Solving Partition Problems Almost Always Requires Pushing Many Vertices Around

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Abstract

A fundamental graph problem is to recognize whether the vertex set of a graph $G$ can be bipartitioned into sets $A$ and $B$ such that $G[A]$ and $G[B]$ satisfy properties $\Pi_A$ and $\Pi_B$, respectively. This so-called $(\Pi_A, \Pi_B)$-Recognition problem generalizes amongst others the recognition of 3-colorable, bipartite, split, and monopolar graphs. A powerful algorithmic technique that can be used to obtain fixed-parameter algorithms for many cases of $(\Pi_A, \Pi_B)$-Recognition, as well as several other problems, is the pushing process. For bipartition problems, the process starts with an “almost correct” bipartition $(A', B')$, and pushes appropriate vertices from $A'$ to $B'$ and vice versa to eventually arrive at a correct bipartition.

In this paper, we study whether $(\Pi_A, \Pi_B)$-Recognition problems for which the pushing process yields fixed-parameter algorithms also admit polynomial problem kernels. In our study, we focus on the first level above triviality, where $\Pi_A$ is the set of $P_3$-free graphs (disjoint unions of cliques, or cluster graphs), the parameter is the number of clusters in the cluster graph $G[A]$, and $\Pi_B$ is characterized by a set $\mathcal{H}$ of connected forbidden induced subgraphs. We prove that, under the assumption that $\text{NP} \not\subseteq \text{coNP}/\text{poly}$, $(\Pi_A, \Pi_B)$-Recognition admits a polynomial kernel if and only if $\mathcal{H}$ contains a graph with at most 2 vertices. In both the kernelization and the lower bound results, we make crucial use of the pushing process.

1 Introduction

A graph $G$ is a $(\Pi_A, \Pi_B)$-\textit{graph}, for two hereditary graph properties $\Pi_A, \Pi_B$, if $V(G)$ can be partitioned into two sets $A, B$ such that $G[A] \in \Pi_A$ and $G[B] \in \Pi_B$. We call $(A, B)$ a $(\Pi_A, \Pi_B)$-\textit{partition} of $G$. The $(\Pi_A, \Pi_B)$-Recognition problem is to recognize whether a given graph is a $(\Pi_A, \Pi_B)$-\textit{graph}. This captures a wealth of famous problems, including the recognition of 3-colorable, bipartite, co-bipartite, and split graphs, and II-Vertex Deletion, which asks for a partition $(A, B)$ such that $G[A] \in \Pi$ and $G[B]$ has order at most $k$ for some given $k$\textsuperscript{1}. In the most interesting (and NP-hard) cases [2, 14, 23].

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\textsuperscript{1}The order of a graph is its number of vertices.
$\Pi_A$ and $\Pi_B$ are both characterized by a (not necessarily finite) set of forbidden connected\footnote{The restriction to connected graphs is probably necessary for NP-hardness: the recognition of unipolar graphs, where $\Pi$ is the set of complete graphs (nonedge-less graphs), can be solved in polynomial time \cite{10,18,27,30}.} induced subgraphs. In other words, $\Pi_A$ and $\Pi_B$ are each closed under the disjoint union of graphs in these cases.

Many such $(\Pi_A, \Pi_B)$-Recognition problems were shown fixed-parameter tractable by Kanj et al. \cite{22}, for example when $\Pi_A$ is the class of graphs that is a disjoint union of $k$ cliques, using parameter $k$. The central algorithmic idea that was employed in \cite{22} is the pushing process. The algorithm empties the input graph, and adds vertices back one by one while maintaining a valid partition. Since adding a vertex might invalidate a previously valid partition, vertices are pushed from one part of the partition to the other part in the hope of obtaining a valid partition again. A similar algorithmic idea, known as iterative localization, was used earlier by Heggernes et al. \cite{21} to show the fixed-parameter tractability of computing the cochromatic number of perfect graphs and the stabbing number of disjoint rectangles with axes-parallel lines (using the standard parameters). Iterative localization was also applied in follow-up work related to the cochromatic number \cite{24}.

A crucial ingredient in applying the pushing process is to understand the avalanches caused by this process. For $(\Pi_A, \Pi_B)$-Recognition, an avalanche is triggered when a vertex is pushed to $A$; this may imply that several other vertices must be pushed to $B$, which, in turn, triggers the pushing of yet more vertices to $A$, and so on. Similar effects are visible in the aforementioned cochromatic number and rectangle stabbing number problems \cite{21,24}. The contribution of the previous works \cite{21,22,24} was to bound the depth of this process by some function of the parameter, leading to fixed-parameter algorithms. However, such a bound does not provide an answer to the question of which vertices trigger avalanches and their continued rolling, and whether the number of such vertices can somehow be limited.

This question can be naturally formalized in terms of the kernelization complexity of problems to which the pushing process applies. A kernel reduces the size of the graph and thus directly reduces the number of vertices triggering or being affected by avalanches when an algorithm based on the pushing process is applied to the kernelized instance. In previous work, Kolay et al. \cite{24} studied the kernelization complexity of computing the cochromatic number of a perfect graph $G$, which is the smallest number $k = r + \ell$ such that $V(G)$ can be partitioned into $r$ sets that each induces a clique and $\ell$ sets that each induces an edgeless graph. This problem has a parameterized algorithm using iterative localization (i.e., a pushing process) \cite{21}, but Kolay et al. \cite{24} showed that, unless $\text{NP} \subseteq \text{coNP/poly}$, this problem does not admit a polynomial kernel parameterized by $r + \ell$. This suggests that, for this problem, one cannot control the number of vertices affected by avalanches. The kernelization complexity of $(\Pi_A, \Pi_B)$-Recognition, however, has not been studied so far. Hence, it is open whether avalanches can be controlled to affect few vertices in this case.

**Our Result** We study the kernelization complexity of $(\Pi_A, \Pi_B)$-Recognition through the lens of the pushing process. To this end, we consider the first level above triviality of the problem. When $\Pi_A$ is characterized by a forbidden induced subgraph of order 2, then $(\Pi_A, \Pi_B)$-Recognition can be solved in linear time \cite{18}, and thus we focus on the NP-hard case when the forbidden induced subgraph has order 3 \cite{2,13,25}. In particular, we let $\Pi_A$ be the class of so-called cluster graphs. These are the graphs that contain no $P_3$—the (simple) path on three vertices—as an induced subgraph, or equivalently, graphs that are disjoint unions of complete graphs. This leads to the following problem:

**Cluster-II-Partition**

**Input:** A graph $G = (V, E)$.

**Question:** Can $V(G)$ be partitioned into $(A, B)$ such that $G[A]$ is a cluster graph and $G[B] \in \Pi$?

**Cluster-II-Partition** generalizes the recognition problem of many graph classes, such as the recognition of monopolar graphs \cite{10,8,9,26} ($\Pi$ is the set of edgeless graphs), 2-subcolorable graphs \cite{5,16,20,29} ($\Pi$ is the set of cluster graphs), and several others \cite{1,4,7}. Unfortunately, **Cluster-II-Partition** is NP-hard in these special cases, and in general when $\Pi$ is characterized by a set of connected forbidden induced subgraphs \cite{2,4,25}. Hence, we consider the number $k$ of clusters in the cluster graph $G[A]$ as a parameter, and study the pushing process with respect to this parameter.

Our result gives a complete characterization of the kernelization complexity of **Cluster-II-Partition** through a deeper understanding of the pushing process. We show that, while for a specific $\Pi$ the pushing process can be used to witness a small vertex set of size $k^{O(1)}$ containing the vertices affected by
Theorem 1.1. Let \( \Pi \) be a graph property characterized by a (not necessarily finite) set \( \mathcal{H} \) of connected forbidden induced subgraphs. Then unless \( \text{NP} \subseteq \text{coNP}/\text{poly} \), \( \text{CLUSTER-II-PARTITION} \) parameterized by the number \( k \) of clusters in the cluster graph \( G[A] \) admits a polynomial kernel if and only if \( \mathcal{H} \) contains a graph of order at most 2.

The positive result corresponds to the recognition of monopolar graphs. Indeed, the graph properties with forbidden induced subgraphs of order 2 are “being edgeless” and “being nonedge-less”, but the latter is not characterized by connected forbidden induced subgraphs.

The pushing process and a deeper understanding of the avalanches it causes are indeed central to the deterministic behavior of the pushing process for monopolar graphs: when an edge in \( G[B] \) is created by pushing a vertex to \( B \), the other endpoint of the edge must be pushed to \( A \) (recall that \( G[B] \) must become edgeless). This limits the avalanches. However, for more complex properties \( \Pi_B \), such a simple correspondence no longer exists. In particular, when the forbidden induced subgraphs have order at least 3, pushing a vertex to \( B \) may create a forbidden induced subgraph in \( G[B] \) that can be repaired in at least two different ways. Then the pushing process starts to behave nondeterministically, and the avalanches grow beyond control. We exploit this intuition to exclude the existence of a polynomial kernel, unless \( \text{NP} \subseteq \text{coNP}/\text{poly} \), by providing a cross-composition.

Other Parameterizations. One might consider two other parameters: the size of a largest cluster in \( G[A] \) and the size of one of the sides. The size of a largest cluster in \( G[A] \) will not lead to tractability, as \( \text{CLUSTER-II-PARTITION} \) is NP-hard on subcubic graphs, even when \( \Pi \) is the set of edgeless graphs [26]. Thus, we consider the number \( k \) of vertices in the graph \( G[B] \), even for the broader \( (\Pi_A, \Pi_B)-\text{RECOGNITION} \) problem. We previously proved a general fixed-parameter tractability result in this case [22]. We observe a very general kernelization result:

Theorem 1.2. \( (\Pi_A, \Pi_B)-\text{RECOGNITION} \) has a kernel of size \( O(k^4) \) parameterized by \( k \), the maximum size of \( B \), when \( \Pi_A \) can be characterized by a collection \( \mathcal{H} \) of forbidden induced subgraphs, each of size at most \( d \), and \( \Pi_B \) is hereditary.

We obtain a better bound in terms of the number of vertices for \( \text{CLUSTER-II-PARTITION} \), the restriction of \( \text{CLUSTER-II-PARTITION} \) to the case when all graphs containing a vertex of degree at least \( \Delta + 1 \) are forbidden induced subgraphs of \( \Pi \).

Theorem 1.3. \( \text{CLUSTER-II}_\Delta\text{-PARTITION} \) parameterized by \( k \), the maximum size of \( B \), has an \( O((\Delta^2 + 1) \cdot k^2) \)-vertex kernel.

2 Preliminaries

Graphs. We follow standard graph-theoretic notation [11]. Let \( G \) be a graph. By \( V(G) \) and \( E(G) \) we denote the vertex-set and the edge-set of \( G \), respectively. Throughout the paper, we use \( n := |V(G)| \) to denote the number of vertices in \( G \) and \( m := |E(G)| \) to denote its number of edges. We also say that \( G \) is of order \( |V(G)| \). We assume \( n = O(m) \) since isolated vertices can be safely removed in the problems that we consider. For \( X \subseteq V(G) \), \( G[X] := (X, \{ e \mid e \in E(G) \cap X \} ) \) denotes the subgraph of \( G \) induced by \( X \). For a vertex \( v \in G \), \( N(v) = \{ u \mid \{ u,v \} \in E(G) \} \) and \( N[v] = N(v) \cup \{ v \} \) denote the open neighborhood and the closed neighborhood of \( v \), respectively. For \( X \subseteq V(G) \), we define \( N(X) := (\bigcup_{v \in X} N(v)) \setminus X \) and \( N[X] := \bigcup_{v \in X} N[v] \), and for a family \( \mathcal{X} \) of subsets \( X \subseteq V(G) \), we define \( N(\mathcal{X}) := (\bigcup_{X \in \mathcal{X}} N(X)) \setminus (\bigcup_{X \in \mathcal{X}} X) \) and \( N[\mathcal{X}] := \bigcup_{X \in \mathcal{X}} N[X] \).
We say that a vertex $v$ is adjacent to a subset $X \subseteq V(G)$ of vertices if $v$ is adjacent to at least one vertex in $X$. Similarly, we say that two vertex sets $X \subseteq V(G)$ and $Y \subseteq V(G)$ are adjacent if there exist $x \in X$ and $y \in Y$ that are adjacent. If $X$ is any set of vertices in $G$, we write $G - X$ for $G[V(G) \setminus X]$. For a vertex $v \in V(G)$, we write $G - v$ for $G \setminus \{v\}$.

**Graph Partitions** We say a partition $(A, B)$ of $V(G)$ is a cluster-$\Pi$ partition if (1) $G[A]$ is a cluster graph and (2) $G[B] \in \Pi$. A monopolar partition of a graph $G$ is a partition of $V(G)$ into a cluster graph and an independent set.

**Monopolar Recognition**

**Input**: A graph $G = (V, E)$ and an integer $k$.

**Question**: Does $G$ admit a monopolar partition $(A, B)$ such that the number of clusters in the cluster graph of the partition is at most $k$?

For an instance $(G, k)$ of Monopolar Recognition, a monopolar partition of $G$ is valid if the number of clusters in the cluster graph of the partition is at most $k$. For $\ell \in \mathbb{N}$, we use $[\ell]$ to denote $\{1, 2, \ldots, \ell\}$.

**Parameterized Complexity** A parameterized problem is a tuple $(P, \kappa)$, where $P \subseteq \Sigma^*$ is a language over some finite alphabet $\Sigma$ and $\kappa : \Sigma^* \to \mathbb{N}$ is a parameterization. For a given instance $x \in \Sigma^*$, we also say $\kappa(x)$ is the parameter. A parameterized problem $(P, \kappa)$ is fixed parameter tractable (FPT), if there exists an algorithm that on input $x \in \Sigma^*$ decides if $x$ is a yes-instance of $P$, that is, $x \in P$, and that runs in time $f(\kappa(x))n^{O(1)}$, where $f$ is a computable function independent of $n = |x|$. We will denote by fpt-time a running time of the form $f(\kappa(x))n^{O(1)}$. A parameterized problem is kernelizable if there exists a polynomial-time reduction that maps an instance $x$ of the problem to another instance $x'$ such that: (1) $|x'| \leq \lambda(\kappa(x))$ for some computable function $\lambda$, (2) $\kappa(x') \leq \lambda(\kappa(k))$, and (3) $x$ is a yes-instance of the problem if and only if $x'$ is. The instance $x'$ is called the kernel of $x$.

Let $Q$ be a language and $(P, \kappa)$ a parameterized problem, i.e., $P$ is a language and $\kappa : \Sigma^* \to \mathbb{N}$ a parameterization. An or-cross-composition from $Q$ into $(P, \kappa)$ is a polynomial-time algorithm that, given $t$ instances $q_1, \ldots, q_t \in \Sigma^*$ of $Q$, computes an instance $r \in \Sigma^*$ such that

$$\kappa(r) \leq \text{poly} \left( \log t + \max_{i=1}^{t} |q_i| \right),$$

and $r \in P$ if and only if $q_i \in Q$ for some $i \in [t]$. We have the following:

**Theorem 2.1** ([3] [19]). Let $Q \subseteq \Sigma^*$ be an NP-hard language and $(P, \kappa)$ be a parameterized problem. If there is an or-cross-composition from $Q$ into $(P, \kappa)$ and $(P, \kappa)$ admits a polynomial-size problem kernel, then $\text{NP} \subseteq \text{coNP}/\text{poly}$.

For more discussion on parameterized complexity, we refer to the literature [10] [12].

3 A Polynomial Kernel for Monopolar Recognition Parameterized by the Number of Clusters

The outline of the kernelization algorithm is as follows. First, we compute a decomposition of the input graph into sets of vertex-disjoint maximal cliques which we call a clique decomposition. This decomposition is used and updated throughout the data-reduction procedure. We also maintain sets of vertices that are determined to belong to $A$ or $B$. We first apply a sequence of reduction rules whose aim is roughly to bound the number of cliques and the number of edges between the cliques in the decomposition, and to restrict the structure of edges between cliques. Then, we build an auxiliary graph to model how the placement of a vertex in $A$ or $B$ implies an avalanche of placements of vertices in $A$ and $B$. If this avalanche creates too many clusters in $A$, then this determines the placement of certain vertices in $A$ or $B$, and triggers another reduction rule. If this reduction rule does not apply anymore, then the size of the auxiliary graph is bounded, which in turn helps bounding the size of the instance.


3.1 Clique Decompositions

Say that a clique \( C \) is a large clique if \(|C| \geq 3\), an edge clique if \(|C| = 2\) (i.e., \( C \) is an edge), and a vertex clique if \(|C| = 1\) (i.e., \( C \) consists of a single vertex). Let \((G,k)\) be an instance of Monopolar Recognition. Suppose that \( A_{\text{true}} \subseteq V(G) \) and \( B_{\text{true}} \subseteq V(G) \) are subsets of vertices that have been determined to be in \( A \) and \( B \), respectively, in any valid monopolar partition of \((G,k)\). We define a decomposition \((C_1,\ldots,C_r)\) of \( V(G) \setminus (A_{\text{true}} \cup B_{\text{true}})\), referred to as a nice clique decomposition, that partitions this set into vertex-disjoint cliques \( C_1,\ldots,C_r, \ r \geq 1\), such that the tuple \((C_1,\ldots,C_r)\) satisfies the following properties:

(i) In the decomposition tuple \((C_1,\ldots,C_r)\), the large cliques appear before the edge cliques, and the edge cliques, in turn, appear before the vertex cliques; that is, for each large clique \( C_i \) and for each edge or vertex clique \( C_j \) we have \( i < j \), and for each edge clique \( C_i \), and for each vertex clique \( C_j \), we have \( i < j \).

(ii) Each clique \( C_i, i \in [r-1] \), is maximal in \( \bigcup_{j=i+1}^{r} C_j \); that is, there does not exist a vertex \( v \in \bigcup_{j=i+1}^{r} C_j \) such that \( C_i \cup \{v\} \) is a clique.

(iii) The subgraph of \( G \) induced by the union of the edge cliques and vertex cliques does not contain any large clique.

The following fact is implied by property (ii) above:

**Fact 3.1.** The vertex cliques in a nice clique decomposition form an independent set in \( G \).

A nice clique decomposition of \( V(G) \setminus (A_{\text{true}} \cup B_{\text{true}}) \) can be computed as follows. Let \( V' = V(G) \setminus (A_{\text{true}} \cup B_{\text{true}}) \neq \emptyset \). We check whether \( G[V'] \) contains a clique \( C \) of size at least three. If this is the case, then we find a maximal clique \( C' \supseteq C \) in \( G[V'] \), add \( C' \) as a large clique to the decomposition, set \( V' \leftarrow V' - C' \) and repeat. Otherwise, \( G[V'] \) does not contain any clique of size 3, we check whether \( G[V'] \) contains an edge clique (i.e., two endpoints of an edge), add \( C \) to the decomposition, set \( V' \leftarrow V' - C \) and repeat. If no edge clique exists in \( G[V'] \), then the remaining vertices in \( V' \) form an independent set, and we add each one of them to the decomposition as a vertex clique. This process can be seen to run in polynomial time, but we will use the following more precise bound.

