ON CONVERGENCE SETS OF FORMAL POWER SERIES

DAOWEI MA AND TEJINDER S. NEELON

Abstract. The convergence set of a divergent formal power series \( f(x_0, \ldots, x_n) \) is the set of all “directions” \( \xi \in \mathbb{P}^n \) along which \( f \) is absolutely convergent. We prove that every countable union of closed complete pluripolar sets in \( \mathbb{P}^n \) is the convergence set of some divergent series \( f \). The (affine) convergence sets of formal power series with polynomial coefficients are also studied. The higher-dimensional analogs of the results of A. Sathaye, P. Lelong, N. Levenberg and R.E. Molzon, and of J. Ribón are obtained.

1. INTRODUCTION

A formal power series \( f(x_0, x_1, \ldots, x_n) \) with coefficients in \( \mathbb{C} \) is said to be convergent if it converges absolutely in a neighborhood of the origin in \( \mathbb{C}^{n+1} \). A classical result of Hartogs (see [5]) states that a series \( f \) converges if and only if \( f_z(t) := f(z_0 t, z_1 t, \ldots, z_n t) \) converges, as a series in \( t \), for all \( z \in \mathbb{C}^{n+1} \). This can be interpreted as a formal analog of Hartogs’ theorem on separate analyticity. Because a divergent power series still may converge in certain directions, it is natural and desirable to consider the set of all \( z \in \mathbb{C}^{n+1} \) for which \( f_z \) converges. Since \( f_z(t) \) converges if and only if \( f_w(t) \) converges for all \( w \in \mathbb{C}^{n+1} \) on the affine line through \( z \), ignoring the trivial case \( z = 0 \), the set of directions along which \( f \) converges can be identified with a subset of \( \mathbb{P}^n \). The convergence set \( \text{Conv}(f) \) of a divergent power series \( f \) is defined to be the set of all directions \( \xi \in \mathbb{P}^n \) such that \( f_z(t) \) is convergent for some \( z \in \pi^{-1}(\xi) \), where \( \pi : \mathbb{C}^{n+1} \setminus \{0\} \rightarrow \mathbb{P}^n \) is the natural projection. For the case \( n = 1 \), P. Lelong [9] proved that the convergence set of a divergent series \( f(x_1, x_2) \) is an \( F_\sigma \) polar set (i.e. a countable union of closed sets of vanishing logarithmic capacity) in \( \mathbb{P}^1 \), and moreover, every \( F_\sigma \) polar subset of \( \mathbb{P}^1 \) is contained in the convergence set of a divergent series \( f(x_1, x_2) \). The optimal result was later obtained by A. Sathaye (see [16]) who showed that the class of convergence sets of divergent power series \( f(x_1, x_2) \) is precisely the class of \( F_\sigma \) polar sets in \( \mathbb{P}^1 \). To study the collection \( \text{Conv}(\mathbb{P}^n) \) of convergence sets of divergent series in higher dimensions we consider the class \( \text{PSH}_\omega(\mathbb{P}^n) \) of \( \omega \)-plurisubharmonic functions on \( \mathbb{P}^n \) with respect to the form \( \omega := dd^c \log |Z| \) on \( \mathbb{P}^n \). We show that \( \text{Conv}(\mathbb{P}^n) \) contains projective hulls of compact pluripolar sets and countable unions of projective varieties. We prove that each convergence set (of divergent power series) is a countable union of projective hulls of compact pluripolar sets. Our main result states that a countable union of closed complete pluripolar sets in \( \mathbb{P}^n \) belongs to \( \text{Conv}(\mathbb{P}^n) \). This generalizes the results of P. Lelong [9], Levenberg and Molzon [10], and Sathaye [16]. Our line of approach was inspired by [16], and influenced by the methods developed in [10], [13], and [15].

2000 Mathematics Subject Classification. Primary: 32A05, 30C85, 40A05.

Key words and phrases. formal power series, convergence sets, pluripolar sets, projective hulls.
We also consider convergence sets of a formal power series of the type
\[
f(t, x) = \sum_{j=1}^{\infty} P_j(x) t^j \in \mathbb{C}[x_1, x_2, \ldots, x_n][[t]],
\]
where deg \(P_j \leq j\). The affine convergence set Conv\(_a(f)\) of a divergent power series \(f(t, x)\) is defined to be the set of all \(x \in \mathbb{C}^n\) for which \(f(t, x)\) is convergent as a series in \(t\). We prove that a countable union of closed complete pluripolar sets is an affine convergence set.

2. Pluripolar sets in \(\mathbb{C}^n\)

For \(z \in \mathbb{C}^n\), let \(|z| := (|z_1|^2 + \cdots + |z_n|^2)^{1/2}\). Let \(\mathbb{C}[[x]] := \mathbb{C}[[x_1, \ldots, x_n]]\) denote the ring of formal power series and let \(\mathbb{C}\{x\}\) be the ring of all power series \(f(x) \in \mathbb{C}[[x]]\) that are absolutely convergent in a neighborhood of the origin in \(\mathbb{C}^n\). Let \(\mathcal{H}(\mathbb{C}^n)\) denote the set of all homogeneous polynomials of positive degrees in \(x_1, \ldots, x_n\) with complex coefficients and, for an integer \(k > 0\), let \(\mathcal{P}_k(\mathbb{C}^n)\) denote the set of polynomials of degree at most \(k\) in \(x_1, \ldots, x_n\) with complex coefficients.

A Borel\(^1\) subset \(E\) of \(\mathbb{C}^n\) is said to be pluripolar (polar when \(n = 1\)) if for each point \(x \in E\) there is a plurisubharmonic function \(u, u \neq -\infty\), defined in a neighborhood \(U\) of \(x\) in \(\mathbb{C}^n\) such that \(u = -\infty\) on \(E \cap U\). A set \(E\) is said to be globally pluripolar if there is a nonconstant plurisubharmonic function \(u\) defined on \(\mathbb{C}^n\) such that \(E \subset \{y : u(y) = -\infty\}\). A theorem of Josefson (see [7]) states that \(E\) is pluripolar if and only if \(E\) is globally pluripolar.

A set \(E \subset \mathbb{C}^n\) is said to be a complete pluripolar set if there is a non-constant plurisubharmonic function \(u\) defined on \(\mathbb{C}^n\) such that \(E = \{y : u(y) = -\infty\}\). So the set \(\{(0, x_2) \in \mathbb{C}^2 : |x_2| < 1\}\) and its closure are pluripolar, but they are not complete pluripolar sets. A countable union of pluripolar sets is pluripolar. So the set of rationals in the interval \([0, 1]\) is polar. It is not a complete polar set, because each complete pluripolar set is \(G_\delta\). In \(\mathbb{C}\) each \(G_\delta\) polar set is a complete polar set, which is Deny’s Theorem (see [3]).

A subset \(E\) of a domain \(D\) in \(\mathbb{C}^n\) is said to be a complete pluripolar set in \(D\) if there is a nonconstant plurisubharmonic function \(u\) defined on \(D\) such that \(E = \{u = -\infty\}\).

**Proposition 2.1.** Let \(w\) be a nonconstant plurisubharmonic function defined on a Stein manifold \(\Omega\), and let \(E = \{w = -\infty\}\). Let \(v\) be a continuous, non-negative, plurisubharmonic exhaustion function of \(\Omega\). Then there is a plurisubharmonic function \(u\) on \(\Omega\) such that \(u \leq v\) on \(\Omega\) and \(E = \{u = -\infty\}\).

**Proof.** Let \(V_j = \{x \in \Omega : v(x) < 2^j\}\) for \(j \in \mathbb{N}\). Let \(w\) be a plurisubharmonic function such that \(E = \{x : w(x) = -\infty\}\). Choose an increasing sequence \(\{M_j\}\) of positive numbers such that \(\lim_{j \to \infty} M_j = \infty\) and \(M_j \geq \sup_{x \in V_j} w(z) \forall j\). For each \(j\), define a function \(u_j\) by
\[
 u_j(x) = \begin{cases} 
 \max(M_j^{-1}w(x) - 1, v(x) - 2^j), & \text{if } x \in V_j, \\
 v(x) - 2^j, & \text{if } x \notin V_j. 
\end{cases}
\]
Then \(u_j\) is plurisubharmonic on \(\Omega\) by the gluing theorem. On each compact subset of \(\Omega\), all but a finite number of \(u_j\) are negative. It follows that the sum \(u(x) := \sum_{j=1}^{\infty} 2^{-j}u_j(x)\) is plurisubharmonic (or identically \(-\infty\)), since the sequence of the partial sums of the series is eventually decreasing. It is clear that \(u_j(x) \leq v(x)\) for each \(j\), so that \(u(x) \leq v(x)\).

\(^1\)All sets considered in this paper are assumed to be Borel.
Suppose that \( x \in E \). Then \( w(x) = -\infty \), and \( u_j(x) = v(x) - 2^j \) for each \( j \). Thus \( u(x) = -\infty \).

Now suppose that \( x \in \Omega \setminus E \). There is an \( m \) such that \( x \in V_m \). Then

\[
\begin{align*}
u(x) & \geq \sum_{j=1}^{m-1} 2^{-j} u_j(x) + \sum_{j=m}^{\infty} 2^{-j}(M_j^{-1}w(x) - 1) \\
& \geq \sum_{j=1}^{m-1} 2^{-j} u_j(x) + \sum_{j=1}^{\infty} 2^{-j}(-M_1^{-1}|w(x)| - 1) \\
& = \sum_{j=1}^{m-1} 2^{-j} u_j(x) + (-M_1^{-1}|w(x)| - 1) > -\infty.
\end{align*}
\]

This implies, in particular, that \( u \) is not identically \( -\infty \). Therefore, \( E = \{ x : u(x) = -\infty \} \). \( \square \)

Remark. We got the idea of the proof from Sadullaev (private discussion) and from Bedford and Taylor [2].

**Corollary 2.2.** Let \( E \) be a complete pluripolar set in \( \mathbb{C}^n \). Then there is a plurisubharmonic function \( u \) on \( \mathbb{C}^n \) such that \( u \leq (1/2) \log(1 + |z|^2) \) on \( \mathbb{C}^n \) and \( E = \{ u = -\infty \} \).

