A Framework for Moment Invariants

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Abstract

For more than half a century, moments have attracted lot ot interest in
the pattern recognition community. The moments of a distribution (an ob-
ject) provide several of its characteristics as center of gravity, orientation,
disparity, volume. Moments can be used to define invariant characteris-
tics to some transformations that an object can undergo, commonly called
moment invariants. This work provides a simple and systematic formalism
to compute geometric moment invariants in n-dimensional space.

keywords:

Geometric moments, affine invariants, n-dimensional space.

1 Introduction

Since they were introduced in 1962 by Hu [1], Moment Invariants have gener-
ated a lot of interest in the pattern recognition field. During more than half
a century, many theoretical frameworks and applications have been developed:
pose estimation [2, 3], character recognition [4], target recognition [5], quality
inspection [6], image matching [7], multi-sensors fusion [8] and visual servin
g [9, 10],

Initially, moment invariants have been mainly applied in 2D space. Subse-
quently, invariant from moments defined in higher dimensional space have been
defined [8, 10, 11]. Although using moments of order higher than 2 is not new
[12, 13, 14], such applications are gaining interest thanks to 3D vision sensors.

The first important result for deriving moment invariants is the fundamental
theorem of moment invariants (FTMI) [1]. Hu employed his theorem to derive
seven 2D moment invariants. In fact, the FTMI contains some mistakes that
have been emphasized by Mamistvalov in [15] in 1970 (in Russian). Despite
this, the FTMI as proposed by Hu has been quoted in several works until 1991
when Reiss established the revised fundamental theorem of moment invariants
(RFTMI) in 2D Space. The FTMI was generalized for the n-dimensional case
and applied in multi-sensor fusion in [8]. Unfortunately, the generalized theo-
rem contains the same mistake as the one given by Hu. Finally, Mamistvalov
proposed the correct generalization of the RFTMI to n-dimensional solids. The idea behind using the FTMI is that invariants of n-ary forms are also invariant in the case where the coefficients of the n-ary form are replaced by the corresponding moments. [16] proposed an algorithm for determining invariants of binary forms. In [17], a systematic method to derive independent moment invariants to orthogonal transformations in n-dimensional space has been proposed. The proposed scheme is based on rotation speed tensor rather than rotation matrix to obtain the invariant to rotations. More recent works in the last decade proposed systematic schemes for 2D and 3D cases [18, 19], [20]. Despite the difference of terminology, the way to construct invariants in [16] and in [18] can be considered as similar.

This paper proposes a unified and novel scheme to derive affine invariants from geometric moments of n-dimensional solids. The idea behind the method is to consider that any affine transformation can be decomposed using SVD into a rotation followed by a non-uniform scale change then by another rotation. Firstly, the method proposed in this paper is quite simple to understand. Second, the method ensures that all possible invariants are obtained and redundancy between different invariants is quite easy to express and to eliminate using matrix algebra. In the next section, basic definitions of moment in n-dimensional space are presented. Then, in the subsequent sections, invariance to scale changes, to rotation and affine transformations will be dealt with.

2 Geometrical moments in n-dimensional space

2.1 Notations

The following notations will be used in the sequel:

- \( n \): the dimension of the considered space.
- \( m_{p_1...p_n} \) and \( \mu_{p_1...p_n} \): a moment and a centered moment of order \( p = p_1 + \ldots + p_n \) of n-dimensional object.
- \( \mathbf{v}^1_p \): vector composed by all moments of order \( p \).
- \( \mathbf{v}^k_p \): vector composed by all monomials of degree \( k \) and using as variable the entries of \( \mathbf{v}^1_p \).
- \( \mathbf{v}^{(kk'k''...)}_{(pp'p''...)} \): vector obtained by all possible products between the entries of \( \mathbf{v}^k_p, \mathbf{v}^{k'}_{p'} \).
- \( \mathbf{S} = \text{diag}(\sigma_1, \sigma_2, \ldots, \sigma_n) \): scale change transformation in n-dimensional space.
- \( \mathbf{R} \): orthogonal transformation in n-dimensional space.
- \( \mathbf{A} \): non-singular \( n \times n \) non singular matrix defining an affine transformation in n-dimensional space.
• $i_s$: invariant to scale change.
• $i_r$: invariant to rotations.
• $i_a$: affine invariant.

2.2 Geometrical moments in n-dimensional space

We first recall some basic definitions of moment functions. Denoting $X = (x_1, \ldots, x_n)$ the coordinates of a point in a n-dimensional space, the moments of the density function $f(X)$ are defined by:

$$m_{p_1 \ldots p_n} = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} x_1^{p_1} \cdots x_n^{p_n} f(x_1, \ldots, x_n) dx_1 dx_2 \cdots dx_n$$

$$= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} x_1^{p_1} \cdots x_n^{p_n} f(X) dX. \quad (1)$$

The moments of the density function $f(X)$ exist if $f(X)$ is piecewise continuous and has nonzero values only in a finite region of the space. The moment $m_{p_1 \ldots p_n}$ is called of order $p = p_1 + \ldots + p_n$.

