FACTORIZATIONS OF INVERTIBLE OPERATORS
AND $K$-THEORY OF $C^*$-ALGEBRAS

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Abstract. Let $\mathcal{A}$ be a unital $C^*$-algebra. We describe $K$-skeleton factorizations of all invertible operators on a Hilbert $C^*$-module $\mathcal{H}_\mathcal{A}$, in particular on $\mathcal{H} = l^2$, with the Fredholm index as an invariant. We then outline the isomorphisms $K_0(\mathcal{A}) \cong \pi_{2k}(GL^0(\mathcal{A}))$ and $K_1(\mathcal{A}) \cong \pi_{2k+1}(GL^0(\mathcal{A}))$ for $k \geq 0$, where $[p]$ denotes the class of all compact perturbations of a projection $p$ in the infinite Grassmann space $Gr^\infty(\mathcal{A})$ and $GL^0(\mathcal{A})$ stands for the group of all those invertible operators on $\mathcal{H}_\mathcal{A}$ essentially commuting with $p$.

1. Introduction

Throughout, we assume that $\mathcal{A}$ is any unital $C^*$-algebra. Let $\mathcal{H}_\mathcal{A}$ be the Hilbert (right) $\mathcal{A}$-module consisting of all $l^2$-sequences in $\mathcal{A}$; i.e., $\mathcal{H}_\mathcal{A} := \{\{a_i\} : \sum_{i=1}^\infty a_i^*a_i \in \mathcal{A}\}$, on which an $\mathcal{A}$-valued inner product and a norm are naturally defined by $\langle \{a_i\}, \{b_i\} \rangle := \sum_{i=1}^\infty a_i^*b_i$ and $\|\{a_i\}\| = \|\sum_{i=1}^\infty a_i^*a_i\|^{1/2}$. Let $\mathcal{L}(\mathcal{H}_\mathcal{A})$ stand for the $C^*$-algebra consisting of all bounded operators on $\mathcal{H}_\mathcal{A}$ whose adjoints exist, and let $\mathcal{K}(\mathcal{H}_\mathcal{A})$ denote the closed linear span of all finite rank operators on $\mathcal{H}_\mathcal{A}$, respectively. In case $\mathcal{A}$ is the algebra $\mathbb{C}$ of all complex numbers, $\mathcal{H}_\mathcal{A}$ is the separable, infinite-dimensional Hilbert space $\mathcal{H} = l^2$; correspondingly, $\mathcal{L}(\mathcal{H}_\mathcal{A})$ reduces to the algebra $\mathcal{L}(\mathcal{H})$ of all bounded operators on $\mathcal{H}$, and $\mathcal{K}(\mathcal{H}_\mathcal{A})$ reduces to the algebra $\mathcal{K}$ of all compact operators on $\mathcal{H}$. Each element in $\mathcal{L}(\mathcal{H}_\mathcal{A})$ can be identified with an infinite, bounded matrix whose entries are elements in $\mathcal{A}$ [Zh4, §1]. This identification can be realized by $C^*$-algebraic techniques and the two important $*$-isomorphisms $\mathcal{L}(\mathcal{H}_\mathcal{A}) \cong M(\mathcal{A} \otimes \mathcal{K})$ and $\mathcal{K}(\mathcal{H}_\mathcal{A}) \cong \mathcal{A} \otimes \mathcal{K} = (\lim_n M_n(\mathcal{A}))^\sim$; where $M(\mathcal{A} \otimes \mathcal{K})$ is the multiplier algebra of $\mathcal{A} \otimes \mathcal{K}$ [Kas]. For more information about multiplier algebras the reader is referred to [APT, Bl, Cu1, El, Br2, Pe1, OP, L, Zh4–5], among others. The set of projections $Gr^\infty(\mathcal{A}) := \{p \in \mathcal{L}(\mathcal{H}_\mathcal{A}) : p = p^2 = p^* \text{ and } p \sim 1 \sim 1 - p\}$

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is called the infinite Grassmann space associated with $A$; where \( q \sim p \) is the well-known Murray-von Neumann equivalence of two projections; i.e., there exists a partial isometry $v \in \mathcal{L}(\mathcal{H}_A)$ such that $vv^* = p$ and $v^*v = q$. If $A = C$, then $Gr^\infty(A)$ reduces to the well-known Grassmann space $Gr^\infty(\mathcal{H})$ consisting of all projections on $\mathcal{H}$ with an infinite dimension and an infinite codimension.

