SECOND ORDER GEOMETRIC FLOWS ON FOLIATED MANIFOLDS
LUCIO BEDULLI, WEIYONG HE, AND LUIGI VEZZONI

Abstract. We prove a general result about the short time existence and uniqueness of second order geometric flows transverse to a Riemannian foliation on a compact manifold. Our result includes some flows already existing in literature, as the transverse Ricci flow, the Sasaki-Ricci flow and the Sasaki $J$-flow and motivates the study of other evolution equations. We also introduce a transverse version of the K"ahler-Ricci flow adapting some classical results to the foliated case.

1. Introduction

In this paper we study transversally parabolic flows on manifolds foliated by Riemannian foliations. The definition of Riemannian foliation was introduced and firstly studied by B. Reinhart in [20] as a natural generalization of Riemannian submersions. Roughly speaking a foliation $F$ on a manifold $M$ is Riemannian if there exists a Riemannian metric on $M$ such that the distance from one leaf of $F$ to another is locally constant. The normal bundle $Q$ to a Riemannian foliation $F$ inherits a metric $g_Q$ along the fibres which is “constant” along the leaves of $F$. Furthermore, $g_Q$ induces a canonical connection $\nabla$ on $Q$ preserving $g_Q$ and having vanishing transverse torsion. This connection can be used to define the transverse curvature and the transverse Ricci tensor of $g_Q$.

Searching for a preferred transverse metric on a manifold foliated by a Riemannian foliation, it is quite natural to follow the nonfoliated case studying the flow of a transverse metric along the transverse Ricci tensor. This was initiated in [16] in the context of Cartan geometry where it is introduced the transverse Ricci flow and it is proved a foliated version of the famous Hamilton’s results for 3-dimensional compact manifolds with positive Ricci tensor (see [12]). Furthermore, the transverse Ricci flow was used in [23] to evolve Sasakian metrics and then investigated in [11, 15, 31, 13, 28]. A similar flow for evolving Riemannian metrics on manifolds foliated by 1-codimensional non-Riemannian foliations was introduced and studied in [21, 22].

For the flows mentioned above, the short-time existence is proved by using an argument ad hoc. For instance, in [16] the short-time existence of the transverse Ricci flow is obtained regarding the flow as a flow of Cartan connections and then applying the original technique of Hamilton for parabolic systems satisfying integrability conditions, whilst in [23] the short time existence of the Sasaki Ricci flow is obtained by modifying the flow with a “parabolic complement”.

The main goal of this paper is to show that a second order quasilinear transversally parabolic flow of basic sections of a vector bundle over a foliated manifold has always a unique short-time solution. This result implies the short time existence of the transverse Ricci flow and of the Sasaki-Ricci flow and motivates the study of other flows. The precise framework is the following:

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we consider a compact manifold \( M \) foliated by a \textit{transversally orientable} Riemannian foliation \( \mathcal{F} \), an \( \mathcal{F} \)-bundle \((E, \nabla)\) over \( M \) and a second order quasilinear basic partial differential operator 
\[
D: C^\infty(E/\mathcal{F}) \to C^\infty(E/\mathcal{F}).
\]

By an "\( \mathcal{F} \)-bundle" we mean a vector bundle \( \pi: E \to M \) with an assigned connection \( \nabla \) whose curvature vanishes along vector fields tangent to the foliations. Furthermore, \( C^\infty(E/\mathcal{F}) \) denotes the set of smooth sections \( u \) of \( E \) satisfying \( \nabla_X u = 0 \) for every vector field \( X \) tangent to \( \mathcal{F} \). Roughly speaking, \( D \) is a \textit{basic partial differential operator} if locally with respect to a foliated atlas it writes as a partial differential operators in the transverse coordinates (see definition 3.3).

In this set-up, we consider the evolution equation 
\[
\partial_t u_t = D(u_t), \quad u_{t=0} = u_0
\]
where \( u_0 \in C^\infty(E/\mathcal{F}) \) is fixed and the solution \( u: M \times [0, \epsilon) \to M \) is required to be smooth and such that \( u_t \in C^\infty(E/\mathcal{F}) \) for every \( t \).

**Theorem 1.1.** Assume that \( D \) is strongly transversally elliptic at \( u_0 \). Then equation (1) has always a unique maximal solution defined for \( t \in [0, \epsilon) \). Moreover, when \( D \) is linear \( u \) is defined for \( t \in [0, \infty) \).

The proof of theorem 1.1 is mainly based on the treatment in [9] of basic differential operators. Indeed, from [17], [9] it follows that if \((E, \nabla)\) is an \( \mathcal{F} \)-bundle over a manifold \( M \) foliated by a transversally oriented Riemannian foliation \( \mathcal{F} \), then there exist a compact smooth manifold \( W \) and an SO\((n)\)-bundle \( \bar{E} \) over \( W \) such that \( C^\infty(E/\mathcal{F}) \) is canonically isomorphic to the space of SO\((n)\)-invariant sections of \( \bar{E} \). Moreover, from [9] it follows that if \( D: C^\infty(E/\mathcal{F}) \to C^\infty(E/\mathcal{F}) \) is a \textit{linear} basic strongly transversally elliptic operator, then it can be regarded as a \( G \)-invariant strongly elliptic differential operator on \( C^\infty(\bar{E}) \). From this result it follows that in the linear case equation (1) can be regarded as a genuine parabolic equation involving sections of a fiber bundle and the existence and uniqueness of a solution follows from the standard parabolic theory. The proof of the nonlinear case follows the same approach, but since the results in [9] are proved only for linear operators, we have to adapt El Kacimi’s theorem to the nonlinear case (see theorem 4.2).

In the second part of the paper we apply theorem 1.1 to some explicit flows on foliated manifolds. In section 5 we consider the \textit{transverse Ricci flow} introduced in [16] and we prove that it is well-posed by applying theorem 1.1. Indeed, as it happens in the non foliated case, the flow is not strongly parabolic and it has to be modified by using a basic vector field. The modified transverse Ricci flow is strongly transversally parabolic and it is well-posed in view of theorem 1.1. The existence of a solution to the transverse Ricci flow follows from the well-posedness of its modification, while for the uniqueness of the solutions we adapt an argument in [15] to the foliated case. In this second part we have to assume that the foliation is \textit{homologically orientable} in order to introduce an integral functional.

In section 6 we take into account Kähler foliations studying a foliated version of the Kähler-Ricci flow. Here the well-posedness of the flow is again implied by theorem 1.1 while the long time existence is obtained adapting some well-known results of the non-foliated case.

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## 2. Preliminaries on Riemannian foliations

Let \( M \) be an \((m+n)\)-dimensional smooth manifold. A codimension \( n \) foliation \( \mathcal{F} \) on \( M \) can be defined as an open cover \( \{U_i\} \) of \( M \) together a family \( f_i: U_i \to T \) of submersions onto
an $m$-dimensional manifold $T$, called the base of the foliation, such that whenever $U_i \cap U_j \neq \emptyset$ there exists a smooth map $\gamma_{ij} : f_i(U_i \cap U_j) \to f_j(U_i \cap U_j)$ satisfying $f_j \circ \gamma_{ij} = f_i$. The pair $(M, \mathcal{F})$ is usually called a foliated manifold. The previous construction is equivalent to assign an involutive distribution $L$ of rank $m$. A leaf of $\mathcal{F}$ is by definition a maximal integral submanifold of $L$. Given a foliated manifold $(M, \mathcal{F})$, we denote by $Q$ the normal bundle $TM/L$ and it is therefore defined the following exact sequence of vector bundles

$$0 \to L \to TM \to Q \to 0.$$  

A transverse structure on a foliated manifold is by definition a geometric structure on the base manifold $T$ which is invariant by the transition maps $\{\gamma_{ij}\}$. The most important class of transverse structures is the one of transverse Riemannian metrics introduced by Reinhart in [20] as a natural generalisation of Riemannian submersions. In contrast to the non-foliated case, the existence of a transverse Riemannian metric is not always guaranteed and imposes some strong conditions on the foliation. A foliation $\mathcal{F}$ is Riemannian if and only if $Q$ inherits a metric $g_Q$ along its fibres satisfying the holonomy invariant condition

$$(2) \quad \mathcal{L}_X g_Q = 0$$

for every $X \in C^\infty(L)$, where $\mathcal{L}$ is the Lie derivative (see e.g. [26, 17]). Condition (2) makes sense since if $X$ is a section of $L$, then its flow $\{\phi_t\}$ preserves the foliation. The metric $g_Q$ is simply defined by gluing together the pull-backs of the metric of the base $T$ via the local submersions and can be regarded as a degenerate symmetric 2-tensor on $M$. Such a $g_Q$ can be always “completed” to a bundle-like Riemannian metric $g$ on $M$, i.e. there always exists a metric $g$ on $M$ whose restriction to the orthogonal complement of $L$ is $g_Q$. A metric $g$ on a foliated manifold is usually called bundle-like if its restriction to $L$ satisfies (2). In this paper we refer to a metric $g_Q$ on $Q$ satisfying (2) as to a transverse metric.

A vector field $X$ on a foliated manifold $(M, \mathcal{F})$ is called foliated if $[X, Y] \in C^\infty(L)$ for every $Y \in C^\infty(L)$. Denoting by $C^\infty(M, \mathcal{F})$ the space of foliated vector fields on $(M, \mathcal{F})$, the quotient $C^\infty(M/\mathcal{F}) := C^\infty(M, \mathcal{F})/C^\infty(L)$ is by definition the set of basic vector fields on $(M, \mathcal{F})$. For every $X \in C^\infty(M, \mathcal{F})$, $X^b$ denotes the correspondent class in $C^\infty(M/\mathcal{F})$ and can be regarded as a section of $Q$. A foliation is called transversally parallelizable if there exist $n$ basic vector fields which are linear independent at any point.

Now we recall the definition of $\mathcal{F}$-fibration. Let $\pi : P \to M$ be a $G$-principal bundle over a foliated manifold $(M, \mathcal{F})$ and let $H \subseteq TP$ be the horizontal distribution defined by a connection on $P$ with connection 1-form $\omega$. Then for every $p \in P$, $H_p$ is isomorphic to $T_{\pi(p)}M$ and consequently $\mathcal{F}$ induces a distribution $\tilde{\mathcal{F}}$ in $P$. The connection given by $H$ is called a basic if $\tilde{\mathcal{F}}$ is a foliation and $\omega$ is basic, i.e. $\mathcal{L}_X \omega = 0$ for every vector field $X$ tangent to $\tilde{\mathcal{F}}$. In this case the pair $(P, H)$ is called an $\mathcal{F}$-fibration. A vector bundle $E$ with an assigned affine connection $\nabla$ is called an $\mathcal{F}$-connection if the induced principal bundle $(P, H)$ is an $\mathcal{F}$-fibration. This is equivalent to require that the curvature $R$ of $\nabla$ satisfies $i_X R = 0$ for every smooth section $X$ of $L$. Moreover, if $(E, \nabla)$ is an $\mathcal{F}$-bundle, then the foliation $\tilde{\mathcal{F}}$ on the principal bundle induces a foliation $\mathcal{F}_E$ on $E$. A map $T$ between two $\mathcal{F}$-bundle $(E, \nabla)$ and $(E', \nabla')$ on $M$ is called foliated if it takes leaves of $\mathcal{F}_E$ to leaves of $\mathcal{F}_{E'}$ and a smooth section $\alpha$ of an $\mathcal{F}$-bundle $(E, \nabla)$ is called basic if $\nabla_X \alpha = 0$ for every $X$ tangent to $\mathcal{F}$. We denote by $C^\infty(E/\mathcal{F})$ the set of smooth basic sections of $(E, \nabla)$.

