On sums of graph eigenvalues

Evans M. Harrell II

School of Mathematics, Georgia Institute of Technology, Atlanta, GA 30332-0160
U.S.A.

Joachim Stubbe

EPFL, MATH-GEOM, Station 8, CH-1015 Lausanne, Switzerland

Abstract

We use two variational techniques to prove upper bounds for sums of the lowest several eigenvalues of matrices associated with finite, simple, combinatorial graphs. These include estimates for the adjacency matrix of a graph and for both the standard combinatorial Laplacian and the renormalized Laplacian. We also provide upper bounds for sums of squares of eigenvalues of these three matrices.

Among our results, we generalize an inequality of Fiedler for the extreme eigenvalues of the graph Laplacian to a bound on the sums of the smallest (or largest) $k$ such eigenvalues, $k < n$.

Furthermore, if $\lambda_j$ are the eigenvalues of the graph Laplacian $H = -\Delta$, in increasing order, on a finite graph with $|V|$ vertices and $|E|$ edges which is isomorphic to a subgraph of the $\nu$-dimensional infinite cubic lattice, then the spectral sums obey a Weyl-type upper bound, a simplification of which reads

$$\sum_{j=1}^{k-1} \lambda_j \leq \frac{\pi^2 |E|}{3} \left( \frac{k}{|V|} \right)^{1+\frac{2}{\nu}}$$

for each $k < |V|$.

This and related estimates for $\sum_{j=1}^{k-1} \lambda_j^2$ provide a family of necessary conditions for the embeddability of the graph in a lattice of dimension $\nu$ or less.

Email addresses: harrell@math.gatech.edu (Evans M. Harrell II), Joachim.Stubbe@epfl.ch (Joachim Stubbe).

Preprint submitted to Elsevier 11 May 2014
1 Introduction

It is possible to discern some structural features of a graph $G$ from the spectra of various matrices associated with $G$. In practice, the most important such matrices are the adjacency matrix, the graph Laplacian, and the renormalized Laplacian favored for example by Chung [4]. (In this article, a graph will be assumed to be finite, simple, connected, and non-directed without further comment. For the definitions of these terms and other general theory, we refer to [3,7].) The eigenvalues of these three matrices have been well studied and are discussed in several monographs, especially [4,5,6,2]. The particular objects of the present study are the (incomplete) sums of the ordered eigenvalues associated with a graph, and related quantities such as sums of powers of eigenvalues. We use two variational methods to obtain upper bounds on the partial sums of eigenvalues, which reflect the topology of the graph and the possibility of embedding it in a regular lattice. The inequalities are for the most part optimal in the sense that, given a little information about the structure of the graph, there are examples in which the inequalities are saturated.

There is a long history in quantum physics (e.g., [16,18]) and in spectral geometry (e.g., [17,15]) of investigation of the sums of the lowest $k$ eigenvalues of operators, and relating them to the nature of the phase space or to the geometry, but eigenvalue sums have received much less attention in the context of graph spectra. In the main, the complete sums of eigenvalues have been recognized as a kind of energy connected to the structure of graphs and have been studied, for example in [19].

The notational conventions of the standard references on graph spectra are, unfortunately, not consistent with one another. Because of this we recall some basic definitions to fix the notation to be used.

**Definition 1.1** Given a graph $G$ with $|\mathcal{V}| = n$ vertices, labeled in some fashion, the adjacency matrix $A$ has elements $a_{uv} = 1$ when vertex $u$ is connected to vertex $v$ and 0 otherwise. The (combinatorial) graph Laplacian is defined as

$$\Delta := A - \text{Deg},$$

where $\text{Deg}$ is the diagonal matrix such that $\text{Deg}_{vv} = d_v$ is the degree of the vertex $v$, i.e., the number of edges connecting to $v$. We prefer to express our results in terms of $H := -\Delta$, noting that $H$ is positive semidefinite, since

$$\langle \phi, H\phi \rangle = \sum_{\text{edges } [uv]} |\phi_u - \phi_v|^2 = \frac{1}{2} \sum_u \sum_v |\phi_u - \phi_v|^2.$$  \hspace{1cm} (1.1)

The null space of $H$ includes the constant vector with all entries equal to 1, which we denote $1$, and is one-dimensional (assuming that the graph is connected).
The renormalized Laplacian, cf. [4], corresponds to the matrix
\[
\hat{H} := \text{Deg}^{-\frac{1}{2}} H \text{Deg}^{-\frac{1}{2}}.
\]

Our notation for the eigenvalues of these three matrices is as follows:

- **A**: \( \alpha_0 > \alpha_1 \geq \ldots \alpha_{n-1} \)
- **H**: \( 0 = \lambda_0 < \lambda_1 \leq \ldots \lambda_{n-1} \)
- **\( \hat{H} \)**: \( 0 = c_0 < c_1 < c_2 \leq \ldots c_{n-1} \leq 2 \)

The indexing scheme ensures that in the case of a regular graph of degree \( d \), \( \lambda_k = d - \alpha_k = d c_k \) for each \( k \). When discussing an arbitrary matrix (usually self-adjoint), we shall call it and its eigenvalues \( (M, \mu_0 \ldots \mu_{n-1}) \). Throughout the article, \( n = |V| \) is reserved for the number of vertices, and \( m = |E| \) designates the number of edges.

For later purposes we recall some basic identities relating the spectra of \( H \) and \( A \) to properties of the graph:

\[
\text{Tr} (H) = \text{Tr} (A^2) = \sum_v d_v = 2m, \quad \text{Tr} (H^2) = 2m + \sum_v d_v^2.
\]  

The topological quantity \( \sum_x d_x^2 \) is known as the first Zagreb index of the graph \( G \), denoted \( M_1(G) \) [9].

2 An extension of a result of Fiedler to sums of graph eigenvalues

The usual variational strategy for estimating the spectrum of an operator is to make shrewd, simplifying guesses at the eigenvectors, and to use them in inequalities deriving from the spectral theorem. In this section we exploit the additive version of the min-max principle for the eigenvalues \( \{\mu_\ell\}_{\ell=0}^{n-1} \) of a self-adjoint matrix \( M \), viz., that for any orthonormal set of vectors \( \{\phi^{(\ell)}\}_{\ell=0}^{k-1} \),

\[
\sum_{\ell=0}^{k-1} \mu_\ell \leq \sum_{\ell=0}^{k-1} \langle M \phi^{(\ell)}, \phi^{(\ell)} \rangle,
\]  

and

\[
\sum_{\ell=k}^{n-1} \mu_\ell \geq \sum_{\ell=k}^{n-1} \langle M \phi^{(\ell)}, \phi^{(\ell)} \rangle.
\]  

cf. [1, §34]. With the aid of a special basis we shall obtain sharp bounds on sums of eigenvalues, which reduce to a result of Fiedler when there is only one summand. In the following section we obtain some different, competing
results on sums of eigenvalues and related quantities, using a novel variational argument that incorporates spectral projectors and an averaging over a family of test functions.

