1. Introduction

Understanding the dynamical evolution of galactic discs over cosmic times is a long-standing endeavour. Self-gravitating discs are cold dynamical systems, for which rotation represents an important reservoir of free energy. Fluctuations of the potential induced by discrete (possibly distant) encounters may be strongly amplified, while resonances tend to confine and localise their dissipation: such small stimuli can lead to discs spontaneous evolution to distinct equilibria. Quantifying the relative importance of the collisional term (Vlasov) but the collisional term displays a logarithmic divergence at large scales (Jeans 1929; Landau 1936; Chandrasekhar 1942). In the case of plasmas, this divergence is due to the neglect of collective effects that are responsible for Debye shielding. Landau (1936) phenomenologically introduced a cut-off at the Debye length to regularize the divergence. In astrophysics, the inhomogeneous Balescu-Lenard equation and its WKB limit may also describe the secular diffusion of giant molecular clouds in galactic discs, the secular migration and segregation of planetesimals in proto-planetary discs, or even the long-term evolution of population of stars within the Galactic center.

Key words. Galaxies: evolution - Galaxies: kinematics and dynamics - Galaxies: spiral - Diffusion - Gravitation

1 For an historical account of the development of kinetic theories in astrophysics, plasma physics, and for systems with long-range interactions, see the introductions of the references Chavanis (2013a,b).
their treatment through the dielectric function (that is absent from the Landau equation). In the case of stellar systems, the divergence at large scales is solved by the spatial inhomogeneity of the system and its finite extent. One can phenomenologically introduce a cut-off at the Jeans scale (Weinberg 1993), i.e. at the system’s size, which would correspond to the analogue of the Debye length in plasma physics, but this ad hoc treatment is not fully satisfactory. Furthermore, it cannot be applied to cold (centrifugally supported) stellar discs where spatial inhomogeneity is more crucial than in 3D.

A more fruitful procedure is to write the kinetic equation with angle-action variables that are the appropriate variables to describe spatially inhomogeneous multi-periodic systems. When collective effects are neglected, one obtains the inhomogeneous Landau equation (Chavanis 2007, 2013b). When collective effects are accounted for, one gets the inhomogeneous Balescu-Lenard equation (Heyvaerts 2010; Chavanis 2012a). For self-gravitating systems, where the interaction is attractive (instead of being repulsive as in Coulombian plasmas), collective effects are responsible for an anti-shielding which tends to increase the effective mass of the stars, hence reducing the relaxation time. The Balescu-Lenard equation is valid at the order 1/N in an expansion of the dynamics in terms of this small parameter, where \( N \gg 1 \) is the number of stars. Therefore, it takes finite-\( N \) effects into account and describes the evolution of the system on a timescale of the order \( N \tau_\text{p} \), where \( \tau_\text{p} \) is the dynamical time. For times \( t < N \tau_\text{p} \), or for \( N \rightarrow +\infty \), it reduces to the Vlasov equation which ignores distant encounters between stars. Although the kinetic theory was initially developed for 3D stellar systems, the final form of the inhomogeneous Balescu-Lenard equation also applies to stellar discs such as those considered in this paper.

Indeed, the Balescu-Lenard non-linear equation accounts for self-driven orbital secular diffusion of a gravitating system induced by the intrinsic shot noise due to its discreteness and the corresponding long range correlations. Even though this kinetic equation was first written down more than fifty years ago, it has hardly ever been applied in its prime context, but only in various limits where it reduces to simpler kinetic equations, as discussed above.

In this paper, we will focus on solving explicitly such an equation describing the self-gravitating response of a tepid thin disc to its own stochastic fluctuating potential induced by its finite number of components. In this cool regime, the self-gravity of the disc can be tracked via a local WKB-like response, which in turn allows us to simplify the a priori 2D formalism to an effective (non degenerate) 1D formalism. We will compare the prediction of the WKB limit to a numerical experiment presented in the literature, and discuss its diagnosis power and possible limitations.

The paper is organized as follows. Section 2 briefly presents the content of the inhomogeneous Balescu-Lenard equation. Section 3 focuses on razor thin axisymmetric galactic discs within the WKB approximation. Section 4 investigates the formation of a narrow resonant ridge in an isolated self-gravitating Mestel disc. Finally, section 5 wraps up. Appendix A provides a short sketch of the derivation of the Balescu-Lenard equation. Appendix B considers the inhomogeneous Balescu-Lenard equation without collective effects. Appendix D compares it to other similar kinetic equations, and in particular its Fokker-Planck limit.

### 2. The inhomogeneous Balescu-Lenard equation

We consider a system made of \( N \) particles. We suppose that the gravitational background, associated to the Hamiltonian \( H_0 \), is stationary and integrable, so that we may always remap the physical phase-space coordinates \((\mathbf{x}, \mathbf{v})\) to the angle-actions coordinates \((\theta, J)\) (Goldstein 1950; Born 1960; Binney & Tremaine 2008). We also introduce the intrinsic frequencies of the system \( \Omega \) defined as

\[
\Omega(J) = \dot{\theta} = \frac{\partial H_0}{\partial J}.
\]

Along the unperturbed trajectories, the angles \( \theta \) are \( 2\pi \)-periodic evolving with the frequency \( \Omega \), whereas the actions \( J \) are conserved. To describe the long-term evolution of such a system, one assumes that there are two decoupled timescales: a short dynamical timescale and a secular timescale of collisional evolution. We assume that the system is always in a virialized stable state (i.e. is a stable stationary solution of the Vlasov equation), so that the distribution function can be written as a quasi-stationary distribution function \( F = F(J, t) \). This is a function of the actions only that slowly evolves in time due to stellar encounters (finite-\( N \) effects). From Heyvaerts (2010) and Chavanis (2012a) (see also Appendix A for a short sketch of the derivation), the secular evolution, induced by collisional finite-\( N \) effects, of such a quasi-stationary distribution function \( F(J, t) \) is given by the inhomogeneous Balescu-Lenard equation which reads

\[
\frac{\partial F}{\partial t} = \pi(2\pi)^d \frac{\partial}{\partial J_1} \sum_{m_1, m_2} \int \frac{\partial \delta_0(m_1, \Omega_1 - m_2, \Omega_2)}{[D_{m_1, m_2}(J_1, J_2, \Omega_1, \Omega_2)]^2} \left( m_1 \cdot \frac{\partial}{\partial J_1} - m_2 \cdot \frac{\partial}{\partial J_2} \right) F(J_1, t) F(J_2, t),
\]

where \( d \) is the dimension of the physical space, and where we used the shortened notation \( \Omega = \Omega(J) \). The r.h.s of equation (2) is the Balescu-Lenard operator which encompasses the secular diffusion due to collisional effects, see figure 1. Because it is the divergence of a flux, this writing ensures that the total number of stars is exactly conserved during the secular diffusion. The Dirac delta \( \delta_0(m_1, \Omega_1 - m_2, \Omega_2) \) is the sharp resonance condition. One must note that this condition allows to describe non-trivial gravitational interactions. Indeed, it can cause non-local resonances by coupling different regions of action-space \( J_1 \) and \( J_2 \). Even for local resonances (i.e. \( J_1 = J_2 \)), it can allow for non-trivial coupling of oscillations, as soon as \( m_1 \) and \( m_2 \) have non-zero components. The coefficients \( 1/[D_{m_1, m_2}(J_1, J_2, \Omega)^2] \) represent the dressed susceptibilities of the system, for which collective effects have been taken into account. To deal with the resolution of the non-local Poisson equation, following Kalnajs’ matrix method (Kalnajs 1976), one has to introduce a complete biorthonormal basis of potentials and densities \( \rho^{(p)}(x) \) and \( \psi^{(p)}(x) \) such that

\[
\Delta \psi^{(p)} = 4\pi G \rho^{(p)}, \quad \int dx |\psi^{(p)}(x)|^2 \rho^{(p)}(x) = -\delta_\rho^p.
\]

1. In this paper, we are not interested in the initial complex mechanism of violent relaxation (Lynden-Bell 1967), during which the system gets virialized, since we intend to describe the long-term evolution of an already and continuously virialized system.

2. In the secular timescale limit, the amplification through the propagation of waves between resonances is assumed to be instantaneous, see Appendix A.
2.1. Content of the diffusion equation

One may also rewrite the Balescu-Lenard equation (2) as an anisotropic Fokker-Planck equation introducing the associated drift and diffusion coefficients. It then reads

$$\frac{\partial F}{\partial t} = -\sum_{m_1} \frac{\partial}{\partial J_{i_1}} \left[ A_{m_1}(J_1) F(J_1) + D_{m_1}(J_1) m_{i_1} \frac{\partial F}{\partial J_{i_1}} \right],$$

(7)

where $A_{m_1}(J_1)$ and $D_{m_1}(J_1)$ are respectively the anisotropic drift and diffusion coefficients associated to a given resonance $m_1$, i.e. to a given Fourier mode $m_1$ in angles. They both secularly depend on the distribution function $F$, but this dependence has not been explicitly written out to shorten the notations. The drift coefficients are given by

$$A_{m_1}(J) = -\pi(2\pi)^{\delta} \sum_{m_{2}} \langle \delta_{m_1,m_{2}}(J_{1,2}) \Omega_{1,2} \rangle J_{m_{1}} \frac{1}{|m_{1}|^2} \frac{\partial F}{\partial J_{1}}.$$

(8)

while the diffusion coefficients are given by

$$D_{m_1}(J_1) = \pi(2\pi)^{\delta} \sum_{m_{2}} \langle \delta_{m_1,m_{2}}(J_{1,2}) \Omega_{1,2} \rangle J_{m_{1}}^{\perp} \frac{1}{|m_{1}|^2} \frac{\partial F}{\partial J_{1}}.$$

(9)

The rewriting from equation (7) allows us to discuss some important properties of such anisotropic diffusion equation. We introduce the total flux, $\mathcal{F}_{\text{tot}}$, associated with this diffusion, which reads

$$\mathcal{F}_{\text{tot}} = \sum_{m_1} m \left( A_{m_1}(J) F(J) + D_{m_1}(J) m \frac{\partial F}{\partial J} \right).$$

(10)

As a consequence, the Balescu-Lenard diffusion equation given by the expressions (2) and (7) takes the shortened form

$$\frac{\partial F}{\partial t} = \text{div}(\mathcal{F}_{\text{tot}}).$$

(11)

We then define as $M(t)$ the mass contained in a volume $\mathcal{V}$ of the action-space at time $t$, so that we have

$$M(t) = \int_{\mathcal{V}} dJ F(J,t).$$

(12)

Thanks to the divergence theorem, the variation of mass in $\mathcal{V}$ due to secular diffusion corresponds to the flux of particles through the boundary $\partial \mathcal{V}$ of this volume so that

$$\frac{dM}{dt} = \int_{\partial \mathcal{V}} \mathcal{F}_{\text{tot}} \cdot d\mathbf{S} = \sum_{m_1} \int_{\partial \mathcal{V}} dS(m,n) \left[ A_{m_1}(J) F(J) + D_{m_1}(J) m \frac{\partial F}{\partial J} \right].$$

(13)

where $n$ is the exterior pointing normal vector. One can note in equation (13) that the contribution from a given resonance $m$ takes the form of a preferential diffusion in the direction $m$. This diffusion is therefore anisotropic because it is maximum for $n \cdot m$ and equal to 0 for $n \cdot m = 0$. To emphasize the anisotropy of the diffusion, one may use the formalism of the slow and fast actions (Lynden-Bell 1979; Earm & Lynden-Bell 1996). For simplicity, we consider the 2D case. For a given resonance $m = (m_1, m_2)$, we consider the change of coordinates

$$J^s_m = \frac{J \cdot m}{|m|}; \quad J^f_m = \frac{J \cdot m^s}{|m|},$$

(14)

where $J^s_m$ and $J^f_m$ are respectively the slow and fast actions associated to the resonance $m$. Here $m^s$ corresponds to the direction perpendicular to the resonance so that $m^s = (m_2, -m_1)$, and
\[ |m| = \sqrt{\mathbf{m} \cdot \mathbf{m}}. \] Thanks to the chain rule, for any function \( X(J) \), one has
\[ \frac{\partial X}{\partial J} = |m| \left. \frac{\partial X}{\partial J} \right|_{J = 0}. \] (15)

Introducing the natural vector basis elements \( e_m^* = m/|m| \) and \( e_m = m^*/|m| \) associated with this change of coordinates, the diffusion flux \( F_m \) associated with a resonance \( m \) takes the form
\[ F_m(J_m, \dot{J}_m) = |m| \left[ A_m(J) F(J) + |m| D_m(J) \right] \frac{\partial F}{\partial J_m} e_m^*. \] (16)

Such a rewriting illustrates the fact that as soon as only one resonance \( m \) dominates the secular evolution, the diffusion flux will be aligned with this resonance. Hence one will observe a narrow response matrix. Moreover, it also entails that all the resonances may implement a WKB approximation (Liouville 1837; Toomre 1964; Kalnajs 1965; Lin & Shu 1966; Palmer et al. 1989) which requires to explicitly deal with the resonance constraint
\[ \Omega_1 - m \omega = \Omega_2. \]

The heart of the epicyclic approximation is to assume that small radial excursions can be approximated as harmonic librations. For a given value of \( J_\phi \), we implicitly introduce the guiding radius \( R_g \) as
\[ \frac{\partial \psi_\text{eff}}{\partial R} \bigg|_{R_g} = \frac{\partial \psi_0}{\partial R} \bigg|_{R_g} + \frac{J_\phi^2}{2R_g^2}. \] (20)

As the radial oscillations are supposed to be small, one may perform a Taylor expansion at first order of the evolution equation (19) in the neighborhood of the minimum \( R = R_g \) so that \( R \) satisfies the differential equation \( \dot{R} = -\kappa^2 (R - R_g) \). Hence one can note that in this limit the evolution of the radius of a star is the one of a harmonic oscillator centered on \( R_g \). Up to an initial phase, one has therefore \( R(t) = R_g + A \cos(\kappa t), \) where \( A \) is the amplitude of the radial oscillations. The associated radial action \( J_r \) is then given by
\[ J_r = \frac{1}{2R_g} \int dR \mathcal{K} = \frac{1}{2} \kappa A^2, \] (23)

For \( J_r = 0 \), the orbit is circular. Within the epicyclic approximation, the frequencies of motion along the action-tori, introduced in equation (1) are given by \( \Omega_1(J) = (\Omega_1(0), \kappa(J)) \). An important dynamical consequence of this approximation is that these two frequencies are only function of \( J_r \) and do not depend on \( J_\phi \), so that the resonance constraint \( m_1 \Omega_1 - m_2 \Omega_2 = 0 \) becomes simpler. Finally, one can explicitly construct the mapping between \( (R, \phi, p_R, p_\phi) \) and \( (\theta_g, \theta_\phi, J_r, J_\phi) \) (Lynden-Bell & Kalnajs 1972; Palmer 1994; Binney & Tremaine 2008), which takes at first order the form
\[ \begin{align*}
R = R_g + A \cos(\theta_\phi), \\
\phi = \theta_\phi - \frac{2\Delta \phi}{R_g} \sin(\theta_\phi).
\end{align*} \] (24)
Thanks to this mapping and the definitions of the actions from equations (18) and (23), the epicyclic approximation allows us to build up an explicit mapping between the physical phase-space coordinates and the angle-actions ones.

