FAULT-TOLERANT CONTROL AGAINST ACTUATOR FAILURES FOR UNCERTAIN SINGULAR FRACTIONAL ORDER SYSTEMS

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Abstract. A method of designing observer-based feedback controller against actuator failures for uncertain singular fractional order systems (SFOS) is presented in this paper. By establishing actuator fault model and state observer, an observer-based fault-tolerant state feedback controller is developed such that the closed-loop SFOS is admissible. The controller designed by the proposed method guarantees that the closed-loop system is regular, impulse-free and stable in the event of actuator failures. Finally, a numerical example is given to illustrate the effectiveness of the proposed design method.

1. Introduction. During the past decades, fault tolerant control research and its application in a wide range of industrial and commercial processes have become the focus of intensive research [2, 12]. Nowadays, it is known that the existence of the fault is one of the main reasons leading to poor or even unstable performance of control systems [2, 3, 10]. In order to realize effective control strategies to make the system run safely and properly under normal or faulty conditions, the design of fault tolerant control (FTC) systems has become more and more interesting in the past decades [4, 8, 9, 14, 19, 23, 30, 35, 39, 7]. The problem of model tracking control for a class of nonlinear systems with model uncertainty and actuator failure is solved in [4]. Based on variable structure control, an attitude control system is developed that guarantees asymptotical stability for a system with the actuators partially lose their effectiveness [8]. In the work of [9], considering the design of decentralized $H_{\infty}$ control, the fault-tolerant problem of the obtained system is studied. For linear systems with time-varying parameter uncertainty, disturbance and actuator faults including outage, loss of effectiveness and stuck, [14] designs a robust adaptive fault-tolerant compensation controller. In the non-linear systems for actuator failure, [19] proposes a new fault detection and adjustment scheme. Besides, many other results on FTC synthesis are obtained via different approaches, such as networked control systems [23], Markovian jump system models [30], fuzzy systems [35] and so on. In [7], the FTC problem for a class of continuous Lipschitz nonlinear systems with actuator fault, mismatched disturbance, and time-varying delay is investigated. An adaptive integral sliding mode fault tolerant control technique is proposed.

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Researchers are more and more interested in the research of fractional order systems (FOS) with the progress of technology. Many practical systems are better represented by fractional differential equations than integer differential equations. In addition, in order to improve the control performance, fractional calculus is a powerful tool which can be applied to state space representation in control theory [24, 27]. For finite-dimensional linear fractional differential systems, stability results of main concern for control theory are given in [22]. Based on such a method, the definition of Mittag-Leffler stability is proposed and the fractional Lyapunov direct method is introduced in [15]. By using the LMI approach, the stability and stabilization criteria for FOS are presented in [41]. When parameter uncertainty appears, the robust stability and stabilization problems of FOS are developed, see in [17, 18, 1, 28] and the references therein. Specifically, in the uncertain polytopic systems, the state of the feedback stabilization problem is solved in [5], but there are some conservatism. To overcome the conservatism, a parameter-dependent matrix variable is introduced to deal with the same problem in [6]. To the best of our knowledge, the case of robust fault tolerant control of uncertain fractional order systems against actuator faults and sensor faults are proposed in [26, 29], which establish fault models and state observers, respectively.

Singular fractional order systems are extensions of fractional order systems. The singular systems are fundamentally different from the normal systems. Transfer function of the singular systems may be inappropriate. Therefore, the study of singular systems is more complex than normal systems. It is worth mentioning that the admissibility and robust stabilization of SFOS with the fractional order $0 < \alpha < 1$ in [42] and the case of $1 \leq \alpha < 2$ in [20] are useful and heavily quoted. Necessary and sufficient conditions for stabilizing singular fractional order systems with partially measurable state are proposed in [34, 16]. For singular fractional order systems, the output feedback control synthesis is raised, where applies normalized method [33]. [40] study the problem of pseudostate and static output feedback stabilization for singular fractional-order linear systems with fractional order $\alpha$ when $0 < \alpha < 1$. However, the above papers do not take into account the situation of actuator faults.

In view of the above motivation, this paper focuses on observer-based fault-tolerant control for uncertain SFOS with order $1 \leq \alpha < 2$. Our purpose is to design state feedback controller so that SFOS with time-varying parameter uncertainty and actuator failures are admissible. Firstly, a sufficient condition is presented for the admissibility of SFOS with actuator faults and equality constraint. Then, a sufficient condition is obtained for observer-based fault tolerant control design by using appropriate matrix variable decoupling technique. Finally, the effectiveness of the proposed new design technique is verified by a numerical example.