A good way to come up with test vectors for use in (2.1) is to consider special graphs on \( n \) or more vertices, with eigenvectors that are known explicitly. Thus we consider a graph \( G_p \) which is the join of a complete graph with a completely disconnected graph. That is, there are \( n \) vertices of which the first \( n - p \) vertices have no edges in common, but the graph is otherwise maximally connected. For \( 1 \leq p \leq n - 1 \) the graph Laplacian \( H_p \) has the form

\[
H_p := \begin{pmatrix}
p & 0 & \ldots & 0 & -1 & -1 & \ldots & -1 \\
0 & p & \ldots & 0 & -1 & -1 & \ldots & -1 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & p & -1 & -1 & \ldots & -1 \\
-1 & -1 & \ldots & -1 & n - 1 & -1 & \ldots & -1 \\
-1 & -1 & \ldots & -1 & -1 & n - 1 & 1 & \ldots & -1 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
-1 & -1 & \ldots & -1 & 1 & -1 & \ldots & n - 1 \\
\end{pmatrix}.
\] (2.3)

Remark 1 In particular, \( H_1 \) is the Laplacian of a star graph, while \( H_{n-1} \) is the Laplacian of a complete graph. For future purposes we observe that

\[
tr(H_p) = p(2n - p - 1), \quad tr(H_p^2) = p(n^2 + pn - p^2 - p).
\] (2.4)

Building a variational estimate for an arbitrary graph from the eigenvectors of this family of graphs leads to an extension of the result of Fiedler \[8\], as we next demonstrate. Letting \( e_j, j = 1, \ldots, n \) denote the canonical orthonormal basis vectors of \( \mathbb{R}^n \), we construct a reduced basis of eigenfunctions as follows.

**Proposition 2.1 (Spectral analysis of \( H_p \))** Let \( \epsilon^{(0)} := \frac{1}{\sqrt{n}} := \frac{1}{\sqrt{n}} \sum_{j=1}^{n} e_j \) and for \( \ell = 1, \ldots, n - 1 \), define

\[
\epsilon^{(\ell)} := \frac{1}{\sqrt{\ell(\ell + 1)}} \left( \epsilon_{\ell+1} - \sum_{j=1}^{\ell} e_j \right),
\] (2.5)

noting that \( \{\epsilon^{(\ell)}\} \) is an orthonormal basis of \( \mathbb{R}^n \). For each \( \ell = 1, \ldots, n - p - 1 \), \( \epsilon^{(\ell)} \) is an eigenvector of \( H_p \) with corresponding eigenvalue \( p \), and for each \( \ell = n - p, \ldots, n - 1 \), \( \epsilon^{(\ell)} \) is an eigenvector of \( H_p \) with corresponding eigenvalue
n. (Trivially, \(e^{(0)}\) is the normalized eigenvector of \(H_p\) with eigenvalue 0.)

The proposition can be verified directly. More details about the spectral analysis of the graphs \(G_p\) are collected in Appendix A.

Matrix elements of a general graph in the reduced basis \(\{e^{(\ell)}\}\). We first compute the matrix elements of a general self-adjoint matrix \(M\) with respect to \(e^{(\ell)}\), \(\ell = 1, \ldots, n - 1\). Let \(C_{j,\ell} = \frac{1}{\sqrt{j(j+1)(\ell+1)}}\). Then

\[
C_{j,\ell}^{-1} \langle e^{(j)}, M e^{(\ell)} \rangle = j\ell m_{j+1,\ell+1} + \sum_{\alpha=1}^{\ell} \sum_{\beta=1}^{\ell} m_{\alpha\beta} - j \sum_{\beta=1}^{\ell} m_{\beta,j+1} - \ell \sum_{\alpha=1}^{\ell} m_{\alpha,\ell+1}.
\]

(2.6)

In particular,

\[
\langle e^{(1)}, M e^{(1)} \rangle = \frac{1}{\ell(\ell+1)} \left( \ell^2 m_{\ell+1,\ell+1} + \sum_{\alpha=1}^{\ell} \sum_{\beta=1}^{\ell} m_{\alpha\beta} - 2\ell \sum_{\alpha=1}^{\ell} m_{\alpha,\ell+1} \right).
\]

(2.7)

If we now specialize so that \(M = H\), a graph Laplacian, then

\[
\langle e^{(1)}, H e^{(1)} \rangle = \frac{1}{2} (d_1 + d_2) - a_{12},
\]

or, using the fact that the sum over rows of \(H\) is equal to zero,

\[
\langle e^{(n-2)}, H e^{(n-2)} \rangle = \frac{(n-1)^2 d_{n-1} - 2(n-1)a_{n-1} + d_n}{(n-2)(n-1)},
\]

(2.8)

\[
\langle e^{(n-1)}, H e^{(n-1)} \rangle = \frac{n}{n-1} d_n.
\]

(2.9)

Similarly, we compute

\[
\langle H e^{(n-1)}, H e^{(n-1)} \rangle = \frac{n}{n-1} \left( d_n^2 + d_n \right).
\]

(2.10)

If the diagonal elements of \(H\) are arranged in decreasing order, then applying the variational principle (2.1) to (2.9) immediately yields an alternative proof of a result of Fiedler [8]:

**Proposition 2 (Fiedler)** For the graph Laplacian,

\[
\lambda_1 \leq \frac{n}{n-1} \min_v d_v, \quad \frac{n}{n-1} \max_v d_v \leq \lambda_{n-1},
\]

(2.11)

with equality for the complete graph and the star graph.
We are now ready to extend Proposition 2 to sums of ordered eigenvalues. Applying the min-max principle for sums of eigenvalues (2.1), choosing $\phi(\ell) = \epsilon^{(\ell)}$ and using the fact that we may relabel vertices, we get the following.

**Proposition 3** The partial sums of the eigenvalues of the graph Laplacian satisfy the following inequalities.

\[
\lambda_1 + \lambda_2 \leq \frac{n-1}{n-2} \min_{u \neq v} \left( d_u + d_v - \frac{2a_{uv}}{n-1} \right), \\
\frac{n-1}{n-2} \max_{u \neq v} \left( d_u + d_v - \frac{2a_{uv}}{n-1} \right) \leq \lambda_{n-2} + \lambda_{n-1}
\]  
with equality for the complete graph and the star graph. Moreover (by averaging over $u$),

\[
\lambda_1 + \lambda_2 \leq \frac{2m}{n-2} + \frac{n(n-3)}{(n-1)(n-2)} \min_v d_v, \\
\frac{2m}{n-2} + \frac{n(n-3)}{(n-1)(n-2)} \max_v d_v \leq \lambda_{n-2} + \lambda_{n-1}.
\]  

