TRANSLATING SURFACES OF THE NON-PARAMETRIC MEAN CURVATURE FLOW IN LORENTZ MANIFOLD $M^2 \times \mathbb{R}$

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Abstract. In this paper, for the Lorentz manifold $M^2 \times \mathbb{R}$, with $M^2$ a 2-dimensional complete surface with nonnegative Gaussian curvature, we investigate its space-like graphs over compact strictly convex domains in $M^2$, which are evolving by the non-parametric mean curvature flow with prescribed contact angle boundary condition, and show that solutions converge to ones moving only by translation.

Keywords: Translating surfaces, mean curvature flow, Lorentz manifolds.

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1. Introduction

In Riemannian (or pseudo-Riemannian) geometry, the mean curvature flow (MCF for short) is actually evolving a family of immersed submanifolds along their mean curvature vectors $\vec{H}$ with a speed $|\vec{H}|$. More precisely, let $X_0 : M^n \to N^{n+m}$ be an isometric immersion from an $n$-dimensional oriented Riemannian manifold $M$ to an $(n + m)$-dimensional Riemannian (or pseudo-Riemannian) manifold $N^{n+m}$ (or with a pseudo-Riemannian metric whose signature is $(p, n+m-p), n \leq p \leq n+m-1$). The MCF corresponds to a one-parameter family $X(\cdot, t) = X_t$ of immersions $X_t : M^n \to N^{n+m}$ whose images $M_t^n = X_t(M^n)$ satisfy

\[
\begin{align*}
\frac{d}{dt} X(x,t) &= \vec{H}, & \text{on } M^n \times [0, T) \\
X(x,0) &= X_0(x), & \text{on } M^n,
\end{align*}
\]

for some $T > 0$. The MCF attracts a lot of attention since Huisken’s significant work [8], where, by using the method of $L^p$ estimates, he proved that if $M^n$ is a compact strictly convex hypersurface in the Euclidean $(n + 1)$-space $\mathbb{R}^{n+1}$, the MCF (1.1) has a unique smooth solution on the finite time interval $[0, T_{\text{max}}]$ with $T_{\text{max}} < \infty$, and the evolving hypersurfaces $M_t^n$ contract to a single point as $t \to T_{\text{max}}$. Moreover, after an area-preserving rescaling, the rescaled hypersurfaces converge in $C^\infty$-topology to a round sphere having the same area as $M^n$. For the MCF (1.1), if

\[\vec{H} = -X^\perp,\]

then the submanifold $X_t : M^n \to N^{n+m}$ is called a self-shrinker, which is a self-similar solution to (1.1). Here $(\cdot)^\perp$ denotes the normal projection of a prescribed vector to the normal bundle of $M^n_t$ in $N^{n+m}$. Self-shrinking solutions are important in the study of type-I singularities of MCF. For instance, by proving the monotonicity formula, at a given type-I singularity of the MCF, Huisken [10] proved that the flow is asymptotically self-similar, which implies that in this situation the flow can be modeled by self-shrinking solutions. If there exists a constant unit vector $V$ such that

\[\vec{H} = V^\perp,\]

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then the submanifold $X_t : M^n \rightarrow N^{n+m}$ is called a translating soliton of the MCF (1.1). It is easy to see that the translating soliton gives an eternal solution $X_t = X_0 + tV$ to (1.1), which is called the translating solution. Translating solitons play an important role in the study of type-II singularities of the MCF. For instance, Angenent and Velazquez [2, 3] gave some examples of convergence which implies that type-II singularities of the MCF there are modeled by translating surfaces.

From the above brief introduction, we know that translating solutions of the MCF are special solutions to the flow equation and are worthy of be investigated for the purpose of understanding type-II singularities of the MCF. There exist some interesting results which we prefer to mention here. For instance, Shahriyari [12] proved that for the MCF, there are only three types of complete translating graphs in $\mathbb{R}^3$, i.e., entire graphs, graphs between two parallel planes, and graphs in one side of a plane. Moreover, in the last two types, graphs are asymptotic to planes next to their boundaries. For the case that $N^{n+m} = \mathbb{R}^{n+m}$, Xin [14] proved that any complete translating soliton has infinite volume and has Euclidean volume growth at least. Moreover, he showed that graph translating soliton hypersurfaces are weighted area-minimizing. Huisken [9] investigated graphs over bounded domains (with $C^2, \alpha$ boundary) in $\mathbb{R}^n (n \geq 2)$, which are evolving by the MCF with vertical contact angle boundary condition, and proved that the evolution exists for all the time and the evolving graphs converge to a constant function as time tends to infinity (i.e., $t \rightarrow \infty$). Altschuler and Wu [1] proved that graphs, defined over strictly convex compact domains in $\mathbb{R}^2$, evolved by the nonparametric MCF with prescribed contact angle (not necessary to be vertical), converge to translating surfaces as $t \rightarrow \infty$. Guan [7] investigated graphs over bounded domains in $\mathbb{R}^n$, which are evolving by the nonparametric MCF with prescribed contact angle, and proved that the flow exists for all the time. But an extra assumption about the prescribed contact angle should be added in order to get the asymptotical behavior of limiting solutions. Zhou [15] has improved Altschuler-Wu’s and Guan’s conclusions to the case of general product spaces $M^n \times \mathbb{R}$ with closed manifold $M^n$ having nonnegative Ricci curvature.

The purpose of this paper is to investigate the case of space-like graphs evolved by the nonparametric MCF with the prescribed contact angle boundary condition, and try to get interesting convergence conclusions.

