Superconductivity in (2+1)-dimensional Relativistic Quantum Field Theory

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Abstract

We investigate the superconductivity in (2+1)-dimensional relativistic quantum field theory. We employ the massless Gross-Neveu model as a model Lagrangian. By using this model, we study the superconductivity and superconducting instability. Our investigation is strongly related to the superconductivity in (2+1)-dimensional two-band systems.

I. INTRODUCTION

Recently, the author published the theory of superconductivity in relativistic quantum field theory [1~3]. This paper is an application of the theory.

Condensed matter physics in low-dimensional systems is an interesting area both experimentally and theoretically [4]. For example, the discovery of the high-\(T_c\) superconductors made the large impact on condensed matter physics, and in such systems, the superconductivity occurs in a quasi-two-dimensional copper oxide CuO\(_2\) plane [5~10]. The recent investigation of the superconductivity in MgB\(_2\) was also a very important event [11], and in this system, people suppose that a plane constructed by B atoms plays the main role in the
superconductivity. Graphite and carbon nanotube also attract much attention to us, and they are plane systems [12]. Therefore, the superconductivity in such kind of low-dimensional systems is a very important subject in condensed matter physics. In theoretical side, Aoki et al. studied the effect of the dimensionality [13], the effect of band structures [14], the multiband effect (two-band model [15], four-band model [16]) and the effect of shapes of the Fermi surfaces [17], in superconductivity of several systems (see also Ref. 18 for two-band model and Ref. 19 for three-band model). The keywords of recent theoretical investigation about superconductivity are “two-dimensional” and “two-band”. The two-band models of superconductivity [20-31] have the origin in the papers of Suhl, Matthias and Walker [20], and Kondo [21,22], and was applied to various systems under various situations. Both MgB$_2$ and graphite have honeycomb lattice structure, and essentially they are two-band systems. Very recently, a theoretical study of two-band superconductivity in MgB$_2$ appeared, and the experimental evidence for two-band superconductivity in MgB$_2$ was also obtained [32]. Some low-energy effective theories like the Ginzburg-Landau model for two-band superconductivity were proposed [30,31]. Some theoreticians consider that the copper oxide high-$T_c$ superconductor can also be understood by a two-band theory [6,7]. Therefore, today, much attentions are paid for two-band superconductivity. There are various two-band models for superconductivity. For example, in the paper of Suhl et al., they used the model which has only an attractive interaction between particles in a two-band system [20]. Yamaji discussed a pairing problem by an interband polarization function arised from a repulsive interaction [23]. Kondo found that a kind of two-band effect enhances the superconductivity [21]. But until now, it is not clear how much the contribution of the lower band presents. There are no quantitative understanding about the strength of the two-band effect, especially the lower-band effect, in a system. In other words, there is a wide variety of models, considerations and interpretations in the two-band effect. Some relations between superconductivity and excitonic state (exciton condensed state) in two-band models were also investigated [28,29].

By the way, relativistic fermion appears in various situations of condensed matter sys-
tems [33~39]. Semenoff studied the (2+1)-dimensional (2 for space and 1 for time) relativistic fermion model in graphite. Based on the character of (2+1)-dimensional relativistic quantum field theory [40~44], he discussed the effect of anomaly (the Chern-Simons term and the fractional fermion numbers) [33]. As a consequence of the honeycomb lattice structure of graphite, the band structure of it has two degeneracy points in the first Brillouin zone (two conical intersections between upper and lower bands), and the relativistic fermion model is obtained in the continuum limit of it [33~37]:

\[
\epsilon = v_F(\gamma^1 p_x + \gamma^2 p_y)
\] (1)

(as illustrated in Figs. 1 and 2). Here, \(v_F\) is the Fermi velocity, \(p_x\) and \(p_y\) are momentum operators. By using a similar model, González et al. discussed that the superconductivity can emerge in graphite [35]. Shankar derived a Dirac fermion model in (2+1)-dimensional doped antiferromagnets [38]. In fact, those relativistic models have two bands (positive and negative energy states), and they can be useful tools to study the low-energy and long-wavelength properties of (2+1)-dimensional two-band systems.

Based on the discussion given above, we investigate the superconductivity in (2+1)-dimensional relativistic quantum field theory. The purposes of this paper are, not only to study fundamental aspects in field theory and many-body theory in (2+1)-dimensions, but also to examine a two-band effect in superconductivity in the (2+1)-dimensional system. We concentrate on the examination of the effect of the lower band, by extracting the strength of its contribution in the superconductivity. In real material, our theory can be applied to systems which have 2-dimensional honeycomb lattice structure like MgB\(_2\), doped-graphite and graphite intercalation compounds (LiC\(_6\), KC\(_8\), etc.), or to systems of the kagome lattice structure [45]. In this paper, we do not consider the Kosterlitz-Thouless transition in two-dimensional superconductivity seriously [46].

This paper is organized as follows. In Sec. II, a relativistic model Lagrangian is introduced, and its characteristic aspects are discussed. By using this model Lagrangian, we study the superconductivity, namely, a phenomenon of dynamical \(U(1)\)-gauge-symmetry
breaking in (2+1)-dimension. In Sec. III, the Gor’kov formalism [47] for a BCS-type contact attractive interaction in our theory is presented. In Sec. IV, the group-theoretical consideration of the mean fields (BCS gap functions) is produced. In Sec. V, by using the Gor’kov formalism, the gap equations in our theory are derived, and they are solved numerically. In 1965, Kohn and Luttinger proved that, whether a two-body interaction is attractive or repulsive, there is a Cooper instability in an interacting many-fermion system (the Kohn-Luttinger effect) [48]. Shankar and Chubukov independently proved the presence of the Kohn-Luttinger effect in a two-dimensional system [49,50]. In Sec. VI, for the pairing problem in the case of a repulsive interaction, the Kohn-Luttinger effect in our theory is examined by using the Bethe-Salpeter (BS) formalism. Finally in Sec VII, we give the conclusion of this work, with further possible investigation in our theory.

II. LAGRANGIAN

In this section, we give a model Lagrangian to study the (2+1)-dimensional relativistic superconductivity. We will give a Lagrangian for the starting point:

\[ \mathcal{L}(x) = \bar{\psi}(x)i\gamma^\mu \partial_\mu \psi(x) + \frac{G_0}{2}(\bar{\psi}(x)\psi(x))^2. \]  

(2)

This is the (2+1)-dimensional Gross-Neveu model [51]. The first term is the kinetic term of the Dirac field, \( \psi \) and \( \bar{\psi} \) are the 2-component relativistic spinors describing the Dirac fields. Here, we do not give a mass to fermion. It is well known in (2+1)-dimensional relativistic field theory [40-44], the Dirac mass term \( m\bar{\psi}\psi \) violates both the parity and the time reversal symmetries. As discussed in the introduction of this paper, we treat the system which has a degeneracy point at zero momentum. We have to consider the massless case. Then, we concentrate on the question as how the generation of the mean fields dynamically breaks the symmetry which the Lagrangian itself has. In the Lagrangian given above, we introduce the four-body contact interaction at the same space-time point. Here, we consider one of the simplest relativistic interaction. When \( G_0 > 0 \), it will give a BCS-like attractive
interaction. We set aside the question of the origin of the attractive interaction, and we regard the Lagrangian (2) as a phenomenological one. On the other hand, when \( G_0 < 0 \), it will become a similar (not the same) interaction to the on-site repulsion of the Hubbard model in its continuum limit. Due to the \( \delta \)-function character of the interaction, the theory becomes unrenormalizable also in (2+1)-dimension and we have to introduce a cutoff. For the purpose of this paper, we do not have to introduce the constant for the Fermi velocity \( v_F \) in the model. This Lagrangian itself has symmetries of Poincaré invariance, \( U(1) \)-gauge invariance, charge conjugation invariance, spatial inversion and time reversal invariance.