The contributions of the paper focus on:

1. An observer-based control for uncertain singular fractional order systems by using appropriate matrix variable decoupling technique is firstly presented.

2. The research on uncertain singular fractional order systems against actuator faults have been well solved in this paper, and we derive a expression of observer-based fault-tolerant controllers against actuator failures.

3. The conditions in this paper are obtained in terms of LMI with less unknown variables, and these conditions are better than other references.

4. There is no equality constraint in Theorem 3.2 of the paper, and non-strict equality limitations in Theorem 3.1 are removed in this theorem.
Notation. Throughout this paper, \( \mathbb{R}^n \) denotes the \( n \) dimension column vector. \( \mathbb{R}^{m \times n} \) is the real matrix space with dimension \( m \times n \). \( \text{spec}(E, A) \) is the spectrum of \( \det(\lambda E - A) = 0 \). \( X \prec 0 \) \((X \succ 0)\) denotes a symmetric negative (positive) definite matrix, respectively. \( A^T \) denote the transpose of matrix \( A \). \( \text{sym}(X) \) is used to denote \( X + X^T \). \( I_n \) denotes the identity matrix. \( \otimes \) stands for the Kronecker product.

2. Preliminaries and problem formulation. In this section, some preliminaries results on fractional integrals, derivatives and the singular fractional order actuator fault model are presented.

The fractional order derivative has two main classes: Riemann-Liouville derivative and Caputo derivative. In this paper, we only use the Caputo definition since its Laplace transform allows using initial values of classical integer order derivatives with clear physical interpretations.

Definition 2.1. \([25]\) The Caputo type fractional derivative with order \( \alpha > 0 \) is defined as

\[
D^\alpha_0 f(t) = \frac{1}{\Gamma([\alpha] - \alpha)} \int_0^t \frac{f^{([\alpha])}(\tau)}{(t - \tau)^{\alpha + 1 - [\alpha]}} d\tau,
\]

where \( \Gamma(\cdot) \) is the Gamma function which is defined by

\[
\Gamma(q) = \int_0^\infty t^{q-1} e^{-t} dt,
\]

and \([\alpha]\) denotes the least integer number greater than or equal to \( \alpha \) (The value of order \( \alpha \) can be practically obtained from the system identification methods \([11, 31]\)).

Considering the following class of uncertain SFOS

\[
ED^\alpha x(t) = (A + \Delta A)x(t) + Bu(t), y(t) = Cx(t),
\]

where \( 1 \leq \alpha < 2 \) is the fractional commensurate order, \( x(t) \in \mathbb{R}^n \) is the state of the plant, \( u(t) \in \mathbb{R}^s \) and \( y(t) \in \mathbb{R}^m \) are the control input and output, respectively. \( A, B \) and \( C \) are systems matrices of appropriate dimensions, matrix \( E \) is singular with \( \text{rank}(E) = r < n \), where \( \Delta A \) represents the following time varying matrix with uncertainty:

\[
\Delta A = DA F(\sigma) E_A,
\]

where \( D_A \) and \( E_A \) are known constant matrices, and \( F(\sigma) \) is unknown matrices with Lebesgue measurable elements satisfying

\[
F^T(\sigma) F(\sigma) \leq I.
\]

To formulate the FTC problem, the following actuator fault model from \([38]\) is adopted in this paper. Specifically, for the control input \( u(t) \), we use \( u_f(t) \) to describe the control signal form actuators, and

\[
u_f(t) = Fu(t),
\]

where \( F = \text{diag}\{f_1, f_2, \ldots, f_s\} \) is the actuator fault matrix, where \( 0 \leq f_{il} \leq f_i \leq f_{iu} < \infty \), and \( f_{il} < 1, f_{iu} \geq 1, i = 1, 2, \ldots, s \) are known real constants. It can be seen that the actuator in complete failure (the outage case) when \( f_i = 0 \), similarly, the actuator is in normal case when \( f_i = 1 \).