Throughout this paper, $M^2$ denotes a 2-dimensional complete Riemannian manifold with a metric $\sigma$ and $\Omega$ is a strictly convex, bounded domain of $M^2$ with smooth boundary $\partial \Omega$. Let $\kappa > 0$ be the curvature function of $\partial \Omega$. Assume that a point on $\Omega$ is described by local coordinates $\{\omega^1, \omega^2\}$. Let $\partial_i, i = 1, 2$, be the corresponding coordinate vector fields and $\sigma_{ij} = \sigma(\partial_i, \partial_j)$. Similar to the basic introduction of geometry of graphs shown in [4, 5], we know that for the space-like graph $\mathcal{G} := \{(x, u(x, \cdot))|x \in \Omega\}$, defined over $\Omega \subset M^2$, in the Lorentz manifold $M^2 \times \mathbb{R}$ with the metric $g := \sigma_{ij} dw^i \otimes dw^j - ds \otimes ds$, tangent vectors are given by

$$\vec{e}_i = \partial_i + D_i u \partial_s, \quad i = 1, 2,$$

and the corresponding upward unit normal vector is given by

$$\vec{n} = \frac{1}{\sqrt{1 - |Du|^2}} \left( \partial_s + D^j u \partial_j \right),$$

where $D^i u = \sigma^{ij} D_i u$ with $D$ the covariant derivative operator on $M^2$. Denote by $\nabla$ the gradient operator on $\mathcal{G}$, and then the second fundamental form $h_{ij} dw^i \otimes dw^j$ of $\mathcal{G}$ is given by

$$h_{ij} = -\langle \nabla \vec{e}_i, \vec{e}_j \rangle = \frac{1}{\sqrt{1 - |Du|^2}} D_i D_j u.$$
Moreover, the scalar mean curvature of \( \mathcal{G} \) is
\[
H = \sum_{i=1}^{2} h_i = \frac{1}{\sqrt{1 - |Du|^2}} \left( \sum_{i,k=1}^{2} g^{ik} D_k D_i u \right) = \frac{\sum_{i,k=1}^{2} \left( \sigma^{ik} + \frac{D^i u D^k u}{1 - |Du|^2} \right) D_k D_i u}{\sqrt{1 - |Du|^2}}.
\]

Let \( T \) be the counterclockwise unit smooth tangent vector of \( \partial \Omega \) and \( N \) be the inward unit normal vector of \( \partial \Omega \). Then one can smoothly extend \( N, T \) to a thin neighborhood of the boundary \( \partial \Omega \) (see Subsection 2.1 for details).

In order to bring convenience to calculations in the sequel and state our main conclusion clearly, we use the following notations.

\[
v = \sqrt{1 - |Du|^2},
\]
\[
g_{ij} = \sigma_{ij} - D_i u D_j u,
\]
\[
g^{ij} = \sigma^{ij} + \frac{D^i u D^j u}{1 - |Du|^2},
\]
\[
u_t = \frac{\partial u}{\partial t}.
\]

For vectors, \( V, W \) or matrices \( A, B \), we will use the shorthand as follows
\[
\langle V, W \rangle_g = g^{ij} V_i W_j; \quad \langle V, W \rangle_\sigma = \sigma^{ij} V_i W_j; \quad \langle A, B \rangle_g = g^{ij} \sigma_{kl} A_{ikl} B_{jkl}.
\]

Define
\[
g^{TN} := g^{ij} T_i N_j, \quad g^{TT} := g^{ij} T_i T_j, \quad \text{and} \quad g^{NN} := g^{ij} N_i N_j \text{ on } \partial \Omega.
\]

For the second-order covariant derivatives of a prescribed function, we have the formula
\[
D_V D_W u = V^i W^j D_{ij} u + \langle D_V W, Du \rangle.
\]

For the space-like graphs \( \mathcal{G} \), consider the following initial value problem (IVP for short)
\[
\begin{cases}
  u_t = \left( \sigma^{ij} + \frac{D^i u D^j u}{1 - |Du|^2} \right) D_i D_j u, & \text{on } \Omega \times [0, T] \\
  D_N u = \phi(x) v, & \text{on } \partial \Omega \times [0, T] \\
  u(\cdot, 0) = u_0(\cdot), & \text{on } \Omega_0
\end{cases}
\]

where \( \Omega_t = \Omega \times \{ t \} \) is a slice in \( \Omega \times [0, T] \), \( \phi \in C^\infty(\partial \Omega) \), and \( u_0 \in C^\infty(\Omega) \). Clearly, the IVP (\( \sharp \)) describes the evolution of space-like graphs \( \mathcal{G} \) by the mean curvature vector with the specified contact angle, since by (1.2) the RHS of the first evolution equation in (\( \sharp \)) equals \( Hv \). For the IVP (\( \sharp \)), we can prove the following.

**Theorem 1.1.** If \( \Omega \) is a strictly convex bounded domain in \( M^2 \) with nonnegative Gaussian curvature, then, for solutions to IVP (\( \sharp \)), we have the followings:

1. there exists some constant \( c_1 := c_1(u_0, \kappa_0, \phi_0, \phi_1, \phi_2) > 0 \) so that \( |Du|^2 \leq c_1 < 1 \) on \( \Omega \times (0, \infty) \), thus \( u(x, t) \in C^\infty(\Omega \times [0, \infty)) \), where

\[
\kappa_0 := \min_{x \in \partial \Omega} \kappa(x), \quad \phi_0 := \min_{x \in \partial \Omega} \phi(x), \quad \phi_1 := \max_{x \in \partial \Omega} \phi(x), \quad \phi_2 := \max_{x \in \partial \Omega} |D_T \phi(x)|;
\]

2. \( u(x, t) \) converges as \( t \to \infty \) to a space-like surface \( u_\infty \) (unique up to translation) which moves at a constant speed \( c_3 \) given by (2.17);

3. if \( \int_{\partial \Omega} \phi = 0 \) then \( c_3 = 0 \), hence \( u_\infty \) is a maximal space-like surface in the Lorentz manifold \( M^2 \times \mathbb{R} \).

**Remark 1.1.** Clearly, if \( M^2 \equiv \mathbb{R}^2 \), Theorem 1.1 would give the existence of translating solutions to the space-like nonparametric MCF with the prescribed contact angle boundary condition in the Minkowski 3-space \( \mathbb{R}^{2,1} \). Besides, it is worth pointing out one thing here, that is, it might be a little surprise to readers that in order to get the existence of translating solutions, our
assumption here (only the strictly convex assumption for the bounded domain \( \Omega \)) is weaker than that in [1, Theorem 1.2].

The paper is organized as follows. The uniform estimates for the time derivative and the gradient of the solution to the IVP (\( \sharp \)) have been given in Section 2, which can be used to get the solvability of the BVP (\( \ast \)), the elliptic version of (\( \sharp \)), and the long-time existence of the IVP (\( \sharp \)). The existence of translating solutions to (\( \sharp \)) has been shown in Section 3.

