Rigidity results for Riemann solitons

William Tokura\textsuperscript{a,1,*}, Marcelo Barboza\textsuperscript{b,2}, Elismar Batista\textsuperscript{b,3}, Ilton Menezes\textsuperscript{b,4}

\textsuperscript{a}Universidade Federal do Acre, CCET, 69920-900, Rio Branco - AC, Brazil
\textsuperscript{b}Universidade Federal de Goiás, 1ME, 74001-970, Goiânia - GO, Brazil

Abstract

In this paper we take a look at conditions that make a Riemann soliton trivial, compacity being one of them. We also show that the behaviour at infinity of the gradient field of a non-compact gradient Riemann soliton might cause the soliton to be an Einstein manifold. Finally, we obtain scalar curvature estimates for complete shrinking or steady gradient Riemann solitons whose scalar curvature is bounded from below at infinity.

Keywords: Riemann flow, Riemann solitons, gradient Riemann solitons, rigidity results, gradient almost Ricci solitons.

2010 MSC: 53C21, 53C25, 53C26

1. Introduction and main results

The Riemann flow, which was introduced by Udriste \cite{1,2}, is the flow associated with the evolution equation

\begin{equation}
\frac{\partial}{\partial t} G(t) = -2 Rm_{g(t)},
\end{equation}

for Riemannian metrics on a given differentiable manifold $M^n$, where $Rm_{g(t)}$ is the Riemann curvature $(0,4)$-tensor of the metric $g(t)$, $G = \frac{1}{2} g \odot g$ and $\odot$ is the Kulkarni-Nomizu product. Recall that, for symmetric $(0,2)$-tensors $A$ and $B$, the Kulkarni-Nomizu product $A \odot B$ is defined by (see \cite{3}):

\begin{align}
(A \odot B)(X_1, X_2, X_3, X_4) &= A(X_1, X_3)B(X_2, X_4) + A(X_2, X_4)B(X_1, X_3) \\
&\quad - A(X_1, X_4)B(X_2, X_3) - A(X_2, X_3)B(X_1, X_4).
\end{align}

Short-time existence and uniqueness of the flow (1) is guaranteed by Hirica and Udriste on compact manifolds in \cite{4}, where the authors utilize the Ricci flow to construct short-time solutions of (1).

\*Corresponding author

Email addresses: willian.tokura@ufac.br (William Tokura), bezerra@ufg.br (Marcelo Barboza), elismar@ufg.br (Elismar Batista), iltonmenezes@ufg.br (Ilton Menezes)

1\https://orcid.org/0000-0001-9063-793X
2\https://orcid.org/0000-0003-4258-4676
3\https://orcid.org/0000-0002-4914-238
4\https://orcid.org/0000-0002-9590-8731
Just like it happens with Ricci solitons, critical metrics for the Riemann flow are self similar solutions of (1), that is, it evolves over time from a given Riemannian metric on the manifold by diffeomorphisms and dilatations [5].

**Definition 1.** A Riemannian manifold \((M^n, g)\), \(n \geq 3\), is a Riemann soliton if there exist a vector field \(X \in \mathfrak{X}(M)\) and a real number \(\lambda \in \mathbb{R}\) satisfying:

\[
Rm + \frac{1}{2} L_X g \circ g = \lambda G,
\]

(2)

where \(L_X g\) is the Lie derivative of \(g\) in the direction of \(X\). We write the soliton in (2) as \((M^n, g, X, \lambda)\) for the sake of simplicity. It may happen that \(X = \nabla f\) for some \(f \in C^\infty(M)\), in which case we say that \((M^n, g, \nabla f, \lambda)\) is a gradient Riemann soliton. Notice that equation (2) for gradient Riemann solitons reads:

\[
Rm + \nabla^2 f \circ g = \lambda G.
\]

(3)

A Riemann soliton is classified into one of three types according to the sign of \(\lambda\), expanding if \(\lambda < 0\), steady if \(\lambda = 0\) and, finally, shrinking if \(\lambda > 0\). Also, a Riemann soliton is called trivial if either \(L_X g = 0\) or \(\nabla^2 f = 0\).