For any $L = 1, \ldots, n-1$ we get

\[
\sum_{i=1}^{L} \lambda_i \leq \frac{L}{L+1} \sum_{x=1}^{L+1} d_x + \frac{1}{L+1} \sum_{u=1}^{L+1} \sum_{v=1}^{L+1} a_{uv} \leq \sum_{j=n-L+1}^{n} \lambda_j \\
\sum_{i=1}^{L} \lambda_i \leq \frac{n-L+1}{n-L} \sum_{x=n-L+1}^{n} d_x + \frac{1}{n-L} \sum_{u=n-L+1}^{n} \sum_{v=n-L+1}^{n} a_{uv} \leq \sum_{j=n-L+1}^{n} \lambda_j.
\]  

3 An averaged variational principle and consequences for spectral sums

In [14] P. Krüger proved an upper bound for sums of eigenvalues of a vibrating free membrane (i.e. the Neumann Laplacian) on a bounded domain. Krüger’s bound is sharp in the sense of having the same dependence on dimension as the classic asymptotic estimate of large eigenvalues due to Weyl. Although as presented in [14] the bound appears to rely on special properties of the Laplacian and of the Fourier transform, in our view the essence of the argument was that it averaged different parts of a variational estimate in different ways, one of which simplified some coefficients. We shall formulate an abstract version of the spectral estimate of [14] and apply it to two situations, in one of which the graph is assumed to be a finite subset of the lattice $\mathbb{Z}^\nu$ equipped with nearest-neighbor edges, which we term the cubic lattice graph $\mathcal{G}^\nu$. Under
this assumption, we obtain an analogue for graphs of what Kröger proved for the Neumann problem on a compact $\Omega \subset \mathbb{R}^\nu$, and in particular we obtain an upper bound with Weyl dependence on dimension. The second situation is more generic, and applies to an arbitrary graph on $n$ vertices.

Suppose that $M$ is a self-adjoint operator on a Hilbert space $\mathcal{H}$, with discrete eigenvalues $-\infty < \mu_0 \leq \mu_1 \leq \ldots$ Let $P_k$ be the spectral projector associated to the eigenvalues 0 through $k$, and let $f$ be in the quadratic-form domain $Q(M) \subset \mathcal{H}$. (We reassure the reader that in this article all operators will be bounded matrices, in which case domain technicalities are avoided, as $Q(M)$ coincides with $\mathcal{H}$, and indeed $\mathcal{H}$ will merely be $\mathbb{C}^n$.)

By the variational principle (2.1),

$$\mu_k \left( \langle f, f \rangle - \langle P_{k-1} f, P_{k-1} f \rangle \right) \leq \langle M f, f \rangle - \langle MP_{k-1} f, P_{k-1} f \rangle. \quad (3.1)$$

Now consider a family of such trial functions $f_z$ indexed by a variable over which we can average. By averaging over two different sets, we get the following variational principle, corresponding to the main theorem of [14].

**Theorem 3.1** Consider a self-adjoint operator $M$ on a Hilbert space $\mathcal{H}$, with ordered, entirely discrete spectrum $-\infty < \mu_0 \leq \mu_1 \leq \ldots$ and corresponding normalized eigenvectors $\{\psi^{(l)}\}$. Let $f_z$ be a family of vectors in $Q(M)$ indexed by a variable $z$ ranging over a measure space $(\mathcal{M}, \Sigma, \sigma)$. Suppose that $\mathcal{M}_0$ is a subset of $\mathcal{M}$. Then for any eigenvalue $\mu_k$ of $M$,

$$\mu_k \left( \int_{\mathcal{M}_0} \langle f_z, f_z \rangle \, d\sigma - \sum_{j=0}^{k-1} \int_{\mathcal{M}} |\langle f_z, \psi^{(j)} \rangle|^2 \, d\sigma \right) \leq \int_{\mathcal{M}_0} \langle H f_z, f_z \rangle \, d\sigma - \sum_{j=0}^{k-1} \mu_j \int_{\mathcal{M}} |\langle f_z, \psi^{(j)} \rangle|^2 \, d\sigma, \quad (3.2)$$

provided that the integrals converge.

**Proof.** By integrating (3.1),

$$\mu_k \int_{\mathcal{M}_0} \left( \langle f_z, f_z \rangle - \langle P_{k-1} f_z, P_{k-1} f_z \rangle \right) \, d\sigma \leq \int_{\mathcal{M}_0} \langle M f_z, f_z \rangle \, d\sigma - \int_{\mathcal{M}_0} \langle MP_{k-1} f_z, P_{k-1} f_z \rangle \, d\sigma, \quad (3.3)$$
or

\[
\mu_k \int_{M_0} \left( \langle f_z, f_z \rangle - \sum_{j=0}^{k-1} |\langle f_z, \psi^{(j)} \rangle|^2 \right) d\sigma
\]

(3.4)

\[
\leq \int_{M_0} \langle Mf_z, f_z \rangle d\sigma - \int_{M_0} \sum_{j=0}^{k-1} \mu_j |\langle f_z, \psi^{(j)} \rangle|^2 d\sigma.
\]

Since \( \mu_k \) is larger than or equal to any weighted average of \( \mu_1 \ldots \mu_{k-1} \), we add to (3.4) the inequality

\[
-\mu_k \int_{M \setminus M_0} \left( \sum_{j=0}^{k-1} |\langle f_z, \psi^{(j)} \rangle|^2 \right) d\sigma \leq -\int_{M \setminus M_0} \sum_{j=0}^{k-1} \mu_j |\langle f_z, \psi^{(j)} \rangle|^2 d\sigma,
\]

(3.5)

and obtain the claim. \( \square \)

Although Theorem 3.1 appears designed to bound \( \mu_k \), its most notable use is to provide an upper bound on \( \mu_0 + \ldots + \mu_{k-1} \) by arranging that the left side be nonnegative, under which condition

\[
\sum_{j=0}^{k-1} \mu_j \int_{M} |\langle f_z, \psi^{(j)} \rangle|^2 d\sigma \leq \int_{M_0} \langle Mf_z, f_z \rangle d\sigma.
\]

(3.6)

In this work, inequalities obtained using Theorem 3.1 will turn out to satisfy the hypotheses of a celebrated theorem of J. Karamata (e.g., see [1, §28]), which we restate here in a slightly extended version:

**Lemma 3.1 (Karamata-Ostrowski)** Let two nondecreasing ordered sequences of real numbers \( \{\mu_j\} \) and \( \{m_j\}, j = 0, \ldots, n - 1 \), satisfy

\[
\sum_{j=0}^{k-1} \mu_j \leq \sum_{j=0}^{k-1} m_j
\]

(3.7)

for each \( k \). Then for any differentiable convex function \( \Psi(x) \),

\[
\sum_{j=0}^{k-1} \Psi(\mu_j) \geq \sum_{j=0}^{k-1} \Psi(m_j) + \Psi'(m_{k-1}) \cdot \sum_{j=0}^{k-1} (\mu_j - m_j).
\]

In particular, assuming either that \( \Psi \) is nonincreasing or that \( \sum_{j=0}^{k-1} \mu_j = \sum_{j=0}^{k-1} m_j \),

\[
\sum_{j=0}^{k-1} \Psi(\mu_j) \geq \sum_{j=0}^{k-1} \Psi(m_j)
\]

for each \( k \). Similarly, for any nondecreasing concave function \( \Phi(x) \) and each
As our first application of Theorem 3.1 using the same stratagem as in [14], we suppose that a graph $G$ is a finite subgraph of the cubic lattice graph $\mathcal{Q}^\nu$. By definition $\mathcal{Q}^\nu$ has countably many vertices, which will be labeled by integer-valued vectors $x \in \mathbb{Z}^\nu$ rather than by a single integer $v$. Two vertices are adjacent precisely when all but one of the coordinates of $x_1$ and $x_2$ are equal, while the remaining coordinate differs by $\pm 1$. As a finite subgraph, the vertex and edge sets of $G$ are subsets of those of $\mathcal{Q}^\nu$.

