On governing equation of Gauged Sigma model for Heisenberg ferromagnet

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Abstract
In this note, we study weak solutions of equation
\[ \Delta u = \frac{4e^u}{1 + e^u} - 4\pi \sum_{i=1}^{N} \delta_{p_i} + 4\pi \sum_{j=1}^{M} \delta_{q_j} \quad \text{in} \quad \mathbb{R}^2, \] (0.1)
where \( \{\delta_{p_i}\}_{i=1}^{N} \) (resp. \( \{\delta_{q_j}\}_{j=1}^{M} \)) are Dirac masses concentrated at the points \( p_i, i = 1, \ldots, N \), (resp. \( q_j, j = 1, \ldots, M \)) and \( N - M > 1 \). Here equation (0.1) presents a governing equation of Gauged Sigma model for Heisenberg ferromagnet and we prove that it has a sequence of solutions \( u_\beta \) having behaviors as \( -2\pi \beta \ln|x| + O(1) \) at infinity with a free parameter \( \beta \in (2, 2(N - M)) \), and our concern in this paper is to study the asymptotic behavior’s estimates in the extremal case that \( \beta \) near 2 and \( 2(N - M) \).

Key words: Gauged Sigma model; Dirac mass; Asymptotic behavior.
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1 Introduction
Vortices appear in various planar condensed-matter systems and have important applications in many fundamental areas of physics including superconductivity [1, 10, 13], particle physics [12], optics [4] and cosmology [22]. The study of multiple charges vortex construction in gauged field theory was studied by Taubes [13, 19, 20], initiated the existence and asymptotic behaviors of static solutions of the Sigma model. Later on, Schroers [18] extended the classical \( O(3) \) sigma model solved by Belavin-Polyakov [3] to incorporate an Abelian gauged field and allow the existence of vortices of opposite local charges so that the vortices of negative local charges viewed as poles of a complex scalar field \( u \) makes contribute to, but those positive local charges viewed as zero of \( u \) do not affect, the total energy, although they give some magnetic manifestation for their existence [23]. In fact, these peculiar properties are all due to the absence of symmetry breaking and in order to obtain vortices of opposite magnetic alignments with an energy that takes account of both type of vortices, it suffices to impose a broken symmetry. After that, Yang in [25] established an Abelian field theory model that allows the coexistence of vortices and anti-vortices, showed how vortices and anti-vortices with the coupling of gravity, namely, cosmic strings and anti-strings, can be constructed in the Abelian gauged field model.

After involving the magnetic field, the Sigma model for Heisenberg ferromagnet, would be transformed into the local \( U(1) \)–invariant action density,
\[ \mathcal{L} = -\frac{1}{4} F_{\mu
u} F^{\mu\nu} + \frac{1}{2} D_\mu \phi D^\mu \phi - \frac{1}{2} (1 - \vec{n} \cdot \phi)^2, \]
where \( \vec{n} = (0, 0, 1) \), \( \phi : S^2 \rightarrow \mathbb{R}^3 \) with \( |\phi| = 1 \), \( D_\mu \) is gauge-covariant derivatives on \( \phi \), defined by
\[ D_\mu \phi = \partial_\mu \phi + A_\mu (\vec{n} \times \phi), \quad \mu = 0, 1, 2 \]
\[ F_{\mu \nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \]

Assuming the temporal gauge \( A_0 = 0 \), the total energy is derived as

\[
E(\phi, A) = \frac{1}{2} \int_{\mathbb{R}^2} \left\{ (D_1 \phi)^2 + (D_2 \phi)^2 + (1 - \bar{n} \cdot \phi)^2 + F_{12}^2 \right\}
= 4\pi |\text{deg}(\phi)| + \frac{1}{2} \int_{\mathbb{R}^2} \left\{ (D_1 \phi \pm \phi \times D_2 \phi)^2 + (F_{12} \mp (1 - \bar{n} \cdot \phi))^2 \right\},
\]

where \( \text{deg}(\phi) \) represents the Brouwer’s degree of \( \phi \).

The related Bogomol’nyi equations could be stated as

\[
\begin{aligned}
D_1 v + i D_2 v &= 0, \\
F_{12} &= \frac{2|v|^2}{1 + |v|^2},
\end{aligned}
\]

then, setting \( u = \ln |v|^2 \), it reduces into the following governing equation of the Gauged Sigma model

\[
-\Delta u + \frac{4e^u}{1 + e^u} = 4\pi \sum_{i=1}^{N} \delta_{p_i} - 4\pi \sum_{j=1}^{M} \delta_{q_j} \quad \text{in } \mathbb{R}^2.
\]

where \( \delta_p \) is the Dirac mass concentrated at \( p \in \mathbb{R}^2 \). This subject has been expanded extensively in recent years, see the works of Chern-Yang [9], Lin-Yang [15], Yang [24] and the references therein. In this paper, we study weak solutions of equation (1.1) and our aim in this paper is to consider the asymptotic decay behavior of the solutions of (1.1) at infinity. For convenience of readers, we use some notations and follow some presentations of known results mainly from the book of Yang [24].

We first introduce some auxiliary functions. Let \( \rho \) be a smooth monotone increasing function over \((0, +\infty)\) such that

\[
\rho(t) = \begin{cases} 
\ln t, & 0 < t \leq 1/2 \\
0, & t \geq 1.
\end{cases}
\]

Set \( v_1(x) = 2 \sum_{i=1}^{N} \rho\left(\frac{|x-p_i|}{\varrho}\right) \) and \( v_2(x) = 2 \sum_{j=1}^{M} \rho\left(\frac{|x-q_j|}{\varrho}\right), \) where \( \varrho \in (0, 1) \) such that any two balls of

\[
\{B_\varrho(p_i) : i = 1, \cdots, N\} \cup \{B_\varrho(q_j) : j = 1, \cdots, M\}
\]
do not intersect. We fix a positive number \( r_0 \geq 4\varrho \) large enough such that \( B_\varrho(p_i), B_\varrho(q_j) \subset B_{r_0}(0) \) for \( i = 1, \cdots, N \) and \( j = 1, \cdots, M \). Let \( \eta_0 : [0, +\infty) \to [0, 2] \) be a smooth, non-increasing function with compact support in \([0, 1]\) such that \( \int_{0}^{1} \eta_0(r)rdr = 1 \), and we take also the notation \( \eta_0(x) = \eta_0(|x|), \forall x \in \mathbb{R}^2 \). Denote \( v_3 = \Gamma * \eta_0 - c_0 \), where \( \Gamma \) means the standard convolution operator and \( \Gamma(x) = -\frac{1}{2\pi} \ln |x| \) is the fundamental solution of Laplacian in \( \mathbb{R}^2 \), i.e.

\[
-\Delta \Gamma = \delta_0 \quad \text{in } \mathcal{D}'(\mathbb{R}^2)
\]

and \( c_0 = \int_{0}^{1} (-\ln r) \eta_0(r)rdr > 0 \). Notice that \( v_3 \leq 0 \) is a smooth function in \( \mathbb{R}^2 \) satisfying \( \Delta v_3 = -\eta_0 \leq 0 \) in \( B_1(0) \) and

\[
|v_3(x) + \ln |x| + c_0| \leq 3\eta_0|x|^{-1} \quad \text{for } |x| \geq 2r_0.
\]

See Section 2 for the proof.

