The Supercharges of Eleven-dimensional Supergraviton on Gravitational Wave Background

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Abstract

We find the explicit expression of the supercharges of eleven dimensional supergraviton on the background geometry of gravitational waves in asymptotically light-like compactified spacetime. We perform the calculations order by order in the fermions $\psi$, while retaining all orders in bosonic degrees of freedom, and get the closed form up to $\psi^5$ order. This should correspond to the supercharge of the effective action of (0+1)-dimensional matrix quantum mechanics for, at least, $\nu^4$ and $\nu^6$ order terms and their superpartners.

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1 Introduction

One of the most striking observations in the recent developments in M/string theory is the realization of the deep connection between the supergravity and super Yang-Mills theory. Among others, one remarkable example is the $AdS$/CFT correspondence\cite{1}. It tells us that the M/String theory or their low energy effective supergravity on $AdS$ spaces are equivalent to the conformal limit of super Yang-Mills theory on the boundary of $AdS$ spaces. This holographic nature of the theory has been clarified in \cite{2}.

Another important example is matrix theory \cite{3} in which $(0+1)$-dimensional $SU(N)$ matrix quantum mechanics is conjectured to give eleven dimensional M/supergravity in the large $N$ limit. In the framework of the discrete light-cone quantization (DLCQ), in which we take light-cone circle with radius $R$, the correspondence seems to hold even for finite $N$ \cite{4} and then one may understand the original matrix theory on full eleven-dimensional non-compact spacetime as the large $N$ limit, while keeping the light-cone momentum $p_-=N/R$ fixed.

From the prescriptions given in \cite{5, 6}, the DLCQ M theory on $T^p$ is described by $(p+1)$-dimensional super Yang-Mills theory on dual torus, for up to $p=3$. By applying their arguments on the supergravity backgrounds, it has been argued in \cite{7} that, as a kind of generalization of $AdS$/CFT correspondence, these microscopic descriptions of DLCQ M theory on $T^p$ via super Yang-Mills theory corresponds to the M/string theory on non-trivial backgrounds such as plane-fronted gravitational waves or $AdS$ spaces with some identifications. This can be inferred from the simple observation that the limit considered in DLCQ M theory on $T^p$ is exactly the same as the limit considered by Maldacena in \cite{1} to get the $AdS$/CFT correspondence. It is the natural limit if one wants to have only D-brane world-volume theory, while decoupling all the bulk degrees of freedom. In \cite{1}, the corresponding supergravity backgrounds are found by near-horizon limit of the supergravity solutions in the presence of source branes. In \cite{7}, those come from taking DLCQ and T-duality on the supergravity solutions of the source branes.

Especially interesting cases are DLCQ M theories on $T^0$ (noncompact ten-dimensional spacetime) and $T^1$, which correspond to the original matrix theory and matrix string theory \cite{8}, respectively. In these cases we get the gravitational waves on the asymptotically light-like compactified spacetime as background geometry \cite{7, 8, 9, 10}. Indeed in \cite{12}, it has been shown that the matrix quantum mechanics at one loop gives the same effective action as those of the eleven-dimensional probe graviton on this background. Subsequently, superpartners of
bosonic $F^4$ term have been explicitly obtained within matrix theory and verified to agree with those of supergraviton on this background [13]-[21].

One may wonder why they give the correct descriptions, without good explanation on the nature of holography. Recent studies on the non-renormalization theorem of matrix model due to 16 supercharges [22, 23, 24] strongly suggests the key role played by the supersymmetry in these correspondences. After all, the regime that we can trust supergravity solutions is different from the regime that we can trust Yang-Mills descriptions of D-branes, and only reasonable amount of supersymmetry would connect those two regions.

In this paper, we want to shed some light on these issues. In explicit, we find the supercharge of the eleven-dimensional supergraviton in the background of gravitational waves, which are eleven-dimensionally lifted D0 solutions. First of all, we consider the superparticle on flat background. The superparticle action we choose is the first quantized version of eleven-dimensional supergravity. If the light-cone coordinate $x^-$ is periodically identified, the light-cone momentum is quantized, $p_\pm = \frac{N}{R}$, and the single supergraviton ($N = 1$) moving in that direction is described by (0+1)-dimensional $U(1)$ matrix quantum mechanics. The original superparticle action has target space supersymmetry, which is apparently very different from the supersymmetry of super Yang-Mills theory. However, the superparticle action has additional local world-line fermionic symmetry, namely $\kappa$-symmetry, and it is shown that they are identical after choosing the light-cone gauge for this local $\kappa$-symmetry and modifying the original supersymmetry by the appropriate $\kappa$-transformations in such a way to preserve the gauge-fixing conditions.

In section 3 we consider the superparticle on the background geometry produced by source gravitational waves on the asymptotically light-like compactified spacetime. As a (0+1)-dimensional $\mathcal{N} = 16$ D0 quantum mechanics, this corresponds to the effective theory of probe D0-branes moving in the background of $N$ source D0-branes. Since the full explicit expressions for the action of the superparticle on non-trivial background are not known, we perform the calculations order by order in fermions $\psi$ and get supercharges up to $\psi^5$ order.

Among others, we find the effective Lagrangian on this background is given by

$$\frac{1}{p_-} \mathcal{L}_{eff} = \frac{1}{1 + \sqrt{1 - h v^2}} v^2 + i \frac{(1 + \sqrt{1 - h v^2})}{4 \sqrt{1 - h v^2}} \psi \dot{\psi} + i \frac{v^2 v_i \partial_j h (\psi \gamma^{ij} \dot{\psi})}{8 \sqrt{1 - h v^2} (1 + \sqrt{1 - h v^2})}$$

$$- \frac{h^2 v^2}{32 (1 - h v^2)^{3/2}} (\psi \dot{\psi})^2 - \frac{(2 - h v^2) v_i \partial_j h}{32 (1 - h v^2)^{3/2}} (\psi \gamma^{ij} \dot{\psi}) \dot{\psi} \dot{\psi} + \cdots ,$$

(1.1)
where $x^i$ are the position coordinates of superparticle (Higgs fields in D0 quantum mechanics), $v^i = \dot{x}^i$ and $h$ is the nine-dimensional harmonic function which will be given later.

