Series expansion of the Gamma function 
and its reciprocal

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Abstract: In this paper we give representations for the coefficients of the Maclaurin series for 
\( \Gamma(z + 1) \) and its reciprocal (where \( \Gamma \) is Euler’s Gamma function) with the help of a differential 
operator \( D \), the exponential function and a linear functional * (in Theorem 3.1). As a result we 
obtain the following representations for \( \Gamma \) (in Theorem 3.2):

\[
\Gamma(z + 1) = (e^{-u(x)}e^{-zD[e^{u(x)}]})^*,
\]
\[
(\Gamma(z + 1))^{-1} = (e^{u(x)}e^{-zD[e^{-u(x)}]})^*.
\]

Theorem 3.1 and Theorem 3.2 are our main results. With the help of the first theorem we give 
our approach for finding the coefficients of Maclaurin series for \( \Gamma(z + 1) \) and its reciprocal in an 
explicit form.

Keywords: Gamma function, Zeta function, Euler’s constant, Maclaurin series.

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1 Introduction

Let \( \mathbb{N} \) be the set of all positive integers, \( \mathbb{C} \) - the complex number field, \( \mathbb{E} \) - the set of all entire 
functions of one complex variable. For \( F \in \mathbb{E} \) the operator \( D^k : \mathbb{E} \to \mathbb{E} \) is defined by:
\[
D^0[F(z)] := F^{(0)}(z) = F(z), \quad z \in \mathbb{C}; \quad D^k[F(z)] := F^{(k)}(z), \quad k \in \mathbb{N}, \quad z \in \mathbb{C}, \text{ i.e., } D = \frac{d}{dz} \text{ and } D^k = \left( \frac{d}{dz} \right)^k.
\]
If \( D^k[F(z)] = H_k(z) \), we define \( D^k_s[F(z)] := H_k(s) \).
Further, we shall use the following notation: \( \zeta \) for Riemann zeta function; \( \gamma \) for Euler’s constant, i.e. \( \gamma = \lim_{n \to \infty} \left( \sum_{i=1}^{n} \frac{1}{i} - \ln n \right) \approx 0.5772156649 \ldots \); \( \zeta(1) = \gamma \) and \( \zeta(k) = \zeta(k) \) for each integer \( k > 1 \).

The Gamma function admits the following basic representations (see [7], pp. 31, 33, 34):

(i) \( \Gamma(z) = \int_{0}^{\infty} e^{-t} t^{z-1} dt \), valid for \( z \in \mathbb{C}, \text{Re} z > 0 \) (Euler)

where \( \Gamma \) is a holomorphic function and \( D^k[\Gamma(z)] = \int_{0}^{\infty} e^{-t} t^{z-1} (\ln t)^k dt \);

(ii) \( \Gamma(z) = \lim_{m \to \infty} \frac{m!m^z}{z(z+1)\ldots(z+m)} \), valid for \( z \in \mathbb{C}, z \neq 0, -1, -2, \ldots \) (Euler–Gauss)

(iii) \( \Gamma(z) = \frac{1}{z} e^{-\gamma z} \prod_{n=1}^{\infty} e^{\frac{\pi}{n}(1 + \frac{z}{n})} \), valid for \( z \in \mathbb{C}, z \neq 0, -1, -2, \ldots \) (Weierstrass)

From (iii) it is seen that \( (\Gamma(z))^{-1} \in \mathbb{E} \) and that \( \Gamma(z) \) is a meromorphic function without zeroes and with simple poles: \( z = 0, -1, -2, \ldots \)

We have \( \Gamma(1) = 1 \) and for any \( z \in \mathbb{C} \) \( \Gamma \) satisfies the functional equation

\[
\Gamma(z+1) = z\Gamma(z). \tag{1}
\]

Hence \( \Gamma(z+1)^{-1} \in \mathbb{E} \) and \( \Gamma(z+1) \) is a meromorphic function without zeroes and with simple poles: \( z = -1, -2, \ldots \)

Also, from (i), the representations:

\[
A(k) = \Gamma^{(k)}(1) = D_0^k[\Gamma(z+1)] = \int_{0}^{\infty} e^{-t} (\ln t)^k dt, \quad k = 0, 1, 2, \ldots, \tag{2}
\]

hold.

Although the Gamma function was introduced by Euler about two hundred and ninety two years ago it still has its secrets. In the present paper we introduce a linear operator \( \mathcal{D} : \mathbb{E} \to \mathbb{E} \) on which our main results are based – Theorem 3.1 and Theorem 3.2. With the help of Theorem 3.1 we find explicit formulae for the coefficients of Maclaurin series of \( \Gamma(z+1) \) and \( (\Gamma(z+1))^{-1} \) (Theorem 3.3.). Here we must note that such type of formulae are given by other authors too. For example formulae for these coefficients are contained in: [3, 4, 6, 8].

