On the convergence of generalized power series solutions of \(q\)-difference equations

Renat Gontsov, Irina Goryuchkina, and Alberto Lastra

Abstract. A sufficient condition for the convergence of a generalized formal power series solution to an algebraic \(q\)-difference equation is provided. The main result leans on a geometric property related to the semi-group of (complex) power exponents of such a series. This property corresponds to the situation in which the small divisors phenomenon does not arise. Some examples illustrating the cases where the obtained sufficient condition can or cannot be applied are also depicted.

Mathematics Subject Classification. 39A13 (Primary), 39A45 (Secondary).

Keywords. Convergence, Generalized formal power series, \(q\)-difference equation.

1. Introduction

Compared to formal solutions of algebraic equations, which are Puiseux power series at most, algebraic ordinary differential equations (ODEs) in general may have formal solutions in the form of power series with complex power exponents. First examples of such formal solutions, without a general approach to computing formal solutions of such a nature to algebraic ODEs though, can be found in papers studying Painlevé equations near their fixed singular points (see [13, 14, 18, 19]). Later, general computational methods were proposed by Della Dora and Richard-Jung [6] and Bruno [4], based on the generalization of the Newton–Puiseux polygonal method for algebraic equations. The first authors have improved the methods of Yu. Grigoriev and Singer [12] and Cano [5] which, in turn, go back to results of Fine [8]. Bruno’s algorithmic methods use the same principles as the papers by Della Dora, Richard-Jung and their predecessors, partially repeating the latter and also extending it. We may also note that complex power exponents in the asymptotic of actual solutions and their computation via the generalized Newton–Puiseux polygonal method already arise at the end of the nineteenth century in the works by Petrovitch.
Power series with complex power exponents thus become fairly familiar objects in the theory of algebraic ODEs. Algebraic \( q \)-difference equations also may have such formal solutions, which are quite new objects to study in the field though. A recent work by Barbe et al. [1] dealing with formal solutions of algebraic \( q \)-difference equations in the form of Hahn series with real power exponents is one of the first contributions in this direction.

In our paper, we study the convergence of formal power series solutions with complex power exponents to an algebraic \( q \)-difference equation. The corresponding problem in the differential case was treated by Malgrange [16], Cano [5] for a formal Taylor series solution, who have proposed sufficient conditions for its convergence and, more generally, estimated the growth of the series coefficients in the case where it diverges (the Maillet–Malgrange theorem). Their results were generalized to formal power series solutions with complex power exponents in [9,10]. As for the \( q \)-difference case, convergence and, more generally, the Maillet–Malgrange type theorem have been studied only for formal Taylor series solutions so far [1–3,7,15,20]. Even for such solutions, an additional phenomenon of small divisors may arise for \(|q| = 1\) (see [2,7]), in contrast to the differential case where one does not meet this phenomenon while studying convergence. For power series solutions with complex power exponents, the small divisors phenomenon may occur not only in the case of \(|q| = 1\). Beginning the study of such formal solutions to algebraic \( q \)-difference equations, we will restrict ourselves here to the case where small divisors do not arise.

We consider a \( q \)-difference equation
\[
F(z, y, \sigma y, \sigma^2 y, \ldots, \sigma^n y) = 0,
\]
where \( F = F(z, y_0, y_1, \ldots, y_n) \) is a polynomial and \( \sigma \) stands for the dilatation operator
\[
\sigma(f(z)) = f(qz),
\]
\( q \neq 0, 1 \) being a fixed complex number.

For any sequence of complex numbers \((\lambda_j)_{j \geq 0}\) that satisfy
(i) \( \Re \lambda_j \leq \Re \lambda_{j+1} \) for all \( j \geq 0 \),
(ii) \( \lim_{j \to \infty} \Re \lambda_j = +\infty \),
one may consider a formal power series with complex power exponents \( \lambda_j, \sum_{j=0}^{\infty} c_j z^{\lambda_j} \), which will be called a generalized formal power series.

We note that the conditions (i), (ii) make the set of all generalized formal power series an algebra over \( \mathbb{C} \). The definition of the dilatation operator extends naturally to this algebra after fixing the value \( \ln q \) by the condition \( 0 \leq \arg q < 2\pi \):
\[
\sigma\left(\sum_{j=0}^{\infty} c_j z^{\lambda_j}\right) = \sum_{j=0}^{\infty} c_j q^{\lambda_j} z^{\lambda_j}.
\]
Thus the notion of a generalized formal power series solution of (1) is correctly defined in view of the above remarks: such a series \( \varphi \) is said to be a formal solution of (1) if the substitution of \( \varphi \) into the polynomial \( F \) leads to a generalized power series with zero coefficients. We will start with an example of such a formal solution (Sect. 2), then give some auxiliary statements concerning generalized power series solutions of algebraic \( q \)-difference equations (Sect. 3) and propose a sufficient condition for the convergence of such solutions (Theorem 1 in Sect. 4). In the last section we give and discuss examples illustrating different situations concerning Theorem 1.

2. Preliminary examples

Let us start with an example of the differential equation Painlevé III with the parameters \( a = b = 0, c = d = 1 \):

\[
d^2y = \frac{1}{y} \left( \frac{dy}{dz} \right)^2 - \frac{1}{z} \frac{dy}{dz} + y^3 + \frac{1}{y}.
\]

Rewritten with respect to the differential operator \( \delta = z(d/dz) \) this becomes

\[
y \delta^2 y - (\delta y)^2 - z^2 y^4 - z^2 = 0. \tag{2}
\]

This equation has a two-parameter family of generalized formal power series solutions \([11,18]\):

\[
\varphi = C z^r + \sum_{\lambda \in K} c_\lambda z^\lambda, \tag{3}
\]

where \( C \neq 0 \) is an arbitrary complex number and \( r \) is any complex number with \(-1 < \text{Re} r < 1\). The other coefficients \( c_\lambda \) are uniquely determined by \( C \), and the set \( K \) of power exponents is of the form

\[
K = \{ r + m_1(2 - 2r) + m_2(2 + 2r) \mid m_1, m_2 \in \mathbb{Z}_+, m_1 + m_2 > 0 \}. \tag{4}
\]

This can be observed by making the change of variable \( y = C z^r + z^r u \), under which (2) is transformed to an equation of the form

\[
C \delta^2 u = z^{2-2r} + C^4 z^{2+2r} + (\delta u)^2 - u \delta^2 u + z^{2+2r} u P_3(u), \tag{5}
\]

\( P_3 \) being a polynomial of the third degree. Thus, searching for its generalized power series solution \( u = \sum_{j=0}^{\infty} c_j z^{\lambda_j} \) for a given non-zero \( \lambda_0 \), one comes to the relation

\[
C \sum_{j=0}^{\infty} c_j \lambda_j^2 z^{\lambda_j} = z^{2-2r} + C^4 z^{2+2r} + (\delta u)^2 - u \delta^2 u + z^{2+2r} u P_3(u).
\]

From this one deduces that \( \lambda_0 \) belongs to the additive semi-group generated by the numbers \( 2 - 2r, 2 + 2r \) and finds \( c_0 \), further proceeding recursively with the other \( \lambda_j \)'s, \( c_j \)'s. Moreover, the obtained generalized power series solution converges in sectors of small radius with vertex at the origin and of opening
less than $2\pi$ (see [9,18]). We also note that in the case of non-zero parameters $a, b$ of the Painlevé III equation, the set $K$ of the power exponents of the solution (3) is more dense, with the generators $1-r, 1+r$.

