ON A SPACE DISCRETIZATION SCHEME FOR THE FRACTIONAL STOCHASTIC HEAT EQUATIONS

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Abstract. In this work, we introduce a new discretization to the fractional Laplacian and use it to elaborate an approximation scheme for fractional heat equations perturbed by a multiplicative cylindrical white noise. In particular, we estimate the rate of convergence.

Keywords: fractional Laplacian, cylindrical Wiener process, Sobolev spaces, heat equation, approximation scheme, finite difference scheme.

Subjclass MSC[2000]: 60H15, 35R11, 35A35, 26A33.

1. Introduction

In this work, we are interested in the space approximation of the solutions of fractional stochastic heat equations. Equations where leading operators are fractional or more general pseudo-differential operators are widely used to model complex phenomena. For example, they are ubiquitous in the study of the quasi geostrophic flow, the fast rotating fluids, the dynamic of the frontogenesis in meteorology, the diffusions in fractal or disordered medium, the pollution problems, the mathematical finance and the transport problems, see e.g. [2, 5, 6, 17, 19, 26, 27, 32] and the references therein. The wellposedness of these equations, in the deterministic and stochastic cases, has been extensively studied see e.g. [2, 4, 5, 6, 9, 18, 30]. Although the numerical approximation of the solution is needed in applications, the number of numerical schemes relevant to such approximations is quite restricted. The main difficulty of the numerical approximation of fractional equations is related to the fractional operator. For example, contrarily to second order differential operators, the fractional operators can not be discretized by three points. Using the classical schemes and as a global operator, all the values on the grid should be used in every step. Via the integro-differential representation of the fractional operator, a direct discretization is based on the discretization of the integrals. This idea has been used in the numerical study of the deterministic conservation law driven by fractional power of the Laplacian in [10]. Unfortunately, as it is mentioned in the paper, the

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2The frontogenesis is the terminology used by atmosphere scientists for describing the formation in finite time of a discontinuous temperature front.
convergence of the scheme elaborated is slow and leads to some unreasonable values. Moreover, in the theoretical study, the author did not give an explicit form to the discretized operator. The discretization of the integrals has been already used for the Liouville and Riemann fractional operators and has yielded the Gr"unwald formula, see for short list \[7, 16, 24\].

During the preparation of this work, we found the work of Westphal \[31\], on an approach to fractional powers of operators via fractional differences. The approximation given by Westphal provides a rigorous mathematical support to a numerical discretization of fractional operators. Westphal defined the fractional operator of the infinitesimal generator \(A\) of a semigroup \(S(t)\), defined on Banach space \(X\), by:

\[
A^r := s - \lim_{\tau \downarrow 0} \tau^{-r} (I - S(\tau))^r, 0 < r < 1,
\]

where

\[
(I - S(\tau))^r := \sum_{j=0}^{\infty} C^{-r-1}_j S(j \tau) = \sum_{j=0}^{\infty} \frac{\Gamma(j - r)}{\Gamma(j + 1)\Gamma(-r)} S(j \tau)
\]

and \(s - \lim\) means the strong limit, i.e. the limit in the space \(X\) of \(\tau^{-r}(I - S(\tau))^r f\), for every \(f \in X\). In particular, she proved:

\[
\lim_{\tau \downarrow 0} \tau^{-r} \sum_{j=0}^{[x/\tau]} C^{-r-1}_j f(x - j \tau) = D^r f(x)
\]

where \([x/\tau]\) is the integer part of \(x/\tau\) and \(D^r\) is the fractional differential Riemann-Liouville operator defined on \(\mathbb{R}_+\). An intuitive way to discretize the operator \(D^r\) can be obtained by taking \(\tau = \frac{1}{n}\):

\[
D^r_n f(.) := n^r \sum_{j=0}^{[nx]} C^{-r-1}_j f(., - \frac{j}{n}).
\]

To encounter the difficulties of the direct discretization of the fractional operator, probabilistic technics have been used. In particular, in \[25\] the authors used the Monte Carlo method to approximate numerically the solution of some deterministic fractional partial differential equations, among them the Burgers equations. In \[23\] the authors used wavelet techniques to approximate the Kolmogorov equation driven by the infinitesimal generator of a Feller process.

Our idea to discretize a fractional operator \(A^r\) is to discretize first the operator \(A\) then to take the fractional power of the discrete operator. As far as the authors know, this idea is new.

In this work, we discretize the fractional Laplacian, in the way described above, and we elaborate a scheme to approximate the fractional stochastic heat equation. Our aim is also to calculate explicitly the rate of the convergence and to show its dependence on the fractional power of the Laplacian.
We are also interested in the critical values of the fractional order which insures the convergence of the scheme as well.

We consider the following fractional stochastic heat equation:

\begin{equation}
\begin{aligned}
\frac{\partial}{\partial t} u(t,x) &= \frac{\partial^\alpha}{\partial x^\alpha} u(t,x) + g(u(t,x)) \frac{\partial^2 W}{\partial x^2}(t,x), \quad t > 0, \quad x \in (0,1), \\
u(0,x) &= u_0(x), \quad x \in (0,1), \\
u(t,0) &= u(t,1) = 0, \quad t > 0,
\end{aligned}
\end{equation}

where $\frac{\partial^\alpha}{\partial x^\alpha} = (-\Delta)^{\frac{\alpha}{2}}$, $\alpha > 1$ is the fractional power of the Laplacian. Let us denote by $A_\alpha = A^{\frac{\alpha}{2}}$, where $A = -\Delta$ is the Laplacian with boundary Dirichlet conditions, defined on $D(A) = H^{2,2}(0,1) \cap H^{1,2}_0(0,1)$. $H^{k,p}(0,1)$, for $k \in \mathbb{N}$, $p \in [1, \infty)$ is the Sobolev space of order $k$. The fractional operator $A_\alpha = A^{\frac{\alpha}{2}}$ is well defined, see e.g. Lemma 2.6.6 in [20] and it is given by the formula (see [20] pp 72-73):

\begin{equation}
A_\alpha = (-\Delta)^{\frac{\alpha}{2}} := \frac{\sin \frac{\alpha \pi}{2}}{\pi} \int_0^\infty z^{\frac{\alpha}{2} - 1} A(Iz + A)^{-1} dz.
\end{equation}

The operator $A_\alpha$ is a closed densely defined operator with domain of definition given via the complex interpolation of order $\frac{\alpha}{2}$: $D(A_\alpha) = [H, D(A)]_{\frac{\alpha}{2}}$, see e.g. [15] [29] and Theorem 4.2 in [28]. More precisely,

\[ D(A_\alpha) = D(A^{\alpha/2}) = \{ v \in L^2(0,1) : \sum_{k=1}^{\infty} \lambda_k^{\alpha/2} v_k^2 < \infty \}, \]

where $v_k = \langle v, e_k \rangle = \sqrt{2} \int_0^1 v(x) \sin k\pi x \, dx$ and $\lambda_k = k^2 \pi^2$, $k \in \mathbb{N}$ are the eigenvalues of the operator $A$ corresponding to the eigenfunctions: $\{ e_k = \sqrt{2} \sin k\pi \cdot \}_{k \in \mathbb{N}}$. The map $g : \mathbb{R} \to \mathbb{R}$, is a bounded Lipschitz continuous function on $\mathbb{R}$. The operator $g$ is regarded as a nonlinear operator from $H = L^2(0,1)$ to $L(H)$, the set of bounded linear operators on $H$, defined by $g(u)(h) = \{(0,1) \ni x \mapsto g(u(x)) h(x) \in \mathbb{R}\}$. In other words, the nonlinear operator $g$ is the Nemytskii map associated with function $g$. For $v \in H$, $g(v)$ is given as a multiplicative operator. From the hypothesis that $g$ is bounded we have $\|g(v)\| \leq b_0$, where $b_0 = \sup_{\mathbb{R}} |g(x)|$. \{ $W(t), t \geq 0$ \} is a cylindrical Wiener process on the probability space $(\Omega, \mathcal{F}, \{\mathcal{F}\}_{t \geq 0}, \mathbb{P})$. The initial condition $u_0$ is a $L^2(0,1)$-valued $\mathcal{F}_0$-measurable function. In section 4 we will suppose stronger condition on the diffusion term $g$ and on the initial condition $u_0$. In particular we will suppose that $A^\delta g$ is a bounded Nemytskii map for some $\delta > 0$; $\|A^\delta g(v)\| \leq b_6$ with $b_6 = \sup_{\mathbb{R}} |A^\delta g(x)|$ and $u_0$ belongs to a given fractional Sobolev space.

We rewrite the equation (1.2) in the following form:

\begin{equation}
\begin{aligned}
du(t) &= -A_\alpha u(t) \, dt + g(u(t)) \, dW(t), \quad t > 0, \\
u(0) &= u_0.
\end{aligned}
\end{equation}
Let us denote by $\{S_\alpha(t), t \geq 0\}$ the semigroup generated by $-A_\alpha$ and by $H^{\theta,2}$ the fractional Sobolev space of order $\nu$. By a solution of the equation (1.4), we mean, see e.g. [8]:

**Definition 1.1.** Suppose that $\alpha > 1$. An $F_t$-adapted $H^{\theta,2}$-valued continuous process $u = \{u(t), t \geq 0\}$ is called a mild solution of equation (1.4) with initial condition $u_0 \in H^{\eta,2}, \eta > 0$, iff for some $p > \frac{2\alpha}{\alpha - 1 - 2\theta}$ where $\theta < \min\{\frac{\alpha - 1}{2}, \eta\}$

$$E \sup_{t \in [0,T]} |u(t)|^p_{H^{\theta,2}} < \infty, \quad T > 0$$

and for all $t \geq 0$, a.s. the following identity holds

$$u(t) = S_\alpha(t)u_0 + \int_0^t S_\alpha(t-s)g(u(s)) \, dW(s).$$

We introduce the space:

**Definition 1.2.** Let $T > 0$ and $p \in [1, \infty]$ be fixed and $H$ a Hilbert space. By $Z_{T,\theta,p}(H)$ we denote the space of all $H^{\theta,2}$-valued continuous and $F_t$-adapted processes $u = \{u(t), t \in [0,T]\}$ such that

$$\|u\|_{Z_{T,\theta,p}}^p := E \sup_{t \in [0,T]} |u(t)|^p_{H^{\theta,2}} = \|u\|_{L^p(\Omega, L^\infty(0,T) \times H^{\theta,2})} < \infty.$$}

If $\theta = 0$, we use shortly the notation $Z_{T,p}(H)$.

The following result of existence and uniqueness of the solution of equation (1.4) can be concluded from the calculus in [4]:

**Theorem 1.3.** Let $\alpha > 1$ and let $u_0$ be a $H^{\eta,2}$-valued $F_0$-measurable function such that

$$E|u_0|^{p}_{H^{\eta,2}} < \infty$$

for some $p > \frac{2\alpha}{\alpha - 1}, \ 0 \leq \theta < \min\{\frac{\alpha - 1}{2}, \frac{\alpha}{p}\}$, and let $T > 0$. Then there exists a unique mild solution $u \in Z_{T,\theta,p}$, of equation (1.4).