Similarly, the centered moments of order $p$ are defined by:

$$\mu_{p_1 \ldots p_n} = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} (x_1 - \overline{x}_1)^{p_1} \cdots (x_n - \overline{x}_n)^{p_n} f(X) dX, \quad (2)$$

where $(\overline{x}_1 = m_{0,0, \ldots, 0}, \overline{x}_2 = m_{0,0, \ldots, 0}, \ldots, \overline{x}_n = m_{0,0, \ldots, 0})$ are the coordinates of the object gravity center. It is well known that the centered moments are invariant to translations in their respective n-dimensional space. In the sequel, the objects are considered centered at the frame origin, which means that $\mu_{p_1 \ldots p_n} = m_{p_1 \ldots p_n}$.

Based on central moments, we propose an automatic scheme for deriving invariants to scale, to orthogonal transformations and finally to affine invariants. The results presented in this paper for continuous case can be extended straightforwardly moments of discrete distribution defined by

$$m_{p_1 \ldots p_n} = \sum \cdots \sum x_1^{p_1} \cdots x_n^{p_n} f(x_1, \ldots, x_n) \quad (3)$$

3 Scale change

A scale change in n-dimensional space is defined by the following transformation:

$$X' = S X \quad (4)$$

where $S = \text{diag}(\sigma_1, \sigma_2, \ldots, \sigma_n)$. The scale change is called uniform if $\sigma_1 = \sigma_2 = \ldots = \sigma_n = \sigma$. Invariants to uniform scale are easy to derive. Using (2) it is easy to show that after a scale change defined by $\sigma$, $\mu_{0,0, \ldots, 0}$ and moments of higher $\mu_{p_1 \ldots p_n}$ are multiplied by $\sigma^n$ and $\sigma^{n+p}$ respectively (remind $p = p_1 + \ldots + p_n$). Therefore, the ratio $\frac{\mu_{p_1 \ldots p_n}}{\mu_{0,0, \ldots, 0}}$ is an invariant to uniform scale change.
Let us now deal with a non-uniform scale change. For that, let define the following product of moments:

\[ p_m = \prod_{i=1}^{l} \mu_{p_i^1 \ldots p_i^n}, \]  

such that

\[ \sum_{i=1}^{l} (p_i^j + 1)k_i = d, \ \forall j = 1, \ldots, n \]  

and where \( d \) is a positive integer. From (5), it can be seen that \( p_m \) can be a product of \( l \) moments of different orders. For each moment \( \mu_{p_i^1 \ldots p_i^n} \) of the product, the power on the coordinate \( x_i \) is given by the integer \( p_i^j \). This means that after the scale change \( S \), \( x_i^{p_i^j} \) and \( x_i^{p_i^j} \, dx_i \) are multiplied by \( \sigma_i^{p_i^j} \) and \( \sigma_i^{p_i^j+1} \) respectively. This results in a multiplication by \( \prod_{j=1}^{N} (\sigma_j^{p_j^j+1})k_i \) on \( \mu_{p_i^1 \ldots p_i^n} \) and \( \mu_{k_i^1 \ldots k_i^n} \) respectively. Therefore, after the scale change \( S \), the whole product \( p_m \) is multiplied by \( \prod_{j=1}^{N} (\sigma_j^{d}) = \prod_{j=1}^{n} \sigma_j^d \) (if the condition (6) is satisfied). Finally, since after (4), \( \mu_{0,0,\ldots} \) is multiplied by \( \prod_{j=1}^{d} \sigma_j \), the ratio:

\[ i_s = \frac{p_m}{\mu_{0,0,\ldots}} \]  

is an invariant to the scale change defined by (4).

Let us now gives some example of invariants to scale in 2D and 3D spaces:

- **2D space:** \( \frac{\mu_{11}}{\mu_{00}}, \frac{\mu_{20}}{\mu_{00}}, \frac{\mu_{30}}{\mu_{00}}, \frac{\mu_{21}}{\mu_{00}}, \frac{\mu_{40}}{\mu_{00}}, \frac{\mu_{31}}{\mu_{00}}, \frac{\mu_{50}}{\mu_{00}}, \frac{\mu_{41}}{\mu_{00}} \)

- **3D space:** \( \frac{\mu_{111}}{\mu_{000}}, \frac{\mu_{200}}{\mu_{000}}, \frac{\mu_{300}}{\mu_{000}}, \frac{\mu_{211}}{\mu_{000}}, \frac{\mu_{400}}{\mu_{000}}, \frac{\mu_{311}}{\mu_{000}}, \frac{\mu_{500}}{\mu_{000}}, \frac{\mu_{411}}{\mu_{000}}, \frac{\mu_{600}}{\mu_{000}}, \frac{\mu_{511}}{\mu_{000}} \)

- **4D space:** \( \frac{\mu_{1111}}{\mu_{0000}}, \frac{\mu_{2000}}{\mu_{0000}}, \frac{\mu_{3000}}{\mu_{0000}}, \frac{\mu_{2111}}{\mu_{0000}}, \frac{\mu_{4000}}{\mu_{0000}}, \frac{\mu_{3111}}{\mu_{0000}}, \frac{\mu_{5000}}{\mu_{0000}}, \frac{\mu_{4111}}{\mu_{0000}}, \frac{\mu_{6000}}{\mu_{0000}}, \frac{\mu_{5111}}{\mu_{0000}} \)

In the next section, a new scheme to obtain invariants to orthogonal transformation is given.