The most natural example of $\mathcal{F}$-fibration is the $\text{SO}(n)$-bundle of transversally oriented frames of a Riemannian foliation defined as follows. Let $\mathcal{F}$ be a Riemannian foliation on a smooth manifold $M$ with transverse metric $g_Q$. Then it is defined the transverse Levi-Civita connection
\[ \nabla \text{on Q as} \]
\[
\nabla_X V = \begin{cases} 
[X, \sigma(V)]_Q & \text{if } X \in \Gamma(L) \\
(\nabla_X^Q \sigma(V))_Q & \text{if } X \in \sigma(Q), 
\end{cases}
\]

for every \( V \in C^\infty(Q) \), where \( g \) is a bundle-like metric on \( M \) inducing \( g_Q \) with Levi-Civita connection \( \nabla^g \), \( \sigma \) is the isomorphism between \( Q \) and \( L^\perp \) and \( X \mapsto X_Q \) is the projection onto \( Q \). Such a \( \nabla \) does not depend on the choice of \( g \) and it is the unique connection on \( Q \) satisfying
\[
X g_Q(V_1, V_2) = g_Q(\nabla_X V_1, V_2) + g_Q(V_1, \nabla_X V_2)
\]
\[(5) \quad \nabla_X Y_Q - \nabla_Y X_Q - [X, Y]_Q = 0, \]

for every vector fields \( X, Y \) on \( M \) and \( V_1 \) and \( V_2 \) in \( C^\infty(Q) \). Moreover, \( \mathcal{F} \) is called \textit{transversally orientable} if there exists a nowhere vanishing transverse volume form \( \nu \). When such a \( \nu \) is fixed, then the principal bundle of linear frames of \( Q \) has a natural \( \text{SO}(n) \)-reduction which we denote as in [17] by \( \rho: M^2 \rightarrow M \). The transverse Levi-Civita connection induces a connection on \( M^2 \) making it an \( \mathcal{F} \)-bundle. The following result is due to Molino and it will be important subsequently:

**Theorem 2.1** (Molino, [17]). The foliation \( \mathcal{F}^2 \) induced by \( \mathcal{F} \) on \( M^2 \) is always transversally parallelizable and invariant by the action of \( \text{SO}(n) \). Moreover, the leaf closures of \( \mathcal{F}^2 \) are the fibres of a locally trivial fibration \( F \rightarrow M^2 \rightarrow W \), where \( W \) is a compact manifold called the basic manifold of \( \mathcal{F} \).

In the last part of this section we recall the definition of the \textit{basic Laplace operator} and \textit{basic cohomology} groups. Let \((M, \mathcal{F})\) be a manifold foliated by a Riemannian foliation and let \( g_Q \) be its transverse metric. A \( p \)-form \( \alpha \) on \( M \) is called \textit{basic} if
\[
\iota_X \alpha = 0, \quad \mathcal{L}_X \alpha = 0
\]
for every smooth section \( X \) of the fibre bundle \( L \) generated by \( \mathcal{F} \), where \( \mathcal{L} \) denotes the Lie derivative. We denote by \( \Omega^p_B(M) \) the set of basic \( p \)-form on \( M \) and by \( C^\infty_B(M) \) the set of basic fuctions (i.e. basic 0-forms). Notice that accordingly to our previous notation we have \( C^\infty_B(M) = C^\infty(M \times \mathbb{R}/\mathcal{F}) \). Then the de Rham differential operator takes basic forms into basic forms and the pair \((\Omega_B, d)\) induces a cohomology \( H_B \) usually called the \textit{basic cohomology} of \((M, \mathcal{F})\). As is usual we will denote by \( d_B \) the restriction \( d|_{\Omega_B} \). When \( \mathcal{F} \) is transversally oriented the basic hodge “star” operator \( *_B \) and the basic codifferential operator \( \delta_B \) are defined in the usual way. Furthermore, it is defined the \textit{basic Laplacian operator} \( \Delta_B = d_B \delta_B + \delta_B d_B \) acting on basic forms of degree at least 1. On the other hand for conventional reasons we put a minus sign in the definition of the basic Laplacian acting on basic functions \( \Delta_B = -d_B \delta_B + \delta_B d_B : C^\infty_B(M) \rightarrow C^\infty_B(M) \). As in the classical Hodge theorem, in the compact case the basic cohomology groups are isomorphic to the kernels of \( \Delta_B \), but, in contrast to the nonfoliated case, they do not always satisfy Poincaré duality. Poincaré duality is guaranteed under some strong topological assumption on \( \mathcal{F} \), for instance when \( \mathcal{F} \) is homological orientable:

**Definition 2.2.** A \textit{transversally oriented Riemannian foliation} \( \mathcal{F} \) is called homologically orientable if there exists an \( m \)-form \( \chi \) on \( M \) restricting to a volume along the leaves of \( \mathcal{F} \) and such that
\[
\iota_X d\chi = 0
\]
for every \( X \in C^\infty(L) \).

It is well-known that when \( \mathcal{F} \) is homologically orientable, the form \( \chi \) can always be written as
\[
\chi(Y_1, \ldots, Y_m) = \det (g(Y_i, E_j)), \quad Y_1, \ldots, Y_m \in \Gamma(TM)
\]
where $g$ is a bundle-like metric on $M$ making the leaves of $\mathcal{F}$ minimal and $\{E_1, \ldots, E_m\}$ is an oriented orthonormal frame of $L$. Furthermore, the existence of $\chi$ allows us to introduce the following scalar product on basic forms

\[(\alpha, \beta) = \int_M \alpha \wedge \ast_B \beta \wedge \chi.\]

which makes $\Delta_B$ self-adjoint.

3. Basic differential operators on foliated manifolds

In order to introduce basic differential operators on foliated manifolds, we briefly recall the non-foliated case. Let $M$ be a manifold and let $\pi: E \to M$ be a vector bundle over $M$. We denote by $C^\infty(E)$ the vector space of smooth global sections of $E$. A quasilinear differential operator of order $r$ is a map $D: C^\infty(E) \to C^\infty(E)$ which can be locally written as

\[D(u) = \left[ a^{i_1..i_r}_{\alpha \beta} (x, u, \nabla u, \ldots, \nabla^{r-1} u) \partial_{x^{i_1} \ldots i_r} u^\alpha + b_\alpha(x, u, \nabla u, \ldots, \nabla^{r-1} u) \right] e_\alpha\]

where $\{x^i\}$ are local coordinates on $M$ and $\{e_\alpha\}$ is a local frame of $E$. In this definition and throughout all the paper we use the Einstein summation convention. When $D$ has even order $r$, it is called strongly elliptic at $u \in C^\infty(E)$ if there exists a constant $\lambda > 0$ such that the differential $L_u = D|_{u=0}$ of $D$ at $u$ satisfies

\[(\lambda^2)\nabla^r \sigma(L_u)(x, \xi, v, v) \geq \lambda |\xi|^2 |v|^2\]

for every $(x, \xi) \in T^*M$, $\xi \neq 0$, and $v \in E_x$, where $h$ is an arbitrary metric along the fibres of $E$. Here $\sigma(L_u)$ denotes the principal symbol of $L_u$, which, for every $(x, \xi) \in T^*M$, is the endomorphism of $E_x$ defined by

\[\sigma(L_u)(x, \xi) v = \frac{r}{r!} L_{u_0}(f^r u)(x)\]

for an $f \in C^\infty(M)$ such that $f(x) = 0$, $f|_{x^c} = \xi$, $u \in C^\infty(E)$, $u(x) = v$. More generally, if $\tau$ is a subbundle of $T^*M$, $D$ is called strongly $\tau$-elliptic if $\sigma(L_u)(x, \xi)$ satisfies (7) for every $(x, \xi) \in \tau$.

We recall the following classical result (see e.g. [1, Chapter 4])

**Theorem 3.1.** Let $D: C^\infty(E) \to C^\infty(E)$ be a second order quasilinear operator which is strongly elliptic at $0$, then the evolution equation

\[\partial_t u_t = D(u_t) \quad u_{t=0} = u_0\]

has a unique maximal solution $u \in C^\infty(M \times [0, \epsilon))$ for some $\epsilon > 0$. Moreover, when $D$ is linear, $u$ is in $C^\infty(M \times [0, \infty))$.

Consider now a Lie group $G$ together a representation of $G$ in $\text{Aut}(E)$. In this case we have also an induced $G$-action on $M$ and $E$ is usually called a $G$-bundle and $C^\infty(E)$ inherits the natural $G$-action $(g \cdot \alpha)(x) := g \cdot \alpha(g^{-1}x)$. We denote, adopting the notation of [9], by $C^\infty_G(E)$ the space of $G$-invariant sections of $E$. A section $u$ of $E$ belongs to $C^\infty_G(E)$ if and only if $L_X u = 0$ for every fundamental vector field $X$ of the action of $G$, where $L$ denotes the Lie derivative. Moreover, a partial differential operator $D: C^\infty(E) \to C^\infty(E)$ is called $G$-invariant if it commutes with $L_X$ for every fundamental vector field $X$. The following lemma will be useful

**Lemma 3.2.** Let $E \to M$ be a $G$-bundle over a compact manifold, $D: C^\infty(E) \to C^\infty(E)$ a quasilinear second order strongly elliptic differential operator and $\tilde{u}_0$ be a $G$-invariant section of $E$. Then the solution to the parabolic system

\[\partial_t u_t = D(u_t), \quad u_{t=0} = \tilde{u}_0\]
stays $G$-invariant for every $t$.

Proof. Let $X$ be a fundamental vector field for the action of $G$ on $M$. Then taking the Lie derivative of \([X]\) and taking into account that $D$ commutes with $L_X$ we get
\[
\partial_t(L_X u_t) = D(L_X u_t), \quad (L_X u_t|_{t=0}) = 0.
\]
Hence $L_X u_t$ is a solution to
\[
\partial_t v_t = D(v_t), \quad v_t|_{t=0} = 0.
\]
and the claim follows. \hfill \Box

Now we can focus on the foliated case. We adopt the following definition:

**Definition 3.3.** Let $(M,F)$ be a foliated manifold and $(E,\nabla)$ an $F$-fibration. A quasilinear basic differential operator of order $r$ is a map $D: C^\infty(E/F) \to C^\infty(E/F)$ such that with respect to local foliated coordinates $\{x^1, \ldots, x^n, y^1, \ldots, y^m\}$ takes the local expression
\[
D(u) = \left[ a^{1\ldots r}_{\alpha\beta} (y, u, \nabla u, \ldots, \nabla^{r-1} u) \partial_{y^1 \ldots y^r} + b_\alpha (y, u, \nabla u, \ldots, \nabla^{r-1} u) \right] e_\alpha
\]
where $\{e_\alpha\}$ is a local trivialisation of $E$.

