JORDAN SZABÓ ALGEBRAIC COVARIANT DERIVATIVE CURVATURE TENSORS

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Abstract. We show that if $R$ is a Jordan Szabó algebraic covariant derivative curvature tensor on a vector space of signature $(p, q)$, where $q \equiv 1 \mod 2$ and $p < q$ or $q \equiv 2 \mod 4$ and $p < q - 1$, then $R = 0$. This algebraic result yields an elementary proof of the geometrical fact that any pointwise totally isotropic pseudo-Riemannian manifold with such a signature $(p, q)$ is locally symmetric.

1. Introduction

Let $V$ be a vector space which is equipped with a non-degenerate inner product $(\cdot, \cdot)$ of signature $(p, q)$. Let $m := p + q = \dim V$. Let $S^\pm(V)$ be the pseudo-spheres of unit spacelike (+) and timelike (−) vectors in $V$:

$$S^\pm(V) := \{ v \in V : (v, v) = \pm 1 \}.$$

We now review briefly some facts from linear algebra which we shall need in what follows. Let $A$ be a linear map from $V$ to $V$. We say $A$ is self-adjoint if $(Av, w) = (v, Aw)$ for all $v, w \in V$; let $A(V)$ be the vector space of all self-adjoint linear maps from $V$ to $V$. If $A$ is self-adjoint and if $\lambda \in \mathbb{C}$, then define real operators $A_\lambda$ on $V$ and associated generalized subspaces $E_\lambda$ by setting:

$$A_\lambda := \begin{cases} A - \lambda \cdot \text{Id} & \text{if } \lambda \in \mathbb{R}, \\ (A - \lambda \cdot \text{Id})(A - \bar{\lambda} \cdot \text{Id}) & \text{if } \lambda \in \mathbb{C} - \mathbb{R}, \end{cases}$$

$$E_\lambda := \ker \{ A_\lambda^m \}. \tag{1.1}$$

Since $A_\lambda = A_{\bar{\lambda}}$, $E_\lambda = E_{\bar{\lambda}}$. Both $A$ and $A_\lambda$ preserve each generalized eigenspace $E_\lambda$. The operator $A$ is said to be Jordan simple if $A_\lambda = 0$ on $E_\lambda$ for all $\lambda$. We say $\lambda$ is an eigenvalue of $A$ if $\dim \{ E_\lambda \} > 0$; the spectrum $\text{Spec} \{ A \} \subset \mathbb{C}$ is the set of complex eigenvalues of $A$.

Lemma 1.1. Let $A$ be a self-adjoint linear map of a vector space of signature $(p, q)$.

1. We may decompose $V = \oplus_{\text{Im} (\lambda) \geq 0} E_\lambda$.
2. We have $E_\lambda \perp E_\mu$ if $\lambda \neq \mu$ and $\lambda \neq \bar{\mu}$.
3. The spaces $E_\lambda$ inherit non-degenerate metrics of signature $(p_\lambda, q_\lambda)$.
4. If $\lambda \notin \mathbb{R}$, then $p_\lambda = q_\lambda$.

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We say that \( R \in \odot^4 V^* \) is an algebraic curvature tensor on \( V \) if \( R \) satisfies the symmetries:
\[
R(x, y, z, w) = R(z, w, x, y) = -R(y, x, z, w),
\]
\[
R(x, y, z, w) + R(y, z, x, w) + R(z, x, y, w) = 0.
\]
We say that \( \mathfrak{R} \in \odot^5 V^* \) is an algebraic covariant derivative curvature tensor on \( V \) if \( \mathfrak{R} \) satisfies the symmetries:
\[
\mathfrak{R}(x, y, z, w; v) = \mathfrak{R}(z, w, x, y; v) = -\mathfrak{R}(y, x, z, w; v),
\]
\[
\mathfrak{R}(x, y, z, w; v) + \mathfrak{R}(y, z, x, w; v) + \mathfrak{R}(z, x, y, w; v) = 0,
\]
\[
\mathfrak{R}(x, y, z, w; v) + \mathfrak{R}(x, y, w, v; z) + \mathfrak{R}(x, y, v, z; w) = 0.
\]

Let \((M, g)\) be a pseudo-Riemannian manifold of signature \((p, q)\). The Riemann curvature tensor \( R_g \) is an algebraic curvature tensor on the tangent space \( T_P M \) for every point \( P \in M \). Similarly, the covariant derivative of the curvature tensor \( \nabla R_g \) is an algebraic covariant derivative curvature tensor on \( T_P M \). Conversely, such tensors are geometrically realizable. Thus tensors with the symmetries of equations (1.2) and (1.3) are important in differential geometry.

Let \( R \) be an algebraic curvature tensor on \( V \). The associated Jacobi operator \( J \) is the linear map of \( V \) characterized by the identity:
\[
(J(v)y, z) = R(y, v, v, z);
\]
\( J(v) \) is self-adjoint by equation (1.2). We say that \( R \) is a spacelike Osserman (resp. timelike Osserman) tensor if \( R \) is an algebraic curvature tensor and if \( \text{Spec} \{ J \} \) is constant on \( S^+(V) \) (resp. on \( S^-(V) \)); these two notions are equivalent \([4]\) and such a tensor is said to be an Osserman tensor.

Analogously, let \( \mathfrak{R} \) be an algebraic covariant derivative curvature tensor on \( V \). The associated Szabó operator \( \mathcal{S} \) is the linear map of \( V \) characterized by the identity:
\[
(\mathcal{S}(v)y, z) = \mathfrak{R}(y, v, v, z);
\]
the symmetries of equation (1.3) show \( \mathcal{S} \) is self-adjoint. We say that \( \mathfrak{R} \) is a spacelike Szabó (resp. timelike Szabó) tensor if \( \mathfrak{R} \) is an algebraic covariant derivative curvature tensor and if \( \text{Spec} \{ \mathcal{S} \} \) is constant on \( S^+(V) \) (resp. \( S^-(V) \)); we denote this common spectrum by \( \text{Spec}^+(\mathcal{S}) \) (resp. \( \text{Spec}^-\{\mathcal{S}\} \)). The notions spacelike Szabó and timelike Szabó are equivalent \([4]\); such a tensor is said to be a Szabó tensor.

