SOBOLEV CLASSES AND HORIZONTAL ENERGY MINIMIZERS BETWEEN CARNOT-CARATHÉODORY SPACES

KANGHAI TAN

Abstract. The notion of horizontal energy minimizers between C-C spaces is introduced. We prove existence of such energy minimizers when the domain is a $C^2$, noncharacteristic bounded open set in a C-C space and the target is a C-C space of Carnot type.

Keywords: Sobolev mappings, Carnot-Carathéodory spaces, Carnot groups, energy minimizers, existence, regularity

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1. Introduction

Recently many people paid their attentions to the study of analysis and geometry in metric measure spaces in particular in Carnot-Carathéodory (written as C-C for brevity) spaces, see [23], [4] and references therein. In this direction, there has been a number of works devoted to the notions of Sobolev functions and mappings on metric spaces. Let us in particular mention the definitions proposed by Korevaar-Schoen in [39], by Hajlasz in [22] and by Reshetnyak in [53], see [6], [57], [15], [43] and [46] for other definitions and generalizations. The Sobolev spaces of [6], [57] and [22] are originally defined in metric measure spaces for real-valued functions. These classes of Sobolev functions are equivalent as sets when the Sobolev exponent is larger than one, and all equivalent to the horizontal Sobolev spaces ([19]) when the domain is a C-C space satisfying suitable conditions, see [14], [57] and [23]. The notions in [6], [22], [39], [53], [57] can be extended to define Sobolev mappings between metric measure spaces in particular C-C spaces, see [29] and [61].

On the other hand, in [39] Korevaar-Schoen used their developed theory of Sobolev mappings to study harmonic mappings from smooth Riemannian manifolds to nonpositive curvature spaces. Let us briefly recall their ideas. Assume that $\Omega$ is a smooth domain in $\mathbb{R}^n$ and $M$ is a separable metric space with a metric $d$. A function $u \in L^\alpha(\Omega,M)$ is in $KS^{1,\alpha}(\Omega,M)$ if

$$E^\alpha(u,\Omega) = \sup_{f \in C_c(\Omega,[0,1])} \limsup_{\epsilon \to 0} \int_\Omega f(x) \int_{B_\epsilon(x)} \left( \frac{d(u(x),u(y))}{\epsilon} \right)^\alpha dydx$$

is finite where $C_c(\Omega,[0,1])$ is the set of all compactly supported functions in $\Omega$ taking values in the interval $[0,1]$. When $\Omega$ is an open set in a smooth Riemannian manifold, the definition is similar. If $u \in KS^{1,\alpha}(\Omega,M)$, $E^\alpha(u,\Omega)$ is called the energy of the mapping $u$. Roughly speaking, the story of [39] is based on a subpartitional lemma ([39], Lemma 1.3.1). It follows from the subpartitional lemma that $E^\alpha(u,\Omega)$ is lower semicontinuous with respect to the topology of $L^\alpha(\Omega,M)$ and $KS^{1,\alpha}(\Omega,M)$ possesses some type of precompactness property ([39], Theorem...
1.13). Korevaar-Schoen proved a satisfactory existence and regularity theory for energy minimizers of \(E^a(u, \Omega)\) when the target is a nonpositive curvature space (in the sense of Alexandrov). In [10] and [13], Eells and Fuglede made a systematic generalization of the Korevaar-Schoen’s results to Riemannian polyhedra. For similar results but with different methods we refer to [33], [34], [35], and [36].

We briefly recall the definition of C-C spaces (or sub-Riemannian manifolds), in particular of Carnot groups. Let \(\Delta\) be a smooth distribution in \(\mathbb{R}^n\), \(\delta x_1, \ldots, \delta x_n\). (in the sense of Alexandrov). In [10] and [18], Eells and Fuglede made a systematic generalization of the Korevaar-Schoen’s results to Riemannian polyhedra. For C-C spaces, which are most interesting C-C spaces. A Carnot group \(G\) admits the grading \(G = V_1 \bigoplus \cdots \bigoplus V_k\), with \([V_1, V_i] = V_{i+1}\), for any \(1 \leq i \leq l - 1\) and \([V_1, V_l] = 0\) (the integer \(l\) is called the step of \(G\)). Let \(\{e_1, \ldots, e_n\}\) be a basis of \(G\) with \(n = \sum_{i=1}^l \dim(V_i)\). Let \(X_i(g) = (L_g)_{*} e_i\) for \(i = 1, \ldots, k := \dim(V_1)\) where \((L_g)_{*}\) is the differential of the left translation \(L_g(g') = gg'\) and let \(X_i(g) = (L_g)_{*} e_{i+k}\) for \(i = 1, \ldots, n - k\). We call the system of left-invariant vector fields \(\Delta := V_1 = \text{span}\{X_1, \ldots, X_k\}\) the horizontal bundle of \(G\). If we equip \(\Delta\) an inner product \(<\cdot, \cdot>\), such that \(\{X_1, \ldots, X_k\}\) is an orthonormal basis of \(\Delta\), \((G, \Delta, <\cdot, \cdot>)\) is an equiregular sub-Riemannian manifold. In \((G, \Delta, <\cdot, \cdot>)\), \(d_c\) is invariant with respect to left translation, that is \(d_c(p_0 p, p_0 q) = d_c(p, q) \forall p_0, p, q \in G\), and is 1-homogeneous with respect to the natural dilations, that is \(d_c(s \delta_s p, \delta_s q) = sd_c(p, q) \forall s > 0, p, q \in G\), where \(\delta_s = \exp(\sum_{i=1}^l s \xi_i)\) for \(p = \exp(\sum_{i=1}^l s \xi_i), \xi_i \in V_i\). We usually identify \(G\) with \(\mathbb{R}^n\) by the exponential map and use \((\mathbb{R}^n, V_1, \delta_s)\) to denote \(G\). \(Q = \sum_{i=1}^l i \dim(V_i)\) is called homogeneous dimension of \(G\) and \(\mathcal{L}^n\) is the Haar measure of \(G\). It is easy to prove that

\[
X_j(x) = \frac{\partial}{\partial x_j} + \sum_{i=k+1}^n a_i(x) \frac{\partial}{\partial x_j}, \quad X_j(0) = e_j = \frac{\partial}{\partial x_j}, \quad j = 1, \ldots, k, \tag{1.1}
\]

where \(a_i(x) = a_i(x_1, \ldots, x_k)\) are polynomials such that \(a_i(\delta_s x) = \lambda^{s} a_i(x)\). The simplest noncommutative Carnot group is the Heisenberg group \(H^m\) which is, by definition, \(R^{2m+1}\) with the group law \(pp' = (z + z', t + t' + 2\omega(z, z'))\) where \(p = (z, t), p' = (z', t') \in R^{2m} \times R\) and \(\omega\) stands for the standard symplectic form in \(R^{2m}\). For more about Carnot groups, see [13] and [35].

In this paper we want to generalize the theory of harmonic mappings to C-C spaces in particular to Carnot groups. In [4] Capogna and Lin made the first step in this direction. Using the energy of Sobolev mappings of Korevaar-Schoen, they considered energy minimizers with smooth Euclidean domain and target Heisenberg group \(H^m\) endowed with a C-C metric. Note that Heisenberg group does not possess any curvature bound in the sense of Alexandrov and the arguments in [39] is not valid in this case. Capogna and Lin made full use of the differential structure of the domain and the target to characterize the Sobolev mappings and explicitly described the energy. It turns out that these Sobolev mappings are weakly contact (satisfying a Legendrian condition) while the energy is not a Dirichlet integral (except the case when \(\alpha = 2\)). Precisely they proved the following:
Theorem 1.1. Let $\Omega$ be a bounded domain in $\mathbb{R}^n$ and $\alpha \geq 2$. Then $u = (z, t) = (x, y, t) \in KS^{1,\alpha}(\Omega, H^m)$ if and only if $z \in W^{1,\alpha}(\Omega)$ and $t \in L^2$ is weakly differentiable, and for a.e. $p \in \Omega$, $i = 1, \cdots, n$, $\partial_p t = 2(y \partial_p x - x \partial_p y) \in L^2(\Omega)$ with $\beta = \frac{\alpha n}{2n-\alpha}$. Moreover the energy can be written as

$$E^\alpha(u, \Omega) = C \int_{\Omega} \int_{B_1} |\nabla z(p) \cdot \omega|^\alpha d\omega dp. \quad (1.2)$$

In general, the energy of Korevaar-Schoen can not be written as a Dirichlet integral

$$C \int_{\Omega} |\nabla \bar{u}(p)|^\alpha dp \quad (1.3)$$

for some $\bar{u}$ related to $u$ when the target is not the real line, see also [39]. It is easily seen that only when $\alpha = 2$, (1.2) has the form of (1.3). In our opinion in general the energy has the form of (1.3) is a necessary condition to make the energy minimizing problem solvable when the target does not possess any curvature bound. We remark that the method in [5] used to characterize Sobolev mappings is not valid for $1 \leq \alpha < 2$ due to the non-isotropic property of the gauge distance.

To generalize the concept of harmonic mapping to C-C spaces we must introduce a natural energy which not only has “good” form (like Dirichlet integral) but also inherits some essential nature from the considered C-C spaces. To this end we first study the energy of Korevaar-Schoen. We will show that the energy of Korevaar-Schoen is not the one we expected. We will give an explicit description of the energy of Korevaar-Schoen when both the domain and the target are Carnot groups, see Theorem 4.1. That is,

$$E^\alpha(u, \Omega) = C \int_{\Omega} \int_{B_c(0,1)} \tilde{\rho}(Du(p)(\omega))^{\alpha} d\omega dp \quad (1.4)$$

where $\Omega \subset G$ is a bounded open set of Carnot group $G$ with a homogeneous norm $\rho$; $u \in KS^{1,\alpha}(\Omega, \tilde{G})$ where $\tilde{G}$ is another Carnot group with homogeneous norm $\tilde{\rho}$; $Du(p) : G \to \tilde{G}$ is the approximate Pansu derivative of $u$ at $p \in \Omega$, see Definition 2.4 and Theorem 3.17. $C$ is a constant and $B_c(0,1)$ is the unit C-C ball centered at 0. Our arguments rely on the equivalence of several Sobolev classes between C-C spaces. Let $R^{1,\alpha}(\Omega, M)$ and $H^{1,\alpha}(\Omega, M)$ denote the Sobolev spaces defined in the sense of Reshetnyak and Hajlasz respectively, see Definition 3.2 and Definition 3.3. When $\alpha > 1$ and $\Omega$ is a bounded open set in a C-C space with some conditions and $M$ is a separable metric space, we prove that

$$KS^{1,\alpha}(\Omega, M) = R^{1,\alpha}(\Omega, M) = H^{1,\alpha}(\Omega, M) \quad (1.5)$$

as sets, see Theorem 3.5. The proof essentially depends on several observations of the theory of real-valued Sobolev classes defined on metric measure spaces which was developed in [30], [14] and [24]. Let us mention that the equivalence of several definitions of Banach space-valued Sobolev classes has been proven in [29] where an important technique, that each metric space $Y$ can be isometrically embedded into a Banach space, for example into $L^\infty(Y)$ or $L^2$ if $Y$ is separable, is trickily adopted. Since such isometric embedding is not good enough (see [56] for the fact that Heisenberg group is not bilipschitz equivalent to any Euclidean space in any scale), we will not use this idea. Compared with the proof suggested in [29], our proof of (1.5) is convenient for our purpose, also direct and simpler due to the differential structure of C-C spaces.
In [61] and [62], Vodop’yanov made a systematic study of \( R^{1,\alpha}(\Omega, \tilde{G}) \) where \( \alpha > 1 \), \( \Omega \) is a bounded open set of a Carnot group \( G \) and \( \tilde{G} \) is another Carnot group. In particular, he gave several equivalent descriptions of \( R^{1,\alpha}(\Omega, \tilde{G}) \), including a characterization using properties of coordinate functions which obviously covers the first statement in Theorem 1.1. Equation (1.4) is deduced from (1.5) and the results in [61] and [62]. When \( \Omega \) is an Euclidean domain and \( \tilde{G} \) is the Heisenberg group, (1.4) is just (1.2) (recall that \( R^{\alpha} \) can be seen as an abelian Carnot group).

Although we can explicitly formulate the energy of Korevaar-Schoen as (1.4), we do not know whether or not \( E^{\alpha}(\Omega, \tilde{G}) \) is lower-semicontinuous with respect to some topology of \( KS^{1,\alpha}(\Omega, \tilde{G}) \). As done in [39], the lower semicontinuity of \( E^{\alpha}(\Omega, M) \) with respect to the topology of \( L^{\alpha}(\Omega, M) \) is a byproduct of a subpartitional lemma when \( \Omega \) is a Riemannian domain, see also [10]. Sturm in [59] generalized this fact to domains which possesses a strong or weak “measure contraction property”, see also [41] and [42]. Unfortunately, in general C-C spaces seem to have no “measure contraction property”. We will illustrate this fact for Heisenberg group in Section 4.