When $D$ is linear, definition \([33]\) agrees to the one given in \([9]\). For a linear basic differential operator $D$ of order $r$ and $(x, \xi) \in T_x^* M$, the principal symbols $\sigma(D)(x, \xi)$ of $D$ at $(x, \xi)$ is defined by
\[
\sigma(D)(x, \xi)v = \frac{\partial^r}{r!} D(f^r u)(x)
\]
for $v \in E_x$ and $f \in C^\infty(M)$ basic and such that $f(x) = 0$, $f|_x = \xi$, $u \in C^\infty(E/F)$, $u(x) = v$. In analogy to the nonfoliated case, $D$ is called strongly transversally elliptic at $u \in C^\infty(E/F)$ if $D$ has even order $r$ and there exists a constant $\lambda > 0$ such that the differential $L_u = D|_{\sigma u}$ of $D$ at $u$ satisfies
\[
(-1)^{r/2} h(\sigma(L_u)(x, \xi)v, v) \geq \lambda |\xi|^2 |v|^2
\]
for every $(x, \xi) \in T^* M$, $\xi \neq 0$, and $v \in E_x$, where $h$ is some metric along the fibres of $E$.

**Example 3.4.** The foremost example of strongly transversally elliptic operator is the basic Laplacian operator $\Delta_B$ acting on basic functions described in the previous section.

In analogy to the non-foliated case, every linear basic differential operator can be described in terms of jets. We briefly recall this description and refer to \([9]\) for details. Let $r$ be a positive integer and let $J_r^r(E/F)$ be the vector bundle whose fiber at a point $x \in M$ is given by
\[
J_r^r(E/F) = \frac{C^\infty(E/F)}{Z_x(E/F)}
\]
$Z_x(E/F)$ being the ring of basic sections $u$ of $E$ satisfying $\nabla^k u = 0$ at $x$ for every $k \leq r$. Let
\[
S^k(Q, E) := S^k(Q^\ast) \otimes E,
\]
where $S^k$ denotes the $k$-symmetric power. Then we have the canonical isomorphism
\[
J^r_r(E/F) \simeq \bigoplus_{k=0} S^k(Q, E)
\]
(see \([9]\), corollary 2.3.7). In particular, $J^r_r(E/F)$ inherits a basic connection $\nabla^J$ since all the bundles involved are indeed $F$-bundles. For every basic section $u$ of $E$ we denote by $J_r(u)_x$ the corresponding class in $J_r(E/F)$. Then it is defined the natural map $J_r: C^\infty(E/F) \to C^\infty(J^r_r(E/F))$ as $J_r(f)(x) := J_r(f)_x$. Every linear basic partial differential operator $D: C^\infty(E/F) \to C^\infty(E/F)$ of order $r$ can be written as $D = T^r \circ J_r$, where $T^r: C^\infty(J^r(E)) \to C^\infty(E)$ is the map induced by the foliated morphism $T: J^r_r(E/F) \to E$. 

4. Proof of theorem 1.1

This section contains the proof of theorem 1.1 and it is subdivided in two parts. The first part is about the linear case and it is obtained as direct consequence of a theorem of El Kacimi proved in [9] (see theorem 4.1 below). For the nonlinear case, we generalise El Kacimi’s theorem to quasilinear operators and then we get the proof of theorem 1.1 as a consequence.

Let us consider a compact manifold \( M \) equipped with an \( n \)-codimensional transversally oriented Riemannian foliation \( \mathcal{F} \) and let \( (E, \nabla) \) be an \( \mathcal{F} \)-bundle over \( M \). Let \( G = \text{SO}(n) \) and \( \rho: M^\sharp \to M \) be the \( G \)-principal bundle of orthonormal oriented frames of \( (M, \mathcal{F}) \) and let \( \mathcal{F}^\sharp \) be the induced transversally parallelizable Riemannian foliation on \( M^\sharp \). Denote, accordingly to Molino’s theorem 2.1 by \( W \) the basic manifold of \( \mathcal{F} \) and by \( F \to M^\sharp \to W \) the locally trivial fibration induced by the the leaf closures of \( \mathcal{F}^\sharp \). Denote by \( E^\sharp \to M^\sharp \) the pull-back of \( E \) via \( \rho \). In view of [9] there always exist a \( G \)-bundle \( \bar{E} \to \bar{W} \) and canonic isomorphisms

\[
\psi^\sharp: C^\infty(E^\sharp/\mathcal{F}^\sharp) \to C^\infty(\bar{E}), \quad \psi: C^\infty(E/\mathcal{F}) \to C^\infty_G(\bar{E}).
\]

The following result is proved in [9]

**Theorem 4.1 (El Kacimi).** Let \( D: C^\infty(E/\mathcal{F}) \to C^\infty(E/\mathcal{F}) \) be a linear strongly transversally elliptic basic differential operator. Then there exists a \( G \)-invariant strongly elliptic differential operator \( \bar{D}: C^\infty(\bar{E}) \to C^\infty(\bar{E}) \) making the following diagram commutative

\[
\begin{array}{ccc}
C^\infty(E/\mathcal{F}) & \xrightarrow{D} & C^\infty(E/\mathcal{F}) \\
\downarrow \psi & & \downarrow \psi \\
C^\infty_G(\bar{E}) & \xrightarrow{\bar{D}} & C^\infty_G(\bar{E}).
\end{array}
\]

**Proof of theorem 4.1 in the linear case.** In the linear case, the statement of theorem 4.1 easily follows from theorem 4.1 and lemma 3.2. Indeed, theorem 4.1 implies that if \( D \) is linear, then there exists \( \bar{D}: C^\infty(\bar{E}) \to C^\infty(\bar{E}) \) as in the statement of theorem 4.1. Let \( \bar{u}_0 = \psi(u_0) \). Then theorem 3.1 implies that the parabolic problem

\[
\partial_t \bar{u}(t) = \bar{D}(\bar{u}_t), \quad \bar{u}_{t=0} = \bar{u}_0
\]

has a unique solution \( \bar{u} \in C^\infty(\bar{E} \times [0, \infty)) \). Since \( \bar{u}_0 \) and \( \bar{D} \) are \( G \)-invariant, then lemma 3.2 ensures that \( \bar{u}_t \) stays \( G \)-invariant for every \( t \). Hence we can write \( \bar{u}_t = \psi(u_t) \) for some smooth curve \( u \) in \( C^\infty(E/\mathcal{F}) \) solving (1). The uniqueness of \( u \) follows from the fact that \( \psi \) is an isomorphism and the uniqueness of standard parabolic problems.

The proof of theorem 4.1 in the nonlinear case works in the same way, but since El Kacimi’s theorem 1.1 is proved in [9] only for linear operators, we have to extend it to the quasilinear case.

**Theorem 4.2.** Let \( D: C^\infty(E/\mathcal{F}) \to C^\infty(E/\mathcal{F}) \) be a quasilinear basic differential operator which is strongly transversally elliptic at \( u \in C^\infty(E/\mathcal{F}) \). Then there exists a \( G \)-invariant differential operator \( \bar{D}: C^\infty(\bar{E}) \to C^\infty(\bar{E}) \) which is strongly elliptic at \( \psi(u) \) and makes the following diagram commutative

\[
\begin{array}{ccc}
C^\infty(E/\mathcal{F}) & \xrightarrow{D} & C^\infty(E/\mathcal{F}) \\
\downarrow \psi & & \downarrow \psi \\
C^\infty_G(\bar{E}) & \xrightarrow{\bar{D}} & C^\infty_G(\bar{E}).
\end{array}
\]
Proof. Assume that $D$ has even degree $r$. In terms of jets $D$ can be written as $D = T \circ J_r$, where we still denote by $T : C^\infty(J^r(E/F)) \to C^\infty(E)$ the map induced by the natural morphism
\[ T : J^r(E/F) \to E \]
(see the discussion at the end of section [3]). It is clear that $D$ is linear if and only if $T_x$ is linear for every $x \in M$. Let $g_Q$ be the transverse metric of $F$; then the total space $M^\sharp$ inherits a transversally oriented foliation $F^\sharp$ which is $SO(n)$-invariant. The connection $H^\sharp$ on $M^\sharp$ induced by the transverse Levi-Civita connection of $g_Q$ gives the splitting $TM^\sharp = H^\sharp \oplus V$, where $V$ is the vertical bundle. Since the induced bundle $L^\sharp$ is by definition a subbundle of $H^\sharp$, then $Q^\sharp = TM^\sharp/L^\sharp$ inherits the splitting
\[ Q^\sharp = H^b \oplus V^b, \]
where $V^b$ is isomorphic to $V$ and $H^b = H^\sharp/L^\sharp$. Let $\pi^\sharp : E^\sharp \to M^\sharp$ be the pull-back of $E$ via $\rho$ and $\rho^\sharp : E^\sharp \to E$ the map induced by $\rho$, i.e.
\[
\begin{array}{ccc}
E^\sharp & \xrightarrow{\pi^\sharp} & M^\sharp \\
\downarrow \rho^\sharp & & \downarrow \rho \\
E & \xrightarrow{\pi} & M
\end{array}
\]
We denote by $S^k(Q, E)^\sharp$ the pull-back of $S^k(Q, E)$ to $M^\sharp$. Then we easily get
\[ S^k(Q, E)^\sharp \simeq S^k(Q^\sharp/V^b, E^\sharp) \simeq S^k(H^b, E^\sharp). \]
Now for every $k$ we can split $S^k(Q^\sharp, E^\sharp)$ as
\[ S^k(Q^\sharp, E^\sharp) = \bigoplus_{i+j=k} S^{i,j}, \]
where
\[ S^{i,j} := S^i(H^b, E^\sharp) \otimes S^j(V^b, E^\sharp). \]
This fact allows us to lift the map $T$ to a map $T^\sharp : \bigoplus_{k=0}^\infty S^k(Q^\sharp, E^\sharp) \to E^\sharp$ making the following diagram commutative
\[
\begin{array}{ccc}
\bigoplus_{k=0}^\infty S^k(Q^\sharp, E^\sharp) & \xrightarrow{T^\sharp} & E^\sharp \\
\downarrow \rho^\sharp & & \downarrow \rho^\sharp \\
\bigoplus_{k=0}^\infty S^k(Q, E) & \xrightarrow{T} & E
\end{array}
\]
where $\rho^\sharp$ is induced by $\rho^\sharp$. The map $T^\sharp$ is defined as follows:
Let $x^\sharp \in M^\sharp$ and $x = \rho(x^\sharp)$; since
\[ \rho^\sharp : E^\sharp_{x^\sharp} \to E_x \]
is an isomorphism, if $\theta \in S^k_{x^\sharp}(H^b, E^\sharp)$, then we can define
\[ T_{x^\sharp}(\theta) = (\rho^\sharp)^{-1}_x (T_x(\rho^\sharp_{s,x}(\theta))) \]
In this way we have a map $T^\sharp$ making the following diagram commutative
\[
\begin{array}{ccc}
\bigoplus_{k=0}^\infty S^k(H^b, E^\sharp) & \xrightarrow{T^\sharp} & E^\sharp \\
\downarrow \rho^\sharp & & \downarrow \rho^\sharp \\
\bigoplus_{k=0}^\infty S^k(Q, E) & \xrightarrow{T} & E
\end{array}
\]
and we extend $T^\sharp$ as the null map in $S^{i,j}$ whenever $j \neq 0$. Keeping in mind the isomorphism
\[
\bigoplus_{k=0}^{r} S^k(Q^\sharp, E^\sharp) \simeq J^r(E^\sharp/F^\sharp)
\]
we can use the map $T^\sharp$ to define the partial differential operator $D^\sharp = T^\sharp \circ J^r$ acting on $C^\infty(E^\sharp/F^\sharp)$. Now $D^\sharp$ induces the genuine partial differential operator $\bar{D}: C^\infty(E) \to C^\infty(\bar{E})$ by defying $\bar{D} = \psi^\sharp D^\sharp(\psi^\sharp)^{-1}$. By construction $\bar{D}$ is $G$-invariant and makes diagram (11) commutative. Assume that $D$ is strongly transversally elliptic at $u$ and fix a complement $H$ of $L^\sharp$ in $H^\sharp$ in order to have the splitting
\[
TM^\sharp = L^\sharp \oplus H \oplus V.
\]
We firstly show that $D^\sharp$ is $H^\ast$-strongly transversally elliptic at $u$. By differentiating the following commutative diagram at $u^\sharp$