If we consider a $q$-difference analogue of Eq. (2) formally changing $\delta$ for $\sigma$, $y_2 \sigma^2 y - (\sigma y)^2 - z^2 y^4 - z^2 = 0$, (6) we will see that the existence of a formal solution of the same form (3), (4) as in the differential case, is not guaranteed for any $r$ with $-1 < \text{Re} \, r < 1$. Indeed, after the change of the unknown $y = C z^r + z^r u$, one comes to a $q$-difference equation

$$C (\sigma - 1)^2 u = q^{-2r} z^{2-2r} + C^4 q^{-2r} z^{2+2r} + (\sigma u)^2 - u \sigma^2 u + z^{2+2r} u P_3(u),$$

whose right-hand side is similar to that of (5). However, under the action of the left-hand side operator $C (\sigma - 1)^2$ on a potential generalized formal power series solution $u = \sum_{j=0}^{\infty} c_j z^{\lambda_j}$, with $\lambda_j$’s belonging to the additive semi-group generated by the numbers $2 - 2r, 2 + 2r$, this $u$ turns into

$$C \sum_{j=0}^{\infty} (q^{\lambda_j} - 1)^2 c_j z^{\lambda_j}.$$

This means that one can guarantee the existence of a formal solution of the form (3), (4) to Eq. (6), if none of the roots $2\pi i k / \ln q, k \in \mathbb{Z}$, of the equation $q^{\lambda} = 1$ belongs to that semi-group. In particular, this holds if the numbers $1 - r, 1 + r, \pi i / \ln q$ are linearly independent over $\mathbb{Z}$ or if the numbers $1 - r, 1 + r$ lie in the open half-plane in $\mathbb{C}$ on one side of the line with the slope $\ln |q| / \arg q$ passing through the origin. As we will see further, in the second case the convergence of such a formal solution in sectors of small radius with vertex at the origin and of opening less than $2\pi$ is guaranteed.

3. Preliminary statements

Let

$$\varphi = \sum_{j=0}^{\infty} c_j z^{\lambda_j}, \quad c_j \in \mathbb{C}^*, \quad (7)$$

be a generalized formal power series solution of (1), that is,

$$F(z, \Phi) = 0, \quad \Phi := (\varphi, \sigma \varphi, \ldots, \sigma^n \varphi).$$

Assumption (A): For each $k = 0, \ldots, n$, the generalized formal power series $F'_{y_k}(z, \Phi)$ is of the form

$$\frac{\partial F}{\partial y_k}(z, \Phi) = A_k z^\lambda + B_k z^{\tilde{\lambda}_k} + \cdots, \quad \text{Re} \, \tilde{\lambda}_k > \text{Re} \, \lambda,$$

$\lambda$ being the same for all $k = 0, \ldots, n$ and at least one of the $A_k$’s being non-zero.
Under the above assumption, let us define a non-zero polynomial

\[ L(\xi) = \sum_{k=0}^{n} A_k \xi^k \]

of degree \( \leq n \). Note that some of \( A_k \)'s may be equal to zero.

**Assumption (B):** There is \( j_0 \in \mathbb{Z}_+ \) such that for the power exponents \( \lambda_j \) of the formal solution (7) of (1) satisfying Assumption (A), there holds

\[ L(q^{\lambda_j}) \neq 0, \quad \text{for all} \quad j \geq j_0. \]

**Remark 1.** In the case where all \( \lambda_j \)'s are real and \( |q| \neq 1 \), this assumption holds automatically, since \( \lim_{j \to \infty} q^{\lambda_j} = \infty \) (if \( |q| > 1 \)) or \( \lim_{j \to \infty} q^{\lambda_j} = 0 \) (if \( |q| < 1 \)).

In this section we prove two lemmas which are auxiliary for deducing the main theorem on the convergence of the formal solution (7). Let us preliminarily fix the following notations.

For any \( m \geq 0 \) we denote by \( \varphi_m \) the truncation of \( \varphi \) at the index \( m \), i. e.,

\[ \varphi_m = \sum_{j=0}^{m} c_j z^{\lambda_j}, \quad \Phi_m := (\varphi_m, \sigma \varphi_m, \ldots, \sigma^n \varphi_m). \]

Then \( \varphi \) is represented in the form

\[ \varphi = \varphi_m + z^{\lambda_m} \psi, \quad \psi = \sum_{j=1}^{\infty} c_{m+j} z^{\lambda_{m+j} - \lambda_m}, \]

and, respectively,

\[ \Phi = \Phi_m + z^{\lambda_m} \Psi, \quad \Psi = (\psi_0, \psi_1, \ldots, \psi_n) := (\psi, (q^{\lambda_m} \sigma) \psi, \ldots, (q^{\lambda_m} \sigma)^n \psi). \]

The first lemma guarantees that Eq. (1) can be reduced to a standard form which will be useful for further considerations.

**Lemma 1.** Under Assumption (A), for any \( m \geq 0 \) such that

\[ \text{Re} \lambda_m > \text{Re} \lambda \quad \text{and} \quad \text{Re} \lambda_{m+1} > \text{Re} \lambda_m, \]

a transformation of the dependent variable

\[ y = \varphi_m + z^{\lambda_m} u \quad (8) \]

transforms (1) into an equation of the form

\[ L(q^{\lambda_m} \sigma) u = M(z, u, \sigma u, \ldots, \sigma^n u), \quad (9) \]

where \( M \) is a finite linear combination of monomials of the form

\[ z^{\alpha} u^{p_0} (\sigma u)^{p_1} \ldots (\sigma^n u)^{p_n}, \quad \alpha \in \mathbb{C}, \quad \text{Re} \alpha > 0, \quad p_i \in \mathbb{Z}_+. \]
Proof. Since for each $k = 0, \ldots, n$ the Taylor formula yields
\[
\frac{\partial F}{\partial y_k}(z, \Phi) - \frac{\partial F}{\partial y_k}(z, \Phi_m) = z^{\lambda_m} \sum_{l=0}^{n} \frac{\partial^2 F}{\partial y_k \partial y_l}(z, \Phi_m) \psi_l + \cdots,
\]
the assumption $\Re \lambda_m > \Re \lambda$ made for $m$ entails that $F_{y_k}'(z, \Phi_m)$ is of the form
\[
\frac{\partial F}{\partial y_k}(z, \Phi_m) = A_k z^{\lambda} + \hat{B}_k z^{\hat{\lambda}_k} + \cdots, \quad \Re \hat{\lambda}_k > \Re \lambda,
\tag{10}
\]
that is, the restriction of the polynomial $F_{y_k}'$ on the truncation $\Phi_m$ begins with the same term as its restriction on the whole $\Phi$ (if $A_k \neq 0$) or with the term whose exponent has the real part greater than $\Re \lambda$ (if $A_k = 0$).

