Bosonic Algebras and Their Inhomogeneous Invariance Quantum Groups

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Abstract. We introduce different types of bosonic algebras and investigate their inhomogeneous quantum invariance groups and the relations between the quantum groups.

1. Introduction
Harmonic oscillator system is extremely important in physics. An algebraic relation which is defined in harmonic oscillator system is boson algebra. The boson algebra can be written as

\[ aa^* - a^* a = 1, \] (1)

here \( a \) is bosonic creation operator and \( a^* \) is bosonic annihilation operator. This algebra defines the behavior of bosons which are force carrying particles. Because of this fact boson algebra is very important. In addition to its physical importance boson algebra, namely oscillator algebra is also important in mathematics since it is connected with representations of semisimple Lie algebras[1].

One way of constructing a quantum group is making a deformation on a Lie algebra. Simply a deformation parameter is defined and a standard algebra becomes a deformed algebra by using the deformation parameter. This is well known Drinfeld’s approach[2]. Since the boson algebra is connected with representations of semisimple Lie algebras, to construct representations of deformed Lie algebras, boson algebra can be deformed. This is known a q-oscillator and it is related with quantum groups. There are different types of deformed boson algebras, but in fact all types are derived from standard boson algebra by deformation.

There is another way to construct a quantum group. In this method we can build a matrix whose elements satisfy Hopf algebra axioms. This type quantum groups are matrix quantum groups [3] and the idea behind this involves noncommutative matrix multiplication. To get inhomogeneous invariance quantum group, we make linear canonical transformation on an algebra and check whether the elements of transformation satisfy Hopf algebra axioms[4].

In this study starting from standard boson algebra, multiparameter deformed boson algebra and q deformed bosonic Newton algebra are introduced. Then their inhomogeneous invariance quantum groups introduced and the relations between them investigated.
2. BISp(2d)

In this section we introduce standard boson algebra and its inhomogeneous invariance quantum group. The boson algebra is:

\[ a_i a_j - a_j a_i = 0, \quad (2) \]
\[ a_i a_j^* - a_j^* a_i = \delta_{ij}. \quad (3) \]

Here \( i, j = 1, 2, \ldots, d \), where \( d \) is number of bosons. To find the inhomogeneous invariance quantum group of boson algebra we make a linear transformation.

\[
\begin{pmatrix}
  a_i \\
  a_i^* \\
  1
\end{pmatrix}' =
\begin{pmatrix}
  \alpha_{ij} & \beta_{ij} & \gamma_i \\
  \beta_{ij}^* & \alpha_{ij}^* & \gamma_i^* \\
  0 & 0 & 1
\end{pmatrix}
\otimes
\begin{pmatrix}
  a_j \\
  a_j^* \\
  1
\end{pmatrix}. \quad (4)
\]

We want that the transformed operators should satisfy the relations of boson algebra. In other words we want the algebra remains invariant under the transformation which is given in equation (4). Then \( \alpha_{ij}, \beta_{ij}, \alpha_{ij}^* \) and \( \beta_{ij}^* \) commute with each other and they commute with \( \gamma_i \) and \( \gamma_i^* \). Then the relation between the inhomogeneous parameters are:

\[ \gamma_i \gamma_j - \gamma_j \gamma_i = \beta_{ik} \alpha_{jk} - \alpha_{ik} \beta_{jk}, \quad (5) \]
\[ \gamma_i \gamma_j^* - \gamma_j \gamma_i^* = \delta_{ij} - \alpha_{ik} \alpha_{jl}^* + \beta_{ik} \beta_{jl}^* \quad (6) \]

In order to say that the transformation is a quantum group we should find the coproduct, counit and the antipode of the transformation. Before going further we should write the transformation matrix and describe its homogeneous and inhomogeneous parts. The transformation matrix is:

\[
T = \begin{pmatrix}
  \alpha_{ij} & \beta_{ij} & \gamma_i \\
  \beta_{ij}^* & \alpha_{ij}^* & \gamma_i^* \\
  0 & 0 & 1
\end{pmatrix} = \begin{pmatrix} A & \Gamma \\
  0 & 1
\end{pmatrix}. \quad (7)
\]

The coproduct of the relations can be found by using tensor matrix multiplication rule,

\[
\Delta(T) = T \otimes T. \quad (8)
\]

The counit and antipode are defined as;

\[
\varepsilon(T) = I, \quad S(T) = T^{-1}. \quad (9) \]
\[
\otimes \quad (10)
\]

In order to find the inverse of transformation matrix \( T \) we use block matrices,

\[
T^{-1} = \begin{pmatrix}
  A^{-1} & -A^{-1} \Gamma \\
  0 & 1
\end{pmatrix}. \quad (11)
\]

Because of the elements of matrix \( A \) commute with each other the inverse of \( A \) is ordinary matrix inverse.