**Proposition 4** Suppose that $G$ is a finite subgraph of $\mathcal{Q}^\nu$. Then for $k \geq 1$ the eigenvalues of the graph Laplacian $H_G$ satisfy

$$
\sum_{j=0}^{k-1} \lambda_j \leq 2m\kappa \left(1 - \text{sinc}(\kappa^{1/\nu})\right),
$$

where $\text{sinc}(x) := \sin(x)/x$, and $\kappa := k/n$. Moreover, the sum of squares can be bounded in terms of simple topological properties of the graph including the number of pairs of neighbors of vertex $p$ that are collinear, which we denote $d_p^\parallel$.

$$
\sum_{j=0}^{k-1} \lambda_j^2 \leq \kappa \left(1 - \text{sinc}(\pi\kappa^{1/\nu})\right)^2 \text{Tr}(H^2) + 2\kappa \text{sinc}(\pi\kappa^{1/\nu})(1 - \text{sinc}(\pi\kappa^{1/\nu})) \sum_{x \in G} d_x^\parallel - 2\kappa \text{sinc}(\pi\kappa^{1/\nu})(1 - \cos(\pi\kappa^{1/\nu})) \sum_{x \in G} d_x^\parallel.
$$

**Remarks 3.2**

1. Because of (1.2), Inequalities (3.8) and (3.9) show that four topological properties of the graph, viz., the dimension of the ambient lattice, the number of edges of $G$, its Zagreb index, and the quantity $\sum_{x \in G} d_x^\parallel$, control the distribution of eigenvalues of subgraphs of a cubic lattice. The upper bounds are increasing functions of the dimension, which means that these estimates provide a family of necessary conditions for embeddability of the graph in a lattice of dimension $\nu$ or less. The authors plan to discuss further spectral conditions for embeddability of graphs in regular lattices in a future article.

2. As a simplification of (3.8) it is true independently of dimension that

$$
\sum_{j=1}^{k-1} \lambda_j \leq 2m\kappa,
$$

which becomes a standard equality when $\kappa = 1$ (i.e., $k = n$). Inequality (3.9)
also yields an equality when $\kappa = 1$. Another upper bound,

$$\sum_{j=1}^{k-1} \lambda_j \leq \frac{\pi^2 m}{3} \kappa^{1+\frac{2}{3}},$$

(3.11)

which has the form of the Weyl law for Laplacians on domains $\Omega \subset \mathbb{R}^\nu$, is both better when $k \ll n$ and correct to leading order in $\kappa$.

3. Complementary lower bounds for $\sum_{j=k}^{n-1} \lambda_j$ are available as usual by calling upon $\sum_{j=1}^{n-1} \lambda_j = 2m$ or by passing from $H$ to $-H$. There are similar bounds when the complementary graph $G'$ is embedded in $\Omega'$ owing to the standard relation among the nontrivial eigenvalues, $\lambda_j = n - \lambda'_j$.

4. When a graph $G$ is embedded in $\Omega'$ its Laplacian energy (see [10]) also satisfies a Weyl-type estimate. Recall that by definition, $LE(G) := \sum_{i=0}^{n-1} | \lambda_i - \frac{2m}{n} | = 2 \sum_{i=0}^{n-1} \left( \frac{2m}{n} - \lambda_i \right)$. Since the variational inequality (3.17) holds when $\lambda_k$ is replaced by $z \in [\lambda_{k-1}, \lambda_k]$ it is equivalent to the following inequality for the Riesz mean of the spectrum,

$$\sum_j (z - \lambda_j)_+ \geq zna^{2\nu} - 2ma^{\nu}(1 - \text{sinc}(\pi a))$$

(3.12)

for all $z \in [0, 2m]$. After simplifying and optimizing with respect to $a \in [0, 1]$, (3.12) becomes

$$\sum_j (z - \lambda_j)_+ \geq \frac{2 \cdot m \pi^2}{\nu} \cdot \frac{\nu}{\nu + 2} \cdot \frac{3nz}{m \pi^2} \cdot \nu/2.$$  (3.13)

In particular, the Laplacian energy of a finite subgraph of $\Omega'$ satisfies

$$LE(G) \geq 4m \max_{0 \leq a \leq 1} a^{\nu} \text{sinc}(\pi a) \geq \frac{8m}{\nu + 2} \frac{6\nu}{\pi^2(\nu + 2)^{\nu/2}}.$$

(3.14)

Proof. We use Theorem 3.1, taking $\mathcal{M}$ as the cube $[-\pi, \pi]^\nu$, with Lebesgue measure, and a vector-valued $z \in \mathcal{M}$; we can then make the same choice of test functions as in [14], viz., $f_z(x) = \exp(i x \cdot z)$. (However, now think of $f_z$ as a function on the graph $G$ consisting of set of vertices $x$ having integer coordinates, with a parameter $z$ ranging over $\mathbb{R}^\nu$.) The discrete Fourier transform on functions on $G$ is normalized as

$$\hat{\phi}(z) := \sum_{x \in G} e^{-ix \cdot z} \phi_x.$$
and we observe that the inversion formula for functions in the range of this transform is
\[ \phi_x = \frac{1}{(2\pi)^\nu} \int_{[-\pi,\pi]^\nu} e^{ix\cdot z} \hat{\phi}(z). \quad (3.15) \]

We begin by calculating from (1.1)
\[ \langle H e^{ix\cdot z}, e^{ix\cdot z} \rangle = \frac{1}{2} \sum_{x \in G} \sum_{q \sim x} |e^{ix\cdot z} - e^{iq\cdot z}|^2. \quad (3.16) \]

