Superfluid transition in disordered dipolar Fermi gases

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We consider a weakly interacting two-component Fermi gas of dipolar particles (magnetic atoms or polar molecules) in the two-dimensional geometry. The dipole-dipole interaction (together with the short-range interaction at Feshbach resonances) for dipoles perpendicular to the plane of translational motion may provide a superfluid transition. The dipole-dipole scattering amplitude is momentum dependent, which violates the Anderson theorem claiming the independence of the transition temperature on the presence of weak disorder. We have shown that the disorder can strongly increase the critical temperature (up to 10 nK at realistic densities). This opens wide possibilities for the studies of the superfluid regime in weakly interacting Fermi gases, which was not observed so far.

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I. INTRODUCTION

The last decades were marked by profound achievements in the physics of ultracold atomic Fermi gases. The key circumstance was the use of Feshbach resonances (magnetic field dependence of the interaction amplitude) allowing one to change the interaction strength in a wide range, even from an infinite repulsion to infinite attraction. Experiments with two-component Fermi gases have reached the strongly interacting regime and identified a superfluid transition in this regime, which brings in analogies with neutron stars and superconductors. However, experimental studies did not achieve the weakly interacting Bardeen-Cooper-Schrieffer (BCS) regime: for common densities \( n \lesssim 10^{14} \text{ cm}^{-3} \) the superfluid transition temperature \( T_c \) would be about a nanokelvin or lower, i.e. beyond experimental reach.

Possibilities to manipulate the superfluid transition temperature, in particular by manipulating the external confining potential, was always at the core of the studies. In the present stage, after the observation of Anderson localization in dilute clouds of neutral atoms in disorder, the behavior of disordered ultracold quantum gases became a rapidly growing domain of research. One of the key questions is how the superfluid transition temperature of a two-component Fermi gas can be modified by introducing a disorder. This question has been the subject of a number of works in condensed matter and in cold atomic gas. As was pointed out by several authors, the critical temperature can be increased when approaching the Anderson transition. In a weak disorder (\( k_F l \gg 1 \), where \( k_F \) is the Fermi momentum, and \( l \) the mean free path) and for the case of weak short-range interaction where the interaction amplitude is momentum independent, one has the Anderson theorem: the BCS transition temperature is disorder independent. In a later stage, this statement was justified by Abrikosov and Gor'kov within the diagrammatic approach. However, the works do not take into account weak localization effects, which, in the presence of interaction, change the fermion self-energy and the density of states. Including these corrections the disorder leads to a moderate increase of the BCS transition temperature (in the absence of Coulomb interactions).

In this paper we consider a two-component two-dimensional (2D) gas of dipolar fermions (magnetic atoms or polar molecules) in a weak disorder, assuming that the dipoles are perpendicular to the plane of the translational motion. This can be a mixture of two different isotopes of magnetic atoms in the lowest Zeeman states (for example, fermionic isotopes of dysprosium which has magnetic moment of \( 10 \mu_B \), and we will omit a small difference in masses of these isotopes). In this geometry the dipole-dipole interaction amplitude by itself consists of a large short-range repulsive contribution and a long-range attractive momentum-dependent contribution, so that the total amplitude is positive. However, the short-range repulsion (complemented by the non-dipole contribution) can be strongly reduced or even converted to attraction by using Feshbach resonances. This can make the total interaction amplitude attractive and provide a superfluid transition like in bilayer dipolar systems. Since the amplitude is now momentum-dependent, in the presence of weak disorder the Anderson theorem does not work.

Strictly speaking, in two dimensions we have the Kosterlitz-Thouless superfluid phase transition. However, in the weakly interacting regime the transition temperature is very close to that calculated in the Bardeen-Cooper-Schrieffer (BCS) approach. We find that the momentum dependence of the interaction amplitude by itself may lead to a significant increase of the BCS transition temperature in the presence of disorder. The weak localization corrections work in the same direction. As a result, the BCS transition temperature can be strongly increased by the disorder, which opens wide possibilities for the observation of superfluidity in weakly interacting Fermi gases of magnetic atoms and/or polar molecules.
The paper is organized as follows. In section II we present a general formalism for studying the Cooper pairing instability in the presence of disorder. Section III contains our derivation of the critical temperature \( T_c \) omitting weak localization corrections. These corrections are taken into account in section IV, where we present the final result for the increase of \( T_c \) by the disorder. In section V we conclude.

