MODULE STRUCTURE OF THE CENTER OF THE UNIVERSAL CENTRAL EXTENSION OF A GENUS ZERO KRICHEVER-NOVIKOV ALGEBRA.

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Abstract. We describe how the center of the universal central extension of the genus zero Krichever-Novikov current algebra decomposes as a direct sum of irreducible modules for automorphism group of the coordinate ring of this algebra.

1. Introduction

Let \( a_1, \ldots, a_N \) be distinct complex numbers. In previous work of the author together with X. Guo, R. Lu, and K. Zhao, we described explicitly the automorphism groups of \( \text{Der}(R) \) for the rings \( R = \mathbb{C}[t, (t - a_1)^{-1}, \ldots, (t - a_N)^{-1}] \) where the possible groups that can appear are the cyclic groups \( C_n \), the dihedral groups \( D_n, S_4, A_4 \) and \( A_5 \) (see [CGLZ14]). This famous list of groups discovered by F. Klein appear naturally in the McKay correspondence and in the study of resolution of singularities.

In the present work we describe in a series of propositions the irreducible subrepresentations that appear in the decomposition of \( \Omega^1_R/dR \) under the action of \( \text{Aut}(R) \) for the various groups listed above. The irreducible representations that do appear all have multiplicity one, but not all irreducible representations make their appearance. The proofs use classical results of Schur and Frobenius. These techniques can be found in nearly any book on the representation theory of finite groups such as Serre’s book [Ser77] or Fulton and Harris [FH91]. We don’t know of any conceptual nor geometric reason why not all irreducible representations appear and when they appear why they occur with multiplicity one.

From the work of S. Block, C. Kassel and J.L. Loday (see [Blo81], [KL82], and [Kas84]) we know if \( \mathfrak{g} \) is a simple Lie algebra and \( R \) is a commutative algebra, both defined over the complex numbers, then the universal central extension \( \hat{\mathfrak{g}} \) of \( \mathfrak{g} \otimes R \) is the vector space \( \hat{\mathfrak{g}} := (\mathfrak{g} \otimes R) \oplus \Omega^1_R/dR \) where \( \Omega^1_R/dR \) is the space of Kähler differentials modulo exact forms (see [Kas84]). The vector space \( \hat{\mathfrak{g}} \) then becomes a Lie algebra by defining

\[
[x \otimes f, y \otimes g] := [xy] \otimes fg + (x, y)f \text{d}g, \quad [x \otimes f, \omega] = 0
\]

for \( x, y \in \mathfrak{g}, f, g \in R, \omega \in \Omega^1_R/dR \) and \((-\cdot,-)\) denotes the Killing form on \( \mathfrak{g} \). In the above \( \pi \) denotes the image of \( a \in \Omega^1_R \) in the quotient space \( \Omega^1_R/dR \). When \( R \) is the ring of meromorphic functions on Riemann surface with a fixed finite number of poles, the algebra \( \hat{\mathfrak{g}} \) is called a current Krichever-Novikov algebra and has been extensively studied (see for example the books [Sch14] and [She12] and their references). There are a number of natural questions that arise in studying such Lie algebras and one of them arises as follows.

It is known that \( l \)-adic cohomology groups tend to be acted on by Galois groups, and the way in which these cohomology groups decompose can give interesting and important number theoretic

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information (see for example R. Taylor’s review of Tate’s conjecture \cite{Tay04}). Moreover it is an interesting and very difficult problem to describe the group $\text{Aut}(R)$ where $R$ is the space of meromorphic functions on a compact Riemann surface $X$ and to determine the module structure of its induced action on the module of holomorphic differentials $\mathcal{H}^1(X)$ (see \cite{Bre04}). Now if one realizes the fact that $\Omega^1_R/dR$ can be identified with the $H_2(\mathfrak{sl}(R), \mathbb{C})$ (see \cite{Blo81}), it is natural to ask how $\Omega^1_R/dR$ decomposes into a direct sum of irreducible modules under the action of the $\text{Aut}(R)$. We answer this question when $R$ is the $N$-point algebra $R = \mathbb{C}[t, (t-a_1)^{-1}, \ldots, (t-a_N)^{-1}]$ with $a_1, \ldots, a_N$ distinct complex numbers giving rise to the appropriate Kleinian groups $\text{Aut}(R)$.

Let us give a bit of background where these algebras appear in conformal field theory. The ring of functions on the Riemann sphere regular everywhere except at a finite number of points appears naturally in Kazhdan and Luszig’s explicit study of the tensor structure of modules for affine Lie algebras (see \cite{KL93} and \cite{KL94}). M. Bremner gave the name of an $N$-point algebra (see \cite{Bre94}). In the monograph \cite[Ch. 12]{FBZ01} algebras of the form $\oplus_{i=1}^N \mathfrak{g}((t-x_i)) \oplus \mathbb{C}c$ appear in the description of the conformal blocks. These contain the $n$-point algebras $\mathfrak{g} \otimes \mathbb{C}[t, (t-a_1)^{-1}, \ldots, (t-a_N)^{-1}] \oplus \mathbb{C}c$ modulo part of the center $\Omega_R/dR$. M. Bremner explicitly described the universal central extension of such an algebra in \cite{Bre94} where the center has basis $(t-a_1)^{-1}dt, \ldots, (t-a_N)^{-1}dt$. The current algebra $\mathfrak{g} \otimes R \oplus \Omega^1_R/dR$ and the derivation algebra $\text{Der}(R)$ are examples of Krichever-Novikov algebras for the genus zero Riemann sphere minus a finite number of points $R = \mathbb{C}[t, (t-a_1)^{-1}, \ldots, (t-a_N)^{-1}]$. In Krichever and Novikov’s original work they only dealt with a particular one dimensional central extension (see for example \cite{KN87}, \cite{KN88}, \cite{KN89}, \cite{Sch14} and \cite{She12}). Also recently M. Schlichenmaier has written a paper giving a different point of view of the universal central extension of the $N$-point Virasoro algebra (see \cite{Sch15}).

We should also mention work of S. Lombardo, A. V. Mikhailov, and J. A. Sanders dealing with the structure of genus zero multi-point Krichever-Novikov current al-gebras (and their modules) fixed under the same type of symmetries as in our work with X. Guo, R. Lu, and K. Zhao (see \cite{LM05}, \cite{LM05b}, \cite{LS10}, and \cite{KLS14}). They do not describe how the center of the universal central extension decomposes into a direct sum of irreducible modules for the automorphism group of the coordinate ring.

