Elliptic soliton solutions of the spin non-chiral intermediate long-wave equation

Bjorn K. Berntson · Edwin Langmann · Jonatan Lenells

Received: 5 December 2022 / Revised: 5 December 2022 / Accepted: 8 March 2023 / Published online: 31 May 2023
© The Author(s) 2023

Abstract
We construct elliptic multi-soliton solutions of the spin non-chiral intermediate long-wave (sncILW) equation with periodic boundary conditions. These solutions are obtained by a spin-pole ansatz including a dynamical background term; we show that this ansatz solves the periodic sncILW equation provided the spins and poles satisfy the elliptic A-type spin Calogero-Moser (sCM) system with certain constraints on the initial conditions. The key to this result is a Bäcklund transformation for the elliptic sCM system which includes a non-trivial dynamical background term. We also present solutions of the sncILW equation on the real line and of the spin Benjamin–Ono equation which generalize previously obtained solutions by allowing for a non-trivial background term.

Keywords Soliton equations · spin Calogero-Moser systems · Exact solutions · Benjamin–Ono-type equations

Mathematics Subject Classification 35Q51 · 37K10 · 35Q70 · 35Q35

1 Introduction
In a recent paper [1], we introduced and solved new soliton equations related to the A-type spin Calogero–Moser (sCM) systems of Gibbons and Hermsen [2] (see also [3]). One of these equations, the spin non-chiral intermediate long wave (sncILW) equation, was shown to have multi-soliton solutions with dynamics described by the hyperbolic

---

1 Department of Physics, KTH Royal Institute of Technology, SE-106 91 Stockholm, Sweden
2 Nordita, KTH Royal Institute of Technology and Stockholm University, SE-106 91 Stockholm, Sweden
3 Department of Mathematics, KTH Royal Institute of Technology, SE-100 44 Stockholm, Sweden
sCM system. In this paper, we generalize these solutions to the periodic case. More specifically, we construct periodic solutions of the sncILW equation with dynamics described by the elliptic sCM system [4]. This generalization is non-trivial in several regards; in particular, our solutions include a dynamical background term which, as we show, provides a non-trivial generalization even in the hyperbolic limit when the spatial period becomes infinite. We also present corresponding generalizations of known solutions to the spin Benjamin-Ono (sBO) equation introduced in Ref. [1] and, in this way, obtain the full correspondence between sCM models and soliton equations conjectured in Ref. [1].

A prominent feature of the sncILW equation is its nonlocality, which arises through integral operators (see (1.4)–(1.5) below). While soliton equations with this feature have been studied for a long time (the classical examples are the Benjamin–Ono [5, 6] and intermediate long wave [7, 8] equations), there has recently been considerable interest in constructing and analyzing novel nonlocal soliton equations; see, for instance, [9–16]. In this context, in addition to resolving a conjecture posed in Ref. [1], this paper serves to exhibit the sncILW equation with periodic boundary conditions as an interesting system worthy of further study.

Throughout this paper, we denote by \( \zeta(z) \) and \( \wp(z) \) the usual Weierstrass \( \zeta \)- and \( \wp \)-functions of a complex variable \( z \) with half-periods \( (\ell, i\delta) \), with \( \ell > 0 \) and \( \delta > 0 \) fixed parameters and \( i := \sqrt{-1} \) (for the convenience of the reader, we give the definitions of these functions in Appendix A.1). We find it convenient to use the following variants of \( \zeta(z) \),

\[
\begin{align*}
\zeta_1(z) &:= \zeta(z) - \frac{\eta_1}{\ell} z, \quad \eta_1 := \zeta(\ell), \\
\zeta_2(z) &:= \zeta(z) - \frac{\eta_2}{i\delta} z, \quad \eta_2 := \zeta(i\delta),
\end{align*}
\tag{1.1}
\]

and the functions

\[
\begin{align*}
\wp_2(z) &:= -\zeta_2'(z) = \wp(z) + \frac{\eta_2}{i\delta}, \\
\kappa(z) &:= \zeta_2(z)^2 - \wp_2(z);
\end{align*}
\tag{1.2}
\]

see Appendix A.1 for details. The function \( \zeta_1(z) \) is \( 2\ell \)-periodic (but not \( 2i\delta \)-periodic) and the function \( \zeta_2(z) \) is \( 2i\delta \)-periodic (but not \( 2\ell \)-periodic). Note that \( \kappa(z) \) reduces to a constant in the limits \( \ell \to \infty \) and/or \( \delta \to \infty \) (as can been seen by evaluating \( \kappa(z) \) in (1.2) using the corresponding degenerations of the functions \( \wp_2(z) \) and \( \zeta_2(z) \) presented below in (2.2) and (2.1), respectively). This is one reason why the cases treated in Ref. [1] are significantly easier than the elliptic case treated in the present paper. We also note that \( \zeta_2(z) = \zeta_1(z) + \gamma_0 z \) with the constant

\[
\gamma_0 := \frac{\pi}{2\ell\delta}; \tag{1.3}
\]

The constant \( \gamma_0 \) is non-zero only if both \( \ell \) and \( \delta \) are finite; this is another reason why the elliptic case is more complicated than the cases treated in Ref. [1].
1.1 Periodic sncILW equation

For $d$ a fixed positive integer, we denote by $\mathbb{C}^{d\times d}$ the algebra of complex $d \times d$ matrices. The periodic sncILW equation describes the time evolution of two $\mathbb{C}^{d\times d}$-valued functions $U = U(x, t)$ and $V = V(x, t)$ of $x \in \mathbb{R}$ and $t \in \mathbb{R}$ as follows,

\begin{align*}
U_t + [U, U_x] + T U_{xx} + i [U, T U_x] + i [U, \tilde{T} V_x] &= 0, \\
V_t - [V, V_x] - T V_{xx} - i [V, T V_x] + i [V, \tilde{T} U_x] &= 0,
\end{align*}

(1.4)

together with the requirement that both functions are $2\ell$-periodic, $U(x + 2\ell, t) = U(x, t)$ and $V(x + 2\ell, t) = V(x, t)$, where $[\cdot, \cdot]$ and $\{\cdot, \cdot\}$ denote the commutator and anti-commutator of square matrices, respectively, and $T$ and $\tilde{T}$ are integral operators acting on $2\ell$-periodic functions $f(x)$ of $x \in \mathbb{R}$ as

\begin{align*}
(Tf)(x) &= \frac{1}{\pi} \int_{-\ell}^{\ell} \zeta_1(x' - x) f(x') \, dx', \\
(\tilde{T}f)(x) &= \frac{1}{\pi} \int_{-\ell}^{\ell} \zeta_1(x' - x + i \delta) f(x') \, dx',
\end{align*}

(1.5)

where the dashed integral indicates a principal value prescription and $T$ and $\tilde{T}$ act component-wise on matrix-valued functions. Note that, for $d = 1$, the sncILW equation reduces to the periodic non-chiral ILW equation introduced and studied by us in Refs. [17, 18].

1.2 Main result

The solutions of the periodic sncILW equation (1.4) that we construct have the form

\begin{align*}
U(x, t) &= e^{i \gamma_0 (P + P^\dagger)} U_0(x, t) e^{-i \gamma_0 (P + P^\dagger)} t, \\
V(x, t) &= e^{i \gamma_0 (P + P^\dagger)} V_0(x, t) e^{-i \gamma_0 (P + P^\dagger)} t,
\end{align*}

(1.6)

where

\begin{align*}
U_0(x, t) &= M(t) + i \sum_{j=1}^{N} P_j(t) \xi_2(x - a_j(t) - i \delta/2) - i \sum_{j=1}^{N} P_j^\dagger(t) \xi_2(x - a_j^*(t) - i \delta/2), \\
V_0(x, t) &= -M(t) - i \sum_{j=1}^{N} P_j(t) \xi_2(x - a_j(t) + i \delta/2) + i \sum_{j=1}^{N} P_j^\dagger(t) \xi_2(x - a_j^*(t) - i \delta/2).
\end{align*}

(1.7)
Here, the time-dependent variables $M(t) \in \mathbb{C}^{d \times d}$, $a_j(t) \in \mathbb{C}$, and $P_j(t) \in \mathbb{C}^{d \times d}$ are such that

$$
P := \sum_{j=1}^{N} P_j(t)$$

is time-independent. We refer to $M(t)$ as the background and to $a_j(t)$ and $P_j(t)$ as pole and spin degrees of freedom, respectively. Our main result is that, by setting (our notation is explained in Sect. 1.4 below)

$$
P_j(t) = |e_j(t)\rangle\langle f_j(t)|$$

with $|e_j(t)\rangle$ and $\langle f_j(t)|$ vectors in a $d$-dimensional complex vector space $\mathcal{V}$ and its dual $\mathcal{V}^*$, respectively, (1.6)–(1.8) gives an exact solution of the periodic sncILW equation (1.4) provided the following conditions are fulfilled:

(i) The dynamical variables $\{a_j, |e_j\rangle, \langle f_j|\}_{j=1}^{N} = \{a_j(t), |e_j(t)\rangle, \langle f_j(t)|\}_{j=1}^{N}$ evolve in time according to the following equations,

$$
\ddot{a}_j = -4 \sum_{k \neq j}^{N} \langle f_j|e_k\rangle\langle f_k|e_j\rangle \varphi_2'(a_j - a_k),
$$

$$
|\dot{e}_j| = 2i \sum_{k \neq j}^{N} |e_k\rangle\langle f_k|e_j\rangle \varphi_2(a_j - a_k),
$$

$$
\langle \dot{f}_j| = -2i \sum_{k \neq j}^{N} \langle f_j|e_k\rangle\langle f_k|e_j\rangle \varphi_2(a_j - a_k)
$$

for $j = 1, \ldots, N$.

(ii) The dynamics of the background $M = M(t)$ is given by

$$
\dot{M} = -\frac{1}{2} \sum_{j=1}^{N} \sum_{k \neq j}^{N} [P_j, P_k] \varphi'(a_j - a_k) + \frac{1}{2} \sum_{j=1}^{N} \sum_{k \neq j}^{N} [P_j^\dagger, P_k^\dagger] \varphi'(a_j^* - a_k^*),
$$

where $P_j = P_j(t)$.

(iii) At time $t = 0$, the following conditions are fulfilled,

$$
\langle e_j|f_j \rangle = 1,
$$

$$
\dot{a}_j \langle f_j| = 2 \langle f_j|M + 2i \sum_{k \neq j}^{N} \langle f_j|P_k \xi_2(a_j - a_k) - 2i \sum_{k=1}^{N} \langle f_j|P_k^\dagger \xi_2(a_j - a_k^* + i\delta),
$$

$$
\langle 1\rangle
\quad\text{(1.14)}
$$
\[-\frac{3\delta}{2} < \text{Im}(a_j) < -\frac{\delta}{2}\]  \hspace{1cm} (1.15)

for \( j = 1, \ldots, N \), together with

\[M^\dagger = M\]  \hspace{1cm} (1.16)

and

\[p^\dagger = p.\]  \hspace{1cm} (1.17)

(iv) The time \( t \) is small enough that the poles neither leave the strip defined in (1.15) nor collide (see Theorem 2.1 for a more precise formulation; as will be discussed, this is a technical condition needed in our proof but which probably can be ignored).

Several remarks are in order.

1. One can check that (1.9) and (1.11) indeed imply that \( P \) in (1.8) is time-independent.
2. It is important to note that (1.10) and (1.11) are the time evolution equations of the elliptic sCM model [2]. (Note that the elliptic sCM model is usually defined with the standard Weierstrass \( \wp \)-function \( \wp(z) \) instead of \( \wp_2(z) \); however, this difference is irrelevant since the system of Eqs. (1.10)–(1.11) is invariant under the transformation

\[|e_j(t)\rangle \rightarrow e^{2i\pi P t}|e_j(t)\rangle, \quad \langle f_j(t)\rangle \rightarrow \langle f_j(t)\rangle e^{-2i\pi P t}, \quad \wp_2(z) \rightarrow \wp_2(z) + c\]  \hspace{1cm} (1.18)

with \( P \) in (1.8) Hermitian, for arbitrary \( c \in \mathbb{R} \).)
3. We emphasize that the conditions (1.13)–(1.17) are constraints on initial conditions. If (1.13)–(1.17) are fulfilled at time \( t = 0 \), then the time evolution Eqs. (1.10)–(1.12) guarantee that (1.13)–(1.17) hold true for all times (this is easy to check for (1.13), (1.16) and (1.17), guaranteed by our assumptions for (1.15), and proved in Proposition 4.1 in Sect. 4 for (1.14)).
4. While the solutions given above are Hermitian, \( U = U^\dagger \) and \( V = V^\dagger \), we will actually prove a more general result providing non-Hermitian solutions of the periodic sncILW equation and obtain the Hermitian solutions as a corollary; see Theorem 2.1 for the general result. We emphasize the Hermitian solutions here since they are easier to state and probably more interesting in physics applications.
5. It is not obvious but true that the functions \( U(x, t) \) and \( V(x, t) \) in (1.6)–(1.8) are \( 2\ell \)-periodic as functions of \( x \); to see this, insert \( \xi_2(z) = \xi_1(z) + \gamma_0 z \) where \( \xi_1(z) \) is \( 2\ell \)-periodic, and observe that the potentially dangerous term \( \propto \gamma_0 x \) in (1.7) arising from this insertion vanishes due to the constraint (1.17).
6. We will show in Sect. 6.2 that the constraints (1.13)–(1.16) can be solved by linear algebra.
7. For \( \ell \to \infty \) and in the special case \( M = 0 \), the solutions above reduce to the multi-soliton solutions of the sncILW equation on the real line obtained in Ref. [1]; it is important to note that one can allow for a non-zero \( M \) also in the limit \( \ell \to \infty \) and, in this way, our solutions here generalize the ones in Ref. [1] even in the case \( \ell \to \infty \).

8. The key to our result is a generalization of the Bäcklund transformations of the sCM model in Ref. [1] to the elliptic case which is non-trivial since it requires the presence of a non-trivial background \( M \); with this generalization, we obtain a complete correspondence between sCM models and soliton equations, as anticipated in Ref. [1] (we discuss this point in more detail in Sect. 2). We also mention closely related earlier work on Bäcklund transformations of (s)CM systems [19, 20].

9. The results in this paper add further support to the conjecture in Ref. [1] that the periodic sncILW equation is an integrable system. It is important to note that this would be an elliptic integrable system whose analysis indispensably involves elliptic functions, and this presents challenges not present in the limiting cases \( \ell \to \infty \) and/or \( \delta \to \infty \) treated in Ref. [1].

10. As discussed, the constraint (1.17) is required for the solutions (1.6)–(1.8) to be \( 2\ell \)-periodic and, in this sense, it can be regarded as a balancing condition. We observe that, in the limiting cases \( \ell \to \infty \) and/or \( \delta \to \infty \), the constraint (1.17) can be ignored; otherwise, the result (with non-zero \( M \)) remains true as it stands in these limits (this can be seen by going through our proof of Theorem 2.1 and replacing \( \varkappa(z) \) by a constant).

1.3 Plan of the paper

In Sect. 2, we formulate and discuss our main result, Theorem 2.1, which provides, in general, non-Hermitian elliptic soliton solutions to the periodic sncILW equation; corresponding results for the sBO equation are also given. In Sect. 3, we show that the functions \( U(x, t) \) and \( V(x, t) \) in (1.6) solve the periodic sncILW equation (1.4) provided that a certain first-order system is satisfied. In Sect. 4, we establish a new Bäcklund transformation for the elliptic sCM system. Using these results, Theorem 2.1 is then proved in Sect. 5. In Sect. 6, we show how to solve the constraints of Theorem 2.1 to generate initial data for our \( N \)-soliton solutions, paying particular attention to the one-soliton case. The definitions and functional identities for the Weierstrass elliptic functions that we use are collected in Appendix A. The proofs of two important lemmas stated in Sect. 4 are deferred to Appendix B.

