Abstract. Randomized linear solvers randomly compress and solve a linear system with compelling theoretical convergence rates and computational complexities. However, such solvers suffer a substantial disconnect between their theoretical rates and actual efficiency in practice. Fortunately, these solvers are quite flexible and can be adapted to specific problems and computing environments to ensure high efficiency in practice, even at the cost of lower effectiveness (i.e., having a slower theoretical rate of convergence). While highly efficient adapted solvers can be readily designed by application experts, will such solvers still converge and at what rate? To answer this, we distill three general criteria for randomized adaptive solvers, which, as we show, will guarantee a worst-case exponential rate of convergence of the solver applied to consistent and inconsistent linear systems irrespective of whether such systems are over-determined, under-determined or rank-deficient. As a result, we enable application experts to design randomized adaptive solvers that achieve efficiency and can be verified for effectiveness using our theory. We demonstrate our theory on twenty-six solvers, nine of which are novel or novel block extensions of existing methods to the best of our knowledge.

Key words. Block Solvers, Adaptive Solvers, Randomized Solvers, Linear Systems, Sketching

AMS subject classifications. 15A06, 15B52, 65F10, 65F25, 65N75, 65Y05, 68W20, 68W40

1. Introduction. Solving linear systems and least squares problems remain critical operations in scientific and engineering applications. As the size of systems or the sheer number of systems that need to be solved grow, faster and approximate linear solvers have become essential to scalability. Recently, randomized linear solvers have become of interest as they can compress the information in the original system in a problem-blind fashion, which can then be used to inexpensively and approximately solve the original system [39]. Moreover, by iterating on this procedure, randomized linear solvers will converge exponentially fast to the solution of the original system [29]. In fact, a rather simple randomized linear system solver was recently shown to achieve a universal exponential rate of convergence for any consistent linear system with high probability [36].

Despite such an incredible result, as we show through a salient example (see Section 2), randomized linear solvers suffer a substantial disconnect between their convergence rate theory and actual efficiency in practice because they often violate simple computing principles (e.g., the locality principle [9]). Briefly, in the example in Section 2, an “oracle” linear solver inspired by [36] is applied to a specific $10^7 \times 100$ system such that it only requires 100 arithmetic operations to find a solution with absolute error of $10^{-16}$, yet is slower than block Kaczmarz—which, in theory, requires over $10^{10}$ arithmetic operations to find such a solution—because of access patterns that violate data locality. Unfortunately, nearly all variations of such linear solvers that exist [2, 21, 37, 4, 41, 14, 24, 25, 26, 5, 15, 22, 13, 32, 35] can be shown to suffer from this disconnect between their theoretical convergence rates and actual efficiency by specific choices of the linear system, software environment or hardware.

A pessimistic view of these solvers would imply that they should be wholly abandoned. An alternative perspective would suggest a better prognosis: because of the
adaptability of such solvers, they can be highly tailored to specific linear systems, software environments, and hardware to achieve high efficiency even at the expense of worse theoretical convergence rates. This latter view is the one adopted in this work.

A bevy of adapted methods can be designed and deployed by atomizing, composing and customizing key components of randomized linear solvers.\(^1\) Owing to the freedom of creating such solvers, understanding whether the efficient highly-adapted method will still converge and at what cost to the rate (e.g., will the solver now converge sub-exponentially?) becomes integral to a practitioner’s decision to implement the method.

To address this consideration, a handful of adaptive solvers were shown to retain exponential convergence by \(^1\), but in a limited context: the set of projections must be finite; and the exactness assumption \([33, \text{Assumption 2}]\) must be satisfied, which is generally difficult to verify in practice.\(^2\) In our previous work \([28, 27]\), adaptive solvers relying on vector operations were shown to retain exponential convergence. While our previous work accounted for a number of existing solvers (e.g., \([37, 41, 14, 35, 2, 21, 5, 15]\)), adaptive solvers using high-efficiency block operations did not fall within our results. As block operations have been critical to achieving high-efficiency in traditional factorizations (e.g., QR \([11, \text{Ch. 5}]\)), classical iterative methods (e.g., Krylov Iterations \([34, \text{Ch. 6}]\)), randomized factorization methods \([18, \S 16.2]\), and on GPUs \([6]\), adaptive solvers using block operations must be shown to retain exponential convergence.

Therefore, in this work, we provide generic sufficient conditions, that if satisfied by an a randomized block adaptive solver (RBAS), will guarantee a worst-case (i.e., with probability one) exponential rate of convergence.\(^3\) In particular, we provide these generic sufficient conditions and consequent worst-case exponential convergence rates in two contexts:

1. for row-action RBASs on consistent linear systems, which may be over-determined, under-determined or rank deficient (see Corollaries 3.8 and 3.10); and
2. for column-action RBASs for linear least squares problems, which may be over-determined, under-determined or rank deficient (see Corollaries 3.18 and 3.20).

We then show how to apply these results to twenty-six different solvers, nine of which—to the best of our knowledge—are either novel or novel block-operation extensions of existing methods. Thus, in this work, we give end-users the tools to design effective solvers for their specific problems and environments.

The remainder of this work is organized as follows. In section 2, we demonstrate the disconnect between rates of convergence and efficiency. In section 3, we present the two archetype RBASs, provide examples for each, state and discuss the refined properties that such solvers satisfy, and state our convergence results for each type. In section 4, we provide a common formulation for the two types of RBASs, prove the convergence of these methods using this common formulation, and interlace numerical experiments that demonstrate key parts of the theory. In section 5, we show how to apply our convergence theory to a variety of existing and novel RBASs, and provide numerical experiments where appropriate. In section 6, we conclude.

\(^1\)We are implementing a software package to enable this approach. See https://github.com/numlinalg/RLinearAlgebra.jl.
\(^2\)See Appendix A.19 on how we can eliminate this assumption for an important class of methods.
\(^3\)Other worst-case rates can be provided using similar ideas that we present herein, but we do not know of a context where such rates are useful.
2. Counter Example. Here, we demonstrate that the theoretical convergence rates of randomized solvers can be quite disconnected from their actual efficiency in practice. Consider a consistent, linear system with \( n = 10^7 \) equations and \( d = 100 \) unknowns represented with double precision. Owing to the size of the system relative to the 4 Gigabytes of memory available on an Intel i5 8th Generation CPU computer, the system is split into 0.5 Gigabyte chunks, which contain at most 66,666 equations each.

Consider an “oracle” solver inspired by [36], which can randomly replace \( d \) equations in the original system in such a way that the coefficients of the resulting replaced \( d \) equations correspond to the rows of the \( d \times d \) identity matrix and the system is still consistent. Then, with knowledge of the index of these \( d \) equations, the solver applies Kaczmarz to these rows to solve the system. As a result, the oracle solver requires \( d \) iterations and \( O(d) \) arithmetic operations. For this specific example, the oracle solver requires about 100 arithmetic operations.

Consider an alternative solver, the random block Kaczmarz solver, which will randomly choose a chunk from the system and perform a block updated to its iterate. In our example, a single block Kaczmarz update requires approximately \( 10^7 \) arithmetic operations, and, with an expected squared error rate of convergence of at least 0.993 [24, Theorem 1.2], will required over 5,500 iterations and, correspondingly, over \( 5 \times 10^{10} \) operations to achieve an expected absolute squared error of \( 10^{-16} \).

Clearly, from a theoretical perspective, the “oracle” solver is substantially faster than the random block Kaczmarz solver as the former requires 10 fold fewer iterations and \( 10^8 \) fewer operations. However, when applied to the system, the “oracle” solver is trounced by random block Kaczmarz (see Figure 1). To understand this, the “oracle” solver needs to read in a new chunk (in expectation and in reality) to access the equations that it has embedded, which is highly expensive as it violates data locality. On the other hand, the block Kaczmarz solver simply does what it can with the information that is given in a single chunk, which turns out to contain sufficient information for finding a high quality solution in one iteration. To summarize, these solvers behave very differently in their theoretical convergence rates and in practice as this example shows.

This observation is motivation to adapt such solvers to ensure that they are efficient for specific problems and computing environments. However, as the next example will demonstrate, such efficient solvers can lose effectiveness (i.e., suffer from arbitrarily slow convergence behavior). As a result, in this work, we provide sufficient conditions that, if satisfied by an adapted solver, will be effective—that is, the solver will have a worst-case exponential rate of convergence.

3. Randomized Block Adaptive Solvers. Consider solving the consistent linear system

\[
Ax = b,
\]

or consider finding the least squares solution for a (possibly) inconsistent system by solving

\[
\min_x \|Ax - b\|_2,
\]

where \( A \in \mathbb{R}^{n \times d} \); \( x \in \mathbb{R}^d \); and \( b \in \mathbb{R}^n \). We emphasize we have not required that \( n < d \), \( n > d \) or that \( A \) has full rank; in other words, we allow for underdetermined systems, over determined systems and rank deficient linear systems. To solve these systems, we
A Comparison of an Optimal Algorithm against Randomized Block Kaczmarz

Fig. 1. A comparison runtime of the “Oracle” Algorithm against Block Kaczmarz for the described system. The optimal algorithm achieves an absolute error of 0 in 100 iterations requiring 2.47 seconds. Block Kaczmarz achieves an absolute error of $10^{-15}$ in 1 iteration requiring 0.68 seconds.

will consider two archetypes of RBAS methods: row-action RBAS methods for (3.1) or column-action RBAS methods for (3.1) and (3.2). We will define each variation below, provide examples, state the assumptions, and present the main convergence results.

### 3.1. Row-action RBASs

For row-action methods, we will need to assume

**Assumption 3.1.** The system, (3.1), is consistent. That is, the set $\mathcal{H} := \{x \in \mathbb{R}^d : Ax = b\}$ is nonempty.

With this assumption, we begin with an iterate $x_0 \in \mathbb{R}^d$ and some prior information, encapsulated by $\zeta_{-1} \in \mathcal{Z}$, where $\mathcal{Z}$ is finite in some sense (e.g., the product of a finite set and a finite dimensional linear space). We then generate a sequence of iterates, $\{x_k : k \in \mathbb{N}\}$, according to

\begin{equation}
    x_{k+1} = x_k - A^T W_k (W_k^T A A^T W_k)^+ W_k^T (Ax_k - b),
\end{equation}

where $\cdot^+$ represents a pseudo-inverse; and $\{W_k \in \mathbb{R}^{n \times n_k}\}$ are possibly random quantities (i.e., vectors or matrices) generated according to a possibly random, adaptive procedure, $\varphi_R$, which supplies

\begin{equation}
    W_k, \zeta_k = \varphi_R(A, b, \{x_j : j \leq k\}, \{W_j : j < k\}, \{\zeta_j : j < k\}) \in \mathbb{R}^{n \times n_k} \times \mathcal{Z}.
\end{equation}

We make several comments about this procedure. First, $n_k$ can be selected adaptively so long as it is known given the arguments of $\varphi_R$. Second, $\zeta_k$ contains information generated from previous iterations that may be essential to the operation of the adaptive procedure (see examples below). Third, we can change the inner product space as is done is [14] without issue (see Appendix A.19). The next examples illustrate this formulation of row-action RBASs.

**Example 3.2** (Cyclic Vector Kaczmarz). The cyclic vector Kaczmarz method cycles through the equations of $Ax = b$ (without reordering) and updates the current iterate by projecting it onto the hyperplane that solves the selected equation. To rephrase the cyclic vector Kaczmarz method in our framework, let $\{e_i : i = 1, 0.5, 1, 1.5, 2, 2.5\}

ht
1, . . . , n} denote the standard basis elements of \( \mathbb{R}^n \). Moreover, let \( \mathfrak{F} = \{0\} \cup \mathbb{N} \), and \( \zeta_{-1} = 0 \). We then define \( \varphi_R \) to be

\[
(3.5) \quad \varphi_R(A, b, \{x_j : j \leq k\}, \{W_j : j < k\}, \{\zeta_j : j < k\}) = (e_{\text{rem}(\zeta_{k-1}, n) + 1}, \zeta_{k-1} + 1).
\]

With this choice of \((W_k, \zeta_k)\), we readily see that the described cyclic vector Kaczmarz method is equivalent to

\[
(3.6) \quad x_{k+1} = x_k - A^T e_{\text{rem}(\zeta_{k-1}, n) + 1} \frac{e_{\text{rem}(\zeta_{k-1}, n) + 1}^T (Ax_k - b)}{\|A^T e_{\text{rem}(\zeta_{k-1}, n) + 1}\|^2_2},
\]

which is exactly (3.3). We highlight that \( \varphi_R \) only depends on \( \zeta_{k-1} \) and the number of equations in the linear system, which will be important in our discussion below. ■

**Example 3.3** (Random Permutation Block Kaczmarz). The random permutation block Kaczmarz method partitions the equations of \( Ax = b \) (not necessarily equal partitions) into blocks of equations, generates a random permutation of the blocks, selects a block by cycling through the permutation, updates the current iterate by projecting it onto the hyperplane that solves all of the equations in the block, and, if the random permutation is exhausted, generates a new random permutation of the blocks.

To rephrase this method in our framework, let \( \{E_i\} \) be matrices whose columns are generated by some partitioning of the identity matrix in \( \mathbb{R}^{n \times n} \), and let \( \epsilon = |\{E_i\}| \). Moreover, let \( \mathfrak{F} \) be product of the set of all permutations of \( \{1, \ldots, \epsilon\} \) with the empty set, and \( \{0\} \cup \mathbb{N} \). Let \( \{Z_k : k + 1 \in \mathbb{N}\} \) be an independent random permutations of \( \{1, \ldots, \epsilon\} \). Let \( \zeta_{-1} = (Z_0, 0) \). Then, we can define \( \varphi_R \) to be

\[
(3.7) \quad \varphi_R(A, b, \{x_j : j \leq k\}, \{W_j : j < k\}, \{\zeta_j : j < k\}) = \begin{cases} 
(E_{\text{rem}(\zeta_{k-1}[1], \epsilon) + 1}[\zeta_{k-1}[1]], \zeta_{k-1}[2] + 1) & \text{rem}(\zeta_{k-1}[2], \epsilon) < \epsilon - 1 \\
(E_{\text{rem}(\zeta_{k-1}[1], \epsilon)}, Z_{\text{div}(\zeta_{k-1}[2] + 1, \epsilon)}, \zeta_{k-1}[2] + 1) & \text{rem}(\zeta_{k-1}[2], \epsilon) = \epsilon - 1,
\end{cases}
\]

where \( \zeta_k[1] \) is the permutation component of \( \zeta_k \); \( \zeta_k[1][j] \) is the \( j \)th element of the permutation; and \( \zeta_k[2] \) is the iteration counter. With this choice of \((W_k, \zeta_k)\), it is easy to see that the random permutation block Kaczmarz method can be equivalently written as (3.3). We highlight that \( \varphi_R \) only depends on \( \zeta_{k-1} \), the partitioning of the identity matrix, and the size of the partition. ■

**Example 3.4** (Greedy Block Selection Kaczmarz). This method partitions the equations of \( Ax = b \), computes the residual norm of each block at the given iteration, selects the block with the largest residual norm, and updates the current iterate by projecting it onto the hyperplane that solves all of the equations in the block.

To rephrase this method in our framework, let \( \{E_i\} \) be matrices whose columns are generated by some partitioning of the \( n \times n \) identity matrix, and let \( \epsilon \) be the size of this set. Moreover, let \( \mathfrak{F} = \{0\} \), \( \zeta_{-1} = 0 \), and let

\[
(3.8) \quad \pi(k) = \arg\max_{i=1, \ldots, \epsilon} \|E_i^T (Ax_k - b)\|_2.
\]

Then, we can define \( \varphi_R \) to be

\[
(3.9) \quad \varphi_R(A, b, \{x_j : j \leq k\}, \{W_j : j < k\}, \{\zeta_j : j < k\}) = (E_{\pi(k)}, 0).
\]

With this choice of \((W_k, \zeta_k)\), it is easy to see that this method is of the form (3.3). We emphasize that \( \varphi_R \) only depends on \( A, b, x_{k-1} \), and the partitioning of the identity matrix. ■
One of the key properties that is apparent in the examples above is that they are forgetful. In other words, the choice of \((W_k, \zeta_k)\) only depends on some finite number of previous iterations. To state this formally, for all \(j + 1 \in \mathbb{N}\) and \(k \in [1, j + 1] \cap \mathbb{N}\), let

\[
\mathcal{F}_k^j = \sigma(\zeta_{j-k}, x_{j-k+1}, W_{j-k+1}, \ldots, W_j, \zeta_j, x_j),
\]

that is, the \(\sigma\)-algebra generated by the random variables indicated. Note, we take \(\mathcal{F}_1^j = \sigma(\zeta_{j-1}, x_j)\) and \(\mathcal{F}_0^j\) to be the trivial \(\sigma\)-algebra. Then, we can formalize this forgetfulness property as follows.

**Definition 3.5 (Markovian).** A row-action RBAS is Markovian if there exists a finite \(M \in \mathbb{N}\) such that for any measurable sets \(\mathcal{W} \subset \mathbb{R}^{n \times n_k}\) and \(\mathcal{Z} \subset \mathcal{F}\),

\[
P\left[W_k \in \mathcal{W}, \zeta_k \in \mathcal{Z} \mid \mathcal{F}_{k+1}^k \right] = P\left[W_k \in \mathcal{W}, \zeta_k \in \mathcal{Z} \mid \mathcal{F}_{\text{min}(M, k+1)}^k \right].
\]

**Remark 3.6.** As discussed in [20, Ch. 3], a Markov process that depends on some extended period of information can be rewritten into a Markov process that only depends on the most recent information only, which can be achieved by expanding the state of the Markov process. For a Markovian RBAS, we can do the same by adding this information in \(\zeta_k\), so long as we ensure that \(\mathcal{F}\) is finite. Thus, the value of \(M\) in the preceding definition can always be taken as 1. We also note that if \(\mathcal{F}\) is finite, then it cannot be used to store all previous iterates.

Another key property of the above examples is that either the iterate will be updated within some reasonable amount of time or the current iterate is the solution. For instance, in the random permutation block Kaczmarsz method, if \(x_0\) is not a solution then within \(\epsilon\) iterations from \(k = 0\), we will find an \(E_1^1(Ax_0 - b) \neq 0\). As a result, \(x_0\) will eventually be updated. We can generalize this property as follows.

**Definition 3.7 (N,\(\pi\)-Exploratory).** A row-action RBAS is \(N, \pi\)-Exploratory for some \(N \in \mathbb{N}\) and \(\pi \in (0,1)\) if

\[
\sup_{x_0 \in \mathbb{R}^d, x_0 \notin \mathcal{P}_\mathcal{H} x_0} \left[\mathbb{P}\left[\bigcap_{j=0}^{N-1} \{\text{col}(A^TW_j) \perp x_0 - \mathcal{P}_\mathcal{H} x_0\} \mid \mathcal{F}_0^1\right]\right] \leq 1 - \pi.
\]

Here, we come to a bifurcation point in the theory of RBAS methods based on whether \(\{\text{col}(A^TW_k)\}\) is a finite set or if it is an infinite set. In all of the examples above, \(\{\text{col}(A^TW_k)\}\) belong to a finite set. In this case, we have the following result.

**Corollary 3.8.** Let \(A \in \mathbb{R}^{n \times d}\) and \(b \in \mathbb{R}^n\), satisfying Assumption 3.1. Let \(x_0 \in \mathbb{R}^d\) and \(\zeta_1 \in \mathcal{F}\). Let \(\{x_k : k \in \mathbb{N}\}\) be a sequence generated by (3.3) and (3.4) satisfying Definition 3.5 and Definition 3.7 for some \(N \in \mathbb{N}\) and \(\pi \in (0,1)\). If the elements of \(\{\text{col}(A^TW_k) : k + 1 \in \mathbb{N}\}\) take value in a finite set, then either

1. there exists a stopping time \(\tau\) with finite expectation such that \(x_\tau = \mathcal{P}_\mathcal{H} x_0\); or

2. there exists a sequence of non-negative stopping times \(\{\tau_j : j + 1 \in \mathbb{N}\}\) for which \(\mathbb{E}[\tau_j] \leq j(\text{rank}(A) - 1)(N/\pi) + 1\), and there exist \(\gamma \in (0,1)\) and a sequence of random variables \(\{\gamma_j : j + 1 \in \mathbb{N}\} \subset (0,\gamma]\), such that

\[
\mathbb{P}\left[\prod_{j=0}^{\infty} \left\{\|x_{\tau_j} - \mathcal{P}_\mathcal{H} x_0\|_2^2 \leq \left(\prod_{\ell=0}^{j-1} \gamma_\ell\right) \|x_0 - \mathcal{P}_\mathcal{H} x_0\|_2^2\right\}\right] = 1.
\]
Comparing Corollary 3.8 to classical results about the convergence of cyclic Kaczmarz-type methods (see [7, Theorem 1]), we see that our result is a probabilistic analogue: rather than guaranteeing a certain amount of convergence within a fixed number of iterations, we offer a certain amount of convergence within a random number of iterations whose expectation is controlled by a regularly increasing value (i.e., $\mathbb{E} \{ \tau_j \} \leq j((\text{rank } (A) - 1)(N/\pi) + 1)$). Moreover, Corollary 3.8 includes the important possibility of the procedure terminating in a finite amount of time. Finally, we have a guaranteed worst case rate (i.e., with probability one) of convergence for all such methods. Of course, this rate is pessimistic, but, given the generality of the methods (e.g., adaptive, deterministic, random, etc.) that fall within the scope of our result, it is quite surprising that such a bound can be found under such few, very general assumptions.

Now, the alternative case to $\{\text{col}(A^TW_k)\}$ belonging to a finite set is that it belongs to an infinite set, for which the canonical example is the row-action analogue to Example 3.14. Unfortunately, our strategy for proving Corollary 3.8 will break down for the infinite set case: in the proof of Corollary 3.8, we set $\gamma$ to be the maximum over a finite set of elements that are all strictly less than one; however, if we attempt to use the same strategy for the infinite set case, we can find systems and methods such that the supremum over the same set produces a $\gamma = 1$ (an explicit example is constructed in subsection 4.5). Thus, rather than looking at the supremum, we can attempt to control the distribution of $\{\gamma_\ell : \ell + 1 \in \mathbb{N}\}$. Surprisingly, we will only need to control the mean behavior of these random quantities rather than the entire distribution.