Again, applying the Taylor formula to the relation $F(z, \Phi) = 0$ we arrive at
\[
0 = F(z, \Phi_m + z^{\lambda_m} \Psi) = F(z, \Phi_m) + z^{\lambda_m} \sum_{k=0}^{n} \frac{\partial F}{\partial y_k}(z, \Phi_m) \psi_k +
\]
\[
+ \frac{1}{2} z^{2\lambda_m} \sum_{k,l=0}^{n} \frac{\partial^2 F}{\partial y_k \partial y_l}(z, \Phi_m) \psi_k \psi_l + \cdots. \tag{11}
\]
In view of (10), (11) and the assumptions on $m$, we get that the monomials of $F(z, \Phi_m)$ all have exponents whose real part is larger than $\Re(\lambda_m + \lambda)$. At this point, one divides (11) by $z^{\lambda_m + \lambda}$ and obtains the equality
\[
L(q^{\lambda_m} \sigma)\psi - N(z, \psi, (q^{\lambda_m} \sigma) \psi, \ldots, (q^{\lambda_m} \sigma)^n \psi) = 0,
\]
where $N(z, u_0, u_1, \ldots, u_n)$ is given by a finite linear combination of monomials of the form
\[
z^{\alpha} u_0^{p_0} u_1^{p_1} \ldots u_n^{p_n}, \quad \alpha \in \mathbb{C}, \quad \Re \alpha > 0, \quad p_i \in \mathbb{Z}_+.
\]
Therefore, (8) transforms equation (1) into (9), whose formal solution is given by $u = \psi$. \hfill \qed

The second lemma describes the structure of the set of the power exponents $\lambda_j$ of the generalized formal power series (7) (see also Th. 3.7.4 in [1] for Hahn series with real power exponents).

Lemma 2. Under Assumption (B), there is $m \geq 0$ such that all the numbers $\lambda_{m+j} - \lambda_m$, $j > 0$, belong to a finitely generated additive semi-group $\Gamma$ whose generators all have a positive real part.

Proof. Apply Lemma 1 with $m \geq j_0$ and obtain Eq. (9). Let $\Gamma$ be an additive semi-group generated by a (finite) set of power exponents $\alpha$ of the variable $z$ contained in the monomials of $M(z, u, \sigma u, \ldots, \sigma^n u)$. Denote by $\alpha_1, \ldots, \alpha_s$ the generators of this semi-group, that is,
\[
\Gamma = \left\{ m_1 \alpha_1 + \cdots + m_s \alpha_s \mid m_i \in \mathbb{Z}_+, \sum_{i=1}^{s} m_i > 0 \right\}, \quad \Re \alpha_i > 0.
\]
Let us use the equality for the generalized formal power series \( \psi = \sum_{j=1}^{\infty} c_{m+j} z^{\lambda_{m+j} - \lambda_m} \):

\[
L(q^{\lambda_m}) \psi = M(z, \psi, \sigma \psi, \ldots, \sigma^n \psi).
\]

Since

\[
L(q^{\lambda_m}) \psi = \sum_{j=1}^{\infty} L(q^{\lambda_{m+j}}) c_{m+j} z^{\lambda_{m+j} - \lambda_m}, \quad L(q^{\lambda_{m+j}}) \neq 0,
\]

the first term of the generalized formal power series on the left-hand side of (12) is

\[
L(q^{\lambda_{m+1}}) c_{m+1} z^{\lambda_{m+1} - \lambda_m} \neq 0.
\]

On the other hand, the first term of the series on the right-hand side of (12) is some monomial \( cz^\alpha \), since any monomial \( cz^\alpha \psi^{p_0} (\sigma \psi)^{p_1} \ldots (\sigma^n \psi)^{p_n} \) with \( \sum_{k=0}^{n} p_k > 0 \) begins with a term whose power exponent has the real part greater than \( \text{Re}(\lambda_{m+1} - \lambda_m) \). Therefore,

\[
\lambda_{m+1} - \lambda_m = \alpha \in \Gamma.
\]

By similar reasoning, for each \( j > 1 \) we have

\[
\lambda_{m+j} - \lambda_m = \bar{\alpha} + k_1 (\lambda_{m+1} - \lambda_m) + \cdots + k_{j-1} (\lambda_{m+j-1} - \lambda_m),
\]

\[
\bar{\alpha} \in \Gamma, \quad k_i \in \mathbb{Z}_+,
\]

which finishes the proof with the use of mathematical induction. \( \square \)

Without loss of generality, one may assume that the generators \( \alpha_1, \ldots, \alpha_s \) of the semi-group \( \Gamma \) are linearly independent over \( \mathbb{Z} \) (see Lemma 3 in [9]). This important property of the generators will indeed be assumed and used further.

4. A result on convergence

In this section we present a sufficient condition for the convergence of the generalized formal power series solution (7) of (1), for which Assumption (B) holds. As we have seen, such a solution can be represented in the form

\[
\varphi = \varphi_m + z^{\lambda_m} \psi,
\]

and the power exponents of the generalized formal power series \( \psi \) belong to a finitely generated additive semi-group \( \Gamma \) whose generators all have a positive real part. Moreover, the generators of \( \Gamma \) are linearly independent over \( \mathbb{Z} \).

**Theorem 1.** Let all the generators \( \alpha_1, \ldots, \alpha_s \) of the semi-group \( \Gamma \) lie in the open half-plane in \( \mathbb{C} \) on one side of the line \( \mathcal{L} \) with the slope \( \ln |q|/ \arg q \) passing through the origin. Let, additionally, in the case where they lie above \( \mathcal{L} \) the coefficient \( A_0 \) of the polynomial \( L \) be non-zero, or in the case where they lie under \( \mathcal{L} \) the coefficient \( A_n \) of \( L \) be non-zero. Then the series \( \psi \), and therefore
the series $\varphi$, converge in sectors of small radius with vertex at the origin and of opening less than $2\pi$.

Remark 2. The case $\ln|q|/\arg q = 0/0$ is excluded, since $q \neq 1$. The cases $\ln|q|/\arg q = -\infty$ and $\ln|q|/\arg q = +\infty$ correspond to a positive real $q$ ($0 < q < 1$ and $q > 1$, respectively). In these cases the line $L$ becomes the imaginary axis of the complex plane. Therefore, as the generators of the semi-group $\Gamma$ lie in the right open half-plane of $\mathbb{C}$, they are automatically placed "above $L$" for $0 < q < 1$ and "under $L$" for $q > 1$.

Remark 3. When all the generators of the semi-group $\Gamma$ are real (and hence lie on the positive real axis) they are placed either

- above $L$ (if $\ln|q|/\arg q < 0$ and hence $|q| < 1$) or
- under $L$ (if $\ln|q|/\arg q > 0$ and hence $|q| > 1$) or
- on $L$ (if $\ln|q|/\arg q = 0$ and hence $|q| = 1$).

Therefore, in the first two cases Theorem 1 becomes Theorem 6.1.1 from [1] or the particular case of convergence in the Main Theorem in [20] (more precisely, a generalization of these theorems to the case of real generators, as they deal with formal Taylor series solutions of analytic $q$-difference equations). The third case, that of $|q| = 1$, is exceptional and requires additional studying since the small divisors phenomenon arises (see [2] and [7]). On the other hand, this case has no specific feature if the generators of $\Gamma$ do not belong to $\mathbb{R}$. Its analogue in the case of complex generators is a situation where they lie on the line $L$ or on opposite sides of this line. We don’t consider here situations that lead to the small divisors phenomenon and deserve further study.