1.4 Notation

We follow [2] and use the Dirac bra-ket notation [21] to write our solutions and relate them to the elliptic sCM system. In particular, we denote vectors in a \( d \)-dimensional complex vector space \( \mathcal{V} \) by \( |e\rangle \) and vectors in the dual space \( \mathcal{V}^* \) by \( \langle f| \). Readers not familiar with this notation can identify \( |e\rangle \) with \( (e_{\mu})_{\mu=1}^{d} \in \mathbb{C}^d \), \( \langle f| \) with \( (f_{\mu}^*)_{\mu=1}^{d} \in \mathbb{C}^d \) where \( * \) is complex conjugation, \( \langle f|e\rangle \) with the scalar product \( \sum_{\mu=1}^{d} f_{\mu}^* e_{\mu} \), and
with the matrix \((e_\mu f^*_\nu)_{\mu,\nu=1}^d\). We denote Hermitian conjugation by \(\dagger\); note that \(|e\rangle\langle f| = |f\rangle\langle e|\) for all \(|e\rangle \in \mathcal{V}\) and \(|f\rangle \in \mathcal{V}^*\).

We use the shorthand notation \(\sum_{k\neq j}^N\) for sums \(\sum_{k=1,k\neq j}^N\), etc. Dots indicate differentiation with respect to time \(t\) and primes indicate differentiation with respect to the argument of the function.

## 2 Results

As mentioned already in the introduction, the periodic sncILW equation is the elliptic case in a general correspondence between sCM models and soliton equations proposed in Ref. [1]. More specifically, there are four cases in this correspondence which, on the sCM side, can be distinguished by the following special functions

\[
V(z) := \begin{cases} 
1/z^2 & \text{(I: rational case)} \\
(\pi/2\ell)^2/\sin^2(\pi z/2\ell) & \text{(II: trigonometric case)} \\
(\pi/2\delta)^2/\sinh^2(\pi z/2\delta) & \text{(III: hyperbolic case)} \\
\wp_2(z) & \text{(IV: elliptic case)}
\end{cases}
\]

which are the well-known two-body interaction potentials in \(A\)-type Calogero-Moser systems (see [22] for review). On the soliton side, the functions

\[
\alpha(z) := \begin{cases} 
1/z & \text{(I: rational case)} \\
(\pi/2\ell) \cot(\pi z/2\ell) & \text{(II: trigonometric case)} \\
(\pi/2\delta) \coth(\pi z/2\delta) & \text{(III: hyperbolic case)} \\
\zeta_2(z) & \text{(IV: elliptic case)}
\end{cases}
\]

are the building blocks in the spin-pole ansatz we use to solve the corresponding soliton equations. Note that \(V(z) = -\alpha'(z)\) in all cases. Moreover, the elliptic case (IV) is most general, and it reduces to the cases I, II and III in the limits \((\ell, \delta) \to (\infty, \infty), \delta \to \infty\) (keeping \(\ell\) finite), and \(\ell \to \infty\) (keeping \(\delta\) finite), respectively. Nevertheless, Cases I–III are interesting in their own right since, first, they are often sufficient in applications, and second, they are significantly simpler and thus allow for more general results that are not directly obtainable as limits of results for Case IV.

We now give the soliton equations corresponding to Cases I–IV. Cases I and II correspond to the sBO equation given by

\[
U_t + \{U, U_x\} + HU_{xx} + i[U, HU_x] = 0
\]

with a \(\mathbb{C}^{d \times d}\)-valued function \(U = U(x, t)\) of \(x \in \mathbb{R}\) and \(t \in \mathbb{R}\), with \(H\) the Hilbert transform; Case I corresponds to the sBO equation on the real line where \(U(x, t)\) has
suitable decaying conditions at $x \to \pm \infty$ and

$$(Hf)(x) := \frac{1}{\pi} \int_{x'}^{x} \frac{1}{x' - x} f(x') \, dx', \quad (2.4)$$

and Case II corresponds to the periodic sBO equation, where $U(x + 2\ell, t) = U(x, t)$, and with the periodic Hilbert transform

$$(Hf)(x) := \frac{1}{2\ell} \int_{-\ell}^{\ell} \cot \left( \frac{\pi}{2\ell} (x' - x) \right) f(x') \, dx'. \quad (2.5)$$

Case III corresponds to sncILW equation (1.4) on the real line, with functions $U(x, t)$ and $V(x, t)$ of $x \in \mathbb{R}$ and $t \in \mathbb{R}$ satisfying suitable decaying conditions at $x \to \pm \infty$, and with the integral operators $T$ and $\tilde{T}$ defined as

$$(Tf)(x) = \frac{1}{2\delta} \int_{\mathbb{R}} \coth \left( \frac{\pi}{2\delta} (x' - x) \right) f(x') \, dx', \quad (2.6)$$

Finally, Case IV, which is the most general, corresponds to the periodic sncILW equation (1.4) with $T$ and $\tilde{T}$ given by (1.5). Note that, since $\tilde{T} \to 0$ and $T \to H$ in the limit $\delta \to \infty$ [23], the first equation in (1.4) reduces to the sBO equation (2.3) in this limit. Moreover, $T$ and $\tilde{T}$ in (1.5) reduce to $T$ and $\tilde{T}$ in (2.6) in the limit $\ell \to \infty$, and $H$ in (2.5) reduces to $H$ in (2.4) in this limit. For future reference, we summarize the discussion in the present paragraph as follows,

$$\begin{cases}
\text{sBO equation on the real line} & \text{(I: rational case)} \\
\text{periodic sBO equation} & \text{(II: trigonometric case)} \\
\text{sncILW equation on the real line} & \text{(III: hyperbolic case)} \\
\text{periodic sncILW equation} & \text{(IV: elliptic case)}.
\end{cases} \quad (2.7)$$

In this section, we present our solutions of the soliton equations in (2.7) in all Cases I–IV. The solutions for the Cases I–III that we present are generalizations of solutions obtained already in Ref. [1]; the simplest way to prove these generalizations is to adapt the proofs in Ref. [1], as recently done in a thesis by Anton Ottosson [24]. This is due to additional constraints appearing in Case IV which prevent a direct derivation of all solutions in Cases I–III as limits of the general solution in Case IV. For this reason, and for clarity, the detailed proofs we present in this paper are restricted to Case IV.

### 2.1 Solutions of the sncILW equation

Throughout this subsection, we consider Cases III and IV in (2.1)–(2.2) and (2.7); in particular, $V(z)$ and $\alpha(z)$ are as in (2.1) and (2.2) for the hyperbolic and elliptic
cases. Note that $\varphi(z) = (\pi/2\delta)^2$ (a constant) in Case III, while $\varphi(z)$ is the non-trivial function in (1.2) in Case IV.

Our general solutions of the sncILW equation (including non-Hermitian ones) are defined in terms of two sets of variables satisfying the time evolution equations of the sCM model:

$$\ddot{a}_j = -4 \sum_{k \neq j}^N \langle f_j | e_k \rangle \langle f_k | e_j \rangle V'(a_j - a_k),$$
$$|\dot{e}_j| = 2i \sum_{k \neq j}^N |e_k\rangle \langle f_k | e_j \rangle V(a_j - a_k),$$
$$\langle \dot{f}_j \rangle = -2i \sum_{k \neq j}^N \langle f_j | e_k \rangle \langle f_k | e_j \rangle V(a_j - a_k)$$

for $j = 1, \ldots, N$ and

$$\ddot{b}_j = -4 \sum_{k \neq j}^M \langle h_j | g_k \rangle \langle h_k | g_j \rangle V'(b_j - b_k),$$
$$|\dot{g}_j| = 2i \sum_{k \neq j}^M |g_k\rangle \langle h_k | g_j \rangle V(b_j - b_k),$$
$$\langle \dot{h}_j \rangle = -2i \sum_{k \neq j}^M \langle h_j | g_k \rangle \langle h_k | g_j \rangle V(b_j - b_k)$$

for $j = 1, \ldots, M$ (we use the notation in Sect. 1.4). More specifically, the spin-pole ansatz providing solutions of the sncILW equation (1.4), both in the real-line case (III) and the periodic case (IV), is given in terms of these dynamical variables as follows,

$$U(x, t) = e^{i\gamma_0(P+Q)t} U_0(x, t) e^{-i\gamma_0(P+Q)t},$$
$$V(x, t) = e^{i\gamma_0(P+Q)t} V_0(x, t) e^{-i\gamma_0(P+Q)t},$$

where

$$U_0(x, t) = M(t) + i \sum_{j=1}^N P_j(t) \alpha(x - a_j(t) - i\delta/2) - i \sum_{j=1}^M Q_j(t) \alpha(x - b_j(t) + i\delta/2),$$
$$V_0(x, t) = -M(t) - i \sum_{j=1}^N P_j(t) \alpha(x - a_j(t) + i\delta/2) + i \sum_{j=1}^M Q_j(t) \alpha(x - b_j(t) - i\delta/2),$$

(2.13)
with
\[ P_j(t) = |e_j(t)\rangle\langle f_j(t)| \quad (j = 1, \ldots, N), \quad Q_j(t) = |g_j(t)\rangle\langle h_j(t)| \quad (j = 1, \ldots, M) \]

such that
\[ P = \sum_{j=1}^{N} P_j(t), \quad Q := \sum_{j=1}^{M} Q_j(t) \]

both are time-independent, together with an additional variable \( M(t) \in \mathbb{C}^{d \times d} \) describing a non-trivial background. In Case III, \( N \) and \( M \) are arbitrary and the variable \( M \) must be constant, whereas in Case IV, we must have \( N = M \) and \( M \) is necessarily dynamical. The precise statement is as follows.

**Theorem 2.1** (Non-Hermitian solutions of the sncILW equation) For fixed \( N, M \in \mathbb{Z}_{\geq 0} \), let \( a_j(t) \in \mathbb{C}, |e_j(t)\rangle \in \mathcal{V}, \langle f_j(t)| \in \mathcal{V}^\ast \) for \( j = 1, \ldots, N \), \( b_j(t) \in \mathbb{C}, |g_j(t)\rangle \in \mathcal{V}, \langle h_j(t)| \in \mathcal{V}^\ast \) for \( j = 1, \ldots, M \), and \( M(t) \in \mathbb{C}^{d \times d} \) be functions of \( t \in \mathbb{R} \) satisfying the following conditions: (i) Both sets of variables \( \{a_j, |e_j\rangle, \langle f_j|\}_{j=1}^{N} \) and \( \{b_j, |g_j\rangle, \langle h_j|\}_{j=1}^{M} \) satisfy the time evolution equations of the sCM model: (2.8)–(2.9) hold true for \( j = 1, \ldots, N \), and (2.10)–(2.11) hold true for \( j = 1, \ldots, M \). (ii) The background \( \dot{M} = M(t) \) satisfies
\[ \dot{\dot{M}} = -\frac{1}{2} \sum_{j=1}^{N} \sum_{k \neq j} [P_j, P_k] \gamma'(a_j - a_k) + \frac{1}{2} \sum_{j=1}^{M} \sum_{k \neq j} [Q_j, Q_k] \gamma'(b_j - b_k) \]

with \( P_j = P_j(t) \) and \( Q_j = Q_j(t) \) given by (2.14). (iii) At time \( t = 0 \), the following conditions are fulfilled: first,
\[ \langle e_j|f_j \rangle = 1 \quad (j = 1, \ldots, N), \quad \langle g_j|h_j \rangle = 1 \quad (j = 1, \ldots, M), \]

second,
\[ \dot{a}_j|f_j \rangle = 2|f_j|\dot{M} + 2i \sum_{k \neq j} \langle f_j|e_k \rangle \langle f_k|\alpha(a_j - a_k) - 2i \sum_{k=1}^{N} \langle f_j|g_k \rangle \langle h_k|\alpha(a_j - b_k + i\delta) \]
\[ (j = 1, \ldots, N), \]
\[ \dot{b}_j|g_j \rangle = 2M|g_j \rangle - 2i \sum_{k \neq j} |g_k \rangle \langle h_k|g_j \rangle \alpha(b_j - b_k) + 2i \sum_{k=1}^{N} |e_k \rangle \langle f_k|g_j \rangle \alpha(b_j - a_k + i\delta) \]
\[ (j = 1, \ldots, M), \]

and third,
Then, in Case III, \( P \) and \( Q \) in (2.15) are both time-independent, and the ansatz (2.12)–(2.15) gives a solution of the sncILW equation on the real line for all times in the interval \( t \in [0, \tau) \) if \( \tau > 0 \) is such that (2.19) and

\[
\begin{align*}
-\frac{3\delta}{2} < \text{Im}(a_j) < -\frac{\delta}{2} \quad (j = 1, \ldots, N), \\
\frac{\delta}{2} < \text{Im}(b_j) < \frac{3\delta}{2} \quad (j = 1, \ldots, M).
\end{align*}
\]

(2.19)

hold true for all times \( t \in [0, \tau) \); in Case IV, the same holds true provided that \( N = M \), the conditions in (2.20) hold mod \( 2\ell \), and the following holds true at time \( t = 0 \),

\[ P = Q. \]

(2.21)

1. Theorem 2.1 gives a Hermitian solution, \( U = U^\dagger \) and \( V = V^\dagger \), if and only if \( N = M \) and the initial conditions satisfy the following further constraints,

\[
M = M^\dagger, \quad b_j = a_j^*, \quad |e_j\rangle = \langle h_j|^\dagger, \quad (f_j| = |g_j\rangle^\dagger \quad (j = 1, \ldots, N),
\]

(2.22)

which imply \( Q_j = P_j^\dagger \). It is easy to check that the reduction (2.22) is consistent; moreover, if we impose it at time \( t = 0 \), it is fulfilled at all times. Thus, in Case IV, we obtain the Hermitian solution of the sncILW equation presented in Sect. 1.

2. It is important to note the following differences between Cases III and IV: First, in Case III, \( \gamma_0 = 0 \), and therefore \( U(x, t) = U_0(x, t) \) and \( V(x, t) = V_0(x, t) \). However, in Case IV, the functions \( U_0(x, t) \) and \( V_0(x, t) \) given by the spin-pole ansatz (2.13) are related to the solutions \( U(x, t) \) and \( V(x, t) \) by a time-dependent similarity transformation determined by the total spin \( P \). Second, in Case IV, the background \( M(t) \) has non-trivial dynamics, while in Case III, (2.16) simplifies to \( \dot{M} = 0 \) (since \( \kappa(z) \) reduces to a constant), i.e., the background is constant in time: \( M(t) = M(0) =: M_0 \). Third, in Case IV, we impose the constraint (2.21) on the initial conditions, but this constraint is absent in Case III; as discussed in Sect. 1.2, this additional constraint in Case IV is required for the \( 2\ell \)-periodicity of \( U \) and \( V \) (the argument given there straightforwardly generalizes to the more general case here).

3. In Case III, our solutions (2.12) obey the boundary conditions

\[
\lim_{x \to \pm \infty} U(x, t) = -\lim_{x \to \pm \infty} V(x, t) = e^{i\gamma_0 (P+Q)t} \left( M_0 \pm \frac{i\pi}{2\delta} (P-Q) \right) e^{-i\gamma_0 (P+Q)t}
\]

(2.23)

\[ \square \] Springer
(this follows from \( \lim_{x \to \pm \infty} \alpha(x-a) \to \pm \pi/2\delta \) for all \( a \in \mathbb{C} \)); thus, the condition (2.21) is equivalent to \( U \) and \(-V\) being equal to \( e^{i\gamma_0(P+Q)t}M_0e^{-i\gamma_0(P+Q)t} \) at \( x \to \pm \infty \); however, there is no need to impose these conditions in Case III. Thus, our solutions suggest that the sncILW equation on the real line is well-defined for the following boundary conditions:

\[
\lim_{x \to \pm \infty} U(x, t) = -\lim_{x \to \pm \infty} V(x, t) = e^{i\gamma_0(P+Q)t}M_{\pm \infty}e^{-i\gamma_0(P+Q)t}, \tag{2.24}
\]

where \( M_{\pm \infty} \) are time-independent and, in general, such that \( M_{+ \infty} \neq M_{- \infty} \).

4. The solutions of the sncILW on the real line (Case III) obtained in Ref. [1] correspond to the special case \( M = 0 \); note that this specialization is only possible in Case III (in Case IV, it is prevented by the non-trivial dynamics (2.16)).