2. The Universal Central Extension of the Current Algebra $\mathfrak{g} \otimes R$.

Let $R$ be a commutative algebra defined over $\mathbb{C}$. Consider the left $R$-module $F = R \otimes R$ with left action given by $f(g \otimes h) = fg \otimes h$ for $f, g, h \in R$ and let $K$ be the submodule generated by the elements $1 \otimes fg - f \otimes g - g \otimes f$. Then $\Omega^1_R = F/K$ is the module of Kähler differentials. The element $f \otimes g + K$ is traditionally denoted by $fdg$. The canonical map $d : R \to \Omega^1_R$ is given by $df' = 1 \otimes f + K$. The exact differentials are the elements of the subspace $dR$. The coset of $fdg$ modulo $dR$ is denoted by $\frac{fdg}{dR}$. As K. Cassil showed the universal central extension of the current algebra $\mathfrak{g} \otimes R$ where $\mathfrak{g}$ is a simple finite dimensional Lie algebra defined over $\mathbb{C}$, is the vector space $\hat{\mathfrak{g}} = (\mathfrak{g} \otimes R) \oplus \Omega^1_R/dR$ with Lie bracket given by

$$[x \otimes f, Y \otimes g] = [xy] \otimes fg + (x, y)\frac{fdg}{dR}, [x \otimes f, \omega] = 0, [\omega, \omega'] = 0,$$

where $x, y \in \mathfrak{g}$, and $\omega, \omega' \in \Omega^1_R/dR$ and $(x, y)$ denotes the Killing form on $\mathfrak{g}$.

Proposition 2.1 (\cite{Bre94}). Let $a = (a_1, \ldots, a_N)$ and $R_a = \mathbb{C}[t, (t-a_1)^{-1}, \ldots, (t-a_N)^{-1}]$ be as above. The set

$$\{\omega_1 := (t-a_1)^{-1}dt, \ldots, \omega_N := (t-a_N)^{-1}dt\}$$

is a basis of $\Omega^1_{R_a}/dR_a$. 

3. Recollections

In [CGLZ14] one of the main results we proved together with Rencai Lu, Xiangqian Guo, and Kaiming Zhao was the following result.

**Theorem 3.1.** Suppose \(\{a_1, a_2, \ldots, a_n\}\) and \(\{a_1', a_2', \ldots, a'_n\}\) are two sets of distinct complex numbers and \(\phi : R_a \to R_{a'}\) is an isomorphism of commutative algebras. Then \(m = n\), \(\phi\) is a fractional linear transformation, and one of the following two cases holds

(a). There exists some constant \(c \in \mathbb{C}\) such that \(a_i - a_1 = c(a_i' - a_1')\) and \(\phi((t - a_1)^k) = c^k(t - a_1')^k\) for all \(k \in \mathbb{Z}\) and \(i = 1, 2, \ldots, n\) after reordering \(a_i'\) if necessary.

(b). There exists some constant \(c \in \mathbb{C}\) such that \((a_i - a_1)(a_i' - a_1') = c\) for all \(i \neq 1\), and

\[
\phi((t - a_1)^k) = \frac{c^k}{(t - a_1')^k},
\]

\[
\phi((t - a_i)^k) = \frac{(a_i - a_1)(t - a_1')^k}{(t - a_i')^k}, \quad \forall \, i > 1,
\]

for all \(k \in \mathbb{Z}\) after reordering \(a_i\) and \(a_i'\) if necessary.

Moreover, the maps given in (a) and (b) indeed define algebra isomorphisms between \(R_a\) and \(R_{a'}\) under the corresponding conditions.

We call isomorphisms defined in (a) and (b) isomorphisms of the first kind and isomorphisms of the second kind respectively.

We also note the following theorems.

**Theorem 3.2** ([Shu97] Theorem 2.3.1). Any finite subgroup of the automorphism group \(\text{Aut}(\hat{\mathbb{C}})\) of the Riemann sphere \(\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}\) is conjugate to a rotation group.

**Theorem 3.3** ([Shu97] Theorem 2.6.1). If \(G\) is a finite rotation group of the group of automorphisms of the Riemann sphere, then it is isomorphic to one of the following:

(a). A cyclic group \(C_n = \{s|s^n = 1\}\) for \(n \geq 1\).

(b). A dihedral group \(D_n = \langle s, t | s^n = 1 = t^2, \ tst = s^{-1}\rangle\).

(c). A platonic rotation group \(A_4, S_4\) or \(A_5\).

These are the only groups that appear as automorphisms groups of \(R_a\) and all of them appear (see the appendix of [CGLZ14] and the lemmas below).

4. Decompositions of the Center under Action of \(\text{Aut}(R_a)\)

Let \(R\) be an associative commutative ring defined over a field \(\mathbb{F}\). We first observe that \(\text{Aut}(R)\) acts on \(\Omega_R/dR\) via the tensor product action on \(R \otimes_\mathbb{F} R\) since this action preserves \(K\) and \(dR\).

4.1. **Decomposition of \(\Omega_{R_a}/dR_a\) under \(C_n\).** We give in this section examples of various \(a\) that give the automorphism groups of Klein listed in Theorem 3.3 as a series of lemmas followed by propositions that describe how the centers decompose under the action of the respective automorphism groups.

**Lemma 4.1** ([CGLZ14], Appendix). Let \(n \geq 4\) and \(a = (a_1, a_2, \ldots, a_n)\) where \(a_k = \zeta^k\) for the primitive \(n\)-th root of unity \(\zeta = \exp(2\pi i/n)\). Then \(\text{Aut}(\mathcal{V}_a) \cong C_n\), the cyclic group of order \(n\).

In the proof of this Lemma it was noted that all automorphisms of \(R_a\) where of the first kind

\[
\phi_k(t) = \zeta^k t, \quad k = 0, 1, \ldots, n-1
\]

with \(\phi_1 = \phi_1^k\).

Fix \(n \geq 2\). Let \(\zeta_i = \exp(2\pi i/n), \omega_i = (t - a_i)^{-1}dt\) and \(R_a = \mathbb{C}[t, (t - a_1)^{-1}, \ldots, (t - a_n)^{-1}]\).
Proposition 4.2. Let \( a = (\zeta, \zeta^2, \ldots, \zeta^{n-1}, 1) \). The center \( \Omega_{R_a}/dR_a \) is the direct sum of each irreducible representation of \( C_n \) with multiplicity one. More precisely

\[
\Omega_{R_a}/dR_a = \oplus_{k=0}^{n-1} \mathbb{C} u_k
\]

where

\[
u_k = \sum_{i=1}^{n} \zeta^{ki} \omega_i
\]

are eigenvectors for \( C_n \) with eigenvalue \( \zeta^k \) for the automorphism \( \phi(z) = \zeta z, 0 \leq k \leq n - 1 \). In particular the subspaces \( \mathbb{C} u_n \) are distinct irreducible one dimensional \( C_n \) submodules of \( \Omega_{R_a}/dR_a \).

Proof. The only automorphisms of \( R_a \) are of the first kind and are powers of \( \phi \) where

\[
\phi(z) = \zeta z.
\]

Hence

\[
\phi(\omega_i) = \begin{cases} 
\omega_{i-1}, & \text{for } 2 \leq i \leq n \\
\omega_n, & \text{for } i = 1.
\end{cases}
\]

As a consequence

\[
\phi(u_k) = \sum_{i=1}^{n} \zeta^{ki} \phi(\omega_i) = \zeta^k \phi(\omega_1) + \sum_{i=2}^{n} \zeta^{ki} \phi(\omega_i)
\]

\[
= \zeta^k \omega_n + \sum_{i=2}^{n} \zeta^{ki} \omega_{i-1} = \zeta^k \omega_n + \sum_{i=1}^{n-1} \zeta^{k(i+1)} \omega_i
\]

\[
= \zeta^k \left( \omega_n + \frac{n-1}{i=1} \zeta^{ki} \omega_i \right) = \zeta^k \left( \sum_{i=1}^{n} \zeta^{ki} \omega_i \right)
\]

\[
= \zeta^k u_k.
\]

(See [Ser77] Section 5.1 for more information on the representation theory of \( C_n \).) \( \square \)

4.2. Decomposition of \( \Omega_{R_a}/dR_a \) under \( S_4 \) and \( D_n \).

Lemma 4.3 ([CGLZ14], Appendix). Let \( n \geq 3 \), and \( a = (0, a_1, a_2, \ldots, a_n) \) where \( a_k = \zeta^k \) for a primitive \( n \)-th root of unity \( \zeta \).