Remark. A stronger version of the above corollary appeared in [2].

Let \( B_n \) be the open unit ball in \( \mathbb{C}^n \).

**Corollary 2.3.** Let \( E \) be a complete pluripolar set in \( B_n \). Then there is a plurisubharmonic function \( u \) on \( B_n \) such that \( u \leq -\log(1 - |z|^2) \) on \( B_n \) and \( E = \{ u = -\infty \} \).

Let

\[
\mathcal{L}(\mathbb{C}^n) = \{ u \in \text{PSH}(\mathbb{C}^n) : \sup_{x \in \mathbb{C}^n} (u(x) - (1/2) \log(1 + |x|^2)) < \infty \}
\]

denote the Lelong class of plurisubharmonic functions. For a polynomial \( P \in \mathcal{P}_k(\mathbb{C}^n) \), where \( k > 0 \), the function \((1/k) \log |P(x)|\) is a prototypical member of \( \mathcal{L}(\mathbb{C}^n) \).

Corollary 2.2 implies that if \( E \) is a complete pluripolar set in \( \mathbb{C}^n \) then there is a \( u \in \mathcal{L}(\mathbb{C}^n) \) such that \( E = \{ u = -\infty \} \).

The pluripolar hull (see [11]) in \( \mathbb{C}^n \) of a pluripolar set \( E \) in \( \mathbb{C}^n \) is defined to be

\[
E^* = \cap \{ z \in \mathbb{C}^n : u(z) = -\infty \},
\]

where the intersection is taken over all plurisubharmonic functions \( u \) on \( \mathbb{C}^n \) that are \( -\infty \) on \( E \).

For a polynomial \( P \in \mathcal{P}_k(\mathbb{C}^n) \) and a subset \( K \) of \( \mathbb{C}^n \) we set

\[
\langle P(x) \rangle_k = \frac{|P(x)|^{1/k}}{\sqrt{1 + |x|^2}}, \quad \langle P \rangle_{k,K} = \sup_{x \in K} \langle P(x) \rangle_k, \quad \langle P \rangle_k = \langle P \rangle_{k,\mathbb{C}^n}.
\]

Note that if \( P \in \mathcal{P}_k(\mathbb{C}^n) \) and \( m \) is a positive integer, then \( \langle P^m(x) \rangle_{k,m} = \langle P(x) \rangle_k \).

**Definition 2.4.** Let \( F \subset \mathbb{C}^n \), \( F \neq \emptyset \), \( x \in \mathbb{C}^n \), and \( 0 \leq r \leq 1 \). Define

\[
\tau(x,F,r) = \inf \{ \langle P \rangle_{k,F} : k \in \mathbb{N}, P \in \mathcal{P}_k(\mathbb{C}^n), \langle P(x) \rangle_k \geq r, \langle P \rangle_k \leq 1 \}
\]
and
\[ T(x, F) = \sup\{r : 0 \leq r \leq 1, \tau(x, F, r) = 0\}. \]

For the empty set, we put \( \tau(x, \emptyset, r) = 0 \) and \( T(x, \emptyset) = 1 \). It is clear that if \( E \subset F \), then \( \tau(x, E, r) \leq \tau(x, F, r) \) and \( T(x, E) \geq T(x, F) \).

**Lemma 2.5.** Let \( u \in \mathcal{L}(\mathbb{C}^n) \). Suppose that the set \( E := \{u = -\infty\} \) is closed. Then for each \( x \in \mathbb{C}^n \setminus E \), and each non-empty compact set \( K \subset E \), we have
\[ T(x, K) \geq (1 + |x|^2)^{-1/2} e^{u(x) - b}, \]
where
\[ b := \sup(u(z) - \frac{1}{2} \log(1 + |z|^2)). \]

**Proof.** Without loss of generality, we assume that \( b = 0 \). Let \( g(x) = e^{u(x)} \). Then \( (1 + |x|^2)^{-1/2} g(x) \leq 1 \) for \( x \in \mathbb{C}^n \). Fix a \( x \in \mathbb{C}^n \setminus E \) and a non-empty compact set \( K \subset E \). Let \( r > 0 \) be such that \( r < (1 + |x|^2)^{-1/2} g(x) \). Let \( \eta \) be a positive number with \( \eta < r \). Let \( \lambda \) be a positive number that is less than the distance between the closed set \( \{ y : g(y) \geq \eta \} \) and the compact set \( K \), and that is so small that
\[ (\lambda + \eta)^{1-\lambda} < \sqrt{\eta}. \]

Let
\[ \omega(y) = \begin{cases} c_n \exp(-1/(1 - |y|^2)), & \text{if } |y| < 1, \\ 0, & \text{if } |y| \geq 1, \end{cases} \]
where \( c_n \) is so chosen that \( \int \omega(t) dt = 1 \). For \( \mu > 0 \), let \( g_\mu(y) = \int g(y + \mu z) \omega(z) dz \). Then \( \log g_\mu \in \mathcal{L}(\mathbb{C}^n) \), \( g_\mu \) is \( C^\infty \), positive, and \( g_\mu \downarrow g \) as \( \mu \downarrow 0 \). If \( y \in K \), and if \( |z| < 1 \), then \( y + \lambda z \not\in \{ y : g(y) \geq \eta \} \), and hence \( g(y + \lambda z) < \eta \). It follows that \( g_\lambda(y) = \int g(y + \lambda z) \omega(z) dz < \int \eta \omega(z) dz = \eta \). For each \( y \in \mathbb{C}^n \),
\[ g_\lambda(y) \leq \int (1 + |y + \lambda z|^2)^{1/2} \omega(z) dz \leq (1 + (|y| + \lambda)^2)^{1/2}, \]
and hence
\[ g_\lambda(y) \leq (1 + \lambda)(1 + |y|^2)^{1/2}. \]

As in [15, p. 17], we define a function \( \phi_\lambda \) on \( \mathbb{C} \times \mathbb{C}^n \) by
\[ \phi_\lambda(y_0, y) = \begin{cases} |y_0|/(\lambda + g_\lambda(y/y_0))^{1-\lambda}, & \text{if } y_0 \neq 0, \\ \lambda |y|, & \text{if } y_0 = 0. \end{cases} \]

Then \( \phi_\lambda \) is continuous and plurisubharmonic. Moreover, it satisfies \( \phi_\lambda(cw) = |c| \phi_\lambda(w) \) for \( c \in \mathbb{C} \) and \( w \in \mathbb{C}^{n+1} \). We then define \( \psi_\lambda \) by
\[ \psi_\lambda(y) = \phi_\lambda(1, y) = (\lambda + g_\lambda(y))^{1-\lambda} + \lambda(1 + |y|^2)^{1/2}. \]

Then \( \psi_\lambda \) is \( C^\infty \) and \( \log \psi_\lambda \in \mathcal{L}(\mathbb{C}^n) \) by [15, Prop. 2.7].

By [15, Prop. 2.10],
\[ \phi_\lambda(y_0, y) = \sup\{|h(y_0, y)|^{1/\deg h}\}, \]
where the supremum is taken over all homogeneous polynomials \( h \) of \( n + 1 \) variables such that \( |h(z_0, z)|^{1/\deg h} \leq \phi_\lambda(z_0, z) \) \( \forall (z_0, z) \in \mathbb{C} \times \mathbb{C}^n \). Since \( \psi_\lambda(z)/\sqrt{1 + |z|^2} \) extends to a continuous function on \( \mathbb{P}^n \), it follows that
\[ \psi_\lambda(x) = \sup\{|P(x)|^{1/k} : k \in \mathbb{N}, P \in \mathcal{P}_k(\mathbb{C}^n), |P(y)|^{1/k} \leq \psi_\lambda(y) \forall y \in \mathbb{C}^n\}. \]
For all $y \in \mathbb{C}^n$, 
\[
\psi_\lambda(y) = (\lambda + g_\lambda(y))^{1-\lambda} + \lambda(1 + |y|^2)^{1/2} \\
\leq (\lambda + (1 + \lambda)(1 + |y|^2)^{1/2})^{1-\lambda} + \lambda(1 + |y|^2)^{1/2} \\
< (1 + 3\lambda)(1 + |y|^2)^{1/2}.
\]
If $y \in K$, then 
\[
\psi_\lambda(y) = (\lambda + g_\lambda(y))^{1-\lambda} + \lambda(1 + |y|^2)^{1/2} \\
\leq (\lambda + \eta)^{1-\lambda} + \lambda(1 + |y|^2)^{1/2} \\
< \sqrt{\eta} + \lambda(1 + |y|^2)^{1/2} \\
\leq (\sqrt{\eta} + \lambda)(1 + |y|^2)^{1/2}.
\]
If $P \in \mathcal{P}_k(\mathbb{C}^n)$ and if $|P(z)|^{1/k} \leq \psi_\lambda(z) \forall z \in \mathbb{C}^n$, then 
\[
\langle P \rangle_k \leq 1 + 3\lambda \quad \text{and} \quad \langle P \rangle_{k,K} \leq \sqrt{\eta} + \lambda.
\]
For sufficiently small $\lambda$, 
\[
(\lambda + g_\lambda(x))^{1-\lambda} + \lambda(1 + |x|^2)^{1/2} > (1 + 3\lambda)r(1 + |x|^2)^{1/2},
\]
since as $\lambda$ approaches 0, the difference of the left side minus the right side tends to $g(x) - r(1 + |x|^2)^{1/2} > 0$. It follows that for sufficiently small $\lambda$, 
\[
\psi_\lambda(x) > (1 + 3\lambda)r(1 + |x|^2)^{1/2}.
\]
By (2), (3) and (4), we have $\tau(x, K, r) \leq (1 + 3\lambda)^{-1}(\sqrt{\eta} + \lambda)$. Letting $\lambda \to 0$, and then $\eta \to 0$, yields that $\tau(x, K, r) = 0$. Since this holds for every $r < g(x)(1 + |x|^2)^{-1/2}$, it follows that $T(x, K) \geq g(x)(1 + |x|^2)^{-1/2}$. 

\[\square\]

Remark. Some comments about (2) may be in order. By [8, Th. 5.1.6(iii)], 
\[
\psi_\lambda = (\limsup_{j \to \infty} |P_j|^{1/j})^* \quad \text{for some sequence } \{P_j\} \text{ of polynomials on } \mathbb{C}^n \text{ such that } \deg P_j \leq j. \text{ Here } w^* \text{ denotes the upper semicontinuous regularization of } w. \text{ The stronger equality (2) depends on the fact that } \psi_\lambda(z)/\sqrt{1 + |z|^2} \text{ extends to a continuous function on } \mathbb{P}^n.
\]

Definition 2.6. A subset $E$ of $\mathbb{C}^n$ is said to have Property J (in $\mathbb{C}^n$) if for each $x \in \mathbb{C}^n \setminus E$ there is a positive number $r_x$ such that $T(x, K \cap E) \geq r_x$ for each compact subset $K$ of $\mathbb{C}^n$.

