IIB PP-Waves with Extra Supersymmetries

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Abstract

We examine Killing spinor equations of the general IIB pp-wave backgrounds, which contain a scalar \( H(x^m, x^-) \) in the metric and a self-dual four-form \( \xi(x^m, x^-) \) in the self-dual five-form flux. Considering non-harmonic extra Killing spinors, we find that if the backgrounds admit at least one extra Killing spinor in addition to 16 standard Killing spinors, backgrounds can be reduced to the form with \( H = A_{mn}(x^-)x^m x^n \) and \( \xi(x^-) \), modulo coordinate transformations. We examine further the cases in which the extra Killing spinors are characterized by a set of Cartan matrices. Solving Killing spinor equations, we find IIB pp-wave backgrounds which admit 18, 20, 24 and 32 Killing spinors.
1 Introduction

After the advent of the maximally supersymmetric IIB pp-wave background [1], pp-wave backgrounds have attracted renewed interests among other supergravity solutions. It was shown [2] that the Green-Schwarz superstring on the maximally supersymmetric IIB pp-wave background is exactly solvable in the light-cone gauge and the full string spectrum has been obtained. Considering the large $N$ limit corresponding to the Penrose limit [3], AdS/CFT correspondence has been examined [4] beyond the supergravity level.

General pp-wave solutions of IIB supergravity admit at least sixteen standard Killing spinors. At a special point in the moduli space, the background turns out to be the maximally supersymmetric pp-wave solution [1] which admits sixteen extra Killing spinors in addition to the sixteen standard Killing spinors, and thus maximal thirty-two supersymmetries. This background is a Penrose limit of the AdS$_5 \times S^5$ background [5], and the corresponding super-isometry algebras are related [6] by an Inönu-Wigner (IW) contraction. In addition to the cases with sixteen and thirty-two supersymmetries, it has been shown that there exist pp-wave backgrounds which admit 20, 24 and 28 Killing spinors [7, 8, 9, 10, 11, 12].

For eleven-dimensional supergravity, the maximally supersymmetric pp-wave solution called the Kowalski-Glikman (KG) solution was obtained in [13, 14]. This background is a Penrose limit of the AdS$_{4/7} \times S^{7/4}$ backgrounds [5], and the corresponding super-isometry algebras [15] are related [16] by IW contractions. It has been shown that there exist non-maximally supersymmetric solutions with 18, 20, 22, 24, 26 supersymmetries [7, 17, 8, 18, 19]. For the type-IIA supergravity theory, the maximally supersymmetric pp-wave solution does not exist [20]. The non-maximally cases were found in [17, 8, 19, 21, 22]. For the lower dimensions, the maximally supersymmetric pp-wave solutions were found in [23] for five- and six-dimensions, and in [24] for four-dimensions.

In [25], a uniqueness of eleven-dimensional pp-wave solutions with extra supersymmetries was discussed. Examining Killing spinor equations of the general eleven-dimensional pp-wave backgrounds, which contain a scalar $H(x^m, x^-)$ in the metric and a three-form $\xi(x^m, x^-)$ in the flux, it was shown that if the backgrounds admit at least one non-harmonic extra Killing spinor characterized by mutually commuting projectors in addition to the standard 16 Killing spinors, backgrounds can be reduced to the form with $H = A_{mn}(x^-) x^m x^n$ and $\xi(x^-)$, modulo coordinate transformations. One of main purpose of this paper is to prove the similar uniqueness theorem for IIB pp-waves.

We examine Killing spinor equations of the IIB pp-wave background with a self-dual Ramond-Ramond (R-R) five-form flux,

$$ds^2 = 2dx^+ dx^- + H(x^m, x^-)(dx^-)^2 + (dx^m)^2,$$
\[ F = dx^- \wedge \xi(x^m, x^-), \quad \xi = *_8 \xi \] (1.1)

where \( \xi \) is a self-dual four-form on the transverse \( \mathbb{E}^8 \) spanned by \( x^m \). We show that the IIB pp-wave background (1.1) is highly restricted if there is at least one non-harmonic extra Killing spinor, so that \( H = A_{mn}(x^-)x^m x^n \) and \( \xi(x^-) \), modulo coordinate transformations.

We further examine Killing spinor equations and consider two types of \( \xi \). One is related to the Kähler form of a Calabi-Yau four-fold with \( SU(4) \) holonomy, and the other is the self-dual Cayley four-form of \( d = 8 \) Riemannian manifold with \( Spin(7) \) holonomy. The former case can combine Neveu-Schwarz-Neveu-Schwarz (NS-NS) and R-R three-forms and has been examined well. Examining the latter case, we find IIB pp-wave solutions which have not been given in the literature yet, as long as we know. Our solutions admit 18, 20, 24 and 32 supersymmetries. The pp-wave solution with 32 supersymmetries is shown to be related to the solution given in [1] by a coordinate transformation.

This paper is organized as follows. In the next section, Killing spinor equations for the background (1.1) is derived. In section 3, we prove a uniqueness theorem which states that \( H(x^-, x^m) \) and \( \xi(x^-, x^m) \) can be reduced to \( A_{mn}(x^-)x^m x^n \) and \( \xi(x^-) \), respectively, modulo coordinate transformations, provided that the background admits at least one non-harmonic extra Killing spinor in addition to the standard sixteen spinors. We further examine Killing spinor equations for the cases in which extra Killing spinors are charac-
terized by mutually commuting projectors in section 4. IIB pp-wave solutions with extra supersymmetries are given in section 5. The last section is devoted to a summary and discussions.

### 2 Killing spinor equations

The general IIB pp-wave background we consider is

\[ ds^2 = 2dx^+ dx^- + H(x^m, x^-)(dx^-)^2 + (dx^m)^2, \] (2.1)

\[ F = dx^- \wedge \xi(x^m, x^-), \quad \xi = *_8 \xi \] (2.2)

where both a scalar \( H \) and a self-dual four-form \( \xi \) on \( \mathbb{E}^8 \) spanned by \( x^m \), are functions of \( x^- \) and \( x^m \). This is a supergravity solution when

\[ \triangle H = - \frac{32}{4!} \xi_{klmn} \xi^{klmn}, \] (2.3)

where \( \triangle \) is the Laplacian on \( \mathbb{E}^8 \). The frame one-forms defined by \( ds^2 = 2e^+ e^- + e^m e^m \) are

\[ e^- = dx^-, \quad e^+ = dx^+ + \frac{1}{2} H(x^m, x^-)dx^- \quad e^m = dx^m \] (2.4)

and thus the spin connection is

\[ w^+_m = \frac{1}{2} \partial_m H dx^- \] (2.5)
Killing spinor equations for general IIB backgrounds with a R-R five-form field strength \( F_{M_1\cdots M_5} \)

\[
\mathcal{D}_M \varepsilon = (\nabla_M - \Omega_M) \varepsilon = 0,
\]

\[
\nabla_M = \partial_M + \frac{i}{4} w^a_M \Gamma_{ab}, \quad \Omega_M = -\frac{i}{24} F_{M L_1\cdots L_4} \Gamma^{L_1\cdots L_4},
\]

(2.6)

reduce on this pp-wave background to

\[
\partial_+\varepsilon = 0, \quad \partial_-\varepsilon - \frac{1}{4} \partial_m H \Gamma^m \Gamma_+ \varepsilon = \Omega_- \varepsilon, \quad \partial_m \varepsilon = \Omega_m \varepsilon,
\]

