GLOBAL REGULARITY CRITERION FOR THE 3D NAVIER-STOKES EQUATIONS WITH LARGE DATA

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ABSTRACT. In this paper, we study the global regularity of strong solution to the Cauchy problem of 3D incompressible Navier-Stokes equations with large data and non-zero force. We prove that the strong solution exists globally for \( \nabla u \in L^\gamma (0, T; L^2 (\Omega)) \) with \( \gamma \geq \frac{16}{7} \).

1. Introduction

Leray (1934) [9] and Hopf (1951) [6] showed the existence of global weak solutions to the three-dimensional Navier-Stokes system, but their uniqueness is still a fundamental open problem [4], [5], [7] and [10]. Furthermore, the strong solutions for the 3D Navier-Stokes equations are unique and can be shown to exist on a certain finite time interval for small initial data and small forcing term (see [2], [4], [5], [7] and references therein). The global existence of small strong solutions has been proved, see Constantin [4] and Younsi [11]. For the 3D Navier-Stokes equations with large data, the question of asymptotic behavior of strong solution, as time variable \( t \) goes to \(+\infty\) is a major problem. Most recently, there has been some progress along this line (see, for example, [2], [3] and [12]).

In this paper, we study the global regularity of strong solutions to the 3D Navier-Stokes problem. We give a sufficient criterion for global existence in time for strong solution to the 3D Navier-Stokes equations with external force and large data.

2. The regularity criterion

In this paper, we consider the three-dimensional Navier-Stokes system

\[
\frac{\partial u}{\partial t} + u \cdot \nabla u = \nu \Delta u + \nabla p + f, \quad t > 0,
\]

\[\text{div} \ u = 0, \ \text{in} \ \Omega \times (0, \infty), \ u = 0 \ \text{on} \ \partial \Omega \times (0, \infty) \ \text{and} \ u (x, 0) = u_0, \ \text{in} \ \Omega,
\]

where \( u = u (x, t) \) is the velocity vector field, \( p (x, t) \) is a scalar pressure, \( f \) is a given force field and \( \nu > 0 \) is the viscosity of the fluid. \( u (x, 0) \) with \( \text{div} u_0 = 0 \) in the sense of distribution is the initial velocity field, and \( \Omega \) is a regular, open, bounded subset of \( \mathbb{R}^3 \) with smooth boundary \( \partial \Omega \). We denote by \( H^m (\Omega) \), the Sobolev space. We define the usual function spaces \( \mathcal{V} = \{ u \in C_0^\infty (\Omega) : \text{div} u = 0 \} \), \( \mathcal{V} = \text{closure of} \ \mathcal{V} \ \text{in} \ H_0^1 (\Omega), \ H = \text{closure of} \ \mathcal{V} \ \text{in} \ L^2 (\Omega) \). The space \( H \) is equipped with the scalar product \( (.,. \) induced by \( L^2 (\Omega) \) and the norm \( \| . \|_{L^2 (\Omega)} \). For the local existence of strong solutions, we have the following result in 3D [4].

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Theorem 2.1. [4, P. 82 Theorem 9.4] Assume that $u_0 \in V$ and $f \in L^2(0,T;H)$ are given. Then there exists a $T^* > 0$ depending on $\nu$, $f$, $u_0$ and $T$, such that there exists a unique solution of (2.1) satisfies $u \in L^\infty(0,T^*;V) \cap L^2(0,T^*;H^2(\Omega))$.

We will denote by $c_{i \in \mathbb{N}}$, positive constants. We begin by establishing the following

Lemma 2.2. Assume that $u_0 \in V$ and $f \in L^2(0,T;L^2(\Omega))$ are given. Let be $u$ a solution of 3D Navier-Stokes equations (2.1), such that if

$$\nabla u \in L^\alpha(0,T;L^2(\Omega))$$

then $\int_0^T \frac{||\nabla u||^2_{L^2}}{||\nabla u||^2_{L^2}} ds$ is finite for $||u(.,t)||_{L^2} \geq c_1$ with $c_1 > 0$.

