MULTIPlicity OF POSITIVE SOLUTIONS FOR NONLINEAR FIELD EQUATIONS IN $\mathbb{R}^N$

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Abstract. In this paper we study the multiplicity of positive solutions for nonlinear elliptic equations on $\mathbb{R}^N$. The number of solutions is greater or equal than the number of disjoint intervals on which the nonlinear term is negative. Applications are given to multiplicity of standing waves for the nonlinear Schrödinger, Klein-Gordon and Klein-Gordon-Maxwell equations.

1. Introduction

In this paper we study the problem of existence of multiple positive solutions for nonlinear elliptic equations. This problem has received much attention in recent years and different kinds of results have been proved. In particular, topological methods have been used to show that the number of solutions may depend on the topological or geometrical properties of the domain of the equation. We recall the results by Dancer ([13]), Benci-Cerami ([6]) and Cerami-Molle-Passaseo ([12]) for domains in $\mathbb{R}^N$, and the results by Benci-Bonanno-Micheletti ([5]) and Hirano ([15]) for what concerns equations on Riemannian manifolds.

In this paper we are interested in multiplicity of solutions for equations on $\mathbb{R}^N$. The existence of multiple solutions is guaranteed by an “oscillating behaviour” of the nonlinear term. This phenomenon has been studied in several papers, but as far as we know only by techniques of bifurcation and on bounded domains. See for example the papers [1] and [16], and references therein.

Our proof is based on a topological argument, indeed we find different solutions as different points of local minimum for a constrained minimization problem. We have put in evidence the properties we need for our multiplicity result in Section 2. The main result is Theorem 2.1 in which we prove the multiplicity of points of local minimum for a rotationally invariant functional $\mathcal{H}$ constrained to a smooth manifold $M$ which is a level set. The functional $\mathcal{H}$ is defined into two terms, $J$ of the form

$$J(u) = \int_{\mathbb{R}^N} \left( \frac{1}{2} |\nabla u(x)|^2 + R(u(x)) \right) dx$$

for a smooth function $R(s)$, and $K$. We show that the number of different positive points of minimum is greater or equal than the number of disjoint intervals of the set $\{ R(s) < 0 \}$.

In Section 3 we give some applications of Theorem 2.1 to nonlinear field equations. Starting from the nonlinear Schrödinger (3.12) or Klein-Gordon (3.19) equation, if one looks for standing waves solutions of the form $\psi(t,x) = u(x)e^{-i\omega t}$, $u \geq 0$, one gets nonlinear elliptic equations for $u$ depending on the frequency $\omega$. See (3.14) and (3.21). Existence of standing waves has been proved under general assumptions in [9].

In Section 3.1 we first study existence of multiple positive solutions $u$ with fixed frequency. This corresponds to study a semi-linear elliptic equation of the kind studied by Berestycki-Lions in [9]. We obtain multiplicity of positive solutions to this equation under slightly different conditions on the nonlinear term.

In Sections 3.2 and 3.3 we apply Theorem 2.1 to obtain multiple existence of standing waves for the nonlinear Schrödinger and Klein-Gordon equations with fixed charge. The charge is the
invariant motion for both the systems which corresponds to the gauge action of $S^1$. In particular, for the Schrödinger equation the charge of a standing wave turns out to be the $L^2$ norm of $u$, and for the Klein-Gordon equation the charge of a standing wave is given by $\omega \|u\|_{L^2}^2$.

The results in Sections 3.1-3.3 are obtained by looking at standing waves as constrained critical points for functionals which satisfy the assumptions of Theorem 2.1. The proofs that these assumptions are satisfied follow along the same arguments. We give the proof in full details only in the first case. The details for the Klein-Gordon equation are skipped since they are already contained in [10].

Our last application in Section 3.4 is about a Klein-Gordon-Maxwell system. We consider multiplicity of electrostatic standing waves, that is with null magnetic potential and electric potential constant in time. Existence of electrostatic standing waves for a Klein-Gordon-Maxwell system has been proved in [7] (and in [8] in an abstract framework). In particular, in [7] the authors show that solutions are found as points of minimum for a constrained minimization problem. We briefly recall the main results of [7] and show that they are enough to guarantee that assumptions of Theorem 2.1 are satisfied for the relative functional.

2. The abstract setting

We consider the space $H^1(\mathbb{R}^N), N \geq 3$, equipped with the usual norm $\|u\| = (\|u\|_2^2 + \|\nabla u\|_2^2)^{1/2}$, and denote it simply as $H^1$. An important subspace is the space of radially symmetric functions, which is denoted by $H^1_r := H^1_r(\mathbb{R}^N)$.

Let $\mathcal{H} : H^1(\mathbb{R}^N) \to \mathbb{R}$ be a $C^1$ functional which is invariant under rotations, that is for all $u \in H^1$

$$\mathcal{H}(ugx) = \mathcal{H}(u(x))$$

for all $g \in SO(N)$ and can be written as

$$(2.1) \quad \mathcal{H}(u) = J(u) + K(u)$$

for two $C^1$ functionals $J$ and $K$. We assume that $J$ is of the form

$$(2.2) \quad J(u) = \int_{\mathbb{R}^N} \left( \frac{1}{2} |\nabla u(x)|^2 + R(u(x)) \right) dx$$

where $R(s) : \mathbb{R} \to \mathbb{R}$ is an even function of class $C^2$, and such that:

(A1) the set $\{ s : R(s) < 0 \}$ is not empty, and is written as

$$(2.3) \quad \{ s : R(s) < 0 \} = C_1 \sqcup \cdots \sqcup C_\ell \quad \ell \in \mathbb{N}$$

where $C_i$ are disjoint open intervals

$$(2.4) \quad C_i = (\xi_i, \eta_i) \quad i = 1, \ldots, \ell$$

with

$$0 \leq \xi_1 < \eta_1 < \xi_2 < \cdots < \xi_i < \eta_i < \xi_{i+1} < \cdots < \eta_\ell \leq \infty$$

In the following, we use modified functions $\tilde{R}_j$ defined as follows: for all $j = 1, \ldots, \ell$ consider a function $f_j(s)$ for which $f_j^2(s) \geq 0$ and let for $s \geq 0$

$$(2.5) \quad \tilde{R}_j(s) = \begin{cases} R(s) & s \leq \eta_j \\ f_j(s) & s \geq \eta_j \end{cases}$$

We assume that $\tilde{R}_j$ are of class $C^2$ and $\tilde{R}_j(s) \geq R(s)$. Another class of useful modified functions is that given by the translated modified terms defined for $s \geq 0$ by

$$(2.6) \quad s \mapsto \tilde{R}_j(s + \eta_{j-1})$$
We shall denote by \( \tilde{J}_j \) and \( \tilde{H}_j \) the functionals defined in \( (2.2) \) and \( (2.1) \), respectively, with \( \tilde{R}_j \) instead of \( R \). We assume that the functions \( f_j \) are such that \( \tilde{J}_j \) are of class \( C^1 \). This is guaranteed by growth estimate on the functions \( f_j \), hence it is obtained for example if \( f_j' = 0 \) for \( s \) big enough.

