A note on the rigidity of marginally outer trapped 2-spheres

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Abstract

As discussed in the paper, in a matter-filled spacetime, perhaps with positive cosmological constant, a stable marginally outer trapped 2-sphere must satisfy a certain area inequality. Namely, its area must be bounded above by $4\pi/c$, where $c > 0$ is a lower bound on a natural energy-momentum term. In this note we consider the rigidity that results for stable, or weakly outermost, marginally outer trapped 2-spheres that achieve this upper bound on the area. The “canonical” dynamical horizon in Vaidya spacetime and certain spacelike hypersurfaces in Nariai spacetime provide illustrations of the main results. These results may be viewed as spacetime analogues of the rigidity results of Bray, Brendle and Neves [10] concerning area minimizing 2-spheres in Riemannian 3-manifolds with scalar curvature having positive lower bound.

1 Introduction

Let $(M, g, K)$ be an $n$-dimensional initial data set in an $(n+1)$-dimensional spacetime $(\bar{M}, \bar{g})$. By this we mean that $M$ is a spacelike hypersurface, and $g$ and $K$ are the induced metric and second fundamental form, respectively, of $M$. Let $\Sigma$ be a marginally outer trapped surface (MOTS) in $M$. Thus, $\Sigma$ is a closed (compact without boundary) 2-sided hypersurface in $M$ such that the expansion of its future-directed outward null normal geodesics vanishes along $\Sigma$, $\theta_+ = 0$. We say that $\Sigma$ is weakly outermost if there are no outer trapped ($\theta_+ < 0$) surfaces outside of, and homologous, to $\Sigma$. (See Section 2 for precise definitions.) In [20] the following result was obtained.

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Theorem 1.1. Let \((M, g, K)\) be an \(n\)-dimensional initial data set, \(n \geq 3\), in a spacetime obeying the dominant energy condition (DEC). If \(\Sigma\) is a weakly outermost MOTS in \(M\) then (apart from certain exceptional circumstances) \(\Sigma\) must be of positive Yamabe type, i.e. must admit a metric of positive scalar curvature.

The ‘exceptional circumstances’ are ruled out if, for example, the DEC holds strictly on \(\Sigma\) or \(\Sigma\) is not Ricci flat. \(\Sigma\) being of positive Yamabe type implies many well-known restrictions on the topology. As such, Theorem 1.1 may be viewed as a higher dimensional generalization of Hawking’s black hole topology theorem; see [19] for a detailed discussion.

As noted in [20], it is sufficient to assume in Theorem 1.1 that \(\Sigma\) be a stable MOTS (see Section 2). Geometrically, MOTSs may be viewed as spacetime analogues of minimal surfaces in Riemannian manifolds. In fact, in the time-symmetric case \((K = 0)\), a MOTS \(\Sigma\) is just a minimal hypersurface in \((M, g)\). From this point of view, Theorem 1.1 may be viewed as a spacetime analogue of results in [25, 26] for compact stable minimal hypersurfaces in manifolds of nonnegative scalar curvature.

Theorem 1.1 leaves open certain possibilities that one would like to rule out. For example, it does not rule out the possibility of a (flat) toroidal black hole in a vacuum spacetime. In [17], the author obtained a rigidity result which rules out such possibilities.

Theorem 1.2. Let \((M, g, K)\) be an \(n\)-dimensional, \(n \geq 3\) initial data set in a spacetime obeying the DEC. Suppose \(\Sigma\) is a weakly outermost MOTS in \(M\). If \(\Sigma\) is not of positive Yamabe type then an outer neighborhood \(U \approx [0, \epsilon) \times \Sigma\) of \(\Sigma\) is foliated by MOTS. Moreover, each such MOTS is Ricci flat and has vanishing outer null second fundamental form, \(\chi_+ = 0\).

Thus, if \(\Sigma\) is an outermost MOTS (i.e, if there are no outer trapped, or marginally outer trapped, surfaces outside of and homologous to \(\Sigma\)) then \(\Sigma\) must be of positive Yamabe type without exception.

Continuing our comparison with Riemannian results, Theorem 1.2 may be viewed as a spacetime analogue of [12, Theorem 1] concerning least area tori in 3-manifolds of nonnegative scalar curvature, and [11, Theorem 3] concerning an analogous result for higher dimensional manifolds (see also Theorem 1.3 in [18]).