### 3.1 Invariants to orthogonal transformation

#### 3.2 Orthogonal transformation and rotation speed

An orthogonal transformation is defined by the relation:

\[ \mathbf{X}' = \mathbf{R} \mathbf{X}, \]  

where \( \mathbf{R} \) is a rotation matrix that have to satisfy \( \mathbf{RR}^\top = \mathbf{I} \) and \( det(\mathbf{R}) = 1 \) (\( \mathbf{I} \) is the identity matrix in the n-dimensional space). One can show that a moment after a rotation can be expressed as linear combination of moments of
the same order, where the coefficients of those combinations are polynomials on rotation matrix entries. The rotation invariance of some functions of moments are obtained thanks to the orthogonality condition of \( \mathbf{R} \). The latter is composed of constraints that are nothing but polynomials of orders 2 on the rotation matrix entries. For this reason, it is not possible to build invariants to rotations from moments of odd orders without putting them to some even power.

Rather basing our reasoning on the rotation matrix and its orthogonality condition, it is possible to build invariants based on rotation speed (Tensor) in very simple way. For that, let us consider that an n-dimensional solid undergoes a transformation in time. If a rotational speed is applied, the speed of each point of an n-dimensional object is given by:

\[
\dot{\mathbf{X}} = \mathbf{LX}
\]

where \( \mathbf{L} \) is an antisymmetric matrix defined by rotational speeds in the 2D planes built by two different axes. For instance \( \mathbf{L} \) is defined as follows:

- In 2D space:
  \[
  \mathbf{L}_{2D} = \begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix}
  \] (10)
  where \( \omega \) is the rotation speed in the 2D plane.

- In 3D space:
  \[
  \mathbf{L}_{3D} = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ -\omega_3 & 0 & -\omega_1 \\ \omega_2 & \omega_1 & 0 \end{bmatrix}
  \] (11)
  where \( \omega_1, \omega_2, \omega_3 \) are respectively the rotation speeds in the planes \( yz \), \( zx \) and \( xy \). The scalars \( \omega_1, \omega_2, \omega_3 \) can be also be called respectively the rotation speeds around the x-axis, the y-axis and the z-axis in the case of 3D space.

- In 4D space, let us consider that the 4D frame has 4 axes \( x, y, z \) and \( w \). The antisymmetric matrix \( \mathbf{L}_{4D} \) can be defined by:

\[
\mathbf{L}_{4D} = \begin{bmatrix} 0 & \omega_1 & -\omega_2 & \omega_3 \\ -\omega_1 & 0 & \omega_4 & -\omega_5 \\ \omega_2 & -\omega_4 & 0 & \omega_6 \\ -\omega_3 & \omega_5 & -\omega_6 & 0 \end{bmatrix}
\] (12)

where \( \omega_1, \omega_2, \omega_3, \omega_4, \omega_5, \omega_6 \) are respectively the rotation speeds in the planes \( xy, xz, xw, yz, yw \) and \( zw \).

Actually, every two frame axis of an n-dimensional space specifies a 2D plane. Therefore, there exist \( \frac{n!}{(n-2)!2} \) different planes and then the same number of rotational speeds.
3.3 Moment time variation and rotational speeds

After taking the derivative of (1), the time variation of a moment $m_{p_1...p_n}$ is obtained by [17], [21]:

$$
\dot{m}_{p_1...p_n} = \sum_{i=0}^{n} \sum_{j=0}^{n} p_i \dot{x}_i \chi_{p_1}^{p_i-1} \prod_{j=1, j \neq i}^{n} x_j^{p_i} \chi_{p_n}^{p_n} f(x_1, ..., x_n) dx_1 dx_2 ... dx_n
$$

+ $\sum_{i=0}^{n} \sum_{j=0}^{n} p_i \dot{x}_i \chi_{p_1}^{p_i-1} \prod_{j=1, j \neq i}^{n} x_j^{p_i} \chi_{p_n}^{p_n} f(x_1, ..., x_n) dx_1 dx_2 ... dx_n$

+ $\sum_{i=0}^{n} \sum_{j=0}^{n} p_i \dot{x}_i \chi_{p_1}^{p_i-1} \prod_{j=1, j \neq i}^{n} x_j^{p_i} \chi_{p_n}^{p_n} f(x_1, ..., x_n) (\sum_{i=0}^{n} \dot{x}_i) dx_1 dx_2 ... dx_n$

(13)

Since the matrix $L$ is antisymmetric (the diagonal entries are null), we have $\frac{\partial L}{\partial x_i} = 0$. This implies that the third term of (13) vanishes when rotational speeds are applied to the object. The second term of (13) vanishes as well if we assume that the time derivative of density function $\dot{f}(x_1, ..., x_n) = 0$. The latter assumption has been also made in [22] and [23] to prove the revised FTMI. Assuming $\dot{f}(x_1, ..., x_n) = 0$ means that applying a rotation on the object point does not change its corresponding density function value. According to (13), the time variation of moment caused by rotational speeds can be obtained by:

$$
\dot{m}_{p_1...p_n} = \sum_{i=0}^{n} \sum_{j=0}^{n} p_i l_{ij} m_{p_1,...,p_{i-1},p_j+1,p_n}.
$$

(14)

Let us now consider for instance moments from 2D, 3D and 4D spaces.

