On properties of skew-framed immersions cobordism groups

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Abstract

In this paper, we introduce geometric technique of working with skew-framed manifolds. It allows us to study stable homotopy groups of some Thom spaces by geometric means. We schematically describe how our results (which are also of independent interest) can be applied to obtain a proof of the Baum-Browder theorem stating non-immersibility of $\mathbb{R}P^{10}$ to $\mathbb{R}^{15}$.

Introduction

Our paper develops ideas of the Lecture course [3] and we devote it to the memory of Yuri Petrovich Solovyov.

In [1], the authors suggested and partly realized a scheme of a new geometric proof of the Baum-Browder theorem concerning non-immersibility of projective space $\mathbb{R}P^{10}$ to euclidean space $\mathbb{R}^{15}$ [4, Corollary (9.9)]. It turns out that the obstruction group for immersibility of $\mathbb{R}P^{10}$ to $\mathbb{R}^{15}$ is closely connected with the cobordism group $Imm^{sf}(5)(3, 5)$ of skew-framed immersions of 3-dimensional manifolds to euclidean space $\mathbb{R}^8$. More precisely: the group $Imm^{sf}(5)(3, 5)$ can be interpreted as a subgroup of the obstruction group for the immersion mentioned above [1]. In this paper we compute this cobordism group and study some of its geometric-type properties.

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Note that the Pontryagin-Thom construction \cite{23}, \cite{26} provides an isomorphism \[ \text{Imm}^s f(5)(3,5) \cong \Pi_8(\mathbb{RP}^\infty_5), \]
where \( \mathbb{RP}^\infty_5 = \mathbb{RP}^\infty / \mathbb{RP}^4 \) is stunted projective space. In these terms Theorem \[ \text{II} \] states:

\[ \Pi_8(\mathbb{RP}^\infty_5) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2; \quad \Pi_8(\mathbb{RP}^\infty_5) \cong \mathbb{Z}/2. \]

Results of Theorems \[ \text{II} \] are more technical. They are used in the proof of the Baum-Browder theorem, which will be given in our subsequent paper.

**Notation, conventions**

A number above an arrow \( \overset{n}{\rightarrow} \) means that this map was defined in formula \( (n) \).

The symbol \( \cong \) has several meanings in the paper: diffeomorphism of manifolds, isomorphism of vector bundles, isomorphism of groups. Its concrete meaning is clear from the context.

All manifolds and maps are supposed to be smooth. Manifolds can have boundaries. Sometimes the dimension \( m \) of a manifold \( M \) is expressed in notation: \( M^m \).

By \( c \) we denote a constant map (to the point \( c \)).

Considering preimages of submanifolds, we assume necessary transversality; see e.g. \cite{10}.

A suspension \( Ef \) of an immersion \( f : M^n \hookrightarrow \mathbb{R}^{n+k} \) is the immersion defined as the composition \( M^n \overset{f}{\rightarrow} \mathbb{R}^{n+k} \subset \mathbb{R}^{n+k+1} \), where the second embedding is the standard embedding onto the hyperplane \( x_{n+k+1} = 0 \).

The normal bundle of an immersion \( f : M \hookrightarrow \mathbb{R}^{n+k} \) is denoted by \( \nu(f) \), or simply by \( \nu(M) \), if it is clear which immersion is meant.

For a submanifold \( M \subset N \) we will use the evident isomorphism \( \nu(M) \cong \nu(M,N) \oplus (\nu(N))|_{M} \).

We make free with \( \varepsilon \) to denote the trivial line bundle (over any space), and with \( \gamma \) to denote the tautological line bundle over projective space (of arbitrary dimension).

For a vector bundle \( \xi \) over a base space \( B \) and a map \( \kappa : M \rightarrow B \), the pullback bundle \( \kappa^*(\xi) \) is defined in a standard way. For \( \xi = k\varepsilon \) we fix an evident isomorphism \( \kappa^*(\xi) \cong k\varepsilon \).

For \( N \in \mathbb{N} \cup \{ \infty \} \), the \( N \)-dimensional real projective space is denoted by \( \mathbb{RP}^N \), and the stunted projective space is denoted by \( \mathbb{RP}^N_k = \mathbb{RP}^N / \mathbb{RP}^{k-1} \).

Points of the total space \( E(k_1\gamma \oplus k_2\varepsilon) \) of the bundle \( k_1\gamma \oplus k_2\varepsilon \) over \( \mathbb{RP}^N \) are written as rows

\[
[x; \lambda_1, \ldots, \lambda_{k_1}, \mu_1, \ldots, \mu_{k_2}], \quad \text{where} \quad x \in S^N, \lambda_1, \ldots, \lambda_{k_1}, \mu_1, \ldots, \mu_{k_2} \in \mathbb{R}. \quad (1)
\]
Such a row is a pair of identified rows

\[(x; \lambda_1, \ldots, \lambda_k, \mu_1, \ldots, \mu_k) \sim (-x; -\lambda_1, \ldots, -\lambda_k, \mu_1, \ldots, \mu_k).\]

By \( \rho : S^N \to \mathbb{R}^N, x \mapsto [x] = \{x, -x\} \) we denote the standard 2-fold covering.

Using rows \((x; \lambda_1, \ldots, \lambda_k)\) to denote points of the space \(E(\rho^*(k\gamma))\), we define the isomorphism \( \rho^*(k\gamma) \cong k\varepsilon \) of bundles over \(S^N\) using the formula

\[E(\rho^*(k\gamma)) \to E(k\varepsilon), \quad (x; \lambda_1, \ldots, \lambda_k) \mapsto (x; \lambda_1, \ldots, \lambda_k). \quad (2)\]

The embeddings \(\mathbb{P}^n \subset \mathbb{P}^{n+k}\) and the isomorphisms \(\nu(\mathbb{P}^n, \mathbb{P}^{n+k}) \cong k\gamma\) are standard ones.

A symbol \(\Pi_n(X)\) denotes the \(n\)-th stable homotopy group of space \(X\); and \(\Pi_n = \Pi_n(S^0)\). We will make use of some stable homotopy groups of \(\mathbb{P}^\infty\) computed in [19].

Finally, let \(I = [0, 1]\).

## 1 Basic Notions

### 1.1 Definition of the group \(Imm^{\xi, B}(n, k)\)

Cobordism groups of framed embeddings were defined and studied by L.S. Pontryagin, see [23]. Investigation of cobordism groups of immersions with additional structure on normal bundle comes back to papers [24], [30], [29], [28], [17]; see also [14, Exercise 7.2.5]. (On some generalizations see e.g. [25], [9].)

Let \(\xi\) be a \(k\)-dimensional vector bundle over a base space \(B\), and \(k \geq 1\). Let \(n \geq 0\).

**Definition 1.** A normal \(\xi\)-structure for an immersion of an \(n\)-dimensional compact manifold \(f : M^n \hookrightarrow \mathbb{R}^{n+k}\) is a continuous map \(\kappa : M \to B\) together with an isomorphism \(\Xi : \nu(f) \cong \kappa^*\xi\).

Such an object is called an immersion with normal \((\xi, B)\)-structure, or shortly a \((\xi, B)\)-immersion. It is denoted by triples \((f_M, \kappa_M, \Xi_M)\). Sometimes is clear which manifold is meant, in these cases we write simply \((f, \kappa, \Xi)\).

**Remark 1.** Equivalently, a \((\xi, B)\)-structure \(\Xi : \nu(f) \cong \kappa^*(\xi)\) over \(M\) can be thought of as a fiberwise isomorphism of bundles

\[
\begin{align*}
\nu(f) & \longrightarrow \xi \\
\downarrow & \\
M & \longrightarrow B
\end{align*}
\]
Remark 2. For a covering $p: \tilde{M} \to M$ (in particular, for a diffeomorphism $\varphi: M \to M$) and an immersion $f: M \hookrightarrow \mathbb{R}^{n+k}$, there is a natural fiberwise isomorphism of normal bundles

$$\nu(f \circ p) \xrightarrow{\sim} \nu(f)$$

The corresponding isomorphism $\nu(f \circ p) \cong p^*(\nu(f))$ of bundles over $\tilde{M}$ will be denoted by $p'$.

Hence an isomorphism $\Xi : \nu(f) \cong \kappa^*(\xi)$ induces an isomorphism $p^*(\Xi) \circ p' : \nu(f \circ p) \cong p^*(\nu(f)) \cong p^*(\kappa^*(\xi))$.

Definition 2. Two immersions with normal $(\xi, B)$-structures $(f_0 : M_0 \hookrightarrow \mathbb{R}^{n+k}, \kappa_0, \Xi_0)$, $(f_1 : M_1 \hookrightarrow \mathbb{R}^{n+k}, \kappa_1, \Xi_1)$ are called cobordant if there exist

1. a compact $(n+1)$-dimensional manifold $\mathcal{M}$,
2. a diffeomorphism $\varphi_0 \sqcup \varphi_1 : M_0 \sqcup M_1 \to \partial \mathcal{M}$,
3. an immersion $\mathcal{F} : \mathcal{M} \hookrightarrow \mathbb{R}^{n+k} \times I$ orthogonal to the boundary $\mathbb{R}^{n+k} \times \partial I$ of the strip $\mathbb{R}^{n+k} \times I$ at any point of $\partial \mathcal{M}$; also, the restriction $\mathcal{F}|_{\varphi_s(M_s)}$ must equal $M_s \rightrightarrows \mathbb{R}^{n+k} \subset \mathbb{R}^{n+k} \times \{s\}$ for $s = 0, 1$;
4. a continuous map $\kappa : \mathcal{M} \to B$ such that the composition map $\kappa \circ \varphi_s$ equals $\kappa_s : M_s \to B$ for $s = 0, 1$;
5. an isomorphism $\Xi : \nu(\mathcal{F}) \cong \kappa^*(\xi)$, such that the bundle $\nu(f_s) = \varphi_s^*(\nu(\mathcal{F})) \cong \varphi_s^*(\kappa^*(\xi)) = \kappa_s^*\xi$ over $M_s$ coincides with $\Xi_s$ for $s = 0, 1$.

In other words, the triple $(\mathcal{F}, \kappa, \Xi)$ provides a cobordism of $(f_0, \kappa_0, \Xi_0)$ and $(f_1, \kappa_1, \Xi_1)$; we write

$$\partial(\mathcal{F}, \kappa, \Xi) = (f_0, \kappa_0, \Xi_0) \sqcup (f_1, \kappa_1, \Xi_1).$$

In what follows, we omit the diffeomorphism symbol $\varphi_0 \sqcup \varphi_1$, identifying $\partial \mathcal{M}$ with $M_0 \sqcup M_1$.

By standard arguments, this cobordism relation is reflexive, symmetric, and transitive.

Denote by $Imm^{\xi,B}(m, k)$ the set of cobordism classes of immersions of closed $m$-dimensional manifolds with normal $\xi$-structure. Sometimes we inaccurately speak of $(\xi, B)$-immersions as elements of this group.
The sum of two cobordism classes is defined as the union of disjoint representatives; thus $\text{Imm}^{\xi,B}(m,k)$ becomes an abelian group.

