On the design of nonautonomous fixed-time controllers with a predefined upper bound of the settling time

David Gómez-Gutiérrez

1Multi-Agent Autonomous Systems Lab, Intel Labs, Intel Tecnología de México, Zapopan, Mexico
2Escuela de Ingeniería y Ciencias, Tecnológico de Monterrey, Zapopan, Mexico

Correspondence
David Gómez–Gutiérrez, Multi-Agent Autonomous Systems Lab, Intel Labs, Intel Tecnología de México, Av. del Bosque 1001, Colonia El Bajío, Zapopan, Jalisco 45019, Mexico.
Email: david.gomez.g@ieee.org

Summary
Recently, there has been a great deal of attention in a class of finite-time stable dynamical systems, called fixed-time stable, that exhibit uniform convergence with respect to its initial condition, that is, there exists an upper bound for the settling-time (UBST) function, independent of the initial condition of the system. Of particular interest is the development of stabilizing controllers where the desired UBST can be selected a priori by the user since it allows the design of controllers to satisfy real-time constraints. Unfortunately, existing methodologies for the design of controllers for fixed-time stability exhibit the following drawbacks: on the one hand, in methods based on autonomous systems, either the UBST is unknown or its estimate is very conservative, leading to over-engineered solutions; on the other hand, in methods based on time-varying gains, the gain tends to infinity, which makes these methods unrealizable in practice. To bridge these gaps, we introduce a design methodology to stabilize a perturbed chain of integrators in a fixed-time, with the desired UBST that can be set arbitrarily tight. Our approach consists of redesigning autonomous stabilizing controllers by adding time-varying gains. However, unlike existing methods, we provide sufficient conditions such that the time-varying gain remains bounded, making our approach realizable in practice.

KEYWORDS
fixed-time control, predefined-time control, predefined-time stabilization, prescribed-time control

1 | INTRODUCTION

Recently, there has been a great deal of attention in the control community on the analysis of a class of systems, known as fixed-time stable systems, because they exhibit finite-time convergence with an upper bound of the settling time (UBST) that is independent of the initial conditions of the system.1-5 This effort has produced many contributions on algorithms with the fixed-time convergence property, such as multiagent coordination,6-9 distributed resource allocation,10 synchronization of complex networks,11,12 stabilizing controllers,1,13-16 state observers,17 and online differentiation algorithms.18,19

The fixed-time stability property is of great interest in the development of algorithms for scenarios where real-time constraints need to be satisfied. In fault detection, isolation, and recovery schemes,20 failing to recover from the fault on time may lead to an unrecoverable mode. In missile guidance,21 the impact time control guidance laws require stabilization in a desired time.22,23 In hybrid dynamical systems, it is frequently required that the observer (respectively, controller)
stabilizes the observation error (respectively, tracking error) before the next switching occurs. In the frequency control of an interconnected power network, not only is the frequency deviation of interest but also how long the frequency stays out of the bounds. Similarly, in power systems, oscillations are acceptable if they can be damped within a limited time, thus for chaos suppression in power systems, the convergence time is an essential performance specification.

Nonetheless, to enable the application to scenarios under real-time constraints, it is additionally required to be able to set a priori (or predefine) the desired UBST. However, it is not always the case that existing approaches, for the development of algorithms with fixed-time convergence, include a methodology for predefining the desired UBST (examples include the online differentiator algorithm from Angulo et al., the controller design by Tian et al. and the consensus protocols in Gómez-Gutiérrez et al. and Zuo et al. As, for instance, using the homogeneity approach, the fixed-time stability property is verified from the asymptotically stability of different homogeneous approximations without obtaining an estimate of the UBST.

Regarding the controller design problem for fixed-time stability, the existing methodologies exhibit the following drawbacks. First, in methodologies like those proposed by Basin et al. and Tian et al. which are based on the homogeneity property, the UBST is unknown. Second, autonomous controllers derived based on Lyapunov analysis may provide non-conservative estimates of the UBST for the scalar case (see, eg, the work from Sanchez-Torres et al. and Aldana-López et al.), but the estimate of the UBST becomes very conservative in high-order systems (see, eg, section 5 in Zimenko et al.). Third, in nonautonomous controllers based on time-varying gains, the origin is reached exactly at the desired time, but the time-varying gain tends to infinity as the time approaches the desired convergence time.

To fill these gaps, in this article, we propose a methodology for the design of stabilizing controllers for a perturbed chain of integrators.† Our approach has the following four properties:

1. the closed-loop system is fixed-time stable;
2. the desired UBST is set a priori explicitly, with one parameter;
3. the UBST can be set arbitrarily tight (ie, the slack between the predefined and the least UBST can be set arbitrarily small);
4. the controller is non-autonomous with bounded time-varying gains.

Because of these four properties, we say that the controller is a fixed-time non-autonomous controller with a predefined UBST that can be made arbitrarily tight. To the best of our knowledge, no other methodology for the design of stabilizing controllers exhibits these four desirable features. In Table 1, we contrast these main features of our approach with the existing literature on (autonomous and non-autonomous) fixed-time controllers, highlighting our contribution.

Our methodology consists of redesigning known autonomous controllers by adding time-varying gains, constructed from time-base generators. However, unlike existing methods based on time-base generators, we provide sufficient conditions such that our time-varying gains remain bounded. Moreover, contrary to Becerra et al., no initial state is explicitly used in the feedback law; and unlike Song et al. and Becerra et al., predefined-time convergence is guaranteed even in the presence of external disturbances.

To illustrate our approach, we show how to redesign the autonomous fixed-time controllers proposed by Aldana-López et al. and Zimenko et al. to obtain nonautonomous fixed-time controllers with predefined UBST, significantly reducing the overestimation of the UBST while maintaining the time-varying gain bounded.

Notation: is the set of real numbers, , are the real part and the imaginary part of . For a time function , we write to stress that for all .

The rest of the manuscript is organized as follows. In Section 2, we present the preliminaries on fixed-time stability and time-scale transformations. In Section 3, we introduce our redesign methodology, which is applied in Section 4 to redesign a linear controller, and two nonlinear controllers for fixed-time stability (the controller from Aldana-López et al. and the controller from Zimenko et al.). Finally, in Section 5, we present the conclusion and future work.

Seeber showed that the UBST derived by Basin et al. is incorrect. Thus, the class of systems presented in Basin et al. are fixed-time stable (which can be shown using the result on homogeneity in the bi-limit), but the UBST is unknown.

