Observations on Symmetric Circuits

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Abstract

We study symmetric arithmetic circuits and improve on lower bounds given by Dawar and Wilsenach (ArXiv 2020). Their result showed an exponential lower bound of the permanent computed by symmetric circuits. We extend this result to show a simpler proof of the permanent lower bound and show that a large class of polynomials have exponential lower bounds in this model. In fact, we prove that all polynomials that contain at least one monomial of the permanent have exponential size lower bounds in the symmetric computation model. We also show super-polynomial lower bounds for smaller groups.

We support our conclusion that the group is much more important than the polynomial by showing that on a random process of choosing polynomials, the probability of not encountering a super-polynomial lower bound is exponentially low.

2012 ACM Subject Classification Theory of Computation → Computational Complexity and Cryptography → Algebraic Complexity Theory

Keywords and phrases Arithmetic Circuit Complexity, Symmetric Arithmetic Circuits, Lower Bound

1 Introduction

The complexity of arithmetic circuits and especially the complexity of the determinant versus the permanent is a long open problem [15] and it is remaining open, despite recent results such as [5, 6, 7, 13]. An approach to this question is to prove lower bounds in restricted models and try to extend these. This includes results in the non-commutative setting by Nisan [8], lower bounds for multilinear circuits such as [2, 10] or the recent VP vs VNP separation in the monotone world by Yehudayoff [16]. The latest model, and our focus of this paper, was given by Dawar and Wilsenach [3]. The authors showed that for circuits that have to be symmetric under the permutation group $S_n \times S_n$ we can already prove super-polynomial lower bounds. We believe that our proof is simpler and hence noteworthy even without the extension to other polynomials and groups.

An interesting concede of [3] is that they study different permutation groups for the determinant and permanent, as the determinant is symmetric under fewer permutations than the permanent. We will show that the permutation group is much more important than the polynomial for a lower bound in this model. We show a similar lower bound for a wider range of polynomial including the permanent. Additionally, we show that for certain subgroups, roughly corresponding to $S_k \times S_k$, we can already prove super-polynomial lower bounds. We believe that our proof is simpler and hence noteworthy even without the extension to other polynomials and groups.

We argue that under reasonable random distribution of polynomials, almost all polynomials symmetric under the permutation group $S_n \times S_n$ have large symmetric circuit size, hence, the permanent is not “special” but the norm.

2 Definitions

We use the common shorthand of $[n] = \{1, \ldots, n\}$. We assume familiarity with arithmetic circuits, their definition, polynomial families, as well as the definition for the permanent.
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Recommended surveys of these topics are [11, 13]. An arithmetic circuit is given as a directed graph labelled with one root, inputs labelled by variables from a set $X$ or constants from a field. In general, the specific field will not matter in this paper, as long as the characteristic is not equal to two because over these fields the permanent coincides with the determinant. Other gates are labelled by either $+$ or $\times$. The polynomial computed by these circuit is given iteratively in the obvious manner. We generally talk about families or circuits and families of polynomials, meaning polynomials parameterized by the number of input variables, such as $X = \{x_1, \ldots, x_n\}$ or in our case more often $X = \{x_{1,1}, x_{1,2}, \ldots, x_{n,n}\}$.

Definition 2.1. A parse-graph of a circuit $C$ is the induced subgraph for a set of vertices $V$ such that the following conditions hold:

- The root is in $V$.
- If a multiplication gate ($\times$) is in $V$ then both its child vertices are in $V$.
- If a summation gate ($+$) is in $V$ then exactly one child vertex is in $V$.

We will generally denote circuits with $C$ and groups with $G$. We restrict ourselves to certain groups describing permutations of $n$ elements. In particular, we have $S_n$, the symmetric group on $n$ elements, $S_n \times S_n$, the group on elements $(i,j) \in [n]^2$ that can permute the first and second elements of the tuple separately.