Finally, throughout our calculation, we will assume that the stationary distribution function of the disc is a Schwarzschild distribution function (or locally isothermal DF) given by

$$F(R_g, J_r) = \frac{\Omega_0(R_g)}{\pi k(R_g)} \frac{\Sigma(R_g)}{\sigma^2 \Sigma(R_g)} \exp \left[ \frac{\kappa(R_g) J_r}{\sigma^2 \Sigma(R_g)} \right],$$

where $\Sigma(R_g)$ is the surface density of the disc and $\sigma^2 \Sigma(R_g)$, which varies within the disc, represents the radial velocity dispersion of the stars at a given radius. Increasing values of $\sigma^2 \Sigma$ correspond to hotter discs that are therefore more stable.

3.2. The WKB basis

As we are considering a 2D case, the potential basis elements $\psi^{(l)}$ introduced in equation (3) must be written as $\psi^{(l)}(R, \phi)$ in the disc polar coordinates and must be orthonormal to the associated surface density $\Sigma^{(l)}(R, \phi)$. Using a WKB approximation amounts to building up local basis elements thanks to which the response matrix will become diagonal.

3.2.1. Definition of the basis elements

We introduce the basis elements

$$\psi^{(k, k, R_0)}(R, \phi, \sigma) = A e^{i(k, \phi + k, R)} B_{R_0}(R),$$

where the window function $B_{R_0}(R)$ is defined as

$$B_{R_0}(R) = \frac{1}{(\pi \sigma^2)^{1/4}} \exp \left[ -\frac{(R - R_0)^2}{2 \sigma^2} \right].$$

The basis elements are indexed by three numbers: $k_0$ is an azimuthal number which parametrizes the angular component of the basis elements, $R_0$ is the radius position in the disc around which the Gaussian window $B_{R_0}$ is centered, and $k$ is the radial frequency of the basis element. We also introduced an additional parameter $\sigma$ of scale-separation, which will ensure the biorthogonality of the basis elements, as detailed later on. Finally, $A$ is an amplitude which will be tuned in order to normalize correctly the basis elements. Thanks to a somewhat unusual normalization of $B_{R_0}$, we will ensure that $A$ is independent of $\sigma$. Figure 2 illustrates the radial dependence of the basis elements. Figure 3 illustrates the shape of these basis elements in the polar $(R, \phi)$–plane. The next steps will be to ensure that these WKB basis elements have all the properties required to allow for the computation of the dressed susceptibility coefficients introduced in equation (4). Therefore, we will successively compute the associated surface density elements $\Sigma^{(k, k, R_0)}$, ensure the biorthogonality of the basis elements and their correct normalization, and finally compute the Fourier transform in angles of the basis elements.

3.2.2. Associated surface densities

In order to ensure the biorthogonality of the basis, we will first build up the surface densities associated to the potential elements introduced in equation (26). We extend the WKB potential in the $z$–direction using the Ansatz

$$\psi^{(k, k, R_0)}(R, \phi, z) = A e^{i(k, \phi + k, R)} B_{R_0}(R) Z(z).$$

Poisson’s equation in vacuum $\Delta \psi^{(k, k, R_0)} = 0$ leads to

$$\frac{Z''}{Z} = k^2 \left[ 1 - \frac{1}{k R} + 2 \frac{R - R_0}{\sigma^2 k} \right] + \frac{1}{(\sigma k)^2} + \frac{k^2}{(k R)^2} + \frac{R - R_0}{\sigma^2 k} \left[ \frac{R - R_0}{\sigma^2} \right].$$

We now explicitly introduce our WKB assumptions. We assume that the spirals are tightly wound so that

$$k, R \gg 1.$$ (30)

Introducing the typical size of the system $R_{sys}$, we also additionally suppose that we have

$$k, R \gg \frac{R_{sys}}{\sigma}.$$ (31)

Fig. 2: Two WKB basis elements. Each Gaussian is centered around a radius $R_0$. The typical extension of the Gaussian is given by the decoupling scale $\sigma$, and they are modulated at the radial frequency $k_r$. 

Fig. 3: Two WKB basis elements in the polar $(R, \phi)$–plane. Each basis element is located around a central radius $R_0$, on a region of size $\sigma$. The winding of the spirals is governed by the radial frequency $k_r$. The number of azimuthal patterns is given by the index $k_0$, e.g. $k_0 = 1$ for the interior dark gray element, whereas $k_0 = 2$ for the exterior light gray one.
Assuming that $k_0$ is of the order of unity, equation (29) becomes
\[
Z'' \equiv \frac{Z^{''}}{Z} = k_0^2. \tag{32}
\]

Hence within the WKB limit, the extended potential from equation (28) takes the form
\[
\psi_{\epsilon(k,k',k_0)}(R, \phi, z) = \psi_{\epsilon(k_0,k_0)}(R, \phi) \exp[-ik_0z], \tag{33}
\]
where we ensured that for $z \to \pm\infty$ the potential tends to 0, therefore introducing a discontinuity for $\partial \phi / \partial z$ in $z=0$. Thanks to Gauss theorem, the associated surface density satisfies
\[
\Sigma(R, \phi) = \frac{1}{4\pi G} \left[ \lim_{z \to -0} \frac{\partial \psi}{\partial z} - \lim_{z \to +0} \frac{\partial \psi}{\partial z} \right], \tag{34}
\]
so that we have
\[
\Sigma_{\epsilon(k,k',k_0)}(R, \phi) = -\frac{|k_1|}{2\pi G} \psi_{\epsilon(k,k',k_0)}(R, \phi). \tag{35}
\]

### 3.2.3. Biothorongality condition and normalization

One must now ensure that the basis elements introduced in equations (26) and (33) form a biorthogonal basis as assumed in equation (3). Indeed, it has to satisfy the property
\[
\delta_{k'}^{k} \delta_{\mathcal{L}}^{\mathcal{L}_0} = \frac{1}{\oint} dR \psi^*_{\epsilon(k',\mathcal{L},\mathcal{L}_0)} \psi_{\epsilon(k,\mathcal{L},\mathcal{L}_0)}, \tag{36}
\]
with
\[
\delta_{k'}^{k} \delta_{\mathcal{L}}^{\mathcal{L}_0} = -\frac{1}{2\pi G} \oint d\phi \exp[i(k' - k)_\phi] \Sigma_{\epsilon(k',\mathcal{L},\mathcal{L}_0)}. \tag{37}
\]

The r.h.s. of this expression takes the form
\[
\frac{|k_1|}{2\pi G} \mathcal{A}_p \mathcal{A}_0 \int d\phi \exp[i(k' - k)_\phi] \exp[-(R - R_0')^2/\sigma^2] \exp[-(R - R_0')^2/\sigma^2]. \tag{38}
\]

The integration on $\phi$ is straightforward and is equal to $2\pi \delta_{k'}^{k} \delta_{\mathcal{L}}^{\mathcal{L}_0}$. In order to perform the integration on $R$, we have to introduce additional assumptions to ensure the biothorongality of the basis. The peaks of the Gaussians in equation (37) can be considered as separated if $\Delta R_0 = R_0' - R_0''$ satisfies the condition
\[
\Delta R_0 \gg \sigma \quad \text{if} \quad R_0' \neq R_0'', \tag{39}
\]
Under this assumption, the term from equation (37) can be assumed to be non-zero only for $R_0'' = R_0'$. The r.h.s of equation (36) then takes the form
\[
\frac{|k_1|}{2\pi G} \mathcal{A}_p \mathcal{A}_0 \frac{1}{\sqrt{4\pi \sigma^2}} \int dR \exp[i(k' - k)_R] \exp[-(R - R_0')^2/\sigma^2]. \tag{40}
\]

The integration on $R$ takes the form of a radial Fourier transform of a Gaussian of spread $\sigma$ at the frequency $\Delta k_r = k_r' - k_r''$. It is therefore proportional to $\exp[-(\Delta k_r)^2/(4\sigma^2)]$. Hence we will suppose that the frequency spread $\Delta k_r$ satisfies
\[
\Delta k_r \gg \frac{1}{\sigma} \quad \text{if} \quad k_r' \neq k_r''. \tag{41}
\]
Under this assumption, the term from equation (39) is non-zero only for $k_r'' = k_r'$. In order to have a biorthogonal basis, one must therefore consider a spreading $\sigma$, central radii $R_0$, and radial frequencies $k_r$ such that
\[
\Delta R_0 \gg \sigma \gg \frac{1}{\Delta k_r}. \tag{42}
\]

With these constraints, one must necessarily have $k_r'' = k_r'$ and $R_0'' = R_0'$ in order to have a non negligible term in equation (36). It then only remains to explicitly estimate the amplitude $\mathcal{A}$ of the basis elements. Equation (36) gives
\[
\mathcal{A}^2 \frac{|k_1|}{G} \frac{1}{\sqrt{4\pi \sigma^2}} \int dR \exp[-(R - R_0')^2/\sigma^2] = 1. \tag{43}
\]
Thanks to the WKB assumption from equation (41), the integration can be straightforwardly computed and leads to
\[
\mathcal{A} = \sqrt{\frac{G}{|k_1| R_0}}. \tag{44}
\]

### 3.2.4. Fourier transform in angles

In order to estimate the susceptibility coefficients and the response matrix from equations (4) and (5), one has to be able to calculate $\psi_m^{\mu}(J)$ for the WKB basis elements. Thanks to the explicit mapping from equation (24), we have to compute

\[
\psi_m^{\mu}(J) = A e^{ik_0 R_0} \int d\theta_0 e^{-im \theta_0} e^{-i \phi \theta_0} \int d\theta \mathcal{A} e^{ik \theta_0} \mathcal{A} e^{i \phi \theta_0} \mathcal{B}(\theta_0 + \phi \theta_0). \tag{45}
\]

The integration on $\theta_0$ is straightforward and equal to $2\pi \delta_{\mu \mu_0}$. Regarding the dependence on $\theta_0$ in the complex exponential, we may write
\[
k_1 A \cos(\theta_0) - k_0 \Omega_0 \frac{A}{k R_0} \sin(\theta_0) = H_s(k_1) \sin(\theta_0 + \phi \theta_0), \tag{46}
\]
where the amplitude $H_s(k_1)$ and the phase shift $\phi \theta_0$ are given by
\[
H_s(k_1) = A \left| k_1 \right| \sqrt{1 + \frac{\Omega_0}{k R_0} \left| k_0 \right|^2}; \quad \phi \theta_0 = \tan^{-1} \left( \frac{k_1 R_0}{\Omega_0 / 2 \pi} \right). \tag{47}
\]

For typical galaxies, we have $1/2 \leq \Omega_0 / k \leq 1$ (Binney & Tremaine 2008). Assuming that the azimuthal number $k_0$ is of the order of unity, one can use the WKB hypothesis introduced in equation (30), so that equations (46) can be approximated as
\[
H_s(k_1) \approx A \left| k_1 \right| \sqrt{\frac{2 \pi}{k \left| k_0 \right|}} \ ; \quad \phi \theta_0 \approx \frac{\pi}{2}. \tag{48}
\]

Because we have assumed that the disc is ternal, the radial oscillations are small so that $A \ll R_0$. We may then get rid of the dependences on $A$ in $B_R(R_0 - A \cos(\theta_0))$ by replacing it with $B_R(R_0)$, so that the only remaining dependence on $A$ will be in the complex exponentials. To be able to explicitly perform the remaining integration on $\theta_0$ in equation (44), we introduce the Bessel functions $J_0$ of the first kind which satisfy the relation
\[
\int e^{i \xi \sin(\theta)} = \sum_{\ell \in \mathbb{Z}} (2J_0[|\xi|] e^{i \ell \theta}). \tag{49}
\]

We then finally obtain the expression of the Fourier transform in angles of the WKB basis elements which reads
\[
\psi_m^{\mu}(k_0, k_1) = \delta_{\mu \mu_0} e^{i \phi \theta_0} \mathcal{A} J_0[H_s(k_1) B_R(k_0)]. \tag{50}
\]
3.3. Estimation of the response matrix

Thanks to the WKB basis introduced in equation (26), one can now explicitly compute the response matrix from equation (5). Indeed, we use the expression (49) of the Fourier transform of the basis elements, and after simplification of the phase-shift terms thanks to the approximation from equation (47), one has to evaluate

\[ \hat{\mathbf{M}}(k_x, k_y, k_z) = (2\pi)^3 \int \mathcal{F}(\mathbf{r}) e^{ik_x x+ik_y y+ik_z z} \, d^3 \mathbf{r} \]

If the radial Fourier transform is given by

\[ \hat{\mathcal{J}}(k_x, k_y, k_z) = (2\pi)^3 \int \mathcal{J}(\mathbf{r}) e^{ik_x x+ik_y y+ik_z z} \, d^3 \mathbf{r} \]

where \( \mathcal{J}(\mathbf{r}) \) is the density of the disc. Then the response matrix \( \hat{\mathbf{M}}(k_x, k_y, k_z) \) can be expressed as

\[ \hat{\mathbf{M}}(k_x, k_y, k_z) = \left( \begin{array}{ccc} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{array} \right) \]

where

\[ M_{ij} = \int \mathcal{F}(\mathbf{r}) \mathcal{J}(\mathbf{r}) e^{ik_x x+ik_y y+ik_z z} \, d^3 \mathbf{r} \]

The first step of the calculation is to show that in the WKB limit, the response matrix becomes diagonal. One should note that the previous expression (50) is similar to equation (37), where we discussed the biradial orthogonality of the WKB basis. In equation (50), the azimuthal Kronecker symbols ensures that \( k_x^2 = k_y^2 \).

Moreover, because of our WKB assumptions from equation (41) on the phase shifts, the product of the two radial Gaussians in \( \mathcal{R} \) imposes that \( R_{0}^2 = R_{0}^2 \) in order to have a non-zero contribution. In order to shorten the notations, we temporarily introduce the function \( h(R_{0}) \) defined as

\[ h(R_{0}) = \frac{J_{0}}{\sigma_{0}} e^{\frac{m \cdot \mathbf{F}}{\sigma_{0}} - \frac{m \cdot \mathbf{F}}{\sigma_{0}}} \]

which encompasses all the additional radial dependences appearing in equation (50). Thanks to the change of variables \( J_{0} \rightarrow R_{0} \), the integral on \( J_{0} \) which has to be evaluated in equation (50), when estimated for \( R_{0} = R_{0}^2 \), is of the form

\[ \int \mathcal{F}(\mathbf{r}) h(R_{0}) \, d^3 \mathbf{r} = \left( \frac{R_{0} - R_{0}^2}{\sigma_{0}^2} \right)^{2} \]

This expression corresponds to a radial Fourier transform \( \mathcal{F} \) at the frequency \( \Delta \). It can be rewritten as a convolution of two radial Fourier transforms so that it becomes

\[ \int d^{3} \mathbf{r} \mathcal{F}(\mathbf{r}) e^{ik_{x} x+ik_{y} y+ik_{z} z} \]

where \( \Delta = k_{x} - k_{y} \). We now rely on the WKB assumption from equation (40). If one has \( \Delta \neq 0 \), because of the Gaussian from equation (52), the contribution from \( \mathcal{F}(\mathbf{r}) \) will come from the region \( k' - \Delta \kappa > 1/\sigma \). We assume that the function \( h \) is such that its Fourier Transform is limited to the frequency region \( |k'| \leq 1/\sigma \). This is consistent with assuming that the properties of the disc are radially slowly varying, and this implies that non-zero contributions to the response matrix can only be obtained when \( \Delta \kappa = k_{x} - k_{y} = 0 \). Therefore, we have shown that within our WKB formalism, the response matrix from equation (5) is diagonal.