Remark 4. The assumption of the theorem “lie on one side of the line $L$” implies that all the numbers $\text{Re} \alpha_i \ln|q| - \text{Im} \alpha_i \arg q$ are

- negative, if the generators $\alpha_1, \ldots, \alpha_s$ lie above the line $L$, and hence
  $$k = \max_{i=1,\ldots,s} (\text{Re} \alpha_i \ln|q| - \text{Im} \alpha_i \arg q) < 0;$$
- positive, if the generators $\alpha_1, \ldots, \alpha_s$ lie under the line $L$, and hence
  $$\tilde{k} = \min_{i=1,\ldots,s} (\text{Re} \alpha_i \ln|q| - \text{Im} \alpha_i \arg q) > 0.$$

Therefore, when $\sum_{i=1}^{s} m_i$ tends to infinity one has

$$\left| q^{\sum_{i=1}^{s} m_i \alpha_i} \right| = e^{\sum_{i=1}^{s} m_i (\text{Re} \alpha_i \ln|q| - \text{Im} \alpha_i \arg q)} \leq e^{k \sum_{i=1}^{s} m_i} \to 0$$

in the first case, and

$$\left| q^{\sum_{i=1}^{s} m_i \alpha_i} \right| = e^{\sum_{i=1}^{s} m_i (\text{Re} \alpha_i \ln|q| - \text{Im} \alpha_i \arg q)} \geq e^{\tilde{k} \sum_{i=1}^{s} m_i} \to \infty$$

in the second case.
Proof. We consider the case where the generators of $\Gamma$ lie above the line $L$ and $A_0 \neq 0$. The second case can be studied similarly. Further, for the simplicity of exposition, we will assume that $\Gamma$ is generated by two numbers $\alpha_1, \alpha_2$ linearly independent over $\mathbb{Z}$.

$$\Gamma = \left\{ m_1 \alpha_1 + m_2 \alpha_2 \mid m_1, m_2 \in \mathbb{Z}_+, m_1 + m_2 > 0 \right\}, \quad \text{Re}\, \alpha_1, \text{Re}\, \alpha_2 > 0. \label{eq:Gamma}$$

In the case of an arbitrary number $s$ of generators all constructions are analogous, only multivariate Taylor series in $s$ rather than in two variables are involved.

We should establish the convergence of the generalized power series $\psi = \sum_{j=1}^{\infty} c_{m+j} z^{m+j} - \lambda_m$ which satisfies equality (12) and whose exponents $\lambda_{m+j} - \lambda_m$, $j > 0$, are of the form

$$\lambda_{m+j} - \lambda_m = m_1 \alpha_1 + m_2 \alpha_2, \quad (m_1, m_2) \in \mathbb{Z},$$

$Z$ being a uniquely determined subset of $\mathbb{Z}^2_+ \setminus \{0\}$ such that the map $j \mapsto (m_1, m_2)$ is a bijection from $\mathbb{N}$ to $Z$. In other words,

$$\psi = \sum_{(m_1, m_2) \in Z} c_{m_1, m_2} z^{m_1 \alpha_1 + m_2 \alpha_2} = \sum_{(m_1, m_2) \in Z^2_+ \setminus \{0\}} c_{m_1, m_2} z^{m_1 \alpha_1 + m_2 \alpha_2} \label{eq:psi}$$

(in the last series one puts $c_{m_1, m_2} = c_{m+j}$ for $(m_1, m_2) = (m_1(j), m_2(j)) \in Z$, and $c_{m_1, m_2} = 0$ for $(m_1, m_2) \notin Z$). We will represent $\psi$ by a bivariate formal Taylor series and prove that the latter has a non-empty bidisc of convergence, whence the convergence of $\psi$ itself and Theorem 1 will follow.

The linear independence of the generators $\alpha_1, \alpha_2$ over $\mathbb{Z}$ allows us to define a bijective linear map $\iota : \mathbb{C}[[z^\Gamma]] \to \mathbb{C}[[z_1, z_2]]$, from the $\mathbb{C}$-algebra of generalized formal power series with exponents in $\Gamma$ to the $\mathbb{C}$-algebra of formal Taylor series in two variables without a constant term,

$$\iota : \sum_{\gamma = m_1 \alpha_1 + m_2 \alpha_2 \in \Gamma} a_\gamma z^\gamma \mapsto \sum_{(m_1, m_2) \in Z^2_+ \setminus \{0\}} a_{m_1, m_2} z_1^{m_1} z_2^{m_2}, \quad a_{m_1, m_2} = a_{m_1 \alpha_1 + m_2 \alpha_2}. \label{eq:iota}$$

Moreover, this map is an isomorphism of algebras, since

$$\iota(\eta_1 \eta_2) = \iota(\eta_1) \iota(\eta_2) \quad \forall \eta_1, \eta_2 \in \mathbb{C}[[z^\Gamma]].$$

The $q$-difference operator $\sigma : \mathbb{C}[[z^\Gamma]] \to \mathbb{C}[[z^\Gamma]]$ naturally induces a linear automorphism $\tilde{\sigma}$ of $\mathbb{C}[[z_1, z_2]]$,

$$\tilde{\sigma} : \sum_{(m_1, m_2) \in Z^2_+ \setminus \{0\}} a_{m_1, m_2} z_1^{m_1} z_2^{m_2} \mapsto \sum_{(m_1, m_2) \in Z^2_+ \setminus \{0\}} q^{m_1 \alpha_1 + m_2 \alpha_2} a_{m_1, m_2} z_1^{m_1} z_2^{m_2}, \label{eq:tildesigma}$$
which clearly satisfies $\tilde{\sigma} \circ \iota = \iota \circ \sigma$, so that the following commutative diagram holds:

$$
\begin{array}{ccc}
\mathbb{C}[[z^\Gamma]] & \xrightarrow{\sigma} & \mathbb{C}[[z^\Gamma]] \\
\downarrow \iota & & \downarrow \iota \\
\mathbb{C}[[z_1, z_2]]_* & \xrightarrow{\tilde{\sigma}} & \mathbb{C}[[z_1, z_2]]_*
\end{array}
$$

The reasoning above gives us the representation

$$
\tilde{\psi} = \iota(\psi) = \sum_{(m_1, m_2) \in \mathbb{Z}^2_+ \{0\}} c_{m_1, m_2} z_1^{m_1} z_2^{m_2}
$$

of the formal solution $\psi$ of (9) by a bivariate formal Taylor series. Applying the map $\iota$ to both sides of equality (12) we obtain the following relation for $\tilde{\psi}$:

$$
L(q^\lambda \tilde{\sigma}) \tilde{\psi} = \tilde{M}(z_1, z_2, \tilde{\psi}, \tilde{\sigma} \tilde{\psi}, \ldots, \tilde{\sigma}^n \tilde{\psi}),
$$

where $\tilde{M}(z_1, z_2, u_0, \ldots, u_n)$ is a polynomial such that

$$
\tilde{M}(0, 0, u_0, \ldots, u_n) \equiv 0.
$$

Now using the relation (13) we will prove that $\tilde{\psi}$ has a non-empty bidisc of convergence.

