Understanding Quantitative Wave-Particle Duality

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The complementary character of wave and particle natures of quantum objects (or quantons) was pointed out by Niels Bohr. This wave-particle duality, in the context of the two-slit experiment, is now quantitatively understood in terms of a duality relation. A very simple and intuitive derivation of the duality relation is presented, which should be understandable to a new student.

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I. INTRODUCTION

The two-slit experiment with particles is considered the most beautiful experiment in physics. Particles going through a double-slit one by one, accumulate to form an interference pattern. With advancement of technology, multislit diffraction has been demonstrated with large molecules like C_{60}. Although the particles (or quantons, as we would like to call them) go through the double-slit one at a time, the accumulated result of many quantons shows an interference. It indicates that each quanton behaves like a wave and interferes with itself.

Objects like electrons and C_{60} molecules are supposed to be “indivisible” particles, and one would naively imagine that these quantons pass through one of the two slits. However, if one tries to find out which of the two slits the quanton passed through, the interference disappears. The quanton passing through one of the two slits, is associated with the particle nature, whereas the interference seen in cumulative results is associated with the wave nature. Wave and particle natures can only be seen one at a time, never simultaneously. This was formalized by Niels Bohr as the principle of complementarity. Einstein had tried to argue against such a principle, and had proposed a thought experiment which he claimed, showed wave and particle natures in the same experiment. Bohr pointed out the flaw in Einstein’s argument and the principle of complementarity stood its ground.

It was recognized later that the wave and particle natures are not mutually exclusive in the strict sense of the word. It is possible to get partial information on which slit a quanton went through, and get an interference pattern which is not sharp. In other words wave-particle duality in Bohr’s principle can be stated more quantitatively. Wave-particle duality is now quantitatively well understood by the following duality relation:

\[ V^2 + D^2 \leq 1, \]  

where \( D \) is a path distinguishability and \( V \) the visibility of the interference pattern. Both these quantities vary between 0 and 1, and the above relation quantifies how visible will the interference be if the two paths through the two slits can be distinguished with a distinguishability \( D \). One might wonder however, what distinguishability means for a single particle, and whether each particle contributes partially to the interference. Or does the duality relation mean that some quantons give full which-path information and do not contribute to interference and some give zero which-path information and contribute fully to interference? These are the questions a student is likely to wonder about since the interference is built up by particles going through one by one. We would like to address this question and suggest a picture to visualize. We will analyze a thought modification of the two-slit experiment with provision for path detection, and give a novel derivation of the duality relation.

II. FULL WHICH-PATH INFORMATION

It can be demonstrated quite simply that if there is a path-detector which gains information about which of the two slits the quanton passed through, the interference will be completely destroyed. Suppose that the state of a quanton, passing through a double-slit, is given by

\[ \Psi(x) = \frac{1}{\sqrt{2}}(\psi_1(x) + \psi_2(x)) \]  

where \( \psi_1(x) \) and \( \psi_2(x) \) are the amplitudes of the quanton passing through slit 1 and 2, respectively. Probability of finding the quanton at a point \( x \) on the screen, is given by

\[ |\Psi(x)|^2 = \frac{1}{2}(|\psi_1(x)|^2 + |\psi_2(x)|^2 + \psi_1^*(x)\psi_2(x) + \psi_2^*(x)\psi_1(x)) \]  

The last two terms represent interference. Now, let us suppose that we have a quantum path-detector also included in the setup. Without going into the details of what such a path-detector might look like, we just assume that a quanton going through slit 1 leaves the detector in a state \( |d_1\rangle \) and that going through slit 2 leaves the detector in a state \( |d_2\rangle \). If this detector is capable of detecting which path the quanton went through, according to von Neumann’s recipe of a quantum measurement, the states of the detector should get correlated with the two paths of the quanton. The state of the quanton and the path-detector combined, will necessarily be entangled, and will be of a form

\[ \Psi(x) = \frac{1}{\sqrt{2}}(\psi_1(x)|d_1\rangle + \psi_2(x)|d_2\rangle) \]
Measuring an observable which has two eigenstates \(|d_1\rangle, |d_2\rangle\), with different eigenvalues, will reveal which slit the quanton went through. A measurement of that observable yielding \(|d_1\rangle\) will lead to a definite conclusion that the quanton passed through slit 1, and likewise for \(|d_2\rangle\). In this case the probability of finding the particle at a position \(x\), is given by

\[|\Psi(x)|^2 = \frac{1}{2}(|\psi_1(x)|^2 + |\psi_2(x)|^2)\] (5)

