Voros coefficients of the third Painelvé equation and parametric Stokes phenomena

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Abstract

We compute all “Voros coefficients” of the third Painlevé equation \((P_{III})_{D_6}\) of the type \(D_6\) (in the sense of [OKSO]) and discuss the “parametric Stokes phenomena” occurring to formal transseries solutions of \((P_{III})_{D_6}\). We derive connection formulas for parametric Stokes phenomena under an assumption for Borel summability of transseries solutions. Furthermore, we also compute the Voros coefficient of the degenerate third Painlevé equation of the type \(D_7\) in Appendix [D].

Key Words : The third Painlevé equation, exact WKB analysis, Voros coefficients, parametric Stokes phenomena.

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1 Introduction

In this article we study the third Painlevé equation with a large parameter \(\eta > 0\)

\[(P_{III})_{D_6} : \frac{d^2 \lambda}{dt^2} = \frac{1}{\lambda}\left(\frac{d\lambda}{dt}\right)^2 - \frac{1}{t} \frac{d\lambda}{dt} + \eta^2 \left(\frac{\lambda^3}{t^2} - \frac{c_\infty \lambda^2}{t^2} + \frac{c_0 \lambda}{t} - \frac{1}{\lambda}\right)\]

from the view point of the exact WKB analysis (cf. [KT3, V]); that is, an asymptotic analysis for large \(\eta\) based on the Borel resummation method. (See [KT3] and series of papers [KT1, AKT1, KT2] for WKB analysis of Painlevé equations) Here \(c_{\infty}\) and \(c_0\) are non-zero complex parameters, and the equation is the type \(D_6\) in the sense of [OKSO]. We are interested in (one-parameter family of) formal transseries solutions of \((P_{III})_{D_6}\):

\[\lambda(t, \eta; \alpha) = \lambda^{(0)}(t, \eta) + \alpha \eta^{-1/2} \lambda^{(1)}(t, \eta) e^{\eta \phi(t)} + (\alpha \eta^{-1/2})^2 \lambda^{(2)}(t, \eta) e^{2\eta \phi(t)} + \cdots,\]

where \(\alpha\) is a free parameter, \(\lambda^{(k)}(t, \eta) = \lambda^{(k)}_0(t) + \eta^{-1} \lambda^{(k)}_1(t) + \eta^{-2} \lambda^{(k)}_2(t) + \cdots (k \geq 0)\) is a formal power series in \(\eta^{-1}\) and \(\phi(t)\) is a certain function.

What we discuss in this article is so-called “parametric Stokes phenomena” occurring to the transseries solutions of \((P_{III})_{D_6}\). These are kinds of Stokes phenomena concerning with continuous variations of the complex parameters \(c_{\infty}\) and \(c_0\).
In [11, 12] we discussed the parametric Stokes phenomena occurring to transseries solutions of the second Painlevé equation with the large parameter $\eta > 0$

\[(P_{II}) : \frac{d^2 \lambda}{dt^2} = \eta^2 (2\lambda^3 + t\lambda + c),\]

where $c$ is a complex parameter. As is shown in [11, 12], when $\arg c = \pi/2$, transseries solutions may not be Borel summable (as a formal series in $\eta^{-1}$), and the Borel sums defined when $\arg c = \pi/2 - \varepsilon$ and $\arg c = \pi/2 + \varepsilon$ give different analytic solutions (after analytic continuation with respect to the parameter $c$ across the imaginary axis $\arg c = \pi/2$) of $(P_{II})$. This kind of phenomenon was firstly observed in [SS] through the analysis of a linear ordinary differential equation, and the analysis in terms of the exact WKB analysis was given by [T2]. (See also [KoT, AT].)

As discussed in [T2, 11], the parametric Stokes phenomena are closely related to the “degeneration of the Stokes geometry” (i.e., two turning points are connected by a Stokes curve), and the “Voros coefficients” (cf. [DDP, T2]) play a central role when we discuss the connection problem for parametric Stokes phenomena. That is, formal solutions may not be Borel summable when the Stokes geometry degenerates, and an explicit connection formula describing the parametric Stokes phenomena can be read off the “jump property” of the Voros coefficients.

![Figure 1.1](image1.png)\[(c_{\infty}, c_0) = (3, 3 - i).\]

![Figure 1.2](image2.png)\[(c_{\infty}, c_0) = (3i, 1 - 2i).\]

The degeneration of the Stokes geometry is also observed for $(P_{III})_{D_6}$. For example, Figure 1.1 and 1.2 describe some Stokes geometry of $(P_{III})_{D_6}$ for some values of parameters $(c_{\infty}, c_0)$, and we can observe that the Stokes geometry degenerate. We observe the following two kinds of degenerations in the Stokes geometry of $(P_{III})_{D_6}$. The first one is “triangle-type degeneration”; that is, there are three pairs of turning points connected by a Stokes curve simultaneously. Figure 1.1 shows an example of the triangle-type degeneration. The second one is “loop-type degeneration”; that is, a Stokes curve form a loop around the singular point $t = 0$. An example the loop-type degeneration is shown in Figure 1.2. Intriguingly enough, as far as we checked, only these two kinds of degeneration can be observed for the Stokes geometry of $(P_{III})_{D_6}$. (See Appendix A.)

As similar to [SS, T2, 11], we can expect that parametric Stokes phenomena also occur for transseries solutions of $(P_{III})_{D_6}$ relevant to these degenerations of
Stokes geometry. The purpose of this article is to compute the all Voros coefficients of \((P_{III})_{D_6}\), and to analyze the parametric Stokes phenomena occurring to the transseries solutions. Voros coefficients of \((P_{III})_{D_6}\) (cf. [I1]) are formal power series defined by the integral

\[
W(c_\infty, c_0, \eta) = \int_\tau^* \left( R_{\text{odd}}(t, \eta) - \eta R_{-1}(t) \right) dt,
\]

where \(R_{\text{odd}}(t, \eta) = \sum_{n=0}^{\infty} \eta^{1-2n} R_{2n-1}(t)\) given by (2.11) is the odd part of a formal solution of the Riccati equation (2.9) associated with the Fréchet derivative (2.7) of \((P_{III})_{D_6}\). \(\tau\) is a turning point and \(*\) (\(= 0\) or \(\infty\)) is a singular point of \((P_{III})_{D_6}\). There are several Voros coefficients depending on the choice of the singular point \(*\), and of the path of integration. One of Voros coefficients is represented as follows (Theorem 4.1):

**Theorem 1.1.** The Voros coefficient for an appropriate path from a turning point to the singular point \(* = \infty\) is given by

\[
W(c_\infty, c_0, \eta) = \int_\tau^\infty \left( R_{\text{odd}}(t, \eta) - \eta R_{-1}(t) \right) dt = \sum_{n=1}^{\infty} \frac{2^{1-2n} - 1}{2n(2n-1)} B_{2n} \left( \frac{c_\infty - c_0}{\eta} \right)^{1-2n}. \tag{1.1}
\]

Here \(B_{2n}\) is the \(2n\)-th Bernoulli number defined by

\[
\frac{w}{e^w - 1} = 1 - \frac{1}{2} w + \sum_{n=1}^{\infty} \frac{B_{2n}}{(2n)!} w^{2n}. \tag{1.2}
\]

All other Voros coefficients can also be expressed in terms of the Bernoulli numbers. They are summarized in Theorem 3.1 and Theorem 3.2. Moreover, from above explicit representation we can compute its Borel sum and check the following jump property of the Voros coefficients (Proposition 5.1 and Corollary 5.1).

**Proposition 1.1.** Let \(W(c_\infty, c_0, \eta)\) be the formal power series (1.1).

(i) \(W(c_\infty, c_0, \eta)\) is not Borel summable (as a formal power series in \(\eta^{-1}\)) when \(c_\infty - c_0 \in i\mathbb{R}\), and Borel summable otherwise.

