CHARACTERIZATION OF RECTIFIABLE MEASURES IN TERMS OF 
\(\alpha\)-NUMBERS

JONAS AZZAM, XAVIER TOLSA, AND TATIANA TORO

ABSTRACT. We characterize Radon measures \(\mu\) in \(\mathbb{R}^n\) that are \(d\)-rectifiable in the sense that their supports are covered up to \(\mu\)-measure zero by countably many \(d\)-dimensional Lipschitz graphs and \(\mu \ll \mathcal{H}^d\). The characterization is in terms of a Jones function involving the so-called \(\alpha\)-numbers. This answers a question left open in a former work by Azzam, David, and Toro.

1. INTRODUCTION

A Borel measure \(\mu\) in \(\mathbb{R}^n\) is called \(d\)-rectifiable if there are countably many Lipschitz images \(\Gamma_i\) of \(\mathbb{R}^d\) such that

\[(1.1) \quad \mu\left(\mathbb{R}^n \backslash \bigcup_i \Gamma_i\right) = 0\]

and additionally \(\mu \ll \mathcal{H}^d\), where \(\mathcal{H}^d\) denotes the \(d\)-dimensional Hausdorff measure. A set \(E\) is called \(d\)-rectifiable if \(\mathcal{H}^d|_E\) is a \(d\)-rectifiable measure.

The goal of this paper is to give sufficient conditions for the \(d\)-rectifiability of a Borel measure \(\mu\) in the above sense. Such conditions are desirable since rectifiable measures and sets enjoy many useful properties and are ubiquitous in analysis. Characterizations of rectifiability usually arise from the study of certain properties that are trivial for the Lebesgue measure in Euclidean space. These properties do not necessarily hold for rectifiable sets and measures except in an approximate way. For example, the property that a measure \(\mu\) satisfies \(\mu(B(x, r)) = r^d\) for all \(x \in \text{supp } \mu\) and \(r > 0\) is trivially satisfied by Lebesgue measure, though not for general rectifiable measures. However, the weaker property that \(\lim_{r \to 0} \frac{\mu(B(x, r))}{r^d} \in (0, \infty)\) for \(\mu\)-almost every \(x\) is satisfied by rectifiable measures, and this also implies \(d\)-rectifiability by the amazing work of Preiss [Pre87]. See also [TT15] and [Tol17a] for related characterizations in terms of densities.

In this paper, we will study \(d\)-rectifiability from the perspective of how well a measure resembles \(d\)-dimensional Lebesgue measure at various scales and locations. It is a classical result that if \(\mu\) is \(d\)-rectifiable, then for \(\mu\)-almost every \(x \in \mathbb{R}^n\), the measures \(\mu_{x,r}\) defined by

\[\mu_{x,r}(A) = r^{-d} \mu(rA + x)\]

converge weakly to a constant times Lebesgue measure restricted to a \(d\)-dimensional plane (see [DL08] and [Pre87]). In particular, the distance between these rescaled measures and...
the class of $d$-dimensionally "flat" measures tends to zero. We can make this distance more precise as follows. For measures $\mu$ and $\nu$ and an open ball $B$ we define
\[ F_B(\sigma, \nu) := \sup \left\{ \left| \int \phi \, d\sigma - \int \phi \, d\nu \right| : \phi \in \text{Lip}_1(B) \right\}, \]
where
\[ \text{Lip}_1(B) = \{ \phi : \text{Lip}(\phi) \leq 1, \supp \sigma \subset B \} \]
and $\text{Lip}(\phi)$ stands for the Lipschitz constant of $\phi$.

It is easy to check that this is indeed a distance in the space of finite Borel measures supported in the open ball $B$. See [Chapter 14, Ma] for other properties of this distance. In fact, this is a variant of the well-known Wasserstein 1-distance from mass transport theory.

For a measure $\mu$ and $d \in \mathbb{N}$, we define
\[ \alpha_d^d(B) := \frac{1}{r_B \mu(B)} \inf_{c \geq 0, L} F_B(\mu, c \mathcal{H}^d|_L), \]
where the infimum is taken over all $c \geq 0$ and all $d$-dimensional planes $L$. Also, if $\mu(B^\circ) > 0$, we denote by $c_B$ and $L_B$ a constant and a plane such that, if we set
\[ \mathcal{L}_B := c_B \mathcal{H}^d|_{L_B}, \]
then
\[ \alpha_d^d(B) = \frac{1}{r_B \mu(B)} F_B(\mu, \mathcal{L}_B). \]

Let us remark that $c_B$ and $L_B$ (and so $\mathcal{L}_B$) may be not unique. Moreover, we may (and will) assume that $L_B \cap B \neq \emptyset$. When $B = B(x, r)$, we will also write $\alpha_d^d(B) = \alpha_d^d(x, r)$, and $c_B = c_{x, r}$. Further we may drop the superindex $d$ quite often, to shorten notation.

These are the so-called $\alpha$ coefficients from [To09]. If $\mu$ is $d$-rectifiable, the convergence of $\mu_{x, r}$ to $d$-dimensional Lebesgue on a $d$-plane as $r \to 0$ for a.e. $x$ implies the weaker property that
\[ \lim_{r \to 0} \alpha_d^d(x, r) = 0 \quad \text{for } \mu\text{-a.e. } x \in \mathbb{R}^n. \]

However, this limit being zero is not enough to imply rectifiability. This can be seen by considering a variant of the Von Koch snowflake such that if $K_k$ denotes the $k$-th stage of the construction, $K_{k+1}$ is obtained from $K_k$ by introducing new edges that make an angle equal to $\frac{1}{\sqrt{k}}$ with the previous edges, and then let $\mu_k = [\mathcal{H}^1(K_k)]^{-1} \mathcal{H}^1|_{K_k}$. These measures converge weakly to a measure $\mu$ for which (1.5) holds (with $d = 1$) yet the measure is singular with respect to $\mathcal{H}^1$. Thus, it is a natural question to ask what additional information is needed aside from (1.5) to imply rectifiability.

In [ADT16], the first author, David and the third author considered some variant of the $\alpha$ coefficients. Denote $T_{x, r}(y) = (y - x)/r$ and let $W_1$ be the 1-Wasserstein distance between probability measures and the infimum is taken over all $d$-planes. Then one sets
\[ \alpha_d^d(x, r) = \inf_L W_1(\mu(B(x, r))^{-1} T_{x, r}[\mu], \mathcal{H}^d(L \cap B(0, 1))^{-1} \mathcal{H}^d|_{L \cap B(0, 1)}), \]
where the infimum is taken over all $d$-planes. In [ADT16] it was shown that if $\mu$ is doubling and
\[ \int_0^1 \alpha_d^d(x, r) \frac{dr}{r} < \infty \quad \text{for } \mu\text{-a.e. } x \in \mathbb{R}^n, \]
then $\mu$ is $d$-rectifiable. In [ADT16] it was also conjectured that the same result should be true if $\tilde{\alpha}^d_{\mu}(x, r)$ were replaced with $\tilde{\alpha}^d_{\mu}(x, r)^2$.

In [Orp17], Orponen showed the conjecture is true for $n=d=1$. In fact, he proved that if $\mu$ and $\nu$ are two Radon measures on the real line (where $\nu$ is doubling) then $\mu \ll \nu$ if $\int_0^1 \alpha^1_{\mu, \nu}(x, r)^2 \frac{dr}{r} < \infty$ holds $\mu$-almost everywhere, where now $\alpha^1_{\mu, \nu}$ measures the 1-Wasserstein distance between $\mu$ and $\nu$, normalized appropriately.

If one assumes absolute continuity a priori, then there are other some related results in the literature. Define the Jones’ $\beta$-numbers

$$\beta^d_{\mu, p}(x, r)^p = \inf_L \frac{1}{r^d} \int_{B(x, r)} \left( \frac{\text{dist}(y, L)}{r} \right)^p d\mu(y),$$

where the infimum is over all $d$-dimensional planes $L$. In a sense, these coefficients are weaker than the $\alpha$-numbers that we described above since they only measure how close the measure is to lying on a $d$-plane, not how much it resembles $d$-dimensional Lebesgue measure (so for example, if $\mu$ is supported in a plane but not supported on a portion inside the ball $B(x, r)$ with positive area in this plane, then the $\beta$-number of $B(x, r)$ is zero while the $\alpha$-number is positive). If $\mu \ll H^dE$ for some set $E$ of finite $H^d$-measure, it has been shown recently by the first and second authors [AT15] that $\mu$ is rectifiable if

$$(1.7) \quad \int_0^1 \beta^d_{\mu, 2}(x, r)^2 \frac{dr}{r} d\mu(x) < \infty,$$

for $\mu$-almost every $x \in \mathbb{R}^n$. More recently, Edelen, Naber, and Valtorta [ENV16] have obtained a related result of more quantitative nature.

The converse to the result obtained in [AT15] also holds, as shown by the second author [Tol15]. That is, if $\mu$ is $d$-rectifiable, then (1.7) holds. Further in the same work it is shown that if $\mu$ is $d$-rectifiable, then (1.7) is satisfied with $\beta^d_{\mu, 2}(x, r)$ replaced by $\alpha^d_{\mu}(x, r)$. This fact motivated the above conjecture about the characterization of rectifiability in terms of the $\alpha$-numbers.

In this paper, we confirm this conjecture for measures that are pointwise doubling. More precisely, we prove the following:

**Theorem I.** Let $\mu$ be a Radon measure in $\mathbb{R}^n$, $0 < d \leq n$, and $E$ a Borel set with $\mu(E) > 0$ such that

$$(1.8) \quad J_{\alpha, 1}(x) := \int_0^1 \alpha^d_{\mu}(x, r)^2 \frac{dr}{r} < \infty \text{ for all } x \in E$$

and

$$(1.9) \quad \limsup_{r \to 0} \frac{\mu(B(x, 2r))}{\mu(B(x, r))} < \infty \text{ for all } x \in E.$$  

Then $\mu|_E$ is $d$-rectifiable.

As stated above, in [Tol15] it is shown that if $\mu$ is any $d$-rectifiable measure (not necessarily doubling), then (1.8) holds. Thus combining this result with the theorem above we obtain a characterization of rectifiable measures in terms of their $\alpha$-coefficients and the doubling condition (1.9).

It is not hard to see using the definition of Wasserstein distance that $\alpha^d_{\mu}(x, r) \leq \tilde{\alpha}^d_{\mu}(x, r)$, and so Theorem I implies the conjecture from [ADT16] for measures satisfying (1.9).
The doubling condition (1.9) is necessary as shown by the following result.

**Theorem II.** There exists a Radon measure \( \mu \) in \( \mathbb{R}^2 \) which satisfies

\[
\int_0^1 \alpha_\mu(x,r)^2 \frac{dr}{r} < \infty \quad \text{for all } x \in \text{supp } \mu,
\]

and such that

\[
\lim_{r \to 0} \frac{\mu(B(x,r))}{r} = 0 \quad \text{for all } x \in \text{supp } \mu.
\]

In particular, \( \mu \) is not 1-rectifiable.

We remark that a related phenomenon occurs for the \( \beta_p \) coefficients when \( p < 2 \) in the absence of doubling conditions. Indeed, it is has been shown recently in [Tol17b] that there exists a set \( E \subset \mathbb{R}^2 \) with \( \mathcal{H}^1(E) < \infty \) which is not 1-rectifiable and such that, for all \( 1 \leq p < 2 \),

\[
\int_0^1 \beta_{\mathcal{H}^1|_E,p}(x,r)^2 \frac{dr}{r} < \infty \quad \text{for } \mathcal{H}^1\text{-a.e. } x \in E.
\]

On the other hand, by a result due to Pajot [Paj97] it follows that, for all \( p \in [1,2] \), the above condition implies the rectifiability of \( E \) under the additional assumption that

\[
\liminf_{r \to 0} \frac{\mathcal{H}^1(E \cap B(x,r))}{r} > 0 \quad \text{for } \mathcal{H}^1\text{-a.e. } x \in E,
\]

which is stronger than the pointwise doubling assumption (1.9) (for \( \mu = \mathcal{H}^1|_E \) with \( \mathcal{H}^1(E) < \infty \)).

We should also mention that there are results that provide necessary and/or sufficient conditions for a different notion of rectifiability of measures introduced by Federer. This notion of rectifiability only asks that condition (1.1) hold, and does not require the absolute continuity with respect to \( \mathcal{H}^d \). The characterization of Federer rectifiability is a more difficult problem. Part of the interest in this topic was motivated by an example of Garnett, Killip, and Schul [GKS10] of a doubling measure \( \mu \) with \( \text{supp } \mu = \mathbb{R}^2 \) that satisfies (1.1). This was a surprising result since doubling measures are considered to be well-behaved apart from possibly being singular, so it was anticipated that, if a doubling measure has support equal to \( \mathbb{R}^2 \), then it should give zero measure to any rectifiable curve. Later on Badger and Schul [BS16] characterized the measures in Euclidean space that can be covered up to measure zero by Lipschitz curves, assuming a positive lower density condition on the measure. Also, the first author and Mourgoglou showed in [AM16] that if a measure is doubling with connected support and positive 1-dimensional lower density, then it is 1-rectifiable. Previously, in [Ler03], Lerman gave sufficient conditions for 1-rectifiability in terms of \( \beta \)-type numbers without any lower density assumption. Thus far, the most general necessary conditions for this kind of rectifiability using \( \beta \)-type numbers is given in [BS15]. Unfortunately, these necessary conditions are not sufficient, as shown by an example of Martikainen and Orponen [MO16]. However, see [BS17] for a characterization for measures with positive lower density using a different \( \beta \)-type quantity.
2. Notation

We will write \( A \lesssim B \) if \( A \leq CB \) for some universal constant \( C \). Throughout this paper, we will assume all such implicit constants depend on the dimension \( n \); otherwise, we will write \( A \lesssim_t B \) if the constant \( C \) depends on some parameter \( t \). We will write \( A \approx B \) to mean \( A \lesssim B \lesssim A \) and define \( A \approx_t B \) similarly.

We denote by \( B(x,r) \) the open ball centered at \( x \) of radius \( r > 0 \) in \( \mathbb{R}^n \). If \( B \) is a ball, we write \( x_B \) for its center and \( r_B \) for its radius. If \( B = B(x,r) \) and \( \lambda > 0 \), we will write \( \lambda B = B(x,\lambda r) \), that is, the ball with same center but \( \lambda \)-times the radius.

For a measure \( \mu \) in \( \mathbb{R}^n \) and a ball \( B = B(x,r) \), we write \( \Theta^d_{\mu}(x,r) = \Theta^d_{\mu}(B) = \frac{\mu(B(x,r))}{r^d} = \frac{\mu(B)}{r_B^d} \).

Given \( E,F \subset \mathbb{R}^n \) closed sets, \( d_H(E,F) \) stands for the Hausdorff distance between \( E \) and \( F \). For \( x \in \mathbb{R}^n \) and \( r > 0 \) we also consider the following local scale invariant version of Hausdorff distance

\[
\text{dist}_{x,r}(E,F) = \frac{1}{r} \max \left( \sup_{y \in E \cap B(x,r)} \text{dist}(y,F); \sup_{y \in F \cap B(x,r)} \text{dist}(y,E) \right).
\]

Given two \( d \)-planes \( L_1 \) and \( L_2 \), let \( L'_1, L'_2 \) be the respective parallel \( d \)-planes passing through the origin. Then we denote

\[
\angle(L_1,L_2) = d_H(L'_1 \cap B(0,1), L'_2 \cap B(0,1)).
\]

In a sense, \( \angle(L_1,L_2) \) is the angle between \( L_1 \) and \( L_2 \).

3. Preliminaries

Below we use constants \( A, \tau, C_1, \) and \( \varepsilon > 0 \). We choose them so that

\[
(3.1) \quad \tau \ll 1 \ll \min\{A,C_1\} \quad \text{and} \quad \varepsilon \ll \min\{A^{-1}, \tau^4, C_1^{-1}\},
\]

We recall Besicovitch covering lemma as we will use it frequently. There exists \( N = N(n) \) depending only on \( n \) such that for any bounded set \( E \subset \mathbb{R}^n \), and any collection of closed balls \( \{B(x,r(x)) : x \in E\} \) with \( \sup \{r(x) : x \in E\} < \infty \) there are \( \mathcal{G}_1 \cdots, \mathcal{G}_N \) countable disjoint subcollections such that

\[
(3.2) \quad E \subset \bigcup_{j=1}^N \bigcup_{B \in \mathcal{G}_j} B \quad \text{consequently} \quad \chi_{E} \leq \sum_{j=1}^N \chi_{B_j} \lesssim_n 1,
\]

where \( B_j = \bigcup_{B \in \mathcal{G}_j} B \). In particular, for a measure \( \mu \), there is \( j_0 \in \{1, \cdots, N\} \) such that

\[
(3.3) \quad \mu(E) \leq \sum_j \mu(B_j) \leq N \mu(B_{j_0}) = N \mu \left( \bigcup_{B \in B_{j_0}} B \right)
\]

Such covering will often be referred to as a Besicovitch subcovering of the collection \( \{B(x,r(x)) : x \in E\} \).
We now go over some basic facts about $\alpha$ numbers. Some of them are proven in [Tol09] for $d$-AD-regular measures and in [Tol17a] for general measures. However we supply some more precise estimates here.

**Lemma 3.1.** For $x, y \in \mathbb{R}^n$, if $B(x, r) \subset B(y, s)$, then

\[
\alpha_\mu(x, r) \leq \frac{s \mu(B(y, s))}{r \mu(B(x, r))} \alpha_\mu(y, s). \tag{3.4}
\]

**Proof.** Let $\varepsilon > 0$ and pick $\mathcal{L}_{y,s} = c\mathcal{H}^d|_L$ so that

\[
\frac{1}{s\mu(B(y, s))} F_{B(y, s)}(\mu, \mathcal{L}_{y,s}) \leq (1 + \varepsilon)\alpha_\mu(y, s).
\]

For $\phi \in \text{Lip}_1(B(x, r))$

\[
\left| \int \phi \, d\mu - \int \phi \, d\mathcal{L}_{y,s} \right| \leq s\mu(B(y, s))(1 + \varepsilon)\alpha_\mu(y, s).
\]

Taking the supremum over $\phi \in \text{Lip}_1(B(x, r))$ and using (1.2) we have

\[
\alpha(x, r)r\mu(B(x, r)) \leq F_{B(x,r)}(\mu, \mathcal{L}_{y,s}) \leq (1 + \varepsilon)s\mu(B(y, s))\alpha_\mu(y, s).
\]

Hence,

\[
\alpha_\mu(x, r) \leq \frac{s\mu(B(y, s))}{r\mu(B(x, r))} (1 + \varepsilon) \alpha_\mu(y, s)
\]

and letting $\varepsilon \to 0$ we obtain (3.4). \qed

**Lemma 3.2.** For $x \in \mathbb{R}^n$, if $y \in B(x, r/2)$, $B(y, 2s) \subset B(x, r)$, $\mathcal{L}$ is a measure supported on a $d$-plane $L$, and $F_{B(x,r)}(\mu, \mathcal{L}) < s\mu(B(y, s))$, then

\[
L \cap B(y, 2s) \neq \emptyset.
\]

In particular, if $\alpha_\mu(x, r) < \frac{\mu(B(x, r/2))}{s\mu(B(x, r))}$, then

\[
L_{x,r} \cap B(x, r/4) \neq \emptyset.
\]

**Proof.** Let $\phi(z) = (2s - |y - z|)_+$. Note that $\phi \in \text{Lip}_1(B(x, r))$ and $\phi \geq s$ on $B(y, s)$. If $L \cap B(y, 2s) = \emptyset$, then

\[
s \mu(B(y, s)) \leq \int \phi \, d\mu \leq \int \phi (\mu - \mathcal{L}) \leq F_{B(x,r)}(\mu, \mathcal{L}) < s\mu(B(y, s)),
\]

which is a contradiction. Thus, dist$(y, L) < 2s$. \qed

**Lemma 3.3.** For $x \in \mathbb{R}^n$, if $\mathcal{L} = c\mathcal{H}^d|_L$ and $F_{B(x,r)}(\mu, \mathcal{L}) < \frac{r}{s} \mu(B(y, s))$, then

\[
\Theta_\mu^d(x, r/2) \leq c \leq \Theta_\mu^d(x, r). \tag{3.6}
\]

In particular, if $\alpha_\mu(x, r) < \frac{\mu(B(x, r/2))}{s\mu(B(x, r))}$, then

\[
\Theta_\mu^d(x, r/2) \leq c_{x,r} := c_{B(x, r)} \leq \Theta_\mu^d(x, r). \tag{3.7}
\]
Proof. Let \( \phi(x) = (r - |x - y|)_+ \), so that \( \phi \in \text{Lip}_1(B(x, r)) \) and \( \phi \geq r/2 \) on \( B(x, r/2) \). Since \( L \cap B(x, r/4) \neq \emptyset \) by the previous lemma, we have

\[
c_d r^{d+1} \approx r \mathcal{L}(B(x, r/2)) \lesssim \int \phi \, d\mathcal{L} \leq F_{B(x, r)}(\mu, \mathcal{L}) + \int \phi \, d\mu \leq 2r \mu(B(x, r))
\]

and hence \( c \lesssim \Theta_d^d(x, r) \). A similar computation reversing the roles of \( \mu \) and \( \mathcal{L} \) yields \( c \gtrsim \Theta_d^d(x, r/2) \).

