Field theory of Ising percolating clusters

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**Abstract**

The clusters of up spins of a two-dimensional Ising ferromagnet undergo a second order percolative transition at temperatures above the Curie point. We show that in the scaling limit the percolation threshold is described by an integrable field theory and identify the non-perturbative mechanism which allows the percolative transition in absence of thermodynamic singularities. The analysis is extended to the Kertész line along which the Coniglio-Klein droplets percolate in a positive magnetic field.
1 Introduction

The Ising model provides the fundamental example of critical behavior produced by the interaction among an infinite number of degrees of freedom, represented by spins on a lattice. Another type of criticality, which requires no interaction and, for this reason, is often called geometrical, finds its simplest illustration in random percolation [1]. Here the sites of the lattice are randomly occupied with probability \( p \), and criticality (the appearance of an infinite cluster of nearest-neighbor occupied sites) arises when \( p \) reaches a lattice-dependent value \( p_c \).

On the other hand, a ferromagnetic system like the Ising model encodes itself a percolation problem: nearest-neighbor sites with spin ‘up’ form clusters which can percolate for specific values of the parameters (temperature \( T \) and magnetic field \( H \)) which determine the interaction among the spins. This kind of percolation problem, which is completely determined by the magnetic properties and does not affect them, is called correlated percolation. The comparison of the magnetic and percolative phase diagrams on the \( T-H \) plane is particularly interesting. It turns out that in two dimensions (not in three) the temperature \( T_p \) at which clusters percolate in zero field coincides with the Curie temperature \( T_c \) [2]. Even in this case, however, a second order percolation line goes from \( (T, H) = (T_c, 0) \) to a finite value of the magnetic field at infinite temperature, a circumstance which can appear paradoxical if one considers that there are no thermodynamic singularities above \( T_c \) and that spin-spin correlations decay exponentially.

Essential insight into this problem comes from the Kasteleyn-Fortuin representation [3] which associates the percolative properties to an auxiliary site variable \( s_i \) which takes \( q \) values. As a result, the magnetic and percolative transitions, being related to different site variables, are in a sense disentangled. The price to pay is that \( s_i \), being auxiliary, cannot remain in the game to the end: everything needs to be evaluated in the limit \( q \to 1 \) in which the Ising spins are the only real degrees of freedom. The solution of the above paradox then requires to deal with the subtleties of this limit.

Field theory, as the natural framework for dealing with second order critical points and the scaling region around them, is in principle the right place where to address this problem and isolate its universal features. The difficulty, however, is in the fact that the answer can hardly come from a perturbative approach. In this paper we show that the percolation line above \( T_c \) in two dimensions actually corresponds, in the scaling limit, to an integrable field theory. On its exact solution, we can perform the limit \( q \to 1 \) analytically, unveiling the presence of \( q - 1 \) massless particles together with a massive unstable particle with lifetime inversely proportional to \( q - 1 \). Percolative properties are determined at first order in \( q - 1 \), where the theory is massless and describes the crossover from the correlated to the random percolation fixed point. The magnetic properties are determined instead at \( q = 1 \), where there are no massless excitations left and the massive particle has become stable, providing the required finite correlation length.

Most of this discussion for the clusters can be repeated for the Coniglio-Klein ‘droplets’ [4] which, for \( H = 0 \) (also in three dimensions), satisfy the requirement of Fisher’s droplet model [5]: \( T_p = T_c \) and percolative exponents equal to the magnetic ones. Also the Coniglio-Klein
droplets exhibit for $H > 0$ a second order percolation line, known as the Kertész line \[6\], going this time from the Curie point to a finite value of the temperature at infinite magnetic field. In the scaling limit this line corresponds again to a renormalization group trajectory within the Ising field theory, which, however, in this case is not integrable, so that its origin through the resonance mechanism observed for the clusters, although very likely, cannot be followed analytically.

The paper is organized as follows. In the next section we review the main results about the phase diagram of Ising clusters before recalling in section 3 the Kasteleyn-Fortuin formulation of the problem and the renormalization group analysis which leads to the identification of two different fixed points for clusters and droplets. The field theoretical discussion of clusters and droplets is then presented in sections 4 and 5, respectively, while section 6 contains a summary of the main conclusions.

2 Percolation of Ising clusters

Consider the ferromagnetic Ising model defined by the reduced Hamiltonian

$$-\mathcal{H}_{\text{Ising}} = \frac{1}{T} \sum_{\langle ij \rangle} \sigma_i \sigma_j + H \sum_i \sigma_i, \quad \sigma_i = \pm 1,$$

where $\sigma_i$ is a spin variable located at the $i$-th site of an infinite regular lattice, $T \geq 0$ and $H$ are couplings that we call temperature and magnetic field, respectively, and the first sum is restricted to nearest-neighbor spins. For $H = 0$ the model is well known to exhibit a non-zero magnetization per site $M = \langle \sigma_i \rangle$ at temperatures below a critical value $T_c$ (the magnetization in zero field is called spontaneous). The magnetization has a discontinuity at $H = 0$ along a path taken at fixed $T < T_c$ on the $T$-$H$–plane (first order transition), and vanishes when $T_c$ is approached from below at $H = 0$ (second order transition). No other magnetic transition (i.e. discontinuity in $M$ or its derivatives) takes place away from $T \leq T_c, H = 0$.

If, given a spin configuration, we draw a link between nearest-neighbor ‘up’ spins (i.e. spins taking the value $+1$), we obtain connected sets of up spins that we call clusters. Of course, clusters of ‘down’ ($-1$) spins are defined analogously; in the following, talking of clusters without further specification we will refer to clusters of up spins, being understood that similar statements hold for the clusters of down spins under the substitution $H \rightarrow -H$.

