Fano deformation rigidity of rational homogeneous spaces of submaximal Picard numbers

Qifeng Li

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Abstract
We study the question whether rational homogeneous spaces are rigid under Fano deformation. In other words, given any smooth connected family $\pi : \mathcal{X} \to Z$ of Fano manifolds, if one fiber is biholomorphic to a rational homogeneous space $S$, is $\pi$ an $S$-fibration? The cases of Picard number one were answered by Hwang and Mok. The manifold $\mathbb{F}(1, Q^5)$ is the unique rational homogeneous space of Picard number one that is not rigid under Fano deformation, and a Fano degeneration of it is constructed by Pasquier and Perrin. For higher Picard number cases, one notices that the Picard number of a rational homogeneous space $G/P$ satisfies $\rho(G/P) \leq \text{rank}(G)$. Weber and Wiśniewski proved that the rational homogeneous spaces $G/P$ with $\rho(G/P) = \text{rank}(G)$ (i.e. complete flag manifolds) are rigid under Fano deformation. In this paper, we show that the rational homogeneous spaces $G/P$ with $\rho(G/P) = \text{rank}(G) - 1$ are rigid under Fano deformation, provided that $G$ is a simple algebraic group of type $ADE$, and $G/P$ is not biholomorphic to $\mathbb{F}(1, 2, \mathbb{P}^3)$ or $\mathbb{F}(1, 2, Q^6)$. We also show that $\mathbb{F}(1, 2, \mathbb{P}^3)$ has a unique Fano degeneration, which is explicitly constructed. Furthermore, the structure of possible Fano degenerations of $\mathbb{F}(1, 2, Q^6)$ is also described explicitly. Our main result is obtained by applying the theory of Cartan connections and symbol algebras.

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1 Introduction

We work over the field $\mathbb{C}$ of complex numbers. A Fano manifold $M$ is said to be rigid under Fano deformation if any smooth connected family $\pi : \mathcal{X} \to \mathcal{Z}$ of Fano manifolds with $M$ being a fiber must be an $M$-fibration. If the fiber $\mathcal{X}_z := \pi^{-1}(z)$ at some point $z \in \mathcal{Z}$ is not biholomorphic to $M$, we say $\mathcal{X}_z$ is a Fano degeneration of $M$.

Our interest in this paper is the Fano deformation rigidity of rational homogeneous spaces. A rational homogeneous space is the family of cosets $G/P$, where $G$ is a semisimple algebraic group and $P$ is a parabolic subgroup of $G$. The Picard number of $G/P$ satisfies that $\rho(G/P) \leq \text{rank}(G)$, where $\text{rank}(G)$ is the dimension of any maximal torus of $G$.

In a series of papers [6,8–10], Hwang and Mok answer the question of Fano deformation rigidity of rational homogenous spaces of Picard number one. Among the rational homogeneous spaces of Picard number one, $\mathbb{F}(1, Q^5)$ is the only variety that is not rigid under Fano deformation, where $\mathbb{F}(1, Q^5)$ is the family of projective lines on a 5-dimensional smooth quadric hypersurface. In [15], B. Pasquier and N. Perrin construct a smooth Fano degeneration of $\mathbb{F}(1, Q^5)$, which is a quasi-homogeneous variety under the action of $G_2$. On the other hand, A. Weber and J. A. Wiśniewski [17] prove Fano deformation rigidity of the cases when $\rho(G/P) = \text{rank}(G)$.

Motivated by these results on extremal cases, one naturally asks what about the intermediate cases? Our main result is on the cases with submaximal Picard numbers, i.e. $\rho(G/P) = \text{rank}(G) - 1$. More precisely, we have the following

**Theorem 1.1** Let $G$ be a simple algebraic group of type $ADE$. Then the rational homogeneous space $G/P$ with Picard number $\text{rank}(G) - 1$ is rigid under Fano deformation, provided that $G/P$ is not biholomorphic to $\mathbb{F}(1, 2, \mathbb{P}^3)$ or $\mathbb{F}(1, 2, Q^6)$.
The variety $\mathbb{F}(1, 2, \mathbb{P}^3)$ is the set of flags of projective lines and planes on $\mathbb{P}^3$, and the variety $\mathbb{F}(1, 2, \mathbb{Q}^6)$ is the set of flags of projective lines and planes on a 6-dimensional smooth quadric hypersurface $\mathbb{Q}^6$.

In the study of the Fano deformation rigidity of rational homogeneous spaces of Picard number one, Hwang and Mok apply the theory of Cartan connections and symbol algebras. We also apply this theory to higher Picard number cases to prove Theorem 1.1.

Before explain this method in details, let us illustrate the strategy to prove Theorem 1.1. We firstly reduce the problem to the cases when $\rho(G/P) \leq 4$, and then verify the latter cases. On the other hand, the analysis on the cases with $\rho(G/P) \leq 4$ depends on the classification of Fano deformations of $\mathbb{F}(1, 2, \mathbb{P}^3)$, which is stated as follows.

**Proposition 1.2** Let $\omega$ be a symplectic form on $\mathbb{C}^4$, and $\mathcal{L}_\omega \subset T\mathbb{P}^3$ be the associated contact distribution on $\mathbb{P}^3 := \mathbb{P}(\mathbb{C}^4)$. Then the variety $F^d(1, 2, \mathbb{C}^4) := \mathbb{P}(\mathcal{L}_\sigma \oplus \mathcal{L}_\omega)$ is the unique Fano degeneration of $\mathbb{F}(1, 2, \mathbb{P}^3)$, where $\mathcal{L}_\sigma := T\mathbb{P}^3/\mathcal{L}_\omega$.

It is first observed by Weber and Wiśniewski [17] that $F^d(1, 2, \mathbb{C}^4)$ is a Fano degeneration $\mathbb{F}(1, 2, \mathbb{P}^3)$. The way how the Fano degeneration of $\mathbb{F}(1, 2, \mathbb{P}^3)$ appears is similar with that of $\mathbb{F}(1, \mathbb{Q}^5)$. The automorphism group of $\mathbb{Q}^5$ is the simple group $\text{PSO}(7)$, and there is a proper subgroup $G_2$ of $\text{PSO}(7)$ that acts transitively on $\mathbb{Q}^5$. The manifold $\mathbb{F}(1, \mathbb{Q}^5)$ is the highest weight orbit in the projectivization of the second fundamental representation $V$ of $\text{SO}(7)$. Under the action of $G_2$, the module $V$ is the direct sum of two irreducible modules $V_1$ and $V_2$, which are of dimensions 7 and 14 respectively. The Fano degeneration of $\mathbb{F}(1, \mathbb{Q}^5)$ constructed by B. Pasquier and N. Perrin is the locus covered by $G_2 \cdot \mathbb{P}(\mathbb{C}^2)$, where $\mathbb{P}(\mathbb{C}^2)$ is a projective line in $\mathbb{P}(V)$. On the other hand, the automorphism group of $\mathbb{P}^3$ is the simple group $\text{PSL}(4)$, and there is a proper subgroup $\text{PSp}(4)$ of $\text{PSL}(4)$ that acts transitively on $\mathbb{P}^3$. The manifold $\mathbb{F}(1, \mathbb{P}^3)$ is the highest weight orbit in the projectivization of $\text{W}_{\omega_1+\omega_2}$, where $\omega_i$ is the $i$-th fundamental weight of $\text{SL}(4)$ and $\text{W}_{\omega_1+\omega_2}$ is the irreducible representation of $\text{SL}(4)$ with highest weight $\omega_1 + \omega_2$. Under the action of $\text{PSp}(4)$, the module $\text{W}_{\omega_1+\omega_2}$ is the direct sum of two irreducible modules $W_1$ and $W_2$, which are of dimensions 4 and 16 respectively. The Fano degeneration of $\mathbb{F}(1, 2, \mathbb{P}^3)$ constructed by Weber and Wiśniewski is the locus covered by $\text{PSp}(4) \cdot \mathbb{P}(\mathbb{W}_{\omega_1+\omega_2})$, where $\mathbb{W}_{\omega_1+\omega_2}$ is a projective line in $\mathbb{P}(\mathbb{W}_{\omega_1+\omega_2})$. The similarity of both Fano degenerations may induce the construction of more Fano manifolds which are closely related to rational homogeneous spaces. This deserves a further study in the future.

Now let us return to sketch the proof of Proposition 1.2. Let $X$ be a Fano deformation of $\mathbb{P}(1, 2, \mathbb{P}^3)$, then there is a $\mathbb{P}^2$-bundle structure $\phi : X \to \mathbb{P}^3$. By the study of minimal rational curves on $X$, we can obtain a symplectic form $\omega$ on $\mathbb{P}^3$ such that $\phi$ coincides with the $\mathbb{P}^2$-bundle $\mathbb{P}(\mathcal{L}_\sigma \oplus \mathcal{L}_\omega) \to \mathbb{P}^3$.

The remaining case of Theorem 1.1 is studied in Sect. 5. More precisely, we describe the structure of possible Fano degenerations of $\mathbb{F}(1, 2, \mathbb{Q}^6)$, see Proposition 5.1.
Now let us explain the proof of Theorem 1.1 in more details. To do this, we need some conventions on notations. Let $G$ be a simple algebraic group and $B$ be a Borel subgroup. Denote by $R$ the set of simple roots and $\Gamma_R$ the Dynkin diagram. There is a one to one correspondence between the subsets $I$ of $R$ and the parabolic subgroups $P_I$ containing $B$ such that $P_R = B$, $P_{\emptyset} = G$ and $P_I \subset P_I'$ whenever $I' \subset I$. There is a one to one correspondence between the rational homogeneous spaces $G/P_I$ and the marked Dynkin diagrams defined by marking nodes $I$ in $\Gamma_R$. One can see the marked Dynkin diagrams in an intuitive way on page 5. The following proposition reduces the Fano deformation rigidity of $G/P_I$ to the properties of its homogeneous submanifolds.

**Proposition 1.3** Let $G$ be a simple algebraic group of type $ADE$, and $I$ be a subset of $R$ with cardinality $|I| \geq 3$. Suppose that given any two distinct elements $\alpha$ and $\beta$ in $I$, there exists a subset $A \subset I$ such that $\alpha, \beta \in A$ and the rational homogeneous space $P_I \setminus A/P_I$ is rigid under Fano deformation. Then $G/P_I$ is rigid under Fano deformation.

Note that the Picard number of $P_I \setminus A/P_I$ is $|A|$, which is less than or equal to $\rho(G/P_I) = |I|$. By Proposition 1.3 and an easy analysis of the marked Dynkin diagrams, Theorem 1.1 is a consequence of the known cases and the following

**Proposition 1.4** The rational homogeneous spaces $A_4/P_I'$ and $D_5/P_I''$ are rigid under Fano deformation, where $|I'| = 3$ and $|I''| = 4$ respectively.

As an example, we verify the Fano deformation rigidity of $D_4/P_I$ with $I = \{\alpha_1, \alpha_3, \alpha_4\}$. Given any two different roots $\alpha, \beta \in I$, $P_I \setminus [\alpha, \beta]/P_I$ is biholomorphic to $A_3/P_{(\alpha_1, \alpha_3)}$, and it is rigid under Fano deformation due to J. A. Wiśniewski [18]. Hence, $D_4/P_I$ is rigid under Fano deformation by Proposition 1.3.

Our argument on the Fano deformation rigidity of $A_4/P_{[\alpha_1, \alpha_2, \alpha_4]}$, which is a special case of Proposition 1.4, works equally well for $A_m/P_{[\alpha_1, \alpha_2, \alpha_m]}$ with $m \geq 3$. In other words, we have

**Proposition 1.5** The manifolds $A_m/P_{[\alpha_1, \alpha_2, \alpha_m]}$ with $m \geq 3$ are rigid under Fano deformation.

Applying Propositions 1.3 and Proposition 1.5, we have the following consequence.

**Theorem 1.6** Let $G$ be a simple algebraic group of type $ADE$. When $G$ is of type $D$ or $E$, let $\bar{\alpha}$ be the node with three branches in the Dynkin diagram $\Gamma_R$. Let $J$ be a subset of $R$ such that

1. $J$ contains no end node of $\Gamma_R$,
2. the subdiagram with the nodes $J$ are connected, and
3. there is at most one $\beta \in J$ such that the Cartan integer $\langle \beta, \bar{\alpha} \rangle \neq 0$.

Then the rational homogeneous space $G/P_{R \setminus J}$ is rigid under Fano deformation.

If $G$ is of type $A$ or $D$ in Theorem 1.6, the manifold $G/P_I$ is $\mathbb{F}(0, 1, \ldots, k_1, k_2, k_2 + 1, \ldots, n - 1, \mathbb{P}^n)$ and $\mathbb{F}(0, 1, \ldots, k_1, k_2, k_2 + 1, \ldots, n, Q^{2n+2})$ for some $0 \leq k_1 < \ldots < \ldots$
$k_2 \leq n - 1$, respectively. Now we turn the case when $G$ is of type $E_n$. The variety $E_6/P_{a_1}$ is known as the complex Cayley plane. To the knowledge of the author the geometric meanings of $E_7/P_{a_1}$ and $E_8/P_{a_1}$ are less clear. The rational homogeneous space $E_n/P_{a_k}$ is an irreducible component of the family of projective subspaces contained in $E_n/P_{a_1} \subset \mathbb{P}(H^0(E_n/P_{a_1}, \mathcal{O}_{E_n/P_{a_1}}(1)))^*$, where $6 \leq n \leq 8$ and $2 \leq k \leq n - 1$. And $E_n/P_I$ is a family of flags on $E_n/P_{a_1} \subset \mathbb{P}(H^0(E_n/P_{a_1}, \mathcal{O}_{E_n/P_{a_1}}(1)))^*$, where $6 \leq n \leq 8$.

Let us explain the proofs of Propositions 1.3, 1.4 and 1.5 now. It is well-known that the local deformation rigidity of rational homogeneous spaces follows from the vanishing $H^1(G/P_I, T_{G/P_I}) = 0$, and the latter is a consequence of the Borel-Weil-Bott theorem. Then we may assume that the situation of Setting 1.7 below holds.

**Setting 1.7** Let $\pi : \mathcal{X} \rightarrow \Delta \ni 0$ be a holomorphic map such that $\mathcal{X}_t \cong S := G/P_I$ for $t \neq 0$ and $\mathcal{X}_0$ is a connected Fano manifold, where $G$ is a connected simple algebraic group of type $ADE$ and $I$ is a subset of $R$.

The key point to prove Propositions 1.3, 1.4 and 1.5 is the study of symbol algebras. Given a distribution $\mathcal{V}$ on a complex manifold $Y$, the weak derived systems give rise to a filtration $\mathcal{V}^0 \subset \mathcal{V}^{-1} \subset \mathcal{V}^{-2} \subset \cdots$, where $\mathcal{V}^0 := 0$, $\mathcal{V}^{-1} := \mathcal{V}$, and $\mathcal{V}^{-k-1} := \mathcal{V}^{-k} + [\mathcal{V}^{-1}, \mathcal{V}^{-k}]$ for $k \geq 1$. The symbol algebra of $\mathcal{V}$ at a point $y \in Y$ is the graded nilpotent Lie algebra

$$\text{Symb}_y(\mathcal{V}) := \oplus_{k \geq 1} \mathcal{V}_y^{-k}/\mathcal{V}_y^{-k+1}.$$ 

Let $\mathfrak{g}_{-1}(S)$ be the sum of all $G$-invariant minimal distributions on $S$. The subscript $-1$ in the notation $\mathfrak{g}_{-1}(S)$ comes from the grading induced by $I$, see Sect. 2.1. There is a meromorphic distribution $\mathfrak{g}_{-1}(\mathcal{X}) \subset T_\pi$ such that its singular locus is a (possibly reducible) proper closed subvariety of $\mathcal{X}_0$, and its restriction on $\mathcal{X}_t$ with $t \neq 0$ coincides with the distribution $\mathfrak{g}_{-1}(\mathcal{S})$.

It is known that $\text{Symb}_p(\mathfrak{g}_{-1}(S)) \cong \mathfrak{g}_{-1}(I)$, where $p$ is any point of $S$, $\mathfrak{g}_{-1}(I)$ is the nilradical of the Lie algebra of $P_I^-$, and $P_I^-$ the opposite parabolic group of $P_I$. By the works on Cartan connections, due to Čap and Schichl [3] and Yamaguchi [20], we can conclude the following

**Proposition 1.8** In the situation of Setting 1.7, suppose that $|I| \geq 3$ and $\text{Symb}_x(\mathfrak{g}_{-1}(\mathcal{X}_0)) \cong \mathfrak{g}_{-1}(I)$ at general points $x \in \mathcal{X}_0$. Then $\mathcal{X}_0 \cong S$. 

Now Propositions 1.3, 1.4 and 1.5 can be implied by Proposition 1.8 and the following

**Proposition 1.9** In the situation of Setting 1.7, if $G/P_I$ also satisfies the assumptions in Propositions 1.3, 1.4 or 1.5, then $\text{Symb}_x(\mathfrak{g}_{-1}(\mathcal{X}_0)) \cong \mathfrak{g}_{-1}(I)$ at general points $x \in \mathcal{X}_0$.

To prove Proposition 1.9, we need the algebraic feature and the geometric feature of each case. As an example, suppose $S = A_m/P_{\{a_1, a_2, a_3\}}$ in Setting 1.7. Then any two points in $\mathcal{X}_0$ can be jointed by chains of rational curves that are tangent to $\mathfrak{g}_{-1}(\mathcal{X}_0)$. 

\[\mathfrak{g}_{-1}(\mathcal{X}_0)\]
Hence the tangent bundle $T\mathcal{X}_0$ is $k$-th weak derivative of $g_{-1}(\mathcal{X}_0)$ for some $k$. In particular, $\dim \text{Symb}_x(g_{-1}(\mathcal{X}_0)) = \dim \mathcal{X}_0 = \dim g_{-}(I)$ at a general point $x \in \mathcal{X}_0$. One the other hand, if the symbol algebra $\text{Symb}_x(g_{-1}(\mathcal{X}_0)) \neq g_{-}(I)$, then an easy calculation of Lie algebras shows that $\dim \text{Symb}_x(g_{-1}(\mathcal{X}_0)) < \dim g_{-}(I)$. The contradiction implies that $\text{Symb}_x(g_{-1}(\mathcal{X}_0)) \cong g_{-}(I)$.

The organization of this paper is as follows. In Sect. 2, we firstly recall some geometric properties of rational homogeneous spaces and then give a characterization of $g_{-}(I)$. In Sect. 3, we firstly verify Proposition 1.8 and then apply it to prove Proposition 1.3. In Sect. 4, we study the Fano deformations of $\mathbb{P}(1, 2, \mathbb{P}^3)$ and prove Proposition 1.2. In Sect. 5, we study the structure of the possible Fano degenerations of $\mathbb{F}(1, 2, \mathbb{P}^3)$. In Sect. 6, we prove Proposition 1.5. In Sect. 7, we prove Proposition 1.4. Finally, we prove Theorems 1.1 and 1.6 in Sect. 8.

2 Geometry on rational homogeneous spaces

2.1 Distributions and the families of lines

In this subsection, we collect the geometric properties on rational homogeneous spaces.

Setting 2.1 Let $G$ be a connected semisimple algebraic group of adjoint type such that each simple factor is of type $ADE$, $B$ be a Borel subgroup, $H$ be a maximal torus, and $R$ be the set of simple roots. Fix a subset $I$ of $R$ and denote by $J := R \setminus I$.

Denote by $P_I := \bigcap_{\alpha \in I} P_{\alpha}$, where $P_{\alpha}$ is the associated maximal parabolic subgroup of $G$ which contains $B$. Denote by $P_I^-$ the opposite parabolic subgroup of $P_I$, which is the conjugation of $P_I$ by the longest element of the Weyl group. The intersection $G_0 := P_I \cap P_I^-$ is the Levi factor of both $P_I$ and $P_I^-.$

Definition 2.2 Let $g$ and $\mathfrak{h}$ be the Lie algebras of $G$ and $H$ respectively. In particular, $\mathfrak{h}$ is a Cartan subalgebra of $g$. Let $\Lambda$ be the set of all roots of $G$. Given $\eta_1, \eta_2 \in \Lambda$, denote by $\langle \eta_1, \eta_2 \rangle$ the corresponding Cartan integer. For each $\eta \in \Lambda$, denote by $g_{\eta}$ the 1-dimensional linear subspace of $g$ with weight $\eta$. We can write $\eta = \sum_{\alpha \in R} n_{\alpha} \alpha$, where either all $n_{\alpha}$ are nonnegative integers or all $n_{\alpha}$ are nonpositive integers. Define $\deg_{\eta} \eta := \sum_{\alpha \in \mathbb{Z}} n_{\alpha}$. For each $k \in \mathbb{Z}$ denote by $\Lambda_k(I)$ the set of elements $\eta \in \Lambda$ with $\deg_{\eta}(\eta) = k$. Consider a grading on $g$ such that

$$g_k(I) := \bigoplus_{\eta \in \Lambda_k(I)} g_{\eta} \text{ for } k \neq 0, \text{ and } g_0(I) := \mathfrak{h} \oplus \bigoplus_{\eta \in \Lambda_0(I)} g_{\eta}.$$ 

Then $g$ becomes a graded Lie algebra. Moreover $g_0$, $p_I := \bigoplus_{k \geq 0} g_k$ and $p_I^- := \bigoplus_{k \leq 0} g_k$ are the Lie algebras of $G_0$, $P_I$ and $P_I^-$ respectively. When there is no confusion, we omit $I$ in the expressions, for example $g_k := g_k(I)$. In case $I = R$, we may also write $g_{-}(G) := g_{-}(R)$ in order to emphasize on the group $G$.

A rational homogeneous space can be expressed by a marked Dynkin diagram. To explain the order of simple roots and the way to express a rational homogeneous space,
we draw the marked Dynkin diagram corresponding to $G/P(\alpha_1, \alpha_2, \alpha_4)$ as follows, where $G = A_n, D_m$ or $E_k$ with $n, m \geq 4$ and $k = 6, 7, 8$ respectively.

\[ \begin{align*}
A_n : & \quad \alpha_1 \quad \bullet \quad \alpha_2 \quad \bullet \quad \alpha_3 \quad \bullet \quad \alpha_4 \quad \bullet \quad \alpha_{n-1} \quad \bullet \quad \alpha_n \\
D_m : & \quad \alpha_1 \quad \bullet \quad \alpha_2 \quad \bullet \quad \alpha_3 \quad \bullet \quad \alpha_4 \quad \bullet \quad \alpha_{m-2} \quad \bullet \quad \alpha_{m-1} \quad \bullet \quad \alpha_m \\
E_k : & \quad \alpha_1 \quad \bullet \quad \alpha_3 \quad \bullet \quad \alpha_4 \quad \bullet \quad \alpha_{k-1} \quad \bullet \quad \alpha_k \quad \bullet \quad \alpha_2 \quad \bullet
\end{align*} \]

Since we assume that $G$ is of adjoint type, the restriction of the adjoint representation induces an injective homomorphism $G_0 \subset \text{GL}(\mathfrak{g}_-(I))$, where $\mathfrak{g}_-(I) := \bigoplus_{k \geq 1} \mathfrak{g}_{-k}(I)$.

**Example 2.3** Let $\mathfrak{g}$ be the simple algebra $\mathfrak{sl}(\mathbb{C}^6)$, which is of type $A_5$. Take $I = \{\alpha_1, \alpha_4\}$. Then the corresponding marked Dynkin diagram is

\[ \begin{align*}
\alpha_1 \quad \bullet \quad \alpha_2 \quad \bullet \quad \alpha_3 \quad \bullet \quad \alpha_4 \quad \bullet \quad \alpha_5
\end{align*} \]
Then \( \Lambda_0(I) = \{ \pm \alpha_2, \pm \alpha_3, \pm \alpha_5, \pm (\alpha_2 + \alpha_3) \} \), \( \Lambda_1(I) = \{ \sum_{i=0}^{k_1} \alpha_{1+i}, \sum_{i=k_2}^{k_3} \alpha_{4+i} | 0 \leq k_1, k_2 \leq 2, 0 \leq k_3 \leq 1 \} \), \( \Lambda_2(I) = \{ \sum_{i=0}^{k} \alpha_{1+i} | k = 3, 4 \} \), \( \Lambda_{-1}(I) = -\Lambda_1(I) \) and \( \Lambda_{-2}(I) = -\Lambda_2(I) \).

**Definition 2.4** There is a unique minimal ample Cartier divisor on \( S := G/P_1 \), denoted by \( O_S(1) \). It is minimal in the sense that given any ample Cartier divisor \( A \) on \( S \), the Cartier divisor \( A \otimes O_S(-1) \) is nef. The induced closed embedding \( S \subset \mathbb{P}(H^0(S, O_S(1))^*) \) is called the minimal embedding. Given any integers \( 0 \leq k_1 \leq k_2 \leq \cdots \leq k_\ell, \) we use \( \mathbb{P}(k_1, \cdots, k_\ell, S) \) to denote the set of flags \( L_{k_1} \subset \cdots \subset L_{k_\ell} \) such that each \( L_{k_i} \) is contained in \( S \), and it is a projective subspace of dimension \( k_i \) under the minimal embedding of \( S \).

**Remark 2.5** In this paper, we use a similar notation \( F(k_1, \cdots, k_\ell, \mathbb{C}^n) \) to denote the set of flags \( V_{k_1} \subset \cdots \subset V_{k_\ell} \) such that each \( V_{k_i} \) is a \( k_i \)-dimensional vector subspace of \( \mathbb{C}^n \), where the integers \( 1 \leq k_1 \leq k_2 \leq \cdots \leq k_\ell \leq n-1 \). In particular, the identification \( \mathbb{P}^{n-1} = \mathbb{P}(\mathbb{C}^n) \) induces an identification \( F(k_1 - 1, \cdots, k_\ell - 1, \mathbb{P}^{n-1}) = F(k_1, \cdots, k_\ell, \mathbb{C}^n) \), where \( \mathbb{P}(k_1 - 1, \cdots, k_\ell - 1, \mathbb{P}^{n-1}) \) is a set of flags defined in Definition 2.4 by regarding \( \mathbb{P}^{n-1} \) as the rational homogeneous space \( S \). We will use these two notations freely according to which one is more suitable for our discussions.

**Notation 2.6** Given any \( \alpha \in R \), denote by \( N(\alpha) \) the set of simple roots that are linked to \( \alpha \) in the Dynkin diagram \( \Gamma_R \) of \( G \), and set \( N_J(\alpha) := N(\alpha) \cap J \). Given any subset \( A \) of \( R \), let \( \Gamma_A \) be the subdiagram supported at the nodes of \( A \), and \( G_A \) be the semisimple subgroup of \( G \) associated to the Dynkin diagram \( \Gamma_A \).

**Definition 2.7** Set \( S := G/P_1 \). Given a subset \( A \subset I \), denote by \( S^A \) the central fiber of \( \Phi^A : S \to G/P_{1\backslash A} \), which is a rational homogeneous space of Picard number \( |A| \). Given any \( \alpha \in I \), each fiber of \( \Phi^\alpha \) is biholomorphic to \( S^\alpha \), and it is covered by lines under its minimal embedding. Denote by \( \mathcal{K}^\alpha(S) \) the family of these lines (associated with \( \alpha \)) on \( S \). One can see \( G/P_{(I\cup N(\alpha))\backslash\{\alpha\}} \subset \mathcal{K}^\alpha(S) \) by the following commutative diagram of fibrations.

\[
\begin{array}{ccc}
G/P_{(I\cup N(\alpha))\backslash\{\alpha\}} & \xrightarrow{\text{ev}} & G/P_1 \\
\mu \downarrow & & \downarrow \\
G/P_{(I\cup N(\alpha))\backslash\{\alpha\}} & \xrightarrow{\text{ev}} & G/P_{1\backslash\{\alpha\}}.
\end{array}
\]

Since \( G \) is of type ADE, we have \( \mathcal{K}^\alpha(S) = G/P_{(I\cup N(\alpha))\backslash\{\alpha\}} \). Denote by \( \mathcal{C}^\alpha(S) \subset \mathbb{P}(TS) \) the variety of tangent directions of \( \mathcal{K}^\alpha(S) \), i.e. at each point \( x \in S \),

\[
\mathcal{C}^\alpha_x(S) = \bigcup_{C \in \mathcal{K}^\alpha_x(S)} \mathbb{P}(T_x C) \subset \mathbb{P}(T_x(S)).
\]

Denote by \( Z^\alpha := \mathcal{C}^\alpha_p(S) \subset \mathbb{P}(T_p S) \), where \( p \) is the base point of \( S \).

Tits [16] studied diagrams in the style of (2.4) and he called \( \mu(\text{ev}^{-1}(z)) \) the shadow of \( z \in G/P_1 \). This variety is biholomorphic to \( \mathcal{C}^\alpha_x(S) \) for \( x \in \text{ev}^{-1}(z) \subset G/P_1 \). The
variety $C_x^\alpha(S)$ is called the variety of minimal rational tangents (VMRT for short) at $x$ of the minimal rational component $K^\alpha(S)$. One could find more details about minimal rational components and VMRT in [7]. For more details about the properties of those $C^\alpha$, one can refer to [12].

The group $G_J$ is the semisimple part of the reductive group $G_0$, and each simple factor of $G_J$ is of type $ADE$. Given $\alpha \in I$, the natural $G_0$-action on $Z^\alpha$ is transitive, and the inclusion $Z^\alpha \subset \mathbb{P}(g_{-1}(\alpha))$ is equivariant. Indeed $g_{-1}(\alpha)$ is an irreducible module of the semisimple algebraic group, see [12, Propositions 2.4 and 2.5] for the proof. It follows that $Z^\alpha$ is the unique closed $G_0$-orbit in $\mathbb{P}(g_{-1}(\alpha))$ and it is nondegenerate in $\mathbb{P}(g_{-1}(\alpha))$. The $G_J$-action on $Z^\alpha$ induces the isomorphism

$$Z^\alpha \cong G_J/P_{NJ}(\alpha) \cong \prod_{\beta \in NJ(\alpha)} G_J/P_\beta.$$ 

Particularly each $Z^\alpha_\beta := G_J/P_\beta \cong G_0/P_\beta$ is a rational homogeneous space of Picard number one.

**Example 2.8** In the setting of Example 2.3, we have $S = F(1, 4, \mathbb{C}^6) = A_5/P_{[\alpha_1, \alpha_4]}$, $J = \{\alpha_2, \alpha_3, \alpha_5\}$, $N_J(\alpha_1) = \{\alpha_2\}$, $N_J(\alpha_4) = \{\alpha_3, \alpha_5\}$, $G_0 = GL(\mathbb{C}^3) \times GL(\mathbb{C}^2)$, and $G_J = SL(\mathbb{C}^3) \times SL(\mathbb{C}^2)$. The marked Dynkin diagram of $S_{\alpha_1}^\alpha$, $Z_{\alpha_1}^\alpha$, and $K_{\alpha_1}^\alpha$ are as follows:

![Dynkin diagrams](image)

Furthermore, we have the following identities:

$$S_{\alpha_1}^\alpha = \mathbb{P}^3, \quad (Z_{\alpha_1}^\alpha \subset \mathbb{P}(g_{-1}(\alpha_1))) = (\mathbb{P}^2 \subset \mathbb{P}(\mathbb{C}^3)), \quad K_{\alpha_1}^\alpha(S) = F(2, 4, \mathbb{C}^6),$$

$$S_{\alpha_4}^\alpha = Gr(3, \mathbb{C}^5), \quad (Z_{\alpha_4}^\alpha \subset \mathbb{P}(g_{-1}(\alpha_4))) = (\mathbb{P}^2 \times \mathbb{P}^1 \subset \mathbb{P}(\mathbb{C}^3 \otimes \mathbb{C}^2)), \quad K_{\alpha_4}^\alpha(S) = F(1, 3, 5, \mathbb{C}^6).$$

In particular, $N_J(\alpha_4)$ consists of two roots, namely $\alpha_3$ and $\alpha_5$. The varieties $Z_{\alpha_3}^{\alpha_4} = \mathbb{P}^2$ and $Z_{\alpha_5}^{\alpha_4} = \mathbb{P}^1$ are the two factors of $Z_{\alpha_4}^\alpha$ respectively.

**Definition 2.9** The tangent bundle of $S$ is identified with $G \times P_I (g/p_I)$. Given $\alpha \in I$, we can define a $G$-invariant holomorphic distribution

$$g_\alpha^\alpha(S) := G \times P_I ((g_{-1}(\alpha) + p_I)/p_I).$$

In particular, $C^\alpha(S) \subset \mathbb{P}(g^\alpha(S))$. The $G$-invariant holomorphic distribution

$$g_{-1}(S) := G \times P_I ((g_{-1}(I) + p_I)/p_I).$$
satisfies that
\[ g_{-1}(S) = \bigoplus_{\alpha \in I} g_{\alpha}(S) = \sum_{\alpha \in I} g_{\alpha}(S) \subset TS. \]

Take \( \alpha \in I \) and \( \beta \in N_J(\alpha) \). Replacing \( S := G/P_J \) by the rational homogeneous space \( Z^\alpha = G_{\beta}/P_{J,\alpha} \), we can apply Definition 2.7 to obtain a family of rational curves \( K(\beta)(Z^\alpha) \) on \( Z^\alpha \) and its associated variety of tangent directions \( C(\beta)(Z^\alpha) \). As in Definition 2.9 we can construct the distribution \( g(\beta)(Z^\alpha) \) on \( Z^\alpha \), which is the minimal \( G_0 \)-invariant (hence \( G_J \)-invariant) distribution associated with the root \( \beta \in N_J(\alpha) \subset I \). Denote by \( \mathcal{Z}^\alpha \subset g_{-1}(\alpha) \) the affine cone of \( Z^\alpha \subset \mathbb{P}(g_{-1}(\alpha)) \). Then can define \( \mathcal{C}^\alpha(S), C(\beta)(\mathcal{Z}^\alpha), C(\beta)(\mathcal{Z}^\alpha) \) and \( g(\beta)(\mathcal{Z}^\alpha) \) in an obvious way.