Now, introducing the following matrices

\[
F_0 = \text{diag}\{f_{01}, f_{02}, \ldots, f_{0s}\},
\]
\[ J = \text{diag}\{j_1,j_2,\cdots,j_s\}, \]
\[ |L| = \text{diag}\{|l_1|,|l_2|,\cdots,|l_s|\}, \]

where \( F_{0i} = \frac{f_{iu}+f_{iu}}{2}, l_i = \frac{f_{iu}-f_{iu}}{f_{iu}+f_{iu}}, j_i = \frac{f_{iu}-f_{iu}}{f_{iu}+f_{iu}} \) (\( i = 1,2,\cdots,s \)), the one can be obtained
\[ F = F_0(I + L), |L| \leq J \leq I. \quad (5) \]

Now, designing the following observer based controller
\[ E\dot{\hat{x}}(t) = (A + \Delta A)\hat{x}(t) + Bu(t) + G(C\hat{x}(t) - y(t)), \quad u(t) = K\hat{x}(t), \quad (6) \]

where \( \hat{x}(t) \in \mathbb{R}^n \) is the estimated pseudo state, \( G \in \mathbb{R}^{n \times p} \) and \( K \in \mathbb{R}^{s \times n} \) are observer gain and controller to be designed, respectively, such that the augmented system
\[ \bar{E}\dot{\bar{x}}(t) = \bar{A}\bar{x}(t), \quad (7) \]
is admissible, where
\[ \bar{E} = \begin{bmatrix} E & 0 \\ 0 & E \end{bmatrix}, \quad \bar{x}(t) = \begin{bmatrix} x(t) \\ e(t) \end{bmatrix}, \quad \bar{A} = \begin{bmatrix} A + \Delta A + BFK & -BFK \\ 0 & A + \Delta A + GC \end{bmatrix}, \quad (8) \]
with error signal \( e(t) = x(t) - \hat{x}(t) \).

Before moving on, some lemmas are given to be used in the proof of our main results.

**Definition 2.2.** [42]
(i) System (7) is said to be regular if \( \text{det}(s^a\bar{E} - \bar{A}) \) is not identically zero for \( s \in \mathbb{C} \).
(ii) System (7) is said to be impulse free if \( \text{deg}(\text{det}(\lambda\bar{E} - \bar{A}))=\text{rank}(\bar{E}) \).
(iii) System (7) is said to be stable if all the finite eigenvalues of \( \text{det}(\lambda\bar{E} - \bar{A}) = 0 \) satisfy \( |\arg(\text{spec}(\bar{E},\bar{A}))| > \frac{\pi}{2} \).
(iv) System (7) is said to be admissible if it is regular, impulse free and stable.

**Lemma 2.3.** [13]
For given real matrices \( S_1, S_2 \) and \( S_3 \), where \( S_1 = S_1^T, S_3 > 0 \),
\[ S_1 + S_2S_3^{-1}S_2^T < 0, \]
if and only if
\[ \begin{bmatrix} S_1 & S_2 \\ S_2^T & -S_3 \end{bmatrix} < 0. \]

**Lemma 2.4.** [36] Given two real appropriate dimension matrices \( X \) and \( Y \)
\[ \Omega + X^TF(\sigma)Y + Y^TF^T(\sigma)X < 0, \]
for all \( F(\sigma) \) satisfying \( F^T(\sigma)F(\sigma) \leq I \) if and only if there exists an \( \varepsilon > 0 \) such that
\[ \Omega + \varepsilon X^TX + \varepsilon^{-1}Y^TY < 0, \]
where \( \sigma \in \Theta \), and \( \Theta \) is a compact set in \( \mathbb{R} \).

Next, we present some preliminaries results on singular fractional order systems.
Lemma 2.5. [20] System (1) with $\Delta A = 0$ and $u(t) = 0$ is admissible if and only if there exists a matrix $P$ satisfying
\begin{align}
& E^T P = P^T E \geq 0, \\
& \text{sym} \{ \Theta \otimes A^T P \} < 0,
\end{align}
where
\[
\Theta = \begin{bmatrix}
\sin(\pi - \alpha \frac{\pi}{2}) & \cos(\pi - \alpha \frac{\pi}{2}) \\
-\cos(\pi - \alpha \frac{\pi}{2}) & \sin(\pi - \alpha \frac{\pi}{2})
\end{bmatrix}.
\]

Lemma 2.6. System (1) with $\Delta A = 0$ and $u(t) = 0$ is admissible if and only if there exists a matrix $P$ satisfying
\begin{align}
& P^T E^T = EP \geq 0, \\
& \text{sym} \{ \Theta \otimes AP \} < 0,
\end{align}
where $\Theta$ is defined in (11).