II. COOPER PAIRING INSTABILITY IN DISORDERED FERMI SYSTEMS. GENERAL FORMALISM

The threshold of the Cooper pairing instability in a system of weakly interacting two-component fermions is determined by a singularity that occurs at a critical temperature \( T_c \) in the susceptibility function \( \chi(\mathbf{r}, \mathbf{r'}, \epsilon) \), which describes the system response \( \langle \psi_\uparrow(\mathbf{r})\psi_\downarrow(\mathbf{r'}) \rangle \) to a perturbation of the form \( \int d\mathbf{r'} d\mathbf{r''} \psi_\downarrow(\mathbf{r'})\psi_\uparrow(\mathbf{r''}) h(\mathbf{r'}, \mathbf{r''}) \). Here \( \psi_\uparrow(\mathbf{r}) \) and \( \psi_\downarrow(\mathbf{r}) \) are annihilation operators of fermionic components, let say spin up and spin down. For weakly interacting fermions the diagrammatic representation of \( \chi \) corresponds to a series of ladder diagrams, where the upper and lower fermionic lines are connected by non-intersecting (wavy) lines associated with the interaction potential \( V(\mathbf{r}_1 - \mathbf{r}_2) \), see Fig. 1. Symbolically, the ladder series corresponds to an infinite sum

\[
B + B(-V)B + B(-V)B(-V)B... = B[I + VB]^{-1},
\]

where

\[
B = T_c \sum_{\epsilon_n} B(\epsilon_n),
\]

and \( B(\epsilon_n) \) is an elementary block of two fermionic Green functions: \( B(\epsilon_n) = G(\epsilon_n)G(-\epsilon_n) \). The summation in Eq. (2) runs over the fermion Matsubara frequencies \( \epsilon_n = 2nT_c(n + 1/2) \), \( n = 0, \pm 1, \ldots \). In a clean system (without any disorder), the instability of the ladder series corresponds to a zero eigenvalue of a linear integral operator \( I + VB \) or, in other words, to the existence of a non-zero eigenfunction \( \Delta \) obeying the (symbolic) equation

\[
\Delta = -VBD\Delta,
\]

which is the standard equation for \( T_c \).

In the presence of disorder, one should associate the instability threshold with the singularity of the susceptibility function \( \chi \) averaged over the disorder. In the case of a weak disorder (such that \( k_F \gg 1 \)), the operator \( B \) in the equation for \( \Delta \) is replaced with a new operator \( \tilde{B} \). There are two kinds of modifications. First, the elementary block \( B(\epsilon) \) is replaced with its disorder averaged value

\[
B_{av}(\epsilon_n) = \langle G(\epsilon_n)G(-\epsilon_n) \rangle,
\]

After the summation over the Matsubara frequencies this block gives a contribution \( \tilde{B}_{av} = T_c \sum_{\epsilon_n} B(\epsilon_n) \) to the integral operator

\[
\tilde{B} = \tilde{B}_{av} + \delta \tilde{B}
\]

of the disordered system. The second contribution

\[
\delta \tilde{B} = \delta \tilde{B}_Z + \delta \tilde{B}_V,
\]

originates not from averaging the elementary block but from the disorder-induced corrections \( \delta \Sigma \) to the fermion self-energy and to the fermion interaction \( V \) (the so called vertex corrections). These "weak localization" (WL) corrections were studied quite some time ago. Corrections to the self-energy and the corresponding WL corrections to the density of states were considered in the pioneer paper\cite{19}. The influence of WL corrections on the critical temperature of superconducting transition was explored in\cite{20} and in the later work\cite{21}. We shall discuss the significance of these corrections later. The relative smallness of the disorder-induced corrections allows one to calculate them independently of each other.

In the present section and in the next one we are returning to the study of the first kind of corrections to \( T_c \), which are caused by the contribution \( \tilde{B}_{av} \) to the kernel \( \tilde{B} \). These corrections are sensitive to the particular spatial dependence of the interaction potential. For instance, they are absent for the contact interaction (in accordance with the Anderson theorem\cite{22}). On the contrary, we will show that for the dipole-dipole interaction that we are interested in the corrections are nonzero and can dominate over the WL corrections. Details of the calculation are presented in the next section. Here we only describe the structure of the averaged block \( B_{av}(\epsilon_n) \). As is well known, the leading correction to the averaged (over a weak disorder) product of two Green functions with opposite frequencies and incident wave vectors is given by the ladder of parallel impurity lines connecting two fermionic lines. Such a "Cooperon" installment bears the total zero momentum and depends on the difference between the two frequencies, \( \epsilon_n \) and \( -\epsilon_n \), i.e., on \( 2\epsilon_n \). Thus, the averaged block \( B_{av}(\epsilon_n) \) has the form

