SO(5) Symmetry in $t$–$J$ and Hubbard Models

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Numerical and analytical results are reviewed, which support SO(5) symmetry as a concept unifying superconductivity and antiferromagnetism in the high-temperature superconductors. Exact cluster diagonalizations verify that the low-energy states of the two-dimensional $t$–$J$ and Hubbard models, widely used microscopic models for the high-$T_c$ cuprates, form SO(5) symmetry multiplets. Apart from a small standard deviation ($\sim J/10$), these multiplets become degenerate at a critical chemical potential $\mu_c$ (transition into doped system). As a consequence, the $d$–wave superconducting states away from half-filling are obtained from the higher spin states at half-filling through SO(5) rotations. Between one and two dimensions, using weak-coupling renormalization, a rather general ladder Hamiltonian including next-nearest-neighbor hopping can be shown to flow to an SO(5) symmetric point. Experimental tests and consequences such as the existence of a $\pi$-Goldstone mode both in the insulator and superconductor and, in particular, the relationship between the photoemission spectra of the insulator and superconductor, are emphasized.

I. INTRODUCTION

As their most prominent universal feature, high-temperature superconductors (HTSC) always display antiferromagnetism (AF) and $d$–wave superconductivity in close proximity, in their phase diagram. Recently, a unifying theory has been proposed, according to which these two at first sight radically different phases are “two faces of one and the same coin”. They are unified by a common symmetry principle, the SO(5) symmetry \([1]\). In the meantime, numerical simulations, i.e. exact cluster diagonalizations, have verified that this symmetry principle is obeyed for widely used microscopic models for the high-$T_c$ cuprates, namely the (two-dimensional) $t$–$J$ and Hubbard models, to within an accuracy which is better than the average superconducting energy gap \([2,3]\).

These microscopic models are known to reproduce salient experimental results of the normal state of the high-$T_c$ cuprates, including in particular the antiferromagnetic state. SO(5) symmetry then implies that these microscopic models also display $d$–wave superconductivity. This gives rise to a microscopic description of the complex phase diagrams of the HTSC from the insulating antiferromagnetic phase over to the metallic normal state and, finally, to the $d$–wave superconducting phase. These results were further corroborated between one and two dimensions by weak-coupling renormalization-group calculations, which demonstrated that rather general ladder Hamiltonians, including longer-ranged interactions \([4,5]\) and hoppings \([6]\), flow to an SO(5)-invariant fixed point \([7]\). In this paper, the basic numerical and analytical calculations are summarized, and recent insights regarding the microscopic principle behind SO(5) symmetry \([8]\) are reviewed.

The paper is organized as follows:

Section \([9]\) describes a new idea and dynamical principle behind SO(5) symmetry. It is shown that a formulation in terms of triplet and hole fluctuations around an “RVB vacuum” allows for a physically transparent demonstration of the cornerstone in SO(5) theory, i.e. that AF and SC are “two faces of one and the same coin”. By starting from this “RVB vacuum” \(|\Omega|\), which represents the spin liquid state at half-filling, we demonstrate that an AF ordered state can be generated by forming a coherent state

$$|\psi\rangle \sim e^{\Delta^d(q=\pi)}|\Omega\rangle,$$

which corresponds to $z$-like triplets condensed into the $q=\pi$, i.e. AF wave-vector state. However, in the SO(5) theory, the $z$-like triplet with momentum $\pi$ and the hole pair with momentum $0$ are components of one and the same SO(5) vector. They are rotated into each other by the SO(5) generating operator $\hat{\pi}$. This implies that the above coherent state with condensed triplets can be SO(5)-rotated into a corresponding coherent state with $t^\dagger(q=\pi)$ replaced by the (hole-) pair creation operator $\Delta^d(q=0)$. This state corresponds to hole pairs condensed into the $q=0$ state, i.e. a superconducting state.

In other words: both the AF and the SC state can be viewed as a kind of condensation out of the RVB state, or the spin liquid. If the so-constructed AF state is the actual ground state at half-filling, then this physically very appealing SO(5) construction yields automatically the ground state in the doped situation, i.e. the SC state \([8]\).

In Section \([10]\), we summarize numerical evidence for the approximate SO(5) symmetry of the two-dimensional (2D) Hubbard and $t$–$J$ model. The SO(5) symmetry organizes the low-energy degrees of freedom and gives a new microscopic picture of the transition from an AF ground state to the $d$–wave SC state as the chemical potential is varied \([12]\). Our results show that the $d$–wave SC ground states away from half-filling are obtained from the higher spin states at half-filling through SO(5) rotations. We use a general and direct recipe for checking microscopic Hamiltonians for SO(5) symmetry, i.e. the
concept of “superspin multiplets” \[^2\]. The basic idea is that the low-energy excited eigenstates of a cluster display a definite structure characteristic of a particular symmetry, a scheme which has already provided convincing evidence for the long-range order of the Heisenberg AF on a triangular lattice \[^3\]. We have numerically diagonalized the low-lying states of the \(t\)–\(J\) model near half-filling and found that they fit into irreducible representations (irreps) of the SO(5) symmetry group. At a critical value \(\mu_c\) of the chemical potential, the superspin multiplets are nearly degenerate and, therefore, higher spin AF states at half-filling can be freely rotated into dSC states away from half-filling. Our overall exact-diagonalization results, when further combined with a detailed spectroscopy of “SO(5)-allowed” and “SO(5)-forbidden” transitions between the superspin multiplets, suggest that the low-energy dynamics of the \(t\)–\(J\) model \[^2\] can be described by a quantum nonlinear \(\sigma\)–model with anisotropic couplings, and the transition is that of a superspin flop transition \[^1\]. It is truly remarkable that, while the physical properties of AF and dSC states appear to be diagonal opposite, and they are characterized by very different form of order, there exists, nevertheless, a fundamental SO(5) symmetry that unifies them.

In these two-dimensional (2D) microscopic models, the undoped situation – in agreement with the experimental situation in the cuprates – corresponds to that of a Mott insulator with broken SO(3) or spin rational symmetry: long-range AF order is realized. The SO(5)-symmetry principle then tells us how this long-range magnetic order and the accompanying low-energy spin excitations are mapped onto the corresponding off-diagonal long-range SC order and the low-energy “Goldstone bosons” (the \(\pi\)–mode) in the doped situation \[^1\].

However, there exists also a second class of Mott-type insulators without long-range AF order, i.e. spin liquids, which have a gap to spin excitation. There is growing experimental evidence that they are also intimately related to the physics of high-\(T_c\) compounds: Not only do these compounds show above the Néel temperature and superconducting transition temperature at small dopings signs of such a spin gap, but there exist also copper-oxides with a CuO\(_2\) plane containing line-defects, which result in ladder-like arrangements of \(Cu\)–atoms (for a summary, see \[^4\]). These systems can be described in terms of coupled two-leg ladders \[^1\], which exhibit a spin gap in the insulating compound and thus belong to the spin-liquid Mott-insulator variety. Also the related “stripe phases” of the 2D CuO\(_2\) planes in the cuprate superconductors have recently received considerable attention \[^1\]. In these systems, the apparent connection between the spin gap and superconductivity must be explained.