Denote

\[
\begin{aligned}
u(x) &= -v_1(x) + v_2(x) + \beta v_3(x),
\end{aligned}
\]

\[ F_{\mu \nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \]
where \( \beta \) is a positive number to be cleared later. Then we observe that a solution \( u \) of (1.1) could be written as \( u = u_0 + v \), where \( u_0 \) contains all singularities at the points \( p_i, q_j \) and the infinity, the remainder term \( v \) is a solution of

\[
\Delta v = \frac{4K_\beta e^{v_2+v}}{e^{v_1} + K_\beta e^{v_2+v}} - g_\beta \quad \text{in } \mathbb{R}^2,
\]

(1.3)

where \( K_\beta = e^{\beta v_3} \), \( g_1 = \sum_{i=1}^{N} 4\pi \delta_{p_i} - \Delta v_1 \), \( g_2 = \sum_{j=1}^{M} 4\pi \delta_{q_j} - \Delta v_2 \) and

\[
g_\beta = g_1 - g_2 + \beta \Delta v_3,
\]

(1.4)

which is a smooth function with compact support in \( B_{\rho_0}(0) \) and verifies that

\[
\int_{\mathbb{R}^2} g_\beta \, dx = 2\pi [2(N-M) - \beta].
\]

Our main results on asymptotic behavior of solutions states as follows.

**Theorem 1.1** Let \( N - M > 1 \), then for any \( \beta \in (2, 2(N-M)) \), problem (1.3) has a unique solution \( v_\beta \) such that

\[
v_\beta(x) = b_\beta + \tilde{v}_\beta(x),
\]

(1.5)

where the constant \( b_\beta \in \mathbb{R} \) depends on \( \beta \) satisfying

\[
\lim_{\beta \to 2(N-M)^-} \frac{b_\beta}{\ln(2(N-M) - \beta)} = 1
\]

(1.6)

and

\[
\lim_{\beta \to 2^+} \frac{b_\beta}{\ln(\beta - 2)} = 1,
\]

(1.7)

and we have

\[
\tilde{v}_\beta(x) = O(|x|^{-\beta-2}) \quad \text{as } |x| \to +\infty.
\]

(1.8)

An interesting phenomena in Theorem 1.1 is that, by (1.6) and (1.7), the asymptotic behavior of \( b_\beta \) as \( \beta \to (2(N-M))^\pm \) (resp. \( \beta \to 2^\pm \)) is of order \( \ln(2(N-M) - \beta) \) (resp. \( \ln(\beta - 2) \)). Back to problem (1.1), a sequence of solutions are constructed with the behaviors as \( 2\pi \beta \ln |x| + O(1) \) at infinity with a free parameter \( \beta \in (2, 2(N-M)) \). Normally, this type of solutions are called as non-topological solutions, for instance [2, 6, 8, 17] on non-topological solutions of Chern-Simon equation or systems. In particular, for \( \beta \in (2, 4) \), the author in [24, Chapter 2] gave an existence result of (1.3) and asserted that the solution converges to a constant at infinity. Precisely, our result extend the existence of solutions for (1.3) with the free parameter \( \beta \) in the range \( (2, 2(N-M)) \) and the uniqueness follows by comparison principle.

Our idea for the estimates (1.6) and (1.7) is to construct suitable super-solution and sub-solutions of (1.3), by the uniqueness to see the asymptotic behavior from the super-solutions and sub-solutions. These super-solutions and sub-solutions are constructed by adding some suitable constants depending on \( \beta \) to the solution of (1.3) with \( \beta = N-M+1 \).

The rest of this paper is organized as follows. In Section 2, we show some estimates at infinity of the convolution function \( \Gamma \ast F \) for function \( F \) satisfying \( \int_{\Omega} F = 0 \) and sketch the proof of the existence in Theorem 1.1. Section 3 is devoted to the estimates for (1.6) and (1.7) by constructing super-solutions and sub-solutions.
2 Existence and Uniqueness

Firstly, we claim that (1.2) holds for $|x| > 2r_0$. In fact, we have that

$$v_3(x) + \ln|x| + c_0 = \frac{1}{2\pi} \int_{B_1(0)} (-\ln|x-y|)\eta_0(y)dy + \frac{1}{2\pi} \int_{B_1(0)} \ln|x|\eta_0(y)dy$$

$$= -\frac{1}{2\pi} \int_{B_1(0)} \ln(|x-y|/|x|)\eta_0(y)dy.$$

For $|x| > 2r_0$ and $|y| \leq r_0$ (since $|y| \leq 1$), we have that

$$1 - \frac{r_0}{|x|} \leq |x-y|/|x| \leq 1 + \frac{r_0}{|x|},$$

so

$$\ln(|x-y|/|x|) \leq \ln(1 + \frac{r_0}{|x|}) - \ln(1 - \frac{r_0}{|x|}) \leq 3r_0/|x|,$$

and thus,

$$|v_3(x) + \ln|x| + c_0| \leq 3r_0/|x|,$$

and the claim is proved. \hfill \Box

Observe that $K_\beta = e^{\beta v_3}$ is a positive smooth function verifying that

$$e^{-2(N-M)c_1|x|^{-\beta}} \leq K_\beta(x) \leq e^{2(N-M)c_1|x|^{-\beta}} \quad \text{for} \quad |x| \geq 2r_0,$$

(2.1)

when $\beta$ varies from 2 to $2(N-M)$ and $-c_1 \ln|x| \leq v_3 \leq c_1 \ln|x|$ for $|x| \geq 2r_0 > 2e$ by (1.2). Let

$$X_\beta = \left\{ w : \mathbb{R}^2 \to \mathbb{R} \mid \| \nabla w \|_{L^2(\mathbb{R}^2)} + \| w \|_{L^2(\mathbb{R}^2, K_\beta dx)} < \infty \right\}.$$

Then we will prove the following

**Proposition 2.1** Let $N - M > 1$ and $2 < \beta < 2(N - M)$. Then problem (1.3) has a unique solution $v_\beta \in X_\beta$ and there exists $b_\beta \in \mathbb{R}$ such that

$$v_\beta(x) = b_\beta + O(|x|^{\frac{\beta-2}{\beta-1}}) \quad \text{as} \quad |x| \to \infty.$$

(2.2)

We start the analysis by doing the decay estimates at infinity.