An important subset of \( H^1 \) turns out to be the set of \( u \) for which \( J(u) < 0 \). We use the notation
\[
J^{<0} := \{ u : J(u) < 0 \}
\]
From (A1) it follows that \( J^{<0} \) is not empty as is shown by the sequence of functions
\[
(2.8) \quad u_n(x) := \begin{cases} 
    s_0 & \text{if } |x| \leq r_n \\
    0 & \text{if } |x| \geq r_n + 1 \\
    s_0(1 + r_n - |x|) & \text{if } r_n \leq |x| \leq r_n + 1
\end{cases}
\]
with \( r_n \to \infty \), and \( R(s_0) < 0 \). Indeed
\[
J(u_n) = \frac{1}{2} \int_{r_n}^{r_n+1} s_0^2 r^{N-1} dr + \int_0^{r_n} R(s_0) r^{N-1} dr + \int_{r_n}^{r_n+1} R(s_0(1 + r_n - r)) r^{N-1} dr
\]
The first and last term are \( O(r_n^{N-1}) \), whereas the second is negative and grows as \( r_n^N \). Hence for \( n \) big enough \( u_n \in J^{<0} \).

A further assumption on \( J \) is:

(A2) if \( R(s) \) is non-negative for \( s \) small, then there exists a constant \( c \geq 0 \) such that \( J(u) < 0 \) implies \( \|u\|^2_{L^2} \geq c \).

We consider constrained minimization problems for functionals \( H \) on a \( C^1 \) manifold \( M \subset H^1 \) defined as a level set of a \( C^1 \) even function \( g : H^1 \to \mathbb{R} \), that is
\[
(2.9) \quad M := \{ u \in H^1 : g(u) = \text{const} \}
\]
We now come to the assumptions on the functional \( K \). We assume that \( K \) is an even functional and that for any function \( R \) satisfying (A1) and (A2) (and in particular for all its modified terms defined in \( (2.5) \) and \( (2.6) \)) the following hold:

(A3) for any open subset \( O \) of \( M \cap J^{<0} \) such that \( \inf_O H > -\infty \) and \( \inf_O H < \inf_{\partial O} H \), there exists \( u \in O \) such that \( H(u) = \inf_O H \);

(A4) if there exists \( s_1 \in \mathbb{R}^+ \) such that \( R'(s) \geq 0 \) for all \( s \geq s_1 \), then if there exists a critical point \( u \) for \( H \) constrained to \( M \) it satisfies \( \|u\|_\infty \leq s_1 \);

(A5) \( \inf_{M \cap J^{<0}} H < \inf_M K \).

Assumption (A3) is necessary for existence of a critical point and corresponds for example to the classical compactness results for minimizing sequences. Assumption (A4) turns out to be important for the multiplicity of solutions, it is easily obtained for elliptic equations by the maximum principle (see Lemma [3, 4] below). Assumption (A5) is inspired by the idea of hylomorphic solitons introduced in [2] and [3]. It turns out to be fundamental for the existence of multiple solutions.

We adapt an argument from [10] to prove that

**Theorem 2.1.** Let (A1)-(A5) hold and let \( H \) be bounded from below on \( M \). If \( \ell \) is the number of disjoint intervals in \( (2.3) \), then \( H \) has at least \( \ell \) distinct non-negative critical points constrained to \( M \).

**Proof.** First by the Palais principle of symmetric criticality [17], since \( H \) is invariant under rotations of \( \mathbb{R}^N \), it follows that critical points for \( H \) restricted to \( M \cap H^1_1 \) are also critical points for \( H \) restricted to \( M \). Hence we restrict ourselves to radially symmetric functions and use the notation \( M_\ell := M \cap H^1 \). Moreover, by evenness of the functionals we can restrict ourselves to non-negative functions \( u \in H^1 \).
The restriction to radially symmetric functions is fundamental. We recall that for functions in \( H^1_r \) there exist positive constants \( \beta \) and \( \gamma \) only depending on \( N \) such that
\[
|u(x)| \leq \gamma \frac{\|u\|_{H^1}}{|x|^{\frac{N-2}{2}}} \quad \text{for} \quad |x| \geq \beta
\]
(2.10)

For a proof of this property see e.g. [9].

Given an interval \( I \subset \mathbb{R}^+ \) and a function \( u \in H^1_r \), we introduce the notation
\[
\chi_{u,I}(x) := \chi_{\{u(x) \in I\}}(x)
\]
for the characteristic function of the set \( \{x : u(x) \in I\} \), and
\[
\text{Pal} := \left\{ \begin{array}{ll}
  u(x) & u(x) \in I \\
  0 & u(x) \notin I
\end{array} \right. \quad u_I \in H^1_r(\{x : u(x) \in I\})
\]
(2.11)

The following lemmas are from [10]. We repeat here the proofs which are short and simple.

**Lemma 2.2** ([10]). *For any interval \( I = (a,b) \) with \( a > 0 \), if a sequence \( \{u_n\} \) converges to \( u \) in the \( H^1_r \) norm then \( \{\chi_{u_n,I}(x)\} \) converges to \( \{\chi_{u,I}(x)\} \) in the \( L^1 \) norm. Hence in particular the symmetric difference \( \{u_n(x) \in I\} \triangle \{u(x) \in I\} \) has vanishing measure.*

**Proof.** Up to the choice of a sub-sequence, the sequence \( \{u_n\} \) converges to \( u \) for almost all \( x \), hence \( \{\chi_{u_n,I}(x)\} \) converges to \( \{\chi_{u,I}(x)\} \) for almost all \( x \). The proof is finished by the Lebesgue theorem of dominated convergence, since by (2.10) there exists \( \tilde{\beta} > \beta \) such that if \( |x| \geq \tilde{\beta} \) then \( |u(x)| < a \) for all \( n \). Hence for all \( n \) it follows \( \chi_{u_n,I}(x) \leq \chi_{\{|x| \leq \beta\}}(x) \in L^1 \).

**Lemma 2.3** ([10]). *For any interval \( I = (a,b) \) with \( a > 0 \), the function \( u \mapsto J(u_I) \) is continuous in the \( H^1_r \) norm.*

**Proof.** We recall that for \( u \in H^1_r \) using notation (2.12)
\[
J(u_I) = \int_{\{u(x) \in I\}} \left( \frac{1}{2} |\nabla u(x)|^2 + R(u(x)) \right) \, dx
\]
(2.13)

Let \( \{u_n\} \) be a sequence converging to \( u \) in the \( H^1 \) norm. Then using notation (2.11)
\[
\left| \int_{\{u(x) \in I\}} |\nabla u_n|^2 - \int_{\{u(x) \in I\}} |\nabla u|^2 \right| = \left| \int_{\mathbb{R}^N} (\chi_{u_n,I} - \chi_{u,I}) |\nabla u_n|^2 - \chi_{u,I} |\nabla u|^2 \right| \leq
\]
\[
\leq \int_{\mathbb{R}^N} |\chi_{u_n,I} - \chi_{u,I}| |\nabla u_n|^2 + \int_{\mathbb{R}^N} |\chi_{u_n,I} - \chi_{u,I}| |\nabla u|^2
\]
The first term in the right-hand side is vanishing since \( \{u_n\} \) converges to \( u \) in the \( H^1 \) norm and \( |\chi_{u_n,I}| \leq 1 \). The second term is vanishing by Lemma 2.2 which implies that the symmetric difference \( \{u_n(x) \in I\} \triangle \{u(x) \in I\} \) has vanishing measure as \( n \to \infty \), and by the absolute continuity of the integral, being \( |\nabla u|^2 \) in \( L^1 \). The same argument applies to the part with \( R(u) \), by using the continuity of \( J \).

