Dichotomy Condition and Periodic Solutions for Two Nonlinear Neutral Systems

Mouataz Billah Mesmouli,1 Adel A. Attiya,1,2 A. A. Elmandouha,3 Ayékotan M. J. Tchalla,4 and Taher S. Hassan1,2,5

1 Mathematics Department, College of Science, University of Ha'il, Ha'il 2440, Saudi Arabia
2 Department of Mathematics, Faculty of Science, Mansoura University, Mansoura 35516, Egypt
3 Department of Mathematics and Statistics, College of Science, King Faisal University, P. O. Box 400, Al-Ahsa 31982, Saudi Arabia
4 Département de Mathématiques, Faculté des Sciences, Université de Lomé, 01 BP 1515 Lomé, Togo
5 Section of Mathematics, International Telematic University Uninettuno, Corso Vittorio Emanuele II, 39, 00186 Roma, Italy

Correspondence should be addressed to Ayékotan M. J. Tchalla; atchalla@univ-lome.tg

Received 13 March 2022; Accepted 26 June 2022; Published 1 August 2022

1. Introduction

Periodic solutions of equations are solutions that describe regularly repeated processes. The periodic solutions of systems of differential equations occupy special importance in branches of science such as the theory of oscillations, dynamical systems, and celestial mechanics, and the analysis of these systems in depth opens up new possibilities and horizons in these sciences. Such a study aids in understanding the geometric behavior of solutions eventually (see [1–4]).

In recent years, several investigators have tried the stability and existence of periodic solutions by using the technique of fixed point, in particular Burton, Furumochi, Zhang, and others (see [5–13]).

By Krasnoselski’s fixed point theorem, Luo et al. [14] investigate the existence of positive periodic solutions for two neutral functional differential equations

\[
\left( y(\zeta) - cy(\zeta - \tau(\zeta)) \right)' = -a(\zeta)y(\zeta) + f(\zeta, y(\zeta - \tau(\zeta))),
\]

\[
\frac{d}{d\zeta} \left[ y(\zeta) - c \int_{-\infty}^{0} Q(\omega)y(\zeta + \omega)d\omega \right] = -a(\zeta)y(\zeta) + \int_{-\infty}^{0} Q(\omega)f(\zeta, y(\zeta + \omega))d\omega,
\]

in which \( y : \mathbb{R} \rightarrow \mathbb{R} \); \( a(\zeta) \in C(\mathbb{R}, (0, \infty)) \); \( f \in C(\mathbb{R} \times \mathbb{R}, \mathbb{R}) \);
\( \tau(\zeta) \in C(\mathbb{R}, \mathbb{R}) \); \( a(\zeta), b(\zeta), \tau(\zeta), \) and \( f(\zeta, y) \) are \( T \)-periodic functions; \( T > 0 \) and \( |c| < 1 \) are constants; \( Q(\omega) \in C([-\infty,0], [0,\infty)) \); and \( \int_{-\infty}^{0} Q(\omega)d\omega = 1 \).

The above functional differential equations (1) and (2) cover many mathematical ecological and population models, for example, hematopoiesis models (see [15, 16]),

[1] Hindawi
[2] Journal of Function Spaces
[3] Volume 2022, Article ID 6319312, 7 pages
[4] https://doi.org/10.1155/2022/6319312
Nicholson’s blowflies models (see [17, 18]), and blood cell production (see [19]).

Sa Ngiamsunthorn [11] considered the differential system

\[(y(\cdot) - \sigma(y(\cdot - \tau(\cdot))))' = A(\cdot)y(\cdot) + f(\cdot,y(\cdot - \sigma_1(\cdot)),\ldots,y(\cdot - \sigma_m(\cdot))),(3)\]

with dichotomy condition (3) periodic coefficients. Similar system of (3) has been studied in [20].

Motivated by the works mentioned above, we are concerned with the existence of periodic solutions for two nonlinear neutral systems of differential equations

\[(y(\cdot) - \eta(y(\cdot - \tau(\cdot))))' = A(\cdot)y(\cdot) + f(\cdot,y(\cdot - \sigma_1(\cdot)),\ldots,y(\cdot - \sigma_m(\cdot))),(4)\]

\[\left(y(\cdot) - \int_{-\infty}^{0} Q(\omega)q(\cdot,y(\cdot + \tau(\omega)))d\omega\right)' = A(\cdot)y(\cdot) + f(\cdot,y(\cdot + \sigma_1(\cdot)),\ldots,y(\cdot + \sigma_m(\cdot)))d\omega,(5)\]