\[
\begin{array}{ccc}
C^\infty(J^r(E^\sharp/F^\sharp)) & \xrightarrow{T^\sharp} & C^\infty(E^\sharp) \\
\downarrow{\rho^\sharp} & & \downarrow{\rho^\sharp} \\
C^\infty(J^r(E/F)) & \xrightarrow{T} & C^\infty(E)
\end{array}
\]

we get

\[
\begin{array}{ccc}
C^\infty(J^r(E^\sharp/F^\sharp)) & \xrightarrow{T^\sharp_{\ast\mid J^r(u^\sharp)}} & C^\infty(E^\sharp) \\
\downarrow{\rho^\sharp} & & \downarrow{\rho^\sharp} \\
C^\infty(J^r(E/F)) & \xrightarrow{T_{\ast\mid J^r(u)}} & C^\infty(E)
\end{array}
\]

where $u = \rho^\sharp(u^\sharp)$. Since $D_{\ast\mid u}$ is strongly transversally elliptic and

\[
D_{\ast\mid u} = T_{\ast\mid J^r(\alpha)} \circ J^r, \quad D_{\ast\mid u^\sharp} = T_{\ast\mid J^r(u^\sharp)} \circ J^r,
\]

then [9, proposition 2.8.5] implies that $D_{\ast\mid \alpha^\sharp}$ is $H^\ast$-strongly transversally elliptic. Although the induced $D^\sharp$ cannot be transversally strongly elliptic, we can correct it with some extra terms according to the construction described in [9]. Let $\{X_1, \ldots, X_N\}$ be a basis of the Lie algebra of $SO(n)$ and let $Q_j$ be the $SO(n)$-invariant differential operator on $\Gamma(E^\sharp)$ defined by

\[
(Q_j \alpha)(x) := \frac{d}{dt} \alpha(\exp(tX_j)x)|_{t=0}.
\]

Then we set

\[
Q := (-1)^{r/2} \left( \sum_{i=1}^{N} Q_j \circ Q_j \right)^{r/2}.
\]

Clearly $Q$ is $V^\ast$-strongly elliptic and null in $C^\infty_G(E^\sharp/F^\sharp)$. Let $D' := D^\sharp + Q$ and $\bar{D}' : C^\infty(\bar{E}) \to C^\infty(\bar{E})$ be the induced operator. Since $\bar{D}'_{\ast\psi(u)}$ is the operator induced by $D^\sharp_{\ast\mid u^\sharp} + Q$, [9, Proposition 2.8.6] implies that $\bar{D}'_{\ast\psi(u)}$ is strongly elliptic and the claim follows. \hfill \Box
5. The transverse Ricci flow

In this section we give a proof of the well-posedness of the of the transverse Ricci flow based on theorem 1.1 the short-time existence is treated in the spirit of \[8\], while the uniqueness is obtained with the energy approach of \[15\].

We briefly recall the definition of the flow introduced in \[16\]. Let \(M\) be a compact \((m+n)\)-dimensional manifold equipped with an \(n\)-codimensionial Riemannian foliation with tangent bundle \(L\). We denote as usual by \(\pi: Q \to M\) the normal bundle \(TM/L\) and by \(g_Q\) the transverse metric. We also assume \(F\) homologically oriented by a form \(\chi\) (see definition \[2.2\]). Let \(g\) be a bundle-like metric on \((M,F)\) inducing \(g_Q\) and let \(\sigma: Q \to L^\perp\) be the map which assigns to each \([v]\) \(v\) component of \(v\) orthogonal to \(L\) with respect to \(g\). We denote by \(\nabla\) the transverse Levi-Civita connection \([3]\) induced by \(g_Q\) and by \(R_Q\) its curvature adopting the sign convention

\[
R_Q(X,Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}. 
\]

The transverse Ricci curvature of \(g_Q\) is then the basic tensor \(Rc_Q \in \Gamma(S^2Q^*)\) defined by

\[
Rc_Q(V,W) = g_Q(R_Q(\sigma(V),\sigma(e_i))e_i,W),
\]

where \(\{e_i\}_{i=1}^n\) is a \(g_Q\)-orthonormal frame of \(Q\). We further denote by \(s_Q\) the transverse scalar curvature of \(g_Q\) which is defined by

\[
s_Q := \sum_{k=1}^n Rc_Q(e_k,e_k).
\]

The flow introduced in \([16]\) is the flow \(g_Q(t)\) of transverse Riemannian metrics governed by the equation \(\partial_t g_Q(t) = -2Rc_Q(t)\), where \(Rc_Q(t)\) is the transverse Ricci curvature induced by the transverse metric \(g_Q(t)\). The main result of this section is the following

**Theorem 5.1.** Let \((M,F)\) be a compact manifold equipped with a homologically oriented Riemannian foliation and let \(\tilde{g}_Q\) be a smooth holonomy invariant metric on the quotient bundle \(Q\). Then the evolution equation

\[
\partial_t g_Q(t) = -2Rc_Q(t), \quad g_Q(0) = \tilde{g}_Q
\]

has a unique short-time solution.

As in the non-foliated setting the evolution equation \([14]\) cannot be parabolic because it is invariant by diffeomorphisms preserving \(F\). However it can be made parabolic by the de Turck-like trick we are going to describe.

We regard \(Rc_Q\) as an operator on the space of transverse Riemannian metrics on \((M,F)\), which is open in the space of basic sections of \(S^2Q^*\). Let \(\{x^1,\ldots,x^m,y^1,\ldots,y^n\}\) be a foliated coordinate system. A local frame of \(Q\) is obtained by taking \(V_i = \pi(\partial_{y^i})\) for \(i = 1,\ldots,n\). Hence locally \(\nabla\) is described by the functions \(\Gamma^k_{ij}\) defined by

\[
\nabla_{\partial_{y^i}} V_j = \Gamma^k_{ij} V_k.
\]

Note that this is equivalent to say \(\nabla_{\partial_{y^i}} \pi(\partial_{y^j}) = \Gamma^k_{ij} \pi(\partial_{y^k})\). Let \(g_{ij} := g_Q(V_i,V_j)\) and let \(g^{rs}\) be the components of the inverse matrix of \((g_{ij})\). Then

\[
Rc_Q(V_i,V_j) = g^{kl} g(R_Q(\partial_{y^i},\partial_{y^k})V_l,V_j) = g^{kl} \partial_{y^i}(\Gamma^r_{kl} g_{rj}) - g^{kl} \partial_{y^k}(\Gamma^r_{il} g_{rj}) + \text{l.o.t}.
\]
Once a background transverse metric \( \hat{g}_Q \) is fixed, every other transverse metric \( g_Q \) induces the basic vector field

\[
(16) \quad X = g^{ij}(\hat{\Gamma}^k_{ij} - \Gamma^k_{ij}) \partial_{y^k},
\]

where the functions \( \hat{\Gamma}^k_{ij} \) are defined by \( \hat{\nabla}_{\partial_{y^i}} V_j = \hat{\Gamma}^k_{ij} V_k \) and \( \hat{\nabla} \) is the transverse Levi-Civita connection of \( \hat{g}_Q \) on \( Q \). Then

\[
(\mathcal{L}_X g_Q)_{ij} = -g^{kr} \partial_{y^r}(\Gamma^l_{kr})g_{ij} - g^{kr} \partial_{y^r}(\Gamma^l_{kr})g_{li} + \text{1.o.t.}
\]

Now from (15), (5) and the definition of the frame \( V_i \), it easily follows

\[
\Gamma^k_{ij} = \frac{1}{2}(\partial_{y^i}(g_{jk}) + \partial_{y^j}(g_{ik}) - \partial_{y^k}(g_{ij})).
\]

Finally we have

\[
(-2Rc_Q - \mathcal{L}_X g_Q)(V_i, V_j) = g^{kl} \partial_{y^l}(g_{ij}) + 1.o.t. = \Delta_B g_{ij} + 1.o.t.,
\]

where \( \Delta_B \) is the basic Laplacian of \( g_Q \). This shows that the operator

\[
g_Q \mapsto -2Rc_Q - \mathcal{L}_X g_Q
\]

is strongly transversally elliptic on an open subset of \( C^\infty(S^2Q^*/F) \). Thus we have the following proposition which is now a consequence of theorem (1).

**Proposition 5.2.** Let \((M, F)\) be a compact manifold equipped with a transversally orientable Riemannian foliation with transverse metric \( \hat{g}_Q \). There exists a \( T > 0 \) and a smooth one-parameter family of transverse metrics \( g_Q(t) \in C^\infty(S^2Q^*/F), \ t \in [0, T), \) solving

\[
(17) \quad \partial_t g_Q(t) = -2Rc_Q(t) - \mathcal{L}_X g_Q(t), \quad g_Q(0) = \hat{g}_Q,
\]

where \( X \) is given by (16).