For $H = +\infty$ all the spins are forced to be up, so that the whole lattice is occupied by a unique infinite cluster. The fraction $P$ of the lattice occupied by this infinite cluster decreases when the magnetic field decreases at fixed $T$, and is certainly zero at $H = -\infty$. We denote by $H_0(T)$ the value of the magnetic field below which the infinite cluster is absent. Since at $T = 0$ all the spins are up for $H = 0^+$, we have $H_0(0) = 0^+$. When $T = \infty$, on the other hand, the spins are uncorrelated and take the value $+1$ with probability $e^H/2 \cosh H$; hence, denoting by $p_c^0$ the percolation threshold for the random site percolation problem, we have

$$\frac{e^{H_0(\infty)}}{2 \cosh H_0(\infty)} = p_c^0.$$  \(2\)
Like $T_c$, $p^0_c$ is non-universal, i.e. depends on the structure of the lattice. Notice for example that, if $p^0_c < 1/2$, $H_0(\infty)$ is negative, so that at sufficiently high temperature an infinite cluster of up spins exists also in a negative field and coexists with an infinite cluster of down spins in an interval of $H$ around zero.

For finite temperatures the Ising model encodes a generalized percolation problem in which the sites are not independent but interact through the Hamiltonian $H$. The probability that a site has spin up is $(M + 1)/2$ in terms of the average magnetization per site. Some early studies on Ising clusters can be found in [7, 8, 10, 11, 2, 12]. Here we summarize the main evidences relying on the following three statements:

i) $H_0(T)$ is a monotonic function;
ii) the existence of a spontaneous magnetization implies the presence of an infinite cluster;
iii) let $p$ be the probability that a site has spin up at $H = 0$, and $p_c$ the critical value above which an infinite cluster appears. Then $p_c$ does not exceed the value $p^0_c$ of random percolation.

Statement i) can be justified observing that, as a consequence of the ferromagnetic interaction, the fraction of up (down) spins increases as we decrease the temperature for a fixed positive (negative) value of the magnetic field, and eventually becomes 1 at $T = 0$. Hence, it is reasonable to expect that, if $H_0(\infty)$ is positive (negative), the strength of the positive (negative) field needed to produce (destroy) the infinite cluster of up spins decreases with the temperature, until it vanishes (it is zero at $T = 0$).

Statement ii) is a rigorous result of [2]. Notice that the opposite is not true: for $H_0(\infty) < 0$ there is an infinite cluster at large enough temperatures in zero field, but there is no spontaneous magnetization above $T_c$. Also, magnetization at $H \neq 0$ does not imply an infinite cluster: if $0 < H < H_0(\infty)$, there is no infinite cluster at infinite temperature in spite of the positive magnetization.

Statement iii) is an observation of [10]. A way of understanding it is to think of the ferromagnetic interaction as inducing an attraction among the clusters, which favors (with respect to the random case) the formation of larger clusters, and eventually of the infinite cluster. Indeed, if $Z_{\text{Ising}}$ is the partition function, the probability of a configuration is $e^{-H_{\text{Ising}}}/Z_{\text{Ising}}$. While configurations with a fixed number of up spins are all equally probable in the random case, in presence of the interaction cluster formation lowers the energy and increases the probability.

We can now distinguish three cases according to the value of the random percolation threshold $p^0_c$.

a) $p^0_c > 1/2$, i.e. $H_0(\infty) > 0$. In this case i) prevents the coexistence of infinite clusters of up spins with infinite clusters of down spins. $H_0(T)$ takes the value $0^+$ at $T = 0$ and is forced by ii) to stay constant up to $T_c$; above $T_c$ it is free to increase and to reach its positive asymptotic value (Fig. 1). So $T_c$ is also the percolation point in zero field; since the magnetization vanishes at this point, one has $p_c = 1/2$, which is consistent with iii).

b) $p^0_c < 1/2$, i.e. $H_0(\infty) < 0$. In this case $H_0(T)$ is zero up to some value $T_p$ of the temperature, above which it becomes negative, so that an infinite cluster of up spins coexists with an infinite cluster of down spins in the region $|H| < -H_0(T)$. At $H = 0^-$, the fraction
**Figure 1:** Qualitative phase diagram expected when the critical probability \( p_c^0 \) of random site percolation is larger than 1/2. The thick line is the curve \( H_0(T) \) above which there is an infinite cluster of up spins. Its dashed portion indicates that the percolative transition is first order below \( T_c \). The Kertész line (dotted) will be discussed in section 5.

\( p \) of sites with spin up, which is zero at \( T = 0 \), increases with the temperature and reaches the critical value \( p_c \) for \( T = T_p \). Since \( p = 1/2 \) for \( T \geq T_c \), iii) requires \( T_p < T_c \). Hence, for \( T_p < T < T_c \) there are infinite clusters, with different origin and different density, on both sides of \( H = 0 \).

c) \( p_c^0 = 1/2 \), i.e. \( H_0(\infty) = 0 \). In this case \( H_0(T) = 0 \) at all temperatures. Continuity with the previous two cases requires that \( T_p \rightarrow T_c^- \) as \( H_0(\infty) \rightarrow 0^- \), so that, like in case a), the percolative transition is first order below \( T_c \) and, presumably, second order above.

This classification according to \( p_c^0 \) leads to a distinction between the two- and three-dimensional cases \[10\]. Indeed, the two-dimensional lattices have \[1] \( p_c^0 \geq 1/2 \) \[13, 14\], and then fall into the cases a), c), which share the same critical pattern with both the magnetic and the percolative transition taking place at \( T_c \) in zero field. For the square lattice Ising model this was confirmed by series expansions in \[12\] and proved rigorously in \[2\]. For three-dimensional lattices the evidence is that \( p_c^0 < 1/2 \) \[15, 16\], so that they fall into the case b), for which the magnetic and percolative transitions in zero field are not simultaneous. For the simple cubic lattice this was first seen in \[8\] using Monte Carlo simulations.

### 3 Kasteleyn-Fortuin representation

#### 3.1 Dilute Potts model

The correlated percolation problem introduced in the previous section admits a formulation in terms of auxiliary Potts variables which generalizes that originally given for random percolation by Kasteleyn and Fortuin \[3\].

Let us first of all rewrite, up to an inessential additive constant, the Hamiltonian \[11\] in the

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\footnote{In a given dimension \( d \), the critical probability \( p_c^0 \) of random site percolation decreases as the coordination number \( C \) of the lattice increases. In \( d = 2 \), the triangular lattice (\( C = 6 \)) has \( p_c^0 = 1/2 \); in \( d = 3 \), the diamond lattice (\( C = 4 \)) has \( p_c^0 \simeq 0.43 \) (see \[4\]).}
Figure 2: Site-bond configuration on the square lattice. The full (empty) circles correspond to occupied (empty) sites of the lattice gas. Bonds (thick segments) between nearest-neighbor occupied sites are present with probability $p_B$ and define the KF clusters. For $p_B = 1$ all segments are thick and the KF clusters coincide with the Ising clusters.