**Lemma 3.2.** A nice clique decomposition of \( G \) can be computed in \( O(nm) \) time.

**Proof.** First, in \( O(nm) \) time, compute a list of all triangles in \( G \). Then, label all vertices as free. Let \( G' \) denote the graph \( G[V'] \). Process the list from head to tail; that is, consider each triangle in the list. If one vertex of the triangle is not labeled as free, then continue with the next triangle. If all vertices in this triangle are labeled as free, then compute a maximal clique in \( G' \) containing this triangle. This can be done in \( O(m) \) time [28]. Add the maximal clique to the decomposition as described above, remove all vertices of the maximal clique from \( G' \), and unlabel all vertices of the maximal clique. Overall this step takes \( O(nm) \) time, since we encounter at most \( n/3 \) triangles whose vertices are labeled free. Once all triangles in the list are processed, compute a set of edge cliques in \( O(m) \) time by computing a maximal matching in \( G[V'] \). Finally, add all remaining vertices as vertex cliques in \( O(n) \) time. \( \square \)

Let \((G,k)\) be an instance of Monopolar Recognition. We initialize \( A_{\text{true}} = B_{\text{true}} = \emptyset \), \( V' = V(G) \setminus (A_{\text{true}} \cup B_{\text{true}}) \), and we compute a nice clique decomposition \((C_1,\ldots,C_r)\) of \( V' \). We will then apply reduction rules to simplify the instance \((G,k)\). During this process, we may identify vertices in \( V' \) to be added to \( A_{\text{true}} \) or \( B_{\text{true}} \). At any point in the process, we will maintain a partition \((A_{\text{true}},B_{\text{true}},C_1,\ldots,C_r)\) of \( V(G) \) such that (1) \( A_{\text{true}} \subseteq A \) and \( B_{\text{true}} \subseteq B \) for any valid monopolar partition \((A,B)\) of \( V(G) \), and (2) \((C_1,\ldots,C_r)\) is a nice clique decomposition of \( V' = V(G) \setminus (A_{\text{true}} \cup B_{\text{true}}) \). We call such a partition \((A_{\text{true}},B_{\text{true}},C_1,\ldots,C_r)\) a normalized partition of \( V(G) \).

3.2 Basic Reduction Rules

We now describe our basic set of reduction rules. After the application of a reduction rule, a normalized partition may change as the result of moving vertices from \( \bigcup_{i=1}^{r} C_i \) to \( A_{\text{true}} \cup B_{\text{true}} \), and we will need to compute a nice clique decomposition of the resulting (new) set \( V(G) \setminus (A_{\text{true}} \cup B_{\text{true}}) \). However, a vertex that has been moved to \( A_{\text{true}} \) (resp. \( B_{\text{true}} \)) will remain in \( A_{\text{true}} \) (resp. \( B_{\text{true}} \)). When a reduction rule is applied, we assume that no reduction rule preceding it, with respect to the order in which the rules are listed, is applicable.

The following rule is straightforward:
Reduction Rule 3.3. Let \((A_{\text{true}}, B_{\text{true}}, C_1, \ldots, C_r)\) be a normalized partition of \(V(G)\). If \(A_{\text{true}}\) is not a cluster graph with at most \(k\) clusters, or \(B_{\text{true}}\) is not an independent set, then reject the instance \((G, k)\).

The following rule is correct because, for every monopolar partition \((A, B)\) of \(G\), \(B_{\text{true}} \subseteq B\) and \(B\) is an independent set.

Reduction Rule 3.4. Let \((A_{\text{true}}, B_{\text{true}}, C_1, \ldots, C_r)\) be a normalized partition of \(V(G)\). If there is a vertex \(v \in V(G) \setminus (A_{\text{true}} \cup B_{\text{true}})\) that is adjacent to \(B_{\text{true}}\), then set \(A_{\text{true}} = A_{\text{true}} \cup \{v\}\).

The following rule is correct, since \(A_{\text{true}} \subseteq A\) for every monopolar partition \((A, B)\) of \(G\):

Reduction Rule 3.5. Let \((A_{\text{true}}, B_{\text{true}}, C_1, \ldots, C_r)\) be a normalized partition of \(V(G)\). If there is a vertex \(v \in V(G) \setminus (A_{\text{true}} \cup B_{\text{true}})\) that is either (1) adjacent to two clusters in \(A_{\text{true}}\), or (2) adjacent to a cluster \(C\) in \(A_{\text{true}}\) but not to all the vertices \(C\), then set \(B_{\text{true}} = B_{\text{true}} \cup \{v\}\).

The proof of the following reduction rule is straightforward, after recalling that the vertex cliques induce an independent set in \(G\) (Fact 3.1), and observing that no two vertices of an independent set can belong to the same cluster in a cluster graph:

Reduction Rule 3.6. Let \((A_{\text{true}}, B_{\text{true}}, C_1, \ldots, C_r)\) be a normalized partition of \(V(G)\). If there is a vertex \(v \in V(G) \setminus (A_{\text{true}} \cup B_{\text{true}})\) with more than \(k\) neighbors that are vertex cliques, then set \(A_{\text{true}} = A_{\text{true}} \cup \{v\}\).

The next two reduction rules restrict the number and type of edges incident to large cliques.

Reduction Rule 3.7. Let \((A_{\text{true}}, B_{\text{true}}, C_1, \ldots, C_r)\) be a normalized partition of \(V(G)\). If there exists a vertex \(v \in V(G) \setminus (A_{\text{true}} \cup B_{\text{true}})\) and a large clique \(C_i\) such that \(1 < |N(v) \cap C_i| \leq |C_i| - 1\), then set \(A_{\text{true}} = A_{\text{true}} \cup (N(v) \cap C_i)\).

Correctness Proof. Since \(1 < |N(v) \cap C_i| \leq |C_i| - 1\), \(v\) has at least two neighbors \(u, w \in C_i\) and at least one nonneighbor \(x \in C_i\). If a vertex \(z \in N(v) \cap C_i\) is in \(B\), for any valid monopolar partition \((A, B)\) of \(V(G)\), then since \(B\) is an independent set, it follows that \(C_i - \{z\} \subseteq A\). In particular, \(v\) is in \(A\), at least one of \(u, w\), say \(u\), is in \(A\), and \(x\) is in \(A\). But this implies that \((v, u, x)\) forms an induced \(P_3\) in \(A\), contradicting that \(A\) is a cluster graph.

Reduction Rule 3.8. Let \((A_{\text{true}}, B_{\text{true}}, C_1, \ldots, C_r)\) be a normalized partition of \(V(G)\), and let \(C_i, C_j\), \(i < j\), be two cliques such that \(C_i\) is a large clique and \(C_j\) is either a large clique or an edge clique. If there are at least two edges between \(C_i, C_j\), then one of the following reductions, considered in the listed order, is applicable:

Case (1) There are two edges \(uu'\) and \(vv'\), where \(u, v \in C_i\) and \(u', v' \in C_j\), such that \(u \neq v\) and \(u' \neq v'\). Let \(w \in C_i\) be such that \(w \notin \{u, v\}\) (note that \(w\) exists because \(|C_i| \geq 3\)). Set \(A_{\text{true}} = A_{\text{true}} \cup \{w\}\).

Case (2) \(N(C_i) \cap C_j = \{v\}\). Set \(B_{\text{true}} = B_{\text{true}} \cup \{v\}\).

Correctness Proof. We first prove that either case (1) or case (2) applies. Suppose that case (1) does not apply, and we show that case (2) does.

By maximality of \(C_i\) in \(\bigcup_{j \geq i} C_j\) (property (ii) in the definition of a nice clique decomposition), no vertex in \(C_j\) can be adjacent to all vertices in \(C_i\). It follows from this fact and from the inapplicability of Reduction Rule 3.7 that each vertex in \(C_j\) has at most one neighbor in \(C_i\). Since case (1) does not apply, the vertices in \(C_j\) that have a neighbor in \(C_i\) must all have the same neighbor, which proves that case (2) applies.

Now suppose that case (1) applies, and we will show the correctness of the reduction rule in this case. Let \((A, B)\) be any valid monopolar partition of \((G, k)\). Since at most one of \(u', v'\) can be in \(B\), at least one of \(u', v', y\), say \(y\), is in \(A\). Suppose, to get a contradiction, that \(w \in B\). Then both \(u\) and \(v\) must be in \(A\). By maximality of \(C_i\) in \(\bigcup_{j \geq i} C_j\), \(u'\) cannot be adjacent to all vertices in \(C_i\). Since Reduction Rule 3.7 is not applicable, \(u\) must be the only neighbor of \(u'\) in \(C_i\). But then \((v, u, u')\) is an induced \(P_3\) in \(A\), contradicting that \(A\) is a cluster graph.

Suppose that case (2) applies, and suppose to get a contradiction that \(v \in A\) in some valid monopolar partition \((A, B)\) of \((G, k)\). Since there are at least two edges between \(C_i, C_j\), \(v\) has at least two neighbors \(u, v' \in C_i\). Again, observe that at least one of \(u', v', y\), say \(v'\), must be in \(A\). Since \(|C_i| \geq 3\), at least one vertex in \(C_i\), say \(w\), must be in \(A\). Since \(v\) is the only neighbor of \(v'\) in \(C_i\) by the premise of case (2), it follows that \((v, v, v')\) is an induced \(P_3\) in \(A\), contradicting that \(A\) is a cluster graph.

\[\square\]
We can now bound the number of large cliques and edge cliques in yes-instances.

**Reduction Rule 3.9.** Let \((G,k)\) be an instance of **Monopolar Recognition**, and let \((A_{\text{true}}, B_{\text{true}}, C_1, \ldots, C_r)\) be a normalized partition of \(V(G)\). If in \((C_1, \ldots, C_r)\) either the number of large cliques is more than \(k\), or the number of large cliques plus the number of edge cliques is more than \(2k\), then reject the instance \((G,k)\).

**Correctness Proof.** Let \((A,B)\) be any monopolar partition of \(V(G)\). Since a large clique \(C\) has size at least \(3\), at least \(|C| - 1 \geq 2\) vertices from \(C\) must belong to the same cluster in \(A\). By Reduction Rule 3.8, the number of edges between any large clique and any other large or edge clique is at most \(1\). It follows from the aforementioned statements that two vertices from two different large cliques, or from a large clique and an edge clique, must belong to different clusters in \(A\). Consequently, if the number of large cliques in \((C_1, \ldots, C_r)\) is more than \(k\), then for any monopolar partition \((A,B)\) of \(G\), the number of clusters in \(A\) is more than \(k\), and hence \((G,k)\) is a no-instance of **Monopolar Recognition**.

Suppose now that the number of large cliques in \((C_1, \ldots, C_r)\) is \(\ell \leq k\), and that the number of edge cliques is \(\ell'\). From above, for any monopolar partition \((A,B)\), no vertex from an edge clique can belong to a cluster in \(A\) containing a vertex from a large clique. Let \(C_i\) and \(C_j\), \(i < j\), be any two edge cliques. Since \(B\) is an independent set, at least one vertex from each edge clique must be in \(A\). By property (iii) of a nice decomposition, no cluster in \(A\) can contain three vertices from three different edge cliques in \((C_1, \ldots, C_r)\). It follows from the aforementioned two statements that the number of clusters in \(A\) that contain vertices from edge cliques in \((C_1, \ldots, C_r)\) is at least \(\ell'/2\). Now the set of clusters in \(A\) containing vertices from large cliques is disjoint from that containing vertices from edge cliques, and hence the number of clusters in \(A\) is at least \(\ell + \ell'/2\). If the number of large cliques plus the number of edge cliques is more than \(2k\), then \(\ell + \ell' > 2k\), and hence \(\ell + \ell'/2 > \ell/2 + \ell'/2 > k\). This means that for any monopolar partition \((A,B)\) of \(G\), the number of clusters in \(A\) is more than \(k\). It follows that \((G,k)\) is a no-instance of **Monopolar Recognition**.

Next, we sanitize the connections between already determined clusters in \(A_{\text{true}}\) and the remaining cliques in the normalized partition.

**Reduction Rule 3.10.** Let \((A_{\text{true}}, B_{\text{true}}, C_1, \ldots, C_r)\) be a normalized partition of \(V(G)\), let \(C\) be a cluster in \(A_{\text{true}}\), and let \(C_i, i \in [r]\), be a large clique. If \(v \in C_i\) is such that: (1) \(v\) is the only vertex in \(C_i\) that is adjacent to \(C\), or (2) \(v \in \text{the only vertex in } C_i\) that is not adjacent to \(C\), then set \(B_{\text{true}} = B_{\text{true}} \cup \{v\}\).

**Correctness Proof.** To prove the correctness of the reduction rule in case (1) holds, suppose that \(v\) is the only vertex in \(C_i\) that is adjacent to \(C\). Let \((A,B)\) be any monopolar partition of \(G\). Let \(w\) be any vertex in \(C\) that is adjacent to \(v\). Since \(C_i\) is a large clique, there exists a vertex \(u \in C_i\), with \(u \neq v\), such that \(u \in A\). Since \(v\) is the only vertex in \(C_i\) that is adjacent to \(C\), \(u\) is not adjacent to \(w\). Now if \(v\) were in \(A\), then since \(C \subseteq A\) and hence \(w \in A\), \((u,v,w)\) would be an induced \(P_3\) in \(A\), contradicting that \(A\) is a cluster graph. It follows that \(v \in B\) for any monopolar partition \((A,B)\) of \(G\). To prove the correctness of the reduction rule in case (2) holds, suppose that \(v\) is the only vertex in \(C_i\) that is not adjacent to \(C\). Let \((A,B)\) be any monopolar partition of \(G\). Since \(C_i\) is a large clique, there exists a vertex \(u \in C_i\), with \(u \neq v\), such that \(u \in A\). Since \(v\) is the only vertex in \(C_i\) that is not adjacent to \(C\), \(u\) is adjacent to some vertex \(w \in C\). Now if \(v\) were in \(A\), then since \(C \subseteq A\) and hence \(w \in A\), \((v,u,w)\) would be an induced \(P_3\) in \(A\), contradicting that \(A\) is a cluster graph. It follows that \(v \in B\) for any monopolar partition \((A,B)\) of \(G\).

Suppose that none of the above reduction rules applies to the instance \((G,k)\). Then, the following lemma holds:

**Lemma 3.11.** Let \((A_{\text{true}}, B_{\text{true}}, C_1, \ldots, C_r)\) be a normalized partition of \(V(G)\), let \(C\) be a cluster in \(A_{\text{true}}\), and let \(C_i, i \in [r]\), be a large clique such that \(C_i\) is adjacent to \(C\). If \(G\) admits a monopolar partition, then \(C \cup C_i\) induces a clique in \(G\).

**Proof.** Suppose, to get a contradiction, that \(C \cup C_i\) does not induce a clique, and hence, there exists a vertex \(x_i \in C_i\) such that \(x_i\) is not adjacent to some vertex in \(C\). Since \(C\) and \(C_i\) are adjacent, there exist vertices \(y_i \in C_i\) and \(v \in C\) such that \(v \neq y_i\) are adjacent. Since Reduction Rule 3.9 is not applicable, \(x_i\) is not adjacent to any vertex in \(C\), and \(y_i\) is adjacent to every vertex in \(C\). Since cases (1) and (2) of Reduction Rule 3.10 are not applicable, there exist vertices \(y'_i \neq y_i\) and \(x'_i \neq x_i\) in \(C_i\) such that \(y'_i\) is
adjacent to C and \( x'_i \) is not adjacent to C. Since Reduction Rule 3.5 is not applicable, \( y'_i \) is adjacent to every vertex in C. Now for any monopolar partition \((A, B)\) of \( G \), since \( B \) is an independent set, at least one vertex \( w \in \{y_i, y'_i\} \) is in \( A \), and at least one vertex of \( u \in \{x_i, x'_i\} \) is in \( A \). But then \((v, w, u)\) is an induced \( P_3 \) in \( A \), contradicting that \( A \) is a cluster graph. □

The above structure allows us to simplify the instance by shrinking already determined clusters in \( A_{\text{true}} \).

**Reduction Rule 3.12.** Let \((G, k)\) be an instance of Monopolar Recognition, and let the tuple \((A_{\text{true}}, B_{\text{true}}, C_1, \ldots, C_r)\) be a normalized partition of \( V(G) \). If either (1) \( B_{\text{true}} \) contains more than \( k + 1 \) vertices or (2) there exists a cluster in \( A_{\text{true}} \) that is not a singleton, then reduce the instance \((G, k)\) to an instance \((G', k)\) with \( G' \) constructed as follows. Let \( V(G') = V_1 \cup V_2 \cup V_3 \), where \( V_1 = \{u_C \mid C \text{ is a cluster in } A_{\text{true}}\}, V_2 = \{v_1, \ldots, v_{k+1}\}, \) and \( V_3 = C_1 \cup \cdots \cup C_r; \) and \( E(G') = \{uv_C \mid v \in V_2 \land uv_C \in V_1\} \cup \{uv_{C'} \mid v \in V_3 \land uv_{C'} \in V_1 \land v \text{ is adjacent to } C_{r+1}\}. \) That is, \( G' \) is constructed from \( G \) by introducing \( k + 1 \) new vertices, replacing each cluster \( C \) in \( A_{\text{true}} \) (if any) by a single vertex \( u_C \) whose neighborhood is the neighborhood of \( C \) in \( C_1, \ldots, C_r \) plus the \( k + 1 \) new vertices, and keeping \( C_1, \ldots, C_r \) the same.

**Correctness Proof.** To prove the correctness of the reduction rule, we need to show that \((G, k)\) is a yes-instance of Monopolar Recognition if and only if \((G', k)\) is. First, observe that by Reduction Rule 3.3 no vertex in \( C_1 \cup \cdots \cup C_r \) is adjacent to any vertex in \( B_{\text{true}} \).

If \( A_{\text{true}} = \emptyset \), then the reduction rule consists of removing the vertices in \( B_{\text{true}} \) from \( G \), and replacing them with \( k + 1 \) isolated vertices \( v_1, \ldots, v_{k+1} \). Since \( A_{\text{true}} = \emptyset \) and no vertex in \( C_1 \cup \cdots \cup C_r \) is adjacent to any vertex in \( B_{\text{true}} \), the vertices in \( B_{\text{true}} \) are isolated vertices in \( G \). Therefore, the reduction rule in this case consists of replacing the isolated vertices in \( B_{\text{true}} \) with \( k + 1 \) isolated vertices that can be safely added to \( B \), for any valid monopolar partition \((A, B)\) of \( G \). Hence, the reduction rule is obviously correct in this case.