The Whitney Theorem [14, Theorem 1.3.5] and the Pontryagin-Thom construction [23, 26] (see also [14, Exercise 7.2.5]) imply

**Statement 1.** Suppose $k \geq n + 2$. Then

$$\text{Imm}^{\xi,B}(n,k) \cong \text{Emb}^{\xi,B}(n,k) \cong \pi_{n+k}(T(\xi)).$$

Together with Hirsch theorem [13, Theorem 6.4] this gives (see [30, Theorem 1], [17, Theorem 1.1]):

**Statement 2.** Let $B$ be a finite complex. Then there is an isomorphism $\text{Imm}^{\xi,B}(n,k) \cong \Pi_{n+k}(T(\xi))$.

### 1.2 Important particular cases

We have given general definition of immersion with normal $(\xi, B)$-structure. Let us enumerate several particular cases which will be investigated in the present work (see [2], [1]).

1. For $B = \text{pt}$, $\xi = k\varepsilon$ the group $\text{Imm}^{k\varepsilon,\text{pt}}(n,k)$ is denoted by $\text{Imm}^{fr}(n,k)$; we call it the group of framed immersions. Elements of this group are represented by pairs of the form $(f : M^n \hookrightarrow \mathbb{R}^{n+k}, \Xi : \nu(f) \cong k\varepsilon)$. The Pontryagin-Thom construction together with Hirsch results provides for $k \geq 1$ an isomorphism

$$\text{Imm}^{fr(k)}(n,k) \cong \Pi_n.$$

2. For $B = \mathbb{R}P^N$, $\xi = k\gamma$ the group $\text{Imm}^{k\gamma,\mathbb{R}P^N}(n,k)$ will be denoted by $\text{Imm}^{sf(k),\mathbb{R}P^N}(n,k)$, we call it the group of skew-framed $\mathbb{R}P^N$-controlled immersions. For $N = \infty$ this group is denoted by $\text{Imm}^{sf(k)}(n,k)$, and we speak simply of the skew-framed immersions group. The chain of standard inclusions $\mathbb{R}P^0 \subset \mathbb{R}P^1 \subset \ldots \subset \mathbb{R}P^\infty$ induces a chain of homomorphisms (by the cell approximation theorem, the first map in the second row is surjective, and all other maps are bijective)

$$\begin{align*}
\text{Imm}^{sf(k),\mathbb{R}P^0}(n,k) &= \text{Imm}^{fr(k)}(n,k) \rightarrow \text{Imm}^{sf(k),\mathbb{R}P^1}(n,k) \rightarrow \ldots \\
\text{Imm}^{sf(k),\mathbb{R}P^n}(n,k) &= \text{Imm}^{sf(k),\mathbb{R}P^{n+1}}(n,k) \cong \ldots \cong \text{Imm}^{sf(k)}(n,k).
\end{align*}$$

**Statement 2** and [15, Theorem 15.1.8] provide the isomorphisms

$$\begin{align*}
\text{Imm}^{sf(k),\mathbb{R}P^N}(n,k) &\cong \Pi_{n+k}(\mathbb{R}P^{k+N}) , \\
\text{Imm}^{sf(k)}(n,k) &\cong \Pi_{n+k}(\mathbb{R}P^\infty).
\end{align*}$$

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3. For $B = \mathbb{RP}^N$, $\xi = k_1\gamma + k_2\varepsilon$, $k_2 \geq 1$ we have a suspension isomorphism

$$E : Imm^{sf(k_1) \times fr(k_2-1), \mathbb{RP}^N} (n, k_1+k_2-1) \rightarrow Imm^{sf(k_1) \times fr(k_2), \mathbb{RP}^N} (n, k_1+k_2),$$

for $k_1 + k_2 \geq 2$ it is an isomorphism by [13, Theorem 6.4]. The group $Imm^{sf(k_1) \times fr(k_2), \mathbb{RP}^\infty} (n, k_1+k_2)$ is denoted shortly by $Imm^{sf(k_1) \times fr(k_2)} (n, k_1+k_2)$.

Injection of the structure groups \{0\} $\subset \mathbb{Z}/2$ induces a homomorphism

$$Imm^{fr(k)} (n, k) \rightarrow Imm^{sf(k)} (n, k)$$

**Example 1.** The map $\iota : Imm^{fr(k)} (n, k) \rightarrow Imm^{fr(k) \times sf(0)} (n, k)$ such that $(f, \Xi) \mapsto (f, \text{const}, \Xi)$ is a monomorphism.

The map $r : Imm^{fr(k) \times sf(0)} (n, k) \rightarrow Imm^{fr(k)} (n, k)$ such that $(f, \kappa, \Xi : \nu(f) \cong \kappa^* (k\varepsilon \oplus 0_1)) \mapsto (f, \Psi : \nu(f) \cong \kappa^* (k\varepsilon) \cong k\varepsilon)$ is an epimorphism.

The composition $r \circ \iota : Imm^{fr(k)} (n, k) \rightarrow Imm^{fr(k)} (n, k)$ is the identity map. Therefore the group $Imm^{fr(k)} (n, k)$ is a direct summand of the group $Imm^{fr(k) \times sf(0)} (n, k)$.

### 1.3 Simple modifications of the triples $(f, \kappa, \Xi)$

In this section we describe several simple procedures which will be applied in the proofs below. The reader can verify their properties himself or herself.

**Statement 3.** For arbitrary $B$, $\xi$ we have:
1) for any diffeomorphism $\varphi : M \rightarrow M$ the $(\xi, B)$-immersions $(f : M^n \leftrightarrow \mathbb{R}^{n+k}, \kappa, \Xi)$ and $(f \circ \varphi, \kappa \circ \varphi, \varphi^* (\Xi) \circ \varphi')$ are cobordant [concerning $\varphi'$ see Remark 2].
2) Suppose $(f, \kappa, \Xi_0)$ is a $(\xi, B)$-immersion, and $\{\Xi_t\}$, $t \in I$ is a family of bundle isomorphisms $\nu(f) \cong \kappa^*(\xi)$. Then the $(\xi, B)$-immersions $(f, \kappa, \Xi_0)$ and $(f, \kappa, \Xi_1)$ are cobordant.
3) Let $H : M \times I \rightarrow \mathbb{R}^N \times I$ be a regular homotopy of immersions $f_0 \simeq f_1$ such that $f_t = f_0$ for $t \in [0, \frac{1}{10}]$, and $f_t = f_1$ for $t \in [\frac{9}{10}, 1]$. Suppose $\kappa : M \rightarrow B$ is a continuous map and $\Xi_0 : \nu(H)|_{M \times \{0\}} = \nu(f_0) \cong \kappa^*(\xi)$ is an isomorphism. Let $\Xi : \nu(H) \cong (\kappa \circ p)^*(\xi)$ be any isomorphism extending the isomorphism $\Xi_0$; here $p : M \times I \rightarrow M$ is the projection map. Let $\Xi_1 = \Xi|_{M \times \{1\}} : \nu(H)|_{M \times \{1\}} = \nu(f_1) \cong \kappa^*(\xi)$ be “the resulting” isomorphism. Then the $(\xi, B)$-immersions $(f_0, \kappa, \Xi_0)$ and $(f_1, \kappa, \Xi_1)$ are cobordant.

Now let us consider the case $B = \mathbb{RP}^N$, $\xi = k_1\gamma + k_2\varepsilon$. In this situation, we can “spread out” the $\xi$-structure following a homotopy of the map $\kappa.$
Such spreading generalizes the procedure described by Pontryagin for framed embeddings.

So, let a \((k_1 \gamma \oplus k_2 \varepsilon, \mathbb{R}P^N)\)-immersion \((f, \kappa_0, \Xi_0)\) and a homotopy \(\{\kappa_t\}\) be given.

We describe a method to construct the corresponding family \(\{\Xi_t : \nu(f) \cong \kappa^*_t(k_1 \gamma \oplus k_2 \varepsilon)\}\).

In accordance with Remark 1 replace the isomorphism \(\Xi_0 : \nu(f) \cong \kappa_0^*(\xi)\) by the diagram

\[
\begin{array}{ccc}
E(\nu(f)) & \xrightarrow{\Psi_0} & E(\xi = k_1 \gamma \oplus k_2 \varepsilon) \\
\downarrow \pi & & \downarrow \\
M & \xrightarrow{\kappa_0} & \mathbb{R}P^N
\end{array}
\]

where \(\Psi_0\) is a fiberwise isomorphism.

Let us define the isomorphism \(\Psi_t\) in the diagram

\[
\begin{array}{ccc}
E(\nu(f)) & \xrightarrow{\Psi_t} & E(\kappa_1 \gamma \oplus k_2 \varepsilon) \\
\downarrow \pi & & \downarrow \\
M & \xrightarrow{\kappa_1} & \mathbb{R}P^N
\end{array}
\]

Take an arbitrary point \(X \in E(\nu(f))\). If

\[
\Psi_0(X) = [\kappa_0(\pi(X)); \lambda_1(X), \ldots, \lambda_{k_1}(X), \mu_1(X), \ldots, \mu_{k_2}(X)]
\]

then put

\[
\Psi_t(X) = [\kappa_t(\pi(X)); \lambda_1(X), \ldots, \lambda_{k_1}(X), \mu_1(X), \ldots, \mu_{k_2}(X)].
\]

Denote by \(\Xi_t\) the corresponding isomorphism \(\nu(f) \cong \kappa^*_t(k_1 \gamma \oplus k_2 \varepsilon)\).

**Statement 4.** Suppose \(B = \mathbb{R}P^N, \xi = k_1 \gamma \oplus k_2 \varepsilon\). Then the above construction provides a \((k_1 \gamma \oplus k_2 \varepsilon, \mathbb{R}P^N)\)-immersion \((f, \kappa_1, \Xi_1)\) which is cobordant to the original immersion \((f, \kappa_0, \Xi_0)\).

**Example 2.** For the standard embedding \(E : S^3 \subset \mathbb{R}^4\), the unit exterior normal vector gives a trivialization \(\Phi : \nu(E) \cong c^*(\gamma)\), where \(c : S^3 \rightarrow c \in \mathbb{R}P^\infty\) is the constant map.

The pair \((E, \Phi)\) represents zero element of the group \(Imm_{fr}(1)(3,1)\); the triple \((E, c, \Phi)\) represents the zero element both in \(Imm_{sf}(1), \mathbb{R}P^3(3,1)\) and in \(Imm_{sf}(1)(3,1)\).

**Example 3.** To continue Example 2, take a homotopy \(H : S^3 \times I \rightarrow \mathbb{R}P^\infty\) between the maps \(c\) and \(S^3 \xrightarrow{\rho} \mathbb{R}P^3 \subset \mathbb{R}P^\infty\); here \(\rho\) is the standard 2-covering.
Statement 4 uses the isomorphism $\Phi : \nu(E) \cong c^*(\gamma)$ to construct the isomorphism $\Xi_E : \nu(E) \cong \rho^*(\gamma)$.

In the group $Imm^{s,f}(1)(3,1)$ the triple $(E, \rho, \Xi_E)$ is cobordant to the original triple $(E, c, \Phi)$, that is, it represents the zero element.

