Circular colorings, orientations, and weighted digraphs

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Abstract

In this paper we prove that if a weighted symmetric digraph \((\vec{G}, c)\) has a mapping \(T : E(\vec{G}) \rightarrow \{0, 1\}\) with \(T(xy) + T(yx) = 1\) for all arcs \(xy\) in \(\vec{G}\) such that for each dicycle \(C\) satisfying \(0 < |C|c(\text{mod } r) < \max_{xy \in E(\vec{G})} c(xy) + c(yx)\) we have \(|C|c/|C|T \leq r\), then \((\vec{G}, c)\) has a circular \(r\)-coloring. Our result generalizes the work of Zhu (J. Comb. Theory, Ser. B, 86 (2002) 109-113) concerning the \((k, d)\)-coloring of a graph, and thus is also a generalization of a corresponding result of Tuza (J. Comb. Theory, Ser. B, 55 (1992) 236-243). Our result also strengthens a result of Goddyn, Tarsi and Zhang (J. Graph Theory 28 (1998) 155-161) concerning the relation between orientation and the \((k, d)\)-coloring of a graph.

1 Introduction

A \textit{weighted digraph} is denoted by \((\vec{G}, c)\), where \(\vec{G}\) is a digraph, and \(c\) is a function which assigns to each arc of \(\vec{G}\) a positive real number. For simplicity of notation, the arc \((u, v)\) is written as \(uv\), and \(c(uv)\) is written as \(c_{uv}\). If arcs \(uv, vu\) both exist and do not exist for all vertices \(u, v\) in \(\vec{G}\) then \((\vec{G}, c)\) is said to be a \textit{weighted symmetric} digraph. A \textit{dicycle} \(C\) of \(\vec{G}\) is a closed directed walk \((v_1, \ldots, v_{k+1})\) in which \(v_1, \ldots, v_k\) are distinct vertices, \(v_1 = v_{k+1}\), and \(v_iv_{i+1} (i = 1, \ldots, k)\) are arcs. A \textit{dipath} \(P\) of \(\vec{G}\) is a directed walk \((v_1, \ldots, v_k)\) in which \(v_1, \ldots, v_k\) are distinct vertices, and \(v_iv_{i+1} (i = 1, \ldots, k-1)\) are arcs. For a graph \(G\) equipped with an orientation \(\omega\), and a cycle

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C of G with a chosen direction of traversal (each cycle has two different directions for traversal), let $|C^+_ω|$ denote the number of edges of C whose direction in ω coincide with the direction of the traversal. Let $|C^-_ω|$ denote the value $|C| - |C^+_ω|$ where $|C|$ is the length of C. Define $τ(C, ω) = \max\{|C|/|C^+_ω|, |C|/|C^-_ω|\}$.

For reals x and r, let $x(\text{mod } r)$ be the unique value $t ∈ [0, r)$ such that $t \equiv x(\text{mod } r)$. A breaker function of a weighted symmetric digraph $(\vec{G}, c)$ is a function $T : E(\vec{G}) \rightarrow \{0, 1\}$ such that $T(xy) + T(yx) = 1$ for each arc $xy ∈ E(\vec{G})$. Henceforth, $T(xy)$ is written as $T_{xy}$. If C is a dicycle of the weighted digraph $(\vec{G}, c)$ having a breaker function $T$, then the two values $\sum_{uv ∈ E(C)} T_{uv}$ and $\sum_{uv ∈ E(C)} c_{uv}$ are denoted by $|C|_T$ and $|C|_c$, respectively, where $E(C)$ is the collection of all arcs in C. If $P$ is a dipath of $(\vec{G}, c)$ then $|P|_T$ and $|P|_c$ are defined in the same way.

Let $G$ be an undirected graph. A symmetric digraph $\vec{G}$ is said to be derived from $G$ if $V(\vec{G}) = V(G)$, and if $xy$ is an edge of $G$ then $xy$ and $yx$ are arcs of $\vec{G}$ and vice versa. If digraph $\vec{G}$ is derived from $G$, then $G$ is called the underlying graph of $\vec{G}$.

