UPPER BOUNDS FOR THE NUMBER OF LIMIT CYCLES OF SOME PLANAR POLYNOMIAL DIFFERENTIAL SYSTEMS

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ABSTRACT. We give an effective method for controlling the maximum number of limit cycles of some planar polynomial systems. It is based on a suitable choice of a Dulac function and the application of the well-known Bendixson-Dulac Criterion for multiple connected regions. The key point is a new approach to control the sign of the functions involved in the criterion. The method is applied to several examples.

1. Main result

One of the few general methods that allows to give upper bounds for the number of limit cycles of planar differential systems
\[ \dot{x} = P(x, y), \quad \dot{y} = Q(x, y) \]
is the use of Dulac functions in multiple connected regions, see [2, 3, 4, 7, 8, 9].
Recall that the primer idea is that when the function \( \text{div}(P, Q) \) does not vanish on a simply connected region \( U \subset \mathbb{R}^2 \), then the above differential system has no periodic orbit totally contained in \( U \). We state the general Bendixson-Dulac Criterion in next section, see Theorem 2.1. The main difficulty for practical uses of this result is that it is needed to find a (Dulac) function \( g \) such that \( \text{div}(gP, gQ) \) does not vanish on a suitable set. This paper gives a quite general result for polynomial differential systems, see Theorem A. Its proof is based on a “good” choice of a Dulac function. As we will see in Section 3 this result provides a constructive way for giving upper and lower bounds for the number of limit cycles of a large class of planar polynomial systems.

Given a polynomial \( p(s) \in \mathbb{R}[s] \), we will say that the couple \((k, w(r)) \in \mathbb{R}^+ \times \mathbb{R}[r] \) is a Dulac pair of \( p(s) \) if
\[ p_{k, w}(r) := rp(r^2)w'(r) - 2k(p(r^2) + r^2p'(r^2))w(r) < 0 \quad \text{for all} \quad r > 0. \]
As we will see in Lemma 2.7 and in the proof of Proposition 2.2, the above inequality implies that the function \( |w(r)|^{-1/k} \) is a Dulac function, in any of the connected components of \( \mathbb{R}^2 \setminus \{w(r) = 0\} \), for the systems that in polar coordinates, \( r \) and \( \theta \), writes as \( \dot{r} = rp(r^2), \ \dot{\theta} = \hat{p}(r^2) \). Here \( \hat{p} \) is any arbitrary polynomial. We remark that there is no need to introduce this Dulac pair to study the phase portrait of this system. For instance their limit cycles are given by the positive zeros of \( p \).

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that are not zeros of \( \hat{p} \). Nevertheless, as we will see in next theorem, Dulac pairs are useful to study more general systems.

**Theorem A.** Consider the polynomial differential system
\[
\begin{align*}
\dot{x} &= P(x, y), \\
\dot{y} &= Q(x, y),
\end{align*}
\]
where \( P \) and \( Q \) are real polynomials of degree \( n \), satisfying that \( P(0, 0) = Q(0, 0) = 0 \). In polar coordinates it writes as
\[
\begin{align*}
\dot{r} &= R(r, \theta) := P(r \cos \theta, r \sin \theta) \cos \theta + Q(r \cos \theta, r \sin \theta) \sin \theta, \\
\dot{\theta} &= \Theta(r, \theta) := \frac{1}{r} \left( Q(r \cos \theta, r \sin \theta) \cos \theta - P(r \cos \theta, r \sin \theta) \sin \theta \right).
\end{align*}
\]
Define the polynomial
\[
p(r^2) := \frac{1}{2\pi r} \int_{0}^{2\pi} R(r, \theta) \, d\theta
\]
and assume that \( p(s) \) has the Dulac pair \( (k, w(r)) \), where \( w \) is a polynomial of degree \( d \). For these \( k \) and \( w \) consider the function
\[
M(r, \theta, k, w) := R(r, \theta)w'(r) - k \left( \frac{\partial R(r, \theta)}{\partial r} + \frac{\partial \Theta(r, \theta)}{\partial \theta} + \frac{R(r, \theta)}{r} \right) w(r)
\]
and, for any \( i \geq 1 \), let \( m_i(k, w) \) be such that \( \max_{\theta \in [0, 2\pi]} M_i(\theta, k, w) \leq m_i(k, w) \).

Let \( m^+ \) be the number of non-negative roots of \( w \). Then, if the polynomial
\[
\Phi_{k, w}(r) := \sum_{i=1}^{n+d-1} m_i(k, w)r^i
\]
is negative for all \( r \in (0, \infty) \), system (1.1) has at most \( m^+ \) limit cycles and all of them are hyperbolic.

Next remarks collect some comments on the above result.

**Remark 1.1.** (i) The above theorem gives a constructive way of testing when a Dulac function of a system of the form \( \dot{r} = rp(r^2), \dot{\theta} = \hat{p}(r^2) \) is also suitable for system (1.1).

(ii) In Proposition 2.5 we will prove that there are many polynomials \( p \) for which a Dulac pair exists. Moreover, in Section 3 we will show many systems of the form (1.1) for which the above theorem can be applied.

(iii) Remark 2.8 proves that the upper bound given in the theorem can not be improved.

