A SYMPLECTIC STRUCTURE FOR THE
SPACE OF QUANTUM FIELD THEORIES

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ABSTRACT

We use the formal Lie algebraic structure in the “space” of hamiltonians provided by equal
time commutators to define a Kirillov-Konstant symplectic structure in the coadjoint orbits
of the associated formal group. The dual is defined via the natural pairing between operators
and states in a Hilbert space.

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Introduction

There has been an increasing interest in the construction of action functionals in what can be called loosely the “space of quantum field theories”. This interest has mainly sprung from the idea that the configuration space for string field theory is the space of all two dimensional conformal field theories ([1] and references therein). The fact that this space, by itself, does not seem to be connected seems to necessitate working in a bigger space which, itself, contains all conformal field theories. In this state of affairs, it seems natural to consider as a candidate for the configuration space the space of all two dimensional quantum field theories.

But certainly the interest for this subject is not limited to the string field theoretical approach. One may conjecture that the existence of hamiltonian structures in the space of quantum field theories could lead to a possible hamiltonian approach to particular kinds of renormalization group flows. In fact, part of our interest in the subject came from the observation that the renormalization group flows for the Sine-Gordon theory are hamiltonian up to two loops [2]. It is reasonable to suppose that if this property is not just an artifact of low-order perturbation theory the partition function itself should provide us with a natural hamiltonian for the flows. The question that follows naturally is how to determine the nonperturbative expression for the symplectic structure.

Here it will be shown, how to equip the space of all quantum field theories (not necessarily two dimensional) with a symplectic structure by a (hopefully) judicious use of the formal Lie algebraic structure provided by the equal time commutator algebra. This infinite-dimensional Lie algebra has a natural pairing induced by the pairing among operators and states. The coadjoint action defined by requiring the invariance of this pairing allows us to define a Kirillov-Konstant two form [3] in the coadjoint orbits of the associated formal group.

Before starting we wish to point out that our construction is quite formal - in the physicist’s sense of the word. Although we are able to define symplectic structures for $D + 1$ dimensional quantum field theories for arbitrary $D$, we expect that a rigorous approach to the subject (if at all possible) would be restricted to $D = 0$, i.e. to ordinary quantum mechanics. Nevertheless, by defining the quantum field theory both on a space lattice and inside a finite box, in order to control ultraviolet and infrared divergences, the problem becomes a quantum mechanical one. For which all our manipulations have some hope to be properly defined. It can then be expected that a smooth continuum limit can be achieved. It may also be possible that, via some standard renormalization procedure, the symplectic structure could be computed perturbatively in the neighborhood of a free field theory. Here we adopt the viewpoint that formal manipulations though sometimes misleading can also be rather illuminating.
The General Setup

First of all, we should be a little more explicit about what we mean by the space of all quantum field theories in \( D + 1 \) dimensions. It will be convenient to start with classical field theories and move later to the quantum case.

In the classical case we need two bits of information to define our classical field theory: one kinematical, consisting of a symplectic phase space which we will denote by \( Y \), and the other dynamical, a hamiltonian function on \( Y \). When we will talk about the space of all classic field theories, we will refer to the “space of all hamiltonian functions on \( Y \)”. That is, we will restrict ourselves to a given symplectic structure. Is this much of a restriction? In the classical case certainly not. In order to understand this a little better, let us first fix our attention to the \( D = 0 \) case, \textit{i.e.} standard classical mechanics with a finite number of degrees of freedom. And let us consider the case in which the configuration space enjoys only one degree of freedom, say, the position of the particle. The phase space is two dimensional and by the Darboux theorem it admits (at least locally) coordinates in which the symplectic structure becomes the canonical one, \textit{i.e.} \( \{ p, q \} = 1 \). Now the space of all \( 0 + 1 \) classical field theories with one degree of freedom will become, in our terminology, the space of all possible hamiltonian functions in that phase space. This essentially exhausts the possible dynamical systems with one degree of freedom. Of course, if we would consider systems with infinite number of degrees of freedom some subtleties would arise. But in the context of our formal approach these complications are ignored.

The first step in our construction should be to define a symplectic phase space which will be the natural arena for defining the space of field theories, or equivalently, the space of hamiltonians. Of course, one can take the approach of taking the starting phase space so big as to accomodate any possible number of fields or degrees of freedom. Perhaps, a more “workable” approach is to start with the desired degrees of freedom and consider all possible associated hamiltonians which fulfill certain symmetry or regularity properties.