The partition function allows the following representation in which the sum over the Potts variables $s_i$ is replaced by one over bond variables [17]

$$Z_q = \sum_{\{t_i\}} \sum_{\{s_i\}} e^{-\mathcal{H}_q}$$

$$= \sum_{\{t_i\}} e^{-\mathcal{H}_{Ising}} q^{N_e} \sum_G p_B^b (1 - p_B)^{\bar{b}} q^{N_c}, \quad p_B \equiv 1 - e^{-J},$$

where $N_e$ is the number of empty ($t_i = 0$) sites, the last sum is performed over the graphs $G$ obtained putting bonds in all possible ways between nearest-neighbor sites belonging to the restricted lattice formed by the occupied ($t_i = 1$) sites only, $b$ is the number of bonds in the graph $G$, $\bar{b}$ is the number of absent bonds on the restricted lattice, and $N_c$ is the number of connected components in $G$. Such connected components are called Kasteleyn-Fortuin (KF) clusters (Fig. 2).

For $q = 1$ there are no Potts degrees of freedom ($Z_1 = Z_{Ising}$) and the second sum in (5) gives 1, showing that $p_B$ is simply the probability that a bond is present on the restricted lattice. By construction, the KF clusters live on the clusters of up spins of the Ising model, and become the Ising clusters for $p_B = 1$.

The number $q$ of Potts colors appears in (5) as a parameter (which can be taken continuous) through power-like terms accounting for the fact that both the empty sites and the KF clusters
can take \( q \) colors. If \( X \) is an observable, the average associated to the partition function (5) with \( q = 1 \) is

\[
\langle X \rangle = Z_{\text{Ising}}^{-1} \sum_{\{t_i\}} e^{-H_{\text{Ising}}} \sum_G X p_B^b (1 - p_B)^{\bar{b}}.
\] (6)

When \( H = +\infty \) there are no vacancies and the average reduces to the weighted sum over bond configurations on the whole lattice, i.e. the usual random bond percolation problem. For generic values of \( T \) and \( H \), instead, (6) corresponds to a generalized percolation problem in presence of vacancies whose distribution is weighted by the Ising lattice gas Hamiltonian. For example, for \( X = N_c \), (6) gives the mean cluster number in this generalized percolation problem, as a function of \( p_B \). Evaluated at \( p_B = 1 \), this quantity is the mean number of spin up clusters in the Ising model at the given values of \( T \) and \( H \).

Notice that, while the correlations among Ising spins affect the associated bond percolation problem, the opposite is not true. At \( q = 1 \) the Hamiltonian (4) reduces to \( H_{\text{Ising}} \) and the parameter \( J \) (or \( p_B \)) plays no dynamical role: the conjugated bond variables over which the second sum in (6) is performed only serve enumeration purposes. In particular, an observable \( X \) which does not depend on the bond variables goes out of the second sum, which then gives 1, leaving us with the usual Ising thermodynamic average.

The role of the parameter \( q \) is further clarified if we consider the Hamiltonian \( \tilde{H}_q \) obtained adding to \( H_q \) the term \( \tilde{H} \sum_i \bar{t}_i \delta_{s_i,1} - 1 \), where the magnetic field \( \tilde{H} \) conjugated to the Potts variable is usually called ‘ghost’ field. The new partition function \( \tilde{Z}_q \) is then given [17] by \( Z_q \) with \( q^{N_c} \) replaced by \( \prod_r \left[ (q - 1)e^{\tilde{H}S_r} + 1 \right] \), where \( r \) labels the KF clusters in \( G \) and \( S_r \) is the number of sites in the \( r \)-th cluster. When we expand the free energy per site around \( q = 1 \),

\[
\tilde{f}_q = -\frac{1}{N} \ln \tilde{Z}_q = f_{\text{Ising}} - (q - 1)F + O((q - 1)^2),
\] (7)

the function \( dF/d\tilde{H} \) depends on \( \tilde{H} \) through terms containing the factor \( 1/N \sum_r S_r e^{\tilde{H}S_r} = \sum_S n_S S e^{\tilde{H}S} \), where \( n_S \) is the number of clusters of size \( S \) per site, so that

\[
\left( \frac{d^k F}{d\tilde{H}^k} \right)_{\tilde{H}=0} = \sum_S S^k \langle n_S \rangle,
\] (8)

i.e. \( F \) is the generating function for the moments of the cluster size distribution. The probability that a set of sites belongs to the same cluster can be obtained along the same lines introducing a site-dependent ghost field [17]. The important message of (7) and (8) is that the dilute \( q \)-state Potts model coincides with the Ising model at \( q = 1 \) and determines the properties of KF clusters within the lattice gas at first order in \( q - 1 \); it describes Ising clusters at first order in \( q - 1 \) when \( p_B = 1 \).

### 3.2 Renormalization group analysis

The Hamiltonian description in terms of the dilute Potts model allows a renormalization group analysis of percolative properties within the lattice gas. The case we are interested in is the
two-dimensional one, for which the magnetic and percolative transitions in the Ising model both take place at $T_c$ for $H = 0$. The renormalization group analysis for this case was performed in [4] (see also [18, 19]). We now recall the main results of this analysis casting them within the field theoretical language needed for our subsequent purposes.

We look for fixed points of the Hamiltonian (11) in $d = 2$ for $q \to 1$. Since the percolative properties do not affect the magnetic ones, we need to be at a magnetic fixed point to start with. The only non-trivial such fixed point is at $T = T_c, H = 0$; there we look for fixed points of the residual coupling $J$.