For the Clifford algebra [40,41] of \( \gamma \)-matrices, using the definition by the Pauli matrices

\[
\gamma^0 = \sigma^3, \quad \gamma^1 = i\sigma^1, \quad \gamma^2 = i\sigma^2, \tag{3}
\]
( here the Chirality \( \eta \equiv \frac{i}{2} \text{tr} \gamma^0 \gamma^1 \gamma^2 = +1 \) ) we obtain the next relations

\[
\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}, \quad \gamma^\dagger = \gamma^0 \gamma^\mu \gamma^0, \quad \gamma^\mu \gamma^\nu = g^{\mu\nu} - i\epsilon^{\mu\nu\lambda} g_{\lambda\rho} \gamma^\rho. \tag{4}
\]

Here, \( \epsilon^{012} = 1 \). We take metric convention as \( g^{\mu\nu} = \text{diag}(1, -1, -1) \). The charge conjugation matrix is given as

\[
C^{-1} \gamma^\mu C = -\gamma^{\mu T}, \quad C^\dagger C = 1. \tag{5}
\]

For the Poincaré algebra, the generator of the Lorentz transformation satisfies the \( SO(2,1) \) algebra:

\[
[j^\mu, j^\nu] = -i\epsilon^{\mu\nu\lambda} g_{\lambda\rho} j^\rho, \tag{6}
\]

where \( j^0 \) is the generator of 2-dimensional rotation same as \( U(1) \) phase transformation, while \( j^1 \) and \( j^2 \) are the boost operators. Especially the representation on the Dirac field is given as

\[
j^\mu = \frac{1}{2} \gamma^\mu, \tag{7}
\]
then \( \psi \) transforms as

\[
\psi(x) \rightarrow e^{i\omega \cdot j} \psi(x) = e^{\frac{i\omega \cdot \gamma}{2}} \psi(x). \tag{8}
\]
III. GOR'KOV FORMALISM

In this section, we derive the BCS-Gor'kov theory [47] for pairing problem under the attractive interaction $G_0 > 0$ in the Lagrangian (2). The formalism given in this section is parallel with the (3+1)-dimensional theory [1~3]. The field equations are obtained from the Lagrangian (2) by the action principle:

$0 = \frac{\partial L}{\partial \psi} - \partial_\mu \frac{\partial L}{\partial (\partial_\mu \psi)} = i \gamma^\mu \partial_\mu \psi + G_0 (\bar{\psi} \psi) \psi, \quad (9)$

$0 = \frac{\partial L}{\partial \psi} - \partial_\mu \frac{\partial L}{\partial (\partial_\mu \psi)} = -i \partial_\mu \bar{\psi} \gamma^\mu + G_0 (\bar{\psi} \psi) \bar{\psi}, \quad (10)$

and the Hamiltonian becomes

$H = \int d^2 x \bar{\psi} (-i \vec{\gamma} \cdot \vec{\nabla} - \gamma^0 \mu) \psi - \frac{G_0}{2} \int d^2 x (\bar{\psi} \psi)^2. \quad (11)$

Here the second term will give the attractive interaction for $G_0 > 0$. We introduce the chemical potential $\mu$ in the above Hamiltonian. In the context to study the relativistic field theory, it describes the finite density at $\mu \neq 0$ as the conjugate of the particle number minus antiparticle number. When we consider a crystal by the model, $\mu$ becomes the conjugate of the electron number in upper band minus hole number in lower band. Throughout this study, we completely neglect temperature dependence of $\mu$, and we treat $\mu$ as a parameter introduced from the outside of the system. Thus we set $\mu = \epsilon_F$ (Fermi energy); $\mu$ is determined by the position of the Fermi energy of a system. In the case of graphite, $\mu$ determines the electron doping concentration.

Next, we make some preparations to study the theory of superconductivity. First, we introduce various propagators. We use the 4-component Nambu notation [52]:

$\hat{\Psi}(x) \equiv \left( \begin{array}{c} \hat{\psi}(x) \\ \hat{\psi}(x) \end{array} \right), \quad \hat{\Psi}(x) \equiv (\hat{\psi}(x), \hat{\psi}^T(x)), \quad (12)$

where $T$ means the transposition. The definition of the one-particle propagator is

$G(x, y) \equiv -i \langle T \hat{\Psi}(x) \hat{\Psi}(y) \rangle$
This is a 4×4 matrix. \( T \) means the time-ordered product, and \( \langle \cdots \rangle \) means the expectation value. \( S_F \) is the Feynman propagator for quasiparticle, while \(-iF\) and \(-i\bar{F}\) are the anomalous propagators. Next, we obtain the equations of motion for the propagators (13).

We employ the Gor’kov factorization in (9) and (10), taking account of only the superconducting pair-correlation by introducing the mean-field approximation. Then we obtain the relativistically generalized (2+1)-dimensional Gor’kov equation written down as a 4×4 matrix equation:

\[
\begin{pmatrix}
\gamma^\mu \partial_\mu + \gamma^0 \mu & \Delta(x) \\
\bar{\Delta}(x) & i(\gamma^\mu)^T \partial_\mu - (\gamma^0)^T \mu
\end{pmatrix}
\begin{pmatrix}
S_F(x,y) \\
i\bar{F}(x,y)
\end{pmatrix}
= \begin{pmatrix}
\delta^{(3)}(x-y) & 0 \\
0 & \delta^{(3)}(x-y)
\end{pmatrix}.
\]

(14)

\( \Delta(x) \) and \( \bar{\Delta}(x) \) are 2×2 matrix mean fields, so called order parameters. The definitions are

\[
\Delta(x_0,x)_{\alpha\beta} \equiv G_0 F(x_0^+, x; x_0, x)_{\alpha\beta} = G_0 (\hat{\psi}_\alpha(x_0^+, x)\hat{\psi}_\beta^T(x_0, x)),
\]

(15)

\[
\bar{\Delta}(x_0,x)_{\alpha\beta} \equiv G_0 F(x_0^+, x; x_0, x)_{\alpha\beta} = G_0 (\hat{\psi}_\alpha^T(x_0^+, x)\hat{\psi}_\beta(x_0, x)).
\]

(16)

This gives the self-consistency condition. In general, the mean field clearly violates the Lorentz symmetry, as well as the gauge symmetry. In other words, the mean field involves quantities other than the scalar.

