Classically and Quantum Integrable Systems
with Boundary
Yi-Xin Chen†, Xu-Dong Luo‡ and Ke Wu‡
† Institute of Zhejiang Modern Physics and Department of Physics,
Zhejiang University, Hangzhou 310027, China
‡ Institute of Theoretical Physics, Academia Sinica, Beijing 100080, China

Abstract

We study two-dimensional classically integrable field theory with independent boundary condition on each end, and obtain three possible generating functions for integrals of motion when this model is an ultralocal one. Classically integrable boundary condition can be found in solving boundary $K_\pm$ equations. In quantum case, we also find that unitarity condition of quantum $R$-matrix is sufficient to construct commutative quantities with boundary, and its reflection equations are obtained.

1 Introduction

Recently, there has been great progress in understanding two-dimensional integrable field theory on a finite interval with independent boundary condition on each end [1]-[5]. The motivation is not only from the necessary of itself, but also from the studies in boundary-related phenomena in statistical systems near criticality [6] and integrable deformations of conformal field theories [7].

In order to deal with integrable models with boundary, relying on previous results of Cherednik [1], Sklyanin [2, 3] introduced a new generating function which originates from the periodic boundary one. In classically integrable models [2], if there has a well-known relation for monodromy matrix [8] as $\{T \otimes T, 1\} = [r, T \otimes T]$, and $r$-matrix satisfies the condition of $r(\alpha) = -r(-\alpha)$, the new generating function defined in \[x-, x+\] can be expressed as:

$$\tau(\alpha) \equiv \text{tr} \left\{ K_+ (\alpha) T(x_+, x_-, t, \alpha) K_- (\alpha) T^{-1}(x_+, x_-, t, -\alpha) \right\}, \tag{1}$$

where $K_\pm$ are boundary reflection matrices.

Expanded as a Laurent series in $\alpha$, all coefficients of $\tau(\alpha)$ make an infinite number of integrals of motion which ensure the completely integrability of the model. From [8], $\tau(\alpha)$ must be in involution between different spectral parameter $\alpha$ and it is independent of time. In other words, $K_\pm$ must satisfy some constraint equations, and existence of nontrivial $K_\pm$ solutions means there are nontrivial classically integrable boundary conditions (CIBC).

There has no condition of $r(\alpha) = -r(-\alpha)$ in affine Toda field theory (ATFT), so P.Bowcock et al. [5] developed a method of modified Lax pair to deal with such models, in which the new generating function in \[(-\infty, x_+)\] reads

$$\tau(\alpha) \equiv \text{tr} \left\{ T^\dagger(-\infty, x_+, t, -\alpha) K_+ (\alpha) T(-\infty, x_+, t, \alpha) \right\}, \tag{2}$$

in which "$\dagger$" denotes conjugation and it has little difference with the original paper [1] according to different definition in $T$ matrix. We must point out that boundary Lax pair in [5] has been modified from the periodic boundary one.

In this paper, we find it is necessary to add a new parameter to the generating function (1) in order to deal with ATFT, and no symmetry condition of $r$-matrix is needed in fact. Besides this modified form, we also construct two other possible generating functions by zero curvature representation. After we extend our results to quantum integrable systems, we find unitarity condition of quantum $R$-matrix is sufficient to construct commutative quantities with boundary too.

The paper is organized as follows. In section 2, three possible generating functions are constructed by zero curvature representation. In order to regard constructed functions as generating functions, algebra
equations (reflection equations) and evolution equations of $K_{±}$ matrices appear in section 3. Then, we study ATFT in section 4 and find the links among these generating functions. In section 5, we extend our results to the quantum case, and demonstrate that unitarity condition of quantum $R$-matrix is sufficient to construct commutative quantities. Then, we compare our commutative quantities with those of paper [13] and find the relation between them. At last, some discussion will be found in section 6.

2 Construction of generating function

2.1 Periodic boundary condition

The zero curvature approach to inverse scattering [8] relies on the existence of a pair of linear partial differential equations in $d \times d$ matrix.

$$\partial_x \Psi = U(x, t, \alpha) \Psi, \quad \partial_t \Psi = V(x, t, \alpha) \Psi,$$

here, Lax pair $(U, V)$ are $d \times d$ matrices whose elements are functions of complex valued field $\phi(x, t)$ and its derivatives, $\alpha \in C$ is a spectral parameter. Zero curvature condition appears from compatibility of the above equation, it is

$$\partial_t U - \partial_x V + [U, V] = 0. \quad (3)$$

By zero curvature representation, we define transition matrix

$$T(x, y, t, \alpha) = \mathcal{P} \exp \left\{ \int_y^x U(x', t, \alpha) dx' \right\}, \quad x \geq y,$$

where $\mathcal{P}$ denotes a path ordering of non-commuting factors. Now, $T$ matrix satisfies

$$\begin{align*}
\partial_x T &= U(x, t, \alpha) T, \\
\partial_t T &= V(x, t, \alpha) T - TV(y, t, \alpha), \\
Id &= T(x, x, t, \alpha),
\end{align*} \quad (5)$$

where $Id$ is $d \times d$ identity matrix.

It is well known that trace of monodromy matrix $T_{L}(t, \alpha) \equiv T(L, -L, t, \alpha)$ is a generating function on periodic boundary condition, so is another more explicit form $\tau(\alpha) = \ln \text{tr} T_{L}(t, \alpha)$. Expanded as a Laurent series in $\alpha$, $\tau(\alpha)$ make an infinite number of integrals of motion. Conservation condition of these integrals can be proved by the second equation of (5) with periodic boundary condition, and the involution condition is proved in Poisson bracket:

$$\{T(x, y, \alpha) \otimes T(x, y, \beta)\} = [r(\alpha, \beta), T(x, y, \alpha) \otimes T(x, y, \beta)] \quad , L \geq x \geq y \geq -L, \quad (6)$$

in which, $T$ is a $d \times d$ matrix, $\mathbb{T} T \otimes Id$ and $\mathbb{F} Id \otimes T$. $r(\alpha, \beta)$ is a $d^2 \times d^2$ matrix whose elements depend on $\alpha$ and $\beta$ only. The Jacobi identity for the bracket holds if and only if $r$-matrix is a solution of classical Yang-Baxter equation.

In periodic boundary condition, it is obvious that $\{\tau(\alpha) \otimes \tau(\beta)\} = 0$. So $\tau(\alpha)$ constructs a family of generating function for integrals of motion.

