The matter power spectrum in f(R) gravity

Tomi Koivisto
Helsinki Institute of Physics, FIN-00014 Helsinki, Finland
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Modified gravity can be considered as an alternative to dark energy. In a generalized theory of gravity, the universe may accelerate while containing only baryonic and dark matter. We study, in particular, the evolution of matter fluctuations in f(R) models within the Palatini approach, and find that the resulting matter power spectrum is sensitive to a nonlinear dependence on the curvature scalar in the gravitational action. The constraints that arise from comparison to the form of the observed matter power spectrum tighten the previous constraints derived from background expansion by several orders of magnitude. Models in the allowed parameter space are practically indistinguishable from general relativity with a cosmological constant when the background expansion is considered.

I. INTRODUCTION

One approach to the current dark energy problem in cosmology is to consider the observed acceleration of the universe as a consequence of gravitational dynamics that deviate from general relativity at cosmological scales. This can be achieved by modifying the linear proportionality to the curvature (Ricci) scalar in the Einstein-Hilbert action to a more general dependence on curvature invariants, see[1] for a recent introduction and[2] for other applications of extended gravities in cosmology. Here we will consider models in which the gravitational Lagrangian is a nonlinear function of the curvature scalar R, and the deviations from general relativity become important at small curvature. Such cases have been shown to lead to effective dark energy[3][4][5]. Furthermore, we adopt the Palatini formulation, in which an f(R) gravity is described by second order field equations[6][7][8][9][10]. The expansion of the universe has been solved in these models and recently also successfully contrasted with various cosmological data sets[11][12], see also[13]. Assuming that the correction to the Einstein gravity can be parameterized in the form $\sim R^{\beta}$, the data excludes the inverse curvature model (with $\beta = -1$), but shows a slight preference for a non-null correction, although the evidence is not compelling[12].

In this paper we will investigate further the viability of the Palatini approach to nonlinear gravity by extending the analysis to the constraints arising from cosmological perturbations. In a previous work[14], we were concerned that the modified matter couplings these models feature would, while causing the effective large negative background pressure, affect the evolution of inhomogeneities in such ways that the primordial fluctuations might not form into similar large scale structure as observed at the present. We solve cosmology to linear order and match the calculated shape of the matter power spectrum with the measurements of Sloan Digital Sky Survey (SDSS)[15]. The resulting constraints indeed reduce the allowed parameter space into a tiny region around the simplest correction to $f(R) = R$, which is the cosmological constant, corresponding to $\beta = 0$ and thus $f(R) = R - 2\Lambda$. For a recent discussion of matter perturbations within the metric formalism of f(R) gravity, see[16].

We begin with a brief review of Palatini approach to f(R) gravity and its cosmology in section II. The scheme for deriving the large scale structure constraints and the results are presented in section III. We conclude in section IV.

II. PALATINI APPROACH TO f(R) GRAVITY

A. Field equations

We will consider the class of gravity theories represented by the action

$$S = \int d^4 x \sqrt{-g} \left[ \frac{1}{2\kappa} f(R(g_{\mu\nu}, \tilde{\Gamma}_{\beta\gamma}^\alpha)) + \mathcal{L}_m(g_{\mu\nu}, \Phi, ...) \right],$$

(1)

where $\kappa = 8\pi G$. The matter action coupling to gravity depends only on some matter fields $\Phi, ...$ and the metric $g_{\mu\nu}$. Our notation emphasizes that the Ricci scalar defining the gravitational sector of the action depends on the two independent fields, the metric and the connection $\tilde{\Gamma}_{\beta\gamma}$, see also[14]. Explicitly, we may write

$$R = g^{\mu\nu} \tilde{R}_{\mu\nu},$$

(2)

where

$$\tilde{R}_{\mu\nu} = \tilde{\Gamma}_{\mu\nu,\alpha} - \tilde{\Gamma}^{\alpha}_{\mu\nu,\alpha} + \tilde{\Gamma}^{\alpha}_{\mu\lambda} \tilde{\Gamma}_{\lambda\nu} - \tilde{\Gamma}^{\alpha}_{\mu\lambda} \tilde{\Gamma}^{\lambda}_{\lambda\nu}.$$ (3)

The field equations which follow from extremization of the action, Eq.(1), with respect to metric variations, are

$$F \delta \tilde{R}_{\mu\nu} - \frac{1}{2} f' \delta g_{\mu\nu} = \kappa T_{\mu\nu},$$

(4)

where the energy momentum tensor is as usually

$$T_{\mu\nu}^{(m)} = - \frac{2}{\sqrt{-g}} \frac{\delta (\sqrt{-g} \mathcal{L}_m)}{\delta (g^{\mu\nu})},$$

(5)

*Electronic address: tomikoiv@pcu.helsinki.fi*
and we have defined for notational convenience

\[ F \equiv \partial f / \partial R. \]  

In general relativity \( F = 1 \). Contraction of the field Eq. (1) gives

\[ RF - 2f = \kappa T, \]  

an algebraic relation between the trace of the energy momentum tensor and the scalar curvature \( R \).