We will also obtain the Fourier transform of the Gor’kov equation:

\[
\begin{pmatrix}
\hat{k} & \Delta \\
\bar{\Delta} & \hat{k}^T
\end{pmatrix}
\begin{pmatrix}
S_F(k) \\
i\bar{F}(k)
\end{pmatrix}
= \begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix}.
\]

(17)

Here \( \hat{k} \equiv (k_0 + \mu, \mathbf{k}) \) and \( \hat{k} \equiv (k_0 - \mu, \mathbf{k}) \). \( \hat{k}^T \) means the transpose of \( \hat{k} \). The self-consistency condition now becomes
\[ \Delta = G_0 \int \frac{d^3p}{(2\pi)^3} F(p), \quad \bar{\Delta} = G_0 \int \frac{d^3p}{(2\pi)^3} \bar{F}(p). \] (18)

Here the mean field has only the internal degrees of freedom. In the nonrelativistic BCS theory, the mean field has no internal degree of freedom. In the case of the relativistic theory, there is a possibility to obtain much more complicated states.

The finite-temperature theory of the Matsubara formalism can be obtained in the same way. We introduce imaginary time \( \tau = it \). The temperature Green’s function is defined as

\[
\mathcal{G}(x, y) \equiv -\langle T_\tau \hat{\Psi}(x) \hat{\Psi}(y) \rangle
\]

\[
= \begin{pmatrix}
-\langle T_\tau \hat{\psi}_\alpha(x) \hat{\psi}_\beta(y) \rangle & -\langle T_\tau \hat{\psi}_\alpha(x) \hat{\psi}_\beta^T(y) \rangle \\
-\langle T_\tau \hat{\psi}_\alpha^T(x) \hat{\psi}(y) \rangle & -\langle T_\tau \hat{\psi}_\alpha^T(x) \hat{\psi}_\beta^T(y) \rangle
\end{pmatrix} = 
\begin{pmatrix}
S(x, y)_{\alpha\beta} & -F(x, y)_{\alpha\beta} \\
-\bar{F}(x, y)_{\alpha\beta} & -S(y, x)_{\beta\alpha}
\end{pmatrix},
\] (19)

where \( \langle \cdots \rangle \) means the statistical average. From the equation of motion of the temperature Green’s function, the Gor’kov equation becomes

\[
\begin{pmatrix}
- \gamma^0 \frac{\partial}{\partial \tau} - \mu + i \gamma^k \partial_k & \Delta(x) \\
\bar{\Delta}(x) & - (\gamma^0)^T \frac{\partial}{\partial \tau} + \mu + i (\gamma^k)^T \partial_k
\end{pmatrix} \begin{pmatrix}
S(x, y)_{\alpha\beta} & -F(x, y)_{\alpha\beta} \\
-\bar{F}(x, y)_{\alpha\beta} & -S(y, x)_{\beta\alpha}
\end{pmatrix} = 
\begin{pmatrix}
\delta^{(3)}(x - y) & 0 \\
0 & \delta^{(3)}(x - y)
\end{pmatrix}.
\] (20)

Here the definition of the mean fields are the simple extension of those for the zero temperature:

\[
\Delta(\tau, x)_{\alpha\beta} \equiv G_0 F(\tau^+, x; \tau, x) = G_0 \langle \hat{\psi}_\alpha(\tau^+, x) \hat{\psi}_\beta^T(\tau, x) \rangle,
\] (21)

\[
\bar{\Delta}(\tau, x)_{\alpha\beta} \equiv G_0 \bar{F}(\tau^+, x; \tau, x) = G_0 \langle \hat{\psi}_\alpha^T(\tau^+, x) \hat{\psi}_\beta(\tau, x) \rangle.
\] (22)

Fourier transform is also obtained as follows:

\[
\begin{pmatrix}
\gamma^0 (i\omega_n + \mu) - \vec{\gamma} \cdot \vec{k} & \Delta \\
\bar{\Delta} & (\gamma^0)^T (i\omega_n - \mu) - (\vec{\gamma})^T \cdot \vec{k}
\end{pmatrix} \begin{pmatrix}
S(\omega_n, \vec{k}) & -F(\omega_n, \vec{k}) \\
-\bar{F}(\omega_n, \vec{k}) & -S(-\omega_n, -\vec{k})^T
\end{pmatrix}
\]
\[
\begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix}.
\]

(23)

Here \( \beta \equiv 1/k_B T \) \((k_B; \text{the Boltzmann constant})\), \( \omega_n = (2n + 1)\pi/\beta \) is a fermion discrete frequency. Solving Eq. (23), we will obtain the solutions of (17) in the same form, except that we need to substitute \( k_0 \to i\omega_n \).

**IV. GROUP-THEORETICAL CONSIDERATION OF THE MEAN-FIELDS**

Now we perform the group-theoretical consideration of the mean fields. The mean field is a 2×2 matrix. Under the Lorentz transformation

\[
\psi'(x') = S\psi(x) = \exp\left(\frac{i}{2} \omega_\nu \gamma^\nu\right)\psi(x),
\]

(24)

then the mean field is transformed as

\[
\Delta'(x') = \langle \psi'(x')\psi'^T(x') \rangle = \langle S\psi(x)\psi^T(x)S^T \rangle \\
= S\Delta(x)S^T \\
\cong (1 + \frac{i}{2} \omega_\nu \gamma^\nu)\Delta(x)(1 + \frac{i}{2} \omega_\nu \gamma^\nu T) \\
= \Delta(x) + \frac{i}{2} \omega_\nu [\gamma^\nu, \Delta(x)C^{-1}]C.
\]

(25)

Thus we can decompose the mean fields as follows:

\[
\Delta = (\Delta^S + \Delta^V_\mu \gamma^\mu)C, \quad \bar{\Delta} = -C^{-1}(\Delta^{S*} + \Delta^{V*_\mu \gamma^\mu}),
\]

(26)

where \( S \) indicates scalar, while \( V \) indicates vector. Under the discrete transformations

\[
\psi \xrightarrow{C} C\bar{\psi}^T = i\gamma^1 \gamma^0 \bar{\psi}^T; \quad \bar{\psi} \xrightarrow{C} -\psi^T C^{-1} = \psi^T i\gamma^1 \gamma^0,
\]

(27)

\[
\psi(x_0, \mathbf{x}) \xrightarrow{P} -i\gamma^1 \psi(x_0, -\mathbf{x}'); \quad \bar{\psi}(x_0, \mathbf{x}) \xrightarrow{P} -\bar{\psi}(x_0, -\mathbf{x}')(i\gamma^1),
\]

(28)

\[
\psi(x_0) \xrightarrow{T} -i\gamma^2 \psi(-x_0); \quad \bar{\psi}(x_0) \xrightarrow{T} -\bar{\psi}(-x_0)(i\gamma^2),
\]

(29)

