A Classification of Isolated Singularities
of Elliptic Monge-Ampère Equations in Dimension Two

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Abstract

Let $\mathcal{M}_1$ denote the space of solutions $z(x, y)$ to an elliptic, real analytic Monge-Ampère equation $\det(D^2z) = \varphi(x, y, z, Dz) > 0$ whose graphs have a non-removable isolated singularity at the origin. We prove that $\mathcal{M}_1$ is in one-to-one correspondence with $\mathcal{M}_2 \times \mathbb{Z}_2$, where $\mathcal{M}_2$ is a suitable subset of the class of regular, real analytic, strictly convex Jordan curves in $\mathbb{R}^2$. We also describe the asymptotic behavior of solutions of the Monge-Ampère equation in the $C^k$-smooth case, and a general existence theorem for isolated singularities of analytic solutions of the more general equation $\det(D^2z + A(x, y, z, Dz)) = \varphi(x, y, z, Dz) > 0$. © 2015 Wiley Periodicals, Inc.

1 Introduction

In 1955 K. Jörgens wrote a seminal paper [22] that initiated the study of isolated singularities of the classical elliptic Monge-Ampère equation in dimension two,

$$\det(D^2z) = \varphi(x, y, z, Dz) > 0,$$

where $D, D^2$ denote the gradient and Hessian operators. Jörgens proved for $\varphi = 1$ a removable singularity theorem and gave a description of the behavior of a solution to $\det(D^2z) = 1$ around a nonremovable isolated singularity.

In this paper we classify the isolated singularities of (1.1) in the case that $\varphi$ is real analytic and give a complete description of the asymptotic behavior of such solutions around an isolated singularity when $\varphi$ is only of class $C^k$. Specifically, we give this classification by explicitly parametrizing the moduli space of solutions.
to (1.1) with a nonremovable isolated singularity at some given point, as we explain next.

By convexity, any solution \( \dot{z} \) to (1.1) defined on a punctured disk extends continuously to the puncture. If the extension is not \( C^2 \), we say that the puncture is an isolated singularity. The gradient of \( \dot{z} \) converges at the singularity to a point, a segment, or a closed convex curve. We call this set the limit gradient of \( \dot{z} \) at the singularity. The limit gradient \( \gamma \subset \mathbb{R}^2 \) describes the asymptotic behavior of \( z(x, y) \) as \( (x, y) \) converges to the puncture.

Without loss of generality, we will assume that the singularity is placed at \( (0, 0, 0) \). We let \( \varphi > 0 \) be defined on an open set \( \mathcal{U} \subset \mathbb{R}^5 \) such that \( \mathcal{H} := \{(p, q) \in \mathbb{R}^2 : (0, 0, p, q) \in \mathcal{U}\} \neq \emptyset \). Note that if \( z \) is a solution to (1.1) with an isolated singularity at the origin, by continuity the limit gradient of \( z \) at the origin is contained in \( \mathcal{H} \subset \mathbb{R}^2 \). As a matter of fact, \( \gamma \) is contained in \( \mathcal{H} \) in many natural situations, for instance, if \( \mathcal{H} = \mathbb{R}^2 \) or, more generally, if \( \mathcal{U} \subset \mathbb{R}^3 \times \mathcal{H} \) where \( \mathcal{H} \subset \mathbb{R}^2 \) is simply connected (see Remark 2.1).

With these conventions, we prove the following:

**Theorem 1.1.** Let \( \varphi \in C^\alpha(\mathcal{U}), \varphi > 0 \). Let \( \mathcal{M}_1 \) denote the class of solutions \( z \) to (1.1) that have an isolated singularity at the origin and whose limit gradient at the singularity is contained in \( \mathcal{H} \subset \mathbb{R}^2 \). Let \( \mathcal{M}_2 \) denote the class of regular, analytic, strictly convex Jordan curves \( \gamma \) in \( \mathcal{H} \subset \mathbb{R}^2 \).

Then the map sending each \( z \in \mathcal{M}_1 \) to \((\gamma, \varepsilon)\), where \( \gamma \) is its limit gradient at the singularity and \( \varepsilon \in \mathbb{Z}_2 \) is 0 (respectively, 1) if \( z_{xx} > 0 \) (respectively, \( z_{xx} < 0 \)) defines a bijective correspondence between \( \mathcal{M}_1 \) and \( \mathcal{M}_2 \times \mathbb{Z}_2 \).

Theorem 1.1 is a consequence of two more general results that we obtain.

In Theorem 2.3 we prove that if \( \varphi \in C^k(\mathcal{U}), k \geq 4 \), the limit gradient of any solution to (1.1) with an isolated singularity at the origin is a \( C^{k,\alpha} \)-smooth regular, strictly convex (i.e., of nowhere-zero curvature) Jordan curve \( \gamma \) in \( \mathbb{R}^2 \). If \( \varphi \) is analytic, we show that \( \gamma \) is also analytic. We also provide a way of parametrizing the graph of any such solution so that the resulting map is defined on an annulus and extends smoothly (or analytically) across the boundary circle in which the parametrization collapses to the singularity. This result provides a complete description of the asymptotic behavior of a solution to (1.1) with \( \varphi \in C^k(\mathcal{U}), k \geq 4 \), at an isolated singularity.

In Theorem 3.1 we give a general existence theorem for isolated singularities of the general elliptic equation of Monge-Ampère type in dimension two,

\[(1.2) \quad \det(D^2z + A(x, y, z, Dz)) = \varphi(x, y, z, Dz) > 0.\]

We prove that if both \( \varphi \) and the symmetric matrix \( A \) are analytic in \( \mathcal{U} \subset \mathbb{R}^5 \), any regular, analytic, strictly convex Jordan curve \( \gamma \in \mathcal{H} \) can be realized as the limit gradient of a solution to (1.2) that has an isolated singularity at the origin. Even more generally, if in our construction process we start with a closed, analytic curve in \( \mathcal{H} \) (not necessarily convex or regular), we obtain a multivalued solution to (1.2).
Figure 1.1. Left: the radial function $z(x, y) = \frac{1}{2}(\sqrt{x^2 + y^2} + \sinh^{-1}(r))$, $r = \sqrt{x^2 + y^2}$, is the simplest solution to $\det(D^2z) = 1$ other than quadratic polynomials. Right: a rotational peaked sphere in $\mathbb{R}^3$; it is the simplest $K = 1$ surface in $\mathbb{R}^3$ other than round spheres. Both examples present nonremovable isolated singularities.

Theorem 1.1 is not true for the more general equation (1.2); see Remark 3.4.

The study of isolated singularities is a fundamental problem in the theory of nonlinear geometric PDEs and has been extensively studied. Several such elliptic equations (including the minimal surface equation [2]) only admit removable isolated singularities; see [24] and references therein. The asymptotic behavior at an isolated singularity of solutions to fully nonlinear, conformally invariant geometric PDEs has been studied in detail in many works; see, for instance, [9, 15, 17, 26, 27] and references therein (see also [6, 23, 28]). Some previous works on isolated singularities of elliptic Monge-Ampère equations can be found in [1, 3, 4, 11–13, 20–22, 31]. The exterior Dirichlet problem as well as the asymptotic behavior at infinity of solutions to $\det(D^2u) = 1$ in $\mathbb{R}^n$ minus a ball has been studied for $n = 2$ in [10] and for $n \geq 2$ in [7].

The appearance of solutions to (1.1) with nonremovable isolated singularities is a very natural phenomenon; see Figure 1.1. This justifies the interest of the study of the asymptotic behavior and classification of such isolated singularities beyond a removable singularity type theorem. It should be emphasized that, typically, solutions to (1.1) with nonremovable isolated singularities do not belong to the usual classes of generalized solutions to (1.1) (viscosity solutions, Alexandrov solutions).

Let us note that some of the arguments that we use here seem specific to the two-dimensional case, since they rely on complex analysis and surface theory. Nonetheless, the basic strategy of our classification—transforming the PDE into a first-order differential system for which the isolated singularity turns into a regular boundary curve, and then studying the Cauchy problem for that system along
the boundary to determine the asymptotic behavior at the singularity—seems applicable to other fully nonlinear PDEs admitting cone singularities, even in arbitrary dimensions.

The PDEs (1.1) and (1.2) appear in a variety of applications, among which we may quote optimal transport problems, isometric embedding of abstract Riemannian metrics, surfaces of prescribed curvature in Riemannian and Lorentzian 3-manifolds, parabolic affine spheres, linear Weingarten surfaces, etc. In this way, the results of this paper frequently admit reformulations in these specific theories. For instance, some solutions with cone singularities that we construct here provide (via the Legendre transform) solutions to an obstacle problem for Monge-Ampère equations, as explained in [30].

We shall give in the Appendix, as an example of a geometric application of this type, a classification of the isolated singularities of embedded surfaces in \( \mathbb{R}^3 \) with prescribed positive Gaussian curvature.