Before starting this search, let us recall that in two dimensions one can associate to a fixed point of the renormalization group a number $c$, called central charge, which grows with the number of dynamical degrees of freedom [20, 21, 22]. A fixed point (i.e. a conformal field theory) with central charge $c$ parameterized as

$$c = 1 - \frac{6}{m(m+1)}$$

contains scalar fields $\varphi_{r,s}$ (called primaries) with scaling dimension

$$X_{r,s} = \frac{[(m+1)r - ms]^2 - 1}{2m(m+1)}$$

(10)

together with infinitely many other less relevant fields (called descendants) whose dimensions exceed (10) by integers. The field $\varphi_{1,1}$ with dimension 0 is the identity $I$. For $m$ integer and larger than 1 the theory admits a ‘minimal’ realization in which the operator product expansion closes on the primaries with $r$ and $s$ positive integers up to $m - 1$ and $m$, respectively, and their descendants [20]. Considering that within this set of primaries each dimension appears twice and excluding the identity, the number of independent and non-trivial primaries within a minimal model is $m(m - 1)/2 - 1$.

Back to our problem, since no dynamical degrees of freedom are associated to the percolative properties, for all the fixed points with $T = T_c, H = 0$ the central charge is that of the critical Ising model, i.e. $c = 1/2$ ($m = 3$). Clearly, the decoupling point $J = 0$ yields a trivial fixed point at which percolation plays no role ($p_B = 0$) and we deal with the pure magnetic fixed point of the Ising model. The latter is well known to be described by the minimal realization of the $m = 3$ conformal theory, which indeed contains two relevant scalar fields: the spin field $\sigma$ with dimension $X_\sigma = X_{1,2} = 1/8$, and the thermal field $\varepsilon$ with dimension $X_\varepsilon = X_{1,3} = 1$. It follows that at this fixed point the coupling $J$ is conjugated to an irrelevant field.

In order to progress, we need to recall that the dilute Potts model (11) (with $T$ which can be fixed to $T_c$ for the time being) admits two distinct lines of fixed points for $q < 4$ [23]. The first one is the critical line of the undilute ($\Delta = +\infty$) Potts model [24], while the second is a tricritical line arising at some $\Delta_c$ yielding a finite concentration of vacancies. Both lines are described by conformal theories with central charge [19] and the following relations between $q$
\[ \sqrt{q} = 2 \sin \frac{\pi(t - 1)}{2(t + 1)}, \quad m = \begin{cases} 
  t & \text{for the critical line}, \\
  t + 1 & \text{for the tricritical line}. 
\end{cases} \] (11)

The two fixed lines meet at \( q = 4 \) \((m = \infty, c = 1)\) and the transition is first order above this value. The Potts spin field has scaling dimension \( X_s(q) \) which coincides with \( X_{(m-1)/2,(m+1)/2} \) along the critical line, and with \( X_{m/2,m/2} \) along the tricritical one. As for the fields invariant under the \( S_q \) symmetry of color permutation, the leading scaling dimension \( X_{t_1}(q) \) along the critical (tricritical) line is \( X_{2,1_1}(X_{1,2}) \); since also the dilution is relevant at tricriticality, a second relevant, \( S_q \)-invariant scaling field is present along the tricritical line and corresponds to \( \varphi_{1,3} \), with dimension \( X_{t_2}(q) = X_{1,3} \).

In the limit \( q \to 1 \) relevant for the percolative properties, the critical line gives \( m = 2, c = 0 \), in agreement with the fact that the undilute case corresponds to random percolation, which carries no dynamical degrees of freedom. Hence, the critical exponents for random percolation are determined by those of the \( q \to 1 \) pure Potts model. In particular, the cluster size exponent coincides with the Potts susceptibility exponent \( \gamma \). This and the correlation length exponent \( \nu \) are given by

\[ 1/\nu = 2 - X_{t_1}, \quad \gamma/\nu = 2 - 2X_s, \] (12)

and at the random percolation point are determined by the values \( X_s(1) = 5/48 \) and \( X_{t_1}(1) = 5/4 \) on the critical line. These values characterize the infinite temperature fixed point of Fig. 1. Notice that the minimal realization of the \( m = 2 \) fixed point contains only the identity: critical percolative properties are not described by minimal conformal field theories.

The limit \( q \to 1 \) along the tricritical line gives \( m = 3, c = 1/2 \). This is also expected because at \( q = 1 \) there are no Potts degrees of freedom and the critical degrees of freedom are those of the lattice gas, which is an Ising model. As a consequence, within the model [4] this fixed point corresponds to \( T = T_c, H = 0 \) and to some \( J = J^* \). Contrary to the case \( J = 0 \) discussed above, it is a non-trivial fixed point for correlated percolation. The percolative exponents are determined here by the tricritical values \( X_s(1) = 5/96, X_{t_1}(1) = 1/8 \) and \( X_{t_2}(1) = 1 \). Once again, the first of these dimensions does not belong to the minimal realization of the \( m = 3 \) theory.

Notice that the dimensions of the the two relevant \( S_q \)-invariant fields on the tricritical line coincide at \( q = 1 \) with those of the Ising spin and thermal fields, which are conjugated to \( H \) and \( T \), respectively. As a consequence, the remaining coupling \( J \) in [4] is irrelevant at \( J^* \). Since it was irrelevant also at \( J = 0 \), consistence of the renormalization group flows requires a third, intermediate fixed point where \( J \) is relevant (Fig. 3). This was located in [4] at \( J = 2/T_c \) as a

\(^2\)We recall that the values of the critical exponents depend on the direction in coupling space along which the fixed point is approached. In [12] we take into account that on the tricritical line there are two \( S_q \)-invariant relevant fields.
which goes like the Potts magnetization per site, i.e. like space dimensionality $d$. The fractal dimension is easily determined considering the number $S_{\infty}(L)$ of sites belonging to the incipient infinite cluster inside a box of side $L$. This goes like $PL^d$, where $P$ is the density of the infinite cluster, namely the percolative order parameter, which goes like the Potts magnetization per site, i.e. like $L^{-X_s}$. Hence $D = d - X_s$. In $d = 2$ the values of $X_s$ given above determine the fractal dimensions $91/48 = 1.89\ldots$ for random percolation, $187/96 = 1.94\ldots$ for Ising clusters, and $15/8 = 1.87\ldots$ for Ising droplets, which are less dense than Ising clusters, as expected.

