Abstract—This is a complete version of the 6-page IEEE TAC technical note [1]. In this paper, we consider the distributed surrounding of a convex target set by a group of agents with switching communication graphs. We propose a distributed controller to surround a given set with the same distance and desired projection angles specified by a complex-value adjacency matrix. Under mild connectivity assumptions, we give results in both consistent and inconsistent cases for the set surrounding in a plane. Also, we provide sufficient conditions for the multi-agent coordination when the convex set contains only the origin.

Index Terms—Multi-agent systems, complex weights, set surrounding, joint connection

I. INTRODUCTION

The distributed coordination and control of multi-agent systems has been investigated from various perspectives due to its various applications. After the study of consensus or formation of multi-agent systems [1, 4, 5, 6, 7, 9], much attention has been paid to set coordination problems of multi-agent systems. Among the studies of multi-agent set coordination, distributed containment control has achieved much, which makes agents reach a convex set maybe spanned by multiple leaders [10, 13, 12, 11]. Moreover, some results were obtained to control a group of agents in order to protect or surround a convex target set. For example, the distributed controller was designed for the agents to surround all stationary leaders in the convex hull spanned by the agents in [18], while a model was provided for multiple robots to protect a target region [2]. However, many theoretical problems to surround a target set remain to be solved.

On the other hand, complex Laplacians or rotation matrices have been applied to consensus and formation (see [22, 21, 23]), partially because the complex representation may significantly simplify the analysis when the state space is a plane. Formation control for directed acyclic graphs with complex Laplacians and related stability analysis were discussed in [21], while new methods were developed for pattern formation with complex-value elements in [22].

The objective of this paper is to study the distributed set surrounding design based on complex adjacency matrices, that is, to design a distributed protocol to make a group of agents protect/surround a convex set in a plane. We first propose a distributed controller to make all agents achieve the set projection with the same distance and different projection angles specified by a given complex-value adjacency matrix. For uniformly jointly strongly connected undirected graph and fixed strongly connected graph, we provide the initial conditions guaranteeing that all agents will not converge to the set. Then we investigate the special case when the set becomes the origin, with a necessary and sufficient condition in the fixed strongly connected graph case. In addition, our results also extend some existing ones including the consensus [3, 4] and bipartite consensus [14].

The contributions of this paper include: 1) we proposed a distributed controller to solve the set surrounding problem under the switching communication graphs; 2) we characterize the relationship between the consistency of directed cycles of the configuration graph and the system dynamic behavior, or roughly speaking, the consistent cycles produce the consistent case, while inconsistent cycles yield the inconsistent case; 3) we extend some existing results of consensus and bipartite consensus when the set contains only one point.

The paper is organized as follows. Section II gives preliminary knowledge and the problem formulation. Section III provides the main results for the distributed set surrounding problems and then considers an important special case when the target becomes the origin. Then Section IV gives a numerical example for illustration. Finally, Section V shows some concluding remarks.

Notation: \( \mathbb{R} \) and \( \mathbb{C} \) denote the real field and complex field, respectively; \(| | \) denotes the modulus of a complex number or the number of elements in a set; \( P_X(z) \) denotes the projection operator onto the closed convex set \( X \); \( z^p \) denotes the projection vector of point \( z \) onto \( X \), i.e., \( z^p = z - P_X(z) \); \( | \cdot | \) denotes the distance between a point and \( X \), i.e., \( |z|_X = |z - P_X(z)| \); \( \varepsilon = \sqrt{-1} \) denotes the imaginary unit; \( \angle z \) denotes the argument of complex number \( z \); \( \langle \cdot, \cdot \rangle \) denotes the inner product of two complex numbers, i.e., \( \langle a_1 + a_2 i, b_1 + b_2 i \rangle = a_1 b_1 + a_2 b_2 \).

II. PRELIMINARIES AND FORMULATION

In this section, we first introduce preliminary knowledge and then formulate the distributed set surrounding problem.

A. Preliminaries

A digraph (or directed graph) \( G = (V, E) \) consists of node set \( V = \{1, 2, ..., n\} \) and arc set \( E \subseteq V \times V \). A weak path in digraph \( G \) is an alternating sequence \( i_1, i_2, ..., i_k \) of nodes \( i_r, r = 1, ..., k + 1 \) and arcs \( e_r = (i_r, i_{r+1}) \in E \) or \( e_r = (i_{r+1}, i_r) \in E \) or \( e_r = (i_r, i_{r+1}) \) for all \( r \), the weak path becomes a directed path if \( i_1 = i_{k+1} \), the weak path is called a weak cycle; A weak cycle containing a directed path is called a directed cycle. Digraph \( G \) is said to be weakly strongly connected if there exists a weak path in \( G \) between every pair of nodes in \( V \), and strongly connected if there exists a directed path in \( G \) between every pair of nodes in \( V \). Moreover, \( G \) is undirected if \( (i, j) \in E \) is equivalent to \( (j, i) \in E \). Undirected graph (digraph) \( G \) is said to be a (directed) tree if there is one node such that there is one and only one (directed) path from any other node to this node. Undirected graph (digraph) \( G \) is said to contain a (directed) spanning tree if it has a (directed) tree containing all nodes of \( G \) as its subgraph. Here we assume \( G \) contains no self-loop, i.e., \( (i, i) \notin E, i \in V \).