## 2 Asymptotic Behavior at the Singularity

In this section we will use the following conventions:

- \( \Omega = \{(x, y) \in \mathbb{R}^2 : 0 < x^2 + y^2 < \rho^2\} \) is a punctured disc centered at the origin.
- \( \mathcal{U} \subset \mathbb{R}^3 \) is an open set such that
  \[
  \mathcal{H} := \{(p, q) \in \mathbb{R}^2 : (0, 0, p, q) \in \mathcal{U}\} \neq \emptyset.
  \]
- \( \varphi \in C^k(\mathcal{U}), \varphi > 0, k \geq 4. \)
- \( \zeta \) is a solution to (1.1) in \( \Omega \) with an isolated singularity at \((0, 0)\), which is of class \( C^{k+1, \alpha} \) on compact sets of \( \Omega \). We will assume without loss of generality from now on that \( \zeta \) has been continuously extended to the origin by \( \zeta(0, 0) = 0. \)
- \( \gamma \subset \mathbb{R}^2 \) will denote the limit gradient of \( \zeta \) at the origin; that is, \( \gamma \subset \mathbb{R}^2 \) is the set of points \( \xi \in \mathbb{R}^2 \) for which there is a sequence \( q_n \to (0, 0) \) in \( \Omega \) such that \( (\zeta_x, \zeta_y)(q_n) \to \xi \). Note that \( \gamma \subset \mathcal{H}. \)
- We will assume without loss of generality that \( \zeta_{xx} > 0. \) Observe that by (1.1) and since \( \Omega \) is connected, either \( \zeta_{xx} > 0 \) or \( \zeta_{xx} < 0 \) on \( \Omega \). If \( \zeta \) is a solution to (1.1) with \( \zeta_{xx} < 0 \) and an isolated singularity at the origin, then \( \tilde{z}(x, y) := -z(-x, -y) \) is a solution to
  \[
  \zeta_{xx} \tilde{z}_{yy} - \zeta_{xy}^2 = \tilde{\varphi}(x, y, z, \tilde{z}, \zeta_{x}, \zeta_{y}),
  \]
  where \( \tilde{\varphi}(x, y, z, p, q) = \varphi(-x, -y, -z, p, q) \), with \( \tilde{z}_{xx} > 0 \) and an isolated singularity at the origin. Note that the limit gradients \( \gamma, \tilde{\gamma} \) of \( \zeta \) and \( \tilde{\zeta} \) at the origin coincide and that \( \mathcal{H} = \tilde{\mathcal{H}}. \)
- \( \gamma \subset \mathcal{H}. \) See the next remark.

**Remark 2.1.** The condition that \( \gamma \subset \mathcal{H} \) (and not just that \( \gamma \subset \tilde{\mathcal{H}}, \) which is always true by continuity) automatically holds if the domain \( \mathcal{U} \subset \mathbb{R}^3 \) where \( \varphi \) is defined.
and positive has a simple geometry. Indeed, observe that $z \in C^2(\Omega)$ is a locally strictly convex graph, as well as a continuous convex graph on the convex planar set $\Omega \cup \{(0,0)\}$. It is then easy to see from the theory of convex sets that given a Jordan curve $\Gamma \subset \Omega$, then $\hat{\Gamma} = (z_x, z_y)|_{\Gamma}$ is a Jordan curve in $\mathbb{R}^2$ with the property that if $(x, y) \in \Omega$ is in the interior of the bounded domain determined by $\Gamma$, then $(z_x(x, y), z_y(x, y))$ is contained in the interior of the bounded domain determined by $\hat{\Gamma}$. Hence, it is clear that $\gamma \subset \mathcal{H}$ if, for instance, $\mathcal{H}$ is simply connected and $\mathcal{U} \subset \mathbb{R}^3 \times \mathcal{H}$.

In the above conditions, the expression
\begin{equation}
(2.1) \quad ds^2 = z_{xx} dx^2 + 2z_{xy} dx dy + z_{yy} dy^2
\end{equation}
is a Riemannian metric on $\Omega$. It is a well-known fact that $ds^2$ admits conformal coordinates $w := u + iv$ such that
\begin{equation}
(2.2) \quad ds^2 = \frac{\sqrt{\psi}}{u_x v_y - u_y v_x} |dw|^2.
\end{equation}
That is, there exists a $C^2$-diffeomorphism
\begin{equation}
(2.3) \quad \Phi : \Omega \to \Lambda := \Phi(\Omega) \subset \mathbb{R}^2, \quad (x, y) \mapsto \Phi(x, y) = (u(x, y), v(x, y)),
\end{equation}
satisfying
\begin{equation}
(2.4) \quad x_u y_v - x_v y_u > 0,
\end{equation}
and the Beltrami system
\begin{equation}
(2.5) \quad \begin{pmatrix} v_x \\ v_y \end{pmatrix} = \frac{1}{\sqrt{\psi}} \begin{pmatrix} z_{xy} & -z_{xx} \\ z_{yy} & -z_{xy} \end{pmatrix} \begin{pmatrix} u_x \\ u_y \end{pmatrix}.
\end{equation}

Here, $\Lambda$ is a domain in $\mathbb{R}^2 \equiv \mathbb{C}$ that is conformally equivalent to either the punctured disc $\mathbb{D}^*$ or an annulus $\mathbb{A}_\rho = \{ \xi \in \mathbb{C} : 1 < |\xi| < \rho \}$. As $z$ does not extend smoothly across the origin, by [20], lemma 3.3) we have:

**Lemma 2.2.** $\Lambda$ is conformally equivalent to some annulus $\mathbb{A}_\rho$.

Thus, in order to study solutions of (1.1) with an isolated singularity, we may assume $\Lambda = \mathbb{A}_\rho$. If we denote $\Sigma_R := \{ w : 0 < \text{Im}(w) < R \}$, then $\mathbb{A}_\rho$ is conformally equivalent to $\Gamma_R := \Sigma_R/(2\pi \mathbb{Z})$ for $R = \log \rho$. So, composing with this conformal equivalence, we will suppose that the map $\Phi$ in (2.3) is a diffeomorphism from $\Omega$ into $\Gamma_R$; in particular, $\Phi$ is $2\pi$-periodic and $(u, v)$ will denote the canonical coordinates of the strip $\Sigma_R$.

Let $G = \{ (x, y, z(x, y)) : (x, y) \in \Omega \} \subset \mathbb{R}^3$ be the graph of $z(x, y)$. By using the parameters $(u, v)$, we may parametrize $G$ as a map
\begin{equation}
(2.6) \quad \psi(u, v) = (x(u, v), y(u, v), z(u, v)) : \Gamma_R \to G \subset \mathbb{R}^3
\end{equation}
such that $\psi$ extends continuously to $\mathbb{R}$ with $\psi(u, 0) = (0, 0, 0)$.
In this section we prove the following result about the asymptotic behavior, parametrization, and uniqueness in terms of the limit gradient of solutions to (1.1) at an isolated singularity.

**Theorem 2.3.** In the previous conditions, assume that \( \varphi \in C^k(U), k \geq 4 \) (respectively, \( \varphi \in C^u(U) \)). Then:

1. \( \gamma \) is a regular, strictly convex Jordan curve in \( \mathbb{R}^2 \), which is \( C^{k,\alpha} \forall \alpha \in (0,1) \) (respectively, analytic).
2. If \((u,v)\) denote conformal coordinates on \( \Sigma_R \) for the metric \( ds^2 \) as explained previously, and \( p = z_x, q = z_y \) are viewed as functions of \((u,v)\), then those functions extend \( C^{k,\alpha}\)-smoothly (respectively, analytically) to \( \Sigma_R \cup \mathbb{R} \) and \( \gamma(u) := (p(u,0), q(u,0)) \) is a \( C^{k,\alpha} \) (respectively, analytic), \( 2\pi \)-periodic, negatively oriented parametrization of \( \gamma \) such that \( \gamma'(u) \neq (0,0) \) for all \( u \in \mathbb{R} \).
3. Let \( \varphi \in C^u(U) \), and consider \( z, z' \in C^u(\Omega) \) to be two solutions to (1.1) with an isolated singularity at \((0,0)\) and \( z(0,0) = z'(0,0) = 0 \), with the same limit gradient \( \gamma \subset H \) at the origin, and such that both \( z_{xx} \) and \( z'_{xx} \) are positive. Then the graphs of \( z \) and \( z' \) agree on an open set containing the origin.

**Proof.** From now on, we will consider all the functions depending on the parameters \((u,v)\) via \((x,y) = \Phi^{-1}(u,v)\). For simplicity, we keep the same notation. From system (2.5) (see, for example, [3]) we have the following equations:

\[
(p_u, p_v, q_u, q_v) = \sqrt{\varphi}(y_v, -y_u, -x_v, x_u).
\]

Moreover, we have that

\[
z_v = px_v + qy_v = \frac{1}{\sqrt{\varphi}}(qp_u - pq_u).
\]

Therefore,

\[
z(u,v) := (x(u,v), y(u,v), z(u,v), p(u,v), q(u,v)) : \Gamma_R \to \mathbb{R}^5
\]

is a solution to system

\[
\begin{pmatrix}
  x \\
  y \\
  z \\
  p \\
  q
\end{pmatrix}_v = \begin{pmatrix}
  x \\
  y \\
  z \\
  p \\
  q
\end{pmatrix}_u,
\quad
M = \frac{1}{\sqrt{\varphi}} \begin{pmatrix}
  0 & 0 & 0 & 0 & -1 \\
  0 & 0 & 0 & 1 & 0 \\
  0 & 0 & 0 & q & -p \\
  0 & -\varphi & 0 & 0 & 0 \\
  \varphi & 0 & 0 & 0 & 0
\end{pmatrix}.
\]

The following claim provides a boundary regularity result for \( z(u,v) \):

**Claim 2.4.** In the above conditions, if \( \varphi \in C^k(U) \) (respectively, \( \varphi \in C^u(U) \)), then \( z(u,v) \) extends as a \( C^{k,\alpha} \) map \( \forall \alpha \in (0,1) \) (respectively, as a real analytic map) to \( \Gamma_R \cup \mathbb{R} \).
Proof of Claim. The first part of the proof follows a bootstrapping method. Consider an arbitrary point of \( \mathbb{R} \), which we will suppose without loss of generality to be the origin. Also, consider for \( 0 < \delta < R \) the domain \( \mathbb{D}^+ = \{ (u,v) : 0 < \|u^2 + v^2\| < \delta^2 \} \cap \Gamma_R \).

From (2.7) it follows that (cf. [20])

\[
\begin{align*}
\Delta x &= h_1(x_u^2 + x_v^2) + h_2(x_u y_u + x_v y_v) + h_3(x_u y_v - x_v y_u), \\
\Delta y &= h_1(x_u y_u + x_v y_v) + h_2(y_u^2 + y_v^2) + h_4(x_u y_v - x_v y_u),
\end{align*}
\]

where the coefficients \( h_1 = h_1(x,y,z,p,q), \ldots, h_4 = h_4(x,y,z,p,q) \) are

\[
\begin{align*}
h_1 &= -\frac{1}{2\varphi}(\varphi_x + \varphi_{z,p}), & h_2 &= -\frac{1}{2\varphi}(\varphi_y + \varphi_{z,q}), \\
h_3 &= \frac{1}{\sqrt{\varphi}}(-\frac{1}{2}\varphi_p), & h_4 &= \frac{1}{\sqrt{\varphi}}(-\frac{1}{2}\varphi_q),
\end{align*}
\]

all evaluated at \((u,v)\). On the other hand, observe that the inequalities

\[
(x_u + y_v)^2 + (x_v - y_u)^2 \geq 0, \quad (x_u - y_u)^2 + (x_v - y_v)^2 \geq 0,
\]

lead, respectively, to \( x_u y_v - x_v y_u \leq \frac{1}{2}(|\nabla x|^2 + |\nabla y|^2) \) and \( x_u y_u + x_v y_v \leq \frac{1}{2}(|\nabla x|^2 + |\nabla y|^2) \).