2.2 Independent boundary condition

As soon as periodic boundary condition is broken, $\tau(\alpha)$ defined before should be not a conservative quantity, so that we have to find a new expression of generating function. As discussed in papers [4, 5], if Lagrangian density in bulk theory is $\mathcal{L}_f$, then, the new Lagrangian density with boundary appears as:

$$\mathcal{L} = \theta(x_+ - x) \theta(x - x_-) \mathcal{L}_f - \delta(x_+ - x) V_+ (\phi(x_+), \partial_\mu \phi(x_+)) - \delta(x - x_-) V_- (\phi(x_-), \partial_\mu \phi(x_-)). \quad (7)$$

By means of principle of the least action associated with [7], we will obtain motion equation in $(x_-, x_+)$ and boundary equations on each end.
If \( L_f \) is expressed as \( L_f = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - V(\phi) \) and \( V_\pm \) depend only on \( \phi(x_\pm) \) (and independent of its derivatives), boundary equations on each end will be

\[
\partial_x \phi = \mp \partial_x V_\pm, \quad x = x_\pm.
\] (8)

On each end, if \( \phi(x) \) is a smooth function of coordinate \( x \), we can extend the motion equation to whole domain \([x_-, x_+]\), so are the boundary Lax pair. In other words, we keep the uniform expression of fundamental Poisson bracket in the whole domain, even if it is on independent boundary condition.

Similar to the definition of transition matrix, there is another matrix function

\[
F(x, t, \alpha) = \mathcal{P} \exp \left\{ \int_{t_0}^t V(x, t', \alpha) dt' \right\}, \quad t \geq t_0.
\] (9)

By zero curvature condition, we construct a quantity which is independent of time:

\[
F^{-1}(x_+, t_1, \alpha) T(x_+, x_-, t_1, \alpha) F(x_-, t_1, \alpha) = F^{-1}(x_+, t_2, \alpha) T(x_+, x_-, t_2, \alpha) F(x_-, t_2, \alpha).
\] (10)

It can be proved easily because both sides in the equation are equal to \( T(x_+, x_-, t_0, \alpha) \).

With another equation of argument \((-\alpha + \delta)\), which can be obtained by the same method, we expect generating function with boundary shall be constructed as follows. For example:

\[
\text{tr} \left\{ [F(x_+, t, -\alpha + \delta) F^{-1}(x_+, t, \alpha)] T(x_+, x_-, t, \alpha) T^{-1}(x_+, x_-, t, -\alpha + \delta) \right\}.
\]

By its product method, it is obvious that this quantity is a conservative quantity. If we regard it as a generating function for integrals of motion, there are the main problem: 1. Does such a constructed quantity satisfy involution condition? 2. It had better be independent of "\( t_0 \)" which comes from \( F \) matrix. Base on these problems, we introduce \( K_\pm \) matrices instead of \( F \) terms and impose involution and conservation conditions on the new form. The new generating function is

\[
\tau(\alpha) = \text{tr} \left\{ K_+(x_+, t, \alpha) T(x_+, x_-, t, \alpha) K_-(x_-, t, \alpha) T^{-1}(x_+, x_-, t, -\alpha + \delta) \right\}.
\] (11)

Moreover, we use "\( T \)" (conjugation) or "\( T^t \)" (transposition) instead of "\(-1\)" (inverse) in order that \( K_+ \) and \( K_- \) depend only on the variables of the boundary \( x_+ \) and \( x_- \), respectively. The results are:

\[
\tau(\alpha) = \text{tr} \left\{ K_+(x_+, t, \alpha) T(x_+, x_-, t, \alpha) K_-(x_-, t, \alpha) T^t(x_+, x_-, t, -\alpha + \delta) \right\},
\] (12)

\[
\tau(\alpha) = \text{tr} \left\{ K_+(x_+, t, \alpha) T(x_+, x_-, t, \alpha) K_-^{-1}(x_-, t, \alpha) T^t(x_+, x_-, t, -\alpha + \delta) \right\}.
\] (13)

In quantities (11), (12), and (13), \( K_\pm \) matrices and \( \delta \) are similar in symbols only. Each of these quantities will be a generating function of integrable systems with boundary, if both involution and conservation conditions are satisfied.

3 \( \tau(\alpha) \) as a generating function

It is well known that generating function for integrals of motion must be in involution (between each other) and independent of time \[3\]. So if we regard quantity (11) as a generating function, some constraint conditions must be imposed on it. Now we study quantity (11) in this section.

3.1 Involution condition

Taking the notations similar to paper [3], we define:

\[
T_+(x, \alpha) = T^{-1}(x_+, x_-, t, -\alpha + \delta) K_+(x_+, t, \alpha) T(x_+, x, t, \alpha),
\]

\[
T_-(x, \alpha) = T(x, x_-, t, \alpha) K_-(x_-, t, \alpha) T^{-1}(x, x_-, t, -\alpha + \delta),
\]

\[
T(x, \alpha) = T_-(x, \alpha) T_+(x, \alpha).
\]
Comparing it with quantity (11), we find $tr T(x, \alpha)$ is just equal to $\tau(\alpha)$. Now, we impose some constraint conditions on $K_\pm$ matrices:

$$
\{K_\pm(x_\pm, t, \alpha) \odot T(x_+, x_-, t, \beta)\} = 0,
$$
$$
\{K_\pm(\alpha) \odot K_\pm(\beta)\} = 0, \quad \{K_\pm(\alpha) \odot K_\pm(\beta)\} = 0. \quad (14)
$$

It means

$$
\{T_+(x, t) \odot T_-(x, t)\} = 0.
$$

If $K_\pm$ matrices are independent of field variances, condition (14) is satisfied naturally. But we must point out that the $K_\pm$ matrices, in general, can depend on the field variables. Poisson bracket on $T_\pm$ is:

**Proposition 1.** If $K_+$ matrix satisfies

$$
0 = -r(-\alpha + \delta, -\gamma + \delta) K_+(t, \alpha) K_+(t, \gamma) + K_+(t, \alpha) r(-\alpha + \delta, -\gamma + \delta) K_+(t, \gamma)
$$

$$
+ K_+(t, \gamma) r(-\alpha + \delta, -\gamma + \delta) K_+(t, \alpha) - K_+(t, \alpha) K_+(t, \gamma)r(-\alpha, \gamma), \quad (15)
$$

then, $T_+$ algebra should obey the following relation:

$$
\begin{align*}
\left\{ \frac{1}{2} T_+(x, \alpha), \frac{2}{2} T_+(x, \gamma) \right\} \\
= -r(-\alpha + \delta, -\gamma + \delta) T_+(x, \alpha) + T_+(x, \alpha) r(-\alpha + \delta, -\gamma + \delta) T_+(x, \gamma)
\end{align*}
$$

$$
+ T_+(x, \gamma) r(-\alpha + \delta, -\gamma + \delta) T_+(x, \alpha) - T_+(x, \alpha) T_+(x, \gamma)r(-\alpha, \gamma). \quad (16)
$$