Lemma 3.4. Let \( x, y \in \mathbb{R}^n, B(x, 2r) \subset B(y, s) \), and \( \mathcal{L}_i = c_i \mathcal{H}^d \mid L_i \) for \( i = 1, 2 \). If \( F_{B(x, r)}(\mu, \mathcal{L}_1) < \frac{s}{2} \mu(B(x, r/8)) \), and \( L_2 \cap B(y, s) \neq \emptyset \), then

\[
\angle(L_1, L_2) + \text{dist}_{x,r/2}(L_1, L_2) \lesssim \frac{F_{B(x, r)}(\mu, \mathcal{L}_1) + F_{B(y, s)}(\mu, \mathcal{L}_2)}{r \mu(B(x, r/2))}.
\]

In particular, if \( \alpha_\mu(x, r) \leq \frac{\mu(B(x, r/8))}{s \mu(B(x, r))} \), then

\[
\angle(L_{x,r}, L_{y,s}) + \text{dist}_{x,r/2}(L_{x,r}, L_{y,s}) \lesssim \frac{s \mu(B(y, s))}{r \mu(B(x, r/2))} \alpha_\mu(y, s).
\]

Proof. Suppose first that \( L_2 \cap B(x, 2r) = \emptyset \). Let \( \phi_0(z) = (2r - |x - z|)_+ \). Then we have

\[
F_{B(y, s)}(\mu, \mathcal{L}_2) \geq \int \phi_0 \, d(\mu - \mathcal{L}_2) = \int \phi_0 \, d\mu \geq r \mu(B(x, r)).
\]

It is also immediate that \( \text{dist}_{x,r/2}(L_1, L_2) \lesssim 1 \), and thus

\[
\angle(L_1, L_2) + \text{dist}_{x,r/2}(L_1, L_2) \lesssim \frac{F_{B(y, s)}(\mu, \mathcal{L}_2)}{r \mu(B(x, r))},
\]

and so (3.8) holds in this case.

Suppose now that \( L_2 \cap B(x, 2r) \neq \emptyset \). Let \( \Phi \) be a \( \frac{2}{r} \)-Lipschitz function that equals 1 on \( B(x, r/2) \) and 0 outside \( B(x, r) \). Also set

\[
\phi(z) = \Phi(z) \cdot \text{dist}(z, L_2).
\]

Using that \( \text{dist}(z, L_2) \leq 3r \) on \( \text{supp} \Phi \), it is immediate to check that \( \phi \) is a \( 7 \)-Lipschitz on \( B(x, r) \). By Lemma 3.2, \( L_1 \cap B(x, r/4) \neq \emptyset \), and so \( \mathcal{H}^d(L_1 \cap B(x, r/2)) \approx r^d \). Thus, using that \( \phi \) vanishes on \( L_2 \),

\[
\int_{B(x, r/2)} \frac{\text{dist}(z, L_2)}{r} \, d\mathcal{H}^d \big|_{L_1} \lesssim \frac{1}{\Theta_d^d(x, r/2) r^{d+1}} \int \phi \, d\mathcal{L}_1 = \frac{1}{r \mu(B(x, r/2))} \int \phi \, d\mu \]

\[
\lesssim \frac{F_{B(x, r)}(\mu, \mathcal{L}_1)}{r \mu(B(x, r/2))} + \frac{1}{r \mu(B(x, r/2))} \int \phi \, d(\mu - \mathcal{L}_2) \]

\[
\lesssim \frac{F_{B(x, r)}(\mu, \mathcal{L}_1) + F_{B(y, s)}(\mu, \mathcal{L}_2)}{r \mu(B(x, r/2))}.
\]

Let \( z_0 \in B(x, r/2) \cap L_1 \) be such that \( \text{dist}(z_0, L_2) = \inf \{ \text{dist}(z, L_2) : z \in B(x, r/2) \cap L_1 \} \). Then since \( L_1 \) and \( L_2 \) are \( d \)-planes, for \( z \in B(x, r/2) \cap L_1 \) we have

\[
\text{dist}(z, L_2) = \text{dist}(z_0, L_2) + \text{dist}(z - z_0, L_2 - z_0) = \text{dist}(z_0, L_2) + |z - z_0| \angle(L_1, L_2)
\]
Integrating (3.11) over $B(x, r/2) \cap L_1$ and using (3.10) we obtain that

\begin{equation}
\text{dist}(z_0, L_2) + r \angle(L_1, L_2) \lesssim \frac{F_{B(x,r)}(\mu, \mathcal{L}_1) + F_{B(y,s)}(\mu, \mathcal{L}_2)}{r \mu(B(x, r/2))}.
\end{equation}

Thus (3.11) and (3.12) yield

\begin{equation}
\frac{1}{r} \sup \{ \text{dist}(z, L_2) : z \in B(x, r/2) \cap L_1 \} \leq \frac{1}{r} \text{dist}(z_0, L_2) + 2 \angle(L_1, L_2)
\end{equation}

\begin{equation}
\lesssim \frac{F_{B(x,r)}(\mu, \mathcal{L}_1) + F_{B(y,s)}(\mu, \mathcal{L}_2)}{\mu(B(x, r/2))},
\end{equation}

and

\begin{equation}
\angle(L_1, L_2) \lesssim \frac{F_{B(x,r)}(\mu, \mathcal{L}_1) + F_{B(y,s)}(\mu, \mathcal{L}_2)}{r \mu(B(x, r/2))}.
\end{equation}

Since $L_1$ and $L_2$ are planes this is enough to conclude (3.8). Now (3.9) follows from (3.8) and (3.4), by taking $L_1 = L_{x,r}$ and $L_2 = L_{y,s}$. Indeed, we derive

\begin{equation}
F_{B(x,r)}(\mu, \mathcal{L}_{x,r}) + F_{B(y,s)}(\mu, \mathcal{L}_{y,s}) \lesssim \alpha_{\mu}(x, r) r \mu(B(x, r)) + \alpha_{\mu}(y, s) s \mu(B(y, s))
\end{equation}

\begin{equation}
\lesssim \alpha_{\mu}(y, s) s \mu(B(y, s)).
\end{equation}

Plugging this estimate into (3.8), we obtain (3.9). \hfill \square

**Lemma 3.5.** Let $x, y \in \mathbb{R}^n$ be such that $B(x, 2r) \subset B(y, s)$, $\mathcal{L}_i = c_i \mathcal{H}^d|_{L_i}$, $F_{B(x,r)}(\mu, \mathcal{L}_1) < \frac{\theta}{\mu(B(x, \frac{r}{2}))}$ and $F_{B(y,s)}(\mu, \mathcal{L}_2) < \frac{\theta}{\mu(B(y, \frac{s}{2}))}$. Then

\begin{equation}
|c_1 - c_2| \lesssim \frac{F_{B(x,r)}(\mu, \mathcal{L}_1) + F_{B(y,s)}(\mu, \mathcal{L}_2)}{r^{d+1}} \left(1 + \frac{\theta_{\mu}(y, s)}{\theta_{\mu}(x, r/2)}\right) \frac{s}{r}.
\end{equation}

In particular, if $\alpha_{\mu}(x, r) < \frac{\mu(B(x, \frac{r}{2}))}{\theta_{\mu}(B(x, r))}$ and $\alpha_{\mu}(y, s) < \frac{\mu(B(y, \frac{s}{2}))}{\theta_{\mu}(B(y, s))}$, then

\begin{equation}
|c_{x,r} - c_{y,s}| \lesssim \alpha_{\mu}(y, s) \theta_{\mu}(y, s) \left(1 + \frac{\theta_{\mu}(y, s)}{\theta_{\mu}(x, r/2)}\right) \frac{s^{d+2}}{r^{d+2}}.
\end{equation}

**Proof.** Let $\phi(z) = (r - |x - z|)_+$. Then, by (3.8) and (3.6), since $2r \leq s$

\begin{align*}
r^{d+1}|c_1 - c_2| \lesssim & \int \phi c_1 \, d\mathcal{H}^d|_{L_1} - \int \phi c_2 \, d\mathcal{H}^d|_{L_1} \\
& \lesssim \int \phi c_1 \, d\mathcal{H}^d|_{L_1} - \int \phi \, d\mu + \int \phi \, d\mu - \int \phi c_2 \, d\mathcal{H}^d|_{L_2} \\
& + c_2 \left| \int \phi \, d\mathcal{H}^d|_{L_2} - \int \phi \, d\mathcal{H}^d|_{L_1} \right| \\
& \lesssim F_{B(x,r)}(\mu, \mathcal{L}_1) + F_{B(y,s)}(\mu, \mathcal{L}_2) \\
& + \theta_{\mu}(y, s) \frac{F_{B(x,r)}(\mu, \mathcal{L}_1) + F_{B(y,s)}(\mu, \mathcal{L}_2)}{\mu(B(x, r))} r^{d-1} s \\
& \lesssim \left( F_{B(x,r)}(\mu, \mathcal{L}_1) + F_{B(y,s)}(\mu, \mathcal{L}_2) \left(1 + \frac{\theta_{\mu}(y, s)}{\theta_{\mu}(x, r/2)}\right) \right) \frac{s}{r},
\end{align*}

which yields (3.16).
To get (3.17) we apply (3.15) using the fact $B(x, 2r) \subset B(y, s)$ and then we obtain
\[
\frac{F_{B(x,r)}(\mu, \mathcal{L}_x, r) + F_{B(y,s)}(\mu, \mathcal{L}_y, s)}{r^{d+1}} \leq \alpha_\mu(y, s) \frac{s \mu(B(y, s))}{r^{d+1}} = \alpha_\mu(y, s) \Theta_\mu(B(y, s)) \frac{s^{d+1}}{r^{d+1}}.
\]
Plugging this estimate into (3.16), we derive (3.17). \qed

4. Outline of Proof

In order to present an outline of the proof to Theorem 1 we first explore the consequences of the hypotheses. Consider a Radon measure $\mu$ and a Borel set $E$, with $\mu(E) > 0$ and satisfying (1.8) and (1.9). Let $E_1 = E \cap B(0, R)$ with $R$ large enough so $0 < \mu(E_1) < \infty$. By (1.9) for $M > 1$ large enough there exists a closed set $\tilde{E} \subset E_1$ such that $\mu(\tilde{E}) > 0$ and for all $x \in \tilde{E}$

\[
\lim_{k \to \infty} \sup_{0 < r < 2^{-k}} \frac{\mu(B(x, 2r))}{\mu(B(x, r))} \leq M \quad \text{and} \quad \lim_{k \to \infty} \int_0^{2^{-k}} \alpha_{\mu}^d(x, r)^2 \frac{dr}{r} = 0.
\]

By Egoroff, there exists a closed set $\tilde{E}_0 \subset \tilde{E}$, with $\mu(\tilde{E}_0) \geq \frac{9}{100} \mu(\tilde{E}) > 0$ so that for $\varepsilon \in (0, 10^{-3})$ there is $k_0 = k_0(M, \varepsilon) > 1$ so that for $k \geq k_0$ and $x \in \tilde{E}_0$

\[
\sup_{0 < r < 2^{-k}} \frac{\mu(B(x, 2r))}{\mu(B(x, r))} \leq M \quad \text{and} \quad \int_0^{2^{-k}} \alpha_{\mu}^d(x, r)^2 \frac{dr}{r} < \varepsilon^2.
\]

Since $0 < \mu(\tilde{E}_0) < \infty$, for $\mu$-a.e. $x \in \tilde{E}_0$ ([Mat95, Corollary 2.14]),

\[
\lim_{r \to 0} \frac{\mu(B(x, r) \cap \tilde{E}_0)}{\mu(B(x, r))} = 1.
\]

By Egoroff, once again, given $\delta \in (0, \frac{1}{10})$ there exists a closed set $\tilde{F}_0 \subset \tilde{E}_0$, with $\mu(\tilde{F}_0) \geq (1 - \delta) \mu(\tilde{E}_0) \geq \frac{81}{100} \mu(\tilde{E}) > 0$ so that for $\varepsilon \in (0, 10^{-3})$ there is $k_1 = k(\varepsilon, \delta) > 1$ so that for $r < 2^{-k_1}$ and $x \in \tilde{F}_0$

\[
\mu(B(x, r) \setminus \tilde{E}_0) \leq \varepsilon \mu(B(x, r)).
\]

Summarizing we have that given $M > 1$ large enough, $\delta \in (0, \frac{1}{10})$ and $\varepsilon \in (0, 10^{-3})$ there exist closed sets $\tilde{F}_0 \subset \tilde{E}_0 \subset E \cap B(0, R)$ and $\rho_o > 0$ such that $\mu(\tilde{F}_0) \geq (1 - \delta) \mu(\tilde{E}_0) > 0$ and for every $x \in \tilde{E}_0$ and every $0 < r < \rho_o$ (see (4.1))

\[
\mu(B(x, 2r)) \leq M \mu(B(x, r)),
\]

\[
J_{\alpha, \rho_o}(x) := \int_0^{\rho_o} \alpha_{\mu}^d(x, r)^2 \frac{dr}{r} < \varepsilon^2,
\]

and for every $x \in \tilde{F}_0$ and every $0 < r < \rho_o$ (see (4.3))

\[
\mu(B(x, r) \setminus \tilde{E}_0) \leq \varepsilon \mu(B(x, r)).
\]

Without loss of generality we may assume that $0 \in \tilde{F}_0$. Moreover note that if $\tilde{\mu}_r(A) = \mu(B(0, r))^{-1} \mu(rA)$ and $c > 0$ then for $y = \frac{x}{r}$ with $x \in \tilde{E}_0$

\[
\alpha_{\tilde{\mu}_r}^d(y, s) = \alpha_{\mu}^d(x, sr).
\]
Letting \( \rho_0 = 4C_1r_0 \) where \( C_1 \) is as in (3.1) and replacing \( \tilde{E}_0 \) by \( E_0 = \frac{1}{r_0}\tilde{E}_0 \), \( \tilde{F}_0 \) by \( F_0 = \frac{1}{r_0}\tilde{E}_0 \), and \( \mu \) by \( \mu(B(0, 3C_1r_0))^{-1}\tilde{\mu}_{r_0} \) and relabeling it \( \mu \) we have that \( 0 \in F_0 \)

\[
\mu(B(0, 3C_1)) = 1,
\]

and for given \( M > 1 \) large enough, \( \delta \in (0, \frac{1}{M}) \), and \( \varepsilon \in (0, 10^{-3}) \) there exist closed bounded sets \( F_0 \subseteq E_0 \subseteq \frac{1}{r_0}E \) such that \( \mu(F_0) \geq (1 - \delta)\mu(E_0) > 0 \) and for every \( x \in E_0 \) and every \( 0 < r < 4C_1 \) (see (4.4) and (4.5))

\[
\mu(B(x, 2r)) \leq M\mu(B(x, r)),
\]

and for every \( x \in F_0 \) and every \( 0 < r < 4C_1 \) (see (4.6))

\[
\mu(B(x, r) \setminus E_0) \leq \varepsilon\mu(B(x, r)).
\]

Note that (4.10) ensures that for \( x \in E_0 \) and \( r \in (0, 2C_1) \) there exits \( t \in [r, 2r] \) such that

\[
\alpha^d_\mu(x, t)^2 \leq 2 \int_r^{2r} \frac{\alpha^d_\mu(x, s)^2 ds}{s} < 2\varepsilon^2.
\]

Then (4.12) and (4.4) combined with (3.4) ensure that for \( x \in E_0 \) and \( r \in (0, 2C_1) \)

\[
\alpha^d_\mu(x, r) \leq \frac{t}{r} \frac{\mu(B(x, t))}{\mu(B(x, r))} \alpha^d_\mu(x, t) \leq 2 \frac{\mu(B(x, 2r))}{\mu(B(x, r))} \sqrt{2\varepsilon} \leq 4M\varepsilon.
\]

Now we outline the plan for the rest of the proof: Note that on the set \( E_0 \) the rescaled measure \( \mu \) is doubling on a range of scales (4.9), the Jones function \( J_\alpha \) and \( \alpha \)-numbers corresponding to \( \mu \) and also small (see (4.10) and (4.12)). For each point in \( E_0 \) we consider the supremum over all radii, less than a fraction of \( 4C_1 \), for which \( \mu \) does not behave like an Ahlfors regular measure above these scales. Hence, for most of these scales, the measure is either too large or too small. Our goal is to show that the subset of \( E_0 \) for which this supremum is not 0, is small. To do this we use techniques from [DT12], to build a Lipschitz graph which approximates \( \text{supp } \mu \) at every good scale and location. Upon this graph, we construct a projection \( \nu \) of the measure \( \mu \). The nice estimates on the \( \alpha \)-numbers for \( \mu \), yield even nicer estimates for \( \nu \). The advantage now is that we have a surrogate \( \nu \) for \( \mu \), supported on a graph, and \( \nu \) is Ahlfors regular (see (7.8)). To estimate the set where the density drops at small scales we use techniques that come from [L99] and which were also used in other works, such as [AT15]. To control the measure of the set where the density increases too much at small scales we use our \( \alpha \)-number estimates to estimate the \( L^2 \)-norm of the density of \( \nu \) into the domain of the graph (that is, \( \mathbb{R}^d \)). This idea is newer and comes from [Tol17a]. Altogether, these techniques give us control on the total mass of the area where \( \nu \) (and thus \( \mu \)) can have low or high density with respect to surface measure. This will show that in fact in most places the density of \( \mu \) stays bounded away from 0 and \( \infty \), implying absolute continuity with respect \( d \)-dimensional Hausdorff measure and rectifiability. It is important to note that this argument proves the rectifiability if a rescaled version of \( \mu \), namely \( [\mu(B(0, 3C_1r_0))^{-1}\tilde{\mu}_{r_0}] \) restricted to the set \( \frac{1}{r_0}E \), which is equivalent to the rectifiability of our original \( \mu \) restricted to the set \( E \).
5. THE STOPPING TIME

The rest of the paper will be devoted to proving the following lemma, which implies Theorem 1 by an exhaustion argument.

**Lemma 5.1.** With the assumptions of Theorem 1, there is $E' \subset E$ with $\mu(E') > 0$ such that $\mu|_{E'}$ is $d$-rectifiable.

Let $E$ and $\mu$ be as in Theorem 1. We assume that there is no set $E' \subset E$ as in the lemma. Using the notation introduced in the previous section we obtain a contradiction as follows. For $\tau$ and $A$ as in (3.1), and $B_0 = B(0,1)$ let

$$ G = \{ x \in E_0 \cap B_0 : \Theta^d_\mu(x,r) \in [2^{-d}\tau, 2^d A] \text{ for all } r \in (0,C_1) \}. $$

Under the hypothesis on $E_0$, $\mu|G$ is $d$-rectifiable (see proof of Lemma 5.8). Therefore $\mu(G) = 0$ by the contradiction assumption. Using [DT12] we construct an approximating Lipschitz surface $\Sigma$ near $E_0$ (see Section 6). We then construct an Ahlfors regular measure $\nu$ on $\Sigma$ which captures the behavior of $\mu$ on $E_0$ (see Section 7). This allows us to conclude in Section 9 that $\mu(G)$ is proportional to $\mu(E_0 \cap B_0) \geq (1-\varepsilon)\mu(B_0) \geq C(\varepsilon,M,C_1)\mu(B(0,3C_1)) > 0$ (see (4.9), (4.11) and (4.8)), which contradicts the fact that $\mu(G) = 0$.

For $x \in E_0 \cap B_0$, we define $\delta(x)$ to be the supremum over all radii $0 < r \leq C_1$ such that either the density ratio of $B(x,r)$ is either too big or too small or the angle between $L_{x,r} = L_{B(x,r)}$ as in (1.3) and (1.4) and $L_{B_0}$ is too big, that is:

- **ND:** $\mu(B(x,r) \setminus E_0) \geq \varepsilon \frac{\mu(B(x,r))}{2}$,
- **LD:** $\Theta^d_\mu(B(x,r)) \leq \tau$,
- **HD:** $\Theta^d_\mu(B(x,r)) \geq A$, or
- **BA:** $\angle(L_{x,r}, L_{B_0}) \geq \varepsilon^{1/4}$.

The abbreviations stand for “not dense”, “low density”, “high density”, and “big angle”, respectively. Note that by (4.11) if $x$ is such that $\mu(B(x,r) \setminus E_0) \geq \varepsilon \mu(B(x,r))$ for some $r \in (0,4C_1)$ then $x \in F_0$.

For $x \in \mathbb{R}^n$ define

$$ d(x) = \inf_{y \in E_0 \cap B_0} \{ \delta(y) + |x-y| \}. $$

Note that $d$ is a continuous function. Indeed, this is a 1-Lipschitz function since this is defined as an infimum over the family of 1-Lipschitz functions $\{ \delta(y) + | \cdot - y| : y \in E_0 \cap B_0 \}$.

**Lemma 5.2.** For $A$ and $\tau^{-1}$ large enough, depending on $C_1$, and $M$,

$$ d(x) \leq \delta(x) \leq 10^{-3} \text{ for all } x \in E_0 \cap B_0. $$

Moreover, for all $r$ such that $\delta(x) \leq r < 2C_1$ and $x \in E_0 \cap B_0$,

$$ \Theta^d_\mu(x,r) \in [\tau,A], \quad \frac{\mu(B(x,r) \setminus E_0)}{\mu(B(x,r))} \leq \varepsilon^{1/4}, \text{ and } \angle(L_{x,r}, L_{B_0}) \leq \varepsilon^{1/4}. $$

**Proof.** First note that since $C_1 > 1$, if $x \in E_0 \cap B_0$ and $10^{-3} \leq r < 2C_1$, then $B(x,r) \subset 3C_1B_0$. Hence,

$$ \Theta^d_\mu(x,r) \leq 10^{3d} \mu(3C_1B_0) \leq 1 $$
and since $3C_1B_0 \subset B(x, 4C_1)$ (4.9) yields

$$\Theta_d^\mu(x, r) \gtrsim_M C_1, \quad \Theta_d^\mu(x, 4C_1) \gtrsim_M \frac{\mu(3C_1B_0)}{(4C_1)^d} \gtrsim C_1^{-d}. \tag{5.6}$$

Thus, for $A, \tau^{-1}$ large enough depending on $C_1$, and $M$, (5.5) and (5.6) imply

$$\Theta_d^\mu(x, r) \in [\tau, A] \quad \text{for all } x \in E_0 \cap B_0 \text{ and } 10^{-3} \leq r < 2C_1. \tag{5.7}$$

Furthermore, by (4.13), (3.9), and (5.7), for the same choice of $x$ and $r$,

$$H(L_{x,r}, L_{B_0}) \lesssim_{\Theta_1, A, \tau, M} \alpha(2C_1B_0)$$

and so for $\varepsilon > 0$ small enough, we can guarantee that $H(L_{x,r}, L_{B_0}) < \varepsilon^{\frac{1}{2}}$ for all $x \in E_0 \cap B_0$ and $10^{-3} \leq r < 2C_1$.