The Noether supercharges, quite surprisingly, turn out to be

$$Q = p \cdot \gamma \psi + \mathcal{O}(\psi^5),$$

where $p_i \equiv \frac{1}{p_-} \frac{\partial \mathcal{L}_{\text{eff}}}{\partial \dot{v}^i}$ is the effective conjugate momentum of $x^i$ in units of $p_-$. Note that there is no correction at the $\psi^3$ order, when written in terms of conjugate momenta $p_i$, and it is tempting to conjecture that it would be the case to all orders in $\psi$. Yet they contain all order corrections in $\psi$ if written in terms of $v^i$. As the effective Lagrangian contains fermions, it has the second class constraints which can be analyzed using Dirac brackets. Since it is non-local, the resultant Dirac brackets are nontrivial, which is another source for non-trivial higher order corrections. The supercharges satisfy the usual supersymmetry algebra:

$$[Q_a, Q_b] = \frac{2}{p_-} \mathcal{H}_{\text{eff}} \delta_{ab} + \mathcal{O}(\psi^4),$$

where $\mathcal{H}_{\text{eff}}$ is the effective Hamiltonian given by

$$\mathcal{H}_{\text{eff}} = p_- \frac{p^2}{1 + \sqrt{1 + hp^2}} - \frac{ip_-}{8} \frac{p^2 p_i \partial_j h \psi \gamma^{ij} \psi}{\sqrt{1 + hp^2}(1 + \sqrt{1 + hp^2})} + \mathcal{O}(\psi^4).$$

2 Supersymmetry of eleven-dimensional supergraviton in the
light-cone gauge

In this section we consider the supersymmetry transformation rules of eleven-dimensional supergraviton on light-like compactified spacetime. The natural gauge choice is the light-cone gauge and the supersymmetries which preserve this gauge choice are identical to those of D0 quantum mechanics as expected.

Consider the eleven-dimensional manifestly spacetime supersymmetric massless point particle action,

$$S = \int d\lambda \mathcal{L} = \frac{1}{2} \int d\lambda e^{-1} (\dot{x}^\mu + i \tilde{\theta} \Gamma^\mu \dot{\theta})^2,$$

where $\theta$ are 32 component real spinors and $\tilde{\theta} = \theta^T \Gamma^0 \Gamma[\Gamma$. This action describes eleven-dimensional supergraviton multiplet in flat Minkowskian spacetime. The equations of motion of the multiplet are given by those of linearized eleven-dimensional supergravity. The model has world-line

\footnote{The eleven-dimensional $32 \times 32$ gamma matrices $\Gamma^r$ ($r = 0, 1, \cdots, 9, 11$) that we use in this paper are given}
reparametrization invariance under which \( x^\mu \) and \( \theta \) transform as world-line scalar,

\[
\delta_\zeta x^\mu = \zeta \dot{x}^\mu, \quad \delta_\zeta \theta = \zeta \dot{\theta}.
\]  

As we take the light-cone coordinate \( x^- \) periodic, it is natural to identify \( x^+ \) as time coordinate. The natural gauge choice for the world-line diffeomorphism in this DLCQ formulation is the static gauge

\[
\dot{x}^+ = 2 \dot{x}^- = 2.
\]  

The target spacetime supersymmetry transformation laws with parameter \( \xi \) are given by

\[
\delta \theta = \xi, \quad \delta x^\mu = -i \bar{\xi} \Gamma^\mu \theta, \\
\delta \bar{\theta} = \bar{\xi}, \quad \delta e = 0.
\]  

In addition, the action has local fermionic symmetry with parameter \( \kappa(\lambda) \), under which the fields transform as

\[
\delta \theta = ie \Gamma \cdot p \kappa, \quad \delta x^\mu = -i \bar{\theta} \Gamma^\mu \delta \theta, \quad \delta e = 4e \dot{\bar{\theta}} \kappa,
\]  

where \( p_\mu = e^{-1} \eta_{\mu\nu} (\dot{x}^\nu + i \bar{\theta} \Gamma^\nu \dot{\theta}) \) denotes the conjugate momentum of \( x^\mu \). As being local gauge symmetry, this \( \kappa \)-symmetry reduces the \( \theta \) degrees of freedom by half. We can fix it by choosing

\[
\Gamma^+ \theta = 0, \quad \theta = \begin{pmatrix} \theta_{(16)} \\ -\theta_{(16)} \end{pmatrix},
\]  

where \( \Gamma^\pm = (\Gamma^{10} \pm \Gamma^0) \). Note that with this gauge fixing the conjugate \( \bar{\theta} \equiv \theta^T \Gamma^0 \) becomes

\[
\bar{\theta} = \theta \frac{1}{2} (\Gamma^+ - \Gamma^-) = \theta \Gamma^r,
\]  

where \( \Gamma^r = \Gamma^+/2 \). This is in accord with the gauge fixing (2.3).