5. It is interesting to note that (2.18), together with (2.9), (2.11), (2.16) and (2.21), is a Bäcklund transformation of the sCM system in the elliptic case (IV); see Sect. 4 for precise statements. Moreover, in the limits \( (\ell, \delta) \to (\infty, \infty), \delta \to \infty, \) and \( \ell \to \infty, \) this Bäcklund transformation reduces to Bäcklund transformations for the sCM system in Cases I, II and III, respectively; it is important to note that, in Cases I–III, the constraint (2.21) can be omitted; this Bäcklund transformation is a generalization of the Bäcklund transformation of sCM model in Cases I–III obtained in Ref. [1]; the latter correspond to the special case \( M = 0 \) where, again, this specialization is possible in Cases I–III, but not in Case IV. We also remark that, in the elliptic case (IV), a similar Bäcklund transformation for a certain singular limit of the elliptic sCM system, where \( d = 2 \) and \( \langle f_j | e_j \rangle = \langle h_j | g_j \rangle = 0, \) was recently found by one of the authors in collaboration with Klabbers [25].

6. The Bäcklund transformation, (2.18) with (2.9), (2.11), (2.16) and (2.21), forms an overdetermined system of ordinary equations (ODEs) whose consistency must be established. This was done for the known Bäcklund transformation for the sCM system [2] in Cases I–III in Ref. [26] by constructing functions that measure the departure of the Bäcklund transformation from consistency and showing that they obey a system of linear homogeneous ODEs; if the initial data is consistent with the Bäcklund transformation at \( t = 0 \), consistency will be preserved at future times, under mild assumptions. While the approach of Ref. [26] can be straightforwardly generalized to the Bäcklund transformation in Case IV, leading to essentially the same equations (see [26, Eq. (2.43)]), in this paper we take a more streamlined approach, which allows us to show that the relevant quantities obey a system of linear homogeneous ordinary differential equations without deriving its precise form.

7. Theorem 2.1 is stated under the assumption that the poles remain in the strip defined in (2.19) and that no pole collisions occur (see (2.20)). As already mentioned, we believe that these assumptions are unnecessary, and we expect that our soliton solutions can be extended to all times \( t \in \mathbb{R} \). To support this expectation, we mention the following known results. First, in the sBO case, a Lax pair is known which can be used to prove that the former condition is satisfied for all times [14]; we hope that it is possible to generalize this argument to the sncILW case. Second, for the scalar Benjamin-Ono equation, it is known that pole collisions occur, but they are no problem for the soliton solutions [27]. A third reason is recent work by Gérard and...
Lenzmann on multi-soliton solutions to a nonlocal nonlinear Schrödinger equation [28] (see also [29]) governed by a complexification of the rational CM system; in this work, an explicit example of a two-soliton solution is given where (i) the poles collide and (ii) the solution remains valid during and after the collision. Clearly, it would be interesting to prove the more general result.

2.2 Solutions of the sBO equation

In this section, we consider Cases I and II in (2.1)–(2.2) and (2.7); in particular, \( V(z) \) and \( \alpha(z) \) are as in (2.1) and (2.2) for the rational and trigonometric cases.

The spin-pole ansatz for solutions of the sBO equation (2.3), both in the real-line and periodic cases, is given by

\[
U(x, t) = M_0 + i \sum_{j=1}^{N} P_j(t) \alpha(x - a_j(t)) - i \sum_{j=1}^{M} Q_j(t) \alpha(x - b_j(t)) \tag{2.25}
\]

with \( P_j(t) \) and \( Q_j(t) \) as in (2.14); as in the hyperbolic case, we can consistently assume that the background \( M(t) = M_0 \) is time-independent.

**Theorem 2.2** (Non-Hermitian solutions of the sBO equation) For fixed \( N, M \in \mathbb{Z}_{\geq 0} \) and \( M_0 \in \mathbb{C}^{d \times d} \), let \( a_j(t) \in \mathbb{C}, |e_j(t)| \in \mathbb{V}, \langle f_j(t) \rangle \in \mathbb{V}^* \) for \( j = 1, \ldots, N \) and \( b_j(t) \in \mathbb{C}, |g_j(t)| \in \mathbb{V}, \langle h_j(t) \rangle \in \mathbb{V}^* \) for \( j = 1, \ldots, M \), satisfy the following conditions: (i) Both sets of variables \( \{a_j, |e_j|\}, \langle f_j \rangle_{j=1}^{N} = \{a_j(t), |e_j(t)|, \langle f_j(t) \rangle \}_{j=1}^{N} \) and \( \{b_j, |g_j|\}, \langle h_j \rangle_{j=1}^{M} = \{b_j(t), |g_j(t)|, \langle h_j(t) \rangle \}_{j=1}^{M} \) satisfy the time evolution equations of the sCM model: (2.8)–(2.9) hold true for \( j = 1, \ldots, N \) and (2.10)–(2.11) hold true for \( j = 1, \ldots, M \). (ii) At time \( t = 0 \), the constraints (2.17), (2.18) with \( M(0) = M_0 \) and \( \delta \to 0 \), and

\[
\text{Im}(a_j) < 0 \quad (j = 1, \ldots, N), \quad \text{Im}(b_j) > 0 \quad (j = 1, \ldots, M) \tag{2.26}
\]

hold. Then, in Case I, the ansatz (2.25) with (2.14) gives a solution of the sBO equation for all times in the interval \( t \in [0, \tau) \) if \( \tau > 0 \) is such that (2.20) holds for all times \( t \in [0, \tau) \). In Case II, the same holds true provided the equalities in (2.20) hold mod \( 2\ell \).

3 From the sncILW equation to a first-order system

In this section, we establish conditions under which the ansatz (2.12) solves the periodic sncILW equation. Throughout this section, \( V(z) = \wp_2(z), \alpha(z) = \zeta_2(z), \) and \( M = N \).

**Proposition 3.1** The functions \( U(x, t) \) and \( V(x, t) \) in (2.12) satisfy the periodic sncILW equation (1.4) provided that the equations (2.16),
\[ \dot{a}_j p_j = 2p_j m + 2 \sum_{k \neq j}^N p_j p_k \alpha(a_j - a_k) - 2i \sum_{k=1}^N p_j Q_k \alpha(a_j - b_k + i\delta), \]
\[ \dot{b}_j q_j = 2mq_j - 2i \sum_{k \neq j}^N q_j q_k \alpha(b_j - b_k) + 2i \sum_{k=1}^N p_k q_j \alpha(b_j - a_k + i\delta) \quad (j = 1, \ldots, N), \]

(3.1)

and

\[ \dot{p}_j = -2i \sum_{k \neq j}^N [p_j, p_k] V(a_j - a_k), \quad (j = 1, \ldots, N), \]
\[ \dot{q}_j = -2i \sum_{k \neq j}^N [q_j, q_k] V(b_j - b_k) \]

are satisfied and the constraints (2.19)–(2.21) and

\[ p_j^2 = p_j, \quad q_j^2 = q_j \quad (j = 1, \ldots, N) \]

are fulfilled.

**Proof** The proof is facilitated by introducing the notation

\[ \left( \begin{array}{c} F_1 \\ F_2 \end{array} \right) \circ \left( \begin{array}{c} G_1 \\ G_2 \end{array} \right) := \left( \begin{array}{c} F_1 G_1 \\ -F_2 G_2 \end{array} \right) \]

(3.4)

for \( \mathbb{C} \)-valued functions \( F_j, G_j \) \( (j = 1, 2) \) and the operator

\[ T := \left( \begin{array}{cc} T & \tilde{T} \\ -\tilde{T} & -T \end{array} \right), \]

(3.5)

interpreted as a linear operator on vector-valued functions, see [17]. In the present paper, we use the product \( \circ \) defined in (3.4) also for vectors \( \mathcal{F}, \mathcal{G} \) whose components \( F_j, G_j \) are in \( \mathbb{C}^d \times d \), and we let

\[ [\mathcal{F} \circ \mathcal{G}] := \mathcal{F} \circ \mathcal{G} - \mathcal{G} \circ \mathcal{F}, \quad \{\mathcal{F} \circ \mathcal{G}\} := \mathcal{F} \circ \mathcal{G} + \mathcal{G} \circ \mathcal{F} \]

(3.6)

be the corresponding generalizations of the commutator and anti-commutator, respectively. With this notation, the periodic sncILW equation (1.4) can be written as

\[ \mathcal{U}_t + [\mathcal{U} \circ \mathcal{U}_x] + \mathcal{T} \mathcal{U}_{xx} + i[\mathcal{U} \circ \mathcal{T} \mathcal{U}_x] = 0, \]

(3.7)
where

\[ U(x, t) := \left( \begin{array}{c} U(x, t) \\ V(x, t) \end{array} \right). \]  

(3.8)

Using the shorthand notation

\[
\begin{align*}
(a_j, |e_j\rangle, \langle f_j|, P_j, r_j) & := \\
& \begin{cases} 
(a_j, |e_j\rangle, \langle f_j|, P_j, +1) & j = 1, \ldots, N, \\
(b_j-N, |g_{j-N}\rangle, \langle h_{j-N}|, Q_{j-N}, -1) & j = N + 1, \ldots, \mathcal{N},
\end{cases}
\end{align*}
\]

\[ \mathcal{N} := 2N, \]  

(3.9)

we write the ansatz (2.12) as

\[ U(x, t) = e^{i\gamma_0(P+Q)t}U_0(x, t)e^{-i\gamma_0(P+Q)t}, \]  

(3.10)

with

\[ U_0(x, t) = M(t)\mathcal{E} + i \sum_{j=1}^{\mathcal{N}} r_j P_j(t) \mathcal{A}_j(x - a_j(t)), \]  

(3.11)

where

\[ \mathcal{E} := \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad \mathcal{A}_\pm(z) := \begin{pmatrix} +\alpha(z \mp i\delta/2) \\ -\alpha(z \pm i\delta/2) \end{pmatrix}. \]  

(3.12)

It is important to note that, provided (3.2) holds, the quantities \( P \) and \( Q \) defined in (2.15) and appearing in (3.10) are conserved quantities,

\[ \dot{P} = \sum_{j=1}^{N} \dot{P}_j = -2i \sum_{j=1}^{N} \sum_{k\neq j}^{N} [P_j, P_k] V(a_j - a_k) = 0 \]  

(3.13a)

and

\[ \dot{Q} = \sum_{j=1}^{N} \dot{Q}_j = -2i \sum_{j=1}^{N} \sum_{k\neq j}^{N} [Q_j, Q_k] V(b_j - b_k) = 0, \]  

(3.13b)

using the anti-symmetry of the commutator and the fact that \( V(z) \) is an even function (A.5). Thus, assuming (3.2), it follows from (3.7) and (3.10)–(3.11) that \( \mathcal{U} \) satisfies (3.7) if and only if \( U_0 \) satisfies

\[ \mathcal{U}_{0,t} + i\gamma_0[(P+Q)\mathcal{E} \mathcal{U}_0] + \{U_0, \mathcal{U}_{0,x}\} + \mathcal{T}_0 U_{0,xx} + i[U_0, \mathcal{T}_0 U_{0,x}] = 0. \]  

(3.14)
We compute each term in (3.14) with $\mathcal{U}_0$ given by (3.11). We start with

$$\mathcal{U}_{0,t} = \dot{\mathcal{M}}\mathcal{E} + i \sum_{j=1}^{\mathcal{N}} r_j (\dot{P}_j \mathcal{A}_{r_j} (x - a_j) - P_j \dot{\alpha}_j \mathcal{A}_{r_j}'(x - a_j)).$$  \tag{3.15}$$

Next, we compute

$$\{\mathcal{U}_0; \mathcal{U}_{0,x}\} = i \sum_{j=1}^{\mathcal{N}} r_j \{M, P_j\} \mathcal{E} \circ \mathcal{A}_{r_j}'(x - a_j) - \sum_{j=1}^{\mathcal{N}} \sum_{k=1}^{\mathcal{N}} r_j r_k \{P_j, P_k\} \mathcal{A}_{r_j} (x - a_j) \circ \mathcal{A}_{r_k}' (x - a_k)$$

$$= i \sum_{j=1}^{\mathcal{N}} r_j \{M, P_j\} \mathcal{A}_{r_j}'(x - a_j) - 2 \sum_{j=1}^{\mathcal{N}} p_j^2 \mathcal{A}_{r_j} (x - a_j) \circ \mathcal{A}_{r_j}' (x - a_j)$$

$$- \sum_{j=1}^{\mathcal{N}} \sum_{k \neq j}^{\mathcal{N}} r_j r_k \{P_j, P_k\} \mathcal{A}_{r_j} (x - a_j) \circ \mathcal{A}_{r_k}' (x - a_k).$$ \tag{3.16}$$

To proceed, we need the identities

$$2\mathcal{A}_{r_j} (x - a_j) \circ \mathcal{A}_{r_j}' (x - a_j) = -\mathcal{A}_{r_j}''' (x - a_j) + \mathcal{F}_{r_j}' (x - a_j)$$ \tag{3.17}$$

and

$$\mathcal{A}_{r_j} (x - a_j) \circ \mathcal{A}_{r_k}' (x - a_k)$$

$$= -\alpha (a_j - a_k + i (r_j - r_k) \delta / 2) \mathcal{A}_{r_k}' (x - a_k)$$

$$- V (a_j - a_k + i (r_j - r_k) \delta / 2) (\mathcal{A}_{r_j} (x - a_j) - \mathcal{A}_{r_k} (x - a_k))$$

$$+ \frac{1}{2} \mathcal{F}_{r_k}' (x - a_k) + \frac{1}{2} \alpha' (a_j - a_k + i (r_j - r_k) \delta / 2) \mathcal{E},$$ \tag{3.18}$$

where

$$\mathcal{F}_\pm (z) := \left( \begin{array}{c} \mathcal{E} (z \mp i \delta / 2) \\ - \mathcal{E} (z \pm i \delta / 2) \end{array} \right).$$ \tag{3.19}$$

The first identity (3.17) can be obtained by differentiating (A.6) with respect to $z$ and setting $z = x - a_j \pm r_j i \delta / 2$ while the second identity (3.18) can be obtained by differentiating (A.7) with respect to $c$, setting $a = x$, $b = a_j \pm r_j i \delta / 2$, and $c = a_k \pm r_k i \delta / 2$, and using the periodicity property $\alpha (z + 2i \delta) = \alpha (z)$ and the fact that $\alpha (z)$ is an odd function. Inserting (3.17) and (3.18) into (3.16) gives

$$\{\mathcal{U}_0; \mathcal{U}_{0,x}\}$$

$$= i \sum_{j=1}^{\mathcal{N}} r_j \{M, P_j\} \mathcal{A}_{r_j}' (x - a_j) + \sum_{j=1}^{\mathcal{N}} p_j^2 \mathcal{A}_{r_j}''' (x - a_j) - \sum_{j=1}^{\mathcal{N}} p_j^2 \mathcal{F}_{r_j}' (x - a_j)$$

$$+ \sum_{j=1}^{\mathcal{N}} \sum_{k \neq j}^{\mathcal{N}} r_j r_k \{P_j, P_k\} \alpha (a_j - a_k + i (r_j - r_k) \delta / 2) \mathcal{A}_{r_k}' (x - a_k).$$
\[ + \sum_{j=1}^{N} \sum_{k \neq j}^{N} r_j r_k \{P_j, P_k\} V(a_j - a_k + i(r_j - r_k)\delta/2)(A_{r_j}(x - a_j) - A_{r_k}(x - a_k)) \]

\[- \frac{1}{2} \sum_{j=1}^{N} \sum_{k \neq j}^{N} r_j r_k \{P_j, P_k\} \mathcal{F}_{r_k}'(x - a_k) \]

\[- \frac{1}{2} \sum_{j=1}^{N} \sum_{k \neq j}^{N} r_j r_k \{P_j, P_k\} \mathcal{F}'(a_j - a_k + i(r_j - r_k)\delta/2)\mathcal{E}. \]

The double sum in the third line and the second double sum in the fourth line vanish by symmetry. Hence, after relabelling summation indices \( j \leftrightarrow k \) in the double sum in the second line and rearranging, we are left with

\[
\{ \mathcal{U}_0, \mathcal{U}_0 \} = i \sum_{j=1}^{N} r_j \{M, P_j\} \mathcal{A}_{r_j}'(x - a_j) + \sum_{j=1}^{N} P_j^2 \mathcal{A}_{r_j}'(x - a_j) - \sum_{j=1}^{N} \sum_{k \neq j}^{N} r_j r_k \{P_j, P_k\} \alpha(a_j - a_k + i(r_j - r_k)\delta/2) \mathcal{A}_{r_j}'(x - a_j) \]

\[- \frac{1}{2} \sum_{j=1}^{N} \sum_{k \neq j}^{N} r_j r_k \{P_j, P_k\} \mathcal{F}_{r_j}'(x - a_j) - \frac{1}{2} \sum_{j=1}^{N} \sum_{k \neq j}^{N} r_j r_k \{P_j, P_k\} \mathcal{F}_{r_k}'(x - a_k). \]

To compute terms involving \( \mathcal{T} \), we need the following lemma.