**Remark 1.2.** (i) It is not difficult to see that any differential system fulfilling the hypotheses of Theorem A has at least \( m^+ - 2 - C \) limit cycles, where \( C \geq 0 \) can be computed as follows: Let \( 0 \leq r_1 < \cdots < r_{m^+} \) be the ordered non-negative roots of \( w(r) \). Define the rings \( R_i = \{(x, y) : r_i < \sqrt{x^2 + y^2} < r_{i+1} \} \), \( i = 1, 2, \ldots, m^+ - 1 \). Then \( C \) is the number of rings among \( R_1, R_2, \ldots, R_{m^+ - 1} \) containing critical points of system (1.1).
(ii) From the proof of the theorem it follows that there is at most one limit cycle in each of the sets $R_1, R_2, \ldots, R_{m^+ - 1}, R_{m^+}$, where $R_{m^+}$ is the unbounded ring $R_{m^+} = \{(x, y) : r_{m^+} < \sqrt{x^2 + y^2}\}$ and that (when $r_1 > 0$) there is no limit cycle in the disc $\{(x, y) : \sqrt{x^2 + y^2} < r_1\}$. Moreover in the rings $R_2, R_3, \ldots, R_{m^+ - 1}$ not containing critical points of the system there exists always a limit cycle and its stability is given by the sign of $w$ on them. The limit cycle on $R_1$ only can exist when $r_1 > 0$ or when $r_1 = 0$ and the sign of $w$ on this set does not coincide with the stability of the origin.

Let us see how the theorem works in a concrete example. We will prove that

$$
\begin{align*}
\dot{x} &= -y + 4x - \frac{49}{10} x^3 - \frac{26}{5} xy^2 + \frac{1}{5} x^2 y^2 + x^5 + 2x^3 y^2 + xy^4 = P(x, y), \\
\dot{y} &= x + 4y - \frac{23}{5} x^2 y - 5y^3 - \frac{1}{5} xy^3 - \frac{2}{15} y^4 + x^4 y + 2x^2 y^3 + y^5 = Q(x, y)
\end{align*}
$$

has exactly two limit cycles. Examples with parameters will be studied in Section 3. Following the notation of the theorem we get that $p(s) = 4 - 79s/16 + s^2$. Moreover, by using Proposition 2.5 we take $w(r) = r^2 p'(r^2) = r^2(2r^2 - 79/16)$. Finally we choose $k = 4/5$. We have

$$p_{k,w}(r) = -\frac{79}{10} r^2 - \frac{1287}{128} r^4 + \frac{237}{40} r^6 - \frac{8}{5} r^8.$$

By using Sturm's rule it is easy to prove that $p_{k,w}(r) < 0$ for all $r \neq 0$ and so ($k, w(r)$) is a Dulac pair for $p$. Straightforward computations give that

$$
\begin{align*}
M(r, \theta, k, w) &= -\frac{79}{10} r^2 + \frac{1}{2560} \left( -25740 + \frac{16432}{5} \cos(2\theta) + 316 \cos(4\theta) \right) r^4 \\
&\quad - \frac{79}{4800} \left( 12 \cos(\theta) + 3 \cos(3\theta) - 15 \cos(5\theta) + 46 \sin(\theta) - 7 \sin(3\theta) - 5 \sin(5\theta) \right) r^5 \\
&\quad + \frac{1}{80} \left( 474 - \frac{128}{5} \cos(2\theta) - 8 \cos(4\theta) \right) r^6 \\
&\quad + \frac{1}{75} \left( 6 \cos(\theta) + 9 \cos(3\theta) - 15 \cos(5\theta) - 2 \sin(\theta) + 9 \sin(3\theta) - 5 \sin(5\theta) \right) r^7 - \frac{8}{5} r^8.
\end{align*}
$$

By using rough bounds like

$$-46 \leq 6 \cos(\theta) + 9 \cos(3\theta) - 15 \cos(5\theta) - 2 \sin(\theta) + 9 \sin(3\theta) - 5 \sin(5\theta) \leq 46$$

we get that

$$M(r, \theta, k, w) \leq \Phi_{k,w}(r) = -\frac{79}{10} r^2 - \frac{3459}{400} r^4 + \frac{869}{600} r^5 + \frac{1269}{200} r^6 + \frac{46}{75} r^7 - \frac{8}{5} r^8$$

and it can be proved by using Sturm's rule that it is negative for all $r > 0$. Therefore, since $w$ has two non-negative roots, $m^+ = 2$ and the system has at most two (hyperbolic) limit cycles.

It is clear that the origin is an unstable focus. By taking the resultant of $P$ and $Q$, for instance with respect to $y$, and applying Sturm's rule to this new
polynomial we prove that the origin is the unique critical point of the system. By studying the flow of the system on the circles \( \{ x^2 + y^2 = R^2 \} \) for \( R \) big enough, and on \( \{ x^2 + y^2 = r_0^2 \} \), where \( r_0 \) is the positive root of \( w \), and using the above information we deduce that the two limit cycles actually exist. Indeed one of them is in \( \mathcal{D} = \{ x^2 + y^2 < r_0^2 \} \) and is stable and the other surrounds \( \mathcal{D} \) and is unstable.

In fact, we have presented the study of system (1.3) in this introduction because it shows at the same time the power and the limitations of the method. By one hand, for a concrete system which has no a priori structure, we can give the exact number of limit cycles. On the other hand, in all the cases that we have studied, for polynomials systems of degree \( 2n \) or \( 2n + 1 \) it only has been applicable for systems having at most \( n \) limit cycles and when all these limit cycles are hyperbolic and nested.

