Supergeometry and Arithmetic Geometry.

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Abstract

We define a superspace over a ring $R$ as a functor on a subcategory of the category of supercommutative $R$-algebras. As an application the notion of a $p$-adic superspace is introduced and used to give a transparent construction of the Frobenius map on $p$-adic cohomology of a smooth projective variety over $\mathbb{Z}_p$ (the ring of $p$-adic integers).

1 Introduction

It is possible that all physical quantities are measured in rational numbers (or even in integers if there exist elementary length, elementary unit of time, etc). All other numbers should be introduced only for mathematical convenience as it is much more practical to work with an algebraically closed field that is complete with respect to some norm.

Let us consider an oversimplified example where time and space are quantized and the motion of a particle satisfies the equation

$$\Delta_2 x(t) = F(x(t), t)$$

(1)

Here $\Delta_2$ stands for the second difference:

$$\Delta_2 x(t) = [(x(t+2) - x(t+1)) - (x(t+1) - x(t))] = x(t+2) - 2x(t+1) + x(t)$$

(2)

The equation (1) which is the finite difference analogue of the Newton’s second law, permits us to calculate recursively $x(t)$ for $t \in \mathbb{Z}$ if we know $x(0), x(1)$ and $F$; if the initial data $x(0), x(1)$ and the “force” $F$ are all integers then all coordinates $x(t)$ will also be integers. In some sense, all physical questions can be answered if we know only integer numbers. However, if we would like to write down an explicit solution of (1) even in the simplest situation when $F$ is a linear function of $x(t)$ and does not depend explicitly on $t$ we need irrational and complex numbers. Indeed, the general solution to the equation

$$\Delta_2 x(t) = ax(t)$$

(3)

has the form

$$x(t) = A_1 \lambda_1^t + A_2 \lambda_2^t$$

(4)

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where $\lambda_1, \lambda_2$ are solutions of a quadratic equation.

Usually we take $\lambda_1, \lambda_2 \in \mathbb{C}$; we are working with the field of complex numbers that is algebraically closed and complete with respect to the natural norm. However it is possible to make another choice. It is well known that every norm on $\mathbb{Q}$ is equivalent to either the standard norm $\|x\| = |x|$ or $p$-adic norm $\|x\|_p = p^{-\text{ord}_p x}$ where $p$ is a prime number. Here $\text{ord}_p x$ is equal to the multiplicity of $p$ in the prime decomposition of $x$ if $x \in \mathbb{Z}$; in general, $\text{ord}_p(x/y) = \text{ord}_p x - \text{ord}_p y$. Completing $\mathbb{Q}$ with respect to the $p$-adic norm we obtain the field of $p$-adic numbers $\mathbb{Q}_p$; the completion of $\mathbb{Z}$ in the $p$-adic norm gives the ring of $p$-adic integers $\mathbb{Z}_p$. Since $\mathbb{Q}_p$ is not algebraically closed, sometimes it is convenient to work with the field $\mathbb{C}_p$ obtained as a completion of the algebraic closure $\overline{\mathbb{Q}}_p$ of the field $\mathbb{Q}_p$ with respect to an appropriate norm. The field $\mathbb{C}_p$ is algebraically closed and complete; in many ways it is as convenient as the field of complex numbers.

Of course, in most cases it is better to work with the familiar complex numbers, but situations do arise where $p$-adic methods are more efficient. Let us suppose, for example, that the space variable $x$ in (3) is periodic with period $N$, or, in other words, $x$ takes values in cyclic group $\mathbb{Z}_N$. (We are using the notation $\mathbb{Z}_p$ for $\mathbb{Z}/p\mathbb{Z}$ to distinguish it from the ring of $p$-adic integers $\mathbb{Z}_p$.) Then the equation (3) splits into independent equations corresponding to prime factors of $N$; it is natural to study the elementary equations by means of $p$-adic methods.

There were numerous attempts to use $p$-adic numbers in quantum field theory and string theory (see, for example, [2] for review of $p$-adic strings). It is clear that taking $p$-adic physics seriously (i.e. considering it not as a formal gadget, but as physical reality) one should work with all prime numbers simultaneously (in other words one should work with adeles). We would like to emphasize that we think that not only $p$-adic numbers, but also irrational and complex numbers should be regarded as a formal tool in the theory based on rational numbers. If we accept this viewpoint there is no necessity to work in the adelic setting.

In general, $p$-adic methods have a good chance to be useful when interesting physical quantities are represented by integer or rational numbers. This is the case for topological sigma-models. It was shown in [1], [5] that the instanton numbers for sigma-models over complex numbers can be expressed in terms of $p$-adic $B$-model, or, in other words, in terms of the variation of Hodge structure on $p$-adic cohomology. More precisely, it was proven that instanton numbers can be expressed in terms of the Frobenius map on $p$-adic cohomology and this fact was used to analyze integrality of instanton numbers.

Recall that the Frobenius map of a field of characteristic $p$ transforms $x$ into $x^p$. This map is an endomorphism of the field, it is furthermore an automorphism for finite fields and an identity map for the field $\mathbb{F}_p$ consisting of $p$ elements. Of course the $p$-th power map is an algebra homomorphism for any
This last observation leads to the Frobenius map on a variety over $\mathbb{F}_p$. One can consider the $p$-th power map also on the $p$-adic numbers, but in this case it is not a homomorphism. We can define the Frobenius map on the $n$-dimensional affine or projective space over $\mathbb{Z}_p$ by raising the coordinates to the $p$-th power. However, a variety sitting inside (an affine or projective variety) will not, in general, be invariant with respect to the Frobenius map. Nevertheless, one can define the Frobenius map on the cohomology with coefficients in the ring of $p$-adic integers $\mathbb{Z}_p$.

