On superconducting instability in non-Fermi liquid: scaling approach

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The superconducting instability in a non-Fermi liquid in $d > 1$ is considered. For a particular form of the single particle spectral function with homogeneous scaling $A(\Lambda k, \Lambda \omega) = \Lambda^\alpha A(k, \omega)$ it is shown that the pair susceptibility is also a scaling function of temperature with power defined by $\alpha$. We find three different regimes depending on the scaling constant. The BCS result is recovered for $\alpha = -1$ and it corresponds to a marginal scaling of the coupling constant. For $\alpha > -1$ the superconducting transition happens above some critical coupling. In the opposite case of $\alpha < -1$ for any fixed coupling the system undergoes a transition at low temperatures. Possible implications for theories of high-$T_c$ with a superconducting transition driven by the interlayer Josephson tunneling are discussed.

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The question of non-Fermi liquid behavior in higher dimensions ($d > 1$) has been addressed recently in the context of possible description of the normal state of high temperature superconductors [1–3]. The solution of this question, apart from understanding the normal state of high-$T_c$ materials also might lead to a better understanding of the possible superconducting and density wave instabilities of the non-Fermi liquid state.

Recently a general class of systems with a nontrivial scaling coefficient has been proposed in [3] as a possible description of the normal state of the cuprate layers. It was pointed out that for the non-Fermi liquid form of $A(k, \omega)$ interlayer single particle tunneling is strongly suppressed and only Josephson tunneling is relevant at low energies. This Josephson interlayer tunneling enhances the effective superconducting transition temperature, driven by intralayer attraction. Although the non-Fermi liquid behavior was assumed inside each layer in [3], the superconducting transition was considered within the BCS theory and the in-plane pair susceptibility was taken in the BSC form $\chi_{\text{pair}} = \text{th}(\epsilon_k/2T)/\epsilon_k$. This choice was argued to give qualitatively correct answer for the large enough transition temperatures. However in general the true pairing susceptibility in a non-Fermi liquid should be used to identify the superconducting instability of the normal state.

In this note we will consider the superconducting instability of a non-Fermi liquid in $d > 1$, using the same set of assumptions made in [3]. Let us leave the most important and interesting question of the origin of the breakdown of Fermi liquid unanswered and consider the effect of attraction between excitations in a non-Fermi liquid and conditions under which this attraction will lead to superconductivity. We will argue that to be consistent one has to calculate the pair susceptibility using non-Fermi liquid Green’s functions. It is shown that for the non-Fermi liquid behavior, characterized by the vanishing quasiparticle residue $Z_\omega$ at low frequency, the pair susceptibility is lower then in the Fermi liquid and is of a non-BCS form for general values of $\alpha$. This results
in a \textit{qualitative} difference with the BCS instability: we find a critical coupling for the superconducting transition for the case of a Luttinger liquid behavior, below which the system remains normal down to \( T = 0 \).

As is well known, in the Luttinger liquid in \( d = 1 \) we always have competing density waves and superconducting interactions coming from the same terms in the Hamiltonian. These interactions do not produce true long range order but they lead to power law correlators. Presumably at \( d > 1 \) interaction will lead to a true instability in one of the channels. The artificial nesting will disappear in \( d > 1 \) (except in some special cases) and we can consider different channels independently. We will use the ladder approximation as was used in \cite{3}, ignoring parquet effects.

