Proof gap in “Sufficient conditions for uniqueness of the Weak Value” by J. Dressel and A. N. Jordan, J. Phys. A 45 (2012) 015304

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Abstract
The commented article attempts to prove a “General theorem” giving sufficient conditions under which a previously introduced “general conditioned average” “converges uniquely to the quantum weak value in the minimal disturbance limit.” The “general conditioned average” is obtained from a positive operator valued measure (POVM) \{\hat{E}_j(g)\}_{j=1}^n depending on a small “weakness” parameter g. We point out that unstated assumptions in the presentation of the “sufficient conditions” make them appear much more general than they actually are. Indeed, the stated “sufficient conditions” strengthened by these unstated assumptions seem very close to an assumption that the POVM operators \hat{E}_j(g) be linear polynomials (i.e., of first order in g). Moreover, there appears to be a critical error or gap in the attempted proof, even assuming a linear POVM. A counterexample to the proof of the “General theorem” (though not to its conclusion) is given. Nevertheless, I conjecture that the conclusion is actually true for linear POVM’s whose contextual values are chosen by the commented article’s “pseudoinverse prescription”.

1 Relation between traditional “weak measurement” theory and the “contextual value” approach of [1]

1.1 General introduction
This is an expanded version of a paper submitted to J. Phys. A commenting on [1] (called DJ below). DJ attempts to refute counterexamples given in [6] to a “General theorem” (GT). The “Comment” paper discusses the validity of these counterexamples and gives a new counterexample to the proof (though not to the conclusion) of the GT.

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The submitted paper had to be written more tersely in order to keep its length appropriate for a “Comment” paper and may be incomprehensible to anyone not already familiar with DJ. This expanded version adds the present Section 1 introduction together with an appendix which analyzes in detail some logical problems with the stated hypotheses of the GT.

For orientation I first give a brief description of my view of the notion of “contextual values”, introduced in [2] (called DAJ below) and expounded by Dressel and Jordan in more detail in [3] and the paper DJ under review [1]. My views are presented in more detail in [5] and [6]. The review assumes that the reader has some familiarity with the ideas of weak measurement. A more leisurely exposition of these can be found in [7] and references cited there. To minimize confusion, I try to use the notation of DJ wherever practical, even though it is not the notation that I would choose.

One minor exception is that I usually write $\langle u| \hat{A}|v \rangle$ instead of $\langle u| \hat{A}|v \rangle$ as in DJ’s (1.1). The reason is that all the type was set before noticing this small difference, and attempting to change it risks more confusion than retaining it in case some instances which should be changed go unnoticed.

1.2 Weak measurement

“Weak measurement”, introduced in [4], is in part a technique for measuring the expectation of a quantum observable $\hat{A}$ in a given state $s$ without appreciably changing the state. This can be accomplished as follows.

Suppose the observable $\hat{A}$ operates on a Hilbert space $S$. Couple $S$ to an auxiliary “meter space” $M$, obtaining a new Hilbert space $S \otimes M$ which is the tensor product of $S$ and $M$.

With each state $s$ of $S$, associate a slightly entangled state $\hat{U}s \in S \otimes M$, where $\hat{U}$ is a isometry $^4$ from $S$ to $S \otimes M$. Find a “meter observable” $\hat{B}$ on $M$ such that the expectation of $\hat{I} \otimes \hat{B}$ in the state $\hat{U}s$ is almost the same as the expectation of $\hat{A}$ in the state $s$, where $\hat{I}$ generically denotes the identity operator on whatever space is relevant in the context (in this case $M$).

To make this precise, introduce a small real “weak measurement” parameter $g$ with $\hat{U} = \hat{U}(g)$ depending on $g$. In terms of this parameter, “slightly

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1. The reader should be warned that DAJ is vaguely written with many errors and omissions of important definitions and hypotheses.

2. [3] is written in an unusual, complicated notation different from both DAJ and DJ. It contains what the authors characterize as a “slight generalization” of the “General theorem” of the first arXiv version of DJ. The discussion of the GT in [3] gives no indication that its validity is disputed in [6].

3. The reader should be warned that the six versions of [5] were written over a period of months as I tried to make sense of the vaguely written and error-ridden DAJ, and an evolution of its ideas from earlier to later versions will be apparent. The presentation may seem unusual in that introductions to the later versions were simply prepended to rewritten earlier versions. The presentation of [6] is more concise.

4. An isometry $\hat{U}$ is a linear transformation which preserves inner products: $\langle \hat{U}v|\hat{U}w \rangle = \langle v|w \rangle$ for all $v, w$. The only difference between an isometry and a unitary operator is that an isometry need not be surjective (i.e., “onto”).
entangled” is interpreted as
\[ \lim_{g \to 0} \hat{U}(g)s = s \otimes m \]  
(1)

(an unentangled product state, where \( m \) is some state of \( m \)).

Denote the projector onto the subspace spanned by a nonzero \( v \) as \( \hat{P}_v \). A routine calculation shows that the preceding paragraph guarantees that the (mixed) state of \( S \) corresponding to \( \hat{U}(g)s \), namely \( \text{Tr}_M \hat{P}_v \hat{U}(g)s \) where \( \text{Tr}_M \) denotes partial trace, approaches \( \hat{P}_v \) (i.e., the original pure state \( s \) written in mixed state notation) as \( g \to 0 \). Similarly “the expectation of \( \hat{f} \otimes \hat{B} \) in the state \( \hat{U}(g)s \) is almost the same as the expectation of \( \hat{A} \) in the state \( s \)” is interpreted as holding exactly in the limit \( g \to 0 \):
\[ \lim_{g \to 0} \langle \hat{U}(g)s, (\hat{f} \otimes \hat{B})\hat{U}(g)s \rangle = \langle s|\hat{A}s \rangle \quad . \quad (2) \]

The mathematics of [1] can easily be made rigorous only under the assumption that all Hilbert spaces occurring are finite dimensional, so that all observables have discrete spectra. Let \( \{f_j\}_{j=1}^n \) denote a collection of orthonormal eigenvectors of \( \hat{B} \), with \( \alpha_j \) the corresponding eigenvalues, which we assume distinct for expositional simplicity. We shall allow the \( \alpha_j = \alpha_j(g) \) to depend on \( g \), which implies that \( \hat{B} = \hat{B}(g) \) also depends on \( g \). In principle, one could also allow the eigenvectors \( f_j \) to depend on \( g \), but for simplicity we assume that they are constant. (This assumption can be justified by making appropriate identifications.)

