On the super-energy radiative gravitational fields

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Abstract
We extend our recent analysis (Ferrando J J and Sáez J A 2012 Class. Quantum Grav. 29 075012) on the Bel radiative gravitational fields to the super-energy radiative gravitational fields defined by García-Parrado (2008 Class. Quantum Grav. 25 015006). We give an intrinsic characterization of the new radiative fields and consider some distinguished classes of both radiative and non-radiative fields. Several super-energy inequalities are improved.

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(Some figures may appear in colour only in the online journal)

1. Introduction

Elsewhere [1], we have analyzed the Bel concept of intrinsic radiative gravitational field [2–4] and have shown that the three radiative types, \(N\), \(III\) and \(II\), correspond with three different physical situations: pure radiation, asymptotic pure radiation and generic (non-pure, non-asymptotic pure) radiation. In the aforementioned paper, we have also shown that, for Bel non-radiative fields, the minimum value of the relative super-energy is acquired by the observers at rest with respect to the field (those seeing a vanishing super-Poynting vector).

Following Bel’s ideas, García-Parrado [5] has introduced new relative super-energy quantities and has written the full set of equations for these super-energy quantities. This study leads naturally to a concept of intrinsic radiation which is less restrictive than that given by Bel.

Here, we extend our analysis [1] on the Bel approach to the García-Parrado radiative gravitational fields. We show that the non-radiative fields correspond to type \(D\) metrics and some classes of type \(I\) metrics already considered in the literature: the \(IM^+\) and the \(IM^\infty\) metrics [6]. These classes appear in a natural way when classifying the Bel–Robinson tensor as an endomorphism [7, 8]. Moreover, in these spacetimes, the four null Debever directions span a 3-plane [6, 9, 10]. On the other hand, we show that three classes of García-Parrado type \(I\) radiative fields can be considered: the \(IM^-\), the \(IM^-\infty\) and the generic type \(I\) metrics.
Our study deepens our understanding of the Bel–Robinson tensor and performs the Bel and García-Parrado concepts. The interest of this subject and the connection between super-energy and gravitational energy have been widely remarked and analyzed in the literature (see [11, 5] and references therein). Now we want to add two brief remarks.

The Bel super-energy density is the leading-order contribution to the quasi-local energy in vacuum [12]. Is there a similar property for the other super-energy quantities? This is a question to be analyzed in the future. The quasi-local energy is associated with a proper energy surface density [13]. Is it possible to define quasi-local quantities associated with a proper momentum surface density or with a spatial stress? The leading-order contribution of these (tensorial) quasi-local quantities could be given by (tensorial) relative super-energy quantities.

The asymptotic behavior of the field created by an isolated system is given by the Sachs peeling theorem [14]. Far from the sources or in the transitional zone, the field has a Bel radiative behavior, type $N$ or types $III$ and $II$, respectively. However, the terms of Bel non-radiative type ($I$ or $D$) are dominant near the sources. To analyze if these terms correspond to García-Parrado radiative or non-radiative fields is a further question to be considered. Another possible approach to this subject could be to study a peeling-like theorem for the Bel–Robinson tensor itself. In this approach, the algebraic study of the Bel–Robinson tensor [7, 8] will play an important role.

This paper is organized as follows. In section 2, we introduce the basic concepts and notation and summarize previous results which help us to understand this paper. The super-energy inequalities presented in [1] are revisited in section 3, where we extend the kind of bounds already known for the super-energy density to all the tensorial quantities defined by contracting the Bel–Robinson tensor with the observer velocity. In section 4, we define the proper super-energy scalars and show that for Bel non-radiative fields, they are acquired by the observers at rest and for Bel radiative fields, it is the infimum for all the observers of the super-energy scalars. We also define the principal super-stresses of a Bel non-radiative field. Section 5 is devoted to studying both the García-Parrado radiative and non-radiative gravitational fields. The relationships between the different considered classes, and both the Debever directions and the algebraic properties of the Bel–Robinson tensor, are outlined. Both radiative and non-radiative classes are characterized in terms of the principal super-stresses. In section 6, we analyze the results of this paper with several diagrams which clarify the relation between the different classes of type $I$ fields, and we introduce a radiation scalar which measures how radiative (in the sense defined by García-Parrado) is a gravitational field at a point. Finally, we present three appendices. The first one summarizes the algebraic classes of the Bel–Robinson tensor, the second one presents some constraints on the relative super-energy quantities and the third one gives accurate proofs of propositions 1 and 2. The notation that we use in this work is the same as that used in [1].

2. The Bel approach to radiative gravitational states

With the purpose of defining intrinsic states of gravitational radiation, Bel [2–4] introduced the super-energy Bel tensor which plays an analogous role for gravitation to that played by the Maxwell–Minkowski tensor for electromagnetism. In the vacuum case, this super-energy Bel tensor is divergence-free and it coincides with the super-energy Bel–Robinson tensor $T$.

Using tensor $T$, Bel defined the relative super-energy density and the super-Poynting vector associated with an observer. Then, following the analogy with electromagnetism, the intrinsic radiative gravitational fields are those for which the Poynting vector does not vanish for any observer [2, 4].
This analogy with electromagnetism also plays a fundamental role in our analysis [1] of the Bel radiative and non-radiative fields. Now we summarize our main results introducing the basic concepts required in this work.