Proof. Multiplying (2.1) by $\Delta u$ and integrate over $\Omega$, we obtain

$$\frac{1}{2} \frac{d}{dt} ||\nabla u(.,t)||_{L^2}^2 + \nu ||\nabla u||_{L^2}^2 = ((u,\nabla u) \cdot \Delta u) + (f,\Delta u).$$

Using the Holder inequality and the Sobolev theorem, we get

$$||((u,\nabla u) \cdot \Delta u)|| \leq c_2 ||\nabla u||_{L^2}^2 ||\Delta u||_{L^2}^\alpha,$$

see [4, P. 79 (2.22)]. Applying Young’s inequality to (2.4) yields

$$||((u,\nabla u) \cdot \Delta u)|| \leq c_3 ||\nabla u||_{L^2}^6 + \frac{\nu}{4} ||\Delta u||_{L^2}^2.$$  

Using the Schwartz and Young inequalities, we obtain that

$$||f, \Delta u)\| \leq ||f||_{L^2} ||\Delta u||_{L^2} \leq c_4 ||f||_{L^2}^2 + \frac{\nu}{4} ||\Delta u||_{L^2}^2.$$ 

Combining (2.5), (2.6) and (2.7), we have

$$\frac{d}{dt} ||\nabla u(.,t)||_{L^2}^2 + \nu ||\nabla u||_{L^2}^2 \leq c_4 ||f||_{L^2}^2 + c_3 ||\nabla u||_{L^2}^6.$$ 

By Poincare inequality we have $||\nabla u||_{L^2} \geq c_5 ||u||_{L^2} > c_1 c_5 = c_6 > 0$.

Dividing (2.8) by $||\nabla u||_{L^2}^2$, we obtain

$$\frac{d}{dt} \frac{||\nabla u(.,t)||_{L^2}^2}{||\nabla u||_{L^2}^2} + \nu \frac{||\Delta u||_{L^2}^2}{||\nabla u||_{L^2}^2} \leq c_4 \frac{||f||_{L^2}^2}{||\nabla u||_{L^2}^2} + c_3 ||\nabla u||_{L^2}^{6-\beta}.$$ 

For $f \in L^2(0,T, L^2(\Omega))$ and for each $T > 0$, integrate (2.8) to get

$$\left(\frac{1}{1 - \frac{\nu}{2}}\right) \left(\frac{1}{||\nabla u(T)||_{L^2}^2} - \frac{1}{||\nabla u(0)||_{L^2}^2}\right) + \nu \int_0^T \frac{||\Delta u||_{L^2}^2}{||\nabla u||_{L^2}^2} ds \leq c_4 \int_0^T \frac{||f||_{L^2}^2}{||\nabla u||_{L^2}^2} ds + c_3 \int_0^T ||\nabla u||_{L^2}^{6-\beta} ds.$$ 

Since $||\nabla u||_{L^2} \geq c_6$, this gives

$$\nu \int_0^T \frac{||\Delta u(s)||_{L^2}^2}{||\nabla u(s)||_{L^2}^2} ds \leq \left(\frac{1}{2} - \frac{\nu}{2}\right) \frac{1}{c_6^{\beta-2}} + \frac{c_4}{c_6^\beta} \int_0^T ||f(s)||_{L^2}^2 ds + \frac{c_3}{c_6} \int_0^T ||\nabla u(s)||_{L^2}^{6-\beta} ds,$$ 

the right-hand side is finite thanks to the assumption (2.2) which finish the proof. □

A particular case of the lemma above can found in Foias et al. [5, P. 105 (B.11)].
Lemma 2.3. Let \( u_0 \in V \) and \( f \in L^2(0,T;L^2(\Omega)) \) be given. Let \( u \) be a solution of (2.11), such that \( \|u(.,t)\|_{L^2} \geq c_1 \). Then if
\[
\|\nabla u\|_{L^2} \in L^m(0,T) \text{ with } 6 - \left( \frac{1}{2} + m \right) \frac{4}{3} \leq m q, \ 1 < m < 4
\]
and \( m, q, p \) are positive real numbers, we have
\[
\frac{\|\Delta u\|_{L^2}^{\frac{3}{2}}}{\|\nabla u\|_{L^2}^{\frac{3}{2}}} \in L^r(0,T) \text{ for } \frac{3(1 - \theta)}{4} + \frac{\theta}{q} = 1, \ 1 \leq q < r < \frac{4}{3} \text{ and } 0 < \theta < 1. \quad (2.12)
\]