( here \( \mathbf{x} = (x_1, x_2) \) and \( \mathbf{x}' = (-x_1, x_2) \) ), where \( C, P \) and \( T \) denote the operations of charge conjugation, spatial inversion and time reversal, respectively. Therefore, the mean fields are transformed as
\[ \langle \psi \psi^T \rangle \rightarrow C \langle \tilde{\psi}^T \tilde{\psi} \rangle C^{-1} = -\gamma^2 \langle \tilde{\psi}^T \tilde{\psi} \rangle \gamma^2, \quad (30) \]
\[ \langle \tilde{\psi}^T \tilde{\psi} \rangle \rightarrow C \langle \psi \psi^T \rangle C^{-1} = -\gamma^2 \langle \psi \psi^T \rangle \gamma^2, \quad (31) \]
\[ \langle \psi(x_0, \mathbf{x}) \psi^T(x_0, \mathbf{x}) \rangle \xrightarrow{P} -\gamma^1 \langle \psi(x_0, \mathbf{x}') \psi^T(x_0, \mathbf{x}') \rangle \gamma^1, \quad (32) \]
\[ \langle \tilde{\psi}^T(x_0, \mathbf{x}) \tilde{\psi}(x_0, \mathbf{x}) \rangle \xrightarrow{P} -\gamma^1 \langle \tilde{\psi}^T(x_0, \mathbf{x}') \tilde{\psi}(x_0, \mathbf{x}') \rangle \gamma^1, \quad (33) \]
\[ \langle \psi \psi^T \rangle \xrightarrow{T} \gamma^2 \langle \psi \psi^T \rangle^* \gamma^2, \quad (34) \]
\[ \langle \tilde{\psi}^T \tilde{\psi} \rangle \xrightarrow{T} \gamma^2 \langle \tilde{\psi}^T \tilde{\psi} \rangle^* \gamma^2. \quad (35) \]

Thus each type of the mean fields is transformed under the spatial inversion and time reversal as

\[ \Delta^S \gamma^2 \xrightarrow{P} -\Delta^S \gamma^2, \quad (36) \]
\[ \xrightarrow{T} -\Delta^S \gamma^2, \quad (37) \]
\[ \Delta_0^V \gamma^0 \gamma^2 \xrightarrow{P} \Delta_0^V \gamma^0 \gamma^2, \quad (38) \]
\[ \xrightarrow{T} \Delta_0^V \gamma^0 \gamma^2, \quad (39) \]
\[ \Delta_1^V \gamma^1 \gamma^2 \xrightarrow{P} -\Delta_0^V \gamma^1 \gamma^2, \quad (40) \]
\[ \xrightarrow{T} \Delta_1^V \gamma^1 \gamma^2, \quad (41) \]
\[ \Delta_2^V \gamma^2 \gamma^2 \xrightarrow{P} \Delta_2^V \gamma^2 \gamma^2, \quad (42) \]
\[ \xrightarrow{T} -\Delta_2^V \gamma^2 \gamma^2. \quad (43) \]

Therefore, with 2-dimensional rotation and parity, the mean field is decomposed into 3 irreducible representations: \( \Delta^S \), \( \Delta_0^V \) and \( \Delta_1^V \). As expected, \( \Delta^S \) violates the parity like the Dirac mass term \( m \tilde{\psi} \psi \).

**V. THE GAP EQUATION**

In this section, we derive gap equations. For this purpose, first we have to solve the Gor’kov equation. Like the case of (3+1)-dimensional theory [1~3], it is difficult to solve Eq. (17) or Eq. (23) completely because of its matrix structure. Therefore we have to solve
the equations assuming the type of the mean field that might be realized. Then we obtain 3 Gor’kov equations for each type of the mean fields. These equations can be solved in the same way as the case of the (3+1)-dimensinal theory [1∼3]. We give the following results.

First the case of the scalar $\Delta^S$

$$
\begin{pmatrix}
S_F(k) & -iF(k) \\
-i\tilde{F}(k) & -S_F(-k)^T
\end{pmatrix}
$$

$$
= \frac{1}{D(k)} \begin{pmatrix}
(\tilde{k}\tilde{k} - |\Delta^S|^2)\tilde{k} & \Delta^S(\tilde{k}\tilde{k} - |\Delta^S|^2)C \\
-\Delta^S* C^{-1}(\tilde{k}\tilde{k} - |\Delta^S|^2) & -C^{-1}(\tilde{k}\tilde{k} - |\Delta^S|^2)\tilde{k}C
\end{pmatrix},
$$

$$
D(k) = (\tilde{k} \cdot \tilde{k})(\tilde{k} \cdot \tilde{k}) - 2|\Delta^S|^2(\tilde{k} \cdot \tilde{k}) + |\Delta^S|^4.
$$

Next the case of 0th-component of vector $\Delta^V_0$

$$
\begin{pmatrix}
S_F(k) & -iF(k) \\
-i\tilde{F}(k) & -S_F(-k)^T
\end{pmatrix}
$$

$$
= \frac{1}{D(k)} \begin{pmatrix}
(\tilde{k}^0\tilde{k}^0 - |\Delta^V_0|^2)\tilde{k}^0 & \Delta^V_0(\tilde{k}^0\tilde{k}^0 - |\Delta^V_0|^2)C \\
-\Delta^V_0* C^{-1}(\tilde{k}^0\tilde{k}^0 - |\Delta^V_0|^2) & -C^{-1}(\tilde{k}^0\tilde{k}^0 - |\Delta^V_0|^2)\tilde{k}^0C
\end{pmatrix},
$$

$$
D(k) = (\tilde{k} \cdot \tilde{k})(\tilde{k} \cdot \tilde{k}) - 2|\Delta^V_0|^2(\tilde{k} \cdot \tilde{k}) + |\Delta^V_0|^4.
$$

Then, the case of 1st-component of vector $\Delta^V$

$$
\begin{pmatrix}
S_F(k) & -iF(k) \\
-i\tilde{F}(k) & -S_F(-k)^T
\end{pmatrix}
$$

$$
= \frac{1}{D(k)} \begin{pmatrix}
(\tilde{k}\tilde{k}^1 - |\Delta^V|^2)\tilde{k}^1 & \Delta^V(\tilde{k}\tilde{k}^1 + |\Delta^V|^2)C \\
-\Delta^V* C^{-1}(\tilde{k}\tilde{k}^1 + |\Delta^V|^2) & -C^{-1}(\tilde{k}\tilde{k}^1 + |\Delta^V|^2)\tilde{k}^1C
\end{pmatrix},
$$

$$
D(k) = (\tilde{k} \cdot \tilde{k})(\tilde{k} \cdot \tilde{k}) - 2|\Delta^V|^2(\tilde{k} \cdot \tilde{k}) + |\Delta^V|^4.
$$
In all cases, \( D(k) \) is second order in \( k_0^2 \), and we can easily factorized it as \( D(k) = (k_0 - E_+)(k_0 + E_+)(k_0 - E_-)(k_0 + E_-) \). Here, \( E_+ \) corresponds to the energy of the quasiparticles coming from positive energy states, while \( E_- \) corresponds to the energy of the quasiparticles coming from negative energy states.

