Exact Solutions to Sourceless Charged Massive Scalar Field Equation on Kerr-Newman Background

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The covariant Klein-Gordon equation in the Kerr-Newman black hole geometry is separated into a radial part and an angular part. It is discovered that in the non-extreme case, these two equations belong to a generalized spin-weighted spheroidal wave equation. Then general exact solutions in integral forms and several special solutions with physical interest are given. While in the extreme case, the radial equation can be transformed into a generalized Whittaker-Hill equation. In both cases, five-term recurrence relations between coefficients in power series expansion of general solutions are presented. Finally, the connection between the radial equations in both cases is discussed.

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I. INTRODUCTION

Since the Hawking effect\(^1\) on black hole was found, the evaporation of black hole has been investigated in several coordinates by miscellaneous methods such as path integral approach,\(^2\) tortoise coordinate \((r_*)\) transformation,\(^3,4\) and \(r_*\)-coordinate analytical extension,\(^4\) etc. Among these methods, the generalized tortoise transformation method has been used widely in the discussions not only on evaporation of static black hole and stationary black hole, but also on that of non-stationary ones.\(^3\) Much more progress has been made. But this method can not give an exact solution of the radial \((r_*)\) equation, the radial wave function, which can be analyzed only in asymptotic expression.

Couch\(^5\) obtained a series of exact solutions by transforming the separated radial equation into modified Whittaker-Hill equation under some special conditions. But these solutions seem to have nothing to do with the discussion on black hole evaporation.

Solutions to generalized spheroidal wave equation have been studied to some extent\(^6,7\) by using power series expansions around regular singular points, so that three-term recurrence relations between coefficients can be manipulated in terms of the continued fraction method. Leaver\(^6\) has shown that Teukolsky master equations in Kerr geometry are, in fact, spin-weighted generalized spheroidal wave equations.

It appears to be more important to obtain an exact solution to the radial equation for this is crucial in discussing Hawking effect of black hole. However, it is very difficult
to do so. It is this motivation that stimulates our present research. The main aim of this paper is to show that the separated radial part of a massive covariant Klein-Gordon equation on the Kerr-Newman black hole (KNBH) background is a generalized spin-weighted spheroidal wave equation of imaginary number order. In this paper, we shall discuss the solutions to a massive complex scalar field in the KNBH geometry with three parameters. In the non-extreme case, its general solutions of the separated parts are spin-weighted generalized spheroidal wave functions$^{6-8}$ and some special solutions to the radial equation with physical interest are given. General solutions to the radial equation in the extreme case shall be briefly discussed. Finally, we show that the radial equation in the extreme case is a confluent equation of that in the non-extreme case.

Sec.II deals with variable separation of a sourceless complex scalar field on KNBH and solutions to the angular part. In Sec.III and IV, the radial equation is solved in both non-extreme and extreme cases respectively. In III(a) we reduce the radial equation to standard form, and in III(b) and (c) we obtain general solutions and special ones including case $(\omega = \mu = 0)$ respectively. Five-term recurrences between coefficients of solutions in power series forms are given in both cases. In addition, we give solutions in integral forms and some special solutions of physical interest in the non-extreme case. Conditions for general solutions exist are given in these two cases. Sec.V is devoted to discussing the connection between the radial equation in the extreme case and that in the non-extreme case. Finally, we point out some probable applications and generalization of exact solutions in Sec.VI.
In appendix part, three-term recurrence relation between coefficients in power series expansions around regular singular points for generalized spheroidal wave equation are presented.

II. SEPARATION OF KLEIN-GORDON EQUATION AND SOLUTION TO THE ANGULAR EQUATION

The Kerr-Newman line element and electromagnetic one-form are given in the Boyer-Lindquist coordinates as follows

\[ ds^2 = g_{\mu\nu}dx^\mu dx^\nu \]
\[ = -\frac{\Delta}{\Sigma}(dt - a \sin^2 \theta d\varphi)^2 + \frac{\sin^2 \theta}{\Sigma}(adt - (r^2 + a^2)d\varphi)^2 + \Sigma \left(\frac{d\theta^2}{\Delta} + d\theta^2\right), \]
\[ A = A_\mu dx^\mu = -\frac{e r}{\Sigma}(dt - a \sin^2 \theta d\varphi) \]

with event horizon function \( \Delta = r^2 - 2Mr + a^2 + e^2 \), and \( \Sigma = r^2 + a^2 \cos^2 \theta \), where mass \( M \), charge \( e \), specific angular momentum \( a = J/M \) being three parameters to describe KNBH. (Use Planck units system \( G = \hbar = c = 1 \), and denote \( \partial_\mu = \frac{\partial}{\partial x^\mu} \)).