Proof. To estimate the non linear term in the right-hand side of (2.4), we put
\[
\frac{\|\Delta u\|_{L^2}^{\frac{3}{2}}}{\|\nabla u\|_{L^2}^{\frac{3}{2}}} = \frac{\|\Delta u\|_{L^2}^{\frac{3}{2} p}}{\|\nabla u\|_{L^2}^{\frac{3}{2} q}} \|\nabla u\|_{L^2}^{m}. \quad (2.13)
\]
Applying the generalized Holder inequality, we find that
\[
\left\| \frac{\|\Delta u(., \tau)\|_{L^2}^{\frac{3}{2}}}{\|\nabla u(., \tau)\|_{L^2}^{\frac{3}{2} + m}} \|\nabla u(., \tau)\|_{L^2}^{m} \right\|_{L^r(0,T)} \leq \left( \int_0^T \frac{\|\Delta u\|_{L^2}^{\frac{3}{2} p}}{\|\nabla u\|_{L^2}^{\frac{3}{2} (\frac{3}{2} + m) p}} d\tau \right)^{\frac{1 - \theta}{p}} \left( \int_0^T \|\nabla u\|_{L^2}^{m q} d\tau \right)^{\frac{\theta}{p}}. \quad (2.14)
\]
with
\[
p = 4 \frac{1}{3}, \ \frac{1}{r} = 1 - \frac{\theta}{q}, \ 1 \leq q < r < p \text{ and } 0 < \theta < 1. \quad (2.15)
\]
Since for \( \|\nabla u\|_{L^2} \) belongs to the space \( L^m(0,T) \) with \( 6 - \left( \frac{1}{2} + m \right) \frac{4}{3} \leq m q, \) we have that \( \frac{\|\Delta u(., \tau)\|_{L^2}^{\frac{3}{2}}}{\|\nabla u(., \tau)\|_{L^2}^{\frac{3}{2} + m}} \) belongs to the space \( L^\frac{4}{3}(0,T) \) (see Lemma 2.2. ). Therefore,
\[
\frac{\|\Delta u(., \tau)\|_{L^2}^{\frac{3}{2}}}{\|\nabla u(., \tau)\|_{L^2}^{\frac{3}{2} + m}} \|\nabla u(., \tau)\|_{L^2}^{m} \in L^r(0,T) \text{ with } r > 1. \quad (2.16)
\]
Finally, from equality (2.13), we deduce that \( \frac{\|\Delta u(., \tau)\|_{L^2}^{\frac{3}{2}}}{\|\nabla u(., \tau)\|_{L^2}^{\frac{3}{2} + m}} \in L^r(0,T). \)

The following Lemmas will be useful

Lemma 2.4. Let \( u \) be a solution of (2.11) with \( \|u\|_{L^2} \geq c_1 \), then we have
\[
\frac{\|\Delta u\|_{L^2}^{\frac{3}{2}}}{\|\nabla u\|_{L^2}^{\frac{3}{2}}} \leq c_7 \frac{\|\Delta u\|_{L^2}^{\frac{3}{2} r}}{\|\nabla u\|_{L^2}^{\frac{3}{2}}} \text{ for } r > 1. \quad (2.17)
\]

Proof. Using a Poincare inequality see [10] II. (1.39)
\[
c_8 \|\nabla u\|_{L^2} \leq \|\Delta u\|_{L^2} \quad (2.18)
\]
and the fact that \( \|u\|_{L^2} \geq c_1 \), we get
\[
c_6 c_8 \|\nabla u\|_{L^2}^{\frac{3}{2}} \leq c_8 \|\nabla u\|_{L^2}^{\frac{3}{2}} \leq \|\Delta u\|_{L^2}^{\frac{3}{2}}. \quad (2.19)
\]
We rewrite (2.18) as follows
\[
c_6 c_8 \leq \frac{\|\Delta u\|_{L^2}^{\frac{3}{2}}}{\|\nabla u\|_{L^2}^{\frac{3}{2}}} \quad (2.19)
\]
Then, it is easy to show that

\[ 1 \leq \left( c_6 c_8^{\frac{3}{2}} \right)^{1-r} \left( \frac{\| \Delta u \|_{L^2}^{\frac{3}{2}}}{\| \nabla u \|_{L^2}^{\frac{3}{2}}} \right)^{r-1} \quad \text{with } r > 1. \tag{2.20} \]