It should be emphasized that $K_+$ is a subalgebra of $T_+$ algebra according to the definition of $T_+$. Proposition 1 can be proved by calculating Poisson bracket on $T_+$ directly. There is another algebra of $T_-$ similar to $T_+$:

**Proposition 2.** If $K_-$ matrix satisfies:

$$
0 = r(\alpha, \gamma) K_-(t, \alpha) K_-(t, \gamma) - K_-(t, \alpha) r(-\alpha + \delta, -\gamma + \delta) K_-(t, \gamma)
$$

$$
- K_-(t, \gamma) r(-\alpha + \delta, -\gamma + \delta) K_-(t, \alpha) + K_-(t, \alpha) K_-(t, \gamma)r(-\alpha + \delta, -\gamma + \delta), \quad (17)
$$

then, it leads to the relation of $T_-$ algebra being:

$$
\begin{align*}
\left\{ \frac{1}{2} T_-(x, \alpha), \frac{2}{2} T_-(x, \gamma) \right\} \\
= r(\alpha, \gamma) T_-(x, \alpha) - T_-(x, \alpha) r(-\alpha + \delta, -\gamma + \delta) T_-(x, \gamma)
\end{align*}
$$

$$
- T_-(x, \gamma) r(-\alpha + \delta, -\gamma + \delta) T_-(x, \alpha) + T_-(x, \alpha) T_-(x, \gamma)r(-\alpha + \delta, -\gamma + \delta). \quad (18)
$$

The proof is similar to proposition 1. Used proposition 1 and 2, Poisson bracket on $T(x, \alpha)$ can be calculated as follows:

$$
\begin{align*}
\left\{ \frac{1}{2} T(x, \alpha), \frac{2}{2} T(x, \gamma) \right\} \\
= \left\{ \frac{1}{2} T_-(x, \alpha) \frac{1}{2} T_+(x, \alpha), \frac{2}{2} T_-(x, \gamma) \frac{2}{2} T_+(x, \gamma) \right\}
\end{align*}
$$

$$
= \frac{1}{2} T_-(x, \alpha) \frac{2}{2} T_-(x, \gamma) \left\{ \frac{1}{2} T_+(x, \alpha), \frac{2}{2} T_+(x, \gamma) \right\} + \left\{ \frac{1}{2} T_-(x, \alpha), \frac{2}{2} T_-(x, \gamma) \right\}
$$

$$
\left[ r(\alpha, \gamma), T_+(x, \alpha) \frac{2}{2} T(x, \gamma) \right] + \left[ \frac{1}{2} T_+(x, \alpha), \frac{2}{2} T_-(x, \gamma) r(-\alpha + \delta, -\gamma + \delta) \frac{2}{2} T_+(x, \gamma) \right]
$$

$$
+ \left[ \frac{2}{2} T_-(x, \gamma), \frac{1}{2} T_-(x, \alpha) r(-\alpha + \delta, -\gamma + \delta) \frac{1}{2} T_+(x, \alpha) \right].
$$
After taking trace on $\mathcal{T}$, we find

$$\left\{ \text{tr} \frac{1}{2} \mathcal{T}(x, \alpha), \text{tr} \frac{2}{3} \mathcal{T}(x, \gamma) \right\} = \text{tr}_1 \text{tr}_2 \left\{ \frac{1}{2} \mathcal{T}(x, \alpha), \frac{2}{3} \mathcal{T}(x, \gamma) \right\} = 0,$$

that is

$$\left\{ \frac{1}{2} \tau(\alpha), \frac{2}{3} \tau(\gamma) \right\} = 0. \tag{19}$$

In other words, $\tau(\alpha)$ constructs a one-parameter involutive family. Here we remark that no symmetry conditions of $r$-matrix is used to obtain equation (19). Consequently it can be applied to general model.

### 3.2 Conservation condition

If $\tau(\alpha)$ is a generating function for integrals of motion, it must be independent of time. We find

$$\partial_t \text{tr} \mathcal{T}(x, \alpha)$$

$$= \partial_t \left\{ K_+(t, \alpha) \mathcal{T}(x_+, x_-, t, \alpha) K_-(t, \alpha) \mathcal{T}^{-1}(x_+, x_-, t, -\alpha + \delta) \right\}$$

$$= \text{tr} \left\{ \partial_t K_+(t, \alpha) - V(x_+, t, -\alpha + \delta) K_+(t, \alpha) + K_+(t, \alpha) V(x_+, t, \alpha) \right\}$$

$$\times \mathcal{T}(x_+, x_-, t, \alpha) K_-(t, \alpha) \mathcal{T}^{-1}(x_+, x_-, t, -\alpha + \delta)$$

$$+ \left[ \partial_t K_-(t, \alpha) - V(x_-, t, \alpha) K_-(t, \alpha) + K_-(t, \alpha) V(x_-, t, -\alpha + \delta) \right]$$

$$\times \mathcal{T}^{-1}(x_+, x_-, t, -\alpha + \delta) K_+(t, \alpha) \mathcal{T}(x_+, x_-, t, \alpha) \right\}. \tag{20}$$

Taking $\partial_t \text{tr} \mathcal{T}(x, \alpha) = 0$, and supposing there has no connection between boundary variances on each end, we obtain the evolution equations of $K_{\pm}$ matrices

$$\partial_t K_+(t, \alpha) - V(x_+, t, -\alpha + \delta) K_+(t, \alpha) + K_+(t, \alpha) V(x_+, t, \alpha) = 0,$$

$$\partial_t K_-(t, \alpha) - V(x_-, t, \alpha) K_-(t, \alpha) + K_-(t, \alpha) V(x_-, t, -\alpha + \delta) = 0. \tag{21}$$

For these equations, we find immediately that there have two isomorphisms between $K_+$ and $K_-$, which are $K_+(\alpha) \rightarrow K_-(\alpha)$ and $K_+(\alpha) \rightarrow K_-(\alpha)$.