2. Factorizations and $K$-theory

Let $p \in Gr^\infty(A)$. If $x$ is any element in $\mathcal{L}(\mathcal{H}_A)$, with respect to the decomposition $p \oplus (1 - p) = 1$ one can write $x$ as a $2 \times 2$ matrix, say $(a \ b; c \ d)$, where $a = pxp$, $b = px(1 - p)$, $c = (1 - p)xp$, and $d = (1 - p)x(1 - p)$. A unitary operator $u = (a \ b; c \ d)$ is called a $K$-skeleton unitary along $A$ if $b$ is some partial isometries in $A \otimes K$. An easy calculation shows that a unitary operator $u$ is a $K$-skeleton unitary if and only if $a$ is a Fredholm partial isometry on the submodule $p \mathcal{H}_A$ and $d$ is a Fredholm partial isometry on the submodule $(1 - p)\mathcal{H}_A$; in other words, all $p - aa^*$, $p - a^*a$, $(1 - p) - dd^*$, $(1 - p) - d^*d$ are projections in $A \otimes K$. The term ‘$K$-skeleton’ is chosen, since $K_0(A)$ is completely described by the homotopy classes of all such unitaries.

Let $GL_p^\infty(A)$ be the topological group consisting of all those invertible operators in $\mathcal{L}(\mathcal{H}_A)$ such that $xp - px \in A \otimes K$, equipped with the norm topology from $\mathcal{L}(\mathcal{H}_A)$. Let $GL^\infty_p(A)$ stand for the path component of $GL^\infty_p(A)$ containing the identity; in the special case when $A = C$, we instead use the notation $GL^\infty_p(\mathcal{H})$ and $GL^\infty_p(\mathcal{H})$, respectively. Let $GL^\infty(A)$ and $GL^\infty_p(\mathcal{A})$ denote the group of all invertible elements in the unitization of $A \otimes K$ and its identity path component, respectively.

2.1. $K$-skeleton factorization theorem [Zh4]. (i) If $x \in GL^\infty_p(A)$, then there exist an element $k \in A \otimes K$, an invertible element $(\begin{smallmatrix} z_1 & 0 \\ 0 & z_2 \end{smallmatrix})$, and a $K$-skeleton unitary $(a \ b; c \ d)$ along $p$ such that $1 + k \in GL^\infty_0(A)$ and

$$x = (1 + k) \begin{pmatrix} z_1 & 0 \\ 0 & z_2 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$ 

A factorization of $x$ with the form above is called a $K$-skeleton factorization along $p$.

(ii) If two $K$-skeleton factorizations of $x$ along $p$ are given, say

$$x = x_0x_p \begin{pmatrix} a & b \\ c & d \end{pmatrix} = x_p'x' \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix},$$

then $[cc^*] - [bb^*] = [c'c'^*] - [b'b'^*] \in K_0(A)$; in other words, $[cc^*] - [bb^*]$ is an invariant independent of all (infinitely many) possible $K$-skeleton factorizations of $x$ along $p$.