Osserman \([1]\) and Szabó \([4]\) wrote the original papers concerning the spectral properties of the operators \( J \) and \( \mathcal{S} \) in the Riemannian setting. That is why their names have become associated with the subject. The Jacobi operator has been studied extensively in this context; we refer to \([3, 4]\) for a more complete bibliography. By contrast, the Szabó operator has received considerably less attention and the present paper is devoted to the study of the spectral properties of \( \mathcal{S} \) in the pseudo-Riemannian setting. Although there are certain formal parallels between the Jacobi operator and the Szabó operator, the fact that \( J(-v) = J(v) \) while \( \mathcal{S}(v) = -\mathcal{S}(-v) \) plays a crucial role.

In the higher signature setting, the eigenvalue structure does not control the Jordan normal form (i.e. the conjugacy class) of a self-adjoint map. We shall say that \( \mathfrak{R} \) is a spacelike (resp. timelike) Jordan Szabó tensor if \( \mathfrak{R} \) is an algebraic
covariant derivative curvature tensor and if the Jordan normal form of $\mathfrak{g}$ is constant
on $S^+(V)$ (resp. $S^-(V)$).

The following result is due in the Riemannian setting ($p=0$) to Szabó [14] and in the
Lorentzian setting ($p=1$) to Gilkey and Stavrov [3]:

**Theorem 1.2.** Let $\mathfrak{K}$ be a Szabó tensor on a vector space of signature $(p,q)$. If
$p=0$ or if $p=1$, then $\mathfrak{K} = 0$.

By replacing $g$ by $-g$, one can interchange the roles of $p$ and $q$. Consequently
Theorem 1.2 implies there are no non-trivial Szabó tensors if $q = 0$ or if $q = 1$ as well.
Therefore, we shall assume $p \geq 2$ and $q \geq 2$ henceforth. Although we are
primarily interested in spacelike or timelike Jordan Szabó tensors, many of
our results hold under the weaker assumption that the tensor is Szabó. We shall
establish the following result in Section 3.

**Theorem 1.3.** Let $\mathfrak{K}$ be a Szabó tensor on a vector space $V$ of signature $(p,q)$,
where $p \geq 2$ and $q \geq 2$. Then

1. $\text{Spec}^\pm \{ \mathfrak{g} \} = -\text{Spec}^\pm \{ \mathfrak{g} \} = \sqrt{-1} \text{Spec}^\mp \{ \mathfrak{g} \}$;
2. $\text{Spec}^\pm \{ \mathfrak{g} \} \subset \mathbb{R} \cup \sqrt{-1} \mathbb{R}$;
3. If $p < q$, then $\text{Spec}^+ \{ \mathfrak{g} \} \subset \sqrt{-1} \mathbb{R}$ and $\text{Spec}^- \{ \mathfrak{g} \} \subset \mathbb{R}$;
4. If $q < p$, then $\text{Spec}^+ \{ \mathfrak{g} \} \subset \mathbb{R}$ and $\text{Spec}^- \{ \mathfrak{g} \} \subset \sqrt{-1} \mathbb{R}$.

A pseudo-Riemannian manifold $(M,g)$ of signature $(p,q)$ is said to be a Szabó
manifold if $\nabla R_g$ is a Szabó tensor on $T_PM$ for all $P \in M$. If $p \geq 2$ and if $q \geq 2$, then
there exist Szabó pseudo-Riemannian manifolds and algebraic covariant derivative
curvature tensors of signature $(p,q)$ which are neither locally symmetric, nor locally
homogeneous, nor pointwise totally isotropic [2, 3].

Let $S^{q-1}$ be the unit sphere in $\mathbb{R}^q$ with the usual positive definite inner product.
If $q$ is odd, then there are no non-vanishing vector fields on $S^{q-1}$. This well known
result has been generalized by Adams [1]. The **Adams number** $\nu(q)$ is defined as
follows. Let $q = 2^\ell q_0$, where $q_0$ is odd. Then $\nu(q) = \nu(2^\ell)$, where

\begin{align}
\nu(1) & = 0, \quad \nu(2) = 1, \quad \nu(4) = 3, \quad \nu(8) = 7, \quad \text{and} \quad \nu(2^{\ell+4}) = 2^{\ell} + 8. \\
\end{align}

Let $\{v_1, \ldots, v_\ell\}$ be linearly independent vector fields on $S^{q-1}$. Adams showed that
$\ell \leq \nu(q)$; thus $\nu(q)$ provides an upper bound to the number of linearly independent
vector fields that can exist on $S^{q-1}$. Furthermore, the estimate is sharp. If $\nu(q) > 0$,
then there exist $\nu(q)$ linearly independent vector fields on $S^{q-1}$. Since $\nu(q) = 0$
if $q$ is odd, the result of Adams contains as a special case the original observation
that there does not exist a nowhere vanishing vector field on an even dimensional
sphere.

If $p \geq 2$ and if $q \geq 2$, then there non-trivial nilpotent Szabó pseudo-Riemannian
manifolds [2, 3]. However, there are as yet no known examples of spacelike Jordan
Szabó tensors $\mathfrak{K}$, where $\mathfrak{K} \neq 0$, and we conjecture none exist. We have some partial
results in this direction. In section 3, we shall establish the following result:

**Theorem 1.4.** Let $\mathfrak{K}$ be a spacelike Jordan Szabó tensor on a vector space $V$ of
signature $(p,q)$, where $p < q$. Let $v \in S^+(V)$. Then:

1. $\mathfrak{g}(v)$ is Jordan simple;
2. If $p < q - \nu(q)$, then $\text{rank} \{ \mathfrak{g}(v) \} \leq 2\nu(q)$;
3. If $q$ is odd, then $\mathfrak{K} = 0$. 