2.1. The Bel–Robinson tensor: algebraic restrictions

In terms of the Weyl tensor $\omega$, the Bel–Robinson tensor takes the expression [2–4]

$$T_{\mu\nu} = \frac{1}{2}(\omega_{\alpha\beta} W_{\mu\nu\rho\sigma} + \ast W_{\mu\nu\rho\sigma})$$

(1)

For any observer $u$, the relative electric and magnetic Weyl fields are given by $E = W(u; u)$ and $H = \ast W(u; u)$, respectively. The following relative super-energy quantities can be defined:

$$\tau = T(u, u, u, u), \quad q_\perp = -T(u, u, u)_\perp, \quad t_\perp = T(u, u)_\perp, \quad Q_\perp = -T(u)_\perp, \quad T_\perp,$$

(2)

where, for a tensor $A$, $A_\perp$ denotes the orthogonal projection defined by the projector $\gamma = u \otimes u + g$.

Bel introduced the super-energy density $\tau$ and the super-Poynting (energy flux) vector $q_\perp$ years ago [2, 4]. Bonilla and Senovilla [15] used $t_\perp$ in studying the causal propagation of gravity and, recently, García-Parrado [5] has considered $Q_\perp$ and $T_\perp$. The last three relative quantities have been called the super-stress tensor, the stress flux tensor and the stress–stress tensor, respectively. The expression of the relative super-energy quantities (2) in terms of the electric and magnetic Weyl tensors can be found in [5].

In vacuum, the Bianchi identities imply that $T$ satisfies $\nabla \cdot T = 0$. For any observer, this equation shows that the relative quantities $q_\perp$ and $Q_\perp$ play the role of fluxes of the relative quantities $\tau$ and $t_\perp$, respectively [5].

The algebraic constraints on the Bel–Robinson tensor playing a similar role to that played by the Rainich conditions [16] for the electromagnetic energy tensor were obtained by Bergqvist and Lankinen [17].

On the other hand, we have studied elsewhere [7, 8] the Bel–Robinson tensor $T$ as an endomorphism on the nine-dimensional space of the traceless symmetric tensors. Its nine eigenvalues depend on the three (complex) Weyl eigenvalues $\{\rho_i\}$ as $t_k = |\rho_k|^2$, $\tau_k = \rho_k \bar{\rho}_k$, $(ijk)$ being an even permutation of $(123)$. Three independent invariant scalars can be associated with $T$. In fact, the nine eigenvalues $\{t_i, \tau_i, \chi\}$ can be written in terms of three scalars $\{\kappa_i\}$ as [7]:

$$t_i = 2(\kappa_j + \kappa_k), \quad \tau_i = -2(\kappa_j + i \kappa), \quad \kappa^2 = \kappa_1 \kappa_2 + \kappa_2 \kappa_3 + \kappa_3 \kappa_1.$$

(3)

We have also intrinsically characterized the algebraic classes of $T$ [7] and have given their Segre type and their canonical form [8]. Some of these results which we need here are summarized in appendix A. In what follows, we will make use of the scalar invariants $\alpha$, $\xi$ and $\chi$ defined by the expressions:

$$\alpha \equiv \frac{1}{2} \sqrt{\langle T, T \rangle}, \quad \langle T, T \rangle = T_{\alpha\beta\gamma\delta} T^{\alpha\beta\gamma\delta} = \frac{1}{64}[(W, W)^2 + (\ast W, W)^2] \geq 0;$$

$$\xi \equiv \frac{1}{4} \sum_{i=1}^{3} t_i^2; \quad \chi \equiv \frac{1}{4} \sum_{i=1}^{3} t_i^2, \quad 8\xi^2 + \alpha^2 = 6\chi,$$

(4)

(5)

where the above constraint between the invariants $\alpha$, $\chi$ and $\xi$ is a consequence of the restrictions (3) on the Bel–Robinson eigenvalues.
2.2. Bel radiative gravitational fields

The super-energy density \( \tau \) vanishes only when the Weyl tensor \( W \) vanishes. Then, if we consider \( \tau \) as a measure of the gravitational field, its flux \( q_\perp \) denotes the presence of gravitational radiation. This is the point of view of Bel [2, 4], who gave the following definition.

**Definition 1** (Intrinsic gravitational radiation, Bel 1958). *In a vacuum spacetime, there exists intrinsic gravitational radiation (at a point) if the super-Poynting vector \( q_\perp \) does not vanish for any observer.*

It is known that the Bel radiative gravitational fields are those of Petrov–Bel types \( \text{N}, \text{III} \) and \( \text{II} \) [4]. Then, also motivated by the Lichnerowicz ideas [18], we have proposed to distinguish three physical situations [1]: the pure gravitational radiation (type \( \text{N} \)), the asymptotic pure gravitational radiation (type \( \text{III} \)) and the generic radiative states (type \( \text{II} \)).

2.3. Bel non-radiative gravitational fields: observer at rest and proper super-energy density

From Bel’s point of view, non-radiative gravitational fields are those for which an observer exists who sees a vanishing relative super-Poynting vector. The following definition naturally arises:

**Definition 2.** The observers for whom the super-Poynting vector vanishes are said to be observers at rest with respect to the gravitational field.

It is known [4] that the non-radiative gravitational fields are the Petrov–Bel-type \( \text{I} \) or \( \text{D} \) spacetimes and the observers at rest with respect to the gravitational field are those for whom the electric and magnetic Weyl tensors simultaneously diagonalize. In a type \( \text{I} \) spacetime, a unique observer \( e_0 \) at rest with respect to the gravitational field exists, and in a type \( \text{D} \) spacetime, the observers \( e_0 \) at rest with respect to the gravitational field are those lying on the Weyl principal plane.

In [1], we have given the following definition and result.

**Definition 3.** We call proper super-energy density of a gravitational field the invariant scalar \( \xi \) given in (5).

**Theorem 1.** For a Bel non-radiative gravitational field (I or D), the minimum value of the relative super-energy density is the proper super-energy density \( \xi \), which is acquired by the observers at rest with respect to the field.