2 A new operator \( \mathcal{D} \) and its basic properties

Definition 2.1. Let \( u(z) \in \mathbb{E} \). The operator \( \mathcal{D} \) is defined by:

- \( \mathcal{D} = \mathcal{D}^1; \quad \mathcal{D}^0[u(z)] = u(z); \)
- \( \mathcal{D}[D^0[u(z)]] = D^1[u(z)]; \)
- \( \mathcal{D}[D^k[u(z)]] = kD^{k+1}[u(z)], \quad k \geq 1; \)
- \( \mathcal{D}^k[u(z)] = \mathcal{D}[\mathcal{D}^{k-1}[u(z)]], \quad k \geq 1. \)

Lemma 2.1. Let \( u(z), F(z), G(z) \in \mathbb{E} \). Then

\( (j_1) \quad \mathcal{D}^k[u(z)] = (k-1)!D^k[u(z)] \quad (\forall k \in \mathbb{N}) \)

\( (j_2) \quad \mathcal{D}^k \) is a linear operator \( (\forall k \in \mathbb{N}) \)

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Proof. (j1) follows by induction from the definition of $\mathcal{D}$.

Let $\lambda, \mu \in \mathbb{C}$. From (j1) and from the linearity of $D^k$ we obtain:

$$D^k[\lambda F(z) + \mu G(z)] = (k-1)!D^k[\lambda F(z) + \mu G(z)] = \lambda (k-1)!D^k[F(z)] + \mu (k-1)!D^k[G(z)],$$

which proves (j2).

Let us prove (j3) by induction. In the case $k = 1$, $\mathcal{D}$ coincides with $D$ and (j3) is obvious. If for $k \geq 1$ (j3) is true, then applying $\mathcal{D}$ to (j3) and using the linearity of $\mathcal{D}$ we obtain:

$$(j_3) \quad \mathcal{D}^k [F(z)G(z)] = \sum_{\nu=0}^{k} \binom{k}{\nu} D^{k-\nu}[F(z)] \mathcal{D}^\nu[G(z)] \quad (\forall k \in \mathbb{N})$$

(analogue of Leibniz formula for $D^k[F(z)G(z)]$)

$$(j_4) \quad \mathcal{D}[F(G(z))] = (DF)(G(z))\mathcal{D}[G(z)]$$

$$(j_5) \quad \text{For } f, g \in \mathbb{E} \text{ and } n \in \mathbb{N} \text{ the following analogue of Faa di Bruno’s formula for } \mathcal{D}^n[f(g(z))]:$$

$$\mathcal{D}^n[f(g(z))] = \sum_{m=0}^{n} (D^n f)(g(z)) \sum_{\alpha} C_n(\alpha)(\mathcal{D}^1[g(z)])^{\alpha_1} \cdots (\mathcal{D}^n[g(z)])^{\alpha_n} \quad (3)$$

is true, where: $C_n(\alpha) = \frac{1}{(\alpha_1)!\cdots(\alpha_n)!}$ and $\sum_{\alpha}$ means that the sum is over all $\alpha = (\alpha_1, \ldots, \alpha_n)$, for which $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{N} \cup \{0\}$, $\alpha_1 + \cdots + \alpha_n = m$ and $1.\alpha_1 + \cdots + n.\alpha_n = n$.

Let us prove (j5). For $f, g \in \mathbb{E}$ and $n \in \mathbb{N}$ the following analogue of Faa di Bruno’s formula for $\mathcal{D}^n[f(g(z))]$:

$$(j_5) \quad \mathcal{D}^n[f(g(z))] = \sum_{m=0}^{n} (D^n f)(g(z)) \sum_{\alpha} C_n(\alpha)(\mathcal{D}^1[g(z)])^{\alpha_1} \cdots (\mathcal{D}^n[g(z)])^{\alpha_n} \quad (3)$$

is true, where: $C_n(\alpha) = \frac{1}{(\alpha_1)!\cdots(\alpha_n)!}$ and $\sum_{\alpha}$ means that the sum is over all $\alpha = (\alpha_1, \ldots, \alpha_n)$, for which $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{N} \cup \{0\}$, $\alpha_1 + \cdots + \alpha_n = m$ and $1.\alpha_1 + \cdots + n.\alpha_n = n$.

Proof. (j1) follows by induction from the definition of $\mathcal{D}$.

Let $\lambda, \mu \in \mathbb{C}$. From (j1) and from the linearity of $D^k$ we obtain:

$$D^k[\lambda F(z) + \mu G(z)] = (k-1)!D^k[\lambda F(z) + \mu G(z)] = \lambda (k-1)!D^k[F(z)] + \mu (k-1)!D^k[G(z)],$$

which proves (j2).

Let us prove (j3) by induction. In the case $k = 1$, $\mathcal{D}$ coincides with $D$ and (j3) is obvious. If for $k \geq 1$ (j3) is true, then applying $\mathcal{D}$ to (j3) and using the linearity of $\mathcal{D}$ we obtain:

$$(j_3) \quad \mathcal{D}^k [F(z)G(z)] = \sum_{\nu=0}^{k} \binom{k}{\nu} D^{k-\nu}[F(z)] \mathcal{D}^\nu[G(z)]$$

The right-hand side $R$ in the above equality is $I_1 + I_2$, where:

$$I_1 = \mathcal{D}^{k+1}[F(z)]\mathcal{D}^0[G(z)] + \sum_{\nu=1}^{k} \binom{k}{\nu} \mathcal{D}^{k+1-\nu}[F(z)]\mathcal{D}^\nu[G(z)];$$

$$I_2 = \sum_{\nu=1}^{k} \binom{k}{\nu-1} \mathcal{D}^{k+1-\nu}[F(z)]\mathcal{D}^\nu[G(z)] + \mathcal{D}^0[F(z)]\mathcal{D}^{k+1}[G(z)].$$

$I_2$ is obtained after substitution $\nu + 1 = t$ and replacing $t$ with $\nu$.