(2.7)

where

\[
\Omega_m = -i \bar{\varepsilon}_m \Gamma_+, \quad \Omega_- = -i \bar{\varepsilon}, \quad \bar{\varepsilon} = \frac{1}{4!} \xi_{pqrs} \Gamma^{pqrs}, \quad \bar{\varepsilon}_m = \frac{1}{3!} \xi_{mnpq} \Gamma^{pq}.
\]

(2.8)

It is convenient to introduce the eight-dimensional gamma matrices \( \gamma^m \in \text{Spin}(8) \)

\[
\Gamma_0 = \mathbb{I}_{16} \otimes i \sigma_2, \quad \Gamma_9 = \mathbb{I}_{16} \otimes \sigma_1, \quad \Gamma_9 = \gamma_m \otimes \sigma_3.
\]

(2.9)

Defining the light-cone projection operator as

\[
P_{\pm} = \frac{1}{2} \Gamma_{\pm} \Gamma_\mp, \quad \Gamma_\pm = \frac{1}{\sqrt{2}} (\Gamma_9 \pm \Gamma_0),
\]

(2.10)

the complex Weyl spinor \( \varepsilon \) decomposes into

\[
\varepsilon = \begin{pmatrix} \varepsilon_+ \\ \varepsilon_- \end{pmatrix}, \quad P_+ \varepsilon = \begin{pmatrix} \varepsilon_+ \\ 0 \end{pmatrix}, \quad P_- \varepsilon = \begin{pmatrix} 0 \\ \varepsilon_- \end{pmatrix}.
\]

(2.11)

Defining the chirality projection operator as

\[
h_+ = \frac{1}{2} (1 + \Gamma_{11}), \quad \Gamma_{11} = \Gamma_{012\cdots 9} = \gamma_{12\cdots 8} \otimes \sigma_3,
\]

(2.12)

the positive chirality condition, \( h_+ \varepsilon = \varepsilon \), implies that

\[
\gamma_{12\cdots 8} \varepsilon_\pm = \pm \varepsilon_\pm,
\]

(2.13)

which halves the 32 complex components of \( \varepsilon \) into 16 complex components. \( \varepsilon_+ \) is called the standard Killing spinor which exists on the general pp-wave backgrounds, while \( \varepsilon_- \) is the extra Killing spinor. Defining \( \xi \) and \( \xi_m \) as

\[
\xi = \frac{1}{4!} \xi_{lmnp} \gamma^{lmnp}, \quad \xi_m = \frac{1}{3!} \xi_{mnpq} \gamma^{npq}, \quad \bar{\varepsilon} = \xi \otimes \mathbb{I}, \quad \bar{\varepsilon}_m = \xi_m \otimes \sigma_3,
\]

(2.14)

the Killing spinor equations are expressed as

\[
\partial_+ \varepsilon_+ = 0,
\]

(2.15)

\[
\partial_- \varepsilon_+ - \frac{\sqrt{2}}{4} \partial_m H \gamma_m \varepsilon_- = -i \xi \varepsilon_+,
\]

(2.16)

\[
\partial_m \varepsilon_+ = -i \sqrt{2} \xi_m \varepsilon_-,
\]

(2.17)

\[
\partial_+ \varepsilon_- = 0,
\]

(2.18)

\[
\partial_- \varepsilon_- = 0,
\]

(2.19)

\[
\partial_m \varepsilon_- = 0,
\]

(2.20)
where we have used the fact \( \xi \varepsilon_- = 0 \), which follows from the self-duality property of \( \xi \), (2.2), and the positive chirality property of \( \varepsilon_- \), (2.13).

### 3 Uniqueness

In this section, we examine Killing spinor equations (2.15)–(2.20) and derive conditions on \( H \) and \( \xi \), providing that there exists at least one extra Killing spinor.

It follows from eqns. (2.18), (2.19) and (2.20) that \( \varepsilon_- \) is a constant spinor. Eqn. (2.15) implies that \( \varepsilon_+ \) must be independent of \( x^+ \). Acting \( \gamma^m \) on (2.17), one finds

\[
\gamma^m \partial_m \varepsilon_+ = 0 \quad (3.1)
\]

because \( \gamma^m \xi_m \varepsilon_- = 4 \xi \varepsilon_- = 0 \). Further acting \( \gamma^n \partial_n \) on this equation, we find that \( \varepsilon_+ \) obeys the Laplace equation

\[
\partial_m \partial^m \varepsilon_+ = 0. \quad (3.2)
\]

It follows that \( \varepsilon_+ \) is linear in \( x^m \) at most, up to a harmonic function. In this paper, we concentrate on the non-harmonic function part.\(^1\) We can thus write \( \varepsilon_+ \) as

\[
\varepsilon_+ = \varepsilon_0(x^-) + \varepsilon_m(x^-)x^m \quad (3.3)
\]

where \( \varepsilon_0 \) and \( \varepsilon_m \) are functions of \( x^- \) only and eqn. (2.17) becomes

\[
\varepsilon_m = -i \sqrt{2} \xi_m \varepsilon_- \quad (3.4)
\]

Because one can show that

\[
\xi_m \varepsilon_- = -\frac{1}{2} \xi \gamma_m \varepsilon_- \quad (3.5)
\]

this equation becomes

\[
\varepsilon_m = i \sqrt{2} \frac{1}{2} \xi \gamma_m \varepsilon_- \quad (3.6)
\]

Since \( \varepsilon_m \) depends only on \( x^- \) and \( \varepsilon_- \) is a constant spinor, \( \xi \gamma_m \mathbb{P} \) must depend only on \( x^- \) for all \( m \), where \( \mathbb{P} \) is a projection operator to non-trivial extra Killing spinors defined by

\[
\mathbb{P} \varepsilon_- = \varepsilon_- , \quad \overline{\mathbb{P}} \varepsilon_- = 0 , \quad \mathbb{P} + \overline{\mathbb{P}} = \mathbb{I} . \quad (3.7)
\]

Because \( \mathbb{P} \) is constructed in terms of gamma matrices as will be seen in the next section, the term \( \xi \gamma_m \mathbb{P} \) is equal either to \( \xi \mathbb{P} \gamma_m \) for a certain set of \( m \) or \( \xi \overline{\mathbb{P}} \gamma_m \) for the rest of \( m \).\(^2\)

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\(^1\)PP-wave backgrounds which admit harmonic Killing spinors were extensively discussed in [26].
The former (latter) case implies that $\xi(P_{\xi})$ depends only on $x^-$, and thus $\xi$ depends only on $x^-$. It follows from (2.16) that $\partial_m H$ must be linear in $x^m$ at most, and thus $H$ can be written as $H = f(x^-) + g_m(x^-)x^m + A_{mn}(x^-)x^m x^n$, where $f$, $g_m$ and $A_{mn}$ are functions of $x^-$. As was done in [25], one can show that $f(x^-)$ and $g_m(x^-)$ can be absorbed by a redefinition

$$x^+ = y^+ - F(x^-) - G_m(x^-)y^m, \quad x^m = y^m - H^m(x^-),$$  \hspace{1cm} (3.8)

where $F$, $G$ and $H$ satisfy

$$-\partial F + \frac{1}{2} f - g_m H^m + \frac{1}{2} A_{mn} H^m H^n + \frac{1}{2} (\partial H^m)^2 = 0,$$  \hspace{1cm} (3.9)

$$-\partial G_m + \frac{1}{2} g_m - A_{mn} H^n = 0,$$  \hspace{1cm} (3.10)

$$-G_m - \partial H_m = 0,$$  \hspace{1cm} (3.11)

and the line element reduces to

$$ds^2 = 2dy^+dx^- + A_{mn}(x^-)y^m y^n (dx^-)^2 + (dy^m)^2.$$  \hspace{1cm} (3.12)