To state this notion of control, we will need some notation. First, for each $\ell + 1 \in \mathbb{N}$, let

$$\chi_\ell = \begin{cases} 1 & x_{\ell+1} \neq x_\ell, \\ 0 & \text{otherwise}, \end{cases}$$

be an indicator of whether we make progress in a given iteration. Moreover, for each $\ell + 1 \in \mathbb{N}$, let $\Omega_\ell$ denote the collection of sets of vectors that are orthonormal and are a basis of $\text{col}(A^TW_\chi)$, and define $G(Q_0, \ldots, Q_\ell)$ to be the set of matrices whose columns are maximal linearly independent subsets of $\bigcup_{s=0}^{\ell} Q_s$ where $Q_s \in \Omega_s$. With this notation, we have the following definition to control the distribution of $\{1 - \gamma_\ell : \ell + 1\}$.

**Definition 3.9 (Uniformly Nontrivial).** A row-action RBAS is uniformly nontrivial if for any $\{A_k : \mathbb{R}^d \times \mathbb{N} \rightarrow F_{k+1}^d\}_{k+1 \in \mathbb{N}}$ such that $\lim_{k \to \infty} \inf_{x_0 : x_0 \neq P_N x_0, \zeta_{-1} \in \mathbb{Z}} \mathbb{P}[A_k(x_0, \zeta_{-1}) | F_0^k] = 1$, there exists a $g_\mathcal{A} \in (0, 1]$ such that

$$\inf_{x_0 : x_0 \neq P_N x_0, \zeta_{-1} \in \mathbb{Z}} \sup_{\zeta_{-1} \in \mathbb{Z}} \mathbb{E} \left[ \left. \sup_{Q_s \in G(Q_0, \ldots, Q_k)} \min_{s \in \{0, \ldots, k\}} \det(G^TG) \right| F_0^k \right] \geq g_\mathcal{A}.$$

Before stating the result, we point out some important connections and features of Definition 3.9. First, so long as $G \in G(Q_0, \ldots, Q_k)$ is nontrivial, $\det(G^TG) > 0$ with probability one. Thus, for each $x_0$ such that $x_0 \neq P_N x_0$ and $\zeta_{-1} \in \mathbb{Z}$, there

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4If $n_k > \text{rank } (A)$, then $\text{col}(A^TW_k) = \text{row}(A)$ with probability one, which is covered by Corollary 3.8.
exists a $k \in \mathbb{N} \cup \{0\}$ such that

$$
\mathbb{E} \left[ \sup_{Q_0, \ldots, Q_k \in \mathcal{Q}, \varphi \in \mathcal{F}(Q_0, \ldots, Q_k)} \min_{\epsilon \in \{0, \ldots, k\}} \det(G^\top G) \mathbf{1} \left[ A_k(x_0, \zeta_{-1}) \right] \begin{bmatrix} F_1^0 \end{bmatrix} \right] > 0.
$$

Unfortunately, when we take the infimum over all allowed values of $x_0$ and $\zeta_{-1}$, we can no longer guarantee that the lower bound is zero, as supplied by Definition 3.9.

Second, Definition 3.9 is closely related, yet complementary to the foundational notion of uniformly integrable random variables. To be specific, when a family of random variables is uniformly integrable, then the expected absolute value of the random variables in the family are uniformly bounded from above. Analogously and quite roughly, when we satisfy Definition 3.9, then the expected value of the random variables in the family are uniformly bounded from below.\(^5\) Thus, we believe Definition 3.9 to be quite a foundational property and will need to be validated on a case-by-case basis (possibly with the help of tools such as analogues to the theorems of [30, 10]).

Finally, we are only controlling the expected behavior in Definition 3.9, and we do not need to make any statements about higher moments, which is surprising as $\{\gamma_\ell : \ell + 1\}$ is a dependent sequence, and usually dependencies require more complex moment statements (e.g., covariance relationships as in stationary processes). With these observations, we are ready for the next statement.

**Corollary 3.10.** Let $A \in \mathbb{R}^{n \times d}$ and $b \in \mathbb{R}^n$ satisfy Assumption 3.1. Let $x_0 \in \mathbb{R}^d$ and $\zeta_{-1} \in \mathcal{F}$. Let $\{x_k : k \in \mathbb{N}\}$ be a sequence generated by (3.3) and (3.4) satisfying Definition 3.5, Definition 3.7 for some $N \in \mathbb{N}$ and $\pi \in (0, 1)$, and Definition 3.9. One of the following is true.

1. There exists a stopping time $\tau$ with finite expectation such that $x_\tau = \mathcal{P}_H x_0$.
2. There exists a sequence of non-negative stopping times $\{\tau_j : j + 1 \in \mathbb{N}\}$ for which $\mathbb{E}[\tau_j] \leq j([\text{rank}(A) - 1](N/\pi) + 1)$, there exists $\bar{\gamma} \in (0, 1)$, and there exists a sequence of random variables $\{\gamma_j : j + 1 \in \mathbb{N}\} \subset (0, 1)$, such that

$$
\mathbb{P} \left[ \bigcap_{j=0}^{\infty} \left\{ \|x_{\tau_j} - \mathcal{P}_H x_0\|_2^2 \leq \left( \prod_{\ell=0}^{j-1} \gamma_\ell \right) \|x_0 - \mathcal{P}_H x_0\|_2^2 \right\} \right] = 1,
$$

where for any $\gamma \in (\bar{\gamma}, 1)$, $\mathbb{P}[\bigcup_{L=0}^{\infty} \cap_{j=L}^{\infty} \left\{ \prod_{\ell=0}^{j-1} \gamma_\ell \leq \gamma^j \right\}] = 1$.

**Remark 3.11.** $\mathbb{P}[\bigcup_{L=0}^{\infty} \cap_{j=L}^{\infty} \left\{ \prod_{\ell=0}^{j-1} \gamma_\ell \leq \gamma^j \right\}] = 1$ is equivalent to: there exists a finite random variable, $L$, such that, for any $j \geq L$, $\prod_{\ell=0}^{j-1} \gamma_\ell \leq \gamma^j$ with probability one.

### 3.2. Column-action RBASs

In contrast to row-action RBASs, column-action RBASs do not need to assume that the system is consistent. Thus, we simply begin with an iterate $x_0 \in \mathbb{R}^d$ and some prior information, encapsulated by $\zeta_{-1} \in \mathcal{F}$, where $\mathcal{F}$ is finite in some sense. We then generate a sequence of iterates, $\{x_k : k \in \mathbb{N}\}$, according to

$$
x_{k+1} = x_k - W_k(W_k^\top A^\top A W_k)^{-1} W_k^\top A^\top (A x_k - b),
$$

\(^5\)We say this roughly as we ignore the supremum over $k$ to demonstrate the parallels between uniformly integrable families and a uniformly nontrivial RBAS.
where \( \dagger \) represents a pseudo-inverse; and \( \{ W_k \in \mathbb{R}^{d \times n_k} \} \) are possibly random quantities (i.e., vectors or matrices) generated according to a possibly random, adaptive procedure, \( \varphi_C \), which supplies

\[
(3.19) \quad W_k, \zeta_k = \varphi_C(A, b, \{ x_j : j \leq k \}, \{ W_j : j < k \}, \{ \zeta_j : j < k \}) \in \mathbb{R}^{d \times n_k} \times \mathbb{R}.
\]

Note, our remarks about row-action RBASs apply here as well. We now present several examples.

**Example 3.12** (Cyclic Vector Coordinate Descent). Let \( \{ e_i : i = 1, \ldots, d \} \) denote the standard basis elements of \( \mathbb{R}^d \). In this method, we update the iterate \( x_k \) to \( x_{k+1} \) by one coordinate at a time according to \( x_{k+1} = x_k + e_i \alpha_k \) where \( \alpha_k \) solves

\[
(3.20) \quad \min_{\alpha \in \mathbb{R}} \| (b - Ax_k) - Ae_i \alpha \|_2,
\]

which produces

\[
(3.21) \quad x_{k+1} = x_k + e_i \frac{e_i^T A^T (b - Ax_k)}{\| A e_i \|_2^2}.
\]

The choice of \( e_i \) is determined by simply cycling through the basis elements in order. To rephrase this method within our formulation, we define \( \mathcal{Z} = \{0\} \cup \mathbb{N} \), \( \zeta_{-1} = 0 \), and

\[
(3.22) \quad \varphi_C(A, b, \{ x_j : j \leq k \}, \{ W_j : j < k \}, \{ \zeta_j : j < k \}) = (e_{\text{rem}(\zeta_{k-1}, d)+1}, \zeta_{k-1} + 1).
\]

With this choice of \( (W_k, \zeta_k) \), we see that the cyclic vector coordinate descent method is equivalent to (3.18). We underscore that \( \varphi_C \) only depends on \( \zeta_{k-1} \) and the standard basis elements. \( \blacksquare \)

**Example 3.13** (Random Permutation Block Coordinate Descent). Let \( \{ E_i : i = 1, \ldots, d \} \) be matrices whose columns are generated by some partitioning of the \( d \times d \) identity matrix. In this method, we have the update \( x_{k+1} = x_k + E_i v_k \), where \( v_k \) solves

\[
(3.23) \quad \min_v \| (b - Ax_k) - AE_i v \|_2,
\]

which produces the update

\[
(3.24) \quad x_{k+1} = x_k + E_i (E_i^T A^T A E_i)^\dagger E_i^T A^T (b - Ax_k).
\]

To choose \( E_i \), we begin by randomly permuting \( \{ E_i : i = 1, \ldots, d \} \), pass through this permutation until it is exhausted, select a new random permutation, pass through this permutation until it is exhausted, and repeat. By following the column-action analogue of **Example 3.3**, we can rephrase this method within our formulation. \( \blacksquare \)

**Example 3.14** (Block Gaussian Column Space Descent). Let \( \{ W_k : k + 1 \in \mathbb{N} \} \) be matrices with independent, identically distributed standard Gaussian components. In this method, we use the update \( x_{k+1} = x_k + W_k v_k \), where \( v_k \) solves

\[
(3.25) \quad \min_v \| (b - Ax_k) - AW_k v \|_2,
\]

which produces the update

\[
(3.26) \quad x_{k+1} = x_k + W_k (W_k^T A^T A W_k)^\dagger W_k^T A^T (b - Ax_k).
\]
It is clear that this update is exactly in the form of (3.18). Moreover, we can choose \( \mathcal{S} = \{\emptyset\}, \zeta_1 = \emptyset \), and we can define
\[
\varphi_C(A, b, \{x_j : j \leq k\}, \{W_j : j < k\}, \{\zeta_j : j < k\}) = (W_k, \emptyset).
\]

Thus, this method fits within our formulation. □

As these example demonstrate, column-action RBAS methods are also forgetful—that is, they satisfy the following analogue of Definition 3.5.

**Definition 3.15** (Markovian). A column-action RBAS is Markovian if there exists a finite \( M \in \mathbb{N} \) such that for any measurable sets \( \mathcal{W} \subset \mathbb{R}^{d \times n_k} \) and \( \mathcal{Z} \subset \mathcal{S} \),
\[
\mathbb{P} \left[ W_k \in \mathcal{W}, \zeta_k \in \mathcal{Z} \mid \mathcal{F}_{k+1}^k \right] = \mathbb{P} \left[ W_k \in \mathcal{W}, \zeta_k \in \mathcal{Z} \mid \mathcal{F}_{\min(M, k+1)}^k \right].
\]

**Remark 3.16.** See Remark 3.6.

Similarly, just as with row-action methods, column-action RBASs are also \( N, \pi \)-Exploratory. To state this definition, define \( r^* = -\mathbb{P}_{\ker(A^T)} b \).

**Definition 3.17** (\( N, \pi \)-Exploratory). A column-action RBAS is \( N, \pi \)-Exploratory for some \( N \in \mathbb{N} \) and \( \pi \in (0, 1] \) if
\[
\sup_{x_0 \in \mathbb{R}^d : Ax_0 - b \neq r^*} \mathbb{P} \left[ \bigcap_{j=0}^{N-1} \{\text{col}(AW_j) \perp Ax_0 - b\} \bigg| \mathcal{F}_1^0 \right] \leq 1 - \pi.
\]

Note, the Block Gaussian Column Space Descent method, Example 3.14, is \( 1, 1 \)-Exploratory.

Just as for row-action methods, we will have a bifurcation of the theory for the convergence of column-action methods based on whether the elements of \( \{\text{col}(AW_k)\} \) take value in a finite set. In the case that they do, we have the following analogue of Corollary 3.8.

**Corollary 3.18.** Let \( A \in \mathbb{R}^{n \times d}, b \in \mathbb{R}^n \), and \( r^* = -\mathbb{P}_{\ker(A^T)} b \). Let \( x_0 \in \mathbb{R}^d \) and \( \zeta_1 \in \mathcal{S} \). Let \( \{x_k : k \in \mathbb{N}\} \) be a sequence generated by (3.18) and (3.19) satisfying Definition 3.15 and Definition 3.17 for some \( N \in \mathbb{N} \) and \( \pi \in (0, 1] \). If the elements of \( \{\text{col}(AW_k) : k + 1 \in \mathbb{N}\} \) take value in a finite set, then either

1. there exists a stopping time \( \tau \) with finite expectation such that \( Ax_\tau - b = r^* \); or

2. there exists a sequence of non-negative stopping times \( \{\tau_j : j + 1 \in \mathbb{N}\} \) for which \( \mathbb{E} [\tau_j] \leq j (\text{rank}(A) - 1)(N/\pi) + 1 \), and there exist \( \gamma \in (0, 1) \) and a sequence of random variables \( \{\gamma_j : j + 1 \in \mathbb{N}\} \subset (0, \gamma) \), such that
\[
\mathbb{P} \left[ \bigcap_{j=0}^{\infty} \left\{ \|Ax_{\tau_j} - b - r^*\|_2^2 \leq \left( \prod_{\ell=0}^{j-1} \gamma_{\ell} \right) \|Ax_0 - b - r^*\|_2^2 \right\} \right] = 1.
\]

The same comments for Corollary 3.8 apply to Corollary 3.18. Also, just as for Corollary 3.8, Corollary 3.18 does not cover Example 3.14 if \( n_k < \text{rank}(A) \). For the infinite set case, we will make use of the same notation as before with the following modifications. First,
\[
\chi_\ell = \begin{cases} 1 & Ax_{\ell+1} - b \neq Ax_\ell - b, \\ 0 & Ax_{\ell+1} - b = Ax_\ell - b. \end{cases}
\]
Second, let $\Omega_\ell$ denote the collection of sets of vectors that are orthonormal and are a basis of $\text{col}(AW_{\ell}\chi_{\ell})$. We now state the analogues of Definition 3.9 and Corollary 3.10.

**Definition 3.19 (Uniformly Nontrivial).** A column-action RBAS is uniformly nontrivial if for any $\{A_k : \mathbb{R}^d \times 3 \rightarrow \mathcal{F}_{k+1}^b\}_{k+1 \in \mathbb{N}}$ such that $\lim_{k \to \infty} \inf_{x_0 : A x_0 \neq b, \zeta_1 \in \mathcal{F}_1} \mathbb{P}[A_k(x_0, \zeta_1) \in \mathcal{F}_1] = 1$, there exists a $g_A \in (0, 1]$ such that

$$
\forall_{x_0 : A x_0 \neq b, r^* \in \mathcal{F}_1} \sup_{k+1 \in \mathbb{N}} \mathbb{E} \left[ \sup_{Q_1 \in \Omega_{\ell} \cap \mathcal{G}_1(Q_{\ell}, \ldots, Q_k)} \min_{s \in \{0, \ldots, k\}} \det(G^T G) [A_k(x_0, \zeta_1)] \left| \mathcal{F}_1^b \right] \right] = g_A.
$$

**Corollary 3.20.** Let $A \in \mathbb{R}^{n \times d}$, $b \in \mathbb{R}^n$, and $r^* = -P\text{bet}(A) b$. Let $x_0 \in \mathbb{R}^d$ and $\zeta_1 \in \mathcal{F}_1$. Let $\{x_k : k \in \mathbb{N}\}$ be a sequence generated by (3.18) and (3.19) satisfying Definition 3.15, Definition 3.17 for some $N \in \mathbb{N}$ and $\pi \in (0, 1]$, and Definition 3.19. One of the following is true.

1. There exists a stopping time $\tau$ with finite expectation such that $Ax_\tau - b = r^*$.
2. There exists a sequence of non-negative stopping times $\{\tau_j : j + 1 \in \mathbb{N}\}$ for which $\mathbb{E}[\tau_j] \leq j[\text{rank}(A) - 1](N/\pi) + 1$, there exists $\gamma_j \in (0, 1)$, and there exists a sequence of random variables $\{\gamma_j : j + 1 \in \mathbb{N}\} \subset (0, 1)$, such that

$$
\forall_{\gamma \in (\gamma_1, 1)} \mathbb{P} \left[ \bigcap_{j=0}^{\infty} \left\{ \|Ax_\tau_j - b - r^*\|_2^2 \leq \left( \prod_{\ell=0}^{j-1} \gamma_\ell \right) \|Ax_0 - b - r^*\|_2^2 \right\} \right] = 1,
$$

where for any $\gamma \in (\gamma_1, 1)$, $\mathbb{P}[\bigcup_{L=0}^{\infty} \gamma_0^L \bigcap_{j=0}^{\infty} \{ \prod_{\ell=0}^{j-1} \gamma_\ell \leq \gamma^j \}] = 1$.

**Remark 3.21.** See Remark 3.11.

**4. Convergence Theory.** We now prove Corollaries 3.8, 3.10, 3.18, and 3.20 by the following steps.

1. In subsection 4.1, we will write row-action and column-action methods using a common form, which reveals that the iterates (in the common form) are generated by products of orthogonal projections, which raises the questions: when will this sequence of products of orthogonal projections produce a reduction in the norms of the iterates and how big will this reduction be?

2. In subsection 4.2, we will answer this question by proving a generalized block Meany inequality, which states that when the iterate is in a space generated by the a sequence of projection matrices, we are guaranteed a certain amount of reduction in the norms of the iterates. Of course, this raises the question: when will the iterate be in this space?

3. In subsection 4.3, we define a stopping time for each iterate that, when finite, implies that the iterate will be in the aforementioned space. We show that when a RBAS is Markovian and $N, \pi$-exploratory, then, starting at any iterate, this stopping time is finite in expectation and we derive an explicit bound on this expectation.

4. Once we have established the finiteness of this stopping time, we can then apply our generalized block Meany’s inequality to guarantee a reduction in the norm of the iterates. However, owing to the possible randomness of the procedure and the stopping times, we will need to find a deterministic control over the reduction constant provided by our generalized block Meany’s inequality. In subsection 4.4, we will find this deterministic value by using the worst case over a finite set, which will prove Corollaries 3.8 and 3.18. In subsection 4.5, we will find this deterministic value by using the uniformly nontrivial property, which will prove Corollaries 3.10 and 3.20.
4.1. Common Formulation. Our first step will be to rewrite row-action and column-action RBASs, and the corresponding definitions using a common formulation. To this end, we define

\[
y_k = \begin{cases} 
  x_k - \mathcal{P}_M x_0 & \text{if (3.3) and Assumption 3.1}, \\
  Ax_k - b - r^* & \text{if (3.18)},
\end{cases}
\]

where \( r^* = -\mathcal{P}_{\ker(A^\top)} b \). Owing to this definition, the update \( y_k \) to \( y_{k+1} \) is

\[
y_{k+1} = (I - \mathcal{P}_k)y_k,
\]

where \( \mathcal{P}_k \) are orthogonal projection matrices defined by

\[
\mathcal{P}_k = \begin{cases} 
  A^\top W_k(W_k^\top A A^\top W_k)^{-1} W_k^\top A & \text{if (3.3)} \\
  AW_k(W_k^\top A A^\top W_k)^{-1} W_k^\top A^\top & \text{if (3.18)}.
\end{cases}
\]

Thus, with these definitions, it is enough to prove convergence and rate of convergence results about \( \{y_k\} \).

Remark 4.1. We can change the inner product space as done in [12], and we would still recover (4.2) with a simple change of variables. See [31].

To focus on \( \{y_k\} \), we can update some of our definitions in terms of \( \{y_k\} \) and \( \{\mathcal{P}_k\} \).

Definition 4.2 (Markovian). An RBAS (see (4.2)) is Markovian if there exists a finite \( M \in \mathbb{N} \) such that for any measurable sets \( \mathcal{W} \) and \( \mathcal{Z} \subset 3 \),

\[
\mathbb{P} \left[ W_k \in \mathcal{W}, \zeta_k \in \mathcal{Z} \mid \mathcal{F}^k_{k+1} \right] = \mathbb{P} \left[ W_k \in \mathcal{W}, \zeta_k \in \mathcal{Z} \mid \mathcal{F}^k_{\min(M,k+1)} \right].
\]

Definition 4.3 (N, \( \pi \)-Exploratory). An RBAS (see (4.2)) is \( N, \pi \)-Exploratory for some \( N \in \mathbb{N} \) and \( \pi \in (0,1] \) if

\[
\sup_{y_0, y_0 \neq 0} \sup_{k \in \mathbb{N}} \mathbb{P} \left[ \bigcap_{j=0}^{N-1} \right. \left. \{ \text{col}(\mathcal{P}_j) \perp y_0 \} \big| \mathcal{F}^0_1 \right] \leq 1 - \pi.
\]

Definition 4.4 (Uniformly Nontrivial). An RBAS (see (4.2)) is uniformly nontrivial if for any \( \{A_k : \mathbb{R}^d \times 3 \rightarrow \mathcal{F}^k_{k+1} : k + 1 \in \mathbb{N} \} \) such that \( \lim_{k \rightarrow \infty} \inf_{y_0, y_0 \neq 0, \zeta_1 \in 3} \mathbb{P}[A_k(x_0, \zeta_1) \mid \mathcal{F}^0_1] = 1 \), there exists a \( g_A \in (0,1] \) such that

\[
\inf_{\zeta_1 \in 3} \sup_{k \in \mathbb{N} \cup \{0\}} \mathbb{E} \left[ \sup_{Q_2 \in Q_1} \min_{g \in G(Q_1, \ldots, Q_k)} \det(G^\top G) \mathbf{1} \left[ A_k(x_0, \zeta_1) \right] \big| \mathcal{F}^0_1 \right] \geq g_A.
\]

4.2. Generalized Block Meany Inequality. From (4.2), we see that \( \{y_k\} \) are updated by applying a sequence of orthogonal projection matrices. We can now ask whether this application of projection matrices will drive \( \|y_k\|_2 \) to zero and at what rate. This question was first answered for products of projections of the form \( I - qq^\top \) in [19], where the \( q \)'s in the product are linearly independent—a result known as Meany’s inequality. Meany’s inequality has been generalized in two ways. First, Meany’s inequality was extended to products of the form \( I - QQ^\top \) in [4, Theorem...
Table 1 and (4.8). We can first ask whether the choice of (4.8) until an absolute error of 10

Then, for any $y \in C_k^j$ with $\|y\|_2 = 1$, $(I - P_j + k \chi_{j+k}) \cdots (I - P_j \chi_i) y$ is no greater than $1 - \max_{i: i \in Q_i \cap \Omega_{i,j}} \min_{Q_j \in G(Q_i, \ldots, Q_j, \Omega_{i,j})} \det(G)$.

