THE DIFFERENTIAL STRUCTURE OF THE BRIESKORN LATTICE

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Abstract. The Brieskorn lattice \( H'' \) of an isolated hypersurface singularity with Milnor number \( \mu \) is a free \( \mathbb{C}\{s\} \)-module of rank \( \mu \) with a differential operator \( t = s^2 \partial_s \). Based on the mixed Hodge structure on the cohomology of the Milnor fibre, M. Saito constructed \( \mathbb{C}\{s\} \)-bases of \( H'' \) for which the matrix of \( t \) has the form \( A = A_0 + A_1 s \). We describe an algorithm to compute the matrices \( A_0 \) and \( A_1 \). They determine the differential structure of the Brieskorn lattice, the spectral pairs and Hodge numbers, and the complex monodromy of the singularity.

1. The Milnor Fibration

Let \( f : (\mathbb{C}^{n+1}, 0) \rightarrow (\mathbb{C}, 0) \) be a holomorphic function germ with an isolated critical point and Milnor number \( \mu = \dim_{\mathbb{C}} \mathbb{C}\{x\}/(\partial(f)) \) where \( \underline{x} = x_0, \ldots, x_n \) is a complex coordinate system of \( (\mathbb{C}^{n+1}, 0) \) and \( \partial = \partial_{x_0}, \ldots, \partial_{x_n} \). By the finite determinacy theorem, we may assume that \( f \in \mathbb{C}[\underline{x}] \). By E.J.N. Looijenga [7, 2.B], for a good representative \( f : X \rightarrow T \) where \( T \subset \mathbb{C} \) is an open disk at the origin, the restriction \( f : X' \rightarrow T' = T \setminus \{0\} \) and \( X' = X \setminus f^{-1}(0) \) is a \( \mathcal{C}^\infty \) fibre bundle unique up to diffeomorphism, the Milnor fibration. By J. Milnor [9, 6.5], the general fibre \( X_t = f^{-1}(t), \ t \in T' \), is homotopy equivalent to a bouquet of \( \mu \) \( n \)-spheres and, in particular, its reduced cohomology is \( \overline{H}^k(X_t) \cong \delta_{k,n} \mathbb{Z}^\mu \) where \( \delta \) is the Kronecker symbol. Since \( T' \) is locally contractible, the \( n \)-th cohomologies \( H(U) = H^n(X_U) \) of \( X_U = f^{-1}(U) \) form a locally free \( \mathbb{Z} \)-sheaf of rank \( \mu \) and \( \mathcal{H} = H \otimes_{\mathbb{Z}} \mathbb{C} \) is a complex local system of dimension \( \mu \). Hence, the sheaf of holomorphic sections \( \mathcal{H} = H \otimes_{\mathbb{Z}} \mathcal{O}_{T'} \) of \( \mathcal{H} \) is a locally free \( \mathcal{O}_{T'} \)-sheaf of rank \( \mu \), the cohomology bundle. By P. Deligne [4, 2.23], there is a natural flat connection \( \nabla : \mathcal{H} \rightarrow \mathcal{H} \otimes_{\mathcal{O}_{T'}} \Omega^1_{T'} \) on \( \mathcal{H} \) with sheaf of flat sections \( H = \ker(\nabla) \), the Gauss-Manin connection.

2. The Monodromy Representation

Let \( t \) be a complex coordinate of \( T \subset \mathbb{C} \), \( i : T' \rightarrow T \) the canonical inclusion, and \( u : T^\infty \rightarrow T' \) the universal covering of \( T' \) defined by \( u(\tau) = \exp(2\pi i \tau) \) for a complex coordinate \( \tau \) of \( T^\infty \subset \mathbb{C} \). Then the covariant derivative \( \nabla_{\partial_t} \) of \( \nabla \) along \( \partial_t \) induces a differential operator \( \partial_t \) on \( i_* \mathcal{H} \) and the pullback \( f^\infty : X^\infty = X' \times_{T'} T^\infty \rightarrow T^\infty \)
is a $C^\infty$ fibre bundle with $X^\infty_t = X_{u(t)}$, the (canonical) Milnor fibre. Since $T^\infty$ is contractible, the $n$-th cohomologies $H(U) = H^n(X^\infty_t)$ of $X^\infty_t = (f^\infty)^{-1}(U)$ form a free $\mathbb{Z}$-sheaf of rank $\mu$ and $u_*H$ is the sheaf of multivalued sections of $H$. Lifting closed paths in $T'$ along sections of $H$ defines the monodromy representation $\pi_1(T', t) \to \text{Aut}(H_t)$ on $H_t$ inducing the monodromy representation $\pi_1(T') \to \text{Aut}(H)$ on the cohomology $H$ of the Milnor fibre. The image $M$ of the counterclockwise generator of $\pi_1(T')$ is called the monodromy operator and fulfills $M(s)(\tau) = s(\tau + 1)$ for $s \in H$. The sheaf $H$ is determined by the monodromy representation up to isomorphism. The following well known theorem is due to E. Brieskorn [2, 0.6] and others.