(a). If \( n \neq 4 \) then \( \text{Aut}(V_a) \cong D_n \), the dihedral group of order \( 2n \).

(b). If \( n = 4 \) then \( \text{Aut}(V_a) \cong S_4 \).

In the proof of this Lemma it was noted that the only automorphisms of \( R_a \) are products of the automorphisms \( \phi \) and \( \psi \) where

\[
\phi(t) = \zeta t, \quad \text{and} \quad \psi(t) = \zeta/t.
\]

Fix \( n \geq 2 \). Let \( \zeta = \exp(2\pi i/n), a_i = \zeta^i, \omega_0 = \frac{t-1}{dt}, \omega_i = \frac{(t-a_i)^{-1}}{dt}, \) and \( R_a = \mathbb{C}[t, (t-a_1)^{-1}, \ldots, (t-a_n)^{-1}] \).

4.2.1. The decomposition for \( D_n \).

Proposition 4.4. Let \( a = (0, \zeta, \zeta^2, \ldots, \zeta^{n-1}, 1) \). The center \( \Omega_{R_a}/dR_a \) is the direct sum of all of the irreducible two dimensional representation of \( D_n \) with multiplicity one together with at most three other distinct irreducible representations. More precisely
i). If \( n = 2m \neq 4 \) is even, then
\[
\Omega_{R_n}/dR_n = C\omega_0 \oplus (\oplus_{k=1}^{m-1} U_k) \oplus, Cu_m \oplus Cu_n
\]
where
\[
U_k = Cu_k \oplus Cu_{n-k}, \ 1 \leq k \leq m-1, \ u_k = \sum_{i=1}^{n} \zeta^{ki} \omega_i, \ 1 \leq k \leq n.
\]

The subspaces \( C\omega_0, Cu_m, Cu_n \) and the \( U_k \) are irreducible subrepresentations for \( D_n = (\phi, \psi) \) for the automorphism \( \phi(t) = \zeta t \) and \( \psi(t) = \zeta/t \). The irreducible modules \( U_k, 1 \leq k \leq m-1 \) exhaust all of the irreducible two dimensional modules for \( D_n \).

ii). If \( n = 2m+1 \) is odd, then
\[
\Omega_{R_n}/dR_n = C\omega_0 \oplus (\oplus_{k=1}^{m} U_k) \oplus, Cu_n
\]
where
\[
U_k = Cu_k \oplus Cu_{n-k}, \ 1 \leq k \leq m-1 \ u_k = \sum_{i=1}^{n} \zeta^{ki} \omega_i, \ 1 \leq k \leq n.
\]

The subspace \( C\omega_0, Cu_n \) and the \( U_k \) are irreducible subrepresentations for \( D_n \). The irreducible modules that appear in the above decomposition exhaust all of the irreducible modules for \( D_n \) with multiplicity one.

**Proof.** Recall only automorphisms of \( R_n \) are products of the automorphims \( \phi \) and \( \psi \) where
\[
\phi(t) = \zeta t, \ \text{and} \ \psi(t) = \zeta/t.
\]

Now we calculate
\[
\phi(\omega_0) = \phi(t)^{-1} d\phi(t) = \zeta^{-1} t^{-1} \zeta dt = \omega_0
\]
\[
\psi(\omega_0) = \psi(t)^{-1} d\psi(t) = \zeta^{-1} t \zeta dt^{-1} = -t^{-1} dt = -\omega_0.
\]

As a consequence the subspace \( C\omega_0 \) is a one dimensional irreducible representation of \( D_n \) occurring as a direct summand of \( \Omega_{R_n}/dR_n \).

As above \( \phi(u_i) = \zeta^i u_i \). On the other hand
\[
\psi(\omega_i) = \psi(t - \zeta^i)^{-1} d\psi(z) = ((\zeta/t) - \zeta^i)^{-1} \zeta dt^{-1}
\]
\[
= -\zeta^{-1-i} (\zeta^{-1-i} - t) t^{-1} dt
\]
\[
= \zeta^{-i} \left(-\zeta^{-i-1} t dt + \zeta^{-i} (t - \zeta^{-i-1}) dt\right)
\]
\[
= t^{-1} dt + \omega_{n+1-i}
\]

and so
\[
\psi(u_k) = \sum_{i=1}^{n} \zeta^{ki} \psi(\omega_i) = -\left(\sum_{i=1}^{n} \zeta^{ki}\right) \omega_0 + \sum_{i=1}^{n} \zeta^{ki} \omega_{n+1-i}
= \sum_{m=1}^{n} \zeta^{k(n-m+1)} \omega_m = \zeta^k \sum_{m=1}^{n} \zeta^{(n-k)m} \omega_m
= \zeta^k u_{n-k}.
\]

For \( k = n \) we have \( \psi(u_n) = u_0 = u_n \). Similarly \( \phi(u_n) = u_n \) and as a consequence \( Cu_n \) is the trivial irreducible representation of \( D_n \).

Suppose \( n \) is even with \( n = 2m, m \in \mathbb{Z} \). Then \( \psi(u_m) = \zeta^m u_m = -u_m \). Similarly \( \phi(u_m) = \zeta^m u_m = -u_m \). This means that \( Cu_m \) is the one dimensional alternating representation.
Let \( n = 2m \) be still even. Now suppose \( 1 \leq k \leq m - 1 \). If we set \( w_k = u_k \) for \( 1 \leq k \leq m - 1 \) and \( w_{n-k} = \zeta^k u_{n-k} \) for \( m \leq k \leq n - 1 \), then the matrices for \( \phi \) and \( \psi \) for the basis \( \{w_k, w_{n-k}\} \), \( 1 \leq k \leq m - 1 \) are

\[
\phi = \begin{pmatrix}
e^{2\pi ik/n} & 0 \\
0 & e^{-2\pi ik/n}
\end{pmatrix}, \quad \psi = \begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix}.
\]

Suppose now \( n = 2m + 1 \) is odd. If we set \( w_k = u_k \) for \( 1 \leq k \leq m \) and \( w_{n-k} = \zeta^k u_{n-k} \) for \( m + 1 \leq k \leq n - 1 \), then the matrices for \( \phi \) and \( \psi \) for the basis \( \{w_k, w_{n-k}\} \) are as in (4.3).

(See [Ser77] Section 5.3 for more information on the representation theory of \( D_n \).) \( \square \)

For \( D_n \) we are left with considering the case of \( D_4 \).