\[
B_{av}(\epsilon_n) = B_0(\epsilon_n) + B_0(\epsilon_n)\Gamma(\epsilon_n)B_0(\epsilon_n),
\]

where \( B_0(\epsilon_n) = \langle G(\epsilon_n)G(-\epsilon_n) \rangle \) is the product of two disorder-averaged Green functions (a more detailed definition is given below in Eq. (13)), and the quantity

\[
\Gamma(\epsilon_n) = \frac{1 + 2\tau|\epsilon_n|}{2\tau|\epsilon_n|},
\]
results from the Cooperon carrying zero total momentum\cite{23}.
The parameter $\gamma$ comes from the correlation function for a short range disorder potential $U(r)$, namely $<U(r)U(r')>=\gamma \delta(r-r')$. The time $\tau$ is the inverse disorder-induced scattering rate $1/\tau = 2\pi \rho F \gamma$, and $\rho_F$ is the density of states on the Fermi surface.

III. DERIVATION WITHOUT WL CORRECTIONS

Taking into account only the averaged block $\vec{k}_{av}$ in Eq.\cite{5} we rewrite the equation $\Delta = -V\delta \Delta$ at $T \to T_c$ in the form

$$\Delta(r-r') = -V(r-r')T \sum_{\omega_n} \int d\vec{r}_1 d\vec{r}_2 \langle G(\vec{r}, \vec{r}_1; \omega_n)G(\vec{r}_2, \vec{r}'; -\omega_n)\rangle \Delta(r_1-r_2),$$

where the averaged Green function is

$$\bar{G}(\vec{k}, \omega_n) = \frac{1}{i\omega_n + \frac{1}{2\pi} \text{sgn} \omega_n - \xi_k},$$

with $\xi_k = \frac{\hbar^2 k^2}{2m} - \mu$, and $\mu$ the chemical potential (hereinafter $\hbar = 1$).

We now represent $\Delta(k')$ in the rhs of Eq.\cite{11} as $\Delta(p) + [\Delta(k') - \Delta(p)]$ and argue later that the second term gives a small contributions and can be neglected.

Then we make a summation over $k'$ by using the identity

$$B_0(\omega_n) = \sum_{\vec{q}} \bar{G}(\vec{q}, \omega_n)\bar{G}(\vec{q}, -\omega_n) = \frac{\gamma^{-1}}{1 + 2|\omega_n|\tau}.$$

or in the momentum representation

$$\Delta(k) = -\sum_{p,\vec{k}} V(k-p)T \sum_{\omega_n} \left[ \bar{G}(p, \omega_n)\bar{G}(p, -\omega_n) \frac{\gamma(1 + 2|\omega_n|\tau)}{2|\omega_n|\tau} \right] \bar{G}(k', \omega_n)\bar{G}(k', -\omega_n) \Delta(k'),$$

where the normalization volume is put equal to unity. After averaging over the disorder and using Eqs. \cite{7} and \cite{8} we obtain

$$\Delta(k) = -\sum_{n>0} \int \frac{d^2 p}{(2\pi)^2} \frac{V(k-p)\Delta(p)(1 + \frac{1}{\omega_n\tau})}{\xi^2 + (\omega_n + \frac{1}{2\pi\tau})^2},$$

with $\omega_n = \pi T_c (2n + 1)$. After the summation over the frequencies we obtain:

$$\Delta(k) = -\int \frac{d^2 p}{(2\pi)^2} \frac{V(k-p)\Delta(p)K(p)}{\xi^2 + (\omega_n + \frac{1}{2\pi\tau})^2} + c.c.,$$

where

$$K(p) = i \frac{\Psi\left(\frac{1}{2} + \frac{i\xi_p}{2\pi T_c}\right) - \Psi\left(\frac{1}{2}\right)}{2\pi}$$

and

$$\Psi(x) \equiv \Gamma'(x)/\Gamma(x)$$

the digamma function, $z_p = \xi_p + \frac{i}{2\pi}$, and near the Fermi surface one has $\xi_p \approx v_F(p-p_F)$ with $v_F$ being the Fermi velocity. We then have

$$\text{Im}\Psi\left(\frac{1}{2} + \frac{i\xi_p}{2\pi T_c}\right) = \frac{\pi}{2} \tanh \frac{\xi_p}{2T_c}.$$
integration contour as $\xi \to \xi - i/2\pi$ in Eq. (13) and use analytical properties of the digamma function. As a result we get the clean case equation.