Consider a multi-agent system consists of \( n \) agents. Let \( \sigma : [0, \infty) \rightarrow G \) be a piecewise constant function to describe the switching graph process with \( G \) the index set of all possible digraphs on \( V \). Denote a switching graph with signal \( \sigma \) as \( G_\sigma = (V, E_\sigma) \), which is called a communication graph to describe the (time-varying)
communication between the agents (regarded as nodes) with taking its connection weight $a_{ij} = 1$ if $(i, j) \in E_G$ for simplicity. Denote $G_G = (\{s_1, s_2\})$ as the union graph with node set $\mathcal{V}$ and arc set $\bigcup_{t \in \mathbb{C}_{t \in \mathbb{Z}}} E_G(t), 0 \leq s_1 < s_2$. Let $(t_0, k \geq 0) := \Delta$ with $t_0 = 0$ denote the set of all switching moments of switching graph $G_G$. The switching digraph $G_G$ is uniformly jointly strongly connected (UJSC) if there exists $T > 0$ such that $G_G([t, t + T])$ is strongly connected for any $t \geq 0$. As usual, we assume there is a dwell time as the lower bound between two consecutive switching moments. That is, there exists $\tau > 0$ such that $t_k + \tau \geq \tau$ for all $k$.

The Dini derivative of a continuous function $f : (a, b) \to \mathbb{R}$ at $t \in (a, b)$ is defined as follows:

$$D^+ f(t) = \limsup_{s \to +0} \frac{f(t + s) - f(t)}{s}.$$ 

Clearly, $f$ is non-increasing on $(a, b)$ if $D^+ f(t) \leq 0, \forall t \in (a, b)$. The following result can be found in [17].

**Lemma 1.** Let $f_i(t, x) : \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}, i = 1, ..., M$ be continuously differentiable and $f(t, x) = \max_{1 \leq i \leq M} f_i(t, x)$. Then $D^+ f(t, x(t)) = \max_{i \in \mathbb{Z}(t)} f_i(t, x(t))$, where $\mathbb{Z}(t) = \{i \mid f_i(t, x(t)) = f(t, x(t)), 1 \leq i \leq M$.\]

As we know, a set $K$ is said to be convex if $(1 - \lambda)z_1 + \lambda z_2 \in K$ whenever $z_1, z_2 \in K$ and $0 \leq \lambda \leq 1$. Moreover, let $P_K(\cdot) : \mathbb{C} \to K$ be the projection operator onto closed convex set $K$, i.e., $P_K(z)$ is the unique element in $K$ satisfying $\inf_{z \in K} |z - y| = |z - P_K(y)| = |y|_K$. In the following convergence analysis, we need to apply the Barbula’s Lemma to the given switching graph.

**Lemma 2.** Let $g : [0, \infty) \to \mathbb{R}$ be a continuous function with $\lim_{t \to \infty} g(t) = g^*$. Suppose $g$ is continuously differentiable except all the switching moments $t_k, t \geq 0$ and the derivative $\dot{g}$ is uniformly continuous with respect to all time intervals $(t_k, t_{k+1})$ (i.e., for any $\varepsilon > 0$, there exists $\delta > 0$ such that for any $k$ and any $s_1, s_2$ satisfying $t_k < s_1 < s_2 < t_{k+1}$, when $|s_2 - s_1| \leq \delta, |\dot{g}(s_2) - \dot{g}(s_1)| \leq \varepsilon$). Then $\lim_{t \to \infty} \dot{g}(t) = 0$.

Proof: This conclusion can be shown by contradiction via the almost same arguments in the proof of the well-known Barbula’s Lemma. Without loss of generality, suppose $\limsup_{t \to \infty} \dot{g}(t) := \dot{g}^* > 0$. Then let $\{s_k\}_{k \geq 0}$ be a subsequence such that $g^* / 2 \leq \dot{g}(s_k) \leq 3g^*/2$ for all $k$. Therefore, for $\varepsilon = g^*/4$, there exists $\delta > 0$ such that for any $k$ and any $r_1, r_2$ satisfying $t_k < r_1 < r_2 < t_{k+1}$, when $|r_2 - r_1| \leq \delta, |\dot{g}(r_2) - \dot{g}(r_1)| \leq \varepsilon$. Since $t_{k+1} - t_k > 0$ for all $k$, we can assume $\delta$ is sufficiently small such that for each $k$, either $s_k + \delta \in (t_k, t_{k+1})$ or $s_k - \delta \in (t_k, t_{k+1})$. First suppose $s_k + \delta \in (t_k, t_{k+1})$. Then $\dot{g}(s_k + \delta) \geq \dot{g}(s_k) - g^*/4 \geq g^*/4$. As a result, $\dot{g}(s_k + \delta) = \dot{g}(s_k) + \int_{s_k}^{s_k + \delta} \dot{g}(t)dt \geq \dot{g}(s_k) + \delta g^*/4$. Similarly, we also have $g(s_k - \delta) \geq g(s_k) - \delta g^*/4$ when $s_k - \delta \in (t_k, t_{k+1})$. This contradicts the hypothesis $\lim_{t \to \infty} \dot{g}(t) = 0$. Thus, the conclusion follows. \)