If $K_{\pm}$ are constant matrices ($\partial_t K_{\pm} = 0$), the equation (21) can be simplified as

$$V(x_+, t, -\alpha + \delta) K_+(t, \alpha) = K_+(t, \alpha) V(x_+, t, \alpha),$$

$$V(x_-, t, \alpha) K_-(t, \alpha) = K_-(t, \alpha) V(x_-, t, -\alpha + \delta). \tag{22}$$

In the present case, we remark that $K_{\pm}$ matrices are not singular matrices, so that determinants of $V$ satisfy

$$\det V(x_+, t, -\alpha + \delta) = \det V(x_+, t, \alpha), \tag{23}$$

by which we can obtain the value of $\delta$. After inserting $\delta$ value into equation (22), we can find some nontrivial CIBC when nontrivial $K_{\pm}$ matrices appear. In other words, a class of $K_{\pm}$ matrices is relate to a class of integrable boundary condition.

We must point out that $K_{\pm}$ matrices depending on field variables take its meaning in fact $[9]$. On one hand, we must use such $K_{\pm}$ matrices in order that quantity (13) can be regarded as a generating function on periodic boundary condition too. On the other hand, studying such $K_{\pm}$ matrices, we understand integrable condition more deeply.

When Sklyanin’s function (10) is regarded as a generating function in ATFT, boundary $K_{\pm}$ matrices will have no constant solution in equation (21) (except for Sine-Gordon theory). So we have to solve equation (21) as differential equations. Besides these difficulty, even when one had found a nontrivial solution, he should have to prove the involution condition again because condition (14) may be broken. In our method, $\delta$ added on spectral parameter guarantees the existence of constant $K_{\pm}$ matrices, and $\delta$ can be solved by means of equation (23), so $K_{\pm}$ matrices solving procedure is simplified effectively.

**Proposition 3.** If $K_{\pm}$ matrices satisfy not only algebra equation (15) and (17), but also evolution equations of (21), $tr \mathcal{T}(x, \alpha)$ is a generating function for integrals of motion.
There has a difficult step in proving this proposition. Are solutions of $K_\pm$ matrices in equation (21) compatible with its algebra equation (12) and (13)? Although one found it is true in Sine-Gordon theory [5] and it has been proved in ATFT [3] with the form of (12), it still keeps an open problem in general theory. If this compatibility is satisfied, the proposition is proved naturally.

3.3 Other generating functions

As exhibited in above subsections, we also obtain other $K_\pm$ matrices’ algebra and evolution equations when quantity (12) or (13) is regarded as generating function for integrals of motion. In the form of (12), it read as

$$0 = r^{t_1 t_2}(-\alpha + \delta, -\gamma + \delta)K_+(t, \alpha)K_+(t, \gamma) + K_+(t, \alpha)r^{t_1}(-\alpha + \delta, \gamma)K_+(t, \alpha) + K_+(t, \alpha)^2 r^{t_1}(-\alpha + \delta, \gamma)K_+(t, \alpha),$$

(24)

where the upper index "$t_i, i = 1, 2$" denote transposition on the "$i$" space. its evolution equations read as

$$0 = \partial_t K_+(t, \alpha) + V^i(x_+, t, -\alpha + \delta)K_+(t, \alpha) + K_+(t, \alpha)V(x_+, t, \alpha),$$

$$0 = \partial_t K_-(t, \alpha) - V(x_-, t, \alpha)K_-(t, \alpha) - K_-(t, \alpha)V^i(x_-, t, -\alpha + \delta).$$

(25)

Let $\partial_t K_\pm = 0$, we can also obtain $\delta$ value in (25) by taking determinants. With constraint conditions of (24) and (25), quantity (12) is a generating function for integrals of motion.

In the form of (13), the similar equations are still balance except that we must use "$\dagger$" (conjugation) instead of "$t$" (transposition). If those modified equations are satisfied, quantity (13) can be regarded as a generating function for integrals of motion too.

Now, we have obtained three forms of generating function as well as their constraint conditions. If all of them are regarded as generating functions, we believe that they are the same one in fact. In the next section, we will prove it explicitly in ATFT.

4 Classically Integrable boundary condition in ATFT

4.1 Links among generating functions

Lagrangian in ATFT on independent boundary condition is [3, 11]:

$$\mathcal{L} = \int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} dt \left\{ \theta(x - x_-)\theta(x_+ - x) \left[ \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{m^2}{\beta^2} \sum_{i=0}^{r} n_i (e^{\beta \alpha_i \phi} - 1) \right] - \delta(x - x_-)V_- (\phi(x_-), \partial_\mu \phi(x_-)) - \delta(x_+ - x) V_+ (\phi(x_+), \partial_\mu \phi(x_+)) \right\},$$

(26)

where $m$ is mass scale and $\beta$ is the coupling constant in real domain; $\alpha_i$ are simple roots of a simple Lie algebra of rank $r$ (included the affine root $\alpha_0$). There are $\sum_{i=0}^{r} n_i = 0$ and $n_0 = 1$. It is a theory of $r$ scalar fields ($\alpha_i \cdot \phi = \sum_{i=0}^{r} \alpha_i \phi$). The potentials $V_+$ and $V_-$ are additions on the ends $x_+$ and $x_-$, respectively. They denote independent boundary conditions. When $V_\pm$ depend on $\phi(x_\pm)$ only (and independent of its derivatives), we obtain:

$$\left( \partial_t^2 - \partial_x^2 \right) \phi = -\frac{m^2}{\beta} \sum_{i=0}^{r} n_i \alpha_i e^{\beta \alpha_i \phi}, \quad x_+ < x < x_-,$$
\[ \partial_x \phi_a = \pm \frac{\partial V_{a \pm}}{\partial \phi_a}, \quad x = x_{\pm}. \]  

Lax pair in ATFT read as \((\lambda = e^n)\)

\[ U(x, t, \lambda) = - \left\{ \frac{1}{2} \beta H \cdot \partial_t \phi + m \sum_{i=0}^r \sqrt{m_i} (\lambda E_{\alpha_i} + \lambda^{-1} E_{-\alpha_i}) e^{\beta \alpha_i \cdot \phi/2} \right\}, \]
\[ V(x, t, \lambda) = - \left\{ \frac{1}{2} \beta H \cdot \partial_x \phi + m \sum_{i=0}^r \sqrt{m_i} (\lambda E_{\alpha_i} - \lambda^{-1} E_{-\alpha_i}) e^{\beta \alpha_i \cdot \phi/2} \right\}. \]

in which \(H\) and \(E_{\pm \alpha_i}\) are the Cartan subalgebra and the generators responding to the simple roots, respectively, of the simple Lie algebra of rank \(r\). The coefficients \(m_i\) are equal to \(n_i \alpha_i^2 / 8\). There are the Lie algebra relation:

\[ [H_i, H_j] = 0, \quad [H, E_{\pm \alpha_i}] = \pm \alpha_i E_{\pm \alpha_i}, \]
\[ [E_{\alpha_i}, E_{-\alpha_i}] = 2 \alpha_i \cdot H / (\alpha_i^2). \]  