The determinant of KNBH metric tensor is \( g = \det(g_{\mu\nu}) = -\Sigma^2 \sin^2 \theta \), while the electromagnetic four-vector potential \( A_\mu \) apparently satisfies the following covariant Lorentz gauge condition:

\[ \nabla_\mu A^\mu = \frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g}g^{\mu\nu}A_\nu) = 0. \]

In curved spacetime, a sourceless scalar field \( \Phi \) with mass \( \mu \) and charge \( q \) obeys the covariant Klein-Gordon equation (KGE):

\[ (\Box_c - \mu^2)\Phi = 0, \]
where d’ Alembert operator $\Box_c$ on KNBH background is given by

\[
\Box_c \equiv \frac{1}{\sqrt{-g}} D_\mu (\sqrt{-g} g^{\mu\nu} D_\nu)
\]

\[
= \frac{1}{\Sigma} \{-\frac{1}{\Delta} [r^2 + a^2] \partial_t + a \partial_\varphi + i q e r]^2 + \partial_r (\Delta \partial_r)
\]

\[
+ (a \sin \theta \partial_t + \frac{1}{\sin \theta} \partial_\varphi)^2 + \frac{1}{\sin \theta} \partial_\theta (\sin \theta \partial_\theta) \}, \quad (5)
\]

here covariant gauge differential operator being $D_\mu = \partial_\mu - iq A_\mu$.

The scalar wave function $\Phi$ for KGE of Eq.(4) has a solution of variables separable form $\Phi(t, r, \theta, \varphi) = R(r) S(\theta) e^{i(m\varphi - \omega t)}$: 

\[
\frac{1}{\Delta} [\omega(r^2 + a^2) - q e r - ma]^2 \Phi + \partial_r (\Delta \partial_r \Phi) - \mu^2 \Sigma \Phi
\]

\[
-(a \omega \sin \theta - \frac{m}{\sin \theta})^2 \Phi + \frac{1}{\sin \theta} \partial_\theta (\sin \theta \partial_\theta \Phi) = 0. \quad (6)
\]

The separated results of the above equation are

\[
\partial_r [\Delta \partial_r R(r)] + \left\{ \frac{[\omega(r^2 + a^2) - q e r - ma]^2}{\Delta} - \mu^2 (r^2 + a^2) - \lambda + 2ma \omega \right\} R(r) = 0 \quad (7)
\]

and

\[
\frac{1}{\sin \theta} \partial_\theta [\sin \theta \partial_\theta S(\theta)] + [\lambda - \frac{m^2}{\sin^2 \theta} + (\mu^2 - \omega^2) a^2 \sin^2 \theta] S(\theta) = 0 \quad (8)
\]

where $\lambda$ is a separation constant.

The general solutions to the angular part are ordinary spheroidal angular wave functions\(^8,10\) with spin-weight $s = 0$. When $a^2(\omega^2 - \mu^2) = 0$, these solutions degenerate to Legendre spherical functions.

Let $x = \cos \theta$, $S(\theta) = S(x) = (1 - x^2)^{m/2} \Theta(x)$, Eq.(8) should take the following forms:

\[
(1 - x^2)S''(x) - 2x S'(x) + [\lambda - \frac{m^2}{1 - x^2} + a^2(\omega^2 - \mu^2)(x^2 - 1)] S(x) = 0 \quad (9)
\]
\[(1-x^2)\Theta''(x) - 2(1+m)x\Theta'(x) + [\lambda - m(m+1) + a^2(\omega^2 - \mu^2)(x^2 - 1)]\Theta(x) = 0. \quad (10)\]

Here and after, \(S'(x) = \partial S(x)/\partial x\), etc.

The eigenfunctions to Eqs. (9) and (10) are generalized spheroidal wave functions\(^6-^8\)
\[S(x) = S_{\ell m}^{m,0}(c, x)\] with eigenvalue \(\lambda = \lambda_{ml} + c^2, c^2 = a^2(\mu^2 - \omega^2).\) When \(\mu = 0\), Eqs. (9) and (10) are special cases \((s = 0)\) of the following spin-weighted spheroidal wave equations\(^6-^8,^10\)
\[(1-x^2)P''(x) - 2xP'(x) + \left[a^2\omega^2x^2 - 2a\omega sx - \frac{(m + sx)^2}{1-x^2}\right] + s + \lambda']P(x) = 0 \quad (11)\]

and
\[(1-x^2)Q''(x) - 2[s+(1+m)x]Q'(x) + [\lambda' - (m-s)(m+s+1) - 2a\omega sx + a^2\omega^2x^2]Q(x) = 0 \quad (12)\]

where \(P(x) = (1-x)^{|m+s|/2}(1+x)^{|m-s|/2}Q(x)\) and \(x = \cos \theta.\)