In order to shorten the notations, we will denote the matrix eigenvalues as

\[ \lambda_{i}(k_{x}, k_{y}, k_{z}) = \hat{\mathbf{M}}(k_{x}, k_{y}, k_{z}) \]

For these diagonal coefficients, the last step is to explicitly compute the integrals over \( J_{0} \) and \( J_{1} \) in equation (50) to obtain the expression of the response matrix eigenvalues. We now detail this calculation. Thanks to our scale-decoupling approach, we may replace the radial Gaussian from equation (51) by a Dirac delta \( \delta_{\Omega}(R_{0} - R_{0}^2) \) while paying a careful attention to the correct normalization of the Gaussian. Hence we have to evaluate

\[ \lambda_{i}(k_{x}, k_{y}, k_{z}) = \int \mathcal{F}(\mathbf{r}) h(R_{0}) \, d^3 \mathbf{r} \]

Because of the presence of the azimuthal Kronecker symbol, we may drop the sum on \( m \). Then we may then only keep the term corresponding to a gradient with respect to the radial action \( J_{1} \),

\[ \int \mathcal{F}(\mathbf{r}) \, d^3 \mathbf{r} = \frac{\lambda_{i}}{\Delta \kappa} \]

Moreover, we assume that the galactic disc is tepid so that \( \Delta F/\Delta J_{1} \approx \Delta F/\Delta J_{1} \). We may then only keep the term corresponding to a gradient with respect to the radial action \( J_{1} \),

\[ \int \mathcal{F}(\mathbf{r}) \, d^3 \mathbf{r} = \frac{\lambda_{i}}{\Delta \kappa} \]

According to the expression of the Schwarzschild distribution function from equation (25) and the expression of the basis amplitude from equation (43), equation (54) becomes after some simple algebra

\[ \lambda_{i}(k_{x}, k_{y}, k_{z}) = \frac{2\pi G \Sigma_{0}}{k^{2}} \kappa^{2} \kappa^{2} \sum_{n} \int \mathcal{F}(\mathbf{r}) \, d^3 \mathbf{r} = \frac{\lambda_{i}}{\Delta \kappa} \]

We may now use the following integration formula (see formula (6.15) from Gradsheteyn & Ryzhik (2007))

\[ \int_{0}^{\infty} \, dJ_{1} \, e^{-\alpha J_{1}} J_{m}^{\frac{\alpha}{2}} = \frac{\Gamma(m + \frac{1}{2})}{\Gamma(m)} \]

where \( \alpha > 0, \beta > 0, m \in \mathbb{Z} \) and \( J_{m} \) are the modified Bessel functions of the first kind. We apply this formula with \( \alpha = k^{2}/\sigma^{2} \) and \( \beta = (2k^{2}/\sigma^{2}) \). We also introduce the notation

\[ \chi = \frac{k^{2}}{\sigma^{2}} \]

so that equation (56) becomes

\[ \lambda_{i}(k_{x}, k_{y}, k_{z}) = \frac{2\pi G \Sigma_{0}}{\kappa^{2}} \chi \sum_{n} \frac{\lambda_{i}}{\Delta \kappa} \]

We now define the dimensionless shifted frequency \( s \) as

\[ s = \frac{\omega - k_{0} \Omega_{0}}{\kappa} \]

Because we have \( I_{m} = [x] \), we may rewrite equation (59) using the reduction factor (Kalnajs 1965; Lin & Shu 1966) defined as

\[ F(s, \chi) = 2 \Re \int_{0}^{\infty} \, dJ_{1} \, e^{-\alpha J_{1}} J_{m}^{\frac{\alpha}{2}} \]

As a conclusion, we obtain that within our WKB formalism the response matrix \( \hat{\mathbf{M}} \) becomes diagonal and in the limit of tepid discs reads

\[ \hat{\mathbf{M}}(k_{x}, k_{y}, k_{z}) = \frac{2\pi G \Sigma_{0}}{\kappa^{2}} \chi \frac{\lambda_{i}}{\Delta \kappa} F(s, \chi) \]

This eigenvalue recovered using the WKB basis introduced in equation (26) is in full agreement with the seminal results from Kalnajs (1965) and Lin & Shu (1966). In order to handle the singularity of the eigenvalue appearing for \( s = n + \pi \eta \), one adds a small imaginary part to the frequency of evaluation, so that \( s = n + \pi \eta \). Indeed, as long as \( \eta \) is small compared to the imaginary part of the least damped mode of the disc, adding this complex part makes a negligible contribution to the expression of \( \Re(\lambda) \).
3.4. Estimation of the susceptibility coefficients

One can now estimate the dressed susceptibility coefficients from equation (4). In order to shorten the notations, we will write the WKB basis elements introduced in equation (26) as
\[ \psi^{(p)} = \psi_{m}^{(p)}(k_{p}^\ell, R^\ell). \] (63)

Using the expression of the Fourier transformed basis elements obtained in equation (49), we obtain
\[ \frac{1}{D_{m,m}(J_1, J_2, \omega)} = \sum_{p} \psi_{m}^{(p)}(J_1) \frac{1}{1 - \lambda_p(\omega)} \psi^{(p)}_{m}(J_2). \]

where the resonance condition \( f(R_{\ell}^2) = 0 \) is given by
\[ f(R_{\ell}^2) = m_1 \cdot \Omega(R_{\ell}) - m_2 \cdot \Omega(R_{\ell}^2). \] (68)

The radii \( R_{\ell}^2 \) therefore correspond to the resonant radii for which the resonance condition is satisfied. When writing equation (67), we have assumed that the zeros of the resonance function are simple, which corresponds to the assumption that for any resonant radius \( R_{\ell}^2 \), we have \( f'(R_{\ell}^2) \neq 0 \). Assuming that \( m_{\ell}^p \neq 0 \), this condition can be rewritten as
\[ \frac{\partial \Omega}{\partial k_{\ell}} \neq \frac{m_{\ell}^p}{m_2^p}. \] (69)

Resonance poles are therefore simple as long as the rates of change of the two intrinsic frequencies are not in a rational ratio. One must note that the Keplerian case for which \( \kappa = \Omega_2 \) and the harmonic case for which \( \kappa = 2 \Omega_2 \) are in this sense degenerate. It can lead to resonant poles of higher multiplicity and would therefore require a more involved evaluation of the Balescu-Lenard collision operator. In what follows we assume that the potential is not degenerate.

Let us now use the properties of the WKB basis to restrict the range of resonant radii, \( R_{\ell}^2 \). The expression (64) of the susceptibility coefficients, thanks to the two Gaussians, imposes that the relevant resonant radius \( R_{\ell}^2 \) must necessarily be close to \( R_1 \). As noted in equation (65), in order to have a non-zero susceptibility, one also has to satisfy the constraint \( m_{\ell}^p = m_{\ell}^p \). The resonant condition which has to be satisfied is therefore given by
\[ m_1^p \Omega_2(R_1) + m_2^p \kappa(R_1) = m_1^p \Omega_2(R_{\ell}^2) + m_2^p \kappa(R_{\ell}^2). \] (70)

where the distance \( \Delta R = R_1 - R_{\ell}^2 \) is such that \(|\Delta R| \lesssim \sigma \). Because the scale-decoupling parameter \( \sigma \) is supposed to be small compared to the size of the system, we may approximate the previous resonant condition as
\[ m_1^p \frac{\partial \Omega_2}{\partial R} + m_2^p \frac{\partial \kappa}{\partial R} \Delta R = m_1^p - m_2^p \kappa(R_1). \] (71)

On the l.h.s of equation (71), the term within bracket is non-zero, because we assumed in equation (69) that the resonant poles are simple. Moreover, \( \Delta R \) is small, because of our scale-decoupling approach. The r.h.s of equation (71) is discrete: it is either zero or at least of the order of \( \kappa(R_1) \). Because the l.h.s is necessarily small, we must have
\[ R_{\ell}^2 = R_1, \quad m_2^p = m_1^p. \] (72)

This result is a crucial consequence of our WKB tightly wound spiral assumption, because it implies that only local resonances are allowed. In particular this implies that the WKB limit does not allow for distant orbits to resonate (through e.g. propagation of swing amplified wave packets, see below). Then the sum \( \sum_{R_{\ell}^2} \) from equation (67) can be limited to the evaluation in \( R_{\ell}^2 = R_1 \). Hence within this WKB limit, the susceptibility coefficients from equation (64) have to be evaluated only for \( m_2 = m_1 \) and \( R_2 = R_1 \), so that we have to deal with the expression
\[ \frac{1}{D_{m,m}(R_1, J_1, J_2, \omega)} = \sum_{k_{\ell}^p R_{\ell}^2} \frac{G}{k_{\ell}^p R_{\ell}^2} \frac{1}{1 - \lambda_p} \frac{1}{\sqrt{\pi \sigma^2}} \exp \left[ -\frac{(R_1 - R_{\ell}^2)^2}{2 \sigma^2} \right]. \] (73)

3.5. Restriction of the loci of resonance

Before proceeding with the evaluation of the susceptibility coefficients obtained in equation (64), let us first emphasize a crucial consequence of the WKB basis from equation (26), which is the restriction to only exactly local resonances. One can note that the expressions (8) and (9) of the drift and diffusion coefficients all involve an integration over the mute variable \( J_2 \). For a given value of \( J_1, m_1 \) and \( m_2 \), this integration should be seen as a scan of the entire action-space, searching for resonant region where the constraint \( m_1 \cdot \Omega_1 - m_2 \cdot \Omega_2 = 0 \) is satisfied. We first recall the rule for the composition of a Dirac delta and a function which reads
\[ \delta_{D}(f(x)) = \sum_{x \neq y} \delta_{D}(x - y) \frac{f(y)}{|f(y)|}, \] (66)

where \( Z_{f} = \{ x | f(y) = 0 \} \), and we have supposed that all the poles of \( f \) are simple. As noted in equation (22), within the epicyclic approximation, the intrinsic frequencies \( \Omega = \Omega_{\ell}(\kappa) \) only depend on \( R_{\ell} = R_{\ell}(J_\ell) \) and are independent of \( J_\ell \). Hence, the resonance condition \( m_1 \cdot \Omega_1 - m_2 \cdot \Omega_2 = 0 \) only depends on \( J_\ell^2 \) and is independent of \( J_\ell^2 \). Hence if we consider fixed \( J_1, m_1 \) and \( m_2 \), the resonant Dirac delta which has to be studied takes the form
\[ \delta_{D}(m_1 \cdot \Omega_1 - m_2 \cdot \Omega_2) = \sum_{R_\ell^2} \delta_{D}(R_\ell^2 - R_{\ell}^2) \left[ \frac{1}{D_{m,m}(m_1 \cdot \Omega_1 - m_2 \cdot \Omega_2)} \right]_{R_\ell^2}. \] (67)

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3.6. Asymptotic continuous limit

One can note that in equation (73) the susceptibility coefficients are still expressed as a discrete sum on the basis index \(k_0^2\) and \(R_0^2\). Our next step is to replace these sums by continuous integrals. The discrete basis elements are separated by the step distances \(\Delta R_0\) and \(\Delta k_0\), which must satisfy the WKB hypothesis detailed in equation (41). We use the Riemann sum formula

\[ \sum f(x)\Delta x \approx \int f(x)\,dx, \]

where \(\Delta x\) controls the distance between the basis elements. This transformation is a subtle stage of the calculation, because one has to consider step distances \(\Delta R_0\) and \(\Delta k_0\), which have to simultaneously be large to comply with the WKB assumption from equation (41) and small to allow the use of the Riemann sum formula. As we are going to transform both the sums on \(k_0^2\) and \(R_0^2\), the exact value of the susceptibility coefficients will depend on our choice for \(\Delta R_0\) and \(\Delta k_0\). One has to consider the case

\[ \Delta R_0 \Delta k_0 = 2\pi \quad (74) \]

This sampling corresponds to a critical sampling condition (Gabor 1946; Daubechies 1990) (See also Fouvry et al. 2015). Equation (73) then takes the form

\[ \frac{1}{D_{m,m}(R_1, J_1, R_2, J_2, \omega)} = \frac{1}{2\pi R_1} \int \frac{dK_{\omega}}{1 - \lambda_{k_0}(R_1, \omega)} \int J_{m_{\omega}}(\sqrt{\frac{2}\pi} k_0) J_{m_{\omega}}(\sqrt{\frac{2}\pi} |\phi|) \exp \left\{ \frac{(R_1 - R_0)^2}{\sigma^2} \right\} \right. \]

One can now assume that the radial Gaussian present in equation (75) is sufficiently peaked. Because it is correctly normalized, we may in this limit replace it by \(\delta_0(R_1 - R_0)\). The integration on \(R_0\) can then be immediately performed to give

\[ \frac{1}{D_{m,m}(R_1, J_1, R_2, J_2, \omega)} = \frac{1}{2\pi R_1} \int \frac{dK_{\omega}}{1 - \lambda_{k_0}(R_1, \omega)} \int J_{m_{\omega}}(\sqrt{\frac{2}\pi} k_0) J_{m_{\omega}}(\sqrt{\frac{2}\pi} |\phi|) \exp \left\{ \frac{(R_1 - R_0)^2}{\sigma^2} \right\} \right. \]

where only \(\lambda_{k_0}\) depends on the frequency of evaluation \(\omega\). One may note that in equation (76), all the dependencies in \(\sigma\) have disappeared, so that the value of the susceptibility coefficients is independent of the precise choice of the WKB basis. The square of the susceptibility coefficients which is required to estimate the drift and diffusion coefficients from equation (8) and (9) is therefore given by

\[ \left| \frac{1}{D_{m,m}(R_1, J_1, R_2, J_2, \omega)} \right|^2 = \frac{G^2}{4\pi^2 R_1} \int \frac{dK_{\omega}}{1 - \lambda_{k_0}(R_1, \omega)} \int J_{m_{\omega}}(\sqrt{\frac{2}\pi} k_0) J_{m_{\omega}}(\sqrt{\frac{2}\pi} |\phi|) \exp \left\{ \frac{(R_1 - R_0)^2}{\sigma^2} \right\} \right. \]

where we introduced a cut-off at \(1/\sigma_2\) for the integration on \(k_0\). This bound is justified by the WKB constraint from equation (41), which imposes that the probed radial frequency region is bounded from below. It is also important to note that these susceptibility coefficients should be evaluated at \(R_0 = R_1\), since we proved in equation (72) that, consistently with our WKB approximation, exactly local resonances are the only ones which have to be considered.