The light-cone momentum \( p_- \), which is conjugate to the periodically identified light-cone coordinate \( x^- \equiv x^- + 2\pi R \), is quantized and given by \( p_- = \frac{N}{R} \). Since the coordinate \( x^- \) by

\[
\Gamma^0 = \begin{pmatrix} 0 & I_{16} \\ -I_{16} & 0 \end{pmatrix}, \quad \Gamma^i = \begin{pmatrix} 0 & \gamma^i \\ \gamma^i & 0 \end{pmatrix},
\]  

\[
\Gamma^{11} = \Gamma^0 \cdots \Gamma^9 = \begin{pmatrix} I_{16} & 0 \\ 0 & -I_{16} \end{pmatrix},
\]  

where \( I_{16} \) is the \( 16 \times 16 \) identity matrix and \( \gamma^i \) are real \( 16 \times 16 \) \( SO(9) \) gamma matrices \( (i = 1, \cdots, 9) \). These gamma matrices satisfy \( (\Gamma^0)^\dagger = -\Gamma^0 \) and \( (\Gamma^i)^\dagger = \Gamma^i \).
is cyclic, $p_-$ is conserved and we consider the fixed $N$-sector of the theory. Therefore the appropriate effective action is given by Routhian

$$\mathcal{L}_{eff} = \mathcal{L} - p_+ \dot{x}^- (p_-) = -p_- \dot{x}^-, \quad (2.7)$$

where the last relation in the above comes from the constraint equation

$$\eta_{\mu \nu} p^\mu p^\nu = 0, \quad (2.8)$$

which is also used to solve $\dot{x}^-$ in terms of $p_-, x^i, \theta$.

The effective Lagrangian after the gauge fixing (2.3) and (2.6) becomes

$$\mathcal{L}_{eff} = p_- \left( \frac{(\psi^i)^2}{2} + 4 i \theta_{(16)} \dot{\theta}_{(16)} \right), \quad (2.9)$$

which corresponds to the (0+1)-dimensional $U(1)$ supersymmetric Yang-Mills quantum mechanics with adjoint fermions $\psi \equiv 2 \sqrt{2} \theta_{(16)}$.

This tells that even though the original supersymmetry transformation laws of the first-quantized supergravity action (2.1) are very different from those of super Yang-Mills theory, they should be the same after the gauge fixing (2.3) and (2.6). Half of the original supersymmetries with $\Gamma^+ \xi = 0$ remain to be true symmetry in which

$$\delta \theta = \xi, \quad \delta x^i = 0,$$

even after gauge fixing. These are just constant shift in spinors which also appear as trivial kinematic supersymmetry in the D0 quantum mechanics.

On the other hand, other half of the form $\xi = \left( \begin{array}{c} \xi_{(16)} \\ \xi_{(16)} \end{array} \right)$, corresponding to $\Gamma^- \xi = 0$, do not preserve the gauge fixing conditions and thus the original supersymmetry transformation laws (2.4) with these parameters should be modified in such a way to preserve those by taking appropriate $\kappa$-transformations (2.5). From the condition that total transformations should preserve the gauge fixing (2.6):

$$\Gamma^+ (\delta \theta) = \Gamma^+ (\xi + i \Gamma \cdot p \kappa) = 0, \quad (2.10)$$

we find the relation between target spacetime supersymmetry parameter $\xi$ and the kappa symmetry parameter $\kappa$ of the form,

$$\kappa = \left( \begin{array}{c} \kappa_{(16)} \\ -\kappa_{(16)} \end{array} \right), \quad (2.11)$$

which is given by

$$\kappa_{(16)} = i \frac{\xi_{(16)}}{2}. \quad (2.11)$$
The total supersymmetry transformations preserve the static gauge (2.3) and are given by

\[ \delta \psi = v^i \gamma^i \epsilon, \]
\[ \delta x^i = -i \epsilon \gamma^i \psi \]

with \(\epsilon \equiv \sqrt{2} \xi(16)\). These are nothing but the supersymmetry transformation laws of (0+1)-dimensional super Yang-Mills quantum mechanics at the tree level.

This suggests that the matrix model, viewed as the large \(N\) limit of DLCQ M theory, in the infrared limit may describe eleven-dimensional second-quantized DLCQ supergravity, i.e. the large \(N\) limit of first-quantized DLCQ supergravity. To confirm this conjecture, one may construct vertex operators [19, 25] which describe generic interactions among various fields and macroscopic objects and compare with super Yang-Mills theory, which is outside the scope of this paper.

In the next section we consider the supergraviton in gravitational wave background, which would correspond to the case of two-body interactions between two clusters of D0-particles in which one plays the role as a source and the other as a probe.

### 3 Supergraviton in gravitational wave background

The action (2.1) can be generalized to describe supergraviton on the general background in the following form:

\[ S = \int d\lambda \mathcal{L} = \frac{1}{2} \int d\lambda e^{-1} \eta_{rs} \Pi^r \Pi^s, \]

where \(Z^M(\lambda) = (x^\mu(\lambda), \theta^a(\lambda))\) are the superspace coordinates\(^2\). The pull-back \(\Pi^A\) of the supervielbein \(E^A_M\) to the particle’s world-line satisfies \(\Pi^A = (dZ^M/d\lambda) E^A_M\). This can be determined order by order in \(\theta\) by following the method in [26] and is given by

\[ \Pi^r = \dot{x}^\mu (e^r_\mu - \frac{1}{4} \bar{\theta} \Gamma^{rst} \theta \omega_{\mu st}) + \bar{\theta} \Gamma^r \dot{\theta} + \mathcal{O}(\theta^4) \]

for the background geometry with vanishing three form gauge field \(C_{\mu \nu \rho}\) and gravitino \(\psi_\mu\).