The inequality $T(x, K \cap E) \geq r_x$ means that there is a sequence $\{k_j\}$ of positive integers, and a sequence $\{P_j\}$ of polynomials, with deg $P_j \leq k_j$, such that 
\[
\langle P_j(x) \rangle_{k_j} \geq r_x, \quad \langle P_j \rangle_{k_j} \leq 1, \quad \lim_{j \to \infty} \langle P_j \rangle_{k_j, K \cap E} = 0.
\]
It is clear that a set that has Property J must be closed.

Theorem 2.7. A subset of $\mathbb{C}^n$ has Property J if and only if it is a closed complete pluripolar set.
Lemma 2.5, that Suppose that Proof. Let $a$ for each positive integer $C$ Thus $u$ or, equivalently, $z$.

Conversely, suppose that $E \subset \mathbb{C}^n$ has Property J. Then $E$ is closed. Set $E_j = E \cap \{ z \in \mathbb{C}^n : |z| \leq j \}$. Let $x \in \mathbb{C}^n \setminus E$. Then there is a positive number $r_x$ such that $T(x, E_j) \geq r_x$ for each positive integer $j$. Thus there is a sequence $\{k_j\}$ of positive integers, and a sequence $\{P_j\}$ of polynomials, with $\deg P_j \leq k_j$, such that

$$
(P_j(x))_{k_j} \geq r_x, \quad (P_j)_{k_j} \leq 1, \quad (P_j)_{k_j, E_j} \leq \exp(-2^j).
$$

Let

$$
u(z) := \sum_{j=1}^{\infty} 2^{-j} \log |P_j(z)|^{1/k_j} = \frac{1}{2} \log(1 + |z|^2) + \sum_{j=1}^{\infty} 2^{-j} \log \langle P_j(z) \rangle_{k_j}.
$$

Then $u$ is plurisubharmonic with $u(z) \leq (1/2) \log(1 + |z|^2)$ on $\mathbb{C}^n$, and $u(x) \geq \log r_x + (1/2) \log(1 + |x|^2)$. Let $y \in E$. Then there is a positive integer $m$ such that $y \in E_j$ for $j \geq m$, hence

$$u(y) = \frac{1}{2} \log(1 + |y|^2) + \sum_{j=m}^{\infty} 2^{-j} \log \langle P_j(y) \rangle_{k_j}
\leq \frac{1}{2} \log(1 + |y|^2) + \sum_{j=m}^{\infty} 2^{-j} (-2^j) = -\infty.
$$

It follows that $u = -\infty$ on $E$. Thus $x$ does not belong to the pluripolar hull of $E$. Therefore, the pluripolar hull of $E$ is $E$. A Theorem of Zeriahi [17] states that if a pluripolar set $F$ is both $F_\delta$ and $G_\delta$, and if the pluripolar hull of $F$ equals $F$, then $F$ is a complete pluripolar set. Since $E$, being a closed set, is both $G_\delta$ and $F_\delta$, it follows that $E$ is a complete pluripolar set.

3. PLURIPOLAR SETS IN $\mathbb{P}^n$

Let $\pi : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{P}^n$ denote the standard projection mapping that maps a point $z = (z_0, z_1, \ldots, z_n) \in \mathbb{C}^{n+1}$ to its corresponding homogeneous coordinates $\pi(z) = [z] = [z_0 : z_1 : \ldots : z_n] \in \mathbb{P}^n$. Suppose that $z = (z_0, z_1, \ldots, z_n) \in \mathbb{C}^{n+1} \setminus \{0\}$ and $Z = [Z_0 : Z_1 : \cdots : Z_n] \in \mathbb{P}^n$. Then $Z = \pi(z)$ if and only if

$$[Z_0 : Z_1 : \cdots : Z_n] = [z_0 : z_1 : \ldots : z_n],$$

or, equivalently,

$$z_j Z_k = z_k Z_j, \quad \text{for } j, k = 0, \ldots, n.
$$

Suppose that $p \in \mathcal{H}(\mathbb{C}^{n+1})$ is a homogeneous polynomial of $n + 1$ variables and that $Z = [z] \in \mathbb{P}^n$, where $z \in \mathbb{C}^{n+1} \setminus \{0\}$. Set

$$\langle p(Z) \rangle := \frac{|p(Z)|^{1/\deg p}}{|Z|} = \frac{|p(z)|^{1/\deg p}}{|z|}.
$$
Note that $\langle p(Z) \rangle$ is independent of the choice of the representative $z$ and is a well-defined function on $\mathbb{P}^n$. Furthermore, if $m$ is a positive integer, then $\langle p^m(Z) \rangle = \langle p(Z) \rangle$. For a set $K \subset \mathbb{P}^n$, put $\langle p \rangle_K = \sup_{z \in K} \langle p(Z) \rangle$, $\langle p \rangle = \langle p \rangle_{\mathbb{P}^n}$.

**Definition 3.1.** The projective hull $\hat{K}$ of a compact set $K \subset \mathbb{P}^n$ is the set of all points $Z \in \mathbb{P}^n$ for which there exists a constant $C = C_Z > 0$ such that
\begin{equation}
\langle p(Z) \rangle \leq C \langle p \rangle_K
\end{equation}
for all homogeneous polynomials $p \in \mathcal{H}(\mathbb{C}^{n+1})$. A compact set $K \subset \mathbb{P}^n$ is said to be projectively convex if $\hat{K} = K$.

Since, for $k \geq 1$, the set $H^0(\mathbb{P}^n, \mathcal{O}(k))$ of global holomorphic sections of the line bundle $\mathcal{O}(k)$ is canonically identified with the set $\mathcal{H}_k(\mathbb{C}^{n+1})$ of homogeneous polynomials of degree $k$, it follows that the above definition of projective hulls is equivalent to that in [6, p. 607].

It is clear that each algebraic variety is projectively convex. In particular, each finite set is projectively convex.

The complement of a hyperplane in $\mathbb{P}^n$ is called an affine open set. An affine open set in $\mathbb{P}^n$ is biholomorphically equivalent to $\mathbb{C}^n$. The sets
\begin{equation}
U_j := \{Z = [Z_0 : Z_1 : \cdots : Z_n] \in \mathbb{P}^n : Z_j \neq 0\}, \quad j = 0, \ldots, n,
\end{equation}
are canonical affine open sets in $\mathbb{P}^n$.

**Definition 3.2.** A subset $F$ of $\mathbb{P}^n$ is said to be a pluripolar set if for each $Z \in F$ there is a neighborhood $V$ of $Z$ in $\mathbb{P}^n$ and a nonconstant plurisubharmonic function $u$ defined on $V$ such that $u$ is identically $-\infty$ on $V \cap F$. A subset $E$ of $\mathbb{P}^n$ is said to be a complete pluripolar set in $\mathbb{P}^n$ if for each affine open set $U$ in $\mathbb{P}^n$ the set $U \cap E$ is a complete pluripolar set in $U$.

A compact set $K$ in $\mathbb{P}^n$ is pluripolar if and only if $\hat{K} \neq \mathbb{P}^n$ (see [6, Corollary 4.4]).

In the above definition, the pluripolar sets and the complete pluripolar sets are defined “locally”, because there are no globally defined nonconstant plurisubharmonic functions on $\mathbb{P}^n$. It is desirable to have an equivalent definition of (complete) pluripolar sets in terms of some kind of substitutes of plurisubharmonic functions that are globally defined on $\mathbb{P}^n$.

We could use the $\omega$-plurisubharmonic functions described below to define pluripolar sets and complete pluripolar sets. Definition 3.2 is to emphasize that the notion of (complete) pluripolar sets is independent of any differential forms.

We fix a Kähler form $\omega := dd^c \log |Z| = i\partial \bar{\partial} \log(|Z_0|^2 + \cdots + |Z_n|^2)$ on $\mathbb{P}^n$, where $d^c = i(\bar{\partial} - \partial)$. Note that $(2\pi)^{-1}\omega$ is the Fubini-Study form on $\mathbb{P}^n$. An upper semicontinuous function $u$ from an open subset of $\mathbb{P}^n$ to $\mathbb{R} \cup \{-\infty\}$ is said to be $\omega$-plurisubharmonic if $dd^c u + \omega \geq 0$ (see, e.g., [4]). Let $\text{PSH}_\omega(\mathbb{P}^n)$ denote the family of $\omega$-plurisubharmonic functions on $\mathbb{P}^n$. For a homogeneous polynomial $p$, the function $Z \mapsto \log \langle p(Z) \rangle$ is a prototypical function in $\text{PSH}_\omega(\mathbb{P}^n)$.

There is a one to one correspondence between $\text{PSH}_\omega(\mathbb{P}^n)$ and the Lelong class $\mathcal{L}(\mathbb{C}^n)$. This can be seen by identifying $\mathbb{C}^n$ with the affine open set $U_0$ (see (6)):
\begin{equation}
\mathbb{C}^n \simeq \{[1 : \xi_1 : \xi_2 : \cdots : \xi_n] \in \mathbb{P}^n : (\xi_1, \ldots, \xi_n) \in \mathbb{C}^n\} = U_0.
\end{equation}
Given \( \varphi \in \text{PSH}_w(\mathbb{P}^n) \), the function

\[
\tilde{\varphi}(z_1, ..., z_n) := \frac{1}{2} \log(1 + |z|^2) + \varphi(1 : z_1 : \cdots : z_n)
\]

belongs to \( \mathcal{L}(\mathbb{C}^n) \), and the map \( \varphi \mapsto \tilde{\varphi} \) is a bijection from \( \text{PSH}_w(\mathbb{P}^n) \) onto \( \mathcal{L}(\mathbb{C}^n) \).