Finally, for $x \in E_0 \cap B_0$ and $10^{-3} \leq r < 2C_1$, by (5.7),

$$\mu(B(x, r) \setminus E_0) \leq \mu(3C_1B_0 \setminus E_0) < \varepsilon \mu(3C_1B_0) \lesssim_M \varepsilon \mu(B(x, r)),$$

and so $\mu(B(x, r) \setminus E_0) < \varepsilon^{\frac{1}{2}} \mu(B(x, r))$ for $\varepsilon$ small enough. These facts imply that $\delta(x) \leq 10^{-3}$, and (4.4) follows immediately.

**Remark 5.3.** Using (4.9) and a similar argument to the one that appears in the proof of Lemma 5.2 we deduce that for any given constant $0 < c_0 \leq 1$, given $r$ such that $c_0 \delta(x) \leq r < 2C_1$ and $x \in E_0 \cap B_0$, we have

$$\tau \lesssim_{c_0, M} \Theta^d_\mu(x, r) \lesssim_{c_0, M} A, \quad \frac{\mu(B(x, r) \setminus E_0)}{\mu(B(x, r))} \lesssim_{c_0, M} \varepsilon^{\frac{1}{2}}, \quad \text{and} \quad H(L_{x,r}, L_{B_0}) \lesssim_{c_0, M} \varepsilon^{\frac{1}{2}}. \tag{5.8}$$

**Lemma 5.4.** For $x \in \mathbb{R}^n$ and $2d(x) \leq r < C_1$,

$$2^{-d} r^d \leq \mu(B(x, r)) \leq 2^d A r^d \tag{5.9}$$

and

$$\frac{\mu(B(x, r) \setminus E_0)}{\mu(B(x, r))} \lesssim_M \varepsilon^{\frac{1}{2}}. \tag{5.10}$$

**Proof.** If $d(x) = 0$, since $E_0$ is closed by (5.2) $x \in E_0 \cap B_0$ thus by (4.9) and (5.4), (5.9) and (5.10) hold. Suppose that $d(x) > 0$. Let $y \in E_0 \cap B_0$ be such that

$$\delta(y) + |x - y| \leq \frac{1}{2} r.$$

Then $r/2 \geq \delta(y)$ and $|x - y| \leq r/2$. Recalling that $r < C_1$, we deduce that $\Theta^d_\mu(y, r/2) \geq \tau$ and $\Theta^d_\mu(y, 3r/2) \leq A$ (this follows from the definition of $\delta(y)$ if $3r/2 < C_1$ and from the fact that $\mu(3C_1B_0) = 1$ otherwise). Hence,

$$\mu(B(x, r)) \geq \mu(B(y, r - |x - y|)) \geq \mu(B(y, \frac{1}{2}r)) \geq 2^{-d} r^d,$$

and also

$$\mu(B(x, r)) \leq \mu(B(y, r + |x - y|)) \leq \mu(B(y, (1 + \frac{1}{2})r)) \leq 2^d A r^d.$$
The following is an immediate consequence of (5.9) and Lemma 3.1.

**Lemma 5.5.** For \(x, y \in \mathbb{R}^n\), \(2d(x) < r < s < C_1\), if \(B(x, 2r) \subset B(y, s)\),

\[
\alpha_\mu(x, r) \lesssim_A \tau \left( \frac{r}{\tau} \right)^{d+1} \alpha_\mu(y, s).
\]

**Lemma 5.6.** For \(\varepsilon > 0\) small enough, \(x \in \mathbb{R}^n\) and \(2d(x) \leq r < C_1\),

\[
L_{x,r} \cap B(x, r/4) \neq \emptyset,
\]

and if \(B(x, 2r) \subset B(y, s)\), then

\[
c_{x,r} \approx \Theta_\mu^d(x, r) \approx_A \tau 1.
\]

Further, if \(x, y \in \mathbb{R}^n\), \(2 \max\{d(x), d(y)\} \leq r < s < C_1\), and \(B(x, 2r) \subset B(y, s)\), then,

\[
|c_{x,r} - c_{y,s}| \lesssim_A \tau \left( \frac{s}{r} \right)^{d+2} \alpha_\mu(y, s).
\]

This lemma follows from (5.9), and Lemmas 3.2, 3.3, 3.4, and 3.5. In fact note that if \(2d(x) < r < C_1\), there is \(z \in E_0 \cap B_0\) such that \(\delta(z) + |x - z| \leq r/2\), then \(B(x, r) \subset B(z, 2r)\) and \(\alpha_\mu^d(z, 2r) \leq 4M\varepsilon\) by (4.13) then as in Lemma 5.5, \(\alpha_\mu(x, r) \lesssim_A \varepsilon\), which by (5.9) ensures that the conclusions to Lemmas 3.2, 3.3, 3.4, and 3.5 hold.

**Remark 5.7.** In the preceding lemma, if we assume that \(x, y \in E_0 \cap B_0\) and we allow \(c_0 d(x) \leq r < C_1\), with \(c_0 < 2\), then (5.14), (5.15), (5.16), and (5.17) also hold, with implicit constants depending on \(A, \tau, M, c_0\), assuming \(\varepsilon\) small enough.

**Lemma 5.8.** Under the contradiction assumption for Lemma 5.1 and using the notation above we have that the set

\[
G = \{x \in E_0 : \Theta_\mu^d(x, r) \in [2^{-d}r, A2^d] \text{ for all } r \in (0, C_1)\}
\]
satisfies \(\mu(G) = \mathcal{H}^d(G) = 0\). In particular, if \(Z = \{x : d(x) = 0\}\), then \(Z \subset G \subset E_0\) and \(\mu(Z) = \mathcal{H}^d(Z) = 0\).

**Proof.** It is easy to see that \(\mu|_G \ll \mathcal{H}^d|_G \ll \mu|_G\) since

\[
2^{-d}\tau \leq \liminf_{r \to 0} \Theta_\mu^d(x, r) \leq \limsup_{r \to 0} \Theta_\mu^d(x, r) \leq 2^d A \text{ for all } x \in G.
\]

See for example [Mat95, Theorem 6.9]. Given \(x \in G\) and \(0 < r < C_1/2\), consider the function \(\phi(y) = \frac{1}{r}(2r - |x - y|)_+\). Then we have

\[
\beta_\mu^d(x, r) := \inf_L \frac{1}{r^d} \int_{B(x,r)} \frac{\text{dist}(y, L)}{r} d\mu|_G(y) \leq \frac{1}{r^d} \int_{B(x,r)} \phi(y) \frac{\text{dist}(y, L_{x,2r})}{r} d\mu(y)
\]

\[
\lesssim \alpha_\mu(x, 2r) \frac{\mu(B(x, 2r))}{r^d} \lesssim_A \alpha_\mu(x, 2r).
\]
Thus, $\int_0^1 \beta^d_{\omega(x,t)}(x,r)^2 \frac{dr}{r} < \infty$ for each $x \in G$, and so $\mu|_G$ is $d$-rectifiable by [BS16, Theorem A]. Therefore, $\mu(G) = 0$ by our assumption at the beginning of the proof that $\mu$ vanishes on any $d$-rectifiable subset of positive measure. Now we just observe that by (5.9), $Z \subset G$, and so the proof is finished. \hfill \square

As explained at beginning of Section 5 the goal of the rest of the paper is to show that in fact $\mu(G) > 0$.

6. The approximating surface

We will rely on the following theorem.

**Theorem 6.1.** [DT12] For $k \in \mathbb{N} \cup \{0\}$, set $r_k = 10^{-k}$ and let $\{x_j,k\}_{j \in J_k}$ be a collection of points so that for some $d$-plane $P_0$,

$$\{x_{j,0}\}_{j \in J_0} \subset P_0,$$

$$|x_{i,j} - x_{j,j}| \geq r_k \quad \text{for all } i, j \in J_k,$$

and, denoting $B_{j,k} = B(x_{j,k}, r_k)$,

$$x_{i,j} \in V^2_{k-1} \quad \text{for all } i \in J_k,$$

where

$$V^k = \bigcup_{j \in J_k} \lambda B_{j,k}.$$

To each point $x_{j,k}$, associate a $d$-plane $P_{j,k} \subset \mathbb{R}^n$ such that $P_{j,k} \ni x_{j,k}$ and set

$$\varepsilon_k(x) = \sup \{ \text{dist}_{x,10^{l-1}}(P_{j,k}, P_{i,l}) : j \in J_k, |l-k| \leq 2, i \in J_l, x \in 100B_{j,k} \cap 100B_{i,l} \}.$$

There is $\varepsilon_1 > 0$ such that if $\varepsilon \in (0, \varepsilon_1)$ and

$$\varepsilon_k(x_{j,k}) < \varepsilon \text{ for all } k \geq 0 \text{ and } j \in J_k,$$

then there is a bijection $g : \mathbb{R}^n \rightarrow \mathbb{R}^n$ so that the following hold

(i) We have

$$E_\infty := \bigcap_{k=1}^\infty \bigcup_{j \in J_k} \{x_{j,k}\}_{j \in J_k} \subset \Sigma := g(\mathbb{R}^d).$$

(ii) $g(z) = z$ when $\text{dist}(z, P_0) > 2$.

(iii) There is some $\tau_0 > 0$ such that, for $x, y \in \mathbb{R}^n$,

$$\frac{1}{4} |x - y|^{1+\tau_0} \leq |g(x) - g(y)| \leq 10|x - y|^{1-\tau_0}.$$

(iv) We have

$$|g(z) - z| \lesssim \varepsilon \text{ for } z \in \mathbb{R}^n.$$

(v) There is a maximal $\frac{\pi}{2}$-separated set $\{x_{j,k}\}_{j \in L_k}$ in $\mathbb{R}^n \setminus V^0_k$ such that setting

$$B_{j,k} = B(x_{j,k}, r_k/10) \text{ for } j \in L_k,$$

we have $g(x) = \lim_k \sigma_k \circ \cdots \circ \sigma_0(x)$ all for $x \in P_0$, where $\sigma_k : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is defined by

$$\sigma_k(y) = \psi_k(y) y + \sum_{j \in J_k} \theta_{j,k}(y) \pi_{j,k}(y),$$

where

$$\psi_k(y) = \left\{ \begin{array}{ll}
0 & \text{if } |y| \leq \varepsilon k^{-1/2} \\
1 & \text{if } |y| > 1.5 \varepsilon k^{-1/2}.
\end{array} \right.$$
and where \( \pi_{j,k} \) is the orthogonal projection onto \( P_{j,k} \), \( \{ \theta_{j,k} \}_{j \in L_k \cup J_k} \) is a partition of unity such that \( \chi_9 B_{j,k} \leq \theta_{j,k} \leq \chi_{10} B_{j,k} \) for all \( k \) and \( j \in L_k \cup J_k \), and \( \psi_k = \sum_{j \in L_k} \theta_{j,k} \).

(vi) [DT12, Equation (4.5)] For \( k \geq 0 \),

\[
\sigma_k(y) = y \quad \text{and} \quad D\sigma_k(y) = Id \quad \text{for} \quad y \in \mathbb{R}^n \setminus \mathbb{V}_k^{10}.
\]

(vii) [DT12, Proposition 5.1] Let \( \Sigma_0 = P_0 \) and

\[
\Sigma_{k+1} = \sigma_k(\Sigma_k).
\]

There is a function \( A_{j,k} : P_{j,k} \cap 49B_{j,k} \to \mathbb{P}_{j,k}^\perp \) of class \( C^2 \) such that \( |A_{j,k}(x_{j,k})| \lesssim \varepsilon \) for \( P_{j,k} \cap 49B_{j,k} \) and if \( \Gamma_{j,k} \) is its graph over \( P_{j,k} \), then

\[
\Sigma_{k+1} \cap D(x_{j,k}, P_{j,k}, 49r_k) = \Gamma_{j,k} \cap D(x_{j,k}, P_{j,k}, 49r_k),
\]

where \( D(x, P, r) = \{ z + w : z \in P \cap B(x, r), w \in P^\perp \cap B(0, r) \} \).

(Above \( P^\perp \) is the \((n - d)\)-plane perpendicular to \( P \) going through 0.) In particular,

\[
\text{dist}_{x_{j,k}, 49r_{j,k}}(\Sigma_{k+1}, P_{j,k}) \lesssim \varepsilon.
\]

(viii) [DT12, Lemma 6.2] For \( k \geq 0 \) and \( y \in \Sigma_k \), there is an affine \( d \)-plane \( P \) through \( y \) and a \( C\varepsilon \)-Lipschitz and \( C^2 \) function \( A : P \to P^\perp \) so that if \( \Gamma \) is the graph of \( A \) over \( P \), then

\[
\Sigma_k \cap B(y, 19r_k) = \Gamma \cap B(y, 19r_k).
\]

(ix) [DT12, Proposition 6.3] \( \Sigma = g(P_0) \) is \( C\varepsilon \)-Reifenberg flat in the sense that for all \( z \in \Sigma \) and \( t \in (0, 1) \), there is a \( d \)-plane \( P = P(z, t) \) so that \( F_{z,t}(\Sigma, P) \lesssim \varepsilon \).

(x) [DT12, Equation (6.7)] For all \( y \in \Sigma_k \),

\[
|\sigma_k(y) - y| \lesssim \varepsilon r_k.
\]

In particular, it follows that

\[
\text{dist}(y, \Sigma) \lesssim \varepsilon r_k \quad \text{for} \quad y \in \Sigma_k.
\]

(xi) [DT12, Lemma 7.2] For \( k \geq 0 \), \( y \in \Sigma_k \cap \mathbb{V}_k^n \), choose \( i \in J_k \) such that \( y \in 10B_{i,k} \).

Then

\[
|\sigma_k(y) - \pi_{i,k}(y)| \lesssim \varepsilon_k(y) r_k.
\]

(xii) [DT12, Proposition 8.3] If \( g_k(x) = \sigma_k \circ \cdots \circ \sigma_0(x) \) and, for all \( x \in P_0 \),

\[
\sum_{k \geq 0} \varepsilon_k(g_k(x))^2 \leq \varepsilon,
\]

then for \( \varepsilon \) small enough, \( g \) is \( \exp(C\varepsilon) \)-bi-Lipschitz, and hence \( (1 + C\varepsilon) \)-bi-Lipschitz (this is not stated as such in [DT12, Proposition 8.3], but it follows from its proof. To observe this, the crucial inequalities are (8.10)-(8.11) and (8.22)-(8.23) in [DT12]).

(xiii) [DT12, Lemma 13.2] Under the assumption (6.14), for \( x \in \Sigma \) and \( r > 0 \),

\[
\mathcal{H}^d_\infty(B(x, r) \cap \Sigma) \geq (1 - C\varepsilon) \omega_d r^d,
\]

where \( \omega_d \) is the volume of the unit ball in \( \mathbb{R}^d \) (this statement is proven in [DT12] with \( \mathcal{H}^d \) in place of \( \mathcal{H}^d_\infty \), but the same proof works for \( \mathcal{H}^d_\infty \)).
We now apply this result to our situation. For \( k \geq 0 \), let \( r_k = 10^{-k} \) and \( \{x'_j,k\}_{j \in J_k} \) be a maximally \((1 + 1/10)r_k\) separated set in \( E_k \), where
\[
E_k := \{ x \in E_0 \cap B_0 : d(x) < r_k \} \subset E_0 \cap B_0.
\]
Here \( E_0, B_0 \) and \( d(x) \) are as in Section 5 and (5.2). Note that by (5.3) \( E_k = E_0 \cap B_0 \) for \( k = 0, 1, 2, 3 \). If \( E_k = \emptyset \) then \( J_k = \emptyset \). Let \( C_2 \) be such that \( 1 < C_2^2 < C_1 \).

\[
P'_j,k = L_{x'_j,k,c_2r_k}, \quad L'_j,k = L_{x'_j,k,C_2r_k}
\]
and \( P_0 = P'_{0,0} \).

These would be good planes and points for the purpose of applying Theorem 6.1 if each \( d \)-plane \( P_{j,k} \) passed through \( x'_j,k \). Since this may fail, some extra care must be taken.

Note that \( r_k > d(x'_j,k) \), and so by Lemma 5.2 and the subsequent remark, arguing as in (5.20), we obtain
\[
\int_{B(x'_j,k,r_k/2) \cap E_0} \frac{\text{dist}(x, P'_j,k)}{r_k} d\mu \lesssim \frac{\alpha(x'_j,k, 2C_2r_k)}{\mu(B(x'_j,k; r_k/2) \cap E_0)} \frac{\mu(B(x'_j,k, 2C_2r_k))}{\mu(B(x'_j,k; r_k/2)) (1 - c(M) \varepsilon^2)} \approx_{A, r, \alpha, C_2} \alpha(x'_j,k, 2C_2r_k) r_k.
\]
Thus, for \( \varepsilon > 0 \) small enough, there is \( x_{j,k} \in B(x'_j,k, r_k/10) \cap E_0 \) so that
\[
\text{dist}(x_{j,k}, P'_j,k) \lesssim_{A, r, \alpha, C_2} \alpha(x'_j,k, 2C_2r_k) r_k.
\]
Let \( B_{j,k} = B(x_{j,k}, r_k) \), \( B'_j,k = B(x'_j,k, r_k) \), and \( V_k^3 \) be as in Theorem 6.1. Notice that since \( \{x'_j,k\}_{j \in J_k} \) is a maximal \((1 + 1/10)r_k\)-net for \( E_k \cap B_0 \), the sequence \( \{x_{j,k}\}_{j \in J_k} \) is now \( r_k \)-separated (because \( \alpha \mu(2C_2B'_j,k) \ll 1 \)), and we have
\[
E_k \cap B_0 \subset V_k^{3/2}.
\]
Moreover, since \( E_{k+1} \subset E_k, x'_{j,k+1} \in \bigcup_i B(x'_{i,k}, r_k) \), and so
\[
x_{j,k+1} \subset \bigcup_i B(x'_{i,k}, r_k + C\varepsilon r_{k+1}) \subset \bigcup_i B(x_{i,k}, r_k + r_k/10 + C\varepsilon r_k) \subset V_k^{3/2},
\]
which ensures that (6.1) holds.

Let \( P_{j,k} \) be the \( d \)-plane parallel with \( P'_j,k \) that passes through \( x_{j,k} \) and let
\[
c_{j,k} := cB(x'_j,k, C_2r_k) = cC_2B'_j,k.
\]
Similarly, let \( L_{j,k} = c_{j,k} \mathcal{H}^d|_{P_{j,k}} \) be the translate of \( L'_j,k \). Note that \( B_{j,k} \subset 2B'_j,k \), and so
\[
F_{C_2B_{j,k}}(\mu, L_{j,k}) \leq F_{C_2B_{j,k}}(\mu, L'_j,k) + F_{C_2B_{j,k}}(L'_j,k, L_{j,k}) \leq A, \tau, C_2 \, F_{C_2B_{j,k}}(\mu, L_{j,k}) + r_k^{d+1} \alpha_\mu(2C_2B'_j,k) \]
\[
\leq A, \tau, C_2 \, r_k^{d+1} \alpha_\mu(2C_2B'_j,k).
\]
In the case \( k = 0 \), since \( B_0 = B(0, 1) \) we may assume that \( \{x_{j,0}\}_{j \in J_0} = \{x_{0,0}\} = \{0\} \) and so \( P_{0,0} \) passes through the center of \( B_0 \).
Lemma 6.2. For $C_2$ large enough and $x \in E_k$,

\begin{equation}
\varepsilon_k(x) \lesssim_{A,\tau,C_2} \alpha_{\mu}(x,C_2^2 r_k).
\end{equation}

Notice that this lemma ensures that (6.2) holds (up to a constant).

Proof. Let $i, j, k, l$ be such that $j \in J_k$, $i \in J_l$, $l \leq k \leq l + 2$, and $x \in 100B_{j,k} \cap 100B_{i,l}$. Then for $C_2$ large enough, $\frac{C_2}{2} B_{j,k} \subset C_2 B_{j,k} \cap C_2 B_{i,l}$, and so

\begin{equation}
F_{\frac{C_2}{2} B_{j,k}}(\mathcal{L}_{j,k},\mu) + F_{\frac{C_2}{2} B_{j,k}}(\mu,\mathcal{L}_{i,l}) \leq F_{C_2 B_{j,k}}(\mathcal{L}_{j,k},\mu) + F_{C_2 B_{i,l}}(\mu,\mathcal{L}_{i,l}) \\
\lesssim_{A,\tau,C_2} r_k^{d+1} \alpha_{\mu}(2C_2 B_{j,k}) + r_k^{d+1} \alpha_{\mu}(2C_2 B_{i,l}) \\
\lesssim_{A,\tau,C_2} r_k^{d+1} \alpha_{\mu}(x,C_2^2 r_k).
\end{equation}

The lemma now follows from (3.8) as $x,j \in B(x',r) \cap E_0$, and $d(x',r) < r_k$ then $d(x,j) < 2r_k$ applying Remark 5.7.

Let $\Sigma_k, \Sigma, \sigma_k, g$, and so forth be the data we obtain from applying Theorem 6.1. If $E_k = \emptyset$ then for all $j \geq kE_j = \emptyset$ and the construction stops.

