PT-Symmetric Matrix Quantum Mechanics

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Recently developed methods for PT-symmetric models are applied to quantum-mechanical matrix models. We consider in detail the case of potentials of the form $V = -(g/N^{p/2-1}) \text{Tr}(M)^p$ and show how the calculation of all singlet wave functions can be reduced to solving a one-dimensional PT-symmetric model. The large-$N$ limit of this class of models exists, and properties of the lowest-lying singlet state can be computed using WKB. For the special case of $p = 4$, we extend recent work on the $-gx^4$ potential to the matrix model: we show that the PT-symmetric matrix model is equivalent to a hermitian matrix model with a potential proportional to $(4g/N) \text{Tr} \Pi$. However, this hermitian equivalent model includes an anomaly term $\hbar^2 g/N \text{Tr} \Pi$. In the large-$N$ limit, the anomaly term does not contribute at leading order to the properties of singlet states.

I. INTRODUCTION

Matrix models appear in many contexts in modern theoretical physics, with applications ranging from condensed matter physics to string theory. Interest in the large-$N$ limit of matrix models was strongly motivated by work on the large-$N_c$ limit of QCD [1], but interest today is much wider. For example, Hermitian matrix quantum mechanics leads to a construction of two-dimensional quantum gravity coupled to $c = 1$ matter [2].

We will show below that the matrix techniques pioneered in [3] for Hermitian matrix quantum mechanics can be extended to PT-symmetric matrix quantum mechanics, where the matrices are normal but not necessarily Hermitian. The large-$N$ limit can then be taken in PT-symmetric matrix theories just as in the Hermitian case. Quantities of interest such as the scaled ground state energy and scaled moments can be calculated using WKB methods. In the special case of a quartic potential with the “wrong” sign, we prove using functional integration for all values of $N$ that the PT-symmetric model is equivalent to a hermitian matrix model with an anomaly, as in the one-component case [4, 5]. Interestingly, the anomaly vanishes to leading order in the large-$N$ limit.

II. FORMALISM

The solution for all $N$ of the quantum mechanics problem associated with the Euclidean Lagrangian

$$L = \frac{1}{2} \text{Tr} \left( \frac{dM}{dt} \right)^2 + \frac{g}{N} \text{Tr} M^4$$

(1)

where $M$ is an $N \times N$ Hermitian matrix was first given by Brezin et al. [3]. The ground state $\psi$ is a symmetric function of the eigenvalues $\lambda_j$ of $M$. The antisymmetric wave function $\phi$ defined by

$$\phi(\lambda_1, \ldots, \lambda_N) = \prod_{j<k} (\lambda_j - \lambda_k) \psi(\lambda_1, \ldots, \lambda_N)$$

(2)

satisfies the Schrodinger equation

$$\sum_j \left[ -\frac{1}{2} \frac{\partial^2}{\partial \lambda_j^2} + \frac{g}{N} \lambda_j^4 \right] \phi = N^2 E^{(0)} \phi$$

(3)

where $E^{(0)}$ is the ground state energy scaled for the large-$N$ limit. This equation separates into $N$ individual Schrodinger equations, one for each eigenvalue, and the antisymmetry of $\phi$ determines $N^2 E^{(0)}$ as the sum of the $N$ lowest eigenvalues.

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Here we solve the corresponding problem where the potential term is $PT$-symmetric but not Hermitian. As shown by Bender and Boettcher \[6\], the one-variable problem may be solved by extending the coordinate variable into the complex plane. This implies that for $PT$-symmetric matrix problems, we must analytically continue the eigenvalues of $M$ into the complex plane, and in general $M$ will be normal rather than Hermitian. We consider the Euclidean Lagrangian

$$L = \frac{1}{2} \text{Tr} \left( \frac{dM}{dt} \right)^2 - \frac{g}{N^{p/2-1}} \text{Tr} (iM)^p$$

with $g > 0$. Making the substitution $M \rightarrow U \Lambda U^+$, with $U$ unitary and $\Lambda$ diagonal, we can write $L$ as

$$L = \frac{1}{2} \sum_j \left( \frac{d\lambda_j}{dt} \right)^2 + \sum_{j,k} \frac{1}{2} (\lambda_j - \lambda_k)^2 \left( \frac{dH}{dt} \right)_{jk} \left( \frac{dH}{dt} \right)_{kj} - \frac{g}{N^{p/2-1}} \sum_j (i\lambda_j)^p$$

where

$$\frac{dH}{dt} = -iU^+ dU.$$ 

In the analysis of conventional matrix models by Brezin et al., a variational argument shows that the ground state is a singlet, with no dependence on $U$. Because the $\lambda_j$'s are in general complex for $PT$-symmetric theories, this argument does not apply. However, in two cases we can prove that the ground state is indeed a singlet: for $p = 2$, which is trivial, and for $p = 4$, where the explicit equivalence with a hermitian matrix model proven below can be used. Henceforth, we will assume that the ground state is a singlet, but our results will apply in any case to the lowest-energy singlet state.

