LINDENBAUM METHOD (PROPOSITIONAL LANGUAGE)*

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Lindenbaum method is named after the Polish logician Adolf Lindenbaum who prematurely and without a clear trace disappeared in the turmoil of the Second World War at the age of about 37. (Cf. [22].) The method is based on the symbolic nature of formalized languages of deductive systems and opens a gate for applications of algebra to logic and, thereby, to Abstract algebraic logic.

Lindenbaum’s Theorem

A formal propositional language, say $\mathcal{L}$, is understood as a nonempty set $\mathcal{V}_\mathcal{L}$ of symbols $p_0, p_1, \ldots p_\gamma, \ldots$ called propositional variables and a finite set $\Pi$ of symbols $F_0, F_1, \ldots, F_n$ called logical connectives. By $|\mathcal{V}_\mathcal{L}|$ we denote the cardinality of $\mathcal{V}_\mathcal{L}$. For each connective $F_i$, there is a natural number #(F_i) called the arity of the connective $F_i$. The notion of a statement (or a formula) is defined as follows:

1. Each variable $p \in \mathcal{V}_\mathcal{L}$ is a formula;
2. If $F_i$ is a connective of the arity 0, then $F_i$ is a formula;
3. If $A_1, A_2, \ldots, A_n$, $n \geq 1$, are formulas, and $F_i$ is a connective of arity $n$, then the symbolic expression $F_i A_1 A_2 \ldots A_n$ is a formula;
4. A formula can be constructed only according to the rules (f1) – (f3).

The set of formulas will be denoted by $Fr_\mathcal{L}$ and $\mathcal{P}(Fr_\mathcal{L})$ denotes the power set of $Fr_\mathcal{L}$. Given a set $X \subseteq Fr_\mathcal{L}$, we

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denote by $\mathcal{V}(X)$ the set of all propositional variables that occur in the formulas of $X$. Two formulas are counted equal if they are represented by two copies of the same string of symbols. (This is the key observation on which Theorem 1 is grounded.) Another key observation (due to Lindenbaum) is that $\mathcal{F}_L$ along with the connectives $\Pi$ can be regarded as an algebra of the similarity type associated with $L$, which exemplifies an $L$-algebra. We denote this algebra by $\mathcal{F}_L$. The importance of $\mathcal{F}_L$ can already be seen from the following statement.

**Theorem 1.** Algebra $\mathcal{F}_L$ is a free algebra of rank $\overline{\mathcal{V}_L}$ with free generators $\mathcal{V}_L$ in the class (variety) of all $L$-algebras. In other words, $\mathcal{F}_L$ is an absolutely free algebra of this class. (Cf. [17], section 4.11).

A useful feature of the set $\mathcal{F}_L$ is that it is closed under (simultaneous) substitution. More than that, any substitution $\sigma$ is an endomorphism

$$\sigma : \mathcal{F}_L \longrightarrow \mathcal{F}_L.$$

A monotone deductive system (or a deductive system or simply a system) is a relation between subsets and elements of $\mathcal{F}_L$. Each such system $\vdash_S$ is subject to the following conditions: For all $X, Y \subseteq \mathcal{F}_L$,

1. if $A \in X$, then $X \vdash_S A$;
2. if $X \vdash_S B$ for all $B \in Y$, and $Y \vdash_S A$, then $X \vdash_S A$;
3. if $X \vdash_S A$, then for every substitution $\sigma$, $\sigma[X] \vdash_S \sigma(A)$.

If $A$ is a formula and $\sigma$ is a substitution, $\sigma(A)$ is called a substitution instance of $A$. Thus, by $\sigma[X]$ above, one means the set of the instances of the formulas of $X$ with respect to $\sigma$.

Given two sets $Y$ and $X$, we write

$$Y \in X$$

if $Y$ is a finite (maybe empty) subset of $X$. 
A deductive system is said to be *finitary* if, in addition, it satisfies the following:

\[(s_4) \text{ if } X \vdash_S A, \text{ then there is } Y \subseteq X \text{ such that } Y \vdash_S A.\]

We note that the monotonicity property

\[\text{if } X \subseteq Y \text{ and } X \vdash_S A, \text{ then } Y \vdash_S A\]

is not postulated, because it follows from \((s_1)\) and \((s_2)\).