(ii) Let \(S_{\pm}[e^W(c_\infty, c_0, \eta)]\) be the Borel sum of the formal power series \(e^W(c_\infty, c_0, \eta)\) when \(\arg(c_\infty - c_0) = \pi/2 \pm \varepsilon\) for a sufficiently small \(\varepsilon > 0\). After the analytic continuation across the axis \(\{\arg(c_\infty - c_0) = \pi/2\}\), the following relation holds:

\[
S_+[e^W(c_\infty, c_0, \eta)] = (1 + e^{\pi i(c_\infty - c_0)\eta}) S_-[e^W(c_\infty, c_0, \eta)]. \tag{1.3}
\]

From this jump property and a certain assumption for the Borel summability of the transseries solution (Conjecture 5.1), when the independent variable \(t\) is fixed at a point, we derive an explicit connection formula describing the parametric Stokes phenomenon relevant to the triangle-type degeneration in Figure 2.4 (Section 5).

**Connection formula for the parametric Stokes phenomenon.** Assume that Conjecture 5.1 holds. Let \(S_{\Pi}^{\pm}[\lambda(t, \eta; \alpha)]\) (resp., \(S_{\Pi}^{\pm}[\lambda(t, \eta; \tilde{\alpha})]\)) be the Borel sum of a transseries solution \(\lambda(t, \eta; \alpha)\) when \((c_\infty, c_0) = (2 - \varepsilon, 2 - i)\) (resp., \(\lambda(t, \eta; \tilde{\alpha})\) when \((c_\infty, c_0) = (2 - \varepsilon, 2 - i)\)).
when \((c_\infty, c_0) = (2 + \varepsilon, 2 - i)\). If they represent the same analytic solution of \((P_{III})_{D_6}\) after the analytic continuation with respect to the parameter \((c_\infty, c_0)\) across \(\{\text{arg}(c_\infty - c_0) = \pi/2\}\), then the parameters related as
\[
\tilde{\alpha} = (1 + e^{\pi i (c_\infty - c_0) \eta}) \alpha.
\] (1.4)

That is, the following relation holds:
\[
S_{II} \left[ \lambda_{\tau_1}(t, \eta; \alpha) \right] = S_{I} \left[ \lambda_{\tau_1}(t, \eta; \tilde{\alpha}) \right] \bigg|_{\tilde{\alpha} = (1 + e^{\pi i (c_\infty - c_0) \eta}) \alpha}.
\] (1.5)

This formula describes the parametric Stokes phenomenon; that is, (1.5) gives the explicit relationship between the Borel sums of the transseries solutions of \((P_{III})_{D_6}\) in different region in the parameter space of \((c_\infty, c_0)\). We also discuss another type connection formula in Section 5. These are our main results. However, we have succeeded to derive the explicit connection formulas only in certain cases; a crucial case (when the loop-type degeneration happens and \(t\) lies inside the loop) remains to be analyzed. (See Section 5.3.)

This article is organized as follows. In Section 2 we recall the fundamental notions in the WKB theory of Painlevé equations. In Section 3 we recall the definition of the Voros coefficients for non-linear differential equations, and state the main results about the explicit representations of the Voros coefficients (Theorem 3.1 and Theorem 3.2). Section 4 is devoted to the proof of these main theorems. Based on these results, we discuss the connection problems for parametric Stokes phenomena in Section 5. Furthermore, we also compute the all Voros coefficients of the degenerate third Painlevé equation
\[
(P_{III})_{D_7} : \frac{d^2 \lambda}{dt^2} = \frac{1}{\lambda} \left( \frac{d\lambda}{dt} \right)^2 - \frac{1}{t} \frac{d\lambda}{dt} + \eta^2 \left( -\frac{2\lambda^2}{t^2} + \frac{c}{t} - \frac{1}{\lambda} \right)
\] of the type \(D_7\) in Appendix D (Theorem D.1). Note that a part of this result is announced in [12].

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2 Transseries solutions and Stokes geometry of \((P_{III})_{D_6}\)

In this section we briefly review a construction of transseries solutions of the third Painlevé equation of the type \(D_6\) with a large parameter \(\eta > 0\) of the following form:
\[
(P_{III})_{D_6} : \frac{d^2 \lambda}{dt^2} = \frac{1}{\lambda} \left( \frac{d\lambda}{dt} \right)^2 - \frac{1}{t} \frac{d\lambda}{dt} + \eta^2 \left( -\frac{\lambda^3}{t^2} + \frac{c_\infty \lambda^2}{t^2} + \frac{c_0}{t} - \frac{1}{\lambda} \right).
\] (2.1)
Here \( c_\infty \) and \( c_0 \) are complex parameters. In this paper we write \( \mathbf{c} = (c_\infty, c_0) \) and always assume

\[
\mathbf{c} \in S = \{(c_\infty, c_0) \in \mathbb{C}^2; c_\infty, c_0, c_\infty^2 - c_0^2, c_\infty^2 + c_0^2 \neq 0\} \tag{2.2}
\]

for genericity. We will explain the meaning of the assumptions \( (2.2) \) in Remark 2.2.

Moreover, we also recall the definitions of turning points and Stokes curves \([\text{KT1}]\) in this section.

Note that the equation \((P_{\text{III}})_D^6\) is obtained from the “original” one \(\text{cf.}[\text{Ok}, \text{OKSO}]\)

\[
\frac{d^2 y}{dz^2} = \frac{1}{y} \left( \frac{dy}{dz} \right)^2 - \frac{1}{z} \frac{dy}{dz} + \frac{\gamma y^3}{4z^2} + \frac{\alpha y^2}{4z^2} + \frac{\beta}{4z} + \frac{\delta}{4y}
\]

through the following scalings: \( y = \eta \lambda, z = \eta^2 t, \alpha = -4\eta c_\infty, \beta = 4\eta c_0, \gamma = 4 \) and \( \delta = -4 \). Therefore, quantities appearing in this paper have a homogeneity. They are summarized in Appendix B.

### 2.1 Transseries solutions

Our main interest consists in the analysis of transseries solutions; that is, formal solutions of \((P_{\text{III}})_D^6\) of the form

\[
\lambda(t, \mathbf{c}, \eta; \alpha) = \sum_{k \geq 0} (\alpha \eta^{-1/2})^k \lambda^{(k)}(t, \mathbf{c}, \eta) e^{k\eta \phi}, \tag{2.3}
\]

where \( \alpha \) is a free parameter, \( \lambda^{(k)}(t, \mathbf{c}, \eta) (k \geq 0) \) is a formal power series in \( \eta^{-1} \) of the form

\[
\lambda^{(k)}(t, \mathbf{c}, \eta) = \sum_{\ell \geq 0} \eta^{-\ell} \lambda^{(k)}_\ell(t, \mathbf{c}),
\]

and \( \phi = \phi(t, \mathbf{c}) \) is some function defined as follows. Let \( F(\lambda, t, \mathbf{c}) \) be the coefficient of \( \eta^2 \) in the right-hand side of \((2.1)\):

\[
F(\lambda, t, \mathbf{c}) = \frac{\lambda^3}{t^2} - \frac{c_\infty \lambda^2}{t^2} + \frac{c_0}{t} - \frac{1}{\lambda}, \tag{2.4}
\]

and \( \lambda_0 = \lambda_0(t, \mathbf{c}) \) be an algebraic function defined by

\[
F(\lambda_0, t, \mathbf{c}) = \frac{\lambda_0^3}{t^2} - \frac{c_\infty \lambda_0^2}{t^2} + \frac{c_0}{t} - \frac{1}{\lambda_0} = 0, \tag{2.5}
\]

which is nothing but the leading term \( \lambda_0^{(0)}(t, \mathbf{c}) \) of the formal power series \( \lambda^{(0)}(t, \mathbf{c}, \eta) \) in \((2.3)\). Then the phase function \( \phi(t, \mathbf{c}) \) is defined by

\[
\phi(t, \mathbf{c}) = \int_t^\infty \sqrt{\Delta(t, \mathbf{c})} \, dt, \quad \Delta(t, \mathbf{c}) = \frac{\partial F}{\partial \lambda}(\lambda_0(t, \mathbf{c}), t, \mathbf{c}). \tag{2.6}
\]

Transseries solutions are considered in a domain where the real part of \( \phi \) is negative; i.e., \( \exp(\eta \phi) \) is exponentially small when \( \eta \to +\infty \). See \([\text{C}, \text{II}]\) for a construction of
such formal solutions. The (generalized) Borel resummation method for transseries are discussed in [C].