The absence of a microscopic description of a non-Fermi liquid for \( d > 1 \) leads us to consider the phenomenology of this state, analogous to the ideas proposed in \cite{1,3}. We assume that this state supports single particle excitations which obey Fermi statistics with a Green’s function:

\[
G(k, \omega_n) = \int_{-\infty}^{+\infty} \frac{A(k, x)}{x - i\omega_n} dx
\]

With the spectral function obeying the following homogeneous equation:

\[
A(\Lambda k, \Lambda \omega) = \Lambda^{\alpha} A(k, \omega)
\]

with some scaling coefficient \( \alpha \). We assume an isotropic dispersion hereafter and only the magnitude of the momentum, counted from the Fermi momentum \( k_F \), enters into the spectral function. This scaling form taken over the whole frequency range violates the spectral sum rule, except for \( \alpha = -1 \). Indeed \( \int_{-\infty}^{+\infty} A(k, \omega) d\omega = \Lambda^{\alpha+1} \int_{-\infty}^{+\infty} A(k, \omega) d\omega = 1 \), where the second equation is obtained by rescaling \( \omega \rightarrow \Lambda \omega \).

We consider this scaling form of the spectral function to be valid only for the low energy part of the spectrum.

The simplest example of the scaling law of this type is a Fermi liquid spectral function:
\[ A(k, \omega) = Z \pi \delta(\omega - v_F k), \quad \alpha = -1 \] (3)

and the scaling coefficient follows immediately from the fact that \( \delta(\Lambda a) = \Lambda^{-1} \delta(a) \).

Another example of a system with \( \alpha = -1 \), up to irrelevant logarithmic factors, is a marginal Fermi liquid with \( A(k, \omega) = -2 \text{Im}(\omega - v_F k - \gamma(\omega \log \omega + i\pi \omega))^{-1} \), proposed in [1]. The case of general scaling is realized for a Luttinger liquid Green’s function, e.g. \( G(k, \omega) \sim (\omega^2 - v_F^2 k^2)^\alpha / (\omega - v_F k) \) with \( \alpha = 2g - 1 > -1 \).

Here we will show that under the scaling assumptions of Eq.(2) the in-plane susceptibility will be a non-BCS form and subsequently the theory of the superconducting transition, driven by an in-plane attraction, will be different from BCS. The main difference will come from the fact that for general scaling exponent \( \alpha \) the temperature dependence of the pair susceptibility will be a power law with the power dependent on \( \alpha \). For the physically interesting case of \( \alpha > -1 \), which leads to a Luttinger liquid behavior in \( d > 1 \), we find a critical coupling, above which the superconducting transition is possible.

We now consider a small s-wave attractive interaction:

\[ H_{\text{int}} = -V \int d\mathbf{r} c_{\uparrow}(\mathbf{r}) c_{\downarrow}(\mathbf{r}) \] (4)

In the ladder approximation within the weak coupling theory we have for a critical \( T_c \):

\[ 1 = gT \sum_{n,k} G(k, \omega_n)G(-k, -\omega_n) \] (5)

where \( g = -V N_0 \). Using the spectral representation Eq.(1) and rescaling variables as \( \mathbf{r} = x\beta, \mathbf{y} = y\beta, v_F \mathbf{k} = v_F k\beta, \beta = 1/T \) in the integral in this equation we find:

\[ 1 = g \beta^{-2(1+\alpha)} \int_0^{\omega_0^{\beta}} d\xi F(\kappa) \] (6)

where,

\[ F(\kappa) = \int \int d\mathbf{r} d\mathbf{y} A(\kappa, \mathbf{r}) A(\kappa, \mathbf{y})(t \mathbf{r}/2 + t \mathbf{y}/2)(\mathbf{r} + \mathbf{y})^{-1} \] (7)
with \( \omega_0 \) being the upper cut off. In deriving Eq.(6) we assume that the scaling form of \( A(k, \omega) \) is valid at all temperatures above \( T_c \) and that the density of states \( N_0 \) is invariant under scaling, which is certainly true in \( 2d \) and it holds in general upon ignoring particle-hole asymmetry. The selfconsistent treatment of the gap equation below \( T_c \) is a more complicated matter due to a possible breakdown of the scaling in Eq.(2) and we will not consider it here.