We obtain the desired result by multiplying \(2.20\) by \( \frac{\| \Delta u \|_{L^2}^{\frac{3}{2}}}{\| \nabla u \|_{L^2}^{\frac{3}{2}}} \),

\[ \left( \frac{\| \Delta u \|_{L^2}^{\frac{3}{2}}}{\| \nabla u \|_{L^2}^{\frac{3}{2}}} \right) \leq \left( c_6 c_8^{\frac{3}{2}} \right)^{1-r} \left( \frac{\| \Delta u \|_{L^2}^{\frac{3}{2}}}{\| \nabla u \|_{L^2}^{\frac{3}{2}}} \right)^r, \tag{2.21} \]

we put \( c_7 = \left( c_6 c_8^{\frac{3}{2}} \right)^{1-r} \), wich finish the proof.

\[ \square \]

For small data, it well known that for \( \| \nabla u_0 \|_{L^2}^2 + \frac{2}{\nu} \int_0^T \| f \|_{L^2}^2 \, ds \leq c_8 \lambda_{1/2}^2 / \nu^2 \) we have global regularity Constantin \cite{4} P. 80 Theorem 9.3 and Younsi \cite{11}. In the work of Beirao 1995 \cite{1}, it has been proved that the regularity is ensured if \( \nabla u \in L^4(0, T; L^2(\Omega)) \). Our main result on the global regularity for strong solutions with large data to the 3D Navier-Stokes equations \(2.1\) reads as follows

**Theorem 2.5.** Let \( u_0 \in V, f \in L^2(0, T; L^2(\Omega)) \) and \( u(x, t) \) is the corresponding strong solution to the system \(2.1\) on \([0, T]\), such that \( \| u \|_{L^2}^2 \geq c_8 \nu^2 / \lambda_{1/2}^2 \). Then if \( \nabla u \) satisfies

\[ \nabla u \in L^\gamma(0, T; L^2(\Omega)) \quad \text{with } \gamma \geq \frac{16}{7}, \tag{2.22} \]

the solution \( u(x, t) \) exists globally in time.

**Proof.** We estimate the nonlinear term in \(2.4\), we obtain

\[ \left| \int_\Omega (u, \nabla u) \cdot \nabla w \, dx \right| \leq c_2 \| \nabla u \|_{L^2}^2 \frac{\| \Delta u \|_{L^2}^{\frac{3}{2}}}{\| \nabla u \|_{L^2}^{\frac{3}{2}}}. \tag{2.23} \]

Thanks to \(2.7\), \(2.23\) and \(2.10\) in Lemma 2.4, we estimate the right-hand side of \(2.3\) to get for \( r > 1 \)

\[ \frac{d}{dt} \| \nabla u (\cdot, t) \|_{L^2}^2 \leq c_9 \| \nabla u \|_{L^2}^2 \| \frac{\| \Delta u \|_{L^2}^{\frac{3}{2}}}{\| \nabla u \|_{L^2}^{\frac{3}{2}}} + c_4 \| f (\cdot, t) \|_{L^2}^2 \tag{2.24} \]

with \( c_9 = c_7 c_2 \) and \( c_1 = c_8 \nu^2 / \lambda_{1/2}^2 \). We assume that \( u \) satisfies the Lemma 2.3., then we prove that the condition \(2.22\) is a sufficient condition for global regularity. Since \( u \) satisfies the condition \(2.14\) and the condition \(2.12\) in Lemma 2.3., then \(2.12\) implies that \( \| \frac{\| \Delta u \|_{L^2}^{\frac{3}{2}}}{\| \nabla u \|_{L^2}^{\frac{3}{2}}} \| \leq 1 \) \( L^1(0, T) \) and as a consequence,

\[ \int_0^T \frac{\| \Delta u (\cdot, t) \|_{L^2}^{\frac{3}{2}}}{\| \nabla u (\cdot, t) \|_{L^2}^{\frac{3}{2}}} \, dt \leq a_1. \tag{2.25} \]

The assumption on \( f \) gives

\[ \int_0^T \| f (\cdot, t) \|_{L^2}^2 \, dt \leq a_2. \tag{2.26} \]
Multiplying (2.1) by $u$ and integrate over $\Omega$, we get
\[
\frac{1}{2} \frac{d}{dt} \| u(\cdot, t) \|_{L^2}^2 + \nu \| \nabla u \|_{L^2}^2 = (\nu, u).
\] (2.27)