Due to the extreme difficulty of the second part of the Hilbert’s sixteenth problem it is never easy to give explicit and realistic upper bounds for the number of limit cycles of a planar polynomial system. Moreover most results in the literature can only be applied to control the limit cycles of particular types of differential systems, namely Liénard systems, quadratic systems, cubic systems, systems with homogeneous nonlinearities, etc, giving usually upper bounds of 0, 1 or 2 limit cycles, see for instance [1, 10, 11]. The criterion proved in Theorem 5 is not subject to these restrictions.

The paper is organized as follows: Section 2 contains some preliminary results and the proof of Theorem A. In Section 3, we will apply this result to different families of differential systems providing in all of them explicit upper bounds for the number of limit cycles. In fact, we present two examples with limit cycles surrounding a unique critical point (see Examples 1 and 2), two situations with a limit cycle surrounding several critical points (see Examples 3 and 4) and finally a family of examples of differential systems showing that, in the set of systems of the form (1.1), there are many open subsets of systems satisfying all the hypotheses of the theorem.

2. Preliminary results and proof of Theorem A

Let \( \mathcal{U} \subset \mathbb{R}^2 \) be an open set with smooth boundary and such that its fundamental group, \( \pi_1(\mathcal{U}) \), is \( \mathbb{Z} \times \cdots \times \mathbb{Z} \) or in other words having \( \ell \) gaps. For short we will say that \( \mathcal{U} \) is an \( \ell \)-punctured open set and we will denote by \( \ell(\mathcal{U}) \) its number of gaps. Notice that with this notation, simply connected sets are 0-punctured sets and multiple connected sets are \( \ell \)-punctured sets with \( \ell \geq 1 \).

The following result is a well known extension of the Bendixson-Dulac Criterion to \( \ell \)-punctured sets. For a proof, see any of the papers [6, 8, 9].

**Theorem 2.1** (Extended Bendixson-Dulac Criterion). **Consider the \( C^1 \)-differential system**

\[
\dot{x} = P(x, y), \quad \dot{y} = Q(x, y) \tag{2.1}
\]
and set $X = (P, Q)$. Let $\mathcal{U}$ be an open $\ell$-punctured subset of $\mathbb{R}^2$ with smooth boundary. Let $g: \mathcal{U} \to \mathbb{R}$ be a $C^1$-function such that

$$M := \text{div}(gX) = \frac{\partial g}{\partial x} P + \frac{\partial g}{\partial y} Q + (\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}) g = \langle \nabla g, X \rangle + g \text{div}(X)$$

does not change sign in $U$ and vanishes only on a null measure Lebesgue set and such that $\{M = 0\} \cap \{g = 0\}$ does not contain periodic orbits of (2.1). Then the maximum number of periodic orbits of (2.1) contained in $U$ is $\ell$. Furthermore each one is a hyperbolic limit cycle that does not cut $\{g = 0\}$ and its stability is given by the sign of $gM$ over it.

We will prove and use the following corollary of the above theorem.

**Proposition 2.2.** Consider the $C^1$-differential system

$$\dot{x} = P(x, y), \quad \dot{y} = Q(x, y)$$

and set $X = (P, Q)$. Assume that there exist a positive real number $k$ and a polynomial $f(x, y)$ such that

$$M_k := \frac{\partial f}{\partial x} P + \frac{\partial f}{\partial y} Q - k(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}) f = \langle \nabla f, X \rangle - kf \text{div}(X)$$

does not vanish in an open region with regular boundary $\mathcal{W} \subset \mathbb{R}^2$. Let $\mathcal{U}_1, \mathcal{U}_2, \ldots, \mathcal{U}_m$, be the connected components of $\mathcal{W} \setminus \{f = 0\}$. Then

(i) The periodic orbits of system (2.2) contained in $\mathcal{W}$ never cut the set $\{f = 0\}$.

(ii) The maximum number of limit cycles of system (2.2) contained in each $\mathcal{U}_j$, $j = 1, 2, \ldots, m$, is $\ell(\mathcal{U}_j)$, and all of them are hyperbolic. Moreover their stability is given by the sign of $-f M_k$ on each region.

(iii) The maximum number of limit cycles of system (2.2) in $\mathcal{W}$ is $\ell(\mathcal{U}_1) + \ell(\mathcal{U}_2) + \cdots + \ell(\mathcal{U}_m)$ and all them are hyperbolic.

**Proof.** Since $M_k$ does not vanish on $\mathcal{W}$ we know that $\langle \nabla f, X \rangle|_{\{f = 0\} \cap \mathcal{W}}$ does not vanish. Therefore the periodic orbits of (2.2) which are totally contained in $\mathcal{W}$ never cut the set $\{f = 0\}$ and (i) follows. To prove items (ii) and (iii) it suffices to apply Theorem 2.1 to each one of the connected components $\mathcal{W} \setminus \{f = 0\}$. Note that, on each of them, the function $g = |f|^{-1/k}$ is smooth and moreover

$$\text{div}(gX) = \frac{\text{sign}(f)}{k} |f|^{-1/k-1} \left[ \langle \nabla f, X \rangle - kf \text{div}(X) \right] = \frac{\text{sign}(f)}{k} |f|^{-1/k-1} M_k.$$

Therefore the upper bound for the number of limit cycles follows.