Now using that $\binom{k}{\nu} + \binom{k}{\nu-1} = \binom{k+1}{\nu}, \nu = 1, \ldots, k$, we obtain

$$R = \sum_{\nu=0}^{k+1} \binom{k+1}{\nu} \mathcal{D}^{k+1-\nu}[F(z)]\mathcal{D}^\nu[G(z)]$$

and (j3) is proved.

The proof of (j4) is obvious since $D[F(G(z))] = (DF)(G(z))D[G(z)]$ and we may replace $D[G(z)]$ with $\mathcal{D}[G(z)]$ (see Definition 2.1).
Let us prove \((j_3)\). We consider the equalities:

\[ D[f(g(z))] = (D f)(g(z)) D[g(z)], \]
\[ \mathcal{D}[f(g(z))] = (D f)(g(z)) \mathcal{D}[g(z)] \]

(see \((j_4)\)).

Applying to the left one \(D^{n-1}\) and to the right one \(\mathcal{D}^{n-1}\) and after that using Leibnitz formula for the right-hand side of the first equality and the analogue of the Leibnitz formula (see \((j)\)) for the right-hand side of the second equality, one may see that \(D^n[f(g(z))]\) and \(\mathcal{D}^n[f(g(z))]\) have the same structure with only one difference: \(D^n[g(z)]\) for the first expression is replaced with \(\mathcal{D}^m[g(z)]\) for the second expression. But since we have the Faa di Bruno’s formula (see [1], p.823):

\[ D^n[f(g(z))] = \sum_{m=0}^{n} (D^m f)(g(z)) \sum_{\alpha} C_n(\alpha) (D^1[g(z)])^{\alpha_1} \cdots (D^n[g(z)])^{\alpha_n}, \]

then replacing \(D^m(g(z))\) with \(\mathcal{D}^m(g(z))\) in it, we obtain exactly (3). \(\square\)

**Corollary 2.1.** Formula (3) admits the representation

\[ \mathcal{D}^n[f(g(z))] = \sum_{m=0}^{n} (D^m f)(g(z)) \sum_{\alpha} C_n(\alpha) \prod_{\nu=1}^{n} ((\nu - 1)!)^{\alpha_\nu} (D^1[g(z)])^{\alpha_1} \cdots (D^n[g(z)])^{\alpha_n} \quad (4) \]

**Proof.** It follows immediately from Lemma 2.1, \((j_1)\). \(\square\)

**Corollary 2.2.** Formula (4) admits the representation

\[ \mathcal{D}^n[f(g(z))] = \sum_{m=0}^{n} (D^m f)(g(z)) \sum_{\alpha} A_n(\alpha) (D^1[g(z)])^{\alpha_1} \cdots (D^n[g(z)])^{\alpha_n}, \quad (5) \]

where

\[ A_n(\alpha) = \frac{n!}{\alpha_1! \alpha_2! \cdots \alpha_n! (\alpha_1 \alpha_2 \cdots \alpha_n)} \quad (6) \]

**Proof.** One may check directly that

\[ C_n(\alpha) \prod_{\nu=1}^{n} ((\nu - 1)!)^{\alpha_\nu} = A_n(\alpha). \] \(\square\)

**Corollary 2.3.** Let \(u \in E\) and \(n \in \mathbb{N}\). Then:

\[ \mathcal{D}^n[e^{u(z)}] = \sum_{m=0}^{n} \sum_{\alpha} A_n(\alpha) (D^1[u(z)])^{\alpha_1} \cdots (D^n[u(z)])^{\alpha_n} \quad (7) \]

\[ \mathcal{D}^n[e^{-u(z)}] = \sum_{m=0}^{n} \sum_{\alpha} (-1)^{\sum_{\nu=1}^{n} \alpha_\nu} A_n(\alpha) (D^1[u(z)])^{\alpha_1} \cdots (D^n[u(z)])^{\alpha_n} \]
3 Maclaurin series for $\Gamma(z + 1)$ and its reciprocal

If $k \geq 0$ we define $a_k$, $d_k$, $b_k$, $\rho_k$ by:

$$k!a_k = D_0^k[\Gamma(z + 1)]$$  \hspace{1cm} (8)

$$k!d_k = D_0^k[(\Gamma(z + 1))^{-1}]$$  \hspace{1cm} (9)

$$b_k = (-1)^kk!a_k$$  \hspace{1cm} (10)

$$\rho_k = k!d_k$$  \hspace{1cm} (11)

Hence:

$$\Gamma(z + 1) = \sum_{k=0}^{\infty} a_k z^k, \quad |z| < 1,$$  \hspace{1cm} (12)

$$(\Gamma(z + 1))^{-1} = \sum_{k=0}^{\infty} d_k z^k, \quad |z| < \infty.$$  \hspace{1cm} (13)