Assume now that \( A_{\text{true}} \neq \emptyset \). It is easy to see that if \((G, k)\) is a yes-instance of Monopolar Recognition then so is \((G', k)\). This can be seen as follows. If \((A, B)\) is a valid monopolar partition of \((G, k)\), then the above reduction rules guarantee that \( A_{\text{true}} \subseteq A \), and hence each cluster of \( A_{\text{true}} \) must be a subset of a single cluster in \( A \). If we (i) remove the vertices in \( B_{\text{true}} \) and add \( k + 1 \) vertices to \( B \) that induce an independent set, and (ii) replace each cluster \( C \) in \( A_{\text{true}} \) by a single vertex \( u_C \) connected to the \( k + 1 \) new vertices in \( B \) and to the vertices of the cluster that \( C \) belongs to \( A \), we still get a valid monopolar partition of \( G \).

To prove the converse, suppose that \((G', k)\) is a yes-instance of Monopolar Recognition, and let \((A', B')\) be a valid monopolar partition of \( V(G') \). Since \((A', B')\) is a valid monopolar partition of \( V(G') \), and every vertex \( u_{C'} \), \( C \) is a cluster in \( A_{\text{true}} \), is adjacent to the \( k + 1 \) independent vertices \( v_1, \ldots, v_{k+1} \), we have \( u_C \in A' \) for every cluster \( C \) in \( A_{\text{true}} \), and \( \{v_1, \ldots, v_{k+1}\} \subseteq B' \). Let \( B = B' \setminus \{v_1, \ldots, v_{k+1}\} \cup B_{\text{true}} \). Since (1) \( B' \) induces an independent set, (2) every vertex \( u_{C'} \), \( C \) is a cluster in \( A_{\text{true}} \), is in \( A' \), and (3) no vertex in \( C_1 \cup \cdots \cup C_r \) is adjacent to any vertex in \( B_{\text{true}} \), it follows that \( B \) is an independent set. Let \( A \) be the set of vertices obtained from \( A' \) by replacing each vertex \( u_C \) by the vertices in the cluster \( C \) in \( A_{\text{true}} \). We claim that \( A \) is a cluster graph with at most \( k \) clusters. Suppose that a vertex \( u_{C'} \) is replaced in \( A' \) by the vertices in cluster \( C \) in \( A_{\text{true}} \); assume that \( u_C \) belongs to cluster \( C' \) in \( A' \). Each vertex in \( C' \), other than \( u_C \), must be a vertex in \( V_3 = C_1 \cup \cdots \cup C_r \). Let \( v' \in C' \setminus \{u_C\} \) be chosen arbitrarily. Since \( v' \) and \( u_C \) belong to the same cluster \( C' \), by definition of \( G' \), \( v' \) must be adjacent to \( C \) in \( G \). By Reduction Rule 3.5, \( v' \) must be adjacent to all vertices in \( C \). Since \( v' \) was an arbitrarily chosen vertex in \( C' \setminus \{u_C\}, (C' \setminus \{u_C\}) \cup C \) induces a cluster in \( A \). It remains to show that no two clusters in \( A \) are adjacent. Suppose, to get a contradiction, that this is not the case. Since \( A' \) induces a cluster graph, there must exist two vertices \( u_{C_1} \) and \( u_{C_2} \) in \( A' \), that belong to clusters \( C_1' \) and \( C_2' \) in \( A' \), respectively, such that cluster \( C_1' \cup \{u_{C_1}\} \) is adjacent to cluster \( C_2' \cup \{u_{C_2}\} \). Since \( A' \) is a cluster graph, this implies that either: (1) \( C_1 \) is adjacent to \( C_2 \), (2) \( C_1 \) is adjacent to \( C_2' \setminus \{u_{C_2}\} \), or (3) \( C_2 \) is adjacent to \( C_1' \setminus \{u_{C_1}\} \). This leads to a contradiction in each of the three cases above: (1) would contradict that \( A_{\text{true}} \) is a cluster graph (Reduction Rule 3.3), (2) would imply that \( u_{C_1} \), and hence \( C_1 \), is adjacent to \( C_2' \) in \( A' \), and (3) would imply that \( u_{C_2} \), and hence \( C_2 \), is adjacent to \( C_1' \) in \( A' \). It follows from the above that the constructed partition \((A, B)\) is a valid monopolar partition for \( V(G) \). Finally, the number of clusters in \( A \) is the same as that in \( A' \), which is at most \( k \). □

If Reduction Rule 3.12 is applied, then after its application, we set \( A_{\text{true}} \) to \( V_1 \) and \( B_{\text{true}} \) to \( \{v_1, \ldots, v_{k+1}\} \). Note that in any valid monopolar partition \((A, B)\) of the graph resulting from the application of Reduction Rule 3.12, \( A \) must contain at least one vertex of degree 0.
duction Rule 3.12 each vertex in $V_1$ must be in $A$, being adjacent to the $k+1$ independent set vertices $v_1, \ldots, v_{k+1}$, whereas the vertices $v_1, \ldots, v_{k+1}$ can be safely assumed to be in $B$ since their only neighbors are in $V_1 \subseteq A$.

### 3.3 Modelling the Pushing Process by a Bipartite Graph

We have now arrived at a stage where we have bounded the number of large and edge cliques, and the size of $A_{true}$ and $B_{true}$. It remains to bound the size of the large cliques and the number of vertex cliques to obtain a polynomial-size problem kernel. The challenge here is that we need to identify vertices such that putting them in $A$ or $B$ will eventually, after a series of pushes, lead either to the creation of too many clusters in $A$, or to the addition of two adjacent vertices in $B$. To model the avalanche of pushes to $A$ or $B$, we introduce the following auxiliary graph.

**Definition 3.13.** For a normalized partition $(A_{true}, B_{true}, C_1, \ldots, C_r)$ of $V(G)$, we define the auxiliary bipartite graph $\Lambda$ as follows. The vertex set of $\Lambda$ is $V(\Lambda) = (V_C, V_I)$, where $V_C$ is the set of all vertices in the large cliques in $C_1, \ldots, C_r$, and $V_I$ is the set of all vertices in the vertex cliques in $C_1, \ldots, C_r$. The edge set of $\Lambda$ is $E(\Lambda) = \{uv \in E(G) \mid u \in V_C$ and $v \in V_I\}$; that is, $E(\Lambda)$ consists of precisely the edges in $E(G)$ that are between $V_C$ and $V_I$.

Recall that $V_I$ is an independent set in $G$ by Fact 3.1. For a vertex $v \in V(\Lambda)$, we write $N_\Lambda(v) := N(v) \cap V(\Lambda)$ for the neighbors of $v$ in $\Lambda$. We have the following lemma:

**Lemma 3.14.** Let $(A_{true}, B_{true}, C_1, \ldots, C_r)$ be a normalized partition of $V(G)$ and consider the auxiliary graph $\Lambda = (V(\Lambda) = (V_C, V_I), E(\Lambda))$. Then the maximum degree of $\Lambda$, $\Delta(\Lambda)$, is at most $k$.

**Proof.** For every vertex $v \in V_C$, we have $|N_\Lambda(v)| \leq k$ because Reduction Rule 3.6 is inapplicable. By property (ii) of a nice decomposition and the inapplicability of Reduction Rule 3.7, every vertex clique that is adjacent to a large clique $C$ is adjacent to exactly one vertex in $C$. Since by Reduction Rule 3.9 the number of large cliques is at most $k$, every vertex in $V_I$, which is a vertex clique by definition of $V_I$, has at most $k$ neighbors in $V_C$. Therefore, for every vertex $v \in V_I$, we have $|N_\Lambda(v)| \leq k$.

Using the following lemma, we now observe that the auxiliary graph $\Lambda$ captures some of the avalanches emanating from vertices in large or vertex cliques. Namely, pushing a vertex $v$ in a large clique to $B$ (or in a vertex clique to $A$) will also require pushing each vertex reachable (in the auxiliary graph) from $v$ from $A$ to $B$ or vice versa.

For two vertices $u, v \in V(\Lambda)$, write $\text{dist}_\Lambda(u, v)$ for the length of a shortest path between $u$ and $v$ in $\Lambda$. For a vertex $v \in V(\Lambda)$ and $i \in \{0, \ldots, n\}$, define $N^i(v) = \{u \in V(\Lambda) \mid \text{dist}_\Lambda(u, v) = i\}$. Write $\overline{\theta}_n$ for the set of even integers in $\{0, \ldots, n\}$, and $\overline{\eta}_n$ for the set of odd integers in $\{0, \ldots, n\}$.

**Lemma 3.15.** Let $(A_{true}, B_{true}, C_1, \ldots, C_r)$ be a normalized partition of $V(G)$, let $\Lambda = (V(\Lambda), E(\Lambda))$ be the associated auxiliary graph where $V(\Lambda) = (V_C, V_I)$, and let $(A, B)$ be any valid monopolar partition of $G$.

(i) For any vertex $v \in V_C$: If $v \in B$ then $N_\Lambda(v) \subseteq A$.

(ii) For any vertex $v \in V_I$: If $v \in A$ then $N_\Lambda(v) \subseteq B$.

(iii) For any vertex $v \in V_C$: If $v \in B$ then $N^i_\Lambda(v) \subseteq B$ for $i \in \overline{\theta}_n$, and $N^i_\Lambda(v) \subseteq A$ for $i \in \overline{\eta}_n$.

(iv) For any vertex $v \in V_I$: If $v \in A$ then $N^i_\Lambda(v) \subseteq A$ for $i \in \overline{\theta}_n$, and $N^i_\Lambda(v) \subseteq B$ for $i \in \overline{\eta}_n$.

**Proof.** (i): This trivially follows because $B$ is an independent set.

(ii): Suppose that $v \in V_I$ is in $A$, and let $u \in N_\Lambda(v)$. Then $u \in V_C$ because $\Lambda$ is bipartite, and hence, by definition, $u$ belongs to a large clique $C_i$ for some $i \in [r]$. Suppose, to get a contradiction, that $u \in A$. Since $C_i$ is a large clique, and hence $|C_i| \geq 3$, there exists a vertex $w \not\in u$ in $C_i$ such that $w \in A$. By property (ii) of the nice decomposition $(C_1, \ldots, C_r)$ and the inapplicability of Reduction Rule 3.7, $v$ is not a neighbor of $w$ in $G$. But this implies that $(v, u, w)$ is an induced $P_3$ in $A$, contradicting that $A$ is a cluster graph. It follows that $N_\Lambda(v) \subseteq B$.

(iii): This follows by repeated alternating applications of (i) and (ii) above.

(iv): This follows by repeated alternating applications of (ii) and (i) above.

The above lemma about the avalanches captured by the auxiliary graph allows us to identify vertices whose push to one side of the partition would lead to avalanches that, in turn, would lead to too many clusters in $A$ or to two adjacent vertices in $B$. We can hence fix them in the corresponding part.
Reduction Rule 3.16. Let \((A_{\text{true}}, B_{\text{true}}, C_1, \ldots, C_r)\) be a normalized partition of \(V(G)\), and let \(\Lambda = (V(\Lambda) = (V_C, V_I), E(\Lambda))\) be the associated auxiliary graph.

(i) For any vertex \(v \in V_C\): If either \(\bigcup_{i \in \Lambda} N^*_A(v)\) contains two adjacent (in \(G\)) vertices or \(|\bigcup_{i \in \Lambda, V} N^*_A(v)| > k\), then set \(A_{\text{true}} = A_{\text{true}} \cup \{v\}\).

(ii) For any vertex \(v \in V_I\): If either \(|\bigcup_{i \in \Lambda, V} N^*_A(v)| > k\) or \(\bigcup_{i \in \Lambda, V} N^*_A(v)\) contains two adjacent (in \(G\)) vertices, then set \(B_{\text{true}} = B_{\text{true}} \cup \{v\}\).

Correctness Proof. (i) Let \(v \in V_C\), and suppose that either \(\bigcup_{i \in \Lambda} N^*_A(v)\) contains two adjacent vertices or \(|\bigcup_{i \in \Lambda, V} N^*_A(v)| > k\). If \(v \in B\) for any valid partition \((A, B)\) of \(G\), then by part (iii) of Lemma 3.15 it would follow that \(\bigcup_{i \in \Lambda} N^*_A(v) \subseteq B\) and \(\bigcup_{i \in \Lambda, V} N^*_A(v) \subseteq A\). In either case this contradicts that \((A, B)\) is valid partition of \(G\): If \(\bigcup_{i \in \Lambda} N^*_A(v)\) contains two adjacent vertices, then \(B\) is not an independent set, and if \(|\bigcup_{i \in \Lambda, V} N^*_A(v)| > k\) then \(A\) contains more than \(k\) clusters since \(\bigcup_{i \in \Lambda, V} N^*_A(v)\) induces an independent set in \(G\).

(ii) Let \(v \in V_I\), and suppose that either \(|\bigcup_{i \in \Lambda} N^*_A(v)| > k\) or \(\bigcup_{i \in \Lambda, V} N^*_A(v)\) contains two adjacent vertices. If \(v \in A\) for any valid partition \((A, B)\) of \(G\), then by part (iv) of Lemma 3.15 it would follow that \(\bigcup_{i \in \Lambda} N^*_A(v) \subseteq A\) and \(\bigcup_{i \in \Lambda, V} N^*_A(v) \subseteq B\). In either case this contrast that \((A, B)\) is valid partition of \(G\): If \(|\bigcup_{i \in \Lambda} N^*_A(v)| > k\) then \(A\) contains at more than \(k\) clusters since the vertices in \(\bigcup_{i \in \Lambda} N^*_A(v)\) induce an independent set in \(G\), and if \(|\bigcup_{i \in \Lambda, V} N^*_A(v)| > k\) then \(B\) contains two adjacent vertices, then \(B\) is not an independent set.

We are now ready to define a set of representative vertices which already capture the remaining structure of avalanches in the instance.

Definition 3.17. Let \((A_{\text{true}}, B_{\text{true}}, C_1, \ldots, C_r)\) be a normalized partition of \(V(G)\), and let \(\Lambda = (V(\Lambda) = (V_C, V_I), E(\Lambda))\) be the associated auxiliary graph. From each large clique \(C_i, i \in [r]\), fix three vertices \(u_i, v_i, w_i\); define \(V_{\text{fixed}} = \{u_i, v_i, w_i | C_i \text{ is a large clique}\}\) to be the set of all fixed vertices. Define \(V_{\text{edge}} = \{C_i | C_i \text{ is an edge clique}\}\) to be the set of vertices of the edge cliques, define \(N_{\text{edge}} = N(V_{\text{edge}}) \cap V(\Lambda)\) to be the neighbors of \(V_{\text{edge}}\) in \(V(\Lambda)\), and define \(N_{\text{edge}}^\cup = \bigcup_{v \in N_{\text{edge}}} N^*_A(v)\) to be the set of all vertices in \(V(\Lambda)\) that are reachable in \(\Lambda\) from the vertices in \(N_{\text{edge}}\). (Note that \(N_{\text{edge}} \subseteq N_{\text{edge}}^\cup\).) Define \(V_{\text{inter}} = \{u, v | u \in C_i, v \in C_j, i \neq j \land uv \in E(G) \land (C_i, C_j \text{ are large cliques})\}\) to be the set of endpoints of edges between large cliques, and define \(N_{\text{inter}}^\cup = \bigcup_{v \in V_{\text{inter}}} N^*_A(v)\) to be the set of all vertices in \(V(\Lambda)\) that are reachable in \(\Lambda\) from the vertices in \(V_{\text{inter}}\). (Note that \(V_{\text{inter}} \subseteq N_{\text{inter}}^\cup\).) Finally, let \(V_{\text{rep}} = A_{\text{true}} \cup B_{\text{true}} \cup V_{\text{fixed}} \cup N_{\text{inter}}^\cup \cup V_{\text{edge}} \cup N_{\text{edge}}^\cup\).

The next reduction rule shrinks the instance to the set of representative vertices defined above.

Reduction Rule 3.18. Let \((G, k)\) be an instance of MONOPOLAR RECOGNITION, and let the tuple \((A_{\text{true}}, B_{\text{true}}, C_1, \ldots, C_r)\) be a normalized partition of \(V(G)\). Let \(V_{\text{rep}}\) be as defined in Definition 3.17. Set \(G = G[V_{\text{rep}}]\).

Correctness Proof. To prove the correctness of the reduction rule, let \(G' = G[V_{\text{rep}}]\) and we need to show that the two instances \((G, k)\) and \((G', k)\) of MONOPOLAR RECOGNITION are equivalent. Since \(G'\) is a subgraph of \(G\) and the property of having a valid monopolar partition is a hereditary graph property, it follows that if \((G, k)\) is a yes-instance of MONOPOLAR RECOGNITION then so is \((G', k)\). Therefore, it suffices to show the converse, namely that if \((G', k)\) is a yes-instance of MONOPOLAR RECOGNITION then so is \((G, k)\).

Suppose that \((G', k)\) is a yes-instance of MONOPOLAR RECOGNITION, and let \((A, B)\) be a valid monopolar partition of \((G', k)\). Let \((A_{\text{true}}, B_{\text{true}}, C_1, \ldots, C_r)\) be the normalized partition of \(V(G)\) with respect to which \(V_{\text{rep}}\), and hence \(G' = G[V_{\text{rep}}]\), were defined, and let \(V_{\text{fixed}}, V_{\text{inter}}, N_{\text{inter}}^\cup, V_{\text{edge}}, N_{\text{edge}}^\cup\) be as in Definition 3.17. Let \(v\) be an arbitrary vertex in \(V(G) \setminus V(G')\). It suffices to show that \(G[V(G') \cup \{v\}]\) has a valid monopolar partition, as we can repeatedly add vertices, one after the other, and the same proof applies. Since the set of vertices forming the edge cliques, \(V_{\text{edge}}\), is a subset of \(V(G')\), and \(A_{\text{true}} \cup B_{\text{true}} \subseteq V(G')\), vertex \(v\) is either a vertex of a large clique of \(G\) that is not in \(V_{\text{fixed}}\), or \(v\) is a vertex clique in \(G\).

We distinguish these two cases.

Case 1. \(v \in V(C_i) \setminus V_{\text{fixed}}\) for some large clique \(C_i\), where \(i \in [r]\). Since three vertices from \(C_i\) are in \(V_{\text{fixed}}\), at least two of these vertices must belong to a cluster, say \(C'_i\), in part \(A\) of the partition \((A, B)\).