Proof. By Lemma 2.5, System (1) with $\Delta A(t)$ and $u(t) = 0$ is admissible if and only if (9) and (10) hold for some existing matrix $P$. From (10), we can know that matrix $P$ is invertible. By using congruent transformation, (9) and (10) can be transformed into
\[
P^{-T} E^T = EP^{-1},
\]
\[
\text{sym} \{ \Theta \otimes AP^{-1} \} < 0.
\]
Letting $P_1 = P^{-1}$, we can obtain (12) and (13), the desired result is obtained. \qed

Lemma 2.7. [21] System (7) without actuator fault and $\Delta A = 0$ is admissible if and only if both the pairs $(E, A + BK)$ and $(E, A + GC)$ are admissible.

In order to eliminate the equality constraint of Theorem 3.2, we propose the following two lemmas.

Lemma 2.8. [20] System (1) with $\Delta A = 0$ and $u(t) = 0$ is admissible if and only if there exist matrices $X > 0$ and $Y$ satisfying
\[
\text{sym} \{ \Theta \otimes A^T (XE + E_0 Y^T) \} < 0,
\]
where $E_0 \in \mathbb{R}^{n \times (n-r)}$ is any matrix of full column rank and satisfies $E^T E_0 = 0$.

Lemma 2.9. [20] System (1) with $\Delta A = 0$ and $u(t) = 0$ is admissible if and only if there exist matrices $X > 0$ and $Y$ satisfying
\[
\text{sym} \{ \Theta \otimes A (XET + E_0 Y^T) \} < 0,
\]
where $E_0 \in \mathbb{R}^{n \times (n-r)}$ is any matrix of full column rank and satisfies $EE_0 = 0$.

3. Main results. In this section, we provide solutions to the fault tolerant control problem formulated in the previous section. We can get two important sufficient condition theorems for system (7).

Theorem 3.1. Design the controller (6) for system (1), consider the resulting closed-loop control system (7), if there exist matrices $P_i \in \mathbb{R}^{n \times n}$, $i = 1, 2$, together with $R_1$, $R_2$, of appropriate dimensions and three real scalars $\varepsilon_i > 0$, $i = 1, 2, 3$, such that (16)-(19) hold, then for all of the admissible uncertainties and actuator faults, the closed-loop system (7) is admissible.
\[
EP_1 = P_1^T E^T \geq 0,
\]
\[
EP_2 = P_2^T E^T \geq 0,
\]
\[
P_1^T E_0 \geq 0,
\]
\[
P_2^T E_0 \geq 0.
\]
The following result is thus established.

Let \( E, A \) be any matrices satisfying

\[
\begin{pmatrix}
\text{sym}(\Theta \otimes (A + \Delta A + BF_0 K)P_1) & \text{sym}(\Theta \otimes (D_A F(\sigma)E_A P_1)) & \text{sym}(\Theta \otimes (BF_0 LKP_1))
\end{pmatrix} < 0.
\]

Then

\[
\text{sym}(\Theta \otimes (A + \Delta A + BF_0 (I + L)K)P_1) < 0,
\]

we obtain

\[
\text{sym}(\Theta \otimes (A + \Delta A + BF_0 K)P_1) < 0.
\]

From (12) and (13) in Lemma 2.6, if (16) and (17) hold, the pair \((E, A + \Delta A + BF_0 K)\) is admissible.

Likewise, suppose (18) and (19) hold, from Lemma 2.3, (19) is equivalent to

\[
\text{sym}(\Theta \otimes (A^T P_2 + C^T R_2)) + \epsilon_3(I_2 \otimes D_A D_A^T) + \epsilon_3^{-1}(I_2 \otimes R_1^T R_1) < 0.
\]

From (3), (25) can be rewritten as

\[
\text{sym}(\Theta \otimes (A + \Delta A + GC)^T P_2) < 0.
\]

From (9) and (10) in Lemma 2.5, it can obtain (18) and (19) make the pair \((E, A + \Delta A + GC)\) is admissible.

For all of the admissible uncertainties with actuator faults, from Lemma 2.7 closed-loop system (7) is admissible.

