SOBOLEV SPACES FOR THE WEIGHTED $\overline{\partial}$-NEUMANN OPERATOR

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ABSTRACT. We discuss compactness of the $\overline{\partial}$-Neumann operator in the setting of weighted $L^2$-spaces on $\mathbb{C}^n$. In addition we describe an approach to obtain the compactness estimates for the $\overline{\partial}$-Neumann operator. For this purpose we have to define appropriate weighted Sobolev spaces and prove an appropriate Rellich - Kondrachov lemma.

1. INTRODUCTION
Let $\Omega$ be a bounded open set in $\mathbb{R}^n$, and $k$ a nonnegative integer. We denote by $W^k(\Omega)$ the Sobolev space
\[ W^k(\Omega) = \{ f \in L^2(\Omega) : \partial^\alpha f \in L^2(\Omega), |\alpha| \leq k \}, \]
where the derivatives are taken in the sense of distributions and endow the space with the norm
\[ \| f \|_{k,\Omega} = \left( \sum_{|\alpha| \leq k} \int_{\Omega} |\partial^\alpha f|^2 d\lambda \right)^{1/2}, \]
where $\alpha = (\alpha_1, \ldots, \alpha_n)$ is a multiindex, $|\alpha| = \sum_{j=1}^{n} \alpha_j$ and
\[ \partial^\alpha f = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}} f. \]
$W^k(\Omega)$ is a Hilbert space. If $\Omega \subset \mathbb{R}^n$, $n \geq 2$, is a bounded domain with a $C^1$ boundary, the Rellich-Kondrachov lemma says that for $n > 2$ one has
\[ W^1(\Omega) \subset L^r(\Omega), r \in [1, 2n/(n-2)) \]
and that the imbedding is also compact; for $n = 2$ one can take $r \in [1, \infty)$ (see for instance [1]), in particular, there exists a constant $C_r$ such that
\[ \| f \|_r \leq C_r \| f \|_{1,\Omega}, \]
for each $f \in W^1(\Omega)$, where
\[ \| f \|_r = \left( \int_{\Omega} |f|^r d\lambda \right)^{1/r}. \]
Now let $\Omega \subseteq \mathbb{C}^n (\cong \mathbb{R}^{2n})$ be a smoothly bounded pseudoconvex domain. We consider the $\overline{\partial}$-complex
\[ L^2(\Omega) \overset{\overline{\partial}}{\rightarrow} L^2_{(0,1)}(\Omega) \overset{\overline{\partial}}{\rightarrow} \cdots \overset{\overline{\partial}}{\rightarrow} L^2_{(0,n)}(\Omega) \overset{\overline{\partial}}{\rightarrow} 0, \]
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where $L_{(0,q)}^2(\Omega)$ denotes the space of $(0,q)$-forms on $\Omega$ with coefficients in $L^2(\Omega)$. The $\overline{\partial}$-operator on $(0,q)$-forms is given by

$$
(1.3) \overline{\partial} \left( \sum_J J' a_J d\bar{z}_J \right) = \sum_{j=1}^n \sum_J J' \frac{\partial a_J}{\partial z_j} d\bar{z}_j \wedge d\bar{z}_j,
$$

where $\sum'$ means that the sum is only taken over strictly increasing multi-indices $J$. The derivatives are taken in the sense of distributions, and the domain of $\overline{\partial}$ consists of those $(0,q)$-forms for which the right hand side belongs to $L_{(0,q+1)}^2(\Omega)$. So $\overline{\partial}$ is a densely defined closed operator, and therefore has an adjoint operator from $L_{(0,q+1)}^2(\Omega)$ into $L_{(0,q)}^2(\Omega)$ denoted by $\overline{\partial}^\ast$.

We consider the $\overline{\partial}$-complex

$$
(1.4) L_{(0,q+1)}^2(\Omega) \xleftarrow{\overline{\partial}} L_{(0,q)}^2(\Omega) \xrightarrow{\overline{\partial}} L_{(0,q)}^2(\Omega),
$$

for $1 \leq q \leq n-1$.

We remark that a $(0,q+1)$-form $u = \sum_J u_J d\bar{z}_J$ belongs to $C^\infty_{(0,q+1)}(\Omega) \cap \text{dom}(\overline{\partial}^\ast)$ if and only if

$$
(1.5) \sum_{k=1}^n u_{kK} \frac{\partial r}{\partial z_k} = 0
$$
on $b\Omega$ for all $K$ with $|K| = q$, where $r$ is a defining function of $\Omega$ with $|\nabla r(z)| = 1$ on the boundary $b\Omega$. (see for instance [9])

The complex Laplacian $\Box = \overline{\partial} \overline{\partial}^\ast + \overline{\partial}^\ast \overline{\partial}$, defined on the domain

$$
\text{dom}(\Box) = \{ u \in L_{(0,q)}^2(\Omega) : u \in \text{dom}(\overline{\partial}) \cap \text{dom}(\overline{\partial}^\ast), \overline{\partial}u \in \text{dom}(\overline{\partial}^\ast), \overline{\partial}^\ast u \in \text{dom}(\overline{\partial}) \}
$$

acts as an unbounded, densely defined, closed and self-adjoint operator on $L_{(0,q)}^2(\Omega)$, for $1 \leq q \leq n$, which means that $\Box = \Box^\ast$ and $\text{dom}(\Box) = \text{dom}(\Box^\ast)$.

