33)

By integrating inequality (33) with respect to \( \tau \) from 0 to \( t \) and by taking into account the inequalities

\[ 2\mu_1(t)u(0, t) \leq \varepsilon u^2(0, t) + (1/\varepsilon)\mu_1^2(t), \quad 2\mu_2(t)u(l, t) \leq \varepsilon u^2(l, t) + (1/\varepsilon)\mu_2^2(t), \]

for \( \varepsilon = \beta, \)

\[ u^2_0(0), u^2_0(l) \leq (1 + 1/l)\|u_0(x)\|_{W^2_{1/2}(0, l)}^2, \]

we obtain

\[ D_{\alpha-1}^{\alpha} \|u_1\|_0^2 + \|u\|_{W^2_{1/2}(0, l)}^2 \leq \]

\[ \leq M_{10} \left( \int_0^t \left( \|u_\tau\|_0^2 + \|u\|_{W^2_{1/2}(0, l)}^2 \right) d\tau + \int_0^t \left( \|f\|_0^2 + \mu_1^2(\tau) + \mu_2^2(\tau) \right) d\tau \right) + \]

\[ + M_{10} \left( \|\mu_1\|_{C[0, T]}^2 + \|\mu_2\|_{C[0, T]}^2 + \|u_1(x)\|_0^2 + \|u_0(x)\|_{W^2_{1/2}(0, l)}^2 \right). \]  

(34)

By analogy with the first boundary value problem, by using first the Gronwall-Bellman lemma [7, p. 152] and then Lemma 2, from inequality (34), we obtain the a priori estimate (32).

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