What about the quantum case? The analogue of the phase space is a Hilbert space \( \mathcal{H} \), and that of Poisson brackets is an equal time commutator algebra. Of course in the quantum case we do not have an analogue of the Darboux theorem, making the choice of the starting commutator algebra more arbitrary. From now on, when we talk about the space of all quantum field theories of dimension \( D + 1 \) the reader should keep in mind that we have already fixed the field content, and the Hilbert space structure. To fix ideas, one could think that we have chosen the Hilbert space associated with a set of free fields with cannonical commutation relations. Then the space of all quantum field theories becomes equivalent to the space of all possible self-adjoint hamiltonians on those free fields, \textit{i.e.} \( H_0 + H_{\text{pert}} \) for all possible perturbations.

In what follows we will completely ignore questions such as how to properly define the hamiltonian “operators”, what is their domain, etc.
The Formal-ism

Let us denote by $M^D$ the space of all quantum field theories in $D + 1$ dimensions. As explained before we will parametrize the points of $M^D$ by their associated Hamiltonian operator $H$. A set of local coordinates on $M^D$ can be defined as follows. Let us assume, as is usual in formal treatments, that there is a countable basis for the algebra (or a big enough subalgebra) of local operators. If we denote by $O_i(x)$, with $i$ belonging to the set of positive integers, the basis elements for the subalgebra of self-adjoint operators, any point in $M^D$ can be written as
\[ H_U \equiv \sum_j \int dx^D u^j O_j(x), \]
and the coupling constants $u^j$ provide us with a set of convenient coordinates.

The space $M^D$ can be given a natural formal Lie algebraic structure
\[ [H_U, H_V] = i H_{[U,V]}, \]
with
\[ [U, V]^l = f^l_{jk} u^j v^k, \]
where the structure constant coefficients are defined by
\[ \left[ \int O_j, O_k(x) \right] = i f^l_{jk} O_l(x), \]
and the commutator is defined through the composition of “operators”. The extra factor of $i$ is there to preserve “self-adjointness”.

The next step is to identify the dual, i.e. the set of linear functionals on the formal Lie algebra defined above. The dual can be identified with $M^D$ by considering the natural pairing among operators and states in a Hilbert space. Let us define
\[ U(V) = \langle \Omega_U | H_V | \Omega_U \rangle, \]
where $| \Omega_U \rangle$ is the ground state associated to the hamiltonian $H_U$. Notice that, as defined, the map that assigns to $U$ an element of the dual $U(\cdot)$ is nonlinear, and therefore $U(V) \neq V(U)$ and $(U + W)(V) \neq U(V) + W(V)$. Nevertheless, as we will show now, this coadjoint action turns out to be the natural one.

The coadjoint action is defined by requiring invariance of (5). For infinitesimal transformations this requirement reads
\[ (U - ad^*_W(U))(V) = U(V) + U(ad_W(V)), \]
where $ad_W(V)$ is defined to be $[W, V]$. It is now simple to check that if
\[ ad^*_W(U) = [W, U] \]
the pairing is invariant. In order to show this, let us first compute the right hand side of (6)
\[ U(V) + U(ad_W(V)) = U(V) - i \langle \Omega_U | [H_W, H_V] | \Omega_U \rangle \]
\[ = U(V) + i \sum_p \langle \Omega_U | H_V | p \rangle \langle p | H_W | \Omega_U \rangle - \langle \Omega_U | H_W | p \rangle \langle p | H_V | \Omega_U \rangle, \]
where $\sum_p | p \rangle \langle p |$ stands for a resolution of the identity in eigenstates of $H_U$. 

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The left hand side of (6) can now be computed by using first order perturbation theory. Notice that at first order in $W$

$$\langle \Omega_{U+[W,U]}|\psi\rangle = \langle \Omega_U|\psi\rangle - \sum_p \langle p|\left[\left[H_W,H_U\right],\Omega_U\right]|p\rangle,$$

where we have assumed that the ground state is nondegenerate. It is now a simple algebraic computation to check that

$$\langle \Omega_{U+[W,U]}|HV|\Omega_{U+[W,U]}\rangle$$

reproduces (8) up to second order terms in $W$.