One might start picturing Riemann solitons by noticing that the trivial ones satisfy \(Rm = \frac{\lambda}{2} g \circ g\), thus having constant sectional curvature. Also, in [6] Catino and Mastrolia define what they call \(X\)-space and \(f\)-space forms. Interestingly, \(X\)-space forms with constant \(\text{div}(X)\) are Riemann solitons.

**Example 1.** Let \((Q^n(c), g)\) be a Riemannian manifold of constant sectional curvature \(c \in \{-1, 0, 1\}\). Then, \((Q^n(c), g, f, c)\) is a trivial Riemann soliton for any constant function \(f\) on \(Q^n(c)\).

**Example 2.** (Gaussian soliton) The Gaussian soliton on \(\mathbb{R}^n\), which is given by

\[
g_{ij} = \delta_{ij} \quad \text{and} \quad f(x) = \frac{\lambda}{4} |x|^2,
\]

is a Riemann soliton since

\[
Rm = 0 \quad \text{and} \quad \nabla^2 f = \frac{\lambda}{2} g.
\]

**Example 3.** Let \((M^{n+1}, g)\) be the Riemannian product manifold of \((0, \infty)\) and \(S^n\). Then, \((M^{n+1}, g, \nabla f, \lambda)\) is a gradient Riemann soliton for

\[
f : (0, \infty) \times S^n \rightarrow \mathbb{R}, \quad (r, p) \mapsto \frac{\lambda}{4r},
\]

where \(\lambda \in \mathbb{R}\).
Example 4. Let \((M^{n+1}, g)\) be the product manifold \((0, \infty) \times \mathbb{S}^n\) equipped with the metric tensor given by
\[
g_{(r,p)}(t_1 \oplus v_1, t_2 \oplus v_2) = r(t_1 t_2 + v_1 v_2),
\]
for every \((r, p) \in (0, \infty) \times \mathbb{S}^n\) and \(t_1 \oplus v_1, t_2 \oplus v_2 \in \mathbb{R} \oplus T_p \mathbb{S}^n \simeq T_{(r, p)}((0, \infty) \times \mathbb{S}^n)\). Then, the quadruple \((M^{n+1}, g, \nabla f, \lambda)\) is a gradient Riemann soliton for
\[
f : (0, \infty) \times \mathbb{S}^n \rightarrow \mathbb{R}, \quad (r, p) \mapsto -\ln \left(\frac{r^2}{4}\right).
\]

Examples (3) and (4) both are locally conformally flat gradient Riemann solitons invariant by the left action of
\[
SO(n, \mathbb{R}) = \{ A \in GL(n, \mathbb{R}) : A^{-1} = A^t \},
\]
on \((0, \infty) \times \mathbb{S}^n\), given by
\[
SO(n, \mathbb{R}) \times ((0, \infty) \times \mathbb{S}^n) \rightarrow (0, \infty) \times \mathbb{S}^n, \quad (A, (r, p)) \mapsto (r, A \cdot p).
\]

We refer the reader to \([9, 11, 12, 8, 10, 7?]\) for examples of other symmetric solitons. When we first met Riemann solitons it occurred to us the results of Ivey \([13]\), showing that expanding or steady compact gradient Ricci solitons are rigid, of Hsu \([14]\), showing that compact gradient Yamabe solitons are trivial, of Barros et al. \([15]\), showing that compact almost Ricci solitons with constant scalar curvature are Einstein manifolds, of Catino et al. \([16]\), showing that compact \(\rho\)-Einstein solitons are trivial, and even one of our own, Tokura et al. \([17]\), showing that compact gradient \(k\)-Yamabe solitons are trivial. On Riemann solitons, specifically, we found a result due to Blaga \([18]\), showing that compact Riemann solitons with potential vector field \(X\) of constant length are trivial. Our first result extends Blaga’s result as it doesn’t impose any additional conditions on the potential vector field.

Theorem 1. Compact Riemann solitons \((M^n, g, X, \lambda)\) are trivial.

Also, we show that non-compact Riemann solitons are Einstein manifolds as long as it has constant scalar curvature and a certain weighted integral of \(|\nabla f|^2\) is finite.

