COMPACTNESS THEOREM OF COMPLETE $k$-CURVATURE MANIFOLDS WITH ISOLATED SINGULARITIES

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Abstract. In this paper, we prove that when $k$-Dilational Pohozaev invariants for $k < n/2$ are positive bounded from below, the set of metrics conformal to the standard metric on $S^n \setminus \{p_1, \cdots, p_l\}$ is locally compact in $C^{m,\alpha}$ topology. Here $k$-Dilational Pohozaev invariants are defined from the Kazdan-Warner type identity for the $\sigma_k$ curvature, which is derived by Viaclovsky [V] and Han [H1]. When $k = 1$, Pollack [Pollack] proved the compactness of the complete metrics of constant positive scalar curvature on $S^n \setminus \{p_1, \cdots, p_l\}$.

1. Introduction

For a Riemannian manifold $(M^n, g)$, let $\text{Ric}$, $R$ and $A$ be the corresponding Ricci curvature, Scalar curvature, and Schouten tensor respectively, where

$$A = \frac{1}{n - 2}(\text{Ric} - \frac{R}{2(n - 1)}g),$$

$$R = g^{ij}\text{Ric}_{ij}.$$  

Let $\{\lambda(A)\}_{i=1}^n$ be the set of eigenvalues of $A$ with respect to $g$. Define

$$\sigma_k(g^{-1}A_g) = \sum_{1 \leq i_1 < \cdots < i_k \leq n} \lambda_{i_1}\lambda_{i_2} \cdots \lambda_{i_k}.$$  

Here $\sigma_1(A) = \text{Tr}A = \frac{1}{2(n-1)}R$.

We sometimes call $(M, g)$ a $k$-curvature manifold if

$$\sigma_k(g^{-1}A_g) = c.$$  

Like classical Yamabe problem, Viaclovsky raised the $\sigma_k$ Yamabe problem as the following: in a conformal class $[g] = \{e^{2u}g | u \in C^\infty(M)\}$, can we find $g_1 \in [g]$ such that $\sigma_k(g_1^{-1}A_{g_1}) = c$ for some positive constant $c$?

To solve the equation, a positive cone condition is introduced as below

$$\Gamma_k^+ := \{\lambda = (\lambda_1, \cdots, \lambda_n), \text{ s.t. } \sigma_1(\lambda) > 0, \cdots, \sigma_k(\lambda) > 0\}.$$  

Note that when $g^{-1}A_g \in \Gamma_k^+$, equation (1.1) is a fully nonlinear elliptic partial differential equation when $k > 1$. In past decades, the $k$-Yamabe problem for smooth manifolds has been studied from various aspects and mathematicians have established many parallel results to classical Yamabe problem. For more information, we refer to [CGY1] [CGY2] [GV0] [GV4] [GW1] [LL1] [LL2] [LN] [STW] [TW] [W1] and references therein.

Singular sets of locally conformally flat metrics with positive $\sigma_k$ curvature are widely studied. For $k = 1$, Schoen and Yau have proved if $\Omega \subset S^n (n \geq 5)$ admits a
complete conformal metric $g$ with constant curvature, then $\dim(\partial \Omega) < (n - 2)/2$. In [CHY2], Chang-Han-Yang have proved that if $\Omega \subset S^n (n \geq 5)$ admits a complete, conformal metric $g$ with $\sigma_1 (A_g) \geq c > 0, \sigma_2 (A_g) \geq 0$ and $|R_g| + |\nabla_g R_g| \leq c_0$, then $\dim(\partial \Omega) < (n - 4)/2$. For $1 \leq k < n/2$, González [G1] and Guan-Lin-Wang [GLW] have proved that under some natural condition for $\sigma_k, \dim(\partial \Omega) < (n - 2k)/2$. When $k > n/2$, there exists no complete manifold with $\sigma_1 (g^{-1} A_g), \cdots, \sigma_k (g^{-1} A_g) > C_0$ on subdomain of $S^n$, which is proved by González [G1].

For singular sets $P = \{p_1, \cdots, p_q\}$, a classical problem is the existence of complete constant scalar curvature manifold, which is conformal to $g$ on $M \setminus P$. This problem has been widely studied and we refer the classical paper of Schoen [Sc2] and its references. The existence of complete manifold with constant $\sigma_k$ curvature attracts many mathematicians. In [L], Li has proved that on $\mathbb{R}^n \setminus \{0\}$, the conformal factor is radial for $1 \leq k \leq n$. When $k < n/2$, Chang-Han-Yang [CHY] have proved that on $\mathbb{R}^n \setminus \{0\}$, there exists a complete manifold with constant positive $k$-curvature. Catino-Mazzieri [MN] constructed the complete manifolds with constant positive $k$-curvature on connected sum $M_1 \sharp M_2$ for $k < \frac{n}{2}$. Mazzieri-Segatti [LS] have constructed complete locally conformally flat manifold with constant positive $k$-curvature for $4 \leq 2k < n$. In [SS], Santos has proved that when $k = 2, n \geq 5$, there exist complete manifolds on $M \setminus P$ with constant positive $k$-curvature, where $(M, g_0)$ has constant $k$-curvature and $\nabla^j g_0 W_{g_0} (P) = 0$ for $j = 0, 1, \cdots, \lfloor \frac{n-2}{2} \rfloor$. With multiplicities of solutions, it is natural to wonder the compactness of the complete manifolds with positive constant $k$-curvature on $S^n \setminus P := \Omega$. When $k = n/2$, the related results are not so abundant. For $k = 2, n = 4$, in [FW], the authors have given a necessary condition for the existence of the conic 4-sphere with positive constant 2-curvature. For $k > \frac{n}{2}$, singularities have been studied by [L] [GV3] [TW09].