Suppose $k ≥ 2d ≥ 1$ are positive integers. A $(k, d)$-coloring of a graph G is a mapping $f : V(G) \rightarrow \{0, 1, \ldots, k - 1\}$ such that for any edge $xy$ of $G$, $d ≤ |f(x) - f(y)| ≤ k - d$. The circular chromatic number $χ_c(G)$ of G is defined as

$$χ_c(G) = \inf\{k/d : G \text{ has a } (k, d)\text{-colorable }\}.$$  

For a real number $r ≥ 1$, a circular $r$-coloring of a graph G is a function $f : V(G) \rightarrow [0, r)$ such that for any edge $xy$ of $G$, $1 ≤ |f(x) - f(y)| ≤ r - 1$. It is known $[10, 12]$ that

$$χ_c(G) = \inf\{r : G \text{ has a circular } r\text{-coloring }\}.$$  

It is clear that G has a $(k, d)$-coloring if and only if G has a circular $k/d$-coloring.

For a positive real $p$, let $S^p$ denote a circle with perimeter $p$ centered at the origin of $R^2$. In the obvious way, we can identify the circle $S^p$ with the interval $[0, p)$. For $x, y ∈ S^p$, let $d_p(x, y)$ denote the length of the arc on $S^p$ from $x$ to $y$ in the clockwise direction if $x \neq y$, and let $d_p(x, y) = 0$ if $x = y$. A circular $p$-coloring of a weighted digraph $(\vec{G}, c)$ is a function $φ : V(\vec{G}) \rightarrow S^p$ such that for each arc $uv$ of $\vec{G}$, $d_p(φ(u), φ(v)) ≥ c_{uv}$. The circular chromatic number $χ_c(\vec{G}, c)$ of a weighted digraph $(\vec{G}, c)$, recently introduced by Mohar $[6]$, is defined as
\[ \chi_c(\vec{G}, c) = \inf \{p : (\vec{G}, c) \text{ has a circular } p\text{-coloring} \}. \]

It is clear that \( \chi_c(G) = \chi_c(\vec{G}, 1) \), where \( \vec{G} \) is a digraph derived from \( G \) and \( 1(xy) = 1 \) for each arc \( xy \) of \( \vec{G} \).

As we can see in [8, 10, 12], the parameter \( \chi_c(G) \) is a refinement of \( \chi(G) \). The concept of circular chromatic number has attracted considerable attention in the past decade (see [10, 12] for a survey of research in this area). Further, the parameter \( \chi_c(\vec{G}, c) \) generalizes \( \chi_c(G) \) from many application points of view. Readers are referred to [9] for a connection of circular colorings of weighted digraphs and parallel computations. It was also shown in [6] that the notion of \( \chi_c(\vec{G}, c) \) generalizes the weighted circular colorings [1], the linear arboricity of a graph and the metric traveling salesman problem.

In this paper we consider weighted symmetric digraph, and explore the relation between circular coloring and breaker function of a weighted symmetric digraph. Our main result is Theorem 1, which generalizes the work of Zhu [11], a result of Tuza [7], and a result of Goddyn, Tarsi and Zhang [3], all these results of [11], [7], and [3] are generalizations of Minty’s work in [5].

For a graph \( G \) and a weighted symmetric digraph \( (\vec{G}, c) \), let \( M(G) \) (resp. \( M(\vec{G}) \)) denote the collection of all cycles (resp. dicycles) in \( G \) (resp. \( \vec{G} \)), and let \( L(\vec{G}, c) = \max \{c_{xy} + c_{yx} : xy \text{ is an arc of } \vec{G} \} \). In this paper, for our convenience, we say that a quotient has a value of infinity if its denominator is zero.

**Theorem 1** Let \( (\vec{G}, c) \) be a weighted symmetric digraph. Suppose \( r \) is a real number with \( r \geq L(\vec{G}, c) \). Then \( (\vec{G}, c) \) has a circular \( r \)-coloring if and only if \( (\vec{G}, c) \) has a breaker function \( T \) such that

\[
\max_C \frac{|C|_c}{|C|_T} \leq r,
\]

where the maximum is taken over all dicycle \( C \) satisfying \( 0 < |C|_c(\text{mod } r) < L(\vec{G}, c) \).