But since the image of the homotopy $H$ is not contained in the subspace $\mathbb{R}P^3 \subset \mathbb{R}P^\infty$, we cannot assert that the triple $(E, \rho, \Xi_E)$ is cobordant to the original triple $(E, c, \Phi)$ in the group $Imm^{s,f}(1), \mathbb{R}P^3(3,1)$. It turns out that the triple $(E, \rho, \Xi_E)$ represents a non-zero element in the group $Imm^{s,f}(1), \mathbb{R}P^3(3,1)$ (see Remark 4).

**Remark 3.** Formally, the triple $(E, \rho, \Xi_E)$ defined in Example 3 depends on the choice of the homotopy $H$. But it turns out that its cobordism class depends on the choice of homotopy. In fact, since $S^3$ is simply connected, for any choice of homotopy $H$ the resulting triple will be cobordant either to the triple $(E, \rho, \Xi_E)$ or to the triple $(E, \rho, -\Xi_E)$, where $-\Xi_E$ is the trivialization obtained from the trivialization $\Xi_E$ by multiplication by $-1$ in each fiber. The triples $(E, \rho, \Xi_E)$ and $(E, \rho, -\Xi_E)$ are cobordant in the group $Imm^{s,f}(1), \mathbb{R}P^3(3,1)$ by Statement 8.

2 Some geometric constructions

2.1 Construction of inverse element

Let $R : \mathbb{R}^d \to \mathbb{R}^d$ be the reflection in the hyperplane $x_d = 0$.

For an arbitrary immersion $F : M \looparrowright \mathbb{R}^d$ define an isomorphism $R_{\nu(F)}$ in a natural way:

\[
\begin{array}{ccc}
\nu(F) & \xrightarrow{R_{\nu(F)}} & \nu(R \circ F) \\
\downarrow & & \downarrow \\
M & \xrightarrow{id} & M
\end{array}
\]

If we are also given an isomorphism $\Xi : \nu(F) \cong \kappa^*(k_1 \gamma \oplus k_2 \varepsilon)$, then we can define the composite isomorphism $\Xi \circ (R_{\nu(F)})^{-1} : \nu(R \circ F) \xrightarrow{(R_{\nu(F)})^{-1}} \nu(F) \xrightarrow{\Xi} \kappa^*(k_1 \gamma \oplus k_2 \varepsilon)$.

Then, the following simple assertion holds.

**Statement 5.** For $k_2 \geq 1$ and $N \in \mathbb{N} \cup \{\infty\}$ the triples $(F, \kappa, \Xi)$ and $(R \circ F, \kappa, \Xi \circ (R_{\nu(F)})^{-1})$ represent mutually inverse elements of the group $Imm^{s,f}(k_1) \times fr(k_2), \mathbb{R}P^N(n, k_1 + k_2)$. 

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Let $k_2 \geq 1$; define by $\tau$ the isomorphism of the bundle $k_1 \gamma \oplus k_2 \varepsilon$ over $\mathbb{R}P^N$ which overturns the last summand $\varepsilon$:

\[ [x; \lambda_1, \ldots, \lambda_{k_1}, \mu_1, \ldots, \mu_{k_2}] \rightarrow [x; \lambda_1, \ldots, \lambda_{k_1}, \mu_1, \ldots, -\mu_{k_2}] \rightarrow [x] \]

**Statement 6.** For $k_2 \geq 1$ and $N \in \mathbb{N} \cup \{\infty\}$ the triples $(F, \kappa, \Xi)$ and $(F, \kappa, \Psi)$, where $\Psi$ is the composition $\nu(F) \cong \kappa^*(k_1 \gamma \oplus k_2 \varepsilon) \cong \kappa^*(k_1 \gamma \oplus k_2 \varepsilon)$, represent mutually inverse elements of the group $\text{Imm}^{sf(k_1) \times fr(k_2), \mathbb{R}P^N}(n, k_1 + k_2)$.

**Proof.** By [13, Theorem 6.4], we can apply regular homotopy and assume that the image $F(M)$ lies in the hyperplane $x_d = 0$ of space $\mathbb{R}^{n+k_1+k_2}$. (We assume that the map $\kappa$ is left unchanged, and the isomorphism $\Xi$ has changed as in item 3) of Statement 3; it is denoted by the same symbol $\Xi$. The cobordism class has not changed.) Let $R : \mathbb{R}^{n+k_1+k_2} \rightarrow \mathbb{R}^{n+k_1+k_2}$ be the reflection in the hyperplane $x_d = 0$. By Statement 5 the element represented by the triple $(R \circ F, \kappa, \Xi \circ (R_{\nu(F)})^{-1})$ is inverse to the original one. But $R \circ F = F$ and $\Xi \circ (R_{\nu(F)})^{-1} = \Psi$. □

### 2.2 Antipodal transformation of a skew-framing keeps an element in the cobordism group unchanged

For arbitrary vector bundle $\xi$ over base space $B$ define the fiberwise antipodal involution. It is the automorphism $\Upsilon_\xi$ such that $\Upsilon_\xi(v) = -v$ for any vector $v \in F_x$ belonging to the fiber over $x \in B$. [If the bundle under consideration is clear from the context, then we will omit the symbol $\xi$.] It is easily seen that for the map $\kappa : M \rightarrow B$ the equality holds $\kappa^* \circ \Upsilon_\xi = \Upsilon_{\kappa^*(\xi)}$.

Statement 6 implies (compare [5, p.206])

**Statement 7.** In the group $\text{Imm}^{fr(k)}(n, k)$ the pairs $(F, \Xi)$ and $(F, \Theta)$, where $\Theta : \nu(F) \cong k \cong k \varepsilon$, represent:

- for odd $k$, mutually inverse elements,
- for even $k$, equal elements.

In this section we prove
Statement 8. In the group $\text{Imm}^{sf(k)}(n, k)$ the triples $(F, \kappa, \Xi)$ and $(F, \kappa, \Theta)$, where $\Theta : \nu(F) \cong \kappa^*(k\gamma) \cong \kappa^*(k\gamma^*)$, represent equal elements.

For odd $n$, the same holds true if these triples are considered as representatives of the group $\text{Imm}^{sf(k),R}P^n(n, k)$.

For the proof we need some preliminary remarks.

Projective space of odd dimension $\mathbb{P}^{2d+1}$ consists of elements of the form $[x] = \{x, -x\}$, where $x \in S^{2d+1}$. Since $S^{2d+1} \subset \mathbb{R}^{2d+2} \cong C^{d+1}$, we can write $x$ as an array of complex numbers: $(z_1, \ldots, z_{d+1})$.

Define an isotopy $h_t : \mathbb{P}^{2d+1} \rightarrow \mathbb{P}^{2d+1}$, $t \in I$ by the formula

$$h_t([z_1, \ldots, z_{d+1}]) = [e^{\pi it} \cdot z_1, \ldots, e^{\pi it} \cdot z_{d+1}].$$

It is clear that $h_0 = h_1 = \text{id}$.

Points of the total space $E(\gamma)$ are written as arrays of the form $[z_1, \ldots, z_{d+1}; \lambda]$, see (1).

For $t \in I$ define a map $H_t : E(\gamma) \rightarrow E(\gamma)$ by the formula

$$H_t([z_1, \ldots, z_{d+1}; \lambda]) = [e^{\pi it} \cdot z_1, \ldots, e^{\pi it} \cdot z_{d+1}; \lambda].$$

Next statement has straightforward proof:

Lemma 1. 1. The map $H_t$ is well-defined.

2. The diagram

$$\begin{array}{c}
E(\gamma) \xrightarrow{H_t} E(\gamma) \\
\downarrow \quad \downarrow \\
\mathbb{P}^{2d+1} \xrightarrow{h_t} \mathbb{P}^{2d+1}
\end{array}$$

commutes for $t \in I$.

3. For induced commutative diagram

$$\begin{array}{c}
E(\gamma) \xrightarrow{L_t} E(h_1^*\gamma) \\
\downarrow \quad \downarrow \\
\mathbb{P}^{2d+1}
\end{array}$$

we have $L_0 = \text{id}$, $L_1 = \Upsilon_\gamma$.

Now let us prove Statement 8.

Let us construct a cobordism $(\hat{F}, \hat{\kappa}, \hat{\Xi})$ of the triples under consideration. Take a manifold $\hat{M} = M \times I$ and the product immersion $\hat{F} = F \times \text{id}_I : M \times I \rightarrow \mathbb{R}^{n+k} \times I$. 
Let
\[ N = \begin{cases} n, & \text{for odd } n; \\ n + 1, & \text{for even } n. \end{cases} \]

By the cell approximation theorem we can assume that \( \kappa(M) \subset \mathbb{P}^n \). (In the case of the \( \mathbb{P}^n \)-controlled group, this inclusion holds automatically.)

Define the map \( \hat{\kappa} : M \times I \to \mathbb{P}^N \) by the formula
\[ \hat{\kappa}|_{M \times \{t\}} = h_t \circ \kappa \circ p, \]
where \( p : M \times I \to M \) is the projection, and \( h_t : \mathbb{P}^N \to \mathbb{P}^N \) is defined by the formula (6).

Note that \( \hat{\kappa}|_{M \times \{0\}} = \kappa \) and \( \hat{\kappa}|_{M \times \{1\}} = \kappa \), as required.

Define the isomorphism \( \hat{\Xi} : \nu(\hat{F}) \cong \kappa^*(k\gamma) \).

To do this, for each \( t \in I \) define an isomorphism
\[ \hat{\Xi}_t : \nu(\hat{F}|_{M \times \{t\}}) \cong (\hat{\kappa}|_{M \times \{t\}})^*(k\gamma). \]

Let \( \hat{\Xi}_t \) be the composition
\[ \nu(\hat{F}|_{M \times \{t\}}) \cong \nu(\hat{F}|_{M \times \{0\}}) \cong \kappa^*(k\gamma) \cong \kappa^*(h_t^*(k\gamma)) = (h_t \circ \kappa)^*(k\gamma) = (\hat{\kappa}|_{M \times \{t\}})^*(k\gamma). \] (7)

Note that \( \hat{\Xi}|_{M \times \{0\}} = \Xi \) and \( \hat{\Xi}|_{M \times \{1\}} = \Theta \), as required.

Statements 7, 8 imply

**Corollary 1.** Let \( k \) be an odd integer. Then the homomorphism (5)
\[ \phi : \text{Imm}^{fr(k)}(n, k) \to \text{Imm}^{sf(k)}(n, k) \]
satisfies \( 2 \text{Im} \phi = \{0\} \).

### 2.3 Transfer homomorphism

Recall the definition of the transfer homomorphism (2)
\[ \text{Imm}^{sf(k), \mathbb{P}^N}(n, k) \to \text{Imm}^{fr(k)}(n, k). \]
(8)

For the bundle \( \gamma \) over \( \mathbb{P}^N \) let \( S(\gamma) \) be the corresponding spherical bundle over \( \mathbb{P}^N \), and \( \pi : S(\gamma) \to \mathbb{P}^N \) the projection map. The pullback \( \pi^*(\gamma) \) is the trivial line bundle over \( S(\gamma) \). Fix an isomorphism \( \tau : \pi^*(\gamma) \cong \varepsilon \).