**Lemma 3.1** The operator \( \mathcal{T} \) defined in (3.5) has the following action on the functions \( \mathcal{A}_{r_j}'(x - a_j) \)

\[
(\mathcal{T} \mathcal{A}_{r_j}'(x - a_j))(x) = i r_j \mathcal{A}_{r_j}'(x - a_j) + (1 + r_j)\gamma_0 \begin{pmatrix} 0 \\ 1 \end{pmatrix} + (1 - r_j)\gamma_0 \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (j = 1, \ldots, N) \]

(3.22)

provided (2.19) holds.

**Proof** Using the definitions (3.12) of \( \mathcal{A}_{\pm}(z) \) and recalling that \( \xi_2(z) = \xi_1(z) + \gamma_0 z \) with \( \gamma_0 \) given by (1.3), we write

\[
\mathcal{A}_{r_j}(x - a_j) = \begin{pmatrix} +\xi_1(x - a_j + i r_j \delta/2) \\ -\xi_1(x - a_j - i r_j \delta/2) \end{pmatrix} + \gamma_0 \begin{pmatrix} x - a_j - i r_j \delta/2 \\ -(x - a_j + i r_j \delta/2) \end{pmatrix}. \]

(3.23)

By differentiating (3.23) with respect to \( x \), we obtain

\[
\mathcal{A}_{r_j}'(x - a_j) = \begin{pmatrix} -\phi_1(x - a_j + i r_j \delta/2) \\ +\phi_1(x - a_j + i r_j \delta/2) \end{pmatrix} + \gamma_0 \begin{pmatrix} 1 \\ -1 \end{pmatrix}. \]

(3.24)
where $\varphi_1(z) := -\zeta_1'(z)$ is a $2\ell$-periodic, zero-mean function (see Appendix A.1). We compute the action of $T$ on the first and second terms in (3.24) separately.

First, we use the following result [17, Appendix 2.a]: for a $2\ell$-periodic, zero-mean function $f(z)$ analytic in a strip $-A < \text{Im}(z) < A$ for some $A > \delta/2$, the vector-valued functions $(f(x \mp i\delta/2), -f(x \pm i\delta/2))^T$ are eigenfunctions of the $T$ operator with eigenvalues $\pm i$. Applied to the functions $f(z) = \varphi_1(z - a_j)$, this gives

$$
\left( T \left( -\varphi_1(\cdot - a_j - ir_j\delta/2) + \varphi_1(\cdot - a_j + ir_j\delta/2) \right) \right)(x) = i r_j \left( -\varphi_1(x - a_j - ir_j\delta/2) + \varphi_1(x - a_j + ir_j\delta/2) \right) = i r_j A_{r_j}'(x - a_j) - i r_j \gamma_0 \left( \frac{1}{1} \right),
$$

(3.25)

where we have used (3.24) in the second step.

Second, we recall from [18, Appendix B] that

$$
T(1) = 0, \quad \tilde{T}(1) = -i
$$

(3.26)

and hence,

$$
T \left( \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right) = i \begin{pmatrix} 1 \\ 1 \end{pmatrix}.
$$

(3.27)

By applying $T$ to (3.24) and using and (3.25) and (3.27), we obtain (3.22). □

Using (3.11), Lemma 3.1, and (2.21) in the form

$$
\sum_{j=1}^{N'} r_j P_j = 0,
$$

(3.28)

we obtain

$$
TU_{0,x} = -\sum_{j=1}^{N'} P_j (A_{r_j}'(x - a_j) - \gamma_0 \mathcal{E}), \quad TU_{0,xx} = -\sum_{j=1}^{N'} P_j A_{r_j}''(x - a_j),
$$

(3.29)

where we have used the fact that $T$ commutes with differentiation [18] to obtain the second identity.

From (3.11) and the first equation in (3.29), we compute

$$
i[U_0 \circ TU_{0,x}] = -i \sum_{j=1}^{N'} [M, P_j] \mathcal{E} \circ A_{r_j}'(x - a_j)
$$

$$
+ \sum_{j=1}^{N'} \sum_{k \neq j} r_j [P_j, P_k] A_{r_j}(x - a_j) \circ A_{r_k}'(x - a_k)
$$

(3.30)
\[ + i \gamma_0 [U_0 \circ (P + Q) \mathcal{E}] \]

\[ = -i \sum_{j=1}^{N} [M, P_j] A'_{r_j} (x - a_j) \]

\[ - \sum_{j=1}^{N} \sum_{k \neq j} r_j [P_j, P_k] \alpha (a_j - a_k + i (r_j - r_k) \delta / 2) A'_{r_k} (x - a_k) \]

\[ - \sum_{j=1}^{N} \sum_{k \neq j} r_j [P_j, P_k] V (a_j - a_k + i (r_j - r_k) \delta / 2) \]

\[ \times (A_{r_j} (x - a_j) - A_{r_k} (x - a_k)) \]

\[ + \frac{1}{2} \sum_{j=1}^{N} \sum_{k \neq j} r_j [P_j, P_k] \mathcal{F}'_{r_j} (x - a_j) \]

\[ + \frac{1}{2} \sum_{j=1}^{N} \sum_{k \neq j} r_j [P_j, P_k] \mathcal{V}'(a_j - a_k + i (r_j - r_k) \delta / 2) \mathcal{E} \]

\[ - i \gamma_0 [(P + Q) \mathcal{E} \circ U_0], \quad (3.30) \]

where we have employed (3.18) and used the anti-symmetry of the generalized commutator (3.6) in the second step. Since \( V(z) \) is an even function, we can rewrite the double sum in the second line of the right-hand side as follows:

\[ \sum_{j=1}^{N} \sum_{k \neq j} r_j [P_j, P_k] V (a_j - a_k + i (r_j - r_k) \delta / 2) (A_{r_j} (x - a_j) - A_{r_k} (x - a_k)) \]

\[ = \frac{1}{2} \sum_{j=1}^{N} \sum_{k \neq j} (r_j + r_k) [P_j, P_k] V (a_j - a_k + i (r_j - r_k) \delta / 2) \]

\[ \times (A_{r_j} (x - a_j) - A_{r_k} (x - a_k)) \]

\[ = \sum_{j=1}^{N} \sum_{k \neq j} (r_j + r_k) [P_j, P_k] V (a_j - a_k + i (r_j - r_k) \delta / 2) A_{r_j} (x - a_j). \quad (3.31) \]

Moreover, changing variables \( j \leftrightarrow k \) in the double sum in the first line and in the first double sum in the third line of the right-hand side of (3.30), we arrive at

\[ i [U_0 \circ T U_0, x] = -i \sum_{j=1}^{N} [M, P_j] A'_{r_j} (x - a_j) \]

\[ - \sum_{j=1}^{N} \sum_{k \neq j} r_k [P_j, P_k] \alpha (a_j - a_k + i (r_j - r_k) \delta / 2) A'_{r_j} (x - a_j) \]

\[ \odot Springe \]
\[- \sum_{j=1}^{N} \sum_{k \neq j}^{N} (r_j + r_k)[P_j, P_k] V(a_j - a_k + i(r_j - r_k)\delta/2) \mathcal{A}_{r_j}(x - a_j) \]

\[- \frac{1}{2} \sum_{j=1}^{N} \sum_{k \neq j}^{N} r_k[P_j, P_k] \mathcal{F}'_{r_j}(x - a_j) \]

\[+ \frac{1}{2} \sum_{j=1}^{N} \sum_{k \neq j}^{N} r_j[P_j, P_k] \mathcal{F}'(a_j - a_k + i(r_j - r_k)\delta/2) \mathcal{E} \]

\[- i\gamma_0[(P + Q)\mathcal{E} \circ U_0]. \tag{3.32} \]

Inserting (3.15), (3.21), the second equation in (3.29), and (3.32) into (3.14) gives

\[0 = \left( \dot{M} + \frac{1}{4} \sum_{j=1}^{N} \sum_{k \neq j}^{N} (r_j + r_k)[P_j, P_k] \mathcal{F}'(a_j - a_k + i(r_j - r_k)\delta/2) \right) \mathcal{E} \]

\[+ \sum_{j=1}^{N} \left( ir_j \dot{P}_j - \sum_{k \neq j}^{N} (r_j + r_k)[P_j, P_k] V(a_j - a_k + i(r_j - r_k)\delta/2) \right) \mathcal{A}_{r_j}(x - a_j) \]

\[+ \sum_{j=1}^{N} \left( - ir_j P_j \dot{a}_j + ir_j \{M, P_j\} - i[M, P_j] \right) \]

\[- \sum_{k \neq j}^{N} r_k(r_j[P_j, P_k] + [P_j, P_k]) \mathcal{A}'(a_j - a_k + i(r_j - r_k)\delta/2) \right) \mathcal{A}'_{r_j}(x - a_j) \]

\[+ \sum_{j=1}^{N} (P_j^2 - P_j) \mathcal{A}''_{r_j}(x - a_j) \]

\[= \sum_{j=1}^{N} \left( 2P_j^2 + \sum_{k \neq j}^{N} (r_j r_k[P_j, P_k] + r_k[P_j, P_k]) \right) \mathcal{F}'_{r_j}(x - a_j), \tag{3.33} \]

where we have symmetrized the double sum in the first line using the fact that \( \mathcal{F}'(z) \) is an odd function.

We first consider the conditions under which the terms proportional to \( \mathcal{F}'_{r_j}(x - a_j) \) vanish. We write

\[2P_j^2 + \sum_{k \neq j}^{N} (r_j r_k[P_j, P_k] + r_k[P_j, P_k]) = 2P_j^2 - (1 - r_j)X_j P_j + (1 + r_j)P_j X_j, \]

where \( X_j := \sum_{k \neq j}^{N} r_k P_k \), and notice that the right-hand side vanishes if \( X_j = -r_j P_j \), i.e., if (3.28) holds.
Thus, the functions $U$ and $V$ defined in (2.12) satisfy the sncILW equation whenever (3.28) holds and the following conditions from (3.33) are fulfilled,
\[
\dot{\mathcal{M}} = -\frac{1}{4} \sum_{j=1}^{N} \sum_{k \neq j}^{N} (r_j + r_k) [P_j, P_k] \mathcal{N} (a_j - a_k + i(r_j - r_k)\delta/2)
\]
and
\[
\begin{align*}
 r_j \dot{P}_j \dot{a}_j &= r_j \{M, P_j\} - [M, P_j] \\
 &\quad + i \sum_{k \neq j}^{N} r_k (r_j [P_j, P_k] + [P_j, P_k]) \mathcal{N} (a_j - a_k + i(r_j - r_k)\delta/2), \\
 r_j \dot{P}_j &= -i \sum_{k \neq j}^{N} (r_j + r_k) [P_j, P_k] V (a_j - a_k + i(r_j - r_k)\delta/2), \\
 P_j^2 &= P_j,
\end{align*}
\]
for $j = 1, \ldots, N$. Recalling the notation (2.14) and (3.9), we see that these are equivalent to (2.16), (3.1)–(3.3) and (2.21). The result follows.

\[\square\]

4 Bäcklund transformation

Throughout this section, $V(z) = \wp_2(z), \mathcal{N}(z) = \zeta_2(z)$, and $M = N$.

In this section, we prove that solutions of the elliptic sCM equations of motion (2.8)–(2.11) are, under certain conditions, also solutions of a Bäcklund transformation for the elliptic sCM system. This Bäcklund transformation is given by

\[
\begin{align*}
\dot{a}_j (f_j) &= 2(f_j|M + 2i \sum_{k \neq j}^{N} \langle f_j | e_k \rangle \langle f_k | \mathcal{N}(a_j - a_k) - 2i \sum_{k=1}^{N} \langle f_j | g_k \rangle \langle h_k | \mathcal{N}(a_j - b_k) \rangle, \\
\dot{b}_j (g_j) &= 2M|g_j - 2i \sum_{k \neq j}^{N} |g_k \rangle \langle h_k | g_j \rangle \mathcal{N}(b_j - b_k) + 2i \sum_{k=1}^{N} \langle e_k | f_k | g_j \rangle \mathcal{N}(b_j - a_k),
\end{align*}
\]
for $j = 1, \ldots, N$, together with (2.9), (2.11), and (2.16). The precise statement is as follows.

**Proposition 4.1** Let $\{a_j, |e_j\rangle, \langle f_j|\}_{j=1}^{N}$, and $\{b_j, |g_j\rangle, \langle h_j|\}_{j=1}^{N}$ be a solution of the system consisting of (2.16) and the sCM systems (2.8)–(2.11) on an interval $[0, \tau)$ for some $\tau \in (0, \infty) \cup \{\infty\}$, with initial conditions satisfying (2.17), (2.18), and (2.21) at $t = 0$, and where (4.2) holds on $[0, \tau)$. Then, the first-order system consisting of (2.9), (2.11), (2.16), and (4.1) is satisfied on $[0, \tau)$.

In order to prove Proposition 4.1, we need two lemmas. The first lemma shows that the first-order system of Proposition 3.1 admits a unique solution under mild assumptions, generalizing a known result in Cases I–III [26].
Lemma 4.1 The initial value problem consisting of (2.9), (2.11), (2.16), and (4.1) with initial conditions satisfying (2.17), (2.21), and (4.1) at \( t = 0 \) has a unique solution on a maximal interval \([0, \tau_{\text{max}})\) for some \( \tau_{\text{max}} \in (0, \infty) \cup \{\infty\} \). On this interval, it holds that

\[
\begin{align*}
  a_j &\neq a_k \mod \{2\ell, 2i\delta\} \quad (1 \leq j < k \leq N), \\
  b_j &\neq b_k \mod \{2\ell, 2i\delta\} \quad (1 \leq j < k \leq N), \\
  a_j &\neq b_k \mod \{2\ell, 2i\delta\} \quad (j, k = 1, \ldots, N).
\end{align*}
\]  

(4.2)

Proof See Appendix B.1.

The second lemma shows that the first-order system of Proposition 4.1 implies the second-order sCM equations of motion (2.8)–(2.11).

Lemma 4.2 The solution of (2.9), (2.11), (2.16), and (4.1) in Lemma 4.1 solves (2.8) and (2.10) on \([0, \tau_{\text{max}})\).

Proof See Appendix B.2.

5 Proof of Theorem 2.1 in Case IV

Throughout this section, \( V(z) = \wp_2(z), \alpha(z) = \zeta_2(z), \) and \( M = N \). We first establish the following lemma.

Lemma 5.1 Let \( \tilde{a}_j := a_j - i\delta/2 \) and \( \tilde{b}_j := b_j + i\delta/2 \) for \( j = 1, \ldots, N \). Suppose that \( M, \{\tilde{a}_j, |e_j\rangle, \langle f_j|\}_{j=1}^N, \) and \( \{\tilde{b}_j, |g_j\rangle, \langle h_j|\}_{j=1}^N \) satisfy the equations (2.9), (2.11), (2.16), and (4.1) on \([0, \tau)\) with initial conditions satisfying (2.17) and (2.21) at \( t = 0 \). Then, (3.1)–(3.3) and (2.21) hold on \([0, \tau)\).