The following result on automorphism groups is straightforward from Definition 2.7.

**Lemma 2.10** The natural homomorphism \( G \to \text{Aut}^\alpha(S) \) is bijective. Take \( \alpha \in I \) and let \( \text{Aut}^\alpha(\mathcal{Z}^\alpha, g_{-1}(\alpha)) \) be the identity component of \( \text{Aut}(\mathcal{Z}^\alpha, g_{-1}(\alpha)) := \{ \varphi \in \text{GL}(g_{-1}(\alpha)) \mid \varphi \cdot \mathcal{Z}^\alpha = \mathcal{Z}^\alpha \} \). Then the natural homomorphism \( G_0 \to \text{Aut}^\alpha(\mathcal{Z}^\alpha) \) is surjective and \( \text{Aut}^\alpha(\mathcal{Z}^\alpha) = \text{Aut}^\alpha(\mathcal{Z}^\alpha, g_{-1}(\alpha)) \). Take any \( \beta \in N_J(\alpha) \). Then the distribution \( g(\beta)(Z^\alpha) \) on \( Z^\alpha \) is \( \text{Aut}^\alpha(Z^\alpha) \)-invariant.

Given \( \alpha, \beta \in I, k \geq 1 \) and \( [C^\alpha] \in K^\alpha(S) \), we can describe the splitting types of the distributions \( g_{-k}(S) \) along \( C^\alpha \cong \mathbb{P}^1 \) on \( S \). To obtain it, we need to apply Grothendieck’s splitting theorem for principal bundles on \( \mathbb{P}^1 \), see [4]. The following statement of Grothendieck’s splitting theorem is not the version given in [4] but a consequence of [8, Proposition 15].

**Proposition 2.11** (Grothendieck). Let \( O(1)^* \) be the \( \mathbb{C}^* \)-principal bundle on \( \mathbb{P}^1 \) corresponding to the line bundle \( O(1) \). Let \( L \) be a reductive complex Lie group. Up to conjugation, any \( L \)-principal bundle on \( \mathbb{P}^1 \) is associated to \( O(1)^* \) by a group homomorphism from \( \mathbb{C}^* \) to a maximal torus of \( L \). If \( E \) is the positive coroot of \( sl(2) \), such a group homomorphism is determined by the image of \( E \) in \( h \), where \( h \) is a fixed Cartan subalgebra of the Lie algebra of \( L \). Given a representation of \( L \) with weights \( \mu_1, \ldots, \mu_k \in h^* \), the associated vector bundle on \( \mathbb{P}^1 \) splits as \( O(\mu_1(E)) \oplus \cdots \oplus O(\mu_k(E)) \), where \( \mu_j(E) \) denotes the value of \( \mu_j \) on the image of \( E \) in \( h \).

Given \( \alpha \in I \) and \( [C^\alpha] \in K^\alpha(S) \), we can identify \( C^\alpha \) with \( \exp(sl_\alpha(2))/\exp(g_\alpha \oplus [g_\alpha, g_\alpha]) \), where \( sl_\alpha(2) := g_\alpha \oplus g_\alpha \oplus [g_\alpha, g_\alpha] \subset g \) is a subalgebra isomorphic to \( sl(2) \), and \( g_\alpha \oplus [g_\alpha, g_\alpha] = p_I \cap sl_\alpha(2) \) is a Borel subalgebra.

**Proposition 2.12** Given \( \alpha, \beta \in I, k \geq 1 \) and \( [C^\alpha] \in K^\alpha(S) \), we have \( g_{-k}(S)|_{C^\alpha} = \bigoplus_{\gamma \in \Lambda_k(\beta)} O_{\mathbb{P}^1}((\gamma, \alpha)) \).

**Proof** Using the notations in Proposition 2.11, we have \( L = \exp(sl_\alpha(2)) \), and the coroot \( E \in [g_\alpha, g_\alpha] \) is the one associated with simple root \( \alpha \). Then the restriction of \( g_{-k}(S) \) on \( C^\alpha \) is the vector bundle induced by the \( \exp(sl_\alpha(2)) \)-module \( g_{-k}(\beta) \). Then the conclusion follows from Proposition 2.11. \( \square \)
2.2 Characterization of the nilradical of a parabolic subalgebra

We want to give a description of the algebra \( \mathfrak{g}_-(I) := \bigoplus_{k \geq 1} \mathfrak{g}_{-k}(I) \). When \( I = R \), it is described by Serre’s theorem on semisimple Lie algebra in the following way.

**Proposition 2.13** [5, Section 18] Let \( R \) be a set of simple roots for \( \mathfrak{g} \) and choose a nonzero element \( x_\alpha \in \mathfrak{g}_- \alpha \) for each \( \alpha \in R \). Then the subalgebra \( \mathfrak{g}_-(R) \) of \( \mathfrak{g} \) is the quotient of the free Lie algebra generated by \( \{x_\alpha \mid \alpha \in R\} \) by the relations \( \text{ad}(x_\alpha)^{-}(-\beta, \alpha) + 1(x_\beta) = 0 \) for all \( \alpha \neq \beta \in R \).

**Proposition 2.14** Denote by \( \mathbb{F}(\mathfrak{g}_-(I)) \) the free graded Lie algebra generated by \( \mathfrak{g}_-(I) \). Fix an arbitrary element \( z_\alpha \in \mathfrak{Z}^\alpha \setminus \{0\} \) for each \( \alpha \in I \). Let \( \mathcal{I} := \mathcal{I}(z_\alpha, \alpha \in I) \) be the ideal of \( \mathbb{F}(\mathfrak{g}_-(I)) \) generated by the following relations:

(i) \( (\text{ad}v')^{-}(\alpha', \alpha'') + 1(v'') = 0 \) for all \( \alpha' \neq \alpha'' \in I \) and all \( (v', v'') \in \mathfrak{g}_0 \cdot (z_{\alpha''}, z_{\alpha''}) \in \mathfrak{Z}^\alpha \times \mathfrak{Z}^{\alpha''} ; \)

(ii) \( (\text{ad}v)^{-}(-\beta, \alpha)(u) = 0 \) for all \( \alpha \in I, \beta \in N_J(\alpha), v \in \mathfrak{Z}^\alpha \setminus \{0\} \), and \( u \in \mathfrak{g}_0^\beta(\mathfrak{Z}^\alpha) \).

Then \( \mathbb{F}(\mathfrak{g}_-(I))/\mathcal{I} \) is isomorphic to \( \mathfrak{g}_-(I) := \bigoplus_{i \geq 1} \mathfrak{g}_{-i} \) as a graded nilpotent Lie algebra. In particular, up to isomorphism, the Lie algebra \( \mathbb{F}(\mathfrak{g}_-(I))/\mathcal{I}(z_\alpha, \alpha \in I) \) is independent of the choice of those \( z_\alpha \in \mathfrak{Z}^\alpha \setminus \{0\} \).

**Proof** Step 1. We will show that \( \mathfrak{g}_-(I) \) satisfies conditions (i) and (ii).

The inclusion \( \mathfrak{g}_-(R) \subset \mathfrak{g}_- \) induces a semidirect product decomposition of the Lie algebra structure \( \mathfrak{g}_-(R) = \mathfrak{n}_0 \rtimes \mathfrak{g}_-(I) \), where \( \mathfrak{n}_0 := \mathfrak{g}_-(R) \cap \mathfrak{g}_0(\mathfrak{I}) \). For each \( \alpha \in R \) we choose a nonzero element \( x_\alpha \in \mathfrak{g}_- \).

For those \( \alpha \in I \), we write \( z_\alpha := x_\alpha \). Since the point \( \mathbb{P}(\mathfrak{g}_-) \subset \mathbb{P}(\mathfrak{g}_-) \subset \mathbb{P}(\mathfrak{g}_-) \), we have \( z_\alpha \in \mathfrak{Z}^\alpha \setminus \{0\} \). For those \( \beta \in J := R \setminus I \), we have \( x_\beta \in \mathfrak{n}_0 \). Denote by \( \pi : \mathfrak{n}_0 \subset \mathfrak{g}_0(\mathfrak{I}) \rightarrow \text{aut}(\mathfrak{g}_-(\mathfrak{I})) \) the homomorphism induced by the adjoint representation, and write \( \eta_\beta := \pi(x_\beta) \in \text{aut}(\mathfrak{g}_-(\mathfrak{I})) \). Then by Proposition 2.13,

\[
\begin{align*}
(\text{ad}z_\alpha)^{-}(-\alpha', \alpha'') + 1(z_{\alpha''}) &= 0, \quad \text{for all } \alpha' \neq \alpha'' \in I, \\
(\text{ad}z_\alpha)^{-}(-\beta, \alpha)(\eta_\beta(z_\alpha)) &= 0, \quad \text{for all } \alpha \in I \text{ and } \beta \neq I.
\end{align*}
\]

Since the Lie algebra \( \mathfrak{g}_-(I) \) is a \( \mathfrak{g}_0 \)-module, the conclusion (2.5) implies the condition (i) in the statement of Proposition 2.14.

Now let us check the condition (ii). By (2.6), \( \eta_\beta(z_\alpha) = 0 \) for \( \beta \in J \setminus N_J(\alpha) \). Now suppose that \( \beta \in N_J(\alpha) \). Then \( \eta_\beta(v_\alpha) = [x_\beta, x_\alpha] \) is a nonzero vector in \( \mathfrak{g}_-(\alpha) = \sum_{\gamma \in \Lambda_1(\alpha)} \mathfrak{g}_\gamma \). Moreover, \( \mathbb{P}(\eta_\beta(z_\alpha)) \) is a point in \( \mathfrak{C}_{\mathfrak{Z}_\alpha}^\beta(\mathfrak{Z}^\alpha) \subset \mathbb{P}(\mathfrak{g}_-(\mathfrak{I})) \). Since \( \mathfrak{Z}_\alpha \) is a \( \mathfrak{g}_0 \)-module, we have \( (\text{ad}(\varphi \cdot z_\alpha))^{-}(-\beta, \alpha)(\varphi \cdot \eta_\beta(z_\alpha)) = 0 \) for all \( \varphi \in \mathfrak{g}_0 \). Denote by \( \mathfrak{P}_\beta := \mathfrak{P}_\beta \cap \mathfrak{g}_0 \). Then \( \mathfrak{P}_\beta \cdot \eta_\beta(z_\alpha) = \mathfrak{C}_{\mathfrak{Z}_\alpha}^\beta(\mathfrak{Z}^\alpha) \). Since \( \mathfrak{Z}_\alpha \) is the nondegenerate in the subspace \( \mathfrak{Z}^\alpha \) of \( \mathfrak{g}_-(\alpha) \), we have \( \text{ad}(z_\alpha)^{-}(-\beta, \alpha)(\mathfrak{U} \cdot u) = 0 \) for all \( \mathfrak{U} \in \mathfrak{Z}_\alpha^\beta(\mathfrak{Z}_\alpha) \). It follows that \( (\text{ad}(\varphi \cdot z_\alpha))^{-}(-\beta, \alpha)(\varphi \cdot u) = 0 \) for all \( \varphi \in \mathfrak{g}_0 \) and all \( \mathfrak{U} \in \mathfrak{Z}_\alpha^\beta(\mathfrak{Z}_\alpha) \). Since \( \mathfrak{Z}^\alpha \) is \( \mathfrak{g}_0 \)-transitive and \( \mathfrak{g}_0^\beta(\mathfrak{Z}^\alpha) \) is \( \mathfrak{g}_0 \)-equivariant, the condition (ii) holds.
Step 2. Show that the Lie algebra $\mathbb{F}(g_{-1}(I))/\mathcal{I}(z_\alpha, \alpha \in I)$ is independent of the choice of $z_\alpha \in \hat{Z}^a \setminus \{0\}$.

Now take any $z_\alpha' \in \hat{Z}^a \setminus \{0\}$ for each $\alpha \in I$. Since the inclusion $\hat{Z}^a \subset g_{-1}(\alpha)$ is $G_0$-equivariant and $\hat{Z}^a \setminus \{0\}$ is a single $G_0$-orbit, there exists an isomorphism $\varphi^a : g_{-1}(\alpha) \rightarrow g_{-1}(\alpha)$ of $G_0$-modules sending $\hat{Z}^a$ onto itself and $\varphi^a(z_\alpha) = z_\alpha'$. These $\varphi^a$ induce an isomorphism $\mathbb{F}(g_{-1}(I)) \rightarrow \mathbb{F}(g_{-1}(I))$, whose restriction sends $\mathcal{I}(z_\alpha, \alpha \in I)$ onto $\mathcal{I}(z_\alpha', \alpha \in I)$. It follows that $\mathbb{F}(g_{-1}(I))/\mathcal{I}(z_\alpha, \alpha \in I) \cong \mathbb{F}(g_{-1}(I))/\mathcal{I}(z_\alpha', \alpha \in I)$.

Step 3. Show the isomorphism $\mathbb{F}(g_{-1}(I))/\mathcal{I} \cong g_{-}(I)$.

By Step 1 and Step 2, we can set $z_\alpha := x_\alpha \in \hat{Z}^a \setminus \{0\}$ for each $\alpha \in I$ and get a surjective homomorphism of $G_0$-modules $\psi : \mathbb{F}(g_{-1}(I))/\mathcal{I} \rightarrow g_{-}(I)$. It should be noticed that $p_\gamma := g_0(I) \oplus g_{-}(I) = \oplus_{i \leq 0} g_i(I)$ and $F := g_0(I) \oplus (\mathbb{F}(g_{-1}(I))/\mathcal{I}) \cong (g_0(I) \oplus \mathbb{F}(g_{-1}(I)))/\mathcal{I}$ are both graded Lie algebras as well as $G_0$-modules. Moreover, $\psi$ induces a surjective homomorphism between Lie algebras as well as between $G_0$-modules: $\psi : F \rightarrow p_\gamma$.

Similarly as in Step 1 let $n_0$ be the Lie subalgebra of $g_0(I)$ generated by those $x_\beta$ with $\beta \in J$. We have $g_{-}(R) = n_0 \oplus g_{-}(I) \subset p_\gamma$, and set $F := n_0 \oplus (\mathbb{F}(g_{-1}(I))/\mathcal{I}) \subset F$. Then the restriction of $\psi$ induces a surjective homomorphism of Lie algebras $\psi : \tilde{F} \rightarrow \mathbb{F}(g_{-}(R))$.

Denote by $\tilde{F}$ the free graded Lie algebra generated by those $x_\gamma$ with $\gamma \in R$. Let $\tilde{I}$ be the ideal of $\tilde{F}$ generated by the set $\{\langle ad x_{\gamma'} \rangle^{-\langle \gamma'', \gamma' \rangle + 1}(x_{\gamma''}) | \gamma' \neq \gamma'' \in R \}$. There is a commutative diagram of Lie algebras as follows:

\[
\begin{array}{ccc}
\tilde{F} & \overset{\theta_1}{\longrightarrow} & \mathbb{F}(g_{-1}(I))/\mathcal{I} \cong g_{-}(I) \\
& \searrow \theta_2 & \\
& & \mathbb{F}(g_{-1}(I))/\mathcal{I} \cong g_{-}(I)
\end{array}
\]

We claim that $\theta_1(\tilde{I}) = 0$. Equivalently, we claim that for all $\gamma' \neq \gamma'' \in R$,

$$\theta_1((\text{ad} x_{\gamma'})^{-\langle \gamma'', \gamma' \rangle + 1}(x_{\gamma''})) = 0. \quad (2.7)$$

Case 1. Suppose $\gamma', \gamma'' \in I$. Recall our definition of $\mathcal{I}$ for $z_\alpha := x_\alpha \in \hat{Z}^a \setminus \{0\}$. Then in this case (2.7) follows from the condition (i) of Proposition 2.14.

Case 2. Suppose $\gamma' \in I$ and $\gamma'' \in J := R \setminus I$. The condition (iii) of Proposition 2.14 implies (2.7) under the additional assumption that $\gamma'' \in N_J(\gamma')$. Here it should be noticed that $\theta_1(x_{\gamma'}) \in \mathbb{F}(g_{-1}(I))$ and $\theta_1(x_{\gamma''}) \in n_0$.

Now for $\gamma' \in I$ and $\gamma'' \in J \setminus N_J(\gamma')$, we have $\langle \gamma'', \gamma' \rangle = 0$. Then (2.7) is equivalent to $[\theta_1(x_{\gamma''}), \theta_1(x_{\gamma'})]_F = 0$. The latter can be deduced from the $g_0$-action (hence the $n_0$-action) on $g_{-1}(I)$.

Case 3. Suppose $\gamma' \in J := R \setminus I$ and $\gamma'' \in I$. Similarly, in this case (2.7) can also be deduced from the $g_0$-action (hence the $n_0$-action) on $g_{-1}(I)$.

Case 4. Suppose $\gamma', \gamma'' \in J := R \setminus I$. In this case (2.7) can also be deduced from the Lie algebra structure of $n_0$ (coming from that of $g_0$).
In summary, the claim $\theta_1(\tilde{I}) = 0$ holds. Then it induces a homomorphism $\tilde{\theta}_1 : \tilde{\mathbb{P}}/\tilde{I} \to \tilde{\mathcal{F}}$.

By the construction of $\tilde{\mathcal{F}}$, the morphism $\theta_1$ is surjective. Hence $\tilde{\theta}_1$ is surjective. By Proposition 2.13, $\theta_2$ induces an isomorphism $\tilde{\theta}_2 : \tilde{\mathbb{P}}/\tilde{I} \cong \mathfrak{g}_- (R)$. Hence $\tilde{\psi}$ is an isomorphism preserving gradings, and its restriction gives an isomorphism of graded nilpotent Lie algebras $\tilde{\mathbb{P}}(\mathfrak{g}_{-1}(I))/\tilde{I} \cong \mathfrak{g}_-(I)$.

\section{3 Reduction to the rigidity of homogeneous submanifolds}

From now on, we study the family $\mathcal{X}$ over $\Delta$ in Setting 1.7, and set $J := R \setminus I$. The organization of this section is as follows. In Sect. 3.1, we study the basic properties of minimal rational curves and Cartier divisors on the family $\mathcal{X}/\Delta$. In Sect. 3.2, we study the property of symbol algebras and prove Proposition 1.8, which is reformulated in Proposition 3.19. In Sect. 3.3, we prove Theorem 3.22, which implies Proposition 1.3 as a corollary.

\subsection{3.1 Minimal rational curves on the family}

The following result is due to Wiśniewski [19].

**Proposition 3.1** [19, Theorem 1] We can identify the Mori cones $\overline{NE}(\mathcal{X}/\Delta) = \overline{NE}(\mathcal{X}_t)$ for all $t \in \Delta$.

The following is a classical result on the rational homogeneous space $S := G/P_I$. This statement can be easily deduced from [14, Proposition 2.6], and one can find a proof of it in [2, Proposition 1.3.6] when $G$ is of type $A_n$. For the readers’ convenience, we provide a proof of it.

**Lemma 3.2** The Mori cone $\overline{NE}(S)$ is a simplicial cone generated by those $R_\alpha := \mathbb{R}^+ [K_\alpha (S)]$ with $\alpha \in I$ i.e. dim $\overline{NE}(S)$ equals to the cardinality of $I$, and $\overline{NE}(S) = \sum_{\alpha \in I} R_\alpha$, where $K_\alpha (S)$ is as in Definition 2.7. The set of extremal faces of $\overline{NE}(S)$ can be identified with the set of subsets of $I$.

**Proof** Since $S$ is Fano, the Mori cone $\overline{NE}(S)$ is equal to $\sum_{i=1}^m \gamma_i$, where $m$ is some positive integer and $\gamma_1, \ldots, \gamma_m$ are all the extremal rays of $\overline{NE}(S)$. Let $\phi_i : S \to S_i$ be the Mori contraction associated with $\gamma_i$. Since $\phi_i$ has connected fibers, the $G$-action on $S$ descends onto $S_i$ and thus $S_i$ is $G$-transitive. There is a proper subset $I_i$ of $I$ such that $S_i = G/P_{I_i}$ and $\phi_i$ is the map induced by the inclusion $P_I \subset P_{I_i}$. The formula $\rho(S_i) = \rho(S) - 1$ on the Picard numbers of $S$ and $S_i$ implies that there is a unique $\alpha \in I$ such that $I_i = I \setminus \{\alpha\}$ and thus $\gamma_i = R_\alpha$.

Now we know $\overline{NE}(S) = \sum_\alpha R_\alpha$ and thus $\rho(S) = \dim \overline{NE}(S) \leq |I|$, where $|I|$ is the cardinality of the set $I$. Now let us show $\rho(S) = |I|$ by induction on $|I|$. When $I$ contains a single element $\alpha$, it is known that $\rho(G/P_\alpha) = 1$. Now suppose $|I| \geq 2$ and take $\alpha \in I$. Then $S = G/P_I \to G/P_{I \setminus \{\alpha\}}$ is a fiber bundle such that it fiber at the base point is a rational homogeneous space of Picard number one. By induction $\rho(G/P_{I \setminus \{\alpha\}}) = |I| - 1$ and thus $\rho(S) = \rho(G/P_{I \setminus \{\alpha\}}) + 1 = |I|$. This completes the proof.

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As a direct consequence of Proposition 3.1 and Lemma 3.2, we have the following result.

**Proposition-Definition 3.3** For each \( A \subset I \), denote by 
\[ \Phi^A: S \to G/P_I \backslash A \]
the Mori contraction associated with the extremal face \( \sum_{\alpha \in A} R_\alpha \) of \( \overline{NE}(S) \). We can extend it to a relative Mori contraction
\[ \pi^A: X \to X^A. \]
We denote by \( \pi^A_t := \pi^A|_{X^A_t} \) for each \( t \in \Delta \).

**Notation 3.4** Given a subset \( A \subset I \) and a point \( x \in X \), denote by \( F^A_x \) the fiber of \( \pi^A: X \to X^A \) passing through \( x \). In particular, if \( x \notin X_0 \) then \( F^A_x \cong S^A \), where \( S^A \) is defined in Definition 2.7.

**Proposition 3.5** Take any \( \alpha \in I \). Then \( F^\alpha_x \cong S^\alpha \) for \( x \in X_0 \) general.

**Proof** The fiber \( F^\alpha_x \) is a smooth Fano deformation of \( S^\alpha \). Then the conclusion follows from the Fano deformation rigidity of \( S^\alpha \), which is obtained by Hwang and Mok [9, Main Theorem]. \( \square \)

By Proposition 3.1, Proposition 3.5 and the intersection theory on rational homogeneous spaces, we have the following result.

**Proposition-Definition 3.6** Take any \( \alpha \in I \). Denote by \( K^\alpha(X) \) the irreducible component of \( \text{Chow}(X) \) extending \( K^\alpha(S) \). Take any \( [C] \in K^\alpha(X) \). Then \( C \) is an irreducible and reduced rational curve on \( X_t \) for a unique \( t \in \Delta \). If either \( t \neq 0 \) or \( [C] \) is general in \( K^\alpha(X_0) \), then \( C \cong \mathbb{P}^1 \). Moreover, there exists a unique \( L^\alpha \in \text{Pic}(X/\Delta) \) such that 
\[ (L^\alpha \cdot K^\beta(X)) = \delta_{\alpha\beta} \text{ for all } \beta \in I. \]
We also have \( \overline{NE}(X_t) = \sum_{\beta \in I} \mathbb{R}^+[K^\beta(X_t)] \) for each \( t \in \Delta \).

**Proof** Take \( [C^\alpha] \in K^\alpha(S) \) for \( \alpha \in I \). There exist nef line bundles \( L^\alpha \in \text{Pic}(S) \) such that 
\[ (L^\alpha \cdot C^\beta) = \delta_{\alpha\beta} \text{ for } \alpha, \beta \in I, \]
and \( L := \bigotimes_{\alpha \in I} L^\alpha \) is an ample line bundle on \( S \). Note that any \( [C] \in K^\alpha(X) \) is a deformation of \( [C^\alpha] \in K^\alpha(X_0) \) with \( t \neq 0 \). There exist \( L^\alpha, L \in \text{Pic}(X/\Delta) \) such that \( L^\alpha|_{X_t} = L^\alpha \) and \( L|_{X_t} = L \) for \( t \neq 0 \). By Proposition 3.1, \( L^\alpha|_{X_0} \) is nef and \( L|_{X_0} \) is ample. Take any \( [C] \in K^\beta(X) \). Then 
\[ (L^\alpha \cdot C) = \delta_{\alpha\beta}. \]
Then the fact \( (L \cdot C) = 1 \) implies that \( C \) is irreducible and reduced. The rest is straightforward. \( \square \)

**Proposition 3.7** Any two points \( x, y \in X_0 \) can be connected by the chains of elements in \( \bigcup_{\alpha \in I} K^\alpha(X_0) \).

**Proof** Consider the dominant rational map \( \psi: X_0 \dasharrow Z \) defined by equivalence relation induced by \( \bigcup_{\alpha \in I} K^\alpha(X_0) \). For the existence and the properties of such a rational map, see [11, Theorem IV.4.16]. Suppose \( \dim Z \geq 1 \). Take a general divisor
$D \subset Z$ and a general point $x \in Z \setminus D$. Then $\psi^{-1}(z)$ is a closed subvariety of $X_0$ which has the empty intersection with the indeterminant locus of $\psi$. Thus $\psi^{-1}(z) \cap E = \emptyset$, where $E := \psi^{-1}(D) \subset X_0$ is an effective divisor. For each $x \in \psi^{-1}(z)$ and each $\alpha \in I$, we have $\mathcal{K}^\alpha_{\psi}(X_0) \neq \emptyset$. By definition $C^\alpha_x \subset \psi^{-1}(z)$ (hence $C^\alpha_x \cap E = \emptyset$) for all $[C^\alpha_x] \in \mathcal{K}^\alpha_x(X_0)$. It follows that $(E \cdot \mathcal{K}^\alpha_{\psi}(X_0)) = 0$ for all $\alpha \in I$. Since $N_1(X_0)$ is generated by $[\mathcal{K}^\alpha_{\psi}(X_0)]$ with $\alpha \in I$, we have $\mathcal{O}_{X_0}(E) = \mathcal{O}_{X_0} \in \text{Pic}(X_0)$ by Proposition-Definition 3.6. It contradicts the choice of $E$. Then the conclusion follows.

3.2 Properties of the symbol algebras

**Definition 3.8** Given a distribution $\mathcal{V}$ on a complex manifold $Y$, define the weak derived system $\mathcal{V}^{−k}$ inductively by

$$
\mathcal{V}^{0} := 0, \quad \mathcal{V}^{−1} := \mathcal{V}, \quad \text{and} \quad \mathcal{V}^{−k−1} := \mathcal{V}^{−k} + [\mathcal{V}^{−1}, \mathcal{V}^{−k}] \quad \text{for} \quad k \geq 1.
$$

Denote by $\mathcal{V}^{−\infty} := \lim_{k \to \infty} \mathcal{V}^{−k}$. There exists a positive integer $d$ such that $\mathcal{V}^{−d−i} = \mathcal{V}^{−d}$ for all $i \geq 0$. In particular, $\mathcal{V}^{−\infty} = \mathcal{V}^{−d}$ and it is integrable on $Y$. In an open neighborhood of a general point $y \in Y$ these $\mathcal{V}^{−k}$’s are subbundles of $TY$. We define the symbol algebra of $\mathcal{V}$ at $y$ as the graded nilpotent Lie algebra

$$
\text{Symb}_y(\mathcal{V}) := \bigoplus_{k=1}^{d} \mathcal{V}^{−k}_y / \mathcal{V}^{−k+1}_y.
$$

We say $\mathcal{V}$ is bracket-generating if $\mathcal{V}^{−\infty} = TY$. In this case, $\dim \text{Symb}_y(\mathcal{V}) = \dim T_y Y = \dim Y$.

**Notation 3.9** Take a subset $A \subset I$. The distribution

$$
g^A_{−1}(S) := G \times_{P_I} \left( \sum_{\alpha \in A} (g_{−1}(\alpha) + p_I) / p_I \right)
$$

on $S$ can be extended to a meromorphic distribution $D^A$ on $\mathcal{X}$, which is holomorphic outside a proper closed subset of $X_0$. Take a general point $x \in X_0$. Denote by $\text{m}_x(A)$ the symbol algebra of $D^A_x$, i.e.

$$
\text{m}_x(A) := \text{Symb}_x(D^A).
$$

We say $\text{m}_x(A)$ is standard if it is isomorphic to the symbol algebra of the distribution $g^A_{−1}(S)$ on $S$. Otherwise, we say $\text{m}_x(A)$ is degenerate. When $A = I$, we omit the superscript $I$ and write $D := D^I$ briefly.

**Proposition 3.10** The unique integrable meromorphic distribution on $X_0$ containing $D$ is the tangent bundle. Hence the distribution $D$ is bracket-generating on $X_0$ and $\dim \text{m}_x(I) = \dim X_0$ for $x \in X_0$ general.
Proof We take an analogue of the proof of [8, Proposition 13]. Let $\mathcal{E}$ be the minimal integrable meromorphic distribution on $\mathcal{X}_0$ containing $\mathcal{D}$ and $U \subset \mathcal{X}_0$ be the regular locus of $\mathcal{E}$. Suppose $\mathcal{E} \neq T\mathcal{X}_0$. Now we write $I = \{\beta_1, \ldots, \beta_\rho\}$, where $\rho$ is the Picard number of $S := G/P_I$. By Proposition-Definition 3.3, there is an elementary Mori contraction $\pi^j_0 : \mathcal{X}_0 \to \mathcal{X}^{\beta_j}_0$ for each $j$. Let $U'$ be the maximal dense open subset of $U$ such that for each $j$ and each $x \in U'$ the fiber of $\pi^j_0$ passing through $x$ is a smooth connected variety whose dimension is equal to $\dim \mathcal{X}_0 - \dim \mathcal{X}^{\beta_j}_0$.

Take any $x \in U'$, and set $V_0(x) := \{x\}$. Now we define a variety $V_k(x)$ for each positive integer $k$ by induction. Let $s$ be the unique integer such that $1 \leq s \leq \rho$ and $k \equiv s$ (modæ). Define $V_k(x)$ to be the Zariski closure of $(\pi_0^{\beta_s})^{-1}(\pi_0^{\beta_s}(V_{k-1}(x) \cap U'))$ in $\mathcal{X}_0$. This gives rise to a chain of irreducible closed subvarieties $V_0(x) \subset V_1(x) \subset V_2(x) \subset \cdots$ such that each $V_k(x)$ is tangent to $\mathcal{E}$ along its intersection with $U$. By dimension counting, there is a positive integer $m_x$ depending on $x \in U'$ such that $V_{m_x+i}(x) = V_{m_x}(x)$ for all $i \geq 1$. Given any $y \in V_{m_x}(x) \cap U'$, then the construction indicates that $V_k(y) \subset V_{m_x}(x)$ for each $k \in \mathbb{N}$ and thus $V_{m_x}(y) \subset V_{m_y}(y) \subset V_{m_x}(x)$. If $x$ is moreover a general point in $\mathcal{X}_0$, then the semicontinuity of the dimension functions on the fibers implies that $V_{m_x}(y) = V_{m_y}(y) = V_{m_x}(x)$ and $m_y \leq m_x$. There is a common positive integer $m$ such that $m_x = m$ for $x \in \mathcal{X}_0$ general.

Now we can define an integrable meromorphic distribution $\mathcal{E}'$ on $U'$ such that $\mathcal{E}'_x$ is the tangent space of $V_{m_x}$ at $x$ for $x \in U'$ general. By the construction we have $\mathcal{E}' = T_x V_{m_x}(x) \subset \mathcal{E}_x$ for $x \in U'$ general. Moreover, we have $D_x = \sum_{\alpha \in I} D^\alpha_x \subset T_x V_\rho(x) \subset \mathcal{E}'_x$. The minimality of $\mathcal{E}$ implies that $\mathcal{E}|_{U'} = \mathcal{E}'$ and the leaf of $\mathcal{E}$ at a general point $x \in \mathcal{X}_0$ is $\mathcal{X}_0$. There is an irreducible closed subvariety $Z$ of the Chow space of $\mathcal{X}_0$ whose general elements representing these varieties $V_{m_x}(x)$ with $x \in \mathcal{X}_0$ being general. Let $\mu : U \to Z$ be the universal family and $e : U \to \mathcal{X}_0$ be the evaluation map. Then the argument above shows that $e$ is birational and we obtain a dominant rational map $\psi := \mu \circ e^{-1}$ from $\mathcal{X}_0$ to $Z$. Let $E \subset \mathcal{X}_0$ be the undetermined locus of $\psi$. Then $E$ has codimension at least two in $\mathcal{X}_0$, and we have a dominant morphism $\psi : \mathcal{X}_0 \setminus E \to Z$.