2. Estimates

2.1. The boundary.

Let \( \{ \theta, r \} \), with \( r(x) \) the Riemannian distance function \( d(x, \partial \Omega) \) from \( x \) to the boundary \( \partial \Omega \), be the local coordinates for a thin neighborhood of \( \partial \Omega \) such that

\[
\begin{bmatrix}
\frac{\partial}{\partial r} & \frac{\partial}{\partial \theta}
\end{bmatrix}
\begin{bmatrix}
\frac{\partial}{\partial r} & \frac{\partial}{\partial r}
\end{bmatrix} = 0, \quad \begin{bmatrix}
\frac{\partial}{\partial r} & \frac{\partial}{\partial \theta}
\end{bmatrix}
\begin{bmatrix}
\frac{\partial}{\partial r} & \frac{\partial}{\partial r}
\end{bmatrix} = 1,
\]

and

\[
\left\langle \frac{\partial}{\partial r}, \frac{\partial}{\partial \theta} \right\rangle_{\partial \Omega} = \{N, T\}.
\]

Define a function \( \varphi \) such that \( |\varphi^{-1} \frac{\partial}{\partial \theta}|^2 = 1 \). Then one can get the extended normal and tangent vectors to be the orthonormal frame \( \left\{ \frac{\partial}{\partial r}, \varphi^{-1} \frac{\partial}{\partial \theta} \right\} \) of the thin neighborhood of \( \Omega \), which, with the abuse of notations, is also denoted by \( \{N, T\} \). That is, \( \left\{ \frac{\partial}{\partial r}, \varphi^{-1} \frac{\partial}{\partial \theta} \right\} = \{N, T\} \). By [1, Lemma 2.1], we have

**Lemma 2.1.** On \( \partial \Omega \), one has

(i) \( \nabla_T T = kN \), \( \nabla_T N = -kT \), \( \nabla_N T = \nabla_N N = 0 \);

(ii) for any \( f \in C^\infty(\bar{\Omega}) \), \( D_N D_T f = D_T D_N f + kD_T f \).

From the boundary condition of (\( \sharp \)), it is not hard to verify the following facts

(2.1) \[ |D_N u|^2 = \phi^2 v^2, \]

(2.2) \[ |D_T u|^2 = 1 - (1 + \phi^2)v^2. \]

Differentiating conditions (2.1) – (2.2) in the time and tangential direction, we know that all the derivatives of \( u \) on \( \partial \Omega \), except \( D_N D_N U \), can be given in terms of the first derivatives of \( u \). More precisely, we have

(2.3) \[ D_N u_t = -\phi \frac{D u D u}{\sqrt{1 - |D u|^2}}, \]

(2.4) \[ D_T D_N u = \phi D_T v + v D_T \phi, \]

(2.5) \[ D_N D_T u = \phi D_T v + v D_T \phi + k D_T u, \]

(2.6) \[ D_T D_T u = \frac{-v(1 + \phi^2)D_T v + v^2 \phi D_T \phi}{D_T u}. \]

Besides, elements of the inverse of the metric matrix of the space-like graphs in \( M^2 \times \mathbb{R} \) are given by

(2.7) \[ g_T^T = \frac{1 - (D_N u)^2}{v^2}, \]
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(2.8) $g^{NN} = 1 + \phi^2$,

(2.9) $g^{NT} = g^{TN} = \frac{D_N u D_T u}{v^2}$.

2.2. Existence of solutions.

The key point of the existence for small time and the uniqueness of solutions to IVP (♯) is to show that the evolution equation in (♯) is uniformly parabolic at $t = 0$, which can be assured by the assumption that the initial graphic surface over $\Omega$ is space-like. In fact, by the linearization theory (see [11]) and the inverse function theorem (see, e.g., [13]), together with the space-like graphic assumption, the short-time existence and the uniqueness of solutions to IVP (♯) can be obtained.

Assume that IVP (♯) has smooth solutions on the time interval $[0, T]$, which means all derivatives of $u$ have bounds on $[0, T]$. In the following, we establish a time independent priori estimate for the gradient of the solution (see Theorem 2.4), which leads to the preserving space-like property for the evolving graphic surfaces in $M^2 \times \mathbb{R}$, and then turn the quasilinear evolution equation into a uniformly parabolic equation. Furthermore, by the standard theory of the second-order parabolic PDE, the higher order regularity follows, which leads to the long-time existence of smooth solutions of the IVP (♯).

2.3. The time derivative estimate.

By the maximum principle of the second-order parabolic PDE, we have the following.

**Lemma 2.2.** $\sup_{\Omega \times [0,T]} |u_t|^2 = \sup_{\Omega} |u_t|^2$. That is, there exists some positive constant $c_0 = c_0(u_0) \in \mathbb{R}^+$ such that for any $(x,t) \in \Omega \times [0,T]$, we have

$$|u_t|^2(x,t) \leq c_0.$$  

**Proof.** We first show that the maximum of $u_t$ must occur on $(\partial \Omega \times [0,T]) \cup \Omega_0$. Let $(g^{ij})'$ be the differential of $g^{ij} = g^{ij}(x,u,Du) = g^{ij}(x,z,p)$ with respect to $p$. By a direct computation, we have

\[
\frac{\partial}{\partial t}|u_t|^2 = 2u_t \frac{\partial u_t}{\partial t}
\]

\[
= 2u_t \left( \frac{\partial g^{ij}}{\partial t} u_{ij} + g^{ij} \frac{\partial u_{ij}}{\partial t} \right)
\]

\[
= 2u_t \left( \frac{\partial g^{ij}}{\partial u^k} \frac{\partial u^k}{\partial t} u_{ij} + g^{ij} D_i D_j u_t \right)
\]

\[
= 2u_t \frac{\partial g^{ij}}{\partial u^k} \frac{\partial u^k}{\partial t} u_{ij} + g^{ij} (D_i D_j |u_t|^2 - 2 D_i u_t D_j u_t)
\]