**Theorem 1** (Monodromy Theorem). The eigenvalues of the monodromy are roots of unity and its Jordan blocks have size at most $(n + 1) \times (n + 1)$ and size at most $n \times n$ for eigenvalue 1.

### 3. The Gauss-Manin Connection

Let $M = M_sM_u$ be the decomposition of $M$ into semisimple part $M_s$ and unipotent part $M_u$ and let $N = -\log M_u$ be the nilpotent part of $M$. Note that $-2\pi iN \in \text{End}_Q(H_Q)$ where $H_Q = H \otimes Z Q$. Let $H_C = \bigoplus_{\lambda} H^\lambda_C$ be the decomposition of $H_C = H \otimes Z C$ into generalized $\lambda$-eigenspaces $H^\lambda_C$ of $M$ and $M^\lambda = M|_{H^\lambda_C}$. Note that $H_Q = H^1_Q \oplus H_Q^\neq 1$ where $H^1_Q \otimes Q C = H^1_C$ and $H_Q^\neq 1 \otimes Q C = \bigoplus_{\lambda \neq 1} H^\lambda_C$. Then there is an inclusion

$$H_C^{e^{-2\pi i a}} \xrightarrow{\psi_\alpha} (i_*\mathcal{H})_0$$

defined by $\psi_\alpha(A) = t^{\alpha+N}A = t^\alpha \exp(N \log(t))$ with image $C^\alpha = \text{im}(\psi_\alpha)$. In particular, the operators $M$ and $N$ act on $C^\alpha$. The following lemma is an immediate consequence of the definition of $\psi_\alpha$.

**Lemma 2.**

1. $t \circ \psi_\alpha = \psi_{\alpha+1}$ and $\partial_t \circ \psi_\alpha = \psi_{\alpha-1} \circ (\alpha + N)$.
2. $t : C^\alpha \to C^{\alpha+1}$ is bijective and $\partial_t : C^\alpha \to C^{\alpha-1}$ is bijective if $\alpha \neq 0$.
3. On $C^\alpha$, $t\partial_t - \alpha = N$ and $\exp(-2\pi it\partial_t) = M^{e^{-2\pi i a}}$.
4. $C^\alpha = \ker(t\partial_t - \alpha)^{n+1}$.

**Definition 3.** We call $G = \bigoplus_{-1 < \alpha \leq 0} C\{t\}[t^{-1}]C^\alpha \subset (i_*\mathcal{H})_0$ the local Gauss-Manin connection.

The local Gauss-Manin connection is a $\mu$-dimensional $C\{t\}[t^{-1}]$-vector space and a regular $C\{t\}[\partial_t]$-module. The generalized $\alpha$-eigenspaces $C^\alpha$ of the operator $t\partial_t$ define the decreasing filtration on $G$ by free $C\{t\}$-modules

$$V^\alpha = \bigoplus_{\alpha \leq \beta < \alpha + 1} C\{t\}C^\beta, \quad V^{> \alpha} = \bigoplus_{\alpha < \beta \leq \alpha + 1} C\{t\}C^\beta$$
of rank $\mu$, the $V$-filtration. In contrast to the $\psi_\alpha$ and $C^\alpha$, the $V^\alpha$ are independent of the coordinate $t$. The $C^\alpha$ define a splitting
\[ C^\alpha \cong V^\alpha / V^\alpha > \alpha = \text{gr}_V G \]
of the $V$-filtration and we denote by lead$_V$ the leading term with respect to this splitting. The ring $C\{t\}$ is a free module of rank 1 over the ring
\[ C\{s\} = \left\{ \sum_{k=0}^\infty a_k s^k \in C[s] \middle| \sum_{k=0}^\infty \frac{a_k}{k!} k^k t^k \in C\{t\} \right\} \]
where $s = \int_0^1 dt$ acts by integration. This fact is generalized by the following lemma [13, 1.3.11].

**Lemma 4.** The action of $s = \partial_t^{-1}$ on $V^\alpha > -1$ extends to a $C\{s\}$-module structure and $V^\alpha > -1$ is a free $C\{s\}$-module of rank $\mu$.