For the next definition we need a notation to combine two partial functions. Let $T$ be a set. For two partial functions $\sigma : T \rightarrow T$, $\pi : [n] \setminus T \rightarrow [n] \setminus T$ we define

$$\sigma \cup \pi(x) = \begin{cases} 
\sigma(x) & \text{if } x \text{ is defined on } \sigma, \\
\pi(x) & \text{otherwise.}
\end{cases}$$

Definition 2.2. We, by slight abuse of notation, define $S'_k \times S'_k$ as the set of all groups $G_T$ such that for any $T \subseteq [n]$ with $|T| = k$, $G_T$ is defined by

$$G_T = \{ \sigma \cup (\text{id}_T, \text{id}_T) \mid \sigma : (T \rightarrow T) \times (T \rightarrow T) \text{ and is bijective} \} \subseteq S_n \times S_n$$

where $\text{id}_T$ is the identity on $[n] \setminus T$.

This is just all permutations on a fixed set of $T$ elements where all other elements in $[n]$ are mapped to the identity.

In the following, let $X = \{x_1, \ldots, x_n\}$. We define the action of an element $\sigma \in G$ as $\sigma(x_i) = x_{\sigma(i)}$ and extend this action on monomials and polynomials in the natural way.

The following two definitions are taken from Dawar and Wilsenach [13]. We define the notion of an automorphism on a circuit that is an extension of the action of a group element $\sigma \in G$.

Definition 2.3 (Circuit Automorphism). Let $C$ be an arithmetic circuit over variables $X = \{x_1, \ldots, x_n\}$, $G$ a group, and $\sigma \in G$ and $\sigma : [n] \rightarrow [n]$. We say $\pi : G \rightarrow G$ is an automorphism extending $\sigma$ if for every $v$ in $C$ the following holds:

- If $v$ is a constant gate then $\pi(v) = v$.
- If $v$ is a non-constant input such as $v = x_i$ then $\pi(v) = x_{\sigma(i)}$.
- If $(u, v)$ is a wire then so is $\pi(u, \pi(v))$.
- If $v$ is labelled by $+$ then $\pi(v)$ is labelled by $+$.
- If $v$ is labelled by $\times$ then $\pi(v)$ is labelled by $\times$. 


This essentially means that we permute the variable indices according to a group which implies a permutation on the circuit while the computation structure is kept intact. We modified this definition slightly to just allow permutations on \([n]\) in contrast to the original which allows arbitrary variable sets and permutations on them.

This leads us to the main definition of a symmetric circuit.

\textbf{Definition 2.4.} For a group \(G\), a circuit \(C\) is said to be symmetric for \(G\) if for every \(\sigma \in G\) the action of \(\sigma\) on \(X\) extends to an automorphism on \(C\) as defined above.

For completeness, we restate the major theorem from [3].

\textbf{Theorem 2.5 ([3]).} Let \(G = S_n \times S_n\) as defined above. Then the determinant can be computed by a family of symmetric circuits (on the group \(G\)) of size \(O(n^3)\). For any \(\epsilon > 0\), the permanent does not have symmetric circuits (on the group \(S_n \times S_n\)) of size \(2^{n^{1-\epsilon}}\).

3 Proof

3.1 Overview of the Proof

Our major insight is that for any monomial \(m\) that changes under the action of the group, it has to be computed weakly monotonely. We use this fact in Lemma 3.9. Our proof now works in two steps.

1. We modify the proof of [12, Theorem 4.1] in Section 3.2. The original result showed a size lower bound for monotone circuits computing the 0-1 permanent. To make this work we notice that the original proof works for a slightly larger class of circuits. We define this class in Definition 3.1 and show the lower bound for this class in Theorem 3.7.

2. In Section 3.3 we show that a large class of circuits fulfills our definition from Item 1. We characterize this in Definition 3.8 which gives us our main lower bound in Theorem 3.10.

We especially note, that our definition includes some smaller groups such as \(G \in S_{\sqrt{n}}' \times S_{\sqrt{n}}'\) which still give super polynomial lower bounds.