Write
\[ \hat{U}(g)s = \sum_j \hat{M}_j(g)s \otimes f_j \quad , \quad (3) \]
where this defines the “measurement” operators \( \hat{M}_j(g) \) on \( S \). These measurement operators define a positive operator valued measure (POVM) \( \{\hat{E}_j(g)\} \) on \( S \) by \( \hat{E}_j(g) := \hat{M}_j^+(g)\hat{M}_j(g) \). The probability \( P(j) \) that a measurement of \( \hat{f} \otimes \hat{B} \) in state \( \hat{U}(g)s \) will produce result \( j \) is the norm-squared of the \( f_j \)-component of \( \hat{E}_j(g) \):
\[ P(j) = |\hat{M}_j(g)s|^2 = \langle s|\hat{E}_j(g)s \rangle \quad . \quad (4) \]

From [4], the expectation \( \langle \hat{U}(g)s, (\hat{f} \otimes \hat{B})\hat{U}(g)s \rangle \) of \( \hat{f} \otimes \hat{B} \) in the state \( \hat{U}(g)s \) is
\[ \langle \hat{U}(g)s, (\hat{f} \otimes \hat{B})\hat{U}(g)s \rangle = \sum_j \alpha_j P(j) = \sum_j \alpha_j(g)\langle s|\hat{E}_j(g)s \rangle \quad . \quad (5) \]

Since the probabilities \( P(j) \) depend only on data in \( S \), by allowing the eigenvalues \( \alpha_j = \alpha_j(g) \) to depend on \( g \), we might hope to choose them to satisfy the desired relation
\[ \langle \hat{U}(g)s, (\hat{f} \otimes \hat{B}(g))\hat{U}(g)s \rangle = \sum_j \alpha_j(g)P(j) = \langle s|\hat{A}s \rangle \quad , \quad (6) \]

\footnote{The slightly subtle reason is discussed in [3].}
which says that the expectation of the system observable $\hat{A}$ in the state $s$ could also be obtained by measuring the expectation of the meter observable $\hat{B}(g)$ in the state $\hat{U}(g)s$. The advantage of measuring $\hat{B}(g)$ instead of $\hat{A}$ is that for small $g$, the (unnormalized) state of $S$ after measurement result $j$ is obtained, namely $\hat{M}_j(g)s$, is very close to $s$ because of (1). Traditional weak measurement theory shows how to choose $\hat{U}(g)$ and $\alpha_j(g)$ to obtain (6) in the limit $g \to 0$, but it does not give (6) as an exact equation for small but nonzero $g$.

The above discussion sketches a formulation of weak measurement theory which gives a more or less direct translation into the language of DAJ and DJ. Weak measurement theory is traditionally formulated in terms of a “system” Hilbert space $S$ and a “meter” space $M$ with a “meter observable” $\hat{B}$. The setup of DAJ and DJ replaces the meter space and meter observable by a set of measurement operators on $S$. The eigenvalues $\alpha_j(g)$ of the meter observable $\hat{B}$ are renamed “contextual values”.

Although it seems clear that any statement about the meter observable can be translated into a statement about measurement operators in $S$ and conversely, there are significant differences between traditional weak measurement theory following [4] and the contextual value theory of DAJ and DJ. For example, unlike contextual value theory, traditional weak measurement theory does not attempt to obtain (6) for all $g$, but only in the limit $g \to 0$:

$$\lim_{g \to 0} \sum_j \alpha_j(g) P(j) = \langle s | \hat{A} s \rangle.$$

DAJ and DJ assume (6) (expressed in terms of measurement operators), i.e.,

$$\sum_j \alpha_j P(j) = \sum_j \alpha_j(g) \langle s | \hat{E}_j(g)s \rangle = \langle s | \hat{A} s \rangle,$$

for all small $g \neq 0$, a very strong assumption. Given $\hat{U}(g)$ (equivalently, given the measurement operators), it is often not possible to choose contextual values $\alpha_j(g)$ satisfying the strong hypothesis (6).

For this and other reasons, the claims of DAJ and DJ that their contextual value formalism “subsumes” the traditional weak value formalism seem open to question. For example, DJ writes:

“The formalism is powerful enough to subsume strong measurements, weak measurements, and any strength of measurement in between.”

This is true only because they have added the strong hypothesis (6) that contextual values can always be chosen for all positive $g$. If the same hypothesis were added to traditional weak measurement theory, that theory would also apply to
strong measurement and “any strength of measurement in between”. In the
other direction, though contextual value theory as formulated in DAJ and DJ
does not apply to cases when contextual values do not exist for all \( g \), it could
probably be reformulated to apply with the weaker hypotheses \( 6 \) in place of
\( 5 \), though at the expense of additional complication.