Theorem 2. Let \((M^n, g, \nabla f, \lambda)\) be a complete non-compact gradient Riemann soliton with constant scalar curvature satisfying
\[
\int_{M^n \setminus B_r(x_0)} \frac{|\nabla f|^2}{d(x, x_0)^2} \, dv_g < \infty,
\]
for some \(r > 0\), where \(d\) is the \(g\)-distance function on \(M^n\) and \(B_r(x_0)\) then is the \(d\)-ball of radius \(r\) centered at \(x_0\). Then, \((M^n, g)\) is an Einstein manifold.

Next, we show that for Riemann solitons the sign of the scalar curvature at infinity determines the sign of the scalar curvature as a whole.
**Theorem 3.** Let \((M^n, g, X, \lambda)\) be a complete non-compact shrinking or steady Riemann soliton. Assume that \(\liminf_{x \to \infty} S(x) \geq 0\). Then, \(M\) has nonnegative scalar curvature. Furthermore, if \(M\) is not scalar flat, then \(S > n\lambda\).

Our last result classifies scalar flat and steady gradient Riemann solitons.

**Theorem 4.** Let \((M^n, g, \nabla f, \lambda)\) be a complete steady gradient Riemann soliton. If \(M^n\) is scalar flat and \(f\) is not constant, then \(M^n\) is isometric to a cylinder \(\mathbb{R} \times \Sigma^{n-1}\) where \(\Sigma^{n-1}\) is a Ricci flat hypersurface of \(M^n\) and \(f(t, x) = at + b\), for certain \(a, b \in \mathbb{R}, a \neq 0\).

### 2. Key lemmas and proofs

Our first lemma provides a link between Riemann solitons and almost Ricci solitons. This relation was also noted by Hirica and Udriste [5] and Blaga [18].

**Lemma 5.** The Riemannian manifold \((M^n, g)\) admits a Riemannian soliton structure \((M^n, g, X, \lambda)\) if and only if \((M^n, g)\) has null Weyl tensor and admits an almost Ricci soliton structure with soliton vector field \((n-2)X\) satisfying

\[
Rc_{jk} + \frac{n-2}{2} (\mathcal{L}_X g)_{jk} = \left[ \frac{(n-2)\lambda}{2} + \frac{S}{2(n-1)} \right] g_{jk},
\]

where \(Rc\) is the Ricci tensor and \(S\) the scalar curvature of \((M^n, g)\).

**Proof of Lemma 5.** From the fundamental equation (2), we have

\[
Rm_{ijkl} + \frac{1}{2} [g_{jl}(\mathcal{L}_X g)_{ik} + g_{jk}(\mathcal{L}_X g)_{il} - g_{ik}(\mathcal{L}_X g)_{jl} - g_{jl}(\mathcal{L}_X g)_{ik}] = \lambda [g_{il}g_{jk} - g_{ik}g_{jl}],
\]

which by contraction over \(i\) and \(l\), gives

\[
Rc_{jk} + \frac{n-2}{2} (\mathcal{L}_X g)_{jk} = [(n-1)\lambda - \text{div}(X)]g_{jk},
\]

or equivalently

\[
Rc_{jk} + \frac{n-2}{2} (\mathcal{L}_X g)_{jk} = \left[ \frac{(n-2)\lambda}{2} + \frac{S}{2(n-1)} \right] g_{jk}.
\]

Now, substituting equation (6) into the decomposition of the Riemann tensor (see [3]):

\[
Rm_{ijkl} = \left[ \frac{1}{n-2} (Rc - \frac{S}{2(n-1)} g) \right]_{ijkl} + W_{ijkl},
\]

and using the Riemann soliton equation (2), we deduce that \(W_{ijkl} = 0\).

On the other hand, if \((M^n, g)\) has null Weyl \(W = 0\) and admits an almost Ricci soliton structure

\[
Rc_{jk} + \frac{n-2}{2} (\mathcal{L}_X g)_{jk} = \left[ \frac{(n-2)\lambda}{2} + \frac{S}{2(n-1)} \right] g_{jk}.
\]
Then from (7) we have
\[
R_{ijkl} = \left[ \frac{1}{n-2} \left( Rc_g - \frac{S}{2(n-1)} g \right) \right]_{ijkl} = \frac{1}{2} \left[ (-\mathcal{L}Xg + \lambda g) \odot g \right]_{ijkl}.
\]
Therefore, \((M^n, g)\) admits a Riemann solitons structure
\[
R_{ijkl} + \frac{1}{2} \left( \mathcal{L}Xg \odot g \right)_{ijkl} = \left( \frac{\lambda}{2} g \odot g \right)_{ijkl}.
\]

**Remark 1.** In the particular case in which \(X\) is a gradient vector field, we have that equation (4) produces a Schouten soliton [16]. Therefore, \((M^n, g)\) is a gradient Riemann soliton with potential \(X\) if and only if \((M^n, g)\) is a Schouten soliton with potential \((n-2)X\) and \(W_{ijkl} = 0\).