Thus we will abandon the energy of Korevaar-Schoen. Instead we will introduce the horizontal energy. Let us first recall the definition of the energy in the theory of harmonic mappings between smooth Riemannian manifolds (e.g. [26], [34]). Let \((M, g)\) and \((N, h)\) be two smooth manifolds with Riemannian metric \( g \) and \( h \) respectively. The energy of a smooth map \( u : M \to N \) is defined as (up to a constant)

\[
E(u) = \int_M \| du \|^\alpha dv
\]

where \( du \) is the induced differential map \( du(p) : T_pM \to T_{f(p)}N \) (\( du \) can be regarded as an element of \( \Gamma(T^*M \otimes u^{-1}TN) \); \( \| du \| \) is the norm with respect to the fiber metric of \( \Gamma(T^*M \otimes u^{-1}TN) \) induced by \( u \) from \( g, h \) and \( dv \) is the volume form in \( M \). If we choose a coordinate chart of \( M \) such that \( (\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n}) \) is orthonormal with respect to \( g \), then (1.7) can be rewritten as

\[
E(u) = \int_M \left( \sum_{i=1}^m h(du(\frac{\partial}{\partial x_i}), du(\frac{\partial}{\partial x_i})) \right)^{\frac{\alpha}{2}} dv.
\]

Now let \((G, \Delta, g_c)\) and \((\tilde{G}, \tilde{\Delta}, \tilde{g}_c)\) be two sub-Riemannian manifolds. By definition, \( G \) and \( \tilde{G} \) are two smooth manifolds endowed with smooth distributions \( \Delta = \text{span}\{X_1, \cdots, X_k\} \), \( \tilde{\Delta} = \text{span}\{Y_1, \cdots, Y_{\tilde{k}}\} \) respectively, and \( g_c \) and \( \tilde{g}_c \) are fiberwise inner products endowed to \( \Delta \), \( \tilde{\Delta} \) respectively, such that \( \{X_1, \cdots, X_k\} \) and \( \{Y_1, \cdots, Y_{\tilde{k}}\} \) are orthonormal with respect to \( g_c \), \( \tilde{g}_c \) respectively. Note that any such \( g_c \) (or \( \tilde{g}_c \)) can be realized as the restriction of a Riemannian metric \( g \) (or \( \tilde{g} \)) on \( G \) (or \( \tilde{G} \)) to \( \Delta \) (or \( \tilde{\Delta} \)). Let \( u : G \to \tilde{G} \) be a smooth map satisfying the following contact condition

\[
du(p)(X_i(p)) \in \tilde{\Delta}_{u(p)} \quad \text{for } i = 1, \cdots, m_1.
\]

We define the horizontal energy of \( u \) as follows:

\[
\text{HE}(u) = \int_G \left( \sum_{i=1}^k \tilde{g}(du(X_i), du(X_i)) \right)^{\frac{\alpha}{2}} dv
\]
where $dv$ is the volume form in $G$ with respect to $\bar{g}$.

Note that $HE(u)$ is dependent on $g$ but independent of any extension of $\bar{g}_c$. In the case $\Delta = TG$, $HE(u)$ only depends on $q_c$ and $\bar{g}_c$. The definition of horizontal energy obviously generalizes the Riemannian energy (1.7) in the sense that if $\Delta = TG$ and $\bar{\Delta} = T\bar{G}$, then (1.9) is just (1.7). Any smooth map satisfying (1.8) is called a contact map, see Definition 5.15. Any map in $R^{1,\alpha}(\Omega, \bar{G})$ satisfies (1.8) in a weak sense, see Remark 3.10 It turns out that $R^{1,\alpha}(\Omega, \bar{G})$ is the natural space to study the minimizing problem with respect to the horizontal energy. In this paper, we will not explore the full general situation, but restrict ourselves to C-C spaces, in particular to Carnot groups. We will give an existence result of horizontal minimizers (see Definition 5.2) when the target is of Carnot type.

In contrast to the easy existence problem of horizontal minimizers, regularity problem is very complicated. By now, we have some results in the case when $\Omega \subset R^2$ is smooth and bounded open set and the target is the Heisenberg group $H^m$. In this case, due to the conformal invariance of the horizontal energy there is a close link to the two dimensional isotropically constrained Plateau problem in $R^{2m}$ investigated in [54] by Schoen-Wolfson when $m = 2$ and in [51] by Qiu Weiyang when $m > 2$. The method of constructing isotropic variations in [51] may be useful to further investigation.

To end this introduction, we sketch the structure of the paper. In Section 2 we give notations, definitions and collect some basic facts about C-C spaces and several definitions of Sobolev classes defined in C-C spaces. The equivalence of several definitions of Sobolev classes from C-C spaces to separable metric spaces will be proven in Section 3.1 see Theorem 3.5. We discuss in 3.2 and 3.3 the properties of $R^{1,\alpha}(\Omega, \bar{G})$ such as several equivalent characterizations (Theorem 3.10, 3.13), precompactness (Theorem 3.18) and the trace problem (Theorem 3.22). In Section 4 we discuss the properties of the energy of Korevaar-Schoen (Theorem 4.1) and give reasons why we abandon it. We conjecture that C-C spaces do not possess any type of “measure contraction property”. We will illustrate an evidence to this conjecture by showing that Heisenberg group does not possess the strong “measure contraction property”. So the method used to prove that the approximate Korevaar-Schoen energies satisfy a subpartitional lemma and then deduce that Korevaar-Schoen energy is lower semicontinuous may not be valid in this case. Section 5 is devoted to defining the horizontal energy, to proving the existence of minimizers of the horizontal energy minimizing problem when the domain is a smooth, noncharacteristic bounded open set in a C-C space and the target is a C-C space of Carnot type. The existence result is immediately from the compactness theorem and the trivial lower semicontinuity of the horizontal energy with respect to the weak topology. In Section 6 we discuss the regularity of the minimizers when the domain is a bounded open set in $R^2$ and the target is the Heisenberg group $H^m$.

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2. Preliminaries and basic results

The aim of this section is to fix the notations and collect some basic results which will be used in the sequel.

2.1. Carnot-Carathéodory spaces. Let $\Delta = \text{span}\{X_1, X_2, \cdots, X_k\}$ be a smooth distribution in $\mathbb{R}^n$. We identify $X_i$ with a first order differential operator in $\mathbb{R}^n$.

Denote by $\nabla_j(p)$ the subspace of $T_p \mathbb{R}^n = \mathbb{R}^n$ spanned by all commutators of $X_i$‘s of order $\leq j$ ($\nabla_1 = \Delta = \text{span}\{X_1, \cdots, X_k\}$ is called the horizontal bundle whose cross sections are called horizontal vector fields). We say that $\Delta$ satisfies the Hörmander condition provided for any $p \in \mathbb{R}^n$ there exists $r_p$ such that $\dim(\nabla_{r_p}(p)) = n$. $\Delta$ is equiregular if for each $j$, $\dim(\nabla_j(p))$ is independent of the point. If $\Delta$ satisfies the Hörmander condition and is equiregular, then the least integer $r$ such that $\dim(\nabla_r) = n$ is called the step of $\Delta$.

An absolutely continuous curve $\gamma : [a, b] \to \mathbb{R}^n$ is horizontal if there exist Borel functions $c_i(t), a \leq t \leq b$, such that $\gamma(t) = \sum_{i=1}^k c_i(t)X_j(\gamma(t))$ for a.e. $t \in [a, b]$. We endow a fiberwise inner product $\langle \cdot, \cdot \rangle$ to $\Delta$ such that $\{X_1(p), \cdots, X_k(p)\}$ is orthonormal at every point $p \in \mathbb{R}^n$. The length of a horizontal curve $\gamma$ is defined as $L(\gamma) = \int_a^b (\sum_{i=1}^k |c_i(t)|^2)^{\frac{1}{2}} dt$. Then the C-C distance $d_c$ between $p$ and $q$ in $\mathbb{R}^n$ is defined as the infimum of the lengths of all horizontal curves connecting $p$ to $q$. $d_c$ is called the C-C distance. $\mathbb{R}^n$ equipped with the C-C distance is called C-C space, denoted by $(\mathbb{R}^n, \Delta, d_c)$. The Chow theorem ([7]) says that if the distribution $\Delta$ satisfies the Hörmander condition then there exists an admissible curve connecting any given pair of points in $\mathbb{R}^n$ and thus $d_c$ is a metric. For other equivalent definitions of the C-C distance, we refer to [31].

Notation. In the remainder of the paper, when we speak of a C-C space $(\mathbb{R}^n, \Delta, d_c)$ we assume that the distribution $\Delta = \text{span}\{X_1, \cdots, X_k\}$ satisfies the Hörmander condition. We will use $\Delta_p$ to denote the fiber of $\Delta$ through $p$. In the sequel $|E|$ will always stand for $\mathcal{L}^n(E)$, where $\mathcal{L}^n$ is the $n$-dimensional Lebesgue measure on $\mathbb{R}^n$. $B_c(p, \delta)$ (or $B_c(p)$) will denote a C-C (Euclidean) open ball centered at $p$ with radius $\delta$. We will use $\overline{\Omega}$ to denote the closure of a subset $\Omega \subset \mathbb{R}^n$. By $\Omega \Subset \overline{\Omega}$ we mean that $\overline{\Omega}$ is contained in $\Omega$. Let $u$ be a Borel function defined on $\Omega \subset \mathbb{R}^n$. The average value of $u$ on $\Omega$ will be denoted by $u_\Omega = \int_{\Omega} u dx = |\Omega|^{-1} \int_{\Omega} u dx$.

Lemma 2.1 ([9]). Let $(\mathbb{R}^n, \Delta, d_c)$ be a C-C space. Then for every bounded open set $\Omega \subset \mathbb{R}^n$ there exists $C \geq 1$ such that one has

$$|B_c(p, 2\delta)| \leq C|B_c(p, \delta)|$$

whenever $p \in \Omega$ and $\delta \leq 5\text{diam}\Omega$.

The condition (2.1) is called the doubling condition and the least constant $C$ such that (2.1) holds is called the doubling constant and $Q := \log_2 C \geq n$ is called the local homogeneous dimension of $\Omega$. According to [18], if $\Delta$ is equiregular, then the constant $Q = \sum_{i=1}^n i(\dim(V^i) - \dim(V^{i-1}))$ is the Hausdorff dimension of $(\mathbb{R}^n, \Delta, d_c)$. We refer to [19] for more about C-C balls.

Let $\Omega$ be a bounded open set in $(\mathbb{R}^n, \Delta, d_c)$. Following [28] we say that a Borel function $g : \Omega \to [0, \infty]$ is an upper gradient of another Borel function $u : \Omega \to R$ if for every 1-Lipschitz curve $\gamma : [0, T] \to \Omega$ we have $|u(\gamma(0)) - u(\gamma(T))| \leq \int_0^T g(\gamma(t)) dt$. We recall that a curve $\gamma$ is called 1-Lipschitz if $d_c(\gamma(t_1), \gamma(t_2)) \leq |t_2 - t_1|$ for all $0 \leq t_1 < t_2 \leq T$. 


Let $u$ and $g \geq 0$ be two Borel functions defined on an open subset $\Omega$. For the pair $(u, g)$ if there exist $C > 0$ and $\lambda \geq 1$ such that
\[
\int_{B_r} |u - u_{B_r}|dx \leq C r \left( \int_{\lambda B_r} g^{\alpha} \right)^{\frac{1}{\alpha}}
\] (2.2)
holds for every metric ball $B_r$ in $\Omega$, where $r$ is the radius of $B_r$, then we say the pair $(u, g)$ satisfies a $(1, \alpha)$-Poincaré inequality for $C$ and $\lambda$. We say $(R^n, \Delta, d_c)$ supports a $(1, \alpha)$-Poincaré inequality, $1 \leq \alpha < \infty$, if for every bounded open set $\Omega$ when $u$ is a continuous function in $\Omega$ and $g$ is an upper gradient of $u$, the pair $(u, g)$ satisfies a $(1, \alpha)$-Poincaré inequality for some choice of constants $C_\Omega > 0$ and $\lambda_\Omega \geq 1$. The following theorem is well known, see [30], [31] and [23].

**Theorem 2.2.** $(R^n, \Delta, d_c)$ supports a $(1, \beta)$-Poincaré inequality for any $\beta \in [1, \infty)$.

For sharp results about Poincaré inequalities in metric measure spaces we refer to [44], [45] and [16].

**Definition 2.3 (G-linear map).** Let $G = (R^n, V_1, \delta_\lambda)$ and $\tilde{G} = (\tilde{R^n}, \tilde{V}_1, \tilde{\delta}_\lambda)$ be two Carnot groups. A mapping $L : G \to \tilde{G}$ is called a G-linear map if
\begin{enumerate}
\item $L$ is a homogeneous with respect to $\delta_\lambda$ and $\tilde{\delta}_\lambda$, that is, $L(\delta_\lambda p) = \tilde{\delta}_\lambda L(p)$ for any $p \in G$ and $\lambda > 0$.
\item $L$ is a group homomorphism, that is, $L(pq) = L(p)L(q)$ for any $p, q \in G$.
\end{enumerate}

Any G-linear map is smooth and contact, for a proof see e.g [47].

**Definition 2.4 (Pansu differential).** Let $G$ and $\tilde{G}$ be two Carnot groups with homogeneous norms $\rho$ and $\tilde{\rho}$ respectively. Let $E$ be a Borel subset of $G$. A G-linear map $L$ is called a Pansu differential of a mapping $u : E \to \tilde{G}$ at a point $p \in E$ if
\[
\lim_{x \to p, x \in E} \frac{\tilde{\rho}(L(a^{-1}x)^{-1}u(x))}{\rho(p^{-1}x)} = 0.
\]
A G-linear map $L$ is called an approximate Pansu differential of $u$ in $E$ at a point $p \in U$ if
\[
ap \lim_{x \to p} \frac{\tilde{\rho}(L(a^{-1}x)^{-1}u(x))}{\rho(p^{-1}x)} = 0
\]
where ap lim$_{x \to p} f(x)$ denotes the approximate limit of $f$ at $p$ (see [12]).

**Remark 2.5.** The notion of derivatives for mappings between Carnot groups was originally introduced by P. Pansu in [50] where the set $E$ in Definition 2.4 is required to be an open set. The version of Definition 2.4 is due to [64] and [47].

2.2. **Sobolev functions defined on Carnot-Carathéodory spaces.** There are several equivalent definitions for Sobolev functions on metric measure spaces. The fundamental references in this topic are [23], [27]. We concentrate on Sobolev functions in C-C spaces. Due to the differential structure of C-C spaces, the theory of Sobolev functions in C-C spaces are more abundant than that in general metric measure spaces.