† For simplicity, we focus on chains of integrators. However, the results can be straightforwardly extended to a controllable linear system and feedback linearizable nonlinear systems, and further extended to the multivariable case.
|                             | Autonomous/Nonautonomous | Does the methodology allow to set a priori the desired UBST? | Does the methodology allow to set the UBST arbitrarily tight? | Does singularities appear in the control law computation? | Predefined UBST under disturbances? |
|-----------------------------|--------------------------|-------------------------------------------------------------|--------------------------------------------------------------|-----------------------------------------------------------|-----------------------------------|
| Polyakov et al\(^1\)        | Autonomous               | Yes (but too conservative)                                  | No                                                           | No                                                        | Yes                               |
| Aldana-López et al\(^34\)  | Autonomous               | Yes (but only for first and second order systems)            | No (least UBST only in first order systems)                   | No                                                        | Yes                               |
| Sánchez-Torres et al\(^40\) | Autonomous               | Yes (but only for first and second order systems)            | No (least UBST only in first order systems)                   | No                                                        | Yes                               |
| Basin et al\(^32\)          | Autonomous               | No (unknown UBST)                                           | No (unknown UBST)                                            | No                                                        | No                                |
| Tian et al\(^28\)           | Autonomous               | No (only second order systems, unknown UBST)                 | No (unknown UBST)                                            | No                                                        | No                                |
| Tian et al\(^29\)           | Autonomous               | No (unknown UBST)                                           | No (unknown UBST)                                            | No                                                        | Yes                               |
| Zimenko et al\(^15\)        | Autonomous               | Yes                                                         | No (UBST too conservative)                                   | No                                                        | No                                |
| Mishra et al\(^26\)         | Autonomous               | Yes                                                         | No (UBST too conservative)                                   | No                                                        | Yes                               |
| Song et al\(^23\)           | Nonautonomous             | Yes                                                         | Settling time exactly the desired one                         | Yes                                                       | No                                |
| Becerra et al\(^39\)        | Nonautonomous             | Yes                                                         | Settling time exactly the desired one                         | No (but uses \(x_0\) explicitly in the control law)       | No                                |
| Pal et al\(^22\)            | Nonautonomous             | Yes                                                         | Settling time exactly the desired one                         | Yes                                                       | Yes (if \(x_0\) is known)         |
| Ours                        | Nonautonomous             | Yes                                                         | Yes                                                          | No                                                        | Yes                               |

*Note: Highlighted aspects in the methods that present some drawbacks.*
2 | PRELIMINARIES

Consider the system
\[ \dot{x} = -f(x, t) + D\delta(t), \quad \forall t \geq t_0, \]
where \( x \in \mathbb{R}^n \) is the state of the system, \( t \in [t_0, +\infty) \) is time, \( D = [0, \ldots, 0, 1]^T \), and \( \delta \) is a disturbance satisfying \( |\delta(t)| \leq L \), for a constant \( L < +\infty \).

The solutions of (1) are understood in the sense of Filipov.\(^{41}\) We assume that \( f(\cdot, \cdot) \) is such that the origin of (1) is asymptotically stable and, except at the origin, (1) has the properties of existence and uniqueness of solutions in forward-time on the interval \([t_0, +\infty)\) (see proposition 5 in Cortes\(^{41}\)). The set of admissible disturbances, on the interval \([t_0, \hat{t}]\), where \( \hat{t} \) is some time satisfying \( \hat{t} > t_0 \), is denoted by \( D_{[t_0, \hat{t}]} \). The solution of (1) for \( t \in [t_0, \hat{t}] \), with disturbance \( \delta_{[t_0, \hat{t}]} \) (i.e., the restriction of \( \delta(t) \) to \([t_0, \hat{t}]\)) and initial condition \( x_0 \) is denoted by \( x(t; x_0, t_0, \delta_{[t_0, \hat{t}]} \) ), and the initial state is given by \( x(t_0; x_0, t_0, \cdot) = x_0 \), when \( t_0 = 0 \), we simply write \( x(t; x_0, \delta_{[t_0, \hat{t}]} \) ). Moreover, if \( \delta(t) \equiv 0 \), we simply write \( x(t; x_0) \).

For simplicity, throughout the article, we assume that the origin is the unique equilibrium point of the systems under consideration. Thus, without ambiguity, we refer to global stability (in the respective sense) of the origin of the system as the stability of the system. The extension to local stability is straightforward.

**Definition 1** (Settling-time function). The settling-time function of system (1) is defined as
\[ T(x_0, t_0) := \inf \left\{ \xi \geq t_0 : \exists \delta_{[t_0, \xi]} \in D_{[t_0, \xi]}, \lim_{t \to \xi} x(t; x_0, t_0, \delta_{[t_0, \xi]} = 0 \right\} - t_0. \]

For autonomous systems (\( f \) in (1) does not depend on \( t \)), the settling-time function is independent of \( t_0 \); in such cases, we simply write \( T(x_0) \). Note that Definition 1 admits \( T(x_0, t_0) = +\infty \).

**Definition 2** (Fixed-time stability\(^{42}\)). System (1) is said to be fixed-time stable if it is asymptotically stable\(^{43}\) and the settling-time function \( T(x_0, t_0) \) is bounded on \( \mathbb{R}^n \times \mathbb{R}_+ \), that is, there exists \( T_{\text{max}} \in \mathbb{R} \setminus \{0\} \) such that \( T(x_0, t_0) \leq T_{\text{max}} \) if \( t_0 \in \mathbb{R}_+ \) and \( x_0 \in \mathbb{R}^n \). Thus, \( T_{\text{max}} \) is a UBST of \( x(t; x_0, t_0, \delta_{[t_0, \infty]} \).

2.1 | Time-scale transformations

As in Picó et al\(^{44}\) and Aldana-López et al\(^{45}\), the trajectories corresponding to the system solutions are interpreted, in the sense of differential geometry\(^{46}\), as regular parametrized curves. Since we apply regular parameter transformations over the time variable, then without ambiguity, this reparametrization is sometimes referred to as time-scaling.

**Definition 3** (Definition 2.1 in Kühnel\(^{46}\)). A regular parametrized curve, with parameter \( t \), is a \( C^1(I) \) immersion \( c : I \to \mathbb{R} \), defined on a real interval \( I \subseteq \mathbb{R} \). This means that \( \frac{dc}{dt} \neq 0 \) holds everywhere.

**Definition 4** (Pg. 8 in Kühnel\(^{46}\)). A regular curve is an equivalence class of regular parametrized curves, where the equivalence relation is given by regular (orientation preserving) parameter transformations \( \varphi \), where \( \varphi : I \to I' \) is \( C^1(I) \), bijective, and \( \frac{d\varphi}{dt} > 0 \). Therefore, if \( c : I \to \mathbb{R} \) is a regular parametrized curve and \( \varphi : I \to I' \) is a regular parameter transformation, then \( c \) and \( c \circ \varphi : I' \to \mathbb{R} \) are considered to be equivalent.