**Lemma 6.** Let \((M^n, g, X, \lambda)\) be a Riemann soliton. Then the following formulas hold
\[
div(X) = \frac{n\lambda}{2} - \frac{S}{2(n-1)},
\]
\[
Rc_{ij} X_l + \frac{S_j}{2(n-1)} + X_{jii} = 0
\]
\[
\frac{n-2}{2} X_j S_j + \frac{n-2}{2} \lambda S + \frac{S^2}{2(n-1)} - |Rc|^2 = 0
\]

**Proof of Lemma 6.** In order to obtain (8) it is enough to contract equation (4). Now, for equation (9), we remember the Ricci identity:
\[
X_{ijk} - X_{ikj} = X_l R_{ltijk}.
\]
Contracting (11) with respect \(i, k\), we have
\[
X_{iji} - X_{iij} = R_{tij} X_l = Rc_{lj} X_l.
\]
Next, differentiating (4) we get
\[
Rc_{ij,i} = -\frac{n-2}{2} (X_{iji} + X_{jii}) + \frac{S_j}{2(n-1)} g_{ij}
\]
\[
= -\frac{n-2}{2} (X_{iji} - X_{iij} + X_{iij} + X_{jii}) + \frac{S_j}{2(n-1)} g_{ij}
\]
\[
= -\frac{n-2}{2} Rc_{lj} X_l - \frac{n-2}{2} (X_{ijj} + X_{jii}) + \frac{S_j}{2(n-1)} g_{ij}.
\]
Using the twice contracted second Bianchi identity and equation (8), we deduce
\[
\frac{1}{2} S_j = Rc_{ij,i} = -\frac{n-2}{2} Rc_{lj} X_l - \frac{n-2}{2} \left[ -\frac{S_j}{2(n-1)} + X_{jii} \right] + \frac{S_j}{2(n-1)},
\]
which enables us obtain equation (9).
Finally, from Lemma 5, the Riemann soliton \((M^n, g)\) admits a gradient almost Ricci solitons structure with vector field \((n - 2)X\) and soliton function \(\frac{(n-2)\lambda}{2} + \frac{S}{2(n-1)}\). Then, according to Lemma 3 of [15], we have

\[
\Delta_{(n-2)X} Rc_{ik} = \left[(n-2)\lambda + \frac{S}{(n-1)}\right] Rc_{ik} - 2Rm_{ijks}Rc_{js} + \frac{n-2}{2} Rc_{is}(X_{sk} - X_{ks}) \\
+ \frac{n-2}{2} Rc_{sk}(X_{si} - X_{is}) + \frac{\Delta S}{2(n-1)} g_{ki} - \frac{S_{ij}}{2(n-1)} g_{kj}.
\]  

(12)

Computing the trace of identity (12), we deduce

\[-\frac{n-2}{2} X_j S_j = \left[\frac{n-2}{2} \lambda + \frac{S}{2(n-1)}\right] S - |\text{Rc}|^2,
\]

which is equation (10).

PROOF OF THEOREM 1. Let \(p, q \in M\) be the points where \(S\) attains its maximum and minimum in \(M\), i.e.,

\[S(q) = \min_{x \in M} S(x), \quad S(p) = \max_{x \in M} S(x), \quad \nabla S(p) = 0 = \nabla S(q),\]

and

\[\lambda - \frac{S(q)}{n(n-1)} \geq \lambda - \frac{S(x)}{n(n-1)} \geq \lambda - \frac{S(p)}{n(n-1)}.\]

From (10) and the fact \(|\text{Rc}|^2 \geq \frac{1}{n} S^2\), we have

\[0 = \frac{n-2}{2} \lambda S(q) + \frac{S(q)^2}{2(n-1)} - |\text{Rc}|^2(q) \leq \frac{n-2}{2} S(q) \left(\lambda - \frac{S(q)}{n(n-1)}\right),\]

