Shear-free Dust Solution in General Covariant Hořava-Lifshitz Gravity

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(Dated: December 21, 2015)

In this paper, we have studied non stationary dust spherically symmetric spacetime, in general covariant theory (U(1) extension) of the Hořava-Lifshitz gravity with the minimally coupling and non-minimum coupling with matter, in the post-newtonian approximation (PPN) in the infrared limit. The Newtonian prepotential \( \varphi \) was assumed null. The aim of this work is to know if we can have the same spacetime, as we know in the General Relativity Theory (GRT), in Hořava-Lifshitz Theory (HLT) in this limit. We have shown that there is not an analogy of the dust solution in HLT with the minimally coupling, as in GRT. Using non-minimum coupling with matter, we have shown that the solution admits a process of gravitational collapse, leaving a singularity at the end. This solution has, qualitatively, the same temporal behaviour as the dust collapse in GRT. However, we have also found a second possible solution, representing a bounce behavior that is not found in GRT.

PACS numbers: 04.50.Kd; 98.80.-k; 98.80.Bp

I. INTRODUCTION

The construction of a Quantum Theory of Gravitation is one of the unsolved problems of modern physics. The biggest challenge faced by all the researchers, who have tried to build a new gravitation theory, is that this new theory must be valid at all scales. Since the gravity force is a relatively weak one, it is not expected to be observed any of its quantum effects. For this reason the criteria that a new candidate quantum gravity theory must fulfill are constrained by mathematical and self-consistency criteria, reproducing observable results, confirming General Relativity Theory, and finally, making non-trivial predictions that may possibly be tested.

Hořava-Lifshitz gravity theory (HLT) [1] [2] has drawn a lot of interest primarily due to its properties to solve the perturbative non-renormalizability of quantized standard Einstein gravity at the expense of breaking relativistic invariance at very high energies, while restoring it at low energies. After that, further deep connections of Hořava-Lifshitz formalism to condensed matter physics were discovered within the framework of gauge/gravity duality (holography).

Unfortunately, the original Hořava formulation was contaminated by a number of serious problems which has caused the development of various non-trivial modifications. One of the promising latter modifications is the one without the so called projectability condition and with an additional Abelian gauge symmetry, which in particular solves the problem with the unphysical scalar graviton. Due to the extensive works in this area, we suggest to the reader the references [3]-[38].

In order to establish the physical relevance of any modification of Hořava-Lifshitz gravity it is very important to check that the latter reproduces the known physically feasible properties of standard Einstein General Relativity (EGR) at low energies. Within this condition, we have checked whether the non-projectable version of Hořava-Lifshitz gravity with the extra U(1) gauge symmetry contains in the low energy limit solutions of Vaidya-type [12]. The answer was negative and, therefore, led us to the important conclusion that in order to establish consistency with EGR at low energies the gauge field associated with the extra U(1) gauge symmetry of the enlarged non-projectable Hořava-Lifshitz gravity should have some interaction with the pure radiation matter generating the Vaidya spacetime geometry.

Lin et al. (2014) [41], have proposed a universal coupling between the gravity and matter in the framework of the Hořava-Lifshitz theory of gravity with an extra U(1) symmetry for both the projectable and non-projectable cases. Then, using this universal coupling they have studied the PPN approximations and they have obtained the PPN parameters in terms of the coupling constants of the theory.

Goldoni et al. (2014) [12], using Lin at al. (2014) [41] approximation, have shown that there is not an analogy of the Vaidya’s solution in Hořava-Lifshitz Theory (HLT) without projectability, as we know in General Relativity Theory (GRT).

In another recent paper, Goldoni et al. (2015) [42], using again Lin at al. (2014) [41] approach, have also shown that there is not an analogy of the Vaidya’s solution in Hořava-Lifshitz Theory (HLT) with the minimally coupling and without projectability, again as we know in GRT.
The goal of this present paper is to study the behavior of a dust fluid solution of GRT in HLT for the infrared limit. We want to know if we can have the same spacetime, as we know in GRT, in HLT in this limit.

Using again the results of Lin et al. (2014) [41] we have studied the spherically symmetric spacetime filled by a dust fluid, in general covariant theory (U(1) extension) of Hořava-Lifshitz gravity with a minimally coupling [41], in the PPN approximation and in the infrared limit. We will analyze if a solution like this one can be described in the general covariant HLT of gravity [22, 23]. In Section II we present a brief introduction to HLT with the minimally coupling constants. For detail, we refer readers to [22, 23, 41].

II. GENERAL COVARIANT HOŘAVA-LIFSHITZ GRAVITY WITH COUPLING WITH MATTER

In this section, we shall give a very brief introduction to the general covariant HLT gravity with the minimally coupling. For detail, we refer readers to [22, 23, 41].

The Arnowitt-Deser-Misner (ADM) form is given by [39],

\[ ds^2 = -N^2 dt^2 + g_{ij} \left( dx^i + N^i dt \right) \left( dx^j + N^j dt \right), \]

(1)

where the non-projectability condition imposes that \( N = N(t, x^i) \).