Proof. Let $n_i = \dim(\text{col}(P_i))$. Begin by fixing $Q_i \in \Omega_i$ for $i = j, \ldots, j + k$, and let \{q_{i,\ell} : \ell = 1, \ldots, n_k\} denote the elements of $Q_i$. We can now follow the strategy of [4]. Letting the product notation indicate terms with increasing index are being multiplied, the supremum term also underscores the importance of block methods, there are only two choices in the set $\Omega_i$, and both produce the same value of Meany’s constant. Thus, for vector methods, Meany’s constant will only differ based on which vectors are seen. To demonstrate this, we run cyclic Kaczmarz and block cyclic Kaczmarz on the coefficient matrix in (4.8) until an absolute error of $10^{-4}$ is
achieved. We plot one minus the ratio in norm error squared for each method, and the corresponding Meany’s constants in Figure 2. Clearly, we see that the block method is substantially superior over the corresponding vector method both in practice and in theory.

4.3. Stopping Times. To apply Theorem 4.5, we need to determine for which \( k, y_j \in C_k^i \). Given that \( C_k^i \) is random, we will have to allow the time at which this occurs to be random, as follows.\(^6\) For \( j + 1 \in \mathbb{N} \), let

\[
\nu(j) = \min \left\{ k \geq 0 : y_j \in C_k^j \right\}.
\]

Thus, when \( \nu(j) \) is finite, Theorem 4.5 implies \( \|y_j + \nu(j) + 1\|_2^2 / \|y_j\|_2^2 \) is no greater than \( 1 - \sup_{Q, j, i \in \mathcal{N}, \nu \geq 0} \min_{Q, j, i} \|Q_j\| \det(G^T G) \). Hence, we need to determine whether \( \nu(j) \) is finite for all \( j \), and, ideally, we want to bound it, at the very least, in expectation. To this end, we will study another stopping time that is an upper bound on \( \nu(j) \), and will find a bound on this new stopping time’s expectation. We will begin by specifying this stopping time and showing that it is an upper bound on \( \nu(j) \).

**Lemma 4.6.** For any \( j + 1 \in \mathbb{N} \), let \( \nu(j) \) be defined as in (4.9). Then, \( \nu(j) \leq \min \{ k \geq 0 : y_{j+k+1} \in \text{span} [y_j, \ldots, y_{j+k}], \chi_{j+k} \neq 0 \} \).

**Proof.** We begin with a key fact. By (4.2), \( y_{k+1} - y_k \in \text{col}(P_k \chi_k) \) for any \( k + 1 \in \mathbb{N} \). It follows that \( y_i \in \text{span} \{C_k^i \cup \{y_k\} \} \) for any \( \ell, i \in [j, j+k+1] \cap \mathbb{N} \).

Now, let \( \nu(j)' = \min \{ k \geq 0 : y_{j+k+1} \in \text{span} [y_j, \ldots, y_{j+k}], \chi_{j+k} \neq 0 \} \). Then, by the preceding fact, if \( y_{j+\nu(j)'} \in C_{\nu(j)'}^j \), then \( y_j \in C_{\nu(j)'}^j \). Thus, \( \nu(j) \leq \nu(j)' \) by the minimality of \( \nu(j) \). So it is enough to show \( y_{j+\nu(j)'} \in C_{\nu(j)'}^j \).

Let \( r \) denote the dimension of \( \text{span} \{C_{\nu(j)'}^j \cup \{y_{j+\nu(j)'}\} \} \). Then, by the Gram-Schmidt procedure, there exist \( \phi_1, \ldots, \phi_{r-1} \in C_{\nu(j)}^j \), such that the set of vectors \( \{y_{j+\nu(j)'}, \phi_1, \ldots, \phi_{r-1}\} \) is an orthogonal basis for \( \text{span} \{C_{\nu(j)}^j \cup \{y_{j+\nu(j)'}\} \} \). Now,

\(^6\)It is understood that if the condition fails to occur then the stopping time is infinite.
by the definition of $\nu(j)'$, there exist scalars $c_0, \ldots, c_{r-1}$ such that $y_{j+\nu(j)'+1} = c_0 y_{j+\nu(j)'} + c_1 \phi_1 + \cdots + c_{r-1} \phi_{r-1}$. Plugging this into (4.2),

$$c_0 y_{j+\nu(j)'} + c_1 \phi_1 + \cdots c_{r-1} \phi_{r-1} = y_{j+\nu(j)'} - \mathcal{P}_{j+\nu(j)'} y_{j+\nu(j)'} \chi_{j+\nu(j)}',$$

which gives rise to two cases. In the first case, we assume that $c_0 \neq 1$. Then, rearranging (4.10), we conclude $y_{j+\nu(j)'} \in \text{span} \{\phi_1, \ldots, \phi_{r-1}\} + \text{col}(\mathcal{P}_{j+\nu(j)'} \chi_{j+\nu(j)'}$) = $C_{\nu(j)'}$. In the second case, $c_0 = 1$. Then, multiplying both sides of (4.10) by $y_{j+\nu(j)'}$,\n
$$\sum_{i=1}^{r-1} c_i y_j^{\top} y_{j+\nu(j)'} \phi_i = - \|\mathcal{P}_{j+\nu(j)'} y_{j+\nu(j)'}\|_2^2 \chi_{j+\nu(j)}'.$$

By the orthogonality of $\phi_i$ and $y_{j+\nu(j)'}$, the left hand side is zero. The right hand side can only be zero if $\chi_{j+\nu(j)'} = 0$, which contradicts the definition of $\nu(j)'$. To summarize these two cases, we showed that $y_{j+\nu(j)'} \in C_{\nu(j)'}$. The result follows.

**Theorem 4.7.** Let $\xi$ be an arbitrary, finite stopping time with respect to $\mathcal{F}^k_{k+1}$, and let $\mathcal{F}^\xi_{k+1}$ denote the stopped $\sigma$-algebra. Given that $\{y_k : k \geq 0\}$ are well-defined (see (4.1)), let $y_k$ be generated by an $N, \pi$-exploratory, Markovian RBAS. If $y_\xi \neq 0$, then $\nu(\xi)$ is finite, and $\mathbb{E}[\nu(\xi) | \mathcal{F}^\xi_{k+1}] \leq (\text{rank}(A) - 1)(N/\pi)$.

**Proof.** We need only bound the upper bound in Lemma 4.6. At any given $k \geq 0$, there are three possible cases, either (Case 1) $\chi_{\xi+k} = 0$; (Case 2) $\chi_{\xi+k} = 1$ and $y_{\xi+k+1} \notin \text{span}[y_{\xi+1}, \ldots, y_{\xi+k}]$; or (Case 3) $\chi_{\xi+k} = 1$ and $y_{\xi+k+1} \in \text{span}[y_{\xi+1}, \ldots, y_{\xi+k}]$. We will show that Cases 1 and 2 cannot hold for all $k \geq 0$ with probability one.

To this end, define $s(j) = \min\{k \geq 0 : \chi_{j+k} \neq 0\}$ and let $s_1 = s(\xi)$ and $s_{j+1} = s(\xi + s_1 + \cdots + s_j)$ for all $j \in \mathbb{N}$. With this notation, the Markovian property and the $N, \pi$-exploratory property,

$$\mathbb{P}\left[s_1 \geq N | \mathcal{F}^\xi_{\xi+1}\right] = \mathbb{P}\left[\bigcap_{j=0}^{N-1} \{\chi_{\xi+j} = 0\} | \mathcal{F}^\xi_{\xi+1}\right] = \mathbb{P}\left[\bigcap_{j=0}^{N-1} \{\text{col}(\mathcal{P}_{\xi+j} \perp y_\xi)\} | \mathcal{F}^\xi_{\xi+1}\right]$$

(4.11)

$$\mathbb{P}\left[\bigcap_{j=0}^{N-1} \{\text{col}(\mathcal{P}_{\xi+j} \perp y_\xi)\} | \mathcal{F}_{1}^\xi\right] \leq 1 - \pi,$$

(4.12)

where the last line is a consequence of Remark 3.6. Now, using induction and the Markovian property, $\mathbb{P}[s_1 \geq N | \mathcal{F}^\xi_{\xi+1}] \leq (1 - \pi)^{\ell}$ for all $\ell \in \mathbb{N}$. Therefore, $s_1$ is finite with probability one and $\mathbb{E}[s_1 | \mathcal{F}^\xi_{\xi+1}] \leq N/\pi$. Moreover, since $\xi$ is an arbitrary stopping time, it follows that $\{s_j\}$ are finite with probability one and $\mathbb{E}[s_j | \mathcal{F}^\xi_{\xi+1}] \leq N/\pi$. Thus, Case 1 cannot occur for all $k \geq 0$, and Cases 2 or 3 must occur infinitely often.

Now, the dimension of $\text{span}\{y_{\xi+1}, \ldots, y_{\xi+s_1+\cdots+s_j}\}$ is $j+1$. Since $\{y_k\}$ are either in row$(A)$ or col$(A)$, $j + 1 \leq \text{rank}(A)$. Thus, Case 2 cannot be the only situation to occur when $\chi_{\xi+k} \neq 0$. In conclusion, the largest value of $j$ is $\text{rank}(A) - 1$, which implies $\nu(\xi) \leq s_1 + \cdots + s_{\text{rank}(A) - 1}$, which are the sum of exponentially distributed random variables. The result follows by using $\mathbb{E}[s_j | \mathcal{F}^\xi_{\xi+1}] \leq N/\pi$.

Now, putting together Theorems 4.5 and 4.7 supplies the following result.

**Corollary 4.8.** Suppose $A \in \mathbb{R}^{n \times d}$ and $b \in \mathbb{R}^n$. Given that $\{y_k : k + 1\}$ are well-defined (see (4.1)), suppose that they are generated by a Markovian, $N, \pi$-exploratory RBAS. Then, one of the following to cases occurs.
1. There exist a stopping time \( \tau \) with finite expectation such that \( y_\tau = 0 \).

2. There exist stopping times \( \tau_j : j + 1 \in \mathbb{N} \) such that \( \mathbb{E}[\tau_j] \leq j [(\text{rank}(A) - 1)(N/\pi) + 1] \) for all \( j + 1 \in \mathbb{N} \), and \( \mathbb{P}[\gamma_{j=1}^\infty \|y_{\tau_j}\|^2 \leq (\prod_{k=1}^{j-1} \gamma_k)\|y_0\|^2] = 1 \), where \( \gamma_k = 1 - \sup_{Q_{\tau_k} \in G_{\tau_k + \nu(\tau_k)}} \min_{\tilde{G} \in (Q_{\tau_k}, \ldots, Q_{\tau_k + \nu(\tau_k)})} \det(G \tilde{G}) \in (0, 1) \).

**Proof.** The proof proceeds by induction. For \( j = 0 \), recall \( \tau_0 = 0 \). Now, either \( y_{\tau_0} = 0 \) or \( y_{\tau_0} \neq 0 \). In the former case, the statement of the result is true. In the latter case, define \( \tau_1 = \nu(\tau_0) + 1 \). Then, \( \tau_1 \) is finite with probability one and \( \mathbb{E}[\tau_1] \leq (\text{rank}(A) - 1)N/\pi + 1 \) by Theorem 4.7. Moreover, by Theorem 4.5, \( \|y_{\tau_1}\|^2 \leq \gamma_0\|y_{\tau_0}\|^2 \).

Thus, we have established the base case.

For the induction hypothesis, suppose that for \( j \in \mathbb{N} \), \( A^j \tau_{j-1} \neq b \), \( \mathbb{E}[\tau_k] \leq k[(\text{rank}(A) - 1)N/\pi + 1] \) for \( k \in [0, j - 1] \cap \mathbb{N} \), and \( \|y_{\tau_k}\|^2 \leq \gamma_k\|y_{\tau_{k-1}}\|^2 \) for \( k \in [1, j - 1] \cap \mathbb{N} \).

To conclude, define \( \tau_j = \tau_{j-1} + \nu(\tau_{j-1}) + 1 \). By Theorem 4.7, \( \tau_j \) is finite and \( \mathbb{E}[\tau_j] \leq (j - 1)(\text{rank}(A) - 1)N/\pi + 1 + (\text{rank}(A) - 1)N/\pi + 1 = j[(\text{rank}(A) - 1)N/\pi + 1] \).

Finally, either \( y_{\tau_j} = 0 \) or \( y_{\tau_j} \neq 0 \). In the latter case, Theorem 4.5 implies \( \|y_{\tau_j}\|^2 \leq \gamma_{j-1}\|y_{\tau_{j-1}}\|^2 \). The result follows.

Our final task is to control the joint behavior of \( \{\gamma_\ell : \ell + 1 \in \mathbb{N} \} \subset (0, 1) \) in the latter case of Corollary 4.8. Depending on our goal, we could require two different types of control. For instance, to ensure convergence of \( \{y_k\} \) to 0, we need to ensure that \( \liminf_{\ell \to \infty} \gamma_\ell < 1 \). However, for a rate of convergence, we need to ensure that \( \limsup_{\ell \to \infty} \gamma_\ell < 1 \). As the latter case is more desirable in practice, we will focus on ensuring that \( \limsup_{\ell \to \infty} \gamma_\ell < 1 \). This will give rise to two separate cases in our theory of convergence of RBAS methods, which we now address one at a time.

4.4. Convergence for a Finite Set. In the first case, we have that \( \{\text{col}(\mathcal{P}_k)\} \) take value in finite sets, as in Examples 3.2 to 3.4, 3.12, and 3.13.

**Theorem 4.9.** Let \( A \in \mathbb{R}^{n \times d} \) and \( b \in \mathbb{R}^n \). Given that \( \{y_k : k + 1\} \) are well-defined (see (4.1)), suppose \( \{y_k : k + 1 \in \mathbb{N} \} \) are generated by a Markovian, \( N, \pi \)-exploratory RBAS. If the elements of \( \{\text{col}(\mathcal{P}_k) : k + 1 \in \mathbb{N} \} \) take value in a finite set, then either

1. There exist a stopping time \( \tau \) with finite expectation such that \( y_\tau = 0 \).

2. There exist stopping times \( \tau_j : j + 1 \in \mathbb{N} \) such that \( \mathbb{E}[\tau_j] \leq j[(\text{rank}(A) - 1)(N/\pi) + 1] \) for all \( j + 1 \in \mathbb{N} \), and there exist \( \gamma \in (0, 1) \) and a sequence of random variables \( \{\gamma_{\ell} : j + 1 \in \mathbb{N} \} \subset (0, \gamma) \), such that \( \mathbb{P}[\gamma_{j=1}^\infty \|y_{\tau_j}\|^2 \leq (\prod_{k=1}^{j-1} \gamma_k)\|y_0\|^2] = 1 \).

**Proof.** By Corollary 4.8, we can focus on the second case and we need only show that there exists a \( \gamma \in (0, 1) \) such that \( \gamma_\ell \leq \gamma \). To this end, let \( \{U_i : i = 1, \ldots, r\} \) denote the set of linear spaces in which \( \{\text{col}(\mathcal{P}_k)\} \) takes value. For each \( U_i \), we can define the set of all orthonormal bases of \( U_i \), denoted \( \mathcal{U}_i \). Let \( \mathcal{P} \) denote the power set of \( \{U_i : i = 1, \ldots, r\} \). For a given element \( \{U_{i_1}, \ldots, U_{i_s}\} \in \mathcal{P} \), we can choose a set \( \{U_{i_1}^s, \ldots, U_{i_s}^s\} \subset \mathcal{U}_{i_1} \) and let \( \mathcal{H} \) denote the set of all matrices whose columns are maximal linearly independent subsets of \( \{U_{i_1}, \ldots, U_{i_s}\} \). Finally, define

(4.13)

\[
\Gamma = \left\{ 1 - \sup_{(U_{i_1}^s, \ldots, U_{i_s}^s) \in \mathcal{H}} \min_{H \in \mathcal{H}} \det(H^TH) : \{U_{i_1}, \ldots, U_{i_s}\} \in \mathcal{P}, s = 1, \ldots, r \right\}.
\]

Now, since \( \{Q_{\ell}\} \) takes value in \( \{U_i : i = 1, \ldots, r\} \) and \( \{Q_\ell : \ell = \tau_\ell, \ldots, \tau_\ell + \nu(\tau_\ell)\} \in \mathcal{P} \) for all \( \ell + 1 \in \mathbb{N} \). Therefore, \( \gamma_\ell \in \Gamma \) for all \( \ell + 1 \in \mathbb{N} \). Thus, \( \gamma_\ell \leq \max(\Gamma) = \gamma \). By Hadamard's inequality, each element of \( \Gamma \) is in \( [0, 1) \), which implies that \( \gamma \in [0, 1) \). As we are only proving the second case, \( \gamma \neq 0 \) (else we would have converged finitely
and would be in the first case), which implies $\gamma \in (0, 1)$.

We make two remarks. First, by substituting in the appropriate definitions of $\{y_k : k + 1 \in \mathbb{N}\}$ and $\{P_k : k + 1 \in \mathbb{N}\}$ into Theorem 4.9, then we have proven Corollaries 3.8 and 3.18. Second, the value of $\gamma$ can be vary depending on how the set to which $\{{\rm col}(P_k)\}$ belongs is designed, which was a central point of discussion in [24]. For instance, consider the three unique partitions of the rows of the coefficient matrix

$$
\begin{bmatrix}
1 & -1 & 1 \\
1 & -1 & 1 + 10^{-5} \\
3 & -1 & 3 \\
0 & 1 & 6
\end{bmatrix}
$$

such that each partition contains two rows. Now, consider a sampling scheme that selects a partition and cycles through the blocks in this partition. For such a method, we compute can $\gamma$. The results for each of the three partitions are in presented in Table 2.

| Partition I | Partition II | Partition III |
|-------------|--------------|---------------|
| 0.880       | 0.372        | 0.372         |

From Table 2, we see that to get the same guaranteed relative reduction in error from Partition I in comparison to Partition II or III requires over seven fold more iterations. Indeed, as shown in Figure 3, we observe exactly this behavior when we implement cyclic block Kaczmarz on (4.14) for the three different partitions up to an absolute error of $10^{-4}$.

**Fig. 3.** A comparison of the absolute errors of cyclic block Kaczmarz methods for the coefficient matrix in (4.14). Corresponding to the theory, Partition I produces the worst convergence rate.

### 4.5. Convergence for an Infinite Set.

In the second case, $\{\{{\rm col}(P_k)\}\}$ can take value over an infinite set, as in Example 3.14 with $n_k < \text{rank}(A)$. Suppose we attempt to prove the convergence result as we did in subsection 4.4. Then, we would need to prove that $\sup\{\Gamma\} < 1$. However, when $\Gamma$ is infinite, we could potentially have
sup{Γ} = 1. For instance, consider a $3 \times 2$ coefficient matrix whose first two rows are the first standard basis element of $\mathbb{R}^2$ and the last row is the second standard basis element, and a procedure that alternates between choosing either the first row of the matrix, or taking a linear combination of the second row and a product of a standard Gaussian random variable with the third row. If we let $\mathcal{N}(0,1)$ denote a standard Gaussian distribution, then $\Gamma$ is made up of all possible values of $(Z^2 + 1)^{-1}$ with $Z \sim \mathcal{N}(0,1)$. Since $Z$ has nonzero density about 0, the supremum of $\Gamma$ would be 1 in this case.

As this example suggests, it is possible to have arbitrarily poor values for $\{\gamma \ell : \ell + 1 \in \mathbb{N}\}$. However, this example also shows that shows that bulk of values of $\{\gamma \ell : \ell + 1 \in \mathbb{N}\}$ are well-behaved (i.e., the mean and standard deviation of $(Z^2 + 1)^{-1}$ is approximately 0.66 and 0.26, respectively), which partially motivates Definition 4.4.

Under Definition 4.4, we have the following result, from which Corollaries 3.10 and 3.20 follow immediately.

**Theorem 4.10.** Let $A \in \mathbb{R}^{n \times d}$ and $b \in \mathbb{R}^n$. Given that $\{y_k : k + 1 \in \mathbb{N}\}$ are well-defined (see (4.1)), suppose $\{y_k : k + 1 \in \mathbb{N}\}$ are generated by a Markovian, $N, \pi$-exploratory, and uniformly nontrivial RBAS. One of the following is true.

1. There exists a stopping time $\tau$ with finite expectation such that $y_\tau = 0$.

2. There exists a sequence of non-negative stopping times $\{\tau_j : j + 1 \in \mathbb{N}\}$ for which $E[\tau_j] \leq j[(\text{rank}(A) - 1)/(N/\pi) + 1]$, there exists $\gamma \in (0,1)$, and there exists a sequence of random variables $\{\gamma_j : j + 1 \in \mathbb{N}\} \subset (0,1)$, such that $P[\gamma_j \in \{y_\tau \leq \gamma_j\}] = 1$, where for any $\gamma \in (\gamma_j,1)$, $P[\bigcup_{\ell=0}^\infty \bigcap_{j=\ell}^\infty \{\prod_{\ell=1}^{j-1} \gamma_i \leq \gamma_j\}] = 1$.