**Lemma 2.1** Let $\Gamma$ be the fundamental solution of $-\Delta$ in $\mathbb{R}^2$, $F \in L^\infty(\mathbb{R}^2)$ with support in $B_R(0)$ with $R > 0$ large and satisfying

$$\int_{\mathbb{R}^2} F(x)dx = 0.$$

(2.3)

Then we have

$$\| \Gamma \ast F \|_{L^\infty(\mathbb{R}^2)} \leq \| F \|_{L^1(\mathbb{R}^2)} + R \ln R \| F \|_{L^\infty(\mathbb{R}^2)}$$

(2.4)

and

$$|\Gamma \ast F(x)| \leq \frac{R}{|x|} \| F \|_{L^1(\mathbb{R}^2)} \quad \text{for} \quad |x| > 4R.$$

(2.5)

**Proof.** Since $\text{supp}(F) \subset B_R(0)$ and $F \in L^\infty(\mathbb{R}^2)$, then $F \in L^1(\mathbb{R}^2)$ and for $|x| > 4R$,

$$|\Gamma \ast F(x)| = \frac{1}{2\pi} \left| \int_{B_R(0)} \ln|x-y|F(y)dy - \int_{B_R(0)} \ln|x|F(y)dy \right|$$

$$= \frac{|x|^2}{2\pi} \left| \int_{B_{R/|x|}(0)} \ln|e_z - z|F(|x|z)dz \right|$$

$$\leq \frac{|x|^2}{\pi} \int_{B_{R/|x|}(0)} |z||F(|x|z)|dz$$

$$\leq \frac{R}{|x|} \| F \|_{L^1(\mathbb{R}^2)},$$
where \( c_x = \frac{x}{|x|} \) and we have used (2.3) and the fact that
\[
|\ln |e_x - z|| \leq 2|z| \leq 2 \frac{R}{|x|} \quad \text{for} \quad z \in B_{R/|x|}(0) \subset B_{1/4}(0).
\]
Therefore (2.5) is proved. On the other hand, for \( |x| \leq 4R \), we have that
\[
|\Gamma * F(x)| = \frac{1}{2\pi} \left| \int_{B_R(0)} F(y) \ln |x-y|dy \right| \leq \|F\|_{L^\infty(\mathbb{R}^2)} R^2 \ln R,
\]
which completes the proof of (2.4). \( \square \)

Replacing the compact support assumption for \( F \) by some decay at infinity, we have the following estimate.

**Lemma 2.2** Let \( \beta > 2 \), \( F \in L^1(\mathbb{R}^N) \) verifies (2.3) and
\[
|F(x)| \leq c_2|x|^{-\beta} \quad \text{for} \quad |x| \geq e,
\]
for some \( c_2 > 0 \). Then we have that
\[
|\Gamma * F(x)| \leq \frac{c_5}{(\beta - 2)^2} |x|^{-\frac{\beta - 2}{\beta - 2 - 1}} \quad \text{for large} \quad |x| > 4e,
\]
where \( c_5 > 0 \) is dependent of \( c_2 \) and \( \|F\|_{L^1(\mathbb{R}^2)} \), but it is independent of \( \beta \).

**Proof.** We observe, since \( F \) satisfies (2.3), that for all \( |x| > 4e \),
\[
2\pi \Gamma * F(x) = |x|^2 \int_{\mathbb{R}^2} \ln |e_x - z| F(|x|z)dz + |x|^2 \ln |x| \int_{\mathbb{R}^2} F(|x|z)dz
\]
\[
= |x|^2 \int_{B_{R/|x|}(0)} \ln |e_x - z| F(|x|z)dz + |x|^2 \int_{B_{1/2}(e_x)} \ln |e_x - z| F(|x|z)dz
\]
\[
+ |x|^2 \int_{(\mathbb{R}^2 \setminus (B_{R/|x|}(0) \cup B_{1/2}(e_x)))} \ln |e_x - z| F(|x|z)dz
\]
\[
=: I_1(x) + I_2(x) + I_3(x),
\]
where \( R \in (e, \frac{4e}{|x|}) \) will be chosen later. By directly computation, we have that
\[
|I_1(x)| \leq |x|^2 \int_{B_{R/|x|}(0)} |z||F(|x|z)|dz
\]
\[
= 2 \frac{R}{|x|} \int_{B_R(0)} |F(y)|dy \leq 2 \frac{R}{|x|} \|F\|_{L^1(\mathbb{R}^2)}.
\]
For \( z \in B_{1/2}(e_x) \), we have that \( |x||z| \geq \frac{1}{2} |x| > e \), then \( |F(|x|z)| \leq c_2 |x|^{-\beta} |z|^{-\beta} \) and
\[
|I_2(x)| \leq c_2 |x|^{2-\beta} \int_{B_{1/2}(e_x)} (-\ln |e_x - z|)|z|^{-\beta}dz
\]
\[
\leq c_2 2^\beta \left( \int_{B_{1/2}(e_x)} (-\ln |e_x - z|)dz \right) |x|^{2-\beta} \leq c_3 |x|^{2-\beta} \leq c_3 R^{2-\beta},
\]
where \( c_3 \geq c_2 2^\beta \left( \int_{B_{1/2}(0)} (-\ln |z|)dz \right) \) is a constant independent of \( \beta \in (2, 2(N - M)) \).

For \( z \in \mathbb{R}^2 \setminus (B_{R/|x|}(0) \cup B_{1/2}(e_x)) \), we have that \( |\ln |e_x - z|| \leq \ln (1 + |z|) \) and \( |F(|x|z)| \leq c_4 |x|^{-\beta} |z|^{-\beta} \), since \( |z| \geq \frac{2}{|x|} > \frac{2}{|x|} \), and thus the integration by parts gives
\[
|I_3(x)| \leq c_4 |x|^{2-\beta} \int_{\mathbb{R}^2 \setminus B_{R/|x|}(0)} \ln (1 + |z|)|z|^{-\beta}dz
\]
\[
\leq \frac{2\pi c_4}{(\beta - 2)^2} R^{2-\beta} \ln (1 + \frac{R}{|x|}) + \frac{2\pi c_4}{(\beta - 2)^2} R^{2-\beta}
\]
\[
\leq \frac{2\pi c_4}{(\beta - 2)^2} \left( (\beta - 2) \ln 2 + 1 \right) R^{2-\beta}.
\]
Thus taking \( R = |x|^{\frac{1}{n-1}} \) and \( |x| \) sufficient large (certainly \( R \in (e, \frac{1}{\sqrt{eta}}) \) is satisfied), we have that

\[
|\Gamma * F(x)| \leq \frac{c}{2\pi |x|} \|F\|_{L^1(\mathbb{R}^2)} + \frac{c^4}{(\beta - 2)^2 \mathbb{R}^2} \left( 2(N - M - 1) \ln(e + 1) + 1 \right) R^{2 - \beta}
\]

where \( c_5 > 0 \) can be chosen independent of \( \beta \). This ends the proof. \( \square \)

Now for \( \sigma \in \mathbb{R} \) and \( s \in \mathbb{N} \), we define \( W_{s,\sigma}^2 \) as the closure of the set of \( C^\infty \) functions over \( \mathbb{R}^2 \) with compact supports under the norm

\[
\|\xi\|^2_{\mathbb{R}^2, \sigma} = \sum_{|\alpha| \leq s} \|(1 + |x|)^{\sigma + |\alpha|} D^\alpha \xi\|^2_{L^2(\mathbb{R}^2)}.
\]

For more details of properties of these weighted Sobolev spaces, see e.g. \[5\], \[16\]. Let \( C_0(\mathbb{R}^2) \) be the set of continuous functions on \( \mathbb{R}^2 \) vanishing at infinity.

**Lemma 2.3** \[24\], Lemma 2.4.5] The following statements hold:

(i) If \( s > 1 \) and \( \sigma > -1 \), then \( W_{s,\sigma}^2 \subset C_0(\mathbb{R}^2) \).