In this paper, we want to prove that under some condition, the compactness of the complete manifold on $S^n \setminus P$ exists.

Following the work of Pollack [Pollack], we can define $k$-Dilatational Pohozaev invariants as the classical Dilational Pohozaev invariants, which are derived from the Kazdan-Warner type identity.

The following Kazdan-Warner type identity is proved by Viaclovsky [V] and Han [HI], which is significant in this paper.

**Theorem 1.** [HI] [V] Let $(N^n, g)$ be a locally conformally flat $n$-dimensional manifold with boundary $\partial N$. For any conformal Killing field $X$ on $N^n$, we have

$$
\frac{n - k}{n} \int_N < X, \nabla \sigma_k (g^{-1} A) > dv_g = \int_{\partial N} H^b_a n_a X^a d\sigma_g,
$$

where

$$
\tilde{H}^b_a = H^b_a - \frac{H^c}{n} \delta^b_a,
$$

$$
H^b_a = T^b_{k-1, c} A^c_a
$$

and

$$
T^b_{k-1, c} = \frac{1}{(k-1)!} \sum_{i_1, \cdots, i_k, j_1, \cdots, j_k-1} \delta(i_1, \cdots, i_{k-1}, j_1, \cdots, j_{k-1}) \cdot (\begin{array}{c}
\cdot \\
\cdot \\
\cdot
\end{array}
) A_{i_1}^{j_1} \cdots A_{i_{k-1}}^{j_{k-1}}
$$

where $\nu$ denotes the outward unit normal vector to $\partial N$, and $dv_g, d\sigma_g$ are volume and surface measure respectively.
The Kazdan-Warner type identity is derived from the divergence structure as below
\[
\frac{n-k}{n} < X, \nabla \sigma_k(g^{-1}A) = \nabla_b(X^a \hat{H}_b^k).
\]
For \( k = 1 \), it has been proved by Schoen [Sc2] and applied to prove the existence of singular Yamabe problem. This was also applied to the Yamabe problem on star-shaped domain by Pohozaev, which is called Pohozaev identity. For \( k = n \), we refer to [HLT] for the Kazdan-Warner type identity, which plays an important role in Yamabe problem. To study the local behavior of the solution near singularities, the following Hölder regularity is key, which is proved in [HLT].

**Theorem 2.** [HLT] Let \( g = v^{\frac{4}{n-2}}g_E \) and \( \sigma_k(g^{-1}A_g) = c \) on \( B_R \setminus \{0\} \) with \( \lambda(A_g) \in \Gamma_+^k \), where \( c \) is a positive constant and \( 2 \leq k \leq n \). Then there exist constants \( \alpha > 0 \) and \( C > 0 \) such that for \( |x| < \frac{1}{4} \),
\[
|v(x) - \bar{v}(|x|)| \leq C|x|^{\alpha} \bar{v}(|x|),
\]
where \( \bar{v} \) is the radial smooth solution to (1.1) on \( \mathbb{R}^n \setminus \{0\} \) in the \( \Gamma_+^k \) class.

Theorem 2 generalizes the classical theorem for \( k = 1 \) proved by Caffarelli-Gidas-Spruck [CGS] and Korevaar-Mazzeo-Pacard-Schoen [KMPS], which provides a powerful tool to deal with singular Yamabe problem.

When \( g = g_u = u^{\frac{4}{n-2}}g_0 \) for a positive function \( u \), we have
\[
A = A_{g_0} - \frac{2}{n-2}u^{-1}v_{g_0}^2 u + \frac{2n}{(n-2)^2} u^{-2} du \otimes du - \frac{2}{(n-2)^2} u^{-2} du^2 g_{g_0, g_0}.
\]

The corresponding equation to (1.1) is
\[
\sigma_k(g_0^{-1}(A_{g_0} - \frac{2}{n-2}u^{-1}v_{g_0}^2 u + \frac{2n}{(n-2)^2} u^{-2} du \otimes du - \frac{2}{(n-2)^2} u^{-2} du^2 g_{g_0, g_0})) = \left( \frac{n}{k} \right) \left( \frac{1}{2} \right)^k u^{\frac{4k}{n-2}}.
\]
Here we always normalize the constant \( c \) to be \( \left( \frac{n}{k} \right) \left( \frac{1}{2} \right)^k \).

Consider the radial solution of (1.2) and write in cylindrical metric as \( g_0 = \frac{u(t)^{\frac{4}{n-2}}}{(dt^2 + d\theta^2)} \), where \( \sigma_k(g_0^{-1}A_{g_0}) = \left( \frac{n}{k} \right) \left( \frac{1}{2} \right)^k \). As [HLT] [CHY], \( \bar{u} \) satisfies
\[
\frac{1}{2} = (1 - \frac{2}{n-2})^2 \left( \frac{u_t}{u} \right)^2 - \frac{k}{n} - \frac{2}{n-2} \left( 1 - \frac{2}{n-2} \right) \left( \frac{u_t}{u} - \frac{u_t^2}{u^2} \right) + \left( \frac{1}{2} - \frac{k}{n} \right) \left( 1 - \frac{2}{n-2} \right) \left( \frac{u_t^2}{u^2} \right) u^{-2k} \frac{n-2}{n-1}
\]
and
\[
|1 - \frac{k}{n} \frac{u_t^2}{u^2} (1 - \frac{2}{n-2} \frac{u_t^2}{u^2}) | u^{-2k} \frac{n-2}{n-1}
\]
is a constant function.