We have now reduced the problem of finding the ground state to the problem of solving for the first $N$ states of the single-variable Hamiltonian

$$H = \frac{1}{2} p^2 - \frac{g}{N^{p/2-1}} (i\lambda)^p.$$ 

This Hamiltonian is $PT$-symmetric but in general not Hermitian. The case $p = 2$ is the simple harmonic oscillator. For $p > 2$, the Schrödinger equation associated with each eigenvalue may be continued into the complex plane as explained in \[6\]. We exclude the case $p < 2$, where $PT$ symmetry is spontaneously broken and the eigenvalues of $H$ are no longer real.

### III. GROUND STATE PROPERTIES

As with Hermitian matrix models, the ground state energy is the sum of the first $N$ eigenenergies of the Hamiltonian $H$. In the large $N$ limit, this sum may be calculated using WKB. A novelty of WKB for $PT$-symmetric models is the extension of classical paths into the complex plane. This topic has been treated extensively in \[6, 7\].

We define the Fermi energy $E_F$ as the energy of the $N$'th state

$$1 = \frac{1}{2\pi} \int dp d\lambda \theta [E_F - H(p, \lambda)]$$

where the path of integration must be a closed, classical path in the complex $p - \lambda$ plane. In order to construct the large-$N$ limit, we perform the rescaling $p \rightarrow \sqrt{N} p$ and $\lambda \rightarrow \sqrt{N} \lambda$ yielding

$$H_{sc}(p, \lambda) = \frac{1}{2} p^2 - g (i\lambda)^p$$

where the scaled Hamiltonian $H_{sc}$ is related to $H$ by $H = NH_{sc}$. We introduce a rescaled Fermi energy $\epsilon_F$ given by $E_F = N \epsilon_F$, which is implicitly defined by

$$1 = \frac{1}{2\pi} \int dp d\lambda \theta [\epsilon_F - H_{sc}(p, \lambda)].$$ 

After carrying out the integration over $p$, we have

$$1 = \frac{1}{\pi} \int d\lambda \sqrt{2\epsilon_F + 2g (i\lambda)^p} \theta [\epsilon_F + g (i\lambda)^p]$$
where the contour of integration is taken along a path between the turning points which are the analytic continuation of the turning points at \( p = 2 \). This equation determines \( \epsilon_F \) as a function of \( g \).

We define a scaled ground state energy \( E^{(0)}_N \) by

\[
E^{(0)}_N = \frac{1}{N^2} \sum_{k=0}^{N-1} E_k. \tag{12}
\]

The WKB result for the sum of the energies less than \( E_F \) can be written as

\[
\sum_{k=0}^{N-1} E_k = N^2 \int dp \int d\lambda H_{sc}(p, \lambda) \theta [\epsilon_F - H_{sc}(p, \lambda)] \tag{13}
\]

so that in the large-\( N \) limit \( E^{(0)}_\infty \) is given by

\[
E^{(0)}_\infty = \frac{1}{2\pi} \int dp \int d\lambda H_{sc}(p, \lambda) \theta [\epsilon_F - H_{sc}(p, \lambda)]. \tag{14}
\]

The integration over \( p \) is facilitated by using equation (10) to insert a factor of \( \epsilon_F \), giving

\[
E^{(0)}_\infty = \epsilon_F - \frac{1}{2\pi} \int dp \lambda [2\epsilon_F + 2g (i\lambda)^p]^{3/2} \theta [\epsilon_F + g (i\lambda)^p]. \tag{15}
\]

The turning points in the complex \( \lambda \) plane are

\[
\lambda_- = \left( \frac{\epsilon_F}{g} \right)^{1/p} e^{i\pi(3/2-1/p)} \tag{17}
\]

\[
\lambda_+ = \left( \frac{\epsilon_F}{g} \right)^{1/p} e^{-i\pi(1/2-1/p)} \tag{18}
\]

We integrate \( \lambda \) along a two-segment, straight-line path connecting the two turning points via the origin \([6]\). Solving equation (10) for \( \epsilon_F \), we find

\[
\epsilon_F = \left[ \left( \frac{\pi}{2} \right)^p \left( \frac{\Gamma(3/2 + 1/p)}{\sin(\pi/p) \Gamma(1 + 1/p)} \right)^{2p} g^2 \right]^{1/p}, \tag{19}
\]

and solving (16) for the scaled ground state energy we have

\[
E^{(0)}_\infty = \frac{p + 2}{3p + 2} \epsilon_F = \frac{p + 2}{3p + 2} \left[ \left( \frac{\pi}{2} \right)^p \left( \frac{\Gamma(3/2 + 1/p)}{\sin(\pi/p) \Gamma(1 + 1/p)} \right)^{2p} g^2 \right]^{1/p}. \tag{20}
\]

For \( p = 2 \), this evaluates to \( E^{(0)}_\infty = \sqrt{g/2} \), in agreement with the explicit result for the harmonic oscillator.