By varying the action with respect to \( \hat{\Gamma}^{\alpha\beta\gamma}_{\gamma\delta\epsilon} \), it is found that the connection is compatible with the conformal metric

\[ \hat{g}_{\mu\nu} \equiv F g_{\mu\nu}. \]  

It follows that the Ricci tensor in Eq. (3) is expressed solely in terms of \( g_{\mu\nu} \) as in [12], we find that the Hubble parameter may be expressed in terms of \( R \) and \( \kappa \rho \),

\[ \dot{H} \equiv H(1 + \frac{\dot{R}}{2F}). \]  

\[ H^2 = \frac{1}{6F} \frac{3f - RF}{1 - \frac{3f}{2F} \frac{RF - 2f}{RF - F}}. \]  

For a known function \( f(R) \), one can be solve \( R \) at a given \( a \) from Eq. (17) and thus get the expansion rate Eq. (12) by just algebraic means. In what follows, we will also need the derivatives \( H \) and \( \dot{H} \). Expressions for these can be derived similarly.

In order to investigate these models quantitatively, we must specify the form the gravitational Lagrangian. We will use the same parameterization as [12],

\[ f(R) = R - \alpha R^\beta, \]  

where \( \alpha \) is positive and has dimensions of \( H^{2-2\beta} \). The (dimensionless) exponent \( \beta \) is less than unity, lest the correction to the Einstein-Hilbert action would interfere with the early cosmology. Given \( \beta \) and the amount of matter in the present universe, \( \Omega_m \equiv \kappa \rho_m / (3H_0^2) \), determines the scale \( \alpha \). Not all combinations of \( \alpha \) and \( \beta \) are consistent with a flat matter dominated universe today, and therefore we rather use \( \Omega_m \) and \( \beta \) as our pair of parameters. We will refer to the limit \( \beta = 0 \) as the \( \Lambda \)CDM case.

To demonstrate that models defined by the Lagrangian [13] can produce a plausible expansion history, one can compute the predicted evolution of the Hubble parameter, Eq. (12) to cosmological data. As an example we compare the predicted evolution of the Hubble parameter, Eq. (12) to cosmological data. As an example we compare the predicted evolution of the Hubble parameter, Eq. (12) to cosmological data. As an example we compare the predicted evolution of the Hubble parameter, Eq. (12) to cosmological data.
The equation may be written as

$$d^2 \delta_m / dx^2 = A(x) \frac{d \delta_m}{dx} + B(x) + \frac{k^2}{a^2 H^2} C(x) \delta_m,$$

where the dimensionless functions $A$, $B$ and $C$ are given by

$$FH^2 (2FH + \dot{F}) A = -2 \left( \dot{H} + 2H^2 \right) F^2 + 2FH \dot{F} + \left( 2FH + F \dot{H} - 2FH^2 \right) \dot{F},$$

$$FH^4 (2FH + \dot{F}) B = 2 \left( \dot{H} + 2H \dot{H} \right) H^2 F^2 + 2\dot{H}HF \dot{F} - \left( 2\dot{HHF} - \ddot{HHF} - 2\dot{H}^2 F - 2H^2 \ddot{H}F \right) \dot{F},$$

$$C = -\frac{\dot{F}}{3F(2FH + \dot{F})}.$$  \hfill (15)

We have also checked that our solutions of the Eq. (15) are consistent with the explicit field equations listed in [14].

We get the initial conditions for the matter perturbation,

$$\delta_m = \frac{\rho_c \delta_c + \rho_b \delta_b}{\rho_c + \rho_b},$$

by evolving the standard Einstein-Boltzmann equations up until $z_i = 200$, corresponding to $x_i = -\log(201)$. It is only at smaller redshifts that the corrections to general relativity begin to have effect. Note also that treating the energy density of the universe as consisting of single dust-like fluid (made of dark matter and baryons) is well justified. At $z_i = 200$ radiation is subdominant and may be neglected, and the imprint it has left on perturbations in the earlier universe is carried to the initial conditions (namely, $\delta_m(k, x_i)$ and $d\delta_m(k, x_i)/dx$). The Compton scattering with photons and pressure of the baryons are completely negligible at $z_i$ and therefore baryons obey the same equations as dark matter. Hence we can, at the linear order, consider just the total fluid.

The effective sound speed [21] is given by Eq. (10), and is nonvanishing except at the $\Lambda$CDM limit. The curvature corrections induce effective pressure fluctuations in matter, leading to the gradient term in Eq. (15). While the other deviations from the perturbation evolution in the $\Lambda$CDM scenario may be small when $|\beta|$ is, the gradient can still be large at small enough scales. As one could expect [14], inhomogeneities are significantly affected by the additional matter couplings, the most sensitive effect being the response of modified gravity to spatial variations in the distribution of matter. The situation bears some resemblance to dark energy models with a coupled dark sector [21, 22, 23]. In fact, in the conformally equivalent Einstein frame, where the line-element [10] is given by the metric [8], one finds a scalar field with non-minimal coupling to matter [24].