The transformation (3.8) does not affect $F = dx^- \wedge \xi(x^-)$. In summary, we have shown that IIB pp-wave backgrounds which admit extra Killing spinors must be of the form

$$ds^2 = 2dx^+dx^- + A_{mn}(x^-)x^m x^n (dx^-)^2 + (dx^m)^2, \quad F = dx^- \wedge \xi(x^-),$$  \hspace{1cm} (3.13)

modulo coordinate transformations.\(^2\)

$A_{mn}(x^-)$ and $\xi(x^-)$ are restricted by (2.16). On the background (3.13), eqn. (2.16) reduces to

$$\partial_- \varepsilon_m - \frac{\sqrt{2}}{2} A_{mn} \gamma^n \varepsilon_- = -i \xi(\gamma)_{\varepsilon_m},$$  \hspace{1cm} (3.14)

which becomes, substituting (3.14) into this equation,

$$[i \partial_- \xi(\gamma)_{(m)} - \xi^2(\gamma)_{(m)} + \gamma_{(m)} - A_m] \varepsilon_- \equiv D_{(m)} \varepsilon_- = 0,$$  \hspace{1cm} (3.15)

where

$$\gamma^m \xi_{(m)} \equiv \xi^m, \quad \gamma_{(m)} \equiv - \sum_{n=1}^8 A_{mn} \gamma^{mn}, \quad A_m \equiv A_{mm}.$$  \hspace{1cm} (3.16)

We examine this condition in the next section for the case in which $D_{(m)}$ is expanded solely in terms of mutually commuting projectors.

\(^2\)It should be noted again that we are considering non-harmonic extra Killing spinors.
4 $D_{(m)}$ expanded in mutually commuting projectors

For non-maximally supersymmetric backgrounds, $D_{(m)}$ must be a linear combination of projection operators. We restrict the study to the case in which $D_{(m)}$ is expanded solely in terms of mutually commuting projectors. These projection operators are composed of a set of Cartan matrices, $H_I$, as $P_I = \frac{1}{2}(\mathbb{I} + H_I)$. Among infinitely many Cartan matrices, we consider Cartan matrices which are monomials of gamma-matrices, such as $H_I = \gamma^{[N]}$ where $\gamma^{[N]}$ is an $N$-th antisymmetrized product of gamma matrices. One finds that mutually commuting matrices $\gamma^{[N]}$ must share a definite number of indices. We indicate the number of common indices shared among two of matrices below.

| $\gamma^{m_1\ldots m_{2i}}$ | $\gamma^{n_1\ldots n_{2j}}$ | $\gamma^{n_1\ldots n_{2j+1}}$ |
|----------------------------|-----------------|-----------------|
| 0, 2, ..., $\min(2i, 2j)$   | 0, 2, ..., $\min(2i, 2j + 1)$ | 1, 3, ..., $\min(2i + 1, 2j + 1)$ |

(4.1)

The projection operators must commute with the chirality projection operator, so that matrices must commute with $\gamma^{12\ldots 8}$. We find that there are two sets of mutually commuting matrices; one is

$\gamma^{12}$, $\gamma^{34}$, $\gamma^{56}$, $\gamma^{78}$

$\gamma^{1234}$, $\gamma^{3456}$, $\gamma^{5678}$, $\gamma^{1256}$, $\gamma^{1278}$, $\gamma^{3478}$

$\gamma^{123456}$, $\gamma^{123478}$, $\gamma^{125678}$, $\gamma^{345678}$

$\gamma^{12345678}$

(4.2)

and the other is

$\gamma^{1238}$, $\gamma^{1458}$, $\gamma^{1678}$, $\gamma^{2346}$, $\gamma^{2578}$, $\gamma^{3478}$, $\gamma^{3568}$

$\gamma^{4567}$, $\gamma^{2367}$, $\gamma^{2345}$, $\gamma^{1357}$, $\gamma^{1346}$, $\gamma^{1256}$, $\gamma^{1247}$

$\gamma^{12345678}$

(4.3)

The first set (4.2) can be related to the Kähler form $J$ of a Calabi-Yau four-fold with $SU(4)$ holonomy. The Kähler form $J$ is covariantly constant $dJ = 0$. The terms in the first line of (4.2) are the constituents of $J$, and those in the second, third and fourth lines are the constituents of $J \wedge J$, $J \wedge J \wedge J$ and $J \wedge J \wedge J \wedge J$, respectively. On the other hand, the second set (4.3) can be related to the self-dual Cayley four-form $\Psi$ of $d = 8$ Riemannian manifold with $Spin(7)$ holonomy, which is covariantly constant $d\Psi = 0$. The terms in the first and second lines of (4.3) are the constituents of $\Psi$, and those in the third line is the constituent of $\Psi \wedge \Psi$. The relation between covariantly constant forms

$^3$More general cases are studied in [27]. We thank the authors for explanation of their work.
and special holonomy groups is well known \cite{28}. We find that there are three other possibilities to construct the self-dual four-form from invariant forms, which are shown to reduce to the self-dual Cayley four-form $\Psi$. One is to use the associative three-form $\phi$ of $d = 7$ Riemannian manifold with $G_2$ holonomy and to make a wedge product of $\phi$ and a one-form, say $e^8$. The self-dual four-form is obtained by adding eight-dimensional Hodge dual, $*e$, of $\phi \wedge e^8$ as $\phi \wedge e^8 + *e(\phi \wedge e^8) = \phi \wedge e^8 + *_7 \phi$. This is nothing but the self-dual Cayley four-form $\Psi$. Another is to use the seven-dimensional Hodge dual, $*_7$, of $\phi$. Again, the self-dual four-form turns out to be the self-dual Cayley four-form $\Psi$, because $*_7 \phi + *e(*_7 \phi) = *_7 \phi + \phi \wedge e^8$. The third is to use the holomorphic $(4,0)$-form $\Omega$ of a Calabi-Yau four-fold with $SU(4)$ holonomy. A self-dual four-form can be constructed as the real or imaginary part of $\Omega$, which turns out to be a part of the self-dual Cayley four-form $\Psi$.