Notice that the conditions of (16)-(19) can be equivalently transformed into strict LMIs by using variable replacements analogous to that in [37, 32]. Let \( E_i, \ i = 1, 2, \) be any matrices satisfying

\[
E_1 \in \mathbb{R}^{n \times (n-r)}, \ \text{rank}(E_1) = n - r, \ EE_1 = 0,
\]

\[
E_2 \in \mathbb{R}^{n \times (n-r)}, \ \text{rank}(E_2) = n - r, \ E^T E_2 = 0.
\]

The following result is thus established.
Theorem 3.2. Design the controller (6) for system (1), consider the resulting closed-loop control system (7), if there exist positive definite matrices $X_i \in \mathbb{R}^{n \times n}$, $i = 1, 2$, and $Y_1$, $Y_2$ together with $Z_1$, $Z_2$, of appropriate dimensions and three real scalars $\eta_i > 0$, $i = 1, 2, 3$ such that (29)-(30) hold, then for all of the admissible uncertainties and actuator faults, the closed-loop system (7) is admissible. 

\[
\begin{bmatrix}
\Phi \\
I_2 \otimes E_A(X_1 E^T + E_1 Y_1^T) \\
\quad - \eta_1 I_{2n \times 2n} \\
\eta_3 I_2 \otimes A_D^T + \eta_2 I_2 \otimes B_F J^T F_0^T B^T \\
\quad \text{sym}(\Theta \otimes A^T(X_2 E + E_2 Y_2^T)) + \text{sym}(\Theta \otimes C^T Z_2) + \eta_3 I_2 \otimes E_A^T E_A \\
I_2 \otimes D_x^T (X_2 E + E_2 Y_2^T) \\
\quad \eta_3 I_2 \otimes D_x^T (X_2 E + E_2 Y_2^T) \\
\quad - \eta_2 I_{2n \times 2n} \\
\end{bmatrix} < 0, 
\]  
(29)

\[
\begin{bmatrix}
\eta_2 I_2 \otimes B_F J^T F_0^T B^T \\
\end{bmatrix} < 0, 
\]  
(30)

where $\Theta$ is defined in (11),

\[\Phi = \text{sym}(\Theta \otimes (A + B F_0 K)(X_1 E^T + E_1 Y_1^T)) + \eta_1 I_2 \otimes A D_A^T + \eta_2 I_2 \otimes B F_0 J^T F_0^T B^T,\]

(31)

the controller gain and the observer gain are given by

\[K = Z_1 (X_1 E^T + E_1 Y_1^T)^{-1}, G = (X_2 E + E_2 Y_2^T)^{-1} Z_2^T.\]

(32)

Proof of Theorem 3.2. This proof is based on Lemma 2.8 and Lemma 2.9. We aim to remove the positive semi-definite terms in (16) and (18). It can be seen from Lemmas 2.8 and 2.9 that both $(X_1 E^T + E_1 Y_1^T)$ and $(X_2 E + E_2 Y_2^T)$ are invertible if (29)-(30) are feasible. The rest of the proof is similar to Theorem 3.1, the details here are thus omitted. \qed

Remark 1. It is easy to see that $E_i$, $i = 1, 2$, satisfying (27) and (28) are not unique. However, the selections of $E_i$ do not alter the solvability of LMIs in Theorem 3.2. In fact, for another $\tilde{E}_i$, $i = 1, 2$, satisfying (27) and (28), one has $\tilde{E}_i = E_i U_i^{-1}$ and $\tilde{E}_2 = E_2 U_2^{-1}$ for some invertible matrices $U_i$, $i = 1, 2$. Hence, one can confirm that (27) and (28) are equivalent to those with replacing the terms of $E_1 M_1$ and $M_2 E_2$, respectively, by $E_1 M_1$ and $M_2 E_2$ for $M_1 = U_1 M_1$ and $M_2 = M_2 U_2$.

Remark 2. With the definitions of $R_1, R_2$ in Theorem 3.1 and $Z_1, Z_2$ in Theorem 3.2, the coupling problem of multiple decision matrices has been solved. Without application of the Kronecker product, Theorems 3.1 and 3.2 are more concise than other papers for SFOS against actuator faults. Compared with the results in [26], the synthesis method in this paper are more general and least conservative.

4. A numerical example. In this section, we provide a numerical example to demonstrate the efficiency of the controller design method.