The Hamiltonian (13) is that of a $(q+1)$-state Potts model and is critical for $2H = \ln q$ and some $q$-dependent value of $T$. As $q \to 1$ we obtain an Ising model with critical point at $H = 0$ and $T = Tc$. It is clear from (13) that $J = 2/T$ is a special case of (14) in which the lattice gas variable and the Potts spin are treated symmetrically, so that they have the same scaling dimension at the fixed point $J = 2/T_c$: $X_s = X_\sigma = 1/8$; similarly, $X_{\langle i \rangle} = X_\varepsilon = 1$. This means that the KF clusters with $p_B = 1 - e^{-2/T}$ percolate at $T_c$ with exponents $\gamma$ and $\nu$ which coincide with those of the magnetic susceptibility and correlation length in the Ising model. Since these are requirements of the droplet model [5] meant to describe magnetic transitions in a cluster language, the KF clusters with $p_B = 1 - e^{-2/T}$ are called Coniglio-Klein Ising droplets [4].

Concerning Ising clusters, the renormalization group pattern of Fig. 3 finally leads to the conclusion that their critical exponents at $T = T_c$, $H = 0$ are those of the fixed point $J^*$, onto which $J = \infty$ ($p_B = 1$) renormalizes at large distances [4]. The mean cluster size exponent in the limit $T \to T_c^-$, $H = 0$ was first evaluated by series expansions in [12], with the result $\gamma \approx 1.91$, quite close to the exact one $\gamma = 91/48 = 1.895\ldots$ coming from $X_s = 5/96$ and $\nu = 1$. This exact value was identified in [19].

We finish this section recalling that at the percolation point clusters behave as fractals, i.e. their size grows with the linear extension $L$ as $L^D$, with a fractal dimension $D$ smaller than the space dimensionality $d$ [1]. The fractal dimension is easily determined considering the number $S_{\infty}(L)$ of sites belonging to the incipient infinite cluster inside a box of side $L$. This goes like $PL^d$, where $P$ is the density of the infinite cluster, namely the percolative order parameter, which goes like the Potts magnetization per site, i.e. like $L^{-X_s}$. Hence $D = d - X_s$. In $d = 2$ the values of $X_s$ given above determine the fractal dimensions $91/48 = 1.89\ldots$ for random percolation, $187/96 = 1.94\ldots$ for Ising clusters, and $15/8 = 1.87\ldots$ for Ising droplets, which are less dense than Ising clusters, as expected.
Figure 4: Phase diagram of the field theory \( \text{(15)} \) describing the scaling dilute Potts model. The ferromagnetic phase transition surface corresponds to \( \lambda = 0 \). On this surface, the trajectories originating from the tricritical line are massive for \( g < 0 \) (first order transition); they are massless and flow into the critical line for \( g > 0 \) (second order transition).

4 Field theory of the scaling limit

The scaling region of lattice models around second order phase transition points can be described in a continuous, field theoretical framework. With the notation for the fields introduced in the previous section, the scaling limit of the Hamiltonian \( \text{(1)} \) in \( d = 2 \) is described by the Ising field theory \([27]\) for a review) with action

\[
A_{\text{Ising}} = A_{\text{CFT}}^{\text{Ising}} - \tau \int d^2 x \varepsilon(x) - h \int d^2 x \sigma(x),
\]

where \( A_{\text{CFT}}^{\text{Ising}} \) is the action of the conformal field theory with central charge \( c = 1/2 \), and \( \tau \sim T - T_c, \ h \sim H \) as the critical point is approached. The action \( \text{(14)} \) encodes all the universal features of the ferromagnetic phase transition. The universal percolative properties of the Ising model are instead contained in the scaling limits of the dilute Potts Hamiltonian \( \text{(1)} \) with \( q \to 1 \). It follows from the discussion of the previous section that, as far as Ising clusters are concerned, the renormalization group trajectories which matter are those originating from the line of tricritical fixed points. These are described by the field theory

\[
A_q = A_{\text{CFT}}^{\text{tricr}} - g \int d^2 x \varphi_{1,3}(x) - \lambda \int d^2 x \varphi_{1,2}(x),
\]

where \( A_{\text{CFT}}^{\text{tricr}} \) is the action of the conformal theory with central charge \( \text{(1)} \), related to \( q \) by \( \text{(11)} \) with \( t = m - 1 \), and, as we saw, \( \varphi_{1,2} \) and \( \varphi_{1,3} \) are the two relevant \( S_q \)-invariant fields at tricriticality.

Let us first discuss some general features of the field theory \( \text{(15)} \) for \( 1 < q \leq 4 \). It follows from general results \([28]\) that the theory is integrable when at least one of the two couplings \( g \) and \( \lambda \) vanishes. Integrability allows to establish that the critical surface separating magnetically
Figure 5: Qualitative behavior (along the dotted path on the right) of the magnetic order parameter $M_q$ and of $U_q = \langle \varphi_{1,2} \rangle$ in the scaling dilute Potts model (15). For $q = 1$ this model becomes the Ising field theory (14) with $g = \tau$ and $\lambda = h$; $M_1$ and $U_1$ become the Ising percolative and magnetic order parameter, respectively.

The ordered and disordered regions of the scaling dilute Potts model corresponds to $\lambda = 0$ (Fig. 4): for $\lambda = 0$, $g < 0$, (15) describes the first order part of the transition surface on which the $q$ ordered ground states are degenerate with the disordered one [29]; for $\lambda = 0$, $g > 0$, instead, the theory is massless [21, 30] and describes the second order part of the transition surface, spanned by the trajectories flowing from the tricritical to the critical line of fixed points.

For $\lambda < 0$, the theory (15) possesses a unique vacuum $|\Omega\rangle$ corresponding to the disordered ground state; for $\lambda > 0$ the degenerate vacua $|\Omega_i\rangle$, $i = 1, \ldots, q$, correspond to the $q$ ordered ground states. The order parameter of the ferromagnetic transition is the Potts magnetization per site $M_q$, i.e. the expectation value of the field $\delta s_i(x) - 1/q$ taken over $|\Omega\rangle$ for $\lambda < 0$, and over $|\Omega_j\rangle$ for $\lambda > 0$. At the first order transition the $q + 1$ vacua are degenerate, i.e. $\langle \varphi_{1,3} \rangle$ is the same on all vacua. The expectation value $U_q \equiv \langle \varphi_{1,2} \rangle$, like $M_q$, is discontinuous across the first order transition surface, its discontinuity corresponding to the latent heat [20]. Both $M_q$ and $U_q$ vanish on the second order transition surface (Fig. 5).