Now $D_{(m)}$ is a linear combination of projection operators, and thus must be constructed from these matrices. Noting that $\xi = *e\xi$, $\xi$ turns out to be of the form

$$\xi = a_1(\gamma^{1234} + \gamma^{5678}) + a_2(\gamma^{3456} + \gamma^{1278}) + a_3(\gamma^{1256} + \gamma^{3478})$$ (4.4)

for the former set (4.2), while

$$\xi = b_1(\gamma^{4567} + \gamma^{1238}) + b_2(\gamma^{2367} + \gamma^{1458}) + b_3(\gamma^{2345} + \gamma^{1678}) + b_4(\gamma^{1357} + \gamma^{2468}) + b_5(\gamma^{1346} + \gamma^{2578}) + b_6(\gamma^{1256} + \gamma^{3478}) + b_7(\gamma^{1247} + \gamma^{3568})$$ (4.5)

for the latter set (4.3). $A_{mn}$ is restricted to be non-trivial only when $m = n$ and $(m, n) = (1, 2), (3, 4), (5, 6), (7, 8)$ for (4.2), while only when $m = n$ for (4.3). In the former case, $\xi_{(m)}$ for (4.3) and $\eta_{(m)}$ are expressed as

$$\xi_{(m)} = a_1^m(\gamma^{1234} - \gamma^{5678}) + a_2^m(\gamma^{3456} - \gamma^{1278}) + a_3^m(\gamma^{1256} - \gamma^{3478}),$$

$$\eta_{(m)} = \mu_1^m \gamma^{12} + \mu_2^m \gamma^{34} + \mu_3^m \gamma^{56} + \mu_4^m \gamma^{78},$$

where

$$\alpha_1 = (-a_1, -a_1, -a_1, -a_1, +a_1, +a_1, +a_1, +a_1),$$

$$\alpha_2 = (+a_2, +a_2, -a_2, -a_2, -a_2, +a_2, +a_2),$$

$$\alpha_3 = (-a_3, -a_3, +a_3, +a_3, -a_3, -a_3, +a_3, +a_3),$$

$$\mu_i^{2i-1} = -\mu_i^i = A_{2i-1, 2i}, \quad i = 1, 2, 3, 4.$$ (4.7)

In the latter case, $\xi_{(m)}$ for (4.3) and $\eta_{(m)}$ are expressed as

$$\xi_{(m)} = \beta_1^m(\gamma^{4567} - \gamma^{1238}) + \beta_2^m(\gamma^{2367} - \gamma^{1458}) + \beta_3^m(\gamma^{2345} - \gamma^{1678}) + \beta_4^m(\gamma^{1357} - \gamma^{2468}) + \beta_5^m(\gamma^{1346} - \gamma^{2578}) + \beta_6^m(\gamma^{1256} - \gamma^{3478}) + \beta_7^m(\gamma^{1247} - \gamma^{3568})$$

$$\eta_{(m)} = 0$$ (4.8)
where

\[ \begin{align*}
\beta_1 &= (+b_1, +b_1, +b_1, -b_1, -b_1, -b_1, -b_1, +b_1), \\
\beta_2 &= (+b_2, -b_2, -b_2, +b_2, +b_2, -b_2, -b_2, +b_2), \\
\beta_3 &= (+b_3, -b_3, -b_3, -b_3, +b_3, +b_3, +b_3, +b_3), \\
\beta_4 &= (-b_4, +b_4, -b_4, +b_4, -b_4, +b_4, -b_4, -b_4), \\
\beta_5 &= (-b_5, +b_5, -b_5, +b_5, -b_5, +b_5, +b_5, +b_5), \\
\beta_6 &= (-b_6, -b_6, +b_6, +b_6, -b_6, -b_6, +b_6, +b_6), \\
\beta_7 &= (-b_7, -b_7, +b_7, +b_7, -b_7, -b_7, +b_7, +b_7). 
\end{align*} \] (4.9)

In the following two subsections, we examine (3.13) for these two cases in turn.

### 4.1 the case (4.16)

In this case, all the matrices (4.12) can be constructed as products of four matrices, \( \gamma^{12}, \gamma^{34}, \gamma^{56} \) and \( \gamma^{78} \). From these matrices, we make four rank-8 projection operators

\[ \begin{align*}
P_1 &= \frac{1}{2}(\mathbb{I} + i\gamma^{12}), \\
P_2 &= \frac{1}{2}(\mathbb{I} + i\gamma^{34}), \\
P_3 &= \frac{1}{2}(\mathbb{I} + i\gamma^{56}), \\
P_4 &= \frac{1}{2}(\mathbb{I} + i\gamma^{78}), 
\end{align*} \] (4.10)

which satisfy

\[ P_A^2 = P_A, \quad P_A P_B = P_B P_A, \quad A, B = 1, 2, 3, 4. \] (4.11)

We rewrite (3.14) in terms of these projection operators. Noting \( \gamma^{12,8} \xi_- = -\xi_- \), one finds that

\[ \begin{align*}
\xi(m)\xi_- &= (2\alpha_1^m\gamma^{1234} + 2\alpha_2^m\gamma^{3456} + 2\alpha_3^m\gamma^{1256})\xi_-,
\eta(m)\xi_- &= (\mu_1^m\gamma^{12} + \mu_2^m\gamma^{34} + \mu_3^m\gamma^{56} + \mu_4^m\gamma^{123456})\xi_-.
\end{align*} \] (4.12)

Because

\[ \begin{align*}
\gamma^{1234} &= -(2P_1 - \mathbb{I})(2P_2 - \mathbb{I}), \quad \gamma^{3456} = -(2P_2 - \mathbb{I})(2P_3 - \mathbb{I}), \\
\gamma^{1256} &= -(2P_1 - \mathbb{I})(2P_3 - \mathbb{I}), \quad \gamma^{12} = -i(2P_1 - \mathbb{I}), \quad \gamma^{34} = -i(2P_2 - \mathbb{I}), \\
\gamma^{56} &= -i(2P_3 - \mathbb{I}), \quad \gamma^{123456} = i(2P_1 - \mathbb{I})(2P_2 - \mathbb{I})(2P_3 - \mathbb{I}),
\end{align*} \] (4.13)

\( \xi(m)\xi_- \) and \( \eta(m)\xi_- \) in (4.12) can be rewritten as

\[ \begin{align*}
\xi(m)\xi_- &= \left[ -2(\alpha_1^m + \alpha_2^m + \alpha_3^m) + 4(\alpha_1^m + \alpha_3^m)P_1 + 4(\alpha_2^m + \alpha_3^m)P_2 + 4(\alpha_2^m + \alpha_3^m)P_3 \\
&\quad - 8\alpha_1^mP_1P_2 - 8\alpha_3^mP_1P_3 - 8\alpha_2^mP_2P_3 \right] \xi_-,
\eta(m)\xi_- &= \left[ i(\mu_1^m + \mu_2^m + \mu_3^m - \mu_4^m)\mathbb{I} - 2i(\mu_1^m - \mu_4^m)P_1 - 2i(\mu_2^m - \mu_4^m)P_2 \\
&\quad - 2i(\mu_3^m - \mu_4^m)P_3 - 4i\mu_4^mP_1P_2 - 4i\mu_4^mP_1P_3 - 4i\mu_4^mP_2P_3 + 8i\mu_4^mP_1P_2P_3 \right] \xi_-.
\] (4.14)
Substituting these into (3.15) yields

\[
\begin{align*}
&\left[ -2i\partial_- (\alpha_1^m + \alpha_2^m + \alpha_3^m)^2 + i(\mu_1^m + \mu_2^m + \mu_3^m - \mu_4^m) - 4(\alpha_1^m + \alpha_2^m + \alpha_3^m)^2 - A_m \right] I \\
&+ \left( 4i\partial_- (\alpha_1^m + \alpha_3^m) - 2i(\mu_1^m - \mu_3^m) + 16\alpha_1^m(\alpha_2^m + \alpha_3^m) \right) P_1 \\
&+ \left( 4i\partial_- (\alpha_1^m + \alpha_2^m) - 2i(\mu_2^m - \mu_4^m) + 16\alpha_2^m(\alpha_1^m + \alpha_2^m) \right) P_2 \\
&+ \left( 4i\partial_- (\alpha_2^m + \alpha_3^m) - 2i(\mu_3^m - \mu_4^m) + 16\alpha_3^m(\alpha_2^m + \alpha_3^m) \right) P_3 \\
&+ \left( -8i\partial_- \alpha_1^m - 4i\mu_4^m - 32\alpha_2^m\alpha_3^m \right) P_1 P_2 + \left( -8i\partial_- \alpha_2^m - 4i\mu_4^m - 32\alpha_1^m\alpha_3^m \right) P_2 P_3 \\
&+ \left( -8i\partial_- \alpha_3^m - 4i\mu_4^m - 32\alpha_1^m\alpha_2^m \right) P_1 P_3 + \left( 8i\mu_4^m \right) P_1 P_2 P_3 \right] \varepsilon_- = 0. \quad (4.15)
\end{align*}
\]