If $\mathcal{R}$ is spacelike and timelike Jordan Szabó, then we let $r_{\pm}$ be the rank of $\mathcal{S}$ on the pseudo-spheres $S^\pm(V)$. In Section 4, we shall prove:

Theorem 1.5. Let $\mathcal{R} \neq 0$ be an algebraic covariant derivative curvature tensor on a vector space $V$ of signature $(p, q)$ which is both spacelike and timelike Jordan Szabó. Then:

1. $r_+ = r_-$.
2. If $p \neq q$, then $\mathcal{S}(v)$ is Jordan simple for $v \in S^+(V)$.

We say $\mathcal{R}$ is null Jordan Szabó if the Jordan normal form of $\mathcal{S}$ is constant on the null cone $\mathcal{N} := \{v \in V : (v, v) = 0\} - \{0\}$; this implies $\mathcal{S}$ is nilpotent and has constant rank, which we shall denote by $r_0$, on $\mathcal{N}$. We say that $\mathcal{R}$ is Jordan Szabó if $\mathcal{R}$ is spacelike, timelike, and null Jordan Szabó. In Section 5, we shall prove:

Theorem 1.6. Let $\mathcal{R}$ be an algebraic covariant derivative curvature tensor on a vector space $V$ of signature $(p, q)$ which is Jordan Szabó. Then

1. If $\mathcal{R} \neq 0$, then $r_0 < r_+$.
2. If $q \equiv 2 \mod 4$ and if $p < q - 1$, then $\mathcal{R} = 0$.

Following Wolf [15], we say that a pseudo-Riemannian manifold $(M, g)$ is locally isotropic if given a point $P \in M$ and nonzero tangent vectors $X$ and $Y$ in $T_PM$ with $(X, X) = (Y, Y)$, there is a local isometry of $(M, g)$, fixing $P$, which sends $X$ to $Y$. Wolf showed that such a manifold is necessarily locally symmetric, see [15] (Theorem 12.3.1). The Szabó operator of a locally isotropic pseudo-Riemannian manifold is necessarily Jordan Szabó. Thus Wolf’s result in certain special cases can be derived from Theorems 1.4 and 1.5.

Corollary 1.7. Let $(M, g)$ be a locally isotropic pseudo-Riemannian manifold of signature $(p, q)$. If $q \equiv 1 \mod 2$ and if $p < q$ or if $q \equiv 2 \mod 4$ and if $p < q - 1$, then $(M, g)$ locally symmetric.

2. Spacelike Szabó Tensors

In this section we prove Theorem 1.3. Let $V$ be a vector space which is equipped with a non-degenerate inner product of signature $(p, q)$, where $p \geq 2$ and $q \geq 2$. We can choose a non-canonical decomposition

\[ V = V^+ \oplus V^-, \]

where $V^+$ is a maximal spacelike subspace and where $V^- := (V^+)\perp$ is the complementary maximal timelike subspace. Let $\pi^\pm$ be orthogonal projection on $V^\pm$. The following is a useful technical observation.

Lemma 2.1. Let $v \in S^+(V^+)$. If $w \perp v$, then $\pi^+w \in V^+ \cap v^\perp = T_v(S^+(V^+))$.

Proof. We may decompose $w = cv + v^+ + v^-$, where $v^\pm \in V^\pm$ and $v \perp v^+$. Because $c = (v, w) = 0$, we have $\pi^+w = v^+$.

The next result is well known, see, for example [3], and generalizes the decomposition of equation (2.1) to the bundle setting:
Lemma 2.2. Let $E$ be a vector bundle over a smooth manifold $M$ which is equipped with a non-degenerate fiber metric. Then we can decompose $E$ as a direct sum $E^+ \oplus E^-$ of complementary orthogonal subbundles, where $E^+$ is a maximal spacelike subbundle and $E^- = (E^+)\perp$ is a maximal timelike subbundle.

Let $\mathfrak{R}$ be a Szabó tensor on $V$. Since $\mathfrak{R}(-v) = -\mathfrak{R}(v)$,

$$\text{Spec} \{ \mathfrak{S}(v) \} = -\text{Spec} \{ \mathfrak{S}(-v) \} \quad \text{so} \quad \text{Spec} \{ \mathfrak{S}^\pm \} = -\text{Spec} \{ \mathfrak{S} \}. $$

We now establish Theorem 1.3 (1). Let $\mathfrak{R}$ be a Szabó tensor on a vector space of signature $(p, q)$ for $p \geq 2$ and $q \geq 2$. Let $V_C := V \otimes \mathbb{C}$ be the complexification of $V$. We extend the inner product and the tensor $\mathfrak{R}$ to be complex multilinear on $V_C$. Let

$$\mathcal{O}_C : = \{ v \in V_C : (v, v) \neq 0 \}, $$

$$\mathfrak{P}(v) : = (v, v)^{-3} \mathfrak{S}(v)^2 \text{ for } v \in \mathcal{O}_C, \text{ and}$$

$$p(t, v) : = \det \{ \mathfrak{P}(v) - t \cdot \text{Id} \} = c_0(v) + \ldots + c_m(v)t^m \text{ for } v \in \mathcal{O}_C.$$ 

The set $\mathcal{O}_C$ is a connected open subset of $V_C$ and the coefficients $c_i(v)$ are holomorphic functions on $\mathcal{O}_C$. The intersection of $\mathcal{O}_C$ with the real subspace $V$ is the disjoint union of the non-empty sets $\mathcal{O}_{\pm}$ of spacelike and timelike vectors in $V$. Since $\mathfrak{R}$ is a Szabó tensor, $\text{Spec} \{ \mathfrak{P} \}$, the characteristic polynomial $p$, and hence the functions $c_i$ are constant on $\mathcal{O}_{\pm}$. By the identity theorem, the functions $c_i$ are constant on $\mathcal{O}_C$. Let $v_{\pm} \in S^\pm(V)$. Because $p(t, v^-) = p(t, v^+)$, the operators $\mathfrak{P}(v^-) = -\mathfrak{S}(v^-)^2$ and $\mathfrak{P}(v^+) = \mathfrak{S}(v^+)^2$ have the same spectrum. This completes the proof of assertion (1) of Theorem 1.3 by establishing the identity:

$$\text{Spec} \{ \mathfrak{S}(v^-) \} = \sqrt{-1} \text{ Spec} \{ \mathfrak{S}(v^+) \}.$$ 

Before continuing to prove the other assertions of Theorem 1.3 we must recall some facts from algebraic topology. Assertion (1) in the following Lemma follows from results of Szabó [14], and assertion (2) follows from the Borsuk-Ulam theorem (see for example Spanier [12], Corollary 8 p. 266). They concern $\mathbb{Z}_2$ equivariant vector fields and vector valued functions on spheres.