**Lemma 2.4.** *There exists a critical point for \( \mathcal{H} \) restricted to \( \mathcal{M}_r \) with \( \|u(x)\|_{L^\infty} < \eta_1 \).*

**Proof.** Let us consider the modified nonlinear term \( \tilde{R}_1(s) \) defined as in (2.5). Then \( \tilde{H}_1 = \tilde{J}_1 + K \) satisfies assumptions (A3), (A4) with \( s_1 \leq \eta_1 \), and (A5).

Let us consider the open set \( \mathcal{O} = \mathcal{M}_r \cap \mathcal{J}^{>0} \). The set \( \mathcal{O} \) is open in the topology induced on \( \mathcal{M}_r \) by the continuity of the functional \( J_1 \). Since \( \mathcal{H} \) is bounded from below on \( \mathcal{M} \) and \( \tilde{R}_1(s) \geq R(s) \), also \( \tilde{H}_1 \) is bounded from below. Then, by assumption (A3), if we show that \( \inf_{\mathcal{O}} \tilde{H}_1 < \inf_{\partial \mathcal{O}} \tilde{H}_1 \), then there exists \( u \in \mathcal{O} \) which satisfies \( \tilde{H}_1(u) = \inf_{\mathcal{O}} \tilde{H}_1 \). This implies that \( u \) is a constrained
critical point, then by assumption (A4) \( \|u(x)\|_{L^\infty} < s_1 \leq \eta_1 \). It remains to show that \( u \in \mathcal{O} \) and not on the boundary. This is immediate since on the boundary of \( \mathcal{O} \) it holds \( \tilde{J}_1 = 0 \), hence
\[
\tilde{H}_1|_{\partial \mathcal{O}} = \tilde{J}_1|_{\partial \mathcal{O}} + K|_{\partial \mathcal{O}} = K|_{\partial \mathcal{O}} \geq \inf_{\mathcal{O}} \tilde{H}_1
\]
by assumption (A5).

To finish the proof, notice that \( d\tilde{H}_1 \) and \( d\mathcal{H} \) coincide on \( u \), since \( \|u(x)\|_{L^\infty} < \eta_1 \).

\begin{lemma}
There exists a critical point for \( \mathcal{H} \) restricted to \( \mathcal{M}_r \) with \( \xi_2 < \|u(x)\|_{L^\infty} < \eta_2 \).
\end{lemma}

\begin{proof}
Let us consider now the modified nonlinear term \( \tilde{R}_2 \) defined as in (2.5). Then \( \tilde{H}_2 = \tilde{J}_2 + K \) satisfies assumptions (A3), (A4) with \( s_1 \leq \eta_2 \), and (A5). Then any critical point of \( \tilde{H}_2 \) we find satisfies \( \|u(x)\|_{L^\infty} < \eta_2 \), hence it is also a critical point for \( \mathcal{H} \). It remains to prove that there exists one critical point for \( \tilde{H}_2 \) with \( \|u(x)\|_{L^\infty} > \xi_2 \).

Let us consider the open set (see (2.13))
\[
\mathcal{O} = \mathcal{M}_r \cap \tilde{J}_2 \leq 0 \cap \left\{ \tilde{J}_2 \left( u(\eta_1, \eta_2) \right) < 0 \right\}
\]
The set \( \mathcal{O} \) is open in the topology induced on \( \mathcal{M}_r \) by the continuity of the functional \( \tilde{J}_2 \) and by Lemma 2.3. Since \( \mathcal{H} \) is bounded from below on \( \mathcal{M} \) and \( \tilde{R}_2(s) \geq R(s) \), also \( \tilde{H}_2 \) is bounded from below. Hence, as in Lemma 2.4, by assumption (A3), if we show that \( \inf_{\mathcal{O}} \tilde{H}_2 < \inf_{\partial \mathcal{O}} \tilde{H}_2 \), there exists a point \( u \in \mathcal{O} \) which realizes the minimum of \( \tilde{H}_2 \) in \( \mathcal{O} \).

First, by assumption (A4), it holds \( \inf_{\mathcal{O}} \tilde{H}_2 < \inf_{\mathcal{M}} K \). Hence the infimum of \( \tilde{H}_2 \) on \( \mathcal{O} \) is not certainly realized by functions \( v \) on \( \partial \mathcal{O} \) for which \( \tilde{J}_2(v) = 0 \). Indeed for these functions \( \tilde{H}_2(v) = K(v) \).

Second, by assumption (A2), there exists \( c > 0 \) such that functions \( u \in \left\{ \tilde{J}_2 \left( u(\eta_1, \eta_2) \right) < 0 \right\} \) satisfy
\[
\int u^2(\eta_1, \eta_2) \geq c
\]
hence by
\[
m\left( \left\{ u(x) \in (\eta_1, \eta_2) \right\} \right) \eta_2^2 \geq \int u^2(\eta_1, \eta_2)
\]
and Lemma 2.2, it follows that \( v \in \partial \mathcal{O} \) implies \( \|v\|_{L^\infty} > \eta_1 \). Hence, the infimum of \( \tilde{H}_2 \) is reached neither on functions \( v \) on the boundary of \( \mathcal{O} \) for which \( \tilde{J}_2 \left( v(\eta_1, \eta_2) \right) = 0 \). Indeed for such functions, letting \( w(x) = v(x) - \eta_1 \), it follows
\[
0 = \int_{\{w(x) \in (0, \eta_2 - \eta_1)\}} \left( \frac{1}{2} |\nabla w(x)|^2 + \tilde{R}_2(w(x) + \eta_1) \right) dx \leq \int_{\mathbb{R}^N} \left( \frac{1}{2} |\nabla w|^2 + \tilde{R}_2(w + \eta_1) \right)
\]
since \( \tilde{R}_2(s + \eta_1) \) is non-negative for \( s \geq \eta_2 - \eta_1 \). However, since \( \tilde{R}_2(s + \eta_1) \) satisfies assumption (A1), it follows from assumption (A5) that there exists \( u \in H^1 \) for which
\[
\int_{\mathbb{R}^N} \left( \frac{1}{2} |\nabla u|^2 + \tilde{R}_2(u + \eta_1) \right) + K(u) < \inf_{\mathcal{M}} K
\]
Perturbing these solutions, by a cut-off on the tails, we find functions \( u \in \left\{ \tilde{J}_2 \left( u(\eta_1, \eta_2) \right) < 0 \right\} \) with \( \tilde{H}_2(u) < K(v) = \tilde{H}_2(v) \) for all \( v \in \partial \mathcal{O} \) with \( \tilde{J}_2 \left( v(\eta_1, \eta_2) \right) = 0 \).

We have thus obtained that the point \( u \) which realizes the minimum of \( \tilde{H}_2 \) in \( \mathcal{O} \) is in the interior part of \( \mathcal{O} \). That this point satisfies \( \|u(x)\|_{L^\infty} > \xi_2 \) is immediate from the definition of \( \mathcal{O} \) and \( \xi_2 \) (see (2.13) and (2.4)).
\end{proof}
The proof of Theorem 2.1 is finished by repeating Lemma 2.5 for all intervals $C_i$. Some modifications are needed for the last interval $C_\ell$ in the case $\eta_\ell = \infty$. There is no need for using a modified nonlinear term, since we only need to show the existence of a point of local minimum with $\|u(x)\|_{L^\infty} > \xi_\ell$. This is achieved as above by slightly modifying the argument. 