Now we construct the gap equations by using the Green’s functions we have obtained. We use the finite-temperature Matsubara formalism. From the self-consistency conditions:

\[
\Delta = G_0 \sum_n \frac{1}{\beta} \int \frac{d^2k}{(2\pi)^2} F(\omega_n, k),
\]

we obtain the gap equation for the case of the scalar \( \Delta^S \):

\[
1 = \frac{G_0}{2} \int \frac{d^2k}{(2\pi)^2} \left( \frac{1}{2E_+} \tanh \frac{\beta}{2} E_+ + \frac{1}{2E_-} \tanh \frac{\beta}{2} E_- \right),
\]

\[
E_\pm = \sqrt{(|k| \mp \mu)^2 + |\Delta^S|^2}.
\]

The second term in the integrand is the contribution coming from negative energy states and/or the lower band. In the context of this paper, the relativistic effect is the two-band effect. For the case of the 0th-component of vector \( \Delta^V_0 \), we get

\[
1 = \frac{G_0}{2} \int \frac{d^2k}{(2\pi)^2} \frac{1}{\sqrt{\mu^2 + |\Delta^V_0|^2}} \left( - \tanh \frac{\beta}{2} E_+ + \tanh \frac{\beta}{2} E_- \right),
\]

\[
E_\pm = |k| \mp \sqrt{\mu^2 + |\Delta^V_0|^2},
\]

and the case of the 1st-component of vector \( \Delta^V_1 \), we get

\[
-1 = \frac{G_0}{2} \int \frac{d^2k}{(2\pi)^2} \left( \sqrt{k^2 \mu^2 + |\Delta^V_1|^2 k_1^2 - k_1^2} \right) \frac{1}{2E_+} \tanh \frac{\beta}{2} E_+ \\
+ \left( \sqrt{k^2 \mu^2 + |\Delta^V_1|^2 k_1^2 + k_1^2} \right) \frac{1}{2E_-} \tanh \frac{\beta}{2} E_-,
\]

\[
E_\pm = \sqrt{k^2 + \mu^2 + |\Delta^V_1|^2 \mp \sqrt{k^2 \mu^2 + |\Delta^V_1|^2 k_1^2}}.
\]

From the results given above, we find that the nontrivial solutions can be obtained for the cases of the scalar \( \Delta^S \) and 0th-component of vector \( \Delta^V_0 \). We do not obtain nontrivial solutions for the case of the spatial components of vector \( \Delta^V_1 \) and \( \Delta^V_2 \), because the gap equations (55) become the form “\(-1 = \text{positive quantity}\)”.  

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We solve our gap equations numerically. For the integration, we take

\[ \int \frac{d^2p}{(2\pi)^2} \cdots = \frac{1}{4\pi^2} \int_{\mu-\frac{\Lambda}{2}}^{\mu+\frac{\Lambda}{2}} pdp \int_{0}^{2\pi} d\theta \cdots. \tag{57} \]

Therefore, there are four parameters (coupling constant \( G_0 \), chemical potential \( \mu \), cutoff \( \Lambda \) and temperature \( T \)) in our gap equations. Because our theory treats a massless case, there is no unit of energy in our theory. To see the effect of the lower band, we also treat the gap equation of the "no sea" (neglect the contribution of the Dirac sea in the gap equation of the scalar) case:

\[ 1 = \frac{G_0}{2} \int \frac{d^2\mathbf{k}}{(2\pi)^2} \frac{1}{2E_+} \tanh \frac{\beta}{2} \frac{1}{2} \tag{58} \]

\[ E_+ = \sqrt{(|k| - \mu)^2 + |\Delta_{no-sea}|^2}. \tag{59} \]

There is a simple scaling relation in our gap equations. When we transform the gap equation in next relations

\[ |k|/\mu = |k'|, \quad \Lambda/\mu = \Lambda', \quad \beta \mu = \beta', \quad |\Delta|/\mu = |\Delta'|, \tag{60} \]

Eq.(51) is transformed as

\[ 1 = \frac{G_0}{16\pi^2 \mu} \int_{-\Lambda'/2}^{1+\Lambda'/2} dk' \frac{1}{\sqrt{(|k'| - 1)^2 + |\Delta'|^2}} \tanh \frac{\beta'}{2} \frac{1}{2} \sqrt{(|k'| - 1)^2 + |\Delta'|^2} \]

\[ + \frac{1}{\sqrt{(|k'| + 1)^2 + |\Delta'|^2}} \tanh \frac{\beta'}{2} \frac{1}{2} \sqrt{(|k'| + 1)^2 + |\Delta'|^2}. \tag{61} \]

Therefore, when the ratio \( \Lambda/\mu \) is fixed and when we take \( G_0' = G_0 \mu = \text{const}. \), we treat the same equation with (61).

Usually, a gap is much smaller than the Fermi energy: \(|\Delta| \ll \mu\). For example, in the case of solid, the ratio \(|\Delta(T = 0)|/\mu\) is \(10^{-4} \sim 10^{-2}\). We have to choose model parameters \( G_0 \) and \( \Lambda \) to satisfy the condition. We choose the range of the variation of the cutoff \( 0 < \Lambda \leq 2\mu \).

After choosing the values of parameters, the integration in the gap equation is performed, and search the self-consistency condition under the variation with respect to the amplitude of the gap. We have performed the integration in our gap equations by using the numerical package Mathematica version 4.1.
Fig. 3 shows the gap at $T = 0$ as a function of the coupling constant $G_0$. Here we take the cutoff $\Lambda$ as $\Lambda/2\mu = 1$ and we set $\mu = 1$. Both the scalar and "no sea" depend exponentially on $G_0$, like usual nonrelativistic BCS theory. The gap of the scalar is always larger than the "no sea" case. The ratio $|\Delta^S(T = 0)|/|\Delta^{\text{no-sea}}(T = 0)|$ is almost always 1.57.

In Fig. 4, we give the dependence of the cutoff $\Lambda$ in the solutions of the scalar and "no sea" cases at $T = 0$. Here we set $\mu = 1$ and $G_0 = 2.2\mu^{-1}$. We take the value of $\Lambda$ so as to integrate the outside and inside the Fermi surface (Fermi circle) symmetrically (see Eq.(57)). The solution of the "no sea" completely linearly depend on $\Lambda$. The solution of the scalar behaves almost as a linear function of $\Lambda$, but it deviates slightly from the linear dependence at the large cutoff region. We can say that the solutions behave almost $|\Delta(T = 0)| \propto \Lambda e^{-1/G_0}$, same as the usual nonrelativistic BCS theory. The ratio $|\Delta^S(T = 0)|/|\Delta^{\text{no-sea}}(T = 0)|$ depends on $\Lambda$. At $\Lambda/2\mu = 1$, this ratio becomes 1.57, and when we squeeze the integration width, the ratio becomes small.

In Fig. 5, we show the temperature $T$ dependence of the solutions in the scalar and "no sea" cases. Here we set $G_0 = 2\mu^{-1}$, $\mu = 1$ and $\Lambda/2\mu = 1$. In graphite, the relativistic model can be applied to the region of the energy width $1 \sim 2\text{eV}$. If we take $\mu = 1\text{eV}$, the critical temperature $T_c$ becomes 38K (0.0033eV) for the scalar and 24K (0.0021eV) for the "no sea". The ratio $T_c^{\text{scalar}}/T_c^{\text{no-sea}}$ is 1.57. Both the scalar and "no sea" cases fulfill the BCS universal constant $|\Delta(T = 0)|/T_c = 1.76$. Therefore, both cases obey the BCS-like temperature dependence.

To summarize our numerical results, we conclude that, the scalar (the two-band case) gives always larger solution than the "no sea" case (the one-band case), and when $\Lambda$ is large enough, the difference of the solutions between the scalar and "no sea" becomes significant. The contribution coming from the negative energy states (the lower band) enhances the superconducting gap.