Lemma 2.3. Let $S^{q-1}$ be the unit sphere in a vector space of signature $(0, q)$.

1. There does not exist a continuous nowhere vanishing vector field $F$ on $S^{q-1}$ so $F(v) = F(-v)$ for every $v \in V$.

2. If $\dim \{ V^- \} = p < q$, then there does not exist a continuous nowhere vanishing map $F$ from $S^{q-1}$ to $V$ so that $F(v) = -F(-v)$ for every $v \in S^{q-1}$.

Recall that $\mathcal{A}(V)$ is the set of all self-adjoint linear maps of $V$. We adopt the notation of Lemma 1.1. Let $V = V^+ \oplus V^-$ be the decomposition described in equation (2.1). The remaining assertions of Theorem 1.3 will be proved, as has become traditional in this subject, using methods from algebraic topology. The essential point is to abstract precisely those properties relevant to the investigation. What will be crucial for us are the facts that:

$$\mathfrak{S}(-v) = -\mathfrak{S}(v), \quad \mathfrak{S}(v)v = 0, \quad \text{and}$$

$$\mathfrak{S}(v) \in \mathcal{A}(V) \quad \text{for all} \quad v \in S^{q-1} := S^+(V^+).$$

Lemma 2.4. Let $V$ be a vector space of signature $(p, q)$. Let $V^+$ be a maximal spacelike subspace of $V$ and let $S^{q-1} := S^+(V^+)$. Let $A : v \rightarrow A_v$ be a continuous
map from $S^{q-1}$ to $\mathcal{A}(V)$ with $A_{-v} = -A_v$ and with $A_v v = 0$ for all $v \in S^{q-1}$. Assume that $A$ has constant spectrum. Then:

1. If $\lambda \in \text{Spec} \{A\}$ with $\lambda \notin \sqrt{-1} \mathbb{R}$, then $q_{\lambda} = 0$;
2. $\text{Spec} \{A\} \subset \mathbb{R} \cup \sqrt{-1} \mathbb{R}$;
3. If $p < q$, then $\text{Spec} \{A\} \subset \sqrt{-1} \mathbb{R}$.

Proof. Let $\pi^\pm$ be orthogonal projection on $V^\pm$. Consider the following vector bundles on $S^{q-1}$:

$$TS^{q-1} := \{(v_1, v_2) \in S^{q-1} \times V^+ : (v_1, v_2) = 0\}, \quad \text{and} \quad TS^{q-1} \oplus V^- := \{(v_1, v_2) \in S^{q-1} \times V : (v_1, v_2) = 0\}.$$

Let $A : S^{q-1} \to \mathcal{A}(V)$ satisfy the hypothesis of the Lemma. Since $A$ has constant spectrum, the generalized eigenspaces $E_{\lambda,v}$ of $A_v$, as described in Lemma 1.1, have constant rank and fit together to define smooth vector bundles $E_\lambda$ over $S^{q-1}$.

Suppose $\lambda \in \text{Spec} \{A\}$ and $\lambda \notin \sqrt{-1} \mathbb{R}$. We apply Lemma 1.1, $0 \neq \lambda$, $-\lambda \neq \lambda$ and $-\lambda \neq \lambda$. Thus $E_{0,v}$, $E_{\lambda,v}$, and $E_{-\lambda,v}$ are mutually orthogonal vector spaces which inherit non-degenerate metrics of signatures $(p_0, q_0)$, $(p_\lambda, q_\lambda)$, and $(p_{-\lambda}, q_{-\lambda})$, respectively. As $A_v v = 0$, $v \in E_{0,v}$ so $v \perp E_{\lambda,v}$. By Lemma 2.2, there exist maximal orthogonal timelike and spacelike sub-bundles $E_{\lambda}^\pm$ of $E_\lambda$ so:

$$E_\lambda = E_\lambda^+ \oplus E_{\lambda}^-.$$

Since $A_{-v} = -A_v$, we have $E_{\lambda,-v} = E_{-\lambda,v}$. Thus we may obtain a similar splitting of the bundle $E_{-\lambda}$ by setting:

$$E_{-\lambda,v}^\pm := E_{\lambda,-v}^\pm.$$

Let $\varepsilon$ be a choice of signs $\pm$. We suppose $\dim \{E_{\lambda}^\varepsilon\} > 0$; this means $p_\lambda > 0$ if $\varepsilon$ is $-$ and $q_\lambda > 0$ if $\varepsilon$ is $+$. Let $N$ be the north pole of $S^{q-1}$. Since $S^{q-1} - \{N\}$ is contractable, there exists a global section $s_\lambda$ to $E_\lambda^\varepsilon$ which only vanishes at $N$. Let $s_{-\lambda}(v) := s_\lambda(-v)$. Then $s_{-\lambda}$ is a section to $E_{-\lambda}^\varepsilon$ which only vanishes at the south pole $S := -N$ and which satisfies $s_{-\lambda}(v) \perp s_{\lambda}(v)$. Let

$$F := s_\lambda + \varepsilon s_{-\lambda}.$$

If $\varepsilon$ is $+$, then $s_\lambda$ and $s_{-\lambda}$ are spacelike; if $\varepsilon$ is $-$, then $s_\lambda$ and $s_{-\lambda}$ are timelike. Since $s_{\lambda} \perp s_{-\lambda}$, $F$ is a nowhere vanishing section to $E_\lambda^\varepsilon \oplus E_{-\lambda}^\varepsilon$.