\(^2\)In the superspace formalism, the indices \((A, B, C, \cdots)\) collectively denote the bosonic and fermionic tangent space indices, while \((M, N, P, \cdots)\) collectively denote the bosonic and fermionic curved space indices. In its component form, we use \((\mu, \nu, \rho, \cdots)\) indices for the curved space bosonic coordinates, and \((r, s, t, \cdots)\) indices for the tangent space bosonic coordinates. Therefore the metric \(\eta_{rs}\) is the Lorentz invariant constant metric. Spinor indices are denoted as \((a, b, c, \cdots)\).
3.1 Effective action

As the background geometry we consider is independent of periodically identified coordinate \( x^- \), the quantized conjugate momentum is still conserved and, in the fixed \( p_- \) sector, the effective action is again of the form (2.7). In [20], along the same steps as described above in the free massless superparticle case, it was shown that the supersymmetric effective action, in the light-cone gauge, corresponding to \( \psi_4 \) term is the same as matrix theory effective action. The action has world-line diffeomorphism and \( \kappa \)-symmetry as local gauge symmetries. We use the same gauge fixing conditions (2.3) and (2.6) as in the case of flat spacetime background.

The eleven-dimensionally lifted D0-solution with asymptotically light-like compactification, \( x^- \equiv x^- + 2\pi R \), is given by [9]

\[
ds_{11}^2 = dx^+ dx^- + h(r)(dx^-)^2 + dx_1^2 + \cdots + dx_9^2 ,
\]

where \( r = (x_1^2 + \cdots x_9^2)^{1/2} \) and \( \psi_\mu = C_{\mu\nu\theta} = 0 \). The harmonic function \( h(r) \), which characterize this background geometry, is of the form

\[
h(r) = \frac{15}{2} \sum_{I=1}^N \frac{l_p^0/R^2}{|r - r_I|^7} ,
\]

where \( l_p \) is eleven-dimensional Planck scale. This geometry represents half BPS states of \( M \) theory, admitting 16 Killing spinors which satisfy the Killing spinor equation

\[
D_\mu \xi = (\partial_\mu - \frac{1}{4} \omega_{\mu}^rs \Gamma_{rs}) \xi = 0 .
\]

These sixteen Killing spinors are of the form

\[
\xi = \begin{pmatrix} 
{\xi}_{(16)} \\
{\xi}_{(16)} 
\end{pmatrix} = \frac{1}{\sqrt{2}} f^{-1/4} \begin{pmatrix} 
{\epsilon} \\
{-\epsilon} 
\end{pmatrix} ,
\]

where \( \epsilon \) denotes a constant spinor\(^3\) and \( f(r) = 1 + h(r) \).

The effective Lagrangian (2.7) on this background can be decided order by order in \( \psi_4 \) using the constraint equation,

\[
\eta_{rs} \Pi^r \Pi^s = 0 .
\]

\(^3\)From now on all spinors denote 16-component spinors and subscripts are omitted.

\(^4\) \( \psi_a \) is defined by \( \psi_a \equiv 2\sqrt{2} f^{-1/4} \theta_a \). These play the role as the superpartner of Higgs field \( x^i \) in the context of Yang-Mills quantum mechanics and thus the supersymmetry transformation laws for these turn out to be much simpler than those of \( \theta_a \). In deriving various formula, we will switch back and forth between \( \psi_a \) and \( \theta_a \), but the final results are presented in terms of \( \psi_a \).
It is given by, up to \( \psi^4 \) order,

\[
\frac{1}{p_-} \mathcal{L}_{\text{eff}} = \frac{v^2}{1 + \sqrt{1 - hv^2}} + i \left( \frac{1 + \sqrt{1 - hv^2}}{4\sqrt{1 - hv^2}} \right) \psi \dot{\psi} + i \frac{p_- v^2 \nu_i h \gamma^i j \psi}{8(1 - hv^2 + \sqrt{1 - hv^2})} + O(\psi^4) \quad (3.6)
\]

In contrast to \([20]\), in which only \( v^4 \) terms and their superpartners are presented, we retain the expression to all orders in \( v^i \). This may correspond to the effective Lagrangian of a D0-brane probe in the background of N D0-branes, relatively moving with the velocity \( v^i \) in the transverse direction. There is a non-renormalization theorem for the \( v^4 \) term and its superpartners in the (0+1)-dimensional super Yang-Mills quantum mechanics \([22, 24]\). Therefore they are completely determined by the one-loop calculations and was shown to agree with the supergravity result \((3.6)\) \([20]\). The same arguments hold true for the \( v^6 \) term and its superpartners and they agree with the supergravity result \((3.6)\) as well \([23]\).

### 3.2 Supercharge: leading order corrections

In the matrix model, it is extremely tedious to determine the supersymmetry transformation laws, though still one can decide the effective action of bosonic \( v^4 \) term and its superpartners without much knowledge on those \([22, 21]\). In the supergravity side, the explicit form of supersymmetry transformation rules are known up to \( \psi^3 \) order. We use these to find supercharges for supergraviton in the gravitational wave background. Up to \( v^6 \) (and possibly to all order in \( v \)) they should correspond to the supercharges in the matrix theory.

The supersymmetry transformation laws of fields in the supergraviton action \((3.1)\), up to \( \theta^3 \) order, are given by

\[
\delta \theta = \xi + i \frac{1}{4} (\bar{\theta} \Gamma^\mu \xi) \omega^s r_{rs} \theta + O(\theta^4) , \\
\delta x^\mu = i \bar{\theta} \Gamma^\mu \xi + O(\theta^3) . \quad (3.7)
\]

As mentioned earlier, the action \((3.1)\) has local fermionic \( \kappa \)-symmetry, whose transformation laws are

\[
\delta \theta = i \Gamma \cdot \Pi \kappa + \frac{1}{4} (\bar{\theta} \Gamma^\mu \kappa \cdot \Pi \kappa) \omega^r s_{rs} \theta + O(\theta^4) , \\
\delta x^\mu = -i \bar{\theta} \Gamma^\mu \delta \theta + O(\theta^3) , \\
\delta e = 4 e \bar{\theta} \kappa + O(\theta^3) . \quad (3.8)
\]