For the Bel radiative gravitational field (N, III or II), the super-energy density decreases and tends to the proper super-energy density \( \xi \) as the velocity vector of the observer approaches the unique fundamental direction \( \ell \).

For pure and asymptotic pure radiation (N or III), the proper super-energy density \( \xi \) is zero. For generic radiation (type II), \( \xi \) is strictly positive.

3. Super-energy inequalities

The Bel–Robinson tensor satisfies the generalized dominant energy condition [11, 19], which implies that for any observer \( u \), the relative quantities \( \tau \) and \( q = -T(u, u, u) \) are subject to the known inequalities, \( \tau \geq 0 \), \( (q, q) \leq 0 \). In [1], we have generalized these super-energy inequalities in two aspects. On the one hand, we have shown that \( \tau \) and \( (q, q) \) are bounded by scalars depending on the main quadratic invariant \( \alpha \) given in (4).
On the other hand, we have extended these kinds of bounds to other spacetime relative quantities (see [1, theorem 2]). Now we present stronger inequalities on these spacetime quantities, restrictions that lead naturally to the concept of super-energy scalars.

From the algebraic properties of the Bel–Robinson tensor and, in particular from the Bergqvist and Lankinen conditions [17], we obtain the following propositions (see the proof in appendix C).

**Proposition 1.** Let $\alpha$ and $\chi$ be the Bel–Robinson invariants defined in (4) and (5), and \( t = T(u,u) \) for any observer \( u \). Then, it holds:

- For type \( N \), \((t,t) = \chi = \frac{1}{2}\alpha^2 = 0.\)
- For type \( III \), \((t,t) > \chi = \frac{1}{2}\alpha^2 = 0.\)
- For type \( II \), \((t,t) > \chi = \frac{1}{2}\alpha^2 > 0.\)
- For type \( D \), \((t,t) > \chi = \frac{1}{2}\alpha^2 > 0.\)
- For type \( I \), \((t,t) > \chi > \frac{1}{2}\alpha^2 > 0.\)

Moreover, (i) for types \( I \) and \( D \), \((t_0,t_0) = \chi, \) with \( t_0 = T(e_0,s_0) \), \( e_0 \) being a principal observer, and (ii) for types \( III \) and \( II \), \((t,t)\) tends to \( \chi \) as the velocity vector of the observer approaches the unique fundamental direction \( \ell \).

**Proposition 2.** Let $\alpha$ and $\xi$ be the Bel–Robinson invariants defined in (4) and (5), and \( q = -T(u,u) \) for any observer \( u \). Then, it holds:

- For type \( N \), \((q,q) = -\xi^2 = -\frac{1}{2}\alpha^2 = 0.\)
- For type \( III \), \((q,q) < -\xi^2 = -\frac{1}{2}\alpha^2 = 0.\)
- For type \( II \), \((q,q) < -\xi^2 = -\frac{1}{2}\alpha^2 < 0.\)
- For type \( D \), \((q,q) < -\xi^2 = -\frac{1}{2}\alpha^2 < 0.\)
- For type \( I \), \((q,q) < -\xi^2 < -\frac{1}{2}\alpha^2 \leq 0.\)

Moreover, (i) for types \( I \) and \( D \), \((q_0,q_0) = -\xi^2, \) with \( q_0 = -T(e_0,s_0) \), \( e_0 \) being a principal observer, and (ii) for types \( III \) and \( II \), \((q,q)\) tends to \( -\xi^2 \) as the velocity vector of the observer approaches the unique fundamental direction \( \ell \).

Now we can state the following theorem.

**Theorem 2** (Super-energy inequalities). Let \( T \) be the Bel–Robinson tensor and for any observer \( u \), let us define the relative spacetime quantities:

\[
Q = -T(u), \quad t = T(u,u), \quad q = -T(u,u,u), \quad \tau = T(u,u,u,u). \tag{6}
\]

Then, the following super-energy inequalities hold:

\[
(T,T) \equiv 4\alpha^2 \geq 0, \quad (Q,Q) = -\alpha^2 \leq 0, \\
(t,t) \geq \chi \geq \frac{1}{2}\alpha^2 \geq 0, \quad (q,q) \leq -\xi^2 \leq -\frac{1}{2}\alpha^2 \leq 0, \quad \tau \geq \xi \geq \frac{1}{2}\alpha \geq 0, \tag{7}
\]

where $\xi$ and $\chi$ are the invariant scalars defined in (5).

The first, second and last conditions in (7) have been stated and shown in [1] (theorems 1 and 2). The third condition in (7) is a consequence of proposition 1 and, finally, the fourth condition in (7) is a consequence of proposition 2.
4. Lower bounds on the super-energy scalars and proper super-energy scalars

For any observer \( u \), the amounts of the super-energy quantities (2) are the super-energy scalars given by

\[
\tau, \quad |q_\perp| = \sqrt{(q_\perp, q_\perp)}, \quad |t_\perp| = \sqrt{(t_\perp, t_\perp)}, \quad |Q_\perp| = \sqrt{(Q_\perp, Q_\perp)}, \quad |T_\perp| = \sqrt{(T_\perp, T_\perp)}.
\]

Note that \( \text{tr} T_\perp = t_\perp \) and \( \text{tr} t_\perp = \tau \) and, consequently, the Weyl tensor vanishes when \( T_\perp \) or \( t_\perp \) vanish [5]. A similar property holds for the super-energy scalars \( |T_\perp| \) and \( |t_\perp| \) because they are the modulus of the spatial tensors \( T_\perp \) and \( t_\perp \), respectively.

Theorem 1 shows that the proper super-energy density \( \xi \) is the infimum for all the observers \( u \) of the super-energy densities \( \tau_u \). This property justifies the following definition.