For the sequences $\{a_k\}$ and $\{d_k\}$ recurrence relations are known (see [5], pp. 12, 17; [2], p. 12) that we give below but for the sequences $\{b_k\}$ and $\{\rho_k\}$:

$$b_{k+1} = \sum_{\nu=0}^{k} \binom{k}{\nu} (k - \nu)!\tilde{\zeta}(k - \nu + 1) b_{\nu}, \quad k \geq 0, \quad b_0 = 1,$$  \hspace{1cm} (14)

$$\rho_{k+1} = \sum_{\nu=0}^{k} \binom{k}{\nu} (k - \nu)!(-1)^{k-\nu}\tilde{\zeta}(k - \nu + 1) \rho_{\nu}, \quad k \geq 0, \quad \rho_0 = 1.$$  \hspace{1cm} (15)

3.1 Connection between the Gamma function and the exponential function

**Definition 3.1.** Let $F(z_1, z_2, \ldots, z_k)$ be a polynomial und $u(z) \in \mathbb{E}$. Then we define the mapping $^*$ by:

$$(F(D^1[u(z)], D^2[u(z)], \ldots, D^k[u(z)]))^* := F(\tilde{\zeta}(1), \tilde{\zeta}(2), \ldots, \tilde{\zeta}(k)).$$

**Remark.** From the above definition it is clear that $^*$ is a linear and multiplicative mapping. The multiplicativity of $^*$ means that if $F(z_1, \ldots, z_k)$ and $G(z_1, \ldots, z_m)$ are polynomials and

$$H(z_1, \ldots, z_n) = F(z_1, \ldots, z_k)G(z_1, \ldots, z_m),$$

then

$$(H(D^1[u(z)], \ldots, D^n[u(z)]))^* = (F(D^1[u(z)], \ldots, D^k[u(z)]))^* (G(D^1[u(z)], \ldots, D^m[u(z)]))^*.$$

Our first main result in this paper is the following.

**Theorem 3.1.** $\forall k \in \mathbb{N} \cup \{0\}$ $b_k$ and $\rho_k$ are given by the formulae:

$$b_k = (e^{-u(z)} D^k[e^{u(z)}])^*,$$  \hspace{1cm} (16)

$$\rho_k = ((-1)^k e^{u(z)} D^k[e^{-u(z)}])^*.$$  \hspace{1cm} (17)
Proof. We shall prove only (16) since one may prove (17) in the same way. We prove (16) by induction. For $k = 0$ we have $b_0 = 1$ and $(e^{-u(z)}\mathcal{D}^k[e^u(z)])^* = (e^{-u(z)}\mathcal{D}^0[e^u(z)])^* = (\{1\})^* = 1$. Assume that (16) is true for some $k \geq 0$. We will show that (16) is true for $k + 1$ too. Let

$$b'_{k+1} = (e^{-u(z)}\mathcal{D}^{k+1}[e^u(z)])^*.$$ 

Then

$$b'_{k+1} = (e^{-u(z)}\mathcal{D}^k\mathcal{D}[e^u(z)])^* = (e^{-u(z)}\mathcal{D}^k[D[e^u(z)]])^*$$

$$= (e^{-u(z)}\mathcal{D}^k[e^u(z)D[u(z)]])^* = (e^{-u(z)}\mathcal{D}^k[e^u(z)\mathcal{D}[u(z)]])^*$$

[now we apply Lemma 2.1, (j3) to $\mathcal{D}^k[e^u(z)\mathcal{D}[u(z)]]$]

$$= (e^{-u(z)}\sum_{\nu=0}^{k} \binom{k}{\nu} \mathcal{D}^{k+1-\nu}[u(z)]\mathcal{D}^\nu[e^u(z)])^*$$

$$= (\sum_{\nu=0}^{k} \binom{k}{\nu} \mathcal{D}^{k+1-\nu}[u(z)](e^{-u(z)}\mathcal{D}^\nu[e^u(z)]))^*$$

[now we use $\mathcal{D}^{k+1-\nu}[u(z)] = (k - \nu)!\mathcal{D}^{k+1-\nu}[u(z)]$, see Lemma 2.1, (j1)]

$$= \left(\sum_{\nu=0}^{k} \binom{k}{\nu} (k - \nu)!\mathcal{D}^{k+1-\nu}[u(z)](e^{-u(z)}\mathcal{D}^\nu[e^u(z)])\right)^*$$

[from the linearity and multiplicativity of $^*$]

$$= \sum_{\nu=0}^{k} \binom{k}{\nu} (k - \nu)!\mathcal{D}^{k+1-\nu}[u(z)](e^{-u(z)}\mathcal{D}^\nu[e^u(z)])^*$$

[from the induction hypothesis we have $(e^{-u(z)}\mathcal{D}^\nu[e^u(z)])^* = b_\nu$, $0 \leq \nu \leq k$]

$$= \sum_{\nu=0}^{k} \binom{k}{\nu} (k - \nu)!\zeta(k - \nu + 1)b_\nu = b_{k+1}$$

(the last from (14)). Thus we proved $b'_{k+1} = b_{k+1}$ and (16) is proved. 

From Theorem 3.1, using the fact that the mapping $^*$ is linear and that the generating function of the operator $\mathcal{D}$:

$$\sum_{k=0}^{\infty} \frac{(-z)^k}{k!} \mathcal{D}^k$$

is equal to $e^{-z\mathcal{D}}$, we obtain our second main result in this paper.

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Theorem 3.2. \( \Gamma(z + 1) \) and its reciprocal have the following important representations:

\[
\Gamma(z + 1) = (e^{-u(x)}e^{-zD[e^u(x)]})^*,
\]
\[
(\Gamma(z + 1))^{-1} = (e^{u(x)}e^{-zD[e^{-u(x)}]})^*,
\]

where \(*\) acts only with respect to the variable \( x \).