In the study of Lin et al. (2014) [41] it is proposed that, in the IR limit, it is possible to have matter fields universally couple to the ADM components through the transformations

\[ \tilde{N} = FN, \]
\[ \tilde{N}^i = N^i + Ng^{ij} \nabla_j \varphi, \]
\[ \tilde{g}_{ij} = \Omega^2 g_{ij}, \]

(2)

with

\[ F = 1 - a_1 \sigma, \quad \Omega = 1 - a_2 \sigma, \]

(3)

where

\[ \sigma \equiv \frac{A - \mathcal{A}}{N}, \]
\[ \mathcal{A} \equiv -\varphi + N^i \nabla_i \varphi + \frac{1}{2} N \left( \nabla^i \varphi \right) \left( \nabla_i \varphi \right), \]

(4)

and where \( A \) and \( \varphi \) are the gauge field and the Newtonian prepotential, respectively, and \( a_1 \) and \( a_2 \) are two arbitrary coupling constants. Note that by setting the first terms in \( F \) and \( \Omega \) to unity, we have used the freedom to rescale the units of time and space. We also have

\[ \tilde{N}_i = \Omega^2 \left( N_i + N \nabla_i \varphi \right), \quad \tilde{g}^{ij} = \Omega^{-2} g^{ij}. \]

(5)

Considering the exposed before, the matter action can be written as

\[ S_m = \int dt \sqrt{-g} \mathcal{L}_m \left( \tilde{N}, \tilde{N}_i, \tilde{g}^{ij}, \psi_n \right), \]

(6)

where \( \psi_n \) collectively stands for matter fields. One can then define the matter stress-energy in the ADM decomposition, with the minimally coupling. The different components are given by (for the details see [41])

\[ \rho_H(EGR) = T_{\mu\nu} n^\mu n^\nu \equiv J^t = -\frac{\delta(\tilde{N} \mathcal{L}_m)}{\delta(N)}, \]
\[ S^i(EGR) = -T_{\mu\nu} h^{(4)\mu\nu} n^\nu \equiv J^i = -\frac{\delta(\tilde{N} \mathcal{L}_m)}{\delta(N^i)}, \]
\[ S^{ij}(EGR) = T_{\mu\nu} h^{(4)\mu\nu} n^\nu \equiv \tau^{ij} = \frac{2}{N \sqrt{g}} \frac{\delta(\tilde{N} \sqrt{g} \mathcal{L}_m)}{\delta(g_{ij})}, \]

(7)

where \( h^{(4)\mu\nu} \) is the projection operator, defined as \( h^{(4)\mu\nu} \equiv g^{(4)\mu\nu} + n^\mu n^\nu \) and \( n^\mu \) is the normal vector to the hypersurface \( t = \) constant, defined as \( n^\mu = \left(1, N^i\right) \). Before, the prime and dot denotes the partial differentiation in relation to the coordinate \( r \) and \( t \), respectively.

Thus, the total action of the theory can be written as

\[ S = \frac{\zeta^2}{2} \int dt dx \sqrt{g} N \left( \mathcal{L}_K - \mathcal{L}_V + \mathcal{L}_A + \mathcal{L}_\varphi + \mathcal{L}_S + \frac{1}{\xi^2} \mathcal{L}_M \right), \]

(8)

where \( g = \text{det}(g_{ij}) \), \( N \) is given in the equation [41] and

\[ \mathcal{L}_K = K_{ij} K^{ij} - \lambda K^2, \]
\[ \mathcal{L}_V = \frac{1}{2} \left( \partial_\alpha a_i \partial^\alpha a_i - \gamma_1 R \right) + \frac{1}{\zeta^2} \left( \gamma_2 R^2 + \gamma_3 R_{ij} R^{ij} \right) \]
\[ + \frac{1}{\xi^2} \left[ \beta_1 (a_i a^i)^2 + \beta_2 (a^i a_i)^2 + \beta_3 (a_i a^i) a^j + \beta_4 a^i a^j a_i j + \beta_5 (a_i a^i) R + \beta_6 a^i a_j R^{ij} + \beta_7 R a^i \right] \]
\[ + \frac{1}{\xi^2} \left[ \gamma_5 C_{ij} a^{ij} + \beta_8 \left( \Delta a^i \right)^2 \right], \]
\[ \mathcal{L}_A = -\frac{AR}{N}, \]
\[ \mathcal{L}_\varphi = \varphi \mathcal{G}_{ij} \left( 2K_{ij} + \nabla_i \nabla_j \varphi + a_i \nabla_j \varphi \right) \]
\[ + (1 - \lambda) \left[ (\Delta \varphi + a_i \nabla_i \varphi)^2 + 2(\Delta \varphi + a_i \nabla_i \varphi) R \right] \]
\[ + \frac{1}{3} \hat{G}^{ij} \left[ 4 \left( \nabla_i \nabla_j \varphi \right) a_{(k} \nabla_{l)} \varphi \\
+ 5 \left( a_{(i} \nabla_j \varphi \right) a_{(k} \nabla_{l)} \varphi + 2 \left( \nabla_{(i} \varphi \right) a_{j)k} \nabla_{l} \varphi \\
+ 6K_{ij}q(l \nabla_k) \varphi \right], \]

\[ \mathcal{L}_S = \sigma \left( a^i \partial_i a_i + \sigma_2 a_i \right), \quad (9) \]

where

\[ \zeta^2 = \frac{1}{16 \pi G}, \quad (10) \]