Example 1. Consider the fault tolerant control problem for uncertain singular fractional order system (1) with the fractional order $\alpha = 1.2$, $u(t) = K \dot{x}(t)$ and the following parameters

\[E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} 2.4 & 0.2 & 1.2 \\ 4 & 1.5 & 3 \\ 0 & 0 & 1 \end{bmatrix},\]
\[ B = \begin{bmatrix} 4 & 1 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}, \quad C = \begin{bmatrix} 1.5 & 2 & 1 \end{bmatrix}, \]

\[ D_A = \begin{bmatrix} 0.1 & 0 & 0 \\ 0.1 & 0.3 & 0.4 \\ 0.2 & 0 & 0 \end{bmatrix}, \quad E_A = \begin{bmatrix} 0 & 0 & 0 \\ 0.2 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \]

\[ f_l = \begin{bmatrix} 0.7 \\ 0.5 \end{bmatrix}, \quad f_u = \begin{bmatrix} 0.9 \\ 0.7 \end{bmatrix}, \]

\[ F(\sigma) = \text{diag}\{\sin(\sigma), \cos(\sigma), \sin(\sigma)\}, \]

\[ E_1 = \text{null}(E), \quad E_2 = \text{null}(E^T). \]

For a given initial condition \( x_0 = \begin{bmatrix} 0.2 & 0.4 & -0.3 \end{bmatrix}^T \), the time response of the system with \( u(t) = 0 \) is shown in Fig. 1, which shows that the uncertain singular fractional-order system is not admissible and the states are not convergent to zero.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure1.png}
\caption{State responses of the closed-loop system in Example 1 with \( u(t) = 0 \).}
\end{figure}

However, using Matlab LMI control toolbox, we find that LMI (29) and (30) in Theorem 3.2 are feasible, which implies that the actuator fault system (1) is admissible under state feedback control input (6). We can obtain the solution as follows:

\[ X_1 = \begin{bmatrix} 9.9560 & -10.3815 & 0 \\ -10.3815 & 25.5456 & 0 \\ 0 & 0 & 40.0437 \end{bmatrix}, \quad Y_1 = \begin{bmatrix} -3.0263 \\ -9.7947 \\ 0.6289 \end{bmatrix}, \]
\[
\begin{align*}
X_2 &= \begin{bmatrix}
16.6084 & -13.8982 & 0 \\
-13.8982 & 25.3065 & 0 \\
0 & 0 & 40.0437
\end{bmatrix}, \\
Y_2 &= \begin{bmatrix}
24.7532 \\
-28.3124 \\
-41.8443
\end{bmatrix}, \\
Z_1 &= \begin{bmatrix}
-14.7471 & 6.7799 & 3.9653 \\
5.6372 & 1.9590 & -17.9020
\end{bmatrix}, \\
Z_2 &= \begin{bmatrix}
-2.7423 & -30.4020 & -0.3411
\end{bmatrix},
\end{align*}
\]

\[
\eta_1 = 40.0215, \quad \eta_2 = 48.5256, \quad \eta_3 = 40.5653.
\]

Therefore, the resulting closed-loop system is admissible under observer-based controller (6), the observer gain and the controller gain are computed by using (32) as

\[
K = \begin{bmatrix}
5.6103 & 4.9629 & 6.3051 \\
-33.6437 & -24.5100 & -28.4653
\end{bmatrix}, \\
G = \begin{bmatrix}
-2.1741 & -2.3862 \\
0.0082
\end{bmatrix}.
\]

For given initial conditions \(x_0 = \begin{bmatrix} 5 & 5 & -3 \end{bmatrix}^T\) and \(e_0 = \begin{bmatrix} 10 & 10 & -2 \end{bmatrix}^T\), the simulation results (Fig. 2 and Fig. 3), which imply that the closed-loop system is admissible and its states converge to zero.

**Figure 2.** Observation errors of the selected system in Example 1 with \(u(t) = Kx(t)\).

5. **Conclusion.** In this paper, the problem of admissibility for uncertain singular fractional order system with actuator faults with order \(1 \leq \alpha < 2\) is worked out. A sufficient LMI condition for the fault-tolerant control of the observer-based SFOS is first proposed. Then, the existence condition and method of designing a observer-based controller is derived in terms of LMIs, which are more convenient to be
achieved in practice. A numerical example has been provided to demonstrate the validity of this approach. In the future, the issue of fault tolerant control for singular fractional order systems with sensor faults can also be extended by the similar approach in Theorem 3.2.

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