We denote \( \bar{u}_h \) as the radial solution \( \bar{u} \), where
\[
h = \left[ 1 - \frac{2}{n-2} \frac{u_t^2}{u^2} (1 - \frac{2}{n-2} \frac{u_t^2}{u^2}) | u^{-2k} \frac{n-2}{n-1} \right].
\]

Following the symbol in Pollack [Pollack], for a conformal Killing field \( X \) on \( S^n \) and \( p \in P \), we denote
\[
D_h(g, p)(X) = \int_{\Sigma} \hat{H}_b^k \nu_b X^a d\sigma_g.
\]
Lemma 3. Let $D_k(1.4)$ be a compactness theorem for complete $k$-curvature manifolds with isolated singularities. For general $k$, we also call it $k$-Dilational Pohozaev invariants.

By the divergence structure of $X, \nabla \sigma_k g^{-1} A >$, we know that $D_k(g, p)(X)$ is independent of the hypersurface $\Sigma$, which is homologous to $\partial B_\delta(p)$. With Theorem A and Theorem B, we can compute the local quantity $D_k(g, p)(X)$ near the singularities. By Theorem C for $S^n \setminus P$,

\[
(1.4) \quad \sum_{i=1}^{q} D_k(g, p_i)(X) = 0.
\]

Let $X_p$ denote the conformal killing field on $S^n$ which fixes the point $p$. Denote $D_k(g, p) = D_k(g, p)(X_p)$.

**Lemma 3.** Let $g = \nu^{\frac{4}{n-2}} g_0$ satisfy (1.3) with singular point $p$, then there exists a constant $h$ such that

\[
D_k(g, p) = C(n-1)_k \frac{n-k}{n} \frac{1}{2n} h w_{n-1},
\]

where $w_{n-1}$ is the volume of the unit $(n-1)$-sphere in $\mathbb{R}^n$ and $g$ is asymptotic to $g_h$, where $g_h = \nu_h^{-\frac{4}{n-2}} (dt^2 + d\theta^2)$.

For $k = n/2$, from Han-Li-Teixeira [HLT] and Chang-Han-Yang [CHY], we know the singularity is conic, which is also described in [FW] for $k = 2, n = 4$. Especially, we assume that at $p_1$, $u(x) = |x - p_1|^\beta_1 + v(x)$ for $-1 < \beta_1 < 0$ with a bounded function $v(x)$ and $g = u^2 g_E$. From the proof in [CHY], $\beta_1 = -1 + \sqrt{1 - \sqrt{k}}$ and then $h = \beta_1^2 (2 + \beta_1)^2$, which also appears in Fang-Wei’s subcritical condition in [FW]. From (1.4), we know that the metric with constant $\sigma_k$ curvature can not admit one singularity on $S^n$.

Now we state our main theorem as below

**Theorem 4.** Let $P = \{p_1, \ldots, p_q\} \subset S^n$ be a set of $q$ distinct points and $k < n/2$. If there exists $p \in P$ such that any sequence of solutions $g_i = u_i^{-\frac{4}{n-2}} g_0$ on $S^n \setminus P$ have positive constant $\sigma_k$ curvature and $k$-Dilational Pohozaev invariants $D_k(g_i, p)$ are uniformly bounded away from $0$, then $u_i$ sub-converges to a positive solution on any compact subsets of $S^n \setminus P$ in $C^\infty$. Furthermore, the corresponding metric is complete on $S^n \setminus P$. Here $g_0$ is the standard sphere metric.

The nonvanishing $k$-Dilational Pohozaev invariants are used to say that the singularities are non-removable. Actually one nonvanishing $k$-Dilational Pohozaev invariant is enough to imply the completeness of the limiting metric. By Han-Li-Teixeira’s theorem [HLT] and Chang-Han-Yang’s radial classification [CHY], the metric on $S^n \setminus P$ with positive constant $\sigma_k$ curvature is complete for $k < n/2$. For $k = n/2$, the metric with positive constant $\sigma_k$ curvature is not complete, but it is conic. This paper is organized as following. We firstly give the growth estimate of the solution to the singularity and then by dilational Pohozaev invariants, we prove Theorem 4 by compactness argument. In this paper, the constant $C, c$ may different from line to line without confusion.

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2. Growth estimate of singular solution

In this section, we will describe the behavior of the solution near the singularity by the contradiction argument. The following behavior near singularity in Theorem 5 is well known in the classical Yamabe problem and we refer to Schoen and Pollack [Pollack].

**Theorem 5.** Let \( g = u^{4/n}g_0 \) be a complete manifolds with smooth metric \( g_0 \) and \( \sigma_k(g^{-1}A_g) = C_0 \), where \( C_0 \) is a positive constant on \( S^n \) with \( \lambda(A_g) \in \Gamma_k^+ \). Then

\[
u(x) \leq Cd(x, P)^{2m} \]

where \( C \) is independent of \( u \), \( d(\cdot, \cdot) \) denotes the distance function on \( (S^n, g_0) \).

In Theorem 1.1' [L], Li proved this type estimate in local case and the difference from there is the constant \( C \) here is independent of \( u \). Also Li-Luc have given a similar decay theorem for a sequence of smooth solutions at the beginning of Part 3 for \( \sigma_k \) Yamabe problem in [LN]. With local volume information, this kind of theorem has also been proved by Han [H], González [G2] and [DNW].