The standard approach to the construction is quite complicated; it is based on the consideration of the so called DP-neighborhood of a variety that has the same cohomology as the original variety and at the same time is invariant with respect to the Frobenius map. One of the main goals of the present paper is to give a simplified construction of the Frobenius map in terms of supergeometry. The logic of our construction remains the same, but the role of a DP-neighborhood is played by a $p$-adic superspace having the same body as the original variety.

We hope that our approach to the Frobenius map will be much more accessible to physicists than the standard one based on the consideration of the crystalline site and DP-neighborhoods. Our considerations, as presented here, are not completely rigorous; we did not want to exceed significantly the level of rigor that is standard for a physics journal. However, we believe that our results are of interest also to pure mathematicians and therefore we have written a rigorous exposition of our construction of the Frobenius map.

It seems that arithmetic geometry should play an essential role in string theory; the paper sketches one way of applying arithmetic geometry, but there are also other ways (let us mention a series of papers by Schimmrigk, for example). The present paper shows that, conversely, some ideas borrowed from physics can be used to clarify some important notions of arithmetic geometry.

As we mentioned, our construction is based on the notion of a $p$-adic superspace. We hope that this notion will have also other applications. In particular, we expect that it can be used to construct supersymmetric sigma-models and topological sigma-models in the $p$-adic setting. The $p$-adic $B$-model used in was defined completely formally as the theory of variations of Hodge structures. However it is natural to conjecture that it can be defined in the framework of Lagrangian formalism and that it is related to the (conjectural) $p$-adic supersymmetric sigma-model.

One should notice that in an appropriate coordinate system the operations

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1. An $R$-algebra is a ring $A$ equipped with a ring homomorphism $R \rightarrow A$. This notion is perhaps more familiar in the case when $R$ is a field. All rings are assumed to be unital and associative.

2. Varieties we consider are defined over the ring of $p$-adic integers $\mathbb{Z}_p$. The elements of this ring can be represented as formal power series $\sum a_k p^k$ where the coefficients $a_0, a_1, ...$ are integers between 0 and $p - 1$. If the series is finite it specifies a conventional integer and the operations on $p$-adic numbers coincide with operations in $\mathbb{Z}$. Assigning the coefficient $a_0$ to a series $\sum a_k p^k$ we obtain a homomorphism of $\mathbb{Z}_p$ onto the field $\mathbb{F}_p$. Using this homomorphism we obtain from every variety over the $p$-adic integers a variety over $\mathbb{F}_p$; we say that this variety is the body of the variety over the $p$-adic integers.
in the (super)group of supersymmetries are given by polynomial formulas with integer coefficients; this means that one can define the supersymmetry group over an arbitrary commutative ring \( R \) assuming that the coordinates belong to this ring (for example, one may consider the supersymmetry group over \( \mathbb{Z} \); then we can talk about supersymmetry over integers).

To construct supersymmetric objects it is natural to introduce a notion of superspace over ring \( R \); this can be done in many ways. In the conventional superspace over real numbers coordinates are considered as even or odd elements of a Grassmann algebra. The algebra of functions on the \((m, n)\)-dimensional affine superspace over \( \mathbb{R} \) can be considered as a tensor product of the algebra of smooth functions of \( m \) real variables and the Grassmann algebra with \( n \) generators. In mathematical formulation [7] the conventional superspace is regarded as a functor on the category of Grassmann algebras with values in the category of sets (we assign to every Grassmann algebra \( \Lambda \) the set of points with coordinates from \( \Lambda \)). A function on a superspace is defined as a map of functors. It is clear that the functions on superspace described above specify maps of functors (natural transformations). It is not so clear that there are no other natural transformations, but this can be proven [11]. One can consider also superspaces where in addition to \( m \) even and \( n \) odd coordinates we have \( r \) even nilpotent coordinates; then there is an additional factor (the algebra of formal power series with \( r \) variables) in the tensor product specifying the algebra of functions (see [3]). It is important to notice that even considering only superspaces over real numbers one can modify the definition of a superspace in a variety of ways to obtain different classes of functions. For example, we can assume that the coordinates are even and odd elements of an arbitrary supercommutative algebra over \( \mathbb{R} \); then only polynomial functions are allowed.

If we take as the starting point the functorial approach, the definition of a superspace over a ring \( R \) becomes very natural: we should fix a subcategory of the category of supercommutative \( R \)-algebras and define a superspace as a functor from this subcategory to the category of sets. The algebra of functions depends on the choice of a subcategory. Notice that the functorial approach is common in algebraic geometry where a variety over a ring \( R \) specifies a functor on the category of \( R \)-algebras with values in the category of sets. (A variety is singled out by polynomial equations with coefficients in \( R \) in affine or projective space over \( R \). These equations make sense if the coordinates are considered as elements of an \( R \)-algebra; we can consider the corresponding set of solutions. Similarly one can define the notion of a supervariety singled out by means of polynomial equations in an affine or projective superspace and construct the corresponding functor.)

We are interested in the case when \( R \) is the ring of \( p \)-adic integers \( \mathbb{Z}_p \) and we define the subcategory \( \Lambda(p) \) as the category of rings of the form \( B \otimes \Lambda \) where \( B \) is a commutative ring such that \( p^{N} B = 0 \) and \( B/pB \) does not contain nilpotent elements; \( \Lambda \) stands for a Grassmann ring. By definition, a \( p \)-adic superspace is a functor on the category \( \Lambda(p) \) with values in the category of sets. As always, we can define an affine superspace as a space with \( m \) even, \( r \) even nilpotent and \( n \) odd coordinates. Functions on such a superspace are described
in [8]. In this paper we need only the case when \( m = n = 0 \). If \( r = 1 \) (one even nilpotent coordinate), then functions have the form \( \sum c_n n! z^n \) where \( c_n \in \mathbb{Z}_p \). The superspace at hand can be considered as an infinitesimal neighborhood \( pt \) of a point on a line; it follows from the description of functions that the cohomology of \( pt \) coincides with the cohomology of a point (this is a crucial observation that is used in the construction of the Frobenius map). The space \( pt^r \) with \( r \) even nilpotent coordinates can be regarded as an infinitesimal neighborhood of a point in the \( r \)-dimensional affine space; its cohomology also coincides with the cohomology of a point. Notice that the superspace \( pt^r \) does not have any odd coordinates (even though Grassmann algebras were used in its construction).