The following proposition can be found in \([4]\) or \([6, \text{Theorem 4.3}]\).

**Proposition 3.3.** Let \( E \) be a subset of \( \mathbb{P}^n \). Then \( E \) is pluripolar if and only if there is a function \( u \in \text{PSH}_w(\mathbb{P}^n) \), \( u \not\equiv -\infty \), such that \( E \subset \{ u = -\infty \} \). \( \square \)

**Lemma 3.4.** Suppose that \( n \geq 2 \) and \( \mathbb{H} \cong \mathbb{P}^{n-1} \) is a hyperplane in \( \mathbb{P}^n \). Let \( u \) be an \( \omega \)-plurisubharmonic function on \( H \). Then there is an \( \omega \)-plurisubharmonic function \( v \) on \( \mathbb{P}^n \) such that

\[
\{ Z \in \mathbb{P}^n : v(Z) = -\infty \} = \{ Z \in H : u(Z) = -\infty \}.
\]

**Proof.** Without loss of generality we assume that \( u \leq 0 \) and \( H = \{ Z = [Z_0 : \cdots : Z_n] : Z_0 = 0 \} \). Let \( O = [1 : 0 : \cdots : 0] \). The function

\[
w(Z) := u(0 : Z_1 : \cdots : Z_n) + \frac{1}{2} \log |Z_1|^2 + \cdots + |Z_n|^2
\]

is an \( \omega \)-plurisubharmonic function on \( \mathbb{P}^n \setminus O \) such that \( w|_H = u \). Let

\[
v(Z) = \begin{cases} \max(w(Z), 1 + \log(|Z_0|/|Z|)), & \text{if } Z \in \mathbb{P}^n \setminus O, \\ 1, & \text{if } Z = O. \end{cases}
\]

Since the function \( 1 + \log(|Z_0|/|Z|) \) is \( \geq 0 \) on \( U \setminus O \) for some neighborhood \( U \) of \( O \), we see that \( v(Z) = 1 + \log(|Z_0|/|Z|) \) on \( U \setminus O \), hence \( v(Z) = 1 + \log(|Z_0|/|Z|) \) on \( U \). It follows that \( v \in \text{PSH}_w(\mathbb{P}^n) \). It is then clear that (8) holds. \( \square \)

**Proposition 3.5.** Let \( E \subset \mathbb{P}^n \). Then \( E \) is a complete pluripolar set in \( \mathbb{P}^n \) if and only if there is a \( u \in \text{PSH}_w(\mathbb{P}^n) \) such that \( E = \{ Z : u(Z) = -\infty \} \).

**Proof.** Suppose that there is a \( u \in \text{PSH}_w(\mathbb{P}^n) \) such that \( E = \{ u = -\infty \} \). Let \( U \) be an affine open set, and let \( H = \mathbb{P}^n \setminus U = \{ Z : \Lambda(Z) = 0 \} \), where \( \Lambda(Z) := a_0Z_0 + a_1Z_1 + \cdots + a_nZ_n \) is a linear form. Then \( v(Z) := u(Z) + \log(|Z|/|\Lambda(Z)|) \) is plurisubharmonic on \( U \) and

\[
U \cap E = \{ Z \in U : u(Z) = -\infty \} = \{ Z \in U : v(Z) = -\infty \}.
\]

It follows that \( U \cap E \) is a complete pluripolar set in \( U \) for each affine open set \( U \). Therefore, \( E \) is a complete pluripolar set in \( \mathbb{P}^n \).

Conversely, suppose that \( E \) is a complete pluripolar set in \( \mathbb{P}^n \). For \( j = 0, \ldots, n \), let \( H_j = \{ Z_j = 0 \} \) and \( U_j = \mathbb{P}^n \setminus H_j \). Since \( E \cap U_j \) is a complete pluripolar set in \( U_j \), there is a plurisubharmonic function \( h_j \) on \( U_j \) with \( E \cap U_j = \{ Z \in U_j : h_j(Z) = -\infty \} \) and \( h_j(Z) \leq -\log(|Z|/|Z_j|) \), by Corollary 2.2. The function \( h_j(Z) - \log(|Z|/|Z_j|) \) is a non-positive \( \omega \)-plurisubharmonic function on \( U_j \), which extends uniquely to an \( \omega \)-plurisubharmonic function \( w_j \) on \( \mathbb{P}^n \).

Let \( w = \max w_j \). For each \( j \),

\[
\{ Z \in U_j : w(Z) = -\infty \} \subset \{ Z \in U_j : w_j(Z) = -\infty \} = E \cap U_j.
\]

It follows that \( \{ w = -\infty \} \subset E \). Let \( \Omega = U_0 \cap \cdots \cap U_n \). Then

\[
\{ Z \in \Omega : w(Z) = -\infty \} = \Omega \cap \{ Z \in U_j : w_j(Z) = -\infty \} = \Omega \cap (E \cap U_j) = E \cap \Omega,
\]
for each \( j \), hence
\[
\{ w = -\infty \} \cap \Omega = \cap_j (\{ w_j = -\infty \} \cap \Omega) = E \cap \Omega.
\]

To summarize, we have \( \{ w = -\infty \} \subset E \) and \( \{ w = -\infty \} \cap \Omega = E \cap \Omega \).

To prove that there is a \( u \in \text{PSH}_\omega(\mathbb{P}^n) \) with \( E = \{ u = -\infty \} \), we proceed by induction on \( n \). Suppose that \( n = 1 \). If \( [0 : 1] \in E \), let \( v_0 = \log(|Z_0|/|Z|) \); if \( [0 : 1] \not\in E \), let \( v_0 = 0 \). Then \( v_0 \in \text{PSH}_\omega(\mathbb{P}^1) \) and \( \{ v_0 = -\infty \} = E \cap H_0 \). We similarly define \( v_1 \in \text{PSH}_\omega(\mathbb{P}^1) \) so that \( \{ v_1 = -\infty \} = E \cap H_1 \). Then \( u := (w + v_0 + v_1)/3 \) belongs to \( \text{PSH}_\omega(\mathbb{P}^1) \) and \( \{ u = -\infty \} = E \). The statement holds for \( n = 1 \).

Suppose that \( n \geq 2 \) and that the statement holds for \( n - 1 \). It is clear that for each \( j \), either \( H_j \subset E \), or \( H_j \cap E \) is a complete pluripolar set in \( H_j \). If \( H_0 \subset E \), let \( v_0 = \log(|Z_0|/|Z|) \). If \( H_0 \not\subset E \), then, by the induction hypothesis, there is a \( u_0 \in \text{PSH}_\omega(H_0) \) with \( \{ Z \in H_0 : u_0(Z) = -\infty \} = E \cap H_0 \), and hence, by Lemma 3.4, there is a \( v_0 \in \text{PSH}_\omega(\mathbb{P}^n) \) with \( \{ Z \in \mathbb{P}^n : v_0(Z) = -\infty \} = E \cap H_0 \). Either case, we have \( v_0 \in \text{PSH}_\omega(\mathbb{P}^n) \) and \( \{ v_0 = -\infty \} = E \cap H_0 \). We similarly obtain \( v_j \) so that \( v_j \in \text{PSH}_\omega(\mathbb{P}^n) \) and \( \{ v_j = -\infty \} = E \cap H_j \), for each \( j \). Then \( u := (w + v_0 + \ldots + v_n)/(n+2) \) belongs to \( \text{PSH}_\omega(\mathbb{P}^n) \) and \( \{ u = -\infty \} = E \), which completes the proof.

Propositions 3.3 and 3.5 can be considered equivalent definitions of pluripolar sets and complete pluripolar sets respectively. It is clear from Definition 3.2 that the union of a countable collection of pluripolar sets is pluripolar. It is clear from Propositions 3.3 and 3.5 that each pluripolar set in \( \mathbb{P} \) is projectively convex.

**Proposition 3.6.** Let \( K \) be a compact pluripolar set in \( \mathbb{P}^n \). Then \( \hat{K} \subset K^* \).

**Proof.** Let
\[
L_K(Z) = \sup \{ \varphi(Z) : \varphi \in \text{PSH}_\omega(\mathbb{P}^n) \text{ and } \varphi|_k \leq 0 \}
\]
and
\[
\Lambda_K(Z) = \sup \{ \log \langle p(Z) \rangle : p \in \mathcal{H}(\mathbb{C}^{n+1}), \langle p \rangle_K \leq 1 \}.
\]
Then \( \Lambda_K(Z) = L_K(Z) \), by [6, Prop. 4.2].

Let \( X \in K^{\text{c}} \), the complement of \( K^* \). Then there is a \( u \in \text{PSH}_\omega(\mathbb{P}^n) \) with \( u(X) > -\infty \) and \( u|_K \equiv -\infty \). Thus \( L_K(X) = \infty \). It follows that \( \Lambda_K(X) = \infty \), which implies that \( X \in \hat{K}^c \), the complement of \( \hat{K} \). Therefore, \( K^{\text{c}} \subset \hat{K}^c \), which is equivalent to \( \hat{K} \subset K^* \). □

**Corollary 3.7.** Each compact complete pluripolar set in \( \mathbb{P}^n \) is projectively convex. □

**Proposition 3.8.** Let \( U \) be an affine open set in \( \mathbb{P}^n \), and let \( E \) be a compact complete pluripolar set in \( U \). Then \( E \) is a complete pluripolar set in \( \mathbb{P}^n \).