An important comment is in order here. As was mentioned above, the scaling form of the spectral function can not be valid in the whole frequency range. We consider the case when \( A(k, \omega) = \omega^\alpha f(\omega/k) + A_{\text{inc}}(k, \omega) \) with normalization \( \int_{-\infty}^{+\infty} dx f(x) = 1 \). The first term is a “scale invariant” part of the spectral function, the second term violates scaling and is small at low frequency. One can show that at \( T \sim \Lambda \to 0 \) the dominant part in the spectral sum comes from \( A_{\text{inc}}(\Lambda k, \Lambda \omega) \). Thus in the integral in Eq.(6) terms, containing \( A_{\text{inc}}(k, \omega) \), will dominate at \( T \sim \Lambda \to 0 \). However, for the general case of not quite small \( \Lambda \), \( 0 < \Lambda < 1 \), we assume that the scale invariant part provides a major contribution, what allows us to get Eq.(7). This also means that Eqs.(6, 7) are invalid at sufficiently low temperatures.

We generally can consider two possibilities: 1) the integral of \( F(k) \) is convergent at the upper limit and 2) the integral is divergent at the upper limit. In the first case we can safely put the upper limit to infinity. And the answer for the dependence of \( T_c \) on \( g \) is universal. The simplest example of this sort will be, say, power law decaying \( F(k) \). In the second case the dependence of the integral on the upper cut off is crucial and obviously the result depends on the specific form of \( F(k) \)[6]. We can still estimate the integral if we would assume some asymptotic form for the integral on the upper limit. However this would require extra input apart from the scaling coefficient \( \alpha \). The simplest example of this sort is the BCS case with \( F(k) = \text{th}(\pi/2)/\pi \) which leads to a

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1 I am indebted to J.R. Schrieffer for pointing out this possibility
logarithmically divergent integral in Eq.(6).

Let us concentrate now on the first case, when the answer is universal and independent of the detailed form of $F(T)$. Then the momentum integral is trivial and equals some number which can be incorporated into the definition of the coupling constant. The final scaling law for coupling constant $g$ vs $T_c$ follows immediately from Eq.(6):

$$1 = g(W/T)^{-2(1+\alpha)}$$

where $W$ is the energy scale, within which the scaling form of the spectral function is assumed. This equation is the main result of this note. It gives the dependence of the critical temperature $T_c$ on the coupling constant $g$. It is useful to write it in the form of a scaling equation:

$$\frac{\partial g}{\partial \log \beta} = 2(1 + \alpha)g + O(g^2)$$

We want to discuss now possible regimes for $g$ vs $T_c$ for different coefficients $\alpha$. From Eqs.(8, 9) follows that the critical value $\alpha_c = -1$ precisely corresponds to a Fermi liquid case, when the BCS equation yields a $\log(\omega_0/T)$ in Eq.(6). The Fermi liquid superconducting instability is a marginal case with a quadratic $\beta$-function.

For $\alpha > \alpha_c$ the solution of the $T_c$ equation, Eq.(8), is $T_c/W = \left(\frac{1}{g}\right)^{\frac{1}{2(1+\alpha)}}$. From this it follows that the smaller $T_c$ is the larger is coupling constant required to produce the instability. For any fixed coupling constant at $T/W < \left(\frac{1}{g}\right)^{\frac{1}{2(1+\alpha)}}$ the state will remain normal and the pair susceptibility is finite. The same result also follows from the $\beta$-function in Eq.(9) which indicates that for lower temperatures one has to go to higher coupling to produce an instability. We find that there always is a critical coupling

$$g_{\text{min}} \sim (W/W')^{2(1+\alpha)} \sim 0(1)$$

for the superconducting transition, $W'$ is some energy scale of the order of $W$. One can show that for the case 2) it is true as well.
This result leads immediately to a question of reentrant superconducting behavior at high temperatures, see Fig.1. The slope of the function $g_c(T)$ is negative at low temperatures. It would correspond to an irrelevant coupling at low temperatures. However one can not exclude the possibility of a superconducting transition at high temperatures which then reenters the normal phase at low temperatures $T/W < (\frac{1}{g})^{\frac{1}{\alpha + 1}}$.

