On Fekete-Szegö problems for a certain subclass defined by $q$- analogue of Ruscheweyh operator

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Abstract. The main objective of this work is to establish the Fekete-Szegö Inequality for a certain class $M_{q,b,λ}^β(φ)$ of analytical functions that is associated with $q$- analogue of Ruscheweyh Operator. Some of our results generalize the related work of some authors.

1. Introduction
We begin by letting $U = \{z \in \mathbb{C} : |z| < 1\}$ be the open unit disk of the complex plane and $A$ denote the class of functions of the form

\[ f(z) = z + \sum_{k=2}^{∞} a_k z^k, \]  

which are analytic in $U$ and satisfy the following usual normalization conditions $f(0) = f'(0) - 1 = 0$.

Also, let $S$ be the class of functions $f \in A$ which are univalent in $U$. Given two analytic functions $f$ and $g$, the subordination between them is written as $f \prec g$ or $f(z) \prec g(z)$, ($z \in U$). In addition, we say $f(z)$ is subordinate to $g(z)$ if there is a Schwarz function $w$ with $w(z) = 0$, $|w(z)| < 1$, ($z \in U$) such that $f(z) = g(w(z))$ for all $z \in U$. Furthermore, if $g(z)$ is univalent in $U$, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(U) \subseteq g(U)$.

In [6], [7], for a function $f \in A$ and $0 < q < 1$ Jackson defined the $q$-derivative operator $D_q$ as follows:

\[ D_q f(z) = \frac{f(qz) - f(z)}{(q - 1)z}, \quad (z \neq 0) \]  

and $D_q f(0) = f'(0)$ and $D_q^2 f(z) = D_q(D_q f(z))$. In case $f(z) = z^k$ for $k$ is a positive integer, the $q$-derivative of $f(z)$ is given by

\[ D_q z^k = \frac{z^k - (qz)^k}{z(1 - q)} = [k]_q z^{k-1}, \]

where $[k]_q$ defined by

\[ [k]_q = \frac{1 - q^k}{1 - q}. \]
As \( q \to 1^- \) and \( k \in \mathbb{N} \), \([k]_q \to k\).
From (1.1) and (1.2) we get that
\[
D_q f(z) = 1 + \sum_{k=2}^{\infty} [k]_q a_k z^{k-1}.
\]
As a right inverse, Jackson [6] introduced the q-integral
\[
\int_0^z f(t) d_q t = z (1-q) \sum_{k=0}^{\infty} q^k f(z^q).
\]
The authors in [1] defined the q-analogue of Ruscheweyh Operator \( R_\lambda^f \) by
\[
R_\lambda^f(z) = z + \sum_{k=2}^{\infty} \frac{[k + \lambda - 1]_q!}{(\lambda)_q! [k - 1]_q!} a_k z^k,
\]
where \([k]_q!\) defined by :
\[
[k]_q! = \begin{cases} 
[k]_q[k-1]_q\ldots[1]_q, & k = 1, 2, \ldots; \\
1, & k = 0.
\end{cases}
\]
All the details about q-calculus used in this paper can be found in [3] and [4].
Also, as \( q \to 1^- \) we have
\[
\lim_{q \to 1^-} R_\lambda^f(z) = z + \lim_{q \to 1^-} \left[ \sum_{k=2}^{\infty} \frac{[k + \lambda - 1]_q!}{(\lambda)_q! [k - 1]_q!} a_k z^k \right]
= z + \sum_{k=2}^{\infty} \frac{(k + \lambda - 1)!}{(\lambda)!(k - 1)!} a_k z^k
= R_\lambda^f(z),
\]
where \( R_\lambda^f(z) \) is Ruscheweyh differential operator which was defined in [13] and has been studied by several authors, for example [9] and [15].
The following class of analytic functions can be defined as a result of full utilization of subordination and q-derivative principle.