(13)

and

\[0 = \frac{n-2}{2} \lambda S(p) + \frac{S(p)^2}{2(n-1)} - |\text{Rc}|^2(p) \leq \frac{n-2}{2} S(p) \left(\lambda - \frac{S(p)}{n(n-1)}\right).\]

(14)

On the other hand, since \(M\) is compact, integrating equation (8), we obtain the identity

\[\lambda = \frac{1}{\text{Vol}(M)} \int_M \frac{S}{n(n-1)} dv_g,\]

(15)

from what we see that

\[\lambda - \frac{S(q)}{n(n-1)} \geq 0, \quad \lambda - \frac{S(p)}{n(n-1)} \leq 0.\]

(16)

Now, we state that

\[\lambda - \frac{S(q)}{n(n-1)} = 0.\]

In fact, if \(\lambda - \frac{S(q)}{n(n-1)} > 0\), then from (13), we have \(S(q) \geq 0\) and \(\lambda > 0\). Consequently, \(S(p) > 0\) from (16).

Now, since \(S(p) > 0\) we deduce from (14) and (16) that \(n(n-1)\lambda = S(p)\) and \(S \equiv S(p)\) which is an absurd from \(\lambda - \frac{S(q)}{n(n-1)} > 0\). Therefore, \(\lambda - \frac{S(q)}{n(n-1)} = 0\). However, this implies \(S \equiv S(q)\) and \(\lambda - \frac{S}{n(n-1)} = 0\).

Hence, from (4) we deduce that \((M^n, g)\) is a Ricci soliton. From Perelman work’s [19] we known that any compact Ricci soliton is a gradient Ricci soliton. Then there exist a smooth function \(f\) on \(M\) such that \(\text{div}(X) = \Delta f = 0\). The compactness of \(M\) implies that \(f\) is constant and the soliton is trivial.
Proof of Theorem 2. We follow the argument in [20]. The Bochner formula states that ([21]):
\[
\frac{1}{2} \Delta |\nabla f|^2 = |\nabla^2 f|^2 + \langle \nabla f, \nabla \Delta f \rangle + Rc(\nabla f, \nabla f).
\]
From equation (9) we conclude that
\[
Rc_{ij} f_i = -\frac{S_j}{2(n-1)} - (\nabla^2 f)_{j,j,i} = -\frac{S_j}{2(n-1)} + \frac{Rc_{ji,i}}{n-2} \frac{S_j}{2(n-1)(n-2)}
\]
\[
= -\frac{S_j}{2(n-1)} + \frac{S_j}{2(n-2)} \frac{S_j}{2(n-1)(n-2)} = 0.
\]
(17)
Since \( S \) is constant, we have that \( \text{div}(X) = \Delta f \) is constant from (8). Hence, from (17), we arrive at
\[
\frac{1}{2} \Delta |\nabla f|^2 = |\nabla^2 f|^2.
\]
(18)
We choose a cut-off function \( \psi_r \) on \( B_{2r}(x_0) \) such that \( 0 \leq \psi_r \leq 1 \), \( \text{supp}(\psi_r) \subset B_{2r}(x_0) \) and
\[
\psi_r = 1 \quad \text{in} \quad B_r(x_0), \quad |\nabla \psi_r|^2 \leq \frac{C}{r^2}, \quad \Delta \psi_r \leq \frac{C}{r^2}.
\]
Then, multiplying (18) by \( \psi_r \) and integrating it over \( B_r(x_0) \), we get
\[
\int_{B_r(x_0)} |\nabla^2 f|^2 \psi_r^2 = \int_{B_r(x_0)} \frac{1}{2} \Delta |\nabla f|^2 \psi_r^2
\]
\[
= \int_{B_r(x_0)} \frac{1}{2} |\nabla f|^2 \Delta \psi_r^2
\]
\[
\leq \int_{B_{2r}(x_0) \setminus B_r(x_0)} \frac{1}{2} \frac{C}{r^2} |\nabla f|^2 \rightarrow 0 \quad \text{as} \quad r \rightarrow \infty.
\]
Hence
\[
\int_M |\nabla^2 f|^2 = 0.
\]
Therefore \( \nabla^2 f = 0 \), and from Lemma 5 \((M^n, g)\) is Einstein.