One way in which contextual value theory might be argued to be more
general is that the original formulation of weak measurement theory in [4] and
much of the subsequent literature assume a particular form for \( \hat{U}(g)s \), namely
\[
\hat{U}(g)s = \exp(ig\hat{A} \otimes \hat{P})s \otimes m ,
\]
where \( \hat{P} \) is a particular operator on a particular meter space \( M \). However, the
formulation of weak measurement theory sketched above does not require this
hypothesis: \( \hat{U}(g)s \) does not need to be of the form \( 9 \).

An advantage of traditional weak measurement theory using \( 9 \) is that it
gives a method to weakly measure any observable. If it were required that \( 6 \)
hold for all \( g \), it would not be obvious that a weak measurement procedure would
exist. The same problem arises in the setup of DAJ and DJ, but although not
yet explicitly addressed by the authors (so far as I know), I would expect it to
to be easy to solve under their assumption of finite dimensionality.

Traditional measurement theory is formulated in a system+meter space \( S \otimes M \). I view contextual value theory of DAJ and DJ as a simpler formulatio
in system space \( S \) alone which is less general as developed in DAJ and DJ
but could probably be reformulated to become essentially equivalent in finite
dimensions. I think claims that contextual value theory is more powerful are
questionable, but it does have the very attractive advantage of simplicity. It
would be unfortunate if inadequately researched and overstated claims turn out
to obscure its genuine merit of conceptual simplicity.

1.3 Postselection

Most applications of weak measurement theory involve more than mere weak
measurement as described above. Typically, after making a weak measurement
one “postselects” to a given final state \( s_f \in S \). This means that one performs
a second projective measurement (with respect to the orthogonal decomposition
\( \{ \hat{P}_{s_f}, \hat{I} - \hat{P}_{s_f} \} \)) to see if after the first measurement, the system is in state \( s_f \)
(“success”) or a state orthogonal to \( s_f \) (“failure”). \( \text{[7]} \) (It would take us too far
afield to explain why one might want to do this.) This is done repeatedly start-
ning with the same initial state \( s \), and only the results of “successful” trials are
retained. The (conditional) expectation of the meter measurement given suc-
cessful postselection is called a “weak value” of the system observable \( \hat{A} \).

\[ 7 \text{For simplicity of exposition we restrict attention to postselection to a pure state, as does DJ. DAJ considers postselection to a mixed state, but does not explain how this could be physically accomplished.} \]

\[ 8 \text{The “weak value” is often confused with the conditional expectation of } \hat{A} \text{ (instead of the meter observable } \hat{B} \text{), and it is important to keep the distinction in mind. The conditional} \]
using given measurement operators”) is DAJ’s “general conditioned average”, which is routinely calculated.

Weak values are not unique; in general they depend on the measurement procedure. The seminal paper [4] calculated (via questionable mathematics) a particular weak value for a particular measurement procedure. Since this calculated weak value is generally nonreal (even though it is supposed to represent the procedure described above which would result in a weak value which is manifestly real), most subsequent authors replace it with its real part. The weak value calculated by [4] is (1.1) of DJ, written in a notation different from that of DJ (and the present paper) which is common in the “weak value” literature:

\[ A_w = \frac{\langle \psi_f | \hat{A} | \psi_i \rangle}{\langle \psi_f | \psi_i \rangle} . \] (1.1)

Here \( \psi_i \) represents the initial state of the system \( S \) (called \( s \) above), \( \psi_f \) the postselected final state (called \( s_f \) above), and \( A_w \) stands for “weak value of \( \hat{A} \).”

DJ’s (1.2) is a generalization of the real part of (1.1) to mixed states:

\[ f(A)_w = \frac{\text{Tr}(\hat{E}_f^{(2)} (\hat{A} \hat{\rho} + \hat{\rho} \hat{A}))}{2\text{Tr}(\hat{E}_f^{(2)} \hat{\rho})} . \] (1.2)

We shall refer to either the real part of (1.1) or (1.2) as the “traditional” weak value (though I’ve not seen (1.2) in the literature prior to DAJ). Most of the “weak value” literature seems to implicitly assume that the traditional weak value is the only possible weak value.

In the contextual value approach, it is natural to ask which collections of measurement operators \( \{ \hat{M}_j(g) \} \) will result in the traditional weak value in the limit \( g \to 0 \). DAJ claims without adequate proof that this will occur when the measurement operators are positive. DJ formulates and attempts to prove a “General theorem” (GT) with this conclusion. One might roughly summarize the GT by the statement that the traditional weak value is essentially inevitable when the measurement operators are positive and commute with each other and the system observable \( \hat{A} \).

Counterexamples to the GT are given in [6]. DJ attempts to refute these counterexamples by reinterpreting (but unfortunately not restating in a logically precise way) the hypotheses of the GT given in the first preprint version of DJ, [arXiv:1106.1871v1], to which [6] replied. The present work will make the reinterpretations explicit and give a new counterexample to the proof of the GT (though not to its conclusion) under its reinterpreted hypotheses.

expectation of \( \hat{A} \) must necessarily be a convex linear combination of the possible values (eigenvalues) for \( \hat{A} \), whereas the conditional expectation of \( \hat{B} \) may lie far outside this set, as the provocative title of [4] suggests.
2 Second introduction for those already familiar with the commented paper DJ [1]

The following, from here to the appendix, is essentially the “Comment” paper currently under review by J. Phys. A. It is not identical because a few expository improvements have been made, but there are no differences of substance.

Notation will be the same as in the article under review [1], called DJ below. To compress this Comment to a traditional length, we must assume that the reader is already familiar with DJ. Its main purpose seems to be to justify a statement of [2] (called DAJ below) that a “general conditioned average” introduced in DAJ “converges uniquely to the quantum weak value in the minimal disturbance limit”. DJ formulates “sufficient conditions” as hypotheses for a “General theorem” (GT) with this statement as its conclusion.