3 | MAIN RESULT

Consider the perturbed systems
\[ \begin{align*}
\dot{x}_1 &= x_2 \\
\vdots \\
\dot{x}_{n-1} &= x_n \\
\dot{x}_n &= u(t) + \delta(t),
\end{align*} \]
(2)
where \( x = [x_1, \ldots, x_n]^T \in \mathbb{R}^n \) is the state, \( u(t) \) is the controller, and \( \delta(t) \) is a bounded disturbance satisfying \( |\delta(t)| \leq L, \forall t \geq t_0 \), for a known constant \( L \).

Let \( T_c > 0 \) be a desired UBST for (2). Our aim is to present a methodology for designing a stabilizing controllers \( u(t) \), such that:

1. the closed-loop system (2) is fixed-time stable;
2. \( T_c \) is a UBST;
3. with an appropriate selection of the control parameters, the UBST can be made arbitrarily tight (ie, the slack between the predefined UBST and the least UBST can be set arbitrarily small).
4. the algorithm is nonautonomous with a bounded time-varying gain.

Because of these features, we say that the proposed controller is a \textit{fixed-time controller with a predefined UBST that can be made arbitrarily tight}.

Our methodology consists of defining \( u(t) \) as a piecewise controller. First, a nonautonomous controller drives the state to the origin, ensuring that the origin is reached, regardless of the initial condition, in time \( t < t_0 + T_c \). This controller maintains the state at the origin until time \( t = t_0 + T_c \) when switching occurs to an autonomous controller designed to maintain the state at the origin despite the disturbance.

**Assumption 1.** The mapping \( w_L : \mathbb{R}^n \to \mathbb{R} \) is such that with \( u(t) = w_L(x) \), the perturbed system (2) is asymptotically stable for all disturbances satisfying \( |\delta(t)| \leq L \), for all \( t \geq t_0 \).

Assumption 1 means that we already have a robust controller for system (1), but such a controller may not satisfy the real-time constraints. This controller is the one that we will use to maintain the state at the origin for all \( t \geq t_0 + T_c \). If \( L = 0 \), then \( w_L(x) \) can be chosen as an appropriate linear state feedback.\(^{47}\) If \( L > 0 \), then \( w_L(x) \) can be chosen, for instance, as a high order sliding mode control.\(^{48}\)

Our approach is based on the following time transformation.

**Lemma 1.** Let \( \eta \) be a constant satisfying \( 0 < \eta \leq 1 \) and \( \alpha > 0 \). Then, the function \( \varphi(t) = \tau = -\alpha^{-1} \ln(1 - \eta(t - t_0)/T_c) \) defines a parameter transformation with \( \varphi^{-1}(\tau) = t = \eta^{-1}T_c(1 - e^{-\alpha \tau}) + t_0 \) as its inverse mapping.

**Proof.** It follows from Definition 4. \( \square \)

To derive the controller designed to drive the state of system (1) to the origin in a fixed-time, with predefined UBST given by \( T_c \), let us introduce the time-varying gain, which is parametrized by \( T_c \) (the desired UBST):

\[
\kappa(t - t_0) := \begin{cases} \eta \frac{\alpha(T_c - \eta(t - t_0))}{1} & \text{if } t \in [t_0, t_0 + T_c) \\ 0 & \text{otherwise,} \end{cases}
\]

(3)

where \( 0 < \eta \leq 1 \); together with the following auxiliary system:

\[
\begin{align*}
\frac{dy_1}{d\tau} &= y_2 \\
\frac{dy_2}{d\tau} &= y_3 - \alpha y_2 \\
&\vdots \\
\frac{dy_{n-1}}{d\tau} &= y_n - \alpha(n - 2)y_{n-1} \\
\frac{dy_n}{d\tau} &= \nu(y) - \alpha(n - 1)y_n + \pi(\tau),
\end{align*}
\]

(4)

where \( y = [y_1, \ldots, y_n]^T \) is the state, \( \alpha > 0 \), \( \nu(y) \) is the controller, \( \tau \) is the new time, associated with \( t \) by the time transformation \( \tau = -\alpha^{-1} \ln(1 - \eta(t - t_0)/T_c) \), and

\[
\pi(\tau) := \left[ \kappa(t - t_0)^{-\eta} \delta(t) \right]_{t_0 + T_c(1 - e^{-\alpha \tau}) + t_0}
\]

is a disturbance.
Note that the disturbance $\pi(\tau)$ is vanishing, since $|\delta(t)| \leq L$, for all $t \geq t_0$ and
\[
\kappa(t-t_0)^{-1}\bigg|_{t=\eta^{-1}T_c(1-e^{-\eta t})+t_0} = a\eta^{-1}T_c e^{-\alpha \tau}.
\]

Thus, $\pi(\tau)$ is bounded and $\pi(\tau) \to 0$ as $\tau \to +\infty$. Moreover, note that if $\eta < 1$, then (3) is bounded.

The controller designed to drive the origin of system (2) in a fixed-time upper bounded by $t_0 + T_c$ consists on redesigning $u(y)$ with the time-varying gain (3), that is,
\[
u(t) = \kappa(t-t_0)^{n-1}v(\Omega(t-t_0)^{-1}x), \quad \text{for} \quad t \in [t_0, t_0 + T_c),
\]
where
\[
\Omega(t-t_0) = \text{diag}(1, \kappa(t-t_0), \ldots, \kappa(t-t_0)^{n-1}) \in \mathbb{R}^{nxn}.
\]

To this end, we make the following assumption on the controller $v(y)$.

**Assumption 2.** The map $v : \mathbb{R}^n \to \mathbb{R}$ is such that the auxiliary system (4) satisfies:

- the system (4) is asymptotically stable with settling time function $\mathcal{T}(y_0)$,
- $T_f$ is the smallest known value such that, for all $y_0 \in \mathbb{R}^n$, $\mathcal{T}(y_0) \leq T_f \in \mathbb{R}$ (Notice that, if $\sup_{y_0 \in \mathbb{R}^n} \mathcal{T}(y_0) = +\infty$ or no upper bound is known for $\sup_{y_0 \in \mathbb{R}^n} \mathcal{T}(y_0)$, then $T_f = +\infty$).
- \[
\lim_{t \to t_0 + T_f} \kappa(t-t_0)^{i-1} y_i'(\tau; y_0, \pi_{(0,T_f)(y_0)}) = 0, \quad \text{for } i = 1, \ldots, n,
\]
where $y_i'(\tau; y_0, \pi_{(0,T_f)(y_0)})$ is the $i$th element of the solution of (4), denoted by $y_i(\tau; y_0, \pi_{(0,T_f)(y_0)})$.

Notice that, if (4) is finite-time stable, the condition (5) is trivially satisfied.

Now, we are ready to present our main result.