Using the relation between the potential $V(k' - k)$ and the off-shell scattering amplitude $f(k', k)$:

$$f(k', k) = V((k' - k) + \int \frac{d^2q}{(2\pi)^2} \frac{V(k' - q)/f(q, k)}{2(E_k - E_q - i0)}.$$ \hspace{1cm} (19)

we find (see, e.g.\textsuperscript{26})

$$\Delta(k) = -\int \frac{d^2k'}{(2\pi)^2} f(k', k)\Delta(k') \left[K(k') - \frac{1}{2(E_{k'} - E_k)}\right].$$ \hspace{1cm} (20)

Expanding the order parameter $\Delta(k)$ and the scattering amplitude in a series over the states with different orbital quantum numbers: $\Delta(k) = \sum_{m=-\infty}^{\infty} \Delta_m(k) \exp(im\phi_k)$; $f(k', k) = \sum_{m=-\infty}^{\infty} f_m(k', k) \exp[im(\phi_{k'} - \phi_k)]$, we focus on the s-wave symmetry ($m = 0$) of the order parameter and (omitting index $m = 0$) obtain from Eq. (20):

$$\Delta(k) = -\int \frac{kdk}{2\pi} f(k', k)\Delta(k') \left[K(k') - \frac{1}{2(E_{k'} - E_k)}\right].$$ \hspace{1cm} (21)

with the amplitude $f(k', k)$ given below.

The scattering amplitude contains two terms, due to the local and nonlocal (dipole-dipole) interactions. For the s-wave scattering the nonlocal part is given by the integral

$$\int \frac{d^2\rho}{r} (J_0(k'r)J_0(kr) - 1)2\pi r dr = 2\pi d^2 \begin{cases} -kF(-1/2, -1/2, 1, k^2/k^2), & k'<k \\ -k'F(-1/2, -1/2, 1, k^2/k^2), & k' < k \end{cases}.$$ \hspace{1cm} (22)

Since the hypergeometric function slowly varies in the interval (0, 1): $1 < F(...) < 4/\pi$, we put approximately $F(...) = 4/\pi$, which is the value on the Fermi surface, so that

$$f(k', k) = F_0 - 8d^2 \max(k,k'),$$ \hspace{1cm} (23)

and $f(k_F, k_F) \equiv f_0 = F_0 - 8d^2 k_F < 0$. The local part $F_0$ is momentum indepenent\textsuperscript{20} and can be varied by the use of Feshbach resonances.

To find the critical temperature we use the ansatz for the order parameter (see\textsuperscript{26}), which follows from Eq. (21) assuming that the main contribution to the integral comes from $k'$ close to $k_F$:

$$\Delta(k) = \Delta(k_F) \frac{f(k_F,k)}{f(k_F,k_F)}.$$ \hspace{1cm} (24)

For $k = k_F$ Eq. (21) takes the form

$$1 = -\int \frac{d^2k}{(2\pi)^2} \frac{f(k_F, k_F)}{f(k_F,k_F)} \left[K(k') - \frac{1}{2(E_{k'} - E_{k_F})}\right].$$ \hspace{1cm} (25)

Near the Fermi surface we have

$$E_k - E_{F_F} \approx \xi_k \approx v_F(k - k_F); \int \frac{d^2k}{(2\pi)^2} \approx \int m\frac{d\xi}{2\pi}.$$ \hspace{1cm} (26)

and

$$f(k', k_F) = f_0 - \frac{8d^2}{v_F} \xi \theta(k' - k_F).$$ \hspace{1cm} (27)

After the integration in Eq. (25) we obtain the equation

$$\lambda \ln \left(\frac{T}{T_c} \right) = \frac{1}{2\pi} \frac{2k_F}{\pi} \left[\ln\left(\frac{\mu}{\pi T_c}\right) - 2\Psi\left(\frac{1}{2}\right)\right] + \frac{4}{\pi} \frac{1}{m^2} \left(\frac{2\mu}{\pi T_c} - 2 - 2\Psi\left(\frac{1}{2}\right)\right) - \frac{1}{4\pi^2} \ln\left(\frac{2\mu c}{\pi T_c}\right).$$ \hspace{1cm} (28)

where the mean free path is $l = v_F \tau$, $C = -\Psi(1) = 0.577$, and $r_c = mu^2$ is the dipole-dipole distance. The quantity $T_{c0}$ is the critical temperature in the absence of disorder, and $\lambda = |f_0|m/2\pi \ll 1$. Detailed calculations leading to Eq. (28) are given in the Appendix.