In [10], Bray, Brendle and Neves obtain a rigidity result similar to [12, Theorem 1] for area minimizing 2-spheres in 3-manifolds with scalar curvature having positive lower bound. They first observe that if \(\Sigma\) is an area minimizing 2-sphere in a 3-manifold \((M, g)\) with scalar curvature bounded below by \(2c > 0\), then \(A(\Sigma) = \text{area of } \Sigma \leq 4\pi/c\). This, in fact, only requires that \(\Sigma\) be a stable minimal surface. They then show that if the area of \(\Sigma\) achieves this upper bound, \((M, g)\) splits metrically along \(\Sigma\), i.e., is (locally) isometric to the product \(I \times S^2\).

A similar area inequality is known to hold for stable marginally outer trapped 2-spheres in initial data sets in spacetime. (See, for example, results of Hayward et al. [23] in a slightly different context.) In this spacetime case, the lower
bound on the scalar curvature is replaced by a lower bound on a certain energy-momentum term associated with the given initial data set. We review this result in Section 3. The main aim of this paper is to examine the rigidity that results when the upper bound on the area of the MOTS is achieved. We first present an *infinitesimal* rigidity result, and then use this to obtain a local rigidity result similar in nature to Theorem 1.2 above. The former result is illustrated by the MOTSs foliating the spherically symmetric dynamical horizon of the Vaidya spacetime (see e.g. [7]), and the latter result is illustrated by the MOTS foliating certain spacelike slices in the Nariai spacetime (see e.g. [9, 8]), cf. Section 3.

In the next section we present some preliminary background material. The main results are presented in Section 3.

## 2 Preliminaries

A marginally outer trapped surface (MOTS) in spacetime represents an extreme gravitational situation: Under suitable circumstances, the occurrence of a MOTS signals the presence of a black hole [21, 13]. For this and other reasons MOTSs have played a fundamental role in quasi-local descriptions of black holes; see e.g., [7]. MOTSs arose in a more purely mathematical context in the work of Schoen and Yau [27] concerning the existence of solutions of Jang’s equation, in connection with their proof of the positive mass theorem. The mathematical theory of MOTSs has been greatly developed in recent years. We refer the reader to the survey article [1] which describes many of these developments.

In this section we recall some basic definitions and facts about MOTSs. Let \((\bar{M}, \bar{g})\) be a 4-dimensional spacetime (time oriented Lorentzian manifold). Consider an initial data set \((M, g, K)\) in \((\bar{M}, \bar{g})\). Hence, \(M\) is a spacelike hypersurface (of dimension three), and \(g\) and \(K\) are the induced metric and second fundamental form, respectively, of \(M\).

Let \(\Sigma\) be a closed (compact without boundary) two-sided surface in \(M\). Then \(\Sigma\) admits a smooth unit normal field \(\nu\) in \(M\), unique up to sign. By convention, refer to such a choice as outward pointing. Then \(l_+ = u + \nu\) and \(l_- = u - \nu\) are future directed outward pointing and inward pointing, respectively, null normal vector fields along \(\Sigma\). Associated to \(l_+\) and \(l_-\) are the null second fundamental forms \(\chi_+\) and \(\chi_-\), respectively, defined by

\[
\chi_{\pm} : T_p\Sigma \times T_p\Sigma \to \mathbb{R}, \quad \chi_{\pm}(X, Y) = \bar{g}(\nabla_X l_\pm, Y). \tag{2.1}
\]

The *null expansion scalars* (or *null mean curvatures*) \(\theta_{\pm}\) of \(\Sigma\) are obtained by tracing \(\chi_{\pm}\) with respect to the induced metric \(h\) on \(\Sigma\),

\[
\theta_{\pm} = \text{tr}_h \chi_{\pm} = h^{AB} \chi_{\pm AB} = \text{div}_\Sigma l_{\pm}, \tag{2.2}
\]
where $\nabla$ is the Levi-Civita connection of $(\bar{M}, \bar{g})$. Physically, $\theta_+ \ (\text{resp., } \theta_-)$ measures the divergence of the outgoing (resp., ingoing) light rays emanating from $\Sigma$. In terms of the initial data $(M, g, K)$,

$$\theta_\pm = \text{tr}_h K \pm H,$$

where $H$ is the mean curvature of $\Sigma$ within $M$. In particular, in the time-symmetric case $(K = 0)$, $\theta_+$ is just the mean curvature of $\Sigma$ in $M$.