**Theorem 1.** Let $\kappa(t-t_0)$ be as in (3) with $\eta := (1 - e^{-\alpha T_f})$ and $T_f$ as defined in Assumption 2. Then, if $w_L : \mathbb{R}^n \to \mathbb{R}$ satisfies Assumption 1, $v : \mathbb{R}^n \to \mathbb{R}$ satisfies Assumption 2, and $u(t)$ is designed as
\[
u(t) = \begin{cases} \kappa(t-t_0)^{n-1}v(\Omega(t-t_0)^{-1}x) & \text{if } t \in [t_0, t_0 + T_c) \\ w_L(x) & \text{otherwise}, \end{cases}
\]

with $\Omega(t-t_0) := \text{diag}(1, \kappa(t-t_0), \ldots, \kappa(t-t_0)^{n-1})$, then the system (2) is fixed-time stable with $T_c$ as the predefined UBST.

**Proof.** Consider the coordinate change $x_1 = \kappa(t-t_0)^{i-1}y_i$, $i = 1, \ldots, n$ (i.e., $x = \Omega(t-t_0)y$) then $\dot{x}_i = \kappa(t-t_0)^{i-1}\dot{y}_i + (i-1)\dot{\kappa}(t-t_0)\kappa(t-t_0)^{i-2}y_i$ and the dynamics in the new coordinates are
\[
\dot{y}_i = \kappa(t-t_0)^{i-1}[-\kappa(t-t_0)^{i-1}y_{i+1} + (i-1)\dot{\kappa}(t-t_0)\kappa(t-t_0)^{i-2}y_i]
\]
\[
= \kappa(t-t_0)[y_{i+1} - (i-1)\dot{\kappa}(t-t_0)\kappa(t-t_0)^{i-2}y_i]
\]
for $i = 1, \ldots, n-1$ and
\[
\dot{y}_n = \kappa(t-t_0)^{1-n}[\kappa(t-t_0)^{n-1}v(\Omega(t-t_0)^{-1}x) + \delta(t) - (n-1)\dot{\kappa}(t-t_0)\kappa(t-t_0)^{n-2}y_n]
\]
\[
= \kappa(t-t_0)[-\dot{v}(y) + \kappa(t-t_0)^{-n}\delta(t) - (n-1)\dot{\kappa}(t-t_0)\kappa(t-t_0)^{n-2}y_n].
\]

Note that $\dot{\kappa}(t-t_0)\kappa(t-t_0)^{n-2} = \alpha$.

Moreover, consider the parameter transformation given in Lemma 1, $\tau = -a^{-1} \ln(1 - \eta(t-t_0)/T_c)$, then $t = \eta^{-1}T_c(1 - e^{-\alpha \tau}) + t_0$ and
\[
\frac{dt}{d\tau} \bigg|_{\tau=-a^{-1} \ln(1 - \eta(t-t_0)/T_c)} = [a\eta^{-1}T_c e^{-\alpha \tau}] \bigg|_{\tau=-a^{-1} \ln(1 - \eta(t-t_0)/T_c)} = a\eta^{-1}T_c(1 - \eta(t-t_0)/T_c) = a\eta^{-1}(T_c - \eta(t-t_0))
\]
\[
= \kappa(t-t_0)^{-1}.
\]
Thus, the dynamic of the system in the \( \tau \) variable is given by (4) where \( \pi(\tau) = \kappa(t - t_0)^{-\eta} \delta(t) \mid_{t=t_0} \). Note that since \( \delta(t) \) satisfies that \( |\delta(t)| \leq L \), for all \( t \geq t_0 \) and \( \kappa(t - t_0)^{-\eta} \mid_{t=t_0} \), this implies that \( |\pi(\tau)| \leq (a_\eta^{-1} T_c e^{-\tau_\eta})^{\eta} \), and \( \pi(\tau) \to 0 \) as \( \tau \to +\infty \).

Let \( \Omega(t - t_0) = \text{diag}(1, \kappa(t - t_0), \ldots, \kappa(t - t_0)^{n-1}) \). Since, the system (4) is asymptotically stable with \( \mathcal{O}(y_0) \) as its settling time function and \( x_0 = \Omega(0)y_0 \), then (2) reaches the origin at

\[
T(x_0, t_0) = \lim_{\tau \to \tau(\Omega(0)^{-1} x_0)} \eta^{-1} T_c(1 - e^{-\tau}) = T_c,
\]

and the control \( w_L(x_1, \ldots, x_n) \) maintains it at the origin for all \( t \geq t_0 + T(x_0, t_0) \).

Thus, (2) is fixed-time stable with \( T_c \) as the predefined UBST.

**Remark 1.** Note that system (2) can be seen as the error dynamics of a closed-loop system. Thus, the results derived in Theorem 1 can be used to design stabilizing algorithms, trajectory tracking controllers, or regulators.

**Remark 2.** Without loss of generality, the results in this manuscript can be applied to controllable linear systems and feedback linearizable nonlinear systems. The extension to the multivariable case is straightforward.

**Corollary 1.** The settling time function of system (2), under the controller (6), is given by \( T(x_0, t_0) = \lim_{\tau \to \tau(\Omega(0)^{-1} x_0)} \eta^{-1} T_c(1 - e^{-\tau}) \). Additionally, the following holds:

1. If \( T(y_0) = +\infty \) for every \( y_0 \neq 0 \), then the settling time of (2) is exactly \( T_c \) for every \( x_0 \neq 0 \).
2. If \( \sup_{y_0 \in \mathbb{R}^n} T(y_0) = +\infty \), then \( T_c \) is the least UBST of (2).
3. If there exists \( T_f < +\infty \) such that, for all \( y_0 \in \mathbb{R}^n \), \( \mathcal{O}(y_0) \leq T_f \), then \( \kappa(t - t_0) \) is bounded for all \( t \in [t_0, t_0 + T(x_0, t_0)] \) and all \( x_0 \in \mathbb{R}^n \).

The next corollary states that if system (4) is fixed-time stable with a known UBST, then the time-varying gain \( \kappa(t - t_0) \) is upper bounded, and the predefined UBST of (2), under the control (6), which is given by \( T_c \), can be set arbitrarily tight.

**Corollary 2.** Assume that \( T_f = T_{\max}^n \) with a known constant \( T_{\max} \) < +\( \infty \) (i.e., system (4) is fixed-time stable with a known UBST). Then, \( \eta < 1 \) and \( \kappa(t - t_0) \) is upper bounded by

\[
\kappa(t - t_0) \leq \frac{\eta}{a (T_c - \eta(t - t_0))} = \frac{\eta}{a T_c (1 - \eta)} < +\infty.
\]

Moreover, let \( \hat{T} = \sup_{y_0 \in \mathbb{R}^n} T(y_0) \) and let \( s_n \) be the slack (parametrized by \( a \)) between the least UBST of (2) and the predefined one given by \( T_c \), that is,

\[
s_n = (\eta^{-1} T_c (1 - e^{-\alpha T}) + t_0 - (T_c + t_0) = T_c (1 - e^{-\alpha T_{\max}})^{-1} (1 - e^{-\alpha T}) - T_c.
\]

Then, for every \( e > 0 \), there exists \( \alpha \) such that \( s_n \leq e \).