Proof of Theorem 3. From equation (10) of Lemma 6 and the inequality \( |Rc|^2 \geq \frac{S^2}{n} \), we deduce that
\[
\langle X, \nabla S \rangle + \lambda S - \frac{S^2}{n(n-1)} \geq 0.
\]
(19)
Using the maximum principle we can prove Theorem 3. In fact, suppose that \( \inf_M S(x) < 0 \), then by the assumption \( \lim_{x \rightarrow \infty} S(x) \geq 0 \), there exists some point \( z \in M \) such that
\[
S(z) = \inf_M S(x) < 0.
\]
Hence,
\[
\Delta S(z) \geq 0, \quad \nabla S(z) = 0.
\]
Then, from (19) we deduce that
\[ \lambda S(z) - \frac{S(z)^2}{n(n-1)} \geq 0. \]
This is impossible since \( \lambda S(z) - \frac{S(z)^2}{n(n-1)} < 0 \) for \( \lambda \geq 0 \). Therefore \( S(x) \geq 0 \). From the maximum principle we have that either \( S \equiv 0 \) or \( S(x) > n(n-1)\lambda \).

The next Lemma is a particular case of theorem 6.3 of Catino et al. [22]. For the sake of completeness we will prove here.

**Lemma 7.** Let \((M^n, g, \nabla f, \lambda)\) be a complete gradient Riemann soliton, \( c \in \mathbb{R} \) be a regular value of \( f \) and \( \Sigma_c = f^{-1}(c) \) be its level surface. Then

a) \( |\nabla f|^2 \) is constant on a connected component of \( \Sigma_c \).

b) \( \Sigma_c \) is totally umbilical and the scalar curvature \( S \) is constant on \( \Sigma_c \).

c) the mean curvature \( H \) is constant on \( \Sigma_c \).

d) \( \Sigma_c \) is Einstein with respect to the induced metric.

e) in any open set of \( \Sigma_c \) in which \( f \) has non critical points, the metric \( g \) can be written as
\[ g = dr^2 + \psi(r)^2 g_{\Sigma}, \]
where \( g_{\Sigma} \) is the metric induced by \( g \) in \( \Sigma_c \) and \( \psi(r) = e^{\frac{1}{n-1} \int_0^r H(s)ds}. \)

**Proof of Lemma 7.** Let \( \{e_1, e_2, \ldots, e_n\} \) be a local orthonormal frame in a neighborhood of a regular point \( p \in \Sigma_c \) such that \( \{e_2, \ldots, e_n\} \) are tangent to \( \Sigma_c \) and \( e_1 = \frac{\nabla f}{|\nabla f|} \). In order to prove (a), note that for any \( X \perp \nabla f \) we have
\[ \nabla_X |\nabla f|^2 = 2 \nabla^2 f(X, \nabla f) = \frac{2}{n-2} Rc(X, \nabla f) = 0. \]
Now, from Lemma 5 the Weyl tensor \( W \) vanishes, then since Riemann solitons are in the class of almost Ricci solitons, we can apply Theorem 4.4 and Proposition 6.1 of [22] with \( \alpha = 1, \beta = n-2, \mu = 0 \) to deduce that
\[ h_{ab} = \frac{H}{n-1} g_{ab}, \quad |\nabla S| = 0, \quad \text{on} \quad \Sigma_c, \]
where \( h_{ab} \) and \( H \) are respectively the second fundamental form and the mean curvature of \( \Sigma_c \). This proves (b).

For item (c) we use the Codazzi equation
\[ Rm_{abc} = \nabla_a h_{bc} - \nabla_b h_{ac}, \quad a, b, c \in \{2, \ldots, n\}. \] \hspace{1cm} (20)
Tracing over \( b \) and \( c \) in (20), we obtain
\[ Rc_{1a} = \nabla_a h_{ab} = \left( 1 - \frac{1}{n-1} \right) \nabla_a H. \] \hspace{1cm} (21)
Then (c) follows since \( R_{c1a} = 0 \).