Obviously the sixteen Killing spinors \((3.5)\), which correspond to the dynamical supersymmetry in the flat background limit, do not respect the gauge fixing condition \((2.6)\). At the
leading order in $\theta$, to preserve the gauge fixing condition (2.6), the transformation should be supplemented by the $\kappa$-transformations of the form (2.11) with

$$\kappa = i \frac{f^{1/2}}{1 + \sqrt{1 - hv^2}} \xi + O(\theta^2).$$ \hspace{1cm} (3.9)

The combined transformation law of (3.5) and (3.9) for $\theta$ is given by

$$\delta \theta = \frac{f^{1/2}}{1 + \sqrt{1 - hv^2}} (v \cdot \gamma) \xi + O(\theta^2).$$ \hspace{1cm} (3.10)

In contrast to the flat background case, the combined transformation laws do not satisfy the gauge fixing (2.3) as $\delta x^+$ is non-vanishing:

$$\delta x^+ = 4i \frac{h}{1 + \sqrt{1 - hv^2}} (\theta v \cdot \gamma \xi) + O(\theta^3).$$ \hspace{1cm} (3.11)

This should be compensated by world-line reparametrization with parameter

$$\zeta = -2i \frac{h}{1 + \sqrt{1 - hv^2}} (\theta v \cdot \gamma \xi) + O(\theta^3)$$ \hspace{1cm} (3.12)

in (2.2). This in turn gives additional transformations on $x^i$ and $\theta^a$. The overall transformation law for $x^i$, in the leading order in $\theta$, is given by

$$\begin{align*}
\delta x^i &= 4i (\theta^i \xi) + \zeta \dot{x}^i + O(\theta^3) \\
&= 4i (\theta^i \xi) - 2i \frac{h (\theta v \cdot \gamma \xi)}{1 + \sqrt{1 - hv^2}} \dot{x}^i + O(\theta^3). \hspace{1cm} (3.13)
\end{align*}$$

The effective Lagrangian (3.7) is indeed invariant under the transformation laws (3.10) and (3.11), in the leading order terms of $\psi$, up to total derivatives,

$$\delta \mathcal{L}_{eff} = -i \frac{\not{p} - d}{2} (\epsilon v \cdot \gamma \psi) + O(\psi^3).$$

The corresponding supercharges can be easily obtained by Noether method and are given by the simple form:

$$Q = p \cdot \gamma \psi + O(\psi^3).$$ \hspace{1cm} (3.14)

### 3.3 Dirac brackets and supersymmetry algebra

In order to see the above supercharges (3.14) give the right transformation laws (3.10) and (3.11), we need to study carefully the commutation relations among fields. From the effective
Lagrangian (3.6) one can read off the effective conjugate momenta $p_i$ of $x^i$ and $\pi_a$ of $\psi_a$:

\[
p_i \equiv \frac{1}{p_-} \frac{\partial L_{\text{eff}}}{\partial v_i} = \frac{v_i}{\sqrt{1 - hv^2}} + \frac{i \hbar (\psi \dot{\psi}) v_i}{4(1 - hv^2)^{3/2}} + \frac{i \nu_k \partial_j h(\psi^T \gamma^j \psi) v_i}{8(1 - hv^2)^{3/2}} + \frac{i v^2 \partial_j h(\psi^T \gamma^j \psi)}{8(1 + \sqrt{1 - hv^2}) \sqrt{1 - hv^2}} + \mathcal{O}(\psi^4),
\]

(3.15)

and

\[
\pi_a \equiv \frac{1}{p_-} \frac{\partial L_{\text{eff}}}{\partial \dot{\psi}_a} = \frac{i(1 + \sqrt{1 - hv^2}) \psi_a}{4 \sqrt{1 - hv^2}} + \mathcal{O}(\psi^3),
\]

(3.16)

which has been defined in units of $p_-$ for simplicity. Note that (3.16) gives the second-class constraints:

\[
\Phi_a = \pi_a - \frac{i}{4} (1 + \sqrt{1 + hp^2}) \psi_a + \mathcal{O}(\psi^3) \approx 0
\]

(3.17)

It is well-known that in order to deal with the second-class constraints in the Hamiltonian formalism, which is typically present in the theory involving fermions, we need to introduce Dirac brackets. Let’s denote $(q^A, p_A)$ as fields and their conjugates collectively for bosons and $(q^a, p_a)$ for fermions. Then Poisson bracket between two functions $F(q, p)$ and $G(q, p)$ is defined as

\[
\{F, G\}_{PB} \equiv \frac{\partial F}{\partial q^A} \frac{\partial G}{\partial p_A} - (-1)^{n_F n_G} \frac{\partial F}{\partial q^A} \frac{\partial G}{\partial p_A},
\]

(3.18)

where $n_F$ is the fermion number of $F$. If the theory contains the second class constraints, $\Phi_a(q^A, p_A) \approx 0$, the Poisson bracket should be replaced by the Dirac bracket which is defined as

\[
\{F, G\} \equiv \{F, G\}_{PB} - \{F, \Phi_a\}_{PB} (A^{-1})_{ab} \{\Phi_b, G\}_{PB},
\]

(3.19)

where $A_{ab} \equiv \{\Phi_a, \Phi_b\}_{PB}$.