The paper is organized in the following way. In section 2 we describe the discretization of the fractional operator. In particular, we apply the idea for the Galerkin approximation and for the finite difference method. In section 3 we elaborate a numerical scheme to approximate the solution of the fractional stochastic heat equation (1.4). In section 4 we give some preliminary estimations of the Green functions corresponding to the fractional operator and to the approximated operator. The section 5 is devoted to prove the convergence of the approximated solution to the solution of the equation (1.4). In the end of this introduction, let us mention the following references where the the approximations of certain stochastic partial differential equations are treated [1, 12, 13, 14, 21]. Let us also mention that, we take in the whole paper $p, \alpha > 1$ and the values of the constants may change from line to line.
2. Discretization of the fractional operator

Let us first recall notions about the approximations of the Laplacian $-A$, see e.g. [3]. We consider the Gelfand triple $(V, H, V')$, where $V \hookrightarrow H = H' \hookrightarrow V'$ densely, where $V = H^1_0(0,1)$ and $V'$ is its dual. The operator $A$ defines a coercive bilinear form on $V$ by

$$a(u,v) := \langle Au, v \rangle = \langle u', v' \rangle, \quad u, v \in V,$$

where $u'$, $v'$ are the first derivatives of $u$ and $v$ in the distribution sense. It is widely accepted that to approximate the Laplacian $-A$ via the stiffness matrix the following formula (see [20] pp 72-73):

$$A_{V_n} := \sum_{i,j=1}^{n} u_{ij} f_{ij},$$

where $A_{V_n}$ defines a coercive bilinear form on $V$. It is known that the operator $A_{V_n}$ is well defined via the double series index $(a(f_i, f_j))_{i,j}$. Using Riesz representation we can rewrite the bilinear form $a_{V_n}$ as:

$$a_{V_n}(u_{V_n}, v_{V_n}) = \langle A_{V_n} u_{V_n}, v_{V_n} \rangle = \sum_{i,j=1}^{n} u_{ij} \langle A_{V_n} f_i, f_j \rangle,$$

where $A_{V_n}$ is a positive bounded linear operator on $V_n$ which is well defined via the stiffness matrix $A_{V_n} := (a(f_i, f_j) = \langle A_{V_n} f_i, f_j \rangle)_{i,j=1}^{n}$.

Now we define the fractional power of the approximated operator $A_{V_n}$ by the following formula (see [20] pp 72-73):

$$A^{\frac{\alpha}{2}}_{V_n} = \sin \frac{\pi \alpha}{2} \int_{0}^{\infty} z^{\frac{\alpha}{2}-1} A_{V_n} (zI + A_{V_n})^{-1} dz$$

and the fractional bilinear form

$$a_{V_n, \alpha}(u_{V_n}, v_{V_n}) = \langle A^{\frac{\alpha}{2}}_{V_n} u_{V_n}, v_{V_n} \rangle.$$

The fractional stiffness matrix $A^{\frac{\alpha}{2}}_{V_n}$ is then given by $(a_{V_n, \alpha}(f_i, f_j) = \langle A^{\frac{\alpha}{2}}_{V_n} f_i, f_j \rangle)_{i,j=1}^{n}$.

Our idea is to investigate how and on what rate the operator $A^{\alpha}_{V_n}$ and the bilinear form $a_{V_n, \alpha}$ are good approximations to the operator $A^{\alpha}$ respectively the bilinear form $a_{\alpha} := \langle A^{\frac{\alpha}{2}} u, v \rangle$.

Before going through this calculus, let us apply this method to calculate the approximation of the fractional operator and of the stiffness matrix corresponding to the following two methods; the Galerkin method and the finite difference method.

2.1. Approximation by Galerkin method. Let $V_n$ be the subspace generated by the basis $(e_j)_{j=1}^{n}$ defined above. Recall that $e_j := \sqrt{2} \sin(j\pi)$. It is known that the operator $A$ and $A^{\frac{\alpha}{2}}$ are diagonalizable under the basis $(e_j)_{j=1}^{\infty}$. Consequently the approximating operator $A^{\alpha}_{V_n}$ and the stiffness matrix $A^{\frac{\alpha}{2}}_{V_n}$ are diagonal with respect to the basis $(e_j)_{j=1}^{n}$ and with corresponding eigenvalues $(\lambda_i)_{i=1}^{n}$. Thanks to the boundness of the approximation
operator $A_{V_n}$, it is also easy to define $A_{V_n}$ on the basis as: $A_{V_n}^\frac{\alpha}{2} e_i = \lambda_i^\frac{\alpha}{2} e_i$, see also Lemma C.1.

2.2. Approximation by finite difference method. Let $\{x_i := \frac{i}{n}, i = 0, \ldots, n\}$ be the set of grid points and let $\phi_i, i \in \{1, \ldots, n-1\}$ be a pyramid function, i.e. a function, which takes value 1 at the grid point $\frac{i}{n}$, vanishes at the other grid points and is linear between the grid points. The approximating space $V_{n-1}$ is then generated by $\{\phi_i\}_{i=1}^{n-1}$. For an implementation reasons, we will focus on the stiffness matrix. It is well known that the stiffness matrix $A_{V_{n-1}}$ corresponding to the finite difference approximation of the operator $A$ is given by:

$$a_{ij} = \begin{cases} 
2, & i = j, \\
-1, & i = j + 1 \text{ or } i = j - 1, \\
0, & \text{otherwise.}
\end{cases}$$

Using the matrix theory it is easy to calculate the fractional power of the $A_{V_{n-1}}$, denoted by $A_{V_{n-1}}^\frac{\alpha}{2}$ via the formula:

$$(2.2) \quad A_{V_{n-1}}^\frac{\alpha}{2} = \sin \frac{\alpha \pi}{2} \int_0^\infty z^{\frac{\alpha}{2}-1} A_{V_{n-1}} (Iz + A_{V_{n-1}})^{-1} dz.$$ 

3. Discretization of the fractional stochastic Heat equations

Let us first observe that the eigenvalues of $A_{V_{n-1}}$ are given by $\lambda_{jn} := j^2 \pi^2 c_{jn}$, $j = 1, 2, \ldots n - 1$ where $c_{jn} := \sin^2 (\frac{j \pi}{2n})/(\frac{j \pi}{2n})^2$, the corresponding eigenvectors $e_j^n = ((e_{jk})_k)$ are given by [1]:

$$e_{jk} = \sqrt{\frac{2}{n}} \sin (\frac{j k}{n}).$$

From Lemma C.1 it is easy to see that $\lambda_{jn} j = 1, 2, \ldots n - 1$ are the eigenvalues of $A_{V_{n-1}}$ corresponding to the eigenvectors $e_j^n = ((\sqrt{\frac{2}{n}} \sin (\frac{j k}{n})))$, $j = 1, 2, \ldots n - 1$. The semi group $S^n_\alpha(t)$ generated by $A_{V_{n-1}}^\frac{\alpha}{2}$ is given by:

$$(3.1) \quad S^n_\alpha(t) x := (\sum_{j=1}^{n-1} G^n_\alpha(t, i, j) x_j)_i,$$

where $G^n_\alpha(t, i, j) := \sum_{k=1}^{n-1} e^{-t \lambda_k^\alpha} e_k^n e_k^n$ and $x = (x_j)_{1 \leq j \leq n-1}$. Let us define now the operators: $P_n : L^2(0, 1) \to \mathbb{R}^{n-1}$ and $E_n : \mathbb{R}^{n-1} \to L^2(0, 1)$, called projection respectively interpolation operators given by the following formula:

For all $f \in L^2(0, 1)$ and for all $x \in \mathbb{R}^{n-1}$:

$$(3.2) \quad P_n f := \sum_{k=1}^{n-1} (P_n f)_k e_k^n,$$
the function $y \in \mathbb{R}$ is the composition of the two operators:

\[(P_nf)_k := \sum_{j=1}^{n-1} (f, e_j)e_j^n(x_k).
\]

(3.4) 
$E_n x := \sum_{k=1}^{n-1} \langle x, e_k^n \rangle e_k$.

It is easy to see that the operators $P_n$ and $E_n$ satisfy the properties:

**Lemma 3.1.**
- $P_n$ and $E_n$ are bounded linear operators such that $||P_n|| \leq 1$ and $||E_n|| = 1$.
- $P_n E_n = I_n$, where $I_n$ is the identity matrix in $\mathbb{R}^{n-1}$.
- $E_n P_n = \hat{P}_n$, where $\hat{P}_n$ is the projection on the finite dimensional space in $L^2$ generated by $e_1, e_2, \ldots, e_{n-1}$. i.e. $\hat{P}_n f := \sum_{j=1}^{n-1} (f, e_j)e_j$.
- $P_n e_j = e_j^n$, if $j = 1, 2, \ldots, n - 1$ and zero if $j \geq n$.
- $E_n e_j^n = e_j$, if $j = 1, 2, \ldots, n - 1$.
- $\{e_j, j = 1, 2, \ldots, n - 1\}$ are the eigenfunctions of the operator $E_n \hat{A} \frac{\phi}{\gamma} e_n P_n$ corresponding to the eigenvalues $\lambda_j^n$.
- The Green function of $E_n \hat{A} \frac{\phi}{\gamma} e_n P_n$, which is also the kernel of the semigroup: $E_n e^{-t \hat{A} \frac{\phi}{\gamma} e_n} P_n$, is given by

\[G_n(t, x, y) := \sum_{k=1}^{n-1} e^{-t \lambda_k^n} e_k(x)e_k(y).\]

**Lemma 3.2.** For $0 \leq \delta < \frac{1}{4} + \frac{3}{4} \alpha$ and $\gamma > \frac{1}{\alpha} - 4 \frac{\delta}{\alpha}$, there exists $K > 0$, such that $\forall t \in (0, T]$,

\[|S_\alpha(t) - E_n e^{-t \hat{A} \frac{\phi}{\gamma} e_n} P_n|_{L(H \to D(A^{-\delta}))} + \|A^{-\delta}(S_\alpha(t) - E_n e^{-t \hat{A} \frac{\phi}{\gamma} e_n} P_n)\|_{HS} \leq K \phi_{\alpha, \delta, \gamma}(t, n),\]

(3.5) where

\[\phi_{\alpha, \delta, \gamma}(t, n) := \begin{cases} 
\frac{n^{-\alpha - \frac{3}{2}} - \frac{2 \delta}{\alpha} t - \frac{1 + 4 \delta - 4 \alpha}{4\alpha}}{n^{-\alpha - \frac{3}{2}} + \frac{2 \delta}{\alpha} t - \frac{1 + 4 \delta - 4 \alpha}{4\alpha}}, & 0 \leq \delta \leq \frac{1}{4}, \\
\frac{n^{-2 \delta} + \frac{3}{4} t - \frac{1 + 4 \delta - 4 \alpha}{2 \alpha}}{n^{-2 \delta} + \frac{3}{4} t - \frac{1 + 4 \delta - 4 \alpha}{2 \alpha}}, & \frac{1}{4} < \delta < \frac{1}{4} + \frac{3}{4} \alpha.
\end{cases}\]

Let us make the following convention to write $\phi_{\alpha, \delta, \gamma}$ shortly as $\phi_{\alpha, \delta}$ when $\gamma$ is not presented, i.e. when $\frac{1}{4} < \delta < \frac{1}{4} + \frac{3}{4} \alpha$.