The terms in the rhs of Eq. (28) should be small (strictly speaking, much smaller than unity). It is this condition that allows us to omit higher order disorder corrections, i.e. terms that are higher order in $1/k_F$. In the BCS regime one has $\ln(\mu/T_c) \sim 1/\lambda \gg 1$ and, hence, the second term in the first line of the rhs of Eq. (28) can be omitted. As we consider the case where $k_F r_c < 1$ and $k_F l \gg 1$, the last two terms in the rhs of Eq. (28) contain additional small parameters $k_F r_c$ and $k_F r_c / k_F l$, and can also be neglected. Thus, equation (28) reduces to

$$\ln \left(\frac{T_c}{T_{c0}}\right) \approx \frac{r_c}{\pi l \lambda^2},$$ \hspace{1cm} (29)

and the rhs of Eq. (29) should be significantly smaller than $1/\lambda$. For $r_c / l = 0.2$, decreasing $\lambda$ from 0.2 to 0.15 we obtain $T_c/T_{c0}$ increasing from 1.3 to 1.8. Importantly, comparing the result of Eq. (29) with that of original equation (25) we see that the former is valid within a few percent of accuracy. Note that we used the simplified equation (14) instead of Eq. (11). A simple but cumbersome calculation shows that omitted terms give only a small contribution to the third line in the rhs of Eq. (28).

IV. INFLUENCE OF WEAK LOCALIZATION CORRECTIONS ON THE DISORDER-INDUCED INCREASE OF $T_c$

The WL corrections for the disorder-induced change of the critical temperature $T_c$ have been calculated in Refs.\textsuperscript{15,16}. The WL corrections by themselves lead to the following ratio of $T_c$ to the critical temperature $T_{c0}$ in the system without disorder:

$$\ln \left(\frac{T_c}{T_{c0}}\right)_{WL} = \frac{3g_F - g_0}{4\pi E_F \pi} \ln^2 \left(\frac{1}{\tau T_c}\right) - \frac{(g_F + g_0)\rho_F}{6\pi E_F \tau} \ln \left(\frac{1}{\tau T_c}\right).$$ \hspace{1cm} (30)

The quantity $g_F$ is defined as $g_F \equiv \bar{V}(k - k')$, where the bar means the angular average of the interaction potential (in the
momentum representation) on the Fermi surface, \( k = k' = k_F \).

The quantity \( g_0 \) is \( V(q = 0) \), i.e., the interaction potential with zero momentum transfer. The first term in Eq. (30) results from the self-energy WL corrections, whereas the second one originates from the vertex WL corrections. Some of the corresponding diagrams are shown in Fig. 2, 3. The paired dashed lines there resemble schematically the ladder diagrams connected by the disorder lines (so-called diffuson and Cooperon diagrams). Equation (30) has been derived under the assumption \( \tau T_c \ll 1 \), where the Cooperon and diffusons are large in the low momentum and low energy limit. The condition \( \tau T_c \ll 1 \) means that the mean free path \( l = v_F \tau \) is small compared to the correlation length \( v_F / T_c \), i.e., the motion has a diffusive character. In this diffusive regime \( \ln \left( \frac{T_c}{T_c^0} \right) \gg 1 \), so that the second term in Eq. (30) should be considered as the leading one.

We first express the quantities \( g_F \) and \( g_0 \) in terms of the scattering amplitude by using the relation between the potential \( V(k' - k) \) and the off-shell scattering amplitude \( f(k', k) \).

In the lowest order (appropriate for the discussed corrections) one has \( V(k' - k) \approx f(k', k) \). Hence, the quantity \( g_F \) coincides with the on-shell amplitude of the \( l = 0 \) channel, i.e.,

\[
g_F = f_{l=0}(k_F, k_F).
\]

This amplitude, denoted as \( f_0 \), is given by Eq. (23) with \( k = k' = k_F \) in the previous section. Hence, we have

\[
g_F \rho_F = \frac{f_0 m}{2 \pi} = -\lambda. \tag{31}
\]

The quantity \( g_0 \) coincides with the on-shell amplitude of the forward scattering: \( g_0 = f(k, k) \) with \( k = k_F \). This amplitude is represented as a sum of partial scattering amplitudes with all momenta:

\[
f(k, k) = f_{l=0}(k, k) + \sum_{l \neq 0} f_l(k, k) \tag{32}
\]