In order to illustrate how the SO(5) theory can, in principle, cope with the challenge, an exactly SO(5) invariant ladder model has recently been constructed \[^7\]. In particular, it was shown that the spin-gap magnon mode of the Mott insulator evolves continuously into the “\(\pi\)–resonance” mode of the superconductor. This SO(5) symmetric model offers a reference point around which departures from the SO(5) symmetry can be studied. Then, two key questions arise, the first being the relationship of the exact SO(5) ladder to the “physical” \(t\)–\(J\) or Hubbard ladders and the second regarding the connection to the other variety of Mott insulators, i.e. the ones with long-range AF order.

With regard to the first question, progress was recently made in the regime of weak-coupling: Using the weak-coupling renormalization group method, two independent works \[^2\,\[^3\]\] have demonstrated that rather generic ladder models at half-filling flow to an SO(5) symmetric fixed point. This work is reviewed in section [\[^4\]].

In section [\[^5\]], we also summarize work, which tries to numerically attack both questions in the experimentally relevant intermediate to strong-coupling regimes. As shown in this recent work \[^8\], SO(5) symmetry has profound implications for the dynamical correlation functions, most notably the single-particle spectrum. Specific predictions of SO(5), like a “generalized rigid band behavior” \[^1\,\[^2\]\] and the appearance of sidebands in the inverse photoemission spectrum \[^3\] may indeed have been observed long ago in the actual 2D \(t\)–\(J\) model and also in recent angular-resolved photoemission experiments \[^4\]. Motivated by the present ladder theory, we have carried out more detailed spectroscopies on the 2D model and obtained results in strong support of SO(5).

Finally, we and other authors \[^1\,\[^2\,\[^3\]\] have demonstrated that, despite the, at first sight, rather unphysical parameter values of the SO(5) symmetric ladder model, a “Landau mapping” to the more realistic \(t\)–\(J\) model is feasible. This may suggest that the SO(5) symmetric ladder is indeed the generic effective Hamiltonian for 2-leg ladder systems and for the above spin-liquid Mott-insulator variety, in general.

\textbf{II. ONE-TO-ONE CORRESPONDENCE OF ANTIFERROMAGNETISM AND SUPERCONDUCTIVITY}

In the HTSC, the dominant charge-carrier dynamics takes place in the two-dimensional (2D) CuO\(_2\)-planes \[^12\]. Each CuO\(_2\) unit cell contains an effective magnetic moment of spin \(\frac{1}{2}\), essentially due to the Cu ion. Neighboring Cu-spins form singlets – the energy win due to the singlet formation, the magnetic exchange \(J\) is relatively large \(\sim 120\text{meV} \sim 1400\text{K}\). On the other hand, the temperatures for the transition into both low-temperature phases, the AF and the SC phases, \(T_{N\text{ell}}\) and \(T_c\), are both significantly lower and of similar magnitude \(\sim 250\text{K}\) for \(T_{N\text{ell}}\) and \(\sim 100\text{K}\) for \(T_c\). Already this order of magnitude suggests that the mechanism of superconductivity does not directly result from the singlet formation, but
is instead related to the mechanism, which results in AF in the undoped, insulating situation.

Indeed, both low temperature phases of the cuprates are “ordered”: In the undoped case, i.e. in the insulator, we have AF order, in the doped case, the phase coherence of the superconductor and both phases are in the HTSC in immediate vicinity (the “spin glass phase”, which occurs in some cuprates, is likely due to disorder). Therefore, it seems tempting to unify these low-temperature phases, despite the fact that on first glance, they appear dramatically different: On the one hand, the insulator and, on the other hand, the ideal conductor, i.e. the superconductor.

Let us consider, at first, the insulator: At high temperatures \( \sim 1000 \text{ K} \), the singlet pairs are completely disordered. This state is termed, therefore, a spin liquid. How does one arrive from this disordered state at high temperatures, at an ordered Néel state at low temperatures? To solve this problem, we shall consider a dynamical principle, which gives a particularly simple and transparent demonstration of the key feature of SO(5) theory, namely the one-to-one correspondence of antiferromagnetism and superconductivity: According to this dynamical principle [8], the ordered AF state can be considered as a kind of Bose-Einstein condensation of triplet excitations. The SC state, on the other hand, corresponds to a Bose-Einstein type of condensation of Cooper pairs, i.e. a SC state. The “rotation” AF\( \rightarrow \)SC, therefore, is described by an operator, the \( \pi \) - operator of SO(5) theory [9], which replaces triplets by hole pairs.

In the following, we give a particularly simple illustration for this key-feature of SO(5) theory, which can be worked out for a ladder [8], and discuss the equivalence of antiferromagnetism and superconductivity for this example. The ground state of the ladder models is actually a resonating valence bond (RVB) type of vacuum without AF long-range-order [10]. However, for illustrative purposes, let us now construct an AF ordered state (which is in general not an eigenstate of the Hamiltonian) by condensing magnons into the RVB ground state. One can express the operator of staggered magnetization in \( z \)-direction as

\[
M_z = \sum_n e^{i\pi n} ( P_n (\uparrow \downarrow) - P_n (\downarrow \uparrow) ),
\]

where e.g. \( P_n (\uparrow \downarrow) \) projects onto states where the \( n \)th rung has the configuration \( \uparrow \downarrow \) (see Fig. 4). It is easy to see that

\[
( P_n (\uparrow \downarrow) - P_n (\downarrow \uparrow) ) \delta_n^\uparrow = t_n^\uparrow a_{n,z}^\dagger,
( P_n (\uparrow \downarrow) - P_n (\downarrow \uparrow) ) t_n^\downarrow a_{n,a}^\dagger = \delta_n^\downarrow a_{n,a}^\dagger,
\]

where \( \delta_n^\uparrow (t_n^\uparrow) \) creates a singlet(\( \alpha \)-triplet) on rung \( n \). Therefore

\[
M_z = \sqrt{\frac{N}{2}} \left[ t_n^\uparrow (q = \pi) + t_n^\downarrow (q = \pi) \right].
\]

If we now form the coherent state

\[
|\Psi_\lambda \rangle = \frac{1}{\sqrt{n}} e^{\lambda \sqrt{N} t_n^\uparrow (q = \pi)} |\Omega \rangle,
\]

which corresponds to \( z \)-like triplets condensed into the \( k = \pi \) state, and treat the \( \pi \) as ordinary Bosons, we obtain

\[
\langle \Psi_\lambda | M_z | \Psi_\lambda \rangle = \sqrt{2} \lambda N.
\]

