Calculation of an flexible gently sloping filament using the generalized finite difference method

Lydia Zakharova and Maxim Aleksandrovskiy

Moscow State University of Civil Engineering, Yaroslavskoe shosse, 26, Moscow, 129337, Russia

E-mail: 1AleksandrovskiyMV@mgsu.ru

Abstract. At the present time, in the construction industry hanging structures, such as, suspension bridges, ropeways, coverings of industrial and civil facilities are widely used. The thread can serve as a settlement scheme of some structural elements of such systems. The aim of the article is to create an algorithm for calculating an elastic flat filament that is convenient for practical application under the action of an arbitrary load. The article presents an algorithm for calculating an elastic flat thread on the effect of a vertical load, arbitrarily varying along its length, as well as vertical and horizontal concentrated forces. To solve this problem, we use the generalized method of final differences allowing to consider ruptures of function of displacement and her derivatives without special increase the number of points and without use the points lying behind the contour. The calculation is divided into two stages. First, we consider the equilibrium of the filament under the action of its own weight. The exact solution of this problem is known. Span of a thread is divided into a number of elements within which the distributed vertical load is continuous, and concentrated forces are applied at the boundaries. The article gives an example of calculating a flexible flat thread based on the presented algorithm. Comparison of the calculation results for some types of loading with similar solutions obtained analytically showed the ease of use and effectiveness of this method with a small number of elements on which the system is divided.

1. Introduction

At present, the construction of objects involving suspension structural systems has become widespread in civil engineering. Examples are flexible parts of cableways, power lines, industrial and civil structures and other facilities, the structural part of which includes suspension cable systems, the elements of which work mainly on tension. Thus, issues related to the improvement of calculations of the elements, in which the filament can serve as a design diagram, become topical.

Depending on the operational and design requirements imposed on the projected suspension structural systems, their design diagrams can be represented both in the form of hard filaments and filaments having elastic properties. Analytical calculation of flexible flat filaments is a very difficult task due to the geometric nonlinearity of the system, but the current level of development of computer technology makes it possible to apply numerical methods for solving it.

Despite the variety of existing methods for calculating flexible filaments, they are usually not interconnected and are used to solve particular problems. For example, in [1], [2] and [3], the finite element method is used to calculate a flexible non-flat filament. In [4], a flexible filament, when calculated for concentrated forces, is modeled by a pin chain consisting of elements with different
geometric parameters, and in [5] a generalized finite difference method is used to solve the differential equilibrium equation for a non-stretch non-flat filament.

In this paper, we present an algorithm for calculating a flexible flat filament for the action of a vertical distributed load that arbitrarily varies along its length, as well as vertical and horizontal concentrated forces.

To solve the problem, a generalized finite difference method [6] is used, which allows to take into account discontinuities in the displacement function and its derivatives without special concentration of calculation points and the use of edge points [7]. The results of numerical calculations carried out for some types of loading were compared with solutions of analytical problems obtained by exact methods [8].

2. Materials and Methods

Calculation of a flexible flat filament can be divided into two stages. At the first stage, the calculation is performed for the action of own weight. “Figure 1” shows the calculated model of a filament with span of $\ell$, which is in equilibrium under the action of its own weight $q_0(x) = \text{const}$. In addition to these parameters, a sag in the center of the span is usually assigned $f_0$. The solution of this problem is known. Following the algorithm presented in [8, 9], one can determine all the physical and geometric characteristics of the initial state of the filament.

![Figure 1. The design model of a flexible flat filament under the action of its own weight](image)

The expression for the initial sag can be represented as:

$$y_0(x) = M_b(x)/T_0,$$

(1)

where

$$M_b(x) = \frac{q_0}{2}(\ell x - x^2).$$

(2)

Here $M_b(x)$ – girder moment of load $q_0$.

The initial tension $T_0$ can be determined from (1).

$$T_0 = q_0\ell^2/8f_0.$$  

(3)

Differentiating (1) twice, taking into account (2) and (3), we obtain the expression for the first derivative of sag and initial curvature $-K$ in the following form:

$$\gamma = \frac{dy_0(x)}{dx} = \frac{4f_0}{\ell^2}(\ell - 2x), \quad K = \frac{d^2y_0(x)}{dx^2} = \frac{-8f_0}{\ell^2}.$$  

(4)

The task of the second stage is to determine the loading in the same filament with a changed external load, the nature and manner of application of which are presented in “Figure 2”. Span of the filament is divided into $n$ elements of length $h$, within which the distributed vertical load is continuous, and vertical and horizontal concentrated forces are applied on their boundaries $\Phi_i$ and $G_j$. 