When \(a\omega = 0\), the solutions to the above equations are Jacobi ultra-sphere D-functions\(^10\)\(D_{\ell m,s}^{\ell}(c, x)\) or spin-weighted spherical harmonic functions\(^11\) with eigenvalue \(\lambda' = \ell(\ell + 1) - s(s + 1), \ell = \max(|m|, |s|).\) In general case, the solutions should be the generalized spin-weighted spheroidal wave functions\(^6-^8\)\(P(x) = P_{\ell m,s}^{\ell}(c, x), c^2 = -a^2\omega^2.\)

By taking account of some reasonable boundary conditions, these solutions could be a set of orthogonal polynomials.

In the following, we shall assume that all parameters, \(M, e, a, \mu, q, m,\) are nonzero, and discuss the radial equation of Eq.(7) according to two cases, namely the non-
extreme case \( M^2 \neq a^2 + e^2 \) and the extreme case \( M^2 = a^2 + e^2 \). Special case \((\omega = \mu = 0)\) will be included in subsection III(c).

III. SOLUTIONS TO THE RADIAL EQUATION IN THE NON-EXTREME CASE \( M^2 \neq a^2 + e^2 \)

(a) Simplification of the radial equation in the case \((\varepsilon \neq 0)\)

In this case, we put \( \varepsilon = \sqrt{M^2 - a^2 - e^2} \), \((0 < \varepsilon < M)\). After making substitutions of \( r = M + \varepsilon z \) and \( R(r) = R(z) = (z - 1)^{i|B+A|/2}(z + 1)^{i|B-A|/2} F(z) \), the exterior horizon and interior horizon are located at points \( r_{\pm} = M \pm \varepsilon \), \((z = \pm 1)\), respectively, the radial equation of Eq.(7) can be reduced to the following standard forms:

\[
(z^2 - 1) R'' + 2z R' + \left[ \varepsilon^2 (\omega^2 - \mu^2)(z^2 - 1) + 2 \varepsilon (A \omega - M \mu^2) z + \frac{(Az + B)^2}{z^2 - 1} \right] R = 0 \quad (1 < z < \infty) \tag{13}
\]

and

\[
(z^2 - 1) F'' + 2[iA + (1 + iB) z] F' + \left[ \varepsilon^2 (\omega^2 - \mu^2)(z^2 - 1) + 2 \varepsilon (A \omega - M \mu^2) z \right] F = 0 \quad (1 < z < \infty) \tag{14}
\]

where \( A = 2M \omega - q e, \varepsilon B = \omega(2M^2 - e^2) - qeM - ma \).

In order to study behaviors of solutions to Eqs.(13) and (14) in the interval \((-1 < z < 1)\), we rotate firstly \( T \) from real axis to imaginary axis \( T = i \tau \) after making substitution of \( z = \cosh T = \cosh(i \tau) = \cos \tau \), then return to real \( z \)-axis \( z = \cos \tau \). Therefore, Eqs.(13) and (14) have corresponding forms in the interval \(|z| < 1\) as
follows

\[(1 - z^2)R'' - 2zR' + [\varepsilon^2(\omega^2 - \mu^2)(z^2 - 1) + 2\varepsilon(A\omega - M\mu^2)z - \frac{(Az + B)^2}{1 - z^2} + (2\omega^2 - \mu^2)(2M^2 - e^2) - 2qeM\omega - \lambda]R = 0 \quad (-1 < z < 1) \quad (15)\]

and

\[(1 - z^2)G'' - 2[A + (1 + B)z]G' + [\varepsilon^2(\omega^2 - \mu^2)(z^2 - 1) + 2\varepsilon(A\omega - M\mu^2)z + (2\omega^2 - \mu^2)(2M^2 - e^2) - 2qeM\omega - \lambda + A^2 - B^2 - B]G = 0 \quad (|z| < 1) \quad (16)\]

where we have made a function transformation

\[R(z) = (1 - z)^{\frac{|B + Ai|}{2}} (1 + z)^{\frac{|B - Ai|}{2}} G(z).\]

The essence of our manipulation is that we extend the domain of \( z \) from the real axis to complex \( z \)-plane \( z = x + iy \) at first and then make an analytical extension on complex \( z \)-plane from region outside unit circle (\( |z| > 1 \)) to that inside it (\( |z| < 1 \)). The handled issue is that only derivative terms change a negative sign, while the non-derivative terms, namely terms in square brackets, make no change in symbol. This method is equivalent to that Eqs.(13-16) are solved initially and then the solutions are made analytical extension on the complex \( z \)-plane.