**Proof.** Let us prove this theorem by a contradiction argument. Choose \( x_0 \in S^n \) and \( \sigma \) sufficiently small so that \( \overline{B}_\sigma(x_0) \subset S^n \). Let \( \rho(x) = d(x, x_0) \) and define

\[
f(x) = (\sigma - \rho(x))^{\frac{n-2}{2}}u(x).
\]

Since \( u \) is smooth up to \( \partial B_\sigma(x_0) \), \( f = 0 \) on \( \partial B_\sigma(x_0) \). Assume that we have a sequence functions \( \{g_i = u_i^{4/n}g_0\} \) such that for all \( i \),

\[
(\sigma - d(x_1, x_0))^{\frac{n-2}{2}}u_i(x_1) = f(x_i) = \max_{B_\sigma(x_0)} f > i.
\]

Since \( (\sigma - d(x_1, x_0))^{\frac{n-2}{2}} \leq \sigma^{\frac{n-2}{2}} \), we have \( u_i(x_i) \to \infty \) as \( i \to \infty \). Let \( y^1, \cdots, y^n \) be normal coordinates centered at \( x_1 \). Denoting \( \lambda_i = u_i(x_i)^{\frac{n-2}{2}} \), we get that \( \lambda_i \to \infty \) as \( i \to \infty \). Let \( z = \lambda_i y \) and

\[
\hat{u}_i(z) = \lambda_i^{\frac{n-2}{2}} u_i(\exp_{x_1} \frac{z}{\lambda_i}),
\]

with \( \hat{u}_i(0) = 1 \) and \( \exp \) is the exponential map.

Let \( g_i = u_i^{4/(n-2)}g_0 \), with \( g_0 \) centered at \( x_1 \), where \( g_0, i(z) = \sum_{k,l=1}^n g_{0,kl} (\hat{\phi}) dz^k dz^l \) converges uniformly to Euclidean metric.

Denote \( r_i = \sigma - \rho(x_i) \). We have \( u_i(x) \leq \frac{2^{\frac{n-2}{2}}}{r_i} u_i(x_i) \) in \( B_{r_i/2}(x_i) \). So

\[
\hat{u}_i(z) \leq 2^{\frac{n-2}{2}} \text{ in } |z| \leq \lambda_i \frac{r_i}{2} = \frac{1}{2} (u_i(x_i)(\sigma - \rho(x_i))^{\frac{n-2}{2}})^{\frac{n-2}{2}} \geq \frac{1}{2} l^{\frac{n-2}{2}}.
\]

Thus \( \hat{u}_i(z) \) are uniformly bounded in any compact set in \( \mathbb{R}^n \) and \( \hat{u}_i(0) = 1 \). Moreover, we know \( \sigma_k(\lambda(A_{\hat{u}_i})) = C_0 \) in \( B_{\lambda_i^{\frac{n-2}{2}}}(0) \).

By the interior gradient and Hessian estimate by [GWT] [LL] [SC] [WT] [LL], we know that for sufficiently large \( i \), for some \( C > 0 \) independent of \( i \) and any compact set \( K \subset \mathbb{R}^n \),

\[
\sup_K |\nabla \ln \hat{u}_i| + |\nabla^2 \ln \hat{u}_i| \leq C
\]
With $\tilde{u}_i(0) = 1$, we have $\tilde{u}_i \geq C$ in $K$. By Schauder theory, we know
$$\sup_K |\ln \tilde{u}_i|_{C^{2,\alpha}} \leq C.$$  

Then by Arzela-Ascoli theorem, there exists a subsequence of $\tilde{u}_i$ (still denoted by $\tilde{u}_i$) and a function $u_\infty$ such that $\tilde{u}_i \to u_\infty$ in $C^{2,\alpha}$ for any compact sets in $\mathbb{R}^n$. Therefore
$$\sigma_k(\lambda(A_{u_\infty\, g_E})) = C_0 \quad \text{in} \quad \mathbb{R}^n.$$  

Here $g_\infty = u_\infty^2 g_E$ is the standard sphere metric, which is proved by Li-Li [LL2].

By Theorem 7 in the following, we know that respect to $g_\infty$, ball in $S^n \setminus P$ has concave boundary respect to the outer normal direction, which is contradicted to the Liouville Theorem proved by Li-Li [LL2].

Now we know there exists a positive $c > 0$ such that $f(x) \leq c$ for $x \in B_\sigma(x_0)$ and we choose $\sigma = d(x_0, P)/2$, which yields
$$f(x_0) = \sigma^{-\frac{2}{n-2}} u(x_0),$$  

and thus
$$u(x_0) \leq c 2^{\frac{4-n}{n-2}} \cdot d(x_0, P)^\frac{2-n}{2}.$$  

The theorem is proved. \[\square\]

Remark 6. The $C_0$ in the theorem is also independent with the position if we replace $x_0$ with $x_{0,j}$ in the argument. When $k = \frac{n}{2}$, the proof for this kind of decay can be simplified by Gauss-Bonnet-Chern formula by some volume restriction. see [FW2].

Theorem 7. If $g$ is a complete metric of constant positive $\sigma_k$ curvature on $\Omega \subset S^n$ ($\Omega \neq S^n$), which is conformal to $g_\infty$, then any ball $B$ (with respect to $g_\infty$) with $B \subset \Omega$ has boundary $\partial B$, which is geodesically convex with respect to $g$ respect to inner direction of $B$. (Here $g_\infty$ is the standard sphere metric.)

For the scalar curvature, Schoen [Sc0] has proved this theorem. For constant $Q$-curvature order $\gamma$, inspired by [QR], González-Mazzeo-Sire [GMS] have proved this type theorem by the moving plane method. For $\sigma_k$ Yamabe problem, the moving plane method was used in [VA] to prove the Liouville theorem for sphere, which indicates many similarities between $\sigma_k$ Yamabe problem and $Q$-curvature. In the below, we will follow González-Mazzeo-Sire’s argument and give a brief description of the proof.

