COHOMOLOGIES OF REGULAR LATTICES
OVER THE KLEINIAN 4-GROUP

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Abstract. We calculate explicitly cohomologies of the lattices over the
Kleinian 4-group belonging to the regular components of the Auslander–
Reiten quiver as well as of their dual modules. We also give a canonical
form of cohomology classes under the action of automorphisms.
The result is applied to the classification of some crystallographic and
Chernikov groups.

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The aim of this paper is to apply the results on cohomologies of the
Kleinian 4-group [9] to the classification of crystallographic and Chernikov
groups. For this purpose it is important to have an explicit presentation of
2-cocycles. We find such presentation for a special class of lattices, called
regular, and for their dual modules. Moreover, we describe the orbits of
the action of automorphisms of modules on cohomologies. From this results
we obtain a complete description of crystallographic and Chernikov groups
with the Kleinian top and regular base.

Key words and phrases. Kleinian 4-group, regular lattices, cohomology, action of au-
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1. Lattices over the Kleinian 4-group.

In what follows $K$ denotes the Kleinian 4-group, $K = \langle a, b \mid a^2 = b^2 = 1, ab = ba \rangle$. We study cohomologies of this group with the values in $K$-lattices, i.e. $K$-modules $M$ such that the additive group of $M$ is free abelian of finite rank, and in their duals, i.e. the modules $\text{Hom}_Z(M, \mathbb{Q}/\mathbb{Z})$. Let $R = \mathbb{Z}K$. We embed it into $R^2 = \mathbb{Z}^4$ identifying $a$ with the quadruple $(1,1,-1,-1)$ and $b$ with $(1,-1,1,-1)$. Note that $R^2$ is the integral closure of $R$ in $\mathbb{Q} \otimes \mathbb{Z}R$. Let $\mathbb{Z}_p$ be the ring of $p$-adic integers, $M_p = M \otimes \mathbb{Z}_p$ for every abelian group $M$. Then $R_p = \mathbb{Z}_p^4$ for $p \neq 2$ and $R_2 \supseteq 4\mathbb{Z}_2^4$. It follows from [8, Th. 3.7] that two $K$-lattices $M, N$ are isomorphic if and only if they are in the same genus, i.e. $M_p \simeq N_p$ for all $p$. Moreover, if $p \neq 2$, the $R_p$-lattice $M_p$ is uniquely defined by the rational envelope $\mathbb{Q} \otimes \mathbb{Z}M$. Therefore, a $K$-lattice $M$ is uniquely determined by its $2$-adic completion, which we denote by $\hat{M}$. We denote by $R$-lat the category of $R$-lattices and by $\hat{R}$-lat the category of $\hat{R}$-lattices, i.e. $\hat{R}$-modules which are finitely generated and torsion free (hence free) as $\mathbb{Z}_2$-modules. The functor $M \mapsto \hat{M}$ is a representation equivalence between the categories $R$-lat and $\hat{R}$-lat, i.e. it maps non-isomorphic modules to non-isomorphic, indecomposable to indecomposable and every $\hat{R}$-lattice is isomorphic to $\hat{M}$ for a uniquely defined $R$-lattice $M$.

Since $4\hat{H}^n(K, M) = 0$ for any $K$-module $M$ [11 Prop. XII.2.5], $\hat{H}^n(K, M) \simeq \check{H}^n(K, \hat{M})$. Let $\mathbb{D} = \mathbb{Q}_2/\mathbb{Z}_2$, where $\mathbb{Q}_2$ is the field of $p$-adic numbers. It is the group of type $2\mathbb{N}$, i.e. the direct limit $\varprojlim_n \mathbb{Z}/2^n\mathbb{Z}$ of finite cyclic $2$-groups with respect to the natural embeddings $\mathbb{Z}/2^n\mathbb{Z} \twoheadrightarrow \mathbb{Z}/2^{n+1}\mathbb{Z}$. We call $K$-modules of the form $\text{Hom}_\mathbb{Z}(M, \mathbb{D}) \simeq \text{Hom}_\mathbb{Z}(\hat{M}, \mathbb{D})$, where $M$ is a $K$-lattice, $K$-colattices.

The ring $R$ is Gorenstein, i.e. $\text{inj.dim}_R R = 1$. Since $R_p$ is a maximal order for $p \neq 2$ and $R_2$ is local, [13 Lem. 2.9] implies that $R$ has a unique minimal proper overring $\Lambda$ and every indecomposable $R$-lattice, except $R$ itself, is an $\Lambda$-lattice. Actually, $\Lambda$ coincides with the subring of $R^2 = \mathbb{Z}^4$ consisting of all quadruples $(z_1, z_2, z_3, z_4)$ such that $z_1 \equiv z_2 \equiv z_3 \equiv z_4 \equiv 0 \pmod{2}$. Let $\mathfrak{m}$ be the ideal of $\Lambda$ consisting of all quadruples $(z_1, z_2, z_3, z_4)$ such that $z_1 \equiv z_2 \equiv z_3 \equiv z_4 \equiv 0 \pmod{2}$. Then $\mathfrak{m} = \text{rad } \Lambda = \text{rad } \hat{R}^2$. So $\Lambda$ is a Backström order in the sense of [11]. Therefore, according to [11], $\hat{\Lambda}$-lattices, hence also $\Lambda$-lattices, are classified by the representations of the quiver

\[ \Lambda = \bullet \]

\[ \begin{array}{c}
\uparrow \\
+ \\
\downarrow \\
- \\
\end{array} \]

Recall the corresponding construction (on the level of $\Lambda$-lattices). For any $K$-module $M$ let $M_{\alpha\beta}$, where $\alpha, \beta \in \{+, -\}$, be the submodule $\{ u \in M \mid$
au = αu, bu = βu}. If M is an A-lattice, set \( M^\sharp = R^\sharp M \) (the product of lattices inside \( \mathbb{Q} \otimes \mathbb{Z} M \), which coincides with the image of the natural map \( R^\sharp \otimes_A M \to \mathbb{Q} \otimes \mathbb{Z} M \)). It is the smallest \( R^\sharp \)-module containing M. Then \( M^\sharp = \bigoplus_{\alpha\beta} M^\sharp_{\alpha\beta} \). Let \( V_\alpha = M/\mathfrak{m}M \) and \( V_{\alpha\beta} = M^\sharp_{\alpha\beta}/2M^\sharp_{\alpha\beta} \). Taking for \( f_{\alpha\beta} \) the natural maps \( V_\alpha \to V_{\alpha\beta} \), we obtain a representation of the quiver \( \Lambda \), which we denote by \( \Phi(V) \). Thus we define a functor \( \Phi \) from the category \( A\)-lat of \( A \)-lattices to the category rep \( \Lambda \) of representations of the quiver \( \Lambda \) over the field \( \mathbb{k} = \mathbb{Z}/2\mathbb{Z} \). We have the following result analogous to that of [11].