Finally we prove the hyperbolicity of all the limit cycles. Let $\gamma = \{(x(t), y(t)) : t \in [0, T]\}$ be one of them. We must prove that $\int_0^T \text{div} X(x(t), y(t)) \, dt \neq 0$, see for instance [5, Thm. 1.23]. Since $\gamma$ does not intersect the set $\{f = 0\}$ we have that over $\gamma$

$$\text{div} X = -\frac{M_k}{kf} + \frac{\langle \nabla f, X \rangle}{kf}.$$

Hence

$$\int_0^T \text{div} X(x(t), y(t)) \, dt = -\int_0^T \frac{M_k(x(t), y(t))}{kf(x(t), y(t))} \, dt \neq 0$$
and its stability is given by the sign of \(-fM_k\) on the region where \(\gamma\) lies. Therefore the result follows. 

To apply the above result, it will be useful to get the expression of the function \(M_k\) given in the Proposition 2.2 in terms of the components of the differential system (2.2), written in polar coordinates.

**Lemma 2.3.** Let \(\dot{r} = R(r, \theta), \dot{\theta} = \Theta(r, \theta)\) be the expression of system (2.2) in polar coordinates. Then the function (2.3) writes as

\[
M_k = \frac{\partial f}{\partial x} P + \frac{\partial f}{\partial y} Q - k \left( \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right) f
\]

\[
= \frac{\partial f}{\partial r} R + \frac{\partial f}{\partial \theta} \Theta - k \left( \frac{\partial R}{\partial r} + \frac{\partial \Theta}{\partial \theta} + \frac{R}{r} \right) f.
\]

**Lemma 2.4.** Let \(p(s)\) be a real polynomial having all its roots real and simple.

(i) For all \(s \in \mathbb{R}\)

\[
p(s)p''(s) - (p'(s))^2 < 0.
\]

(ii) For each \(k \in \mathbb{R}\) define the new polynomial

\[
q_k(r) := 2r^4 \left( p(r^2) p''(r^2) - k(p'(r^2))^2 \right) + 2(1-k)r^2 p(r^2) p'(r^2).
\]

Then there exists \(k\), with \(|k-1|\) small enough, such that \(q_k(r) < 0\) for all \(r \neq 0\).

**Proof.** It is not restrictive to write \(p(s) = \prod_{j=1}^l (s-s_j)\) with \(s_i \neq s_j\) for \(i \neq j\). From the equalities

\[
p(s)p''(s) - (p'(s))^2 = p^2(s) \left( \frac{p'(s)}{p(s)} \right)' = p^2(s) \left( \sum_{i=1}^l \frac{1}{(s-s_j)} \right)'
\]

\[
= p^2(s) \left( \sum_{i=1}^l \frac{1}{s-s_i} \right)' = -p^2(s) \left( \sum_{i=1}^l \frac{1}{(s-s_i)^2} \right) < 0
\]

item (i) follows.

To prove (ii) notice that by item (i)

\[
q_k(r) = q_k(r) + (1-k)q(r) \text{ for some polynomial } q \text{ of degree } 4l \text{ and}
\]

\[
q_k(r) = 2c_0c_1 (1-k)r^2 + \sum_{i=1}^l \left( c_i^2 + 4c_0c_2 - 2k(c_i^2 + c_0c_2) \right) r^4
\]

where \(p(s) = c_0+c_1s+c_2s^2+\cdots+s^l\). Note that \(p(0)p''(0) - (p'(0))^2 = 2c_0c_2 - c_1^2 < 0\) and thus, for \(|k-1|\) small enough, \((1-k)l^2 - kl\) and \(c_i^2 + 4c_0c_2 - 2k(c_i^2 + c_0c_2)\) are both negative. Hence when \(c_0c_1 = 0\), taking \(|k-1|\) small enough, we can always ensure that the sign of \(q_k(r)\) is negative in \(\mathbb{R} \setminus \{0\}\). When \(c_0c_1 \neq 0\), in addition we have to take \(k\) such that \(c_0c_1(1-k) < 0\). 

\(\square\)
Proposition 2.5. Let \( p(s) \) be a real polynomial. Then

(i) If \( p(s) \) has all its roots real and simple by taking \( w(r) = r^2 p'(r^2) \) and some \( k \) near 1 then \((k, w(r))\) is a Dulac pair of \( p(s) \).

(ii) If \( p(s) \) has some real multiple positive root then it has no Dulac pair.

(iii) There are many polynomials \( p(s) \) with complex roots also having Dulac pairs.

Proof. (i) Note that

\[
p_{k,w}(r) = rp(r^2) \left( r^2 p'(r^2) \right)' - 2k \left( p(r^2) + r^2 p'(r^2) \right) r^2 p'(r^2) = q_k(r),
\]

where \( q_k \) is given in Lemma 2.4. By using this lemma the result follows.

(ii) Let \( s^* > 0 \) be one of these roots. It is easy to see that for any couple \( k \) and \( w \), the polynomial \( p_{k,w}(r) \) also vanishes at \( \sqrt{s^*} \). Hence the result follows.

