ON THE FINITENESS OF MINIMAL AND MAXIMAL SPECTRA

ABOLFAZL TARIZADEH

Abstract. In this article, it is proved that the minimal spectrum of a commutative ring is quasi-compact with respect to the flat topology. Consequently, all of the related results of Kaplansky, Anderson, Gilmer-Heinzer and Bahmanpour et al on minimal primes are deduced as special cases of this result. Dually, some similar results are also obtained for maximal ideals.

1. Introduction

Recently in [9] we have rediscovered the Hochster’s inverse topology (see [3, Prop. 8]) on the prime spectrum by a new and purely algebraic method. We call it the flat topology (it is worthy to mention that during the writing [9] we were not aware of Hochster’s work). Hence the flat topology and Hochster’s inverse topology are exactly the same things. Roughly speaking, for a given ring $R$, then the collection of subsets $V(I) = \{ p \in \text{Spec}(R) : I \subseteq p \}$ where $I$ runs through the set of f.g. ideals of $R$ forms a basis for the opens of the flat topology, see [9, Theorem 3.2]. We use f.g. in place of “finitely generated”.

Certainly Hochster’s work is seminal. But some aspects of the inverse topology are hidden in the voluminous of Hochster’s work. For instance, there are some major results in the literature on the finiteness of minimal primes which have been proved independently while many of these results are easily implied from the basic properties of the inverse topology. One of the main aims of this article is to show the simplicity of some of these results. Indeed, we obtain Theorem 2.1 which is the first main result of this note. All of the results [7, Theorem 88], [5, Theorem 1.6] and [4, Theorem 2.1] are special cases and can be easily deduced from this Theorem. The machinery of deep results of commutative algebra are applied to prove the final main result of this note, Theorem 2.7. All of the rings which are considered in this...
note are commutative.

2. Main results

**Theorem 2.1.** Let $R$ be a ring and consider the flat topology over $\text{Spec}(R)$. Then $\text{Min}(R)$ the minimal spectrum of $R$ is quasi-compact.

**Proof.** Suppose $\text{Min}(R) \subseteq \bigcup U_\alpha$ where for each $\alpha$, $U_\alpha$ is a flat open subset of $\text{Spec}(R)$. We claim that $\text{Spec}(R) = \bigcup U_\alpha$. Let $p$ be a prime ideal of $R$. There is a minimal prime $p'$ of $R$ such that $p' \subseteq p$. There exists some $\alpha$ such that $p' \in U_\alpha$. We have $p \in U_\alpha$. If not, then $\overline{\{p\}} \subseteq U_\alpha^c = \text{Spec}(R) \setminus U_\alpha$. By [9, Corollary 3.6], $\overline{\{p\}} = \{q \in \text{Spec}(R) : q \subseteq p\}$. This is a contradiction. This establishes the claim. By [9, Remark 3.5], the flat topology is quasi-compact. □

The main result of [4, Theorem 2.1] is a special case of Theorem 2.1.

**Corollary 2.2.** Let $R$ be a ring and consider the flat topology over $\text{Spec}(R)$. Then the following conditions are equivalent.

(i) The set $\text{Min}(R)$ is finite.

(ii) For every minimal prime $p$ there is an element $f \in p$ such that $\overline{\{p\}} = \text{Min}(R) \cap V(f)$.

(iii) Every minimal prime is an open point of the subspace $\text{Min}(R)$.

**Proof.** (i) $\Rightarrow$ (ii) : By the Prime avoidance theorem [1, Tag 00DS], there is an element $f \in p$ such that $f \notin \bigcup_{p' \in \text{Min}(R), p' \neq p} p'$.

(ii) $\Rightarrow$ (iii) : There is nothing to prove.

(iii) $\Rightarrow$ (i) : It is an immediate consequence of Theorem 2.1. □

Theorem 2.1 vastly generalizes [7, Theorem 88], [5] and [6, Theorem 1.6]:

**Corollary 2.3.** Let $R$ be a ring such that every minimal prime is the radical of a f.g. ideal. Then $R$ has a finitely many minimal primes.

**First proof.** It implies from Theorem 2.1 or Corollary 2.2.
Second proof. Here, we present another proof for it without using the previous results. Let \( p \) be a minimal prime of \( R \). By the hypothesis, there is a f.g. ideal \( I \) of \( R \) such that \( V(p) = V(I) \). Therefore \( V(p) \) is an open subset of \( \text{Spec}(R) \) w.r.t. the patch (constructible) topology. We have \( \text{Spec}(R) = \bigcup_{p \in \text{Min}(R)} V(p) \). The patch topology is compact, see [9, Proposition 2.4]. Therefore \( \text{Min}(R) \) is a finite set. \( \square \)

The converse of Corollary 2.3 does not necessarily hold, see [4, Example 2.14].