Let us now discretize the diffusion term $g$. We denote by $g_n$ the matrix which is given by the column vectors: $(P_n((g \circ E_n)e_j))_j, 1 \leq j \leq n - 1$, where $\circ$ is the composition of the two operators: $E_n$ and the Nymetsky map $g$. For $y \in \mathbb{R}^{n-1}$, the operator $(g \circ E_n)y$ acts as the Nymetsky map associated with the function $g(E_n y \cdot)$, i.e. $(g \circ E_n)y h = \{(0, 1) \ni x \mapsto g(E_n y(x)) h(x) \in \mathbb{R}\}$. We denote by $W_n(t)$ the vector $(B_1(t), B_2(t), \ldots, B_{n-1}(t))$ of independent
Furthermore, for all \( T > 0 \) Brownian motions. We introduce the following multidimensional stochastic differential equation, where \( u_n(t) := (u^k_n(t))_{1 \leq k \leq n-1} \):

\[
\begin{aligned}
\left\{ \begin{array}{ll}
du_n(t) &= -\frac{\partial}{\partial V_{n-1}} u_n(t) \, dt + g_n(u_n(t)) \, dW_n(t), \; t > 0, \\
\quad u_n(0) &= P_n u_0.
\end{array} \right.
\end{aligned}
\tag{3.7}
\]

For the existence and uniqueness of the solution of the stochastic differential equation (3.7), we refer e.g. to Theorem 2.3 in [11]:

**Theorem 3.3.** There exists a continuous \( \mathbb{R}^{n-1} \)-valued \( F_t \)-adapted process \( u_n = \{u_n(t), t \geq 0\} \) solution of the problem (3.7) such that:

\[
u_n(t) = e^{-\alpha V_{n-1}} u_n(0) + \int_0^t e^{-(t-s)\lambda V_{n-1}} g_n(u_n(s)) dW_n(s), \; \text{a.s.}
\tag{3.8}
\]

Furthermore, there exists a constant \( C_{T,n,||g||} \) such that

\[
E \sup_{t \in [0,T]} |u_n(t)|_{L^p}^p \leq C_{T,n,||g||}(1+E|u_0|_{L^2}^p), \; \text{for all} \; T > 0 \; \text{and} \; p \in [1, \infty).
\tag{3.9}
\]

We define the \( L^2 \)-valued stochastic process \( u^n(t) := E_n u_n(t) \). We prove, see Appendix A, that:

**Lemma 3.4.** The process \( u^n(t) := E_n u_n(t) \) satisfies the following stochastic integral equation:

\[
u^n(t) = E_n e^{-\alpha V_{n-1} t} P_n u_0 + \int_0^t e^{-(t-s)\lambda V_{n-1}} P_n g(u^n(s)) dW^n(s), \; \text{a.s.}
\tag{3.10}
\]

where

\[
\int_0^t E_n e^{-(t-s)\lambda V_{n-1}} P_n g(u^n(s)) dW^n(s) := \sum_{j=1}^{n-1} \int_0^t E_n (e^{-(t-s)\lambda V_{n-1}} g_n(u_n(s)))_j dB_j(s)
\]

\[
= \sum_{j=1}^{n-1} \int_0^t E_n e^{-(t-s)\lambda V_{n-1}} P_n g(u^n(s)) e_j dB_j(s)
\]

and \( (e^{-(t-s)\lambda V_{n-1}} g_n(u_n(s)))_j \) is the \( j \)’s column of the matrix \( e^{-(t-s)\lambda V_{n-1}} g_n(u_n(s)) \).

Furthermore, for all \( T > 0 \) and \( p \in [1, \infty) \),

\[
E \sup_{t \in [0,T]} |u^n(t)|_{L^p}^p \leq C_{T,n,||g||}(1+E|u_0|_{L^2}^p).
\]

In other words, \( u^n(t) \) satisfies the stochastic partial differential equation:

\[
\left\{ \begin{array}{ll}
du^n(t) &= -E_n \frac{\partial}{\partial V_{n-1}} P_n u^n(t) \, dt + g(u^n(t)) dW^n(t), \; t > 0, \\
\quad u^n(0) &= E_n P_n u_0.
\end{array} \right.
\]
4. Preliminary Estimates

In this section we give a priori estimations to the Green functions, \( G_\alpha^n \) and \( G_\alpha \), corresponding to \( \partial_t - A^\alpha_t \) respectively \( \partial_t - A^\frac{\alpha}{2n-1} \) and to their difference.

Lemma 4.1. For \( 0 \leq \delta < \frac{1}{4} + \frac{3}{4}\alpha \) and for all \( \gamma > \frac{1}{\alpha} - 4\frac{\delta}{\alpha} \), there exists \( K > 0 \), such that \( \forall t \in (0, T] \),

\[
(4.1) \quad \sum_{j=1}^{n-1} \lambda_j^{-2\delta} |e^{-t\lambda_j^\alpha} - e^{-t\lambda_j^\beta}|^2 \leq Kn^{-\alpha} t^{-1 - \frac{1}{\alpha} + 4\frac{\delta}{\alpha}}
\]

\[
(4.2) \quad \sum_{j=n}^\infty \lambda_j^{-2\delta} e^{-2t\lambda_j^\beta} \leq K \left( n^{-\alpha\gamma - 4\delta} t^{-\gamma} \mathcal{X}_{[0, \frac{1}{4}]}(\delta) + n^{-\delta} \mathcal{X}_{[\frac{1}{4}, \frac{1}{4} + \frac{\alpha}{4}]}(\delta) \right),
\]

where \( \mathcal{X}_B \) is the characteristic function of the set \( B \):

\[
\mathcal{X}_B(b) := \begin{cases} 
1, & b \in B \\
0, & b \notin B.
\end{cases}
\]

Proof. To get the estimation (4.1), we use the mean value theorem. We obtain,

\[
\sum_{j=1}^{n-1} \lambda_j^{-2\delta} |e^{-t\lambda_j^\alpha} - e^{-t\lambda_j^\beta}|^2 = \sum_{j=1}^{n-1} \lambda_j^{-2\delta} e^{-2t(\lambda_j^\alpha - \lambda_j^\beta)} |1 - e^{(1-c_{jn})t(\lambda_j^\alpha - \lambda_j^\beta)}|^2
\]

\[
\leq \sum_{j=1}^{n-1} \lambda_j^{-2\delta} e^{-2t(\lambda_j^\alpha - \lambda_j^\beta)} |1 - c_{jn}^{\alpha/\lambda_j^\alpha}|^2 (\lambda_j^\alpha)^2 e^{2t(1-c_{jn})t^*} t^*, \quad t^* \in [0, 1]
\]

\[
\leq \sum_{j=1}^{n-1} (j\pi)^{-4\delta} |1 - c_{jn}^{\alpha/\lambda_j^\alpha}|^2 (\lambda_j^\alpha)^2 e^{2t(1-c_{jn})t^*} t^*, \quad t^* \in [0, 1]
\]

We know that: \( \left( \frac{\pi}{n} \right)^{\alpha/\lambda_j^\alpha} \leq c_{jn}^{\alpha/\lambda_j^\alpha} := |\sin(\frac{j\pi}{2n})/(\frac{j\pi}{2n})|^{\alpha} \leq 1 \). Taking \( 1 - c_{jn}^{\alpha/\lambda_j^\alpha} = O(\frac{\pi}{2n})^{2\alpha} \), we obtain

\[
\sum_{j=1}^{n-1} \lambda_j^{-2\delta} |e^{-t\lambda_j^\alpha} - e^{-t\lambda_j^\beta}|^2 \leq K t^2 n^{-4\alpha} \sum_{j=1}^{n-1} j^{6\alpha - 4\delta} e^{-2\alpha + 1} j^\alpha t
\]

\[
\leq K t^2 n^{-\alpha} \sum_{j=1}^{n-1} j^{3\alpha - 4\delta} e^{-2\alpha + 1} j^\alpha t
\]

\[
\leq K t^2 n^{-\alpha} \int_0^\infty x^{3\alpha - 4\delta} e^{-2\alpha + 1} x^\alpha dx
\]

\[
\leq K t^{-1 - \frac{1}{\alpha} + 4\delta} n^{-\alpha} \int_0^\infty y^{3\alpha - 4\delta} e^{-2\alpha + 1} y^\alpha dy.
\]
The integral \( \int_0^\infty y^{3\alpha-4\delta} e^{-2\alpha y} dy \) converges provided \( 0 \leq \delta < \frac{1}{4} + \frac{3}{4}\alpha \). Hence, there exists \( K > 0 \) such that:

\[
\sum_{j=1}^{n-1} \lambda_j^{-2\delta} |e^{-\lambda_j^\frac{2}{\alpha} t} - e^{-\lambda_j^\frac{2}{\alpha} t}|^2 \leq K t^{-1-\frac{1}{\alpha} + \frac{4\delta}{\alpha}} n^{-\alpha}.
\]

To get the second estimation (4.2), let us first consider the case \( \delta \in [0, \frac{1}{4}] \). We use the known result: for all \( \gamma > 0 \), there exists a constant \( k = k_\gamma \) such that \( e^{-x} \leq K x^{-\gamma} \), we get

\[
\sum_{j=n}^{\infty} \lambda_j^{-2\delta} e^{-2\lambda_j^\frac{2}{\alpha} t} \leq \sum_{j=n}^{\infty} \frac{K}{(2\lambda_j^\frac{2}{\alpha})^\gamma} \leq Kn^{-\alpha\gamma - 4\delta} t^{-\gamma} \sum_{j=n}^{\infty} \frac{n^\gamma}{j^\gamma}.
\]

But \( \sum_{j=n}^{\infty} \frac{n^\gamma}{j^\gamma} \leq K \int_1^{+\infty} x^{-\alpha\gamma - 4\delta} dx < \infty \), provided \( \frac{1}{\alpha} - 4\frac{\delta}{\alpha} < \gamma \). Hence

\[
\sum_{j=n}^{\infty} \lambda_j^{-2\delta} e^{-2\lambda_j^\frac{2}{\alpha} t} \leq Kn^{-\alpha\gamma - 4\delta} t^{-\gamma}.
\]

For \( \delta \in (\frac{1}{4}, \frac{1}{4} + \frac{3}{4}\alpha) \), we use the inequality: \( \sum_{j=n}^{\infty} \lambda_j^{-2\delta} e^{-2\lambda_j^\frac{2}{\alpha} t} \leq \sum_{j=n}^{\infty} \lambda_j^{-2\delta} \), than we arguing as above and using the condition \( \delta > \frac{1}{4} \), we obtain

\[
\sum_{j=n}^{\infty} \lambda_j^{-2\delta} e^{-2\lambda_j^\frac{2}{\alpha} t} \leq \sum_{j=n}^{\infty} \lambda_j^{-2\delta} \leq Kn^{-4\delta} \sum_{j=n}^{\infty} \frac{n^\delta}{j^\delta} \leq Kn^{-4\delta}.
\]

\[\square\]

**Lemma 4.2.** For \( 0 \leq \delta < \frac{1}{4} + \frac{3}{4}\alpha \) and \( \gamma > \frac{1}{\alpha} - 4\frac{\delta}{\alpha} \), there exists \( K > 0 \), such that \( \forall t \in (0, T] \), we have

\[
\sup_{x \in [0,1]} |A^{-\delta}(G_\alpha(t, x, .) - G_\alpha^n(t, x, .))|_H \leq K \phi_{\alpha, \delta, \gamma}(t, n),
\]

where \( \phi_{\alpha, \delta, \gamma}(t, n) \) is given by (3.6).
Proof. Using the definitions of the functions $G_\alpha(t, x, y)$ and $G^n_\alpha(t, x, y)$ and the fact that the orthonormal basis $(e_k)_{k \geq 1} \subset L^\infty(0, 1)$, we get

$$|A_x^{-\delta}(G_\alpha(t, x, .) - G^n_\alpha(t, x, .))|_H = \left| \sum_{j=1}^{n-1} (e^{-\lambda_j^2 t} - e^{-\lambda_j^2 n t}) A_x^{-\delta} e_j(x)e_j \right|_H + \left| \sum_{j=n}^{\infty} e^{-\lambda_j^2 t} A_x^{-\delta} e_j(x)e_j \right|_H$$

$$\leq \left| \sum_{j=1}^{n-1} (e^{-\lambda_j^2 t} - e^{-\lambda_j^2 n t}) A_x^{-\delta} e_j(x)e_j \right|_H + \left| \sum_{j=n}^{\infty} e^{-\lambda_j^2 t} A_x^{-\delta} e_j(x)e_j \right|_H$$

$$\leq \left( \sum_{j=1}^{n-1} (e^{-\lambda_j^2 t} - e^{-\lambda_j^2 n t})^2 \lambda_j^{-\delta} e_j(x)e_j(x) \right)^{1/2} + \left( \sum_{j=n}^{\infty} e^{-2\delta t} \lambda_j^{-2\delta} e_j(x)e_j(x) \right)^{1/2}$$

$$\leq \left( \sum_{j=1}^{n-1} \lambda_j^{-2\delta} (e^{-\lambda_j^2 t} - e^{-\lambda_j^2 n t})^2 \right)^{1/2} + \left( \sum_{j=n}^{\infty} \lambda_j^{-2\delta} e^{-2\delta t} \right)^{1/2}.$$

(4.8)

Replacing (4.2) and (4.1) in (4.8), we get the result. □

As a consequence of Lemma 4.2, we obtain:

**Corollary 4.3.** Under the same conditions in Lemma 4.2

(4.9) \[|A^{-\delta}(G_\alpha(t, ., .) - G^n_\alpha(t, ., .))|_{H \times H} \leq K_{\phi, \delta, \gamma}(t, n).\]

**Proof of Lemma 3.2.** Let $f \in H$. The semigroups $S_\alpha$ and $E_n e^{-t\alpha V_n^{-1}} P_n$ are acting on an element $f$ via their Green functions in the following:

$$(S_\alpha(t)f)(x) = \int_0^1 G_\alpha(t, x, y)f(y)dy$$

and

$$(E_n e^{-t\alpha V_n^{-1}} P_n f)(x) = \int_0^1 G^n_\alpha(t, x, y)f(y)dy.$$

Applying the Hölder inequality, we get:
\[ |A^{-\delta}(S_\alpha(t) - E_n e^{-t\kappa_{\alpha \gamma} P_n})f|_H = \left[ \int_0^1 |A^{-\delta}((S_\alpha(t) - E_n e^{-t\kappa_{\alpha \gamma} P_n})f)(x)|^2 \, dx \right]^{\frac{1}{2}} \]

\[ = \left[ \int_0^1 \int_0^1 A^{-\delta}_x(G_\alpha(t, x, y) - G^n_\alpha(t, x, y))f(y)dy \, dx \right]^{\frac{1}{2}} \]

\[ \leq \left[ \int_0^1 dx \left( \int_0^1 |A^{-\delta}_x(G_\alpha(t, x, y) - G^n_\alpha(t, x, y))|^2 dy \right) \right]^{\frac{1}{2}} |f|_H \]

\[ \leq \left| A^{-\delta}_x(G_\alpha(t, \cdot, \cdot) - G^n_\alpha(t, \cdot, \cdot)) \right|_{H \times H} \frac{1}{2} |f|_H. \]

Using (4.9), we get the result. For the second estimation, we have by a direct application of the definitions of Hilbert-Schmidt norm and the properties of the semigroups \( S_\alpha(t) \) and \( E_n e^{-t\kappa_{\alpha \gamma} P_n} \),

\[ \| A^{-\delta}(S_\alpha(t) - E_n e^{-t\kappa_{\alpha \gamma} P_n}) \|_{HS}^2 := \sum_{j=1}^{\infty} |A^{-\delta}(S_\alpha(t) - E_n e^{-t\kappa_{\alpha \gamma} P_n})e_j|_H^2 \]

\[ \leq \sum_{j=1}^{n-1} \lambda^{-2\delta}_j e^{-t\lambda^{\frac{\alpha}{2}}_j} - e^{-t\lambda^{\frac{\alpha}{2}}_j} 2 \| e_j \|_H \]

\[ \sum_{j=n}^{\infty} \lambda^{-2\delta}_j e^{-t\lambda^{\frac{\alpha}{2}}_j} - e^{-t\lambda^{\frac{\alpha}{2}}_j} 2 \| e_j \|_H \]

\[ \leq \sum_{j=1}^{n-1} \lambda^{-2\delta}_j e^{-t\lambda^{\frac{\alpha}{2}}_j} - e^{-t\lambda^{\frac{\alpha}{2}}_j} 2 \| e_j \|_H + \sum_{j=n}^{\infty} \lambda^{-2\delta}_j e^{-t\lambda^{\frac{\alpha}{2}}_j}. \]

(4.10)

Thanks to the basic inequality \((a + b)^{\frac{1}{2}} \leq a^{\frac{1}{2}} + b^{\frac{1}{2}}\) for \(a, b \geq 0\) and by the formula (4.1) and (4.2), we get the result.

5. Convergence of the scheme

Now we are ready to give the main result of this work.

**Theorem 5.1.** For \( \alpha > 1 \),

\[ \frac{1}{4} + \frac{\alpha}{4} < \eta < \frac{1}{4} + \frac{3\alpha}{4} \]

(5.1)

\[ \max \left\{ \frac{1}{2}, \frac{1}{4} + \frac{\alpha}{8} \right\} < \delta < \frac{1}{4} + \frac{3\alpha}{4} \]

(5.2)

\[ p > \max \left\{ \frac{2\alpha}{\alpha - 2}, \frac{\alpha}{2\delta - 1}, \frac{2\alpha}{8\delta - \alpha - 2} \right\} \]

(5.3)

assume that:
• \((H_g)\) the diffusion term is the Nemytski operator defined by the Lipschitz function \(g \in D(A^\delta)\) and such that \(b_\delta := \sup_{x \in \mathbb{R}} |A^\delta g(x)| < \infty\).

• \((H_{u_0})\) the initial condition \(u_0\) is an \(\mathcal{D}(A^\eta)\)-valued \(L^p\) random variable i.e. \(u_0 \in L^p(\Omega, \mathcal{D}(A^\eta))\).

Then \(u^n = \{u^n(t), t \in [0, T]\}\) converges to \(u := \{u(t), t \in [0, T]\}\) in the space \(Z_{T,p}(L^2(0,1))\). Furthermore, there exists a constant \(K > 0\), such that,

\[
\|u(t) - u^n(t)\|_{Z_{T,p}(L^2(0,1))} \leq K_{T,|u_0|_{\mathcal{D}(A^\eta)},b_\delta} n^{-\xi},
\]

(5.4)

where \(\xi\) is given by

\[
\xi = \min\{\frac{\alpha}{2}, 2\delta\}.
\]

(5.5)

In particular, for \(1 < \alpha \leq 2\), the rate of convergence \(\xi = \frac{\alpha}{2}\).

**Remark 1.** Let us remark that the rate of convergence is independent of the regularity of the diffusion term when the dissipation order \(\alpha\) is less than the Laplacian dissipation. In this case, it is enough to take \(g \in H^1\).

**Theorem 5.2.** Assume that \(\alpha > 2\), \(p > \frac{2\alpha}{\alpha - 2}\), \(g\) and \(u_0\) satisfy respectively \((H_g)\) with \(\delta = 0\) and \((H_{u_0})\), with \(\eta\) satisfying (5.1). Then \(u^n = \{u^n(t), t \in [0, T]\}\) converges to \(u := \{u(t), t \in [0, T]\}\) in the space \(Z_{T,p}(L^2(0,1))\) and

\[
\|u(t) - u^n(t)\|_{Z_{T,p}(L^2(0,1))} \leq K_{T,|u_0|_{\mathcal{D}(A^\eta)},b_\delta} n^{-(\frac{\alpha}{2} - \frac{1}{2} - \frac{\alpha}{p})}.
\]

(5.6)

First let us introduce some lemmata which we will use in the proof of the convergence.

**Lemma 5.3.** The operator \(A^{-\delta}\) commutes with \(S_\alpha(t)\) and with \(E_n e^{-\frac{A^\eta}{2} P_n} Z\) for all \(t \geq 0\).

**Proof.** For the proof see the Appendix B. \(\square\)

**Lemma 5.4.** Suppose that \(z \in L^q(0,1)\) for \(q \in [2, \infty]\). Let \(Z\) denote the multiplication operator by \(z\). Then, for \(\beta < 1 - \frac{1}{\alpha}\), there exists a constant \(K > 0\), such that

\[
\int_0^\infty s^{-\beta} \|E_n e^{-\frac{A^\eta}{2} P_n} Z\|^2_{HS} ds \leq K \left( \sum_{k=1}^{+\infty} k^{-\alpha(1-\beta)} \right) \|z\|^2_{L^q} < \infty.
\]

(5.7)

**Proof.** Let us first estimate the term \(\|E_n e^{-\frac{A^\eta}{2} P_n} Z\|^2_{HS}\). Using Lemma 3.1 we have
\[
\|E_n e^{-\frac{\Phi}{h V_{n-1}}} P_n Z\|_{HS}^2 = \sum_{k=1}^{\infty} |E_n e^{-\frac{\Phi}{h V_{n-1}}} P_n Z e_k|_{L^2}^2 = \sum_{k=1}^{\infty} |Z E_n e^{-\frac{\Phi}{h V_{n-1}}} P_n e_k|_{L^2}^2 = \sum_{k=1}^{n-1} |Z E_n e^{-\frac{\Phi}{h V_{n-1}}} e_k|_{L^2}^2 = \sum_{k=1}^{n-1} |Z e^{-\frac{\Phi}{h V_{n-1}}} E_n e_k|_{L^2}^2 = \sum_{k=1}^{n-1} e^{-\lambda^{\alpha}_{kn}s} |Z e_k|_{L^2}^2.
\]

Let us observe that by the Hölder inequality, \(|Z e_k|_{L^2} \leq |Z|_{L^q}|e_k|_{L^r}\), where \(\frac{1}{r} + \frac{1}{q} = \frac{1}{2}\). Moreover, since \(|e_k|_{L^2} = 1\) and \(|e_k|_{L^\infty} = 2^{1/2}\) it follows by applying the Hölder inequality that \(|e_k|_{L^r} \leq 2^{1/q}\). Let us recall that \(\lambda^{\alpha}_{kn} := (c_{kn}(k\pi)^2)^\frac{\alpha}{2}\), where \(c_{kn} := \sin^2\left(\frac{k\pi}{2n}\right)\), we get,

\[
\|E_n e^{-\frac{\Phi}{h V_{n-1}}} P_n Z\|_{HS}^2 \leq 2^2 |Z|_{L^q}^2 \sum_{k=1}^{n-1} e^{-\lambda^{\alpha}_{kn}s} \leq 2^2 |Z|_{L^q}^2 \sum_{k=1}^{n-1} e^{-\lambda_{kn}|k\pi|^\alpha s}.
\]

Therefore and thanks to the fact that: \((\frac{\alpha}{2})^\alpha \leq |c_{kn}|^{\frac{\alpha}{2}} = |\sin(\frac{k\pi}{2n})/(\frac{k\pi}{2n})|^{\alpha} \leq 1,

\[
\int_0^\infty s^{-\beta} \|E_n e^{\frac{\Phi}{h V_{n-1}}} P_n Z\|_{HS}^2 ds \leq 2^2 |Z|_{L^q}^2 \int_0^\infty s^{-\beta} \sum_{k=1}^{n-1} e^{-\lambda_{kn}|k\pi|^\alpha s} ds = 2^2 |Z|_{L^q}^2 \sum_{k=1}^{n-1} (c_{kn})^{\frac{\alpha}{2}} (k\pi)^{\alpha - \beta} \int_0^\infty \tau^{-\beta} e^{-\tau} d\tau \leq K |Z|_{L^q}^2 \sum_{k=1}^{n-1} (k\pi)^{\alpha - \beta} \int_0^\infty \tau^{-\beta} e^{-\tau} d\tau \leq K |Z|_{L^q}^2 \sum_{k=1}^{n-1} k^{-\alpha(1-\beta)}.
\]

Since by our assumptions \(\alpha(1-\beta) > 1\) the series on the RHS above is convergent and the result follows.