Theorem 1 says that to show a weighted symmetric digraph \( (\vec{G}, c) \) has a circular \( r \)-coloring, it suffices to check those dicycles \( C \) of \( \vec{G} \) for which \( 0 < |C|_c(\text{mod } r) < L(\vec{G}, c) \). Three simple consequences of Theorem 1 are the following.

**Corollary 2** Suppose \( r \) is a real number with \( r \geq 2 \). If \( G \) has an orientation \( \omega \) such that \( \max_C \tau(C, \omega) \leq r \), where the maximum is taken over all cycle \( C \) satisfying \( 0 < |C|(\text{mod } r) < 2 \), then \( G \) has a circular \( r \)-coloring.
Corollary 3 Suppose \( k \) and \( d \) are integers with \( k \geq 2d \geq 1 \). If \( G \) has an orientation \( \omega \) such that \( \max_C \tau(C, \omega) \leq k/d \), where the maximum is taken over all cycle \( C \) satisfying \( 1 \leq d|C|(\mod k) \leq 2d - 1 \), then \( G \) has a \((k, d)\)-coloring.

Corollary 4 If for each cycle \( C \) of \( G \), \( |C|(\mod r) \notin (0, 2) \), then \( G \) has a circular \( r \)-coloring.

It is clear that Tuza's result in [7] is a special case of Corollary 3 as \( d = 1 \). It had been shown in [11] that Minty's work of [5] and the following Goddyn, Tarsi and Zhang's result for \((k, d)\)-coloring both are special cases of Corollary 3: if \( G \) has an acyclic orientation \( \omega \) such that \( \max_{C \in M(G)} \tau(C, \omega) \leq k/d \), then \( G \) has a \((k, d)\)-coloring. In section 2 we prove the main result: Theorem 1 and derive the first two corollaries mentioned above.

2 The proof of the main result

Proof of the ‘if’ part of Theorem 1. Suppose \((\vec{G}, c)\) has a breaker function \( T \) such that

\[
\max_C \frac{|C|_c}{|C|_T} \leq r,
\]

where the maximum is taken over all dicycle \( C \) satisfying \( 0 < |C|_c(\mod r) < L(\vec{G}, c) \).

Let \( w \) be a function which assigns to each arc \( xy \) of \( \vec{G} \) a weight \( w_{xy} = c_{xy} - rT_{xy} \).

Let \( G \) be the underlying graph of \( \vec{G} \). For a spanning tree \( T \) of \( G \) and two vertices \( x \) and \( y \) in \( G \), clearly there is a unique path \( v_1v_2\ldots v_k \) in \( G \) having \( v_1 = x \), \( v_k = y \), and \( v_iv_{i+1} \) \((i = 1, \ldots, k - 1)\) are edges of \( T \). The \( x, y \)-dipath \((v_1, \ldots, v_k)\) of \( \vec{G} \) generated in this way is called the dipath of \( \vec{G} \) from \( x \) to \( y \) in \( T \). Let \( s \) be a fixed vertex in \( G \). Given a spanning tree \( T \) of \( G \), we define the function \( f_T : V(\vec{G}) \to \mathbb{R} \) as follows:

- \( f_T(s) = 0; \)
- If \( x \) is a vertex other than \( s \) then \( f_T(x) = \sum_{xy} w_{xy} \), where the summation is taken over all arcs in the dipath of \( \vec{G} \) from \( s \) to \( x \) in \( T \).

The weight of \( T \) is defined to be \( \sum_{v \in V(\vec{G})} f_T(v) \) and is denoted by \( f(T) \). In the following, let \( T \) be a spanning tree of \( G \) with maximum weight. Let \( \varphi \) be a function which assigns to each vertex of \( \vec{G} \) a color \( f_T(v)(\mod r) \) in \([0, r)\) (and hence in \( S^r \)).
We shall show that $\varphi$ is a circular $r$-coloring of $(\vec{G}, c)$. To prove this, let $xy$ and $yx$ be a pair of arcs in $\vec{G}$ and consider the following cases. In these cases, we consider $T$ as a rooted tree with root $s$, let $x'$ and $y'$ be the fathers of $x$ and $y$ respectively.