Each deductive system \(\vdash_S\) induces a *(monotone structural)* consequence operator \(Cn_S\) defined on the power set of \(\text{Fr}_L\) as follows: For every \(X \subseteq \text{Fr}_L\),

\[A \in Cn_S(X) \iff X \vdash_S A,\]

so that the following conditions are fulfilled: For all \(X, Y \subseteq \text{Fr}_L\) and any substitution \(\sigma\),

\[(c_1) \quad X \subseteq Cn_S(X); \quad \text{(reflexivity)}\]
\[(c_2) \quad Cn_S(Cn_S(X)) = Cn_S(X); \quad \text{(idempotency)}\]
\[(c_3) \quad \text{if } X \subseteq Y, \text{ then } Cn_S(X) \subseteq Cn_S(Y); \quad \text{(monotonicity)}\]
\[(c_4) \quad \sigma[Cn_S(X)] \subseteq Cn_S(\sigma[X]). \quad \text{(structurality or substitution invariance)}\]

If \(\vdash_S\) is finitary, then

\[(c_5) \quad Cn_S(X) = \bigcup\{Cn_S(Y) \mid Y \subseteq X\},\]

in which case \(Cn_S\) is called *finitary*.

Conversely, if an operator \(Cn : \mathcal{P}(\text{Fr}_L) \rightarrow \mathcal{P}(\text{Fr}_L)\) satisfies the conditions \((c_1)-(c_4)\) (with \(Cn\) instead of \(Cn_S\)), then the equivalence

\[X \vdash_S A \iff A \in Cn(X)\]

defines a deductive system, \(S\). Thus \((1)\) allows one to use the deductive system and consequence operator (in a fixed formal language) interchangeably or even in one and the same context. For instance, we call

\[T_S = Cn_S(\emptyset)\]
the set of theorems of the system ⊢S (i.e. S-theorems), and
given a subset X ⊆ FrL, CnS(X) is called the S-theory
generated by X. A subset X ⊆ FrL, as well as the theory
CnS(X), is called inconsistent if CnS(X) = FrL; otherwise
both are consistent. Thus, given a system ⊢S, T_S is one of
the system’s theories; that is to say, if X ⊆ T_S and X ⊢ S A,
then A ∈ T_S. This simple observation sheds light on the
central idea of Lindenbaum method, which will be ex-
plained soon. For now, let us fix the ordered pair ⟨FrL, T_S⟩
and call it a Lindenbaum matrix. (The full definition will
be given later.) We note that an operator Cn satisfying
\((c_1) - (c_3)\) can be obtained from a closure system over FrL;
that is for any subset A ⊆ P(FrL), which is closed under
arbitrary intersection, we define:

\[ Cn_A(X) = \cap\{Y \mid X \subseteq Y \text{ and } Y \in A\}. \]

It is well known that any consequence operator can be
defined in this way. (Cf. [27], section 1.2.)

Another way of defining deductive systems is through
the use of logical matrices. Given a language L, a logical
L-matrix (or simply a matrix) is a pair \(M = \langle \mathfrak{A}, \mathcal{F}\rangle\), where
\(\mathfrak{A}\) is an L-algebra and \(\mathcal{F} \subseteq |\mathfrak{A}|\), where the latter is the
universe of \(\mathfrak{A}\). The (nonempty) set \(\mathcal{F}\) is called a filter of
the matrix \(M\) and the elements of \(\mathcal{F}\) are called designated.
Given a matrix \(M = \langle \mathfrak{A}, \mathcal{F}\rangle\), the cardinality of |\(\mathfrak{A}|\) is also
the cardinality of \(M\).

Given a matrix \(M = \langle \mathfrak{A}, \mathcal{F}\rangle\), any homomorphism of \(\mathfrak{L}\)
into \(\mathfrak{A}\) is called a valuation (or an assignment). Each such
homomorphism can be obtained simply by assigning ele-
ments of |\(\mathfrak{A}|\) to the variables of \(V_L\), since, by virtue of
Theorem 1 any \(v : V_L \to |\mathfrak{A}|\) can be extended uniquely
to a homomorphism \(v : \mathfrak{L} \to \mathfrak{A}\). Usually, v is meant un-
der a valuation (or an assignment) of variables in a matrix.
Now let $\sigma$ be a substitution and $v$ be any assignment in an algebra $\mathfrak{A}$. Then, defining

$$v_\sigma = v \circ \sigma,$$  \hspace{1cm} (2)

we observe that $v_\sigma$ is also an assignment in $\mathfrak{A}$.