The formal expansion of the form (2.3) is called “instanton-type expansion” (or “non-perturbative expansion”) in physical literatures. We follow this terminology and call the \(k\)-th formal series \((\alpha_\eta^{-1/2})^k \lambda^{(k)}(t, c, \eta)e^{k\eta(t)}\) “\(k\)-instanton part” of (2.3) (for \(k \geq 0\)). We note that the 0-instanton part \(\lambda^{(0)}(t, c, \eta)\) is itself a formal power series solution of (\(P_{III}'\))\(_{D_6}\) called a 0-parameter solution ([KT1]). The coefficients of 0-parameter solution are determined recursively once we fix the branch of the algebraic function \(\lambda_0\). For example, the coefficients of first few terms are given as follows:

\[
\lambda^{(0)}_1(t, c) = 0, \\
\lambda^{(0)}_2(t, c) = \frac{1}{\Delta} \left( \frac{d^2 \lambda_0}{dt^2} - \frac{1}{\lambda_0} \left( \frac{d\lambda_0}{dt} \right)^2 + \frac{1}{t} \frac{d\lambda_0}{dt} \right), \\
\lambda^{(0)}_3(t, c) = 0, \\
\lambda^{(0)}_4(t, c) = \frac{1}{\Delta} \left( \frac{d^2 \lambda_2^{(0)}}{dt^2} - \frac{2}{\lambda_0} \frac{d\lambda_0}{dt} \frac{d\lambda_2^{(0)}}{dt} + \frac{\lambda_2^{(0)}}{\lambda_0^2} \left( \frac{d\lambda_0}{dt} \right)^2 + \frac{1}{t} \frac{d\lambda_2^{(0)}}{dt} \\
- \frac{3\lambda_0 \lambda_2^{(0)} t^2}{t^2} + c_\infty \lambda_2^{(0)2} t^2 + \lambda_2^{(0)2} \right),
\]

\[\vdots\]

We can verify that \(\lambda^{(0)}_\ell(t, c) = 0\) for each odd number \(\ell\). The Borel summability of 0-parameter solution is discussed in [KamKO].

Next we discuss normalizations of transseries solutions. We can easily confirm that, if we denote the 1-instanton part of (2.3) by

\[
\tilde{\lambda}^{(1)} = \alpha \eta^{-1/2} \lambda^{(1)}(t, c, \eta)e^{\eta \phi},
\]

then \(\tilde{\lambda}^{(1)}\) satisfies the following second order linear differential equation

\[
\frac{d^2 \tilde{\lambda}^{(1)}}{dt^2} = \left( \frac{2}{\lambda^{(0)}} \frac{d\lambda^{(0)}}{dt} - \frac{1}{t} \right) \frac{d\tilde{\lambda}^{(1)}}{dt} + \eta^2 \left\{ \frac{\partial F}{\partial \lambda} \left( \lambda^{(0)}, t, c \right) - \eta^{-2} \left( \frac{1}{\lambda^{(0)}} \frac{d\lambda^{(0)}}{dt} \right)^2 \right\} \tilde{\lambda}^{(1)}, \quad (2.7)
\]

which is a Fréchet derivative of (\(P_{III}'\))\(_{D_6}\) at \(\lambda = \lambda^{(0)}\). Thus we can take a WKB solution [KT3, §2] of (2.7) for the 1-instanton part \(\tilde{\lambda}^{(1)}\); that is,

\[
\tilde{\lambda}^{(1)} = \alpha \exp \left( \int^t R(t, c, \eta) dt \right), \quad (2.8)
\]

where \(\alpha\) is a free parameter, and

\[
R(t, c, \eta) = \sum_{\ell \geq -1} \eta^{-\ell} R_\ell(t, c) = \eta R_{-1}(t, c) + R_0(t, c) + \eta^{-1} R_1(t, c) + \cdots
\]

is a formal solution of the following Riccati equation associated with (2.7):

\[
R^2 + \frac{dR}{dt} = \left( \frac{2}{\lambda^{(0)}} \frac{d\lambda^{(0)}}{dt} - \frac{1}{t} \right) R + \eta^2 \left\{ \frac{\partial F}{\partial \lambda} \left( \lambda^{(0)}, t, c \right) - \eta^{-2} \left( \frac{1}{\lambda^{(0)}} \frac{d\lambda^{(0)}}{dt} \right)^2 \right\}. \quad (2.9)
\]
Remark 2.1. We can easily confirm that, once we fix the square root
\[
R_{-1}(t, c) = \sqrt{\Delta(t, c)},
\]
then coefficients \(R_{\ell}(t, c)\) \((\ell \geq 0)\) are determined recursively. For example, the coefficients of first few terms are given by
\[
R_0(t, c) &= -\frac{1}{2R_{-1}} \frac{dR_{-1}}{dt} + \frac{1}{\lambda_0} \frac{d\lambda_0}{dt} - \frac{1}{2t}, \\
R_1(t, c) &= \frac{1}{2R_{-1}} \left( -R_0^2 - \frac{dR_0}{dt} + \left( \frac{2}{\lambda_0} \frac{d\lambda_0}{dt} - \frac{1}{t} \right) R_0 \\
&\quad + \left( \frac{6\lambda_0}{t^2} - \frac{2c_\infty}{t^2} - \frac{2}{\lambda_0^2} \right) \lambda_2^0 - \left( \frac{1}{\lambda_0^2} \frac{d\lambda_0}{dt} \right)^2 \right).
\]
\[\vdots\]
Thus we have two formal solutions \(R_+\) and \(R_-\) of (2.9) with \(R_{\pm}(t, c, \eta) = \pm \eta \sqrt{\Delta(t, c)} + \cdots\). We define \(R_{\text{odd}}\) and \(R_{\text{even}}\) by
\[
R_{\text{odd}}(t, c, \eta) = \frac{1}{2} \left( R_+(t, c, \eta) - R_-(t, c, \eta) \right), \\
R_{\text{even}}(t, c, \eta) = \frac{1}{2} \left( R_+(t, c, \eta) + R_-(t, c, \eta) \right).
\]
We can easily confirm that all the coefficients of \(\eta^{2n}\) in \(R_{\text{odd}}(t, c, \eta)\) vanish for \(n \geq 0\), and \(R_{\text{odd}}(t, c, \eta)\) has the form
\[
R_{\text{odd}}(t, c, \eta) = \sum_{n \geq 0} \eta^{1-2n} R_{2n-1}(t, c) = \eta R_{-1}(t, c) + \eta^{-1} R_1(t, c) + \cdots. 
\]
Moreover, since
\[
R_{\text{even}} = -\frac{1}{2} \frac{1}{R_{\text{odd}}} \frac{dR_{\text{odd}}}{dt} + \left( \frac{1}{\lambda_0} \frac{d\lambda_0}{dt} - \frac{1}{2t} \right), 
\]
holds by (2.9), the WKB solution \(\tilde{\lambda}^{(1)}\) of (2.7) can be written as
\[
\tilde{\lambda}^{(1)} = \alpha \frac{\lambda_0^{(0)}(t, c, \eta)}{\sqrt{t R_{\text{odd}}(t, c, \eta)}} \exp \left( \int^t R_{\text{odd}}(t, c, \eta) dt \right) \\
= \alpha \eta^{-1/2} \left( \lambda_0^{(1)}(t, c) + \eta^{-1} \lambda_1^{(1)}(t, c) + \eta^{-2} \lambda_2^{(1)}(t, c) + \cdots \right) e^{\eta \phi}.
\]
We mainly use the expression (2.15) rather than (2.16) for the convenience of descriptions of normalizations.