(ii) For \( -1 < \sigma < 0 \), the Laplace operator \( \Delta : W_{2,\sigma}^2 \rightarrow W_{0,\sigma + 2}^2 \) is one to one and the range of \( \Delta \) has the characterisation

\[
\Delta(W_{2,\sigma}^2) = \left\{ F \in W_{0,\sigma + 2}^2 \mid \int_{\mathbb{R}^2} F dx = 0 \right\}.
\]

(iii) If \( \xi \in \mathbb{X}_\beta \) and \( \Delta \xi = 0 \), then \( \xi \) is a constant.

**Proof of Proposition 2.1** The key point to study \( (1.3) \) is the following equation

\[
\Delta w = \frac{4K_\beta e^w}{e^{v_1} + K_\beta e^w} - h \quad \text{in} \quad \mathbb{R}^2,
\]

where \( K_\beta \) is a positive smooth function, \( h \geq 0 \) is a function in \( C^\infty_c(\mathbb{R}^2) \), i.e. with compact support such that

\[
\int_{\mathbb{R}^2} h(x) dx = 2\pi [2(N - M) - \beta].
\]

**Existence:** As it is proved in \[24\], Chapter 2, Section 2.4.2, Problem (2.8) has a solution \( w_1 \) which is derived by considering the critical point of the energy functional

\[
I(w) = \int_{\mathbb{R}^2} \left\{ \frac{1}{2} |\nabla w|^2 + 4 \ln(e^{v_1} + K_\beta e^w) - hw \right\} dx
\]

in the admissible space

\[
\mathcal{A} = \left\{ w \in \mathbb{X}_\beta \left| \int_{\mathbb{R}^2} \frac{4K_\beta e^w}{e^{v_1} + K_\beta e^w} dx = \int_{\mathbb{R}^2} h dx \right. \right\}.
\]

Moreover, \( w_1 \) is a classical solution of (2.8).

The subsolution of (1.3) could be constructed as \( w_- = w_1 - \Gamma * (h - g) - c_6 \), where \( c_6 > 0 \) is a constant such that \( w_1 - c_6 \leq 0 \). The supersolution is given by \( w^+ = \tilde{w}_1 - \tilde{w}_2 - v_2 \), where \( \tilde{w}_1 \) is the solution of

\[
\Delta w = \frac{4K_\beta e^w}{1 + K_\beta e^w} - \tilde{h},
\]

\( \tilde{w}_2 = \Gamma * (\tilde{h} - \bar{g}) \) with \( \bar{h} \geq 0 \) is a function in \( C^\infty_c(\mathbb{R}^2) \) such that

\[
\int_{\mathbb{R}^2} \tilde{h} dx = \int_{\mathbb{R}^2} \bar{g} dx = 2\pi (2N - \beta),
\]

\[6\]
and $\tilde{g} = (\sum_{i=1}^{N} 4\pi \delta_{p_i} - \Delta v_1) + \beta \Delta v_3$. Then a solution $v$ of (1.3) is derived by the method of super- and subsolutions. Furthermore,

$$\int_{\mathbb{R}^2} \frac{4K\beta e^v}{e^{v_1} + K\beta e^v} \, dx = \int_{\mathbb{R}^2} h \, dx,$$

by Lemma 2.3 it is known that there is a constant $b$ such that

$$v(x) \to b \quad \text{as} \quad |x| \to +\infty.$$

**Uniqueness.** Assume that $w_i$ with $i = 1, 2$ are two solutions of (1.3), by Lemma 2.4.5 in [24], verifying that

$$w_i(x) \to b_i \quad \text{as} \quad |x| \to +\infty,$$

where we may assume that $b_1 \geq b_2$. We claim that

$$w_1 \geq w_2 \quad \text{in} \quad \mathbb{R}^2. \quad (2.10)$$

Otherwise, it follows by $b_1 \geq b_2$, there exists $x_0 \in \mathbb{R}^2$ such that

$$w_1(x_0) - w_2(x_0) = \min_{x \in \mathbb{R}^2} (w_1 - w_2)(x) < 0,$$

then

$$\Delta(w_1 - w_2)(x_0) \geq 0,$$

which contradicts the fact that

$$\Delta(w_1 - w_2)(x_0) = \frac{4K\beta e^{v_1(x_0)}}{e^{v_1(x_0)} - v_2(x_0) + K\beta e^{v_1(x_0)}} - \frac{4K\beta e^{v_2(x_0)}}{e^{v_1(x_0)} - v_2(x_0) + K\beta e^{v_2(x_0)}} < 0.$$

Thus, $w_1 \geq w_2$ in $\mathbb{R}^2$. If $w_1 \not= w_2$, then it implies from (2.10) that

$$\int_{\mathbb{R}^2} g_\beta \, dx = \int_{\mathbb{R}^2} \frac{4K\beta e^{v_1}}{e^{v_1} - v_2 + K\beta e^{v_1}} \, dx > \int_{\mathbb{R}^2} \frac{4K\beta e^{v_2}}{e^{v_1} - v_2 + K\beta e^{v_2}} \, dx = \int_{\mathbb{R}^2} g_\beta \, dx,$$

which is impossible. Therefore, we have that $w_1 \equiv w_2$ in $\mathbb{R}^2$.

We conclude that for $\beta \in (2, 2(N - M))$, problem (1.3) has a unique solution $v_\beta$ and $v_\beta(x) \to b_\beta$ as $|x| \to +\infty$. Then we may rewrite that

$$v_\beta = b_\beta + \Gamma * \left( \frac{4K\beta e^{v_\beta}}{e^{v_1} + K\beta e^{v_\beta}} - g_\beta \right),$$

where $\int_{\mathbb{R}^2} \left( \frac{4K\beta e^{v_\beta}}{e^{v_1} + K\beta e^{v_\beta}} - g_\beta \right) \, dx = 0$. Then by applying Lemma 2.2 that

$$\left| \Gamma * \left( \frac{4K\beta e^{v_\beta}}{e^{v_1} + K\beta e^{v_\beta}} - g_\beta \right) \right| \leq c_6 |x|^\frac{\beta - 2}{\beta - 1} \quad \text{for} \quad |x| > 1$$

for some constant $c_6$ independent of $\beta$. This completes the proof. 

\[\square\]

**Remark 2.1** (i) Since the map $t \mapsto \frac{4K\beta e^t}{e^{v_1(t)} + K\beta e^t}$ is increasing, the method of super and sub solutions is valid to find out the solution. So by the uniqueness and constructing a super solution $w_1$ and a sub solution $w_2$ such that $w_1 \geq w_2$, then the unique solution of (1.3) stays between $w_1$ and $w_2$. Furthermore, we have that

$$\int_{\mathbb{R}^2} \frac{4K\beta e^{v_1}}{e^{v_1} - v_2 + K\beta e^{v_1}} \, dx \geq \int_{\mathbb{R}^2} g_\beta \, dx = 2\pi [2(N - M) - \beta]$$

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and
\[ \int_{\mathbb{R}^2} \frac{4K_\beta e^{w_2}}{e^{v_1-v_2} + K_\beta e^{w_2}} \, dx \leq 2\pi [2(N-M) - \beta]. \]

(ii) From the proof of uniqueness in Proposition 2.1, we conclude a type of Comparison Principle: Let \( w_1, w_2 \) be super and sub solutions of (1.3) respectively, verifying that \( b_1 \geq b_2 \), where
\[ w_i(x) \to b_i \quad \text{as} \quad |x| \to +\infty, \quad i = 1, 2. \]

Then
\[ w_1 \geq w_2 \quad \text{in} \quad \mathbb{R}^2. \]

3 Analysis of \( b_\beta \) and \( \Gamma \ast v_\beta \)

In this section, we refine the estimates of \( b_\beta \) and \( \Gamma \ast v_\beta \) by constructing suitable super and sub solutions of (1.3). Our constructing is to shift the solution \( v_{\beta_0} \) of (1.3) with \( \beta_0 = N - M + 1 \), \( \int_{\mathbb{R}^2} g_{\beta_0} \, dx = 2\pi [2(N-M) - \beta_0] \)
by some constant depending on \( \beta \).