**Case I.** Suppose that $x$ is not on the $s, y$-path of $T$ and $y$ is not on the $s, x$-path of $T$. Let $T'$ be the spanning tree of $G$ obtained from $T$ by deleting its edge $x'x$ and adding the edge $xy$. Since $f(T') \leq f(T)$, we have $f_{T'}(x) \leq f_T(x)$, and hence $f_T(y) + wy_x \leq f_T(x)$ because $y$ is the father of $x$ in $T'$. Then by symmetry we have $f_T(x) + wy_y \leq f_T(y)$. Therefore

$$w_yx \leq f_T(x) - f_T(y) \leq -w_xy.$$  

If $T_{xy} = 1$ then we have $c_{yx} \leq f_T(x) - f_T(y) \leq r - c_{xy}$. If $T_{xy} = 0$ then we have $c_{xy} \leq f_T(y) - f_T(x) \leq r - c_{yx}$. In both subcases we arrive at $d_r(\varphi(x), \varphi(y)) \geq c_{xy}$, and hence $d_r(\varphi(y), \varphi(x)) \geq c_{yx}$ by symmetry.

**Case II.** Suppose that either the $s, y$-path of $T$ contains $x$ or the $s, x$-path of $T$ contains $y$. Since $xy$ and $yx$ both are arcs of $\vec{G}$, it suffices to consider the case that $y$ is on the $s, x$-path of $T$. Let $P$ be the dipath of $\vec{G}$ from $y$ to $x$ in $T$ and $C$ be the dicycle of $\vec{G}$ consisting of $P$ and the arc $xy$. Using the same method as in the previous case, we have

$$w_yx \leq f_T(x) - f_T(y).$$

Note that $f_T(x) = f_T(y) + \sum_{uv \in E(P)} w_{uv}$. That is

$$f_T(x) - f_T(y) = |P|_c - r|P|_T.$$  

Suppose $|P|_c \equiv \ell \pmod{r}$, where $0 \leq \ell < r$. Consider the following three scenarios.

**Subcase II(a).** If $T_{xy} = 1$ and $\ell > -w_{xy}$, then $r - c_{xy} < \ell < r$, and hence $0 < \{ |P|_c + c_{xy} \} \pmod{r} < c_{xy}$. It follows that $0 < |C|_c \pmod{r} < c_{xy}$. We see at once that $0 < |C|_c \pmod{r} < L(\vec{G}, c)$. By the hypothesis, we have $|C|_c/|C|_T \leq r$ which is equivalent to $|P|_c - r|P|_T \leq rT_{xy} - c_{xy}$. Therefore $f_T(x) - f_T(y) \leq -w_{xy}$.

**Subcase II(b).** If $T_{xy} = 1$ and $\ell < w_{yx}$, then $0 < \ell < c_{yx}$, and hence $c_{xy} < |C|_c \pmod{r} < c_{xy} + c_{yx}$. We see at once that $0 < |C|_c \pmod{r} < L(\vec{G}, c)$. By the same argument as in the Subcase II(a), we arrive at $f_T(x) - f_T(y) \leq -w_{xy}$.

**Subcase II(c).** If $T_{xy} = 1$ and $w_{yx} \leq \ell \leq -w_{xy}$, then $c_{yx} \leq \ell \leq r - c_{xy}$. Since $f_T(x) - f_T(y) = |P|_c - r|P|_T$ and $|P|_c = q'r + \ell$ for some integer $q'$, there exists an
Now we can conclude that $q$ for some integer $\phi$ and $d_r(\phi(x), \phi(y)) \geq c_{xy}$ and $d_r(\phi(y), \phi(x)) \geq c_{yx}$.