It is pointed out by Hollowood [12] that the complex affine Toda theories have soliton solution (in which coupling constant \(\beta\) is purely imaginary), in contrast with the real coupling constant ones. And its Lagrangian, motion equation and Lax pair can be expressed similarly to equation (26)-(28) except for taking \(\beta \rightarrow i \tilde{\beta}\) (\(\tilde{\beta} \in \text{Re}\)). In our paper, we use (26)-(28) equations in general, and distinguish them only when the real and imaginary cases can’t be treated in the same way.

For those generators in Lax pair (28), we can find a representation in which they satisfy:

\[ H_i^1 = H_i^\dagger = H_i, \quad E_{\pm \alpha_i}^1 = E_{\pm \alpha_i}^\dagger = E_{\mp \alpha_i}. \]

So there has an automorphism map:

\[ H_i \rightarrow H_i^\dagger = \Omega^{-1} H_i \Omega = -H_i, \]
\[ E_{\alpha_i} \rightarrow E_{\alpha_i}^\dagger = \Omega^{-1} E_{\alpha_i} \Omega = E_{-\alpha_i}, \]
\[ E_{-\alpha_i} \rightarrow E_{-\alpha_i}^\dagger = \Omega^{-1} E_{-\alpha_i} \Omega = E_{\alpha_i}. \]

The new generators satisfy the same Lie algebra relation (29). In other words, there are

\[ U^t(x, \lambda) = U(x, \lambda^{-1}) = -\Omega^{-1} U(x, -\lambda) \Omega, \]
\[ V^t(x, \lambda) = V(x, -\lambda^{-1}) = -\Omega^{-1} V(x, -\lambda) \Omega. \]  

We remark that equation (30) can be applied to both real and imaginary coupling constant cases. But if one uses "\(\dagger\)" instead of "\(\dagger\)", equation (30) must be modified because of its complex fields.

From definition of \(T(x, y, t, \lambda)\), there has

\[ \partial_x T^t(x, y, \lambda) = T^t(x, y, \lambda) U^t(x, \lambda) \]
\[ = T^t(x, y, \lambda) \left[ -\Omega^{-1} U(x, -\lambda) \Omega \right], \]

or

\[ \partial_x [\Omega T^t(x, y, \lambda) \Omega^{-1}] = - [\Omega T^t(x, y, \lambda) \Omega^{-1}] U(x, -\lambda). \]

Comparing it with \(\partial_x T^{-1}(x, y, \lambda) = -T^{-1}(x, y, \lambda) U(x, \lambda)\) and the initial condition in (31), we obtain

\[ \Omega T^t(x, y, \lambda) \Omega^{-1} = T^{-1}(x, y, -\lambda), \]

or

\[ \Omega T^t(x, y, \alpha) \Omega^{-1} = T^{-1}(x, y, \alpha + i\pi). \]  

It means

\[ tr \left\{ K_-(\alpha) T^{-1}(\alpha - \alpha + \delta) K_+(\alpha) T(\alpha) \right\} \]
\[ = tr \left\{ K_-(\alpha) \Omega T^t(\alpha - \alpha + \delta + i\pi) \Omega^{-1} K_+(\alpha) T(\alpha) \right\} \]
\[ = tr \left\{ \bar{K}_-(\alpha) T^t(\alpha - \alpha + \delta') \bar{K}_+(\alpha) T(\alpha) \right\}. \] (32)
in which $K_{\pm}$ matrices in (11) and (12) are distinguished by $K_{\pm}$ and $\bar{K}_{\pm}$ now, and the quantities added on spectral parameter become $\delta$ and $\delta'$ respectively. In other words, quantity (11) is equal to (12), if $\delta'$ is equal to $\delta + i\pi$ and reflection matrices satisfy

$$\bar{K}_{-}(\alpha) = K_{-}(\alpha)\Omega, \quad \bar{K}_{+}(\alpha) = \Omega^{-1}K_{+}(\alpha).$$

(33)

Using the second equation of (30) and comparing equation (21) with (25), we find these relations appear again. So quantities (11) and (12) are the same one in fact, when both of them are regarded as generating functions for integrals of motion.

In real coupling constant and real fields case, if we use "\(t\)" instead of "\(\tau\)", equation (30) is still balance when spectral parameter is real. We obtain a relation similar to (32) again. In this case, when we rewrite (13) as $r\bar{K}_{-}(\alpha)T^{\dagger}(\alpha + \delta')\bar{K}_{+}(\alpha)T(\alpha)$, then

$$\bar{K}_{\pm}(\alpha) = \bar{K}_{\pm}(\alpha), \quad \delta' = \delta'' = \delta + i\pi.$$  

(34)

Now, we have proved quantity (11) is equal to (12) when both of them are regarded as generating functions in ATFT. When coupling constant is real, they are equal to generating function (13) too. But when coupling constant is purely imaginary, equation (30) may be not satisfied, so $K_{\pm}$ matrices in (13) may have no constant solution.

4.2 Classically Integrable Boundary Condition

In real coupling constant ATFT, if we regard $r\bar{K}_{-}(\alpha)T^{\dagger}(\alpha + \delta)\bar{K}_{+}(\alpha)T(\alpha)$ as a generating function, we will obtain the evolution equation of $\bar{K}_{+}$ in $x_{+}$ boundary:

$$V^{\dagger}(x_{+}, -\alpha + \delta)\bar{K}_{+}(\alpha) + \bar{K}_{+}(\alpha)V(x_{+}, \alpha) = 0.$$  

(35)

After taking $\delta = 0$, we obtain equation of $\bar{K}_{+}^{-1}(\lambda)$ (in which $\lambda = e^{\alpha}$):

$$\frac{1}{2} \left[ \frac{\beta}{m} \partial_{x} \phi \cdot H \right]_{+} + \left[ \bar{K}_{+}^{-1}(\lambda), \sum_{0}^{r} \sqrt{m_{i}}(\lambda E_{\alpha_{i}} - \lambda^{-1} E_{\alpha_{i}})e^{\beta_{\alpha_{i}}\phi/2} \right]_{-}.$$  