Let us restrict the functor defining a \( p \)-adic superspace to the subcategory of \( \mathbb{F}_p \)-algebras without nilpotent elements. If there exists a variety over \( \mathbb{F}_p \) that specifies the same functor we say that this variety is the body of the superspace at hand.

Consider a projective variety \( V \) over \( \mathbb{Z}_p \). We mentioned already that the Frobenius map acts on the projective space sending a point with homogeneous coordinates \((x_0 : ... : x_n)\) to the point with coordinates \((x_0^p : ... : x_n^p)\). The variety \( V \) is not invariant with respect to the Frobenius map, but its body is invariant. (We can define the body considering the functor \( X_{\mathbb{V}} \) corresponding to the variety \( V \) or we can use the simpler definition in the footnote 2; both definitions lead to the same result.) Now we consider the maximal subsuperspace \( \tilde{X}_{\mathbb{V}} \) of the projective space having the same body as \( V \) (and call it the infinitesimal neighborhood of \( X_{\mathbb{V}} \)). It is obvious that \( \tilde{X}_{\mathbb{V}} \) is invariant with respect to the Frobenius map. This means that the Frobenius map induces a homomorphism on the cohomology groups of the \( p \)-adic superspace \( \tilde{X}_{\mathbb{V}} \). If \( V \) is smooth one can prove that the cohomology groups of \( \tilde{X}_{\mathbb{V}} \) are isomorphic to the cohomology groups of \( V \). (The comparison of cohomology groups can be reduced to local calculations and locally we use the isomorphism between the cohomology of \( pt^r \) and the cohomology of a point.) We thus obtain an action of Frobenius map on cohomology groups of a non-singular projective variety over \( \mathbb{Z}_p \) (Sec. 6). In the last section we demonstrate the properties of Frobenius map that were used in [4]. In the paper [9] the same properties were used to express the Frobenius map on the cohomology of a Calabi-Yau threefold in terms of the mirror map and instanton numbers.

## 2 Superspaces

Let us denote by \( A \) a supercommutative ring (i.e. a \( \mathbb{Z}_2 \)-graded ring \( A = A_0 \oplus A_1 \) where \( a_0 b = ba_0, a_1 a'_1 = -a'_1 a_1 \) if \( a_0 \in A_0, a_1, a'_1 \in A_1, b \in A \)).

One can say that \( A^{p,q} = \{(x_1, ..., x_p, \xi_1, ..., \xi_q) | x_i \in A_0, \xi_i \in A_1 \} \) is a set of \( A \)-points of \( (p,q) \)-dimensional superspace (of space with \( p \) commuting and \( q \) anticommuting coordinates). One can consider \( A^{p,q} \) as an abelian group or as an \( A_0 \)-module, but these structures will not be important in what follows.

A parity preserving homomorphism \( \varphi : A \to \tilde{A} \) induces a map \( F(\varphi) : A^{p,q} \to \tilde{A}^{p,q} \). It is obvious that \( F(\varphi \psi) = F(\varphi) \circ F(\psi) \) and \( F(id) = id \). This means that
we can consider a functor $\mathbb{A}^{p,q}$ on the category of supercommutative rings taking values in the category of sets and assigning $A^{p,q}$ to the ring $A$.

One can define a superspace as a functor from the category of supercommutative rings into the category of sets; then $\mathbb{A}^{p,q}$ represents the $(p, q)$-dimensional affine superspace (over $\mathbb{Z}$).

We define a map of superspaces as a map of functors. In particular a map of $(p, q)$-dimensional superspace into a $(p', q')$-dimensional superspace is defined as a collection of maps $\psi_A : A^{p,q} \to A^{p',q'}$ that are compatible with parity preserving homomorphisms of rings. In other words, for every parity preserving homomorphism $\varphi : A \to \tilde{A}$ the diagram

$$
\begin{array}{ccc}
A^{p,q} & \xrightarrow{\psi_A} & A^{p',q'} \\
F(\varphi) \downarrow & & \downarrow F(\varphi) \\
\tilde{A}^{p,q} & \xrightarrow{\tilde{\psi}_A} & \tilde{A}^{p',q'}
\end{array}
$$

should be commutative.

Let us denote by $\mathcal{O}_{\mathbb{Z}}^{p,q}$ a $\mathbb{Z}_2$-graded ring of polynomials depending on $p$ commuting and $q$ anticommuting variables and having integer coefficients. In other words:

$$\mathcal{O}_{\mathbb{Z}}^{p,q} = \mathbb{Z}[x^1, \ldots, x^p] \otimes \Lambda_{\mathbb{Z}}[\xi^1, \ldots, \xi^q]$$

is a tensor product of the polynomial ring $\mathbb{Z}[x^1, \ldots, x^p]$ and the Grassmann ring $\Lambda_{\mathbb{Z}}[\xi^1, \ldots, \xi^q]$; the $\mathbb{Z}_2$-grading comes from the $\mathbb{Z}_2$-grading of the Grassmann ring.