Hence, if we denote \( Y = (x,y) : \mathbb{D}^+ \rightarrow \Omega \), formula (2.9) and the fact that \( h_1, \ldots, h_4 \) are bounded (since \( \gamma \subset \mathcal{H} \)) yield

\[
|\Delta Y| \leq c(|\nabla x|^2 + |\nabla y|^2)
\]

for a certain constant \( c > 0 \).

Observe that \( Y \in C^2(\mathbb{D}^+) \cap C^0(\overline{\mathbb{D}^+}) \) with \( Y(u,0) = (0,0) \) for all \( u \). Hence, we can apply Heinz’s theorem in [19] to deduce that \( Y \in C^{1,\alpha}(\overline{\mathbb{D}^+_\epsilon}) \) for all \( \alpha \in (0,1) \), where \( \mathbb{D}^+_\epsilon = \mathbb{D}^+ \cap B(0,\epsilon) \) for a certain \( 0 < \epsilon < \delta \).

Now, the right-hand side terms in (2.7) are bounded in \( \overline{\mathbb{D}^+}_\epsilon \) and so \( p,q \in W^{1,\infty}(\overline{\mathbb{D}^+_\epsilon}) \). Hence \( p,q \in C^{0,1}(\overline{\mathbb{D}^+_\epsilon}) \) (cf. [14] p. 154).

Taking into account

\[
z_u = px_u + qy_u, \quad z_v = px_v + qy_v,
\]

we obtain \( z \in C^{1,\alpha}(\overline{\mathbb{D}^+_\epsilon}) \) \( \forall \alpha \in (0,1) \). Then, the right-hand side functions in (2.7) are Hölder continuous of any order in \( \overline{\mathbb{D}^+_\epsilon} \). That is, \( p,q \in C^{1,\alpha}(\overline{\mathbb{D}^+_\epsilon}) \) \( \forall \alpha \in (0,1) \).

With this, we have from (2.9) that \( \Delta Y \) is Hölder-continuous in \( \mathbb{D}^+_\epsilon \). Then, a standard potential analysis argument (cf. [14] lemma 4.10) ensures that \( x,y \in C^{2,\alpha}(\overline{\mathbb{D}^+_\epsilon/2}) \). Again, by formula (2.7) we have that \( p,q \in C^{2,\alpha}(\overline{\mathbb{D}^+_\epsilon/2}) \) and so, from (2.12), that \( z \in C^{2,\alpha}(\overline{\mathbb{D}^+_\epsilon/2}) \).

At this point we may apply the same argument to \( Y_u \) and \( Y_v \) in order to obtain that \( x,y,z,p,q \in C^{3,\alpha}(\overline{\mathbb{D}^+_\epsilon/4}) \). A recursive process leads to the fact that \( z =
(x, y, z, p, q) is $C^{k, \alpha}$ for all $\alpha \in (0, 1)$ (respectively, $C^\infty$) at the origin. As we can do the same argument for all points of $\mathbb{R}$ and not just the origin, we conclude that $z(u, v) \in C^{k, \alpha}(\Gamma_R \cup \mathbb{R})$ (respectively, $z(u, v) \in C^\infty(\Gamma_R \cup \mathbb{R})$).

Finally, suppose that $\gamma \in C^\infty(\Gamma_R \cup \mathbb{R})$. From (2.7),

$$\Delta p = (\sqrt{q \circ z} z_y y_v - (\sqrt{q \circ z}) z_z y_u, \Delta q = -(\sqrt{q \circ z}) z_x x_v + (\sqrt{q \circ z}) z_v x_u, \Delta z = p u x^2 + p v y^2 + q u y u + q v y v + p \Delta x + q \Delta y.$$ (2.13)

Therefore, $z(u, v)$ satisfies

$$\Delta z = h(z, z_u, z_v)$$ (2.14)

where $h : O \subset \mathbb{R}^{15} \to \mathbb{R}^5$ is a real analytic function on an open set $O$ of $\mathbb{R}^{15}$ containing the closure of the bounded set $\{ (z, z_u, z_v) : (u, v) \in \Gamma_R \}$.

Moreover, if we write

$$z(u, v) = (\psi(u, v), \phi(u, v)) : \Gamma_R \to \mathbb{R}^3 \times \mathbb{R}^2 \equiv \mathbb{R}^5$$

where $\psi(u, v)$ is given by (2.6) and $\phi(u, v) = (p(u, v), q(u, v))$, then we see that $z(u, v)$ is a solution to (2.14) that meets the mixed initial conditions

$$\begin{cases} 
\psi(u, 0) = (0, 0, 0), \\
\phi(u, 0) = (0, 0).
\end{cases}$$

As $z \in C^\infty(\Gamma_R \cup \mathbb{R})$ by the previous bootstrapping argument, we have the conditions to apply theorem 3 in [29] to $z$ around every point in $\mathbb{R}$. Thus, we deduce that $z$ is real analytic in $\mathbb{R}^2$, which concludes the proof of Claim 2.4.

It follows from Claim 2.4 that the functions $p(u, v)$ and $q(u, v)$ extend $C^{k, \alpha}$ smoothly for all $\alpha \in (0, 1)$ (respectively, analytically) to $\Gamma_R \cup \mathbb{R}$, so that

$$(\alpha(u), \beta(u)) : (p(u, 0), q(u, 0))$$
is a $2\pi$-periodic map. Let now $\gamma \subset \mathbb{R}^2$ denote the limit gradient of $z(x, y)$. Then, clearly $\gamma = \{ (\alpha(u), \beta(u)) : u \in \mathbb{R} \}$, and so we get that $\gamma$ is a closed curve in $\mathbb{R}^2$, possibly with singularities, that can be parametrized as a $2\pi$-periodic $C^{k, \alpha}$ (respectively, analytic) function as $\gamma(u) = (\alpha(u), \beta(u))$ in terms of the conformal parameters $(u, v)$ associated to the solution $z(x, y)$.

**Claim 2.5.** $\gamma'(u) \neq (0, 0)$ for every $u \in \mathbb{R}$.

**Proof.** We start by proving that $\gamma'(u)$ can vanish at only at most two points in $[0, 2\pi)$. Indeed, assume by contradiction that $\gamma'(u_1) = \gamma'(u_2) = \gamma'(u_3) = 0$ for three distinct values $u_1, u_2, u_3 \in [0, 2\pi)$. Since $x(u, 0) = 0$ for every $u \in \mathbb{R}$, by (2.7) we see that $x(u_i, 0) = 0$, $i = 1, 2, 3$. Noting then that $x(u, v)$ satisfies the elliptic PDE (2.9) and that the zero function is another solution to the same PDE, we deduce by theorem $\dag$ in [18] that $x_{uv} x_{uv} - x_{uv}^2 < 0$ in a punctured neighborhood of each $(u_i, 0)$ in the $uv$-plane. In other words, the
axis \( v = 0 \) is a nodal curve of \( x(u, v) \) that is crossed at \( u_1, u_2, u_3 \) by three other nodal curves \( \delta_1, \delta_2, \delta_3 \) at a positive angle.

Next, observe that the map \((2.6)\) is a diffeomorphism from \( \Gamma_R := \Sigma_R / (2\pi \mathbb{Z}) \) into the graph \( G = \{(x, y, z(x, y)) : (x, y) \in \Omega \} \). As \( G \) is a graph, \( G \cap \{x = 0\} \subset \mathbb{R}^3 \) is formed by exactly two regular curves with an endpoint at the origin. Thus, there cannot exist three nodal curves of \( x(u, v) \) in \( \Gamma_R \). This contradiction shows that \( \gamma'(u) \) vanishes at two points in \([0, 2\pi)\) at most.

As a consequence, as \( \gamma(\mathbb{R}) \) is convex, we can choose \( u_1, u_2 \in [0, 2\pi) \) with \( \gamma'(u_i) \neq (0, 0) \) for \( i = 1, 2 \) and such that the respective support lines to \( \gamma \) passing through \( \gamma(u_1) \) and \( \gamma(u_2) \) are both tangent to a certain direction \( v_\theta = (\cos \theta, \sin \theta) \in S^1 \). In particular, \( -\sin \theta \alpha'(u_i) + \cos \theta \beta'(u_i) = 0 \) holds for \( i = 1, 2 \). Using \((2.7)\) and the fact that \( x(u, 0) = y(u, 0) = 0 \) for every \( u \in \mathbb{R} \), we deduce that \( x_\theta(u, v) := \cos \theta x(u, v) + \sin \theta y(u, v) \) satisfies \( x_\theta(u, 0) = 0 \) for every \( u \in \mathbb{R} \), and \( D x_\theta(u_1, 0) = D x_\theta(u_2, 0) = (0, 0) \). Also, a computation using \((2.9)\) shows that, if we denote \( y_\theta(u, v) := -\sin \theta x(u, v) + \cos \theta y(u, v) \), then \( x_\theta(u, v) \) satisfies the elliptic PDE

\[
\Delta x_\theta = H_1((x_\theta)_u^2 + (x_\theta)_v^2) + H_2((x_\theta)_u (y_\theta)_u + (x_\theta)_v (y_\theta)_v) \\
+ H_3((x_\theta)_u (y_\theta)_v - (x_\theta)_v (y_\theta)_u)
\]

where the coefficients \( H_i := H_i(u, v) \in C^{k-1}(\Gamma_R \cup \mathbb{R}) \) are given in terms of the functions in \((2.10)\) by

\[
\begin{align*}
H_1 &= (h_1 \circ z) \cos \theta + (h_2 \circ z) \sin \theta, \\
H_2 &= (h_2 \circ z) \cos \theta - (h_1 \circ z) \sin \theta, \\
H_3 &= (h_3 \circ z) \cos \theta + (h_4 \circ z) \sin \theta.
\end{align*}
\]

As the zero function is also a solution of this PDE, we can deduce again by theorem \( \dagger \) in \([18] \) that \( x_\theta(u, v) \) has two nodal curves \( \gamma_1, \gamma_2 \) that intersect at a positive angle the nodal curve \( v = 0 \) at the points \( (u_1, 0) \) and \( (u_2, 0) \). Geometrically, the restriction of these two nodal curves \( \gamma_1, \gamma_2 \) to \( \Gamma_R \) corresponds (as explained above for the case \( \theta = 0 \)) to the intersection of the graph \( G \) with the plane \( \cos \theta x + \sin \theta y = 0 \) in \( \mathbb{R}^3 \). In particular, the axis \( v = 0 \) cannot be crossed by any other nodal curve of \( x_\theta(u, v) \).