In order to see which Killing spinor survives, it is convenient to introduce rank-1 projection operators of a 16-component spinor onto the $I$-th component:

\[
P_I = \text{diag}(0, \ldots, 0, 1, 0, \ldots, 0).
\]

(4.16)

The rank-8 projection operators $P_A$ can then be expressed in terms of these rank-1 projection operators as

\[
P_1 = \sum_{I=1,2,\ldots,8} P_I, \quad P_2 = \sum_{I=1,2,3,4,9,10,11,12} P_I, \\
P_3 = \sum_{I=1,2,5,6,9,10,13,14} P_I, \quad P_4 = \sum_{I=1,3,5,7,9,11,13,15} P_I.
\]

(4.17)

The chirality condition, $\gamma^{12\ldots8}\varepsilon_- = -\varepsilon_-$, reduces to

\[
\frac{1}{2}(\gamma^{12\ldots8} + I)\varepsilon_- = \frac{1}{2}\left( (2P_1 - I)(2P_2 - I)(2P_3 - I)(2P_4 - I) + I \right) \\
= \sum_{I=1,4,6,7,10,11,13,16} P_I \varepsilon_- = 0
\]

(4.18)

and thus non-trivial spinors are $P_I \varepsilon_-$ with $I = 2, 3, 5, 8, 9, 12, 14$ and 15. Taking this into account, (4.16) becomes

\[
\begin{align*}
&\left[ -2i\partial_- (\alpha_1^m + \alpha_2^m + \alpha_3^m)^2 + i(\mu_1^m + \mu_2^m + \mu_3^m - \mu_4^m) - 4(\alpha_1^m + \alpha_2^m + \alpha_3^m)^2 - A_m \right] I \\
&+ \left( -2i(\mu_1^m + \mu_2^m + \mu_3^m - \mu_4^m) \right) \varepsilon_- = 0. \quad (4.19)
\end{align*}
\]
It follows that the coefficient of $I$ in this equation must vanish in order to get a complex extra Killing spinor, $P_{15}\varepsilon$.. Because $\alpha_i^m$, $\mu_i^m$ and $A_m$ are real, this implies two equations

\begin{align}
-2\partial_-(\alpha_1^m + \alpha_2^m + \alpha_3^m) + (\mu_1^m + \mu_2^m + \mu_3^m - \mu_4^m) = 0, & \quad (4.20) \\
A_m = -4(\alpha_1^m + \alpha_2^m + \alpha_3^m)^2. & \quad (4.21)
\end{align}

Without loss of generality, we can take these equations as the conditions for the existence of a complex extra Killing spinor, because the condition for another spinor to be a complex extra Killing spinor is simply obtained by changing signs of $\alpha_i^m$ and $\mu_i^m$. The first equation (4.20) leads to

\begin{align}
-2\partial_-(a_1^{2n-1} + a_2^{2n-1} + a_3^{2n-1}) + \mu_n^{2n-1} = 0, & \quad n = 1, 2, 3, 4, \quad (4.22) \\
-2\partial_-(a_1^{2n} + a_2^{2n} + a_3^{2n}) + \mu_n^{2n} = 0,
\end{align}

from which we find that $\mu_i^m = 0$, so that $A_{12} = A_{34} = A_{56} = A_{78} = 0$, and

$$\partial_-(\alpha_1^m + \alpha_2^m + \alpha_3^m) = 0,$$  

(4.23)

because, from (4.7), $a_i^{2n} = a_i^{2n-1}$ while $\mu_n^{2n} = -\mu_n^{2n-1}$. Consequently $A_m$ must be independent of $x^-$, because the right hand side of (4.21) is independent of $x^-$ from eq. (4.23).

Noting that $\alpha_i^m$ is related to $a_i$ in (4.7), we find that eq. (4.23) leads to four differential equations for $a_i$,

\begin{align}
\partial_-(a_1 - a_2 - a_3) = 0, & \quad \partial_-(a_1 + a_2 - a_3) = 0, \\
\partial_-(a_1 - a_2 + a_3) = 0, & \quad \partial_-(a_1 + a_2 + a_3) = 0, \quad (4.24)
\end{align}

which imply

$$\partial_-a_1 = \partial_-a_2 = \partial_-a_3 = 0.$$  \hspace{0.2cm} (4.25)

This means that $\xi$ is independent of $x^-$. The extra Killing spinors are determined as a non-trivial solution of

$$\left[ -4(\alpha_1^m + \alpha_2^m + \alpha_3^m)^2 - A_m \right] \mathbb{I} + 16\alpha_1^m(\alpha_2^m + \alpha_3^m) (P_3 + P_{14}) + 16\alpha_3^m(\alpha_1^m + \alpha_2^m) (P_5 + P_{12}) + 16\alpha_2^m(\alpha_1^m + \alpha_3^m)P_8 + P_9 \right] \varepsilon = 0. \hspace{0.2cm} (4.26)$$

which reveals the four-fold degeneracy of the extra Killing spinors. If (4.21) is satisfied, $P_{2}\varepsilon$ and $P_{15}\varepsilon$ are a pair of complex extra Killing spinors and the background admits 20 Killing spinors, 16 standard and 4 extra Killing spinors. If, in addition, the coefficient of $(P_I + P_{17-I})$, $I = 3, 5, 8$, vanishes, then $P_I\varepsilon$, and $P_{17-I}\varepsilon$ give a pair of additional complex extra Killing spinors.
The backgrounds with extra supersymmetries automatically satisfy the supergravity
equation of motion (2.3) because (4.21) and (4.4) lead to
\[
\Delta H = \sum_{m=1}^{8} 2A_m = -64(a_1^2 + a_2^2 + a_3^2), \quad (4.27)
\]
\[-\frac{32}{4!}\xi_{mnpq}\xi^{mnpq} = -64(a_1^2 + a_2^2 + a_3^2). \quad (4.28)\]