- In 2D space, using (13) leads to:
  $$
  \dot{m}_{p_1p_2} = (-p_1 m_{p_1-1,p_2+1} + p_2 m_{p_1+1,p_2-1}) \omega.
  $$

  (15)

- In 3D space, using (13) leads to:
  $$
  \dot{m}_{p_1p_2p_3} = (p_2 m_{p_1,p_2-1,p_3+1} - p_3 m_{p_1,p_2+1,p_3-1}) \omega_1
  + (p_3 m_{p_1-1,p_2,p_3-1} - p_1 m_{p_1-4,p_2,p_3+1}) \omega_2
  + (p_1 m_{p_1-1,p_2+1,p_3} - p_2 m_{p_1+1,p_2-1,p_3}) \omega_3.
  $$

  (16)

- In 4D space, using (13) leads to:
  $$
  \dot{m}_{p_1p_2p_3p_4} = (p_1 m_{p_1-1,p_2,p_3,p_4} - p_2 m_{p_1+1,p_2-1,p_3,p_4}) \omega_1
  + (p_3 m_{p_1+1,p_2-1,p_3,p_4} - p_1 m_{p_1-1,p_2,p_3+1,p_4}) \omega_2
  + (p_1 m_{p_1-1,p_2,p_3+1} - p_4 m_{p_1+1,p_2,p_3-1}) \omega_3
  + (p_2 m_{p_1-1,p_2,p_3} - p_3 m_{p_1+1,p_2-1,p_3,p_4}) \omega_4
  + (p_4 m_{p_1+1,p_2,p_3+1} - p_2 m_{p_1-1,p_2,p_3+1}) \omega_5
  + (p_3 m_{p_1+1,p_2,p_3} - p_4 m_{p_1+1,p_2,p_3+1}) \omega_6.
  $$

(17)
In the next section, a simple method to determine invariant to rotations using the relation between the time variation of moments and the rotational speeds.

3.4 Time variation of Moment vectors

Let $v_p^1$ be the vector composed by all moments of order $p$. Let $v_p^k$ be the monomials vector of degree $k$ computed from $v_p^1$. As an example from 2D space for $p = 2$ and $k = 2$:

$$v_2^1 = [m_{20}, m_{11}, m_{02}]^T, \quad v_2^2 = [m_{20}^2, m_{20}m_{11}, m_{02}m_{20}, m_{11}^2, m_{02}m_{11}, m_{02}^2]^T$$

Let us also define moment vectors mixing moments of different orders $v_p^k$ and $v_p^{k'}$. For instance, by multiplying the entries of $v_2^1$ and $v_2^3$, a new moment vector can be obtained as

$$m_{1(2,3)} = \begin{bmatrix} m_{20}m_{20} & m_{20}m_{30}m_{21} & m_{20}m_{30}m_{12} & \ldots & m_{02}m_{02}^3 \end{bmatrix}^T.$$}

Moment vectors from three different orders or more can also be defined. For instance $m_{(k,k',k'')}^{k,k',k''}$ by all possible products between the entries of the vectors $m_p^k, m_p^{k'}$ and $m_p^{k''}$.

Using (14), it can be concluded that the time variation of each entry of $v_p^1$ is a linear combination of the other entries and the rotational speed. This implies that the time variation of the moment vector $v_p^1$ can be written as follow:

$$\dot{v}_p^1 = \sum_{i=0}^{\frac{n!}{(n-2)!2!}} L_{\omega_i} v_p^1 \omega_i$$

(18)

where $L_{\omega_i}^p$ are matrices of integers. As a simple example from 2D space, let us consider $v_2^1 = [m_{20}, m_{11}, m_{02}]^T$. For the latter vector, using (15), we obtain:

$$\dot{v}_2^1 = L_{\omega_2}^1 v_2^1 = \begin{bmatrix} 0 & -2 & 0 \\ 1 & 0 & -1 \\ 0 & 2 & 0 \end{bmatrix} v_2^1 \omega.$$ 

(19)

Since $m_p^k$ is defined by the monomials on $m_p^1$, the time derivative $\dot{m}_p^k$ can be expressed as linear combination of the $m_p^k$ entries. We have then

$$\dot{v}_p^k = \sum_{i=0}^{\frac{n!}{(n-2)!2!}} L_{\omega_i}^k v_p^k \omega_i. $$

(20)

Since $L_{\omega_i}^p$ are matrices of integers, $L_{\omega_i}^q$ are also of the same nature. This is also true for any vectors $v_{(p,p',p''})$ mixing moments of different orders. In the following, it will be shown that it is easy to build invariants to rotations as linear combination of the entries of the previous kinds of moment vectors.
3.4.1 Rotation invariants