Now, using (4) we deduce the following expression for the second fundamental form on \( \Sigma_c \)

\[
h_{ab} = \langle \nabla_a e_1, e_b \rangle = \frac{\langle \nabla^2 f \rangle_{ab}}{\langle \nabla f \rangle} = \frac{\frac{\lambda}{2} + \frac{S}{2(n-1)(n-2)}}{\langle \nabla f \rangle} g_{ab} - \frac{R_{cab}}{n-2} = \frac{H}{n-1} g_{ab}. \tag{22}
\]

Hence

\[
R_{cab} = \left[ \frac{\lambda(n-2)}{2} + \frac{S}{2(n-1)} - \frac{H(n-2)}{n-1} \frac{\langle \nabla f \rangle}{n-1} \right] g_{ab}. \tag{23}
\]

Since \( S, H \) and \( |\nabla f| \) are constant on \( \Sigma_c \), we have that \( \Sigma_c \) is Einstein.

Finally, since \( \nabla f \) and the level surface of \( f \) are orthogonal to each other, we can express the metric \( g \) in the form

\[
g = dr^2 + g_{ab}(r, \theta) d\theta^a d\theta^b,
\]

where \( \theta = (\theta^2, \ldots, \theta^n) \) denotes a local coordinates on \( \Sigma_c \), and \( r(x) = \int \frac{df}{|\nabla f|} \). A good survey on level set structures can be found in [23, 24, 22, 25].

Fixing local coordinates system

\[
(x^1, \ldots, x^n) = (r, \theta^1, \ldots, \theta^n),
\]

we can express the second fundamental form of \( \Sigma_c \) in terms of the Christoffel symbol \( \Gamma^1_{ab} \), that is

\[
h_{ab} = -(\partial_r, \nabla_a \partial_b) = -(\partial_r, \Gamma^1_{ab} \partial_r) = -\Gamma^1_{ab} = -g^{11} \left( -\frac{\partial g_{ab}}{\partial r} \right) = \frac{1}{2} \frac{\partial g_{ab}}{\partial r}.
\]

Then

\[
\frac{1}{2} \frac{\partial g_{ab}}{\partial r} = \frac{H}{n-1} g_{ab},
\]

which implies that

\[
g_{ab}(r, \theta) = e^{\frac{3}{n-1} \int_{r_0}^{r} H(s) ds} g_{ab}(r_0, \theta).
\]

Here the level surface \( \{r = r_0\} \) corresponds to \( \Sigma_{c_0} \). Therefore,

\[
g = dr^2 + e^{\frac{3}{n-1} \int_{r_0}^{r} H(s) ds} g_{ab}(r_0, \theta) d\theta^a d\theta^b,
\]

which proves item (e).

**Proof of Theorem 4.** From equation (10) of Lemma 6, we deduce that \( R_c = 0 \), so, from (4) we deduce that \( \nabla^2 f = 0 \). In particular \( |\nabla f| \) is constant. Then either \( f \) is constant, or \( f \) has no critical point at all. Since \( f \) is not constant, this latter case occurs. Now, Lemma 7 implies that \((M^n, g)\) is isometric to the warped product \( \mathbb{R} \times \psi \Sigma^{n-1} \) of the entire real line with a \((n-1)\)-dimensional complete Riemannian manifold.
We now prove that $\psi$ is constant and $\Sigma^{n-1}$ is Ricci flat. Indeed, since $\nabla^2 f = 0$, we have from equation (22) that

$$h_{ab} = \langle \nabla_a e_1, e_b \rangle = \frac{(\nabla^2 f)_{ab}}{|\nabla f|} = 0,$$

which implies that $H = 0$. Then

$$\psi(r)^2 = e^{\frac{2}{n-1} \int_0^r H(s) \, ds} = 1.$$

This fact shows that $M^n = \mathbb{R} \times \Sigma^{n-1}$ is a Riemannian product. On the other hand, from the expression (23), we have that $Rc_{ab} = 0$, i.e., $\Sigma^{n-1}$ is Ricci flat. Furthermore, since $\nabla^2 f = 0$ on $\mathbb{R} \times \Sigma^{n-1}$, we derive that $f(t, x) = at + b$, $a, b \in \mathbb{R}$, $a \neq 0$.

Conflict of interest

The authors declare that there is no conflict of interest.

Data Availability Statements

Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

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