**FIGURE 1** Example of a time-scaling with \( \eta = 1 \) and \( T_c = 10 \) [Colour figure can be viewed at wileyonlinelibrary.com]
Remark 3. Note that since $\eta$ is a function of the parameter $\alpha$, by tuning $\alpha$, we can select the bound of $\kappa(t - t_0)$. Alternatively, by tuning $\alpha$, we can make the UBST arbitrarily tight (i.e., the slack $s_a$ arbitrarily small). However, note that as $\alpha$ increases, the bound in (7) increases, and the slack $s_a$ decreases. Thus, one needs to establish a tradeoff between the size of the upper bound for $\kappa(t - t_0)$ and how small the slack $s_a$ is. Clearly, the bigger the bound on the time-varying gain, the higher the tolerance required for simulation, and the more sensitive the controller becomes to measurement noise.

Example 1. A plot of the time scale transformation with $t_0 = 0$, $\eta = 1$ and $T_c = 10$ is shown in Figure 1. Note that $\alpha$ provides a degree of freedom to select how a time in $\tau$ maps to a time in $t$. By Corollary 2, by an appropriate selection of $\alpha$, the UBST can be set arbitrarily tight.

4 EXAMPLES: REDESIGNING AUTONOMOUS CONTROLLERS TO OBTAIN FIXED-TIME NONAUTONOMOUS SYSTEM WITH PREDEFINED UBST

Let $a_i$, $i = 1, \ldots, n$ be such that $s^n + a_1 s^{n-1} + \ldots + a_n = 0$ has roots $\lambda_i = \alpha(1 - i)$, $i = 1, \ldots, n$, and let $Q$ be the similarity transformation taking the linear system $\frac{dy}{d\tau} = Ay + Bu$, where

$$A = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & -\alpha & 1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 1 \\ 0 & 0 & 0 & \ldots & -\alpha(n-2) \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix},$$

into the controller canonical form,\(^{47}\) that is, $y = Qz$, where

$$Q := C(A, B)V,$$

with $C(A, B)$ as the controllability matrix of the pair $(A, B)$ and

$$V = \begin{bmatrix} a_{n-1} & a_{n-2} & \ldots & a_1 & 1 \\ a_{n-2} & a_{n-3} & \ldots & 1 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_1 & 1 & \ldots & 0 & 0 \\ 1 & 0 & \ldots & 0 & 0 \end{bmatrix}.$$ 

**Proposition 1.** Let $w_{L_a}(z)$ be such that the system given by

$$\frac{dz_i}{d\tau} = z_{i+1},$$

for $i = 1, \ldots, n-1$, and

$$\frac{dz_n}{d\tau} = w_{L_a}(z) + \pi(\tau),$$

where $z = [z_1, \ldots, z_n]^T$ is asymptotically stable with settling-time function given by $T_z(z_0)$. Then, with

$$v(y) = \left[ w_{L_a}(z) + a_1 z_1 + \ldots + a_n z_n \right]_{z=Q^{-1}y},$$

where $a_i$, $i = 1, \ldots, n$ are the coefficients of the characteristic polynomial of $A$ and $Q$ is defined in (8), the system (4) is asymptotically stable, and the settling time function of (4) is $T(y_0) = T_z(Q^{-1}y_0)$. Thus, under Assumptions 1 and 2, the controller (6) where $v(y)$ is given by (10) is fixed-time stable with $T_c$ as the predefined UBST.

**Proof.** Note that under the coordinate change $y = Qz$, the system (4) is given by
\[
\frac{dz_i}{d\tau} = z_{i+1},
\]
for \(i = 1, \ldots, n - 1\), and
\[
\frac{dz_n}{d\tau} = v(Qz) - a_nz_1 - \ldots - a_1z_n + \pi(\tau).
\]

Thus, if \(v(y)\) is given by (10), then, the closed-loop system becomes (9). Thus, by Assumption 1, (9) is asymptotically stable with settling-time function given by \(T_{\text{set}}(z_0)\), where \(z_0 = Q^{-1}\Omega(0)x_0\). Since \(T(x_0, t_0) = \lim_{\tau \to \tau_{\text{eq}}(z_0)}\eta^{-1}\), \(T_c(1 - e^{-\epsilon T_c}) \leq T_c\).

Note that if (9) is finite-time stable (respectively, fixed-time stable), then if \(v(y)\) is given by (10), the system (4) is also finite-time stable (respectively, fixed-time stable). Moreover, if there exists \(T_c^- < +\infty\) such that, for all \(z_0 \in \mathbb{R}^n\), \(T_c(z_0) \leq T_c^-\), then for all \(y_0 \in \mathbb{R}^n\), \(T(y_0) \leq T_c^-\). This result is interesting because it allows to derive nonautonomous controllers with the desired properties based on autonomous controllers from the literature. For instance, fixed-time controllers for (9) have been proposed, for instance, in References 1, 15, 26, 32, 34.

### 4.1 Deriving fixed-time controllers with predefined UBST

In this subsection, we illustrate our main result by deriving different controllers with predefined UBST. These results are derived by applying Proposition 1. We illustrate our methodology by redesigning three controllers: a linear feedback control, the fixed-time controller proposed in by Aldana-López et al., and the homogeneous fixed-time controller proposed in Zimenko et al. The first controller is based on linear feedback control and allows us to obtain a fixed-time stable closed-loop system with settling time exactly at \(T_c\). To introduce this result, let us first provide a sufficient condition such that (5) holds for the case when \(v(\cdot)\) is linear.

**Lemma 2.** Let \(\pi(\tau) \equiv 0\) and let (4) be a linear system with eigenvalues \(\lambda_i\), \(i = 1, \ldots, n\). Then, if \(\min_{\lambda_i}(|\text{Re}(\lambda_i)|^{a^{-1}}) > n\) then (5) holds.

**Proof.** Note that \(y(t; x_0)\) is a linear combination of terms of the form:

- \(e^{\text{Re}(\lambda_i)t}\) for distinct \(\lambda_i\),
- \(t^je^{\text{Re}(\lambda_i)t}\), \(i = 0, \ldots, j - 1\) for \(\lambda_i\) with algebraic multiplicity \(j\),
- \(e^{\text{Re}(\lambda_i)t}\sin(\text{Im}(\lambda)t + \theta)\) for complex complex conjugate \(\lambda_i\) with and \(\tan(\theta) = \frac{\text{Im}(\lambda_i)}{\text{Re}(\lambda_i)}\).