At this stage, there is an arbitration to make between two possible behaviors depending on the physical properties of the underlying disc. First of all, if the amplification function \(k_0 \rightarrow 1/(1 - \lambda_{k_0})\) is asymptotically a sharp function reaching a maximum value \(\lambda_{\text{max}}\) for \(k_0 = \lambda_{\text{max}}\), one can assume that the susceptibility coefficients are dominated by the contribution from the peak in \(\lambda_{k_0}\). In this situation, we can perform an approximation of the small denominators. The second possible behavior arises if the function \(k_0 \rightarrow 1/(1 - \lambda_{k_0})\) is asymptotically flat, so that there is no characteristic scale of blow-up of the amplification eigenvalues. In such a situation, the susceptibility coefficients are mostly dominated by the behavior at the boundaries of integration from equation (77) where \(k_0 \rightarrow 1/\sigma_2\). The detailed response structure of the self-gravitating disc then does not play a significant role.

We place ourselves within the approximation of the small denominators, assuming that the biggest contribution to the susceptibility coefficients comes from waves which yield the largest \(\lambda_{k_0}\). Therefore, one has to suppose that the function \(k_0 \rightarrow 1/(1 - \lambda_{k_0})\) is a sharp function reaching a maximum value \(\lambda_{\text{max}}(R_1, \omega)\), for \(k_0 = \lambda_{\text{max}}(R_1, \omega)\), with a characteristic spread \(\Delta k_0(R_1, \omega)\). The expression (77) then becomes

\[ \left| \frac{1}{D_{m,m}(R_1, J_1, R_2, J_2, \omega)} \right|^2 = \frac{G^2}{4\pi^2 R_1^2} \left( \frac{\sigma^2}{\sigma_{\text{max}}^2} \right)^2 \]

While still focusing on the contribution to the susceptibility coefficients due to the waves with the largest amplification \(\lambda(k_0)\), one can improve the approximation of the small denominators from equation (78). Indeed, starting from equation (77), one can instead perform the \(k_0\)-integration for \(k_0 \in [k_{\text{min}}; k_{\text{sup}}]\), where these bounds are given by \(\lambda(k_{\text{min/sup}}) = \lambda_{\text{max}}/2\). This approach is numerically more demanding but does not alter the principal conclusions drawn in this paper, while allowing a more precise determination of the secular flux structure. All the calculations presented in section 4 were performed with this improved approximation. Finally, in Appendix B, we detail how this same WKB formalism may be applied to the inhomogeneous Balescu-Lenard equation without collective effects (Chavanis 2013b).

3.7. Estimation of the drift and diffusion coefficients

The drift and diffusion coefficients are given by equations (8) and (9). Within the WKB approximation, we have shown in equations (65) and (72) that the susceptibility coefficients have to be evaluated only for \(m_1 = m_2\), so that the sum on \(m_1\) in the expressions of the drift and diffusion coefficients may be dropped. As the resonances are exactly local, using the formula from equation (67) adds a prefactor of the form \(1/|d(m_1, \Omega)|/\partial J_0\), so that the drift coefficients from equation (8) become

\[ A_{m}(J_1) = - \frac{4\pi^3}{|d_{J_0}^*|} \frac{m_1}{|d_{J_0}^*|} \frac{\partial F(J_1, J_2)}{|d_{m,m}(J_1, J_0, J_2, m_1, J_0, \Omega)|^2}. \]

Similarly, the diffusion coefficients are given by

\[ D_{m}(J_1) = \left| \frac{4\pi^3}{|d_{J_0}^*|} \frac{m_1}{|d_{J_0}^*|} \frac{\partial F(J_1, J_2)}{|d_{m,m}(J_1, J_0, J_2, m_1, J_0, \Omega)|^2} \right|^2 \]

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In both equations \((79)\) and \((80)\), the susceptibility coefficients are given by equation \((77)\) or, within the approximation of the small denominators, by equation \((78)\).

Such simple expressions of the drift and diffusion coefficients along with the required expression of the susceptibility coefficients constitute a main result of this paper. Note importantly that this WKB formalism is self contained and is obtained without any ad hoc assumptions or fittings. Except for the explicit recovery of the amplification eigenvalues from equation \((62)\), the calculations presented previously are not limited to the Schwarzschild distribution function from equation \((25)\). Indeed, the drift and diffusion coefficients from equations \((79)\) and \((80)\) are valid for any tepid disc, as long as the epicyclic angles-actions mapping from equation \((24)\) may be used. In Appendix C, we show how the drift and diffusion coefficients from equations \((79)\) and \((80)\) can be explicitly computed, when one considers a Schwarzschild distribution function as in equation \((25)\) and when the susceptibility coefficients are estimated thanks to the approximation of the small denominators from equation \((78)\). Finally, in Appendix D, we compare this 2D WKB Balescu-Lenard equation with other similar kinetic equations.

4. Application

Let us now illustrate how this WKB approximation of the inhomogeneous Balescu-Lenard equation can be implemented to recover some results obtained via well-crafted numerical simulations. Indeed, Sellwood (2012) (hereafter S12) using careful numerical simulations studied the long-term evolution of an isolated stable and stationary Mestel disc (Mestel 1963) sampled by pointwise particles. When evolved for hundreds of dynamical times, such a disc would secularly diffuse in action-space through the spontaneous generation of transient spiral structures. Figure 7 of S12 shows for instance the late time formation of resonant ridges along very specific resonant directions. Such features are possible signatures of secular evolution, which results in a long-term aperiodic evolution of a self-gravitating system, during which small resonant and cumulative effects can add up in a coherent way. These small effects, which are then amplified through the self-gravity of the system originate from finite-\(N\) effects. Indeed, the distribution function of the system is made of a finite number \(N\) of pointwise particles. Even with a perfect numerical integrator, the system would necessarily undergo encounters during which orbits feel the discreteness of the joint DF through its two point correlation. Note that these interactions need not be local but assume that the potential fluctuations are resonant so as to build up a secular evolution of the system. This effect, which is still present even in the absence of any numerical noise, is the effect captured by the Balescu-Lenard equation (see figure 1).

4.1. The disc model

The disc considered by S12 is an infinitely thin Mestel disc, for which the circular speed \(v_0\) is a constant \(V_0\) independent of the radius. Such a model represents fairly well the observed rotation curve of real galaxies. The stationary background potential \(\psi_M\) of such a disc and its associated surface density \(\Sigma_M\) are given by

\[
\psi_M(R) = V_0^2 \ln \frac{R}{R_i} ; \quad \Sigma_M(R) = \frac{V_0^2}{2\pi c G R},
\]

where \(R_i\) is a scale parameter. Because of this scale invariance, the relationship between the angular momentum \(J_\phi\) and the guiding radius \(R_g\) is straightforwardly given by

\[
J_\phi = R_g V_0.
\]

Within the epicyclic approximation, the intrinsic frequencies \(\Omega_\phi\) and \(\kappa\) can be computed thanks to equations \((22)\) and read

\[
\Omega_\phi(J_\phi) = \frac{V_0^2}{J_\phi} ; \quad \kappa(J_\phi) = \sqrt{2} \Omega_\phi(J_\phi).
\]

We note that \(\kappa/\Omega_\phi = \sqrt{2}\), so that the Mestel disc could be seen as an intermediate case between the Keplerian case for which \(\kappa/\Omega_\phi = 1\) and the harmonic case for which \(\kappa/\Omega_\phi = 2\). The ratio of the intrinsic frequencies is a important parameter for the system since it will determine the location of the resonances and a constant ratio may introduce dynamical degeneracies. This is the case for the Keplerian and harmonic discs for which \(\kappa/\Omega_\phi\) is a rational number, as discussed below equation \((69)\). By contrast, for the Mestel disc, the non-rational ratio \(\kappa/\Omega_\phi = \sqrt{2}\) ensures that the potential is non-degenerate. Using the epicyclic approximation, the DF considered by S12 takes, as in equation \((25)\), the form of...
where $\nu$ is the intrinsic frequencies are given by equation (83), the velocity dispersion $\sigma_r$ is constant throughout the entire disc, and the surface density is given by $\Sigma_\star$, i.e., the active surface density of the disc. Indeed, in order to accommodate the central singularity and the infinite extent of the Mestel disc, one introduces tapering functions $T_{\text{inner}}$ and $T_{\text{outer}}$ to damp out the inner and outer regions, which read

$$
\begin{align}
T_{\text{inner}}(J_\phi) &= \frac{J_\phi}{(R_0V_0 \nu_1 J_\phi)}, \\
T_{\text{outer}}(J_\phi) &= \left[ 1 + \frac{J_\phi}{R_0V_0} \right]^{-1},
\end{align}
$$

where $\nu$ and $\mu$ control the sharpness of the two tapers and $R_0$ is an additional scale parameter. The two tapers are physically motivated by the presence of a bulge and an outer truncation for the disc. Moreover, in order to resist the susceptibility coefficient of the disc, we also suppose that only a fraction $\xi$ of the disc is self-gravitating, with $0 \leq \xi \leq 1$, so that the rest of the gravitational field is provided by the static halo. As a conclusion, the active surface density $\Sigma_\star$ is given by

$$
\Sigma_\star(J_\phi) = \xi \Sigma_M(J_\phi) T_{\text{inner}}(J_\phi) T_{\text{outer}}(J_\phi),
$$

where $\Sigma_M$ is the surface density of the Mestel disc from equation (81). We place ourselves in the same units system as in S12, so that we have $V_0 = G = R_0 = 1$. The other numerical factors are given by $\sigma_r = 0.284$, $\nu = 4$, $\mu = 5$, $\xi = 0.5$ and $R_0 = 11.5$. The shape of the active surface density is illustrated in figure 4. The initial contours of the Schwarzschild DF from equation (25) are shown in figure 5. For such an almost scale invariant disc, the local Toomre parameter, $Q$ (Toomre 1964)

$$
Q(J_\phi) = \frac{\sigma_r k(J_\phi)}{3.36 G \Sigma_\star(J_\phi)},
$$

which for $Q > 1$ ensures the stability of the disc with respect to local axisymmetric disturbances, becomes almost independent of the radius, especially in the intermediate regions of the disc. As illustrated in figure 6, $Q \simeq 1.5$ between the tapers and increases strongly in the tapered regions.

The expression (10) of the secular diffusion flux requires to sum on all the resonances $m$. S12 restricted perturbations forces to $m_0 = 2$, so that we may impose this same restriction on the considered azimuthal number $m_0$. Throughout our numerical calculations, we will restrict ourselves to only three different resonances which are: the inner Lindblad resonance (ILR) corresponding to $(m_0^{\text{ILR}}, m_0^{\text{OLR}}) = (-1, 2)$, the outer Lindblad resonance (OLR) given by $(m_0^{\text{OLR}}, m_0^{\text{OLR}}) = (1, 2)$ and the corotation resonance (COR) for which $(m_0^{\text{COR}}, m_0^{\text{COR}}) = (0, 2)$. Moreover, all the calculations in the upcoming sections have also been performed while taking into account the contributions from the resonances with $m_0 = 2$, which were checked to be subdominant. Being able to perform such a restriction to the relevant resonances appearing in the secular flux $\mathcal{F}_{\text{tot}}$ from equation (10) is an important step of the calculation.

Returning to the fast and slow coordinates from equation (14), note that the diffusion associated to the COR resonance amounts to diffusion along the $J_\phi$-axis. Such diffusion brings stars from one quasi-circular orbit to another of a different radius and is called radial migration. Conversely, the diffusion associated to the ILR and OLR resonances exhibits a non-zero diffusion component in the $J_\nu$-direction. It therefore increases the velocity dispersion within the disc so as to heat it, while either decreasing (ILR) or increasing (OLR) star’s angular momentum.

**4.2. Disc amplification**

One may now study the behavior of the amplification eigenvalues $\lambda_k$ given by equation (62), thanks to which one can perform the improved approximation of the small denominators. For a given resonance $m$ and angular momentum $J_\phi$, the amplification function $k_\nu \mapsto \lambda_k$ is presented in figure 7. As equation (62) only depend on $\nu$, the ILR and OLR resonances always have the same response matrix eigenvalues. One can also note that the eigenvalues $\lambda_k$ are maximum for a frequency $k_{\text{max}}(J_\phi)$, where $\lambda_k = \lambda_{\text{max}}$, in a region whose size is given by the width at half maximum $\Delta k$. Because of the scale-invariance property of the Mestel disc, it is straightforward to show that $\Delta k_1 \propto 1/J_\phi$, $k_{\text{max}} \propto 1/J_\phi$ and $k_{\text{max}}(\nu_{\text{tap}}) \propto 1/J_\phi$. One can then consider the behavior of the amplification factor $1/(1-\lambda_{\text{max}})$, which encodes the strength of the self-gravitating amplification, as shown in figure 8. Note that the COR resonance is always more amplified than the ILR and OLR resonances, but the maximum amplification (~3 for the COR and ~1.5 for the ILR and OLR) remains sufficiently small, so that the susceptibility coefficients

![Fig. 6: Dependence of the local $Q$ Toomre parameter with the angular momentum. It is scale invariant except in the inner/outer regions because of the presence of the tapering functions $T_{\text{inner}}$ and $T_{\text{outer}}$. The unit system has been chosen so that $V_0 = G = R_0 = 1$.](image1)

![Fig. 7: Variations of the response matrix eigenvalues $\lambda$ with the WKB-frequency $k_\nu$ for $m = m_0^{\text{COR}}$ and two values of $J_\phi$. The curve that peaks at large $k_\nu$ is for the smaller value of $J_\phi$.](image2)
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4.3. Computing the diffusion flux

Given the knowledge of the eigenvalues, it is now straightforward to compute the susceptibility coefficients within the improved approximation of the small denominators thanks to equation (77), where the integration on $k_r$ is performed for $k_r \in [k_{\text{inf}} ; k_{\text{sup}}]$. One can then compute the associated drift and diffusion coefficients respectively given by equations (79) and (80). The diffusion flux $F_{\text{tot}}$ defined in equation (10) immediately follows, where the sum on $m$ is restricted only to the three resonances ILR, COR and OLR. In Appendices E and F, we discuss two specific properties of such a truncated Mestel disc, namely the cancellation between the radial components of the diffusion and drift elements (the Schwarzschild conspiracy, Appendix E) and the natural and intrinsic presence of a temporal frequency bias (Appendix F), which both enlighten the subtle arbitrations between the different resonances.

Finally let us compute the divergence of this flux, $\text{div}(F_{\text{tot}})$ given by equation (11) in order to compare quantitatively the WKB predictions with the results from S12’s simulations. Figure 9 represents the contours of $\text{div}(F_{\text{tot}})$. A comparison of our WKB predictions with the results from S12’s simulation are illustrated in the figures 10 and 11. In figure 9, red contours correspond to regions for which $\text{div}(F_{\text{tot}}) < 0$, so that thanks to equation (11) they are associated to action-space regions where the WKB Balescu-Lenard equation predicts a decrement of the DF during secular evolution. In contrast, blue contours from equation (4) are not dominated only by the self-gravitating amplification.

Fig. 9: Map of the divergence of the total flux $F_{\text{tot}}$ summed over the three resonances (ILR, COR and OLR). Red contours, for which $\text{div}(F_{\text{tot}}) < 0$, correspond to regions from which the orbits will be depleted, whereas blue contours, for which $\text{div}(F_{\text{tot}}) > 0$, correspond to regions where secular diffusion will tend to increase the value of the DF. The net fluxes involve simultaneously radial migration near $(J_\phi, J_r) \sim (1.8, 0)$, and heating near $(J_\phi, J_r) \sim (1, 0.1)$.