Before our analysis, we introduce the following notations: for given \( \tau_\beta \in \mathbb{R} \), which is chosen later in different case and depends on \( \beta \in (2, 2(N-M)) \), denote
\[ w_\beta := v_{\beta_0} + \tau_\beta \quad \text{and} \quad F_\beta := \frac{4 e^{\tau_\beta} K_\beta e^{v_{\beta_0}}}{e^{v_1-v_2} + e^{\tau_\beta} K_\beta e^{v_{\beta_0}}} - g_\beta, \]
where \( g_\beta \) is given by (1.4). In particular, we always choose that \( \tau_{\beta_0} = 0 \) and then \( \int_{\mathbb{R}^2} F_{\beta_0} \, dx = 0 \). Let
\[ t_\beta := \frac{1}{2\pi} \int_{\mathbb{R}^2} F_\beta \, dx \quad \text{and} \quad \tilde{F}_\beta := -F_\beta + t_\beta (-\Delta v_3), \]
then we have that \( \int_{\mathbb{R}^2} \tilde{F}_\beta \, dx = 0 \).

We highlight that the parameters and functions vary as the choices of \( \tau_\beta \).

**Proposition 3.1** Let \( \beta \in [\beta_0, 2(N-M)) \) and \( b_\beta \) be derived by Proposition 2.1, then there exists a positive constant \( c_7 > 0 \) independent of \( \beta \) such that
\[ \left| b_\beta - \ln(2(N-M) - \beta) \right| \leq c_7. \] (3.1)

**Proof.** Construction of subsolution for \( \beta \in (\beta_0, 2(N-M)) \). For \( \beta \in [\beta_0, 2(N-M)) \), we choose
\[ \tau_\beta = \left( \ln(2(N-M) - \beta) + \ln \frac{2\pi}{d_1} \right)_-, \] (3.2)
where \( a_- = \min\{0, a\} \) and \( d_1 = \int_{\mathbb{R}^2} 4K_{\beta_0} e^{v_{\beta_0}-v_3} \, dx \). Then we have that
\begin{align*}
-\Delta w_\beta + \frac{4K_\beta e^{w_\beta}}{e^{v_1-v_2} + K_\beta e^{w_\beta}} - g_\beta &= \left( \frac{4 e^{\tau_\beta} K_\beta e^{v_{\beta_0}}}{e^{v_1-v_2} + e^{\tau_\beta} K_\beta e^{v_{\beta_0}}} - g_\beta \right) - \left( \frac{4K_{\beta_0} e^{v_{\beta_0}}}{e^{v_1-v_2} + K_{\beta_0} e^{v_{\beta_0}}} - g_{\beta_0} \right) \\
&=: F_\beta - F_{\beta_0}. \tag{3.3}
\end{align*}
Since the map $\beta \mapsto K_\beta$ is decreasing, $v_2 \leq 0$, then
\[
0 < \int_{\mathbb{R}^2} \frac{4e^\tau_\beta K_\beta e^{y_\beta}}{e^{v_1-v_2} + e^{\tau_\beta} K_\beta e^{y_\beta}} \, dx \leq e^{\tau_\beta} \int_{\mathbb{R}^2} 4K_\beta e^{v_\beta}e^{v_1} \, dx \leq e^{\tau_\beta} \int_{\mathbb{R}^2} 4K_\beta e^{v_\beta}e^{-v_1} \, dx = 2\pi[2(N-M) - \beta].
\]

From the fact that \(\int_{\mathbb{R}^2} g_\beta \, dx = 2\pi[2(N-M) - \beta]\), we have that
\[
\int_{\mathbb{R}^2} F_\beta \, dx \leq 0 \quad \text{and} \quad -4\pi(N-M) < t_\beta \leq 0. \tag{3.4}
\]

**Claim 1:** For $\beta \in [\beta_0, 2(N-M))$ and $\tau_\beta$ is chosen by (3.2), there exist $\nu_1, c_8 > 0$ independent of $\beta$ such that
\[
\|\Gamma \ast \tilde{F}_\beta\|_{L^\infty(\mathbb{R}^2)} \leq \nu_1. \tag{3.5}
\]

and moreover
\[
|\Gamma \ast \tilde{F}_\beta(x)| \leq c_8|x|^{-\frac{\beta - 2}{\beta}} \quad \text{for } |x| > 4e \text{ large enough.} \tag{3.6}
\]

Since the function $t \mapsto \frac{4e^\tau_\beta K_\beta e^{y_\beta}}{e^{v_1-v_2} + e^{\tau_\beta} K_\beta e^{y_\beta}}$ is increasing, then we have that
\[
\frac{4e^\tau_\beta K_\beta e^{y_\beta}}{e^{v_1-v_2} + e^{\tau_\beta} K_\beta e^{y_\beta}} \leq \frac{4K_\beta e^{y_\beta}}{e^{v_1-v_2} + K_\beta e^{y_\beta}}.
\]

thus,
\[
-e^{2(N-M)c_1}|x|^{-\beta} \leq \tilde{F}_\beta(x) \leq -e^{2(N-M)c_1}|x|^{-\beta}, \quad \forall |x| > 2r_0
\]

and
\[
\|\tilde{F}_\beta\|_{L^\infty(\mathbb{R}^2)} \leq 4 + \|g_1\|_{L^\infty(\mathbb{R}^2)} + \|g_2\|_{L^\infty(\mathbb{R}^2)} + (4\pi + 2)(N - M)\|\Delta v_\beta\|_{L^\infty(\mathbb{R}^2)} := a_0.
\]

From Lemma 2.2 we have that (3.6) holds true for $x \in \mathbb{R}^N \setminus B_{4e}(0)$ and then $\Gamma \ast \tilde{F}_\beta$ is bounded in $\mathbb{R}^N \setminus B_{4e}(0)$. So we only have to prove that
\[
2\pi|\Gamma \ast \tilde{F}_\beta(x)| = \left| \int_{\mathbb{R}^2} \ln |x - y| \tilde{F}_\beta(y) \, dy \right| \leq \nu_0, \quad \text{for } |x| \leq 4e. \tag{3.7}
\]

In fact, for $|x| \leq 4e$
\[
\left| \int_{B_{r_0}(0)} \ln |x - y| \tilde{F}_\beta(y) \, dy \right| \leq \|\tilde{F}_\beta\|_{L^\infty(\mathbb{R}^2)} \int_{B_{r_0}(0)} |\ln |x - y|| \, dy \leq \pi a_0 \left( r_0^2 (|\ln r_0| + 1) + 4e \right)
\]
and
\[
\left| \int_{B_{r_0}(0)} \ln |x - y| \tilde{F}_\beta(y) \, dy \right| \leq e^{2(N-M)c_1} \int_{B_{r_0}(0)} \ln (4e + |y|) |y|^{-\beta} \, dy \leq c_8,
\]

which imply (3.5). Thus, Claim 1 holds true.