However, since
\[
e^{\text{Re}(\lambda_i)t} = e^{\ln(1-\eta(t-t_0)/T_c)(\text{Re}(\lambda_i)|^{a^{-1}})} = (1 - \eta(t-t_0)/T_c)^{\text{Re}(\lambda_i)|^{a^{-1}}},
\]
then
\[
k(t-t_0)^{a^{-1}} e^{\text{Re}(\lambda_i)t} = \eta^{a^{-1}}(1 - \eta(t-t_0)/T_c)^{\text{Re}(\lambda_i)|^{a^{-1}}} (a(T_c - \eta(t-t_0))^a)^{-1}, \quad i = 1, \ldots, n.
\]
Thus, if \(\min_{\lambda_i}(|\text{Re}(\lambda_i)|^{a^{-1}}) > n\), then (5) holds. ■

**Corollary 3.** Let \(\delta(t) \equiv 0\), if \(v(y)\) is given as in (10) with
\[
w_{l_0}(z) = -k_nz_1 - \ldots - k_1z_n,
\]
where \(s^n + k_1s^{n-1} + \ldots + k_n\) is a Hurwitz polynomial with roots \(\lambda_i\) satisfying \(\min_{\lambda_i}(|\text{Re}(\lambda_i)|^{a^{-1}}) > n\), then (2) is fixed-time stable with \(T_c\) as the settling time for every nonzero trajectory.
Proof. The proof follows trivially by Lemma 2 and Corollary 1, item 1), and by noticing that (9) is asymptotically stable with eigenvalues $\lambda_i, i = 1, \ldots, n$.

Example 2. Consider a chain of three integrators and let $\alpha = 1$ and $\eta = 1$. The simulation of the proposed fixed-time control (6) with $\nu(y)$ as in (10) with $T_c = 10$, $a_1 = 3$, $a_2 = 2$, $a_3 = 0$; $z_1 = y_1$, $z_2 = y_2$, and $z_3 = (y_3 - y_2)$, $w_{L_0}(z)$ as in (11) with $k_1 = 21$, $k_2 = 134.75$, and $k_3 = 257.25$ and $w_L(x) = -6x_1 - 11x_2 - 6x_3$ is shown in Figure 2.

Remark 4. Note that if system (4) is such that for every $y_0 \neq 0$, $T(y_0) = +\infty$ (such as, in the linear case), then $\eta = 1$. Thus, $\lim_{t \to t_0 + T_c^*} \kappa(t - t_0) = +\infty$. Moreover, note that, even if $\lim_{t \to t_0 + T_c^*} u(t) = 0$, to compute the control law, one needs to compute the time-varying gain $\kappa(t - t_0)$. Thus, the controller is not realizable in practice. These are drawbacks also present in the nonautonomous controllers proposed by Song et al.\textsuperscript{23,37} and Pal et al.\textsuperscript{22} and as stated by Song et al.\textsuperscript{23} also the finite-horizon optimal control approach with a terminal constraint, inevitably yields gains that go to infinity. However, unlike such methods, our methodology allows us to design controllers with bounded time-varying gains and the application to perturbed systems, as we illustrate in our next case.

The second controller is based on the autonomous control for perturbed second-order systems proposed in Aldana-López et al.\textsuperscript{34} which provides an estimation of the UBST. We show that the overestimation of the UBST is significantly reduced in our proposal, while the time-varying gain remains bounded. Compared with previous fixed-time controllers based on time-varying gains,\textsuperscript{22,23} our approach allows us to guarantee fixed-time convergence with predefined UBST and bounded gains even if the system is affected by disturbances.

Corollary 4. Let $n = 2$ and let $\delta(t)$ be such that $|\delta(t)| \leq L$ holds for a known constant $L$. Then, if $\nu(y)$ is given as in (10) with $w_{L_0}(z)$ given as in (A1) and $\eta$ is selected as $\eta = (1 - e^{-\alpha T_{\text{max}}})$, where $T_{\text{max}}$ is given in (A2). Then (2) is fixed-time stable with $T_c$ as the predefined UBST. Moreover, $\kappa(t - t_0)$ is bounded for all $t \in [t_0, t_0 + T_c]$.

Example 3. Consider a chain of two integrators with disturbance $\delta(t) = \sin(2\pi t/5)$. For comparison, consider $u(t) = w_{L_0}(x)$ with $w_{L_0}(\cdot)$ given by (A1) where $\zeta = 1$, $p = 0.5$, $q = 3$, $k = 1.5$ and $a_1 = a_2 = 1/\beta_1 = 1/\beta_2 = 4$. It follows from Reference \textsuperscript{34} that with $T_c = T_\gamma = 5$, $T_{\text{max}} = 10$ is a UBST. The simulation under such autonomous control is given in the first two rows of Figure 3.

Now, consider the system under the proposed controller with $\nu(y)$ as in (10) with $a_1 = 1$, $a_2 = 0$; $z_1 = y_1$, $z_2 = y_2$, $w_{L_0}(z)$ as in (A1) where $\zeta = 1$, $p = 0.5$, $q = 3$, $k = 1.5$, and $a_1 = a_2 = 1/\beta_1 = 1/\beta_2 = 4$. A simulation of such closed-loop system, with $T_c = 10$, $\alpha = 1$ and $\eta = 1 - e^{-10}$ as the parameters for $\kappa(t - t_0)$, is given in the last two rows of Figure 3. Note that compared with the autonomous control, the overestimation of the UBST is significantly reduced.

The third proposed controller is based on the autonomous fixed-time control for arbitrary order integrator chains proposed by Zimenko et al.\textsuperscript{15} Since Zimenko et al.\textsuperscript{15} provide a UBST (which is very conservative), in our proposed controller, the time-varying gain remains bounded. As discussed previously, we can easily tune our controller to significantly reduce the over-estimation, which additionally results in a reduced control effort. According to Remark 3, one needs to do a trade-off between the size of the upper bound for $\kappa(t - t_0)$ and how tight the UBST is.

Corollary 5. Let $\delta(t) \equiv 0$. Then, if $\nu(y)$ is given as in (10) with $w_{L_0}(z)$ given as in (B2) and $\eta$ is selected as $\eta = (1 - e^{-\alpha T_{\text{max}}})$ where $T_{\text{max}}$ is given in (B3), then (2) is fixed-time stable with $T_c$ as the predefined UBST. Moreover, $\kappa(t - t_0)$ is bounded for all $t \in [t_0, t_0 + T_c]$.