Fig. 10: Overlay of the WKB predictions for the divergence of the diffusion flux $\text{div}(F_{\text{tot}})$ and the differences between the initial and the evolved DF in S12’s simulation. The opaque contours correspond to the differences in the action-space for the DF in S12 between the time $t_{\text{S12}} = 1000$ and $t_{\text{S12}} = 0$ (see the upper panel of S12’s figure 10). The red opaque contours correspond to negative differences, so that these regions are emptied from their orbits, whereas blue opaque contours correspond to positive differences, i.e. regions where the DF has increased through diffusion. The transparent contours correspond to the predicted values of $\text{div}(F_{\text{tot}})$ using the same conventions as in figure 9. Note the overlap of the predicted transparent red and blue contours with the measured solid ones.
are associated to regions for which \( \text{div}(\mathcal{F}_{\text{tot}}) > 0 \), so that the DF will increase there. The overall picture involves two competing processes: i) the beginning of a ridge forming towards \((J_0, J_r) \sim (1, 0, 1)\), and ii) the formation of an over density near \((J_0, J_r) \sim (1.8, 0)\). Point i) is in fact consistent with both the early time measurement of S12 as shown in figure 10 and the late time measurement of S12 as shown in figure 11. These qualitative agreements are in fact surprisingly good, given that the WKB theory is approximate and was only estimated for \( t = 0^* \). Interestingly, the early time measurement from figure 10 also displays a hint of an over-density on the \( J_r = 0 \) axis, in agreement with point ii), while the late time measurement suggests that the over density has split, with a hint of a second ridge forming. The time evolution of equation (2) is likely to explain why this over density on the axis seems to split, and why the ridge gets amplified.

From figure 9, we explicitly compute \( \text{div}(\mathcal{F}_{\text{tot}}) \), so that we may now study the typical timescale of collisional relaxation predicted by the WKB Balescu-Lenard equation. This is the purpose of the next section.

### 4.4. Physical timescales

Given estimates of the diffusion flux, one can explicitly compute the collisional timescale, i.e. the timescale for which the finite-\( N \) effects become significant. Indeed, the larger the number \( N \) of particles, the later these effects will come into play. One should note that our writing of the Balescu-Lenard equation (2) is independent of the number \( N \) of particles. However, when correctly dimensionalized, this kinetic equation takes the form

\[
\frac{\partial F}{\partial t} + L[F] = \frac{1}{N} C_{\text{BL}}[F],
\]

where \( L \) is the operator of pure advection, and \( C_{\text{BL}} \) is the Balescu-Lenard collisional operator, i.e. the r.h.s of equation (2). Equation (87) underlines the fact that the collisional term is associated to a kinetic Taylor expansion in the parameter \( \varepsilon = 1/N \ll 1 \).

Within the angle-actions coordinates, the advection operator is immediately given by

\[
L = \mathbf{\Omega} \frac{\partial}{\partial \phi}.
\]

Because we have assumed that \( F \) is always quasi-stationary, so that \( F = F(J, t) \) (adiabatic approximation), one has \( L[F] = 0 \). We now introduce the time

\[
\tau = \frac{t}{N},
\]

so that equation (87) immediately becomes

\[
\frac{\partial F}{\partial \tau} = C_{\text{BL}}[F].
\]

Equation (90) corresponds to a rewriting of the Balescu-Lenard equation, where \( N \) is not present anymore. This will allow us to quantitatively compare the time during which the S12 simulation was run to the diffusion time predicted by our WKB Balescu-Lenard formalism. In order to ease this comparison, we place ourselves in the same units system as the one used by S12. Figure 7 of S12 for which the ridge was observed was obtained with the parameters \( N = 50 \times 10^0 \) and \( \Delta S_{12} = 1400 \). Using the rescaled time introduced in equation (89), one obtains that S12 observed the resonant ridge after a time \( \Delta t_{S12} = \Delta S_{12}/N \approx 3 \times 10^{-5} \). One can then compare this time, with the typical time required to obtain a resonant ridge within our WKB formalism. Given the map of \( \text{div}(\mathcal{F}_{\text{tot}}) \) described in section 4.3, one can estimate the typical time for which this flux could lead to the features observed in S12. The contours presented in the figure 7 of S12 are separated by a value of \( 0.1 \times F_{\text{max}} \), where \( F_{\text{max}} \approx 0.12 \) corresponds to the maximum of the normalized DF from equation (25). As a consequence, to observe the resonant ridge, the DF should typically change by a value of the order of \( \Delta F = 0.1 \times F_{\text{max}} \). From figure 9, one can note that the maximum value of the divergence of the flux is given by \( \text{div}(\mathcal{F}_{\text{tot}}) \approx 0.06 \). Finally, thanks to equation (11), one may write the relation \( \Delta t_{S12} = \Delta t_{\text{WKB}} \cdot \text{div}(\mathcal{F}_{\text{tot}}) \), where \( \Delta t_{\text{WKB}} \) is the minimal time during which the WKB Balescu-Lenard equation has to be considered in order to develop a ridge. Thanks to the previous typical numerical values, one obtains that \( \Delta t_{\text{WKB}} \approx 3 \times 10^{-1} \). When comparing these two typical times, \( \Delta t_{S12} \) the duration during which S12 simulation was performed and \( \Delta t_{\text{WKB}} \) the duration required to observe secular diffusion in the WKB Balescu-Lenard formalism, one obtains the order of magnitude

\[
\frac{\Delta t_{S12}}{\Delta t_{\text{WKB}}} \approx 10^{-4}.
\]

Hence the direct application of the WKB-limited Balescu-Lenard equation does not allow us to predict the observed timescale for the diffusion features in simulations. Indeed, the timescale of collisional diffusion predicted by this WKB formalism seems much larger than the time during which the numerical simulation was performed. This discrepancy is also strengthened by the use of a softening length in numerical simulations, which induces an effective thickening of the disc, so as to slow down the collisional relaxation. A possible explanation for this timescale discrepancy is discussed in the next section.

### 4.5. Interpretation

In order to interpret S12 simulation under the light of a collisional secular diffusion equation, such as the Balescu-Lenard
equation and its WKB limit, one should first note the undisputed presence of collisional effects in S12’s simulation. Indeed, figure 2 of S12 shows that when the number of particles of the simulation is increased, the strength of the density fluctuations are delayed, which in turn is likely to be related to the amplitude of the secular features. The larger the number of particles, the later the effect of secular diffusion. Such dependence illustrates the fact that discreteness effects do play a role in the secular diffusion observed in S12.

Sellwood & Kahn (1991) have argued that a sequence of causally connected swing amplified transients could occur subject to a (possibly non local) resonant condition between successive spirals waves. The Balescu-Lenard formalism captures precisely such sequences – in as much as it integrates over dressed correlated potential fluctuations subject to relative resonant conditions, but does not preserve causality nor resolve them on dynamical timescales. The exact initial phases are not relevant in the Balescu-Lenard formalism: see Appendix A for a sketch of a full derivation which makes this point clear.

The timescale discrepancy observed in equation (91) might be driven by the incompleteness of the WKB basis. Indeed, equation (26)’s basis – thanks to which the susceptibility coefficients were evaluated in equation (77) – does not form a complete set, as it can only represent correctly tightly wound spirals. It also enforces local resonances, and does not allow for remote orbits to resonate, or wave packets to propagate between such non local resonances. The seminal works from Goldreich & Lynden-Bell (1965); Julian & Toomre (1966); Toomre (1981) showed that any leading spiral wave during its unwinding to a trailing wave undergoes a significant amplification, coined swing amplification. Because it involves open spirals this mechanism is not captured by the WKB formalism. This additional amplification is expected to increase the susceptibility of the disc and therefore accelerate secular diffusion (both drift and diffusion), so that the timescales discrepancy from equation (91) will become less restrictive. Following the notations from Toomre (1981), the truncated Mestel disc considered in the S12 simulation corresponds to $Q = 1.5$ and $X = 2$, so that figure 7 from Toomre (1981) shows that significant swing amplification (of order $\sim 10$) may be expected. It has also been claimed (Toomre & Kalnajs 1991) that swing amplified shot noise in the shearing sheet approximation would behave like significantly heavier macro-particles. Such an amplification would keep a dependence of the secular response with the total number $N$ of particles, but would reduce significantly the effective number of particles.

The Balescu-Lenard WKB limit seems to capture qualitatively the main features of the initial diffusion process in action space (as discussed in Section 4.3), but falls short in predicting the timescale. The remaining questions are therefore: what is the exact impact of swing amplification? Can it explain the timescale discrepancy?

5. Conclusion

We implemented the inhomogeneous Balescu-Lenard equation (2) for an infinitely thin galactic disc using two main approximations. We first assumed the disc to be tepid. We could then use the epicyclic approximation which allowed for an explicit mapping between the physical coordinates $(x, v)$ and the angle-actions coordinates $(\theta, J)$ via equation (24). Our second approximation relied on the introduction of the tightly wound basis elements from equation (26). Because of the corresponding WKB approximation, we obtained in equation (62) a diagonal response matrix, so that gravity is effectively treated locally. The associated scale-decoupling hypothesis yields a crucial restriction to only local resonances, as shown in equation (72). We then derived in equation (77) a simple quadrature for the susceptibility coefficients, given by equations (B.7) and (B.8) for the bare ones. Thanks to this restriction to local resonances, we were also able to write the drift and diffusion coefficients as simple quadratures in equations (79) and (80).

These simple expressions derived within the WKB formalism yield, to our knowledge, a first non trivial explicit expression for the Balescu-Lenard diffusion and drift coefficients. They are certainly useful to provide insight into the physical processes at work during the secular diffusion of a self-gravitating discrete disc. Moreover, modulo the restriction to the three physically motivated resonances ILR, COR and OLR, our WKB formalism can be used for quantitative comparisons to numerical experiments such as the one presented in Sellwood (2012). It considered a stable isolated Mestel disc sampled by pointwise particles, whose secular evolution is induced by finite-$N$ effect ideally captured by the Balescu-Lenard equation.

The straightforward calculation in the WKB limit of the divergence of the full diffusion flux, $div(F_{tot})$, (illustrated in figures 9, 10 and 11), recovered most of the secular features observed in S12. This qualitative agreement is impressive, given the level of approximation involved in the WKB limit. The hints for the formation of a ridge – depletion and enrichment of orbits along a preferred direction – is qualitatively consistent with the findings of S12 and Fouvry & Pichon (2015); Fouvry et al. (2015), without postulating additional assumptions about the source of fluctuations.

The comparison of the collisional time predicted in the WKB limit (equation (91)) to the diffusion time of S12 simulation, highlights nonetheless a significant quantitative overestimation. We provided a possible explanation which relies on the intrinsic limitations of the WKB formalism, as it cannot account for swing amplification, during which unwinding transient spirals are strongly amplified. This additional amplification, which involves explicitly non local wave absorption and emission, may be the missing contribution required to reconcile quantitatively our predictions and the simulation. One venue will be to compute numerically exactly equations (8) and (9) in action space – without assuming tightly wound spirals or epicyclic orbits – with a complete basis. This is the topic of an upcoming numerical investigation (Fouvry et al. 2015).

Should this confirmatory investigation explain the timescale mismatch, one would be in a stronger position to validate the accuracy of $N$-body schemes to correctly capture secular evolution of discrete self-gravitating cold discs over very long timescales. This would clearly be a worthy assessment of such schemes relying on the Balescu-Lenard theory. Once the above described conundrum is resolved, we also will be able to evolve over secular times the Balescu-Lenard equation and predict the full cosmic time evolution of such discrete discs. This may also contribute to solving the timescale discrepancy.

In closing, beyond the application presented in section 4, the above developed tightly wound Balescu-Lenard formalism may for instance describe the secular diffusion of giant molecular clouds in galactic discs (which in turn could play a role in migration driven metallicity gradients and disc thickening), the second...
ular migration of planetesimals in partially self-gravitating protoplanetary discs, or even the long-term evolution of population of stars, gas blobs and debris near the Galactic center. Such topics will be subject to further investigations.

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Appendix A: Sketch of Balescu-Lenard Derivation

Two derivations of the inhomogeneous Balescu-Lenard equation have been presented in the literature. The first one (Heyvaerts 2010) is based on the appropriate truncation at the order 1/N of the BBGKY hierarchy. The second (Chavanis 2012a) relies on the Klimontovich equation, using a quasilinear approximation. We now briefly sketch the derivation presented in Chavanis (2012a). We consider an isolated system of N particles in interaction, of mass m = 1, in a physical space of dimension d. Their dynamics is entirely described by Hamilton’s equations

\[
\frac{dx_i}{dt} = \frac{\partial H}{\partial p_i}, \quad \frac{dp_i}{dt} = -\frac{\partial H}{\partial x_i},
\]

where the Hamiltonian of the system is given by

\[
H = \sum_{i=1}^{N} \left( \frac{1}{2} p_i^2 + \sum_{j<i} u(x_i-x_j_i) \right).
\]

Here u(|x_i-x_j|) is the binary potential of interaction. In the gravitational case, it satisfies u(x) = -G/x. One can now introduce the discrete distribution function F_d(x,v,t) defined as

\[
F_d(x,v,t) = \sum_{i=1}^{N} \delta(x-x_i(t)) \delta(v-v_i(t)),
\]

along with the corresponding potential

\[
\psi_d(x,t) = \int dx' \langle v' \rangle u(|x-x'|) F_d(x',v',t).
\]

One can show that F_d satisfies the Klimontovich equation (Klimontovich 1967)

\[
\frac{\partial F_d}{\partial t} + \frac{\partial H_d}{\partial v} - \frac{\partial F_d}{\partial x} \frac{\partial H_d}{\partial p} + \frac{\partial F_d}{\partial p} = 0,
\]

where we have defined the Hamiltonian H_d as

\[
H_d \equiv \frac{1}{2} v^2 + \psi_d(x,t).
\]

At this stage, it is important to note that the Klimontovich equation (A.5) contains exactly the same information as the Hamilton equation (A.1). We now introduce the smooth distribution function F(x,v,t) = (F_d(x,v,t)), corresponding to an average of F_d over a large number of initial conditions. One can then write F_d = F + \delta F, where \delta F denotes fluctuations around the smooth distribution. In a similar way, we introduce \psi(x,v,t) = (\psi_d(x,v,t)), so that \psi_d = \psi + \delta \psi. We have therefore de-composed the discrete distribution function F_d into a smooth component F that evolves slowly with time, whereas the fluctuating component \delta F evolves more rapidly. As a consequence, when considering the evolution of the fluctuations, one can assume the smooth distribution to be frozen. Using this timescale-decoupling approach, one can use the angle-actions coordinates (\theta_i, J_i) associated with the quasi-stationary smooth potential \psi to describe the fast evolution of the fluctuations. Using these decompositions and this change of coordinates, equation (A.5) takes the form of two evolution equations

\[
\frac{\partial F}{\partial t} + \Omega \frac{\partial F}{\partial \theta} + \frac{\partial \delta F}{\partial \theta} = 0,
\]

and

\[
\frac{\partial F}{\partial t} = \frac{\partial}{\partial J} \left( \frac{\partial \psi + \delta \psi}{\partial \theta} \right).
\]
where we have introduced the intrinsic frequencies of the system \(\Omega_a = \Omega(J_1)\), as in equation (1), and where \(\langle \cdot \rangle\) denotes an angle average. Because of our timescale-decoupling approach, we may neglect the time variation of \(F(J_1, t)\) in the calculation of the collision term (adiabatic approximation). It implies that \(F\) may be treated as a constant in equation (A.7), because \(F\) evolves on a (relaxation) timescale much larger than the (dynamical) time corresponding to the evolution of \(\delta F\). In order to be valid, this approximation requires to have \(N \gg 1\). Finally, we also assume that the distribution \(F\) remains Vlasov stable, so that its evolution is only governed by correlations and not by dynamical instabilities.