By the assumption that $\mathcal{E} \neq T\mathcal{X}_0$, we have $\dim Z \geq 1$. Take a prime divisor $D$ on $Z$ such that $\psi^{-1}(D)$ is a nonempty subset of $\mathcal{X}_0 \setminus E$. Let $H$ be the Zariski closure of $\psi^{-1}(D)$ in $\mathcal{X}_0$. For each $\beta_j \in I$, we take a general element $[C_j] \in \mathcal{K}^{\beta_j}(\mathcal{X}_0)$. Recall the fact that general members of a family of unbending rational curves on a uniruled projective manifold lie in the complement of any fixed closed subset of codimension at least two, see [7, Lemma 2.1]. Then $C_j \cap E = \emptyset$ and $\psi$ is well-defined on $C_j$. Take a general point $x_j \in C_j$. Then $z_j := \psi(x_j)$ is a general point in $Z$. As a point in the Chow space, $z_j$ represents the closed subvariety $V_m(x_j)$ of $\mathcal{X}_0$. The fiber $\psi^{-1}(z_j)$ is an open dense subset of $V_m(x_j)$, and the irreducible rational curve $C_j \subset V_j(x_j) \subset V_m(x_j)$. Then $\psi$ sends $C_j$ to a single point $z_j$, implying that $C_j \cap \psi^{-1}(D) = \emptyset$. It follows that $C_j \cap H = \emptyset$ and $(H \cdot \mathcal{K}^{\beta_j}(\mathcal{X}_0)) = 0$ for each $\beta_j \in I$. Then by Proposition-Definition 3.6 we have $\mathcal{O}_{\mathcal{X}_0}(H) = \mathcal{O}_{\mathcal{X}_0} \in \text{Pic}(\mathcal{X}_0)$, which contradicts with the choice of $H$. This implies that $\mathcal{E}$ must be the tangent bundle of $\mathcal{X}_0$. \qed

To continue, we need to recall some concepts and results related with Cartan connections.
Definition 3.11  Fix a positive integer \( n \). Let \( \mathfrak{l}_n = \mathfrak{l}_{-1} \oplus \cdots \oplus \mathfrak{l}_{-n} \) be a graded nilpotent Lie algebra. Denote by \( \text{grAut}(\mathfrak{l}_n) \) the group of Lie algebra automorphisms of \( \mathfrak{l}_n \) preserving the gradation. Let \( \text{graut}(\mathfrak{l}_n) \) be its Lie algebra. Fix a connected algebraic subgroup \( L_0 \subset \text{grAut}(\mathfrak{l}_n) \) and its Lie algebra \( \mathfrak{l}_0 \subset \text{graut}(\mathfrak{l}_n) \). By induction, we define the \( i \)-th prolongation \( \mathfrak{l}_i \) of \( \mathfrak{l}_0 \) to be the set of \( \phi \in \bigoplus_{k=1}^{n} \text{Hom}(\mathfrak{l}_{-k}, \mathfrak{l}_{-k+i}) \) such that \( \phi([v_1, v_2]) = \phi(v_1)(v_2) - \phi(v_2)(v_1) \) for any \( v_1, v_2 \in \mathfrak{l}_n \), where \( i \geq 1 \), and \([ , , ]\) denotes the Lie bracket on \( \mathfrak{l}_n \). In particular, if \( \phi(v_1) \in \mathfrak{l}_n \), then \( \phi(v_1)(v_2) := [\phi(v_1), v_2]_{\mathfrak{l}_n} \). For convenience, we put \( \mathfrak{l}_{-0} = \mathfrak{l}_0 \) for every positive integer \( j \) and write \( \mathfrak{l}_n = \bigoplus_{k \in \mathbb{N}} \mathfrak{l}_{-k} \). The graded vector space \( \mathfrak{l} := \bigoplus_{k \in \mathbb{Z}} \mathfrak{l}_k \) is a graded Lie algebra and called the universal prolongation of \( (\mathfrak{l}_0, \mathfrak{l}_n) \).

The following result on prolongations is due to Yamaguchi [20].

Proposition 3.12  [20, Theorem 5.2] In the situation of Setting 2.1, suppose that \( G \) is simple.

(i) Suppose \( G/P_1 \) is not a projective space. Then \( \mathfrak{g} \) is the universal prolongation of \( (\mathfrak{g}_0(I), \mathfrak{g}_0(I)). \)

(ii) Suppose \( |I| \geq 2 \) and \( (G, I) \) is not one of the following:

\[
(A_m, \{\alpha_i, \alpha_i\}), \quad 2 \leq i \leq m; \tag{3.1}
\]

\[
(A_m, \{\alpha_i, \alpha_m\}), \quad 1 \leq i \leq m - 1. \tag{3.2}
\]

Then \( \mathfrak{g}_0(I) \) is isomorphic to \( \text{aut}(\mathfrak{g}_0(I)) \), the Lie algebra of \( \text{grAut}(\mathfrak{g}_0(I)). \)

Definition 3.13  Let \( L \) be a connected algebraic group, \( L_0 \subset L \) be a connected algebraic subgroup, and \( \mathfrak{l}_0 \subset \mathfrak{l} \) be their Lie algebras. A Cartan connection of type \( (L, L_0) \) on a complex manifold \( M \) with \( \dim M = \dim L/L_0 \) is a principal \( L_0 \)-bundle \( E \to M \) with an \( \mathfrak{l} \)-valued 1-form \( \gamma \) on \( E \) satisfying the following properties.

(i) For \( A \in \mathfrak{l}_0 \), denote by \( \xi_A \) the fundamental vector field on \( E \) induced by the right \( L_0 \)-action on \( E \). Then \( \gamma(\xi_A) = A \) for each \( A \in \mathfrak{l}_0 \).

(ii) For \( a \in L_0 \), denote by \( R_a : E \to E \) the right action of \( a \). Then \( R_a^* \gamma = \text{Ad}(a^{-1}) \circ \gamma \) for each \( a \in L_0 \).

(iii) The linear map \( \gamma_y : T_y E \to \mathfrak{l} \) is an isomorphism for each \( y \in E \).

The Cartan connection \( (E \to M, \gamma) \) is flat if the curvature \( \kappa := d\gamma + \frac{1}{2}[[\gamma, \gamma]] \) vanishes.

Example 3.14  Let \( L \) and \( L_0 \) be as in Definition 3.13, and denote by \( \omega^{MC} \) the Maurer–Cartan form on \( L \). Then \( (L \to L/L_0, \omega^{MC}) \) is a flat Cartan connection of type \( (L, L_0) \).

Definition 3.15  Let \( \mathfrak{l}_n = \bigoplus_{k \in \mathbb{N}} \mathfrak{l}_{-k} \) be a graded nilpotent Lie algebra with \( \mathfrak{l}_{-j} = 0 \) for all \( j \) larger than a fixed positive integer \( n \). A filtration of type \( \mathfrak{l}_n \) on a complex manifold \( M \) is a filtration \( (F^j M, j \in \mathbb{Z}) \) on \( M \) such that

(i) \( F^j M \subset F^j M \) for \( i \geq j \);

(ii) \( F^k M = 0 \) for all \( k \geq 0 \);
(iii) $F^{-k}M = TM$ for all $k \geq v$; and
(iv) for any $x \in M$, the symbol algebra $\text{gr}_x(M) := \bigoplus_{i \in \mathbb{N}} F^{-i}x/M/F^{-i+1}xM$ is isomorphic to $\mathfrak{l}_-$ as graded Lie algebras.

The graded frame bundle of the manifold $M$ with a filtration of type $\mathfrak{l}_-$ is the $\text{grAut}(\mathfrak{l}_-)$-principal bundle $\text{grFr}(M)$ on $M$, whose fiber at $x$ is the set of graded Lie algebra isomorphisms from $\mathfrak{l}_-$ to $\mathfrak{g}_x(M)$. Let $L_0 \subset \text{grAut}(\mathfrak{l}_-)$ be a connected algebraic subgroup. An $L_0$-structure (subordinate to the filtration) on $M$ means an $L_0$-principal subbundle $E \subset \text{grFr}(M)$.

**Remark 3.16** Now let us summarize the work of Čap and Schichl [3] on the construction of the Cartan connections of type $(G, P_I)$. For more details, see Sects. 3.20–3.23 in [3]. Let $G/P_I$ be as in Setting 2.1 and suppose that $\mathfrak{g}$ is the universal prolongation of $(\mathfrak{g}_-(I), \mathfrak{g}_0(I))$. Suppose there is a differential system $\mathcal{V}$ and a principal bundle $E$ on a complex manifold $M$ such that the weak derivatives of $\mathcal{V}$ induces a filtration of type $\mathfrak{g}_-(I)$ and $E \subset \text{grFr}(M)$ is a $G_0$-structure on $M$. Then we can construct a Cartan connection of type $(G, P_I)$ on $M$. The construction is canonical in the sense that it works well for a family, which will be explained in the proof of Proposition 3.18, and that the Cartan connection we construct on $G/P_I$ itself is $(G \to G/P_I, \omega^{MC})$.

Now we introduce a setting that is slightly more general than Setting 1.7.

**Setting 3.17** In the situation of Setting 2.1, suppose that $G$ is simple and $G/P_I$ is not biholomorphic to a projective space. Let $\psi : \mathcal{V} \to \Delta \ni 0$ be a holomorphic map from an irreducible analytic variety $\mathcal{V}$ to $\Delta$ such that $\mathcal{V}_t \cong G/P_I$ for $t \neq 0$ and $\mathcal{V}_0$ is an irreducible reduced projective variety.

**Proposition 3.18** In the situation of Setting 3.17, suppose that there exists a proper closed algebraic subset $Z \subset \mathcal{V}_0$ and a holomorphic fiber bundle $E \to \mathcal{V}_0 \setminus Z$ such that $\mathfrak{m}_x(I) \cong \mathfrak{g}_-(I)$ for all $x \in \mathcal{V}_0 \setminus Z$ and $E_t \to \mathcal{V}_t \setminus Z$ is a $G_0$-structure for all $t \in \Delta$. Then $\mathcal{V}_0 \cong G/P_I$.

**Proof** By Proposition 3.12, the Lie algebra $\mathfrak{g}$ is the universal prolongation of $(\mathfrak{g}_-, \mathfrak{g}_0)$. By Sects. 3.20–3.23 [3], there exists a principal $P_I$-bundle $\Psi : \mathcal{P} \to \mathcal{V}' := \mathcal{V}/Z$ and a holomorphic 1-form $\omega : T\mathcal{P} \to \mathfrak{g}$ such that $(\Psi_t, \omega_t)$ is the Cartan connection $(G \to G/P_I, \omega^{MC})$ for each $t \neq 0$, and $(\Psi_0, \omega_0)$ is also a Cartan connection of type $(G, P_I)$. By the continuity on $t \in \Delta$ of the curvature $\kappa_t := d\omega_t + \frac{1}{2}[\omega_t, \omega_t]$, the Cartan connection $(\Psi_0, \omega_0)$ is also flat.

Given $A \in \mathfrak{g}$, we define $\tilde{A} \in H^0(\mathcal{P}, T_{\mathcal{P}/\Delta})$ such that $\tilde{A}_p = \omega_p^{-1}(A)$ for each $p \in \mathcal{P}$. The vector field $\tilde{A}$ descends to be a vector field $\eta^A_\Delta$ on the family $\mathcal{V}'/\Delta$, i.e. $\eta^A_\Delta \in H^0(\mathcal{V}', T_{\mathcal{V}'/\Delta})$. By Hargtogs extension theorem, $\eta^A_\Delta$ extends to a global vector field $\eta_A$ on the family $\mathcal{V}/\Delta$. It induces an injective homomorphism of Lie algebras $\phi : \mathfrak{g} \to H^0(\mathcal{V}_0, T\mathcal{V}_0) = \text{aut}(\mathcal{V}_0)$ by sending each $A \in \mathfrak{g}$ to $\eta_A|_{\mathcal{V}_0}$, where the preservation of the Lie bracket is implied by the flatness of Cartan connections. The isotropic subalgebra of $\mathfrak{g}$ at any point $y \in \mathcal{V}_0 \setminus Z$ is conjugate to $p_I$. It follows that $G/P_I \subset \mathcal{V}_0$, implying $\mathcal{V}_0 = G/P_I$ by dimension counting. \hfill \Box

**Proposition 3.19** In the situation of Setting 3.17, suppose that $|I| \geq 2$ and $(G, I)$ is neither (3.1) nor (3.2) listed in Proposition 3.12. Then the followings are equivalent:
\( \mathcal{Y}_0 \cong G/P_1; \)
\( m_x(I) \) is standard at general points \( x \in \mathcal{Y}_0. \)

**Proof** It is straightforward to see that (i) implies (ii). Now let us prove that (ii) implies (i). Let \( \mathcal{Y}^0 \) be the open subset of \( \mathcal{Y} \) where the symbol algebras of \( \mathcal{D} \) are isomorphic to \( g_-(I) \). In particular, \( \mathcal{Y}_t \subset \mathcal{Y}^0 \) for all \( t \neq 0 \) and \( \mathcal{Y}_0^0 \) is a dense Zariski open subset of \( \mathcal{Y}_0 \). Denote by \( \mathcal{F} \) a connected component of the graded frame bundle of the family \( \mathcal{Y}_t \) with \( t \neq 0 \) is holomorphically extended to the \( G_0 \)-structure \( \mathcal{F}_0 \) on \( \mathcal{Y}_0^0 \). Then the conclusion follows from Proposition 3.18.

The key point to obtain \( \mathcal{Y}_0 \cong G/P_1 \) in Setting 3.17 is invariance of symbol algebras. Once this is done, it is not hard to extend the \( G_0 \)-structure \( E \subset \text{grFr}(G/P_1) \) holomorphically to general points on \( \mathcal{Y}_0 \), even in case (3.1) or (3.2) listed in Proposition 3.12.

**Proposition 3.20** In the situation of Setting 3.17, suppose that \( S \cong A_m/P_1 \) and \( m_x(\alpha_1, \alpha_2) \cong g_-(I) \), where \( m \geq 2 \), \( I = \{\alpha_1, \alpha_2\} \), and \( x \in \mathcal{Y}_0 \) is general. Then \( \mathcal{Y}_0 \cong A_m/P_1. \)

**Proof** Being the relative tangent sheaves of fibrations, the distributions \( \mathcal{D}^{\alpha_1} \) and \( \mathcal{D}^{\alpha_2} \) are integrable on \( \mathcal{Y}_0 \). Thus the isomorphism \( m_x(\alpha_1, \alpha_2) \cong g_-(I) \) implies that \( F : \mathcal{D}^{\alpha_1} \otimes \mathcal{D}^{\alpha_2} \to T_x/\mathcal{D} \) is surjective on the general point \( x \in \mathcal{Y}_0 \), where \( F \) is the restriction of the Frobenius bracket of \( \mathcal{D} = \mathcal{D}^{\alpha_1} + \mathcal{D}^{\alpha_2} \subset T_x. \)

Denote by \( Z \subset \mathcal{Y} \) the set of points \( z \) such that \( m_z(I) \not\cong g_-(I) \). Then \( Z \) is a proper closed algebraic subset of \( \mathcal{Y}_0 \). Take any \( y \in \mathcal{Y}\backslash Z \), and define \( \mathcal{E}_y \) to be the set of grading preserving isomorphisms \( \varphi : m_y(I) \to g_-(I) \) such that \( \varphi(D_y^{\alpha_i}) = g_{-1}(\alpha_i) \) for \( i = 1, 2 \). Then \( \mathcal{E} \) is a \( G_0 \)-structure on the family \( \mathcal{Y}\backslash Z \) over \( \Delta \), and the conclusion follows from Proposition 3.18.

### 3.3 Proof of Proposition 1.3

The following is straightforward.

**Lemma-Definition 3.21** In the situation of Setting 2.1, take \( \alpha \neq \beta \in I \). Then the following are equivalent:

(i) the manifold \( S^{\alpha \cdot \beta, \beta} \cong S^\alpha \times S^\beta; \)

(ii) the roots \( \alpha \) and \( \beta \) lie in different connected components of the Dynkin diagram of \( G_{J \cup \{\alpha, \beta\}}. \)

If (i) and (ii) do not hold, we say \( (\alpha, \beta) \) is a \( J \)-connected pair.

Our main aim in this subsection is to show that

**Theorem 3.22** In the situation of Setting 1.7, suppose that \( |I| \geq 3 \) and \( F_x^{\alpha, \beta} \cong S^{\alpha, \beta} \) for any \( J \)-connected pair \( \alpha \neq \beta \in I \) and general \( x \in X_0 \). Then the manifold \( X_0 \cong S. \)

As a direct consequence of Theorem 3.22, we have the following result.
Corollary 3.23  In the situation of Setting 2.1, suppose that \(|I| \geq 2\) and for any \(\alpha \neq \beta \in I\) there exists a subset \(A \subset I\) such that \(\alpha, \beta \in A\) and the rational homogeneous space \(S^A\) is rigid under Fano deformation. Then \(G/P_I\) is rigid under Fano deformation.

Proof  By Proposition 3.24 below, we can assume that the group \(G\) is simple. Then we may assume that we are in the situation of Setting 1.7. Given any subset \(A \subset I\), a general fiber of \(\pi_0^A : X_0 \to X_0^A\) is a Fano deformation of \(S^A\). Then the conclusion follows from Theorem 3.22.

Proposition 3.24  [13, Theorem 1] Let \(\phi : Z \to \Delta \ni 0\) be a holomorphic map with all fiber being connected Fano manifolds. Suppose that \(Z_0 \cong Z'_0 \times Z''_0\). Then there are holomorphic maps \(\phi' : W' \to \Delta\) and \(\phi'' : W'' \to \Delta\) such that all fibers of \(\phi'\) and \(\phi''\) are connected Fano manifolds, \(W'_0 \cong Z'_0\), \(W''_0 \cong Z''_0\), and \(Z = W' \times \Delta W''\).

Now we turn to the proof of Theorem 3.22. By Proposition 3.19, it suffices to show that the symbol algebra \(m_x(I)\) is standard for \(x \in X_0\) general. To verify it, we will apply Proposition 2.14.

Lemma 3.25  Take a general point \(x \in X_0\) in Setting 1.7. We have \(\sum_{\alpha \in I} T_x F_\alpha^x \subset T_x X_0\), and \(D_x = \sum_{\alpha \in I} D_\alpha^x = \bigoplus_{\alpha \in I} D_\alpha^x \subset T_x X_0\), where the distributions \(D_\alpha^x\) and \(D_\alpha^x\) are as in Notation 3.9.

Proof  The relative Mori contractions \(\pi^\alpha : X \to X^\alpha\) and \(\pi^I\backslash\{\alpha\} : X \to X^I\backslash\{\alpha\}\) induce a morphism

\[
\pi' : X_0 \to X^\alpha_0 \times X^I\backslash\{\alpha\}, \\
x \mapsto (\pi^\alpha(x), \pi^I\backslash\{\alpha\}(x)),
\]

which contracts no curves. Then \(T_x F_\alpha^x \cap T_x F_\alpha^{I\backslash\{\alpha\}} = \{0\}\) for \(\alpha \in I\) and \(x \in X_0\), which implies the first assertion. Now the second assertion follows from the first one and the inclusion \(D_\alpha^x \subset T_x F_\alpha^x\).

Lemma 3.26  Take \(\alpha \in I\) and \(x \in X_0\) general in the setting of Theorem 3.22. Then \(C_\alpha^x \subset \mathbb{P}(D_\alpha^x)\) is projectively equivalent to \(Z_\alpha^x \subset \mathbb{P}(g_{-1}(\alpha))\).

Proof  By Proposition 3.5, the general fiber \(F_\alpha^x\) is biholomorphic to \(S^\alpha\). Thus \(C_\alpha^x \cong Z_\alpha^x\).

Lemma 3.27  In the setting of Theorem 3.22, take \(\alpha \in I\), \(\beta \in N_J(\alpha)\) and a general point \(x \in X_0\). Then the distribution \(g^\beta(\hat{Z}_x^\alpha)\) extends to a holomorphic distribution \(D^\beta(\hat{C}_x^\alpha)\) on \(C_x^\alpha\). Moreover, under the identification \(\hat{Z}_x^\alpha = \hat{C}_x^\alpha\) we have \(g^\beta(\hat{Z}_x^\alpha) = D^\beta(\hat{C}_x^\alpha)\).

Proof  It is a direct consequence of Lemmas 3.26 and 2.10.

Now we are ready to check condition \((ii)\) of Proposition 2.14, while condition \((i)\) is to be checked later.
Lemma 3.28 In the setting of Theorem 3.22, take \( x \in \mathcal{X}_0 \) general, and any \( \alpha \in I, \beta \in N_I(\alpha) \), \( v \in \tilde{Z}_x^{\alpha} \setminus \{0\} \) and us \( \in \mathfrak{g}_v^\beta (\tilde{Z}_x^{\alpha}) \). Then we have \( (\text{adv})^{-1}(\beta, \alpha)(u) = 0 \) in \( \mathfrak{m}_x(I) \).

**Proof** Let \( \gamma \in I \setminus \{\alpha\} \) be any root that is \( J \)-connected with \( \alpha \). By the assumption of Theorem 3.22, \( (\text{adv})^{-1}(\beta, \alpha)(u) = 0 \) in \( \mathfrak{m}_x(\alpha, \gamma) \). Then the conclusion follows from the fact \( D^{\alpha, \gamma} \subset D^I \). \( \square \)

To check the condition (i) of Proposition 2.14, let us write \( I \) as a disjoint union \( I(j) \) in a special way.

**Construction 3.29** Fix any element \( \tilde{\alpha} \in I \) and define \( I(1) := \{\tilde{\alpha}\} \).

Now for each \( j \geq 1 \) define by induction that \( I(j + 1) := \{\alpha \in I \cup \bigcup_{s \leq j} I(s) \mid (\alpha, \beta) \text{ is } J \text{-connected for some } \beta \in I(j)\} \).

**Lemma 3.30** In the setting of Construction 3.29, the following holds.

1. The set \( I \) is the disjoint union of \( I(j), j \geq 1 \).
2. Given any \( j \geq 2 \) with \( I(j) \neq \emptyset \) and any \( \alpha \in I(j) \), there exists a unique \( \beta \in (\bigcup_{s \leq j} I(s)) \setminus \{\alpha\} \) such that \( (\alpha, \beta) \text{ is } J \text{-connected}. \) Moreover, this unique \( \beta \) belongs to \( I(j - 1) \).
3. Given any \( J \)-connected pair \( (\alpha, \beta) \), there exists a unique \( j \geq 1 \) such that \( \{\alpha, \beta\} \subset I(j) \cup I(j + 1) \). Moreover, either \( \alpha \in I(j), \beta \in I(j + 1) \) or \( \beta \in I(j), \alpha \in I(j + 1) \).

**Proof** The assertion (1) holds because the Dynkin diagram \( \Gamma_{I \cup J} \) is connected. To prove (2), it suffices to notice that \( \Gamma_{I \cup J} \) contains no loop and each element in \( \bigcup_{s=2}^j I(s) \) is connected with the unique element \( \tilde{\alpha} \in I(1) \) by the elements in \( J \cup (\bigcup_{s \leq j} I(s)) \). The assertion (3) is a direct consequence of (1) and (2). \( \square \)

Now we are ready to check the condition (i) of Proposition 2.14 in our situation.

**Lemma 3.31** In the setting of Theorem 3.22, take a general point \( x \in \mathcal{X}_0 \). We can define a \( G_0 \)-representation on \( D_x^{\alpha} \) and fix some \( 0 \neq v_\alpha \in \widehat{C}_x^{\alpha} \) for each \( \alpha \in I \) such that

\[
(\text{adv}')^{-1}(\alpha'', \alpha') + 1(v'') = 0 \in \mathfrak{m}_x(I) \text{ for } \alpha' \neq \alpha'' \in I \text{ and } (v', v'') \in G_0 \cdot (v_{\alpha'}, v_{\alpha''}) \in \tilde{Z}_x^{\alpha'} \times \tilde{Z}_x^{\alpha''}.
\]

**Proof** Now we will define a \( G_0 \)-representation on \( D_x^{\alpha} \) and fix some \( 0 \neq v_\alpha \in \widehat{C}_x^{\alpha} \) for each \( \alpha \in I = \bigcup_{j \geq 1} I(j) \) and show they satisfy (3.3) by induction on \( j \geq 1 \).

By our construction, \( I(1) \) consists of a unique element, which is \( \tilde{\alpha} \). Since \( |I| > 1 \), the set \( I(2) \neq \emptyset \). Fix any \( \tilde{\beta} \in I(2) \). By definition \( (\tilde{\alpha}, \tilde{\beta}) \) is \( J \)-connected.

By the assumption of Theorem 3.22, \( F_{x\tilde{\alpha}, \tilde{\beta}} \) with \( x \in \mathcal{X}_0 \) general is biholomorphic to \( P_{\tilde{\alpha}, \tilde{\beta}}/P_1 \).

This is also biholomorphic to \( G_{J \cup \{\tilde{\alpha}, \tilde{\beta}\}}/P_{\{\tilde{\alpha}, \tilde{\beta}\}} \), see Notation 2.6. By Lemma 2.10, \( G_{J \cup \{\tilde{\alpha}, \tilde{\beta}\}} \rightarrow \text{Aut}^o(F_{x\tilde{\alpha}, \tilde{\beta}}) \) is a surjective homomorphism with finite kernel. Then we obtain the \( G_0(J \cup \{\tilde{\alpha}, \tilde{\beta}\}) \) representations on \( D_y^\alpha \) and \( D_y^\beta \) on any point \( y \in F_{x\tilde{\alpha}, \tilde{\beta}} \), which preserve \( \widehat{C}_y^\alpha \) and \( \widehat{C}_y^\beta \). Here \( G_0(J \cup \{\tilde{\alpha}, \tilde{\beta}\}) \) is the Lie subgroup

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of $G_{J \cup \{\alpha, \beta\}}$ associated with Lie subalgebra $\mathfrak{g}_0 \subset \text{Lie}(G_{J \cup \{\alpha, \beta\}})$. The $G_0(J \cup \{\alpha, \beta\})$ representations induce the required $G_0(= G_0(R))$ representations on $\mathcal{D}_{\alpha}$ and $\mathcal{D}_{\beta}$ respectively. Note that $G_0(J \cup \{\alpha, \beta\})$ is the quotient group of $G_0 := G_0(R)$ by some torus in the center. This torus acts trivially on $\mathcal{D}_{\alpha}$ and $\mathcal{D}_{\beta}$. Denote by $\varphi_{[\alpha, \beta]} : G_0 \rightarrow \text{Aut}^o(\mathcal{C}_{\alpha}, \mathcal{D}_{\alpha}) \subset \text{GL}(\mathcal{D}_{\alpha})$ and $\varphi_{\beta} : G_0 \rightarrow \text{Aut}^o(\mathcal{C}_{\beta}, \mathcal{D}_{\beta}) \subset \text{GL}(\mathcal{D}_{\beta})$.

Applying Proposition 2.14 to $F_{x, \beta} \cong P_{I \setminus \{\alpha\}} / P_I$, we conclude that there exists $0 \neq v_{\alpha} \in \mathcal{C}_{\alpha}$ and $0 \neq v_{\beta} \in \mathcal{C}_{\beta}$ such that $(\text{ad}w_{\alpha})^{-(\beta, \beta)+1}(w_{\beta}) = 0 \in m_x(\alpha, \beta)$ and $(\text{ad}w_{\beta})^{-(\alpha, \beta)+1}(w_{\alpha}) = 0 \in m_x(\alpha, \beta)$ for any $(w_{\alpha}, w_{\beta}) \in G_0 \cdot (v_{\alpha}, v_{\beta}) \in \mathcal{C}_{\alpha} \times \mathcal{C}_{\beta}$. Then the inclusion $\mathcal{D}_{\alpha} \subset \mathcal{D} := \mathcal{D}^I$ implies that $(\text{ad}w_{\alpha})^{-(\beta, \beta)+1}(w_{\beta}) = 0$ and $(\text{ad}w_{\beta})^{-(\alpha, \beta)+1}(w_{\alpha}) = 0 \in m_x(I)$.

In case $I(2) = \{\beta\}$, we have constructed the $G_0$-representation for both $I(1)$ and $I(2)$.

Now suppose (for the moment) that $|I(2)| \geq 2$. By Lemma 2.10, $G_0 \rightarrow \text{Aut}^o(\mathcal{C}_{\alpha}) = \text{Aut}^o(\mathcal{C}_{\bar{\alpha}}, \mathfrak{g}_{-1})$ is surjective. Take any $\gamma \in I(2) \setminus \{\beta\}$. Then as the previous argument for $(\alpha, \beta)$ we get $G_0$-representations $\varphi_{[\alpha, \gamma]} : G_0 \rightarrow \text{Aut}^o(\mathcal{C}_{\alpha}, \mathcal{D}_{\alpha}) \subset \text{GL}(\mathcal{D}_{\alpha})$ and $\varphi_{\gamma, [\alpha]} : G_0 \rightarrow \text{Aut}^o(\mathcal{C}_{\gamma}, \mathcal{D}_{\gamma}) \subset \text{GL}(\mathcal{D}_{\gamma})$. There is an automorphism $\psi(\alpha; \beta, \gamma) : \text{Aut}^o(\mathcal{C}_{\alpha}, \mathcal{D}_{\alpha}) \rightarrow \text{Aut}^o(\mathcal{C}_{\alpha}, \mathcal{D}_{\alpha})$ such that the following diagram commutes:

\[
\begin{array}{ccc}
G_0 & \xrightarrow{\varphi_{[\bar{\alpha}, \beta]}} & \text{Aut}^o(\mathcal{C}_{\alpha}, \mathcal{D}_{\alpha}) \\
& \searrow \varphi_{\alpha, \gamma} & \downarrow \psi(\alpha; \beta, \gamma) \\
& \text{Aut}^o(\mathcal{C}_{\alpha}, \mathcal{D}_{\alpha}) & \\
\end{array}
\]

Since $G_0$ is reductive, there is an automorphism $\theta(\alpha; \beta, \gamma) : G_0 \rightarrow G_0$ such that the following diagram commutes:

\[
\begin{array}{ccc}
G_0 & \xrightarrow{\varphi_{\alpha, \gamma}} & \text{Aut}^o(\mathcal{C}_{\alpha}, \mathcal{D}_{\alpha}) \\
\downarrow \theta(\alpha; \beta, \gamma) & & \downarrow \varphi_{\alpha, \beta} \\
G_0 & & \\
\end{array}
\]

In other words, we lift the automorphism $\psi(\alpha; \beta, \gamma)$ of $\text{Aut}^o(\mathcal{C}_{\alpha}, \mathcal{D}_{\alpha})$ to an automorphism $\theta(\alpha; \beta, \gamma)$ of $G_0$.