We are now ready to discuss the limit we are actually interested in, $q \to 1^+$, which we know a priori is peculiar. Indeed, it follows from the scaling dimensions given in the previous section that, as expected, $A_1 = A_{Ising}$, with $g = \tau$ and $\lambda = h$. Since $A_{Ising}$ is massive for any value of the couplings away from $\tau = h = 0$, the massless trajectories of (15) for $\lambda = 0$, $g > 0$ have to become massive as $q \to 1$. Remarkably, this phenomenon can be described analytically exploiting the integrability of (15) with $\lambda = 0$.

Let us recall that use of (non-conformal) integrability in two-dimensional field theory requires switching to a particle language. Indeed, integrable field theories are solved by exact determination of the $S$-matrix of the associated scattering theory in (1 + 1)-dimensional space-time [31].

In the ordered phase $\lambda > 0$, the elementary excitations of (15) on which the scattering theory is

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3Field theoretically, due to the reflection positivity of (14), this conclusion follows from Zamolodchikov’s $c$-theorem [21].
built are the \(q(q - 1)/2\) kinks which interpolate between the \(q\) degenerate vacua, together with their antikinks; due to \(S_q\) symmetry all these kinks have the same mass \([32]\). In the disordered phase \(\lambda < 0\), instead, the elementary excitations are ordinary particles \(A_k, k = 1, \ldots, q - 1\), forming a degenerate multiplet in which the antiparticle \(\bar{A}_k\) coincides with \(A_{q-k}\) \([33]\). The mass of all these excitations goes to zero when the second order transition surface is approached (\(\lambda \to 0, g > 0\)). Since the \(q\) vacua of the spontaneously broken phase coalesce in this limit, the elementary excitations on the transition surface are to be identified with the massless limit of the particles \(A_k\), which, in \((1 + 1)\) dimensions, are right/left movers with energy \(p^0 = (\mu/2)e^{\pm \theta}\) and momentum \(p^1 = \pm p^0\), \(\mu\) being a mass scale and \(\theta\) a rapidity parameter.

While the number of these massless degrees of freedom clearly vanishes as \(q \to 1\), the emergence of a massive particle in the same limit follows from the study of the right-left scattering amplitudes. The latter are known exactly \([30]\) within a particle basis which is not the Potts basis we are discussing. This is, however, immaterial as far as the analytic properties we are interested in are concerned (see e.g. \([37]\)), and the only think we need to know is that the poles of the amplitudes are determined by the factor \([30]\]

\[
\frac{1}{\cosh \rho(i\pi - \theta)} \exp \left[ -i \int_0^\infty \frac{dx}{x} \frac{\sin \frac{x}{2\rho}}{\sinh \frac{x}{2\rho}} \sin \frac{x}{\pi} \right],
\]

where \(\rho = 1/(m - 1)\) and \(\theta\) determines the square of the center of mass energy as \(s = \mu^2 e^\theta\). As usual, the scattering amplitudes have a multi-sheet structure in the complex \(s\)-plane. The ‘physical sheet’ on this plane corresponds to \(\text{Im}\theta \in (0, \pi)\), and the ‘second sheet’ to \(\text{Im}\theta \in (0, -\pi)\). Writing the exponential part of (16) as \(S_{-1/2}(\theta)/S_{1/2}(\theta)\), with

\[
S_1(\theta) = \prod_{n=0}^{\infty} \frac{\Gamma \left( \frac{1}{2} + \left( 2n + \frac{3}{2} - \gamma \right) \rho - \frac{\theta}{i\pi} \right) \Gamma \left( \frac{1}{2} + \left( 2n + \frac{3}{2} - \gamma \right) \rho + \frac{\theta}{i\pi} \right)}{\Gamma \left( \frac{1}{2} + \left( 2n + \frac{3}{2} - \gamma \right) \rho - \frac{\theta}{i\pi} \right) \Gamma \left( \frac{1}{2} + \left( 2n + \frac{3}{2} - \gamma \right) \rho + \frac{\theta}{i\pi} \right)^2},
\]

one can see that for \(q > 1\) (i.e. \(m > 3\)) \([16]\) has no poles on the physical sheet, and possesses a single pole, coming from the \(1/\cosh\) prefactor, at \(\theta_0 = -i\pi(m - 3)/2\), which is on the second sheet for \(m \in (3, 5)\). This pole at \(s_0 = \mu^2 e^{\theta_0}\) on the second sheet (Fig. 6) corresponds to a resonant particle \(A\) in the right-left scattering channel. When \(q \to 1^+\), the pole approaches the physical sheet and gives a narrow resonance with square mass \(\text{Re} s_0 \simeq \mu^2\) and inverse lifetime \(\text{Im} s_0/\mu \propto (q - 1)\mu\), which coexists with the \(q - 1\) massless particles. At \(q = 1\), no massless particles are left and \(A\) provides the stable massive particle of the Ising field theory \([14]\) with \(h = 0, \tau > 0\).

Since the number of components of \(S_q\)-multiplets vanishes as \(q \to 1\), the particle \(A\) we are talking about must be an \(S_q\)-singlet, i.e. arises in the right-left scattering channels \(A_k A_{q-k}\). This prevents the correlators of \(S_q\)-invariant fields from vanishing as \(q \to 1\). Indeed, let us denote by \(\phi\) such a field (scalar, for simplicity) and consider the spectral decomposition of its

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4See \([34, 35]\) for the theory of integrable massless scattering.