We calculate the matter power spectrum,

$$P(k) = (2\pi k)^{-3} \delta_m^2(k, x = 0),$$

where $\delta$ is considered in the Fourier space. To fit to the shape of the observed power spectrum, we use the best-fit normalization for each model. We assume that the spectral index lies between $n_S \in [0.8, 1.2]$ and marginalize over these values. We make a $\chi^2$ fit to the SDSS data [12]. Conservatively, we use only measurements at scales $k < 0.2h$ Mpc$^{-1}$, since at smaller scales there are nonlinear effects in the measured $P(k)$. We set the

\[\text{FIG. 2: The matter power spectra for } \beta = -0.00001 \text{ (solid), } \beta = -0.00005 \text{ (dash-dotted), } \beta = 0.00001 \text{ (dotted) and } \beta = 0.00005 \text{ (dashed). In all the cases } \Omega_m = 0.3 \text{ and } n_S = 1. \text{ (We do not use the data points at the three smallest scales, and instead of the error bars plotted here we use the exact window functions.)} \]

\[\text{FIG. 1: Projecting to the } (\alpha, \beta) \text{ plane would reproduce Fig. 1 of [12], where are presented also further constraints derived from multiple data sets.} \]

\[\text{III. CONSTRAINTS FROM THE LARGE SCALE STRUCTURE} \]

The foundations of cosmological perturbation theory in the Palatini approach to modified gravity have been laid in [14]. There was also derived an evolution equation for the comoving energy density perturbation $\delta_m$ in a matter dominated universe. Using the time variable $x \equiv \log(a)$, the equation may be written as

$$D_\alpha^2 \delta_m = A(x) D_\alpha \delta_m + B(x) + \frac{k^2}{a^2 H^2} C(x) \delta_m,$$

where $D_\alpha$ is the parallel transport derivative. A matter dominated universe. Using the time variable $x \equiv \log(a)$, the equation may be written as

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\[\text{1 We use the window functions provided at } \text{http://space.mit.edu/home/tegmark/sdss.html} \]
Hubble parameter to $h = 0.72$ and fix the baryon density to $\Omega_b = 0.044$ (since especially the latter is well determined\cite{24}, and now these parameters have only a slight effect on the initial conditions).

The resulting power spectra is shown for a few choices of the parameter $\beta$ in Fig. 2. For these small $|\beta|$, the gradient term begins to affect perturbations at just about $k \sim 0.1$ Mpc$^{-1}$. For much smaller $|\beta|$, the gradient could be neglected at non-linear scales. For much larger $|\beta|$, there would be appreciable deviation from $\Lambda$CDM cosmology also at very large scales (due to the $k$-independent coefficients $A$ and $B$ in the evolution equation Eq.\,(15)), and the effect of the gradient would be large enough to render these models completely incompatible with SDSS data. Thus we can understand why fitting the models to the data provides the tight confidence limits plotted in Fig. 3. For $\beta > 0$, red tilt for the primordial spectrum is preferred, since the power at small scales is enhanced due to modified gravity. Correspondingly, when $\beta < 0$ the best fit is achieved with blue tilt, $n_S > 1$. The prior we have assumed, $n_S \in [0.8, 1.2]$, is rather loose, since the WMAP experiment is able to rule out large deviations from scale invariance\cite{20}, at least when standard assumptions hold in the pre-recombination universe. For this reason we plot also, in the lower panel of Fig. 3, the somewhat tighter constraints which follow from fixing the spectral index to $n_S = 1$. Marginalizing over $\Omega_m$, we find that the favored values of the exponent $\beta$ are smaller than $\sim 3 \cdot 10^{-5}$ in magnitude. Clearly, inclusion of the smallest scale SDSS data points would have tightened these constraints even further.

**IV. CONCLUSION**

We calculated the matter power spectrum in the Palatini formulation of modified gravity, and found that the observational constraints reduce the allowed parameter space to a tiny region around the $\Lambda$CDM cosmology. We investigated a specific form for the gravitational Lagrangian, Eq.\,(13), but similar conclusions would probably hold regardless of the parameterization employed. As already found in\cite{14}, the problematic features in the perturbation evolution, governed by Eq.\,(15), are generic to these models.

It would seem difficult to find a way to produce the observed large scale structure in the presence of the modified matter couplings peculiar to these alternative gravity theories when $F$ is not very nearly a constant. If this could be accomplished by invoking exotic properties to the matter components (isocurvature initial conditions, or may be entropic perturbations to cancel the effects of the couplings), it would not be without adding fine tuning and ad hoc assumptions to these models.

Determining consequences of modified gravity at scales of the Solar system\cite{25} or even particle physics\cite{26} experiments present different challenges as, while exact predictions may be incalculable, there can be ambiguities in how to parameterize and interpret the possible deviations from Einstein gravity\cite{27,28}. Furthermore, it seems that by simple arguments broad classes of models can altogether evade the Solar system constraints\cite{25,28}. In contrast, such implications within cosmological perturbation theory as discussed here are well understood and robust. Hence it might be worthwhile to study to which extent the results obtained here can be generalized to other forms of alternative gravity theories. This requires detailed work, but would enable probing the possibilities of a more fundamental theory of gravitation by the present and up-coming high-precision measurements of cosmological large-scale structure.
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