It is easy to check that an even element of $\mathcal{O}_{\mathbb{Z}}^{p,q}$ determines a map of the $(p, q)$-dimensional affine superspace into the $(1, 0)$-dimensional affine superspace and an odd element of $\mathcal{O}_{\mathbb{Z}}^{p,q}$ determines a map into $(0, 1)$-dimensional affine superspace. More generally, a row $(f^1, \ldots, f^{p'}, \varphi^1, \ldots, \varphi^{q'})$ of $p'$ even elements of $\mathcal{O}_{\mathbb{Z}}^{p,q}$ and $q'$ odd elements of $\mathcal{O}_{\mathbb{Z}}^{p,q}$ specifies a map of $(p, q)$-dimensional superspace into $(p', q')$-dimensional superspace (a map of functors $\mathbb{A}^{p,q} \to \mathbb{A}^{p',q'}$).

The construction is obvious: we substitute elements of the ring $A$ instead of generators of $\mathcal{O}_{\mathbb{Z}}^{p,q}$. One can prove that all maps of functors $\mathbb{A}^{p,q} \to \mathbb{A}^{p',q'}$ are described by means of this construction.

More examples of superspaces can be constructed as affine supervarieties. Affine supervariety (as usual affine variety) can be defined by means of polynomial equations:

$$f_1(x^1, \ldots, x^p, \xi^1, \ldots, \xi^q) = 0$$

$$\ldots$$

$$f_{p'}(x^1, \ldots, x^p, \xi^1, \ldots, \xi^q) = 0$$

$$\varphi_1(x^1, \ldots, x^p, \xi^1, \ldots, \xi^q) = 0$$

$$\ldots$$

$$\varphi_{q'}(x^1, \ldots, x^p, \xi^1, \ldots, \xi^q) = 0$$

More formally, we consider a map of functors $F : \mathbb{A}^{p,q} \to \mathbb{A}^{p',q'}$ corresponding to a row $(f_1, \ldots, f_{p'}, \varphi_1, \ldots, \varphi_{q'})$ and define a set $\mathcal{B}_A$ of $A$-points of the supervariety
$B$ as a preimage of zero by the map $F(A) : A^{p,q} \to A^{p',q'}$. It is obvious that every parity preserving homomorphism $\varphi : A \to \tilde{A}$ induces a map $B_A \to B_{\tilde{A}}$.

Notice that to every affine supervariety we can assign an affine variety (underlying variety) by dropping the equations $\varphi_i = 0$ and setting $\xi^i = 0$ in the equations $f_i = 0$, i.e. we neglect odd variables and equations.

One can easily move from the affine to the projective picture as in the usual commutative case. Recall that the functor of points corresponding to the projective space $\mathbb{P}^n$ can be defined by the formula $\mathbb{P}^n(A) = (A^{n+1})^\times / A^\times$, where $(A^{n+1})^\times = \{(x_0, \ldots, x_n) \in A^{n+1} | \sum Ax_i = A\}$, the group $A^\times$ of invertible elements of the ring $A$ acts by means of componentwise multiplication. We can define the super version of the projective space by setting $\mathbb{P}^{n,m}(A) = \{(x_0, \ldots, x_n, \xi_1, \ldots, \xi_m) \in (A^{n+1})^\times \times A^m_0 \} / A^\times_0$ where $A$ is a supercommutative ring. One easily checks that this results in a functor thereby defining the projective superspace.

As before we may get a more general object, namely a projective supervariety by considering the solutions of graded homogeneous equations in the variables $x^i$ and $\xi^i$, that is solutions of a system:

\[
\begin{align*}
&f_1(x^0, \ldots, x^n, \xi^1, \ldots, \xi^m) = 0 \\
&\ldots \\
&f_s(x^0, \ldots, x^n, \xi^1, \ldots, \xi^m) = 0 \\
&\varphi_1(x^0, \ldots, x^n, \xi^1, \ldots, \xi^m) = 0 \\
&\ldots \\
&\varphi_t(x^0, \ldots, x^n, \xi^1, \ldots, \xi^m) = 0
\end{align*}
\]

where $f_i \in O^{n+1,m}_x$ is an even homogeneous element and $\varphi_i \in O^{n+1,m}_x$ is an odd homogeneous element.

Again we can associate to the projective supervariety its underlying variety: a projective variety obtained by neglecting the odd variables and equations.\(^3\)

The above constructions produce affine and projective supervarieties as functors on the category of supercommutative rings, hence they can be considered as superspaces. A map of superspaces is a natural transformation of the functors. Similarly, a function on a superspace is a natural map to the affine superspace $\mathbb{A}^{1,1}$.\(^3\)

There are some important modifications to these constructions that will be useful to us. First of all instead of rings one can consider algebras over a field or more generally $R$-algebras, where $R$ is some fixed commutative ring. Modifying the source category for our functors in this way leads to the notion of a superspace over $R$. Furthermore, we can consider instead of the category of

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\(^3\)In the terminology of algebraic geometry a system of polynomial equations in affine or projective space specifies an affine or projective scheme. The term variety is reserved for schemes where local rings do not have nilpotent elements; we will assume that this condition is satisfied. Similarly one should talk about affine superschemes reserving the word supervariety for the case when underlying scheme is a variety.
all supercommutative \( R \)-algebras its various subcategories. It is worthwhile to note that the functions on a superspace depend on the particular subcategory that we choose. The most important example for us will be the subcategory of supercommutative \( R \)-algebras that consists of rings of the type \( B \otimes \Lambda[\xi^1, ..., \xi^q] \) where \( B \) is a commutative \( R \)-algebra. The concepts of affine and projective supervarieties modify readily to this more general setting.

The construction of \( \mathbb{A}^{p,q} \) can be generalized in the following way. Given an arbitrary \( \mathbb{Z}_2 \)-graded \( R \)-module \( E = E_0 \oplus E_1 \), we can define a superspace over \( R \), by setting \( E(A) = E_0 \otimes A_0 \oplus E_1 \otimes A_1 \). When \( R = \mathbb{Z} \) and \( E \) is a free abelian group we obtain \( \mathbb{A}^{p,q} \).