Note that

$$
(1.6) \Box u, u = (\overline{\partial} \overline{\partial}^\ast u + \overline{\partial}^\ast \overline{\partial} u, u) = \|\overline{\partial} u\|^2 + \|\overline{\partial}^\ast u\|^2,
$$

for $u \in \text{dom}(\Box)$.

If $\Omega$ is a smoothly bounded pseudoconvex domain in $\mathbb{C}^n$, the so-called basic estimate says that

$$
(1.7) \|\overline{\partial} u\|^2 + \|\overline{\partial}^\ast u\|^2 \geq c \|u\|^2,
$$

for each $u \in \text{dom}(\overline{\partial}) \cap \text{dom}(\overline{\partial}^\ast)$, $c > 0$.

This estimate implies that $\Box : \text{dom}(\Box) \rightarrow L_{(0,q)}^2(\Omega)$ is bijective and has a bounded inverse

$$
N_{(0,q)} : L_{(0,q)}^2(\Omega) \rightarrow \text{dom}(\Box).
$$

$N_{(0,q)}$ is called $\overline{\partial}$-Neumann operator. In addition

$$
(1.8) \|N_{(0,q)} u\| \leq \frac{1}{c} \|u\|.
$$

Hence the $\overline{\partial}$-Neumann operator $N_{(0,q)}$ is continuous from $L_{(0,q)}^2(\Omega)$ into itself. Compactness of the $\overline{\partial}$-Neumann operator is relevant for a number of circumstances ([9]). From the point of view of the $L^2$-Sobolev theory of the $\overline{\partial}$-Neumann operator, an important application of compactness is that it implies global regularity. Kohn and Nirenberg ([8])
proved that compactness of $N_{(0,q)}$ on $L^2_{(0,q)}(\Omega)$ implies compactness (in particular, continuity) of $N_{(0,q)}$ from the Sobolev spaces $W^s_{(0,q)}(\Omega)$ into itself for all $s \geq 0$, see also [9]. For this result the Rellich - Kondrachov lemma is important, it holds as $\Omega$ is a bounded domain.

The aim of this paper is to study similar properties for the weighted $\mathcal{J}$-Neumann operator on $\mathbb{C}^n$.

Let $\varphi : \mathbb{C}^n \to \mathbb{R}$ be a plurisubharmonic $C^2$-function and let
\[
L^2(\mathbb{C}^n, e^{-\varphi}) = \{ g : \mathbb{C}^n \to \mathbb{C} \text{ measurable} : \| g \|_\varphi^2 = (g, g)_\varphi = \int_{\mathbb{C}^n} |g|^2 e^{-\varphi} d\lambda < \infty \}.
\]

Let $1 \leq q \leq n$ and
\[
f = \sum_{|J|=q} f_J \, d\bar{z}_J,
\]
where the sum is taken only over increasing multiindices $J = (j_1, \ldots, j_q)$ and $d\bar{z}_J = d\bar{z}_{j_1} \wedge \cdots \wedge d\bar{z}_{j_q}$ and $f_J \in L^2(\mathbb{C}^n, e^{-\varphi})$.

We write $f \in L^2_{(0,q)}(\mathbb{C}^n, e^{-\varphi})$ and define
\[
\overline{\partial} f = \sum_{|J|=q} \sum_{j=1}^n \frac{\partial f_J}{\partial \bar{z}_j} d\bar{z}_j \wedge d\bar{z}_J
\]
for $1 \leq q \leq n - 1$ and
\[
\text{dom}(\overline{\partial}) = \{ f \in L^2_{(0,q)}(\mathbb{C}^n, e^{-\varphi}) : \overline{\partial} f \in L^2_{(0,q+1)}(\mathbb{C}^n, e^{-\varphi}) \}.
\]
In this way the $\mathcal{J}$ becomes a densely defined closed operator and its adjoint $\overline{\partial}^* \varphi$ depends on the weight $\varphi$.