**Theorem 1.1.** Let \( \mathcal{R} \) be the category of representations

\[
\begin{array}{c}
\text{V} \\
\begin{array}{c}
\text{V}_+ \quad \text{V}_- \\
\text{f}_+ \quad \text{f}_- \\
\text{V}_+ \quad \text{V}_- \\
\end{array}
\end{array}
\]

of the quiver \( \Lambda \) over \( \mathbb{k} \) such that all maps \( f_{\alpha\beta} \) are surjective and the induced map \( f_{\oplus} : V_\bullet \to V_{\oplus} = \bigoplus_{\alpha\beta} V_{\alpha\beta} \) is injective. The functor \( \Phi \) is a representation equivalence \( A\)-lat \( \to \mathcal{R} \) such that all induced maps \( \Phi_{M,N} : \text{Hom}_A(M,N) \to \text{Hom}_A(\Phi(M),\Phi(N)) \) are surjective and

\[
\text{Ker} \Phi_{M,N} = \text{Hom}_A(M,\mathfrak{m}N) = 2\text{Hom}_{R^\sharp}(M^\sharp,N^\sharp).
\]

**Proof.** Obviously, always \( \Phi(M) \in \mathcal{R} \). Let \( V \in \mathcal{R} \), \( d_{\alpha\beta} = \text{dim} V_{\alpha\beta} \) and \( d_\bullet = \text{dim} V_\bullet \). Denote by \( \mathbb{Z}_{\alpha\beta} \) the \( K \)-module \( \mathbb{Z} \), where \( a \) acts as \( \alpha 1 \) and \( b \) acts as \( \beta 1 \). Thus \( R^\sharp = \bigoplus_{\alpha\beta} \mathbb{Z}_{\alpha\beta} \). Set \( M^\sharp = \bigoplus_{\alpha\beta} \mathbb{Z}_{\alpha\beta}^d \) and define \( M(V) \) as the preimage of \( \text{Im} f_{\oplus} \) in \( M^\sharp \) under the epimorphism \( M^\sharp \to M^\sharp/2M^\sharp \simeq V_{\oplus} \). Then \( M(V) \) is an \( A \)-lattice such that \( \Phi(M(V)) \simeq V \) and \( M(V)^\sharp = M^\sharp \). It is also evident that \( M(\Phi(M)) \simeq M \). Hence \( \Phi \) is a representation equivalence.

A morphism \( \phi : V \to V' \), where

\[
\begin{array}{c}
\text{V}' \\
\begin{array}{c}
\text{V}'_+ \quad \text{V}'_- \\
\text{f}'_+ \quad \text{f}'_- \\
\text{V}'_+ \quad \text{V}'_- \\
\end{array}
\end{array}
\]
is given by a quintuple of homomorphisms \( \{ \phi_\alpha f_\ast \} \) for all \( \alpha, \beta \). If \( V = \Phi(M) \) and \( V' = \Phi(N) \), these homomorphisms give a homomorphism \( \tilde{\phi} : M^2/2M^2 \to N^2/2N^2 \) such that \( \tilde{\phi} f_{\alpha} = f'_{\alpha} \tilde{\phi} \). If we lift \( \tilde{\phi} \) to a homomorphism \( \psi^2 : M^2 \to N^2 \), it implies that \( \psi^2(M) \subseteq N \), so we obtain a homomorphism \( \psi : M \to N \) such that \( \Phi(\psi) = \phi \). Obviously, \( \Phi(\psi) = 0 \) if and only if \( \text{Im} \psi \subseteq \mathfrak{m}N = 2N^2 \). \( \square \)

We call the quintuple \((d_\ast, d_{\alpha\beta})\) (\( \alpha, \beta \in \{+, -\} \)) the \textit{dimension} of the representation \( V \) or of the corresponding lattice \( M = M(V) \), denote it by \( \dim V \) or \( \dim M \) and usually present it in the form

\[
\begin{array}{c|c|c|c|c}
    & d_{++} & d_{+-} & d_{-+} & d_{--} \\
\hline
d_\ast & & & & \\
\end{array}
\]

We also denote, if necessary, \( d_\ast = d_\ast(M) \), \( d_{\alpha\beta} = d_{\alpha\beta}(M) \) and \( d_{\ominus} = d_{\ominus}(M) = \sum_{\alpha, \beta} d_{\alpha\beta}(M) \). Note that the rank of \( M \) as of \( \mathbb{Z} \)-module equals \( d_{\ominus}(M) \), while \( d_\ast = \dim \_\mathbb{Z}_M/\mathfrak{m}M \).

We also need analogues of some results from \([12]\). For this purpose we establish a lemma. For any \( \mathbb{Z} \)-lattice or \( \mathbb{R}^2 \)-lattice \( M \) we set \( \tilde{M} = M/2M \) and if \( \alpha : M \to N \) is a homomorphism of lattices, we denote by \( \tilde{\alpha} \) the induced map \( \tilde{M} \to \tilde{N} \).

**Lemma 1.2.** Any exact sequence \( 0 \to \tilde{M} \xrightarrow{\tilde{\alpha}} \tilde{N} \xrightarrow{\tilde{\beta}} \tilde{L} \to 0 \) of \( \mathbb{Z} \)-lattices or of \( \mathbb{R}^2 \)-lattices can be lifted to an exact sequence \( 0 \to M \xrightarrow{\alpha} N \xrightarrow{\beta} L \to 0 \).

**Proof.** As \( \mathbb{R}^2 = \mathbb{Z}^4 \), we only have to consider the case of \( \mathbb{Z} \)-lattices. We choose bases in \( M, N, L \) and the corresponding bases in \( \tilde{M}, \tilde{N}, \tilde{L} \) and identify \( \tilde{\alpha} \) and \( \tilde{\beta} \) with their matrices with respect to these bases. There are invertible matrices \( S, T \) of appropriate sizes over the field \( \mathbb{k} \) such that \( S^{-1} \tilde{\alpha} T = \left( \begin{array}{c} I \end{array} \right) \), where \( I \) is the unit matrix. Then \( \tilde{\beta} S = \left( \begin{array}{c} 0 & C \end{array} \right) \) for some invertible matrix \( C \). Since the maps \( \text{GL}(n, \mathbb{Z}) \to \text{GL}(n, \mathbb{k}) \) are surjective, we can lift \( \tilde{S}, \tilde{T}, \tilde{C} \) to invertible matrices \( S, T, C \) over \( \mathbb{Z} \). Then the homomorphisms \( M \to N \) and \( N \to L \) respectively given by the matrices \( \alpha = S \left( \begin{array}{c} I \end{array} \right) T^{-1} \) and \( \beta = \left( \begin{array}{c} 0 & C \end{array} \right) S^{-1} \) are the necessary liftings of \( \tilde{\alpha} \) and \( \tilde{\beta} \). \( \square \)

**Corollary 1.3.** Let \( 0 \to V' \xrightarrow{\alpha} V \xrightarrow{\beta} V'' \to 0 \) be an exact sequence of representations from \( \mathcal{R} \). It can be lifted to an exact sequence of \( A \)-lattices \( 0 \to M(V') \xrightarrow{\alpha} M(V) \xrightarrow{\beta} M(V'') \to 0 \).

\[1\]This result does not follow directly from \([12\text{, Lem.} 4]\), where the case of algebras over complete discrete valuation rings is considered. Moreover, it highly depends on Lemma 1.2 and hence on the “smallness” of the residue field \( \mathbb{k} \).
Proof. We denote \( \tilde{V} = \bigoplus_{\alpha \beta} V_{\alpha \beta} \). Then we have a commutative diagram