**Proof.** Without loss of generality, we assume that \( U = \{ Z_0 \neq 0 \} \). By the proof of [10, Lemma 5.4], there are numbers \( a > b > 0 \) and a plurisubharmonic function \( u \) defined on \( U \) such that \( E = \{ X \in U : u(X) = -\infty \} \) and \( u(Z) = \log(\sqrt{|Z_1|^2 + \cdots + |Z_n|^2}/|Z_0|) - \log b \) on the set \( \{ Z \in U : \sqrt{|Z_1|^2 + \cdots + |Z_n|^2}/|Z_0| \geq a \} \). Let
\[
v(Z) = \begin{cases} 
  u(Z) - \log(|Z|/|Z_0|), & \text{if } Z \not\in U, \\
  -\log b, & \text{if } Z \in U.
\end{cases}
\]
Then \( v \in \text{PSH}_\omega (\mathbb{P}^n) \) and \( E = \{ Z \in \mathbb{P}^n : v(Z) = -\infty \} \). Therefore, \( E \) is a complete pluripolar set in \( \mathbb{P}^n \).

4. Convergence Sets in \( \mathbb{P}^n \)

For a given \( f(x) \in \mathbb{C}[[x]] = \mathbb{C}[[x_0, x_1, \ldots, x_n]] \), we are interested in the set of all \( z \in \mathbb{C}^{n+1} \) for which the restriction \( f_z(t) := f(z_0 t, z_1 t, \ldots, z_n t) \in \mathbb{C} \{ t \} \). Since for \( \lambda \in \mathbb{C} \setminus \{0\} \), the series \( f_z(t) \) and \( f_{\lambda z} \) either both converge or both diverge, it is more appropriate to consider the set of affine lines along which \( f \) converges. Thus the set of affine lines along which \( f \) converges can be identified as a subset of the projective space \( \mathbb{P}^n \). Since a \( f \in \mathbb{C}[[x]] \) converges if and only if \( f_z(t) \in \mathbb{C} \{ t \}, \forall z \in \mathbb{C}^{n+1} \), our focus will be on the divergent power series.

The convergence set of a power series \( f \in \mathbb{C}[[x]] \), denoted by \( \text{Conv}(f) \), is the set of all \( Z \in \mathbb{P}^n \) such that \( f_z(t) \in \mathbb{C} \{ t \} \) for some (and hence all) \( z \in \pi^{-1}(Z) \). It is clear that \( f \) converges if and only if \( \text{Conv}(f) = \mathbb{P}^n \).

**Definition 4.1.** A subset \( E \) of \( \mathbb{P}^n \) is said to be a convergence set (in \( \mathbb{P}^n \)) if \( E = \text{Conv}(f) \) for some divergent power series \( f \). Let \( \text{Conv}(\mathbb{P}^n) \) denote the collection of all convergence sets in \( \mathbb{P}^n \):

\[
\text{Conv}(\mathbb{P}^n) := \{ \text{Conv}(f) : f \in \mathbb{C}[[x_0, x_1, \ldots, x_n]], \ f \text{ diverges} \}. \]

Convergence sets were first studied in [1].

Consider a divergent series \( f \in \mathbb{C}[[x_0, x_1, \ldots, x_n]] \). Since

\[
f_z(t) := f(z_0 t, z_1 t, \ldots, z_n t) = \sum_{j=1}^{\infty} p_j(z) t^j, \quad p_j \in \mathbb{H}(\mathbb{C}^{n+1}), \ \deg p_j = j,
\]

we see that

\[
\text{Conv}(f) = \{ Z \in \mathbb{P}^n : \sup_{j} (p_j(Z)) < \infty \}.
\]

In fact, we have the following lemma (see [16]).

**Lemma 4.2.** Suppose that \( E \subsetneq \mathbb{P}^n \). Then \( E \in \text{Conv}(\mathbb{P}^n) \) if and only if there exists a countable family \( \mathcal{F} \) of homogeneous polynomials in \( \mathbb{H}(\mathbb{C}^{n+1}) \) such that

\[
E = \{ Z \in \mathbb{P}^n : \sup_{p \in \mathcal{F}} (p(Z)) < \infty \}. \]

Note that there is no need to require the degrees of the polynomials in \( \mathcal{F} \) to form a strictly increasing sequence. Suppose that \( E \subsetneq \mathbb{P}^n \), and that there exists a countable family \( \mathcal{F} \subset \mathbb{H}(\mathbb{C}^{n+1}) \) such that (9) holds. Let \( \{p_j\} \) be an enumeration of \( \mathcal{F} \). Raise \( p_j \) to suitable powers to obtain \( q_j := p_j^{k_j} \) so that the sequence \( \{\deg q_j\} \) is strictly increasing. Since \( \langle p_j(Z) \rangle = \langle q_j(Z) \rangle \), we see that

\[
E = \{ Z \in \mathbb{P}^n : \sup_{j} (q_j(Z)) < \infty \}.
\]

It follows that \( E = \text{Conv}(g) \), where \( g(x) = \sum_{j=1}^{\infty} q_j(x) \).

**Proposition 4.3.** If \( E \in \text{Conv}(\mathbb{P}^n) \) and \( F \in \text{Conv}(\mathbb{P}^n) \) then \( E \cap F \in \text{Conv}(\mathbb{P}^n) \).
Proof. Suppose that $E$ and $F$ are convergence sets. By Lemma 4.2, there are countable families $\mathcal{E}$ and $\mathcal{F}$ of homogeneous polynomials such that

$$E = \{Z \in \mathbb{P}^n : \sup_{p \in \mathcal{E}} \langle p(Z) \rangle < \infty \}, \quad F = \{Z \in \mathbb{P}^n : \sup_{p \in \mathcal{F}} \langle p(Z) \rangle < \infty \}.$$ 

It follows that

$$E \cap F = \{Z \in \mathbb{P}^n : \sup_{p \in \mathcal{E} \cup \mathcal{F}} \langle p(Z) \rangle < \infty \}.$$ 

Therefore, $E \cap F \in \text{Conv}(\mathbb{P}^n)$.

We do not know whether the union of two convergence sets is necessarily a convergence set.

**Proposition 4.4.** Suppose that $K$ is a compact pluripolar subset of $\mathbb{P}^n$. Then $\hat{K} \in \text{Conv}(\mathbb{P}^n)$. In particular, each projectively convex compact pluripolar set in $\mathbb{P}^n$ is a convergence set.

**Proof.** The proposition is proved by following the approach of [10, Thm 5.6]. Let

$$\mathcal{F} = \{p \in \mathcal{H}(\mathbb{C}^{n+1}) : \langle p \rangle_K \leq 1\}.$$ 

Then

$$\hat{K} = \{Z \in \mathbb{P}^n : \sup_{p \in \mathcal{F}} \langle p(Z) \rangle < \infty \}.$$ 

Let $\mathcal{E}$ be the set of polynomials in $\mathcal{F}$ whose coefficients belong to $\mathbb{Q} + i\mathbb{Q}$, the set of complex numbers whose real and imaginary parts are rational numbers. Since $\mathbb{Q} + i\mathbb{Q}$ is dense in $\mathbb{C}$, we see that

$$\hat{K} = \{Z \in \mathbb{P}^n : \sup_{p \in \mathcal{E}} \langle p(Z) \rangle < \infty \}.$$ 

Since $\mathcal{E}$ is countable, the set $\hat{K}$ is a convergence set in $\mathbb{P}^n$ by Lemma 4.2.

**Remark.** Proposition 4.4 is motivated by Theorem 5.6 in [10], which states that if $U$ is an affine open set in $\mathbb{P}^n$ and if $K$ is a compact complete pluripolar set in $U$ then $\hat{K}$ is the intersection of $U$ and a convergence set in $\mathbb{P}^n$. We observe that the same reasoning which proves that theorem can be applied to prove the more general statement Proposition 4.4. Proposition 4.4 is more general than Theorem 5.6 in [10] because (a) $\hat{K}$ may be non-compact, (b) $\hat{K}$ is not necessarily a complete pluripolar set, and (c) $\hat{K}$ does not have to lie in an affine open set.

**Corollary 4.5.** Each algebraic variety in $\mathbb{P}^n$ is a convergence set. In particular, each finite set is a convergence set.

**Theorem 4.6.** Let $E$ be a convergence set in $\mathbb{P}^n$. Then there exists an ascending sequence $\{K_j\}$ of compact pluripolar sets such that $E = \bigcup K_j = \bigcup \hat{K}_j$. In particular, $E$ is a pluripolar $F_\sigma$ set.

**Proof.** Put $E = \text{Conv}(f)$ and $f(z) = \sum_{m=1}^{\infty} f_m(z)$, where $f_m \in \mathcal{H}(\mathbb{C}^{n+1})$ with $\text{deg} f_m = m$. Then $E = \bigcup_j K_j$ where

$$K_j = \{Z \in \mathbb{P}^n : \langle f_m(Z) \rangle \leq j, \forall m\}.$$ 

Fix a $j$ and suppose that $X \in \hat{K}_j$. There is a positive integer $\ell$ such that

$$\langle p(X) \rangle \leq \ell \langle p \rangle_{K_j}.$$
for each \( p \in \mathcal{H}(\mathbb{C}^{n+1}) \), and in particular for \( p = f_m \). Thus \( X \in K_{j_k} \subseteq E \). It follows that 
\( K_j \subseteq E \) for each \( j \) and \( E = \bigcup_j K_j \). For a compact set \( K \subseteq \mathbb{P}^n \), \( \hat{K} \neq \mathbb{P}^n \) if and only if \( K \) and \( \hat{K} \) are pluripolar (see, e.g., [6, Cor. 4.4]). Since \( E \neq \mathbb{P}^n \), we see that \( K_j \) and \( \hat{K}_j \) are pluripolar for each \( j \).

**Lemma 4.7.** Let \( \{K_j\} \) be a sequence of compact pluripolar sets in \( \mathbb{P}^n \) such that \( \bigcup_{j=1}^k K_j \) is projectively convex for each \( k \), and let \( E = \bigcup K_j \). Let \( \{U_j\} \) be a sequence of open sets such that \( U_k \supseteq \bigcup_{j=1}^k K_j \) and \( \bigcap_{j=k}^\infty U_j \subseteq E \) for each \( k \). Then \( E \in \text{Conv}(\mathbb{P}^n) \).