Observe that $V_k^{10} \subset V_0^{10} = B(0,10)$, and so by (6.6),

\begin{equation}
\Sigma \setminus B(0,10) = P_0 \setminus B(0,10).
\end{equation}

Observe also that in our scenario (recalling $B_0$ is closed)

\begin{equation}
E_\infty = Z \subset B_0,
\end{equation}

which might a priori be empty. Note that if $x \in V_k^{40}$ for infinitely many $k$, then $d(x) = 0$ thus we define $k(x)$ for $x \in \Sigma \cap 40B_0 \setminus Z$ as follows:

\begin{equation}
\text{For } x \in \Sigma \cap 40B_0 \setminus Z, \text{ let } k(x) \text{ be the smallest integer } k \text{ for which } x \not\in V_k^{40}.
\end{equation}

Since $0 \in \{x,j\}_{j \in J_0}$, we know $10B_0 \subset V_0^{40}$ and hence $k(\cdot) > 0$ is well defined.

Since $k(x)$ is minimal, $x \in V_k^{40}$ and so

\begin{equation}
x \in 40B_{j,k(x)-1} \text{ for some } j \in J_k(x)-1.
\end{equation}

Thus, $B(x,r_{k(x)}) \subset 41B_{j,k(x)-1}$.

Lemma 6.3. For $x \in \Sigma \cap 40B_0 \setminus Z$, and recalling the notation from Theorem 6.1,

\begin{equation}
B(x,r_{k(x)}) \cap \Sigma = B(x,r_{k(x)}) \cap \Sigma_{k(x)} = B(x,r_{k(x)}) \cap \Gamma_{j,k(x)-1}.
\end{equation}

for some $j \in J_{k(x)-1}$.

Proof. Indeed, the second equality is from (6.7) (keep in mind for later that $\Gamma_{j,k(x)-1}$ is a $C\varepsilon$-Lipschitz graph over $P_{j,k(x)-1}$). To show the first identity, notice that since $x \not\in V_k^{40}$,

\begin{equation}
B(x,r_{k(x)}) \subset \mathbb{R}^n \setminus V_k^{39}.
\end{equation}

Note that by (6.1), for $k > k(x)$,

$$\{x_i \}_{i \in J_k} \subset V_k^{2} = \{ y \in \mathbb{R}^n : \text{dist}(y,\{x_i \}_{i \in J_{k-1}}) < 2r_{k-1}\}.$$ 

By iterating this via the triangle inequality and recalling that $r_k = 10^{-k}$, we get

$$\{x_i \}_{i \in J_k} \subset \{ y \in \mathbb{R}^n : \text{dist}(y,\{x_i,k(x) \}_{i \in J_{k(x)}}) < 2r_{k-1} + \cdots + 2r_{k(x)}\}$$

$$\subset \{ y \in \mathbb{R}^n : \text{dist}(y,\{x_i,k(x) \}_{i \in J_{k(x)}}) < 2\delta \} \subset V_k^{3}.$$
In particular, $V_k^{10} \subset V_{k(x)}^{13} \subset V_k^{29}$, hence $B(x, r_k(x)) \subset \mathbb{R}^n \setminus V_k^{10}$ and by (6.6), $\sigma_k$ is the identity on $B(x, r_k(x))$ for all $k \geq k(x)$. By Theorem 6.1 (6.3), (6.5), (6.6), the first equality of (6.27) holds and this finishes the claim.

\[ \Box \]

**Lemma 6.4.** We have

\[ \frac{1}{10} r_k(x) \leq d(x) \leq 60 r_k(x) \quad \text{for } x \in \Sigma \cap 40B_0 \setminus Z. \]

**Proof.** Since $x \in \Sigma \cap 40B_0 \setminus Z$, $d(x) > 0$ and there is $k = k(x)$ as in (6.25). Assume $d(x) < r_k/10$. Let $y \in E_0 \cap B_0$ be such that $\delta(y) + |x - y| < 2d(x)$. Since $d(y) \leq \delta(y)$ then

\[ d(y) \leq 2d(x) < \frac{1}{5} r_k < r_k. \]

Hence, $y \in E_k$ and by (6.19), there is $x_{j,k}$ so that $|x_{j,k} - y| \leq \frac{3}{2} r_k$, thus

\[ |x - x_{j,k}| \leq |x - y| + |y - x_{j,k}| \leq \frac{1}{5} r_k + \frac{3}{2} r_k < 2r_k, \]

which is a contradiction since $x \notin V_k^{40}$ (by the definition of $k(x)$). Thus, $d(x) \geq r_k/10$.

To prove the upper bound, recall that $x \in 40B_{i,k(x)-1}$ for some $i \in J_{k-1}$ (see the paragraph before Lemma 6.3). Thus, there is some $x' \in \{x'_{i,k-1}\}_{i \in J_{k-1}}$ such that $|x - x'| \leq 41r_{k-1}$. Since $\{x'_{i,k-1}\}_{i \in J_{k-1}} \subset E_{k-1}$, we have $d(x') \leq r_{k-1} = 10r_k$ by definition. Since $d(\cdot)$ is 1-Lipschitz, then we get

\[ d(x) \leq |x - x'| + d(x') \leq 41r_{k-1} + 10r_k \leq 60r_k. \]

\[ \Box \]

Let $\eta = 1/1000$ and $\{B_j\}_{j=1}^N$ be a Besicovitch subcovering (see (3.2) and (3.3)) of the collection

\[ \{B(x, \eta d(x)) : x \in \Sigma \setminus Z\} \]

where, by the previous lemma, for our choice of $\eta$,

\[ s_j := \eta d(x_j) < \frac{3r_{k(x_j)}}{10} = 3r_{k(x_j)-1} \quad \text{if } x_j \in 40B_0 \cap \Sigma \setminus Z. \]

Since $d(x_j) \leq |d(0)| + |x_j| \leq 1 + 40$, we also have

\[ s_j \leq \frac{41}{1000} < \frac{1}{20}. \]

For $x_j \in 40B_0$, let $B_{i,k(x_j)-1}$ be a ball such that $x_j \in 40B_{i,k(x_j)-1}$ (recall (6.26)), so that $3B_j \subset 49B_{i,k(x_j)-1}$. Denote

\[ \tilde{B}_j = C_2 B_{i,k(x_j)-1}, \quad \xi_j' = x_{i,k(x_j)-1}. \]

Also, set

\[ P_j = P_{i,k(x_j)-1}, \quad \Gamma_j = \Gamma_{i,k(x_j)-1}, \quad \text{and } \mathcal{L}_j = \mathcal{L}_{i,k(x_j)-1} = c_{B_j} \mathcal{H}^d|_{P_j} \]

so that by (6.7), $\Gamma_j$ is a graph of a $C\varepsilon$-Lipschitz function $A_j$ over $P_j$ so that

\[ 3B_j \cap \Sigma = 3B_j \cap \Gamma_j \]
and since $\xi_j' \in P_j$ and $A_j$ is $C\varepsilon$-Lipschitz, \( \text{dist}(\xi_j; P_j) \lesssim \varepsilon r_k(\xi_j) \). These facts imply that, for

$$\sigma := \mathcal{H}^d|_\Sigma,$$

we have

$$(6.34) \quad F_{3B_j}(\sigma, \mathcal{H}^d|_{P_j}) \lesssim \varepsilon r_{k(\xi_j)} \lesssim (6.28) \quad \varepsilon d(\xi_j)^{d+1} \approx \varepsilon s_j^{d+1} \quad \text{if } \xi_j \in 40B_0.$$  

Note also that, by (6.21),

$$(6.35) \quad F_{3B_j}(\mu, \mathcal{L}_j) \lesssim A \tau \varepsilon r_{k(\xi_j)} \lesssim \varepsilon r_k^{d+1}. $$

**Remark 6.5.** Since $\Sigma$ coincides with $P_0$ in $B(0,10)^c$, we do not need to define $\bar{B}_j$ and other related terms for $\xi_j \notin 40B_0$.

Next we record the following lemma for later.

**Lemma 6.6.** If $2B_j \cap 39B_0 \neq \emptyset$, then $2B_j \subset 40B_0$.

**Proof.** Since $2B_j \cap B(0,39) \neq \emptyset$, by (6.31)

$$4s_j = 4\eta d(\xi_j) \leq 4\eta(|\xi_j| + d(0)) \overset{(5.3)}{\leq} 4\eta(39 + 2s_j + 1) = 160\eta + \frac{8}{20}\eta \leq \frac{1}{5}$$

and since $\eta = 1/1000$. Thus $\text{diam } 2B_j = 4s_j < 1$, and so $2B_j \subset B(0,40)$. \( \square \)

**Remark 6.7.** It may seem like overkill to invoke Theorem 6.1 to construct a Lipschitz graph. We could instead construct a graph directly as in [DS91]. However, our approach is not very harmful because the condition in (xi) of Theorem 6.1 will allow to get nice bounds on the $L^2$-norm of the gradient of the graph which will be useful to deal with the stopping condition $\text{BA}$.

**Lemma 6.8.** Let

$$\sigma := \mathcal{H}^d|_\Sigma.$$  

For $\varepsilon > 0$ small and $C_1$ large enough (depending on $C_2$ but independent of $\varepsilon$), the map $g$ is $(1 + C\varepsilon)$-bi-Lipschitz (with $C$ depending on $A, \tau, \text{and } C_1$). In particular, $\sigma$ is $A\tau$-regular with constant close to 1. For $\varepsilon > 0$ small enough,

$$(6.36) \quad 2^{-1} \cdot (2r)^d < (1 - C\varepsilon)(2r)^d \leq \sigma(B(x,r)) \leq (1 + C\varepsilon)(2r)^d < 2 \cdot (2r)^d \quad \text{for all } x \in \Sigma.$$

Recall that $\mathcal{H}^d(B(x,r) \cap \mathbb{R}^d) = (2r)^d$ for $x \in \mathbb{R}^d$, so (6.36) is saying that surface measure is very close to being uniform like planar surface measure.

**Proof.** By Theorem 6.1, to prove the lemma it suffices to show that

$$(6.37) \quad \sum_{k \geq 0} \varepsilon_k(g_k(y))^2 \lesssim \varepsilon^2,$$

for $g_k(y) = \sigma_k \circ \cdots \circ \sigma_0(x)$ and for all $y \in P_0$.

Suppose first that $x : = g(y) \in \Sigma \cap 10B_0 \setminus Z$. Then $k(x) < \infty$. By (6.11), $x_k = g_k(y)$ satisfies $x = \lim x_k$ and

$$|x_k - x| \lesssim \varepsilon r_k.$$  

Note that for $k \geq k(x)$, $x_k = x$ by (6.6), taking also into account that $x \notin V^{40}_{k(x)}$ by the definition of $k(x)$. In fact, all $z \in B(x, \varepsilon r_k)$ satisfy $z \notin V^{39}_{k(x)}$ and thus $\sigma_k$ is the identity.
map in $B(x, r_{k(x)})$. So it follows that $\varepsilon_k(x_k) \neq 0$ only for $k \leq Ck(x)$ for some universal constant $C$. Let $k \leq Ck(x)$.

Let $z \in E_0 \cap B_0$ be such that $|x - z| < 2d(x) \lesssim r_{k(x)}$ by Lemma 6.4. Then

$$|x_k - z| \leq |x_k - x| + |x - z| \lesssim \varepsilon_k + r_{k(x)} \lesssim r_k$$

where the implicit constant is universal, and so we can pick $C_2$ large enough so that

$$B(x_k, C_2^2 r_k) \subset B(z, C_2^2 r_k + C r_k) \subset B(z, 2C_2^2 r_k).$$

Thus,

$$\varepsilon_k(x_k) \lesssim \alpha_{\mu}(x_k, C_2^2 r_k) \lesssim \alpha_{\mu}(z, 2C_2^2 r_k).$$

Hence, for $C_1 > 2C_2^2$ large enough, using (4.10) we obtain

$$\sum_{k=0}^{C_1} \varepsilon_k(x_k)^2 \lesssim \sum_{k=0}^{C_1} \alpha_{\mu}(z, C_1 r_k)^2 \lesssim \int_0^{C_1} \alpha_{\mu}(z, r)^2 dr < \varepsilon^2.$$

When $x \in \Sigma \setminus 10B_0 = P_0 \setminus B(0, 10)$ (see (6.23)), then $\varepsilon_k(x_k) = 0$ for $k \geq C$ for some large $C > 0$ because $V_{10}^0 = 10B_0$ and since by (6.20) $x_{j,k+1} \in V_{2}^{1/2}$ for all $k$, hence $x_{j,k} \in V_{2}^0$ for all $k \geq 0$. But $(V_{10}^1)^c \subset (V_{2}^3)^c \subset (10B_0)^c$, which means that $\varepsilon_k = 0$ for $k$ large enough. This proves (6.14) in the case that $x \notin Z$.

If $x \in Z$, then $d(x) = 0$ and for each $k$ there are $x_{j,k} \in E_k$ such that $|x - x_{j,k}| < r_k(1 + 1/10)$, and $x_{j,k}$ such that $|x - x_{j,k}| < 2r_k$, thus $x \in E_\infty$. Let $y \in \mathbb{R}^d$ be such that $x = g(y)$ and let $x_k = g_k(y)$, recall that $|x - x_k| \lesssim r_k$. Then by (6.38) and (6.39) we have that

$$\varepsilon_k(x_k) \lesssim \alpha_{\mu}(x, 2C_2^2 r_k).$$

We conclude (6.41) as before with $x$ instead of $z$. Thus (6.37) follows.

\[
\text{Lemma 6.9. There is a constant } C = C(n) > 0 \text{ such that the surface } \Sigma \text{ is a } C\varepsilon^{1/4}\text{-Lipschitz graph over } P_0, \text{ that is, there is a } C\varepsilon^{1/4}\text{-Lipschitz function } h : P_0 \to P_0^+ \text{ such that }
\]

$$\Sigma = \{x + h(x) : x \in P_0\}.$$ 

\[
\text{Proof. By Lemma 5.8 } \mathcal{H}^d(Z) = 0. \text{ Moreover } Z \text{ is a closed subset of } \Sigma \text{ since } d \text{ is continuous. In particular, we infer that for } \sigma\text{-almost every } x \in \Sigma, d(x) > 0. \text{ Hence, for } \sigma\text{-almost every } x \in 10B_0 \cap \Sigma, \text{ } B(x, r_{k(x)}) \cap \Sigma \text{ is a } C\varepsilon^{1/4}\text{-Lipschitz graph over } P_{j,k(x)-1} \text{ by (6.27).}
\]

Recalling that $d(x) \approx r_{k(x)}$, by (5.8) $P_{j,k(x)-1}$ is a $C\varepsilon^{1/4}$-Lipschitz graph over $P_0$, and hence so is $B(x, r_{k(x)}) \cap \Sigma$ (with another constant $C$). On the other hand, if $x \in \Sigma \setminus 10B_0$, then $x \in P_0$ by (6.23). Thus, we can cover $\Sigma$ up to a set of surface measure zero by balls $B_j$ in which $\Sigma$ is a $C\varepsilon^{1/4}$-Lipschitz graph over $P_0$. By the previous lemma, $g : P_0 \to \Sigma$ is bi-Lipschitz, and so for a.e. $z \in P_0$, $g(z) \in \bigcup_{j=1}^{N} B_j$.

The initial goal is to show that for any $x, y \in \Sigma$, $|g(x) - g(y)| \lesssim \varepsilon^{1/4} |x - y|$, which would guarantee that $\Sigma$ is included in a Lipschitz graph with constant bounded above by a constant times $\varepsilon^{1/4}$. Let $x, y \in \Sigma$ and $x', y' \in P_0$ be such that $g(x') = x$ and $g(y') = y$. 

Note that (6.11) implies that \(|x' - g(x')| = |x' - x| \lesssim \varepsilon\) and \(|y' - g(y')| = |y' - y| \lesssim \varepsilon\). If \(|x - y| \geq 1/10\) then

\[
|\pi_{P^\perp}(x - y)| \leq |\pi_{P^\perp}(x)| + |\pi_{P^\perp}(y)| \leq |x' - x| + |y' - y| \lesssim \varepsilon \lesssim |x - y|.
\]

Thus we assume that \(|x - y| < 1/10 = r_1\). Hence there exists \(k \geq 1\) such that \(r_{k+1} \leq |x - y| < r_k\). We consider two cases: either \(\max\{k(x), k(y)\} > k\) or \(\max\{k(x), k(y)\} \leq k\).

In the first case we assume without loss of generality we assume that \(k(x) > k\). If \(x \notin Z\) and \(y \in B(x, r_{k(x)}) \cap \Sigma\), then

\[
|\pi_{P^\perp_{j,k(x)-1}}(x) - \pi_{P^\perp_{j,k(x)-1}}(y)| \lesssim \varepsilon |\pi_{P_{j,k(x)-1}}(x) - \pi_{P_{j,k(x)-1}}(y)| \lesssim \varepsilon |x - y|
\]
as \(B(x, r_{k(x)}) \cap \Sigma\) is a \(C\varepsilon\) Lipschitz graph over \(P_{j,k(x)-1}\) for some \(j \in J_k\). By Lemma 6.3. Since \(x_{j,k(x)-1} \in E_0 \cap B_0\) by the choice of \(x_j, k\), \(P_{j,k}\) and \(x_{j,k}\) (see (6.17) and line above (6.18)) we have by Lemma 5.2 that \(\angle(P_{j,k(x)-1}, L_{B_0}) \leq \varepsilon^{\frac{1}{4}}\). Thus a simple geometric argument ensures that

\[
|\pi_{P^\perp_{0}}(x) - \pi_{P^\perp_{0}}(y)| \lesssim \varepsilon^{\frac{1}{4}}|\pi_{P_{0}}(x) - \pi_{P_{0}}(y)|,
\]
provided \(x, y \in \Sigma \setminus Z\) and \(\max\{k(x), k(y)\} > k\).

In the case when \(\max\{k(x), k(y)\} \leq k\), for \(g_k\) as in (xii) in Theorem 6.1, denote by \(x_k = g_k(x') \in \Sigma_{k+1}\) and \(y_k = g_k(y') \in \Sigma_{k+1}\). Iterating (6.11) we have \(|x_k - x| \lesssim \varepsilon r_k\) and \(|y_k - y| \lesssim \varepsilon r_k\). Thus, for \(\varepsilon > 0\) small enough,

\[
|x_k - y_k| \leq |x - y| + |x_k - x| + |y_k - y| \leq r_k + C\varepsilon r_k \leq 2r_k.
\]

By the construction there is \(x_{j,k}\) such that \(|x_k - x_{j,k}| \leq 10r_k\) and therefore \(|y_k - x_{j,k}| \leq 12r_k\). Hence

\[
x_k, y_k \in \Sigma_{k+1} \cap B(x, k, 12r_k) \subset \Sigma_{k+1} \cap D(x_{j,k}, P_{j,k}, 49r_k) = \Gamma_{j,k} \cap D(x_{j,k}, P_{j,k}, 49r_k)
\]
as in (6.7) where by (vii) in Theorem 6.1 \(\Gamma_{j,k}\) is a graph over \(P_{j,k}\) with constant less than \(C\varepsilon\). A similar argument to the one used above yields \(\angle(P_{j,k}, L_{B_0}) \leq \varepsilon^{\frac{1}{4}}\) which we also appeal to Remark 5.3 with \(c_0 = 1/100\), which ensures that (6.45) also holds in this case.

The inequality (6.45) proves that there exists \(C(n)\varepsilon^{\frac{1}{4}}\)-Lipschitz function \(h : P_0 \rightarrow P_0^\perp\) such that

\[
\Sigma \setminus Z \subset \{x + h(x) : x \in P_0\} = \Gamma.
\]

For \(x \in Z \subset \Sigma\) by (6.36) since \(\mu(Z) = 0\) there exists a sequence \(x_k \in \Sigma \setminus Z\) such that \(x_k \rightarrow x\) as \(k \rightarrow \infty\). By (6.46) there is \(y_k \in P_0\) such that \(x_k = y_k + h(y_k) \rightarrow x\), thus \(|y_k - y| - l| \leq |y_k + h(y_k) - (y_l + h(y_l))| + |h(y_k) - h(y_l)| \leq |x_k - x_l| + C\varepsilon l|y_k - y| \leq 2|x_k - x|\) which ensures that \(\{y_k\}\) is a Cauchy sequence. Let \(y = \lim_{k \rightarrow \infty} y_k \in P_0\). Since \(h\) is Lipschitz continuous \(x_k = y_k + h(y_k) \rightarrow y + h(y) = x\). Thus \(\Sigma \subset \Gamma\). Since \(\Sigma\) and \(\Gamma\) are both closed if here is \(x + h(x) \in \Gamma \setminus \Sigma\) with \(x \in P_0\) then since \(\Sigma \cap 10B_0 = P_0 \cap 10B_0\) there exists \(\rho > 0\) such that \(B(x + h(x), \rho) \cap (\Sigma \cup (10B_0)^c) = \emptyset\). Then the map \(\pi_{P_0} \circ g : P_0 \rightarrow P_0^\perp\) is bi-Lipschitz and satisfies \(\pi_{P_0} \circ g = I\) on \(P_0 \cap 10B_0\) which is a contradiction (via a minor degree argument).

It is worth emphasizing that the reason why in this case \(\Sigma\) is a Lipschitz graph in contrast with the general \(\Sigma\) constructed in Theorem 6.1 is that since we have that \(\mathcal{H}^d(Z) = 0\).
the construction always stops before the tilt between the original plane \( P_0 \) and the good approximating plan at a given scale gets larger than \( \varepsilon^{1/4} \).