To prove assertion (1) of Lemma 2.4, we suppose that $q_{\lambda} > 0$ and argue for a contradiction. We take $\varepsilon$ to be $+$. Then $F(v)$ is nowhere vanishing, spacelike, and perpendicular to $v$ so Lemma 2.3 shows that $\pi^+ F(v)$ is a nowhere vanishing vector field on $S^{q-1}$ with the equivariance property $\pi^+ F(v) = \pi^+ F(-v)$. This contradicts Lemma 2.3 (1) and proves Lemma 2.4 (1).

If $\lambda \in \text{Spec} \{A\}$ and if $\lambda \notin \mathbb{R}$, then $p_{\lambda} = q_{\lambda} \neq 0$. Assertion (1) then implies $\lambda \in \sqrt{-1} \mathbb{R}$. Thus $\text{Spec} \{A\} \subset \mathbb{R} \cup \sqrt{-1} \mathbb{R}$ which establishes assertion (2).

Let $p < q$. To prove assertion (3), we assume that there exists an eigenvalue $\lambda$ belonging to $\mathbb{R} - \{0\}$ and argue for a contradiction. We take $\varepsilon$ equal to $-$. As $F(v)$ is timelike and non-vanishing, $\pi^- F(v)$ is a nowhere vanishing continuous map from $S^{q-1}$ to $V$ with the equivariance property $\pi^- F(v) = -\pi^- F(-v)$. This contradicts Lemma 2.3 (2) and proves assertion (3) of Lemma 2.4. \qed
Proof of Theorem 1.3. Let \( \mathcal{R} \) be a spacelike Szabó tensor on \( V \). We have already established assertion (1) of the Theorem. Assertions (2), (3), and (4) follow from Lemma 2.4, where, if necessary, we change the sign of the quadratic form to interchange the roles of \( p \) and \( q \) and the notions spacelike and timelike. \( \square \)

3. Spacelike Jordan Szabó Tensors

We begin the preparation of Theorem 1.4 by recalling some topological results. Let \( \mathbb{RP}^{q-1} \) be real projective space; this is the space of lines thru the origin in \( \mathbb{R}^q \). If \( v \in S^{q-1} \), then we denote the corresponding element of \( \mathbb{RP}^{q-1} \) by \( \langle v \rangle = v \cdot \mathbb{R} \). We may identify \( \mathbb{RP}^{q-1} = S^{q-1}/\mathbb{Z}_2 \). The classifying line bundle and orthogonal complement bundle over projective space are defined by

\[
L := \{ (v), w) \in \mathbb{RP}^{q-1} \times \mathbb{R}^q : w \in \langle v \rangle \}
\]

\[
L^\perp := \{ (v), w) \in \mathbb{RP}^{q-1} \times \mathbb{R}^q : v \perp w \}.
\]

Note that we may identify \( T(\mathbb{RP}^{q-1}) \) with \( L \oplus L^\perp \). Let \( \nu(q) \) be the Adams number which was defined in equation (1.4). We refer to [5] for the proof of the first assertion and to Adams [1] for the proof of the second assertion in the following Lemma:

Lemma 3.1.

1. Let \( U_1 \) be non-trivial vector bundles over \( \mathbb{RP}^{q-1} \). If \( U_1 \) is a sub-bundle of \( L^\perp \) and if \( U_2 \) is a sub-bundle of a trivial bundle of dimension \( p < q \), then \( U_1 \) is not isomorphic to \( U_2 \).
2. Let \( S^{q-1} \) be the unit sphere in \( \mathbb{R}^q \). Let \( \nu(q) \) be the Adams number. Then:
   a. If \( E \) is a rank \( r \) sub-bundle of \( TS^{q-1} \), then either \( q - 1 - \nu(q) \leq r \) or \( r \leq \nu(q) \);
   b. If \( \{ \chi_1, ..., \chi_\ell \} \) are linearly independent vector fields on \( S^{q-1} \), then we have \( \ell \leq \nu(q) \).

Again, the properties of equation (2.3) are crucial. We begin the proof of Theorem 1.4 by using Lemma 3.1 to establish a related result in the abstract setting. As in Lemma 2.4, we decompose \( V = V^- \oplus V^+ \) and identify \( S^{q-1} \) with the unit sphere in \( V^+ \).

Lemma 3.2. Let \( V \) be a vector space of signature \( (p, q) \), where \( p < q \). Let \( V^+ \) be a maximal spacelike subspace of \( V \) and let \( S^{q-1} := S^+(V^+) \). Let \( A : v \to A_v \) be a continuous map from \( S^{q-1} \) to \( A(V) \) with \( A_{-v} = -A_v \) and with \( A_v v = 0 \) for all \( v \in S^{q-1} \). Assume that \( A \) has constant Jordan normal form. Then:

1. The operator \( A_v \) is Jordan simple for any \( v \in S^{q-1} \);
2. If \( p < q - \nu(q) \), then \( \text{rank} \{ A \} \leq 2 \cdot \nu(q) \).
3. If \( q \) is odd, then \( A = 0 \).

Proof. By Lemma 2.4, \( \text{Spec} \{ A \} \subset \sqrt{-1} \mathbb{R} \). We use equation (1.1) to define the operators \( A_{\lambda,v} \) for \( \lambda \in \text{Spec} \{ A_v \} \). Since \( \lambda \) is purely imaginary,

\[
A_{\lambda,v} = \begin{cases} 
A_v^2 + |\lambda|^2 & \text{if } \lambda \neq 0, \\
A_v & \text{if } \lambda = 0.
\end{cases}
\]

Since \( A_v v = 0 \), \( A_v \) preserves \( v^\perp \). We therefore introduce the reduced generalized eigenspaces

\[
\tilde{E}_{v,\lambda} := \ker \{ A_{\lambda,v}^m \} \cap v^\perp.
\]
To prove assertion (1), we suppose, to the contrary, that \( A_{\lambda,v} \) is not Jordan simple. Since \( A_v = 0 \), this means that \( A_{\lambda,v} \not\equiv 0 \) on the generalized eigenspaces \( E_{v,\lambda} \). Choose \( k \geq 1 \) maximal so \( A_{\lambda,v}^k \not\equiv 0 \) on \( E_{v,\lambda} \). Then \( A_{\lambda,v}^{k+1} = 0 \). Let
\[
U_{\lambda,v} := A_{\lambda,v}^k E_{v,\lambda}.
\]
Because \( A_v \) has constant Jordan normal form, the vector spaces \( U_{\lambda,v} \) glue together smoothly to define a smooth vector bundle \( U_{\lambda} \) over \( S^{q-1} \). This bundle descends to a smooth bundle \( W_{\lambda} \) over \( \mathbb{R}P^{q-1} = S^{q-1}/\mathbb{Z}_2 \) because, by equation (3.1)
\[
A_{\lambda,v} = \pm A_{\lambda,-v}.
\]