This calculation shows that by starting from a spin liquid, i.e. an RVB vacuum, an antiferromagnetically ordered state with \( M_z \) in \( z \)-direction can be generated by condensing \( z \)-like triplet-excitations into the \( k = \pi \) state. At this point, we can invoke the SO(5) symmetry of the model [8], which tells us that since the \( z \)-like triplet with
momentum $\pi$ and the hole pair with momentum 0 are two different components of a 5-vector, they are dynamically indistinguishable. This means that the AF state, with condensed triplets, can be SO(5)-rotated into a state with condensed hole pairs. It follows that, if the antiferromagnetic state were the ground state at half-filling (which is the case for 2D materials and physical ladder systems), we can replace all $z$-like triplets by hole pairs with momentum 0 and by SO(5) symmetry automatically obtain the ground state in the doped case. The latter then consists of hole-pairs along the rungs condensed into the $k=0$ states and thus is necessarily superconducting. In other words: both the antiferromagnetic and the superconducting state may be viewed as some kind of condensate ‘on top of’ the rung-RVB state. SO(5) symmetry then simply implies that the condensed objects are combined into a single vector, whence the unification of antiferromagnetism and superconductivity follows in a most natural way.

The above derivation makes sense only in a strong coupling limit, where a ground-state description starting out from rung-singlets is appropriate. One might expect, however, that similar considerations will apply also for cases with a weak coupling within the rungs (see also section [III]).

In two dimensions (2D) the above interpretation of SO(5) symmetry hinges crucially on two points: first, the excitation spectrum of a 2D spin-liquid or RVB state must consist of Bosonic triplets (rather than e.g. Fermionic ‘spinons’), whose condensation into the minimum of their dispersion at $(\pi, \pi)$ leads to antiferromagnetic ordering. This would imply that the antiferromagnetic state in 2D also could be interpreted as a condensate of triplet Bosons. Work along this line is in progress and promising [1].

Second, these triplet-like Bosons with momentum $(\pi, \pi)$ must be dynamically equivalent to a $d_{x^2-y^2}$ Cooper pair. Having established this equivalence one could immediately conclude that the antiferromagnetic phase (viz ‘condensate of triplets’) is identical to the $d$-wave superconducting phase (viz ‘condensate of hole pairs’) in the same sense as two nuclei belonging to the same isomultiplet are ‘identical’. Since the $\pi$-operator in 2D precisely converts a triplet with momentum $(\pi, \pi)$ into a $d$-wave hole pair (see section [II]), the latter requirement is equivalent to $[H, \pi] = i\omega_0 \pi$. This commutation relation, which will be discussed in detail in the next section, tells us whether the dynamics of charge carriers respects SO(5) symmetry, i.e. the Hamiltonian is in accord with the SO(5) rotation $\pi$. Ideally, we should then have zero for the commutator. This will happen for a critical chemical potential $\mu_c$, right where the transition into the doped system takes place. Further changing $\mu$ gives rise to a precession frequency, or energy to perform the AF $\rightarrow d$–SC rotation $\omega_0 \neq 0$. The energy shift $\omega_0$ here would correspond to the mass difference of proton and neutron in the Isospin algebra. The validity of this eigenoperator relationship has already been verified numerically [1].

### III. SO(5) Symmetry and Microscopic Models for HTSC

The dynamical principle of the previous section suggests to unify the triplet excitations $t^\dagger$ (corresponding to a vector) and hole-pair excitations $\Delta$ (described by $(\text{Re}(\Delta)), \text{Im}(\Delta))$ into a 5–dimensional vector, the so-called superspin vector. More precisely, Zhang suggested to group the AF order parameter $S(Q)$ with $Q = (\pi, \pi)$ and $d$-wave SC order parameter $\Delta$ into a single five-component vector, the superspin $\hat{\pi}$

$$n_a = (\Delta^\dagger + \Delta, S(Q), -i (\Delta^\dagger - \Delta)),$$

where $\Delta = (i/2)\sum_p (\cos p_x - \cos p_y) c_p^\dagger \sigma_c c_{-p}$ denotes the $d_{x^2-y^2}$ superconducting order parameter. $S(Q) = \sum_p Q^\dagger c_p \sigma_p c_p^\dagger$ stands for the AF Néel vector and $\sigma_p$ are the Pauli spin matrices. The transition from AF to dSC is then viewed as a kind of “superspin flop” transition as a function of the chemical potential or doping, where the direction of the superspin changes abruptly. This transition or SO(5) rotation is formally described by the so-called $\pi$–operator.

The “superspin flop” transition is analogous to the problem of a spinning top in a uniform gravitational field, or a magnetic moment in a uniform magnetic field, as in nuclear magnetic resonance (NMR). As we shall see, in the presence of the symmetry-breaking field (which, in our case is the chemical potential or doping), the order parameter is forced to precess, and this explicit breaking of the SO(5) invariance induces rotations between AF and dSC states (with frequency $\omega_0$) and governs the competition between these two phenomena.

Mathematically, the $\pi$–operator can straightforwardly be constructed [13]: as a rotation from AF to dSC, it has to patch up the differences in the quantum numbers of the corresponding order parameters. Since $\Delta$ has spin $S=0$, whereas $S(Q)$ has $S=1$ (which is obvious from our triplet excitation picture in the last section), $\pi$ has to carry $S=1$, i.e. it must be a triplet operator. $S(Q)$ has no charge, $\Delta$ has charge $\pm 2$, therefore, $\pi$ must create charge $\pm 2$. Finally, $S(Q)$ has momentum $Q = (\pi, \pi)$, $\Delta$ instead has $Q = 0$, thus, $\pi$ must have momentum $Q$. Combination of these requirements fixes the operator uniquely up to a form factor, which is given by $d$-wave symmetry [38]. This results in the $\pi$ operators (we just give one of them)

$$\pi^\dagger = \sum_p (\cos p_x - \cos p_y) c_p^\dagger \sigma_p Q^\dagger c^\dagger_{-p}.$$

Charge conjugation and spin lowering operation gives five other similar operators. In real-space representation, the
\( \hat{\pi} (\hat{\pi}^\dagger) \) operator does precisely what we discussed in the previous section, namely it replaces triplets oriented in \( x \) and \( y \) planar directions by \( d_{x^2-y^2} \) hole (electron) pairs. These \( \hat{\pi} \) operators were first introduced by Demler and Zhang \cite{24} to explain the resonant neutron scattering peaks in the \( YBCO \) superconductors. Together with the total spin \( S_\alpha \), which is the generator of the SO(3) spin rotation forming a subgroup of SO(5), and the charge operator \( Q \) generating the \( U(1) \) charge symmetry (which is also a subgroup of SO(5)), the six \( \hat{\pi} (\hat{\pi}^\dagger) \) operators form the ten generators \( L_{ab} \) of the SO(5) algebra (for definition of \( L_{ab} \), see \cite{1} ). As constructed, the \( \hat{\pi} \) operators indeed rotate the AF order parameter into the dSC order parameter, i.e.

\[
\left[ \hat{\pi}^\dagger_{\alpha}, S(Q)_\beta \right] = i\delta_{\alpha\beta}\Delta^1, \tag{8}
\]

and vice versa.