We note that, once the normalization of the 1-instanton part \(\tilde{\lambda}^{(1)}\) (i.e. the end point and the path of the integral of \(R_{\text{odd}}\) in (2.15)) is fixed, then the formal power series \(\lambda^{(k)}(t, c, \eta)\) \((k \geq 2)\) are determined uniquely in a recursive manner. Then the all coefficients \(\lambda^{(k)}_\ell(t, c)\) appears in (2.3) are holomorphic in \(t\) on the universal covering of
\[
\Omega_{D_6} = \mathbb{C}^{\times}/\{\text{turning points}\}.
\]
Here turning points of \((P_{\text{HP}})_{D_6}\) are defined in Definition 2.1 below. In Section 3 we will introduce some special normalizations of (2.15), and analyze parametric Stokes phenomena for those transseries solutions in Section 5.
2.2 Stokes geometry

Now we recall the definitions of turning points and Stokes curves of \((P_{III})_{D_6}\).

**Definition 2.1.** [KT3, Definition 4.5] (i) A point \(t = \tau\) is called a turning point if \(\tau \neq 0\) and it is a zero of the function \(\Delta(t, c)\).

(ii) For a turning point \(t = \tau\), real one-dimensional curves defined by

\[
\text{Im} \int_{\tau}^{t} \sqrt{\Delta(t, c)} \, dt = 0 \tag{2.18}
\]

are called Stokes curves (emanating from \(\tau\)).

In Definition 2.1 the branch of \(\lambda_0\) is fixed. But later we consider all branches of \(\lambda_0\) and lift turning points and Stokes curves onto the Riemann surface of \(\lambda_0\), which is a four-fold covering of \(\mathbb{C}^\times\) branching at turning points. See Remark 2.3 for the lift.

**Remark 2.2.** Turning points are, by definition, zeros of the discriminant of the following algebraic equation for \(\lambda\):

\[
(t^2 \lambda F(\lambda, t, c) =) \lambda^4 - c_\infty \lambda^3 + c_0 t \lambda - t^2 = 0. \tag{2.19}
\]

The discriminant \(\text{Disc}(t, c)\) of the above algebraic equation is given by

\[
\text{Disc}(t, c) = t^3(-256t^3 + 192t^2c_\infty c_0 + 6tc_\infty^2 c_0^2 - 27tc_\infty^4 - 27t^4 c_0^4 + 4c_\infty^3 c_0^3). \tag{2.20}
\]

Moreover, the discriminant of the cubic equation \(\text{Disc}(t, c)/t^3 = 0\) for \(t\) is factorized as

\[
-20155392(c_\infty^2 - c_0^2)^4(c_\infty^2 + c_0^2)^2. \tag{2.21}
\]

Therefore, under the assumptions (2.2), there are three turning points on \(\mathbb{C}^\times\), and all of them are simple (in the sense of [KT3, Definition 4.5]). We always assume (2.2) in this paper.

Unfortunately, Definition 2.1 is not enough to describe the “complete” Stokes geometry because the singular point \(t = 0\) of \((P_{III})_{D_6}\) may play the same role of turning point ([TW, T3, KamKo]). We know that there are following four asymptotic behaviors of \(\lambda_0\) as \(t \to 0\):

\[
\lambda_0(t, c) = c_\infty + O(t), \tag{2.22}
\]

\[
\lambda_0(t, c) = t/c_0 + O(t^2), \tag{2.23}
\]

\[
\lambda_0(t, c) = \pm \sqrt{c_0/c_\infty} t^{1/2}(1 + O(t^{1/2})). \tag{2.24}
\]

Then, as shown in [TW, KamKo], for the branch (2.24) of \(\lambda_0\), the singular point \(t = 0\) plays the same role as turning points; that is, there exist a Stokes curve emanating from \(t = 0\) on which transseries solutions may not be Borel summable. (A similar phenomenon occurs for WKB solutions of Schrödinger equations whose potential function has a simple-pole [Ko].) Thus we define the following:
Definition 2.2. [TW, KamKo] (i) The singular point $t = 0$ for the branch (2.24) of $\lambda_0$ is called the simple-pole of $(P_{III})_{D_6}$. (See (2.29) below for the reason why we call (2.24) simple-pole.) The simple-pole is denoted by $\tau_{sp}$.

(ii) For the above branch of $\lambda_0$, the real one-dimensional curve defined by

$$\text{Im} \int_{\tau_{sp}}^{t} \sqrt{\Delta(t,c)} \, dt = 0$$

is also called a Stokes curve (emanating from the simple-pole $t = \tau_{sp}$).

Figures 2.1∼2.5 describe the Stokes curves of $(P_{III})_{D_6}$ on the $t$-plane for several values of $c$. Note that the quadratic differential $\Delta(t,c) \, dt^2$ which defines the Stokes curves behaves as

$$\Delta(t,c) \, dt^2 = C_\tau (t - \tau)^{1/2} (1 + O((t - \tau)^{1/2})) \, dt^2 \text{ as } t \to \tau \neq \tau_{sp}, \quad (2.26)$$

$$\Delta(t,c) \, dt^2 = C_{sp} t^{-3/2} (1 + O(t^{1/2})) \, dt^2 \text{ as } t \to \tau_{sp}, \quad (2.27)$$

with some constants $C_\tau$ and $C_{sp}$ which are non-zero under the assumption (2.2).

Hence, five Stokes curves emanate from each turning points, and one Stokes curve
emanates from the simple-pole. Connection problems on Stokes curves of Painlevé equations in terms of the exact WKB analysis are discussed in [KT3, TI, TI, TW].

In Figure 1.1, 1.2, 2.2 and 2.4 we can observe that there exists a “bounded Stokes curve”, which connect two different turning points, or form a loop around the singular point $t = 0$. We call such situation “degeneration of the Stokes geometry”, and it is closely related to parametric Stokes phenomena for transseries solutions, as we pointed in Section I. Connection problems for the parametric Stokes phenomena are discussed in the subsequent sections.

Remark 2.3. Since $\lambda_0(t, c)$ is multivalued function of $t$, we should consider the lift of Stokes geometry onto the Riemann surface of $\lambda_0$ (i.e., the Riemann surface defined by (2.5)). Since $\lambda_0$ satisfies the algebraic equation (2.19) of degree 4, we need four sheets (Sheet 1 ∼ Sheet 4) to consider the lift. For example, Figure 2.6
Figure 2.9 describe the lift of Stokes geometry when \( c = (2, 2 - i) \). The wiggly line of the type \((j, k)\) \((1 \leq j, k \leq 4)\), solid lines and dotted lines in these figures represent the branch cut for between the Sheet \( j \) and Sheet \( k \), Stokes curves on the sheet under consideration and Stokes curves on the other sheets, respectively. The origin of the type (2.22) and (2.23) correspond to the origin of the Sheet 1 and Sheet 2 respectively. Thus the Riemann surface of \( \lambda_0 \) has genus 0.