in which \( y : \mathbb{R} \to \mathbb{R}^n, \tau(\cdot), \) and \( \sigma_i(\cdot), i = 1,\ldots,m, \) are real continuous \( T \)-periodic functions on \( \mathbb{R} \), \( T > 0 \). \( A(\cdot) \) is a \( n \times n \) real continuous matrix \( T \)-periodic function defined on \( \mathbb{R} \). \( Q(\omega) \) is a \( n \times n \) real continuous matrix periodic function defined on \( (-\infty,0] \) with \( \int_{-\infty}^{0} Q(\omega)d\omega = I \). The functions \( q(\cdot,u) \) and \( f(\cdot,u_1,\ldots,u_m) \) are real continuous vector functions defined on \( \mathbb{R} \times \mathbb{R}^n \) and \( \mathbb{R} \times (\mathbb{R}^n)^m \), respectively, such that

\[f(\cdot + T,u_1,u_2,\ldots,u_m) = f(\cdot,u_1,u_2,\ldots,u_m),\]

\[q(\cdot + T,u) = q(\cdot,u).\]

Note that the functional \( y(\cdot - \tau(\cdot)) \) and function \( y(\cdot) \) are in different spaces because \( y(\cdot - \tau(\cdot)) \) is in the phase space, but their norms are equivalent (for more details on space theory, we refer the reader to the following papers) [21, 22].

This paper is arranged as follows: after this introduction, we list a set of definitions and previous results related to integrable dichotomies and fixed point theorems in Section 2. Sections 3 and 4 deal with the existence and uniqueness of periodic solutions of systems (4) and (5), respectively, and are followed by a conclusion.

2. Preliminaries

In this section, we outline some results and definitions of integrable dichotomy that will be crucial in the proof of our results (see [23, 24]). Consider the following linear differential system:

\[z' = A(\cdot)z(\cdot),\]

in which \( A(\cdot) \) is a continuous \( n \times n \) matrix function. Let \( \Psi(\cdot) \) be the fundamental matrix solution of system (7) with \( \Psi(0) = I \). Assume \( P \) is a projection matrix. We let a green matrix \( G := G_p \) be associated with \( P \) by

\[G(\cdot,\omega) = \begin{cases} \Psi(\cdot)^{P\Psi^{-1}(-1)}(\cdot), & \text{for } \omega \geq 0, \\ -\Psi(\cdot)(1-P)^{P\Psi^{-1}(-1)}(\cdot), & \text{for } \omega < 0. \end{cases}\]

Definition 1 (see [23]). If a projection matrix \( P \) and a positive constant \( \mu \) exist for which the associated Green matrix \( G = G_p \) satisfies

\[\sup_{\omega \in \mathbb{R}} ||G(\cdot,\omega)||d\omega = \mu,(9)\]

the linear differential system (7) has an integrable dichotomy.

Proposition 2 (see [23]). Assume that system (7) has an integrable dichotomy. Then, \( z(\cdot) = 0 \) is the only bounded solution to (7).

Now, the set of bounded and continuous functions is designated as \( BC(\mathbb{R},\mathbb{R}^n) \). If we consider the nonhomogeneous linear system

\[z'(\cdot) = A(\cdot)z(\cdot) + f(\cdot),\]

under an integrable dichotomy condition, we take the following theorem from [23].

Theorem 3. Assume that system (7) has an integrable dichotomy. If \( f \in BC(\mathbb{R},\mathbb{R}^n) \), then system (10) has a unique bounded solution \( z \in BC(\mathbb{R},\mathbb{R}^n) \). Furthermore,

\[z(\cdot) = \int_{-\infty}^{\omega} G(\cdot,\omega)f(\cdot)d\omega.(11)\]

Theorem 4 (see [23]). Assume that the homogeneous system (7) has an integrable dichotomy for which \( \Psi(\cdot)^{P\Psi^{-1}(-1)}(\cdot) \) is bounded. If \( A \) is \( T \)-periodic, then \( \Psi(\cdot)^{P\Psi^{-1}(-1)}(\cdot) \) is also \( T \)-periodic. In addition, if \( f \in BC(\mathbb{R},\mathbb{R}^n) \) is \( T \)-periodic, then (10) has a unique periodic solution satisfying (11).

We present the fixed point theorems that we utilize to demonstrate the existence and uniqueness of periodic solutions to system (4) (see [5, 25]).

Theorem 5 (Banach). Assume that \( (Y,\rho) \) is a complete metric space and \( \Gamma : Y \to Y \). If there is a constant \( \gamma < 1 \) such that for \( u,v \in Y \),

\[\rho(\Gamma u,\Gamma v) \leq \gamma \rho(u,v),(12)\]

then there is one and only one point \( z \in Y \) with \( \Gamma z = z \).