Let the polynomial $\tilde{M}$ be written in the form

$$
\tilde{M}(z_1, z_2, u_0, \ldots, u_n) = \sum_{k_1, k_2, \mathbf{p}} A_{k_1, k_2, \mathbf{p}} z_1^{k_1} z_2^{k_2} u_0^{p_0} \ldots u_n^{p_n}, \quad \mathbf{p} = (p_0, \ldots, p_n).
$$

To prove the convergence of $\tilde{\psi} \in \mathbb{C}[[z_1, z_2]]_*$ in some neighbourhood of the origin, we construct an equation

$$
\nu W = \sum_{k_1, k_2, \mathbf{p}} |A_{k_1, k_2, \mathbf{p}}| z_1^{k_1} z_2^{k_2} W^{p_0} \ldots W^{p_n},
$$

whose right-hand side is obtained from the polynomial $\tilde{M}$ by the change of its coefficients $A_{k_1, k_2, \mathbf{p}}$ to their absolute values and all the $u_j$’s to the one variable $W$. The number $\nu$ is defined by the formula

$$
\nu = \inf_{j \geq 1} |L(q^{\lambda_j + m})| = \inf_{(m_1, m_2) \in \mathbb{Z}} |L(q^{m_1 \alpha_1 + m_2 \alpha_2})|,
$$

which is strictly positive, since $q^{m_1 \alpha_1 + m_2 \alpha_2} \to 0$ as $m_1 + m_2 \to \infty$ (see Remark 4) whereas $L(0) = A_0 \neq 0$.

Equation (15) possesses a unique solution $W(z_1, z_2)$ holomorphic near the origin,

$$
W = \sum_{(m_1, m_2) \in \mathbb{Z}^2_+ \{0\}} C_{m_1, m_2} z_1^{m_1} z_2^{m_2},
$$

which satisfies the condition $W(0, 0) = 0$. This follows from the implicit function theorem. Indeed, the polynomial $H(z_1, z_2, W) := \nu W - \sum_{k_1, k_2, \mathbf{p}} |A_{k_1, k_2, \mathbf{p}}$
|zk1 z2| Wp1 Wp2 is such that $H(0, 0, 0) = |A_{0,0,0}| = 0$ in view of (14). Condition (14) also implies that $\partial H / \partial W(0, 0, 0) = \nu - \sum_{|p|=1} |A_{0,0,p}| = \nu > 0$.

We prove that the power series $W$ is majorant for $\tilde{\psi}$:

$$C_{m_1, m_2} \in \mathbb{R}_+, \quad |c_{m_1, m_2}| \leq C_{m_1, m_2} \forall (m_1, m_2) \in \mathbb{Z}_+^2 \setminus \{0\},$$

which will imply the convergence of $\tilde{\psi}$ in a neighbourhood of the origin in $\mathbb{C}^2$.

First we use equality (13) to obtain recursive expressions for the coefficients $c_{m_1, m_2}$. Denote by $\phi$ the formal power series from the right-hand side of this equality,

$$\phi = \sum_{k_1, k_2, p} A_{k_1, k_2, p} z_1^{k_1} z_2^{k_2} \tilde{\psi}^p (\tilde{\sigma} \tilde{\psi})^{p_1} \ldots (\tilde{\sigma}^n \tilde{\psi})^{p_n} \in \mathbb{C}[[z_1, z_2]].$$

Then (13) implies

$$L(q^{\lambda m_1 + m_1 \alpha_1 + m_2 \alpha_2}) c_{m_1, m_2} = \left. \frac{\partial^{m_1} \partial^{m_2} \phi}{m_1! m_2!} \right|_{z_1=z_2=0} = \sum_{k_1, k_2, p} A_{k_1, k_2, p}$$

where $\partial_1, \partial_2$ are the partial derivatives with respect to $z_1, z_2$. For the right-hand side of (16) we have

$$\left. \frac{\partial^{m_1} \partial^{m_2} \phi}{m_1! m_2!} \right|_{z_1=z_2=0} = \sum_{k_1, k_2, p} A_{k_1, k_2, p}$$

$$\times \sum_{l_0^{(1)} + \ldots + l_n^{(1)} = m_1 - k_1} \frac{\partial^{l_0^{(1)}} \partial^{l_2^{(2)}} \tilde{\psi}^p}{l_0^{(1)}! l_2^{(2)}!}$$

$$\tilde{\psi}^p (\tilde{\sigma} \tilde{\psi})^{p_1} \ldots (\tilde{\sigma}^n \tilde{\psi})^{p_n} \left. \right|_{z_1=z_2=0}$$

(17)

The summations in the formulae (17), (18) go over non-negative integers $l_j^{(1)}$, $l_j^{(2)}$ and $\lambda_i$, $\mu_i$. These formulae are similar to the formulae (18), (19) in [9] for the differential case and follow from the Leibniz rule of differentiation (for more technical details see [9]). Since $\tilde{M}(0, u_0, \ldots, u_n) \equiv 0$, for each coefficient $A_{k_1, k_2, p}$ in (17) at least one of the indices $k_1, k_2$ is non-zero. Therefore, for each fixed triple $(k_1, k_2, p)$ either all the summation indices $l_j^{(1)}$ are less than
where the summation indices \( j_0^{(2)} \) are less than \( m_2 \). Hence, the formula \((16)\) can be written in the form

\[
L(q^{\lambda_1+m_1\alpha_1+m_2\alpha_2}) c_{m_1,m_2} = p_{m_1,m_2}(\{A_{k_1,k_2,\mathbf{p}}\}, \{c_{\lambda,\mu}\}),
\]

where \( p_{m_1,m_2} \) is a polynomial of the variables \( \{A_{k_1,k_2,\mathbf{p}}\}, \{c_{\lambda,\mu}\} \), with \( \lambda \leq m_1, \mu \leq m_2, \) and \( \lambda + \mu < m_1 + m_2 \) (which is determined by the formulae \((17), (18)\)).

Now we similarly use equality \((15)\) to obtain recursive expressions for the coefficients \( C_{m_1,m_2} \) of the series \( W = \sum_{(m_1,m_2)\in \mathbb{Z}_+^2 \setminus \{0\}} C_{m_1,m_2} z_1^{m_1} z_2^{m_2} \) convergent near the origin in \( \mathbb{C}^2 \). Denote by \( \Theta \) the power series from the right-hand side of this equality,

\[
\Theta = \sum_{k_1,k_2,\mathbf{p}} |A_{k_1,k_2,\mathbf{p}}| z_1^{k_1} z_2^{k_2} W^{p_0} \cdots W^{p_n} \in \mathbb{C}\{z_1, z_2\}.
\]

Then \((15)\) implies

\[
\nu C_{m_1,m_2} = \frac{\partial_1^{m_1} \partial_2^{m_2} \Theta}{m_1! m_2!} \bigg|_{z_1=z_2=0},
\]

and by analogy with \( \phi \) one has

\[
\frac{\partial_1^{m_1} \partial_2^{m_2} \Theta}{m_1! m_2!} \bigg|_{z_1=z_2=0} = \sum_{k_1,k_2,\mathbf{p}} |A_{k_1,k_2,\mathbf{p}}| \times \sum_{l_0^{(1)}+l_0^{(2)}=m_1-k_1, l_0^{(2)}=m_2-k_2} \frac{\partial_1^{l_0^{(1)}} \partial_2^{l_0^{(2)}} W^{p_0}}{l_0^{(1)}! l_0^{(2)}!} \cdots \frac{\partial_1^{l_n^{(1)}} \partial_2^{l_n^{(2)}} W^{p_n}}{l_n^{(1)}! l_n^{(2)}!} \bigg|_{z_1=z_2=0},
\]

where

\[
\frac{\partial_1^{j_1} \partial_2^{j_2} W^{p_j}}{l_1^{(1)}! l_1^{(2)}!} \bigg|_{z_1=z_2=0} = \sum_{\lambda_1+\ldots+\lambda_{p_j}=l_1^{(1)}, \mu_1+\ldots+\mu_{p_j}=l_1^{(2)}} C_{\lambda_1,\mu_1} \cdots C_{\lambda_{p_j},\mu_{p_j}}.
\]