For simplicity, we shall only consider the special case of DAJ and DJ’s “minimal disturbance” condition for which all measurement operators \{\hat{M}_j\} are positive. (All statements will also hold for DJ’s slightly more general definition.) The associated positive operator valued measure (POVM) is \{\hat{E}_j\} with \(\hat{E}_j := \hat{M}_j^\dagger \hat{M}_j\). The measurement operators \(\hat{M}_j = \hat{M}_j(g)\) depend on a small “weakness” parameter \(g\) which quantifies the degree to which the measurement affects the system being measured. Our “minimal disturbance limit” will refer to the so-called “weak limit” \(g \to 0\) for positive measurement operators.

DAJ claims that under these assumptions, its “general conditioned average” (corresponding to what is more usually called a “weak” measurement followed by a postselection) is given by the traditional “quantum weak value” (the real part of DJ’s (1.1)) in the weak limit \(g \to 0\):

“This technique leads to a natural definition of a general conditioned average that converges uniquely to the quantum weak value in the minimal disturbance limit.”

Counterexamples to this claim were given in [6], examples which DJ attempts to refute by reinterpreting the hypotheses of its “General theorem” (GT) given in the first version of DJ [arXiv:1106.1871v1]. Unfortunately, DJ does not make explicit this reinterpretation, but some such reinterpretation is necessary for their objection to make sense.

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9The term “minimally disturbing measurement” for a measurement with positive measurement operators was used (and perhaps coined) in the recent book [8] of Wiseman and Milburn. This reference was unfortunately not cited in DAJ, which uses the term “minimal disturbance limit” without definition or intuitive explanation. I’ve not seen the term used elsewhere in the literature outside of DAJ and subsequent papers by its authors. The technical definition of the phrase “minimally disturbing measurement” as referring to a measurement with positive measurement operators does not correspond to the meaning which one might assume from ordinary usage of the words “minimally disturbing”. (This is discussed in more detail in [5], Section 11.)

For simplicity of exposition, our definition of “minimal disturbance limit” will be essentially that of Wiseman and Milburn: the limit \(g \to 0\) for positive measurement operators, even though DJ uses a slightly more general definition. All statements will also hold for DJ’s definition.
The mathematics of these counterexamples is undisputed\textsuperscript{10}; the only issue is whether they satisfy the hypotheses of DAJ or DJ. DJ correctly notes that the first counterexample using $2 \times 2$ measurement matrices does not satisfy what they call the “pseudoinverse prescription”, but DAJ does not clearly state this prescription as a hypothesis. The second counterexample using $3 \times 3$ matrices does satisfy the pseudoinverse prescription, so the following will deal exclusively with this counterexample.

Contrary to claims of DJ, this counterexample is valid when the hypotheses of the “General theorem” (GT) are interpreted as written, according to standard usage of logical language. However, DJ’s attempted refutation of the counterexamples requires a great strengthening of one of these hypotheses, a strengthening not noted in DJ. We shall see that when so strengthened, the hypotheses of the GT seem very close to the assumption that the POVM must be a linear polynomial in the weak measurement parameter $g$, i.e.,

$$
\hat{E}_j(g) = \hat{E}_j^{(0)} + g \hat{E}_j^{(1)} \text{ where } \hat{E}_j^{(0)} \text{ and } \hat{E}_j^{(1)} \text{ are constant operators. (10)}
$$

The analysis leading to this conclusion will be straightforward and simple. DJ’s attempted proof of the GT is densely written, and our analysis of it must be correspondingly technical. Although probably few readers will be sufficiently familiar with the proof to convince themselves either of its truth or of the claim that there is a major error, I hope that the analysis may motivate anyone tempted to employ (or cite without comment) the “General theorem” to first carefully scrutinize its proof.

3 Unstated hypotheses for the “General theorem”

The hypotheses of the “General theorem” (GT) which will concern us are:

- (iii) The equality $\hat{A} = \sum_j \alpha_j(g) \hat{E}_j(g)$ must be satisfied, where the contextual values $\alpha_j(g)$ are selected according to the pseudoinverse prescription.

- (iv) The minimum nonzero order in $g$ for all $\hat{E}_j(g)$ is $g^n$ such that (iii) is satisfied.”

“Minimum nonzero order” is not a standard mathematical phrase, but I take its occurrence in (iv) to mean that

$$
\hat{E}_j(g) = \hat{E}_j^{(0)} + g^n \sum_{k=0}^{\infty} \hat{E}_j^{(k+n)} g^k
$$

\textsuperscript{10}However, DJ does correctly note a typo in the definition of one of the measurement operators in \cite{6}, a $\sqrt{1/3}$ had been mistakenly written as $1/3$. However the correct value was used in the subsequent calculations, so apart from this single substitution, no other alterations in the argument of \cite{6} are necessary. I thank the authors for this helpful correction.

\textsuperscript{11}This is discussed in more detail in \cite{0}.
with $\hat{E}_j^{(k+n)}$ constant operators and $\hat{E}_j^{(n)} \neq 0$. This is the way the phrase is used (just after DJ’s equation (5.2)) in the attempted proof of the GT. Then the logical content of (iv) is that all $\hat{E}_j$ have the same minimum nonzero order, which is to be denoted $n$. This is a strange and quite restrictive assumption for a "General theorem", but it will not be our main concern.

When the hypotheses of the GT are given the standard logical interpretation just described, the counterexample using $3 \times 3$ matrices which DJ attempts to refute is indeed a counterexample to the GT. However, DJ’s attempt to refute the counterexample appears to assume something like the following.