About the existence of a solution to (14), we reconstruct a solution of the transverse Ricci flow (14) from the modified flow (17). In order to do this, we firstly integrate the time-dependent vector field \( X_t \) to a 1-parameter group \( \phi_t \) of diffeomorphisms of \( M \) and observe that by definition (16) these diffeomorphisms preserve the foliation, i.e. \( (\phi_t)_*(L_x) = L_{\phi_t(x)} \) for any \( x \in M \) and any \( t \). Hence if \( g_Q(t) \) is a solution of (17), we can define \( \phi^*_t(g_Q) \) by means of

\[
\phi^*_t(g_Q)(V, W) = g_Q(\pi(\phi_t \tilde{V}), \pi(\phi_t \tilde{W})),
\]

where \( V, W \in Q \) and \( \pi(\tilde{V}) = V \) and \( \pi(\tilde{W}) = W \). It is immediate to verify that \( \phi^*_t(g_Q) \) is a solution of the original transverse Ricci flow (14).

In order to prove the uniqueness of the transverse Ricci flow, we adapt the argument of (15) to the compact foliated case. Assume that \( g_Q \) and \( \tilde{g}_Q \) are two solutions in the interval \( [0, T] \) of the transverse Ricci-flow with the same initial value \( \hat{g}_Q \). We denote by \( \nabla \) and \( \tilde{\nabla} \) the induced transverse Levi-Civita connections and by \( R^\nabla \) and \( R^{\tilde{\nabla}} \) the transverse curvature tensors. Let \( \{e_1, \ldots, e_n\} \) and \( \{\bar{e}_1, \ldots, \bar{e}_n\} \) be two local frames of \( Q \) orthonormal with respect to \( g_Q \) and \( \tilde{g}_Q \) respectively. Then we define the following smooth tensors on \( M \times [0, T] \)

\[
h = g_Q - \tilde{g}_Q, \quad A = \nabla - \tilde{\nabla}, \quad S = R^\nabla - R^{\tilde{\nabla}}.
\]
and their norms with respect to $g_Q$

$$|h|^2 = |g_Q - \bar{g}_Q|^2_{g_Q} = \sum_{i,j=1}^n (\delta_{ij} - \bar{g}_Q(e_i, e_j))^2$$

$$|A|^2 = |\nabla - \bar{\nabla}|^2_{g_Q} = \sum_{i,j,k=1}^n (g_Q(\nabla e_i, e_j) - \bar{g}_Q(\bar{\nabla} e_i, e_j))^2$$

$$|S|^2 = |R^\nabla - R^\bar{\nabla}|^2_{g_Q} = \sum_{i,j,k,l=1}^n \left( g_Q(R^\nabla (\bar{e}_i, \bar{e}_j)e_k, e_l) - \bar{g}_Q(R^\bar{\nabla} (\bar{e}_i, \bar{e}_j)e_k, e_l) \right)^2$$

and consider the function $\mathcal{E} : [0, T] \rightarrow \mathbb{R}^+$

$$\mathcal{E}(t) = \int_M (|h|^2 + |A|^2 + |S|^2) \chi \wedge d\mu ,$$

where $d\mu$ is the time dependent family of transverse volume forms induced by $g_Q(t)$ and the transverse orientation. To conclude that indeed $\mathcal{E}(t)$ vanishes identically on $[0, T]$, we only need to prove the following proposition and apply Gronwall’s lemma. This is analogous to proposition 7 in [15]

**Proposition 5.3.** There exists a constant $C_0$ depending on $n$ and an upper bound on $R^\nabla$ and $R^\bar{\nabla}$ and their first derivatives, such that $\mathcal{E}'(t) \leq C_0 \mathcal{E}(t)$, for all $t \in [0, T]$.

The necessary ingredients are contained in the following lemmas

**Lemma 5.4.** The following estimates hold

(18) $|\partial_t h| \leq C|S|$, 

(19) $|\partial_t A| \leq C(|\bar{g}_Q|^{-1}||\nabla R^\nabla||h| + |R^\nabla||A| + |\nabla S|)$.

Moreover, if $U = g^{ab}\nabla_b R^\nabla - \bar{g}^{ab}\bar{\nabla}_b R^\bar{\nabla}$, then

(20) $|\partial_t S - \Delta S - \text{div} U| \leq C \left( |\bar{g}_Q|^{-1}||\nabla R^\nabla||A| + |\bar{g}^{-1}||R^\bar{\nabla}|^2|h| + (|R_Q| + |R^\nabla|)|S| \right)$

and

(21) $|U| \leq C(|\bar{g}^{-1}||\bar{\nabla} R^\bar{\nabla}|h| + |A||R^\bar{\nabla}|)$.

where $\text{div} U|_{ijkl} = \nabla_a U^a_{ijkl}$ and in all the inequalities the constant $C$ depends only on the codimension of the foliation.

**Proof.** The estimates are proved in [15] for the Ricci flow in the non-foliated case. Since all the estimates in [15] are local and a solution of the transverse Ricci flow can be regarded as a collection of solutions to the Ricci flow on open sets in $\mathbb{R}^n$, the claim follows.

**Lemma 5.5.** The metrics $g_Q(t)$, $\bar{g}_Q(t)$, $\bar{g}_Q$ are all uniformly equivalent in $[0, T]$.

**Proof.** The statement follows from [12] Theorem 14.1.

**Corollary 5.6.** The following estimates hold

(22) $|\partial_t A| \leq C(|h| + |A| + |\nabla S|)$,

(23) $|\partial_t S - \Delta S - \text{div} U| \leq C (|A| + |h| + |S|)$,

(24) $|U| \leq C(|h| + |A|)$,

where the constants depend on $n$, $T$ and an upper bound of the curvatures and its first derivatives.
Proof. The inequalities are obtained by combining lemma 5.4 and lemma 5.5.

Proof of Proposition 5.3. Let us define
\[ H = \int_M |h|^2 \chi \wedge d\mu, \quad I = \int_M |A|^2 \chi \wedge d\mu, \quad J = \int_M |\nabla S|^2 \chi \wedge d\mu, \quad G = \int_M |S|^2 \chi \wedge d\mu \]
so that
\[ E = H + I + G. \]
Then
\[ \partial_t H \leq C H + \int_M \langle \partial_t h, h \rangle \chi \wedge d\mu \leq C H + \int_M |S||h| \chi \wedge d\mu \]
and using
\[ C|S||h| \leq C|h|^2 + C|S|^2 \]
we get
\[ \partial_t H \leq C H + C G. \]
Moreover,
\[ \partial_t I \leq C I + \int_M \langle \partial_t A, A \rangle \chi \wedge d\mu \]
\[ \leq C I + \int_M C \left( |\tilde{g}^{-1}||\tilde{R}||h| + |\tilde{R}||A| + |\nabla S| \right) |A| \chi \wedge d\mu \]
\[ \leq C I + \int_M C|h||A| + |\nabla S||A| \chi \wedge d\mu \]
and using
\[ C|h||A| + |\nabla S||A| \leq C|h|^2 + C|A|^2 + |\nabla S|^2 \]
we get
\[ \partial_t I \leq C H + J + C I. \]
Moreover,
\[ \partial_t G \leq C G + \int_M 2\langle \partial_t S, S \rangle \chi \wedge d\mu \]
\[ \leq C G + \int_M (2(|\Delta S + \text{div} U, S) + C|\tilde{g}^{-1}||\tilde{R}||A||S| + C|\tilde{g}^{-1}||\tilde{R}||h||S| + (|\tilde{R}| + |\tilde{R}||S|^2)|S| \chi \wedge d\mu \]
\[ \leq C G + \int_M (2(|\Delta S + \text{div} U, S) + C|A||S| + C|h||S| + C|S|^2) \chi \wedge d\mu \]
\[ \leq C G + C I + C H + \int_M 2(|\Delta S + \text{div} U, S) \chi \wedge d\mu. \]
Now
\[ \int_M 2(|\Delta S + \text{div} U, S) \chi \wedge d\mu \leq -2 J + 2 \int_M |\nabla S||U| \chi \wedge d\mu \leq - J + 3 \int_M |U|^2 \chi \wedge d\mu \]
and then we get
\[ \int_M 2(|\Delta S + \text{div} U, S) \chi \wedge d\mu \leq - J + C H + C I. \]
Therefore
\[ \partial_t G \leq C G + C I + C H - J, \]
which implies
\[ \partial_t E \leq C E. \]
and the statement follows. 

6. The transverse Kähler-Ricci flow

This section is about the generalization of the Kähler-Ricci flow to transverse geometry. The Kähler-Ricci flow is a powerful tool for studying Kähler manifolds which was introduced by Cao in [3]. In [23] Smoczyk, Wang and Zhang generalized the flow to Sasaki manifolds proving a “Sasaki version” of Cao’s theorem.

A Kähler foliation is by definition a foliation $\mathcal{F}$ provided with a transverse Kähler structure (see e.g. [2] and [11]). Such a structure can be regarded as a pair $(g_Q, J)$ of tensors on the normal bundle of the foliation $Q$, where $g_Q$ is a transverse metric making the foliation Riemannian and $J$ is an endomorphism of $Q$ satisfying $J^2 = -\text{Id}$. $g_Q(J\cdot, J\cdot) = g_Q(\cdot, \cdot)$ and an integrability condition. The pair $(g_Q, J)$ induces a closed basic 2-form $\omega$ on $M$ defined as the pull-back of $g_Q(J\cdot, \cdot)$.

We refer to $\omega$ as to the fundamental form of the transverse Kähler structure. The transverse complex structure $J$ induces a natural splitting of the space $\Omega^r_B(M, \mathbb{C})$ of complex basic $r$-forms on $M$ into $\Omega^r_B(M, \mathbb{C}) = \bigoplus_{p+q=r} \Omega^p_B \otimes \Omega^q_B$ and the restriction of $\partial_B$ to basic complex $(p, q)$-forms splits accordingly as $d_B = \partial_B + \bar{\partial}_B$. As in the non-foliated case $\partial_B^2 = \bar{\partial}_B^2 = 0$ and these operators define some cohomology groups (see e.g. [2] for details). From the local point of view it is useful to recall that we can always find coordinates $\{x^1, \ldots, x^m, z^1, \ldots, z^n\}$ taking values in $\mathbb{R}^m \times \mathbb{C}^n$, such that $\{x^1, \ldots, x^m\}$ are coordinates on the leaves and $\{V_k := \pi(\partial_{z^k})\}$ is a local $(1, 0)$-frame of $Q$. Such coordinates are usually called complex foliated.

From now on we assume $M$ compact and $\mathcal{F}$ homologically oriented by a form $\chi$ on $M$. The existence of $\chi$ allows us to generalize many results about Kähler manifolds to the non-foliated case. For instance, El Kacimi proved in [20] a foliated version of the $\partial \bar{\partial}$-lemma (called the $\partial_B \bar{\partial}_B$-lemma) and gave a generalization of the Calabi-Yau theorem. Indeed, accordingly to the non-foliated case, it is defined the transverse Ricci form of $(\omega, J)$ as a closed basic form $\rho_B$ on $M$ obtained as pull-back of $\text{Rc}_Q(J\cdot, \cdot)$ to $M$. Such a form locally writes as $\rho_B = -i\partial_B \bar{\partial}_B \log \det(g_{kr})$, where we locally write $g_Q = g_{kr} dz^r d\bar{z}^k$, and allows us to define the basic first Chern class as

$$c^1_B(M) := \frac{1}{2\pi} [\rho_B] \in H^2_B(M).$$

We recall the following

**Theorem 6.1** (El Kacimi [20]). For every $\beta$ representing $c^1_B(M)$ there exists a unique Kähler form in the same basic cohomology class as $\omega$ whose transverse Ricci form is $2\pi \beta$.