Proof Consider the set of equations (4.1) with the replacements

\[
\begin{align*}
  \{a_j\}_{j=1}^N &\rightarrow \{\tilde{a}_j\}_{j=1}^N, \\
  \{b_j\}_{j=1}^N &\rightarrow \{\tilde{b}_j\}_{j=1}^N.
\end{align*}
\]  

(5.1)
By left-multiplying the first set of the resulting equations by $|e_j\rangle$ and right-multiplying the second set of the resulting equations by $\langle h_j|$, we obtain

$$
\dot{a}_j|e_j\rangle\langle f_j| = 2|e_j\rangle\langle f_j|M + 2i \sum_{k \neq j}^N |e_j\rangle\langle f_j|e_k\rangle\langle f_k|\alpha(a_j - \tilde{a}_k)
$$

$$
- 2i \sum_{k=1}^N |e_j\rangle\langle f_j|g_k\rangle\langle h_k|\alpha(\tilde{a}_j - \tilde{b}_k),
$$

(5.2)

$$
\dot{b}_j|g_j\rangle\langle h_j| = 2M|g_j\rangle\langle h_j| - 2i \sum_{k \neq j}^N |g_k\rangle\langle h_k|g_j\rangle\langle h_j|\alpha(\tilde{b}_j - \tilde{b}_k)
$$

$$
+ 2i \sum_{k=1}^N |e_k\rangle\langle f_k|g_j\rangle\langle h_j|\alpha(\tilde{b}_j - \tilde{a}_k)
$$

for $j = 1, \ldots, N$. Recalling the definitions (2.14) of $P_j$ and $Q_j$ and those of $\tilde{a}_j$ and $\tilde{b}_j$, we see that (5.2) is equivalent to (3.1) which thus holds on $[0, \tau)$.

By differentiating (2.14) with respect to time and inserting (2.9) and (2.11) with (5.1), we find

$$
\dot{P}_j = |\dot{e}_j\rangle\langle f_j| + |e_j\rangle\langle \dot{f}_j|
$$

$$
= 2i \sum_{k \neq j}^N |e_k\rangle\langle f_k|e_j\rangle\langle f_j|V(\tilde{a}_j - \tilde{a}_k) - 2i \sum_{k \neq j}^N |e_j\rangle\langle f_j|e_k\rangle\langle f_k|V(\tilde{a}_j - \tilde{a}_k)
$$

(5.3a)

and

$$
\dot{Q}_j = |\dot{g}_j\rangle\langle h_j| + |g_j\rangle\langle \dot{h}_j|
$$

$$
= 2i \sum_{k \neq j}^N |g_k\rangle\langle h_k|g_j\rangle\langle h_j|V(\tilde{b}_j - \tilde{b}_k) - 2i \sum_{k \neq j}^N |g_j\rangle\langle h_j|g_k\rangle\langle h_k|V(\tilde{b}_j - \tilde{b}_k)
$$

(5.3b)

for $j = 1, \ldots, N$. Recalling the definitions of $P_j$ and $Q_j$ in (2.14) and those of $\tilde{a}_j$ and $\tilde{b}_j$ we see that (5.3) is equivalent to (3.2) which thus holds on $[0, \tau)$. 
By differentiating the quantities \( \langle f_j | e_j \rangle \) and \( \langle h_j | g_j \rangle \) with respect to time and inserting (2.8) and (2.11) with (5.1), we find

\[
\frac{d}{dr} \langle f_j | e_j \rangle = \langle \dot{f}_j | e_j \rangle + \langle f_j | \dot{e}_j \rangle = -2i \sum_{k \neq j} \langle f_j | e_k \rangle \langle f_k | e_j \rangle V(\bar{a}_j - \bar{a}_k) + 2i \sum_{k \neq j} \langle f_j | e_k \rangle \langle f_k | e_j \rangle V(\bar{a}_j - \bar{a}_k) = 0
\]

and

\[
\frac{d}{dr} \langle h_j | g_j \rangle = \langle \dot{h}_j | g_j \rangle + \langle h_j | \dot{g}_j \rangle = -2i \sum_{k \neq j} \langle h_j | g_k \rangle \langle h_k | g_j \rangle V(\bar{b}_j - \bar{b}_k) + 2i \sum_{k \neq j} \langle h_j | g_k \rangle \langle h_k | g_j \rangle V(\bar{b}_j - \bar{b}_k) = 0
\]

for \( j = 1, \ldots, N \). Because (2.17) holds at \( t = 0 \) by assumption, (5.4) guarantees it holds on \([0, \tau)\). Thus, by writing \( P_j^2 = |e_j \rangle \langle f_j | e_j \rangle \langle f_j | = \langle f_j | e_j \rangle P_j \) and \( Q_j^2 = |g_j \rangle \langle h_j | g_j \rangle \langle h_j | = \langle h_j | g_j \rangle Q_j \), we see that (3.3) holds on \([0, \tau)\).

To show that (2.21) holds on \([0, \tau)\), we use (3.13) with (5.1); hence \( \hat{P} = \hat{Q} = 0 = \hat{P} - \hat{Q} \) which implies that (2.21) holds on \([0, \tau)\) because it holds at \( t = 0 \).

We are now ready to prove Theorem 2.1.

**Proof of Theorem 2.1** Suppose that \( M(t), \{a_j(t),|e_j(t)\rangle,\langle f_j(t)\rangle\}_{j=1}^N \) and \( \{b_j(t),|g_j(t)\rangle,\langle h_j(t)\rangle\}_{j=1}^N \) defined for \( t \in [0, \tau) \) satisfy the assumptions in the statement of Theorem 2.1 for some \( \tau > 0 \). For \( j = 1, \ldots, N \), let \( \tilde{a}_j(t) := a_j(t) - i\delta/2 \) and \( \tilde{b}_j(t) := b_j(t) + i\delta/2 \). By assumption, \( M, \{a_j,|e_j\rangle,\langle f_j\rangle\}_{j=1}^N \), and \( \{b_j,|g_j\rangle,\langle h_j\rangle\}_{j=1}^N \) obey (2.8)–(2.11) and (2.16) on \([0, \tau)\). The definitions of \( \tilde{a}_j \) and \( \tilde{b}_j \) imply that (2.8)–(2.11), and (2.16) hold on \([0, \tau)\) also with \( \{a_j, b_j\}_{j=1}^N \) replaced by \( \{\tilde{a}_j, \tilde{b}_j\}_{j=1}^N \). Moreover, by assumption, the relations (2.18) hold at time \( t = 0 \). Using the \( 2i\delta \)-periodicity of \( \alpha \), it follows that the relations (4.1) with \( \{a_j, b_j\}_{j=1}^N \) replaced by \( \{\tilde{a}_j, \tilde{b}_j\}_{j=1}^N \) hold at \( t = 0 \). We conclude that the functions \( M, \{\tilde{a}_j,|e_j\rangle,\langle f_j\rangle\}_{j=1}^N \), and \( \{\tilde{b}_j,|g_j\rangle,\langle h_j\rangle\}_{j=1}^N \) solve the initial value problem of Proposition 4.1, so by Proposition 4.1 these functions satisfy the first-order system of Lemma 4.1 on \([0, \tau)\). In other words, the equations (4.1) with \( \{a_j, b_j\}_{j=1}^N \) replaced by \( \{\tilde{a}_j, \tilde{b}_j\}_{j=1}^N \) hold for all \( t \in [0, \tau) \), so we can use Lemma 5.1 to deduce that the functions \( M, \{a_j,|e_j\rangle,\langle f_j\rangle\}_{j=1}^N \), and \( \{b_j, |g_j\rangle, \langle h_j\rangle\}_{j=1}^N \) satisfy (3.1)–(3.3) and (2.21) on \([0, \tau)\). In particular, \( M, \{a_j,|e_j\rangle,\langle f_j\rangle\}_{j=1}^N \), and \( \{b_j, |g_j\rangle, \langle h_j\rangle\}_{j=1}^N \) fulfill the assumptions of Proposition 3.1. Thus we can employ Proposition 3.1 to infer that the ansatz (2.12) provides a solution to the periodic sncILW equation (1.4). This completes the proof of Theorem 2.1.  

\[ \square \]
6 Construction of soliton solutions

In this section, we show how to solve the nonlinear constraints on the initial data in Theorem 2.1. While we specifically reference the constraints in Case IV, the following results can be straightforwardly adapted to Cases III in Theorem 2.1 and Cases I–II in Theorem 2.2. One-soliton solutions are derived in Sect. 6.1 and a linear algebra problem whose solutions parameterize the corresponding multi-soliton solutions is presented in Sect. 6.2.

6.1 One-soliton solution

Throughout this subsection, \( \alpha(z) = \zeta_2(z) \). For \( N = M = 1 \), the time evolution Eqs. (2.8)–(2.11) and (2.16) simplify to \( \dot{M} = 0 \),
\[
\ddot{a}_1 = 0, \quad |\dot{e}_1| = 0, \quad \langle f_1 | = 0, \quad (6.1)
\]
and
\[
\ddot{b}_1 = 0, \quad |\dot{g}_1| = 0, \quad \langle h_1 | = 0. \quad (6.2)
\]
Given initial conditions \( M(0) = M_0 \),
\[
a_1(0) = a_{1,0}, \quad \dot{a}_1(0) = v_1, \quad |e_1(0)| = |e_{1,0}|, \quad \langle f_1 |(0) = \langle f_{1,0} |, \quad (6.3)
\]
and
\[
b_1(0) = b_{1,0}, \quad \dot{b}_1(0) = w_1, \quad |g_1(0)| = |g_{1,0}|, \quad \langle h_1 |(0) = \langle h_{1,0} |, \quad (6.4)
\]
we find the solution \( M = M_0 \),
\[
a_1 = a_{1,0} + v_1 t, \quad |e_1| = |e_{1,0}|, \quad \langle f_1 | = \langle f_{1,0} |. \quad (6.5)
\]
and
\[
b_1 = b_{1,0} + w_1 t, \quad |g_1| = |g_{1,0}|, \quad \langle h_1 | = \langle h_{1,0} |. \quad (6.6)
\]
The constraints (2.17) and (2.21) reduce to \( \langle f_{1,0} | e_{1,0} \rangle = 1 = \langle h_{1,0} | g_{1,0} \rangle \) and \( |e_{1,0} \rangle \langle f_{1,0} | = |g_{1,0} \rangle \langle h_{1,0} | \), respectively, which together imply \( |g_{1,0} \rangle = c |e_{1,0} \rangle \) and \( \langle h_{1,0} | = c^{-1} \langle f_{1,0} | \) for some \( c \in \mathbb{C}\setminus\{0\} \). The constraint (2.18) then reduces to
\[
v_1 \langle f_{1,0} | = 2 \langle f_{1,0} | M_0 - 2 i \langle f_{1,0} | \alpha(a_{1,0} - b_{1,0} + i \delta), \quad (6.7)
\]
i.e., \( \langle f_{1,0} | \) and \( |e_{1,0} \rangle \) are left- and right-eigenvectors of \( M_0 \), respectively. By right-multiplying the first equation in (6.7) by \( |e_{1,0} \rangle \) and left-multiplying the second equation
in (6.7) by \( \langle f_{1,0} \rangle \), we find

\[
v_1 = w_1 = 2\langle f_{1,0} | M_0 | e_{1,0} \rangle - 2i\alpha(a_{1,0} - b_{1,0} + i\delta). \tag{6.8}
\]

Collecting the observations above and using Theorem 2.1, we see that

\[
\begin{align*}
U(x, t) &= e^{2i\gamma_0 |e_{1,0} \rangle\langle f_{1,0}|} U_0(x, t) e^{-2i\gamma_0 |e_{1,0} \rangle\langle f_{1,0}|}, \\
V(x, t) &= e^{2i\gamma_0 |e_{1,0} \rangle\langle f_{1,0}|} V_0(x, t) e^{-2i\gamma_0 |e_{1,0} \rangle\langle f_{1,0}|},
\end{align*}
\tag{6.9}
\]

with

\[
\begin{align*}
U_0(x, t) &= M_0 + i |e_{1,0} \rangle\langle f_{1,0}| (\alpha(x - a_{1,0} - v_1 t - i\delta/2) - \alpha(x - b_{1,0} - v_1 t + i\delta/2)), \\
V_0(x, t) &= -M_0 - i |e_{1,0} \rangle\langle f_{1,0}| (\alpha(x - a_{1,0} - v_1 t + i\delta/2) - \alpha(x - b_{1,0} - v_1 t - i\delta/2)),
\end{align*}
\tag{6.10}
\]

provides a solution of the sncILW equation (1.4) when \( \langle f_{1,0} \rangle \) and \( |e_{1,0} \rangle \) are left- and right-eigenvectors, respectively of \( M_0 \) corresponding to the same eigenvalue and normalized to satisfy \( \langle f_{1,0} | e_{1,0} \rangle = 1 \) and \( v_1 \) is given by (6.8).

**Remark 6.1** In the generic case where \( v_1 \) in (6.8) has a nonzero imaginary part, (6.10) does not provide a traveling wave solution and (2.19) will be violated in finite time, after which Theorem 2.1 does not guarantee (6.10) solves the sncILW equation (1.4).

For \( v_1 \) to be real, in which case (6.10) provides a traveling wave solution of the sncILW equation (1.4) on \([0, \infty)\), it suffices for \( M_0 \) to be Hermitian (in this case \( |f_{1,0} \rangle = |e_{1,0} \rangle \) is a possibility but not a requirement unless all eigenvalues of \( M_0 \) are simple) and \( b_{1,0} = a_{1,0}^\ast \). It is interesting to note that in the singular limit of the sncILW equation studied in Ref. [25], no such one-soliton, traveling wave solutions exist.

### 6.2 Solution of constraints: Case IV

Throughout this subsection, \( V(z) = \phi_2(z), \alpha(z) = \zeta_2(z) \), and \( M = N \). For each \( j = 1, \ldots, N \), let us identify the vector \( |e_j \rangle \in \mathcal{V} \) with the vector \( e_j \in \mathbb{C}^d \) whose components \( (e_j)_\mu, \mu = 1, \ldots, d \) are the components of \( |e_j \rangle \) with respect to some given basis of \( \mathcal{V} \). Next, let us identify the collection of \( N \) vectors \( e_j, j = 1, \ldots, N \), with the single vector \( e \in \mathbb{C}^{Nd} \) whose components \( e_{j,\mu} := (e_j)_\mu \) are indexed by \( j = 1, \ldots, N \) and \( \mu = 1, \ldots, d \). Similarly, let us identify the three collections of vectors \( \{ |f_j \rangle \}_{j=1}^N, \{ |g_j \rangle \}_{j=1}^N, \) and \( \{ |h_j \rangle \}_{j=1}^N \) with the vectors \( f = (f_{j,\mu}) \in \mathbb{C}^{Nd}, g = (g_{j,\mu}) \in \mathbb{C}^{Nd}, \) and \( h = (h_{j,\mu}) \in \mathbb{C}^{Nd} \), respectively. Moreover, consider the matrix representation of \( M \) with respect to the same basis of \( \mathcal{V} \). We identify \( M \) with its vectorization \( \mathbf{M} \in \mathbb{C}^{d^2} \), i.e., the concatenation of the columns of \( M \). Then, the constraints (2.18) and (2.21) can be written as the \((2N + d)d \times (2N + d)d \) linear system

\[
\begin{pmatrix}
A^1 & A^2 & A^3 \\
B^1 & B^2 & B^3 \\
C^1 & C^2 & 0
\end{pmatrix}
\begin{pmatrix}
h \\
e \\
\mathbf{M}
\end{pmatrix}
= 
\begin{pmatrix}
D^1 f \\
-D^2 g \\
0
\end{pmatrix},
\tag{6.11}
\]
where $A^1 \in \mathbb{C}^{Nd \times Nd}$, $A^2 \in \mathbb{C}^{Nd \times Nd}$, $A^3 \in \mathbb{C}^{Nd \times d^2}$, $B^1 \in \mathbb{C}^{Nd \times Nd}$, $B^2 \in \mathbb{C}^{Nd \times Nd}$, $B^3 \in \mathbb{C}^{Nd \times d^2}$, $C^1 \in \mathbb{C}^{d^2 \times Nd}$, $C^2 \in \mathbb{C}^{d^2 \times Nd}$, $D^1 \in \mathbb{C}^{Nd \times Nd}$, and $D^2 \in \mathbb{C}^{Nd \times Nd}$ are defined by

\begin{align}
A^1_{j,\mu;\kappa;\nu} &= -2i(f_j | g_k)\delta_{\mu,\nu} \alpha(a_j - b_k + i\delta), \quad A^2_{j,\mu;\kappa;\nu} = 2i(1 - \delta_{j,k}) f_j \nu f_k \mu \alpha(a_j - a_k), \\
A^3_{j,\mu;\kappa;\nu} &= 2f_j \mu \delta_{\mu,\nu}, \\
B^1_{j,\mu;\kappa;\nu} &= 2i(1 - \delta_{j,k}) g_j \nu g_k \mu \alpha(b_j - b_k), \quad B^2_{j,\mu;\kappa;\nu} = -2i(f_k | g_j)\delta_{\mu,\nu} \alpha(b_j - a_k + i\delta), \\
B^3_{j,\mu;\kappa;\nu} &= -2g_j \nu g_k \mu \delta_{\mu,\nu}, \\
C^1_{j,\mu;\nu;\kappa;\sigma} &= 2g_j \nu \delta_{\mu,\kappa} \delta_{\nu,\sigma}, \quad C^2_{j,\mu;\nu;\kappa;\sigma} = -2f_j \mu \delta_{\nu,\sigma}, \\
D^1_{j,\mu;\kappa;\nu} &= v_j \delta_{j,k} \delta_{\mu,\nu} \delta_{\kappa,\mu}, \quad D^2_{j,\mu;\kappa;\nu} = w_j \delta_{j,k} \delta_{\mu,\nu} \delta_{\kappa,\mu}.
\end{align}