Thus, the formula \((19)\) can be written in the form

\[
\nu C_{m_1,m_2} = P_{m_1,m_2}(\{|A_{k_1,k_2,\mathbf{p}}|\}, \{C_{\lambda,\mu}\}),
\]

where \( P_{m_1,m_2} \) is a polynomial of the variables \( \{|A_{k_1,k_2,\mathbf{p}}|\}, \{C_{\lambda,\mu}\} \), with \( \lambda \leq m_1, \mu \leq m_2, \) and \( \lambda + \mu < m_1 + m_2 \) (which is determined by the formulae \((20), (21)\) and has real positive coefficients). Since for \((m_1,m_2)\) equal to \((1,0)\) and \((0,1)\) we have, respectively,

\[
\nu C_{1,0} = \partial_1 \Theta(0,0) = |A_{1,0,0}|, \quad \nu C_{0,1} = \partial_2 \Theta(0,0) = |A_{0,1,0}|,
\]
all the coefficients $C_{m_1,m_2}$ are real non-negative numbers.

Finally we come to a concluding part of the proof, the estimates
\[ |c_{m_1,m_2}| \leq C_{m_1,m_2} \quad \forall (m_1, m_2) \in \mathbb{Z}_+^2 \setminus \{0\}. \]

We prove them by induction on the sum $m_1 + m_2$ of indices.

For $m_1 + m_2 = 1$ according to (16) we have
\[ L(q^{\lambda_m+\alpha_1}) c_{1,0} = \partial_1 \phi(0,0) = A_{1,0,0}, \quad L(q^{\lambda_m+\alpha_2}) c_{0,1} = \partial_2 \phi(0,0) = A_{0,1,0}, \]

hence
\[ |c_{1,0}| = \left| \frac{A_{1,0,0}}{L(q^{\lambda_m+\alpha_1})} \right| = \frac{\nu C_{1,0}}{|L(q^{\lambda_m+\alpha_1})|} \leq C_{1,0}, \]
\[ |c_{0,1}| = \left| \frac{A_{0,1,0}}{L(q^{\lambda_m+\alpha_2})} \right| = \frac{\nu C_{0,1}}{|L(q^{\lambda_m+\alpha_2})|} \leq C_{0,1} \]

(for $(1,0) \in Z$ we have $|L(q^{\lambda_m+\alpha_1})| \geq \nu$, whereas $c_{1,0} = 0$ for $(1,0) \notin Z$; the analogous situation remains valid for the index $(0,1)$).

Further, by the definition of the polynomials $p_{m_1,m_2}$ and $P_{m_1,m_2}$, for any $(m_1, m_2) \in Z$ we have
\[ |p_{m_1,m_2}(\{A_{k_1,k_2,p}\}, \{c_{\lambda,\mu}\})| \leq P_{m_1,m_2}(\{|A_{k_1,k_2,p}\}, \{|c_{\lambda,\mu}\}|), \]
since the estimate $|q^{l_1\alpha_1+l_2\alpha_2}| \leq 1$ holds for any non-negative integers $l_1, l_2$ (see Remark 4). Therefore, the induction hypothesis (the second inequality below) implies
\[ |L(q^{\lambda_m+m_1\alpha_1+m_2\alpha_2})| |c_{m_1,m_2}| = |p_{m_1,m_2}(\{A_{k_1,k_2,p}\}, \{c_{\lambda,\mu}\})| \leq P_{m_1,m_2}(\{|A_{k_1,k_2,p}\}, \{|c_{\lambda,\mu}\}|) \leq P_{m_1,m_2}(\{|A_{k_1,k_2,p}\}, \{C_{\lambda,\mu}\}) = \nu C_{m_1,m_2}, \]

whence the required estimate follows:
\[ |c_{m_1,m_2}| \leq \frac{\nu C_{m_1,m_2}}{|L(q^{\lambda_m+m_1\alpha_1+m_2\alpha_2})|} \leq C_{m_1,m_2} \]

(again, for $(m_1, m_2) \in Z$ we have $|L(q^{\lambda_m+m_1\alpha_1+m_2\alpha_2})| \geq \nu$, whereas $c_{m_1,m_2} = 0$ for $(m_1, m_2) \notin Z$).

Now it remains to note that for any sector $S \subset \mathbb{C}$ with vertex at the origin and of opening less than $2\pi$, the terms of a generalized power series are holomorphic single-valued functions in $S$, and to pass from the convergence of $\tilde{\psi} = \sum_{(m_1,m_2)\in \mathbb{Z}_+^2 \setminus \{0\}} c_{m_1,m_2} z_1^{m_1} z_2^{m_2}$ to the convergence of $\tilde{\psi} = \sum_{j=1}^\infty c_{m+j} z^{m+j-\lambda_m}$ and $\varphi = \sum_{j=0}^\infty c_j z^{\lambda_j}$. Indeed, let the power series $\tilde{\psi}$ converge in a bidisc $\{|z_1| < r, |z_2| < r\}$. If $z \in S$ is small enough for the inequalities
\[ |z^{\alpha_1}| = |z|^{\text{Re} \alpha_1} e^{-\text{Im} \alpha_1 \text{ arg } z} < r, \quad |z^{\alpha_2}| = |z|^{\text{Re} \alpha_2} e^{-\text{Im} \alpha_2 \text{ arg } z} < r \]
to hold (recall that $\operatorname{Re} \alpha_1, \alpha_2 > 0$), then
$$|c_{m+j} z^{\lambda m+j-\lambda m}| = |c_{m_1, m_2}| \cdot |z^{\alpha_1}|^{m_1} \cdot |z^{\alpha_2}|^{m_2} < |c_{m_1, m_2}| r^{m_1} r^{m_2},$$

hence the series $\psi = \sum_{j=1}^{\infty} c_{m+j} z^{\lambda m+j-\lambda m}$ and $\varphi = \sum_{j=0}^{\infty} c_j z^{\lambda j}$ converge uniformly in $S$ for sufficiently small $|z|$.

\[\square\]

5. Concluding examples

We recall that one of the $q$-analogues of the Euler equation,

$$\sigma y - zy - z = 0,$$

possesses a formal Taylor series solution $\varphi = \sum_{m=1}^{\infty} q^{-m(m+1)/2} z^m$, and with null radius of convergence, if $|q| < 1$. Let us give an example of an algebraic $q$-difference equation and its divergent generalized power series solution whose power exponents form a semi-group generated by two numbers.