\[
= 2u_t \frac{\partial g^{ij}}{\partial u^k} \frac{\partial u^k}{\partial t} u_{ij} + g^{ij} D_i D_j |u_t|^2 - 2 \langle D u_t, D u_t \rangle
\]

\[
= 2u_t \frac{\partial g^{ij}}{\partial u^k} \frac{\partial (\sigma^{ki} u_i)}{\partial t} u_{ij} + g^{ij} D_i D_j |u_t|^2 - 2 \langle D u_t, D u_t \rangle
\]

\[
= \frac{\partial g^{ij}}{\partial u^k} \sigma^{ki} u_{ij} D_i |u_t|^2 + g^{ij} D_i D_j |u_t|^2 - 2 \langle D u_t, D u_t \rangle
\]

\[
= D_i D_j u \left( (g^{ij})', \nabla |u_t|^2 \right)_\sigma + g^{ij} D_i D_j |u_t|^2 - 2 \langle D u_t, D u_t \rangle.
\]
where \( D_i D_j u \left((g^{ij}), \nabla |u_t|^2\right) = \sum_{i,j,k=1}^{2} D_i D_j u \left(\frac{\partial g^{ij}}{\partial \xi^k} \nabla_k |u_t|^2\right) \). The boundedness of all the coefficients of the above evolution equation in the bounded domain \( \bar{\Omega} \times [0, T]\) follows by the continuity, which implies that
\[
\sup_{\bar{\Omega} \times [0, T]} |u_t|^2 = \sup_{(\partial \Omega \times [0, T]) \cup \Omega_0} |u_t|^2
\]
by directly applying the weak maximum principle.

Next, we expel the possibility that the maximum occurs at \((\xi, \tau) \in \partial \Omega \times [0, T]\). Assume that \( \max |u_t|^2 = |u_t|^2(\xi, \tau) > 0 \), we have \((D_t u_t)(\xi, \tau) = 0\). By Lemma 2.2, it follows that
\[
(D_N u_t)(\xi, \tau) = -\phi \sqrt{1 - |D u|^2} \frac{D N u D N u + D t u D t u}{\sqrt{1 - |D u|^2}} (\xi, \tau)
\]
which implies \((1 + \phi^2)(D_N u_t)(\xi, \tau) = 0\), i.e., \((D_N u_t)(\xi, \tau) = 0\). Therefore, by the Hopf Lemma, \(\frac{\partial}{\partial t} |u_t|^2(\xi, \tau) \leq 0\), and then the conclusion of Lemma 2.2 follows. \(\square\)

2.4. The gradient estimate.
First, we need the evolution equation of \(|D u|^2\).

**Lemma 2.3.** We have the evolution equation of \(|D u|^2\) as follows
\[
\frac{\partial}{\partial t} |D u|^2 = \frac{D k |D u|^2 D_i |D u|^2}{v^2} g^{ik} + g^{ij} D_i D_j |D u|^2 - |D^2 u|^2 - |D |D u|^2|^2 - K |D u|^2,
\]
where \( K \) denotes the Gaussian curvature of \( M^2 \).

**Proof.** First, by direct computations, we have
\[
\frac{\partial}{\partial t} |D u|^2 = \frac{\partial}{\partial t} (u^m u_m) = 2 u^m (u_m) = 2 u^m (u_t) = (u_t)_m = (g^{ij} u_{ij})_m = (g^{ij})_m u_{ij} + g^{ij} (u_{ij})_m,
\]
and
\[
(g^{ij})_m = \left(\sigma^{ij} + \frac{u^i u^j}{1 - |D u|^2}\right)_m = \frac{(u^i)_m u^j}{1 - |D u|^2} + \frac{w^i (w^j)_m}{1 - |D u|^2} + \frac{2 u^k (u_k)_m u^i u^j}{(1 - |D u|^2)^2},
\]
and
\[
(|D u|^2)_{ij} = 2 \sigma^{mk} u_{kj} u_{mi} + 2 \sigma^{mk} u_{k} u_{mij}.
\]
Therefore, the evolution equation for the gradient is given as follows
\[
\frac{\partial}{\partial t} |D u|^2 = 2 u^m \left[\frac{(u^i)_m u^j}{1 - |D u|^2} + \frac{w^i (w^j)_m}{1 - |D u|^2} + \frac{2 u^k (u_k)_m u^i u^j}{(1 - |D u|^2)^2}\right] u_{ij} + 2 g^{ij} u^m (u_{ij})_m
\]
and
\[
\frac{\partial}{\partial t} |D u|^2 = 2 u^m \left[\frac{(\sigma^{ik} u_k)_m u^j}{v^2} + \frac{u^i (\sigma^{ik} u_k)_m}{v^2} + \frac{2 u^k (u_k)_m u^i u^j}{v^4}\right] u_{ij} + 2 g^{ij} u^m (u_{ij})_m
\]
Theorem 2.4. Let $u$ be a solution to the non-parametric mean curvature flow in $M^2 \times \mathbb{R}$.

Proof. We first show that the maximum of $|Du|^2$ must occur on $(\partial \Omega \times [0, T]) \cup \Omega_0$. Applying Lemma 2.3, we can get the following estimate:

$$
\frac{\partial}{\partial t} |Du|^2 \leq \frac{D_k |Du|^2 D_i |Du|^2}{v^2} g^{ik} + g^{ij} D_i D_j |Du|^2 - 2 |D^2 u|^2 - 2 \sigma^{ij} u^m u^l R_{limj}
$$

where $R_{limj}$, $1 \leq l, i, m, j \leq 2$, are the components of the curvature tensor on $M^2$. 

Then, we begin to estimate the gradient of $u$ by a clever use of the boundary algebra.