Now, each term \( |e^{ix\cdot z} - e^{iq\cdot z}|^2 \) simplifies to \( |e^{\pm iz_\ell} - 1|^2 = 4\sin^2 \left( \frac{z_\ell}{2} \right) \) for one of the Cartesian coordinates \( z_\ell \). If we integrate over a cube of the form \( M_0 := [-a\pi, a\pi]^\nu \), then these terms are replaced by \( 4(a\pi - \sin(a\pi)) \), and thus the quantity in (3.16) evaluates to
\[ 2(a\pi - \sin(a\pi))(2a\pi)^{\nu-1} \sum_{x \in G} d_x = (2a\pi)\nu 2 \left( 1 - \frac{\sin(a\pi)}{a\pi} \right) m, \]
drawing upon (1.2). To evaluate the other quantity on the right side of (3.2), we note that by the completeness relation associated with (3.15),
\[ \int_{[-\pi,\pi]^\nu} |\langle e^{ix\cdot z}, \psi^{(j)} \rangle|^2 = (2\pi)^\nu \|\psi^{(j)}\|^2 = (2\pi)^\nu. \]
Meanwhile, after integration, the lesser side of (3.2) becomes
\[ \lambda_k (n(2a\pi)^\nu - k(2\pi)^\nu), \]
and therefore, after division by \( (2\pi)^\nu \), we obtain
\[ \lambda_k \left( na^\nu - k \right) \leq 2ma^\nu (1 - \text{sinc}(\pi a)) - \sum_{j=0}^{k-1} \lambda_j \quad (3.17) \]
for all \( 0 \leq a \leq 1 \). Letting \( a^\nu \to \kappa \), we obtain (3.8).

For the inequality on sums of squares we observe that
\[ H f_x \big|_x = d_x e^{ix\cdot z} - \sum_{q \sim x} e^{iq\cdot z} = e^{ix\cdot z} \left( d_x - \sum_{q \sim x} e^{\pm iz_\ell} \right), \]
where as before the Cartesian direction \( \ell \) and the sign depend on \( x \) and \( q \). Thus
\[ \langle f_x, H^2 f_x \rangle = \|H f_x\|^2 = \sum_x \left( d_x^2 - 2d_x \Re \left( \sum_{q \sim x} e^{\pm iz_\ell} \right) + \left| \sum_{q \sim x} e^{\pm iz_\ell} \right|^2 \right). \quad (3.18) \]
When integrated in $z$ over the cube $[-a\pi, a\pi]^\nu$, the first two contributions to this equation become $(2a\pi)^\nu - 4(2a\pi)^{\nu-1}\sin a\pi \sum_x d_x^2$. The final term in (3.18) reflects the way in which the graph is embedded in $Q$: With $d_x^\parallel$ as defined in the Theorem,

$$\int_{[-a\pi,a\pi]^\nu} \left| \sum_{q_x \sim m_x} e^{\pm i z} \right|^2 d\text{Vol}_x = d_x(2a\pi)^\nu + \text{cross terms},$$

where the latter amount to

$$2(2a\pi)^\nu - 2 \left( (2a\pi) d_x^\parallel \int_{-a\pi}^{a\pi} \cos(2z)dz + \left( \left( d_x \right)^2 - d_x^\parallel \right) \left( \int_{-a\pi}^{a\pi} \cos(z)dz \right)^2 \right)$$

$$= 4(2a\pi)^{\nu-2} \left( (a\pi) d_x^\parallel \sin(2a\pi) + \left( d_x^\parallel - d_x \right) 2 \sin^2(a\pi) \right).$$

Summing all the contributions, the inequality corresponding to (3.6) reads

$$(2\pi)^\nu \sum_{j=1}^{k-1} \lambda_j^2 \leq (2a\pi)^\nu \left[ \left( 1 - (\sin (a\pi))^2 \right) \sum_{x \in G} d_x \right.$$

$$+ \left( 1 - 2 \sin (a\pi) + (\sin (a\pi))^2 \right) \sum_{x \in G} d_x^2$$

$$\left. + 2 \left( \sin (2a\pi) - (\sin (a\pi))^2 \right) \sum_{x \in G} d_x^\parallel \right]$$

Again letting $a^\nu \to \kappa$, we obtain (3.9).

The upper bound in inequality (3.8) is an increasing convex function of $\kappa = k/n$. Defining this upper bound as $S_k$, it follows that $m_k = S_k - S_{k-1}$ is a sequence satisfying the hypotheses of Karamata’s inequality. As a consequence, Corollary 5 Under the same conditions as in Proposition 4, for any nondecreasing concave function $\Phi(x)$,

$$\sum_{j=0}^{k-1} \Phi(\lambda_j) \leq \sum_{j=0}^{k-1} \Phi \left( 1 + \frac{2}{\nu} \right) \frac{\pi^2 m_j^{2/\nu}}{3n^{1+2/\nu}},$$

and for any nonincreasing convex function $\Psi(x)$,

$$\sum_{j=0}^{k-1} \Psi(\mu_j) \geq \sum_{j=0}^{k-1} \Psi \left( 1 + \frac{2}{\nu} \right) \frac{\pi^2 m_j^{2/\nu}}{3n^{1+2/\nu}}. \quad (3.19)$$

The statements in the Corollary are simply applications of Karamata’s Lemma 3.1 to the bound (3.11). Lemma 3.1 can be applied directly to (3.8) with a
slight improvement, but the form is complicated. Interesting choices for \( \Psi \) in (3.19) include \( x \to e^{-tx}, t \geq 0 \), which corresponds to the partition function of statistical physics, and \( x \to (t - x)_+^p \), which when summed on the spectrum becomes its Riesz mean.

Two simple examples are offered to illustrate Proposition 4.

**Examples 6**

1. A path with \( n \) vertices is a one-dimensional graph with eigenvalues \( \lambda_j = 4 \sin^2 \frac{\pi j}{2n} \). Therefore \( \sum_{j=1}^{k-1} \lambda_j = (2k - 1) \left( 1 - \frac{\sin(\pi \frac{k-1}{2n})}{\sin(\pi \frac{1}{2n})} \right) \), which admits the asymptotic expansion \( \frac{1}{k} \sum_{j=1}^{k-1} \lambda_j = \frac{\pi^2}{3} \kappa^2 + O(\kappa^3) \), thereby proving the sharpness of the bound (3.11) for \( \nu = 1 \). Considering its Laplacian energy the lower bound (3.14) yields \( \text{LE}(G) \geq \frac{4n - 4}{\pi} \). A simple upper bound is given by

\[
\text{LE}(G) \leq 2 \sum_{j=0}^{n/2} 2 - 4 \sin^2 \frac{\pi j}{2n} = 2 \sum_{j=1} \frac{\pi (n+1)}{2n} \sin \frac{\pi}{2n}
\]

(for \( n \) even, but a similar expression holds for \( n \) odd). This behaves like \( \frac{4n}{\pi} \) for \( n \) large, proving the sharpness of the lower bound (3.14).

