A simple counterexample related to the Lie–Trotter product formula

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Abstract
In this note a very simple example is given which shows that if the sum of two semigroup generators is itself a generator, the generated semigroup in general can not be represented by the Lie-Trotter product formula.

1 Introduction
In 1959, H.F. Trotter [8] extended the Lie product formula for matrices
\[ e^{A+B} = \lim_{n \to \infty} \left( e^{\frac{t}{n}A} e^{\frac{t}{n}B} \right)^n \]
to unbounded operator in Banach spaces. The original result can be generalized and one can prove the following theorem (see [4], Chapter III, Corollary 5.8)

**Theorem 1.1** Let \((S(t))_{t \geq 0}\) and \((T(t))_{t \geq 0}\) be strongly continuous semigroups on a Banach space \(X\) satisfying the stability condition

\[ \left\| \left[ S \left( \frac{t}{n} \right) T \left( \frac{t}{n} \right) \right]^n \right\| \leq M e^{\omega t}, \quad \text{for all } t \geq 0, \ n \in \mathbb{N} \setminus \{0\} \quad (1.1) \]

and for constants \(M \geq 1, \ w \in \mathbb{R}\). Consider the “sum” \(A + B\) on \(\mathcal{D} := \mathcal{D}(A) \cap \mathcal{D}(B)\) of the generators \((A, \mathcal{D}(A))\) of \((S(t))_{t \geq 0}\) and \((B, \mathcal{D}(B))\) of \((T(t))_{t \geq 0}\), and assume that \(\mathcal{D}\) and \((\lambda_0 - A - B)\mathcal{D}\) are dense in \(X\) for some \(\lambda_0 > w\). Then \(C := A + B\) generates a strongly continuous semigroup \((U(t))_{t \geq 0}\) given by the Trotter product formula:

\[ U(t)x = \lim_{n \to \infty} \left[ S \left( \frac{t}{n} \right) T \left( \frac{t}{n} \right) \right]^n x, \quad \text{for all } x \in X, \quad (1.2) \]

with uniform convergence for \(t\) in compact intervals.
From this theorem, the following corollary can be deduced

Corollary 1.2 Let \((S(t))_{t \geq 0}\) and \((T(t))_{t \geq 0}\) be strongly continuous semigroups on a Banach space \(X\), with generators \((A, D(A))\) and \((B, D(B))\) respectively, such that:

a) the stability condition \((1.1)\) holds;

b) the closure of the sum of the generators \(C = A + B\) generates a strongly continuous semigroup \((U(t))_{t \geq 0}\).

Then \((U(t))_{t \geq 0}\) is given by the Trotter product formula \((1.2)\) with uniform convergence for \(t\) in compact intervals.

This raises the question whether \((a)\) is enough to guarantee the convergence of the Trotter product formula \((1.2)\) to the generated semigroup \((U(t))_{t \geq 0}\).

In 2000 F. Kühnemund and M. Wacker [5] answered negatively to this question providing a counterexample. However, their counterexample is quite elaborated and makes use of results on strongly continuous evolution families.

Aim of this note is to show that very simple and natural counterexamples come from linear hyperbolic systems of partial differential equations. In particular we will show that there exist very simple examples of semigroups such that the sum of their generators generates itself (without need of taking any closure) a semigroup, but this semigroup can not be obtained through the Trotter product formula \((1.2)\).

A counterexample showing that \((a)\) is not sufficient for the Trotter formula to hold is already present in the original paper by Trotter [8]. A counterexample relevant for nonlinear semigroups is given in [7].

In [2, 3] there are extensions of Corollary 1.2 to nonlinear semigroups in metric spaces requiring commutator conditions instead of \((b)\). A commutator condition was also used in a linear setting in [6].

2 The counterexample

Let \(X\) be the Banach space of all vector valued bounded and uniformly continuous functions on \(\mathbb{R}\), \(X = C_{ub}(\mathbb{R}, \mathbb{R}^n)\), provided with the supremum norm

\[
\|f\|_X = \sup_{x \in \mathbb{R}} \|f(x)\|,
\]

where \(\| \cdot \|\) denotes the standard Euclidean norm in \(\mathbb{R}^n\).