Since the elements of transformation matrix \( T \) have coproduct, counit and antipode one can state that the transformation is a quantum group and this quantum group is called as the Bosonic Inhomogeneous Symplectic Quantum Group, BISp(2d)[5],[6].

One question is what quantum subgroups does BISp(2d) have. The subgroups are obtained by imposing additional relations on the matrix elements of \( T \). These additional relations are:

\( (a) \) \( \delta_{ij} - \alpha_{ik} \alpha_{jk}^* \pm \beta_{ik} \beta_{jk}^* = \pm \beta_{ik} \alpha_{jk} - \alpha_{ik} \beta_{jk} = 0 \)
(b) $\gamma_i = 0$

(c) $\beta_{ij} = 0$

(d) $\alpha_{ij} = 0$

(e) $\hbar \to 0$

and the sub(quantum)group diagram:

\[
\begin{align*}
BISp(2d, \mathbb{R}) & \xrightarrow{(a)} ISp(2d, \mathbb{R}) \xrightarrow{(b)} Sp(2d, \mathbb{R}) \\
& \xrightarrow{(c)} ISp(2d, \mathbb{R}) \xrightarrow{(b)} Sp(2d, \mathbb{R}) \\
& \xrightarrow{(c)} ISp(2d, \mathbb{R}) \xrightarrow{(b)} Sp(2d, \mathbb{R}) \\
BHU(d) & \xrightarrow{(a)} IU(d) \xrightarrow{(b)} U(d) \\
& \xrightarrow{(d)} IU(d) \xrightarrow{(b)} U(d) \\
& \xrightarrow{(d)} IU(d) \xrightarrow{(b)} U(d)
\end{align*}
\]

3. $BIGHL_{q_{ij},q_{-1}}(2d)$

A quantum group is can be constructed from the usual Lie group by defining a deformation parameter. The procedure is very similar to quantization of a classical system. Using the same deformation procedure on the boson algebra one can define the multidimensional q-deformed boson algebra as;

\[
a_i a_j^* - q_{ij} a_j^* a_i = \delta_{ij}, \quad (12)
\]

\[
a_i a_j = a_j a_i \quad (13)
\]

where

\[
q_{ij} = (q - 1)\delta_{ij} + 1. \quad (14)
\]

Here $i, j = 1, 2, \ldots, n$ where $n$ is number of bosons. To find the inhomogeneous invariance quantum group of multiparameter deformed boson algebra we make a linear canonical transformation on the algebra and and the elements of the transformation matrix have to satisfy Hopf algebra relations. The transformation is;

\[
T = \begin{pmatrix}
\alpha_{ik} & \beta_{ik} & \gamma_i \\
\beta_{ik}^* & \alpha_{ik}^* & \gamma_i^* \\
0 & 0 & 1
\end{pmatrix} = \begin{pmatrix} A & \Gamma \\
0 & 1
\end{pmatrix}. \quad (15)
\]
Since we want that, after transformation the algebra remains unchanged, the elements of transformation matrix should satisfy the following relations.

$$\alpha_{ik} \alpha_{jm} = \alpha_{jm} \alpha_{ik},$$  \hspace{1cm} (16)  

$$\alpha_{ik} \alpha_{jm}^* = q_{km} \alpha_{jm}^* \alpha_{ik},$$  \hspace{1cm} (17)  

$$\alpha_{ik} \beta_{jm} = \frac{1}{q_{km}} \beta_{jm} \alpha_{ik},$$  \hspace{1cm} (18)  

$$\alpha_{ik} \beta_{jm}^* = q_{jk} \beta_{jm}^* \alpha_{ik},$$  \hspace{1cm} (19)  

$$\alpha_{ik} \gamma_j = \gamma_j \alpha_{ik},$$  \hspace{1cm} (20)  

$$\alpha_{ik} \gamma_j^* = q_{ij} \gamma_j^* \alpha_{ik},$$  \hspace{1cm} (21)  

$$\beta_{ik} \beta_{jm} = \beta_{jm} \beta_{ik},$$  \hspace{1cm} (22)  

$$\beta_{ik} \beta_{jm}^* = q_{jk} \beta_{jm}^* \beta_{ik},$$  \hspace{1cm} (23)  

$$\beta_{ik} \gamma_j = \gamma_j \beta_{ik},$$  \hspace{1cm} (24)  