Note that since \(A_{\text{true}} \subseteq A\), if \(C_i\) has a neighbor in \(A_{\text{true}}\), which must be a neighbor of all the vertices in \(C_i\),
including \( v \), by Lemma 3.11, then this neighbor must be in \( C'_i \). We first claim that \( C'_i \cup \{ v \} \) is a clique. Observe that since \( C'_i \) contains two vertices from \( V_{\text{fixed}} \), and hence from \( C_i \), by Reduction Rule 3.5, \( C'_i \) does not contain any vertices from a large clique other than \( C_i \) or from an edge clique. Moreover, by property (ii) of the nice decomposition \((C_1, \ldots, C_r)\) and Reduction Rule 3.7, \( C'_i \) does not contain any vertex clique. Therefore, \( C'_i \) consists only of vertices in \( C_i \), plus possibly a single vertex in \( A_{\text{true}} \) that must be adjacent to all the vertices in \( C_i \). Since \( v \in C_i \), it follows that \( C'_i \cup \{ v \} \) is a clique.

Let \( S \) be the set of vertex cliques in \( A \), and note that \( S \) is an independent set. Define the following layered structure. The root of this structure is \( v \). The first layer contains the set of vertices (possibly empty), denoted \( N_i(v) \), that are the neighbors of \( v \) in \( S \), that is, \( N_i(v) = N(v) \cap S \); and the second layer contains the set of vertices, denoted \( N_2(v) \), that are the neighbors in \( B \) of the vertices of \( N_i(v) \), that is \( N_2(v) = N(N_i(v)) \cap B \). For \( i > 2 \), layer \( i \) contains the set of vertices \( N_i(v) = N(N_{i-1}(v)) \cap S \) if \( i \) is odd, and the set of vertices \( N_i(v) = N(N_{i-1}(v)) \cap B \) if \( i \) is even. Let \( N_0(v) = \{ v \} \). We claim that the partition \((A', B')\) obtained from \((A, B)\) by placing \( v \) in \( A \), moving the vertices in \( N_i(v) \) for odd \( i \) from \( A \) to \( B \), is a valid monopolar partition; that is, \((A', B')\), where \( A' = (A \cup \bigcup_{i \in 0} N_i(v)) \setminus \bigcup_{i \in 1} N_i(v) \) and \( B' = (B \cup \bigcup_{i \in 1} N_i(v)) \setminus \bigcup_{i \in 0} N_i(v) \) is a valid monopolar partition of \( G[V(G') \cup \{ v \}] \). Since \( S \) is an independent set, so is \( \bigcup_{i \in 1} N_i(v) \subseteq S \). Since the set of neighbors of \( \bigcup_{i \in 1} N_i(v) \) is precisely \( \bigcup_{i \in 0} N_i(v) \) and \( B \) is an independent set, \( B' = (B \cup \bigcup_{i \in 1} N_i(v)) \setminus \bigcup_{i \in 0} N_i(v) \) is an independent set as well. Therefore, to show that \((A', B')\) is a valid monopolar partition, it suffices to show that \( A' \) is the cluster graph of at most \( k \) clusters.

First, we claim that each vertex in \( N_{\text{even}} = \bigcup_{i \in 0} N_i(v) \) belongs to a large clique in \( C_1, \ldots, C_r \). This is certainly true for the vertex \( v \), which is in \( C_i \), where \( C_i \) is a large clique. Now for any other vertex \( u \in N_{\text{even}} \), by construction, \( u \) is the neighbor of a vertex clique in \( S \). Since the set of all vertex cliques induces an independent set, \( u \) itself cannot be a vertex clique, being a neighbor of a vertex clique. Since \( u \in N_i(v) \), for some \( i \), and hence \( v \) is reachable from \( u \), \( u \) cannot be an endpoint of an edge clique; otherwise, \( v \) would belong to \( N_{\text{edge}} \) and hence, would belong to \( V_{\text{rep}} \). Since \( A_{\text{true}} \subseteq A \), and no vertex in \( V \setminus (A_{\text{true}} \cup B_{\text{true}}) \) is adjacent to a vertex in \( B_{\text{true}} \) by Reduction Rule 3.4, \( u \notin A_{\text{true}} \cup B_{\text{true}} \). It follows from the preceding that each vertex in \( N_{\text{even}} \) belongs to a large clique in \( C_1, \ldots, C_r \). From each large clique in \( C_1, \ldots, C_r \), at least two fixed vertices are in \( A' \); denote by \( C'_j \) the cluster in \( A' \) that contains the two fixed vertices from a large clique \( C_j \). As shown at the beginning of Case 1 about \( C'_j \), the same holds true for any \( C'_j \); \( C'_j \) consists of a subset of \( C_j \), plus possibly a vertex in \( A_{\text{true}} \) that is adjacent to all vertices in \( C_j \). Now add each vertex in \( N_{\text{even}} \) that belongs to a (large clique) \( C_j \) in \( G \) to the corresponding cluster \( C'_j \) in \( A' \). We claim that the resulting partition is a valid monopolar partition. Since each vertex \( u \) in \( N_{\text{even}} \) was added to the cluster \( C'_j \) such that \( u \in C_j \) and \( C'_j \) consists of a subset of \( C_j \) plus possibly a vertex in \( A_{\text{true}} \) that is adjacent to all vertices in \( C_j \), \( C'_j \cup \{ u \} \) is a clique. Moreover, since each vertex \( u \in N_{\text{even}} \) was added to an existing cluster, this addition does not increase the number of clusters in \( A \), and hence, \( A' \) has at most \( k \) clusters. It remains to show that this addition does not create an edge between two different clusters. Suppose that this is not the case. Since \( N_{\text{even}} \subseteq B \) is an independent set, this implies that there exists a vertex \( u \in N_{\text{even}} \) that is added to a cluster \( C'_j \) in \( A \) such that \( u \) is adjacent to some vertex \( w \) in \( A \). Since all the neighbors of \( u \) that are vertex cliques are in \( \bigcup_{i \in 1} N_i(v) \subseteq B' \), \( w \) is not a vertex clique. Since \( v \) is reachable from \( u \), and hence from \( w \), and \( v \notin N_{\text{edge}} \), \( w \) cannot be a vertex of an edge clique. By the same token, since \( v \) is reachable from \( w \) and \( v \notin N_{\text{inter}} \), \( w \) cannot be a vertex of a large clique. Finally, \( w \) cannot be in \( B_{\text{true}} \) because no vertex in \( V(G) \setminus (A_{\text{true}} \cup B_{\text{true}}) \) is adjacent to a vertex in \( B_{\text{true}} \), and \( w \) cannot be in \( A_{\text{true}} \) because \( w \) would be adjacent to all vertices of \( C'_j \). This completes the proof of Case 1.

Case 2. \( v \) is a vertex clique. The treatment of this case is very similar to Case 1. We define \( N_0(v) = \{ v \} \), \( N_i(v) = N(N_{i-1}(v)) \cap B \) if \( i \geq 1 \) is odd, and \( N_i(v) = N(N_{i-1}(v)) \cap S \) if \( i \geq 2 \) is even, where \( S \) is the set of vertex cliques in \( A \). It can then be shown—using very similar arguments to those made in Case 1—that the partition \((A', B')\), where \( A' = (A \cup \bigcup_{i \in 1} N_i(v)) \setminus \bigcup_{i \in 0} N_i(v) \) and \( B' = (B \cup \bigcup_{i \in 0} N_i(v)) \setminus \bigcup_{i \in 1} N_i(v) \) is a valid monopolar partition of \( G[V(G') \cup \{ v \}] \). The proof is omitted to avoid repetition. \( \square \)

We now give the polynomial kernel whose existence was promised in Theorem 1.1.

**Theorem 3.19.** Monopolar Recognition has a kernel of size at most \( 9k^4 + 9k + 1 \) which can be computed in \( O(n^2m) \) time.

**Proof.** Given an instance \((G, k)\) of Monopolar Recognition, we apply Reduction Rules 3.3–3.18 exhaustively to \((G, k)\). Clearly, the above rules can be applied in polynomial time. Let \((G', k')\) be the
resulting instance, let \( (A_{\text{true}}, B_{\text{true}}, C_1, \ldots, C_r) \) be a normalized partition of \( V(G') \) with respect to which none of Reduction Rules 3.3–3.18 applies, and let \( \Lambda = (V(\Lambda) = (V_C, V_I), E(\Lambda)) \) be the auxiliary graph. Note that, by Reduction Rule 3.18, \( V(G') = V_{\text{rep}} = A_{\text{true}} \cup B_{\text{true}} \cup V_{\text{fixed}} \cup \Lambda_{\text{inter}} \cup \Lambda_{\text{edge}} \cup \Lambda_{\text{edge}}^\prime \). By Reduction Rule 3.9, the number of large cliques is at most \( k \), and the number of edge cliques is at most \( 2k \). It follows that \( |V_{\text{fixed}}| \leq 3k \) and \( |\Lambda_{\text{edge}}| \leq 4k \). For a vertex \( v \in V_{\text{edge}} \), by Reduction Rule 3.9, \( v \) has at most \( k \) neighbors in \( V_I \). Moreover, by Reduction Rule 3.8, \( v \) can have at most \( k \) neighbors in \( V_C \), and therefore, \( |N_\Lambda(v)| \leq 2k \), and \( |\Lambda_{\text{edge}}| \leq 4k \cdot 2k = 8k^2 \). Since Reduction Rule 3.16 does not apply and \( \Delta(\Lambda) \leq k \) by Lemma 3.14, we have that, for any \( v \in V(\Lambda) \), we have \( \sum_{i \in N_\Lambda(v)} |N_i^{\Lambda}(v)| \leq \Delta(\Lambda) \cdot k \leq k^2 \). This implies that \( |\Lambda_{\text{edge}}| \leq 8k^2 \cdot k^2 \leq 8k^4 \). Now since the number of large cliques is at most \( k \), by Reduction Rule 3.9, it follows that \( |\Lambda_{\text{inter}}| \leq \left\lfloor \frac{k^2}{2} \right\rfloor < k^2 \). Since for a vertex \( v \in V(\Lambda) \) we have \( \sum_{i \in N_\Lambda(v)} |N_i^{\Lambda}(v)| \leq k^2 \) as argued above, it follows that \( |\Lambda_{\text{inter}}| \leq k^4 \). Since \( |A_{\text{true}}| \leq k \) and \( |B_{\text{true}}| \leq k+1 \), putting everything together, we conclude that the number of vertices in \( V(G') \), \( |V_{\text{rep}}| \), is at most \( k + k + 1 + 3k + k^4 + 4k + 8k^4 \leq 9k^4 + 9k + 1 \).

It remains to show the running time. First, observe that the overall number of applications of the reduction rules is \( O(n) \), since each application either moves a vertex to \( A_{\text{true}} \) or \( B_{\text{true}} \), or reduces the number of vertices in \( G \). To obtain the overall running time bound, we first bound the time to check the applicability of each reduction rule.

For Reduction Rules 3.3–3.6 it is obvious that their applicability can be checked in \( O(m) \) time (recall that we assume \( n \in O(m) \)).

For Reduction Rule 3.7, its applicability can be checked in \( O(m) \) time, if we assign to each vertex a label indicating the number of its clique and an additional “counter-variable” for each cluster. Then, we consider the vertices of the clique decomposition one by one. When considering a vertex \( v \), we reset all clique counters to 0. Then we scan through the adjacency list of \( v \), and increase the counter of a clique \( C_i \) for each edge between \( v \) and \( C_i \) (the cluster for each edge can be checked in \( O(1) \) time using the clique labels of the vertices). After scanning through the adjacency list, we check for each clique \( C_i \) that \( v \) is adjacent to, if the number of edges between \( v \) and \( C_i \) and the size of \( C_i \) meet the conditions in Reduction Rule 3.7.

For Reduction Rule 3.8, we create once in \( O(n^2) \) time an \( n' \times n' \) matrix \( M \) where \( n' \) is the number of large and edge cliques. Entry \( M[i,j] = M[j,i] \) is used to count the number of edges between the \( i \)th large or edge clique \( C_i \) and the \( j \)th large or edge clique \( C_j \). Before and after we test the applicability of the rule, \( M[i,j] = 0 \) for all \( i, j \in [n'] \). To test applicability, we scan through a list containing each edge of \( G \) exactly once and increment \( M[i,j] \) each time we encounter an edge between \( C_i \) and \( C_j \). If at some point in this procedure \( M[i,j] = 2 \) for some \( i \) and \( j \), then the rule applies. After the check, we reset \( M \) to 0 by keeping a list of all pairs of modified matrix indices.

It is clear that we can check in \( O(m) \) time whether Reduction Rule 3.9 applies.

Reduction Rule 3.10 can be checked in a similar manner to Reduction Rule 3.8. We use a \( n_1 \times n_2 \) matrix \( M' \) where \( n_1 \) is the number of large cliques in the current normalized partition and \( n_2 \) is the number of clusters in \( A_{\text{true}} \). We use entry \( M'[i,j] \) to count the number of vertices in large clique \( C_i \) adjacent to the \( j \)th cluster \( D_j \) in \( A_{\text{true}} \). After a one-time \( O(n^2) \) initialization, we will ensure that before and after the test of applicability \( M'[i,j] = 0 \) for all \( i \in [n_1], j \in [n_2] \). Additionally, we use a vertex labeling for all vertices in \( G \), which we initialize for every vertex as \( \text{unlabeled} \). We iterate in \( O(m) \) time over the list of edges in \( G \) and whenever we encounter an edge such that one endpoint \( v \) of which is unlabeled and in large clique \( C_i \), and the other endpoint is in the \( j \)th cluster in \( A_{\text{true}} \), then we increment \( M[i,j] \) and remove the labeling from \( v \). If, after processing some edge, \( M[i,j] \) now equals 1, or \( |C_i| - 1 \), then we label \( C_i \) as \( \text{amenable} \) and otherwise remove the amenable-label from \( C_i \) (if any).

Reduction Rule 3.10 applies if and only if a large clique is amenable after processing each edge. As before, after the check for applicability, we restore all entries \( M[i,j] = 0 \) by tracking the pairs of indices which changed during the applicability test.

Reduction Rule 3.12 can obviously be checked in \( O(m) \) time. For the remaining reduction rules, it is necessary to compute the auxiliary bipartite graph \( \Lambda \) which can be done in \( O(m) \) time by iterating over all edges and checking whether they are incident with a large clique or vertex clique. To check whether Reduction Rule 3.16 applies, it is enough to compute the connected components of \( \Lambda \), and compute for each component the size of each part and the subgraph of \( G \) that is induced by each part. This can clearly be done in \( O(m) \) time. For Reduction Rule 3.18, we first need to compute \( V_{\text{rep}} \) in \( O(m) \) time—we iterate over all edges and check whether one of the corresponding conditions applies to the endpoints—and then compute the subgraph \( G[V_{\text{rep}}] \) also in \( O(m) \) time.
The time to perform each reduction rule is $O(m)$, plus the time needed to update the clique decomposition. We update the clique decomposition $O(n)$ times, by Lemma 3.2, this takes $O(nm)$ time. Thus, the latter step has a total running time of $O(n^2m)$, which gives the overall running time for computing the kernel.

4 Kernel-size lower bound

This section is dedicated to proving the “only if” direction of Theorem 1.1, which completes its proof together with Theorem 3.19. More precisely, we prove the following:

**Theorem 4.1.** Let $\Pi$ be a graph property characterized by a (not necessarily finite) set $H$ of connected forbidden induced subgraphs, each of order at least 3. Then unless $\text{NP} \subseteq \text{coNP}/\text{poly}$, Cluster-$\Pi$-Partition parameterized by the number $k$ of clusters in the cluster graph $G[A]$ does not admit a polynomial kernel.

Throughout, let $\Pi$ be any graph property satisfying the conditions of Theorem 4.1. We show Theorem 4.1 by giving a cross-composition from the NP-hard problem Colorful Independent Set [15], defined below.

**Colorful Independent Set**

**Input:** A graph $G = (V, E)$, $k \in \mathbb{N}$, and a proper $k$-coloring $c : V \rightarrow \{1, \ldots, k\}$.

**Question:** Is there an independent set with $k$ vertices in $G$ that contains exactly one vertex of each color?

In the remainder of this section, we explain the construction behind the cross-composition and prove its correctness. We start by describing the intuition behind the construction, and why the avalanches in the case of properties $\Pi$ as above cannot be contained.

In contrast to Monopolar Recognition, the avalanches caused by the pushing process for the general Cluster-$\Pi$-Partition problem are much more uncontrollable: If some push to the $B$-side $B$ creates a forbidden induced subgraph $M$ for $\Pi$ in $G[B]$, we can repair the partition and “break” $M$ by moving a vertex of $M$ to the cluster graph side $A$. However, each move of a vertex in $M$ may lead—through further necessary pushes from $A$ to $B$—to distinct forbidden induced subgraphs in $G[B]$, again with multiple possible ways of breaking them in order to repair the partition. These avalanches cannot be contained, and lead to many possible paths along which they can be repaired, which can be modeled using a tree-like structure.

It is precisely the above-described behavior of avalanches that we exploit to obtain a cross-composition: The main gadgets select a Colorful Independent Set instance and independent-set vertices within that instance. Each such selection gadget has a trivial cluster-$\Pi$ partition with one caveat: It has one (singleton) cluster too many in $G[A]$, and only this vertex can be pushed into the $\Pi$-side $B$. We call this vertex the activator vertex of the gadget. Pushing the activator vertex into $B$ creates a forbidden induced subgraph for $\Pi$, requiring further pushes that propagate along a root-leaf path in a binary-tree-like structure. In the end, exactly one vertex corresponding to a leaf in this structure will be pushed from $A$ to $B$, transmitting the choice to further gadgets.

Next, in Section 4.1, we give some definitions, scaffolds for the construction, and operations that we need. Then, in Sections 4.2 and 4.3, we show how to construct a selection gadget and use it to create instance-selection and vertex-selection gadgets. Finally, in Section 4.4 we construct verification gadgets that ensure that the selected vertices in the selected instance form an independent set.

4.1 Setup

Let $t$ instances of Colorful Independent Set be given, with graphs $G_1, \ldots, G_t$, respectively. Below, we use an instance and its index in $[t]$ interchangeably. Without loss of generality, assume that the following properties hold; they can be achieved by simple padding techniques:

- Each instance asks for an independent set of size $k$ (otherwise, introduce new colors and isolated vertices as needed);
- each color class in each graph has $n$ vertices and $n$ is a power of two (otherwise, add universal vertices as needed); and
We construct an instance of Cluster-II-Partition as described below. The instance consists of the graph $G$ and asks for a cluster-II partition $(A, B)$ with at most $d$ clusters in $G[A]$ (we specify $d$ below). The graph $G$ is constructed by first adding $d$ vertices which we call anchors (see below). The clusters in any cluster-II partition $(A, B)$ of $G$ with $d$ clusters in $G[A]$ will extend these anchor vertices into larger cliques. We then successively add gadgets that are attached to these anchors. We first construct an instance-selection gadget that selects one of the given $t$ instances. Then we add a vertex-selection gadget for each instance which selects $k$ vertices in its corresponding instance if it has been selected. Finally, we add verification gadgets that ensure that the selected vertices are pairwise nonadjacent in the graph of the selected instance.