Take now a \(n \times n\) hyperbolic matrix \(A\), that is a real diagonalizable matrix with real eigenvalues, and denote by \(\lambda_1^A \leq \ldots \leq \lambda_n^A\), \(r_1^A, \ldots, r_n^A\)
and \( l_1^A, \ldots, l_n^A \) its eigenvalues, right eigenvectors and left eigenvectors respectively, normalized in such a way that \( \| r_i^A \| = 1 \) for \( i = 1, \ldots, n \) and \( \left\langle t_j^A, r_i^A \right\rangle = \delta_{ij} \) for \( i, j = 1, \ldots, n \), where the symbol \( \left\langle \cdot, \cdot \right\rangle \) denotes the scalar product in \( \mathbb{R}^n \) and \( \delta_{ij} \) is the Kronecker delta.

Then, one can define the following strongly continuous semigroup (it is a group actually) on \( X \):

\[
\left[ S(t)f \right](x) = \sum_{i=1}^n \left\langle t_i^A, f(x + \lambda_i^A t) \right\rangle r_i^A, \quad t \geq 0, \ x \in \mathbb{R} \quad f \in X. \tag{2.3}
\]

Indeed we have the following proposition.

**Proposition 2.1** If the hyperbolic matrix \( A \) is invertible, then \( \left( S(t) \right)_{t \geq 0} \) defined in (2.3) is a strongly continuous group on the Banach space \( X \) and its generator \( (A, D(A)) \) is given by

\[
D(A) = \{ f \in X : f' \in X \},
\]

\[
Af = A f' \quad \text{for all } f \in D(A). \tag{2.4}
\]

If \( f \in D(A) \) the function \( u(t, x) = \left[ S(t)f \right](x) \) is the unique classical solution to the following Cauchy problem for a hyperbolic system of first order partial differential equations

\[
\begin{align*}
\frac{\partial}{\partial t} u(t, x) &= A \frac{\partial}{\partial x} u(t, x) \\
u(0, x) &= f(x).
\end{align*}
\]

**Proof.** If \( f \in X \), \( S(t)f \in X \) for all \( t \in \mathbb{R} \) since any linear combination of uniformly continuous bounded functions is a uniformly continuous bounded function. The group property follows from the relation \( \left\langle t_i^A, r_j^A \right\rangle = \delta_{ij} \) by direct computations. Finally the uniform continuity of \( f \in X \) implies that \( \lim_{t \to 0} S(t)f = f \) in \( X \). This concludes the proof that \( S(t) \) is a strongly continuous group in \( X \).

Denote now by \( (\tilde{A}, D(\tilde{A})) \) its generator. We have to show that \( \tilde{A} = A \) as defined in (2.4). Take first \( f \in D(A) \) and note that \( f = \sum_{i=1}^n \left\langle t_i^A, f \right\rangle r_i^A \) and that the action of the hyperbolic matrix \( A \) can be expressed by \( A f' = \)
\[ \sum_{i=1}^{n} \langle t_i^A, f' \rangle \lambda_i^A r_i^A. \] We can compute
\[
\lim_{t \to 0} \left\| \frac{S(t)f - f}{t} - Af' \right\| = \lim_{t \to 0} x \left\| \left[ \frac{S(t)f}{t} - f(x) \right] - Af'(x) \right\|
\]
\[
= \lim_{t \to 0} x \left\| \frac{1}{t} \sum_{i=1}^{n} \langle t_i^A, f(x + \lambda_i^A t) - f(x) \rangle r_i^A - \sum_{i=1}^{n} \langle t_i^A, f'(x) \rangle \lambda_i^A r_i^A \right\|
\]
\[
= \lim_{t \to 0} x \left\| \sum_{i=1}^{n} \left( \frac{t_i^A f(x + \lambda_i^A t) - f(x)}{t} - \lambda_i^A f'(x) \right) r_i^A \right\|
\]
\[
= \lim_{t \to 0} x \left\| \sum_{i=1}^{n} \left( t_i^A, \lambda_i^A \int_{1}^{t} \left( f'(x + \lambda_i^A st) - f'(x) \right) ds \right) r_i^A \right\| = 0,
\]
where we have used the uniform continuity of \( f' \). Hence if \( f \in \mathcal{D}(A) \) then \( f \in \mathcal{D}(\tilde{A}) \) and \( \tilde{A}f = Af \), that is \( A \subset \tilde{A} \).