The scattering amplitudes with \( l \neq 0 \) can be omitted for the short-range interaction, and for the dipole-dipole tail in 2D they have been calculated in[20]. In the limit \( k_F r_s \ll 1 \), the leading contribution to these partial amplitudes comes from large distances where the interaction can be treated in the first Born approximation:

\[
f_{l=0}(k; k) \approx \frac{8k r_s}{m} \frac{1}{4l^2 - 1}. \tag{33}
\]

Making a summation over \( l \) in Eq. (33) we arrive at the expression for the quantity \( g_0 \):

\[
g_0 = f_0(k_F) + \frac{8k r_s}{m} = f_0(k_F) + 8k r_s d^2, \tag{34}
\]

and, respectively,

\[
g_0 \rho_F = -\lambda + \frac{4k r_s}{\pi}. \tag{35}
\]

Relative smallness of the WL corrections allows one to replace \( T_c \) by \( T_c^0 \) in the rhs of Eq. (30) and to represent the latter in the form

\[
\ln \left( \frac{T_c}{T_c^0} \right)_{WL} \approx \frac{2\lambda - 4k r_s}{3\pi k_F l} \ln^3 \left( \frac{1}{\tau T_c^0} \right), \tag{36}
\]

where we have kept only the leading term with the third power of the large logarithm. Equation (36) originates from the vertex corrections and can be interpreted as a renormalization of the coupling constant: \( \lambda \to \lambda + \delta \lambda \), where \( \frac{\delta \lambda}{\lambda} \sim \frac{1}{k_F l} \ln \left( \frac{1}{\tau T_c} \right) \).

To provide the validity of the approach, the relative correction \( \delta \lambda / \lambda \) should be small. This requirement results in the condition

\[
k_F l \gg \ln \left( \frac{1}{\tau T_c^0} \right) = \frac{1}{\lambda} + \ln \left( \frac{\pi e^\gamma}{k_F l} \right) \gg 1. \tag{37}
\]

Making a summation of the correction to \( \ln(T_c/T_c^0) \) due to the momentum dependence of the dipole-dipole scattering amplitude and the WL correction we arrive at the final result:

\[
\ln \left( \frac{T_c}{T_c^0} \right) \approx \frac{2r_s}{\pi^2 k_F l} + \frac{2\lambda - 4k r_s}/\pi k_F l \left[ \frac{1}{\lambda} + \ln \left( \frac{\pi e^\gamma}{k_F l} \right) \right]^3. \tag{38}
\]

The validity of our approach requires several conditions, including Eq. (37) and \( k_F r_s \ll 1 \).

For instance, for the choice \( \lambda = 0.2 \), \( r_s / l = 0.01 \), \( k_F r_s = 0.1 \), and \( k_F l = 10 \) we obtain an appreciable increase of the critical temperature: \( T_c \approx 1.4 T_c^0 \). Moreover, decreasing \( \lambda \) to 0.15 we find \( T_c \approx 2.3 T_c^0 \). In these cases and also for intermediate values of \( \lambda \) the correction to \( T_c \) caused by the momentum dependence of the dipole-dipole interaction amplitude exceeds the WL correction.
To calculate the last integral we use analytical properties of the digamma function \(\Psi(z)\):

\[
\Psi\left(\frac{1}{2} + ix\right) - \Psi\left(\frac{1}{2} - ix\right) = i\pi \tanh(\pi x), \quad (A.1)
\]

\[
\Psi\left(\frac{1}{2} + ix\right) + \Psi\left(\frac{1}{2} - ix\right) = \begin{cases} 16.8\chi^2, & x \ll 1 \\ 2\ln x - 2\Psi\left(\frac{1}{2}\right) - \frac{1}{12x^2}, & x \gg 1. \end{cases} \quad (A.2)
\]

\[
\int_0^\infty \left(\Psi\left(\frac{1}{2} + ix\right) + \Psi\left(\frac{1}{2} - ix\right)\right) dx = i\ln\left(\frac{\Gamma\left(\frac{1}{2} - ix\right)}{\Gamma\left(\frac{1}{2} + ix\right)}\right). \quad (A.3)
\]

\[
\int_0^\infty \left(\Psi\left(\frac{1}{2} + ix\right) + \Psi\left(\frac{1}{2} - ix\right) - \Psi\left(\frac{1}{2}\right)\right) dx = \ln^2 a - 2\Psi\left(\frac{1}{2}\right) \ln a, \quad (A.4)
\]

We then rewrite Eq. (25) in the form

\[
1 = -\int \frac{d^2k}{(2\pi)^2} \frac{(f(k, k_F))^2}{f(k_F, k_F)} \left[K_0(k) - \frac{1}{2(\xi F + E_F)}\right] - \int \frac{d^2k}{(2\pi)^2} \frac{(f(k, k_F))^2}{f(k_F, k_F)} \left[K(k) - K_0(k)\right]. \quad (A.5)
\]