The following Lemma is a special case of Lemma 2.7 from [4]:

**Lemma 5.5.** Provided that \(\nu > p^{-1}\) the operator \(R_\nu : L^p(0, T; L^2(0, 1)) \to C([0, T]; L^2(0, 1))\) given by

\[
R_\nu h(t) = \int_0^t (t-s)^{\nu-1} S_\alpha(t-s) h(s) ds, \ h \in L^p(0, T; L^2(0, 1))
\]
Lemma 5.6. Let \( \frac{1}{4} < \delta < \frac{1}{4} + \frac{3}{4}\alpha \) and \( p > \max\{1, \frac{2\alpha}{\alpha + 1 + \delta}\} \), then there exists \( \nu \), satisfying \( \max\{0, \frac{1}{2} + \frac{\alpha}{2\alpha} - \frac{2\alpha}{\alpha + 1 + \delta} + \frac{1}{p}\} < \nu < 1 \), such that the operator \( G_{n,\nu} : L^p(0, T; D(A^\delta)) \rightarrow C([0, T]; L^2) \) given by

\[
G_{n,\nu}h(t) = \int_0^t (t-s)^{\nu-1} \left[ S_\alpha(t-s) - E_n e^{-(t-s)A^{1/\nu} P_n} \right] h(s) \, ds, \quad h \in L^p(0, T; D(A^\delta))
\]

is well defined, linear and bounded. Moreover, there exists a constant \( K_{p,\nu} > 0 \) such that for all \( 0 < \nu < 1 \) and \( t \in (0, T) \) and thanks to Lemma 5.3, we have

\[
|G_{n,\nu}h(t)|_{L^2} \leq K_{T,p}(n^{-2\delta} + n^{-2\nu})|h|_{L^p(0, T; D(A^\delta))}.
\]

Proof. Let us fix \( h \in L^p(0, T; D(A^\delta)) \). Then for \( t \in (0, T) \) and thanks to Lemma 5.3, we have

\[
|G_{n,\nu}h(t)|_{L^2} = \left| \int_0^t (t-s)^{\nu-1} \left[ S_\alpha(t-s) - E_n e^{-(t-s)A^{1/\nu} P_n} \right] A^{\delta} h(s) \, ds \right|_{L^2}
\]

\[
\leq \int_0^t (t-s)^{\nu-1} |A^{\delta} [S_\alpha(t-s) - E_n e^{-(t-s)A^{1/\nu} P_n}] h(s)|_{L^2} \, ds
\]

\[
\leq \int_0^t (t-s)^{\nu-1} |S_\alpha(t-s) - E_n e^{-(t-s)A^{1/\nu} P_n} h(s)|_{L^2} \, ds.
\]

From Lemma 5.2 (5.8) and (5.9) and by applying the Hölder inequality, we get

\[
|G_{n,\nu}h(t)|_{L^2} \leq K \int_0^t (t-s)^{\nu-1} \phi_{\alpha,\delta}(t-s, n) |A^{\delta} h(s)|_{L^2} \, ds
\]

\[
\leq K \left( \int_0^T (s^{\nu-1} \phi_{\alpha,\delta}(s, n))^{\frac{p}{\nu+1}} \, ds \right)^{\frac{\nu-1}{\nu}} \left( \int_0^T |A^{\delta} h(s)|_{L^2}^p \, ds \right)^{\frac{1}{p}}.
\]

Thanks to the basic inequality: \( (x+y)^\theta \leq c_\theta(x^\theta + y^\theta) \), for \( \theta > 1 \) and \( x, y \geq 0 \), we have

\[
\int_0^T (s^{\nu-1} \phi_{\alpha,\delta}(s, n))^{\frac{p}{\nu+1}} \, ds = \int_0^T (s^{\nu-1} (n^{-2\delta} + n^{-\frac{\alpha}{2\alpha} - \frac{1+\alpha-4\delta}{2\alpha}}))^{\frac{p}{\nu+1}} \, ds
\]

\[
\leq c_p (n^{-2\delta} T^{\frac{p}{\nu+1}} + n^{-\frac{\alpha}{2\alpha} - \frac{1+\alpha-4\delta}{2\alpha}} T^{\frac{p}{\nu+1}} \int_0^T s^{(\nu-1 - \frac{1+\alpha-4\delta}{2\alpha})} \, ds).
\]

(5.10)
The last integral in the RHS of (5.10) converges provided \( \nu > \frac{1}{2} + \frac{1}{2\alpha} - \frac{2\delta}{\alpha} + \frac{1}{p} \).

Hence
\[
\left( \int_0^T (s^{\nu-1} \phi_{\alpha,\delta}(s,n))^{\frac{p}{p-1}} ds \right)^{\frac{p-1}{p}} \leq c_{p,T}(n^{-2\delta} + n^{-\frac{\alpha}{2}}).
\]

The choice of \( \nu \) such that \( \frac{1}{2} + \frac{1}{2\alpha} - \frac{2\delta}{\alpha} + \frac{1}{p} < \nu < 1 \) is possible thanks to the condition \( p > \frac{2\alpha}{\alpha - 1 + 4\delta} \). Finally, we have for all \( t \geq 0 \)
\[
|G_{\nu}h(t)|_{L^2} \leq K_{T,p}(n^{-2\delta} + n^{-\frac{\alpha}{2}})|h|_{L^p(0,T;D(A^\delta))}.
\]

**Proof of Theorem 5.1.** Let
\[
M_n := ||u - u^n||_{T,p} = \mathbb{E} \sup_{t \in [0,T]} |u(t) - u^n(t)|_H^p.
\]

We have from equations (1.6) and (3.10),
\[
|u(t) - u^n(t)|_H^p \leq c_p(A(t) + B(t))
\]
where
\[
A(t) := |S_\alpha(t)u_0 - E_n e^{-t\Lambda_n^{\frac{\alpha}{2}}} P_n u_0|_H^p
\]
(5.12)
\[
B(t) := |\int_0^t S_\alpha(t-s)g(u(s))dW(s) - \int_0^t E_n e^{-(t-s)\Lambda_n^{\frac{\alpha}{2}}} P_n g(u^n(s))dW^n(s)|_H^p
\]
(5.13)

**Estimation of \( A(t) \):** Using Lemmata 5.3 and 3.2 ((3.5) and (3.6)), We obtain,
\[
A(t) := |(S_\alpha(t) - E_n e^{-t\Lambda_n^{\frac{\alpha}{2}}} P_n)u_0|_H^p
\]
\[
= |A^{-\eta}(S_\alpha(t) - E_n e^{-t\Lambda_n^{\frac{\alpha}{2}}} P_n)A^\eta u_0|_H^p
\]
\[
\leq |(S_\alpha(t) - E_n e^{-t\Lambda_n^{\frac{\alpha}{2}}} P_n)|_{L(H \to D(A^{-\eta}))}^p A^\eta u_0|_H^p
\]
\[
\leq |u_0|_{D(A^\eta)}^p \phi_{\alpha,\eta}(t,n)
\]
\[
\leq |u_0|_{D(A^\eta)}^p \left( n^{-2\eta} + \frac{\frac{\alpha}{2} - 4\delta}{2\alpha} \right)^p.
\]

Thanks to the condition: \( \eta > \frac{1}{2} + \frac{\alpha}{2} \), the power \( \frac{1}{2} + \frac{\alpha}{2} - \frac{2\delta}{\alpha} \) is positive, hence
\[
A(t) \leq K_T |u_0|_{D(A^\eta)}^p \left( n^{-2\eta} + n^{-\frac{\alpha}{2}} \right)^p.
\]
Consequently:

\[
\mathbb{E}\left[ \sup_{[0,T]} A(t) \right] \leq K_T \mathbb{E}|u_0|_{D^{(A^\alpha)}}^p \left( n^{-2\eta} + n^{-\frac{\eta}{2}} \right)^p
\]

\[
\leq K_T \mathbb{E}|u_0|_{D^{(A^\alpha)}}^p \left( n^{-2\eta_0} + n^{-\frac{\eta_0}{2}} \right).
\]

(5.14)

**Estimation of \( B(t) \):** Let us first introduce the transformations \( n \) defined on the set of Nemytsky maps \( N \), such that for \( h \in N \):

\[
\hat{n} h(u) e_j := \begin{cases} h(u)e_j & j < n, \\ 0 & j \geq n \end{cases}
\]

(5.15)

Then we write the second stochastic integral in RHS of (5.13), as

\[
\int_0^t E_n e^{-(t-s)\frac{\alpha}{\nu} A^\alpha_{n-1} P_n g(u^n(s))} dW^n(s) := \sum_{j=1}^{n-1} \int_0^t E_n e^{-(t-s)\frac{\alpha}{\nu} A^\alpha_{n-1} P_n g(u^n(s))} e_j dB_j(s)
\]

\[
= \int_0^t E_n e^{-(t-s)\frac{\alpha}{\nu} A^\alpha_{n-1} P_n g(u^n(s))} dW(s).
\]

(5.16)

Using the factorization method see e.g. [4] and [8], we can again rewrite the integrals in (5.13) for \( 0 < \nu < 1 \) as:

\[
\int_0^t S_\alpha(t-s) g(u(s)) dW(s) = \int_0^t (t-s)^{\nu-1} S_\alpha(t-s) Y(s) ds
\]

and

\[
\int_0^t E_n e^{-(t-s)\frac{\alpha}{\nu} A^\alpha_{n-1} P_n g(u^n(s))} dW(s) = \int_0^t (t-s)^{\nu-1} E_n e^{-(t-s)\frac{\alpha}{\nu} A^\alpha_{n-1} P_n Y_n(s)} ds,
\]

where

\[
Y(s) := \int_0^s (s-r)^{-\nu} S_\alpha(s-r) g(u(r)) dW(r)
\]

and

\[
Y_n(s) := \int_0^s (s-r)^{-\nu} E_n e^{\frac{\alpha}{\nu} A^\alpha_{n-1} P_n g(u^n(r))} dW(r)
\]

(5.17) \( Y(s) \) and (5.18) \( Y_n(s) \)
Consequently,

\[
B(t) = \left| J_t^0 (t-s)^{-1} \alpha (t-s) Y(s) ds - J_t^0 (t-s)^{-1} E_n e^{-(t-s)\frac{\alpha}{T_n^{\frac{2}{\nu}}}} P_n Y_n(s) ds \right|_H^p
\]

\[
\leq c_p \left( J_t^0 (t-s)^{-1} \alpha (t-s) (Y(s) - Y_n(s)) ds \right|_H^p
\]

\[
+ J_t^0 (t-s)^{-1} \left[ \alpha (t-s) - E_n e^{-(t-s)\frac{\alpha}{T_n^{\frac{2}{\nu}}}} P_n \right] Y_n(s) ds \right|_H^p
\]

(5.19)

Using Lemmata 5.5 and 5.6 and taking

\[
\max \{ p^{-1}, \frac{1}{2} + \frac{1}{2\alpha} - \frac{2\delta}{\alpha} + \frac{1}{p} \} < \nu < 1
\]

we deduce that

\[
\mathbb{E} \left[ \sup_{[0,T]} B(t) \right] \leq c_p \mathbb{E} \sup_{[0,T]} \left( J_t^0 (t-s)^{-1} \alpha (t-s) (Y(s) - Y_n(s)) ds \right|_H^p
\]

\[
+ J_t^0 (t-s)^{-1} \left[ \alpha (t-s) - E_n e^{-(t-s)\frac{\alpha}{T_n^{\frac{2}{\nu}}}} P_n \right] Y_n(s) ds \right|_H^p
\]

\[
\leq K_{T,p} \left[ \mathbb{E} \| Y - Y_n \|^p_{L^p(0,T;L^2(0,1))} + (n^{-2\delta} + n^{-\frac{2}{p}}) \mathbb{E} \| Y_n \|^p_{L^p(0,T;D(A^\delta))} \right]
\]

(5.20)

**Calculation of \( \mathbb{E} \| Y_n \|^p_{L^p(0,T;D(A^\delta))} \).** By the Burkholder’s inequality and Lemma 5.7, there exists a constant \( C_p > 0 \) such that

\[
\int_0^T \mathbb{E} \| Y_n(s) \|^p_{D(A^\delta)} ds \leq C_p \int_0^T \mathbb{E} \left( \int_0^s (s-r)^{-2\nu} \| A^\delta e^{-(s-r)\frac{\alpha}{T_n^{\frac{2}{\nu}}}} P_n g(u(r)) \|^2_{HS} dr \right)^{\frac{p}{2}} ds
\]

\[
\leq K_p \int_0^T \mathbb{E} \left( \int_0^s (s-r)^{-2\nu} \| A^\delta e^{-(s-r)\frac{\alpha}{T_n^{\frac{2}{\nu}}}} P_n \|_{HS}^2 \sup_{r \in [0,T]} | A^\delta g(u(r)) |_{L^\infty(H)}^2 dr \right)^{\frac{p}{2}} ds
\]

\[
\leq K_p \left( \int_0^T \int_0^s (s-r)^{-2\nu} \| A^\delta e^{-(s-r)\frac{\alpha}{T_n^{\frac{2}{\nu}}}} P_n \|_{HS}^2 \sup_{r \in [0,T]} | A^\delta g(u(r)) |_{L^\infty(H)}^2 dr \right)^{\frac{p}{2}} ds
\]

\[
\leq K_p \left( \int_0^T \int_0^s (s-r)^{-2\nu} \| A^\delta e^{-(s-r)\frac{\alpha}{T_n^{\frac{2}{\nu}}}} P_n \|_{HS}^2 \sup_{r \in [0,T]} | A^\delta g(u(r)) |_{L^\infty(H)}^2 dr \right)^{\frac{p}{2}} ds
\]