It is very interesting to compare the large-\( N \) result with results for finite \( N \). The low-lying eigenvalues for the Hamiltonian \( p^2 - (ix)^p \) have been calculated by Bender and Boettcher in [6] for the cases \( p = 3 \) and \( p = 4 \); the case \( p = 2 \) is trivial. We can use their results by noting that the eigenvalues of our Hamiltonian \( H \) are related to theirs by

\[
E_j = \frac{g^{2/(p+2)}}{2p/(p+2) N^{(p-2)/(p+2)}} E^{BB}_j. \tag{21}
\]

Results for \( p = 3 \) and 4 and small values of \( N \) are compared with the large-\( N \) limit in Table 1. The energies for finite values of \( N \) rapidly approach the \( N \to \infty \) limit. The approach to the limit appears monotonic in both cases, but with opposite sign.
The expected value of $\langle TrM \rangle$ for large $N$ is given by

$$
\langle TrM \rangle = \frac{1}{2\pi} \int dp d\lambda \theta \left[ E_F - H(p, \lambda) \right].
$$

Calculations of higher moments $\langle TrM^n \rangle$ are carried out in the same manner. Upon rescaling, we find that $\langle TrM \rangle$ grows as $N^{3/2}$, and the scaled expectation value is given by

$$
\mu = \lim_{N \to \infty} \frac{1}{N^{3/2}} \langle TrM \rangle = \frac{1}{2\pi} \int dp d\lambda \theta \left[ \epsilon_F - H_{sc}(p, \lambda) \right]
$$

which reduces to

$$
\mu = \frac{1}{\pi} \int d\lambda \lambda \sqrt{2\epsilon_F + 2g (i\lambda)^p} \theta \left[ 2\epsilon_F + 2g (i\lambda)^p \right].
$$

Using the same two-segment straight line path as before, we find that

$$
\mu = -i \left( \frac{\pi}{2g} \right)^{1/2} \sin \left( \frac{\pi}{p} \right) \left[ \Gamma(3/2 + 1/p) \right]^{1/2} \left[ \Gamma(1 + 2/p) \right]^{1/2}. \tag{25}
$$

For $p = 2$, $\mu = 0$, as expected for a harmonic oscillator. For $p > 2$, the expectation value $\mu$ is imaginary because $\langle \lambda_j \rangle$ for each eigenstate of the reduced problem is imaginary. For $p = 3$, $\mu = -0.52006i$. For $p = 4$, $\mu = -0.772539i$. In the limit $p \to \infty$, $\mu$ goes to $-i$. This behavior is easy to understand, because in this limit, the turning points become degenerate at $-i$.

### IV. SPECIAL CASE OF $TrM^4$

For the case of a $TrM^4$ interaction, we can explicitly exhibit the equivalence of the PT-symmetric matrix model with a conventional Hermitian quantum mechanical system. As in the single-variable case, there is a parity-violating anomaly, in the form of an extra term in the Hermitian form of the Hamiltonian, proportional to $\hbar$. We show below that the anomaly term does not contribute at leading order in the large-$N$ limit.

The derivation of the equivalence closely follows the path integral derivation for the single-variable case [4, 5]. The Euclidean Lagrangian is

$$
L = \frac{1}{2} Tr \left( \frac{dM}{dt} \right)^2 + \frac{1}{2} m^2 Tr M^2 - \frac{g}{N} Tr M^4 \tag{26}
$$

and the path integral expression for the partition function is

$$
Z = \int [dM] \exp \left\{ - \int dt L \right\}. \tag{27}
$$
Motivated by the case of a single variable, we make the substitution

\[ M = -2i\sqrt{1 + iH} \tag{28} \]

where \( H \) is an Hermitian matrix. Because \( M \) and \( H \) are simultaneously diagonalizable, this transformation is tantamount to the relation

\[ \lambda_j = -2i\sqrt{1 + ih_j} \tag{29} \]

between the eigenvalues of \( M \) and the eigenvalues \( h_j \) of \( H \). The change of variables induces a measure factor

\[ [dM] = \frac{[dH]}{\text{Det} \sqrt{1 + iH}} \tag{30} \]

where the functional determinant depends only on the eigenvalues of \( H \). The Lagrangian becomes

\[ L = \frac{1}{2} \text{Tr} \left( \frac{(dH/dt)^2}{1 + iH} \right) - 2m^2 \text{Tr} (1 + iH) - 16 \frac{g}{N} \text{Tr} (1 + iH)^2 \tag{31} \]

at the classical level. However, following [5], we note that in the matrix case the change of variables introduces an extra term in the potential of the form

\[ \Delta V = \sum_j \frac{1}{8} \left( \frac{d}{dh_j} \left( \frac{dh_j}{d\lambda_j} \right) \right)^2 \tag{32} \]

which can be written as

\[ \Delta V = -\frac{1}{32} \sum_j \frac{1}{1 + ih_j} = -\frac{1}{32} \text{Tr} \left( \frac{1}{1 + iH} \right). \tag{33} \]