**Definition 1.1** For \( \phi \in \mathcal{P} \), \( b \in \mathbb{C}^* = \mathbb{C} \setminus \{0\} \), \( \beta \geq 0 \) and \( \lambda > -1 \). The function \( f \) is said to be in the class \( \mathcal{M}_{q,b,\lambda}^\beta(\phi) \) if it satisfies the condition :
\[
1 + \frac{1}{b} \left[ \frac{z D_q (R_\lambda^f(z)) + \beta z^2 D_q (D_q (R_\lambda^f(z)))}{R_\lambda^f(z)} - 1 \right] < \phi(z). \tag{1.3}
\]

**Remark 1.1** It can be seen that, by specializing the parameters, the class \( \mathcal{M}_{q,b,\lambda}^\beta(\phi) \) is reduced to numerous known subclasses of analytic functions, for example:

- If \( b = 1 \) and \( \beta = 0 \) we obtain \( \mathcal{M}_{q,1,\lambda}^\beta(\phi) \equiv \mathcal{S}_{q,\lambda}^* (\phi) \) (Aldweby and Darus [2]).
If $\lambda = 0$ and $\beta = 0$ we obtain $M_{q,b,\lambda}^3(\phi) \equiv S_{q,b}(\phi)$ (Seoudy and Aouf [14]).

If $\lambda = 0$, $\lim_{q \to 1} M_{q,b,0}^3(\phi) = M_{a,b}(\phi)$ (Suchithra et al. [16]).

If $\lambda = 0$ and $\beta = 0$, $\lim_{q \to 1} M_{q,b,0}^0(\phi) = S_b(\phi)$ (Ravichandran et al. [11]).

If $\lambda = 0$ and $\beta = 0$, $\lim_{q \to 1} M_{q,b,0}^0(\frac{1 + z}{1 - z}) = S^*(b)$ (Nasr and Aouf [10]).

If $\lambda = 0$ and $\beta = 0$, $\lim_{q \to 1} M_{q,b,0}^0(\frac{1 - z}{1 + z}) = S^*(\eta)$ (Robertson [12]).

In 1933 a study conducted by Fekete and Szegö [5] revealed that the maximum value of $|a_3 - \mu a_2^2|$ as a function of the real parameters $\mu$, for functions belonging to the class $S$. Following this, several attempts and researchers solved the Fekete-Szegö problem for various classes of $S$ class in addition to subclasses of functions in $A$. Similar work is shown in [2], [14] and [16]. In this paper, the Fekete-Szegö inequality is obtained for functions in a more general class of $M_{q,b,\lambda}^3(\phi)$ for functions defined above.

In order to prove and validate our results, following preliminary results are required.

2. Preliminary Results

Lemma 2.1 [8] If $p(z) = 1 + c_1z + c_2z^2 + \ldots \in P$ of positive real part is in $U$ and $\mu$ a complex number, then

$$|c_2 - \mu c_1^2| \leq 2 \max \{1; |2\mu - 1|\}.$$  

The result is sharp given by

$$p(z) = \frac{1 + z}{1 - z} \quad \text{and} \quad p(z) = \frac{1 + z^2}{1 - z^2}.$$  

Lemma 2.2 [8] If $p(z) = 1 + c_1z + c_2z^2 + \ldots$ is a function with positive real part, then

$$|c_2 - vc_1^2| = \begin{cases} -4v + 2, & \text{if } v \leq 0; \\
2, & \text{if } 0 \leq v \leq 1; \\
4v - 2, & \text{if } v \geq 1. \end{cases}$$

3. Main Results

Now is our theorems using similar methods studied by Aldweby and Darus in [2].

Theorem 3.1 Let $\phi(z) = 1 + B_1z + B_2z^2 + \ldots \in P$. If $f$ given by (1.1) is in the class $M_{q,b,\lambda}^3(\phi)$ and $\mu$ is a complex number, then

$$|a_3 - \mu a_2^2| \leq \frac{\frac{2}{q}bB_1}{(1 + [2]_q}\frac{[3]_q - 1}{[\lambda + 1]_q\lambda + 2}q_{bB_1}}, \max \left\{1, \frac{B_2}{B_1} + \frac{\frac{2}{q}[(1 + \beta)(2]_q - 1}{[2]_q[(1 + \beta)[2]_q - 1]^2\lambda + 1}\frac{1 + [2]_q\beta}{(3)_{q - 1}\lambda - 1}\frac{[\lambda + 2]}{q_{bB_1}} \right\}. \quad (3.1)$$

The result is sharp.