**Lemma 4.5** ([CGLZ14], Example 3. Case 2). Let \( a = (0,1,-1) \). Then we have the automorphism of the first kind \( \tau(t) = -t \) and an automorphism of the second kind

\[
\sigma(t) = \frac{t + 1}{t - 1}
\]

Then \( \text{Aut}(V_a) = H \cup H \tau \cong D_4 = \langle x, b \mid x^4 = b^2 = 1, \ bxb = x^{-1} \rangle \) where for example \( x = \sigma \tau \) and \( b = \tau \), in this case.

**Proposition 4.6.** Again suppose \( a = (0,1,-1) \). Let \( \omega_0 = \frac{t-1}{dt}, \omega_1 = \frac{(t-1)^{-1}}{dt} \) and \( \omega_2 = \frac{(t+1)^{-1}}{dt} \). Then \( \Omega_{R_1}/dR_a \), is the direct sum of a one dimensional representation \( C u_1 \) where \( u_1 = \omega_0 - \omega_1 - \omega_2 \) and the unique irreducible representation of \( D_4 \) of dimension 2 which we denote by \( U_2 = C u_2 + C u_3 \) where \( u_2 = \omega_0, u_3 = \omega_1 - \omega_2 \).

**Proof.** There are four irreducible one dimensional representations which we denote by \( \psi_i, \, i = 1, 2, 3, 4 \) and one irreducible two dimensional representation which we denote by \( \chi_2 \). The character table of \( D_4 \) is

|               | \( x^k \) | \( \tau x^k \) |
|---------------|----------|--------------|
| \( \psi_1 \)  | 1        | 1            |
| \( \psi_2 \)  | \( (1)^k \) | -1          |
| \( \psi_3 \)  | \( (-1)^k \) | \( (-1)^k \) |
| \( \psi_4 \)  | \( (-1)^k \) | \( (-1)^{k+1} \) |
| \( \chi_2 \)  | 2(\( \delta_{k,0} - \delta_{k,2} \)) | 0           |

for \( 0 \leq k \leq 3 \). (See [Ser77] Section 5.3.) Here \( \delta_{k,l} \) is the Kronecker delta function.

The one irreducible representation \( \rho \) of \( D_4 \) of dimension 2 defined on \( C^2 \) is given by

\[
\rho(x^k) = \begin{pmatrix}
e^{k\pi/2} & 0 \\
0 & e^{-k\pi/2}
\end{pmatrix}, \quad \rho(\tau x^k) = \begin{pmatrix}
0 & e^{-k\pi/2} \\
e^{k\pi/2} & 0
\end{pmatrix}.
\]

There are five conjugacy classes of \( D_4 \) and they are \( \{I\}, \{x^2\}, \{\tau, x^2 \tau\}, \{x\tau, x^3 \tau\}, \) and \( \{x, x^3\} \).
On the representation $\Omega_{R_a}/dR_a$ we have

$$\tau(\omega_0) = \frac{\tau(t^{-1}) d\tau(t)}{t^{-1} dt} = \omega_0,$$

$$\tau(\omega_1) = \frac{\tau((t-1)^{-1}) d\tau(t)}{(t+1)^{-1} dt} = \omega_2,$$

$$\tau(\omega_2) = \frac{\tau((t+1)^{-1}) d\tau(t)}{(t-1)^{-1} dt} = \omega_1,$$

$$x\tau(\omega_0) = \sigma(\omega_0) = \frac{\sigma(t^{-1}) d\sigma(t)}{t^{-1} dt} = \frac{t-1}{t+1} d \left( \frac{t+1}{t-1} \right) = \omega_2 - \omega_1,$$

$$x\tau(\omega_1) = \sigma(\omega_1) = \frac{\sigma((t-1)^{-1}) d\sigma(t)}{(t+1)^{-1} dt} = \frac{(t+1)^{-1} (t+1)}{t-1} d \left( \frac{t+1}{t-1} \right) = -\omega_1,$$

$$x\tau(\omega_2) = \sigma(\omega_2) = \frac{\sigma((t+1)^{-1}) d\sigma(t)}{(t-1)^{-1} dt} = \frac{t+1}{t-1} d \left( \frac{t+1}{t-1} \right) = \omega_0 - \omega_1.$$

Thus

$$x(\omega_0) = \sigma \tau(\omega_0) = \sigma(\omega_0) = \omega_2 - \omega_1,$$

$$x(\omega_1) = \sigma \tau(\omega_1) = \sigma(\omega_2) = \omega_0 - \omega_1,$$

$$x(\omega_2) = \sigma \tau(\omega_2) = \sigma(\omega_1) = -\omega_1,$$

$$x^2(\omega_0) = x(\omega_2 - \omega_1) = -\omega_1 - (\omega_0 - \omega_1) = -\omega_0,$$

$$x^2(\omega_1) = x(\omega_0 - \omega_1) = \omega_2 - \omega_1 - (\omega_0 - \omega_1) = \omega_2 - \omega_0,$$

$$x^2(\omega_2) = x(-\omega_1) = \omega_1 - \omega_0.$$

As a consequence we have $\chi_{\Omega_{R_a}/dR_a}(\tau) = 1$, $\chi_{\Omega_{R_a}/dR_a}(x\tau) = -1 = \chi_{\Omega_{R_a}/dR_a}(x) = \chi_{\Omega_{R_a}/dR_a}(x^2)$.

We know that there exists a unique set of nonnegative integers $n_1, n_2, n_3, n_4, m$ such that

$$(4.4) \quad \chi_{\Omega_{R_a}/dR_a} = n_1 \psi_1 + n_2 \psi_2 + n_3 \psi_3 + n_4 \psi_4 + m \chi_2.$$

Then

$$3 = n_1 + n_2 + n_3 + n_4 + 2m$$

$$1 = n_1 \psi_1(\tau) + n_2 \psi_2(\tau) + n_3 \psi_3(\tau) + n_4 \psi_4(\tau) + m \chi_2(\tau)$$

$$= n_1 - n_2 + n_3 - n_4$$

$$-1 = n_1 \psi_1(x\tau) + n_2 \psi_2(x\tau) + n_3 \psi_3(x\tau) + n_4 \psi_4(x\tau) + m \chi_2(x\tau)$$

$$= n_1 - n_2 + n_3 - n_4$$

$$-1 = n_1 \psi_1(x^3) + n_2 \psi_2(x^2) + n_3 \psi_3(x^2) + n_4 \psi_4(x^2) + m \chi_2(x^2)$$

$$= n_1 + n_2 + n_3 + n_4 - 2m.$$
If \( m = 0 \) then the last equation is inconsistent. Moreover as \( \dim \Omega_{R_a}/dR_a = 3 \), \( m \leq 1 \) so we must have \( m = 1 \). Then the only solution to the above equations is \( n_1 = n_2 = n_4 = 0 \) and \( n_3 = 1 \). Thus \( \chi_{\Omega_{R_a}/dR_a} = \psi_3 + \chi_2 \).

