We consider the equal area laws in the charge-electric potential plane for the charged Anti-de Sitter black holes. Besides the conventional area law, a novel one is presented and studied in detail. Based on these two different area laws, we obtain the analytical coexistence curve in the charge-electric potential plane. Moreover, the phase diagram, critical exponents, and the order parameter are analytically investigated.

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I. INTRODUCTION

Since the establishment of the four laws of black hole thermodynamics, phase transition and critical phenomena continue to be an exciting and challenging topic in black hole physics. It was Hawking and Page, who first discovered the existence of phase transition between the stable large black hole and thermal gas in a AdS space [1]. Especially, motivated by the AdS/CFT correspondence [2–4], this Hawking-Page phase transition was explained as the confinement/deconfinement phase transition of the gauge field [5]. Subsequently, the study of the black hole phase transition in an AdS space attracts more attention.

Interestingly, there exist stable small and large charged or rotating black hole in AdS space. Among them, there was a phase transition of the van der Waals (vdW) type [6–8]. The similar critical exponents were also found.

Very recently, the study of the black hole thermodynamics has been further generalized to the extended phase space. In this parameter space, the cosmological constant is interpreted as the thermodynamic pressure [9–12]. The corresponding conjugate quantity as the thermodynamic volume. Then the small-large black hole phase transition was reconsidered in Ref. [13] for the charged AdS black holes. Significantly, the results stated that the black hole systems and the vdW fluid have the similar phase structure and $P-V$ (pressure-volume) criticality. Or probably they have the similar microstructure [14]. The study has been extended to other black hole backgrounds. Besides the small-large black hole phase transition, other more interesting phase transitions were found, for review see [15, 16].

Comparing with the $P-V$ criticality, the thermodynamics of the $Q-\Phi$ (charge-electric potential) criticality also have attracted much more attention. It allows a low electric potential black hole transits to a high electric potential black hole triggered by the temperature or pressure. The phase diagram and critical phenomena were studied for the different AdS black holes [17–22].

On the other hand, as an alternative method of Gibbs free energy to determine the phase transition point, the equal area laws were investigated. As early as in [7], the authors started form the first law and showed that there is the equal area law in the $T-S$ (temperature-entropy) plane for the charged AdS black holes. Employing this equal area law, the first analytical coexistence curve was obtained for the four dimensional charged AdS black holes [23]. Based on it, the analytical study of the critical phenomena becomes possible. In Ref. [24], we started from the first law of the charged AdS black hole, and showed that there exist three kinds of the equal area laws in $T-S$, $P-V$, and $Q-\Phi$ planes. Moreover, for the rotating black holes, the equal area law also holds in $J-\Omega$ (angular momentum-angular velocity) plane [25]. This result also clarifies that the equal area law does not valid in the $P-\nu$ (pressure-specific volume) plane. Other work concerning the equal area law can be found in [26–29].
Although it is shown that the equal area laws must hold in different planes, one needs to construct them in
detailed. For the four dimensional charged AdS black hole, equal area laws in $T$-$S$, $P$-$V$ planes have been well
constructed. However, in the $Q$-$\Phi$ plane, the detailed construction is still not given. So in this paper, we will carry
out the investigation of the equal area law in the $Q$-$\Phi$ plane for the charged AdS black hole. When plotting the
isothermal and isobaric lines in the $Q$-$\Phi$ plane, we find that the equal area law has quite difference behaviors when the
thermodynamic parameters is very near or far from the critical point. One is the conventional case, while another is
a novel one. For these two different cases, we, respectively, construct the equal area laws. Based on them, we obtain
analytically the coexistence curve in the $Q$-$\Phi$ plane. Moreover, employing the form of the coexistence curve, we study
the phase diagram, order parameter, and $Q$-$\Phi$ criticality.

The paper is structured as follows. In Sec. II, we give a brief review of the thermodynamics and phase transition
for the charged AdS black holes. In Sec. III, we start from the first law and show the equal area law in the
$Q$-$\Phi$ plane. Besides the conventional one, a novel case appears when the parameters far from their critical values. Based
on them, we obtained analytically the coexistence curve in the $Q$-$\Phi$ plane. Then we carry out the analytical study for
the phase diagram, order parameter, and the critical exponent in Sec. IV. Finally, a brief summary is given in Sec. V.