Define $\tau(\gamma) := \varphi_{\alpha, \gamma} \circ \theta(\alpha; \beta, \gamma)^{-1} : G_0 \rightarrow G_0 \rightarrow \text{Aut}^o(\mathcal{C}_{\gamma}, \mathcal{D}_{\gamma}) \subset \text{GL}(\mathcal{D}_{\gamma})$ and $\tau(\alpha) := \psi_{[\alpha, \gamma]} \circ \theta(\alpha; \beta, \gamma)^{-1}$. In particular, we have $\tau(\alpha) = \psi_{[\alpha, \beta]}$. Applying Proposition 2.14 to $F_{x, \gamma} \cong P_{I \setminus \{\alpha\}} / P_I$, we can conclude that there exists $0 \neq v_{\alpha} \in \mathcal{C}_{\alpha}$ and $0 \neq v_{\gamma} \in \mathcal{C}_{\gamma}$ such that for any $(w_{\alpha}, w_{\gamma}) \in G_0 \cdot (v_{\alpha}, v_{\gamma}) \in \mathcal{C}_{\alpha} \times \mathcal{C}_{\gamma}$ (under
the representation $\varphi_{[\alpha, \gamma]}$ and $\varphi_{[\alpha, \gamma]}^\gamma$,)

$$(\mathrm{ad}w_{\alpha})^{-\langle\gamma, \alpha\rangle + 1}(w_{\gamma}) = 0 \quad \text{and} \quad (\mathrm{ad}w_{\gamma})^{-\langle\alpha, \gamma\rangle + 1}(w_{\alpha}) = 0 \quad \text{in} \quad m_x(\alpha, \gamma). \quad (3.4)$$

Denote by $R(\bar{\alpha}, \gamma) := \varphi_{[\bar{\alpha}, \gamma]}(G_0) \cdot ([v_{\alpha}'], [v_{\gamma}']) \subset H_x^\alpha \times H_x^\gamma$. Then $R(\bar{\alpha}, \gamma)$ is a closed $G_0$-orbit, and the two projections $R(\bar{\alpha}, \gamma) \rightarrow H_x^\alpha$ and $R(\bar{\alpha}, \gamma) \rightarrow H_x^\gamma$ are surjective. In particular, for the previously chosen element $0 \neq v_{\bar{\alpha}} \in \hat{C}_x^\alpha$ there exists $0 \neq v_{\gamma} \in \hat{C}_x^\gamma$ such that $([v_{\alpha}'], [v_{\gamma}']) \in R(\bar{\alpha}, \gamma)$. Furthermore,

$$R(\bar{\alpha}, \gamma) = \varphi_{[\bar{\alpha}, \gamma]}(G_0) \cdot ([v_{\alpha}'], [v_{\gamma}']) = \varphi_{[\bar{\alpha}, \gamma]}(G_0) \cdot ([v_{\bar{\alpha}}], [v_{\gamma}]) \subset H_x^\alpha \times H_x^\gamma.$$

Since $\theta := \theta(\bar{\alpha}; \bar{\beta}, \gamma)$ is an automorphism of $G_0$, we know that $\tau_{[\bar{\alpha}, \gamma]}(G_0) = \varphi_{[\bar{\alpha}, \gamma]}(\theta^{-1}(G_0)) = \varphi_{[\bar{\alpha}, \gamma]}(G_0)$, where $\tau_{[\bar{\alpha}, \gamma]}(G_0) := (\tau(\bar{\alpha}), \tau(\gamma))$. It follows that $\tau_{[\bar{\alpha}, \gamma]}(G_0) \cdot ([v_{\alpha}], [v_{\gamma}]) = \varphi_{[\bar{\alpha}, \gamma]}(G_0) \cdot ([v_{\bar{\alpha}}], [v_{\gamma}]) = R(\bar{\alpha}, \gamma)$. Hence the formula (3.4) holds for all $(w_{\alpha}, w_{\gamma}) \in \tau_{[\bar{\alpha}, \gamma]}(G_0) \cdot (v_{\bar{\alpha}}, v_{\gamma}) \subset \hat{C}_x^\alpha \times \hat{C}_x^\gamma$. Then the inclusion $D_{\bar{\alpha}, \gamma} \subset D := D^I$ implies that for all $(w_{\alpha}, w_{\gamma}) \in \tau_{[\bar{\alpha}, \gamma]}(G_0) \cdot (v_{\bar{\alpha}}, v_{\gamma}) \subset \hat{C}_x^\alpha \times \hat{C}_x^\gamma$, $\mathrm{ad}w_{\alpha}^{-\langle\gamma, \alpha\rangle + 1}(w_{\gamma}) = 0$ and $\mathrm{ad}w_{\gamma}^{-\langle\alpha, \gamma\rangle + 1}(w_{\alpha}) = 0$ in $m_x(I)$.

Now we have obtained $G_0$-representations on $D_{x, \alpha}^{\bar{\alpha}}$ and chosen $0 \neq v_{\alpha} \in \hat{C}_x^\alpha$ for all $\alpha \in I(1) \cup I(2)$ such that (3.3) holds for $J$-connected pair $\alpha', \alpha'' \in I(1) \cup I(2)$. Repeating the argument above, we can obtain $\tau_\alpha : G_0 \rightarrow \mathrm{Aut}^x(\hat{C}_x^\alpha, D_{x, \alpha}^{\bar{\alpha}}) \subset \mathrm{GL}(D_{x, \alpha}^{\bar{\alpha}})$ and choose $0 \neq v_{\alpha} \in \hat{C}_x^\alpha$ for all $\alpha \in I = \bigcup_{j \geq 1} I(j)$ such that (3.3) holds for all $J$-connected pair $(\alpha', \alpha'') \in I \times I$.

Now take any pair $\alpha \neq \beta \in I \times I$ which is not $J$-connected. By Lemma-Definition 3.21, $\alpha = F_{x, y}^{\alpha, \beta} = F_{x}^{\alpha} \times F_{y}^{\beta}$ at any $y \in \bigcup_{I \neq 0} X_t$. By Proposition 3.24,

$$F_{x, y}^{\alpha, \beta} = F_{x}^{\alpha} \times F_{y}^{\beta} \quad \text{at any} \quad x \in X_0. \quad (3.5)$$

Now for $x \in X_0$ general, $D_{x, \alpha}^{\bar{\alpha}}, D_{x, \beta}^{\bar{\beta}}$ and $D_x$ are well-extended. By (3.5) the Levi bracket of vector fields satisfies $[D_{x, \alpha}^{\bar{\alpha}}, D_{x, \beta}^{\bar{\beta}}] \subset D_{x, \alpha}^{\bar{\alpha}} + D_{x, \beta}^{\bar{\beta}} \subset D_x$, which implies that $[w_{\alpha}, w_{\beta}] = 0$ in $m_x(I)$ for any $(w_{\alpha}, w_{\beta}) \in \hat{C}_x^\alpha \times \hat{C}_x^\beta \subset D_{x, \alpha}^{\bar{\alpha}} \times D_{x, \beta}^{\bar{\beta}}$. In summary, (3.3) holds for all pairs $(\alpha', \alpha'') \in I \times I$ with $\alpha' \neq \alpha''$. \hfill $\Box$

Now we are ready to complete the proof of Theorem 3.22.

**Proof of Theorem 3.22** Take a general point $x \in X_0$. By Lemmas 3.28 and 3.31, the symbol algebra $m_x(I)$ satisfies the conditions (i) and (ii) in Proposition 2.14. Then by Proposition 2.14 the symbol algebra $m_x(I)$ is a quotient algebra of $g_{-}(I)$. By Proposition 3.10, $\dim m_x(I) = \dim g_{-}(I)$, which implies $m_x(I) \cong g_{-}(I)$. Then the conclusion follows from Proposition 3.19. \hfill $\Box$

### 4 The Fano deformations of $\mathbb{F}(1,2, \mathbb{P}^3)$

The aim of this section is to prove Proposition 1.2. Throughout Section 4, we assume the following assumption.
Assumption 4.1 Let \( \pi : \mathcal{X} \to \Delta \ni 0 \) be a holomorphic map such that \( \mathcal{X}_t \cong A_3/P_{(a_1, a_2)} \) for \( t \neq 0 \), \( \mathcal{X}_0 \) is a connected Fano manifold, and \( \mathcal{X}_0 \cong A_3/P_{(a_1, a_2)} \).

The statement of Proposition 1.2 is about the deformation of \( \mathbb{P}(1, 2, \mathbb{P}^3) \), and we have isomorphisms \( \mathbb{P}(1, 2, \mathbb{P}^3) \cong A_3/P_{(a_2, a_3)} \cong A_3/P_{(a_1, a_2)} = F(1, 2, \mathbb{C}^4) \). By definition of \( F^d(1, 2, \mathbb{C}^4) \), the restriction of the \( \mathbb{P}^2 \)-bundle \( F^d(1, 2, \mathbb{C}^4) \to \mathbb{P}^3 \) gives a biholomorphic map \( \mathbb{P}(\mathcal{L}_\sigma) \cong \mathbb{P}^3 \), and the hyperplane bundle \( \mathbb{P}(\mathcal{L}_\sigma) \) is biholomorphic to the complete flag manifold \( C_2/B \), where \( B \) is a Borel subgroup of the simple group \( C_2 \).

Now let us give the outline to prove Proposition 1.2.

Outline of the proof of Proposition 1.2 Equivalently, we will show \( \mathcal{X}_0 \cong F^d(1, 2, \mathbb{C}^4) \) in the situation of Assumption 4.1.

Step 1. The Mori contraction \( \pi_0^{a_2} : \mathcal{X}_0 \to \mathcal{X}_0^{a_2} \) is a \( \mathbb{P}^2 \)-bundle over \( \mathbb{P}^3 \), which is obtained by applying a result of Weber and Wiśniewski [17].

Step 2. At a general point \( x \in \mathcal{X}_0 \), the family \( \mathcal{K}_x^{a_1}(\mathcal{X}_0) \) consists of a single element, denoted by \( \{ C_x \} \). An irreducible component of the locus \( \{ x \in \mathcal{X}_0 \mid \dim \mathcal{K}_x^{a_1}(\mathcal{X}_0) \geq 1 \} \) gives rise to a meromorphic section \( \sigma : \mathbb{P}^3 \to \mathcal{X}_0 \).

Step 3. Let \( H \) be an effective divisor on \( \mathcal{X}_0 \), which is a general element in a linear system satisfying \( (H \cdot \mathcal{K}^{a_1}) = 0 \) and \( (H \cdot \mathcal{K}^{a_2}) = 1 \). The restriction of \( \pi_0^{a_2} \) on \( H \) is a fibration over \( \mathbb{P}^3 \), whose general fiber is a line in \( \mathbb{P}^2 \). Then we show that \( \sigma \) is a holomorphic section, and \( H \to \mathbb{P}^3 \) is a \( \mathbb{P}^1 \)-bundle satisfying \( H \cap \sigma(\mathbb{P}^3) = \emptyset \).

Step 4. Denote by \( \mathcal{K}^{a_1}(\mathcal{X}_0/\mathbb{P}^3) \subset \text{Chow}(\mathbb{P}^3) \) the closure of the set of those \( \pi_0^{a_2}(C_x) \), where \( x \in \mathcal{X}_0 \) is a general point and \( \mathcal{K}_x^{a_1}(\mathcal{X}_0) = \{ [C_x] \} \). By considering the symbol algebra of \( \mathcal{D} = D^{a_1} + D^{a_2} \) on \( \mathcal{X}_0 \), we obtain a meromorphic distribution \( \mathcal{E} \) of rank two on \( \mathbb{P}^3 \) such that \( \mathcal{K}^{a_1}(\mathcal{X}_0/\mathbb{P}^3) \) is the family of lines tangent to \( \mathcal{E} \). This gives an antisymmetric form \( \omega \) on \( \mathbb{C}^4 \) – which is shown to be a symplectic form later – such that \( \mathcal{E} \) coincides with the induced contact form \( \mathcal{L}_\omega \) on \( \mathbb{P}^3 = \mathbb{P}(\mathbb{C}^4) \). It follows that \( H \cong C_2/B \) and \( \mathcal{X}_0 \cong F^d(1, 2, \mathbb{C}^4) \).

This section is organized as follows. In Sect. 4.1, by studying the splitting types of various meromorphic vector bundles along a general element in \( \mathcal{K}^{a_2}(\mathcal{X}_0) \), we obtain the structure of the symbol algebra of \( \mathcal{D} = D^{a_1} + D^{a_2} \) on \( \mathcal{X}_0 \). In Sect. 4.2, a similar analysis for meromorphic vector bundles over a general element of \( \mathcal{K}^{a_1}(\mathcal{X}_0) \) provides the section \( \sigma \). In Sect. 4.3, we study the properties of the family \( \mathcal{K}^{a_1}(\mathcal{X}_0/\mathbb{P}^3) \). In Sect. 4.4, we complete the proof of Proposition 1.2 by studying the properties of divisor \( H \) explained above. In Sect. 4.5, we summarize some properties of the manifold \( F^d(1, 2, \mathbb{C}^4) \) which will be used in later sections.

4.1 Type of the symbol algebra

Convention 4.2 In Section 4, we denote by \( \mathcal{D}_0 \), \( (\mathcal{D}_0)^{a_i} \), and \( (\mathcal{D}_0)^{-1} \) the restriction of \( \mathcal{D} \), \( D^{a_i} \), and \( D^{-1} \) on \( \mathcal{X}_0 \) respectively, where the latter is defined in Notation 3.9.
Lemma 4.3  In the situation of Assumption 4.1, there exists a unique meromorphic line bundle $\mathcal{N} \subset T_{\pi_2}X_0$ such that $[\mathcal{N}, D_0] \subset D_0$. Moreover, $\text{rank}(D_0)^{-2} = 4$ and $(D_0)^{-3} = TX_0$.

Proof The restriction of the Frobenius bracket of $D_0$ induces a homomorphism $F : (D_0)^{a_1} \otimes (D_0)^{a_2} \to TX_0/D_0$. The image of $F$ is $(D_0)^{-2}/D_0$ on $X_0$, whose rank is at most two. By Proposition 3.10, $\text{rank}((D_0)^{-2}/D_0) \geq 1$. If $\text{rank}((D_0)^{-2}/D_0) = 2$, then $m_x(\alpha_1, \alpha_2) \cong g_{\alpha_2}(\alpha_1, \alpha_2)$ for $x \in X_0$. Then by Proposition 3.20 $X_0 \cong A_3/P_{[\alpha_1, \alpha_2]}$, contradicting Assumption 4.1. Hence $\text{rank}((D_0)^{-2}/D_0) = 1$. By Proposition 3.10 $\text{rank}((D_0)^{-3}/(D_0)^{-2}) \geq 1$, implying that $(D_0)^{-3} = TX_0$. □

Lemma 4.4  In the situation of Assumption 4.1, there exists a unique meromorphic vector subbundle $\mathcal{W} \subset (D_0)^{-1}$ of rank two such that $[\mathcal{W}, (D_0)^{-2}] \subset (D_0)^{-2}$. Furthermore, $\mathcal{N} \subset \mathcal{W}$.

Proof The conclusion follows from the fact that $\text{rank}(D_0)^{-3} = \text{rank}(D_0)^{-2} + 1$ and that $[\mathcal{N}, (D_0)^{-1}] \subset (D_0)^{-1}$. □

The following result is important to the proof of Proposition 1.2.

Proposition 4.5  We have $\mathcal{W} = (D_0)^{a_2}$.

As a summarization of the descriptions of the symbol algebras $\text{Symb}(D_0)_x$ studied in Lemma 4.3, Lemma 4.4 and Proposition 4.5, we have the following result.

Corollary 4.6  The symbol algebra $m_x(\alpha_1, \alpha_2) := \text{Symb}(D_0)_x$ at general point $x \in X_0$ is isomorphic to $g_{\alpha_2}(C_2 \times A_1)$, where $g_{\alpha_2}(C_2 \times A_1)$ is defined in Definition 2.2. More precisely, there exists a nonempty Zariski open subset $\Omega$ of $X_0$ such that

(i) there is an isomorphism on $\Omega$:

\[
(D_0) \cong (D_0)^{a_1} \oplus (D_0)^{a_2};
\]  \hspace{1cm} (4.1)

(ii) the Frobenius bracket of $D_0$ induces a surjective homomorphism on $\Omega$:

\[
\wedge^2 (D_0)^{-1} \to (D_0)^{a_1} \otimes ((D_0)^{a_2}/\mathcal{N}) \cong ((D_0)^{-2}/(D_0)^{-1}),
\]  \hspace{1cm} (4.2)

where $(D_0)^{-1} := D_0$ by definition;

(iii) the restriction of the Frobenius bracket of $(D_0)^{-2}$ induces a surjective homomorphism on $\Omega$:

\[
(D_0)^{-1} \otimes ((D_0)^{-2}/(D_0)^{-1}) \to (D_0)^{a_1} \otimes ((D_0)^{-2}/(D_0)^{-1}) \cong ((D_0)^{-3}/(D_0)^{-2}),
\]  \hspace{1cm} (4.3)

(iv) the weak derivative $(D_0)^{-3}$ of $D_0$ is equal to the whole tangent bundle of $X_0$, i.e.

$(D_0)^{-3} = TX_0$. 
Remark 4.7  (i) The isomorphisms in (4.1) (4.2) and (4.3) hold on \( \Omega \) instead of on the whole holomorphic loci of the corresponding meromorphic vector bundles. Meanwhile, as meromorphic vector bundles over \( \chi_0 \), we have injective homomorphisms

\[
(\mathcal{D}_0)^{\alpha_1} \oplus (\mathcal{D}_0)^{\alpha_2} \hookrightarrow \mathcal{D}_0,
\]

\[
(\mathcal{D}_0)^{\alpha_1} \otimes ((\mathcal{D}_0)^{\alpha_2}/\mathcal{N}) \hookrightarrow (\mathcal{D}_0)^{-2}/(\mathcal{D}_0)^{-1},
\]

\[
(\mathcal{D}_0)^{\alpha_1} \otimes ((\mathcal{D}_0)^{-2}/(\mathcal{D}_0)^{-1}) \hookrightarrow (\mathcal{D}_0)^{-3}/(\mathcal{D}_0)^{-2}.
\]

(ii) The Lie algebra \( \text{Symb}(\mathcal{D}_0)_x \cong \mathfrak{g}_- (C_2 \times A_1) \) can be described explicitly as the graded Lie algebra \( m_- := \bigoplus m_{-k} \) such that \( m_{-1} := \mathbb{C}v_1 \oplus \mathbb{C}v_2 \oplus \mathbb{C}v_3, m_{-2} := \mathbb{C}v_{12} \), \( m_{-3} := \mathbb{C}v_{121} \) and \( m_{-k} := 0 \) for all \( k \geq 4 \), where \( v_{12} := [v_1, v_2] \) and \( v_{121} := [v_{12}, v_1] \). In the identification \( \text{Symb}(\mathcal{D}_0)_x = m_- \), we have \( m_x(\alpha_1) = (\mathcal{D}_0)^{\alpha_1}_x = \mathbb{C}v_1, \mathcal{N}_x = \mathbb{C}v_3 \subset (\mathcal{D}_0)^{\alpha_2}_x \), and \( m_x(\alpha_2) = (\mathcal{D}_0)^{\alpha_2}_x = \mathbb{C}v_2 \oplus \mathbb{C}v_3 \).

The rest of Sect. 4.1 is devoted to the proof of Proposition 4.5. Firstly, the following conclusion is straightforward.

Lemma 4.8 There exists a closed variety \( Y_1 \subset \chi_0 \) such that the codimension of \( Y_1 \) in \( \chi_0 \) is at least two, \( (\mathcal{D}_0)^{\alpha_1}, (\mathcal{D}_0)^{\alpha_2}, \mathcal{N}, \mathcal{W} \) and \( \mathcal{D}_0 \) are holomorphic vector bundles over \( \chi_0 \setminus Y_1 \). Moreover, \( C_1 \cap Y_1 = \emptyset \) and \( C_2 \cap Y_1 = \emptyset \) for \( [C_1] \in \mathcal{K}^{\alpha_1} (\chi_0) \) general and \( [C_2] \in \mathcal{K}^{\alpha_2} (\chi_0) \) general.

To continue, we need a useful result in [1] due to L. Bonavero, C. Casagrande and S. Druel.

Proposition 4.9 [1, Proposition 1] Let \( Y \) be a normal \( \mathbb{Q} \)-factorial projective variety, and \( \mathcal{F} \) be a quasi-unsplit covering family of 1-cycles on \( Y \). Denote by \( E_{\mathcal{F}} \subset Y \) the union of all \( \mathcal{F} \)-equivalence classes of dimension larger than \( m \), where \( m \) is the dimension of a general \( \mathcal{F} \)-equivalence class. Then

(i) \( E_{\mathcal{F}} \) is a Zariski closed subset of \( Y \), and \( \dim E_{\mathcal{F}} \leq \dim Y - 2 \);

(ii) there exists a normal variety \( Z \) and a surjective morphism \( \varphi : Y \setminus E_{\mathcal{F}} \to Z \) such that fiber \( \varphi^{-1}(z) \) at each point \( z \in Z \) is an \( \mathcal{F} \)-equivalence class on \( Y \).

Definition 4.10 In the setting of Proposition 4.9, we call \( \varphi \) the \( \mathcal{F} \)-equivalence map on \( Y \).

Remark 4.11  (i) In the setting of Proposition 4.9, the meaning of \( \mathcal{F} \) being a quasi-unsplit family is that all irreducible components of the cycles parameterized by \( \mathcal{F} \) are numerically proportional.

(ii) In the setting of Proposition 4.9, two points in \( Y \) are defined to be \( \mathcal{F} \)-equivalent if they are connected by a chain of elements in \( \mathcal{F} \).

(iii) In our situation of Assumption 4.1, both \( \mathcal{K}^{\alpha_1} (\chi_0) \) and \( \mathcal{K}^{\alpha_2} (\chi_0) \) are unsplit (hence quasi-unsplit) covering families of rational curves on the complex projective manifold \( \chi_1 \). In particular, the conditions in Proposition 4.9 are satisfied by both families \( \mathcal{K}^{\alpha_1} (\chi_0) \) and \( \mathcal{K}^{\alpha_2} (\chi_0) \).
Applying Proposition 4.9 to $\lambda_0$, we obtain the following result immediately.

**Corollary 4.12** Let $\Psi_{0}^{a_1}: \lambda_0 \setminus \mathcal{E}_{0}^{a_1} \to \mathcal{Z}_0^{a_1}$ and $\Psi_{0}^{a_2}: \lambda_0 \setminus \mathcal{E}_{0}^{a_2} \to \mathcal{Z}_0^{a_2}$ be the $\mathcal{K}_{a_1}(\lambda_0)$-equivalence map and the $\mathcal{K}_{a_2}(\lambda_0)$-equivalence map on $\lambda_0$ respectively. Denote by $\text{sing}(\Psi_{0}^{a_1})$ and $\text{sing}(\Psi_{0}^{a_2})$ the singular loci of $\Psi_{0}^{a_1}$ and $\Psi_{0}^{a_2}$ respectively. Let $Y_2$ be the union of $Y_1$, $\text{sing}(\Psi_{0}^{a_1})$, $\text{sing}(\Psi_{0}^{a_2})$, $\mathcal{E}_{0}^{a_1}$ and $\mathcal{E}_{0}^{a_2}$, where $Y_1$ is as in Lemma 4.8. Then $\dim Y_2 \leq \dim \lambda_0 - 2 = 3$.

**Proof** The existence of $\Psi_{0}^{a_1}$ and $\Psi_{0}^{a_2}$ follows from Proposition 4.9. The rest follows from the generic smoothness and the equi-dimensionality of $\Psi_{0}^{a_i}$, $i = 1, 2$. □

**Proposition 4.13** Take $[C_2] \in \mathcal{K}_{a_2}(\lambda_0)$ general. Then $(D_0)^{a_1}|_{C_2} \cong \mathcal{O}_{\mathbb{P}^1}(-1)$ and $(D_0)^{a_2}|_{C_2} \cong \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(1)$.

**Proof** The curve $C_2$ is a line in a general fiber $F_{\pi_{a_2}}^{a_2} \cong \mathbb{P}^2$ of the elementary Mori contraction $\pi_{a_2}$, where $x$ is a general point in $C_2$. Thus, $(D_0)^{a_2}|_{C_2} = TF_{\pi_{a_2}}^{a_2}|_{C_2} = \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(1)$.

Now take a general local section of $\mathcal{K}_{a_2}(\lambda_0) \to \Delta$ passing through $[C_2] \in \mathcal{K}_{a_2}(\lambda_0) \subset \mathcal{K}_{a_2}(\lambda)$. We obtain a holomorphic family $\{A^t\}_{t \in \Delta}$ (by shrinking $\Delta$ if necessary) such that $S := \bigcup_{t \in \Delta} A^t \subset \lambda$ is a complex manifold of dimension two, and $A^0 = C_2 \subset \lambda_0$, $A^t \subset \lambda_t$. Moreover, $S \cap Y_2 = C_2 \cap Y_2 = \emptyset$ by Corollary 4.12. Thus for any $x \in S$, there exists a unique $[l_x] \in \mathcal{K}_{a_2}(\lambda)$ such that $x \in l_x$. Furthermore, $x$ is a smooth point of $l_x$. Denote by $\mathcal{L} := \bigcup_{x \in S} T_{l_x,x}$ which is a holomorphic line bundle over $S$. By Proposition 2.12 we know that for any $t \neq 0$, $\mathcal{L}|_{A^t} = T_{\pi_{a_2}}^{a_2}|_{A^t} \cong \mathcal{O}_{\mathbb{P}^1}(\langle \alpha_1, \alpha_2 \rangle) = \mathcal{O}_{\mathbb{P}^1}(-1)$. It follows that $\mathcal{L}|_{C_2} \cong \mathcal{O}_{\mathbb{P}^1}(-1)$. Thus $(D_0)^{a_1}|_{C_2} \cong \mathcal{L}|_{C_2} \cong \mathcal{O}_{\mathbb{P}^1}(-1)$. □

**Proposition 4.14** Take $[C_2] \in \mathcal{K}_{a_2}(\lambda_0)$ general. Then $(D_0)^{a_1}$, $(D_0)^{a_2}$, $D_0$, $(D_0)^{-2}$, $(D_0)^{-3}$, $N$, $W$ are holomorphic vector bundles in an open neighborhood of $C_2 \subset \lambda_0$. Moreover,

\[
\begin{align*}
N|_{C_2} &= \mathcal{O}_{\mathbb{P}^1}(1), \\
(D_0)^{a_2}/N|_{C_2} &= \mathcal{O}_{\mathbb{P}^1}(2), \\
(D_0)^{-2}/D_0|_{C_2} &= \mathcal{O}_{\mathbb{P}^1}(1), \\
(D_0)^{-3}/(D_0)^{-2}|_{C_2} &= \mathcal{O}_{\mathbb{P}^1}, \\
D_0|_{C_2} &= (D_0)^{a_1}|_{C_2} \oplus (D_0)^{a_2}|_{C_2} = \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1), \\
(D_0)^{-2}|_{C_2} &= \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}^2, \\
(D_0)^{-3}|_{C_2} &= T_{\lambda_0}|_{C_2} = \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}^3.
\end{align*}
\]

**Proof** By the generality of $[C_2] \in \mathcal{K}_{a_2}(\lambda_0)$, $T_{\lambda_0}|_{C_2} = \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}^2$. Then by Proposition 4.13 and the injectivity of $(D_0)^{a_1} \oplus (D_0)^{a_2} \to D_0 \subset T_{\lambda_0}$ in an open neighborhood of $C_2 \subset \lambda_0$,

either $D_0|_{C_2} = \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}$, $D_0/(D_0)^{a_2}|_{C_2} = \mathcal{O}_{\mathbb{P}^1}$, 
or $D_0|_{C_2} = \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$, $D_0/(D_0)^{a_2}|_{C_2} = \mathcal{O}_{\mathbb{P}^1}(-1)$,
By Lemma 4.3, $|N, D_0] \subset D_0, [(D_0)^{\alpha_1}, (D_0)^{\alpha_2}] \subset (D_0)^{\alpha_2} \subset D_0$ and $[D_0, D_0] \not\subset D_0$. Then the Frobenius bracket $\Lambda^2 D_0 \to T X_0/D_0$ induces a nonzero homomorphism:

$$f: (D_0/(D_0)^{\alpha_2}) \otimes ((D_0)^{\alpha_2}/N) \to TX_0/D_0.$$  (4.4)

Note that $\deg((D_0)/(D_0)^{\alpha_2})|_{C_2} \geq -1, \deg((D_0)^{\alpha_2}/N)|_{C_2} \geq 2$ and that the degree of each factor of $TX_0/D_0$ is at most one. Since $f$ in (4.4) is a nonzero morphism, $D_0/(D_0)^{\alpha_2}|_{C_2} = O_{P^2}(1), (D_0)^{\alpha_2}/N|_{C_2} = O_{P^2}(2)$ and $(D_0)^{2}/D_0|_{C_2} = O_{P^2}(1)$. It follows that $N|_{C_2} = O_{P^1}(1)$, and $D_0|_{C_2} = O_{P^1}(2) \oplus O_{P^1}(1) \oplus O_{P^1}(-1)$. Since $(D_0)^{-2}/(D_0)^{\alpha_2}|_{C_2} \subset TX_0/(D_0)^{\alpha_2} = O_{P^2}$, and $\deg((D_0)^{2}/(D_0)^{\alpha_2}|_{C_2}) = \deg((D_0)^{2}/D_0|_{C_2}) + \deg((D_0)/(D_0)^{\alpha_2}|_{C_2}) = 0$, we have $(D_0)^{-2}|_{C_2} = O_{P^1}(2) \oplus O_{P^2}(1) \oplus O_{P^1}(1)$, and $(D_0)^{-3}/(D_0)^{-2}|_{C_2} = O_{P^1}$.

Now we can complete the proof of Proposition 4.5.

**Proof of Proposition 4.5** By definition of $(D_0)^{-2}$, we have $[(D_0)^{\alpha_2}, (D_0)^{-1}] \subset (D_0)^{-2}$. Then the Frobenius bracket of $(D_0)^{-2}$ induces $\psi: (D_0)^{\alpha_2} \otimes ((D_0)^{\alpha_2}/D_0) \to TX_0/(D_0)^{-2}$, which is a homomorphism of meromorphic vector bundles over $X_0$. Recall that $(D_0)^{\alpha_2}, (D_0)^{-2}/D_0, TX_0/(D_0)^{-2}$ are holomorphic in an open neighborhood of $C_2 \subset X_0$, where $[C_2] \in K^{\alpha_2}(X_0)$ is a general element. By Propositions 4.13 and 4.14, $(D_0)^{\alpha_2}|_{C_2} = O_{P^1}(2) \oplus O_{P^1}(1), ((D_0)^{-2}/(D_0)^{-1})|_{C_2} = O_{P^1}(1)$ and $(TX_0/(D_0)^{-2})|_{C_2} = O_{P^1}$. Thus, $\psi|_{C_2} = 0$. By the general choice of $C_2$, $\psi = 0$ on $X_0$. In other words, $[(D_0)^{\alpha_2}, (D_0)^{-1}] \subset (D_0)^{-2}$. By the uniqueness of $W$ in Lemma 4.4, we have $W = (D_0)^{\alpha_2}$.

### 4.2 The meromorphic section $\sigma$

Let us firstly recall a result of A. Weber and J. A. Wiśniewski in [17], where they studied the Fano deformation rigidity of complete flag manifolds.

**Proposition 4.15** [17, Corollary 1.4, Corollary 3.3] In the situation of Setting 1.7 let $\alpha$ be an element of $I$ such that $\Phi^{\alpha}: G/P_1 \to G/P_{1\setminus[a]}$ is a $\mathbb{P}^{k}$-bundle for some $k \geq 1$. Suppose either

(i) $H^*(G/P_{1\setminus[a]}, \mathbb{Q})$ is generated by $H^2(G/P_{1\setminus[a]}, \mathbb{Q})$; or
(ii) $X_0^{\alpha}$ is smooth.

Then $\pi_0^{\alpha}: X_0 \to X_0^{\alpha}$ is also a $\mathbb{P}^{k}$-bundle.

As a consequence of Proposition 4.15, we have the following result.

**Proposition 4.16** There exists a unique vector bundle of rank 3 over $\mathbb{P}^3$, denoted by $\mathcal{V}$, such that

(i) $X_0$ is biholomorphic to $\mathbb{P}(\mathcal{V})$ and $X_0^{\alpha_2}$ is biholomorphic to $\mathbb{P}^3$;
(ii) under the isomorphism given in (i), $\pi_0^{\alpha_2}: X_0 \to X_0^{\alpha_2}$ coincides with the projective bundle $\phi: \mathbb{P}(\mathcal{V}) \to \mathbb{P}^3$.

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under the isomorphism given in (i), the distribution \((D_0)^{a_2}\) corresponds to \(T_\phi\), and it is holomorphic on \(\mathcal{X}_0\);

(iv) \(\phi(C_1)\) is a line in \(\mathbb{P}^3\) for each \([C_1] \in K^{a_1}(\mathcal{X}_0)\);

(v) along any line \(l\) in \(\mathbb{P}^3\), \(4 \leq \deg(V|_l) \leq 6\).

**Proof** By Proposition 4.15, there exists a vector bundle \(V\) on \(\mathbb{P}^3\) satisfying the properties (i) and (ii). Hence \((D_0)^{a_2} = T_\phi\), verifying (iii). By Proposition 3.6, \(\phi(C_1)\) is a line in \(\mathbb{P}^3\), verifying (iv). Since \(\deg(V \otimes O_{\mathbb{P}^1}(k))|_l = \deg(V|_l) + 3k\), we obtain the uniqueness of \(V\) with the property (v).

\[\square\]

**Notation 4.17** In the rest of Sect. 4, we fix the vector bundle \(V\) as in Proposition 4.16. We use \(\phi : \mathbb{P}(V) \to \mathbb{P}^3\) to represent \(\pi_0^{a_2} : \mathcal{X}_0 \to \mathcal{X}_0^{a_2}\). For \(t \in \mathbb{P}^3\) general, we denote by \(\mathbb{P}^2_t := \phi^{-1}(t)\).

Now let us check the splitting types of various meromorphic distributions and their quotients along general elements in \(K^{a_1}(\mathcal{X}_0)\).