We continue to construct a subsolution. Let
\[
v_\beta = w_\beta + \Gamma \ast (\tilde{F}_\beta + F_{\beta_0}) - 2\nu_1.
\]
From Claim 1, we have that \( \Gamma \ast (\tilde{F}_\beta + F_{\beta_0}) - 2\nu_1 \leq 0 \), then
\[
-\Delta \nu_3 + \frac{4K_\beta e^\nu_3}{e^{v_1-v_2} + K_\beta e^{\nu_3}} - g_\beta \leq -\Delta w_3 + \tilde{F}_\beta + F_{\beta_0} + \frac{4K_\beta e^{w_3}}{e^{v_1-v_2} + K_\beta e^{w_3}} - g_\beta \\
= t_\beta(-\Delta \nu_3) \leq 0,
\]
where we used (3.4). Then \( \nu_3 \) is a subsolution of (1.3) for \( \beta \in [\beta_0, 2(N-M)) \).

**Construction of supersolution for \( \beta \in [\beta_0, 2(N-M)) \).** We choose that
\[
\tau_\beta = \left( \ln(2(N-M) - \beta) + \frac{2\pi}{d_2} \right)^+ 
\]
with \( d_2 = \int_{\mathbb{R}^2} \frac{4K_\beta e^{w_3}}{e^{v_1-v_2} + K_\beta e^{w_3}} \, dx \). We see that for \( \beta \in [\beta_0, 2(N-M)) \),
\[
-\Delta w_3 + \frac{4K_\beta e^{w_3}}{e^{v_1-v_2} + K_\beta e^{w_3}} - g_\beta = F_\beta - F_{\beta_0},
\]
see e.g. (3.3). Again since the function \( \beta \mapsto K_\beta \) is decreasing, then
\[
\int_{\mathbb{R}^2} 4e^{\eta_3} K_\beta e^{v_3} \, dx \geq \int_{\mathbb{R}^2} 4e^{\eta_3} K_\beta e^{v_3} \, dx \\
\geq e^{\eta_3} \int_{\mathbb{R}^2} 4K_\beta e^{v_3} \, dx \\
= 2\pi |2(N-M) - \beta|,
\]
which, together with \( \int_{\mathbb{R}^2} g_\beta dx = 2\pi |2(N-M) - \beta| \), implies that
\[
\int_{\mathbb{R}^2} F_\beta dx \geq 0.
\]
Thus,
\[
t_\beta = \frac{1}{2\pi} \int_{\mathbb{R}^2} F_\beta dx \geq 0. 
\]
From the choice of \( \tau_\beta \) in (3.8), Claim 1 holds true, that is, there exists \( \nu_2 > 0 \) such that for \( \beta \in [\beta_0, 2(N-M)) \),
\[
\|\tilde{F}_\beta\|_{L^\infty(\mathbb{R}^2)} \leq \nu_2
\]
Let
\[
\tilde{\nu}_3 = w_3 + \nu_1 + \nu_2 + \Gamma \ast (\tilde{F}_\beta + F_{\beta_0}) + \nu_3,
\]
where \( \nu_3 = \max\{0, \ln \frac{d_2}{d_1}\} \). Then we have that
\[
\nu_3 \leq \tilde{\nu}_3 \text{ in } \mathbb{R}^2.
\]
Since \( \nu_1 + \nu_2 + \Gamma \ast (\tilde{F}_\beta + F_{\beta_0}) \geq 0 \), then we have that
\[
-\Delta \tilde{v}_3 + \frac{4K_\beta e^{\tilde{v}_3}}{e^{v_1-v_2} + K_\beta e^{\tilde{v}_3}} - g_\beta \\
\geq -\Delta w_3 + \tilde{F}_\beta + F_{\beta_0} + \frac{4K_\beta e^{w_3}}{e^{v_1-v_2} + K_\beta e^{w_3}} - g_\beta \\
= t_\beta(-\Delta \nu_3) \\
\geq 0,
\]
where \( t_\beta \geq 0 \) by (3.9). Then \( \tilde{v}_\beta \) is a supersolution of (1.3) for \( \beta \in [\beta_0, 2(N - M)) \).

From Remark 2.1 (i), we have that for any \( \beta \in [\beta_0, 2(N - M)) \), we have that
\[
\underline{v}_\beta \leq v_\beta \leq \tilde{v}_\beta \quad \text{in} \quad \mathbb{R}^2,
\]
which implies that (3.1) holds. We complete the proof. \( \square \)

**Proposition 3.2** Let \( \beta \in (2, \beta_0) \) and \( b_\beta \) be derived by Proposition 2.1, then there exists a positive constant \( c_0 > 0 \) independent of \( \beta \) such that
\[
|b_\beta - \ln(\beta - 2)| \leq c_0.
\]

**Proof.** Construction of subsolution for \( \beta \in (2, \beta_0) \). To this end, we now choose
\[
\tau_\beta = \left( \ln(\beta - 2) + \ln\left( \frac{2\pi(\beta_0 - 2)}{d_3} \right) \right)_-,
\]
where \( d_3 = 4e^{\|v_{\beta_0}\|_{L^{\infty}(\mathbb{R}^2)}} \left( (\beta_0 - 2) \int_{B_{\rho_0}(0)} e^{-v_1} dx + 2\pi \right) \) is independent of \( \beta \).

We recall that
\[
-\Delta w_\beta + \frac{4K_\beta e^{w_\beta}}{e^{v_1 - v_2 + K_\beta e^{v_\beta}}} - g_\beta = F_\beta - F_{\beta_0},
\]
where \( F_\beta, F_{\beta_0} \) are given in (3.3), and
\[
0 < \int_{\mathbb{R}^2} \frac{4e^{\tau_\beta} K_\beta e^{v_\beta_0}}{e^{v_1 - v_2} + e^{v_1} K_\beta e^{v_\beta_0}} dx
\]
\[
\leq 4e^{\tau_\beta} e^{\|v_{\beta_0}\|_{L^{\infty}(\mathbb{R}^2)}} \int_{\mathbb{R}^2} K_\beta e^{-v_1} dx
\]
\[
\leq 4e^{\tau_\beta} e^{\|v_{\beta_0}\|_{L^{\infty}(\mathbb{R}^2)}} \left( \int_{B_{\rho_0}(0)} e^{-v_1} dx \right)
\]
\[
\leq 4e^{\tau_\beta} e^{\|v_{\beta_0}\|_{L^{\infty}(\mathbb{R}^2)}} \left( \int_{B_{\rho_0}(0)} e^{-v_1} dx + \frac{2\pi}{\beta - 2} \right)
\]
\[
\leq 2\pi(\beta_0 - 2).
\]

So we have that for \( \beta \in (2, \beta_0) \),
\[
-2\pi[2(N - M) - \beta] < \int_{\mathbb{R}^2} F_\beta \, dx \leq 2\pi(\beta_0 - 2) - 2\pi[2(N - M) - \beta] < 0.
\]