With each matrix $M = \langle \mathfrak{A}, F \rangle$, we associate a relation $\models_M$ between subsets of $Fr_L$ and formulas of $Fr_L$. Namely we define

$$X \models_M A \iff \text{for every assignment } v, \text{ if } v[X] \subseteq F, \text{ then } v(A) \in F.$$  

Then, we observe that the following properties hold:

- (m$_1$) if $A \in X$, then $X \models_M A$;
- (m$_2$) if $X \models_M B$ for all $B \in Y$, and $Y \models_M A$, then $X \models_M A$.

Also, with help of the definition (2), we derive the following:

- (m$_3$) if $X \models_M A$, then for every substitution $\sigma$, $\sigma[X] \models_M \sigma(A)$.

Comparing the condition (m$_1$) – (m$_3$) with (s$_1$) – (s$_3$), we conclude that every matrix defines a structural deductive system and hence, in view of (1), a structural consequence operator.

Given a system $S$, suppose a matrix $M = \langle \mathfrak{A}, F \rangle$ satisfies the condition

$$\text{if } X \vdash_S A \text{ and } v[X] \subseteq F, \text{ then } v(A) \in F.$$  \hspace{1cm} (3)

Then the filter $F$ is called an $S$-filter and the matrix $M$ is called an $S$-matrix (or an $S$-model). In view of (3), $S$-matrices are an important tool in showing that $X \vdash_S A$ does not hold. This idea has been employed in proving that one axiom is independent from a group of others in the search for an independent axiomatic system, as well as for semantic completeness results.

As Lindenbaum’s famous theorem below explains, every structural system $S$ has an $S$-model.
Theorem 2 (Lindenbaum). For any structural deductive system \( S \), the matrix \( \langle \text{Fr}_L, \text{Cn}_S(\emptyset) \rangle \) is an \( S \)-model. Moreover, for any formula \( A \),

\[
A \in T_S \iff v(A) \in \text{Cn}_S(\emptyset) \quad \text{for any valuation } v.
\]

A matrix \( \langle \mathcal{A}, \mathcal{F} \rangle \) is said to be weakly adequate for a deductive system \( S \) if for any formula \( A \),

\[
A \in T_S \iff v(A) \in \mathcal{F} \quad \text{for any valuation } v.
\]

Thus, according to Theorem 2, every structural system \( S \) has a weakly adequate \( S \)-matrix of cardinality less than or equal to \( V^+ + \aleph_0 \). In general, in the last assessment, \( \aleph_0 \) cannot be omitted. For instance, if \( S = \text{IPC} \) (intuitionistic propositional calculus), \( S \) has no finite weakly adequate matrix. (Cf. [9].)

An \( S \)-matrix is called strongly adequate for \( S \) if for any set \( X \subseteq \text{Fr}_L \) and any formula \( A \),

\[
X \vdash_{-S} A \iff X \models_{M} A.
\]

We note that Theorem 2 cannot be improved to include strong adequacy. Also, if \( V \leq \aleph_0 \) and \( S = \text{IPC} \), there is no denumerable matrix \( M \) with (4). (Cf. [28].)

Historical remarks
A. Tarski seems to be the first who promoted “the view of matrix formation as a general method of constructing systems” [14]. However, matrices had been employed earlier, e.g., by P. Bernays [1] and others either in the search for an independent axiomatic system or for defining a system different from classical logic. Also, later on J.C.C. McKinsey [15] used matrices to prove independence of logical connectives in intuitionistic propositional logic.

Theorem 2 was discovered by A. Lindenbaum. Although this theorem was not published by the author, it had been
known in Warsaw-Lvov logic circles at the time. In a published form it appeared for the first time in \cite{14} without proof. Its proof appeared later on in the two independent publications \cite{13} and \cite{11}. McKinsey and Tarski \cite{16} gave an example of a deductive system with $V \leq \aleph_0$ but without any finite weakly adequate matrix.

**Wójcicki’s Theorems**

We get more $S$-matrices, noticing the following. Let $\Sigma_S$ be an $S$-theory. The pair $\langle Fr_L, \Sigma_S \rangle$ is called a \textit{Lindenbaum matrix} relative to $S$. We observe that for any substitution $\sigma$,

\[
\text{if } X_{\vdash} S A \text{ and } \sigma[X] \subseteq \Sigma_S, \text{ then } \sigma(A) \in \Sigma_S.
\]

That is to say, any Lindenbaum matrix relative to a system $S$ is an $S$-model.