Next suppose $-|P|_c \equiv \ell'(\mod r)$, where $0 \leq \ell' < r$. Consider the following three scenarios.

**Subcase II(d).** If $T_{xy} = 0$ and $\ell > -w_{yx}$, then $-r < -\ell' < c_{yx} - r$, and hence $-r + c_{xy} < -\ell' + c_{xy} < c_{xy} + c_{yx} - r$ which implies $c_{xy} < |C|_c(\mod r) < c_{xy} + c_{yx}$. Now, by the same argument as in the Subcase II(b), we come to $f_T(x) - f_T(y) \leq -w_{xy}$.

**Subcase II(e).** If $T_{xy} = 0$ and $\ell' < w_{xy}$, then $0 < -\ell' + c_{xy} < c_{xy}$, and hence $0 < \{|P|_c + c_{xy}\}(\mod r) < c_{xy}$. By the same argument as in the Subcase II(a), we arrive at $f_T(x) - f_T(y) \leq -w_{xy}$.

**Subcase II(f).** If $T_{xy} = 0$ and $w_{xy} \leq \ell' \leq -w_{yx}$, then $c_{yx} \leq r - \ell' < r - c_{xy}$. Similarly as in the Subcase II(c), since $f_T(x) - f_T(y) = |P|_c - r|P|_T$ and $-|P|_c = q'r + \ell'$ for some integer $q'$, there exists an integer $q$ such that $f_T(x) - f_T(y) = qr + (r - \ell')$. Now we can conclude that $d_r(\phi(x), \phi(y)) \geq c_{xy}$ and $d_r(\phi(y), \phi(x)) \geq c_{yx}$.

As we can see from the above cases, no matter what is the value of $T_{xy}$, we always have $d_r(\phi(x), \phi(y)) \geq c_{xy}$ and $d_r(\phi(y), \phi(x)) \geq c_{yx}$. This completes the proof of the ‘if’ part.

**Proof of the ‘only if’ part of Theorem 1.** Suppose that $(\bar{G}, c)$ has a circular $r$-coloring $\phi : V(\bar{G}) \to [0, r)$. Note that here we view $[0, r)$ as $S^r$. We will show that $(\bar{G}, c)$ has a breaker function $T$ such that

$$\max_C \frac{|C|_c}{|C|_T} \leq r,$$

where the maximum is taken over all dicycles $C$ of $\bar{G}$, which is a stronger result than what we state in Theorem 1. Define a mapping $T$ which assigns to each arc $xy$ of $\bar{G}$ a value from $\{0, 1\}$ such that $T(xy) = 1$ as $\phi(x) > \phi(y)$, and $T(xy) = 0$ as $\phi(x) < \phi(y)$. Clearly, $T$ is a breaker function of $(\bar{G}, c)$. Consider the two possibilities, $T_{xy} = 1$ and $T_{xy} = 0$, separately. It is easy to check that for each arc $xy$ of $\bar{G}$ we have $\phi(x) + c_{xy} \leq \phi(y) + rT_{xy}$.

Let $\hat{C}$ be a dicycle of $\bar{G}$ with $|\hat{C}|_c/|\hat{C}|_T = \max_{C \in M(\bar{G})} |C|_c/|C|_T$, say $\hat{C} = (v_1, v_2, \ldots, v_k)$. From the result proved in the last paragraph, we have

$$\phi(v_i) + c_{v_iv_{i+1}} \leq \phi(v_{i+1}) + rT_{v_iv_{i+1}} \ (i = 1, 2, \ldots, k),$$
where we let \( v_{k+1} = v_1 \). Adding up both side of the inequalities separately, we shall arrive at \(|\hat{C}|_c \leq r|\hat{C}|_T\). Therefore \( \max_{C \in M(\hat{G})} |C|_c/|C|_T \leq r \), that completes the proof of the ‘only if’ part.

As a byproduct, the proof of Theorem 1 also provides a direct proof of the following result which was proved by Mohar [6] through using a result of Hoffman [4] and Ghouila-Houri [2]: 
\[
\chi_c(\vec{G}, c) = \min_T \max_C |C|_c/|C|_T,
\]
where the minimum is taken over all breaker function \( T \) of \((\vec{G}, c)\) and the maximum is taken over all dicycles \( C \) of \( \vec{G} \).