**Definition 4.** We call proper super-energy scalars the infimum for all the observers of the super-energy scalars.

The two last inequalities in (7) imply

\[
\tau^2 \geq \tau^2 - |q_\perp|^2 \geq \xi^2, \quad |q_\perp| \geq 0.
\]

On the other hand, from (8) and the quadratic scalar constraints (B.3), we obtain

\[
|t_\perp|^2 = \frac{1}{2} \tau^2 + \frac{1}{3} \alpha^2 + \frac{1}{3} |q_\perp|^2 \geq \frac{1}{6} \xi^2 + \frac{1}{6} \alpha^2, \quad |Q_\perp|^2 = 2 \tau^2 - \frac{1}{3} \alpha^2 - |q_\perp|^2 \geq 2 \xi^2 - \frac{1}{3} \alpha^2, \quad |T_\perp|^2 = 5 \tau^2 + \alpha^2 - 4 |q_\perp|^2 \geq 5 \xi^2 + \alpha^2.
\]

Moreover, as a consequence of proposition 2, equalities in (8) and (9) hold for the observers at rest in the case of Bel non-radiative fields and for Bel radiative fields, inequalities (9) approach an equality as the velocity vector of the observer approaches the unique fundamental direction \( \ell \). Consequently, we can state the following theorem.

**Theorem 3.** The proper scalars of radiated super-energy \( \xi_q \), super-stress \( \xi_t \), radiated super-stress \( \xi_Q \) and stress–stress \( \xi_T \) are, respectively, the invariant scalars:

\[
\xi_q = 0, \quad \xi_t = \sqrt{\frac{1}{6} \xi^2 + \frac{1}{6} \alpha^2}, \quad \xi_Q = \sqrt{2 \xi^2 - \frac{1}{3} \alpha^2}, \quad \xi_T = \sqrt{5 \xi^2 + \alpha^2},
\]

where \( \xi \) and \( \alpha \) are the proper energy density (5) and the main quadratic scalar (4).

For a Bel non-radiative gravitational field (I or D), the proper super-energy scalars are acquired by the observers at rest with respect to the field: if \( e_0 \) is such an observer, then \( \tau_0 = \xi, \quad |q_{0\perp}| = \xi_q = 0, \quad |t_{0\perp}| = \xi_t, \quad |Q_{0\perp}| = \xi_Q, \quad |T_{0\perp}| = \xi_T \).

For a Bel radiative gravitational field (N, III or II), the scalar associated with each relative super-energy quantity decreases and tends to the proper scalar of this quantity as the velocity vector of the observer approaches the unique fundamental direction \( \ell \).

On the other hand, from expressions (9), we obtain a result which extends a previous one given in [15].

**Proposition 3.** The super-energy scalars are subject to the following constraints:

\[
|Q_\perp|^2 - |q_\perp|^2 = 3(\tau^2 - |t_\perp|^2) \geq 0, \quad 3 \tau - |T_\perp| \geq 0.
\]

Note that for Bel non-radiative fields (types I and D), we can consider the super-energy quantities relative to the observers \( e_0 \) at rest with respect to the field: the proper super-energy \( \tau_0 = \xi \), the proper Poynting vector \( q_{0\perp} = 0 \), the proper super-stress tensor \( t_{0\perp} \), the proper super-stress flux tensor \( Q_{0\perp} \) and the proper stress–stress tensor \( T_{0\perp} \). If \( \{e_0, e_1\} \) is a Weyl
canonical frame of a type I or type D spacetime, from the Bel–Robinson canonical form (see [1]), we obtain

\[ t_{0\perp} = \sum_{i=1}^{3} \kappa_i e_i \otimes e_i, \quad 4\kappa_i = t_k + t_j - t_i, \quad i \neq j \neq k \neq i, \quad (11) \]

\[ Q_{0\perp} = \kappa \sum_{\sigma} (e_{\sigma(1)} \otimes e_{\sigma(2)} \otimes e_{\sigma(3)}), \quad \kappa^2 = \kappa_1\kappa_2 + \kappa_2\kappa_3 + \kappa_3\kappa_1, \quad (12) \]

where \( \sigma \) is a permutation of \((123)\).

The three eigenvalues \( \kappa_i \) of the super-stress tensor \( t_{0\perp} \) play for gravitation the same role played by the principal stresses for electromagnetism. Then, they are the principal super-stresses, which are subject to the constraint

\[ \kappa_1 + \kappa_2 + \kappa_3 = \xi. \quad (13) \]

It is worth remarking that in type D, the proper super-stress tensor \( t_{0\perp} \) depends on the chosen principal observer. Nevertheless, the principal super-stresses do not.

5. García-Parrado radiative gravitational fields

The super-stress tensor \( t_{\perp} \) vanishes only when the Weyl tensor \( W \) vanishes. Thus, we can also consider \( t_{\perp} \) as a measure of the gravitational field. Then, its flux \( Q_{\perp} \) denotes the presence of gravitational radiation. This fact has been pointed out by García-Parrado [5], who has given the following definition.

**Definition 5** (Intrinsic super-energy radiation, García-Parrado 2008). In a vacuum spacetime, there exists intrinsic super-energy radiation (at a point) if the stress flux tensor \( Q_{\perp} \) does not vanish for any observer.

For any observer, we have \( \text{tr } Q_{\perp} = q_{\perp} \). Then, \( q_{\perp} \) vanishes when \( Q_{\perp} \) vanishes. Consequently,

**Proposition 4.** Every Bel radiative gravitational field is a García-Parrado radiative gravitational field.

Every García-Parrado non-radiative gravitational field is a Bel non-radiative gravitational field and the observers who do not see stress flux (\( Q_{\perp} = 0 \)) are the observers at rest with respect to the field.