We consider the weighted $\mathcal{J}$-complex
\[
L^2_{(0,q-1)}(\mathbb{C}^n, e^{-\varphi}) \xrightarrow{\overline{\partial} \varphi} L^2_{(0,q)}(\mathbb{C}^n, e^{-\varphi}) \xrightarrow{\overline{\partial} \varphi} L^2_{(0,q+1)}(\mathbb{C}^n, e^{-\varphi})
\]
and we set
\[
\square_{\varphi}^{(0,q)} = \overline{\partial} \overline{\partial} \varphi + \overline{\partial}^* \varphi \overline{\partial},
\]
where
\[
\text{dom}(\square_{\varphi}^{(0,q)}) = \{ u \in \text{dom}(\overline{\partial}) \cap \text{dom}(\overline{\partial}^* \varphi) : \overline{\partial} u \in \text{dom}(\overline{\partial} \varphi), \overline{\partial}^* \varphi u \in \text{dom}(\overline{\partial}) \}.
\]
It turns out that $\square_{\varphi}^{(0,q)}$ is a densely defined, non-negative self-adjoint operator, which has a uniquely determined self-adjoint square root $(\square_{\varphi}^{(0,q)})^{1/2}$. The domain of $(\square_{\varphi}^{(0,q)})^{1/2}$ coincides with $\text{dom}(\overline{\partial}) \cap \text{dom}(\overline{\partial}^* \varphi)$, which is also the domain of the corresponding quadratic form
\[
Q_{\varphi}(u, v) := (\overline{\partial} u, \overline{\partial} v)_\varphi + (\overline{\partial}^* \varphi u, \overline{\partial}^* \varphi v)_\varphi,
\]
see for instance [3].

Next we consider the Levi matrix
\[
M_{\varphi} = \left( \frac{\partial^2 \varphi}{\partial z_j \partial \bar{z}_k} \right)_{j,k=1}^n
\]
and suppose that the lowest eigenvalue $\mu_{\varphi}$ of $M_{\varphi}$ satisfies
\[
(1.9) \quad \liminf_{|z| \to \infty} \frac{\mu_{\varphi}(z)}{1} > 0.
\]
implies that \( \Box^{(0,1)}_\varphi \) is injective and that the bottom of the essential spectrum \( \sigma_e(\Box^{(0,1)}_\varphi) \) is positive (Persson’s Theorem), see [5]. Now it follows that \( \Box^{(0,1)}_\varphi \) has a bounded inverse, which we denote by
\[
N^{(0,1)}_\varphi : L^2_{(0,1)}(\mathbb{C}^n, e^{-\varphi}) \longrightarrow L^2_{(0,1)}(\mathbb{C}^n, e^{-\varphi}).
\]
Using the square root of \( N^{(0,1)}_\varphi \) we get the basic estimates
\[
\|u\|_{\varphi}^2 \leq C(\|\overline{\partial} u\|_{\varphi}^2 + \|\overline{\partial}^* \varphi u\|_{\varphi}^2),
\]
for all \( u \in \text{dom}(\overline{\partial}) \cap \text{dom}(\overline{\partial}^* \varphi) \).

Now we will study compactness of the weighted \( \overline{\partial} \)-Neumann operator \( N^{(0,1)}_\varphi \). For this purpose we will use the description of compact subsets in \( L^2 \)-spaces, as it is done in [3] Chapter 11, to derive a sufficient condition for compactness in terms of the weight function. It turns out that compactness of the \( \overline{\partial} \)-Neumann operator \( N^{(0,1)}_\varphi \) is equivalent to compactness of the embedding of a certain complex Sobolev space into \( L^2_{(0,1)}(\mathbb{C}^n, e^{-\varphi}) \).

**Definition 1.1.** Let
\[
\mathcal{W}^{Q}_\varphi = \{ u \in L^2_{(0,1)}(\mathbb{C}^n, e^{-\varphi}) : u \in \text{dom}(\overline{\partial}) \cap \text{dom}(\overline{\partial}^* \varphi) \}
\]
with norm
\[
\|u\|_{Q,\varphi} = (\|\overline{\partial} u\|_{\varphi}^2 + \|\overline{\partial}^* \varphi u\|_{\varphi}^2)^{1/2}.
\]
So \( \mathcal{W}^{Q}_\varphi \) is the form domain of \( Q_\varphi \).

**Theorem 1.2.** Suppose that the weight function \( \varphi \) is plurisubharmonic and that the lowest eigenvalue \( \mu_\varphi \) of the Levi - matrix \( M_\varphi \) satisfies
\[
\lim_{|z| \to \infty} \mu_\varphi(z) = +\infty.
\]
Then the embedding
\[
j_\varphi : \mathcal{W}^{Q}_\varphi \hookrightarrow L^2_{(0,1)}(\mathbb{C}^n, e^{-\varphi})
\]
is compact. Consequently, the \( \overline{\partial} \)-Neumann operator \( N^{(0,1)}_\varphi \) is compact.