Let $(R^n, \Delta, d_c)$ be a C-C space and let $\Omega$ be an open set in $R^N$. Let $\alpha$ be in $[1, \infty]$. The horizontal Sobolev space is the Banach space $W^{1,\alpha}_X(\Omega) = \{u \in L^\alpha(\Omega)|X_j u \in L^\alpha(\Omega), j = 1, \cdots, k\}$
endowed with the norm \( \|u\|_{W^{1,p}_X(\Omega)} = \|u\|_{L^p(\Omega)} + \sum_{i=1}^k \|X_i u\|_{L^p(\Omega)} \). In the above definition, \( X_i u \) is understood in the distributional sense. Another way to define the space \( W^{1,\alpha}_X(\Omega) \) for \( 1 \leq \alpha < \infty \) is to take the closure of \( C^\infty \) functions in the norm \( \| \cdot \|_{W^{1,\alpha}_X(\Omega)} \). As in the Euclidean case, the two approaches are equivalent. This was obtained independently in [17] and [20].

For \( 1 \leq \alpha < \infty \), the Sobolev space \( H^{1,\alpha}(\Omega) \) is defined as the set of all \( u \in L^\alpha(\Omega) \) for which there exists \( 0 \leq g \in L^\alpha(\Omega) \) such that the inequality

\[
|u(x) - u(y)| \leq d_c(x, y) (g(x) + g(y))
\]

holds a.e. \( x, y \in \Omega \). \( H^{1,\alpha}(\Omega) \) is firstly introduced by Hajłasz in [22]. By \( P^{1,\alpha}(\Omega) \) we denote the set of all functions \( u \in L^\alpha(\Omega) \) such that there exists \( 0 \leq g \in L^\alpha(\Omega) \) such that the pair \((u, g)\) satisfies a \((1, \alpha)\)-Poincaré inequality. Roughly speaking, the function \( g \) in (2.3) corresponds to the maximal function of the gradient, while the function \( g \) in (2.2) looks more like the norm of the gradient (see the Introduction of [23]). For other notions of Sobolev functions in C-C spaces or general metric measure spaces we refer the reader to [6], [57], [15], [43] and [46].

The following theorem, which follows from Theorem 2.2 and Theorem 1, Corollary 13 in [14] (see also [40]), is crucial to Theorem 3.5 in Section 3.

**Theorem 2.6.** Let \((\mathbb{R}^n, \Delta, d_c)\) be a C-C space and \( \Omega \subset \mathbb{R}^n \) be a bounded open set. Assume \( 1 < \alpha < \infty \). Then the following four conditions are equivalent.

1. \( u \in W^{1,\alpha}_X(\Omega) \).
2. \( u \in H^{1,\alpha}(\Omega) \).
3. \( u \in P^{1,\alpha}(\Omega) \).
4. \( u \in L^\alpha(\Omega) \) and there exist \( 0 \leq g \in L^\alpha(\Omega) \), constants \( C > 0 \), \( \lambda \geq 1 \) such that \((u, g)\) satisfies a \((1, \beta)\)-Poincaré inequality for \( C \), \( \lambda \) where \( \beta \in [1, \alpha) \).

Moreover

(i) If \( u \in L^\alpha(\Omega) \) and there exist \( 0 \leq g \in L^\alpha(\Omega) \), constants \( C > 0 \), \( \lambda \geq 1 \) such that \((u, g)\) satisfies a \((1, \beta)\)-Poincaré inequality for \( C \) and \( \lambda \geq 1 \) where \( \beta \in [1, \alpha) \), then \( \omega(x) = C(\sup_{r>0} \int_{B_r(x, r)} g^\beta(x) dx)^{\frac{1}{\beta}} \) is in \( L^\alpha(\Omega) \) and the pair \((u, \omega)\) satisfies (2.3) where \( g \) is replaced by \( \omega \).

(ii) If \( u \in W^{1,\alpha}_X(\Omega) \) and \((u, g)\) satisfies a \((1, \alpha)\)-Poincaré inequality, then \( |X u| \leq C g \) a.e. for some constant \( C \) independent of \( u \) and \( g \).

3. **Sobolev classes from Carnot-Carathéodory spaces to separable metric spaces**

In this section we study Sobolev classes from C-C spaces to separable metric spaces. In Section 3.1 we define \( H^{1,\alpha}, KS^{1,\alpha} \) and \( R^{1,\alpha} \), then we prove that they are equivalent as sets when \( 1 < \alpha < \infty \). In Section 3.2 we study properties of Sobolev mappings from a C-C space to another C-C space of Carnot type by giving several equivalent descriptions of \( R^{1,\alpha} \) which slightly generalizes some corresponding results in [61].

**3.1. Equivalence of Sobolev classes.** Let \((\mathbb{R}^n, \Delta, d_c)\) be a C-C space, \( \Omega \subset \mathbb{R}^n \) be a bounded open set with smooth boundary and \((M, d)\) be a complete metric space with a (quasi-)metric \( d \). Assume \( 1 \leq \alpha < \infty \). Let \( u : \Omega \to M \) be a measurable map. \( u \) is called in \( L^\alpha(\Omega, M) \) if \( \int_\Omega d(m_0, u(p))^{\alpha} dp < \infty \) for some \( m_0 \in M \). Since \( \Omega \) is bounded, the definition is independent of the choice of \( m_0 \) by the (quasi-)triangle
inequality of $d$. We identify two mappings which coincide $L^n$-almost everywhere. It is easily proved that $L^n(\Omega, M)$ is a complete metric space with the distance $d_{L^n(\Omega, M)}(u, v) = \int_{\Omega} d(u(p), v(p))^\alpha dp$, see e.g. [52].

For $\epsilon > 0$, let $\Omega_\epsilon := \{ p \in \Omega : \text{dist}_c(p, \partial \Omega) > \epsilon \}$ with
$$\text{dist}_c(p, \partial \Omega) = \inf_{q \in \partial \Omega} d_c(p, q).$$

For a map $u : \Omega \to M$ and for a point $p \in \Omega$, we define the averaged $\epsilon$-approximate density function
$$e^\alpha(p; u) = \frac{1}{\Omega} \int_{B_\epsilon(p, c)} \left( \frac{d(u(p), u(q))}{\epsilon} \right)^\alpha dq$$
where $1 \leq \alpha < \infty$ and $p \in \Omega$. If $\varphi \in C_c(\Omega, [0, 1])$ and $\epsilon < \text{dist}_c(\text{supp} \varphi, \partial \Omega)$, we define the approximate energy
$$E^\alpha_\epsilon(\varphi; u) = \int_{\Omega} \varphi(p) e^\alpha(p; u) dp. \quad (3.1)$$

We now define the class $KS^{1,\alpha}(\Omega, M)$.

**Definition 3.1.** Let $u \in L^n(\Omega, M)$. We say $u$ is in $KS^{1,\alpha}(\Omega, M)$ if
$$E^\alpha(u, \Omega) = \sup_{\varphi \in C_c(\Omega, [0, 1])} \limsup_{\epsilon \to 0} E^\alpha_\epsilon(\varphi; u)$$
is finite. If $u \in KS^{1,\alpha}(\Omega, M)$, $E^\alpha(u, \Omega)$ is called the energy of Korevaar-Schoen.

The above definition is firstly introduced in [39] by Korevaar and Schoen in the case where $\Omega$ is a Riemannian domain. Later it is generalized to general metric measure spaces, see [52] and [29].

**Definition 3.2.** Let $u \in L^n(\Omega, M)$. We say $u \in H^{1,\alpha}(\Omega, M)$ if there exists $0 \leq \omega \in L^n(\Omega)$ such that
$$d(u(p), u(q)) \leq d_c(p, q)(\omega(p) + \omega(q)) \quad (3.2)$$
holds for a.e. $p, q \in \Omega$. We set
$$E^\alpha_H(u, \Omega) = \inf_{\omega} \|\omega\|^\alpha_{L^n(\Omega)}$$
where the infimum is taken among all nonnegative functions $\omega$ in $L^n(\Omega)$ such that (3.2) holds.

$H^{1,\alpha}(\Omega, M)$ is a natural generalization of $H^{1,\alpha}(\Omega)$ and can also be extended to more general metric measure spaces ([29]).

**Definition 3.3.** Let $u \in L^n(\Omega, M)$. We say $u \in R^{1,\alpha}(\Omega, M)$ if for any $m \in M$, the scalar function $\theta_m(p)$ defined by $\theta_m(p) := d(m, u(p))$ is in $W^{1,\alpha}_X(\Omega)$ and there exists $0 \leq g \in L^n(\Omega)$ (independent of $m$) such that
$$|X \theta_m(p)| \leq g(p)$$
a.e. $p \in \Omega$ for any $m \in M$. We call $g$ is a dominant function of $u$. We set
$$E^\alpha_R(u, \Omega) = \inf_{g} \|g\|^\alpha_{L^n(\Omega)}$$
where the infimum is taken among all dominant functions $g$ of $u$.

When $\Omega$ is an Euclidean domain (that is $\Delta = \text{span}(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n})$), this definition coincides with that in [53].
Lemma 3.4. If \( u \in KS^{1,\alpha}(\Omega, M) \), then \( \theta_m \in KS^{1,\alpha}(\Omega, R) \) and \( e_\alpha(p, \theta_m) \leq Ce_\gamma(p, u) \) for any \( m \in M \) and some constant \( C \) (independent of \( m \)). Thus there exists \( 0 \leq q \in L^\alpha(\Omega) \) independent of \( m \) such that the pair \((\theta_m, g)\) satisfies a \((1, \beta)\)-Poincaré inequality for any \( m \in M \) and \( \beta \in [1, \alpha) \).

Lemma 3.4 is from Theorem 2.2 and a careful examination of the proof of Theorem 4.5 in [40].

Various seemly different definitions of Sobolev classes are equivalent. Precisely, we have

Theorem 3.5. Let \((R^n, \Delta, d_c)\) be a C-C space and \( \Omega \) be a bounded open set in \( R^n \). Assume that \( M \) is a separable metric space with a (quasi-) metric \( d \), \( d \) is dense in \( M \), \( M \) is separable, we can choose a sequence of points \( \{m_i\}_{i=1}^\infty \) such that \( \{m_i\}_{i=1}^\infty \) is dense in \( M \). From Theorem 2.6 we conclude that for any \( \Omega \in \mathcal{M}(\Omega) \) and \( \alpha \in [0, 1] \), \( H^{1,\alpha}(\Omega, \Omega_0) \) is a separable metric space with a (quasi-) metric \( d \).

Step 1. \( KS^{1,\alpha}(\Omega, M) \subset R^{1,\alpha}(\Omega, M) \).

Let \( u \in KS^{1,\alpha}(\Omega, M) \), then from Theorem 2.6 and Lemma 3.4 we conclude that \( \theta_m \in W^{1,\alpha}_X(\Omega) \) and there exists \( g' \in L^\alpha(\Omega) \) independent of \( m \) such that \( |X\theta_m(p)| \leq g'(p) \) for any \( m \in M \) and a.e. \( p \in \Omega \). Thus \( KS^{1,\alpha}(\Omega, M) \subset R^{1,\alpha}(\Omega, M) \).

Step 2. \( R^{1,\alpha}(\Omega, M) \subset H^{1,\alpha}(\Omega, M) \).

Let \( u \in R^{1,\alpha}(\Omega, M) \). By definition, \( u \in L^\alpha(\Omega, M) \), \( \theta_m \in W^{1,\alpha}_X(\Omega) \) for any \( m \in M \) and there exists \( g \in L^\alpha(\Omega) \) independent of \( m \) such that \( |X\theta_m(p)| \leq g(p) \) holds a.e. \( p \in \Omega \) for any \( m \in M \). Since the pair \((\theta_m, X\theta_m)\) satisfies a \((1, \alpha)\)-Poincaré inequality for some choice of constants \( C > 0 \) and \( \lambda \geq 1 \) for any \( m \in M \) (see e.g. [19] \( \), the pair \((\theta_m, g)\) also satisfies a \((1, \alpha)\)-Poincaré inequality for \( C \) and \( \lambda \) and any \( m \in M \). For \( \beta \in [1, \alpha) \) we let

\[
\omega(x) = C \left( \sup_{r > 0} \int_{B_r(x, r)} g^\beta(x)dx \right)^{\frac{1}{\beta}}.
\]

Since \( M \) is separable, we can choose a sequence of points \( \{m_i\}_{i=1}^\infty \) such that \( \{m_i\}_{i=1}^\infty \) is dense in \( M \). From Theorem 2.6 we conclude that for any \( i \) there exists a set \( \Omega_i \) of measure zero such that the inequality

\[
|\theta_{m_i}(p) - \theta_{m_i}(q)| \leq d_c(p, q)(\omega(p) + \omega(q))
\]

holds for any \( p, q \in \Omega \setminus \Omega_i \). Let us set \( \Omega' := \bigcup_{i=1}^\infty \Omega_i \), then \( |\Omega'| = 0 \) and

\[
|d(m_i, u(p)) - d(m_i, u(q))| \leq d_c(p, q)(\omega(p) + \omega(q))
\]

holds for any \( p, q \in \Omega \setminus \Omega' \). Thus \( u \in H^{1,\alpha}(\Omega, M) \).

Step 3. \( H^{1,\alpha}(\Omega, M) \subset KS^{1,\alpha}(\Omega, M) \).