Proof. If $f \in M_{q,b,\lambda}^3(\phi)$, then there is a function $w(z)$ in $U$ with $w(0) = 0$ and $|w(z)| < 1$ in $U$ such that

$$1 + \frac{1}{b} \left[ zD_q(\lambda^2 f(z)) + \beta z^2 D_q(\lambda^2 f(z)) \right] = \phi(w(z)). \quad (3.2)$$
Define the function $p(z)$ by

$$p(z) = \frac{1 + w(z)}{1 - w(z)} = 1 + d_1 z + d_2 z^2 + ... .$$

Since $w(z)$ is a Schwarz function, immediately $\text{Re}(p(z)) > 0$ and $p(0) = 1$. Therefore,

$$\phi(w(z)) = \phi\left( \frac{p(z) - 1}{p(z) + 1} \right) = \phi\left( \frac{1}{2} d_1 z + \left( d_2 - \frac{d_1^2}{2} \right) z^2 + \left( d_3 - d_1 d_2 + \frac{d_1^3}{4} \right) z^3 + ... \right)$$

$$= 1 + \frac{1}{2} B_1 d_1 z + \left[ \frac{1}{2} B_1 \left( d_2 - \frac{d_1^2}{2} \right) + \frac{1}{4} B_2 d_1^2 \right] z^2 + ...,$$

since

$$1 + \frac{1}{b} \left[ \frac{z D_q(R_q^\lambda f(z)) + \beta z^2 D_q(D_q(R_q^\lambda f(z))))}{R_q^\lambda f(z)} - 1 \right] = 1 + \frac{(1 + \beta) [2]_q - 1 \left[ \lambda + 1 \right]_q a_2 z}{b}$$

$$+ \left[ \frac{(1 + [2]_q \beta) [3]_q - 1 \left[ \lambda + 1 \right]_q \lambda + 2]_q a_3 - \frac{(1 + \beta) [2]_q - 1 \left[ \lambda + 1 \right]_q^2 a_2}{b} \right] z^2 + ... . \quad (3.3)$$

From (3.2) and (3.3), we obtain

$$\frac{(1 + \beta) [2]_q - 1 \left[ \lambda + 1 \right]_q a_2}{b} = \frac{B_1 d_1}{2},$$

$$\frac{(1 + [2]_q \beta) [3]_q - 1 \left[ \lambda + 1 \right]_q \lambda + 2]_q a_3 - \frac{(1 + \beta) [2]_q - 1 \left[ \lambda + 1 \right]_q^2 a_2}{b} = \frac{1}{2} B_1 \left( d_2 - \frac{d_1^2}{2} \right) + \frac{1}{4} B_2 d_1^2,$$

or, equivalently,

$$a_2 = \frac{b B_1 d_1}{2 (1 + \beta) [2]_q - 1 \left[ \lambda + 1 \right]_q},$$

$$a_3 = \frac{[2]_q b B_1 d_2}{2 (1 + [2]_q \beta) [3]_q - 1 \left[ \lambda + 1 \right]_q \lambda + 2]_q} + \frac{[2]_q d_1^2}{4 (1 + [2]_q \beta) [3]_q - 1 \left[ \lambda + 1 \right]_q \lambda + 2]_q} \left[ \frac{b^2 B_1^2}{(1 + \beta) [2]_q - 1} - b (B_1 - B_2) \right].$$

Therefore,

$$a_3 - \mu^2 a_2^2 = \frac{[2]_q b B_1}{2 (1 + [2]_q \beta) [3]_q - 1 \left[ \lambda + 1 \right]_q \lambda + 2]_q} \left( d_2 - \nu d_1^2 \right),$$

where

$$\nu = \frac{1}{2} \left[ 1 - \frac{B_2}{B_1} - \frac{b [2]_q (1 + \beta) [2]_q - 1 \left[ \lambda + 1 \right]_q - \mu b [1 + [2]_q \beta) [3]_q - 1 \left[ \lambda + 2]_q B_1}{[2]_q (1 + \beta) [2]_q - 1 \left[ \lambda + 2]_q B_1} \right].$$
Our result now follows by an application of Lemma 2.1. Again by Lemma 2.1, the equality in (3.1) is gained for

\[ p(z) = \frac{1 + z}{1 - z} \quad \text{or} \quad p(z) = \frac{1 + z^2}{1 - z^2}. \]

Thus the proof of Theorem (3.1) is complete.