Proof. We prove only (18) since one may prove (19) analogically.

\[
(e^{-u(x)}e^{-zD[e^u(x)]})^* = e^{-u(x)} \left( \sum_{k=0}^{\infty} \frac{(-z)^k}{k!} D^k [e^u(x)] \right)^* = e^{-u(x)} \sum_{k=0}^{\infty} \frac{(-z)^k}{k!} D^k [e^u(x)]^* = \sum_{k=0}^{\infty} \frac{(-z)^k}{k!} e^{-u(x)} D^k [e^u(x)]^*
\]

[here we suppose that \(*\) is defined not only for polynomials but for power series, too]

\[
= \sum_{k=0}^{\infty} \frac{(-z)^k}{k!} \left( e^{-u(x)} D^k [e^u(x)] \right)^* = \sum_{k=0}^{\infty} \frac{(-z)^k}{k!} b_k e^{-u(x)} D^k [e^u(x)] = \sum_{k=0}^{\infty} a_k z^k = \Gamma(z + 1). \square
\]

3.2 Explicit formulae for the coefficients

From Theorem 3.1 and Corollary 2.3 we obtain using (16) and (17):

\[
b_n = \sum_{m=0}^{n} \sum_{\alpha} A_n(\alpha) (\tilde{\zeta}(1))^\alpha_1 \cdots (\tilde{\zeta}(n))^\alpha_n \quad \text{(compare with [9], the formula for } \Gamma(n)(1))
\]

\[
\rho_n = (-1)^n \sum_{m=0}^{n} \sum_{\alpha} (-1)^{\sum_{\nu=1}^{m} \alpha_\nu} A_n(\alpha) (\tilde{\zeta}(1))^\alpha_1 \cdots (\tilde{\zeta}(n))^\alpha_n
\]

In [6], p. 43, (10) and (11), explicit formulae for \( a_n \) and \( d_n \) are given.

Further we shall find formulae for \( b_n \) and \( \rho_n \) in a polynomial form, of one variable, using instead of this variable Euler’s constant \( \gamma \). For this purpose we need two lemmas.

Lemma 3.1. For \( k \geq 2 \) the following identities are valid

\[
e^{-u(z)} D^k [e^u(z)] = (D^1[u(z)])^k + \sum_{p=2}^{k} \binom{k}{p} \Delta_p.(D^1[u(z)])^{k-p}, \quad (20)
\]

\[
(-1)^k e^{u(z)} D^k [e^{-u(z)}] = (D^1[u(z)])^k + \sum_{p=2}^{k} \binom{k}{p} \sigma_p.(D^1[u(z)])^{k-p}, \quad (21)
\]
where:

\[ \Delta_2 = D^2[u(z)], \quad \Delta_3 = D[D^2[u(z)]], \quad \Delta_{p+1} = p.(D^2[u(z)]).\Delta_{p-1} + D[\Delta_p], \quad p > 3, \quad (22) \]

\[ \sigma_2 = -D^2[u(z)], \quad \sigma_3 = D[D^2[u(z)]], \quad \sigma_{p+1} = -p.(D^2[u(z)]).\sigma_{p-1} - D[\sigma_p], \quad p > 3. \quad (23) \]

**Proof.** We prove (20) by induction. For \( k = 2 \) and \( k = 3 \) we check directly the validity of (20). Let \( R_k \) the right-hand side of (20) and let (20) be true for some \( k \geq 3 \). For \( k + 1 \) we must verify that \( e^{-u}D^{k+1}[e^u] = R_{k+1}, \) i.e., \( R_{k+1} = e^{-u}D[D^k[e^u]] = e^{-u}D[e^u e^{-u} D^k[e^u]] = (\text{from our assumption for } k) = e^{-u}D[e^u R_k] = (\text{after computation}) = D[u]R_k + D[R_k]. \) Thus, to prove (20), it remains to prove that \( R_{k+1} = D[u]R_k + D[R_k] \).

It is easy to see that the coefficient in front of \( (D^1[u(z)])^{k+1-p} \) in the left-hand side of the last equality equals to \( \binom{k+1}{p} \Delta_p \) and in the right-hand side equals to \( (k+2-p)\binom{k}{p-2} \Delta_{p-2}.D^2[u(z)] + \binom{k}{p} \Delta_p + \binom{k}{p-1} D[\Delta_{p-1}] \). So, to prove (20), we must check that

\[ \binom{k+1}{p} \Delta_p = (k+2-p)\binom{k}{p-2} \Delta_{p-2}.D^2[u(z)] + \binom{k}{p} \Delta_p + \binom{k}{p-1} D[\Delta_{p-1}]. \]

One may easily check the above equality using (22) and the well-known relations:

\[ \binom{k+1}{p} = \binom{k}{p} + \binom{k}{p-1}; \quad t\binom{k}{t} = k\binom{k-1}{t-1}.\]

In the same way, one may prove (21) (using (23)) and Lemma 3.1 is proved. \( \square \)