**Lemma 6.10.** We have

\[
\int_{P_0} |Dh|^2 \, d\mathcal{H}^d \lesssim \varepsilon. \tag{6.47}
\]

**Proof.** Since \( \sigma(x) = x \) outside \( B(0,10) \) by (6.6), using some simple degree theory as in the proof of [DT12, Theorem 13.1], we know that

\[
\pi_{P_0}(B(0,10) \cap \Sigma) = B(0,10) \cap P_0.
\]

and \( h|_{P_0 \setminus B(0,10)} \equiv 0 \). Let the function \( f : P_0 \to \mathbb{R}^n \) be defined by \( f(y) = (y, h(y)) \).

Since \( h \) is a \( C{\varepsilon}^{1/4} \) Lipschitz function, by the area formula the generalized Jacobian, \( J_f \) of \( f \) is given by:

\[
J_f = \sqrt{\det \left( \delta_{ij} + \frac{\partial h}{\partial x_i} \frac{\partial h}{\partial x_j} \right)} \geq 1 + C |Dh|^2, \tag{6.48}
\]

where we have used the fact that \( |Dh| \leq C\varepsilon^{1/4} \) and a Taylor expansion for this type determinant.

From (6.48), we get

\[
\int_{P_0} |Dh|^2 \, d\mathcal{H}^d \lesssim \int_{B(0,10) \cap P_0} (J_f - 1) \, d\mathcal{H}^d = \mathcal{H}^d(B(0,10) \cap \Sigma) - \mathcal{H}^d(B(0,10) \cap P_0) \lesssim \varepsilon, \tag{6.36}
\]

as wished. \( \square \)

Note that the argument above can also be reduced by using standard results, see for example the proof of Lemma 23.10 [Mag12].

Notice that the estimate (6.47) follows from the \((1 + C\varepsilon)\)-bilipschitz character of \( f(y) = (y, h(y)) \), which in turn comes from the smallness of the \( \alpha \)-numbers ensured by the condition (4.10). If, instead, we use the property that \( h \) is \( C\varepsilon^{1/4} \)-Lipschitz coming from the stopping condition \( BA \) involving the angles that the approximating \( d \)-planes form with \( P_0 \), we get the worse estimate

\[
\int_{P_0} |Dh|^2 \, d\mathcal{H}^d \lesssim \varepsilon^{1/2},
\]

which is not useful for our purposes. The sharper inequality (6.47) plays a key role later to show that the set of points where \( BA \) holds has small measure.

**Lemma 6.11.** Let \( B \) be a ball centered on \( \Sigma \), and \( f \) a function such that

\[
||f - f(x_B)||_{L^\infty(3B \cap \Sigma)} \lesssim \varepsilon^{1/4} \quad \text{and} \quad f(x) \in (1/C, C)
\]

uniformly for all \( x \in 3B \cap \Sigma \) for some constant \( C > 0 \). Then

\[
\int_{B} \int_{J_0}^{r_B} \alpha_{f_\sigma}(x,r)^2 \frac{dr \, d\sigma(x)}{r} \lesssim \varepsilon^{1/4} r_B^{d-1}, \tag{6.49}
\]

and there is a plane \( P_B \) such that

\[
\alpha_{f_\sigma}(B) \lesssim r_B^{-d-1} F_B(f_\sigma, \mathcal{H}^d|_{P_B}) \lesssim \varepsilon^{1/4}. \tag{6.50}
\]
This result is a direct consequence of [Tol09, Theorem 1.1] and Remark 4.1 immediately proceeding it, taking into account that \( \Sigma \) is a \( C \varepsilon^{1/2} \)-Lipschitz graph. The original statement was for \( f \equiv 1 \), but the same proof works for the preceding lemma. We sketch the adjustments below, using the notation of [Tol09]. First, since \( \Sigma \) is a \( C \varepsilon^{1/2} \)-Lipschitz graph, by Whitney extension we can replace it with a \( C \varepsilon^{1/2} \)-graph \( \Gamma \) that agrees with \( \Sigma \) in \( 3B \) but is constant outside \( 4B \), and also set \( f = f(x_B) \) outside \( 3B \). By rotating, we can assume \( \Gamma \) is a graph along \( \mathbb{R}^d \).

At the beginning of the proof of [Tol09, Theorem 1.1], replace the function 
\[
g(x) := \rho(\tilde{A}(x))|J(\tilde{A})(x)|.
\]
with \( \tilde{g} = f(\tilde{A})g \). There, the graph \( \Gamma \) is a graph of a function \( A \) and \( \tilde{A}(x) = (x, A(x)) \), and in our case \( A \) is \( C \varepsilon^{1/4} \)-Lipschitz. Then the proof continues verbatim. As in [Tol09, Remark 4.1], we obtain from the proof that 
\[
\sum_{Q \in \mathcal{D}_{2n}} \alpha(Q)^2 \mu(Q) \lesssim \sum_{Q \in \mathcal{D}_{2n}} \beta_1(2Q)^2 \mu(Q) + \sum_{I \in \mathcal{D}_{2d}} ||\Delta_I \tilde{g}||^2_2
\]
and again, as in [Tol09, Remark 4.1],
\[
\sum_{Q \in \mathcal{D}_{2n}} \beta_1(2Q)^2 \mu(Q) \lesssim ||\nabla A||^2_2.
\]

In our situation, if \( \tilde{A}(x) \in 3B \), then since \( |g| \lesssim 1 \),
\[
|\tilde{g}(x) - 1| = |f(\tilde{A}(x))g(x) - f(x_B)| \leq |f(\tilde{A}(x)) - f(x_B)| \cdot |g(x)| + |f(x_B)| \cdot |1 - g(x)| 
\lesssim \varepsilon \frac{1}{2} |\mathbb{1}_{3B} + |1 - g(x)||.
\]
Hence, since \( g \lesssim 1 \) and \( |f(x_B)| \lesssim 1 \), and because \( f \equiv f(x_B) \) outside \( 3B \) and \( \nabla A = 0 \) outside the projection of \( 4B \) into \( \mathbb{R}^d \),
\[
\sum_{I \in \mathcal{D}_{2d}} ||\Delta_I \tilde{g}||^2_2 \sim ||\tilde{g} - f(x_B)||^2 \lesssim \varepsilon^{1/2} r_B^d + \int_{\mathbb{R}^d} |1 - g|^2 
\lesssim \varepsilon^{1/2} r_B^d + ||\nabla A||^2_2 \lesssim \varepsilon^{1/2} r_B^d.
\]
Now (6.50) follows by applying the same argument to a slightly larger ball, say \( \frac{3}{2} B \), then (6.49) and Chebychev’s inequality imply there must be \( r \in (\frac{4}{9} r_B, \frac{3}{2} r_B) \) and \( x \in \frac{1}{3} B \) such that
\[
\alpha_{f, \sigma}(B) \lesssim \alpha_{f, \sigma}(x, r) \lesssim \varepsilon^{1/2}.
\]

7. THE APPROXIMATING MEASURE

Let \( \theta_j \) be a partition of unity subordinated to the balls \( B_j \), belonging to the Besicovitch subcovering of balls as in (6.29), and satisfying
\[
0 \leq \theta_j \leq 1, \quad \text{Lip}(\theta_j) \approx s_j^{-1}, \quad \frac{3}{2} B_j \subseteq \text{supp} \theta_j \subseteq 2B_j
\]
and
\[
\theta := \sum_j \theta_j \equiv 1 \quad \text{on} \quad \bigcup \frac{3}{2} B_j =: O.
\]
Note that, by the finite superposition of the balls $B_j$ we may assume that
\[ \theta_j(x) \approx 1 \quad \text{for all } x \in B_j. \]

Define
\begin{equation}
(7.3) \quad c_j := \begin{cases} 
\int \frac{\theta_j d\mu}{\mathcal{C}_{2B_0}} / \int \theta_j d\sigma & \text{if } 2B_j \subset B(0, 10), \\
\text{if } 2B_j \not\subset B(0, 10), 
\end{cases}
\end{equation}
and
\begin{equation}
(7.4) \quad d\nu := \sum_j c_j \theta_j d\sigma.
\end{equation}

Note that $\nu \ll \sigma = \mathcal{H}^d|_\Sigma$. Note that by the way we have chosen the $c_j$, we have
\begin{equation}
(7.5) \quad \nu|_{B(0, 10)^c} = c_{C, B_0} \mathcal{H}^d|_{P_j \setminus B(0, 10)} = \mathcal{C}_{B_0}|_{B(0, 10)^c}.
\end{equation}

**Lemma 7.1.** For all $j$ such that $2B_j \subset B(0, 10)$,
\begin{equation}
(7.6) \quad |c_j - c_{\tilde{B}_j}| \lesssim_{A, \tau} \varepsilon.
\end{equation}

**Proof.** Recall that by (6.33), $3B_j \cap \Sigma$ is a $C\varepsilon$-Lipschitz graph over $P_j$, and also that $3B_j \subset \tilde{B}_j$, $r_{\tilde{B}_j} \approx r_{B_j}$ (as in (6.32)), and $P_j$ passes through the center of $\tilde{B}_j$. Then as in (6.34)
\begin{equation}
(7.7) \quad F_{3B_j}(\sigma, \mathcal{H}^d|_{P_j}) \lesssim_{A, \tau} \varepsilon s_j^{d+1}.
\end{equation}

Recalling that $\text{Lip}_1(\theta_j) \lesssim s_j^{-1}$, we have
\[ s_j |c_j - c_{\tilde{B}_j}| \lesssim |c_j - c_{\tilde{B}_j}| \int \theta_j d\sigma = \left| \int \theta_j d\mu - \int \theta_j d\sigma \right| \leq \left| \int \theta_j d\mu - \int \theta_j c_{\tilde{B}_j} d\sigma \right| + \left| \int \theta_j c_{\tilde{B}_j} d\sigma - \int \theta_j d\sigma \right| \lesssim_{A, \tau} \varepsilon s_j^{d}. \]

**Lemma 7.2.** The measure $\nu$ is $d$-AD-regular with constants depending on $A$ and $\tau$, that is,
\begin{equation}
(7.8) \quad \nu(B(x, r)) \approx_{A, \tau} r^d \quad \text{for all } x \in \Sigma \text{ and } r > 0.
\end{equation}

**Proof.** Note that by (5.15) and the definition of $c_j$, using Lemma 5.6, Remark 5.3 and Remark 5.7, we have $c_j \approx_{A, \tau} 1$ for all $j$. Thus,
\begin{equation}
(7.9) \quad d\sigma \lesssim_{A, \tau} \sum_j c_j \theta_j d\sigma \lesssim_{A, \tau} d\sigma,
\end{equation}
which by Lemma 6.8 ensures that $\nu$ is AD-regular.

**Lemma 7.3.** If $2B_i \cap 2B_j \neq \emptyset$, then
\begin{equation}
(7.10) \quad |c_i - c_j| \lesssim_{A, \tau} \varepsilon.
\end{equation}

**Thus,**
\begin{equation}
(7.11) \quad \left| \sum_j c_j \theta_j(x) - c_i \right| \lesssim \varepsilon \quad \text{for all } x \in \Sigma \cap 2B_i.
\end{equation}
Proof. Let \( B_i \) and \( B_j \) be such that \( 2B_i \cap 2B_j \neq \emptyset \). Then we have

\[
s_i = \eta d(\xi_i) \leq \eta d(\xi_j) + \eta |\xi_i - \xi_j| \leq s_j + 2\eta (s_i + s_j)
\]

and since \( \eta = 1/1000 \) we get \( s_i \leq (2 + 4\eta)s_j \leq 3s_j \). Therefore, by symmetry,

\[
\frac{1}{3} s_j \leq s_i \leq 3s_j.
\]

Since \( \tilde{B}_i \supset 2B_i \) and \( \tilde{B}_j \supset 2B_j \) for \( C_2 \) large enough, we derive

\[
\tilde{B}_i \subset 4\tilde{B}_j \quad \text{and} \quad \tilde{B}_j \subset 4\tilde{B}_i.
\]

First assume both \( 2B_i \) and \( 2B_j \) are contained in \( B(0, 10) \). Then,

\[
|c_i - c_j| \leq |c_i - c_{\tilde{B}_i}| + |c_{\tilde{B}_i} - c_{4\tilde{B}_j}| + |c_{4\tilde{B}_j} - c_{\tilde{B}_j}| + |c_{\tilde{B}_j} - c_j|
\]

\[
\lesssim_{A, r} \alpha_\mu(4\tilde{B}_j) + \varepsilon \lesssim \varepsilon.
\]

Suppose now that \( 2B_i \subset B(0, 10) \) and \( 2B_j \supset B(0, 10) \). From the fact that \( s_j \leq 1/20 \) by (6.31), it follows that \( B_j \cap B(0, 9) = \emptyset \), and thus \( s_j \approx d(\xi_j) \approx 1 \). So we have \( s_i \approx s_j \approx 1 \). Thus, (7.10) follows by similar estimates to above using the fact that \( c_j = c_{C_2 B_0} \).

Finally, if both \( 2B_i \) and \( 2B_j \) are not contained in \( B(0, 10) \), then \( c_i = c_j = c_{C_2 B_0} \) and so (7.10) is trivial. \( \square \)

Lemma 7.4. For all \( x \in B(0, 20) \) with \( 2d(x) < r < 20 \), and \( \theta \) as in (7.1)

(7.12) \[
F_{B(x,r)}(\mu, \theta \mu) \lesssim_A \varepsilon^\frac{1}{\tau} r^{d+1}.
\]

Proof. Let \( \phi \in \text{Lip}_1(B(x, r)) \). Since \( 4d(x) < r < C_1 \), we have

\[
\frac{\mu(B(x, r) \setminus E_0)}{\mu(B(x, r))} \lesssim_M \varepsilon^\frac{1}{2}.
\]

\[
\left| \int \phi \, d\mu - \int \phi \, d\mu|_{E_0} \right| \leq r \mu(B(x, r) \setminus E_0) \lesssim_M r \varepsilon^\frac{1}{2} \mu(B(x, r)) \lesssim_A \varepsilon^\frac{1}{\tau} r^{d+1}
\]

which implies \( F_{B(x,r)}(\mu, \mu|_{E_0}) \lesssim_{A, M} \varepsilon^\frac{1}{\tau} r^{d+1} \). Similarly, \( F_{B(x,r)}(\theta \mu, \theta \mu|_{E_0}) \lesssim_A \varepsilon^\frac{1}{\tau} r^{d+1} \). So it suffices to show that \( \mu|_{E_0} = \theta \mu|_{E_0} \), which is equivalent to saying that \( \theta \equiv 1 \) \( \mu \)-a.e. in \( E_0 \).

Let \( y \in E_0 \cap B(x, r) \setminus Z \subset B(0, 40) \). We wish to show that \( y \in \frac{3}{2} B_j \) for some \( j \). Let \( k = k(y) \). Then \( k > 0 \) since \( y \in 40B_0 \), and so \( y \in V_{k-1}^{40} \), hence \( y \in 40B_{i,k-1} \) for some \( j \in J_{k-1} \). For \( \varepsilon > 0 \) small enough depending on \( M \) and \( \eta \), by Lemma 3.2, recalling that \( d(y) \approx r_{k-1} \),

\[
\text{dist}(y, P_{j,k-1}) \leq \min\{\eta^2 d(y), r_{k-1}\}.
\]

Also, since \( y \in 40B_{j,k-1} \), we have \( \pi_{j,k-1}(y) \in 40B_{j,k-1} \). Then, for \( \varepsilon > 0 \) small,

\[
\text{dist}(y, \Sigma) \leq |y - \pi_{j,k-1}(y)| + \text{dist}(\pi_{j,k-1}(y), \Sigma)
\]

\[
\leq \eta^2 d(y) + C\varepsilon r_{k-1}
\]

\[
\leq \eta^2 d(y) + C\varepsilon^2 d(y) < 2\eta^2 d(y).
\]
Let \( z \in \Sigma \) be such that
\[
\text{dist}(y, \Sigma) = |y - z| < 2\eta^2 d(y) \leq \frac{d(y)}{2}.
\]
Then,
\[
d(z) \geq d(y) - |y - z| \geq d(y)/2 > 0.
\]
Hence, \( z \in B_j \) for some \( j \) and
\[
s_j = \eta d(\xi_j) \geq \eta(d(y) - |\xi_j - z| - |z - y|) \geq \eta \left( d(y) - s_j - \frac{d(y)}{2} \right) \geq \frac{\eta}{2} d(y) - \eta s_j.
\]
Therefore,
\[
(7.13) \quad s_j \geq \frac{\eta}{2(1 + \eta)} d(y) \geq \frac{\eta}{4} d(y).
\]
In particular,
\[
y \in B(z, 2\eta^2 d(y)) \subset B(\xi_j, s_j + 2\eta^2 d(y)) \subset B(\xi_j, s_j(1 + 8\eta)) \subset \frac{3}{2} B_j.
\]
This proves
\[
(7.14) \quad E_0 \cap 40 B_0 \subset \bigcup_j \frac{3}{2} B_j = O.
\]
So \( \theta \equiv 1 \) on \( O \) and, since \( \mu(Z) = 0 \), we deduce that thus \( \theta \equiv 1 \) \( \mu \)-a.e. on \( E_0 \), as wished. \( \square \)

**Lemma 7.5.** For \( x \in \Sigma \cap B(0, 20) \) and \( 0 < r \leq 15 \),
\[
(7.15) \quad F_{B(x,r)}(\nu,\theta\mu) \lesssim_A, r \sum_{2B_j \cap B(x,r) \neq \emptyset} \xi_j \frac{s_j^{d+1}}{}.
\]

**Proof.** Let
\[
J(x,r) = \{ j : 2B_j \cap B(x,r) \neq \emptyset \}.
\]
Observe that for \( j \in J(x,r) \), since \( x \in B(0,20) \) and \( r \leq 15 \), we have \( 2B_j \cap B(0,35) \neq \emptyset \), so \( \xi_j \in 40 B_0 \) by Lemma 6.6. Let \( \phi \in \text{Lip}_1(B(x,r)) \). Then
\[
\int \phi \, d\nu = \sum_{j \in J(x,r)} \int \phi \theta_j c_j \, d\sigma = \sum_{j \in J(x,r)} (\phi(\xi_j)) \theta_j c_j \, d\sigma + \sum_{j \in J(x,r)} \phi(\xi_j) \int \theta_j c_j \, d\sigma =: I_1 + I_2.
\]
We will estimate the two sums separately. Note that \( \text{Lip}((\phi - \phi(\xi_j))\theta_j) \lesssim 1 \) and \( c_j \approx A, r \approx 1 \), and so, with constants \( C \) depending on \( A \) and \( r \),

\[
I_1 = \sum_{j \in J(x,r)} \int (\phi - \phi(\xi_j))\theta_j c_j d\sigma 
\leq \sum_{j \in J(x,r)} \left( \int (\phi - \phi(\xi_j))\theta_j c_B^d d\mathcal{H}^d|_{P_j} + C\varepsilon s_j^{d+1} \right) 
\leq \sum_{j \in J(x,r)} \left( \int (\phi - \phi(\xi_j))\theta_j c_B^d d\mathcal{H}^d|_{P_j} + C\varepsilon s_j^{d+1} \right) 
\leq \sum_{j \in J(x,r)} \left( \int (\phi - \phi(\xi_j))\theta_j d\mu + C\varepsilon s_j^{d+1} \right) 
\]

Let \( J_1 \) be those \( j \in J(x,r) \) for which \( 2B_j \subset B(0,1) \) and \( J_2 = J(x,r) \setminus J_1 \). We split

\[
I_2 = \sum_{j \in J_1} \phi(\xi_j) \int \theta_j c_j d\sigma + \sum_{j \in J_2} \phi(\xi_j) \int \theta_j c_j d\sigma =: I_{21} + I_{22}.
\]

We now estimate these two terms separately. First,

\[
I_{21} = \sum_{j \in J_1} \phi(\xi_j) \int \theta_j d\mu.
\]

For \( I_{22} \), note that if \( 2B_j \cap B(0,1)c \neq \emptyset \), then \( s_j \approx 1 \) and so \( r \approx s_j \) and there number of such \( j \)'s is bounded above by a constant only depending on \( n \). Thus, \( |\phi(\xi_j)| \lesssim r \lesssim 1 \) and moreover, for \( C_2 \) large enough, \( \bigcup_{j \in J_2} 2B_j \subset C_2B_0 \). Also, since \( \Sigma \) is a \( C\varepsilon^{\frac{1}{4}} \)-Lipschitz graph over \( P_0 \), we know

\[
F_{C_2B_0}(\sigma, \mathcal{H}^d|_{P_0}) \lesssim \varepsilon^\frac{1}{4}.
\]

We can thus estimate

\[
I_{22} = \sum_{j \in J_2} \phi(\xi_j) \int \theta_j c_{C_2B_0} d\sigma 
\leq \sum_{j \in J_2} \phi(\xi_j) \int \theta_j c_{C_2B_0} d\mathcal{H}^d|_{P_0} + C \sum_{j \in J_2} |\phi(\xi_j)| \varepsilon^\frac{1}{4} 
\leq \sum_{j \in J_2} \phi(\xi_j) \int \theta_j d\mu + C \sum_{j \in J_2} |\phi(\xi_j)| (\alpha_\mu(C_2B_0) + \varepsilon^\frac{1}{4}) 
\leq \sum_{j \in J_2} \phi(\xi_j) \int \theta_j d\mu + C \varepsilon^\frac{1}{4} s_j^{d+1}.
\]

Thus,

\[
I_2 \leq I_{21} + I_{22} = \sum_{j \in J(x,r)} \left( \phi(\xi_j) \int \theta_j d\mu + \varepsilon^\frac{1}{4} s_j^{d+1} \right).
\]
Hence,
\[
\int \phi \, d\nu = I_1 + I_2 \\
\leq \sum_{j \in J(x,r)} \int (\phi - \phi(\xi_j)) \theta_j \, d\mu + C \sum_{j \in J(x,r)} \varepsilon \frac{1}{2} s_j^{d+1} + \sum_{j \in J(x,r)} \phi(\xi_j) \int \theta_j \, d\mu \\
= \int \phi \, d\nu + C \sum_{j \in J(x,r)} \varepsilon \frac{1}{2} s_j^{d+1}.
\]

We can similarly show a converse inequality, and this proves the lemma. \(\square\)

An immediate consequence of the previous two lemmas is the following.