This result can be seen as a Rellich Kondrachov lemma for Sobolev spaces defined by complex derivatives. Notice that
\[
N^{(0,1)}_\varphi : L^2_{(0,1)}(\mathbb{C}^n, e^{-\varphi}) \longrightarrow L^2_{(0,1)}(\mathbb{C}^n, e^{-\varphi})
\]
can be written in the form
\[
N^{(0,1)}_\varphi = j_\varphi \circ j^*_\varphi,
\]
where
\[
j^*_\varphi : L^2_{(0,1)}(\mathbb{C}^n, e^{-\varphi}) \longrightarrow \mathcal{W}^{Q}_\varphi
\]
is the adjoint operator to \( j_\varphi \), see [3] Section 6.2, or [3] Section 2.8.

It is now clear that \( N^{(0,1)}_\varphi \) is compact if and only if \( j_\varphi \) is compact.
We have to show that the unit ball in \( \mathcal{W}^{Q}_\varphi \) is relatively compact in \( L^2_{(0,1)}(\mathbb{C}^n, e^{-\varphi}) \). For this purpose we use the characterization of compact subsets in \( L^2 \)-spaces (see [3] Chapter 11).

For \( u \in \mathcal{W}^{Q}_\varphi \) we have
\[
\|\overline{\partial} u\|_{\varphi}^2 + \|\overline{\partial}^* \varphi u\|_{\varphi}^2 \geq (M_\varphi u, u)_\varphi.
\]
This implies
\[
\|\overline{\partial} u\|_{L^2}^2 + \|\partial \varphi u\|_{L^2}^2 \geq \int_{\mathbb{C}^n} \mu_\varphi(z) |u(z)|^2 e^{-\varphi(z)} d\lambda(z) \geq \int_{\mathbb{C}^n \setminus B_R} \mu_\varphi(z) |u(z)|^2 e^{-\varphi(z)} d\lambda(z),
\]
where $B_R$ is the ball with center 0 and radius $R > 0$.

Consequently, assumption (1.12) implies that for each $\epsilon > 0$ there is $R > 0$ such that
\[
(1.15) \quad \int_{\mathbb{C}^n \setminus B_R} |u(z)|^2 e^{-\varphi(z)} d\lambda(z) < \epsilon,
\]
for all $u$ in the unit ball of $\mathcal{W}^{Q_\varphi}$. Also, the map $u \mapsto u|_{B_R}$ is compact from $\mathcal{W}^{Q_\varphi}$ to $L^2_{(0,1)}(\mathbb{B}_R, e^{-\varphi})$, in view of the ellipticity of $\overline{\partial} \oplus \partial^* \varphi$. Together with (1.15), this latter fact shows that the image of a bounded set in $\mathcal{W}^{Q_\varphi}$ is pre-compact in $L^2_{(0,1)}(\mathbb{C}^n, e^{-\varphi})$.

In the following we will describe an approach to obtain the so-called compactness estimates for the $\overline{\partial}$-Neumann operator $N_{(0,1)}^{2(\varphi)}$, where we follow [9], Propostion 4.2. For this purpose we have to define appropriate weighted Sobolev spaces and we need an appropriate Rellich - Kondrachov lemma.

2. Weighted $L^2$-Sobolev spaces

Let $z = (z_1, \ldots, z_n) = (x_1 + iy_1, \ldots, x_n + iy_n) \in \mathbb{C}^n$ and write for a multiindex
\[
\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_2n-1, \gamma_{2n})
\]
and an appropriate function
\[
\partial^\gamma f = \partial_{x_1}^{\gamma_1} \partial_{y_1}^{\gamma_2} \ldots \partial_{x_n}^{\gamma_{2n-1}} \partial_{y_n}^{\gamma_{2n}}.
\]

**Definition 2.1.** We denote by $W^k(\mathbb{C}^n)$ the Sobolev space
\[
W^k(\mathbb{C}^n) = \{ f \in L^2(\mathbb{C}^n) : \partial^\gamma f \in L^2(\mathbb{C}^n), \ |\gamma| \leq k \},
\]
where the derivatives are taken in the sense of distributions and endow the space with the norm
\[
\|f\|_k = \left( \sum_{|\gamma| \leq k} \int_{\mathbb{C}^n} |\partial^\gamma f|^2 d\lambda \right)^{1/2}.
\]

$W^k(\mathbb{C}^n)$ is a Hilbert space. It is well-known that the embedding $i : W^1(\mathbb{C}^n) \hookrightarrow L^2(\mathbb{C}^n)$ fails to be compact. In sake of completeness we recall the easy proof: let $\psi \in C_0^\infty(\mathbb{C}^n)$ be a smooth function with compact support such that $\text{Tr} \psi \subset B_{1/2}(0)$ and $\int_{\mathbb{C}^n} |\psi(z)|^2 d\lambda(z) = 1$. For $k \in \mathbb{N}$ let $\psi_k(z) = \psi(z - \overrightarrow{k})$, where $\overrightarrow{k} = (k, 0, \ldots, 0) \in \mathbb{C}^n$. Then $\text{Tr} \psi_k \subset B_1(\overrightarrow{k})$.
and \((\psi_k)_k\) is a bounded sequence in \(W^1(\mathbb{C}^n)\). Now let \(k, m \in \mathbb{N}\) with \(k \neq m\). Due to the fact that \(\psi_k\) and \(\psi_m\) have non-overlapping supports we have
\[
\|\psi_k - \psi_m\|^2 = \|\psi_k\|^2 + \|\psi_m\|^2 = 2,
\]
and the sequence \((\psi_k)_k\) has no convergent subsequence in \(L^2(\mathbb{C}^n)\).