Here we want to study the transverse version of the Kähler-Ricci flow for Kähler foliations. Let $M$ be a compact manifold equipped with an initial Kähler foliation $(\mathcal{F}, \bar{g}_Q, J)$ and consider the transverse Ricci flow

$$\partial_t g_Q(t) = -\text{Rc}_Q(t), \quad g_Q(0) = \bar{g}_Q. \quad (27)$$

In this case we can prove the following two results

**Theorem 6.2.** There exists a unique smooth family of transverse Kähler metrics $g_Q(t)$, defined for $t \in [0, T)$, such that $g_Q(t)$ solves (27) where

$$T = \sup_{t > 0} \left\{ [\bar{\omega}]_B - 2\pi t c^1_B(M, J) > 0 \right\},$$

and $\bar{\omega}$ is the fundamental form of $\bar{g}_Q$. Moreover if $c^1_B(M, J) = 0$, then $g_Q(t)$ converges to a transversally Ricci-flat metric.
In the statement above, when we write that a class $\gamma \in H^{1,1}_B(M)$ is positive we mean that there exists a form $\kappa \in \gamma$ which is the fundamental form of a transverse Hermitian metric on $(M, F, J)$.

**Theorem 6.3.** If $c^B_t(M, J) = \nu [\tilde{\omega}]_B$ with $\nu < 0$, then there exists a unique smooth family of transverse Kähler metrics $g_Q(t)$ defined for $t \in [0, \infty)$, whose fundamental form $\omega_t$ solves

$$\partial_t \omega_t = -\rho_B(\omega_t) - \nu \omega_t, \quad \omega_{t=0} = \tilde{\omega},$$

and $g_Q(t)$ converges to a transversally Kähler-Einstein metric.

The short-time existence and the uniqueness for the solutions to the transverse Kähler-Ricci flow will be obtained by using theorem [1] while the long time behaviour will be studied working as in Kähler geometry. For the long time existence we follow the description in [24] omitting those computations which totally agree to the non-foliated case.

### 6.1. Some known results in open Kähler Manifolds

Since the transverse Kähler-Ricci flow looks locally as a collection of Kähler-Ricci flows on open sets of $\mathbb{C}^n$, we can use the local estimates for the Kähler-Ricci flow to study the transverse case. In this subsection we recall some results involving Kähler structures on non-compact Kähler manifolds. The first of them is the following easy-to-prove lemma of linear algebra

**Lemma 6.4.** Let $V$ be an $n$-dimensional complex vector space and let $\omega_1$ and $\omega_2$ be two positive $(1,1)$-forms and let $A$ and $B$ two positive constants.

i. If $\text{tr}_{\omega_2} \omega_1 + A \log \frac{\omega_1}{\omega_1} \leq B$, then there exists a constant $C > 0$ depending only on $A$, $B$ and $n$ such that $\text{tr}_{\omega_1} \omega_2 \leq C$;

ii. Assume $\omega_2 \leq A \omega_1$ and $\omega_1^n \leq B \omega_2^n$, then there exists a constant $C > 0$ depending only on $A$, $B$ and $n$ such that $\omega_1 \leq C \omega_2$.

Let us consider now a Kähler manifold $(X, \tilde{\omega})$ and let $\omega_t$, $t \in [0, T]$, be a solution to the normalised Kähler-Ricci flow

$$\partial_t \omega_t = -\text{Ric}(\omega_t) - \nu \omega_t, \quad \omega_{t=0} = \tilde{\omega}$$

where $\nu$ is a non-negative real constant. The next lemma can be for instance easily deduced from theorem 2.2 and corollary 2.3 in [24]. Here and throughout this subsection the symbol $\Delta_t$ will stand for the complex Laplacian of the form $\omega_t$, i.e. $\Delta_t f = g^{j\bar{i}}_t \partial_j \partial_{\bar{j}} f$, where $f \in C^\infty(M)$.

**Lemma 6.5.** Let $s_t$ be the scalar curvature of $\omega_t$, then $(\partial_t - \Delta_t) e^{\nu t} (s_t + \nu n) \geq 0$.

Moreover, assume that there exists a uniform constant $C$ such that $s_t \geq -\nu n - Ce^{-\nu t}$, then

- if $\nu = 0$, then $\omega^n_t \leq e^{Ct} \tilde{\omega}^n$;
- if $\nu = 1$, then $\omega^n_t \leq e^{C(1-e^{-t})} \tilde{\omega}^n$.

Now we recall the following results involving the Kähler-Ricci flow (for the proofs we still refer to [24])

**Theorem 6.6.** Assume that there exists a uniform constant $C$ such that $\frac{1}{\nu} \tilde{\omega} \leq \omega_t \leq C \tilde{\omega}$. Then any point $x \in X$ has a neighborhood $U$ where the $C^\infty$ norm of $\omega$ with respect to $\tilde{\omega}$ is uniformly bounded.

**Lemma 6.7.** Let $\kappa$ be a Kähler structure on $X$ having bisectional curvature bounded from below, then there exists a uniform constant $C$ such that

$$(\partial_t - \Delta_t) \log \text{tr}_\kappa \omega_t \leq C \text{tr}_\omega \kappa - \nu.$$
Let us consider now on \((X, \tilde{\omega})\) a solution \(\varphi\) to the parabolic Monge-Ampère equation
\[
\partial_t \varphi = \log \left( \frac{\tilde{\omega} + i \partial \bar{\partial} \varphi}{\tilde{\omega}} \right)^n - \varphi_t, \quad \tilde{\omega} + i \partial \bar{\partial} \varphi > 0, \quad \varphi(0) = 0
\]
defined in \(X \times [0, \infty)\).

**Lemma 6.8.** Assume \(\|\partial_t \varphi_t\|_{C^0} \leq Ce^{-t}\) for a uniform constant \(C\). Then
1. there exists a smooth map \(\varphi_\infty\) on \(X\) such that such that \(\|\varphi_t - \varphi_\infty\|_{C^0} \leq Ce^{-t}\);
2. \(\frac{1}{\varphi_t} \tilde{\omega}^n \leq (\tilde{\omega} + i \partial \bar{\partial} \varphi)^n \leq C' \tilde{\omega}^n\) for a uniform constant \(C'\).

6.2. A maximum principle in foliated manifolds. Here we prove a general maximum principle involving basic functions on compact manifolds foliated by Riemannian foliations. The result can be seen as an extension of \([27\text{ Proposition 5.1}]\) to the foliated non-Sasakian case.

By a smooth family of linear basic partial differential operators \(\{E\}_{t \in [0, \epsilon)}\) we mean a smooth family of linear basic differential operators \(E(\cdot, t): C^\infty_B(M) \to C^\infty_B(M)\) whose coefficients depend smoothly on \(t\).

**Proposition 6.9** (Maximum principle for basic maps). Let \((M, F, g_Q, J)\) be a compact manifold with a Kähler foliation. Let \(\{E\}_{t \in [0, \epsilon)}\) be a smooth family of linear basic partial differential operators such that \(E(\cdot, t)\) is transversally strongly elliptic for every \(t \in [0, \epsilon)\) and satisfies
\[
E(h(x, t), t) \leq 0
\]
whenever \(h \in C^\infty_B(M \times [0, \epsilon))\) is such that \(i \partial \bar{\partial} B h(x, t) \leq 0\). Then if \(h \in C^\infty_B(M \times [0, \epsilon), \mathbb{R})\) satisfies
\[
\partial_t h(x, t) - E(h(x, t), t) \leq 0,
\]
we have
\[
\sup_{(x, t) \in M \times [0, \epsilon]} h(x, t) \leq \sup_{x \in M} h(x, 0).
\]

**Proof.** Fix \(\epsilon_0 \in (0, \epsilon)\) and let \(h_\lambda: M \times [0, \epsilon_0] \to \mathbb{R}\) be \(h_\lambda(x, t) = h(x, t) - \lambda t\). Assume that \(h_\lambda\) achieves its global maximum at \((x_0, t_0)\) and assume by contradiction \(t_0 \neq 0\). Then
\[
\partial_t h_\lambda(x_0, t_0) \geq 0, \quad i \partial \bar{\partial} B h_\lambda(x_0, t_0) \leq 0.
\]
Therefore condition (30) implies \(E(h_\lambda(x_0, t_0), t_0) \leq 0\) and then
\[
\partial_t h_\lambda(x_0, t_0) - E(h_\lambda(x_0, t_0), t_0) \geq 0.
\]
Since \(\partial_t h_\lambda = \partial_t h - \lambda\) and \(E(h_\lambda(x, t), t) = E(h(x, t), t)\), we have
\[
0 \leq \partial_t h(x_0, t_0) - E(h(x_0, t_0), t_0) - \lambda \leq -\lambda,
\]
which is a contradiction. Therefore \(h_\lambda\) achieves its global maximum at a point \((x_0, 0)\) and
\[
\sup_{M \times [0, \epsilon_0]} h \leq \sup_{M \times [0, \epsilon_0]} h_\lambda + \lambda \epsilon_0 \leq \sup_{x \in M} h(x, 0) + \lambda \epsilon_0.
\]
Since the above inequality holds for every \(\epsilon_0 \in (0, \epsilon)\) and \(\lambda > 0\), the claim follows. \(\square\)

**Corollary 6.10.** Let \((M, F, g_Q(t), J)\) be a manifold with a family of Kähler foliations. Let \(h \in C^\infty_B(M \times [0, T])\) which is basic for very \(t\). Assume
\[
(\partial_t - \Delta_{B,t}) h_t \leq 0,
\]
where \(\Delta_{B,t}\) is the basic Laplacian operator of \(g_Q(t)\), then
\[
\sup_{M \times [0, T]} h \leq \max_{M} h_0.
\]
6.3. Proof of theorem 6.2. In this subsection we prove theorem 6.2.

Every transverse volume form \( \Omega \) on a manifold \( M \) foliated by a Kähler foliation can be written as

\[
\Omega = (i)^n f d\bar{z}^1 \wedge d\bar{z}^2 \wedge \cdots \wedge d\bar{z}^n,
\]
where the map \( f \) depends only on the transverse complex coordinates. Then when we write \( \log \Omega \), we mean \( \log f \).

Proof of theorem 6.2. First of all we show that (29) has a unique transversally Kähler maximal solution \( g_Q(t) \) defined in \( M \times [0, T_{\text{max}}) \), where \( T_{\text{max}} \leq T \).