where \( G \) denotes the Newtonian constant, \( L_M \) is the Lagrangian of matter fields, \( \hat{G}^{ij} = g^{ij} - g^{ij} g^{kl} \delta_{kl} \). Here \( \Delta \equiv g^{ij} \nabla_i \nabla_j \) and all the coefficients, \( \beta_n \) and \( \gamma_n \), are dimensionless and arbitrary. \( C_{ij} \) denotes the Cotton tensor, defined by

\[ C_{ij} = \frac{\epsilon^{ijkl}}{\sqrt{g}} \nabla_k \left( R^i_l - \frac{1}{4} R \delta^i_l \right), \quad (11) \]

with \( \epsilon^{123} = 1 \). Using the Bianchi identities, one can show that \( C_{ij} \) can be written in terms of the five independent sixth-order derivative terms in the form

\[ C_{ij} = \frac{1}{2} R^i - \frac{5}{2} R R_{ij} R^j + 3 R^i R^j R_k \delta^i_l + \frac{3}{2} R \Delta R \\
+ \left( \nabla_i R_{jk} \right) \left( \nabla^j R^k \right) + \nabla_k G^k, \quad (12) \]

where

\[ G^k = \frac{1}{2} R^{jk} \nabla_j R - R_{ij} \nabla^j R^k - \frac{3}{4} R \nabla R. \quad (13) \]

The Ricci and Riemann tensors \( R_{ij} \) and \( R^{i} g_{kl} \) all refer to the 3-metric \( g_{ij} \), with \( R_{ij} = R^{k} g_{ik} \) and

\[ R_{ij} = g_{ik} R_{jl} + g_{jl} R_{ik} - g_{aj} R_{ik} - R_{il} g_{jk} \\
- \frac{1}{2} \left( g_{ik} g_{jl} - g_{il} g_{jk} \right) R, \]

\[ K_{ij} = \frac{1}{2N} \left( -g_{ij} + \nabla_i N_j + \nabla_j N_i \right), \]

\[ \hat{g}_{ij} = R_{ij} - \frac{1}{2} g_{ij} R, \]

\[ a_i = \frac{N_i}{N}, \quad a_{ij} = \nabla_j a_i, \quad (14) \]

where \( N_i \) is defined in the ADM form of the metric \[ g_{ij} \], given by equation \( (3) \). The variations of the action \( S \) \[ S \] with respect to \( N \) and \( N^i \) give rise to the Hamiltonian and momentum constraints,

\[ \mathcal{L}_K + \mathcal{L}_V + F_V - F_\varphi - F_\lambda + \mathcal{H}_S = 8 \pi G J^i, \quad (15) \]

\[ M_S^i + \nabla_j \left\{ \pi^{ij} - \left( 1 - \lambda \right) g^{ij} \left( \nabla^2 \varphi + a_k \nabla^k \varphi \right) \\
- \varphi \hat{g}^{ij} - \hat{g}^{ijkl} a_l \nabla_k \varphi \right\} = 8 \pi G J^i, \quad (16) \]

where

\[ \mathcal{H}_S = \frac{2 \sigma_1}{N} \left[ a^i \left( A - A \right) \right] - \frac{\sigma_2}{N} \nabla^2 \left( A - A \right) \\
+ \frac{1}{2} a_S \nabla_j \varphi \nabla^j \varphi, \]

\[ M_S^i = - \frac{1}{2} a_S \nabla^i \varphi, \]

\[ J^i = - N \delta \mathcal{L}_M \delta N_i, \quad J^i = 2 \delta \left( N \mathcal{L}_M \right) \delta N_i \]

\[ \pi^{ij} = - K^{ij} + \lambda K \delta \varphi, \quad (17) \]

with

\[ a_S = \sigma_1 a^i a_i + \sigma_2 a_i, \quad (18) \]

and \( F_V, F_\varphi \) and \( F_\lambda \) are given in the Appendix A. Variations of \( S \) with respect to \( \varphi \) and \( A \), yield, respectively,
where

$$\tau^{ij} = \frac{2}{\sqrt{\delta N}} \frac{\delta (\sqrt{\delta N} L_M)}{\delta g_{ij}},$$

(24)

and $F^{ij}, F_S^{ij}, F_a^{ij}$ and $F_\varphi^{ij}$ are given in the Appendix A.

From reference [41], we have that

$$\dot{N} = \ddot{N}(N, N_i, g_{ij}, A, \varphi),$$

$$\ddot{N}_i = \ddot{N}_i(N, N_i, g_{ij}, A, \varphi),$$

$$\ddot{g}_{ij} = \ddot{g}_{ij}(N, N_i, g_{ij}, A, \varphi).$$

(25)

Thus,

$$J^t = 2\Omega^3 \left\{ -\rho \frac{\delta \dot{N}}{\delta N} + \frac{\delta \dot{N}_i}{\delta N} S^i + \frac{1}{2} \ddot{N} \frac{\delta \ddot{g}_{ij}}{\delta N} S^{ij} \right\}.\tag{26}$$