Thus, the definition given by García-Parrado is less restrictive than Bel’s definition and it allows type I radiative gravitational fields [5]. Now we analyze them and consider several significant classes of both radiative and non-radiative García-Parrado gravitational fields.

5.1. Super-energy non-radiative gravitational fields

According to the proposition above, the García-Parrado non-radiative fields are type I or type D metrics with a vanishing proper stress flux tensor \( Q_{0\perp,0\perp} \), or equivalently, with a vanishing \( \xi_Q \). This condition is equivalent to the Bel–Robinson tensor having real eigenvalues as a consequence of expressions (12) and (3). On the other hand, García-Parrado [5] showed that \( Q_{0\perp} = 0 \) if and only if the proper electric and magnetic Weyl tensors are linearly dependent. Thus, we can state the following theorem.

**Theorem 4.** The García-Parrado non-radiative gravitational fields are the type I or type D metrics which satisfy one of the following equivalent conditions:
(i) The proper stress flux tensor vanishes: \( Q_{0\perp} = 0 \).
(ii) The proper electric and magnetic Weyl tensors are linearly dependent:
\[ E_0 \otimes H_0 = H_0 \otimes E_0. \]
(iii) The proper scalar of the stress flux vanishes:
\[ \xi_Q \equiv \sqrt{2\xi^2 - \frac{1}{2} \alpha^2} = 0. \]
(iv) The Bel–Robinson tensor has real eigenvalues.

Note that characterizations (i) and (ii) of the above theorem make reference to relative quantities, namely the stress flux tensor and the electric and magnetic parts of the Weyl tensor. Nevertheless, conditions (iii) and (iv) are intrinsic in the Bel–Robinson tensor \( T \) and they impose, respectively, the vanishing of an invariant scalar and an algebraic property of \( T \).

As pointed out previously [5], all the type \( D \) gravitational fields satisfy conditions of theorem 4. Now we study the type \( I \) metrics which satisfy them.

Elsewhere [10], we have studied the aligned Weyl fields, i.e. the spacetimes with linearly dependent electric and magnetic Weyl tensors, and we have shown that they correspond to metrics of types \( IM^+ \) or \( IM^\infty \) in the classification of McIntosh and Arianrhod [6]. This means that the scalar invariant \( M \) defined in (A.1) is, respectively, a positive real number or infinity, and they are the type \( I \) spacetimes where the four null Debever directions span a 3-plane [6, 9, 10]. Moreover, the (spatial) direction orthogonal to this 3-plane is the Weyl principal direction associated with the Weyl eigenvalue with the shortest modulus.

We can state the last condition in terms of the principal super-stresses \( \kappa_i \). Indeed, say \( \rho_1 \) is the shortest Weyl eigenvalue, then \( t_1 \) is the shortest Bel–Robinson real eigenvalue and, from (11), \( \kappa_1 \) is the largest principal super-stress. Thus, we have

**Proposition 5.** A type \( I \) spacetime defines a super-energy non-radiative gravitational field if and only if the four null Debever directions span a 3-plane.

Moreover, the direction orthogonal to this 3-plane is that defined by the eigenvector associated with the largest principal super-stress.

Finally, we analyze how to distinguish the three types of non-radiative fields in terms of relative super-energy quantities. The above results imply that all the three cases, \( IM^+, IM^\infty \) and \( D \), are subject to the same constraints for the super-energy scalars:

\[ \xi_q = \xi_Q = 0, \quad \xi_T = 3\xi_t = 3\xi. \]

Nevertheless, we can distinguish these three types by using the principal super-stresses \( \kappa_i \). In type \( D \), we have \( t_2 = t_3 \neq 0, t_1 = 4t_2 \) and then \( \kappa_2 = \kappa_3 = -2\kappa_1 \neq 0 \) as a consequence of (11). In type \( IM^\infty \), we have \( t_1 = t_2 \neq 0, t_3 = 0 \) and then \( \kappa_1 = \kappa_2 = 0 \).

**Proposition 6.** In a super-energy non-radiative spacetime, the principal super-stresses satisfy
\[ \kappa_1 \leq 0 \leq \kappa_2 \leq \kappa_3, \quad |\kappa_1| \leq |\kappa_2|. \]

Moreover, the spacetime is
- Type \( D \) iff \( \kappa_1 = -\frac{1}{2} \kappa_2 < 0 < \kappa_2 = \kappa_3 \).
- Type \( IM^\infty \) iff \( \kappa_1 = 0 = \kappa_2 < \kappa_3 \).
- Type \( IM^+ \) otherwise, and then \( \kappa_1 < 0 < \kappa_2 < \kappa_3 \).

We can quote some examples of non-radiative fields:

(i) All the type \( D \) vacuum solutions are known [20, 21].
(ii) A result inferred by Szekeres [22] and confirmed by Brans [23] states that no vacuum solutions of type \( IM^\infty \) exist (see [25] for a generalization).
(iii) The purely electric or purely magnetic solutions are of type $IM^+$ [26]. Every static solution has a purely electric Weyl tensor and, consequently, is of type $IM^+$. Both static and non-static purely electric metrics can be found in the Kasner vacuum solutions [24, 21]. Some restrictions are known on the existence of other $IM^+$ vacuum solutions (see [10] and references therein).