**B. Problem Formulation**

Consider the $n$ agents described by the first-order integrator

$$\dot{x}_i(t) = u_i(t), \quad i = 1, ..., n, \quad (1)$$

where $x_i, u_i \in \mathbb{C}$ are the state and control input of agent $i$ in the plane, respectively. Consider a 2-dimensional bounded closed convex set $X \subseteq \mathbb{R}^2$ to be surrounded. For a desired surrounding configuration or pattern, we need to assign the desired relative projection angles between the agents when they surround $X$. To this end, we give a complex-value adjacency matrix $W = (w_{ij}) \in \mathbb{C}^{n \times n}$ to describe the desired relative angles of projections for agents to $X$ as follows: $w_{ii} = 0, i = 1, ..., n$ and either $w_{ij} = 1$ or $w_{ij} = 0$ for $i \neq j$. In this way, we get a digraph $G^w = (\mathcal{V}, \mathcal{E}^w)$ with $\mathcal{E}^w = \{(i, j) | w_{ij} \neq 0\}$, which is called a configuration graph. Meanwhile, $w_{ij}$ is called the configuration weight of arc $(i, j) \in \mathcal{E}^w$. A weak cycle $i_1, e_1 i_2 e_2 \cdots i_k e_k i_1$ in $G^w$ is said to be consistent if

$$\prod_{t = 1}^{k} w(e_t) = 1,$$

where $w(e_r) = w_{i_r i_{r+1}}$ for $e_r = (i_r, i_{r+1}), w(e_r) = w_{i_{r+1} i_r}^{-1}$ for $e_r = (i_{r+1}, i_r)$; otherwise, it is said to be inconsistent. Clearly, $w_{ij} w_{ji} = 1$ in the consistent case when $w_{ij} \neq 0$ and $w_{ji} \neq 0$.

**Remark 1.** Although no convex set gets involved in the control design in multi-agent formation [17, 22], its design is directly based on the desired formation configuration determined by the desired relative distances or positions. Sometimes, the desired formation can be described by a set of desired relative position vectors $d_{ij}$ to show the desired position of agent $j$ relative to that of agent $i$ for $i = 1, ..., n$. In this case, for a given weak cycle $i_1 e_1 i_2 e_2 \cdots i_k e_k i_1$, we also have the consistent case with $\sum_{r=1}^{k} d(e_r) = 0$, and inconsistent case with $\sum_{r=1}^{k} d(e_r) \neq 0$, where $d(i_1 + 1) = i_1, d(e_r) = d(i_{r+1} i_r), d(e_r) = d(i_{r+1} i_r)$ for $r = (i_r, i_{r+1}), d(e_r) = -d(i_{r+1} i_r), d(e_r) = (i_{r+1}, i_r)$. It is known that the formation may fail in the inconsistent case. In our problem, the desired relative projection angles are described by $w_{ij}$ to achieve the desired surrounding configuration, which plays a similar role as $d_{ij}$ in the formation. Therefore, in both formation and surrounding problems, the agents’ indexes are given in the desired configuration.

In this paper, we consider how to surround the given set $X$ with the same distance by the $n$ agents from different projection angles (that is, the rotation angles of projection vectors) specified by $W$. To be strict, we introduce the following definition.

**Definition 1.** The distributed set surrounding is achieved for system (1) with a distributed control $u_i$ if, for any initial condition $x_i(0), 1 \leq i \leq n$, $\lim_{t \to \infty} w_{ij}(x_j(t) - P_X(x_j(t))) - (x_i(t) - P_X(x_i(t))) = 0$ for $(i, j) \in \mathcal{E}^w$.

In fact, there are two cases for the set surrounding:

- **Consistent case:** All agents surround the convex set $X$ with the same nonzero distance to $X$ and desired projection angles between each other determined by the entries $w_{ij}$ of $W$.

- **Inconsistent case:** All agents converge to the convex set $X$.

In what follows, we will show: if the weights given in the configuration graph are inconsistent, the inconsistent case appears; if the weights are consistent, we can achieve the consistent case somehow.

**Remark 2.** Different from the surrounding formulation given in [17], the agents in our problem not only surround the target set but also keep the same distance from the target set (potentially for balance or coordination concerns). Additionally, the inconsistent case resulting from the inconsistency of configuration graph is related to containment problems [21, 22, 23].

In practice, node $i$ may not receive the information from node $j$ sometimes due to communication failure or energy saving. Denote the set of all arcs transiting information successfully at time $t$ as $E_G(t)$, which is a subset of $\mathcal{E}^0$, and the resulting graph is $G_G(t) : (\mathcal{V}, E_G(t))$, which is the communication graph of the multi-agent system. Note that the configuration graph $G^w$ shows the desired relative projection angles of the agents, while the communication graph $G_G$, a subgraph of $G^w$, describes the communication topology of the agents. Let $N_i =$
\( \{j \mid (i, j) \in E^w \} \) and \( \mathcal{N}_i(\sigma(t)) \subseteq \mathcal{N}_i \) denote the neighbor set of node \( i \) in communication graph \( G_\sigma(t) \). Then we take the following control

\[
u_i(t) = \sum_{j \in \mathcal{N}_i(\sigma(t))} \left[ w_{ij}(x_j(t) - P_X(x_i(t))) - (x_i(t) - P_X(x_i(t))) \right]. \tag{2}
\]

As usual, in the design of controller (3), agents only count in the received information from their neighbors.

**Remark 3.** Let us check the role of complex-value adjacency matrix \( W \). In Definition 7 the complex-value configuration weight \( w_{ij} = e^{i\lambda_{ij}} \) indicates that \( \lambda_{ij} \) is the desired angle difference between projection vector of agent \( i \) onto set \( X \) and that of agent \( j \). Because \( |w_{ij}| = 1 \) for \( (i, j) \in E^w \), \( w_{ij}x_j^p(t) - x_i^p(t) \rightarrow 0 \) implies \( |x_i(t)| \leq |x_j(t)| \rightarrow 0 \), and therefore, all agents will have the same distance to the convex set when the (consistent) set surrounding is achieved. If \( |w_{ij}| \neq 1 \), we may get the set surrounding with different distances from the agents to the target set.

### III. Main Results

In this section, we will solve the following basic surrounding problems: (i) How to design distributed controllers to achieve the set surrounding? (ii) What initial conditions can guarantee the consistent case? (iii) What happens when the target set consists of only one point?