Now choose \( f \in \mathcal{D}(\tilde{A}) \), then \( \lim_{t \to 0} \frac{S(t)f - f}{t} = g \) in \( X \) for some \( g = \tilde{A}f \in X \). This implies the following pointwise limit
\[
\lim_{t \to 0} \sum_{i=1}^{n} \left( \frac{t_i^A f(x + \lambda_i^A t)}{t} - \frac{t_i^A f(x)}{t} \right) r_i^A = \sum_{i=1}^{n} \left( t_i^A, \tilde{A}f(x) \right) r_i^A.
\]
Since \( \lambda_i^A \neq 0 \) for \( i = 1, \ldots, n \) (\( A \) is invertible), we obtain that \( \langle t_i^A, f(x) \rangle \) is differentiable with derivative \( \frac{1}{\lambda_i^A} \left( t_i^A, \tilde{A}f(x) \right) \). Then \( f(x) = \sum_{i=1}^{n} \left( t_i^A, f(x) \right) r_i^A \) is differentiable with derivative \( f'(x) = \sum_{i=1}^{n} \frac{1}{\lambda_i^A} \left( t_i^A, \tilde{A}f(x) \right) r_i^A \in X \), which implies \( f \in \mathcal{D}(A) \), \( Af = Af' = \tilde{A}f \), \( \tilde{A} \subset A \) concluding the proof that \( A \) is the generator of \( S(t) \).

The last statement of the theorem follows by direct computations (see for instance [1]).

\[ \square \]

Take now \( A, B \) and \( C \) three \( n \times n \) matrices such that:

**H** they are invertible hyperbolic matrices, \( C = A + B \) and the greatest eigenvalue of the matrix \( C \) is bigger than the sum of the greatest eigenvalues of the matrices \( A, B \): \( \lambda_n^C > \lambda_n^A + \lambda_n^B \).

**Remark 2.2** Given two arbitrary invertible hyperbolic matrices, their sum needs not to be even hyperbolic, but three matrices satisfying the property
above exist, take for example
\[ A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 \\ 4 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 2 \\ 5 & 0 \end{pmatrix}. \]

For these matrices we have \( C = A + B, \lambda^A_1 = -1, \lambda^B_1 = 1; \lambda^C_1 = -2, \lambda^C_2 = 2; \lambda^C_2 = -\sqrt{10}, \lambda^C_2 = \sqrt{10}, \) so that \( \lambda^C_2 = \sqrt{10} > 3 = 1 + 2 = \lambda^A_2 + \lambda^B_2. \)

Given three matrices \( A, B \) and \( C \) satisfying \((H)\), define now the following three strongly continuous semigroups on \( X \):

\[
\begin{align*}
[S(t)f](x) &= \sum_{i=1}^{n} \langle t^A_i, f(x + \lambda^A_i t) \rangle r^A_i, \quad t \geq 0, x \in \mathbb{R} \quad f \in X, \\
[T(t)f](x) &= \sum_{i=1}^{n} \langle t^B_i, f(x + \lambda^B_i t) \rangle r^B_i, \quad t \geq 0, x \in \mathbb{R} \quad f \in X, \quad (2.5) \\
[U(t)f](x) &= \sum_{i=1}^{n} \langle t^C_i, f(x + \lambda^C_i t) \rangle r^C_i, \quad t \geq 0, x \in \mathbb{R} \quad f \in X.
\end{align*}
\]

Observe that, by Proposition 2.1, their generators \((A, D(A)), (B, D(B))\) and \((C, D(C))\) satisfy

\[ D(A) = D(B) = D(C) = D = \{ f \in X : f' \in X \}, \]

so that the sum of the generators \( A \) and \( B \),

\[ (A + B) f = Af + Bf = Af' + Bf' = (A + B) f' = Cf' = Cf, \]

for any \( f \in D \),

is already closed and generates \((U(t))_{t \geq 0}\).

We can now state our main result.