Finally, note that if \( \gamma'(\xi) = (0, 0) \) for some \( \xi \in [0, 2\pi) \), then by \((2.7)\) we would have \( x_\theta(\xi) = 0 \) and \( D x_\theta(\xi) = (0, 0) \). Therefore, there would exist a nodal curve of \( x_\theta(u, v) \) crossing the \( v = 0 \) axis at \( \xi \). Thus, \( \xi = u_1 \) or \( \xi = u_2 \), which is a contradiction since we initially chose \( u_1, u_2 \) to be regular points of \( \gamma \). Thus, \( \gamma'(u) \neq (0, 0) \) for every \( u \in \mathbb{R} \), which proves Claim \( 2.5 \). \( \square \)

**Claim 2.6.** The regular curve \( \gamma(u) \) is strictly locally convex and negatively oriented; i.e., it holds \( \alpha''(u) \beta'(u) - \alpha'(u) \beta''(u) > 0 \) for every \( u \in \mathbb{R} \).
Proof. Let \( \psi : \Gamma_R \to \mathbb{R}^3 \) be the conformal parametrization of the graph \( z = z(x, y) \) given in (2.6). So, \( \psi \) is an immersion with unit normal

\[
N(u, v) = \frac{(-p(u, v), -q(u, v), 1)}{\sqrt{1 + p(u, v)^2 + q(u, v)^2}} : \Gamma_R \to S^2.
\]

By Claim 2.4, \( \psi, N \in C^{k, \alpha}(\Gamma_R \cup \mathbb{R}) \), with

\[
N(u, 0) = \frac{(-\alpha(u), -\beta(u), 1)}{\sqrt{1 + \alpha(u)^2 + \beta(u)^2}}
\]

and \( \psi(u, 0) = (0, 0, 0) \). In particular, it follows from Claim 2.5 that \( N(u, 0) \) is a \( 2\pi \)-periodic regular curve in \( S^2 \). Moreover, a simple computation shows that \( N(u, 0) \) has negative geodesic curvature in \( S^2 \) at every point if and only if \( \alpha''(u)\beta'(u) - \alpha'(u)\beta''(u) > 0 \) for every \( u \).

Note that the metric \( ds^2 \) in (2.1) is conformally equivalent to the second fundamental form of the graph \( z = z(x, y) \). Thus, if we write \( w = u + iv \), the first and second fundamental forms of \( z = z(x, y) \) with respect to this parametrization are written as

\[
\begin{align*}
I &= \langle d\psi, d\psi \rangle = Q dw^2 + 2\mu|dw|^2 + \bar{Q}d\bar{w}^2, \\
II &= -\langle d\psi, dN \rangle = 2\rho|dw|^2,
\end{align*}
\]

where \( Q := \langle \psi_w, \psi_w \rangle : \Gamma_R \cup \mathbb{R} \to \mathbb{C} \) (recall that \( \partial_w := (\partial_u - i \partial_v)/2 \) and \( \mu, \rho : \Gamma_R \cup \mathbb{R} \to (0, \infty) \) are positive real functions. By Claim 2.4 \( Q, \mu, \rho \) are \( C^{k-1, \alpha} \)-smooth in \( \Gamma_R \cup \mathbb{R} \).

Also, note that by (1.1) the Gaussian curvature \( K \) of \( z = z(x, y) \) is

\[
K = \frac{zz_{xx}z_{yy} - z_{xy}^2}{(1 + z_x^2 + z_y^2)^2} = \frac{\varphi(x, y, z, z_x, z_y)}{(1 + z_x^2 + z_y^2)^2},
\]

that is, in terms of the conformal parameters \((u, v)\) we have

\[
K(u, v) = \frac{\varphi \circ z(u, v)}{(1 + p(u, v)^2 + q(u, v)^2)^2} \in C^k(\Gamma_R \cup \mathbb{R}),
\]

and so \( K(u, v) > 0 \) for all \((u, v) \in \Gamma_R \cup \mathbb{R}\) since \( \gamma \subset \mathcal{H} \). A direct computation using (2.16) shows that, in \( \Gamma_R \cup \mathbb{R} \),

\[
N \times N_u = -\sqrt{K}\psi_v, \quad N \times N_v = \sqrt{K}\psi_u,
\]

where \( \times \) denotes the cross product in \( \mathbb{R}^3 \). From here,

\[
Q(u, 0) = \frac{1}{4} \left( \langle \psi_u, \psi_u \rangle - \langle \psi_v, \psi_v \rangle - 2i \langle \psi_u, \psi_v \rangle \right)(u, 0) = \frac{-1}{4K} (N \times N_u, N \times N_u)(u, 0)
\]

(2.19)
In particular, since \( (N_u, N_u)(u, 0) > 0 \) for every \( u \) as we explained above, we may assume by choosing a smaller \( R > 0 \) if necessary that \( Q \) does not vanish on \( \Gamma_R \cup \mathbb{R} \). Using now that \( K = \det(II)/\det(I) \) on \( \Gamma_R \) and the previous boundary regularity, we get from (2.17) that
\[
\rho^2 = K(\mu^2 - |Q|^2) \quad \text{in} \quad \Gamma_R \cup \mathbb{R}.
\]
Since \( Q \neq 0 \), (2.20) implies the existence of a function \( \omega \in C^{k-1}(\Gamma_R \cup \mathbb{R}) \) such that \( \mu = |Q| \cosh \omega \) and \( \rho = \sqrt{K}|Q| \sinh \omega \). Note that \( \omega > 0 \) on \( \Gamma_R \) and \( \omega(u, 0) = 0 \) for every \( u \in \mathbb{R} \). In particular, we can rewrite (2.17) as
\[
\omega w \overline{w} + Uw - Vw + K|Q| \sinh \omega = 0,
\]
where
\[
U = \frac{-Kw \overline{Q}}{2K|Q|} \sinh \omega, \quad V = \frac{Kw \overline{Q}}{2K|Q|} \sinh \omega.
\]
In terms of the parameters \((u, v)\), (2.22) is a quasilinear elliptic PDE for \( \omega \) of the type
\[
\Delta \omega + a_1 \omega_u \cosh \omega + a_2 \omega_v \cosh \omega + a_3 \sinh \omega = 0,
\]
where \( a_i = a_i(u, v) \in C^{k-2}(\Gamma_R \cup \mathbb{R}) \). Observe that \( \omega = 0 \) is a trivial solution to (2.23).

Moreover, if we denote \( \sigma(u) := N(u, 0) : \mathbb{R}/(2\pi \mathbb{Z}) \to S^2 \), we have, using (2.18) and (2.19),
\[
\langle \sigma'', \sigma \times \sigma' \rangle = \langle N'_{uu}, N \times N_u \rangle(u, 0) = \sqrt{K}(N \times \psi_{uv}, N \times N_u)(u, 0)
\]
\[
= \sqrt{K}(\psi_{uv}, N_u)(u, 0) = \sqrt{K}\left( \frac{\partial}{\partial v}(\psi_u, N_u) - \langle \psi_u, N_{uv} \rangle \right)(u, 0)
\]
\[
= \sqrt{K}\frac{\partial}{\partial v}(\langle \psi_u, N_u \rangle)(u, 0) = -2K|Q|\omega_v \cosh \omega(u, 0)
\]
\[
= \frac{1}{2}\langle \sigma', \sigma' \rangle \omega_v(u, 0).
\]
Therefore,
\[
\omega_v(u, 0) = -\frac{2\langle \sigma''(u), \sigma(u) \times \sigma'(u) \rangle}{\langle \sigma'(u), \sigma'(u) \rangle} = -2\|\sigma'(u)\|\kappa_\sigma(u).
\]
where $\kappa_\sigma$ denotes the geodesic curvature of $\sigma$ in $S^2$. Let us recall at this point that the real axis is a nodal curve of $\omega$. Since $\omega$ is a solution to the elliptic PDE (2.23), by theorem $\dagger$ in [18] we deduce that, at the points $(u,0)$ where $\omega_v(u,0) = 0$ there exists at least one nodal curve of $\omega$ that crosses the real axis at a definite angle. But this situation is impossible, since $\omega > 0$ in $\Gamma_R$. Therefore we see that $\omega_v(u,0) > 0$ for every $u$. Consequently, the geodesic curvature of $N(u,0)$ in $S^2$ is strictly negative. As explained previously, this condition implies that $\alpha''(u)\beta'(u) - \alpha'(u)\beta''(u) > 0$ for every $u \in \mathbb{R}$. This proves Claim 2.6.

We observe that these three claims together with the paragraph above Claim 2.5 prove the first two items in Theorem 2.3.