In the case we consider, the Poisson brackets among the fields and their conjugates are given by

\[
\{x^i, p_j\}_{PB} = \delta^i_j, \quad \{\psi_a, \pi_b\}_{PB} = \delta_{ab}
\]

(3.20)

and all others vanish. For the constraint (3.17), $A_{ab}$ read

\[
A_{ab} = -\frac{i}{2} (1 + \sqrt{1 + hp^2}) \delta_{ab} + \mathcal{O}(\psi^2),
\]

(3.21)
from which the Dirac brackets among fields and their conjugates can be read as

\[ \{x^i, p_j\} = \delta^j_i + O(\psi^2), \]
\[ \{\psi_a, \pi_b\} = \frac{1}{2} \delta_{ab} + O(\psi^2), \]
\[ \{\psi_a, \psi_b\} = -\frac{2i}{(1 + \sqrt{1 + h\psi^2})} \delta_{ab} + O(\psi^2), \]
\[ \{x^i, \psi_a\} = -\frac{hp_i}{2\sqrt{1 + h\psi^2}(1 + \sqrt{1 + h\psi^2})} \psi_a + O(\psi^3), \]
\[ \{p_i, \psi_a\} = \frac{p^2 \partial_i h}{4\sqrt{1 + h\psi^2}(1 + \sqrt{1 + h\psi^2})} \psi_a + O(\psi^3). \]

(3.22)

Note that the Dirac brackets between bosonic and fermionic variables are highly nontrivial representing the nonlocal nature of the effective theory.

Indeed using these Dirac brackets (3.22) one finds the transformation laws:

\[ \delta x^i = -i\epsilon_a \{Q_a, x^i\} = i\psi \gamma^i \epsilon - i\frac{hv^i(\psi v \cdot \gamma\epsilon)}{2(1 + \sqrt{1 - hv^2})} + O(\psi^3), \]
\[ \delta \psi_a = -i\epsilon_b \{Q_b, \psi_a\} = \frac{2}{(1 + \sqrt{1 - hv^2})} (v \cdot \gamma\epsilon)_a + O(\psi^2), \]

(3.23)

which agree with (3.10) and (3.13) for \( \psi = 2\sqrt{2}/f^{-1/4}\theta \).

The supersymmetry algebra can be also read using these Dirac brackets as

\[ \{Q_a, Q_b\} = -i\frac{2}{p_-} H_{\text{eff}} \delta_{ab} + O(\psi^2). \]

(3.24)

By replacing the Dirac bracket with the (anti-)commutator, \([Q_a, Q_b]_+ = i\{Q_a, Q_b\}\), we get (1.3) up to \(\psi^2\) order.

### 3.4 Higher order corrections

In this section we find the next-to-leading order corrections to the supercharge from the transformation laws (3.7). After some calculations, we find the supersymmetry should be modified by \(\kappa\)-transformation (2.11) \(\kappa^{(2)}\) at the \(\psi^2\) order of the form:

\[ \kappa^{(2)} = \frac{2(\theta v \cdot \gamma\xi)}{(1 + \sqrt{1 - hv^2})^2} \partial_i h \gamma^i \theta - \frac{hv^2 v_i \partial_j h(\theta \gamma^i \gamma^j \theta)}{(1 + \sqrt{1 - hv^2})^3 \sqrt{1 - hv^2}} \xi - \frac{2h(\theta \dot{\theta})}{(1 + \sqrt{1 - hv^2})^2} \xi. \]

(3.25)
The resultant transformation law for $\theta$ at the next-to-leading order, which includes the transformation due to the above $\kappa$-transformation \((3.25)\) as well as world-line diffeomorphism by $\zeta$ \((3.12)\), becomes

$$
\delta^{(2)} \theta = if^{-1}\partial_i h(\theta^i \xi)\theta + iv^2 v_i \partial_j h(\theta^j \gamma^i \xi)(v \cdot \gamma)\xi \frac{1}{1 + \sqrt{1 - hv^2}} + i\frac{2h(\theta \dot{\theta})(v \cdot \gamma)\xi}{(1 + \sqrt{1 - hv^2})\sqrt{1 - hv^2}} + i \frac{v^2 \partial_i h(\theta \gamma^i \theta)\gamma^i \xi - 2(\theta v \cdot \gamma \xi)v^i \partial_j h \gamma^i \gamma^j \theta}{(1 + \sqrt{1 - hv^2})^2} - i \frac{2h(\theta v \cdot \gamma \xi)}{(1 + \sqrt{1 - hv^2})}\dot{\theta}.
$$

This becomes much simpler if we impose the effective equations of motion:

$$
\dot{\psi} = -\frac{v^2}{2(1 + \sqrt{1 - hv^2})^2} v_i \partial_j h \gamma^i \gamma^j \psi - \frac{d}{dt}(hv^2)\psi + O(\psi^3),
$$

$$
\frac{d}{dt}(hv^2) = \frac{2(1 - hv^2)v^2}{1 + \sqrt{1 - hv^2}} \frac{dh}{dt} + O(\psi^2),
$$

which may be considered as the full quantum corrected equations of motion of the matrix quantum mechanics. Up to these equations of motion, the on-shell transformation laws for Yang-Mills fermions $\psi$, at the $\psi^2$ order, becomes

$$
\delta^{(2)} \psi = i \frac{v^2 \partial_j h(\psi \gamma^j \psi)}{4(1 + \sqrt{1 - hv^2})^2} \gamma^i \epsilon + i \frac{(\epsilon v \cdot \gamma \psi)}{4(1 + \sqrt{1 - hv^2})} (v_i \partial_j h \gamma^i \gamma^j + \frac{dh}{dt})\psi.
$$

Now we would like to determine the next-to-leading order corrections to the supercharges which gives the above transformation law \((3.27)\). As a commutation relation with supercharge, the $\psi^2$ order corrections in \((3.27)\) come from the $\psi^3$ order corrections of the supercharge and/or the constraints \((3.17)\) which give rise to the higher order corrections in the Dirac brackets.