(36)

By boundary equation (27), it is just the reflection equation appears in paper [3]. We find $K_{+}$ and $T$ matrices defined in that paper are just the quantities of $\bar{K}_{+}^{-1}$ and $T^{-1}$ in our paper, according to different definition of Lax pair. Analogy with method in paper [3], we solve equation (36) and obtain CIBC in ATFT. In simple-laced case, it is the same as paper [3]

$$\frac{\beta}{m} \partial_{x} \phi = -\sum_{0}^{r} B_{i} \sqrt{\frac{n_{i}}{2|\alpha_{i}|^{2}}} \alpha_{i} e^{\beta_{\alpha_{i}}\phi/2},$$

in which $|B_{i}| = 2$, $i = 0, 1, \cdots, r$,

or $B_{i} = 0$, $i = 0, 1, \cdots, r.$

(37)

In imaginary coupling constant one, we regard $r\bar{K}_{-}(\alpha)T^{\dagger}(\alpha + \delta)\bar{K}_{+}(\alpha)T(\alpha)$ as a generating function. The results in real coupling constant can be used, thanks to section 4.1. In other words, the new CIBC can be obtained by analytic continuation by $\beta \rightarrow i\bar{\beta}$ ($\bar{\beta} \in Re$). It reads:

$$\frac{i\bar{\beta}}{m} \partial_{x} \phi = -\sum_{0}^{r} \bar{B}_{i} \sqrt{\frac{n_{i}}{2|\alpha_{i}|^{2}}} \alpha_{i} e^{i\beta_{\alpha_{i}}\phi/2},$$

in which $|B_{i}| = 2$, $i = 0, 1, \cdots, r$,

or $B_{i} = 0$, $i = 0, 1, \cdots, r.$

(38)

We remark that Sklyanin’s method [3] can’t be used in ATFT except for Sine-Gorden theory, this conclusion comes from the fact that $\delta \neq 0$ on independent boundary condition if we regard (11) as a generating function. Now, we must take $\delta = -i\pi$ according to equation (33) and (34). Sine-Gordon theory is an exception in which it is satisfied both $\delta = 0$ and $\delta = -i\pi$.

As we discussed in section 3, in the classical case, no symmetry conditions of $r$- matrix is necessary in constructing generating function for integrals of motion. So it is interesting to study whether commutative quantities can be constructed with less symmetry of $R$- matrix in the quantum case.
5 Quantum integrable systems with boundary

There are many papers (for example, [8] and [9] - [14]) in which the authors deal with quantum integrable boundary condition in two-dimensional lattice models. As far as we know, both unitarity and crossing unitarity conditions (or the weaker property [14]) are used in constructing commutative quantities. Since finding crossing unitarity condition of a given $R$- matrix is a difficult problem, it is useful to construct commutative quantities without this symmetry.

In this section, we explore how to obtain commutative quantities by means of unitarity condition only. Unitarity condition read as:

$$R_{12}(u)R_{21}(-u) = \xi(u),$$  \hspace{1cm} (39)

where $\xi(u)$ is some even scalar function and $R$- matrix is a solution of quantum Yang-Baxter equation (YBE):

$$R_{12}(u-v)R_{13}(u)R_{23}(v) = R_{23}(v)R_{13}(u)R_{12}(u-v).$$

As usual, the transfer matrix $t(u)$ is defined as

$$t(u) = \text{tr} \ T_+(u)T_-(u),$$  \hspace{1cm} (40)

and each entry of $T_+(u)$ commutes with $T_-(u)$.

**Proposition 4.** If $T_\pm$ satisfy such equations

$$R_{21}^{t_1}(u)R_{21}^{t_2}(u)R_{21}^{t_1}(u - \delta) = R_{21}^{t_2}(u)R_{21}^{t_1}(u - \delta) R_{21}^{t_1}(u - \delta),$$

and the quantum $R$- matrix obeys the unitarity condition, then, transfer matrix $t(u)$ defines a one-parameter commutative family.

For explicity, we use $\xi_1^{-1}$ and $\xi_2^{-1}$ replace $\xi^{-1}(u_+ - \delta)$ and $\xi^{-1}(u_-)$ respectively, as well as $u_\pm = u \pm v$.

The proof is directly:

$$t(u)t(v) = tr_1 \ T_+(u) \ T_-(u) tr_2 \ T_+(v) \ T_-(v)$$

$$= tr_{12} \ T_+(u) \ T_+(v) \ T_-(u) \ T_-(v)$$

$$= \xi_1^{-1} tr_{12} \ T_+ \ T_+ \ T_+ \ T_+ \ T_-(u_+ - \delta)R_{12}(-u_+ + \delta) T_+ \ T_-$$

$$= \xi_1^{-1} tr_{12} \ T_+ \ T_+ \ T_+ \ T_+ \ T_-(u_+ - \delta) R_{21}^{t_2}(u_+ - \delta) R_{21}^{t_1}(u_-) R_{21}^{t_1}(u_-) \ T_+ \ T_- \ T_+ \ T_+ \ T_+$$

$$= \xi_1^{-1} \xi_2^{-1} tr_{12} \ T_+ \ T_+ \ T_+ \ T_+ \ T_-(u_+ - \delta) R_{21}^{t_2}(u_+ - \delta) R_{21}^{t_1}(u_-) R_{21}^{t_1}(u_-) \ T_+ \ T_- \ T_+ \ T_+ \ T_+$$

$$= \xi_1^{-1} \xi_2^{-1} tr_{12} \ T_+ \ T_+ \ T_+ \ T_+ \ T_-(u_+ - \delta) R_{21}^{t_2}(u_+ - \delta) R_{21}^{t_1}(u_-) R_{21}^{t_1}(u_-) \ T_+ \ T_- \ T_+ \ T_+ \ T_+$$

using equations (41), we find

$$t(u)t(v) = \xi_1^{-1} \xi_2^{-1} tr_{12} \ T_+ \ T_+ \ T_+ \ T_+ \ T_-(u_+ - \delta) R_{21}^{t_2}(u_+ - \delta) R_{21}^{t_1}(u_-) R_{21}^{t_1}(u_-) \ T_+ \ T_- \ T_+ \ T_+ \ T_+$$

$$= \xi_1^{-1} tr_{12} \ T_+ \ T_+ \ T_+ \ T_+ \ T_-(u_+ - \delta) R_{21}^{t_2}(u_+ - \delta) R_{21}^{t_1}(u_-) R_{21}^{t_1}(u_-) \ T_+ \ T_- \ T_+ \ T_+ \ T_+$$