Let \(U_\varphi : L^2(\mathbb{C}^n) \to L^2(\mathbb{C}^n, e^{-\varphi})\) denote the isometry given by \(U_\varphi(f) = f e^{\varphi/2}\), for \(f \in L^2(\mathbb{C}^n)\). The inverse is given by \(U_{-\varphi}(g) = g e^{-\varphi/2}\), for \(g \in L^2(\mathbb{C}^n, e^{-\varphi})\). The appropriate weighted Sobolev spaces are determined as the images of \(W^k(\mathbb{C}^n)\) under the isometry \(U_\varphi\). In the following we consider only Sobolev spaces of order 1. Let \(f \in W^1(\mathbb{C}^n)\). Then \(f e^{\varphi/2}, (\partial_j f) e^{\varphi/2} \in L^2(\mathbb{C}^n, e^{-\varphi})\), where \(\partial_j f\) denotes all first order derivatives of \(f\) with respect to \(x_j\) and \(y_j\) for \(j = 1, \ldots, n\). Set \(h = f e^{\varphi/2}\). Then
\[
\partial_j h = (\partial_j f) e^{\varphi/2} + \frac{1}{2} f (\partial_j \varphi) e^{\varphi/2}
= (\partial_j f) e^{\varphi/2} + \frac{1}{2} (\partial_j \varphi) h,
\]
which implies \((\partial_j f) e^{\varphi/2} = \partial_j h - \frac{1}{2} (\partial_j \varphi) h\) and
\[
U_\varphi(W^1(\mathbb{C}^n)) = \{ h \in L^2(\mathbb{C}^n, e^{-\varphi}) : \partial_j h - \frac{1}{2} (\partial_j \varphi) h \in L^2(\mathbb{C}^n, e^{-\varphi}), j = 1, \ldots, 2n \}.
\]
For reasons which will become clear later, we denote
\[
W^1_0(\mathbb{C}^n, e^{-\varphi}) := U_\varphi(W^1(\mathbb{C}^n)),
\]
and we endow the space \(W^1_0(\mathbb{C}^n, e^{-\varphi})\) with the norm \(h \mapsto (\|h\|_\varphi^2 + \sum_j \|\partial_j h - \frac{1}{2} (\partial_j \varphi) h\|_\varphi^2)^{1/2}\). in this way \(U_\varphi : W^1(\mathbb{C}^n) \to W^1_0(\mathbb{C}^n, e^{-\varphi})\) is again isometric and we have the following commutative diagram
\[
\begin{array}{ccc}
W^1(\mathbb{C}^n) & \xrightarrow{\iota} & L^2(\mathbb{C}^n) \\
U_\varphi \downarrow & & \downarrow U_\varphi \\
W^1_0(\mathbb{C}^n, e^{-\varphi}) & \xrightarrow{\iota_\varphi} & L^2(\mathbb{C}^n, e^{-\varphi})
\end{array}
\]
where \(\iota_\varphi : W^1_0(\mathbb{C}^n, e^{-\varphi}) \to L^2(\mathbb{C}^n, e^{-\varphi})\) is the canonical embeddings. As \(U_\varphi \iota = \iota_\varphi U_\varphi\) and \(\iota\) fails to be compact, \(\iota_\varphi\) is also not compact.

**Definition 2.2.** Let \(\eta \in \mathbb{R}\). We denote by \(W^1_\eta(\mathbb{C}^n, e^{-\varphi})\) the Sobolev space
\[
W^1_\eta(\mathbb{C}^n, e^{-\varphi}) = \{ h \in L^2(\mathbb{C}^n, e^{-\varphi}) : \partial_j h - \frac{1 + \eta}{2} (\partial_j \varphi) h \in L^2(\mathbb{C}^n, e^{-\varphi}), j = 1, \ldots, 2n \},
\]
endowed with the norm \(h \mapsto (\|h\|_\varphi^2 + \sum_j \|\partial_j h - \frac{1 + \eta}{2} (\partial_j \varphi) h\|_\varphi^2)^{1/2}\). We use the notation
\[
X_j = \frac{\partial}{\partial x_j} - \frac{1 + \eta}{2} \frac{\partial \varphi}{\partial x_j} \quad \text{and} \quad Y_j = \frac{\partial}{\partial y_j} - \frac{1 + \eta}{2} \frac{\partial \varphi}{\partial y_j},
\]
for \(j = 1, \ldots, n\). Then
\[
W^1_\eta(\mathbb{C}^n, e^{-\varphi}) = \{ f \in L^2(\mathbb{C}^n, e^{-\varphi}) : X_j f, Y_j f \in L^2(\mathbb{C}^n, e^{-\varphi}), j = 1, \ldots, n \},
\]
with norm
\[
\|f\|_{\eta, \varphi}^2 = \|f\|_\varphi^2 + \sum_{j=1}^n (\|X_j f\|_\varphi^2 + \|Y_j f\|_\varphi^2).
\]
For suitable weight functions \(\varphi\), we can prove an analogous result to the Rellich Kondrachov lemma.
Theorem 2.3. Suppose that \( \varphi \) is a \( C^2 \)-function satisfying