Setting \( q \to 1^- \) and \( \lambda = 0 \) in Theorem 3.1, we obtain the following result.

**Corollary 3.1** [16] Let \( \phi(z) = 1 + B_1z + B_2z^2 + \ldots \in \mathcal{P} \). If \( f \) given by (1.1) is in the class \( \mathcal{M}_{\beta,\lambda}(\phi) \) and \( \mu \) is a complex number, then

\[ |a_3 - \mu a_2^2| \leq \frac{|B_1b|}{2(1 + 3\beta)} \max\left\{ 1, \left| \frac{B_2}{B_1} + \left( \frac{1 + 2\beta - 2\mu(1 + 3\beta)}{(1 + 2\beta)^2} \right) bB_1 \right| \right\}. \]

The result is sharp.

Taking \( \lambda = 0 \) and \( \beta = 0 \) in Theorem 3.1, we obtain the following result.

**Corollary 3.2** [14] Let \( \phi(z) = 1 + B_1z + B_2z^2 + \ldots \in \mathcal{P} \). If \( f \) given by (1.1) is in the class \( S_{q,b}(\phi) \) and \( \mu \) is a complex number, then

\[ |a_3 - \mu a_2^2| \leq \frac{|B_1b|}{|3| q - 1} \max\left\{ 1, \left| \frac{B_2}{B_1} + \left( \frac{(2|q| - 1) - \mu(3|q| - 1)}{(2|q| - 1)^2} \right) bB_1 \right| \right\}. \]

The result is sharp.

By using Lemma 2.2, we have the following theorem.

**Theorem 3.2** Let \( \phi(z) = 1 + B_1z + B_2z^2 + \ldots \) with \( B_1 > 0 \) and \( B_2 \geq 0 \). Let

\[
\rho_1 = \frac{|2|q|(1 + \beta)(2|q| - 1)\lambda + 1|q| [bB_1^2 + (1 + \beta)(2|q| - 1)(B_2 - B_1)]}{bB_1^2(1 + 3\beta)[1 + (1 + \beta)(2|q| - 1)[1 + (2|q|\beta)(3|q| - 1)]},
\]

\[
\rho_2 = \frac{|2|q|(1 + \beta)(2|q| - 1)\lambda + 1|q| [bB_1^2 + (1 + \beta)(2|q| - 1)(B_2 + B_1)]}{bB_1^2(1 + 3\beta)[1 + (1 + \beta)(2|q| - 1)[1 + (2|q|\beta)(3|q| - 1)]}.
\]

Let \( f \) given by (1.1) be in the class \( \mathcal{M}_{q,b,\lambda}(\phi) \). Then

\[
|a_3 - \mu a_2^2| \leq \begin{cases}
\frac{|2|q|bB_1^2}{\gamma} + \frac{|2|q|bB_1^2}{\gamma} \left( \frac{b|q|(1 + \beta)(2|q| - 1)[\lambda + 1]q - \mu b\gamma/[\lambda + 1]q}{b|q|(1 + \beta)(2|q| - 1)[\lambda + 1]q} \right), & \text{if } \mu \leq \rho_1; \\
\frac{|2|q|bB_1^2}{\gamma} - \frac{|2|q|bB_1^2}{\gamma} \left( \frac{b|q|(1 + \beta)(2|q| - 1)[\lambda + 1]q - \mu b\gamma/[\lambda + 1]q}{b|q|(1 + \beta)(2|q| - 1)[\lambda + 1]q} \right), & \text{if } \rho_1 \leq \mu \leq \rho_2; \\
\frac{|2|q|bB_1^2}{\gamma} \left( \frac{b|q|(1 + \beta)(2|q| - 1)[\lambda + 1]q - \mu b\gamma/[\lambda + 1]q}{b|q|(1 + \beta)(2|q| - 1)[\lambda + 1]q} \right), & \text{if } \mu \geq \rho_2,
\end{cases}
\]

where \( \gamma = [(1 + [2] q \beta)][3] q - 1][\lambda + 1][\lambda + 2] q \).