II. THERMODYNAMICS AND PHASE TRANSITION

Now, it is widely known that there exists the small-large black hole phase transition in the background of charged
AdS black hole. Here we would like to give a brief review of it.

The line element to describe this charged AdS black hole is
\[
ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2(d\theta^2 + \sin^2 \theta d\phi^2),
\]
where the function is given by
\[
f(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2} + \frac{r^2}{l^2}.
\]
Here the parameter $M$ and $Q$ are the black hole mass and charge, respectively. The AdS radius $l$ is related with the
pressure in the way given by (1). The temperature $T$, entropy $S$, electric potential $\Phi$, thermodynamic volume $V$, and
Gibbs free energy $G$ are
\[
T = 2Pr_h + \frac{1}{4\pi r_h} - \frac{Q^2}{4\pi r_h^3}, \quad S = \pi r_h^2, \quad \Phi = \frac{Q}{r_h},
\]
\[
V = \frac{4}{3} \pi r_h^3, \quad G = \frac{r_h^4}{4} - \frac{2\pi Pr_h^3}{3} + \frac{3Q^2}{4r_h},
\]
with $r_h$ being the horizon radius of the black hole. Solving from the equation of the temperature, one can get the
state equation for the black hole, which reads
\[
P = \frac{T}{v} - \frac{1}{2\pi v^2} + \frac{2Q^2}{\pi v^4},
\]
where $v = 2r_h$ is the specific volume of the black hole. Using the condition $(\partial_v P) = (\partial_{v,P} P) = 0$, one can easily obtain
the critical point
\[
T_c = \frac{\sqrt{6}}{18\pi Q}, \quad v_c = 2\sqrt{6}Q, \quad P_c = \frac{1}{96\pi Q^2}.
\]
This point corresponds to a second order phase transition. Below the point, the system will encounter a first order
phase transition. The isotherms and isobaric lines also show the non-monotonous behavior with $v$ or $r_h$. More
interestingly, the Gibbs free energy demonstrates the swallow tail behavior.

III. EQUAL AREA LAW AND PHASE DIAGRAM

The equal area law for the charged AdS black hole has been well studied in the $P$-$V$ plane and $T$-$S$ plane. Now
would like to examine the equal area law in the $Q$-$\Phi$ plane in detailed.
First, let us start with the first law, which is

$$dM = TdS + dQ + V dP.$$  

(8)

Here the mass $M$ should be treated as the enthalpy rather than the energy of the black hole system. Then the Gibbs free energy $G = M - TS$ will have the relation

$$dG = -SdT + dQ + V dP.$$  \(9\)

In the following, we would like to investigate the $Q$-$\Phi$ criticality for the black hole, so we fix the temperature $T$ and pressure $P$. Then we have

$$dG = \Phi dQ.$$  \(10\)

Considering that the system transforms from a small black hole phase to a large black hole phase, we have the change of the Gibbs free energy

$$\Delta G = \int_{Q_s}^{Q_l} \Phi dQ.$$  \(11\)

If both the small and large black hole phases are in the coexistence phase, their free energy will be the same, so one has $\Delta G = 0$. Therefore, we arrive

$$\int_{Q_s}^{Q_l} \Phi dQ = 0,$$  \(12\)

where $Q_1 = Q_s = Q^*$ denotes the charge at the phase transition point. In general, we can express (12) in the form

$$\int_{\Phi_s}^{\Phi_l} Q(\Phi) d\Phi = Q^*(\Phi_l - \Phi_s).$$  \(13\)

This is the conventional equal area law. However, we will show in the next that this does not always hold.