Figure 2. Design diagram of the filament for a specified external load

Since on the sections between concentrated horizontal forces the horizontal projection of the longitudinal force $T$ is constant, the differential equation of equilibrium relative to the normal to the initial position of the filament has the form:

$$T(K + \frac{d^2w}{dx^2}) + q(x) = 0.$$  \hspace{1cm} (5)

Where $w$ – the normal displacement, measured from the initial position, $q(x)$ – the total external load including the own weight.

We represent equation (5) in the following form:

$$\frac{d^2w}{dx^2} = -(\frac{q(x)}{T} + K) = -p(x)$$  \hspace{1cm} (6)

Where $p(x)$ performs the function of a fictitious load. To solve equation (6), we use the generalized finite difference method, which ideas are presented in [6].

We introduce the following labeling:

$$\frac{dw}{dx} = \varphi \quad ; \quad \frac{d^2w}{dx^2} = \frac{d\varphi}{dx} = f$$ \hspace{1cm} (7)

$$f = -p(x)$$ \hspace{1cm} (8)

Consider the two neighboring elements shown in “Figure 3”. The number of the element will be determined by the number of its right juncture. The function $w$ is continuous at the junctures and the functions $\varphi$ and $f$ can have jumps.

Figure 3. Design diagram of two adjacent elements

Within each element, the function $w$ is approximated by a square parabola

$$w(\xi) = a_0 + a_1\xi + a_2\xi^2,$$ \hspace{1cm} (9)

and its derivatives have the form:

$$\varphi = w'(\xi) = a_1 + 2a_2\xi; \quad f = w''(\xi) = 2a_2.$$ \hspace{1cm} (10)

The coefficients $a_i$ are expressed in terms of the values of $w$ at the boundaries of the elements and
through the derivative $\varphi$ at the juncture $i$. For the element $\langle i \rangle$ the following labeling is used: $\varphi_{i}^{left}$ — is the derivative in the left section, and $\varphi_{i}^{rig}$ — is the derivative in the right section.

Then, taking (7) into account for the element $\langle i \rangle$:

$$a_{0} = w_{i-1}, \quad a_{1} = -\varphi_{i}^{left} + \frac{2}{h} (w_{i} - w_{i-1}); \quad a_{2} = \frac{\varphi_{i}^{left}}{h} - \frac{1}{h^{2}} (w_{i} - w_{i-1}), \quad f_{i} = \frac{2}{h} \varphi_{i}^{left} - \frac{2}{h^{2}} (w_{i} - w_{i-1}) \quad (11)$$

For the element $\langle i + 1 \rangle$:

$$\tilde{a}_{0} = w_{i}, \quad \tilde{a}_{1} = \varphi_{i}^{rig}; \quad \tilde{a}_{2} = -\frac{\varphi_{i}^{rig}}{h} - \frac{1}{h^{2}} (w_{i} - w_{i+1}), \quad f_{i+1} = -\frac{2}{h} \varphi_{i}^{rig} - \frac{2}{h^{2}} (w_{i} - w_{i+1}) \quad (12)$$

We satisfy equation (8) in the middle part of each of the elements $\langle i \rangle$ and $\langle i + 1 \rangle$. Substituting (11) and (12) into (8), we obtain the expressions for the $\langle i \rangle$th element:

$$\varphi_{i}^{left} = -\frac{h}{2} \rho_{i}^{mean} + \frac{1}{h} (w_{i} - w_{i-1}), \quad -\varphi_{i}^{rig} = -\frac{h}{2} \rho_{i}^{mean} + \frac{1}{h} (w_{i} - w_{i+1}) \quad (13)$$

where $\rho_{i}^{mean}$ and $\rho_{i}^{mean}$ — fictitious load in the middle of the elements $\langle i \rangle$ and $\langle i + 1 \rangle$.

Consider the equilibrium of the juncture $i$, shown in “Figure 4”. Here $N_{i}^{left}$ and $N_{i}^{rig}$ — longitudinal forces in the juncture to the left and right, and $T_{i}$ and $T_{i+1}$ are their horizontal projections.