Since $[\partial_t, t] = 1$, $[t, s] = s^2$ and hence
\[ t = s^2 \partial_s, \quad \partial_t t = s \partial_s. \]
We call a free $C\{s\}$-submodule of $V^\alpha > -1$ of rank $\mu$ a $C\{s\}$-lattice and call a $t \partial_t$-invariant $C\{s\}$-lattice saturated. A basis $e$ of a $C\{s\}$-lattice defines a matrix $A = \sum_{k \geq 0} A_k s^k$ of $t$ by $t e = e A$ such that
\[ t \cong A + s^2 \partial_s \]
is the basis representation of $t$.

## 4. The Brieskorn Lattice

The description of cohomology in terms of holomorphic differential forms by the de Rham isomorphism leads to the definition of the Brieskorn lattice
\[ H'' = \Omega_{X,0}^{n+1} / df \wedge d\Omega_{X,0}^{n-1}. \]
By E. Brieskorn [2, 1.5] and M. Sebastiani [15], the Brieskorn lattice is the stalk at 0 of a locally free $O_T$-sheaf $\mathcal{H}''$ of rank $\mu$ with $\mathcal{H}'' |_{T'} \cong \mathcal{H}$ and hence $H'' \subset (i_* \mathcal{H})_0$. The regularity of the Gauss-Manin connection proved by E. Brieskorn [2, 2.2] implies that $H'' \subset G$. B. Malgrange [8, 4.5] improved this result by the following theorem.

**Theorem 5.** $H'' \subset V^{-1}$.

By E. Brieskorn [2, 1.5], the Leray residue formula can be used to express the action of $\partial_t$ in terms of differential forms by $\partial_t [df \wedge \omega] = [d\omega]$. In particular, $s H'' \subset H''$ and
\[ H'' / s H'' \cong \Omega_{X,0}^{n+1} / df \wedge \Omega_{X,0}^{n} \cong C\{\omega\} / \langle \Omega(f) \rangle. \]
Since the $V^\alpha > -1$ is a $C\{s\}$-module, theorem 5 implies that $H''$ is a free $C\{s\}$-module of rank $\mu$ and the action of $s$ can be expressed in terms of differential forms by
\[ s [d\omega] = [df \wedge \omega]. \]
For computational purposes, we may restrict our attention to the completion of the Brieskorn lattice. E. Brieskorn [2, 3.4] proved the following theorem.

**Theorem 6.** The $m_{X,0}$- and $m_{T,0}$-adic topologies on $H^n$ coincide.

While the proof of theorem [6] is highly non-trivial, the analogous statement for the $C\{\{s\}\}$-structure of the Brieskorn lattice is quite elementary [13, 1.5.4].

**Proposition 7.** The $m_{X,0}$- and $m_{C\{\{s\}\}}$-adic topologies on $H^n$ coincide.

We call the completion $\hat{H}^n$ of $H^n$ the formal Brieskorn lattice. Since completion is faithfully flat, $\hat{H}^n$ is a free $C[s]$-module of rank $\mu$ with a differential operator $t = s^2\partial_s$. The equality $[\partial(f)\overline{y}dx] = s[\partial(g)dx]$ motivates to consider the differential relation $\partial(f) - s\partial$. It is not difficult to prove that it defines the formal Brieskorn lattice as a quotient of $C[s,\overline{x}]$ [13, 1.5.6].

**Proposition 8.**

$$C[s,\overline{x}] \xrightarrow{\pi_H} C[s,\overline{x}]/\langle \partial(f) - s\partial \rangle C[s,\overline{x}] \cong C[s] : \hat{H}^n.$$  

Proposition 8 is the starting point for an algorithmic approach to the local Gauss-Manin connection. Let $<_s$ be a local degree ordering on $C[s]$ such that $\deg(\overline{x}) < 0$ and $\deg(\overline{y}) = -\deg(\overline{x}) > 0$. One can compute a polynomial standard basis $\overline{g}$ of the Jacobian ideal $\langle \partial(f) \rangle$ and a polynomial transformation matrix $\overline{B} = (\overline{b}^j)$ such that $\overline{g} = \overline{\partial(f)} \overline{B}$. By Nakayama’s lemma, $\overline{m} = (\overline{x}^\beta)_{\overline{x}^\beta \in \langle \text{lead}(\overline{g}) \rangle}$ represents a $C[s]$-basis $[\overline{m}]$ of $\hat{H}^n$. Let $<_s$ be the local degree ordering on $C[s]$ and let $<= (<_s, <_\overline{x})$ be the block ordering of $<_s$ and $<_\overline{x}$ on $C[s,\overline{x}]$.