Denote by $\hat{E}_j'(g)$ the truncation of the series (11) to order $n$, namely,

$$\hat{E}_j'(g) := \hat{E}_j^{(0)} + \hat{E}_j^{(n)} g^n.$$  \hfill (12)

Then (iv) assumes (iii) with the $\hat{E}_j(g)$ in (iii) replaced by $\hat{E}_j'(g)$, but with the contextual values $\alpha_j(g)$ unchanged (i.e., the contextual values for the truncated POVM $\{\hat{E}_j'(g)\}$ are the same as for the original POVM $\{\hat{E}_j(g)\}$). More explicitly, it assumes that

$$\sum_j \alpha_j(g) \hat{E}_j'(g) = \hat{A},$$  \hfill (13)

where the $\alpha_j(g)$ satisfy the pseudo-inverse prescription for the truncated POVM.

DJ’s objection to the counterexample, given after its equation (7.4), is that it does not satisfy (13). DJ does not explicitly say that the contextual values for the truncated POVM are the same as for the original POVM, but that seems suggested by the fact that it uses the same symbols, $\alpha_j(g)$ for both. Also, the details of DJ’s attempted proof support that interpretation.

Next recall that (iii) assumes that the contextual values $\vec{\alpha}(g) = (\alpha_1(g), \ldots, \alpha_n(g))$ satisfy the “pseudo-inverse prescription”

$$\vec{\alpha} = F^+ \vec{a},$$  \hfill (14)

where $\vec{a}$ is a list of eigenvalues for the system observable $\hat{A}$, and $F^+$ is the Moore-Penrose pseudoinverse for the matrix

$$F = F(g) := [\hat{E}_1(g), \ldots, \hat{E}_n(g)].$$  \hfill (15)

Here the column vector $\vec{E}_j(g)$ is the list of eigenvalues for $\hat{E}_j(g)$, and $F$ is the matrix composed of those columns. Note that $F$ is $g$-dependent, but we write

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\footnote{The restrictive phrase “such that (iii) is satisfied” is logically redundant, since (iii) has already been assumed. If the authors mean that some alteration of (iii) is to be assumed, such as (iii) with the $\hat{E}_j$ replaced by their truncations to order $n$ or (iii) with the original contextual values previously denoted $\alpha_j(g)$ replaced by others or some combination of these, then standard logical language requires that this be explicitly stated. I have considered several alternative interpretations of (iv), but all have led to inconsistencies with other parts of DJ. In the absence of requested clarification from the authors, I selected the one which seems most nearly consistent with the rest of DJ.}
$F = F(g)$ only when necessary to emphasize this point, to avoid possible confusion with the result of applying the matrix $F$ to a vector. If contextual values exist (in general, they don’t), they are uniquely determined by the “pseudoinverse prescription” (14).

Since the contextual values for the truncated POVM $\{\hat{E}_j'(g)\}$ are assumed the same as those for the original and to also satisfy the pseudo-inverse prescription for the truncated POVM, we also have

$$\vec{\alpha} = F'^{+} \vec{a}$$

with

$$F' := [\hat{E}_1'(g), \ldots, \hat{E}_n'(g)],$$

where the $\hat{E}_j'(g)$ are the column vectors of eigenvalues for $\hat{E}_j'(g)$. Equation (16) also uniquely determines the contextual values $\alpha_j'(g)$, so it would be surprising if both (14) and (16) would hold except in the trivial case in which $\hat{E}_j = \hat{E}_j'$ for all $j$. In that case, we can make $\{\hat{E}_j\}$ linear (i.e., of form (1)) by replacing the parameter $g$ by a new parameter $h := g^n$, so for brevity we shall refer to this as the “linear case”. The hypothesis that both do hold seems very close to a hypothesis that the original POVM be linear. Indeed, I do not know of any example of a nonlinear POVM for which both (14) and (16) can hold.

4 Error or gap in proof

Readers thinking of building on the work of DAJ and DJ may need to convince themselves of the validity of its “General theorem”. Since its attempted proof is densely written, it may help to pinpoint what I think is a critical error (or at least a serious gap), even under the strong hypothesis that the POVM is linear.

This hypothesis is equivalent to the assumption that the matrix $F = F(g)$ determining the contextual values $\vec{\alpha}$ is first order in $g$, in which case the minimum nonzero order of $F$ which the proof calls $n$ is $n = 1$. To expose the gap, we use these assumptions to rewrite the questionable part of the proof in a simplified form. It applies to a matrix $F$ with singular value decomposition $F = U\Sigma V^T$, where $\Sigma$ is a diagonal matrix and “$U$ and $V$ are orthogonal matrices”. All of these matrices depend on the weak limit parameter $g$.

The contextual values $\vec{\alpha}$ (which the proof renames $\vec{\alpha}_0$) are determined by the pseudoinverse prescription $\vec{\alpha} = \vec{\alpha}_0 = F^{+} \vec{a}$, where $\vec{a}$ is the vector of eigenvalues of $A$. Here $F^{+}$ is the Moore-Penrose pseudoinverse of $F$, given by $F^{+} = V\Sigma^{+}U^T$, where $\Sigma^{+}$ is the diagonal matrix obtained from $\Sigma$ by inverting all its nonzero elements.

In reading the following, please keep in mind that if correct, it should apply to any matrix function $F = F(g)$. Although an $F = F(g)$ derived from a POVM has a special form given in part by DJ’s preceding equation (5.9), nothing in the following proof fragment uses this special form.

The proof mentions “relevant” singular values, but for brevity I have omitted the definition of “relevant” (which does not involve the special form of $F$)
because for the simple counterexample to be given, all we have to know is that a “relevant” singular value is a particular kind of singular value, as the syntax implies. The simplified proof fragment is:

Since the orthogonal matrices $U$ and $V$ have nonzero orthogonal limits $\lim_{g \to 0} U = U_0$ and $\lim_{g \to 0} V = V_0$, such that $U_0^T U_0 = V_0 V_0^T = 1$, and since $\bar{a}$ is $g$-independent, then the only poles in the solution $\tilde{a}_0 = F^+ \bar{a} = V \Sigma^+ U^T \bar{a}$ must come from the inverses of the relevant singular values in $\Sigma^+$.