Both Eqs.(13) and (14) are generalized spin-weighted spheroidal wave equation\(^6\) with imaginary number order, while both Eqs.(15) and (16) with real number order. The formers are suitable especially to study problems about scattering state, whereas the latters more convenient to investigate energy levels of bound states. Furthermore,
we can apparently find out connection between poles of scattering amplitudes and energy levels of bound states. Actually, domain in which $z$ takes values in Eqs.(13) and (14) is on the $x$-axis, while those in Eqs.(15) and (16) on $y$-axis. So these equations can be thought of equivalence. However, to be convenient, we have made a restriction on intervals that $z$ takes values of $1 < z < \infty$ in Eqs.(13) and (14), while of $|z| < 1$ in Eqs.(15) and (16).

When $\mu = 0$ or $M = 0$, if taking $R_1(z_1)$ as the first solution to Eq.(15) in the interval of $|z| < 1$, then $R_2(z_2)$, the second one to the same equation in that of $|z| > 1$, might be:

$$R_2(z_2) = (z_2 - 1)\frac{|B + A|}{2} (z_2 + 1)\frac{|B - A|}{2} \int_{-1}^{+1} e^{i\omega z_2 z_1} (1 - z_1)\frac{|B + A|}{2} (1 + z_1)\frac{|B - A|}{2} R(z_1) dz_1. \quad (17)$$

Here, we have assumed that $\omega > 0$. Integral equation of Eq.(17) connects irregular solution $R_2(z_2)$ with regular solution $R_1(z_1)$.

Comparing Eqs.(9-12) with Eqs.(13-16), especially Eqs.(11, 12) with Eqs.(15, 16), we can draw a conclusion that the separated angular and radial equations are ordinary differential equations of the same type, generalized spin-weighted spheroidal wave equations. Furthermore, we discover that $\varepsilon, A, B$ correspond to $-a, -s, -m$ respectively when $\mu = 0$. There may exist three pairs of power series solutions to generalized spheroidal wave equations around singular points $z = \pm 1, \infty$ respectively. Added with some proper boundary conditions, these power series expansions of spheroidal wave functions can be cut off to be polynomials.

Therefore, in the following subsection, we shall only study the generalized spheroidal
wave equation. The reader who has more interest in this equation, can find more information in Refs. 6 and 7 (and references cited therein).

(b) General solutions to the radial equation ($\varepsilon \neq 0$)

The standard generalized spin-weighted spheroidal wave equation that we reduce to study is as follows

$$(1 - z^2)W''(z) - 2[\alpha + (\beta + 1)z]W'(z) + [\gamma^2(z^2 - 1) + 2\delta z + \bar{\lambda} - \beta]W(z) = 0, \quad (18)$$

where $\bar{\lambda}$ is a redefined eigenvalue which could make $W(z)$ finite at $z = \pm 1$, and region of $z'$ taking values could be the whole complex $z$-plane.

(i) For the radial equation of Eq.(16), we have

$$\alpha = A, \beta = B, \gamma^2 = \varepsilon^2(\omega^2 - \mu^2), \delta = \varepsilon(A\omega - M\mu^2);$$

(ii) For the angular equation of Eq.(12), we have

$$\alpha = s, \beta = m, \gamma^2 = a^2\omega^2, \delta = -as\omega;$$

(iii) For the angular equation of Eq.(10), we have

$$\alpha = 0, \beta = m, \gamma^2 = a^2(\omega^2 - \mu^2), \delta = 0.$$

The form of Eq.(18) is invariant both under Laplace-transformation and by changing parameters $\alpha, \beta, \gamma^2, \delta, z$ into $-\alpha, \beta, \gamma^2, -\delta, -z$ respectively. Namely, $W(z) = W(\alpha, \beta, \gamma, \delta; z)$ satisfies the following integral equation

$$\int_0^{+\infty} e^{-tz}W(\alpha, \beta, \gamma, \delta; z)dz = W(\frac{\delta}{\gamma}, -\beta, \gamma, -\alpha\gamma; \frac{t}{\gamma})$$

$$= W(-\frac{\delta}{\gamma}, -\beta, \gamma, \alpha\gamma; \frac{t}{\gamma}), \quad (19)$$
\begin{align*}
W(\alpha, \beta, \gamma, \delta; z) &= W(-\alpha, \beta, \gamma, -\delta; -z) \\
&= \int_{0}^{+\infty} e^{-\gamma z t} W(\frac{-\delta}{\gamma}, -\beta, \gamma, \alpha \gamma; t) dt. \tag{20}
\end{align*}

The above formulae are integral solutions to Eq.(18). If one knows a solution, then he can obtain another by integral transformations of Eqs.(19) and (20). It is obvious that solutions are symmetry under the following condition:

\[ \alpha = \delta = 0, \quad (\gamma \neq 0). \]