For the charged AdS black hole, the state equation can be reexpressed as

$$\Phi^4 - \Phi^2 + 4\pi T Q \Phi - 8\pi P Q^2 = 0.$$  \(14\)

Solving $Q$ from the equation, we have two solutions

$$Q_1 = \frac{\sqrt{\pi T} + \sqrt{2P\Phi^2 + \pi T^2 - 2P}}{4\sqrt{\pi P}} \Phi,$$  \(15\)

$$Q_2 = \frac{\sqrt{\pi T} - \sqrt{2P\Phi^2 + \pi T^2 - 2P}}{4\sqrt{\pi P}} \Phi.$$  \(16\)

These two curves meet each other at

$$Q_0 = \frac{\Phi \sqrt{1 - \Phi^2}}{2\sqrt{2\pi P}}.$$  \(17\)

A simple calculation shows that $Q_1$ has no extremal point. While $Q_2$ has one minimum and one maximum. Both of them have the same expression

$$Q_2^m = \frac{\Phi \sqrt{1 - 3\Phi^2}}{2\sqrt{2\pi P}},$$  \(18\)

and the minimum and maximum values of the potential are divided by the critical point $\Phi_c = \frac{1}{\sqrt{6}}$. Moreover, the minimum vanishes when $T > \sqrt{\frac{2P}{\pi}}$. In Fig. 1(a), we plot the charge $Q$ as a function of $\Phi$ with fixing $P$ and $T$. The pressure $P = 1$, and the temperature $T$ varies from 0.74 to 0.8 from top to bottom. From the critical point (7), one can find that the critical temperature corresponded to $P = 1$ is $T_c = 0.7523$. It is obvious that $Q_1$ always increases with $\Phi$. While for $Q_2$, it has a different behavior. If the temperature $T < T_c$, $Q_2$ decreases with $\Phi$. However, when $\sqrt{\frac{2P}{\pi}} > T > T_c$, $Q_2$ firstly decreases, then increases, and finally decreases with $\Phi$. While when $T > \sqrt{\frac{2P}{\pi}}$, $Q_2$ firstly increases, then decreases with $\Phi$.

In order to determine the phase transition point with constructing the equal area laws, we show two cases according to the behavior of the charge, see Fig. 1(b). Case I is very near the critical point, and case II is far from the critical point. In the first case, the constant phase transition charge only intersects with $Q_2$. For the second case, the constant phase transition charge intersects $Q_1$ at the left and intersects $Q_2$ at the right. In the following, we will construct the equal area law for these two different cases.
FIG. 1: (a) The behavior of the charge $Q$ as a function of the electric potential $\Phi$ with $P=1$. The temperature $T=0.74$ to $0.80$ from top to bottom. The dashed and solid lines are for $Q_1$ and $Q_2$, respectively. The thin red line denotes the connection point $Q_0$ of the curves $Q_1$ and $Q_2$ and the blue thin line is for the minimum point $Q_2^m$. (b) Two different behaviors of charge $Q_2$. Case I is very near the critical point, and case II is far from the critical point.

FIG. 2: Two different cases of the equal area laws in the $Q$-$\Phi$ plane. (a) Case I: very near the critical point. (b) Case II: far from the critical point. In order to avoid the confuse, it is worthwhile noting that $\Phi_0$ denotes the potential of the connection point of the $Q_1$ and $Q_2$ rather than the minimum point of $Q_2$.

A. Case I: near the critical point

For this case described in Fig. 2(a), we can find that only the solution $Q_2$ is enough to construct these two equal areas marked in shadow. This is the conventional case for the Maxwell equal area law. So it can be expressed in the form of Eq. 13. Plunging $Q_2$ into it, we will have

$$Q^* = \frac{1}{(\Phi_1 - \Phi_s)} \left( \frac{T}{8P} (\Phi_1^2 - \Phi_s^2) + \frac{(\pi T^2 + 2P(-1 + \Phi_s^2))^2}{24\sqrt{\pi P^2}} - \frac{(\pi T^2 + 2P(-1 + \Phi_1^2))^2}{24\sqrt{\pi P^2}} \right).$$

(19)

For the phase transition points $\Phi_s$ and $\Phi_1$, they also satisfy the state equation, so we have

$$Q^* = \frac{\pi T \Phi_s - \sqrt{\pi} \sqrt{-2P\Phi_s^2 + \pi T^2\Phi_s^2 + 2P\Phi_s^4}}{4P\pi},$$

(20)

$$Q^* = \frac{\pi T \Phi_1 - \sqrt{\pi} \sqrt{-2P\Phi_1^2 + \pi T^2\Phi_1^2 + 2P\Phi_1^4}}{4P\pi}.$$

(21)