(5.21)

Since \( p > \max \{ \frac{\alpha}{2\nu}, \frac{2\alpha}{\nu} \} \) and \( \delta > \frac{1}{2} \), we can chose \( \nu \), such that

\[
\max \{ p^{-1}, \frac{1}{2} + \frac{1}{2\alpha} - \frac{2\delta}{\alpha} + \frac{1}{p} \} < \nu < \frac{1}{2} - \frac{1}{2\alpha}
\]

what implies that \( \alpha(1-2\nu) > 1 \) we infer that the last term in (5.21) is finite.
Calculation of $\mathbb{E}|Y - Y_n|^p_{L^p(0,T;L^2(0,1))}$. Using the formula (5.17) and (5.18), we have:

$$
|Y(s) - Y_n(s)|^p_H \leq c_p \left( \left| \int_0^s (s - r)^{-\nu} (S_\alpha(s - r) - E_n e^{-(s-r)\Delta Y_{V_{n-1}} P_n}) g(u(r)) dW(r) \right|^p_H + \left| \int_0^s (s - r)^{-\nu} E_n e^{-(s-r)\Delta Y_{V_{n-1}} P_n} (g(u(r)) - g(u^n(r))) dW(r) \right|^p_H \right).
$$

By Burkholder’s inequality, we have

$$
\mathbb{E}|Y(s) - Y_n(s)|^p_H \leq c_p \left( \mathbb{E} \left( \int_0^s (s - r)^{-2\nu} \| (S_\alpha(s - r) - E_n e^{-(s-r)\Delta Y_{V_{n-1}} P_n}) g(u(r)) \|_{H_S}^2 dr \right)^{\frac{p}{2}} + \mathbb{E} \left( \int_0^s (s - r)^{-2\nu} \| E_n e^{-(s-r)\Delta Y_{V_{n-1}} P_n} (g(u(r)) - g(u^n(r))) \|_{H_S}^2 dr \right)^{\frac{p}{2}} \right).
$$

Using the well known functional inequality $\| AB \|_{H_S} \leq \| A \|_{H_S} \| B \|_{L(H)}$, we estimate the first term in the RHS of (5.22) as follow:

$$
\int_0^s (s - r)^{-2\nu} \| (S_\alpha(s - r) - E_n e^{-(s-r)\Delta Y_{V_{n-1}} P_n}) g(u(r)) \|_{H_S}^2 dr
\leq \int_0^s (s - r)^{-2\nu} \| A^{-\delta} (S_\alpha(s - r) - E_n e^{-(s-r)\Delta Y_{V_{n-1}} P_n}) A^{\delta} g(u(r)) \|_{H_S}^2 dr
\leq \int_0^s (s - r)^{-2\nu} \| A^{-\delta} (S_\alpha(s - r) - E_n e^{-(s-r)\Delta Y_{V_{n-1}} P_n}) \|_{H_S}^2 \| A^{\delta} g(u(r)) \|_{L(H)}^2 dr
\leq b_3^2 \int_0^s (s - r)^{-2\nu} \| A^{-\delta} (S_\alpha(s - r) - E_n e^{-(s-r)\Delta Y_{V_{n-1}} P_n}) \|_{H_S}^2 dr.
$$

(5.23)

Thanks to (3.3) (3.6), we have

$$
\int_0^s (s - r)^{-2\nu} \| A^{-\delta} (S_\alpha(s - r) - E_n e^{-(s-r)\Delta Y_{V_{n-1}} P_n}) \|_{H_S}^2 dr
\leq K \int_0^s (s - r)^{-2\nu} \phi_{\alpha,\delta}^2 (s - r, n) dr
\leq K \left( n^{-4\delta} \int_0^s r^{-2\nu} dr + n^{-\alpha} \int_0^s r^{-2\nu - 1 - \frac{4\delta}{\alpha}} dr \right).
$$

The integral $\int_0^s r^{-2\nu} dr$ is finite thanks to the condition $\nu < \frac{1}{2} - \frac{1}{2\alpha} < \frac{1}{2}$ and the integral $\int_0^s r^{-2\nu - 1 - \frac{4\delta}{\alpha}} dr$ converges thanks to the conditions $\nu < \min\left(\frac{2\delta}{\alpha} - \frac{1}{2\alpha}, \frac{1}{2} - \frac{1}{2\alpha}\right)$. Hence, we take the parameter $\nu$ which satisfies the
following inequalities:

\[
\max\{p^{-1} \frac{1}{p} + \frac{1}{2} + \frac{1}{2\alpha} - \frac{2\delta}{\alpha}\} < \nu < \min\{\frac{2\delta}{\alpha} - \frac{1}{2\alpha}, \frac{1}{2}\}.
\]

(5.24)

The parameter \(\nu\) exists thanks to the conditions: \(\delta > \frac{1}{4} + \frac{\alpha}{8}\) and \(p > \max\{\frac{2\alpha}{8\delta - 2\alpha}, \frac{\alpha}{2\delta - 1}\}\). Hence

\[
\int_s^0 (s - r)^{-2\nu} \|S_\alpha(s - r) - E_n e^{-(s-r)\frac{\alpha}{\nu} P_n} g(u(r))\|^2_{HS} dr \\
\leq K_T \left(n^{-4\delta} + n^{-\alpha}\right)
\]

(5.25)

By replacing (5.25) in (5.23), we get

\[
\int_s^0 (s - r)^{-2\nu} \|\left(S_\alpha(s - r) - E_n e^{-(s-r)\frac{\alpha}{\nu} P_n} g(u(r))\right)\|^2_{HS} dr \\
\leq b_\delta^2 K_T \left(n^{-4\delta} + n^{-\alpha}\right).
\]

Hence,

\[
E\left(\int_s^0 (s - r)^{-2\nu} \|\left(S_\alpha(s - r) - E_n e^{-(s-r)\frac{\alpha}{\nu} P_n} g(u(r))\right)\|^2_{HS} dr\right)^{\frac{1}{2}} \\
\leq b_\delta^2 K_T \left(n^{-4\delta} + n^{-\alpha}\right)^{\frac{1}{2}} \\
\leq b_\delta^2 K_T \left(n^{-2\delta} + n^{-\alpha}\right)
\]

(5.26)

Now we estimate the second term in (5.22). Arguing as in the proof of Lemma 3.2, we have

\[
\|E_n e^{-(s-r)\frac{\alpha}{\nu} P_n} P_n g(u(r)) - g(u^n(r))\|^2_{HS} = \|E_n e^{-(s-r)\frac{\alpha}{\nu} P_n} P_n^2 g(u(r)) - g(u^n(r))\|^2_{HS} \\
= \inf_{j=1}^{\infty} \|\sum_{j=1}^{n}(g(u(r)) - g(u^n(r))) e_n e^{-(s-r)\frac{\alpha}{\nu} P_n e_j} - g(u^n(r))\|^2_{H}.
\]

From Lemma 3.1, we have

\[
E_n e^{-(s-r)\frac{\alpha}{\nu} P_n e_j} = \begin{cases} e^{-(s-r)\frac{\alpha}{\nu} e_j}, & j \in \{1, \ldots, n - 1\} \\
0, & j \geq n \end{cases}
\]
using the definition of $g(u^n(r))$, the Lipschitz property of $g$ and Lemma 3.1, we get

\[
\|E_n e^{-(s-r)\frac{\lambda}{\alpha} V_n^{\alpha-1}} P_n \left( g(u(r)) - g(u^n(r)) \right) \|_{HS}^2 = \sum_{j=1}^{n-1} e^{-2(s-r)\lambda_j} \left| \left( g(u(r)) - g(u^n(r)) \right) e_j \right|_H^2
\]

\[
\leq \sum_{j=1}^{n-1} e^{-2(s-r)\lambda_j} |g(u(r)) - g(u^n(r))|_{L^2(H)}^2
\]

\[
\leq \sum_{j=1}^{n-1} e^{-2(s-r)\lambda_j} |u(r) - u^n(r)|_H^2.
\]

Hence,

\[
\mathbb{E}\left( \int_0^s (s-r)^{-2\nu} \|E_n e^{-(s-r)\frac{\lambda}{\alpha} V_n^{\alpha-1}} P_n \left( g(u(r)) - g(u^n(r)) \right) \|_{HS}^2 dr \right)^{\frac{\nu}{2}} \leq \mathbb{E} \sup_{r \in [0,s]} |u(r) - u^n(r)|_H^p \left( \sum_{j=1}^{n-1} \int_0^s e^{-2r\lambda_j} dr \right)^{\frac{\nu}{2}}.
\]

(5.27)

Arguing as in the proof of Lemma 5.4 we get a constant $K$ which depends only on $\nu$, such that

\[
\mathbb{E}\left( \int_0^s (s-r)^{-2\nu} \|E_n e^{-(s-r)\frac{\lambda}{\alpha} V_n^{\alpha-1}} P_n \left( g(u(r)) - g(u^n(r)) \right) \|_{HS}^2 dr \right)^{\frac{\nu}{2}} \leq K \mathbb{E} \sup_{r \in [0,s]} |u(r) - u^n(r)|_H^p \left( \sum_{j=1}^{\infty} j^{-\alpha(1-2\nu)} \right)^{\frac{\nu}{2}}
\]

(5.28)

since $0 < \nu < \frac{1}{2} - \frac{1}{2\alpha}$. Now replacing (5.20) and (5.28) in (5.22), we obtain

\[
\mathbb{E}|Y(s) - Y_n(s)|_H^p \leq K_{p,T,b,s} \left( \left( n^{-\delta} + n^{-\alpha} \right)^{\frac{\nu}{2}} + \mathbb{E} \sup_{r \in [0,s]} |u(r) - u^n(r)|_H^p \right)
\]

(5.29)
consequently,

\[
\mathbb{E}[Y - Y_n]_{L^p(0,T;L^2(0,1))}^p = \int_0^T \mathbb{E}[|Y(s) - Y_n(s)|^p_{L^2}] \, ds
\]

\[
\leq K_{p,T,b\delta} \left( \left( n^{-4\delta} + n^{-\alpha} \right)^\frac{p}{2} + \int_0^T \mathbb{E} \sup_{r \in [0,s]} |u(r) - u^n(r)|^p_{H^1} \, ds \right)
\]

\[
\leq K_{p,T,b\delta} \left( n^{-2\delta p} + n^{-\frac{\alpha p}{2}} + \int_0^T \mathbb{E} \sup_{r \in [0,s]} |u(r) - u^n(r)|^p_{H^1} \, ds \right)
\]

(5.30)

and by replacing (5.21) and (5.30) in (5.20), we obtain

\[
\mathbb{E} \left[ \sup_{[0,T]} B(t) \right] \leq K_{p,T,b\delta} \left[ n^{-2\delta p} + n^{-\frac{\alpha p}{2}} + \int_0^T \mathbb{E} \sup_{r \in [0,s]} |u(r) - u^n(r)|^p_{H^1} \, ds \right].
\]

(5.31)

Finally, we join the estimations in (5.31) and (5.14), we obtain,

\[
\mathbb{E} \left[ \sup_{[0,T]} |u(t) - u^n(t)|^p_{H^1} \right] \leq K^p_{p,T,|u_0|_{D(\lambda^\alpha)},b\delta} \left( \int_0^T \mathbb{E} \sup_{r \in [0,s]} |u(r) - u^n(r)|^p_{H^1} \, ds \right)
\]

\[
+ n^{-2\eta p} + n^{-2\delta p} + n^{-\frac{\alpha p}{2}}
\]

(5.32)

Thanks to Gronwall Lemma,

\[
\mathbb{E} \left[ \sup_{[0,T]} |u(t) - u^n(t)|^p_{H^1} \right] \leq K^p_{p,T,|u_0|_{D(\lambda^\alpha)},b\delta} \left( n^{-2\eta p} + n^{-2\delta p} + n^{-\frac{\alpha p}{2}} \right).
\]

Now it is easy to see that from the conditions on \( \eta \), that \( \frac{\alpha}{2} \leq 2\eta \), hence

\[
\mathbb{E} \left[ \sup_{[0,T]} |u(t) - u^n(t)|^p_{H^1} \right] \leq K^p_{p,T,|u_0|_{D(\lambda^\alpha)},b\delta} n^{-\xi p},
\]

(5.33)

where \( \xi \) is given by (5.5). Furthermore, if \( 1 < \alpha \leq 2 \), then \( \frac{\alpha}{2} \leq \frac{1}{2} + \frac{\alpha}{4} \leq 1 < 2\delta \), which implies that \( \xi = \frac{\alpha}{2} \).