Outline of a proof. There is a shorter proof solely for this theorem. For the sake of clarifying some internal relations among $\pi_0(GL^\infty_p(A))$, $\pi_0([p]_0)$, and $K_0(A)$, we outline a proof as follows. First, every element in $GL^\infty_p(A)$ can be written as a product of the form $x_0x_p$ for some invertible $x_0 \in GL^\infty_0(A)$ with $x_0 - 1 \in A \otimes K$ and another invertible $x_p$ with $x_p = px_p$ [Zh4]. Secondly, write the polar decomposition $x = (xx^*)^{1/2}u$, where $(xx^*)^{1/2} \in GL^\infty_p(A)$ and $u$ is a unitary in $GL^\infty_p(A)$. Then consider the following subsets of $Gr^\infty(A)$:

$$[wpu^*] := \{ wpuu^*w^* : w \in GL^\infty_0(A) \text{ with } wu^*u = w^*w = 1 \}$$
are two bijections, which induce the following isomorphisms:

\[ [p]_0 := \{ vpu^* : v \in GL_r^p(A) \quad vv^* = v^*v = 1 \}. \]

Technical arguments show that \([upu^*]_r\) is precisely the path component of \([p]_0\) containing \(upu^*\). Thirdly, there is a representative in \([upu^*]_r\) with the form \((p - r_1) \oplus r_2\) for some projections \(r_1, r_2 \in A \otimes K\). It follows that there exists a unitary \(u_0 \in GL_0^\infty(A)\) such that

\[ u^*_0 upu^*_0 = (p - r_1) \oplus r_2. \]

Then one obtains a \(K\)-skeleton unitary \(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\) such that \(u = u_0 \begin{pmatrix} a & b \\ c & d \end{pmatrix}\), where \(bb^* = r_1\) and \(cc^* = r_2\). Since \((xx^*)^{1/2}u_0 \in GL_0^\infty(A)\), we can rewrite it as a product in the desired form \(x_0(\begin{smallmatrix} z_1 & 0 \\ 0 & z_2 \end{smallmatrix})\). The details are contained in [Zh4].

It follows from Theorem 2.1 that \(x \cdot GL_0^\infty(A) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot GL_0^\infty(A)\) (cosets) for each \(x \in GL_r^p(A)\). The invariant \([cc^*] - [bb^*]\) associated with the \(K\)-skeleton factorization of \(x \in GL_r^p(A)\) yields the bijection

\[ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot GL_0^\infty(A) \leftrightarrow [(p - bb^*) \oplus (cc^*)]_r. \]

It can be shown that \([ (p - r_1) \oplus r_1^\prime \] \(= [(p - r_2) \oplus r_2^\prime]_r\) iff \([r_1^\prime] - [r_1] = [r_2^\prime] - [r_2] = 0\) in \(K_0(A)\). Therefore, we conclude the following theorem whose details are given in [Zh4].

**2.2. Theorem [Zh4].** The maps defined by

\[ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot GL_0^\infty(A) \rightarrow [(p - r_1) \oplus r_2]_r \rightarrow [r_2] - [r_1] \]

are two bijections, which induce the following isomorphisms:

\[ GL_r^p(A) / GL_0^\infty(A) \cong D_h([p]_0) \cong K_0(A), \]

where \(GL_r^p(A) / GL_0^\infty(A)\) is the quotient group with the induced multiplication and

\[ D_h([p]_0) = \{ [upu^*]_r : u \in GL_r^p(A) \quad \text{with} \quad uu^* = u^*u = 1 \} \]

is the set of all path components of \([p]_0\). The group operation on \(D_h([p]_0)\) is defined by

\[ [(p - r_1) \oplus r_1^\prime]_r + [(p - r_2) \oplus r_2^\prime]_r = [(p - r_1 - s_2) \oplus (r_1^\prime + s_2^\prime)]_r \]

for some projections \(s_2 \in p(A \otimes K)p\) and \(s_2^\prime \in (1 - p)(A \otimes K)(1 - p)\) such that \(s_2 \sim r_2, s_2r_1 = 0, s_2^\prime \sim r_2^\prime,\) and \(s_2^\prime r_1 = 0\).
**2.3. Theorem.** Let the base point of \([p]_0\) be \(p\) and the base point of \(GL_p^\infty(A)\) be the identity. Then

\[
\pi_{2k+1}([p]_0) \cong \pi_{2k+1}(GL_p^\infty(A)) \cong K_1(A),
\]

and

\[
\pi_{2k+2}([p]_0) \cong \pi_{2k+2}(GL_p^\infty(A)) \cong K_0(A) \quad \forall k \geq 0.
\]