\[
(2.1) \quad \lim_{|z| \to \infty} (\eta^2 |\nabla \varphi(z)|^2 + (1 + \epsilon)\eta \Delta \varphi(z)) = +\infty,
\]

for some \( \epsilon > 0 \), where

\[
|\nabla \varphi(z)|^2 = \sum_{k=1}^{n} \left( \left| \frac{\partial \varphi}{\partial x_k} \right|^2 + \left| \frac{\partial \varphi}{\partial y_k} \right|^2 \right).
\]

Then the canonical embedding \( \iota_{\varphi,\eta} : W^1_\eta(\mathbb{C}^n, e^{-\varphi}) \hookrightarrow L^2(\mathbb{C}^n, e^{-\varphi}) \) is compact.

Proof. We adapt methods from \([2], [6]\) and \([7]\) and use the general result that an operator between Hilbert spaces is compact if and only if the image of a weakly convergent sequence is strongly convergent.

In addition we remark that \( C^\infty_0(\mathbb{C}^n) \) is dense in all spaces which are involved. For the vector fields \( X_j \) and their adjoints \( X_j^* \) in the weighted space \( L^2(\mathbb{C}^n, e^{-\varphi}) \) we have \( X_j^* = -\frac{\partial}{\partial x_j} + \frac{1-\eta}{2\epsilon} \frac{\partial^2 \varphi}{\partial x_j^2} \) and

\[
(2.2) \quad (X_j + X_j^*)f = -\eta \frac{\partial \varphi}{\partial x_j} f \quad \text{and} \quad [X_j, X_j^*]f = -\eta \frac{\partial^2 \varphi}{\partial x_j^2} f,
\]

for \( f \in C^\infty_0(\mathbb{C}^n) \), and

\[
(2.3) \quad ([X_j, X_j^*]f, f)_\varphi = \|X_j^* f\|^2_\varphi - \|X_j f\|^2_\varphi,
\]

\[
(2.4) \quad \|(X_j + X_j^*)f\|^2_\varphi \leq (1 + 1/\epsilon)\|X_j f\|^2_\varphi + (1 + \epsilon)\|X_j^* f\|^2_\varphi
\]

for each \( \epsilon > 0 \), where we used the inequality

\[
|a + b|^2 \leq |a|^2 + |b|^2 + 1/\epsilon |a|^2 + \epsilon |b|^2.
\]

Similar relations hold for the vector fields \( Y_j \). Now we set

\[
\Psi(z) = \eta^2 |\nabla \varphi(z)|^2 + (1 + \epsilon)\eta \Delta \varphi(z).
\]

By (2.2), (2.3) and (2.4), it follows that

\[
(\Psi f, f)_\varphi \leq (2 + \epsilon + 1/\epsilon) \sum_{j=1}^{n} (\|X_j f\|^2_\varphi + \|Y_j f\|^2_\varphi).
\]

Since \( C^\infty_0(\mathbb{C}^n) \) is dense in \( W^1_\eta(\mathbb{C}^n, e^{-\varphi}) \) by definition, this inequality holds for all \( f \in W^1_\eta(\mathbb{C}^n, e^{-\varphi}) \).

If \( (f_k)_k \) is a sequence in \( W^1_\eta(\mathbb{C}^n, e^{-\varphi}) \) converging weakly to 0, then \( (f_k)_k \) is a bounded sequence in \( W^1_\eta(\mathbb{C}^n, e^{-\varphi}) \) and our assumption implies that

\[
\Psi(z) = \eta^2 |\nabla \varphi(z)|^2 + (1 + \epsilon)\eta \Delta \varphi(z)
\]

is positive in a neighborhood of \( \infty \). So we obtain

\[
\int_{\mathbb{C}^n} |f_k(z)|^2 e^{-\varphi(z)} d\lambda(z) \leq \int_{|z| < R} |f_k(z)|^2 e^{-\varphi(z)} d\lambda(z)
\]

\[
+ \int_{|z| \geq R} \frac{\Psi(z) |f_k(z)|^2}{\inf\{\Psi(z) : |z| \geq R\}} e^{-\varphi(z)} d\lambda(z)
\]