\( \square \)

To prove Theorem 5.2, we will use the same scheme as before, but with different estimations:
Lemma 5.7. For \( \gamma > \frac{1}{\alpha} \), \( 0 < \beta < 1 \), there exists \( K > 0 \), such that \( \forall t \in (0, T) \),
\[
|S_\alpha(t) - E_n e^{-th\nu_{n-1} P_n}|_{L(H)} + \|S_\alpha(t) - E_n e^{-th\nu_{n-1} P_n}\|_{HS} \leq K\left( n^{-\frac{\alpha + \beta}{2}} t^{-\frac{\beta}{2}} + n^{-\frac{\alpha - \beta}{2}} t^{-\frac{1 + \alpha - \beta}{2\alpha}} \right),
\]
(5.34)

Proof. The proof follows the same steps as in Lemmata 3.2 and 4.1 and by considering the following key estimation:
\[
\sum_{j=1}^{n-1} |e^{-\lambda_j^2 t} - e^{-\lambda_j^2 t}|^2 \leq Kt^2 n^{-\frac{4\alpha}{\gamma}} \sum_{j=1}^{n-1} j^{6\alpha} e^{-2^{\alpha+1} j^\alpha t}
\leq Kt^2 n^{\alpha + \beta \alpha} \sum_{j=1}^{n-1} j^{3\alpha - \alpha \beta} e^{-2^{\alpha+1} j^\alpha t}
\leq Kt^2 n^{\alpha + \beta} \int_{0}^{\infty} x^{3\alpha - \alpha \beta} e^{-2^{\alpha+1} x^\alpha} dx
\leq Kt^{-1 + \frac{1}{\gamma} + \beta} n^{\alpha + \beta} \int_{0}^{\infty} y^{3\alpha - \alpha \beta} e^{-2^{\alpha+1} y^\alpha} dy
\leq Kt^{-1 + \frac{1}{\gamma} + \beta} n^{\alpha + \beta}.
\]
(5.35)

Lemma 5.8. Provided \( p > \frac{2\alpha}{2\alpha - 1} \), \( \frac{1}{\alpha} < \gamma < \frac{2p-1}{p} \) and \( \frac{2}{p} + \frac{1}{\alpha} - 1 < \beta \), there exists \( \nu \) satisfying: \( \max\{1+\alpha - \alpha \beta, \frac{1}{p} + \frac{1}{\alpha} \} < \nu < 1 \), such that the operator \( G_{n,\nu} : L^p(0, T; L^2(0, 1)) \rightarrow C([0, T]; L^2(0, 1)) \) given by
\[
G_{n,\nu} h(t) = \int_{0}^{t} (t-s)^{\nu-1} \left[ S_\alpha(t-s) - E_n e^{-(t-s)\nu_{n-1} P_n} \right] h(s) \, ds, \, h \in L^p(0, T; L^2(0, 1))
\]
is well defined, linear and bounded. Moreover, there exists a constant \( K_{p,\nu} > 0 \) such that for all \( h \in L^p(0, T; L^2(0, 1)) \),
\[
|G_{n,\nu} h|_{C([0, T]; L^2(0, 1))} \leq K_{p,\nu} (n^{-\frac{\alpha + \beta}{2}} + n^{-\frac{\alpha - \beta}{2}}) \|h\|_{L^p(0, T; L^2(0, 1))}.
\]
(5.36)

Proof. Let \( h \in L^p(0, T; L^2(0, 1)) \). Then for \( t \in (0, T) \) we have
\[
|G_{n,\nu} h(t)|_{L^2} \leq \int_{0}^{T} (t-s)^{\nu-1} \left\| S_\alpha(t-s) - E_n e^{-(t-s)\nu_{n-1} P_n} \right\|_{L(H)} \|h(s)\|_{L^2} \, ds.
\]
From Lemma 5.7 and by applying the Hölder inequality, we get
\[
|G_{n,\nu} h(t)|_{L^2} \leq K \int_{0}^{T} (t-s)^{\nu-1} \left( n^{-\frac{\gamma}{2}} (t-s)^{-\frac{\gamma}{2}} + n^{-\frac{\alpha - \beta}{2}} (t-s)^{-\frac{1 + \alpha - \beta}{2\alpha}} \right) \|h(s)\|_{L^2} \, ds
\leq K \left( \int_{0}^{T} (s^{\nu-1} \left( n^{-\frac{\gamma}{2}} s^{-\frac{\gamma}{2}} + n^{-\frac{\alpha - \beta}{2}} s^{-\frac{1 + \alpha - \beta}{2\alpha}} \right))^{\frac{p}{p-1}} \, ds \right)^{\frac{p-1}{p}} \left( \int_{0}^{T} |h(s)|_{L^2}^{p} \, ds \right)^{\frac{1}{p}}.
\]
Thanks to the basic inequality: \((x+y)\theta \leq c_{\theta} x^\theta + y^\theta\), for \(\theta > 1\) and \(x, y \geq 0\), we have

\[
\int_0^T \left( s^{\nu-1} \left( n^{-\frac{\alpha}{2}} s^{-\frac{\gamma}{2}} + n^{-\frac{\alpha}{2} - \frac{\alpha - \beta}{2\alpha}} s^{-\frac{\alpha + \alpha - \beta}{2\alpha}} \right) \right)^{\frac{p}{p-1}} ds \leq c_{p}(n^{-\frac{\alpha}{2}} + n^{-\frac{\alpha}{2} - \frac{\alpha - \beta}{2\alpha}}) \int_0^T \left( s^{\nu-1} \right)^{\frac{p}{p-1}} ds + n^{-\frac{\alpha}{2} - \frac{\alpha - \beta}{2\alpha}} \int_0^T \left( s^{\nu-1} \right)^{\frac{p}{p-1}} ds.
\]

(5.37)

For \(p > \frac{2\alpha}{2\alpha - 1}\), \(\frac{1}{\alpha} < \gamma < \frac{2p-1}{p}\) there exists \(\nu > \frac{\gamma}{2} + \frac{1}{p}\), such that the first integral in the RHS of (5.37) converges. Furthermore, if \(\beta > \frac{2}{\gamma} + \frac{1}{p} - 1\) then we can chose \(\nu\) satisfying also the inequality \(\nu > \frac{1+\alpha-\alpha\beta}{2\alpha} + \frac{1}{p}\) and hence the second integral in the RHS of (5.37) converges also. Hence for \(\nu > \max\left\{\frac{1+\alpha-\alpha\beta}{2\alpha}, \frac{1}{p}\right\}\), we have:

\[
\left( \int_0^T \left( s^{\nu-1} \left( n^{-\frac{\alpha}{2}} s^{-\frac{\gamma}{2}} + n^{-\frac{\alpha}{2} - \frac{\alpha - \beta}{2\alpha}} s^{-\frac{1+\alpha-\alpha\beta}{2\alpha}} \right) \right)^{\frac{p}{p-1}} ds \right)^{\frac{p-1}{p}} \leq c_{p}(n^{-\frac{\alpha}{2}} + n^{-\frac{\alpha}{2} - \frac{\alpha - \beta}{2\alpha}}).
\]

Hence for all \(t \geq 0\)

\[
|G_{\nu} h(t)|_{L^2} \leq K_{T,p}(n^{-\frac{\alpha}{2}} + n^{-\frac{\alpha}{2} - \frac{\alpha - \beta}{2\alpha}})|h|_{L^p(0,T;L^2(0,1))}.
\]

\[\Box\]

**Proof of Theorem 5.2.** We arguing as in the proof of Theorem 5.1. We define

\[(5.38)\]

\[M_n := |u-u^n|_{T,P}^p = \mathbb{E} \sup_{t \in [0,T]} |u(t) - u^n(t)|_H^p \leq c_{p}\left( \mathbb{E} \sup_{t \in [0,T]} A(t) + \mathbb{E} \sup_{t \in [0,T]} B(t) \right),\]

where \(A(t)\) and \(B(t)\) are given respectively by (5.12) and (5.13). The term \(\mathbb{E} \sup_{t \in [0,T]} A(t)\) is estimated thanks to the inequality (5.14). To estimate the term \(B(t)\), we use Lemmata 5.5 and 5.8

\[
\mathbb{E} \left[ \sup_{[0,T]} B(t) \right] \leq c_{p}\mathbb{E} \sup_{[0,T]} \left( \int_0^t (t-s)^{\nu-1} S_{\alpha}(t-s)(Y(s) - Y_n(s)) ds \right)_H^p
\]

\[
+ \int_0^t (t-s)^{\nu-1} \left[ S_{\alpha}(t-s) - E_n e^{-\frac{\alpha}{\nu} s} - n^{-\frac{\alpha-\alpha\beta}{2\alpha}} P_n \right] Y_n(s) ds \right)_H^p \leq K_{T,p} \left[ \mathbb{E}[Y - Y_n]_{L^p(0,T;L^2(0,1))}^p \right] \left[ \mathbb{E}[Y - Y_n]_{L^p(0,T;L^2(0,1))}^p \right] \leq K_{T,p} \left[ \mathbb{E}[Y - Y_n]_{L^p(0,T;L^2(0,1))}^p \right] \left[ \mathbb{E}[Y - Y_n]_{L^p(0,T;L^2(0,1))}^p \right]
\]

(5.39)

Now we calculate \(\mathbb{E}[Y - Y_n]_{L^p(0,T;L^2(0,1))} \) and \(\mathbb{E}[Y_n]_{L^p(0,T;L^2(0,1))} \). By the Burkholder’s inequality and Lemma 5.4 there exists a constant \(C_p > 0\) such
that

\[ \int_0^T \mathbb{E}\left| Y_n(s) \right|_{L^2}^p \, ds \leq K_p \int_0^T \mathbb{E} \left( \int_0^T (s-r)^{-2\nu} \| E_n \, e^{-(s-r)A_{\nu} T} P_n g(u(r)) \|_{H^2}^2 \right)^{\frac{p}{2}} \, ds \]

\[ \leq K_p \int_0^T \mathbb{E} \left( \int_0^T (s-r)^{-2\nu} \| E_n \, e^{-(s-r)A_{\nu} T} P_n \|_{H^2}^2 \| g(\frac{1}{L} dr) \|_{L^\infty}^2 \right)^{\frac{p}{2}} \, ds \]

\[ \leq K_p b_0^p \int_0^T \left( \int_0^T (s-r)^{-2\nu} \| E_n \, e^{-(s-r)A_{\nu} T} P_n \|_{H^2}^2 \right)^{\frac{p}{2}} \, ds \]

\[ \leq K_p b_0^p \left( \sum_{k=1}^{+\infty} k^{-\alpha(1-2\nu)} \right)^{\frac{p}{2}}. \]