**Proof.** By Corollary 4.8, we can focus on the second case and we need only prove that there exists a $\bar{\gamma} \in (0,1)$ such that for any $\gamma \in (\gamma_j,1)$, $P[\bigcup_{\ell=0}^\infty \bigcap_{j=\ell}^\infty \{\prod_{\ell=1}^{j-1} \gamma_i \leq \gamma_j\}] = 1$. To show this, we need to prove $E[\prod_{\ell=0}^{j-1} \gamma_i | F_j^0] \leq \bar{\gamma}_j$ for each $j$, which we will do by induction. For the base case, $j = 0$,

$$E[1 - \gamma_0 | F_0^0]$$

(4.15)

$$= E \left[ \sup_{Q \in \Omega, i \in \{\tau_0, \ldots, \tau_0 + v(\tau_0)\}} \min_{G \in \mathcal{G}(Q_{\tau_0}, \ldots, Q_{\tau_0 + v(\tau_0)})} \left| \det(G^TG) \right| F_0^0 \right]$$

(4.16)

$$= E \left[ \sum_{k=1}^\infty \mathbb{1}[\nu(\tau_0) = k] \sup_{Q \in \Omega, i \in \{\tau_0, \ldots, \tau_0 + k\}} \min_{G \in \mathcal{G}(Q_{\tau_0}, \ldots, Q_{\tau_0 + k})} \left| \det(G^TG) \right| F_0^0 \right].$$

Since $\min_{Q \in \mathcal{G}(Q_{\tau_0}, \ldots, Q_{\tau_0 + k})} \det(G^TG) \geq \min_{Q \in \mathcal{G}(Q_{\tau_0}, \ldots, Q_{\tau_0 + k},Q)} \det(G^TG)$ for any $Q \in \Omega_{\tau_0 + k + 1}$ and any $k + 1 \in \mathbb{N}$, then, for every $k + 1 \in \mathbb{N}$, $E[1 - \gamma_0 | F_0^0]$ is bounded below by

$$E \left[ \mathbb{1}[\nu(\tau_0) \leq k] \sup_{Q \in \Omega, i \in \{\tau_0, \ldots, \tau_0 + k\}} \min_{G \in \mathcal{G}(Q_{\tau_0}, \ldots, Q_{\tau_0 + k})} \left| \det(G^TG) \right| F_0^0 \right].$$

(4.17)

Now, by Theorem 4.7 and Markov’s inequality, for any $y_0 \neq 0$ and any $\zeta_1 \in \mathcal{S}$, $P[\nu(\tau_0) \leq k | F_0^0] \geq 1 - N(\text{rank}(A) - 1)/(N/\pi)$. Hence, we can apply Definition 4.4 to conclude that there exists a $g \in (0,1]$ such that $E[1 - \gamma_0 | F_0^0] \geq g$. If we let $\bar{\gamma} = 1 - g$, then $E[\gamma_0 | F_0^0] \leq \bar{\gamma}$. Now, for the induction hypothesis, suppose that $E[\prod_{\ell=0}^{j-2} \gamma_i | F_j^0] \leq \bar{\gamma}_j^{j-1}$. To conclude, we note that by the Markovian property and the base case, $E[\gamma_j - 1 | F_{\tau_j}^j] \leq \bar{\gamma}$. Therefore, $E[\prod_{\ell=0}^{j-1} \gamma_i | F_j^0] = E[E[\gamma_j - 1 | F_{\tau_j}^j] \prod_{\ell=0}^{j-1} \gamma_i | F_j^0] \leq \bar{\gamma}_j^{j-1}$. 


Now, for any $\gamma \in (\bar{\gamma}, 1)$, the preceding proof and Markov’s inequality provide

$$\sum_{j=1}^{\infty} \mathbb{P} \left[ \prod_{\ell=0}^{j-1} \gamma_{\ell} > \gamma^j \right] F_0^j \leq \sum_{j=1}^{\infty} \left( \frac{\bar{\gamma}}{\gamma} \right)^j < \infty. \tag{4.18}$$

By the Borel-Cantelli lemma, $\mathbb{P}\left[ \bigcup_{L=0}^{\infty} \bigcap_{j=L}^{\infty} \{ \prod_{\ell=0}^{j-1} \gamma_{\ell} \leq \gamma^j \} \right] = 1$. \hfill \qed

**Remark 4.11.** Our proof readily allows us to bound the convergence rates of the moments of $\{y_k : k + 1\}$.

5. **Examples.** We provide a series of examples to demonstrate how we can apply our theory to a variety of methods. Of particular practical value, we will show how to verify the relevant properties (e.g., Exploratory). We summarize these examples, references, and reference the convergence result for the given method based on our theory in Table 3.

| Method                  | References          | Details          | Convergence Result |
|-------------------------|---------------------|------------------|--------------------|
| Cyclic Vector Kaczmarz  | [17, 16, 7]         | Appendix A.1     | Theorem A.1.3      |
| Gaussian Vector Kaczmarz| [14, 33]            | Appendix A.2     | Theorem A.2.4      |
| Strohmer-Vershynin       | [37]                | Appendix A.3     | Theorem A.3.4      |
| Vector Kaczmarz          | [35]                | Appendix A.4     | Theorem A.4.3      |
| Steinerberger Vector Kaczmarz | [14, 33]      | Appendix A.5     | Theorem A.5.3      |
| Motzkin’s Method         | [21, 2]             | Appendix A.6     | Theorem A.6.3      |
| Agmon’s Method           | [2]                 | Appendix A.7     | Theorem A.7.3      |
| Greedy Randomized Sampling | [5]               | Appendix A.8     | Theorem A.8.4      |
| Kaczmarz-Motzkin Method  | [15]                | Appendix A.9     | Theorem A.9.4      |
| Streaming Vector Kaczmarz| [27]                | Appendix A.10    | Theorem A.10.3     |
| Cyclic Vector Coordinate Descent | [38]           | Appendix A.11    | Theorem A.11.4     |
| Gaussian Vector Column Space Descent | [38]          | Appendix A.12    | Theorem A.12.3     |
| Max Residual Coordinate Descent | [41]             | Appendix A.13    | Theorem A.13.3     |
| Max Distance Coordinate Descent |                   | Appendix A.14    | Theorem A.14.3     |
| Random Permutation       | [24, 22]            | Appendix A.15    | Theorem A.15.3     |
| Block Kaczmarz           | [33, 13]            | Appendix A.16    | Theorem A.16.3     |
| Steinberger Block Kaczmarz |                  | Appendix A.17    | Theorem A.17.3     |
| Motzkin’s Block Method   |                     | Appendix A.18    | Theorem A.18.3     |
| Agmon’s Block Method     |                     | Appendix A.19    | Theorem A.19.3     |
| Sketch-and-Project       |                     |                  |                    |
6. Conclusion. In order to enable broader use of highly tailored randomized methods for solving linear systems, we began with the challenge of providing a unifying theory for randomized block adaptive solvers (RBASs) for linear systems—regardless of whether the linear systems are underdetermined, overdetermined, or rank deficient. To this end, we studied two archetypes of RBAS solvers—row-action methods for consistent linear systems and column-action methods for arbitrary linear systems—and showed that under very general conditions both archetypes will converge exponentially fast to a solution. Specifically, we had two results.

1. When a RBAS is Markovian, $N, \pi$-exploratory, and projects either the absolute error (for row-action methods) or residual (for column-action methods) onto only a finite number of spaces, then the RBAS will converge exponentially fast to a solution of the linear system.

2. When a RBAS is Markovian, $N, \pi$-exploratory, and uniformly nontrivial, then, after some finite number of iterations, the RBAS will converge exponentially fast to a solution of the linear system.

We further provided numerical evidence to elucidate key aspects of theory at key points. In particular, we demonstrated the value of the supremum in our generalization of Meany’s inequality (see Theorem 4.5 and Figure 2), and we discussed the importance of finding appropriate partitions when using block cyclic solvers (see Figure 3), which was quite carefully studied in [24]. Finally, we provided a host of examples of how to apply our theory to existing methods and some novel methods, which we complemented with appropriate numerical experiments.

In completing the above tasks, we have provided practitioners with a powerful theory and demonstrations of how to use the theory to rigorously analyze a wide variety of RBASs. Thus, we hope that practitioners will be empowered to use this theory and create novel RBASs that are optimized to their specific applications and computing environments.
Appendix A. Worked Examples.

A.1. Cyclic Vector Kaczmarz. For a description of this method, see Example 3.2. We assume that \( A \) and \( b \) are arbitrary so long as they form a consistent system. Recall, \( \zeta_{-1} = 0 \),

\[
\phi_R(A, b, \{x_j : j \leq k\}, \{W_j : j < k\}, \{\zeta_j : j < k\}) = (e_{\text{rem}(\zeta_{-1}, n)+1}, \zeta_{-1}+1).
\]

and

\[
x_{k+1} = x_k - A^T e_{\text{rem}(\zeta_{k-1}, n)+1} \frac{e_{\text{rem}(\zeta_{k-1}, n)+1}(Ax_k - b)}{\|A^T e_{\text{rem}(\zeta_{k-1}, n)+1}\|_2},
\]

where \( \{e_1, \ldots, e_n\} \) are the standard basis elements of \( \mathbb{R}^n \).

**Lemma A.1.1. Cyclic Vector Kaczmarz is Markovian.**

**Proof.** Note, \( W_k = e_{\text{rem}(\zeta_{k-1}, n), n}+1 \) and \( \zeta_k = \zeta_{k-1}+1 \). That is, \( (W_k, \zeta_k) \) are fully determined by \( \zeta_{k-1} \). Therefore, \( \mathbb{P}[W_k \in W, \zeta_k \in \mathcal{F}_{k+1}^\mathcal{H}] = \mathbb{P}[W_k \in W, \zeta_k \in \mathcal{F}_{k}^\mathcal{H}] \).

**Lemma A.1.2. Cyclic Vector Kaczmarz is \( n, 1 \)-Exploratory.**

**Proof.** If \( x_0 \) is not a solution to \( Ax = b \), then there exists a row of \( A \), denoted by \( A_{x_0} \in \mathbb{R}^d \), and corresponding constant, \( b_{x_0} \), such that \( A^T_{x_0}x_0 \neq b_{x_0} \). Hence,

\[
\bigcap_{j=0}^{n-1} \left\{ e_{j+1}^T A \perp x_0 - \mathcal{P}_H x_0 \bigcup \{ A_{x_0} \perp x_0 - \mathcal{P}_H x_0 \} = \{ A^T_{x_0}x_0 - b_{x_0} = 0 \} \right\}.
\]

Clearly, the last event must be empty. The conclusion follows.

Since the set of \( |\{\text{col}(A^T e_j) : j = 1, \ldots, n\}| \leq n \), we can apply Corollary 3.8 to conclude as follows.

**Theorem A.1.3.** Let \( A \in \mathbb{R}^{n \times d} \) and \( b \in \mathbb{R}^n \) such that the linear system’s solution set, \( \mathcal{H} \), is nonempty. Let \( x_0 \in \mathbb{R}^d \). Let \( \{x_k : k \in \mathbb{N}\} \) be a sequence generated by the Cyclic Vector Kaczmarz method. Then, there exists a stopping time \( \tau \) with finite expectation such that \( x_\tau \in \mathcal{H} \); or there exists a sequence of non-negative stopping times \( \{\tau_j : j+1 \in \mathbb{N}\} \) for which \( E[\tau_j] \leq j(\text{rank}(A) - 1)n + 1 \), and there exist \( \gamma \in (0, 1) \) and a sequence of random variables \( \{\gamma_j : j+1 \in \mathbb{N}\} \subset (0, \gamma] \), such that

\[
\mathbb{P} \left[ \bigcap_{j=0}^{\infty} \|x_{\tau_j} - \mathcal{P}_H x_0\|_2^2 \leq \left( \prod_{\ell=0}^{j-1} \gamma_\ell \right) \|x_0 - \mathcal{P}_H x_0\|_2^2 \right] = 1.
\]

A.2. Gaussian Vector Kaczmarz. Suppose \((A, b)\) form a consistent linear system. In Gaussian Vector Kaczmarz, we generate independent standard normal vectors \( \{w_k : k+1 \in \mathbb{N}\} \subset \mathbb{R}^n \) at each iteration and then apply

\[
x_{k+1} = x_k - A^T w_k \frac{w_k^T(Ax_k - b)}{\|A^T w_k\|_2^2}.
\]

Therefore, we can define

\[
\phi_R(A, b, \{x_j : j \leq k\}, \{W_j : j < k\}, \{\zeta_j : j < k\}) = (w_k, \emptyset).
\]

**Lemma A.2.1.** Gaussian Vector Kaczmarz is Markovian.
Proof. Since \( w_k \) is independently generated at each iteration, \( \mathbb{P}[W_k \in \mathcal{W}, \zeta_{-1} \in \mathcal{Z}|\mathcal{F}_k^{k+1}] = \mathbb{P}[w_k \in \mathcal{W}] = \mathbb{P}[W_k \in \mathcal{W}, \zeta_{-1} \in \mathcal{Z}|\mathcal{F}_1^1] \). \( \square \)

Lemma A.2.2. Gaussian Vector Kaczmarz is 1,1-Exploratory.

Proof. For any vector \( v \neq 0 \), \( \mathbb{P}[w_0^Tv = 0] = 0 \) since \( w_0 \) is a continuous random variable. In particular, when \( x_0 \notin \mathcal{P}_H x_0 \), \( \mathbb{P}[w_0^T(Ax_0 - \mathcal{P}_H x_0) = 0|\mathcal{F}_1^0] = 0 \). \( \square \)

Lemma A.2.3. Gaussian Vector Kaczmarz is Uniformly Nontrivial.

Proof. Suppose \( x_k \notin \mathcal{P}_H x_0 \). Then, \( w_k^T(Ax_k - b) \neq 0 \) and \( A^T w_k \neq 0 \) with probability one since \( w_k \) is a continuous random variable. Hence, \( \chi_k = 1 \) with probability one so long as \( x_k \notin \mathcal{P}_H x_0 \). We now have two cases: either \( \text{rank}(A) = 1 \) or \( \text{rank}(A) > 1 \).

If \( \text{rank}(A) = 1 \), then \( x_0 - \mathcal{P}_H x_0 \in \text{row}(A) \). Hence, \( \text{span}[x_0 - \mathcal{P}_H x_0] = \text{span}[A^Tw_0] \) with probability one. Therefore,

\[
(\text{A.2.3}) \quad x_1 - \mathcal{P}_H x_0 = x_0 - \mathcal{P}_H x_0 - \frac{(x_0 - \mathcal{P}_H x_0)(x_0 - \mathcal{P}_H x_0)^T}{\|x_0 - \mathcal{P}_H x_0\|^2_2} = 0.
\]

Hence, \( x_1 \) solves the system, \( \chi_0 = 1 \) and \( \chi_k = 0 \) for all \( k \in \mathbb{N} \). Thus, \( \mathcal{G}(Q_0, \ldots, Q_s) \) equals \( \mathcal{G}(Q_0) \), which is simply the \( \mathbb{R}^{d \times 1} \) matrix whose column is a unit vector in \( \text{row}(A) \). To conclude this case, \( \det(G^T G) = 1 \) and

\[
(\text{A.2.4}) \quad \inf_{x_0:Ax_0 \neq b \in \mathcal{N}(0)} \sup_{\zeta_{-1} \in \mathcal{F}} \mathbb{E} \left[ \sup_{Q_k \in \mathcal{Q}_k} \min_{s \in \{0, \ldots, k\}} \det(G^T G) \mathbf{1}_{\left[ A_k(x_0, \zeta_{-1}) \right]} \right] = \mathbb{P}\left[ A_k(x_0, \zeta_{-1}) \in \mathcal{F}_1^0 \right] = 1.
\]

In other words, when rank \((A) = 1 \), Gaussian Vector Kaczmarz is uniformly nontrivial.

If \( \text{rank}(A) > 1 \), we will proceed by induction. Since \( A^T w_0 \) is a continuous random variable with co-domain of dimension at least two, \( x_0 - \mathcal{P}_H x_0 \in \text{span}[A^T w_0] \) with probability zero. Thus, \( x_1 \notin \mathcal{P}_H x_0 \). Suppose that \( x_{k-1} \notin \mathcal{P}_H x_0 \) with probability one. Then, by the same reasoning as the base case, \( x_k \notin \mathcal{P}_H x_0 \). Hence, \( \chi_k = 1 \) for all \( k + 1 \in \mathbb{N} \) with probability one. It follows that

\[
(\text{A.2.5}) \quad \sup_{Q_k \in \mathcal{Q}_k} \min_{s \in \{0, \ldots, k\}} \det(G^T G) \text{ is independent of } x_0 \text{. Moreover, } (\text{A.2.5}) \text{ is positive with probability one. Hence, } \exists \{\epsilon_k : k + 1 \in \mathbb{N}\} \in \mathbb{R}_{\geq 0} \text{ such that}
\]

\[
(\text{A.2.6}) \quad \mathbb{P} \left[ \sup_{Q_k \in \mathcal{Q}_k} \min_{s \in \{0, \ldots, k\}} \det(G^T G) > \epsilon_k \right] \geq 1/2.
\]

Therefore, by Markov’s inequality,

\[
(\text{A.2.7}) \quad \inf_{x_0:Ax_0 \neq b \in \mathcal{N}(0)} \sup_{\zeta_{-1} \in \mathcal{F}} \mathbb{E} \left[ \sup_{Q_k \in \mathcal{Q}_k} \min_{s \in \{0, \ldots, k\}} \det(G^T G) \mathbf{1}_{\left[ A_k(x_0, \zeta_{-1}) \right]} \right] \geq \epsilon_k \mathbb{P} \left[ \sup_{Q_k \in \mathcal{Q}_k} \min_{s \in \{0, \ldots, k\}} \det(G^T G) > \epsilon_k, A_k(x_0, \zeta_{-1}) \in \mathcal{F}_1^0 \right].
\]
Now, \( \exists K \in \mathbb{N} \) such that, for all \( k \geq K \), \( \mathbb{P}[A_k(x_0, \zeta_{-1})|\mathcal{F}_1^0] \geq 3/4 \). By the inclusion-exclusion principle, for \( k \geq K \),

\[
(A.2.8) \quad \mathbb{P} \left[ \sup_{Q_i \in \Omega_i} \min_{s \in \{0, \ldots, k\}} \det(G^T G) > \epsilon_k, A_k(x_0, \zeta_{-1}) \right| \mathcal{F}_1^0 \right] \geq 1/4.
\]

Therefore, for all \( k \geq K \),

\[
(A.2.9) \quad \inf_{x_0: A x_0 \neq b \in \mathbb{R}^n} \sup_{\zeta_{-1} \in \mathcal{E}} \mathbb{E} \left[ \sup_{Q_i \in \Omega_i} \min_{s \in \{0, \ldots, k\}} \det(G^T G) 1_{A_k(x_0, \zeta_{-1})} \right| \mathcal{F}_1^0 \right] \geq \epsilon_k/4.
\]

The result follows. \( \square \)

We can now apply Corollary 3.10 to conclude as follows

**Theorem A.2.4.** Let \( A \in \mathbb{R}^{n \times d} \) and \( b \in \mathbb{R}^n \) such that the linear system’s solution set, \( \mathcal{H} \), is nonempty. Let \( x_0 \in \mathbb{R}^d \). Let \( \{x_k : k \in \mathbb{N}\} \) be a sequence generated by Gaussian Vector Kaczmarz. Then, there exists a stopping time \( \tau \) with finite expectation such that \( x_\tau \in \mathcal{H} \); or there exists a sequence of non-negative stopping times \( \{\tau_j : j + 1 \in \mathbb{N}\} \) for which \( \mathbb{E}[\tau_j] \leq j \text{rank}(A) \), and there exists \( \tilde{\gamma} \in (0, 1) \) such that for any \( \gamma \in (\tilde{\gamma}, 1) \),

\[
(A.2.10) \quad \mathbb{P} \left[ \bigcap_{L=0}^{\infty} \bigcap_{j=L}^{\infty} \left\{ \|x_{\tau_j} - P_{\mathcal{H}} x_0\|_2^2 \leq \gamma^j \|x_0 - P_{\mathcal{H}} x_0\|_2^2 \right\} \right] = 1.
\]

**A.3. Strohmer-Vershynin Vector Kaczmarz.** Suppose we are given \( A \in \mathbb{R}^{n \times d} \) and \( b \in \mathbb{R}^n \) such that \( Ax = b \) has a solution and every row of \( A \) has at least one nonzero entry. In Strohmer-Vershynin Vector Kaczmarz method, we select an equation by sampling it with replacement from the set of all equations with a probability proportional to the sum of squares of the coefficients of the equation. Once this equation is selected, the regular Kaczmarz update is used. In our notation, \( \varphi_R(A, b) = (e_{i_k}, \emptyset) \), where

\[
(A.3.1) \quad \mathbb{P}[i_k = j] \propto \begin{cases} \|e_j^T A\|_2^2 & j = 1, \ldots, n \\ 0 & \text{otherwise.} \end{cases}
\]

**Lemma A.3.1.** Strohmer-Vershynin Vector Kaczmarz is Markovian.

**Proof.** By the independence of \( \{i_k : k + 1 \in \mathbb{N}\} \), \( \mathbb{P}[e_{i_k} \in \mathcal{W}, \zeta_{-1} \in \mathcal{Z}|\mathcal{F}_{k+1}^0] = \mathbb{P}[e_{i_k} \in \mathcal{W}] = \mathbb{P}[e_{i_k} \in \mathcal{W}, \zeta_{-1} \in \mathcal{Z}|\mathcal{F}_{k}^0]. \) \( \square \)

**Lemma A.3.2.** Let \( \pi_{\text{min}} = \min_{j=1, \ldots, n} \|e_j^T A\|/\|A\|_F^2 \). Strohmer-Vershynin Vector Kaczmarz is \( 1, \pi_{\text{min}} \)-Exploratory.

**Proof.** Let \( x_0 \neq P_{\mathcal{H}} x_0 \). Then, there is an equation of \( Ax = b \) which is not satisfied by \( x_0 \). The probability of selecting this row on the first iteration is at least \( \pi_{\text{min}} \). Hence,

\[
(A.3.2) \quad \sup_{x_0 \in \mathbb{R}^d: x_0 \neq P_{\mathcal{H}} x_0, \zeta_{-1} \in \mathcal{E}} \mathbb{P}[e_{i_0}^T A(x_0 - P_{\mathcal{H}} x_0) = 0|\mathcal{F}_1^0] \leq 1 - \pi_{\text{min}}.
\]

\( \square \)
Remark A.3.3. Depending on the rows of $A$, we can always play with constants in the $N, \pi$-Exploratory definition to choose one that will have the smallest ratio. However, for demonstration, our selection is sufficient.

Now, we apply Corollary 3.8 to conclude as follows.

