Congruences modulo powers of 5 for three-colored Frobenius partitions

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Abstract

Motivated by a question of Lovejoy [8], we show that three-colored Frobenius partition function $c\phi_3$ and related arithmetic function $\tau\phi_3$ vanish modulo some powers of 5 in certain arithmetic progressions. To be more specific, we show that for every nonnegative integer $n$,

$$c\phi_3(45n + 23) \equiv 0 \pmod{625},$$

$$c\phi_3(45n + 41) \equiv 0 \pmod{625},$$

$$c\phi_3(75n + 22) \equiv 0 \pmod{25},$$

$$\tau\phi_3(75n + 72) \equiv 0 \pmod{25}.$$

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1 Introduction

The ordinary partition of a non-negative integer $n$ is a non-increasing sequence of integers whose sum is $n$. A generalized Frobenius partition of $n$ is a two rowed array of non-negative integers of the form

$$(a_1 \ a_2 \ \cdots \ a_k \ b_1 \ b_2 \ \cdots \ b_k),$$

where the entries in each row are in non-increasing order and the integer $n$ that is partitioned is $\sum_{i=1}^{k}(a_i + b_i + 1)$. A 3-colored Frobenius partition is an array of the above where the integer entries are taken from 3 distinct copies of the non-negative integers distinguished by color and the rows ordered first by size and then by color with no two consecutive like entries in any row. [9] gave such an example. If we denote by $p(n)$ the number of the ordinary partitions of $n$ and by $c\phi_3(n)$ the 3-colored Frobenius partitions of $n$ with the convention that $p(\alpha) = 0$, if $\alpha \notin \mathbb{Z}$, Kotlitsch proved that the generating function for $\tau\phi_3(n) := c\phi_3(n) - p(n/3)$ is

$$\sum_{n=0}^{\infty} \tau\phi_3(n) q^n = 9q \prod_{n \geq 1} \frac{(1 - q^{3n})^3}{(1 - q^{3n})(1 - q^{3n}3^3)}.$$

Since Ramanujan found his famous partition congruences, their generalizations for various partition functions have been the subject of much investigation see [5] [6] [12]. Recently,
Ono [9] and Lovejoy [8] proved the following congruences for small primes and every non-negative integer $n$
\[
\begin{align*}
\phi_3(45n + 23) &\equiv 0 \pmod{5}, \\
\phi_3(45n + 41) &\equiv 0 \pmod{5}, \\
\phi_3(63n + 50) &\equiv 0 \pmod{7}, \\
\phi_3(99n + 95) &\equiv 0 \pmod{11}, \\
\phi_3(171n + 50) &\equiv 0 \pmod{19}.
\end{align*}
\]

Moreover, Lovejoy asked the question: are there generalizations of the congruences (1.1) to powers of 5, 7, 11, 19 analogous to the generalizations of Ramanujan’s congruences for the ordinary partition function? In this note, we prove

**Theorem 1.1.** For every nonnegative integer $n$, we have
\[
\begin{align*}
\phi_3(45n + 23) &\equiv 0 \pmod{625}, \\
\phi_3(45n + 41) &\equiv 0 \pmod{625}.
\end{align*}
\]

Also we give a theorem of another type of theorem 1.1:

**Theorem 1.2.** For every nonnegative integer $n$, we have
\[
\begin{align*}
\phi_3(75n + 22) &\equiv 0 \pmod{25}, \\
\phi_3(75n + 72) &\equiv 0 \pmod{25}.
\end{align*}
\]

We note that there are no similar congruences for the arithmetic progressions and modulus listed in 1.1.

## 2 Preliminaries

When proving congruences we need to determine when the Fourier expansion of a modular form has coefficients which are all multiples of $M$. So we define the $M$-adic order of a formal power series.

**Definition** Let $M$ be a positive integer and $f(z) = \sum_{n \geq N} a(n)q^n$ be a formal power series in the variable $q$ with rational integer coefficients. The $M$-adic order of $f$ is defined by
\[
\text{Ord}_M(f) = \inf\{n \mid a(n) \not\equiv 0 \pmod{M}\}.
\]

Sturm proved the following criterion for determining whether two modular forms are congruent for primes, Ono [10] noted the criterion holds for general integers as modulus.

**Proposition 2.1.** Let $M$ be a positive integer and $f(z), g(z) \in M_k(\Gamma_0(N))$ with rational integers satisfying
\[
\text{ord}_M(f(z) - g(z)) \geq 1 + \frac{kN}{12} \prod_p (1 + \frac{1}{p}),
\]
where the product is over the prime divisors $p$ of $N$. Then $f(z) \equiv g(z) \pmod{M}$, i.e., $\text{ord}_M(f(z) - g(z)) = \infty$. 

Our proofs depend on special holomorphic modular forms on some congruence subgroups. We use the following facts to construct such modular forms. Let
\[
\eta(z) := q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n)
\]
denote the Dedekind’s Eta-function, where \( q = e^{2\pi i z} \) and \( \text{Im}(z) > 0 \). We know that it is a holomorphic modular form of weight \( \frac{1}{2} \) which does not vanish on complex upper half plane. A function \( f(z) \) is called an Eta-product if it can be written in the form of
\[
f(z) = \prod_{\delta | N} \eta^{r_\delta}(\delta z),
\]
where \( N \) and \( \delta \) are positive integers and \( r_\delta \in \mathbb{Z} \).