Let \( w_i = A_{\lambda,v}^k v_i \) for \( v_i \in E_{v,\lambda} \). Since \( k \geq 1 \), \( 2k > k \) so:
\[
(3.2) \quad 0 = (A_{\lambda,v}^k v_1, v_2) = (A_{\lambda,v}^k v_1, A_{\lambda,v}^k v_2) = (w_1, w_2).
\]

By equation (3.2), the bundle \( W_{\lambda} \) is totally isotropic. Since \( W_{\lambda} \) contains no spacelike or timelike vectors, the projections \( \pi^\pm \) are non-singular on \( W_{\lambda} \). Set
\[
W_{\lambda}^\pm := \pi^\pm(W_{\lambda}).
\]

If \( w \in W_{\lambda,v} \), then \( w \perp v \). Consequently, by Lemma 2.1, \( \pi^+ w \in V^+ \cap v^\perp \). It is immediate that \( \pi^- w \in V^- \). Therefore,
\[
W_{\lambda}^+ \subset L^+ \quad \text{and} \quad W_{\lambda}^- \subset V^-.
\]

As \( W_{\lambda}^+ \), \( W_{\lambda} \), and \( W_{\lambda}^- \) are isomorphic, this contradicts Lemma 3.1 and proves Lemma 3.2 (1).

By assertion (1), \( A_v = 0 \) on \( E_{\lambda,v} \). Thus
\[
(3.3) \quad \text{rank } \{A\} = 2\sigma \quad \text{where } \sigma := \sum_{\Im(\lambda) > 0} p_\lambda = \sum_{\Im(\lambda) > 0} q_\lambda.
\]

By Lemma 2.2, choose a maximal spacelike sub-bundle \( \tilde{E}_{v,\lambda}^+ \) of \( \tilde{E}_{v,\lambda} \) of rank \( q_\lambda \). Let
\[
(3.4) \quad \tilde{E}^+ = \bigoplus_{\Im(\lambda) > 0} \tilde{E}_{v,\lambda}^+
\]
be a sub-bundle of \( v^+ \) of rank \( \sigma \). Since elements of \( \tilde{E}^+ \) are all spacelike, the projection \( \pi^+ \) is injective and, by Lemma 2.2, \( \tilde{E}^+ \) is a sub-bundle of \( TS^{q-1} \) of rank \( \sigma \). By Lemma 3.2 (2), either \( \sigma \leq \nu(q) \) or \( \sigma \geq q - 1 - \nu(q) \). Since the first inequality implies assertion (2) of Lemma 3.2, we assume, to the contrary, that
\[
(3.5) \quad \sigma \geq q - 1 - \nu(q).
\]

Displays (3.3) and (3.5) imply
\[
(3.6) \quad p \geq \sigma \geq q - 1 - \nu(q).
\]

We assumed that \( p < q - \nu(q) \). Therefore, by (3.6), we have:
\[
(3.7) \quad \sigma = p = q - 1 - \nu(q).
\]

Let \( e \in \tilde{E}_{\lambda,v}^+ \). By assertion (1), \( A_v \) is Jordan simple. Thus \( (A_v^2 + |\lambda|^2)e = 0 \) so
\[
(3.8) \quad (A_v e, A_v e) = (A_v^2 e, e) = -|\lambda|^2(e, e).
\]

The decomposition of equation (3.8) is an orthogonal direct sum. Equation (3.7) shows that \( A_v \tilde{E}_{\lambda,v}^+ \) is a timelike subspace. Because \( \text{rank } \{A_v \tilde{E}_{\lambda,v}^+ \} = p \), \( \pi^- \) is an isomorphism from \( A_v \tilde{E}_{\lambda,v}^+ \) onto \( V^- \). Consequently, \( \pi^+ \tilde{E}^+ \) is isomorphic to a trivial bundle so there exist \( p \) linearly independent vector fields on \( S^{q-1} \). This shows \( \sigma \leq p \leq \nu(q) \) and completes the proof of assertion (2).
Assertion (3) follows directly from assertion (2) since \( \nu(q) = 0 \) if \( q \) is odd. \( \square \)

The following observation is well known, see, for example, [7]:

**Lemma 3.3.** Let \( \mathcal{R} \) be an algebraic covariant derivative curvature tensor. If \( \mathcal{S} \) vanishes identically, then \( \mathcal{R} = 0 \).

**Proof of Theorem 1.4.** Let \( \mathcal{R} \) be a spacelike Jordan Szabó tensor on a vector space \( V \) of signature \((p,q)\), where \( p < q \). By equation (2.2), we may apply Lemma 3.2. Assertions (1) and (2) of Theorem 1.4 now follow directly from assertions (1) and (2) of Lemma 3.2. If \( q \) is odd, we apply Lemma 3.2 (3) to see \( \mathcal{S} = 0 \). Assertion (3) now follows from Lemma 3.3. \( \square \)

4. **Timelike and spacelike Jordan Szabó Tensors**

Let \( \mathcal{R} \) be both timelike and spacelike Jordan Szabó. Let \( r_\pm := \text{rank} \{ \mathcal{S}(\cdot,\cdot) \} \) on \( S^\pm(V) \). To establish Theorem 1.5 (1), we must show that \( r_+ = r_- \). Fix a vector \( v^+ \in S^+(V) \). Choose vectors \( \{ v_1, ..., v_{r_+} \} \) in \( V \) and choose dual elements \( \{ v^*_1, ..., v^*_{r_+} \} \) in the dual vector space \( V^* \) so

\[
v^*_i \mathcal{S}(v^+) = \delta_{ij}.
\]