**Definition 9.**

1. $h = \left((g_j - s\overline{\partial(f)}\overline{x}^\beta)_{j,\beta}\right)$.
2. $\deg(s) = \min \deg(m_j) + 2 \min \deg(x) < 0$.
3. $N = (N_K)_{K \geq 0}$ with $N_K = K \deg(s) - 2 \min \deg(x)$.
4. $V = (V_K)_{K \geq 0}$ with $V_K = \{p \in C[s,\overline{x}] | \deg(p) < N_K \} + \langle s \rangle^K \subset C[s,\overline{x}]$.

Since $\hat{H}^n$ is a free $C[s]$-module, $h$ is a standard basis of the $C[s]$-module $\langle \partial(f) - s\partial \rangle C[s,\overline{x}]$. The following lemma is technical but not very deep and can be generalized to formal differential deformations [13, 2.2.10].

**Lemma 10.** $V = (V_K)_{K \geq 0}$ is a basis of the $\langle s,\overline{x} \rangle$-adic topology of $C[s,\overline{x}]$ with $\pi_H(V_K) = \langle s \rangle^K \hat{H}^n$. If $s^a \text{lead}(h_{j,\beta}) \in V_K$ then $s^a h_{j,\beta} \in V_K$.

Lemma 10 leads to a normal form algorithm for the Brieskorn lattice [13, 2.2.12]. It computes a normal form with respect to $h$ and hence
the $[m]$-basis representation in $H''$. The normal form computation up to a given degree can be continued up to any higher degree without additional computational effort. The normal form algorithm for the Brieskorn lattice is a special case of a modification of Buchberger’s normal form algorithm [3] for power series rings where termination is replaced by adic convergence [13, 2.1.19].

5. Mixed Hodge Structure

By lemma 2 there is a $C$-isomorphism

$$H_C = \bigoplus_{-1 < \alpha \leq 0} H_C^{e^{-2\pi i \alpha}} \xrightarrow{\psi} \bigoplus_{-1 < \alpha \leq 0} C^\alpha \cong V^{>-1}/sV^{>-1}$$

defined by $\psi = \bigoplus_{-1 < \alpha \leq 0} \psi_\alpha$ and the monodromy $M$ on $H_C$ corresponds to $\exp(-2\pi i t \partial_t)$ on $\bigoplus_{-1 < \alpha \leq 0} C^\alpha$.

The Hodge filtration $F = (F_k)_{k \in \mathbb{Z}}$ on $V^{>-1}$ defined by J. Scherk and J.H.M. Steenbrink [14] is the increasing filtration by the free $C\{\{s\}\}$-modules $F_k = F_n-k = (s^{-k} H^n) \cap V^{>-1}$ of rank $\mu$. Via the splitting $C^\alpha \cong \text{gr}_V V^{>-1}$, the Hodge filtration induces an increasing Hodge filtration $FC^\alpha$ by $C$-vectorspaces on $C^\alpha$ and, via $\psi$, on $H_C$. The nilpotent operator $-2\pi i N \in \text{End}_Q(H_Q)$ defines an increasing weight filtration $W = (W_k)_{k \in \mathbb{Z}}$ centered at $n$ resp. $n+1$ on $H_{Q^1}$ resp. $H_{Q^1}^1$.

Theorem 11. The weight filtration $W$ on $H_Q$ and the Hodge filtration $F$ on $H_C$ define a mixed Hodge structure on the cohomology $H$ of the Milnor fibre and the operator $N$ is a morphism of mixed Hodge structures of type $(-1, -1)$.

The mixed Hodge structure on the cohomology of the Milnor fibre was discovered by J.H.M. Steenbrink [16] and described in terms of the Brieskorn lattice by A.N. Varchenko [17].

The nilpotent operator $N$ on $C^\alpha$ defines an increasing weight filtration $W = (W_k)_{k \in \mathbb{Z}}$ centered at $n$ on $C^\alpha$. By definition $N$ commutes with $\psi_\alpha$ and hence

$$\psi_\alpha(W H_C^{e^{-2\pi i \alpha}}) = \begin{cases} WC^\alpha, & \alpha \notin \mathbb{Z}, \\ W[-1] C^\alpha, & \alpha \in \mathbb{Z}. \end{cases}$$

The weight filtration $W = \bigoplus_{-1 < \alpha \leq 0} C\{\{s\}\} WC^\alpha$ on $V^{>-1}$ by free $C\{\{s\}\}$-modules induces $WC^\alpha$ via the splitting $C^\alpha \cong \text{gr}_V V^{>-1}$.