![Figure 4. Diagram of forces acting on the juncture $\langle i \rangle$](image)

From the sum of the projections of the forces acting on the horizontal axis it follows that

$$T_{i+1} = T_{i} - G_{i} \quad (14)$$

The sum of the projections of all forces in the juncture $i$ on the vertical axis leads to the equation

$$\Phi_{i} = T_{i} \gamma_{i}^{left} - T_{i+1} \gamma_{i}^{rig} \quad (15)$$

It should be noted that $\gamma_{i} = \frac{dy}{dx} = \varphi + \gamma$, then (14) taking into account (12) and (13) takes the form:

$$-w_{i-1} + 2w_{i} - w_{i+1} = \frac{G_{i}}{T_{i}} (w_{i} - w_{i+1}) + \frac{h}{T_{i}} \Phi_{i} - \frac{G_{i}}{T_{i}} \gamma_{i} (w_{i} + \frac{h^{2}}{2} \rho_{i+1}^{mean}) + \frac{h^{2}}{2} (\rho_{i}^{mean} + \rho_{i+1}^{mean}) \quad (16)$$

Equation (16) is written for each internal juncture and, for non-displacing abutment, leads to system $(n-1)$ of the equation for determining $w_{i}$. It should be noted that the value $T_{i}$ is constant within the element, and $T_{i} = T_{A}$. Then

$$T_{i} = T_{A} - \sum_{j=1}^{i-1} G_{j}, \quad (i = 2, 3, \ldots, h) \quad (17)$$

Here $G_{j}$ is the horizontal force in the juncture $j$.

In order to solve this problem, in addition to the equations of statics (16), it is necessary to consider the geometric and physical side of the problem.

With additional loading, the longitudinal deformation is determined as follows with respect to the initial state:

$$\varepsilon = \frac{du}{dy} = Kw + \frac{1}{2} \varphi^{2}, \quad (18)$$

where $u$ — the displacement is tangent to the initial position of the filament.

Identifying, due to the flatness of the filament, the longitudinal force with its projection on the
horizontal axis, we represent the equation relating the longitudinal deformations and forces on each section, in the following form:

\[ \omega(x) = \frac{1}{2} \varphi^2 - Kw. \]  
(19)

Where \( E \) – is the elasticity modulus, \( F \) – is the area of cross section.

Integrating equation (19) within the sag, with displacing abutments, we obtain:

\[ \int_0^\ell Tdx - T_0 \ell - EF \int_0^\ell \omega(x)dx = 0 \]  
(20)

Since \( T \) is a piecewise constant function, taking (17) into account, the first integral in (20) is reduced to the following form:

\[ \int_0^\ell Tdx = T_A \ell - h \sum_{i=1}^{n-1} G_i(n-i). \]  
(21)

The Simpson formula is used to calculate the second integral in (20) within the element [10]. Summation over all elements leads to the following expression

\[ \int_0^\ell \omega(x)dx = \frac{h}{6} \sum_{i=1}^{n} (\omega_{i-\frac{1}{2}} + 4\omega_{i\text{ mean}} + \omega_{i\text{ left}}) \]  
(22)

Taking into account (20) and (21), the equation of nonseparability of deformations (19) takes the form

\[ T_A = T_0 + \frac{EFh}{6\ell} \sum_{i=1}^{n} (\omega_{i\text{ rig}} + 4\omega_{i\text{ mean}} + \omega_{i\text{ left}}) + \frac{h}{6} \sum_{i=1}^{n-1} G_i(n-i). \]  
(23)

The additional quantities used in equation (23) are determined from (9) and (10) at \( \xi = h/2 \). Taking into account (11) and (13), they can be represented in the following form:

\[ \omega_{i\text{ mean}} = \frac{1}{2} (w_{i-1} + w_i) + \frac{h^2}{8} \phi_{i\text{ mean}}, \quad \omega_{i\text{ mean}} = \frac{1}{h} (w_i - w_{i-1}) \]  
(24)

For the numerical implementation of the calculation, an iterative method can be used [11].

3. Results

As an example illustrating the practical application of the above calculation algorithm, consider the filament shown in “Figure 5”.

![Figure 5](image_url)

The initial tension and the initial curvature, determined by the formulas (3) and (4) are equal \( T_0 = 200kH \) and \( K = -5 \cdot 10^{-3}f/m \) respectively.

In the absence of horizontal forces, the tension is constant along the span and, consequently, \( T_A = T_R = T \). The fictitious load, determined by formula (6), is also the same for all elements and is...
equal to \( p = -5 \cdot 10^{-3} + 1/T \).

Writing the resolving equation (16) for points \( i = 1, 2 \) with allowance for the fact that \( G_i = 0 \), \( w_0 = 0 \), and \( w_1 = w_3 \), we arrive at the following system of algebraic equations

\[
\begin{align*}
2w_1 - w_2 &= h^2 p + h \Phi/T, \\
2w_1 + 2w_2 &= h^2 p + h \Phi/T.
\end{align*}
\]

Solving the system taking into account the values \( h \), \( p \), \( \Phi \), we obtain: \( w_1 = 6600/T - 3 \), \( w_2 = 8800/T - 4 \).