As first defined by Penrose, $\Sigma$ is said to be a trapped surface if both $\theta_-$ and $\theta_+$ are negative. Focusing attention on the outward null normal only, we say that $\Sigma$ is an outer trapped surface if $\theta_+ < 0$. Finally, we define $\Sigma$ to be a marginally outer trapped surface (MOTS) if $\theta_+$ vanishes identically.

In [2, 3], Andersson, Mars and Simon introduced a notion of stability for MOTSs, analogous in a certain sense to that for minimal surfaces, which we now recall.

Let $\Sigma$ be a MOTS in the initial data set $(M, g, K)$ with outward unit normal $\nu$. We consider a normal variation of $\Sigma$ in $M$, i.e., a variation $t \to \Sigma_t$ of $\Sigma = \Sigma_0$ with variation vector field $V = \frac{\partial}{\partial t} \big|_{t=0} = \phi \nu$, $\phi \in C^\infty(\Sigma)$. Let $\theta(t)$ denote the null expansion of $\Sigma_t$ with respect to $l_t = u + \nu_t$, where $u$ is the future directed timelike unit normal to $M$ and $\nu_t$ is the outer unit normal to $\Sigma_t$ in $M$. A computation as in [3] gives,

$$\frac{\partial \theta}{\partial t} \bigg|_{t=0} = L(\phi),$$

where $L : C^\infty(\Sigma) \to C^\infty(\Sigma)$ is the operator,

$$L(\phi) = -\Delta \phi + 2\langle X, \nabla \phi \rangle + \left(\frac{1}{2} S_\Sigma - (\mu + J(\nu)) - \frac{1}{2} |\chi|^2 + \text{div} X - |X|^2 \right) \phi. \quad (2.5)$$

Here, $\Delta$, $\nabla$ and div are the Laplacian, gradient and divergence operator, respectively, on $\Sigma$, $S_\Sigma$ is the scalar curvature of $\Sigma$, $\chi$ is the null second form of $\Sigma$ with respect to the null normal $\ell = \ell_0$, $X$ is the vector field on $\Sigma$ dual to the one form $K(\nu, \cdot)|_\Sigma$, $\langle \cdot, \cdot \rangle = h$ is the induced metric on $\Sigma$, and $\mu$ and $J$ are defined in terms of the Einstein tensor $G = \text{Ric}_M - \frac{1}{2} R_g \bar{g}$: $\mu = G(u, u)$, $J = G(u, \cdot)$. When the Einstein equations are assumed to hold, $\mu$ and $J$ represent the energy density and linear momentum density along $M$. As a consequence of the Gauss-Codazzi equations, the quantities $\mu$ and $J$ can be expressed solely in terms of initial data, $\mu = \frac{1}{2} (S_\Sigma + (\text{tr } K)^2 - |K|^2)$ and $J = \text{div} K - d(\text{tr } K)$.

We wish to record some properties of the operator (2.5). Actually, for our purposes, it will be convenient to consider the slightly more general operator $\tilde{L} : C^\infty(\Sigma) \to C^\infty(\Sigma)$, defined by,

$$\tilde{L}(\phi) = -\Delta \phi + 2\langle X, \nabla \phi \rangle + \left( Q + \text{div} X - |X|^2 \right) \phi,$$

where $Q$ is any smooth function on $\Sigma$ and $X$ is any smooth vector field on $\Sigma$. The operator $\tilde{L}$ is not self-adjoint in general. However, the following is known; see [3] and references therein.
Lemma 2.1. The following holds for the operator $\tilde{L}$.

(i) There is a real eigenvalue $\lambda_1 = \lambda_1(\tilde{L})$, called the principal eigenvalue, such that for any other eigenvalue $\mu$, $\text{Re}(\mu) \geq \lambda_1$. The associated eigenfunction $\phi$, $\tilde{L}(\phi) = \lambda_1 \phi$, is unique up to a multiplicative constant, and can be chosen to be strictly positive.

(ii) $\lambda_1 \geq 0$ (resp., $\lambda_1 > 0$) if and only if there exists $\psi \in C^\infty(\Sigma)$, $\psi > 0$, such that $\tilde{L}(\psi) \geq 0$ (resp., $L(\psi) > 0$).