2. Consider next a cycle with \( n = 2n' \) vertices, which embeds minimally in \( \mathcal{Q}_\nu \) with \( \nu = 2 \). The example of the cycle is an interesting test case for Proposition 4 because it is in a sense only slightly two-dimensional, and because it has many different realizations in \( \mathcal{Q}_2 \), for example either with many collinear neighbors or with none. Its eigenvalues are \( \lambda_0 = 0 \), \( \lambda_j' = 4 \sin^2 \frac{\pi j'}{n} \), \( j' = 1 \ldots n' - 1 \) with multiplicity 2 and \( \lambda_{n-1} = \lambda_{2n'-1} = 4 \). We consider the sum over an even number of eigenvalues. Let \( k = 2k' + 1 \), \( k' \in \mathbb{N} \). Then

\[
\sum_{j=1}^{k-1} \lambda_j = 8 \sum_{j'=1}^{k'} \sin^2 \frac{\pi j'}{n} \]

Therefore

\[
\sum_{j=1}^{k-1} \lambda_j = 2k \left( 1 - \frac{\sin(\pi \frac{k}{n})}{\sin(\pi \frac{1}{n})} \right) = 2mk \left( 1 - \frac{\sin (\pi \kappa)}{\sin (\pi \frac{1}{n})} \right),
\]

which for \( 1 \ll k \ll n \) agrees asymptotically with the upper bound (3.11) for \( \nu = 1 \).

The next application of Theorem 3.1 makes no assumption on \( G \) other than finiteness.

**Corollary 7** Let \( G \) be a finite graph on \( n \) vertices, and let \( \mathcal{M}_0 \) be any set of
\( n(k - 1) \) (ordered) pairs of vertices \( \{u, v\} \). Then for \( k < n \) the eigenvalues \( \lambda_k \) of the graph Laplacian \( H_G \) satisfy

\[
\sum_{j=1}^{k-1} \lambda_j \leq \frac{1}{2n} \sum_{\{u,v\} \in \mathcal{M}_0} (d_u + d_v + 2a_{uv}). \tag{3.20}
\]

**Remark 3.3** Ideally, one would optimize the choice of \( \mathcal{M}_0 \), whether by favoring vertices with low values of \( d_u \) or by choosing a subset where \( a_{uv} = 0 \) as often as possible. For example, if there is a large coloring subset, choosing pairs only from it will by definition guarantee that \( a_{uv} = 0 \). The extreme case of a graph with a large coloring subset is the star graph on \( n \) vertices, and it can be verified that for such graphs, \( (3.20) \) becomes an equality. Yet in the other extreme case, of the complete graph \( K_n \), \( (3.20) \) also becomes an equality.

**Proof.** It is helpful to apply Theorem 3.1 thinking of the Hilbert space as the orthogonal complement of the constant vector \( 1 = \sqrt{n} \psi(0) \). That is, \( \mathcal{H} \) consists of the vectors of mean 0. We take \( \mathcal{M} \) as the set of all ordered pairs \( \{u, v\} \) of vertices, the labels \( u, v \) each being identified with integers \( 1 \ldots n \), and in this case we can simply begin the sum in (3.2) with \( j = 1 \). We use the counting measure on the elements of \( \mathcal{M} \) or respectively of \( \mathcal{M}_0 \), a subset of \( \mathcal{M} \) to be chosen. For each such pair, define the vector \( b_{u,v} := e_u - e_v \). As before we calculate the quantities appearing in (3.2), beginning with

\[
\langle Hb_{u,v}, b_{u,v} \rangle = d_u + d_v + 2a_{uv}.
\]

(This formula is easy to see from (1.1) by considering separately the cases where \( u \) and \( v \) are connected and where they are not connected.) For any eigenvector \( \psi^{(\ell)} \) other than for \( \ell = 0 \), the orthogonality of \( \psi^{(\ell)} \) to \( \psi(0) \propto 1 \) implies that

\[
\int_{\mathcal{M}} |\langle b_{u,v}, \psi^{(\ell)} \rangle|^2 \, d\sigma = \sum_{u,v=1}^{n} \left( |\psi^{(\ell)}_u|^2 - 2\Re(\psi^{(\ell)}_u \overline{\psi^{(\ell)}_v}) + |\psi^{(\ell)}_v|^2 \right) = 2n\|\psi^{(\ell)}\| = 2n,
\]

and therefore from (3.6) it follows that

\[
2n \sum_{j=1}^{k-1} \lambda_j \leq \sum_{\{u,v\} \in \mathcal{M}_0} (d_u + d_v + 2a_{uv}), \tag{3.22}
\]

provided that the coefficient of \( \lambda_k \) coming from (3.2) is nonnegative, i.e., we require that \( 0 \leq 2|\mathcal{M}_0| - 2n(k - 1) \) (again calling on (3.21)). This establishes Corollary 7. \( \square \)

Next we apply the same ideas to the renormalized Laplacian:

**Corollary 8** Let \( G \) be a finite graph on \( n \) vertices, and let \( \mathcal{M}_0 \) be any set of \( p \) vertices...
pairs of vertices \{u, v\} with \(\sum_{2\mathfrak{M}_0} (d_u + d_v) \geq 4(k - 1)m\). Then the eigenvalues of the renormalized Laplacian \(\hat{H}_G\) satisfy

\[
\sum_{j=1}^{k-1} c_j \leq \frac{1}{4m} \sum_{2\mathfrak{M}_0} (d_u + d_v + 2a_{uv}),
\]  

(3.23)

and

\[
\sum_{j=1}^{k-1} c_j^2 \leq \frac{1}{4m} \sum_{2\mathfrak{M}_0} \left( d_u + d_v + 4a_{uv} + \sum_x \frac{1}{d_x} \left( \frac{a_{xu}d_u}{d_v} - \frac{a_{xv}d_v}{d_u} \right)^2 \right).
\]  

(3.24)

The final term in (3.24) is a measure of the deviation of \(G\) from regularity.

**Proof.** We use Theorem 3.1, again choosing \(\mathcal{H}\) as the orthogonal complement of \(1\), and taking \(\mathfrak{M}\) as the set of all pairs \(\{u, v\}\). For each such pair, this time we define the vector \(b_{u,v} := \sqrt{d_v}e_u - \sqrt{d_u}e_v\). As before we calculate the quantities on the right side of (3.2), beginning with

\[
\langle H b_{v,w}, b_{v,w} \rangle = \langle H_G \text{Deg}^{-1/2} b_{v,w}, \text{Deg}^{-1/2} b_{v,w} \rangle
\]

\[
= \sum_{x,y} \left( (\text{Deg}^{-1/2} b_{u,v})_x - (\text{Deg}^{-1/2} b_{u,v})_y \right)^2
\]

\[
= a_{uv} \left( \frac{d_v}{d_u} + \frac{d_u}{d_v} \right)^2 + (d_u - a_{uv}) \left( \frac{d_u}{d_v} \right) + (d_v - a_{uv}) \left( \frac{d_v}{d_u} \right)
\]

\[
= d_u + d_v + 2a_{uv},
\]

just as in Corollary 7. This quantity is an upper bound for

\[
\sum_{j=1}^{k-1} c_j \sum_{u,v} |\langle b_{v,w}, \psi^{(j)} \rangle|^2.
\]