Suppose we are given a triple \( (f : M \hookrightarrow \mathbb{P}^{n+k}, \kappa : M \to \mathbb{P}^N, \Xi : \nu(f) \cong \kappa^*(k\gamma)) \).
Take $\tilde{M} = S(\kappa^*(\gamma))$. It is an $n$-manifold, and $p = \kappa^*(\pi) : \tilde{M} \to M$ is its 2-fold covering:

$$
\begin{array}{ccc}
\tilde{M} = S(\kappa^*(\gamma)) & \xrightarrow{\tilde{\kappa}} & S(\gamma) \\
p = \kappa^*(\pi) & \downarrow & \pi \\
M & \xrightarrow{\kappa} & \mathbb{R}P^N
\end{array}
$$

There is a sequence of isomorphisms

$$
p^*\kappa^*(\gamma) \cong (\kappa \circ p)^*(\gamma) \cong (\pi \circ \tilde{\kappa})^*(\gamma) = \tilde{\kappa}^*(\pi^*(\gamma)) \cong \tilde{\kappa}^*(\varepsilon) \cong \varepsilon.
$$

(9)

In Remark 2, for a given isomorphism $\Xi : \nu(f) \cong \kappa^*(k\gamma)$, we give the method to construct the isomorphism $p^*(\Xi) \circ p' : \nu(f \circ p) \cong p^*(\kappa^*(k\gamma))$. Taking its composition with $k$-th iterate of the isomorphism (9), we obtain the isomorphism $\tilde{\Xi} : \nu(\tilde{f}) \cong k\varepsilon$.

Now, to a given triple $(f : M \to \mathbb{R}^{n+k}, \kappa : M \to \mathbb{R}P^N, \Xi : \nu(f) \cong \kappa^*(k\gamma))$, assign the pair $(\tilde{f} = f \circ p : M \to \mathbb{R}^{n+k}, \tilde{\Xi} : \nu(\tilde{f}) \cong k\varepsilon)$.

This formula defines the homomorphism (9).

**Example 4.** The transfer homomorphism $\mathbb{Z}/2 \cong Imm^{sf(1)}(0, 1) \to Imm^{fr(1)}(0, 1) \cong \mathbb{Z}$ is zero.

**Example 5.** Composition of the homomorphism (5) and transfer

$$
Imm^{fr(k)}(n, k) \to Imm^{sf(k)}(n, k) \to Imm^{fr(k)}(n, k)
$$

is for even $k$ a multiplication by 2; and for odd $k$, a zero map.

**Example 6.** The transfer homomorphism

$$
Imm^{sf(5)}(3, 5) \to Imm^{fr(5)}(3, 5)
$$

is zero.

In fact, by the previous example the composition

$$
Imm^{fr(5)}(3, 5) \to Imm^{sf(5)}(3, 5) \to Imm^{fr(5)}(3, 5)
$$

is zero. But the first homomorphism $\mathbb{Z}/24 \cong Imm^{fr(5)}(3, 5) \to Imm^{sf(5)}(3, 5) \cong \mathbb{Z}/2$ is surjective (see Theorem 1 and [1, Theorem 1]). Hence, the considered transfer is zero.
3 Low-dimensional examples

3.1 Dimension 0

Example 7. \( Imm_{fr}^{(1)}(0, 1) \cong \Pi_0 \cong \mathbb{Z}, \) (10)

however \( Imm_{sf}^{(1)}(0, 1) \cong \Pi_1(\mathbb{RP}^\infty) \cong \mathbb{Z}/2, \) (11)
since \( \mathbb{RP}^\infty \) is the Thom space of the universal line \( O(1) \)-bundle.

Although the elements of both group are finite sets of points, and cobordisms are sets of segments (over which each line bundle is trivial), in the case of the group of skew-framed immersions we are to construct a characteristic map \( \kappa : M \to \mathbb{RP}^\infty \) on the cobordism. Let us underline that here additional possibilities arise, in distinction from the case of framed immersions (where \( \kappa \) is a constant map).

Let us explain shortly the formulae (10) and (11).

The generator \( a_0 \) of the first group can be represented by an immersion of the point, i.e. by the pair \( (f_0 : p_0 \mapsto 0 \in \mathbb{R}^1, \Xi_0 : \nu(f_0) \cong \varepsilon) \), where \( \Xi_0 \) is an isomorphism (arbitrary, but fixed).

The homomorphism defined in (5) \( \phi : Imm_{fr}^{(1)}(0, 1) \to Imm_{sf}^{(1)}(0, 1) \) is surjective, and \( \phi(a_0) \neq 0 \). Statement \( \Xi \) implies that \( \phi(a) = \phi(-a) \); hence \( Imm_{sf}^{(1)}(0, 1) \cong \mathbb{Z}/2. \)

3.2 Dimension 1

Example 8. The homomorphism (5)

\[ \mathbb{Z}/2 \cong \Pi_1 \cong Imm_{fr}^{(1)}(1, 1) \xrightarrow{\phi} Imm_{sf}^{(1)}(1, 1) \cong \Pi_2(\mathbb{RP}^\infty) \cong \mathbb{Z}/2 \]

is an isomorphism.

Argument 1: Each compact 1-manifold is a union of circles. Let an element \( a_1 \in Imm_{sf}^{(1)}(1, 1) \) be represented by the triple \( (F : S^1 \to \mathbb{R}^2, \kappa : S^1 \to \mathbb{RP}^\infty, \Xi : \nu(F) \cong \kappa^*(\gamma)) \). Since the bundle \( \nu(F) \) is trivial, the map \( \kappa \) is homotopic to a constant map. Thus \( a_1 \in \text{Im} \phi \), hence \( \phi \) is an isomorphism.

Argument 2: Consider the commutative diagram

\[ \begin{array}{ccc}
\mathbb{Z}/2 &=& Imm_{fr}^{(1)}(1, 1) \\
&\xrightarrow{\phi} & Imm_{sf}^{(1)}(1, 1) = \mathbb{Z}/2 \\
&\xleftarrow{q_{fr}} & \mathbb{Z}/2 \\
&\xrightarrow{q_{sf}} & \mathbb{Z}/2
\end{array} \]
The homomorphisms $q^{fr}$, $q^{sf}$ assign to a given general position immersion a number of its double points mod 2 [7, Proposition 2.2]. Investigation of the “figure eight” immersion shows that both $q^{fr}$ and $q^{sf}$ are epimorphisms, hence, are isomorphisms. Hence $\phi$ is also an isomorphism.

Let us introduce a notation for the two next examples. For a line bundle $\zeta$ over a closed 1-dimensional manifold $M = S^1 \sqcup \ldots \sqcup S^1$ we define a modulo 2 residue

$$s(\zeta) = w_1(\zeta|_{S^1}) + \ldots + w_1(\zeta|_{S^1}) \in \mathbb{Z}/2.$$ 

Recall that elements of the group $\text{Imm}^{fr(1) \times sf(0)}(1, 1)$ are represented by triples of the form $(f, \kappa, \Xi)$, where $f : M \to \mathbb{R}^2$ is an immersion of a closed 1-manifold, $\kappa : M \to \mathbb{RP}^\infty$ is a continuous map, and $\Xi : \nu(f) \cong \kappa^*(1\varepsilon \oplus 0\gamma)$ is an isomorphism.

**Example 9.** There is an isomorphism

$$\text{Imm}^{fr(1) \times sf(0)}(1, 1) \to \text{Imm}^{fr(1)}(1, 1) \oplus \mathbb{Z}/2,$$

it can be defined by the formula

$$(f, \kappa, \Xi : \nu(f) \cong \kappa^*(1\varepsilon \oplus 0\gamma)) \mapsto ((f, \Psi : \nu(f) \cong \kappa^*(1\varepsilon) \cong \varepsilon), s(\kappa^*(\gamma)))$$

(Compare with Example 1.)

**Example 10.** The transfer homomorphism

$$\text{Imm}^{fr(1) \times sf(0)}(1, 1) \rightarrow \text{Imm}^{fr(1) \times sf(0)}(1, 1) \cong \text{Imm}^{fr(1)}(1, 1) \cong \mathbb{Z}/2$$

is computed by the map

$$(f, \kappa) \mapsto s(\kappa^*(\gamma));$$

in particular, it is surjective.

**Example 11.** The diagram

$$\begin{array}{ccc}
\text{Imm}^{fr(1) \times sf(0)}(1, 1) & \rightarrow & \text{Imm}^{fr(1)}(1, 1) \\
\cong & \Downarrow & \cong \\
\text{Imm}^{fr(1) \times sf(4k)}(1, 1 + 4k) & \rightarrow & \text{Imm}^{fr(1+4k)}(1, 1 + 4k) \\
\cong & \Downarrow & \cong \\
\text{Imm}^{sf(4k)}(1, 4k) & \rightarrow & \text{Imm}^{fr(4k)}(1, 4k)
\end{array}$$

commutes (here all horizontal arrows are transfer homomorphisms [8], right vertical arrows are suspension isomorphisms [4], left vertical arrows are isomorphisms from Corollary [2]).

Thus Example 10 implies surjectivity of the transfer homomorphism $\text{Imm}^{sf(4k)}(1, 4k) \to \text{Imm}^{fr(4k)}(1, 4k).$
3.3 Dimension 2

Investigation of both immersion cobordism group of $n$-dimensional manifolds in $\mathbb{R}^{n+k}$ (without additional structures) and its oriented variant comes back to papers [24], [30], [27]. Here we denote these groups by $\text{Imm}^O(n, k)$ and $\text{Imm}^{SO}(n, k)$ correspondingly. The diagram

$$
\Pi_n \cong \text{Imm}^{fr(k)}(n, k) \longrightarrow \text{Imm}^{sf(k)}(n, k) \cong \Pi_{n+k}(\mathbb{R}P^\infty)
$$

commutes.

For $k = 1$ vertical arrows are isomorphisms (see Statement 6).

In particular case, for $k = 1$ and $n = 2$, results of [22], [12] and Statement 8 imply the following. Here Arf denotes the isomorphism given by virtue of the Arf-invariant [22]; and Brown stands for its generalization [6] (see also [11], [21]).

**Statement 9.** 1) The homomorphism

$$
\mathbb{Z}/2 \cong \text{Imm}^{fr(1)}(2, 1) \xrightarrow{\text{Arf}} \text{Imm}^{sf(1)}(2, 1) \cong \mathbb{Z}/8
$$

has zero kernel.

2) The Boy surface $B : \mathbb{H}^2 \hookrightarrow \mathbb{R}^3$ (with arbitrary skew-framing) serves as a generator of the group $\text{Imm}^{sf(1)}(2, 1) \cong \mathbb{Z}/8$. The same holds true for the mirror image $\bar{B}$ of the Boy surface.

3) Each immersion $\mathbb{H}^2 \hookrightarrow \mathbb{R}^3$ is regularly homotopic either to the Boy surface or to its mirror image.

3.4 Dimension 3

In dimension 3, we have already considered Examples 2, 3.

**Example 12.** From [8, Remark 1] it follows that there exists a generic immersion $F : S^3 \hookrightarrow \mathbb{R}^4$ such that the pair $(F, \Xi_F)$, where the isomorphism $\Xi_F : \nu(F) \cong \varepsilon$ is defined via external normal vector, represents a generator of the group $\text{Imm}^{fr(1)}(3, 1) \cong \Pi_3 \cong \mathbb{Z}/24$.