**Theorem A.3.4.** Let $A \in \mathbb{R}^{n \times d}$ and $b \in \mathbb{R}^n$ such that the linear system’s solution set, $\mathcal{H}$, is nonempty. Let $\pi_{\min} = \min_{j=1, \ldots, n} \|e_j^T A\|_2^2/\|A\|_F^2$. Let $x_0 \in \mathbb{R}^d$. Let $\{x_k : k \in \mathbb{N}\}$ be a sequence generated by the Strohmer-Vershynin Vector Kaczmarz method. Then, there exists a stopping time $\tau$ with finite expectation such that $x_\tau \in \mathcal{H}$; or there exists a sequence of non-negative stopping times $\{\tau_j : j+1 \in \mathbb{N}\}$ for which $\mathbb{E}[\tau_j] \leq j[(\text{rank}(A) - 1)/\pi_{\min} + 1]$, and there exist $\gamma \in (0, 1)$ and a sequence of random variables $\{\gamma_j : j+1 \in \mathbb{N}\} \subset (0, \gamma)$, such that

\[
\Pr\left(\bigcap_{j=0}^{\infty} \left\{ \|x_{\tau_j} - \mathcal{P}_\mathcal{H} x_0\|_2^2 \leq \left( \prod_{\ell=0}^{j-1} \gamma_\ell \right) \|x_0 - \mathcal{P}_\mathcal{H} x_0\|_2^2 \right\} \right) = 1.
\]

**A.4. Steinerberger’s Vector Kaczmarz.** Suppose $Ax = b$ is a consistent system. Steinerberger’s method selects an equation from the system by using an $l^p$ weighted residual, and then performing the Kaczmarz update with this equation. In our notation, $\varphi_R(A, b, x_k) = (e_{i_k}, 0)$ where

\[
\Pr[e_{i_k} = j] \propto \begin{cases} |e_j^T (Ax_k - b)|^p & j = 1, \ldots, n \\ 0 & \text{otherwise,} \end{cases}
\]

and

\[
x_{k+1} = x_k - A^T e_{i_k} e_j^T (Ax_k - b)/\|A^T e_{i_k}\|_2^2.
\]

**Lemma A.4.1.** Steinerberger’s Vector Kaczmarz method is Markovian.

**Proof.** As $e_{i_k}$’s distribution only depends on $x_k$, $\Pr[e_{i_k} \in \mathcal{W}|F_{k+1}] = \Pr[e_{i_k} \in \mathcal{W}|F_1]$.

**Lemma A.4.2.** Steinerberger’s Vector Kaczmarz method is $1, 1$-Exploratory.

**Proof.** For any $x_0 \not\in \mathcal{P}_\mathcal{H} x_0$, $e_{i_0}^T (Ax_0 - b) \not= 0$ with probability one. Hence,

\[
\sup_{x_0: x_0 \not\in \mathcal{P}_\mathcal{H} x_0} \Pr[e_{i_0}^T (Ax_0 - b) = 0 | F_1] = 0.
\]

The claim follows.

We can conclude using Corollary 3.8.

**Theorem A.4.3.** Let $A \in \mathbb{R}^{n \times d}$ and $b \in \mathbb{R}^n$ such that the linear system’s solution set, $\mathcal{H}$, is nonempty. Let $x_0 \in \mathbb{R}^d$. Let $\{x_k : k \in \mathbb{N}\}$ be a sequence generated by the Steinerberger’s Vector Kaczmarz method. Then, there exists a stopping time $\tau$ with finite expectation such that $x_\tau \in \mathcal{H}$; or there exists a sequence of non-negative stopping times $\{\tau_j : j+1 \in \mathbb{N}\}$ for which $\mathbb{E}[\tau_j] \leq j\text{rank}(A)$, and there exist $\gamma \in (0, 1)$ and a sequence of random variables $\{\gamma_j : j+1 \in \mathbb{N}\} \subset (0, \gamma)$, such that

\[
\Pr\left(\bigcap_{j=0}^{\infty} \left\{ \|x_{\tau_j} - \mathcal{P}_\mathcal{H} x_0\|_2^2 \leq \left( \prod_{\ell=0}^{j-1} \gamma_\ell \right) \|x_0 - \mathcal{P}_\mathcal{H} x_0\|_2^2 \right\} \right) = 1.
\]
A.5. Motzkin’s Method. Given a consistent linear systems, \( Ax = b \), Motzkin’s method starts by choosing an equation from the system whose solution hyperplane is the furthest from the current iterate, and then performs the Kaczmarz update. In our notation, \( \varphi_R(A, b, x_k) = (e_{i_k}, 0) \) where

\[
A.5.1 \quad i_k \in \arg\max_{j \in \{1, \ldots, n\}} \frac{|e_j^T(Ax_k - b)|}{\|A^T e_j\|_2^2},
\]

and

\[
A.5.2 \quad x_{k+1} = x_k - A^T e_{i_k} \frac{e_{i_k}^T(Ax_k - b)}{\|A^T e_j\|_2^2}.
\]

**Lemma A.5.1.** Motzkin’s method is Markovian.

**Proof.** The method is deterministic and only depends on the information in \( F_1^k \). \( \square \)

**Lemma A.5.2.** Motzkin’s method is 1,1-Exploratory

**Proof.** For any \( x_0 \neq \mathcal{P}_H x_0, Ax_0 - b \neq 0 \). Therefore, there is an equation that is not satisfied by \( x_0 \). Hence, the set of equations that maximize the distance between \( x_0 \) and the solution set of the equation is nonempty. The result follows. \( \square \)

We can now apply **Corollary 3.8** to conclude.

**Theorem A.5.3.** Let \( A \in \mathbb{R}^{n \times d} \) and \( b \in \mathbb{R}^n \) such that the linear system’s solution set, \( \mathcal{H} \), is nonempty. Let \( x_0 \in \mathbb{R}^d \). Let \( \{x_k : k \in \mathbb{N}\} \) be a sequence generated by Motzkin’s method. Then, there exists a stopping time \( \tau \) with finite expectation such that \( x_\tau \in \mathcal{H} \); or there exists a sequence of non-negative stopping times \( \{\tau_j : j + 1 \in \mathbb{N}\} \) for which \( \mathbb{E}[\tau_j] \leq j \text{rank} (A) \), and there exist \( \gamma \in (0, 1) \) and a sequence of random variables \( \{\gamma_j : j + 1 \in \mathbb{N}\} \subset (0, \gamma] \), such that

\[
A.5.3 \quad \mathbb{P} \left[ \bigcap_{j=0}^{\infty} \left\{ \|x_{\tau_j} - \mathcal{P}_H x_0\|_2^2 \leq \left( \prod_{\ell=0}^{j-1} \gamma_\ell \right) \|x_0 - \mathcal{P}_H x_0\|_2^2 \right\} \right] = 1.
\]

A.6. Agmon’s Method. Given a consistent linear systems, \( Ax = b \), Agmon’s method starts by choosing an equation from the system with the largest absolute residual at the current iterate, and then performs the Kaczmarz update. In our notation, \( \varphi_R(A, b, x_k) = (e_{i_k}, 0) \) where

\[
A.6.1 \quad i_k \in \arg\max_{j \in \{1, \ldots, n\}} |e_j^T(Ax_k - b)|,
\]

and

\[
A.6.2 \quad x_{k+1} = x_k - A^T e_{i_k} \frac{e_{i_k}^T(Ax_k - b)}{\|A^T e_j\|_2^2}.
\]

**Lemma A.6.1.** Agmon’s method is Markovian.

**Proof.** The method is deterministic and only depends on the information in \( F_1^k \). \( \square \)

**Lemma A.6.2.** Agmon’s method is 1,1-Exploratory

**Proof.** For any \( x_0 \neq \mathcal{P}_H x_0, Ax_0 - b \neq 0 \). Therefore, there is an equation that is not satisfied by \( x_0 \). Hence, the set of equations that maximize the absolute residual at \( x_0 \) is nonempty. The result follows. \( \square \)
We can now apply Corollary 3.8 to conclude.

**Theorem A.6.3.** Let $A \in \mathbb{R}^{n \times d}$ and $b \in \mathbb{R}^n$ such that the linear system’s solution set, $\mathcal{H}$, is nonempty. Let $x_0 \in \mathbb{R}^d$. Let $\{x_k : k \in \mathbb{N}\}$ be a sequence generated by Agmon’s method. Then, there exists a stopping time $\tau$ with finite expectation such that $x_\tau \in \mathcal{H}$; or there exists a sequence of non-negative stopping times $\{\tau_j : j + 1 \in \mathbb{N}\}$ for which $\mathbb{E}[\tau_j] \leq j \text{rank}(A)$, and there exist $\gamma \in (0,1)$ and a sequence of random variables $\{\gamma_j : j + 1 \in \mathbb{N}\} \subset (0,\gamma)$, such that

$$\mathbb{P} \left[ \bigcap_{j=0}^{\infty} \left\{ \|x_{\tau_j} - P_H x_0\|_2^2 \leq \prod_{\ell=0}^{j-1} \gamma_{\ell} \|x_0 - P_H x_0\|_2^2 \right\} \right] = 1.$$  

**A.7. Greedy Randomized Vector Kaczmarz.** In this method, at each iteration, a threshold value is calculated based on the current iterate value

$$\epsilon_k = \frac{1}{2} \max_{j \in \{1,\ldots,n\}} \frac{|e_j^T (Ax_k - b)|^2}{\|Ax_k - b\|_2^2 \|A^T e_j\|_2^2} + \frac{1}{2} \frac{1}{\|A\|_F^2};$$

then a subset of equations whose residual surpassing this threshold is generated,

$$\mathcal{U}_k = \left\{ j : \frac{|e_j^T (Ax_k - b)|^2}{\|A^T e_j\|_2^2 \|Ax_k - b\|_2^2} \geq \epsilon_k \right\};$$

an equation, $i_k$, is then selected from this subset with a probability distribution

$$\mathbb{P}[i_k = j] \propto \begin{cases} |e_j^T (Ax_k - b)|^2 & j \in \mathcal{U}_k \\ 0 & \text{otherwise} \end{cases};$$

and a Kaczmarz update is then performed.

**Lemma A.7.1.** Greedy Randomized Vector Kaczmarz is Markovian.

**Proof.** The probability distribution of $e_{i_k}$ only depends on the current iterate, $x_k$. Hence, $\mathbb{P}[e_{i_k} \in \mathcal{W}| \mathcal{F}_k] = \mathbb{P}[e_{i_k} \in \mathcal{W}| \mathcal{F}_k^k]$. \[\square\]

**Lemma A.7.2.** Greedy Randomized Vector Kaczmarz is 1,1-Exploratory.

**Proof.** Let $x_0 \neq P_H x_0$. Then, $e_0$ is a positive real number, and, if $\mathcal{U}_0$ is nonempty, for any $j \in \mathcal{U}_0$, $e_j^T (Ax_0 - b) \neq 0$. Therefore, if $\mathcal{U}_0$ is nonempty for all $x_0 \neq P_H x_0$, then

$$\sup_{x_0 : x_0 \neq P_H x_0} \mathbb{P}[e_{i_0}^T (Ax_0 - b) = 0] = 0.$$  

Therefore, we need only verify that $\mathcal{U}_0$ is nonempty for all $x_0 \neq P_H x_0$. Note,

$$\max_{j \in \{1,\ldots,n\}} \frac{|e_j^T (Ax_0 - b)|^2}{\|A^T e_j\|_2^2 \|Ax_0 - b\|_2^2} \geq \sum_{j=1}^{n} \frac{\|A^T e_j\|_2^2}{\|A\|_F^2} \frac{|e_j^T (Ax_0 - b)|^2}{\|A^T e_j\|_2^2 \|Ax_0 - b\|_2^2} = \frac{1}{\|A\|_F^2}.$$  

Hence,

$$\max_{j \in \{1,\ldots,n\}} \frac{|e_j^T (Ax_0 - b)|^2}{\|A^T e_j\|_2^2 \|Ax_0 - b\|_2^2} \geq \frac{1}{2} \max_{j \in \{1,\ldots,n\}} \frac{|e_j^T (Ax_0 - b)|^2}{\|A^T e_j\|_2^2 \|Ax_0 - b\|_2^2} + \frac{1}{2} \frac{1}{\|A\|_F^2}.$$  

Therefore, for every $x_0 \neq P_H x_0$, $\mathcal{U}_0 \neq \emptyset$. \[\square\]
We can now apply Corollary 3.8 to conclude as follows.

**Theorem A.7.3.** Let \( A \in \mathbb{R}^{n \times d} \) and \( b \in \mathbb{R}^n \) such that the linear system’s solution set, \( \mathcal{H} \), is nonempty. Let \( x_0 \in \mathbb{R}^d \). Let \( \{x_k : k \in \mathbb{N}\} \) be a sequence generated by the Greedy Randomized Vector Kaczmarz method. Then, there exists a stopping time \( \tau \) with finite expectation such that \( x_\tau \in \mathcal{H} \); or there exists a sequence of non-negative stopping times \( \{\tau_j : j + 1 \in \mathbb{N}\} \) for which \( \mathbb{E}[\tau_j] \leq j[(\text{rank}(A) - 1)/\pi_{\min} + 1] \), and there exist \( \gamma \in (0, 1) \) and a sequence of random variables \( \{\gamma_j : j + 1 \in \mathbb{N}\} \subset (0, \gamma] \), such that

\[
\mathbb{P}\left[ \bigcap_{j=0}^{\infty} \left\{ \|x_{\tau_j} - \mathcal{P}_H x_0\|_2^2 \leq \left( \prod_{\ell=0}^{j-1} \gamma_\ell \right) \|x_0 - \mathcal{P}_H x_0\|_2^2 \right\} \right] = 1.
\]

**A.8. Sampling Kaczmarz-Motzkin Method.** Let \( Ax = b \) be a consistent linear system. In this method, a subset of equations of a fixed size is randomly selected from the linear system at each iteration (independently and with uniform probability); then the equation with the largest absolute residual is selected from this subset; and the Kaczmarz update is then applied with this equation.

**Lemma A.8.1.** The Sampling Kaczmarz-Motzkin Method is Markovian.

**Proof.** The subset is selected independently and then the equation with the largest absolute residual is selected from this subset. Hence, the selection process only depends on knowledge of the current iterate. \( \Box \)

**Lemma A.8.2.** The Sampling Kaczmarz-Motzkin Method is \( 1, n^{-1} \)-Exploratory.

**Proof.** If \( x_0 \neq \mathcal{P}_H x_0 \), then there is at least one equation in the system that is not satisfied by \( x_0 \). The probability that this equation is included in a sample is \( 1/n \). By the second step of the selection process, this equation is selected or one with a larger absolute residual is selected. Therefore, letting \( e_{i_0} \) denote the standard basis element corresponding to the equation selected,

\[
\sup_{x_0 : x_0 \neq \mathcal{P}_H x_0} \mathbb{P}\left[ e_{i_0}^T (Ax_0 - b) = 0 \big| \mathcal{F}_\tau \right] \leq 1 - \frac{1}{n}.
\]

**Remark A.8.3.** The ratio of \( N/\pi = n \) can be improved if we have more information about the system, but our choice applies quite generally.

We can now apply Corollary 3.8 to conclude as follows.

**Theorem A.8.4.** Let \( A \in \mathbb{R}^{n \times d} \) and \( b \in \mathbb{R}^n \) such that the linear system’s solution set, \( \mathcal{H} \), is nonempty. Let \( x_0 \in \mathbb{R}^d \). Let \( \{x_k : k \in \mathbb{N}\} \) be a sequence generated by the Sampling Kaczmarz-Motzkin method. Then, there exists a stopping time \( \tau \) with finite expectation such that \( x_\tau \in \mathcal{H} \); or there exists a sequence of non-negative stopping times \( \{\tau_j : j + 1 \in \mathbb{N}\} \) for which \( \mathbb{E}[\tau_j] \leq j[(\text{rank}(A) - 1)n + 1] \), and there exist \( \gamma \in (0, 1) \) and a sequence of random variables \( \{\gamma_j : j + 1 \in \mathbb{N}\} \subset (0, \gamma] \), such that

\[
\mathbb{P}\left[ \bigcap_{j=0}^{\infty} \left\{ \|x_{\tau_j} - \mathcal{P}_H x_0\|_2^2 \leq \left( \prod_{\ell=0}^{j-1} \gamma_\ell \right) \|x_0 - \mathcal{P}_H x_0\|_2^2 \right\} \right] = 1.
\]

**A.9. Streaming Vector Kaczmarz.** In this case, we apply Kaczmarz method to a stream of equations that are assumed to be independent and identically distributed. We will refer to this method as the Streaming Vector Kaczmarz method. We
The set $\mathcal{H} \subset \mathbb{R}^d$ of vectors that satisfy an equation with probability one is nonempty. To encapsulate this method in our framework, we assume, at iteration $k$, that $\varphi_R() = (W_k, \theta)$ and that we only observe $\alpha_k = A^T W_k$ and $\beta_k = b^T W_k$. Then, we perform the update

$$x_{k+1} = x_k - \alpha_k \frac{\alpha_k^T x_k - \beta_k}{\|\alpha_k\|_2^2}. \tag{A.9.1}$$

**Lemma A.9.1.** The Streaming Vector Kaczmarz method is Markovian. 

**Proof.** Since $\{ (\alpha_k, \beta_k) : k + 1 \in \mathbb{N} \}$ are independent and identically distributed, then $P[\alpha_k \in W | F_k^{+1}] = P[\alpha_k \in W] = P[\alpha_k \in W | F_k^+]$. \(\square\)

**Lemma A.9.2.** There exists $\pi \in (0, 1)$ such that the Streaming Vector Kaczmarz method is $1, \pi$-Exploratory.

**Proof.** For any $x$, let $\mathcal{N} = \{ z - P_H x : z \in \mathcal{H} \}$, and define $\mathcal{R} = \mathcal{N} \perp$. Of course, $\mathcal{N}$ is independent of the choice of $x$, and, for any $x$, $x - P_H x \in \mathcal{R}$. Let $S$ denote the unit sphere in $\mathbb{R}^d$. For a contradiction, suppose $\sup_{v \in \mathcal{R} \setminus \{0\}} P[\alpha_0^T v = 0] = 1$. Then, there is a sequence $\{ v_k : k \in \mathbb{N} \} \subset \mathcal{R} \cap S$, such that $\lim_{k \to \infty} P[\alpha_0^T v_k = 0] = 1$. By the compactness of $\mathcal{R} \cap S$, there exists a subsequence, $\{ v_{k_j} \}$, and a vector, $w \in \mathcal{R} \cap S$, such that $\lim_{j \to \infty} v_{k_j} = w$. Note, $w \in \text{span}(\bigcup_{j=1}^\infty \{ v_{k_j} \})$ for all $\ell \in \mathbb{N}$, and $\text{span}[w] = \mathcal{R} \cap \mathcal{N} \cap S$. Therefore,

$$1 = \lim_{j \to \infty} P[\alpha_0^T v_{k_j} = 0] = \lim_{j \to \infty} P[\alpha_0 \perp \text{span}(\bigcup_{\ell=j}^\infty \{ v_{k_j} \})] \tag{A.9.2}$$

$$= P[\alpha_0 \perp \bigcap_{j=1}^\infty \text{span}(\bigcup_{\ell=j}^\infty \{ v_{k_j} \})] = P[\alpha_0 \perp w] = P[w \in \mathcal{R} \cap \mathcal{N} \cap S]. \tag{A.9.3}$$

Hence, we have a contradiction as the last probability must be $0$ since $\mathcal{R} \perp \mathcal{N}$. It follows that there exists a $\pi \in (0, 1)$ such that $\sup_{v \in \mathcal{R} \setminus \{0\}} P[\alpha_0 \perp v] \leq 1 - \pi$. The result follows. \(\square\)

**Lemma A.9.3.** The Streaming Vector Kaczmarz is Uniformly Nontrivial.

**Proof.** Let $\Omega_k$ be the set of all orthonormal bases of $\text{col}(\alpha_k)$ (c.f., $\Omega_k$ is the set of all orthonormal bases of $\text{col}(\alpha_k \perp \chi_k)$). Then, for all $k + 1 \in \mathbb{N}$,

$$\sup_{Q_k \in \Omega_k, s \in \{1, \ldots, k\}} \min_{G \in \mathcal{G}(Q_0, \ldots, Q_k)} \det(G^T G) \geq \sup_{Q_k \in \Omega_k, s \in \{1, \ldots, k\}} \min_{G \in \mathcal{G}(Q_0, \ldots, Q_k)} \det(G^T G), \tag{A.9.4}$$

where the latter quantity is independent of $(x_0, \zeta_1)$ and is positive with probability one. Therefore, for every $k + 1 \in \mathbb{N}$, there exists $\epsilon_k > 0$ such that

$$P\left[\sup_{Q_k \in \Omega_k, s \in \{1, \ldots, k\}} \min_{G \in \mathcal{G}(Q_0, \ldots, Q_k)} \det(G^T G) > \epsilon_k\right] \geq \frac{1}{2}. \tag{A.9.5}$$

Moreover, there exists a $K \in \mathbb{N}$ such that for $k \geq K$, $P[A_k(x_0, \zeta_1) | F_k^0] \geq 3/4$ for all $x_0 \neq P_H x_0$ and $\zeta_1 \in \mathcal{S}$, which implies that, for $k \geq K$,

$$P\left[\sup_{Q_k \in \Omega_k, s \in \{1, \ldots, k\}} \min_{G \in \mathcal{G}(Q_0, \ldots, Q_k)} \det(G^T G) > \epsilon_k, A_k(x_0, \zeta_1) | F_k^0\right] \geq \frac{1}{4}. \tag{A.9.6}$$

Hence, by Markov’s Inequality, we can find $g_A \geq \epsilon_k/4 > 0$. \(\square\)
We can conclude by Corollary 3.10.

**Theorem A.9.4.** Let \( \{(\alpha_k, \beta_k) : k + 1 \in \mathbb{N}\} \subset \mathbb{R}^d \times \mathbb{R} \) be a sequence of independent, identically distributed random variables such that \( \mathcal{H} = \{x \in \mathbb{R}^d : P[\alpha_0^T x = b] = 1\} \neq 0 \). Let \( x_0 \in \mathbb{R}^d \) and let \( \{x_k : k \in \mathbb{N}\} \) be generated by (A.9.1). Then, there exists a stopping time \( \tau \) with finite expectation such that \( x_\tau \in \mathcal{H} \); or there exists \( \pi \in (0, 1] \),

there exists \( \tau \) such that \( \mathbb{E}[\tau] \leq j((\dim \mathcal{H} - 1)/\pi + 1) \), and there exists \( \gamma \in (0, 1) \) such that for any \( \gamma \in (\gamma, 1) \),

\[
(A.9.7) \quad \mathbb{P} \left[ \bigcup_{k=0}^{\infty} \bigcap_{j=L}^{\infty} \left\{ \|x_{\tau_j} - \mathcal{P}_\mathcal{H} x_0\|_2^2 \leq \gamma_j^2 \|x_0 - \mathcal{P}_\mathcal{H} x_0\|_2^2 \right\} \right] = 1.
\]

**A.10. Cyclic Vector Coordinate Descent.** For a description of this method, see Example 3.12. We assume that \( A \in \mathbb{R}^{n \times d} \) and \( b \in \mathbb{R}^n \) are arbitrary—that is, we do not require that they form a consistent system, and we let \( r^* = -\mathcal{P}_\ker(A^\top)b \).