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & V' & \xrightarrow{\tilde{\alpha}} & \tilde{V} & \xrightarrow{\tilde{\beta}} & V'' & \longrightarrow & 0 \\
\downarrow f_{\nu}' & & \downarrow f_{\nu} & & \downarrow f_{\nu}'' & & \downarrow f_{\nu}'' & & \downarrow 0 \\
0 & \longrightarrow & \tilde{V}' & \xrightarrow{\tilde{\alpha}} & \tilde{V} & \xrightarrow{\tilde{\beta}} & \tilde{V}'' & \longrightarrow & 0
\end{array}
\]

By Lemma 1.2, the second row can be lifted to an exact sequence

\[
0 \rightarrow M(V')^\sharp \xrightarrow{\alpha^\sharp} M(V)^\sharp \xrightarrow{\beta^\sharp} M(V'')^\sharp \rightarrow 0.
\]

Recall that \( M(V) \supseteq 2M(V)^\sharp \) and \( M(V)/2M(V)^\sharp \simeq V \). Since the diagram (1.3) is commutative, it implies that \( \alpha^\sharp(M(V')) \subseteq M(V) \) and \( \beta^\sharp(M(V)) \subseteq M(V'') \). So we obtain the necessary lifting

\[
0 \rightarrow M(V') \xrightarrow{\alpha} M(V) \xrightarrow{\beta} M(V'') \rightarrow 0.
\]

It is exact by the 3 \times 3 lemma, \( \square \).

We say that a monomorphism of lattices \( \phi : M \rightarrow N \) is pure if \( \text{Coker } \phi \) is torsion free (hence, also a lattice).

Corollary 1.4. Every epimorphism (monomorphism, isomorphism) \( V \rightarrow V' \) of representations from \( \mathcal{R} \) can be lifted to an epimorphism (respectively, pure monomorphism, isomorphism) of \( A \)-lattices \( \Phi(V) \rightarrow \Phi(V') \).

Corollary 1.5. Given a chain of subrepresentations

\[
V = V_0 \supset V_1 \supset V_2 \supset \ldots \supset V_{m-1} \supset V_m = 0,
\]

there is a chain of sublattices in \( M = M(V) \)

\[
M = M_0 \supset M_1 \supset M_2 \supset \ldots \supset M_{m-1} \supset M_m = 0
\]

such that \( M_k \simeq M(V_k) \) and \( M_k/M_{k+l} \simeq M(V_k/V_{k+l}) \) for all possible values of \( k,l \).

2. Regular lattices

Recall the structure of the Auslander–Reiten quiver \( \mathcal{Q} \) of the category \( \hat{\mathcal{A}} \)-lat [10]. According to [12], it is obtained from the Auslander–Reiten quiver of the category rep \( \Lambda \) by gluing the preprojective and the preinjective components into one component. The resulting preprojective-preinjective component is shown at Figure 1. Here all lattices are uniquely determined by their dimensions. The Auslander–Reiten transpose \( \tau \) of the category \( \hat{\mathcal{A}} \)-lat acts on this component as the shift to the left. Note that, for every \( A \)-lattice \( M \), \( \tau \hat{M} \simeq \Omega \hat{M} \), the syzygy of \( \hat{M} \) as of \( \hat{R} \)-module [9, Prop.1.1]. Hence, \( \Omega M \simeq N \) if \( \tau \hat{M} = \hat{N} \).

The other components, called regular, are tubes. They are parametrized by the closed points of the projective line \( \mathbb{P}^1_k \) (considered as a scheme), which consist of monic irreducible polynomials from \( k[t] \) and the symbol \( \infty \). We denote the tube corresponding to the polynomial \( f(t) \) by \( T^f \) and the
tube corresponding to \( \infty \) by \( T^\infty \). We also write \( T^0 \) instead of \( T^t \) and \( T^1 \) instead of \( T^{t-1} \). When describing tubes, we substitute \( A \)-lattices for their completions and say that \( T \) belongs to a tube \( T^f \) if \( \tilde{T} \) belongs to this tube. Then we call \( T \) a regular \( K \)-lattice. We call a \( K \)-lattice \( M \) regular if all its indecomposable direct summands are regular.

All tubes except \( T^\lambda \) (\( \lambda \in \{0, 1, \infty\} \)) are homogeneous, i.e. \( \tau M \simeq M \) for all \( M \) from this tube. They are of the form

\[
T_1^f \quad T_2^f \quad T_3^f \quad \cdots
\]

(2.1)

\[
\dim T_m^f = \begin{bmatrix} d_m & 0 \\ d_m & d_m \\ 0 & d_m \\ I \end{bmatrix},
\]

where \( d = \deg f(t) \). Actually, \( T_m^f = M(T_m^f) \), where \( T_m^f \)
is the following representation of the quiver \( \Lambda \):

\[
\begin{array}{c}
\xrightarrow{(1 0)}
\xrightarrow{(0 1)}
\xrightarrow{(I 1)}
\xrightarrow{(I F)}
\end{array}
\]

Here \( I \) is the \( dm \times dm \) unit matrix and \( F \) is the Frobenius matrix with the characteristic polynomial \( f(t)^m \).

The tube \( T^\lambda \) for \( \lambda \in \{0, 1, \infty\} \) is of the form

\[
T_1^{\lambda_1} \quad T_2^{\lambda_1} \quad T_3^{\lambda_1} \quad T_4^{\lambda_1} \quad \cdots
\]

(2.3)

Here \( \tau T_n^{\lambda_1} \simeq T_n^{\lambda_2} \) and \( \tau T_n^{\lambda_2} \simeq T_n^{\lambda_1} \). For \( \lambda = 1 \) we have

\[
\dim T_{2m}^{1j} = \begin{bmatrix} m & m \\ m & m \\ 2m-1 & 2m-1 \\ m-1 & m-1 \\ m & m \end{bmatrix},
\]

for both \( j = 1 \) and \( j = 2 \),

\[
\dim T_{2m-1}^{11} = \begin{bmatrix} m & m \\ m & m \\ 2m-1 & 2m-1 \\ m-1 & m-1 \\ m & m \end{bmatrix},
\]

Actually, \( T_{2m}^{11} = M(T_{2m}^{11}) \), where \( T_{2m}^{11} \) is of the form (2.2) with \( d = 1 \) and \( F = J_1 \) being the Jordan \( m \times m \) matrix with eigenvalue 1. \( T_{2m-1}^{11} = M(T_{2m-1}^{11}) \),
where $T_{2m-1}^{11}$ is of the form

\[
\begin{align*}
\mathbb{K}^{2m-1} & \xrightarrow{f_1} \mathbb{K}^m \\
\mathbb{K}^m & \xrightarrow{f_2} \mathbb{K}^{2m-1} \\
\mathbb{K}^{2m-1} & \xrightarrow{f_3} \mathbb{K}^{m-1} \\
\mathbb{K}^{m-1} & \xrightarrow{f_4} \mathbb{K}^m
\end{align*}
\]

Here

\[
\begin{align*}
f_1 &= \begin{pmatrix} I & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \\
f_2 &= \begin{pmatrix} 0 & I & 0 \\ 0 & 0 & 1 \end{pmatrix}, \\
f_3 &= \begin{pmatrix} I & I & 0 \end{pmatrix}, \\
f_4 &= \begin{pmatrix} I & J_1 & e \end{pmatrix}, \text{ with } e = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix},
\end{align*}
\]

where the matrices $I$ and $J_1$ are of size $(m - 1) \times (m - 1)$.