Similarly, it can be shown that

$$J^i = -\Omega^3 \left\{ -\rho \frac{\delta \dot{N}_i}{\delta N_i} + \frac{\delta \ddot{N}_k}{\delta N_i} S^k + \frac{1}{2} \ddot{N} \frac{\delta \ddot{g}_{kl}}{\delta N_i} S^{kl} \right\},$$

$$\tau^{ij} = \frac{2}{N} \Omega^3 \left\{ -\rho \frac{\delta \dot{N}_i}{\delta g_{ij}} + \frac{\delta \ddot{N}_k}{\delta g_{ij}} S^k + \frac{1}{2} \ddot{N} \frac{\delta \ddot{g}_{kl}}{\delta g_{ij}} S^{kl} \right\},$$

$$J_A = 2\Omega^3 \left\{ -\rho \frac{\delta \dot{N}}{\delta A} + \frac{\delta \ddot{N}_k}{\delta A} S^k + \frac{1}{2} \ddot{N} \frac{\delta \ddot{g}_{kl}}{\delta A} S^{kl} \right\},$$

$$J_\varphi = -\frac{1}{N} \left\{ \frac{1}{\sqrt{g}} \langle B \sqrt{g} \rangle_t - \nabla_i \left[ B \left( N^i + N \nabla^i \varphi \right) \right] - \nabla_i \left( N \Omega^5 S^i \right) \right\}.\tag{27}$$

where

$$B = -\Omega^3 \left\{ a_1 \rho - \frac{2 \alpha_2 (1 - a_2 \sigma)}{N} S_k (N_k + N \nabla_k \varphi) - a_2 (1 - a_1 \sigma) (1 - a_2 \sigma) g_{ij} S^{ij} \right\}.\tag{28}$$

A. Shear-free Dust Solution with the Minimally Coupling in the Infrared Limit

In order to be consistent with observations in the infrared limit [41], we assume that

$$\beta_1 = \beta_2 = \beta_3 = \beta_4 = \beta_5 = \beta_6 = \beta_7 = \beta_8 = \beta_9 = 0,$$

(29)

$$\gamma_0 = \gamma_2 = \gamma_3 = \gamma_4 = \gamma_5 = \gamma_6 = \gamma_7 = \gamma_8 = \gamma_9 = 0,$$

(30)

and for the PPN approximation in minimally coupling theory, we have

$$\beta_0 = -2(\gamma_1 + 1),$$

(31)

$$a_1 = a_2 = 0,$$

(32)

$$\sigma_1 = 0,$$

(33)

$$\sigma_2 = 4(1 - a_1) = 4.$$ \tag{34}

Thus, we have the vanishing of the cosmological constant, as follows

$$\Lambda_g = \frac{1}{2} \rho \gamma_0 = 0.$$ \tag{35}

Besides, In the infrared limit we must have

$$J^t = -2 \rho.$$ \tag{36}

Now we consider the case of a shear-free dust [40]. Thus, the metric can be rewritten as

$$ds^2 = -dt^2 + Y(r, t)^2 \left[ f(r)^2 dr^2 + d\Omega^2 \right].$$ \tag{37}

Using the equations (11) and (14), we have

$$K_{rr} = -\dot{Y} Y f^2,$$ \tag{38}

$$K_{\theta \theta} = -\dot{Y} Y,$$ \tag{39}

$$K_{\phi \phi} = \sin(\theta)^2 K_{\theta \theta},$$ \tag{40}

$$K = -3 \frac{\dot{Y}}{Y},$$ \tag{41}

$$R_{rr} = 2 f' Y' Y - f Y' Y + f Y'^2,$$ \tag{42}

$$R_{\theta \theta} = \frac{f' Y' - Y'' f - f'^3}{f Y'},$$ \tag{43}

$$R_{\phi \phi} = \sin(\theta)^2 R_{\theta \theta}.$$ \tag{44}
\[ R = 2 \frac{2f'Y''Y - 2Y''fY - f'fY'^2 + f^3Y^2}{f^3Y^4}, \]  
\[ \mathcal{L}_K = \frac{3(1 - 3\lambda)}{Y^2}, \]  
\[ \mathcal{L}_V = 2\gamma_1 \frac{2f'Y'Y - 2fY''Y - fY'^2 + f^3Y^2}{f^3Y^4}, \]  
\[ F_V = 0, \]  
\[ H = 8\pi J = \frac{1}{fY^8} \times \]  
\[ [f^4Y^5A'(f'Y - fY') - 16f''f^2Y'^3 + 36f''f'Y'^2Y^3 + 144f''fY''Y^3 - 64f''f^2Y'^3 + 122f''f^2Y'^2Y^2 - 192f''f^2Y'Y^3 + 336f''f^2Y'^2Y^2 - 80f'f^2Y''Y^2] = \frac{\gamma_1}{Y^2}, \]  
\[ J_r = (1 - 3\lambda)(Y'Y' - Y'Y) \]  
\[ J_A = \frac{2}{f^3Y^4} \frac{2f'Y'Y - 2fY''Y - fY'^2 + f^3Y^2}{f^3Y^4}, \]  
\[ J_\varphi = \frac{1}{fY^5} \times \]  
\[ [3(1 - \lambda)(f'Y'Y'^2 + fY''Y - fY'^2Y) + fY''Y + 3\lambda(f'Y'Y'Y - fY''YY + fY'^2Y - 3f^3YY^4) - f'Y''Y + fY''Y - 2fY'^2Y + f^3Y^2]. \]  
From the dynamical equations we have
\[ D'' = 8\pi \tau = \frac{d''}{2f^3Y^6}, \]  
where
\[ d'' = 2[(A - \gamma_1)Y'^2 + (A + \gamma_1)f^2Y'^2] - 4A'Y''Y + (1 - 3\lambda)(2f^3YY^3 + f^3Y^2Y^2), \]  
\[ D'' = 8\pi \tau = \frac{d''}{2f^3Y^6}, \]  
\[ d'' = -2A''fY'^2 + 2A'f^2Y'^2 + 2(A + \gamma_1) \times \]  
\[ (f^2Y'Y - f^2Y'^2 + f^3Y'^3) + (1 - 3\lambda) \times \]  
\[ (2f^3YY^3 + f^3Y^2Y^2), \]  
and
\[ D'' = 8\pi \tau = \sin(\theta)^2D'' \]  
For a spacetime filled by a dust fluid without pressure we have that
\[ J_r = 0. \]  
Using the equation \[ J_A = 0, \]  
we have that
\[ Y(r, t) = y_1(r)y_2(t). \]  
Taking the above condition and imposing that \( J_A = 0 \), from equation \[ y_1(t) = y_1(t + c_1), \]  
we obtain
\[ 2f'\int_1^0 y_1y_2 - 2f\int_1^0 y_1y_2 - f(y_1y_2) + f^3(y_1y_2)^2 = 0. \]  
Solving the above equation for \( f)(r) \) we have that
\[ f(r) = \frac{(y_1)^2y_2}{\sqrt{2\int_1^0 y_1y_2 + c_1}}, \]  
where \( c_1 \) is a constant of integration.