**Lemma A.10.1.** Cyclic Vector Coordinate Descent is Markovian.

**Proof.** Note, the search coordinate at iteration \( k \) is fully determined by \( \zeta_{k-1} \). Hence, the result follows. \( \square \)

**Lemma A.10.2.** Cyclic Vector Coordinate Descent is \( d, 1 \)-Exploratory.

**Proof.** Suppose \( x_0 \in \mathbb{R}^d \) such that \( Ax_0 \neq b \). Then, \( A^\top(Ax_0 - b) \neq 0 \). Hence, there is some \( e_i \) for \( i \in \{1, \ldots, d\} \) such that \( e_i^\top A^\top(Ax_0 - b) \neq 0 \). Therefore,

\[
(A.10.1) \quad \sup_{x_0:Ax_0 = b \neq r^*} \mathbb{P} \left[ \bigcap_{j=0}^{d-1} \{e_i^\top A^\top(Ax_0 - b) = 0\} \big| \mathcal{F}_1 \right] = 0.
\]

The conclusion follows. \( \square \)

Since \( \{\text{col}(Ae_i) : i = 1, \ldots, d\} \) is a finite set, we can apply Corollary 3.18 to conclude as follows.

**Theorem A.10.3.** Let \( A \in \mathbb{R}^{n \times d} \) and \( b \in \mathbb{R}^n \), and define \( r^* = -\mathcal{P}_\ker(A^\top)b \). Let \( x_0 \in \mathbb{R}^d \) and \( \{x_k : k \in \mathbb{N}\} \) be a sequence generated by cyclic vector coordinate descent. Then, either there exists a stopping time \( \tau \) with finite expectation such that \( Ax_\tau - b = r^* \); or there exists a sequence of non-negative stopping times \( \{\tau_j : j+1 \in \mathbb{N}\} \) for which \( \mathbb{E}[\tau_j] \leq j((\rank(A) - 1)d + 1) \), and there exist \( \gamma \in (0, 1) \) and a sequence of random variables \( \{\gamma_j : j+1 \in \mathbb{N}\} \subset (0, \gamma] \), such that

\[
(A.10.2) \quad \mathbb{P} \left[ \bigcup_{j=0}^{\infty} \left\{ \|Ax_{\tau_j} - b - r^*\|_2^2 \leq \left( \prod_{\ell=0}^{j-1} \gamma_\ell \right) \|Ax_0 - b - r^*\|_2^2 \right\} \right] = 1.
\]

**A.11. Gaussian Vector Column Space Descent.** Let \( A \in \mathbb{R}^{n \times d} \) and \( b \in \mathbb{R}^n \) be arbitrary, and let \( r^* = -\mathcal{P}_\ker(A^\top)b \). The Gaussian Vector Column Space method is specified as follows. Let \( \{w_k : k+1 \in \mathbb{N}\} \subset \mathbb{R}^d \) be a sequence of Gaussian random variables, and consider an update scheme \( x_{k+1} = x_k + \alpha_k w_k \), where

\[
(A.11.1) \quad \alpha_k \in \arg\min_{\alpha} \|Ax_k - b + \alpha Aw_k\|_2^2.
\]

By solving for \( \alpha_k \), the update is

\[
(A.11.2) \quad x_{k+1} = x_k + w_k \frac{w_k^\top A^\top (b - Ax_k)}{\|Aw_k\|_2^2},
\]
which we see is of the form (3.18) with \( \phi_C() = (w_k, \emptyset) \).

**Lemma A.11.1.** The Gaussian Vector Column Space method is Markovian.

*Proof.* This follows from the independence of \( \{w_k\} \).

**Lemma A.11.2.** The Gaussian Vector Column Space method is 1,1-Exploratory.

*Proof.* Since \( w_0 \) is a continuous random variable, \( \Pr[w_0^\top A^\top v = 0] = 0 \) for any \( v \neq 0 \). In particular, for any \( x_0 \) such that \( Ax_0 - b \neq r^* \), \( \Pr[w_0^\top A^\top (Ax_0 - b) = 0|\mathcal{F}_1^0] = 0 \).

**Lemma A.11.3.** The Gaussian Vector Column Space method is Uniformly Non-trivial.

*Proof.* Because \( \{w_k\} \) are continuous random variables, \( \Pr[w_k^\top A^\top (Ax_k - b) = 0|\mathcal{F}_k^0] \) is 0 whenever \( Ax_k - b \neq r^* \). Therefore, \( \chi_k = 1 \) whenever \( Ax_k - b \neq r^* \). We now have two cases.

For the first case, rank \( (A) = 1 \). Then, for any \( Ax_0 - b - r^*, Aw_0 \in \text{col}(A) \setminus \{0\} \). Therefore, we can substitute \( Aw_0/\|Aw_0\|_2 \) with \( (Ax_0 - b - r^*)/\|Ax_0 - b - r^*\|_2 \) in the update for \( x_1 \), which implies \( Ax_1 - b = r^* \). Therefore, \( \chi_k = 0 \) for all \( k \in \mathbb{N} \). So \( G(Q_0, \ldots, Q_k) = G(Q_0) \) for all \( k \in \mathbb{N} \), and for all \( G \in G(Q_0) \), \( \det(G^\top G) = 1 \). Therefore, (A.11.3)

\[
\inf_{x_0:Ax_0 - b \neq r^*} \sup_{k \in \mathbb{N} \setminus \{0\}} \mathbb{E} \left[ \sup_{Q_s \in Q_s} \min_{g \in G(Q_s, \ldots, Q_k)} \det(G^\top G) 1 \left[ A_k(x_0, \zeta_{-1}) \right] \right] = 1.
\]

In other words, when rank \( (A) = 1 \), the Gaussian Vector Column Space method is uniformly nontrivial.

For the second case, rank \( (A) > 1 \). We now proceed by induction. Note, by the continuity of the random variables, \( Ax_{k+1} - b \neq r^* \) if \( Ax_k - b \neq r^* \), which implies \( \chi_k = 1 \). By induction, we can conclude that \( \chi_k = 1 \) for all \( k + 1 \in \mathbb{N} \). Therefore, owing to their independence of \( x_0 \) and positivity, for every \( k + 1 \in \mathbb{N} \) there exists \( \epsilon_k > 0 \) such that

(A.11.4) \[ \Pr \left[ \sup_{Q_s \in Q_s} \min_{g \in G(Q_s, \ldots, Q_k)} \det(G^\top G) > \epsilon_k \right] \geq \frac{1}{2}. \]

Moreover, \( \exists K \in \mathbb{N} \) such that for all \( k \geq K \), \( \Pr[A_k(x_0, \zeta_{-1})|\mathcal{F}_k^0] \geq 3/4 \). By the inclusion-exclusion principles and Markov’s Inequality, for all \( k \geq K \),

(A.11.5) \[ \mathbb{E} \left[ \sup_{Q_s \in Q_s} \min_{g \in G(Q_s, \ldots, Q_k)} \det(G^\top G) 1 \left[ A_k(x_0, \zeta_{-1}) \right] \right] \geq \epsilon_k/4. \]

The conclusion follows.

From Corollary 3.20, we can conclude as follows.

**Theorem A.11.4.** Let \( A \in \mathbb{R}^{n \times d} \) and \( b \in \mathbb{R}^n \), and define \( r^* = -P_{\ker(A^\top)} b \). Let \( x_0 \in \mathbb{R}^d \) and \( \{x_k : k \in \mathbb{N}\} \) be a sequence generated by the Gaussian Vector Column Space method. Then, either there exists a stopping time \( \tau \) with finite expectation such that \( Ax_\tau - b = r^* \); or there exists a sequence of non-negative stopping times
\{\tau_j : j + 1 \in \mathbb{N}\} for which \(\mathbb{E}[\tau_j] \leq j \text{rank}(A)\), and there exists \(\bar{\gamma} \in (0, 1)\) such that for any \(\gamma \in (\bar{\gamma}, 1)\),

\[
\mathbb{P}\left[ \bigcup_{L=0}^{\infty} \bigcap_{j=L}^{\infty} \left\{ \|Ax_{\tau_j} - b - r^*\|_2^2 \leq \gamma^j \|Ax_0 - b - r^*\|_2^2 \right\} \right] = 1.
\]

A.12. Zouzias-Freris Vector Coordinate Descent. Let \(A \in \mathbb{R}^{n \times d}\) and \(b \in \mathbb{R}^n\) be arbitrary. Note, if \(A\) has a column that is entirely zero, we can eliminate it, so we will assume this case does not happen. In Zouzias-Freris Vector Coordinate Descent,\(^7\) at iteration \(k\), we select independently sample an element, \(e_{i_k}\), from the standard basis of \(\mathbb{R}^d\) from distribution

\[
P[i_k = j] \propto \begin{cases} \|Ae_j\|_2^2 & j = 1, \ldots, n \\ 0 & \text{otherwise} \end{cases}
\]

and then apply coordinate descent with this row.

**Lemma A.12.1.** Zouzias-Freris Vector Coordinate Descent is Markovian.

**Proof.** This follows immediately from the fact that \(\{i_k\}\) are selected independently at each iteration. \(\square\)

**Lemma A.12.2.** Let \(\pi_{\min} = \min_j \{\|Ae_j\|_2^2/\|A\|_F^2\}\). Zouzias-Freris Vector Coordinate Descent is in \(\pi_{\min}\)-Exploratory.

**Proof.** Suppose \(x_0 \in \mathbb{R}^d\) such that \(Ax_0 - b \neq r^*\), where \(r^* = -P_{\ker(A^\top)}b\). Then, for some \(e_j\), \(e_j^\top A^\top (Ax_0 - b) \neq 0\). The probability that we select this basis element on the first iteration (i.e., \(k = 0\)) is at least \(\pi_{\min}\). The conclusion follows. \(\square\)

We now apply Corollary 3.18 to conclude.

**Theorem A.12.3.** Let \(A \in \mathbb{R}^{n \times d}\) and \(b \in \mathbb{R}^n\), and define \(r^* = -P_{\ker(A^\top)}b\). Let \(x_0 \in \mathbb{R}^d\) and \(\{x_k : k \in \mathbb{N}\}\) be a sequence generated by Zouzias-Freris vector coordinate descent. Then, either there exists a stopping time \(\tau\) with finite expectation such that \(Ax_\tau - b = r^*\); or there exists a sequence of non-negative stopping times \(\{\tau_j : j + 1 \in \mathbb{N}\}\) for which \(\mathbb{E}[\tau_j] \leq j(\text{rank}(A) - 1)/\pi_{\min} + 1\), and there exist \(\gamma \in (0, 1)\) and a sequence of random variables \(\{\gamma_j : j + 1 \in \mathbb{N}\} \subset (0, \gamma]\), such that

\[
P\left[ \bigcap_{j=0}^{\infty} \left\{ \|Ax_{\tau_j} - b - r^*\|_2^2 \leq \prod_{\ell=0}^{j-1} \gamma_\ell \|Ax_0 - b - r^*\|_2^2 \right\} \right] = 1.
\]

The methods described in in this section, Appendices A.10 and A.11 are compared on a simple statistical regression problem in Figure 4.

A.13. Max Residual Vector Coordinate Descent. In this approach, we have \(e_{i_k}\) at an iteration \(k\) such that

\[
i_k \in \arg\max_{j \in \{1, \ldots, d\}} |e_j^\top A^\top (Ax_k - b)|.
\]

\(^7\)In [41], the authors propose a randomized extended Kaczmarz method, not a coordinate descent method. However, the authors did propose the distribution for the coordinate descent method that we discuss here, which is why we have named this method as we have.
A Comparison of Some Vector Column-action Methods

![Graph showing the comparison of different methods.](image)

Fig. 4. A comparison of three vector column-action methods on a linear regression problem where the design matrix is derived from a balanced design of 50 treatments with twenty replicates each. All methods are stopped when the residual-norm of the normal equation is less than $10^{-8}$.

where ties are broken by choosing the smallest index. We then do the coordinate descent update with this choice of $i_k$.

**Lemma A.13.1. Max Residual Vector Coordinate Descent is Markovian.**

**Proof.** The choice of $e_{i_k}$ only depends on the most recent iterate, $x_k$. Thus, the procedure is Markovian. 

**Lemma A.13.2. Max Residual Vector Coordinate Descent is 1,1-Exploratory.**

**Proof.** Let $x_0$ be such that $Ax_0 - b \neq r^*$ where $r^* = -P_{\ker(A^T)}b$. Then, there exists a $j \in \{1, \ldots, d\}$ such that $Ae_j \not\perp Ax_0 - b$. As a result, $e_{i_k}^T A^T(Ax_0 - b) \neq 0$. The conclusion follows.

We can now apply Corollary 3.18 to conclude.

**Theorem A.13.3.** Let $A \in \mathbb{R}^{n \times d}$ and $b \in \mathbb{R}^n$, and define $r^* = -P_{\ker(A^T)}b$. Let $x_0 \in \mathbb{R}^d$ and $\{x_k : k \in \mathbb{N}\}$ be a sequence generated by Max Residual vector coordinate descent. Then, either there exists a stopping time $\tau$ with finite expectation such that $Ax_\tau - b = r^*$; or there exists a sequence of non-negative stopping times $\{\tau_j : j + 1 \in \mathbb{N}\}$ for which $\mathbb{E}[\tau_j] \leq j \text{rank}(A)$, and there exist $\gamma \in (0, 1)$ and a sequence of random variables $\{\gamma_j : j + 1 \in \mathbb{N}\} \subset (0, \gamma]$, such that

\[
(A.13.2) \quad \mathbb{P} \left[ \bigcap_{j=0}^{\infty} \left\{ \|Ax_{\tau_j} - b - r^*\|_2^2 \leq (\prod_{\ell=0}^{j-1} \gamma_{\ell}) \|Ax_0 - b - r^*\|_2^2 \right\} \right] = 1.
\]

**A.14. Max Distance Vector Coordinate Descent.** In this approach, we have $e_{i_k}$ such that

\[
(A.14.1) \quad i_k \in \arg\max_{j \in \{1, \ldots, d\}} \frac{|e_j^T A^T(Ax_k - b)|}{\|A^T A e_j\|_2^2},
\]

This selection procedure is an analogue of Agmon’s method, but it is not Agmon’s method applied to the normal equations.
where ties are broken by choosing the smallest index. Then, we update $x_k$ using coordinate descent at coordinate $i_k$.\footnote{The selection process is analogous to Motzkin’s method, but the update is not equivalent to applying Motzkin’s method to the normal equations.}

**Lemma A.14.1.** Max Distance Vector Coordinate Descent is Markovian.

**Proof.** The choice of $e_{i_k}$ only depends on $x_k$, which makes this procedure Markovian.

**Lemma A.14.2.** Max Distance Vector Coordinate Descent is 1,1-Exploratory.

**Proof.** For any $x_0$ such that $Ax_0 - b \neq r^*$ where $r^* = -P_{\ker(A^T)}b$, there exists an $e_j$ such that $e_j^T A^T (Ax_0 - b) \neq 0$. Hence, $e_j^T A^T (Ax_0 - b) \neq 0$.

We can now conclude by Corollary 3.18.

**Theorem A.14.3.** Let $A \in \mathbb{R}^{n \times d}$ and $b \in \mathbb{R}^n$, and define $r^* = -P_{\ker(A^T)}b$. Let $x_0 \in \mathbb{R}^d$ and $\{x_k : k \in \mathbb{N}\}$ be a sequence generated by Max Distance vector coordinate descent. Then, either there exists a stopping time $\tau$ with finite expectation such that $Ax_\tau - b = r^*$; or there exists a sequence of non-negative stopping times $\{\tau_j : j+1 \in \mathbb{N}\}$ for which $\mathbb{E}[\tau_j] \leq j \text{rank}(A)$, and there exist $\gamma \in (0,1)$ and a sequence of random variables $\{\gamma_j : j+1 \in \mathbb{N}\} \subset (0,\gamma]$, such that

\[
P\left[ \bigcap_{j=0}^{\infty} \left\{ \|Ax_{\tau_j} - b - r^*\|_2^2 \leq \left( \prod_{\ell=0}^{j-1} \gamma_\ell \right) \|Ax_0 - b - r^*\|_2^2 \right\} \right] = 1. \tag{A.14.2}
\]

**A.15. Random Permutation Block Kaczmarz.** For details on this method, see Example 3.3. Suppose $A \in \mathbb{R}^{n \times d}$ and $b \in \mathbb{R}^n$ form a consistent system with solution set $\mathcal{H}$. Suppose we split the system up into $\epsilon$ blocks.

**Lemma A.15.1.** Random Permutation Block Kaczmarz is Markovian.

**Proof.** The value of $E_{i_k}$ and $\zeta_k$ are dependent only on $\zeta_{k-1}$ and, periodically, on an independently generated permutation. Hence, the procedure is Markovian.

**Lemma A.15.2.** Random Permutation Block Kaczmarz is $1,\epsilon^{-1}$-Exploratory.

**Proof.** If $x_0 \neq P_{\mathcal{H}}x_0$, then for some $j \in \{1, \ldots, n\}$, $E_j^T (Ax_0 - b) \neq 0$. The probability of selecting this particular $E_j$ on the first sample is $1/\epsilon$. The conclusion follows.

We can conclude now using Corollary 3.8.

**Theorem A.15.3.** Let $A \in \mathbb{R}^{n \times d}$ and $b \in \mathbb{R}^n$ such that the linear system’s solution set, $\mathcal{H}$, is nonempty. Let $x_0 \in \mathbb{R}^d$ and let $\{x_k : k \in \mathbb{N}\}$ be generated by Random Permutation Block Kaczmarz. Then either there exists a stopping time $\tau$ with finite expectation such that $x_\tau \in \mathcal{H}$; or there exist a sequence of non-negative stopping times, $\{\tau_j : j+1 \in \mathbb{N}\}$, such that $\mathbb{E}[\tau_j] \leq j[(\text{rank}(A) - 1)/\epsilon + 1]$, and $\exists \gamma \in (0,1)$ and a sequence of random variables $\{\gamma_j : j+1 \in \mathbb{N}\} \subset (0,\gamma]$ such that

\[
P\left[ \bigcap_{j=0}^{\infty} \left\{ \|x_{\tau_j} - P_{\mathcal{H}}x_0\|_2^2 \leq \left( \prod_{\ell=0}^{j-1} \gamma_\ell \right) \|x_0 - P_{\mathcal{H}}x_0\|_2^2 \right\} \right] = 1. \tag{A.15.1}
\]
A.16. Steinerberger’s Block Kaczmarz. Though not explicitly discussed, this method is an example of sketch-and-project [14, 33, 13]. Let \( \{E_j : j = 1, \ldots, \epsilon\} \) where \([E_1 \cdots E_\epsilon]\) is a column permutation of the \(\mathbb{R}^{n \times n}\) identity matrix. In Steinerberger’s block method, at iteration \(x_k\), we select \(E_{i_k}\) such that

\[
\mathbb{P} [i_k = j] \propto \begin{cases} \|E_j^\top(Ax_k - b)\|_p^p & j = 1, \ldots, \epsilon \\ 0 & \text{otherwise.} \end{cases}
\]

(A.16.1)

With this choice of \(E_{i_k}\), we compute \(x_{k+1}\) using the block Kaczmarz update.

**Lemma A.16.1.** Steinerberger’s Block Kaczmarz is Markovian.

*Proof.* The choice of \(E_{i_k}\) only depends on \(x_k\), which implies the conclusion. \(\blacksquare\)

**Lemma A.16.2.** Steinerberger’s block Kaczmarz is 1,1-Exploratory.

*Proof.* Let \(\mathcal{H}\) denote the solution set of the linear system. For any \(x_0 \neq \mathcal{P}_\mathcal{H}x_0, \exists j \in \{1, \ldots, \epsilon\}\) such that \(E_j^\top(Ax_0 - b) \neq 0\). Therefore, \(\mathbb{P}[E_{i_0}^\top(Ax_0 - b) = 0|\mathcal{F}_1^\epsilon] = 0\). The result follows.

We can conclude using Corollary 3.8 as follows.

**Theorem A.16.3.** Let \(A \in \mathbb{R}^{n \times d}\) and \(b \in \mathbb{R}^n\) such that the linear system’s solution set, \(\mathcal{H}\), is nonempty. Let \(x_0 \in \mathbb{R}^d\). Let \(\{x_k : k \in \mathbb{N}\}\) be a sequence generated by the Steinerberger’s Block Kaczmarz method. Then, there exists a stopping time \(\tau\) with finite expectation such that \(x_\tau \in \mathcal{H}\); or there exists a sequence of non-negative stopping times \(\{\tau_j : j + 1 \in \mathbb{N}\}\) for which \(\mathbb{E}[\tau_j] \leq j \text{rank}(A)\), and there exist \(\gamma \in (0, 1)\) and a sequence of random variables \(\{\gamma_j : j + 1 \in \mathbb{N}\} \subset (0, \gamma]\), such that

\[
\mathbb{P} \left[ \bigcap_{j=0}^{\infty} \left\{ \|x_{\tau_j} - \mathcal{P}_\mathcal{H}x_0\|_2^2 \leq \left( \prod_{\ell=0}^{j-1} \gamma_\ell \right) \|x_0 - \mathcal{P}_\mathcal{H}x_0\|_2^2 \right\} \right] = 1.
\]

(A.16.2)

A.17. Motzkin’s Block Method. Let \(A \in \mathbb{R}^{n \times d}\) and \(b \in \mathbb{R}^n\) form a consistent linear system. In this method, we begin with \(\{E_j : j = 1, \ldots, \epsilon\}\) where \([E_1 \cdots E_\epsilon]\) is a column permutation of the \(\mathbb{R}^{n \times n}\) identity matrix. At iteration \(k\), we choose \(i_k\) such that

\[
i_k \in \text{argmax}_{j \in \{1, \ldots, \epsilon\}} \| (E_j^\top AA^\top E_j)^\dagger E_j^\top (b - Ax_k) \|_2.
\]

(A.17.1)

Then, we perform the block Kaczmarz update using \(E_{i_k}\). Note, this update rule is not the same as the max distance adaptive sketch-and-project [13].

**Lemma A.17.1.** Motzkin’s Block Method is Markovian.

*Proof.* The selection of \(E_{i_k}\) only depends on \(x_k\). Hence, the procedure is Markovian. \(\blacksquare\)

**Lemma A.17.2.** Motzkin’s Block Method is 1,1-Exploratory.

*Proof.* Let \(\mathcal{H} = \{x : Ax = b\}\). For any \(x_0 \neq \mathcal{P}_\mathcal{H}x_0\), there exists \(j \in \{1, \ldots, \epsilon\}\) such that \(E_j^\top (b - Ax_0) \neq 0\). Hence, \((E_j^\top AA^\top E_j)^\dagger E_j^\top (b - Ax_0) \neq 0\). Therefore, \(E_{i_0}^\top (Ax_0 - b) \neq 0\).