**Proof.** First, let \( \mu \leq \rho_1 \)

\[
|a_3 - \mu a_2^2| \leq \frac{|2|q|bB_1}{2 ([1 + (2|q|\beta)][3] q - 1][\lambda + 1][2] q [-4\nu + 2]}
\]

\[
\leq \frac{|2|q|bB_2}{\gamma} + \frac{|2|q|bB_1^2}{\gamma} \left( \frac{b|q|(1 + \beta)(2|q| - 1)[\lambda + 1]q - \mu b\gamma/[\lambda + 1]q}{b|q|(1 + \beta)(2|q| - 1)[\lambda + 1]q} \right).
\]
Now, let $\rho_1 \leq \mu \leq \rho_2$, then using the above calculation, we get
\[
|a_3 - \mu a_2^2| \leq \frac{[2]_q b B_1}{[(1 + [2]_q) [3]_q - 1][\lambda + 1]_q [\lambda + 2]_q}.
\]
Finally, if $\mu \geq \rho_2$, then
\[
|a_3 - \mu a_2^2| \leq \frac{[2]_q b B_1}{2[(1 + [2]_q) [3]_q - 1][\lambda + 1]_q [\lambda + 2]_q [4\nu - 2]}
\leq \frac{[2]_q b B_2}{\gamma} \left(\frac{[2]_q [(1 + \beta) [2]_q - 1][\lambda + 1]_q - \mu b \gamma/[\lambda + 1]_q}{[2]_q [(1 + \beta) [2]_q - 1]^2 [\lambda + 1]_q}\right),
\]
where $\gamma = [(1 + [2]_q) [3]_q - 1][\lambda + 1]_q [\lambda + 2]_q$.

Taking $\beta = 0$ in Theorem 3.2 and knowing that $[n]_q - 1 = q^n - 1$, $[n + 1]_q = [n]_q + q^n$ and $[n + 2]_q = [n]_q + q^n [2]_q$, we obtain the following result.

**Corollary 3.3** ([2]) Let $\varphi(z) = 1 + B_1 z + B_2 z^2 + \ldots$, with $B_1 > 0$ and $B_2 \geq 0$. Let
\[
\sigma_1 = \frac{([\lambda]_q + q^\lambda ([B_1^1 + q(B_2 - B_1))}{([\lambda]_q + q^\lambda [2]_q) B_1^2}, \quad \sigma_2 = \frac{([\lambda]_q + q^\lambda ([B_2^1 + q(B_2 + B_1))}{([\lambda]_q + q^\lambda [2]_q) B_2^2}.
\]
Let $f$ given by (1.1) be in the class $S^{\alpha,\lambda}_{q,\varphi}$. Then
\[
|a_3 - \mu a_2^2| \leq \left\{
\begin{array}{ll}
\frac{B_2}{[\lambda+1]_q[\lambda+2]_q} + \frac{B_2^2}{q[\lambda+1]_q[\lambda+2]_q} \frac{([\lambda]_q + q^\lambda ([\lambda]_q + q^\lambda [2]_q)\mu}{q([\lambda]_q + q^\lambda)}, & \text{if } \mu \leq \sigma_1; \\
-\frac{B_2}{[\lambda+1]_q[\lambda+2]_q} - \frac{B_2^2}{q[\lambda+1]_q[\lambda+2]_q} \frac{([\lambda]_q + q^\lambda ([\lambda]_q + q^\lambda [2]_q)\mu}{q([\lambda]_q + q^\lambda)}, & \text{if } \sigma_1 \leq \mu \leq \sigma_2;
\end{array}
\right.
\]
if $\mu \geq \sigma_2$.

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