3. Applications

In the applications we shall study solutions of elliptic problems. It is useful to recall that

**Lemma 3.1.** Let $u$ be a solution of

$$- \Delta u + G'(u) = 0$$

for a $C^1$ function $G : \mathbb{R}^+ \to \mathbb{R}$ for which there exist $\tilde{s} > 0$ such that $G'(s) \geq 0$ for $s \geq \tilde{s}$. Then

$$\|u(x)\|_{L^\infty(\mathbb{R}^N)} \leq \tilde{s}$$

**Proof.** Let $u$ be a solution of (3.1) and set $u = \tilde{s} + v$. It is sufficient to prove that $v \leq 0$. Let $A := \{x : v(x) \geq 0\}$. By (3.1) we have that

$$-\Delta v + G' (\tilde{s} + v) = 0 \text{ in } A$$

$$v = 0 \text{ on } \partial A$$

Multiplying both sides of the above equation by $v$ and integrating in $A$, we get

$$0 = \int_A \left( |\nabla v|^2 + G'(\tilde{s} + v) v \right) dx \geq \int_A |\nabla v|^2 dx$$

where we have used $G'(s) \geq 0$ for $s \geq \tilde{s}$. From this it follows that $v = 0$ in $A$. 

**3.1. Semi-linear elliptic problems.** In this section we apply Theorem 2.1 to the case of semi-linear elliptic problems studied in [9]. We consider the problem

$$- \Delta u + F'(u) = 0$$

where $u \in H^1(\mathbb{R}^N, \mathbb{R})$ with $N \geq 3$, and $F : \mathbb{R} \to \mathbb{R}$ is an even function of class $C^2$ such that

**(H1)** $F$ can be written as

$$F(s) = \frac{\Omega_2}{2} s^2 + T(s)$$

with $T(0) = T'(0) = T''(0) = 0$;

**(H2)** there exists $s_0 \in \mathbb{R}^+$ such that $F(s_0) < 0$;

**(H3)** there exist positive constants $c_1, c_2$ such that for all $s$

$$|T''(s)| \leq c_1 s^{p-2} + c_2 s^{q-2}$$

with $2 < p, q < 2^* = \frac{2N}{N-2}$.

It is well known that solutions of (3.2) can be found as constrained critical points of the functional

$$\mathcal{H}(u) = \int_{\mathbb{R}^N} \left( \frac{1}{2} |\nabla u(x)|^2 + F(u(x)) \right) dx$$

on the manifold

$$\mathcal{M}_c = \left\{ u : \int_{\mathbb{R}^N} F(u(x)) dx = c \right\}$$

Hence we are reduced to a minimization problem of the form studied in Section 2. In this case $\mathcal{H}$ is as in (2.1) with $K \equiv 0$ and $R(s) = F(s)$, and $\mathcal{M}_c$ is as in (2.9). We now show that the assumptions of Theorem 2.1 are verified for the functional $\mathcal{H}$ in (3.3) restricted to the manifold $\mathcal{M}_c$ in (3.4) for the choice of a constant negative and large in absolute value.
Proposition 3.2. If $c$ in (3.4) is negative and sufficiently small, then conditions (H1)-(H3) imply assumptions (A1)-(A5).

Proof. (A1). Condition (H2) implies (A1) with $\ell \geq 1$, hence the set $J^{<0}$ defined in (2.1) is not empty.

(A2). Recall the result

Lemma 3.3 ([10]). Let $G : \mathbb{R}^+ \to \mathbb{R}$ be a $C^2$ function satisfying conditions (H1)-(H3) with $\Omega^2 = 0$. Then there exists $k > 0$ such that

$$\inf_{\|u\|_{L^2} = k} \int_{\mathbb{R}^N} \left( \frac{1}{2} |\nabla u(x)|^2 + G(u(x)) \right) \, dx \begin{cases} < 0 & \text{for } k > \bar{k} \\ = 0 & \text{for } k < \bar{k} \end{cases}$$

but the infimum is not attained for $k < \bar{k}$. Moreover $\bar{k} = 0$ if and only if $G(s)$ is negative for small.

Now we show that Lemma 3.3 implies (A2). Indeed, by condition (H1) it follows that $F(s)$ is non-negative in a small interval $(0, \varepsilon)$, hence there exists $G$ which satisfies conditions (H1)-(H3) with $\Omega^2 = 0$ and such that $F(s) \geq G(s)$ for all $s \geq 0$. If $T(s)$ is non-negative in $(0, \varepsilon)$ we can choose $G = T$. Let $J(u) = H(u) < 0$, then

$$\int_{\mathbb{R}^N} \left( \frac{1}{2} |\nabla u(x)|^2 + G(u(x)) \right) \, dx \leq J(u) < 0$$

Hence, by Lemma 3.3 $\|u\|_{L^2}^2 \geq \bar{k} > 0$.

(A3). For any $c < 0$, the functional $H$ in (3.3) is bounded from below on $M^c$. Let now $\{u_n\}$ be a minimizing sequence on an open set $O \subset M^c \cap J^{<0}$, that is

$$\lim_n H(u_n) = \inf_O H < \inf_{\partial O} H \quad \{u_n\} \in O$$

and by invariance under rotations of $H$ and evenness of $F(s)$, we can assume that the $u_n$ are non-negative radially symmetric functions. Then a simple argument shows that, up to the choice of a sub-sequence, there exists $u \in H^1_+$ such that $\{u_n\}$ converges to $u$ in the $H^1$ norm, hence $u \in O$ and $H(u) = \inf_O H$.

The first step is to show that the $H^1$ norm of the functions $u_n$ is bounded. First $\|\nabla u_n\|_{L^2}$ is bounded since $H(u_n)$ is bounded and $\int F(u_n) = c$. Second, if $\|u_n\|_{L^2}$ is not bounded, we get a contradiction. Indeed, by (H1) and (H3) we get that, using the notation $T^+$ and $T^-$ for the positive and negative part of $T$, there exists a positive constant such that

$$T^-(s) \leq \frac{\Omega^2}{4} s^2 + \text{const} |s|^{2^*}$$

hence for all $u \in M^c$ it holds

$$\int_{\mathbb{R}^N} \left( \frac{\Omega^2}{2} u^2 + T^+(u) \right) = c + \int_{\mathbb{R}^N} T^-(u) \leq c + \int_{\mathbb{R}^N} \left( \frac{\Omega^2}{4} u^2 + T^+(u) + \text{const} |u|^{2^*} \right)$$