Let the conjugate momenta of the $\psi$'s are of the form

$$
\pi_a = \frac{i}{4}(1 + \sqrt{1 + hp^2})\psi_a + l_{abcd}\psi_b \psi_c \psi_d + O(\psi^5),
$$

where $l_{abcd} = l_{a[bc]}(x^i, p_i)$ are totally antisymmetric in last three indices. We assumed that $\pi_a$ do not depend on $\dot{\psi}$ when written in terms of phase space variables $x^i, p_i$. If not, these do not give the constraints, invalidating the analysis in the lower order in $\psi$. This is justified by the consistency of the results followed. Of course, when rewritten in terms of $x^i$ and $v^i$, they have $\dot{\psi}$ dependence. It will be interesting to justify these by obtaining the higher order corrections in $\psi$ to the effective action from direct calculations following the method outlined in [26].

The relations \((3.28)\) give $\psi^3$ order corrections to the second class constraints and the Poisson brackets between the constraints become

$$
A_{ab} = -\frac{i}{2}(1 + \sqrt{1 + hp^2})\delta_{ab} - 3(l_{abcd} + l_{b[cd]}(x^i, p_i)\psi_d + O(\psi^4).\n$$

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The Dirac brackets between the spinors $\psi_a$, which include the $\psi^2$ order corrections, are given by

$$\{\psi_a, \psi_b\} = -\frac{2i}{(1 + \sqrt{1 + h^2})^2} \delta_{ab} + \frac{12(l_{abcd} + l_{baed})}{(1 + \sqrt{1 + h^2})^2} \psi_c \psi_d + O(\psi^4). \quad (3.30)$$

The supercharge may also have the $\psi^3$ order corrections and thus generically can be written as

$$Q_a = p \cdot \gamma_{ab} \psi_b + A_{abcd} \psi_b \psi_c \psi_d + O(\psi^5), \quad (3.31)$$

where $A_{abcd} = A_{a[bc]}(x^i, p_i)$ are also totally antisymmetric in last three indices. Note that the momenta $p_i$ in (3.28) and (3.31) have $\psi^2$ order corrections when written in terms of $v^i$.

In order to decide $A_{abcd}$ and $l_{abcd}$, we calculate $\{Q_a, \psi_b\}$ and $\{Q_a, Q_b\}$ and compare with (3.27) and (1.3). These consistency conditions give the unique choice for $A_{abcd}$ and $l_{abcd}$. The transformation laws for $\psi_a$ with the supercharge (3.31) can be read readily using (3.22) and (3.30) and are given by

$$\{Q_a, \psi_b\} = -i \frac{2p \cdot \gamma_{ab}}{(1 + \sqrt{1 + h^2})^2} + \frac{p^2 \partial_j \gamma_{ac} \psi_c}{4 \sqrt{1 + h^2}(1 + \sqrt{1 + h^2})} \psi_b \quad (3.32)$$

$$+ 6\left[\frac{\sqrt{1 + h^2}A_{abcd} + 2p \cdot \gamma_{ac}(l_{becd} + l_{ebcd})}{(1 + \sqrt{1 + h^2})^2}\right] \psi_c \psi_d + O(\psi^4).$$

By comparing with (3.27), we find a relation:

$$24\sqrt{1 - hv^2} \left[A_{abcd} + \frac{2v \cdot \gamma_{ac}}{(1 + \sqrt{1 - hv^2})(l_{becd} + l_{ebcd})}\right] \psi_c \psi_d \quad (3.33)$$

$$= \left[v \cdot \gamma_{ac}(v_i \partial_j \gamma_{bd} + \frac{dh}{dt} \delta_{bd}) - v \cdot \gamma_{ab}v_i \partial_j \gamma_{cd} - v^2 \partial_j \gamma_{ac} \delta_{bd}\right] \psi_c \psi_d.$$

The commutation relations among $Q_a$ 's read

$$\{Q_a, Q_b\} = -2i \frac{p^2 \delta_{ab}}{(1 + \sqrt{1 + h^2})^2} - \frac{p^2}{4 \sqrt{1 + h^2}(1 + \sqrt{1 + h^2})} \left[p \cdot \gamma_{ac} \partial_j \gamma_{bd} + (a \leftrightarrow b)\right] \psi_c \psi_d$$

$$+ \frac{p \cdot \gamma_{be}}{(1 + \sqrt{1 + h^2})^2} \left[A_{aecd} + p \cdot \gamma_{af}(l_{efcd} + l_{efcd})\right] + (a \leftrightarrow b) \psi_c \psi_d + O(\psi^4).$$

These satisfy (1.3) provided that

$$\left[v \cdot \gamma_{bd} \left(A_{aecd} + \frac{v \cdot \gamma_{af}}{(1 + \sqrt{1 - hv^2})(l_{efcd} + l_{fecd})}\right) + (a \leftrightarrow b)\right] \psi_c \psi_d$$

$$= \frac{v^2}{24\sqrt{1 - hv^2}} \left[v \cdot \gamma_{ac} \partial_j \gamma_{bd} + (a \leftrightarrow b)\right] - v_i \partial_j \gamma_{cd} \delta_{ab}\right] \psi_c \psi_d. \quad (3.34)$$
From (3.33) and (3.34), we find the unique solution
\[ A_{abcd} = 0 , \]
\[ l_{abcd} = -\frac{1 + \sqrt{1 + hp^2}}{32\sqrt{1 + hp^2}} p_i \partial_j h \delta_{[a} [b} \gamma_{c d]} ; \tag{3.35} \]
and therefore the supercharges are given by (1.2).