Let \( T' < T \) and consider a transversally Kähler form \( \beta \) such that

\[
[\beta]_B = [\tilde{\omega}]_B - 2\pi T' c_B(M, J).
\]

Then we define

\[
\hat{\omega}_t = \frac{1}{T'}((T' - t)\tilde{\omega} + t\beta)
\]
for \( t \in [0, T') \) and we consider the scalar flow

\[
\partial_t \varphi_t = \log \left( \frac{\hat{\omega}_t + i\partial_B \bar{\partial}_B \varphi_t}{\Omega} \right), \quad \hat{\omega}_t + i\partial_B \bar{\partial}_B \varphi_t > 0, \quad \varphi|_{t=0} = 0
\]
where \( \varphi_t \) is smooth family of basic functions and \( \Omega \) is a transverse volume form satisfying

\[
i\partial_B \bar{\partial}_B \log \Omega = \frac{1}{T}(\beta - \tilde{\omega}).
\]

Since (31) is transversally parabolic, theorem 1 implies that it has a unique maximal short time solution \( \varphi \in C_B^\infty(M \times [0, T_{\text{max}})) \). Moreover the curve of metrics corresponding to the path of fundamental forms

\[
\omega_t = \hat{\omega}_t + i\partial_B \bar{\partial}_B \varphi_t
\]
solves (27) and the transverse \( \partial_B \bar{\partial}_B \)-lemma for Kähler foliations implies that every solution to (27) induces a solution to (31). This implies the existence of a maximal solution \( g_Q \) defined in \( M \times [0, T_{\text{max}}) \). Since \( d/dt(\varphi(t))_B = -(2\pi) c_B^1(M) \), we necessarily have \( T_{\text{max}} \leq T \).

Next we study the long time behavior of the maximal solution \( \omega_t \). Assume by contradiction \( T_{\text{max}} < T \) and for a fixed \( T' \) such that \( T_{\text{max}} < T' < T \) and define \( \hat{\omega}_t \) as above. Note that with our assumptions \( T_{\text{max}} \) is necessarily finite. Then we can write \( \omega_t = \hat{\omega}_t + i\partial_B \bar{\partial}_B \varphi_t \), where \( \varphi \) solves (31). In order to apply theorem 6.6 we have to show that there exists a uniform constant \( C \) such that \( \frac{1}{C} \hat{\omega}_t \leq \omega_t \leq C \hat{\omega}_t \). That is equivalent to require \( \frac{1}{C} \leq \text{tr}_\omega \omega_t \leq C \) and it can be proved by providing some a priori uniform estimates involving \( \varphi \).

- \( \| \varphi_t \|_{C^2} \) is uniformly bounded in \( [0, T_{\text{max}}) \). Keeping in mind that \( \varphi \) is a solution to (31), it is not difficult to show that \( \partial_t(\varphi_t - At) \) is negative for a constant \( A \) sufficiently large and the maximum principle implies that \( \partial_t(\varphi_t - At) \) achieves its maximum at \( t = 0 \). Therefore \( \varphi_t \leq T_{\text{max}} A \). A similar argument yields a lower bound for \( \varphi \).
- \( \| \partial_t \varphi_t \|_{C^0} \) is uniformly bounded in \( [0, T_{\text{max}}) \). This is equivalent to \( \frac{1}{C^1} \hat{\omega}_t \leq \omega_t \leq C \hat{\omega}_t \), for a uniform constant \( C^1 \). Keeping in mind that the basic scalar curvature \( s_B(t) \) and the basic Laplacian operator \( \Delta_B(t) \) of \( g_Q(t) \) are locally the scalar curvature and the Laplacian of the Kähler base manifold \( X \), then lemma 6.5 implies \( (\partial_t - \Delta_B(t))(e^t s_B(t)) \geq 0 \). Therefore corollary 6.10 implies \( s_B(t) \geq -\nu n - C_2 e^{-\nu t} \) for a uniform constant \( C_2 \) and the second part of lemma 6.5 together with the compactness of \( M \) implies \( \omega_t \leq C_1 \hat{\omega}_t \) for a constant \( C_1 \). For the lower bound we have

\[
((\partial_t - \Delta_B(t))(T' - t)\partial_t \varphi_t + \varphi_t + nt) = \text{tr}_\omega \beta \geq 0
\]
and the maximum principle implies

\((T' - t)\partial_t \varphi_t + \varphi_t + nt \geq T' \min_{M} \partial_t \varphi|_{t=0} = 0\).

Since \(\varphi\) is bounded, we get a lower bound for \(\partial_t \varphi\).

• \(\text{tr}_\omega \omega_t\) is uniformly bounded from above in \([0, T_{\max}]\). In view of lemma 6.4 we have

\[(\partial_t - \Delta_{B,t})(\log(\text{tr}_\omega \omega_t)) \leq C_3 \text{tr}_\omega \omega\]

for a uniform constant \(C_3\). Let \(A\) be a fixed constant such that \(A\omega_t - (C_3 + 1)\omega\) is a transversally Kähler form for every \(t \in [0, T_{\max}]\). Then

\[
\text{tr}_{\omega_t}(C_3 \omega - A\omega_t) \leq -\text{tr}_\omega \omega, \quad \text{in } M \times [0, T_{\max}].
\]

Hence

\[(\partial_t - \Delta_{B,t})(\log(\text{tr}_\omega \omega_t) - A\varphi_t) \leq C_3 \text{tr}_\omega \omega - A\partial_t \varphi_t + A \Delta_{B,t} \varphi_t = \text{tr}_{\omega_t}(C_3 \omega - A\omega_t) - A\partial_t \varphi_t + An\]

implies

\[(32) \quad (\partial_t - \Delta_{B,t})(\log(\text{tr}_\omega \omega_t) - A\varphi_t) \leq -\text{tr}_\omega \omega - A\partial_t \varphi_t + An.\]

Let \(\tau \in (0, T_{\max})\) be fixed and let \((x_0, t_0)\) be a point where \(\log(\text{tr}_\omega \omega) - A\varphi\) achieves the maximum in \(M \times [0, \tau]\). If \(t_0 = 0\), then

\[
\max_{M \times [0, \tau]} (\log(\text{tr}_\omega \omega) - A\varphi) \leq \log n.
\]

If \(t_0 > 0\), then \((32)\) implies

\[
\text{tr}_{\omega_{t_0}} \omega \leq An - A\partial_t \varphi|_{t=t_0} = An - A \log \frac{\omega^n_t}{\Omega} \quad \text{at } x_0,
\]

i.e.

\[
\text{tr}_{\omega_{t_0}} \omega + A \log \frac{\omega^n_t}{\Omega} \leq An \quad \text{at } x_0,
\]

and lemma 6.4 implies that \(\text{tr}_\omega \omega\) is uniformly bounded in \((x_0, t_0)\). Hence, since \(\|\varphi\|_{C^0}\) is uniformly bounded, we have

\[
\max_{M \times [0, \tau]} (\log(\text{tr}_\omega \omega) - A\varphi) \leq (\log \text{tr}_\omega \omega_{t_0})(x_0 + A\|\varphi_{t_0}\|_{C^0} \leq C
\]

where \(C\) does not depend on \(\tau\). Thus \(\log(\text{tr}_\omega \omega)\) is uniformly bounded from above in \([0, T_{\max}]\) and the claim follows.

The three facts proved above together with lemma 6.4 imply that \(\frac{1}{\tau} \leq \text{tr}_\omega \omega_t \leq C\) for a uniform constant \(C\) and theorem 6.10 together with the compactness of \(M\) implies that the \(C^\infty\) norm of \(\omega\) is uniformly bounded in \(M \times [0, T_{\max}]\). Therefore as \(t \to T_{\max}\) the solution \(g_Q(t)\) converges to a transversally Kähler metric \(g_Q(T_{\max})\) and the flow can be extended after \(T_{\max}\) contradicting the maximality of the solution.

In particular when \(c_B^1(M) = 0\), the maximal solution \(g_Q\) is defined in \(M \times [0, \infty)\). Now we focus on this last case. The fundamental form \(\omega_t\) of \(g_Q(t)\) can be written in this case as \(\omega = \tilde{\omega} + i\partial_B \bar{\partial}_B \psi_t\), where \(\psi_t\) solves

\[
\partial_t \psi = \log (\tilde{\omega} + i\partial_B \bar{\partial}_B \psi_t^n), \quad \tilde{\omega} + i\partial_B \bar{\partial}_B \psi_t > 0, \quad \psi|_{t=0} = 0
\]

• \(\|\partial_t \psi_t\|_{C^0}\) is uniformly bounded in \([0, \infty)\). The function \(\psi\) solves \((\partial_t - \Delta_{B,t}) \partial_t \psi_t = 0\) and the maximum principle for basic maps implies this claim.
• max $\psi_t - \min \psi_t$ is uniformly bounded in $[0, \infty)$. From the El Kacimi’s paper it follows that the solutions to the transverse Monge-Ampère equation
\[
(\omega + i\partial_B \bar{\partial}_B f)^n = e^F \omega^n, \quad \omega + i\partial_B \bar{\partial}_B f > 0
\]
satisfies the a priori estimate $\max f - \min f < C$ where $C$ depends only on $F$ and $\bar{\omega}$. Now for every fixed $t$, $\psi_t$ solves
\[
(\bar{\omega} + i\partial_B \bar{\partial}_B \psi_t)^n = \left(e^{\partial_t \psi_t}\right) \bar{\omega}^n
\]
and the previous bound on $\partial_t \psi_t$ implies this claim.

• $\operatorname{tr}_{\bar{\omega}} \omega_t$ is uniformly bounded from above in $[0, \infty)$. Lemma 6.7 together with the compactness of $M$ implies that $(\partial_t - \Delta_{B,t}) \log \operatorname{tr}_{\bar{\omega}} \omega_t \leq C \operatorname{tr}_{\bar{\omega}} \bar{\omega}$ for a uniform constant $C$. It follows
\[
(\partial_t - \Delta_{B,t})(\log \operatorname{tr}_{\bar{\omega}} \omega_t - (C + 1) \psi_t) \leq C \operatorname{tr}_{\bar{\omega}} \bar{\omega} - (C + 1) \partial_t \psi_t - (C + 1) \operatorname{tr}_{\bar{\omega}} \bar{\omega} + Cn + n
\]