We have, thus, apparently accomplished the task of unifying AF with SC: the corresponding order parameters are grouped into a five dimensional object, and SC is “nothing” but AF viewed in some rotated coordinates. Nevertheless, SO(5) symmetry overcomes this major obstacle. This construction looks a bit similar to the unification of E and B by the Lorentz group. But, so far, this is only a mathematical construction, we haven’t asked if “Mother Nature” approves the SO(5) construct or not.

In the high-\( T_c \) problem, “Mother Nature” is very complicated, but we can check the SO(5) symmetry within some microscopic models, which are known to reproduce salient features of the phase diagram of HTSC. Such microscopic Hamiltonians, i.e. Hubbard and \( t-J \) models, successfully model the Mott-Hubbard insulator to metal transition \cite{22, 23}, which is driven by the Coulomb correlation \( U \sim 10eV \), i.e. by “high-energy” physics. They also model prominent features of the magnetic interactions on an energy scale of order \( J \sim 0.1eV \) \cite{13, 14, 24, 25}. However, their low-energy content of order of the average SC gap (\( \sim J/5 - J/10 \)) has so far eluded both analytical and numerical investigations \cite{1}. We shall demonstrate that SO(5) symmetry overcomes this major obstacle; it clarifies the role of competing orders and gives a microscopic description of the transition from AF to dSC ground states, as the chemical potential is varied.

One can check the SO(5) symmetry by evaluating the commutator between the Hamiltonian with the \( \hat{\pi} \)-operators. In particular, numerical \cite{3} works, summarized below, show that the \( \hat{\pi} \) operators are approximate eigenoperators of the Hubbard Hamiltonian, in the sense that

\[
[H, \hat{\pi}^\dagger_\alpha] = \omega_0 \hat{\pi}^\dagger_\alpha, \tag{9}
\]

where the eigen-frequency \( \omega_0 \), which is the energy to perform the AF→dSC rotation, is of the order of \( J \), and proportional to the number of holes. This relation \cite{3} reminds us of the commutation relation between the transverse spin components \( S_\alpha \) and \( S_\beta \) in a magnetic field and the Zeeman Hamiltonian, with \( \omega_0 \) being proportional to the B-field. In our case, the SO(5) symmetry is broken explicitly by the chemical potential, i.e. \( \omega_0 \) is proportional to the hole count or doping. Thus, the pattern of explicit symmetry breaking is simple and familiar, and, therefore, easy to handle.

In the following, we employ a kind of “computer spectroscopy” to test whether the dynamics of the charge carriers, i.e. the Hamiltonian, respects SO(5) symmetry and \cite{1} is fulfilled. Ever since the early days of quantum dynamics, group theoretical interpretation of spectroscopy revealed deep symmetry and profound unity of Nature. Atomic spectra can be fitted into irreducible representations (irreps) of SO(3), and the regular patterns which emerged from this classification offered fundamental understanding of the periodic table. After the discovery of a large number of hadrons, the “embarrassment of riches” was removed by the classification of hadronic spectra into irreps of SU(3) and this hidden regularity inspired the predictions of quarks, the fundamental building block of the universe. In our quest for understanding the fundamental design of Nature, the importance of symmetry can never be over-emphasized.

In our work, we used a different kind of spectroscopy and its classification into a different kind of symmetry. The spectroscopy is performed on a computer, which numerically diagonalizes microscopic Hamiltonians, i.e. Hubbard and \( t-J \) models widely believed \cite{20, 21} to model high-\( T_c \) superconductors.

As the simplest 2D lattice model for correlated electrons, the one-band Hubbard model is defined as

\[
H = -t \sum_{\langle i,j \rangle, \sigma} \left( c^\dagger_{i\sigma} c_{j\sigma} + h.c. \right) + U \sum n_{i\uparrow} n_{i\downarrow}, \tag{10}
\]

with nearest-neighbor hopping \( t \) and Coulomb correlation \( U \). On the other hand, the \( t-J \) model has the Hamiltonian

\[
H = P \left[ -t \sum_{\langle i,j \rangle, \sigma} \left( c^\dagger_{i\sigma} c_{j\sigma} + h.c. \right) + J \sum_{\langle i,j \rangle} S_i S_j \right] P. \tag{11}
\]

As in the Hubbard model, \( \langle ij \rangle \) denotes a summation over nearest-neighbors on the 2D square lattice and \( P \) projects onto the subspace with no doubly occupied sites. The latter constraint reflects the strong correlations in the \( U/t \rightarrow \infty \) limit of the Hubbard model.

First numerical evidence for the approximate SO(5) symmetry of the Hubbard model came recently from exact diagonalizations of small-sized (10 sites) clusters \cite{3}, studying dynamic correlation functions involving the AF/dSC rotation \( \hat{\pi} \) operator. We observe that \cite{2} is nothing but the ladder operator relation familiar from standard Quantum Mechanics problems such as the harmonic oscillator or the spin rising (\( S_+ \)) and lowering (\( S_- \))
operators. If this equation is fulfilled, then the equal level distance \( \omega_0 \) between the “rungs” of the ladder should appear – like in NMR or optical spectroscopy – as a sharp peak in the \( \tilde{\tau} - \tilde{\pi} \) correlation function, well separated from a higher-energy incoherent background. This indeed is verified in Fig. 1(a),(b), which displays a typical result for the \( \tilde{\pi}_d \) correlation function in a ten-site Hubbard cluster with \( U/t = 8 \) for dopings of one hole-pair (\( \langle n \rangle = 0.8 \)) and two hole-pairs (\( \langle n \rangle = 0.6 \)). We note that \( \omega_0 \) is a small energy, scaling with the hole count away from half-filling (\( \langle n \rangle = 1 \)), i.e. \( \omega_0 \approx J/2(1 - \langle n \rangle) - 2\mu \).

The approximate relation (1) is highly nontrivial. One could ask if a similar relation would exist for a modified \( \tilde{\pi} \) operator which rotates AF into \( s \)-wave SC order parameters. The answer is negative (3), as seen in Fig. 1(c) which demonstrates that a \( \tilde{\pi} \) -rotation with \( s \)-wave symmetry just generates an incoherent background and no sharp “\( \pi \)-resonance”. Therefore, there is only an approximate symmetry between AF and \( d \)-wave SC (and not \( s \)-wave SC) near half-filling.