A deductive system $S$ is said to be \textit{uniform} if, given a set $X \subseteq Fr_L$ and a consistent set $Y \subseteq Fr_L$, $X \cup Y \vdash S A$ and $V(Y) \cap V(A) = \emptyset$ imply $X_{\vdash} S A$. A system $S$ is \textit{couniform} if for any collection $\{X_i\}_{i \in I}$ of formulas with $V(X_i) \cap V(X_j) = \emptyset$, providing $i \neq j$, if the set $\cup \{X_i\}_{i \in I}$ is inconsistent, then at least one $X_i$ is inconsistent as well.

**Theorem 3** (Wójcicki). A structural deductive system $S$ has a strongly adequate matrix if and only if $S$ is both uniform and couniform.

For the “if” implication of the statement, the matrix of Theorem 2 is not enough. However, it is possible to extend the original language $L$ to $L^+$ in such a way that the \textit{natural extension} $Cn_{S^+}$ of $Cn_S$ onto $L^+$ allows one to define a Lindenbaum matrix $\langle Fr_{L^+}, Cn_{S} \cdot (X) \rangle$, for some $X \subseteq Fr_{L^+}$, which is strongly adequate for $S$. (Cf. \cite{27} for detail.)

A pair $\langle A, \{F_i\}_{i \in I} \rangle$, where $A$ is an $L$-algebra and each $F_i \subseteq \mathcal{A}$, is called a \textit{generalized matrix} (or a $g$-matrix for short).
A $g$-matrix is a $g$-$S$-model (or a $g$-$S$-matrix) if each $\langle \mathcal{A}, F_i \rangle$ is an $S$-model. (In [5] a $g$-matrix is called an atlas.)

**Theorem 4** (Wójcicki). For every structural deductive system $S$, there is a $g$-$S$-matrix $M$ of cardinality $\aleph_0 + \aleph_0$, which is strongly adequate for $S$.

Indeed, let $\{\Sigma_S\}$ be the collection of all $S$-theories. Then the $g$-matrix $\langle \text{Fr}_L, \{\Sigma_S\} \rangle$ is strongly adequate for $S$. (Cf. [27], [5] for detail.)

We note that, alternatively, one could use the notion of a bundle of matrices; a bundle is a set $\{\langle \mathcal{A}, F_i \rangle \mid i \in I\}$, where $\mathcal{A}$ is an $L$-algebra and each $F_i$ is a filter of $\mathcal{A}$. (Cf. [27], section 3.2.11.)

**Historical remarks**

Theorem 3 was the result of the correction by R. Wójcicki of an erroneous assertion in [12], where the important question on the strong adequacy of a system was raised. A number of algebraic equivalents of uniformity is discussed in [6].

T. Smiley [21] was perhaps the first to propose $g$-matrices (known also as Smiley matrices) defined as pairs $\langle \mathcal{A}, Cn \rangle$, where $\mathcal{A}$ is an $L$-algebra and an operator $Cn : \mathcal{P}(|\mathcal{A}|) \to \mathcal{P}(|\mathcal{A}|)$ satisfies the conditions $(c_1) - (c_3)$ (with $Cn$ instead if $Cn_S$). Then, Smiley defined $x_1, \ldots, x_n \vdash y$ if and only of $y \in Cn(\{x_1, \ldots, x_n\})$, where it is assumed that $|\mathcal{A}| \subseteq U$, where $U$ is a universal set of sentences.

**Lindenbaum-Tarski Algebra**

The question of the possibility to decide, whether $X \vdash_S A$ is true or not is central in theory of deduction. Although the notion we are about to introduce is less general than that of $S$-matrix, it points out at a way, following which
this question can be often fruitfully discussed.

An $S$-matrix $\langle \mathcal{A}, F \rangle$ is said to be univalent (or an $S_u$-matrix) if the $S$-filter $F$ consists of one value, say $F = \{1\}$, where $1 \in |\mathcal{A}|$. Let us restrict our original question to the following: How can the property $\emptyset \vdash SA$ be characterized in matrix terms?