Throughout, we use the following notation. We denote by $(A, B)$ an arbitrary fixed cluster-II partition of $G$. We fix $M$ to be a forbidden induced subgraph of $\Pi$ with minimum number of vertices. By the properties of $\Pi$, $M$ contains at least three vertices. The vertices that we introduce will be in three disjoint categories: helper vertices, dial vertices, and volatile vertices. Their meaning is as follows. Helper vertices will always be contained in $B$ and only serve to impose certain properties on other vertices. Dial vertices are normally in $A$ and belong to a cluster extending around an anchor; some of these vertices may be pushed to $B$ by an avalanche. On the other hand, volatile vertices are normally in $B$ and may be pushed to $A$ by an avalanche.

As mentioned, the construction begins by adding anchor vertices. We introduce $d$ anchor vertices, divided into $5+2k$ groups: $a_1^d, a_2^d, \ldots, a_{2\log t}^d$, $a_{2\log t+1}^d, \ldots, a_{k+1}^d$, for each $i \in [k]$, $a_{3^i+1}^d, \ldots, a_{3^{i+1}}^d$; for each $i \in [k]$, $a_{3^i+k+1}^d, \ldots, a_{3^{i+1}+k}^d$; and $a_{3^i+2k}^d, \ldots, a_{5+2k}^d$. Hence, we put $d := 2 + 2\log(t) + k + k\log n + kn + 2m$. Each anchor vertex is a dial vertex. We fix each of the anchors into $A$ by introducing, for each anchor $a_i^d$, $d+1$ copies of $M$ and, for each copy, identifying an arbitrary vertex of that copy with $a_i^d$. The vertices different from $a_i^d$ in the copies of $M$ are helper vertices. If $a_i^d \in B$, then out of each of the $d$ incident copies of $M$, at least one vertex is in $A$, and since these vertices are pairwise nonadjacent, $G[A]$ would contain at least $d+1$ clusters, which is a contradiction. Thus, each anchor must be in $A$.

The meaning of the different groups of the anchors is as follows. The first group, $a_1^d, a_2^d$, simply serves to fix helper vertices into $B$. The second group, $a_3^d, \ldots, a_{2\log t}^d$, will be used in the instance-selection gadget. The third group, $a_{2\log t+1}^d, \ldots, a_{k+1}^d$, serves to connect the instance-selection gadget with the vertex-selection gadgets: A vertex corresponding to an instance will be pushed to the cluster containing $a_i^d$ and this will require to move the activator vertices of the $k$ corresponding vertex-selection gadgets out of the clusters of the remaining $k$ anchors. The groups $a_{3^i+1}^d, \ldots, a_{3^{i+1}}^d$ are used by the verification gadgets. The groups $a_{3^i+k+1}^d, \ldots, a_{3^{i+1}+k}^d$ and $a_{3^i+2k}^d, \ldots, a_{5+2k}^d$ are used by the verification gadgets and correspond to vertices and edges in the selected instance, respectively.

When we construct cluster-II partitions in the following we always tacitly assume that anchors are in $A$ and all helper vertices are in $B$. More generally, we maintain the following invariant throughout the construction.

**Invariant 4.2.** In each cluster-II partition $(A, B)$ of $G$ with at most $d$ clusters in $G[A]$ it holds that all anchors are in $A$ and all helper vertices are in $B$.

We will construct cliques, each containing an anchor $a_i^d$. Most of the vertices of these cliques will be in $A$, but they may contain parts of different gadgets. For ease of notation, we associate each anchor $a_i^d$ with a vertex set $D_i^d$ that contains $a_i^d$ and induces a clique in $G$ (throughout the construction). We say that $D_i^d$ is the dial of $a_i^d$. Initially, $D_i^d = \{a_i^d\}$. Later on, other vertices may join $D_i^d$; by saying a vertex $v$ joins $D_i^d$, we mean that we put $v$ into $D_i^d$ and make $v$ adjacent to all other vertices in $D_i^d$. Throughout, we maintain the following invariant:

**Invariant 4.3.**

(i) For each $j \in \{2\} \cup \{4, \ldots, 3+k\}$ and each odd $i$, the dial $D_i^d$ is a singleton.

(ii) No two dial vertices in different dials are adjacent.

(iii) For each anchor $a_i^d$, each volatile vertex is either nonadjacent to $a_i^d$ or adjacent to all vertices in $a_i^d$ ’s dial $D_i^d$.

A particular corollary will be that each cluster $C$ in $G[A]$ either contains an anchor $a_i^d$ with a dial $D_i^d = \{a_i^d\}$, or $C$ contains only vertices of $D_i^d$. This will help in the correctness proof, where we build a cluster-II partition for $G$ piece-by-piece.
We use the following notation.

**Definition 4.4 (Friendly partition).** Let \((A, B)\) be a cluster-II partition for \(G\) and \(D\) be a set of dials. Partition \((A, B)\) is *friendly* with respect to \(D\) if each singleton dial in \(D\) is a singleton cluster in \(G[A]\).

Next, we introduce the operation of making three vertices \(u, v, w\) exclusive. Intuitively, this operation is our main tool to fan out the possible pushes in avalanches according to a binary tree: When \(u\) is pushed to \(B\), either \(v\) or \(w\) can be pushed to \(A\) to repair the partition. We use this construction extensively in the selection gadget described below.

**Definition 4.5 (Exclusive vertices).** Given three vertices \(u, v, w \in V(G)\), by making \(u, v, \) and \(w\) exclusive we mean:

(i) to introduce a copy of \(M\) into \(G\),
(ii) to identify three distinct vertices of \(M\) with \(u, v,\) and \(w\), respectively, and
(iii) to fix all remaining vertices of \(M\) (if any) into \(B\) by making each of them adjacent to both \(a^1\) and \(a^2\).

The vertices in \(V(M) \setminus \{u, v, w\}\) are helper vertices.

Observe that \(V(M) \setminus \{u, v, w\} \subseteq B\), because, otherwise, there would be a \(P_3\) in \(G[A]\) involving \(a^1\) and \(a^2\). Hence, Invariant [4.2](#) is maintained by this operation (clearly, Invariant [4.3](#) is maintained as well). Furthermore, not all three \(u, v, w \in B\) since otherwise \(G[B]\) contains a copy of \(M\). When constructing cluster-II partitions we will always tacitly assume that \(V(M) \setminus \{u, v, w\} \subseteq B\) and ignore the vertices in \(V(M) \setminus \{u, v, w\}\). Furthermore, to simplify showing that the constructed partition \((A, B)\) is a cluster-II partition we will show that \(G[A]\) is a cluster graph, that \(G[B] - \{u, v, w\} \in \Pi\), that at least one of \(u, v, w\) is in \(A\) and that \(\{u, v, w\} \cap B\) do not have any neighbors in \(G[B]\) other than \(\{u, v, w\}\). Since \(\Pi\) is characterized by connected forbidden induced subgraphs and the helper vertices will not receive further neighbors, this suffices to prove that \(G[B] \in \Pi\). Thus, below we will show only these properties. For easy reference, we state this consideration in the following lemma:

**Lemma 4.6.** Let \(G\) be a graph in which \(u, v, w \in V(G)\) were made exclusive using a copy of \(M\) and let \((A, B)\) be a bipartition of \(V(G)\). If \(G[A]\) is a cluster graph, \(G[B] \setminus V(M) \in \Pi\), at least one of \(u, v, w\) is in \(A\), and \(u, v, w\) are adjacent only to \(\{u, v, w\} \cap B\) in \(G[B]\), then \((A, B \cup (V(M) \setminus \{u, v, w\}))\) is a cluster-II partition for \(G\).

We next construct an instance-selection gadget and then, for each instance, vertex-selection and verification gadgets.

### 4.2 Instance Selection

The inner workings of the gadget use the necessary pushes along a binary-tree-like structure outlined above. That is, the gadget is constructed such that it has a trivial cluster-II partition with one cluster too many, compared to the number of clusters in \(A\) allocated to it (each gadget requires a minimum number of clusters in \(A\), summing up precisely to the number of anchors). This cluster is a singleton, called the activator vertex. Pushing it to \(B\) results in a forbidden induced subgraph in \(G[B]\) requiring subsequent pushes to \(A\). Each of these pushes to \(A\) will create a \(P_3\) involving two anchors, meaning that the third vertex has to be pushed to \(B\). This again creates a forbidden subgraph in \(B\) and so on. The leaves in the resulting tree-like structure correspond to the selection to be made. That is, there is a set of dial vertices, which we call choice vertices below, which are normally in \(A\). Through a path of pushes in the binary-tree-like structure, one of the choice vertices will be pushed to \(B\). This push will in turn activate other gadgets.

For use as an instance-selection gadget, we need to take special care so that the number of clusters used is roughly logarithmic in the number of instances. We achieve this by using only two clusters (represented by anchors and their dials) per level in the binary-tree-like structure of pushes. For use as a vertex-selection gadget, to bound the number of clusters in the size of the largest instance, we need to ensure that all the vertex-selection gadgets share their corresponding clusters. We achieve this by grouping the gadgets according to the groups of anchors above; each gadget uses only anchors in their corresponding group and shares these anchors with all other gadgets in this group. Essentially, the operation of vertices joining dials makes it possible to define the selection gadgets in a relatively local way.
We will use the following (generic) construction both for selecting an instance and for selecting the independent-set vertices in that instance. For this purpose, fix two construction parameters \( p, q \in \mathbb{N} \), where \( p \) specifies which anchors (and dials) we use when constructing the gadget and \( q \) specifies how many possible choices shall be modeled. Herein, we require that \( q \) be a power of two. For example, in the instance-selection gadget we will set \( p = 2 \) and \( q = t \).

We introduce a new vertex \( v^* \). Our goal is to construct a structure in which, starting from a trivial cluster-II partition \((A, B)\), putting \( v^* \in B \) triggers an avalanche of pushes according to a path in a binary-tree-like structure. To this end, fix a rooted binary tree \( T \) with \( q \) leaves (corresponding to the \( q = t \) instances of \textsc{Colorful Independent Set} for the instance-selection gadget). Say a vertex in \( T \) is on level \( i \in [\log q] \) if its distance from the root is \( i \). For \( i \in [\log q] \), \( L_i \) denotes the set of vertices at level \( i \). The tree \( T \) will not be part of the constructed graph, we use it only as a scaffold to define the actual vertices in the graph.

For each vertex \( v \in V(T) \) except the root, introduce two vertices \( \alpha(v), \beta(v) \) into \( G \). Let \( i \) be the level of \( v \). Connect \( \alpha(v) \) to both \( a_{2i-1}^v \) and \( \beta(v) \). Make \( \beta(v) \) join \( D_i^v \). Furthermore, for each vertex \( u \in L_i \), \( i \in \{0, \ldots, \log q\} \), let \( u, w \) be the two children of \( u \) in \( T \) and make \( \beta(u), \alpha(v), \alpha(w) \) exclusive. If \( i = 0 \), then let \( u, w \) be the two vertices in level 1 in \( T \) and make \( v^*, \alpha(v), \alpha(w) \) exclusive instead. This completes the construction of the selection gadget. Vertex \( v^* \) is a volatile vertex, as is \( \alpha(v) \) for \( v \in V(T) \). Each \( \beta(v), v \in V(T) \), is a dial vertex. Observe that Invariants 4.2 and 4.3 are preserved. Call the constructed gadget \textbf{selection}(\( p, q \)), and say that \( v^* \) is the \textbf{activator vertex}, and \( \alpha(v) \) and \( \beta(v) \) are the \textbf{choice vertices}. We fix an arbitrary order of the choice vertices, so that we may speak of the \textbf{activator choice vertex} without confusion.

Lemma 4.7. Let \( G' \) be the graph before applying \textbf{selection}(\( p, q \)) and \( G \) the graph afterwards.

(i) If cluster-II partition \((A, B)\) has at most \( d \) clusters in \( G[A] \) and the activator vertex is in \( B \), then at least one choice vertex is in \( B \).

(ii) If there is a cluster-II partition \((A', B')\) for \( G' \) with \( d \) clusters in \( G'[A'] \), then there is a cluster-II partition \((A, B)\) for \( G \) with \( d+1 \) clusters, where the activator vertex is a singleton cluster and each choice vertex is in \( A \). If \( (A', B') \) is friendly with respect to the dials \( D_i^v \), then \((A, B)\) is friendly with respect to the dials \( D_i^v \).

(iii) If \( G' \) has a cluster-II partition \((A', B')\) that is friendly with respect to the dials \( D_i^v \) and such that \( G'[A'] \) contains at most \( d \) clusters, and out of all choice vertices only the \( i \)th one is in \( B \) (and, necessarily, the activator vertex is in \( B \)). Moreover, the choice vertex that is contained in \( B \) is isolated in \( G[B] \).

Proof. (i). Note that there are \( d \) anchors and each anchor is in \( A \). Hence, each cluster in \( G[A] \) consists of an anchor and possibly further vertices. By assumption, we have \( v^* \in B \). We now prove by induction that for each \( i \in [\log q] \), there is at least one vertex \( v \in L_i \) with \( \beta(v) \in B \), yielding the statement. Consider the case \( i = 1 \). Let \( u, v \in L_1 \). As \( v^* \in B \), we have that either \( \alpha(u) \) or \( \alpha(v) \) is in \( A \); say \( \alpha(u) \in A \), the other case is symmetric. Since \( \alpha(u) \) is adjacent to both \( a_{2i-1}^u \) and \( \beta(u) \), we have \( \beta(u) \in B \) as, otherwise, vertices \( a_{2i-1}^u, \alpha(u), \beta(u) \) would form an induced \( P_3 \) in \( G[A] \). That is, the statement holds if \( i = 1 \). Now suppose that for some \( u \in L_{i-1} \), \( i > 1 \), we have \( \beta(u) \in B \). Consider the children \( v, w \) of \( u \) in \( T \). Since \( \beta(u), \alpha(v), \alpha(w) \) are made exclusive, either \( \alpha(v) \) or \( \alpha(w) \) is in \( A \). Say \( \alpha(v) \in A \), the other case is symmetric. Note that \( \alpha(v) \) is adjacent to both \( a_{2i-1}^u \) and \( \beta(v) \). Hence, \( \beta(v) \in B \) since, otherwise, \( a_{2i-1}^u, \alpha(v), \beta(v) \) would induce a \( P_3 \) in \( G[A] \). Thus, indeed, for some \( v \in L_i \) we have \( \beta(v) \in B \).

(ii). For the second statement, let \((A', B')\) be a cluster-II partition for \( G' \). By the properties of helper vertices and of exclusive vertices (Lemma 4.6), it is enough to assign to \( A \) and \( B \) the vertices \( \alpha(v), \beta(v) \) for \( v \in V(T) \) such that \( G[A] \) is a cluster graph with at most \( d \) clusters and for each three vertices \( u, v, w \) that were made exclusive, at least one is in \( A \) and \{\( u, v, w \)\} \( \cap B \) are adjacent in \( G[B] \) only to \{\( u, v, w \)\}.

Construct a cluster-II partition \((A, B)\) for \( G \) as follows. Put \( v^* \in A \). For each \( v \in T \) at level \( i > 0 \), put \( \alpha(v) \in B \) and \( \beta(v) \in A \).

Clearly, each choice vertex is in \( A \), as required.

We first claim that \( G[A] \) is a cluster graph with \( d + 1 \) clusters. First, note that \( v^* \) is not adjacent to any vertex in \( A \) and hence constitutes a singleton cluster. By Invariant 4.3, each anchor \( a_i^v \) whose dial \( D_i^v \) is not a singleton is contained in a cluster in \( G[A] \) whose vertex set is contained in \( D_i^v \). Apart from \( v^* \), the only vertices from the construction placed into \( A \) are contained in dials which are not singletons, and hence, \( G[A] \) is a cluster graph with \( d + 1 \) clusters.
We apply Lemma 4.6 to show that \( G[B] \in \Pi \). Note that each triple of exclusive vertices contains one vertex from \( A \). Furthermore, for each vertex \( v \in T \), \( \alpha(v) \) is connected in \( B \) only to \( \alpha(w) \) where \( w \) is the sibling of \( v \) in \( T \). Thus, \( G[B] \in \Pi \).

(iii) For the third statement, let \((A', B')\) be a cluster-II partition for \( G' \). Given \( i \in [q] \), we construct a cluster-II partition \((A, B)\) for \( G \) as follows (as before, we ignore helper vertices). Set \( A = A', B = B' \), and note that \( v^* \in B \). Pick a path \( P \) in \( T \) from the root \( r \) to the leaf \( \ell \) corresponding to the \( i \)th choice vertex \( \beta(\ell) \). For each vertex \( v \in V(T) \setminus \{r\} \), if \( v \in V(P) \), put \( \alpha(v) \in A \) and \( \beta(v) \in B \). Otherwise, if \( v \notin V(P) \), put \( \alpha(v) \in B \) and \( \beta(v) \in A \). Clearly, \( \beta(\ell) \in B \) and \( \beta(\ell) \) is isolated in \( G[B] \), as required.

We first show that \( G[A] \) is a cluster graph with at most \( d \) clusters. Suppose that \( G[A] \) contains an induced \( P_d \), say \( Q \). Clearly, \( Q \) contains at least one vertex introduced by the construction. As all helper vertices are in \( B \), path \( Q \) does not involve helper vertices. By Invariant 4.3 \( Q \) involves a volatile vertex; that is, \( \alpha(\ell) \in V(Q) \) for some \( v \in V(T) \). Moreover, \( v \in V(P) \). By construction, \( \alpha(\ell) \) is adjacent only to \( \beta(\ell) \), \( \alpha(w) \) (where \( w \) is \( v \)'s sibling in \( T \)), and \( \alpha(\ell) \in A \). As \( \beta(\ell) \in B \) by definition of \((A, B)\), path \( Q \) contains \( a^p_{2i-1} \). As \( D^p_{2i-1} \) is a singleton by Invariant 4.3 that is, \( D^p_{2i-1} = \{a^p_{2i-1}\} \), and since \((A', B')\) is friendly with respect to the dials \( D^p_{2i-1} \), we have that \( D^p_{2i-1} \) is a singleton cluster in \( G[A] \). Recall that, by construction, the only new vertices adjacent to \( a^p_{2i-1} \) are vertices \( \alpha(x) \) for \( x \in L_i \). By definition of \((A, B)\), only one of these vertices \( \alpha(x) \) is in \( A \), namely \( \alpha(v) \).

Hence, \( \alpha(v) \) is the only neighbor of \( a^p_{2i-1} \), a contradiction to \( Q \) being an induced \( P_d \). To see that there are at most \( d \) clusters, observe that each vertex \( A \) in \( V \) adjacent to one of the anchors and thus, there are at most \( d \) connected components.