The first line in (A.5) is the equation for the critical temperature \(T_0\) in the absence of a disorder. The second line contains \(1/\tau\) corrections originating from the momentum dependence of the interaction amplitude. Near the Fermi surface we have relations (26) and (27) and rewrite the first term in the second line of Eq. (A.5) as

\[
\Psi(w) \text{ in the complex plane } w = Rew + iImw. \text{ This function}
\]

We thus see that the momentum dependence of the amplitude of long-range dipolar interaction is crucial for the disorder-induced increase of the critical temperature, and the overall ratio \(T_c/T_{c0}\) may exceed factor 2 for realistic parameters.

V. CONCLUSIONS AND OUTLOOK

In conclusion, we have shown that the superfluid transition temperature of a weakly interacting two-component dipolar Fermi gas can be strongly increased by introducing disorder in the system. The origin of this phenomenon lies in the density fluctuations caused by the disorder. Our results can be tested in experiments with magnetic atoms and/or polar molecules. Consider a mixture of dysprosium fermionic isotopes, \(^{161}\)Dy and \(^{163}\)Dy, with equal concentrations and in the lowest Zee-

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Appendix: Calculation of the disorder-induced increase of the critical temperature due to momentum dependence of the interaction amplitude

In order to obtain Eq. (28) of the main text we will rely on equations (16), (25) - (27), and use the following relations for digamma function \(\Psi(z)\):

\[
\Psi\left(\frac{1}{2} + ix\right) - \Psi\left(\frac{1}{2} - ix\right) = i\pi \tanh(\pi x), \quad (A.1)
\]

\[
\Psi\left(\frac{1}{2} + ix\right) + \Psi\left(\frac{1}{2} - ix\right) = \begin{cases} 16.8\chi^2, & x \ll 1 \\ 2\ln x - 2\Psi\left(\frac{1}{2}\right) - \frac{1}{12x^2}, & x \gg 1. \end{cases} \quad (A.2)
\]

\[
\int_0^\infty \left(\Psi\left(\frac{1}{2} + ix\right) + \Psi\left(\frac{1}{2} - ix\right)\right) dx = i\ln\left(\frac{\Gamma\left(\frac{1}{2} - ix\right)}{\Gamma\left(\frac{1}{2} + ix\right)}\right). \quad (A.3)
\]

\[
\int_0^\infty \left(\Psi\left(\frac{1}{2} + ix\right) + \Psi\left(\frac{1}{2} - ix\right) - \Psi\left(\frac{1}{2}\right)\right) dx = \ln^2 a - 2\Psi\left(\frac{1}{2}\right) \ln a, \quad (A.4)
\]

We then rewrite Eq. (25) in the form

\[
1 = -\int \frac{d^2k}{(2\pi)^2} \frac{(f(k, k_F))^2}{f(k_F, k_F)} \left[K_0(k) - \frac{1}{2(\xi F + E_F)}\right] - \int \frac{d^2k}{(2\pi)^2} \frac{(f(k, k_F))^2}{f(k_F, k_F)} \left[K(k) - K_0(k)\right]. \quad (A.5)
\]

The first line in (A.5) is the equation for the critical temperature \(T_0\) in the absence of a disorder. The second line contains \(1/\tau\) corrections originating from the momentum dependence of the interaction amplitude. Near the Fermi surface we have relations (26) and (27) and rewrite the first term in the second line of Eq. (A.5) as

\[
\Psi(w) \text{ in the complex plane } w = Rew + iImw. \text{ This function}
\]
is holomorphic in the right semi-plane. The integration over \( d\xi \) can be considered as the integration along the line \( AB \) in Fig. 3 where \( w = 1/2 + 1/4\pi T_C \tau - i\xi/2\pi T_C \). As the integral along the closed contour \( AB \rightarrow BB_1 \rightarrow B_1 A_1 \rightarrow A_1 A \) is equal to zero and the integrals along the lines \( BB_1 \) and \( A_1 A \) can be omitted, the integral along the line \( AB \) is equal to the integral along the line \( B_1 A_1 \). This is equivalent to the change \( w \rightarrow w - 1/4\pi T_C \tau \), or \( \xi \rightarrow \xi - i/2\tau \) in Eq. (A.6) which then takes the form

\[
f_0^\Lambda \frac{m}{2\pi} \frac{d\xi}{|f_{0i}|} \left[ f_0 - \frac{8d^2}{v_F} \left( \frac{\xi - i}{2\tau} \right)^2 - f_0^2 \right] \int \left[ \frac{1}{2\pi} \Psi \left( \frac{1}{2} - \frac{i}{2\tau} \right) - \Psi \left( \frac{1}{2} \right) \right] + c.c., \tag{A.7}
\]

where we put an upper bound \( \Lambda \sim \mu \) having in mind that the integral will be logarithmically divergent.