Then solving the pressure $P$ from $2 * 19 = 20 + 21$, we obtain

$$P = \frac{3\pi T^2}{2(3 + \Phi_1^2 - 4\Phi_1\Phi_s + \Phi_s^2)}.$$

(22)
Further, we change (20) and (21) into the following forms

\[ P = \frac{\Phi_s(4\pi Q^*T - \Phi_s + \Phi_s^3)}{8\pi Q^*^2}, \]  
\[ P = \frac{\Phi_l(4\pi Q^*T - \Phi_l + \Phi_l^3)}{8\pi Q^*^2}. \]  

Thus, by solving the three equations (22), (23), and (24), we can obtain the coexistence curve in the \( Q-\Phi \) plane. Taking (23)-(24)=0 and \( 2\pi Q^* = (23) + (24) \), we arrive

\[-(\Phi_l + \Phi_s) + (\Phi_l + \Phi_s)(-2\Phi_l \Phi_s + (\Phi_l + \Phi_s)^2) + 4\pi Q^*T = 0,\]  
\[ \frac{2\Phi_l \Phi_s - 2\Phi_l^2 \Phi_s^2 - (\Phi_l + \Phi_s)^2 + (-2\Phi_l \Phi_s + (\Phi_l + \Phi_s)^2)^2 + 4(\Phi_l + \Phi_s)\pi Q^*T}{8\pi Q^*^2} - \frac{3\pi T^2}{3 - 6\Phi_l \Phi_s + (\Phi_l + \Phi_s)^2} = 0. \]  

Solving these equations, we can express \( \Phi_s \) and \( \Phi_l \) in terms of the charge \( Q \) and temperature \( T \)

\[ \Phi_s = \left( \frac{1}{24A} \right)^{\frac{1}{3}} + \left( \frac{A}{72} \right)^{\frac{1}{3}} - \frac{B}{6A^{\frac{1}{3}}}, \]  
\[ \Phi_l = \left( \frac{1}{24A} \right)^{\frac{1}{3}} + \left( \frac{A}{72} \right)^{\frac{1}{3}} + \frac{B}{6A^{\frac{1}{3}}}, \]  

where

\[ A = -9\pi Q^*T + \sqrt{81\pi^2 Q^*^2 T^2 - 3}, \]  
\[ B = \sqrt{\frac{3 \times 3^{2/3} + 9 \times (3A^2)^{1/3} + 9A^{4/3} + 3^{2/3}A^2 - 36 \times 3^{2/3} A\pi Q^*T}{3^{1/3} + A^{2/3}}}. \]  

Plugging \( \Phi_s \) and \( \Phi_l \) into (22), one can obtain a relation between the pressure, temperature and charge

\[ T^2 = \frac{8P(3 - \sqrt{96\pi P Q^*^2})}{9\pi}. \]  

Making using this equation, \( \Phi_s \) and \( \Phi_l \) can be further expressed as

\[ \Phi_s = \sqrt{\frac{3 - 8Q^*\sqrt{6\pi P} - \sqrt{9 - 48Q^*\sqrt{6\pi P} + 288\pi Q^*^2 P}}{6}}, \]  
\[ \Phi_l = \sqrt{\frac{3 - 8Q^*\sqrt{6\pi P} + \sqrt{9 - 48Q^*\sqrt{6\pi P} + 288\pi Q^*^2 P}}{6}}. \]  

Moreover, we can solve the charge of the phase transition from the equation, which gives

\[ Q^* = \frac{\sqrt{3\Phi_{s,l}}(\sqrt{3\Phi_{s,l}^2 + 1} - 2\Phi_{s,l})}{2\sqrt{2\pi P}}. \]  

Actually, this describes the coexistence curve of the phase transition, and the phase diagram can be well determined by it. However, we need to note that this equation only effective for the black hole system very near the critical case.

When the thermodynamic quantities are far from the critical point, we need to consider the second case.