Additionally, we show that a random polynomial that is symmetric under the group \(S_n \times S_n\) fulfills this property and has large enough degree with high probability.

3.2 Modifying [12, Theorem 4.1]

Let \(T_n\) be a family of sets of parse-graphs of a family of circuits \(C_n\). For a given parse-graph we will sometimes use \(\alpha_t\) for a parse-graph \(t \in T_n\) to denote the real valued component of the monomial computed by this parse-graph. We will also use the shorthand of \(t \in C_n\) if we mean a parse-graph occurring in \(C_n\).

In the following we sometimes threat polynomials as sets of monomials, i.e., writing \(m \in f_n\) or \(m \in C_n\) and identify the circuit with the polynomial. The support of a monomial, \(\text{supp}(m)\) is the set of variables in the monomial. We will also identify the support of a parse-graph by its monomial without the coefficient, i.e.,

\[
\text{supp}(t) = \prod_{x \in \text{supp}(m)} x.
\]

\textbf{Definition 3.1.} We call \(T_n\) a parse-set\(^1\) if \(T_n\) is a subset of all parse-graphs of \(C_n\) for all \(n\). We say \(T_n\) monotonely supports a polynomial family \((f_n)\) if the following holds for all \(n\):

\(^1\) The author strongly considered the term parset.
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- For all \( t \in T_n \), \( \alpha_t \cdot \text{supp}(t) \in f_n \).
- \( f_n = \sum_{t \in T_n} \alpha_t \cdot \text{supp}(t) \) where \( \alpha_t > 0 \).
- For all \( t \in T_n \) and \( S = \text{supp}(t) \) it holds that \( \sum_{\text{supp}(t') = S} \alpha_{t'} \neq 0 \).

We will sometimes omit the index \( n \) of the polynomial family. We also speak about the permanent polynomial family which means the permanent on \( n \times n \) matrices with variables \( X = \{ x_{1,1}, x_{1,2}, \ldots, x_{n,n} \} \). We will extend notions defined on parse-graphs to parse-sets if there exists a parse-graph in the parse-set for which the condition holds.

\[ \text{Definition 3.2.} \] We call a vertex \((r,d)\)-onesided if \( v \) appears in a parse-graph in which the parse-graph computes a term with \( r \) variables and \( d \) variables are contributed by one child of the vertex.

\[ \text{Lemma 3.3.} \] If \( v \) is a \((r,d)\)-onesided vertex of a parse-set \( T \) that monotonely supports the permanent then \( m(v) \leq d!(r-d)!(n-r)! \).

**Proof.** Let \( v \) be a multiplication vertex in \( T \) with children \( u, u' \). We can now characterize every monomial computed by the circuit by three sets. Namely, \( C_1 = X \setminus \text{supp}(v) \), \( A_1 = \text{supp}(u) \cup C_1 \), \( B_1 = \text{supp}(u') \cup C_1 \). The monomial \( m \) now computed has \( \text{supp}(m) \subseteq A_1 \cdot B_1 \cdot C_1 \). Now if we have a different parse-graph \( t' \) in \( T \) containing \( v \) with triplets \( A_2, B_2, C_2 \) defined similarly. Then all monomials \( m \), with \( \text{supp}(m) \subseteq \text{supp}(A_i \cdot B_j \cdot C_k) \) for \( i, j, k \in \{2\} \) occur in the support of \( T \), hence, these must be valid permutations.

Thus, \( \text{supp}(A_1) = \text{supp}(A_2) \) and in fact they have to be a column/row of the matrix by the requirement on the group. This can be done in at most \(|A||B||C| \) many ways.

Hence, for any \((r,d)\)-onesided vertex, without loss of generality, \(|A| \leq d, |B| \leq r-d, |C| \leq n-r \).