Applying (3.3) to the minimizing sequence $\{u_n\}$ and using the Sobolev inequality $\|u\|_{L^{2^*}} \leq \text{const} \|\nabla u\|_{L^2}$, we get

$$\int_{\mathbb{R}^N} \frac{\Omega^2}{4} u_n^2 \leq c + \text{const} \|\nabla u_n\|_{L^2}^{2^*} \leq \text{const}$$

hence $\|u_n\|_{H^1_1}$ is bounded. It follows that there exists $u \in H^1_+$ such that $u_n$ weakly converges to $u$ in $H^1_+$. Since the spaces $L^p$ for $2 < p < 2^*$ are compactly embedded in $H^1_+$, we also get that up to a sub-sequence

$$u_n \overset{L^p}{\to} u \quad \text{for} \quad 2 < p < 2^*$$
The second step is to show that the convergence to $u$ is strong in the $H^1$ norm. We explicitly write that $\{u_n\}$ is a minimizing sequence for the constrained minimization problem. We get that there exists a sequence $\{\lambda_n\}$ of real numbers such that

$$< d\mathcal{H}(u_n), v > - \lambda_n \int F'(u_n) v = < \varepsilon_n, v > \rightarrow 0$$

for all $v \in H^1$, that is

(3.7) $$\int (\nabla u_n \nabla v + (1 - \lambda_n) F'(u_n) v) = < \varepsilon_n, v > \rightarrow 0$$

where $\varepsilon_n \in H^{-1}$. Now, applying (3.7) to $u_n$ and using boundedness of $\|u_n\|_{H^1}$ and (H3), it follows that the sequence $\{\lambda_n\}$ is bounded. Hence, up to a sub-sequence, it converges to a real number $\lambda$. Hence, by weak convergence of $u_n$ to $u$, we get that the function $u$ satisfies

(3.8) $$- \Delta u + (1 - \lambda) F'(u) = 0$$

hence it satisfies the Derrick-Pohozaev identity (see [9])

(3.9) $$\int_{\mathbb{R}^N} |\nabla u|^2 + \frac{2N}{N-2} \int_{\mathbb{R}^N} (1 - \lambda) F(u) = 0$$

Moreover, since $u_n \in \mathcal{M}_c$ with $c < 0$, we write

(3.10) $$\int F(u) = c + \int (F(u) - F(u_n)) = c + \int \frac{\Omega^2}{2} (u^2 - u_n^2) + \int (T(u) - T(u_n))$$

From the weak convergence of $u_n$ to $u$ in $H^1$ it follows that

$$\int \frac{\Omega^2}{2} (u^2 - u_n^2) \leq 0$$

Further, writing

$$\left| \int (T(u) - T(u_n)) \right| \leq \int |T'(u + \theta (u_n - u))| |u - u_n|$$

for some $\theta \in (0,1)$, we get, using (H3), the inequality

$$\int |u + \theta |u_n - u| |u - u_n| \leq 2^{p-1} \int \left( |u|^{p-1} + \theta |u_n - u|^{p-1} \right) |u - u_n| \leq$$

$$\leq 2^{p-1} \left( \int |u|^p \right)^{\frac{1}{p}} \left( \int |u_n - u|^p \right)^{\frac{1}{p}} + 2^{p-1} \int |u - u_n|^p$$

and the convergence (3.6), that

$$\int (T(u) - T(u_n)) \rightarrow 0$$

From (3.10) it follows that $\int F(u) < 0$. This implies by (3.9), that in (3.8) we have $\lambda < 1$.

The proof is finished by writing for two functions $u_n$ and $u_m$

$$< d\mathcal{H}(u_n) - d\mathcal{H}(u_m), v > - \lambda \int (F'(u_n) - F'(u_m)) v =$$

$$= < \varepsilon_n - \varepsilon_m, v > + (\lambda_n - \lambda) \int F'(u_n) v - (\lambda_m - \lambda) \int F'(u_m) v \rightarrow 0$$

Since $\|u_n - u_m\|_{H^1}$ is bounded, we can write $v = u_n - u_m$ and get

(3.11) $$\int |\nabla u_n - \nabla u_m|^2 + (1 - \lambda) \int \frac{\Omega^2}{2} |u_n - u_m|^2 + (1 - \lambda) \int (T'(u_n) - T'(u_m)) (u_n - u_m) \rightarrow 0$$
Writing
\[ \left| \int (T'(u_n) - T'(u_m)) (u_n - u_m) \right| \leq \int |T''(u_m + \theta (u_n - u_m))| |u_n - u_m|^2 \]
for some \( \theta \in (0, 1) \), and using (H3) and the inequality
\[ \int (|u_m| + \theta |u_n - u_m|)^{p-2} |u_n - u_m|^2 \leq 2^{p-2} \int (|u_m|^{p-2} + \theta |u_n - u_m|^{p-2}) |u_n - u_m|^2 \leq 2^{p-2} \left( \int |u_m|^p \right)^{1 - \frac{2}{p}} \left( \int |u_n - u_m|^p \right)^{\frac{2}{p}} + 2^{p-2} \int |u_n - u_m|^p \]
from (3.6) it follows that
\[ \int_{\mathbb{R}^N} (T'(u_n) - T'(u_m)) (u_n - u_m) \, dx \longrightarrow_{n,m\to\infty} 0 \]
Hence from (3.11) we obtain
\[ \|u_n - u_m\|_{H^1} \longrightarrow_{n,m\to\infty} 0 \]
since \( (1 - \lambda) > 0 \). Hence \( \{u_n\} \) is a Cauchy sequence in \( H^1 \), and it follows that it has a sub-sequence strongly convergent to \( u \) in the \( H^1 \) norm. This finishes the proof of (A3).

(A4). From the proof of (A3) it follows that constrained critical points for \( \mathcal{H} \) on \( \mathcal{M}_c \) satisfy (3.8) with \( \lambda < 1 \). If there exists \( s_1 \) such that \( F'(s) \geq 0 \) for \( s \geq s_1 \), then we can apply Lemma 3.1 with \( G(s) = (1 - \lambda)F(s) \). This implies (A4).

(A5). We have to prove that \( \inf_{\mathcal{M}_c \cap J < \infty} \mathcal{H} < \inf_{\mathcal{M}_c} K = 0 \). For the sequence of functions \( u_n \in H^1 \) defined in (2.5) with \( F(s_1) < 0 \), we proved that for \( n \) big enough \( \mathcal{H}(u_n) = J(u_n) < 0 \), hence \( \mathcal{M}_c \cap J < \infty \) is not empty for \( c \) smaller than a negative constant \( c_0 \).

However, we have to prove that (A3), (A4) and (A5) hold for all the modified nonlinear terms \( \tilde{F}_j \) as in (3.4). Assumptions (A3) and (A4) hold for \( \mathcal{H}_j \) as above. Indeed the nonlinear terms \( \tilde{F}_j \) can be chosen as to satisfy the conditions (H1)-(H3) from which (A3) and (A4) follow. For what concerns (A5), in the proof above we have only used the existence of a point \( s_1 \) with \( F(s_1) < 0 \). Hence, the same follows for \( \mathcal{H}_j \), for \( j = 1, \ldots, \ell \), for the same negative constant \( c_0 \) by using \( s_1 \in (\xi_j, \eta_j) \). For the translated modified terms \( \tilde{F}_j(s + \eta_{j-1}) \), using points \( s_j \in (\xi_j, \eta_j) \), the thesis follows for \( c \) smaller than negative constants \( c_j \). By choosing \( c \) smaller than \( \min \{c_0, c_1, \ldots, c_\ell\} \), assumption (A5) follows for \( F \) and all the modified terms \( \tilde{F}_j \), \( j = 1, \ldots, \ell \).

**Theorem 3.4.** Under conditions (H1)-(H3), if the set \( \{F(s) < 0\} \) has \( \ell \) disjoint intervals, then the problem (3.2) has at least \( \ell \) distinct non-negative solutions.