5See \([36]\) for a discussion on \(S_q\)-invariant scattering theories and change of particle basis.
two-point function \( G(x) = \langle \phi(x)\phi(0) \rangle \) over asymptotic particle states, in the limit \( q \to 1 \). The contribution to \( G(x) \) with the lowest number of intermediate particles is

\[
\lim_{q \to 1} (q - 1) \int d\theta_1 d\theta_2 |\langle \Omega |\phi(0)\rangle A_k(\theta_1)A_{q-k}(\theta_2)|^2 e^{-|x|E_{2,0}(\theta_1,\theta_2)},
\]

where the prefactor \( q - 1 \) comes from the summation over \( k \), and \( E_{2,0}(\theta_1,\theta_2) = \mu(e^{\theta_1} + e^{-\theta_2})/2 \) is the energy of a right-left pair. Right-right and left-left pairs do not contribute to the limit because only the right-left matrix element inherits from the right-left scattering amplitude the resonance pole at \( \theta_1 - \theta_2 = \theta_0 \), with \( \theta_0 \propto -i(q - 1) \) for \( q \to 1 \). Hence, calling \( R_\phi \) the residue of the matrix element on this pole, (18) can be written as

\[
\lim_{q \to 1} (q - 1) \int d\beta d\theta |R_\phi|^2 e^{-|x|E_{1,\mu}(\beta)} \propto \int d\beta e^{-|x|E_{1,\mu}(\beta)},
\]

where \( E_{1,\mu}(\beta) = \mu \cosh(\beta/2) \) is the energy of a particle with mass \( \mu \). We see in this way how correlations mediated by massless particles on the second order surface are replaced by correlations mediated by the particle \( A \) along the massive trajectory at \( q = 1 \) (Fig. 7). Fields which are not \( S_q \)-invariant do not couple to \( A \) and have zero correlations (i.e. are absent) at \( q = 1 \) (as required by \( S_1 = I \)). Recalling that the \( S_q \)-invariant fields along the tricritical Potts line are \( I, \varphi_{1,2}, \varphi_{1,3} \) and their descendants, which become \( I, \sigma, \varepsilon \) and their descendants at \( q = 1 \), the mechanism we illustrated explains in particular how the theory becomes minimal for this value of \( q \).

Summarizing, the percolation line \( H_0(T) \) of Fig. 1 is mapped in the scaling limit onto the case \( h = 0 \) of the field theory \( [14] \). More precisely, as expected from \( [7] \), this field theory describes only the magnetic properties of the Ising model; the universal percolative properties of Ising clusters are described by the ‘embedding’ of \( [14] \) into the dilute Potts field theory \( [15] \) with \( q \to 1 \). In this limit, the expectation values \( M_q \) and \( U_q \) shown in Fig. 5 become the Ising

\[
\text{Figure 6: Location } s_0 \text{ of the resonant pole on the second sheet of the } s\text{-plane. The pole describes an anticlockwise motion along the dashed path as } m \text{ decreases from 5 to 3: } s_0 \text{ is purely imaginary at } m = 4 \text{ (} q = 2 \text{) and real and positive at } m = 3 \text{ (} q = 1 \text{), where the associated particle becomes stable.}
\]
Figure 7: Correlations mediated by two massless particles at $q > 1$ are mediated by one massive particle at $q = 1$. Fields $\phi$ which are not $S_q$-invariant do not couple to the massive particle and are absent at $q = 1$.

percolative and magnetic order parameter, respectively. While for $\tau < 0$ both transition are first order, for $\tau > 0$ there is only a continuous percolative transition, a circumstance which is explained analytically by the evolution of the resonance pole discussed above.

Notice that for the transition at $\tau < 0$ the situation is simpler. On the first order part of the transition surface of (15) the $q$ ordered vacua $|\Omega_i\rangle$ are degenerate with the disordered one $|\Omega\rangle$. The elementary excitations are $q$ kinks interpolating from $|\Omega\rangle$ to $|\Omega_i\rangle$, together with their antikinks [29]. When $q \to 1$ one recovers straightforwardly the single kink of the Ising model.

5 Universal scaling limit of the Kertész line

We saw in section 3 that for $H = 0$ the Coniglio-Klein droplets, i.e. the KF clusters with $p_B = 1 - e^{-2/T}$, percolate at $T_c$ (also in $d = 3$) with critical exponents coinciding with the magnetic exponents. Also, contrary to the case of Ising clusters, an infinite droplet above $T_c$ in zero field is excluded by the vanishing of $p_B$ at infinite temperature. It is easy to see, however, that when $H \neq 0$ the critical properties of the magnetic and percolative degrees of freedom no longer coincide. Indeed, for $H = +\infty$ all the spins are up and we are left with a random bond percolation problem with occupation probability $p_B$, which is critical for some value $p_B'$. This means that there is a line $T_K(H)$, called the Kertész line [6], going from $T_K(0) = T_c$ to $T_K(+\infty) = -2/\ln(1 - p_B)$, which is a percolation line for the Coniglio-Klein droplets (Fig. 1).

The universal features of these droplets are described by the scaling limit of the Hamiltonian (13), namely, in $d = 2$, by the field theory

$$A_{\text{droplets}} = A_{\text{CFT}}^{(q+1)} - \tau_q \int \! d^2 x \varphi_{2,1}(x) + 2h_q \int \! d^2 x \delta_{\nu(x),0}, \quad \nu(x) = 0, 1, \ldots, q, \quad (20)$$

where $A_{\text{CFT}}^{(q+1)}$ accounts for the critical line of a $(q+1)$-state Potts model, $\tau_q$ measures the deviation from the critical Potts temperature, and $h_q$ is a magnetic field pointing in the $\nu = 0$ direction. Since in two dimensions the Potts transition is second order as long as the number of states does not exceed 4, the above action is intended for $q \leq 3$. For $q = 1$, (20) with $\tau_1 = \tau$ and $h_1 = h$ gives back the Ising field theory (14).