The first step of the derivation of the Balescu-Lenard equation is then to study the short timescale evolution equation (A.7). One defines the Fourier-Laplacian transform of the fluctuation \(\delta F\) as

\[
\delta \hat{F}_{m_1}(J_1, \omega_1) = \int_0^{+\infty} \frac{\partial \theta_1}{(2\pi)^2} e^{-i m_1 \cdot \omega_1} \delta F(\theta_1, J_1, t) ,
\]

valid for \(\text{Im}(\omega_1)\) sufficiently large. Similarly to equation (6), one also defines the spatial Fourier Transform of the initial value as

\[
\delta \hat{F}_{m_1}(J_1, 0) = \int_0^{+\infty} \frac{\partial \theta_1}{(2\pi)^2} e^{-i m_1 \cdot \omega_1} \delta F(\theta_1, J_1, 0) .
\]

Thanks to these transformations, equation (A.7) may be rewritten under the form

\[
\frac{d}{dt} \delta \hat{F}_{m_1}(J_1, \omega_1) = \frac{m_1 \cdot \partial F / \partial J_1}{m_1 \cdot \Omega_a - \omega_1} \delta \hat{F}_{m_1}(J_1, 0) + \frac{1}{i(m_1 \cdot \Omega_a - \omega_1)} \delta \hat{F}_{m_1}(J_1, 0) .
\]

We now use the basis elements introduced in equation (3), so that we may decompose the potential fluctuations under the form

\[
\delta \psi(\theta_1, J_1, t) = \sum_p a_p(t) \psi^{(p)}(\theta_1, J_1) .
\]

We introduce the Laplace transform of \(a_p(t)\) as

\[
\tilde{a}_p(\omega_1) = \int_0^{+\infty} dt a_p(t) e^{-i \omega_1 t} .
\]

Let us then take the inverse Fourier transform of equation (A.11), multiply by \(\psi^{(p)}(\theta_1, J_1)\) and integrate over \(\theta_1\) and \(J_1\) (using the property that \(d\omega d\theta = d\theta_1 dJ_1\)). One gets

\[
\tilde{a}_p(\omega_1) = -\langle 2\pi \rangle^{2} \int \hat{M}(\omega_1) \psi^{(p)}(\omega_1) \delta \hat{F}_{m_1}(J_2, 0) ,
\]

where the response matrix \(\hat{M}\) is given by equation (5). Equation (A.14) can be rewritten under the form

\[
\delta \hat{F}_{m_1}(J_1, \omega_1) = -\langle 2\pi \rangle^{2} \int \frac{1}{D_{m_1 m_a}(J_1, J_2, \omega_1)} \delta \hat{F}_{m_1}(J_2, 0) ,
\]

where the susceptibility coefficients have been introduced in equation (4). One can now compute the collision term appearing in the r.h.s. of equation (A.8). It requires us to evaluate

\[
\left[ \frac{d}{d\theta_1} \delta \hat{F}_{m_1}(J_1, \omega_1) \right] = \sum_{m_1, m_a} \int \frac{d\omega_1 \, d\omega_2}{2\pi} \cdot \frac{2\pi}{i(m_1 \cdot \Omega_a - \omega_1)} \times \left[ \delta \hat{F}_{m_1}(J_1, \omega_1) \delta \hat{F}_{m_a}(J_1, \omega_2) \right] .
\]

Using equation (A.11), one immediately obtains that

\[
\langle \delta \hat{F}_{m_1}(J_1, \omega_1) \delta \hat{F}_{m_a}(J_1, \omega_2) \rangle = \frac{m_1 \cdot \partial F / \partial J_1}{m_1 \cdot \Omega_a - \omega_1} \langle \delta \hat{F}_{m_1}(J_1, \omega_1) \delta \hat{F}_{m_a}(J_1, \omega_2) \rangle + \frac{\langle \delta \hat{F}_{m_1}(J_1, 0) \delta \hat{F}_{m_a}(J_1, \omega_2) \rangle}{i(m_1 \cdot \Omega_a - \omega_1)} .
\]

In equation (A.17), the first term corresponds to the self-correlation of the potential whereas the second term corresponds to the correlations between the potential fluctuations and the distribution function at time \(t = 0\). Each of these terms must then be considered one at a time. Assuming that there is no correlation in the initial phases, one can show (see Appendix C from Chavanis 2012a) that

\[
\langle \delta \hat{F}_{m_1}(J_1, 0) \delta \hat{F}_{m_a}(J_2, 0) \rangle = \frac{1}{(2\pi)^2} \langle 2\pi \rangle^{2} \delta_{m_1 m_a} \delta_0(J_1 - J_2) F(J_1) .
\]

Hence, given equation (A.15), one can rewrite the first term of equation (A.17) under the form

\[
\langle \delta \hat{F}_{m_1}(J_1, 0) \delta \hat{F}_{m_a}(J_1, \omega_2) \rangle = \frac{1}{(2\pi)^2} \sum_m \int \frac{d\omega_1}{D_{m_1 m_a}(J_1, J_1, \omega_1)} \frac{1}{D_{m_1 m_a}(J_1, J_1, \omega_1)} F(J_1) ,
\]

where \(D_{m_1 m_a}(J_1, J_1, \omega_1) = \langle m_1 \cdot \Omega_a - \omega_1 \rangle \). If we consider only the contributions that do not decay in time, one can perform the substitution

\[
\frac{1}{(m_1 \cdot \Omega_a - \omega_1) \langle m_1 \cdot \Omega_a - \omega_1 \rangle} \rightarrow \frac{1}{2\pi} \delta_0(\omega_1 + 2\pi) \delta_0(m_1 \cdot \Omega_a - \omega_1) .
\]

Starting from equation (A.19), thanks to the previous substitution and using the fact that \(D_{m_1 m_a}(J_1, J_1, \omega_1) = D_{m_1 m_a}(J_1, J_1, \omega_1)^*\), one can show that the first contribution from equation (A.16) takes the form

\[
\left[ \frac{d}{d\theta_1} \delta \hat{F}_{m_1}(J_1, \omega_1) \right] = i \langle 2\pi \rangle^{2} \int \frac{d\omega_1}{2\pi} \langle m_1 \cdot \Omega_a - \omega_1 \rangle \delta_0(m_1 \cdot \Omega_a - \omega_1) \delta_0(m_1 \cdot \Omega_a - \omega_1) F(J_1) ,
\]

where \(\delta_0\) is the principal value. One can then rewrite equation (A.20) under the form

\[
\left[ \frac{d}{d\theta_1} \delta \hat{F}_{m_1}(J_1, \omega_1) \right] = \frac{\pi (2\pi)^2}{\langle m_1 \cdot \Omega_a - \omega_1 \rangle} \times \frac{\delta_0(m_1 \cdot \Omega_a - \omega_1)}{\delta_0(m_1 \cdot \Omega_a - \omega_1)} F(J_1) ,
\]

Similarly, one can rewrite the second term of equation (A.17) under the form

\[
\langle \delta \hat{F}_{m_1}(J_1, 0) \delta \hat{F}_{m_a}(J_1, \omega_2) \rangle = \frac{1}{i(m_1 \cdot \Omega_a - \omega_1)^2} \langle 2\pi \rangle^{2} \langle 2\pi \rangle^{2} \delta_0(\omega_1 + 2\pi) \delta_0(m_1 \cdot \Omega_a - \omega_1) \delta_0(m_1 \cdot \Omega_a - \omega_1) \delta_0(m_1 \cdot \Omega_a - \omega_1) F(J_1) .
\]
Using symmetries of the matrix $[I - \hat{M}]^{-1}$, starting from equation (A.23), one can show that the second contribution from equation (A.16) finally takes the form
\[
\left\langle \partial F / \partial \theta \right\rangle_{I} = -\pi (2\pi)^d \sum_{m_{1},m_{2}} \int \mathcal{D}m_{2} m_{1} \frac{\partial \Omega}{\partial J_{1}} \frac{\delta \Omega_{1}(m_{1},\Omega_{1} - m_{2},\Omega_{2})}{\mathcal{D}m_{1}(J_{1},J_{2},m_{1},\Omega_{1})} \times \left( m_{1} \frac{\partial F}{\partial J_{1}} + m_{2} \frac{\partial F}{\partial J_{2}} \right)(J_{1},J_{2}) \, .
\]
Combining the two contributions obtained in equations (A.22) and (A.24), one can rewrite equation (A.16) under the form
\[
\left\langle \partial F / \partial \theta \right\rangle_{I} = \pi (2\pi)^d \sum_{m_{1},m_{2}} \int \mathcal{D}m_{2} m_{1} \frac{\delta \Omega_{1}(m_{1},\Omega_{1} - m_{2},\Omega_{2})}{\mathcal{D}m_{1}(J_{1},J_{2},m_{1},\Omega_{1})} \times \left( m_{1} \frac{\partial F}{\partial J_{1}} + m_{2} \frac{\partial F}{\partial J_{2}} \right)(J_{1},J_{2}) \, .
\]

Hence, using the slow evolution equation (A.8), we recover the Balescu-Lenard equation introduced in equation (2). As a final remark, one must note that on the short dynamical timescale, the evolution is governed by equation (A.7), which involves the fluctuating components $\delta F$ and $\delta \Omega$ of the distribution function and the potential. In contrast, on the long secular timescale, the evolution is governed by equation (A.8), which after an angle-average only involves the mean distribution function $F$. Indeed, thanks to the ensemble average, all the cross-correlations between the fluctuations $\delta F$ and $\delta \Omega$, as in equations (A.18), (A.19) and (A.23) can be expressed in terms of the underlying smooth distribution function $F$ only, so that the fluctuating components are absent from the secular Balescu-Lenard collision operator from equation (A.25).

**Appendix B: Turning off collective effects**

The reference Chavanis (2013b) (see also Appendix A of Chavanis (2012a)) considers the inhomogeneous Balescu-Lenard equation without collective effects. This collisional kinetic equation is the equivalent of the Landau equation for inhomogeneous systems. It can be straightforwardly obtained as an approximation of the Balescu-Lenard equation (2). Indeed, one has to make the substitution from the dressed susceptibility coefficients $1/\mathcal{D}m_{1}(J_{1},J_{2},\omega)$ to the bare ones given by $\lambda_{m_{1},m_{2}}(J_{1},J_{2})$, so that the inhomogeneous Balescu-Lenard equation without collective effects (i.e. the inhomogeneous Landau equation) is given by
\[
\frac{\partial F}{\partial t} = \pi (2\pi)^d \frac{\partial}{\partial J_{1}} \left( \sum_{m_{1},m_{2}} \int \mathcal{D}m_{2} m_{1} \frac{\partial \Omega_{1}(m_{1},\Omega_{1} - m_{2},\Omega_{2})}{\mathcal{D}m_{1}(J_{1},J_{2},m_{1},\Omega_{1})} \times \left( m_{1} \frac{\partial F}{\partial J_{1}} + m_{2} \frac{\partial F}{\partial J_{2}} \right)(J_{1},J_{2}) \right) \, ,
\]

where the coefficients $\lambda_{m_{1},m_{2}}$ are associated to the Fourier transform in angles of the interaction potential (Pichon 1994; Chavanis 2013b) and read
\[
\lambda_{m_{1},m_{2}}(J_{1},J_{2}) = \frac{1}{(2\pi)^d} \int d\theta_{1} d\theta_{2} \frac{u(x(\theta_{1},J_{1}) - x(\theta_{2},J_{2}))e^{-im_{1}\theta_{1} - im_{2}\theta_{2}}}{k_{1} k_{2}} \, ,
\]
where $u(x)$ is the binary interaction given by $u(x) = -G/|x|$ in the gravitational case. A key remark at this stage is that the expression (B.2) does not require to introduce biorthogonal basis elements, as in equation (3), whereas in order to estimate the dressed susceptibility coefficients from equation (4), one must necessarily rely on Kalnajs’ matrix method (Kalnajs 1976). It is however possible to express $\lambda_{m_{1},m_{2}}$ using the potential basis. Indeed, for a fixed value of $x_{2}$, we consider the function $x_{1} \mapsto u(x_{1} - x_{2})$. One can then decompose this function on the basis elements $\psi^{(p)}(x_{1})$, so that we may write
\[
u(x_{1} - x_{2}) = \sum_{p} a_{p}(x_{2}) \psi^{(p)}(x_{1}) \, ,
\]
where it is important to note that the basis coefficients $a_{p}(x_{2})$ are functions of $x_{2}$. Thanks to the biorthogonality property of the basis detailed in equation (3), one can obtain the expression of the coefficients $a_{p}(x_{2})$ which reads
\[
a_{p}(x_{2}) = -\int \mathcal{D}x_{1} u(x_{1} - x_{2}) \rho^{(p)}(x_{1}) \, .
\]

This fairly simple relation allows us to express the bare susceptibility coefficients $\lambda_{m_{1},m_{2}}$ using the potential basis. One does not need anymore to perform a Fourier transform in angles of the interaction potential, because the resolution of Poisson’s equation has been implicitly hidden in the effective construction of the basis elements.\(^{7}\) Neglecting the collective effects in the expression (4) of the dressed susceptibility coefficients amounts to taking $\hat{M}(\omega) = 0$, so that we obtain
\[
\frac{1}{\mathcal{D}m_{1}(J_{1},J_{2},\omega)} = \lambda_{m_{1},m_{2}}(J_{1},J_{2}) \, .
\]

The negative sign in equation (B.7) plays no significant role in the kinetic equations, since one has to make the substitution of the square modulus $1/|\mathcal{D}|^{2} \mapsto |\mathcal{A}|^{2}$ in the Balescu-Lenard equation (2), to obtain the inhomogeneous Landau equation (B.1). Using our WKB basis, one can proceed similarly as in the dressed case, by limiting oneself only to local resonances. Equation (76) therefore becomes in the bare case
\[
\frac{1}{\mathcal{D}m_{1}(R_{1},J_{1},J_{2},\omega)} = \frac{1}{2\pi R_{1}} \int d\kappa_{1} \int d\kappa_{2} \frac{1}{k_{1} k_{2}} \mathcal{F}_{\kappa_{1}} \mathcal{F}_{\kappa_{2}} \sqrt{\frac{2\pi}{\kappa_{1}}} \, \sqrt{\frac{2\pi}{\kappa_{2}}} \, .
\]

This expression of the bare susceptibility coefficients can be straightforwardly obtained from the dressed ones by imposing