(iii) We construct a class of polynomials for which the result holds. Let \( p(s) \) be a polynomial such that \( p(s) = p_1(s)p_2(s) \) where \( p_1(r^2) + r^2 p_1'(r^2) > 0 \) for all \( r \in \mathbb{R} \) and \( p_2(s) = \prod_{i=1}^n(s - r_i^2) \), being \( r_i^2 \) different positive numbers. Consider \( w(r) = p_2(r^2) \) and \( k = 1 \). Then

\[
p_{1,w}(r) = -2p_2'(r^2) \left( p_1(r^2) + r^2 p_1'(r^2) \right) \leq 0 \quad \text{for all } r \in \mathbb{R}.
\]

Take now \( w(r) = p_2(r^2) + \varepsilon^2 p_2'(r^2) \). Then

\[
p_{1,w_\varepsilon}(r) = p_{1,w}(r) + \varepsilon^2 W(r)
\]

for some polynomial \( W \) of the same degree that \( p_{1,w} \). Moreover

\[
W(r_i) = -2r_i^2 p_1(r_i^2)(p_2'(r_i^2))^2 < 0
\]

for all \( i = 1, \ldots, j \). Hence for \( |\varepsilon| \) small enough the polynomial \( p_{1,w_\varepsilon}(r) \) is negative for all \( r \in \mathbb{R} \), as we wanted to prove. \( \square \)

Remark 2.6. Notice that, given a polynomial \( p \) under suitable hypotheses, item (i) of Proposition 2.5 provides a constructive way of finding Dulac pairs. On the other hand item (iii) provides a theoretical way to see that the same situation holds for other polynomials. Nevertheless, for a given polynomial \( p \), even with complex roots, it is not difficult to find a Dulac pair. We give a couple of examples and some intuition of how we get them.

(a) For \( p(s) = (s-2)(s-4)(s^2+4)(s+3) \) take \( w(r) = (r^2-1)(r^2-3) \) and \( k \) any of the values 1/2, 1 or 2.

(b) For \( p(s) = -35 - 36s + 49s^2/2 - 14s^3/3 + s^4/4 \), which has a unique positive root \( s_0 \approx 11.12 \) take \( k = 1 \) and \( w(r) = r^2 - \alpha \) for any \( \alpha \) for instance in the interval \([5.5, 11] \).

By Remark 1.1 the existence of \( k > 0 \) and \( w \) such that \( p_{k,w} < 0 \) for all \( r > 0 \) implies that we can apply Proposition 2.2 with \( f = w(r) \). Therefore each one of the hyperbolic limit cycles of system \( \dot{r} = rp(r^2), \dot{\theta} = \dot{p}(r^2) \), which are given by the positive simple zeros of \( p \), has to be contained in one of the 1-punctured regions of \( \mathbb{R}^2 \setminus \{ w = 0 \} \). For instance in case (b), there is only one limit cycle \( r^2 \approx s_0 \) and we can try with a function \( w(r) = r^2 - \alpha \) with \( \alpha \) smaller than this value. We construct the function \( w \) of item (a) in a similar way.
Next lemma will be useful to get systems for which the hypotheses of Theorem A hold and to prove that the upper bound given by the theorem is optimal.

**Lemma 2.7.** Consider system

\[
\begin{align*}
\dot{x} &= xu(x^2 + y^2) - yv(x^2 + y^2), \\
\dot{y} &= xv(x^2 + y^2) + yu(x^2 + y^2),
\end{align*}
\]

where \( u \) and \( v \) are arbitrary real polynomials. If \( u \) has all their roots real and simple then, taking \( w(r) = r^2u'(r^2) \), the function \( \Phi_{1,w}(r) \) introduced in Theorem A is negative for all \( r \in (0, \infty) \).

**Proof.** Writing the system in polar coordinates we obtain

\[
\begin{align*}
\dot{r} &= R(r, \theta) = ru(r^2), \\
\dot{\theta} &= \Theta(r, \theta) = v(r^2).
\end{align*}
\]

Clearly, from the above expression we get \( p = u \), where \( p \) is the polynomial introduced in Theorem A. Thus, taking \( k = 1 \) and \( w(r) = r^2u'(r^2) \), we obtain

\[
M(r, \theta, 1, w) = R(r, \theta) (r^2u'(r^2))' - \left( \frac{\partial R(r, \theta)}{\partial r} + \frac{\partial \Theta(r, \theta)}{\partial \theta} + \frac{R(r, \theta)}{r} \right) r^2u'(r^2)
\]

\[
= ru(r^2) (r^2u'(r^2))' - \left( (ru(r^2))' + u(r^2) \right) r^2u'(r^2)
\]

\[
= 2r^4 \left( u(r^2)u''(r^2) - (u'(r^2))^2 \right) = \Phi_{1,w}(r).
\]

By using Lemma 2.4(i) the result follows. \( \square \)

**Remark 2.8.** Notice that the above lemma implies that the upper bound given in Theorem A can not be improved. Consider in system (2.4) a polynomial \( u \) such that all their roots are real and has no common roots with \( v \). Then system (2.4) has as many limit cycles (indeed invariant circles) as number of positive roots, say \( m^* \). It is easy to take a polynomial \( u \) under the above hypotheses such that \( u' \) has exactly \( m^* - 1 \) positive roots. Hence the number of non-negative roots of \( w(r) = r^2u'(r^2) = r^2u'(r^2) \) is \( m^* = m^* \). By applying Theorem A we get an upper bound of \( m^* = m^* \) limit cycles, which is indeed the actual number of limit cycles of the system.

**Proof of Theorem A** We apply Proposition 2.2 to system (1.1) with \( f(r, \theta) = w(r) \) and the value \( k \) given in the statement of the Theorem. Then, by using Lemma 2.3 we get that the expression of \( M_k \) given in Proposition 2.2 is

\[
M_k = R(r, \theta)w'(r) - k \left( \frac{\partial R(r, \theta)}{\partial r} + \frac{\partial \Theta(r, \theta)}{\partial \theta} + \frac{R(r, \theta)}{r} \right) w(r) = M(r, \theta, k, w),
\]

where \( M(r, k, \theta, w) \) is the function given in Theorem A.