**Proof.** Let \( E_j = \bigcup_{i=1}^j K_i \) and \( \Gamma_j = \mathbb{P}^n \setminus U_j \) for \( j = 1, 2, \ldots \). Fix a \( j \) and let \( X \in \Gamma_j \). Since \( X \) does not belong to \( E_j \), a projectively convex set, there exists a homogeneous polynomial \( p_X \) such that

\[
\langle p_X(X) \rangle > j \quad \text{and} \quad \langle p_X \rangle_{E_j} \leq 1.
\]

Since the sets \( \{Z \in \mathbb{P}^n : \langle p_X(Z) \rangle > j\} \), where \( X \in \Gamma_j \), form an open cover of \( \Gamma_j \), there exist \( X_1, \ldots, X_{\ell_j} \in \Gamma_j \) such that

\[
\bigcup_{i=1}^{\ell_j} \{Z : \langle p_{X_i}(Z) \rangle > j\} \subseteq \Gamma_j.
\]

Put \( p_{ji} = p_{X_i} \). Then

\[
\langle p_{ji} \rangle_{E_j} \leq 1,
\]

and

\[
\max_{1 \leq i \leq \ell_j} \langle p_{ji}(X) \rangle > j, \quad \text{for} \quad X \in \Gamma_j.
\]

Suppose that \( X \in E \). Then there is a positive integer \( k \) such that \( X \in E_k \). Thus \( \langle p_{ji}(X) \rangle \leq 1 \) for \( j \geq k \). Since the set \( \{p_{ji} : j < k, 1 \leq i \leq \ell_j\} \) is finite, we see that

\[
\sup_{j,i} \langle p_{ji}(X) \rangle < \infty.
\]

Conversely, suppose that \( X \in \mathbb{P}^n \) and (11) holds. Then \( \sup_{j,i} \langle p_{ji}(X) \rangle \leq k \) for some positive integer \( k \). It follows from (10) that \( X \not\in \Gamma_j \) for \( j \geq k \). Thus \( X \in \bigcap_{j=k}^\infty U_j \subseteq E \).

To summarize, \( X \in E \) if and only if (11) holds. By Lemma 4.2, \( E \in \text{Conv}(\mathbb{P}^n) \).

**Proposition 4.8.** Let \( \{K_j\} \) be an ascending sequence of projectively convex, compact, pluripolar sets such that \( E := \bigcup_j K_j \) is \( G_\delta \). Then \( E \in \text{Conv}(\mathbb{P}^n) \).

**Proof.** Since \( E \) is \( G_\delta \), there is a descending sequence \( \{U_j\} \) of open sets such that \( E = \bigcup K_j = \bigcap U_j \). It follows from Lemma 4.7 that \( E \in \text{Conv}(\mathbb{P}^n) \).

**Proposition 4.9.** Let \( \{K_j\} \) be a sequence of pairwise disjoint compact pluripolar sets in \( \mathbb{P}^n \) such that for each positive integer \( k \) the set \( \bigcup_{j=1}^k K_j \) is projectively convex. Then \( K := \bigcup K_j \) is a convergence set in \( \mathbb{P}^n \).

**Proof.** Let \( d \) denote a distance function on \( \mathbb{P}^n \) that induces the standard topology. Let

\[
r_k = (1/2) \min(1/k, \min\{d(K_i, K_j) : 1 \leq i < j \leq k + 1\}).
\]

Let \( U_k \) be the \( r_k \)-neighborhood of \( \bigcup_{j=1}^k K_j \).

Suppose that \( m \) is a positive integer and that \( X \in \bigcap_{k=m}^\infty U_k \). We claim that

\[
d(X, \bigcup_{j=1}^m K_j) < r_\ell, \quad \ell = m, m + 1, \ldots.
\]
We will prove (12) by induction on \( \ell \). If \( \ell = m \), then (12) holds because \( X \in U_m \). Suppose that \( \ell > m \) and (12) holds with \( \ell \) replaced by \( \ell - 1 \). Since \( X \in U_\ell \), we have

\[
d(X, \bigcup_{j=1}^\ell K_j) < r_\ell.
\]

On the other hand,

\[
d(X, \bigcup_{j=m+1}^\ell K_j) \geq (\bigcup_{j=1}^m K_j, \bigcup_{j=m+1}^\ell K_j) - d(X, K_j) \geq 2r_{\ell-1} - r_{\ell-1} = r_{\ell-1} \geq r_\ell,
\]

hence

\[
d(X, \bigcup_{j=m+1}^\ell K_j) \geq r_\ell.
\]

The inequalities (13) and (14) imply that \( d(X, \bigcup_{j=1}^\ell K_j) < r_\ell \). Therefore, (12) holds for all \( \ell \geq m \). This implies that \( X \in \bigcup_{j=1}^m K_j \) for each \( X \in \bigcap_{k=m}^\infty U_k \). It follows that \( \bigcap_{k=m}^\infty U_k = \bigcup_{j=1}^m K_j \). By Lemma 4.7, \( K := \bigcup K_j \) is a convergence set in \( \mathbb{P}^n \).

**Corollary 4.10.** Each countable set in \( \mathbb{P}^n \) is a convergence set.

Let \( \varphi(z) \) be a non-polynomial entire function defined on the complex plane. For \( S \subset \mathbb{C} \), let \( \tilde{S} \) be the subset of \( \mathbb{P}^2 \) defined by

\[
\tilde{S} = \{ [1 : \varphi(z)] : z \in \mathbb{P}^2 : z \in S \}.
\]

By Theorem 9.2 in [6], which depends on a deep theorem in [14], for a compact set \( K \subset \mathbb{C} \), the set \( \tilde{K} \) is projectively convex if and only if \( \mathbb{C} \setminus K \) is connected.

**Example 4.11.** The set \( \tilde{C} \) is a convergence set. To see this, write \( \tilde{C} = \bigcup \tilde{\Delta}_j \), where \( \Delta_j = \{ z \in \mathbb{C} : |z| \leq j \} \). Since \( \tilde{C} \) is a \( G_\delta \) set in \( \mathbb{P}^2 \) and since each \( \tilde{\Delta}_j \) is projectively convex, we see that \( \tilde{C} \in \text{Conv}(\mathbb{P}^2) \) by Proposition 4.8.

**Example 4.12.** Let \( \Gamma \) be the unit circle in \( \mathbb{C} \). Then \( \tilde{\Gamma} \in \text{Conv}(\mathbb{P}^2) \). To see this, write \( \tilde{\Gamma} = \bigcup \tilde{S}_j \), where

\[
\tilde{S}_j = \{ e^{it} : 0 \leq t \leq 2\pi - 1/j \}.
\]

Since \( \tilde{\Gamma} \) is \( G_\delta \) and since each \( \tilde{S}_j \) is projectively convex, we see that \( \tilde{\Gamma} \) is a convergence set by Proposition 4.8.

**Example 4.13.** Let \( S \) be the subset of \( \mathbb{C} \) obtained by removing an open triangle from a closed triangle. Then \( \tilde{S} \) is a convergence set. This follows from an argument very similar to the previous example.

**Example 4.14.** Let \( S \) be the subset of \( \mathbb{C} \) obtained by removing a finite number of open triangles from a closed triangle. Then \( \tilde{S} \) is a convergence set. This follows from the previous example and Proposition 4.3.

Let \( \Lambda \) be a closed triangle in \( \mathbb{C} \). The \textit{“open middle triangle”} of \( \Lambda \) is the open triangle whose vertices are the midpoints of the sides of \( \Lambda \). Let \( V_1 \) be the open middle triangle of \( \Lambda \). The set \( \Lambda \setminus V_1 \) is the union of three congruent closed triangles. Let \( V_2, V_3, V_4 \) denote the open middle triangles of the three closed triangles whose union is \( \Lambda \setminus V_1 \). Similarly, let \( V_5, \ldots, V_{13} \) denote the open middle triangles of the nine closed triangles whose union is \( \Lambda \setminus \bigcup_{j=1}^{13} V_j \). Continuing in this way, we obtain a sequence \( \{V_j\} \) of open triangles. The set \( E := \Lambda \setminus \bigcup_{j=1}^\infty V_j \) is called Sierpinsky’s triangle. Let \( E_k = \Lambda \setminus \bigcup_{j=1}^k V_j \) for \( k = 1, 2, \ldots \). By Example 4.14, each \( E_k \) is a convergence set in \( \mathbb{P}^2 \). Note that \( E = \cap_{k=1}^\infty E_k \).
Proposition 4.15. The set $\tilde{E}$ is not a convergence set.

Proof. Seeking for a contradiction, suppose that $\tilde{E} = \text{Conv}(f)$ and $f = \sum p_m$, where $p_m$ is a homogeneous polynomial of degree $m$ in three variables $z_0, z_1, z_2$. Let

$$T_k = \{z \in \mathbb{C} : |p_m(1, z, \varphi(z))|^{1/m} \leq k \ \forall m\}.$$ 

Then each $T_k$ is a closed subset of $\mathbb{C}$ and $E = \bigcup T_k$. By the maximum modulus principle, $\mathbb{C} \setminus T_k$ is connected for each $k$. We observe that the set $E$ has the following property: if $S$ is a closed set in the topological space $E$ with nonempty interior, then $S$ contains the sides of a small triangle and hence $\mathbb{C} \setminus S$ is disconnected. It follows that each $T_k$ has empty interior, which contradicts the Baire category theorem. \hfill \square

Let $\{\tilde{E}_k\}$ be the sequence defined above. Note that each $\tilde{E}_k \in \text{Conv}(\mathbb{P}^2)$, but $\cap \tilde{E}_k \notin \text{Conv}(\mathbb{P}^2)$.

Fix a positive number $M$. Let

$$S_0 = \{|1 : 0 : \cdots : 0\}, \text{ and for } k = 1, \ldots, n,$$

$$S_k = \{X \in \mathbb{P}^n : |X_0|^2 + \cdots + |X_{k-1}|^2 \leq M^2|X_k|^2, X_{k+1} = \cdots = X_n = 0\}.$$

Put

$$K_M = \bigcup_{k=0}^n S_k. \tag{15}$$

Then $\{K_m\}$ is an ascending sequence of closed sets with $\mathbb{P}^n = \bigcup_{m=1}^\infty K_m$.