**Outline of a proof.** Let \(U_\infty(A)\) be the unitary group of the unitization of \(A \otimes K\), and let \(U_p(A)\) be the subgroup of \(U_\infty(A)\) consisting of all those unitaries commuting with \(p\). First, the map \(\psi_p : U_\infty(A) \rightarrow [p]_r\) defined by \(\psi_p(u) = upu^*\) is a Serre (weak) fibration with a standard fiber \(U_p(A)\) [Zh6, §2]. Secondly, the long exact sequence of homotopy groups associated with this fibration breaks into short exact sequences [Zh6, 2.5, 2.8]:

\[
0 \rightarrow \pi_{k+1}([p]_r) \rightarrow \pi_k(U_p(A)) \rightarrow \pi_k(U_\infty(A)) \rightarrow 0 \quad (k \geq 0).
\]

Thirdly, by an analysis on this short exact sequence one concludes

\[
\pi_{2k+2}([p]_0) \cong K_0(A) \quad \text{and} \quad \pi_{2k+1}([p]_0) \cong K_1(A) \quad (k \geq 0).
\]

It is well known that the subgroup \(U_p^\infty(A)\) consisting of all unitary elements in \(GL_p^\infty(A)\) is homotopy equivalent to \(GL_p^\infty(A)\). We consider the maps \(U_p^\infty(A) \rightarrow [p]_0\) defined by \(\phi_p(u) = upu^*\). It can be shown that \(\phi_p\) is a weak fibration with a standard fiber \(U_p^p(A)\), where \(U_p(A)\) is the group consisting of all those unitaries in \(U_p^\infty(A)\) commuting with \(p\). An argument similar to that above applies to this fibration. One can show that \(\pi_{2k+1}(U_p^\infty(A)) \cong K_1(A)\) and \(\pi_{2k+2}(U_p^\infty(A)) \cong K_0(A)\) for \(k \geq 0\). The details are given in [Zh6, §4].

**2.4. Special case.** Let \(A = C(X)\). In particular, if \(A\) is taken to be the commutative \(C^*\)-algebra \(C(X)\) consisting of all complex-valued continuous functions on a compact Hausdorff space \(X\), then each element in \(\mathcal{L}(H_{C(X)})\) can be identified with a norm-bounded, \(*\)-strong continuous map from \(X\) to \(\mathcal{L}(H)\) [APT]. Here \(\mathcal{L}(H) \supset \{x_\lambda\}\) converges to \(x\) in the \(*\)-strong operator topology iff

\[
||((x_\lambda - x)k|| + ||k(x_\lambda - x)|| \rightarrow 0 \quad \text{for any} \ k \in K.
\]

Obviously, \(\mathcal{L}(H_{C(X)})\) contains the \(C^*\)-tensor product \(\mathcal{L}(H) \otimes C(X)\) consisting of all norm-continuous maps from \(X\) to \(\mathcal{L}(H)\) as a \(C^*\)-subalgebra. Then Theorems 2.1 and 2.2 in this special case are interpreted as follows.

**2.5. Corollary.** Let \(GL^\infty(H)\) be the group of all invertible operators in \(\mathcal{L}(H)\).

(i) If \(f : X \rightarrow GL^\infty(H)\) is a norm-bounded, \(*\)-strong continuous map and \(p\) is a projection in the infinite Grassmann space \(Gr^\infty(H)\) such that \(pf - fp \in K \otimes C(X)\), then \(f\) can be factored as the following product of three invertible maps