To find the irreducible subspaces we need the projection formula with respect to the basis \( \omega_0, \omega_1, \omega_2 \) for the representation \( \rho_{D_4} \) of \( D_4 \) on \( \Omega_{R_a}/dR_a \):

\[
\pi_{\chi_2} = \frac{1}{|D_4|} \sum_{g \in S_4} \chi_2(g) \rho_{D_4}(g)
\]

\[
= \frac{1}{8} \left( \begin{array}{ccc}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{array} \right) - 2 \left( \begin{array}{ccc}
-1 & -1 & -1 \\
0 & 0 & 1 \\
0 & 1 & 0
\end{array} \right)
\]

\[
= \frac{1}{4} \left( \begin{array}{ccc}
2 & 1 & 1 \\
0 & 1 & -1 \\
0 & -1 & 1
\end{array} \right).
\]

Thus a basis for \( U_2 \) is \( u_2 = \omega_0, u_3 = \omega_1 - \omega_2 \).

The projection formula for \( \psi_3 \) with respect to the basis \( \omega_0, \omega_1, \omega_2 \) is given by

\[
\pi_{\psi_3} = \frac{1}{|D_4|} \sum_{g \in S_4} \psi_3(g) \rho_{D_4}(g)
\]

\[
= \frac{1}{8} \left( \begin{array}{ccc}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{array} \right) + \left( \begin{array}{ccc}
-1 & -1 & -1 \\
0 & 0 & 1 \\
0 & 1 & 0
\end{array} \right) + \left( \begin{array}{ccc}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{array} \right) - \left( \begin{array}{ccc}
0 & 0 & 1 \\
1 & 0 & 0 \\
1 & 0 & 0
\end{array} \right) - \left( \begin{array}{ccc}
-1 & -1 & -1 \\
0 & 1 & 0 \\
-1 & 1 & -1
\end{array} \right) - \left( \begin{array}{ccc}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{array} \right)
\]

\[
= \frac{1}{8} \left( \begin{array}{ccc}
0 & -4 & -4 \\
0 & 4 & 4 \\
0 & 4 & 4
\end{array} \right).
\]

Thus \( \omega_0 - \omega_1 - \omega_2 \) is a basis of the one dimensional irreducible subrepresentation whose character is \( \psi_3 \).

\[\square\]

4.2.2. The decomposition of \( \Omega_{R_a}/dR_a \) for \( S_4 \).

**Lemma 4.7** ([Kle56], [Shu97] and [CGLZ14] Appendix). The automorphism group of \( V_{(0,1,1,-1,-1)} \) is \( S_4 \).

Here the automorphism group of \( V_{(0,1,1,-1,-1)} \) is generated by an automorphism of the first kind

\[
\phi(t) = at
\]

and also an automorphism of the second kind given by

\[
\psi(t) = \frac{t + 1}{t - 1}.
\]

**Proposition 4.8.** Set \( a = (0,1,1,-1,-1) \). For the automorphism group of \( V_a, S_4 \), we have

\[
(4.5) \quad \Omega_{R_a}/dR_a = U_0 \oplus U_{\psi t}
\]
where
\[ U_\theta = \mathbb{C}(-\omega_0 + \omega_1 + \omega_3) + \mathbb{C}(-\omega_0 + \omega_2 + \omega_4) \]
\[ U_{\rho \epsilon} = \mathbb{C}\omega_0 + \mathbb{C}(-\omega_1 + \omega_3) + \mathbb{C}(-\omega_2 + \omega_4) \]
are the irreducible \( S_4 \)-subrepresentations of \( \Omega_{R_a}/dR_a \).

**Proof.** First observe
\[
\phi(\omega_0) = -\frac{\omega_1}{t-1} dt = \omega_0,
\phi(\omega_1) = -\frac{t}{t-1} dt = \omega_4,
\phi(\omega_2) = -\frac{(-t)}{t-1} dt = \omega_1,
\phi(\omega_3) = -\frac{(t-1)}{t-1} dt = \omega_2,
\phi(\omega_4) = -\frac{(1-1)}{t-1} dt = \omega_3,
\]
and
\[
\psi(\omega_0) = -\omega_2 + \omega_4,
\psi(\omega_1) = -\omega_2,
\psi(\omega_2) = -\omega_2 + \omega_1,
\psi(\omega_3) = -\omega_2 + \omega_0,
\psi(\omega_4), = -\omega_2 + \omega_3.
\]

Thus the linear transformations \( \phi \) and \( \psi \) have with respect to this basis the matrices
\[
(4.6) \quad \phi = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
\end{pmatrix}, \quad \psi = \begin{pmatrix}
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 \\
-1 & -1 & -1 & -1 & -1 \\
0 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 \\
\end{pmatrix}.
\]

We let \( \chi_{\Omega_{R/dR}} = a\chi_0 + b\chi_\epsilon + c\chi_\theta + d\chi_\rho + e\chi_{\epsilon\rho} \) where \( \chi_0 \) is the character of the trivial representation, \( \chi_\epsilon \) is the character of the alternating representation, \( \chi_\theta \) is the character of the unique two dimensional irreducible representation \( \chi_\rho \), \( \chi_{\epsilon\rho} \) is the character of the natural representation on \( \mathbb{C}^3 \) and \( \chi_{\epsilon\rho} \) is the character of the tensor product of the alternating representation and the natural representation.

We then have the following values of the trace of the conjugacy classes

**Table 2. Character of \( S_4 \) on \( \Omega_{R/dR} \)**

| \( \chi_{\Omega_{R/dR}} \) | 1 | \( \psi_0 \) | \( \phi^2 \) | \( \psi \) | \( \phi \) |
|--------------------------|---|-----------|-----------|------------|------------|
| \( f \)                  | 5 | -1        | 1         | -1         | 1          |
| \( \chi_\epsilon \)      | 1 | -1        | 1         | 1          | 1          |
| \( \chi_\theta \)        | 1 | -1        | 1         | 1          | 1          |
| \( \chi_\rho \)          | 1 | -1        | 1         | 1          | 1          |
| \( \chi_{\epsilon\rho} \)| 1 | -1        | 1         | 1          | 1          |

Note \( \phi \) has order 4 while \( \psi \) has order 3. The number of elements in the conjugacy class of \( \psi_0 \) is 6: \( \psi_0, \phi^3 \psi_0^2, \phi^2 \psi_0 \phi^2, \phi \psi_0, \psi^2 \phi \psi_0, \phi \psi_0 \psi_0 \phi^2 \phi^3 \). The number of elements in the conjugacy class
of $\phi^2$ is $3$: $\phi^2$, $\psi\phi^2\psi^2$, $\psi^2\phi^2\psi$. The number of elements in the conjugacy class of $\psi$ is $8$: $\psi$, $\phi\psi\phi^3$, $\phi^3\psi\phi$, $\phi^2\psi\phi^2\psi^2$, $\phi^2\psi^2\phi\psi^2\psi$, $\phi\psi^2\phi^2\psi^3$, $\phi^3\psi^2\phi^2\psi\phi^2\psi$. The number of elements in the conjugacy class of $\phi$ is $6$: $\phi$, $\psi\phi^2\psi^2$, $\phi^2\psi\phi^2\phi\psi^2$, $\phi^2\psi^2\phi\psi^2\phi\psi^2\phi\psi^2\psi$. Solving

\[
5 = a + b + 2c + 3d + 3e \\
-1 = a - b + d - e \\
1 = a + b + 2c - d - e \\
-1 = a + b - c \\
1 = a - b - d + e
\]

we get $a = b = d = 0$ and $c = 1 = e$.