The two cross-terms which would have given interference, are killed by the orthogonality of \(|d_1\rangle\) and \(|d_2\rangle\). An important point to be noted here is that the actual path measurement is not even necessary here - mere existence of which-way information, or mere possibility of a path measurement, is enough to destroy interference.

### III. PARTIAL WHICH-PATH INFORMATION

#### A. Unambiguous quantum state discrimination

Suppose now that the path-detector has states \(|d_1\rangle\) and \(|d_2\rangle\) which are not orthogonal. In such a situation, no observable exists for which \(|d_1\rangle\) and \(|d_2\rangle\) are eigenstates with different eigenvalues. Hence it is not possible to distinguish between the two states perfectly. In the following we will describe a strategy which is considered to be the best to distinguish between two non-orthogonal states. It goes by the name of unambiguous quantum state discrimination (UQSD).\(^{12-10}\)

The state of the quanton plus the path-detector is still given by (3), but now \(|\langle d_1 |d_2 \rangle| \neq 0\). Let there be a two-state ancilla which interacts with the path-detector. This interaction is characterized by the following properties,

\[
U_a|d_1\rangle|a_0\rangle = \alpha|p_1\rangle|a_1\rangle + \beta|q_1\rangle|a_2\rangle
\]

\[
U_a|d_2\rangle|a_0\rangle = \gamma|p_2\rangle|a_1\rangle + \delta|q_2\rangle|a_2\rangle
\] (6)

where the path-detector states \(|p_1|p_2\rangle = 0\), and the ancilla states \(|a_1|a_2\rangle = 0\). The other path-detector states \(|q_1\rangle, |q_2\rangle\) are not orthogonal, and may even be identical. It can be shown that such an interaction always exists.

Now, if in the measurement of the ancilla, one gets the state \(|a_1\rangle\), the corresponding path-detector states will be \(|p_1\rangle \text{ or } |p_2\rangle\), which are orthogonal. Measuring a suitable observable of the path-detector will unambiguously tell us whether the state is \(|p_1\rangle\) or \(|p_2\rangle\), and consequently, whether the original state was \(|d_1\rangle\) or \(|d_2\rangle\). However, if the ancilla measurement yields \(|a_2\rangle\), the corresponding path-detector states \(|q_1\rangle\) and \(|q_2\rangle\) are not orthogonal, and one cannot distinguish between the two. In this case the process of distinguishing between \(|d_1\rangle\) and \(|d_2\rangle\) fails. If the states \(|d_1\rangle\) and \(|d_2\rangle\) occur with probability 1/2 each, the probability of failure to distinguish is just \((|\beta|^2 + |\delta|^2)/2\).

Hence the probability of successfully unambiguously distinguishing between \(|d_1\rangle\) and \(|d_2\rangle\) is thus

\[P = 1 - \frac{1}{2}(|\beta|^2 + |\delta|^2)\] (7)

Note that

\[|\langle d_1 |d_2 \rangle| \leq \sqrt{|\beta|^2 + |\delta|^2}\] (8)

using which one can conclude that minimum value of \(|\beta|^2 + |\delta|^2\) occurs when \(|\beta|^2 = |\delta|^2 = |\langle d_1 |d_2 \rangle|\). So we find that the probability of unambiguously distinguishing between \(|d_1\rangle\) and \(|d_2\rangle\) is

\[P \leq 1 - |\langle d_1 |d_2 \rangle|\] (9)

The equality in the above relation will be achieved if \(|q_1\rangle = |q_2\rangle\), rather up to an ignorable phase factor.