\[
f(. ) = \begin{pmatrix}
1 + k_{11}(.) & k_{12}(.) \\
k_{21}(.) & 1 + k_{22}(.)
\end{pmatrix}
\begin{pmatrix}
g_1(.) & 0 \\
0 & g_2(.)
\end{pmatrix}
\begin{pmatrix}
a(.) & b(.) \\
c(.) & d(.)
\end{pmatrix};
\]
where \( k_{ij}(\cdot)'s \) are norm-continuous maps from \( X \) to \( K \), \( g_1(\cdot) \oplus g_2(\cdot) \) is a norm-bounded, *-strong continuous map from \( X \) to \( GL^\infty(\mathcal{H}) \), \( a(\cdot), d(\cdot) \) are *-strong continuous maps from \( X \) to the set of Fredholm partial isometries on \( pH \) and \((1-p)\mathcal{H} \), respectively, and \( c(\cdot), b(\cdot) \) are norm-continuous maps from \( X \) to the set of partial isometries in \( K \). Furthermore,

\[
[c(\cdot)c(\cdot)^*] - [b(\cdot)b(\cdot)^*] \in K_0(C(X)) \; (\cong K^0(X))
\]
is an invariant independent of all possible factorization with the above form.

(ii) The groups \([X,GL_p^s(\mathcal{H})], [X,[p]_0], \text{and } K_0(C(X))\) are isomorphic, where \([X,\cdot]\) is the set of homotopy classes of norm-bounded, *-strong continuous maps from \( X \) to \( \cdot \).

2.6. **Invertible dilations of a Fredholm operator.** Let us illustrate a \( K \)-skeleton factorization of any invertible dilation of a Fredholm operator \( x \in \mathcal{L}(\mathcal{H}_A) \). There are of course infinitely many invertible \( 2 \times 2 \) matrices with the form

\[
D_2(x) := \begin{pmatrix} x & y_1 \\ y_2 & z \end{pmatrix} \in M_2(\mathcal{L}(\mathcal{H}_A)).
\]

Each such \( 2 \times 2 \) invertible matrix is called an invertible dilation of \( x \). Specific constructions of such a dilation were given by P. Halmos [Ho, 222] and A. Connes [Co]. For each invertible dilation of \( x \) it follows from the \( K \)-skeleton Factorization Theorem 2.1 that

\[
\begin{pmatrix} x & y_1 \\ y_2 & z \end{pmatrix} = \begin{pmatrix} 1 + a_{11} & a_{12} \\ a_{21} & 1 + a_{22} \end{pmatrix} \begin{pmatrix} z_1 & 0 \\ 0 & z_2 \end{pmatrix} \begin{pmatrix} v & 1 - vv^* \\ 1 - v^*v & -v^* \end{pmatrix},
\]

where \( a_{ij}'s \) are some elements in \( A \otimes K \), \( z_1, z_2 \in GL^\infty(\mathcal{A}) \), and the above matrix on the right, say \( w \), is a familiar unitary matrix occurring in the index map in \( K \)-theory [Bl, 8.3] in which \( v \) is a Fredholm partial isometry in \( \mathcal{L}(\mathcal{H}_A) \). Set \( p = \text{diag}(1,0) \). It is well known that

\[
[1 - v^*v] - [1 - vv^*] \in K_0(\mathcal{A})
\]
is precisely the Fredholm index \( \text{Ind}(v) = \text{Ind}(pxp) \) (on \( p\mathcal{H}_A \)). It follows from Theorem 2.1(ii) that those \( K \)-skeleton unitaries associated with all possible invertible dilations of \( x \) in \( M_2(\mathcal{L}(\mathcal{H}_A)) \) only differ from \( w \) by a factor in \( GL_{2n}(\mathcal{A}) \).

3. **Factorizations of invertible operators with integer indices**

Now we consider some special cases such that \( K_0(\mathcal{A}) \cong Z \) (the group of all integers); for example, \( \mathcal{A} = \mathbb{C} \), or \( \mathcal{A} = C(S^{2n+1}) \) where \( S^m \) is the standard \( m \)-sphere, or \( \mathcal{A} = \mathcal{O}_\infty \), the Cuntz algebra generated by isometries \( \{s_i\}_{i=1}^\infty \subset \mathcal{L}(\mathcal{H}) \) such that \( \sum_{i=1}^\infty s_i s_i^* \leq 1 \).