5.2. Super-energy radiative gravitational fields

According to propositions 4 and 5, the García-Parrado radiative fields are the Bel radiative fields (types $N$, $III$ and $II$) and the type $I$ metrics with a non-vanishing proper stress flux tensor $Q_{0L0M}$ [26]. According to theorem 4, the radiative type $I$ fields can be characterized by the following equivalent conditions: (i) the proper electric and magnetic Weyl tensors are linearly independent, (ii) the scalar of the stress flux does not vanish, (iii) the Bel–Robinson tensor has some complex eigenvalue and (iv) the four null Debever directions define a frame. Now we consider some relevant classes of type $I$ radiative fields.

The invariant classification of the Bel–Robinson tensor [7, 8] leads to three classes with non-real eigenvalues (see appendix A): the most regular one $\mathcal{I}$, and those with a double or a triple degeneration, which correspond to types $IM^−$ or $IM^{−6}$, respectively, in the classification of McIntosh and Arianrhod [6]. Moreover, they are the type $I$ spacetimes where the four null Debever directions define a frame. A frame of vectors is said to be symmetric if they cannot be distinguished by their mutual metric products [27]. In [10], we showed that the four null Debever directions define a symmetric frame for the case $IM^{−6}$ and a partially symmetric frame for $IM^{−6}$. Thus, we have

**Proposition 7.** A type $I$ spacetime defines a super-energy radiative gravitational field if and only if the four null Debever vectors define a frame.

Moreover, this frame is symmetric for type $IM^{−6}$ spacetimes and it is partially symmetric (symmetric by pairs) for type $IM^{−6}$ spacetimes. Otherwise, the spacetime is of regular type $\mathcal{I}_r$.

We can label the two degenerate classes in terms of the principal super-stresses $\kappa_i$. Indeed, from (11), condition $t_1 = t_2 = t_3$ implies three equal principal super-stresses, $\kappa_1 = \kappa_2 = \kappa_3$, and $t_1 \neq t_2 = t_3$ implies two equal principal super-stresses, $\kappa_1 \neq \kappa_2 = \kappa_3$.

**Proposition 8.** A super-energy radiative type $I$ spacetime is

Type $IM^{−6}$ iff $\kappa_1 = \kappa_2 = \kappa_3$.
Type $IM^{−6}$ iff $\kappa_1 = \kappa_2 \neq \kappa_3$.
Type $\mathcal{I}_r$ otherwise.

We can quote some examples of radiative fields:

(i) The Petrov homogeneous vacuum solution [28, 21] is of type $IM^{−6}$.
(ii) The windmill metric [29, 21] is a family of type $IM^{−6}$ vacuum solutions.
(iii) Vacuum solutions of generic radiative type $I_r$ are quite abundant. We have, for example, the Taub family of metrics [30, 21] and its counterpart with time-like orbits or its related windmill solutions [25].

6. An analysis of type $I$ classes

The ‘degenerate’ type $I$ classes defined in [6] in terms of the adimensional-invariant $M$ have a nice interpretation in terms of the null Debever directions [6, 9, 10]. Moreover, these
classes also appear when classifying the Bel–Robinson tensor as an endomorphism on the nine-dimensional space of the traceless symmetric tensors [7, 8].

In this paper, we have shown how García-Parrado radiative and non-radiative fields can be distinguished in terms of the invariant $M$. Figure 1(a) presents the complex plane where we can plot the different values of $M$. Out of the real axes, points correspond to generic radiative fields $I_r$. For real negative values of $M$, we have the degenerate radiative fields $IM^-$ and $IM^{-6}$. The real non-negative values of $M$ correspond to non-radiative fields: type $D$ when $M = 0$, type $IM^+$ when $M > 0$ and type $IM^\infty$ when $M$ is not defined.

The study of the different classes using the Bel–Robinson tensor [7, 8] implies analyzing the Bel–Robinson real eigenvalues $t_i$. An approach using the super-energy relative magnitudes can be built with the principal super-stresses $\kappa_i$.

### 6.1. A diagram approach using Bel–Robinson real eigenvalues

Constraints on the real eigenvalues $t_i$:

$$t_1 + t_2 + t_3 = 4\xi, \quad t_1^2 + t_2^2 + t_3^2 = 4\chi \leq 8\xi^2.$$  

Then, the adimensional parameters $\bar{t}_i = \frac{t_i}{4\xi}$ satisfy

$$\bar{t}_1 + \bar{t}_2 + \bar{t}_3 = 1, \quad \bar{t}_1^2 + \bar{t}_2^2 + \bar{t}_3^2 \leq \frac{1}{2},$$

conditions which represent the points on a plane $\Pi$, and in a sphere $S$, respectively, in the parameter space $[\bar{t}_1, \bar{t}_2, \bar{t}_3]$. Every type $I$ metric corresponds to a point on the circle surrounded by the intersection circumference $C = \Pi \cap S$. This $C$ is the incircle of triangle $T$ on $\Pi$ defined by coordinate planes $\bar{t}_1\bar{t}_2\bar{t}_3 = 0$ (see figure 1(b)). The non-radiative fields belong to the circumference $C$ and the radiative fields are in its interior. Both the degenerate non-radiative and radiative fields are located on the medians of the triangle $T$: type $IM^\infty$ on the base points, type $D$ on the three opposing points in $C$, type $IM^{-6}$ on the barycenter and $IM^+$ in any other point on the medians.