Let \( u \in H^{1,\alpha}(\Omega, M) \). By definition, \( u \in L^\alpha(\Omega, M) \) and there exists \( 0 \leq \omega \in L^\alpha(\Omega) \) such that

\[
d(u(p), u(q)) \leq d_c(p, q)(\omega(p) + \omega(q))
\]

(3.4)
holds for a.e. $p, q \in \Omega$. Let $\varphi \in C_c(\Omega, [0, 1])$ and $\epsilon < \frac{1}{3} \text{dist}_c(\text{supp} \varphi, \partial \Omega)$. We have

$$E^\alpha_c(\varphi; u) \leq \int_{\Omega} \varphi \int_{B_c(p, \epsilon)} \left( \frac{d(u(p), u(q))}{\epsilon} \right)^\alpha dq dp$$

(3.5)

$$\leq \int_{\Omega} \varphi \int_{B_c(p, \epsilon)} \left( \frac{d(u(p), u(q))}{d_c(p, q)} \right)^\alpha dq dp$$

$$\leq C \int_{\Omega} \varphi \int_{B_c(p, \epsilon)} (|\omega(p)|^\alpha + |\omega(q)|^\alpha) dq dp$$

(3.6)

$$\leq C \|\omega\|^1_{L^\alpha(\Omega)} + C \int_{\Omega_2 \epsilon} \left( \int_{B_c(p, \epsilon)} |\omega(q)|^\alpha dq \right) dp$$

$$= C \|\omega\|^1_{L^\alpha(\Omega)} + C \int_{\Omega_2 \epsilon} \left( \int_{B_c(p, \epsilon)} |\omega(q)|^\alpha \frac{\chi_{B_c(p, \epsilon)}(q)}{|B_c(p, \epsilon)|} dq \right) dp$$

(3.7)

$$\leq C \|\omega\|^1_{L^\alpha(\Omega)} + C \int_{\Omega_2 \epsilon} |\omega(q)|^\alpha \left( \int_{B_c(q, \epsilon)} \frac{1}{|B_c(p, \epsilon)|} dp \right) dq$$

(3.8)

$$= C \|\omega\|^1_{L^\alpha(\Omega)} + C \int_{\Omega_2 \epsilon} |\omega(q)|^\alpha \left( \int_{B_c(q, \epsilon)} \frac{|B_c(q, \epsilon)|}{|B_c(p, \epsilon)|} dp \right) dq$$

(3.9)

$$\leq C \|\omega\|^1_{L^\alpha(\Omega)} + C \int_{\Omega_2 \epsilon} |\omega(q)|^\alpha \left( \int_{B_c(q, \epsilon)} \frac{|B_c(p, 2\epsilon)|}{|B_c(p, \epsilon)|} dp \right) dq$$

(3.10)

where in (3.9) we used (3.4); (3.7) is from the fact that if $p \in B_c(q, \epsilon)$ then $B_c(q, \epsilon) \subset B_c(p, 2\epsilon)$; (3.10) is from the doubling condition (2.1).

So $u \in KS^{1, \alpha}(\Omega, M)$. \hfill \Box

**Corollary 3.6.** Let $(\mathbb{R}^n, \Delta, d_c)$ be a C-C space and $\Omega$ be a bounded open set in $\mathbb{R}^n$. Assume that $M$ is a separable metric space with a (quasi-) metric $d$. If $1 < \alpha < \infty$ and $u \in R^{1, \alpha}(\Omega, M)$, then $E^\alpha(u, \Omega)$, $E^\alpha_H(u, \Omega)$ and $E^\alpha_B(u, \Omega)$ are equivalent in the sense that each one can be dominated by a constant multiple of another.

### 3.2. Basic properties of Sobolev mappings

In this section we slightly generalize the results in [61] of some equivalent descriptions of $R^{1, \alpha}(\Omega, \tilde{G})$ where $\Omega \subset G$ is a bounded open set and $G, \tilde{G}$ are two Carnot groups to the case when $G$ is a C-C space and $\tilde{G}$ is a C-C space of Carnot type (see Definition 3.7).

In the sequel we will assume that $\Delta$ is equiregular.
Definition 3.7. A C-C space \((\mathbb{R}^n, \Delta, d_c)\) is of Carnot type if the system \(\Delta = \text{span}\{X_1, \cdots , X_k\}\) is of the form
\[
X_i(p) = \frac{\partial}{\partial x_i} + \sum_{j=k+1}^n a_{ij}(p) \frac{\partial}{\partial x_j}\]  
\(i = 1, \cdots , k,\)  
(3.11)
where \(a_{ij}\) are smooth.

This definition is motivated by the analogy with the canonical generating vector fields of a Carnot group (see [13]).

Next we will use the concept of “some property holds for a.e. curves”. Let us briefly describe it, for details see [38]. Let \((\mathbb{R}^n, \Delta, d_c)\) be a C-C space and \(A \subset \mathbb{R}^n\) be a bounded open set. Let \(\Gamma\) be a fibration of \(A\) satisfying that the role of a fiber \(\gamma \in \Gamma\) is played by integral curves of a vector field \(\tau \in \text{span}\{X_1, \cdots , X_k\}\). If we denote the flow induced by the field by the symbol \(f_s\) then the fiber has the form \(\gamma(s) = f_s(p)\), where \(p\) belongs to a hypersurface \(\Sigma\) transversal to \(\tau\) (such \(\Sigma\) exists obviously). We can endow a measure \(d\gamma\) to \(\Gamma\) as follows
\[
d\gamma = F_{f_s} d\gamma \]  
where \(F_{f_s}\) is the Jacobian of the flow \(f_s\), \(i(\tau)\) is the interior product of the vector field \(\tau\) and \(dx\) is the standard volume form in \(\mathbb{R}^n\), such that
\[
c_0|B|^{\frac{Q-1}{Q}} \leq \int_{\gamma \in \Gamma \cap B_c(x, r) \neq \emptyset} d\gamma \leq c_1|B|^{\frac{Q-1}{Q}}.
\]
for sufficiently small balls \(B = B_c(x, r) \subset \mathbb{R}^n\) with constants \(c_0\) and \(c_1\), where \(Q\) is the homogeneous dimension of \((\mathbb{R}^n, \Delta, d_c)\). We can identify a fiber of \(\Gamma\) with a point in \(\Sigma\) through the canonical projection. Roughly speaking, saying that some property holds for a.e. curves in \(\Gamma\) is the same as saying that this property holds for \(d\sigma\) a.e. points in \(\Sigma\) where \(d\sigma\) denotes the Riemannian measure on \(\Sigma\) induced from the standard Euclidean metric in \(\mathbb{R}^n\).

Definition 3.8 (ACL(\(\Omega, M\))). Let \((\mathbb{R}^n, \Delta, d_c)\) be a C-C space and \(\Omega \subset \mathbb{R}^n\) be a bounded open set. Let \(M\) be a metric space with a (quasi-)metric \(d\). A mapping \(u : \Omega \to M\) is absolutely continuous on lines (denoted by ACL for brevity) if for every fibration \(\Gamma_i\) of \(\Omega\) determined by \(X_i\), \(i = 1, \cdots , k\), the curve \(u(\gamma) : \gamma \cap \Omega \to M\), is absolutely continuous in the parameter \(t\) for \(d\gamma\)-almost every curve \(\gamma \in \Gamma_i\).

In Definition 3.8 any element \(\gamma\) in \(\Gamma_i\) is a flow induced by \(X_i\). Since \(\Omega\) is bounded, \(\gamma \cap \Omega\) has the form \(\exp_p(tX_i)\) where \(p \in \Omega\) and vice-versa.

Definition 3.9 (\(R_{1, \alpha}^1(\Omega, M)\) and \(R_{1, \alpha}^2(\Omega, M)\)). Let \((\mathbb{R}^n, \Delta, d_c)\) be a C-C space and \(\Omega \subset \mathbb{R}^n\) be a bounded open set. Let \(M\) be a separable metric space with a (quasi-)metric \(d\). Assume \(u : \Omega \to M\) be a mapping and \(1 \leq \alpha < \infty\). We say \(u \in R_{1, \alpha}^1(\Omega, M)\) if
(1) \(b_m \in L^\alpha(\Omega)\) for any \(m \in M\).
(2) up to a modification on a set of measure zero, \(u \in ACL(\Omega, M)\); moreover the length of the curve \(u(\gamma) : \gamma \cap \Omega \to M\) is absolutely continuous in the parameter \(t\) for \(d\gamma\) a.e. curve \(\gamma \in \Gamma_i\) where \(\Gamma_i\) is a fibration of \(\Omega\) determined by \(X_i\).
(3) the derivative $X_i u(p) = \lim_{t \to 0} \frac{\langle (u(p), u(exp_p(tX_i))) \rangle}{\left|\{ u(exp_p(tX_i)) \} : [0, t] \to M, \right.}$ of the length of the curve $\mathcal{T}(\tau) := u(exp_p(tX_i)) : [0, t] \to M$, which exists almost everywhere in $\Omega$, belongs to $L^\alpha(\Omega)$ for all $i = 1, \ldots, k$. Here $l(u(p), \exp_p(tX_i))$ is the length of the path $\mathcal{T}[0, t]$. We say $u \in R^1_{\alpha}(\Omega, M)$ if

1. for any function $f \in \operatorname{Lip}(M)$, $f \circ u \in W^1_{\alpha}(\Omega)$;
2. there exists $0 \leq g \in L^\alpha(\Omega)$ such that $|X(f \circ u)| \leq \operatorname{Lip}_f g$ holds a.e. for all $f \in \operatorname{Lip}(\Omega)$.

The following theorem is a slight generalization of Proposition 4.1 in [61] (see also [64]).

**Theorem 3.10.** Let $(R^n, \Delta, d_c)$ be a C-C space and $\Omega \subset R^n$ be a bounded open set. Let $M$ be a separable metric space with a (quasi-)metric $d$. Assume $u : \Omega \to M$ be a mapping and $1 \leq \alpha < \infty$, then

$$R^1_{\alpha}(\Omega, M) = R^1_{\alpha}(\Omega, M) = R^2_{\alpha}(\Omega, M)$$

as sets.

**Definition 3.11 (HW$^{1,\alpha}(\Omega, M)$ and HW$^{1,\alpha}(\Omega, M)$).** Let $(R^n, \Delta, d_c)$ be a C-C space and $\Omega \subset R^n$ be a bounded open set. Let $M = (R^n, \tilde{\Delta}, d)$ be a C-C space with $\Delta = \text{span}\{e_1, \ldots, e_k\}$ and a (quasi-)metric $d$ which is equivalent to the C-C metric $d_c$ (that is, there exist constants $C_1$ and $C_2$ such that $C_1 \tilde{d} \leq d_c \leq C_2 \tilde{d}$). Assume $u = (u^1, \ldots, u^k) : \Omega \to M$ be a mapping and $1 \leq \alpha < \infty$. We say $u \in HW^{1,\alpha}(\Omega, M)$ if

1. $\tilde{d}(u) \in L^\alpha(\Omega)$;
2. up to a redefinition on a set of measure zero, $u^i \in \operatorname{ACL}(\Omega, R)$ for $i = 1, \ldots, n$;
3. $X_j u(x) = \sum_{i=1}^n X_j u^i(x) \frac{\partial}{\partial x_i} \in \tilde{\Delta}_u(x)$ which exists for a.e $x \in \Omega$, belongs to $L^\alpha(\Omega)$, that is, $\int_{\Omega} \|X_j u(x)\|_{\alpha < \infty} \ dx < \infty$, $j = 1, \ldots, k$, where $<\cdot, \cdot>_{\tilde{\Delta}}$ denotes the fiberwise inner product in $\tilde{\Delta}$.

We say $u \in HW_{1,\alpha}(\Omega, M)$ if

1. $\tilde{d}(u) \in L^\alpha(\Omega)$;
2. up to a redefinition on a set of measure zero, $u \in \operatorname{ACL}(\Omega, M)$;
3. the derivative $X_j u(x) = \frac{\partial}{\partial t} u(x)$, which exists a.e. in $\Omega$, belongs to $L^\alpha(\Omega)$, $j = 1, \ldots, k$.

In Definition 3.11 we abuse the notation $\tilde{d}(\bar{x}) = \tilde{d}(\bar{x}, 0)$ for $\bar{x} \in R^n$.

The following lemma, draw from [24], is crucial to prove Theorem 3.13.

**Lemma 3.12 (Carathéodory).** Suppose $D$ is an open set in $R^{N+1}$, $f(t, x) : D \to R^n$ satisfies the Carathéodory conditions on $D$, that is, $f$ is Borel measurable in $t$ and for each compact set $D'$ of $D$, there is an integrable function $m_{D'}$ such that $|f(t, x)| \leq m_{D'}(t, x) \in D'$. Moreover for each compact set $U$ in $D$, there exists an integrable function $k_U(t)$ such that $|f(t, x) - f(t, y)| \leq k_U(t)|x - y|$, $(t, x) \in U$, $(t, y) \in U$. Then, for any $(t_0, x_0) \in U$, there exists a unique solution $x(t, t_0, x_0)$ of $\frac{dx(t)}{dt} = f(t, x)$ a.e. passing through $(t_0, x_0)$. Moreover the domain $E$ in $R^{N+2}$ of the function $x(t, t_0, x_0)$ is open and $x(t, t_0, x_0)$ is continuous in $E$. 

The following theorem, which can be proved by using Lemma 3.12 and a similar argument of S. K. Vodop’yanov (Proposition 4.2 in [61]), is of paramount importance for our purpose.

**Theorem 3.13.** Let \( (R^n, \Delta, d_c) \) be a C-C space and \( \Omega \subset R^n \) be a bounded open set. Let \( M = (R^n, \tilde{\Delta}, \tilde{d}) \) be a C-C space of Carnot type where \( \Delta = \text{span}\{X_1, \ldots, X_k\} \) and \( \tilde{d} \) is a metric equivalent to the C-C metric \( \tilde{d}_c \). Assume \( u = (u^1, \cdots, u^\tilde{n}) : \Omega \to M \) be a mapping and \( 1 \leq \alpha < \infty \). Then

\[
R^{1,\alpha}(\Omega, M) = R^1_{1,\alpha}(\Omega, M) = R^1_{2,\alpha}(\Omega, M) = HW^{1,\alpha}(\Omega, M) = HW^1_{1,\alpha}(\Omega, M)
\]
as sets.

**Remark 3.14.** Note that if \( u(\exp_p(tX_j)) \) is a horizontal curve in \( \Omega = (R^n, \tilde{\Delta}, \tilde{d}_c) \), then it follows from

\[
l_c(u(\exp_p(t_1X_j)), u(\exp_p(t_2X_j))) = \int_{t_1}^{t_2} \|X_j u(\exp_p(sX_j))\|_{\tilde{d}_c} ds
\]
that \( X_j u(t_1X_j) = \|X_j u(\exp_p(t_1X_j))\|_{\tilde{d}_c} \) a.e. \( t_1 \), where the length \( l_c \) of \( u(\exp_p(tX_j)) \) is computed with respect to the C-C metric \( \tilde{d}_c \). So if \( u(\exp_p(tX_j)) \) is horizontal, then \( C_1 X_j u(p) \leq \|X_j u(p)\|_{\tilde{d}_c} \leq C_2 X_j u(p) \) trivially holds for a.e. \( p \in \Omega \) where \( C_1, C_2 \) are constants only depends on the (quasi)-metric \( \tilde{d} \).