**Example 13.** The main theorem of the paper [8] implies that the homomorphism

$$
\text{Imm}^{fr(1)}(3, 1) \xrightarrow{\text{Arf}} \text{Imm}^{sf(1)}(3, 1) \cong \Pi_4(\mathbb{H}P^\infty) \cong \mathbb{Z}/2
$$

is surjective. Thus the triple $(F, c = \text{const}, \Xi_F : \nu(F) \cong c^*(\gamma))$ represents a generator of the group $\text{Imm}^{sf(1)}(3, 1)$. 

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Example 14. The triple \((F, c, \varXi)\) considered as an element of the group \(Imm_{sf}^{f(1), \mathbb{R}P^3}(3, 1)\) differs from zero, because the natural homomorphism \(Imm_{sf}^{f(1), \mathbb{R}P^3}(3, 1) \to Imm_{sf}^{f(1)}(3, 1)\) (see formula (3)) is surjective.

By Corollary [1] the triple \((F, c, \varXi)\) considered as an element of the group \(Imm_{sf}^{f(1), \mathbb{R}P^3}(3, 1)\) has order 2.

4 Twisting and untwisting of skew-framed immersions with control in \(\mathbb{R}P^3\)

Let \((f : M^n \leftrightarrow \mathbb{R}^{n+k}, \kappa : M^n \to \mathbb{R}P^3, \varXi : \nu(f) \cong \kappa^*(k\gamma))\) be a \((k\gamma, \mathbb{R}P^3)\)-immersion.

We will define two operations: twisting, that is, transforming of a \((k\gamma \oplus 4\varepsilon, \mathbb{R}P^3)\)-immersion of an \(n\)-manifold into a \(((k + 4)\gamma, \mathbb{R}P^3)\)-immersion of the same manifold, and untwisting, that is, transforming of a \(((k + 4)\gamma, \mathbb{R}P^3)\)-immersion of an \(n\)-manifold into a \((k\gamma \oplus 4\varepsilon, \mathbb{R}P^3)\)-immersion of the same manifold.

Points of the sphere \(S^3\) can be interpreted as quaternions of unit norm, in particular, they can be multiplied. According to notation in the introduction, points of the total space of \(4\gamma\) over \(\mathbb{R}P^3\) are written as arrays of the form \([x; \lambda_1, \lambda_2, \lambda_3, \lambda_4]\), where \(x \in S^3, \lambda_k \in \mathbb{R}\); such array is a pair of identified arrays
\[
(x; \lambda_1, \lambda_2, \lambda_3, \lambda_4) \sim (-x; -\lambda_1, -\lambda_2, -\lambda_3, -\lambda_4).
\]

It will be convenient to identify \(\mathbb{R}^4\) with quaternions; therefore to replace a 4-tuple of numbers \(\lambda_1, \ldots, \lambda_4\) by a quaternion \(w\), and to write points of the total space of \(4\gamma\) over \(\mathbb{R}P^3\) as arrays \([x; w]\), where \(x \in S^3, w \in \mathbb{H}\).

Using this notation, define an isomorphism \(I_{\mathbb{R}P^3} : 4\varepsilon \cong 4\gamma\) over \(\mathbb{R}P^3\) by the formula
\[
([x; \lambda_1, \lambda_2, \lambda_3, \lambda_4]) \mapsto [x; (\lambda_1 + \lambda_2i + \lambda_3j + \lambda_4k) \cdot x].
\]

4.1 Twisting procedure \(\mathbb{T}\)

Suppose we are given a \((k\gamma + 4\varepsilon, \mathbb{R}P^3)\)-immersion of an \(n\)-dimensional manifold:
\[
(f : M^n \leftrightarrow \mathbb{R}^{n+k+4}, \kappa : M \to \mathbb{R}P^3, \varXi : \nu(f) \cong \kappa^*(k\gamma \oplus 4\varepsilon)).
\]

Denote by \(\varXi^T\) the composite isomorphism
\[
\nu(f) \cong \varXi^T \cong \kappa^*(k\gamma \oplus 4\varepsilon) \cong \kappa^*(k\gamma) \oplus \kappa^*(4\varepsilon) \cong \kappa^*(k\gamma) \oplus \kappa^*(4\gamma) = \kappa^*((k+4)\gamma).
\]
The triple
\((f : M^n \hookrightarrow \mathbb{R}^{n+k+4}, \kappa : M \rightarrow \mathbb{RP}^3, \Xi T : \nu(f) \cong \kappa^*((k + 4)\gamma))\)

is a \(((k + 4)\gamma, \mathbb{RP}^3)\)-immersion.

Define the twisting \(T\) by the formula

\[ T(f, \kappa, \Xi) = (f, \kappa, \Xi^T). \]

### 4.2 Untwisting procedure \(U\)

Now let a \(((k + 4)\gamma, \mathbb{RP}^3)\)-immersion of an \(n\)-manifold be given:

\[(f : M^n \hookrightarrow \mathbb{R}^{n+k+4}, \kappa : M \rightarrow \mathbb{RP}^3, \Xi : \nu(f) \cong \kappa^*((k + 4)\gamma)).\]

Define an isomorphism \(\Xi^U\) as the composition

\[ \nu(f) \cong \kappa^*((k+4)\gamma) = \kappa^*(k\gamma) \oplus \kappa^*(4\gamma) \cong \kappa^*(k\gamma) \oplus \kappa^*(4\xi) = \kappa^*(k\gamma \oplus 4\xi). \]

The triple

\[(f : M^n \hookrightarrow \mathbb{R}^{n+k+4}, \kappa : M \rightarrow \mathbb{RP}^3, U(\Xi) : \nu(f) \cong \kappa^*(k\gamma + 4\xi))\]

is a \((k\gamma + 4\xi, \mathbb{RP}^3)\)-immersion.

Define the untwisting \(U\) by the formula

\[ U(f, \kappa, \Xi) = (f, \kappa, \Xi^U). \]

The following two statements are evident:

**Statement 10.** For each \((k\gamma + 4\xi, \mathbb{RP}^3)\)-immersion \((f : M^n \hookrightarrow \mathbb{R}^{n+k+4}, \kappa : M \rightarrow \mathbb{RP}^3, \Xi : \nu(f) \cong \kappa^*(k\gamma + 4\xi))\) we have

\[ U \circ T(f, \kappa, \Xi) = (f, \kappa, \Xi). \]

For each \(((k + 4)\gamma, \mathbb{RP}^3)\)-immersion \((f : M^n \hookrightarrow \mathbb{R}^{n+k+4}, \kappa : M \rightarrow \mathbb{RP}^3, \Xi : \nu(f) \cong \kappa^*((k + 4)\gamma))\) we have

\[ T \circ U(f, \kappa, \Xi) = (f, \kappa, \Xi). \]

**Statement 11.** Procedures \(T\) and \(U\) induce mutually inverse isomorphisms

\[ \text{Imm}_{sf}^{st(k) \times fr(4), \mathbb{RP}^3}(n, k + 4) \xrightarrow{T, U} \text{Imm}_{sf}^{st(k + 4), \mathbb{RP}^3}(n, k + 4). \]
We now obtain from (4), (3) the following known interpretation of the James isomorphism [16]. It was given and used by Mahowald [20]. We used it repeatedly in [1]:

**Corollary 2.** For \( n = 0, 1, 2 \) there are isomorphisms

\[
\text{Imm}^{sf(k)}(n, k) \cong \text{Imm}^{sf(k+4)}(n, k + 4).
\]

There is no analogue of Corollary 2 for three-dimensional case. Moreover, twisting of cobordant elements can be non-cobordant.

**Example 15.** Take the 4-suspension of the triple \((E, c, \Phi)\) described in Example 2. This triple \(E^4(E, c, \Phi)\) represents zero element of the group \(\text{Imm}^{sf(1)} \times \text{fr}(4)(3, 5)\).

Since \(c\) is a constant map, the twisting does not change the triple \(E^4(E, c, \Phi)\). Thus

\[
T(E^4(E, c, \Phi)) = 0 \quad \text{in the group} \quad \text{Imm}^{sf(5)}(3, 5).
\]

**Example 16.** Take the 4-suspension of the triple \((F, c, \Xi_F)\) described in Example 13. This triple \(E^4(F, c, \Xi_F)\) generates the group \(\text{Imm}^{sf(1)} \times \text{fr}(4)(3, 5) \cong \text{Imm}^{sf(1)}(3, 1) \cong \mathbb{Z}/2\). Since \(c\) is a constant map, the twisting does not change this triple.

Further, the result of twisting

\[
E^4(F, c, \Xi_F) \neq 0 \quad \text{in the group} \quad \text{Imm}^{sf(5)}(3, 5).
\]

In fact, the triple \(E^4(F, c, \Xi_F)\) is the image of a generator \(E^4(F, \Xi_F)\) of the group \(\text{Imm}^{fr(5)}(3, 5) \cong \mathbb{Z}/24\) under the homomorphism \(\text{Imm}^{fr(5)}(3, 5) \rightarrow \text{Imm}^{sf(5)}(3, 5)\), see [5]. This homomorphism is non-zero [1, Theorem 1].

[Below we prove that \(\text{Imm}^{sf(5)}(3, 5) \cong \mathbb{Z}/2\), see Theorem 1.]

**Example 17.** In Example 3 we constructed the triple \((E, \rho, \Xi_E)\). Treated as an element of the group \(\text{Imm}^{sf(1)}(3, 1)\), it equals zero. Consider the image of this triple under the composition of homomorphisms

\[
\text{Imm}^{sf(1), \text{RP}^3}(3, 1) \xrightarrow{E^4} \text{Imm}^{sf(1)} \times \text{fr}(4), \text{RP}^3(3, 5) \xrightarrow{T} \cong \text{Imm}^{sf(5), \text{RP}^3}(3, 5) \rightarrow \text{Imm}^{sf(5)}(3, 5).
\]

The embedding \(E\) remains unchanged. Only the structure of normal bundle changes. It is easy to verify that we will obtain, up to a sign, the element \(t_1(\nu)\), where \(\nu\) is a generator of the group \(\text{Imm}^{fr(5)}(3, 5) \cong \mathbb{Z}/24\) (see [1] Statements 4,5), and the homomorphism \(t_1 : \text{Imm}^{fr(5)}(3, 5) \rightarrow \text{Imm}^{sf(5)}(3, 5)\) is defined by the formula (5). This homomorphism is non-zero [1, Theorem 1].
5 Computation of the group \( \text{Imm}^{sf(5)}(3, 5) \)

**Theorem 1.** There are isomorphisms

\[
\text{Imm}^{sf(5)}(3, 5) \cong \Pi_8(\mathbb{RP}^5) \cong \pi_8(\mathbb{RP}^5) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2.
\]

**Proof.** First isomorphisms in each row are well-known \([30]\). Second arrows in each row are isomorphisms by dimensional reasons, see Statement \([1]\).

Let us prove that \( \text{Imm}^{sf(1), \mathbb{R}P^3}(3, 1) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2 \). Therefore the isomorphic group \( \text{Imm}^{sf(5), \mathbb{R}P^3}(3, 5) \) will also be calculated.