We have performed numerical experiments in the cases $N = 2$ and $N = 3$ with $d = 2$ using the following algorithm (i) fix values for $\{a_j, b_j, |g_j|, \langle f_j \rangle\}_{j=1}^N$, (ii) use (6.11) to determine $\{|e_j\}, \{h_j\}_{j=1}^N$ and $M$ as functions of $\{v_j, w_j\}_{j=1}^N$ (when a solution exists), and (iii) use the constraints (2.17) (which become a linear system for $\{v_j, w_j\}_{j=1}^N$) to determine numerical values for $\{v_j, w_j\}_{j=1}^N$ (when a solution exists). These experiments have uncovered examples of a variety of possibilities for (6.11) with (2.17):

1. The matrix in (6.11) is invertible and the linear system for $\{v_j, w_j\}_{j=1}^N$ is consistent.
2. The matrix in (6.11) is invertible and the linear system for $\{v_j, w_j\}_{j=1}^N$ is inconsistent.
3. The matrix (6.11) is rank-deficient (which imposes extra linear constraints on $\{v_j, w_j\}_{j=1}^N$) and the linear system (6.11) together with the (overdetermined) linear system for $\{v_j, w_j\}_{j=1}^N$ is consistent.
4. The matrix (6.11) is rank-deficient and the linear system (6.11) together with the (overdetermined) linear system for $\{v_j, w_j\}_{j=1}^N$ is inconsistent.

It is interesting to note that we found the cases $N = 2$ and $N = 3$ to be quite different. For $N = 2$, we often encountered possibility 3, though possibility 4 was achievable by fine-tuning the input data. For $N = 3$, we often encountered possibility 1, though possibility 2 was achievable by fine-tuning the input data.

While a full analysis and characterization of these different possibilities above is beyond the scope of the paper, we give an example of a three-soliton solution resulting from possibility 1. In the case $d = 2$, $N = 3$ with $\ell = \delta \pi$, we have obtained the solution$^2$

$^1$ Note that by the scaling property of the $\xi_2$-function, $\xi_2(\epsilon \zeta; \epsilon \ell, \epsilon (i\delta)) = e^{-1} \xi_2(\zeta; \ell, i\delta)$ (which follows from the same identity for the $\zeta$-function [31, Eq. 23.10.18]), the system (6.12) admits the symmetry ($\epsilon \in \mathbb{R}$)

\((\ell, \delta) \rightarrow (\epsilon \ell, \delta), \quad (a_j, b_j) \rightarrow c(a_j, b_j), \quad (v_j, w_j) \rightarrow c^{-1}(v_j, w_j) \quad (j = 1, \ldots, N), \quad \text{and} \quad M \rightarrow c^{-1} M,\)

which allows us to employ units of $\delta$ and $\delta^{-1}$.

$^2$ We give exact values for the input data and write the output values to five significant digits.
\[ a_1 = -\frac{\pi \delta}{2} - \frac{5\delta}{4}i, \quad a_2 = -\frac{3\delta}{5}i, \quad a_3 = \frac{\pi \delta}{3} - \frac{5\delta}{6}i, \]
\[ b_1 = -\frac{\pi \delta}{3} + \delta i, \quad b_2 = \frac{7\delta}{5}i, \quad b_3 = \frac{5\delta}{4}i, \]
\[ v_1 = (1.2919 + 2.7109i)\delta^{-1}, \quad v_2 = (1.2517 - 1.5427i)\delta^{-1}, \]
\[ v_3 = (1.7295 - 2.6349i)\delta^{-1}, \]
\[ w_1 = (1.5498 + 2.9260i)\delta^{-1}, \quad w_2 = 0, \quad w_3 = (2.7232 - 4.3927i)\delta^{-1}, \]
\[ |e_1\rangle = \begin{pmatrix} 0.32838 - 0.65641i \\ 0.67162 + 0.65641i \end{pmatrix}, \quad |e_2\rangle = \begin{pmatrix} 0.39430 + 0.38042i \\ -0.38042 - 0.60571i \end{pmatrix}, \]
\[ |e_3\rangle = \begin{pmatrix} 0.049996 - 0.67171i \\ -0.049996 - 0.32829i \end{pmatrix}, \quad \langle f_1 | = (1, 1), \quad \langle f_2 | = (1, i), \quad \langle f_3 | = (i, i), \]
\[ |g_1\rangle = \begin{pmatrix} 1 \\ 1+i \end{pmatrix}, \quad |g_2\rangle = \begin{pmatrix} 1+i \\ 1-i \end{pmatrix}, \quad |g_3\rangle = \begin{pmatrix} 1+i \\ 1+i \end{pmatrix}, \]
\[ (h_1 | = (0.41181 - 0.24610i, 0.41715 - 0.17105i), \]
\[ (h_2 | = (0.092548 - 0.51049i, -0.010488 + 0.40745i), \]
\[ (h_3 | = (0.40879 + 0.029250i, 0.091213 - 0.52925i), \]
\[ M = \delta^{-1} \begin{pmatrix} -0.11465 - 0.32465i & -1.0266 + 0.33804i \\ 1.1526 + 0.52215i & 2.0111 - 0.11410i \end{pmatrix}. \]

**Remark 6.2** Our conventions in this section differ slightly from those in [1, Section 3.1.3], where Hermitian solutions of the sBO equation with \( M = 0 \) are considered. There, bold vectors are always identified with collections of kets, i.e., \( \mathbf{f} \) is identified with \( \{|f_j\rangle\}_{j=1}^N \). We obtain Hermitian solutions of the sncILW equation from (6.11)–(6.12) by setting \( b_j = a_j^*, \quad f_{j,\mu} = e_{j,\mu}^*, \) and \( h_{j,\mu} = g_{j,\mu}^* \) for \( j = 1, \ldots, N \) and \( \mu = 1, \ldots, d \). Note that in this case, the \((2N + d)(2N + d)\) matrix in (6.11) is itself Hermitian.

### 6.3 Solution of constraints: Cases I–III

In this subsection, \( V(z) \) and \( \alpha(z) \) are as in (2.1) and (2.2) for the rational, trigonometric and hyperbolic cases. The idea of the previous subsection can be adapted to Cases I–III. In these cases, the constraint (2.14) is not present, i.e., we may set \( C_1 = 0 \) and \( C_2 = 0 \). This yields an underdetermined system in the variables \( \{h, e, M\} \) in (6.11). To generate a (generically) consistent system, we rearrange (6.11) to

\[
\begin{pmatrix} A^1 & A^2 \\ B^1 & B^2 \end{pmatrix} \begin{pmatrix} h \\ e \end{pmatrix} = \begin{pmatrix} D^1 \mathbf{f} - A^3 M \\ -D^2 \mathbf{g} - B^3 M \end{pmatrix}. \tag{6.14}
\]

In Case III, the submatrices appearing in (6.14) are given by (6.12) with \( \alpha(z) \) in Case III (2.2). In Cases I–II, the submatrices appearing in (6.14) are given by (6.12) with \( \delta \to 0 \) and \( \alpha(z) \) in Cases I–II (2.2). Then, the method described in [1, Section 3.1.3] can be...
straightforwardly applied to the generate admissible initial data for Theorems 2.1 and 2.3.

Acknowledgements We thank Patrick Gérard, Rob Klabbers, Enno Lenzmann, Masatoshi Noumi, Anton Otosson, and Junichi Shiraishi for helpful discussions. We are grateful to Anton Otosson for carefully reading the manuscript and suggesting improvements. The work of B.K.B. was supported by the Olle Engkvist Byggmästare Foundation, Grant 211-0122. E.L. gratefully acknowledges support from the European Research Council, Grant Agreement No. 2020-810451. J.L. acknowledges support from the Ruth and Nils-Erik Stenbäck Foundation, the Swedish Research Council, Grant No. 2021-03877, and the European Research Council, Grant Agreement No. 682537.

Funding Open access funding provided by Royal Institute of Technology.

Data availability Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

Declarations

Conflict of interest On behalf of all authors, the corresponding author states that there is no conflict of interest.

Open Access This article is licensed under a Creative Commons Attribution 4.0 International License, which permits use, sharing, adaptation, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons licence, and indicate if changes were made. The images or other third party material in this article are included in the article’s Creative Commons licence, unless indicated otherwise in a credit line to the material. If material is not included in the article’s Creative Commons licence and your intended use is not permitted by statutory regulation or exceeds the permitted use, you will need to obtain permission directly from the copyright holder. To view a copy of this licence, visit http://creativecommons.org/licenses/by/4.0/.

A Special functions

A.1 Elliptic functions

We recall the standard definitions of the Weierstrass $\zeta$- and $\wp$-functions with half-periods $(\omega_1, \omega_2)$ [31, Section 23.2],

$$\zeta(z) := 1/z + \sum_{(n,m) \in \mathbb{Z}^2 \setminus \{0,0\}} \left( \frac{1}{z - 2n\omega_1 - 2m\omega_2} + \frac{1}{2n\omega_1 + 2m\omega_2} + \frac{z}{(2n\omega_1 + 2m\omega_2)^2} \right)$$

(A.1)

and $\wp(z) = -\partial_z \zeta(z)$. The corresponding modified functions are

$$\zeta_j(z) := \zeta(z) - \frac{\eta_j}{\omega_j} z, \quad \wp_j(z) := -\partial_z \zeta_j(z) = \wp(z) + \frac{\eta_j}{\omega_j} (j = 1, 2) \quad (A.2)$$

where $\eta_j := \zeta(\omega_j)$. Since $\zeta(z + 2\omega_j) = \zeta(z) + 2\eta_j$ for $j = 1, 2$ (see [31, Eq. (23.2.11)]), $\zeta_j(z + 2\omega_j) = \zeta_j(z)$ for $j = 1, 2$. In the main text we have $(\omega_1, \omega_2) = (\ell, i\delta)$.  

Springer
Our definitions imply
\[ \zeta_2(z) = \zeta_1(z) + \gamma_0 z \]  
(A.3)

with \( \gamma_0 = \eta_1/\omega_1 - \eta_2/\omega_2 = (\eta_1 \omega_2 - \eta_2 \omega_1)/(\omega_1 \omega_2) \). Using the well-known identity
\[ \eta_1 \omega_2 - \eta_2 \omega_1 = \frac{1}{2} \pi i \]  
[31, Eq. (23.2.14)], we obtain
\[ \gamma_0 = \frac{\pi i}{2 \omega_1 \omega_2} \]  
(A.4)

which for \((\omega_1, \omega_2) = (\ell, i \delta)\) gives (1.3).

### A.2 Functional identities

We state and outline proofs of several well-known identities involving the special functions \( \zeta_2(z) \), \( \wp_2(z) \), and \( \kappa(z) \) that we use (more detailed proofs of these identities can be found in [18], for example). Throughout this subsection, \( z \) is a complex variable.

First, the functions \( \zeta_2(z) \) and \( \wp_2(z) \) are odd and even, respectively:
\[ \zeta_2(-z) = -\zeta_2(z), \quad \wp_2(-z) = \wp_2(z) \]  
(A.5)

(this is obvious from the definitions). Second,
\[ \wp_2(z) = -\partial_z \zeta_2(z) = \zeta_2(z)^2 - \kappa(z) \]  
(A.6)

(this is implied by the definitions (1.2)). Third,
\[ \zeta_2(a-b)\zeta_2(b-c) + \zeta_2(b-c)\zeta_2(c-a) + \zeta_2(c-a)\zeta_2(a-b) \]
\[ = -\frac{1}{2} \left( \kappa(a-b) + \kappa(b-c) + \kappa(c-a) \right) - \frac{3\eta_2}{2i\delta} \quad (a, b, c \in \mathbb{C}) \]  
(A.7)

(to get this, start with the well-known identity \( (\zeta(x) + \zeta(y) + \zeta(z))^2 = \wp(x) + \wp(y) + \wp(z) \) for \( x + y + z = 0 \), specialize to \( (x, y, z) = (a - b, b - c, c - a) \), use definitions (1.1) and (1.2) to write this as
\[ (\zeta_2(a-b) + \zeta_2(b-c) + \zeta_2(c-a))^2 = \wp_2(a-b) + \wp_2(b-c) + \wp_2(c-a) - \frac{3\eta_2}{i\delta} \]  
(A.8)

and use the identity \( \wp_2(z) = \zeta_2(z)^2 - \kappa(z) \) to obtain (A.7)). Finally, the function \( \zeta_2(z) \) is quasi-periodic with respect to \( 2\ell \) and periodic with respect to \( 2i\delta \):
\[ \zeta_2(z + 2\ell) = \zeta_2(z) + \frac{\pi}{\delta}, \quad \zeta_2(z + 2i\delta) = \zeta_2(z) \]  
(A.9)
(this follows from the well-known identities $\zeta(z + 2\omega_j) = \zeta(z) + 2\eta_j$ for $j = 1, 2$ and the definition of $\varphi_2(z)$ while $\varphi_2(z)$ is $2\ell$- and $2i\delta$-periodic:

$$\varphi_2(z + 2\ell) = \varphi_2(z), \quad \varphi_2(z + 2i\delta) = \varphi_2(z). \quad (A.10)$$

**B Proofs**

**B.1 Proof of Lemma 4.1**

We use the shorthand notation (3.9) and

$$B_j := M + i \sum_{k \neq j}^{N} r_k P_k \alpha(a_j - a_k) \quad (j = 1, \ldots, N) \quad (B.1)$$

to write (4.1) as

$$\dot{a}_j \langle f_j | = 2 \langle f_j | B_j \quad (j = 1, \ldots, N),$$

$$\dot{a}_j \langle e_j | = 2 B_j \langle e_j | \quad (j = N + 1, \ldots, N). \quad (B.2)$$

Moreover, (3.2) becomes

$$\dot{P}_j = -i \sum_{j=1}^{N} (1 + r_j r_k) [P_j, P_k] V(a_j - a_k); \quad (B.3)$$

(2.9) and (2.11) become

$$\langle \dot{e}_j | = i \sum_{k \neq j}^{N} (1 + r_j r_k) P_k \langle e_j | V(a_j - a_k), \quad (j = 1, \ldots, N); \quad (B.4)$$

$$\langle \dot{f}_j | = -i \sum_{k \neq j}^{N} (1 + r_j r_k) \langle f_j | P_k V(a_j - a_k).$$

(2.17) becomes

$$\langle e_j | f_j | = 1 \quad (j = 1, \ldots, N); \quad (B.5)$$

and (2.16) becomes

$$\dot{M} = -\frac{1}{4} \sum_{k=1}^{N} \sum_{l \neq k}^{N} (r_k + r_l) [P_k, P_l] \alpha'(a_k - a_l). \quad (B.6)$$
By right-multiplying the first set of equations in (B.2) by \(|e_j|\) and left-multiplying the second set of equations in (B.2) by \(|f_j|\), we obtain

\[
\dot{a}_j = 2\langle f_j|B_j|e_j \rangle \quad (j = 1, \ldots, N).
\] (B.7)

By the Picard-Lindelöf theorem, the system of equations consisting of (B.4), (B.6), and (B.7) with the given initial data has a unique local solution. This solution may be extended as long as (i) no solution variable tends to infinity and (ii) (2.20) holds (see, e.g., [30, Corollary 3.2]). Moreover, this maximal solution is unique on \([0, \tau_{\text{max}}]\) (see, e.g., [30, Theorem 8.1]).