Therefore, to have a pole of order higher than $1/g$, there must be at least one relevant singular value with a leading order greater than $g^1$.

[This much seems all right, though many details are omitted, but I cannot follow the next and last paragraph of DJ’s attempted proof.]

However, if that were the case then the expansion of $F$ to order $g^1$ would have a relevant singular value of zero and therefore could not satisfy (5.12), contradicting the assumption (iv) about the minimum nonzero order of the POVM. Therefore, the pseudoinverse solution $\tilde{a}_0 = F^+ \bar{a}$ can have no pole with order higher than $O(1/g)$ and the theorem is proved.

If correct, the above proof fragment would imply that if a linear matrix function $F(g) = P + gQ$, with $P$ and $Q$ constant matrices, has a singular value with a leading order greater than $g^1$, then it also has a singular value which is identically zero. (Put differently, the proof claims that if no singular value is identically zero for all $g$, then all singular values are $O(g^1)$.) However, it is easy to construct counterexamples such as

$$F := \begin{bmatrix} 1 + g & 1 \\ -1 & -1 + g \end{bmatrix},$$

which has singular values $[g^2 + 2 - 2\sqrt{g^2 + 1}]^{1/2} = g^2/2 + O(g^4)$ and $[g^2 + 2 + 2\sqrt{g^2 + 1}]^{1/2} = 2 + g^2/2 + O(g^4)$.

Without performing the somewhat messy calculation of the singular values, one can see directly from Cramer’s rule that since $\det F(g) = g^2$, $F(g)^{-1} \sim g^{-2}$ which would make the contextual values $\vec{a} = F^{-1} \bar{a}$ asymptotic to $g^{-2}$. The essence of the full proof of the GT is to show (continuing to assume $n = 1$ for simplicity) that the contextual values are $O(1/g)$, which implies that the “numerator correction” of DJ’s (5.7) vanishes in the limit $g \to 0$.

Let us try to follow in detail the last paragraph of the proof in the context of the counterexample. Applied to the $F$ of (18), the last paragraph asserts that if $F$ has a singular value of order greater than $g^1$ (which it does), then “the expansion of $F$ to order $g^1$ would have a relevant singular value of zero . . .”. However, this is wrong because the expansion of $F$ to order $g^1$ is $F$ itself, and all singular values are positive for $g \neq 0$.

I suspect that the last paragraph of the attempted proof may be based on an erroneous implicit assumption that truncating a $\Sigma$ corresponding to $F(g)$ will produce the $\Sigma$ for the truncated $F$, i.e., that truncation commutes with taking
of singular values. Otherwise, how could one possibly relate the $\Sigma$ corresponding to $F$ in (5.12) to the different $\Sigma$ corresponding to the linear truncation of $F$? (Even assuming this relation, additional argument seems required to justify the last paragraph of the proof fragment.)

To make the above more explicit, write $\Sigma = \Sigma(F)$ to indicate the dependence of the matrix $\Sigma$ of singular values on $F$, and write $\tau(F)$ for the linear truncation of $F$. I can begin to make sense of the last paragraph only by assuming that

$$\Sigma(\tau(F)) = \tau(\Sigma(F)),$$

which says that the singular values for the truncated $F$ are the truncations of the singular values for $F$. The counterexample shows that this is false for its $F$ which satisfies $\tau(F) = F$:

$$\Sigma(\tau(F)) = \Sigma(F) = \begin{bmatrix} 2 + g^2/2 + O(g^4) & 0 \\ 0 & g^2/2 + O(g^4) \end{bmatrix} \neq \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} = \tau(\Sigma(F)).$$

Recall that (13) was my best guess at the intended expansion of DJ’s hypothesis (iv) from its logical meaning. (A direct request to the authors to confirm or correct this was ignored.) My next best guess would be that the $\alpha_j(g)$ in (13) might represent contextual values for the truncated POVM $\left\{ \hat{E}_j' \right\}$ that would not necessarily be contextual values for the original POVM $\left\{ \hat{E}_j \right\}$. However, the above objection to the proof would still apply.

Whatever the intended meaning of DJ’s (iv), in view of DJ’s objection to the counterexample, it presumably imposes some condition on the linear truncation (still taking $n = 1$ for simplicity) of the original POVM $\left\{ \hat{E}_j(g) \right\}$. This seems an unreasonable hypothesis for a theorem billed as “General”. Certainly the original claim of DAJ that its “general conditioned average” “converges uniquely to the quantum weak value in the minimal disturbance limit” gives no hint that unstated hypotheses necessary to validate the claim would fail to apply to simple cases such as the counterexample with POVM which is quadratic in $g$.

DAJ gives the strong impression that the traditional weak value is essentially inevitable when the measurement operators are positive. DJ gives the same impression under the additional hypothesis that the measurement operators commute with each other and the system observable $\hat{A}$. A main point of both [6] and the present Comment is to dispel any such false impressions.

It should be emphasized that (18) is only a counterexample to DJ’s attempted proof, not a counterexample to the conclusion of the GT under the assumption that the POVM is linear, i.e., of the form (10). For a counterexample to the conclusion, one would need an $F$ which is derived from a POVM.

Actually, I conjecture that the conclusion that the “general conditioned average” is given by DJ’s (1.2) (i.e., the traditional weak value generalized to mixed states) is true for linear POVM’s under the pseudoinverse prescription. If so, its proof will surely have to use in some essential way the special form of an $F$ which comes from a POVM (e.g., all rows sum to 1).