In the next step, we use a most general and direct recipe for checking microscopic Hamiltonians for SO(5) symmetry, i.e. the concept of “superspin multiplets” (2). We consider the \( t-J \) model, which, because of its more limited Hilbert space (no double occupancies), allows the exact diagonalization of larger systems (18, 20 sites). Since the \( t-J \) model explicitly projects out the states in the upper Hubbard band, some of the questions raised recently about the compatibility between the Mott Hubbard gap and SO(5) symmetry can also be answered explicitly. In particular, if there is an approximate SO(5) symmetry of the microscopic model, the low-energy states of this model should fall into irreducible representations (irreps) of SO(5). In a given quantum mechanical system, the direction of the SO(5) superspin vector is quantized in a way similar to an ordinary SO(3) spin, and the classically intuitive picture of the precessions of the SO(5) superspin vector under the influence of the chemical potential (4) can be identified with the equal level-spacing between the members of SO(5) multiplets carrying different charge. Therefore, numerically identifying the low-lying states of the microscopic model with the SO(5) irreps can lead to detailed understanding of the one-to-one correspondence and the level crossing between the excited states of the AF and the dSC states, and thereby lead us to the microscopic mechanism by which the AF changes into the dSC state. While finite-size calculations cannot generally be used to prove the existence of long-range order in infinite systems, the spectroscopic information about the SO(5) symmetry can be used as input for the effective field theory (5) which captures the low-energy and long-distance physics of the problem.

Exact diagonalizations (e.g. (4)) commonly study ground-state correlations, but their spatial decay is often inconclusive as a test of order due to small system size. Yet it is possible that the (excited) eigenstates show a well-defined structure characteristic of a particular symmetry; this provided the convincing evidence for long-range order in the spin-1/2 triangular lattice AF (4). In our work, we have pursued exactly such a program in exact diagonalizations of the \( t-J \) model.

Consider as a simplest example the precession of a spin-1/2 system in a homogeneous \( \mathbf{B} \) field in \( z \)-direction. \( \mathbf{B} \) breaks the SO(3) spin-rotation symmetry and the spin (expectation value) precesses with Larmor frequency around the spatially fixed \( \mathbf{B} \)-direction. This can be read off in a spectroscopic experiment from the multiplet structure of the possible spin states (4 \( \leftrightarrow \frac{1}{2} \)), the degeneracy of which is lifted by the \( \mathbf{B} \)-field (Zeeman effect). In this simplest case, the multiplet structure is one-dimensional, extending in \( S_z \)-“direction”.

In our “computer spectroscopy”, we employ a formally analogous recipe for checking the microscopic Hamiltonian for SO(5) symmetry:

In SO(5), the multiplet structure is two-dimensional, i.e. spanning both \( S_z \)-direction and \( Q \)-direction (4). \( Q = L_{15} \) is the total charge, and, in our \( t-J \) calculation, it stands for the number of doped hole pairs, and, thus, for the transition from AF to dSC states. Formally, the multiplets are constructed by observing that \( \{ S_z, Q, C \} \) form a set of commuting operators with their quantum numbers labeling states of an SO(5) invariant Hamiltonian (4). \( S_z \) is the \( z \)-component of the total spin \( S_z = -L_{23} \) and \( C \) the Casimir operator \( \sum_{a,b} \hat{L}_a^2 \hat{L}_b^2 \), which is a natural generalization of the total spin operator \( S_z^2 \). Like \( S^2 \) in the familiar SO(3) spin-rotation symmetry, \( C \) fixes
the level \( \nu \) of an irreducible representation. Instead of \( S(S+1) \) in \( \text{SO}(3) \), it takes the value \( \nu(\nu+3) \) for an \( \text{SO}(5) \) level \( \nu \) irreps. Fig. 3 shows the first four \( (\nu=0 \text{ to } \nu=3) \) irreps of \( \text{SO}(5) \) with the low-lying states of an 18-site \( t-J \) model with \( J/t=0.5 \), which is a typical \( J \) value. 

![Diagram of SO(5) irreps](image)

**FIG. 2.** The upper diagram illustrates general level \( \nu \) irreps of \( \text{SO}(5) \). Every state can be labeled by \( Q \) and \( S_z \). The maximal charge is \( Q=\pm \nu \). The states labeled by a \( \times \) form the shape of a diamond, while states inside the nested diamonds are labeled by \( \circ \) and \( \Delta \). Overlapping states with same \( Q \) and \( S_z \) are distinguished by their \( S \) quantum numbers. The lower diagrams are for \( \nu=1,2,3 \) irreps of \( \text{SO}(5) \). The figure shows the energies of some low energy states for the 18-site cluster with \( J/t=0.5 \). The states are grouped into different multiplets and are labeled by the spin, point group symmetry, and total momentum. \( A_1 \) denotes the totally symmetric, \( B_1 \) the \( d_{x^2-y^2} \)-like representation of the \( \text{C}_4v \) symmetry group. The () symbol denotes as yet unidentified members of the respective multiplet.

What is the physical meaning of these multiplets? In fact, there is a definite physical meaning behind them, which is closely related to our coherent-state description of the AF\( \rightarrow \text{dSC} \) transition in section 3. The AF state is constructed from the linear superposition of the \( Q=0 \), \( S_z=0 \) states in each of the \( \nu \) multiplets. At a given level \( \nu \) this state contains \( \nu \) magnons or triplet excitations, which correspond to the \( \nu \)-th term in the power-series expansion of the coherent-state operator, i.e., \( e^{\lambda t^1} = 1 + \lambda (t^1)^{\nu=1} + \frac{\lambda^2}{2!} (t^1)^{\nu=2} \ldots \)

Correspondingly, the dSC state is constructed from a linear superposition of the lowest corner states of each level \( \nu \) irreps, and may be viewed as stemming from the power-series expansion, \( e^{\Delta A} = 1 + \Delta (A)^{\nu=1} + \frac{\Delta^2}{2!} (A)^{\nu=2} \ldots \)

From Fig. 2 we note that the low-lying states indeed fit into the irreps of \( \text{SO}(5) \): all the different quantum numbers of the states are naturally accounted for by the quantum numbers of the superspin. However, most importantly, the levels with different charge \( Q \) are nearly equally spaced! This is explicitly indicated by the symbol \( \Delta E \) in Fig. 2. More precisely, the mean-level spacing within each multiplet (up to \( Q=2 \)) is \(-2.9886 \) with a standard deviation of 0.0769. This standard deviation is much smaller \((\sim J/8)\) than the natural energy scale \( J \) of the \( t-J \) model and comparable to or even smaller than the average SC gap. If one now adds the chemical potential term, \( H_\mu = -2\mu Q \), then at a chemical potential \( \mu \) comparable to the mean-level spacing, the superspin multiplets are nearly degenerate. In other words, (each of the coherent-state contributions to) the AF state can freely be rotated in (the corresponding contribution to) the SC state.