The spectral pairs are those pairs $(\alpha, l) \in \mathbb{Q} \times \mathbb{Z}$ with positive multiplicity

$$d^\alpha_l = \dim_C \text{gr}_l^W \text{gr}_V^\alpha \text{gr}_0^F V^{>-1}.$$  

Via the isomorphism $\psi$, they correspond to the Hodge numbers

$$h^{p, l-p}_\lambda = \dim_C \text{gr}_F^p \text{gr}_l^W H^\lambda_C.$$
by $d_l^{\alpha+p} = h_{e^{-2\pi i \alpha}}^{n-p,l-n+p}$ for $-1 < \alpha < 0$ and $d_l^p = h_1^{n-p,l+1-n+p}$ and inherit the symmetry properties

$$d_l^{\alpha} = d_l^{2n-l-1-\alpha}, \quad d_l^{n} = d_{2n-l}^{n-l}, \quad d_l^{\alpha} = d_{2n-l}^{n-1-\alpha}$$

from the mixed Hodge structure. The spectral numbers are those numbers $\alpha \in \mathbb{Q}$ with positive multiplicity

$$d^\alpha = \dim \text{gr}_V \text{gr}_0^F V^{>1} = \sum_{l \in \mathbb{Z}} d_l^\alpha$$

and have the symmetry property $d^\alpha = d^{n-1-\alpha}$.

### 6. M. Saito’s Basis

By P. Deligne [5, 1.2.8], a morphism of mixed Hodge structures is strict for the Hodge filtration. In particular, by theorem 11, $N$ is strict for the Hodge filtration on $H_c$ and on $\text{gr}_V V^{>1}$. Hence, there is a direct sum decomposition $F_k C^\alpha = \bigoplus_{j \leq k} C^{\alpha,j}$ such that $N(C^{\alpha,k}) \subset C^{\alpha,k+1}$, and $sC^{\alpha,k} \subset C^{\alpha+1,k-1}$. By definition of the Hodge filtration,

$$\text{lead}_V(H'') = \sum_{\alpha \in \mathbb{Q}} \sum_{k \leq 0} C\{s\} C^{\alpha,k} = \bigoplus_{\alpha \in \mathbb{Q}} C\{s\} G^\alpha$$

where $G^\alpha = C^{\alpha,0}$. Let $<\mathbb{Q} \times \mathbb{Z} = (\mathbb{Q}, >\mathbb{Z})$ be the block ordering of $>\mathbb{Q}$ and $>\mathbb{Z}$ on the index set $\mathbb{Q} \times \mathbb{Z}$. Then the Hodge filtration defines a refinement of the $V$-filtration on $V^{>1}$ by free $C\{s\}$-modules $V^{\alpha,k} = F_k C^\alpha \oplus V^{>\alpha}$ of rank $\mu$ and the $C^{\alpha,k}$ define a splitting of this refined filtration compatible with $s$. We call the refinement the Hodge refinement and the splitting a Hodge splitting. The following lemma follows essentially from the fact that $C\{s\}$ is a discrete valuation ring [13, 1.10.5, 1.10.10].

**Lemma 12.** Let $H$ be a $C\{s\}$-lattice and $C^{\alpha,k}$ a splitting of a refinement of the $V$-filtration compatible with $s$. Then a minimal standard basis of $H$ is a $C\{s\}$-basis and there is a reduced minimal standard basis of $H$.

In particular, there is a reduced minimal standard basis of $H''$ for a Hodge splitting. The following proposition follows essentially from lemma 2.3 [13, 1.10.12].

**Proposition 13.** Let $h$ be a reduced minimal standard basis of $H''$ for a Hodge splitting. Then the $h$-matrix $A$ of $t$ has degree 1. In particular,

$$(H'', t) \xrightarrow{h} (C\{s\})^\mu, A_0 + A_1 s + s^2 \partial_s$$

is an isomorphism. Moreover, $A_1$ is semisimple with eigenvalues the spectral numbers of $f$ added by 1 and $\text{gr}_V(A_0)$ can be identified with $N$. 
Note that the matrices $A_0$ and $A_1$ in proposition 13 determine the differential structure of the Brieskorn lattice. M. Saito [10] first constructed a $\mathbb{C}\{\{s\}\}$-basis of $H''$ as in proposition 13 without calling it a reduced minimal standard basis.