Let \(M\) and \(\{a_j, |e_j|, (f_j|)_{j=1}^N\}\) be this maximal solution. It follows from (B.7) that

\[
\dot{a}_j|f_j| = 2\langle f_j|B_jP_j \rangle \quad (j = 1, \ldots, N),
\]

\[
\dot{a}_j|e_j| = 2P_jB_j|e_j| \quad (j = N + 1, \ldots, N).
\] (B.8)

holds on \([0, \tau_{\text{max}}]\). The overdetermined system of equations for \(\{a_j\}_{j=1}^N\) (B.2) will also be satisfied on \([0, \tau_{\text{max}}]\) if the difference between the right-hand sides of (B.2) and (B.8) vanish on \([0, \tau_{\text{max}}]\). These differences are given by

\[
\langle F_j \rangle := \langle f_j|B_j - (f_j|B_j|e_j)\rangle (f_j|B_j(1 - P_j) \quad (j = 1, \ldots, N),
\]

\[
|E_j \rangle := B_j|e_j| - |e_j\rangle (f_j|B_j|e_j) = (1 - P_j)B_j|e_j| \quad (j = N + 1, \ldots, N).
\] (B.9)

We will show that the time evolution of the quantities \(\{\langle F_j \rangle\}_{j=1}^N\) and \(\{|E_{N+j}\rangle\}_{j=1}^N\) is determined by a linear, homogeneous (in both \(\{\langle F_j \rangle\}_{j=1}^N\) and \(\{|E_{N+j}\rangle\}_{j=1}^N\)) system of ordinary differential equations. The precise form of this system is not needed to establish our result, and so we introduce an equivalence relation \(\simeq\) between two expressions that differ only by terms linear in \(\{\langle F_j \rangle\}_{j=1}^N\) and \(\{|E_{N+j}\rangle\}_{j=1}^N\) with regular coefficients (the regularity of all such coefficients is guaranteed by conditions (i), (ii) in the discussion of maximal solutions above). Thus we only need to show that

\[
\dot{\langle F_j \rangle} \simeq 0 \quad (j = 1, \ldots, N),
\]

\[
|\dot{E}_j \rangle \simeq 0 \quad (j = N + 1, \ldots, N).
\] (B.10)

However, in the system of equations given by (2.9), (2.11), (2.16), (2.17), (2.21), and (4.1), the variables \(\{a_j, |e_j|, (f_j|)_{j=1}^N\) and \(\{b_j, (h_j|, |g_j\rangle\}_{j=1}^N\) can be swapped by Hermitian conjugation and consequently, \(\{\langle F_j \rangle\}_{j=1}^N\) and \(\{|E_{N+j}\rangle\}_{j=1}^N\) can be interchanged using this same symmetry. For this reason, it suffices to show that the first set of equations in (B.10) holds.

We differentiate the first equation in (B.9) with respect to time, which gives (for notational simplicity, we suppress the \(j\)-dependence of the quantities \(\{C_1, C_2, C_3\}\)

\[
\dot{\langle F_j \rangle} = \langle C_1 \rangle + \langle C_2 \rangle + \langle C_3 \rangle,
\] (B.11)
where

\[
\langle C_1 \rangle = \langle \dot{f}_j | B_j (1 - P_j) \rangle,
\]

\[
\langle C_2 \rangle = -\langle f_j | B_j \dot{P}_j \rangle,
\]

\[
\langle C_3 \rangle = \langle f_j | \dot{B}_j (1 - P_j) \rangle.
\]

We compute each of these quantities in turn (note that \( r_j = 1 \) in what follows).

By inserting (B.4) into \( \langle C_1 \rangle \) in (B.12), we find

\[
\langle C_1 \rangle = -i \sum_{k \neq j}^N (1 + r_k) \langle f_j | P_k B_j (1 - P_j) V (a_j - a_k) \rangle
\]

and, similarly, by inserting (B.3) into \( \langle C_2 \rangle \) in (B.12), we compute

\[
\langle C_2 \rangle = i \sum_{k \neq j}^N (1 + r_k) \langle f_j | B_j [1 - P_j, P_k] V (a_j - a_k) \rangle
\]

\[
= -i \sum_{k \neq j}^N (1 + r_k) \langle f_j | B_j (1 - P_j) V (a_j - a_k) \rangle + i \sum_{k \neq j}^N (1 + r_k) \langle f_j | B_j P_k (1 - P_j) V (a_j - a_k) \rangle.
\]

Then, from (B.13) and (B.14), it follows that

\[
\langle C_1 \rangle + \langle C_2 \rangle \simeq -i \sum_{k \neq j}^N (1 + r_k) \langle f_j | [P_k, B_j] (1 - P_j) V (a_j - a_k) \rangle.
\]

To compute \( \langle C_3 \rangle \) in (B.12), we differentiate the expression for \( B_j \) in (B.1) to write

\[
\langle C_3 \rangle = \langle C_{3,1} \rangle + \langle C_{3,2} \rangle + \langle C_{3,3} \rangle,
\]

where

\[
\langle C_{3,1} \rangle := \langle f_j | \dot{M} (1 - P_j) \rangle,
\]

\[
\langle C_{3,2} \rangle := i \sum_{k \neq j}^N r_k \langle f_j | \dot{P}_k (1 - P_j) \alpha (a_j - a_k) \rangle,
\]

\[
\langle C_{3,3} \rangle := -i \sum_{k \neq j}^N r_k \langle f_j | P_k (1 - P_j) (\dot{a}_j - \dot{a}_k) V (a_j - a_k) \rangle.
\]
Using (B.6) in \(\langle C_{3,1}\rangle\) in (B.17) gives

\[
\langle C_{3,1}\rangle = -\frac{1}{4} \sum_{k=1}^{N} \sum_{l \neq k}^{N} (r_k + r_l) \langle f_j | [P_k, P_l] | (1 - P_j) \rangle \alpha'(a_k - a_l). \tag{B.18}
\]

Next, by inserting (B.3) into \(\langle C_{3,2}\rangle\) in (B.17), we compute

\[
\langle C_{3,2}\rangle = \sum_{k \neq j}^{N} \sum_{l \neq k}^{N} r_k (1 + r_k r_l) \langle f_j | [P_k, P_l] | (1 - P_j) \alpha(a_j - a_k) V(a_k - a_l)

= -\sum_{k \neq j}^{N} (1 + r_k) \langle f_j | [P_j, P_k] | (1 - P_j) \alpha(a_j - a_k) V(a_j - a_k)

+ \sum_{k \neq j}^{N} \sum_{l \neq j, k}^{N} (r_k + r_l) \langle f_j | [P_k, P_l] | (1 - P_j) \alpha(a_j - a_k) V(a_k - a_l)

= -\sum_{k \neq j}^{N} (1 + r_k) \langle f_j | P_j P_k | (1 - P_j) \alpha(a_j - a_k) V(a_j - a_k)

+ \sum_{k \neq j}^{N} \sum_{l \neq j, k}^{N} (r_k + r_l) \langle f_j | [P_k, P_l] | (1 - P_j) \alpha(a_j - a_k) V(a_k - a_l). \tag{B.19}
\]

We rewrite the double sum as follows

\[
\sum_{k \neq j}^{N} \sum_{l \neq j, k}^{N} (r_k + r_l) \langle f_j | [P_k, P_l] | (1 - P_j) \alpha(a_j - a_k) V(a_k - a_l)

= \frac{1}{2} \sum_{k \neq j}^{N} \sum_{l \neq j, k}^{N} (r_k + r_l) \langle f_j | [P_k, P_l] | (1 - P_j) \rangle \left( \alpha(a_j - a_k) - \alpha(a_j - a_l) \right) V(a_k - a_l)

= \sum_{k \neq j}^{N} \sum_{l \neq j, k}^{N} r_l \langle f_j | [P_k, P_l] | (1 - P_j) \rangle \left( \alpha(a_j - a_k) - \alpha(a_j - a_l) \right) V(a_k - a_l); \tag{B.20}
\]
using also that $\langle f_j | P_j \rangle = \langle f_j \rangle$ this gives

$$
\langle C_{3,2} \rangle = - \sum_{k \neq j}^N (1 + r_k) \langle f_j | P_k (1 - P_j) \alpha (a_j - a_k) V (a_j - a_k) \\
+ \sum_{k \neq j}^N \sum_{l \neq j, k} r_l \langle f_j | [P_k, P_l] (1 - P_j) \alpha (a_j - a_k) - \alpha (a_j - a_l) V (a_k - a_l) \rangle.
$$

(B.21)

To compute $\langle C_{3,3} \rangle$, we use the following relations, which follow from (B.7) and (B.9),

$$
\dot{a}_j \langle f_j \rangle = - 2 \langle F_j \rangle + 2 \langle f_j | B_j \rangle \simeq 2 \langle f_j | B_j \rangle \quad (j = 1, \ldots, N), \\
\dot{a}_j \langle e_j \rangle = - 2 \langle E_j \rangle + 2 B_j \langle e_j \rangle \simeq 2 B_j \langle e_j \rangle \quad (j = N + 1, \ldots, N')
$$

(B.22)

Note that (B.22) implies

$$
\dot{a}_j P_j \simeq (1 + r_j) P_j B_j + (1 - r_j) B_j P_j \quad (j = 1, \ldots, N).
$$

(B.23)

Using (B.22) and (B.23) in $\langle C_{3,3} \rangle$ in (B.17), we compute

$$
\langle C_{3,3} \rangle \simeq - 2i \sum_{k \neq j}^N r_k \langle f_j | B_j P_k (1 - P_j) V (a_j - a_k) \\
+ i \sum_{k \neq j}^N \langle f_j | ((1 + r_k) P_k - (1 - r_k) B_k P_k) (1 - P_j) V (a_j - a_k) \\
= - 2i \sum_{k \neq j}^N r_k \langle f_j | (B_j - B_k) P_k (1 - P_j) V (a_j - a_k) \\
+ i \sum_{k \neq j}^N (1 + r_k) \langle f_j | [P_k, B_k] (1 - P_j) V (a_j - a_k).
$$

(B.24)
By combining (B.15), (B.21), and (B.24), we arrive at

\[
\langle C_1 \rangle + \langle C_2 \rangle + \langle C_{3,2} \rangle + \langle C_{3,3} \rangle \\
\simeq - \sum_{k \neq j}^N (1 + r_k) \langle f_j | P_k (1 - P_j) \alpha (a_j - a_k) \rangle V (a_j - a_k) \\
+ \sum_{k \neq j}^N \sum_{l \neq j, k}^N r_l \langle f_j | [P_k, P_l] (1 - P_j) (\alpha (a_j - a_k) - \alpha (a_j - a_l)) \rangle V (a_k - a_l) \\
- 2i \sum_{k \neq j}^N r_k \langle f_j | (B_j - B_k) P_k (1 - P_j) \rangle V (a_j - a_k) \\
- i \sum_{k \neq j}^N (1 + r_k) \langle f_j | [P_k, B_j - B_k] (1 - P_j) \rangle V (a_j - a_k). \tag{B.25}
\]

To proceed, we compute a convenient expression for \(B_j - B_k\) directly from (B.1),

\[
B_j - B_k = i \sum_{l \neq j}^N r_l P_l \alpha (a_j - a_l) - i \sum_{l \neq k}^N r_l P_l \alpha (a_k - a_l) \\
= i (r_k P_k + P_j) \alpha (a_j - a_k) + i \sum_{l \neq j, k}^N r_l P_l (\alpha (a_j - a_l) - \alpha (a_k - a_l)). \tag{B.26}
\]

The final two lines of (B.25) can be written, using (B.26) and the relations \(\langle f_j | P_j = \langle f_j \rangle\) and \(P_j (1 - P_j) = 0\), as

\[
2 \sum_{k \neq j}^N (1 + r_k) \langle f_j | P_k (1 - P_j) \alpha (a_j - a_k) \rangle V (a_j - a_k) \\
+ 2 \sum_{k \neq j}^N \sum_{l \neq j, k}^N r_k r_l \langle f_j | P_l P_k (1 - P_j) (\alpha (a_j - a_l) - \alpha (a_k - a_l)) \rangle V (a_j - a_k) \\
- \sum_{k \neq j}^N (1 + r_k) \langle f_j | P_k (1 - P_j) \alpha (a_j - a_k) \rangle V (a_j - a_k) \\
+ \sum_{k \neq j}^N \sum_{l \neq j, k}^N (1 + r_k) r_l \langle f_j | [P_k, P_l] (1 - P_j) (\alpha (a_j - a_l) - \alpha (a_k - a_l)) \rangle V (a_j - a_k); \tag{B.27}
\]
inserting this into (B.25) leads to cancellation of diagonal terms,
\[ \langle C_1 \rangle + \langle C_2 \rangle + \langle C_{3,2} \rangle + \langle C_{3,3} \rangle \]
\[ \simeq \sum_{k \neq j} \sum_{l \neq j, k} \frac{r_l}{\langle f_j \rangle \langle P_k, P_l \rangle} (1 - P_j) (\alpha(a_j - a_k) - \alpha(a_j - a_l)) V(a_k - a_l) \]
\[ + \sum_{k \neq j} \sum_{l \neq j, k} (1 + r_k) r_l (\langle f_j \rangle \langle P_k P_l \rangle (1 - P_j) (\alpha(a_j - a_l) - \alpha(a_k - a_l)) V(a_j - a_k) \]
\[ - \sum_{k \neq j} \sum_{l \neq j, k} (1 - r_k) r_l (\langle f_j \rangle \langle P_l P_k \rangle (1 - P_j) (\alpha(a_j - a_l) - \alpha(a_k - a_l)) V(a_j - a_k) \]
\[ = \sum_{k \neq j} \sum_{l \neq j, k} \frac{r_l}{\langle f_j \rangle \langle P_k, P_l \rangle} (1 - P_j) \]
\[ \times ((\alpha(a_j - a_k) - \alpha(a_j - a_l)) V(a_k - a_l) + (\alpha(a_j - a_l) - \alpha(a_k - a_l)) V(a_j - a_k)) \]
\[ + \sum_{k \neq j} \sum_{l \neq j, k} r_k r_l (\langle f_j \rangle \langle P_k, P_l \rangle (1 - P_j) (\alpha(a_j - a_l) - \alpha(a_k - a_l)) V(a_j - a_k). \] (B.28)

Using the identity
\[ (\alpha(a_j - a_l) - \alpha(a_k - a_l)) V(a_j - a_k) = - (\alpha(a_j - a_k) - \alpha(a_j - a_l)) V(a_k - a_l) \]
\[ - \frac{1}{2} \left( \dot{\varphi}(a_j - a_k) - \dot{\varphi}(a_k - a_l) \right), \]
(B.29)

which can be obtained by differentiating (A.7) with respect to \( b \) and setting \( a = a_j \), \( b = a_k \), and \( c = a_l \), in (B.28) gives
\[ \langle C_1 \rangle + \langle C_2 \rangle + \langle C_{3,2} \rangle + \langle C_{3,3} \rangle \]
\[ \simeq - \frac{1}{2} \sum_{k \neq j} \sum_{l \neq j, k} r_l (\langle f_j \rangle \langle P_k, P_l \rangle (1 - P_j) (\dot{\varphi}(a_j - a_k) - \dot{\varphi}(a_k - a_l)) \]
\[ - \sum_{k \neq j} \sum_{l \neq j, k} r_k r_l (\langle f_j \rangle \langle P_k P_l \rangle (1 - P_j) (\alpha(a_j - a_k) - \alpha(a_j - a_l)) V(a_k - a_l) \]
\[ - \frac{1}{2} \sum_{k \neq j} \sum_{l \neq j, k} r_k r_l (\langle f_j \rangle \langle P_l P_k \rangle (1 - P_j) (\dot{\varphi}(a_j - a_k) - \dot{\varphi}(a_k - a_l)). \] (B.30)