Smart [25] established a hybrid result by combining Banach’s theorem and Schauder’s theorem as follows:
**Theorem 6** (Krasnoselskii). Let $\Omega$ be a closed bounded convex nonempty subset of a Banach space $Y$. Assume that $\Gamma_1$ and $\Gamma_2$ map $\Omega$ into $Y$ such that

(i) $\Gamma_1$ is a contraction mapping on $\Omega$

(ii) $\Gamma_2$ is completely continuous on $\Omega$

(iii) $u, v \in \Omega$ implies $\Gamma_1 u + \Gamma_2 v \in \Omega$

Then, there exists $z \in \Omega$ with $z = \Gamma_1 z + \Gamma_2 z$.

Assume $M > 0$ be a constant. Denote

$$\Omega = \{ u \in BC(\mathbb{R}, \mathbb{R}^n) : \|u\| \leq M \text{ and } u(\xi + T) = u(\xi) \text{ for all } \xi \in \mathbb{R} \}.$$  

(13)

Clearly, the set $\Omega$ is a bounded nonempty and convex subset of $BC(\mathbb{R}, \mathbb{R}^n)$.

Assume that, for $u, v \in \Omega$, there exists $L_1 \in (0, 1)$ such that

$$|q(\xi, u) - q(\xi, v)| \leq L_1 |u - v|,$$  

for all $\xi \in \mathbb{R}$, and for $u_1, u_2, \ldots, u_m, v_1, v_2, \ldots, v_m \in \Omega$, there exists $L_2 > 0$ such that

$$|f(\xi, u_1, u_2, \ldots, u_m) - f(\xi, v_1, v_2, \ldots, v_m)| \leq L_2 (|u_1 - v_1| + \cdots + |u_m - v_m|),$$  

for all $\xi \in \mathbb{R}$.

Denote $\sup_{\xi \in [0, T]} |q(\xi, 0)| = \alpha$, $\sup_{\xi \in [0, T]} |f(\xi, 0, \cdots, 0)| = \beta$, and $\sup_{\xi \in [0, T]} \|A(\xi)\| = \lambda$, and we assume also

$$L_1 M + \alpha + \mu [\lambda (L_1 M + \alpha) + (L_2 M M + \beta)] \leq M.$$  

(16)

3. **Existence of Periodic Solutions for** (4)

In this section, we show the existence and the uniqueness of solution of (4) under the conditions stated in the previous section. So, let $z(\xi) = y(\xi) - q(\xi, y(\xi - \tau(\xi))).$

Hence,

$$z' = A(\xi) z + A(\xi)q(\xi, y(\xi - \tau(\xi))) + f(\xi, y(\xi - \sigma_1(\xi)), \cdots, y(\xi - \sigma_m(\xi))).$$  

(18)

By Theorem 3, system (4) holds the integral equation

$$z(\xi) = \int_{-\infty}^{\infty} G(\xi, \omega) [A(\omega) q(\omega, y(\omega - \tau(\omega))) + f(\omega, y(\omega - \sigma_1(\omega)), \cdots, y(\omega - \sigma_m(\omega)))] d\omega.$$  

(19)

The above equation is equivalent to

$$y(\xi) = q(\xi, y(\xi - \tau(\xi))) + \int_{-\infty}^{\infty} G(\xi, \omega) [A(\omega) q(\omega, y(\omega - \tau(\omega))) + f(\omega, y(\omega - \sigma_1(\omega)), \cdots, y(\omega - \sigma_m(\omega)))] d\omega.$$  

(20)

Define the operators $\Gamma_1$ and $\Gamma_2$ by

$$(\Gamma_1 u)(\xi) = q(\xi, u(\xi - \tau(\xi))) \quad \text{for} \quad u \in BC(\mathbb{R}, \mathbb{R}^n),$$  

(21)

$$(\Gamma_2 u)(\xi) = \int_{-\infty}^{\infty} G(\xi, \omega) f(\omega, u(\omega - \sigma_1(\omega)), \cdots, u(\omega - \sigma_m(\omega))) d\omega.$$  

(22)

Note that if the operator $\Gamma_1 + \Gamma_2$ has a fixed point, then this fixed point is a periodic solution of (4).

**Lemma 7.** If (14) and (15) hold, then the operators $\Gamma_1$ and $\Gamma_2$ are defined by (21) and (22), respectively, from $\Omega$ into $BC(\mathbb{R}, \mathbb{R}^n)$, that is, $\Gamma_1, \Gamma_2 : \Omega \longrightarrow BC(\mathbb{R}, \mathbb{R}^n)$.