Before we study the set surrounding problem, we first show that the consistent case of the set surrounding problem is well-defined, which can be achieved in some situations.

**Theorem 1.** Consider a complex-value adjacency matrix \( W \) and the resulting configuration graph \( G^w = (\mathcal{V}, E^w) \). If all weak cycles of \( G^w \) are consistent, then there are \( z_i, i = 1, \ldots, n \) such that \( z_i | X \neq 0 \), \( \forall i \) and \( z_i - P_X(z_i) = w_{ij}(z_j - P_X(z_j)), \forall (i, j) \in E^w \).

**Proof:** For any \((i, i_0) \in E^w\) and \( z_{i_0} \not\in X \), take \( e_{i_1i_0} = w_{i_1i_0}(z_{i_0} - P_X(z_{i_0})) \). Define a hyperplane

\[ H^X_\lambda = \{ z \mid \langle z, e_{i_1i_0} \rangle = \langle P_X(z_{i_0}) + \lambda e_{i_1i_0}, e_{i_1i_0} \rangle, \lambda \geq 0 \}. \]

and denote the two corresponding closed half spaces as \( H^+_X \) and \( H^-_X \), respectively. Let

\[ \lambda^* = \sup \{ \lambda | \lambda \geq 0, X \cap H^-_X \neq \emptyset \}. \]

Since \( P_X(z_{i_0}) \in H^+_X \) and \( X \cap H^-_X = \emptyset \) for sufficiently large \( \lambda \), \( \lambda^* < \infty \). It is easy to see that \( X \cap \bigcup H^+_X \neq \emptyset \) and \( X \subseteq \bigcup H^-_X \). Let \( z_{i1} = y_{i1} + e_{i_1i_0} + y_{i_1} \in X \cap \bigcup H^-_X \). Clearly, \( z_{i1} \not\in X \), \( P_X(z_{i1}) = y_{i1} \) and \( z_{i1} - P_X(z_{i1}) = e_{i_1i_0} = w_{i_1i_0}(z_{i_0} - P_X(z_{i_0})) \).

If \( G^w \) contains no weak cycle, we can apply the similar arguments to all the other arcs in \( E^w \) to obtain the conclusion; if \( G^w \) contains weak cycles and all its weak cycles are consistent, we can continue the above procedures until there are \( z_1, \ldots, z_n \) such that \( z_i - P_X(z_i) = w_{ij}(z_j - P_X(z_j)) \) for all \((i, j) \in E^w \), where \((\mathcal{V}, E^w) \) is a maximal spanning subgraph of \( G^w \) containing no weak cycle. Because all weak cycles of \( G^w \) are consistent, \( z_i - P_X(z_i) = w_{ij}(z_j - P_X(z_j)) \) also holds for all the other \( w_{ij} \) when \((i, j) \in E^w \setminus E^w \). Thus, the proof is completed.

In the following two subsections, we will show that inconsistent cycles yield the inconsistent case for any initial conditions and the consistent cycles imply the consistent case for all initial conditions except a bounded set, respectively. Then in the third subsection, we will reveal the inherent relationships between consensus and our problem with the set containing only one point.

### A. Set surrounding

The following results provide sufficient conditions for the considered set surrounding problem.

**Theorem 2.** (i) The distributed set surrounding is achieved for system (1) with control law (2) if the communication graph \( G_\sigma \) and all directed cycles of the configuration graph \( G^w \) are consistent; (ii) \( \lim_{t \to \infty} |x_i(t)|_X = 0, i = 1, \ldots, n \) for any initial conditions if the communication graph \( G_\sigma(t) \equiv G^w \) is fixed, strongly connected and there are inconsistent weak cycles in \( G^w \).

**Proof:** (i) Define the arc set connecting infinitely long time

\[ E_\infty = \{ (i, j) \mid \exists \{ s_k \}_{k \geq 0}, s_k \to \infty \text{ such that } (i, j) \in E_{\sigma(s_k)} \} \]

and corresponding graph \( G_\infty = (\mathcal{V}, E_\infty) \). Clearly, \( G_\infty \) is a subgraph of the configuration graph \( G^w \).

Define

\[ d(t) = \max_{1 \leq i \leq n} d_i(t), \quad d_i(t) = \frac{1}{2} |x_i(t)|_X^2, \quad i \in \mathcal{V}, t \geq 0, \]

which are nonnegative. According to Proposition 1 in [16] (page 24), \( |x_i(t)|^2_X \) is continuously differentiable and its derivative is

\[ 2 \langle x_i^p(t), \dot{x}_i(t) \rangle. \]

Applying Lemma [1] gives

\[ D^+ d(t) = \max_{i \in \mathcal{T}(t)} \langle x_i^p(t), \dot{x}_i(t) \rangle = \max_{i \in \mathcal{T}(t)} \langle x_i^p(t), \sum_{j \in \mathcal{N}_i(\sigma(t))} (w_{ij}x_j^p(t) - x_i^p(t)) \rangle \]

\[ \leq \max_{i \in \mathcal{T}(t)} \sum_{j \in \mathcal{N}_i(\sigma(t))} \left( |x_i(t)| |x_j(t)| \right) \]

\[ \leq 0 \]

with \( \mathcal{T}(t) = \{ j \mid j \in \mathcal{V}, d_j(t) = d(t) \} \) and \( t \not\in \Delta \). Therefore, it follows from (3) that \( d(t) \) is non-increasing and then converges to a finite number, that is,

\[ \lim_{t \to \infty} d(t) = d^*. \]

As a result, the agent states \( x_i(t), i \in \mathcal{V}, t \geq 0 \) are bounded because \( X \) is bounded. In addition, if \( d^* = 0 \), the conclusion is obvious. Suppose \( d^* > 0 \) in the following proof of this part.