We can now conclude using Corollary 3.8 as follows.

**Theorem A.17.3.** Let \(A \in \mathbb{R}^{n \times d}\) and \(b \in \mathbb{R}^n\) such that the linear system’s solution set, \(\mathcal{H}\), is nonempty. Let \(x_0 \in \mathbb{R}^d\). Let \(\{x_k : k \in \mathbb{N}\}\) be a sequence generated
Corollary 3.8 as follows. We have seen previously.

To fix a procedure, we consider selecting \((A.19.2)\)

\[ \text{maximum approach. Note, random procedures will follow a similar pattern to what} \]

such that \( \gamma \in (0, 1) \) and a sequence of random variables \( \{\tau_j : j + 1 \in \mathbb{N}\} \subset (0, \gamma] \), such that

\[
\tag{A.17.2}
\mathbb{P} \left[ \bigcap_{j=0}^{\infty} \left\{ \|x_{\tau_j} - P_{\mathcal{H}} x_0\|_2^2 \leq \left( \prod_{t=0}^{j-1} \gamma_t \right) \|x_0 - P_{\mathcal{H}} x_0\|_2^2 \right\} \right] = 1.
\]

**A.18. Agmon’s Block Method.** Let \( A \in \mathbb{R}^{n \times d} \) and \( b \in \mathbb{R}^n \) form a consistent linear system. In this method, we begin with \( \{E_j : j = 1, \ldots, \epsilon\} \) where \( \{E_1, \ldots, E_\epsilon\} \) is a column permutation of the \( \mathbb{R}^{n \times n} \) identity matrix. At iteration \( k \), we choose \( i_k \) such that

\[
\tag{A.18.1}
i_k \in \arg\max_{j \in \{1, \ldots, \epsilon\}} \|E_j^T (b - Ax_k)\|_2.
\]

Then, we perform the block Kaczmarz update using \( E_{i_k} \). Note, this update rule is not the same as the max distance adaptive sketch-and-project [13].

**Lemma A.18.1.** Agmon’s Block Method is Markovian.

**Proof.** The selection of \( E_{i_k} \) only depends on \( x_k \). Hence, the procedure is Markovian.

**Lemma A.18.2.** Agmon’s Block Method is 1,1-Exploratory.

**Proof.** Let \( \mathcal{H} = \{x : Ax = b\} \). For any \( x_0 \neq P_{\mathcal{H}} x_0 \), there exists \( j \in \{1, \ldots, \epsilon\} \) such that \( E_j^T (b - Ax_0) \neq 0 \). Therefore, \( E_j^T (Ax_0 - b) \neq 0 \).

We can now conclude using Corollary 3.8 as follows.

**Theorem A.18.3.** Let \( A \in \mathbb{R}^{n \times d} \) and \( b \in \mathbb{R}^n \) such that the linear system’s solution set, \( \mathcal{H} \), is nonempty. Let \( x_0 \in \mathbb{R}^d \). Let \( \{x_k : k \in \mathbb{N}\} \) be a sequence generated by the Agmon’s block method. Then, there exists a stopping time \( \tau \) with finite expectation such that \( x_\tau \in \mathcal{H} \); or there exists a sequence of non-negative stopping times \( \{\tau_j : j + 1 \in \mathbb{N}\} \) for which \( \mathbb{E}[\tau_j] \leq \text{rank}(A) \), and there exist \( \gamma \in (0, 1) \) and a sequence of random variables \( \{\gamma_j : j + 1 \in \mathbb{N}\} \subset (0, \gamma] \), such that

\[
\tag{A.18.2}
\mathbb{P} \left[ \bigcap_{j=0}^{\infty} \left\{ \|x_{\tau_j} - P_{\mathcal{H}} x_0\|_2^2 \leq \left( \prod_{t=0}^{j-1} \gamma_t \right) \|x_0 - P_{\mathcal{H}} x_0\|_2^2 \right\} \right] = 1.
\]

**A.19. Adaptive Sketch-and-Project.** Let \( \tilde{A} \in \mathbb{R}^{n \times d} \) and \( b \in \mathbb{R}^n \) form a consistent system. Let \( B \) be a positive definite symmetric matrix. In adaptive sketch-and-project, we first generate a set of sketching matrices \( \{S_j : j = 1, \ldots, \epsilon\} \subset \mathbb{R}^{n \times p} \). Then, we initialize with a vector \( z_0 \in \mathbb{R}^d \) and perform the update

\[
\tag{A.19.1}
z_{k+1} = z_k - B^{-1} \tilde{A}^T S_{i_k} (S_{i_k}^T \tilde{A} B^{-1} \tilde{A}^T S_{i_k})^T (\tilde{A} z_k - b),
\]

where \( i_k \) is selected either deterministically or by sampling from a distribution that depends on the functions

\[
\tag{A.19.2}f_j(z_k) = (\tilde{A} z_k - b)^T S_j (S_j^T \tilde{A} B^{-1} \tilde{A}^T S_{i_k})^T S_{i_k}^T (\tilde{A} z_k - b), \quad j = 1, \ldots, \epsilon.
\]

To fix a procedure, we consider selecting \( i_k \in \arg\max_j f_j(z_k) \), which we refer to as the maximum approach. Note, random procedures will follow a similar pattern to what we have seen previously.
To rewrite this in our notation, we let $x_k = B^{1/2}z_k$ and $A = AB^{-1/2}$. Then, the update of $x_k$ is

$$x_{k+1} = x_k - A^TS_k(A^T S_k A)^{-1}A^T S_k(Ax_k - b),$$

and

$$f_j(x_k) = (Ax_k - b)^T S_j(A^T S_k A)^{-1}A^T S_k(Ax_k - b), \ j = 1, \ldots, \epsilon.$$  

Thus, we see that the inner product is readily accounted for in our framework.

While the sketch-and-project framework is quite general, we have already covered many of the interesting special cases already. Therefore, we will focus on a special set of sketching matrices that allow us to effectively do away with the exactness assumption for the sketch-and-project framework (see [13, Assumption 1] and [33, Assumption 2]). That is, we will focus on the case where \{\text{S}_j : j = 1, \ldots, \epsilon\} \subset \mathbb{R}^{n \times p}$ are independently drawn from a distribution that satisfies the Johnson-Lindenstrauss Property: there exist $C, w > 0$ such that for all $\delta \geq 0$ and for any $r \in \mathbb{R}^n$

$$
\mathbb{P}\left[\|S_j^T r\|_2^2 - \|r\|_2^2 > \delta \|r\|_2^2\right] < 2 \exp\left(-C\rho \delta \min\{\delta, w^{-1}\}\right).
$$

\textbf{Remark A.19.1.} This condition is satisfied by many interesting distributions such as Gaussian Matrices [8], Achlioptas Sketches [1], and Fast Johnson Lindenstrauss Sketches [3]. Moreover, values of $C$ and $w$ are summarized in [31, Table 2].

Moreover, for such matrices, the exactness condition [33, Assumption 2] is effectively moot as we now explain.

\textbf{Lemma A.19.2.} Suppose \{\text{S}_j : j = 1, \ldots, \epsilon\} \subset \mathbb{R}^{n \times p}$ are drawn independently from a distribution that satisfies the Johnson-Lindenstrauss Property. If for some $\rho > 0$

$$p > \frac{(\rho + 1)\log(2)}{0.999C} \max\left\{\frac{1}{0.999}, w\right\},$$

then

$$\sup_{v \in \mathbb{R}^n \setminus \{0\}} \mathbb{P}\left[\bigcap_{j=1}^\epsilon \left\{\|S_j^T v\|_2^2 = 0\right\}\right] \leq 2^{-\epsilon p}.$$ 

\textbf{Proof.} For any $v \in \mathbb{R}^n$ such that $v \neq 0$, we apply independence and the Johnson-Lindenstrauss Property with $\delta = 0.999$ to proceed as follows.

$$\mathbb{P}\left[\bigcap_{j=1}^\epsilon \left\{\|S_j^T v\|_2^2 = 0\right\}\right] = \prod_{j=1}^\epsilon \mathbb{P}\left[\|S_j^T v\|_2^2 = 0\right]$$

$$\leq \prod_{j=1}^\epsilon \mathbb{P}\left[\|S_j^T v\|_2^2 - \|v\|_2^2 > 0.999\|v\|_2^2\right]$$

$$\leq \prod_{j=1}^\epsilon \exp(-0.999C \rho \min\{0.999, w^{-1}\} + \log(2))$$

$$\leq \exp(-\epsilon \rho \log(2)),$$

where the last line follows for the constraint on $p$. 

To demonstrate how we use this, consider Achlioptas Sketches for which $C = 0.23467$ and $w = 0.1127$. If we choose $\rho = 4$, then we can choose $p = 15$. Moreover, if we sample $\epsilon = 20$ such matrices, then the probability bound in the preceding lemma is bounded by $10^{-30}$. Thus, if we used 20 independently sampled Achlioptas Sketches of embedding dimension 15, then the probability that each of them would find $S_m^T(Ax_0 - b) = 0$ when $Ax_0 \neq b$ is less than $10^{-30}$. To explain the scale of this, in expectation, we would need to repeat this process (independently) a trillion times a second for the remaining life of the sun (8 billion years, conservatively) before we generated a sample (in exact arithmetic) for which we could find an $x_0$ such that $Ax_0 \neq b$ and $S_m^T(Ax_0 - b) = 0$ for all $j = 1, \ldots, 20$.

We can now verify the relevant properties to analyze this case of maximum adaptive sketch-and-project.

**Lemma A.19.3.** For a fixed $\{S_1, \ldots, S_\epsilon\}$, Maximum Adaptive Sketch-and-Project is Markovian.

**Proof.** The selection method for $S_k$ only depends on $x_k$, hence it is Markovian.

**Lemma A.19.4.** Suppose $\{S_1, \ldots, S_\epsilon\}$ are sampled independently from a distribution satisfying the Johnson-Lindenstrauss Property with

$$p > \frac{(\rho + 1)\log(2)}{0.999C} \max \left\{ \frac{1}{0.999}, w \right\},$$

for some choice of $\rho > 0$. Maximum Adaptive Sketch-and-Project is 1,1-Exploratory with probability at least $1 - 2^{-\epsilon \rho}$.

**Proof.** Suppose $x_0$ is such that $Ax_0 \neq b$. Then, off of an event of probability at most $2^{-\epsilon \rho}$, we will choose $\{S_1, \ldots, S_\epsilon\}$ such that $\exists j \in \{1, \ldots, \epsilon\}$ for which $S_m^T(Ax_0 - b) \neq 0$ on this event (up to measure zero). Therefore, $S_m^T(Ax_0 - b) \neq 0$ with probability one on this event (up to measure zero). The claim follows.

We can now apply Corollary 3.8 to conclude as follows.

**Theorem A.19.5.** Suppose $\{S_1, \ldots, S_\epsilon\}$ are sampled independently from a distribution satisfying the Johnson-Lindenstrauss Property with

$$p > \frac{(\rho + 1)\log(2)}{0.999C} \max \left\{ \frac{1}{0.999}, w \right\},$$

for some choice of $\rho > 0$. Let $A \in \mathbb{R}^{n \times d}$ and $b \in \mathbb{R}^n$ such that the linear system’s solution set, $\mathcal{H}$, is nonempty. Let $x_0 \in \mathbb{R}^d$. Let $\{x_k : k \in \mathbb{N}\}$ be a sequence generated by Maximum Adaptive Sketch-and-Project. Then, on an event of probability at least $1 - 2^{-\epsilon \rho}$, either there exists a stopping time $\tau$ such that $\mathbb{E}[\tau] = \sigma(S_1, \ldots, S_\epsilon) < \infty$ and such that $x_\tau \in \mathcal{H}$; or there exists a sequence of stopping times $\{\tau_j : j + 1 \in \mathbb{N}\}$ such that $\mathbb{E}[\tau_j] = \sigma(S_1, \ldots, S_\epsilon) \leq j \rank(A)$ and $\exists \gamma \in (0, 1)$ (depending on $\{S_1, \ldots, S_\epsilon\}$) such that

$$\|x_{\tau_j} - \mathcal{P}_H x_0\|_2^2 \leq \gamma^j \|x_0 - \mathcal{P}_H x_0\|_2^2$$

up to a set of measure zero.

**A.20. Greedy Randomized Block Kaczmarz.** Suppose $A \in \mathbb{R}^{n \times d}$ and $b \in \mathbb{R}^n$ form a consistent linear system with a solution set $\mathcal{H}$. Moreover, let $\{E_j : j = 1, \ldots, \epsilon\}$ be such that $[E_1 \quad \cdots \quad E_\epsilon]$ is a column permutation of the $\mathbb{R}^{n \times n}$ identity
matrix. For the Greedy Randomized Block Kaczmarz method at iteration $k$, we first compute a threshold,

\begin{equation}
\epsilon_k = \max_{j \in \{1, \ldots, \ell\}} \frac{\|E_j^T (Ax_k - b)\|_2^2}{2 \|Ax_k - b\|_2^2 \|A^T E_j\|_F^2} + \frac{1}{2 \|A\|_F^2}.
\end{equation}

and identify the set

\begin{equation}
\mathcal{U}_k = \left\{ j : \frac{\|E_j^T (Ax_k - b)\|_2^2}{\|Ax_k - b\|_2^2 \|A^T E_j\|_F^2} \geq \epsilon_k \right\}.
\end{equation}

Then, we randomly choose an index $i_k$ according to the distribution

\begin{equation}
P[i_k = j] \propto \begin{cases} \frac{\|E_j^T (Ax_k - b)\|_2^2}{\|Ax_k - b\|_2^2 \|A^T E_j\|_F^2} & j \in \mathcal{U}_k \\
0 & \text{otherwise.} \end{cases}
\end{equation}

We then perform the Kaczmarz update with $E_{i_k}$ to compute $x_{k+1}$ from $x_k$.

**Lemma A.20.1.** Greedy Randomized Block Kaczmarz is Markovian.

**Proof.** The selection of $E_{i_k}$ only depends on knowing $x_k$. Hence, the procedure is Markovian. \hfill \square

**Lemma A.20.2.** Greedy Randomized Block Kaczmarz is $1,1$-Exploratory.

**Proof.** Let $x_0 \neq \mathcal{P}_H x_0$. If we show that $\mathcal{U}_0$ is nonempty, then it follows that for any $j \in \mathcal{U}_0$, $E_j^T (Ax_0 - b) \neq 0$, and, in particular, $E_j^T (Ax_0 - b) \neq 0$. To verify that $\mathcal{U}_0$ is nonempty, note

\begin{equation}
\max_{j \in \{1, \ldots, \ell\}} \frac{\|E_j^T (Ax_0 - b)\|_2^2}{\|Ax_0 - b\|_2^2 \|A^T E_j\|_F^2} = \sum_{j=1}^\ell \frac{\|E_j^T (Ax_0 - b)\|_2^2}{\|Ax_0 - b\|_2^2 \|A^T E_j\|_F^2} = \frac{1}{\|A\|_F^2}.
\end{equation}

Then, we can conclude using Corollary 3.8 as follows.

**Theorem A.20.3.** Let $A \in \mathbb{R}^{n \times d}$ and $b \in \mathbb{R}^n$ such that the linear system’s solution set, $\mathcal{H}$, is nonempty. Let $x_0 \in \mathbb{R}^d$. Let $\{x_k : k \in \mathbb{N}\}$ be a sequence generated by Greedy Block Randomized Kaczmarz. Then, there exists a stopping time $\tau$ with finite expectation such that $x_\tau \in \mathcal{H}$; or there exists a sequence of non-negative stopping times $\{\tau_j : j + 1 \in \mathbb{N}\}$ for which $\mathbb{E} [\tau_j] \leq \text{rank}(A)$, and there exist $\gamma \in (0, 1)$ and a sequence of random variables $\{\gamma_j : j + 1 \in \mathbb{N}\}$ in $(0, \gamma)$, such that

\begin{equation}
P \left[ \bigcap_{j=0}^\infty \left\{ \|x_{\tau_j} - \mathcal{P}_H x_0\|_2 \leq (\prod_{\ell=0}^{j-1} \gamma_\ell) \|x_0 - \mathcal{P}_H x_0\|_2 \right\} \right] = 1.
\end{equation}

A.21. **Streaming Block Kaczmarz.** We assume that we have a sequence of independent, identically distributed random variables, $\{(\alpha_k, \beta_k) : k + 1 \in \mathbb{N}\} \subset \mathbb{R}^{d \times p} \times \mathbb{R}^p$ (for $p > 1$), such that $\mathcal{H} = \{x \in \mathbb{R}^d : \mathbb{P}[\alpha_k^T x = \beta_t] = 1\} \neq \emptyset$. Given $x_0 \in \mathbb{R}^d$, we generate $\{x_k : k \in \mathbb{N}\}$ according to $x_{k+1} = x_k - \alpha_k (\alpha_k^T x_k - \beta_k)$. 

Lemma A.21.1. Streaming Block Kaczmarz is Markovian.

Proof. By independence, \( \mathbb{P} [\alpha_k \in \mathcal{W} | \mathcal{F}_{k+1}^k] = \mathbb{P} [\alpha_k \in \mathcal{W}] = \mathbb{P} [\alpha_k \in \mathcal{W} | \mathcal{F}_1^k] \). \( \square \)

Lemma A.21.2. There exists a \( \pi \in (0, 1) \) such that Streaming Block Kaczmarz is 1, \( \pi \)-Exploratory.

Proof. For any \( x \), let \( \mathcal{N} = \{ z - \mathcal{P}_H x : z \in \mathcal{H} \} \) and \( \mathcal{R} = \mathcal{N}^\perp \). Note, \( \mathcal{N} \) does not depend on the choice of \( x \) and, for any \( x, x - \mathcal{P}_H x \in \mathcal{R} \). Let \( \mathcal{S} \) denote the unit sphere in \( \mathbb{R}^d \).

For a contradiction, suppose there exists a sequence \( \{v_k : k \in \mathbb{N}\} \subset \mathcal{R} \cap \mathcal{S} \) such that \( \lim_{k \to \infty} \mathbb{P} [\alpha_0 \perp v_k] = 1 \). Then, there exists a \( w \in \mathcal{R} \cap \mathcal{S} \) and a subsequence \( \{v_{k_j} \} \) such that \( \lim_{j \to \infty} v_{k_j} = w \). Note, \( w \in \text{span}[\cup_{j=1}^{\infty} \{v_{k_j}\}] \) for all \( j \in \mathbb{N} \), and, consequently, \( \text{span}[\{w\}] = \cap_{j=1}^{\infty} \text{span}[\cup_{j=1}^{\infty} \{v_{k_j}\}] \). Therefore,

\[
(A.21.1) \\
1 = \lim_{k \to \infty} \mathbb{P} [\alpha_0 \perp v_k] = \lim_{j \to \infty} \mathbb{P} [\alpha_0 \perp \text{span} \left( \bigcup_{t=j}^{\infty} \{v_{k_t}\} \right)] \\
= \mathbb{P} [\alpha_0 \perp \text{span} \left( \bigcup_{t=1}^{\infty} \{v_{k_t}\} \right)] = \mathbb{P} [\alpha_0 \perp w] = \mathbb{P} [w \in \mathcal{R} \cap \mathcal{N} \cap \mathcal{S}].
\]

The ultimate probability is zero, which supplies the contradiction. Thus, no such sequence \( \{v_k\} \), which implies \( \exists \pi \in (0, 1) \) such that \( \sup_{v \in \mathcal{R} \setminus \{0\}} \mathbb{P} [\alpha_0 \perp v] \leq 1 - \pi \).

Lemma A.21.3. The Streaming Vector Kaczmarz is Uniformly Nontrivial.

Proof. Let \( \Omega_k' \) be the set of all orthonormal bases of \( \text{col} (\alpha_k) \) (c.f., \( \Omega_k \) is the set of all orthonormal bases of \( \text{col} (\alpha_k \chi_k) \)). Then, for all \( k + 1 \in \mathbb{N} \),

\[
(A.21.2) \\
\sup_{Q_s \in \Omega_s', s \in \{1, \ldots, k\}} \min_{G \in G(Q_0, \ldots, Q_k)} \det(G^T G) \geq \sup_{Q_s \in \Omega_s', s \in \{1, \ldots, k\}} \min_{G \in G(Q_0, \ldots, Q_k)} \det(G^T),
\]

where the latter quantity is independent of \( (x_0, \zeta_{-1}) \) and is positive with probability one. Therefore, for every \( k + 1 \in \mathbb{N} \), there exists \( \epsilon_k > 0 \) such that

\[
(A.21.3) \quad \mathbb{P} \left[ \sup_{Q_s \in \Omega_s', s \in \{1, \ldots, k\}} \min_{G \in G(Q_0, \ldots, Q_k)} \det(G^T G) > \epsilon_k \right] \geq \frac{1}{2}.
\]

Moreover, there exists a \( K \in \mathbb{N} \) such that for \( k \geq K \), \( \mathbb{P} [A_k(x_0, \zeta_{-1}) | \mathcal{F}_0] \geq 3/4 \) for all \( x_0 \neq \mathcal{P}_H x_0 \) and \( \zeta_{-1} \in \mathcal{S} \), which implies that, for \( k \geq K \),

\[
(A.21.4) \quad \mathbb{P} \left[ \sup_{Q_s \in \Omega_s', s \in \{1, \ldots, k\}} \min_{G \in G(Q_0, \ldots, Q_k)} \det(G^T G) > \epsilon_k, A_k(x_0, \zeta_{-1}) \right] \geq \frac{1}{4}.
\]

Hence, by Markov’s Inequality, we can find \( g_A \geq \epsilon_k / 4 > 0 \).

We can conclude by Corollary 3.10.