**Lemma 3.2.** For \( p \geq 2 \) the following representations hold:

\[ \Delta_p = \sum_{\alpha} A_p(\tilde{\alpha})(D^2[u(z)])^{\alpha_2} \cdots (D^p[u(z)])^{\alpha_p}, \quad (24) \]

\[ \sigma_p = \sum_{\tilde{\alpha}} (-1)^{p+k} \sigma^p_{\alpha} A_p(\tilde{\alpha})(D^2[u(z)])^{\alpha_2} \cdots (D^p[u(z)])^{\alpha_p}, \quad (25) \]

where, for \( \tilde{\alpha} := (\alpha_2, \ldots, \alpha_p) \), \( A_p(\tilde{\alpha}) \) is given by:

\[ A_p(\tilde{\alpha}) = p! \prod_{\nu=2}^p (\nu - 1)!^{\alpha_\nu} \prod_{\nu=2}^p (\alpha_\nu, (\nu!)^{\alpha_\nu})^{-1} = \frac{p!}{(2^{\alpha_2} 3^{\alpha_3} \cdots p^{\alpha_p})(\alpha_2, \alpha_3, \ldots, \alpha_p)!}, \]

and \( \sum_{\tilde{\alpha}} \) means that the sum is over all nonnegative integers \( \alpha_\nu \) such that \( \sum_{\nu=2}^p \nu \alpha_\nu = p. \)

**Proof.** We shall prove only (24) since (25) may be proved in the same way. First in (22) we write \( \Delta_t^D \) instead of \( \Delta_t, \forall t \geq 2. \) Second we replace in (22) \( D \) with \( D. \) As a result we obtain:

\[ \Delta_2^D = D^2[u(z)], \quad \Delta_3^D = D[D^2[u(z)]], \quad \Delta_{p+1}^D = p.(D^2[u(z)]).\Delta_{p-1}^D + D[\Delta_p^D], \quad p > 3. \]

Now we shall prove that

\[ \Delta_p^D = \sum_{\tilde{\alpha}} \frac{p!}{(2!)^{\alpha_2} \alpha_2! \cdots (p!)^{\alpha_p} \alpha_p!} (D^2[u(z)])^{\alpha_2} \cdots (D^p[u(z)])^{\alpha_p} \]

using induction with respect to \( p. \) For \( p = 2, 3 \) the validity of the above equality is a matter of direct check. Let \( \tilde{\alpha} = \tilde{\alpha}(p), \quad \tilde{C}_p(\tilde{\alpha}(p)) = \frac{p!}{(2!)^{\alpha_2} \alpha_2! \cdots (p!)^{\alpha_p} \alpha_p!} \) and the equality is checked for \( \nu = 2, 3, \ldots, p. \) For the sake of brevity, let \( \tilde{C}_p := \tilde{C}_p(\tilde{\alpha}(p)). \)
We must prove that
\[ \sum_{\tilde{\alpha}(p+1)} \tilde{C}_{p+1} \prod_{\nu=2}^{p+1} (D^\nu [u(z)])^{\alpha_{\nu}} = \Delta_{p+1}^D = p (D^2 [u(z)]) \Delta_{p-1}^D + D[\Delta_p^D], \]
i.e. that the equality (further denoted by \( H \)):
\[ \sum_{\tilde{\alpha}(p+1)} \tilde{C}_{p+1} \prod_{\nu=2}^{p+1} (D^\nu [u(z)])^{\alpha_{\nu}} = \sum_{\tilde{\alpha}(p-1)} p \tilde{C}_{p-1} (D^2 [u(z)])^{\alpha_2+1} \prod_{\nu=3}^{p-1} (D^\nu [u(z)])^{\alpha_{\nu}} + \sum_{j=2}^{p-1} \sum_{\tilde{\alpha}(p)} \alpha_j \tilde{C}_p P (D^j [u(z)])^{\alpha_j+1} (D^{j+1} [u(z)])^{\alpha_{j+1}+1} Q + D^{p+1} [u(z)], \]
holds, where \( P \) and \( Q \) are given by:
\[ P = \prod_{\nu=2}^{j-1} (D^\nu [u(z)])^{\alpha_{\nu}}; \quad Q = \prod_{\nu=j+2}^{p-1} (D^\nu [u(z)])^{\alpha_{\nu}}. \]
Let \( \prod_{\nu=2}^{p-1} (D^\nu [u(z)])^{\gamma_{\nu}} \) be an arbitrary monomial from the right-hand side of \( H \). Below we calculate the coefficient \( C \) in front of this monomial:
\[ C = p \tilde{C}_{p-1} (\gamma_2 - 1, \gamma_3, \ldots, \gamma_{p+1}) + \sum_{j=2}^{p-1} (\gamma_j + 1) \tilde{C}_p (\gamma_2, \ldots, \gamma_j + 1, \gamma_{j+1} - 1, \ldots, \gamma_{p+1}) \]
\[ = p! \frac{\sum_{\nu=2}^{p+1} \nu \gamma_{\nu} \gamma_{\nu}}{\prod_{\nu=2}^{p+1} (\nu!)^{\gamma_{\nu}}!} = \frac{(p+1)!}{\prod_{\nu=2}^{p+1} (\nu!)^{\gamma_{\nu}}!} = \tilde{C}_{p+1} (\gamma_2, \ldots, \gamma_{p+1}). \]
Hence, the right-hand side of \( H \) is a sum of the terms \( \tilde{C}_{p+1} (\gamma_2, \ldots, \gamma_{p+1}) \prod_{\nu=2}^{p+1} (D^\nu [u(z)])^{\gamma_{\nu}} \), for which \( \sum_{\nu=2}^{p+1} \nu \gamma_{\nu} = p + 1 \). Then, to prove \( H \), it remains to show that in the right-hand side of \( H \) all partitions \( \sum_{\nu=2}^{p+1} \nu \gamma_{\nu} = p + 1 \) are met.
Indeed, let \( \prod_{\nu=2}^{p+1} (D^\nu [u(z)])^{\gamma_{\nu}} \) be an arbitrary monomial from the left-hand side of \( H \). If \( \gamma_{p+1} = 1 \), then \( \gamma_2 = \gamma_3 = \cdots = \gamma_p = 0 \), so this monomial is \( D^{p+1} [u(z)] \) and it is contained in the right-hand side of \( H \), too. If \( \gamma_{p+1} = 0 \) but for some \( j \), such that \( 2 < j \leq p \), \( \gamma_j \neq 0 \), we consider in the right-hand side of \( H \) the sum:
\[ \sum_{\tilde{\alpha}(p)} \alpha_{j-1} \tilde{C}_p (\alpha_2, \ldots, \alpha_p) \left( \prod_{\nu=2}^{j-2} (D^\nu [u(z)])^{\alpha_{\nu}} \right) (D^{j-1} [u(z)])^{\alpha_{j-1}+1} (D^j [u(z)])^{\alpha_j+1} \prod_{\nu=j+1}^{p} (D^\nu [u(z)])^{\alpha_{\nu}}. \]
Now we set: \( \alpha_2 = \gamma_2, \ldots, \alpha_{j-2} = \gamma_{j-2}; \alpha_{j-1} = \gamma_{j-1} + 1; \alpha_j = \gamma_j - 1; \alpha_{j+1} = \gamma_{j+1} + \ldots, \alpha_p = \gamma_p. \) Then \( \sum_{\nu=2}^{p} \nu \alpha_{\nu} = (\sum_{\nu=2}^{p+1} \nu \gamma_{\nu}) - 1 = (p + 1) - 1 = p \). Hence, according to the given definition of \( \sum \) the monomial \( \prod_{\nu=2}^{p+1} (D^\nu [u(z)])^{\gamma_{\nu}} \) is contained in the right-hand side of \( H \).
If \( \gamma_2 \neq 0 \), then we consider in the right-hand side of \( H \) the sum:
\[ \sum_{\tilde{\alpha}(p-1)} p \tilde{C}_{p-1} (\alpha_2, \ldots, \alpha_{p-1}) (D^2 [u(z)])^{\alpha_{2}+1} \prod_{\nu=3}^{p-1} (D^\nu [u(z)])^{\alpha_{\nu}}. \]
Letting \( \alpha_2 = \gamma_2 - 1, \alpha_3 = \gamma_3, \ldots, \alpha_{p-1} = \gamma_{p-1} \), we obtain \( \sum_{\nu=2}^{p-1} \nu \alpha_{\nu} = \left( \sum_{\nu=2}^{p} \nu \gamma_{\nu} \right) - 2 = p - 1 \).