Let $\langle \mathcal{A}, \{1\} \rangle$ be an $S_u$-matrix and $A$ be an $S$-theorem. Then, in view of (3), $v(A) = 1$ for every valuation $v$ in $\mathcal{A}$. It would be interesting to know when the converse is true too. Thus the main problem is: How can one obtain an $S_u$-matrix?

**Definition 1** (Lindenbaum-Tarski algebra). Let $\Sigma_S$ be an $S$-theory and let $\Theta(\Sigma_S)$ be the congruence on $\mathcal{F}_L$ generated by $\Sigma_S$; cf. [3]. The quotient algebra $\mathcal{F}_L/\Theta(\Sigma_S)$ is called a Lindenbaum-Tarski algebra of $S$ relative to $\Sigma_S$. If $\Sigma_S = T_S$, then we call this quotient simply a Lindenbaum-Tarski algebra.

An important conclusion from this definition is the following.

**Theorem 5.** Let $S$ be a structural deductive system and $\Sigma_S$ be a nonempty $S$-theory. Assume that $\Sigma_S$ is a congruence class with respect to $\Theta(\Sigma_S)$. Then $\langle \mathcal{F}_L/\Theta(\Sigma_S), \{\Sigma_S\} \rangle$ is an $S_u$-matrix; that is to say, denoting $1 = \Sigma_S$, if $X_S A$ and $v$ is a valuation in $\mathcal{F}_L/\Theta(\Sigma_S)$, then

$$v[X] = \{1\} \implies v(A) = 1.$$  \hspace{1cm} (5)

Moreover, if $\Sigma_S = T_S$, then

$$A \in T_S \iff v(A) = 1 \text{ for any valuation } v \text{ in } \mathcal{F}_L/\Theta(T_S).$$  \hspace{1cm} (6)

Let the valuation $v_0(p) = p/\Theta(T_S)$ for every $p \in V$. Then

$$A \in T_S \iff v_0(A) = 1.$$  \hspace{1cm} (7)

**Definition 2.** Let $S$ be a structural deductive system. We say that $S$ admits the Lindenbaum-Tarski algebra (relative to $\Sigma_S$) if $T_S$ ($\Sigma_S$ respectively) is a congruence class with respect to $\Theta(T_S)$ (with respect to $\Theta(\Sigma_S)$) on $\mathcal{F}_L$. 


Now let us convert the propositional language $L$ into a first order language $L^*$ with equality so that the propositional variables and the logical connectives of $L$ become the individual variables and functional constants of $L^*$, respectively. The set of individual variables is denoted by $\mathcal{V}_{L^*}$. Also, $L^*$ has an individual constant $1$, the equality symbol ‘=’ and universal and existential quantifiers. (Actually, we will need only the former.) We can assume that there is no logical connectives in $L^*$. Since the formulas of $L$ now become terms of $L^*$, each atomic formula of $L^*$ is an expression of the form:

$$A(p, 1, \ldots) = B(q, 1, \ldots),$$

where variables $p$ and $q$ are not necessarily distinct and they, as well as the constant $1$, may or may not occur in the equality.

A universal closure (in the sense of first order logic) of an atomic formula of $L^*$ is often referred to as an identity. We will deal with interpretations of identities only. Therefore, we semantically treat atomic formulas and their universal closures equally. An unspecified identity will be denoted by $\varphi$.

The $L^*$-formulas are interpreted in algebras $\mathfrak{B}$ of the type $L$ endowed with a 0-ary operation $1$. Then, for instance, an identity

$$A(p, 1, \ldots) = 1$$

is said to be valid (or to hold) in $\mathfrak{B}$, in symbols $\mathfrak{B} \models A(p, 1, \ldots) = 1$, if for any assignment $v : \mathcal{V}_L \to |\mathfrak{B}|$

$$A(v(p), 1, \ldots) = 1.$$

Given a system $S$, we denote

$$\bar{\mathcal{S}}_S = \langle \tilde{\mathcal{S}}_L/\Theta(T_S), 1 \rangle,$$

where $1$ is the congruence class generated by $T_S$. Thus $\bar{\mathcal{S}}_S$ is the expansion of $\tilde{\mathcal{S}}_L/\Theta(T_S)$ obtained by adding the
constant \(1\) to the signature of the latter. Then, we define:

\[
\Phi_S = \{ A = 1 \mid A \in T_S \}
\]

and

\[
K_S = \{ \mathcal{B} \mid \mathcal{B} \models \varphi \text{ for all } \varphi \in \Phi_S \}.
\]

It is obvious that the class \(K_S\) is a variety.