Now we need to find a basis for these two irreducible components for $\chi_{\theta}$, $\chi_{\psi\phi}$. Let $\rho : S_4 \to \text{GL}(\Omega_{R_3}/dR_3)$ is the given induced representation. To find the irreducible component for $\chi_{\theta}$, we need the projection formula:

\[
\pi_{\chi_{\theta}} = \frac{1}{|S_4|} \sum_{g \in S_4} \chi_{\theta}(g)\rho(g) \\
= \frac{1}{24} \sum_{g \in S_4} \chi_{\theta}(g)\rho(g) \\
= \frac{1}{24} \begin{pmatrix}
0 & -6 & -6 & -6 & -6 \\
0 & 6 & 0 & 6 & 0 \\
0 & 0 & 6 & 0 & 6 \\
0 & 6 & 0 & 6 & 0 \\
0 & 0 & 6 & 0 & 6
\end{pmatrix}
= \frac{1}{4} \begin{pmatrix}
0 & -1 & -1 & -1 & -1 \\
0 & 1 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 1 \\
0 & 1 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 1
\end{pmatrix}.
\]

Thus the subspace spanned $U_{\theta}$ by $-\omega_0 + \omega_1 + \omega_3$ and $-\omega_0 + \omega_2 + \omega_4$ is an irreducible subrepresentation of $\Omega_{R_3}/dR_3$.

For the irreducible component of type $\chi_{\psi\phi}$ we have

\[
\pi_{\chi_{\psi\phi}} = \frac{1}{|S_4|} \sum_{g \in S_4} \chi_{\psi\phi}(g)\rho(g) \\
= \frac{1}{24} \sum_{g \in S_4} \chi_{\psi\phi}(g)\rho(g) \\
= \frac{1}{6} \begin{pmatrix}
2 & 1 & 1 & 1 & 1 \\
0 & 1 & 0 & -1 & 0 \\
0 & 0 & 1 & 0 & -1 \\
0 & -1 & 0 & 1 & 0 \\
0 & 0 & -1 & 0 & 1
\end{pmatrix}
\]

Thus a basis for the three dimensional irreducible subrepresentation with character $\chi_{\psi\phi}$ is $\omega_0$, $-\omega_1 + \omega_3$, $-\omega_2 + \omega_4$. \(\square\)

(In the Propositions below the author will not be as detailed about the calculations, but they all are of a similar level of difficulty and if the reader is so interested in the details, please feel free to email him - the author will send you the tex file with nearly all calculations written out explicitly.)
4.3. Decomposition of $\Omega_{R_a}/dR_a$ under $A_4$.

**Lemma 4.9 ([CGLZ14], Example 3. Case 3).** Let $a = (0, 1, x)$ where $x = \frac{1 + i\sqrt{3}}{2}$. Then $\text{Aut}(V_a) \cong A_4$, the alternating group on 4 letters. Here

$$\text{Aut}(V_a) = H \cup H\tau_2 \cup H\tau_2$$

where $H = \{1, \sigma_1, \sigma_2, \sigma_3\}$ and the automorphisms are defined by

$$\sigma_1(t) = \frac{x}{t}, \quad \sigma_2(t) = \frac{t - x}{t - 1}, \quad \sigma_3(t) = \frac{x(t - 1)}{t - x}, \quad \tau_2(t) = -(t - 1).$$

**Proposition 4.10.** Again suppose $a = (0, 1, x)$ where $x = \frac{1 + i\sqrt{3}}{2}$. Let $\omega_0 = \frac{t - 1}{d} dt$, $\omega_1 = (t - 1)^{-1} dt$ and $\omega_2 = (t - x)^{-1} dt$. Then $\Omega_{R_a}/dR_a$, is the unique irreducible representation of $A_4$ of dimension 3.

**Proof.** Set $\zeta = \exp(2\pi i/3)$. Let $\chi_3$ be the character of the unique irreducible 3-dimensional representation of $A_4$ and $\chi_1$ and $\chi_2$ the characters of the two unique non-trivial representations of $A_4$. The character table of $A_4$ is

|    | 1  | $\sigma_1$ | $\tau_2$ | $\tau_2^2$ |
|----|----|------------|----------|------------|
| $\chi_1$ | 1 | 1 | $\zeta$ | $\zeta^2$ |
| $\chi_2$ | 1 | 1 | $\zeta^2$ | $\zeta$ |
| $\chi_3$ | 3 | -1 | 0 | 0 |

Now one can calculate

$$\sigma_1(\omega_0) = \overline{\sigma_1(t^{-1})} d\sigma_1(t) = xt dt(x/t) = \frac{-t - 1}{st} dt = -\omega_0,$$

$$\sigma_1(\omega_1) = -\omega_0 + \omega_2,$$

$$\sigma_1(\omega_2) = -\omega_0 + \omega_1.$$

Thus $\chi_{\Omega_{R_a}/dR_a}(\sigma_1) = -1.$

Moreover

$$\tau_2(\omega_0) = \overline{\tau_2(t^{-1})} d\tau_2(t) = (-(x(t - 1))^{-1} dt(-x(t - 1)) = (t - 1)^{-1} dt = \omega_1,$$

$$\tau_2(\omega_1) = \omega_2,$$

$$\tau_2(\omega_2) = \omega_0,$$

and $\tau_2^2(\omega_0) = \omega_2$, $\tau_2^2(\omega_1) = \omega_0$, $\tau_2^2(\omega_2) = \omega_1$. As a consequence $\chi_{\Omega_{R_a}/dR_a}(\tau_2) = 0 = \chi_{\Omega_{R_a}/dR_a}(\tau_2^2)$. Thus the character of $\Omega_{R_a}/dR_a$ is the same as the three dimensional irreducible representation for $A_4$ and thus it must be isomorphic to it. □
4.4. Decomposition of $\Omega_{R_a}/dR_a$ under $A_5$.

Lemma 4.11 ([Kle56], [Tot02] and [Shm97], [CGLZ14] Appendix). Let $\zeta = \exp(2\pi i/5)$ and 

$$a_0 = 0, \quad a_i = \zeta^{i-1}(\zeta + \zeta^4), \quad 1 \leq i \leq 5, \quad a_i = \zeta^{i-6}(\zeta^2 + \zeta^3), \quad 6 \leq i \leq 10.$$

The automorphism group of $\mathcal{V}_{(a_0,a_1,\ldots,a_{10})}$ is $A_5$ where the automorphisms are

$$t \mapsto \zeta^j t, \quad t \mapsto -\frac{1}{\zeta^j t},$$

$$t \mapsto \zeta^j \frac{(\zeta - \zeta^4)\zeta^j t + (\zeta^2 - \zeta^3)}{\zeta^2 - \zeta^4} \zeta^j t + (\zeta - \zeta^4),$$

$$t \mapsto \zeta^j \frac{(\zeta^2 - \zeta^3)\zeta^j t + (\zeta - \zeta^4)}{\zeta - \zeta^4} \zeta^j t - (\zeta^2 - \zeta^3).$$

for $j, l = 0, \ldots, 4$.