![Figure 2.10: \( c = (2 + i, 3) \). Figure 2.11: \( c = (2, 2 - i) \). Figure 2.12: \( c = (5 + i, 2i) \).](image)

It is convenient to change the variable from \( t \) to \( u \) which is given by

\[
u = \frac{1 - \mu_0}{\mu_0}, \tag{2.28}\]

where \( \mu_0 = \mu_0(t, c) \) is given by (4.15) in Section 4. Both \( t \) and \( \lambda_0(t, c) \) can be written in terms of \( u \) as

\[
\lambda_0 = \frac{u + 1}{4u}((c_\infty + c_0)u - (c_\infty - c_0)),
\]

\[
t = \frac{(u + 1)^2}{16u^2}((c_\infty + c_0)^2u^2 - (c_\infty - c_0)^2).
\]

Stokes geometry can be described without any intersections on the \( u \)-plane. In fact, the quadratic differential \( \Delta(t, c)dt^2 \) which determine the Stokes geometry is written as

\[
\Delta(t, c)dt^2 = q(u, c)du^2,
\]

\[
q(u, c) = \frac{((c_\infty + c_0)^2u^3 + (c_\infty - c_0)^2)^3}{(u + 1)u^4((c_\infty + c_0)^2u^2 - (c_\infty - c_0)^2)^2}. \tag{2.29}
\]

The turning points correspond to zeros

\[
\frac{(c_\infty - c_0)^{2/3}}{(c_\infty + c_0)^{2/3}}(-1)^{1/3}\omega^j \quad (\omega = e^{2\pi i/3}, j = 0, 1, 2)
\]

of (2.29), and the simple-pole \( t = \tau_{sp} \) of (2.24) corresponds to \( u = -1 \); i.e., the simple-pole of (2.29). This is the reason why we call (2.24) simple-pole. Similarly, we will call the singular points \( t = 0 \) of (2.22) and (2.23) are double-poles since they
correspond to double-poles \( u = (c_\infty - c_0)/(c_\infty + c_0) \) and \( u = -(c_\infty - c_0)/(c_\infty + c_0) \) of (2.29) respectively. Moreover, (2.28) gives a coordinate on the Riemann surface of \( \lambda_0 \) because

\[
\frac{du}{dt} = \frac{8u^3}{1 + u} \left( \frac{c_\infty + c_0}{(c_\infty + c_0)^2 + (c_\infty - c_0)^2} \right)
\]

never vanish when \( t \in \Omega_{D_6} \). Figure 2.10 \( \sim \) 2.12 are the Stokes geometry on the \( u \)-plane for several values of \( c \). In Appendix A we show more examples of the Stokes geometry on the \( u \)-plane. The residues of the 1-form \( \sqrt{q(u, c)} \, du \) at singular points are summarized as follows (the sign \( \pm \) depends on the choice of the square root of \( q(u, c) \)):

\[
\text{Res}_{u=\infty} \sqrt{q(u, c)} \, du = \pm (c_\infty + c_0)/2, \tag{2.30}
\]

\[
\text{Res}_{u=0} \sqrt{q(u, c)} \, du = \pm (c_\infty - c_0)/2, \tag{2.31}
\]

\[
\text{Res}_{u=(c_\infty-c_0)/(c_\infty+c_0)} \sqrt{q(u, c)} \, du = \pm c_\infty, \tag{2.32}
\]

\[
\text{Res}_{u=-(c_\infty-c_0)/(c_\infty+c_0)} \sqrt{q(u, c)} \, du = \pm c_0. \tag{2.33}
\]

In the figures above and in Appendix A we observe some interesting characteristic features of the Stokes geometry of \((P_{\text{III}}')_{D_6}\). We find two kinds of degenerations. The first one is “triangle-type degeneration”; that is, there are three pairs of turning points connected by Stokes curves simultaneously. Figure 2.11 shows an example of the triangle-type degeneration. This kind of degeneration is also observed in the case of the second, forth and sixth Painlevé equation (see [11] for the second Painlevé equation). The second one is “loop-type degeneration”; that is, a Stokes curve form a loop around the double-pole type singular point \( t = 0 \) and, at the same time, the turning point which is the end point of the loop and the simple-pole are connected by a Stokes curve. An example the loop-type degeneration is shown in Figure 2.12. It is known that appearances of these loops are caused by the fact that the residue of the 1-form \( \sqrt{\Delta(t, c)} \, dt \) at the double-pole inside the loop takes a pure imaginary number \([51, \S 7]\). See (2.32) and (2.33). The same kind of degeneration is also observed for the degenerate third Painlevé equation of the type \( D_7 \):

\[
\frac{d^2 \lambda}{dt^2} = \frac{1}{\lambda} \left( \frac{d \lambda}{dt} \right)^2 - \frac{1}{t} \frac{d \lambda}{dt} + \eta^2 \left( -\frac{2\lambda^2}{t^2} + \frac{c}{t} - \frac{1}{\lambda} \right), \tag{2.34}
\]

when \( c \) is pure imaginary (see Figure D.2 in Appendix D). Intriguingly enough, as far as we checked, only these two kinds of degeneration can be observed. (See Appendix A.) We expect that these are common characteristic features of Stokes geometries of all Painlevé equations. But we have not understood the mechanism of these phenomena yet.

### 3 Voros coefficients of \((P_{\text{III}}')_{D_6}\)

In this section we compute Voros coefficients of \((P_{\text{III}}')_{D_6}\). Those are formal power series in \( \eta^{-1} \) which describe differences of normalizations of transseries solutions. Voros coefficients play a essential role when we discuss the parametric Stokes phenomena in Section 5.
3.1 Voros coefficients $W_\infty$

We recall the definition of Voros coefficients of $(P_{\text{IV}})_{D_0}$ (cf. [11, 12]).

**Definition 3.1.** For a path $\Gamma(\tau, \infty)$ from a turning point (or the simple-pole) $\tau$ to $\infty$, the Voros coefficient for the path $\Gamma(\tau, \infty)$ is a formal power series in $\eta^{-1}$ defined by the integral

$$W_\infty(c, \eta) = \int_{\Gamma(\tau, \infty)} \left( R_{\text{odd}}(t, c, \eta) - \eta R_{-1}(t, c) \right) dt. \quad (3.1)$$

**Remark 3.1.** A priori, the Voros coefficients may depend on the choice of the path $\Gamma(\tau, \infty)$, hence we should use the notation $W_{\Gamma(\tau, \infty)}$. However, as is shown in the proof of Theorem 3.1 Voros coefficients depends only on the asymptotic behavior of $\lambda_0$ as $t \to \infty$, and the choice of the square root $R_{-1} = \sqrt{\Delta}$. They are independent of the lower end point $\tau$. For this reason, we write them simply as (3.1).

The Voros coefficients represent a difference between several normalizations of transseries solutions. To see this, we introduce the following two special normalizations. Note that, as we mentioned in the end of Section 2.1, giving a normalization of transseries solutions is equivalent to giving that of its 1-instanton part (2.15). (In this section we do not discuss the precise location of the independent variable $t$, and the choice of the path of the integration of (2.15). It will be specified in Section 5 when we describe the connection formula concretely.)

The first one is the normalization at $t = \tau$, ($\tau$ is a turning point or the simple-pole):

$$\lambda_\tau(t, c, \eta; \alpha) = \lambda^{(0)}(t, c, \eta) + \alpha \eta^{-1/2} \lambda^{(1)}_\tau(t, c, \eta)e^{\eta \phi} + \alpha \eta^{-1/2} \lambda^{(2)}_\tau(t, c, \eta)e^{2\eta \phi} + \cdots, \quad (3.2)$$

where its 1-instanton part is normalized as

$$\tilde{\lambda}^{(1)}_\tau(t, c, \eta; \alpha) = \alpha \frac{\lambda^{(0)}_\tau}{\sqrt{t} R_{\text{odd}}} \exp \left( \int_{\tau}^{t} R_{\text{odd}}(t, c, \eta) dt \right). \quad (3.3)$$