### B. Case II: far from the critical point

In this subsection, we would like to consider the case that these thermodynamic quantities are far from their critical values, see Fig. 2(b)
From Fig. 2(b) it is obvious that the equal area law described in (13) does not hold any more. However, the law shown in (12) is still valid. According to it, we can construct the equal two areas marked in light blue and red colors. The areas shown in the figure can be calculated with

\[
\text{area}(a) = -\int_{\Phi_0}^{\Phi_s} Q_1(\Phi) d\Phi - \int_{\Phi_0}^{\Phi_i} Q_2(\Phi) d\Phi, \tag{35}
\]

\[
\text{area}(b) = -\int_{\Phi_s}^{\Phi_i'} Q_2(\Phi) d\Phi + Q^*(\Phi' - \Phi_s), \tag{36}
\]

\[
\text{area}(c) = \int_{\Phi_i'}^{\Phi_i} Q_2(\Phi) d\Phi - Q^*(\Phi_1 - \Phi'). \tag{37}
\]

Since the areas shown in light blue and red colors are equal, one must have area(a) + area(b) = area(c). Solving it, we obtain

\[
Q^*(\Phi_1 - \Phi_0) = \int_{\Phi_0}^{\Phi_i} Q_1(\Phi) d\Phi + \int_{\Phi_0}^{\Phi_i} Q_2(\Phi) d\Phi. \tag{38}
\]

This is the formula of the equal area law for the case II, which is obviously different from the conventional one (13). It is also worthwhile pointing out that \(\Phi_s > \Phi_0\), so the first term in the right side is negative.

Plugging \(Q_1\) and \(Q_2\) into (38), we integrate and get

\[
Q^*(\Phi_1 - \Phi_s) = \frac{T}{8P} (\Phi_1^2 - \Phi_s^2) - \frac{(\pi T^2 + 2P(-1 + \Phi_s^2))^2}{24\sqrt{\pi}P^2} - \frac{(\pi T^2 + 2P(-1 + \Phi_1^2))^2}{24\sqrt{\pi}P^2}, \tag{39}
\]

where we have used \(\Phi_0 = \sqrt{\frac{2P - \pi T^2}{2P}}\) corresponding to the connection point of \(Q_1\) and \(Q_2\). At the phase transition points \(\Phi_s\) and \(\Phi_1\), they, respectively, satisfy

\[
Q^* = \frac{\pi T \Phi_s + \sqrt{\pi} \sqrt{-2P\Phi_s^2 + \pi T^2\Phi_s^2} + 2P\Phi_s^4}{4P\pi}, \tag{40}
\]

\[
Q^* = \frac{\pi T \Phi_1 - \sqrt{\pi} \sqrt{-2P\Phi_1^2 + \pi T^2\Phi_1^2} + 2P\Phi_1^4}{4P\pi}. \tag{41}
\]

Making use (39)-(41), we have

\[
3P(\Phi_1 - \Phi_s) \left( \Phi_s \sqrt{-2P + \pi T^2 + 2P\Phi_s^2} - \Phi_1 \sqrt{-2P + \pi T^2 + 2P\Phi_1^2} \right) = -(-2P + \pi T^2 + 2P\Phi_s^2)^{\frac{3}{2}} - (-2P + \pi T^2 + 2P\Phi_1^2)^{\frac{3}{2}}. \tag{42}
\]

Solving the pressure from it, one can obtain

\[
P = \frac{3\pi T^2}{2(3 + \Phi_1^2 - 4\Phi_1\Phi_s + \Phi_s^2)}. \tag{43}
\]

Interestingly, this pressure for the case II is exactly the same as (22) for the case I. Moreover, from (40) and (41), we arrive

\[
P = \frac{\Phi_s(4\pi Q^* T - \Phi_s + \Phi_s^3)}{8\pi Q^*^2}, \tag{44}
\]

\[
P = \frac{\Phi_1(4\pi Q^* T - \Phi_1 + \Phi_1^3)}{8\pi Q^*^2}, \tag{45}
\]

which also the same as (23) and (24). Thus, it is clear that the equations determining the coexistence curve are exactly the same for both the cases, so the results (27)-(34) for the first case also valid for the second case.
IV. PHASE DIAGRAM AND CRITICAL EXPONENT

As shown above, the coexistence curve [34] obtained for the thermodynamic quantities near their critical values also valids when these quantities are far from their critical values. Based on the result, we in this section would like to study the phase diagram and critical exponent in the \( Q-\Phi \) plane.