In order to prove item (3) of Theorem 2.3, assume that $\varphi$ (and hence any solution to (1.1)) is analytic. Observe that the map $z(u,v)$ can be recovered in terms of an analytic, $2\pi$-periodic curve $\gamma(u) = (\alpha(u),\beta(u))$ as the unique solution to the Cauchy problem for the system (2.8) with the initial condition
\begin{equation}
(2.24) \quad z(u,0) = (0,0,0,\alpha(u),\beta(u)).
\end{equation}

Also, observe that the parameters $(u,v) \in \Gamma_R$ associated to the solution $\gamma$ of (1.1) are defined up to $2\pi$-periodic conformal changes of $\Gamma_R$ that simply yield regular, analytic reparametrizations of the limit gradient $\gamma$.

Taking this into account, we deduce by the uniqueness of the solution to the Cauchy problem for system (2.8) that if two solutions $\gamma,\gamma'\in C^\omega(\mathcal{U})$ satisfy: (i) $z_{xx} > 0$, $z'_{xx} > 0$, (ii) $z$ and $z'$ have an isolated singularity at the origin, and (iii) both $z$ and $z'$ have the same limit gradient $\gamma \subset \mathcal{H}$ at the singularity, then their graphs agree on a neighborhood of the origin. This finishes the proof of Theorem 2.3.

\section{Existence of Isolated Singularities and the Proof of Theorem 1.1}

In this section we consider the general elliptic equation of Monge-Ampère type in dimension 2, i.e., the fully nonlinear PDE (1.2). Note that (1.2) can be rewritten as
\begin{equation}
(3.1) \quad A z_{xx} + 2 B z_{xy} + C z_{yy} + z_{xx} z_{yy} - z_{xy}^2 = E,
\end{equation}
where $A = A(x,y,z,z_x,z_y), \ldots, E = E(x,y,z,z_x,z_y)$ are defined on an open set $\mathcal{U} \subset \mathbb{R}^5$ and satisfy on $\mathcal{U}$ the ellipticity condition
\begin{equation}
(3.2) \quad \mathcal{D} := AC - B^2 + E > 0.
\end{equation}

We can also rewrite (3.1) as $(A + z_{yy})(C + z_{xx}) - (B - z_{xy})^2 = \mathcal{D} > 0$, from where we see that $C + z_{xx}$ is never 0.

The next theorem provides a general existence result for solutions to (1.2) with an isolated singularity at the origin and a prescribed limit gradient at the singularity. We recall the definitions of $\Gamma_R$ and $\Sigma_R$ in Section 2 and denote $\Gamma_{\mathcal{R}} := \Sigma_R/(2\pi \mathbb{Z})$ where $\Sigma_R = \{ w \in \mathbb{C} : -R < \mathrm{Im}(w) < R \}$. 

\begin{theorem}
\end{theorem}

\begin{proof}
\end{proof}
**Theorem 3.1.** Assume that the coefficients $A,\ldots,E$ are real analytic in $U$. Let

$$\gamma(u) = (\alpha(u),\beta(u))$$

be a real analytic, $2\pi$-periodic curve such that $(0, 0, 0, \gamma(\mathbb{R})) \subset U$.

Then, there exists a real analytic map $\psi : \Gamma_R \to \mathbb{R}^3$ such that:

1. $\psi(u, 0) = (0, 0, 0)$ for every $u \in \mathbb{R}$.
2. There exists a real analytic map $(p, q) : \Gamma_R \to \mathbb{R}^2$ such that $(p, q)(u, 0) = \gamma(u)$ for every $u \in \mathbb{R}$ and $(\psi, p, q)(\Gamma_R) \subset U$. Moreover, the map $N(u, v) : \Gamma_R \to \mathbb{S}^2$ defined by

$$N(u, v) = \frac{(-p, -q, 1)}{\sqrt{1 + p^2 + q^2}}(u, v)$$

satisfies that $\langle \psi_u, N \rangle = \langle \psi_v, N \rangle = 0$ in $\Gamma_R$.
3. Assume that the map $(x(u, v), y(u, v))$ is an orientation-preserving local diffeomorphism at some point $(u_0, v_0) \in \Gamma_R$. Then the image of $\psi$ around that point is the graph $G \subset \mathbb{R}^3$ of some real analytic solution $z = z(x, y)$ to (3.1) for the coefficients $A,\ldots,E$ such that $C + z_{xx} > 0$.
4. If $\gamma(u)$ is a regular, negatively oriented, strictly convex parametrized Jordan curve (so both $-\|\gamma'(u)\|$ and the curvature of $\gamma(u)$ are strictly negative for every $u$); then for $R > 0$ small enough, $\psi(\Gamma_R)$ is the graph of a solution $z$ to (3.1) for the coefficients $A,\ldots,E$ that is defined on a punctured neighborhood around the origin and has an isolated singularity at the puncture. Moreover, the limit gradient of this solution is the curve $\gamma = \gamma(\mathbb{R})$ and $C + z_{xx} > 0$.

**Remark 3.2.** The first three items of Theorem 3.1 prove that, if we start from a $2\pi$-periodic, real analytic curve $\gamma(u)$ in $\mathbb{R}^2$, we can construct from $\gamma(u)$ a multivalued solution to (3.1) with a singularity at the origin. Here by a “multivalued solution” we mean a surface such that whenever it is transverse to the vertical direction around one point, it is a local solution to (3.1) around this point. If $\gamma(u)$ is regular and strictly locally convex but nonembedded, the singularity at the origin of the corresponding multivalued solution is isolated. See [8] for a different study of multivalued solutions to Monge-Ampère equations.

We also observe that Theorem 3.1 implies the following corollary.

**Corollary 3.3.** Assume that the coefficients $A,\ldots,E$ are real analytic in $U \subset \mathbb{R}^5$. Let $\gamma \subset \mathbb{R}^2$ be a real analytic, regular, strictly convex Jordan curve such that $(0, 0, 0, \gamma(\mathbb{R})) \subset U$. Then there exists a solution $z$ to (3.1) for these coefficients that has an isolated singularity at the origin and whose limit gradient at the singularity is $\gamma$.

Before proving Theorem 3.1 let us make some comments about solutions to the general equation (3.1). Let $z$ be a solution to (3.1) on some domain $W \subset \mathbb{R}^2$, where $A,\ldots,E$ are of class $C^2$, so that $z$ is of class $C^{3,\alpha}$ on compact sets of $W$. 

Assume $z_{xx} + C > 0$. By the ellipticity condition (3.2),
\[
\begin{aligned}
\frac{d^2 s^2}{d\alpha^2} = \frac{d^2 s^2}{d\alpha^2} + 2(z_{xy} - B) \frac{d\alpha}{d\alpha} dy + (z_{yy} + A) dy^2
\end{aligned}
\]
is a Riemannian metric on $W$. Then, $(W, ds^2)$ admits in a neighborhood of each point of $W$ conformal parameters $w := u + iv$ of class $C^2$ such that (cf. [20])
\[
\begin{aligned}
ds^2 = \frac{\sqrt{D}}{u_xv_y - u_yv_x} |dw|^2,
\end{aligned}
\]
where $(x, y)$ satisfy $x_u y_v - x_v y_u > 0$. From [3] we have the equations
\[
\begin{aligned}
p_u &= \sqrt{D} y_u + B y_u - C x_u, \\
p_v &= -\sqrt{D} y_v + B y_v - C x_v, \\
q_u &= -\sqrt{D} x_u + B x_u - A y_u, \\
q_v &= \sqrt{D} x_v + B x_v - A y_v,
\end{aligned}
\]
from where, since
\[
\begin{aligned}
z_v &= p x_u + q y_u \\
&= -\frac{p}{\sqrt{D}} (q_u - B x_u + A y_u) + \frac{q}{\sqrt{D}} (p_u - B y_u + C x_u) \\
&= \frac{1}{\sqrt{D}} (x_u (Bp + Cq) - y_u (Ap + Bq) + q p_u - p q_u),
\end{aligned}
\]
we arrive at the following system which generalizes (2.8):
\[
\begin{aligned}
\begin{pmatrix}
x \\
y \\
z \\
p \\
q
\end{pmatrix}
= \tilde{M}
\begin{pmatrix}
x \\
y \\
z \\
p \\
q
\end{pmatrix},
\end{aligned}
\]
where (3.6)
\[
\begin{aligned}
\tilde{M} = \frac{1}{\sqrt{D}}
\begin{pmatrix}
B & -A & 0 & 0 & -1 \\
C & -B & 0 & 1 & 0 \\
0 & -E & 0 & B & C \\
0 & 0 & -A & -B
\end{pmatrix}.
\end{aligned}
\]

**Proof of Theorem 3.1.** Let $\gamma(u) = (\alpha(u), \beta(u))$ be a real analytic, $2\pi$-periodic curve in $\mathbb{R}^2$, and assume that $A, \ldots, E$ are real analytic functions on an open set $U \subset \mathbb{R}^5$ that contains $(0, 0, 0, y(\mathbb{R}))$ and that satisfy the ellipticity condition (3.2). Let us consider the $2\pi$-periodic initial data $(0, 0, 0, \alpha(u), \beta(u))$ along the axis $v = 0$ in the $uv$-plane for the system (3.6). By the Cauchy-Kowalevsky theorem, there exists a unique real analytic solution $(x, y, z, p, q)$ to (3.6), defined on a neighborhood $\Sigma_R = \{(u, v) : -R < v < R\}$ of the axis $v = 0$, such that
\[
\begin{aligned}
(x, y, z, p, q)(u, 0) &= (0, 0, 0, \alpha(u), \beta(u)).
\end{aligned}
\]
Observe that $\Psi := (x, y, z, p, q) : \Sigma_R \to \mathbb{R}^5$ is $2\pi$-periodic with respect to $u$; i.e., it is well-defined on the quotient $\Gamma_R := \Sigma_R / (2\pi \mathbb{Z})$.