Example 1. Consider the algebraic $q$-difference equation

$$\sigma^2 y - 2^{-\sqrt{2}} \sigma y + y^2 - 5z = 0, \quad q = \frac{1}{2}, (23)$$

which has a generalized formal power series solution

$$\varphi = \sum_{m_1+m_2 \geq 0} c_{m_1, m_2} z^{1+m_1+m_2(\sqrt{2}-1)}, \quad c_{m_1, m_2} \in \mathbb{C}, (24)$$

where $c_{01}$ is an arbitrary constant and the other coefficients are determined uniquely. This formal solution does not satisfy the assumptions of Theorem 1, since the exponents $1, \sqrt{2} - 1$ lie above the line $L$ but the polynomial $L(\xi) = \xi^2 - 2^{-\sqrt{2}} \xi$ vanishes at $\xi = 0$. Let us prove that the series (2) has zero radius of convergence. Substituting it in Eq. (23) we obtain the equality

$$\sum_{m_1+m_2 \geq 0} c_{m_1, m_2} \left( q^2(1+m_1+m_2(\sqrt{2}-1)) - 2^{-\sqrt{2}} q^{1+m_1+m_2(\sqrt{2}-1)} \right) z^{1+m_1+m_2(\sqrt{2}-1)} = 5z - \left( \sum_{m_1+m_2 \geq 0} c_{m_1, m_2} z^{1+m_1+m_2(\sqrt{2}-1)} \right)^2. (25)$$

Put

$$z_1 = z, \quad z_2 = z^{\sqrt{2}-1},$$

then equality (3) can be rewritten in the form

$$\sum_{m_1+m_2 \geq 0} c_{m_1, m_2} \left( 2^{-\sqrt{2}} q^{1+m_1+m_2(\sqrt{2}-1)} - q^2(1+m_1+m_2(\sqrt{2}-1)) \right) z_1^{m_1} z_2^{m_2} =$$

$$= -5 + z_1 \left( \sum_{m_1+m_2 \geq 0} c_{m_1, m_2} z_1^{m_1} z_2^{m_2} \right)^2, \quad (26)$$
whence
\[ \varphi = \sum_{m_1, m_2 \geq 0} c_{m_1, m_2} z_1^{m_1+1} z_2^{m_2}. \]  
(27)

Then it follows from (26) that
\[ c_{00} = \frac{5}{q^2 - 2 - \sqrt{2}q} = \frac{10}{2 - 1 - 2 - \sqrt{2}} > 80 \]
and
\[ \sum_{m_1, m_2 \geq 1} c_{m_1, m_2} \left( 2 - \sqrt{2} \right) q^{1 + m_1 + m_2(\sqrt{2} - 1)} - q^{2(1 + m_1 + m_2(\sqrt{2} - 1))} z_1^{m_1} z_2^{m_2} \]
\[ = z_1 \left( c_{00} + \sum_{m_1, m_2 \geq 1} c_{m_1, m_2} z_1^{m_1} z_2^{m_2} \right)^2. \]

Hence \( c_{01} \) is a free parameter and \( c_{0, m_2} = 0 \) for all \( m_2 \geq 2 \). Further,
\[ c_{10} = \frac{c_{00}^2}{2 - \sqrt{2} q^2 - q^4} = \frac{4c_{00}^2}{2 - \sqrt{2} - 2 - 2} \]
and
\[ \sum_{m_1, m_2 \geq 2} c_{m_1, m_2} q^{1 + m_1 + m_2(\sqrt{2} - 1)} q^{\sqrt{2}} q^{1 + m_1 + m_2(\sqrt{2} - 1)} z_1^{m_1} z_2^{m_2} \]
\[ = 2c_{00} z_1 \sum_{m_1, m_2 \geq 1} c_{m_1, m_2} z_1^{m_1} z_2^{m_2} + z_1 \left( \sum_{m_1, m_2 \geq 1} c_{m_1, m_2} z_1^{m_1} z_2^{m_2} \right)^2. \]

If we take an arbitrary positive real number \( c_{01} \), the last equality will imply that all the coefficients \( c_{m_1, m_2} \) are non-negative real numbers and
\[ c_{m_1, m_2} q^{1 + m_1 + m_2(\sqrt{2} - 1)} q^{\sqrt{2}} - q^{1 + m_1 + m_2(\sqrt{2} - 1)} \geq 2c_{00} c_{m_1-1, m_2}, \quad m_1 \geq 1, \]
whence
\[ c_{m_1, 0} q^{1 + m_1} q^{\sqrt{2}} - q^{1 + m_1} \geq 2c_{00} c_{m_1-1, 0}, \quad m_1 \geq 1. \]

Therefore,
\[ c_{m_1, 0} > c_{m_1-1, 0} q^{-m_1} > c_{m_1-2, 0} q^{-(m_1-1) - m_1} > \ldots > c_{00} q^{-(m_1+1)m_1/2}. \]

It follows that the series \( \sum_{m_1=1}^{\infty} c_{m_1, 0} z_1^{m_1} \) has zero radius of convergence, hence the series (27) diverges for any positive real \( z_1, z_2 \).

In the next example of a second order \( q \)-difference equation possessing a generalized power series solution whose power exponents form a semi-group generated by two numbers again, these generators lie on the opposite sides of the line \( L \), though \( |q| > 1 \) and the coefficient \( A_2 \) of the polynomial \( L(\xi) \) is non-zero. Thus one of the assumptions of Theorem 1 is not satisfied again and it seems that one also has divergence here.
Example 2. The algebraic $q$-difference equation
\[ \sigma^2 y - q^{1+i} \sigma y = y^2 + z^2, \quad q = 1.0001 i, \quad i = \sqrt{-1}, \] (28)
possesses a generalized formal power series solution
\[ \varphi = \sum_{m_1, m_2 \geq 0} c_{m_1, m_2} z^{(m_1+1)(1+i)+m_2(1-i)}, \quad c_{m_1, m_2} \in \mathbb{C}, \] (29)
where $c_{00}$ is an arbitrary constant and the other coefficients are determined uniquely. This formal solution does also not satisfy the assumptions of Theorem 1, since the line $L$ is almost horizontal ($\ln |q|/\arg q \approx 0.00006 << 1$) and the generators $1 - i, 1 + i$ of the semi-group $\Gamma$ thus lie on the opposite sides of this line. Note that, at the same time, $|q| > 1$ and the coefficient $A_2$ of the polynomial $L(\xi) = \xi(\xi - q^{1+i})$ is non-zero.

Similarly to the previous example, define
\[ z_1 = z^{1+i}, \quad z_2 = z^{1-i}. \]
Then
\[
\begin{align*}
\varphi &= z_1 \sum_{m_1, m_2 \geq 0} c_{m_1, m_2} z_1^{m_1} z_2^{m_2}, \\
\sigma \varphi &= z_1 \sum_{m_1, m_2 \geq 0} q^{(m_1+1)(1+i)+m_2(1-i)} c_{m_1, m_2} z_1^{m_1} z_2^{m_2}, \\
\sigma^2 \varphi &= z_1 \sum_{m_1, m_2 \geq 0} q^{2(m_1+1)(1+i)+2m_2(1-i)} c_{m_1, m_2} z_1^{m_1} z_2^{m_2}.
\end{align*}
\]
Substituting the above bivariate formal Taylor series into Eq. (28) one comes to the equality
\[
\sum_{m_1, m_2 \geq 0} q^{(m_1+2)(1+i)+m_2(1-i)} \left( q^{m_1(1+i)+m_2(1-i)} - 1 \right) c_{m_1, m_2} z_1^{m_1} z_2^{m_2}
= z_1 \left( \sum_{m_1, m_2 \geq 0} c_{m_1, m_2} z_1^{m_1} z_2^{m_2} \right)^2 + z_2.
\]
Setting $m_2 = 0$ in it we will have
\[
\sum_{m_1 = 0}^{\infty} q^{(m_1+2)(1+i)} \left( q^{m_1(1+i)} - 1 \right) c_{m_1, 0} z_1^{m_1} = z_1 \left( \sum_{m_1 = 0}^{\infty} c_{m_1, 0} z_1^{m_1} \right)^2,
\]
and whence deduce that $c_{00} \in \mathbb{C}$ is a free parameter,
\[
c_{10} = \frac{c_{00}^2}{q^{3(1+i)}(q^{1+i} - 1)}, \quad c_{m_1, 0} = \frac{2 c_{00} c_{m_1-1, 0} + \sum_{l=1}^{m_1-2} c_{l, 0} c_{m_1-1-l, 0}}{q^{(m_1+2)(1+i)} \left( q^{m_1(1+i)} - 1 \right)} \bigg|_{m_1 > 1}.
\]
We note that $|q^{1+i}| \approx 0.2$ and the denominators in the above recurrent formulae for the coefficients descend to zero exponentially as $m_1$ tends to infinity. Numerical evidence indicates that the series $\sum_{m_1 \geq 0} |c_{m_1, 0}| z_1^{m_1}$, with $c_{00} = 1$, is rapidly diverging:

$$|c_{m_1, 0}| > |q^{1+i}|^{-(m_1+2)(m_1+3)/2} > 5^{(m_1+2)(m_1+3)/2},$$

beginning with $m_1 = 8$, which allows us to suppose that the corresponding series (29) has zero radius of convergence.

**Example 3.** Let us return to the $q$-difference equation (6) from Sect. 2,

$$y \sigma^2 y - (\sigma y)^2 - z^2 y^4 - z^2 = 0.$$

Under the assumption that the numbers $1 - r$, $1 + r$, $\pi i / \ln q$ are linearly independent over $\mathbb{Z}$, its generalized power series solution (3) is of the form $\varphi = C z^r + z^r \psi$, where the power exponents of the generalized power series $\psi$ belong to the additive semi-group generated by $2 - 2r$, $2 + 2r$. In the case where $|q| = 1$ we cannot apply Theorem 1 for studying the convergence of such a formal solution, since the line $L$ coincides with the real axis in this case and the numbers $2 - 2r$, $2 + 2r$ lie on different sides of this line. On the other hand, for any $q \in \mathbb{C}^*$ with $|q| \neq 1$, the line $L$ has non-zero slope, hence there is an open set $U_q$ of values of the parameter $r$ for which the numbers $2 - 2r$, $2 + 2r$ lie on one side of this line. Since the polynomial $L$ in this example is $L(\xi) = C(\xi - 1)^2$, both of its coefficients $A_0$, $A_2$ are non-zero and, by Theorem 1, for $r \in U_q$ the series $\varphi$ converges in small sectors $S \subset \mathbb{C}$ with vertex at the origin.

**Acknowledgements**

A. Lastra is partially supported by the project PID2019-105621GB-I00 of Ministerio de Ciencia e Innovación, Spain, and by Dirección General de Investigación e Innovación, Consejería de Educación e Investigación de la Comunidad de Madrid (Spain), and Universidad de Alcalá under grant CM/JIN/2019-010, Proyectos de I+D para Jóvenes Investigadores de la Universidad de Alcalá 2019.

**Publisher’s Note** Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

**References**

[1] Barbe, P., Cano, J., Fortuny Ayuso, P., McCormick, W. P.; $q$-Algebraic equations, their power series solutions and the asymptotic behavior of their coefficients (2020) arXiv:2006.09527 [math.CA]
[2] Bézivin, J.-P.: Sur les équations fonctionnelles aux \( q \)-différences. Aequat. Math. 43, 159–176 (1992)

[3] Bézivin, J.-P.: Convergence des solutions formelles de certaines équations fonctionnelles. Aequat. Math. 44, 84–99 (1992)

[4] Bruno, A.D.: Asymptotic behaviour and expansions of solutions of an ordinary differential equation. Russ. Math. Surv. 59(3), 429–480 (2004)

[5] Cano, J.: On the series defined by differential equations, with an extension of the Puiseux polygon construction to these equations. Analysis 13, 103–119 (1993)

[6] Della Dora, J., Richard-Jung, F.: About the Newton algorithm for non-linear ordinary differential equations. In: Proceedings of the 1997 International Symposium on Symbolic and Algebraic Computation, United States, pp. 298–304

[7] Di Vizio, L.: An ultrametric version of the Maillet–Malgrange theorem for nonlinear \( q \)-difference equations. Proc. Am. Math. Soc. 136(8), 2803–2814 (2008)

[8] Fine, H.: On the functions defined by differential equations, with an extension of the Puiseux polygon construction to these equations. Am. J. Math. 11(4), 317–328 (1889)

[9] Gontsov, R.R., Goryuchkina, I.V.: On the convergence of generalized power series satisfying an algebraic ODE. Asympt. Anal. 93(4), 311–325 (2015)

[10] Gontsov, R.R., Goryuchkina, I.V.: The Maillet–Malgrange type theorem for generalized power series. Manuscr. Math. 156(1–2), 171–185 (2018)

[11] Gridnev, A.V.: Power expansions of solutions to the modified third Painlevé equation in a neighborhood of zero. J. Math. Sci. 145, 5180–5187 (2007)

[12] Grigor'ev, DYu., Singer, M.F.: Solving ordinary differential equations in terms of series with real exponents. Trans. Am. Math. Soc. 327(1), 329–351 (1991)

[13] Jimbo, M.: Monodromy problem and the boundary condition for some Painlevé equations. Publ. RIMS Kyoto Univ. 18, 1137–1161 (1982)

[14] Kimura, H.: The construction of a general solution of a Hamiltonian system with regular type singularity and its application to Painlevé equations. Ann. Mat. Pura Appl. 134, 363–392 (1983)

[15] Li, X., Zhang, C.: Existence of analytic solutions to analytic nonlinear \( q \)-difference equations. J. Math. Anal. Appl. 375, 412–417 (2011)

[16] Malgrange, B.: Sur le théorème de Maillet. Asympt. Anal. 2(1), 1–4 (1989)

[17] Petrovitch, M.: Thèses: Sur les zéro et les infinis des intégrales des équations différentielles algébraiques. Propositions données par la Faculté. Paris (1894)

[18] Shimomura, S.: A family of solutions of a nonlinear ordinary differential equation and its application to Painlevé equations (III), (V) and (VI). J. Math. Soc. Jpn. 39, 649–662 (1987)

[19] Takano, K.: Reduction for Painlevé equations at the fixed singular points of the first kind. Funkc. Ekvac. 29, 99–119 (1986)

[20] Zhang, C.: Sur un théorème du type de Maillet–Malgrange pour les équations \( q \)-différences-différentielles. Asympt. Anal. 17(4), 309–314 (1998)
Irina Goryuchkina  
Keldysh Institute of Applied Mathematics  
Russian Academy of Sciences  
Miusskaya sq. 4  
Moscow  
Russia 125047  
e-mail: igoryuchkina@gmail.com

Alberto Lastra  
Universidad de Alcalá  
Departamento de Física y Matemáticas  
Ap. de Correos 20  
28871 Alcalá de Henares, Madrid  
Spain  
e-mail: alberto.lastra@uah.es

Received: January 29, 2021  
Revised: May 19, 2021  
Accepted: May 24, 2021