\[
\begin{array}{c}
\Lambda \\
2\pi T_C
\end{array}
\]

\[
\begin{array}{c}
1/2 \\
1/2 + 1/4\pi T_C \tau
\end{array}
\]

\[
\begin{array}{c}
\text{A1} \\
\text{A}
\end{array}
\]

\[
\begin{array}{c}
\text{B1} \\
\text{B}
\end{array}
\]

\[
\text{W}
\]

FIG. 4. Contours of integration in the complex plane \( w \)

Substituting the integral of Eq. (A.7) and its complex conjugated value into (A.5) we obtain

\[
1 - \int_0^\Lambda \frac{m}{2\pi} \frac{d\xi}{|f_{0i}|} \left[ K_0(k) - \frac{1}{2\pi} \right] = + \int_0^\Lambda \frac{m}{2\pi} \frac{d\xi}{\gamma} \frac{8d^2}{v_F} \left( \frac{1}{2\pi} - \frac{1}{2\pi} \right)^2 - 2\Psi \left( \frac{1}{2} \right) + \int_0^\Lambda \frac{m}{2\pi} \frac{d\xi}{\gamma} \frac{64d^4}{v_F^2} \left( \frac{1}{2\pi} - \frac{1}{2\pi} \right)^2 - 2\Psi \left( \frac{1}{2} \right)

- \int_0^\Lambda \frac{m}{2\pi} \frac{d\xi}{|f_{0i}|} \frac{8d^4}{v_F^2} \tanh \frac{2\pi}{\xi} \tag{A.8}
\]

The first line of Eq. (A.5) gives a standard expression \( \lambda \ln(T_c/T_c^0) \), where \( T_c^0 = \frac{2\mu e^C}{\pi} \exp(-1/\lambda) \) is the critical temperature in the absence of disorder. The second line is calculated using Eq. (A.4) and it gives:

\[
\frac{1}{k_F} \frac{2k_F r_s}{\pi^2} \ln^2 \frac{\mu}{2\pi T_c} - 2\Psi \left( \frac{1}{2} \right) \ln \frac{\mu}{2\pi T_c} \tag{A.9}
\]

The third line is calculated by the use of Eq. (A.3):

\[
\frac{1}{k_F} \frac{8(k_F r_s)^2}{\pi^2 \lambda} \frac{\mu}{v_F p_F} \left[ 2\ln \frac{\mu}{2\pi T_c} - 2 - 2\Psi \left( \frac{1}{2} \right) \right] \tag{A.10}
\]

The forth line contains a standard integral and it gives

\[
- \frac{1}{(k_F r_s)^2} \frac{2\mu e^C}{\pi^2 \lambda} \ln \frac{2\mu e^C}{\pi T_c} \tag{A.11}
\]

Substituting relations (A.9)-(A.11) into Eq. (A.8) we obtain Eq. (28) of the main text.

Note that we used approximate relations (26) and (27) valid near the Fermi surface. However, we checked that the use of exact expressions for \( \xi \) and \( F(k, k_F) \) leads to practically the same results.

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In two dimensions the s-wave short-range contribution to the interaction amplitude contains a logarithmic momentum dependence. However, in the quasi2D case obtained by strongly confining the motion of atoms in one direction to zero point oscillations, as in all 2D experiments with cold atoms, this logarithmic dependence is not important for any relevant momenta, and the short-range (local) contribution to the interaction amplitude can be put momentum independent.

The quantity $\Gamma/\gamma$ is given by the series $1 + \gamma B_0(\epsilon_n) + [\gamma B_0(\epsilon_n)]^2 + ...$, where $B_0(\epsilon_n)$ is determined by Eq. (13).

The integral along the line $BB_1$ is much smaller than the integral along the line $A_1A$. The ratio of the latter to the main term of Eq. (28) is smaller than $\ln^2(1/4\pi T_c) / \ln^2(E_F/2\pi T_c)$. For all realistic parameters $\lambda$ and $k_F l$ this ratio is much smaller than unity. Moreover, the integral along the line $A_1A$ is also smaller than (or of the order of) the other terms of Eq. (28).

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