Since \( 0 < \nu < \frac{1}{2} - \frac{1}{2\alpha} \), what implies that \( \alpha(1-2\nu) > 1 \) we infer that the last term is finite. In the aim to get an estimation to \( \mathbb{E}\ | Y - Y_n |_{L^p(0,T;L^2(0,1))}^p \) we use the inequality \((5.22)\). Let us remark that the estimation \((5.28)\) remains true for the second term in the RHS of \((5.22)\). Let us now estimate the first term in RHS of this inequality. We have:

\[ \int_0^s (s-r)^{-2\nu} \| S_\alpha(s-r) - E_n e^{-(s-r)A_{\nu} T} P_n g(u(r)) \|_{H^2}^2 \, dr \]

\[ \leq \int_0^s (s-r)^{-2\nu} \| S_\alpha(s-r) - E_n e^{-(s-r)A_{\nu} T} P_n \|_{H^2}^2 \| g(u(r)) \|_{L(H)}^2 \, dr \]

\[ \leq b_0^2 \int_0^s (s-r)^{-2\nu} \| S_\alpha(s-r) - E_n e^{-(s-r)A_{\nu} T} P_n \|_{H^2}^2 \, dr \]

\[ \leq b_0^2 K \left( \int_0^s (s-r)^{-2\nu} \, dr + \int_0^s (s-r)^{-2\nu} \, dr \right). \]

The last two integrals converge provided \( \nu < \min \{ \frac{\alpha}{2}, \frac{1}{2} - \frac{1}{2\alpha}, \frac{1}{2} - \frac{1}{2} \} \). Hence,

\[ \mathbb{E} \left( \int_0^s (s-r)^{-2\nu} \| (S_\alpha(s-r) - E_n e^{-(s-r)A_{\nu} T} P_n) g(u(r)) \|_{H^2}^2 \, dr \right)^{\frac{p}{2}} \]

\[ \leq b_0^p K \left( \int_0^s (s-r)^{-2\nu} \, dr + \int_0^s (s-r)^{-2\nu} \, dr \right). \]

\[ \leq b_0^p K T \left( \int_0^s (s-r)^{-2\nu} \, dr + \int_0^s (s-r)^{-2\nu} \, dr \right). \]
By accumulating the conditions, the parameter \( \nu \) should satisfy:

\[
\max\left\{ \frac{1 + \alpha - \alpha \beta}{2\alpha} + \frac{1}{p} \frac{\gamma}{2} + \frac{1}{p}\right\} < \nu < \min\left\{ \frac{\beta}{2} - \frac{1}{2\alpha}, \frac{1 - \gamma}{2} - \frac{1}{2}\right\}.
\]

(5.43)

Since \( \alpha > 2 \) and \( p > \frac{2\alpha}{\alpha - 2} \), it is possible to choose \( \gamma \in (\frac{1}{\alpha}, \frac{1}{2} - \frac{1}{p}) \) and \( \beta \in (\frac{1}{\alpha} + \frac{1}{p} + \frac{1}{2}, 1) \) (notice that \( p > \frac{2\alpha}{\alpha - 2} \iff \frac{1}{\alpha} < \frac{1}{2} - \frac{1}{p} \iff \frac{1}{\alpha} + \frac{1}{p} + \frac{1}{2} < 1 \)). With such choice of \( \gamma \) and \( \beta \), we can find \( \nu \) satisfying (5.43). In fact \( \frac{2}{\alpha} + \frac{1}{p} - 1 < \frac{2}{\alpha} + \frac{1}{p} < \frac{1}{\alpha} + \frac{1}{p} + \frac{1}{2} < \beta < 1 \) and \( \gamma + \frac{1}{\alpha} + \frac{1}{p} < \frac{1}{\alpha} + \frac{1}{p} + \frac{1}{2} < \beta \).

From (5.28) and (5.42), we have

\[
\mathbb{E}\left[ \sup_{[0,T]} B(t) \right] \leq K_{T,p,b_0} \left[ n^{-\alpha\beta_{\frac{p}{2}} + \alpha\beta_{\frac{p}{2}}} + n^{-\alpha\gamma_{\frac{p}{2}}} + \int_0^T \mathbb{E} \sup_{r \in [0,T]} |u(r) - u^n(r)|_{H^s}^p ds \right].
\]

(5.44)

Finally, we join the estimations in (5.44) and (5.14) and applying the Gronwall Lemma, we get

\[
||u - u^n||_{T,p} := \left( \mathbb{E}\left[ \sup_{[0,T]} |u(t) - u^n(t)|_{H^s}^p \right] \right)^{\frac{1}{p}} \leq K_{T,p,b_0} \left( n^{-2p\eta} + n^{-\alpha\beta_{\frac{p}{2}}} + n^{-\alpha\gamma_{\frac{p}{2}}} \right)^{\frac{1}{p}}. 
\]

(5.45)

To get a good precision of estimation, we push \( \beta \) to its lower bound and \( \gamma \) to its upper bound, i.e. we take \( \beta := \frac{1}{2} + \frac{1}{\alpha} + \frac{1}{p} \) and \( \gamma = \frac{1}{2} - \frac{1}{p} \), then we get:

\[
||u - u^n||_{T,p} \leq K_{T,p,b_0} \left( n^{-2p\eta} + n^{-\alpha\beta_{\frac{p}{2}}} + n^{-\alpha\gamma_{\frac{p}{2}}} \right)^{\frac{1}{p}} \leq K_{T,p,b_0} \left( n^{-2p\eta} + n^{-\frac{\alpha\beta_{\frac{p}{2}} + \alpha\beta_{\frac{p}{2}}}{2}} \right)^{\frac{1}{p}} 
\]

(5.46)

Thanks to the condition (5.1), we have \( \frac{\alpha\beta_{\frac{p}{2}}}{2} - \frac{\alpha p}{2} < \frac{p}{2} < \frac{\alpha\beta_{\frac{p}{2}}}{2} + \frac{\alpha\beta_{\frac{p}{2}}}{2} \). Now, it is easy to get the inequality (5.46).

**APPENDIX A. PROOF OF LEMMA 3.3**

We apply on the both sides of equation (A.8) the operator \( E_n \), we get

(A.1)

\[
E_n u_n(t) = E_n \left( e^{-t\mathcal{A}_{V_n-1}^\frac{2}{p}} u_n(0) \right) + E_n \left( \int_0^t e^{-(t-s)\mathcal{A}_{V_n-1}^\frac{2}{p}} g_n(u_n(s))dW_n(s) \right).
\]

Using the definitions of the two operators \( P_n \) and \( E_n \) and by the fact that \( P_n E_n = I \), the RHS in (A.1) is equal to \( u^n(t) \). Moreover

\[
E_n e^{-t\mathcal{A}_{V_n-1}^\frac{2}{p}} u_n(0) = E_n e^{-t\mathcal{A}_{V_n-1}^\frac{2}{p}} P_n u_0.
\]
Let us now explain how to get the stochastic term. We denote by \((\cdot)_k\) the component of a vector and by \((\cdot)_{kj}\) the component of a matrix. We have

\[
E_n \left( \int_0^t e^{-(t-s)\dot{A}_{V_{n-1}}^n} g_n(u_n(s)) \, dW_n(s) \right) = \sum_{k=1}^{n-1} \left( \int_0^t e^{-(t-s)\dot{A}_{V_{n-1}}^n} g_n(u_n(s)) \, dB_j(s) \right) \, e_k
\]

But

\[
\sum_{k=1}^{n-1} \left( e^{-(t-s)\dot{A}_{V_{n-1}}^n} g_n(u_n(s)) \right)_{kj} e_k = E_n \left( e^{-(t-s)\dot{A}_{V_{n-1}}^n} g_n(u_n(s)) \right)_j,
\]

where \(e^{-(t-s)\dot{A}_{V_{n-1}}^n} g_n(u_n(s))\)_j is the column "j" of the matrix \(e^{-(t-s)\dot{A}_{V_{n-1}}^n} g_n(u_n(s))\).

Hence we have the first result:

\[
E_n \left( \int_0^t e^{-(t-s)\dot{A}_{V_{n-1}}^n} g_n(u_n(s)) \, dW_n(s) \right) = \sum_{j=1}^{n-1} \int_0^t E_n \left( e^{-(t-s)\dot{A}_{V_{n-1}}^n} g_n(u_n(s)) \right)_j dB_j(s).
\]

We know from the basic calculus on matrices that:

\[
\left( e^{-(t-s)\dot{A}_{V_{n-1}}^n} g_n(u_n(s)) \right)_{kj} = e^{-(t-s)\dot{A}_{V_{n-1}}^n} \left( g_n(u_n(s)) \right)_{kj}.
\]

By the definition of the matrix \(g_n\), the column "j" of the matrix \(g_n(u_n(s))\):

\((g_n(u_n(s)))_j\) is equal to \(P_n((g \circ E_n)e_j)\), hence we have

\[
E_n \left( \int_0^t e^{-(t-s)\dot{A}_{V_{n-1}}^n} g_n(u_n(s)) \, dW_n(s) \right) = \sum_{j=1}^{n-1} \int_0^t E_n \left( e^{-(t-s)\dot{A}_{V_{n-1}}^n} g_n(u_n(s)) \right)_j dB_j(s)
\]

and we denote these integrals by

\[
\int_0^t E_n e^{-(t-s)\dot{A}_{V_{n-1}}^n} P_n g(u_n(s)) \, dW_n(s).
\]

To get the estimation of \(u^n(t)\), we use Lemma 3.1 and Theorem 3.3, then we have

\[
\mathbb{E} \sup_{t \in [0,T]} |u^n(t)|_{L^2}^p \leq \mathbb{E} \sup_{t \in [0,T]} ||E_n|| |u_n(t)|_{R_{n-1}}^p \leq C_{T,n,||g||}(1+\mathbb{E}|u_0|_{L^2}), \text{ for each } T > 0.
\]
APPENDIX B. PROOF OF Lemma 5.3

In fact, the operators $S_\alpha(t)$, $E_n e^{-\hat{\mathcal{A}}_{n-1} t} P_n$, and $A^{-\delta}$ are bounded and we have $A^{-\delta} S_\alpha(t) e_j = S_\alpha(t) A^{-\delta} e_j$ and $E_n e^{-\hat{\mathcal{A}}_{n-1} t} P_n A^{-\delta} e_j = A^{-\delta} E_n e^{-\hat{\mathcal{A}}_{n-1} t} P_n e_j$, for all $e_j, j \in \mathbb{N}$.

APPENDIX C. Lemma C.1

Lemma C.1. Let $(\lambda_k)_{k \geq 0}$ be the sequence of the eigenvalues corresponding to the eigenfunctions $(e_k)_{k \geq 0}$ of the positive operator $A$. Then $(e_k)_{k \geq 0}$ are also eigenfunctions of $A^\alpha$ corresponding to the eigenvalues $(\lambda_k^\alpha)_{k \geq 0}$.

Proof. Using the definition of the fractional operator (1.3):
\[
A^\alpha e_k = \sin \frac{\alpha \pi}{2} \int_0^\infty t^{\frac{\alpha}{2} - 1} A(tI + A)^{-1} e_k dt \\
= \sin \frac{\alpha \pi}{2} \int_0^\infty t^{\frac{\alpha}{2} - 1} \lambda_k (t + \lambda_k)^{-1} dte_k \\
= \lambda_k^\alpha \left( \frac{\sin \frac{\alpha \pi}{2}}{\pi} \int_0^\infty \xi^{\frac{\alpha}{2} - 1} (t + \xi)^{-1} d\xi \right) e_k.
\]

(C.1)

By the residues theory, we have
\[
(1 - e^{i\pi \alpha}) \int_0^\infty \xi^{\frac{\alpha}{2} - 1} (t + \xi)^{-1} d\xi = 2\pi i \text{Res}(-1, \xi^{\frac{\alpha}{2} - 1} (t + \xi)^{-1}) = -2\pi i e^{i\pi \alpha}
\]
so,
\[
\int_0^\infty \xi^{\frac{\alpha}{2} - 1} (t + \xi)^{-1} d\xi = \frac{\pi}{\sin \frac{\alpha \pi}{2}}.
\]

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