A computation from (3.6) proves the relation
\[
\begin{aligned}
p_v x_u + q_v y_u = p_u x_v + q_u y_v,
\end{aligned}
\]
which is the integrability condition needed for the existence of some smooth function $z_0$ on $\Sigma_R$, unique up to an additive constant, such that
\[
\begin{aligned}
(z_0)_u = p x_u + q y_u, \\
(z_0)_v = p x_v + q y_v.
\end{aligned}
\]
It follows from (3.6) that \((z_0)_v = z_v\) and so \(z(u, v) = z_0(u, v) + f(u)\) for some \(f \in C^\omega(\Gamma_R \cup \mathbb{R})\). Also, observe that (3.7) implies that \(z(u, 0) \equiv 0\) and \((z_0)_u(u, 0) \equiv 0\). Thus, \(f(u)\) must be constant, and as \(z_0\) was defined up to additive constants, we may assume that \(z(u, v) = z_0(u, v)\). In particular, it holds that

\[
(3.9) \quad z_u = p x_u + q y_u, \quad z_v = p x_v + q y_v.
\]

Defining now

\[
(3.10) \quad \psi(u, v) := (x(u, v), y(u, v), z(u, v)) : \Gamma_R \to \mathbb{R}^3
\]

and

\[
(3.11) \quad N(u, v) := \frac{(-p(u, v), -q(u, v), 1)}{\sqrt{1 + p(u, v)^2 + q(u, v)^2}} : \Gamma_R \to S^2,
\]

we see that the first two items of Theorem 3.1 hold.

To prove item (3), suppose now that the map \((x(u, v), y(u, v))\) is an orientation-preserving local diffeomorphism at some point \((u_0, v_0) \in \Sigma_R\), i.e., the condition

\[
(3.12) \quad J := x_u y_v - x_v y_u > 0
\]

holds at this point. Thus, around \((u_0, v_0)\) the image of the map \(\psi(u, v)\) is the graph \(G\) in \(\mathbb{R}^3\) of a real analytic function \(z = z(x, y)\), and from formula (3.9) the relations \(z_x = p\) and \(z_y = q\) hold. We prove next that \(z(x, y)\) is a solution to (3.1) for the coefficients \(A, \ldots, E\) we started with.

If we define \(r := z_{xx}\), \(s := z_{xy}\), and \(t := z_{yy}\), then using (3.5) and working in terms of the \((u, v)\) coordinates we obtain

\[
(3.13) \quad \sqrt{D} y_v = p_u - B y_u + C x_u = (C + r) x_u - (B - s) y_u,
\]

and working similarly,

\[
\sqrt{D} y_u = -(C + r) x_v + (B - s) y_v,
\]

\[
\sqrt{D} x_v = (B - s) x_u - (A + t) y_u,
\]

\[
\sqrt{D} x_u = -(B - s) x_v + (A + t) y_v.
\]

After the change of coordinates \((u, v) \mapsto (x, y)\), these expressions yield

\[
(3.15) \quad u_x = \frac{(C + r) y + (B - s) v}{\sqrt{D}}, \quad v_x = \frac{-(C + r) y - (B - s) u}{\sqrt{D}},
\]

\[
\sqrt{D} y_v = \frac{-(B - s) v - (A + t) y}{\sqrt{D}}, \quad v_y = \frac{(B - s) u + (A + t) u}{\sqrt{D}}.
\]

We deduce then from the second and fourth equation in (3.15) that the system

\[
(3.16) \quad \begin{pmatrix} v_x \\ v_y \end{pmatrix} = \mathfrak{m}_1 \begin{pmatrix} u_x \\ u_y \end{pmatrix}
\]
holds, where

$$\mathfrak{M}_1 = \frac{1}{\sqrt{D}} \begin{pmatrix} -(B - s) & -(C + r) \\ A + t & B - s \end{pmatrix}.$$ 

Similarly, from the first and the third equation in (3.15) we get

(3.17) \[
\begin{pmatrix} u_x \\ u_y \end{pmatrix} = \mathfrak{M}_2 \begin{pmatrix} v_x \\ v_y \end{pmatrix},
\]

where

$$\mathfrak{M}_2 = \frac{1}{\sqrt{D}} \begin{pmatrix} B - s & C + r \\ -(A + t) & -(B - s) \end{pmatrix}.$$ 

Clearly, $\mathfrak{M}_1$ is proportional to $\mathfrak{M}_2^{-1}$, i.e., $\mathfrak{M}_1 \mathfrak{M}_2 = \lambda(x, y) \text{Id}$ for some function $\lambda$. Hence, from (3.16) and (3.17) we obtain $\lambda = 1$, i.e., $\mathfrak{M}_1 \mathfrak{M}_2 = \text{Id}$, and so

$$(A + t)(C + r) - (B - s)^2 = D.$$ 

That is, $z(x, y)$ is a solution to (3.1), as we wanted to show. Besides, a computation from (3.13) and (3.14) shows that

$$C + r = \frac{\sqrt{D}(y_u^2 + y_v^2)}{x_u y_v - x_v y_u},$$

which is positive by (3.12). This completes the proof of item (3).

To prove item (4), assume that $\gamma(u) = (\alpha(u), \beta(u))$ is also regular, embedded, negatively oriented, and strictly convex, i.e., $\alpha''(u) \beta'(u) - \beta''(u) \alpha'(u) > 0$ for every $u$. If we let $J$ be the function in (3.12), then $J(u, 0) = 0$ for every $u$, and a computation from (3.10) at $(u, 0)$ yields

(3.18) \[
J_v(u, 0) = (x_u y_v - x_v y_u)(u, 0) \\
= \left(\frac{-\beta'}{\sqrt{D}}\right) u \frac{\alpha'}{\sqrt{D}} + \frac{\beta'}{\sqrt{D}} \left(\frac{\alpha'}{\sqrt{D}}\right)_u (u, 0) \\
= \frac{1}{D^{3/2}} \left((\beta'' \sqrt{D} - \beta'(\sqrt{D})_u) \alpha' - \beta'(\alpha'' \sqrt{D} - \alpha'(\sqrt{D})_u)\right)(u, 0) \\
= \frac{1}{D} \left(\beta'(u) \alpha''(u) - \beta''(u) \alpha'(u)\right) > 0.
\]

Consequently, since $J$ is $2\pi$-periodic, there is some $R > 0$ such that $J > 0$ on $\Gamma_R$. In particular, the map $\psi(u, v) : \hat{\Gamma}_R \to \mathbb{R}^3$ given by (3.10) satisfies:

1. The projection $(x(u, v), y(u, v)) : \Gamma_R \to \mathbb{R}^2$ is an orientation-preserving local diffeomorphism.
2. $\psi(u, 0) = 0$ for every $u \in \mathbb{R}$.
3. The upward-pointing unit normal $N : \Gamma_R \to S^2_+ \subset \mathbb{R}^3$ of $\psi$ restricted to $\Gamma_R$ extends analytically to $\hat{\Gamma}_R$, and (3.11) holds.
4. For every point $(u_0, v_0) \in \Gamma_R$ there exists some $\delta > 0$ such that the restriction of $\psi$ to the disk of radius $\delta$ centered at $(u_0, v_0)$ is a graph $z = z(x, y)$ that satisfies (3.1) and $C + z_{xx} > 0$. 


We need to prove now that for $R > 0$ small enough, $\psi(\Gamma_R)$ is a graph of a function $z = z(x, y)$ over a punctured disc $\Omega \subset \mathbb{R}^2$.

Since the set of points $(0, 0, 0, y(u))$, $u \in \mathbb{R}$, is a compact set contained in $\mathcal{U}$, and $A, B, C$ are continuous functions in $\mathcal{U}$, there exist sufficiently large real constants $a, c$ such that the inequalities

$$a - A > 0, \quad c - C > 0, \quad (c - C)(a - A) - B^2 > 0,$$

are satisfied in $\Gamma_R$ for a certain positive real number $R' \leq R$. With no loss of generality we will assume $R = R'$.

Using item (4) above we can view $\psi(\Gamma_R)$ locally as a graph $z = z(x, y)$ around any point $(u_0, v_0) \in \Gamma_R$ in such a way that the expression $ds^2$ given by (3.3) defines a Riemannian metric around $(u_0, v_0)$. Hence, we obtain that the matrix

$$\begin{pmatrix} r + c & s \\ s & t + a \end{pmatrix} = \begin{pmatrix} r + C & s - B \\ s - B & t + A \end{pmatrix} + \begin{pmatrix} c - C & B \\ B & a - A \end{pmatrix}$$

is positive definite around $(u_0, v_0)$ because it is the sum of two positive definite matrices. As $(u_0, v_0) \in \Gamma_R$ is arbitrary, this means that the map $\psi^* : \Gamma_R \to \mathbb{R}^3$ given by

$$\psi^*(u, v) = \left(x(u, v), y(u, v), z(u, v) + \frac{c}{2} x(u, v)^2 + \frac{a}{2} y(u, v)^2\right)$$

is a regular, strictly convex surface in $\mathbb{R}^3$ when restricted to $\Gamma_R$ because

$$z_{xx}z_{yy} - z_{xy}^2 = (r + c)(t + a) - s^2 > 0,$$

where $z^* = z + \frac{c}{2} x^2 + \frac{a}{2} y^2$.

Also, $\psi^*(u, 0) = 0$ for every $u$, and the projection of $\psi^*|_{\Gamma_R}$ into $\mathbb{R}^2$ is a local diffeomorphism. The unit normal of $\psi^*$ in $\Gamma_R$ is

$$N^*(u, v) = \frac{1}{\sqrt{1 + (p + cx)^2 + (q + ay)^2}}(-p - cx, -q - ay, 1),$$

where $x, y, p, q$ are evaluated at $(u, v)$. We remark that

$$N^*(u, 0) = N(u, 0) = \frac{(-\alpha(u), -\beta(u), 1)}{\sqrt{1 + \alpha(u)^2 + \beta(u)^2}},$$

which is a regular, strictly convex Jordan curve in the upper hemisphere of $\mathbb{S}^2$.