This just is the case (iii). At this moment, the symmetric solutions are ordinary spheroidal angular wave functions.\(^{10}\)

Now, we consider a solution to generalized spheroidal equation of Eq.(18) which is in power series form in the interval of \(-1 < z < 1\). According to the knowledge of ordinary differential equation, one can know that Eq.(18) has two regular singularities \((z = \pm1)\) and one confluently irregular singular point \((z = \infty)\). As \(z = 0\) is its ordinary point, we can make a Taylor expansion of \(W(z)\) in the vicinity of ordinary point \((z = 0)\):

\[ W(z) = W_n(z) = \sum_{n=0}^{\infty} a_n z^n, \quad (|z| < 1). \tag{21} \]

Substituting power series of Eq.(21) into Eq.(18), we obtain five-term recurrence relations between coefficients as follows

\begin{align*}
(\lambda - \gamma^2 - \beta)a_0 - 2\alpha a_1 + 2a_2 &= 0, \\
2\delta a_0 + [\lambda - \gamma^2 - \beta - (2 + 2\beta)]a_1 - 4\alpha a_2 + a_3 &= 0, \\
\gamma^2 a_0 + 2\delta a_1 + [\lambda - \gamma^2 - \beta - 2(3 + 2\beta)]a_2 - 6\alpha a_3 + 12a_4 &= 0,
\end{align*}
\[
\gamma^2 a_{n-2} + 2\delta a_{n-1} + [\bar{\lambda} - \gamma^2 - \beta - n(n + 1 + 2\beta)]a_n
\]
\[-2(n + 1)\alpha a_{n+1} + (n + 2)(n + 1)a_{n+2} = 0.\]

Redefine coefficients:

\[
A_n = \frac{\gamma^2}{\bar{\lambda} - \gamma^2 - \beta - n(n + 1 + 2\beta)},
\]
\[
B_n = \frac{2\delta}{\bar{\lambda} - \gamma^2 - \beta - n(n + 1 + 2\beta)},
\]
\[
C_n = \frac{-2(n + 1)\alpha}{\bar{\lambda} - \gamma^2 - \beta - n(n + 1 + 2\beta)},
\]
\[
D_n = \frac{(n + 2)(n + 1)}{\bar{\lambda} - \gamma^2 - \beta - n(n + 1 + 2\beta)}.
\]

When taking limits \(n \to \infty\), we have \(A_n, B_n, C_n \to 0\), and \(D_n \to -1\).

Then, five-term recurrence relations become

\[
A_n a_{n-2} + B_n a_{n-1} + a_n + C_n a_{n+1} + D_n a_{n+2} = 0. \tag{22}
\]

After arranging coefficients \(A_n, B_n, C_n, D_n\) and making up them into a quasi-diagonal band matrix \(\Lambda\) and \(a_0, a_1, \ldots, a_n, \ldots\) into a column vector \(\vec{a} = (a_0, a_1, \ldots, a_n, \ldots)\), the above recurrence relations become an infinite tridiagonal matrix equation:

\[
\Lambda \vec{a} = h \vec{a}. \tag{23}
\]

The condition for solutions of Eq.(23) exist is that determinant of \(\Lambda\) is zero,

\[
\det(\Lambda) = 0. \tag{24}
\]
In fact, this condition could be satisfied, and we have \( \det(\Lambda) \to 0 \) when \( n \to \infty \).

Matrix equation of Eq.(23), together with determinant equation of Eq.(24) determines coefficients \( a_0, a_1, \ldots, a_n, \ldots \), and eigenvalue \( \bar{\lambda} \), hence, eigenvalue \( \bar{\lambda} \) will be a complicated function of \( \alpha, \beta, \gamma, \delta \), as well as \( n \). The second power series solution around the same point \( z = 0 \) can be obtained by Frobenius’s method. To be finite at \( z = \pm 1 \), power series \( W(z) \) could be truncated to be polynomial, and \( \alpha, \beta, \gamma, \delta \) could be integers or half-integers. While in general case, solutions to spin-weighted generalized spheroidal equation of Eq.(18) are transcendental functions.\(^6\),\(^7\)

Absolutely, solution \( W_n(z) = W_n(\alpha, \beta, \gamma, \delta; z) \) of Eq.(18) can be orthonormalized to constitute a set of complete functions.

\[
\int_{-1}^{1} (1 - z)^{\beta + \alpha}(1 + z)^{\beta - \alpha} W_n(z) W_{n'}(z) dz = \delta_{n,n'}.
\]

Solutions \( W_n(z) \) at infinity can have asymptotic forms \( W_n(z) \to e^{\pm \gamma z}, (z \to \pm \infty, \gamma > 0) \). This is consistent with that the Minkowski spacetime is an asymptotic spacetime of the Kerr-Newmann black hole. Thus, in-going wave and out-going wave at infinity can take form of plane waves.