Since \( f \) depends only on the coordinate \( r \), then there is always a combination of the functions \( y_1 \) and \( y_2 \) in the equation \[ f(r) \text{ that probably transforms its right side independent of the coordinate } t. \] Substituting now, the function \( f(r) \) into \( J_r = 0 \), from equation \[ \text{Substitute } c_1, \]  
we get that
\[ y_2(t) = c_2, \]  
where \( c_2 \) is an arbitrary constant.

This last equation shows that the function \( Y \) does not depend on \( t \), i.e., \( Y(r, t) = Y(r) \). Thus, we can rewrite the equation \[ f(r) \]  
as
\[ f(r) = \frac{(Y')^2}{\sqrt{2\int_1^0 (Y')^2y_2^2} + c_1}, \]  
where \( c_2 \) is an arbitrary constant.
From this equation we can derive $A$ and the following conditions, minimum coupling with matter \([41]\). In this case we have fluid in GRT.

contradicting the behavior of a such model with the same metric of the coordinate static metric with a fluid density of dust confirmed again in this Section, as our results have led to a dust fluid, for which the metric is projectable, it was con-

Weimann\([41]\] that the projectable version of HLT excludes the minimally coupling with matter. Therefore, in the case of spacetime is stationary.

However, it had already been shown in the reference \([41]\) that the projectable version of HLT excludes the minimally coupling with matter. Therefore, in the case of dust fluid, for which the metric is projectable, it was confirmed again in this Section, as our results have led to a static metric with a fluid density of dust $\rho(r,t) = \rho(r)$, contradicting the behavior of a such model with the same fluid in GRT.

**B. Shear-free Dust Solution with the Non-Minimum Coupling in the Infrared Limit**

Let us now analyze the case where it exists a non-minimum coupling with matter \([41]\). In this case we have the following conditions,

\[
\gamma_1 = -1, \quad (68)
\]

\[
a_1 = 1, \quad (69)
\]

\[
a_2 = 0, \quad (70)
\]

\[
F = 1 - A, \quad (71)
\]

\[
\Omega = 1, \quad (72)
\]

\[
\tilde{N} = 1 - A, \quad (73)
\]

\[
\tilde{N}^i = N^i, \quad (74)
\]

\[
\tilde{g}_{ij} = g_{ij}, \quad (75)
\]

\[
A = 0, \quad (76)
\]

\[
\sigma = A. \quad (77)
\]

With these new parameters we have found that

\[
J^t = -2\rho(1 - A), \quad (78)
\]

\[
J^i = 0, \quad (79)
\]

\[
\tau^{ij} = 0, \quad (80)
\]

\[
J_A = -2\rho, \quad (81)
\]

\[
J_\varphi = \frac{1}{\tilde{Y}}(Y\tilde{\rho} + 3\tilde{Y}\rho). \quad (82)
\]

Thus,

\[
H = 8\pi J^t = \frac{1}{f^3Y^2}[\vartriangledown(1 - 3\lambda)(3\tilde{Y}^2f^3Y^2) - 4f'Y'fY + 4Y''fY - 2f^2Y'^2 - 2f^3Y^2], \quad (83)
\]

\[
J_r = \frac{1}{f^2Y^4}(1 - 3\lambda)(Y' - \tilde{Y}''), \quad (84)
\]

\[
J_A = \frac{2}{f^3Y^4}(2f'Y'Y - 2f''Y'Y + fY'^2 + f^3Y^2), \quad (85)
\]

\[
J_\varphi = \frac{1}{f^3Y^3}[3(1 - \lambda)(f'\tilde{Y}'Y^2 + f\tilde{Y}'Y'Y - f\tilde{Y}'Y^2) + fY''Y + (1 - 3\lambda)(fY''\tilde{Y}Y - f'\tilde{Y}'Y\tilde{Y}) + (3\lambda - 2)fY'^2 + f^3\tilde{Y}Y^2]. \quad (86)
\]