Our main results will rely on the following key fact. Consider the “symmetrized” operator $L_0 : C^\infty(\Sigma) \rightarrow C^\infty(\Sigma),

$$L_0(\phi) = -\Delta \phi + Q\phi.$$  \hfill (2.7)

obtained formally from (2.6) by setting $X = 0$.

Lemma 2.2. $\lambda_1(L_0) \geq \lambda_1(\tilde{L})$. Hence, if $\lambda_1(\tilde{L}) \geq 0$,

$$\int_\Sigma |\nabla f|^2 + Qf^2 dA \geq 0,$$  \hfill (2.8)

for all $f \in C^\infty(\Sigma)$.

The assertion $\lambda_1(L_0) \geq \lambda_1(\tilde{L})$ follows from the main argument in [20]; see also [3], [18]. The inequality (2.8) then follows from the Rayleigh formula characterizing the principal eigenvalue of the operator $L_0$,

$$\lambda_1(L_0) = \inf_{f \neq 0} \frac{\int_\Sigma |\nabla f|^2 + Qf^2 d\mu}{\int_\Sigma f^2 d\mu}.$$  \hfill (2.9)

An inequality similar to (2.8) has been obtained in [24].

Return now to the operator (2.5) associated with the variation $t \rightarrow \Sigma_t$ of the MOTS $\Sigma = \Sigma_0$. In the time-symmetric case, $\theta$ in (2.4) becomes the mean curvature $H$, the vector field $X$ vanishes and $L$ reduces to the stability operator of minimal surface theory. As such, we refer to $L$ in (2.5) as the MOTS stability operator associated with variations in the null expansion $\theta$.

In analogy with the minimal surface case, we say that a MOTS is stable provided $\lambda_1(L) \geq 0$. (In the minimal surface case this is equivalent to the second variation of area being nonnegative.) Heuristically, a MOTS $\Sigma$ is stable if it is infinitesimally outermost. Stable MOTSs arise in various situations. For example, weakly outermost MOTSs are stable. Indeed, if $\lambda_1(L) < 0$, (2.4) implies that $\Sigma$ can be deformed outward to an outer trapped surface. Weakly outermost MOTSs include, in particular, compact cross sections of the event horizon in stationary black hole spacetimes obeying the null energy condition. More generally, results of Andersson and Metzger [5, 4], and of Eichmair [15, 16] establish natural criteria for the existence of outermost (and hence weakly outermost) MOTSs; see also [1].
3 Main results

We first present an infinitesimal rigidity result.

**Proposition 3.1.** Let $\Sigma$ be a stable spherical (topologically $S^2$) MOTS in a 3-dimensional initial data $(M, g, K)$. Suppose there exists $c > 0$, such that $\mu + J(\nu) \geq c$ on $\Sigma$, where $\nu$ is the outward unit normal to $\Sigma$. Then the area of $\Sigma$ satisfies,

$$A(\Sigma) \leq \frac{4\pi}{c}.$$  \hspace{1cm} (3.10)

Moreover, if equality holds, $\Sigma$ is a round 2-sphere, with Gaussian curvature $k_\Sigma = c$, the null second fundamental form $\chi$ (with respect the outward null normal) of $\Sigma$ vanishes, and $\mu + J(\nu) = c$ on $\Sigma$.

Proposition 3.1 is very closely related to results in [23] for spacetimes with positive cosmological constant, in which stability is expressed in terms of variations of the null expansion along a null hypersurface.

In the presence of matter fields and/or a positive cosmological constant, a positive lower bound on $\mu + J(\nu)$ like that assumed in Proposition 3.1 is expected. Indeed, suppose the initial data set $(M, g, K)$ comes from a spacetime $(\bar{M}, \bar{g})$ which satisfies the Einstein equation,

$$G + \Lambda \bar{g} = T$$  \hspace{1cm} (3.11)

where, as in Section 2, $G = \text{Ric}_M - \frac{1}{2}R_M \bar{g}$ is the Einstein tensor, and $T$ is the energy-momentum tensor. Then, setting $\ell = u + \nu$, where $\nu$ is any unit vector tangent to $M$ and $u$ is the future directed unit normal to $M$, we have along $\Sigma$ in $M$,

$$\mu + J(\nu) = G(u, \ell) = T(u, \ell) + \Lambda.$$  \hspace{1cm} (3.12)

Thus, in the presence of ordinary matter fields one will have $\mu + J(\nu) > 0$ even if $\Lambda = 0$. Moreover, if one assumes $T$ obeys the dominant energy condition (which includes the matter vacuum case $T = 0$) and $\Lambda > 0$, then one has

$$\mu + J(\nu) \geq \Lambda,$$  \hspace{1cm} (3.13)

in which case $\mu + J(\nu)$ has a positive lower bound on all of $M$.