Let \( p \) be any projection in \( G_{r}^\infty(\mathcal{H}) \subset G_{r}^\infty(\mathcal{A}) \) [the inclusion holds because \( \mathcal{H} \subset \mathcal{H}_A \) and \( \mathcal{L}(\mathcal{H}) \subset \mathcal{L}(\mathcal{H}_A) \)]. Let \( \{\xi_i\}_{i=0}^\infty \) be any orthonormal basis of the subspace \( p\mathcal{H} \) and \( \{\xi_i\}_{i=-\infty}^{-1} \) be any orthonormal basis of the subspace \((1-p)\mathcal{H} \). Then \( \{\xi_i\}_{i=-\infty}^\infty \) is an orthonormal basis of both \( \mathcal{H} \) and \( \mathcal{H}_A \). Let \( u_0 \) denote the bilateral shift associated with the basis \( \{\xi_i\}_{i=-\infty}^\infty \) of \( \mathcal{H} \), defined by \( u_0(\xi_i) = \xi_{i+1} \) for all \( i \in Z \). Clearly, \( u_0 \) is a \( K \)-skeleton unitary of \( \mathcal{L}(\mathcal{H}_A) \) along \( p \). Applying the \( K \)-skeleton Factorization Theorem 2.1 to the above special cases, we have the following factorizations of invertible operators orientated by the integer-valued Fredholm index:
3.1. **Corollary.** Suppose that $K_0(\mathcal{A}) \cong \mathbb{Z}$ is generated by $[1]$ where $1$ is the identity of $\mathcal{A}$. If $x$ is an invertible operator on $\mathcal{H}_\mathcal{A}$ such that $px - xp \in \mathcal{A} \otimes \mathcal{K}$, then $x = (1 + k)x_{p}u_{n}$, where $k \in \mathcal{A} \otimes \mathcal{K}$, $x_{p}$ is an invertible operator commuting with $p$, and the integer $n$ is the Fredholm index of $pxp$ on the submodule $p\mathcal{H}_\mathcal{A}$, say $\text{Ind}(pxp)$, which is independent of the choice of $\{\xi_i\}^\infty_{i=0}$, $\{\xi_i\}^{-\infty}_{-\infty}$ and all possible factorizations along $p$ with the same form above.

Outline of a proof. It is obvious that $\text{Ind}(pu_0^n p) = -n$. Let $G$ be the group $\{u_i^n : n \in \mathbb{Z}\}$ in which every element is a $K$-skeleton unitary along $p$. As a special case of Theorem 2.1 one can show that the map from $G$ to $GL^p(\mathcal{A})/GL^p_{\infty}(\mathcal{A})$ defined by $u_0^n \mapsto u_0^n \cdot GL^p_{\infty}(\mathcal{A})$ is a group isomorphism. It follows that $\pi_0(GL^p(\mathcal{A})) = \{u_0^n \cdot GL^p_{\infty}(\mathcal{A}) : n \in \mathbb{Z}\}$. Then the factorization follows. The reader may want to consider the extreme case $\mathcal{A} = \mathbb{C}$ and then generalize the conclusion to a larger class of $C^*$-algebras.

A similar proof yields the following alternative factorization of $x$ as a product of three invertibles under the same assumptions as of Corollary 3.1:

$$x = \begin{cases} (1 + k_1)x_1 & \text{if } \text{Ind}(pxp) = 0, \\ (1 + k_2)x_2(u_1 \oplus u_2 \oplus \cdots \oplus u_{-n} \oplus w_1) & \text{if } \text{Ind}(pxp) = n < 0, \\ (1 + k_3)x_3(u_1^* \oplus u_2^* \oplus \cdots \oplus u_n^* \oplus w_2) & \text{if } \text{Ind}(pxp) = n > 0, \end{cases}$$

where $u_i$ is a bilateral shift on a subspace $\mathcal{H}_i$ of $\mathcal{H}$ for $1 \leq i \leq n$, $w_j$’s are unitary operators on $(\bigoplus_{i=1}^n \mathcal{H}_i)^\perp$, $k_j \in \mathcal{A} \otimes \mathcal{K}$, and $x_j$’s are invertible operators commuting with $p$.