$$\beta_{ik} \gamma_j^* = q_{ij} \gamma_j^* \beta_{ik},$$  \hspace{1cm} (25)  

$$\gamma_i \gamma_j - \gamma_j \gamma_i = \alpha_{jk} \beta_{ik} - \alpha_{ik} \beta_{jk},$$  \hspace{1cm} (26)  

$$\gamma_i \gamma_j^* - q_{ij} \gamma_j^* \gamma_i = \delta_{ij} - \alpha_{ik} \alpha_{jk}^* - q_{ij} \beta_{jk} \beta_{ik},$$  \hspace{1cm} (27)  

together with their hermitian conjugates. In order to say that the transformation is a quantum group we should find the coproduct, the counit and the coinverse of the transformation. The coproduct of the relations can be found by using tensor matrix multiplication rule,

$$\Delta(T) = T \otimes T$$  \hspace{1cm} (28)  

The counit and antipode are defined as:

$$\epsilon(T) = I,$$  \hspace{1cm} (29)  

$$S(T) = T^{-1}.$$  \hspace{1cm} (30)  

Inverse of transformation can be written by using equation (11). In that equation $A^{-1}$ has to be found. We can easily say that the elements of matrix $A$ are the elements of multiparameter deformed GL(n)by writing following deformation.

$$q_{ij} = p_{ij}^{-1} = \begin{cases} 
1 & i < j \text{ and } j \neq i + d, \\
q & i < j \text{ and } j \neq i + d.
\end{cases}$$  \hspace{1cm} (31)  

Thus, one can write the inverse of matrix A in light of Schirrmachers multiparametric deformed GL(n)[7]. Since the elements of transformation matrix T have coproduct, counit and antipode one can state that the transformation is a quantum group and this quantum group is called as the Bosonic Inhomogeneous Deformed General Linear Quantum Group $BIGL_{q_{ij}, q_{ij}^{-1}}(2d)$[8]. It can be easily seen that homogeneous part is also a quantum group. Also by taking all $q_{ij}$’s are equal to one $BIGL_{q_{ij}, q_{ij}^{-1}}(2d)$ becomes $BISp2d$.

4. $BIGL_{q,q^{-2}}(2d)$

It is known that q-algebras can be constructed by using deformed oscillator algebras. One of the deformed oscillator algebra is a multidimensional q-deformed bosonic Newton oscillator algebra. It was obtained from the standard quantum harmonic oscillator Newton oscillator. The algebra is:

$$a_i a_i^* - q^2 a_i^* a_i = q^{2N} \delta_{ij},$$  \hspace{1cm} (32)  

$$a_i N - N a_i = a_i,$$  \hspace{1cm} (33)  

$$a_i a_j = a_j a_i.$$  \hspace{1cm} (34)
Here $N$ is the usual number operator. The indices $i$ and $j$ take values from 1 to $d$. To find the inhomogeneous invariance quantum group of this algebra first we make little change in the relations. We write $H$ instead of $q^{2N}$, then the algebra becomes:

$$a_ia_j^* - q^2a_j^*a_i = q^{2N}\delta_{ij},$$
$$a_iH = q^2Ha_i,$$
$$a_ia_j = a_ja_i.$$ (35, 36, 37)

Then we make a linear canonical transformation on the elements of algebra and we want that the algebra remains invariant after the transformation. The transformation matrix is

$$T = \begin{pmatrix}
\alpha_{ij} & \beta_{ij} & \eta_i & \gamma_i \\
\beta_{ij}^* & \alpha_{ij}^* & \eta_i^* & \gamma_i^* \\
0 & 0 & \chi_3 & \chi_4 \\
0 & 0 & 0 & 1
\end{pmatrix} = \begin{pmatrix} A & \Gamma \\
0 & B \end{pmatrix}. $$ (38)

The relations between the elements of matrix $A$ is as the same as multiparametric deformed $GL(n)$ by the following deformation

$$p_{ij} = q_{ij}^{-1} = \begin{cases} 
1 & i < j \leq d \quad \text{or} \quad d < i < j \leq 2d, \\
q & i \leq d \quad \text{and} \quad d < j \leq 2d.
\end{cases}$$ (39)