Since the each coefficient $R_{2n-1}$ of $R_{\text{odd}}$ has a branch point at $\tau$, the integral (3.3) should be considered as a contour integral. (See Remark 3.2.) The second one is the normalization at $t = \infty$:

$$\lambda_\infty(t, c, \eta; \alpha) = \lambda^{(0)}(t, c, \eta) + \alpha \eta^{-1/2} \lambda^{(1)}_\infty(t, c, \eta)e^{\eta \phi} + \alpha \eta^{-1/2} \lambda^{(2)}_\infty(t, c, \eta)e^{2\eta \phi} + \cdots, \quad (3.4)$$

where its 1-instanton part is normalized as

$$\tilde{\lambda}^{(1)}_\infty(t, c, \eta; \alpha) = \alpha \frac{\lambda^{(0)}_\infty}{\sqrt{t} R_{\text{odd}}} \exp \left( \eta \int_{\tau}^{t} R_{-1}(t, c) dt + \int_{\infty}^{t} \left( R_{\text{odd}}(t, c, \eta) - \eta R_{-1}(t, c) \right) dt \right). \quad (3.5)$$

Since all coefficients of $R_{\text{odd}}$ are integrable at $t = \infty$ except for the leading term $R_{-1}$ (see Appendix C), the integral in (3.5) is well-defined. These two normalizations (3.2) and (3.4) are related as

$$\tilde{\lambda}^{(1)}_\tau(t, c, \eta; \alpha) = e^{W_\infty} \tilde{\lambda}^{(1)}_\infty(t, c, \eta; \alpha), \quad (3.6)$$

$$\lambda_\tau(t, c, \eta; \alpha) = \lambda_\infty(t, c, \eta; \alpha e^{W_\infty}), \quad (3.7)$$

by the Voros coefficient $W_\infty$ for a suitable path $\Gamma(\tau, \infty)$ from $\tau$ to $\infty$. 

Remark 3.2. Here we give a remark for integrals of $R_{\text{odd}}$ from turning points or the simple-pole. Since we can easily verify that $R_{2n-1}$ has a Puiseux expansion

$$R_{2n-1} = (t - \tau)^{N/4} \sum_{k \geq 0} r_k (t - \tau)^{k/2}$$

(3.8)

near $t = \tau$ (where $\tau$ is a turning point or the simple-pole) with some odd integer $N$, we can define the integral from $\tau$ in terms of contour integral on the Riemann surface of $\sqrt{\Delta}$ (i.e., the double cover of the Riemann surface of $\lambda_0$ branching at turning points and the simple-pole). In addition to Figure 2.6 ∼ 2.9, we need further branch cuts to determine the square root $\sqrt{\Delta}$ as in Figure 3.1 ∼ 3.4, which describe the case $c = (2, 2 - i)$. In these figures the spiral lines designate the new branch cuts, and we fix the square root so that the real part of the integral

$$\int_{\tau}^{t} \sqrt{\Delta}(t, c) \, dt$$

along a Stokes curve is positive if the integral is taken in parallel with the arrow on the Stokes curve. Then, for example, if $t$ is fixed at the point in Figure 3.2, we define the integral from $\tau_1$ to $t$ by the following contour integral:

$$\int_{\tau_1}^{t} R_{\text{odd}}(t, c, \eta) \, dt = \frac{1}{2} \int_{\Gamma_t} R_{\text{odd}}(t, c, \eta) \, dt,$$  

(3.9)

where $\Gamma_t$ is a path on the Riemann surface of $\sqrt{\Delta}$ described in Figure 3.1 and 3.2. Here $\tilde{t}$ in Figure 3.2 represents the point on the other sheet of square root $\sqrt{\Delta}$ satisfying $\lambda_0(\tilde{t}, c) = \lambda_0(t, c)$ and $R_{\ell}(\tilde{t}, c) = (-1)^{\ell} R_{\ell}(t, c)$ ($\ell \geq -1$) holds. The dotted part of $\Gamma_t$ represents the path on the other sheet of square root $\sqrt{\Delta}$. We can also define integrals from other turning points or the simple-pole in the same manner.

There are several Voros coefficients in accordance with choices of a path $\Gamma(\tau, \infty)$ of the integration (3.1), and we should distinguish the branch of $\lambda_0$ near $t = \infty$. We know from the algebraic equation (2.5) that $\lambda_0$ has following four asymptotic behaviors as $t$ tends to infinity:

$$\lambda_0 = +t^{1/2} \left(1 + O(t^{-1/2})\right) \quad \text{as} \quad t \to \infty_1,$$  

(3.10)

$$\lambda_0 = -t^{1/2} \left(1 + O(t^{-1/2})\right) \quad \text{as} \quad t \to \infty_2,$$  

(3.11)

$$\lambda_0 = +it^{1/2} \left(1 + O(t^{-1/2})\right) \quad \text{as} \quad t \to \infty_3,$$  

(3.12)

$$\lambda_0 = -it^{1/2} \left(1 + O(t^{-1/2})\right) \quad \text{as} \quad t \to \infty_4.$$  

(3.13)

Here we use the symbols $\infty_j$ ($1 \leq j \leq 4$) to distinguish the asymptotic behaviors of $\lambda_0$. Note that both $t = \infty_1$ and $\infty_2$ correspond to $u = \infty$, while both $t = \infty_3$ and $\infty_4$ correspond to $u = 0$ in the coordinate (2.28). We use another symbols $\infty_{j, \pm}$ ($1 \leq j \leq 4$) to specify the choice of the square root $R_{-1} = \sqrt{\Delta(t, c)}$ as follows:
\[ \lambda_0 = \pm t^{1/2}(1 + O(t^{-1/2})) , \quad R_{-1} = \pm 2t^{-1/2}(1 + O(t^{-1/2})) \quad \text{as } t \to \infty_{1, \pm} \quad \text{(3.14)} \]

\[ \lambda_0 = -t^{1/2}(1 + O(t^{-1/2})) , \quad R_{-1} = \mp 2t^{-1/2}(1 + O(t^{-1/2})) \quad \text{as } t \to \infty_{2, \pm} \quad \text{(3.15)} \]

\[ \lambda_0 = \pm it^{1/2}(1 + O(t^{-1/2})) , \quad R_{-1} = \pm 2it^{-1/2}(1 + O(t^{-1/2})) \quad \text{as } t \to \infty_{3, \pm} \quad \text{(3.16)} \]

\[ \lambda_0 = -it^{1/2}(1 + O(t^{-1/2})) , \quad R_{-1} = \mp 2it^{-1/2}(1 + O(t^{-1/2})) \quad \text{as } t \to \infty_{4, \pm} \quad \text{(3.17)} \]

See Appendix C for higher order asymptotic behaviors. Then, our first main result are stated as follows.
Theorem 3.1. Let $F(c, \eta)$ be a formal power series in $\eta^{-1}$ defined by

$$F(c, \eta) = \sum_{n=1}^{\infty} \frac{2^{1-2n}-1}{2n(2n-1)} B_{2n}(c\eta)^{1-2n}, \quad (3.18)$$

where $B_{2n}$ is the $2n$-th Bernoulli number defined by (1.2). Then, the Voros coefficient $W_\infty(c, \eta)$ for a path $\Gamma(\tau, \infty)$ is represented explicitly as follows:

$$W_{\infty,1}(c, \eta) = W_{\infty,2}(c, \eta) = \pm F(c_\tau, \eta). \quad (3.19)$$

$$W_{\infty,3}(c, \eta) = W_{\infty,4}(c, \eta) = \pm F(c_m, \eta). \quad (3.20)$$

Here $c_\tau$ and $c_m$ are given by

$$c_\tau = \frac{c_\infty + c_0}{2}, \quad c_m = \frac{c_\infty - c_0}{2}. \quad (3.21)$$

The proof of Theorem 3.1 will be given in Section 4 together with that of Theorem 3.2 below.

### 3.2 Voros coefficients $W_{0,\infty}$ and $W_{0,\infty}$

In the previous subsection we defined the Voros coefficient for $t = \infty$. However, since $(P_{HF})_{\mathcal{D}_0}$ also has double-pole type singular points (2.22) and (2.23) at $t = 0$ (cf. Remark 2.3), we can consider Voros coefficients relevant to these double-poles.