Count the formulas (24) we determine:

\[
\begin{align*}
w_1^{\text{mean}} &= \frac{1}{2} w_1 + \frac{h^2}{8} p = 3350/T - 1.75; \\
w_2^{\text{mean}} &= \frac{1}{2} (w_1 + w_2) + \frac{h^2}{8} p = 7750/T - 3.75; \\
\phi_0^{\text{rig}} &= \frac{1}{h} w_i + \frac{h}{2} p = 340/T - 0.2, \\
\phi_1^{\text{rig}} &= 120/T - 0.1, \\
\phi_2^{\text{mean}} &= 110/T - 0.05, \\
\phi_2^{\text{left}} &= 100/T, \\
\phi_i^{\text{mean}} &= \frac{1}{h} w_i = \frac{1}{h} w_3 = 330/T - 0.15; \\
\phi_i^{\text{left}} &= \frac{1}{h} w_1 - \frac{h}{2} p = 320/T - 0.1.
\end{align*}
\]

Equation (23), taking into account the symmetry of the system, reduces to the form

\[ T = T_0 + \frac{EFh}{6\ell} [2(\omega_0^{\text{rig}} + \omega_1^{\text{mean}} + \omega_1^{\text{left}}) + (\omega_1^{\text{rig}} + 4\omega_2^{\text{mean}} + \omega_2^{\text{left}})]. \]

Using (19) for calculation, \( w_i \), we obtain a nonlinear equation with respect to \( T \).

Solving it, we find \( T = 1973.05 \). The obtained result completely coincides with the analytical solution [8]. Obviously, the increase in the number of elements does not result to refine the solution in this example.

The displacements in this case of loading will be:

\[ w_1^{\text{mean}} = -0.05; \quad w_1 = 0.345; \quad w_2^{\text{mean}} = 0.178; \quad w_2 = 0.46. \]

The results of calculations performed for several other variants also give a complete coincidence with the analytical calculation.

4. Conclusions

The paper presents an algorithm for calculating a flexible flat filament for the action of vertical and horizontal concentrated forces and a distributed load using the generalized finite difference method (FDM). It should be noted that accounting for the horizontal components of the external load is important when improving the methods of calculating the suspension cable systems. The presented paper, in our opinion, can be regarded as a certain development of methods for calculating flexible filaments. The given algorithm can be recommended for use in engineering calculations, as an alternative to existing methods, since its simplicity and efficiency can produce results that are in good agreement with those obtained using analytical calculations.

References

[1] Leonard J and W Recker (1972). Nonlinear dynamics of cables with low reference initial tension. // Journal of Engineering Mechanics Division, American Society of Civil Engineers, Vol. 98, No. EM2, pp 293-309
[2] Zakharova L, Aleksandrovsky M On the algorithm for calculating an elastic non-flat filament using the variational method. Scientific Review 2017 No 6 pp 33-39
[3] Aleksandrovsky M, Zakharova L Peculiarities of the algorithm of the variational method for the non-linear formulation of the problem of calculating an elastic non-flat filament [Electronic resource] // Internet-journal "NAUKOVEDENIE" ISSN 2223-5167 http://naukovedenie.ru/ Volume 9, No 3 (2017) URL: http://naukovedenie.ru/PDF/78TVN317.pdf
[4] Skvortsov A Calculation of a non-flat flexible linearly deformed filament for concentrated effects: Proc. of scientific and practical conf. "Week of Science - 99". - Moscow: MSURE, 1999 – pp II – 22 – II–23

[5] Zakharova L, Uvarova N To the calculation of a non-flat inextensible filament using the generalized finite difference method // Scientific Review. – 2016 No 12 pp 72–75

[6] Gabbasov R, Gabbasov A, and Filatov V Numerical construction of discontinuous solutions to the problems of structural mechanics Moscow: ACB Publishing House, 2008 p 277

[7] Varvak P and Varvak L. The method of nets in the problems of calculating building structures. - Moscow: Stroizdat, 1977 p 160

[8] Kachurin V Static calculation of cable systems - Leningrad: Stroiizdat, 1969 p 141

[9] Kirienko V, Shimanovsky V, Korshunov D and Smirnov Yu Hanging crossings. Kiev. Budivelnik 1968 p 159

[10] Korn G, Korn T Reference book of Mathematics. - Moscow: Nauka, 1984 p 831

[11] Zakharova L, Uvarova N Calculation of rigid filaments by a numerical method of successive approximations to the effect of arbitrary discontinuous loads // Proceedings of universities. Construction and architecture. 1994 No. 1 pp 21-23