About the 0th-component of vector case, we could not find a reasonable solution in our numerical calculation. The reason can be understood in the following way. We rewrite Eq.
(53) at $T_c$:

$$1 = \frac{G_0}{8\pi} \int_{\mu - \Lambda/2}^{\mu + \Lambda/2} p dp \frac{1}{\mu} \left( - \tanh \frac{\beta}{2} (p - \mu) + \tanh \frac{\beta}{2} (p + \mu) \right)$$

$$\approx \frac{G_0}{4\mu} \rho(\varepsilon_F) \int_{-\Lambda/2}^{\Lambda/2} d\xi \left( - \tanh \frac{\beta}{2} \xi + 1 \right)$$

$$= \frac{G_0}{4} \Lambda.$$

Here $\rho(\varepsilon_F)$ is the density of states at the Fermi level. We used the approximation $\mu = \varepsilon_F$. In the above expression, the equation for determination of $T_c$ has no temperature dependence. Therefore, we cannot determine $T_c$ by Eq.(62). In the (3+1)-dimensional massive theory, the 0th-component of vector is a meaningful state, and we discussed various aspects of the relation between the scalar, "no-sea", "nonrelativistic" and 0th-component of vector [1∼3]. This fact contrast with the results of the (2+1)-dimensional theory.

VI. THE KOHN-LUTTINGER EFFECT

Now we consider the pairing problem with the repulsive interaction $G_0 < 0$ in the Lagrangian (2). For this problem, we should examine the appearance of a pole in the 4-point function (the 2-particle Green’s function, see Fig. 6(a) ) by using the Bethe-Salpeter (BS) formalism [47,53], like the work of Kohn and Luttinger [48]. We start with introducing the fermion-fermion BS equation [54] in the finite-temperature Matsubara formalism:

$$\chi(p) = \sum_n \frac{1}{\beta} \int \frac{d^2p'}{(2\pi)^2} \tilde{\Gamma}(p, p') S(\omega_n, p') \chi(p') S(-\omega_n, -p')^T.$$  (63)

We treat (63) as a function of temperature $T$, and the critical temperature $T_c$ is determined when a self-consistent solution of $\chi(p)$ appear in Eq.(63). The normal-state fermion propagators are given as follows:

$$S(\omega_n, p) = -\frac{1}{\hat{p}}, \quad S(-\omega_n, -p)^T = -C^{-1} \frac{1}{\hat{p}} C.$$  (64)

Here $\hat{p} = (i\omega_n + \mu, p)$ and $\hat{p} = (i\omega_n - \mu, p)$. $\tilde{\Gamma}(p, p')$ is the irreducible vertex part. The BS amplitude $\chi(p)$ is decomposed as follows:
\[ \chi(p) = (\chi^S(p) + \chi^V_\mu(p)\gamma^\mu)C \equiv \left( \sum_{A=1}^{4} \chi_A \Gamma^A \right)C. \]  

(65)

\( \chi(p) \) has to fulfill the Pauli principle:

\[ \chi_{\alpha\beta}(p) = -\chi_{\beta\alpha}(-p). \]  

(66)

Then, each component should obey the following relations

\[ \chi^S(p)C_{\alpha\beta} = -\chi^S(-p)C_{\beta\alpha}, \quad \chi^V_\mu(p)(\gamma^\mu C)_{\alpha\beta} = -\chi^V_\mu(-p)(\gamma^\mu C)_{\beta\alpha}. \]  

(67)

Therefore we get

\[ \chi^S(-p) = \chi^S(p), \quad \chi^V_\mu(-p) = -\chi^V_\mu(p). \]  

(68)

In Eq. (63), replace \( \chi(p) \) to the expanded form ( Eq. (65) ), and take trace for both sides, we obtain the following form:

\[ \chi_A(p)\frac{1}{2} \text{tr}(\Gamma^A \Gamma^A) = \sum_n \frac{1}{\beta} \int \frac{d^2p'}{(2\pi)^2} \tilde{\Gamma}(p,p') \frac{1}{2} \text{tr}(\Gamma^A S(\omega_n,p')\chi_A(p')\Gamma^A S(-\omega_n,-p')^T). \]  

(69)

Similar to the treatment of the Gor’kov equation in Sec.V, we completely neglect couplings between different types of pairing functions. Then, for specific \( A \), we obtain the following equations:

\[ \chi^S(p) = \frac{1}{2} \int \frac{d^2p'}{(2\pi)^2} \tilde{\Gamma}(p,p') \chi^S(p') \times \left( \frac{1}{2(|p'| - \mu)} \tanh \frac{\beta}{2}(|p'| - \mu) + \frac{1}{2(|p'| + \mu)} \tanh \frac{\beta}{2}(|p'| + \mu) \right), \]  

(70)

\[ \chi^V_0(p) = \frac{1}{2} \int \frac{d^2p'}{(2\pi)^2} \tilde{\Gamma}(p,p') \chi^V_0(p') \frac{1}{2\mu} \left( -\tanh \frac{\beta}{2}(|p'| - \mu) + \tanh \frac{\beta}{2}(|p'| + \mu) \right), \]  

(71)

\[ -\chi^V_1(p) = \frac{1}{2} \int \frac{d^2p'}{(2\pi)^2} \tilde{\Gamma}(p,p') \chi^V_1(p') \frac{1}{2\mu|p'|} \times \left( \frac{(p'|^2 - \mu|p'|)}{|p'| - \mu} \tanh \frac{\beta}{2}(|p'| - \mu) - \frac{(p'|^2 + \mu|p'|)}{|p'| + \mu} \tanh \frac{\beta}{2}(|p'| + \mu) \right). \]  

(72)

In this paper, we estimate \( \tilde{\Gamma}(p,p') \) by using a random phase approximation ( RPA ) ( as illustrated in Fig. 6(b) ). Therefore, Eqs. (63), (70)~(72) and then the criteria of the generation of the superconducting states are determined only by the normal state property of a system. The polarization is given as
\[
\Pi(\omega_l, q) = \sum_n \frac{1}{\beta} \int \frac{d^2 p}{(2\pi)^2} \frac{1}{\epsilon_{p+q}^2 - p^2} \text{tr} S(\omega_l + \omega_n, p + q) S(\omega_n, p).
\]  

(73)

Here, \( \omega_l \) is a boson discrete frequency. We neglect \( \omega_l \) dependence of \( \Pi(\omega_l, q) \) to obtain the static polarization:

\[
\Pi(0, q) = \int \frac{d^2 p}{(2\pi)^2} \frac{1}{\epsilon_{p+q}^2 - p^2} \left\{ \frac{p \cdot q}{\epsilon_p} \left( \frac{1}{e^{\beta(p-\mu)} + 1} + \frac{1}{e^{\beta(p+\mu)} + 1} - 1 \right) \right.
\]

\[
+ \left( \frac{p + q}{\epsilon_{p+q}} \left( \frac{1}{e^{\beta(p+q-\mu)} + 1} + \frac{1}{e^{\beta(p+q+\mu)} + 1} - 1 \right) \right) \}
\]

(74)

where, \( \epsilon_p = |p| \) is the relativistic dispersion for massless particle. The integration in (74) has an ultraviolet divergence, and we have to introduce a cutoff \( \Lambda \). At \( T = 0 \), the momentum \( |p| \) integration in Eq. (74) is easily performed with a cutoff, and we obtain

\[
\Pi(0, q)_{T=0} = \frac{1}{2\pi}(\Lambda - p_F) - \frac{|q|}{8\pi^2} \int_0^{2\pi} d\phi \left\{ \frac{1}{\cos \phi} \ln \left| \frac{\cos \phi + |q|/2\Lambda}{\cos \phi + |q|/2p_F} \right| \right\}
\]