**Proof.** Let $u \in \Omega$, by (14), therefore,

$$|((\Gamma_1 u)(\xi))| = \|q(\xi, u(\xi - \tau(\xi)))\| \leq L_1 |u(\xi - \tau(\xi))| + |q(\xi, 0)| \leq L_1 ||u|| + \sup_{\xi \in [0, T]} |q(\xi, 0)| \leq L_1 M + \alpha.$$  

(23)

Secondly, for $u \in \Omega$, by (14) and (15), we get

$$|((\Gamma_2 u)(\xi))| = \int_{-\infty}^{\infty} ||G(\xi, \omega)|| \|A(\omega)\| ||q(\omega, u(\omega - \tau(\omega)))|| d\omega \leq [\lambda (L_1 M + \alpha) + (L_2 M M + \beta)] \int_{-\infty}^{\infty} ||G(\xi, \omega)|| d\omega \leq \rho \lambda [L_1 M + \alpha] + (L_2 M M + \beta).$$  

(24)

Since all quantities in $\Gamma_1$ and $\Gamma_2$ are periodic, then $\Gamma_1$, $\Gamma_2 : \Omega \longrightarrow BC(\mathbb{R}, \mathbb{R}^n)$.

**Lemma 8.** If (14) holds, then the operator $\Gamma_1 : \Omega \longrightarrow BC(\mathbb{R}, \mathbb{R}^n)$ defined by (21) is a contraction.

**Proof.** Let $u, v \in \Omega$. By using (14), we get

$$|((\Gamma_1 u)(\xi)) - ((\Gamma_1 v)(\xi))| = |q(\xi, u(\xi - \tau(\xi))) - q(\xi, v(\xi - \tau(\xi)))| \leq L_1 |u(\xi - \tau(\xi)) - v(\xi - \tau(\xi))| \leq L_1 |u - v|.$$  

(25)
Then,

\[ \| \Gamma_1 u - \Gamma_1 v \| \leq L_1 \| u - v \|. \]  

(26)

Therefore, \( \Gamma_1 \) is a contraction because \( L_1 \in (0, 1) \).

**Lemma 9.** If (14) and (15) hold, then the operator \( \Gamma_2 : \Omega \rightarrow BC(\mathbb{R}, \mathbb{R}^n) \) defined by (22) is completely continuous.

**Proof.** To prove the operator \( \Gamma_2 : \Omega \rightarrow BC(\mathbb{R}, \mathbb{R}^n) \) completely continuous, we must prove that \( \Gamma_2 \) is continuous and \( \Gamma_1(\Omega) \) is contained in a compact set; for this purpose, let \( u_n \in \Omega \) where \( n \) is a positive integer such that \( u_n \rightarrow u \) as \( n \rightarrow \infty \). Then,

\[ \|(\Gamma_2 u_n) - (\Gamma_2 u)\| \leq \int_{-\infty}^{\infty} \| G(\xi, \omega) \| A(\xi, \omega) \| q(\xi, u_n(\omega - \tau(\omega))) \| d\omega \]

\[ - \| q(\xi, u(\omega - \tau(\omega))) \| d\omega \]

\[ \times \| f(\omega, u_n(\omega - \xi(\xi)), \ldots, u_n(\omega - \sigma_n(\xi))) \| d\omega \]

\[ - \| f(\omega, u(\omega - \xi(\xi)), \ldots, u(\omega - \sigma_n(\xi))) \| d\omega \]

\[ \leq \mu(\lambda L_1 + L_2 m) \| u_n - u \|. \]

(27)

So, the dominated convergence theorem implies

\[ \lim_{n \rightarrow \infty} \|(\Gamma_2 u_n) - (\Gamma_2 u)\| = 0, \]

(28)

which implies that \( \Gamma_2 \) is continuous. Next, we show that the image of \( \Gamma_2 \) is contained in a compact set. Let \( u_n \in \Omega \), and by (24), we have

\[ \| \Gamma_2 u_n \| \leq \mu(\lambda L_1 + \alpha) + (L_2 m + \beta). \]

(29)

Second, we calculate \( (\Gamma_2 u_n) \)'(\( \xi \)) and show that it is uniformly bounded.

\[ (\Gamma_2 u_n)'(\xi) = \left( \int_{-\infty}^{\infty} G(\xi, \omega) A(\xi, \omega) q(\xi, u_n(\omega - \tau(\omega))) \right) \]

\[ + f(\omega, u_n(\omega - \xi(\xi)), \ldots, u_n(\omega - \sigma_n(\xi))) d\omega \]

\[ = u_n'(\xi) - q(\xi, u_n(\xi - \tau(\xi))) \]

\[ = A(\xi, u(\omega - \xi(\xi)), \ldots, u_n(\omega - \sigma_n(\xi))) \]

\[ = \lambda(\xi)(\Gamma_1 u_n + \Gamma_2 u_n) + f(\xi, u_n(\xi - \xi(\xi)), \ldots, u_n(\omega - \sigma_n(\xi))). \]

(30)

Then,

\[ \| (\Gamma_2 u_n)' \| \leq (\lambda + L_2 m)M + \beta. \]

(31)

Thus, the sequence \( (\Gamma_2 u_n)' \) is uniformly bounded and equicontinuous. As a result, by Ascoli-Arzela’s theorem \( \Gamma_2(\Omega) \) is relatively compact.