**Proof.** By Proposition 3.2 assumptions (A1)-(A5) are satisfied for the functional \( \mathcal{H} \) defined in (3.3) on manifolds \( \mathcal{M}_c \) defined in (3.4) for \( c \) negative and small enough. Moreover, by definition, \( \mathcal{H} \) is bound from below on \( \mathcal{M}_c \). Hence we can apply Theorem 2.1 and obtain \( \ell \) different non-negative functions \( u_j \in \mathcal{M}_c \), for \( j = 1, \ldots, \ell \), which are constrained critical points for \( \mathcal{H} \).

In the proof of (A3) in Proposition 3.2 we proved that the constrained critical points \( u_j \) satisfy (3.8) with \( \lambda_j < 1 \). By the re-scaling \( \tilde{u}_j(x) = u_j(x/\sqrt{1 - \lambda_j}) \), we obtain \( \ell \) different non-negative solutions of (3.2).}

### 3.2. Nonlinear Schrödinger equations

We now apply Theorem 2.1 to the case of nonlinear Schrödinger equations
\[ i \frac{\partial \psi}{\partial t} + \Delta \psi - T(|\psi|) \frac{\psi}{|\psi|} = 0 \]
where \( \psi(t, x) \in H^1(\mathbb{R} \times \mathbb{R}^N, \mathbb{C}) \) with \( N \geq 3 \), and \( T : \mathbb{R} \to \mathbb{R} \) is an even function of class \( C^2 \) such that

\[ \text{...} \]
(H1) \( T(0) = T'(0) = T''(0) = 0; \)
(H2) there exists \( s_0 \in \mathbb{R}^+ \) such that \( T(s_0) < 0; \)
(H3) there exist positive constants \( c_1, c_2 \) such that for all \( s \)
\[ |T''(s)| \leq c_1 s^{p-2} + c_2 s^{q-2} \]
with \( 2 < p, q < 2^* = \frac{2N}{N-2}; \)
(H4) there exist positive constants \( c_3, c_4 \) such that for all \( s \)
\[ T(s) \geq -c_3 s^2 - c_4 |s|^\gamma \]
with \( 2 \leq \gamma < 2 + \frac{4}{N}. \)

An important class of solutions of (3.12) is given by standing waves. A standing wave is a finite energy solution of the form
\[ \psi(t, x) = u(x)e^{-i\omega t}, \quad u \geq 0, \quad \omega \in \mathbb{R} \]
for which (3.12) takes the form
\[ -\Delta u + T'(u) = \omega u \]

It is well known that standing waves of the form (3.13) are obtained as critical points of the functional
\[ \mathcal{H}(u) = \int_{\mathbb{R}^N} \left( \frac{1}{2} |\nabla u(x)|^2 + T(u(x)) \right) dx \]
on the manifold
\[ \mathcal{M}_c = \{ u : \|u\|_{L^2} = c \} \]
where \( \omega \) is the Lagrange multiplier. In [11] and [4], it is proved that if the standing waves are obtained as points of minimum of \( \mathcal{H} \) on \( \mathcal{M}_c \) then they are also orbitally stable, hence solitons.

We are then reduced to a minimization problem of the form studied in Section 2. In this case \( \mathcal{H} \) is as in (2.1) with \( K \equiv 0 \) and \( R(s) = T(s) \), and \( \mathcal{M}_c \) is as in (2.9). We now show that the assumptions of Theorem 2.1 are verified for the functional \( \mathcal{H} \) in (3.15) restricted to the manifold \( \mathcal{M}_c \) in (3.16) for the choice of a constant large enough.

Proposition 3.5. If \( c \) in (3.16) is large enough, then conditions (H1)-(H4) imply assumptions (A1)-(A5).

Proof. (A1). It follows from condition (H2) with \( \ell \geq 1 \). In particular the set \( J^{<0} \) defined in (2.7) is not empty.

(A2). It follows from Lemma 3.3.

(A3). We follow the same argument as in the proof of (A3) in Proposition 3.2.

The functional \( \mathcal{H} \) is bounded from below on \( \mathcal{M}_c \) for any \( c > 0 \). This follows as in [4] from the Sobolev inequality
\[ \|u\|_{L^q} \leq \text{const} \|u\|_{L^2}^{1 - \frac{N}{2} + \frac{N}{q}} \|\nabla u\|_{L^2}^{\frac{N}{2} - \frac{N}{q}} \quad 2 \leq q \leq 2^* \]
and from (H4). Writing
\[ \mathcal{H}(u) \geq \int \left( \frac{1}{2} |\nabla u(x)|^2 - c_3 u^2 - c_4 u^\gamma \right) \geq \frac{1}{2} \|\nabla u\|_{L^2}^2 - \text{const} \|\nabla u\|_{L^2}^{\frac{2N}{N-2} - N} - c_3 c \]
where we have used \( \|u\|_{L^2} = c \), it follows that
\[ \mathcal{H}(u) \geq \frac{1}{2} \|\nabla u\|_{L^2}^2 + o \left( \|\nabla u\|_{L^2}^2 \right) \]
for $\|\nabla u\|_{L^2}$ going to infinity. From this we also get that any minimizing sequence $\{u_n\}$ in $\mathcal{M}_c$ is bounded in $H^1$. Hence restricting to non-negative radially symmetric functions, we get strong convergence of $u_n$ to a function $u \in H^1$ in $L^p$ for all $2 < p < 2^*$.

It remains to prove strong convergence in the $H^1$ norm. Repeating the same argument as in the proof of (A3) in Proposition 3.2, we get existence of a couple $(u, \omega) \in H^1 \times \mathbb{R}$, $u \neq 0$, which satisfies (3.14). The Derrick-Pohozaev identity in this case becomes

$$
\int_{\mathbb{R}^N} |\nabla u|^2 + \frac{2N}{N-2} \int_{\mathbb{R}^N} T(u) = \frac{N}{N-2} \int_{\mathbb{R}^N} \omega u^2
$$

(3.17)

Since we are minimizing $\mathcal{H}$ on open subsets of $\mathcal{M}_c \cap J^{<0}$, it follows $J(u) < 0$, hence $\int T(u) < 0$. Moreover, since $\frac{N}{N-2} > 1$, we get from (3.17)

$$
\beta = \frac{N}{N-2} \int_{\mathbb{R}^N} \omega u^2 < 2 J(u) < 0
$$

It follows that $\omega < 0$.

The last step is obtained by writing the analogous of equation (3.11). In this case, the constrained minimization problem implies that for two functions $u_n$ and $u_m$ of a minimizing sequence we get

$$
<d\mathcal{H}(u_n) - d\mathcal{H}(u_m), v> = -\omega \int (u_n - u_m) v = 
$$

$$
= \varepsilon_n - \varepsilon_m, v> + (\omega_n - \omega) \int u_n v - (\omega_m - \omega) \int u_m v \rightarrow 0
$$

Since $\|u_n - u_m\|_{H^1}$ is bounded, we can write $v = u_n - u_m$ and get

$$
\int |\nabla u_n - \nabla u_m|^2 - \omega \int |u_n - u_m|^2 + \int (T'(u_n) - T'(u_m))(u_n - u_m) \rightarrow 0
$$

(3.18)

Since

$$
\int_{\mathbb{R}^N} (T'(u_n) - T'(u_m))(u_n - u_m) \ dx \longrightarrow_{n,m \rightarrow \infty} 0
$$

from (3.18) we obtain

$$
\|u_n - u_m\|_{H^1} \longrightarrow_{n,m \rightarrow \infty} 0
$$

since $\omega < 0$. Hence $\{u_n\}$ is a Cauchy sequence in $H^1$, and it follows that it has a sub-sequence strongly convergent to $u$ in the $H^1$ norm. This finishes the proof of (A3).