The lattices $T_{n}^{12}$ are obtained from $T_{n}^{11}$ by the permutations, respectively, of $f_1$ with $f_3$ and of $f_2$ with $f_4$.

The tube $T^0 (T^\infty)$ is obtained from the tube $T^1$ by the permutation of $f_2$ with $f_4$ (respectively, of $f_2$ with $f_3$).

Note that an indecomposable lattice $M$ belongs to a tube if and only if

\[2d_\bullet(M) = \sum_{\alpha \beta} d_{\alpha \beta}(M). \tag{2.4}\]

In this case

\[d_\bullet(\Omega M) = d_\bullet(M) \text{ and } d_{\alpha \beta}(\Omega M) = d_\bullet(M) - d_{\alpha \beta}(M). \tag{2.5}\]

The structure of the representations of quivers belonging to tubes is described in [3, 4] (see also [5, Thm. 31 and 36]). Together with Corollary 1.5 it gives the following result for lattices. From now on, writing $T_m^f$, we always suppose that $f \notin \{t, t - 1\}$, and writing $T_m^{\lambda j}$, we always suppose that $\lambda \in \{0, 1, \infty\}$ and $\lambda \in \{1, 2\}$.

**Theorem 2.1.** Every module $T_m^f$ or $T_m^{\lambda j}$ has a chain of submodules

\[M = M_0 \supset M_1 \supset M_2 \supset \ldots \supset M_{m-1} \supset M_m = \{0\} \tag{2.6}\]

such that

\[M_k / M_{k+1} \simeq \begin{cases} 
T_m^f & \text{if } M = T_m^f, \\
T_m^{\lambda j} & \text{if } M = T_m^{\lambda j} \text{ and } k \text{ is even}, \\
T_m^{\lambda i} & \text{where } i \neq j, \text{ if } M = T_m^{\lambda j} \text{ and } k \text{ is odd}.
\end{cases}\]
(2) The maps \( T^f_m \to T^f_{m+1} \) and \( T^{λ_i}_m \to T^{λ_j}_{m+1} \) in the diagrams, respectively, and \( (2.1) \) and \( (2.2) \) can be chosen injective, with the quotients, respectively, \( T^f_1 \) and \( T^{λ_j}_1 \).

(3) The maps \( T^f_{m+1} \to T^f_m \) and \( T^{λ_i}_{m+1} \to T^{λ_j}_m \) \((i \neq j)\) in the diagrams, respectively, \( (2.1) \) and \( (2.3) \) can be chosen surjective, with the kernels, respectively, \( T^f_1 \) and \( T^{λ_i}_1 \).

(4) If \( M \) and \( M' \) belong to different tubes, then \( \text{Im} \varphi \subseteq 2N \) for every homomorphism \( \varphi : M \to N \).

One can also get a description of endomorphisms of indecomposable lattices belonging to tubes.

For any commutative ring \( K \) and any monic polynomial \( f(t) \) of degree \( d \) there is a natural embedding \( ε_m : K[t]/(f(t)^m) \to \text{Mat}(dm,K) \) arising from the regular representation of this quotient algebra over \( K \). We denote by \( K^f_m \) the image of the embedding \( K[t]/(f(t)^m) \to \text{Mat}(dm,K)^4 \) such that all its components are \( ε_m \). We also denote by \( K^f_m \) the image of the embedding \( K[t]/(t^{m}) \to \text{Mat}(m,K)^2 \times \text{Mat}(m-1,K)^2 \) such that the first two components are \( ε_m \) and the other two components are the compositions \( ε_{m-1} \pi_m \), where \( π_m \) is the surjection \( K[t]/(t^{m}) \to K[t]/(t^{m-1}) \). Certainly, if \( m = 1 \), it is just the diagonal embedding \( K \to K \times K \).

If \( M \) is an \( A \)-lattice and \( \text{dim} M = \begin{pmatrix} d_+ & d_- \\ d_+ & d_- \end{pmatrix} \), the endomorphism ring \( \text{End}_A M \) naturally embeds into \( \text{End}_R M^2 = \prod_{αβ} \text{Mat}(d_{αβ},\mathbb{Z}) \) and we identify it with the image of this embedding.

**Theorem 2.2.** For an irreducible monic polynomial \( f(t) \in \mathbb{k}[t] \) choose a monic polynomial \( \tilde{f}(t) \in \mathbb{Z}[t] \) such that \( f(t) = \tilde{f}(t) \) (mod 2).

1. \( \text{End}_A T^f_m = \mathbb{Z}^f_m + 2 \text{Mat}(dm,\mathbb{Z})^4 \), where \( d = \deg f(t) \).
2. \( \text{End}_A T^{λ_i}_m = \mathbb{Z}^{λ_i}_m + 2 \text{Mat}(m,\mathbb{Z})^4 \) and
\[ \text{End}_A T^{λ_j}_{2m-1} = \mathbb{Z}^{λ_j}_{2m-1} + 2 \left( \text{Mat}(m,K)^2 \times \text{Mat}(m-1,K)^2 \right) \).

Obviously, it does not depend on the choice of \( \tilde{f}(t) \).

**Proof.** It is well known (and can be verified by straightforward calculations) that \( \text{End}_A T^f_m = k^f_m \), while then \( \text{End}_A T^{λ_j}_m = k^{λ_j}_m \) and \( \text{End}_A T^{λ_j}_{2m-1} = k^{λ_j}_{2m-1} \).

It remains to apply Theorem \( [\square] \).

---

**3. Cohomologies**

We will give an explicit description of the cohomologies \( H^n(K,M) \) and \( H^n(K,DM) \) for \( n > 0 \) and regular \( K \)-lattices \( M \). Obviously, we only have to calculate them for indecomposable lattices.

It follows from \( [8] \) that a free resolution \( P \) for the trivial \( K \)-module \( \mathbb{Z} \) can be chosen as follows: \( P_n \) is the set of homogeneous polynomials of degree \( n \),
from $R[x,y]$ and

$$d(x^ky^l) = (a + (-1)^k)x^{k-1}y^l + (-1)^k(b + (-1)^l)x^ky^{l-1}.$$ 

So an $n$-cocycle $\gamma$ is given by the values $\gamma(x^my^{n-m})$ ($0 \leq m \leq n$).

Let $M$ be an indecomposable regular lattice. Set

$$M(n) = \begin{cases} M_{++} & \text{if } n \text{ is even}, \\ M_{--} & \text{if } n \text{ is odd and } \not\in T^{\infty}, \\ M_{+-} & \text{if } n \text{ is odd and } M \in T^{\infty}, \end{cases}$$

We define a homomorphism $\xi : M(n) \rightarrow H^n(K,M)$ ($n > 0$). It sends an element $v \in M(n)$ to the class of the cocycle $\xi_v$ which is defined as follows.