We next prove for any \( u, v \in \Omega \) that \( \Gamma_1 u + \Gamma_2 v \in \Omega \).

**Lemma 10.** If (14)–(16) hold, then for any \( u, v \in \Omega \), we have \( \Gamma_1 u + \Gamma_2 v \in \Omega \).

**Proof.** Let \( u, v \in \Omega \). Then, \( \| u \|, \| v \| \leq M \). By (16), we have

\[ \| \Gamma_1 u(\xi) + \Gamma_2 v(\xi) \| \leq \| q(\xi, u(\omega - \tau(\omega))) \| + \int_{-\infty}^{\infty} \| G(\xi, \omega) \| A(\xi, \omega) \| q(\xi, u(\omega - \tau(\omega))) \| d\omega \]

\[ + \int_{-\infty}^{\infty} \| G(\xi, \omega) \| f(\omega, u(\omega - \sigma_1(\xi)), \ldots, u(\omega - \sigma_n(\xi))) d\omega \]

\[ \leq \lambda L_1 + \alpha + (L_2 m + \beta) \int_{-\infty}^{\infty} \| G(\xi, \omega) \| d\omega \]

\[ \leq \lambda L_1 + \alpha + (L_2 m + \beta) \leq M. \]

(32)

It follows that

\[ \| \Gamma_1 u + \Gamma_2 v \| \leq M, \]

(33)

for all \( u, v \in \Omega \). Hence, \( \Gamma_1 u + \Gamma_2 v \in \Omega \).

**Theorem 11.** Assume that system (7) has an integrable dichotomy. If conditions (14)–(16) hold, then system (4) has at least one \( T \)-periodic solution.

**Proof.** Clearly, by Lemmas 7–10, all the requirements of the Krasnoselskii’s theorem are satisfied. Thus, there exists a fixed point \( z \in \Omega \) such that \( z = \Gamma_1 z + \Gamma_2 z \); this fixed point is a solution of (4). Hence, (4) has a \( T \)-periodic solution.

**Theorem 12.** Assume that system (7) has an integrable dichotomy. If conditions (14) and (15) and

\[ L_1 + \mu(\lambda L_1 + L_2 m) < 1, \]

(34)

hold, then system (4) has a unique \( T \)-periodic solution.

**Proof.** Let the mapping \( \Gamma \) be presented by

\[ (\Gamma u)(\xi) = q(\xi, u(\xi - \tau(\xi))) + \int_{-\infty}^{\infty} G(\xi, \omega) A(\xi, \omega) q(\xi, u(\omega - \tau(\omega))) \]

\[ + f(\omega, u(\omega - \sigma_1(\xi)), \ldots, u(\omega - \sigma_n(\xi))) d\omega. \]

(35)

For \( u_1, u_2 \in BC(\mathbb{R}, \mathbb{R}^n) \), we obtain

\[ \|(\Gamma u_1)'(\xi) - (\Gamma u_2)'(\xi)\| \leq \| q(\xi, u_1(\omega - \tau(\omega))) - q(\xi, u_2(\omega - \tau(\omega))) \| \]

\[ + \int_{-\infty}^{\infty} \| G(\xi, \omega) \| A(\xi, \omega) \| q(\xi, u_1(\omega - \tau(\omega))) \| d\omega \]

\[ - \| q(\xi, u_2(\omega - \tau(\omega))) \| d\omega + \int_{-\infty}^{\infty} \| G(\xi, \omega) \| f(\omega, u_1(\omega - \sigma_1(\xi)), \ldots, u_1(\omega - \sigma_n(\xi))) d\omega \]

\[ - f(\omega, u_2(\omega - \sigma_1(\xi)), \ldots, u_2(\omega - \sigma_n(\xi))) d\omega \]

\[ = (L_1 + \mu(\lambda L_1 + L_2 m))\| u_1 - u_2 \|. \]