![Figure 2](https://example.com/image.png) Simulation of the proposed fixed-time control of Example 2, which is based on linear control. The simulations are performed in the OpenModelica software, using the DASSL solver with Tolerance: 1e-10 and Number of Intervals: 50000 [Colour figure can be viewed at wileyonlinelibrary.com]
FIGURE 3  Simulation of Example 3. In the first two rows, the time response of the autonomous control (A1) from Reference 34 applied to a second-order integrator system affected by a disturbance $\delta(t) = \sin(2\pi t/5)$. In the last two rows, the time response of the proposed non-autonomous fixed-time control derived from (A1) and applied to the same system. The simulations are performed in the OpenModelica software, using the DASSL solver with Tolerance: 0.05 and Number of Intervals: 50 000 [Colour figure can be viewed at wileyonlinelibrary.com]

Example 4. Let $\delta(t) \equiv 0$ and consider a third-order integrator system. The parameters of the stabilizing controller (B2) were taken from the example given in section 5 of Zimenko et al.15 These parameters are $l = 45$, $\mu = 0.4$, $k = [-19.9578, -20.9022, -6.7877]$. With these parameters, if the autonomous control (B2) is applied to the system (9), (9) is fixed-time stable with UBST given by $T_{\text{max}} = 922.84/\rho$. Thus, with $\rho = 922.84/37$, $T_{\text{max}}$ becomes $T_{\text{max}} = 37$. The simulation of the autonomous control is given in the first two rows of Figure 4. As can be observed, the autonomous control is too conservative since, even if $T_{\text{max}} = 37$, the convergence occurs before 0.6 unit of time. In the last two rows of Figure 4, we show the time response of the proposed nonautonomous fixed-time control derived from (B2) and applied to the same system. For this simulation, we set $T_c = 37$, $\alpha = 1$, $\eta = 1 - e^{-37}$. As it can be observed that the overestimation of the UBST was significantly reduced.

5 | CONCLUSION

This manuscript presented new controllers based on time-varying gains, constructed from time-base generators, for the stabilization of a chain of integrators in a fixed-time, with predefined UBST, which allows the application of the results to scenarios with real-time constraints.

We showed how to derive a non-autonomous fixed-time controller with a predefined UBST, based on an autonomous fixed-time controller (which typically has a very conservative UBST, resulting in a UBST that can be set arbitrarily tight. Additionally, we provide conditions under which the time-varying gain is guaranteed to be bounded, which is a significant
contribution relative to existing controllers based on time-varying gains. Moreover, in contrast to existing such controllers, our design is more straightforward and guarantees predefined convergence even in the presence of external disturbances affecting the system. Finally, our approach does not require explicit use of the initial conditions in the feedback law.

We presented numerical simulations and comparisons with existing autonomous fixed-time controllers to demonstrate our contribution. Future work is concerned with the extension of these results to consider a broader class of time-varying gains that have been proposed in the literature.45

ACKNOWLEDGEMENT
The author would like to thank Rodrigo Aldana-López for the fruitful discussion on fixed-time systems and Nilesh Ahuja for proofreading the manuscript.

ORCID
David Gómez-Gutiérrez https://orcid.org/0000-0002-2113-3369

REFERENCES
1. Polyakov A. Nonlinear feedback design for fixed-time stabilization of linear control systems. IEEE Tran Automat Contr. 2012;57(8):2106-2110.
2. Sánchez-Torres JD, Gómez-Gutiérrez D, López E, Loukianov AG. A class of predefined-time stable dynamical systems. IMA J Math Control Inf. 2018;35(Suppl 1):i1-i29. https://doi.org/10.1093/imamci/dnx004.
3. Andrieu V, Praly L, Astolfi A. Homogeneous approximation, recursive observer design, and output feedback. SIAM J Control Optim. 2008;47(4):1814-1850.
4. Aldana-López R, Gómez-Gutiérrez D, Jiménez-Rodriguez E, Sánchez-Torres JD, Defoort M. On the design of new classes of fixed-time stable systems with predefined upper bound for the settling time; 2019. arXiv preprint arXiv.
39. Becerra HM, Vázquez CR, Arechavaleta G, Delfin J. Predefined-time convergence control for high-order integrator systems using time base generators. *IEEE T Contr Syst T*. 2018;26(5):1866-1873.

40. Sánchez-Torres JD, Muñoz-Vázquez AJ, Defoort M, Jiménez-Rodríguez E, Loukianov AG. A class of predefined-time controllers for uncertain second-order systems. *European J Control*. 2019.

41. Cortes J. Discontinuous dynamical systems. *IEEE Control Syst Mag*. 2008;28(3):36-73.

42. Polyakov A, Fridman L. Stability notions and Lyapunov functions for sliding mode control systems. *J Frankl Inst*. 2014;351(4):1831-1865.

43. Khalil HK, Grizzle JW. *Nonlinear Systems*. Upper Saddle River, NJ: Prentice-Hall; 2002.

44. Picó J, Picó-Marco E, Vignoni A, DeBattista H. Stability preserving maps for finite-time convergence: super-twisting sliding-mode algorithm. *Automatica*. 2013;49(2):534-539.

45. Aldana-López R, Gómez-Gutiérrez D, Jiménez-Rodríguez E, Sánchez-Torres JD, Defoort M. On the design of new classes of fixed-time stable systems with predefined upper bound for the settling time; 2019. arXiv preprint arXiv:1901.02782v2.

46. Kühnel W. *Differential Geometry*. 3rd ed. Rhode Island: American Mathematical Society Providence; 2015.

47. Kailath T. *Linear Systems*. Upper Saddle River, NJ: Prentice-Hall; 1980.

48. Ding S, Levant A, Li S. New families of high-order sliding-mode controllers. Paper presented at: Proceedings of the 54th IEEE Conference on Decision and Control (CDC); 2015; 4752-4757; IEEE.

49. Isidori A. *Nonlinear Control Systems*. London: Springer-Verlag; 1995.

**SUPPORTING INFORMATION**

Additional supporting information may be found online in the Supporting Information section at the end of this article.

**How to cite this article:** Gómez-Gutiérrez D. On the design of nonautonomous fixed-time controllers with a predefined upper bound of the settling time. *Int J Robust Nonlinear Control*. 2020;30:3871–3885.