Consider now the analytic Legendre transform of $\psi^*(u, v)$, given by (see [25, p. 89])

$$\mathcal{L}(u, v) = \left(-\frac{N_1^*}{N_3^*}, -\frac{N_2^*}{N_3^*}, -x\frac{N_1^*}{N_3^*} - y\frac{N_2^*}{N_3^*} - z^*\right) : \widehat{\Gamma_R} \to \mathbb{R}^3,$$

where we are denoting $N^* = (N_1^*, N_2^*, N_3^*)$. It is well-known that, since $\psi^*|_{\Gamma_R}$ is a regular, locally strictly convex surface in $\mathbb{R}^3$ whose projection to the $xy$-plane
is a local diffeomorphism, then so is $\mathcal{L}|_{\Gamma}$. Its upward-pointing unit normal is

$$
(3.20) \quad \mathbf{N}_\mathcal{L} = \frac{(-x, -y, 1)}{\sqrt{1 + x^2 + y^2}} : \Gamma \rightarrow \mathbb{S}^2_+,
$$

where $x$ and $y$ are evaluated at $(u, v)$; hence, $\mathbf{N}_\mathcal{L}$ can be analytically extended to $\Gamma$. Since $\mathcal{L}|_{\Gamma}$ is locally strictly convex with $\mathcal{L}(u, 0)$ lying on the horizontal plane $z = 0$, $\mathbf{N}_\mathcal{L}(u, 0) = (0, 0, 1)$, and $z^*_{xx} = r + c > r + C > 0$, we have that there exists $R' > 0$ small enough such that $\mathcal{L}(\Gamma_{R'})$ lies on the upper half-space of $\mathbb{R}^3$. Now, let us see that the intersection of $\mathcal{L}(\Gamma_{R'})$ with each plane $z = \varepsilon$, for $0 < \varepsilon \leq \varepsilon_0$ small enough, is a regular convex Jordan curve in that plane.

Since $\mathcal{L}(u, 0) = (\alpha(u), \beta(u), 0)$ is a horizontal regular curve, we have that $\mathcal{L}_u(u, 0)$ is a nonvanishing tangent horizontal vector. Thus, from the compactness of the set $\mathbb{R}/(2\pi \mathbb{Z})$, there exists $R'' > 0$ small enough such that the horizontal projection $\pi(u, v)$ of the vector $\mathcal{L}_u(u, v)$ does not vanish for $0 \leq v < R''$ and, in addition, $\pi(u)$ is not a normal vector to the horizontal curve $\gamma_\varepsilon$ given by the intersection of $\mathcal{L}(\Gamma_{R'})$ with the plane $z = \varepsilon$, for $0 \leq \varepsilon \leq \varepsilon_0$ with $\varepsilon_0$ small enough. In other words, $\pi(u)$ is a continuous function, $\pi(u, 0)$ agrees with the derivative of the strictly convex Jordan curve $(\alpha(u), \beta(u), 0)$, and $\pi(u)$ is not normal to $\gamma_\varepsilon$. Hence, the rotation index of $\gamma_\varepsilon$ is constant for all $\varepsilon \in [0, \varepsilon_0]$, and so it must be equal to $1$. In particular, the locally strictly convex curve $\gamma_\varepsilon$ must be embedded.

Now, the piece of the surface $\mathcal{L}(\Gamma_{R'})$ lying between two of those parallel planes associated to $0 < \varepsilon_1 < \varepsilon_2$ is strictly convex and bounded by two regular convex Jordan curves, one on each plane. In these conditions, the unit normal of $\mathcal{L}(\Gamma_{R'})$ defines a global diffeomorphism onto some annular domain of $\mathbb{S}^2_+$. Letting $\varepsilon_1 \to 0$ we conclude that there exists some $R > 0$ small enough such that the unit normal $\mathbf{N}_\mathcal{L}$ restricted to $\Gamma$ is a diffeomorphism onto a domain of $\mathbb{S}^2$. But now, in view of the expression (3.20), this means that the map $(x(u, v), y(u, v))$ restricted to this domain $\Gamma$ is a global diffeomorphism onto its image. Thus, both $\psi(\Gamma)$ and $\psi^*(\Gamma)$ are graphs of functions $z(x, y)$ and $z^*(x, y)$ over a punctured disc $\Omega \subset \mathbb{R}^2$.

Observe that by item (3), the function $z(x, y)$ is a solution to (3.1) with an isolated singularity at the origin. Moreover, it is clear from the construction process we followed that its limit gradient at the singularity is the curve $\gamma$ we started with and $C + z_{xx} > 0$. This concludes the proof of item (4) and Theorem 3.1. □

**Proof of Theorem 1.1.** We are now ready to complete the proof of Theorem 1.1.

Assume that $\varphi$ is analytic, and consider the map sending each $z \in \mathcal{M}_1$ to the pair $(\gamma, \varepsilon)$ given by its limit gradient at the origin, and by $\varepsilon = 0$ if $z_{xx} > 0$ and $\varepsilon = 1$ if $z_{xx} < 0$. As explained in Section 2, if $z$ is a solution to (1.1) with $z_{xx} < 0$ and an isolated singularity at the origin, then $\overline{z}(x, y) := -\overline{z}(-x, -y)$ is a solution to $z_{xx}z_{yy} - z_{xy}^2 = \overline{\varphi}(x, y, z, \overline{z}, z_x, z_y)$, where $\overline{\varphi}(x, y, z, p, q) = \varphi(-x, -y, -z, p, q)$,
with $\z_{xx} > 0$ and an isolated singularity at the origin. Moreover, the limit gradients $\gamma, \tilde{\gamma}$ of $z$ and $\tilde{z}$ at the origin coincide, and $\mathcal{H} = \tilde{\mathcal{H}}$.

If $z_{xx} > 0$, the fact that $\gamma \in \mathcal{M}_2$ follows by item (1) of Theorem 2.3. If $z_{xx} < 0$, the same conclusion holds since the function $\tilde{z}$ defined above has the same limit gradient as $z$, and satisfies $\tilde{z}_{xx} > 0$. Thus, the map $z \in \mathcal{M}_1 \mapsto (\gamma, \varepsilon) \in \mathcal{M}_2 \times \mathbb{Z}_2$ is well-defined. This map is also injective by item (3) of Theorem 2.3, with an argument again using the function $\tilde{z}$ instead of $z$ if $z_{xx} < 0$.

Finally, let us prove that this map is surjective. Let $\gamma \in \mathcal{M}_2$. By Corollary 3.3 for $A = B = C = 0, E = \varphi$, we see that $\gamma \in \mathcal{M}_2$ is the limit gradient at the origin of some solution $z \in \mathcal{M}_1$ to (1.1) for $\varphi$ such that $z_{xx} > 0$. If we again apply Corollary 3.3 but this time for $A = B = C = 0$ and $E = \tilde{\varphi}$ with $\tilde{\varphi}$ as above, and define in terms of the obtained solution $\tilde{z}$ a new function $\tilde{z}(x, y) := -\tilde{z}(-x, -y)$, we obtain a solution to (1.1) for $\varphi$ such that $\tilde{z}_{xx} < 0$, with an isolated singularity at the origin and $\gamma$ as its limit gradient at the singularity. Thus, the map $z \mapsto (\gamma, \varepsilon)$ is surjective. This proves Theorem 1.1.

Remark 3.4. Theorem 1.1 does not hold for the general equation of Monge-Ampère type (1.2), as the next two examples highlight.

1. The gradient of a solution to (1.2) can blow up at an isolated singularity. For example, Figure 3.1 shows such behavior on a rotational graph that satisfies a linear Weingarten relation $aH + bK = c$, where $H$ and $K$ are the mean and Gaussian curvature of the graph, respectively, and the constants $a, b, c$ satisfy $a^2 + bc > 0, b \neq 0$. By the standard formulas of $H, K$ for graphs in $\mathbb{R}^3$, it follows that such a rotational graph satisfies an equation of type (1.2).

2. Even when the limit gradient of a solution to (1.2) at an isolated singularity is a convex Jordan curve, it might not be strictly convex. For example, consider the PDE of type (1.2)

$$(3.21) \quad 2u^2_{x}u_{xy} + u_{xx}u_{yy} - u^2_{xy} = 1 + u^4_x,$$
and the regular curve
\[ \gamma(u) = \frac{1}{8}(4 \sin(2u), 4 \cos(2u) + 4 \sin(2u) - \cos(4u)) : \mathbb{R}/(2\pi \mathbb{Z}) \to \mathbb{R}^2. \]

The curvature of \( \gamma(u) \) is always positive except at \( u = 0 \), where it is 0. If we now apply the construction procedure explained in Theorem 3.1 to \( (3.21) \) and \( \gamma(u) \), it can be checked that the corresponding function \( J(u, v) \) is positive on a strip \( \{0 < v < R\} \) for \( R \) small enough, and from there one can prove that a solution to \( (3.21) \) is obtained with an isolated singularity at the origin whose limit gradient is \( \gamma(0) \).

Appendix: Isolated Singularities of Prescribed Curvature in \( \mathbb{R}^3 \)

Let \( \psi : \Omega \to \mathbb{R}^3 \) be an immersion of the punctured disc \( \Omega = \{(x, y) \in \mathbb{R}^2 : 0 < x^2 + y^2 < \rho^2\} \) into \( \mathbb{R}^3 \), and assume that \( \psi \) extends continuously but not \( C^1 \)-smoothly to the origin. Following [13], we say in these conditions that \( \psi \) has an embedded isolated singularity at \( p_0 = \psi(0) \in \mathbb{R}^3 \) if there is a punctured neighborhood \( U \subset \Omega \) of the origin such that \( \psi(U) \) is an embedded surface.

In these conditions, assume moreover that \( \psi : \Omega \to \mathbb{R}^3 \) has positive curvature (not necessarily constant) at every point. Then we can orient it by choosing the unique unit normal \( N \) with respect to which the second fundamental form of \( \psi \) is positive definite. We call this orientation the canonical orientation of the surface. It then follows by theorem 13 in [13] that \( \psi(\Omega) \) can be viewed around the singularity as a convex graph over a punctured disc in some direction of \( \mathbb{R}^3 \). Specifically, there is a punctured neighborhood \( U^* \subset \Omega \) of the origin and an isometry \( \Psi \) of \( \mathbb{R}^3 \) such that if \( (x', y', z') = \Psi(x, y, z) \), then \( \psi(U^*) \) is a convex graph \( z' = z'(x', y') \) with an isolated singularity at the origin, and for which the unit normal \( N \) associated to its canonical orientation is
\[
N = \left(-z'_x, \partial_x z'_y, \partial_y z'_x + \partial_x z'_y\right) / \sqrt{1 + (z'_x)^2 + (z'_y)^2}.
\]

Let \( \sigma \subset S^2 \) denote the limit unit normal of \( \psi \) at the singularity, i.e. the set of points \( w_0 \in S^2 \) for which there exist points \( q_n \in \Omega \) converging to \( (0, 0) \) such that \( N(q_n) \) converge to \( w_0 \). It follows from the previous discussion that \( \sigma \) is explicitly related to the limit gradient of the surface at the singularity, when we view \( \psi(U^*) \) as a graph \( z' = z'(x', y') \) as explained above.