Hence according to the definition of \( \sum_{\hat{\alpha}(p-1)} \) the monomial \( \prod_{\nu=2}^{p+1} (D^\nu[u(z)])^{\gamma_\nu} \) is contained in the right-hand side of \( H \).

Thus we proved by induction that

\[
\Delta_p^D = \sum_{\hat{\alpha}} \frac{D!}{(2!)^{\alpha_2} \cdot \alpha_2! \cdot \cdots \cdot (p!)^{\alpha_p} \cdot \alpha_p!} (D^2[u(z)])^{\alpha_2} \cdots (D^p[u(z)])^{\alpha_p}.
\]

In the above equality we replace \( D \) with \( \mathcal{O} \) (using the fact that Lemma 3.1 remains valid if we replace in (22) and (23) \( \mathcal{O} \) with \( D \)) and obtain

\[
\Delta_p^\mathcal{O} = \sum_{\hat{\alpha}} \frac{D!}{(2!)^{\alpha_2} \cdot \alpha_2! \cdot \cdots \cdot (p!)^{\alpha_p} \cdot \alpha_p!} (\mathcal{O}^2[u(z)])^{\alpha_2} \cdots (\mathcal{O}^p[u(z)])^{\alpha_p}.
\]

Now, using the fact that \( \mathcal{O}^k[u(z)] = (k - 1)!D^k[u(z)] \) (see Lemma 2.1, (j1)), we obtain

\[
\Delta_p^D = \sum_{\hat{\alpha}} \frac{D!}{(2!)^{\alpha_2} \cdot \alpha_2! \cdot \cdots \cdot (p!)^{\alpha_p} \cdot \alpha_p!} (D^2[u(z)])^{\alpha_2} \cdots (D^p[u(z)])^{\alpha_p}.
\]

Hence

\[
\Delta_p = \Delta_p^D = \sum_{\hat{\alpha}} \frac{D!}{(2^{\alpha_2} \cdot \alpha_2! \cdots p^{\alpha_p}) \cdot \alpha_2! \cdot \cdots \cdot (p!)^{\alpha_p} \cdot \alpha_p!} (D^2[u(z)])^{\alpha_2} \cdots (D^p[u(z)])^{\alpha_p}.
\]

Thus (24) and therefore Lemma 3.2 are proved.