To evaluate the coefficient of the summand, recall that for \(j > 0\), the eigenvectors \(\psi^{(j)}\) are orthogonal to \(\psi^{(0)} = \text{Deg}^{1/2} 1\). Hence

\[
\sum_{u,v} |\langle b_{u,v}, \psi^{(j)} \rangle|^2 = \sum_{u,v} |d_v^{1/2} \psi^{(j)}_u - d_u^{1/2} \psi^{(j)}_v|^2 = \sum_{u,v} \left( d_v |\psi^{(j)}_u|^2 + d_u |\psi^{(j)}_v|^2 \right) = 4m.
\]

The coefficient of \(\lambda_k\) coming from this application of (3.2) works out to be \(\sum_{2\mathfrak{M}_0} (d_u + d_v) - 2(k - 1)m\). It follows that if this quantity is nonnegative, then

\[
\sum_{j=1}^{k-1} c_j \leq \frac{1}{4m} \sum_{2\mathfrak{M}_0} (d_u + d_v + 2a_{uv}),
\]

as claimed.
For the sum of the squares, we instead calculate

\[ H\text{Deg}^{-1/2}b_{u,v} = \sqrt{d_ud_v} + a_{uv} \sqrt{\frac{d_u}{d_v}} + a_{uv} \sqrt{\frac{d_v}{d_u}}, \]

from which the expectation value of \( \hat{H}^2 \) becomes

\[ d_u + d_v + 4a_{uv} + \sum_x \frac{1}{d_x} \left( \frac{a_{xu}d_u}{d_v} - \frac{a_{xv}d_v}{d_u} \right)^2, \]

and the rest of the calculation goes as before. \( \square \)

For the adjacency matrix, Theorem 3.1 reduces to an elementary inequality for sums of eigenvalues, but an inequality reflecting somewhat more of the graph structure emerges for the sum of the \( k \) smallest values of \( \{ \alpha_j^2 \} \). (\textit{A priori} the selection of the smallest squares is very different from the ordering of \( \{ \alpha_j \} \).)

**Corollary 9** Let \( G \) be a finite connected graph on \( n \) vertices. Then for \( 1 \leq k \leq n-1 \), the eigenvalues \( \alpha_0 \geq \alpha_1 \geq \cdots \geq \alpha_{n-1} \) of the adjacency matrix \( A_G \) satisfy the elementary inequalities

\[ \sum_{j=0}^{n-k-1} \alpha_j \geq k, \]

\[ \sum_{j=n-k}^{n-1} \alpha_j \leq -k. \]

(3.25)

Now let \( \{ \alpha_{\ell_j} \} \), \( \ell = 0, \ldots, n-1 \) denote the eigenvalues \( \alpha_j \) reordered by magnitude, so that \( |\alpha_{\ell_0}| \leq |\alpha_{\ell_1}| \leq \ldots \). Then for any set \( \mathfrak{M}_0 \) of \( nk \) ordered pairs of vertices,

\[ \sum_{j=0}^{k-1} \alpha_{\ell_j}^2 \leq \frac{1}{2n} \sum_{(u,v) \in \mathfrak{M}_0} (d_u + d_v - 2(A^2)_{uv}). \]

(3.26)

(If the graph is not assumed connected, then \( k \) should be replaced by \( \min(k, m) \) in (3.25).) We note that (3.25) and (3.26) become equalities for complete graphs.

**Proof.** The two statements in (3.25) are equivalent, because \( \text{tr} A = 0 \). We choose to prove the second statement because it fits more comfortably the schema of Theorem 3.1. In this instance the Hilbert space \( \mathcal{H} \) is all of \( \mathbb{C}^n \), and the set \( \mathfrak{M} \) includes all the pairs \( \{u,v\} \) and one additional element which we shall call \( \omega \). As before, for each pair of vertices we define the vector \( b_{u,v} := e_u - e_v \), supplemented with the constant vector \( b_\omega = 1 \). For any vector \( \phi \), \( |\langle \phi, 1 \rangle|^2 = |\sum_u \phi_u|^2 \), and with a calculation similar to that of the proof of
Corollary 7, we find that
\[
\sum_{u,v} | \langle \phi, e_u - e_v \rangle |^2 + 2 | \langle \phi, 1 \rangle |^2 = 2n \| \phi \|^2.
\] (3.27)

We calculate that
\[
\langle A(e_u - e_v), e_u - e_v \rangle = -2a_{uv},
\]
and recall
\[
\langle A1, 1 \rangle = \sum_u d_u = 2m,
\]
cf. (1.2). If \(|\mathcal{M}_0| = nk < n(n-1)\), then the quantity coming from the left side of (3.2) vanishes, and we can conclude that
\[
n \sum_{j=n-k}^{n-1} \alpha_j \leq -\sum_{\mathcal{M}_0} a_{uv}.
\]
For any \(k < n-1\) we can in fact always find a set \(\mathcal{M}_0\) of size \(nk\), and we may furthermore preferentially include pairs \(\{u, v\}\) that are connected before pairs that are not connected. Thus the upper bound for the sum of eigenvalues is \(-k\) (unless \(k \geq m\), which does not occur for connected graphs). The result is
\[
\sum_{j=n-k}^{n-1} \alpha_j \leq -k.
\]
as claimed.

Applying Theorem 3.1 to the square of \(A\) will give a bound on the sum of the \(k\) smallest values of \(|\alpha_j|^2\). We calculate that
\[
\|A(e_u - e_v)\|^2 = \sum_x (a_{xu} - a_{xv})^2 = \sum_x (a_{xu} + a_{xv} - 2a_{xu}a_{xv}) = d_u + d_v - 2(A^2)_{uv},
\]
and with the same condition that \(|\mathcal{M}_0| = nk < n(n-1)\), we sum to obtain 3.26.

\[\square\]

A Spectral analysis of the graph \(G_p\).