Besides, it is easy to observe that for any direction \( v_0 \in S^2 \) a curve \( \sigma(u) \) in the hemisphere \( S^2 \cap \{x \in \mathbb{R}^3 : \langle x, v_0 \rangle > 0\} \) is regular and strictly convex if and only if so is the planar curve \( \gamma(u) \) contained in the plane \( \{v_0\}^\perp \subset \mathbb{R}^3 \) given by
\[
\gamma(u) = \frac{\langle \sigma(u), e_1 \rangle}{\langle \sigma(u), v_0 \rangle} e_1 + \frac{\langle \sigma(u), e_2 \rangle}{\langle \sigma(u), v_0 \rangle} e_2,
\]
where \( \{e_1, e_2, v_0\} \) is a positively oriented orthonormal basis of \( \mathbb{R}^3 \). Also, recall that any regular, strictly convex Jordan curve in \( S^2 \) is contained in some open hemisphere of \( S^2 \).
With all of this, and recalling that the equation for the curvature $K = K(x, y)$ of a graph $z = z(x, y)$ in $\mathbb{R}^3$ is given by
\[
\text{det}(D^2 z) = K(1 + |Dz|^2)^2,
\]
and is invariant by isometries of $\mathbb{R}^3$, it is elementary to obtain the following theorem as a corollary of Theorem 1.1.

**THEOREM A.1.** Let $K : \mathcal{O} \subset \mathbb{R}^3 \to (0, \infty)$ be a positive real analytic function defined on an open set $\mathcal{O} \subset \mathbb{R}^3$ containing a given point $p_0 \in \mathbb{R}^3$. Let $\mathcal{A}_1$ denote the class of all the canonically oriented surfaces $\Sigma$ in $\mathbb{R}^3$ that have $p_0$ as an embedded isolated singularity, and whose extrinsic curvature at every point $(x, y, z) \in \Sigma \cap \mathcal{O}$ is given by $K(x, y, z)$; here, we identify $\Sigma_1, \Sigma_2 \in \mathcal{A}_1$ if they overlap on an open set containing the singularity $p_0$.

Then, the map that sends each surface in $\mathcal{A}_1$ to its limit unit normal at the singularity provides a one-to-one correspondence between $\mathcal{A}_1$ and the class $\mathcal{A}_2$ of regular, analytic, strictly convex Jordan curves in $S^2$.

Let us point out that the $\mathbb{Z}_2$ factor appearing in the correspondence of Theorem 1.1 does not appear in Theorem A.1 by our choice of the canonical orientation for surfaces in $\mathcal{A}_1$.

Theorem A.1 generalizes [11, cor. 13], which covers the case $K = \text{const}$.

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**Bibliography**

[1] Aledo, J. A.; Chaves, R. M. B.; Gálvez, J. A. The Cauchy problem for improper affine spheres and the Hessian one equation. *Trans. Amer. Math. Soc.* 359 (2007), no. 9, 4183–4208 (electronic). doi:10.1090/S0002-9947-07-04378-4

[2] Bers, L. Isolated singularities of minimal surfaces. *Ann. of Math. (2)* 53 (1951), 364–386.

[3] Beyerstedt, R. Removable singularities of solutions to elliptic Monge-Ampère equations. *Math. Z.* 208 (1991), no. 3, 363–373. doi:10.1007/BF02571533

[4] Beyerstedt, R. The behaviour of solutions to elliptic Monge-Ampère equations at singular points. *Math. Z.* 216 (1994), no. 2, 243–256. doi:10.1007/BF02572320

[5] Bobenko, A. I. Surfaces in terms of 2 by 2 matrices. Old and new integrable cases. *Harmonic maps and integrable systems*, 83–127. Aspects of Mathematics, E23. Friedrich Vieweg, Braunschweig, 1994.

[6] Caffarelli, L. A.; Gidas, B.; Spruck, J. Asymptotic symmetry and local behavior of semilinear elliptic equations with critical Sobolev growth. *Comm. Pure Appl. Math.* 42 (1989), no. 3, 271–297. doi:10.1002/cpa.3160420304

[7] Caffarelli, L.; Li, Y. An extension to a theorem of Jörgens, Calabi, and Pogorelov. *Comm. Pure Appl. Math.* 56 (2003), no. 5, 549–583. doi:10.1002/cpa.10067

[8] Caffarelli, L.; Li, Y. Y. Some multi-valued solutions to Monge-Ampère equations. *Comm. Anal. Geom.* 14 (2006), no. 3, 411–441.
[9] Chang, S.-Y. A.; Han, Z.-C.; Yang, P. Classification of singular radial solutions to the $\sigma_k$-Yamabe equation on annular domains. *J. Differential Equations* **216** (2005), no. 2, 482–501. doi:10.1016/j.jde.2005.05.005

[10] Ferrer, L.; Martínez, A.; Milán, F. The space of parabolic affine spheres with fixed compact boundary. *Monatsh. Math.* **130** (2000), no. 1, 19–27. doi:10.1007/s006050050084

[11] Gálvez, J. A.; Hauswirth, L.; Mira, P. Surfaces of constant curvature in $\mathbb{R}^3$ with isolated singularities. *Adv. Math.* **241** (2013), 103–126. doi:10.1016/j.aim.2012.11.019

[12] Gálvez, J. A.; Martínez, A.; Mira, P. The space of solutions to the Hessian one equation in the finitely punctured plane. *J. Math. Pures Appl. (9)* **84** (2005), no. 12, 1744–1757. doi:10.1016/j.matpur.2005.07.007

[13] Gálvez, J. A.; Mira, P. Embedded isolated singularities of flat surfaces in hyperbolic 3-space. *Calc. Var. Partial Differential Equations* **24** (2005), no. 2, 239–260. doi:10.1007/s00526-004-0321-6

[14] Gilbarg, D.; Trudinger, N. S. *Elliptic partial differential equations of second order*. Classics in Mathematics. Springer, Berlin, 2001.

[15] González, M. d. M. Classification of singularities for a subcritical fully nonlinear problem. *Pacific J. Math.* **226** (2006), no. 1, 83–102. doi:10.2140/pjm.2006.226.83

[16] Gursky, M. J.; Viaclovsky, J. Convexity and singularities of curvature equations in conformal geometry. *Int. Math. Res. Not.* (2006), Art. ID 96890, 43 pp. doi:10.1155/IMRN/2006/96890

[17] Han, Z.-C.; Li, Y. Y.; Teixeira, E. V. Asymptotic behavior of solutions to the $\sigma_k$-Yamabe equation near isolated singularities. *Invent. Math.* **182** (2010), no. 3, 635–684. doi:10.1007/s00222-010-0274-7

[18] Hartman, P.; Wintner, A. On the local behavior of solutions of non-parabolic partial differential equations. *Amer. J. Math.* **75** (1953), 449–476.

[19] Heinz, E. Über das Randverhalten quasilinearer elliptischer Systeme mit isothermen Parametern. *Math. Z.* **113** (1970), 99–105.

[20] Heinz, E.; Beyerstedt, R. Isolated singularities of Monge-Ampère equations. *Calc. Var. Partial Differential Equations* **2** (1994), no. 2, 241–247. doi:10.1007/BF01191344

[21] Jin, T.; Xiong, J. Solutions of some Monge-Ampère equations with isolated and line singularities. Preprint, 2012. arXiv:1212.4206 [math.AP]

[22] Jörgens, K. Harmonische Abbildungen und die Differentialgleichung $rt - z^2 = 1$. *Math. Ann.* **129** (1955), 330–344.

[23] Korevaar, N.; Mazzeo, R.; Pacard, F.; Schoen, R. Refined asymptotics for constant scalar curvature metrics with isolated singularities. *Invent. Math.* **135** (1999), no. 2, 233–272. doi:10.1007/s002220050285

[24] Leandro, C.; Rosenberg, H. Removable singularities for sections of Riemannian submersions of prescribed mean curvature. *Bull. Sci. Math.* **133** (2009), no. 4, 445–452. doi:10.1016/j.bulsci.2008.04.002

[25] Li, A. M.; Simon, U.; Zhao, G. S. Global affine differential geometry of hypersurfaces. de Gruyter Expositions in Mathematics, 11. Walter de Gruyter, Berlin, 1993. doi:10.1515/9783110870428

[26] Li, Y. Y. Conformally invariant fully nonlinear elliptic equations and isolated singularities. *J. Funct. Anal.* **233** (2006), no. 2, 380–425. doi:10.1016/j.jfa.2005.08.009

[27] Li, Y. Y.; Nguyen, L. Harnack inequalities and Bôcher-type theorems for conformally invariant fully nonlinear degenerate elliptic equations. Preprint, 2012. arXiv:1206.6264 [math.AP]

[28] Mazzeo, R.; Pacard, F. Constant scalar curvature metrics with isolated singularities. *Duke Math. J.* **99** (1999), no. 3, 353–418. doi:10.1215/S0012-7094-99-09913-1

[29] Müller, F. Analyticity of solutions for semilinear elliptic systems of second order. *Calc. Var. Partial Differential Equations* **15** (2002), no. 2, 257–288. doi:10.1007/s005260100127

[30] Savin, O. The obstacle problem for Monge Ampère equation. *Calc. Var. Partial Differential Equations* **22** (2005), no. 3, 303–320. doi:10.1007/s00526-004-0275-8
[31] Schulz, F.; Wang, L. Isolated singularities of Monge-Ampère equations. *Proc. Amer. Math. Soc.* 123 (1995), no. 12, 3705–3708. doi:10.2307/2161897

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