For any multiplication vertex \( v \), let \( m(v) \) be the number of parse-graphs in which \( v \) appears. We call the weight of a subgraph as \( W(H) = \sum_{v \in \times-\text{vertices in } H} \frac{1}{m(v)} \).

\[ \text{Definition 3.4 (Definition 4.4 \[12\]).} \] For any parse-graph \( G \) a subgraph \( H \) of \( G \) is said to be a stub-graph if it satisfies the following properties:

- \( H \) is a directed graph rooted on any vertex of \( G \) and its associated graph is connected.
- The vertices of \( H \) with in-degree 0 are a subset of the vertices of \( G \) with in-degree 0.
- For any \( \times \)-vertex \( v \) in \( H \) that has both its children in \( H \), if \( v \) is \((r,d)\)-onesided in \( G \), then the number of leaves in the subgraph of \( H \) rooted at \( v \) equals \( r \), exactly \( d \) of which are contributed by one of \( v \)'s children alone.

\[ \text{Lemma 3.5.} \] Let \( \text{supp}(H) \) be the variables occurring in the in-degree zero vertices of \( H \). Let \( C \) be a circuit and \( T \) be parse-set that monotonely supports the permanent. For any stub-graph \( H \) of any parse-graph \( t \in T \), \( W(H) \geq \sum_{i=2}^{\text{supp}(H)} \frac{1}{(r-1)!(n-r)!} \).

**Proof.** The proof is by induction on the number of vertices in \( H \). We will ignore every vertex that is only in \( G \setminus H \). For the base case, \( H \) has a single vertex. Since it must be in input vertex, \( |\text{supp}(H)| = 1 \) and, since \( H \) as no multiplication vertices, \( W(H) = 0 \).

We can ignore addition vertices as they do not increase the support. Let \( H \) have at least one multiplication vertex \( v \) with children \( v_1, v_2 \). This vertex must be \((r,d)\)-onesided for some \( d \). Let \( v_1 \) be the child with the support of \( d \) variables. Let \( H_{v_2} \) be the vertex induced subgraph of \( v_2 \). By definition, \( H_{v_2} \) is a subgraph of \( H \) and \( \text{supp}(H_{v_2}) = r - d \). Construct now \( H_{v_1} \) analogously. Construct \( H'_{v_1} \) from \( H_{v_1} \) as follows:

1. Delete any input vertex from \( H_{v_1} \) that does not belong to the set of \( d \) inputs contributed by \( v_1 \) alone and delete their edges.
2. Delete iteratively every vertex that has no children and is not a circuit input.

$H'_{v_1}$ is now a stub graph rooted at $v_1$ with $\supp(H'_{v_1}) = d$. The $(r, d)$-onesided property is maintained as we do not add any other children. Since $H'_{v_1}$ and $H_{v_2}$ have no multiplication gates in common, as we have just deleted them in Item 1, we have that:

$$W(H) \geq W(H'_{v_1}) + W(H_{v_2}) + \frac{1}{m(v)}.$$  

Remember, that we removed one multiplication vertex $v$ from the graph. Applying the inductive hypothesis results in

$$W(H) \geq \sum_{i=2}^{d} \frac{1}{(i-1)!(n-i)!} + \sum_{i=2}^{r-d} \frac{1}{(i-1)!(n-i)!} + \frac{1}{m(v)}.$$  

Hence,

$$W(H) \geq \sum_{i=2}^{d} \frac{1}{(i-1)!(n-i)!} + \sum_{i=2}^{r-d} \frac{1}{(i-1)!(n-i)!} + \frac{1}{d!(r-d)!(n-r)!}$$  

(by Lemma 3.3)

for some $d$. Let us denote this expression by $\phi(d)$. It is easy to see that $\phi(1)$ is minimal for the range of $0 \leq d \leq \lfloor r/2 \rfloor$ (as $\phi(0) \geq \phi(1)$). Noting that

$$\phi(1) = \sum_{i=2}^{r-1} \frac{1}{(i-1)!(n-i)!} + \frac{1}{1!(r-1)!(n-r)!}$$

finishes the proof. 