### 6.2. A diagram approach using principal super-stress scalars

Constraints on the principal super-stress scalars $\kappa_i$:

$$\kappa_1 + \kappa_2 + \kappa_3 = \xi, \quad \kappa_1^2 + \kappa_2^2 + \kappa_3^2 = \xi^2 - 2\kappa^2 \leq \xi^2.$$
Then, the adimensional parameters $\tilde{k}_1 = \frac{a}{b}$ satisfy
\[
\tilde{k}_1 + \tilde{k}_2 + \tilde{k}_3 = 1, \quad \tilde{k}_1^2 + \tilde{k}_2^2 + \tilde{k}_3^2 \leq 1,
\]
conditions which represent the points on a plane $\Pi$, and in a sphere $S$, respectively, in the parameter space $\{\tilde{k}_1, \tilde{k}_2, \tilde{k}_3\}$. Every type $I$ metric corresponds to a point on the circle surrounded by the intersection circumference $C = \Pi \cap S$. This $C$ is the circumsphere of triangle $T$ on $\Pi$ defined by coordinate planes $\tilde{k}_1 \tilde{k}_2 \tilde{k}_3 = 0$ (see figure 1(c)). The non-radiative fields belong to the circumference $C$ and the radiative fields are in its interior. Both non-radiative and radiative degenerate fields are located on the medians of triangle $T$: type $IM^\infty$ on the vertex points, type $D$ on the three opposing points in $C$, type $IM^{-6}$ on the barycenter and $IM^-$ in any other point on the medians.

We can see in the diagram that non-radiative types $D$ and $IM^\infty$ can be considered as the limit of the ‘generic’ non-radiative type $IM^+$, but can also be the limit of radiative cases, in particular of the ‘degenerate’ radiative type $IM^-$. Type $IM^{-6}$ is the farthest from the non-radiative types, that is to say, it could be considered the most radiative type $I$ case. This statement, based on geometric considerations, can also be supported by an analytical approach. Indeed, if we define the adimensional radiation parameter $\tilde{\kappa}^2 = \frac{\kappa^2}{\kappa^2_0}$, we have: $\tilde{\kappa}^2$ vanishes for non-radiative types and it reaches the maximum value for the radiative type $IM^{-6}$. The study of this or other similar radiation parameters for known vacuum solutions would be an interesting task which we will undertake elsewhere.

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Appendix A. Algebraic classification of the Bel–Robinson tensor

The classification of the Bel–Robinson tensor $T$ as an endomorphism [7, 8] leads to nine classes: the Petrov–Bel types $O, N, III$ and $II$ and five subclasses of type $I$ metrics: types $I_r$, $IM^-, IM^{6-}$, $IM^+$ and $IM^\infty$. These ‘degenerate’ type $I$ metrics may be characterized in terms of the adimensional Weyl scalar invariant $[6, 9]$:
\[
M = \frac{a^3}{b^2} - 6 \tag{A.1}
\]
where $a$ and $b$ are the quadratic and the cubic Weyl complex scalar invariants. Now we summarize for every class the degeneration of the nine Bel–Robinson eigenvalues $\{t_1, t_2, t_3, \tau_1, \tau_2, \tau_3, \tilde{t}_1, \tilde{t}_2, \tilde{t}_3\}$:

Type $I_r$. This is the more regular case. The scalar $M$ is not real and $T$ has nine different eigenvalues, three real ones and three pairs of complex conjugate.

Type $IM^-$. In this case, $M$ is a negative real number different from $-6$ and $T$ has six different eigenvalues, two real ones, a simple and a double and two double and two simple complex conjugate eigenvalues: $t_1 = t_2$ and $t_1 = \tau_2 \neq 0$.

Type $IM^{6-}$. In this case, $M$ is the real number $-6$ and $T$ has three triple eigenvalues, one real and a pair of complex conjugate: $t_1 = t_2 = t_3$ and $\tau_1 = \tau_2 = \tau_3$.

Type $IM^+$. In this case, $M$ is a positive real number and $T$ has six different real eigenvalues, three simple ones and three double ones: $\tau_i = \tilde{\tau}_i$. 

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Type $M^\infty$. In this case, $M$ is infinity and $T$ has three different real eigenvalues with multiplicities 2, 2, 5: $t_1 = t_2 \neq 0$, $t_3 = \bar{t}_3 = -t_1$ and $t_4 = t_1 = t_2 = 0$.

Type $D$. In this case, $M$ vanishes and the eigenvalues of $T$ are restricted by $t_2 = t_3 = t_1 \neq 0$, $t_1 = 4t_2$ and $t_2 = t_3 = -2t_2$.

Type $II$. This case only differs from type $D$ in the minimal polynomial, with eigenvalues of $T$ restricted by $t_2 = t_3 = t_1 \neq 0$, $t_1 = 4t_2$ and $t_2 = t_3 = -2t_2$.

Type $III$. In this case, all the Bel–Robinson eigenvalues vanish.

Type $N$. This case only differs of type $III$ in the minimal polynomial and all the eigenvalues vanish.

Appendix B. Some constraints on the super-energy scalars

From expressions (2) and (6), we easily obtain the following relations between the associated quadratic scalars:

\[ (T, T) = \tau^2 - 4 |q_\perp|^2 + 6 |t_\perp|^2 - 4 |Q_\perp|^2 + |T_\perp|^2, \]
\[ (Q, Q) = -\tau^2 + 3 |q_\perp|^2 - 3 |t_\perp|^2 + |Q_\perp|^2, \]
\[ (t, t) = \tau^2 - 2 |q_\perp|^2 + |t_\perp|^2, \]
\[ (q, q) = -\tau^2 + |q_\perp|^2. \]

The scalars $(T, T)$ and $(Q, Q)$ are invariants as stated in (7). Moreover, from the Bergqvist and Lankinen conditions [17], we also obtain

\[ 3 (t, t) + 4 (q, q) = \frac{1}{2} \alpha^2. \]

Consequently, the super-energy scalars are subject to the

Quadratic scalar constraints

\[ 4 \alpha^2 = \tau^2 - 4 |q_\perp|^2 + 6 |t_\perp|^2 - 4 |Q_\perp|^2 + |T_\perp|^2, \]
\[ \alpha^2 = \tau^2 - 3 |q_\perp|^2 + 3 |t_\perp|^2 - |Q_\perp|^2, \]
\[ \frac{1}{2} \alpha^2 = -\tau^2 - 2 |q_\perp|^2 + 3 |t_\perp|^2. \]