https://doi.org/10.1002/rnc.4976

**APPENDIX A. FIXED-TIME AUTONOMOUS CONTROL FOR PERTURBED SECOND ORDER INTEGRATOR SYSTEMS**

**Lemma 3** (Theorem 4 of Aldana-López et al34). Let $\delta(t)$ be such that $|\delta(t)| \leq L$ holds for a known constant $L$ and let $\alpha_1, \alpha_2, \beta_1, \beta_2, p, q, k > 0$, $kp < 1$, $kq > 1$, $T_{c1}, T_{c2} > 0$, $\zeta > 0$, and

$$
\gamma_1 = \frac{\Gamma\left(\frac{1}{2}\right)^2}{2\alpha_1^{1/2}}\Gamma\left(\frac{1}{2}\right)\alpha_1^{1/4}, \quad \text{and} \quad \gamma_2 = \frac{\Gamma(m_p)\Gamma(m_q)}{a_k^2\Gamma(k)(q-p)}\alpha_2^{m_p},
$$

with $m_p = \frac{1-kp}{q-p}$, $m_q = \frac{kq-1}{q-p}$, and $\Gamma(\cdot)$ the Gamma function defined for $z \in \mathbb{R}_+$, as $\Gamma(z) = \int_0^{+\infty} e^{-x}x^{z-1}dx$. Then, if $n = 2$, under the autonomous control

$$
u = -\left[\frac{\gamma_2}{T_{c2}}(a_2|\sigma|^p + \beta_2|\sigma|^q) + \frac{\gamma_1^2}{2T_{c2}^2} (a_1 + 3\beta_1 z_1^2) + \zeta \right] \text{sign}(\sigma), \quad (A1)
$$

where $\sigma$ is defined as

$$
\sigma = z_2 + \left[|z_2|^2 + \frac{2\gamma_1^2}{T_{c1}^2} (a_1|z_1|^1 + \beta_1|z_1|^3) \right]^{1/2},
$$

the system (2) is fixed-time stable with predefined UBST given by

$$
T_{\text{max}} = T_{c1} + T_{c2}. \quad (A2)
$$
APPENDIX B. FIXED-TIME AUTONOMOUS CONTROL FOR ARBITRARY ORDER INTEGRATOR SYSTEMS

To present the control from Reference \(^15\), let us introduce the following notation. \(\mathbf{r} = [r_1, \ldots, r_n]^T\) and \(x \in \mathbb{R}^n\) where \(r_i \in \mathbb{R}_+, i = 1, \ldots, n\), \(r_{\min} = \min_i r_i\), \(r_{\max} = \max_i r_i\). For \(k = [k_1, \ldots, k_n]\), \(a\) is obtained from \(r\) as

\[
a(r) = \left(1 + \sum_{i=1}^n |k_i| \frac{\phi}{r_{\min}} \right) \alpha_{\min} \left( \sum_{i=1}^n \left( \frac{1}{r_i} \right) \right) \alpha_{\max_{\min}} \frac{1}{\alpha_{\max}}.
\]

Moreover, \(\|x\|_r = \left( \sum_{i=1}^n |x_i|^\frac{\phi}{r_{\min}} \right)^{1/\phi}\) where \(\phi = 2r_{\max}\). For \(\lambda > 0\), \(D_\lambda(\lambda) = \text{diag}(\lambda^\mu, \ldots, \lambda^{n\mu})\); \(A = [a_{ij}] \in \mathbb{R}^{nxn}\) where \(a_{ij} = 1\) if \(i = j + 1\) and \(a_{ij} = 0\) otherwise; \(B = [b_j] \in \mathbb{R}^n\), where \(b_i = 1\) if \(i = n\) and \(b_i = 0\) otherwise; \(m_1, m_2 \in \mathbb{R}_+, m_1 = m_2 + 2\mu, m_1 > n\mu, l > (n - 1)\mu, \mu \in (0, 1); H_1 = \text{diag}(-m_1 + n\mu, \ldots, -m_1 + \mu), H_2 = \text{diag}(-m_1 - n\mu, \ldots, -m_1 - \mu), H_3 = \text{diag}(\lambda^{-m_1 + n\mu}, \ldots, \lambda^{-m_1 + \mu}), H_4 = \text{diag}(\lambda^{m_1 - m_1 + n\mu}, \ldots, \lambda^{m_1 - m_1 + \mu})\), \(r_1 = [l + (n - 1)\mu, l + (n - 2)\mu, \ldots, l]^T\), \(r_2 = [l + (n - 1)\mu, l - (n - 2)\mu, \ldots, l/l]^T\).

**Lemma 4** (Theorem 4.3 of Zimenko et al\(^15\)). If the system of matrix inequalities

\[
0 < X, \quad a(r_1)(H_1 X + X H_1) + A X + X A^T + B y + y^T B^T \leq -\beta_1 X
\]

\[
-a(r_1)(H_1 X + X H_1) + A X + X A^T + B y + y^T B^T \leq -\beta_2 X
\]

\[
a(r_2)(H_2 X + X H_2) + A X + X A^T + B y + y^T B^T \leq -\beta_3 X
\]

\[
-A X + X A^T + B y + y^T B \leq -\beta_4 X
\]

is feasible for some \(X \in \mathbb{R}^{nxn}, y \in \mathbb{R}^{nx1}, \beta_1, \beta_2, \beta_3 \in \mathbb{R}_+, \) where \(a(r_1)\) and \(a_2(r_1)\) are calculated for \(r_1\) and \(r_2\) respectively, from (B1), \(P = X^{-1}\) and \(k = y X^{-1}\). Then, if \(\delta = 0\), under the continuous control

\[
u = \begin{cases} 
\rho^n \|Y x\|_{r_2}^{\left(\frac{n}{\rho}\right)} k D_{\gamma} \left( \frac{1}{\|Y x\|_{r_2}} \right) Y x & \text{for } \|Y x\|_{r_2} \geq 1, \\
\rho^n \|Y x\|_{r_1}^{\left(\frac{n}{\rho}\right)} k D_{\gamma} \left( \frac{1}{\|Y x\|_{r_1}} \right) Y x & \text{for } \|Y x\|_{r_2} < 1, \\
\rho^n k Y x & \text{otherwise},
\end{cases}
\]

where \(Y = \text{diag}(1, 1/\rho, \ldots, 1/\rho^{n-1})\) with \(\rho > 0\); the origin of (2) is fixed-time stable with predefined UBST given by

\[
T(x_0) \leq T_{\max} = \frac{2(m_1 + l - \mu)}{\rho} \left( \frac{c_1^\mu}{c_2^\mu} + \frac{c_1^\mu}{\mu c_1^\mu} + \frac{\ln(c_1 / c_1 1)}{\beta_3} \right)
\]

where \(c_1, c_{12}, c_2, \) and \(c_2\) are such that

\[
\begin{align*}
&c_1 \|x\|_{r_1} \leq V_1(x) \leq c_{12} \|x\|_{r_1} \quad \text{and} \quad c_2 \|x\|_{r_2} \leq V_2(x) \leq c_{22} \|x\|_{r_2},
\end{align*}
\]

where

\[
V_1(x) = \left( x^T H_1 \left( \frac{1}{\|x\|_{r_1}} \right) P H_1 \left( \frac{1}{\|x\|_{r_1}} \right) x \right)^{\frac{1}{2m_1 + l - \mu}}
\]

and

\[
V_2(x) = \left( x^T H_2 \left( \frac{1}{\|x\|_{r_2}} \right) P H_2 \left( \frac{1}{\|x\|_{r_2}} \right) x \right)^{\frac{1}{2m_2 + l - \mu}}.
\]