**Proposition 4.18** Take \([C_1] \in K^{a_1}(\mathcal{X}_0)\) general. Then \((D_0)^{a_1}, (D_0)^{a_2}, D_0, (D_0)^{-2}, (D_0)^{-3}, N\) are holomorphic vector bundles in an open neighborhood of \([C_1] \subset \mathcal{X}_0\). Moreover,

\[
\begin{align*}
\mathcal{N}|_{C_1} &= O_{\mathbb{P}^1}, \\
(D_0)^{a_1}|_{C_1} &= O_{\mathbb{P}^1}(2), \\
(D_0)^{a_2}|_{C_1} &= O_{\mathbb{P}^1}(-2) \oplus O_{\mathbb{P}^1}, \\
(D_0)^{a_2}/\mathcal{N}|_{C_1} &= O_{\mathbb{P}^1}(-2), \\
(D_0)^{-2}/D_0|_{C_1} &= O_{\mathbb{P}^1}, \\
(D_0)^{-3}/(D_0)^{-2}|_{C_1} &= O_{\mathbb{P}^1}(2), \\
D_0|_{C_1} &= (D_0)^{a_1} \oplus (D_0)^{a_2}|_{C_1} = O_{\mathbb{P}^1}(2) \oplus O_{\mathbb{P}^1}(-2) \oplus O_{\mathbb{P}^1}, \\
(D_0)^{-3}|_{C_1} &= T\mathcal{X}_0|_{C_1} = O_{\mathbb{P}^1}(2) \oplus O_{\mathbb{P}^1}^4.
\end{align*}
\]

**Proof** The restriction \((D_0)^{a_2}|_{C_1} = TC_1 = O_{\mathbb{P}^1}(2)\). Choose a holomorphic family \([l_t] \in K^{a_1}(\mathcal{X}_t), t \in \Delta\) satisfying \([l_0] = [C_1] \in K^{a_1}(\mathcal{X}_0)\). By Proposition 4.16(ii), \(\deg((D_0)^{a_2}|_{l_t}) = \deg(T_{\pi_0^{a_2}}|_{l_t}) = \deg(T_{\pi_t^{a_2}}|_{l_t})\) for all \(t \in \Delta\). By Proposition 2.12, we have \(\deg(T_{\pi_t^{a_2}}|_{l_t}) = \langle \alpha_2, \alpha_1 \rangle + \langle \alpha_2 + \alpha_3, \alpha_1 \rangle = -2\) for \(t \neq 0\). It follows that \(\deg((D_0)^{a_2}|_{C_1}) = -2\). Then can write \((D_0)^{a_2}|_{C_1} = O_{\mathbb{P}^1}(a_1) \oplus O_{\mathbb{P}^1}(a_2)\), where \(a_1 + a_2 = -2\). Since \((D_0)^{-3}|_{C_1} = T\mathcal{X}_0|_{C_1} = O_{\mathbb{P}^1}(2) \oplus O_{\mathbb{P}^1}^4\), and \((D_0)^{a_1}|_{C_1} = O_{\mathbb{P}^1}(2)\), we know that \(a_1 \leq 0, a_2 \leq 0\). Hence

\[
\text{either } (D_0)^2|_{C_1} = O_{\mathbb{P}^1}(-1)^2, \text{or } (D_0)^2|_{C_1} = O_{\mathbb{P}^1}(-2) \oplus O_{\mathbb{P}^1}.
\]

It follows that

\[
((D_0)^{a_2}/\mathcal{N})|_{C_1} = O_{\mathbb{P}^1}(a), \text{ where } a \geq -2.
\]
The injectivity of the homomorphism \((D_0)^{a_1} \otimes ((D_0)^{a_2}/\mathcal{N}) \to (D_0)^{-2}/D_0 \subset T_X_0/D_0\) in an open neighborhood of \(C_1 \subset X_0\) implies that

\[
(D_0)^{-2}/D_0|_{C_1} = \mathcal{O}_{\mathbb{P}^1}(b), \text{ where } b \geq a + 2. \tag{4.7}
\]

The injectivity of \((D_0)^{a_1} \otimes ((D_0)^{-2}/D_0) \to T_X_0/(D_0)^{-2}\) in an open neighborhood of \(C_1 \subset X_0\) implies that

\[
(D_0)^{-3}/(D_0)^{-2}|_{C_1} = T_X_0/(D_0)^{-2}|_{C_1} = \mathcal{O}_{\mathbb{P}^1}(c), \text{ where } c \geq b + 2. \tag{4.8}
\]

On the other hand, the injectivity of \((D_0)^{a_2} \to D_0/(D_0)^{a_1} \subset T_X_0/(D_0)^{a_1}\) in an open neighborhood of \(C_1 \subset X_0\) implies that

\[
\deg(D_0/(D_0)^{a_1})|_{C_1} \geq \deg((D_0)^{a_2}|_{C_1}) = -2. \tag{4.9}
\]

We also have

\[
\deg(T_X|_{C_1}) - \deg((D_0)^{a_1})|_{C_1} - \deg((D_0)^{-1}/(D_0)^{a_1})|_{C_1} = \deg((D_0)^{-2}/D_0)|_{C_1} + \deg((D_0)^{-3}/(D_0)^{-2})|_{C_1}.
\]

By (4.6)–(4.10), we have

\[
2 \geq -\deg(D_0/(D_0)^{a_1})|_{C_1} = \deg((D_0)^{-2}/D_0)|_{C_1} + \deg((D_0)^{-3}/(D_0)^{-2})|_{C_1} = b + c \geq 2b + 2 \geq 2a + 6 \geq 2.
\]

Hence \(\deg(D_0/(D_0)^{a_1})|_{C_1} = -2, a = -2, b = 0\) and \(c = 2\). Then we know \((D_0)^{a_2}|_{C_1} = \mathcal{O}_{\mathbb{P}^1}(-2) \oplus \mathcal{O}_{\mathbb{P}^1} \cong (D_0/(D_0)^{a_1})|_{C_1}\) from the formula (4.5) and the fact \(\deg((D_0)^{a_2}/\mathcal{N})|_{C_1} = a = -2 = \deg((D_0/(D_0)^{a_1})|_{C_1}\). The rest of the conclusion follows immediately. \(\square\)

**Proposition 4.19** In the setting of Proposition 4.16, \(\mathcal{V}|_{\pi_0^{a_2}(C_1)} = \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}\) for \([C_1] \in \mathcal{K}^{a_1}(X_0)\) general.

**Proof** Take \([C_1] \in \mathcal{K}^{a_1}(X_0)\) general. Let \(\mathcal{L}(C_1)\) be the line subbundle of \(\mathcal{V}\) over the line \(\pi_0^{a_2}(C_1) \subset \mathbb{P}^3\) such that \(C_1 = \mathbb{P}(\mathcal{L}(C_1)) \subset \mathbb{P}(\mathcal{V}) = X_0\). Then the relative tangent bundle \(T_{\pi_0^{a_2}}|_{C_1} = \mathcal{L}(C_1)^* \otimes (\mathcal{V}|_{C_1}/\mathcal{L}(C_1))\). By Proposition 4.16(iii) and Proposition 4.18, \(T_{\pi_0^{a_2}}|_{C_1} = (D_0)^{a_2}|_{C_1} = \mathcal{O}_{\mathbb{P}^1}(-2) \oplus \mathcal{O}_{\mathbb{P}^1}\). Then \(\mathcal{V}|_{\pi_0^{a_2}(C_1)} = \mathcal{O}_{\mathbb{P}^1}(k)^2 \oplus \mathcal{O}_{\mathbb{P}^1}(k - 2)\), where \(k := \deg \mathcal{L}(C_1)\). By Proposition 4.16(v), \(k = 2\) and the conclusion follows. \(\square\)

**Proposition 4.20** Let \(\mathcal{V}\) be as in Proposition 4.16. Then the followings hold.

(i) **Along any line** \(l \subset \mathbb{P}^3\), either \(\mathcal{V}|_l = \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(1)^2\) or \(\mathcal{V}|_l = \mathcal{O}_{\mathbb{P}^1}(2)^2 \oplus \mathcal{O}_{\mathbb{P}^1}\).

(ii) **Take any** \([C_1] \in \mathcal{K}^{a_1}(X_0)\). Then \(\mathcal{L}(C_1) = \mathcal{O}_{\mathbb{P}^1}(2)\), where \(\mathcal{L}(C_1)\) is the unique line subbundle of \(\mathcal{V}|_{\pi_0^{a_2}(C_1)}\) such that \(C_1 = \mathbb{P}(\mathcal{L}(C_1)) \subset \mathbb{P}(\mathcal{V}) = X_0\).
Proof Since $X_0 \cong \mathbb{P}(\mathcal{V})$ by Proposition 4.16, any $[C_1] \in \mathcal{K}^{a_2}(X_0)$ must be a section over the line $\pi_0^{a_2}(C_1) \subset \mathbb{P}^3$ with the largest degree. The degree of this section over the line $\pi_0^{a_2}(C_1)$ is independent of the choice of $[C_1] \in \mathcal{K}^{a_2}(X_0)$. Then the assertion (ii) follows from Proposition 4.19.

Take any line $l \subset \mathbb{P}^3$. Then by Proposition 4.16, $\mathcal{V}|_l$ is a deformation of $\mathcal{V}|_{\pi_0^{a_2}(C_1)} = O_{\mathbb{P}^1}(2)^2 \oplus O_{\mathbb{P}^1}$. Thus we can write $\mathcal{V}|_l = O_{\mathbb{P}^1}(a_1) \oplus O_{\mathbb{P}^1}(a_2) \oplus O_{\mathbb{P}^1}(a_3)$, where

$$a_1 + a_2 + a_3 = 4, \quad \text{and} \quad a_1 \geq a_2 \geq a_3.$$  \hfill (4.10)

By the maximality of $\deg \mathcal{L}(C_1)$ among the sections of $\mathcal{V}$ over lines in $\mathbb{P}^3$, we have

$$a_1 \leq \deg \mathcal{L}(C_1) = 2.$$  \hfill (4.11)

The assertion (i) follows from (4.10) and (4.11). \hfill \Box

Corollary 4.21 Let $\mathcal{V}$ be as in Proposition 4.16. Then there exists a nonempty Zariski open subset $U \subset \mathbb{P}^3$ and a section of $\pi_0^{a_2} : X_0 = \mathbb{P}(\mathcal{V}) \to \mathbb{P}^3$ over $U$, denoted by $\sigma : U \to X_0$, such that for any $x \in (\pi_0^{a_2})^{-1}(U) \setminus \sigma(U)$,

(i) $\mathcal{N}$ is holomorphic at $x$;

(ii) $\mathcal{N}_x = T_x l_x$, where $l_x := \langle x, \sigma(\pi_0^{a_2}(x)) \rangle$ is the line in $(\pi_0^{a_2})^{-1}(\pi_0^{a_2}(x)) \cong \mathbb{P}^2$ joining $x$ and $\sigma(\pi_0^{a_2}(x))$;

(iii) the leaf of $\mathcal{N}$ at $x$ is the affine line $l_x \setminus \sigma(\pi_0^{a_2}(x))$.

Proof Take $[C_1] \in \mathcal{K}^{a_2}(X_0)$ general. Then $\pi_0^{a_2}(C_1)$ is a line in $\mathbb{P}^3$ and $\mathcal{V}|_{\pi_0^{a_2}(C_1)} = O_{\mathbb{P}^1}(2)^2 \oplus O_{\mathbb{P}^1}$. The curve $C_1$ is identified with

$$\mathbb{P}(\mathcal{L}(C_1)) \subset \mathbb{P}(\mathcal{V}|_{\pi_0^{a_2}(C_1)}) = (\pi_0^{a_2})^{-1}(\pi_0^{a_2}(C_1)),$$

where $\mathcal{L}(C_1) \cong O_{\mathbb{P}^1}(2) \subset \mathcal{V}|_{\pi_0^{a_2}(C_1)}$ is as in Proposition 4.20(ii). We know

$$\mathcal{N} \subset (\mathbb{D}_0)^{a_2}, \quad \mathcal{N}|_{C_1} = O_{\mathbb{P}^1} \text{ and } O_{\mathbb{P}^1}(-2) \oplus O_{\mathbb{P}^1} = (\mathbb{D}_0)^{a_2}|_{C_1} \cong \mathcal{L}(C_1)^* \otimes (\mathcal{V}|_{\pi_0^{a_2}(C_1)}/\mathcal{L}(C_1)).$$

It follows that

$$\mathcal{N}|_{C_1} = \bigcup_{x \in C_1} T_x \mathbb{P}(\mathcal{V}|_{\pi_0^{a_2}(x)}^{+}),$$

where $\mathcal{V}|_{\pi_0^{a_2}(C_1)} = O_{\mathbb{P}^1}(2)^2 \subset \mathcal{V}|_{\pi_0^{a_2}(C_1)} = O_{\mathbb{P}^1}(2)^2 \oplus O_{\mathbb{P}^1} \text{ and } T_{C_1} \mathbb{P}(\mathcal{V}|_{\pi_0^{a_2}(C_1)}^{+})$ is the relative tangent bundle of $\mathbb{P}(\mathcal{V}|_{\pi_0^{a_2}(C_1)}^{+}) \to \pi_0^{a_2}(C_1)$ along $C_1 \subset \mathbb{P}(\mathcal{V}|_{\pi_0^{a_2}(C_1)}^{+})$. In other words, at any point $x \in C_1$,

$$\mathcal{N}_x = T_x \mathbb{P}(O_{\pi_0^{a_2}(C_1)}(2)^2|_{\pi_0^{a_2}(x)}),$$
where $O_{\pi_0^{a_2}(C_1)}(2) |_{\pi_0^{a_2}(x)} \subset V_{\pi_0^{a_2}(x)}$ is the fiber of $O_{\mathbb{P}^1}(2) |_{\pi_0^{a_2}(C_1)}$ at the point $\pi_0^{a_2}(x) \in \pi_0^{a_2}(C_1)$.

Note that $\mathbb{P}_{\pi_0^{a_2}(C_1)}(O_{\mathbb{P}^1}(2)) \cong \mathbb{P}^1 \times \pi_0^{a_2}(C_1) \cong \mathbb{P}^1$. It follows that given any $x \in C_1$ and any $y \in \mathbb{P}(O_{\pi_0^{a_2}(C_1)}(2) |_{\pi_0^{a_2}(x)})$ lying in the regular locus of $\mathcal{N}$, there exists $[C_y] \in \mathcal{K}_{\pi_0^{a_2}}(\mathcal{X}_0)$ such that

$$\pi_0^{a_2}(C_y) = \pi_0^{a_2}(C_1) \text{ and } \mathcal{N}_y = T_y \mathbb{P}(O_{\pi_0^{a_2}(C_1)}(2) |_{\pi_0^{a_2}(x)}).$$

Hence, the closure of the leaf at $x \in C_1 \subset \mathcal{X}_0$ is the line $l_x = \mathbb{P}(O_{\pi_0^{a_2}(C_1)}(2) |_{\pi_0^{a_2}(x)})$.

Take $t \in \mathbb{P}^3$ general and denote by $\mathbb{P}^2_t := (\pi_0^{a_2})^{-1}(t) \equiv \mathbb{P}^2 \subset \mathcal{X}_0$. Let $A \subset \mathbb{F}(1, \mathbb{P}^2_t)$ be the closure of the family of lines $l_x := \mathbb{P}(O_{\pi_0^{a_2}(C_x)}(2) |_{\pi_0^{a_2}(x)})$, where $x$ runs over the set of general points on $\mathbb{P}^2_t$ such that $\mathcal{K}_{\mathcal{X}_0}^t(\mathcal{X}_0)$ consists of a unique element $[C_x]$ and $\mathcal{N}$ is holomorphic at $x$. For a general point $x \in \mathbb{P}^2_t$, $E_x := \{[l] \in \mathbb{F}(1, \mathbb{P}^2_t) \}$ is a line in $\mathbb{F}(1, \mathbb{P}^2_t)$ and $E_x \cap A$ consist of a single point in $\mathbb{F}(1, \mathbb{P}^2_t)$, namely $[l_x]$. Since $E_x$ could be a general line in $\mathbb{F}(1, \mathbb{P}^2_t)$, the intersection number $(E_x \cdot A) = 1$. It follows that $A$ is a line in $\mathbb{F}(1, \mathbb{P}^2_t)$ and there exists a unique point $\sigma(t) \in \mathbb{P}^2_t$ such that $A = \{[l] \in \mathbb{F}(1, \mathbb{P}^2_t) \mid \sigma(t) \in l\}$.

It turns out that $\mathcal{N}$ is well-defined on $\mathbb{P}^2_t \setminus \{\sigma(t)\}$, and at any $x \in \mathbb{P}^2_t \setminus \{\sigma(t)\}$, the line $\langle x, \sigma(t) \rangle$ is the leaf closure of $\mathcal{N}$ at $x$. The conclusion follows. \(\square\)

### 4.3 A subset of the family of lines on $\mathbb{P}^3$

**Notation 4.22** For $t \in \mathbb{P}^3$ general, denote by $\mathcal{K}_{\mathcal{X}_0}^t(\mathcal{X}_0)$ the Zariski closure of

$$\{[\pi_0^{a_2}(C_x)] \in \mathbb{F}(1, \mathbb{P}^3) \mid x \in (\pi_0^{a_2})^{-1}(t) \text{ general, } \mathcal{K}_{\mathcal{X}_0}^t(\mathcal{X}_0) = \{[C_x]\}\}$$

in $\mathbb{F}(1, \mathbb{P}^3)$, and set

$$C_{t}^\alpha(\mathcal{X}_0/\mathbb{P}^3) := \bigcup_{[l] \in \mathcal{K}_{\mathcal{X}_0}^t(\mathcal{X}_0/\mathbb{P}^3)} \mathbb{P}(T_l l) \subset \mathbb{P}(T_l \mathbb{P}^3).$$

Denote by

$$\mathcal{K}^\alpha(\mathcal{X}_0/\mathbb{P}^3) := \text{Zariski closure of } \bigcup_{t \in \mathbb{P}^3 \text{ general}} \mathcal{K}_{\mathcal{X}_0}^t(\mathcal{X}_0/\mathbb{P}^3) \text{ in } \mathbb{F}(1, \mathbb{P}^3),$$

$$C^\alpha(\mathcal{X}_0/\mathbb{P}^3) := \text{Zariski closure of } \bigcup_{t \in \mathbb{P}^3 \text{ general}} C_{t}^\alpha(\mathcal{X}_0/\mathbb{P}^3) \text{ in } \mathbb{P}(T \mathbb{P}^3).$$

Let $U^\alpha(\mathcal{X}_0/\mathbb{P}^3)$ be the inverse image of $\mathcal{K}^\alpha(\mathcal{X}_0/\mathbb{P}^3)$ under the natural morphism $F(1, 2, \mathbb{C}^*) \to \mathbb{F}(1, \mathbb{P}^3) \supset \mathcal{K}^\alpha(\mathcal{X}_0/\mathbb{P}^3)$.

**Lemma 4.23** Take $t \in \mathbb{P}^3$ general. Then $\mathcal{K}_{\pi_0^{a_2}}(\mathcal{X}_0/\mathbb{P}^3)$ is an irreducible rational curve. Take any $[l] \in \mathcal{K}_{\pi_0^{a_2}}(\mathcal{X}_0/\mathbb{P}^3)$. There exists $[C] \in \mathcal{K}_{\pi_0^{a_2}}(\mathcal{X}_0)$ such that $\pi_0^{a_2}(C) = l$. 

\(\square\)
**Proof** Take \( t \in \mathbb{P}^3 \) general and \( x \in \mathbb{P}^2_t := (\pi_0^{|\alpha_2|})^{-1}(t) \) general. Then \( K_{\sigma_1}^x \) consists of a single element, written as \([C_x]\). Furthermore, \( C_x \cong \mathbb{P}^1 \) and \( \pi_0^{|\alpha_2|} \) sends \( C_x \) biholomorphically onto a line in \( \mathbb{P}^3 \). Since \( \mathcal{V}|_{\pi_0^{|\alpha_2|}(C_x)} = \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1} \) and the line \( \langle x, \sigma(t) \rangle \) in \( \mathbb{P}^2_t \) coincides with the fiber \( \mathbb{P}(\mathcal{O}_{\pi_0^{|\alpha_2|}(C_x)}(2)|_t) \), there exists a unique \([C_{t,x}] \in K_{\sigma_1}^x(\mathcal{X}_0)\) such that \( \pi_0^{|\alpha_2|}(C_{t,x}) = \pi_0^{|\alpha_2|}(C_x) \). Take \( y \in \mathbb{P}_t^2 \setminus \{x, \sigma(x)\} \) general. Then the fact that

\[
\mathbb{P}(\mathcal{O}_{\pi_0^{|\alpha_2|}(C_x)}(2)|_t) \cap \mathbb{P}(\mathcal{O}_{\pi_0^{|\alpha_2|}(C_y)}(2)|_t) = \langle x, \sigma(t) \rangle \cap \langle y, \sigma(t) \rangle = \{\sigma(t)\}
\]

implies that \( \pi_0^{|\alpha_2|}(C_x) \neq \pi_0^{|\alpha_2|}(C_y) \) (and hence \( C_{t,x} \neq C_{t,y} \)). This induces injective rational maps (hence injective morphisms)

\[
\xi : \mathbb{P}^1 \cong \{[l] \in \mathbb{P}(1, \mathbb{P}^2) \mid \sigma(t) \in l\} \rightarrow K_{\sigma_1}(\mathcal{X}_0)
\]

\[
\langle x, \sigma(t) \rangle \mapsto [C_{t,x}],
\]

\[
\eta : \mathbb{P}^1 \cong \{[l] \in \mathbb{P}(1, \mathbb{P}^2) \mid \sigma(t) \in l\} \rightarrow K_{\sigma_1}(\mathcal{X}_0/\mathbb{P}^3)
\]

\[
\langle x, \sigma(t) \rangle \mapsto [\pi_0^{|\alpha_2|}(C_x)].
\]

By definition \( K_{\sigma_1}(\mathcal{X}_0/\mathbb{P}^3) \) is the closure of the image of \( \eta \). Then the conclusion follows immediately from the morphisms \( \xi \) and \( \eta \) obtained above.

The following can also be deduced from the proof of Lemma 4.23.

**Lemma 4.24** Take \( t \in \mathbb{P}^3 \) general. Define

\[
\psi : \mathbb{P}_t^2 \rightarrow K_{\sigma_1}(\mathcal{X}_0/\mathbb{P}^3) \subset \mathbb{P}(1, \mathbb{P}^3)
\]

\[
x \longmapsto [\pi_0^{|\alpha_2|}(C_x)],
\]

where \( x \in \mathbb{P}_t^2 \) general and \([C_x]\) is the unique element of \( K_{\sigma_1}^x(\mathcal{X}_0) \). Then \( \psi \) coincides with the linear projection of \( \mathbb{P}_t^2 \) with center \( \sigma(t) \). In other words, given \( x \) and \( y \) in the domain of \( \psi \), \( \psi(x) = \psi(y) \) if and only if \( \langle x, \sigma(t) \rangle = \langle y, \sigma(t) \rangle \).

**Construction 4.25** Take \( x \in \mathcal{X}_0 \) general. Recall that there are two elementary Mori contractions, namely \( \pi_0^{|\alpha_1|} : \mathcal{X}_0 \rightarrow \mathcal{X}_0^{\alpha_1} \) and \( \pi_0^{|\alpha_2|} : \mathcal{X}_0 = \mathcal{V} \rightarrow \mathcal{X}_0^{\alpha_2} = \mathbb{P}^3 \). Set \( \Sigma_0(x) := \{x\} \). For each \( k \geq 0 \) let \( \Sigma_{2k+1}(x) \) be the unique irreducible component of \((\pi_0^{|\alpha_2|})^{-1}(\pi_0^{|\alpha_2|}(\Sigma_{2k}(x)))\) dominating \( \pi_0^{|\alpha_2|}(\Sigma_{2k}(x)) \), and \( \Sigma_{2k+2}(x) \) be the unique irreducible component of \((\pi_0^{|\alpha_1|})^{-1}(\pi_0^{|\alpha_1|}(\Sigma_{2k+1}(x)))\) dominating \( \pi_0^{|\alpha_1|}(\Sigma_{2k+1}(x)) \).

**Lemma 4.26** In the setting of Construction 4.25, \( \dim \Sigma_k(x) = k + 1 \) for \( 1 \leq k \leq 4 \). In particular, \( \Sigma_4(x) = \mathcal{X}_0 \).

**Proof** By the construction, \( \Sigma_1(x) = (\pi_0^{|\alpha_2|})^{-1}(\pi_0^{|\alpha_2|}(x)) \cong \mathbb{P}^2 \), which has dimension 2. Now we claim that for each \( k \geq 1 \), either \( \Sigma_k(x) = \mathcal{X}_0 \) or \( \dim \Sigma_{k+1}(x) \geq \dim \Sigma_k(x) + 1 \).

Suppose \( \dim \Sigma_{k+1}(x) = \dim \Sigma_k(x) \) for some \( k \geq 1 \). Then \( \Sigma_{k+1}(x) = \Sigma_k(x) \). By the construction of \( \Sigma_k(x) \) and \( \Sigma_{k+1}(x) \), \( C_1^y \subset \Sigma_k(x) \) and \( C_2^y \subset \Sigma_k(x) \) for \( y \in \Sigma_k(x) \) general, \([C_y]\) \( \in K_{\sigma_1}^y(\mathcal{X}_0) \) and \([C_y^2]\) \( \in K_{\sigma_1}^y(\mathcal{X}_0) \) general. By Proposition 3.7, we have \( \Sigma_k(x) = \mathcal{X}_0 \), and the claim holds.
By the general choice of \( x \in \mathcal{X}_0 \) and the construction of \( \Sigma_k(x) \), for each \( i \geq 1 \) we have
\[
\begin{align*}
\dim \Sigma_{2i+1}(x) &\leq \dim \pi_0^{\alpha_2}(\Sigma_{2i}(x)) + 2 \leq \dim \Sigma_{2i}(x) + 2, \\
\dim \Sigma_{2i}(x) &\leq \dim \pi_0^{\alpha_1}(\Sigma_{2i-1}(x)) + 1 \leq \dim \Sigma_{2i-1}(x) + 1.
\end{align*}
\]

Note that
\[
\pi_0^{\alpha_2}(\Sigma_2(x)) = \bigcup_{[l] \in K_{\pi_0^{\alpha_2}(x)} \Lambda_1(\mathcal{X}_0/\mathbb{P}^3)} l,
\]
which has dimension 2 by Lemma 4.23. Then the conclusion follows from the inequalities above. \( \square \)

**Lemma 4.27** Take \( t \in \mathbb{P}^3 \) general. Denote by \( \Lambda_1(t) := \bigcup_{[l] \in K_{\pi_0^{\alpha_2}(\mathcal{X}_0/\mathbb{P}^3)}} l \subset \mathbb{P}^3 \), and let \( \Lambda_2(t) \) be the Zariski closure of \( \bigcup_{[l] \in K_{\pi_0^{\alpha_2}(\mathcal{X}_0/\mathbb{P}^3)}} l \) in \( \mathbb{P}^3 \), where we define \( K_{\Lambda_1(t)}^r \) to be the union of \( K_{\alpha_1}(\mathcal{X}_0/\mathbb{P}^3) \) with \( z \) runs over the set of general points in \( \Lambda_1(t) \). Then \( \Lambda_2(t) = \mathbb{P}^3 \).

**Proof** Take \( x \in \mathbb{P}^2 \) general, then by construction we have \( \pi_0^{\alpha_2}(\Sigma_{2k}(x)) = \Lambda_k(t) \) for \( 1 \leq k \leq 2 \), where \( \Sigma_{2k}(x) \) is as in Construction 4.25. By Lemma 4.26, \( \Sigma_4(x) = \mathcal{X}_0 \), which implies the conclusion. \( \square \)

**Lemma 4.28** Let \( \mathcal{L}_\sigma \subset \mathcal{V} \) be the meromorphic line subbundle of \( \mathcal{V} \) over \( \mathbb{P}^3 \) defining the meromorphic section \( \sigma \) of \( \pi_0^{\alpha_2} : \mathcal{X}_0 = \mathbb{P}(\mathcal{V}) \rightarrow \mathbb{P}^3 \), and \( S_\sigma \) be the singular locus of \( \sigma \). Then \( \dim S_\sigma \leq 1 \) and there exist nonempty Zariski open subsets \( U'' \subset U' \subset \mathbb{P}^3 \setminus S_\sigma \) such that

(i) \( C_1 \subset \mathbb{P}(\mathcal{L}_\sigma) \) for any \( t \in U' \) and any \([C_1] \in K_{\sigma_1(t)}^r(\mathcal{X}_0)\);

(ii) given any \( t \in U'' \) we have \( M_2(t) = \mathbb{P}(\mathcal{L}_\sigma) \), where we set \( M_1(t) := \bigcup_{[C] \in K_{\sigma_1(t)}^r(\mathcal{X}_0)} C \subset \mathbb{P}(\mathcal{L}_\sigma) \), let \( M_2(t) \) be the Zariski closure of \( \bigcup_{[C] \in K_{\sigma_1(t)}^r(\mathcal{X}_0)} C \) in \( \mathbb{P}(\mathcal{L}_\sigma) \), where we denote \( K_{\sigma_1(t) \cap \sigma}(U') := \bigcup_{x \in M_1(t) \cap \sigma(U')} K_{\pi_0^{\alpha_2}(U')} \).

**Proof** Being the singular locus of a meromorphic section, the dimension of \( S_\sigma \) is less or equal to \( \dim \mathbb{P}^3 - 2 = 1 \). By Lemma 4.23, \( \dim K_{\sigma_1(t)}^r(\mathcal{X}_0) \geq 1 \) for \( t \in \mathbb{P}^3 \) general. By the semicontinuity of the dimension function, \( \dim K_{\sigma_1}^r(\mathcal{X}_0) \geq 1 \) for all \( x \in \mathbb{P}(\mathcal{L}_\sigma) \). Hence \( \mathbb{P}(\mathcal{L}_\sigma) \subset E(K_{\sigma_1}^r) \), where \( E(K_{\sigma_1}^r) \subset \mathcal{X}_0 \) is the union of \( K_{\sigma_1}(\mathcal{X}_0) \)-equivalence classes that are of dimension at least two. By Proposition 4.9, \( E(K_{\sigma_1}^r) \) is a Zariski closed subset of \( \mathcal{X}_0 \) and \( \dim E(K_{\sigma_1}^r) \leq \dim \mathcal{X}_0 - 2 = 3 \). By dimensional reasons, the variety \( \mathbb{P}(\mathcal{L}_\sigma) \) is an irreducible component of \( E(K_{\sigma_1}^r) \).

Denote by \( U \) the nonempty Zariski open subset of \( \mathbb{P}(\mathcal{L}_\sigma) \) such that at any \( x \in U \), \( \mathbb{P}(\mathcal{L}_\sigma) \) is the unique irreducible component of \( E(K_{\sigma_1}^r) \) containing \( x \). Set \( U' := \pi_0^{\alpha_2}(U) \setminus S_\sigma \), then the assertion (i) of Lemma 4.28 holds.

By Lemma 4.23, \( \pi_0^{\alpha_2}(M_2(t)) = \Lambda_k(t) \) for \( k = 1, 2 \). Then by Lemma 4.27 \( \phi(M_2(t)) = \mathbb{P}^3 \), implying \( \dim M_2(t) \geq 3 \). Since \( M_2(t) \subset \mathbb{P}(\mathcal{L}_\sigma) \) by (i), we have \( M_2(t) = \mathbb{P}(\mathcal{L}_\sigma) \), verifying (ii). \( \square \)
Lemma 4.29  For \( t \in \mathbb{P}^3 \) general, \( C_{t}^{\alpha i}(X_0/\mathbb{P}^3) \) is a line in \( \mathbb{P}(T_t\mathbb{P}^3) \). Furthermore, \( K^{\alpha i}(X_0/\mathbb{P}^3) \) is a hyperplane section of \( \mathbb{P}(1, \mathbb{P}^3) \subset \mathbb{P}^5 \).