**Proposition 2.2.** If \( f(z) = \prod_{\delta | N} \eta^{r_\delta}(\delta z) \) is an Eta-product satisfying the following conditions:
\[
k = \frac{1}{2} \sum_{\delta | N} r_\delta \in \mathbb{Z},
\]
\[
\sum_{\delta | N} \delta r_\delta \equiv 0 \pmod{24},
\]
\[
\sum_{\delta | N} \frac{N}{\delta} r_\delta \equiv 0 \pmod{24},
\]
then \( f(z) \) satisfies
\[
f(\frac{az + b}{cz + d}) = \chi(d)(cz + d)^k f(z)
\]
for each \( \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in \Gamma_0(N) \). Here the character \( \chi \) is defined by \( \chi(d) := \left( \frac{-1}{d} \right)^k \), where
\[
s := \prod_{\delta | N} \delta^{r_\delta}
\]
and \( \left( \frac{m}{n} \right) \) is Kronecker symbol.

The analytic orders of an Eta-product at the cusps of \( \Gamma_0(N) \) was calculated by Ligozat.

**Proposition 2.3.** Let \( c, d \) and \( N \) be positive integers with \( d | N \) and \( (c, d) = 1 \). If \( f(z) \) is an Eta-product satisfying the conditions in Proposition 2.2 for \( N \), then the order of vanishing of \( f(z) \) at the cusp \( \frac{c}{d} \) is
\[
\frac{N}{24} \sum_{\delta | N} \frac{(d, \delta)^2 r_\delta}{(d, \frac{N}{d})d\delta}.
\]

We also use the following proposition to construct modular forms, see Koblitz.
Proposition 2.4. Let \( \chi_1 \) be a Dirichlet character modulo \( N \), and let \( \chi_2 \) be a primitive Dirichlet character modulo \( M \). Suppose \( f(z) \in M_k(\Gamma_0(N), \chi_1) \) with Fourier expansion
\[
f(z) = \sum_{n=0}^{\infty} u(n) q^n.
\]
Then for any positive integer \( t \mid N \),
\[
f(z)\|U(t) := \sum_{n=0}^{\infty} u(tn) q^n
\]
is the Fourier expansion of a modular form in \( M_k(\Gamma_0(N), \chi_1) \). For any positive integer \( t \mid N \),
\[
f(z)\|V(t) := \sum_{n=0}^{\infty} u(n) q^{tn}
\]
is the Fourier expansion of a modular form in \( M_k(\Gamma_0(tN), \chi_1) \). Let
\[
g(z) = \sum_{n=0}^{\infty} u(n) \chi_2(n) q^n.
\]
Then \( g(z) \in M_k(M^2N, \chi_1 \chi_2^2) \).

The last proposition we shall use is due to Eichhorn and Ono [2].

Proposition 2.5. If \( l \) is a prime and \( s \geq 1 \), then the Eta-product \( \frac{\eta^{ls}(z)}{\eta^{s-1}(lz)} \) satisfies
\[
\frac{\eta^{ls}(z)}{\eta^{s-1}(lz)} \equiv 1 \pmod{l^s}.
\]

3 The Congruence mod 625

In this section, we apply propositions above to prove Theorem 1.1.

Theorem 3.1. Define \( a(n) \) by the infinite product
\[
\sum_{n=0}^{\infty} a(n) q^n = \prod_{n=1}^{\infty} \frac{1}{(1-q^{3n})(1-q^n)^3},
\]
then the coefficients of \( a(n) \) satisfies
\[
a(n) \equiv 0 \pmod{625} \text{ if } n \equiv 13, 22, 31, 40 \pmod{45}.
\]
Proof. Define the following Eta-product
\[ g(z) = \frac{\eta^{13}(45z)\eta^3(135z)}{\eta(3z)\eta^3(z)} \left( \frac{\eta^{625}(z)}{\eta^{125}(5z)} \right)^2 \]
\[ = q^{41} \left( \prod_{n=1}^{\infty} \frac{1}{(1 - q^{3n})(1 - q^n)^3} \right) \]
\[ \cdot \left( \prod_{n=1}^{\infty} (1 - q^{45n})^{13}(1 - q^{135n})^3 \right) \]
\[ \cdot \left( \prod_{n=1}^{\infty} (1 - q^{n})^{1250} \right) \]
\[ \cdot \left( \prod_{n=1}^{\infty} (1 - q^{5n})^{250} \right) \]
\[ := \sum_{n \geq 41} r(n)q^n. \]

By Propositions 2.2 and 2.3 it turns out that \( g(z) \in S_{506}(\Gamma_0(135), Id) \), where \( Id \) is the trivial Dirichlet character mod 135. By Proposition 2.5 the product of the first 3 factors of the middle expression above possesses the congruence properties of the Fourier coefficients of \( g(z) \) mod 625. We note that Theorem 3.1 is equivalent to the congruences
\[ r(n) \equiv 0 \pmod{625} \text{ if } n \equiv 9, 18, 27, 36 \pmod{45}. \]