7. The Algorithm

We describe an algorithm to compute $A_0$ and $A_1$ as in proposition 13. This algorithm can be simplified to compute the complex monodromy, the spectral numbers, or the spectral pairs only [13].

The normal form algorithm for the Brieskorn lattice in section 4 computes the $[m]$-matrix $A = \sum_{k \geq 0} A_k s^k$ of $t$ defined by $t[m] = [f[m]] = [m]A$ up to any degree. We identify the columns of a matrix $H$ with the generators of a submodule $\langle H \rangle \subset \mathbb{C}[s]^\mu$ and denote by $E$ the unit matrix. Then $\langle E \rangle$ is the $[m]$-basis representation of $\hat{H}''$. Hence, the following two statements hold for $h$.

(H$_h$) One can compute $\kappa \geq 0$ and a $\mu \times \mu$-matrix $H$ with coefficients in $\mathbb{C}[s]$ of degree at most $\kappa$ such that $\langle H \rangle$ is the $h$-basis representation of $\hat{H}''$ and $s^\kappa \langle E \rangle \subset \langle H \rangle$.

(A$_h$) One can compute the $h$-matrix $A$ of $t$ up to any degree.

Step by step, we improve the $\mathbb{C}[s]$-basis $h$ and show that (H$_h$) and (A$_h$) hold. After the last step, $A_0$ and $A_1$ as in proposition 13 can be computed by a basis transformation of $A$ to a reduced minimal standard basis of $\langle H \rangle$ up to a certain degree bound.

We call the canonical projection $\text{jet}_k : \mathbb{C}[s] \rightarrow \bigoplus_{j=0}^k \mathbb{C}s^j$ the $k$-jet. Let the monomial ordering on $\mathbb{C}[s]^\mu = \mathbb{C}[s] \otimes_{\mathbb{C}} \mathbb{C}^\mu$ be the block ordering $\leq (\langle s, >_\mu)$ of the local degree ordering $<_s$ on $\mathbb{C}[s]$ and the inverse ordering $>_\mu$ on the indices of the basis elements of $\mathbb{C}^\mu$.

7.1. The Saturation of $H''$. In this step, we show that (H$_h$) and (A$_h$) hold for a $\mathbb{C}[s]$-basis $h$ of a saturated $\mathbb{C}[s]$-lattice.

The increasing sequence of $\mathbb{C}[s]$-lattices defined by

$\hat{H}''_0 = \hat{H}''$, $\hat{H}''_{k+1} = s\hat{H}''_k + t\hat{H}''_k \subset \hat{H}''$

is stationary since $\hat{H}''$ is noetherian. Hence, the saturation $\hat{H}''_\infty = \bigcup_{k \geq 0} \hat{H}''_k$ of $\hat{H}''$ is a saturated $\mathbb{C}[s]$-lattice. The $[m]$-basis representation $\langle H_k \rangle$ of $\hat{H}''$ can be computed by

$H_0 = Q_{-1} = E$, $Q_k = (\text{jet}_k(A) + s^2 \partial_s)Q_{k-1}$, $H_{k+1} = (sH_k|Q_k)$.

We successively compute the $H_k$ and check in each step if $\langle Q_k \rangle \subset \langle H_k \rangle$ by a standard basis and normal form computation. If $\langle Q_k \rangle \subset \langle H_k \rangle$ then we stop the computation and set $\kappa = k$ and $H_\infty = H_k$. Then $\langle H_\infty \rangle$ is the $[m]$-basis representation of $\hat{H}''$. We replace $H_\infty$ by a minimal standard basis of $\langle H_k \rangle$. Then $h = s^{-\kappa}hH_\infty$ is a $\mathbb{C}[s]$-basis of a saturated $\mathbb{C}[s]$-lattice. By a normal form computation with respect to
Let \( H_{\infty} \) up to degree \( \kappa \), we compute the \( h \)-basis representation \( \langle H_{\infty}^{-1}s^{\kappa}E \rangle = \langle \text{jet}_k(H_{\infty}^{-1}s^{\kappa}E) \rangle \) of \( \tilde{H}^\nu \). Since \( (H_{\infty}) \subset \langle E \rangle \), \( s^{\kappa}(E) \subset \langle H_{\infty}^{-1}s^{\kappa}E \rangle \). By a normal form computation with respect to \( H_{\infty} \) up to degree \( \kappa + k \), one can compute the \( k \)-jet
\[
\text{jet}_k(H_{\infty}^{-1}(A - \text{\kappa}sE + s^2\partial_s)H_{\infty}) = \text{jet}_k(H_{\infty}^{-1}(\text{\kappa+k}(A - \text{\kappa}sE) + s^2\partial_s)H_{\infty})
\]
of the \( h \)-matrix of \( t \) for any \( k \geq 0 \).