\(^{7}\) This also explains why the factorization assumption $\lambda_{m_{1},m_{2}}(J_{1},J_{2}) = \lambda_{m_{1}}(J_{1}) \lambda_{m_{2}}(J_{2})$ used in Chavanis (2007) does not hold.
\[ \lambda = 0 \]. One should note that because of the absence of the amplification term \((1/1 - \lambda)\), the approximation of the small denominators cannot be used. Hence a physically motivated evaluation of the expression (B.8) becomes more subtle to perform, especially because of the possibly important role that the Coulomb logarithm \(1/\kappa_{r}\) might play. Finally, one can note that even for exactly local resonances, \(i.e. m_1 = m_2\) and \(R_1 = R_2\), the use of our WKB formalism allowed us to obtain non-diverging bare susceptibility coefficients. On the other hand, one could try to estimate the bare susceptibility coefficients starting from equation (B.2), \(i.e.\) without using any potential basis. Using the polar coordinates \((\mathbf{R}, \phi)\), one has to compute

\[ A_{m_1, m_2}(J_1, J_2) = \frac{G}{(2\pi)^2} \int d\Omega_1 d\Omega_2 d\phi_1 d\phi_2 e^{i m_1 \phi_1 - i m_2 \phi_2} \frac{\sqrt{R_1^2 + R_2^2 - 2R_1R_2 \cos(\phi_1 - \phi_2)}}{R_1^2 + R_2^2 - 2R_1R_2 \cos(\phi_1 - \phi_2)} . \]  

(B.9)

By azimuthal symmetry, it is straightforward to show (Pichon & Cannon 1997) that

\[ A_{m_1, m_2} \propto \delta_{m_1}^{m_2} . \]  

(B.10)

Hence the different \(m_1\)-modes of the \(A_{m_1, m_2}\) coefficients are independent. This result is identical to what was obtained in equation (65) for the dressed case. In order to illustrate the regularizing role of the WKB basis from equation (26), we will place ourselves in the context of an extremely tepid disc, and therefore assume \(J_1 = J_2 = 0\). Thanks to the epicyclic mapping from equation (24), one can drop all the dependences on \(b_0\) appearing in \(R\) and \(\phi\). Equation (B.9) then immediately implies that

\[ m_1^\ast = m_2^\ast = 0 , \]  

(B.11)

and the bare susceptibility coefficients are given by

\[ A_{m_1, m_2}(R_1, 0, R_2, 0) = -\frac{G}{(2\pi)^2} \delta_{0}^{m_1} \delta_{0}^{m_2} \int d\Omega_1 d\Omega_2 e^{i m_1 \phi_1 - i m_2 \phi_2} \frac{\sqrt{R_1^2 + R_2^2 - 2R_1R_2 \cos(\phi_1 - \phi_2)}}{R_1^2 + R_2^2 - 2R_1R_2 \cos(\phi_1 - \phi_2)} . \]

(B.12)

In this illustrative limit, we have by construction no contributions from the ILR and OLR resonances for which \(m_1 \neq 0\). After an immediate change of variables, using \(\Delta = \theta_1^2 - \theta_2^2\), it becomes

\[ A_{0, m_1, 0, m_2}(R_1, 0, R_2, 0) = -\frac{G}{2\pi} \int_{-\pi}^{\pi} d\sigma e^{-i m_{1, \Delta}} \frac{\sqrt{R_1^2 - R_2^2 - 2R_1R_2 (1 - \cos(\Delta))}}{R_1^2 - R_2^2 - 2R_1R_2 (1 - \cos(\Delta))} . \]

(B.13)

The resonance condition from equation (68) takes the form

\[ m_1 \Omega_1(R_1) = m_2 \Omega_2(R_2) , \]

because we restricted ourselves in equation (B.11) to the case \(m_1 = 0\). Hence assuming that the function \(R_0 \mapsto \Omega_0(R_0)\) is a monotonic function, one has to satisfy the constraint \(R_1 = R_2\), so that we are restricting ourselves only to local resonances as in equation (72). For such resonances, equation (B.13) becomes

\[ A_{0, m_1, 0, m_2}(R_1, 0, R_2, 0) = -\frac{G}{2\pi \sqrt{R_1}} \int_{-\pi}^{\pi} d\sigma e^{i m_{1, \Delta}} \frac{1}{\sqrt{1 - \cos(\Delta)}} . \]

(B.14)

At this stage, one must note that this integral is divergent. Indeed, for \(\Delta \to 0\), one has \(1/\sqrt{1 - \cos(\Delta)} \sim \sqrt{2}/\Delta\). Hence the expression of the bare susceptibility coefficients derived from the Fourier transform of the interaction potential as in equation (B.2), when restricted to exactly local resonances becomes logarithmically divergent. This divergence, which is observed in the case of local resonances, is induced by the interaction of singular orbits. It is important to note that this divergence was not observed in equation (B.8) when computing the bare susceptibility coefficients using the WKB basis from equation (26). This implies that the WKB basis is not complete. For a complete biorthogonal basis, the expression (B.6) of the \(A_{m_1, m_2}\) coefficients is an exact expression. However, our calculation shows that the scale-decoupled WKB basis we used, possesses a subtle regularizing incompleteness which allows to get rid of the diverging contributions to the coefficients \(A_{m_1, m_2}\) in the limit of exactly local resonances.

**Appendix C: The Schwarzschild DF case**

When considering a Schwarzschild distribution function as in equation (25), while relying on the approximation of the small denominators from equation (78), one can explicitly perform the remaining integration on the radial action \(J_2\) in the expressions (79) and (80) of the drift and diffusion coefficients. We now detail this explicit calculation. For such a Schwarzschild distribution function, it is straightforward to check that the gradients of the distribution function with respect to the actions are given by

\[ \frac{\partial F}{\partial J_r} = -\frac{\kappa}{\sigma_r^2} F ; \quad \frac{\partial F}{\partial J_\phi} = F \frac{\partial}{\partial J_\phi} \left[ \left( -\frac{\Omega_2}{\kappa_2} \right) - \frac{J_2}{\sigma_2^2} \right] . \]  

(C.1)

Using the expression of the susceptibility coefficients from equation (78), after some simple algebra, one can rewrite the drift coefficients from equation (79) under the form

\[ A_{m_1}(J_1) = -g_{m_1}(J_1, J_2) \int dJ_2^2 \exp \left( -\frac{\kappa}{\sigma_2^2} J_2 \int_{m_1}^{\infty} \left[ \frac{\Omega_2}{\kappa_2} \right] - \frac{J_2}{\sigma_2^2} \right) \]  

\[ \times \left[ m_1 \left( -\frac{\partial}{\partial J_2} \ln \left( \frac{\Omega_2}{\kappa_2} \right) \right) - \frac{J_2}{\sigma_2^2} \right] . \]  

(C.2)

Similarly, the diffusion coefficients from equation (80) take the form

\[ D_{m_1}(J_1) = g_{m_1}(J_1, J_2) \int dJ_2^2 \exp \left( -\frac{\kappa}{\sigma_2^2} J_2 \int_{m_1}^{\infty} \left[ \frac{\Omega_2}{\kappa_2} \right] - \frac{J_2}{\sigma_2^2} \right) \]  

\[ \times \left[ m_1 \left( -\frac{\partial}{\partial J_2} \right) \int_{m_1}^{\infty} \left[ \frac{\Omega_2}{\kappa_2} \right] - \frac{J_2}{\sigma_2^2} \right] . \]  

(C.3)

In equations (C.2) and (C.3), in order to shorten the notations, we introduced the function \(g_{m_1}(J_1, J_2)\) defined as

\[ g_{m_1}(J_1, J_2) = \frac{1}{(m_1 \cdot \Omega_1)^2} \frac{G^2}{R_1^2} \frac{\Omega_2}{\kappa_2} \left( \frac{(\Delta k_{\|}^2)}{k_{\max}^2} \right) \]  

\[ \times \left[ \left( -\frac{\beta_1^2}{2a} + 1 + |m_1| \right) \int_{m_1}^{\infty} \left( \frac{\beta_2^2}{2a} + \frac{\beta_1^2}{2a} I_{|m_1|} \right) \right] . \]  

(C.4)

where we used the same shortened notation for the resonant factor \(1/(m_1 \cdot \Omega_2)\) as in equation (E.3). In addition to the integration formula (57), we may also rely on the additional identity

\[ \int_{0}^{\infty} dJ_r J_r e^{-\alpha J_r} J_m^{\alpha J_r} = \left( \frac{\beta_2}{2a} \right)^2 + \frac{\beta_1^2}{2a} I_{|m_1|} \left( \frac{\beta_2^2}{2a} \right) , \]  

(C.5)

\[ \alpha > 0, \beta > 0, \text{and } m_1 \in \mathbb{Z} \]  

In analogy with the definition from equation (58), we also introduce \(\chi_{\max}\) as

\[ \chi_{\max} = \frac{\sigma_r^2}{k_{\max}^2} . \]  

(C.5)
One can then immediately perform the integration on $J_0^2$ from equation (C.3), so that the diffusion coefficients are given by

$$D_m(J_1) = h_m^d(J_0) J_0^2 m \left( 2 \frac{\Omega}{r} k_{\text{max}} \right),$$

where the function $h_m^d(J_0)$ is defined as

$$h_m^d(J_0) = \frac{1}{g_m} \frac{G^2 \Omega \sum (\Delta k)^2}{r^2} \left[ 1 - \frac{1}{k_{\text{max}}} \right] e^{-\chi_{\text{max}}} \int \frac{dJ_0}{J_0}.$$

After some algebra, the drift coefficients from equation (C.2) are given by

$$A_m(J_1) = -h_m^d(J_0) J_0^2 m \left( \frac{\Omega}{r} k_{\text{max}} \right),$$

where the function $h_m^d(J_0)$ is defined as

$$h_m^d(J_0) = \frac{1}{g_m} \frac{G^2 \Omega \sum (\Delta k)^2}{r^2} \left[ 1 - \frac{1}{k_{\text{max}}} \right] e^{-\chi_{\text{max}}} \int \frac{dJ_0}{J_0}.$$

These explicit expressions of the diffusion and drift coefficients obtained in equations (C.6) and (C.7) allow to estimate in a simple way the secular flux in the entire action space $J = (J_0, J_1)$, once we assume that the distribution function is a Schwarzschid DF given by equation (25) and that the susceptibility coefficients can be approximated by equation (78).

**Appendix D: Relation to other kinetic equations**

The kinetic equation governing the collisional evolution of a system of $N$ stars at the order $1/N$ is the inhomogeneous Balescu-Lenard equation (2). This equation conserves the total number of stars and the energy, and monotonically increases the Boltzmann entropy (H-Theorem). We note that the collisional evolution of the system is due to a condition of resonance encapsulated in the term $\delta \Omega(m_1, \Omega_1 - m_2, \Omega_2)$. In general, this condition can allow for local and non local resonances. In the case of tepid discs considered in the present paper, assuming that only tightly wound spirals are sustained by the disc, we justified in equation (72) the fact that the resonances are purely local, so that $m_1 = m_2$ and $J_0 = J_0^2$. Furthermore, because of the epicyclic approximation, the intrinsic frequencies of the system, given by equation (22), depend only on $J_0$, so that $J_0 = \Omega(J_0)$. Under these conditions, the Balescu-Lenard equation giving the collisional evolution of $F = F(J_0, J_1, t)$ may be rewritten as

$$\frac{\partial F}{\partial t} = 4\pi^3 \frac{\partial}{\partial J} \left\{ \sum_m \frac{1}{\delta \Omega[m, \Omega | J_0]} \int dJ_0 dJ_1^2 \right\}$$

$$\times \delta \Omega(J_0 - J_0^2)$$

$$\times \frac{1}{|\delta \Omega[m, \Omega | J_0]|^2}$$

$$\times m \left[ F(J_0, J_1, t) \frac{\partial F}{\partial J} (J_0, J_1, t) - F(J_0, J_1, t) \frac{\partial F}{\partial J} (J_0^2, J_1, t) \right].$$

The integration on $J_0^2$ is straightforward because of the $\delta\Omega$ function (local resonance), and we are left with

$$\frac{\partial F}{\partial t} = 4\pi^3 \frac{\partial}{\partial J} \left\{ \sum_m \frac{1}{\delta \Omega[m, \Omega]} \int dJ_0 \right\}$$

$$\times \frac{1}{|\delta \Omega[m, \Omega | J_0]|^2}$$

$$\times m \left[ F(J_0, J_1, t) \frac{\partial F}{\partial J} (J_0, J_1, t) - F(J_0, J_1, t) \frac{\partial F}{\partial J} (J_0^2, J_1, t) \right],$$

where the susceptibility coefficients are generally given by equation (77).

The kinetic equation (D.2) is an integro-differential equation that governs the evolution of the system as a whole. It describes the effects of encounters between any test particle characterized by the angle-action coordinates $(J_0, J_1)$, and the field particles characterized by the (running) angle-action coordinates $(J_0', J_1')$. Actually, there is no distinction between test and field particles, so that they are characterized by the same distribution $F(x, t)$ that evolves in a self-consistent manner, hence the integro-differential character of the kinetic equation. This is a characteristic of the Balescu-Lenard equation describing the evolution of the system as a whole.

**Appendix D.1: Fokker-Planck limit**

We can also use this formalism to directly obtain the Fokker-Planck equation governing the relaxation of a test star in a bath of field stars, assumed to be in a steady state with a distribution function $F_0(J_0, J_1')$. Proceeding as in Chavanis (2012a), we just have to replace in equation (D.2) the distribution function of the field particles $F(J_0, J_1, t)$ by the static distribution $F_0(J_0, J_1')$, while the time evolving distribution function of the test particle is rewritten as $P(J_0, J_1, t)$ for clarity. This heuristic procedure is justified in Chavanis (2012a) by an explicit calculation of the diffusion and drift coefficients of the Fokker-Planck equation. It transforms the integro-differential equation (D.2) into a differential equation

$$\frac{\partial P}{\partial t} = 4\pi^3 \frac{\partial}{\partial J} \left\{ \sum_m \frac{1}{\delta \Omega[m, \Omega]} \int dJ_0 \right\}$$

$$\times \frac{1}{|\delta \Omega[m, \Omega | J_0]|^2}$$

$$\times m \left[ F_0(J_0, J_1') \frac{\partial P}{\partial J} (J_0, J_1, t) - P(J_0, J_1, t) \frac{\partial F_0}{\partial J} (J_0, J_1') \right],$$

which can be interpreted as a Fokker-Planck equation. If we assume that the field particles are at statistical equilibrium (thermal bath), described by the Boltzmann distribution

$$F_0(J) = C e^{-\beta H(J)},$$

then, using the relation $\partial F_0 / \partial J = -\beta F_0 \Omega$ (see the definition of $\Omega$ in equation (1)), we can reduce the Fokker-Planck equation (D.3) to the form

$$\frac{\partial P}{\partial t} = \frac{\partial}{\partial J} \left\{ \sum_m D_m(J) \left[ \frac{\partial P}{\partial J} + \beta \Omega(J) P \right] \right\},$$

where

$$D_m(J) = \sum_J \int \frac{dJ_0}{J_0} \left[ \frac{1}{\delta \Omega[m, \Omega | J_0]} \int dJ_0 dJ_1^2 \right]$$

$$\times \delta \Omega(J_0 - J_0^2)$$

$$\times \frac{1}{|\delta \Omega[m, \Omega | J_0]|^2}$$

$$\times m \left[ F_0(J_0, J_1') \frac{\partial F_0}{\partial J} (J_0, J_1, t) - F_0(J_0, J_1, t) \frac{\partial F_0}{\partial J} (J_0^2, J_1, t) \right].$$
with the diffusion coefficient
\[ D_m(J) = \frac{4\pi^3}{|\partial_{J_i} [m \cdot \Omega]|} \int dJ_i' \frac{F_0(J_0, J_i')}{|D_m(J_0, J_i', J_0, J_i', m \cdot \Omega)|^2}. \] (D.6)

We note that the friction term in the Fokker-Planck equation (D.5) is proportional and opposite to the intrinsic frequency \( \Omega \), and that the friction coefficient \( \xi \) satisfies a (generalized) Einstein relation \( \xi = D\beta \) (for each resonance) (Chavanis 2012a).