Thus,
\[
t_\beta = \frac{1}{2\pi} \int_{\mathbb{R}^2} F_\beta \, dx \in (2 - 2(N - M), 0).
\]

**Claim 2:** Let \( \tau_\beta \) be chosen by (3.12), then there exists \( \nu_3 > 0 \) such that for any \( \beta \in (2, \beta_0) \),
\[
\Gamma \ast \tilde{F}_\beta(x) \geq -\nu_3, \quad \forall x \in \mathbb{R}^2.
\]

The proof of **Claim 2** is postponed and we continue to prove Proposition 3.2. Reset
\[
\underline{w}_3 = w_\beta - \nu_3 - \nu_1 + \Gamma \ast (\tilde{F}_\beta + F_{\beta_0}).
\]
Note that $-\nu_3 - \nu_1 + \Gamma \ast (\tilde{F}_\beta + F_{\beta_0}) \leq 0$ and then by (3.14),

$$
-\Delta u_\beta + \frac{4K_\beta e^{2\nu_3}}{e^{v_1-v_2} + K_\beta e^{v_3}} - g_\beta \\
\leq -\Delta u_\beta + \tilde{F}_\beta + F_{\beta_0} + \frac{4K_\beta e^{v_3}}{e^{v_1-v_2} + K_\beta e^{v_3}} - g_\beta \\
\leq t_\beta (-\Delta v_3) \leq 0.
$$

Then $u_\beta$ is a subsolution of (1.3) for $\beta \in (2, \beta_0)$. From Lemma 2.2 we have that

$$
\lim_{|x| \to +\infty} \Gamma \ast (\tilde{F}_\beta + F_{\beta_0})(x) = 0
$$

then we have that

$$
\lim_{|x| \to +\infty} u_\beta(x) = b_{\beta_0} + \ln(\beta - 2) + \ln(2\pi(\beta_0 - 1)/d_3) - \nu_3 - \nu_1. \quad (3.16)
$$

**Construction of supersolution.** Reset $\tau_\beta$ in (3.13) by

$$
\tau_\beta = (\ln(\beta - 2) - \ln d_4)_-
$$

with $d_4 = \frac{e^{-2(N-M)c_1}}{N-M-1} \cdot \frac{e^{b_{\beta_0}(2r_0)^2}}{1+2(\beta_0-2)e^{2\beta_0}}$.

Since $\lim_{|x| \to +\infty} v_{\beta_0}(x) = b_{\beta_0}$, there exists $R_0 > 2r_0 > 1$ independent of $\beta$ such that there hold,

$$
\frac{1}{2} e^{b_{\beta_0}} \leq e^{\nu_{\beta_0}} \leq 2 e^{b_{\beta_0}}, \quad \forall |x| \geq R_0
$$

and

$$
e^{\tau_\beta} K_\beta e^{\nu_{\beta_0}} \leq 2(\beta_0 - 2) e^{b_{\beta_0}}, \quad \forall |x| \geq 2r_0,
$$

then for $R \geq 2R_0$ large enough,

$$
\int_{B_R(0)} \frac{4e^{\tau_\beta} K_\beta e^{v_{\beta_0}}}{e^{v_1-v_2} + e^{\tau_\beta} K_\beta e^{v_{\beta_0}}} dx > \int_{B_R(0) \setminus B_{2r_0}(0)} \frac{4e^{\tau_\beta} K_\beta e^{v_{\beta_0}}}{1 + e^{\tau_\beta} K_\beta e^{v_{\beta_0}}} dx \\
\geq \frac{1}{1 + 2(\beta_0 - 2)e^{b_{\beta_0}} \int_{B_R(0) \setminus B_{2r_0}(0)} e^{-2(N-M)c_1}|x|^{-\beta} dx} \\
\geq \frac{e^{-2(N-M)c_1}}{d_4} \int_{B_R(0) \setminus B_{2r_0}(0)} e^{-2(N-M)c_1} |x|^{-\beta} dx \\
\geq 4\pi [(N-M) - 1] \left( 1 - \left( \frac{R}{2r_0} \right)^{2-\beta} \right),
$$

where we have used the estimate (2.1). Thus, passing to the limit as $R \to +\infty$, we have that

$$
\int_{R^2} \frac{4e^{\tau_\beta} K_\beta e^{v_{\beta_0}}}{e^{v_1-v_2} + e^{\tau_\beta} K_\beta e^{v_{\beta_0}}} dx \geq 4\pi [(N-M) - 1].
$$

From $\int_{R^2} g_\beta dx = 2\pi [2(N-M) - \beta]$, we have that

$$
t_\beta = \frac{1}{2\pi} \int_{R^2} F_\beta dx \geq \beta - 2 > 0. \quad (3.18)
$$

It is worth noting that the following construction of supersolution is different from the one in Proposition 3.1 since there is no uniform $L^\infty$ bound for $\tilde{F}_\beta$ as $\beta \to 2^+$, see the estimate (3.20) below, although there is a uniform lower bound (3.15).
Let 
\[ s_\beta = \frac{1}{2\pi} \int_{B_{2\sigma}(0)} F_\beta \, dx \] and 
\[ E_\beta = -F_\beta \chi_{B_{2\sigma}(0)} + s_\beta (\Delta v_3) \]
then \( \int_{\mathbb{R}^2} E_\beta \, dx = 0 \). From \textit{Claim 1} and Lemma 2.1, we have that
\[ \| \Gamma \ast F_{\beta_0} \|_{L^\infty(\mathbb{R}^2)} \leq \nu_1 \quad \text{and} \quad \| \Gamma \ast E_\beta \|_{L^\infty(\mathbb{R}^2)} \leq \nu_4, \] (3.19)
where \( \nu_4 > 0 \) is independent of \( \beta \).
Denote 
\[ \bar{v}_\beta = w_\beta + \nu_1 + \nu_4 + \nu_5 + \Gamma \ast (E_\beta + F_{\beta_0}), \]
where \( \nu_5 = \ln(2\pi(\beta_0 - 2)/d_3) + \ln d_4. \)
Note that 
\[ \nu_1 + \nu_4 + \Gamma \ast (E_\beta + F_{\beta_0}) \geq 0, \]
then by (3.18)
\[ -\Delta \tilde{v}_\beta + \frac{4K_\beta e^{\bar{v}_\beta}}{e^{\nu_1-v_2} + K_\beta e^{\bar{v}_\beta}} - g_\beta \geq -\Delta w_\beta + E_\beta + F_{\beta_0} + \frac{4K_\beta e^{v_\beta}}{e^{\nu_1-v_2} + K_\beta e^{v_\beta}} - g_\beta = s_\beta (\Delta v_3) + F_\beta \chi_{\mathbb{R}^2 \setminus B_{2\sigma}(0)} \geq 0. \]
Then \( \tilde{v}_\beta \) is a supersolution of (1.3) for \( \beta \in (2, \beta_0). \) By the definition of \( \nu_5, \) we have that
\[ \lim_{|x| \to +\infty} v_\beta(x) \leq \lim_{|x| \to +\infty} \tilde{v}_\beta(x), \]
then it infers by Remark 2.1 (ii) that
\[ v_\beta \leq \tilde{v}_\beta \quad \text{in} \quad \mathbb{R}^2. \]
From Remark 2.1 (i), we have that for any \( \beta \in (2, \beta_1), \) we have that
\[ v_\beta \leq \bar{v}_\beta \quad \text{in} \quad \mathbb{R}^2 \]
which implies (3.11).