It remains to show that each three vertices that were made exclusive include one vertex in \( A \), and that they are adjacent in \( G[B] \) only to themselves. By construction, the only exclusive vertices are \( \beta(u) \), \( \alpha(v) \), \( \alpha(w) \) for some \( u \in V(T) \) and its children \( v \), \( w \). (The case of \( v^* \) is analogous.) Either \( v \) or \( w \) is not in \( V(P) \), and hence, either \( \alpha(v) \) or \( \alpha(w) \) is in \( A \). The only connections of \( \beta(u) \) are to \( \alpha(u), \alpha(v), \alpha(w) \) (ignoring copies of \( M \) as per the property of exclusive vertices). If \( \beta(u) \in B \), then \( \alpha(u) \in A \) and, hence, regardless of whether \( \beta(u) \in B \), a possible connection outside must involve \( \alpha(v) \) or \( \alpha(w) \), say \( \alpha(v) \). Vertex \( \alpha(v) \) is only adjacent to \( \beta(u) \), to some anchor, and to \( \beta(v) \). If \( \alpha(v) \in B \), then \( \beta(v) \in A \). Thus, indeed \( \beta(u), \alpha(v), \alpha(w) \) are only connected within themselves in \( G[B] \). This shows that \((A, B)\) is a cluster-II partition with \( d \) clusters in \( G[A] \).

As mentioned, to construct the instance-selection gadget, we carry out \text{selection}(2, t)\). For further reference, fix a bijection \( \phi \) from the set of instances \( [t] \) to the choice vertices produced by the construction. We use \( \phi \) later to denote the choice vertex corresponding to an instance.

### 4.3 Vertex Selection

We now use the above construction \text{selection}(\cdot, \cdot)\) to create vertex-selection gadgets for each instance and each color. Each vertex-selection gadget selects one vertex of the gadget’s color into an independent set when activated by putting the activator vertex into \( B \) (which will be effected by the instance-selection gadget). The vertex-selection gadgets for each instance are distinct, but they use dials which are shared by all instances.

In the first part of the construction of the vertex-selection gadgets, for each instance \( r \in [t] \) and color \( i \in [k] \), carry out \text{selection}(3 + i, n)\). Let \( \psi^*_{r,i} \) be the corresponding activator vertex and fix a bijection \( \psi^*_{r,i} \) from the vertices \( V(G_r) \) of color \( i \) to the choice vertices. Make \( \psi^*_{r,i} \) join \( D^3_{3+i} \). Intuitively, if the activator vertex \( \psi^*_{r,i} \) is put into \( B \), the subgraph constructed by \text{selection}(3 + i, n)\) enforces the push of a choice vertex into \( B \), which by bijection \( \psi^*_{r,i} \) correspond one-to-one to the vertices of color \( i \) in instance \( r \). In this way, we model the selection of an independent-set vertex.

In the second part of the construction of the vertex-selection gadgets, we introduce a way to activate the vertex-selection gadgets of all colors if some instance \( r \in [t] \) has been chosen. For this, carry out the following steps for each \( r \in [t] \). Introduce two vertices \( u_r, v_r \). Make \( \phi(r), u_r \), and \( v_r \) exclusive. Fix \( u_r \in B \) by making it adjacent to both \( a_{11}, a_{22} \). Make \( v_r \) adjacent to \( a^3_{11} \) and, for each \( i \in [k] \), make \( v_r \) adjacent to \( \psi^*_{r,i} \). Vertex \( u_r \) is a helper vertex and \( v_r \) is a volatile vertex. This concludes the construction of the vertex-selection gadgets.

Intuitively, the selection of instance \( r \) is indicated by the fact that \( \phi(r) \in B \). Since \( u_r \in B \) and \( \phi(r) \), \( u_r \), and \( v_r \) are exclusive, \( v_r \in A \). Vertex \( v_r \) forms a \( P_3 \) with \( a^3_{11} \) and each \( \psi^*_{r,i} \). Hence, the activator
vertices $\psi^*_{r,i}$ of each vertex-selection gadget for instance $r$ are in $B$. This enforces the selection of an independent-set vertex of each color.

It is clear that Invariant 4.2 is maintained. Invariant 4.3 is maintained in the first part of the construction because selection(·,·) maintains this invariant. In the second part of the construction, no dial vertices are added, giving the first and second part of Invariant 4.3. The third part holds for $a^2_t$ since $D^3_t$ is a singleton. For all the other anchors, the third part of Invariant 4.3 holds because the invariant was satisfied before the second part of the construction, and because $v_r$ is only made adjacent to $a^2_t$.
Thus, the construction of the vertex-selection gadgets maintains Invariants 4.2 and 4.3.

**Lemma 4.8.** Let $G$ be the graph after constructing the vertex-selection gadgets.

(i) If $G$ admits a cluster-II partition $(A,B)$ with $d$ clusters in $G[A]$, then there is some instance $r \in [t]$ such that for each color $i \in [k]$ there is at least one vertex $v \in V(G_r)$ of color $i$ such that $\psi^*_{r,i}(v) \in B$.

(ii) For each instance $s \in [t]$ and each vertex subset $V' \subseteq V(G_s)$ containing exactly one vertex of each color, there is a cluster-II partition $(A,B)$ for $G$ such that $G[A]$ contains at most $d$ clusters, $\psi^*_{s,i}(V') \subseteq B$, and all other choice vertices of each vertex-selection gadget are in $A$. Moreover, the choice vertices that are contained in $B$ are isolated in $G[B]$.

**Proof.** (i). There are $d$ anchors in $G[A]$ and the activator vertex of the instance-selection gadget is not adjacent to any of the anchors. Thus, the activator vertex is in $B$ and from Lemma 4.7(i) it follows that for at least one instance $r \in [t]$ we have $\phi_r(v) \in B$. Since $\phi_r$, $u_r$, and $v_r$ are exclusive and $u_r \in B$ we have $v_r \in A$. Since, for each instance $s \in [t]$ and each color $i \in [k]$, vertices $a^2_t, v_r$, and $\psi^*_{r,i}$ form a $P_3$, we have for each $i \in [k]$ that $\psi^*_{r,i} \in B$.

By Lemma 4.7(iii) it follows that for each $i \in [k]$ there is one choice vertex of the $i$th vertex-selection gadget in $B$. Thus, for each $i \in [k]$ there is a vertex $v \in V(G_s)$ of color $i$ such that $\psi^*_{r,i}(v) \in B$, as required.

(ii). Without loss of generality, assume that the instance-selection gadget has been constructed first, and the vertex-selection gadgets have been constructed in ascending order of instances and then colors. We show that a partial cluster-II partition with the required properties exists after each call to selection(·,·).

Let $G_0$ be the graph obtained after introducing the instance-selection gadget. Before introducing any selection gadget, the graph has a trivial cluster-II partition $(A, B)$ that is friendly with respect to each dial.

By Lemma 4.7(iii), there is a cluster-II partition $(A_0, B_0)$ for $G_0$ such that $G_0[A_0]$ has $d$ clusters, and out of all choice vertices only the $r$th one is in $B$. Furthermore, this cluster-II partition is friendly with respect to the dials $D^{3+1}_r$, $i \in [k]$.

In the following, let $s \in [t]$ be the instance for which we want to construct a cluster-II partition. Let $G_{s-1}$ be the graph obtained after introducing all vertex-selection gadgets for instances in $[s-1]$. By iteratively applying Lemma 4.7(ii), starting with $G_0$ and $(A_0, B_0)$, we obtain that there is a cluster-II partition $(A_{s-1}, B_{s-1})$ for $G_{s-1}$ such that, for each vertex-selection gadget, each activator vertex is in $A$ (in a cluster together with the dial it was joining) and each choice vertex is in $A$. Since we joined the activator vertices to some dials, $G_{s-1}[A_{s-1}]$ has $d$ clusters. Moreover, since $(A_0, B_0)$ is friendly with respect to the dials $D^{3+1}_r$, $i \in [k]$, $(A_{s-1}, B_{s-1})$ is friendly with respect to these dials as well.

Let $V' \subseteq V(G_s)$ as in the statement of the lemma. For each $i \in [k]$, denote by $v'_i \in V'$ the vertex of color $i$ in $V'$ and let $G_{s,i}$ be the graph obtained after introducing the vertex-selection gadget for instance $s$ and color $i$. By induction over $i$ and by Lemma 4.7(iii), we obtain that $G_{s,i}$ admits a cluster-II partition $(A_{s,i}, B_{s,i})$ with $d$ clusters in $G_{s,i}[A_{s,i}]$ such that, for each $j \in [i]$, we have $\psi^*_{s,j}(v'_j) \in B$, $\psi_{s,j}(v'_j) \in B$, and such that all other choice vertices in any vertex-selection gadget are in $A$. Moreover, $(A_{s,i}, B_{s,i})$ is friendly with respect to the dials $D^{3+1}_r$, $j \in [k] \setminus [i]$ (whence we can apply induction).

Let $G_r$ be the graph obtained after introducing all vertex-selection gadgets for instances in $[t] \setminus [s-1]$. By applying iteratively Lemma 4.7(ii) to $G_{s,k}$ and $(A_{s,k}, B_{s,k})$ we obtain a cluster-II partition $(A_t, B_t)$ for $G_t$ analogously to the cluster-II partition for $G_{s-1}$. Hence, the statement of the lemma holds after the first part of the construction of the vertex-selection gadgets. It remains to incorporate $u_r$ and $v_r$, $r \in [t]$, into $(A_t, B_t)$.

Construct a cluster-II partition $(A,B)$ for $G$ from $(A_t,B_t)$ as follows. Put $A = A_t$, $B = B_t$. For each $r \in [t] \setminus [s]$, put $u_r \in B$ and $v_r \in B$. Put $u_s \in B$ and $v_s \in A$.

We claim that $G[A]$ is a cluster graph with at most $d$ clusters. Recall that $G_t[A]$ contains at most $d$ clusters (corresponding to the $d$ anchors). Thus, $G[A]$ has at most $d$ connected components since, for each $r \in [t]$, $u_r \in B$, and $v_r$ is connected to some anchor. To show that $G[A]$ does not contain an
induced $P_3$, it is enough to show that, for each $r \in [t]$, either $v_r \in B$, or that, for all $i \in [k]$, $\psi^*_{r,i} \in B$. The fact that $v_r \in B$ is trivial for $r \neq s$; otherwise, if $r = s$, we have $\psi^*_{r,i} \in B$ by the construction of $(A_{s,i},B_{s,i})$.

Note that, for each $r \in [t]$, either $\phi(r)$, $u_r$, or $v_r$ is in $A$. Hence, by Lemma 4.6, to show that $G[B] \in \Pi$, it suffices to prove, for each $r \in [t]$, the property $P(r)$ that vertices $\phi(r)$, $u_r$, or $v_r$ are adjacent in $G[B]$ only to themselves. Note that any edge to other vertices can only involve $\phi(r)$ or $v_r$. If $r \neq s$, we have $v_r \in A$ and, thus, by construction of $(A_0,B_0)$ according to Lemma 4.7 (iii), that $\phi(r)$ is an isolated vertex in $G_0[B_0]$, giving property $P(r)$. If $r = s$, then $\phi(s) = \phi(r) \in A$. For the incident edges of $v_r$, by construction of $(A_{s,i},B_{s,i})$, $i \in [k]$, according to Lemma 4.7 (ii), for each $i \in [k]$ we have $\psi^*_{s,i} \in A$. Thus indeed, property $P(r)$ holds, finishing the proof. □

4.4 Verification

We now construct the verification gadgets. It is again crucial to share clusters (anchors) between many gadgets to keep the overall number of clusters in $A$ small. For this, we use $|V| = k \cdot n$ anchors that each represent for each instance one fixed vertex, and $m$ pairs of anchors that each represent for each instance one fixed edge.

The working principle is as follows. Selecting a vertex $v$ via a vertex-selection gadget will make it necessary to push a vertex corresponding to $v$ into the cluster of its associated anchor. This push creates a $P_3$ in $A$ for each incident edge $e$, necessitating a further push. Namely, we are required to push a vertex out the cluster in $A$ corresponding to one anchor associated with $e$. Pushing the corresponding vertex for the other endpoint of $e$ into $B$ will complete a forbidden induced subgraph, yielding that no two endpoints of an edge are selected.

For each $r \in [t]$, let $E(G_r) = \{e_1, \ldots, e_m\}$. (If there are less than $m$ edges, repeat an arbitrary edge as needed.) For each $j \in [m]$, perform the following steps towards constructing the $j$th edge gadget. Let $e_j = \{u, v\}$: Introduce three vertices $w^u_{r,j}, w^v_{r,j}, w^0_{r,j}$. Make $w^u_{r,j}, w^v_{r,j}, w^0_{r,j}$ exclusive. Herein, when identifying $w^u_{r,j}, w^v_{r,j}, w^0_{r,j}$ with vertices of $M$, pick two vertices which are adjacent in $M$ for $w^u_{r,j}$ and $w^0_{r,j}$. Note that this is possible since $M$ is connected. Make $w^u_{r,j}$ and $w^v_{r,j}$ join $D^j_{r,i}$. (Note that this does not introduce edges into the copy of $M$ used for making $w^u_{r,j}, w^v_{r,j}, w^0_{r,j}$ exclusive.) Fix $w^0_{r,j}$ into $B$ by making it adjacent to both $a^1_i$ and $a^2_i$. Vertices $w^u_{r,j}$ and $w^v_{r,j}$ are volatile vertices and $w^0_{r,j}$ is a helper vertex.

We furthermore need for each vertex a vertex gadget, which is constructed for each instance $r \in [t]$, and each color $i \in [k]$ as follows. Fix an arbitrary ordering of the vertices of color $i$ in $G_r$ and say the index of a vertex is its index in that ordering. For each vertex $v \in V(G_r)$ of color $i$, introduce two vertices $x^1_{r,i,v}$ and $x^2_{r,i,v}$. Fix $x^1_{r,i,v}$ into $B$ by making it adjacent to both $a^1_i$ and $a^2_i$. Make $\phi_{r,i}(v), x^1_{r,i,v}$, and $x^2_{r,i,v}$ exclusive. Make $x^1_{r,i,v}$ adjacent to $a^k_i + \ell$, where $\ell$ is the index of $v$. Vertex $x^2_{r,i,v}$ is volatile and $x^2_{r,i,v}$ is a helper vertex.

Finally, connect the edge-gadgets and vertex-gadgets as follows. For each instance $r \in [t]$, perform the following steps. Recall that $E(G_r) = \{e_1, \ldots, e_m\}$. For each $j \in [m]$, let $i_1,i_2 \in [k]$ be the colors of the endpoints $v_1, v_2 \in V(G_r)$ of $e_j$. Make $x^1_{r,i_1,v_1}$ adjacent to $w^0_{r,j}$ and make $x^2_{r,i_2,v_2}$ adjacent to $w^0_{r,j}$. This finishes the construction of the verification gadgets and concludes the construction of the graph $G$ in our instance of CLUSTER-II-PARTITION. Clearly, Invariants 4.2 and 4.3 remain valid.

Lemma 4.9. Let $G$ be the graph constructed above. The graph $G$ admits a cluster-II partition $(A,B)$ with $d$ clusters in $G[A]$ if and only if there exists an instance $s \in [t]$ such that $G_s$ admits an independent set with exactly one vertex of each color.

Proof. Assume that $G$ admits a cluster-II partition $(A,B)$ with $d$ clusters in $G[A]$. Note that Lemma 4.8 refers to a subgraph of $G$. By restricting $(A,B)$ to that subgraph, from Lemma 4.8 (i) we infer that there is an instance $s \in [t]$ such that, for each color $i \in [k]$, there is a vertex $v_i \in V(G_s)$ such that $\psi_{s,i}(v_i) \in B$. We claim that $V' := \{v_i \mid i \in [k]\}$ is an independent set in $G_s$. Suppose $V'$ is not an independent set and let $e_j \in E(G_s)$ be such that $e_j \subseteq V'$. Let $e_j = \{u, v\}$ and let $i, i'$ be the colors of $u$ and $v$, respectively. Since $\psi_{s,i}(u), x^1_{s,i,u}$, and $x^2_{s,i,u}$ are exclusive and $x^2_{s,i,u} \in B$, we have $x^1_{s,i,u} \in A$. Thus, $w^0_{s,i} \in B$ as, otherwise, $a^k_{i} + \ell, x^1_{i,u}$, and $w^0_{s,i}$, would form an induced $P_3$ in $G[A]$, where $\ell$ is the index of $u$. Similarly, $w^0_{s,i} \in B$. However, $w^0_{s,i}, w^0_{s,j}$ are exclusive and each of them is contained in $B$. This contradicts the fact that $G[B] \in \Pi$. Hence, $V'$ is an independent set. Clearly, $V'$ contains exactly one vertex of each color.

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Now assume that for some instance $s \in [t]$, there is an independent set $V' = \{v_i \mid i \in [k]\} \subseteq V(G_s)$ with exactly one vertex $v_i$ of each color $i \in [k]$. Let $G'$ be the graph before constructing the verification gadgets. By Lemma 4.8(ii), there is a cluster-II partition $(A', B')$ for $G'$ with $d$ clusters in $G'[A']$ such that, for each $i \in [k]$, we have $\psi_{s,i}(v_i) \in B'$ (and these vertices are isolated in $G[B']$), and all other choice vertices of each vertex-selection gadget are in $A'$.

We now construct a cluster-II partition $(A,B)$ for $G$ from $(A',B')$. Put $A = A'$ and $B = B'$. For each instance $r \in [t]$, including $s$, and for each $v \in V(G_r)$, let $i$ be the color of $v$ and put $x_{r,s,i,v}^1, x_{r,s,i,v}^2 \in B$. If $v \in V'$, then put $x_{r,s,i,v}^1 \in A$ instead. For each edge $e_j \in E(G_r)$, let $e_j = \{u,v\}$ and put $w_{r,j}^u, w_{r,j}^v \in A$. If one endpoint of $e_j$, say $u$, is in the independent set $V'$, then put $w_{r,j}^v \in B$ instead. Clearly, not both endpoints can be in the independent set.

Observe that $(A,B)$ is a partition of $V(G)$. We claim that $(A,B)$ is a cluster-II partition for $G$ with at most $d$ clusters in $G[A]$. We first show that $G[A]$ is a cluster graph. Suppose that $G[A]$ contains an induced $P_3$, say $Q$. Since $G'[A']$ is a cluster graph, $Q$ contains a vertex in $V(G) \setminus V(G')$. By Invariant 4.3, $Q$ involves a non-dial vertex, that is, a vertex $v$ from a vertex gadget. Since $v \in A$, by definition of $(A,B)$, we have $v = x_{r,s,i,v}^1 \in V(Q)$ for some $v_i \in V'$. The only neighbors of $x_{r,s,i,v}^1$ in $G$ are $\psi_{r,i}(i), x_{r,s,i,v}^2$, and $(w_{r,j}^u)_{u \in J}$ for some set $J \subseteq [m]$. By definition of $(A,B)$, each of these vertices is in $B$, a contradiction to the existence of $Q$. Hence, $G[A]$ is a cluster graph. To see that $G[A]$ contains at most $d$ connected components, observe that $G'[A']$ has at most $d$ connected components, one for each anchor, and each vertex in $A \setminus A'$ is connected to an anchor in $G[A]$.