**Proof of Proposition 3.1.** We have that $\lambda_1(L) \geq 0$, where $L$ is the MOTS stability operator. We apply Lemma 2.2 to $\bar{L} = L$. Thus, taking

$$Q = \frac{1}{2}S_\Sigma - (\mu + J(\nu)) - \frac{1}{2}|\chi|^2$$  \hspace{1cm} (3.14)

in (2.6), and noting that $k_\Sigma = \frac{1}{2}S_\Sigma$, inequality (2.8), with $f = 1$, implies,

$$\int_{\Sigma} \left(\mu + J(\nu) + \frac{1}{2}|\chi|^2\right) dA \leq \int_{\Sigma} k_\Sigma dA = 4\pi.$$  \hspace{1cm} (3.15)
On the other hand, by the definition of the constant $c$, 

$$
\int_{\Sigma} \left( \mu + J(\nu) + \frac{1}{2}|\chi|^2 \right) dA \geq \int_{S} c \ dA = cA(\Sigma).
$$

(3.16)

Inequalities (3.15) and (3.16) now imply (3.10).

Now assume $A(\Sigma) = 4\pi/c$. Then inequalities (3.15) and (3.16) combine to give,

$$
\int_{\Sigma} \left( \mu + J(\nu) + \frac{1}{2}|\chi|^2 \right) dA = 4\pi,
$$

(3.17)

or, equivalently,

$$
\int_{\Sigma} \left( (\mu + J(\nu) - c) + \frac{1}{2}|\chi|^2 \right) dA = 0.
$$

(3.18)

Since $\mu + J(\nu) \geq c$ on $\Sigma$, this implies that $\mu + J(\nu) \equiv c$ and $\chi \equiv 0$.

We now have $Q = k_{\Sigma} - c$. By Lemma 2.2, $\lambda_1(L_0) \geq 0$. But setting $f = 1$ in the right hand side of (2.9) gives zero, which implies that $\lambda_1(L_0) \leq 0$. Thus, $\lambda_1(L_0) = 0$ and $\phi = 1$ is an associated eigenfunction, i.e. is a solution to

$$
- \triangle \phi + (k_{\Sigma} - c)\phi = 0,
$$

(3.19)

and hence $k_{\Sigma} = c$.

Remark: Note that the proof also shows that $\lambda_1(L) = 0$. Indeed, we have $0 = \lambda_1(L_0) \geq \lambda_1(L) \geq 0$.

Dynamical horizons: The notion of a dynamical horizon was studied extensively in [7]. By definition, a dynamical horizon (DH) is a spacelike hypersurface foliated by MOTS, subject to the additional requirement that along each such MOTS, one has $\theta_- < 0$, i.e. the future directed ingoing light rays are converging. The view put forth in [7] (see also [22]) is that a DH should be viewed as a quasi-local version of a dynamical black hole. The condition, $\theta_- < 0$, along each MOTS in the foliation is a physical requirement that, roughly speaking, distinguishes a DH as a black hole, rather than a white hole. As shown in [6], the foliation of a spacelike hypersurface by MOTS, if such a foliation exists, is unique. Moreover each such MOTS is stable, in fact, weakly outermost.

Vaidya spacetime, which is a spherically symmetric spacetime containing a null fluid, is a well-known example of a black hole spacetime containing DHs; cf. [7, Appendix A]. There is a canonical DH $M_{can}$ in Vaidya spacetime which inherits the spherical symmetry. Using the formulas in [7, Appendix A], one easily verifies that equality holds in (3.10) for each MOTS in $M_{can}$, where $c$ is taken to be the greatest lower bound.\footnote{This can also be seen from general considerations.} Now, there is a well-known nonuniqueness feature of DHs [6]: $M_{can}$ can be associated with a family of spherically symmetric spacelike hypersurfaces, each
cutting $M_{can}$ transversely in a MOTS. Smoothly perturbing this family in a nonspherically symmetric manner, will produce, in general, a nonspherically symmetric DH, in which the foliating MOTSs are no longer round, and hence do not saturate the area bound in (3.10). Thus, Proposition 3.1 provides a criterion for singling out the canonical DH in Vaidya spacetime without making explicit reference to the underlying spherical symmetry.