Theorem 2.4. Under the assumptions Theorem 1.7, there exists a positive constant $c_1 = c_1(u_0, \kappa_0, \phi_0, \phi_1, \phi_2)$ such that

$$
\sup_{\Omega \times [0, T]} |Du|^2 \leq c_1 < 1.
$$

Proof. We first show that the maximum of $|Du|^2$ must occur on $(\partial \Omega \times [0, T]) \cup \Omega_0$. Applying Lemma 2.3, we can get the following estimate:

$$
\frac{\partial}{\partial t} |Du|^2 \leq \frac{D_k |Du|^2 D_i |Du|^2}{v^2} g^{ik} + g^{ij} D_i D_j |Du|^2 - 2 |D^2 u|^2 - 2 \sigma^{ij} u^m u^l R_{limj}
$$
since $M^2$ has nonnegative Gaussian curvature. Applying the weak maximum principle to the above evolution inequality, we have

$$\sup_{\overline{\Omega} \times [0,T]} |Du|^2 = \sup_{(\partial \Omega \times [0,T]) \cup \Omega_0} |Du|^2.$$ 

If the maximum of $|Du|^2$ occurs at $\Omega_0$, then $\sup_{\overline{\Omega} \times [0,T]} |Du|^2 \leq \sup_{\Omega_0} |Du|^2 < 1$, where the last inequality holds since $\{(x, u(x, 0)) | x \in M^2\}$ is a space-like graph of $M^2 \times \mathbb{R}$. Now, assume that the maximum of $|Du|^2$ occurs at $(\xi, \tau) \in \partial \Omega \times [0,T]$. We divide the argument into two cases:

Case (1). If $|DTu|(\xi, \tau) \leq \frac{1}{2}$, then applying the fact $(1 + \phi^2)v^2 = 1 - |DTu|^2$, we have

$$|Du|^2(\xi, \tau) \leq 1 - \frac{3}{4(1 + \phi^2)} < 1,$$

which establishes an upper bound for $|Du|^2$ on $\overline{\Omega} \times [0,T]$ already.

Case (2). If $|DTu|(\xi, \tau) > \frac{1}{2}$, then at $(\xi, \tau)$, one has

$$DN|Du|^2(\xi, \tau) \leq 0,$$

$$DT|Du|^2(\xi, \tau) = 0 = DTv(\xi, \tau).$$

Therefore, at $(\xi, \tau)$, (2.4)-(2.6) can be simplified as follows

(2.10) \hspace{1cm} DTDNu = vDT\phi,

(2.11) \hspace{1cm} DNDTu = vDT\phi + \kappa DTu,

(2.12) \hspace{1cm} DTDT(u) = \frac{-v^2\phi DT\phi}{DTu}.

Our target is to show the following

(2.13) \hspace{1cm} (DNu)(DN(DNu(\xi, \tau))) + (DTu)(DNDTU)(\xi, \tau) \leq 0.

In order to get this, we need to consider the evolution equation of $u$. In fact, by using the assumption that $|Du|^2$ gets its maximum at $(\xi, \tau)$, (2.1)-(2.3), and (2.10)-(2.12), we get that at $(\xi, \tau)$, the following identity

$$u_t = g^{TT} DTDTu + g^{TN} DTDNu + g^{NT} DNDTu + g^{NN} DN^2 Nu$$

$$- g^{TT}(DTT, Du) - g^{TN}(DTN, Du) - g^{NT}(DTN, Du) - g^{NN}(DN^2 N, Du)$$

$$= \frac{1 - |DNu|^2}{v^2} \cdot \left( -vDtv(1 + \phi^2) - v^2\phi DT\phi \right) + \frac{DNuDTu}{v^2}(DT\phi \cdot v + \phi \cdot DTv)$$

$$+ \frac{DNuDTu}{v^2}(DT\phi \cdot v + \phi \cdot DTv + \kappa DTu) + (1 + \phi^2)DN^2 Nu - \frac{1 - |DNu|^2}{v^2}(\kappa N, Du)$$

$$- \frac{DNuDTu}{v^2}(\kappa T, Du)$$

$$= \frac{1 - \phi^2}{v^2} \left[ -v \cdot DTv(1 + \phi^2) - v^2\phi DT\phi - \kappa \phi v \right] + (1 + \phi^2)DN^2 Nu$$

$$+ \frac{\phi v DTu}{v^2}(DT\phi \cdot v + \phi \cdot DTv + \kappa \cdot DTu)$$

$$= \frac{1 - \phi^2}{v^2} \left( \frac{-v^2\phi DT\phi}{DTu} - \kappa \phi v \right) + (1 + \phi^2)DN^2 Nu + \frac{\phi v DTu}{v^2}(DT\phi \cdot v + \kappa \cdot DTu)
holds. That is,
\[
(1 + \phi^2)D_N D_N u = u_t - \frac{\phi^2 v^2}{v^2} \left( -v^2 \frac{D_T \phi}{D_T u} - \kappa \phi v \right) - 2 \frac{\phi v D_T u}{v^2} (D_T \phi \cdot v + \kappa \cdot D_T u)
\]
\[
= u_t + \frac{\phi D_T \phi}{D_T u} (1 - \phi^2 v^2 - 2 |D_T u|^2) + \frac{(1 - \phi^2 v^2) \kappa \phi}{v} - \frac{2 \kappa \phi |D_T u|^2}{v}
\]
\[
= u_t + \frac{\phi D_T \phi}{D_T u} (v^2 - |D_T u|^2) - \frac{2 \kappa \phi |D_T u|^2}{v} + \frac{\kappa \phi (1 - \phi^2 v^2)}{v}.
\]

Substituting the above identity into (2.13), together with (2.11), yields
\[
\phi v \left[ u_t + \frac{\phi D_T \phi}{D_T u} (v^2 - |D_T u|^2) - \frac{2 \kappa \phi u^2}{v} + \frac{\kappa \phi (1 - \phi^2 v^2)}{v} \right] + (1 + \phi^2)D_T u (v D_T \phi + \kappa u_T) \leq 0,
\]
which is equivalent with
\[
\phi v u_t + \frac{\phi^2 v D_T \phi}{D_T u} (v^2 - |D_T u|^2) - 2 \kappa \phi^2 |D_T u|^2 + \kappa \phi^2 (1 - \phi^2 v^2) + (1 + \phi^2)D_T u (v D_T \phi + \kappa D_T u) \leq 0.
\]

(2.14)

It is easy to verify that
\[
-2 \kappa \phi^2 |D_T u|^2 + \kappa \phi^2 (1 - \phi^2 v^2) + (1 + \phi^2) \kappa |D_T u|^2 = \kappa (1 - v^2),
\]
and
\[
\frac{\phi^2 v D_T \phi}{D_T u} (v^2 - |D_T u|^2) + (1 + \phi^2)D_T u D_T \phi v = \frac{v D_T \phi}{D_T u} (1 - v^2).
\]