Let us consider a scalar $i_{kr}^p = \alpha^k_p v^k_p$, where $\alpha^k_p$ is a vector of coefficients. Therefore, the time derivative of $i_{kr}^p$ is defined by:

$$\frac{di_{kr}^p}{dt} = \alpha^k_p \nabla v^k_p.$$  \hfill (21)

Combining (21) with (20) gives:

$$\frac{di_{kr}^p}{dt} = \sum_{i=0}^{(n-2)!2!} \frac{n!}{(n-2)!2!} \alpha^k_p L_{v^p_i} v^k_p \omega_i.$$  \hfill (22)

The scalar $i_{kr}^p$ is invariant rotation if:

$$\alpha^k_p L_{v^p_i} v^k_p = 0, \ \forall i = 1, \ldots, \frac{n!}{(n-2)!2!}.$$  \hfill (23)

This implies the following condition:

$$L_{v^p_i}^\top \alpha^k_p = 0, \ \forall i = 1, \ldots, \frac{n!}{(n-2)!2!}.$$  \hfill (23)

If we take again the simple moment vector $v^1_2 = [m_{20}, m_{11}, m_{02}]^\top$, we have:

$$L_{v^p_2} = \begin{bmatrix} 0 & 1 & 0 \\ -2 & 0 & 2 \\ 0 & -1 & 0 \end{bmatrix}$$

The null space of $L_{v^p_2}$ is:

$$\alpha_2^1 = [1, 0, 1]^\top$$

Which gives as invariant $i_{p1} = m_{20} + m_{02}$.

In the next section, the results about both invariant to rotations and those to non-uniform scale change will be used to derive affine invariants in a very straightforward manner.

4 Affine invariants

4.1 Proposed method

Since rotation transformation is a subgroup of affine ones, one can conclude easily that an affine invariant is an invariant to rotation. Let us consider that the affine transformation is defined by a matrix $A$, such that:

$$X' = AX.$$  \hfill (24)
Here, the translation has not been considered since we consider that the object is centered. Actually, using singular value decomposition, any non-singular matrix $A$ can be decomposed as:

$$A = R_2 S R_1,$$

where $R_1$ and $R_2$ are rotations and $S$ is non uniform scale change. This means that an affine transformation is equivalent to a rotation followed by a non-uniform scale change then by another rotation. To explain the proposed approach, let us consider building affine invariants from:

$$v_2^2 = [m_{20}^2, m_{20}m_{11}, m_{20}m_{02}, m_{11}^2, m_{02}m_{11}, m_{02}^2]^T$$

In the vector of $v_2^2$, two entries hold the condition to be invariant to non-uniform scale change (6), which are $m_{20}^2$ and $m_{20}m_{02}$. The vector $v_s^2 = \frac{1}{m_{00}} [m_{20}m_{02}, m_{11}^2]^T$ composed by these moments products is an invariant to non-uniform scale change. Using (15), their time derivatives are given by:

$$\left\{ \begin{array}{l}
d\frac{(m_{20}m_{02})}{dt} = m_{20}m_{02} + m_{20}m_{02} = (-2m_{02}m_{11} + 2m_{20}m_{11})\omega \\
d\frac{(m_{11}^2)}{dt} = 2m_{11}m_{11} = (2m_{11}m_{20} - 2m_{11}m_{02})\omega 
\end{array} \right. \quad (25)$$

Since $m_{40}^4$ is invariant to rotations, from (26) we obtain:

$$v_s^2 = \frac{1}{m_{00}} \left[ \frac{d(m_{20}m_{02})}{dt} \frac{d(m_{11}^2)}{dt} \right] = L_{v_s^2} v_2^2 \omega = \begin{bmatrix} 0 & 2 & 0 & 0 & -2 & 0 \\ 0 & 2 & 0 & 0 & -2 & 0 \end{bmatrix} \frac{v_2^2}{m_{00}^4} \omega$$

Let us consider $i_{a_2} = \alpha_2^T v_s^2$, where $\alpha_2^T$ is vector of coefficients of the same size as $v_s^2$. As it has been shown in the previous section, $i_{a_2}$ is rotation invariant if $\alpha_2^T L_{v_s^2} = 0$. Therefore, it can be obtained that $\alpha = ker(L_{v_s^2}) = [1, -1]^T$, which leads to the invariant:

$$i_{a_2} = \frac{m_{20}m_{02} - m_{11}^2}{m_{00}^4} \quad (27)$$

which is the well known and the simplest affine invariant in 2D space. Actually, this scheme can be applied to any moment vector in a n-dimensional space as follow:

- Consider a moment vector $v_p^k$ (or one combining moments of different orders $v^{(k,k',k'')}_{(p,p',p'')}$),

- Build a new moment vector $v_s$ by selecting only entries of $v_p^k$ if divided by the required power of $m_{0...0}^d$ become invariants to non-uniform scale change.
• The time derivative of \( \mathbf{v}_s \) when rotational speeds are applied to the object can be written as:

\[
\dot{\mathbf{v}}_s = \frac{1}{m_0^{d...0}} \sum_{i=0}^{n-1} \mathbf{L}^i_{\mathbf{v}_s} \mathbf{v}_p^k \omega_i. \tag{28}
\]