![Spectral functions with final states in the Q=-1 subspace](image)

**FIG. 3.** Spectral functions with final states in the \( Q=-1 \) subspace: dynamical spin correlation function for momentum transfer \( Q \), calculated for the \( 1B_1(0,0) \) ground state; spectrum of the \( \pi^+ \)-operator, calculated for the \( 1A_1(0,0) \) ground state in the \( Q=-2 \) sector; spectra of the \( \pi^- \)-operator, calculated for the half-filled \( 1A_1(0,0) \) ground state and the lowest half-filled \( 5A_1(0,0) \) state.

In Fig. 3 this is verified in terms of another “diagnostic tool”, which is the spectral function

\[
A(\omega) = \Im \frac{1}{\pi} \langle \psi | \hat{O}^\dagger \hat{O} | \psi \rangle \frac{1}{\omega - (H - E_{ref}) - i\delta},
\]

We see that, if we apply as the operator \( \hat{O} \) the \( \pi^+ \) operator to the “dSC” state in the \( \nu = 2 \) irreps (the \( Q = -2, S_z = 0 \) state \( 1A_1(0,0) \)), we end up in precisely the same final state (to within computer accuracy) as if we apply the \( \pi^- \) operator to the “AF” state (\( Q = 0, S_z = 0 \)). In other words, two successive \( \pi^- \)-rotations rotate us from the AF state to the d-SC state. The energy required to perform this rotation, \( \omega_0 \), is again seen to scale with \( J \) (in contrast to a
false argument in ref. [24]).

Finally, we note that the dynamical spin correlation function – where we take as the operator $\hat{\mathbf{O}}$ in [12] the magnon operator $\mathbf{S}(\mathbf{Q})$ – has a peak at precisely the same position as the $\pi$–resonance. This confirms the earlier conjecture by Demler and Zhang [25] to interpret a well-known spin resonance detected in neutron-scattering experiments in HTSL, as a “$\pi$–excitation”. More generally, this fingerprint of SO(5) symmetry can be described as follows: Away from $\mu_c$, the chemical potential, which controls the changes in the total charge, breaks SO(5) symmetry. As already mentioned, the effect of this chemical potential is formally analogous to the effect of the $\mathbf{B}$–field on a spin. In the presence of this symmetry-breaking field, the order parameter is forced to precess and this explicit breaking of the SO(5) invariance induces transitions between AF and SC states, and governs the competition between these two phenomena. The SO(5) theory – in good accord with the experimental data [31] – then identifies this precession frequency $\omega_0$ with a resonance in neutron scattering.

IV. SO(5) SYMMETRY IN LADDERS

A. Introduction

High-$T_c$ materials are antiferromagnetic Mott insulators displaying long-range AF order at half filling. The antiferromagnetic phase is rapidly destroyed upon doping and is replaced by the superconducting phase. Below optimal doping (the doping with maximum $T_c$) and above the superconducting temperature there are clear experimental indications for the opening of a spin gap [30,32]. This phase is termed spin-liquid with properties which are quite difficult to reproduce on the theoretical level for a two-dimensional system. On the other hand, at and close to half filling this spin-gap variety of Mott insulators is obtained quite well in half-filled ladder systems: Upon doping the ladder, the spin gap persists and the system exhibits quasi-long-range d-wave superconducting correlations, which become dominating in some parameter range [10]. In this sense, ladders systems show properties very similar to the phase diagram of cuprate materials, the main difference being the fact that correlations are “quasi-long-ranged”, i.e. they show power-law behavior, since they cannot be truly “long-ranged” because of one-dimensionality. Moreover, there also exist copper-oxides with CuO$_2$ planes containing line defects, which result in ladder-like arrangements of Cu-atoms [33]. These systems can be described in terms of coupled two-leg ladders exhibiting a spin gap and thus belong to the spin-liquid Mott-insulator variety. Also the related “stripe phases” of the 2D CuO$_2$ planes in the cuprate superconductors [11], which have recently received considerable attention, can be mapped onto ladder systems.

It is thus interesting to study the occurrence of SO(5) symmetry in ladder systems. As we will discuss, there is in fact a natural way to construct an SO(5) symmetric model for a two-leg ladder, which has only local interactions on a rung of the ladder [3]. We can thus use the ladder system as a theoretical laboratory to check some ideas of the SO(5) theory. On the other hand, we want to study whether generic models, which are not SO(5) invariant at the bare starting (microscopic) level may show SO(5) symmetry in their low-energy regime. An appropriate tool to study this low-energy regime, starting from a microscopic Hamiltonian, is the renormalization group (RG). This method has proven to be particularly suited to study systems between one and two dimensions, for example ladder systems [34-36].

In the following, we consider generalized Hubbard-type ladders with Hamiltonians of the form

$$H = -t \sum_{\langle i,j \rangle, \sigma} \left( c_{i,\sigma}^\dagger c_{j,\sigma} + \text{h.c.} \right) - t_\perp \sum_{i, \sigma} c_{i,\sigma}^\dagger c_{i,\perp \sigma} + U \sum_i n_{i,\uparrow} n_{i,\downarrow} + V \sum_{\langle i,j \rangle} n_{i} n_{j} + \frac{V_i}{2} \sum_i n_{i} n_{i,\perp} + J_\| \sum_i \mathbf{S}_i \mathbf{S}_{i,\perp} .$$

(13)

(see Figure 4). The sum over $i$ in (13) runs over the sites of a two-chain ladder, as before, $\langle i,j \rangle$ covers pairs of nearest-neighbor sites on the same chain and $i \pm \hat{y}$ is the nearest-neighbor site to $i$ on the other chain.

FIG. 4. A generalized Hubbard-type ladder model: the Hamiltonian consists of electron hopping terms ($t$), on-site and off-site Coulomb interactions ($U$, $V$), and spin-spin exchange terms ($J$)

B. Exact SO(5) ladder

In this paragraph we present “numerical spectroscopy experiments” for a recently proposed exactly SO(5) symmetric ladder model based on a simplified version of the more general Hamiltonian [13]: The non-local Coulomb and spin-spin interactions (the $V$ and $J$ terms in (13)) are restricted to act within the rungs, but not along the legs of the ladder, and hopping is only allowed between nearest-neighbor sites (with hopping $t$ along the legs and $t_\perp$ within the rungs) (see Fig. 4). Setting $J_\perp = 4(U+V_\perp)$, the elementary magnetic and “SC” excitations of a half-filled single rung, namely the formation
of a triplet and the formation of an electron (or hole) pair, become degenerate in energy. In this case, one obtains an exactly SO(5) symmetric Hamiltonian $Q$, $S_z$. In a recent work $Q$, this ladder model was studied in detail and related to the triplet-excitation picture and coherent-state description of section $Q$. Furthermore, this work gives the most general construction of the SO(5) irreducible representations not only for even numbers of holes (or electrons) as in section $Q$, but also for odd numbers, encountered for example in direct or inverse photoemission (PE). Exact diagonalization was used to extract the practical application of the selection rules for photoemission implied by SO(5) symmetry.