Proof of Corollary 2. Suppose \( G \) has an orientation \( \omega \) such that \( \max_C \tau(C, \omega) \leq r \), where the maximum is taken over all cycles \( C \) of \( G \) satisfying \( 0 < |C|(\text{mod } r) < 2 \). Let \( \vec{G} \) be the symmetric digraph derived from \( G \). Let \( c \) denote the all 1’s function on the arcs of \( \vec{G} \). Define a mapping \( T : E(\vec{G}) \to \{0, 1\} \) in the following way: If \( xy \) is an edge of \( G \) oriented from \( x \) to \( y \) in the orientation \( \omega \) then let \( T_{xy} = 1 \) and \( T_{yx} = 0 \). Clearly, \( T \) is a breaker function of \((\vec{G}, c)\), and \( L(\vec{G}, c) = 2 \). Thus the corollary follows from Theorem 1 immediately.

Proof of Corollary 3. Suppose \( G \) has an orientation \( \omega \) such that \( \max_C \tau(C, \omega) \leq k/d \), where the maximum is taken over all cycles \( C \) of \( G \) satisfying \( 1 \leq d|C|(\text{mod } k) \leq 2d - 1 \). Let \( r = k/d \). Let \( C \) be a cycle of \( G \) having \( 0 < |C|(\text{mod } r) < 2 \), that is \( |C| = rq + \ell \) for some integer \( q \) and real \( \ell \) satisfying \( 0 < \ell < 2 \). Clearly, \( d\ell \) is an integer such that \( 0 < d\ell < 2d \leq k \). It follows that \( 1 \leq d|C|(\text{mod } k) \leq 2d - 1 \). Our hypothesis implies that \( \tau(C, \omega) \leq k/d \). Corollary 2 now shows that \( G \) has a circular \( k/d \)-coloring, and hence \( G \) has a \((k, d)\)-coloring (see Zhu’s survey [12] for a proof of the last assertion).

References

[1] W. Deuber and X. Zhu, Circular coloring of weighted graphs, J. Graph Theory 23(1996) 365-376.

[2] A. Ghouila-Houri, Sur l’existence d’un flot ou d’une tension prenant ses valeurs dans un groupe abélien, C.R. Acad Sciences 250 (1960) 3931-3932.

[3] L. A. Goddyn, M. Tarsi and C. Q. Zhang, On \((k, d)\)-colorings and fractional nowhere zero flows, J. Graph Theory 28 (1998) 155-161.

[4] A. J. Hoffman, Some recent applications of the theory of linear inequalities to extremal combinatorial analysis, In: “Combinatorial Analysis: Proc. of the Tenth
[5] G. J. Minty, *A theorem on n-coloring the points of a linear graph*, Amer. Math. Monthly 69 (1962) 623-624.

[6] B. Mohar, *Circular colorings of edge-weighted graphs*, Journal of Graph Theory 43 (2003) 107-116.

[7] Z. Tuza, *Graph coloring in linear time*, J. Comb. Theory, Ser. B, 55 (1992) 236-243.

[8] Hong-Gwa Yeh and Xudig Zhu, *Resource-sharing system scheduling and circular chromatic number*, Theoretical Computer Science 332 (2005) 447-460.

[9] Hong-Gwa Yeh, *A dynamic view of circular colorings*, Preprint, 2006. http://arxiv.org/abs/math/0604226

[10] Xuding Zhu, *Circular chromatic number: a survey*, Discrete Math. 229 (2001) 371-410.

[11] Xuding Zhu, *Circular colouring and orientation of graphs*, J. Comb. Theory, Ser. B, 86 (2002) 109-113.

[12] X. Zhu, *Recent developments in circular colouring of graphs*, In M. Klazar, J. Kratochvil, J. Matousek, R. Thomas, and P. Valtr, editors, *Topics in Discrete Mathematics*, pages 497-550, Springer, 2006.