Using the above two identities, (2.14) can be simplified as follows
\[
\phi v u_t + \kappa (1 - v^2) + \frac{v D_T \phi}{D_T u} (1 - v^2) \leq 0.
\]

Note that
\[
1 - v^2 = \frac{\phi^2}{1 + \phi^2} + \frac{|D_T u|^2}{1 + \phi^2}.
\]

Hence,
\[
\phi v u_t + \kappa (1 - v^2) + \frac{v D_T \phi}{D_T u} \frac{\phi^2}{1 + \phi^2} + \frac{v D_T \phi D_T u}{1 + \phi^2} \leq 0.
\]

In view of Lemma 2.2, $|D_T u| (\xi, \tau) > \frac{1}{2}$ and $\Omega$ is strictly convex, we can find $\kappa_0$ such that
\[
0 < \kappa_0 (1 - v^2) \leq \kappa (1 - v^2) \leq c_2 v,
\]
where $c_2$ is a positive constant depending on $c_0, \phi_0, \phi_1, \phi_2$. Therefore
\[
|D u|^2 \leq c_1 := \frac{\sqrt{c_0^4 + 4 c_1^2 \kappa_0^2} - c_2^2}{2 \kappa_0^2} < 1.
\]

Our proof is finished. \qed
2.5. Boundary value problems.

Applying the above gradient estimate, Theorem 2.4 one can solve the following boundary value problem (BVP for short)
\[
(*) \quad \begin{cases}
  \sigma^{ij} + \frac{D_i u D_j u}{1 - |Du|^2} D_i D_j u = c_3 & \text{on } \Omega \\
  D_N u = \phi(x) v & \text{on } \partial\Omega,
\end{cases}
\]
where \(c_3\) is a constant determined uniquely by (2.15) below. Clearly, the BVP (*) can be seen as the elliptic version of IVP (*).

In fact, since the LHS of the first equation in BVP (*) can be written as
\[
\sqrt{1 - |Du|^2} D_i \left( \frac{D_i u}{\sqrt{1 - |Du|^2}} \right),
\]
integrating by parts one can easily get
\[
(2.15) \quad c_3 = - \frac{\int_{\partial\Omega} \phi}{\int_{\Omega} (1 - |Du|^2)^{-\frac{1}{2}}},
\]
where, for convenience, we have dropped volume elements of the domain \(\Omega\) and its boundary \(\partial\Omega\) simultaneously.

One method for solving BVP (*) is to consider the solvability the following BVP.
\[
(**) \quad \begin{cases}
  \sigma^{ij} + \frac{D^i u D^j u}{1 - |Du|^2} D_i D_j u = \varepsilon u_\varepsilon & \text{on } \Omega \\
  D_N u_\varepsilon = \phi(x) \sqrt{1 - |Du_\varepsilon|^2} & \text{on } \partial\Omega.
\end{cases}
\]

**Theorem 2.5.** The BVP (*) has a unique, smooth solution.

**Proof.** We will use an argument similar to those in [11]. For BVP (**), it is known that it has solutions for \(\varepsilon > 0\). Therefore, one can replace \(u_t\) with \(\varepsilon u_\varepsilon\) in the gradient estimate of Theorem 2.4 and get a conclusion that a limit solution to (**) exists as \(\varepsilon \to 0\), provided there exists some \(c_0\), independent of \(\varepsilon\), such that \(|\varepsilon u_\varepsilon|^2 \leq c_0\).

Let \(\psi\) be a smooth function defined on \(\Omega\) satisfying \(D_N \psi < \phi \sqrt{1 - |D\psi|^2}\) on \(\partial\Omega\). This kind of smooth functions can always be constructed. For instance, let \(d\) be the distance function to \(\partial\Omega\) and \(A\) be a constant such that \(\frac{1}{\sqrt{1 - A^2}} < \phi\) on \(\partial\Omega\). It is easy to check that a function \(\psi\) defined to be \(Ad\) near \(\partial\Omega\) and extended to be a smooth function on all of \(\Omega\) would satisfy the requirements that \(\psi \in C^\infty(\partial\Omega)\), \(D_N \psi < \phi \sqrt{1 - |D\psi|^2}\). Assume that \(\psi - u_\varepsilon\) attains its minimum at some point \(\xi \in \Omega\).

If \(\xi \in \partial\Omega\), then \(D_T \psi(\xi) = D_T u_\varepsilon(\xi)\) and \(D_N \psi(\xi) \geq D_N u_\varepsilon(\xi)\). One can get
\[
\phi(\xi) > \frac{D_N \psi}{\sqrt{1 - |D_T \psi|^2 - |D_N \psi|^2}}(\xi) \geq \frac{D_N u_\varepsilon}{\sqrt{1 - |D_T u_\varepsilon|^2 - |D_N u_\varepsilon|^2}}(\xi) = \phi(\xi),
\]
since the function \(\frac{q}{\sqrt{1 - b^2 - q^2}}\) with \(b\) a fixed constant is monotone nondecreasing in \(q\). This is a contradiction.

Therefore, \(\xi \in \Omega\), \(D_T \psi(\xi) = D_T u_\varepsilon(\xi)\) and \(D^2 \psi(\xi) \geq D^2 u_\varepsilon(\xi)\). There exists a constant \(c_4 = c_4(\psi)\) such that
\[
c_4 \geq \left( \frac{\sigma^{ij} + D^i \psi(\xi) D^j \psi(\xi)}{1 - |D\psi(\xi)|^2} \right) D_i D_j \psi(\xi) \geq \left( \frac{D^i u_\varepsilon(\xi) D^j u_\varepsilon(\xi)}{1 - |D u_\varepsilon(\xi)|^2} \right) D_i D_j u_\varepsilon(\xi) = \varepsilon u_\varepsilon(\xi).
\]
Together with the fact that \(\varepsilon \psi(z) - \varepsilon u_\varepsilon(z) \geq \varepsilon \psi(\xi) - \varepsilon u_\varepsilon(\xi)\) for any \(z \in \Omega\), we have
\[
\varepsilon u_\varepsilon(z) \leq \varepsilon \psi(z) - \varepsilon \psi(\xi) + \varepsilon u_\varepsilon(\xi) \leq \varepsilon \psi(z) - \varepsilon \psi(\xi) + c_4.
\]
for any \( z \in \Omega \). By a similar barrier argument, one can get a lower bound for \( \varepsilon u_\varepsilon \). As in [11], \( |Du_\varepsilon|^2 \leq c_1 \) implies \( |D(\varepsilon u_\varepsilon)|^2 \to 0 \) as \( \varepsilon \to 0 \), and then we have \( \varepsilon u_\varepsilon \to c_3 \). This gives the existence of solutions to BVP (*).