3.2. **Corollary.** Suppose that $K_0(\mathcal{A}) \cong \mathbb{Z}$ is generated by $[1]$. If $x$ is an arbitrary element $\mathcal{L}(\mathcal{H}_\mathcal{A})$ and $p \in G^{r\infty}(\mathcal{H})$ (as above) such that $px - xp \in \mathcal{A} \otimes \mathcal{K}$, then there exists a unique norm-continuous map $x(\lambda)$ from $C \setminus (\sigma(x) \cup \{0\})$ to $GL^p_{\infty}(\mathcal{A})$, where $\sigma(x)$ is the spectrum of $x$, such that $x - \lambda = x(\lambda)u_0^{-n}$, where $n_i = \text{Ind}(p(x - \lambda_i)p)$ and $\lambda_i$ is any complex number in the $i$th path component $O_i$ of $C \setminus \sigma(x)$. An alternative $K$-skeleton factorization of $x - \lambda$ for $\lambda \in O_i$ is as follows (when $n_i \neq 0$):

$$x - \lambda = \begin{cases} y_i(\lambda)(u_1 \oplus u_2 \oplus \cdots \oplus u_{n_i} \oplus w_i) & \text{if } \text{Ind}(p(x - \lambda_i)p) = n_i < 0, \\ y_i(\lambda)(u_1^* \oplus u_2^* \oplus \cdots \oplus u_n^* \oplus v_i) & \text{if } \text{Ind}(p(x - \lambda_i)p) = n_i > 0, \end{cases}$$

where $u_i$’s are bilateral shifts on mutually orthogonal closed subspaces $\mathcal{H}_i$’s of $\mathcal{H}$, $w_i$’s, $v_i$’s are unitary operators on the subspace $(\bigoplus_{i=1}^{n_i} \mathcal{H}_i)^\perp$, and $y_i(\lambda), y_i^*(\lambda)$ are norm-continuous maps from $O_i$ to $GL^p_{\infty}(\mathcal{A})$.

3.3. **Winding numbers of invertible operators.** Using the first factorization in Corollary 3.2, we assign an integer $n_i$ to each path component $O_i$ of $C \setminus \sigma(x)$, which is precisely the minus winding number of $w_0^{-n}$ as a continuous map from $S^1$ to $S^1$ (via the Gel’fand transformation). We call $n_i$ the *winding number of $x$ along $p$ over $O_i$*. As a particular case, if $x$ is an operator whose essential spectrum, the spectrum of $\pi(x)$ in the generalized Calkin algebra $\mathcal{L}(\mathcal{H}_\mathcal{A})/K(\mathcal{H}_\mathcal{A})$, does not separate the plane, then all winding numbers of $x$ along any $p \in G^{r\infty}(\mathcal{A})$ are zero as long as $px - xp \in \mathcal{A} \otimes \mathcal{K}$. There is another way to describe the integer $n_i$.

3.4. **Corollary.** Let $G_i(x)$ denote the subgroup of $GL^p(\mathcal{A})$ generated by $GL^p_{\infty}(\mathcal{A})$ and $x - \lambda_i$ where $\lambda_i \in O_i$. Then $G_i(x)/GL^p_{\infty}(\mathcal{A}) \cong \mathbb{Z}_{n_i}$, and hence $GL^p(\mathcal{A})/G_i(x) \cong \mathbb{Z}_{n_i}$, the finite cyclic group of order $n_i$. 
In particular, one can apply the above factorizations to an invertible dilation of a pseudodifferential operator of order zero on a compact manifold and classical multiplication operators. Let us spend few lines to look at the following familiar examples.