The second double sum in (B.30) vanishes by symmetry, as does the part of the third double sum proportional to \( \dot{\varphi}(a_k - a_l) \). We use
\[ \sum_{l \neq j, k} r_l P_l = -P_j - r_k P_k. \] (B.31)
a consequence of (3.28), to simplify what remains. This allows us to compute

\[
\langle C_1 \rangle + \langle C_2 \rangle + \langle C_{3,2} \rangle + \langle C_{3,3} \rangle \\
\simeq \frac{1}{2} \sum_{k \neq j}^{N} (f_j | [P_k, P_j + r_k P_k] | (1 - P_j) \zeta'(a_j - a_k) \\
+ \frac{1}{2} \sum_{k \neq j}^{N} \sum_{l \neq j,k}^{N} r_l (f_j | [P_k, P_l] | (1 - P_j) \zeta'(a_k - a_l) \\
+ \frac{1}{2} \sum_{k \neq j}^{N} r_k (f_j | [P_k, P_j + r_k P_k] | (1 - P_j) \zeta'(a_j - a_k) \\
= -\frac{1}{2} \sum_{k \neq j}^{N} (f_j | P_k | (1 - P_j) \zeta'(a_j - a_k) \\
+ \frac{1}{2} \sum_{k \neq j}^{N} \sum_{l \neq j,k}^{N} r_l (f_j | [P_k, P_l] | (1 - P_j) \zeta'(a_k - a_l) \\
+ \frac{1}{2} \sum_{k \neq j}^{N} (2 + r_k) (f_j | P_k | (1 - P_j) \zeta'(a_j - a_k) \\
= \frac{1}{2} \sum_{k \neq j}^{N} (1 + r_k) (f_j | P_k | (1 - P_j) \zeta'(a_j - a_k) \\
+ \frac{1}{2} \sum_{k \neq j}^{N} \sum_{l \neq j,k}^{N} r_l (f_j | [P_k, P_l] | (1 - P_j) \zeta'(a_k - a_l)).
\]

(B.32)

Hence, using that \( r_j = 1 \), we get

\[
\langle C_1 \rangle + \langle C_2 \rangle + \langle C_{3,2} \rangle + \langle C_{3,3} \rangle \\
\simeq \frac{1}{2} \sum_{k \neq j}^{N} (r_j + r_k) (f_j | P_k | (1 - P_j) \zeta'(a_j - a_k) \\
+ \frac{1}{2} \sum_{k \neq j}^{N} \sum_{l \neq j,k}^{N} r_l (f_j | [P_k, P_l] | (1 - P_j) \zeta'(a_k - a_l) \\
= \frac{1}{2} \sum_{k=1}^{N} \sum_{l \neq k}^{N} r_l (f_j | [P_k, P_l] | (1 - P_j) \zeta'(a_k - a_l) \\
= \frac{1}{4} \sum_{k=1}^{N} \sum_{l \neq k}^{N} (r_k + r_l) (f_j | [P_k, P_l] | (1 - P_j) \zeta'(a_k - a_l)).
\]

(B.33)
and from (B.18) it follows that

\[ \langle C_1 \rangle + \langle C_2 \rangle + \langle C_{3,1} \rangle + \langle C_{3,2} \rangle + \langle C_{3,3} \rangle = \langle C_1 \rangle + \langle C_2 \rangle + \langle C_3 \rangle \simeq 0. \quad (B.34) \]

We have thus shown that (B.10) holds; a unique solution to the initial value problem consisting of (B.10) with initial conditions \( \langle F_j(0) \rangle = 0, |E_{N+j}(0)| = 0, j = 1, \ldots, N \) is given by

\[
\langle F_j(t) \rangle = 0, |E_{N+j}(t)| = 0, j = 1, \ldots, N, \quad \text{for } t \in [0, \tau_{\text{max}}). \]

It follows that \( M \) and \( \{a_j, |e_j\rangle, \langle f_j|\}_{j=1}^N \) uniquely solves the initial value problem of Lemma 4.1 on \([0, \tau_{\text{max}}).

B.2 Proof of Lemma 4.2

In the system of equations given by (2.8)–(2.11), (2.16), (2.17), (2.21), and (4.1), the variables \( \{a_j, |e_j\rangle, \langle f_j|\}_{j=1}^N \) and \( \{b_j, \langle h_j|, |g_j\rangle\}_{j=1}^N \) can be swapped by Hermitian conjugation. Due to this symmetry, it suffices to verify the claim for the first set of variables, i.e., it is enough to show that (2.8) follows from (2.9), (2.11), (2.16), and (4.1) subject to (2.17) and (2.21).

By differentiating the first set of equations in (B.2) with respect to time, we obtain

\[ \ddot{a}_j \langle f_j| = \langle f_j| (2B_j - \dot{a}_j) + 2 \langle f_j| \dot{B}_j \quad (B.35) \]

where, here and below in this section, \( j = 1, \ldots, N \) (note that \( r_j = +1 \)). Using (B.1), (B.2), and (B.4) and by nearly-identical calculations as in the proof of [1, Proposition 2.1], (B.35) may be written as,

\[
\langle f_j| (\ddot{a}_j - 2\dot{M}) = \sum_{k \neq j}^N \sum_{l \neq j, k}^N (r_k + r_l) \langle f_j| [P_k, P_l] (\alpha(a_j - a_k) - \alpha(a_j - a_l)) V(a_k - a_l)
\]

\[ + 4 \sum_{k \neq j}^N (1 + r_k) \langle f_j| P_k P_j \alpha(a_j - a_k) V(a_j - a_k)
\]

\[ + 2 \sum_{k \neq j}^N \sum_{l \neq j, k}^N \langle f_j| (r_k r_l [P_k, P_l] + r_l [P_k, P_l]) (\alpha(a_j - a_l) - \alpha(a_k - a_l)) \]

\[ \times V(a_j - a_k). \quad (B.36) \]

We will rewrite (B.36) using the identities

\[ \alpha(z)V(z) = -\frac{1}{2}(V'(z) + \alpha'(z)), \quad (B.37) \]

\footnote{Using [1, Eq. (2.31)] the right-hand side of of (B.36) is seen to match that of [1, Eq. (2.29)]. The new addition on the left-hand side, the term proportional to \( \dot{M} \), comes from differentiating (B.1) with respect to time and substituting the resulting expression into (B.35).}
which can be obtained by differentiating (A.6) with respect to \( z \), and (B.29). By inserting (B.37) and (B.29) into (B.36) and rearranging, we obtain

\[
\ddot{a}_j\langle f_j | = 2\langle f_j | \dot{M} - 2 \sum_{k \neq j}^N (1 + r_k)\langle f_j | P_k P_j V'(a_j - a_k) \\
+ \sum_{k \neq j}^N \sum_{l \neq k}^N (r_k - r_l)\langle f_j | [P_k, P_l](\alpha(a_j - a_k) - \alpha(a_j - a_l))V(a_k - a_l) \\
- 2 \sum_{k \neq j}^N \sum_{l \neq j, k}^N r_k r_l \langle f_j | [P_k, P_l](\alpha(a_j - a_k) - \alpha(a_j - a_l))V(a_k - a_l) \\
- 2 \sum_{k \neq j}^N (1 + r_k)\langle f_j | P_k P_j \varphi'(a_j - a_k) \\
- \sum_{k \neq j}^N \sum_{l \neq j, k}^N (\langle f_j | (r_k r_l [P_k, P_l] + r_l [P_k, P_l]) \varphi'(a_j - a_k) \\
+ \sum_{k \neq j}^N \sum_{l \neq j, k}^N r_k r_l \langle f_j | [P_k, P_l] \varphi'(a_k - a_l) + \sum_{k \neq j}^N \sum_{l \neq j, k}^N r_l \langle f_j | [P_k, P_l] \varphi'(a_k - a_l). \\
\]

(B.38)

Since \( V(z) \) is even, the double sums in the second and third lines of (B.38) each vanish by symmetry and since \( \varphi'(z) \) is odd, the first double sum in the final line of (B.38) vanishes by symmetry.

To simplify further, we recall the constraint (B.31), which inserted into the double sum in the fourth line of (B.38) gives

\[
\sum_{k \neq j}^N \sum_{l \neq j, k}^N (\langle f_j | (r_k r_l [P_k, P_l] + r_l [P_k, P_l]) \varphi'(a_j - a_k) \\
= \sum_{k \neq j}^N (\langle f_j | (r_k [P_k, P_j] + r_k P_k) + [P_k, P_j + r_k P_k]) \varphi'(a_j - a_k) \\
= \sum_{k \neq j}^N (\langle f_j | (2P_k^2 + r_k [P_j, P_k] - [P_j, P_k]) \varphi'(a_j - a_k) \\
= \sum_{k \neq j}^N (1 + r_k)\langle f_j | [P_j, P_k] \varphi'(a_j - a_k). \\
\]

(B.39)
where we have used $P_k^2 = P_k$ and $\langle f_j | P_j = \langle f_j |$ in the final step (both are consequences of (B.5)). Hence, inserting (B.39) into (B.38) and simplifying, we have

$$\ddot{a}_j \langle f_j | = 2 \langle f_j | \dot{M} - 2 \sum_{k \neq j}^N (1 + r_k) \langle f_j | P_k P_j V'(a_j - a_k)$$

$$+ \sum_{k \neq j}^N (1 + r_k) \langle f_j | [P_j, P_k] \chi'(a_j - a_k) + \sum_{k \neq j}^N \sum_{l \neq j, k}^N r_l \langle f_j | [P_k, P_l] \chi'(a_k - a_l).$$

(B.40)

By symmetrizing the remaining double sum and using the fact that $\chi'(z)$ is an odd function, we see that

$$\sum_{k \neq j}^N \sum_{l \neq j, k}^N r_l \langle f_j | [P_k, P_l] \chi'(a_k - a_l) = \frac{1}{2} \sum_{k \neq j}^N \sum_{l \neq j, k}^N (r_k + r_l) \langle f_j | [P_k, P_l] \chi'(a_k - a_l)$$

$$= - \sum_{k \neq j}^N (1 + r_k) \langle f_j | [P_j, P_k] \chi'(a_j - a_k) + \frac{1}{2} \sum_{k = 1}^N \sum_{l \neq k}^N (r_k + r_l) \langle f_j | [P_k, P_l] \chi'(a_k - a_l),$$

(B.41)

where we have used $r_j = +1$. We arrive at

$$\ddot{a}_j \langle f_j | = - 2 \sum_{k \neq j}^N (1 + r_k) \langle f_j | P_k P_j V'(a_j - a_k)$$

$$+ 2 \langle f_j | \dot{M} + \frac{1}{2} \sum_{k = 1}^N \sum_{l \neq k}^N (r_k + r_l) \langle f_j | [P_k, P_l] \chi'(a_k - a_l).$$

(B.42)

The second line vanishes after inserting (B.6). Using that $(1 + r_k) = 2$ for $k = 1, \ldots, N$ and 0 otherwise, and multiplying (B.42) on the right by $|e_j\rangle$, we obtain (2.8).

References

1. Berntson, B.K., Langmann, E., Lenells, J.: Spin generalizations of the Benjamin–Ono equation. Lett. Math. Phys. 112, 50 (2022)
2. Gibbons, J., Hermsen, T.: A generalisation of the Calogero-Moser system. Phys. D 11(3), 337–348 (1984)
3. Wojciechowski, S.: An integrable marriage of the Euler equations with the Calogero-Moser system. Phys. Lett. A 111(3), 101–103 (1985)
4. Krichever, I., Babelon, O., Billey, E., Talon, M.: Spin generalization of the Calogero-Moser system and the matrix KP equation. In: Novikov, S.P. (ed.) Topics in topology and mathematical physics, pp. 83–120. American Mathematical Society (1995)
5. Benjamin, T.B.: Internal waves of permanent form in fluids of great depth. J. Fluid Mech. 29, 559 (1967)
6. Ono, H.: Algebraic Solitary Waves in Stratified Fluids. J. Phys. Soc. Jpn. 39, 1082 (1975)
7. Joseph, R.I.: Solitary waves in a finite depth fluid. J. Phys. A Math. Gen. 10(12), L225–L227 (1977)
8. Kubota, T., Ko, D.R.S., Dobbs, L.D.: Weakly-nonlinear, long internal gravity waves in stratified fluids of finite depth. J. Hydronaut. 12(4), 157–165 (1978)
9. Abanov, A.G., Bettelheim, E., Wiegmann, P.: Integrable hydrodynamics of Calogero-Sutherland model: bidirectional Benjamin–Ono equation. J. Phys. A Math. Theor. 42, 135201 (2009)
10. Gérard, P., Grellier, S.: The cubic Szegö equation. Ann. Sci. Ec. Norm. Supérieur 43(5), 761–810 (2010)
11. Zhou, T., Stone, M.: Solitons in a continuous classical Haldane-Shastry spin chain. Phys. Lett. A 379(43), 2817–2825 (2015)
12. Lenzmann, E., Schikorra, A.: On energy-critical half-wave maps into $S^2$. Invent. Math. 213(1), 1–82 (2018)
13. Gormley, B., Ferapontov, E.V., Novikov, V.S., Pavlov, M.V.: Integrable systems of the intermediate long wave type in 2+1 dimensions. Phys. D 435, 133310 (2022)
14. Gérard, P.: The Lax pair structure for the spin Benjamin–Ono equation. Adv. Cont. Discr. Mod. 2023(21), (2023)
15. Rui, W., Cheng, J.: Nonlocal integrable equations from the mKP hierarchy. Anal. Math. Phys. 12(6), 134 (2022)
16. Takasaki, K.: Generalized ILW hierarchy: solutions and limit to extended lattice GD hierarchy. J. Phys. A: Math. Theor. 56, 165201 (2023)
17. Berntson, B.K., Langmann, E., Lenells, J.: Non-chiral intermediate long wave equation and inter-edge effects in narrow quantum Hall systems. Phys. Rev. B 102, 155308 (2020)
18. Berntson, B.K., Langmann, E., Lenells, J.: On the non-chiral intermediate long wave equation: II periodic case. Nonlinearity 35(8), 4517–4548 (2022)
19. Wojciechowski, S.: The analogue of the Bäcklund transformation for integrable many-body systems. J. Phys. A Math. Theor. 15(12), L653–L657 (1982)
20. Gibbons, J., Hermsen, T., Wojciechowski, S.: A Bäcklund transformation for a generalised Calogero-Moser system. Phys. Lett. A 94, 251–253 (1983)
21. Dirac, P.A.M.: A new notation for quantum mechanics. Math. Proc. Camb. Philos. Soc. 35(3), 416–418 (1939)
22. Olshanetsky, M.A., Perelomov, A.M.: Classical integrable finite-dimensional systems related to Lie algebras. Phys. Rep. 71(5), 313–400 (1981)
23. Berntson, B.K., Langmann, E., Lenells, J.: On the non-chiral intermediate long wave equation. Nonlinearity 35(8), 4549–4584 (2022)
24. Ottosson, A.: A unified view of a family of soliton equations related to spin Calogero-Moser systems. Master’s thesis, KTH Royal Institute of Technology, (2022)
25. Berntson, B.K., Klabbers, R.: Periodic solutions of the non-chiral intermediate Heisenberg ferromagnet equation described by elliptic spin-pole Calogero-Moser dynamics. Nonlinearity 36, 3068 (2023)
26. Berntson, B.K.: Consistency of the Bäcklund transformation for the spin Calogero-Moser system. Math. Phys. Anal. Geom. 26(12), (2023)
27. Gérard, P., Kappeler, T.: On the integrability of the Benjamin–Ono equation on the Torus. Comm. Pure Appl. Math. 74(8), 1685–1747 (2021)
28. Gérard, P., Lenzmann, E.: The Calogero–Moser Derivative Nonlinear Schrödinger Equation. arXiv:2208.0415 (math.AP), (2022)
29. Matsuno, Y.: Calogero-Moser-Sutherland dynamical systems associated with nonlocal nonlinear Schrödinger equation for envelope waves. J. Phys. Soc. Jpn. 71(6), 1415–1418 (2002)
30. Hartman, P.: Ordinary differential equations. Birkhäuser, Massachusetts, Boston (1982)
31. Olver, F.W.J., Olde Daalhuis, A.B., Lozier, D.W., Schneider, B.I., Boisvert, R.F., Clark, C.W., Miller, B.R., Saunders, B.V., Cohl, H.S., McClain, M.A.: NIST Digital Library of Mathematical Functions. http://dlmf.nist.gov/ (2020). Accessed 15 Mar 2020

Publisher’s Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.