\[
\leq C_{\varphi,R} \|f_k\|^2_{L^2(B(0,R))} + \frac{C_\epsilon \|f_k\|^2_{L^2_\eta}}{\inf\{\Psi(z) : |z| \geq R\}}.
\]
Notice that in the last estimate the expression $\Psi(z)$ plays a similar role as $\mu_\varphi(z)$ in (1.14). It is now easily seen that the sequence $(f_k)_k$ converges also weakly to zero in $W^1(B(0,R))$. Hence the assumption and the fact that the embedding

$$W^1(B(0,R)) \hookrightarrow L^2(B(0,R))$$

is compact (classical Rellich Kondrachov Lemma, see for instance [1]) show that $(f_k)_k$ tends to 0 in $L^2(\mathbb{C}^n,e^{-\varphi})$.

\[ \square \]

**Remark 2.4.** If $\eta = 0$, we get the case corresponding to $W^1(\mathbb{C}^n)$, whereas $\eta = -1$ corresponds to the Sobolev space of all functions $h \in L^2(\mathbb{C}^n,e^{-\varphi})$ such that all derivatives of order 1 satisfy $\partial_j h \in L^2(\mathbb{C}^n,e^{-\varphi})$; in this case the higher order Sobolev spaces are defined as the spaces of all functions $h \in L^2(\mathbb{C}^n,e^{-\varphi})$ such that all derivatives of order $k \geq 1$ belong to $L^2(\mathbb{C}^n,e^{-\varphi})$.

From Theorem 2.3 we can also derive compactness for embeddings in Sobolev spaces without weights. For this purpose we define

**Definition 2.5.** Let $\eta \in \mathbb{R}$. We define

$$W^1_\eta(\mathbb{C}^n,\nabla \varphi) := \{ f \in L^2(\mathbb{C}^n) : \partial_j f - \frac{\partial \varphi}{\partial x_j} f \in L^2(\mathbb{C}^n), j = 1, \ldots, 2n \}.$$  

Then $U_\varphi : W^1_1(\mathbb{C}^n,\nabla \varphi) \rightarrow W^1_1(\mathbb{C}^n,e^{-\varphi})$ is an isometry. We consider the canonical embedding $\iota_\eta : W^1_\eta(\mathbb{C}^n,\nabla \varphi) \hookrightarrow L^2(\mathbb{C}^n)$ and we have the following commutative diagram

$$\begin{array}{ccc}
W^1_\eta(\mathbb{C}^n,\nabla \varphi) & \xrightarrow{\iota_\eta} & L^2(\mathbb{C}^n) \\
\downarrow U_\varphi & & \downarrow U_\varphi \\
W^1_\eta(\mathbb{C}^n,e^{-\varphi}) & \xrightarrow{\iota_\eta,\eta} & L^2(\mathbb{C}^n,e^{-\varphi})
\end{array}$$

Hence the condition (2.1) implies that the canonical embedding $\iota_\eta : W^1_\eta(\mathbb{C}^n,\nabla \varphi) \hookrightarrow L^2(\mathbb{C}^n)$ is compact.

Now we return to compactness of the $\overline{\partial}$-Neumann operator $N^{(0,1)}_\varphi$. We consider the weighted Sobolev space

$$W^1(\mathbb{C}^n,e^{-\varphi}) = \{ h \in L^2(\mathbb{C}^n,e^{-\varphi}) : \partial_j h - (\partial_j \varphi) h \in L^2(\mathbb{C}^n,e^{-\varphi}), j = 1, \ldots, 2n \},$$

and use

$$X_j = \frac{\partial}{\partial x_j} - \frac{\partial \varphi}{\partial x_j} \quad \text{and} \quad Y_j = \frac{\partial}{\partial y_j} - \frac{\partial \varphi}{\partial y_j},$$

for $j = 1, \ldots, n$. Then

$$W^1(\mathbb{C}^n,e^{-\varphi}) = \{ f \in L^2(\mathbb{C}^n,e^{-\varphi}) : X_j f, Y_j f \in L^2(\mathbb{C}^n,e^{-\varphi}), j = 1, \ldots, n \},$$

with norm

$$\| f \|_{W^1(\mathbb{C}^n,e^{-\varphi})}^2 = \| f \|^2_{L^2} + \sum_{j=1}^n (\| X_j f \|_{L^2}^2 + \| Y_j f \|_{L^2}^2).$$