The renormalization group trajectories flowing out of the fixed point at $\tau_q = h_q = 0$ for $h_q \geq 0$ are labelled by the dimensionless parameter

$$\eta_q = \tau_q / h_q^{(2-X_{2,1})/(2-X_s)}, \quad (21)$$

where the thermal and magnetic scaling dimensions $X_{2,1}$ and $X_s$ are those given in section 3 for the Potts critical line, up to the substitution $q \to q + 1$; $\eta_1 \equiv \eta = \tau / h^{8/15}$ labels the Ising
Figure 8: Scaling limit of Fig. 1 within the parameter space of the Ising field theory (14). The percolative order parameter is non-zero in the whole upper half-plane for Ising clusters, and to the left of the Kertész trajectory $\eta_K$ for Coniglio-Klein Ising droplets.

trajectories. For $h_q = +\infty$ the state $\nu = 0$ is forbidden and one is left with a $q$-state Potts model. This means that for $1 < q \leq 3$ there is a critical trajectory $\eta^c_q$ which flows from the $(q + 1)$-to the $q$-state Potts fixed point. The $S_q$-invariance of (20) is spontaneously broken for $\eta^c_q < \eta^c_q$. Since the Potts Curie temperature increases as $q$ decreases, $\eta^c_q$ is expected to be positive.

The massless trajectories $\eta^c_q$ span, as a function of $q$, a critical surface which plays for the droplets exactly the same role the second order surface considered in the previous section played for the Ising clusters. In the present case, however, while it can be seen that the number of massless particles indeed vanishes as $q \to 1$ [33], lack of integrability does not allow to follow analytically the origin of the mass gap at $q = 1$. The most likely mechanism remains the one seen in the previous section, namely at least one neutral massive resonance which becomes stable at $q = 1$. The possibility that this neutral particle is stable already for $q > 1$ cannot be excluded, but requires a mechanism which prevents the decay into the massless excitations.

Either way, it follows from the same line of arguments developed for the clusters that the value $\eta^c_1 \equiv \eta_K$ of the Ising parameter $\eta$ determines the universal scaling limit of the Kertész line (Fig. 8). An infinite Coniglio-Klein droplet exists in the sector of the $\tau$-$h$ plane with $h > 0$, $\eta < \eta_K$. Percolative critical exponents measured along the trajectory $\eta_K$ (as along the trajectory $\eta = +\infty$ for clusters) are those of the percolative infrared fixed point, i.e. the random percolation fixed point with central charge $c = 0$.

Numerical data for the Kertész line of the square lattice Ising model in the vicinity of the magnetic critical point are given in [38]. Using them, together with the known relations between lattice and continuum parameters (see e.g. [39]), we find $\eta_K \simeq 0.12$. We recall that, as a result of a series of theoretical and numerical studies (see [27] for references), very much is known

\footnote{The field theory [20] has been considered in [33] with the purpose of describing the qualitative evolution of the particle spectrum in parameter space. The actual value of $\eta^c_q$ was not essential there and was naively identified with zero.}

\footnote{This and the subsequent values of $\eta$ refer to the normalization

$$\lim_{|x| \to 0} |x|^{1/4} \langle \sigma(x)\sigma(0) \rangle = \lim_{|x| \to 0} |x|^2 \langle \epsilon(x)\epsilon(0) \rangle = 1$$

(22)

of the fields in (14).}
about the $\eta$-dependence of the particle spectrum of the field theory $^{14}$. In particular, the theory possesses a single stable particle for $\eta > \eta(2)$, and two stable particles for $\eta(2) > \eta > \eta(3)$, with $\eta(2) \simeq 0.33$ and $\eta(3) \simeq 0.022$. It follows that $\eta_K$ falls inside the second region, so that, if the resonance scenario for the production of the mass gap as $q \rightarrow 1$ along the critical surface applies also to this case, two particles need to become simultaneously stable at $q = 1$.

6 Conclusion

In this paper we showed explicitly the non-perturbative analytic mechanism through which clusters of up spins in the two-dimensional Ising model undergo a second order percolation transition at values of temperature $T$ and magnetic field $H$ for which there are no thermodynamic singularities. The Kasteleyn-Fortuin formulation of percolation leads to consider a dilute $q$-state Potts model in which the Ising degrees of freedom are associated to the dilution and the Potts spins are auxiliary site variables. In the three-dimensional parameter space of $T$, $H$ and $q$ there is for $q > 1$ a surface of second order phase transition which, in the scaling limit, corresponds to an integrable field theory. This allows to show that the $q - 1$ massless particles on the surface produce in their scattering a massive resonance whose lifetime is proportional to $1/(q-1)$. When the limit $q \rightarrow 1$, needed to get rid of the auxiliary Potts variables, is taken, the percolative correlations, which are determined at first order in $q-1$, are mediated by the massless particles, giving rise to the second order transition line. The thermodynamic observables, instead, are determined at $q$ strictly equal to 1, where no massless particle is left and the correlations are mediated by the massive particle which becomes stable. The fact that the resonant particle is a singlet under the Potts permutational symmetry $S_q$ leads to a smooth transition from the massless to the massive regime for the thermodynamic observables as $q \rightarrow 1$.

We have analyzed in the same framework also the Coniglio-Klein droplets, which are obtained from the Ising point clusters by a partial, temperature-dependent depletion and produce at the Curie point percolative exponents which coincide with the magnetic ones. The corresponding field theory in the Kasteleyn-Fortuin representation describes the scaling limit of a $(q+1)$-state Potts model with a magnetic field which reduces the symmetry to $S_q$. Again, for $q > 1$ there is a massless surface bounded at $q = 1$ by a massive trajectory which corresponds to the universal scaling limit of the second order percolation line for the droplets (the Kertész line). In this case the massless surface is not integrable, but a mechanism analogous to that observed for the clusters is likely to account for mass generation at $q = 1$.

On the lattice, the second order percolative line for the clusters (droplets) goes from the Curie point to a finite value of the magnetic field (temperature) at infinite temperature (magnetic field). The asymptotic value is determined by the critical probability of random site (bond) percolation on the given lattice, and is non-universal. Our field theoretical description of the scaling limit retains only the universal features. The second order critical lines correspond to renormalization group trajectories flowing from the correlated percolation fixed point, with central charge $c = 1/2$ and different dimensions of the percolative order parameter for clusters
and droplets, to the random percolation fixed point with \( c = 0 \). These universal trajectories are identified by two values of the Ising field theory parameter \( \eta = \tau/h^{8/15} \): \( \eta = +\infty \) for the clusters and \( \eta \simeq 0.12 \) for the droplets. The latter value is obtained from existing numerical data and implies that the Kertész trajectory falls in the sector of the Ising field theory with two stable massive particles.

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