Proof. Denote the boundary of the geodesic ball $B_r(p) \subset \Omega$ as $S$. By Stereographic projection for $x_0 \in S$, with the antipode on the plane, the boundary $S$ corresponds to the hyperplane in $\mathbb{R}^n$ and we denote it as $H_0 = \{(x_1, \cdots, x_n) | x_n = 0\}$. $\Omega$ becomes $\Omega_1$ and the singularities ($\partial \Omega_1$) are below $H_0$, which is located on $\{(x_1, \cdots, x_n) | x_n < 0\}$. By Stereographic projection, the metric is $g = v^{\frac{4}{n-2}} g_E$ and the corresponding equation is on $\Omega_1 = \mathbb{R}^n \setminus \{p_1, \cdots, p_\eta\}$,
$$\sigma_k(-\frac{2}{n-2} v^{-1}\nabla^2 v + \frac{2n}{(n-2)^2} v^{-2} dv \otimes dv - \frac{2}{(n-2)^2} v^{-2} |dv|^2) = (\frac{n}{k})(\frac{1}{2})^k v^{-\frac{4k}{n-2}}.$$  

To prove that $S$ is geodesically convex with respect to $g$, we only need to prove that the hyperplane $H_0$ is geodesically convex with respect to $g = v^{\frac{4}{n-2}} g_E$. The second fundamental form of the hyperplane $H_0$ is
$$-\frac{4}{n-2} v^{-1} \frac{\partial v}{\partial x_n} I \big|_{H_0}.$$
Next we will use the moving plane method to show that $\frac{\partial u}{\partial x_n} < 0$ on $H_0$, which yields that conclusion. Denote \( v_\lambda(x) = v(x_1, \cdots , x_{n-1}, 2\lambda - x_n) \) and \( H_\lambda = \{(x_1, \cdots , x_n) | x_n = \lambda \} \). The proof is standard and we refer to [V4] and [GNN] for more details. For completeness, we just write part of the proof.

We claim that there exists a constant $\lambda_0 > 0$ such that for $\lambda > \lambda_0$, we have $w_\lambda(x) = v(x) - v_\lambda(x) < 0$ for $\Sigma_\lambda = \{(x_1, \cdots , x_n) | x_n > \lambda \}$ and $\frac{\partial w}{\partial x_n}|_{H_\lambda} < 0$.

Let $u(y) = v\left(\frac{y}{|y|}\right)|y|^{2-n}$. When $y \to 0$, $\frac{u(y)}{|y|^n} \to \infty$. For $|x|$ big enough, $v(x)|x|^{-2}$ is smooth and we know that $u(y) = v\left(\frac{y}{|y|}\right)|y|^{2-n}$ is smooth near $y = 0$. By Taylor expansion, we have

\[
u(y) = a_0 + u_i(0)y_i + \frac{1}{2}u_{ij}(0)y_iy_j + o(|y|^2).
\]

For more details of the Taylor expansion, we refer to [V4].

Actually same as the classical paper Gidas-Ni-Nirenberg [GNN], we have the following

\[
v(x) = \frac{1}{|x|^{n-2}}(a_0 + \frac{a_ix_i}{|x|^2} + \frac{a_{ij}x_ix_j}{|x|^4} + o(\frac{1}{|x|^2})).
\]

As [GNN] [V4], there exists a constant $\lambda_0 > 0$ such that for $\lambda > \lambda_0$, we have $w_\lambda(x) = v(x) - v_\lambda(x) < 0$ for $\Sigma_\lambda = \{(x_1, \cdots , x_n) | x_n > \lambda \}$. By Hopf’s lemma, on every compact sets of $\Sigma_\lambda$, $\frac{\partial w}{\partial x_n}|_{H_\lambda} < 0$. By the argument of [V4] GNN, we know if there exists a $\lambda > 0$ such that $w_\lambda < 0$ in $\Sigma_\lambda$, then there exists some $\varepsilon > 0$ such that $w_{\lambda_1} < 0$ for $\lambda_1 \in [\lambda - \varepsilon , \lambda]$. Then the moving plane can be continued until touching the singularities. Above all, $\frac{\partial w}{\partial x_n}|_{H_\lambda} = 2 \frac{\partial w}{\partial x_n}|_{H_\lambda} < 0$.

By the interior gradient and hessian estimate by [GW1] [LL1] [SC] [WT1] [L1], we get

**Corollary 8.** For any $K \subset \subset S^n \setminus P$ and $\sigma_k(g^{-1}A_g)$ is a positive constant, where $g = u^{\frac{n-2}{4}}g_0$ with $\lambda(A_g) \in \Gamma_k^+$ and $g_0$ is smooth metric on $S^n$, we have

\[|\nabla_{g_0} \ln u|_{C^0(K)} \leq C, \quad |\nabla^2_{g_0} \ln u|_{C^0(K)} \leq C,\]

where $C$ is dependent on $K, \text{dist}(K, S^n \setminus P), g_0$.

3. **Proof of the Main theorem**

In this section, we give the proof of the main theorem.

**Proof.** By Corollary [S] $||u_i||_{C^2(K)} \leq C_K$ for any compact set in $S^n \setminus P$. We know that there exists a subsequence (still denoted as $u_i$) converging to a function $u \in C^{1,1}(S^n \setminus P)$ in $C^{1,1}$ sense. We still need to prove that $u$ is actually smooth and positive on $S^n \setminus P := \Omega$.