### 7.2. The V-Filtration

In this step, we show that \( (H_{\omega}) \) and \( (A_{\omega}) \) hold for a \( <\Omega \)-increasingly ordered \( \mathbb{C}[s] \)-basis \( \nu \) of a \( \hat{V}^\alpha \) compatible with the direct sum decomposition \( \hat{V}^\alpha/s\hat{V}^\alpha \cong \bigoplus_{\alpha \leq \beta < \alpha + 1} C^\beta \).

Since \( h \) is a \( \mathbb{C}[s] \)-basis of a saturated \( \mathbb{C}[\{s\}] \)-lattice, \( A_0 = 0 \) and, by theorem \( \parallel \) the eigenvalues of \( A_1 \) are rational. In order to compute the eigenvalues of \( A_1 \), we transform \( A_1 \) to Hessenberg form and factorize the characteristic polynomials of its blocks. Then we compute a constant \( \mathbb{C}[s] \)-basis transformation such that \( A_1 = \text{diag}(\alpha_1, \ldots, \alpha_\mu) + N \) with \( \alpha_1 \leq \cdots \leq \alpha_\mu \) where \( \text{diag}(\alpha_1, \ldots, \alpha_\mu) \) denotes the diagonal matrix with entries \( \alpha_1, \ldots, \alpha_\mu \). If \( \alpha_\mu - \alpha_1 < 1 \) then \( v = h \) is a \( <\Omega \)-increasingly ordered \( \mathbb{C}[s] \)-basis \( \nu \) of a \( \hat{V}^\alpha \) compatible with the direct sum decomposition \( \hat{V}^\alpha/s\hat{V}^\alpha \cong \bigoplus_{\alpha \leq \beta < \alpha + 1} C^\beta \). If \( \alpha_\mu - \alpha_1 \geq 1 \) then we proceed as follows. Let
\[
A = \begin{pmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \end{pmatrix}
\]
such that \( A_0 = 0, A_{1,2} = 0, A_{2,1} = 0 \), and the eigenvalues of \( A_{1,1} \) are the eigenvalues \( \alpha \) of \( A_1 \) with \( \alpha < \alpha_1 + 1 \). Then the \( \mathbb{C}[s][s^{-1}] \)-basis transformation
\[
H \mapsto \begin{pmatrix} \frac{1}{s} & 0 \\ 0 & 1 \end{pmatrix} H, \quad A \mapsto \begin{pmatrix} \frac{1}{s} & 0 \\ 0 & 1 \end{pmatrix} (A + s^2\partial_s) \begin{pmatrix} s & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} A_{1,1} + s & 1/s A_{1,2} \\ s A_{2,1} & A_{2,2} \end{pmatrix}
\]
decreases \( \alpha_\mu - \alpha_1 \) and the degree up to which \( A \) is computed by \( 1 \) and increases \( \kappa \) by \( 1 \). After at most \( n \) such transformations, \( \alpha_\mu - \alpha_1 < 1 \).

### 7.3. The Canonical V-Splitting

In this step, we show that \( (H_{\omega}) \) and \( (A_{\omega}) \) hold for a \( <\Omega \)-increasingly ordered \( C \)-basis \( \nu \) of a direct sum \( \bigoplus_{\alpha \leq \beta < \alpha + 1} C^\beta \) compatible with the direct sum.

Let \( \nu \) be the image of \( [\nu] \) under the splitting \( \hat{V}^\alpha/s\hat{V}^\alpha \cong \bigoplus_{\alpha \leq \beta < \alpha + 1} C^\beta \). By Nakayama’s lemma, \( \nu \) is a \( C \)-basis of \( \bigoplus_{\alpha \leq \beta < \alpha + 1} C^\beta \) compatible with the direct sum. The eigenvalues of the commutator \([\cdot, A_1] \in \text{End}_C(C^{\nu})\) are the differences of the eigenvalues of \( A_1 \). Since \( \alpha_\mu - \alpha_1 < 1 \), \([\cdot, A_1] - k \in \text{GL}_{\mu^2}(C) \) for \( k \geq 1 \). Let \( U = \sum_{j=0}^{\infty} U_j s^j \) be the \( \mathbb{C}[s] \)-basis transformation defined by \( \nu = v U \). Then \( U_0 = E \) and \( U A_1 s = (A + s^2\partial_s)U \) or equivalently
\[
U_k = ([\cdot, A_1] - k)^{-1} \sum_{j=0}^{k-1} A_{k-j+1} U_j
\]
for $k \geq 1$ and hence one can compute $U$ up to any degree. Since $U_0 = E$ and $\kappa \geq 0$, jet$_{\kappa}(U)$ is a minimal standard basis of $\langle E \rangle$. By a normal form computation with respect to $U$ up to degree $\kappa$, we compute the $\mathcal{C}$-basis representation $\langle U^{-1}H \rangle = \langle \text{jet}_\kappa(\text{jet}_\kappa(U)^{-1}H) \rangle$ of $\hat{H}''$ and $A_1$ is the $\mathcal{C}$-matrix of $t$.