The partition function is now

\[ Z = \int \frac{[dH]}{\det \sqrt{1 + iH}} \exp \left\{ -\int dt \left[ \frac{1}{2} \text{Tr} \left( \frac{(dH/dt)^2}{1 + iH} \right) - 2m^2 \text{Tr} (1 + iH) - 16 \frac{g}{N} \text{Tr} (1 + iH)^2 - \frac{1}{32} \text{Tr} \left( \frac{1}{1 + iH} \right) \right] \right\} \tag{34} \]

We introduce a hermitian matrix-valued field \( \Pi \) using the identity

\[ \frac{1}{\det \sqrt{1 + iH}} = \int [d\Pi] \exp \left\{ -\int dt \text{Tr} \left[ \frac{1}{2} (1 + iH) \left( \Pi - \frac{iH + 1/4}{1 + iH} \right)^2 \right] \right\}. \tag{35} \]

Dropping and adding appropriate total derivatives and integrating by parts yields

\[ Z = \int [dH] [d\Pi] \exp \left\{ -\int dt \text{Tr} \left[ -2m^2 (1 + iH) - 16 \frac{g}{N} (1 + iH)^2 + \frac{1}{2} (1 + iH) \Pi^2 + \Pi (1 + iH) - \frac{1}{4} \Pi \right] \right\} \tag{36} \]

The integration over \( H \) is Gaussian, and the shift \( H \to H + i \) gives

\[ Z = \int [d\Pi] \exp \left\{ -\int dt \text{Tr} \left[ \frac{N}{64g} \left( \Pi^2 - 2m^2 \Pi^2 + \frac{1}{4} \Pi^4 \right) - \frac{1}{4} \Pi \right] \right\}. \tag{37} \]

After the rescaling \( \Pi \to \sqrt{32g/N} \Pi \) we have finally

\[ Z = \int [d\Pi] \exp \left\{ -\int dt \text{Tr} \left[ \frac{1}{2} \left( \Pi^2 - 2m^2 \Pi^2 \right) + \frac{4g}{N} \Pi^4 - \sqrt{2g/N} \Pi \right] \right\} \tag{38} \]

We have now proven the equivalence of the PT-symmetric matrix model defined by

\[ L = \frac{1}{2} \text{Tr} \left( \frac{dM}{dt} \right)^2 + \frac{1}{2} m^2 \text{Tr} M^2 - \frac{g}{N} \text{Tr} \Pi^4 \tag{39} \]
to the conventional quantum mechanics matrix model given by

$$L' = \frac{1}{2} \text{Tr} \left( \frac{d\Pi}{dt} \right)^2 - \sqrt{\frac{2g}{N}} \text{Tr}\Pi - m^2 \text{Tr}\Pi^2 + \frac{4g}{N} \text{Tr}\Pi^4. \quad (40)$$

This equivalence implies that the energy eigenvalues of the corresponding Hamiltonians are the same. This could also be proven using the single-variable equivalence for the special case of singlet states, but the functional integral proof encompasses both singlet and non-singlet states at once. The equivalence of these two models also allows for an easy proof of the singlet nature of the ground state. Standard variational arguments show that the ground state of the Hermitian form is a singlet. The direct quantum mechanical equivalence of the single-variable case is then sufficient to prove that the ground state of the \( PT \)-symmetric form is also a singlet.

As in the single-variable case, there is a linear term of order \( \hbar \) appearing in the Lagrangian and Hamiltonian of the Hermitian form of the model. This term represents a quantum mechanical anomaly special to the \( TrM^4 \) model. To determine the fate of the anomaly in the large-\( N \) limit, we construct the scaled Hamiltonian of the Hermitian form in exactly the same way as for the \( PT \)-symmetric form. It is given by

$$H_{sc} = \frac{1}{2} p^2 - \frac{1}{N} \sqrt{2g} x - m^2 x^2 + 4g x^4, \quad (41)$$

indicating that the effect of the anomaly is absent in leading order of the large-\( N \) expansion. One easily checks for the \( m = 0 \) case that the Hermitian form without the linear term reproduces the \( PT \)-symmetric prediction for \( E_{\infty}^{(0)} \) at \( p = 4 \).

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