An important example of a superspace that does not belong to the types described above is defined by setting \( \tilde{spt}(A) = A_{\text{nilp}} \) where \( A_{\text{nilp}} \) is the subring of nilpotent elements. We will actually be more interested in \( \tilde{pt}(A) = A_{0,1}^{\text{nilp}} \).

This is an example of a more general and key construction to be discussed later.

Using the functorial approach to supergeometry it is easy to define the notion of supergroup, super Lie algebra and their actions. Namely one should replace the target category of sets in the definition of the superspace with the category of groups, Lie algebras, etc. \( \mathbb{A}^{0,1} \) for instance can be given the structure of a supergroup or that of a super Lie algebra. A differential on a \( \mathbb{Z}_2 \)-graded \( R \)-module generates an action of \( A_{0,1} \) on the corresponding superspace, conversely an action of \( A_{0,1} \) gives a differential. (Recall that a differential on a \( \mathbb{Z}_2 \)-graded \( R \)-module is a parity reversing \( R \)-linear operator \( d \), with \( d^2 = 0 \), satisfying the (graded) Leibnitz rule.)

### 3 Body of a superspace

Let us consider a subcategory \( C \) of the category of supercommutative \( R \)-algebras having the property that for every \( A \in C \) the quotient \( A/A_{\text{nilp}} \) of the ring \( A \) by the ideal of nilpotent elements is also in \( C \). The rings \( A/A_{\text{nilp}} \) where \( A \in C \) form a subcategory of the category of commutative \( R \)-algebras. Note that by assumption, \( C \) is a subcategory of \( C_{\text{red}} \) as well, denote it by \( C_{\text{red}} \).

Recall that a superspace is a functor on \( C \). Define the body of the superspace to be its restriction to \( C_{\text{red}} \). We will always assume that the body of a superspace is an algebraic variety. This means that there exists a variety over \( R \) such that the restriction of the corresponding functor on \( R \)-algebras to \( C_{\text{red}} \) coincides with the restriction of the superspace to \( C_{\text{red}} \). This variety is not unique, however in the situation we consider later, there exists such an ideal \( \mathfrak{m} \subset R \) that \( C_{\text{red}} \) can be identified with the subcategory of commutative \( R \)-algebras consisting of \( R/\mathfrak{m} \)-rings without nilpotent elements, thus the body is a unique \( R/\mathfrak{m} \)-variety.

If our superspace is an affine or projective supervariety then the notion of body agrees with the notion of underlying variety (if the body is considered as a variety over \( R \)).

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4A different (in general) superspace can be obtained by considering \( E'(A) = \text{Hom}_R(E, A) \) where we require the morphisms to be parity preserving. When \( E \) is free over \( R \), this is the same as the tensor construction.
If $X$ is a superspace we will denote its body considered as a variety by $X_{\text{body}}$. Every subvariety $Z \subset X_{\text{body}}$ determines a subsuperspace $X|_Z$ of $X$ that can be thought of as the largest subsuperspace of $X$ with body $Z$. More precisely, for every $A \in \mathcal{C}$ we have a map $\pi_A : X(A) \to X(A/A^{nilp})$ induced by the morphism $A \to A/A^{nilp}$. The subvariety $Z$ specifies a subset $Z(A/A^{nilp}) \subset X(A/A^{nilp})$, we define $X|_Z(A) = \pi_A^{-1}(Z(A/A^{nilp}))$.

Consider an inclusion of superspaces $X \subset Y$, that is a natural transformation $i$ such that for every $A$, $i_A : X(A) \hookrightarrow Y(A)$ identifies $X(A)$ with a subset of $Y(A)$. Note that the variety $X_{\text{body}}$ is a subvariety of $Y_{\text{body}}$. Define the infinitesimal neighborhood $\tilde{X}$ of $X$ in $Y$, by setting $\tilde{X} = Y|_{X_{\text{body}}}$. Clearly $X \subset \tilde{X}$. For example, if $X$ is the origin in $Y = \mathbb{A}^1$, the infinitesimal neighborhood of $X$ is the superspace $pt$ defined earlier.

4 De Rham cohomology of a superspace

Given a general superspace $X$, we associate to it a new superspace $\Pi TX$ called the odd tangent space, defined by $\Pi TX(A) = X(A \otimes \Lambda_2[\varepsilon])$. It should be thought of as the superspace parameterizing the maps from $TX$.

To make the definition more transparent, consider the following more familiar situation. Let us assume that $X$ is an affine algebraic supersuperspace defined by polynomial equations $f_i(x^i, \xi^i) = 0$ and $\varphi_j(x^i, \xi^i) = 0$ as before. Then $\Pi TX$ is given by polynomial equations obtained by setting $x^i = y^i + \epsilon \eta^i$ and $\xi^j = \zeta^j + \epsilon \zeta^j$ in $f_i = 0$ and $\varphi_j = 0$. (Here $y^i, z^j$ are even variables and $\eta^i, \zeta^j, \epsilon$ are odd.) $\Pi TX$ is then an affine supersuperspace with coordinates $y^i, z^j, \eta^i, \zeta^j$ constrained by the even equations $f_k(y^i, \zeta^j) = 0$ and $\sum \eta^i \partial_{y^i} \varphi_k(y^i, \zeta^j) + \sum z^j \partial_{\zeta^j} \varphi_k(y^i, \zeta^j) = 0$ as well as the odd equations $\varphi_k(y^i, \zeta^j) = 0$ and $\sum \eta^i \partial_{y^i} f_k(y^i, \zeta^j) + \sum z^j \partial_{\zeta^j} f_k(y^i, \zeta^j) = 0$.