Recall the skew-framed immersions \((\mathbb{F}, c, \Xi_\mathbb{F})\) and \((\mathbb{E}, \rho, \Xi_\mathbb{E})\) introduced in Examples \([13]\) and \([2]\).

**Lemma 2.** For arbitrary skew-framed immersion of a 3-dimensional manifold \((f_M : M^3 \hookrightarrow \mathbb{R}^4, \kappa_M : M \rightarrow \mathbb{R}P^3, \Xi_M : \nu(f_M) \cong \kappa_M^*(\gamma))\) there exists a skew-framed cobordism \((F_W : W \hookrightarrow \mathbb{R}^4 \times I, \kappa_W : W \rightarrow \mathbb{R}P^3, \Xi_W : \nu(F_W) \cong \kappa_W^*(\gamma))\) between \((f_M, \kappa_M, \Xi_M)\) and a skew-framed immersion of the form

\[
\delta \cdot (\mathbb{F}, c, \Xi_\mathbb{F}) \sqcup m \cdot (\mathbb{E}, \rho, \Xi_\mathbb{E}),
\]

where \(\delta \in \{0; 1\}\) and \(m \geq 0\).

Let us underline that here we strive for obtaining the inclusion \(\kappa_W(W) \subset \mathbb{R}P^3\).

**Proof of Lemma 2**

Since \( \text{Imm}^{sf(1)}(3, 1) \cong \mathbb{Z}/2 \), there exists a skew-framed cobordism \((F_V : V \hookrightarrow \mathbb{R}^4 \times I, \kappa_V : V \rightarrow \mathbb{R}P^4, \Xi_V : \nu(F_V) \cong \kappa_V^*(\gamma))\) between \((f_M, \kappa_M, \Xi_M)\) and one of the two skew-framed immersions: either \((\mathbb{F}, c, \Xi_\mathbb{F})\) or \((\mathbb{E}, c, \Phi)\).

Let us underline that the image \(\kappa_V(V) \subset \mathbb{R}P^4\) in general can not be moved by homotopy into \(\mathbb{R}P^3 \subset \mathbb{R}P^4\).

We change the cobordism \(V\); the new cobordism \(W\) will satisfy the inclusion \(\kappa_W(W) \subset \mathbb{R}P^3\); but doing this, we in general change the boundary: \(\partial W \neq \partial V\).

We assume that the map \(\kappa_V\) is smooth. Let \(pt \in \mathbb{R}P^4\) be a point choosed so that it is a regular value for \(\kappa_V\) and \(pt \notin \kappa_V(\partial V)\). Then \(\kappa_V^{-1}(pt) = \{p_1, \ldots, p_m\}\). There exists an open disk neighborhood \(O(pt)\) of \(pt \in \mathbb{R}P^4\) such that the preimages \(O(p_k) = \kappa_V^{-1}(O(pt))\) are pairwise disjoint.

**Remark 4.** Arguments of Example \([17]\) also show that the triple \((\mathbb{E}, \rho, \Xi_\mathbb{E})\) represents a non-zero element in the group \( \text{Imm}^{sf(1), \mathbb{R}P^3}(3, 1) \).
disk neighborhoods of the points \( p_k \) for \( k = 1, \ldots, m \), and that \( \kappa_V \) maps each of them diffeomorphically onto \( O(pt) \).

Replace the cobordism \( V \) by \( W = V - \bigcup_{k=1}^m O(p_k) \). Put \( f_W = f_V|_W \), \( \kappa_W = \kappa_V|_W \), \( \Xi_W = \Xi_V|_W \).

Homotopying the map \( \kappa_W : W \to \mathbb{R}^4 - O(pt) \), we can assume that the inclusion \( \kappa_W(W) \subset \mathbb{R}^3 \subset \mathbb{R}^4 - O(pt) \) holds. (Moreover, the homotopy can be chosen so that the map \( \kappa_W \) is unchanged nearby \( M \subset \partial W \).) Reference to Example 3 and Remark 3 finish the proof.

By Lemma 2 the group \( \text{Imm}^{sf(1), \mathbb{R}^3}(3,1) \) is generated by two elements: \((F, c, \Xi)\) and \((E, \rho, \Xi)\).

Each of them is non-zero (see Example 14, Remark 4).

The element \((F, c, \Xi)\) has order 2 (Corollary 1). Using Statement 8 one can show that the triple \((E, \rho, \Xi)\) also has order 2 in the group \( \text{Imm}^{sf(1), \mathbb{R}^3}(3,1) \).

Let us show that the triples \((E, \rho, \Xi)\) and \((F, c, \Xi)\) represent different elements in the group \( \text{Imm}^{sf(1), \mathbb{R}^3}(3,1) \). For this purpose, consider the epimorphism \( \text{Imm}^{sf(1), \mathbb{R}^3}(3,1) \to \text{Imm}^{sf(1)}(1,3) \cong \mathbb{Z}/2 \).

According to Example 3 the first triple maps to zero. By Example 10 the second triple maps to non-zero element. Hence they are not cobordant in the group \( \text{Imm}^{sf(1), \mathbb{R}^3}(3,1) \).

Finally, this group is abelian. So, we have proved that \( \text{Imm}^{sf(1), \mathbb{R}^3}(3,1) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2 \). The first part of Theorem 1 is proved.

Let us prove that \( \text{Imm}^{sf(5)}(3,5) \cong \mathbb{Z}/2 \). By Theorem 1, the element \( t_1(\nu) \) of this group is non-zero; here \( \nu \in \text{Imm}^{fr(5)}(3,5) \cong \Pi_3 \cong \mathbb{Z}/24 \) is a generator, and \( t_1 : \text{Imm}^{fr(5)}(3,5) \to \text{Imm}^{sf(5)}(3,5) \) is the homomorphism defined by the formula 5.

Consider the epimorphism

\[
\text{Imm}^{sf(1), \mathbb{R}^3}(3,1) \cong \text{Imm}^{sf(5), \mathbb{R}^3}(3,5) \to \text{Imm}^{sf(5)}(3,5).
\]

By Examples 16 17 this composition maps the generators \((F, c, \Xi)\) and \((E, \rho, \Xi)\) of the group \( \text{Imm}^{sf(1), \mathbb{R}^3}(3,1) \) into \( t_1(\nu) \) up to a sign. Hence the group \( \text{Imm}^{sf(5)}(3,5) \) is cyclic, and \( t_1(\nu) \) is its generator. Since \( \text{Imm}^{sf(1), \mathbb{R}^3}(3,1) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2 \), we obtain \( \text{Imm}^{sf(5)}(3,5) \cong \mathbb{Z}/2 \).

The Theorem is proved.

6 Transformation Theorem

Take a 5-skew-framed immersion of a 3-manifold \((f : M^3 \to \mathbb{R}^8, \kappa : M \to \mathbb{R}^3, \Xi : \nu(f) \cong \kappa^*(5\gamma))\).
Let $\mathcal{A}: 5\gamma \to 5\gamma$ be an isomorphism over $\mathbb{RP}^3$. Denote the isomorphism 

$$\nu(f) \cong \kappa^*(5\gamma) \cong \kappa^*(5\gamma)$$

by $\Xi^A$.

**Theorem 2.** Suppose $\mathcal{A}: 5\gamma \to 5\gamma$ is an isomorphism over $\mathbb{RP}^3$ that preserves orientation in each fiber. Then for each 5-skew-framed immersion of a 3-manifold $(f : M^3 \hookrightarrow \mathbb{R}^8, \kappa : M \to \mathbb{RP}^3, \Xi : \nu(f) \cong \kappa^*(5\gamma))$ the triples $(f, \kappa, \Xi)$ and $(f, \kappa, \Xi)$ are cobordant in the group $\text{Imm}^{5f(5)}(3, 5)$.

**Proof.** Lemma 2 implies that there exists a cobordism $(F_W : W \hookrightarrow \mathbb{R}^8 \times I, \kappa_W : W \to \mathbb{RP}^3, \Xi_W : \nu(F_W) \cong \kappa_W^*(5\gamma))$ between the original triple $(f, \kappa, \Xi)$ and a triple of the form 

$$\delta \cdot \mathbb{T}(E^4(F, c, \Xi_F)) \sqcup m \cdot \mathbb{T}(E^4(E, \rho, \Xi_E)), \quad \text{where} \quad \delta \in \{0; 1\}, \ m \geq 0.$$ 

We have $\mathbb{T}(E^4(F, c, \Xi_F)) = E^4(F, c, \Xi_F)$ (Example 16). For convenience denote the triple $\mathbb{T}(E^4(E, \rho, \Xi_E))$ by $(E^4E, \rho, \Psi_E)$. Thus the triple $(F_W, \kappa_W, \Xi_W)$ gives a cobordism between $(f, \kappa, \Xi)$ and a triple of the form 

$$\delta \cdot E^4(F, c, \Xi_F) \sqcup m \cdot (E^4E, \rho, \Psi_E), \quad \text{where} \quad \delta \in \{0; 1\}, \ m \geq 0.$$ 

Since $\kappa_W(W) \subset \mathbb{RP}^3$, the substitution $\mathcal{A}$ can be done over the whole cobordism $W$. In other words, the triple $(f_W, \kappa_W, \Xi_W^A)$ is a cobordism between $(f, \kappa, \Xi^A)$ and an immersion of the form 

$$\delta \cdot (E^4F, c, (E^4\Xi_F)^A) \sqcup m \cdot (E^4E, \rho, \Psi_E^A), \quad \text{where} \quad \delta \in \{0; 1\}, \ m \geq 0.$$ 

Idea of proof: show that the triples $(E^4F, c, (E^4\Xi_F)^A)$ and $(E^4F, c, E^4\Xi_F)$ are cobordant, and also the triples $(E^4E, \rho, \Psi_E^A)$ and $(E^4E, \rho, \Psi_E)$ are cobordant. Therefore, gluing two cobordisms together, we obtain a cobordism between the triples $(f, \kappa, \Xi^A)$ and $(f, \kappa, \Xi)$.

The proof for the triple $(E^4F, c, E^4\Xi_F)$ is simple: we can assume that the isomorphism $\mathcal{A}$ is identical over the point $c$ which is the image of the characteristic map, hence 

$$(E^4F, c, (E^4\Xi_F)^A) = (E^4F, c, E^4\Xi_F) = E^4(F, c, \Xi_F).$$

The triple $(E^4E, \rho, \Psi_E^A)$ needs a more detailed investigation. This will be done in several steps.

**Step 1.** Each automorphism $\mathcal{A} : 5\gamma \to 5\gamma$ over $\mathbb{RP}^3$ that preserves orientation in fibers is fiberwise homotopic to an automorphism of the form...
$\mathcal{B} \oplus \text{id}_\gamma$, where $\text{id} : \gamma \to \gamma$ is the identity, and $\mathcal{B} : 4\gamma \to 4\gamma$ preserves orientation in fibers.