(75)

In (75), \( \Pi(0, q)_{T=0} \) diverges linearly on \( \Lambda \). It is clear from this expression, when the ratio \( \Lambda/2\mu = \Lambda/2p_F \) is fixed, \( \Pi(0, q)_{T=0} \) depends linearly on \( p_F \): \( \Pi(0, |q|)_{T=0} = \Pi(0, p_F|q|)_{T=0} = p_F \Pi(0, |q'|)_{T=0} \). To obtain \( \Pi(0, q) \), we perform the integration numerically in Eq. (74). Fig. 7 shows the results of numerical integration for \( \Pi(0, |q|) \) with several \( T \). Here we set \( \Lambda = 2\mu \) and \( \mu = 1 \). We find a peak near \( |q| = 2p_F \). This reflects the sharpness of the Fermi surface. The peak decreases when \( T \) increases. To treat the Cooper problem, we concentrate on the behavior of \( \Pi(0, |q|) \) at the Fermi surface. To see this, we define \( q = p' - p \) and substitute \( |q| \) to \( |p' - p| \) in \( \Pi(0, |q|) \). Then we set \( |p| = |p'| = p_F \), we get \( \Pi(0, p_F, p_F, \cos \theta, \theta''p'') \). Here \( \theta''p'' = \theta p - \theta p' \). We calculate \( \Pi(0, p_F, p_F, \cos \theta, \theta''p'') \) numerically. Fig. 8 shows the angular dependence of \( \Pi(0, p_F, p_F, \cos \theta, \theta''p'') \) with several \( T \). The dent at \( \theta'' = \pi \) grows when \( T \) increases. At enough low temperature, the angular dependence is almost \( \cos \theta \), and then we can use the following expression to examine the pairing properties of the system:

\[
\Pi(0, p_F, p_F, \cos \theta, \theta''p'')_{T=0} \simeq \frac{1}{2\pi}(\Lambda - p_F) - \alpha \cos \theta, \theta''p''
\]

\[
= \frac{1}{2\pi}(\Lambda - p_F) - \alpha (\cos \theta p \cos \theta p' + \sin \theta p \sin \theta p').
\]

(76)
From Fig. 8, we find $\alpha \sim 0.358$ for $\mu = 1$ and $\Lambda = 2\mu$. Then we obtain the expression for $\tilde{\Gamma}(p, p')$ at low temperature within the RPA:

$$\tilde{\Gamma}(p, p') \rightarrow \frac{G_0}{1 - G_0 \Pi(0, p_F, p_F, \cos \theta p \cdot p')}.$$  

To obtain the BS equation for specific symmetry of pairing, we employ the angular decomposition to $\chi(p)$ and $\tilde{\Gamma}(p, p')$. The two-dimensional angular momentum eigenfunction is given as

$$y_l(\theta) = \frac{1}{\sqrt{2\pi}} e^{i l \theta},$$  

where, $l$ is an angular momentum quantum number in a two-dimensional system. In this section, we only consider the time-reversal invariant pairings because of simplicity. Therefore, we take a linear combination:

$$\frac{1}{\sqrt{2}} \frac{1}{\sqrt{2\pi}} (e^{i l \theta} + e^{-i l \theta}) = \frac{1}{\sqrt{\pi}} \cos l \theta.$$  

Each component of the amplitude is decomposed by this function:

$$\chi^S(p) = \sum_{l:even} \chi^S(|p|)_l \frac{1}{\sqrt{\pi}} \cos l \theta \hat{p} \rightarrow \sum_{l:even} \chi^S_l \frac{1}{\sqrt{\pi}} \cos l \theta \hat{p},$$  

$$\chi^V_0(p) = \sum_{l:odd} \chi^V_0(|p|)_l \frac{1}{\sqrt{\pi}} \cos l \theta \hat{p} \rightarrow \sum_{l:odd} (\chi^V_0)_l \frac{1}{\sqrt{\pi}} \cos l \theta \hat{p},$$  

$$|\chi^V(p)| = \sum_{j:odd} |\chi^V(|p|)|_j \frac{1}{\sqrt{\pi}} \cos j \theta \hat{p} \rightarrow \sum_{j:odd} |\chi^V|_j \frac{1}{\sqrt{\pi}} \cos j \theta \hat{p}. \quad (j = l \pm 1).$$  

Here, we neglect $|p|$ dependence of the amplitudes for weak coupling approximation: $|\chi(|p|)|_l \approx |\chi|_l$. Because of (68), we choose even $l$ for scalar in Eq.(80), while we choose odd $l$ ($j$) for vector pairings in Eqs.(81) and (82). The irreducible vertex part is also decomposed as follows:

$$\frac{1}{G_0^{-1} - (\Lambda - p_F)/2\pi + \alpha \cos \theta p \cdot p'} = \frac{1}{G_0^{-1} - (\Lambda - p_F)/2\pi}
- \frac{\alpha}{(G_0^{-1} - (\Lambda - p_F)/2\pi)^2} \cos \theta \hat{p} \cdot \hat{p}'
+ \frac{\alpha^2}{(G_0^{-1} - (\Lambda - p_F)/2\pi)^3} \cos^2 \theta \hat{p} \cdot \hat{p}'
- \frac{\alpha^3}{(G_0^{-1} - (\Lambda - p_F)/2\pi)^4} \cos^3 \theta \hat{p} \cdot \hat{p}'
+ \cdots.$$  

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Therefore, when we decide the quantum number $l$ (or $j$) of a BS amplitude, the components coupled with the effective interaction $\tilde{\Gamma}(p, p')$ are determined \textit{a priori}: From the form of the expansion given in Eq. (83), we find the fact that, 1st, 3rd, ..., terms in Eq. (83) couple only with components of even $l$, while 2nd, 4th, ..., terms in Eq. (83) couple with components of odd $l(j)$. It is an important fact that, the sign in each term in the expansion for $\tilde{\Gamma}$ alternates between plus and minus.

Based on the preparation given above, we obtain the BS equations for specific $l$ (or $j$) of several types of pairings. For the scalar:

$$\begin{align*}
1 &= \frac{1}{8\pi^2} \tilde{I} \int_{\mu - \Lambda/2}^{\mu + \Lambda/2} p' dp' \\
&\times \left( \frac{1}{2(|p'| - \mu)} \tanh \frac{\beta}{2}(|p'| - \mu) + \frac{1}{2(|p'| + \mu)} \tanh \frac{\beta}{2}(|p'| + \mu) \right). 
\end{align*}$$

(84)

For the 0th-component of vector:

$$\begin{align*}
1 &= \frac{1}{8\pi^2} \tilde{I} \int_{\mu - \Lambda/2}^{\mu + \Lambda/2} p' dp' \frac{1}{2\mu} \left( - \tanh \frac{\beta}{2}(|p'| - \mu) + \tanh \frac{\beta}{2}(|p'| + \mu) \right). 
\end{align*}$$

(85)