(c) Special solutions to the radial equation in the case \( (\varepsilon \neq 0) \)

In this subsection, we will base our discussion upon Eq.(15), namely

\[
(1 - z^2)R''(z) - 2zR'(z) + [\gamma^2(z^2 - 1) + 2\delta z - \frac{(\beta + \alpha z)^2}{z^2 - 1}]
\]

\[
+(2\omega^2 - \mu^2)(2M^2 - e^2) - 2qeM\omega - \lambda]R(z) = 0, \quad (|z| < 1)
\]

where

\[
\gamma^2 = \varepsilon^2(\omega^2 - \mu^2), \quad \delta = \varepsilon(A\omega - M\mu^2),
\]
\[ \alpha = A = 2M\omega - qe, \beta = B = \frac{\omega(2M^2 - e^2) - qeM - ma}{\varepsilon}. \]

Case-1: when \( \gamma = \delta = 0 \), there exist three situations:

i) \( \omega = \pm \mu = qe/M \neq 0, (\alpha \neq 0) \);

ii) \( \omega = \mu = qe/M = 0, (\alpha = 0) \) (This case can be thought as special one in the above-head case.);

iii) \( \omega = \mu = 0, qeM \neq 0, (\alpha \neq 0) \).

Solutions in situations i) and iii) are Jacobi ultra-sphere functions \( R(z) = P_n^{(\beta + \alpha, \beta - \alpha)}(z) \), whereas solutions in situation ii) degenerate to be Legendre functions, \( R(z) = P_n^\beta(z) \), or \( Q_n^\beta(z) \).

Case-2: when \( M\mu^2 = 0, \omega \neq 0, \delta/\varepsilon \omega = \alpha \), this case has been considered in detail by E. W. Leaver.

Case-3: when \( \alpha = \delta = 0, \gamma \neq 0 \), Eq.(26) is an ordinary spheroidal wave equation, and its solutions are Prolate spheroidal angular wave functions \( R(z) = S_n^{\beta,0}(\gamma, z) \).

Obviously, all these solutions are special cases of general solutions \( R_n^{(\beta + \alpha, \beta - \alpha)}(\gamma, \delta; z) \) = \( (1 - z)^{(\beta + \alpha)/2}(1 + z)^{(\beta - \alpha)/2}W_n(\alpha, \beta, \gamma, \delta; z) \).

Solutions in case-1 will be particular important in physics to black hole evaporation, as scattering cross section, stationary state energy levels, emission coefficients of black hole radiation, etc., could be analytically computed at exact theoretical level by use of Jacobi polynomials. Furthermore, there maybe exist special symmetry in such case.

IV. SOLUTIONS TO THE RADIAL EQUATION IN THE EXTREME CASE \( (M^2 = a^2 + e^2) \)
In the extreme KNBH case ($\varepsilon = 0$), we make substitution $r = M(1 + x)$, then event horizon is located at a single point ($r_h = M$), namely $x = 0$, hence the radial equation of Eq.(7) can be transformed into the following confluent equation:

$$x^2 R''(x) + 2x R'(x) + \left[(\omega^2 - \mu^2)M^2x^2 + 2(A\omega - M\mu^2)Mx\right.$$  
$$+ (A + \frac{B}{x})^2 + (2\omega^2 - \mu^2)(2M^2 - e^2) - 2qeM\omega - \lambda]R(x) = 0 \quad (27)$$

where $A = 2M\omega - qe$, $MB = B\varepsilon = \omega(2M^2 - e^2) - qeM - ma$.

Defining

$$C^2 = M^2(\omega^2 - \mu^2), \quad D = M(A\omega - M\mu^2),$$

$$\lambda_e = (2\omega^2 - \mu^2)(2M^2 - e^2) - 2qeM\omega - \lambda - \frac{1}{4} + A^2.$$  
and making substitutions:

$$x = e^{i\nu \xi}, \quad R(x) = R(\xi) = e^{-i\nu \xi/2}H(\xi)$$

then, Eq.(27) is transformed into the generalized Whittaker-Hill equation (GWHE)

$$- \nu^{-2}H''(\xi) + [C^2e^{2i\nu \xi} + 2De^{i\nu \xi} + 2ABe^{-i\nu \xi} + B^2e^{-2i\nu \xi} + \lambda_e]H(\xi) = 0. \quad (28)$$