7.4. A Hodge Splitting.
In this step, we show that $(H_f)$ and $(A_f)$ hold for a decreasingly ordered $\mathcal{C}$-basis $f$ of a direct sum $\bigoplus_{\alpha \leq \beta < \alpha + 1} \bigoplus_{k \in \mathbb{Z}} \mathcal{C}^{\beta,k}$ compatible with the direct sum and that one can compute $A_0$ and $A_1$ as in proposition 13.

We compute a standard basis of $H$ up to degree $\kappa$ in order to compute the $\mathcal{C}$-basis representation of the Hodge filtration $F$. The nilpotent part of $A_1$ is the $\mathcal{C}$-basis representation of $N$. By computing images and quotients of $\mathcal{C}$-vectorspaces, we compute the $\mathcal{C}$-basis representation of a Hodge splitting $F_k \mathcal{C}^\beta = \bigoplus_{j \leq k} \mathcal{C}^{\beta,j}$. Then we compute a constant $\mathbb{C}[s]$-basis transformation $f = \mathcal{C}U$ such that $f$ is a decreasingly ordered $\mathcal{C}$-basis of the direct sum $\bigoplus_{\alpha \leq \beta < \alpha + 1} \bigoplus_{k \in \mathbb{Z}} \mathcal{C}^{\beta,k}$ compatible with the direct sum.

We replace $H$ by a reduced minimal standard basis of $\langle H \rangle$ up to degree $\kappa + 1$. By a normal form computation with respect to $H$ up to degree $\kappa + 1$, we compute the 1-jet

$$\text{jet}_1(H^{-1}(A + s^2 \partial_s)H) = \text{jet}_1((\text{jet}_{\kappa+1}(H))^{-1}(\text{jet}_{\kappa+1}(A) + s^2 \partial_s)\text{jet}_{\kappa+1}(H))$$

of the $\mathcal{C}H$-matrix $A$ of $t$ in order to compute $A_0$ and $A_1$ as in proposition 10.

8. An Example
The algorithm in section 7 is implemented in the computer algebra system SINGULAR [6] in the procedure tmatrix in the library gaussman.lib [12]. In an example SINGULAR session, we compute the differential structure of the Brieskorn lattice of the singularity of type $T_{2,5,5}$ defined by the polynomial $f = x^2y^2 + x^5 + y^5$.

First, we load the SINGULAR library gaussman.lib:

```plaintext
> LIB "gaussman.lib";
```

Then, we define the local ring $R = \mathbb{Q}[x, y]_{(x, y)}$ with the local degree ordering $\mathcal{d}s$ as monomial ordering and the polynomial $f = x^2y^2 + x^5 + y^5 \in R$:

```plaintext
> ring R=0,(x,y),ds;
> poly f=x2y2+x5+y5;
```

Finally, we compute $A_0$ and $A_1$ as in proposition 10:

```plaintext
> list A=tmatrix(f);
```
The result is the list \( A = A[1], A[2] \) such that \( A[i + 1] = A[i] \) and
\[
A_0 = \begin{pmatrix}
0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 \\
1 & 0 & \cdots & 0
\end{pmatrix}, \quad A_1 = \text{diag}(\frac{1}{2}, 10, 10, 10, 10, 10, 10, 10, 10, 10, 10, 10, 10, 10, 10)
\]

By proposition \( 10 \), \( (H'', t) \cong (C\{s\}^\mu, A_0 + sA_1 + s^2\partial_s) \) and the spectral pairs are \((-\frac{1}{2}, 2), (-\frac{3}{10}, 1)^2, (-\frac{1}{10}, 1)^2, (0, 1), (\frac{1}{10}, 1)^2, (\frac{1}{2}, 0)\).

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