- If $n$ is even, $M \not\in T^{\infty}$ and $v \in M_{++},$
  $$\xi_v(x^m y^{n-m}) = \begin{cases} v & \text{if } m = n, \\ 0 & \text{otherwise}. \end{cases}$$

- If $n$ is even, $M \in T^{\infty}$ and $v \in M_{++},$
  $$\xi_v(x^m y^{n-m}) = \begin{cases} v & \text{if } m = 0, \\ 0 & \text{otherwise}. \end{cases}$$

- If $n$ is odd, $M \not\in T^{\infty}$ and $v \in M_{--},$
  $$\xi_v(x^m y^{n-m}) = \begin{cases} v & \text{if } m = n, \\ 0 & \text{otherwise}. \end{cases}$$

- If $n$ is odd, $M \in T^{\infty}$ and $v \in M_{+-},$
  $$\xi_v(x^m y^{n-m}) = \begin{cases} v & \text{if } m = 0, \\ 0 & \text{otherwise}. \end{cases}$$

One easily verifies that $\xi_v$ is indeed a cocycle.

**Theorem 3.1.** For every indecomposable regular $K$-lattice $M$ and every $n > 0$ the map $\xi$ induces an isomorphism $\bar{\xi} : M(n)/2M(n) \cong H^n(K,M)$.

For the proof we use the following results.

**Lemma 3.2.** If $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$ is an exact sequence of regular $K$-lattices, then the induced sequences

$$0 \rightarrow \Omega M' \rightarrow \Omega M \rightarrow \Omega M'' \rightarrow 0,$$

$$0 \rightarrow \Omega^{-1} M' \rightarrow \Omega^{-1} M \rightarrow \Omega^{-1} M'' \rightarrow 0$$

are also exact.

**Proof.** From the properties of syzygies it follows that there is an exact sequence

$$0 \rightarrow \Omega M' \rightarrow \Omega M \oplus P \rightarrow \Omega M'' \rightarrow 0$$
for some projective $R$-module $P$. But, as $\Omega M = \tau M$, the formulae (2.4) and (2.5) show that $d_{\alpha\beta}(\Omega M) = d_{\alpha\beta}(\Omega M') + d_{\alpha\beta}(\Omega M'')$. Therefore, $P = 0$ and we get the exact sequence (6.1). Then (5.2) follows by duality. \hfill $\Box$

**Corollary 3.3.** If $0 \to M' \to M \to M'' \to 0$ is an exact sequence of regular $K$-lattices, the induced sequence of cohomologies

$$0 \to \hat{H}^n(K, M') \to \hat{H}^n(K, M) \to \hat{H}^n(K, M'') \to 0$$

is also exact for every $n \in \mathbb{Z}$.

**Proof.** It is known [9, Lem. 2.2] that if $M$ contains no direct summands isomorphic to $\mathbb{Z}_{++}$, then $\hat{H}^0(K, M) \simeq \mathbb{k}^{d_{++}(M)}$. It implies the claim for $n = 0$. The general case follows from Lemma 3.2 and the known fact that $\hat{H}^n(K, M) \simeq \hat{H}^{n+1}(K, \Omega M) \simeq \hat{H}^{n-1}(K, \Omega^{-1} M)$. \hfill $\Box$

**Proof of Theorem 3.1.** Note that [9, Th. 2.3] shows that $M(n)/2M(n) \simeq H^n(K, M)$. Hence we only have to check that $\xi$ is injective. First, we check the claim for the lattices $T^j_1$ and $T^{\lambda j}_1$. As the calculations are quite similar, we only consider the case of $M = T^1_{j1}$ and $n$ even (it seems the most complicated). Then $M$ is the submodule of $\mathbb{Z}_{++} \oplus \mathbb{Z}_{--}$ consisting of the pairs $(z, z')$ such that $z \equiv z' \pmod{2}$. Thus the basic element of $M(n)$ is $v = (2, 0)$. We have to check that $\xi_v \neq \partial \gamma$ for any map $\gamma : \mathbb{P}_{n-1} \to M$. Suppose that $\xi_v = \partial \gamma$. Note that if $\gamma(x^{n-1}) = (z, z')$, then $\partial \gamma(x^n) = (2z, 0)$, whence $z = 1$. Let $\gamma(x^{n-k-1}y^k) = (z_k, z'_{k+1})$ (0 < $k$ < $n$). Then

$$\partial \gamma(x^{n-1}y) = (a-1)(z_1, z'_1) - (b-1)(z, z') = (0, -2z' + 2z'_1) = (0, 0),$$

hence $z'_1 = z' \equiv 1 \pmod{2}$;

$$\partial \gamma(x^{n-2}y^2) = (a+1)(z_2, z'_2) + (b+1)(z_1, z'_1) = (2z_2 + 2z_1, 0) = (0, 0),$$

hence $z_2 = -z_1 \equiv 1 \pmod{2}$. Repeating this process, we obtain that all $z_k \equiv 1 \pmod{2}$, so $z_k \neq 0$. Then $\partial \gamma(y^n) = (2z_{n-1}, 0) \neq 0 = \xi_v(y^n)$ and we get a contradiction.

For the lattice $M = T^j_m$ or $T^{\lambda j}_m$ ($m > 1$) we have an exact sequence $0 \to M' \to M \to M'' \to 0$, where $M' \simeq T^j_1$ or $T^{\lambda j}_1$ and $M'' \simeq T^j_{m-1}$ or $T^{\lambda j}_{m-1}$. It gives a commutative diagram with exact rows

$$\begin{array}{cccccc}
0 & \rightarrow & M'(n) & \rightarrow & M(n) & \rightarrow & M''(n) & \rightarrow & 0 \\
& & \uparrow \xi & & \uparrow \xi & & \uparrow \xi & \\
0 & \rightarrow & H^n(K, M') & \rightarrow & H^n(K, M) & \rightarrow & H^n(K, M'') & \rightarrow & 0
\end{array}$$

Using induction, we can suppose that the first and the third vertical maps satisfy the assertion of the theorem. Then the same is true for the second vertical map. \hfill $\Box$
Analogous considerations give an explicit description of the cohomologies for regular colattices, i.e. the dual modules of regular lattices. For an indecomposable regular colattice \( N = DM \) set \( \bar{N} = \{ u \in N \mid 2u = 0 \} \) and

\[
N(n) = \begin{cases} 
\bar{N}^+ & \text{if } n \text{ is odd} \\
\bar{N}^- & \text{if } n \text{ is even and } M \notin T^\infty, \\
\bar{N}^+ & \text{if } n \text{ is even and } M \in T^\infty.
\end{cases}
\]

We define a homomorphism \( \eta : N(n) \to H^n(K, N) \) \((n > 0)\). It sends an element \( u \in N(n) \) to the class of the cocycle \( \eta u \) which is defined as follows.