**Definition 3.15** (contact mapping). Let \( (R^n, \Delta, d_c) \) and \( (R^\tilde{n}, \tilde{\Delta}, \tilde{d}_c) \) be two C-C spaces. Let \( \Omega \) be a bounded open set of \( R^n \). Assume \( u : \Omega \to R^\tilde{n} \) be any measurable mapping. We say \( u \) is a weakly contact mapping if

1. \( X_j u^i(p) = \left. \frac{du^i(\exp_p X_j)}{dt} \right|_{t=0} \) exists a.e. \( p \in \Omega \) for \( j = 1, \cdots, k \) and \( i = 1, \cdots, \tilde{n} \);
2. \( X_j u^i(p) = \sum_{l=1}^{\tilde{n}} X_j u^l(p) \frac{\partial}{\partial x_l} \in \tilde{\Delta}_u(p) \) a.e. \( p \in \Omega \).

If \( u \) is smooth and satisfies (2), then \( u \) is called a contact map. If \( u \) is a weakly contact mapping, then \( u \) induces a linear map \( D_h u(p) : \Delta_p \to \tilde{\Delta}_u(p) \).

**Remark 3.16.** Under the same conditions of Theorem 3.13 two observations are in order:

1. If \( u \in R^{1,\alpha}(\Omega, M) \), then by Theorem 3.13 \( u \) is a weakly contact mapping and the induced map \( D_h u \) can be represented by the matrix \( (X_j u^i(p))_{k \times \tilde{k}} \) of which each entry belongs to \( L^\alpha(\Omega) \). It is easily inferred from (3.11) that \( X_j u(p) = \sum_{l=1}^{\tilde{k}} X_j u^l(p) X_l \).
2. Let \( u \in R^{1,\alpha}(\Omega, M) \). By Theorem 3.13 we have \( u \in L^\alpha(\Omega, M) \), \( u^i \in ACL(\Omega, R) \) for \( i = 1, \cdots, \tilde{n} \) and \( X_j u^i \in L^\alpha(\Omega) \) for \( i = 1, \cdots, \tilde{k} \). We can not verify

\[
u^i \in L^\alpha(\Omega) \quad \text{for } i = 1, \cdots, \tilde{k}.
\]

But if \( M \) is a Carnot group with a homogeneous norm \( \rho \), then (3.12) holds. In general we do not have that \( u^i \in W^{1,\alpha}_X(\Omega) \) for \( i = \tilde{k} + 1, \cdots, \tilde{n} \) even if \( M \) is a Carnot group.

In section 4 we will use the results about Pansu differentiability of Sobolev mappings between Carnot groups with respect to the topology of \( L^\alpha(\Omega) \) to get the explicit form of the Korevaar-Schoen energy. Pansu differentiability with respect
to several topology for Sobolev mappings between Carnot groups has been studied in details in [61], [62], [63] and [64].

Theorem 3.17. Let $G = (\mathbb{R}^n, V_1, \delta_\lambda, \rho)$ and $G = (\mathbb{R}^n, \tilde{V}_1, \tilde{\delta}_\lambda, \tilde{\rho})$ be two Carnot groups where $\rho$ and $\tilde{\rho}$ homogeneous norms endowed to $G$, $\tilde{G}$ respectively. Let $\Omega$ be a bounded open set of $G$. Let $1 \leq \alpha < \infty$. If $u \in \text{R}^{1,\alpha}(\Omega, \tilde{G})$, then

1. $u$ is approximate Pansu differentiable a.e. in $\Omega$. Let $Du(p)$ be the approximate Pansu differential at $p \in \Omega$. The linear map $D_n u$ from $V_1$ to $\tilde{V}_1$ can be extended to a homomorphism $\tilde{D}_u(p)$ of Lie algebras such that

$$Du(p) = \widetilde{\exp} \circ \tilde{D}_u(p) \circ \exp^{-1}.$$ 

2. If $\tilde{\rho}$ is a homogeneous norm of the class of $C^\infty$ on $G \setminus \{0\}$, then for a.e. $p \in \Omega$, $Du(p)$ is the Pansu differential of $u$ in the topology of $L^\alpha(\Omega)$. That is,

$$\lim_{\epsilon \to 0} \int_{\rho(p) \leq 1} \left( \tilde{\rho}(\tilde{\rho}(Du(p)(\omega))) - \delta_{\rho}(u(p)^{-1}u(\delta,\omega))) \right)^\alpha d\omega = 0. \quad (3.13)$$

3.3. Precompactness and the trace theorem for Sobolev mappings. In this section we first give a compactness theorem and then develop a trace theorem, which will be needed in Section 5. The trace theorem for Sobolev mappings between metric spaces is delicate. In [39], a satisfactory trace theorem was developed for mappings in $KS^{1,\alpha}(\Omega, M)$ when $\Omega$ is a Lipschitz Riemannian domain and $M$ is a complete metric space. In the case $\Omega$ is a sub-Riemannian domain, whether an analogue can be developed is the problem we are going to investigate. Note that even for scalar valued Sobolev functions the trace theorem is not trivial when the analogue can be developed is the problem we are going to investigate. We first have the following precompactness theorem. Since its proof is standard (see [1], Theorem 2.4 and [39], Theorem 1.13), we omit it.

Theorem 3.18. Let $(\mathbb{R}^n, \Delta, d_c)$ be a C-C space and $\Omega \subset \mathbb{R}^n$ be a bounded open set. Let $M$ be a separable complete metric space with a (quasi-)metric $\rho$. Assume $\{u_\mu\}_{\mu=1}^\infty$ be a sequence of mappings in $R^{1,\alpha}(\Omega, M)$ such that

$$\sup_\mu \left\{ \int_\Omega d^\alpha(u_\mu(p), m_0) dp + \int_\Omega g^\alpha_\mu(p) dp \right\} \leq C$$

where $0 \leq g_\mu \in L^\alpha(\Omega)$ is a dominant function of the horizontal derivatives of $\theta^m_\mu(p) := d(m, u_\mu(p))$ for any $m \in M$, that is, $|X \theta^m_\mu(p)| \leq g(p)$ a.e. $p \in \Omega$ for any $m \in M$ (see Definition 3.3); $C > 0$ is an absolute constant; $m_0$ is a fixed point in $M$. Then there exists a subsequence $\{u_{\mu'}\}_{\mu'=1}^\infty$ of $\{u_\mu\}_{\mu=1}^\infty$ and a mapping $u \in R^{1,\alpha}(\Omega, M)$ such that

1. $\lim_{\mu' \to \infty} \int_\Omega d^\alpha(u_{\mu'}, p) dp = 0$;
2. there exists a dominant function $0 \leq g \in L^\alpha(\Omega)$ of the horizontal derivatives of $\theta_m(p) := d(m, u(p))$ ($m \in M$) satisfies $\int_\Omega g^\alpha(p) dp \leq \lim_{\mu' \to 0} \int_\Omega g^\alpha_{\mu'}(p) dp \leq C$. 

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Let \((R^n, \Delta, d_e)\) be a C-C space and \(\Omega \subset R^n\) be a \(C^2\) smooth bounded domain whose boundary does not possess characteristic points. We recall that a characteristic point \(p \in \partial \Omega\) is a point where the tangent space \(T_p \partial \Omega\) contains the horizontal space \(\Delta_p\). Let \(\bar{n}\) be the unit Euclidean exterior normal vector field of \(\partial \Omega\). Since \(\Omega\) is \(C^2\), there exists a neighborhood \(\bar{U}\) of \(\partial \Omega\) such that the signed distance function

\[
d_e(p) = \begin{cases} \text{dist}(p, \partial \Omega) := \inf_{q \in \partial \Omega} |p - q| & \text{if } p \in U \cap \Omega, \\ \text{dist}(p, \partial \Omega) & \text{if } p \in U \cap \Omega^c. \end{cases}
\]

is a defining function of \(\Omega\) (near the boundary), that is, \(d_e(p) = C_0\) for a.e. \(p \in \partial \Omega\) for some choice of constants \(C_0\). Then \(\gamma\) exists a representative \(d\sigma\) for some choice of constants \(\rho > \rho^0\) for some choice of constants \(\rho > \rho^0\). The horizontal transverse vector field \(Z(p) = Xd_e(p) = \sum_{i=1}^{k} X_i, \bar{n} > X_i\) satisfies

\[
|Z(p)| = \|Z(p)\|_{\partial \Omega} \geq \rho \quad \text{for any } p \in U \subset \bar{U}
\]

where \(U\) is a neighborhood of \(\partial \Omega\). The horizontal transverse vector field \(Z\) induces a fibration \(\Gamma_Z\) of \(U \cap \Omega\), that is, \(\Gamma_Z = \{\gamma(t) = \exp_p(tZ) : [0, t_0] \rightarrow \Omega, p \in \partial \Omega\}\). Then \(\gamma_p(t)\) satisfies

\[
\gamma_p(t) \in \Omega \quad \text{if} \quad 0 < t < t_0,
\]

\[
\gamma_p(t) \notin \Omega \quad \text{if} \quad -t_0 < t < 0,
\]

\[
|d_e(\gamma_p(t_1)) - d_e(\gamma_p(t_2))| > \rho |t_1 - t_2| \quad \text{if} \quad |t_1|, |t_2| < t_0
\]

for some choice of constants \(\rho > 0\) and \(t_0 > 0\).

We recall the definition of the measure \(d\gamma\) on \(\Gamma_Z\), \(d\gamma = \mathcal{F}_t d\bar{v}(\gamma(t))\), where \(\mathcal{F}_t\) is the Jacobian of the flow \(\exp_p(tZ)\), \(p \in \Omega\) and \(dv\) is the standard volume form of \(\Omega\). Since \(Z\) is transversal to \(\partial \Omega\), the area form \(d\sigma\) of \(\partial \Omega\), up to a normalization, is \(i(Z)dv\) where \(Z\) is understood as the restriction on \(\partial \Omega\) of \(Z\). Note that \(\mathcal{F}_t\) is always bounded in \(U\).

Let \(M\) be a separable metric space with a metric \(d\). Let \(u \in R^{1,\alpha}(\Omega, M)(\alpha \geq 1)\). We define the trace \(Tu \in L^a(\partial \Omega, M)\) of \(u\) on \(\partial \Omega\) as follows. By Theorem 3.10 there exists a representative \(\bar{u}\) of \(u\) such that \(\bar{u}\) is absolutely continuous on \(\gamma\) almost all curves in \(\Gamma_Z\), that is, \(\bar{u}\) is absolutely continuous on \(\gamma_p(t) = \exp_p(tZ) (0 < t \leq t_0)\) for \(d\sigma\) almost all \(p \in \partial \Omega\). Thus the map

\[
Tu(p) = \lim_{t \to 0^+} \bar{u}(\gamma_p(t))
\]

can be defined for a.e. \(p \in \partial \Omega\). Furthermore from the proof of Theorem 3.10 (see (4.1) in Page 641 of [6]) and using Hölder inequality we have

\[
d^\alpha(Tu(p), \bar{u}(\gamma_p(t))) \leq t^{\alpha-1} \int_{[p, \gamma_p(t)]} g^\alpha ds \tag{3.15}
\]

for a.e. \(p \in \partial \Omega\) where \(0 \leq g \in L^\alpha(\Omega)\). Integrating (3.15) with respect to \(p\) we infer

\[
\int_{\partial \Omega} d^\alpha(Tu(p), \bar{u}(\gamma_p(t))) d\sigma(p) \leq t^{\alpha-1} \int_{\partial \Omega} \int_{[p, \gamma_p(t)]} g^\alpha ds d\sigma(p)
\]

\[
\leq Ct^{\alpha-1} \int_{\Omega^c} g^\alpha dv \tag{3.16}
\]

where \(C\) is a constant independent of \(t\) and \(\Omega^c\) denotes the set of points in \(\Omega\) whose \(C-C\) distance to \(\partial \Omega\) is at most \(\epsilon\). Since \(\bar{u} \in L^\alpha(\Omega, M)\), by the Fubini’s theorem the maps \(\bar{u}(\gamma_p(t))\) are in \(L^\alpha(\partial \Omega, M)\) for almost all \(t \in [0, t_0]\). We conclude from (3.16)
that the trace map \( T(u) \) is the \( L^\alpha(\partial \Omega, M) \) limit of the maps \( \bar{u}(\gamma_p(t)) \) as \( t \to 0 \), so is itself an \( L^\alpha \) map. Since \( T(u) \) is the \( L^\alpha \) limit of almost all of the maps \( \bar{u}(\gamma_p(t)) \), as \( t \to 0 \), \( T(u) \) is independent of the choice of the representative of \( u \).

Thus we have proven

**Proposition 3.19.** Let \( (R^n, \Delta, d_r) \) be a C-C space and \( \Omega \subset R^n \) be a \( C^2 \) smooth bounded domain whose boundary \( \partial \Omega \) does not possess characteristic points. Let \( M \) be a separable metric space with a (quasi-)metric \( d \). Assume \( u \in R^{1,\alpha}(\Omega, M) \), \( \alpha \geq 1 \). Then \( \text{the energy of Korevaar-Schoen is not of the form of the Dirichlet integral, though it can be represented by} \)

**Remark 3.20.** In Proposition 3.19 the noncharacteristic condition is restrictive in the sense that “most” smooth bounded domains in a C-C space are characteristic. For examples of noncharacteristic smooth bounded domains we refer the reader to [9].

The following lemma can be easily deduced from Theorem 2.6, Theorem 3.5, Corollary 3.6 and Corollary 1.6.3 in [39].