Now, in what follows, we would like to show the uniqueness of the solutions. Assume that the BVP (*) has two solutions \( u_1, u_2 \) with constants \( c_5, c_6 \) on the RHS of (*) and \( c_5 < c_6 \).

Without loss of generality, assume \( u_1 \geq u_2 \). By the linearization process, one easily knows that \( U := u_1 - u_2 \) satisfies a linear elliptic differential inequality \( \mathcal{L}(U) < 0 \). By the maximum principle, the minimum of \( U \) must be achieved at some point \( \zeta \in \partial \Omega \), which implies that \( |Du_1|^2(\zeta) = |Du_2|^2(\zeta) = a^2 \) for some \( a \in \mathbb{R}^+ \). Since

\[
\frac{D_Nu_1}{\sqrt{1 - a^2 - |D_Nu_1|^2}}(\zeta) = \frac{D_Nu_2}{\sqrt{1 - a^2 - |D_Nu_2|^2}}(\zeta),
\]

it follows that \( D_Nu_1(\zeta) = D_Nu_2(\zeta) \) at \( \zeta \in \partial \Omega \) by using the fact that the function \( \sqrt{1 - a^2 - q^2} \) is monotone nondecreasing in \( q \). However, this is contradict with the Hopf boundary point lemma. So, \( c_5 \geq c_6 \). Reversing the roles of \( c_5 \) and \( c_6 \), one has \( c_5 \leq c_6 \). Therefore, one can get \( c_5 = c_6 \).

By a similar argument, one can also obtain \( u_1 = u_2 \). This gives the uniqueness of solutions to BVP (*).

Our proof is finished. \( \Box \)

**Remark 2.1.** Clearly, if \( u = u(x) \) is a solution to the BVP (*), then \( \bar{u}(x, t) = u(x) + c_3 t \) is a solution to the IVP (*). That is to say \( \bar{u} \) is a translating solution with constant speed \( |c_3| \).

### 3. Convergence

Now, we can show the following uniqueness conclusion of limit solutions to the IVP (*) (up to translation) by applying the strong maximum principle of the second-order linear parabolic PDE.

**Lemma 3.1.** Let \( \tilde{u}_1 \) and \( \tilde{u}_2 \) be any two solution to IVP (*) and let \( \tilde{U} = \tilde{u}_1 - \tilde{u}_2 \). Then \( \tilde{U} \) becomes a constant function as \( t \to \infty \). In particular, if \( u \) is a solution to the BVP (*), then all limit solutions to the IVP (*) are of the form \( u + c_3 t \).

**Proof.** By the linearization process, one can easily get that \( \tilde{U} \) satisfies the following linear parabolic equation

\[
\frac{\partial}{\partial t} \tilde{U} = \tilde{g}^{ij} D_i D_j \tilde{U} + \tilde{b}^i D_i \tilde{U}, \quad \text{on} \quad \Omega \times [0, T]
\]

with the boundary condition

\[
0 = \left( \frac{Du_1}{\sqrt{1 - |Du_1|^2}} - \frac{Du_2}{\sqrt{1 - |Du_2|^2}}, N \right) := \tilde{c}^{ij} N_j D_i \tilde{U}, \quad \text{on} \quad \partial \Omega \times [0, T],
\]

where

\[
\tilde{g}^{ij} = \int_0^1 \tilde{g}^{ij}(\theta Du_1 + (1 - \theta) Du_2) \, d\theta,
\]

and \( \tilde{b}^i, \tilde{c}^{ij} \) are similarly determined (see, e.g., [2]), \( N_j \)'s are components of the unit normal vector \( N \). Note that \( \tilde{c}^{ij} \) is a positive definite matrix. By the strong maximum principle, we know that the oscillation function \( \text{osc}(t) := \max \tilde{U}(\cdot, t) - \min \tilde{U}(\cdot, t) \geq 0 \) is strictly decreasing in \( t \) unless \( \tilde{U} \) is constant.
The long-time existence of solutions to the IVP (2) has been explained in Subsection 2.3 provided the time-independent priori gradient estimate can be obtained. Therefore, we have $T = \infty$ here.

We claim that $|\tilde{U}|$ must be uniformly bounded on $\overline{\Omega} \times [0, \infty)$. By the maximum principle, we know that the minimum of $\tilde{U}$ should be achieved at some point $(\xi, t_0) \in (\partial \Omega \times [0, \infty)) \cup \Omega_0$. If $(\xi, t_0) \in \partial \Omega \times [0, \infty)$, then $D_T \tilde{U}(\xi, t_0) = 0$ and $D_N \tilde{U}(\xi, t_0) \geq 0$. That is, $D_T u_1(\xi, t_0) = D_T u_1(\xi, t_0)$ and $D_N u_1(\xi, t_0) \geq D_N u_1(\xi, t_0)$. Therefore, one has

$$\frac{D_N u_1}{\sqrt{1 - |D_T u_1|^2 - |D_N u_1|^2}}(\xi, t_0) > \frac{D_N u_2}{\sqrt{1 - |D_T u_2|^2 - |D_N u_2|^2}}(\xi, t_0),$$

since the function $\frac{q}{\sqrt{1 - b^2 - q^2}}$ with $b$ a fixed constant is strictly increasing in $q$. However, this is contradict with the boundary condition

$$\frac{D_N u_1}{\sqrt{1 - |D_T u_1|^2 - |D_N u_1|^2}}(\xi, t_0) = \frac{D_N u_2}{\sqrt{1 - |D_T u_2|^2 - |D_N u_2|^2}}(\xi, t_0) = \phi(\xi).$$

Therefore, $(\xi, t_0) \in \Omega_0$, i.e., $\xi \in \Omega$ and $t_0 = 0$. This is to say that $\tilde{U}$ attains its minimum on $\Omega_0$. The same situation happens to the maximum of $U$. Hence, we have $|\tilde{U}| = |u_1 - u_2| \leq c_7(u_0)$ for some nonnegative constant $c_7(u_0)$ only depending on $u_0$.