**Lemma 7.6.** For all \(x \in \Sigma \cap B(0,20)\) with \(2d(x) < r \leq 15\),

\[
F_{B(x,r)}(\nu,\mu) \lesssim_{A,\tau} \varepsilon \frac{1}{2} r^{d+1} + \varepsilon \frac{1}{2} \sum_{2B_j \cap B(x,r) \neq \emptyset} s_j^{d+1}.
\]

**Lemma 7.7.** If \(1 < r, x \in \Sigma,\) and \(B(x, r) \cap B(0,10) \neq \emptyset,\) then

\[
\alpha_\nu(x, r) \lesssim_{A,\tau} \varepsilon \frac{1}{2} r^{-d}.
\]

**Proof.** Let \(\psi\) be a 1-Lipschitz function that is zero on \(B(0,11)^c\) and 1 on \(B(0,10)\). Set

\[
\tilde{c} = \frac{\int \psi \, d\nu}{\int \psi \, d\mathcal{H}^d|_{P_0}}.
\]

Note that the collection \(\{B_j\}\) has finite overlap depending only on \(n\). Moreover if \(2B_j \cap B(x, r) \neq \emptyset,\) with \(B_j = B(\xi_j, s_j),\) \(\xi_j \in \Sigma, s_j = \eta d(\xi_j) \leq \eta(d(x) + 2s_j + r) \leq \eta(3r/2 + 2s_j)\) with \(\eta = 10^{-3}\) which yields \(s_j \leq 2\eta r \) and therefore using (7.8) we have

\[
\sum_{2B_j \cap B(x,r) \neq \emptyset} s_j^{d+1} \lesssim r \sum_{2B_j \cap B(x,r) \neq \emptyset} \sigma(B_j) \lesssim r \sigma(B(x,2r)) \lesssim r^{d+1}.
\]

Then using the fact that \(C_2 \leq C_1\) (see line above (6.17)), (4.13) and (7.16), we have

\[
\tilde{c} \int \psi \, d\mathcal{H}^d|_{P_0} = \int \psi \, d\nu \overset{(7.16)}{\leq} \int \psi \, d\mu + C \varepsilon \frac{1}{2}
\]

\[
\leq \int \psi \, dL_{C_2B_0} + C \alpha_0(0, C_2) + C \varepsilon \frac{1}{2} \leq C_{C_2B_0} \int \psi \, d\mathcal{H}^d|_{P_0} + C \varepsilon \frac{1}{2}.
\]

Since \(\int \psi \, d\mathcal{H}^d|_{P_0} \approx 1,\) this gives \(\tilde{c} \leq C_{C_2B_0} + C \varepsilon \frac{1}{2}\) for \(\varepsilon > 0\) small enough, where \(C\) depends on \(A\) and \(\tau.\) An opposite inequality can be proved by a similar argument. Thus, \(|\tilde{c} - C_{C_2B_0}| \lesssim_{A,\tau} \varepsilon \frac{1}{2}.\) Hence, for \(\phi \in \text{Lip}_1(B(x, r)),\) and since \(\nu = L_{C_2B_0}\) in \(B(0,10)^c\) and \(1 < r\) by (7.5),

\[
\left| \int \phi (d\nu - dL_{C_2B_0}) \right| = \left| \int [(\phi - \phi(0))\psi + \phi(0)\psi + \phi(1 - \psi)] (d\nu - dL_{C_2B_0}) \right| \\
\lesssim \alpha_\nu(0, C_2) + |\phi(0)| \left| \int \psi (d\nu - dL_{C_2B_0}) \right| + 0 \\
\lesssim \alpha_\nu(0, C_2) + r \left( \varepsilon \frac{1}{2} + \left| \int \psi (d\nu - \tilde{c} d\mathcal{H}^d|_{P_0}) \right| \right) \lesssim r \varepsilon \frac{1}{2}.
\]
Thus, (7.17) follows by this and (7.8). □

8. $\Lambda$-estimates

For the rest of the paper we denote by $\phi: \mathbb{R}^n \to \mathbb{R}$ a radial $C^\infty$ function such that $\chi_{B(0,1/2)} \leq \phi \leq \chi_{B(0,1)}$. We also set

$$\phi_r(x) = r^{-d} \phi(r^{-1}x)$$

and

$$\psi_r(x) = \phi_r(x) - \phi_{2r}(x).$$

Let $\pi$ be the orthogonal projection onto $P_0$ and let $\pi[\nu]$ denote the image measure of $\nu$ by $\pi$, that is, the measure such that

$$\pi[\nu](G) = \nu(\pi^{-1}(G))$$

for any Borel subset $G \subset P_0$.

Note that (7.9) ensures that $\nu$ and $\sigma$ are comparable measures on $\Sigma$. Since $\Sigma$ is a Lipschitz graph over $P_0$ (see Lemma 6.9) then $\mathcal{H}^d|_{P_0}$ and $\pi[\nu]$ are mutually absolutely continuous, in fact they are comparable.

The goal of this section is to prove the following.

**Lemma 8.1.** Let $f = \frac{d\nu}{d\mathcal{H}^d|_{P_0}}$. Then

$$\|f - cC_2B_0\|^2_{L^2(\mathcal{H}^d|_{P_0})} \approx \int_{P_0} \int_0^\infty |\psi_r * \pi[\nu](z)|^2 \frac{dr}{r} d\mathcal{H}^d(z) \lesssim_{A,r} \varepsilon^{\frac{1}{2}}. \tag{8.2}$$

The first comparison above is a classical result from harmonic analysis (see [Ste93, Section I.6.3]), so we just will focus on the second inequality.

For a measure $\lambda$ and $x \in \mathbb{R}^n$, we define

$$\tilde{\psi}_r(x) = \psi_r \circ \pi(x) \cdot \phi((5r)^{-1}x)$$

and

$$\Lambda_\lambda(x,r) = \left| \int \tilde{\psi}_r(y-x) \, d\lambda(y) \right|.$$

Since $\Sigma$ is a $C\varepsilon^{\frac{1}{2}}$-Lipschitz graph over $P_0$, we claim that for $\varepsilon > 0$ small enough, then

$$\Lambda_\nu(x,r) = |\psi_r * \pi[\nu](\pi(x))| \quad \text{for all } x \in \Sigma. \tag{8.3}$$

Indeed, it suffices to show that

$$\tilde{\psi}_r(y-x) = \psi_r(\pi(y-x)) \quad \text{for all } x, y \in \Sigma. \tag{8.4}$$

To this end, by the definition of $\tilde{\psi}_r$ it suffices to check that $\phi((5r)^{-1}(y-x)) = 1$ whenever $\psi_r(\pi(y-x)) \neq 0$. Note that the latter condition implies that $\phi_{2r}(\pi(x-y)) \neq 0$ and so $|\pi(x) - \pi(y)| \leq 2r$. In fact if $\phi_{2r}(\pi(x-y)) = 0$ then $\psi_r(\pi(x-y)) = 0$ and $\psi_r(\pi(y-x)) = 0$. Thus, since $\Sigma$ is a $C\varepsilon^{\frac{1}{2}}$-Lipschitz graph,

$$|x-y| \leq 2(1 + C\varepsilon^{\frac{1}{2}})r < \frac{5}{2}r.$$

Thus, $y \in B(x, 5r/2)$, which implies that $\phi((5r)^{-1}(y-x)) = 1$, as wished.
Consequently, since \( \pi \) is bi-Lipschitz between \( \Sigma \) and \( P_0 \), we have
\[
\int_{P_0} \int_0^\infty |\psi_r * \pi [\nu](z)|^2 \frac{dr}{r} d\mathcal{H}^d(z) \approx \int_{P_0} \int_0^\infty |\psi_r * \pi [\nu](z)|^2 \frac{dr}{r} d\pi [\sigma](z)
\]
\[
= \int_\Sigma \int_0^\infty \Lambda_\nu(x,r)^2 \frac{dr}{r} d\sigma(z).
\]
Therefore, to complete the proof of Lemma 8.1, it suffices to show that
\[
\int_\Sigma \int_0^\infty \Lambda_\nu(x,r)^2 \frac{dr}{r} d\sigma(z) \lesssim A_\tau \epsilon^{\frac{1}{4}}.
\]
The rest of this section is devoted to proving this estimate.

First we need the following auxiliary result.

**Lemma 8.2.** For a finite Borel measure \( \lambda \), denote
\[
T_\lambda(x) = \left( \int_0^\infty |\tilde{\psi}_r * \lambda(x)|^2 \frac{dr}{r} \right)^{\frac{1}{2}},
\]
and for \( f \in L^2(\sigma) \), set \( T_\sigma f = T(f \sigma) \). Then \( T_\sigma \) is bounded in \( L^p(\sigma) \) for \( 1 < p < \infty \) and \( T \) is bounded from \( M(\mathbb{R}^n) \) to \( L^{1,\infty}(\sigma) \). Further, the norms \( \|T_\sigma\|_{L^p(\sigma) \to L^p(\sigma)} \) and \( \|T\|_{M(\mathbb{R}^n) \to L^{1,\infty}(\sigma)} \) are bounded above by some absolute constants depending only on \( p, n, \) and \( d \).

The proof of this lemma is quite standard in Calderón-Zygmund theory. First one shows that \( T_\sigma \) is bounded in \( L^2(\sigma) \), taking into account (8.4). By a suitable Calderón-Zygmund decomposition, one can derive then the boundedness of \( T \) from \( M(\mathbb{R}^n) \) to \( L^{1,\infty}(\sigma) \), which implies the boundedness of \( T_\sigma \) in \( L^p(\sigma) \) for \( 1 < p < \infty \) by interpolation. The boundedness in \( L^p(\sigma) \) for \( 2 < p < \infty \) can be deduced by interpolation from its boundedness from \( L^{\infty}(\sigma) \) to \( BMO(\sigma) \). See [TT15, Theorem 5.1] and [Tol17a, Proposition 13.7] for quite similar (but somewhat more difficult) results. We skip the details.

**Lemma 8.3.** We have
\[
\int_{\Sigma \cap 20B_0} \int_0^1 \Lambda_\nu(x,r)^2 \frac{dr}{r} d\sigma(x) \lesssim \epsilon^{\frac{1}{2}}.
\]
**Proof.** For each \( x \in \Sigma \cap 20B_0 \), we split
\[
\int_0^1 \Lambda_\nu(x,r)^2 \frac{dr}{r} \lesssim \int_0^1 \left( \Lambda_\nu(x,r) - \Lambda_{\theta \mu}(x,r) \right)^2 \frac{dr}{r} + \int_0^1 \Lambda_{(1-\theta) \mu}(x,r)^2 \frac{dr}{r} + \int_0^1 \Lambda_\mu(x,r)^2 \frac{dr}{r},
\]
and denote
\[
H = \left\{ x \in \Sigma \cap 20B_0 : \int_0^1 \Lambda_{(1-\theta) \mu}(x,r)^2 \frac{dr}{r} > \frac{1}{\epsilon^2} \right\}.
\]
Now write
\[ \int_{\Sigma \cap 20B_0} \int_0^1 \Lambda_{\nu}(x, r)^2 \frac{dr}{r} \, d\sigma(x) = \int_H \int_{\eta^2 d(x)} \Lambda_{\nu}(x, r)^2 \frac{dr}{r} \, d\sigma(x) \]
\[ + \int_{\Sigma \cap 20B_0 \setminus H} \int_0^1 \Lambda_{\nu}(x, r)^2 \frac{dr}{r} \, d\sigma(x) \]
(8.8)
To estimate the first integral on the right hand side, note that if \( x \in 20B_0 \) and \( r < 1 \), then \( B(x, r) \subset 40B_0 \). So applying Lemma 8.2 with \( \lambda = (\mu - \theta \mu) 40B_0 \), using the fact that \( \theta \equiv 1 \) on \( O \supset E_0 \cap 40B_0 \) by (7.14), by the definition of \( E_0 \), and (4.11) (provided \( C_1 > 40 \)) we get
\[ \sigma(H) \leq \sigma \left( \left\{ x \in \Sigma : T(\chi_{40B_0} (\mu - \theta \mu)) > \varepsilon \right\} \right) \]
\[ \lesssim \varepsilon^{-\frac{1}{2}} \|\mu - \theta \mu\|_{40B_0} \leq \varepsilon^{-\frac{1}{2}} \mu(40B_0 \setminus E_0) \leq \varepsilon^2 \mu(40B_0) \lesssim \varepsilon^2. \]
Consider the function \( q = \frac{d\nu_{C_1 \eta}}{d\sigma} \). Taking into account that \( \|q\|_{L^4(\sigma)} \lesssim A \), and the \( L^4(\sigma) \) boundedness of \( T_\sigma \) (and recalling \( T_\sigma(q) = T(\nu) \)), we obtain
\[ \int_H \int_{\eta^2 d(x)} \Lambda_{\nu}(x, r)^2 \frac{dr}{r} \, d\sigma(x) \leq \int_H |T_\sigma q(x)|^2 \, d\sigma(x) \lesssim \sigma(H)^{\frac{1}{2}} \|T_\sigma q\|_{L^4(\sigma)}^2 \lesssim A \varepsilon. \]
Next we consider the second integral on the right hand side of (8.8). By the definition of \( H \), we have
\[ \int_{\Sigma \cap 20B_0 \setminus H} \int_0^1 \Lambda_{(1-\theta)\mu}(x, r)^2 \frac{dr}{r} \lesssim \varepsilon^\frac{1}{4}, \]
and so, by (8.7),
\[ \int_{\Sigma \cap 20B_0 \setminus H} \int_0^1 \Lambda_{\mu}(x, r)^2 \frac{dr}{r} \, d\sigma(x) \lesssim \int_{\Sigma \cap 20B_0} \int_0^1 \Lambda_{\nu}(x, r)^2 \frac{dr}{r} \, d\sigma(x) \]
\[ + \int_{\Sigma \cap 20B_0 \setminus H} \int_0^1 \Lambda_{\mu}(x, r)^2 \frac{dr}{r} \, d\sigma(x) + \varepsilon^\frac{1}{4} \]
(8.10)
We will now bound \( I_2 \). Given \( x \in \Sigma \cap 20B_0 \setminus Z \), by Lemma 6.4 we have \( d(x) \approx r_k(x) \) and by the definition of \( k(x) \), \( x \in V_k^{40}(x) \). Thus there exists some ball \( B_{j,k}(x) \) such that \( x \in 40B_{j,k}(x) \). So for all \( r \in (\eta^2 d(x), 1) \) there exists some ball \( B_{j,k} \) such that \( B(x, 5r) \subset 3B_{j,k} \) and \( r \approx r_{B_{j,k}} \). Then, taking into account that \( |\nabla \tilde{\psi}_r| \lesssim r^{-d-1} \) and (6.21),
\[ \Lambda_{\mu}(x, r) \lesssim \left| \int \tilde{\psi}_r(y - x) \, d\mu(y) - dC_{j,k}(y) \right| \]
\[ \lesssim \frac{r_{2C_2} B_{j,k} \mu(2C_2 B_{j,k})}{r^{d+1}} \right| \alpha_{j}(2C_2 B_{j,k}) + \int \tilde{\psi}_r(y - x) \, dC_{j,k}(y) \right| \]
(8.11)
The first term on the right hand side satisfies
\[ \frac{r_{2C_2} B_{j,k} \mu(2C_2 B_{j,k})}{r^{d+1}} \alpha_{j}(2C_2 B_{j,k}) \lesssim A \alpha_{j}(2C_2 B_{j,k}) \lesssim A \alpha_{j}(x, k, C_2 r), \]
for a suitable constant $C_3 > 1$, so that in particular $C_3 r > d(x,j,k)$.

Now we claim that the last integral on the right hand side of (8.11) vanishes. To prove
this, first we will check that $\tilde{\psi}_r(y-x) = \psi_r \circ \pi(y-x)$ for $x \in \Sigma$ and $y \in P_{j,k}$ with
$y - x \in \text{supp } \psi_r \circ \pi$. Indeed, the latter condition implies that
\[
|\pi(y) - \pi(x)| \leq 2r.
\]
Also using that $\Sigma$ is a $C \varepsilon^{\frac{1}{4}}$-Lipschitz graph, that $\angle(P_{j,k}, P_0) < \varepsilon^{\frac{1}{4}}$, that $x \in 3B_{j,k}$, (6.9),
and (6.12), it easily follows that
\[
|\pi^\perp(y) - \pi^\perp(x)| \lesssim \varepsilon^{\frac{1}{4}} r,
\]
asuming $\varepsilon$ small enough. Then we infer that
\[
|x - y| \leq 2r + C \varepsilon^{\frac{1}{4}} r \leq \frac{5}{2} r.
\]
Thus $\phi((5r)^{-1}(x - y)) = 1$ and so
\[
\tilde{\psi}_r(y-x) = \phi((5r)^{-1}(x - y)) \psi_r \circ \pi(y-x) = \psi_r \circ \pi(y-x).
\]
Now we derive
(8.12)
\[
\int \tilde{\psi}_r(y-x) \, dL_{j,k}(y) = \int \psi_r(\pi(y)-\pi(x)) \, dL_{j,k}(y) = \int \psi_r(y'-\pi(x)) \, d\pi[L_{j,k}](y') = 0,
\]
taking into account the definition of $\psi_r$ and that $\pi[L_{j,k}]$ coincides with $d$-dimensional
Lebesgue measure on $P_0$ modulo a constant factor.

Consequently, from (8.11) we deduce that
\[
\Lambda_\mu(x, r) \lesssim_A \alpha_\mu(x,j,k, C_3 r).
\]
Therefore, for $C_1$ big enough,
\[
\int_{\eta^2 d(x)}^1 \Lambda_\mu(x, r)^2 \frac{dr}{r} \lesssim \int_0^1 \alpha_\mu(x,j,k, C_3 r)^2 \frac{dr}{r} \lesssim \int_0^{C_1} \alpha_\mu(x,j,k, t)^2 \frac{dt}{t} \lesssim \varepsilon,
\]
and thus using that fact that $\mu(B_0) = 1$, (4.9) and taking $C_1 > 40$ we have
\[
I_2 = \int_{\Sigma \cap 2B_0} \int_{\eta^2 d(x)}^1 \Lambda_\mu(x, r)^2 \frac{dr}{r} \, d\sigma(x) \lesssim \varepsilon.
\]

Finally we handle the integral $I_1$ in (8.10). Recall that
\[
|\Lambda_\nu(x, r) - \Lambda_\theta_\mu(x, r)|^2 \lesssim \left( \frac{\eta^{-d-1} F_{B(x, 5r)}(\nu, \theta \mu)}{r} \right)^2 \lesssim \left( \sum_{2B_j \cap B(x, 5r) \neq \emptyset} \frac{\varepsilon^{\frac{1}{4}} s_j^{d+1}}{r^{d+1}} \right)^2.
\]
Observe that if $B(x, 5r) \cap 2B_j \neq \emptyset$, for $r \in (\eta^2 d(x), 1)$ (recall $\eta = 10^{-3}$) then
\[
s_j = \eta d(\xi_j) \leq \eta(d(x) + 2s_j + 5r) \leq 10^3 r + 2\eta s_j + r = 1001 r + 2\eta s_j
\]
and so $s_j \lesssim 1100 r$ since $\eta < 1/4$. Then, it follows easily that $B_j \subset C_4 B_0$, for a large
enough constant $C_4 > 1$. Hence, by the Cauchy-Schwarz inequality, plus an argument
along the lines of the one used to show (7.18), we have
\[
\left( \sum_{2B_j \cap B(x, 5r) \neq \emptyset} \frac{s_j^{d+1}}{r^{d+1}} \right)^2 \leq \left( \sum_{2B_j \cap B(x, 5r) \neq \emptyset} \frac{s_j^{d+2}}{r^{d+2}} \right) \left( \sum_{2B_j \cap B(x, 5r) \neq \emptyset} \frac{s_j^d}{r^d} \right) \leq \sum_{2B_j \cap B(x, 5r) \neq \emptyset} \frac{s_j^{d+2}}{r^{d+2}}.
\]
This and the fact that \( s_j \leq 1100r \) imply
\[
I_1 = \int_{\Sigma \cap 20B_0} \int_0^1 (\Lambda_{\nu}(x, r) - \Lambda_{\theta\mu}(x, r))^2 \frac{dr}{r} d\sigma(x)
\leq \varepsilon \int_{\Sigma \cap 20B_0} \int_0^1 \frac{dr}{r} d\sigma(x)
\sum \frac{s_j^{d+2}}{r^{d+3}} d\sigma(x)
\leq \varepsilon \sum \frac{s_j^{d+2}}{r^{d+3}} \int_{\Sigma \cap 20B_0} \int_0^1 \frac{dr}{r^{d+3}} d\sigma(x)
\leq \varepsilon \sum \frac{s_j^{d+2}}{r^{d+3}} \int_{\Sigma \cap 20B_0} \int_0^1 \frac{dr}{r^{d+3}} d\sigma(x).
\]
Observe now that if \( B(x, 5r) \cap 2B_j \neq \emptyset \), then
\[
x \in B(\xi_j, 5r + 2s_j) \subset B(\xi_j, 2250r),
\]
recalling that \( s_j \leq 1100r \). Therefore,
\[
I_1 \leq \varepsilon \sum \frac{s_j^{d+2}}{r^{d+3}} \int_{\Sigma \cap 20B_0} \int_0^1 \frac{dr}{r^{d+3}} d\sigma(x)
\leq A \varepsilon \sum \frac{s_j^{d+2}}{r^{d+3}} \int_{\Sigma \cap 20B_0} \int_0^1 \frac{dr}{r^{d+3}} d\sigma(x)
\leq \varepsilon \sum \frac{s_j^{d+2}}{r^{d+3}} \int_{\Sigma \cap 20B_0} \int_0^1 \frac{dr}{r^{d+3}} d\sigma(x)
\leq \varepsilon \sigma(C_4 B_0) \leq \varepsilon.
\]
Gathering (8.7), (8.9), and the estimates obtained for \( I_1 \) and \( I_2 \), the lemma follows. \( \Box \)

**Lemma 8.4.** For all \( j \),
\[
(8.13) \quad \int_{B_j} \int_0^{\eta^2 d(x)} |\Lambda_{\nu}(x, r)|^2 \frac{dr}{r} d\sigma(x) \leq A \varepsilon \frac{1}{4} s_j^d.
\]

**Proof.** Let \( g = \sum c_i \theta_i(x) \). Note that for \( x \in \frac{1}{2} B_j \),
\[
|g(x) - c_j| = \left| \sum (c_i - c_j) \theta_i \right| \overset{(7.10)}{\leq} \varepsilon.
\]
For \( x \in B_j \) we have
\[
\eta^2 d(x) \leq \eta^2 (d(\xi_j) + |x - \xi_j|) < 2\eta s_j.
\]
Also, using the $d$-AD-regularity of $\nu$ and that $\Lambda_\nu(x, r) = |\psi_r * \pi(x)|$ for all $x \in \Sigma$ (see (8.3)), it is easy to check, by an argument similar to the one used in (8.11), that

\begin{equation}
(8.14) \quad \Lambda_\nu(x, r) \lesssim_{A, r} \alpha_\nu(x, 5r).
\end{equation}

Thus, by Lemma (6.11), and recalling Lemma 8.1 we get

$$
\int_{B_j} \int_0^{\eta^2 d(x)} |\Lambda_\nu(x, r)|^2 \frac{dr}{r} \, d\sigma(x) \lesssim_{A, r} \int_{B_j} \int_0^{2\eta_j} \alpha_\nu(x, 5r)^2 \frac{dr}{r} \, d\sigma(x) \lesssim \varepsilon \frac{4^d}{s_j}.
$$

\[ \Box \]

We now complete the proof of (8.5). To this end, we split the domain of integration by setting

$$
A_1 = \{(x, r) : B(x, r) \cap B(0, 10) = \emptyset\},
$$

$$
A_2 = \{(x, r) : B(x, r) \cap B(0, 10) \neq \emptyset, r > 1\},
$$

$$
A_3 = \{(x, r) : B(x, r) \cap B(0, 10) \neq \emptyset, \eta^2 d(x) < r \leq 1\},
$$

$$
A_4 = \{(x, r) : B(x, r) \cap B(0, 10) \neq \emptyset, r \leq \eta^2 d(x)\},
$$

and then we write

$$
I_i = \int \int_{A_i} \Lambda_\nu(x, r)^2 \frac{dr}{r} \, d\sigma(x).
$$

Note that for $(x, r) \in A_1$,

$$
\Lambda_\nu(x, r) = \Lambda_{C_2 B_0}(x, r) = 0.
$$

and so $I_1 = 0$.