As usual set $\omega_0 = \frac{1}{\zeta - 1} dt$ and $\omega_i = (t - a_i)^{-1} dt$ for $1 \leq i \leq 10$.

Proposition 4.12. The $\text{Aut}(R_a)$ module $\Omega_{R_a}/dR_a$ decomposes into a direct sum of irreducible submodules as

$$\Omega_{R_a}/dR_a = U_3 \oplus U'_3 \oplus U_5$$

where $U_3$ and $U'_3$ are the two distinct irreducible representations of $A_5$ of dimension 3 and $U_5$ is the unique irreducible representation of dimension 5. A basis of $U_5$ is $\{-\omega_0 + \omega_i + \omega_{i+5} | 1 \leq i \leq 5\}$ and bases of $U_3$ and $U'_3$ are respectively

$$\sqrt{3}\omega_0 + \sum_{i=1}^{5} (\omega_i - \omega_{i+5}),$$

$$\xi\omega_0 + \xi\omega_1 + \omega_2 + \omega_5 - \xi\omega_6 - \omega_7 - \omega_8,$$

$$\xi\omega_0 + \omega_1 + \xi\omega_2 + \omega_3 - \omega_6 - \xi\omega_7 - \omega_8,$$

and

$$\sqrt{5}\omega_0 - \sum_{i=1}^{5} (\omega_i - \omega_{i+5}),$$

$$-\bar{\xi}\omega_0 - \bar{\xi}\omega_1 - \omega_2 - \omega_5 + \bar{\xi}\omega_6 + \omega_7 + \omega_8,$$

$$-\xi\omega_0 - \bar{\xi}\omega_2 - \omega_3 - \omega_6 + \bar{\xi}\omega_7 + \omega_8,$$

where $\xi = \frac{1 + \sqrt{5}}{2}$ and $\bar{\xi} = \frac{1 - \sqrt{5}}{2}$.

Proof. Let $\phi$ be the automorphism given by $\phi(t) = \zeta t$. Then a straightforward calculation yields

$$\phi(\omega_0) = \omega_0, \quad \phi(\omega_1) = \omega_5, \quad \phi(\omega_2) = \omega_1, \quad \phi(\omega_3) = \omega_2, \quad \phi(\omega_4) = \omega_3, \quad \phi(\omega_5) = \omega_4,$$

$$\phi(\omega_6) = \omega_10, \quad \phi(\omega_7) = \omega_6, \quad \phi(\omega_8) = \omega_7, \quad \phi(\omega_9) = \omega_8, \quad \phi(\omega_{10}) = \omega_9.$$

Thus the $\chi_{\Omega_{R_a}/dR_a}(\phi^l) = 1$ for $1 \leq l \leq 4$. 

Let $\psi$ be the automorphism given by $\psi(t) = \frac{1}{\zeta t}$. In the calculations below note that $(\zeta + \zeta^4)(\zeta^2 + \zeta^3) = -1$. Then we have

\[
\psi(\omega_1) = -\omega_0 + \omega_{10}, \quad \psi(\omega_2) = -\omega_0 + \omega_9, \quad \psi(\omega_3) = -\omega_0 + \omega_8 \\
\psi(\omega_4) = -\omega_0 + \omega_7, \quad \psi(\omega_5) = -\omega_0 + \omega_6, \quad \psi(\omega_6) = -\omega_0 + \omega_5 \\
\psi(\omega_7) = -\omega_0 + \omega_4, \quad \psi(\omega_8) = -\omega_0 + \omega_3, \quad \psi(\omega_9) = -\omega_0 + \omega_2 \\
\psi(\omega_{10}) = -\omega_0 + \omega_1, \quad \psi(\omega_0) = -\omega_0
\]

Thus $\chi_{\Omega_{R^a}/DR_a}(\psi) = -1$ and $\psi^2 = 1$.

Next we observe that

\[
\frac{\zeta - \zeta^4}{\zeta^2 - \zeta^3} = -(\zeta^2 + \zeta^3) \\
\frac{\zeta^2 - \zeta^3}{\zeta - \zeta^4} = \zeta + \zeta^4,
\]

so that

\[
\zeta^j \frac{(\zeta - \zeta^4)(\zeta^2 - \zeta^3)\zeta^{l-1} + (\zeta^2 - \zeta^3)(\zeta - \zeta^4)}{(\zeta^2 - \zeta^3)(\zeta^2 - \zeta^3)\zeta^{l-1} + (\zeta - \zeta^4)} = \zeta^j \frac{(\zeta^2 + \zeta^3)(\zeta^2 + \zeta^3)\zeta^{l-1} + (\zeta^2 + \zeta^3)(\zeta^2 + \zeta^3)}{(\zeta^2 + \zeta^3)(\zeta^2 + \zeta^3)\zeta^{l-1} + (\zeta^2 + \zeta^3)(\zeta^2 + \zeta^3)}
\]

for $j, l = 1, \ldots, 5$. Thus if we set

\[
\beta(t) := \frac{(\zeta^2 + \zeta^3)t + 1}{t - (\zeta^2 + \zeta^3)},
\]

then

\[
\phi^j \circ \beta \circ \phi^{l-1}(t) = \zeta^j \frac{(\zeta^2 + \zeta^3)t + \zeta^{l-1}}{t - \zeta^{l-1}(\zeta^2 + \zeta^3)} \\
\psi \circ \beta(t) = -\zeta^4 \frac{(\zeta + \zeta^4)t + 1}{t - (\zeta + \zeta^4)}
\]

and

\[
\beta(t) - \zeta^i(\zeta + \zeta^4) = \frac{(1 - \zeta^i)(\zeta^2 + \zeta^3)t + 1 + \zeta^i(\zeta^2 + \zeta^3)^2}{t - (\zeta^2 + \zeta^3)}
\]

So we will only calculate $\beta$ evaluated on the basis elements

\[
\beta(\omega_0) = \omega_1 - \omega_5 \\
\beta(\omega_1) = \omega_0 - \omega_5 \\
\beta(\omega_2) = \omega_5 - \omega_5 \\
\beta(\omega_3) = \omega_8 - \omega_5 \\
\beta(\omega_4) = \omega_9 - \omega_6 \\
\beta(\omega_5) = \omega_2 - \omega_6 \\
\beta(\omega_6) = -\omega_6 \\
\beta(\omega_7) = \omega_{10} - \omega_6
\]
The automorphism $\beta \circ \phi$ has order 3 and thus is in the conjugacy class of $(123)$. Moreover its trace is $-1$. The character table of $A_5$ is the following.

| $\lambda \Omega_{R_a/dR_a}$ | $I$ | $I(123)$ | $I(12)(34)$ | $I(12345)$ | $I(21345)$ |
|-----------------------------|-----|----------|-------------|-------------|-------------|
| $\chi_2$                    | 1   | 1        | 1           | 1           | 1           |
| $\chi_3$                    | 3   | 0        | -1          | $1+\sqrt{5}$ | $1-\sqrt{5}$ |
| $\chi_4$                    | 4   | 1        | 1           | 0           | -1          |
| $\chi_5$                    | 5   | -1       | 1           | 0           | 0           |

Solving

$$11 = a + 3b + 3c + 4d + 5e$$
$$-1 = a + d - e$$
$$-1 = a - b - c + e$$
$$1 = a + \frac{1 + \sqrt{5}}{2} b + \frac{1 - \sqrt{5}}{2} c - d$$
$$1 = a + \frac{1 - \sqrt{5}}{2} b + \frac{1 + \sqrt{5}}{2} c - d$$

we get $a = d = 0$ and $b = c = e = 1$. That is to say $\chi_{\Omega_{R_a/dR_a}} = \chi_2 + \chi_3 + \chi_5$.