FIG. 5. SO(5) multiplets connected by allowed photoemission (PES) and inverse photoemission (IPES) transitions. As we inject/remove a $\downarrow$-electron, PES corresponds to an arrow pointing south-east and IPES to one pointing north-west. Here, the initial state is a half-filled state with $S_z=0$ in the (1,1) multiplet (open circles). SO(5) selection rules only allow transitions to the (2,1) and the (1,0) multiplets (black circles), which are superimposed to the initial (1,1)-multiplet.

FIG. 6. PES/IPES from a spin-polarized half-filled state ($Q=0$, $S_z=+1$) and from a doped state ($Q=-1$, $S_z=0$). Both states are members of the (1,1) multiplet like the initial state in Fig. 5. Note that there is no allowed photoemission transition to the (1,0) multiplet for either of the two states.

The classification of SO(3) symmetric states into spin multiplets, e.g. a triplet with $S_z=1$, is well known. For SO(5) symmetry a similar classification holds, but to characterize a multiplet one needs two integer quantum numbers $(p,q)$. Within a given $(p,q)$ multiplet the states are characterized both by $S_z$ and the charge $Q$ (instead of only $S_z$ in SO(3)). Fig. 5 and 6 display some of the lowest SO(5) multiplets as well as some allowed PE transitions between them (note that for $p=q$, one gets the diamond-like structures shown in Fig. 5).

These theoretical assertions can be checked numerically using the Lanczos technique $Q$. The method permits to calculate ground state properties like energy and spin expectation values, but also Green’s functions (cf. $Q$), and, in particular, the photoemission and inverse photoemission spectra (here for a spin-down electron):

\[ A_{PES}(k,\omega) = \frac{1}{\pi} \Im \langle 0 | c_{k\downarrow}^\dagger | \omega + H - \epsilon_0 - i\delta c_{k\downarrow} | 0 \rangle, \]

\[ A_{IPES}(k,\omega) = \frac{1}{\pi} \Im \langle k | c_{k\downarrow}^\dagger | \omega + H - \epsilon_0 - i\delta c_{k\downarrow} | 0 \rangle. \]

$|0\rangle$ is an initial energy-eigenstate found by Lanczos-diagonalization and $\epsilon_0$ its energy.

With this “computer experiment” one can study the direct and inverse PE within a multiplet structure and observe crucial selection rules. The SO(5) multiplets are easily identified, because the energies of states belonging to one multiplet are degenerate (i.e. identical to computer accuracy of ca. $10^{-15}$). This allows to study the evolution of the single-particle spectral function as we pass from one multiplet $(p,q)$ to the other, and within one multiplet through the different doping and spin levels $(Q,S_z)$.

The dotted line in Fig. 6 shows the single-particle spectrum for the half-filled ground state $^1(0,0)_0$ (where the group theoretical notation is a short hand for degeneracy=$S^z$ (momentum)=$Q$): this is an RVB-state of rung-singlets (see section $Q$) which actually forms a one-dimensional (0,0) multiplet. Final states can only belong to the 4-dimensional (1,0) irreps $Q$. Despite the fact that we are using very strong interaction parameters, there is just one single electron-like band in PES, whose cosine-type-of dispersion closely follows the dispersion of a non-interacting electron. The center of gravity of this band is given by the energy difference between a rung-singlet and a singly occupied rung $Q$.

The solid line in Fig. 6 represents the PES for the half-filled $^3(\pi,\pi)_0$ state (with $S_z=0$), which carries one triplet excitation (magnon) and which belongs to the fivefold degenerate $(p=1,q=1)$ multiplet, i.e. the open circles in Fig. 5. From Fig. 6 we expect in this case final states belonging to both the (2,1) and the (1,0) irreps. Indeed, in addition to the band seen in the ground state spectra, which remains practically unchanged, there appears a “sideband” close to $\mu$ both in PES an in IPES; thus...
we have precisely the 4 bands expected from Fig. 8. The physical interpretation is as follows: the initial $^3\pi_0$ state carries a single triplet-like Boson in the “RVB-vacuum”. The “main bands” result from a creation of the photohole in a singlet rung; then the final state, consisting of the initial triplet plus a propagating singly-occupied rung carrying the entire momentum transfer, belongs to the (2, 1) irreps. As the two excitations, triplet and hole, can scatter from each other, the main band becomes broadened. But the photohole can also be created in the initial triplet, which results in a singly-occupied rung propagating in the RVB-vacuum, thus the new state belongs to the (1, 0) irreps. Since in this second case the photohole has to absorb the momentum of the initial triplet, $(\pi, \pi)$, the sidebands’ dispersion is shifted by $(\pi, \pi)$ with respect to the main bands.

Proceeding to a $^3\pi_0$ initial state with $S_z=1$ (instead of $S_z=0$) (Fig. 8, full line), we get a PES/IPES spectrum similar to the one for $S_z=0$, but the sideband in PES has disappeared, whereas the one in IPES has gained some additional weight. The interpretation follows from Fig. 8. As the initial state belongs to the (1, 1) irreps, a PES transition into the (1, 0) multiplet is now impossible.

At this point, we remember that the SO(5) symmetry of our model implies that spin polarization and hole doping are equivalent in that the empty rung is the “SO(5) partner” of the triplet rung. Consequently, Fig. 8 also shows the spectra originating from the two-hole ground state $^3\pi_0$ (dotted line). Both states belong to the (1, 1) irreps. Accordingly, their energies agree to computer accuracy. Since also the allowed final states of these two initial states belong to one and the same multiplet, namely (2, 1), we expect their PES/IPES amplitudes to be directly related. A more detailed consideration shows that their amplitudes are in fact identical – just as Fig. 8 demonstrates. The physical reason for this identity is that the photohole cannot be created in the triplet rung, because the latter only contains two spin-up electrons.

In summary our numerical “experiments” demonstrated, that SO(5) symmetry implies that the single particle spectra in the doped ground state are identical to those of certain higher-spin states at half-filling.