• Compute \( i_a = \mathbf{\alpha}^\top \mathbf{v}_s \) where:

\[
\mathbf{\alpha}^\top \mathbf{L}^i_{\mathbf{v}_s} \mathbf{v}_p^k = 0, \quad \forall i = 1, \ldots, \frac{n!}{(n-2)!2!}
\]

Since each entry of \( \mathbf{v}_s \) is an invariant to non-uniform scale change and the linear combination \( i_a = \mathbf{\alpha}^\top \mathbf{v}_s \) is invariant to rotation, therefore \( i_a \) is an affine invariant. In the following, some example invariant to rotations and to affine transformations in 4D space.

\[
i_{a(1)}^1 = m_0^2 - m_0 m_1 m_2 + m_0 m_3 m_4 + m_1^2 \]

\[
i_{a(2)}^1 = 3m_0^2 - 6m_0 m_1 + 3m_1^2 + 3m_2^2 + 3m_3^2 + 3m_4^2
\]

\[
i_{a(1)}^2 = \frac{1}{m_{0000}^2} (m_{0111}^2 m_{1100}^2 - m_{0200}^2 m_{2000}^2 - m_{0011}^2 m_{0101}^2 m_{0011}^2 + 2m_{2000}^2 m_{0011}^2 m_{0101}^2)
\]

\[
i_{a(2)}^2 = 2m_{0101}^2 m_{1100}^2 - 2m_{0200}^2 m_{2000}^2 - 2m_{0011}^2 m_{0110}^2 m_{1100}^2 + 2m_{0011}^2 m_{0110}^2 m_{1100}^2
\]

\[
i_{a(3)}^2 = 2m_{0020}^2 m_{0101}^2 m_{1010}^2 m_{1010}^2 + 2m_{0002} m_{0011} m_{0101} m_{1010}^2 m_{1100}^2 + m_{0002}^2 m_{0011} m_{0102} m_{1010} m_{1100}^2
\]

\[
i_{a(4)}^2 = m_{0002}^2 m_{0011}^2 m_{0101}^2 + 2m_{0002} m_{0011} m_{0102} m_{1010} m_{1100}^2 m_{1100}^2
\]

\[
i_{a(5)}^2 = m_{0002}^2 m_{0011}^2 m_{0102} m_{1010} m_{1100}^2 m_{1100}^2
\]

\[
i_{a(6)}^2 = m_{0002}^2 m_{0011}^2 m_{0102} m_{1010} m_{1100}^2 m_{1100}^2
\]
4.2 Discussion

The set of constraints given by (23) means that if \(i_r\) is invariant to rotation, its time variation caused by any of the \(\frac{n!}{(n-2)!2}\) rotational speeds is null. Let us consider the case of 3D space. In that case, as it has been mentioned above, there exist \(\omega_1, \omega_2, \omega_3\) are respectively the rotation speeds in the planes \(yz, zx\) and \(xy\) respectively. Let us consider a vector of 3D moments \(\mathbf{v}_p^k\) and \(i_{rk} = \alpha_p^k \mathbf{v}_p^k\) an invariant to rotations. According to (23), \(\alpha\) satisfies the three conditions:

\[
\begin{align*}
(\text{c1}) \quad & \mathbf{L}_{\mathbf{v}_p^k}^{\omega_1^T} \alpha_p^k = 0 \\
(\text{c2}) \quad & \mathbf{L}_{\mathbf{v}_p^k}^{\omega_2^T} \alpha_p^k = 0 \\
(\text{c3}) \quad & \mathbf{L}_{\mathbf{v}_p^k}^{\omega_3^T} \alpha_p^k = 0.
\end{align*}
\]

Each of the three constraints implies the invariance of \(i_r\) with respect to the rotations in the plane corresponding to \(\omega_i\). This means that the constraints \(c_1\) and \(c_2\) implies invariance with respect to rotations in the planes \(yz\) and \(zx\). Actually, if \(c_1\) and \(c_2\) are satisfied, there is no need to check \(c_3\) since a rotation in the \(xy\) plane can be expressed as two consecutive rotations in the plane \(xz\) to bring the \(x\)-axis to the \(z\)-axis position then a rotation in the \(yz\) plane. In \(n\)-dimensional space as well, the vector of coefficients \(\alpha\) has to ensure invariance with respect to rotations only in \(n-1\) well chosen planes instead of \(\frac{n!}{(n-2)!2}\).

To show that, let us consider that \(n\)-dimensional has \(n\) axis named by \(n\) Latin letter \([x \ y \ z \ w, \ldots]\). The \(x\)-axis combined to each of the \(n-1\) other ones form \(n-1\) planes \((xy, xz, xw, \ldots)\) and offer \(n-1\) possible planar rotations. Actually, any other rotation can be obtained as applying two consecutive rotations from \(n-1\) considered above. For instance, a rotation in the in the \(zw\) plane can be obtained as two consecutive rotations in the plane \(xz\) followed another in \(xw\) plane.