The functions on $\Pi TX$ are (by definition) differential forms on $X$. The differential on the ring of functions on $\Pi TX$ (the ring $\Omega(X)$ of differential forms on $X$) can be defined by $dy^i = \eta^i$, $d\eta^i = 0$, $d\zeta^j = z^j$, and $dz^j = 0$.

The differential on $\Pi TX$ has geometric origin that permits one to define it for every superspace $X$. Namely $\mathbb{A}^{0,1}$ has a structure of a supergroup and therefore it acts on itself in an obvious way. Viewing $\Pi TX$ as the superspace parameterizing maps from $\mathbb{A}^{0,1}$ to $X$ we obtain an action of $\mathbb{A}^{0,1}$ on it. Similarly we have an action of $\mathbb{A}^{\times}$ on $\Pi TX$ derived from the action of $\mathbb{A}^{\times}$ on $\mathbb{A}^{0,1}$ via $\mathbb{A}^{\times}(A) \times \mathbb{A}^{0,1}(A) \to \mathbb{A}^{0,1}(A)$ which is just the multiplication map $A^{\times} \times A_1 \to A_1$. The actions of $\mathbb{A}^{0,1}$ and $\mathbb{A}^{\times}$ on $\Pi TX$ are compatible in the sense that the semidirect product $\mathbb{A}^{0,1} \rtimes \mathbb{A}^{\times}$ acts on $\Pi TX$. Furthermore this construction is functorial, i.e. a natural transformation $\psi : X \to Y$ induces a natural transformation between the odd tangent spaces $d\psi : \Pi TX \to \Pi TY$ that is compatible with the action of $\mathbb{A}^{0,1} \rtimes \mathbb{A}^{\times}$.

The action of $\mathbb{A}^{0,1}$ on $\Pi TX$ and thus on the differential forms on $X$ (= functions on $\Pi TX$) specifies a differential $d$ on $\Omega(X)$ that can be used to define the cohomology groups $\ker d/\img d$. These groups are in fact $R$-modules. The action of $\mathbb{A}^{\times}$ descends to the subquotient and gives a grading on the cohomology.
In the case when $X$ is a smooth affine algebraic variety (considered as a functor) and the source category is the full category of supercommutative $R$-algebras, the cohomology groups coincide with the familiar de Rham cohomology. They may or may not coincide with them if the source category is different.

To define the de Rham cohomology in a non-affine (but still smooth case) recall that we assumed that the body of the superspace $X$ is an algebraic variety. For every open subvariety $U \subset X_{\text{body}}$, consider the restriction $X|_U$ of $X$ to $U$. Define $\Omega^\bullet_{X/R}(U)$ as the $R$-algebra of differential forms on $X|_U$, i.e. functions on $\Pi T(X|_U)$. As observed above $\Omega^\bullet_{X/R}(U)$ is a differential $R$-algebra with the differential given by the action of $A^{0,1}$ and the grading given by the action of $A^\times$.

The collection of $\Omega^\bullet_{X/R}(U)$ for every $U$ specify a sheaf of differential $R$-modules on $X_{\text{body}}$. We define the de Rham cohomology of $X$ as hypercohomology of this sheaf. That is we consider the $R$-module of Čech cochains with values in $\Omega^\bullet_{X/R}$ and define the differential on these as the sum of the differential on $\Omega^\bullet_{X/R}(U)$ and the Čech differential. Corresponding cohomology groups are by definition the de Rham cohomology of $X$. If $X$ is a smooth projective variety this definition will produce cohomology groups isomorphic to the familiar de Rham cohomology groups for a choice of the source category that will interest us.

It is easy to see from the definitions that the De Rham cohomology is a contravariant functor from the category of superspaces to the category of graded $R$-algebras. Thus any endomorphism of $X$ induces an endomorphism on the cohomology.

## 5 Cohomology of an infinitesimal neighborhood.

As we mentioned already an infinitesimal neighborhood of a point $pt$ in the affine line $\mathbb{A}^1$ is the superspace $\overline{pt}$ given by a functor that assigns to a supercommutative ring $A$, the nilpotent part $A^{\text{nilp}}_0$ of $A_0$. If the source category is the category of all supercommutative $Q$-algebras, then the ring of functions on $\overline{pt}$ is easily seen to be the ring of formal power series over $Q$, namely $Q[[z]] = \{ \sum a_n z^n \}$. The ring of differential forms is then $Q[[z]][dz] = \{ \sum a_n z^n + \sum b_n z^n dz \}$. (Here $z$ is even and $dz$ odd as usual.) The differential transforms $\sum a_n z^n + \sum b_n z^n dz$ into $\sum n a_n z^{n-1} dz$. We see immediately that any element of $\ker d$ belongs to $\text{im} d$ except $a_0$, thus the map on cohomology induced by the embedding of $pt$ into $\overline{pt}$ is an isomorphism.

It was important to work with $Q$-algebras in the above considerations. If we instead consider the category of all supercommutative rings then we should replace $Q[[z]]$ and $Q[[z]][dz]$ with $\mathbb{Z}[[z]]$ and $\mathbb{Z}[[z]][dz]$ respectively. It is clear that $\overline{pt}$ now has non-trivial higher cohomology.\footnote{In the non-smooth case one follows the procedure below applied to $Y|_X$ instead of $X$ itself, where $Y$ is “smooth” and contains $X$. For this to work one has to choose an appropriate source category for the functors specifying the superspaces.}

\footnote{Naturally because we can no longer divide by $n$.}
However one can define a subcategory of the category of supercommutative rings in such a way that the cohomology of \( pt \) and \( pt \) are naturally isomorphic. Namely we should use the category \( \Lambda(p) \) consisting of objects \( \Lambda_B = B \otimes \Lambda_\mathbb{Z}[[\xi^1, \ldots, \xi^q]] \) where \( B \) is a commutative ring such that \( p^{N>0}B = 0 \) and \( B/pB \) has no nilpotent elements.\(^7\) Here \( p \) is a fixed prime (sometimes we need to assume that \( p > 2 \)). A superspace with the source category \( \Lambda(p) \) will be called a \( p \)-adic superspace. In what follows we work with \( p \)-adic superspaces.