Since the switching communication graph \( G_\sigma \) is UJSC, by similar procedures in the proof of Lemma 4.3 in [20], we can show that \( \lim_{t \to \infty} d_i(t) = d^*, i = 1, \ldots, n \). Based on the boundedness of system states and the non-expansive property of projection operator, we can find that \( d_i \) is uniformly continuous with respect to time intervals \([t_k, t_{k+1})\) for \( k \geq 0 \). Then by Lemma [2] we have \( \lim_{t \to \infty} d_i(t) = 0 \), that is,

\[ \lim_{t \to \infty} d_i(t) = \lim_{t \to \infty} \sum_{j=1}^{n} \chi_{ij}(t) \left( w_{ij}x_j^p(t) - x_i^p(t) \right) \]

\[ = \lim_{t \to \infty} \sum_{j=1}^{n} \chi_{ij}(t) \left( -|x_i(t)|^2_X \right. \]

\[ + \left. |x_j(t)| |x_i(t)| \right) \]

\[ \leq \lim_{t \to \infty} \sum_{j=1}^{n} \chi_{ij}(t) \left( -2d^* \right. \]

\[ + \left. 2d^* \right) \]

\[ = 0, \]

which implies

\[ \lim_{t \to \infty, \tau \in \Xi} \sum_{j \in \Xi} w_{ij}x_j^p(t) - x_i^p(t) = 0, \]
where $\Xi_{i,j} = \{t|(i,j) \in E_\sigma(t)\}$,

$$\chi_{ij}(t) = \begin{cases} 1, & \text{if } (i,j) \in E_\sigma(t); \\ 0, & \text{otherwise.} \end{cases}$$

It follows from (3) that for any $\varepsilon > 0$, there is $T_1 > 0$ such that, when $t \geq T_1$, $|x_{ij}(t) - x_{ij}(t)| \leq \varepsilon$ for all $(i,j) \in E_\sigma(t)$, and then $|x_{ij}(t)| \leq (n-1)\varepsilon$. Thus, $|x_{ij}(t)| \leq \int_{T_1}^t |\dot{x}_{ij}(s)|ds \leq (n-1)(t_2 - t_1)\varepsilon$, and then $|x_{ij}^p(t_2) - x_{ij}^p(t_1)| \leq 2(n-1)(t_2 - t_1)\varepsilon$ for $t_2 \geq t_1 \geq T_1$, where the last inequality follows from the non-expansive property of projection operator: $|P_{X_0}(z_1) - P_{X_0}(z_2)| \leq (z_1 - z_2), \forall z_1, z_2$. Without loss of generality, we assume $T_1$ is sufficiently large such that $E_\sigma(t) \subseteq \mathcal{E}_s$, $\forall t \geq T_1$.

Take $(i_0,j_0) \in \mathcal{E}_s$ arbitrarily. Since the union graph $G_\sigma([t, t + T))$ is strongly connected, there exist nodes $i_1, \ldots, i_k, k \leq n - 2$ and time instants $t \leq s_0, s_1, \ldots, s_k < t + T$ such that $(i_r, i_{r+1}) \in E_\sigma(s_r), r = 0, \ldots, k - 1$ and $(i_k, j_0) \in E_\sigma(s_k)$. At the same time, there also exists a directed path $P$ from $j_0$ to $i_0$ in $G_\sigma([t, t + T))$. Denote the product of all configuration weights on $P$ as $w_P$.

Since $G_\sigma([t, t + T))$ is a subgraph of $G^w$ and all directed cycles of $G^w$ are consistent, all directed cycles of $G_\sigma([t, t + T))$ are also consistent. Therefore, $\prod_{i=0}^{k} w_{i_r, i_{r+1}} w_{i_k, j_0} = 1$. Moreover, since $i_0(i_0, j_0)j_0P$ is a directed cycle in $w_{i_0, j_0}w_{i_0} = 1$. Thus, $\prod_{i=0}^{k} w_{i_r, i_{r+1}} w_{i_k, j_0} = w_{i_0, j_0}$ and

$$|x_{i_0}(t) - w_{i_0, j_0}x_{j_0}(t)| = \left|\prod_{r=0}^{k-1} w_{i_r, i_{r+1}} w_{i_k, j_0}x_{j_0}(t)\right|
\leq \left|\prod_{r=0}^{k-1} w_{i_r, i_{r+1}} x_{i_1}(s_0)\right| + \left|\prod_{r=0}^{k-1} w_{i_r, i_{r+1}} x_{i_1}(s_1)\right| + \cdots + \left|\prod_{r=0}^{k-1} w_{i_r, i_{r+1}} x_{i_1}(s_{k-1})\right|
\leq (k + 1)\varepsilon + 2(k + 2)(n - 1)^2T\varepsilon
\leq (n - 1)^2\varepsilon + 2(n - 1)^2T\varepsilon,$$

where $w_{i_0} = \prod_{r=0}^{k} w_{i_r, i_{r+1}}$. Since $\varepsilon$ can be sufficiently small, we can further obtain

$$\lim_{t \to \infty} x_{i_0}x_{j_0}(t) = x_{i_0}(t) = 0, \forall (i,j) \in \mathcal{E}_s. \quad (6)$$

Clearly, due to the uniformly strong connectivity of $G_\sigma$, $G^w$ is strongly connected. Combining the previous conclusion, (6) and the consistency of directed cycles of $G^w$, we have $\lim_{t \to \infty} w_{i,j}x_{i,j}(t) = x_{i,j}(t) = 0$ for all the other $w_{ij}$ with $(i,j) \in E^w \setminus \mathcal{E}_s$. Thus, the proof of part (i) is completed.