#### B. Distinguishing between the quanton paths

Coming back to our problem of distinguishing between the two paths of the quanton, let us start from the state (3), and let the ancilla interact with the path-detector. The starting state can be written as

\[\Psi_i(x) = \frac{1}{\sqrt{2}}(|\psi_1(x)|d_1\rangle + |\psi_2(x)|d_2\rangle)|a_0\rangle\] (10)

The interaction operator, for the ancilla and the path-detector, acts on the full entangled state, and the result is

\[\Psi_f(x) = U_a \Psi_i(x)\]

\[= \frac{1}{\sqrt{2}} U_a(|\psi_1(x)|d_1\rangle + |\psi_2(x)|d_2\rangle)|a_0\rangle\]

\[= \frac{1}{\sqrt{2}} |\psi_1(x)(\alpha|p_1\rangle|a_1\rangle + \beta|q_1\rangle|a_2\rangle)\]

\[+ \frac{1}{\sqrt{2}} |\psi_2(x)(\gamma|p_2\rangle|a_1\rangle + \delta|q_2\rangle|a_2\rangle)\] (11)

Let us concentrate on the case when the probability of unambiguously distinguishing between \(|d_1\rangle\) and \(|d_1\rangle\) is maximum, i.e., \(|\beta|^2 = |\delta|^2 = |\langle d_1 |d_2 \rangle|\) and \(|q_1\rangle = |q_2\rangle\). In such a situation the entangled state assumes the form

\[\Psi_f(x) = \frac{1}{\sqrt{2}} (\alpha|\psi_1(x)|p_1\rangle + \gamma|\psi_2(x)|p_2\rangle)|a_1\rangle + \sqrt{|\langle d_1 |d_2 \rangle|^2}|\frac{1}{\sqrt{2}} (\psi_1(x) + \psi_2(x))|q_1\rangle|a_2\rangle\] (12)

The quantons hitting the screen can be divided into two sub-ensembles according the ancilla states \(|a_1\rangle\) and \(|a_2\rangle\). Quantons correlated with the ancilla state \(|a_1\rangle\) are in the state

\[|a_1\rangle \Psi_f(x) = \frac{1}{\sqrt{2}} (\alpha|\psi_1(x)|p_1\rangle + \gamma|\psi_2(x)|p_2\rangle).\] (13)
In this state the quanton path amplitudes are correlated with orthogonal states of the path-detector. Hence these quantons will not show any interference, as can be checked by evaluating \(|\langle a_1|\Psi_f(x)\rangle|^2\). However, for each of these quantons, measuring an observable whose non-degenerate eigenstates are \(|p_1\rangle, |p_2\rangle\), unambiguously tells us which slit the quanton passed through.

Quantons correlated with the ancilla state \(|a_2\rangle\) are in the state

\[
\langle a_2|\Psi_f(x) = \frac{1}{\sqrt{2}} (\psi_1(x) + \psi_2(x))|q_1\rangle
\]

(14)

The probability distribution of these quantons on the screen is given by

\[
|\langle a_2|\Psi_f(x)\rangle|^2 = \frac{1}{2} |\psi_1(x) + \psi_2(x)|^2.
\]

(15)

These quantons will show full interference.

C. Quantitative wave-particle duality

From (12) one can see that the fraction of quantons which contribute to interference, is \(|\langle d_1|d_2\rangle\). But the fraction of quantons giving rise to interference should precisely be the visibility of the interference pattern. So the interference visibility, denoted by \(V\) is given by

\[
V = |\langle d_1|d_2\rangle|.
\]

(16)

We can define path-distinguishability \(D_Q\) as the fraction of quantons for whom one can unambiguously tell which slit they passed through, in the best case. For a single quanton, path-distinguishability \(D_Q\) is defined as the maximum probability with which one can unambiguously tell which slit it passed through. From (12), that fraction is just \(1 - |\langle d_1|d_2\rangle|\). Hence path-distinguishability is given by

\[
D_Q = 1 - |\langle d_1|d_2\rangle|.
\]

(17)

From (16) and (17) one can write

\[
D_Q + V = 1.
\]

(18)

The above result is a direct consequence of the fact that the fraction of quantons which give rise to interference is \(|\langle d_1|d_2\rangle|\), and the fraction of quantons for which one can tell for sure which slit they came through, is \(1 - |\langle d_1|d_2\rangle|\).