We adopt the notation established in the proof of Lemma 2.2 and complexify. If \( v \in V_C \), then we consider the matrix \( A_{ij}(v) := v^*_i \mathcal{S}(v)v_j \). We define a holomorphic function on \( V_C \) by setting:

\[
f(v) := \det \{ v^*_i \mathcal{S}(v)v_j \}.
\]

Since \( f(v^+) = 1 \), \( f \) does not vanish identically on \( V_C \). Thus by the identity theorem, there must exist a real timelike element \( v^- \) of \( V \) so \( f(v^-) \neq 0 \). This implies that the elements \( \{ \mathcal{S}(v^-)v_1, ..., \mathcal{S}(v^-)v_{r_+} \} \) are linearly independent so

\[
r_- = \text{rank} \{ \mathcal{S}(v^-) \} \geq r_+.
\]

Similarly we can show \( r_+ \geq r_- \). Thus \( r_+ = r_- \).

To prove the second assertion, we may suppose without loss of generality that \( p < q \). If \( v \in V_C \), then we shall let \( E_\lambda^- \subset V_C \) be the generalized complex eigenspaces of \( \mathcal{S}(v) \). Let \( v_\pm \in S^\pm(V) \). As, by Theorem 1.4, \( \mathcal{S}(v^+) \) is Jordan simple,

\[
r_+ = \sum_{\lambda \neq 0} \text{dim}_C \{ E_\lambda^+ \}.
\]

In Section 2, we showed that the characteristic polynomials of the two operators \( \mathcal{S}(v^+) \) and \( \sqrt{-1} \mathcal{S}(v^-) \) were the same. Thus

\[
r_+ = \sum_{\lambda \neq 0} \text{dim}_C \{ E_\lambda^+ \mathcal{S}(v^-) \}).
\]

Since \( r_+ = r_- \), we conclude that \( \mathcal{S}(v^-) \) must have rank 0 on \( E_0^-(\mathcal{S}(v^-)) \) and consequently 0 is a Jordan simple eigenvalue for \( \mathcal{S}(v^-) \). By Theorem 1.3, the eigenvalues of \( \mathcal{S}(v^-) \) are real. By Lemma 2.4, the generalized eigenspaces for \( \lambda \in \mathbb{R} - \{0\} \) are spacelike. Since \( \mathcal{S}(v^-) \) is self-adjoint with respect to a definite metric on \( E_\lambda \), \( \mathcal{S}(v^-) \) is diagonalizable on \( E_\lambda \). This shows \( \mathcal{S}(v^-) \) is Jordan simple and establishes assertion (2) and completes the proof of Theorem 1.5.
5. Jordan Szabó Tensors

Proof of Theorem 5.6 (1). Let $\mathfrak{R}$ be a non-trivial Jordan Szabó tensor whose rank on the null cone is $r_0$. We must show that $r_0 < r_+$. Let $v_0 \in \mathcal{N}$. As in the proof of Theorem 4.3, choose vectors $\{v_1, ..., v_n\}$ in $V$ and $\{v_1^*, ..., v_n^*\}$ in $V^*$ so $v_i^*\mathfrak{S}(v_0)v_j = \delta_{ij}$. Let $f(v) := \det\{v_i^*\mathfrak{S}(v)v_j\}$. Since $f(v_0) \neq 0$, the holomorphic function $f$ does not vanish identically and thus there exists a timelike vector $v^-$ in $V$ so $f(v^-) \neq 0$. This proves $r_- \geq r_0$.

To show that $r_0 < r_-$, we suppose to the contrary that $r_0 = r_+ = r_-$ and argue for a contradiction; this implies $\mathfrak{S}(v)$ has constant rank $r$ on $V \setminus \{0\}$. Let $V_+$ be a maximal space like subspace of $V$ and let $V_- := V_+^\perp$ be the complementary maximal timelike subspace. If $v \in V$, then we may decompose $v = v^+ + v^-$ for $v^\pm \in V^\pm$. We define a self-adjoint map linear map $\psi$ of $V$ and a positive definite inner product on $V$ by setting:

$$
\psi(v) := v^+ - v^- \quad \text{and} \\
(\psi(v), \psi(w)) := (\psi(v), w) = -(v_-, w_-) + (v_+, w_+).
$$

(5.1)

Define $A(v) := \mathfrak{S}(\psi v)\psi$. We show that $A$ is self-adjoint with respect $(\cdot, \cdot)_e$:

$$
(A(v)v_1, v_2)_e = (\mathfrak{S}(\psi v)v_1, v_2)_e = (\psi \mathfrak{S}(\psi v)v_1, v_2)_e \\
= (v_1, \psi \mathfrak{S}(\psi v_1)) = (v_1, A(v)v_2)_e.
$$

Let $S^{p+q-1} := \{v \in V : (v, v)_e = 1\}$ be the associated unit sphere. Because we have that $A(v)v = \mathfrak{S}(\psi v)\psi v = 0$, we have an orthogonal direct sum decomposition

$$
T_v(S^{p+q-1}) = E_+(v) \oplus E_0(v) \oplus E_-(v),
$$

where $E_+(v)$, $E_-(v)$, and $E_0(v)$ are the span of the eigenvectors of $A(v)$ corresponding to positive eigenvalues, the zero eigenvalue, and negative eigenvalues, respectively. Since $A(-v) = -A(v)$, $E_+(v) = E_-(v)$. As $A$ has constant rank, $\dim(E_\pm)$ is constant so these spaces fit together to define smooth bundles on $S^{p+q-1}$.

Since $\mathfrak{S} \neq 0$, $E_\pm(v) \neq 0$. We choose a section $s_+$ to $E_+$ only vanishing at the north pole. Then $s_-(v) := s_+(v)$ is a section which only vanishes at the south pole and which satisfies $s_+ \perp s_-$. Setting $s := s_+ + s_-$ then constructs a nowhere vanishing section to $T(S^{p+q-1})$ with $s(-v) = s(v)$; this contradicts Lemma 5.3 and completes the proof of Theorem 5.6 (1).

The following is a useful technical lemma.