First, we specify the branch of $\lambda_0$ near the double poles. We use symbols $0_{c_\infty}$ and $0_{c_0}$ depending on the asymptotic behaviors (2.22) and (2.23) of $\lambda_0$; that is, if $t$ tends to $0_{c_\infty}$ (resp., to $0_{c_0}$), then $\lambda_0$ behaves as (2.22) (resp., (2.23)). Then, the Voros coefficients for double-poles are defined as follows:

**Definition 3.2.** For a path $\Gamma(\tau, 0_{c_\infty})$ from a turning point (or the simple-pole) $\tau$ to $0_{c_\infty}$ ($* = \infty$ or 0), the Voros coefficient for the path $\Gamma(\tau, 0_{c_\infty})$ is defined by

$$W_{0,\infty}(c, \eta) = \int_{\Gamma(\tau, 0_{c_\infty})} (R_{odd}(t, c, \eta) - \eta R_{-1}(t, c)) dt. \quad (3.22)$$

It turns out to be that the right-hand side of (3.22) is independent of the choice of the path $\Gamma(\tau, 0_{c_\infty})$ (see Theorem 3.2). As well as (3.7), the Voros coefficients $W_{0,\infty}$ also describe a difference between $\lambda_0(t, c, \eta; \alpha)$ and the transseries solution $\lambda_{0,\infty}(t, c, \eta; \alpha)$ normalized at double-poles;

$$\lambda_{0,\infty}(t, c, \eta; \alpha) = \lambda^{(0)}(t, c, \eta; \alpha) + \alpha \eta^{-1/2} \lambda^{(1)}_{0,\infty}(t, c, \eta) e^{\eta \phi} + (\alpha \eta^{-1/2})^2 \lambda^{(2)}_{0,\infty}(t, c, \eta) e^{2\eta \phi} + \cdots, \quad (3.23)$$

$$\lambda_{0,\infty}(t, c, \eta; \alpha) = \lambda^{(0)}(t, c, \eta; \alpha) + \alpha \eta^{-1/2} \lambda^{(1)}_{0,\infty}(t, c, \eta) e^{\eta \phi} + (\alpha \eta^{-1/2})^2 \lambda^{(2)}_{0,\infty}(t, c, \eta) e^{2\eta \phi} + \cdots, \quad (3.24)$$

where the 1-instanton part is normalized as

$$\tilde{\lambda}^{(1)}_{0,\infty}(t, c, \eta; \alpha) = \alpha \frac{\lambda^{(0)}(t, c, \eta; \alpha)}{\sqrt{t R_{odd}}} \exp \left( \eta \int_{\tau}^{t} R_{-1}(t, c) dt + \int_{0_{\infty}}^{t} (R_{odd}(t, c, \eta) - \eta R_{-1}(t, c)) dt \right), \quad (3.25)$$

16
The choice of the square root $R$ the leading term $R$

where $B$

Since the all coefficient $R_{2n-1}$ of $R_{\text{odd}}$ are integrable at these double-poles except for the leading term $R_{-1}$ (see Appendix C), the above integrals are well-defined.

To state our second main result, we introduce another symbols in order to specify the choice of the square root $R_{-1} = \sqrt{\Delta}$ at double-poles:

$$
\lambda_0 = c_\infty (1 + O(t)), \quad R_{-1} = \pm \frac{c_\infty}{t} (1 + O(t)) \quad \text{as } t \to 0_{c, \pm}, \quad (3.27)
$$

$$
\lambda_0 = \frac{t}{c_0} (1 + O(t)), \quad R_{-1} = \pm \frac{c_0}{t} (1 + O(t)) \quad \text{as } t \to 0_{c, \pm}. \quad (3.28)
$$

Then our second main result which is shown in the next section is stated as follows.

**Theorem 3.2.** Let $F(c, \eta)$ and $G(c, \eta)$ be the formal power series in $\eta^{-1}$ defined by (3.18) and

$$
G(c, \eta) = \sum_{n=1}^{\infty} \frac{B_{2n}}{2n(2n-1)} (c\eta)^{1-2n}, \quad (3.29)
$$

where $B_{2n}$ is the $2n$-th Bernoulli number defined by (1.2). Then, the Voros coefficient $W_{0, \infty}(c, \eta)$ for a path $\Gamma(\tau, 0_c)$ ($*=\infty, 0$) is represented explicitly as follows:

$$
W_{0, \infty, \pm}(c, \eta) = \pm \left\{ F(c_p, \eta) + F(c_m, \eta) - 3G(c_\infty, \eta) \right\}, \quad (3.30)
$$

$$
W_{0, 0, \pm}(c, \eta) = \pm \left\{ F(c_p, \eta) - F(c_m, \eta) - 3G(c_0, \eta) \right\}. \quad (3.31)
$$

Here $c_p$ and $c_m$ are given by (3.21).

**Theorem 3.2** is also proved in in Section 4.

**Remark 3.3.** As you see in the list of Theorem 3.1 and Theorem 3.2, the formal power series $F(c, \eta)$ and $G(c, \eta)$ are fundamental pieces of the Voros coefficients. These formal power series also appear as Voros coefficients of other equations. For example, $F(c, \eta)$ appears as the Voros coefficient of the Weber equation (12) which is linear, the second Painlevé equation (11), and a fourth order analogue of the second Painlevé equation (13)

$$
\frac{d^4 \lambda}{dt^4} = \eta^2 \left( 10\lambda^2 \frac{d^2 \lambda}{dt^2} + 10\lambda \left( \frac{d\lambda}{dt} \right)^2 \right) + \eta^4 \left( -6\lambda^5 + t\lambda + c \right) \quad (3.32)
$$

which is considered in [KT4]. On the other hand, $G(c, \eta)$ appears in the Voros coefficients of the hypergeometric equation (AT) and the Bessel equation (14) (see Proposition 1.1). Note also that $G(c, \eta)$ coincides with the Voros coefficient of the degenerate third Painlevé equation of the type $D_7$. See Theorem D.1 in Appendix D.

**Remark 3.4.** Note that, when $c$ is pure imaginary, $F(c, \eta)$ and $G(c, \eta)$ are not Borel summable as a formal power series in $\eta^{-1}$ (see Proposition 5.1). Intriguingly, when one of Voros coefficients is not Borel summable (i.e., one of $c_p$, $c_m$, $c_\infty$ or $c_0$ is pure
imaginary), then the Stokes geometry degenerates, as far as we have checked; see Appendix A. Moreover, the Borel sums of \( F(c, \eta) \) and \( G(c, \eta) \) jump at the imaginary axis of \( c \)-plane. Since the Borel sum of \( F(c, \eta) \) and \( G(c, \eta) \) are computed explicitly as in Proposition 5.1, we can describe connection formulas for parametric Stokes phenomena exactly. See Section 5 for details.

4 Proof of the Main theorems

In this section we give proofs of the main theorems about explicit representations of the Voros coefficients (Theorem 3.1 and Theorem 3.2). To compute the Voros coefficients, we adopt a similar method used in [T2] and [KoT]. Especially, here we consider the Voros coefficient \( W_\Gamma(c, \eta) \) defined by the integral (3.1) along the Stokes curve \( \Gamma \) in Figure 2.3 which emanates from \( \tau_1 \) and flows to \( \infty_{3,+} \). That is, \( W_\Gamma \) is defined by the following contour integral

\[
W_\Gamma(c, \eta) = \frac{1}{2} \int_{\Gamma_{\text{contour}}} \left( R_{\text{odd}}(t, c, \eta) - \eta R_{-1}(t, c) \right) dt,
\]

where the path \( \Gamma_{\text{contour}} \) is a path on the Riemann surface of \( \sqrt{\Delta} \) taken as in Figure 4.1 and 4.2. For this Voros coefficient, we will show the following.