(36)
Example 1. Consider system (4) with \( n = 2, m = 2, T = 2\pi, \) and \( y = (y_1, y_2)' \), and

\[
y(\zeta) = \begin{pmatrix} y_1(\zeta) \\ y_2(\zeta) \end{pmatrix}, \quad q(\zeta, y(\zeta - \tau(\zeta))) = 10^{-4} \sin(\zeta) \begin{pmatrix} y_2(\zeta - \cos(\zeta)) \\ y_1(\zeta - \cos(\zeta)) \end{pmatrix},
\]

\[
A(\zeta) = \begin{pmatrix} 10^{-2} \sin(t) & -0.99 \\ 0.99 & 10^{-3} \sin(t) \end{pmatrix},
\]

\[
f(\zeta, y(\zeta - \sigma_1(\zeta)), y(\zeta - \sigma_2(\zeta))) = 10^{-5} \cos(\zeta) \begin{pmatrix} y_1(\zeta - 10^{-2}) + y_2(\zeta - \sin(\zeta)) \\ y_2(\zeta - 10^{-2}) + y_1(\zeta - \sin(\zeta)) \end{pmatrix}.
\]

Let the set

\[
\Omega = \{ u \in BC([0, 2\pi], \mathbb{R}^n) : \|u\| \leq M \text{ and } u(\zeta + 2\pi) = u(\zeta) \text{ for all } \zeta \in \mathbb{R} \}.
\]

Clearly, the set \( \Omega \) is a bounded nonempty closed and convex subset of \( BC([0, 2\pi], \mathbb{R}^n) \) for any positive constant \( M \).

Note that \( L_c = 10^{-4}, L_d = 10^{-5}, \alpha = 0, \) and \( \beta = 0, \) and we use \( \|A\| = \max_{1 \leq j \leq 2} \sum_{i=1}^{2} a_{ij} \) to get

\[
\|A(\zeta)\| = \left\| \begin{pmatrix} 10^{-2} \sin(\zeta) & -0.99 \\ 0.99 & 10^{-3} \sin(\zeta) \end{pmatrix} \right\| = \max \left\{ |10^{-2} \sin(\zeta)| + 0.99, |10^{-3} \sin(\zeta)| + 0.99 \right\}
\]

\[
= |10^{-2} \sin(\zeta)| + 0.99.
\]

Then, \( \lambda = 10^{-3}. \)

We can see that conditions (14) and (15) hold.

We substitute all quantities in the inequality (16), and we have

\[
10^{-4} + \mu [10^{-5} + 2 \times 10^{-5}] \leq 1.
\]

Now, since the matrix \( A \) is continuous and periodic, then system (4) has an integrable dichotomy, and we have two cases: if \( \mu \leq (1 - 10^{-4})/(3 \times 10^{-5}) \), then (16) holds for any positive constant \( M \), and by Theorem 11, system (4) has at least one \( 2\pi \)-periodic solution.

If \( \mu < (1 - 10^{-4})/(3 \times 10^{-5}) \), then condition (34) holds, and by Theorem 11, system (4) has a unique \( 2\pi \)-periodic solution.

4. Existence of Periodic Solutions for (5)

In this section, we show the existence and the uniqueness of the solution of (5) under the conditions stated in the previ-
Proof. Let \( u \in \Omega \); by (14), we get

\[
|\langle \{Y_1 u\} (\zeta) \rangle| = \left| \int_{-\infty}^{0} Q(r) q(\zeta, u(\tau + r)) dr \right| \\
\leq (L_1 |u(\zeta - \tau(\zeta))| + |q(\zeta, 0)|) \left| \int_{-\infty}^{0} Q(r) dr \right| \\
\leq \left( L_1 \|u\| + \sup_{\zeta \in [0,1]} |q(\zeta, 0)| \right) \|I\| \leq L_1 M + a.
\]

(47)

Secondly, for \( u \in \Omega \), by (14) and (15), we get

\[
|\langle \{Y_2 u\} (\zeta) \rangle| = \left| \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} Q(r) q(\zeta, u(\omega(\tau + r))) dr dw \right| \\
\leq \left| \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} Q(r) f(\omega, u(\omega + \sigma_1(\tau)), \cdots, u(\omega + \sigma_m(\tau))) dr dw \right| \\
\leq \mu \left[ L_1 M + a + (L_2 M + \beta) \right] \left| \int_{-\infty}^{\infty} Q(r) dr \right| \\
\leq \mu \left[ L_1 M + a + (L_2 M + \beta) \right].
\]

(48)

Since all quantities in \( Y_1 \) and \( Y_2 \) are periodic, then \( Y_1, Y_2 : \Omega \rightarrow BC(\mathbb{R}, \mathbb{R}^n) \).

By the same technique proofs in Lemmas 8–10, we state the following lemmas without proofs.

**Lemma 14.** If (14) holds, then the operator \( Y_1 : \Omega \rightarrow BC(\mathbb{R}, \mathbb{R}^n) \) defined by (45) is a contraction.