Moreover, by hypothesis, we have

\[
M_k = M(r, \theta, k, w) = \sum_{i=1}^{n+d-1} M_i(\theta, k, w)r^i \leq \sum_{i=1}^{n+d-1} m_i(k, w)r^i = \Phi_{k,w}(r) < 0
\]
for all $r \in (0, \infty)$. Therefore, by Proposition 2.2, the maximum number of limit cycles can be bounded above by studying the topology of the connected components of the set $W := \mathbb{R}^2 \setminus \{w(r) = 0\}$. Clearly it has $m^+$ connected components, all of them indeed 1-punctured when $w(0) = 0$ and it has $m^+ + 1$ connected components when $w(0) \neq 0$. Notice that in this later case one of them is simply connected and the other $m^+$ are 1-punctured sets. In any case, again by Proposition 2.2, there is no limit cycle in the simply connected component and there is at most one limit cycle in the 1-punctured components, which is hyperbolic when it exists. Moreover, since $M_k < 0$, its stability is given by the sign of $w$ on each component. As it is already said in Remark 1.2.(ii), it can be proved that the bounded 1-punctured components of $W$ not having critical points of system (1.1) contain effectively a limit cycle. This result holds because each one of these rings is either positively or negatively invariant by the flow of the system. □

3. Examples

3.1. Example 1. Consider the system

$$\begin{align*}
\dot{x} &= x(1 - (x^2 + y^2)) - y(1 + 2(x^2 + y^2)) + axy + bxy^2, \\
\dot{y} &= x(1 + 2(x^2 + y^2)) + y(1 - (x^2 + y^2)) + cy^2 + dx^3,
\end{align*}$$

(3.1)

which has at the origin an unstable focus. Let us see that when $b < 8$ and $a$, $c$ and $d$ are such that

$$\Psi_{a,b,c,d}(r) := -12 + (2|2a - c| + 10|c - a|) r + (2b - 16 + 12|b| + 15|d|) r^2 < 0$$

for all $r > 0$, then it has at most one limit cycle, which when exists is hyperbolic and stable.

We apply Theorem A. Then $p(s) = ((b - 8)s + 8)/8$. By using Proposition 2.5 we take $w(r) = r^2p'(r^2) = (b - 8)r^2/8$ and we choose $k = 7/10$. Then

$$M(r, \theta, k, w) = \frac{3(b - 8)}{40}r^2 + \frac{8 - b}{16} \left( \frac{2a - c}{5} \sin(\theta) + (c - a) \sin(3\theta) \right) r^3$$

$$+ \frac{b - 8}{32} \left( \frac{16 - 2b}{5} + b \left( \frac{7}{5} \cos(2\theta) - \cos(4\theta) \right) + d \left( 2 \sin(2\theta) + \sin(4\theta) \right) \right) r^4.$$

Analogously that in the example given in the introduction, we have

$$M(r, \theta, k, w) \leq \frac{8 - b}{160}r^2 \Psi_{a,b,c,d}(r) < 0$$

for all $r > 0$ and we are under the hypotheses of the theorem. Since $m^+ = 1$ the uniqueness of the limit cycle follows. It is easy to see that for $a, b, c$ and $d$ small enough the condition on $\Psi_{a,b,c,d}$ holds and the limit cycle exists.

Note that for this system we can give a simple and explicit condition on the parameters of the system under which we can prove that there is at most one limit cycle. For instance the condition holds for $a = c = 1$ and $-2b = 2d = 1$. For these values we have also checked numerically that the limit cycle actually exists.
3.2. Example 2. Consider the system
\[
\begin{align*}
\dot{x} &= x(1 - (x^2 + y^2))(2 - (x^2 + y^2)) - y + ax^2y + bx^2y^2, \\
\dot{y} &= x + y(1 - (x^2 + y^2))(2 - (x^2 + y^2)) + cxy^2. \\
\end{align*}
\tag{3.2}
\]
If \(a, b\) and \(c\) are such that
\[
\Psi_{a,b,c}(r) := -10 + \frac{9}{4}(|a| + |c|) + \frac{9}{4}|b|r + (12 + |a| + |c|)r^2 + |b|r^3 - 4r^4 < 0
\]
for all \(r > 0\), then system (3.2) has at most two (hyperbolic) limit cycles. Moreover, when they exist, one is included in the disc \(D := \{x^2 + y^2 \leq 3/2\}\) and is stable and the other one is outside the disc and is unstable.

The proof follows again by using Theorem A. We take \(p(s) = 2 - 3s + s^2\), and by Proposition 2.5, we consider \(w(r) = r^2p'(r^2) = r^2(-3 + 2r^2)\) and \(k = 1\). Then
\[
M(r, \theta, k, w) = \frac{1}{4}(-40 + (6\sin(2\theta) - 3\sin(4\theta)) + c(6\sin(2\theta) + 3\sin(4\theta)))r^4
\]
\[
+ \frac{3}{8}b(2\cos(\theta) - 3\cos(3\theta) + \cos(5\theta))r^5
\]
\[
+ (12 + a\sin(4\theta) - c\sin(4\theta))r^6 - \frac{b}{2}(-\cos(3\theta) + \cos(5\theta))r^7 - 4r^8.
\]
Hence, for the values of the parameters considered, we have
\[
M(r, \theta, k, w) \leq r^4\Psi_{a,b,c}(r) < 0
\]
for all \(r > 0\). Thus we can apply Theorem A with \(m^+ = 2\), proving the existence of at most two (hyperbolic) limit cycles.