The conjugacy class of $\beta \circ \phi$ has 20 elements in it. The conjugacy class of the two cycle $\psi$ has 15 elements in it. There are two conjugacy classes that have 5-cycles in them and they both have 12 elements in them.
The matrices for the above linear transformations with respect to the ordered basis \( \{ \omega_0, \ldots, \omega_{10} \} \) are given as follows:

The linear transformations \( \phi, \beta \) and \( \psi \) have with respect to this basis have matrix representation

\[
\phi = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 
\end{pmatrix},
\]

\[
\psi = \begin{pmatrix}
-1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 
\end{pmatrix},
\]

\[
\beta = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 
\end{pmatrix}.
\]
To get bases for the three irreducible subrepresentations we need to use the projection formulae and sum over the conjugacy classes (using Mathematica).

\[
\pi_{\chi_5} = \frac{1}{|A_5|} \sum_{g \in A_5} \chi_5(g)\rho(g) = \frac{1}{60} \sum_{g \in A_5} \chi_5(g)\rho(g)
\]

\[
\begin{pmatrix}
0 & -6 & -6 & -6 & -6 & -6 & -6 & -6 & -6 & -6 & 0 \\
0 & 0 & 6 & 0 & 0 & 0 & 0 & 6 & 0 & 0 & 0 \\
0 & 0 & 0 & 6 & 0 & 0 & 0 & 0 & 6 & 0 & 0 \\
0 & 0 & 0 & 0 & 6 & 0 & 0 & 0 & 0 & 6 & 0 \\
0 & 0 & 0 & 0 & 0 & 6 & 0 & 0 & 0 & 6 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 6 & 0 & 0 & 0 & 6 \\
0 & 6 & 0 & 0 & 0 & 0 & 0 & 6 & 0 & 0 & 0 \\
0 & 0 & 6 & 0 & 0 & 0 & 0 & 0 & 6 & 0 & 0 \\
0 & 0 & 0 & 6 & 0 & 0 & 0 & 0 & 6 & 0 & 0 \\
0 & 0 & 0 & 0 & 6 & 0 & 0 & 0 & 0 & 6 & 0 \\
0 & 0 & 0 & 0 & 0 & 6 & 0 & 0 & 0 & 0 & 6 \\
0 & 0 & 0 & 0 & 0 & 0 & 6 & 0 & 0 & 0 & 6 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 6 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 6 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 6 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 6
\end{pmatrix}
\]

\[
= \frac{1}{60} \begin{pmatrix}
\sqrt{5} & \xi & \xi & \xi & \xi & -\xi & -\xi & -\xi & -\xi \\
1 & \xi & 1 & 0 & 0 & 1 & \xi & 0 & 1 & 1 & 0 \\
1 & 1 & \xi & 1 & 0 & 0 & \xi & 0 & 1 & 1 \\
1 & 0 & 1 & \xi & 1 & 0 & 1 & 0 & \xi & 0 & 1 \\
1 & 0 & 0 & 1 & \xi & 1 & 1 & 1 & 0 & \xi & 0 \\
1 & 1 & 0 & 0 & 1 & \xi & 0 & 1 & 1 & 0 & \xi \\
-1 & -\xi & -1 & 0 & 0 & -1 & -\xi & 0 & -1 & -1 & 0 \\
-1 & -1 & -\xi & -1 & 0 & 0 & 0 & -\xi & 0 & -1 & -1 \\
-1 & 0 & -1 & -\xi & -1 & 0 & -1 & 0 & -\xi & 0 & -1 \\
-1 & 0 & 0 & -1 & -\xi & -1 & -1 & 0 & -\xi & 0 & -1 \\
-1 & -1 & 0 & 0 & -1 & -\xi & 0 & -1 & -1 & 0 & -\xi
\end{pmatrix}
\]

\[
= \frac{\sqrt{5}}{30} \begin{pmatrix}
\xi & \xi & \xi & \xi & -\xi & -\xi & -\xi & -\xi \\
1 & \xi & 1 & 0 & 0 & 1 & \xi & 0 & 1 & 1 & 0 \\
1 & 1 & \xi & 1 & 0 & 0 & \xi & 0 & 1 & 1 \\
1 & 0 & 1 & \xi & 1 & 0 & 1 & 0 & \xi & 0 & 1 \\
1 & 0 & 0 & 1 & \xi & 1 & 1 & 1 & 0 & \xi & 0 \\
1 & 1 & 0 & 0 & 1 & \xi & 0 & 1 & 1 & 0 & \xi \\
-1 & -\xi & -1 & 0 & 0 & -1 & -\xi & 0 & -1 & -1 & 0 \\
-1 & -1 & -\xi & -1 & 0 & 0 & 0 & -\xi & 0 & -1 & -1 \\
-1 & 0 & -1 & -\xi & -1 & 0 & -1 & 0 & -\xi & 0 & -1 \\
-1 & 0 & 0 & -1 & -\xi & -1 & -1 & 0 & -\xi & 0 & -1 \\
-1 & -1 & 0 & 0 & -1 & -\xi & 0 & -1 & -1 & 0 & -\xi
\end{pmatrix}
\]

This means that a basis for \(U_3\) is

\[
\sqrt{5} \omega_0 + \sum_{i=1}^{5} (\omega_i - \omega_{i+5}),
\]

\[
\xi \omega_0 + \xi \omega_1 + \omega_2 + \omega_5 - \xi \omega_6 - \omega_7 - \omega_{10},
\]

\[
\xi \omega_0 + \omega_1 + \xi \omega_2 + \omega_3 - \omega_6 - \xi \omega_7 - \omega_8.
\]
The projection operator for the other three dimensional subrepresentation is
\[
\pi_{\chi_3} = \frac{1}{|A_5|} \sum_{g \in A_5} \chi_3(g) \rho(g) = \frac{1}{60} \sum_{g \in A_5} \chi_3(g) \rho(g)
\]
\[
= \sqrt{5} \frac{\omega_0 - \sum_{i=1}^{5} (\omega_i - \omega_{i+5})}{30},
\]
\[
-\xi \omega_0 - \bar{\xi} \omega_1 - \omega_2 - \omega_5 + \bar{\xi} \omega_6 + \omega_7 + \omega_{10},
\]
\[
-\xi \omega_0 - \omega_1 - \bar{\xi} \omega_2 - \omega_3 + \omega_6 + \bar{\xi} \omega_7 + \omega_8.
\]

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6. Concluding Remark.

There are other rings \( R \) that arise as rings of meromorphic functions on a Riemann surface with a finite number of points removed. We plan to investigate how \( \Omega_R/dR \) decomposes under the action of the automorphism group of \( R \).

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