Here $\tilde{I}$ is determined as

$$\tilde{I} = \int_0^{2\pi} d\theta_1 \int_0^{2\pi} d\theta_2 \frac{\cos l_1 \cos l_2}{G_0^{-1} - (\Lambda - p_F)/2\pi + \alpha \cos \theta_{12}},$$

(86)

where $\theta_1 = \theta_p$, $\theta_2 = \theta_{p'}$ and $\theta_{12} = \theta_1 - \theta_2$. We have to choose $\Lambda$ in our BS equations (84) and (85) as the same value for the polarization in Eq.(74). By using the expansion (83), we find the fact that, in the angular integration of $\tilde{I}$, there are only negative contribution for $\tilde{I}$ ($\tilde{I} < 0$) of the scalar case, while there are only positive contribution for $\tilde{I}$ ($\tilde{I} > 0$) of the vector case. Therefore, from Eq.(84), we recognize that there is no solution in the scalar pairing. From the same reason given in the discussion for Eq. (62), we cannot find solution for the 0th-component of vector case. For the spatial-component of vector, we get

$$\begin{align*}
-1 &= \frac{1}{8\pi^2} \int_{\mu - \Lambda/2}^{\mu + \Lambda/2} p' dp' \int_0^{2\pi} d\theta_1 \int_0^{2\pi} d\theta_2 \\
&\times \frac{\cos j \theta_1 \cos j \theta_2}{G_0^{-1} - (\Lambda - p_F)/2\pi + \alpha \cos \theta_{12}} \\
&\times \frac{1}{2|p'|} \left( |p'|^2 \cos^2 \theta_2 - \mu |p'| \right) \tanh \frac{\beta}{2}(|p'| - \mu) - \frac{|p'|^2 \cos^2 \theta_2 + \mu |p'|}{|p'| + \mu} \tanh \frac{\beta}{2}(|p'| + \mu)). 
\end{align*}$$

(87)
In Eq. (87), the angular integration gives the negative quantity, but the integrand of momentum integration is also always negative with respect to the variation of $T$. Therefore, it is impossible to find $T_c$, and we have no solution in the spatial-component of vector. The summary of the results of the solutions of various types of our gap equations and BS equations are presented in table I. At least in our treatment, only the parity-violating scalar pairing in BCS-type attractive interaction can have nontrivial solution. At least in our treatment (BS-RPA), we could not find a Kohn-Luttinger effect in our model.

VII. CONCLUDING REMARKS

In this paper, we have investigated the theory of superconductivity in (2+1)-dimensional relativistic quantum field theory. After we have introduced the model Lagrangian, we have derived the (2+1)-dimensional relativistic Gor’kov equation for the pairing problem under an attractive interaction. We have performed the group-theoretical consideration of the BCS gap function. The characteristic aspects of the gap function were revealed. By using the Gor’kov formalism, we have derived the gap equations for several types of pairings, and have solved them numerically. We have understood the effect of the lower band (the negative energy states) in the two-band superconductivity. We also have examined the pairing problem under the repulsive interaction (the Kohn-Luttinger effect), by using the Bethe-Salpeter formalism.

Now, we discuss some remaining problems and/or further possible investigations in our theory.

It is interesting to perform the calculations of the response function or polarization function under the presence of electromagnetic field, because of the reason that those functions can become the Chern-Simons (CS) term or not. Goryo and Ishikawa discussed the induction of the CS term in (2+1)-dimensional nonrelativistic theory, with parity and time-reversal violating superconductors [55]. Hosotani showed the (2+1)-dimensional QED with the CS term dynamically generate a magnetic field [56]. Miransky et al. studied the fact
that an external magnetic field enhances a fermion dynamical mass, and this phenomenon is universal in any models of (2+1) and (3+1) dimensional field theories [57]. We suppose these studies should have an intrinsic relation. We have a plan to investigate these physics in our model in near future, and publish elsewhere.

The feedback effect [58] is also interesting in our theory. There is no attempt in the literature for studying the feedback effect of collective modes in a two-band model. By using the Green’s functions given in Sec. V, we can easily estimate and examine this effect.

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The kagome lattice is a three-band system, but it has cone-like dispersions similar to the case of the honeycomb lattice.

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FIGURES

FIG. 1. (a) The structure of the honeycomb lattice. A and B denote the two different sublattice sites. (b) A schematic figure of the band structure of the honeycomb lattice. By symmetry, the hexagonal two-dimensional Brillouin zone of it has two degeneracy points.

FIG. 2. A schematic figure of the relativistic dispersion in a two-dimensional system. \( \epsilon_F \) denotes the Fermi energy of the system.

FIG. 3. The \( G_0 \) dependence of the pairing gap at \( T = 0 \). We set \( \Lambda/2\mu = 1 \) and \( \mu = 1 \).

FIG. 4. The cutoff dependence of the pairing gap at \( T = 0 \). We set \( \mu = 1 \) and \( G_0 = 2.2\mu^{-1} \).

FIG. 5. The temperature dependence of the pairing gap. We set \( G_0 = 2.2\mu^{-1} \), \( \mu = 1 \) and \( \Lambda/2\mu = 1 \).

FIG. 6. (a) The diagrammatic representation for the two-particle Green’s function \( G^{(2)} \). \( \tilde{\Gamma} \) denotes the irreducible vertex part. (b) The diagram for the irreducible vertex part within the RPA. The solid points represent the bare vertices \( G_0 \).

FIG. 7. The \( q \)-dependence of the static polarization \( \Pi(0, q) \) under various temperature. Here we set \( \mu = 1 \) and \( \Lambda/2\mu = 1 \).

FIG. 8. The \( \theta \)-dependence of the static polarization \( \Pi(0, k_F, k_F, \cos \theta) \) under various temperature. Here we set \( \mu = 1 \) and \( \Lambda/2\mu = 1 \).
TABLE I. List of Solutions for various pairings.

| pairing symmetry         | Gor’kov-BCS ($G_0 > 0$) | Bethe-Salpeter-RPA ($G_0 < 0$) |
|-------------------------|--------------------------|--------------------------------|
| scalar                  | possible                 | no solution                    |
| 0th-component of vector | no solution              | no solution                    |
| spatial-component of vector | no solution             | no solution                    |
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first Brillouin zone

wave number

Fig. 1(b)
Fig. 2

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Fig. 3

The figure shows the gap $\text{Gap} [\mu]$ as a function of $G_0 [\mu^{-1}]$ for two cases: scalar and no sea. The red line represents the scalar case, and the blue dashed line represents the no sea case.
Fig. 4

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title: Superconductivity in (2+1)-dimensional Relativistic Quantum Field Theory
Fig. 5

Gap[$\mu$] vs. $T[\mu]$

- **scalar**
- **no sea**

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\[ \Gamma \approx G^{(2)} = \Gamma \approx G^{(2)} \]

**Fig. 6(a)**

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Fig. 6(b)
Fig. 7

T=0.1\mu
T=0.01\mu
T=0.001\mu
T=0.0001\mu
Fig. 8

The figure shows the dependence of $2\pi \Pi(0,k_F,\cos \theta) [\mu]$ on $\theta [\pi]$ for different temperatures $T = 0.1\mu$, $T = 0.01\mu$, $T = 0.001\mu$, and $T = 0.0001\mu$. The curves illustrate how the phase structure of superconductivity changes with varying temperature.