### 4.2 the case (4.8)

In this case, all the matrices (4.3) can be constructed as products of four matrices, \(\gamma^{2367}\), \(\gamma^{1256}\), \(\gamma^{1247}\) and \(\gamma^{12...8}\). From these matrices, we make four rank-8 projection operators
\[
P_1 = \frac{1}{2}(I + \gamma^{2367}), \quad P_2 = \frac{1}{2}(I + \gamma^{1256}),
\]
\[
P_3 = \frac{1}{2}(I + \gamma^{1247}), \quad P_4 = \frac{1}{2}(I + \gamma^{12...8}). \quad (4.29)
\]
We rewrite (3.15) in terms of these projection operators. Because \(\gamma^{12...8}\varepsilon_- = -\varepsilon_-\), one finds that
\[
\xi_{(m)}\varepsilon_- = \left( 2\beta_1^m\gamma^{4567} + 2\beta_2^m\gamma^{2367} + 2\beta_3^m\gamma^{2345} + 2\beta_4^m\gamma^{1357} + 2\beta_5^m\gamma^{1346} + 2\beta_6^m\gamma^{1256} + 2\beta_7^m\gamma^{1247} \right)\varepsilon_- \quad (4.30)
\]
Because
\[
\gamma^{4567} = -(2P_2 - I)(2P_3 - I), \quad \gamma^{2367} = (2P_1 - I),
\]
\[
\gamma^{2345} = (2P_1 - I)(2P_2 - I)(2P_3 - I), \quad \gamma^{1357} = (2P_1 - I)(2P_2 - I),
\]
\[
\gamma^{1346} = -(2P_1 - I)(2P_3 - I), \quad \gamma^{1256} = (2P_2 - I), \quad \gamma^{1247} = (2P_3 - I) \quad (4.31)
\]

\(\xi_{(m)}\varepsilon_-\) in (4.30) can be rewritten as
\[
\xi_{(m)}\varepsilon_- = \left[ -2(\beta_1^m + \beta_2^m + \beta_3^m - \beta_4^m + \beta_5^m + \beta_6^m + \beta_7^m)I + 4(\beta_2^m + \beta_3^m - \beta_4^m + \beta_5^m)P_1 + 4(\beta_1^m + \beta_3^m - \beta_4^m + \beta_6^m)P_2 + 4(\beta_1^m + \beta_3^m + \beta_5^m + \beta_7^m)P_3 - 8(\beta_3^m - \beta_4^m)P_1P_2 - 8(\beta_3^m + \beta_5^m)P_1P_3 - 8(\beta_1^m + \beta_3^m)(P_2P_3 + 16\beta_5^m) \right]\varepsilon_- \quad (4.32)
\]
Substituting this into (3.15) yields
\[
\left[ -2i\partial_-(\beta_1^m + \beta_2^m + \beta_3^m - \beta_4^m + \beta_5^m + \beta_6^m + \beta_7^m) - 4(\beta_1^m + \beta_2^m + \beta_3^m - \beta_4^m + \beta_5^m + \beta_6^m + \beta_7^m)^2 - A_m \right]I
\]
\[+ \left( 4i\partial_-(\beta_2^m + \beta_3^m - \beta_4^m + \beta_5^m) + 16(\beta_2^m + \beta_3^m - \beta_4^m + \beta_5^m)(\beta_1^m + \beta_6^m + \beta_7^m) \right)P_1
\]
Introducing rank-1 projection operators of a 16-component spinor onto the \( I \)-th component,

\[
P_I = \text{diag}(0, \ldots, 0, 1, 0, \ldots, 0),
\]

the rank-8 projection operators \( P_A \) can be expressed in terms of these rank-1 projection operators as

\[
P_1 = \sum_{I=1,2,\ldots,8} P_I, \quad P_2 = \sum_{I=1,2,3,4,9,10,11,12} P_I, \quad P_3 = \sum_{I=1,2,5,6,9,10,13,14} P_I, \quad P_4 = \sum_{I=1,3,5,7,9,11,13,15} P_I.
\]

Noting that non-trivial spinors are \( P_I \varepsilon_- \) with \( I = 2, 4, 6, 8, 10, 12, 14 \) and 16, because the chirality condition is

\[
\frac{1}{2} (\gamma^{12\ldots8} + \mathbb{1}) \varepsilon_- = P_4 \varepsilon_- = 0,
\]

(4.36) reduces to

\[
\left[ \left( -2i \partial_-(\beta_1^m + \beta_2^m + \beta_3^m - \beta_4^m + \beta_5^m + \beta_6^m + \beta_7^m) \\
- 4(\beta_1^m + \beta_2^m + \beta_3^m - \beta_4^m + \beta_5^m + \beta_6^m + \beta_7^m)^2 - A_m \right) \mathbb{1} \right. \\
+ \left( 4 \partial_-(\beta_2^m + \beta_3^m + \beta_6^m + \beta_7^m) + 16(\beta_2^m + \beta_3^m + \beta_6^m + \beta_7^m)(\beta_1^m - \beta_4^m + \beta_5^m) \right) P_2 \\
+ \left( 4 \partial_-(\beta_1^m + \beta_2^m + \beta_5^m + \beta_6^m) + 16(\beta_1^m + \beta_2^m + \beta_5^m + \beta_6^m)(\beta_3^m - \beta_4^m + \beta_7^m) \right) P_4 \\
+ \left( 4 \partial_-(\beta_1^m + \beta_2^m - \beta_4^m + \beta_7^m) + 16(\beta_1^m + \beta_2^m - \beta_4^m + \beta_7^m)(\beta_3^m + \beta_5^m + \beta_6^m) \right) P_6 \\
+ \left( 4 \partial_-(\beta_2^m + \beta_3^m - \beta_4^m + \beta_5^m) + 16(\beta_2^m + \beta_3^m - \beta_4^m + \beta_5^m)(\beta_1^m + \beta_6^m + \beta_7^m) \right) P_8 \\
+ \left( 4 \partial_-(\beta_4^m + \beta_5^m + \beta_6^m + \beta_7^m) + 16(\beta_4^m + \beta_5^m + \beta_6^m + \beta_7^m)(\beta_1^m + \beta_2^m + \beta_3^m) \right) P_{10} \\
+ \left( 4 \partial_-(\beta_1^m + \beta_3^m - \beta_4^m + \beta_6^m) + 16(\beta_1^m + \beta_3^m - \beta_4^m + \beta_6^m)(\beta_2^m + \beta_5^m + \beta_7^m) \right) P_{12} \\
+ \left( 4 \partial_-(\beta_1^m + \beta_3^m + \beta_5^m + \beta_7^m) + 16(\beta_1^m + \beta_3^m + \beta_5^m + \beta_7^m)(\beta_2^m - \beta_4^m + \beta_6^m) \right) P_{14} \\
\right] \varepsilon_- = 0.
\]
Again the coefficient of $I$ in this equation must vanish in order to give a complex extra Killing spinor. Because $\beta_i^m$ and $A_m$ are real, this implies that

$$\partial_-(\beta_1^m + \beta_2^m + \beta_3^m - \beta_4^m + \beta_5^m + \beta_6^m + \beta_7^m) = 0, \quad (4.38)$$

$$A_m = -4(\beta_1^m + \beta_2^m + \beta_3^m - \beta_4^m + \beta_5^m + \beta_6^m + \beta_7^m)^2. \quad (4.39)$$