Since $\partial_t \psi_t$ is uniformly bounded we have
\[
(33) \quad (\partial_t - \Delta_{B,t})(\log \operatorname{tr}_{\bar{\omega}} \omega_t - (C + 1) \psi_t) \leq -\operatorname{tr}_{\bar{\omega}} \bar{\omega} + C_2
\]
for a uniform constant $C_2$. Now let us fix $\tau > 0$ and let $(x_0, t_0)$ be a point in $M \times [0, \tau]$ where
\[
\log \operatorname{tr}_{\bar{\omega}} \omega_t - (C + 1) \psi_t
\]
achieves the maximum. If $t_0 > 0$, then inequality (33) implies $\operatorname{tr}_{\omega_t} \bar{\omega} \leq C_2$ at $x_0$ and from
\[
(\operatorname{tr}_{\bar{\omega}} \omega^n) \bar{\omega}^n \wedge \chi \leq \frac{1}{(n-1)!} (\operatorname{tr}_{\bar{\omega}} \omega^n)^{n-1} \omega^n \wedge \chi = \frac{1}{(n-1)!} (\operatorname{tr}_{\bar{\omega}} \omega^n)^{n-1} e^{\partial_t \psi} \bar{\omega}^n \wedge \chi
\]
and the bound on $\partial_t \psi$ it follows
\[
\operatorname{tr}_{\omega_{t_0}} \omega \leq C_3 \quad \text{at } x_0,
\]
where $C_3$ does not depend on $\tau$. Moreover, since $\log \operatorname{tr}_{\bar{\omega}} \omega_t - (C + 1) \psi_t$ achieves the maximum at $(x_0, t_0)$, then we have
\[
\log \operatorname{tr}_{\bar{\omega}} \omega \leq C_3 + (C + 1) \psi - (C + 1) \psi_{t_0}(x_0), \quad \text{in } M \times [0, \tau]
\]
and so
\[
\log \operatorname{tr}_{\bar{\omega}} \omega \leq C_3 + (C + 1) \psi - (C + 1) \min_{M \times [0, \tau]} \psi, \quad \text{in } M \times [0, \tau].
\]
Let $V = \int_M \bar{\omega}^n \wedge \chi$ and
\[
\tilde{\psi} = \psi - \frac{1}{V} \int_M \psi \bar{\omega}^n \wedge \chi.
\]
Then
\[
\log \operatorname{tr}_{\bar{\omega}} \omega \leq C_3 + (C + 1) \tilde{\psi} + \frac{C + 1}{V} \int_M \psi \bar{\omega}^n \wedge \chi - (C + 1) \inf_{M \times [0, \tau]} \tilde{\psi} - \frac{C + 1}{V} \inf_{[0, \tau]} \int_M \psi \bar{\omega}^n \wedge \chi
\]
in $M \times [0, \tau]$. Since $\tilde{\psi}$ is bounded in view of the previous point, we get
\[
\log \operatorname{tr}_{\bar{\omega}} \omega \leq C_4 + \frac{C + 1}{V} \int_M \psi \bar{\omega}^n \wedge \chi - \frac{C + 1}{V} \inf_{[0, \tau]} \int_M \psi \bar{\omega}^n \wedge \chi, \quad \text{in } M \times [0, \tau].
\]
Now
\[
\frac{d}{dt} \int_M \psi \bar{\omega}^n \wedge \chi = \int_M \log \left(\frac{\omega^n}{\bar{\omega}^n}\right) \bar{\omega}^n \wedge \chi \leq V \log \left(\frac{1}{V} \int_M \omega^n \wedge \chi\right) = 0
\]
shows that $\int_M \psi \bar{\omega}^n \wedge \chi$ is decreasing in $t$ and thus for every $(x, \tau) \in M \times [0, \infty)$ we have
\[
\log \operatorname{tr}_{\bar{\omega}} \omega(x, \tau) \leq C_4 + \frac{C + 1}{V} \int_M \psi \bar{\omega}^n \wedge \chi - \frac{C + 1}{V} \int_M \psi \bar{\omega}^n \wedge \chi = C_4.
\]
On the other hand if $t_0 = 0$, we have
\[
\log \text{tr}_\omega \omega_{t_0}(x_0) \leq \log n + (C + 1)\psi_0(x)
\]
and we can prove the item by working in the same way.

Now, since the $C^0$-norm of $\partial_t \varphi$ is uniformly bounded, taking into account that $M$ is compact, the previous item and lemma 6.4 imply $\frac{1}{\lambda} \bar{\omega} \leq \omega \leq C \omega$ in $[0, \infty)$ for a uniform constant $C$; therefore 6.6 implies that the $C^\infty$ norm of $\omega$ is uniformly bounded in $[0, \infty)$. It follows that $\psi_t$ converges to a basic smooth map $\psi_\infty$ on $M$ such that $\omega_\infty = \bar{\omega} + i \partial_B \bar{\partial}_B \psi_\infty > 0$.

It remains to prove that $\omega_\infty$ converges to a basic smooth map $\bar{\omega}$ and let $h \in C^\infty_B(M, \mathbb{R})$ be such that $\bar{\rho}_B = i \partial_B \bar{\partial}_B h$. A direct computation yields that if
\[
f(t) = \int_M \log \frac{\omega^n_t \wedge \chi}{\omega^n \wedge \chi} \omega_t^n \wedge \chi - \int_M h(\omega_t^n - \omega^n) \wedge \chi,
\]
then
\[
\dot{f}(t) = -(\partial_B \dot{\psi}_t, \bar{\partial}_B \dot{\psi}_t)_\omega, \quad \ddot{f}(t) \leq 0,
\]
where $(\cdot, \cdot)_\omega$ is the scalar product (5) computed with respect to $\omega$. Equations (34) implies that $\dot{f}(t) \to 0$ as $t \to \infty$. Since $\partial_t \psi_t = \log \frac{(\bar{\omega} + i \partial_B \bar{\partial}_B \psi_t)^n}{\bar{\omega}^n}$, we obtain that $\partial_B \log \frac{\omega^n_t}{\omega^n} = 0$ which implies $\bar{\rho}_B(\omega_\infty) = 0$. $\square$

6.4. Proof of theorem 6.3. We may assume $\nu = 1$ without loss of generality. About the short time existence it is enough to observe that if $g_Q$ solves (27), then the fundamental form of $\frac{1}{\sqrt{g_Q}} g_Q(e^t - 1)$ solves (28). Therefore (28) has a unique solution $\omega$ defined in $[0, \infty)$. By using the transverse $\partial_B \bar{\partial}_B$-lemma, we can write $\omega_t = \bar{\omega} + i \partial_B \bar{\partial}_B \varphi_t$, where $\varphi_t$ solves
\[
\partial_t \varphi_t = \log \frac{(\bar{\omega} + i \partial_B \bar{\partial}_B \varphi_t)^n}{\bar{\omega}^n} - \varphi_t, \quad \bar{\omega} + i \partial_B \bar{\partial}_B \varphi_t > 0, \quad \varphi_{t=0} = 0
\]
Now we have
\begin{itemize}
  \item $\|\partial_t \varphi_t\|_{C^0} \leq C e^{-t}$ for a uniform constant $C$. Since $\partial_t^2 \varphi_t = \Delta_B \partial_t(\partial_t \varphi_t) - \partial_t \varphi_t$, then $\partial_t(e^t \varphi_t) = \Delta_B e^t(\partial_t \varphi_t)$ and the transverse maximum principle implies the item.
  
  Using lemma 6.8 together with the compactness of $M$ we have that there exists a basic smooth map $\varphi_\infty$ such that $\|\varphi_t - \varphi_\infty\|_{C^0} \leq C' e^{-t}$ and $\frac{1}{C} \bar{\omega}^n \leq \omega_t \leq C \bar{\omega}^n$ for uniform constants $C'$ and $C$. Moreover we have
  
  \item $\text{tr}_\omega \omega_t$ is uniformly upper bounded. Lemma 6.7 and the compactness of $M$ imply that
    \[
    (\partial_t - \Delta_B, t) \log \text{tr}_\omega \omega_t - (C + 1) \varphi_t \leq -\text{tr}_\omega \bar{\omega} - 1 - (C + 1) \partial_t \varphi_t + (C + 1) n
    \]
    for a uniform constant $C$. Let $\tau > 0$ be fixed and let $(x_0, t_0)$ be a point in $M \times [0, \tau]$ where $\log \text{tr}_\omega \omega - (C + 1) \varphi$ achieves the maximum. If $t_0 = 0$, then
    \[
    \log \text{tr}_\omega \omega - (C + 1) \varphi \leq \log n \in M \times [0, \tau]
    \]
    and therefore
    \[
    \log \text{tr}_\omega \omega \leq \log n + (C + 1)\|\varphi\|_{C^0} \in M \times [0, \tau].
    \]
  On the other hand, if $t_0 > 0$, then
    \[
    \text{tr}_{\omega_{t_0}} \bar{\omega} \leq -1 - (C + 1) \partial_t \varphi_{t_0} + (C + 1) n \quad \text{at } x_0,
    \]
    and therefore $\text{tr}_{\omega_{t_0}} \bar{\omega}(x_0)$ is uniformly bounded in $(x_0, t_0)$. Since
    \[
    (\text{tr}_{\omega_{t_0}} \bar{\omega}) \bar{\omega}^n \wedge \chi \leq \frac{1}{(n - 1)!} (\text{tr}_{\omega_{t_0}} \omega)^{-1} \omega_{t_0}^n \wedge \chi,
    \]
tr \omega is uniformly bounded in \((x_0, t_0)\) and since \(\varphi\) is uniformly bounded we get the item.

The two items above imply that the maximal solution \(\omega_t = \tilde{\omega} + i \partial_B \bar{\partial}_B \varphi_t\) to (28) satisfies
\[
\frac{1}{C} \tilde{\omega} \leq \omega_t \leq C \omega
\]
for a uniform constant and theorem 6.4 ensures that the \(C^\infty\) norm of \(\omega_t\) and of \(\varphi_t\) are uniformly bounded. This implies that \(\omega_t\) converges to a transverse Kähler-Einstein structure, as required.

6.5. The case of Sasaki manifolds. In the case of Sasaki metrics, theorem 6.2 and theorem 6.3 provide an alternative proof of the main results of [23] on Sasaki-Ricci flow.

We recall that a Sasaki structure on a \((2n+1)\)-dimensional manifold is given by a 1-dimensional foliation generated by a vector field \(\xi\) together with the following triple of tensors: a bundle-like metric \(g\), a 1-form \(\eta\) such that \(\text{ker} \; \eta = \xi^\perp\) and an endomorphism \(\Phi\) of \(TM\) such that
\[
\Phi^2 = -\text{Id} + \eta \otimes \xi.
\]
We denote by \(D\) the kernel of \(\eta\) and by \(g^T\) the restriction of \(g\) to \(D\). Clearly the pair \((D, g^T)\) is identified with \((Q, g_Q)\). If \(g_Q(t)\) is a solution of the flow
\[
\partial_t g_Q(t) = -\text{Ric}_Q(t) - \nu g_Q(t), \quad g_Q(0) = \tilde{g}_Q
\]
with \(\nu = 0, -1\), we can reconstruct the Sasaki structure at any time by setting \(g(t) := g^T(t) + \xi \otimes \xi\) and taking \(\eta\) as the \(g(t)\)-dual of \(\xi\).

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Dipartimento di Ingegneria e Scienze dell’Informazione e Matematica, Università dell’Aquila, via Vetoio, 67100 L’Aquila, Italy
E-mail address, L. Bedulli: bedulli@math.unifi.it
Department of Mathematics, University of Oregon, Eugene, Oregon, 97403
E-mail address, W. He: whe@uoregon.edu
Dipartimento di Matematica, Università di Torino, Via Carlo Alberto 10, 10123 Torino, Italy
E-mail address, L. Vezzoni: luigi.vezzoni@unito.it