In fact, it suffices to show that each map $\mu : \mathbb{R}P^3 \to SO(5)$ can be homotoped to a map whose image lies in $SO(4)$. For this purpose recall that there is a bundle $\pi : SO(5) \to S^4$ with fiber $SO(4)$. The composition $\pi \circ \mu : \mathbb{R}P^3 \to S^4$ is homotopic to a cell map, but the 3-skeleton of $S^4$ is trivial, so $\pi \circ \mu : \mathbb{R}P^3 \to S^4$ is homotopic to zero. This homotopy lifts to the total space $SO(5)$ and gives a homotopy of the original map $\mu$ to a map whose image is contained in a fiber, that is, in $SO(4)$.

So, we assume that $\mathcal{A} = \mathcal{B} \oplus \text{id}_\gamma$, where $\mathcal{B} : 4\gamma \to 4\gamma$ preserves orientation in fibers.

**Step 2.** Construct the isomorphism $\Theta$ as a composition

$$\nu(E^4E) \cong \rho^*(5\gamma) \cong 5\varepsilon,$$

and the isomorphism $\Theta^4$ as a composition

$$\nu(E^4E) \cong \rho^*(5\gamma) \cong 5\varepsilon.$$

2.1. It follows from Statements 4-8 that the image of the pair $(E^4E, \Theta)$ under the homomorphism $t_1 : \text{Imm}^{fr}(5)(3, 5) \to \text{Imm}^{sf}(5)(3, 5)$ (see formula (5)) is cobordant to the triple $(E^4E, \rho, \Psi_E)$, and the image of the pair $(E^4E, \Theta^4)$ to the triple $(E^4E, \rho, \Psi_E^\Theta)$.

2.2. In the framings $\Theta, \Theta^4$ the subbundle generated by vectors from second to fifth is transformed by twisting procedure. It is convenient to assume instead that the subbundle generated by vectors from first to fourth is transformed. More precisely, define the isomorphism $\Theta_1$ as the composition

$$\nu(E^4E) \cong \nu(E) \oplus 4\varepsilon = 5\varepsilon = 4\varepsilon \oplus \varepsilon \cong 4\varepsilon \oplus \varepsilon,$$

and the isomorphism $\Theta_1^4$ as the composition

$$\nu(E^4E) \cong \nu(E) \oplus 4\varepsilon = 5\varepsilon = 4\varepsilon \oplus \varepsilon \cong 4\varepsilon \oplus \varepsilon.$$

Here $I_{S^3} : 4\varepsilon \to 4\varepsilon$ is the automorphism of the trivial bundle over $S^3$ defined by the formula

$$(x; a_1, a_2, a_3, a_4) \mapsto (x; (a_1 + a_2i + a_3j + a_4k) \cdot x),$$

compare with (12).
Then the pairs \((E^4E, \Theta)\) mS \((E^4E, \Theta_1)\) are cobordant up to a sign, and also are the pairs \((E^4E, \Theta^4)\) and \((E^4E, \Theta_1^4)\).

So, for the homomorphism \(t_1 : \text{Imm}^{fr(5)}(3, 5) \rightarrow \text{Imm}^{sf(5)}(3, 5)\) we have

\[
t_1(E^4E, \Theta_1) \quad \text{is cobordant to} \quad (E^4E, \rho, \Psi_E),
\]

\[
t_1(E^4E, \Theta_1^4) \quad \text{is cobordant to} \quad (E^4E, \rho, \Psi_E^4).
\]

2.3. \(\Theta_1^4 = \tilde{A} \circ \Theta_1\), where \(\tilde{A}\) is the composition of the isomorphisms

\[
5\varepsilon \cong \rho^*(5\gamma) \cong \rho^*(4\gamma) \cong 5\varepsilon
\]

over \(S^3\).

2.4. \(\tilde{A}\) decomposes into a direct sum \(\tilde{B} \oplus \text{id}_\varepsilon\), where \(\tilde{B} : 4\varepsilon \rightarrow 4\varepsilon\) is an isomorphism which preserves orientation in each fiber over \(S^3\).

2.5. The isomorphism \(\tilde{B} : 4\varepsilon \rightarrow 4\varepsilon\) possesses a symmetry property: for the corresponding map \(\mu_{\tilde{B}} : S^3 \rightarrow SO(4)\) we have

\[
\mu_{\tilde{B}}(x) = \mu_{\tilde{B}}(-x), \quad x \in S^3.
\]

Step 3. Desuspension. Instead of the embedding \(E^4E : S^3 \rightarrow \mathbb{R}^8\) consider the standard embedding \(E^3E : S^3 \rightarrow \mathbb{R}^7\). We construct two framings of it.

Let us return to item 2.2 and denote by \(\Gamma\) the composite isomorphism

\[
\nu(E^3E) \cong \nu(\mathbb{E}) \oplus 3\varepsilon = 4\varepsilon \cong 4\varepsilon,
\]

and by \(\Gamma^\tilde{B}\) the composition \(\tilde{B} \circ \Gamma : \nu(E^3E) \cong 4\varepsilon\).

Then after suspension \((4)\)

\[
E : \text{Imm}^{fr(4)}(3, 4) \cong \text{Imm}^{fr(5)}(3, 5)
\]

we obtain

\[
(E^3E, \Gamma) \mapsto (E^4E, \Theta_1), \quad (E^3E, \Gamma^\tilde{B}) \mapsto (E^4E, \Theta_1^4).
\]

Step 4. Both stable Hopf invariant \(h^{st} : \Pi_3 \rightarrow \mathbb{Z}/2\) and homomorphism \(t_1\) are surjective, hence there exists an isomorphism \(h^{sf}\) that makes the diagram

\[
\begin{array}{ccc}
\mathbb{Z}/24 & \cong & \mathbb{Z}/2 \\
\Pi_3 & \cong & \text{Imm}^{fr(5)}(3, 5) \xrightarrow{t_1} \text{Imm}^{sf(5)}(3, 5) \\
& & \cong \mathbb{Z}/2
\end{array}
\]
commutative. So, for the elements \( a, b \in Imm^{fr(4)}(3, 4) \) we have

\[
t_1(a) = t_1(b) \iff h_1^{st}(a) = h_1^{st}(b).
\]

**Step 5.** The diagram

\[
\begin{array}{ccc}
Imm^{fr(4)}(3, 4) & \xrightarrow{E} & Imm^{fr(5)}(3, 5) \\
h \downarrow & & h^{st} \downarrow \\
\mathbb{Z} & \xrightarrow{\text{mod } 2} & \mathbb{Z}/2
\end{array}
\]

commutes.

Therefore it remains to show that

\[
h(E^3\mathbb{E}, \Gamma) \equiv h(E^3\mathbb{E}, \Gamma^B) \mod 2. \tag{14}
\]

**Step 6.** It is known that \((E^3\mathbb{E}, \Gamma)\) generates the group \(Imm^{fr(4)}(3, 4)\), and its Hopf invariant equals 1.

Let us prove that \(h(E^3\mathbb{E}, \Gamma^B)\) is odd.

The difference \(h(E^3\mathbb{E}, \Gamma) - h(E^3\mathbb{E}, \Gamma^B)\) coincides, up to a sign, with degree of the map defined by the first basis vector of the framing \(\Gamma^B\), i.e. of the map \(\Delta_B : S^3 \to S^3\), where the image sphere is the fiber of spherization of the normal bundle defined by \(\Gamma\) \cite{23}. From the symmetry property of the map \(\mu_B\) (item 2.5) follows the symmetry property of the map \(\Delta_B\), i.e. \(\Delta_B(x) = \Delta_B(-x)\) for all \(x \in S^3\). The symmetric map \(\Delta_B : S^3 \to S^3\) factorizes through the projection map \(\rho : S^3 \to \mathbb{RP}^3\). Degree of this projection equals 2, hence \(\deg \Delta_B\) is even.

Thus the congruence \(14\) is proved.

**Step 7.** The above arguments show that the triples \((E^4\mathbb{E}, \rho, \Psi_{\mathbb{E}})\) and \((E^4\mathbb{E}, \rho, \Psi_{\mathbb{E}}^A)\) are cobordant in \(Imm^{sf(5)}(3, 5)\). As already said, this enables us to glue together two cobordisms: \((F_W, \kappa_W, \Xi_W^A)\) and \((F_W, \kappa_W, \Xi_W)\). As a result, we obtain a cobordism of the triples \((f, \kappa, \Xi)\) and \((f, \kappa, \Xi^A)\). Theorem is proved.

\[\square\]

### 7 Product and Transfer Theorem

The invariant trivialization (constructed with the help of Cayley octonions) \(T\mathbb{RP}^7 \cong 7\varepsilon\) induces by the Hirsch theorem \cite{13}, Theorem 6.3 an immersion \(f : \mathbb{RP}^7 \rightarrow \mathbb{R}^8\). Fix an isomorphism \(\Xi : \nu(f) \cong \varepsilon\). Also, take the identity map \(\kappa : \mathbb{RP}^7 \to \mathbb{RP}^7 \subset \mathbb{R}^{\infty}\). The triple \((f, \kappa, \Xi)\) represents an element of the group \(Imm^{sf(0) \times fr(1)}(7, 1)\); denote it by \(p\).
Recall that
\[ \text{Imm}^{sf(5) \times fr(1)}(2, 6) \cong \text{Imm}^{sf(1)}(2, 1) \cong \mathbb{Z}/8 \]
and that the Boy surface \( \mathbb{RP}^2 \leftrightarrow \mathbb{R}^3 \) with arbitrary skew-framing can serve as a generator of the last group. In [1] we chose such a generator and denoted it by \( \nu \in \text{Imm}^{sf(1)}(2, 1) \). Its twisting \( T(E^4(\nu)) \) is a generator of the group \( \text{Imm}^{sf(5)}(2, 5) \).

Define the homomorphism
\[ \text{Imm}^{sf(k)}(n, k) \to \text{Imm}^{sf(k+m)}(n-m, m+k) \]  \hspace{1cm} (15)
(for \( m = 1 \) see [2]). Suppose we are given a representative of the first group, that is, a triple \( (f_M : M^n \leftrightarrow \mathbb{R}^{n+k}, \kappa_M : M^n \to \mathbb{RP}^n, \Xi_M : \nu(f) \cong \kappa_M^*(k\gamma)) \). Assuming \( \kappa_M \) to be transversal along \( \mathbb{RP}^{n-m} \subset \mathbb{RP}^n \), put \( N = \kappa_M^{-1}(\mathbb{RP}^{n-m}) \).

It is a \( (n-m) \)-dimensional submanifold of \( M \). Besides, there is a natural isomorphism \( \nu(N, M) \cong \kappa_M^*(\nu(\mathbb{RP}^{n-m}, \mathbb{RP}^n)) \cong \kappa_M^*(m\gamma) \). Let \( f_N : N \subset M \leftrightarrow \mathbb{R}^{n+k} \) be the restriction of the immersion \( f_M ; \kappa_N : N \subset M \to \mathbb{RP}^n \) the restriction of the map \( \kappa_M \); and let the isomorphism \( \Xi_N : \nu(f_N) \cong \kappa_N^*((k+m)\gamma) \) be obtained by the formula
\[ \nu(f_N) \cong \nu(N, M) \oplus \nu(f_M) \cong \kappa_N^*(m\gamma) \oplus \kappa_N^*(k\gamma) = \kappa_N^*((m+k)\gamma). \]
The correspondence
\( (f_M, \kappa_M, \Xi_M) \mapsto (f_N, \kappa_N, \Xi_N) \)
defines the homomorphism (15).