We consider a graph \(G_p\) with \(n\) vertices such that \(p\) vertices, \(p = 1, \ldots, n-1\), are each the center of a star graph with \(n\) vertices, and the centers of the stars are all connected to one another. When \(p = 1\), \(G_p\) is a star graph. When, \(p = n-1\) it is the complete graph. We introduce the following notation: Let \(I_p, 0_p\) the \(p \times p\) identity matrix and zero matrix, respectively. Let \(J_{r,s}\) be the \(r \times s\) matrix whose entries are all equal to 1. Let \(\vec{1}_p\) and \(\vec{0}_p\) be the \(p\)
dimensional vectors with all entries equal to 1 and 0, respectively. Note that $J_{r,s}$ has the properties that $J_{r,s}1_s = s1_r$ and $J_{r,s}J_{s,r} = sJ_{r,r}$. We recall that $e_k$, $k = 1, \ldots, n$ denote the standard orthonormal basis vectors of $\mathbb{R}^n$. The positive graph Laplacian $H_p$, the normalized Laplacian $\hat{H}_p$, and the adjacency matrix $A_p$ of $G_p$ are given by

$$H_p = \begin{pmatrix} pI_{n-p} & -J_{n-p,p} \\ -J_{p,n-p} & nI_p - J_{p,p} \end{pmatrix},$$  \hspace{1cm} (A.1)$$

$$\hat{H}_p = \begin{pmatrix} I_{n-p} & -c_{n,p}J_{n-p,p} \\ -c_{n,p}J_{p,n-p} & \frac{n}{n-1}I_p - \frac{1}{n-1}J_{p,p} \end{pmatrix},$$  \hspace{1cm} (A.2)$$

where $c_{n,p} = p^{-\frac{1}{2}}(n-1)^{-\frac{1}{2}}$, and

$$A_p = \begin{pmatrix} 0_{n-p} & J_{n-p,p} \\ J_{p,n-p} & J_{p,p} - I_p \end{pmatrix}.$$  \hspace{1cm} (A.3)$$

The eigenspaces $H_p$ and the other operators can be represented as follows:
|      | Eigenvalue | Multiplicity | Eigenvectors/eigenspaces |
|------|------------|--------------|--------------------------|
| $H_p$ | 0          | 1            | $\vec{1}_n$              |
|      | $p$        | $n-p-1$      | $\psi = \sum_{k=1}^{n-p} v_k e_k$ such that $\sum_{k=1}^{n-p} v_k = 0$. |
|      | $n$        | $p$          | $\psi = \sum_{k=n-p+1}^{n} v_k e_k$ such that $\sum_{k=n-p+1}^{n} v_k = 0$ and $\psi = \frac{1}{\sqrt{np(n-p)}} \begin{pmatrix} p \vec{1}_{n-p} \\ (p-n) \vec{1}_p \end{pmatrix}$. |

|      | $\tilde{H}_p$ | 0          | 1            |
|------|----------------|------------|--------------|
|      | $1$            | $n-p-1$    | $\psi = \sum_{k=1}^{n-p} v_k e_k$ such that $\sum_{k=1}^{n-p} v_k = 0$. |
|      | $\frac{n}{n-1}$ | $p-1$     | $\psi = \sum_{k=n-p+1}^{n} v_k e_k$ such that $\sum_{k=n-p+1}^{n} v_k = 0$. |
|      | $\frac{2n-1-p}{n-1}$ | 1         | $\psi = \frac{1}{\sqrt{p(2n-p-1)}} \begin{pmatrix} \sqrt{p(n-1)} \vec{1}_{n-p} \\ -\sqrt{n-1} \vec{1}_p \end{pmatrix}$. |
\[
\psi = \sum_{k=1}^{n-p} v_k e_k \quad \text{such that} \quad \sum_{k=1}^{n-p} v_k = 0.
\]

\[
\psi = \sum_{k=n-p+1}^{n} v_k e_k \quad \text{such that} \quad \sum_{k=n-p+1}^{n} v_k = 0.
\]

Here \( d_{n,p} = \sqrt{(p-1)^2 + 4p(n-p)} \) and the eigenvectors for \( A_p \) corresponding to \( \rho_\pm \) are not normalized.

**Acknowledgments** We wish to acknowledge the assistance of Mr. Thomas Boutin for numerical studies of some of the inequalities reported here. E.H. is also grateful to the École Polytechnique Fédérale de Lausanne for hospitality that supported this collaboration.

**References**

[1] Beckenbach, B. and Bellman, R., *Inequalities*. 2nd revised printing, Springer 1965, §34, p.77

[2] Bıyıkoğlu, T., Leydold, J., and Stadler, P. F., *Laplacian Eigenvalues of Graphs, Perron-Frobenius and Faber-Krahn Type Theorems*, Springer Lecture Notes in Mathematics 1915, Springer-Verlag, Berlin, Heidelberg, New York, 2007.

[3] Bollobás, B., *Modern graph theory*, Graduate Texts in Mathematics 184. New York, Springer-Verlag, 1998.

[4] Chung, F. R. K., *Spectral Graph Theory*, Conference Board of the Mathematical Sciences Regional Conference Series in Mathematics 92, Amer. Math. Soc., Providence, 1997.

[5] Cvetković, D., Rowlinson, P., and Simić, S., *Eigenspaces of Graphs*, Cambridge: Cambridge University Press, 1997.

[6] Cvetković, D., Rowlinson, P., and Simić, S., *An Introduction to the Theory of Graph Spectra*, 75. Cambridge: Cambridge University Press, 2010.
[7] Diestel, R., *Graph theory*, Graduate texts in Mathematics 173 (4th Ed.). New York: Springer-Verlag, 2010.

[8] Fiedler, M., *Algebraic connectivity of graphs*, Czechoslovak Mathematical Journal, Vol. 23 (1973), 298–305. Persistent URL: [http://dml.cz/dmlcz/101168](http://dml.cz/dmlcz/101168)

[9] Gutman, I. and Trinajstić, N., *Graph theory and molecular orbitals. Total π-electron energy of alternant hydrocarbons*, Chem. Phys. Lett. 17 (1972), 535–538.

[10] Gutman, I. and Zhou, B., *Laplacian energy of a graph*, Lin. Alg. Appl. 414 (2006), 27–39.

[11] Harrell II, E. M. and Stubbe, J., *On trace identities and universal eigenvalue estimates for some partial differential operators*, Trans. Amer. Math. Soc. 349 (1997), 1797–1809.

[12] Harrell II, E. M. and Stubbe, J., *Trace Identities for Commutators, with Applications to the Distribution of Eigenvalues*, Trans. Amer. Math. Soc. S 0002-9947(2011)05252-9

[13] Harrell II, E. M. and Stubbe, J., *Inequalities for Riesz means and some consequences for graph spectra*, manuscript in prep.

[14] Kröger, P., *Upper bounds for the Neumann eigenvalues on a bounded domain in Euclidean space* J. Funct. Analysis 106(1992), 353-357. MR1165859 (93d:47091)

[15] Li, P. and Yau, S.-T., *On the Schrödinger equation and the eigenvalue problem*. Commun. Math. Phys. 88(1983)309–318.

[16] Lieb E.H. and Thirring, W., *Inequalities for the moments of the eigenvalues of the Schrödinger Hamiltonian and their relation to Sobolev inequalities. Studies in Math. Phys., Essays in Honor of Valentine Bargmann*. Princeton, 269-303 (1976)

[17] Pólya, G., *On the eigenvalues of vibrating membranes* Proc. London Math. Soc. 11(1961)419–433.

[18] Reed, M. and Simon, B., *Methods of Modern mathematical Physics. IV: Analysis of Operators*. New York: Academic Press, 1978.

[19] Zhou, B., *On sum of powers of the Laplacian eigenvalues of a graph*, Lin. Alg. Appl. 429 (2008), 2239–2246.