**Proof**  By Proposition 4.5, \((D_0)_{\alpha^2}, (D_0)^{-2}\) \( \subset (D_0)^{-2}\). It follows that \( E := d\pi_{0, \alpha^2}((D_0)^{-2}) \) is a meromorphic distribution \( E \) on \( \mathbb{P}^3 \) of rank 2, where \( d\pi_{0, \alpha^2} : T_X \to T\mathbb{P}^3 \) is the tangent map of \( \pi_{0, \alpha^2} \). Take a general element \([C_1] \in K^{\alpha i}(X_0)\). Then \( TC_1 = (D_0)^{\alpha^2}_{\alpha^2}|C_1 \subset (D_0)^{-2}\), which implies that \( T(\pi_{0, \alpha^2}^{-1}(C_1)) \subset E|_{\pi_{0, \alpha^2}^{-1}(C_1)} \). Hence at a general point \( t \in \mathbb{P}^3 \), we have \( C_{t}^{\alpha i}(X_0/\mathbb{P}^3) \subset \mathbb{P}(E_t) \). Since \( K^{\alpha i}(X_0/\mathbb{P}^3) \) is a set of lines on \( \mathbb{P}^3 \), we have \( C_{t}^{\alpha i}(X_0/\mathbb{P}^3) \cong K_{t}^{\alpha i}(X_0/\mathbb{P}^3) \), which is an irreducible rational curve by Lemma 4.23. Hence \( C_{t}^{\alpha i}(X_0/\mathbb{P}^3) = \mathbb{P}(E_t) \) is a line in \( \mathbb{P}(T_t\mathbb{P}^3) \). Moreover,

\[
\dim K^{\alpha i}(X_0/\mathbb{P}^3) = \dim \mathbb{P}^3 + \dim K_{t}^{\alpha i}(X_0/\mathbb{P}^3) - 1 = 3.
\]

Thus the variety \( K^{\alpha i}(X_0/\mathbb{P}^3) \) is an effective divisor on \( \mathbb{P}(1, \mathbb{P}^3) \). Consider

\[
\begin{CD}
U_{\alpha i}(X_0/\mathbb{P}^3) @>>> K^{\alpha i}(X_0/\mathbb{P}^3) \\
\downarrow @VVV \downarrow @VVV \\
\mathbb{P}^3 @<<< \mathbb{P}(T\mathbb{P}^3) @>>> \mathbb{P}(1, \mathbb{P}^3).
\end{CD}
\]

Since for \( t \in \mathbb{P}^3 \) general, \( C_{t}^{\alpha i}(X_0/\mathbb{P}^3) = U_{\alpha i}(X_0/\mathbb{P}^3) \cap \mathbb{P}(T_t\mathbb{P}^3) \) is a line in \( \mathbb{P}(T_t\mathbb{P}^3) \), we can conclude that \( K^{\alpha i}(X_0/\mathbb{P}^3) \in |\pi^*\mathcal{O}_{\mathbb{P}^3}(1)| \), where \( \pi : \mathbb{P}(1, \mathbb{P}^3) \to \mathbb{P}^5 \) is the Plücker embedding. Since \( \mathbb{P}(1, \mathbb{P}^3) \subset \mathbb{P}^5 \) is linearly normal, \( K^{\alpha i}(X_0/\mathbb{P}^3) \) is a hyperplane section of \( \mathbb{P}(1, \mathbb{P}^3) \subset \mathbb{P}^5 \).

\[ \square \]

4.4 Hyperplane bundles of \( \mathbb{P}(\mathcal{V}) \) over \( \mathbb{P}^3 \)

**Notation 4.30**  Let \( L_{0}^{\alpha i} \) be the Cartier divisor on \( X_0 \) such that the intersection number \( (L_{0}^{\alpha i} \cdot C_j) = \delta_{ij} \), where \([C_j] \in K^{\alpha j}(X_0)\) and \( 1 \leq i, j \leq 2 \). In other words, \( L_{0}^{\alpha i} := L^{\alpha i}|_{X_0} \), where \( L^{\alpha i} \) is as in Proposition-Definition 3.6. Denote by \( |L_{0}^{\alpha i}| \) the corresponding linear system of effective Weil divisors on \( X_0 \).

**Lemma 4.31**  We have \( \dim |L_{0}^{\alpha i}| = 3 \) and \( \dim |L_{0}^{\alpha^2}| \geq 5 \).

**Proof**  Since \( X_0^{\alpha^2} = \mathbb{P}^3 \), we have \( L_{0}^{\alpha^2} = (\pi_{0, \alpha^2}^*)^*\mathcal{O}_{\mathbb{P}^3}(1) \) and \( \dim |L_{0}^{\alpha^2}| = \dim \mathbb{P}^3 = 3 \). There exists a holomorphic line bundle \( L^{\alpha^2} \) on \( X \) such that \( L_{0}^{\alpha^2} \cong L^{\alpha^2}|_{X_0} \) and for \( 0 \neq t \in \Delta \), the linear system \( |L_{t}^{\alpha^2}| \) induces the morphism \( X_t \cong F(1, 2, \mathbb{C}^4) \to Gr(2, \mathbb{C}^4) \subset \mathbb{P}^5 \), where \( L_{t}^{\alpha^2} := L^{\alpha^2}|_{X_t} \). By the semicontinuity, we have \( \dim |L_{0}^{\alpha^2}| \geq 5 \).

\[ \square \]

**Notation 4.32**  Take any \( W \in |L_{0}^{\alpha^2}| \). Denote by \( K_{t}^{\alpha i}(W/\mathbb{P}^3) := \psi(W_t) \subset K_{t}^{\alpha i}(X_0/\mathbb{P}^3) \) for \( t \in \mathbb{P}^3 \) general, where \( \psi \) is as in Lemma 4.24. Let \( K_{t}^{\alpha i}(W/\mathbb{P}^3) \subset K_{t}^{\alpha i}(X_0/\mathbb{P}^3) \) be the Zariski closure of the union of those \( K_{t}^{\alpha i}(W/\mathbb{P}^3) \) such that \( t \) runs over the set of general points in \( \mathbb{P}^3 \).
Lemma 4.33  In the setting of Notation 4.32, there is an injective map

\[ \theta : \{W \in |L_0^{\alpha_2}| \mid \mathbb{P}(L_{\sigma}) \subset W\} \to \{\text{hyperplane sections of } K^{\alpha_1}(X_0/\mathbb{P}^3)\} \]

\[ W \mapsto K^{\alpha_1}(W/\mathbb{P}^3). \]

Proof  Take \( t \in \mathbb{P}^3 \) general. Then the fact \( \sigma(t) \in W \) implies that \( W_t \) is a line in \( \mathbb{P}^2_t := (\pi_{0}^{\alpha_2})^{-1}(t) \) passing through \( \sigma(t) \). By Lemma 4.24, \( K^{\alpha_1}(W/\mathbb{P}^3) \) consists of a single element. Then \( K^{\alpha_1}(W/\mathbb{P}^3) \) is an effective divisor on \( K^{\alpha_1}(X_0/\mathbb{P}^3) \). Similarly to the analysis for diagram (4.12), we know that \( K^{\alpha_1}(W/\mathbb{P}^3) \) is a hyperplane section of \( K^{\alpha_1}(X_0/\mathbb{P}^3) \). \( \square \)

Lemma 4.34  Take \( W \in |L_0^{\alpha_2}| \) general. Then \( \sigma(t) \not\in W \) for \( t \in \mathbb{P}^3 \) general.

Proof  By Lemma 4.29 and Lemma 4.33, the space \( \{W \in |L_0^{\alpha_2}| \mid \mathbb{P}(L_{\sigma}) \subset W\} \) has dimension at most 4. On the other hand, \( dim |L_0^{\alpha_2}| \geq 5 \) by Lemma 4.31. Then the conclusion follows. \( \square \)

Lemma 4.35  Take \( W \in |L_0^{\alpha_2}| \) general, and denote by \( S(W) := \{t \in \mathbb{P}^3 \mid (\pi_{0}^{\alpha_2})^{-1}(t) \subset W\} \). Then \( dim(S(W) \leq 1 \) and \( W|_{\mathbb{P}^3\setminus S(W)} \to \mathbb{P}^3\setminus S(W) \) is a \( \mathbb{P}^1 \)-bundle.

Proof  We have the isomorphism of Cartier divisors \( \mathcal{O}_{X_0}(W)|_{\mathbb{P}^3_t} \cong \mathcal{O}_{\mathbb{P}^2}(1) \) for any \( t \in \mathbb{P}^3 \). Thus for any \( t \in \mathbb{P}^3\setminus S(W) \), the scheme-theoretic intersection of \( W \) with \( \mathbb{P}^2_t \) is a line. By the dimension counting, \( dim(S(W) \leq dim W - 2 = 2 \). If \( dim S(W) = 2 \), then the intersection number \( (W \cdot C_1) > 0 \) for \( [C_1] \in K^{\alpha_1}(X_0) \), contradicting our definition of \( L_0^{\alpha_2} \) in Notation 4.30. \( \square \)

Lemma 4.36  Take \( W \in |L_0^{\alpha_2}| \) general, and denote by \( S_W := \pi_{0}^{\alpha_2}(\mathbb{P}(L_{\sigma}) \cap W) \subset \mathbb{P}^3 \). Then \( dim S_W \leq 1 \).

Proof  Now suppose \( dim S_W \geq 2 \). By Lemma 4.34, \( S_W \neq \mathbb{P}^3 \). Choose any irreducible component \( \widetilde{S}_W \) of \( S_W \) such that \( dim \widetilde{S}_W = 2 \).

We claim that for \( \tilde{t} \in \widetilde{S}_W \) general, there exists \( t \in U'' \) and \( [l] \in K^{\alpha_1}_{\tilde{t}}(X_0/\mathbb{P}^3) \) such that \( \tilde{t} \in l \), where \( U'' \) is as in Lemma 4.28 (ii).

Suppose the claim holds. By Lemma 4.23 there exists \( [C] \in K^{\alpha_1}_{\sigma(t)}(X_0) \) such that \( \pi_{0}^{\alpha_2}(C) = l \). By Lemma 4.28, \( dim S_{\sigma} \leq 1 \), where \( S_{\sigma} \subset \mathbb{P}^3 \) is the singular locus of the section \( \sigma \). Then the general choice of \( \tilde{t} \) in the divisor \( \widetilde{S}_W \subset \mathbb{P}^3 \) implies that \( \tilde{t} \notin S_{\sigma} \). In particular, \( \mathbb{P}(L_{\sigma}) \widetilde{t} = \sigma(\tilde{t}) \in C \cap W \). Since the intersection number \( (W \cdot C) = 0 \), we have \( C \subset W \), implying that \( \sigma(t) \in W \). By Lemma 4.28(ii) and the fact \( (W \cdot C_1) = 0 \) for any \([C_1] \in K^{\alpha_1}(X_0)\), we have \( \mathbb{P}(L_{\sigma}) \subset W \). This contradicts Lemma 4.34. Hence we obtain the conclusion of Lemma 4.36.

Now we turn to prove the claim. Suppose it fails. Let \( A \) be the Zariski closure of the union \( \bigcup_{t \in \mathbb{P}^3 \text{ general}} K^{\alpha_1}_{\sigma(t)}(X_0/\mathbb{P}^3) \).

By the assumption, \( dim A \leq dim \widetilde{S}_W - 1 = 1 \).
Since every element in $K^a_3(\mathcal{X}_0/\mathbb{P}^3)$ has a nonempty intersection with $\tilde{S}_W$, there is an irreducible component $\tilde{A}$ of $A$ such that $\dim K^a_3(\mathcal{X}_0/\mathbb{P}^3) \geq \dim K^a_3(\mathcal{X}_0/\mathbb{P}^3) - \dim \tilde{A} \geq 2$ for each $s \in \tilde{A}$. Since $K^a_3(\mathcal{X}_0/\mathbb{P}^3) \cong C^a_3(\mathcal{X}_0/\mathbb{P}^3) \subset \mathbb{P}(T_s\mathbb{P}^3)$, we know that $\dim \tilde{A} = 1$, $K^a_3(\mathcal{X}_0/\mathbb{P}^3) \cong C^a_3(\mathcal{X}_0/\mathbb{P}^3) = \mathbb{P}(T_s\mathbb{P}^3)$, and $[(t, s)] \in K^a_3(\mathcal{X}_0/\mathbb{P}^3)$ for all $s \in \tilde{A}$ and all $t \in \mathbb{P}^3\setminus \{s\}$.

Take $t \in \mathbb{P}^3$ general. By Lemma 4.29 and the conclusions above, $K^a_t(\mathcal{X}_0/\mathbb{P}^3) = \{(t, s) \mid s \in A\}$, and the join variety $J(t, \tilde{A}) := \bigcup_{s \in \tilde{A}}(t, s)$ is a plane in $\mathbb{P}^3$. Thus in the notations of Lemma 4.27, we have $\Lambda_1(t) = J(t, \tilde{A})$. For $t' \in \Lambda_1(t)$ general, the same reason implies that $\Lambda_1(t') = J(t', \tilde{A}) = J(t, \tilde{A}) = \Lambda_1(t)$. It follows that $\Lambda_2(t) = \Lambda_1(t) \subset \mathbb{P}^3$, contradicting Lemma 4.27. Hence, the claim holds. \hfill \Box

**Lemma 4.37** There exists a meromorphic vector subbundle $L_W \subset V$ of rank two over $\mathbb{P}^3$ and a closed subvariety $S_W \subset \mathbb{P}^3$ such that

(i) $\dim S_W \leq 1$;
(ii) both $L_{\sigma}$ and $L_W$ are holomorphic vector bundles on $\mathbb{P}^3\setminus S_W$, where $L_{\sigma}$ is as in Lemma 4.28;
(iii) there is a direct sum decomposition $V|_{\mathbb{P}^3\setminus S_W} = L_{\sigma}|_{\mathbb{P}^3\setminus S_W} \oplus L_W|_{\mathbb{P}^3\setminus S_W}$;
(iv) $\mathbb{P}(L_W) \in |L_W^0|$ is a chosen general divisor.

**Proof** It is a direct consequence of Lemma 4.35, Lemma 4.36 and the fact $\dim S_\sigma \leq 1$, where $S_\sigma$ is the singular locus of the section $\mathbb{P}(L_{\sigma})$. \hfill \Box

To continue, we need to recall a result of decomposition of vector bundles, which can be found on page 409 in [8]. See also [13, Proposition 5] for an explicit statement with a brief proof.

**Proposition 4.38** [8, page 409] Let $E$ be a vector bundle over a connected complex manifold $Y$. Suppose there is a complex subvariety $A \subset Y$ and vector bundles $E_1$ and $E_2$ over $Y \setminus A$ such that $\dim A \leq \dim Y - 2$ and $E|_{Y \setminus A} = E_1 \oplus E_2$. Then $E_1$ and $E_2$ can be extended uniquely as vector bundles $E'_1$ and $E'_2$ over $Y$ such that $E = E'_1 \oplus E'_2$.

As a direct consequence of Lemma 4.37 and Proposition 4.38, we have the following result.

**Proposition 4.39** In the setting of Lemma 4.37, both $L_{\sigma}$ and $L_W$ are holomorphic vector bundles on $\mathbb{P}^3$, and $V = L_{\sigma} \oplus L_W$.

**Lemma 4.40** In the setting of Proposition 4.39, the followings hold.

(i) For any $[l] \in K^a_3(\mathcal{X}_0/\mathbb{P}^3)$, $L_{\sigma}|_l = \mathcal{O}_{\mathbb{P}^1}(2)$ and $L_W|_l = \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}$.
(ii) For any $[l] \in K^a_3(\mathcal{X}_0/\mathbb{P}^3)$, there exists a unique $[C_l] \in K^a_3(\mathcal{X}_0)$ such that $C_l \subset W$ and $\pi^a_3(C_l) = l$. Moreover, this curve $C_l \cong \mathbb{P}^1$.
(iii) For any $x \in W$, $K^a_3(\mathcal{X}_0)$ consists of a single element, denoted by $[C_x]$. Moreover, this curve $C_x \subset W$ and $C_x \cong \mathbb{P}^1$.

**Proof** By Proposition 4.19,

\[ V|_l = \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1} \quad \text{for } [l] \in K^a_3(\mathcal{X}_0/\mathbb{P}^3) \quad \text{general}. \quad (4.13) \]
By Proposition 4.20(i), the restriction of $\mathcal{V}$ on any line of $\mathbb{P}^3$ is either $\mathcal{O}_{\mathbb{P}^1}(2)^2 \oplus \mathcal{O}_{\mathbb{P}^1}$ or $\mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(1)^2$. Then by (4.13), we conclude that

$$\mathcal{V}|_l = \mathcal{O}_{\mathbb{P}^1}(2)^2 \oplus \mathcal{O}_{\mathbb{P}^1} \text{ for } |l| \in K^1(X_0/\mathbb{P}^3) = K^1(\mathbb{P}/\mathbb{P}^3). \quad (4.14)$$

This is because a positive dimensional family of vector bundles over $\mathbb{P}^1$ of type $\mathcal{O}_{\mathbb{P}^1}(2)^2 \oplus \mathcal{O}_{\mathbb{P}^1}$ can not have a limit of type $\mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(1)^2$.

Now take any $|l| \in K^{\alpha_1}(X_0/\mathbb{P}^3)$, we have $L_\sigma|_l = \mathcal{O}_{\mathbb{P}^1}(2)$ by Proposition 4.20(ii). Thus by (4.14) and Proposition 4.39, $L_W|_l = \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}$, verifying the assertion (i). It follows that there exists a unique $[C_I] \in K^{\alpha_1}(X_0)$ such that $C_I \subset W = \mathbb{P}(L_W)$, and $\pi^{\alpha_2}_0(C_I) = l$. In fact $C_I = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(2)|_l) \subset \mathbb{P}((\mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1})|_l) = \mathbb{P}(L_W|_l)$. Moreover $C_I \cong \mathbb{P}^1$, verifying the assertion (ii).

Take any $[C] \in K^{\alpha_1}(X_0)$. Since $(W \cdot C) = 0$, either $C \subset W$ or $C \cap W = \emptyset$. Then the assertion (iii) follows from (i) and (ii).

\begin{lemma}
In the setting of Proposition 4.39, $K^{\alpha_1}(X_0/\mathbb{P}^3)$ is a smooth hyperplane section of $\mathbb{P}(1, \mathbb{P}^3) \subset \mathbb{P}^5$, and $W \cong C_2/(\beta_1 \cap \beta_2)$, where $\beta_1$ and $\beta_2$ are the short and long simple roots of the simple group $C_2$ respectively.
\end{lemma}

\begin{proof}
By Lemma 4.29, $K^{\alpha_1}(X_0/\mathbb{P}^3)$ is a hyperplane section of $\mathbb{P}(1, \mathbb{P}^3) \subset \mathbb{P}^5$. By Proposition 4.9 and Lemma 4.40, there is a $\mathbb{P}^1$-fibration $\varphi : W \rightarrow K^{\alpha_1}(W) = K^{\alpha_1}(X_0/\mathbb{P}^3)$, where $K^{\alpha_1}(W)$ is the set of $[C] \in K^{\alpha_1}(X_0)$ such that $C \subset W$. The variety $K^{\alpha_1}(X_0/\mathbb{P}^3)$ is smooth because so is $W$. Then there exists a nondegenerate form $\omega \in \wedge^2(\mathbb{C}^4)^*$ such that $X_0^{\omega^2} = \mathbb{P}^3 = \mathbb{P}(\mathbb{C}^4)$,

$$K^{\alpha_1}(X_0/\mathbb{P}^3) = \{ [A] \in Gr(2, \mathbb{C}^4) \mid \omega(A, A) = 0 \}. \quad (4.15)$$

and $\pi^{\alpha_2}_0|_W : W \rightarrow \mathbb{P}^3$ is the evaluation morphism of the family $K^{\alpha_1}(X_0/\mathbb{P}^3)$. The conclusion follows.
\end{proof}

Now we can complete the proof of Proposition 1.2.

\begin{proof}[Proof of Proposition 1.2]
By Proposition 4.16 and Proposition 4.39, $X_0 \cong \mathbb{P}(\mathcal{V})$, and $\mathcal{V} \cong \mathcal{L}^{\sigma} \oplus \mathcal{L}_W$. By Proposition 4.20(ii), $\mathcal{L}_{\sigma} \cong \mathcal{O}_{\mathbb{P}^1}(2)$. By Lemma 4.41, $\mathcal{L}_W \cong \mathcal{L}^{\omega} \otimes \mathcal{O}_{\mathbb{P}^1}(k)$ for some $k \in \mathbb{Z}$, where $\omega$ is the symplectic form on $\mathbb{C}^4$ satisfying (4.15). Take any line $l \subset \mathbb{P}^3$. Then $\text{deg}(\mathcal{L}_W|_l) = \text{deg}(\mathcal{V}|_l) - \text{deg}(\mathcal{L}_{\sigma}|_l) = 2$, and $\text{deg}(\mathcal{L}^{\omega}|_l) = \text{deg}(\mathcal{T}\mathbb{P}^3|_l) - \text{deg}(\mathcal{O}_{\mathbb{P}^1}(2)|_l) = 2$. It follows that $k = 0$ and $\mathcal{L}_W \cong \mathcal{L}^{\omega}$. Hence $\mathcal{V} \cong \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{L}^{\omega}$ and $X_0 \cong F^d(1, 2, \mathbb{C}^4)$.
\end{proof}

\section{4.5 Properties of $F^d(1, 2, \mathbb{C}^4)$}

For the convenience of the discussions later, we give several basic properties of the manifold $F^d(1, 2, \mathbb{C}^4)$. All these properties are straightforward from the construction. They have also been proved in a more involved way in the previous arguments in Sect. 4.1–4.4 by realizing $F^d(1, 2, \mathbb{C}^4)$ as the a priori unclear Fano degeneration of $A_3/\mathbb{P}(\alpha_1, \alpha_2)$, see Lemma 4.4, Corollary 4.6, Corollary 4.21, Lemma 4.24 for the corresponding statements of them.

\[\square\] Springer
Notation 4.42 Using the notations in Proposition 1.2, we denote by $\phi : F^d(1, 2, \mathbb{C}^4) \to \mathbb{P}^3$ the $\mathbb{P}^2$-bundle, and let $\sigma : \mathbb{P}^3 \to \mathbb{P}(\mathcal{L}_\sigma) \subset F^d(1, 2, \mathbb{C}^4)$ be the holomorphic section. Given a point $x \in F^d(1, 2, \mathbb{C}^4) \setminus \mathbb{P}(\mathcal{L}_\sigma)$, denote by $l_x$ the line $(x, \sigma(\phi(x)))$ in the projective plane $\phi^{-1}(\phi(x)) \cong \mathbb{P}^2$. By abuse of notations (to be compatible with those in Sects. 4.1–4.4), we denote by $(\mathcal{D}_0)^{\alpha_1}$ the meromorphic distribution of rank one on $F^d(1, 2, \mathbb{C}^4)$ whose general leaves are the minimal rational curves biholomorphically sent to the isotropic lines in $\mathbb{P}^3$, and by $\mathcal{K}^{\alpha_1}(F^d(1, 2, \mathbb{C}^4))$ the closure this family of minimal rational curves. Set $(\mathcal{D}_0)^{\alpha_2} := T_\phi$ and $\mathcal{D}_0 := (\mathcal{D}_0)^{\alpha_1} + (\mathcal{D}_0)^{\alpha_2}$. Denote by $\mathcal{K}^{\alpha_2}(F^d(1, 2, \mathbb{C}^4))$ the family of minimal rational curves which are lines in the fibers of $\phi$.

The following two propositions are immediate from the constructions.

Proposition 4.43 At any point $x \in F^d(1, 2, \mathbb{C}^4) \setminus \mathbb{P}(\mathcal{L}_\sigma)$, $\mathcal{K}_x^{\alpha_1}(X_0)$ consists of a unique element, denoted by $[C_x]$. Two points $y, z \in \mathbb{P}^2 \setminus \{\sigma(t)\}$ satisfy $\phi(C_y) = \phi(C_z)$ if and only if the two lines $\langle y, \sigma(t) \rangle$ and $\langle z, \sigma(t) \rangle$ in $\mathbb{P}^2$ coincide, where $t \in X_0^{\alpha_2}$ is an arbitrary point and $\mathbb{P}^2 := \phi^{-1}(t)$.

Proposition 4.44 Using the notations in Proposition 1.2, we have a rational map $F^d(1, 2, \mathbb{C}^4) \dashrightarrow \mathbb{P}(\mathcal{L}_\sigma) \cong C_2/B$ over $\mathbb{P}^3$ induced by the surjective homomorphism $\mathcal{L}_\sigma \oplus \mathcal{L}^{\alpha} \to \mathcal{L}_\sigma$, where $B$ is a Borel subgroup of the simple group $C_2$. It is a linear projection from $\mathbb{P}^2 := \phi^{-1}(t)$ with center $\sigma(t)$ over each $t \in \mathbb{P}^3$.

Proposition 4.45 In the setting of Notation 4.42, define a meromorphic distribution $\mathcal{N}$ on $F^d(1, 2, \mathbb{C}^4)$ such that $\mathcal{N}_x = T_x(l_x)$ at any point $x \in F^d(1, 2, \mathbb{C}^4) \setminus \mathbb{P}(\mathcal{L}_\sigma)$. Then $\mathcal{N}$ is the unique meromorphic line subbundle of $\mathcal{D}_0$ on $F^d(1, 2, \mathbb{C}^4)$ such that $[\mathcal{N}, \mathcal{D}_0] \subset \mathcal{D}_0$. Moreover, $[\mathcal{N}, (\mathcal{D}_0)^{\alpha_1}] \subset [\mathcal{N}, (\mathcal{D}_0)^{\alpha_2}]$.

Proof The leaf of $\mathcal{N}$ passing through a point $x \in F^d(1, 2, \mathbb{C}^4) \setminus \mathbb{P}(\mathcal{L}_\sigma)$ is $l_x := l_x \setminus \{\sigma(t)\}$, where $t := \phi(x)$ and $l_x := \langle x, \sigma(t) \rangle$. The leaf of $\mathcal{D}^{\alpha_1}$ passing through a point $y \in l_x$ is $C_y$, where $[C_y]$ is the unique element of $\mathcal{K}_y^{\alpha_1}(X_0)$. Since $\bigcup_{y \in l_x} y \cong \phi(C_x) \times l_x$, we have $[\mathcal{N}, (\mathcal{D}_0)^{\alpha_1}] \subset [\mathcal{N}, (\mathcal{D}_0)^{\alpha_2}]$. Since $(\mathcal{D}_0)^{\alpha_2}$ is integrable and $\mathcal{N} \subset (\mathcal{D}_0)^{\alpha_2}$, we have $[\mathcal{N}, (\mathcal{D}_0)^{\alpha_2}] \subset (\mathcal{D}_0)^{\alpha_2}$. It follows that $[\mathcal{N}, \mathcal{D}_0] \subset \mathcal{D}_0$.

If the uniqueness of $\mathcal{N}$ fails, then the rank three distribution $\mathcal{D}$ has to be integrable. However one can easily check that $F^d(1, 2, \mathbb{C}^4)$ is chained-connected by the family $\bigcup_{i=1, 2} \mathcal{K}_i^{\alpha_2}(F^d(1, 2, \mathbb{C}^4))$. It is a contradiction. Hence $\mathcal{N}$ is unique.

Proposition 4.46 At each point $x \in F^d(1, 2, \mathbb{C}^4) \setminus \mathbb{P}(\mathcal{L}_\sigma)$, the symbol algebra $\text{Symb}_x(\mathcal{D}_0) \cong \mathfrak{g}_-(C_2) \oplus \mathfrak{g}_-(A_1)$. Moreover, this isomorphism is induced by the identification $\mathfrak{g}_-(A_1) = \mathcal{N}_x$, $(\mathcal{D}_0)^{\alpha_2}_x = \mathfrak{g}_-(\alpha_2) + \mathfrak{g}_-(A_1)$ and $(\mathcal{D}_0)^{\alpha_1}_x = \mathfrak{g}_-(\alpha_1)$, where $\alpha_1$ and $\alpha_2$ is the long and short simple root of the simple group $C_2$ respectively.

Proof It follows from Proposition 4.45 directly.

\section{The Fano deformations of $\mathbb{P}(1, 2, Q^6)$}

The aim in this section is to show the following
Proposition 5.1 In the situation of Setting 1.7, suppose that $X_t \cong D_4/P_{[\alpha_2, \alpha_3, \alpha_4]}$ for $t \neq 0$ and $X_0 \cong D_4/P_{[\alpha_2, \alpha_3, \alpha_4]}$. Then at a general point $x \in X_0$, the fibers $F_{x}^{\alpha_2} \cong \mathbb{P}^1$, $F_{x}^{\alpha_3} \cong \mathbb{P}^1$, $F_{x}^{\alpha_4} \cong \mathbb{P}^1$, $F_{x}^{\alpha_2, \alpha_4} \cong \mathbb{P}^1 \times \mathbb{P}^1$, $F_{x}^{\alpha_3, \alpha_4} \cong F^{d}(1, 2, \mathbb{C}^4)$ and $F_{x}^{\alpha_2, \alpha_3, \alpha_4} \cong F^{d}(1, 2, \mathbb{C}^4)$.

Throughout Sect. 5, we assume that we are in the situation of the setting below.

Setting 5.2 Let $\pi : X \to \Delta \ni 0$ be a holomorphic family of connected Fano manifolds such that $X_t \cong D_4/P_{[\alpha_2, \alpha_3, \alpha_4]}$ for $t \neq 0$.

Firstly we have four possibilities as follows.

Proposition 5.3 In the situation of Setting 5.2, take $x \in X_0$ general. Then $F_{x}^{\alpha_2} \cong \mathbb{P}^1$, $F_{x}^{\alpha_3} \cong \mathbb{P}^1$, $F_{x}^{\alpha_4} \cong \mathbb{P}^1$ and $F_{x}^{\alpha_2, \alpha_4} \cong \mathbb{P}^1 \times \mathbb{P}^1$. Moreover, one of the following cases occur:

(A) $F_{x}^{\alpha_2, \alpha_3} \cong F(2, 3, \mathbb{C}^4)$ and $F_{x}^{\alpha_2, \alpha_4} \cong F(2, 3, \mathbb{C}^4)$;
(B) $F_{x}^{\alpha_2, \alpha_3} \cong F^{d}(1, 2, \mathbb{C}^4)$ and $F_{x}^{\alpha_2, \alpha_4} \cong F^{d}(1, 2, \mathbb{C}^4)$;
(C) $F_{x}^{\alpha_2, \alpha_3} \cong F(2, 3, \mathbb{C}^4)$ and $F_{x}^{\alpha_2, \alpha_4} \cong F^{d}(1, 2, \mathbb{C}^4)$;
(D) $F_{x}^{\alpha_2, \alpha_3} \cong F^{d}(1, 2, \mathbb{C}^4)$ and $F_{x}^{\alpha_2, \alpha_4} \cong F(2, 3, \mathbb{C}^4)$.

Proof The description of $F_{x}^{\alpha_2, \alpha_3}$ and $F_{x}^{\alpha_2, \alpha_4}$ follows from the Fano deformation rigidity of projective spaces and Proposition 3.24. The description of $F_{x}^{\alpha_2, \alpha_3}$ and $F_{x}^{\alpha_2, \alpha_4}$ follows from Proposition 1.2.\qed

Remark 5.4 The positive roots of $D_4$ are as follows:

$$
\alpha_1, \alpha_2, \alpha_3, \alpha_4; \quad \alpha_1 + \alpha_2, \alpha_2 + \alpha_3, \alpha_2 + \alpha_4;
\alpha_1 + \alpha_2 + \alpha_3, \alpha_1 + \alpha_2 + \alpha_4, \alpha_2 + \alpha_3 + \alpha_4;
\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4; \quad \alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4.
$$

Take $G = D_4$ and $I = \{\alpha_2, \alpha_3, \alpha_4\}$ in Definition 2.2, then the graded Lie algebra $\mathfrak{g}_{-}(I) = \bigoplus_{k \geq 1} \mathfrak{g}_{-k}(I)$ satisfies $\mathfrak{g}_{-1}(I) = \mathfrak{g}_{-\alpha_1-\alpha_2} \oplus \mathfrak{g}_{-\alpha_2} \oplus \mathfrak{g}_{-\alpha_3} \oplus \mathfrak{g}_{-\alpha_4}$, $\mathfrak{g}_{-2}(I) = \mathfrak{g}_{-\alpha_1-\alpha_2-\alpha_3} \oplus \mathfrak{g}_{-\alpha_2-\alpha_3} \oplus \mathfrak{g}_{-\alpha_1-\alpha_2-\alpha_4} \oplus \mathfrak{g}_{-\alpha_2-\alpha_4}$, $\mathfrak{g}_{-3}(I) = \mathfrak{g}_{-\alpha_1-\alpha_2-\alpha_3-\alpha_4} \oplus \mathfrak{g}_{-\alpha_2-\alpha_3-\alpha_4}$, $\mathfrak{g}_{-4}(I) = \mathfrak{g}_{-\alpha_1-2\alpha_2-\alpha_3-\alpha_4}$ and $\mathfrak{g}_{-k}(I) = 0$ for $k \geq 5$.

Now we fix nonzero vectors $w_1 \in \mathfrak{g}_{-\alpha_1-\alpha_2}$, $w_2 \in \mathfrak{g}_{-\alpha_2}$, $w_3 \in \mathfrak{g}_{-\alpha_3}$, and $w_4 \in \mathfrak{g}_{-\alpha_4}$ respectively. Then we can rewrite the homogeneous component of $\mathfrak{g}_{-}$ such that $\mathfrak{g}_{-1}(I) = \mathfrak{C}w_1 \oplus \mathfrak{C}w_2 \oplus \mathfrak{C}w_3 \oplus \mathfrak{C}w_4$, $\mathfrak{g}_{-2}(I) = \mathfrak{C}w_{13} \oplus \mathfrak{C}w_{23} \oplus \mathfrak{C}w_{14} \oplus \mathfrak{C}w_{24}$, $\mathfrak{g}_{-3}(I) = \mathfrak{C}w_{134} \oplus \mathfrak{C}w_{234}$, $\mathfrak{g}_{-4}(I) = \mathfrak{C}w_{1342}$, and $\mathfrak{g}_{-k}(I) = 0$ for $k \geq 5$, where $w_{i_1...i_m} := \{w_{i_1...i_{m-1}}, w_{i_m}\}$ by inductive definition.