Assume that there exists a point $Q \in S^n \setminus P$ such that $u(Q) = 0$. Without loss of generality, let $\varepsilon_i = u_i(Q)$ and $\varepsilon_i \to 0$ as $i \to \infty$. Define $v_i = \varepsilon_i^{-1} u_i$ and $\inf_{S^n \setminus P} v_i = 1$. Since $u_i$ satisfy $\sigma_k(g_{u_i}^{-1}A_{g_{u_i}}) = \left(\frac{n}{k}\right)(\frac{1}{2})^k$, we have

\[
s_k(g_{v_i}^{-1}A_{g_{v_i}}) = \left(\frac{n}{k}\right)(\frac{1}{2})^{2k} \varepsilon_i^2.
\]

Since $u_i$ satisfy Lemma [S] and Corollary [S] for any $B_{2R}(y) \subset \subset S^n \setminus P$, we have

\[
sup_{B_R(y)} u_i \leq c \inf_{B_{\frac{R}{2}}(y)} u_i,
\]

where $c = c(n, R)$. Then $\sup_{B_R(y)} u_i \leq c$ and $\ln v_i$ are
locally $C^2$ uniformly bounded on $\Omega$. Now there exists a $v_{\infty} \in C^{1,1}(\Omega)$ such that $v_i \rightarrow v_{\infty}$ in $C^{1,\alpha}_\text{loc}(\Omega)$.

At $p_1 \in P$, we can write the metric $g_i$ on cylindrical coordinate. Here the lower bound of $D_k(g_i, p_1)$ is positive. Let $g_i = (wv_i)^{4/(n-2)}g_{c}$ and $g_0 = w^{4/(n-2)}g_{c}$, where $g_{c} = dt^2 + d\theta^2$. Denote

$$\bar{g}_i = \varepsilon_i^{-4/(n-2)}g_i = (wv_i)^{4/(n-2)}g_{c},$$

and

$$\bar{g}_i \rightarrow g_{\infty} = (wv)^{4/(n-2)}g_{c} \quad \text{in} \quad C^{1,\alpha}_\text{loc}$$

As

$$A_k^j = \bar{g}^{jl}A_{lk} = \varepsilon_i^{4/(n-2)}A_k^j,$$

we have

$$\bar{H}_{i,1}^{1} = \bar{H}_{i,1}^{1}\varepsilon_i^{4k/(n-2)}.$$

Therefore

$$\int_{\Sigma_0} \bar{H}_{i,1}^{1}(u_i w_i)^{2n/(n-2)}u_i^{2n/(n-2)}d\theta = \int_{\Sigma_0} \bar{H}_{i,1}^{1}\varepsilon_i^{4k/(n-2)}u_i^{2n/(n-2)}w_i^{2n/(n-2)}d\theta$$

$$= \int_{\Sigma_0} \bar{H}_{i,1}^{1}u_i^{2n/(n-2)}w_i^{2n/(n-2)}d\theta,$$

where

$$\int_{\Sigma_0} \bar{H}_{i,1}^{1}(u_i w_i)^{2n/(n-2)}d\theta,$$

is bounded.

When $k < n/2$, $\lim_{\varepsilon \rightarrow 0} \int_{\Sigma_0} \bar{H}_{i,1}^{1}(u_i w_i)^{2n/(n-2)}w_i^{2n/(n-2)}d\theta = 0$, which is contradicted to the lower bound of $D_k(g_i, p_1)$. We obtain that $u$ is positive on $\Omega$ and furthermore $u_i$ have positive lower bound. Therefore by the classical Schauder theory, we get the smoothness of $u$ and $u$ satisfy $\sigma_k(\lambda(A_u)) = (\frac{n}{k})(\frac{1}{2})$ with non-vanishing $D_k(g_u, p_1)$, which is the limit of $D_k(g_i, p_1)$ and indicates that $g_u$ has non-removable singularities and with [CHY, HLT]'s results, we know that the metric is complete. \hfill $\square$

4. Appendix

In this section, following the argument for scalar curvature by Pollack [Pollack], we compute the accurate $D_k(g, p)$ by Theorem \[2\].

**Lemma 9.** Let $g = \sqrt[n-1]{g_0}$ satisfy Td with singular point $p$, then there exists a constant $h$ such that

$$D_k(g, p) = C_{n-1}^{k-1}\frac{n-k}{n}\frac{n}{2k}\frac{1}{2}h^{k-1}w_{n-1},$$

where $w_{n-1}$ is the volume of the unit $(n-1)$-sphere in $\mathbb{R}^n$ and $h$ is corresponding to $g_h = \sqrt[n-1]{h}(dt^2 + d\theta^2)$ and $g$ is asymptotic to $g_h$. 


Proof. Let \((t, \theta)\) be the cylindrical coordinates about \(p\) and \(X_p = \frac{\partial}{\partial \theta}\). In local coordinates, we write \(g = u^{4/(n-2)}g_c := u^{4/(n-2)}(dt^2 + d\theta^2)\).

In \(B_{r_0}(p)\backslash\{p\}\), half cylinder corresponds to
\[
C_{t_0}^n = \{(t, \theta) : \theta \in S^{n-1}, t \geq t_0 = -\ln r_0\},
\]
\[
\Sigma_{t_0} = \{(t, \theta) : \theta \in S^{n-1}, t = t_0\}.
\]
Define \(\nu_g = u^{-2/(n-2)}\frac{\partial}{\partial t}\) and \(d\sigma_g = u^{2(n-1)/(n-2)}d\theta\).

\[
\tilde{H}_a^b u_b X^a = \tilde{H}_1^1 \nu_1 X^1 = \tilde{H}_1^1 u^{\frac{2}{n-2}},
\]
where
\[
\tilde{H}_1^1 = H_1^1 - \frac{H_c^c}{n} = \frac{1}{(k-1)!} \sum_{i_1,\ldots,i_{k-1} = 1}^n \delta(i_1 \ldots i_{k-1} = 1) \delta(j_1 \ldots j_{k-1} = 1) A_{i_1}^1 \cdots A_{i_{k-1}}^1
\]
\[
- \frac{1}{n(k-1)!} \sum_{i_1,\ldots,i_{k-1} = 1}^n \delta(i_1 \ldots i_{k-1} = 1) \delta(j_1 \ldots j_{k-1} = 1) A_{i_1}^1 \cdots A_{i_{k-1}}^1.
\]