For $(x, r) \in A_2$, since $B(x, r) \cap B(0, 10) \neq \emptyset$ and $r > 1$,

$$
|x| \leq r + 10 < 11r.
$$

and so

$$
r \geq \max\{\frac{1}{11} |x|, 1\}.
$$

Using again (8.14), which still holds in this case, plus the fact that $\sigma$ is $d$-AD regular we get

\[ (7.17) \]

\begin{align*}
I_2 & \leq \int \int_{\max\{\frac{1}{11} |x|, 1\}}^{\infty} \Lambda_\nu(x, r)^2 \frac{dr}{r} \, d\sigma(x) \\
& \lesssim \int \int_{\max\{\frac{1}{11} |x|, 1\}}^{\infty} \alpha_\nu(x, 5r)^2 \frac{dr}{r} \, d\sigma(x) \\
& \lesssim \varepsilon \frac{1}{2} \int \int_{\max\{\frac{1}{11} |x|, 1\}}^{\infty} \frac{dr}{r^{2d+1}} \, d\sigma(x) \\
& \lesssim \varepsilon \frac{1}{2} \left( \int_{\Sigma \cap B_0} \, d\sigma(x) + \sum_{k=1}^{\infty} 2^{-2dk} \int_{\Sigma \cap (B_{2^k} \setminus B_{2^{k-1}})} \, d\sigma(x) \right) \lesssim \varepsilon \frac{1}{2}.
\end{align*}

If $B(x, r) \cap B(0, 10) \neq \emptyset$ and $r < 1$, then $x \in B(0, 20)$, and so

$$
I_3 \leq \int_{B(0, 20)} \int_0^1 \Lambda_\nu(x, r)^2 \frac{dr}{r} \, d\sigma(x) \lesssim \varepsilon \frac{1}{4}.
$$
Thus, all that is left is $I_4$. Note that if $r < \eta^2 d(x)$ and $x \in B_j$, then $B(x, r) \subset 2B_j$, hence

$$I_4 = \int_{B(0, 20)} \int_{0}^{\eta^2 d(x)} \Lambda_\nu(x, r)^2 \frac{dr}{r} d\sigma(x) = \sum_{B_j \cap B(0, 20) \neq \emptyset} \int_{B_j} \int_{0}^{\eta^2 d(x)} \Lambda_\nu(x, r)^2 \frac{dr}{r} d\sigma(x)$$

(8.13) \[ \lesssim \sum_{B_j \cap B(0, 20) \neq \emptyset} \varepsilon^{4} s_j^d \lesssim \varepsilon^{4} \sum_{B_j \cap B(0, 20) \neq \emptyset} \sigma(B_j) \lesssim \varepsilon^{4} \sigma(B(0, 40)) \lesssim \varepsilon^{4}. \]

Combining the estimates above finishes the proof of Lemma 8.1.

9. THE END OF THE PROOF

Recall that by our choice to $\tau$ and $A$, Lemma 5.2, $\delta(x) \leq 10^{-3}$ for all points $x \in E_0 \cap B_0$. If moreover $\delta(x) > 0$, let

$$B_x = B(x, \delta(x))$$

Recall that for $x \in E_0 \cap B_0 \setminus Z$ with $Z$ as in Lemma 5.8 $d(x) > 0$ and therefore $\delta(x) > 0$ since $E_0 \cap B_0$ is a closed set. Hence $x \in E_0 \cap B_0 \setminus Z$ satisfies one of the following conditions:

- ND: $\mu(B_x \setminus E_0) \geq \varepsilon^2 \mu(B_x)$,
- LD: $\Theta_\mu^{d}(B_x) \leq \tau$,
- HD: $\Theta_\mu^{d}(B_x) \geq A$, or
- BA: $\angle(L_{x,r}, P_{B_0}) \geq \varepsilon^4$.

Recall the abbreviations stand for “low density”, “high density”, and “big angle”, respectively.

What we show now is that each of these sets has measure much smaller than $\mu(E_0 \cap B_0)$, and thus there must be a subset $G' \subset E_0 \cap B_0$ of positive measure for which $\Theta_\mu^{d}(x, r) \in [\tau, A]$ for all $r > 0$ small. This contradicts the conclusion of Lemma 5.8, obtained under the assumption that there is no set $E' \subset E$ with $\mu(E') > 0$ such that $\mu|_{E'}$ is $d$-rectifiable. This will conclude the proof of Lemma 5.1 and hence of Theorem 1.

**Lemma 9.1.** $\mu(ND) \lesssim \varepsilon^{\frac{1}{2}}$.

**Proof.** Let $\{B_j\}_{j=1}^N$ be a Besicovitch subcovering of the collection $\{B(x, \delta(x)) : x \in ND\}$. Since $\delta(x) \leq C_1$ by definition, we have $B \subset 3C_1B_0$ for each $B \in B_j$ and every $j$, and so by our definition of $E_0$,

$$\mu(ND) \leq \sum_{j=1}^{N} \sum_{B \in B_j} \mu(B) \leq \varepsilon^{-\frac{1}{2}} \sum_{j} \mu(B(j) \setminus E_0) \leq \varepsilon^{-\frac{1}{2}} \mu(3C_1B_0 \setminus E_0) \leq \varepsilon^{-\frac{1}{2}} \varepsilon \mu(3C_1B_0) = \varepsilon^{\frac{1}{2}}. \square$$

**Lemma 9.2.** $\mu(LD) \lesssim M, C_1 \tau$.

**Proof.** For any $x \in LD$ we have $\delta(x) \geq d(x) > 0$. Then there exists some $k$ such that $r_{k-1} \leq \delta(x) < r_k$ and so $x \in V_k^2$. For $i \in J_k$ such that $x \in 2B_{i,k}$ we have

$$\text{dist}(x, \Sigma) \leq |x - x_{i,k}| + \text{dist}(x_{i,k}, \Sigma) \lesssim r_k + \varepsilon r_k \lesssim \delta(x).$$
Lemma 9.3. \( \mu \) Lipschitz and, by (9.5)
\[
\delta \quad (9.6)
\]
there is a point \( B \) so that
\[
(9.7)
\]
\[
B(x, \delta(x)) \subset B(z(x), (C_5 + 1)\delta(x)) \subset B(z(x), 2C_5\delta(x)) \subset B(x, 4C_5\delta(x)),
\]
so that
\[
(9.8)
\]
\[
B(x, \delta(x)) \subset B(z(x), (C_5 + 1)\delta(x)) \subset B(z(x), 2C_5\delta(x)) \subset B(x, 4C_5\delta(x)),
\]
so that
\[
(9.9)
\]
\[
\mu(\mathcal{B}(z(x), 2C_5\delta(x))) \leq \mu(B(x, 4C_5\delta(x))) \leq_M \mu(\mathcal{B}(x, \delta(x))) \leq_M \tau \delta(x)^d.
\]

Let \( \{B_j\}_{j=1}^N \) be a Besicovitch subcovering from the collection \( \{B(x, 4C_5\delta(x)) : x \in \mathcal{L}\} \), and let \( x_B \) be the center of each \( B \in \cup_{j=1}^N B_j \). We deduce that

\[
\mu(\mathcal{L}) \leq \sum_{j=1}^N \sum_{B \in B_j} \mu(B) \leq_M \tau \sum_{j=1}^N \sum_{B \in B_j} \delta(x_B)^d
\]

\[
\leq_M \tau \sum_{j=1}^N \sum_{B \in B_j} \sigma(B) \leq \tau \sigma(3C_1 B_0) \leq_M C_1 \tau,
\]
by (9.4) and because the balls \( B(z(x)), 2C_5\delta(x)) \) have finite superposition. \( \square \)

Lemma 9.3. \( \mu(\mathcal{L}) \leq_M C_1 \tau \)

\( \square \)

\[
\mu(\mathcal{L}) \leq_M C_1 \tau.
\]

Proof. Take \( x \in \mathcal{L} \) and note again that \( \delta(x) \geq d(x) > 0 \). Then arguing as in Lemma 9.2, there is a point \( z(x) \in \Sigma \) satisfying (9.2) and (9.3).

Observe that for any ball \( B_j \) such that \( 2B_j \cap B(z(x), 4C_5\delta(x)) \neq \emptyset \), by (9.2) we have

\[
\delta(x) \geq d(x) \geq d(x_j) - |x - z(x)| - |z(x) - x_j| \geq d(x_j) - C_5\delta(x) - 2s_j - 4C_5\delta(x)
\]

and so

\[
\delta(x) \geq \frac{d(x_j) - 2s_j}{1 + 5C_5} = \frac{\eta^{-1}s_j - 2s_j}{1 + 5C_5} > \frac{s_j}{C_5}.
\]

This and the fact that \( 2B_j \cap B(z(x), 4C_5\delta(x)) \neq \emptyset \) imply

\[
B_j \subset B(z(x), 4C_5\delta(x) + 4s_j) \subset B(z(x), 8C_5\delta(x)).
\]

Consider now the function \( \lambda(y) = (4C_5\delta(x) - |y - z(x)|)_+ \). Observe that this is 1-Lipschitz and, by (9.3), satisfies

\[
2C_5\delta(x) \chi_{B(z(x), \delta(x))} \leq 2C_5\delta(x) \chi_{B(z(x), 2C_5\delta(x))} \leq \lambda \leq 4C_5\delta(x) \chi_{B(z(x), 4C_5\delta(x))}.
\]
Thus, with constants \( C \) depending only on \( A \) and \( \tau \), and for \( \varepsilon > 0 \) small enough, we get

\[
4C_5\delta(x) \nu(B(z(x), 4C_5\delta(x))) \geq \int \lambda(y) \, d\nu(y) \\
\geq \int \lambda(y) \theta(y) \, d\mu(y) - \sum_{2B_j \cap B(z(x), 4C_5\delta(x)) \neq \emptyset} \frac{1}{\varepsilon^d} s_{2j}^{d+1}
\]

\[
\geq \int \lambda(y) \, d\mu(y) - C\varepsilon^{\frac{d}{2}} \delta(x)^{d+1} - C\varepsilon^{\frac{d}{4}} \sum_{2B_j \cap B(z(x), 4C_5\delta(x)) \neq \emptyset} \delta(x) \, \sigma(B_j)
\]

\[
\geq \delta(x) \mu(B(x, \delta(x))) - C\varepsilon^{\frac{d}{2}} \delta(x)^{d+1} - C\varepsilon^{\frac{d}{4}} \delta(x) \sigma(B(z(x), 8C_5\delta(x)))
\]

\[
\geq A\delta(x)^{d+1} - C\varepsilon^{\frac{d}{4}} \delta(x)^{d+1} \geq \frac{A}{2} \delta(x)^{d+1}.
\]

Hence, by (8.1)

\[
\pi[\nu](B(y, 5C_5\delta(x))) \geq \nu(B(z(x), 4C_5\delta(x))) \geq \frac{A}{8C_5} \delta(x)^d.
\]

In particular, for \( y \in B(\pi(z(x)), C_5\delta(x)) \cap P_0 \),

\[
\frac{\pi[\nu](B(y, 5C_5\delta(x)))}{\mathcal{H}^d(P_0 \cap B(y, 5C_5\delta(x)))} \geq \frac{\pi[\nu](B(\pi(z(x)), 4C_5\delta(x)))}{\delta(x)^d} \geq A.
\]

So choosing \( A \gg c_{C_2}B_0 \) (recall \( c_{C_1}B_0 \approx \Theta_\mu^d(C_1B_0) = 1 \) by assumption), and since

\[
f = \frac{d\pi[\nu]}{d\mathcal{H}^d|\mathcal{P}_0},
\]

\[
\mathcal{M}(f - c_{C_2}B_0)(y) \geq \frac{\pi[\nu](B(y, 5C_5\delta(x)))}{\mathcal{H}^d(P_0 \cap B(y, 5C_5\delta(x)))} - c_{C_1}B_0 \geq A,
\]

where \( \mathcal{M} \) is the Hardy-Littlewood maximal function on \( P_0 \). Therefore, for any \( x \in \text{HD} \) we have

\[
B(\pi(z(x)), C_5\delta(x)) \cap P_0 \subset \{ y \in P_0 : \mathcal{M}(f - c_{C_1}B_0)(y) > 1 \} =: \Gamma.
\]

On the other hand, since \( \mathcal{M} \) is of weak type \((2, 2)\) with respect to \( \mathcal{H}^d|\mathcal{P}_0 \), by Lemma 8.1 we have

\[
\mathcal{H}^d(\Gamma) \lesssim \| \mathcal{M}(f - c_{C_2}B_0)\|_{L^2(\mathcal{H}^d|\mathcal{P}_0)} \lesssim \| f - c_{C_2}B_0\|_{L^2(\mathcal{H}^d|\mathcal{P}_0)} \lesssim A, \varepsilon \leq \frac{1}{4}.
\]

Let \( \{B_j\}_{j=1}^N \) be a Besicovitch subcovering from the collection \( \{B(x, 4C_5\delta(x)) \:: \ x \in \text{HD}\} \). There is \( j_0 \in \{1, \cdots, N\} \) such that the disjoint subcollection \( \mathcal{B}_{j_0} \) satisfies

\[
\mu(\text{HD}) \lesssim \sum_{B \in \mathcal{B}_{j_0}} \mu(B).
\]

Denote by \( x_B \) the center of each \( B \), and note that the balls \( B(z(x_B), 2C_5\delta(x_B)), B \in \mathcal{B}_{j_0}, \) are also pairwise disjoint, by (9.3). Since \( \Sigma \) is a Lipschitz graph with a very small constant and the balls \( B(z(x_B), 2C_5\delta(x_B)) \) are centered in \( \Sigma \) are pairwise disjoint, it follows that
the $d$-dimensional balls $B(\pi(z(x_B)), C_5\delta(x_B)), B \in B_{i0}$, are also pairwise disjoint. Then we get
\[
\mu(\text{HD}) \lesssim \sum_{B \in B_{i0}} \mu(B) \lesssim \sum_{B \in B_{i0}} t_B^d \lesssim \sum_{B \in B_{i0}} \mathcal{H}^d(B(\pi(z(x_B)), C_5\delta(x_B)))
\]
\[
\leq \mathcal{H}^d\left(\bigcup_{B \in B_{i0}} B(\pi(z(x_B)), C_5\delta(x_B))\right) \leq \mathcal{H}^d(\Gamma) \lesssim A, \tau \varepsilon^\frac{1}{2}.
\]

\[\square\]

**Lemma 9.4.** $\mu(BA) \lesssim A, \tau, M \varepsilon^\frac{1}{2}$. 

**Proof.** As in Lemma 9.1, for each $x \in BA$ there is a point $z(x) \in \Sigma$ satisfying (9.2) and (9.3). Let $\{B_j\}_{j=1}^N$ be a Besicovitch subcovering from the collection $\{B(x, 4C_5\delta(x)) : x \in BA\}$ with centers $x_B$ and radii $t_B$. Again as in Lemma 9.1, for each $x_B$ we take $k$ such that $r_{k-1} \leq \delta(x_B) < r_k$, and so there exists some $i \in J_k$ such that $x_B \in 2B_{i,k}$. Then, using (5.16) and the subsequent Remark 5.7,
\[
\angle(P_{i,k}, P_0) \geq \angle(L_{x_B, \delta(x_B)}, P_0) - \angle(P_{i,k}, L_{x_B, \delta(x_B)}) \gtrsim \varepsilon^\frac{1}{2} - C\varepsilon \gtrsim \varepsilon^\frac{1}{2}.
\]

By (6.9) and (6.11), taking into account (9.3),
\[
\text{dist}_H(B \cap \Sigma, B \cap P_{i,k}) \lesssim \varepsilon r_k \approx \varepsilon \delta(x_B) \approx \varepsilon r_B.
\]

Thus, by (6.33), Lemma 6.9, (9.8) and (9.9), we have
\[
\int_{\pi(B \cap \Sigma)} |Dh|^2 d\mathcal{H}^d \gtrsim \int_{\pi(B \cap \Sigma)} \frac{|h - h(\pi(x_B))|^2}{r_B^2} d\mathcal{H}^d
\]
\[
\gtrsim \int_{\pi(B \cap \Sigma)} \frac{|A_{j,k} - h(\pi(x_B))|^2}{r_B^2} d\mathcal{H}^d - C\varepsilon^2 \gtrsim \varepsilon^\frac{1}{2} - C\varepsilon^2 \gtrsim \varepsilon^\frac{1}{2}.
\]

Then
\[
\mu(BA) \lesssim \sum_{j=1}^N \sum_{B \in B_j} \mu(B) \lesssim A \sum_{j=1}^N \sum_{B \in B_j} t_B^d \lesssim \sum_{j=1}^N \sum_{B \in B_j} \mathcal{H}^d(\pi(B \cap \Sigma))
\]
\[
\lesssim \varepsilon^{-\frac{1}{2}} \sum_{j=1}^N \sum_{B \in B_j} \int_{\pi(B \cap \Sigma)} |Dh|^2 d\mathcal{H}^d \lesssim \varepsilon^{-\frac{1}{2}} \int |Dh|^2 \lesssim \varepsilon^{-\frac{1}{2}} \varepsilon \lesssim \varepsilon^\frac{1}{2},
\]
by (6.47).

\[\square\]

Putting these lemmas together, we obtain that for $\tau$ and $\varepsilon$ small enough,
\[
\mu(\text{HD} \cup LD \cup ND \cup BA) \leq C\tau + C(A, \tau)\varepsilon^\frac{1}{2} < \mu(E_0),
\]
which implies that there exists a subset $G' \subset E_0 \cap B_0$ of positive measure for which $\Theta_\mu(x, r) \in [\tau, A]$ for all $r > 0$ small. This contradicts Lemma 5.8 and concludes the proof of Lemma 5.1 and of Theorem 1.
10. Proof of Theorem II

In this section we assume \( n = 2 \) and \( d = 1 \). That is, we are in the plane and we consider 1-dimensional \( \alpha \)-numbers. Our objective is to construct a measure \( \mu \) such that

\[
\int_0^1 \alpha \mu(x,r)^2 \frac{dr}{r} < \infty \quad \text{for all } x \in \text{supp } \mu,
\]

and such that

\[
\lim_{r \to 0} \frac{\mu(B(x,r))}{r} = 0 \quad \text{for all } x \in \text{supp } \mu.
\]

Given \( h > 0 \) and a horizontal line \( L \subset \mathbb{R}^2 \), we denote by \( L(h) \) the line parallel to \( L \) and above \( L \) which is at a distance \( h \) from \( L \). That is, \( L(h) = h e_2 + L \), where \( e_2 = (0, 1) \). Our measure \( \mu \) will be a weak limit of measures \( \mu_k \) of the form

\[
\mu_k = \sum_{j=1}^{n_k} c_j^k \mathcal{H}^1|_{L_j^k},
\]

where \( L_j^k, j = 1, \ldots, n_k \) are horizontal lines.