Lemma 5.1. Let $V$ be a vector space of signature $(p, q)$. Let $X$ be a connected topological space on which $\mathbb{Z}_2$ acts. Let $f$ be a continuous map from $X$ to the space of self-adjoint linear transformations of $V$. Assume that $r = \rank(f)$ is constant and that $f(-x) = -f(x)$. Then $r$ is even.

Proof. We adopt the notation of equation (5.1) to define $(\cdot, \cdot)_e$ and $\psi$. We shall let $\tilde{f}(x) := f(x)\psi$. Decompose the trivial bundle $X \times V = E_- \oplus E_0 \oplus E_+$ as the span of the eigenvectors of $\tilde{f}$ corresponding to positive eigenvalues, the zero eigenvalue, and negative eigenvalues, respectively. Since $f$ has constant rank, these define vector bundles over $X$. Since $E_-(x) = E_+(x)$ and as $X$ is connected, $\dim E_- = \dim E_+$. Since $\tilde{f}$ is self-adjoint with respect to a positive definite inner product, $\tilde{f}$ vanishes on $E_0$. Thus $r = \rank(f) = \rank(\tilde{f}) = \dim E_- + \dim E_+ = 2 \dim E_-$ is even. □
Proof of Theorem 1.6 (2). Let $V$ be a vector space of signature $(p, q)$ where $q \equiv 2 \mod 4$ and where $p < q - 1$. By Theorem 1.2, we may assume $p \geq 2$. Let $\mathfrak{R}$ be a Jordan Szabó tensor on $V$. We must show $\mathfrak{R} = 0$. By equation 1.4, $\nu(q) = 1$.

Because $p < q - 1 = q - \nu(q)$, Theorem 1.4 (2) shows $r_+ \leq 2$. Thus, by Theorem 1.6 (1), $r_0 < 2$. Since by Lemma 5.1 we have $r_0$ is even, we conclude therefore that $r_0 = 0$. Choose a basis $e_i$ for $V$ and expand $\mathcal{G}(x)v_i = \sum_j S_{ij}(x)v_j$. The functions $S_{ij}$ are cubic polynomials in $x$ which vanish identically on the real null-cone. Complexification then extends this relationship to the complex null-cone. Since the polynomial $(x, x)$ is irreducible as $\dim(V) \geq 3$, we may use the Hilbert Nullstellensatz to see there exists a linear function $f(x)$ so that:

$$\mathcal{G}(x) = (x, x)f(x).$$

Since $\mathcal{G}(x)x = 0$, we have $f(x)x = 0$ for $(x, x) \neq 0$ and hence $f(x)x = 0$ for all $x$ by continuity. We polarize this relation to see $f(x)y + f(y)x = 0$ for all $x, y \in V$. Dotting this relationship with $y$ then yields

$$0 = (f(x)y, y) + (f(y)x, y) = (f(x)y, y) + (x, f(y)y) = (f(x)y, y)$$

for all $x, y \in V$. Polarization then yields $(f(x)y, z) + (f(x)z, y) = 0$ for all $x, y, z$ in $V$ and hence, as $f$ is symmetric, $2(f(x)y, z) = 0$ for all $x, y, z$ in $V$. Since the metric on $V$ is non-degenerate, we may conclude that $f(x)$ and hence $\mathcal{G}(x)$ vanishes identically. Theorem 1.6 (2) now follows from Lemma 1.3. 

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**References**

[1] J. Adams, *Vector fields on spheres*, Annals of Math., **75**, (1962), 603-632.

[2] B. Fiedler and P. Gilkey, *Nilpotent Szabó, Osserman and Ivanov-Petrova pseudo-Riemannian manifolds*, (preprint) http://arXiv.org/abs/math.DG/0211080.

[3] E. García-Río, D. Kupeli, and R. Vázquez-Lorenzo, *Osserman Manifolds in Semi-Riemannian Geometry*, Lecture notes in Mathematics, 1777, Springer Verlag (2002), ISBN 3-540-43144-6.

[4] P. Gilkey, *Geometric Properties of Natural Operators Defined by the Riemann Curvature Tensor*, World Scientific Press (2001), ISBN 981-02-04752-4.

[5] P. Gilkey and R. Ivanova, *Spacelike Jordan Osserman algebraic curvature tensors in signature $(p, q)$ for $p < q$*, (preprint) http://arXiv.org/abs/math.DG/0205072.

[6] P. Gilkey, R. Ivanova, and T. Zhang, *Szabó Osserman IP Pseudo-Riemannian manifolds*, (preprint: http://arXiv.org/abs/math.DG/0205087).

[7] P. Gilkey and I. Stavrov, *Curvature tensors whose Jacobi or Szabó operator is nilpotent on null vectors*, Bull. London Math. Soc. (to appear); http://arXiv.org/abs/math.DG/0205071.

[8] P. Gilkey and T. Zhang, *Algebraic curvature tensors for indefinite metrics whose skew-symmetric curvature operator has constant Jordan normal form*, Houston Math J. **28** (2002), 311-328.

[9] S. Ivanov and I. Petrova, *Riemannian manifold in which the skew-symmetric curvature operator has pointwise constant eigenvalues*, Geom. Dedicata, **70**, (1998), 269–282.

[10] R. Ivanova and G. Stanilov, *A skew-symmetric curvature operator in Riemannian geometry*, in *Sympos. Gaussiana*, Conf A, ed. Behara, Fritsch, and Lintz (1995), 391–395.

[11] R. Osserman, *Curvature in the eighties*, Amer. Math. Monthly, **97**, (1990), 731–756.

[12] E. Spanier, *Algebraic Topology*, McGraw-Hill (1966); corrected reprint Springer-Verlag (1981) ISBN: 0-387-90646-0.

[13] I. Stavrov, Ph.D. Thesis, University of Oregon (2003).

[14] Z. I. Szabó, *A short topological proof for the symmetry of 2 point homogeneous spaces*, Invent. Math., **106**, (1991), 61–64.

[15] J. Wolf, *Spaces of constant curvature*, 5th ed., Publish or Perish Inc. (1984), ISBN 0-914098-07-1
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