(A4). Let $G(s) = T(s) - \omega s^2$. Since $\omega < 0$, if $T'(s) \geq 0$ for $s \geq s_1$, then $G'(s) = T'(s) - 2\omega s \geq 0$. Hence we can apply Lemma 3.3 to $G$. This implies (A4).

(A5). We have to prove that $\inf_{\mathcal{M}_c \cap J^{<0}} \mathcal{H} < \inf_{\mathcal{M}_c} K = 0$. For the sequence of functions $u_n \in H^1_c$ defined in (2.5) with $T(s_0) < 0$, we proved that for $n$ big enough $\mathcal{H}(u_n) = J(u_n) < 0$, hence $\mathcal{M}_c \cap J^{<0}$ is not empty for $c$ larger than a constant $c_0$.

Again, we have to prove that (A3), (A4) and (A5) hold for all the modified nonlinear terms $\tilde{T}_j$ as in (2.5). Assumptions (A3) and (A4) hold for $\tilde{H}_j$ as above. Indeed the nonlinear terms $\tilde{T}_j$ can be chosen as to satisfy the conditions (H1)-(H4) from which (A3) and (A4) follow. For what concerns (A5), in the proof above we have only used the existence of a point $s_1$ with $T(s_1) < 0$. Hence, the same follows for $\tilde{H}_j$, for $j = 1, \ldots, \ell$, for the same negative constant $c_0$ by using $s_1 \in (\xi_j, \eta_j)$. For the translated modified terms $\tilde{T}_j(s + \eta_{j-1})$, using points $s_j \in (\xi_j, \eta_j)$, the thesis follows for $c$ larger than constants $c_j$. By choosing $c$ larger than $c_0$, assumption (A5) follows for $T$ and all the modified terms $\tilde{T}_j$, $j = 1, \ldots, \ell$. 

Theorem 3.6. Under conditions (H1)-(H4) and for constants $c$ large enough, if the set $\{T(s) < 0\}$ has $\ell$ disjoint intervals, then the problem (3.12) admits at least $\ell$ distinct standing waves with $L^2$ norm equal to $c$. 

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Proof. By Proposition 3.5, assumptions (A1)-(A5) are satisfied for the functional $\mathcal{H}$ defined in (3.15) on manifolds $\mathcal{M}_c$ defined in (3.16) for $c$ large enough. Moreover, $\mathcal{H}$ is bounded from below on $\mathcal{M}_c$. Hence we can apply Theorem 2.1 and obtain $\ell$ different non-negative functions $u_j \in \mathcal{M}_c$, for $j = 1, \ldots, \ell$, which are constrained critical points for $\mathcal{H}$. They satisfy (3.14) for some negative $\omega$, hence correspond to standing waves for (3.12). □

3.3. Nonlinear Klein-Gordon equations. We now apply Theorem 2.1 to the case of nonlinear Klein-Gordon equations

\begin{equation}
\frac{\partial^2 \psi}{\partial t^2} - \Delta \psi + W'(|\psi|) \frac{\psi}{|\psi|} = 0
\end{equation}

where $\psi(t, x) \in H^1(\mathbb{R} \times \mathbb{R}^N, \mathbb{C})$ with $N \geq 3$, and $W: \mathbb{R} \rightarrow \mathbb{R}$ is an even function of class $C^2$ such that

(H1) $W$ is non-negative and can be written as

$$W(s) = \frac{\Omega^2}{2} s^2 + T(s)$$

with $T(0) = T'(0) = T''(0) = 0$;

(H2) there exists $s_0 \in \mathbb{R}^+$ such that $T(s_0) < 0$;

(H3) there exist positive constants $c_1, c_2$ such that for all $s$

$$|T''(s)| \leq c_1 s^{p-2} + c_2 s^{q-2}$$

with $2 < p, q < 2^* = \frac{2N}{N-2}$.

Again we consider standing waves solutions

\begin{equation}
\psi(t, x) = u(x)e^{-i\omega t}, \quad u \geq 0, \quad \omega \in \mathbb{R}
\end{equation}

for which (3.19) takes the form

\begin{equation}
-\Delta u + W'(u) = \omega^2 u
\end{equation}

A variational principle for finding standing waves has been introduced in [2], where it is proved that they are obtained as critical points of the two-variables functional

$$E(u, \omega) = \int_{\mathbb{R}^N} \left( \frac{1}{2} |\nabla u(x)|^2 + \frac{\Omega^2}{2} u^2(x) + T(u(x)) + \frac{\omega^2}{2} u^2(x) \right) \, dx$$

on the manifold

$$C_\sigma = \{(u, \omega) \in H^1 \times \mathbb{R}^+ : \omega \|u\|^2_{L^2} = \sigma\}$$

Moreover isolated points of minimum for $E$ are proved in [2] to correspond to orbitally stable standing waves, hence solitons. See also [14].

To use the abstract setting of Section 2, notice that, fixed $\sigma$, the functional $E$ restricted to $C_\sigma$ can be written as dependent only on $u$ and it writes

\begin{equation}
\mathcal{H}(u) := E|_{C_\sigma}(u, \omega) = \int_{\mathbb{R}^N} \left( \frac{1}{2} |\nabla u(x)|^2 + T(u(x)) \right) \, dx + \frac{\Omega^2}{2} \|u\|^2_{L^2} + \frac{\sigma^2}{2} \|u\|^2_{L^2}
\end{equation}

with $u \in H^1$, $u \not\equiv 0$.

We are then reduced to the minimization problem of the functional $\mathcal{H}$, which is as is in (2.1) with

\begin{equation}
J(u) = \int_{\mathbb{R}^N} \left( \frac{1}{2} |\nabla u(x)|^2 + T(u(x)) \right) \, dx
\end{equation}

and

\begin{equation}
K(u) = \frac{\Omega^2}{2} \|u\|^2_{L^2} + \frac{1}{2} \|u\|^2_{L^2}
\end{equation}
and \( M = H^1 \). Points \( u \) of local minimum of \( H \) correspond to points \((u, \omega(u))\) of local minimum of \( E \) on the manifold \( C_\sigma \), with \( \omega(u) = \frac{\sigma}{\|u\|_{L^2}^2} \).

In [10], we proved that for \( \sigma \) large enough problem (3.19) has at least \( \ell \) standing wave solutions with \((u, \omega) \in C_\sigma \). In particular we proved that they are points of local minimum, hence they are indeed solitons. We now briefly recall the arguments used in [10] to show that the assumptions of Theorem 2.1 are verified for the functional \( H \) in (3.22). Hence we obtain as in Theorems 3.4 and 3.6 at least \( \ell \) standing waves for (3.19).

Assumptions (A1) and (A2) are easily obtained as for the nonlinear Schrödinger equations. Assumption (A3) is proved with an argument similar to that used in the proofs of Propositions 3.2 and 3.5, but in two variables. In particular one gets that for \( \sigma \) large enough solutions \((u, \omega)\) of (3.21) satisfy \( u \neq 0 \) and \( 0 < \omega < \Omega \).