We can say that the matrix $A$ is $GL_{q^2, q^{-2}}$. The other relations are:

$$\alpha_{ik}\eta_j = q^{-2}\eta_j\alpha_{ik},$$ (40)
$$\alpha_{ik}\gamma_j = \gamma_j\alpha_{ik},$$ (41)
$$\beta_{ik}\eta_j = q^2\eta_j\beta_{ik},$$ (42)
$$\beta_{ik}\gamma_j = \gamma_j\beta_{ik},$$ (43)
$$\alpha_{ik}\chi_3 = \chi_3\alpha_{ik},$$ (44)
$$\alpha_{ik}\chi_4 = q^2\chi_4\alpha_{ik},$$ (45)
$$\beta_{ik}\chi_3 = q^4\chi_3\beta_{ik},$$ (46)
$$\beta_{ik}\chi_4 = q^2\chi_4\beta_{ik},$$ (47)
$$\alpha_{ik}\eta_j^* = \eta_j^*\alpha_{ik},$$ (48)
$$\alpha_{ik}\gamma_j^* = q^2\gamma_j^*\alpha_{ik},$$ (49)
$$\beta_{ik}\eta_j^* = q^4\eta_j^*\beta_{ik},$$ (50)
$$\beta_{ik}\gamma_j^* = q^2\gamma_j^*\beta_{ik},$$ (51)
$$\eta_i\eta_j = \eta_j\eta_i,$$ (52)
$$\eta_i\gamma_j - \gamma_j\eta_i = \frac{1}{2}(\alpha_{jk}\beta_{ik} - \alpha_{ik})\beta_{ij},$$ (53)
$$\gamma_i\gamma_j = \gamma_j\gamma_i,$$ (54)
$$\eta_i\eta_j^* = q^2\eta_j^*\eta_i,$$ (55)
$$\eta_i\gamma_j^* - q^2\gamma_j^*\eta_i = \frac{1}{2}(\chi_3\delta_{ij} + q^2)\beta_{jk}\beta_{ik} - \alpha_{ik}\alpha_{jk}^*,$$ (56)
$$\gamma_i\gamma_j^* - q^2\gamma_j^*\gamma_i = \chi_4\delta_{ij},$$ (57)
$$\chi_3\chi_4 = \chi_4\chi_3.$$ (58)
The Hopf algebra structure of this system can be studied by defining coproduct, counit, and antipode. The coproduct is matrix tensor multiplication and the counit is defined in equation 29. Now we should find the antipode of the transformation.

\[ S(T) = T^{-1} = \begin{pmatrix} A^{-1} & -A^{-1}B^{-1} \\ 0 & B^{-1} \end{pmatrix}. \] (59)

Since the elements of \( B \) commutes among themselves and the matrix \( A \) is \( GL_{q_2, q^{-2}} \), \( T^{-1} \) can be found by using above equation. Because of all Hopf algebra axioms are satisfied we can conclude that our transformation is a quantum group and we may call it the Bosonic Inhomogeneous two parameter deformed General Linear quantum group \( B\text{IGL}_{q, q^{-2}}(2d) \).[9] By taking \( q \) and \( H \to 1 \) \( B\text{IGL}_{q, q^{-2}}(2d) \) becomes \( \text{BISp}(2d) \).

5. Conclusion

Quantum groups are algebraic objects and its widely used in many areas of mathematics and physics especially to solve quantum integrable systems. There are two ways to obtain quantum groups. The fist one ,may be the most known, is Drinfeld’s approach. In that approach a deformation parameter is defined and a Lie algebra is deformed by using the deformation parameter. By using this approach deformed algebras are constructed. Instead of a Lie algebra one may use particle algebras. In this study we gave some examples of deformed boson algebras.

The second way to get a quantum group is Woronowicz’s matrix quantum group. On the other hand Manin has shown that matrix quantum groups can be derived as linear transformation on a quantum plane. In that approach a linear canonical transformation is made on an algebra and the elements of the transformation have to satisfy Hopf algebra relations.

In this study we have given examples of deformed algebras which are generated from boson algebra and their inhomogeneous invariance quantum groups. Now we can make a concluding list. For boson algebra its inhomogeneous invariance quantum group is \( \text{BISp}(2d) \). If we make multidimensional deformation on the boson algebra we get multidimensional \( q \) deformed boson algebra and its inhomogeneous invariance quantum group is \( B\text{IGL}_{q_{ij}, q_{ij}^{-1}}(2d) \). It can be shown that by taking \( q_{ij} = 1 \) we get usual boson algebra and \( B\text{IGL}_{q_{ij}, q_{ij}^{-1}}(2d) \) becomes \( \text{BISp}(2d) \). The last example was multidimensional \( q \) deformed bosonic Newton algebra and it was constructed from quantum harmonic oscillator algebra. Its inhomogeneous invariance quantum group is \( B\text{IGL}_{q, q^{-2}}(2d) \). By taking \( q \) and \( H \to 1 \) the algebra becomes usual boson algebra and \( B\text{IGL}_{q, q^{-2}}(2d) \) becomes \( \text{BISp}(2d) \).

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