Take $G = D_4$ and $I = \{\alpha_2, \alpha_3\}$ in Definition 2.2, then $\mathfrak{g}_{-}(I) = \bigoplus_{k \geq 1} \mathfrak{g}_{-k}(I)$ is as follows:

$$
\mathfrak{g}_{-1}(I') = \mathfrak{g}_{-\alpha_1-\alpha_2} \oplus \mathfrak{g}_{-\alpha_2} \oplus \mathfrak{g}_{-\alpha_1-\alpha_2-\alpha_4} \oplus \mathfrak{g}_{-\alpha_2-\alpha_4} \oplus \mathfrak{g}_{-\alpha_3}, \quad \mathfrak{g}_{-2}(I') = \mathfrak{g}_{-\alpha_1-2\alpha_2-\alpha_3-\alpha_4},
\mathfrak{g}_{-3}(I') = \mathfrak{g}_{-\alpha_1-\alpha_2-\alpha_3} \oplus \mathfrak{g}_{-\alpha_2-\alpha_3} \oplus \mathfrak{g}_{-\alpha_1-\alpha_2-\alpha_1-\alpha_4} \oplus \mathfrak{g}_{-\alpha_2-\alpha_1-\alpha_4} \oplus \mathfrak{g}_{-\alpha_2-\alpha_3-\alpha_4}, \quad \mathfrak{g}_{-k}(I') = 0 \text{ for } k \geq 4.
(5.1)
$$
The choice of \( w_i \) is kept unchanged. Then (5.1) can be written explicitly as follows:

\[
\begin{align*}
\mathfrak{g}_{-1}(I') &= \mathbb{C}w_1 \oplus \mathbb{C}w_2 \oplus \mathbb{C}w_{14} \oplus \mathbb{C}w_{24} \oplus \mathbb{C}w_3, \\
\mathfrak{g}_{-2}(I') &= \mathbb{C}w_{13} \oplus \mathbb{C}w_{23} \oplus \mathbb{C}w_{134} \oplus \mathbb{C}w_{234}, \\
\mathfrak{g}_{-k}(I') &= 0 \text{ for } k \geq 4.
\end{align*}
\]

(5.2)

**Convention 5.5** In Sect. 5, we denote by \((\mathcal{D}_0)^{\alpha_i}, \mathcal{D}_0\) and \((\mathcal{D}_0)^{-i}\) the restriction of \(\mathcal{D}^{\alpha_i}, \mathcal{D}\) and \(\mathcal{D}^{-i}\) on \(\mathcal{X}_0\) respectively, where the latter is defined in Notation 3.9. For simplicity we write \((m_-)_x := m_x(\alpha_2, \alpha_3, \alpha_4)\) and \((m_{-k})_x := (m_{-k}(\alpha_2, \alpha_3, \alpha_4))_x\), where \(k \geq 1\) and \(x \in \mathcal{X}_0\) is general.

**Lemma 5.6** At \(x \in \mathcal{X}_0\) general \(\dim(m_-)_x = \dim \mathcal{X}_0 = 11\).

**Proof** It is a special case of Proposition 3.10.

**Lemma 5.7** Suppose the case (C) of Proposition 5.3 occurs. Then there exists a unique meromorphic line subbundle \(\mathcal{N}'\) of \((\mathcal{D}_0)^{\alpha_4}\) such that \([\mathcal{N}', (\mathcal{D}_0)^{\alpha_4}] \subset \mathcal{N} + (\mathcal{D}_0)^{\alpha_4}\). Consequently, \([\mathcal{N}', (\mathcal{D}_0)^{\alpha_2} + (\mathcal{D}_0)^{\alpha_4}] \subset (\mathcal{D}_0)^{\alpha_2} + (\mathcal{D}_0)^{\alpha_4} \subset \mathcal{D}_0\).

**Proof** It follows from Proposition 4.46 and the assumptions in the case (C) directly.

**Construction 5.8** In the setting of Lemma 5.7, take \(x \in \mathcal{X}_0\) general. Choose a local section \(\tilde{v}_1\) (resp. \(\tilde{v}_3, \tilde{v}_4\)) of \(\mathcal{N}'\) (resp. \((\mathcal{D}_0)^{\alpha_3}, (\mathcal{D}_0)^{\alpha_4}\)), which is nonzero in an open neighborhood of \(x\) in \(\mathcal{X}_0\). Take a local section \(\tilde{v}_2\) of \((\mathcal{D}_0)^{\alpha_2}\) such that \((\tilde{v}_2)_y \not\in \mathcal{C}(\tilde{v}_1)_y\) at any point \(y\) in an open neighborhood of \(x\) in \(\mathcal{X}_0\). Define by induction \(k \geq 1\) that \(\tilde{v}_{i_1\ldots i_k+1} := [\tilde{v}_{i_1\ldots i_k}, \tilde{v}_k+1]\), which is a local vector field in an open neighborhood of \(x\) in \(\mathcal{X}_0\). Take a subset \(A \subset I := \{\alpha_2, \alpha_3, \alpha_4\}\). When all \(\tilde{v}_{i_j}\) are local sections of \((\mathcal{D}_0)^{A} := \sum_{\beta \in A} (\mathcal{D}_0)^{\beta}\) we denote by \(v_{i_1\ldots i_k}^A\) the class of \(\tilde{v}_{i_1\ldots i_k}\) in \(\text{Symb}(B_0)^{A}\). When \(A = I\) we omit the superscript \(I\), i.e. denote by \(v_{i_1\ldots i_k} \in \text{Symb}(B_0)\) of class of \(\tilde{v}_{i_1\ldots i_k}\). For simplicity we also use \(v_{i_1\ldots i_k}^A\) and \(v_{i_1\ldots i_k}\) to represent the corresponding class in the symbol algebras \(\text{Symb}_x((\mathcal{D}_0)^{A})\) and \(\text{Symb}_x(\mathcal{D}_0)\) at a chosen general point \(x\).

**Proposition 5.9** In the setting of Construction 5.8, the symbol algebra of \(\mathcal{D}_0\) at a general point \(x \in \mathcal{X}_0\) is a quotient algebra of \(\mathfrak{g}_-(B_4)\), denoted by \(\mathfrak{g}_-(B_4)/\mathfrak{q}\). More precisely, under the isomorphism \((m_-)_x \cong \mathfrak{g}_-(B_4)/\mathfrak{q}\) the elements \(v_1, v_2, v_3, v_4\) have weights \(-\beta_1, -\beta_3, -\beta_2, -\beta_4\) respectively, where \(\beta_1, \ldots, \beta_3\) are the three long simple roots of \(B_4\), and \(\beta_4\) is the short one. The ideal \(\mathfrak{q}\) is generated by \(\mathfrak{g}_-\beta_1-\beta_2-\beta_3\) in \(\mathfrak{g}_-(B_4)\). We can write \((m_-)_x\) in an explicit way such that \((m_{-1})_x = \mathbb{C}v_1 \oplus \mathbb{C}v_3 \oplus \mathbb{C}v_2 \oplus \mathbb{C}v_4\), \((m_{-2})_x = \mathbb{C}v_{13} \oplus \mathbb{C}v_{32} \oplus \mathbb{C}v_{24}\), \((m_{-3})_x = \mathbb{C}v_{324} \oplus \mathbb{C}v_{244}\), \((m_{-4})_x = \mathbb{C}v_{3244}\), \((m_{-5})_x = \mathbb{C}v_{32442}\), and \((m_{-k})_x = 0\) for \(k \geq 6\), where \(\dim(m_{-k})_x = 4, 3, 2, 1, 1\) for \(k = 1, \ldots, 5\) respectively.

**Proof** By our assumption, the case (C) of Proposition 5.3 occurs. Then both \((m_-\alpha_2, \alpha_3))_x\) and \((m_-\alpha_3, \alpha_4))_x\) are standard. By Remark 5.4, we have

\[
\text{ad} v_1(v_2) = 0, \quad \text{ad} v_3(v_4) = 0, \quad (\text{ad} v_1)^2(v_3) = 0, \quad (\text{ad} v_3)^2(v_1) = 0 \text{ in } (m_-)_x,
\]
where \( i = 1, 2 \). Since \( F^\alpha_{\alpha^2} \cong F^d(1, 2, \mathbb{C}^4) \), we know from Lemma 5.7 and Proposition 4.46 that
\[
\text{ad} v_1(v_2) = 0, \text{ad} v_1(v_4) = 0, (\text{ad} v_2)^2(v_4) = 0, (\text{ad} v_4)^3(v_2) = 0 \text{ in } (m_-)_x.
\]

In summary, \( (m_-)_x \) is a quotient algebra of \( g_- (B_4) \), where we write the four simple roots \( \beta_1, \ldots, \beta_4 \) of \( B_4 \) in order with \( \beta_4 \) being the short simple root, and the elements \( v_1, v_2, v_3, v_4 \) have weights \( -\beta_1, -\beta_3, -\beta_2, -\beta_4 \) respectively. Since \( (m_k(\alpha_2, \alpha_3))_x = 0 \) for all \( k \geq 3 \), \( [v_{13}, v_2] = 0 \) in \( (m_-)_x \). It follows that \( (m_-)_x \) is a quotient algebra of \( g_- (B_4)/q \), where \( q \) is the ideal in \( g_- (B_4) \) generated by \( g_- (\beta_1 - \beta_2 - \beta_3) \). It is straightforward to see that \( g_- (B_4)/q \) is isomorphic to the graded Lie algebra described in the statement of this Proposition. By Lemma 5.6, \( \dim (m_-)_x = \dim g_- (B_4)/q = 11 \). Hence \( (m_-)_x \cong g_- (B_4)/q \).

\[\Box\]

**Proposition 5.10** The case (C) of Proposition 5.3 does not occur.

**Proof** Suppose that the case (C) of Proposition 5.3 occurs. Denote by \( E \) the meromorphic distribution on \( \mathcal{X}^\alpha_{\alpha^2} \) such that \( E|_\mathcal{X}^\alpha_{\alpha^4} \) coincides with \( g_- (D_4/\mathcal{P}_{(\alpha_2, \alpha_3)}) \) under the identification \( \mathcal{X}^\alpha_t \cong D_4/\mathcal{P}_{(\alpha_2, \alpha_3)} \) for each \( t \neq 0 \). Then the singular locus on \( \mathcal{X}^\alpha_{\alpha^4} \) of \( E \) is a proper closed algebraic subset of \( \mathcal{X}^\alpha_0 \). By Remark 5.4 and Proposition 5.9, \( \mathcal{E} = d\pi^{\alpha^4}(\mathcal{D} + T_{\alpha^2, \alpha^4}) \), where \( d\pi^{\alpha^4} : T\mathcal{X} \to T\mathcal{X}^{\alpha^4} \) is the tangent map of \( \pi^{\alpha^4} : \mathcal{X} \to \mathcal{X}^\alpha_{\alpha^4} \).

Take \( x \in \mathcal{X}_0 \) general. Denote by \( \mathcal{E}_0 := E|_\mathcal{X}^\alpha_0 \), and \( y := \pi^{\alpha^4}(x) \in \mathcal{X}^\alpha_0 \). We claim that \( \text{Symb}_y (\mathcal{E}_0) \cong g_- (\alpha_2, \alpha_3) \), where \( g_- (\alpha_2, \alpha_3) \subset g = \text{Lie}(D_4) \) as in Definition 2.2. Note that \( g_- (\alpha_2, \alpha_3) \) has been explicitly described in (5.1) and (5.2).

By abuse of notation, we denote by \( v_{i_1 \ldots i_k} \in \text{Symb}_y (\mathcal{E}_0) \) the class of the local vector field \( d\pi^{\alpha^4}(\tilde{v}_{i_1 \ldots i_k}) \) on \( \mathcal{X}^\alpha_0 \). Now \( v_1, v_2, v_3, v_4, v_{244} \) form a basis of \( (\mathcal{E}_0)_y \). There is a unique linear isomorphism \( \psi : (\mathcal{E}_0)_y \to g_- (\alpha_2, \alpha_3) \) such that \( \psi (v_1) = w_1, \psi (v_2) = w_2, \psi (v_3) = w_3, \psi (v_{244}) = w_{24} \) and \( \psi (v_{13}) = w_{13} \). By a direct calculation, \( \psi \) induces an isomorphism \( \Psi : \text{Symb}(\mathcal{E}_0)_y \to (g_- (D_4/\mathcal{P}_{(\alpha_2, \alpha_3)}))_q \) satisfying that \( \Psi (v_{13}) = w_{123}, \Psi (v_{32}) = -w_{23}, \Psi (v_{324}) = -w_{234}, \Psi (v_{323}) = w_{134} \), and \( \Psi (v_{2332}) = -w_{1342} \).

By Proposition 3.19 the variety \( \mathcal{X}^\alpha_0 \cong D_4/\mathcal{P}_{(\alpha_2, \alpha_3)} \). Thus \( \pi^{\alpha^4}_0 : \mathcal{X}_0 \to \mathcal{X}^\alpha_0 \) is a \( \mathbb{P}^1 \)-fibration by Proposition 4.15.

On the other hand, by the assumption we have \( F^\alpha_{\alpha^2} \cong F^d(1, 2, \mathbb{C}^4) \). The restriction of \( \pi^{\alpha^4}_0 \) on \( F^\alpha_{\alpha^2, \alpha^4} \) coincides with the morphism \( F^d(1, 2, \mathbb{C}^4) \to \text{cone}(\text{pt}, \mathbb{Q}^3) \). In particular, a fiber of \( \pi^{\alpha^4}_0 \) is biholomorphic to \( \mathbb{P}^3 \), contracting the assertion that \( \pi^{\alpha^4}_0 \) is a \( \mathbb{P}^1 \)-fibration. Hence case (C) of Proposition 5.3 does not occur.

\[\Box\]

Now we can complete the proof of Proposition 5.1

**Proof of Proposition 5.1** By Proposition 5.3, there are four possibilities (A) – (D). By Proposition 5.10, the case (C) does not occur. By the symmetry of the Dynkin diagram, the case (D) is also impossible. If the case (A) occurs, then by Theorem 3.22 the manifold \( \mathcal{X}_0 \cong D_4/\mathcal{P}_{(\alpha_2, \alpha_3, \alpha_4)} \), contradicting to our assumption. Hence only the case (B) is possible, verifying the conclusion.

\[\Box\]
The Fano deformation rigidity of $\mathbb{F}(0, 1, m - 1, \mathbb{P}^m)$

The aim of this section is to prove Proposition 1.5. It suffices to show $X_0 \cong \mathbb{A}_m / \mathbb{P}_{\{\alpha_1, \alpha_2, \alpha_m\}}$ when we are in the situation of Setting 6.1 below.

**Setting 6.1** Set $S := \mathbb{A}_m / \mathbb{P}_{\{\alpha_1, \alpha_2, \alpha_m\}}$ with $m \geq 3$. Let $\pi : X \to \Delta \ni \alpha$ be a holomorphic map such that $X_t \cong S$ for $t \neq 0$ and $X_0$ is a connected Fano manifold.

Let us firstly recall a deformation rigidity result due to Wiśniewski.

**Theorem 6.2** [18] The variety $\mathbb{F}(0, n - 1, \mathbb{P}^n)$ is rigid under Fano deformation, where $\mathbb{F}(0, n - 1, \mathbb{P}^n)$ is the set of flags of points and hyperplanes in $\mathbb{P}^n$.

In the situation of Setting 6.1, we have the following rigidity result on the fibers.

**Proposition 6.3** In the situation of Setting 6.1, the followings hold for $x \in X_0$ general:

1. $F_{x}^{\alpha_1} \cong \mathbb{P}^1$, $F_{x}^{\alpha_2} \cong \mathbb{P}^{m-2}$, $F_{x}^{\alpha_m} \cong \mathbb{P}^{m-2}$;  
2. $F_{x}^{\alpha_1, \alpha_m} \cong F_{x}^{\alpha_1} \times F_{x}^{\alpha_m} \cong \mathbb{P}^1 \times \mathbb{P}^{m-2}$;  
3. $F_{x}^{\alpha_2, \alpha_m} \cong \mathbb{P}_{\alpha_1} / \mathbb{P}_{\{\alpha_1, \alpha_2, \alpha_m\}}$.

**Proof** The conclusions (6.1) and (6.3) follow from the Fano deformation rigidity of projective spaces and $\mathbb{A}_k / \mathbb{P}_{\{\alpha_1, \alpha_k\}}$ respectively, see Theorem 6.2. The conclusion (6.2) follows from (6.1) and Proposition 3.24. □

As a direct consequence of Proposition 6.3, we have the following result.

**Corollary 6.4** In the situation of Setting 6.1, the followings hold for $x \in X_0$ general.

1. The symbol algebras $m_x(\alpha_1)$, $m_x(\alpha_2)$ and $m_x(\alpha_m)$ are standard. More precisely, they are abelian algebras of dimension $1$, $m - 2$ and $m - 2$ respectively.
2. The symbol algebras $m_x(\alpha_1, \alpha_m)$ and $m_x(\alpha_2, \alpha_m)$ are standard. More precisely,
   - (i) there is a decomposition of abelian algebra $m_x(\alpha_1, \alpha_m) = m_x(\alpha_1) \oplus m_x(\alpha_m);
   - (ii) dim $m_{-2}(\alpha_2, \alpha_m) = 1$ and the bilinear map
     \[ m_x(\alpha_2) \times m_x(\alpha_m) \to (m_x(\alpha_2, \alpha_m))_{-2} \]
     \[ (x, y) \mapsto [x, y] \]
     induces an isomorphism of vector spaces $m_x(\alpha_2) \cong \text{Hom}(m_x(\alpha_m), (m_x(\alpha_2, \alpha_m))_{-2})$.

**Proposition 6.5** In the situation of Setting 6.1, we have $F_{x}^{\alpha_1, \alpha_2} \cong \mathbb{P}_I \setminus \{\alpha_1, \alpha_2\} / \mathbb{P}_I$ for $x \in X_0$ general.

**Proof** Take $x \in X_0$ general. We claim that

the symbol algebra $m_x(\alpha_1, \alpha_2)$ is standard. (6.4)
For the simplicity of discussions, we omit the subscript \( x \) in the notations of symbol algebras such as \( m_\text{x}(\alpha_1, \alpha_2) \) and \( m_\text{\$}(\alpha_m) \).

Now suppose that \( m(\alpha_1, \alpha_2) \) is not standard. Then there exists \( 0 \neq v_2 \in m(\alpha_2) \) such that \([m(\alpha_1), v_2] = 0\). Since \( m(\alpha_2, \alpha_m) \) is standard, there exists \( 0 \neq v_3 \in m(\alpha_m) \) such that \( v_4 := [v_2, v_3] \neq 0 \) and \( m_{-2}(\alpha_2, \alpha_m) = \mathbb{C} v_4 \). In particular, there is a decomposition of vector spaces \( m(\alpha_2, \alpha_m) = m(\alpha_2) \oplus m(\alpha_m) \oplus \mathbb{C} v_4 \).

Take \( 0 \neq v_1 \in m(\alpha_1) \). Then we have
\[
[v_1, v_4] = [v_1, [v_2, v_3]] = [[v_1, v_2], v_3] + [v_2, [v_1, v_3]] = 0. \tag{6.5}
\]

In other words, \([m(\alpha_1), \mathbb{C} v_4] = 0\). Let \( A(\alpha_1, \alpha_2, \alpha_m) \) be the vector subspace of \( m(\alpha_1, \alpha_2, \alpha_m) \) generated by \( m(\alpha_1, \alpha_2), m(\alpha_m) \) and \( \mathbb{C} v_4 \). Denote by
\[
m(1; \alpha_1, \alpha_2) := m(\alpha_1) \oplus m(\alpha_2) \quad \text{and} \quad m(k; \alpha_1, \alpha_2) := [m(1; \alpha_1, \alpha_2), m(k - 1; \alpha_1, \alpha_2)] \quad \text{for each} \quad k \geq 2.
\]

Thus \( m(\alpha_1, \alpha_2) = \sum_{k=1}^{\infty} m(k; \alpha_1, \alpha_2) \). We claim that
\[
\text{when (6.4) fails, } A(\alpha_1, \alpha_2, \alpha_m) \text{ is a Lie subalgebra of } m(\alpha_1, \alpha_2, \alpha_m). \tag{6.6}
\]

Indeed by Corollary 6.4 we already know that \( A(\alpha_1, \alpha_2, \alpha_m) = m(\alpha_1, \alpha_2) + m(\alpha_1, \alpha_m) + m(\alpha_2, \alpha_m) \). It follows that \([m(\alpha_m) + \mathbb{C} v_4, m(\alpha_m) + \mathbb{C} v_4] \subset m(\alpha_2, \alpha_m) \subset A(\alpha_1, \alpha_2, \alpha_m) \). Hence to prove the claim (6.6) it remains to show that
\[
[m(k; \alpha_1, \alpha_2), m(\alpha_m) + \mathbb{C} v_4] \subset A(\alpha_1, \alpha_2, \alpha_m) \quad \text{for all} \quad k \geq 1. \tag{6.7}
\]

Now let us prove (6.7) by induction on \( k \). The case \( k = 1 \) of (6.7) follows from the assertions
\[
[m(\alpha_1), m(\alpha_m) + \mathbb{C} v_4] = 0 \quad \text{and} \quad [m(\alpha_2), m(\alpha_m) + \mathbb{C} v_4] \subset m(\alpha_2, \alpha_m) \subset A(\alpha_1, \alpha_2, \alpha_m). \tag{6.8}
\]

where in the first equality we apply Corollary 6.4 and (6.5).

Now we assume that \( k \geq 2 \) and \([m(i; \alpha_1, \alpha_2), m(\alpha_m) + \mathbb{C} v_4] \subset A(\alpha_1, \alpha_2, \alpha_m) \) for all \( 1 \leq i \leq k - 1 \). Then by the definition of \( m(k; \alpha_1, \alpha_2) \) we have
\[
[m(k; \alpha_1, \alpha_2), m(\alpha_m) + \mathbb{C} v_4]
\subset \sum_{j=1,2}[[m(\alpha_j), m(k - 1; \alpha_1, \alpha_2)], m(\alpha_m) + \mathbb{C} v_4]
\subset \sum_{j=1,2} \left( [[m(\alpha_j), m(\alpha_m) + \mathbb{C} v_4], m(k - 1; \alpha_1, \alpha_2)]
+ [m(\alpha_j), [m(k - 1; \alpha_1, \alpha_2), m(\alpha_m) + \mathbb{C} v_4]] \right). \tag{6.9}
\]
We analyse term by term. By (6.8) we have
\[
[[m(\alpha_1), m(\alpha_m) + C_{v_4}], m(k - 1; \alpha_1, \alpha_2)] = 0.
\]
(6.10)

On one hand, we have
\[
\begin{align*}
&m(\alpha_1), [m(k - 1; \alpha_1, \alpha_2), m(\alpha_m) + C_{v_4}] \\
&
\subseteq [m(\alpha_1), A(\alpha_1, \alpha_2, \alpha_m)] \\
&= [m(\alpha_1), m(\alpha_1, \alpha_2)] + [m(\alpha_1), m(\alpha_m)] + [m(\alpha_1), C_{v_4}] \\
&\subseteq m(\alpha_1, \alpha_2) \\
&\subseteq A(\alpha_1, \alpha_2, \alpha_m).
\end{align*}
\]
(6.11)

By Corollary 6.4 we have \([m(\alpha_2), m(\alpha_m) + C_{v_4}] \subseteq m(\alpha_2) + m(\alpha_m) + C_{v_4}\), which implies that
\[
\begin{align*}
&[[m(\alpha_2), m(\alpha_m) + C_{v_4}], m(k - 1; \alpha_1, \alpha_2)] \\
&\subseteq [m(\alpha_2), m(k - 1; \alpha_1, \alpha_2)] + [m(\alpha_m) + C_{v_4}, m(k - 1; \alpha_1, \alpha_2)] \\
&\subseteq m(k; \alpha_1, \alpha_2) + A(\alpha_1, \alpha_2, \alpha_m) \\
&= A(\alpha_1, \alpha_2, \alpha_m).
\end{align*}
\]
(6.12)

Meanwhile, \([m(k - 1; \alpha_1, \alpha_2), m(\alpha_m) + C_{v_4}] \subseteq A(\alpha_1, \alpha_2, \alpha_m) = m(\alpha_1, \alpha_2) + m(\alpha_m) + C_{v_4}\) by induction, which implies that
\[
\begin{align*}
&m(\alpha_2), [m(k - 1; \alpha_1, \alpha_2), m(\alpha_m) + C_{v_4}] \\
&\subseteq [m(\alpha_2), m(\alpha_1, \alpha_2)] + [m(\alpha_2), m(\alpha_m) + C_{v_4}] + [m(\alpha_2), C_{v_4}] \\
&\subseteq m(\alpha_1, \alpha_2) + m(\alpha_2, \alpha_m) \\
&\subseteq A(\alpha_1, \alpha_2, \alpha_m).
\end{align*}
\]
(6.13)

By (6.9)–(6.13) we have \([m(k; \alpha_1, \alpha_2), m(\alpha_m) + C_{v_4}] \subseteq A(\alpha_1, \alpha_2, \alpha_m)\), which implies the claim (6.6).

Now \(A(\alpha_1, \alpha_2, \alpha_m)\) is a Lie subalgebra of \(m(\alpha_1, \alpha_2, \alpha_m)\) that contains \(m(\alpha_1) + m(\alpha_2) + m(\alpha_m)\). Recall that \(m(\alpha_1, \alpha_2, \alpha_m)\) is a Lie algebra generated by \(m(\alpha_1) + m(\alpha_2) + m(\alpha_m)\). Then we have \(A(\alpha_1, \alpha_2, \alpha_m) = m(\alpha_1, \alpha_2, \alpha_m)\). This contradicts the fact that \(\dim A(\alpha_1, \alpha_2, \alpha_m) = 3m - 4 = \dim m(\alpha_1, \alpha_2, \alpha_m) - 1\), where the dimension of \(m(\alpha_1, \alpha_2, \alpha_m)\) is obtained by Proposition 3.10. Hence \(m_x(\alpha_1, \alpha_2)\) is standard for \(x \in \chi_0\) general, and we get the claim (6.4). Then the conclusion follows from Proposition 3.20.

Now we are ready to prove Proposition 1.5.

**Proof of Proposition 1.5** Suppose we are in the situation of Setting 6.1. By Proposition 6.3 and Proposition 6.5, \(E^\alpha_\beta \cong P_{1\setminus \{\alpha, \beta\}}/P_I\) for all \(\alpha \neq \beta \in I\) and general points \(x \in \chi_0\). Then \(\chi_0 \cong A_m/P_{[\alpha_1, \alpha_2, \alpha_m]}\) by Theorem 3.22. In other words, the manifold \(A_m/P_{[\alpha_1, \alpha_2, \alpha_m]}\) is rigid under Fano deformation. □
Rigidity of certain homogeneous manifolds associated to $A_4$ and $D_5$

The aim of this section is to prove Proposition 1.4. In Sect. 7.1, we study the Fano deformation rigidity of $F(2, 3, 4, \mathbb{C}^5)$, which is a special case of Proposition 1.4. In Sect. 7.2, we settle the remaining cases of Proposition 1.4.

7.1 The case of $F(2, 3, 4, \mathbb{C}^5)$

The aim of this subsection is to prove the following result.

**Proposition 7.1** The manifold $A_4 / P_{(\alpha_2, \alpha_3, \alpha_4)}$ is rigid under Fano deformation.

Let us introduce the setting below. We will deduce a contradiction, and thus Proposition 7.1 holds.

**Setting 7.2** Let $\pi : X \to \Delta_1$ be a holomorphic map such that $X_t \cong S$ for all $t \neq 0$, $X_0$ is a connected Fano manifold and $X_0 \not\cong S$, where $S := A_4 / P_{\{\alpha_2, \alpha_3, \alpha_4\}}$.

**Remark 7.3** The main idea of the proof of Proposition 7.1 is the following. In the situation of Setting 7.2, $X_0$ has to be a compactification of the total space of the normal bundle $NU / S$, where $U$ is the inverse image of some hyperplane section of $A_4 / P_{\alpha_2} = \text{Gr}(2, \mathbb{C}^5) \subset \mathbb{P}^9$ under the natural morphism $S \to A_4 / P_{\alpha_2}$. On the other hand, we can show that any Fano deformation of $S$ must be a $\mathbb{P}^2$-bundle over $A_4 / P_{(\alpha_3, \alpha_4)} = F(3, 4, \mathbb{C}^5)$, while the compactification $X_0$ of $NU / S$ does not have such a projective bundle structure.

The following result on the deformation rigidity is due to Weber and Wiśniewski [17].

**Theorem 7.4** [17] The rational homogenous space $G / B$ is rigid under Fano deformation, where $G$ is a semisimple algebraic group and $B$ is a Borel subgroup.

**Proposition 7.5** In the situation of Setting 7.2, take a general point $x \in X_0$. Then $F_{x_{\alpha_2}} \cong \mathbb{P}^2$, $F_{x_{\alpha_3}} \cong \mathbb{P}^1$, $F_{x_{\alpha_4}} \cong \mathbb{P}^1$, $F_{x_{\alpha_2, \alpha_4}} \cong \mathbb{P}^2 \times \mathbb{P}^1$ and $F_{x_{\alpha_3, \alpha_4}} \cong \mathbb{P}(TP^2) = F(1, 2, \mathbb{C}^3)$.

**Proof** The assertions for $F_{x_{\alpha_2}}$, $F_{x_{\alpha_2, \alpha_4}}$ and $F_{x_{\alpha_3, \alpha_4}}$ follow from the rigidity of projective spaces, Proposition 3.24 and Theorem 7.4 respectively.

**Proposition 7.6** In the situation of Setting 7.2, take a general point $x \in X_0$. Then $F_{x_{\alpha_2, \alpha_3}} \cong F^d(1, 2, \mathbb{C}^4)$, where $F^d(1, 2, \mathbb{C}^4)$ is defined in Proposition 1.2.

**Proof** By Proposition 1.2, $F_{x_{\alpha_2, \alpha_3}}$ is biholomorphic to $F(1, 2, \mathbb{C}^4)$ or $F^d(1, 2, \mathbb{C}^4)$. In the former case, $X_0 \cong A_4 / P_{(\alpha_2, \alpha_3, \alpha_4)}$ by Theorem 3.22 and Proposition 7.5. It contradicts with our assumptions in Setting 7.2.

**Proposition 7.7** In the situation of Setting 7.2, the morphism $\pi_{x_{\alpha_2}}^* : X_0 \to X_0^{\alpha_2}$ is a $\mathbb{P}^2$-bundle. In particular, the variety $X_0^{\alpha_2}$ is smooth.

[Springer]
Proposition 4.15 immediately.

Convention 7.8 In the situation of Setting 7.2, we denote by $D_0$, $(D_0)^{a_i}$, $(D_0)^{a_i, a_j}$ and $(D_0)^{-k}$ the restriction of $D$, $D^{a_i}$, $D^{a_i, a_j}$ and $D^{-k}$ on $X_0$ respectively, where the latter is defined in Notation 3.9.

Now let us turn to analyze the symbol algebra $\text{Symb}(D_0)$ on $X_0$.

Lemma 7.9 In the situation of Setting 7.2, there exists a unique meromorphic distribution $\mathcal{N}' \subset (D_0)^{a_2}$ of rank one over $X_0$ such that the Levi bracket of vector fields $[\mathcal{N}', D_0] \subset D_0$.

Proof By Proposition 7.6, $F_{x}^{a_2, a_3} \cong F^d(1, 2, \mathbb{C}^4)$ for $x \in X_0$ general. Then by Proposition 4.45 there exists a unique 1-dimensional meromorphic distribution $\mathcal{N}'$ over $X_0$ such that $\mathcal{N}' \subset (D_0)^{a_2}$ and $[\mathcal{N}', (D_0)^{a_3}] \subset \mathcal{N} + (D_0)^{a_3}$. By Proposition 7.5, $F_{x}^{a_2, a_4} \cong F_{x}^{a_2} \times F_{x}^{a_4}$ for $x \in X_0$ general. Then $[(D_0)^{a_2}, (D_0)^{a_4}] \subset (D_0)^{a_2} + (D_0)^{a_4}$, implying the conclusion.

Notation 7.10 Define a graded nilpotent Lie algebra $m_\_ := \bigoplus_{k \geq 1} m_{-k}$ such that $m_{-1} = \bigoplus_{i=1}^{4} \mathbb{C} v_i$, $m_{-2} = \mathbb{C} v_{23} \oplus \mathbb{C} v_{34}$, $m_{-3} = \mathbb{C} v_{233} \oplus \mathbb{C} v_{34}$, $m_{-4} = \mathbb{C} v_{2334}$, and $m_{-k} = 0$ for $k \geq 5$, where $v_{i_1...i_m} := [v_{i_1...i_{m-1}}, v_{i_m}]$. The Lie algebra structure on $m_\_$ is defined uniquely by the rules $[m_{-i}, m_{-j}] \subset m_{-i-j}$, $[v_1, m_{-}] = 0$, and $[v_{23}, v_{34}] = \frac{1}{2} v_{2334}$. There is a table of Lie brackets as follows:

|       | $v_{23}$ | $v_{34}$ | $v_{233}$ | $v_{234}$ |
|-------|----------|----------|-----------|-----------|
| $v_2$ | 0        | $v_{23}$ | 0         | 0         |
| $v_3$ | $-v_{23}$| 0        | 0         | $-\frac{1}{2} v_{2334}$|
| $v_4$ | $-v_{234}$| 0        | $-v_{2334}$| 0         |

In the table above, we compute the Lie bracket of the left end entry with the top end entry. For example, $[v_4, v_{23}] = -v_{234}$ and $[v_3, v_{234}] = -\frac{1}{2} v_{2334}$.