By Theorem 2, we know that for \(u\), there exists a corresponding singular solution \(u_h(t)\) and for some positive \(\alpha\),
\[
u(t) = u_h(t) + O(e^{-\alpha t}).
\]

By
\[
A = A_c - \frac{2}{n-2} u^{-1} \nabla_c^2 u + \frac{2n}{(n-2)^2} u^{-2} du \otimes du - \frac{2}{(n-2)^2} u^{-2} |du|_{g_c}^2,
\]
\[
A_c = -\frac{1}{2} dt^2 + \frac{1}{2} d\theta^2,
\]
we have
\[
A(\frac{\partial}{\partial t}, \frac{\partial}{\partial \theta}) = -\frac{1}{2} - \frac{2}{n-2} u^{-1} \nabla_c^2 u_h(\frac{\partial}{\partial t}, \frac{\partial}{\partial \theta}) + \frac{2n}{(n-2)^2} u_h^{-2} \frac{\partial u_h}{\partial t} \frac{\partial u_h}{\partial \theta} - \frac{2}{(n-2)^2} u_h^{-2} |du_h|_{g_c}^2 + (u_h^{-2} + u_h^{-1}) O(e^{-\alpha t})
\]
\[
= -\frac{1}{2} - \frac{2}{n-2} u_h^{-1} \frac{\partial^2 u_h}{\partial t^2} + \frac{2n}{(n-2)^2} u_h^{-2} \frac{\partial u_h}{\partial t} \frac{\partial u_h}{\partial \theta} + (u_h^{-2} + u_h^{-1}) O(e^{-\alpha t}),
\]
\[
A(\frac{\partial}{\partial t}, \frac{\partial}{\partial \theta_i}) = O(e^{-\alpha t})(u_h^{-1} + u_h^{-2}),
\]
\[
A(\frac{\partial}{\partial \theta_i}, \frac{\partial}{\partial \theta_j}) = \frac{1}{2} \delta_{ij} - \frac{2}{(n-2)^2} u_h^{-2} (\frac{\partial u_h}{\partial \theta_i})^2 \delta_{ij} + O(e^{-\alpha t}).
\]

Therefore
\[
\frac{1}{(k-1)!} \sum_{i_1, \ldots, i_{k-1} = 1}^{n} \delta(\frac{i_1}{j_1}, \ldots, \frac{1}{l}) A_{i_1}^{j_1} \cdots A_{i_{k-1}}^{j_{k-1}} A_{i_l}^l
\]

\[
= O(e^{-at}) + \frac{1}{(k-1)!} \sum_{i_1, \ldots, i_{k-1} = 1}^{n} \delta(\frac{i_1}{j_1}, \ldots, \frac{1}{l}) A_{i_1}^{j_1} \cdots A_{i_{k-1}}^{j_{k-1}} A_{i_l}^l
\]

\[
= O(e^{-at}) + \frac{1}{(k-1)!} \sum_{i_1, \ldots, i_{k-1} \neq 1}^{n} \delta(\frac{i_1}{j_1}, \ldots, \frac{i_{k-1}}{l}) A_{i_1}^{j_1} \cdots A_{i_{k-1}}^{j_{k-1}} A_{i_l}^l
\]

\[
= O(e^{-at}) + A_1^1 \sigma_{k-1}(g_{u_k} A_{g_{u_k} | i=2, \ldots, n}),
\]

and

\[
- \frac{1}{n} \frac{1}{(k-1)!} \sum_{i_1, \ldots, i_{k-1} = 1}^{n} \delta(\frac{i_1}{j_1}, \ldots, \frac{c}{l}) A_{i_1}^{j_1} \cdots A_{i_{k-1}}^{j_{k-1}} A_{i_l}^l
\]

\[
= O(e^{-at}) - \frac{1}{n} \frac{1}{(k-1)!} \sum_{i_1, \ldots, i_{k-1} = 1}^{n} \delta(\frac{i_1}{j_1}, \ldots, \frac{c}{l}) A_{i_1}^{j_1} \cdots A_{i_{k-1}}^{j_{k-1}} A_{i_l}^l
\]

\[
= O(e^{-at}) - \frac{1}{n} k(\sigma_{k-1}(g_{u_k} A_{g_{u_k} | i=2, \ldots, n}) A_1^1 + \sigma_k(g_{u_k} A_{g_{u_k} | i=2, \ldots, n})).
\]

Furthermore, we get

\[
\hat{H}_1 = O(e^{-at}) + A_1^1 \sigma_{k-1}(g_{u_k} A_{g_{u_k} | i=2, \ldots, n})(1 - \frac{k}{n}) - \frac{1}{n} k \sigma_k(g_{u_k} A_{g_{u_k} | i=2, \ldots, n})
\]

\[
= O(e^{-at}) + (1 - \frac{k}{n}) u_h^{-k(\frac{1}{n-1})} A_{11} \sigma_{k-1}(g_{u_k} A_{g_{u_k} | i=2, \ldots, n}) - \frac{k}{n} u_h^{-k(\frac{1}{n-1})} \sigma_k(g_{u_k} A_{g_{u_k} | i=2, \ldots, n})
\]

\[
= O(e^{-at}) + (1 - \frac{k}{n}) u_h^{-k(\frac{1}{n-1})} \left( - \frac{1}{2} - \frac{2}{n-2} u_h^{-2} \frac{\partial^2 u_h}{\partial t^2} + \frac{2n-2}{(n-2)^2} u_h^{-2} \frac{\partial u_h}{\partial t} \right)
\]

\[
\times \frac{1}{2} - \frac{2}{(n-2)^2} u_h^{-2} \left( \frac{\partial^2 u_h}{\partial t^2} \right)^2
\]