(ii) We first show this conclusion for the case when there exist inconsistent directed cycles in $G^w$. Let $i_{1,1}i_{2,2} \cdots i_{k,1}i_{k,1}$ be an inconsistent directed cycle in $G^w$ with

$$\prod_{r=1}^{k} w_{i_r, i_{r+1}} \neq 1, \quad (7)$$

$i_{k+1} = i_1$. From (3), we have $\lim_{t \to \infty} w_{i_r, i_{r+1}} x_{i_r, i_{r+1}}(t) = x_{i_r, i_{r+1}}(t) = 0$ for $1 \leq r \leq k$. Therefore,

$$\lim_{t \to \infty} x_{i_1}(t) \left(1 - \prod_{r=1}^{k} w_{i_r, i_{r+1}}\right) = 0,$$

which implies $\lim_{t \to \infty} d_{i_1}(t) = 0$ and then $d^* = 0$. For the case of existing weak cycles (not directed cycles) in $G^w$, we can similarly show this conclusion by replacing the configuration weight $w_{i_r, i_{r+1}}$ in (7) with $w_{i_r, i_{r+1}}^{-1}$ corresponding to arc $e_{r+1}$.

Thus, we complete the proof.

From the proof of Theorem 3, we can find that the conclusion (ii) also holds under the following relaxed connectivity condition: the communication graph $G_\sigma$ is $UJ_SC$ and there exist a time sequence $\{\delta_k\}_{k=0}^{\infty}$, $\delta_k \to \infty$ and $b_0 > 0$ such that $E^w \subseteq \bigcup_{\varepsilon = \delta_k}^{b_0} E_\sigma(t)$ for $k \geq 0$.

Remark 4. In the consistent setting, the distance of all agents surrounding the convex set $X$ is same. In fact, we can consider a more general set surrounding problem that all agents surround the convex set $X$ with the pre-specified relative distance relationship to $X$ among agents determined by some matrix $(d_{ij})$ $(d_{ij}, (i,j) \in E$ describes the proportion of the desired distance of agent $i$ onto set $X$ and that of agent $j$ onto set $X$) and desired projection angles between each other determined by the entries $w_{ij}$ of $W$. We can find that this new set surrounding problem can be solved by distributed controller (3) by replacing $w_{ij}$ with $d_{ij}w_{ij}$ if the communication graph $G_\sigma$ is $UJ_SC$, and the consistency defined of directed cycles of the configuration graph $G^w$ with replacing $w_{ij}$ with $d_{ij}w_{ij}$ holds.

Remark 5. Here we provide another simple proof of Theorem 2(i) when $w_{ij} = 1$ for all $i, j$ and the communication graph is a fixed undirected connected graph. First of all, by $\nabla |x_{i}^2| = 2(x - P_{X_0}(x))$, we have

$$\frac{d|x_{i}(t)|^2}{dt} = 2(x_{i}(t), \sum_{j \in N_i} (x_j^2 - x_i^2(t))).$$

Taking the sum for the two sides of the above equality over $i = 1, \ldots, n$, by the undirectedness of the graph $(\forall, E)$, we have

$$\frac{d}{dt} \sum_{i=1}^{n} |x_i(t)|^2 = -2 \sum_{(i,j) \in E} |x_j^2 - x_i^2(t)|^2 \leq 0,$$

which implies that $\sum_{i=1}^{n} |x_i(t)|^2$ is non-increasing and then its limit exists. As a result, the system states $x_i(t), i, t \geq 0$ are bounded. This combines with (3) lead to

$$\int_{0}^{\infty} \sum_{(i,j) \in E} |x_j^2 - x_i^2(t)|^2 dt < \infty.$$

From the boundedness of system states, we can see that $\sum_{(i,j) \in E} |x_j^2 - x_i^2(t)|^2$ is uniformly continuous on $[0, \infty)$. Then it follows from the Babalat’s Lemma that $\lim_{t \to \infty} \sum_{(i,j) \in E} |x_j^2 - x_i^2(t)|^2 = 0$. This implies that $\lim_{t \to \infty} (x_j^2 - x_i^2(t)) = 0$ for all $(i,j) \in E$ and then the set surrounding is achieved.

B. Consistent Case

Theorem 3 showed that the distributed set surrounding can be achieved under $UJSC$ communication graph condition. In this subsection, we further show under the case without inconsistent cycles of $G^w$. How to select initial conditions such that the consistent case (that is, $d^* > 0$ given in (4)) can be guaranteed.

Let $\Lambda_{(\varepsilon,t)}$ be the matrix with entries

$$(\Lambda_{(\varepsilon,t)})_{ij} = \begin{cases} |N_i(\sigma(t))|, & \text{if } i = j; \\ -w_{ij}, & \text{if } i \neq j, j \in N_i(\sigma(t)); \\ 0, & \text{otherwise}. \end{cases} \quad (9)$$

Then system (11) with control law (2) can be written in the following compact form

$$\dot{x}(t) = -\Lambda_{(\varepsilon,t)}x(t), \quad (10)$$
where \( x(t) = (x_1(t), \ldots, x_n(t))^T \) is the stack vector of agents’ states, \( x^*(t) = (x_1^*(t), \ldots, x_n^*(t))^T \) is the stack vector of agents’ projection vectors.

We first consider system (10) with a UJSC undirected graph \( G_σ \), where all directed cycles of the configuration graph \( G^w \) are consistent.