Consequently $A_m$ must be independent of $x^-$, because the right hand side of (4.39) is independent of $x^-$ from eq. (4.38). Noting that $\beta_i^m$ is related to $b_i$ in (4.4), we find that eq. (4.38) leads to four differential equations for $b_i$ which imply

$$\partial_- b_1 = \partial_- b_2 = \ldots = \partial_- b_7 = 0, \quad (4.40)$$

which implies that $\xi$ is independent of $x^-$. The extra Killing spinors are determined as a non-trivial solution of

$$\left[ -4(\beta_1^m + \beta_2^m + \beta_3^m - \beta_4^m + \beta_5^m + \beta_6^m + \beta_7^m)^2 - A_m \right] \mathbb{I} + 16(\beta_2^m + \beta_3^m + \beta_6^m + \beta_7^m)(\beta_1^m - \beta_4^m + \beta_5^m) \mathbb{P}_2 + 16(\beta_1^m + \beta_2^m + \beta_5^m + \beta_6^m)(\beta_3^m - \beta_4^m + \beta_7^m) \mathbb{P}_4 + 16(\beta_1^m + \beta_2^m - \beta_4^m + \beta_7^m)(\beta_3^m + \beta_5^m + \beta_6^m) \mathbb{P}_6 + 16(\beta_2^m + \beta_3^m - \beta_4^m + \beta_5^m)(\beta_1^m + \beta_6^m + \beta_7^m) \mathbb{P}_8 + 16(-\beta_4^m + \beta_5^m + \beta_6^m + \beta_7^m)(\beta_1^m + \beta_2^m + \beta_3^m) \mathbb{P}_{10} + 16(\beta_1^m + \beta_3^m - \beta_4^m + \beta_6^m)(\beta_2^m + \beta_5^m + \beta_7^m) \mathbb{P}_{12} + 16(\beta_1^m + \beta_3^m + \beta_5^m + \beta_7^m)(\beta_2^m - \beta_4^m + \beta_6^m) \mathbb{P}_{14} \right] \varepsilon_- = 0. \quad (4.41)$$

which again shows the two-fold degeneracy of the extra Killing spinors. If (4.39) is satisfied, $\mathbb{P}_{16} \varepsilon_-$ is a complex extra Killing spinor and the background admits 18 Killing spinors, 16 standard and 2 extra Killing spinors. If, in addition, the coefficient of $\mathbb{P}_I, I = 2, 4, 6, 8, 10, 12, 14$, is zero, then $\mathbb{P}_I \varepsilon_-$ becomes an additional complex extra Killing spinor.

Again, the backgrounds with extra supersymmetries automatically satisfy the supergravity equation of motion (2.3) because (4.39) and (4.5) lead to

$$\triangle H = \sum_{m=1}^{8} 2A_m = -64(b_1^2 + b_2^2 + b_3^2 + b_4^2 + b_5^2 + b_6^2 + b_7^2), \quad (4.42)$$

$$-\frac{32}{4!}\xi_{mnpq}^{\varepsilon_{mnpq}} = -64(b_1^2 + b_2^2 + b_3^2 + b_4^2 + b_5^2 + b_6^2 + b_7^2). \quad (4.43)$$

In summary, we have shown that IIB pp-wave backgrounds with a self-dual five-form R-R flux can be reduced to the form modulo coordinate transformations

$$ds^2 = 2dx^+dx^- + A_m x^m x^m(dx^-)^2 + (dx^m)^2, \quad F = dx^- \wedge \xi, \quad (4.44)$$
where $A_m$ and $\xi$ are constants, if the backgrounds admit extra Killing spinors characterized by (4.2) and (4.3). It is interesting to examine the cases in which the extra Killing spinors are characterized by more general Cartan matrices.

5 IIB pp-wave backgrounds with extra supersymmetries

In the previous section, we have shown that Killing spinor equations reduce to (4.26) and (4.41). The former case admits NS-NS and R-R three-forms in addition to a self-dual R-R five-form and has been examined in the literature, whereas the latter case does not admit them and has not been examined well. We provide pp-wave backgrounds obtained by solving (4.26) and (4.41) in this section.

For the former case, (4.26), one finds three classes of solutions. One is the maximally supersymmetric pp-wave background found in [1]

$$A_m = -4\mu^2, \ m = 1, \ldots, 8, \quad \xi = \mu(\gamma^{1234} + \gamma^{5678}), \quad (5.1)$$

which means

$$\xi = \mu(dx^1 \wedge dx^2 \wedge dx^3 \wedge dx^4 + dx^5 \wedge dx^6 \wedge dx^7 \wedge dx^8) \quad (5.2)$$

because $\xi = \frac{1}{4!}\xi_{lmpn}\gamma^{lmpn}$. We will use $\xi$ below to indicate $\xi$ for compact expressions. Another is the background with 24 supersymmetries

$$A_m = \begin{cases} 
-4(\mu_1 - \mu_2)^2, & m = 1, 2, 5, 6, \\
-4(\mu_1 + \mu_2)^2, & m = 3, 4, 7, 8, \end{cases} \quad \xi = \mu_1(\gamma^{1234} + \gamma^{5678}) + \mu_2(\gamma^{3456} + \gamma^{1278}), \quad (5.3)$$

which is a subclass of solutions found in [11]. The third is the 20 supersymmetric background

$$A_m = \begin{cases} 
-4(-\mu_1 + \mu_2 - \mu_3)^2, & m = 1, 2, \\
-4(-\mu_1 - \mu_2 + \mu_3)^2, & m = 3, 4, \\
-4(\mu_1 - \mu_2 - \mu_3)^2, & m = 5, 6, \\
-4(\mu_1 + \mu_2 + \mu_3)^2, & m = 7, 8, \end{cases} \quad \xi = \mu_1(\gamma^{1234} + \gamma^{5678}) + \mu_2(\gamma^{3456} + \gamma^{1278}) + \mu_3(\gamma^{1256} + \gamma^{3478}), \quad (5.4)$$

which contains the above two backgrounds, (5.1) and (5.3), as special cases.

For the latter case, (4.41), we find IIB pp-wave solutions, which have not been given in the literature as long as we know. Eqn. (4.41) leads to two maximally supersymmetric
pp-wave backgrounds. One is the same as (5.1), and the other is

\[ A_m = -4\mu^2, \ m = 1, \ldots, 8, \]
\[ \xi = \frac{\mu}{2}\left( -\left( \gamma^{4567} + \gamma^{1238} \right) + \left( \gamma^{2367} + \gamma^{1458} \right) + \left( \gamma^{1346} + \gamma^{2578} \right) + \left( \gamma^{1256} + \gamma^{3478} \right) \right). \]  

(5.5)

The self-dual four-form can be regarded as the real or imaginary part of the holomorphic \((4,0)\)-form of a Calabi-Yau four-fold with \(SU(4)\) holonomy. It can be shown that there are twelve Lorentz generators in this background, which is the same as in the \(SO(4)\times SO(4)\) case \([54]\). In addition, \(\xi\) in \((5.5)\) can be related to \((5.1)\) by a coordinate transformation. We show that this is the case\(^4\).