**Axisymmetry:** Suppose, in Proposition 3.1, one assumes that $\Sigma$ is axisymmetric in the sense of [24], and hence admits a suitable rotational Killing vector field $\eta$. Then, if $\Sigma$ is axisymmetric-stable in the sense of [24], the inequality (3.10) can be refined. Using [24 Lemma 1], which refines the inequality (2.8) for such MOTSs and for axisymmetric functions $f$, one obtains in a manner similar to the proof of Proposition 3.1 (but where $Q$ in (3.14) now acquires an additional nonnegative term) the area inequality,

$$A(\Sigma) \leq \frac{4\pi}{c + \omega},$$

where $\omega$ is a nonnegative constant which is strictly positive if the angular momentum $J$ of $\Sigma$ (see e.g. [24, 14, 7]) is nonzero. The constant $\omega$ is, in the notation used here, the average value over $\Sigma$ of the quantity $|K(\eta/|\eta|, \nu)|^2$. Thus, while the angular momentum determines a lower bound for the area [24], it also influences the upper bound. If equality holds in (3.20) then, by similar reasoning as before, one sees that $\mu + J(\nu) = c$ and $\chi = 0$.

We now establish a local rigidity result. For this purpose, we fix some notation and terminology. If $\Sigma$ is a separating MOTSs in $(M, g, K)$, let $M_+$ be the region consisting of $\Sigma$ and the region outside of $\Sigma$. Then $\Sigma$ is weakly outermost if there is no outer trapped surface in $M_+$ homologous to $\Sigma$. $\Sigma$ is outer area minimizing if its area is greater than or equal to the area of any surface in $M_+$ homologous to $\Sigma$.

**Theorem 3.2.** Let $(M, g, K)$ be a 3-dimensional initial data set, with $M$ having mean curvature $\tau \leq 0$. Let $\Sigma \subset M$ be a separating spherical MOTS in $M$ which is weakly outermost and outer area minimizing. Suppose there exists $c > 0$, such that $\mu - |J| \geq c$ on $M_+$. Then, if $A(\Sigma) = 4\pi/c$, there exists an outer neighborhood $U \approx [0, \epsilon) \times \Sigma$ of $\Sigma$ such that each slice $\Sigma_t = \{t\} \times \Sigma$, $t \in [0, \epsilon)$ is a MOTS. In fact each such slice has vanishing outward null second fundamental form and is a round sphere of constant Gaussian curvature $k_{\Sigma_t} = c$. Moreover, $\mu + J(\nu) = c$ on $U$, where $\nu$ is the outward unit normal field to the $\Sigma_t$’s.

**Remark:** We wish to emphasize that this is a local result: Since there is no completeness assumption on $M$, one can think of $M$ as a small tubular neighborhood of $\Sigma$. Then, as discussed after the proof, under slightly stronger conditions, the mean curvature condition on $M$ can be dropped. Also, note that, in distinction to Proposition 3.1, the lower bound on $\mu - |J|$ is assumed to hold on $M_+$, not just $\Sigma$.

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\[^{2}\text{Here we take the axisymmetric variation vector field } X \text{ in [24] to be } V = \phi \nu, \text{ as in the sentences above (2.4).}\]
Proof. As observed in Section 2, weakly outermost MOTSs are stable. Hence, since by assumption \( A(\Sigma) = 4\pi/c \), the remark following the proof of Proposition 3.1 implies that \( \lambda_1(L) = 0 \), where \( L \) is the MOTS stability operator of \( \Sigma \).

We now recall an argument from [17] to show that an outer neighborhood of \( \Sigma \) is foliated by constant null expansion hypersurfaces with respect to a suitable scaling of the future directed outward null normals.