From this one can read off the \( \psi^3 \) order corrections to the supersymmetry transformation laws of \( x^i \). The Dirac brackets between \( x^i \) and \( \psi_a \) are given by
\[ \{ x^i , \psi_a \} = -\frac{hp_i}{2\sqrt{1 + hp^2}(1 + \sqrt{1 + hp^2})} \psi_a - i \frac{\partial_j h (\psi \gamma^{ij} \psi)}{\sqrt{1 + hp^2}} \psi_a + \frac{(2 + \sqrt{1 + hp^2}) hp_k (p_k \partial_j h \psi \gamma^{kj} \psi)}{16(1 + \sqrt{1 + hp^2})(1 + hp^2)^{3/2}} \psi_a + O(\psi^5) , \tag{3.36} \]
and therefore \( x^i \) transform as
\[ \delta x^i = i \psi^i \gamma^j \epsilon - i \frac{hv^i (\psi v \cdot \gamma \epsilon)}{2(1 + \sqrt{1 - hv^2})} - \frac{h(v_k \partial_j h \psi \gamma^{kj} \psi)(\psi v \cdot \gamma \epsilon)}{8(1 + \sqrt{1 - hv^2})^2} v^i + \frac{(\partial_j h \psi \gamma^{ij} \psi)(\psi v \cdot \gamma \epsilon)}{8(1 + \sqrt{1 - hv^2})} + \frac{h^2 \partial_j h (\psi \gamma^{ij} \psi)(\psi \gamma^k \epsilon)}{16(1 + \sqrt{1 - hv^2})^2} v^i + O(\psi^5) . \tag{3.37} \]

4 Discussions

In this paper we found an explicit form of the supercharges, up to the \( \psi^5 \) order, of the supergraviton in the background of eleven-dimensionally lifted D0 geometry. They should correspond to the supercharges of the effective action of (0+1)-dimensional matrix quantum mechanics for, at least, \( v^4 \) and \( v^6 \) order terms and their superpartners. From the perspectives of matrix quantum mechanics, the simple form of the supercharges (1.2) is quite striking. Note also that this simple form makes it an easy step to go from classical, represented by Dirac brackets, to quantum, represented by commutators, without operator ordering ambiguity. It would be very interesting to see whether it holds true to all orders in \( \psi \). We expect similar results hold for higher-dimensional Yang-Mills theories with 16 supercharges which correspond to DLCQ M theory on torus compactification. These are under investigations.

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References

[1] J. Maldacena, Adv. Theor. Math. Phys. 2, 231 (1998), hep-th/9711200.

[2] L. Susskind and E. Witten, ”The Holographic Bound in Anti-de Sitter Space”, hep-th/9805114.

[3] T. Banks, W. Fischler, S. H. Shenker, L. Susskind, Phys. Rev. D55, 5112 (1997).

[4] L. Susskind, “Another Conjecture about M(atrix) Theory”, hep-th/9704080.

[5] N. Seiberg, Phys. Rev. Lett. 79, 3577 (1997), hep-th/9710009.

[6] A. Sen, Adv. Theor. Math. Phys. 2, 51 (1998), hep-th/9709220.

[7] S. Hyun, Phys. Lett. B441, 116 (1998), hep-th/9802026.

[8] R. Dijgraaff, E. Verlinde and H. Verlinde, Nucl. Phys. B500, 43 (1997), hep-th/9703030.

[9] S. Hyun, Y. Kiem and H. Shin, Phys. Rev. D57, 4856 (1998), hep-th/9712021.

[10] V. Balasubramanian, R. Gopakumar and F. Larsen, Nucl. Phys. B526, 415 (1998).

[11] S. Hyun and Y. Kiem, Phys. Rev. D59, 026003 (1999), hep-th/9805136.

[12] K. Becker, M. Becker, J. Polchinski, A. Tseytlin, Phys. Rev. D56, 3174 (1997), hep-th/9706072.

[13] J. A. Harvey, Nucl. Phys. Proc. Suppl. 68, 113 (1998), hep-th/9706039.

[14] J. F. Morales, C. A. Scrucca, and M. Serone, Phys. Lett. B417, 233 (1998), hep-th/9709063; Nucl. Phys. B534, 223 (1998), hep-th/9801183.

[15] P. Kraus, Phys. Lett. B419, 73 (1998), hep-th/9709199.

[16] I.N. McArthur, Nucl. Phys. B534, 183 (1998), hep-th/9806082.

[17] M. Barrio, R. Helling and G. Polhemus, J. High Energy Phys. 05, 012 (1998), hep-th/9801189.

[18] J. F. Morales, J. Plefka, C. A. Scrucca, M. Serone, A. Waldron, “Spin dependent D-brane interactions and scattering amplitudes in matrix theory”, hep-th/9812039.
[19] W. Taylor and M. Van Raamsdonk, JHEP 9904, 013 (1999), hep-th/9812239.

[20] S. Hyun, Y. Kiem and H. Shin, “Eleven-dimensional massless superparticles and matrix theory spin-orbit couplings revisited”, hep-th/9901152, to appear in Phys. Rev. D.

[21] S. Hyun, Y. Kiem and H. Shin, “Supersymmetric completion of supersymmetric quantum mechanics”, hep-th/9903022, to appear in Nucl. Phys. B.

[22] S. Paban, S. Sethi and M. Stern, Nucl. Phys. B534, 137 (1998), hep-th/9805018.

[23] S. Paban, S. Sethi and M. Stern, JHEP 9806, 012 (1998), hep-th/9806028.

[24] D. Lowe, JHEP 9811, 009 (1998), hep-th/9810075.

[25] M. B. Green, M. Gutperle, H.-h. Kwon “Light-cone Quantum Mechanics of the Eleven-dimensional Superparticle”, hep-th/9907155.

[26] B. de Wit, K. Peeters and J. Plefka, Nucl. Phys. B532, 99 (1998), hep-th/9803209.