**Statement 12.** The image of the element \( p \) under the homomorphism (15)
\[ \text{Imm}^{sf(0) \times fr(1)}(7, 1) \to \text{Imm}^{sf(5) \times fr(1)}(2, 6) \]
generates the group \( \text{Imm}^{sf(5) \times fr(1)}(2, 6) \cong \mathbb{Z}/8 \).

**Proof.** By definition, the image of the element \( p \) is represented by the triple \( (f_0, \kappa_0, \Xi_0) \), where \( f_0 : M_0 = \kappa_0^{-1}(\mathbb{RP}^2) \subset \mathbb{RP}^7 \leftrightarrow \mathbb{R}^8, \kappa_0 = \kappa|_{M_0} : M_0 \to \mathbb{RP}^\infty, \Xi_0 = \Xi_{M_0} \oplus \kappa_0^*(\nu(\mathbb{RP}^2, \mathbb{RP}^7)) \) : \( \nu(f_0) \cong \varepsilon \oplus 5\kappa_0^*\gamma \).

It is clear that \( M_0 = \kappa_0^{-1}(\mathbb{RP}^2) = \mathbb{RP}^2 \). So, the image of the element \( p \) in the group \( \text{Imm}^{sf(5) \times fr(1)}(2, 6) \) is represented by an immersion of the projective plane. To show that it is cobordant with \( \pm E(T(E^4(\nu))) \), apply the inverse isomorphism \( U \circ E^{-1} : \text{Imm}^{sf(5) \times fr(1)}(2, 6) \cong \text{Imm}^{sf(1)}(2, 1) \). By construction, we are to replace the skew-framing \( \Xi_0 \) by
\[ \Psi : \nu(f_0) \oplus \varepsilon \oplus 5\kappa_0^*\gamma \cong \varepsilon \oplus \kappa_0^*\gamma \oplus 4\varepsilon, \]

\[ \begin{array}{c}
\Psi : \nu(f_0) \\
\oplus \\
\varepsilon \\
\oplus \\
5\kappa_0^*\gamma \\
\cong \\
\varepsilon \\
\oplus \\
\kappa_0^*\gamma \\
\oplus \\
4\varepsilon,
\end{array} \]
and, in accordance with the Hirsch theorem, replace the immersion $f_0 : \mathbb{R}P^2 \hookrightarrow \mathbb{R}^8$ by a regularly homotopic immersion of the form $g : \mathbb{R}P^2 \hookrightarrow \mathbb{R}^3 \subset \mathbb{R}^8$.

By items 3), 2) of Statement \ref{Statement9} the resulting element is cobordant to $\pm \nu$.

To state Theorem \ref{Theorem3}, we introduce elements $c_1, c_2 \in Imm^{sf}(8) (1, 8) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$, they are represented correspondingly by the triples:

$$c_1 = (f_1, \kappa_1, \Xi_1), \quad c_2 = (f_2, \kappa_2, \Xi_2),$$

where $f_1 : S^1 \subset \mathbb{R}^2 \subset \mathbb{R}^9$ is the standard embedding, $\kappa_1 : S^1 \to \mathbb{R}P^\infty$ is homotopically non-trivial, $\Xi_1 : \nu(f_1) \cong \kappa_1^*(8\gamma)$ is an (arbitrary) isomorphism; $f_2 : S^1 \subset \mathbb{R}^2 \subset \mathbb{R}^9$ is the “figure eight” immersion, $\kappa_2 : S^1 \to \mathbb{R}P^\infty$ is homotopically non-trivial, $\Xi_2 : \nu(f_2) \cong \kappa_2^*(8\gamma)$ is an (arbitrary) isomorphism.

By $\eta \in Imm^{fr}(1, 1) \cong \mathbb{Z}/2$ the generator is denoted.

Here symbol $\times$ denotes the product homomorphism, see, for example, (8).

**Theorem 3.** 1) Under the composition of the homomorphisms

$$Imm^{sf}(0) \times^{fr}(1)(7, 1) \otimes Imm^{fr}(1, 1) \xrightarrow{\chi} Imm^{sf}(0) \times^{fr}(2)(8, 2) \xrightarrow{(15)}$$

$$\to Imm^{sf}(5) \times^{fr}(2)(3, 7) \xrightarrow{(E^3)^{-1}} Imm^{sf}(5)(3, 5)$$

the image of the element $p \otimes \eta$ is a generator of the group $Imm^{sf}(5)(3, 5) \cong \mathbb{Z}/2$.

2) Under the composition of the homomorphisms

$$Imm^{sf}(0) \times^{fr}(1)(7, 1) \otimes Imm^{sf}(8)(1, 8) \xrightarrow{\chi} Imm^{sf}(0) \times^{sf}(8)(8, 8) \xrightarrow{(8)}$$

$$\to Imm^{sf}(5) \times^{fr}(8)(8, 8) \xrightarrow{(15)} Imm^{sf}(5)(3, 13) \xrightarrow{(E^8)^{-1}} Imm^{sf}(5)(3, 5)$$

both the image of the element $p \otimes c_1$ and the image of the element $p \otimes c_2$ are equal to the generator of the group $Imm^{sf}(5)(3, 5)$.

**Proof.** Both assertions follow from investigation of the same commutative diagram. It is too large to be placed on a page, and we will consider it in parts; moreover, in this proof groups $Imm$ are denoted shortly by $I$. 

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1) We need to show that going through the diagram down-right

\[ I^{sf}(0) \times fr(1)(7, 1) \otimes I^{fr}(1)(1, 1) \xrightarrow{\text{[15] \times \text{id}}} I^{sf}(5) \times fr(1)(2, 6) \otimes I^{fr}(1)(1, 1) \]

\[ \times \]

\[ I^{sf}(0) \times fr(2)(8, 2) \]

\[ \xrightarrow{\text{[15]}} \]

\[ I^{sf}(5) \times fr(2)(3, 7) \cong I^{sf}(5)(3, 5) \]

the element \( p \otimes \eta \) maps to the generator. Let us go right-down instead. Going right, we obtain (up to a sign) \( E(T(E^4(\nu))) \otimes \eta \), and going down this gives the generator \( E^2(t_1(\nu)) \) [1, Theorem 3].

2) The diagram

\[ I^{sf}(0) \times fr(1)(7, 1) \otimes I^{sf}(8)(1, 8) \xrightarrow{\text{id} \times \text{[5]}} I^{sf}(0) \times fr(8)(1, 8) \]

\[ \times \]

\[ I^{sf}(0) \times sf(8)(8, 8) \]

\[ \xrightarrow{\text{[5]}} \]

\[ I^{sf}(0) \times fr(8)(8, 8) \cong I^{sf}(0) \times fr(2)(8, 2) \]

commutes, and it can be considered as an extension to the right for the diagram of item 1). Hence, in order to prove 2) it suffices to check that the transfer \( Imm^{sf}(8)(1, 8) \rightarrow Imm^{fr}(8)(1, 8) \) maps the elements \( c_1, c_2 \) to the generator of the group \( Imm^{fr}(8)(1, 8) \cong Imm^{fr}(1)(1, 1) \). This is verified directly.

\[ \square \]

8 Sketch of proof of the Theorem on non-immersion of \( \mathbb{R}P^{10} \) into \( \mathbb{R}^{15} \)

In [1], the authors suggested and partly realised a scheme of a geometric proof of the Baum-Browder theorem on non-immersion of \( \mathbb{R}P^{10} \) into \( \mathbb{R}^{15} \). By the Sanderson lemma [4, Lemma (9.7)], it suffices to show that \( T\mathbb{R}P^{15} \) does not allow 10 linearly independent vector fields over \( \mathbb{R}P^{10} \subset \mathbb{R}P^{15} \). A stronger fact is valid: \( T\mathbb{R}P^{15} \) does not allow 9 linearly independent vector fields over \( \mathbb{R}P^{10} \subset \mathbb{R}P^{15} \) [1, Theorem (9.5)].

Koschorke developed singularity approach for the problem of existence of a monomorphism between two arbitrary vector bundles over a manifold [18]. The method prescribes to take a general position morphism \( 9\varepsilon \rightarrow T\mathbb{R}P^{15} \mid_{\mathbb{R}P^{10}} \) over \( \mathbb{R}P^{10} \) and study its singular points — the points where this morphism has non-zero kernel. It turns out that the normal bundle of the singular manifold has special structure; Koschorke defines a special cobordism group and shows that in this group the singular manifold is zero-cobordant if and only if there exists a monomorphism of the given bundles.
In our problem it is inconvenient to take a morphism \( 9 \varepsilon \rightarrow T \mathbb{P}^{15} \) over \( \mathbb{P}^{10} \).

Instead we will prove that there is no monomorphism \( 10 \varepsilon \rightarrow T \mathbb{P}^{15} \oplus \varepsilon \) over \( \mathbb{P}^{10} \). Secondly, we temporarily change the basis space — instead \( \mathbb{P}^{10} \) we will consider \( \mathbb{P}^{15} \) (the possibility to construct a convenient general position morphism arises due to availability of the Hopf bundle \( \mathbb{P}^{15} \rightarrow S^8 \)). The auxiliary problem \( 10 \varepsilon \rightarrow T \mathbb{P}^{15} \oplus \varepsilon \) over \( \mathbb{P}^{15} \) does not satisfy conditions of the Koschorke theorem. Nevertheless, we can consider the singular manifold and its normal bundle. The singularity happens to be represented by an 8-manifold with special structure of normal bundle; this singularity manifold can be interpreted as an element \( x \in Imm^{sf(0)\times fr(8)}(8,8) \), call it an auxiliary obstruction. Returning to the original problem \( 10 \varepsilon \rightarrow T \mathbb{P}^{15} \oplus \varepsilon \) over \( \mathbb{P}^{10} \) corresponds to the homomorphism of the “obstruction” groups (here, inverted commas are used to mention that this word is not fully exact for the problem over \( \mathbb{P}^{15} \)); this homomorphism decomposes as

\[
Imm^{sf(0)\times fr(8)}(8,8) \rightarrow Imm^{sf(5)}(3,5) \hookrightarrow Imm^{sf(5)\times sf(6)}(3,11).
\]

In our next paper we will show that in the group \( Imm^{sf(0)\times fr(8)}(8,8) \) holds the equation \( x = y + 2z + t \), where \( y \) is the image of the product (see formula (16)) \( p \otimes c_1 \in Imm^{sf(0)\times fr(1)}(7,1) \otimes Imm^{sf(8)}(1,8) \) as in item 2) of Theorem 3; \( t \) is an element which admits a clear description. By Theorem 1 the even element \( 2z \) maps to zero. Usage of Theorem 2 will allow us to show that the element \( t \) also maps to zero. By Theorem 3 and [1, Theorem 3] the element \( y \) maps to a non-zero element. Hence, the image of the element \( x \) also differs from zero — but it is exactly the obstruction to existence of 10 linearly independent vector fields in the bundle \( T \mathbb{P}^{15} \oplus \varepsilon \) over \( \mathbb{P}^{10} \).

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