**The Coadjoint Orbit Method**

Thanks to our previous construction, it is now easy to define a $G$-invariant symplectic structure in the coadjoint orbits of the formal group $G$ associated to the formal Lie algebraic structure defined by (3).

Let us briefly recall the general construction in a way that will suit our needs. Let $G$ be a Lie group and $\mathcal{G}$ its Lie algebra. Let us denote by $\mathcal{G}^*$ the space of linear functionals on $\mathcal{G}$. For the time being we will denote the elements of $\mathcal{G}$ with greek characters, and the ones of $\mathcal{G}^*$ by latin characters. The coadjoint representation is defined by requiring invariance of the pairing among elements of $\mathcal{G}^*$ and $\mathcal{G}$, i.e. $\forall g \in G$

$$(Ad^*_{g^{-1}}b)(\xi) = b(Ad_g\xi).$$

If we now fix an element $b$ of $\mathcal{G}^*$, we will denote by $O_b$ its orbit under $G$.

Vector fields on $O_b$ are naturally parametrized by elements of $\mathcal{G}$. Let us define $\partial_\xi \in TO_b$ at a point $a \in O_b$ by

$$\partial_\xi f(a) = \frac{d}{de} f(a + \epsilon ad_\xi^e(a)) \bigg|_{\epsilon=0},$$

with $f$ an arbitrary function on $O_b$. Notice that these vectors are defined up to elements in the stability subalgebra of $a$, elements $\eta \in \mathcal{G}$ such that $ad_\eta^e(a) = 0$. Therefore $\partial_\xi + \eta$ and $\partial_\xi$ define the same tangent vector at that point.

We can now define a symplectic form on any point $a \in O_b$ by

$$\omega(\partial_\xi, \partial_\chi) = a([\xi, \chi]),$$

which is obviously antisymmetric and $G$-invariant. It is also well defined because if $\eta$ belongs to the stability subalgebra of $a$

$$\omega(\partial_{\xi+\eta}, \partial_\chi) = \omega(\partial_\xi, \partial_\chi) + a([\eta, \chi])$$

$$= \omega(\partial_\xi, \partial_\chi) + a(ad_\eta^*(a)(\chi) - a(\chi))$$

$$= \omega(\partial_\xi, \partial_\chi).$$

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In order to show that $\omega$ is nondegenerate, let us suppose that there exists a vector $\partial_\beta \in TO_b$ such that, at point $a \in O_b$,

$$\omega(\partial_\beta, \partial_\chi) = 0 \ \forall \partial_\chi.$$  \hspace{1cm} (15)

This clearly implies that $\beta$ belongs to the stability subalgebra $^1$ at that point and therefore parametrizes the zero vector.

Finally, closedness of $\omega$ is easily proved as follows

$$d\omega(\partial_\alpha, \partial_\beta, \partial_\gamma) = \partial_\alpha \omega(\partial_\beta, \partial_\gamma) - \omega([\partial_\alpha, \partial_\beta], \partial_\gamma) + \text{cyclic permutations} \hspace{1cm} (16)$$

The first term in (16) is zero because of the invariance of the pairing, while the second term is zero when we sum up over cyclic permutations because of Jacobi idenities in $G$.

We can now return to our case and define a symplectic form in the coadjoint orbits of the formal group $G$ by defining $\omega$ at the point $U$ to be

$$\omega(\partial_V, \partial_W) = U([V, W]) = -i\langle \Omega_U | [H_V, H_W] | \Omega_U \rangle.$$  \hspace{1cm} (17)

This expression enjoys all the properties described above except nondegeneracy. This can be seen by considering the one form at the point $U$ defined by $\omega(\partial_V, \cdot)$ which is identically zero as long as $| \Omega_U \rangle$ is an eigenstate of $H_V$, even if $V$ does not belong to the stability subalgebra of $U$. Notice that the hamiltonians with such a property form a closed subalgebra. The problem of characterizing the symplectic leaves of $\omega$ is left open.

Final Comments

We hope to have convinced the reader that the method of coadjoint orbits can play an important role in the search for a symplectic structure in the space of all quantum field theories, and that further investigation of the subject is warranted. Of course, it is of primary importance to find some simple examples where this formalism could be applied and where its relevance could be checked.

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\footnote{Notice that in order to show this property we have to use the linear structure in the dual.}
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