Recall that the set $\Pi$ of all hyperplanes in $\mathbb{P}^n$ is naturally isomorphic to $\mathbb{P}^n$. The set of all hyperplanes in $\mathbb{P}^n$ passing through a fixed point is a hyperplane in $\Pi$.

Lemma 4.16. If $K$ is a closed subset of $\mathbb{P}^n$ contained in an affine open set, and if $\xi \in \mathbb{P}^n$, then $K \cup \{\xi\}$ is contained in an affine open set.

Proof. Let $R = \{H \in \Pi : H \cap K = \emptyset\}$ and $S = \{H \in \Pi : \xi \in H\}$. Then $R$ is a non-empty open set in $\Pi$ and $S$ is a hyperplane in $\Pi$. Thus $R \setminus S$ is non-empty. This means that there is a hyperplane $H$ with $H \cap (K \cup \{\xi\}) = \emptyset$. Therefore, $K \cup \{\xi\}$ is contained in the affine open set $\mathbb{P}^n \setminus H$. \hfill \square

If $1 \leq \nu \leq n$ and if $u_0, \ldots, u_\nu$ are linearly independent vectors in $\mathbb{C}^{n+1}$, let $\text{span}(u_0, \ldots, u_\nu)$ be the $\nu$-dimensional linear space in $\mathbb{P}^n$ defined by

$$\text{span}(u_0, \ldots, u_\nu) := \{\pi(c_0 u_0 + \cdots + c_\nu u_\nu) : (c_0, \ldots, c_\nu) \in \mathbb{C}^{\nu+1} \setminus \{0\}\},$$

where $\pi : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{P}^n$ is the standard projection.

Lemma 4.17. For each $M > 0$, the set $K_M$ is contained in an affine open set.

Proof. Let $e_0, \ldots, e_n$ be the standard basis of $\mathbb{C}^{n+1}$, and let $\varepsilon$ be a sufficiently small positive number. Let $v_j = e_j + \varepsilon e_{j+1}$ for $j = 0, \ldots, n - 1$.

Put $V_j = \text{span}(v_0, \ldots, v_j)$ for $j = 0, \ldots, n - 1$, and $V = V_{n-1}$. Also, let $W_j = \text{span}(e_0, \ldots, e_j)$. Note that

$$V \cap W_j \subset V_{j-1}, \text{ for } j = 1, \ldots, n.$$
Since $S_j \subset W_j$, it follows that $V \cap S_j \subset V_{j-1}$ for $j \geq 1$. It is clear that $V \cap S_0 = \emptyset$. For $j \geq 1$, since $W_{j-1} \cap S_j = \emptyset$, and since $V_{j-1}$ is close to $W_{j-1}$ for sufficiently small $\varepsilon$, we see that $V_{j-1} \cap S_j = \emptyset$. It follows that

$$V \cap K_M = \bigcup_{j=0}^n (V \cap S_j) \subset \bigcup_{j=1}^n (V_{j-1} \cap S_j) = \emptyset.$$ 

Therefore $K_M$ is contained in the affine open set $\mathbb{P}^n \setminus V$. \hfill \Box

**Theorem 4.18.** The union of a countable collection of closed complete pluripolar sets in $\mathbb{P}^n$ is a convergence set in $\mathbb{P}^n$.

**Proof.** Let $\{E_m\}$ be a sequence of closed complete pluripolar sets in $\mathbb{P}^n$ and let $E = \bigcup E_m$. Without loss of generality, we assume that the sequence $\{E_m\}$ is ascending, since the union of a finite number of closed complete pluripolar sets in $\mathbb{P}^n$ is a closed complete pluripolar set. Recall that $\mathbb{P}^n = \bigcup K_m$, where $\{K_m\}$ is ascending and each $K_m$ is a compact set contained in an affine open set. We have $E = \bigcup (E_m \cap K_m)$.

For each positive integer $m$, we shall construct a sequence $\{h_{mk}\}_{k=1}^\infty$ of homogeneous polynomials such that for all $k$,

1. $\langle h_{mk} \rangle_{K_m \cap E_m} \leq 1$,
2. $\langle h_{mk} \rangle \leq m$,

and

3. $\bigcup_{k=1}^\infty \{X \in \mathbb{P}^n : \langle h_{mk}(X) \rangle > m/2\} \supset \mathbb{P}^n \setminus E_m$.

Fix a positive integer $m$. Let $Y \in \mathbb{P}^n \setminus E_m$. By Lemmas 4.16 and 4.17, we see that $(K_m \cap E_m) \cup \{Y\}$ is contained in an affine open set $V$. Since $V \cap E_m$ is a (relatively) closed complete pluripolar set in $V \approx \mathbb{C}^n$, the set $V \cap E_m$ has Property J in $V$ by Theorem 2.7. Hence there is a number $r$ with $0 < r < 1$ such that $\tau(Y, K_m \cap E_m, r) = 0$, which means, in terms of homogeneous coordinates, that there is a sequence $\{p_j\}$ in $\mathcal{H}((\mathbb{C}^{n+1})$ such that

$$\lim_{j \to \infty} \langle p_j \rangle_{K_m \cap E_m} = 0,$$

and for all $j$, $\langle p_j(Y) \rangle \geq r$, $\langle p_j \rangle \leq 1$.

Choose a positive rational number $\beta = a/b < 1$, where $a, b$ are positive integers, such that $(r/m)^\beta > 1/2$. There is a homogeneous polynomial $p \in \mathcal{H}((\mathbb{C}^{n+1})$ such that

$$\langle p \rangle_{K_m \cap E_m} < m^{-1/\beta}, \langle p(Y) \rangle \geq r, \text{ and } \langle p \rangle_{\mathbb{P}^n} \leq 1.$$

Let $v = \deg p$ and $y \in \pi^{-1}(Y)$. Define $g \in \mathcal{H}((\mathbb{C}^{n+1})$ by

$$q(x) = (mx \cdot \overline{y}/|y|)^v = [m|y|^{-1}(x_0\overline{y}_0 + \cdots + x_n\overline{y}_n)]^v.$$ 

By the Cauchy-Schwarz inequality, we have

$$\langle q \rangle_{\mathbb{P}^n} = m = \langle g(Y) \rangle.$$ 

Let $h = p^a q^{b-a}$. Then $h \in \mathcal{H}((\mathbb{C}^{n+1})$ and $\deg h = bv$. For each $X \in \mathbb{P}^n$, $\langle h(X) \rangle = \langle p(X) \rangle^\beta \langle g(X) \rangle^{1-\beta}$. Hence,

$$\langle h \rangle_{\mathbb{P}^n} \leq m^{1-\beta} \leq m,$$

$$\langle h \rangle_{E_m \cap K_m} < m^{-1}m^{1-\beta} = m^{-\beta} \leq 1,$$

$$\langle h(Y) \rangle \geq r^\beta m^{1-\beta} = (r/m)^\beta m > m/2.$$

To summarize, there is an $h_Y \in \mathcal{H}((\mathbb{C}^{n+1})$ such that

$$\langle h_Y \rangle_{\mathbb{P}^n} \leq m, \langle h_Y \rangle_{E_m \cap K_m} < 1, \text{ and } \langle h_Y(Y) \rangle > m/2.$$
Let \( U_Y := \{ X : \langle h_Y(X) \rangle > m/2 \} \). Then \( U_Y \) is a neighborhood of \( Y \). The open cover \( \{ U_Y : Y \in \mathbb{P}^n \setminus E_m \} \) of \( \mathbb{P}^n \setminus E_m \) contains a countable subcover \( \{ U_{Y_k} : k = 1, 2, \ldots \} \). Put \( h_{mk} = h_{Y_k} \). Then the sequence \( \{ h_{mk} \} \) satisfies (i), (ii) and (iii).

Suppose that \( X \in E \). Then there is an \( m_0 \geq 2 \) such that \( X \in (E_{m_0} \cap K_{m_0}) \). We have \( \langle h_{mk}(X) \rangle \leq 1 \) for \( m \geq m_0 \), and \( \langle h_{mk}(X) \rangle \leq m_0 - 1 \) for \( m < m_0 \). Hence, \( \sup_{m,k} \langle h_{mk}(X) \rangle \leq m_0 - 1 < \infty \).

Suppose that \( X \in \mathbb{P}^n \setminus E \). For each \( m, X \in \mathbb{P}^n \setminus E_m \), hence \( \sup_k \langle h_{mk}(X) \rangle \geq m/2 \). Thus \( \sup_{m,k} \langle h_{mk}(X) \rangle = \infty \).

Therefore,

\[
E = \{ X \in \mathbb{P}^n : \sup_{m,k} \langle h_{mk}(X) \rangle < \infty \}.
\]

By Lemma 4.2, \( E \subseteq \text{Conv}(\mathbb{P}^n) \).

**Corollary 4.19.** The union of a countable collection of algebraic varieties in \( \mathbb{P}^n \) is a convergence set in \( \mathbb{P}^n \).

The converse of Theorem 4.18 is not true. The set \( \Lambda := \{ [1 : z : \varphi(z) ] : z \in \mathbb{C}, |z| = 1 \} \) in Example 4.12 is a convergence set, but it is not a countable union of complete pluripolar sets. Recall that \( \varphi \) is a non-polynomial entire function defined on \( \mathbb{C} \). Seeking for a contradiction, suppose that \( \Lambda = \cup F_j \), where \( F_j \) are complete pluripolar sets in \( \mathbb{P}^2 \). Let \( Q := \{ [1 : z : \varphi(z) ] : z \in \mathbb{C} \} \). For each \( j \), since \( F_j \not\subseteq Q \), and since \( F_j \) is a complete pluripolar set in \( \mathbb{P}^2 \), it follows that \( F_j \) is polar in \( Q \). Therefore, \( \Lambda = \cup F_j \) is polar in \( Q \), which is clearly false.