To complete the proof of Proposition 3.2, we are left to prove \textit{Claim 2}. Precisely it is proved directly from the following estimate.
Let \( \tau_\beta \) be chosen by (3.12), then there exists \( \nu_3 > 0 \) such that for \( \beta \in (2, \beta_0), \)
\[ \frac{1}{c_9(\beta - 2)}(1 + |x|)^{-\frac{\beta - 2}{\beta + 1}} - \nu_3 \leq \Gamma \ast \tilde{F}_\beta(x) \leq \frac{c_9}{\beta - 2}(1 + |x|)^{-\frac{\beta - 2}{\beta + 1}} + \nu_3, \quad \forall \, x \in \mathbb{R}^2. \] (3.20)
In fact, since \( g_\beta \) has compact support, then
\[ \| \tilde{F}_\beta \|_{L^\infty(\mathbb{R}^2)} \leq 4 + \| g_\beta \|_{L^\infty(\mathbb{R}^2)} = 4 + \| g_1 \|_{L^\infty(\mathbb{R}^2)} + \| g_2 \|_{L^\infty(\mathbb{R}^2)} + 2(N - M) \| \eta_0 \|_{L^\infty(\mathbb{R}^2)} \]
and for \( x \in \mathbb{R}^2 \setminus B_{\tau_0}(0), \)
\[ \frac{1}{c_{10}} e^{\tau_\beta} |x|^{-\beta} \leq \tilde{F}_\beta(x) \leq c_{10} e^{\tau_\beta} |x|^{-\beta}, \]
where \( c_{10} > 1 \) is independent of \( \beta. \)
Estimates for $|x| \leq 4\epsilon$. Observe that

$$
\int_{\mathbb{R}^2} \ln |x - y| \tilde{F}_\beta(y) \, dy = \left( \int_{B_{\epsilon^0}(0)} + \int_{B_{2}(x) \setminus B_{\epsilon^0}(0)} + \int_{\mathbb{R}^2 \setminus (B_{\epsilon^0}(0) \cup B_{2}(x))} \right) \ln |x - y| \tilde{F}_\beta(y) \, dy,
$$

where

$$
\left| \int_{B_{\epsilon^0}(0)} \ln |x - y| \tilde{F}_\beta(y) \, dy \right| \leq \| \tilde{F}_\beta \|_{L^\infty(\mathbb{R}^2)} \int_{B_{\epsilon^0}(0)} \ln |x - y| \, dy \leq \pi r_0^2 (| \ln r_0 | + 1) \| \tilde{F}_\beta \|_{L^\infty(\mathbb{R}^2)},
$$

$$
\left| \int_{B_{2}(x) \setminus B_{\epsilon^0}(0)} \ln |x - y| \tilde{F}_\beta(y) \, dy \right| = \| \tilde{F}_\beta \|_{L^\infty(\mathbb{R}^2)} \int_{B_{2}(x)} \ln |x - y| \, dy \leq 4(1 + \ln 2) \pi \| \tilde{F}_\beta \|_{L^\infty(\mathbb{R}^2)}
$$

and

$$
0 \leq \int_{\mathbb{R}^2 \setminus (B_{\epsilon^0}(0) \cup B_{2}(x))} \ln |x - y| \tilde{F}_\beta(y) \, dy \leq c_{10} e^{\tau_0} \int_{\mathbb{R}^2 \setminus B_{\epsilon^0}(0)} |y|^{-\beta} \ln |y| \, dy \leq c_{11} (\beta - 2)^{-1},
$$

where $c_{11} > 1$ is independent of $\beta$. So (3.15) holds true for $|x| \leq 4\epsilon$.

Estimates for $|x| > 4\epsilon$. This is very similar to the proof of Lemma 2.2. We rewrite

$$
2 \pi \Gamma \ast \tilde{F}_\beta(x) =: I_1(x) + I_2(x) + I_3(x), \quad \text{for } |x| > 4\epsilon.
$$

We have then

$$
|I_1(x)| \leq \frac{2R}{|x|} \int_{B_{\epsilon^0}(0)} |\tilde{F}_\beta(y)| \, dy \leq 2\pi \frac{R^3}{|x|} \| \tilde{F}_\beta \|_{L^\infty(\mathbb{R}^2)}.
$$

For $z \in B_{1/2}(e_x)$, we have that $0 < \tilde{F}_\beta(|x| z) \leq c_{10} e^{\tau_0} |x|^{-\beta} |z|^{-\beta}$ and then

$$
|I_2(x)| \leq c_{10} e^{\tau_0} |x|^{2-\beta} \int_{B_{1/2}(e_x)} \ln |e_x - z| |z|^{-\beta} \, dz \leq c_{12} |x|^{2-\beta},
$$

where $c_{12} > 0$ is independent of $\beta$.

For $z \in \mathbb{R}^2 \setminus (B_{R/|x|}(0) \cup B_{1/2}(e_x))$, we have that $0 < \tilde{F}_\beta(|x| z) \leq c_{10} e^{\tau_0} |x|^{-\beta} |z|^{-\beta}$ and then

$$
0 \leq I_3(x) \leq 2\pi c_{13} e^{\tau_0} \left( \frac{R^2}{\beta - 2} \ln \frac{R}{|x|} + \frac{2\pi c_{13}}{(\beta - 2)^2} R^{2-\beta} \right) \leq c_{14} (\beta - 2)^{-1} R^{2-\beta},
$$

where $c_{13}, c_{14} > 0$ are independent of $\beta$.

Thus, taking $R = |x|^{\frac{\beta-1}{\beta+1}}$ and $|x| > 4\epsilon$, we have that

$$
2 \pi \Gamma \ast \tilde{F}_\beta(x) \geq -2 \frac{R^3}{|x|} \| \tilde{F}_\beta \|_{L^\infty(\mathbb{R}^2)} - c_{12} |x|^{2-\beta} \geq -c_{15} |x|^{-\frac{\beta-2}{\beta+1}},
$$

and

$$
2 \pi \Gamma \ast \tilde{F}_\beta(x) \leq c_{15} (\beta - 2)^{-1} |x|^{-\frac{\beta-2}{\beta+1}}
$$
where $c_{15} > 0$ is independent of $\beta$. Therefore, (3.20) holds true. □

Proof of Theorem 1.1. From Proposition 2.1, problem (1.3) has a unique solution $w_\beta$ verifies (2.2) and then $w_\beta = b_\beta + \tilde{w}_\beta$, $\tilde{w}_\beta$ verifies (1.8). Proposition 3.1 and Proposition 3.2 show that

$$|b_\beta - \ln(2(N - M) - \beta)| \leq c_7 \text{ for } \beta \in (\beta_0, 2(N - M))$$  \hspace{1cm} (3.21)

and

$$|b_\beta - \ln(\beta - 2)| \leq c_8 \text{ for } \beta \in (2, \beta_0),$$  \hspace{1cm} (3.22)

which imply (1.6) and (1.7) respectively. □

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