**Lemma 15.** If (14) and (15) hold, then the operator \( Y_2 : \Omega \rightarrow BC(\mathbb{R}, \mathbb{R}^n) \) defined by (46) is completely continuous.

**Lemma 16.** If (14)–(16) hold, then for any \( u, v \in \Omega \), we have \( Y_1 u + Y_2 v \in \Omega \).

**Theorem 17.** Assume that system (7) has an integrable dichotomy. If conditions (14)–(16) hold, then system (5) has at least one \( T \)-periodic solution.

Proof. By Lemmas 13–16, all the requirements of the Krasnoselskii’s theorem are satisfied. Thus, there exists a fixed point \( z \in \Omega \) such that \( z = Y_1 z + Y_2 z \); this fixed point is a solution of (5). Hence, (5) has a \( \zeta \)-periodic solution.

**Theorem 18.** Assume that system (7) has an integrable dichotomy. If conditions (14), (15), and (34) hold, then system (5) has a unique \( T \)-periodic solution.

Proof. Let the mapping \( Y \) be presented by

\[
\langle Yu (\zeta) \rangle = \left[ \int_{-\infty}^{0} Q(r) q(\zeta, u(\tau + r)) dr \right] + \left[ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} G(\zeta, \omega) \left( \int_{-\infty}^{0} Q(r) q(\omega, u(\omega + \tau)) dr \right) dw \right] \\
\leq \left( L_1 |u(\zeta - \tau(\zeta))| + |q(\zeta, 0)| \right) \left| \int_{-\infty}^{0} Q(r) dr \right| \\
\leq \left( L_1 \|u\| + \sup_{\zeta \in [0,1]} |q(\zeta, 0)| \right) \|I\| \leq L_1 M + a + (L_2 M + \beta).
\]

(49)

For \( u_1, u_2 \in BC(\mathbb{R}, \mathbb{R}^n) \), we have

\[
\langle (Y u_1) (\zeta) - (Y u_2) (\zeta) \rangle \leq \left| \int_{-\infty}^{0} Q(r) q(\zeta, u_1(\tau + r)) dr - \int_{-\infty}^{0} Q(r) q(\zeta, u_2(\tau + r)) dr \right| \\
\leq \left| \int_{-\infty}^{0} \|G(\zeta, \omega)\| \|A(\omega)\| \left( \int_{-\infty}^{0} Q(r) q(\omega, u_1(\omega + \tau)) dr \right) dw \right| \\
\leq \mu \left[ L_1 M + a + (L_2 M + \beta) \right].
\]

(50)

Since (34) holds, the contraction mapping completes the proof.

\[ \square \]

5. Conclusion

In this paper, we dealt with the study of types of neutral equations more generally, represented in nonlinear systems with several delays under the dichotomy condition, where the fixed point theorems were used to prove existence and uniqueness.

The benefit of this paper is to generalize several well-known researches such as [11, 14]. So that if \( q(\zeta, y(\zeta - \tau(\zeta))) = c y(\zeta - \tau(\zeta)) \), then our results will apply to systems (3) of [11]. Also, the periodicity of (1) and (2) in [14] is generalized by our systems (4) and (5) in \( n \)-dimensional case.

**Data Availability**

The numerical data used to support the findings of this study are included in the article.

**Conflicts of Interest**

The authors declare that they have no competing interests.

There are no any nonfinancial competing interests (political, personal, religious, ideological, academic, intellectual, commercial, or any other) to declare in relation to this manuscript.

**Authors’ Contributions**

Mesmouli directed the study and helped inspection. All the authors carried out the main results of this article and drafted the manuscript and read and approved the final manuscript.
Acknowledgments

This research has been funded by the Scientific Research Deanship at University of Hail, Saudi Arabia, through project number RG-20 125.

References

[1] E. M. Elabbasy, T. S. Hassan, and S. H. Saker, "Oscillation criteria for first-order nonlinear neutral delay differential equations," Electronic Journal of Differential Equations, vol. 2005, 2005.

[2] E. M. Elabbasy, T. S. Hassan, and S. H. Saker, "Oscillation of nonlinear neutral delay differential equations," Journal of Applied Mathematics and Computing, vol. 21, no. 1-2, pp. 99–118, 2006.

[3] L. Erbe, T. S. Hassan, and A. Peterson, "Oscillation of second order neutral delay differential equations," Advances in Dynamical Systems and Applications, vol. 3, no. 1, pp. 53–71, 2008.

[4] L. Erbe, T. S. Hassan, and A. Peterson, "Oscillation criteria for nonlinear functional neutral dynamic equations on time scales," Journal of Difference Equations and Applications, vol. 15, no. 11-12, pp. 1097–1116, 2009.