5. **Affine convergence sets**

Let \( \Lambda_n \) be the set of series \( f(t,x) = \sum_{j=0}^{\infty} P_j(x)t^j \in \mathbb{C}[x_1, \ldots, x_n][[t]] \) such that \( \deg P_j \leq j \) for all \( j \). For \( f \in \Lambda_n \), let \( \text{Conv}_a(f) \) be the set of \( x \in \mathbb{C}^n \) for which \( f(t,x) \) converges as a series of a single indeterminate \( t \):

\[
\text{Conv}_a(f) := \{ x \in \mathbb{C}^n : f(t,x) \in \mathbb{C}\{t\} \}.
\]

By Hartogs’ theorem, \( \text{Conv}_a(f) = \mathbb{C}^n \) if and only if \( f \in \mathbb{C}\{t,x_1, \ldots, x_n\} \), the set of convergent series in \( t, x_1, \ldots, x_n \).

**Definition 5.1.** A subset \( E \) of \( \mathbb{C}^n \) is said to be an affine convergence set (in \( \mathbb{C}^n \)) if \( E = \text{Conv}_a(f) \) for some divergent power series \( f \in \Lambda_n \). Let \( \text{Conv}_a(\mathbb{C}^n) \) denote the collection of all affine convergence sets in \( \mathbb{C}^n \):

\[
\text{Conv}_a(\mathbb{C}^n) := \{ \text{Conv}_a(f) : f \in \Lambda_n, \ f \text{ diverges} \}.
\]

Affine convergence sets were studied in [12] and [13].

Note that if \( x \in \text{Conv}_a(f) \) for a divergent series \( f \) it does not follow that \( \lambda x \in \text{Conv}_a(f) \forall \lambda \in \mathbb{C} \), and therefore \( \pi(\text{Conv}_a(f)) \) in general is not a convergence set in \( \mathbb{P}^{n-1} \).

In analogy with Lemma 4.2, we have the following lemma.

**Lemma 5.2.** Suppose that \( E \subseteq \mathbb{C}^n \). Then \( E \subseteq \text{Conv}_a(\mathbb{C}^n) \) if and only if there exists a sequence \( \{ k_j \} \subset \mathbb{N} \) and a sequence \( \{ P_j \} \) of polynomials with \( P_j \in \mathbb{P}_{k_j}(\mathbb{C}^n) \) for all \( j \) such that

\[
E = \{ x \in \mathbb{C}^n : \sup_j \langle P_j(x) \rangle_{k_j} < \infty \}.
\]

\( \square \)
Let \( \iota : \mathbb{C}^n \to \mathbb{P}^n \) be defined by \( \iota(x_1, \ldots, x_n) = [1 : x_1 : \cdots : x_n] \). Then \( \iota \) embeds \( \mathbb{C}^n \) into \( \mathbb{P}^n \) and identifies \( \mathbb{C}^n \) with the affine open set \( \iota(\mathbb{C}^n) = U_0 := \{ Z \in \mathbb{P}^n : Z_0 \neq 0 \} \). Define \( \tau : \Lambda_n \to \mathbb{C}[[x_0, \ldots, x_n]] \) by

\[
\tau\left(\sum_{j=0}^{\infty} P_j(x)t^j\right) = \sum_{j=0}^{\infty} x_0^j P_j(x/x_0).
\]

Note that \( p_j(x_0, x) := x_0^j P_j(x/x_0) \) belong to \( \mathcal{H}(\mathbb{C}^{n+1}) \) and \( \deg p_j = j \). We also have \( \langle P_j(x) \rangle_j = \langle p_j(\iota(x)) \rangle \) for \( j \geq 1 \). It follows from Lemmas 4.2 and 5.2 that

\[
\iota(\text{Conv}_a(f)) = U_0 \cap \text{Conv}(\tau(f)).
\]

Consequently, we have the following proposition.

**Proposition 5.3.** Suppose that \( E \subseteq \mathbb{C}^n \). Then \( E \in \text{Conv}_a(\mathbb{C}^n) \) if and only if \( \iota(E) = U_0 \cap F \) for some \( F \in \text{Conv}(\mathbb{P}^n) \).

For an affine convergence set \( E \) in \( \mathbb{C}^n \), the set \( \iota(E) \) may or may not be a convergence set in \( \mathbb{P}^n \).

**Example 5.4.** Let \( E = \{(x, e^x) : x \in \mathbb{C}\} \subset \mathbb{C}^2 \). Then the set \( \iota(E) \) equals the set \( \hat{C} \) in Example 4.11 when \( \varphi(x) = e^x \). It follows that \( \iota(E) \) is a convergence set in \( \mathbb{P}^2 \), and hence \( E \) is an affine convergence set in \( \mathbb{C}^2 \).

**Example 5.5.** Let \( E = \{(x, 0) : x \in \mathbb{C}\} \subset \mathbb{C}^2 \). Then \( \iota(E) = U_0 \cap H \), where \( H = \{[Z_0 : Z_1 : Z_2] : Z_2 = 0\} \). Since \( H \) is a convergence set in \( \mathbb{P}^2 \) by Corollary 4.5, we see that \( E \) is an affine convergence set in \( \mathbb{C}^2 \). Since \( H \) is a copy of \( \mathbb{P}^1 \), it follows from Theorem 4.6 that for each convergence set \( Q \) in \( \mathbb{P}^2 \), either \( H \subset Q \), or \( H \cap Q \) is polar in \( H \). Therefore, \( \iota(E) = U_0 \cap H \) is not a convergence set in \( \mathbb{P}^2 \).

For a compact subset \( K \) of \( \mathbb{C}^n \), let

\[
\hat{K}_a = \iota^{-1}(U_0 \cap \overline{\iota(K)}).
\]

Intuitively, \( \hat{K}_a \) is the projective hull of \( K \) minus the part at \( \infty \). Equivalently, \( \hat{K}_a \) is the set of all \( x \in \mathbb{C}^n \) for which there is a constant \( C = C_x > 0 \) such that \( \langle P(x) \rangle_j \leq C \langle P \rangle_{j,K} \) for all \( j \in \mathbb{N} \) and all \( P \in \mathcal{P}_j(\mathbb{C}^n) \).

**Theorem 5.6.** Let \( E \) be an affine convergence set in \( \mathbb{C}^n \). Then there exists an ascending sequence \( \{K_j\} \) of compact pluripolar sets such that \( E = \bigcup K_j = \bigcup \hat{K}_{j,a} \). In particular, \( E \) is a pluripolar \( F_\sigma \) set.

**Theorem 5.7.** The union of a countable collection of closed complete pluripolar sets in \( \mathbb{C}^n \) is an affine convergence set in \( \mathbb{C}^n \).

**Corollary 5.8.** The union of a countable collection of analytic varieties in \( \mathbb{C}^n \) is an affine convergence set in \( \mathbb{C}^n \).

**Corollary 5.9.** The finite sets and the countable sets in \( \mathbb{C}^n \) are affine convergence sets.

The proofs of Theorems 5.6 and 5.7 are routine modifications of the respective proofs of Theorems 4.6 and 4.18. Note that Theorem 5.7 is not a consequence of Theorem 4.18, since a complete pluripolar set in an affine open set in \( \mathbb{P}^n \) in general does not extend to a complete pluripolar set in \( \mathbb{P}^n \).
Acknowledgment. We thank the referee for his comments and suggestions that greatly improved the paper. We thank B. Fridman, N. Levenberg, J. Ribón and A. Sadullaev for helpful discussions. We are very grateful to E. Poletsky for patiently answering our many questions.

References

[1] S.S. Abhyankar, T.T. Moh, A reduction theorem for divergent power series, *J. Reine Angew. Math.*, **241** (1970), 27–33.

[2] E. Bedford, B.A. Taylor, Plurisubharmonic functions with logarithmic singularities, *Ann. Inst. Fourier (Grenoble)*, **38** (1988), no. 4, 133-171.

[3] J. Deny, Sur les infinis d’un potential, *C. R. Acad. Sci. Paris Sér. I Math.*, **224**(1947), 524–525.

[4] V. Guedj, A. Zeriahi, Intrinsic capacities on compact Kähler manifolds, *J. Geom. Anal.*, **15** (2005), 607–639.

[5] F. Hartogs, Zur Theorie der analytischen Funktionen mehrerer unabhängiger Veränderlichen, *Math. Ann.*, **62** (1906), 1–88.

[6] F.R. Harvey, B. Lawson, Projective hulls and projective Gelfand transform, *Asian J. Math.*, **10** (2006) 607–646.

[7] B. Josefson, On the equivalence between locally polar and globally polar sets for plurisubharmonic functions on \(\mathbb{C}^n\), *Arkiv för Mat.*, **16** (1978), 109–115.

[8] M. Klimek, *Pluripotential theory*, Clarendon Press, New York, 1991.

[9] P. Lelong, On a problem of M.A. Zorn, *Proc. Amer. Math. Soc.*, **2** (1951), 11–19.

[10] N. Levenberg, and R.E. Molzon, Convergence sets of a formal power series, *Math. Z.*, **197** (1988), 411–420.

[11] N. Levenberg, E. Poletsky, Pluripolar hulls, *Michigan Math. J.*, **46** (1999), no. 1, 151–162.

[12] R. Pérez-Marco, A note on holomorphic extensions, preprint, 2000.

[13] J. Ribón, Holomorphic extensions of formal objects, *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (5)*, **3** (2004), 657–680.

[14] A. Sadullaev, An estimate for polynomials on analytic sets, *Math. USSR Izvestia*, **20** (1983), 493–502.

[15] J. Siciak, Extremal plurisubharmonic functions and capacities in \(\mathbb{C}^n\), *Sophia Kokyuroku Math.*, **14** (1982), Sophia University, Tokyo.

[16] A. Sathaye, Convergence sets of divergent power series, *J. Reine Angew. Math.*, **283** (1976), 86–98.

[17] A. Zeriahi, Ensembles pluripolaires exceptionnels pour la croissance partielle des fonctions holomorphes, *Ann. Polon. Math.*, **50** (1989), 81–91.

DMA@MATH.WICHITA.EDU, DEPARTMENT OF MATHEMATICS, WICHITA STATE UNIVERSITY, WICHITA, KS 67260-0033, USA

NEELON@CSUSM.EDU, DEPARTMENT OF MATHEMATICS, CALIFORNIA STATE UNIVERSITY SAN MARCOS, CA 92096-0001, USA