On the other hand, from the Bergqvist and Lankinen conditions [17], we obtain the following:

Quadratic vectorial constraints

\[ T_\perp (Q_\perp) - 3 Q_\perp (t_\perp) - \tau q_\perp = 0. \]

Quadratic second-order tensorial constraints

\[ T_\perp (t_\perp) + Q_\perp^2 Q_\perp + 2 Q_\perp (q_\perp) - 3 \tau t_\perp + 2 t_\perp \cdot t_\perp - 3 q_\perp \otimes q_\perp = 0, \]
\[ Q_\perp^2 Q_\perp - q_\perp \otimes q_\perp - (\tau^2 - |t_\perp|^2) \gamma = 0, \]
\[ T_\perp^3 T_\perp - 3 Q_\perp^2 Q_\perp + 3 t_\perp \cdot t_\perp - q_\perp \otimes q_\perp - \alpha^2 \gamma = 0. \]

A dot · denotes the contraction of adjacent indices. Similarly, $^2$ and $^3$ denote, respectively, a double and a triple contraction.

Appendix C

Proof of proposition 1. This proposition states that the following relation holds:

\[ (t, t) \geq \chi \geq \frac{1}{2} \alpha^2 \geq 0, \]

and moreover, it specifies when each of the three involved inequalities becomes strict or is an equality depending on the different Petrov–Bel types. \[\square\]
In types $N$ and $III$, the Bel–Robinson eigenvalues vanish. Then, expressions (5) imply \( \chi = 1/\alpha = 0 \).

In types $D$ and $II$, the real Bel–Robinson eigenvalues satisfy \( t_2 = t_3 = \tau_1 \neq 0, t_1 = 4t_2 \).

Then, expressions (5) imply \( \chi = \frac{9}{4}t_2^2 = \frac{1}{4} \alpha \neq 0 \).

In type $I$, we obtain, from the Bel–Robinson canonical form (see [1]),

\[
16\chi^2 = \left( \sum_{i=1}^{3} t_i^2 \right)^2 = 3\sum_{i=1}^{3} t_i^4 + 2 \sum_{i<j} t_i^2 t_j^2 = \sum_{i=1}^{3} t_i^4 + 2 \sum_{k=1}^{3} \left| \tau_k \right|^4 \geq 3 \left[ t_1^4 + t_2^4 + t_3^4 \right]
\]

\[
= \text{tr} \ T^4 = 4\alpha^4,
\]

where the last equality has been proved in [7], \( \text{tr} \ T^4 \) denoting the trace of the fourth power of \( T \) as an endomorphism.

From definition (6) of \( t \), the first inequality in (C.1) writes \( (t, t) = T^2(u, u, u, u) \geq \chi \).

In type $N$, \( T^2 = 0 \), and then \( T^2(u, u, u, u) = 0 = \chi \).

In type $III$, the canonical form (see [1]) implies \( T^2(u, u, u, u) = (l, x)^2 > 0 = \chi \).

In type $II$, an arbitrary observer \( u \) in terms of the Bel–Robinson canonical frame takes the expression \( u = \lambda(e^\phi \ell + e^{-\phi}k) + \mu(e^{i}m + e^{-i}\bar{m}), 2(\lambda^2 - \mu^2) = 1 \). Then, from the Bel–Robinson canonical form (see [1]), we obtain\( T^2(u, u, u, u) = \chi \left[ 1 + 4\mu^2 + 2\mu^4 \sin^2 2\sigma + 8\left( \frac{1}{2} \lambda^2 e^{-2\phi} - \mu^2 \cos 2\sigma \right)^2 \right] > \chi \).

Finally, we study types $I$ and $D$. In [7], we have introduced a second-order super-energy tensor \( T_{(2)} \) associated with the traceless part \( W_{(2)} \) of the square \( W^2 \) of the Weyl tensor \( W \). That is, \( T_{(2)} \) is defined as (1) by changing \( W \) by \( W_{(2)} \). It follows that \( T_{(2)} \) has the same properties as \( T \) [7]. Then, we can apply to it the last inequality in expression (7) of theorem 2: if \( e_0 \) is a principal observer, then for any observer \( u \) we have

\[
T_{(2)}(u, u, u, u) = T^2(u, u, u, u) = \frac{1}{4} \sum_{i=1}^{3} t_i^2 = \chi.
\]

But, for any observer \( u \), \( T_{(2)}(u, u, u, u) = T^2(u, u, u, u) = \frac{1}{4} \alpha^2 \). Thus, (C.2) holds by substituting \( T_{(2)} \) by \( T^2 \) and we obtain

\[
T^2(u, u, u, u) \geq T^2(e_0, e_0, e_0, e_0) = \frac{1}{4} \sum_{i=1}^{3} t_i^2 = \chi.
\]

**Proof of proposition 2.** This proposition states that the following relation holds:

\[
-(q, q) \geq \varepsilon^2 \geq \frac{1}{4} \alpha^2 \geq 0,
\]

and moreover, it specifies when each of the three involved inequalities becomes strict or is an equality depending on the different Petrov–Bel types. \( \square \)

From (5), (B.2) and (C.1), we obtain

\[
-4(q, q) = 3(t, t) - \frac{1}{4} \alpha^2 \geq 3\chi - \frac{1}{4} \alpha = 4\varepsilon^2 \geq \alpha^2.
\]

Moreover, every inequality becomes strict (or an equality) when the corresponding inequality in (C.1) becomes strict (or an equality).

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