From the dynamical equations we get

\[
D^{rr} = 8\pi \tau^{rr} = \frac{d^{rr}}{2f^3Y^6}, \quad (87)
\]

where

\[
d^{rr} = 2\vartriangledown[(1 - A)(Y'^2 - f^2Y^2)] - 4AYY' + (1 - 3\lambda)(2f^2\tilde{Y}^3 + f^2\tilde{Y}^2Y'Y'), \quad (88)
\]

\[
D^{\theta\theta} = 8\pi \tau^{\theta\theta} = \frac{d^{\theta\theta}}{2f^3Y^6}; \quad (89)
\]
where
\[ d\theta = 2A'f'Y^2 - 2A''fY^2 + 2(1 - A) \times (f'Y'Y + fY''Y - fY'^2) + (1 - 3\lambda)(2f^3\dot{Y}^3 + f^3\ddot{Y}^2Y^2), \] (90)
and
\[ D\phi = 8\pi r\phi = D\theta \sin^2 \theta. \] (91)

Therefore, for a spacetime filled by a fluid of dust with zero pressure and shear-free, we have \( J_r = 0 \), as in Section IV. Thus, solving equation (88) for the metric function \( Y(r,t) \), we have found that the solution admits separation of variables, i.e.,
\[ Y(r,t) = y_1(r)y_2(t). \] (92)

From equations (78) and (81), we have that
\[ A = 1 - \frac{J^t}{J^A}, \] (93)
thus
\[ A = \left[ 3(3\lambda - 1)y_1'y_2f^3 + 4y_1''y_1f + 4y_2'r^2f + 8f'y_1y_2 \left[ 2(f^3y_3^2 + 2f'y_1'y_1 - 2y_1''y_1 + y_2'^2f) \right]^{-1} \] (94)

Then, we derive the above equation once and twice in relation to \( r \), in order to find the expressions for \( A' \) and \( A'' \). With these results, we can use to the dynamic equations (87) and (89), which are identically zero. Although we know the expression for \( A \), the resulting equations are very difficult to be solved individually. However, we can notice that
\[ fY'' = 2[f(1 - A)(Y'^2 - f^2Y^2)] - 4fA'Y'Y + (1 - 3\lambda)(2f^3\dot{Y}^3 + f^3\ddot{Y}^2Y^2) = 0. \] (95)

Thus, we can write that
\[ 4fA'Y'Y - 2[f(1 - A)(Y'^2 - f^2Y^2)] = (1 - 3\lambda)(2f^3\dot{Y}^3 + f^3\ddot{Y}^2Y^2), \] (96)
where the right hand of this equation can be substituted in equation (95), giving
\[ 2A'f'Y^2 - 2A''fY^2 + 2(1 - A)(f'Y'Y + fY''Y - fY'^2 + 4fA'Y'Y - 2[f(1 - A)(Y'^2 - f^2Y^2)] = 0. \] (97)

Again, as in previous Section, solving this last equation in relation to \( y_1(r) \) and \( y_2(t) \), we get two possible solutions
\[ Y(r,t) = 0, \] (98)
or
\[ Y(r,t) = y_1(r)(c_1 t + c_2), \] (99)
where \( c_1 \) and \( c_2 \) are arbitrary constants.

Since the first solution \( Y(r,t) = 0 \) is not possible, the only solution is the second one. Substituting this solution into the equation (88) we obtain that \( A(r,t) = A(r) \).

Now, solving the equation for \( J_\phi \) and using the equations (78) to (80) we have that
\[ \rho(r,t) = \frac{c_1\rho_1(r)}{c_1 t + c_2^3}, \] (100)
where the function \( \rho_1(r) \) and the arbitrary function \( F(r) \) are dependent only to \( r \).

Using the equation (81) for \( J_A \) we get that
\[ \rho = -\frac{2f'Y'Y - 2fY''Y + fY'^2 + f^3Y^2}{f^3Y^2}. \] (101)

Using the equations (83) and (100) we get three possible solutions:
1. \( f(r) = 0 \), which is physically impossible,
2. \( c_1 = 0 \), which gives the same results without the minimally coupling,
3. \( F(r) = 0 \), which allows the density \( \rho \) dependent of \( t \), i.e., \( \rho(r,t) \).

With the third condition, we obtain that
\[ \rho(r,t) = \frac{c_1\rho_1(r)t}{(c_1 t + c_2)^2}. \] (102)

We can see that the non-minimum coupling produces interesting results. It is still necessary analyze the temporal behavior of the energy density, in order to verify if this solution admits a gravitational collapse process, as expected in GRT. Thus, deriving the energy density given in the equation (102), we get
\[ \dot{\rho}(r,t) = \frac{c_1\rho_1(r)(c_2 - 2c_1 t)}{(c_1 t + c_2)^2}. \] (103)

Analyzing \( \rho(r,t) \) at \( t = 0 \), we have
\[ \dot{\rho}(r,t) = 0 = \frac{c_1\rho_1(r)}{c_2}. \] (104)

Since the energy density as well as its temporal rate of change should be always positive, in order to insure a physically acceptable collapsing fluid, we have to impose some conditions on the constants \( c_1 \), \( c_2 \) and on the function \( \rho_1(r) \).

Let us study two cases: Case (1) \( \rho_1(r) > 0 \) and Case (2) \( \rho_1(r) < 0 \) (see Figure 1).