\[
= O(e^{-at}) + u_h^{-k(\frac{1}{n-1})} C_{n-1}^{k-1} - \frac{k}{n} u_h^{-k(\frac{1}{n-1})} C_{n-1}^{k-1} \left( - \frac{1}{2} - \frac{2}{(n-2)^2} u_h^{-2} \frac{\partial u_h}{\partial t} \right)
\]

\[
= O(e^{-at}) + u_h^{-k(\frac{1}{n-1})} C_{n-1}^{k-1} - \frac{k}{n} u_h^{-k(\frac{1}{n-1})} C_{n-1}^{k-1} \left( - \frac{1}{2} - \frac{2}{(n-2)^2} u_h^{-2} \frac{\partial u_h}{\partial t} \right)
\]

\[
= O(e^{-at}) + u_h^{-k(\frac{1}{n-1})} C_{n-1}^{k-1} - \frac{k}{n} u_h^{-k(\frac{1}{n-1})} C_{n-1}^{k-1} \left( - \frac{1}{2} - \frac{2}{(n-2)^2} u_h^{-2} \frac{\partial u_h}{\partial t} \right)
\]

\[
= O(e^{-at}) + u_h^{-k(\frac{1}{n-1})} C_{n-1}^{k-1} - \frac{k}{n} u_h^{-k(\frac{1}{n-1})} C_{n-1}^{k-1} \left( - \frac{1}{2} - \frac{2}{(n-2)^2} u_h^{-2} \frac{\partial u_h}{\partial t} \right)
\]

\[
= O(e^{-at}) + u_h^{-k(\frac{1}{n-1})} C_{n-1}^{k-1} - \frac{k}{n} u_h^{-k(\frac{1}{n-1})} C_{n-1}^{k-1} \left( - \frac{1}{2} - \frac{2}{(n-2)^2} u_h^{-2} \frac{\partial u_h}{\partial t} \right)
\]

\[
= O(e^{-at}) + u_h^{-k(\frac{1}{n-1})} C_{n-1}^{k-1} - \frac{k}{n} u_h^{-k(\frac{1}{n-1})} C_{n-1}^{k-1} \left( - \frac{1}{2} - \frac{2}{(n-2)^2} u_h^{-2} \frac{\partial u_h}{\partial t} \right)
\]

\[
= O(e^{-at}) + u_h^{-k(\frac{1}{n-1})} C_{n-1}^{k-1} - \frac{k}{n} u_h^{-k(\frac{1}{n-1})} C_{n-1}^{k-1} \left( - \frac{1}{2} - \frac{2}{(n-2)^2} u_h^{-2} \frac{\partial u_h}{\partial t} \right)
\]

Where \( K_h(t) = \frac{1}{2} - \frac{2}{n-2} u_h^{-1} \frac{\partial^2 u_h}{\partial t^2} + \frac{2n-2}{(n-2)^2} u_h^{-2} \left( \frac{\partial u_h}{\partial t} \right)^2 - \frac{1}{2} + \frac{2}{(n-2)^2} u_h^{-2} \left( \frac{\partial u_h}{\partial t} \right)^2.

As \( u_h \) satisfy

\[
\frac{1}{2} = (1 - \frac{2}{n-2} \left( u_h^{-1} \right)^2) \left( \frac{k}{n} - \frac{1}{2} \right) - \frac{2}{n-2} u_h^{-1} \frac{\partial^2 u_h}{\partial t^2} + \frac{2n-2}{(n-2)^2} u_h^{-2} \left( \frac{\partial u_h}{\partial t} \right)^2 - \frac{1}{2} + \frac{2}{(n-2)^2} u_h^{-2} \left( \frac{\partial u_h}{\partial t} \right)^2.
\]

we obtain
Furthermore, it holds that

\[ (4.1) \]

By (4.1),

\[ K_h(t) = -1 + \frac{2n}{(n-2)^2} u_h^2 \left( \frac{\partial u_h}{\partial t} \right)^2 \]

\[ + \frac{n}{2k} \left( \frac{u_h}{2} \right)^{2} (1 - \frac{2}{n-2} (\frac{u_h}{u_0})^2)^{-k+1} \]

\[ - \left( \frac{u_h}{u_0} \right)^2 (1 - \frac{2}{n-2} (\frac{u_h}{u_0})^2)^{-k+1} + \frac{u_{h,t}^2}{u_h^2} \frac{2n}{k} \frac{1}{(n-2)^2}. \]

So

\[ \hat{H}_1 = O(e^{-\alpha t}) + C_{n-1} \frac{2n}{(n-2)^2} u_h^2 \left( \frac{\partial u_h}{\partial t} \right)^2 \]

\[ + \left( \frac{u_h}{2} \right)^{2} (1 - \frac{2}{n-2} (\frac{u_h}{u_0})^2)^{-k+1} \frac{2n}{k} \frac{1}{(n-2)^2}. \]

Furthermore, it holds that

\[ \int_{\Sigma_t} \hat{H}_1^n u_h X^n d\sigma_g = \left[ O(e^{-\alpha t}) + C_{n-1} \frac{2n}{(n-2)^2} u_h^2 \left( \frac{\partial u_h}{\partial t} \right)^2 \right] u_h^n \]

\[ \int_{\Sigma_t} \left[ O(e^{-\alpha t}) + C_{n-1} \frac{2n}{(n-2)^2} u_h^2 \left( \frac{\partial u_h}{\partial t} \right)^2 \right] u_h^n \]

As \( D_k(p, g) \) is independent of \( t_0 \), we know

\[ D_k(p, g) = C_{n-1} \frac{2n}{(n-2)^2} u_h^2 \left( \frac{\partial u_h}{\partial t} \right)^2 u_h^n. \]

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