We consider two decreasing sequences of positive numbers \( \{a_k\}_k, \{h_k\}_k \), tending to 0, which will be chosen later, and so that \( 0 < a_k, h_k < 1/2 \). The reader should think that \( h_k \) will tend to zero much faster than \( a_k \). First we set \( \mu_0 = \mathcal{H}^1|_{L_0^0} \), where \( L_0^0 \) is just the horizontal axis. Inductively, \( \mu_{k+1} \) is constructed from \( \mu_k \) as in (10.3), as follows:

\[
\mu_{k+1} = \sum_{j=1}^{n_k} c_j^k \left[ (1 - a_{k+1}) \mathcal{H}^1|_{L_j^k} + a_{k+1} \mathcal{H}^1|_{L_j^k(h_{k+1})} \right].
\]

So roughly speaking, at each step \( k + 1 \), each line \( L_j^k \) is split into the two lines \( L_j^k \) and \( L_j^k(h_{k+1}) \) and the total mass is distributed so that a fraction \( 1 - a_{k+1} \) is kept in \( L_j^k \), while the other fraction \( a_{k+1} \) is transferred to \( L_j^k(h_{k+1}) \). Further, one should think that \( h_k \) goes to 0 very quickly, and \( h_{k+1} \) is much smaller than any of the mutual distances among the lines \( L_j^k, j = 1, \ldots, n_k \). We claim that \( a_k \) and \( h_k \) can be chosen so that (10.1) holds but \( \mu \) is singular with respect to \( \mathcal{H}^1 \).

First we just analyse how the \( \alpha \) coefficients evolve from \( \mu_0 \) to \( \mu_1 \). Consider the measure

\[
\mu_1 = (1 - a) \mathcal{H}^1|_L + a \mathcal{H}^1|_{L'},
\]

where \( a = a_1, L = L_0^0, \) and \( L' = L(h) \), with \( h = h_1 \). Consider a 1-Lipschitz function \( \phi \) supported on \( B(x,r) \), with \( x \in \text{supp } \mu_1 \) and estimate the following integral using a change of variable

\[
\left| \int \phi d(\mathcal{H}^1|_L - \mu_1) \right| = \left| \int \phi d(a \mathcal{H}^1|_L - a \mathcal{H}^1|_{L'}) \right|
= a \left| \int (\phi(x) - \phi(x + h)) \ d\mathcal{H}^1|_L \right| \lesssim a h r.
\]

First we estimate \( \alpha_{\mu_1}(x,r) \) for \( x \in L \). To this end, note that since \( a < 1/2 \), \( \mu(B(x,r)) \approx r \) for all \( r > 0 \). Further, \( \alpha_{\mu_1}(x,r) = 0 \) if \( r \leq h \). For \( r > h \), we write

\[
\alpha_{\mu_1}(x,r) \lesssim \frac{1}{r^2} \text{dist}_{B(x,r)}(\mu_1, \mathcal{H}^1|_L).
\]
Hence by (10.5) we obtain

\[
\alpha_{\mu_1}(x, r) \lesssim a \frac{h}{r},
\]

and thus

\[
\int_0^\infty \alpha_{\mu}(x, r)^2 \frac{dr}{r} \lesssim a^2 \int_0^\infty \frac{h^2}{r^2} \frac{dr}{r} \approx a^2.
\]

Next we turn our attention to the case when \(x \in L'\). Again, for \(r \leq h\), \(\alpha_{\mu_1}(x, r) = 0\). On the other hand, for \(r > 2h\), \(\mu_{\mu_1}(B(x, r)) \approx r\), and almost the same calculations as above (i.e. for \(x \in L\) and \(r > h\)) show that

\[
\alpha_{\mu_1}(x, r) \lesssim a \frac{h}{r},
\]
as in (10.6). This yields

\[
\int_{2h}^{\infty} \alpha_{\mu}(x, r)^2 \frac{dr}{r} \lesssim a^2 \int_{2h}^{\infty} \frac{h^2}{r^2} \frac{dr}{r} \approx a^2.
\]

Assume now that \(x \in L'\) and \(h < r < 2h\). An easy geometric argument shows that

\[
\mathcal{H}^1(L \cap B(x, r)) = 2\sqrt{r^2 - h^2}.
\]

Hence

\[
\mu_1(B(x, r)) = 2(1 - a)\sqrt{r^2 - h^2} + 2ar.
\]

By (10.5) we have

\[
\alpha_{\mu_1}(x, r) \lesssim \frac{ahr}{r \mu_1(B(x, r))} \lesssim \frac{ah}{2(1 - a)\sqrt{r^2 - h^2} + 2ar} \lesssim \frac{ah}{\sqrt{r^2 - h^2} + ar},
\]

and so

\[
\alpha_{\mu_1}(x, r)^2 \lesssim \frac{a^2 h^2}{r^2 - h^2 + (ar)^2} = \frac{a^2 h^2}{(1 + a^2)r^2 - h^2}.
\]

Therefore,

\[
\int_h^{2h} \alpha_{\mu}(x, r)^2 \frac{dr}{r} \lesssim \int_h^{2h} \frac{a^2 h^2}{(1 + a^2)r^2 - h^2} \frac{dr}{r} \lesssim \int_h^{2h} \frac{a^2 r}{(1 + a^2)r^2 - h^2} \frac{dr}{r} = \frac{a^2}{2(1 + a^2)} \log((1 + a^2)r^2 - h^2) \Big|_h^{2h} = \frac{a^2}{2(1 + a^2)} \log \frac{3 + 4a^2}{a^2} \lesssim a^2 \log \frac{4}{a^2} \approx a^2 \log |a|.
\]

Together with (10.8), this yields for \(x \in L'\) since \(0 < a < \frac{1}{2}\) that

\[
\int_0^\infty \alpha_{\mu}(x, r)^2 \frac{dr}{r} \lesssim a^2 |\log a|.
\]

Further, because of (10.7), this estimate is also valid for \(x \in L\).
By the same arguments, in the step \( k + 1 \), denoting \( d_k \) the minimal distance among the lines \( L_j^k \), \( j = 1, \ldots, n_k \), that form the support of \( \mu_k \), and assuming that \( h_{k+1} \ll d_k \), we obtain as before that

\[
\int_0^{d_k/2} \alpha_{\mu_{k+1}}(x, r)^2 \frac{dr}{r} \lesssim d_k^2 |\log a_{k+1}| \quad \text{for all } x \in \text{supp} \mu_{k+1}.
\]

On the other hand, for \( r \geq d_k/2 \), we claim that if we choose \( h_{k+1} \) small enough so that

\[
h_{k+1} \leq a^{4(k+1)}_{k+1} d_{k+1}/4,
\]

then we have

\[
\alpha_{\mu_{k+1}}(x, r) \leq \left(1 + \frac{h_{k+1}^{1/4}}{d_k^{1/4}}\right) \alpha_{\mu_k}(x', r') + C \frac{h_{k+1}}{r},
\]

where \( x' \) is the closest point to \( x \) from \( \text{supp} \mu_k \) and \( r' = r + h_{k+1} \). We defer the proof of this estimate and show how obtain (10.1) provided (10.10) and (10.11) hold.

For any \( 0 < \varepsilon_k < 1/2 \), using that \( h_{k+1} \ll d_k \), a change of variable and the fact that the function \( s \rightarrow h_{k+1} - d_{k+1} s \) is decreasing, we have

\[
\int_{d_k/2}^{\infty} \alpha_{\mu_{k+1}}(x, r)^2 \frac{dr}{r} \leq (1 + \varepsilon_k) \left(1 + \frac{h_{k+1}^{1/4}}{d_k^{1/4}}\right)^2 \int_{d_k/2}^{\infty} \alpha_{\mu_k}(x', r + h_{k+1})^2 \frac{dr}{r} + C \frac{h_{k+1}^2}{\varepsilon_k d_k^2}.
\]

Together with (10.10), and using that

\[
\frac{1}{2} d_k - h_{k+1} \leq \frac{1}{2} d_k \leq 1 + C \frac{h_{k+1}}{d_k} \leq 1 + C \frac{h_{k+1}^{1/4}}{d_k^{1/4}},
\]

this gives

\[
\int_0^{\infty} \alpha_{\mu_{k+1}}(x, r)^2 \frac{dr}{r} \leq C a_{k+1}^2 |\log a_{k+1}| + (1 + \varepsilon_k) \left(1 + C \frac{h_{k+1}^{1/4}}{d_k^{1/4}}\right) \int_0^{\infty} \alpha_{\mu_k}(x', r)^2 \frac{dr}{r} + C \frac{h_{k+1}^2}{\varepsilon_k d_k^2}.
\]

Choosing \( \varepsilon_k = 2^{-k} \) and assuming also that

\[
\frac{h_{k+1}^{1/4}}{d_k^{1/4}} \leq 2^{-k},
\]

we have

\[
\frac{h_{k+1}^{1/4}}{d_k^{1/4}} \leq \frac{1}{2}.
\]
iterating the estimate (10.13) it follows that 

\[
\int_0^\infty \alpha_{\mu_{k+1}}(x,r)^2 \frac{dr}{r} \lesssim \sum_{j=1}^{k+1} a_j^2 \log a_j + \sum_{j=1}^k \frac{2 j h_{j+1}^2}{a_j^2} \lesssim 1 + \sum_{j=1}^{k+1} a_j^2 \log a_j.
\]

Since this estimate is uniform on \(k\), we derive

(10.15) \[
\int_0^\infty \alpha_{\mu}(x,r)^2 \frac{dr}{r} \lesssim 1 + \sum_{j=1}^{\infty} a_j^2 |\log a_j|.
\]

Let \(\{a_j\}_j\) be a sequence such that the last sum in (10.15) is finite (which guaranties that (10.1) holds) but so that \(\sum_j a_j = \infty\) (such as \(a_j = 1/j\), for example). Further choose \(\{h_k\}\) inductively so that both (10.14) and (10.11) hold. Recall that by construction, \(\mu_k\) has constant 1-dimensional density on each line \(L^j_k\), \(i = 1, \ldots, n_k\), and further this density is at most \(\prod_{j=1}^k (1 - a_j)\). The condition \(\sum_j a_j = \infty\) implies that this product tends to 0 as \(k \to \infty\). In turn this ensures that \(\mu\) has vanishing upper density at all points, and thus \(\mu\) is singular with respect to \(\mathcal{H}^1\). This completes the construction of the counterexample, modulo the proof of claim (10.12).

**Proof of (10.12).** Consider a 1-Lipschitz function \(\phi\) supported on \(B(x,r)\) and denote \(B = B(x,r)\) and \(B' = (x', r')\), with \(r' = r + h_{k+1}\) (note that \(|x - x'| \leq h_{k+1}\)). Let \(c_{B'}, L_{B'}\) some pair for which the minimum is attained in the definition of \(\alpha_{\mu_k}(B')\) as in (1.4). Then we have

(10.16) \[
\left| \int \phi \, d(c_{B'} \mathcal{H}^n_{|L_{B'}} - \mu_{k+1}) \right| \leq \left| \int \phi \, d(c_{B'} \mathcal{H}^n_{|L_{B'}} - \mu_k) \right| + \left| \int \phi \, d(\mu_k - \mu_{k+1}) \right| \\
\leq F_{B'}(c_{B'} \mathcal{H}^n_{|L_{B'}}, \mu_k) + \left| \int \phi \, d(\mu_k - \mu_{k+1}) \right|,
\]

taking into account that \(\text{supp } \phi \subset B \subset B'\) in the last inequality. To deal with the last integral note that

\[
\mu_k - \mu_{k+1} = \sum_{j=1}^{n_k} c_j^k \mathcal{H}^1_{|L_j^k} - \sum_{j=1}^{n_k} c_j^k \left[ (1 - a_{k+1}) \mathcal{H}^1_{|L_j^k} + a_{k+1} \mathcal{H}^1_{|L_j^k(h_{k+1})} \right] \\
= a_{k+1} \sum_{j=1}^{n_k} c_j^k \left[ \mathcal{H}^1_{|L_j^k} - \mathcal{H}^1_{|L_j^k(h_{k+1})} \right] = a_{k+1} \left( \mu_k - P_k(\mu_k) \right),
\]

where \(P_k\) stands for the translation \(P_k(y) = y + h_{k+1}\). Therefore,

\[
\left| \int \phi \, d(\mu_k - \mu_{k+1}) \right| = a_{k+1} \left| \int \left( \phi(y) - \phi(y + h_{k+1}) \right) d\mu_k \right| \leq a_{k+1} h_{k+1} \mu_k(B'),
\]
where we used that $\phi$ is 1-Lipschitz and $\text{supp} \phi \cup \text{supp} \phi(\cdot + h_{k+1}) \subset B'$. From this inequality and (10.16) we deduce that

\[(10.17) \quad \alpha_{\mu_{k+1}}(x, r) \leq \frac{F_{B'}(c_{B'}H^n_{|L_{B'}}, \mu_k) + a_{k+1} h_{k+1} \mu_k(B')}{r \mu_{k+1}(B)} = \frac{r' \mu_k(B')}{r \mu_{k+1}(B)} \alpha_{\mu_k}(x', r') + \frac{a_{k+1} h_{k+1} \mu_k(B')}{r \mu_{k+1}(B)}.
\]

Next we need to estimate $\mu_k(B')$ in terms of $\mu_{k+1}(B)$. To this end, recall that

\[
\text{supp} \mu_k = \bigcup_{j=1}^{n_k} (L_j^k \cup L_j^k(h_{k+1}))
\]

and denote by $T_k : \text{supp} \mu_{k+1} \to \text{supp} \mu_k$ the map which equals the identity on each line $L_j^k$ and coincides with the orthogonal projection from $L_j^k(h_{k+1})$ to $L_j^k$ on each $L_j^k(h_{k+1})$. By construction, $\mu_k = T_k[\mu_{k+1}]$, and thus $\mu_k(B') = \mu_{k+1}(T_k^{-1}(B'))$. Moreover note that

\[
T_k^{-1}(B' \cap \text{supp} \mu_k) \subset B' \cup (B' - h_{k+1}) \subset B(x, r + 2h_{k+1}).
\]

Therefore,

\[(10.18) \quad \mu_k(B') \leq \mu_{k+1}(B(x, r + 2h_{k+1})) = \mu_{k+1}(B) + \mu_{k+1}(A(x, r, r + 2h_{k+1})),
\]

where $A(x, r, r + 2h_{k+1})$ denotes the annular region of center $x$ and between radii $r$ and $r + 2h_{k+1}$. By geometric considerations, it is immediate to check that for any line $L \subset \mathbb{R}^d$,

\[
\mathcal{H}^1(L \cap A(x, r, r + 2h_{k+1})) \leq 2\sqrt{(r + 2h_{k+1})^2 - r^2} = 2\sqrt{4h_{k+1}r + 4h_{k+1}^2} \leq 2\sqrt{8h_{k+1}r},
\]

since we have $h_{k+1} \ll d_k \leq r$. Therefore,

\[(10.19) \quad \mu_{k+1}(A(x, r, r + 2h_{k+1})) \leq 2 \sum_{i=1}^{n_{k+1}} c_i^{k+1} \sqrt{8h_{k+1}r} \leq 2\sqrt{8h_{k+1}r},
\]

since by construction $\sum_{i=1}^{n_{k+1}} c_i^{k+1} = 1$. Combining (10.18) and (10.19), we derive

\[
\mu_k(B') \leq \mu_{k+1}(B) + C \sqrt{h_{k+1}r}.
\]

Plugging this estimate into (10.17) we obtain

\[(10.20) \quad \alpha_{\mu_{k+1}}(x, r) \leq \frac{r + h_{k+1}}{r} \left(1 + C \sqrt{\frac{h_{k+1}r}{\mu_{k+1}(B)}}\right) \alpha_{\mu_k}(x', r') + \frac{a_{k+1} h_{k+1}}{r \mu_{k+1}(B)} \left(1 + \sqrt{\frac{h_{k+1}r}{\mu_{k+1}(B)}}\right).
\]

Since $B$ is centered on some line $L_i^{k+1}$, we can use the following brutal estimate for $\mu_{k+1}(B)$:

\[
\mu_{k+1}(B) \geq a_1 \ldots a_{k+1} r \geq a_{k+1}^{k+1} r.
\]

So by the assumption (10.11) we infer that

\[
\sqrt{\frac{h_{k+1}r}{\mu_{k+1}(B)}} \leq \frac{h_{k+1}^{1/2}}{a_{k+1}^{1/2} r^{1/2}} \leq \frac{h_{k+1}^{1/4}}{d_k^{1/4}}.
\]

Our claim (10.12) is an immediate consequence of this estimate and (10.20).
REFERENCES

[ADT16] J. Azzam, G. David, and T. Toro. Wasserstein distance and the rectifiability of doubling measures: part I. Math. Ann., 364(1-2):151–224, 2016.

[AM16] J. Azzam and M. Mourougoglou. A characterization of 1-rectifiable measures with connected supports. Anal. PDE, 9(1):99–109, 2016.

[AT15] J. Azzam and X. Tolsa. Characterization of $\alpha$-rectifiability in terms of Jones’ square function: Part II. Geom. Funct. Anal., 25(5):1371–1412, 2015.

[BS15] M. Badger and R. Schul. Multiscale analysis of 1-rectifiable measures: necessary conditions. Mathematische Annalen, 361(3-4):1055–1072, 2015.

[BS16] M. Badger and R. Schul. Two sufficient conditions for rectifiable measures. Proc. Amer. Math. Soc., 144(6):2445–2454, 2016.

[BS17] M. Badger and R. Schul. Research article. Multiscale analysis of 1-rectifiable measures II: Characterizations. Anal. Geom. Metr. Spaces, 5:1–39, 2017.

[DL08] C. De Lellis. Rectifiable sets, densities and tangent measures. Zurich Lectures in Advanced Mathematics. European Mathematical Society (EMS), Zürich, 2008.

[DS91] G. David and S. W. Semmes. Singular integrals and rectifiable sets in $\mathbb{R}^n$: Beyond Lipschitz graphs. Astérisque, (193):152, 1991.

[DT12] G. David and T. Toro. Reifenberg parameterizations for sets with holes. American Mathematical Soc., 2012.

[ENV16] Nick Edelen, Aaron Naber, and Daniele Valtorta. Quantitative reifenberg theorem for measures. arXiv preprint arXiv:1612.08052, 2016.

[GKS10] J. Garnett, R. Killip, and R. Schul. A doubling measure on $\mathbb{R}^d$ can charge a rectifiable curve. Proc. Amer. Math. Soc., 138(5):1673–1679, 2010.

[LP99] J. C. Léger. Menger curvature and rectifiability. Ann. of Math. (2), 149(3):831–869, 1999.

[Ler03] Gilad Lerman. Quantifying curvelike structures of measures by using $L_2$ Jones quantities. Comm. Pure Appl. Math., 56(9):1294–1365, 2003.

[Mag12] F. Maggi. Sets of finite perimeter and geometric variational problems, volume 135 of Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, 2012. An introduction to geometric measure theory.

[Mat95] P. Mattila. Geometry of sets and measures in Euclidean spaces, volume 44 of Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, 1995. Fractals and rectifiability.

[MO16] H. Martikainen and T. Orponen. Boundedness of the density normalised Jones’ square function does not imply 1-rectifiability. arXiv preprint arXiv:1604.04091, 2016.

[Orp17] T. Orponen. Absolute continuity and $\alpha$-numbers on the real line. ArXiv e-prints, March 2017.

[Paj97] Hervé Pajot. Conditions quantitatives de rectifiabilité. Bull. Soc. Math. France, 125(1):15–53, 1997.

[Pre87] D. Preiss. Geometry of measures in $\mathbb{R}^n$: distribution, rectifiability, and densities. Ann. of Math. (2), 125(3):537–643, 1987.

[Ste93] E. M. Stein. Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals, volume 43 of Princeton Mathematical Series. Princeton University Press, Princeton, NJ, 1993. With the assistance of Timothy S. Murphy. Monographs in Harmonic Analysis, III.

[Tol09] X. Tolsa. Uniform rectifiability, Calderón-Zygmund operators with odd kernel, and quasiorthogonality. Proc. Lond. Math. Soc. (3), 98(2):393–426, 2009.

[Tol15] X. Tolsa. Characterization of $n$-rectifiability in terms of Jones’ square function: part I. Calc. Var. Partial Differential Equations, 54(4):3643–3665, 2015.

[Tol17a] X. Tolsa. Rectifiable measures, square functions involving densities, and the Cauchy transform. Mem. Amer. Math. Soc., 245(1158):v+130, 2017.

[Tol17b] Xavier Tolsa. Rectifiability of measures and the $\beta_p$ coefficient in. To appear in Publ. Mat., 2017.

[TT15] X. Tolsa and T. Toro. Rectifiability via a square function and Preiss’ theorem. Int. Math. Res. Not. IMRN, (13):4638–4662, 2015.
SCHOOL OF MATHEMATICS, UNIVERSITY OF EDINBURGH, JCMB, KINGS BUILDINGS, MAYFIELD ROAD, EDINBURGH, EH9 3JZ, SCOTLAND.
E-mail address: j.azzam “at” ed.ac.uk

ICREA, PASSEIG LLUÍS COMPANYS 23 08010 BARCELONA, CATALONIA, AND, DEPARTAMENT DE MATEMÀTIQUES AND BGSMATH, UNIVERSITAT AUTÒNOMA DE BARCELONA, 08193 BELLATERRA, CATALONIA
E-mail address: xtolsa@mat.uab.cat

UNIVERSITY OF WASHINGTON, DEPARTMENT OF MATHEMATICS, SEATTLE, WA 98195-4350, USA
E-mail address: toro@uw.edu