Theorem A.21.4. Let \( \{(\alpha_k, \beta_k) : k + 1 \in \mathbb{N}\} \subset \mathbb{R}^{d \times p} \times \mathbb{R}^p \) be a sequence of independent, identically distributed random variables such that \( \mathcal{H} = \{ x \in \mathbb{R}^d : \mathbb{P} [\alpha_0^T x = b] = 1 \} \neq \emptyset \). Let \( x_0 \in \mathbb{R}^d \) and let \( \{x_k : k \in \mathbb{N}\} \) be generated by Streaming Block Kaczmarz. Then, there exists a stopping time \( \tau \) with finite expectation such
that \( x_\tau \in \mathcal{H} \); or there exists \( \pi \in (0, 1) \), there exists a sequence of non-negative stopping times \( \{ \tau_j : j + 1 \in \mathbb{N} \} \) for which \( \mathbb{E} [\tau_j] \leq j (\dim \mathcal{H} - 1)/\pi + 1 \), and there exists \( \bar{\gamma} \in (0, 1) \) such that for any \( \gamma \in (\bar{\gamma}, 1) \),

\[
\mathbb{P} \left[ \bigcup_{L=0}^{\infty} \bigcap_{J=L}^{\infty} \left\{ \left\| x_{\tau_j} - \mathcal{P}_H x_0 \right\|_2^2 \leq \gamma^j \left\| x_0 - \mathcal{P}_H x_0 \right\|_2^2 \right\} \right] = 1.
\]

(A.21.5)

**A.22. Random Permutation Block Coordinate Descent.** For details of this method, see Example 3.13. Let \( \mathbb{A}_{n \times d} \) and \( b \in \mathbb{R}^n \), and let \( r^* = -\mathcal{P}_{\ker(A)} b \).

**Lemma A.22.1.** Random Permutation Block Coordinate Descent is Markovian.

**Proof.** The selection of \((W_k, \zeta_{k-1})\) only depends on \( \zeta_{k-1} \). Hence, the method is Markovian.

**Lemma A.22.2.** Random Permutation Block Coordinate Descent is 1, \( \epsilon^{-1} \)-Exploratory.

**Proof.** If \( Ax_0 - b \neq r^* \), then \( \exists j \in \{1, \ldots, \epsilon \} \) such that \( E_j^\top A^\top (Ax_0 - b) \neq 0 \). The probability that we observe this \( E_j \) as the first element of a random permutation is \( \epsilon^{-1} \). The result follows.

We now conclude using Corollary 3.18 as follows.

**Theorem A.22.3.** Let \( A \in \mathbb{R}^{n \times d} \) and \( b \in \mathbb{R}^n \), and define \( r^* = -\mathcal{P}_{\ker(A)} b \). Let \( x_0 \in \mathbb{R}^d \) and \( \{x_k : k \in \mathbb{N} \} \) be a sequence generated by Random Permutation Block Coordinate Descent. Then, either there exists a stopping time \( \tau \) with finite expectation such that \( Ax_\tau - b = r^* \); or there exists a sequence of non-negative stopping times \( \{\tau_j : j + 1 \in \mathbb{N} \} \) for which \( \mathbb{E} [\tau_j] \leq j (\text{rank} (A) - 1) \epsilon + 1 \), and there exist \( \gamma \in (0, 1) \) and a sequence of random variables \( \{\gamma_j : j + 1 \in \mathbb{N} \} \subset (0, \gamma] \), such that

\[
\mathbb{P} \left[ \bigcap_{J=0}^{\infty} \left\{ \left\| Ax_{\tau_j} - b - r^* \right\|_2^2 \leq \prod_{j=0}^{J-1} \gamma_j \left\| Ax_0 - b - r^* \right\|_2^2 \right\} \right] = 1.
\]

(A.22.1)

**A.23. Gaussian Block Column Space Descent.** Let \( A \in \mathbb{R}^{n \times d} \), \( b \in \mathbb{R}^n \), and \( r^* = -\mathcal{P}_{\ker(A)} b \). This method proceeds at each iteration by independently sampling a Gaussian matrix, \( W_k \), that maps into \( \mathbb{R}^d \), and computes \( x_{k+1} = x_k + W_k \alpha_k \) where \( \alpha_k \in \text{argmin}_{\alpha} \|Ax_k - b + AW_k\alpha\|_2^2 \).

**Lemma A.23.1.** Gaussian Block Column Space Descent is Markovian.

**Proof.** This follows from the independence of \( W_k \).

**Lemma A.23.2.** Gaussian Block Column Space Descent is 1, 1-Exploratory.

**Proof.** Since \( W_0 \) is a continuous random variable, \( \mathbb{P}[W_0^\top A^\top (Ax_0 - b) = 0] = 0 \) for any \( Ax_0 - b \neq r^* \). The conclusion follows.

**Lemma A.23.3.** Gaussian Block Column Space Descent is Uniformly Nontrivial.

**Proof.** Let \( \Omega_k \) be the set of all orthonormal bases of \( \text{col}(\alpha_k) \) (c.f., \( \Omega_k \) is the set of all orthonormal bases of \( \text{col}(\alpha_k \chi_k) \)). Then, for all \( k + 1 \in \mathbb{N} \),

\[
\sup_{Q_k, r \in \Omega_k, s \in \{1, \ldots, k\}} \min_{G \in \Gamma(Q_2, \ldots, Q_k)} \det(G^\top G) \geq \sup_{Q_k, r \in \Omega_k, s \in \{1, \ldots, k\}} \min_{G \in \Gamma(Q_0, \ldots, Q_k)} \det(G^\top G),
\]

(A.23.1)
where the latter quantity is independent of \((x_0, \zeta_{-1})\) and is positive with probability one. Therefore, for every \(k + 1 \in \mathbb{N}\), there exists \(\epsilon_k > 0\) such that

\[
\Pr \left[ \sup_{Q_s \in Q_s', s \in \{1,...,k\}} \min_{G \in G(Q_0,\ldots,Q_k)} \det(G^TG) > \epsilon_k \right] \geq \frac{1}{2}.
\]

Moreover, there exists a \(K \in \mathbb{N}\) such that for \(k \geq K\),

\[
\Pr \left[ A_k(x_0, \zeta_{-1}) | \mathcal{F}_0^k \right] \geq \frac{3}{4}
\]

for all \(x_0 \neq \mathcal{P}_R x_0\) and \(\zeta_{-1} \in \mathcal{Z}\), which implies that, for \(k \geq K\),

\[
\Pr \left[ \sup_{Q_s \in Q_s', s \in \{1,...,k\}} \min_{G \in G(Q_0,\ldots,Q_k)} \det(G^TG) > \epsilon_k, A_k(x_0, \zeta_{-1}) \right] | \mathcal{F}_0^k \geq \frac{1}{4}.
\]

Hence, by Markov’s Inequality, \(g_A \geq \epsilon_k/4 > 0\) for all \(k \geq K\).

We conclude by Corollary 3.20 as follows.

**Theorem A.23.4.** Let \(A \in \mathbb{R}^{n \times d}\) and \(b \in \mathbb{R}^n\), and define \(r^* = -\mathcal{P}_{\ker(A^T)} b\). Let \(x_0 \in \mathbb{R}^d\) and \(\{x_k : k \in \mathbb{N}\}\) be a sequence generated by Gaussian Block Column Space Descent. Then, either there exists a stopping time \(\tau\) with finite expectation such that \(Ax_\tau - b = r^*\); or there exists a sequence of non-negative stopping times \(\{\tau_j : j+1 \in \mathbb{N}\}\) for which \(E[\tau_j] \leq \bar{\gamma} j \in (0,1)\) such that for any \(\gamma \in (\bar{\gamma},1)\),

\[
\mathbb{P} \left[ \bigcup_{j=1}^{\infty} \bigcup_{L=0}^{\infty} \left\{ \|Ax_{\tau_j} - b - r^*\|_2^2 \leq \gamma^j \|Ax_0 - b - r^*\|_2^2 \right\} \right] = 1.
\]

**A.24.** Zouzias-Freris Block Coordinate Descent. Let \(A \in \mathbb{R}^{n \times d}\), \(b \in \mathbb{R}^n\) and \(r^* = -\mathcal{P}_{\ker(A^T)} b\). Let \(\{E_j : j = 1,\ldots,\epsilon\}\) be such that \([E_1 \cdots E_\epsilon]\) is a column permutation of the \(R^{d \times d}\) identity matrix. In this method, at iteration \(k\), we select \(E_{i_k}\) independently according to

\[
\Pr[i_k = j] \propto \frac{\|AE_j\|_F^2}{\|A\|_F^2} \quad j = 1,\ldots,\epsilon
\]

and we compute \(x_{k+1} = x_k + E_{i_k} \alpha_k\) where \(\alpha_k \in \arg\min_{\alpha} \|Ax_k - b - AE_{i_k} \alpha\|_2\).

**Lemma A.24.1.** Zouzias-Freris Block Coordinate Descent is Markovian.

**Proof.** This follows from the independence of \(\{i_k\}\).

**Lemma A.24.2.** Let \(\pi_{\text{min}} = \min_j \{\|AE_j\|_F^2/\|A\|_F^2 : AE_j \neq 0\}\). Zouzias-Freris Block Coordinate Descent is Markovian-Exploratory.

**Proof.** If \(Ax_0 - b \neq r^*\), then \(3j \in \{1,\ldots,\epsilon\}\) such that \(E_j^T A^T (Ax_0 - b) \neq 0\). Therefore, \(\Pr[i_0 = j] \geq \pi_{\text{min}}\). The conclusion follows.

By Corollary 3.18, we conclude as follows.

**Theorem A.24.3.** Let \(A \in \mathbb{R}^{n \times d}\) and \(b \in \mathbb{R}^n\), and define \(r^* = -\mathcal{P}_{\ker(A^T)} b\). Let \(x_0 \in \mathbb{R}^d\) and \(\{x_k : k \in \mathbb{N}\}\) be a sequence generated by Zouzias-Freris Block Coordinate Descent. Then, either there exists a stopping time \(\tau\) with finite expectation such that \(Ax_\tau - b = r^*\); or there exists a sequence of non-negative stopping times \(\{\tau_j : j+1 \in \mathbb{N}\}\) for which \(E[\tau_j] \leq \gamma j (\text{rank}(A) - 1)/\pi_{\text{min}} + 1\), and there exist \(\gamma \in (0,1)\) and a sequence of random variables \(\gamma_j : j+1 \in \mathbb{N}\) \(\subset (0,\gamma)\), such that

\[
\mathbb{P} \left[ \prod_{j=0}^{\infty} \left\{ \|Ax_{\tau_j} - b - r^*\|_2^2 \leq \prod_{j=0}^{j-1} \gamma_j \|Ax_0 - b - r^*\|_2^2 \right\} \right] = 1.
\]
A.25. Max Residual Block Coordinate Descent. Let $A \in \mathbb{R}^{n \times d}$, $b \in \mathbb{R}^n$ and $r^* = -P_{\ker(A^\top)}b$. Let $\{E_j : j = 1, \ldots, \epsilon\}$ be such that $[E_1 \; \cdots \; E_\epsilon]$ is a column permutation of the $R^{d \times d}$ identity matrix. In this method, at iteration $k$, we select $E_{i_k}$ according to

\begin{equation}
    i_k \in \operatorname{argmax}_{j \in \{1, \ldots, \epsilon\}} \|E_j^\top A^\top(Ax_k - b)\|_2,
\end{equation}

and we compute $x_{k+1} = x_k + E_{i_k} \alpha_k$ where $\alpha_k \in \operatorname{argmin}_{\alpha} \|Ax_k - b - AE_{i_k} \alpha\|_2$.

**Lemma A.25.1.** Max Residual Block Coordinate Descent is Markovian.

**Proof.** The selection of $\{i_k\}$ depends only on $x_k$, which implies the conclusion. □

**Lemma A.25.2.** Max Residual Block Coordinate Descent is 1,1-Exploratory.

**Proof.** If $Ax_0 - b \neq r^*$, then $\exists j \in \{1, \ldots, \epsilon\}$ such that $E_j^\top A^\top(Ax_0 - b) \neq 0$. The conclusion follows. □

By Corollary 3.18, we conclude as follows.

**Theorem A.25.3.** Let $A \in \mathbb{R}^{n \times d}$ and $b \in \mathbb{R}^n$, and define $r^* = -P_{\ker(A^\top)}b$. Let $x_0 \in \mathbb{R}^d$ and $\{x_k : k \in \mathbb{N}\}$ be a sequence generated by Max Residual Block Coordinate Descent. Then, either there exists a stopping time $\tau$ with finite expectation such that $Ax_\tau - b = r^*$; or there exists a sequence of non-negative stopping times $\{\tau_j : j+1 \in \mathbb{N}\}$ for which $\mathbb{E}[\tau_j] \leq j \operatorname{rank}(A)$, and there exist $\gamma \in (0,1)$ and a sequence of random variables $\{\gamma_j : j + 1 \in \mathbb{N}\} \subset [0, \gamma]$, such that

\begin{equation}
    \mathbb{P} \left( \bigcap_{j=0}^{\infty} \left\{ \|Ax_{\tau_j} - b - r^*\|_2^2 \leq \left( \prod_{\ell=0}^{j-1} \gamma_\ell \right) \|Ax_0 - b - r^*\|_2^2 \right\} \right) = 1.
\end{equation}

A.26. Max Distance Block Coordinate Descent. Let $A \in \mathbb{R}^{n \times d}$, $b \in \mathbb{R}^n$ and $r^* = -P_{\ker(A^\top)}b$. Let $\{E_j : j = 1, \ldots, \epsilon\}$ be such that $[E_1 \; \cdots \; E_\epsilon]$ is a column permutation of the $R^{d \times d}$ identity matrix. In this method, at iteration $k$, we select $E_{i_k}$ according to

\begin{equation}
    i_k \in \operatorname{argmax}_{j \in \{1, \ldots, \epsilon\}} \|(E_j^\top A^\top E_j)^\dagger E_j^\top A^\top(Ax_k - b)\|_2,
\end{equation}

and we compute $x_{k+1} = x_k + E_{i_k} \alpha_k$ where $\alpha_k \in \operatorname{argmin}_{\alpha} \|Ax_k - b - AE_{i_k} \alpha\|_2$.

**Lemma A.26.1.** Max Distance Block Coordinate Descent is Markovian.

**Proof.** The selection of $\{i_k\}$ depends only on $x_k$, which implies the conclusion. □

**Lemma A.26.2.** Max Distance Block Coordinate Descent is 1,1-Exploratory.

**Proof.** If $Ax_0 - b \neq r^*$, then $\exists j \in \{1, \ldots, \epsilon\}$ such that $E_j^\top A^\top(Ax_0 - b) \neq 0$. Hence, $(E_j^\top A^\top E_j)^\dagger E_j^\top A^\top(Ax_0 - b) \neq 0$. The conclusion follows. □

By Corollary 3.18, we conclude as follows.

**Theorem A.26.3.** Let $A \in \mathbb{R}^{n \times d}$ and $b \in \mathbb{R}^n$, and define $r^* = -P_{\ker(A^\top)}b$. Let $x_0 \in \mathbb{R}^d$ and $\{x_k : k \in \mathbb{N}\}$ be a sequence generated by Max Distance Block Coordinate Descent. Then, either there exists a stopping time $\tau$ with finite expectation such that $Ax_\tau - b = r^*$; or there exists a sequence of non-negative stopping times $\{\tau_j : j+1 \in \mathbb{N}\}$ for which $\mathbb{E}[\tau_j] \leq j \operatorname{rank}(A)$, and there exist $\gamma \in (0,1)$ and a sequence of random variables $\{\gamma_j : j + 1 \in \mathbb{N}\} \subset [0, \gamma]$, such that
variables \( \{ \gamma_j : j + 1 \in \mathbb{N} \} \subset (0, \gamma] \), such that

\[
A.26.2 \quad \mathbb{P} \left[ \bigcap_{j=0}^{\infty} \left\{ \| Ax_j - b - r^* \|_2^2 \leq \left( \prod_{\ell=0}^{j-1} \gamma_\ell \right) \| Ax_0 - b - r^* \|_2^2 \right\} \right] = 1.
\]

The methods described in this section, Appendices A.23 and A.26 are compared on a simple statistical regression problem in Figure 5.

REFERENCES

[1] D. Achlioptas, Database-friendly random projections, in Proceedings of the twentieth ACM SIGMOD-SIGACT-SIGART symposium on Principles of database systems, 2001, pp. 274–281.
[2] S. Agmon, The relaxation method for linear inequalities, Canadian Journal of Mathematics, 6 (1954), pp. 382–392.
[3] N. Ailon and B. Chazelle, The fast johnson–lindenstrauss transform and approximate nearest neighbors, SIAM Journal on computing, 39 (2009), pp. 302–322.
[4] Z.-Z. Bai and X.-G. Liu, On the meany inequality with applications to convergence analysis of several row-action iteration methods, Numerische Mathematik, 124 (2013), pp. 215–236.
[5] Z.-Z. Bai and W.-T. Wu, On greedy randomized kaczmarz method for solving large sparse linear systems, SIAM Journal on Scientific Computing, 40 (2018), pp. A592–A606.
[6] A. H. Baker, J. M. Dennis, and E. R. Jessup, On improving linear solver performance: A block variant of gmres, SIAM Journal on Scientific Computing, 27 (2006), pp. 1608–1626.
[7] L. Dai and T. B. Schön, On the exponential convergence of the kaczmarz algorithm, IEEE Signal Processing Letters, 22 (2015), pp. 1571–1574.
[8] S. Dasgupta and A. Gupta, An elementary proof of a theorem of johnson and lindenstrauss, Random Structures & Algorithms, 22 (2003), pp. 60–65.
[9] P. J. Denning, The locality principle, Communications of the ACM, 48 (2005), pp. 19–24.
[10] N. Dunford, A mean ergodic theorem, Duke Mathematical Journal, 5 (1939), pp. 635–646.
[11] G. H. Golub and C. F. Van Loan, Matrix computations, vol. 3, JHU Press, 2012.
[12] R. Gower, D. Molitor, J. Moorman, and D. Needell, Adaptive sketch-and-project methods for solving linear systems, arXiv preprint arXiv:1909.03604, (2019).
[13] R. M. Gower, D. Molitor, J. Moorman, and D. Needell, On adaptive sketch-and-project for solving linear systems, SIAM Journal on Matrix Analysis and Applications, 42 (2021), pp. 954–989.
[14] R. M. Gower and P. Richtárik, Randomized iterative methods for linear systems, SIAM Journal on Matrix Analysis and Applications, 36 (2015), pp. 1660–1690.
[15] J. Haddock and A. Ma, *Greed works: An improved analysis of sampling kaczmarz-motzkin*, arXiv preprint arXiv:1912.03544, (2019).

[16] S. Kaczmarz, *Approximate solution of systems of linear equations*, International Journal of Control, 57 (1993), pp. 1269–1271.

[17] S. Karczmarz, *Angenaherte auflosung von systemen linearer glei-chungen*, Bull. Int. Acad. Pol. Sci. Let., Cl. Sci. Math. Nat., (1937), pp. 355–357.

[18] P.-G. Martinsson and J. A. Tropp, *Randomized numerical linear algebra: Foundations and algorithms*, Acta Numerica, 29 (2020), pp. 403–572.

[19] R. K. Meany, *A matrix inequality*, SIAM Journal on Numerical Analysis, 6 (1969), pp. 104–107.

[20] S. P. Meyn and R. L. Tweedie, *Markov chains and stochastic stability*, Springer Science & Business Media, 2012.

[21] T. S. Motzkin and I. J. Schoenberg, *The relaxation method for linear inequalities*, Canadian Journal of Mathematics, 6 (1954), pp. 393–404.

[22] I. Necoara, *Faster randomized block kaczmarz algorithms*, SIAM Journal on Matrix Analysis and Applications, 40 (2019), pp. 1425–1452.

[23] I. Necoara and D. Clipici, *Parallel random coordinate descent method for composite minimization: Convergence analysis and error bounds*, SIAM Journal on Optimization, 26 (2016), pp. 197–226.

[24] D. Needell and J. A. Tropp, *Paved with good intentions: analysis of a randomized block kaczmarz method*, Linear Algebra and its Applications, 441 (2014), pp. 199–221.

[25] D. Needell, R. Zhao, and A. Zouzias, *Randomized block kaczmarz method with projection for solving least squares*, Linear Algebra and its Applications, 484 (2015), pp. 322–343.

[26] J. Nutini, B. Sephry, I. Laradji, M. Schmidt, H. Koepke, and A. Virani, *Convergence rates for greedy kaczmarz algorithms, and faster randomized kaczmarz rules using the orthogonality graph*, arXiv preprint arXiv:1612.07838, (2016).

[27] V. Patel, M. Jahangoshahi, and D. A. Maldonado, *Convergence of adaptive, randomized, iterative linear solvers*, arXiv preprint arXiv:2104.04816, (2021).

[28] V. Patel, M. Jahangoshahi, and D. A. Maldonado, *An implicit representation and iterative solution of randomly sketched linear systems*, SIAM Journal on Matrix Analysis and Applications, 42 (2021), pp. 800–831.

[29] M. Pilanci and M. J. Wainwright, *Iterative Hessian sketch: Fast and accurate solution approximation for constrained least-squares*, Journal of Machine Learning Research, 17 (2016), pp. 800–831.

[30] C. D. L. V. Poussin, *Sur l’intégrale de lebesgue*, Transactions of the American Mathematical Society, (1915), pp. 435–501.

[31] N. Pritchard and V. Patel, *Residual tracking and stopping for iterative random sketching*, arXiv preprint arXiv:2201.05741, (2022).

[32] E. Rebrova and D. Needell, *On block gaussian sketching for the kaczmarz method*, Numerical Algorithms, 86 (2021), pp. 443–473.

[33] P. Richtárik and M. Takáč, *Stochastic reformulations of linear systems: algorithms and convergence theory*, SIAM Journal on Matrix Analysis and Applications, 41 (2020), pp. 487–524.

[34] Y. Saad, *Iterative methods for sparse linear systems*, vol. 82, siam, 2003.

[35] S. Steinerberger, *A weighted randomized kaczmarz method for solving linear systems*, Mathematics of Computation, (2021).

[36] S. Steinerberger, *Approximate solutions of linear systems at a universal rate*, arXiv preprint arXiv:2207.03388, (2022).

[37] T. Strohmer and R. Vershynin, *A randomized Kaczmarz algorithm with exponential convergence*, Journal of Fourier Analysis and Applications, 15 (2009), p. 262.

[38] J. Warga, *Minimizing certain convex functions*, Journal of the Society for Industrial and Applied Mathematics, 11 (1963), pp. 588–593.

[39] D. P. Woodruff, *Sketching as a tool for numerical linear algebra*, Foundations and Trends® in Theoretical Computer Science, 10 (2014), pp. 1–157.

[40] S. Wright and C.-p. Lee, *Analyzing random permutations for cyclic coordinate descent*, Mathematics of Computation, (2020).

[41] A. Zouzias and N. M. Freris, *Randomized extended kaczmarz for solving least squares*, SIAM Journal on Matrix Analysis and Applications, 34 (2013), pp. 773–793.