Solutions of GWHE of Eq.(28) can be regarded formally as

$$H(\xi) = \sum_{n=-\infty}^{+\infty} g_n e^{in\nu \xi}, \quad n = 0, \pm 1, \pm 2, \ldots. \quad (29)$$

Substituting Eq.(29) into Eq.(28), we obtain five-term recurrence relations between coefficients

$$C^2g_{n-2} + 2Dg_{n-1} + (\lambda_e + n^2)g_n + 2ABg_{n+1} + B^2g_{n+2} = 0,$$
\[ E \vec{g} = h \vec{g} = \sum_{n=-\infty}^{+\infty} \sum_{m=-n-2}^{n+2} E_{m,n} g_n, \]

where we have recast recurrence relations in matrix form in Eq.(30), and defined matrix elements

\[ E_{n,n-2} = \frac{C^2}{\lambda_e + n^2}, \]
\[ E_{n,n-1} = \frac{2D}{\lambda_e + n^2}, \]
\[ E_{n,n} = 1, \]
\[ E_{n,n+1} = \frac{2AB}{\lambda_e + n^2}, \]
\[ E_{n,n+2} = \frac{B^2}{\lambda_e + n^2}. \]

Condition that solutions of simultaneous equations in Eq.(30) exist is that determinant \( \det(E) \) must be zero, that is

\[ \det(E) = 0, \]
\[ \det[E - h I] = 0. \]

Secular equation of Eq.(32) is a characteristic equation that determines the existence of periodic solutions of Eq.(29). Solutions \( R(\xi) \) could be functions with period \( 4\pi/\xi \). There exist four series of periodic functions according to period being odd or even. Eq.(27) has two confluently irregular singular points \( x = 0, \infty \). Behaviors of its solutions at event horizon \( r_h = 0(x = 0) \) depend upon that of \( R(\xi) \) at \( \xi \to \pm i\infty \) (according to \( \nu \) being negative number or positive number).

V. CONNECTION BETWEEN THE RADIAL EQUATION IN NON-EXTREME CASE AND THAT IN EXTREME CASE
In this section, we illustrate that the radial equation of Eq.(27) in the extreme case
is a confluent form of Eq.(13) in the non-extreme case, and give expression to the first
thermodynamic law in the extreme KNBH case.

After making substitutions of \( \varepsilon = M \epsilon \), \( \varepsilon z = x \), \( \varepsilon z = Mx \), \( B = \epsilon B \), \( 0 < \epsilon < 1 \) in
Eq.(13), we have

\[
 r = M + \varepsilon z = M(1 + x), \quad \Delta = \epsilon^2(z^2 - 1) = M^2(x^2 - \epsilon^2),
\]

\[
 A = 2M\omega - qe, \quad \varepsilon B = MB = \omega(2M^2 - \epsilon^2) - qeM - ma.
\]

then, Eq.(13) is equivalent to the following one in the non-extreme case:

\[
 \frac{\partial}{\partial x}[(x^2 - \epsilon^2)\frac{\partial R(x)}{\partial x}] + [M^2(\omega^2 - \mu^2)(x^2 - \epsilon^2) + 2M(\lambda \omega - M\mu^2)x
 + \frac{(Ax + B)^2}{x^2 - \epsilon^2} + (2\omega^2 - \mu^2)(2M^2 - \epsilon^2) - 2qeM\omega - \lambda]R(x) = 0. \tag{33}
\]

Eq.(33) has two regular singular points \( x = \pm \epsilon, \, (z = \pm 1) \) which are located at
exterior horizon and interior horizon (Cauchy surface) \( r_\pm = M \pm \epsilon = M(1 \pm \epsilon) \)
respectively, along with another irregular singular point \( x = \infty \). After taking limits
\( \epsilon \to 0, \, x^2 - \epsilon^2 \to x^2 \), Eq.(33) in the non-extreme case tends to Eq.(27) in the extreme
case. The latter has two conflually irregular singular points \( x = 0, \infty \). The irregular
singular point \( x = 0 \) which is located at event horizon \( r_h = M \) in the extreme case is
just one to which two irregular singular points \( x = \epsilon \) and \( x = -\epsilon \) in the non-extreme
case concur when \( \epsilon \) or \( \varepsilon \to 0 \).

In the extreme KNBH case \( (M^2 = a^2 + e^2) \), surface gravity \( \kappa_h = 0 \), event horizon
\( r_h = M \), reduced event horizon area \( A_h = M^2 + a^2 = 2M^2 - e^2 \), the first thermodynamic
law of extreme Kerr-Newman black hole is expressed as follows

\[ dM = \Omega_h dJ + \Phi_h de \]  \hspace{1cm} (34)

where \( \Phi_h = (er_h)/A_h \), and \( \Omega_h = a/A_h \) are electric potential, angular velocity at event horizon \( (r_h = M) \) respectively.