Because \(\xi\) in \((5.5)\) can be rewritten as

\[ \xi = \frac{\mu}{4}\left( (\gamma^1 - \gamma^7)(\gamma^2 - \gamma^4)(\gamma^5 + \gamma^3)(\gamma^6 - \gamma^8) + (\gamma^1 + \gamma^7)(\gamma^2 - \gamma^4)(\gamma^5 - \gamma^3)(\gamma^6 + \gamma^8) \right), \]  

(5.6)

\(\xi\) in \((5.5)\) is transformed to \(\xi\) in \((5.1)\) by the coordinate transformation

\[ y^{1,5} = \frac{1}{\sqrt{2}}(x^1 \mp x^7), \ y^{2,6} = \frac{1}{\sqrt{2}}(x^2 \mp x^4), \ y^{3,7} = \frac{1}{\sqrt{2}}(x^5 \pm x^3), \ y^{4,8} = \frac{1}{\sqrt{2}}(x^6 \mp x^8). \]  

(5.7)

This transformation does not change the metric, and thus we have rewritten \((5.5)\) as \((5.1)\).

As non-maximally supersymmetric backgrounds, we find pp-wave solutions with 24, 20 and 18 supersymmetries. The background

\[ A_m = \begin{cases} 
-64\mu^2, & m = 1, 3, 5, 7, \\
-16\mu^2, & m = 2, 4, 6, 8,
\end{cases} \]
\[ \xi = \mu\left( -\left( \gamma^{4567} + \gamma^{1238} \right) + \left( \gamma^{2367} + \gamma^{1458} \right) - \left( \gamma^{2345} + \gamma^{1678} \right) - 2\left( \gamma^{1346} + \gamma^{2578} \right) + \left( \gamma^{1256} + \gamma^{3478} \right) + \left( \gamma^{1247} + \gamma^{3568} \right) \right), \]  

(5.8)

admits 24 supersymmetries. We find three classes of 20 supersymmetric backgrounds, a four-parameter family

\[ A_m = \begin{cases} 
-4(\mu_1 + \mu_2 - \mu_3 - \mu_4)^2, & m = 1, 3, \\
-4(\mu_1 + \mu_2 + \mu_3 + \mu_4)^2, & m = 2, 8, \\
-4(\mu_1 - \mu_2 + \mu_3 - \mu_4)^2, & m = 4, 6, \\
-4(\mu_1 - \mu_2 - \mu_3 + \mu_4)^2, & m = 5, 7,
\end{cases} \]
\[ \xi = \mu_1\left( \gamma^{2367} + \gamma^{1458} \right) + \mu_2\left( \gamma^{2345} + \gamma^{1678} \right) + \mu_3\left( \gamma^{1256} + \gamma^{3478} \right) + \mu_4\left( \gamma^{1247} + \gamma^{3568} \right), \]  

(5.9)

and two three parameter families,

\[ A_m = \begin{cases} 
-4(2\mu_1 + 2\mu_2)^2, & m = 1, 3, \\
-16\mu_3^2, & m = 2, 5, 7, 8, \\
-4(2\mu_1 - 2\mu_2)^2, & m = 4, 6,
\end{cases} \]

\(^4\)We thank José Figueroa-O’Farrill for correspondence.
\[ \xi = -\mu_3(\gamma^{4567} + \gamma^{1238}) + \mu_1(\gamma^{2367} + \gamma^{1458}) + \mu_2(\gamma^{2345} + \gamma^{1678}) + \mu_3(\gamma^{1357} + \gamma^{2468}) - \mu_1(\gamma^{1247} + \gamma^{3568}) - \mu_2(\gamma^{1256} + \gamma^{3478}) \]  \hspace{1cm} (5.10)

and

\[ A_m = \begin{cases} 
-64\mu_1^2, & m = 1, 3, \\
-16(-\mu_2 + \mu_3)^2, & m = 2, 8, \\
-16\mu_2^2, & m = 4, 6, \\
-16\mu_2^2, & m = 5, 7, 
\end{cases} \]

\[ \xi = -\mu_2(\gamma^{4567} + \gamma^{1238}) - \mu_1(\gamma^{2367} + \gamma^{1458}) - \mu_1(\gamma^{2345} + \gamma^{1678}) + \mu_2(\gamma^{1357} + \gamma^{2468}) + \mu_3(\gamma^{1346} + \gamma^{2578}) + \mu_1(\gamma^{1256} + \gamma^{3478}) + \mu_1(\gamma^{1247} + \gamma^{3568}). \]  \hspace{1cm} (5.11)

These backgrounds can be obtained as special cases of the backgrounds with 18 Killing spinors

\[ A_m = \begin{cases} 
-4(\mu_1 + \mu_2 + \mu_3 + \mu_4 - \mu_5 - \mu_6 - \mu_7)^2, & m = 1, 3, \\
-4(\mu_1 - \mu_2 - \mu_3 - \mu_4 + \mu_5 - \mu_6 - \mu_7)^2, & m = 2, 8, \\
-4(-\mu_1 + \mu_2 + \mu_3 + \mu_4 - \mu_5 - \mu_6 + \mu_7)^2, & m = 4, 6, \\
-4(-\mu_1 - \mu_2 + \mu_3 + \mu_4 - \mu_5 - \mu_6 + \mu_7)^2, & m = 5, 7, 
\end{cases} \]

\[ \xi = \mu_1(\gamma^{4567} + \gamma^{1238}) + \mu_2(\gamma^{2367} + \gamma^{1458}) + \mu_3(\gamma^{2345} + \gamma^{1678}) + \mu_4(\gamma^{1357} + \gamma^{2468}) + \mu_5(\gamma^{1346} + \gamma^{2578}) + \mu_6(\gamma^{1256} + \gamma^{3478}) + \mu_7(\gamma^{1247} + \gamma^{3568}). \]  \hspace{1cm} (5.12)

6 Summary and Discussions

We have established a uniqueness theorem which states that any IIB pp-wave background of the form (2.2) can be reduced to the form (8.13) modulo coordinate transformations, if there exist at least one non-harmonic extra Killing spinor. We examined further the cases in which extra Killing spinors are characterized by (4.2) and (4.3). Examining Killing spinor equations, we found IIB pp-wave backgrounds which admit 18, 20, 24 and 32 Killing spinors.

It is interesting to examine the similar uniqueness theorem for pp-wave backgrounds of IIA supergravity and supergravities in lower dimensions. We expect that the similar uniqueness theorems can be established.

We have seen that two sets of mutually commuting matrices (4.2) and (4.3) can be related to the Kähler form \( J \) of a Calabi-Yau four-fold with \( SU(4) \) holonomy and the self-dual Cayley four-form \( \Psi \) of \( d = 8 \) Riemannian manifold with \( Spin(7) \) holonomy, respectively. For eleven-dimensions [25], two sets of mutually commuting matrices were shown to be related to the Kähler form \( J \) of a Calabi-Yau four-fold with \( SU(4) \) holonomy.
and the associative three-form of $d = 7$ Riemannian manifold with $G_2$ holonomy. These may suggest that Killing spinor equations for $d$-dimensional pp-wave backgrounds with flux can be reduced to those for $(d - 2)$-dimensional backgrounds without flux. If this is the case, a classification of pp-wave backgrounds with flux gets simplified. Further, it is interesting to try to construct $(d - 2)$-dimensional backgrounds with special holonomy from $d$-dimensional pp-wave backgrounds with flux.

We have seen that the pp-wave backgrounds which admit at least one extra Killing spinor automatically satisfy the supergravity equation of motion, which suggests that Killing spinor equations for backgrounds with extra supersymmetries have rich algebraic structures, just as those for maximally supersymmetric backgrounds. It is interesting to classify all non-maximally supersymmetric backgrounds which admit more than 16 supersymmetries, as was achieved for the maximally supersymmetric case in [20].

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