**Theorem 6.** Let a structural deductive system \(S\) admit the Lindenbaum-Tarski algebra. Then the algebra \(\mathfrak{F}_S\) belongs in the variety \(K_S\). More than that,

\[
\mathfrak{F}_S \models A = 1 \iff A \in T_S.
\]

Moreover, \(\mathfrak{F}_S \models A(p, 1, \ldots) = 1\) if and only if \(A(p/\Theta(T_S), 1, \ldots) = 1\) in \(\mathfrak{F}_S\), that is \(A(p/\Theta(T_S), T_S, \ldots) = T_S\) in \(\mathfrak{F}_L/\Theta(T_S)\).

Theorem 6 gives rise to the following questions: When is \(\mathfrak{F}_S\) functionally free \([25]\) in \(K_S\)? When is \(\mathfrak{F}_S\) a free algebra in \(K_S\)?

**Historical remarks**

In two parts, \([23]\) and \([24]\), of one paper, the English translation of which constitutes one chapter, *Foundations of the Calculus of Systems*, of \([26]\), A. Tarski showed that the Lindenbaum-Tarski algebra of the system based on classical propositional calculus is a Boolean algebra.

**Alternative Approach**

Let \(\langle \mathfrak{A}, \mathcal{F} \rangle\) be a matrix. A congruence (or an equivalence) \(\theta\) on \(\mathfrak{A}\) is said to be compatible with \(\mathcal{F}\) if \(\cup\{x/\theta \mid x \in \mathcal{F}\} = \mathcal{F}\). Since the identity relation is compatible with any \(\mathcal{F}\), the set of compatible congruences (or equivalences) is not empty for any matrix. Then, it can be proven \([2]\) that for any matrix \(M = \langle \mathfrak{A}, \mathcal{F} \rangle\), there is a largest congruence of \(\mathfrak{A}\) compatible with \(\mathcal{F}\). This congruence is called the Leibniz congruence of \(M\); it is denoted by \(\Omega_{\mathfrak{A}}\mathcal{F}\) and can be defined as follows:

\[
\Omega_{\mathfrak{A}}\mathcal{F} = \{(a, b) \mid \forall A(p, p_0, \ldots, p_n)\forall c_0, \ldots, c_n \in |\mathfrak{A}|. A(a, c_0, \ldots, c_n) \in \mathcal{F} \iff A(b, c_0, \ldots, c_n) \in \mathcal{F} \}.
\]
If the matrix in question is a Lindenbaum one, say \( \langle \mathcal{F}_L, \Sigma_S \rangle \), then an example of a compatible equivalence on this matrix is a Frege relation \( \Lambda \Sigma_S \) defined as follows:

\[(A, B) \in \Lambda \Sigma_S \iff \Sigma_S, A \vdash_S B \text{ and } \Sigma_S, B \vdash_S A\]  
(Frege relation relative to \( \Sigma_S \))

A system \( S \) is called Fregean if each \( \Lambda \Sigma_S \) is a congruence on \( \mathcal{F}_L \). Obviously, if \( S \) is Fregean, it admits the Lindenbaum-Tarski algebra relative to any \( \Sigma_S \).

Another example of a compatible relation on \( \langle \mathcal{F}_L, \Sigma_S \rangle \) is the largest congruence of \( \mathcal{F}_L \) contained in \( \Lambda \Sigma_S \), which is referred to as a Suszko congruence:

\[(A, B) \in \tilde{\Omega} \Sigma_S \iff \text{for every } C(p), \Sigma_S, C(A/p) \vdash_S C(B/p) \text{ and } \Sigma_S, C(B/p) \vdash_S C(A/p).\]  
(Suszko congruence relative to \( \Sigma_S \))

Obviously, a system \( S \) is Fregean if and only if \( \Lambda \Sigma_S = \tilde{\Omega} \Sigma_S \) for all \( \Sigma_S \).