I have sketched such a proof but have not written it in detail, so I make no claims. I will be happy to share the ideas of the proof with any qualified
person who might be interested in expanding on them. They are not difficult, but annoyingly detailed. If I decide not to write them up in journal-ready detail myself, I may put a sketch of a proof on my website, www.math.umb.edu/~sp.

A main aim of this Comment is to focus attention on the case of linear POVM’s. If the conjecture is true, it might help to explain why (to my knowledge) actual experiments have only observed the traditional weak value $\Re(\langle \psi_f | A | \psi_i \rangle / \langle \psi_f | \psi_i \rangle)$, despite the fact that arbitrary weak values can be obtained from different measurement procedures, as stressed by DJ. Since these experiments are difficult and have only recently been performed, perhaps they correspond to the simplest POVM’s, e.g., linear POVM’s arising from positive measurement operators.

5 Appendix 1: Guesses at the meaning of hypothesis (iv)

When I saw the grounds on which DJ disputed the counterexample of [6], I was stunned. Never had I even considered the possibility that hypothesis (iv) might refer to truncations, and had I considered it, I would have rejected it as implausible. I still find it hard to imagine that any careful reader could confidently assert that (iv) referred to truncations, much less be confident of any definite meaning regarding truncations.

After DJ was accepted and I began to prepare this Comment, I have thought a great deal about possible interpretations of (iv). The authors’ intended meaning is still not clear to me. All interpretations which I have considered are either logically unacceptable or inconsistent with some part of DJ.

This appendix analyzes the interpretations which I have considered. I have debated whether it would be worth while to include it, since I imagine that few readers will be interested in investing their time in a detailed logical deconstruction of (iv). I decided to include it for three reasons.

First, it may serve to alert some readers to logical problems with the statement of the GT even if they choose not to study them in detail. Second, I hope that it may motivate the authors of DJ to state (iv) precisely in correct logical language in any reply to the Comment, so that readers can make informed decisions based on a definite knowledge of what DJ intended to assume. Third, since apparently no referees’ report has yet been received over two months after submission, I hope it may assist the referee. I assume that the referee who recommended acceptance of DJ without clarification of (iv) had not thought carefully about its logical meaning.

Recall the hypotheses (iii) and (iv) of DJ’s “General theorem” (GT) both as originally posted in arXiv:1106.1871v1 and subsequently in DJ:

\begin{enumerate}
\item[] 13 See [7] or the list of references [10] of DJ.
\item[] 14 The referee has my sincere sympathy. If he is conscientious enough to try to actually determine the correctness of DJ’s densely written proof based on unclearly stated hypotheses, it will take far more time than is reasonable to ask of an unpaid volunteer.
\end{enumerate}
“(iii) The equality \( \hat{A} = \sum_j \alpha_j(g) \hat{E}_j(g) \) must be satisfied, where the contextual values \( \alpha_j(g) \) are selected according to the pseudo-inverse prescription.

(iv) The minimum nonzero order in \( g \) for all \( \hat{E}_j(g) \) is \( g^n \) such that (iii) is satisfied.”

The statement of hypothesis (iv) is very peculiar, certainly not correct logical language. To analyze it, we need to review a few elementary principles of logic.

What is the meaning of the statement:

“\( x = 2 \)”  

I surely hope that the reader mentally replied that it is meaningless in isolation. It is a so-called “open sentence” to which something must be added to give it meaning, i.e., to convert it into a logical statement which is either true or false.

It can be given meaning by defining \( x \) before stating “\( x = 2 \)”. For example, if \( x \) were previously defined as:

Let \( x \) denote the largest positive integer which satisfies the equation

\[ x^5 - x^3 - 24 = 0 \]

then “\( x = 2 \)” would be a logically meaningful statement which would be definitely true or false (though we might not know which).

To phrase the definition of \( x \) just above as

\[ \forall \text{integers } x, \ x = 2 \]

would not be correct logical language for a definition. Though some readers might be able to guess that it was intended as a definition of the symbol \( x \), it is so far from accepted logical language that any logically trained person would have to question what was meant. It could not be justified as a logical shorthand because the previous correctly stated definition is no more complicated. This is analogous to (iv) with an inessential change of word order: \( g^n \) is the minimum nonzero order in \( g \ldots \).

In (iv), the symbol \( n \) has not been previously defined, so if (iv) is not to be treated as meaningless, the best guess at the authors’ meaning is probably that (iv) is intended as a definition of \( n \). But (iv) is supposed to be a hypothesis (i.e., a logical statement assumed true), not a definition.

Let us put (iv) aside for the moment to examine another logical principle. We noted that in isolation, “\( x = 2 \)” is meaningless as a logical statement (because there is no way, even in principle, to assign it a truth value). One way to make it meaningful is to predefine \( x \). Another is to prepend one of the so-called logical “quantifiers” \( \forall \) (“for all” or “for every”) and \( \exists \) (“there exists”), e.g.,

\[ \forall \text{ integers } x, \ x = 2 \] (a meaningful statement which happens to be false)
or

\[ \exists \text{ an integer } x \text{ such that } x = 2 \text{ (a meaningful though somewhat silly statement which happens to be true).} \]

In the second statement, the phrase “such that” is logically unnecessary (and customarily omitted in formal logic), but is added to make the sentence read well in English. Also, a necessary specification of the so-called “universe of discourse” (in this case that we are talking about integers) has been added to both statements. (If the universe of discourse had been previously specified, this would be unnecessary.)