VI. CONCLUSION

In this paper, a sourceless charged massive scalar Klein-Gordon field equation has been separated into the angular and radial parts. The separated equations are all generalized spin-weighted spheroidal wave equations. In the non-extreme case, we present general solutions in power series expansion and that of integral forms, as well as several special solutions with physical interest for the radial equation. These solutions can be orthonormalized to a set of complete functions. In addition, they have asymptotic behaviors of plane waves at infinity. On the base of these orthogonal functions or polynomials, we can expand wave function of a complex scalar field to a quantized Klein-Gordon field on the Kerr-Newman background. In the extreme case, the radial equation can be reduced to modified Whittaker-Hill equation. In both cases, we obtain five-term recurrence relations between coefficients in power series expansions.

At base of this work, the quantum conservation laws about Hawking process and probable generalization to black hole thermodynamic laws can be discussed further. It is anticipated that the separated parts of Dirac equation in the Kerr-Newman geometry could be reduced to the forms of generalized spheroidal wave equation.

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APPENDIX

In this appendix, we present three-term recurrence relations between coefficients in power series expansions around regular singular points \((z = \pm 1)\) for spin-weighted spheroidal wave equation of Eq.(18), namely

\[
(1 - z^2)W''_n(z) - 2[a + (b + 1)z]W'_n(z) + [c^2(z^2 - 1) + 2dz + \lambda_n - b]W_n = 0, \\
(-1 < z < 1) \tag{A1}
\]

Eq.(A1) has two regular singular points \(z = 1\) and \(z = -1\), with indices \(\rho_+ = 0, -a - b\) and \(\rho_- = 0, a - b\) respectively. When \(c, d \to 0\), \(W_n(z)\) must tend to Jacobi polynomials.

Introducing a symbol \(\epsilon = \mp 1\), we denote these two regular singular points \(z = \pm 1 = -\epsilon\). Then, we make power series expansions around regular singular points \(z = -\epsilon\) respectively, where we have written them in an united manner:

\[
W_n(z) = e^{-\epsilon z} \sum_{n=0}^{\infty} f_n(1 + \epsilon z)^n. \tag{A2}
\]

Substituting the above regular solutions of Eq.(A2) into Eq.(A1), we obtain three-term recurrence relations between coefficients as follows

\[
(1 + b - \epsilon a)f_1 + [\lambda_0 + 2ac - b - 2\epsilon(bc + c + d)]f_0 = 0,
\]

\(\ldots\)
(n + 1)(n + 1 + b - εa)f_{n+1} + [λ_n + 2ac - b - 2ε(bc + c + d)]f_n + 4ε(nc + bc + d)f_{n-1} = 0.

After defining coefficients,

\begin{align*}
A_n &= λ_n + 2ac - b - 2ε(bc + c + d) - n(n + 1 + 2b + 4εc), \\
B_n &= (n + 1)(n + 1 + b - εa), \\
C_n &= 4ε[(n + b)c + d],
\end{align*}

recurrence relations for first term and n-th term can be written as

\begin{align*}
B_0f_1 + A_0f_0 &= 0, \\
B_nf_{n+1} + A_nf_n + C_nf_{n-1} &= 0. \quad \text{(A3)}
\end{align*}

Three-term recurrence relations of Eq.(A3) can handled by continued fraction method,\textsuperscript{6,7} or by matrix method (see J. W. Liu's paper in Ref.7) as we can array \( A_n, B_n, C_n \) to make up a generalized Jacobi tridiagonal band matrix. Similar three-term recurrence relations can also be obtained by expansions in the light of Jacobi polynomials, but the coefficients \( A_n, B_n, C_n \) will be more complicated than those presented here.

The second regular solutions around the same points can be easily obtained by Frobenius’s method, and we have not presented them here. Irregular solutions are connected with these regular ones by integrals similar to those in Eqs.(19) and (20).

In order to make \( W_n(z) \) finite at \( z = \pm 1 \), \( W_n(z) \) must be truncated to be polynomials, then \( W_n(z) \) is orthonormalized with eigenvalue \( λ_n \) and weight \( (1 - z)^{b+a}(1 + z)^{b-a} \).
Hence we have

\[
\int_{-1}^{+1} (1 - z)^{b+a}(1 + z)^{b-a}W_m(z)W_n(z)dz = \delta_{m,n}.
\] (A4)

Battle-Lemarié wavelet or Daubechies’ compact support wavelets\textsuperscript{12} can be used in numerical computation for matrix equation of Eq.(A3) and to prove convergence of polynomials \(W_n(z)\), but we don’t pursue this goal here.

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