- If \( n \) is even, \( M \notin T^\infty \) and \( u \in \bar{N}^- \),
  
  \[
  \eta_u(x^m y^{n-m}) = \begin{cases} 
u & \text{if } m = n, \\
 0 & \text{otherwise.}
  \end{cases}
  \]

- If \( n \) is even, \( M \in T^\infty \) and \( u \in \bar{N}^+ \),
  
  \[
  \eta_u(x^m y^{n-m}) = \begin{cases} 
u & \text{if } m = 0, \\
 0 & \text{otherwise.}
  \end{cases}
  \]

- If \( n \) is odd, \( u \in \bar{N}^+ \),
  
  \[
  \eta_u(x^m y^{n-m}) = \begin{cases} 
u & \text{if } m = n, \\
 0 & \text{otherwise.}
  \end{cases}
  \]

One easily verifies that \( \eta_u \) is indeed a cocycle.

**Theorem 3.4.** For every indecomposable regular colattice \( N \) and every \( n > 0 \) the map \( \eta \) induces an isomorphism \( N(n) \to H^n(K, N) \).

We omit the proof since it is quite analogous to that of Theorem 3.1 (and even easier).

### 4. Action of Automorphisms

#### 4.1. Lattices

We also need to know how automorphisms of lattices and of the group act on cohomologies. Let \( M = T^f_m \) or \( M = T^\lambda_j m \). Consider the chain of submodules \( M_k \subset M \) from Theorem 2.1. We denote by \( E^f_{k,m,n} \) or, respectively, by \( E^\lambda_{k,m,n} \), where \( 0 \leq k < m \), the set \( M_k(n) \setminus \{2M_k(n) + M_{k+1}(n)\} \). Note that \( E^\lambda_{k,m,n} \neq \emptyset \) if and only if \( k < m \) and

\[
k \equiv \begin{cases} 
j & \text{if } n \text{ is even} \\
 j + 1 & \text{if } n \text{ is odd (mod 2)}
\end{cases}
\]

Theorems 2.1 and 2.2 easily imply the following result.

**Theorem 4.1.** Let \( e \in E^f_{k,m,n} \) or \( e \in E^\lambda_{k,m,n} \), and \( e' \in E^f_{k',m',n} \) or, respectively, \( e' \in E^\lambda_{k',m',n} \). There is a homomorphism \( \theta : T^f_m \to T^f_{m'} \) or, respectively, \( \theta : T^\lambda_m \to T^\lambda_{m'} \) such that \( \theta(e) = e' \) if and only if either \( m \geq m' \) and
\(k \leq k'\) or \(m \leq m'\) and \(k \leq k' - m' + m\). If \(m = m'\) and \(k = k'\), \(\theta\) can be chosen as an isomorphism.

**Definition 4.2.**

1. We fix for every quadruple \((f, m, k, n)\), where \(k < m\), an element \(e^f_{m,k,n} \in E^f_{m,k,n}\) and for every quintuple \((\lambda, j, m, k, n)\), where \(k < m\) and \(k, j\) satisfy the condition \([\lambda]\), an element \(e^\lambda_j_{m,k,n} \in E^\lambda_{m,k,n}\).

2. For a homogeneous tube \(T^f\) we call a standard sequence a sequence \(\sigma = (m_i, k_i) (1 \leq i \leq s)\), where \(m_1 > m_2 > \cdots > m_s\), \(1 \leq k_i < m_i\) and \(k_i < k_i - m_i + m_{i'}\) for \(i < i'\). We set \(M_\sigma^f = \bigoplus_{i=1}^s T^f_{m_i}\) and \(e^\lambda_{\sigma,n} = \sum_{i=1}^s e^\lambda_j_{m_i,k_i,n}\).

3. For a special tube \(T^\lambda\) we call a standard sequence a sequence \(\sigma = (j_i, m_i, k_i) (1 \leq i \leq s)\), where \(j_i \in \{1, 2\}\), \(m_1 > m_2 > \cdots > m_s\), \(1 \leq k_i < m_i\) and \(k_i < k_i + m_i - m_{i'}\) for \(i < i'\). We set \(M_\sigma^\lambda = \bigoplus_{i=1}^s T^\lambda_{j_i}\). We call such sequence
   - even if \(k_i \equiv j_i (\text{mod } 2)\) for all \(i\),
   - odd if \(k_i \equiv j_i + 1 (\text{mod } 2)\) for all \(i\).

   For an even (odd) standard sequence and even (respectively, odd) \(n\) we set \(e^\lambda_{\sigma,n} = \sum_{i=1}^s e^\lambda_j_{m_i,k_i,n}\).

4. We define standard data as a pair \(\Delta = (\Sigma, S)\), where \(\Sigma = \{T^f_q\} 1 \leq q \leq r\) is a set of different tubes and \(S = \{\sigma_q\} (1 \leq q \leq r)\) is a set of standard sequences \(\sigma_q\) for each tube \(T^f_q\). We call such data special if at least one of the tubes \(T^f_q\) is special. Special standard data are said to be even or odd if all standard sequences \(\sigma_q\) for special tubes \(T^f_q\) are so. We set \(M_\Delta = \bigoplus_{q=1}^r M_{\sigma_q}^f\) and \(e_{\Delta,n} = \sum_{q=1}^r e_{\sigma_q,n}^f\). In the latter definition we suppose that, if \(\Delta\) is special, it is even if \(n\) is even and it is odd if \(n\) is odd (otherwise the element \(e_{\Delta,n}\) is not defined).

   If the tube \(T^\infty\) occurs in \(\Sigma\), say \(f_k = \infty\), we denote by \(e^\infty_{\Delta,n} = e^\infty_{\sigma_k,n}\) and by \(e^0_{\Delta,n}\) the rest of the sum defining \(e_{\Delta,n}\). Of course, it is possible that \(e^\infty_{\Delta,n} = 0\) or \(e^0_{\Delta,n} = 0\).

Theorem [4.1] implies the following result.

**Theorem 4.3.** Let \(M\) be a regular \(R\)-lattice and \(\varepsilon \in H^n(K, M) (n > 0)\). There are standard data \(\Delta\) and a decomposition \(\theta : M \rightarrow M_0 \oplus M_\Delta\) such that the projection of \(\theta(\varepsilon)\) onto \(H^n(K, M_0)\) is zero and the projection of \(\theta(\varepsilon)\) onto \(H^n(K, M_\Delta)\) equals \(\xi(e_{\Delta,n})\) (see page [11] for the definition of \(\xi\)).

If \(\varepsilon = 0\), \(M_\Delta = 0\).

In particular, we obtain a description of orbits of automorphisms of indecomposable regular lattices on cohomologies.
Corollary 4.4. Let $M$ be an indecomposable regular lattice. Consider the chain (2.6) of its submodules and denote by $H^n_k(K, M)$ the image in $H^n(K, M)$ of $H^n(K, M_k)$. Then the orbits of $\text{Aut}_K M$ on $H^n(K, M)$ ($n > 0$) are $H^n_k(K, M) \setminus H^n_{k+1}(K, M)$ ($0 \leq k < m$) and $\{0\}$.

The group of automorphisms of the group $K$ is the symmetric group $S_3$; it just permutes the elements $a, b$ and $c = ab$. Its generators are the transpositions $\tau_2 : a \leftrightarrow b$ and $\tau_3 : a \leftrightarrow c$. They permute the $-+$ component of the diagram (1.1), respectively, with the $--$ component and with the $-+$ component. Thus $\tau_2$ permutes $\mathcal{T}^1$ and $\mathcal{T}^0$, while $\tau_3$ permutes $\mathcal{T}^1$ and $\mathcal{T}^\infty$.