**Lemma 3.21.** Let \( (R^n, \Delta, d_r) \) be a C-C space and \( \Omega \subset R^n \) be a bounded open set. Let \( M \) be a separable metric space with a (quasi-)metric \( d \) and let \( \alpha > 1 \). Assume \( u, v \in R^{1,\alpha}(\Omega, M) \) with dominant functions \( g_u, g_v \) respectively, then \( d(u,v) \in W^{1,\alpha}_X(\Omega) \) and

\[
\|Xd(u,v)\|_{L^\alpha(\Omega)} \leq C(\|g_u\|_{L^\alpha(\Omega)} + \|g_v\|_{L^\alpha(\Omega)})
\]

for some constant \( C > 0 \).

The following lemma can be easily deduced from Theorem 2.6, Theorem 3.5, Corollary 3.6 and Corollary 1.6.3 in [39].

**Theorem 3.22.** Let \( (R^n, \Delta, d_r) \) be a C-C space and \( \Omega \subset R^n \) be a \( C^2 \) smooth bounded domain whose boundary \( \partial \Omega \) does not possess characteristic points. Let \( M \) be a separable complete metric space with a (quasi-)metric \( d \) and let \( \alpha > 1 \). If the sequence \( \{u_\mu\}_{\mu=1}^\infty \subset R^{1,\alpha}(\Omega, M) \) has a sequence of dominant functions \( \{g_\mu\}_{\mu=1}^\infty \subset L^\alpha(\Omega) \) (that is, \( g_\mu \) is a dominant function for \( u_\mu \)) such that \( \{\|g_\mu\|_{L^\alpha(\Omega)}\}_{\mu=1}^\infty \) has uniform bound, and if \( \{u_\mu\} \) converges in \( L^\alpha(\Omega, M) \) to a mapping \( u : \Omega \to M \), then the trace functions of \( u_\mu \) converge in \( L^\alpha(\partial \Omega, M) \) to the trace of \( u \). Two mapping \( u, v \in R^{1,\alpha}(\Omega, M) \) have the same trace if and only if \( d(u,v) \in W^{1,\alpha}_X(\Omega) \) has trace zero.

**Proof.** It follows from Theorem 3.18 that the \( L^\alpha(\Omega, M) \) limit map \( u \) belongs to \( R^{1,\alpha}(\Omega, M) \). Since \( (3.10) \) holds, Theorem 3.22 follows almost verbatim from the arguments in the proof of Theorem 1.12.2 in [39] (for the existence of \( d_r \)-Lipschitz cut-off functions see [20]).

\[\square\]

4. Energy of Korevaar-Schoen

This section and Section 5 are devoted to making a choice of a reasonable energy which should be natural and compatible to the structures of the considered C-C spaces. Since the energy of Korevaar-Schoen has been extensively studied, a natural question is that whether it is the one we expected. When the target does not possess any curvature bound in the sense of Alexandrov, we want the energy to be of “good” form, for example, it is a Dirichlet integral. Unfortunately the energy of Korevaar-Schoen is not of the form of the Dirichlet integral, though it can be represented by an integral (see [41]). By now we can not prove or disprove that the energy of Korevaar-Schoen is lower semicontinuous with respect to some topology. Note that
C-C spaces may not possess “measure contraction property” which Riemannian manifolds possess (see \[59\]). Thus we can not adopt the idea in \[39\] and \[59\].

**Theorem 4.1.** Let \(G = (R^n, V_1, \delta_\lambda, \rho)\) and \(G = (\tilde{R^n}, \tilde{V}_1, \tilde{\delta}_\lambda, \tilde{\rho})\) be two Carnot groups where \(\rho\) and \(\tilde{\rho}\) are homogeneous norms endowed to \(G\) and \(\tilde{G}\) respectively. Let \(\Omega\) be a bounded open set of \(G\). Let \(\alpha \in (1, \infty)\). If \(\tilde{\rho}\) is of the class of \(C^\infty\) on \(\tilde{G}\setminus\{0\}\) and \(u \in KS^{1, \alpha}(\Omega, G)\), then the energy of Korevaar-Schoen can be written as:

\[
E^\alpha(u, \Omega) = \int_\Omega \int_{B_\rho(0, 1)} (\tilde{\rho}(Du(p)(\omega)))^\alpha \, d\omega dp.
\]  

where \(B_\rho(0, 1) = \{ \omega : \rho(\omega) \leq 1 \}\) and \(Du(p) : G \to \tilde{G}\) is the approximate Pansu derivative of \(u\) at \(p\).

**Proof.** We abuse the notation \(\rho(p, q) := d_\rho(p, q) = \rho(p^{-1} q)\). By Theorem 4.1 \(u\) is approximately Pansu differentiable a.e. \(p \in \Omega\). Fix \(p \in \Omega\) at which \(u\) is approximately Pansu differentiable in \(\Omega\). Recalling that the Lebesgue measure \(\mathcal{L}^\alpha\) in \(R^n\) is the Haar measure of \(G\) and the definition of homogeneous norms, by a change of variables we have

\[
e_\epsilon(p; u) = \int_{B_\rho(0, 1)} \frac{\tilde{\rho}(u(p), u(q)) - \tilde{\rho}(u(p), u(q)))}{\epsilon} \, dq
\]

\[
= \int_{B_\rho(0, 1)} (\tilde{\rho}(\delta_\frac{1}{\epsilon}(u(p)^{-1} u(p\delta_\epsilon \omega)))^\alpha \, d\omega.
\]  

By Theorem 3.5 we have \(u \in R^{1, \alpha}(\Omega, \tilde{G})\). Now we can use (4.13) to deduce

\[
\lim_{\epsilon \to 0} e_\epsilon(p; u) = \int_{B_\rho(0, 1)} (\tilde{\rho}(Du(p))(\omega))^\alpha \, d\omega.
\]  

In fact, by a well known inequality

\[
||a + b||^\alpha - ||b||^\alpha \leq C(\delta)||a||^\alpha + \delta||b||^\alpha
\]  

where \(a, b \in R\), \(\delta > 0\) and \(C(\delta)\) only depends on \(\delta\) and \(\alpha\), we obtain

\[
(\tilde{\rho}(\delta_\frac{1}{\epsilon}(u(p)^{-1} u(p\delta_\epsilon \omega)))^\alpha - (\tilde{\rho}(Du(p)(\omega)))^\alpha
\]

\[
= (\tilde{\rho}(\delta_\frac{1}{\epsilon}(u(p)^{-1} u(p\delta_\epsilon \omega))) - \tilde{\rho}(Du(p)(\omega)) + (\tilde{\rho}(Du(p)(\omega)))^\alpha - (\tilde{\rho}(Du(p)(\omega)))^\alpha
\]

\[
\leq \delta (\tilde{\rho}(Du(p)(\omega)))^\alpha + C(\delta) (\tilde{\rho}(\delta_\frac{1}{\epsilon}(u(p)^{-1} u(p\delta_\epsilon \omega))) - \tilde{\rho}(Du(p)(\omega)))^\alpha
\]

\[
\leq \delta (\tilde{\rho}(Du(p)(\omega)))^\alpha + C(\delta)C(\delta) (\rho((Du(p)(\omega))^{-1} \delta_\frac{1}{\epsilon}(u(p)^{-1} u(p\delta_\epsilon \omega)))^\alpha
\]  

where (4.5) is from (4.4) and in (4.6) we have used the quasi-triangle inequality property of \(\tilde{\rho}\). Thus (4.3) follows from (4.2), (4.6), (3.13) and the arbitrariness of \(\delta\) in (4.4).

On the other hand, since by Theorem 3.3 \(u \in H^{1, \alpha}(\Omega, \tilde{G})\), there exists \(0 \leq g \in L^\alpha(\Omega)\) such that

\[
\tilde{\rho}(u(p), u(q)) \leq \rho(p, q)(g(p) + g(q))
\]  

holds for a.e. \(p, q \in \Omega\). Let \(\varphi \in C_{c}(\Omega, [0, 1])\) and \(\epsilon < \text{dist}_\rho(\text{supp}\varphi, \partial \Omega)\). Assume

\[
F_\epsilon(u, p) := \varphi(p)c_\epsilon(p; u),
\]  

\[
\epsilon < \text{dist}_\rho(\text{supp}\varphi, \partial \Omega)\]
then by (4.7) we have

$$F_{c}(u, p) \leq G_{c}(u, p) := C\varphi(p)|g(p)|^\alpha + C\varphi(p)\int_{B_{p}(p, r)}|g(q)|^\alpha dq \tag{4.9}$$

for a.e. $p \in \Omega$ where $C$ only depends on $\alpha$. Since $g \in L^{\alpha}(\Omega)$, by Lebesgue differentiation theorem (see e.g. [27], Chapter 2)

$$\lim_{\epsilon \to 0} G_{c}(u, p) = G(u, p) := 2C\varphi(p)|g(p)|^\alpha \text{ a.e. } p \in \Omega. \tag{4.10}$$

From

$$\int_{\Omega} G_{c}(u, p) dp = C \int_{\Omega} \varphi(p)|g(p)|^\alpha dp + C \int_{\Omega} \varphi(p) \int_{B_{p}(p, r)} |g(q)|^\alpha dq dp \tag{4.11}$$

$$= C \int_{\Omega} \varphi(p)|g(p)|^\alpha dp + C \int_{B_{p}(0,1)} \varphi(p) \int_{\Omega} |g(p)|^\alpha dq dp \tag{4.12}$$

$$= C \int_{\Omega} \varphi(p)|g(p)|^\alpha dp + C \int_{B_{p}(0,1)} \varphi(p') |g(p')|^\alpha dp dp' \tag{4.13}$$

where in (4.11) we have made the change of variables $q' = \delta_{\frac{1}{p}}(p^{-1} q)$, in (4.12) used the Fubini’s Theorem, in (4.13) used the change of variables $p' = p\delta_{\frac{1}{p}}q'$,

$$\varphi(p') |g(p')|^\alpha \leq |g(p')|^\alpha$$

and

$$\int_{\Omega} \varphi(p') |g(p')|^\alpha dp dp' \leq \|g\|_{L^{\infty}(\Omega)}^\alpha,$$

by dominated convergence theorem we infer that

$$\lim_{\epsilon \to 0} \int_{\Omega} G_{c}(u, p) dp = C \int_{\Omega} \varphi(p)|g(p)|^\alpha dp + C \int_{B_{p}(0,1)} \lim_{\epsilon \to 0} \varphi(p') |g(p')|^\alpha dp dp' \tag{4.14}$$

$$= 2C \int_{\Omega} \varphi(p)|g(p)|^\alpha dp$$

$$= \int_{\Omega} G(u, p) dp$$

since $\varphi \in C_{c}(\Omega, [0, 1])$. From (4.3), (4.8), (4.9), (4.10), (4.14) and a variant dominated convergence theorem (see e.g. [11], p21), we have

$$\lim_{\epsilon \to 0} \int_{\Omega} F_{c}(u, p) dp = \int_{\Omega} \varphi(p) \int_{B_{p}(0,1)} (\rho(Du(p)\omega))^\alpha d\omega dp.$$

for any $\varphi \in C_{c}(\Omega, [0, 1])$. Consequently (1.2) follows. 

In Theorem 4.2 we give a representation of the energy of Korevaar-Schoen for mappings in $KS^{1,\alpha}(\Omega, \tilde{G})$ from a Carnot group to another Carnot group. One may ask whether $E^{\alpha}(u, \Omega)$ is lower semicontinuous with respect to some topology of $KS^{1,\alpha}(\Omega, \tilde{G})$. When $\Omega$ is a smooth Riemannian domain and $\tilde{G}$ is a metric space with a metric $\tilde{d}$, in [9] Korevaar-Schoen proved $E^{\alpha}(u, \Omega)$ is lower semicontinuous with respect to the topology of $L^{\alpha}(\Omega, \tilde{G})$, that is, if $\sup_{\mu} E^{\alpha}(u, \Omega) < \infty$ and

$$\lim_{\mu \to \infty} \int_{\Omega} d(u_{\mu}(p), u(p))^{\alpha} dp = 0,$$

then $E^{\alpha}(u, \Omega) \leq \liminf_{\mu \to \infty} E^{\alpha}(u, \Omega)$. This
Let’s first recall some fundamental facts about C-C geodesics in Heisenberg groups $H^m$ (see [3] or [60]).

**Lemma 4.2.** Let $g_0 = (x_0, y_0, t_0) \neq 0$ be a point in $H^m$. We have

1. if $x_0^2 + y_0^2 \neq 0$, then there exists a unique C-C geodesic connecting 0 to $g_0$.
2. otherwise, there exist infinitely many C-C geodesics connecting 0 to $g_0$.