Since $|\tilde{U}|$ is uniformly bounded on $\overline{\Omega} \times [0, \infty)$, we can take a sequence $\{t_n\}$, $n \in \mathbb{Z}^+$ with $\mathbb{Z}^+$ the set of all positive integers, such that the limit $\lim_{t \to \infty} \tilde{U}(\cdot, t_n)$ exists. If $\lim_{t \to \infty} \tilde{U}(\cdot, t)$ were not a constant function, then a limit of $\tilde{U}_n(\cdot, t) := \tilde{U}(\cdot, t + t_n)$ as $t_n \to \infty$ would yield a solution on $\Omega \times [0, \infty)$ which would not be constant but on which $\text{osc}(t)$ would be constant. But this is contradict with the strict monotonicity of $\text{osc}(t)$. Therefore, $\lim_{t \to \infty} \tilde{U}(\cdot, t)$ should be a constant function, which implies the first assertion.

By Remark 2.1, we know that $u + c_3 t$ is a solution to the IVP (2) provided $u$ is a solution to the BVP (\#). Hence, for any solution $\omega$ of (2), by the first assertion, one has $\omega - (u + c_3 t)$ tends to a constant as $t \to \infty$, which implies that $\omega$ tends to $u + c_3 t$ for a different $t$. This completes the proof of the second assertion.

By applying Lemma 3.1 directly, we have the following.

**Corollary 3.2.** For a solution $u = u(x, t)$ of the IVP (2), there exists some positive constant $c_8 \in \mathbb{R}^+$ such that $|u(x, t) - c_3 t| \leq c_8$.

Now, we show that if $\int_{\partial \Omega} \phi = 0$, the limiting surface $u_\infty := \lim_{t \to \infty} u(\cdot, t)$, with $u(\cdot, t)$ the solution to IVP (2), should be maximal space-like.

**Lemma 3.3.** If $\int_{\partial \Omega} \phi = 0$, then $c_3 = 0$ and $\lim_{t \to \infty} u_t = 0$. That is, solutions $u(\cdot, t)$ to the IVP (2) converge to a maximal space-like surface $u_\infty$ in the Lorentz manifold $M^2 \times \mathbb{R}$.

**Proof.** The first assertion follows directly from (2.13). By a direct calculation, we have

$$\frac{d}{dt} \int_{\Omega} v = - \int_{\Omega} \frac{D_i u_1 D_i u}{v} = \int_{\Omega} u_t^2 \frac{v}{v} + \int_{\partial \Omega} u_t \phi,$$

which implies

$$\frac{d}{dt} \left( \int_{\Omega} v - \int_{\partial \Omega} \phi \right) = \int_{\Omega} u_t^2 \frac{v}{v}.$$
Applying Theorem 2.4 and Corollary 3.2, we know that there exists a positive constant $c_9 \in \mathbb{R}^+$ depending on $c_1, c_8$ such that

$$\int_{\Omega} \frac{u^2}{v} \leq c_9,$$

which implies the second assertion of Lemma 3.3.

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\section*{References}

[1] S.-J. Altschuler and L.-F. Wu, \textit{Translating surfaces of the non-parametric mean curvature flow with prescribed contact angle}, Calc. Var. Partial Differential Equations 2 (1994) 101–111.

[2] S.-B. Angenent and J.-J.-L. Velazquez, \textit{Asymptotic shape of cusp singularities in curve shortening}, Duke Math. J. 77 (1995) 71–110.

[3] S.-B. Angenent and J.-J.-L. Velazquez, \textit{Degenerate neckpinches in mean curvature flow}, J. Reine Angew. Math. 482 (1997) 15–66.

[4] L. Chen and J. Mao, \textit{Non-parametric inverse curvature flows in the AdS-Schwarzschild manifold}, J. Geom. Anal. 28 (2018) 921–949.

[5] L. Chen, J. Mao, N. Xiang and C. Xu, \textit{Inverse mean curvature flow inside a cone in warped products}, submitted and available online at arXiv:1705.04865v3.

[6] D. Gilbarg and N. Trudinger, \textit{Elliptic Partial Differential Equations}, Grundlehren Math. Wiss., vol. 224, Springer, Berlin, Heidelberg, New York, 1983.

[7] B. Guan, \textit{Mean curvature motion of nonparametric hypersurfaces with contact angle condition}, Elliptic and parabolic method in geometry, Peters, A.K., Wellesley (MA), 1996, pp. 47–56.

[8] G. Huisken, \textit{Flow by mean curvature of convex surfaces into spheres}, J. Differential Geom. 20 (1984) 237–266.

[9] G. Huisken, \textit{Non-parametric mean curvature evolution with boundary conditions}, J. Differential Geom. 31 (1990) 285–299.

[10] G. Huisken, \textit{Asymptotic behavior for singularities of the mean curvature flow}, J. Differential Geom. 31 (1990) 285–299.

[11] P. Lions, N. Trudinger and J. Urbas, \textit{The Neumann problem for equations of Monge-Ampère type}, Commun. Pure Appl. Math. 39 (1986) 539–563.

[12] L. Shahriyari, \textit{Translating graphs by mean curvature flow}, Geom. Dedicata 175 (2015) 57–64.

[13] A. Stone, \textit{The mean curvature evolution of graphs}, Honour’s Thesis, ANU 31, 1989.

[14] Y.-L. Xin, \textit{Translating solitons of the mean curvature flow}, Calc. Var. Partial Differential Equations 54 (2015) 1995–2016.

[15] H.-Y. Zhou, \textit{Nonparametric mean curvature type flows of graphs with contact angle conditions}, available online at arXiv:1702.02449v1.

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