Without loss of generality (otherwise we can relabel the index of nodes), we take a spanning tree \( T \) of \( G^w \) as follows: \( T = \bigcup_{k=1}^q T_k \), where the initial and the terminal nodes of the path \( T_k \) are \( i_k \) and 1, respectively; the nodes in the path from \( i_k \) to 1 are in the order \( i_k, i_k - 1, \ldots, i_1 - 1, 1 \). Associated with the \( n \) nodes, we define \( n \) nonzero complex numbers

\[
p_{1} = 1, \quad p_{j} = w_{i_{k_{j}}-1+1}^{-1} \prod_{r=i_{k_{j-1}}+1}^{j-1} w_{r+1}^{-1}, \quad \text{for } i_{k_{j-1}} + 1 + 1 \leq j \leq i_{k_{j}}.
\]

Denote a diagonal matrix

\[
P = \text{diag}(p_{1}, \ldots, p_{n}) \tag{11}
\]

with diagonal elements \( p_{i}, 1 \leq i \leq n \). It is easy to see that \( \tilde{L}_σ(t) = PL_σ(t)P^{-1} \) is the Laplacian of undirected graph \( G_σ(t) \). Then we have

**Theorem 3.** *In the switching UJSC undirected graph case, \( d^* > 0 \) if the initial condition \( x(0) \) satisfies \( |T^* P x(0)/n| > \sup_{z\in X} |z| \).

Proof: Let \( \tilde{x}(t) = P x(t) \). Clearly, system (10) can be written as

\[
\dot{\tilde{x}}(t) = -\tilde{L}_σ(t) P x(t).
\]

Because \( 1^T \tilde{L}_σ(t) \equiv 0 \) with \( 1 = (1, \ldots, 1)^T \), \( 1^T \tilde{x}(t)/n \) is time-invariant. Note that \( \sup_{z\in X} |z| \) is a finite number since \( X \) is bounded.

We prove the conclusion by contradiction. Hence suppose \( d^* = 0 \).

Since \( \lim_{t \to \infty} d_i(t) = d^* \) under the UJSC assumption, \( \lim_{t \to \infty} |x_i(t)| = 0 \). Therefore, \( \lim_{t \to \infty} |x_i(t)| \leq \sup_{z\in X} |z| \), and then \( |T^* P x(t)/n| \leq \sup_{z\in X} |z| \), which yields a contradiction due to \( 1^T \tilde{x}(t)/n \equiv 1^T \tilde{x}(t)/n \).

Next we consider system (10) under a fixed strongly connected digraph \( G^w_σ(t) \equiv G^w \) with all its directed cycles being consistent. Since any strongly connected digraph contains a directed spanning tree, \( G^w \) contains a directed spanning tree \( T^d = \bigcup_{k=1}^q T_k^d \), where the initial and the terminal node of the directed path \( T_k^d \) are \( i_k \) and 1, respectively. Moreover, the nodes in the directed path from \( i_k \) to 1 are in the order \( i_k, i_k - 1, \ldots, i_1 - 1, 1 \). Associated with the \( n \) nodes, we can similarly define

\[
q_{1} = 1, \quad q_{j} = w_{i_{k_{j}}-1+1}^{-1} \prod_{r=i_{k_{j-1}}+1}^{j-1} w_{r+1}^{-1}, \quad \text{for } i_{k_{j-1}} + 1 + 1 \leq j \leq i_{k_{j}}.
\]

Let \( Q = \text{diag}(q_{1}, \ldots, q_{n}) \) denote the diagonal matrix with diagonal entries \( q_{1}, \ldots, q_{n} \). It is easy to see that \( Q L^w_σ(t) Q^{-1} \equiv Q L^w_{σ(0)} Q^{-1} \) is the Laplacian of the fixed digraph \( G^w \) and

\[
\alpha^T Q x(t) \tag{12}
\]

is time-invariant, where \( \alpha = (\alpha_1, \ldots, \alpha_n)^T \) with \( \alpha_i > 0 \), \( \sum_{i=1}^n \alpha_i = 1 \) is the left eigenvector of \( Q L^w Q^{-1} \) associated with eigenvalue 0, that is, \( \alpha^T Q L^w Q^{-1} = 0 \). Similar to the undirected graph case, we can show the following result, whose proof is omitted due to space limitations.

**Theorem 4.** *In the fixed strongly-connected digraph case, \( d^* > 0 \) if the initial condition \( x(0) \) satisfies \( |\alpha^T Q x(0)| > \sup_{z\in X} |z| \).*
Lemma 3. 0 is an eigenvalue of $L$ if and only if all directed cycles of $G^w$ are consistent.

Proof: Sufficiency. If all directed cycles of $G^w$ are consistent, then by the discussions in Subsection III. B, there is an invertible diagonal matrix $Q$ such that $QLQ^{-1}$ is the Laplacian of digraph $G^w$. Since all row sums of any Laplacian are zero, any Laplacian has an eigenvalue zero. The sufficiency follows from that similar matrices have the same eigenvalues.

Necessity. Let us show it by contradiction. Suppose that $G^w$ contains inconsistent directed cycles. On one hand, by Theorem 2(ii), \( \lim_{t \to \infty} x_i(t) = 0, i = 1, ..., n \) for any initial conditions (noticing that $X = \{(0,0)\}$). On the other hand, let $\xi \neq 0$ be the eigenvector of $L$ with eigenvalue 0, that is, $L\xi = 0$. Clearly, $x(t) \equiv \xi$ for initial condition $x(0) = \xi$, which yields a contradiction. Thus, the necessity follows.