1. Since \( \rho \geq 0 \), then \( c_1 \) and \( c_2 \) must have the same signs. Besides, if \( \dot{\rho} > 0 \) then \( 0 \leq t < t_{c_1} \) and \( \dot{\rho} < 0 \) then \( t > t_{c_1} \), where \( t = t_{c_1} = |c_1|/(2|c_2|) \) represents the time of the inversion of sign of \( \dot{\rho} \). Thus, this case describes a situation of an initial contraction followed by an expansion, without the formation of a singularity.
2. Again, since $\rho \geq 0$, thus $c_1$ and $c_2$ must have the opposite signs. Besides, when $\dot{\rho} > 0$ then $0 < t < t_{c2}$ and $\rho \to \infty$ at $t = t_{c2} = |c_2|/|c_1|$. Thus, this case describes a typical gravitational collapse situation, which is consistent with the results of GRT.

### III. CONCLUSION

In this present work, using again the results of Lin et al. (2014) \cite{lin2014} we have studied the spherically symmetric spacetime filled by a dust fluid, in general covariant theory ($U(1)$ extension) of HLT with the minimally coupling \cite{lin2014}, in the PPN approximation in the infrared limit. We have analyzed if a solution like this one can be described in the general covariant HLT of gravity \cite{lin2014, lin2015}.

Although we do not have a realistic model, we can describe the gravitational collapse as we can see in GRT. Besides, confirming what was found in the reference \cite{lin2014}, the projectable HLT excludes the minimally coupling, and it does not reproduce the well known results in GRT.

However, when using non-minimum coupling with matter, we have shown that the solution admits a process of gravitational collapse, leaving a singularity at the end.

Note that we have also found a second possible solution, representing a bounce behavior that is not expected in GRT.

### Acknowledgments

The financial assistance from FAPERJ/UERJ (MFAS) are gratefully acknowledged. The author (RC) acknowledges the financial support from FAPERJ (no. E-26/171.754/2000, E-26/171.533/2002, E-26/170.951/2006, E-26/110.432/2009 and E-26/111.714/2010). The authors (RC and MFAS) also acknowledge the financial support from Conselho Nacional de Desenvolvimento Científico e Tecnológico - CNPq - Brazil (no. 450572/2009-9, 301973/2009-1 and 477268/2010-2). The author (MFAS) also acknowledges the financial support from Financiadora de Estudos e Projetos - FINEP - Brazil (Ref. 2399/03). The authors (OG and MFAS) also thank the financial support from CAPES/Science without Borders (no. A 045/2013). We also would like to thank Dr. Anzhong Wang for helpful discussions and comments about this work.

### IV. APPENDIX A: DEFINITION OF $F^{ij}$, $F^{ij}_S$, $F^{ij}_\varphi$ AND $F^{ij}_S$

The quantities $F^{ij}$, $F^{ij}_S$, $F^{ij}_\varphi$, $F^{ij}_S$ and $F^{ij}_S$ are given by

\begin{align}
F^{ij} &= \frac{1}{\sqrt{g_N}} \frac{\delta (-\sqrt{g} N L^R_{ij})}{\delta g_{ij}} \\
&= \sum_{s=0}^{\infty} \beta_s \zeta^s (F^s_{ij}) \sigma + n_b \qquad (105) \\
F^{ij}_S &= -\sigma (\sigma a^i a^j + \sigma_2 a^{ij}) \\
&\quad + \frac{a_s}{2} \left[ (\nabla^i \varphi)(\nabla^j \varphi) + 2 N^{(i)}(\nabla^j) \varphi \right] \\
&\quad + \frac{\sigma_2}{2N} \nabla^k [a_k (A - \varphi)], \\
F^{ij}_\varphi &= \frac{1}{\sqrt{g_N}} \frac{\delta (-\sqrt{g} N L^R_{ij})}{\delta g_{ij}} \\
&= \sum_{s=0}^{\infty} \beta_s \zeta^s (F^s_{\varphi})^{ij} \sigma, \qquad (106) \\
F^{ij}_S &= \frac{1}{\sqrt{g_N}} \frac{\delta (-\sqrt{g} N L^R_{ij})}{\delta g_{ij}} \\
&= \sum_{s=0}^{\infty} \mu_s (F^s_{\varphi})^{ij}, \\
F^{ij}_S &= -\sigma (\sigma a^i a^j + \sigma_2 a^{ij}) \\
&\quad + \frac{a_s}{2} \left[ (\nabla^i \varphi)(\nabla^j \varphi) + 2 N^{(i)}(\nabla^j) \varphi \right] \\
&\quad + \frac{\sigma_2}{2N} \nabla^k [a_k (A - \varphi)], \qquad (107)
\end{align}
with
\[
\dot{\gamma}_s = \left(\gamma_0, \gamma_1, \gamma_2, \ldots, \gamma_N, \mu_s - \lambda, \gamma_0 \right),
\]
\[
n_s = (2, 0, -2, -2, -4, -4, -4, -4, -4, -4),
\]
\[
m_s = (0, -2, -2, -2, -2, -2, -2, -2, -4),
\]
\[
\mu_s = \left(2, 1, 1, 2, \frac{4}{3}, \frac{5}{3}, \frac{2}{3}, 1 - \lambda, 2 - 2\lambda \right).
\]