Rather simple matrix calculations show that $\tau_2$ permutes $\mathcal{T}^f$ with $\mathcal{T}^{f(2)}$, while $\tau_3$ permutes $\mathcal{T}^f$ with $\mathcal{T}^{f(3)}$, where

$$f^{(2)}(t) = f(1)^{-1}(t - 1)^d f \left( \frac{t}{t - 1} \right),$$

$$f^{(3)}(t) = (-1)^d f(1 - t),$$

where $d = \text{deg } f$. It induces the action of $S_3$ on the set of standard data. Note that, if $\psi \in \text{Aut } K$, there is an automorphism $\varphi \in \text{Aut } M_{\Delta}$ such that $\psi \xi(e_{\Delta, n}) = \varphi \xi(e_{\Delta, n})$.

4.2. Colattices. Let now $N = DT$, where $M$ is a $K$-lattice. If $M$ is regular, we call $N$ regular too. If $N = DT$, where $M = T_m^f$ or $M = T_m^{\lambda_j}$, there is a chain of submodules, dual to the chain (2.6) from Theorem 2.1:

$$0 = N_0 \subset N_1 \subset N_2 \subset \ldots \subset N_{m - 1} \subset N_m = N,$$

where $N_k = M^\perp_k$ and

$$N_{k+1}/N_k \simeq \begin{cases} DT_m^f, & \text{if } N = DT_m^f, \\ DT_m^{\lambda_j}, & \text{if } N = DT_m^{\lambda_j} \text{ and } k \text{ is even}, \\ DT_m^{\lambda_i}, & \text{where } i \neq j, \text{if } N = DT_m^{\lambda_j} \text{ and } k \text{ is odd}. \end{cases}$$

We denote by $Z_{k, m, n}^f$ or, respectively, by $Z_{k, m, n}^{\lambda_j}$ the set $N_k \setminus N_k$. Again, $Z_{k, m, n}^{\lambda_j} \neq \emptyset$ if and only if $k < m$ and the condition (4.1) holds.

The duality gives analogues of Theorems (4.1) and (4.3).

**Theorem 4.5.** Let $z \in Z_{k, m, n}^f$ or $z \in Z_{k, m, n}^{\lambda_j}$ and $z' \in Z_{k', m', n}^f$ or, respectively, $z' \in Z_{k', m', n}^{\lambda_j'}$ and $z' \in Z_{k', m', n}^{\lambda_j'}$. There is a homomorphism $\theta : DT_m^f \to DT_{m'}^f$, respectively, $\theta : DT_m^{\lambda_j} \to DT_{m'}^{\lambda_j}$ such that $\theta(z) = z'$ if and only if either $m \leq m'$ and $k \geq k'$ or $m \geq m'$ and $k \geq k' - m + m$. If $m = m'$ and $k = k'$, $\theta$ can be chosen as an isomorphism.

**Definition 4.6.** (1) We fix for every quadruple $(f, m, k, n)$, where $k < m$, an element $z_{m, k}^f \in Z_{m, k}^f$ and for every quintuple $(\lambda, j, m, k, n)$, where $k < m$ and $k, j$ satisfy the condition (4.1), an element $z_{m, k}^{\lambda_j} \in Z_{m, k}^{\lambda_j}$. 
(2) For a homogeneous tube $T^f$ we call a costandard sequence a sequence $\sigma = (m_i, k_i) \ (1 \leq i \leq s)$, where $m_1 < m_2 < \cdots < m_s$, $1 \leq k_i < m_i$ and $k_{i'} > k_i > k_{i'} + m_i - m_{i'}$ for $i < i'$. We set $N^f_\sigma = \bigoplus_{i=1}^s T^f_{m_i}$ and $z^f_{\sigma,n} = \sum_{i=1}^s e^f_{m_i,k_i,n}$.

(3) For a special tube $T^\lambda$ we call a costandard sequence a sequence $\sigma = (j_i, m_i, k_i) \ (1 \leq i \leq s)$, where $j_i \in \{1, 2\}$, $m_1 < m_2 < \cdots < m_s$, $1 \leq k_i < m_i$, $1 \leq k_i < m_i$ and $k_i < k_{i'} < k_i + m_i - m_{i'}$ for $i' < i$. We set $N^\lambda_\sigma = \bigoplus_{i=1}^s T^\lambda_{m_i,j_i}$. We call such sequence

- even if $k_i \equiv j_i \pmod{2}$ for all $i$,
- odd if $k_i \equiv j_i + 1 \pmod{2}$ for all $i$.

For an even (odd) costandard sequence and even (respectively, odd) $n$ we set $z^\lambda_{\sigma,n} = \sum_{i=1}^s z^\lambda_{m_i,k_i,n}$.

(4) We define costandard data as a pair $\Delta = (\Sigma, S)$, where $\Sigma = \{T^f_q\ (1 \leq q \leq r)\}$ is a set of different tubes and $S = \{\sigma_q\} \ (1 \leq q \leq r)$ is a set of costandard sequences $\sigma_q$ for each tube $T^f_q$. We call such data special if at least one of the tubes $T^f_q$ is special. Special costandard data are said to be even or odd if all costandard sequences $\sigma_q$ for special tubes $T^f_q$ are so. We set $N^\Delta = \bigoplus_{q=1}^r M^f_q$ and $z^\Delta_{n} = \sum_{q=1}^s z^f_{\sigma_q,n}$. In the latter definition we suppose that, if $\Delta$ is special, it is even if $n$ is even and it is odd if $n$ is odd (otherwise the elements $z^f_{\sigma_q,n}$ and hence $z^\Delta_n$ are not defined).

If the tube $T^\infty$ occurs in $\Sigma$, say, $f_k = \infty$, we denote by $z^\infty_{\sigma,n} = z^\infty_{\sigma_k,n}$ and by $z^0_{\Delta,n}$, the rest of the sum defining $z^\Delta_n$. Of course, it is possible that $z^\infty_{\sigma,n} = 0$ or $z^0_{\Delta,n} = 0$.

**Theorem 4.7.** Let $N = DM$, where $M$ is a regular $R$-lattice, $\varepsilon \in H^n(K, N)$ ($n > 0$). There are costandard data $\Delta$ and a decomposition $\theta : N \rightarrow N_0 \oplus N_\Delta$ such that the projection of $\theta(\varepsilon)$ onto $H^n(K, N_0)$ is zero and the projection of $\theta(\varepsilon)$ onto $H^n(K, N_\Delta)$ equals $\eta(z_{\Delta,n})$ (see page 12 for the definition of $\eta$).

If $\varepsilon = 0$, $M_\Delta = 0$.

**Corollary 4.8.** Let $N$ be an indecomposable regular colattice. Consider the chain of its submodules and denote by $H^p_k(K, N)$ the image in $H^n(K, N)$ of $H^n(K, N_k)$. Then the orbits of $\text{Aut}_K N$ on $H^n(K, N)$ ($n > 0$) are $H^p_k(K, N) \setminus H^p_{k-1}(K, N)$ ($0 < k \leq m$) and $\{0\}$.