Having made these points explicit, let us return to (iv). Suppose we temporarily ignore the last clause in (iv) and for purposes of examination write the remainder by itself:

(iv) The minimum nonzero order in \( g \) for all \( \hat{E}_j(g) \) is \( g^n \) such that

\[ \ldots \quad . \]

This is a strange wording which no logician would use, so I hate to analyze further without changing the word order to obtain more nearly correct logical language which (so far as I can guess at the authors’ intention) carries the same logical meaning:

(iv) For all \( \hat{E}_j(g) \), the minimum nonzero order in \( g \) is \( g^n \) such that

\[ \ldots \quad . \]

For this to begin to make sense, we would have to know what is meant by “minimum nonzero order”, which is not a standard mathematical phrase. From the way it is used in the proof of the DJ’s “General theorem” (GT) just after (5.2), I think that the only reasonable guess is that it means that

\[ \hat{E}_j(g) = \hat{E}_j^{(0)} + g^n \hat{E}_j^{(n)} + O(g^{n+1}) \quad . \]  \tag{19}

with the \( \hat{E}_j^{(k)} \) constant operators and \( \hat{E}_j^{(n)} \neq 0 \)\(^{15} \). The truncated statement (iv) just above is still not quite meaningful because \( n \) remains undefined, but no matter what integer \( n \) stands for, the statement does imply that \textit{all} the \( \hat{E}_j(g) \) have the \textit{same} minimum nonzero order \( g^n \). If we take this common minimum nonzero order as the \textit{definition} of \( n \), then the statement becomes meaningful. Though still strangely worded, it could reasonably be interpreted as saying that all the \( \hat{E}_j(g) \) have the \textit{same} minimum nonzero order, which is to be denoted \( n \).

But what of the restrictive clause beginning “such that” which we suppressed? This clause is

“\ldots such that (iii) is satisfied”.

But we have \textit{already} assumed that (iii) is satisfied, so the restrictive clause implies no restriction at all.

\(^{15}\text{A request to the authors to confirm or correct this was ignored.}\)
At this point, an experienced reader will become uneasy, and indeed I did. I asked myself why the authors would add a restrictive clause which was no restriction at all. Could the authors have some other meaning in mind, but have expressed it in a logically incorrect way? In trying to guess other meanings, I came up with only one plausible possibility, but it turned out to be inconsistent with something else in DJ. Next we will examine this possibility. If a referee did not think *very* carefully about possible meanings for (iv), it might superficially seem a reasonable possibility.

One sometimes sees statements in the physics and mathematics literature similar to:

\[ e^x = 1 + x + x^2 / 2 \text{ to order } x^2. \]

Could (iv) carry a similar meaning?

Well, (iii) is already assumed to hold *exactly*, so it holds to all orders in \( g \), so if we interpret (iv) as defining \( n \) as the smallest nonzero order to which (iii) holds, then (iv) would define \( n := 1 \) no matter what the POVM \( \{\hat{E}_j(g)\} \) was. It doesn’t seem as if that would be the authors’ intention. Otherwise, why not simply define \( n := 1 \)?

Now we enter the realm of real guesswork. Could (iv) be intended to mean the following, or something like it?

There exists a positive integer \( n \) such that (iii) holds with each \( \hat{E}_j(g) \) replaced by its truncation to order \( n \), i.e., if

\[
\hat{E}_j(g) = \sum_{k=0}^{\infty} \hat{E}_j^{(k)}(g)^k
\]

and we define \( \hat{E}_j'(g) \) by

\[
\hat{E}_j'(g) := \sum_{k=0}^{n} \hat{E}_j^{(k)}(g)^k
\]

then (iii) holds with the original \( \hat{E}_j(g) \) replaced by their truncations \( \hat{E}_j'(g) \),

\[
\sum_j \alpha_j \hat{E}_j'(g) = \hat{A} \quad , \quad (iii)'
\]

and moreover, \( n \) is defined to be the *least* positive integer for which equation \( (iii)' \) holds.

This is still not logically definite because we have to guess if the \( \alpha_j \) are the same as already defined by the original (iii), or are defined by the pseudoinverse prescription (which is part of (iii)) applied to the new, truncated POVM \( \{\hat{E}_j'(g)\} \), or both.

By now the reader’s head is probably spinning at the multiplicity of conceivable interpretations, but mercifully, we do not have to consider all of them in
detail. That is because for the counterexample, there are no \( \alpha_j(g) \) which satisfy \( (iii)' \) for \( n = 1 \), as DJ shows.

Therefore, the least \( n \) for which \( (iii)' \) could hold is \( n = 2 \), and it does hold for \( n = 2 \) because the counterexample is quadratic in \( g \) and satisfies \( (iii) \). Therefore, according to the interpretation of \( (iv) \) being considered, \( n = 2 \) for the counterexample.

But DJ’s objection to the counterexample requires that \( n = 1 \). The problem, I suspect, is that DJ may be simultaneously using two inconsistent definitions for \( n \), the original definition of \( (19) \) and the different definition introduced just above. DJ’s objection to the counterexample is valid only if \( n = 1 \), but the objection also requires some interpretation of \( (iv) \) in terms of truncations. If \( (iv) \) is interpreted in terms of truncations as above, then \( n = 2 \).

Of course, I cannot rule out the possibility that DJ may be using some wild interpretation of \( (iv) \) which I haven’t even considered. But in terms of the above, DJ’s objection to the counterexample is invalid. Having published “Sufficient conditions . . .”, unless the authors withdraw their objection to the counterexample, they have a professional obligation to furnish an unexceptionable statement of \( (iv) \), one which is clear and logically correct. Only then will readers will have the tools necessary to evaluate the counterexample and DJ’s objection to it.

All this would become moot if the authors recognize that DJ’s attempted proof of the GT is in error or incomplete. But if they come up with a revised proof, they should give first priority to restating \( (iv) \) in a clear and logically correct way.

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