Recalling the matrix $Q$ and vector $\alpha$ defined in (12) along with Lemma 3 and Theorem 2.13 in [4], we have the following theorem.

Theorem 6. Consider system (12) with a fixed strongly connected digraph $G^w$. Then $\lim_{t \to \infty} x_i(t) = 0, i = 1, ..., n$ for any initial conditions if and only if $G^w$ contains inconsistent directed cycles. Moreover, if all directed cycles of $G^w$ are consistent, then, for any initial condition $x_i(0), i = 1, ..., n$,

$$\lim_{t \to \infty} x_i(t) = \frac{\sum_{j=1}^{n} \alpha_j q_{ij} x_j(0)}{q_i}, 1 \leq i \leq n.$$ 

Remark 9. Theorem 2 stated that, in the fixed graph case, if there exist irregular weak cycles in $G$, then, for any initial condition, all agents converge to the target set (which is the origin here), which implies that all eigenvalues of generalized Laplacian $L$ have positive real parts. In fact, as shown in Theorems 5 and 6, the converse is also true.

Theorem 7. Suppose $G^w$ is strongly connected. Then all directed cycles of $G^w$ are consistent if and only if all weak cycles of $G^w$ are consistent.

Proof: The sufficiency part is straightforward. We focus on the necessity part. Without loss of generality, let the weak cycle in $G^w$ take the following form

$$i_1 e_1 i_2 e_2 \cdots e_{k-1} i_{k-1} e_{k-1} i_{k-1} + 1 \cdots i_k e_k^{-1} i_1,$$

where $e_r = (i, e_i, i_{r+1})$, $r = 1, ..., k; e_r^{-1} = (i_{r+1}, i)$. Since $G^w$ is strongly connected, for each $r = k_1, ..., k$, there is a directed path $P_i$ from $i_r$ to $i_{r+1}$. Because the directed cycles $P_i e_r^{-1}, r = k_1, ..., k$ are consistent, $w(P_i)w_{i_{r+1} i_r} = 1$, where $w(P_i)$ is the product of all configuration weights on directed path $P_i$. Then, from the consistency of the directed cycle $i_1 i_2 i_3 \cdots i_{k-1} i_{k} P_{k-1} \cdots P_1$, we have

$$w_{i_{k+1} i_1} \cdots w_{i_{k+1} i_2} w_{i_{k+1} i_3} \cdots w_{i_{k+1} i_k} = w_{i_{k+1} i_1} \cdots w_{i_{k+1} i_2} w(P_{k-1}) \cdots w(P_1) = 1.$$

Thus, the conclusion follows.

IV. Numerical Example

In this section, we provide an example to illustrate the results obtained in this paper.

Consider a network of five agents with node set $V = \{1, 2, 3, 4, 5\}$ and the complex-value adjacency matrix $W = (w_{ij})$. The convex set to be surrounded is the unit ball in $\mathbb{C}^2$. The initial conditions are $x_1(0) = 2 + 4i, x_2(0) = 4 + 3i, x_3(0) = -4 - 3i, x_4(0) = -4 + 2i, x_5(0) = 2 + 3i$ (marked as o in Figures 1 and 2).

Two cases are shown as follows:

- **Consistent case**: Take $w_{12} = w_{23} = w_{34} = e^{\pi i}/4, w_{45} = e^{\pi i}/4, w_{51} = e^{\pi i}/4$, and all other configuration weights are zero. Then the resulting configuration graph is $G^w = (V, E^w)$ with arc set $E^w = \{(1, 2), (2, 3), (3, 4), (4, 5), (5, 1)\}$. The communication graph of the multi-agent system is periodically switched between two graphs $G_1 = (V, E_1), G_2 = (V, E_2)$ with $E_1 = \{(1, 2), (3, 4), (5, 1)\}$ and $E_2 = \{(2, 3), (4, 5)\}$ in the following order: $G_1, G_2, G_1, G_2, \ldots$ with switching period 5. Clearly, $G_1$ is $UJSC$ and all directed cycles of $G^w$ are consistent. Figure 1 demonstrates that all agents accomplish the consistent set surrounding at time $t = 2000$, where the five agent trajectories are described by the solid lines and the projection vectors of the final positions of the agents are described by dashed lines.

- **Inconsistent case**: Take $w_{12} = w_{23} = w_{34} = e^{\pi i}/4, w_{45} = e^{\pi i}/4, w_{51} = e^{\pi i}/4, w_{14} = e^{-\pi i}/4$, and all other configuration weights are zero. Suppose the communication graph is fixed, that is, $G_\sigma \equiv G^w = (V, E^w)$ with $E^w = \{(1, 2), (2, 3), (3, 4), (4, 5), (5, 1), (1, 4)\}$. Note that the configuration graph $G^w$ defined by the new configuration weights is clearly inconsistent. Figure 2 shows that all agents converge to the unit ball, where the final positions of the five agents at time $t = 2000$ are marked with *.

V. Conclusion

In this paper, we proposed a formulation and a distributed controller for set surrounding problems. We discussed both consistent and inconsistent cases, and obtained the necessary/sufficient conditions for multi-agent systems with communication topologies described by joint-connected graphs. Moreover, we showed when the consistent case can be guaranteed, and also provided conditions on the leader-following consensus when the target set becomes one point.

Many interesting problems about set surrounding of multi-agent systems remain to be done, such as the analysis in the case with various uncertainties, the construction of configuration graphs based on given optimization indexes, and surrounding control design for agents moving in a high-dimensional space with the help of rotation matrices.
Figure 1: The consistent cycles yield the consistent set surrounding.

Figure 2: The inconsistent cycles yield the inconsistent set surrounding.

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