For instance the condition on \(\Psi_{a,b,c}\) holds for \(a = 1/8\), \(b = 1/15\) and \(c = 1/20\). Moreover for these parameters it is not difficult to prove, by using resultants and the Sturm’s rule, that the origin is the unique critical point, which is unstable. Finally, by studying the flow on \(\{x^2 + y^2 = R^2\}\), for \(R\) big enough, and on \(\{x^2 + y^2 = 3/2\}\), we prove the existence of both limit cycles.

3.3. Example 3. Consider the system
\[
\begin{align*}
\dot{x} &= x(1 - (x^2 + y^2))(2 - (x^2 + y^2)) - y(1 - (x^2 + y^2)) + ax^4 + bx^2y^2, \\
\dot{y} &= x(1 - (x^2 + y^2)) + y(1 - (x^2 + y^2))(2 - (x^2 + y^2)). \\
\end{align*}
\tag{3.3}
\]
If \(a\) and \(b\) are such that
\[
\Psi_{a,b}(r) := -10 + \frac{27|a| + 9|b|}{4}r + 12r^2 + (2|a| + |b|)r^3 - 4r^4 < 0
\]
for all \(r > 0\), Theorem A will allow us to show that system (3.3) has at most two limit cycles. Moreover we will see that when there is no critical point outside the disc \(D := \{x^2 + y^2 \leq 3/2\}\) an unstable hyperbolic limit cycle always exists outside the disc \(D\) and it surrounds several critical points.
We take $p$, $w$ and $k$ as in the previous example. Then
\[
M(r, \theta, k, w) = -10r^4 + 12r^6 - 4r^8 \\
\quad + \frac{3}{8}(a(14\cos(\theta) + 3\cos(3\theta) - \cos(5\theta)) + b(2\cos(\theta) - 3\cos(3\theta) + \cos(5\theta))) \quad r^5 \\
\quad + \frac{1}{2}(a(-2\cos(\theta) + \cos(3\theta) + \cos(5\theta)) + b(\cos(3\theta) - \cos(5\theta))) \quad r^7.
\]
Hence, when the conditions on the parameters hold we have
\[
M(r, \theta, k, w) \leq r^4\Psi_{a,b}(r) < 0
\]
for all $r > 0$ and we can apply Theorem A. In this case since $m^+ = 2$ we know that system (3.3) has at most two limit cycles, and whenever they exist, one is inside the disc $D$ and is hyperbolic and stable and the other one is outside the disc, and is hyperbolic and unstable. Since system (3.3) has several critical points in the disc $D$, $(0,0)$ and $(0,\pm 1)$ among them, the unstable limit cycle, when it exists, surrounds these points.

An example of parameters for which $\Psi_{a,b}$ is negative is $a = 1/20$ and $b = 1/15$. Using the same tools that in the previous case we can prove that the system has exactly five critical points and all them are inside $D$. Hence by studying the flow on the boundary of $D$ and on $\{x^2 + y^2 = r^2\}$, for $R$ big enough, we prove the existence of an unstable limit cycle surrounding $D$.

In short, for these values of the parameters we have proved that system (3.3) has at most two limit cycles and the existence of a limit cycle, which surrounds the five real critical points of the system, which are in $D$. Our numerical explorations indicate that this limit cycle is unique.

We want to stress that there are very few results in the literature giving upper bounds for the number of limit cycles surrounding several critical points. Notice that the critical points different of the origin, surrounded by the limit cycle, come from the continua of critical points $\{x^2 + y^2 = 1\}$ that system (3.3) possesses when $a = b = 0$. The limit cycle is born in $\{x^2 + y^2 = 2\}$.

3.4. Example 4. Consider the system
\[
\dot{x} = x(1 - (x^2 + y^2))(2 - (x^2 + y^2))(3 - (x^2 + y^2)) - y(2 - (x^2 + y^2)) + ax^2y^3, \\
\dot{y} = x(2 - (x^2 + y^2)) + y(1 - (x^2 + y^2))(2 - (x^2 + y^2))(3 - (x^2 + y^2)).
\]
Let us prove that when $a$ is such that
\[
\Psi_a(r) := -98 + \left(192 + \frac{55}{8}|a|\right) \quad r^2 + (-144 + 6|a|) \quad r^4 + \left(48 + \frac{3}{2}|a|\right) \quad r^6 - 6r^8 < 0
\]
for all $r > 0$, it has at most 3 limit cycles. Once more we use Theorem A. In this occasion we take $k = 1$, $p(s) = (1 - s)(2 - s)(3 - s)$ and $w(r) = r^2 p'(r^2) =
\[ r^2(-11 + 12r^2 - 3r^4) = -3r^2(r^2 - 2 - \frac{\sqrt{3}}{3})(r^2 - 2 + \frac{\sqrt{3}}{3}). \] Then
\[ M(r, \theta, k, w) = -98r^4 + \frac{1}{16} (3072 + 55a \sin(2\theta) - 44a \sin(4\theta) + 11a \sin(6\theta)) r^6 \]
\[ + \frac{1}{2}(-288 - 3a \sin(2\theta) + 6a \sin(4\theta) - 3a \sin(6\theta)) r^8 \]
\[ + \frac{3}{16}(256 - a \sin(2\theta) - 4a \sin(4\theta) + 3a \sin(6\theta)) r^{10} - 6r^{12}. \]

Since \( m^+ = 3 \) and \( M(r, \theta, k, w) \leq r^4 \Psi_a(r) < 0 \) for all \( r > 0 \), the existence of at most 3 limit cycles follows. For instance the above hypothesis holds for \( a = 1/34 \).