For \( f \in C^\infty(\Sigma), f \) small, let \( \Theta(f) \) denote the null expansion of the hypersurface \( \Sigma_f : x \rightarrow exp_x f(x)\nu \) with respect to the (suitably normalized) future directed outward null normal field to \( \Sigma_f \). \( \Theta \) has linearization, \( \Theta'(0) = L \). We introduce the operator,

\[
\Theta^* : C^\infty(\Sigma) \times \mathbb{R} \rightarrow C^\infty(\Sigma) \times \mathbb{R}, \quad \Theta^*(f,k) = \left( \Theta(f) - k, \int_{\Sigma} f \right).
\] (3.21)

Since, by Lemma 2.1(i), \( \lambda_1 = 0 \) is a simple eigenvalue, the kernel of \( \Theta'(0) = L \) consists only of constant multiples of the positive eigenfunction \( \phi \). We note that \( \lambda_1 = 0 \) is also a simple eigenvalue for the adjoint \( L^* \) of \( L \) (with respect to the standard \( L^2 \) inner product on \( \Sigma \)), for which there exists a positive eigenfunction \( \phi^* \); cf. [3]. Then the equation \( Lf = v \) is solvable if and only if \( \int v \phi^* = 0 \). From these facts it follows easily that \( \Theta^* \) has invertible linearization about \( (0,0) \). Thus, by the inverse function theorem, for \( s \in \mathbb{R} \) sufficiently small there exists \( f(s) \in C^\infty(\Sigma) \) and \( k(s) \in \mathbb{R} \) such that,

\[
\Theta(f(s)) = k(s) \quad \text{and} \quad \int_{\Sigma} f(s)dA = s.
\] (3.22)

By the chain rule, \( \Theta'(0)(f''(0)) = L(f'(0)) = k'(0) \). The fact that \( k'(0) \) is orthogonal to \( \phi^* \) implies that \( k'(0) = 0 \). Hence \( f'(0) \in \ker \Theta'(0) \). The second equation in (3.22) then implies that \( f'(0) = \text{const} \cdot \phi > 0 \).

It follows that for \( s \) sufficiently small, the hypersurfaces \( \Sigma_{f_s} \), form a smooth foliation of a neighborhood of \( \Sigma \) in \( M \) by hypersurfaces of constant null expansion. Thus, one can introduce coordinates \( (t,x^i) \) in a neighborhood \( W \) of \( \Sigma \) in \( M \) such that, with respect to these coordinates, \( W = (-t_0,t_0) \times \Sigma \), and for each \( t \in (-t_0,t_0) \), the \( t- \) slice \( \Sigma_t = \{t\} \times \Sigma \) has constant null expansion \( \theta(t) \) with respect to \( \ell_t \), where \( \ell_t = u + \nu_t \), and \( \nu \) is the outward unit normal field to the \( \Sigma_t \)'s in \( M \). In addition, the coordinates \( (t,x^i) \) can be chosen so that \( \frac{\partial}{\partial t} = \phi \nu \), for some positive function \( \phi = \phi(t,x^i) \) on \( W \).

A computation similar to that leading to (2.14) (but where we can no longer assume \( \theta \) vanishes) shows that the null expansion function \( \theta = \theta(t) \) of the foliation obeys the evolution equation,

\[
\frac{d\theta}{dt} = \tilde{L}_t(\phi),
\] (3.23)

where, for each \( t \in (-t_0,t_0) \), \( \tilde{L}_t \) is the operator on \( \Sigma_t \) acting on \( \phi \) according to,

\[
\tilde{L}_t(\phi) = -\Delta \phi + 2\langle X,\nabla \phi \rangle + \left( \frac{1}{2}S - (\mu + J(\nu) + \theta \tau - \frac{1}{2}\theta^2 - \frac{1}{2}|\chi|^2 + \text{div } X - |X|^2 \right) \phi.
\] (3.24)
It is to be understood that, for each \( t \), the above terms live on \( \Sigma_t \), e.g., \( \Delta = \Delta_t \) is the Laplacian on \( \Sigma_t \), \( S = S_t \) is the scalar curvature of \( \Sigma_t \), and so on.

The assumption that \( \Sigma \) is weakly outermost, together with the constancy of \( \theta(t) \), implies that \( \theta(t) \geq 0 \) for all \( t \in [0, t_0) \). Hence, since \( \theta(0) = 0 \), in order to show that \( \theta(t) = 0 \) for all \( t \in [0, t_0) \), it is sufficient to show that \( \theta'(t) \leq 0 \) for all \( t \in [0, t_0) \).

Suppose there exists \( t \in (0, t_0) \) such that \( \theta'(t) > 0 \). For this value of \( t \), (3.23) implies \( \tilde{L}_t(\phi) > 0 \). Then Lemma 2.1(ii) implies that \( \lambda_1(\tilde{L}_t) > 0 \).