First, we plot the coexistence curve in the \( Q-\Phi \) plane in Fig. 3(a). To avoiding the confusion, here we name the low-high electric potential black hole phase transition rather than the small-large black hole phase transition. The light blue shadow region denotes the coexistence phase of the low and high electric potential black holes. The low and high electric potential black hole phases are located in the left and right of the figure, respectively. A simple calculation also shows that the boundaries of the coexistence low and high electric potential black hole phases are at \( \Phi = 0 \) and \( 1 \), which is independent of the pressure of the black hole system. The critical point is the top point in the coexistence curve. Solving it, we have

\[
\Phi_c = \frac{1}{\sqrt{6}}, \quad Q_c = \frac{1}{4\sqrt{6\pi P}}.
\]

It is clear that \( \Phi_c \) is a constant and independent of the pressure \( P \). While the critical charge \( Q_c \) decreases with the pressure.

Moreover, we also show the behavior of \( \Delta \Phi = \Phi_l - \Phi_s \) as a function of the charge \( Q \) in Fig. 3(b). At \( Q = 0 \), \( \Delta \Phi \) has maximum value 1. Then \( \Delta \Phi \) decreases with the charge, and approaches zero when the critical charge is achieved. So \( \Delta \Phi \) acts as an order parameter, which can be used to describe the low-high electric potential black hole phase transition.

Next, we would like to examine the critical exponents. Since \( \Phi_s \) and \( \Phi_l \) have analytical forms, see [32] and [33], we can expand them near the critical point, the results are

\[
(\Phi_s - \Phi_c) = -(6\pi P)^{\frac{1}{2}} (Q_c - Q)^{\frac{3}{2}} + \sqrt{\pi P} (Q_c - Q) - \sqrt{\frac{3}{2}} \pi P (Q_c - Q)^2 + O(Q_c - Q)^3,
\]

\[
(\Phi_l - \Phi_c) = (6\pi P)^{\frac{1}{2}} (Q_c - Q)^{\frac{3}{2}} + \sqrt{\pi P} (Q_c - Q) - \sqrt{\frac{3}{2}} \pi P (Q_c - Q)^2 + O(Q_c - Q)^3.
\]

So near the critical point, \( (\Phi_s - \Phi_c) \) and \( (\Phi_l - \Phi_c) \) have the same exponent of \( \frac{3}{2} \). Interestingly, one can find that the first term of these expansions has opposite coefficient. While other coefficients are exactly the same. Therefore, \( \Delta \Phi = (\Phi_l - \Phi_s) \) must have the same exponent \( \frac{3}{2} \). Actually, with a simple calculation, we have

\[
\Delta \Phi = 2(6\pi P)^{\frac{1}{2}} \times (Q_c - Q)^{\frac{3}{2}}.
\]

Obviously, this confirms the exponent. More importantly, this result is an exact one. So it is also valid even for the quantities are far from their critical values. Therefore, we have an analytical form for the order parameter \( \Delta \Phi \).
V. SUMMARY

In this paper, we have considered analytically the equal area law and the thermodynamic criticality in the $Q$-$\Phi$ plane for the charged AdS black holes.

It is widely known that there exists the black hole phase transition for the charged AdS black holes. The phase transition point can be well determined by constructing the equal area law. In the $Q$-$\Phi$ plane, we showed that besides the conventional equal area law, see case I, there is another novel one, case II. Case I is for the parameters near the critical point, while case II for the parameters far from the critical point. Based on the different equal area laws, see (13) and (38), we, respectively, obtained the analytical coexistence curve. Although these two laws behave quite different, both of them confirm the same result, see (34).

Making use the analytical coexistence curve, we explored the phase diagram and critical exponents in the $Q$-$\Phi$ plane. Different black hole phases are clearly displayed in Fig. 3(a). Near the critical points, we found that $(\Phi_s - \Phi_c)$ and $(\Phi_l - \Phi_c)$ have the same exponent of $\frac{1}{2}$. The order parameter $\Delta \Phi$ also has an exponent of $\frac{1}{2}$. More interestingly, the order parameter has a compact expression $\Delta \Phi = 2(6\pi P)^{\frac{1}{4}} \times (Q_c - Q)^{\frac{1}{2}}$ for all the range of the charge.

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