$$= \xi_1^{-1} tr_{12} \ T_+ \ T_+ \ T_+ \ T_+ \ T_-(u_+ - \delta) R_{21}^{t_2}(u_+ - \delta) R_{21}^{t_1}(u_-) R_{21}^{t_1}(u_-) \ T_+ \ T_- \ T_+ \ T_+ \ T_+$$

$$= tr_{12} \ T_+ \ T_+ \ T_+ \ T_+ \ T_-(u_+ - \delta) R_{21}^{t_2}(u_+ - \delta) R_{21}^{t_1}(u_-) R_{21}^{t_1}(u_-) \ T_+ \ T_- \ T_+ \ T_+ \ T_+$$

$$= t(v)t(u).$$
In quantum spin chain model, it is convenient that $\mathcal{T}_\pm$ take such representations

\begin{align}
\mathcal{T}_+(u) &= K_+(u), \\
\mathcal{T}_-(u) &= T(u)K_-(-u)T^t(-u + \delta) \\
&= L_N(u) \cdots L_2(u)K_-(u)L_1^t(-u + \delta)L_2^t(-u + \delta) \cdots L_N^t(-u + \delta),
\end{align}

in which transposition "$t$" acts on the auxiliary space and $n = 1, 2, \ldots, N$ denote quantum space. There has a relation between $R$ and $L$ operators

\begin{align}
R_{ab}(u - v)L_a(u)L_b(v) &= L_b(v)L_a(u)R_{ab}(u - v). \tag{43}
\end{align}

Let $\mathcal{T}_-(u) = L_N(u)\mathcal{T}_-^t(u)L_N^t(-u + \delta)$ and insert equation (42) into equation (41), we find the second equation of (41) becomes

\begin{align}
l.h.s. &= R_{ab}(u_-)L_{a_1N}(u)\mathcal{T}_-^{tb}L_{b_1N}^t(-u + \delta)R_{ba}^t(-u + \delta)L_{bN}(v)\mathcal{T}_-^tL_{aN}^t(-u + \delta) \\
&= R_{ab}(u_-)L_{a_1N}(u)\mathcal{T}_-^{ta}L_{b_1N}(v)R_{ab}^t(-u + \delta)L_{aN}^t(-u + \delta)\mathcal{T}_-^tL_{bN}^t(-u + \delta) \\
&= L_{bN}(v)L_{a_1N}(u)R_{ab}(u_-)\mathcal{T}_-^{ta}R_{ab}^t(-u + \delta)\mathcal{T}_-^tL_{aN}^t(-u + \delta)L_{bN}^t(-u + \delta),
\end{align}

r.h.s. \begin{align}
&= L_{bN}(v)\mathcal{T}_-^{tb}L_{b_1N}^t(-u + \delta)R_{ba}^t(-u + \delta)L_{aN}(u)\mathcal{T}_-^t\mathcal{T}_-^{ta}L_{b_1N}^t(-u + \delta)R_{ba}^t(-u + \delta)L_{aN}^t(-u + \delta) \\
&= L_{bN}(v)\mathcal{T}_-^{ta}L_{a_1N}(u)R_{ab}^t(-u + \delta)L_{bN}^t(-u + \delta)\mathcal{T}_-^t\mathcal{T}_-^{tb}L_{a_1N}^t(-u + \delta)R_{ab}^t(-u + \delta)L_{bN}^t(-u + \delta).
\end{align}

In other words, this equation is reduced to

\begin{align}
R_{ab}(u_-)\mathcal{T}_-^{ta}R_{ba}^t(-u + \delta)\mathcal{T}_-^tL_{aN}^t(-u + \delta)L_{bN}^t(-u + \delta).
\end{align}

We proceed to do the above reduction repeatedly until all of $L$ operators beside $K_-$ matrix disappear. At last, we obtain the reflection equation about $K_-$ only. Now, reflection equations of $K_\pm$ are

\begin{align}
R_{21}^{t_1t_2}(-u_-)K_+(u)R_{21}^{t_2}(-u_- - \delta)K_+(v) &= R_{21}^{t_2}(-u_-)K_+(u)R_{12}^{t_1t_2}(-u_- - \delta)K_+(v), \\
R_{12}(-u_-)K_-(-u_-)R_{12}(-u_- + \delta)K_-(-v) &= R_{12}(-u_-)R_{21}^{t_1t_2}(-u_- - \delta)K_-(-u_- + \delta)K_-(v), \tag{44}
\end{align}

and transfer matrix $t(u)$ becomes

\begin{align}
t(u) &= \text{tr} K_+(u)T(u)K_-(u)T^t(-u + \delta). \tag{45}
\end{align}

We remark that there has no obvious relation between $K_\pm$ matrices. If some symmetry conditions are used, relation between $K_+$ and $K_-$ matrices may be found.

For example, $R$-matrix in paper [3] has $PT$ symmetry and crossing unitarity

\begin{align}
R_{12}(u) &= R_{21}^{t_1t_2}(u), \\
R_{12}(u) &= \frac{1}{V} R_{12}^{t_1t_2}(-u - \rho) \frac{1}{V}. \tag{46}
\end{align}

By $PT$ symmetry, we find there has an isomorphism between boundary matrices:

\begin{align}
K_-(u) = K_+^t(-u + \delta). \tag{47}
\end{align}

If both $PT$ symmetry and crossing unitarity are considered, there is another relation as

\begin{align}
K_-(u) = K_+^{-1}(u + \rho)M^{-1}, \quad M = V^tV. \tag{48}
\end{align}
From (47) and (48), it means
\[ K^\ell_+(u + \delta) M K^\ell_+(u + \rho) = \text{Id}, \quad K^\ell_-(u + \delta) M K^\ell_-(u - \rho) = \text{Id}. \] (49)
These equations may be regarded as constraint conditions on \( \delta \).