Moreover, let $\gamma(s) = (x(s), y(s), t(s))(0 \leq s \leq 1)$ be any C-C geodesic connecting 0 to $g_0$, we have

\[
\begin{align*}
    x_i(s) &= A_i (\cos(s\phi) - 1) + B_i \sin(s\phi), & i = 1, \cdots, m, \\
    y_i(s) &= B_i (\cos(s\phi) - 1) - A_i \sin(s\phi), & i = 1, \cdots, m, \\
    t(s) &= 2 \frac{s\phi - \sin(s\phi)}{\phi^2},
\end{align*}
\]

where $\tau = \phi \in [-2\pi, 2\pi]$ is the unique solution in $[-2\pi, 2\pi]$ of the equation

\[
\frac{1 - \cos \tau}{\tau - \sin \tau} = \frac{|x_0|^2 + |y_0|^2}{t_0}
\]

with

\[
\begin{align*}
    \tau &= 0 \quad \text{if } t_0 = 0, \\
    |\tau| &= 2\pi \quad \text{if } |x_0|^2 + |y_0|^2 = 0, \\
    \tau &\in (0, 2\pi) \quad \text{if } t_0 > 0, \\
    \tau &\in (-2\pi, 0) \quad \text{otherwise};
\end{align*}
\]

$\rho = d_c(0, g_0)$ is the arc length of $\gamma$ determined by

\[
\rho = \sqrt{\frac{x_0^2 t_0}{2|\tau - \sin \tau|}}, \quad \text{if } t_0 \neq 0,
\]

\[
\rho = \sqrt{|x_0|^2 + |y_0|^2}, \quad \text{if } t_0 = 0;
\]

if $|x_0|^2 + |y_0|^2 \neq 0 \{A_1, \ldots, A_m, B_1, \ldots, B_m\}$ is subject to

\[
\sum_{i=1}^n (A_i^2 + B_i^2) = 1,
\]

\[
x_0i = A_i (\cos(\phi) - 1) + B_i \sin(\phi), \quad i = 1, \cdots, m,
\]

\[
y_0i = B_i (\cos(\phi) - 1) - A_i \sin(\phi), \quad i = 1, \cdots, m;
\]
if \(|x_0|^2 + |y_0|^2 = 0\) then \(\{A_1, \ldots, A_m, B_1, \ldots, B_m\}\) is only subject to

\[\sum_{i=1}^{m} (A_i^2 + B_i^2) = 1.\]

**Remark 4.3.** By the left-invariant property of the C-C metric we easily deduce that the \(\gamma_{p_0, p}(s)\) is a C-C geodesic connecting \(p_0\) to \(p\) if and only if \(p_0^{-1}\gamma_{p_0, p}(s)\) is a C-C geodesic connecting \(0\) to \(p_0^{-1}p\), that is, \(\gamma_{p_0, p}(s) = p_0\gamma_{p_0^{-1}p}(s)\) where \(\gamma_{p_0^{-1}p}(s)\) is the C-C geodesic connecting \(0\) to \(p_0^{-1}p\).

To simplify some computation we fix \(m = 1\). Let \(H_1^1 = H_1 \setminus \{(0, 0, t) : t \in \mathbb{R}\}\). Set \(S = \{(\theta, \phi, \rho) : 0 \leq \theta < 2\pi, |\phi| \leq 2\pi, \rho \geq 0\}\), \(S_* = \{(\theta, \phi, \rho) : 0 \leq \theta < 2\pi, |\phi| \leq 2\pi, \rho \geq 0\}\) and \(A(\theta, \phi, \rho) : S \to H_1^1\) by \(A(\theta, \phi, \rho) = (x(\theta, \phi, \rho), y(\theta, \phi, \rho), t(\theta, \phi, \rho))\), where

\[
\begin{align*}
x(\theta, \phi, \rho) &= \cos \theta (\cos(\phi \rho) - 1) + \sin \theta \sin(\phi \rho) \\
y(\theta, \phi, \rho) &= \sin \theta (\cos(\phi \rho) - 1) - \cos \theta \sin(\phi \rho) \\
t(\theta, \phi, \rho) &= 2 \frac{\rho \sin(\phi \rho)}{\phi \rho} 
\end{align*}
\]

By Lemma 4.2 we know that the map \(A : S_* \to H_1^1\) is bijective and equation (4.16) parameterizes \(\partial B_c(0, \rho) = \{p \in H^1 : d_c(0, p) = \rho\}\). We can compute the Jacobian of \(A\) by

\[
\det J_A(\theta, \phi, \rho) = \det \left(\frac{\partial(x, y, t)}{\partial(\theta, \phi, \rho)}\right) = 4 \rho \sin(\phi \rho) - 2(1 - \cos(\phi \rho)) \phi^2
\]

Let \(\bar{s} \in [0, 1]\) and \(p = (x, y, t) \in H_1^1\), we consider the Jacobian of the map of changing variables \(B_{\bar{s}}^\rho(p) : p = (x, y, t) \to p' = (x', y', t') = \gamma_{0, \rho}(\bar{s})\) where \(\gamma_{0, \rho}(s)(0 \leq s \leq 1)\) is the C-C geodesic joining \(0\) and \(p\). Since \(p \in H_1^1\), we can parameterize \(p\) by \((\theta, \phi, \rho)\) through equation (4.16) and \((x', y', t') = A(\theta, \phi, \bar{s} \rho)\). Now we can compute the Jacobian of \(B_{\bar{s}}^\rho\) by

\[
\det J_{B_{\bar{s}}^\rho}(x, y, t) = \det \left(\frac{\partial(x', y', t')}{\partial(x, y, t)}\right) = \det \left(\frac{\partial(x, y, t)}{\partial(\theta, \phi, \rho)}\right) \det \left(\frac{\partial(\theta, \phi, \bar{s} \rho)}{\partial(\theta, \phi, \rho)}\right) \det \left(\frac{\partial(\theta, \phi, \rho)}{\partial(x, y, t)}\right) = \det J_A(\theta, \phi, \bar{s} \rho) \bar{s} \det J_A(\theta, \phi, \rho)^{-1}
\]

\[
= \bar{s} \frac{\rho \sin(\bar{s} \rho) - 2(1 - \cos(\bar{s} \rho))}{\rho \sin(\rho) - 2(1 - \cos(\rho))}
\]

where we have used (4.17). From (4.15) we have if \(t_0 \to 0\), then \(\tau = \rho \to 0\). Thus from (4.18) we obtain

\[
\lim_{t_0 \to 0} \det J_{B_{\bar{s}}^\rho}(x, y, t) = \lim_{\tau \to 0} \bar{s} \tau \sin(\bar{s} \tau) - 2(1 - \cos(\bar{s} \tau)) = \bar{s} \bar{s} \bar{s}
\]

In general case for any \(p_0 = (x_0, y_0, t_0) \in H^1, \bar{s} \in [0, 1]\), we define the map \(B_{\bar{s}}^\rho(p) = \gamma_{p_0, \rho}(\bar{s}) = p_0 \gamma_{p_0^{-1}p}(\bar{s})\) where \(p \in H^1 = \{g \in H^1 : p_0^{-1}g \in H^1\}\), \(\gamma_{p_0, \rho}(s)\) is the C-C geodesic connecting \(p_0\) to \(p\) and \(\gamma_{p_0^{-1}p}(s)\) is the C-C geodesic connecting \(0\) to \(p_0^{-1}p\) (see Remark 4.3). Since the Jacobian of left translation is 1, from (4.18) we can easily infer that the Jacobian of \(B_{\bar{s}}^\rho\) is

\[
\det J_{B_{\bar{s}}^\rho}(p) = \bar{s} \frac{\rho \sin(\bar{s} \rho) - 2(1 - \cos(\bar{s} \rho))}{\rho \sin(\rho) - 2(1 - \cos(\rho))}
\]
where \((\theta, \phi, \rho)\) parameterizes the point \(p_{t_0}^{-1}p\) through (4.16). Thus we deduce from (4.20) and (4.15) that

\[
\lim_{t-t_0+2(x_0-y_0)\to 0} \det \hat{J}_{\bar{s}p_0}(p) = \bar{s}^5
\]

where \(p = (x, y, t) \in \ast H^1_{p_0}\).

It is well known that in a \(m\) dimensional smooth Riemannian manifold \(M\) with Riemannian metric \(g\) the Jacobian of the map of changing variables along Riemannian geodesics can be well estimated. More precisely, if \(\epsilon > 0\) is sufficiently small such that the geodesic ball \(B_{d_g}(p_0, \epsilon)\) is in a normal coordinate neighborhood of \(p_0\), then \(\det \hat{J}_{\bar{s}p_0}(p) \geq C\bar{s}^{m+1+o(\epsilon)}\) for any \(p \in B_g(p_0, \epsilon)\) and \(\bar{s} \in [0,1]\) where the map \(\hat{J}_{\bar{s}p_0}(p)\) is defined similarly as above and \(C \geq 0\) is a constant dependent on the Riemannian metric (Ricci curvature). In the case of Heisenberg group, since the Hausdorff dimension of \(H^m\) is \(Q = 2m + 2\), one would like to guess that \(\det \hat{J}_{\bar{s}p_0}(p) \geq C\bar{s}^Q\) in a neighborhood of \(p_0\) (4.22) for some constant \(C\). We remark that if (4.22) was true, then several problems in analysis on Heisenberg groups could be solved by standard methods, for example to prove an inequality conjectured by [2] and to prove the semicontinuity of the energy of Korevaar-Schoen when the domain space is a Heisenberg group by repeating the story of [39] or [59]. Unfortunately as we have shown above, (4.22) is impossible to hold.

5. Horizontal energy and existence of minimizers

As we have indicated in the Introduction, the concept of the Horizontal energy is a natural generalization of the ordinary energy for mappings between Riemannian manifolds.

**Definition 5.1.** Let \((R^n, \Delta, d_c)\) and \(M = (R^n, \tilde{\Delta}, \tilde{d}_c)\) be two C-C space spaces. Let \(1 \leq \alpha < \infty\) and \(\Omega\) be a bounded open set of \(R^n\). Let \(u \in R^{1,\alpha}(\Omega, M)\), we call the following quantity

\[
HE^\alpha(u, \Omega) = \int_{\Omega} \left( \sum_{i=1}^k \|X_i u(p)\|_{\tilde{\Delta}, \tilde{d}_c}^2 \right)^{\frac{\alpha}{2}} dp
\]

(5.1)

the \(\alpha\)-horizontal energy of \(u\).

Note that if \(M\) is of Carnot type, from Remark 3.16 we have

\[
HE^\alpha(u, \Omega) = \int_{\Omega} \left( \sum_{j=1}^k \sum_{i=1}^{\tilde{k}} |X_j u(p)|_{\tilde{\Delta}, \tilde{d}_c}^2 \right)^{\frac{\alpha}{2}} dp.
\]

Let \((R^n, \Delta, d_c)\) be a C-C space and \(\Omega \subset R^n\) be a \(C^2\) bounded open set whose boundary is noncharacteristic with respect to \(\Delta\). Let \(M = (R^n, \tilde{\Delta}, \tilde{d}_c)\) be another C-C space and let \(\alpha \geq 1\). Fix \(\phi \in R^{1,\alpha}(\Omega, M)\) and set

\[
R_{\phi}^{1,\alpha}(\Omega, M) := \{ u \in R^{1,\alpha}(\Omega, M) : T(u) = T(\phi) \},
\]
where $T(u)$ denotes the trace map of $u$, see Subsection 3.3. We consider the following Dirichlet problem of minimizing $\alpha$-horizontal energy among all mappings in $R^{1,\alpha}(\Omega, M)$ whose traces are equivalent to the trace of $\phi$:

$$\text{find a } u \in R^{1,\alpha}_\phi(\Omega, M) \text{ such that } HE^\alpha(u, \Omega) = \inf_{v \in R^{1,\alpha}_\phi(\Omega, M)} HE^\alpha(v, \Omega). \quad (5.2)$$

**Definition 5.2.** Any solution to Problem (5.2) is called a horizontal energy minimizer.

**Theorem 5.3.** Let $(\mathbb{R}^n, \Delta, d_c)$ be a $C$-$C$ space and $\Omega \subset \mathbb{R}^n$ be a $C^2$ bounded open set whose boundary is noncharacteristic with respect to $\Delta$. Let $M = (\mathbb{R}^n, \Delta, d)$ is a $C$-$C$ space of Carnot type with a (quasi-)metric $d$ which is equivalent to $d_c$ and let $\alpha > 1$. Then Problem (5.2) has a solution.

**Proof.** Let $\{u_\mu\}_{\mu=1}^\infty$ be a minimizing sequence, that is,

$$\lim_{\mu \to \infty} HE^\alpha(u_\mu, \Omega) = \inf_{v \in R^{1,\alpha}_\phi(\Omega, M)} HE^\alpha(v, \Omega) := C_0 \leq HE^\alpha(\phi, \Omega) < \infty. \quad (5.3)$$

From Theorem 3.13 and Remark 3.14 we easily get a sequence of dominant functions $\{g_\mu\}_{\mu=1}^\infty$ of $\{u_\mu\}_{\mu=1}^\infty$ such that $\|g_\mu\|_{L^\infty(\Omega)} \leq C HE^\alpha(u_\mu, \Omega)$ for any $\mu$ where $C$ is a constant. From (5.3) the sequence $\{g_\mu\}$ is uniformly bounded in $L^\infty(\Omega)$. On the other hand, since $T(u_\mu) = T(\phi)$, by Theorem 3.22 we get $d(u_\mu, \phi) \in W^{1,\alpha}_X(\Omega)$ has trace zero for any $\mu$. Applying the Poincaré inequality, (quasi-)triangle inequality property of $d$ and (3.17) we get

$$\int_{\Omega} d^\alpha(u_\mu(p), m_0) dp \leq C \left( \int_{\Omega} d^\alpha(u_\mu(p), \phi(p)) dp + \int_{\Omega} d^\alpha(m_0, \phi) dp \right) \leq C_1 \int_{\Omega} |Xd(u_\mu, \phi)|^\alpha(p) dp + C \int_{\Omega} d^\alpha(m_0, \phi) dp \leq C_3(\|g_\mu\|_{L^\infty(\Omega)}^\alpha + \|g_\phi\|_{L^\infty(\Omega)}^\alpha) + C \int_{\Omega} d^\alpha(m_0, \phi) dp$$

for any $\mu$ where $m_0$ is a fixed point in $M$ and $g_\phi$ is a dominant function of $\phi$. The last inequalities show that $\int_{\Omega} d^\alpha(u_\mu(p), m_0) dp$ is uniformly bounded. So we have

$$\sup_{\mu} \left\{ \int_{\Omega} d^\alpha(u_\mu(p), m_0) dp + \|g_\mu\|_{L^\infty(\Omega)}^\alpha \right\} \leq C$$

for a constant $C > 0$ depending on $\phi$. Now we can use Theorem 3.18 to get a subsequence $\{u_\mu\}$ of $\{u_\mu\}$ and $u \in R^{1,\alpha}(\Omega, M)$ such that

$$\lim_{\mu \to \infty} \int_{\Omega} d^\alpha(u_\mu(p), u(p)) dp = 0 \quad (5.4)$$

and

$$X_1 u_\mu^i \text{ converges weakly in } L^\alpha(\Omega) \text{ to } X_1 u^i \text{ for } i = 1, \cdots, \bar{k} \text{ and } l = 1, \cdots, k. \quad (5.5)$$

From (3.4) and Theorem 3.22 we have $T(u) = T(\phi)$. Thus $u \in R^1_\phi(\Omega, M)$. From the lower semicontinuity of $HE^\alpha(u, \Omega)$ with respect to weak convergence and (5.5), we conclude that $u$ is a minimizer. $\square$
6. Some remarks on the regularity of minimizers: Heisenberg group

TARGET

In this section we briefly mention the known results to the regularity problem. The regularity problem is still quite open and new methods and tools should be developed to tackle it.