3.5. Multiplication operators. Let \( M_f \) be the invertible multiplication operator with symbol \( f \) in \( L^\infty(S^1) \), where \( S^1 \) is the unit circle; i.e., \( M_f(g) = fg \) for any \( g \in L^2(S^1) \). If \( p \) is a projection on \( L^2(S^1) \) such that \( \dim(p) = \text{codim}(1 - p) = \infty \), and \( pM_f - M_f p \) is a compact operator, then it follows from Corollary 3.1 that \( M_f = (1 + k)x_pu_0^{-n} \), where \( n = \text{Ind}(pM_f p) \), \( k \) is a compact operator on \( L^2(S^1) \), \( x_p \) is an invertible operator on \( L^2(S^1) \) commuting with \( p \), and \( u_0 \) is a bilateral shift operator associated with a fixed orthonormal basis of \( L^2(S^1) \). It is well known that \( pM_f p \) is a familiar Toeplitz operator on the subspace \( pL^2(S^1) \).

3.6. Restricted loop group along \( p \in Gr^\infty(H) \). Consider the following restricted loop group along \( p \) consisting of all norm-bounded, \(*\)-strong continuous maps from \( S^1 \) to \( GL_p(H) \), denoted by \( \text{Map}(S^1, GL_p(H))_\beta \). Since \( K_0(C(S^1)) = Z \), each \( f \in \text{Map}(S^1, GL_p(H))_\beta \) can be factored as \( f = (1 + f_0)f_1u_0^{-n} \), where \( n = \text{Ind}(pfp) \), \( f_0 \) is a norm-continuous map from \( S^1 \) to \( K \), \( f_1 \) is a \(*\)-strong continuous map from \( S^1 \) to \( GL_\infty(H) \) such that \( f_1(z)p = pf_1(z) \) for any \( z \in S^1 \), and \( u_0 \) is a bilateral shift with respect to a fixed orthonormal basis of \( H \). If \( f \) is norm-continuous, then \( f_1 \) is also norm continuous. Furthermore, \( [S^1, GL_p(H)] \cong [X, [p]_0] \cong Z \). The same conclusions also hold, if \( S^1 \) is replaced by \( S^{2n+1} \) for any \( n \geq 1 \).

3.7. Remarks. (i) Theorems 2.1–2.3 still hold, if \( A \) is any stably unital \( C^* \)-algebra; i.e., \( A \otimes K \) has an approximate identity consisting of a sequence of projections \([B1, 5.5.4; Z Hab7]\).

(ii) Let \( \text{Index}(x, p) \) denote the invariant \([cc^*] - [bb^*] \in K_0(A) \) in Theorem 2.1(ii). If \( p \) is fixed, then \( \text{Index}(x, p) \) is precisely the Fredholm index of \( pxp \) as an operator on \( pH_A \) and fits into the established theory of the \( K_0(A) \)-valued Fredholm index. However, some new results do arise from invariants of \( \text{Index}(x, p) \) as the variable \( p \) runs in \( \{ p \in Gr^\infty(A) : xp - px \in A \otimes K \} \) or as \( x \) and \( p \) jointly change \([Zhab7]\). As a matter of fact, \( \text{Index}(x, p) \) is an invariant under homotopy and perturbation by elements in \( A \otimes K \) with respect to both variables \( x \) and \( p \). For example, by the combination of the \( K \)-skeleton Factorization Theorem and certain invariants of \( \text{Index}(x, p) \), we proved \([Zhab7]\) the following:

**Theorem.**

\[
\pi_0(GL(M_n(C)c)_e) \cong \{ k \in K_0(A) : n \cdot k = 0 \} \quad \text{for any } n \geq 2;
\]

where \( GL(M_n(C)c)_e \) denotes the group of all invertibles in the essential commutant \( M_n(C)c \) of \( M_n(C) \) which is naturally embedded in \( M_n(L(H_A)) \).

(iii) The reader may want to compare (3.1)–(3.3) and the famous BDF theory \([BDF1, 2]\) to see their obvious relations; we work with invariants on \( H_A \), while the BDF theory dealt with Fredholm operators.

(iv) In \([PS]\) Pressley and Segal have studied the restricted general linear group

\[
GL_{res}(H) := \{ x \in GL^\infty(H) : xp - px \text{ is Hilbert-Schmidt} \}
\]

and given some applications to the Kdv equations. It is a hope that our results will shed some light in the same direction.
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