If real experimental factors are taken into account both visibility and path-distinguishability will have reduced values. Hence we can write the following inequality

\[
D_Q + V \leq 1.
\]

(19)

This is a duality relation which quantifies wave-particle duality, or complementarity, in a two-slit interference experiment.

In terms of the path-detector states \(|d_1\rangle\) and \(|d_2\rangle\), the distinguishability introduced by Englert has the form

\[
D = \sqrt{1 - |\langle d_1|d_2\rangle|^2}.
\]

(20)

Englert’s distinguishability \(D\) can be related to \(D_Q\) by the relation

\[
D_Q = 1 - \sqrt{1 - D^2}
\]

(21)

Substituting the above in (19) gives

\[
V \leq \sqrt{1 - D^2}.
\]

(22)

Squaring both side, one get

\[
V^2 + D^2 \leq 1,
\]

(23)

which is identical to (1). So, the apparently different looking duality relation (19) is, in essence, the same as well-known duality relation (1).

IV. DISCUSSION

We have discussed a thought modification of a two-slit interference experiment, where the which-path information of quantons is extracted using UQSD. The analysis shows that all the quantons passing through the double-slit can be split in two sub-ensembles. For quantons falling in the first sub-ensemble, one can unambiguously tell for each quanton which slit it passed through. These quantons do not contribute to interference.

For the quantons falling in the second sub-ensemble, one cannot tell which slit each of them passed through, but they all contribute to interference. Interference visibility intuitively should be just the fraction of quantons contributing fully to interference.

Using just the above arguments we have derived a bound on the sum of the path-distinguishability and interference visibility. The derived duality relation is completely equivalent to the well-known duality relation. UQSD can also be done for more than two states. Using UQSD, the duality relation has also been extended to interference experiment involving more than 2 slits.

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1 C. Jönsson, “Electron diffraction at multiple slits,” Am. J.
Phys. 42, 4 (1974).

2 O. Nairz M. Arndt, A. Zeilinger, “Quantum interference experiments with large molecules,” Phys. Rev. Lett. 71, 319 (2003).

N. Bohr, “The quantum postulate and the recent development of atomic theory,” Nature (London) 121, 580-591 (1928).

4 T. Qureshi, R. Vathsan, “Einstein’s Recolling Slit Experiment, Complementarity and Uncertainty,” Quanta 2, 58-65 (2013).

5 W. K. Wootters and W. H. Zurek, “Complementarity in the double-slit experiment: Quantum nonseparability and a quantitative statement of Bohr’s principle”, Phys. Rev. D 19, 473 (1979).

6 D.M. Greenberger, A. Yasin, “Simultaneous wave and particle knowledge in a neutron interferometer”, Phys. Lett. A 128, 391 (1988).

7 G. Jaeger, A. Shimony, L. Vaidman, “Two interferometric complementarities,” Phys. Rev. A 51, 54 (1995).

8 B-G. Englert, “Fringe visibility and which-way information: an inequality”, Phys. Rev. Lett. 77, 2154 (1996).

9 M. O. Scully, B.-G. Englert and H. Walther, “Quantum optical tests of complementarity,” Nature (London) 351, 111 (1991).

10 S. Durr, T. Nonn, G. Rempe, “Origin of quantum-mechanical complementarity probed by a ‘which-way’ experiment in an atom interferometer,” Nature 395, 33 (1998).

11 J. von Neumann, Mathematical Foundations of Quantum Mechanics (Princeton University Press, 1955).

12 I.D. Ivanovic, “How to differentiate between non-orthogonal states”, Phys. Lett. A 123, 257 (1987).

13 D. Dieks, “Overlap and distinguishability of quantum states,” Phys. Lett. A 126, 303 (1988).

14 A. Peres, “How to differentiate between non-orthogonal states,” Phys. Lett. A 128, 19 (1988).

15 G. Jaeger, A. Shimony, “Optimal distinction between two non-orthogonal quantum states,” Phys. Lett. A 197, 83 (1995).

16 J.A. Bergou, U. Herzog, M. Hillery, “Discrimination of quantum states,” Lect. Notes Phys. 649, 417-465 (2004).

17 M.A. Siddiqui, T. Qureshi, “Three-Slit Interference: A duality relation”, [arXiv:1405.3721] [quant-ph].