The functions on \( pt \) can then be identified with formal series \( \sum a_n z^n/n! \) and the forms on \( pt \) with \( \sum a_n z^n/n! + \sum b_n z^n/n!dz \) where \( a_n, b_n \in \mathbb{Z}_p \).

It is not difficult to prove that every series of this kind specifies a function (or a form). We notice first that \( \Lambda_B^{ndp} = pB + \Lambda_B^p \), where we denote by \( \Lambda_B^p \) the ideal generated by \( \xi \)'s. Then the claim above follows from these observations:

A. For every nilpotent element \( \zeta \in \Lambda_\mathbb{Z} \) we can consider \( \zeta^n/n! \) as an element of \( \Lambda_\mathbb{Z} \), furthermore \( \zeta^n/n! = 0 \) for \( n \) large, thus \( \sum a_n z^n/n! \) can be evaluated at \( \zeta \).

B. The ring \( \Lambda_B \) can be considered as a \( \mathbb{Z}_p \)-algebra since multiplication by an infinite series \( \sum a_n p^n \) makes sense since we have \( p^{N>0}B = 0 \).

C. For every \( \eta \in pB \) the expression \( \eta^n/n! \) makes sense as an element of \( B \) since \( p^n/n! \) in the reduced form has no \( p \) factors in the denominator. Furthermore, if \( p > 2 \) then \( \sum a_n z^n/n! \) can be evaluated at \( p \) to obtain an element of \( \mathbb{Z}_p \) and so \( \sum a_n z^n/n! \) can be evaluated at \( \eta \).

The statement that these are all the functions on \( \tilde{pt} \) is more complicated. The proof is given in \([8]\), Sec. 4.1. From the viewpoint of a physicist this proof is irrelevant: we can restrict ourselves to the functions described above and not worry about other functions.

The differential on forms on \( \tilde{pt} \) is given by the formula \( d(\sum a_n z^n/n! + b_n z^n/n!dz) = \sum a_n z^{n-1}/(n-1)!dz \). It follows immediately from this formula that the cohomology of \( pt \) coincides with that of \( pt \).

Let us now consider the cohomology of the infinitesimal neighborhood \( \tilde{X} \) of a smooth variety \( X \) singled out by a single equation \( f(z) = 0 \). Locally the infinitesimal neighborhood is a direct product of \( X \) with \( pt \); hence its cohomology coincides with the cohomology of \( X \). More precisely we have a local expression for a differential form of degree \( s \) on \( \tilde{X} \): \( w = \sum a_n f^n/n! + \sum b_n f^n/n!df \), where \( a_n \) and \( b_n \) are differential forms on \( X \) of degree \( s \) and \( s-1 \) respectively. One can derive the fact we need from the formula

\[
h(w) = (-1)^{s-1} \sum_{n=0}^{\infty} b_n f^{n+1}/(n+1)!.
\]

This expression establishes a homotopy equivalence between sheaves of differential forms on \( X \) and \( \tilde{X} \).

The above statements are particular cases of a general theorem valid for a smooth subvariety \( X \) of a smooth variety \( Y \). (More generally, \( X \) and \( Y \) can be supervarieties.) That is: If \( X \) and \( Y \) are considered as superspaces

\(^7\) Examples of such rings \( B \) are \( \mathbb{Z}_p[x] \) and \( \mathbb{Z}_p[[x]] \)
over the category $\Lambda(p)$ or over the category of all supercommutative $\mathbb{Q}$-algebras then the cohomology of the infinitesimal neighborhood $\tilde{X}$ of $X$ in $Y$ is naturally isomorphic to the cohomology of $X$.\footnote{Recall that we defined cohomology as hypercohomology of the complex of sheaves. The embedding of $X$ into $Y$ induces a map of the corresponding complexes of sheaves that one can show, through local analysis, is a quasi-isomorphism by constructing an explicit homotopy. In the case when $X$ has codimension 1 this homotopy was constructed above.}

It follows from the above theorem that the cohomology of an infinitesimal neighborhood of a smooth $X$ does not depend on which particular embedding into a smooth $Y$ one chooses.\footnote{It seems one can prove that this remains true even in the case of a singular $X$. This prompts a definition of the cohomology of a singular $X$ as the cohomology of its infinitesimal neighborhood in some smooth $Y$. In the case of a $p$-adic superspace $X$ obtained from a variety over $\mathbb{F}_p$ we should obtain the crystalline cohomology of the variety as the cohomology of $X$. In the case of $\mathbb{Q}$-algebras over $\mathbb{Z}/p\mathbb{Z}$, one would obtain the crystalline cohomology of $V$ as the cohomology of $\Lambda(p)$.}

6 Frobenius map on the $p$-adic cohomology.

Let $V$ be a smooth projective variety over the ring $\mathbb{Z}_p$ of $p$-adic integers. There is a standard way to define its De Rham cohomology $H^\bullet(V; \mathbb{Z}_p)$ with coefficients in $\mathbb{Z}_p$. One can define the action of the Frobenius morphism on $H^\bullet(V; \mathbb{Z}_p)$. The usual way to obtain this action is based on crystalline cohomology. We will demonstrate a simpler construction in terms of supergeometry.