We now apply Lemma 2.2 to \( \tilde{L} = \tilde{L}_t \). Since \( \lambda_1(\tilde{L}_t) > 0 \), inequality (2.8) holds strictly for \( f \neq 0 \). Thus, setting \( f = 1 \) into (2.8) leads to,

\[
\int_{\Sigma_t} \left( \mu + J(\nu) + \theta |\tau| + \frac{1}{2} \theta^2 + \frac{1}{2} |\chi|^2 \right) dA < \int_{\Sigma_t} k dA = 4\pi.
\]

where we have used that \( k = \frac{1}{2} S \) is the Gaussian curvature of \( \Sigma_t \) and that \( \tau \leq 0 \). But, by the definition of \( c \) the left hand side of the above is greater than or equal to \( cA(\Sigma_t) \). These inequalities then give, \( A(\Sigma_t) < 4\pi/c \), which contradicts that \( \Sigma \) is outer area minimizing.

Thus, \( \theta(t) = 0 \) for all \( t \in [0, t_0) \), i.e., each \( \Sigma_t \), with \( t \in [0, t_0) \), is a MOTS. Since, by (3.23), \( \tilde{L}_t(\phi) = \theta' = 0 \), Lemma 2.1 implies \( \lambda_1(\tilde{L}_t) > 0 \) for each \( t \in [0, t_0) \), and hence each \( \Sigma_t \) is stable. Then by (3.10) and the area minimizing assumption on \( \Sigma \), we have \( A(\Sigma_t) = 4\pi/c \). Thus, Proposition 3.1 implies that for \( t \in [0, t_0) \), \( \Sigma_t \) is a round 2-sphere, with Gaussian curvature \( k_{\Sigma} = c \), the null second form of \( \Sigma_t \) vanishes and \( \mu + J(\nu) = c \) along \( \Sigma_t \).

**Remark:** As noted above, by slightly strengthening some assumptions in Theorem 3.2 the mean curvature assumption, \( \tau \leq 0 \), can be dropped. As in [17], one can smoothly bend \( M \) slightly to the past in a small spacetime neighborhood of \( \Sigma \), while keeping \( \Sigma \) fixed, to obtain a spacelike hypersurface \( M' \) containing the MOTS \( \Sigma \) and satisfying the mean curvature assumption. As shown in [17], this can be done so that \( \Sigma \) is (suitably) weakly outermost in \( M' \). Now, suppose, in the setting of Theorem 3.2

1. The null energy condition holds, i.e. \( G(\ell, \ell) \geq 0 \) for all null vectors \( \ell \).
2. There exists \( c > 0 \) such that \( G(u, \ell) \geq c \) for all future directed unit timelike vectors \( u \) and null vectors \( \ell \) such that \( \tilde{g}(u, \ell) = -1 \) in a neighborhood of \( TM \subset T \Sigma \) (see e.g. [3.143.13]).
3. \( \Sigma \) is outer area minimizing in \( M' \) (rather than \( M \)).

Then, as in the proof of Theorem 1.2 in [17], one can show that the conclusion of Theorem 3.2 remains valid without the mean curvature assumption on \( M \).

The Nariai spacetime gives an illustration of Theorem 3.2. This spacetime is a simple exact solution to the vacuum (\( T = 0 \)) Einstein equation (3.11) with positive cosmological constant \( \Lambda > 0 \). It is a metric product of 2-dimensional de Sitter space and \( S^2 \),

\[
\tilde{M} = (\mathbb{R} \times S^1) \times S^2, \quad \tilde{g} = \frac{1}{\Lambda} \left( -dt^2 + \cosh t d\theta^2 + d\Omega^2 \right).
\]
The time-symmetric slice $t = 0$ is foliated by round 2-spheres of radius $\frac{1}{\sqrt{\Lambda}}$, each of which is a MOTS (actually, a totally geodesic surface in this case). As discussed in [9] (see also [23]), the Nariai spacetime may be viewed as a limit of Schwarzschild-de Sitter space, as the size of the black hole increases and its area approaches the upper bound in (3.10), with $c = \Lambda$. Non-time-symmetric spacelike hypersurfaces of the form $t = t(\theta)$ in the Nariai spacetime are similarly foliated by round MOTS with vanishing outer null second fundamental form, consistent with Theorem 3.2 and the remark above.

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