The Leibniz congruence of a matrix \( \langle \mathcal{F}_L, \Sigma_S \rangle \) is referred to as Leibniz congruence relative to \( \Sigma_S \). It turns out that

\[\Omega \Sigma_S = \cap \{ \tilde{\Omega} \Sigma'_S \mid \Sigma_S \subseteq \Sigma'_S \}\]

and, therefore, each Suszko congruence \( \tilde{\Omega} \Sigma_S \) is compatible with \( \Sigma_S \). Also, given a system \( S \), one defines

\[\tilde{\Omega}_S = \cap \{ \tilde{\Omega} \Sigma_S \mid \Sigma_S \text{ is an } S\text{-theory} \}.\]  
(Tarski congruence)

Thus we have:

\[\tilde{\Omega}_S \subseteq \tilde{\Omega} \Sigma_S \subseteq \Lambda \Sigma_S \cap \Omega \Sigma_S.\]

Suszko, Leibniz and Tarski congruences give rise to the \( S \)-matrices \( \langle \mathcal{F}_L/\Omega \Sigma_S, \Sigma_S/\Omega \Sigma_S \rangle, \langle \mathcal{F}_L/\tilde{\Omega} \Sigma_S, \Sigma_S/\tilde{\Omega} \Sigma_S \rangle, \) and the \( g \)-\( S \)-matrix \( \langle \mathcal{F}_L/\tilde{\Omega}_S, \{ \Sigma_S/\tilde{\Omega}_S \mid \Sigma_S \text{ is an } S\text{-theory} \} \rangle \), whose first components, \( \mathcal{F}_L/\Omega \Sigma_S, \mathcal{F}_L/\tilde{\Omega} \Sigma_S \) and \( \mathcal{F}_L/\tilde{\Omega}_S \), in Algebraic abstract logic are also referred to as Lindenbaum-Tarski
algebras. (See [7] and [8] for comprehensive surveys.)

Specifications and Applications

A structural deductive system \( \mathcal{S} \) is called implicative extensional if its language \( \mathcal{L} \) contains a binary connective \( \to \) (will be written in the infix notation), and for any \( \mathcal{S} \)-theory \( \Sigma_S \) and any \( A, B, C \in Fr_\mathcal{L} \), the following conditions hold:

1. \( A \to A \in \Sigma_S \);
2. \( B \in \Sigma_S \implies A \to B \in \Sigma_S \);
3. \( A \to B, B \to C \in \Sigma_S \implies A \to C \in \Sigma_S \);
4. \( A, A \to B \in \Sigma_S \implies B \in \Sigma_S \);
5. \( A_i \to B_i, B_i \to A_i \in \Sigma_S, 1 \leq i \leq n, \implies \Pi A_1 \ldots A_n \to \Pi B_1 \ldots B_n \)
   for each \( n \)-ary connective \( \Pi \).

Now, given \( \mathcal{S} \), we consider the following relation on \( Fr_\mathcal{L} \):

\[
A \approx_S B \iff A \to B, B \to A \in T_S. \quad \text{(Rasiowa relation)}
\]

**Theorem 7** (Rasiowa). If \( \mathcal{S} \) is an implicative extensional system, then the relation \( \approx_S \) is a congruence on \( \mathcal{L} \). Moreover, \( T_S \) is a congruence class with respect to \( \approx_S \).

Applying Theorem 7 to \( IPC \), one can observe (actually, it was shown in [20]) that \( \mathcal{L}_\mathcal{L}/\approx_{IPC} \) is the free algebra of rank \( \forall \) in the variety of Heyting algebras. Using the Tarski relation \( \approx_{IPC} \), Nishimura [18] gave an elegant description of the Lindenbaum-Tarski algebra of \( IPC \) in a language with a single propositional variable. This algebra is also the free algebra of rank 1 in the variety of Heyting algebras. See Free algebra.

Also, it is worth noticing that, using a Lindenbaum-Tarski algebra as defined above, one can prove that there is an algorithm which decides whether two finite \( g \)-matrices define the same deductive system; this result is due to A.
Citkin (unpublished) and J. Zygmunt [29]. In this connection see *Decision problem*.

**Historical remarks**

In [23], [24] (see [26], chapter XII), Tarski gave the first specification of a system which admits the Lindenbaum-Tarski algebra. Later on, Rasiowa [19] summarized the work that had been done by the time in the notion of “the class of standard systems of implicative extensional propositional calculi,” which is a simplified version of that we use above.

Also, if $S$ is an implicative extensional system, then $\mathfrak{A}_S$ as defined above is Rasiowa’s $\mathcal{P}$-algebra [19], or nowadays known [2] as Hilbert algebra with compatible operations.

In [29] Zygmunt credits Citkin for the decidability result mentioned above. Recently, it was rediscovered by L. Devyatkin [4].

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