Lemma 7.11 In the situation of Setting 7.2, the symbol algebra of $D_0$ at a general point $x \in X_0$ is isomorphic to the graded Lie algebra $m_\_$ defined in Notation 7.10, where we have identifications $\mathcal{N}_x = \mathbb{C} v_1$, $(D_0)^{a_2}_x = \mathbb{C} v_1 + \mathbb{C} v_2$, $(D_0)^{a_3}_x = \mathbb{C} v_3$ and $(D_0)^{a_4}_x = \mathbb{C} v_4$.

Proof By Propositions 7.5, 7.6 and 4.46 (see also Remark 4.7(ii)), we have the description of $m_x(a_i)$ and $m_x(a_i, a_j)$ for $2 \leq i \neq j \leq 4$. In particular, in $m_x(a_2, a_3, a_4) := \text{Symb}_x(D_0)$ we have $(ad v_2)^2(v_3) = 0$, $(ad v_2)^3(v_2) = 0, (v_2, v_4) = 0$, $(ad v_3)^2(v_4) = 0$, $(ad v_4)^2(v_2) = 0$, and $[v_1, v_2] = 0$ for $2 \leq i \leq 4$. Then by Proposition 2.13, $\text{Symb}_x(D_0)$ is a quotient algebra of $g_\_ := g_-(C_3) \oplus g_{-}(A_1)$, where $g_\_ -$ is the graded Lie algebra $\bigoplus_{k \geq 1} g_{-k}$ such that $g_{-1} = \bigoplus_{i=1}^{4} \mathbb{C} v_i$, $g_{-2} = \mathbb{C} v_{23} \oplus \mathbb{C} v_{34}$, $g_{-3} = \mathbb{C} v_{233} \oplus \mathbb{C} v_{34}$, $g_{-4} = \mathbb{C} v_{2334}$, $g_{-5} = \mathbb{C} v_{2334}$, and $g_{-k} = 0$ for $k \geq 6$. Let $q$ be the ideal of $g_{-}$ such that $\text{Symb}_x(D_0) = g_{-}/q$ as a graded nilpotent Lie
algebra. By Proposition 3.10, \( \dim \text{Symb}_x(D_0) = \dim T_x X_0 = 9 \), which implies that \( \dim q = \dim g_- - \dim \text{Symb}_x(D_0) = 1 \).

It remains to show the claim that \( q = \mathbb{C}v_0 \), where \( v_0 := v_{23444} + \lambda v_1 \) for some \( \lambda \in \mathbb{C} \). Note that the graded Lie algebra structure on \( g_-/Cw_0 \) is independent of the choice of \( \lambda \in \mathbb{C} \).

Suppose the claim fails. Then there exists \( 1 \leq k_0 \leq 4 \) such that \( q = \mathbb{C}v_0 \) and \( v_0 = \lambda v_1 + v'_0 + v''_0 \), where \( v'_0 \in \bigoplus_{k \geq k_0 + 1} g_- k \), \( 0 \neq v'_0 \in g_- k_0 \) if \( k_0 \geq 2 \), and \( 0 \neq v''_0 \in \mathbb{C}v_2 \oplus \mathbb{C}v_3 \oplus \mathbb{C}v_4 \) if \( k_0 = 1 \). There exists \( 2 \leq j \leq 4 \) such that \( [v_j, v'_{0}] \neq 0 \), see table (7.1). Then \( 0 \neq [v_j, v_0] \in \bigoplus_{k \geq k_0 + 1} g_- k \). Since \( q \) is an ideal of \( g_- \), we have \( 0 \neq [v_j, v_0] \in q = \mathbb{C}v_0 \). In particular, \( [v_j, v_0] \) has a nonzero component in \( g_- k_0 \). It is a contradiction. Hence the claim holds.

Lemma 7.12 In the situation of Setting 7.2, the Frobenius bracket of \( (T_{\pi^{a_2,a_3}} + T_{\pi^{a_2,a_4}})|_{X_0} \) induces a homomorphism of meromorphic vector bundles over \( X_0 \):

\[
(T_{\pi^{a_2,a_3}}/T_{\pi^{a_2,a_4}})|_{X_0} \times (T_{\pi^{a_2,a_4}}/T_{\pi^{a_2,a_3}})|_{X_0} \rightarrow T X_0/(T_{\pi^{a_2,a_3}} + T_{\pi^{a_2,a_4}})|_{X_0},
\]

which is a surjective homomorphism over a nonempty Zariski open subset of \( X_0 \).

Proof It is a direct consequence of Lemma 7.11. More precisely, the symbol algebras at a general point \( x \in X_0 \) defined by the weak derivatives of \( (D_0)^{a_2}, (D_0)^{a_2} + (D_0)^{a_3} \), and \( (D_0)^{a_2} + (D_0)^{a_4} \) are \( g(T_{\pi^{a_2,a_3}}) = \mathbb{C}v_1 \oplus \mathbb{C}v_2, g(T_{\pi^{a_2,a_3}}) = \mathbb{C}v_1 \oplus \mathbb{C}v_2 \oplus \mathbb{C}v_3 \oplus \mathbb{C}v_4 \), and \( g(T_{\pi^{a_2,a_4}}) = \mathbb{C}v_1 \oplus \mathbb{C}v_2 \oplus \mathbb{C}v_3 \oplus \mathbb{C}v_4 \). Then it is straightforward to deduce the conclusion from the Lie algebra structure of \( g_- \) defined in Notation 7.10.

Proposition 7.13 In the situation of Setting 7.2, the variety \( X_0^{a_2} \) is biholomorphic to \( F(3, 4, \mathbb{C}^5) \).

Proof By Proposition 7.7, the variety \( X_0^{a_2} \) is smooth. Being the smooth deformation of \( F(3, 4, \mathbb{C}^5) \cong X_0^{a_2} \) with \( t \neq 0 \), \( X_0^{a_2} \) is of Picard number two. The relative Mori contraction \( \pi_{a_2,a_2} : X \rightarrow X_0^{a_2,a_2} \) induces a relative Mori contraction \( \psi_0 : X_0^{a_2} \rightarrow X_0^{a_2,a_2} \) which extends \( \psi_0 : A_4/P(a_3,a_4) \rightarrow A_4/P(a_4) \), where \( i \neq k \in \{3, 4\} \). The existence of two elementary contractions of fiber type implies that \( X_0^{a_2} \) is a Fano manifold.

For each \( k \in \{3, 4\} \), the relative tangent sheaf \( T_{\psi_0} \) of \( \psi_0 \) is a meromorphic distribution on \( X^{a_2} \), whose singular locus is a proper closed subvariety of \( X_0^{a_2} \). Denote by \( E_0 = T_{\psi_0}|_{X_0^{a_2}} \), and \( E := E_0 + E_0 \subset T X_0^{a_2} \). The Frobenius bracket of the meromorphic distribution \( E \) on \( X_0^{a_2} \) induces \( F : E_0 \otimes E_0 \rightarrow T X_0^{a_2} \), which is a homomorphism of meromorphic vector bundles over \( X_0^{a_2} \).

It is easy to see that \( F = d\pi_0^{a_2}(T_{\pi^{a_2,a_3}} + T_{\pi^{a_2,a_4}}) \) and \( E_0 = d\pi_0^{a_2}(T_{\pi^{a_2,a_3}}) \) for \( k = 3, 4 \), where \( d\pi_0^{a_2} \) is the tangent map of \( \pi_0^{a_2} \). By Lemma 7.12, \( F \) is surjective at general points of \( X_0^{a_2} \). Then the conclusion follows from Proposition 3.20.

Corollary 7.14 In the situation of Setting 7.2, the varieties \( X_0^{a_2,a_3} \) and \( X_0^{a_2,a_4} \) are biholomorphic to \( A_4/P(a_3) \) and \( A_4/P(a_4) \), respectively. The morphisms \( \pi_0^{a_2,a_3} : X_0 \rightarrow X_0^{a_2,a_3} \) and \( \pi_0^{a_2,a_4} : X_0 \rightarrow X_0^{a_2,a_4} \) are \( F^d(1, 2, \mathbb{C}^4) \)-bundle and \( (\mathbb{P}^2 \times \mathbb{P}^1) \)-bundle respectively.
Proof By Proposition 7.13, \( X_{0}^{a_{2}} \cong A_{4}/P_{(a_{3}, a_{4})} \). Hence \( X_{0}^{a_{2}, a_{3}} \cong A_{4}/P_{a_{3}} \) and \( X_{0}^{a_{2}, a_{4}} \cong A_{4}/P_{a_{4}} \). Furthermore, the two elementary Mori contractions \( \psi_{0}^{a_{3}} : X_{0}^{a_{3}} \to X_{0}^{a_{2}, a_{3}} \) and \( \psi_{0}^{a_{4}} : X_{0}^{a_{4}} \to X_{0}^{a_{2}, a_{4}} \) are \( \mathbb{P}^{3} \)-bundle and \( \mathbb{P}^{1} \)-bundle respectively. Then by Proposition 7.7, \( \pi^{a_{2}, a_{3}} : X_{0} \to \mathbb{P}^{4} \) (resp. \( \pi^{a_{2}, a_{4}} : X_{0} \to Gr(3, \mathbb{C}^{5}) \)) is a smooth morphism such that each fiber is a Fano manifold admitting a \( \mathbb{P}^{2} \)-bundle structure over \( \mathbb{P}^{3} \) (resp. over \( \mathbb{P}^{1} \)). By the rigidity of projective space and Proposition 3.24, the morphism \( \pi_{0}^{a_{2}, a_{3}} \) is a \( (\mathbb{P}^{2} \times \mathbb{P}^{1}) \)-bundle. By Proposition 1.2, each fiber of \( \pi_{0}^{a_{2}, a_{3}} \) is biholomorphic to either \( F(2, 3, \mathbb{C}^{4}) \) or \( F^{d}(1, 2, \mathbb{C}^{4}) \). By the local rigidity of \( F(2, 3, \mathbb{C}^{4}) \) and Proposition 7.6, the morphism \( \pi_{0}^{a_{2}, a_{3}} \) is an \( F^{d}(1, 2, \mathbb{C}^{4}) \)-bundle. \( \square \)

Now we are ready to complete the proof of Proposition 7.1. As a trivial analogue to the construction of \( F^{d}(1, 2, \mathbb{C}^{4}) \), we can define \( F^{d}(2, 3, \mathbb{C}^{4}) \) by using the contact distribution on \( A_{3}/P_{a_{3}} \) instead of that on \( A_{3}/P_{a_{1}} \). Although \( F^{d}(2, 3, \mathbb{C}^{4}) \cong F(1, 2, \mathbb{C}^{4}) \), we use \( F^{d}(2, 3, \mathbb{C}^{4}) \) in the following to make our discussion compatible with the involved simple roots of \( A_{4} \).

Proof of Proposition 7.1 Assuming that we are in the situation of Setting 7.2, we will show how to get a contradiction. In summary of Proposition 7.7, Proposition 7.13 and Corollary 7.14, \( \pi^{a_{2}} : X_{0} \to X_{0}^{a_{2}} = F(3, 4, \mathbb{C}^{5}) \) is a \( \mathbb{P}^{2} \)-bundle and \( \pi_{0}^{a_{2}, a_{3}} : X_{0} \to X_{0}^{a_{2}, a_{3}} = \mathbb{P}^{4} \) is an \( F^{d}(2, 3, \mathbb{C}^{4}) \)-bundle. By Proposition 4.43 there exists a holomorphic section \( \sigma : X_{0}^{a_{2}} \to F(3, 4, \mathbb{C}^{5}) \), \( \sigma^{a_{3}}(X_{0}) \) consists of a unique element, denoted by \( [C_{x}] \).

(i) at any point \( x \in X_{0}\setminus\sigma(X_{0}^{a_{2}}) \), \( K_{x}^{a_{3}}(X_{0}) \) consists of a unique element, denoted by \( [C_{x}] \);

(ii) at any point \( x \in X_{0}\setminus\sigma(X_{0}^{a_{2}}) \), \( C_{x} \cong \mathbb{P}^{1} \) and \( \pi_{0}^{a_{2}} \) sends \( C_{x} \) biholomorphically to a line in a fiber of \( \psi_{0}^{a_{3}} : X_{0}^{a_{2}} \to F(3, 4, \mathbb{C}^{5}) \to X_{0}^{a_{2}, a_{3}} \)

(iii) two points \( x, y \in \mathbb{P}_{1}^{2}(\sigma(t)) \) satisfy \( \pi_{0}^{a_{2}}(C_{x}) = \pi_{0}^{a_{2}}(C_{y}) \) if and only if the two lines \( \langle x, \sigma(t) \rangle \) and \( \langle x, \sigma(t) \rangle \) in \( \mathbb{P}_{1}^{2} \) coincide, where \( t \in X_{0}^{a_{2}} \) is an arbitrary point and \( \mathbb{P}_{1}^{2} := (\pi_{0}^{a_{2}})^{-1}(t) \). Set \( K^{a_{3}}(X_{0}/X_{0}^{a_{2}}) := \bigcup_{t \in X_{0}^{a_{2}}} [\pi_{0}^{a_{2}}(C_{x})] \subset A_{4}/P_{a_{2}, a_{4}} = K^{a_{3}}(X_{0}^{a_{2}}) \). Denote by

\[
X_{0}^{a_{2}} \leftarrow U^{a_{3}}(X_{0}/X_{0}^{a_{2}}) \rightarrow K^{a_{3}}(X_{0}/X_{0}^{a_{2}})
\]

the restriction of the universal family \( X_{0}^{a_{2}} \cong A_{4}/P_{(a_{3}, a_{4})} \leftarrow A_{4}/P_{a_{3}, a_{4}} \rightarrow K^{a_{3}}(X_{0}^{a_{2}}) = A_{4}/P_{a_{2}, a_{4}} \). Since \( \pi_{0}^{a_{2}, a_{3}} : X_{0} \to X_{0}^{a_{2}, a_{3}} = \mathbb{P}^{4} \) is an \( F^{d}(2, 3, \mathbb{C}^{4}) \)-bundle, we can apply Proposition 4.44 to obtain a commutative diagram over \( X_{0}^{a_{2}} \) as follows:

\[
\begin{array}{ccc}
X_{0} & \overset{\theta}{\to} & U^{a_{3}}(X_{0}/X_{0}^{a_{2}}) \cong A_{4}/P_{a_{2}, a_{3}, a_{4}} \\
\pi_{0}^{a_{2}} \downarrow & & \\
X_{0}^{a_{2}} = F(3, 4, \mathbb{C}^{5}) \downarrow & & Gr(2, \mathbb{C}^{5})
\end{array}
\]

where at any point \( t \in X_{0}^{a_{2}} \) the horizontal rational map \( \theta_{t} \) is the linear projection from \( \mathbb{P}_{1}^{2} := (\pi_{0}^{a_{2}})^{-1}(t) \) with center \( \sigma(t) \). In particular,
(iv) \( \gamma : U^{\alpha_3}(\mathcal{X}_0/\mathcal{X}_0^{\alpha_2}) \to \mathcal{X}_0^{\alpha_2} \) is a \( \mathbb{P}^1 \)-bundle.

Now we claim that

(v) under the natural morphism \( A_4/P_{(\alpha_2, \alpha_3, \alpha_4)} \to A_4/P_{\alpha_2} = Gr(2, \mathbb{C}^5) \), the variety \( U^{\alpha_3}(\mathcal{X}_0/\mathcal{X}_0^{\alpha_2}) \subset A_4/P_{(\alpha_2, \alpha_3, \alpha_4)} \) is the inverse image of a hyperplane section of \( Gr(2, \mathbb{C}^5) \).

To verify the claim (v), it suffices to show that the divisor \( D := U^{\alpha_3}(\mathcal{X}_0/\mathcal{X}_0^{\alpha_2}) \) on \( S := A_4/P_{(\alpha_2, \alpha_3, \alpha_4)} \) satisfies

\[
(D \cdot C_i) = \delta_{i2}, \quad [C_i] \in K^{\alpha_i}(S), \ 2 \leq i \leq 4.
\]

Take a point \([V_4] \in \mathcal{X}_0^{\alpha_2, \alpha_4} = A_4/P_{\alpha_4}\), where \( V_4 \) is the corresponding 4-dimensional linear subspace of \( \mathbb{C}^5 \). The restriction \( U^{\alpha_3}(\mathcal{X}_0/\mathcal{X}_0^{\alpha_2}) \subset A_4/P_{(\alpha_2, \alpha_3, \alpha_4)} \to Gr(2, \mathbb{C}^5) \) on the fiber \((\pi_0^{\alpha_2, \alpha_3})^{-1}([V_4]) \cong F^d(2, 3, V_4)\) is \( C_2/B \subset A_3/P_{(\alpha_2, \alpha_3)} \to Gr(2, \mathbb{C}^4) \), where \( B \) is a Borel subgroup of the simple group \( C_2 \). Hence (7.3) holds for \( i = 2 \) and 3.

Now consider a part of (7.2), which is a commutative diagram as follows:

\[
\begin{array}{ccc}
\mathcal{X}_0 & \xrightarrow{\gamma} & U^{\alpha_3}(\mathcal{X}_0/\mathcal{X}_0^{\alpha_2}) \\
& \downarrow & \downarrow \\
\mathcal{X}_0^{\alpha_2} = F(3, 4, \mathbb{C}^5) & \xleftarrow{\delta} & S
\end{array}
\]

Take any \([l_4] \in K^{\alpha_4}(\mathcal{X}_0^{\alpha_2})\). Restricting on \( l_4 \subset \mathcal{X}_0^{\alpha_2} \), we obtain a commutative diagram:

\[
\begin{array}{ccc}
\mathbb{P}^2 \times l_4 & \xrightarrow{\sigma} & \mathbb{P}^1 \times l_4 \\
\downarrow \quad \downarrow \quad \downarrow & & \downarrow \quad \downarrow \\
l_4 & \xleftarrow{\varphi_2} & \mathbb{P}^2 \times l_4
\end{array}
\]

where the horizontal rational map \( \varphi_1 : \mathbb{P}^2 \times l_4 \dashrightarrow \mathbb{P}^1 \times l_4 \) is the linear projection from \( \mathbb{P}^2 \times \{t\}, \ t \in l_4 \) with center \( \sigma(t) \in \mathbb{P}^2 := \mathbb{P}^2 \times \{t\} \), and the vertical morphism \( \varphi_2 : \mathbb{P}^1 \times l_4 \to \mathbb{P}^2 \times l_4 \) is a hyperplane bundle over \( l_4 \). By this diagram we can choose \([\zeta_4] \in K^{\alpha_4}(S)\) such that \( \zeta_4 \subset \mathbb{P}^2 \times l_4 \subset S \) is a section of \( l_4 \) and \( \zeta_4 \cap U^{\alpha_3}(\mathcal{X}_0/\mathcal{X}_0^{\alpha_2}) = \emptyset \). In particular, \( (D \cdot \zeta_4) = 0 \), verifying (7.3) and claim (v) too.

Denote by \( 0 \neq \omega \in \wedge^2(\mathbb{C}^5)^* \) the antisymmetric form on \( \mathbb{C}^5 \) such that

\[
Gr_{\omega}(2, \mathbb{C}^5) := \{[V] \in Gr(2, \mathbb{C}^5) \mid \omega(V, V) = 0\}
\]

is the hyperplane section of \( Gr(2, \mathbb{C}^5) \subset \mathbb{P}^9 \) mentioned in claim (v). The assertion \( \omega \neq 0 \) follows from the fact \( U^{\alpha_3}(\mathcal{X}_0/\mathcal{X}_0^{\alpha_2}) \subsetneq S \).

Then we can conclude that

\[\text{ Springer}\]
(vi) at any point \( t = ([V_3], [V_4]) \in X_0^{a_2} = F(3, 4, \mathbb{C}^5) \) the fiber \( U_{a_3} \) is identified with the space \( M_t := \{ [V_3] \in Gr_2(2, \mathbb{C}^5) | V_2 \subset V_3 \} \).

Denote by \( \omega' \in \wedge^2 V_4^* \) the restriction of \( \omega \) on \( V_4 = \mathbb{C}^4 \subset \mathbb{C}^5 \). If the point \( t = ([V_3], [V_4]) \) is general in \( X_0^{a_2} = F(3, 4, \mathbb{C}^5) \), then \( V_3^{a_2} := \{ v \in V_4 | \omega'(v, V_3) = 0 \} \subset V_4 \) is a linear subspace of dimension one and \( M_t \) is exact \( ([V_2] \in Gr(2, \mathbb{C}^5) | V_3^{a_2} \subset V_2 \subset V_3) \), which is isomorphic to \( \mathbb{P}^1 \).

However, \( \text{Null}(\omega') \neq 0 \) by the dimension reason, where \( \text{Null}(\omega) := \{ v \in \mathbb{C}^5 | \omega(v, \mathbb{C}^5) = 0 \} \). Then there exists \( \tilde{V}_3 \in Gr(3, \mathbb{C}^5) \) such that \( \text{Null}(\omega) \cap \tilde{V}_3 \neq 0 \) and \( \tilde{V}_3 \subset \tilde{V}_3^{a_2} \subset \mathbb{C}^5 \), where \( \tilde{V}_3^{a_2} := \{ v \in \mathbb{C}^5 | \omega(v, \tilde{V}_3) = 0 \} \). Choose \( \tilde{t} := ([\tilde{V}_3], [\tilde{V}_4]) \in X_0^{a_2} \). Then by definition we have \( M_{\tilde{t}} = ([V_2] \in Gr(2, \mathbb{C}^5) | V_2 \subset \tilde{V}_3) \cong \mathbb{P}^2 \). It contradicts with the assertion (vi). This completes the proof of Proposition 7.1.

\[ \square \]

7.2 The remaining cases

In the subsection, we will prove Proposition 1.4. Several special cases of it have been settled in the previous arguments. Now let us deal with another special case.

Proposition 7.15 The manifold \( D_5/P_{[a_1, a_2, a_3, a_4]} \) is rigid under Fano deformation.

Proof Suppose we are in the situation of Setting 1.7, and suppose that \( G = D_5 \) and \( I = \{ a_1, a_2, a_3, a_4 \} \). Take a general point \( x \in X_0 \). By Proposition-Definition 3.3, the Mori contraction \( \Phi^A : S = D_5/P_I \to D_5/P_{I\setminus A} \) is extended to a relative Mori contraction \( \pi^A : X' \to X^A \) for any subset \( A \) of \( I \). Moreover, the fiber \( F_x^A \) of \( \pi^A \) passing through \( x \) is isomorphic to \( S^A \), where \( S^A \) is the fiber of \( \Phi^A \) passing through the base point. By Proposition 7.1, \( F_{x}^{a_1,a_2,a_3} \) is biholomorphic to \( A_4/P_{[a_1, a_2, a_3]} \). Since \( \pi_{a_1,a_2,a_3} \) factors through \( \pi_{a_2,a_3} \), it implies that \( F_{x}^{a_2,a_3} \) is biholomorphic to \( A_3/P_{[a_2, a_3]} \). Note that \( \pi_{a_2,a_3} \) factors through \( \pi_{a_2,a_4} \) and \( F_{x}^{a_2,a_3,a_4} \) is a smooth Fano deformation of \( D_4/P_{[a_2, a_3, a_4]} \). Then by Proposition 5.1, \( F_{x}^{a_2,a_3,a_4} \) is biholomorphic to \( D_4/P_{[a_2, a_3, a_4]} \).

Take any pair \( \beta_1 \neq \beta_2 \in I \) that is \( J \)-connected in the sense of Lemma-Definition 3.21, where \( J := R \setminus \{ a_5 \} \). Note that either \( \pi_{a_1,a_2,a_3} \) or \( \pi_{a_2,a_3,a_4} \) factors through \( \pi_{\beta_1,\beta_2} \). The arguments above imply that the fiber \( F_{x}^{\beta_1,\beta_2} \) is biholomorphic to \( S^{\beta_1,\beta_2} \). By Theorem 3.22, the manifold \( X_0 \) is biholomorphic to \( D_5/P_I \), and thus \( D_5/P_I \) is rigid under Fano deformation.

\[ \square \]

Now we are ready to prove Proposition 1.4.

Proof of Proposition 1.4 (i) Consider the Fano deformation rigidity of \( A_4/P_I \) with \( |I| = 3 \). The set of simple roots is \( R = \{ a_1, \ldots, a_4 \} \). The manifolds \( A_4/P_{R\setminus\{a_1\}} \) and \( A_4/P_{R\setminus\{a_4\}} \) are biholomorphic to each other, which are rigid under Fano deformation by Proposition 7.1. The manifolds \( A_4/P_{R\setminus\{a_2\}} \) and \( A_4/P_{R\setminus\{a_3\}} \) are biholomorphic to each other, which are rigid under Fano deformation by Proposition 1.5.
(ii) Consider the Fano deformation rigidity of \( S := D_5/P_I \) with \(|I| = 4\). Set 
\[ J := R \setminus I = \{\alpha_i\} \] for some \( i \), where \( R \) is the set of simple roots. Note that 
\( D_5/P_{\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}} \) is biholomorphic to \( D_5/P_{\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}} \). When \( i = 4 \) or 5, the 
manifold \( S \) is rigid under Fano deformation by Proposition 7.15.

Now we may assume \( i \in \{1, 2, 3\} \). Take any pair \( \beta_1 \neq \beta_2 \in I \). If \( (\beta_1, \beta_2) \) is not 
\( J \)-connected, then \( S^{\beta_1, \beta_2} \cong S^{\beta_1} \times S^{\beta_2} \), which is rigid under Fano deformation 
by Proposition 3.24.

Consider the case when \( (\beta_1, \beta_2) \) is a \( J \)-connected pair. Either \( S^{\beta_1, \beta_2} \) is biholo-

8 Proofs of the main results

In this section we will complete the proofs of Theorems 1.1 and 1.6.

Proof of Theorem 1.1

We can write the rational homogeneous space \( G/P \) in the state-

When \( \rho(S) \leq 3 \), the manifold \( S \) is biholomorphic to \( A_2/P_{\{\alpha_1, \alpha_2, \alpha_3\}}, A_3/P_{\{\alpha_1, \alpha_3\}}, A_4/P_{\{\alpha_1, \alpha_2, \alpha_4\}}, A_4/P_{\{\alpha_2, \alpha_3, \alpha_4\}}, D_4/P_{\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}}, \) or \( D_4/P_{\{\alpha_2, \alpha_3, \alpha_4\}} \). The two 
manifolds \( A_3/P_{\{\alpha_1, \alpha_3\}} \cong \mathbb{F}(1, 2, \mathbb{P}^3) \) and \( D_4/P_{\{\alpha_2, \alpha_3, \alpha_4\}} \cong \mathbb{F}(1, 2, Q^6) \) are excluded 
in the statement of Theorem 1.1. The rigidity of \( A_2/P_{\{\alpha_i\}} \cong \mathbb{P}^2 \) under Fano deformation 
is clear, since it is the unique 2-dimensional Fano manifold of Picard number 
one. By Theorem 6.2, \( A_3/P_{\{\alpha_1, \alpha_3\}} \cong \mathbb{F}(0, 2, \mathbb{P}^3) \) is rigid under Fano deformation. 
By Proposition 1.4, the manifolds \( A_4/P_{\{\alpha_1, \alpha_2, \alpha_4\}} \) and \( A_4/P_{\{\alpha_2, \alpha_3, \alpha_4\}} \) are rigid under Fano deformation. It remains to check the Fano deformation rigidity of \( D_4/P_{\{\alpha_1, \alpha_3, \alpha_4\}} \).

Consider the case when \( (\beta_1, \beta_2) \) is a \( J \)-connected pair. Either \( S^{\beta_1, \beta_2} \) is biholo-

The condition (i) holds. Now we may assume \( \beta_1 \neq \beta_2 \) is linked to \( \beta_0 \) in the Dynkin diagram 
\( \Gamma_R \) of \( G \), then the condition (i) holds. Now we may assume that \( \beta_1 \) or \( \beta_2 \) is linked to \( \beta_0 \). Since \( (\beta_1, \beta_2) \) is a \( J \)-connected pair, the Dynkin diagram \( \Gamma_{\beta_0, \beta_1, \beta_2} \) is of type \( A_3 \). Since \( \rho(S) \geq 4 \), there exists \( \beta_3 \in I \setminus \{\beta_1, \beta_2\} \) such that \( \Gamma_{\beta_0, \beta_1, \beta_2, \beta_3} \) is connected.

Now the Dynkin diagram \( \Gamma_{\beta_0, \beta_1, \beta_2, \beta_3, \beta_4} \) is of type \( D_5 \) for some \( \beta_3, \beta_4 \in I \setminus \{\beta_1, \beta_2\} \). If it is of type \( D_4 \), then the condition (ii) holds. If it is of type \( D_4 \), then the assumption \( \rho(S) \geq 4 \) implies the existence of \( \beta_3 \in I \setminus \{\beta_1, \beta_2, \beta_3\} \) such that \( \Gamma_{\beta_0, \beta_1, \beta_2, \beta_3, \beta_4} \) is connected. In this case, the condition (iii) holds. In summary, the claim above is true.
If the condition (i) holds, then the manifold $S^\beta_1, \beta_2$ is biholomorphic to $A_2/P_{\{\alpha_1, \alpha_2\}}$, and thus it is rigid under Fano deformation by Theorem 7.4. If the condition (ii) holds, then the manifold $S^\beta_1, \beta_2, \beta_3$ is biholomorphic to $A_4/P_{I'}$ with $|I'| = 3$, and thus it is rigid under Fano deformation by Proposition 1.4. If the condition (iii) holds, then the manifold $S^\beta_1, \beta_2, \beta_3, \beta_4$ is biholomorphic to $D_5/P_{I''}$ with $|I''| = 4$, and thus it is rigid under Fano deformation by Proposition 1.4.

On the other hand, if the pair $\beta_1 \neq \beta_2 \in I$ is not $J$-connected, then $S^\beta_1, \beta_2 \cong S^\beta_1 \times S^\beta_2$. The manifold $S^\beta_1, \beta_2$ is rigid under Fano deformation by Proposition 3.24.

In summary, given any pair $\beta_1 \neq \beta_2 \in I$, there is a subset $A$ of $I$ such that $\beta_1, \beta_2 \in A$ and the manifold $S^A$ is rigid under Fano deformation. Hence, $S$ is rigid under Fano deformation by Corollary 3.23.

\textbf{Proof of Theorem 1.6} In the situation of Theorem 1.6, if $(\alpha, \beta) \in I \times I$ is an arbitrary $J$-connected pair, then the unique connected component of the Dynkin diagram $\Gamma_{J \cup \{\alpha, \beta\}}$ containing both $\alpha$ and $\beta$ is one of the following types:

(i) $(A_m, \{\alpha_1, \alpha_m\})$ with $m \geq 2$;

(ii) $(A_m, \{\alpha_1, \alpha_2\})$ with $m \geq 3$.

By our assumption, in the situation of (ii) there exists $\gamma \in I \setminus \{\alpha, \beta\}$ such that the unique connected component of the Dynkin diagram $\Gamma_{J \cup \{\alpha, \beta, \gamma\}}$ containing all of $\alpha, \beta$ and $\gamma$ is of type $(A_{m+1}, \{\alpha_1, \alpha_2, \alpha_{m+1}\})$ up to symmetry. Recall that $A_m/P_{\{\alpha_1, \alpha_m\}}$ and $A_{m+1}/P_{\{\alpha_1, \alpha_2, \alpha_{m+1}\}}$ are rigid under Fano deformation by Theorem 6.2 and Proposition 1.5 respectively.

On the other hand, if the pair $\alpha \neq \beta \in I$ is not $J$-connected, then $S^\alpha, \beta \cong S^\alpha \times S^\beta$. The manifold $S^\alpha, \beta$ is rigid under Fano deformation by Proposition 3.24.

In summary, given any pair $\alpha \neq \beta \in I$, there is a subset $A$ of $I$ such that $\alpha, \beta \in A$ and the manifold $S^A$ is rigid under Fano deformation. Hence, $S$ is rigid under Fano deformation by Corollary 3.23.

Indeed by a careful analysis of the Dynkin diagrams we can apply the same proof to deduce the following rigidity result.

\textbf{Theorem 8.1} Let $G$ be a simple algebraic group of type $ADE$, $I \subset R$ be a subset and $J := R \setminus I$. Write $I$ as the disjoint union $\bigcup I_i$, where each $\Gamma_{I_i}$ is a connected component of $\Gamma_I$. Suppose that

\begin{enumerate}
\item the end nodes of the Dynkin diagram of $G$ are contained in $I$,
\item each $I_i$ satisfies that either $I_i \cap \partial R \neq \emptyset$ or its cardinality $|I_i| \geq 3$,
\item when $G$ is of type $D$ or $E$, there exists at most one $\beta \in J$ such that $(\beta, \bar{\alpha}) \neq 0$, where $\bar{\alpha}$ is the node in Dynkin diagram of $G$ with three branches.
\end{enumerate}

Then the rational homogeneous space $G/P_I$ is rigid under Fano deformation.

\textbf{Remark 8.2} As a direct consequence of Proposition 3.24, we know that $S$ is rigid under Fano deformation if $S = S_1 \times \cdots \times S_k$ and each $S_i$ satisfies the assumptions of Theorems 1.1, 1.6, 6.2, 7.4 or 8.1.
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Declaration

Conflict of interest  The author declares that there is no conflict of interest.

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