It should be stressed that these results depend on the strong assumption of the convergence of the integral in Eq.(6). Otherwise the dependence on the upper cut off can change the function $g_c(T)$. We found examples of this sort, where this dependence even for $\alpha > \alpha_c$ leads to a nonreentrant phase diagram with $g_c(T)$ monotonically increasing. In these cases the transition is still possible only at a coupling constant greater then some critical value, as is natural for a noninfrared theory.

For the opposite case $\alpha < \alpha_c$ the transition temperature is always nonzero for any coupling $g$ with $g_c(T)$ monotonically increasing. This regime corresponds to a relevant superconducting coupling. The case $\alpha < \alpha_c$ appears to have no analog in a 1d Luttinger liquid and we will not discuss it here.

In the recent theory [3] the non-Fermi liquid behavior in the spectral form Eq. (2) with $\alpha > 0$ was considered as a model of a normal state inside each layer in high-$T_c$ materials. Our results shows that the superconducting transition in each layer will require some critical coupling, for the physically most interesting case of $\alpha > -1$, in order to have a superconducting transition inside each layer. This result is qualitatively different from the BCS pairing theory in which for any coupling $T_c > 0$.

There is one interesting possibility to consider when the superconducting coupling inside each layer is below critical and Josephson tunneling between different $Cu-O$ planes, enhancing attraction in each layer, provides this critical coupling. The simplest example is a model of two planes with non-Fermi liquid behavior, coupled via a single particle tunneling matrix element $t_{\perp}$. If one has an in-plane coupling $g < g_{\text{min}}$ then strong enough $t_{\perp}$ will produce superconductivity in these two planes simultaneously.
Whereas, if to take $t_\perp = 0$, each plane separately will become superconducting at much lower temperature. At this point we go beyond the considered model and assume that at low enough temperatures Fermi liquid behavior sets in and a small in-plane attraction will produce a BCS instability.

**Conclusion** We considered the superconducting transition in the non-Fermi liquid with special scaling properties Eqs. (1,2). In general the pair susceptibility is found to be of non-BCS form. For the scaling coefficient $\alpha = -1$ we recover the BCS results with the marginal scaling of the coupling constant. For the most interesting case of the Luttinger liquid Green’s functions with $\alpha > -1$ we find that superconducting transition requires a threshold value of the coupling constant and is qualitatively different from the BCS case where the instability is caused by arbitrarily small coupling.

In general the results depend on the upper cut off in the momentum sum. However, for a special case of the convergent momentum sum the results are universal and depend on the index $\alpha$. In the opposite case of $\alpha < -1$ the normal ground state is always unstable for all values of the coupling constant.

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[2] See for example P.W. Anderson, in “Superconductivity”, Proceedings of the ICTP Spring College, 1992, P. Butcher and Y. Lu, Eds. (World Scientific, in press).

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[4] After this manuscript was finished, I learned about the Chapter 7 of the book by P. W. Anderson (unpublished), where the pairing susceptibility of a Luttinger liquid in $d > 1$ was calculated. The pairing susceptibility we found is similar to what Anderson and co-workers found. However the existence of the critical coupling for the superconducting transition in the Luttinger liquid and the equation for $T_c$ (see Eqs.(8, 9)), to the best of my knowledge, are shown for the first time in the current article.
FIGURES

FIG. 1. Possible phase diagram for the superconducting transition at $\alpha > -1$. Reentrant evolution of the system from normal to superconducting to normal state at $T_1$ and $T_2$ temperatures is shown. The minimal coupling $g_{\text{min}}$ is required for a superconducting transition. The regime in which Eq. (8) is valid is violated at $T \to 0$ and $T \sim W$ and the dependence of $g$ on $T$ is unknown. However we expect that as $T \to 0$ the critical coupling is finite.