In practice, the matrices \(\mathbf{L}_{\mathbf{v}_p^k}^{\omega_i^T}\) are very sparse. First, when they are calculated and stored, one can consider only non-null values. Second, the vector of coefficients can be computed using null space adapted to sparse matrix. This becomes necessary for large moment vectors. To give an idea how sparse are those matrix, let us consider the 3D moment vector \(\mathbf{v}_3^4\), which is of size equal to 715. Let us consider computing an affine invariant from this vector. In that case, the entries of \(\mathbf{v}_3^4\) satisfying the condition to be invariant to non-uniform scale change (8) are only 25. From these entries, an invariant vector to non-uniform
scale can be defined by:

\[ \mathbf{v}_{s_4} = [m_{003}m_{021}m_{120}m_{300}, m_{012}m_{120}m_{300}, m_{003}m_{030}m_{111}m_{300}, m_{012}m_{021}m_{111}m_{300}, m_{012}m_{030}m_{102}m_{300}, m_{021}m_{102}m_{300}, m_{003}m_{021}m_{210}, m_{003}m_{030}m_{201}m_{210}, m_{012}m_{021}m_{210}, m_{003}m_{111}m_{120}m_{210}, m_{012}m_{102}m_{120}m_{210}, m_{012}m_{111}m_{120}m_{210}, m_{021}m_{102}m_{111}m_{210}, m_{012}m_{102}m_{201}, m_{012}m_{021}m_{201}, m_{003}m_{102}m_{111}m_{201}, m_{021}m_{111}m_{201}, m_{012}m_{102}m_{201}, m_{012}m_{120}m_{201}, m_{102}m_{111}m_{120}m_{201}, m_{012}m_{120}m_{201}, m_{102}m_{111}m_{120}m_{201}, m_{102}m_{111}m_{120}m_{201}, m_{102}m_{111}m_{120}m_{201}]^T \frac{1}{m_{000}}. \]

The time derivative of \( \mathbf{v}_{s_4} \) can be written under the form:

\[ \dot{\mathbf{v}}_{s_4} = \frac{1}{m_{000}} \left( \mathbf{L}^{\omega_1}_s \mathbf{v}^4_s \omega_1 + \mathbf{L}^{\omega_2}_s \mathbf{v}^4_s \omega_2 + \mathbf{L}^{\omega_3}_s \mathbf{v}^4_s \omega_3 \right), \quad (31) \]

where \( \mathbf{L}^{\omega_i}_s \) are matrices of integers of size 25 \( \times \) 715. If we consider obtaining from \( \mathbf{v}_{s_4} \) an affine invariant under the form of \( i_a = \mathbf{a}^T \mathbf{v}_s \), the coefficient vector will be obtained by:

\[ \mathbf{a} = \ker(\mathbf{L}^{\omega_1,\omega_2}_s), \text{ where: } \mathbf{L}^{\omega_1,\omega_2}_s = \begin{bmatrix} \mathbf{L}^{\omega_1}_s^T \\ \mathbf{L}^{\omega_2}_s^T \end{bmatrix}. \]

The matrix \( \mathbf{L}^{\omega_1,\omega_2}_s \) is of size 1430 \( \times \) 25, but it is very sparse. Actually, after removing all the rows of \( \mathbf{L}^{\omega_1,\omega_2}_s \) that contain only null values, the vector of coefficients can be obtained as the null space of a matrix of reduced size equal to 84 \( \times \) 25. Besides, even after this consequent size reduction, the resulting reduced size matrix is still very sparse. The obtained formula of \( i_a \) is given by:

\[ i_a = \frac{1}{m_{000}} \left( -m_{300}m_{012}m_{120} + m_{012}m_{210} + m_{300}m_{012}m_{021}m_{111} - m_{012}m_{021}m_{201}m_{210} - m_{012}m_{102}m_{120}m_{210} + m_{030}m_{300}m_{012}m_{102} - 2m_{012}m_{111}m_{210} + 3m_{012}m_{111}m_{120}m_{201} - m_{300}m_{012}m_{201} - m_{300}m_{012}m_{021}m_{102} + m_{012}m_{111}m_{120}m_{210} - 2m_{012}m_{111}m_{120}m_{201} + m_{003}m_{300}m_{012}m_{111}m_{210} - m_{003}m_{021}m_{201}m_{111}m_{210} + m_{003}m_{300}m_{012}m_{120}m_{210} - m_{003}m_{021}m_{201}m_{111}m_{210} + m_{003}m_{300}m_{012}m_{120}m_{201} - m_{003}m_{021}m_{201}m_{111}m_{201} + m_{003}m_{300}m_{012}m_{111}m_{210} - m_{003}m_{021}m_{201}m_{111}m_{210} - m_{003}m_{021}m_{201}m_{111}. \right) \]

From the formula of the previous affine invariant, it can be noticed the coefficients of many entries of \( \mathbf{v}_{s_4} \) have the same values. Actually, the entries of \( \mathbf{v}_{s_4} \) can be divided to 10 groups:

- **Group 1**: \( m_{300}m_{012}m_{120}, m_{300}m_{012}m_{102}, m_{003}m_{120}m_{201}, m_{030}m_{012}m_{201}, m_{003}m_{021}m_{210}, m_{030}m_{021}m_{210} \)
- **Group 2**: \( m_{012}m_{201}, m_{012}m_{210}, m_{102}m_{120} \)
- **Group 3**: \( m_{030}m_{102}m_{111}m_{201}, m_{003}m_{111}m_{120}m_{210}, m_{300}m_{012}m_{210}m_{111} \)


Group 4: \( m_{012}m_{111}m_{210}, m_{021}m_{111}m_{201}, m_{102}m_{111}m_{120}; \)

Group 5: \( m_{003}m_{300}m_{021}m_{120}, m_{003}m_{300}m_{021}m_{120}, m_{030}m_{300}m_{012}m_{102}; \)

Group 6: \( m_{012}m_{021}m_{201}m_{210}, m_{012}m_{021}m_{201}m_{210}; \)

Group 7: \( m_{021}m_{102}m_{111}m_{210}, m_{012}m_{111}m_{120}m_{201}; \)

Group 8: \( m_{012}m_{111}m_{120}m_{201}; \)

Group 9: \( m_{111}; \)

Group 10: \( m_{003}m_{030}m_{300}m_{111}; \)

Any member of a group can be obtained by coordinates switching from another of the same group. Since \( i_{143} \) is an invariant to rotations, the entries of the same group will have the same coefficient. This reduces the number of unknown to \( 10 \) coefficients instead of \( 25 \).

5 Redundancy between invariants

As it has been shown previously, invariants to rotations or to affine transformation can be obtained as linear combination of the entries of moment vectors. The vectors of coefficients \( \alpha \) defining the linear combinations are obtained as the null space of a matrix of integers. Therefore, by construction, the set of invariants that can be obtained from a vector of moment are linearly independent. However, as dealt with in [19] for instance, dependency between invariants of higher order and of lower should be removed.

Actually, the form and the proposed method used to obtain invariants simplify redundancy elimination. Let us consider an example from 2D space where rotation invariants are to be computed from the moment vectors \( v_1^2, v_5^2 \) and \( v_2^2, v_5^2 \). The products of two invariants computed respectively from \( v_1^2 \) and from \( v_5^2 \) are invariants of the same kind as those computed from \( v_{(1,2)}^2 \). Therefore, while computing invariant from the latter moment vector, it is necessary to eliminate those obtained from the two first. The elimination of such dependency can be achieved as follow:

- Compute the vector of coefficients \( \alpha_1^2 \) from \( v_1^2 \). We obtain
  \[
  i_{r_2} = m_{20} + m_{02}
  \] (32)

- Compute the vector of coefficients \( \alpha_2^2 \) from \( v_5^2 \). We obtain 3 invariants:
  \[
  i_{r_2}^{(1)} = 3m_{23}^2 - 4m_{41}m_{23} + 3m_{32}^2 - 4m_{14}m_{32} + m_{05}m_{41} + m_{14}m_{50}
  \]
  \[
  i_{r_2}^{(2)} = m_{14}m_{32} - m_{05}m_{41} - m_{05}m_{23} - m_{14}m_{50} + m_{23}m_{41} - m_{32}m_{50} + m_{14}^2 + m_{41}^2
  \]
  \[
  i_{r_2}^{(3)} = 15m_{05}m_{23} + 5m_{05}m_{41} + 25m_{14}m_{32} + 5m_{14}m_{50} + 25m_{23}m_{41} + 15m_{32}m_{50} + 3m_{05}^2 + 3m_{50}^2
  \] (33)
• The product of invariants given by (32) and (33) leads to 3 rotation invariants of the same kind as those that can be obtained from $\mathbf{v}_{(1,2)}$. They can be written under the form $\beta^\top \mathbf{v}_{(1,2)}^{(1,2)}$. Without taking into account this dependency, the null space of $L_{(1,2)}^\top$ allows to obtain 9 invariants to rotations $i_{r_{(1,2)}}$ from $\mathbf{v}_{(2,5)}$. The dependency can be easily removed by computing the null space of:

$$\mathbf{\alpha} = \ker \left( \begin{bmatrix} \beta^\top \\ L_{(1,2)}^\top \mathbf{v}_{(1,2)} \end{bmatrix} \right)$$

which gives only 6 invariants independent of $i_{r_{1}}$ and $i_{r_{2}}$. From those 6 invariants, only 1 affine invariant can be derived:

$$i_{a_{(1,2)}} = \frac{1}{m_{69}} (3m_{20}m_{23}^2 - 2m_{11}m_{23}m_{32} - 4m_{02}m_{41}m_{23} + 3m_{02}m_{32}^2 - 4m_{14}m_{20}m_{32} + m_{02}m_{14}m_{50} - m_{05}m_{11}m_{50} + m_{05}m_{20}m_{41} + 3m_{11}m_{14}m_{41})$$

6 Conclusion

In this paper, a novel and systematic scheme to determine moment invariants in n-dimensional space has been proposed. More precisely, affine invariants are written under the form of linear combinations of invariants to non-uniform scale. The coefficient vectors of those linear combinations are obtained as null space of some matrices of integers. This ensures that all possible invariants from a moment vector are obtained. It also allows to remove easily redundancy between invariants using matrix algebra.

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