Thus, \(F_V\), \(F_\varphi\) and \(F_\lambda\) are given, respectively, by
\[
F_V = \beta_0(2a^i + a^i a^j) - \frac{\beta_1}{\zeta^2} \left[3(a_i a^i)^2 + 4\nabla_i(a_k a^k a^i)\right] + \frac{\beta_2}{\zeta^2} \left[(a_i a^j)^2 + \frac{2}{N} \nabla^2(N a_k^j)\right] + \frac{\beta_3}{\zeta^2} \left[\frac{2}{N} \nabla_j \nabla_i(N a^j)\right] + \frac{\beta_4}{\zeta^2} \left[-R(a_i a^i) - 2\nabla_i(R a^i)\right] + \frac{\beta_5}{\zeta^2} \left[\nabla_i a^j - \nabla_j (a_i R^{ij}) - \nabla_j (a_i R^{ij})\right] + \frac{\mu_s}{\zeta^2} \left[\nabla_i \left(\Delta a^i\right)^2 - \frac{2}{N} \nabla_i[N \nabla_i a_i]\right],
\]
\(F_\varphi = -g^{ij} \nabla_i \varphi \nabla_j \varphi - \frac{2}{N} \tilde{g}_{ijkl} \nabla_i (N K_{ij} \nabla_k \varphi),\)
\[
\frac{4}{3} \tilde{g}_{ijkl} \nabla_i (\nabla_k \varphi \nabla_j \varphi)
\]
\[
+ \frac{5}{3} \tilde{g}_{ijkl} \left[(a_i \nabla_j \varphi)(a_k \nabla_l \varphi) + \nabla_i (a_k \nabla_j \varphi) \nabla_l \varphi\right] + \nabla_k (a_i \nabla_j \varphi \nabla_l \varphi)
\]
\[
+ \frac{2}{3} \tilde{g}_{ijkl} \left[a_{ik} \nabla_j \varphi \nabla_l \varphi + \frac{1}{N} \nabla_i \nabla_k (N \nabla_j \varphi \nabla_l \varphi)\right],
\]
\(F_\lambda = (1 - \lambda) \left\{ (\nabla^2 \varphi + a_i \nabla^j \varphi)^2 - \frac{2}{N} \nabla_i (N K \nabla^j \varphi)\right\} - \frac{2}{N} \nabla_i [N (\nabla^2 \varphi + a_i \nabla^j \varphi) \nabla^i \varphi].
\]
\[
-\frac{1}{2} \left[ a_i R^{km} \nabla_m R_{kj} + a_j R^{km} \nabla_m R_{ki} \right] \\
-\frac{3}{8} a_i R_{ij} R + \frac{3}{8} \left\{ R \nabla_k (N a^k) R_{ij} \right\} \\
+ g_{ij} \nabla^2 \left[ R \nabla_k (N a^k) \right] - \nabla_i \nabla_j \left[ R \nabla_k (N a^k) \right] \}
\]
\[
+ \frac{1}{4N} \left\{ - \frac{1}{2} \nabla^m \left[ \nabla_i (N a_j) \nabla_m R + \nabla_i (\nabla_j R) R_m a_m \right] \\
+ \nabla^2 (N a_i (\nabla_j R)) + g_{ij} \nabla^m \nabla^m (N a_m \nabla_n R) \\
+ \nabla^m \left[ \nabla_i (\nabla_j R^k) R_{mk} + \nabla_i (\nabla_m R^k) N a_k \right] \\
- 2 \nabla^2 (N a_k \nabla_i (R^k)) - 2 g_{ij} \nabla^m \nabla^m (N a_k \nabla_m R^k) \\
- \nabla^m \left[ \nabla_i \nabla_p (N a_i R^p_m + N a_m R^p_i) \right] \\
+ \nabla_j \nabla_p (N a_i R^m_p + N a_m R^i_p) \\
+ 2 \nabla^2 \nabla_p (N a_i (R^i_p)) \\
+ 2 g_{ij} \nabla^m \nabla^m \nabla^m (N a_i (R^m_p)) \right\},
\]
\[ + \frac{1}{2N} \nabla^a \left[ a_a N(i \nabla_j \varphi + N(i) a_j \nabla \alpha \varphi \right] - N_a a_i \nabla \varphi + 2g_{ij} N_a a^k \nabla k \varphi \right] + \frac{1}{2} \nabla \left[ (N N_j) a^k \nabla k \varphi \right] + a^k K_i \nabla j \varphi \right] + a_i (K_j) k \nabla k \varphi - K_a (\nabla j) \varphi - K_j a^k (\nabla k \varphi),
\]

\[ (F_{ij}^\varphi) = - \frac{1}{2} g_{ij} F^{mnk}|[a(k_m \nabla \varphi)] [a(k_l \nabla \varphi)] - \left( a^k a^j \varphi - a_i a_k \nabla j \varphi \right) + \frac{1}{2} \nabla k (N, \phi_k \nabla \varphi)\right],
\]

\[ (F_{ij}^\varphi) = - \frac{1}{2} g_{ij} F^{mnk}|(\nabla (a_m) [a(k_m \nabla \varphi)] + \frac{1}{2} g_{ij} \nabla \phi \phi_k \nabla \varphi - \frac{1}{2} g_{ij} \nabla \phi \phi_k \nabla \varphi - \frac{1}{2} g_{ij} \nabla \phi \phi_k \nabla \varphi
\]
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