Let $X = \{\times\text{-vertex } v \mid m(v) \geq 1\}$. Note, that any parse-set supporting the permanent monotonely has $n!$ many elements. We enumerate them by a simple index in the next lemma.

**Lemma 3.6.** Let $H$ be a parse-set supporting the permanent monotonely. Then $|X| = \sum_{i=1}^{n!} W(H_i)$.

**Proof.** This follows from the proof of [12, Lemma 4.2]. Restricting ourselves to $H$, we have that $\sum_{i=1}^{n!} W(H_i) = \sum_{i=1}^{n!} \sum_{v \in \times\text{-vertices in } H_i} \frac{1}{m(v)}$. Now fixing any vertex $v$, its contribution for each $H_i$ is either 0 or $m(v)$. Hence, $\sum_{i=1}^{n!} W(H_i) = \sum_{i=1}^{n!} \sum_{v \in \times\text{-vertex in } H_i} \frac{m(v)}{m(v)} = |X|$. 

We can now give the main ingredient. Informally, for any graph that has a set of parse-graphs that compute the permanent monotonely we can infer a size lower bound.

**Theorem 3.7.** Let $C_n$ be a circuit and $H$ be a parse-set supporting the permanent monotonely. Then the size of the circuit is lower bounded by $n(2^{n-1} - 1)$.

**Proof.** From Lemmas 3.5 and 3.6, we know that the size of the graph is at least

$$\sum_{j=1}^{n!} \sum_{i=2}^{r-1} \frac{1}{(r-1)!(n-i)!}.$$  

Every parse-graph that computes a permutation has $|\supp(H_j)| = n$. Hence, the above equation is equivalent to $n! \sum_{i=2}^{2n-1} \frac{1}{(r-1)!(n-i)!}$. This is exactly $\frac{2^{n-1} - 1}{n-1}$. As we sum over $n!$ sets, we get our bound of $n! \frac{2^{n-1} - 1}{(n-1)!} = n2^{n-1} - n$. 

\[\mathbb{Q.E.D.}\]
3.3 Main Lemma

Definition 3.8. We say a polynomial family \((f_n)\) on \(n^2\) variables is \(k\)-permanent adjacent if the following holds:

- There exists a monomial \(m\) such that a projection of \(n^2 - k^2\) of the coordinates results in a monomial of the permanent, i.e.,

\[
\exists m \in f_n, \pi, \text{ such that } \pi(m) \in \text{perm}_k
\]

where \(\pi_k(m)\) is a projection that sets a set \(n^2 - k^2\) variables to 1 and \(k^2\) variables to the identity.

- \(f_n\) is symmetric on a group \(G_T \in S'_k \times S'_k\) such that \(\pi\) maps elements not mapped to the identity to corresponding elements in \(G_T\).

Notice that by the requirement of the symmetry on the polynomial, any polynomial that is \(k\)-permanent adjacent contains all monomials of the \(k\times k\) permanent.

Lemma 3.9. Let \(f\) be a polynomial that is \(k\)-permanent adjacent that is computed by a symmetric circuit on the group \(G\) as in Definition 3.8. Then there exists a parse-set \(H\) that supports the \(k\times k\) permanent.

Proof. First notice that the following holds. Given a multilinear monomial \(m_1 = \alpha x_{i_1} \cdots x_{i_k}\) of degree \(k\). Assume that there exists a \(\sigma \in S'_k \times S'_k\) such that \(x_{\sigma(i)} = x_{i'}\). Then we know that a parse-graph containing \(x_i\) and a parse-graph containing \(x_{i'}\) have to occur in the circuit, as otherwise we would violate the symmetric property.

Now, assume that \(x_i \in m\) and \(m = f - g\) and \(f \neq \beta g\), hence, \(m\) is not computed by a monotone minimal subcircuit. Then \(\text{supp}(f) \neq \text{supp}(g)\). This violates the symmetric property of the circuit, as exchanging the variable \(x_{i} \in \text{supp}(f) \setminus \text{supp}(g)\) would change the monomial \(m\) that is computed. Hence, all monomials containing variables that are not identical under all permutations have to be computed "monotonely".