C. Recovering SO(5) symmetry at low energies

The renormalization-group (RG) route has been proven to be particularly suited to describe the low-energy behavior of systems between one and two dimensions. For this reason, we have carried out a RG study of a rather general ladder model of the Hubbard type. The main result is that the effective Hamiltonian of the system, i.e. the Hamiltonian which describes
excitations below a certain energy $\omega^*$, becomes SO(5) invariant for a wide range of bare models. These include interactions with quite arbitrary values of $U$, $V$, and $V_\perp$ (cf. Fig. 3), provided they are weak, as well as a moderate next-nearest-neighbor hopping $t_2$. It is, thus, not necessary to introduce unphysical values for the parameters by hand in the model, since the exact SO(5) ladder discussed in section I.2 will be eventually recovered at low energies. This result is remarkable, since it shows that quite general Hubbard-like models, which are relevant for the description of the cuprate materials between 1D and 2D, although not explicitly SO(5) invariant on the microscopic level, display an exact SO(5) symmetry when observed on a macroscopic length scale $\propto E_F/\omega^*$ (in units of the lattice spacing) $\mathcal{O}(t_2)$ (for the $t_2 = 0$ case, see also Ref. 3). Physically, this means that the “standard deviation” between the multiplet splitting (cf. Fig. 3) goes to zero in these models for low energies or large length scales. An important issue here is the introduction of a next-nearest-neighbor term $t_2$, which is known to be essential in order to correctly describe AF correlations and Fermi-surface (FS) topology in cuprate materials.

Specifically, we have considered two coupled chains with total low-energy Hamiltonian $H = H_0 + H_I$, where $H_0$ represents the non-interacting part and $H_I$ the interaction. This Hamiltonian describes interacting Fermions (expressed by creation and destruction operators $c_{k\sigma}$ and $c_{k\sigma}^\dagger$) close to the FS. The FS consists of four points: two bands $k_\perp = 0$ and $k_\perp = \pi$, each one with two Fermi points corresponding to right- and left-moving Fermions. Since we are interested in low-energy properties very close to the FS, the Fermion dispersion can safely be taken as constant to where the corresponding processes take place as well as on the spin of the scattered particles [33,34,35].

The SO(5) invariant part $H_I^{(SO(5))}$ of $H_I$ can conveniently be written as a sum of products of SO(5) scalar contractions in the form $\Psi_{k_1}^\dagger \Psi_{k_2} \Psi_{k_3}^\dagger \Psi_{k_4}$ where $\Psi_{k}$ are SO(5) four-spinor Fermi operator (see Refs. 36,37 for details). At $\omega \approx E_F$, the bare Hamiltonian is not SO(5) invariant in general and $H_I$ consists of two parts $H_I = H_I^{(SO(5))} + H_I^{(breaking)}$, where $H_I^{(breaking)}$ is a symmetry-breaking term. On the other hand, for $t_2 = 0$, the non-interacting part $H_0$ is SO(5) symmetric, whereas a finite $t_2$ breaks PH (and thus SO(5)) symmetry also in $H_0$ through a difference $\Delta v_F$ between the Fermi velocities of the two bands.

The idea of the RG is to divide the electronic excitations within the Brillouin zone into high-energy and low-energy modes, the latter being modes restricted within an energy $\omega \ll E_F$ from the Fermi energy. One then eliminates the high-energy modes by integrating them out and constructs an effective Hamiltonian which is restricted to the low-energy excitations. The parameters of the new effective low-energy Hamiltonian thus depend on the energy cutoff $\omega$. Also the total spectral weights at the FS (quasiparticle weights) $Z_{k\perp,\omega}$ of the two bands are reduced due to the reduction of the cutoff and depend on $\omega$. For this reason, in order to recover the canonical Fermi operators within the low-energy subspace, one should reabsorb  

$$Z_{k\perp,\omega}$$

into the definition of the Fermi operators and transform to $c_{k\sigma} = \sqrt{Z_{k\perp,\omega}} c_{k\sigma}^\dagger$. This transformation has to be done in order to preserve the sum rules for the total integrated spectral weight $Z = 1$ within the restricted subspace. The $c_{k\sigma}$ now acquire the meaning of canonical operators with the correct anticommutation relations within the low-energy subspace.

In practice, the integration of the high-energy modes is carried out by decreasing $\omega$ via infinitesimal steps, starting at $\omega = E_F$ which corresponds to the bare (microscopic) Hamiltonian (see, e.g. 38). Due to the restriction of the modes to a small “shell” around the FS, the scattering amplitudes in the interaction $H_I$ can also be considered as dependent only on the Fermi momenta closest to where the corresponding processes take place as well as on the spin of the scattered particles [33,34,35].

The striking result of our calculation is that, by integrating away the high-energy modes, from some energy scale $\omega = \omega^*$ on, the SO(5)-breaking part of the Hamiltonian $H_I^{(breaking)}$ eventually vanishes with respect to the SO(5)symmetric part. In the inset of Fig. 3, we show the RG flow for the $t_2 = 0$ case (here the SO(5)-breaking term appears only in the interacting part of the Hamiltonian). In the bulk of the same figure, the flow for the $t_2 \neq 0$ case is displayed, for which there are three types of SO(5)-breaking terms, namely, $\Delta v_F$, a PH-breaking term $H_I^{(PH-breaking)}$ and a plain SO(5)-breaking term $H_I^{(breaking)}$. All three terms go to zero at $\omega^*$.

The result is of extreme importance, since it means that even though the system is not SO(5) invariant at the bare level, i.e. at $\omega = E_F$, the SO(5) invariant
part of the Hamiltonian eventually dominates with respect to the symmetry-breaking part at energies smaller than \( \omega^* \). We have checked that this occurs for very general values of the Hamiltonian, including on-site (\( U \)) and nearest-neighbor (\( V \) and \( V_\perp \)) interactions, provided they are weak.

An interesting result is that for the \( t_2 \neq 0 \) case, the quasiparticle weights \( Z_{k,\omega} \) for the two bands renormalize differently. These quasiparticle weights have to be reabsorbed into the definition of the canonical Fermi operators through the transformation to the \( \tilde{c}_{k\sigma} \) variables, as explained above. This has the important consequence that the new low-energy SO(5) invariant Hamiltonian \( H_{1}^{(SO(5))} \) is now invariant under a renormalized SO(5) symmetry in terms of new \( \pi \) operators (cf. (5)), whereby the Fermi operators \( c_{k\sigma} \) are replaced with \( \tilde{c}_{k\sigma} \). This extended concept of SO(5) symmetry makes it possible for this generalized symmetry to occur in a larger and more generic class of physical systems than the ordinary SO(5). Moreover, a SO(5) theory unifying antiferromagnetism and superconductivity in terms of this generalized representation can admit possible asymmetries between the antiferromagnetic and superconducting phase, like for example the difference in \( T_c \) or in the order parameter.

In conclusion, the renormalization-group study shows that the effective low-energy Hamiltonian of a quite generic Hubbard-like ladder with weak interaction is SO(5) symmetric. This holds true also with the inclusion of a next-nearest-neighbor hopping \( t_2 \), provided it is written in terms of the appropriate canonical Fermi operators \( \tilde{c} \) for the low-energy subspace.

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