Remark 2.4. Let \( \{R_i : i \in I\} \) be a family of domains. For each \( i \), \( \pi_i^{-1}(0) \) is a minimal prime of \( R = \prod_{i \in I} R_i \) where \( \pi_i : R \to R_i \) is the canonical projection. Because, suppose there exists a prime ideal \( p \) of \( R \) such that \( p \subset \pi_i^{-1}(0) \). Pick \( a \in \pi_i^{-1}(0) \setminus p \). Then \( ab = 0 \) where \( b = (\delta_{i,j})_{j \in I} \). It follows that \( b \in p \), a contradiction. The minimal prime \( \pi_i^{-1}(0) \) is principal since it is generated by the sequence \( (1 - \delta_{i,j})_{j \in I} \). Using this and Corollary 2.3 then we find a minimal prime \( p \) in every infinite direct product ring \( \prod \mathbb{Z} \) such that \( p \neq \pi_i^{-1}(0) \) for all \( i \). Moreover, \( p \) is not the radical of a f.g. ideal. It is worthy to mention that the set of minimal primes of \( \prod \mathbb{Z} \) is in bijective correspondence to the set of all ultrafilters on the index set, see [8, Proposition 1].

As a dual of Theorem 2.1 we have:

**Proposition 2.5.** Let \( R \) be a ring and consider the Zariski topology over \( \text{Spec}(R) \). Then \( \text{Max}(R) \) the maximal spectrum of \( R \) is quasi-compact.

**Proof.** It is well-known. Indeed, it is proved exactly like Theorem 2.1 \( \square \)

**Corollary 2.6.** Let \( R \) be a ring and consider the Zariski topology over \( \text{Spec}(R) \). Then the following conditions are equivalent.

(i) The set \( \text{Max}(R) \) is finite.

(ii) For each maximal ideal \( m \), \( m + \bigcap_{m' \in \text{Max}(R), m' \neq m} m' = R \).

(iii) For each maximal ideal \( m \) there is an element \( f \in R \setminus m \) such that \( \{m\} = \text{Max}(R) \cap D(f) \).
(iv) Each maximal ideal is an open point of the subspace $\text{Max}(R)$.

Proof. All of the implications (i) $\Rightarrow$ (ii), (ii) $\Rightarrow$ (iii) and (iii) $\Rightarrow$ (iv) are obvious. The implication (iv) $\Rightarrow$ (i) implies from Proposition 2.5. □

As a dual of Corollary 2.3, we have the following non-trivial result.

Theorem 2.7. Let $R$ be a ring with the property that for each maximal ideal $m$ the canonical map $\pi : R \to R_m$ is injective and of finite type. Then $R$ has a finitely many maximal ideals.

Proof. By [2, Theorem 1.1], $\pi$ is of finite presentation. Every flat ring homomorphism which is also of finite presentation then it induces a Zariski open map between the corresponding prime spectra, see [1, Tag 0011]. Therefore there are finite subsets $\{f_1, ..., f_n\} \subseteq R$ and $\{m_1, ..., m_n\} \subseteq \text{Max}(R)$ such that $R = (f_1, ..., f_n)$ and $D(f_i) = \text{Im} \pi_i^*$ for all $i$ where $\pi_i : R \to R_{m_i}$ is the canonical map. It follows that $\text{Max}(R) = \{m_1, ..., m_n\}$. □

References

1. Aise Johan de Jong et al, Stacks Project, see http://stacks.math.columbia.edu
2. Cox. Jr. S, Rush. D, Finiteness in flat modules and algebras, Journal of Algebra, Volume 32, Issue 1, 1974, p. 44-50.
3. Melvin. Hochster, Prime ideal structure in commutative rings, Trans. Amer. Math. Soc. 142 (1969), 43-60.
4. Kamal. Bahmanpour et al, A note on minimal prime divisors of an ideal, Algebra Colloquium. 18 (2011) 727-732.
5. D. D. Anderson, A note on minimal prime ideals, Proc. Amer. Math. Soc., 122, No. 1. (1994), 13-14.
6. R. Gilmer, W. Heinzer, Primary ideals with finitely generated radical in a commutative ring, Manuscripta Math. 78 (1993) 201-221.
7. Irving. Kaplansky, Commutative rings, revised edition, Univ. of Chicago Press, Chicago, 1974.
8. Ronnie. Levy et al, The prime spectrum of an infinite product of copies of $\mathbb{Z}$, Fundamenta Mathematicae. 138 (1991) 155-164.
9. Abolfazl. Tarizadeh, Flat topology and its dual aspects, submitted, arXiv:1503.04299v9 [math.AC].

Department of Mathematics, Faculty of Basic Sciences, University of Maragheh, P. O. Box 55136-553, Maragheh, Iran.

E-mail address: ebulfes1978@gmail.com