Because the family is \(k\)-permanent adjacent, by the symmetry requirement, we can gather all parse-graphs of \(C\) that compute the \(k\) permanent monomials into the parse-set \(H\).

\[\square\]

Combining Lemma 3.9 and Theorem 3.7 immediately gives us the following theorem.

Theorem 3.10. Any polynomial family \((f_n)\) that is \(k\)-permanent adjacent has symmetric circuit over the \(G \in S'_k \times S'_k\) requires circuit size at least \(k(2^{k-1} - 1)\).

This includes a large fraction of polynomials that are symmetric as given by the next lemma. Especially because we allow smaller symmetric groups, such as elements from \(S'_k \times S'_k\). We define the following random process which we deem reasonable. Let us choose uniformly with probability \(1/n\) the degree \(\delta\) from the range of \([n]\). Now we choose uniformly at random \(2\delta\) indices given us the monomial \(x_{i_1, i_2} \cdots x_{i_2\delta - 1, 2\delta}\). We choose all other monomials by adding the minimal amount of monomials necessary to have a polynomial that is symmetric on \(S_n \times S_n\).

Lemma 3.11. The probability that a polynomial chosen by the above random process has size less than \(\sqrt{n}(2^{\sqrt{n} - 1} - 1)\) is less than \(\exp(-O(n))\).
Proof. Decide with probability $\frac{1}{n}$ which degree the monomial has. Then pick the variable indices at random.

$$\Pr[\text{we pick a monomial of degree } i \text{ that is part of the permanent}]$$

$$= \frac{1}{n} \prod_{j=1}^{i} \frac{2n - 2j}{2n}$$

$$\leq \frac{1}{n} \prod_{j=1}^{i} \frac{2n - 2i}{2n}$$

$$= \frac{1}{n} \prod_{j=1}^{i} \frac{n - i}{n}$$

$$= \frac{1}{n} \left(1 - \frac{i}{n}\right)^{i}$$

$$= \frac{1}{ne^{i}}.$$ 

This is the probability that we pick a degree $i$ monomial successful of the permanent. Now we want to estimate the probability that we pick at least $\sqrt{n}$ degree monomial with high probability. It is easy to see that the median is now given by

$$\sum_{i=1}^{n} \frac{1}{ne^{i}}$$

$$= \frac{1 - e^{-n}}{n - en}.$$ 

A simple Chernoff bound will now give us the probability that less than $\sqrt{n}$ variables are chosen correctly and hence the probability that a random monomial is not a monomial of the $\sqrt{n} \times \sqrt{n}$ permanent.

$$\Pr[X < (1 - \delta)\mu] \leq \exp\left(\frac{-\delta^{2}\mu}{2}\right)$$

Setting $\delta = \frac{(n - \sqrt{n})(n - en)}{1 - e^{-n}}$ results in a bound of

$$\exp\left(\frac{- (n - \sqrt{n})^{2}(n - en)^{2}(1 - e^{-n})}{2(1 - e^{-n})^{2}(n - en)}\right)$$

$$= \exp\left(\frac{- (n - \sqrt{n})^{2}(n - en)}{2(1 - e^{-n})}\right)$$

$$= \exp\left(\frac{- (n^{2} - 2n^{3/2} + n) \cdot n \cdot (1 - e)^{2}}{2(1 - e^{-n})}\right)$$

$$\leq \exp\left(\frac{- n}{2}\right).$$ 

Hence, the probability is exponentially small, that we do not chose a valid permanent monomial of degree at least $\sqrt{n}$. ◀

We notice that our proofs also work for groups slightly larger groups, i.e., any group $G$ that contains $G' \in S_{k}^{l} \times S_{k}^{l}$ but $G' \notin S_{k+1}^{l} \times S_{k+1}^{l}$. 
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