We point out that each continuous linear functional $L$ on $W^1(\mathbb{C}^n,e^{-\varphi})$ is represented by

$$L(f) = \int_{\mathbb{C}^n} f g_0 e^{-\varphi} d\lambda + \sum_{j=1}^n \int_{\mathbb{C}^n} ((X_j f) g_j + (Y_j f) h_j) e^{-\varphi} d\lambda,$$

for $f \in W^1(\mathbb{C}^n,e^{-\varphi})$ and for some $g_0, g_j, h_j \in L^2(\mathbb{C}^n,e^{-\varphi}), j = 1, \ldots, n$. In particular, each function in $L^2(\mathbb{C}^n,e^{-\varphi})$ can be identified with an element of the dual space.
then the condition of the Rellich-Kondrachov lemma (2.5) (see [4] and since for any invertible $\phi$, Remark 2.7. If we suppose that $\phi$ is a $C^2$-function satisfying

$$(2.5) \quad \lim_{|z| \to \infty} (|\nabla z(z)|^2 + (1 + \epsilon) \Delta z(z)) = +\infty,$$

for some $\epsilon > 0$, then the embedding

$$L^2_{(0,1)}(\mathbb{C}^n, e^{-\phi}) \hookrightarrow W^{-1}_{1,0}(\mathbb{C}^n, e^{-\phi})$$

is compact by Theorem 2.3 and duality. So, as in [9], Proposition 4.2 or [4], Proposition 11.20, we get the compactness estimates

**Theorem 2.6.**

Suppose that the weight function $\phi$ satisfies (1.9) and

$$\lim_{|z| \to \infty} (|\nabla z(z)|^2 + (1 + \epsilon) \Delta z(z)) = +\infty,$$

for some $\epsilon > 0$, then the following statements are equivalent.

1. The $\overline{\partial}$-Neumann operator $N^{(0,1)}_\phi$ is a compact operator from $L^2_{(0,1)}(\mathbb{C}^n, e^{-\phi})$ into itself.
2. The embedding of the space $\text{dom}(\overline{\partial}) \cap \text{dom}(\overline{T\partial})$, provided with the graph norm $u \mapsto (|u|^2_2 + |\overline{T\partial} u|^2_2)^{1/2}$, into $L^2_{(0,1)}(\mathbb{C}^n, e^{-\phi})$ is compact.
3. For every positive $\epsilon'$ there exists a constant $C_{\epsilon'}$ such that

$$|u|_2^2 \leq \epsilon'(|\overline{T\partial} u|_2^2 + |\overline{T\partial} u|^2_2) + C_{\epsilon'} |u|_2^{2,-1},$$

for all $u \in \text{dom}(\overline{\partial}) \cap \text{dom}(\overline{T\partial})$.
4. For every positive $\epsilon'$ there exists $R > 0$ such that

$$\int_{\mathbb{C}^n \setminus B_R} |u(z)|^2 e^{-\phi(z)} d\lambda(z) \leq \epsilon'(|\overline{T\partial} u|_2^2 + |\overline{T\partial} u|^2_2)$$

for all $u \in \text{dom}(\overline{\partial}) \cap \text{dom}(\overline{T\partial})$.
5. The operators

$$\overline{T\partial}_\phi N^{(0,1)}_\phi : L^2_{(0,1)}(\mathbb{C}^n, e^{-\phi}) \cap \text{ker}(\overline{\partial}) \to L^2(\mathbb{C}^n, e^{-\phi}) \quad \text{and} \quad \overline{T\partial}_\phi N^{(0,2)}_\phi : L^2_{(0,2)}(\mathbb{C}^n, e^{-\phi}) \cap \text{ker}(\overline{\partial}) \to L^2_{(0,1)}(\mathbb{C}^n, e^{-\phi})$$

are both compact.

**Remark 2.7.** If

$$\lim_{|z| \to \infty} \mu_\phi(z) = +\infty,$$

then the condition of the Rellich-Kondrachov lemma (2.5) is satisfied. This follows from the fact that we have for the trace $\text{tr}(M_\phi)$ of the Levi - matrix

$$\text{tr}(M_\phi) = \frac{1}{4} \Delta \phi,$$

and since for any invertible $(n \times n)$-matrix $T$

$$\text{tr}(M_\phi) = \text{tr}(TM_\phi T^{-1}),$$

it follows that $\text{tr}(M_\phi)$ equals the sum of all eigenvalues of $M_\phi$. We mention that for the weight $\phi(z) = |z|^2$ the $\overline{T\partial}$-Neumann operator fails to be compact (see [4] Chapter 15), but condition (2.5) is satisfied.
In view of Theorem 2.3 it is clear that for any weight satisfying (1.9) and (2.1) for \( \eta \in \mathbb{R}, \eta \neq 0 \), and for some \( \epsilon > 0 \), the restriction of the \( \overline{\partial} \)-Neumann operator \( N^{(0,1)}_{\epsilon} \) to \( W^{1}_{\eta,(0,1)}(\mathbb{C}^n, e^{-\varphi}) \) is compact as an operator from \( W^{1}_{\eta,(0,1)}(\mathbb{C}^n, e^{-\varphi}) \) to \( L^2_{(0,1)}(\mathbb{C}^n, e^{-\varphi}) \).

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