Now, it is interesting to compare our commutative quantities with those of Mezincescu and Nepomechie \([13]\). Using conditions of \([13]\) and unitarity condition \( R_{12}(u) R_{21}(u) = \xi(u) \), we obtain
\[
R^\ell_{12}(u - \rho) = (V)^t_{11} R^\ell_{12}(u) (V^{-1})^t_{11},
\]
\[
\xi(-u) (V)^t_{11} R^\ell_{12}(-u) (V^{-1})^t_{11},
\]
or
\[
R^\ell_{12}(u + \delta) = \xi(-u + \rho + \delta) (V)^t_{11} R^\ell_{12}(-u + \rho + \delta) (V^{-1})^t_{11}.
\]
If \( L_n(u) \) is defined as \( L_n(u) \equiv L_n^\ell(u) = R_n^\ell(u) \), we obtain
\[
L_n^\ell(-u + \rho + \delta) V^t L_n^\ell_n(-u + \rho + \delta) (V^{-1})^t,
\]
\[
T^t(-u + \delta) = L_n^\ell(-u + \delta) L_2^\ell(-u + \delta) \cdots L_N^\ell(-u + \delta)
\]
\[
\xi^N(-u + \rho + \delta) V^t T^{-1}(-u + \rho + \delta) (V^{-1})^t.
\]
In other words, transfer matrix \([13]\) becomes
\[
t(u) = \text{tr} K^\ell_+(u) T(u) K^\ell_+(u) \xi^N(-u + \rho + \delta) V^t T^{-1}(-u + \rho + \delta) (V^{-1})^t.
\]
It is convenient to multiply \([24]\) by \( \xi^{-N}(-u + \rho + \delta) \) before we regard it as the commutative quantities, i. e.,
\[
t(u) = \text{tr} K^\ell_+(u) T(u) K^\ell_+(u) (-u + \rho + \delta).
\]
If \( \delta \) is equal to \( -\rho \), it is just the one in paper \([13]\).

As one of the main results in our paper, we have constructed the commutative quantities with unitarity condition of quantum \( R_\cdot \) matrix only. As discussed in the classical case, with symmetry conditions of \([13]\), we can find another form of commutative quantities, and these two forms are the same one in fact when both of them are regarded as commutative quantities.

Finally, we study the classical counterparts of reflection equations \([14]\) by modifying unitarity condition to \( R_{12}(u) R_{21}(u) = \text{Id} \). In the classical limit, as \( \hbar \to 0 \), one has \([1,20]\):
\[
[\ , \ ] = -i\hbar \{\ , \ }; \quad R(u) = \text{Id} + i\hbar r(u) + o(\hbar^2).
\]
So unitarity condition means \( r_{12}(u) = -r_{21}(-u) \) and quantum YBE goes over into the classical YBE. We find reflection equations \([14]\) just turn into equation \([24]\) in which \( r(\alpha, \beta) \) is equal to \( r(\alpha - \beta) \) now. In contrast with the quantum case, there has an isomorphism between \( K_+ \) and \( K_- \), which is \( K_+(\alpha) \to K_-^\ell(\alpha) \).

6 Conclusion and discussion

In this paper, we obtain three possible generating functions for integrals of motion in classically integrable field theory on a finite interval with independent boundary condition on each end. As constraint conditions, we find \( K_\pm \) matrices’ algebra and evolution equations. In contrast with other’s methods, a new parameter is added on spectral parameter, and we expect it shall simplify the procedure of solving \( K_\pm \) matrices effectively. In ATFT, we prove these generating functions are equivalent to each other and its links are discussed too. Our results show that two of these generating functions are always valid in both real and imaginary coupling constant cases.

It is remarkable that no symmetry condition of \( r_\cdot \) matrix is used when we regard quantities \([11]\), \([12]\) and \([13]\) as generating functions for integrals of motion, so we expect it shall be applied to more integrable
models than [3, 5]. As demonstrated in section 4, the added parameter $\delta$ improves this possibility.

We also extend our results to quantum spin chain, we have proved that unitarity condition of quantum $R$-matrix is sufficient to construct commutative quantities with boundary. Reflect equations of $K_\pm$ are obtained. Relation between boundary $K_\pm$ matrices is found when $PT$ symmetry and crossing unitarity condition of $R$-matrix are considered. With these symmetry conditions, we also find another form of commutative quantities as the one defined in [3]. Finally, we find that classical counterparts of quantum reflection equations is just the one which we obtain by classical quantity [12].

Acknowledgement

This work was supported by Climbing Up Project, NSCC, Natural Scientific Foundation of Chinese Academy of Sciences and Foundation of NSF.

References

[1] I.V.Cherednik Theor. Math. Fiz. 61 No.1 (1984) 35
[2] E.K.Sklyanin. Funct. Anal. Appl. 21 (1987) 164;
[3] E.K.Sklyanin. J. Phys. A21 (1988) 2375
[4] S.Ghoshal and A.B.Zamolodchikov, Int. J. Mod. phys. A9 (1994) 3841
[5] P.Bowcock, E.Corrigan. P.E.Dorey and R.H.Rietdijk. Nucl. Phys. B445 (1995) 469
[6] Binder. in ”Phase Transitions and critical Phenomena” Vol. 8. eds. C.Dumb and J.Lebowitz (Academic, London, 1983)
[7] J.Cardy. Nucl. Phys. B324 (1989) 581 ; J.Cardy and D.Lewellen. Nucl. Phys. B259 (1991) 274
[8] L.D.Faddeev and A.B.Takhtajan. in ”Hamiltonian methods in the theory of soliton”. (Springer-Verlag 1987)
[9] Y.X.Chen and X.D.Luo, High Energy Phys. and Nucl. Phys., 22 (1998) 413 ( in Chinese )
[10] A.MacIntyre, J. Phys. A28 (1995) 1089
[11] A.V.Mikhailov, M.A.Olshanetsky and A.M.Perelomov, Commun. Math. Phys. 79 (1981) 473
[12] T.Hollwood. Nucl. Phys. B384 (1992) 523; Int. J. Mod. Phys. A8 (1993) 947
[13] L.Mezincescu and R.I.Nepomechie J. Phys. A24 (1991) L17; Int. J. Mod. Phys. A6 (1991) 5231
[14] H.J. de Vega and A.González Ruiz J. Phys. A26 (1993) L519
[15] T.Inami and H.Konno J. Phys. A27 (1994) 211
[16] N.Reshetikhin and M.Semenov-Tian-Shansky Lett. Math. Phys. 19 (1990) 133
[17] R.H.Yue and Y.X.Chen J. Phys. A26 (1993) 2989
[18] H.Fan, B.Y.Hou, K.J.Shi and Z.X.Yang Phys. Lett. A200 (1995) 109
[19] P.P.Kulish Yang-Baxter equation and reflection equations in integrable models hep-th/9507070
[20] L.D.Faddeev Les Houches 1982 ed J.B.Zuber and R.Stora (Amsterdam: North-Holland) pp 561-608