Let us consider $V$ as a functor on the category $\Lambda(p)$ and denote it by $X_V$.\footnote{V being a variety over $\mathbb{Z}_p$ defines a functor from commutative $\mathbb{Z}_p$-algebras to sets. For any $\Lambda_B \in \Lambda(p)$ define $X_V(\Lambda_B) = V(\Lambda^{even}_B)$. Since $\Lambda^{even}_B$ is a commutative $\mathbb{Z}_p$-algebra, this is well defined. We note that $V$ considered as a functor on $\Lambda(p)$ is not the same thing, for a general $V$, as $V$ considered as a variety, rather it is the $p$-adic completion. However it is known that the De Rham cohomology of $V$ is isomorphic to the De Rham cohomology of $X_V$ in the smooth projective case.}

We noticed previously that a ring belonging to $\Lambda(p)$ can be considered as a $\mathbb{Z}_p$-algebra, thus the de Rham cohomology of $X_V$ as defined previously yields a graded $\mathbb{Z}_p$-algebra. As mentioned previously, the de Rham cohomology of $X_V$ coincides with the de Rham cohomology of its infinitesimal neighborhood $\tilde{X}_V$ in the projective space $\mathbb{P}^n$ (considered as a functor on $\Lambda(p)$). Note that while $X_V$ is obtained from a variety $V$, there is no variety that yields $\tilde{X}_V$. The body of $\tilde{X}_V$ coincides with the body of $X_V$ and is a variety over the field $\mathbb{F}_p$. (The body is defined as a functor on the category $\Lambda(p)$ of rings of the form $B/pB$ without nilpotent elements. This is exactly the category of $\mathbb{F}_p$-algebras without nilpotent elements.)

The Frobenius map on the projective space over $\mathbb{Z}_p$ transforms a point with homogeneous coordinates $(x_0 : ... : x_n)$ into the point with homogeneous coordinates $(x_0^p : ... : x_n^p)$, this is clearly a natural transformation on the $p$-adic superspace $\mathbb{P}^n$. The subfunctor $X_V$ is not preserved by this map, but its infinitesimal neighborhood $\tilde{X}_V$ is invariant. This follows immediately from the definition of $\tilde{X}_V$ as the maximal subsuperspace with the same body as $X_V$ since the body of $X_V$ is invariant inside the body of $\mathbb{P}^n$ under the Frobenius map.\footnote{The invariance of the body of $X_V$ inside the body of $\mathbb{P}^n$ is most easily explained by...}
Now the action of the Frobenius on $\tilde{X}_V$ induces an action on the de Rham cohomology of $\tilde{X}_V$ which is the same as the de Rham cohomology of $X_V$ which in the smooth projective case is the same as the usual de Rham cohomology of $V$. We thereby obtain the celebrated lifting of Frobenius to characteristic 0.\(^{12}\)

7 Hodge filtration

The grading on the sheaf of differential forms on a variety $V$ determines a descending filtration $F^k$ (namely $F^k$ consists of forms of degree at least $k$). The filtration on forms specifies a filtration on the cohomology, called the Hodge filtration, denoted by $F^k$ as well. To analyze the behavior of the Frobenius map with respect to the Hodge filtration on $H^\bullet(V; \mathbb{Z}_p)$, where $V$ is a smooth projective variety, one should define a filtration $\tilde{F}^k$ on the sheaf of differential forms on the infinitesimal neighborhood $\tilde{V}$ of $V$ in such a way that the isomorphism between $H^\bullet(V; \mathbb{Z}_p)$ and $H^\bullet(\tilde{V}; \mathbb{Z}_p)$ identifies the two filtrations. (The exposition given here is inspired by \cite{3}.)

Let us assume for simplicity that $V$ can be singled out locally by one equation $f = 0$; as we have mentioned already, there is the following local expression for a differential form on $\tilde{V}$: $w = \sum a_n f^n/n! + \sum b_n f^n/n! df$, where $a_n$ and $b_n$ are differential forms on $V$. We say that $w$ above belongs to $\tilde{F}^k$ if $a_n \in F^{k-n}$ and $b_n \in F^{k-n-1}$. The filtration $\tilde{F}^k$ has the desired property.\(^{13}\)

Unfortunately the Frobenius does not preserve the Hodge filtration, however certain $p$-divisibility conditions are satisfied. These divisibility estimates were crucial in the proof of integrality of instanton numbers \cite{4,5}.

It is easy to check that if $a$ is a local function then $Fr(a) = a^p + pb$ where $b$ is some other local function, thus $Fr(f^k/k! da_1...da_n) = (f^p + pg)^k/k!d(a_1^p + pb_1)...d(a_n^p + pb_n)$ is divisible by $p^k/k! \cdot p^*$. Using this one can show that

$$Fr(F^k) \subset p^k H^\bullet(V; \mathbb{Z}_p), \text{ if } p > \dim V.$$  

This follows from the estimate $Fr(F^k) \subset p^{[k]} H^\bullet(V; \mathbb{Z}_p)$ where $[k]$ is maximal such that $p^{[k]}$ divides all numbers $p^n/n!$ with $n \geq k$. If $k < p$ then $p^{[k]} = p^k$ and in the case when $k > \dim V$, $F^k$ is trivial, thus one can assume that $k \leq \dim V$.

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pointing out that the Frobenius map on $\mathbb{P}^n$ as defined above has a more natural (equivalent) definition when restricted to the subcategory of $\Lambda(p)$ consisting of $F_p$-algebras (i.e. we consider only those $B$ with $pB = 0$). It can be constructed by observing that since the $p^th$ power map is a homomorphism of $F_p$-algebras (as $a^p = a$ for $a \in F_p$) it induces an endomorphism of any functor on the subcategory of $F_p$-algebras. Because the body is defined as the restriction of the functor to the subcategory $\Lambda(p)_{red}$ which contains only $F_p$-algebras, this constructs an action of Frobenius on the body of any $p$-adic superspace that evidently preserves any subbody.

\(^{12}\)As a bonus we see that the cohomology of $V$ depends only on its restriction to $F_p$.

\(^{13}\)This is due to the fact that this filtration is preserved by the homotopy that establishes the isomorphism of cohomologies.
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