Use the Proposition 2.4 we find that the modular form
\[ g(z) | U(9) - g(z) | U(45)V(5) \]
\[ = \sum_{n \geq 0} r(9n)q^n - \sum_{n \geq 0} r(45n)q^{5n} \]
\[ = \sum_{n \not\equiv 0 \pmod{5}} r(9n)q^n \]
is in \( M_{506}(\Gamma_0(675), Id) \). By Sturm’s criterion, if it can be shown that \( r(9n) \equiv 0 \pmod{625} \) when \( n \leq 45541 \) and \( n \not\equiv 0 \pmod{5} \), then our theorem follows. These have been verified by machine computation so we proved the theorem.

Proof of Theorem 1.1. By Jacobi’s triple product identity, we have
\[ c\phi_3(5n + 23) \]
\[ = \overline{c\phi_3}(5n + 23) \]
\[ = 9 \sum_{k \geq 0} (-1)^k (2k + 1)(a(45n + 23 - (1 + \frac{9k^2 + 9k}{2})))), \]
\[ c\phi_3(45n + 41) \]
\[ = \overline{c\phi_3}(45n + 41) \]
\[ = 9 \sum_{k \geq 0} (-1)^k (2k + 1)(a(45n + 41 - (1 + \frac{9k^2 + 9k}{2})))). \]

Since modulo 45, \( 1 + \frac{9k^2 + 9k}{2} \) is 1, 10, 28, so 45n + 23 - (1 + \frac{9k^2 + 9k}{2}) and 45n + 41 - (1 + \frac{9k^2 + 9k}{2}) are congruence to 13, 22, 31, 40 modulo 45. By Theorem 3.1, Theorem 1.1 is proved.
4 The Congruence mod 25

In this section we prove Theorem 1.2.

Proof of Theorem 1.2 Define the following Eta-product

\[ f(z) = 9 \frac{\eta^3(9z)\eta(75z)}{\eta(3z)\eta^3(z)} \left( \frac{\eta^{25}(z)}{\eta^5(5z)} \right)^2 \]

\[ = 9q^4 \left( \prod_{n=1}^{\infty} \frac{(1-q^{9n})^3}{(1-q^{4n})(1-q^n)^3} \right) \left( \prod_{n=1}^{\infty} (1-q^{75n}) \right) \]

\[ \cdot \left( \prod_{n=1}^{\infty} \frac{(1-q^n)^{50}}{(1-q^{5n})^{10}} \right) \]

\[ = q^3 \left( \sum_{n=0}^{\infty} c_{\phi_3}(n)q^n \right) \left( \prod_{n=1}^{\infty} (1-q^{75n}) \right) \left( \prod_{n=1}^{\infty} \frac{(1-q^n)^{50}}{(1-q^{5n})^{10}} \right) \]

\[ := \sum_{n \geq 4} c(n)q^n. \]

By Propositions 2.2 and 2.3, We find \( f(z) \) is in \( S_{20}(\Gamma_0(225), Id) \), where \( Id \) is the trivial character modulo 225. We note that our Theorem 1.2 is equivalent to the following congruences: For every nonnegative integer \( n \)

\[ c(75n + 25) \equiv 0 \pmod{25}, \]
\[ c(75n) \equiv 0 \pmod{25}. \]

(4.1)

Let

\[ f_2(z) = \sum_{n \geq 4} c(n) \left( \frac{n}{3} \right) q^n. \]

By Proposition 2.4, \( f_2(z) \) is in \( S_{20}(\Gamma_0(2025), Id) \). Define another modular form \( F(z) \in S_{20}(\Gamma_0(2025), Id) \) by

\[ F(z) := \sum_{n=0}^{\infty} d(n)q^n \]

\[ = f(z) + f_2(z) \]

\[ = \sum_{(\frac{n}{4})=1} 2c(n)q^n + \sum_{n \equiv 0 \pmod{3}} c(n)q^n. \]

Apply Hecke operator \( U(25) \), we obtain

\[ F(z) \mid U(25) = \sum_{n \geq 1} d(75n)q^n. \]

If \( F(z) \mid U(25) \equiv 0 \pmod{25} \), then 4.1 holds. By Sturm’s criterion, we need to check that the congruence holds for the first \( 25 \cdot (20/12)[SL_2(Z) : \Gamma_0(2025)] + 1 = 135001 \) terms which is easily verified. ■
5 Conclusion

We find there are no similar congruences for powers of 7, 11, 19 in arithmetic progressions given in 1.1. So we ask: are there congruences for higher powers of five or other primes in the arithmetic progressions of type $3^a5^b \cdot n + \gamma(\alpha, \beta)$? Second, find identities for $c_{\phi_3}$, which interpret congruences above for $c_{\phi_3}$, analogous to the Ramanujan-type identities for the ordinary partition function interpret Ramanujan’s congruences. We note that similar results were obtained by Sellers [12].

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