It remains to show that $G[B] \in \Pi$. Recall that $G'[B'] \in \Pi$. The only edges in $G$ between vertices in $V(G')$ and newly-introduced vertices in $V(G) \setminus V(G')$ are incident with either an anchor or some choice vertex of some vertex-selection gadget. The anchors are in $A$ and if some of the the choice vertices are in $B'$, then they are isolated in $G'[B']$ by Lemma 4.8(ii). Thus, it is enough to show that these choice vertices and the newly-introduced vertices induce a subgraph of $G$ that satisfies $\Pi$. Since all of these vertices have been made exclusive, it is enough to show that the conditions of Lemma 4.6 are satisfied for each triple that has been made exclusive. Each such triple has the form, (i), $(w_{r,j}^u, w_{r,j}^v, r,i,j)$ or, (ii), $(\psi_{r,i}(v), x_{r,s,i,v}^1, x_{r,s,i,v}^2)$. By definition of $(A,B)$, out of each triple, at least one vertex is in $A$. Thus, it remains to prove the adjacency condition of Lemma 4.6. As $\psi_{r,i}(v)$, if contained in $B$, is a singleton in $G'[B]$, by construction, there is no edge in $G[B]$ between any two triples of form (ii). There is no edge between two triples of form (i) because, by definition of $B$, for each edge gadget $j \in [m]$, there is exactly one triple of form (ii) containing a vertex in $B$ and there is no edge between any two triples of form (ii) for distinct edge gadgets $j$. Finally, there is no edge in $G[B]$ between two triples of form (i) and (ii): Assume there is and let $v \in V(G_r)$ and $j \in [m]$ corresponding to the two triples. By construction, $v \in e_j$ for $e_j \in E(G_r)$. That is, $e = (w_{r,j}^v, x_{r,s,i,v}^1)$. We have $w_{r,j}^v \in B$ only if $v \in V'$. However, $x_{r,s,i,v}^1 \in A$ by definition and, thus, $e \not\subseteq B$. Thus, the conditions of Lemma 4.6 are satisfied, meaning that $G[B] \in \Pi$. We thus found that $(A,B)$ is the required cluster-II partition.

It is not hard to verify that the construction can be carried out in polynomial time. Since, moreover, $d \leq \text{poly}(\log t + \max_{i \in [t]} |V(G_i)|)$, we thus have shown that all the conditions of cross-compositions are satisfied, yielding Theorem 4.1.

5 Kernels for Parameterization by the Size of One of the Parts

In this section, we prove that $(\Pi_A, \Pi_B)$-RECOGNITION has a polynomial kernel parameterized by the size of one of the parts of the bipartition when $\Pi_A$ and $\Pi_B$ satisfy certain general technical conditions. To simplify the presentation, we pick $B$ to be the part whose size is at most the parameter $k$. We then consider the conditions that $\Pi_A$ is characterized by forbidden induced subgraphs, each of size at most $d$, and $\Pi_B$ is hereditary (closed under taking induced subgraphs). In the first subsection, we prove a kernel of size $O(d!k(k+1)^d)$ in this general setting. In the second subsection, we consider the restricted setting of CLUSTER-$\Pi_A$-PARTITION: $\Pi_A$ is the set of all cluster graphs ($P_3$-free graphs) and $\Pi_B$ a hereditary property that contains only graphs of degree at most $\Delta$. Although the result of the first subsection implies a kernel of size $O(k^3)$ in this setting, we prove that CLUSTER-$\Pi_A$-PARTITION actually has a smaller kernel, of size $O((\Delta^2+1)k^2)$.
5.1 A Kernel in the Generic Setting

In this subsection, we prove that \((\Pi_A, \Pi_B)\)-RECOGNITION has a polynomial kernel, of size \(O(d!(k+1)^d)\), parameterized by the size \(k\) of \(B\) when \(\Pi_A\) can be characterized by forbidden induced subgraphs, each of size at most \(d\), and \(\Pi_B\) is hereditary. The kernel is obtained by a similar approach to that for the \(d\)-HITTING SET problem using the Sunflower Lemma (see [10, 17]). We start by describing two straightforward reduction rules.

Reduction Rule 5.1. Let \((G,k)\) be an instance of \((\Pi_A, \Pi_B)\)-RECOGNITION where \(\Pi_A\) can be characterized by a collection \(\mathcal{H}\) of forbidden induced subgraphs, each of constant size, and \(\Pi_B\) is hereditary. Then remove from \(G\) any vertex that is not in an induced subgraph of \(G\) that is isomorphic to a member of \(\mathcal{H}\).

Proof. Let \(G'\) be the graph obtained from \(G\) by removing any vertex from \(G\) that is not in an induced subgraph of \(G\) isomorphic to a member of \(\mathcal{H}\). Let \(R\) denote the set of removed vertices. We now prove that \((G,k)\) is a yes-instance if and only if \((G',k)\) is.

Suppose that \((G,k)\) is a yes-instance, and let \((A,B)\) be a partition of \(V(G)\) such that \(G[A] \in \Pi_A, G[B] \in \Pi_B\), and \(|B| \leq k\). Since \(\Pi_A\) can be characterized by a collection of forbidden induced subgraphs, it is hereditary. Hence, \(G[A \setminus R] \in \Pi_A\) and \(G[B \setminus R] \in \Pi_B\). Therefore, \((G',k)\) is a yes-instance.

Suppose that \((G',k)\) is a yes-instance, and let \((A',B')\) be a partition of \(V(G')\) such that \(G[A'] \in \Pi_A, G[B'] \in \Pi_B\), and \(|B'| \leq k\). Then \(G[A' \cup R] \in \Pi_A\), because no vertex of \(R\) is in an induced subgraph of \(G\) isomorphic to a member of \(\mathcal{H}\). Hence, \((A' \cup R, B')\) is a partition of \(V(G)\) such that \(G[A' \cup R] \in \Pi_A, G[B'] \in \Pi_B\), and \(|B'| \leq k\). Therefore, \((G,k)\) is a yes-instance.

In the second reduction rule, we need the Sunflower Lemma (see [10, 17]). A sunflower is a collection of sets \(S_1, \ldots, S_\ell\) for which there exists a set \(C\) (the core) such that \(S_i \setminus C \neq \emptyset\) for all \(1 \leq i \leq \ell\) and \(S_i \cap S_j = C\) for all \(1 \leq i < j \leq \ell\). We call \(\ell\) the size of the sunflower.

Theorem 5.2 (Sunflower Lemma). Let \(\ell \in \mathbb{N}\), and let \(F\) be a set of sets over a universe \(U\) such that each set in \(F\) has size at most \(d\). If \(|F| > d!(\ell - 1)^d\), then \(F\) contains a sunflower of size \(\ell\). Moreover, there is an algorithm running in time polynomial in \(|F|, |U|, \ell\) that computes such a sunflower.

In the remainder, let \((G,k)\) be an instance of \((\Pi_A, \Pi_B)\)-RECOGNITION where \(\Pi_A\) can be characterized by a collection \(\mathcal{H}\) of forbidden induced subgraphs, each of size at most \(d\), and \(\Pi_B\) is hereditary. Throughout, we maintain a set \(F\) of subsets of \(V(G)\) such that each subset induces a subgraph of \(G\) isomorphic to a member of \(\mathcal{H}\). Initially, \(F\) contains all such subsets.

Reduction Rule 5.3. Suppose that \(|F| > d!(k+1)^d\), and let \(S_1, \ldots, S_\ell\) be the sunflower of size \(\ell \geq k+2\) and core \(C\) returned by Theorem 5.2.

- If \(C\) is empty, then reject \((G,k)\) as a no-instance.
- Otherwise, remove \(S_k+1, \ldots, S_\ell\) from \(F\) and remove from \(G\) any vertex that is in \(S_i \setminus C\) for some \(k+2 \leq i \leq \ell\) and is in no other set of \(F\) than \(S_i\).

Proof. If \(C\) is empty, then \(S_1,\ldots,S_k\) are pairwise disjoint subsets of \(V(G)\) that each induce a subgraph of \(G\) isomorphic to a member of \(\mathcal{H}\). Let \((A,B)\) be any partition of \(V(G)\) such that \(G[A] \in \Pi_A, G[B] \in \Pi_B\). Then \(B\) must contain at least one vertex from each of \(S_1,\ldots,S_k\), and thus \(|B| > k\). Hence, \((G,k)\) is a no-instance.

Otherwise, let \(R\) be the set of vertices removed, and let \(G'\) be obtained from \(G\) by removing \(R\). We now prove that \((G,k)\) is a yes-instance if and only if \((G',k)\) is.

Suppose that \((G,k)\) is a yes-instance, and let \((A,B)\) be a partition of \(V(G)\) such that \(G[A] \in \Pi_A, G[B] \in \Pi_B\), and \(|B| \leq k\). Since \(\Pi_A\) can be characterized by a collection of forbidden induced subgraphs, it is hereditary. Hence, \(G[A \setminus R] \in \Pi_A\) and \(G[B \setminus R] \in \Pi_B\). Therefore, \((G',k)\) is a yes-instance.

Conversely, suppose that \((G',k)\) is a yes-instance, and let \((A',B')\) be a partition of \(V(G')\) such that \(G[A'] \in \Pi_A, G[B'] \in \Pi_B\), and \(|B'| \leq k\). Note that \(C \subseteq V(G')\). Suppose that \(C \cap B' = \emptyset\). Then \(S_1 \setminus C, \ldots, S_k+1 \setminus C\) are pairwise disjoint subsets of \(V(G)\) that each, together with \(C\), induces a subgraph of \(G\) isomorphic to a member of \(\mathcal{H}\). Hence, \(|B'| > k\), a contradiction. Therefore, \(C \cap B' \neq \emptyset\). It follows that \(B' \cap S_i \neq \emptyset\) for every \(1 \leq i \leq \ell\). Then, \(G[A' \cup R] \in \Pi_B\), because every vertex of \(R\) is in exactly
one induced subgraph of \( G \) isomorphic to a member of \( \mathcal{H} \), which has a vertex in \( B' \). Hence, \( (A' \cup R, B') \) is a partition of \( V(G) \) such that \( G[A'] \in \Pi_A \), \( G[B' \cup R] \in \Pi_B \), and \( |B'| \leq k \). Therefore, \( (G, k) \) is a yes-instance.

**Proof of Theorem 5.2** Let \( (G, k) \) be an instance of \((\Pi_A, \Pi_B)\)-RECOGNITION where \( \Pi_A \) can be characterized by a collection \( \mathcal{H} \) of forbidden induced subgraphs, each of constant size, and \( \Pi_B \) is hereditary. First apply Rule 5.1 so that each remaining vertex is in some induced subgraph isomorphic to a member of \( \mathcal{H} \). This takes polynomial time (for constant \( d \)). Then exhaustively apply Reduction Rule 5.3 and let \( G' \) denote the resulting graph. By Theorem 5.2, this also takes polynomial time, since the number of sets in \( F \) is initially a polynomial in \( |V(G)|^d \) and \( k \), which decreases by at least 1 after every application of the rule. If the rule returns that the instance is a no-instance, then we return a trivial no-instance. Otherwise, it follows from the rule that \( |F| \leq d! (k+1)^d \). Since each remaining vertex is in some induced subgraph isomorphic to a member of \( \mathcal{H} \) after the application of Rule 5.1 and Rule 5.3 does not change this, it follows that \( |V(G')| \leq d! (k+1)^d \). By the correctness of Reduction Rule 5.1 and Reduction Rule 5.3 this is indeed a polynomial kernel.

### 5.2 Smaller Kernels for a Restricted Setting: Cluster-\( \Pi_\Delta \)-Partition

In this subsection, we prove that Cluster-\( \Pi_\Delta \)-Partition, parameterized by the size \( k \) of \( B \), has a kernel with \( O((\Delta^2 + 1)k^2) \) vertices. This improves on Theorem 1.2 which implies a kernel with \( O((k + 1)^3) \) vertices. Throughout, we say that a cluster-\( \Delta \) partition of \( G \) is valid if \( |B| \leq k \).

The first step of the kernel is to compute a maximal set \( \mathcal{P} \) of vertex-disjoint induced \( P_3 \)s. We call \( \mathcal{P} \) a \( P_3 \)-packing. We let \( V(\mathcal{P}) \) denote the set of vertices of the \( P_3 \)s in \( \mathcal{P} \).

**Reduction Rule 5.4.** Let \( (G, k) \) be an instance of Cluster-\( \Pi_\Delta \)-Partition, and let \( \mathcal{P} \) be a \( P_3 \)-packing. If \( |\mathcal{P}| > k \), then reject.

**Proof.** For each \( P_3 \), at least one vertex must be in \( B \). Therefore, if \( |\mathcal{P}| > k \), then \( |B| > k \) for any valid cluster-\( \Pi_\Delta \) partition \( (A, B) \) of \( G \).

Since \( \mathcal{P} \) is a maximal set of \( P_3 \)s, \( G - V(\mathcal{P}) \) is a cluster graph. The first step of the kernelization is to identify vertices of \( V(\mathcal{P}) \) that are in \( B \) in every valid cluster-\( \Pi_\Delta \) partition.

**Definition 5.5.** For a vertex \( u \in V(\mathcal{P}) \), we say that \( u \) is fixed if either:

1. \( u \) has neighbors in at least \( k + 2 \) different clusters of \( G - V(\mathcal{P}) \); or
2. there is a cluster \( C \) in \( G - V(\mathcal{P}) \) such that \( u \) has (at least) \( \Delta + 2 \) neighbors and (at least) \( \Delta + 2 \) nonneighbors in \( C \).

A fixed vertex \( u \) is said to be heavy if it has neighbors in at least \( k + 2 \) different clusters of \( G - V(\mathcal{P}) \) (i.e., satisfies condition 1 above); otherwise, \( u \) is nonheavy.

**Lemma 5.6.** Let \( (G, k) \) be an instance of Cluster-\( \Pi_\Delta \)-Partition, let \( \mathcal{P} \) be a \( P_3 \)-packing, and let \( u \) be a fixed vertex. If \( G \) has a valid cluster-\( \Pi_\Delta \) partition \( (A, B) \), then \( u \in B \).

**Proof.** Case 1: \( u \) is heavy. If \( u \in A \), then there is at most one cluster \( C \) of \( G - V(\mathcal{P}) \) such that \( A \) contains vertices of \( N(u) \cup C \). Therefore, \( B \) contains vertices of \( k \) + 1 clusters of \( G - V(\mathcal{P}) \) and thus \( |B| > k \).

Case 2: \( u \) is nonheavy. Since \( u \) is fixed, there is a cluster \( C \) in \( G - V(\mathcal{P}) \) such that \( u \) has (at least) \( \Delta + 2 \) neighbors and (at least) \( \Delta + 2 \) nonneighbors in \( C \). Let \( v_1, v_2, \ldots, v_{\Delta+2} \) be \( \Delta + 2 \) neighbors of \( u \) in \( C \), and let \( w_1, w_2, \ldots, w_{\Delta+2} \) be \( \Delta + 2 \) nonneighbors of \( u \) in \( C \). Assume, towards a contradiction, that there is a cluster-\( \Pi_\Delta \) partition \( (A, B) \) with \( u \in A \). Since each of \( G[\{v_1, v_2, \ldots, v_{\Delta+2}\}] \) and \( G[\{w_1, w_2, \ldots, w_{\Delta+2}\}] \) is a clique on \( \Delta + 2 \) vertices (and hence of degree \( \Delta + 1 \), \( A \) must contain at least one vertex \( v_j \in \{v_1, v_2, \ldots, v_{\Delta+2}\} \) and at least one vertex \( w_j \in \{w_1, w_2, \ldots, w_{\Delta+2}\} \). But then \( (u, v_j, w_j) \) forms an induced \( P_3 \) in \( A \).

Next, we label certain vertices in \( V \setminus V(\mathcal{P}) \) as important using the following scheme.

**Labeling Scheme**
(i) For each (fixed) heavy vertex \( u \) of \( V(P) \), pick \( k + 2 \) (distinct) clusters \( C_1, \ldots, C_{k+2} \) in \( G - V(P) \) such that \( C_i \) contains a neighbor \( v_i \) of \( u \), for \( i \in [k+2] \), and label \( v_i \) as important.

(ii) For each (fixed) nonheavy vertex \( u \) of \( V(P) \), pick an arbitrary cluster \( C \) of \( G - V(P) \) such that \( u \) has \( \Delta + 2 \) neighbors \( v_1, v_2, \ldots, v_{\Delta+2} \) and \( \Delta + 2 \) nonneighbors \( w_1, w_2, \ldots, w_{\Delta+2} \) in \( C \), and label \( v_1, v_2, \ldots, v_{\Delta+2}, w_1, w_2, \ldots, w_{\Delta+2} \) as important.

(iii) For each nonfixed vertex \( u \) of \( V(P) \), each cluster \( C \) of \( G - V(P) \) containing at least one neighbor of \( u \), label \( \min \{ \Delta + 2, |N(u) \cap C| \} \) (arbitrary) neighbors of \( u \) in \( C \) and \( \min \{ \Delta + 2, |C - N(u)| \} \) (arbitrary) nonneighbors of \( u \) in \( C \) as important.

Any vertex in \( V \setminus V(P) \) that was not labeled in this scheme is called \textit{unimportant}.

**Observation 5.7.** If \((G, k)\) is reduced with respect to Reduction Rule 5.4, then the number of vertices that are marked as important is \( O((\Delta + 1) \cdot k^2) \).

*Proof.* After Reduction Rule 5.4, we have \(|V(P)| \leq 3k\). Each heavy vertex in \( V(P) \) labels \( k + 2 \) vertices in \( V \setminus V(P) \) as important, according to condition (i) of the labeling scheme. Therefore, the total number of vertices in \( V \setminus V(P) \) labeled as important by heavy vertices is \( O(k^2) \). Each fixed nonheavy vertex in \( V(P) \) labels \( \Delta + 4 \) vertices in \( V \setminus V(P) \) as important, according to condition (ii) of the labeling scheme. Therefore, the total number of vertices in \( V \setminus V(P) \) labeled as important by fixed nonheavy vertices is \( O(\Delta \cdot k + k) \). Each nonfixed vertex \( v \) in \( V(P) \) is adjacent to at most \( k + 1 \) clusters in \( G - V(P) \) (otherwise \( v \) would be fixed), and can label at most \( 2\Delta + 4 \) vertices in each adjacent cluster as important (according to condition (iii) of the labeling scheme). Therefore, a nonfixed vertex \( v \) labels \( O(\Delta \cdot k + k) \) many vertices in \( V \setminus V(P) \) as important. It follows that the at most \( 3k \) vertices in \( V(P) \) label \( O(\Delta \cdot k^2 + k^2) = O((\Delta + 1) \cdot k^2) \) many vertices of \( V \setminus V(P) \) as important. \( \blacksquare \)

We now present several reduction rules that use the above labeling scheme.