Using Schwartz and Poincaré inequality, we obtain
\[
\frac{1}{2} \frac{d}{dt} \| u(\cdot, t) \|_{L^2}^2 + \nu \| \nabla u \|_{L^2}^2 \leq \| u \|_{L^2} \| f \|_{L^2}
\] (2.28)

\[
\leq \frac{\nu}{2} \| \nabla u \|_{L^2}^2 + c_{10} \| f \|_{L^2}^2.
\]

Integrate (2.28) over $[0, T]$, we get
\[
\| u(\cdot, T) \|_{L^2}^2 + \nu \int_0^T \| \nabla u (s) \|_{L^2}^2 ds \leq c_{10} \int_0^T \| f (s) \|_{L^2}^2 ds + \| u (0) \|_{L^2}^2.
\] (2.29)

If we drop the positive term $\| u (\cdot, t) \|_{L^2}^2$ in (2.29) then we have
\[
\int_0^T \| \nabla u (s) \|_{L^2}^2 ds \leq c_{11} \int_0^T \| f \|_{L^2}^2 ds + c_{12} \| u (0) \|_{L^2}^2 = a_3
\] (2.30)

where $a_{i=1,2,3}$ are constants independent of times. We apply the uniform Gronwall lemma [10, P.91 Lemma 1.1.] to (2.24) we obtain for $t \geq 0$
\[
\| \nabla u (\cdot, t) \|_{L^2}^2 \leq c_0 \| \nabla u_0 \|_{L^2}^2 \exp \int_0^t \| \nabla u (\tau) \|_{L^2}^2 d\tau + c_4 \int_0^t \| f (\tau) \|_{L^2}^2 \exp \left( -\int_\tau^t \| \nabla u (\sigma) \|_{L^2}^2 d\sigma \right) d\tau.
\] (2.31)

The estimates (2.25), (2.24) and (2.30) imply that the second member in (2.31) is finite and $\nabla u (\cdot, t) \in L^\infty (0, T; L^2 (\Omega))$. To improve the result in [11] we add the conditions $mq < 4$ and $(\frac{1}{2} + m) p > 2$. For using Lemma 2.2. we put $\frac{2}{7} p = 2$. This gives the system
\[
\begin{cases}
\frac{2}{7} p = 2, \quad mq < 4, \quad (\frac{1}{2} + m) p > 2 \quad \text{and} \quad 6 - (\frac{1}{2} + m) p \leq mq \\
\frac{\theta}{p} + \frac{\theta}{q} = \frac{1}{r}, \quad 1 \leq q < r < p \quad \text{and} \quad 0 < \theta < 1,
\end{cases}
\] (2.32)

which is equivalent to
\[
\begin{cases}
4 > qm, \quad m \left( \frac{1}{2} + \frac{4}{\theta} \right) \geq \frac{4p}{4} \quad \text{and} \quad 4 > m > 1 \\
\frac{3 (1 - \theta)}{4} + \frac{\theta}{q} = \frac{1}{r}, \quad 1 \leq q < r < \frac{4}{3} \quad \text{and} \quad 0 < \theta < 1.
\end{cases}
\] (2.33)

The crucial part, then, is to obtain a suitable estimate for the smallest value of $\gamma = \min (qm)$ such that $q$ and $m$ satisfy
\[
\frac{4}{q} > m \geq \frac{16}{3 (q + \frac{4}{3})} \quad \text{with} \quad 1 \leq q < \frac{4}{3}
\] (2.34)

and the following conditions
\[
\frac{3 (1 - \theta)}{4} + \frac{\theta}{q} = \frac{1}{r}, \quad 1 \leq q < r < \frac{4}{3} \quad \text{and} \quad 0 < \theta < 1.
\] (2.35)

Moreover, the minimum value of $\gamma$, occurs at $q = 1$, then we get from (2.34) that $m = \frac{16}{7}$. Therefore, we find that $\gamma = \frac{16}{7}$. We can chose $\theta = \frac{1}{2}$ and we obtain from
that \( r = \frac{8}{7} \). We may summarize the preceding results as follows

\[
q = 1, \quad m = \frac{16}{7}, \quad \gamma = \frac{16}{7}, \quad r = \frac{8}{7}, \quad p = \frac{4}{3} \quad \text{and} \quad \theta = \frac{1}{2}.
\] (2.36)

This proves Theorem 2.5.

\[\square\]

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