For this value of the parameter we prove, by using the same tools that in the previous examples, that the system has several critical points and that, apart from the origin, all of them are contained in the ring \( \mathcal{C} = \{2 - \frac{\sqrt{3}}{3} < x^2 + y^2 < 2 + \frac{\sqrt{3}}{3}\} \).

Finally, again similarly that in the previous cases, we prove that there is exactly one hyperbolic and stable limit cycle inside the disc \( \{x^2 + y^2 < 2 - \frac{\sqrt{3}}{3}\} \) and another hyperbolic and also stable limit cycle outside the disc \( \{x^2 + y^2 \geq 2 + \frac{\sqrt{3}}{3}\} \).

The novelty of this example is the existence of a non-trivial system for which the maximum number of limit cycles is known (it is 3). Moreover it has at least two hyperbolic limit cycles, one of them surrounding only the origin and the other one surrounding the first limit cycle and having several critical points between them.

We want to comment that our numerical exploration seems to indicate that there is no limit cycle contained in \( \mathcal{C} \), and so that the maximum number of limit cycles of the system for this value of the parameter is two.

3.5. Example 5. This last example is interesting from a theoretical point of view. Consider the system
\[
\begin{align*}
\dot{x} &= xu(x^2 + y^2) - yv(x^2 + y^2) + \varepsilon \tilde{P}(x, y), \\
\dot{y} &= xv(x^2 + y^2) + yu(x^2 + y^2) + \varepsilon \tilde{Q}(x, y),
\end{align*}
\]

(3.4)

where \( u \) and \( v \) are given real polynomials of degree \( j \) and assume \( u \) is such that all their roots are real and simple. Let \( m^+ - 1 \) denote the number of positive real roots of \( u' \). Then, for any couple of polynomials \( \tilde{P}(x, y) \) and \( \tilde{Q}(x, y) \) whose monomials have degrees between 2 and 2\( j + 1 \), both included, there exists \( \varepsilon_0 = \varepsilon_0(\tilde{P}, \tilde{Q}) > 0 \) such that if \( |\varepsilon| < \varepsilon_0 \) then system (3.4) is under the hypotheses of Theorem \( \mathbb{A} \). Moreover, under these conditions, it has at most \( m^+ \) limit cycles and all the existing limit cycles are hyperbolic.

When \( \varepsilon = 0 \), the above assertions follow from Lemma 2.4 and Theorem 3. Consider \( p(s) \equiv u(s) \). By using Proposition 2.5, we can take \( w(r) = r^2 u'(r^2) \) and fix a value \( k > 0 \) for which
\[ p_{k, w}(r) := 2r^4 \left( p(r^2)p''(r^2) - k(p'(r^2))^2 \right) + 2(1 - k)r^2 p(r^2)p'(r^2) < 0 \]
for all \( r \in (0, \infty) \). The function \( p(s, \varepsilon) \) for system (3.4) defined in Theorem \( \mathbb{A} \) writes as \( p(s, \varepsilon) = u(s) + \varepsilon \tilde{p}(s) \), for some new polynomial \( \tilde{p} \), also of degree \( j \). It is clear that for \( |\varepsilon| \) small enough, the polynomial \( u(s) + \varepsilon \tilde{p}(s) \) has also all its roots
real and simple and the number of positive roots of $u'(s) + \varepsilon \tilde{p}'(s)$ is $m^+ - 1$. By taking the same value of $k$ and $w(r, \varepsilon) = r^2 (u'(r^2) + \varepsilon \tilde{p}'(r^2))$ we get that the function $M$ in the theorem writes as

$$M(r, \theta, k, w(r, \varepsilon), \varepsilon) = R(r, \theta, \varepsilon) \frac{\partial w(r, \varepsilon)}{\partial r} - k \left( \frac{\partial R(r, \theta, \varepsilon)}{\partial r} + \frac{\partial \Theta(r, \theta, \varepsilon)}{\partial \theta} + \frac{R(r, \theta, \varepsilon)}{r} \right) w(r, \varepsilon)$$

$$=: \sum_{i=2}^{4j} M_i(\theta, k, \varepsilon)r^i = p_{k,w}(r) + \varepsilon \sum_{i=2}^{4j} \tilde{M}_i(\theta, k, \varepsilon)r^i,$$

where the functions $\tilde{M}_i$ are smooth and $2\pi$-periodic in $\theta$. Since $p_{k,w}(r)$ is a negative polynomial in $(0, \infty)$ of the form $p_{k,w}(r) = b_2 r^2 + b_4 r^4 + \cdots + b_{4j} r^{4j}$, for some $b_i$ real numbers and, moreover $b_2 < 0$ and $b_{4j} < 0$, it is clear that for $|\varepsilon|$ small enough the function $\Phi_k(r, \varepsilon)$ given in Theorem [A] is negative as we wanted to prove. Moreover it follows that the maximum number of limit cycles of system (3.4) is $m^+$, and whenever they exist they are hyperbolic.

Notice that it is always true that when a planar system has $m^+$ hyperbolic limit cycles any small perturbation also has at least the same number of hyperbolic limit cycles. The point of the above example is not only to prove the existence of at most this number of limit cycles, but to prove that there are planar polynomial systems under the hypotheses of Theorem [A] for which all planar systems of the same degree near them are also under the hypotheses of the theorem. Moreover, as we can see in the study of the previous examples, our approach allows to get explicit bounds for the size of the perturbation under which our theorem can be applied.

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