**Reduction Rule 5.8.** If there is a cluster \( C \) in \( G - V(P) \) such that all vertices in \( C \) are unimportant, remove \( C \) from \( G \).

*Proof.* If \( G \) has a valid cluster-\( \Pi_\Delta \) partition \((A, B)\), then obviously so does \( G - C \). To prove the converse, suppose that \((A, B)\) is a valid cluster-\( \Pi_\Delta \) partition of \( G - C \). We claim that \((A \cup C, B)\) is a cluster-\( \Pi_\Delta \) partition, which obviously satisfies \(|B| \leq k\), and hence, is valid.

Suppose not. Then there must exist a vertex \( u \in A \) that has a neighbor in \( C \). Clearly, \( u \in V(P) \) because \( G - V(P) \) is a cluster graph containing cluster \( C \) and \( u \notin C \). Vertex \( u \) cannot be fixed; otherwise, since no vertex in \( C \) is important, \( u \) would remain fixed in \( G - C \), and hence, \( u \) would not belong to \( A \) by Lemma 5.6. Since \( u \) is adjacent to \( C \), it follows from condition (iii) of the labeling scheme that \( \min \{ \Delta + 2, |N(u) \cap C| \} > 0 \) (since \( u \) is adjacent to \( C \)) neighbors of \( u \) in \( C \) are labeled important. This, however, contradicts the assumption of the reduction rule. \( \blacksquare \)

**Reduction Rule 5.9.** If there is a cluster \( C \) in \( G - V(P) \) such that \( C \) contains (at least) \( \Delta + 3 \) unimportant vertices, then remove one of these unimportant vertices.

*Proof.* Let \( u \) be an unimportant vertex in \( C \) that is removed by an application of this rule. If \( G \) has a valid cluster-\( \Pi_\Delta \) partition \((A, B)\), then clearly so does \( G - w \). To prove the converse, suppose that \( G - w \) has a valid cluster-\( \Pi_\Delta \) partition \((A, B)\). We claim that \((A \cup \{w\}, B)\) is a cluster-\( \Pi_\Delta \) partition of \( G \), which obviously will be valid.

Since \( C \) contains \( \Delta + 2 \) neighbors \( w_1, \ldots, w_{\Delta+2} \) of \( w \) that are unimportant and the maximum degree of \( B \) is at most \( \Delta \), at least one of these vertices, say \( w_1 \), belongs to a cluster \( C' \) in \( A \). Every vertex in \( C' \) that is in \( V(P) \) must be in \( C \), and hence, is adjacent to \( w \). Now suppose that a vertex \( u \in V(P) \) is in \( C' \). We will show that \( u \) must be adjacent to \( w \). Suppose, towards a contradiction, that \( u \) is not adjacent to \( w \). Since \( w \) is unimportant, \( u \) cannot be fixed (otherwise, \( u \) would be fixed in \( G - w \), and would belong to \( B \) by Lemma 5.6). Since \( u \) is adjacent to \( w_1 \), \( u \) is nonfixed condition (iii) of the labeling scheme applies to \( u \), and in particular, \( \min \{ \Delta + 2, |C - N(u)| \} \) nonneighbors of \( u \) in \( C \) are labeled important. Since \( w \) is a nonneighbor of \( u \) in \( C \), and \( w \) is unimportant, it follows that there are \( \Delta + 2 \) nonneighbors of \( u \) in \( C \) that are different from \( w \), and that are labeled important. At least one of these vertices, say \( x \) must be in \( A \). But then \((u, w_1, x)\) forms an induced \( P_3 \) in \( A \) (note that \( w_1 \) is
adjacent to $x$ since both of them are in $C$). This is a contradiction. It follows that each vertex in $C'$ is adjacent to $w$, and hence, $C' \cup \{w\}$ is a cluster in $A \cup \{w\}$.

To conclude that $G[A \cup \{w\}]$ is a cluster graph, it remains to show that no vertex $u$ that belongs to another cluster $C'' \neq C'$ in $G[A \cup \{w\}]$ is adjacent to $w$. Suppose not. Then clearly $u \in V(P)$, and by the same arguments as above, $u$ cannot be fixed. Since $u$ is adjacent to $w \in C$, and $u$ is nonfixed, condition (iii) of the labeling scheme applies to $u$, and in particular, $\min\{\Delta + 2, |C \cap N(u)|\}$ neighbors of $u$ in $C$ are labeled as important. Since $w$ is unimportant, it follows that there are $\Delta + 2$ neighbors of $u$ in $C$ that are different from $w$, and that are labeled important. One of these neighbors, say $x$, must be in $A$, and hence, must belong to the same cluster as both $u$ and $w_1$ (because $w_1 \in C$). But then this implies that $C' = C''$, contradicting our assumption that $u$ belongs to a different cluster than $C'$.

It follows that $(A \cup \{w\}, B)$ is a valid cluster-$\Pi_{\Delta}$ partition of $G$. □

**Lemma 5.10.** Let $(G, k)$ be an instance of CLUSTER-$\Pi_{\Delta}$-PARTITION that is reduced with respect to the above rules, then $G$ has $O((\Delta^2 + 1) \cdot k^2)$ vertices.

Proof. Since $(G, k)$ is reduced with respect to Reduction Rule 5.4, $|V(P)| \leq 3k$. By Observation 5.7, the number of important vertices in $V \setminus V(P)$ is $O((\Delta + 1) \cdot k^2)$. Thus, to show the upper bound on the kernel size, it remains to upper bound the number of unimportant vertices in $V \setminus V(P)$.

To this end, we first upper-bound the number of clusters in $G \setminus V(P)$. Since $(G, k)$ is reduced with respect to Reduction Rule 5.8, every cluster in $G \setminus V(P)$ contains at least one important vertex.

By Observation 5.7, the number of important vertices in $G$ is $O((\Delta + 1) \cdot k^2)$. Thus, the total number of clusters in $G \setminus V(P)$ is $O((\Delta + 1) \cdot k^2)$.

Now, observe that since $(G, k)$ is reduced with respect to Reduction Rule 5.9, there are at most $\Delta + 3$ unimportant vertices in each cluster, and thus $O((\Delta^2 + 1) \cdot k^2)$ unimportant vertices overall. □

**Theorem 1.3.** CLUSTER-$\Pi_{\Delta}$-PARTITION, parameterized by the size $k$ of $B$, has a kernel of size $O((\Delta^2 + 1) \cdot k^2)$, that is computable in time $O(k \cdot (m + n))$, where $n$ and $m$ are the number of vertices and edges, respectively, in the graph.

Proof. Given an instance $(G, k)$ of CLUSTER-$\Pi_{\Delta}$-PARTITION, we start by computing a $P_3$-packing $P$. Afterwards, we apply Reduction Rule 5.4-Reduction Rule 5.9. If after the application of these reduction rules the instance $(G, k)$ is not rejected, then these reduction rules result in an equivalent instance $(G', k)$ of CLUSTER-$\Pi_{\Delta}$-PARTITION satisfying $|V(G')| = O((\Delta^2 + 1) \cdot k^2)$ by Lemma 5.10. Therefore, what is left is analyzing the running time taken to apply Reduction Rule 5.4-Reduction Rule 5.9.

First, it is important to observe that each reduction rule is applied exhaustively once, meaning that we apply a particular reduction rule exhaustively, but no more after any of the other reduction rules have been applied. In particular, after applying any of the reduction rules, $G \setminus V(P)$ is still a cluster graph, because the reduction rules only remove vertices. Moreover, the reduction rules leave unchanged the status of each vertex $u \in V(P)$ as (fixed) heavy, (fixed) nonheavy, or nonfixed, because only (edges to) unimportant vertices are removed and the important vertices maintain the status of $u$. The reduction rules also leave unchanged the label of each vertex in $V \setminus V(P)$ as important or unimportant, for the same reason. Therefore, it suffices to analyze the running time of a single, exhaustive application of each of the reduction rules.

To apply Reduction Rule 5.4, we observe that, as is well known, a $P_3$ in $G$ can be recognized in $O(m + n)$ time. (For instance, this can be done by computing the connected components of $G$, and the degree of each vertex in $G$. We can then identify a connected component that is not a clique, which must exist if a $P_3$ exists. A $P_3$ in such a component can then be computed in linear time.) Therefore, $P$ can be greedily computed in time $O(k \cdot (m + n))$ (note that if more than $k$ $P_3$’s are identified in $G$, then the instance can be immediately rejected). It follows from the preceding that Reduction Rule 5.4 can be applied in $O(k \cdot (m + n))$ time.

Next, we show that we can classify the vertices in $V(P)$ into fixed heavy, fixed nonheavy, and nonfixed in $O(m + n)$ time. To do so, we first compute the clusters in $G \setminus V(P)$, and color the vertices of different clusters with different colors, i.e., each vertex in the $i$-th cluster receives color $i$, for some arbitrary numbering of the clusters. We then iterate through the vertices in $V(P)$, and for each vertex $v \in V(P)$, we iterate through its neighbors in $G \setminus V(P)$. If $v$ has at least $k + 2$ neighbors in $G \setminus V(P)$ with different colors (this can be determined in time $O(deg(v))$ by sorting the colors of the neighbors of $v$ using Counting Sort), then we define $v$ to be fixed and heavy. For each vertex in $V(P)$ that has not been
classified yet, we iterate through its neighbors in \( G - V(P) \), and partition its neighbors into subsets, such that all neighbors in the same subset have the same color (belong to the same cluster); for each such subset of neighbors of size \( s \geq \Delta + 2 \) that belong to a cluster \( C \), we check if \( |C| \geq s + \Delta + 2 \), and if it is, we classify \( v \) as fixed but nonheavy. All the remaining vertices in \( V(P) \) are defined to be nonfixed. Clearly, this whole process can be done in \( O(m + n) \) time.

Afterwards, we label the vertices in \( G - V(P) \) as important or unimportant. To do so, for each heavy vertex \( v \) in \( V(P) \), we iterate through its neighbors in \( G - V(P) \) to pick \( k + 2 \) neighbors of distinct colors, and label them important. This can be done in time \( O(\deg(v)) \), and hence, \( O(m + n) \) time overall. For each fixed nonheavy vertex \( v \) in \( V(P) \), we iterate through its neighbors to determine a cluster \( C \) such that \( v \) has \( \Delta + 2 \) neighbors in \( C \) and \( \Delta + 2 \) nonneighbors in \( C \), and label those vertices as important. Again, this can be done in time \( O(\deg(v)) \), and hence, \( O(m + n) \) time overall. Finally, for each nonfixed vertex \( v \) in \( V(P) \), we iterate through its neighbors to partition them into subsets of the same color; for each subset of neighbors of the same color that belong to a cluster \( C \), we label \( \min\{\Delta + 2, |N(u) \cap C|\} \) (arbitrary) neighbors of \( u \) in \( C \) and \( \min\{\Delta + 2, |C - N(u)|\} \) (arbitrary) nonneighbors of \( v \) in \( C \) as important. This can be done in time \( O(\Delta + \deg(v)) \), and hence in time \( O(\Delta \cdot (m + n)) \) overall.

To apply Reduction Rule 5.8, we go over every cluster \( C \) in \( G - V(P) \), checking if it contains any important vertices, and if not, we remove \( C \) from \( G \). This can be done in \( O(m + n) \) time.

Finally, to apply Reduction Rule 5.9, we again go over every cluster \( C \) in \( G - V(P) \), and remove all but \( \Delta + 2 \) unimportant vertices from \( C \). Again, this can be done in \( O(m + n) \) time. It follows that the kernelization algorithm runs in \( O(k \cdot (m + n)) \) time.

### 6 Conclusion and Outlook

As we have seen in this paper, the pushing process is not only useful for finding efficient algorithms for \((\Pi_A, \Pi_B)\)-Recognition as demonstrated by Kanj et al.\(^\text{[22]}\), but can also be used to classify when or when not such problems admit polynomial kernels. Herein, we focused on the well-motivated case when \( \Pi_A \) is the set of cluster graphs on the first level above triviality; when \( \Pi_A \) is characterized by forbidden induced subgraphs of order at least three. A natural next step is to carry over to other sets of forbidden subgraphs for \( \Pi_A \).

The lower bound given in Theorem 1.1 should in a straightforward manner extend to graph classes \( \Pi_A \) that are closed under disjoint union, have neighborhood diversity\(^a\) at most \( k \), and contain cluster graphs. A more challenging avenue is to try and apply our techniques to related partitioning problems such as Rectangle Stabbing.

Finally, when we parameterized by the size \( k \) of one of the parts, we obtained an \( O(k^d) \)-size kernel (Theorem 1.2), where \( d \) is the largest order of a forbidden subgraph of the other part. Since the techniques used herein are similar to the ones for \( d \)-Hitting Set, it is natural to ask whether this upper bound can be improved to \( O(k^{d-1}) \).

### References

[1] F. N. Abu-Khzam, C. Feghali, and H. Müller. Partitioning a graph into disjoint cliques and a triangle-free graph. *Discrete Appl. Math.*, 190-191:1–12, 2015.

[2] D. Achlioptas. The complexity of G-free colourability. *Discrete Math.*, 165–166(0):21 – 30, 1997.

[3] H. L. Bodlaender, B. M. P. Jansen, and S. Kratsch. Kernelization lower bounds by cross-composition. *SIAM J. Discrete Math.*, 28(1):277–305, 2014.

[4] M. Bougeret and P. Ochem. The complexity of partitioning into disjoint cliques and a triangle-free graph. *Discrete Appl. Math.*, 217:438–445, 2017.

[5] H. Broersma, F. V. Fomin, J. Nešetřil, and G. J. Woeginger. More about subcolorings. *Computing*, 69(3):187–203, 2002.

\(^a\)The neighborhood diversity of a graph is the number of different open neighborhoods.
[6] S. Bruckner, F. Hüffner, and C. Komusiewicz. A graph modification approach for finding core-periphery structures in protein interaction networks. *Algorithms Mol. Biol.*, 10:16, 2015.

[7] Z. A. Chernyak and A. A. Chernyak. About recognizing $(\alpha, \beta)$ classes of polar graphs. *Discrete Math.*, 62(2):133–138, 1986.

[8] R. Churchley and J. Huang. Solving partition problems with colour-bipartitions. *Graph. Combinator.*, 30(2):353–364, 2014.

[9] R. Churchley and J. Huang. On the polarity and monopolarity of graphs. *J. Graph Theory*, 76(2):138–148, 2014.

[10] M. Cygan, F. V. Fomin, L. Kowalik, D. Lokshtanov, D. Marx, M. Pilipczuk, M. Pilipczuk, and S. Saurabh. *Parameterized Algorithms*. Springer, 2015.

[11] R. Diestel. *Graph Theory, 4th Edition*. Springer, 2012.

[12] R. G. Downey and M. R. Fellows. *Fundamentals of Parameterized Complexity*. Texts in Computer Science. Springer, Berlin, Heidelberg, 2013.

[13] E. M. Eschen and X. Wang. Algorithms for unipolar and generalized split graphs. *Discrete Appl. Math.*, 162:195–201, 2014.

[14] A. Farrugia. Vertex-partitioning into fixed additive induced-hereditary properties is NP-hard. *Electron. J. Comb.*, 11(1):R46, 2004.

[15] M. R. Fellows, D. Hermelin, F. Rosamond, and S. Vialette. On the parameterized complexity of multiple-interval graph problems. *Theor. Comput. Sci.*, 410(1):53–61, Jan. 2009.

[16] J. Fiala, K. Jansen, V. B. Le, and E. Seidel. Graph subcolorings: Complexity and algorithms. *SIAM J. Discrete Math.*, 16(4):635–650, 2003.

[17] J. Flum and M. Grohe. *Parameterized Complexity Theory*. Springer, 2006.

[18] S. Foldes and P. L. Hammer. Split graphs. *Congr. Numer.*, 19:311–315, 1977.

[19] L. Fortnow and R. Santhanam. Infeasibility of instance compression and succinct PCPs for NP. *J. Comput. Syst. Sci.*, 77(1):91–106, 2011.

[20] J. Gimbel and C. Hartman. Subcolorings and the subchromatic number of a graph. *Discrete Math.*, 272:139–154, 2003.

[21] P. Heged\'{u}res, D. Kratsch, D. Lokshtanov, V. Raman, and S. Saurabh. Fixed-parameter algorithms for Cochromatic Number and Disjoint Rectangle Stabbing via iterative localization. *Infor. Comput.*, 231:109–116, 2013.

[22] I. Kanj, C. Komusiewicz, M. Sorge, and E. J. van Leeuwen. Parameterized algorithms for recognizing monopolar and 2-subcolorable graphs. *J. Comput. Syst. Sci.*, 92:22–47, 2018.

[23] I. Kanj, C. Komusiewicz, M. Sorge, and E. J. van Leeuwen. Solving Partition Problems Almost Always Requires Pushing Many Vertices Around. In *Proceedings of the 26th Annual European Symposium on Algorithms (ESA ’18)*, volume 112 of *Leibniz International Proceedings in Informatics (LIPIcs)*, pages 51:1–51:14. Schloss Dagstuhl–Leibniz-Zentrum fuer Informatik, 2018.

[24] S. Kolay, F. Panolan, V. Raman, and S. Saurabh. Parameterized Algorithms on Perfect Graphs for Deletion to $(r,l)$-Graphs. In *Proc. 41st MFCS*, volume 58 of *LIPIcs*, pages 75:1–75:13. Schloss Dagstuhl–Leibniz-Zentrum fuer Informatik, 2016.

[25] J. Kratochv\'{i}l and I. Schiermeyer. On the computational complexity of $(\mathcal{O},\mathcal{P})$-partition problems. *Discuss. Math. Graph Theory*, 17(2):253–258, 1997.

[26] V. B. Le and R. Nevries. Complexity and algorithms for recognizing polar and monopolar graphs. *Theor. Comput. Sci.*, 528:1–11, 2014.
[27] C. McDiarmid and N. Yolov. Recognition of unipolar and generalised split graphs. *Algorithms*, 8 (1):46–59, 2015.

[28] S. Skiena. *The Algorithm Design Manual*. Springer, 2008.

[29] J. Stacho. On 2-subcolourings of chordal graphs. In *Proc. 8th LATIN*, volume 4957 of *LNCS*, pages 544–554. Springer, 2008.

[30] R. I. Tyshkevich and A. A. Chernyak. Algorithms for the canonical decomposition of a graph and recognizing polarity. *Izvestia Akad. Nauk BSSR, ser. Fiz. Mat. Nauk*, 6:16–23, 1985. In Russian.