Assumption (A4) follows from Lemma 3.1. It is obtained by setting \( G(s) := \frac{1}{4} (\Omega^2 - \omega^2) s^2 + T(s) \) and using \( 0 < \omega < \Omega \).

Finally assumption (A5) is equivalent to
\[
(3.25) \quad \inf_{(u, \omega) \in C_\sigma} E(u, \omega) < \Omega \sigma
\]

since \( \inf_{H^1} K(u) = \Omega \sigma \). Condition (3.25) is called the hylomorphy condition in [2] and [3], and standing waves \( \psi(t, x) \) as in (3.20) with \((u, \omega)\) satisfying the hylomorphy condition are proved to be orbitally stable, and are called hylomorphic solitons. In [10] it is proved that there exists a threshold \( \sigma_g \) such that for all \( \sigma > \sigma_g \), condition (3.25) is verified. Since we need condition (3.25) to be verified also for all the modified functionals \( H_j \), with \( T_j \) as in (2.5) and (2.6), we have to choose \( \sigma \) larger than max \( \{ \sigma_g, (\sigma_g)_1, \ldots, (\sigma_g)_\ell \} \).

3.4. Nonlinear Klein-Gordon-Maxwell systems. As a final application we consider the nonlinear Klein-Gordon-Maxwell system

\[
(3.26) \quad (\partial_t + i e \phi)^2 \psi - (\nabla - i e A)^2 \psi + W'(|\psi|) \frac{\psi}{|\psi|} = 0
\]

\[
(3.27) \quad \nabla \cdot (\partial_t A + \nabla \phi) = e \text{Im}(\bar{\psi} \partial_t \psi) + e^2 \phi |\psi|^2
\]

\[
(3.28) \quad \nabla \times (\nabla \times A) + \partial_t (\partial_t A + \nabla \phi) = e \text{Im}(\bar{\psi} \nabla \psi) - e^2 A |\psi|^2
\]

where \( \psi(t, x) \in H^1(\mathbb{R} \times \mathbb{R}^3, \mathbb{C}) \), \((\phi(t, x), A(t, x))\) are the electric and magnetic potential and \( e \) is a parameter which can be interpreted as the electron charge. Moreover as before \( W : \mathbb{R} \to \mathbb{R} \) is an even function of class \( C^2 \) which satisfies assumptions (H1)-(H3) of Section 3.3. We refer to [7] and [8] for discussion of the system (3.26)-(3.28) in abelian gauge theory.

Following [7] and [8], we consider electrostatic standing waves for (3.26)-(3.28), that is solutions of the form

\[
(3.29) \quad \psi(t, x) = u(x) e^{-i \omega t}, \quad u \geq 0, \ \omega \in \mathbb{R}
\]

\[
(3.30) \quad A = 0, \ \partial_t \phi = 0
\]

Substituting (3.29) and (3.30) into (3.26)-(3.28) one gets

\[
(3.31) \quad -\Delta u + W'(u) = (\omega - e \phi)^2 u
\]

\[
(3.32) \quad \partial_t ((\omega - e \phi) u^2) = 0
\]

\[
(3.33) \quad -\Delta \phi = e (\omega - e \phi) u
\]

where (3.31) is the continuity equation for the “electric” charge density \( \rho = (\omega - e \phi) u^2 \). Solutions of (3.31)-(3.33) are found in [7] and [8] by first solving (3.33) for \( u \) fixed and finding solutions
$\phi_u \in D^{1,2}(\mathbb{R}^3)$, and then looking for solutions $u$ of (3.31) with $\phi = \phi_u$. Hence one is led to look for solutions of the elliptic equation

(3.34) \[ -\Delta u + W'(u) = \omega^2 (1 - e \Phi_u)^2 u \]

where $\Phi_u = \frac{\phi_u}{\omega}$ and satisfies $0 \leq \Phi_u \leq \frac{1}{e}$. Referring to [7] and [8] for details, we recall that solutions of (3.34) are obtained as critical points of the functional

(3.35) \[ \mathcal{H}(u) = \int_{\mathbb{R}^3} \left( \frac{1}{2} |\nabla u|^2 + W(u) \right) \, dx \]

constrained to the $C^1$ manifold of constant electric charge

(3.36) \[ \mathcal{M}_c := \left\{ u \in H^1_\Omega : \frac{1}{2} \int_{\mathbb{R}^3} (1 - e \Phi_u)u^2 = c^2 \right\} \]

We are then reduced to the minimization problem of the functional $\mathcal{H}$, which is as in (2.1) with

(3.37) \[ J(u) = \int_{\mathbb{R}^3} \left( \frac{1}{2} |\nabla u(x)|^2 + T(u(x)) \right) \, dx \]

and

(3.38) \[ K(u) = \frac{\Omega^2}{2} \|u\|_{L^2}^2 \]

We now show that we can apply Theorem 2.1 to this minimization problem. Indeed $\mathcal{H}$ is certainly bounded from below on $\mathcal{M}_c$ for all $c \in \mathbb{R}$ since $W$ is assumed to be non-negative. Assumptions (A1) and (A2) follow as in Section 3.3.

For the following we recall what is proved in [8] (and in [7] for a slightly different functional). Using $J$ and $K$ as defined in (3.37) and (3.38), the fundamental result is that for $e > 0$ small enough

(3.39) \[ \inf_{u \in H^1_\Omega} \left( \frac{1}{2} \int_{\mathbb{R}^3} (1 - e \Phi_u)u^2 \right) < \Omega^2 \]

which implies that for $c$ in an open subset of $\mathbb{R}$ all minimizing sequences $\{u_n\} \subset \mathcal{M}_c$ for $\mathcal{H}$ are, up to a sub-sequence, convergent in the $H^1$ norm to a constrained point of minimum $u \in \mathcal{M}_c$, with Lagrange multiplier $0 < \omega < \Omega$. This implies (A3).

Assumption (A4) follows choosing $G(u) := \frac{1}{2} (\Omega^2 - \omega^2 (1 - e \Phi_u))u^2 + T(u)$. Indeed $G'(u) = (\Omega^2 - \omega^2 (1 - e \Phi_u)^2)u + T'(u)$ and it is positive if $T'(u) \geq 0$, since $0 \leq \Phi_u \leq \frac{1}{e}$ and $0 < \omega < \Omega$.

Finally, assumption (A5) follows for $e > 0$ small enough from (3.39) writing

(3.39) \[ \inf_{\mathcal{M}_c \cap J < 0} \mathcal{H}(u) \leq \Omega^2 c^2 \leq \inf_{\mathcal{M}_c} \frac{1}{2} \Omega^2 \|u\|_{L^2}^2 \]

where in the last inequality we have used $\int_{\mathbb{R}^3} (1 - e \Phi_u)u^2 \leq \int_{\mathbb{R}^3} u^2$ for all $u \in H^1_\Omega$.

Hence, if the set $\{T(s) < 0\}$ has $\ell$ disjoint intervals, for all $c$ in an open subset the functional $\mathcal{H}$ defined in (3.35) on the manifold $\mathcal{M}_c$ defined in (3.36), admits at least $\ell$ different local points of minimum. In particular, there exist at least $\ell$ different electrostatic standing waves (3.29)-(3.30) with electric charge equal to $c$.

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