From Theorem 3.1, Lemma 3.1 and Lemma 3.2 we obtain the following explicit formulae for the coefficients.

**Theorem 3.3.** The following representations are valid:

\[
b_k = \gamma^k + \sum_{p=2}^{k} \binom{k}{p} \gamma^{k-p} \sum_{\hat{\alpha}} A_p(\hat{\alpha}) \prod_{\nu=2}^{p} (\zeta(\nu))^{\alpha_\nu}, \tag{26}
\]

\[
\rho_k = \gamma^k + \sum_{p=2}^{k} \binom{k}{p} \gamma^{k-p} \sum_{\hat{\alpha}} (-1)^{p+\sum_{\nu=2}^{p} \alpha_\nu} A_p(\hat{\alpha}) \prod_{\nu=2}^{p} (\zeta(\nu))^{\alpha_\nu}, \tag{27}
\]

where \( \sum_{\hat{\alpha}} \) means that we sum over all nonnegative integers \( \alpha_\nu \) such that \( \sum_{\nu=2}^{p} \nu \alpha_\nu = p \).

**Corollary 3.1.** From (10)–(13) and from (26), (27) we obtain:

\[
\Gamma(z + 1) = 1 - \gamma z + \sum_{k=2}^{\infty} a_k z^k, \quad |z| < 1 \tag{28}
\]

where

\[
a_k = \frac{(-1)^k}{k!} \left( \gamma^k + \sum_{p=2}^{k} \binom{k}{p} \gamma^{k-p} \sum_{\hat{\alpha}} \frac{1}{(2^{\alpha_2} \cdot \alpha_2! \cdots p^{\alpha_p}) \cdot \alpha_2! \cdot \cdots \cdot (p!)^{\alpha_p} \cdot \alpha_p!} \prod_{\nu=2}^{p} (\zeta(\nu))^{\alpha_\nu} \right). \tag{29}
\]

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where

\[ d_k = \frac{\gamma^k}{k!} + \sum_{p=2}^{k} \frac{\gamma^{k-p}}{(k-p)!} \sum_{\alpha} (-1)^{p+\sum_{\nu=2}^{p} \alpha_{\nu}} \frac{1}{(2^{\alpha_2}3^{\alpha_3} \cdots p^{\alpha_p})(\alpha_2!\alpha_3! \cdots \alpha_p!)} \prod_{\nu=2}^{p} \zeta(\nu)^{\alpha(\nu)} \]  

(31)

and \( \sum_{\alpha} \) means that we take the sum over all nonnegative integers \( \alpha_{\nu} \) such that \( \sum_{\nu=2}^{p} \nu \alpha_{\nu} = p \).

In particular (see (2)):

\[ A(k) = \int_{0}^{\infty} e^{-t}(\ln t)^k dt = (-1)^k \left( \sum_{p=2}^{k} \frac{1}{p!} (2^{\alpha_2}3^{\alpha_3} \cdots p^{\alpha_p})(\alpha_2!\alpha_3! \cdots \alpha_p!) \prod_{\nu=2}^{p} \zeta(\nu)^{\alpha(\nu)} \right) \]

and: \( a_0 = 1, a_1 = -\gamma, a_2 = \frac{1}{2}(\gamma^2 + \zeta(2)) = \frac{1}{2}(\gamma^2 + \frac{\pi^2}{6}), a_3 = -\frac{1}{6}(\gamma^3 + 3\zeta(2)\gamma + 2\zeta(3)) = -\frac{1}{6}(\gamma^3 + \frac{\pi^2}{2}\gamma + 2\zeta(3)); d_0 = 1, d_1 = \gamma, d_2 = \frac{1}{2}(\gamma^2 - \zeta(2)) = \frac{1}{2}(\gamma^2 - \frac{\pi^2}{6}), d_3 = \frac{1}{6}\gamma^3 - \frac{1}{2}\zeta(2)\gamma + \frac{1}{3}\zeta(3) = \frac{1}{6}\gamma^3 - \frac{\pi^2}{12}\gamma + \frac{1}{3}\zeta(3) \)

**Remark.** Using (1) and (28), one may observe that the Laurent series of \( \Gamma(z) \) around \( z = 0 \) is:

\[ \Gamma(z) = \frac{1}{z} - \gamma + \sum_{k=2}^{\infty} a_k z^{k-1}, \quad |z| < 1, \]

where \( a_k \) are given by (29). Also, using (1) and (30), one may observe the Maclaurin series of \( (\Gamma(z))^{-1} \) is

\[ \frac{1}{\Gamma(z)} = z + \gamma z^2 + \sum_{k=2}^{\infty} d_k z^{k+1}, \]

where \( d_k \) are given by (31).

We must note that the representation (30)–(31) for \( (\Gamma(z+1))^{-1} \) is given in another form by Mika Sakata’s formula for \( (\Gamma(z))^{-1} \) in [8], where Sakata uses multiple zeta values.

**4 Conclusion**

In the paper two important results are found in Theorem 3.1 and Theorem 3.2. With the help of Theorem 3.1 we obtain explicit formulae for the coefficients of \( \Gamma(z+1) \) and \( (\Gamma(z+1))^{-1} \) in their Maclaurin series.

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