This Fokker-Planck formalism may have other applications. For example, if we consider a system with two species of particles (e.g. characterized by different masses \( m_A \) and \( m_B \)), and if species B has reached an equilibrium state with a distribution \( \Psi(t) \), we can use equation (D.3) to describe the relaxation of particles of species A due to the encounters with particles of species B (but neglecting the encounters between particles of species A). This approach is further discussed in Appendix F of Chavanis (2013b). On the other hand, for a single species system, we may describe the early dynamics of the system as a whole with a good approximation by replacing \( F(J_0, J_i', t) \) in the Balescu-Lenard equation (D.2) by the initial distribution function \( F_0(J_0, J_i') \), leading again to an equation of the form (D.3).

**Appendix D.2: Other kinetic equations**

We now compare the previous results to other related kinetic equations. For spatially homogeneous systems with long-range interactions, the Balescu-Lenard equation reads (see Chavanis (2012c)):
\[ \frac{\partial F}{\partial t} = \pi(2\pi)^d \epsilon(k, \omega) \frac{\partial}{\partial \omega} \left\{ \int dk' dW' k \cdot \frac{\hat{u}(k)}{\epsilon(k, k') \cdot \delta_D[k \cdot (v-v')]} \varepsilon_F(v', t) - F(v, t) \frac{\partial F}{\partial v'} \right\}, \] (D.7)

where
\[ \epsilon(k, \omega) = 1 - (2\pi)^d \hat{u}(k) \int dW' k \cdot \frac{\partial F/\partial \omega}{k \cdot v - \omega} \] (D.8)
is the dielectric function. For one dimensional systems, it reduces to the trivial form
\[ \frac{\partial F}{\partial t} = 2\pi^2 \frac{\partial}{\partial \omega} \left\{ \int dk' dW' k \cdot \frac{\hat{u}(k)}{\epsilon(k, k') \cdot \delta_D[v-v']} \varepsilon_F(v', t) - F(v, t) \frac{\partial F}{\partial v'} \right\} = 0. \] (D.9)

In \( d > 1 \), there are always resonances between particles with different velocities, implying that the Balescu-Lenard equation relaxes towards the Boltzmann distribution on a timescale of the order \( N_{D\Omega} \), where \( t_{D\Omega} \) is the dynamical time. By contrast, in \( d = 1 \), the resonances become local (in velocity space) and since the term in brackets is anti-symmetric with respect to the interchange of \( v \) and \( v' \), the Balescu-Lenard diffusion current vanishes exactly. This implies that the relaxation time is larger than \( N_{D\Omega} \), presumably of the order \( N_{D\Omega} \), corresponding to the next order term in the expansion of the dynamics in powers of \( 1/N \). The Fokker-Planck equation describing the evolution of a test particle in a bath of field particles is discussed in Chavanis (2012c, 2013a).

The kinetic equation governing the collisional evolution of a system of \( N \) point vortices in two dimensional hydrodynamics at the order \( 1/N \) can be written, in the axisymmetric case, as (see Chavanis (2012b)):
\[ \frac{\partial \omega}{\partial t} = 2\pi^2 \gamma \int \frac{dr'}{r'} \chi(r, r', \Omega(t)) \delta_D[\Omega(t) - \Omega(t')] \] (D.10)

where \( \omega(r, t) \) is the profile of vorticity, \( \Omega(t) \) the profile of angular velocity, and \( \chi(r, r', \Omega(t)) \) is related to the dressed potential of interaction between the point vortices (see Chavanis (2012b) for more details). This equation conserves the total number of point vortices and the energy, and monotonically increases the Boltzmann entropy (H-Theorem). If the profile of angular velocity is monotonic, the kinetic equation reduces to the form
\[ \frac{\partial \omega}{\partial t} = 2\pi^2 \gamma \int \frac{dr'}{r'} \chi(r, r', \Omega(t)) \frac{1}{\Omega(t)} \delta_D(r-r') \] (D.11)

For non-monotonic profile of angular velocity, one can have non-local resonances (i.e. distant collisions between point vortices), as studied in Chavanis & Lemou (2007). This produces a diffusion current. If the profile of angular velocity is, or becomes, monotonic, the resonances are purely local and, since the term in brackets is anti-symmetric with respect to the interchange of \( r \) and \( r' \), the diffusion current also vanishes. This implies that the relaxation time is larger than \( N_{D\Omega} \) as discussed above. The Fokker-Planck equation describing the evolution of a test vortex in a sea of field vortices is discussed in Chavanis (2012b).

If we focus on purely local resonances, we note that the inhomogeneous Balescu-Lenard equation (D.1) is different from equations (D.9) and (D.11) because it is two-dimensional in \( J_i \) and \( J_\Omega \), and the resonances act only on \( J_0 \). Therefore, purely local resonances do not yield a zero flux, contrary to equations (D.9) and (D.11). This really is an effect of the two-dimensionality of the system. Indeed, for local resonances, the \( 1D \) inhomogeneous equation also yields a zero flux:
\[ \frac{\partial F}{\partial t} = 2\pi^2 \frac{\partial}{\partial J_i} \left\{ \sum_m \frac{m \Omega(J)}{|\Omega|} \int dJ_i' \frac{\delta_D(J-J_i')}{|D_m(J, J_i, m \Omega)|^2} \right\} = 0. \] (D.12)

**Appendix E: The Schwarzschild conspiracy**

The Schwarzschild distribution function introduced in equation (25) and considered in S12 simulation has the specificity to be exponential in the \( J_i \)-direction, so that \( F \) is Boltzmannian with respect to the \( J_i \) variable. It is known (e.g. Chavanis 2012a) that the (complete) Boltzmann distribution is the steady state of the Balescu-Lenard equation (2). Therefore, we can expect that the (partial) exponential behavior of the Schwarzschild distribution will induce simplifications that we now detail. In analogy with equation (10), the flux associated to a given resonance \( m \) is defined as
\[ \mathcal{F}_m = m \left[ A_m(J) F(J) + D_m(J) m \frac{\partial F}{\partial J} \right] \equiv m \mathcal{F}_m, \] (E.1)

where the non-bold \( \mathcal{F}_m \) is a scalar. Using shortened notations and forgetting numerical prefactors, we may rewrite the drift and diffusion coefficients from equations (79) and (80) under the form
The distribution function is given by the Schwarzschild distribution. Because of the exponential dependence in \( F \), the distribution function is given by the Schwarzschild conspiracy is not exactly satisfied as observed in equation (E.5) of the drift and diffusion coefficients. The flux can then be decomposed as \( F_m = F_m^\text{ff} + F_m^\text{diff} \) with

\[
F_m^\text{ff} = \frac{m_r}{(m \cdot \Omega_f)} \int \frac{dJ_1}{|D_m|^2} \left( F(J_1^r) \frac{\partial F}{\partial J_1} - F(J_1^s) \frac{\partial F}{\partial J_1} \right),
\]

and

\[
F_m^\text{diff} = \frac{m_\phi}{(m \cdot \Omega_f)} \int \frac{dJ_1^r}{|D_m|^2} \left( F(J_1^r) \frac{\partial F}{\partial J_1^s} - F(J_1^s) \frac{\partial F}{\partial J_1^r} \right).
\]

We are interested in the value of the flux at the initial time, where the distribution function is given by the Schwarzschild distribution. Because of the exponential dependence in \( J_1 \), the Schwarzschild distribution function, one has \( \partial F/\partial J_1 = -\kappa/\sigma_f^2 F \). As a result, the radial component (E.4) of the flux cancels out and the flux is simply given by equation (E.5). This coincidence could be called the Schwarzschild conspiracy and has important consequences on the properties of the collisional distribution. Indeed, for a tidal disc, one has \( |\partial F/\partial J_1^s| \gg |\partial F/\partial J_1^s| \). Hence one would expect the gradients with respect to \( J_1 \) to be the major contributors to the diffusion. When considering independently the drift and diffusion coefficients, the gradients in \( \partial F/\partial J_1^s \) dominate the diffusion current. Thus for \( m_r \neq 0 \), the diffusion-only flux can be approximated by

\[
F_m^\text{diff} \approx \frac{m_r}{(m \cdot \Omega_f)} \int \frac{dJ_1^r}{|D_m|^2} F(J_1^r) \frac{\partial F}{\partial J_1^s}.
\]

As the ILR and the OLR have a non-zero \( m_r \) compared to the COR, these resonances should dominate independently the drift and diffusion components. However, when considering the full flux made of the contributions from the drift and diffusion coefficients, because of the Schwarzschild conspiracy, there is a simplification of the dominant terms in \( \partial F/\partial J_1 \), so that one recovers as in equation (E.5) that only the smaller gradients \( \partial F/\partial J_1^s \) remain present. The Schwarzschild conspiracy between drift and diffusion will therefore tend to slightly reduce the magnitude of the full diffusion flux, so as to moderately slow down the collisional relaxation. More importantly, the Schwarzschild conspiracy will favor the COR resonance (radial migration) over the ILR resonance (\( J_1 \) – heating). One should note that the DF which is effectively sampled in S12 is of the form \( F = F(E, J_0) \exp[-E/\sigma_f^2] \), with \( q = V_0/\sigma_f^2 - 1 \) (Toomre 1977; Binney & Tremaine 2008). It is only within the epicyclic approximation that this DF takes the form of the Schwarzschild DF from equation (25). As a consequence, in S12 simulation, the Schwarzschild conspiracy is not exactly satisfied as observed in equation (E.4), but the residual difference driving the secular diffusion is likely to be subdominant, as illustrated in figure E.1.

**Appendix F: Temporal frequency selection**

An important feature of the diffusion equation (7) is that the diffusion takes place along specific resonance directions associated to the vectors \( m \) as discussed in equation (16). Hence being able to determine the dominant resonance is crucial in order to estimate the direction of the secular diffusion in action space. The temporal frequency associated to a given resonance \( m \) in a location \( J \) of action-space is given by \( \omega = m \cdot \Omega \). Thanks to the expression (83), one immediately notes that for a Mestel disc, one has

\[
0 < \omega_{\text{ILR}} < \omega_{\text{COR}} < \omega_{\text{OLR}}.
\]

In Fouvy & Pichon (2015); Fouvy et al. (2015), we studied the same S12 simulation using the WKB limit of the secular diffusion equation, which intends to describe the secular forcing of a collisionless self-gravitating system perturbed by an external source. An essential assumption of this approach was to consider the external perturbation as originating from numerical Poisson shot noise, and therefore assume it to be proportional to the local active surface density. The autocorrelation of the external perturbation \( \delta \rho_{\text{ext}} \) was taken to be equal to

\[
\left\langle \delta \rho_{\text{ext}}(\omega, k, J, s) \right\rangle \propto \Sigma(k, J_s).
\]

One should note that this crude assumption on the noise properties has no \( \omega \) dependence, so that all resonances are equally favored by the Poisson shot noise and are perturbed similarly whatever their associated intrinsic frequencies \( \omega = m \cdot \Omega \). This ad hoc and simple noise approximation is one of the limitations of the formalism presented in Fouvy & Pichon (2015); Fouvy et al. (2015). In contrast, in the WKB Balescu-Lenard equation described in this paper, this preferential selection of the resonances based on their intrinsic frequency is naturally present. Indeed, one can note in the expression (E.5) of the flux associated to a resonance \( m \), the presence of the prefactor \( 1/(m \cdot \Omega) \) which arose in equation (67) when handling the resonance condition. For the Mestel disc, whose intrinsic frequencies are given by equation (83), this term can be straightforwardly computed and reads

\[
\frac{1}{(m \cdot \Omega)^2} = \frac{1}{|m_\phi + \sqrt{2} m_r| |\partial \Omega_\phi/\partial J_1^s|}.
\]
Comparing the ILR resonance to the OLR and COR, one immediately obtains that

\[
\frac{(m_{\text{OLR}} \cdot \Omega)' \cdot \sqrt{2}}{(m_{\text{ILR}} \cdot \Omega)' \cdot \sqrt{2}} = 5.8, \quad \frac{(m_{\text{COR}} \cdot \Omega)' \cdot \sqrt{2}}{(m_{\text{ILR}} \cdot \Omega)' \cdot \sqrt{2}} = 3.4.
\]

(F.4)

Hence because the ILR resonance is associated to lower intrinsic temporal frequency \( \omega = m \cdot \Omega \), the resonant factor \( 1/(m \cdot \Omega)' \) naturally tends to favor the ILR resonance with respect to the OLR and COR, and therefore performs natively a temporal frequency biasing which was absent from the ad hoc assumption of equation (F.2) describing the external forcing considered in Fouvry & Pichon (2015); Fouvry et al. (2015).

One can even be more specific when comparing the ILR and OLR resonances. The only difference between these two resonances is the sign of \( m_r \). The expression (62) of the amplification eigenvalues shows that its value only depends on \( s^2 \), so that \( \lambda_{\text{ILR}} = \lambda_{\text{OLR}} \). The expression of the susceptibility coefficients from equation (77) is also independent of the sign of \( m_r \) so that \( 1/|D_{\text{ILR}}|^2 = 1/|D_{\text{OLR}}|^2 \). Hence, when considering the flux \( F_m \) given by equation (E.5), one notes that between the ILR and OLR resonances, \( F_m \) only changes through the factor \( 1/(m \cdot \Omega)' \).

Thanks to equation (F.4), one immediately obtains

\[
\frac{F_{\text{ILR}}}{F_{\text{OLR}}} \approx 5.8.
\]

(F.5)

The secular diffusion flux associated to the OLR resonance is therefore always much smaller than the one associated to the ILR resonance, because of this effect of temporal frequency biasing.

As a conclusion, the temporal frequency selection effect described in equation (F.4) will tend to favor the ILR resonance because it is associated to a smaller intrinsic frequency. However, one can note from figure 8 that the COR resonance is always more amplified than the ILR resonance. Finally, the crucial remark is to note that the susceptibility coefficients from equation (77) involve Bessel functions \( J_m \), which are such that \( \lim_{x \to 0} J_m(x) = 1 \) if \( m_r = 0 \), or \( 0 \) otherwise. As a consequence, close to the \( J_r = 0 \) axis, the COR resonance will always tend to become the dominant resonance. There is therefore a non trivial arbitration between these opposite effects when considering the respective contributions of the various resonances to the full secular diffusion flux.