[5] T. A. Burton, Stability by fixed point theory for functional differential equations, Dover Publications, New York, 2006.

[6] S. Deng, X.-B. Shu, and J. Mao, "Existence and exponential stability for impulsive neutral stochastic functional differential equations driven by fBm with noncompact semigroup via M6650inch fixed point," Journal of Mathematical Analysis and Applications, vol. 467, no. 1, pp. 398–420, 2018.

[7] M. B. Mesmouli, A. Ardjouni, and A. Djoudi, "Periodicity of solutions for a system of nonlinear integro-differential equations," Sarajevo Journal of Mathematics, vol. 11, no. 1, pp. 49–63, 2015.

[8] M. B. Mesmouli, A. Ardjouni, and A. Djoudi, "Periodic solutions and stability in a nonlinear neutral system of differential equations with infinite delay," Boletín de la Sociedad Matemática Mexicana, vol. 24, no. 1, pp. 239–255, 2018.

[9] M. B. Mesmouli, "Stability in system of impulsive neutral functional differential equations," Mediterranean Journal of Mathematics, vol. 18, no. 1, pp. 1–9, 2021.

[10] M. B. Mesmouli and C. Tunç, "Matrix measure and asymptotic behaviors of linear advanced systems of differential equations," Boletín de la Sociedad Matemática Mexicana, vol. 27, no. 2, pp. 1–12, 2021.

[11] P. Sa Ngiamsunthorn, "Existence of periodic solutions for differential equations with multiple delays under dichotomy condition," Advances in Difference Equations, vol. 2015, no. 1, Article ID 598, 2015.

[12] L. Shu, X. Shu, and J. Mao, "Approximate controllability and existence of mild solutions for Riemann-Liouville fractional stochastic evolution equations with nonlocal conditions of order 1 < α ≤ 2," Fractional Calculus and Applied Analysis, vol. 22, no. 4, pp. 1086–1112, 2019.

[13] Z. Cheng, L. Lv, J. Liu, and School of Mathematics and Information Science, Henan Polytechnic University, Jiaozuo 454000, China, "Positive periodic solution of first-order neutral differential equation with infinite distributed delay and applications," AIMS Mathematics, vol. 5, no. 6, pp. 7372–7386, 2020.

[14] Y. Luo, W. Wang, and J. H. Shen, "Existence of positive periodic solutions for two kinds of neutral functional differential equations," Applied Mathematics Letters, vol. 21, no. 6, pp. 581–587, 2008.

[15] J. Luo and J. Yu, "Global asymptotic stability of nonautonomous mathematical ecological equations with distributed deviating arguments," Acta Mathematica Sinica, vol. 41, pp. 1273–1282, 1998.

[16] P. Weng and M. Liang, "The existence and behavior of periodic solution of hematopoiesis model," Mathématiques Appliquées, vol. 4, pp. 434–439, 1995.

[17] W. S. C. Gurney, S. P. Blythe, and R. M. Nisbet, "Nicholson’s blowflies revisited," Nature, vol. 287, no. 5777, pp. 17–21, 1980.

[18] W. Joseph, H. So, and J. Yu, "Global attractivity and uniform persistence in Nicholson’s blowflies," Differential Equations and Dynamical Systems, vol. 1, pp. 11–18, 1994.

[19] K. Gopalsamy, Stability and oscillation in delay differential equations of population dynamics, Kluwer Academic Press, Boston, 1992.

[20] C. J. Guo, G. Q. Wang, and S. S. Cheng, "Periodic solutions for a neutral functional differential equation with multiple variable lags," Archiv der Mathematik, vol. 42, no. 1, pp. 1–10, 2006.

[21] Y. Guo, M. Chen, X.-B. Shu, and F. Xu, "The existence and Hyers-Ulam stability of solution for almost periodical fractional stochastic differential equation with fBm," Stochastic Analysis and Applications, vol. 39, no. 4, pp. 643–666, 2021.

[22] X.-B. Shu, F. Xu, and Y. Shi, "S-asymptotically ω-positive periodic solutions for a class of neutral fractional differential equations," Applied Mathematics and Computation, vol. 270, pp. 768–776, 2015.

[23] M. Pinto, "Dichotomy and existence of periodic solutions of quasilinear functional differential equations," Nonlinear Analysis: Theory Methods & Applications, vol. 72, no. 3-4, pp. 1227–1234, 2010.

[24] M. Pinto, "Dichotomies and asymptotic formulas for the solutions of differential equations," Journal of Mathematical Analysis and Applications, vol. 195, no. 1, pp. 16–31, 1995.

[25] D. R. Smart, Fixed Point Theorems, Cambridge University Press, 1980.