For the case when the domain space is a C-C space and the target is Euclidean, the Hölder regularities were obtained in [24] and in [37] using different methods.

If the target \( M = (R^3, \Delta, d_\Delta) \) is a C-C space (\( \Delta \) is non-integrable), then since \( R^{1, \alpha}(\Omega, M) \) is not a linear space (because of the contact condition), it is not trivial to construct contact variations of the minimizer to deduce Euler-Lagrangean equations. The simple example is the case studied by Capogna and Lin in [5] where \( \Omega \) is an Euclidean smooth bounded domain and \( M \) is the Heisenberg group \( H^m \) with a homogeneous metric \( \rho \). We denote by \( u = (z, t) = (x, y, t) \) elements in \( R^{1, \alpha}(\Omega, H^m) \) (\( \alpha \geq 1 \)). Then from Definition 3.11, Theorem 3.13 and (2) of Remark 3.16 we have \( u \in R^{1, \alpha}(\Omega, H^m) \) if and only if

\[
\begin{align*}
(1) & \quad z \in W^{1, \alpha}(\Omega, R^{2m}); \\
(2) & \quad t \in L^{\frac{2}{3}} \cap ACL(\Omega, R); \\
(3) & \quad u \text{ satisfies the Legendrian condition, that is, } \partial_p t = 2(y \cdot \partial_p x - x \cdot \partial_p y) \text{ a.e. } p \in \Omega \text{ for } i = 1, \ldots, n.
\end{align*}
\]

Here and in the sequel we denote by \( x_i \) or \( \partial_i x \) the partial derivative \( \frac{\partial x}{\partial p_i} \) and \( \cdot \) denotes the inner product in \( R^n \). Note that if \( u \) satisfies (1), (2) and (3), then from Sobolev inequality and Hölder inequality, \( \partial_p t \in L^\beta(\Omega) \) (\( \beta = \frac{2n-2}{2n-3} \)) automatically holds for \( i = 1, \ldots, n \). Moreover if \( \alpha \geq 2 \), then \( t \in W^{1, \beta}(\Omega) \). The horizontal energy is

\[
HE^\alpha(u, \Omega) = \int_\Omega |\nabla z(p)|^\alpha dp.
\]

**Lemma 6.1.** Let \( \alpha \geq 2 \) and \( \Omega = (z_u, t_u, v) = (z_v, t_v) \in R^{1, \alpha}(\Omega, H^m) \). Then \( T(u) = T(v) \) if and only if \( T(z_u) = T(z_v) \) and \( T(t_u) = T(t_v) \) where \( T(u) \) denotes the trace of \( u \) on \( \partial \Omega \).

**Proof.** The proof is straightforward. Since any homogeneous metrics are equivalent,

\[
C_1 |z_u - z_v|^\alpha + |t_u - t_v - 2\omega(z_u, z_v)|^\frac{2n}{n-2}
\]

\[
\leq C_2 \|u^{-1} v\|^\alpha \leq \rho^\alpha (u, v) \leq C_3 \|u^{-1} v\|^\alpha
\]

\[
\leq C_4 |z_u - z_v|^\alpha + |t_u - t_v - 2\omega(z_u, z_v)|^\frac{2n}{n-2}
\]

where \( \| \cdot \| \) is the gauge norm: \( \| (z, t) \| = (|z|^4 + t^2)^{\frac{n}{4}} \); \( \omega(\cdot, \cdot) \) is the standard symplectic form in \( R^{2n} \), and \( C_i, i = 1, 2, 3, 4 \) are constants. From Theorem 3.22 \( T(u) = T(v) \) if and only if \( \rho(u, v) \in W^{1, \alpha}(\Omega) \). Since \( \alpha \geq 2 \), the term in (6.1) belongs to \( W^{1, 1}(\Omega) \). Thus the statement follows from (6.1)-(6.3) and the fact that \( \omega(z_u, z_v)|_{\partial \Omega} = 0 \) a.e if \( T(z_u) = T(z_v) \).

The following lemma tells us that the projection of a weakly contact map \( u : \Omega \to H^m = R^{2m} \times R \) to \( R^{2m} \) is a weakly isotropic map and conversely any weakly isotropic map \( z : \Omega \to R^{2m} \) can be lifted to be a weakly contact map.
Lemma 6.2. Let $\alpha \geq 2$ and $\Omega \subset U \subset \mathbb{R}^n$. If $u = (z, t) = (x, y, t) \in W^{1,\alpha}(\Omega, H^m)$, then $z = (x, y) \in W^{1,\alpha}(\Omega, R^{2m})$ and satisfies the following weakly isotropic condition:

$$z^*(\omega) = 0 \text{ a.e in } \Omega,$$

that is,

$$x_i \cdot y_j = x_j \cdot y_i \text{ a.e. for } i, j = 1, \cdots, n. \quad (6.4)$$

Conversely if $\phi = (z_\phi, t_\phi) \in R^{1,\alpha}(U, H^m)$ and

$$z \in W_{z_\phi}^{1,\alpha}(\Omega, R^{2m}) =: \{ \bar{z} \in W^{1,\alpha}(\Omega, R^{2m}) : T(\bar{z}) = T(z_\phi) \}$$

and satisfies (6.4), then there exists $t \in W^{1,\beta}(\Omega)$ such that $u = (z, t) \in W^{1,\alpha}(\Omega, H^m)$ and $t = t_\phi$ at $\partial \Omega$.

Proof. The first statement follows (essentially) from the inequality (2.12) in [5], see Lemma 2.12 and Theorem 2.16 in [3] for details.

We prove the second statement. Let $z = (x, y) \in W^{1,\alpha}(\Omega, R^{2m})$ and satisfies (6.4). Let $\eta$ be the primitive form of the standard symplectic form $\omega$ in $R^{2n}$, that is, $d\eta = \omega$. We first prove that the 1-form

$$\zeta(p) = z^*(\eta) = \frac{1}{2} \sum_{i=1}^{n} (x \cdot y_i - y \cdot x_i) dp_i$$

belongs to $L^2(\Omega)$ and satisfies

$$d\zeta = 0 \quad (6.5)$$

in the sense of distribution. It suffices to prove

$$\int_{\Omega} (x \cdot y_i - y \cdot x_i) \varphi_j - (x \cdot y_j - y \cdot x_j) \varphi_i dp = 0 \quad (6.6)$$

for any $\varphi \in C_0^\infty$ and $i \neq j$. We mollify $z$ and let $z^\varepsilon = (x^\varepsilon, y^\varepsilon) = z * \delta_\varepsilon$ where $\delta_\varepsilon$ is a standard mollifier. We have

$$I = \int_{\Omega} (x^\varepsilon \cdot y_i^\varepsilon - y^\varepsilon \cdot x_i^\varepsilon) \varphi_j - (x^\varepsilon \cdot y_j^\varepsilon - y^\varepsilon \cdot x_j^\varepsilon) \varphi_i dp = - \int_{\Omega} (x^\varepsilon \cdot y_i^\varepsilon - y^\varepsilon \cdot x_i^\varepsilon) \varphi - (x^\varepsilon \cdot y_j^\varepsilon - y^\varepsilon \cdot x_j^\varepsilon) \varphi dp = \Pi + \Pi \Pi$$

where

$$\Pi = \int_{\Omega} (x^\varepsilon \cdot y_i^\varepsilon - y^\varepsilon \cdot x_i^\varepsilon) \varphi - (x^\varepsilon \cdot y_j^\varepsilon - y^\varepsilon \cdot x_j^\varepsilon) \varphi dp$$

and

$$\Pi \Pi = \int_{\Omega} x^\varepsilon \cdot ((y_j^\varepsilon)_{i, j} - (y_i^\varepsilon)_{i, j}) \varphi + y^\varepsilon \cdot ((x_i^\varepsilon)_{i, j} - (x_j^\varepsilon)_{i, j}) \varphi dp = 0.$$

Since $z$ satisfies (6.4), we have $\lim_{\varepsilon \to 0} \Pi = 0$. Consequently (6.6) follows and (6.5) holds. Thus we can apply a well known result about boundary value problem involving differential forms, see e.g. Chapter 3 of [55], to get a $t \in W^{1,\beta}(\Omega)$ of the following equation

$$\left\{ \begin{array}{l}
dt = 4\zeta, \quad \text{in } \Omega; \\
t = t_\phi, \quad \text{in } \partial \Omega.
\end{array} \right. \quad (6.7)$$

Note that (6.7) means $\partial_p t = 2(y \partial_p x - x \partial_p y)$ a.e. $p \in \Omega$ for $i = 1, \cdots, n$. □
Let
\[ W_{I}^{1,\alpha}(\Omega, R^{2m}) = \{ z \in W^{1,\alpha}(\Omega, R^{2m}) : z \text{ satisfies (6.4)} \} \]
be the Sobolev space of weakly isotropic mappings. If \( \varphi \in W_{I}^{1,\alpha}(\Omega, R^{2m}) \), we denote by
\[ W_{I,\varphi}^{1,\alpha}(\Omega, R^{2m}) = W_{I}^{1,\alpha}(\Omega, R^{2m}) \cap W_{\varphi}^{1,\alpha}(\Omega, R^{2m}) \]
the Sobolev space of weakly isotropic mappings with the trace of \( \varphi \).

From Lemma 6.1 and 6.2, we get the following theorem.

**Theorem 6.3.** Let \( \alpha \geq 2 \) and \( \Omega \subset U \subset R^{n} \) be two bounded open sets. Let \( \varphi = (z_{\varphi}, t_{\varphi}) \in R_{1}^{1,\alpha}(U, H^{m}) \). Then \( u = (z, t) \in R_{1}^{1,\alpha}(\Omega, H^{m}) \) is a solution of Problem (5.2), that is,
\[ HE^{\alpha}(u, \Omega) = \inf_{v \in R_{1}^{1,\alpha}(\Omega, M)} HE^{\alpha}(v, \Omega), \]
if and only if \( z \) is a solution of the following isotropically constrained variational problem:
\[ \text{to find } z_{0} \in W_{I,\varphi}^{1,\alpha}(\Omega, R^{2m}) \text{ such that } E(z_{0}, \Omega) = \inf_{z \in W_{I,\varphi}^{1,\alpha}(\Omega, R^{2m})} E(z, \Omega) \]  
(6.8)
where \( E(z, \Omega) = \int_{\Omega} |\nabla z(p)|^{\alpha} dp \).

**Remark 6.4.** (1) The existence of solutions to Problem (6.8) can be easily established.
(2) When \( n = \alpha = 2 \), due to the conformal invariance of the Dirichlet integral, Problem (6.8) is closely related to the following isotropically constrained Plateau problem studied by [54] and [51]:
\[ \text{to find } l_{0} \in X_{I,\Gamma} \text{ such that } \text{Area}(l_{0}, B_{1}) = \inf_{l \in X_{I,\Gamma}} \text{Area}(l, B_{1}) \]  
(6.9)
where \( \Gamma \) is a piecewise \( C^{1} \) closed Jordan curve such that \( \int_{\Gamma} \eta = 0; B_{1} \) is the ball centered at 0 with radius 1; \( X_{I,\Gamma} = \{ l \in W_{I}^{1,2}(B_{1}, R^{2m}) : l_{\partial B_{1}} \text{ is continuous and is a monotone map onto } \Gamma \} \); \( \text{Area}(l, B_{1}) \) is the area of the image of \( l \).

**Proposition 6.5.** Let \( \Omega \subset R^{2} \) be a bounded open set and let \( \alpha \geq 2 \). Let \( \phi \in R_{1}^{1,\alpha}(\Omega, H^{1}) \). If \( u \in R_{1}^{1,\alpha}(\Omega, H^{1}) \) such that
\[ HE^{\alpha}(u, \Omega) = \inf_{v \in R_{1}^{1,\alpha}(\Omega, H^{1})} HE^{\alpha}(v, \Omega), \]
then \( u \) is Lipschitz continuous (with respect to the C-C metric) in the interior of \( \Omega \).

Proposition 6.5 was proven in Theorem 4.3 of [5] where the authors asserted that Proposition 6.5 should hold for \( m \geq 1 \). In our opinion, since the equation (4.2) in [5] holds only for \( m = 1 \), the case when \( m > 1 \) remains open.

Keeping Theorem 6.3 in mind we see the following theorem is just a copy of the main result in [51].

**Theorem 6.6.** Let \( U \) be a neighborhood of a smooth, connected and simply connected, bounded open set \( \Omega \) such that \( \Omega \subset U \subset R^{2} \) and let \( m > 1 \). Let \( \phi = (z_{\phi}, t_{\phi}) \in R_{1}^{1,2}(U, H^{m}) \) such that \( \phi|_{\partial \Omega} \) is continuous and monotone onto \( \phi(\partial \Omega) \)
which is a piecewise $C^1$ closed Jordan curve in $R^{2m}$. Let $u = (z,t) \in R^1_{\phi}(\Omega, H^m)$ be a solution of Problem (5.2), that is

$$HE^2(u,\Omega) = \inf_{v \in R^1_{\phi}(\Omega,M)} HE^2(v,\Omega).$$

Then $z$ is Hölder continuous in $\Omega$ and $u$ is smooth in $\Omega$ with possibly isolated singularities.

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Department of Applied Mathematics, Nanjing University of Science and Technology, 210094, Nanjing, The People’s Republic of China
E-mail address: tankanghai2000@yahoo.com.cn