5. Applications

5.1. Crystallographic groups. Recall that a crystallographic group $G$ is a discontinuous group of isometries of an Euclidean space having a compact fundamental domain [13]. Equivalently, $G$ contains a maximal commutative subgroup $M$ of finite index, which is normal and is a free abelian group of finite rank. Then the group $\Gamma = \Gamma/M$ acts on $M$ by the rule $gv = \bar{g}vg^{-1}$,
where $\bar{g}$ is a preimage of $g$ in $G$, and $G$ is given by a class $\varepsilon \in H^2(\Gamma, M)$. One easily sees that actually $M$ is a unique maximal abelian subgroup of $G$ of finite index. We call the group $\Gamma$ the top and the $\Gamma$-module $M$ the base of the crystallographic group $G$. If $\varphi : G \rightarrow G'$, where $G'$ is another crystallographic group, then $M' = \varphi(M)$ is the maximal commutative subgroup of $G'$, i.e. the base of $G'$. Hence $\Gamma' = G'/M'$ is the top of $G'$ and we have a commutative diagram

$$
\begin{array}{ccccccc}
1 & \longrightarrow & M & \longrightarrow & G & \longrightarrow & \Gamma & \longrightarrow & 1 \\
\theta & & \varphi & & \psi & & \phi & & \psi \\
1 & \longrightarrow & M' & \longrightarrow & G' & \longrightarrow & \Gamma' & \longrightarrow & 1
\end{array}
$$

where $\theta$ and $\psi$ are isomorphisms. Let $\psi M'$ be the group $M'$ considered as $G$-module by the rule $g u = \psi(g) u$ for $g \in G$, $u \in M'$. Then $\theta$ is an isomorphism $M \cong \psi M'$ and the cohomology class defining the group $G'$ is $\theta \varepsilon \psi^{-1}$. Therefore, isomorphism classes of crystallographic groups with the top $\Gamma$ and the base $M$ are in one-to-one correspondence with the orbits of the action of the group $\text{Aut}_G \times \text{Aut}_M$ on $H^2(G, M)$. Therefore, Theorem 4.3 implies a classification result for crystallographic groups with the Kleinian top and regular base.

**Definition 5.1.** Let $\Delta$ be standard data, even if they are special. We call the group $\text{Cr}(\Delta)$ that is the extension of $K$ with the kernel $M$ corresponding to the cohomology class $\varepsilon = \xi(e^{\Delta})$ a standard crystallographic group.

Note that $\text{Cr}(\Delta)$ is generated by the group $M$ and two elements $\bar{a}$ and $\bar{b}$ subject to the relations

$$
\bar{a} w = (a) w \bar{a} \text{ for every } w \in M,
\bar{b} w = (b) w \bar{b} \text{ for every } w \in M,
\bar{a} \bar{b} = \bar{b} \bar{a},
\bar{a}^2 = e^0_{\Delta_2},
\bar{b}^2 = e^\infty_\Delta.
$$

**Theorem 5.2.** Let $G$ be a crystallographic group with the Kleinian top $K$ such that its base $M$ is a regular $K$-lattice.

1. There are standard data $\Delta$, a direct decomposition $M \simeq M_0 \oplus M_\Delta$ and a semidirect decomposition $G \simeq M_0 \times \text{Cr}(\Delta)$, where $\text{Cr}(\Delta)$ acts on $M_0$ as its quotient $\text{Cr}(\Delta)/M_\Delta \simeq K$.

2. If $G \simeq M'_0 \times \text{Cr}(\Delta')$ is another such decomposition, there is an automorphism $\psi$ of the group $K$ such that $M'_0 \simeq \psi M_0$ and $\Delta' = \psi \Delta$.

**Remark 5.3.** $G$ is crystallographic if and only if $M_{\alpha\beta} \neq 0$ for at least two of the pairs $(+-), (-+), (--).$ For a regular $K$-lattice $M$ it means that it is not a multiple of some lattice $T^\lambda_1$ ($\lambda \in \{0,1,\infty\}$).
5.2. Chernikov groups. Recall that a Chernikov group is a locally finite group with minimality condition on subgroups [2]. Such a group $G$ has a maximal divisible subgroup $N$ which is a finite direct sum of quasicyclic groups and $N$ is normal in $G$ with the finite quotient $\Gamma = G/N$. We consider the case when $G$ is a 2-group. Then $N$ is a direct sum of groups $\mathbb{D}$ of type $2^\infty$ and $\Gamma$ is a finite 2-group. It is known that $\text{End} \mathbb{D} \simeq \mathbb{Z}_2$. Therefore, if $N = \mathbb{D}^d$, then $\text{Aut}_\mathbb{Z} N \simeq \text{GL}(d, \mathbb{Z}_2)$. Hence $N \simeq DM$ for some $\Gamma$-lattice $M$. The group $\Gamma$ and the $G$-module $N$ are defined up to an isomorphism. We call $\Gamma$ the top and $N$ the base of the Chernikov group $G$. Again, the isomorphism classes of Chernikov groups with the top $\Gamma$ and the base $N$ are in one-to-one correspondence with the orbits of the group $\text{Aut} \Gamma \times \text{Aut}_{\Gamma} N$ on the cohomology group $H^2(\Gamma, N)$.

Theorem 4.7 implies the following description of Chernikov groups with the Kleinian top and regular bottom.

**Definition 5.4.** Let $\Delta$ be costandard data, even if they are special. We call the group $\text{Ch}(\Delta)$ that is the extension of $K$ with the kernel $N$ corresponding to the cohomology class $\varepsilon = \eta(z_{\Delta,2})$ a standard Chernikov group.

Note that $\text{Ch}(\Delta)$ is generated by the group $N$ and two elements $\bar{a}$ and $\bar{b}$ subject to the relations

$$
\bar{a}w = (^aw)\bar{a} \text{ for every } w \in N,
$$
$$
\bar{b}w = (^bw)\bar{b} \text{ for every } w \in N,
$$
$$
\bar{a}\bar{b} = \bar{b}\bar{a},
$$
$$
\bar{a}^2 = z_{\Delta,2}^0
$$
$$
\bar{b}^2 = z_{\Delta,2}^\infty.
$$

**Theorem 5.5.** Let $G$ be a Chernikov group with the Kleinian top $K$ such that its base $N$ is a regular $K$-colattice.

1. There are costandard data $\Delta$, a direct decomposition $N = N_0 \oplus N_\Delta$ and a semidirect decomposition $G \simeq N_0 \rtimes \text{Ch}(\Delta)$, where $\text{Ch}(\Delta)$ acts on $N_0$ as its quotient $\text{Ch}(\Delta)/N_\Delta \simeq K$.
2. If $G \simeq N'_0 \rtimes \text{Ch}(\Delta')$ is another such decomposition, there is an automorphism $\psi$ of the group $K$ such that $N'_0 \simeq \psi N_0$ and $\Delta' = \psi \Delta$.

Note that there is an amazing parallelism between chrystallographic and Chernikov groups. Perhaps, it is of general nature, and is worth to study.

Aknowledgement

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