**Theorem 2.3** If the matrices \( A, B, C \) satisfy \((H)\), then the semigroup \((U(t))_{t \geq 0}\) is not given by the Trotter product formula of \((S(t))_{t \geq 0}\) and \((T(t))_{t \geq 0}\), i.e. there exists a function \( f \in X \) such that

\[ U(t)f \neq \lim_{n \to +\infty} \left( S \left( \frac{t}{n} \right) T \left( \frac{t}{n} \right) \right)^n f. \quad (2.6) \]

**Proof.** Take the eigenvector \( r^C_n \) corresponding to the greatest eigenvalue \( \lambda^C_n \) of \( C \). Take now a function \( \varphi \in C^\infty(\mathbb{R}, \mathbb{R}) \) satisfying \( \varphi(x) > 0 \) for \( x \in (0, 1) \) and \( \varphi(x) = 0 \) for \( x \notin (0, 1) \). Our function, which will be shown to satisfy \((2.6)\), is given by \( f = \varphi r^C_n \). Indeed we can compute

\[
[U(t)f](x) = \sum_{i=1}^{n} \langle t^C_i, \varphi(x + \lambda^C_i t) r^C_n \rangle r^C_i = \varphi(x + \lambda^C_i t) r^C_n,
\]
therefore $(U(t)f)(x) \neq 0$ for $x \in (-\lambda_n^C t, t)$. Observe now that if $g \in X$ is such that $g(x) = 0$ for all $x \leq a$ for some $a \in \mathbb{R}$, then

$$[S(t)g](x) = \sum_{i=1}^{n} \langle t_i^A, g(x + \lambda_i^A t) \rangle r_i^A = 0 \text{ for all } x \leq a - \lambda_i^A t \quad (2.7)$$

since when $x \leq a - \lambda_i^A t$, then $x + \lambda_i^A t \leq a - \left(\lambda_i^A - \lambda_i^A t\right) t \leq a$ and hence $g(x + \lambda_i^A t) = 0$ for $i = 1, \ldots, n$. Analogously we have

$$[T(t)g](x) = \sum_{i=1}^{n} \langle t_i^B, g(x + \lambda_i^B t) \rangle r_i^B = 0 \text{ for all } x \leq a - \lambda_i^B t. \quad (2.8)$$

Putting (2.7) and (2.8) together we obtain

$$[S(t)T(t)g](x) = 0 \text{ for all } x \leq a - \left(\lambda_n^A + \lambda_n^B\right) t. \quad (2.9)$$

Using (2.9) in the Trotter product one gets

$$\left[ \left( S \left( \frac{t}{n} \right) T \left( \frac{t}{n} \right) \right)^n g \right](x) = 0$$

for all

$$x \leq a - \left(\lambda_n^A + \lambda_n^B\right) t + \cdots + \left(\lambda_n^A + \lambda_n^B\right) t = a - \left(\lambda_n^A + \lambda_n^B\right) t$$

Applying this to $f$ (which vanishes for $x \leq 0$) we get

$$\left[ \left( S \left( \frac{t}{n} \right) T \left( \frac{t}{n} \right) \right)^n f \right](x) = 0 \text{ for all } x \leq a - \left(\lambda_n^A + \lambda_n^B\right) t,$$

therefore

$$\lim_{n \to \infty} \left[ \left( S \left( \frac{t}{n} \right) T \left( \frac{t}{n} \right) \right)^n f \right](x) = 0 \text{ for all } x \leq a - \left(\lambda_n^A + \lambda_n^B\right) t.$$

But since $\lambda_n^C > \lambda_n^A + \lambda_n^B$, we define $\xi(t) = \min \left\{ 1 - \lambda_n^C t, -\left(\lambda_n^A + \lambda_n^B\right) t \right\}$, such that for $x$ in the non empty (for $t > 0$) interval $(-\lambda_n^C t, \xi(t))$

$$\lim_{n \to \infty} \left[ \left( S \left( \frac{t}{n} \right) T \left( \frac{t}{n} \right) \right)^n f \right](x) = 0,$$

while $[U(t)f](x) \neq 0$ proving the Theorem. \qed
Remark 2.4  This example shows that, in general, the solutions to the first order linear hyperbolic system

$$\frac{\partial}{\partial t} u = (A + B) \frac{\partial}{\partial x} u$$

can not be obtained through the operator splitting composition of solutions to the two systems

$$\frac{\partial}{\partial t} u = A \frac{\partial}{\partial x} u, \quad \frac{\partial}{\partial t} u = B \frac{\partial}{\partial x} u.$$ 

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