**Theorem 4.1.** The Voros coefficient \( W_\Gamma(c, \eta) \) is represented explicitly as follows:

\[
W_\Gamma(c, \eta) = \sum_{n=1}^{\infty} \frac{2^{1-2n} - 1}{2n(2n-1)} B_{2n} \left( \frac{c_{\infty} - c_0}{2} \eta \right)^{1-2n}.
\]

This statement is included in the statement of Theorem 3.1. The other statements in Theorem 3.1 and 3.2 can be proved in the same manner presented here.
4.1 Proof of Theorem 4.1 (and Theorem 3.1)

The following lemma is a key for the proof of our main theorems.

Lemma 4.1 (cf. [12, Lemma 1.2] and [KoT, §3]). (i) The formal power series $F(c, \eta)$ and $G(c, \eta)$ defined in (3.18) and (3.29) satisfy the following difference equations:

\[ F(c + \eta^{-1}, \eta) - F(c, \eta) = 1 - (c\eta + 1)\log\left(1 + \frac{1}{c\eta}\right) + \log\left(1 + \frac{1}{2c\eta}\right), \]  

(4.3)

\[ G(c + \eta^{-1}, \eta) - G(c, \eta) = 1 - (c\eta + \frac{1}{2})\log\left(1 + \frac{1}{c\eta}\right). \]  

(4.4)

(ii) Conversely, if there exist a formal power series solution of the equation (4.3) or (4.4) of the form

\[ \sum_{\ell \geq 1} a_\ell (c\eta)^{-\ell} \]

with some $a_\ell \in \mathbb{C}$ which is independent of both $c$ and $\eta$ ($\ell \geq 1$), then it coincides with $F(c, \eta)$ or $G(c, \eta)$, respectively.

To apply Lemma 4.1 for the proof of our main theorems, we derive difference equations satisfied by Voros coefficients. For the purpose, we use the following two Bäcklund transformations $T_1$ and $T_2$, where $T_1$ induces the shift of the parameters of $(P_{IW})_{D_6}$ as $(c_\infty, c_0) \mapsto (c_\infty + \eta^{-1}, c_0 + \eta^{-1})$, while $T_2$ induces $(c_\infty, c_0) \mapsto (c_\infty + \eta^{-1}, c_0 - \eta^{-1})$. It is convenient to use the following Hamiltonian system $(H_{IW})_{D_6}$ which is equivalent to $(P_{IW})_{D_6}$ (cf. [OK]):

\[ (H_{IW})_{D_6} : \frac{d\lambda}{dt} = \eta \frac{\partial H}{\partial \mu}, \quad \frac{d\mu}{dt} = -\eta \frac{\partial H}{\partial \lambda}, \]  

(4.5)

where the Hamiltonian $H = H(\lambda, \mu, t)$ is defined by

\[ tH = \lambda^2 \mu^2 - (\lambda^2 + (c_0 - \eta^{-1})\lambda - t)\mu + \frac{1}{2}(c_\infty + c_0 - \eta^{-1})\lambda. \]

(4.6)

Then the explicit form of Bäcklund transformations $T_1$ and $T_2$ are given by the following:

Lemma 4.2 (e.g., [JM, OK]). Let $(\lambda, \nu)$ be a solution of $(H_{IW})_{D_6}$. Then, $(\Lambda, M) = (\Lambda_j(\lambda, \nu), M_j(\lambda, \nu)) \ (j = 1, 2)$ defined by

\[
\begin{align*}
\Lambda_1 &= -\frac{t}{\lambda} + \frac{(c_\infty + c_0 + \eta^{-1})t}{2\lambda^2(\mu - 1) + (c_\infty - c_0 + \eta^{-1})\lambda + 2t}, \\
M_1 &= \frac{\lambda^2(\mu - 1)}{t} + \frac{(c_\infty - c_0 + \eta^{-1})\lambda}{2t} + 1,
\end{align*}
\]

(4.7)

\[
\begin{align*}
\Lambda_2 &= \frac{2t(\mu - 1)}{2\lambda(\mu - 1) + (c_\infty - c_0 + \eta^{-1})}, \\
M_2 &= \frac{1}{t}\left\{\frac{c_\infty + c_0 - \eta^{-1}}{2}\left(\lambda + \frac{c_\infty - c_0 + \eta^{-1}}{2(\mu - 1)}\right) - \left(\lambda + \frac{c_\infty - c_0 + \eta^{-1}}{2(\mu - 1)}\right)^2\right\},
\end{align*}
\]

(4.8)
is a solution of \((H_{III})_6\) with the parameter \(c\) is shifted by \(T_j\) \((j = 1, 2)\); that is, it is a solution of
\[
\frac{d \Lambda}{dt} = \eta \frac{\partial \mathcal{H}_j}{\partial \mu}, \quad \frac{d M}{dt} = -\eta \frac{\partial \mathcal{H}_j}{\partial \Lambda} \quad (j = 1, 2),
\]
(4.9)
where \(\mathcal{H}_j = \mathcal{H}_j(\Lambda, M, t) \quad (j = 1, 2)\) is given by
\[
t \mathcal{H}_1 = \Lambda^2 M^2 - (\Lambda^2 + c_0 \Lambda - t) M + \frac{1}{2} (c_\infty + c_0 + \eta^{-1}) \Lambda,
\]
(4.10)
\[
t \mathcal{H}_2 = \Lambda^2 M^2 - (\Lambda^2 + (c_0 - 2\eta^{-1}) \Lambda - t) M + \frac{1}{2} (c_\infty + c_0 - \eta^{-1}) \Lambda.
\]
(4.11)

Lemma 4.2 can be shown by straightforward computations. Next we consider a transseries solution
\[
(\lambda(t, c, \eta; \alpha), \mu(t, c, \eta; \alpha)) = \left( \sum_{k \geq 0} (\alpha \eta^{-1/2})^k \lambda^{(k)}(t, c, \eta) e^{k\eta \phi}, \sum_{k \geq 0} (\alpha \eta^{-1/2})^k \mu^{(k)}(t, c, \eta) e^{k\eta \phi} \right)
\]
of \((H_{III})_6\). Here the above transseries expansion of \(\mu(t, c, \eta; \alpha)\) is obtained from that of \(\lambda(t, c, \eta; \alpha)\) by the equality
\[
\mu = \frac{1}{2 \lambda^2} \left( \eta^{-1} t \frac{d \lambda}{dt} + \lambda^2 + (c_0 - \eta^{-1}) \lambda - t \right).
\]
(4.13)
Especially, the formal power series \(\mu^{(0)}(t, c, \eta)\) and its leading term \(\mu_0(t, c)\) is given by
\[
\mu^{(0)}(t, c, \eta) = \frac{1}{2 \lambda^{(0)}^2} \left( \eta^{-1} t \frac{d \lambda^{(0)}}{dt} + \lambda^{(0)} + (c_0 - \eta^{-1}) \lambda^{(0)} - t \right),
\]
(4.14)
\[
\mu_0(t, c) = \frac{1}{2} + \frac{c_0}{2 \lambda_0} - \frac{t}{2 \lambda_0^2}.
\]
(4.15)
Applying the above Bäcklund transformations to this transseries solution, we have the following:

**Lemma 4.3.** Let \(T_1(c) = (c_\infty + \eta^{-1}, c_0 + \eta^{-1})\) and \(T_2(c) = (c_\infty + \eta^{-1}, c_0 - \eta^{-1})\), and \(R = R(t, c, \eta)\) be a formal solution \(R = R(t, c, \eta)\) of the Riccati equation \((2.9)\); that is, \(R = R_+ \) or \(R_-\) in Remark 2.1. Then we have the following:

(i) \(R(t, T_1(c), \eta) - R(t, c, \eta) = \frac{d}{dt} \log t \)
\[
+ \frac{d}{dt} \log \left( \frac{1}{\lambda^{(0)}^2} - \frac{(c_\infty + c_0 + \eta^{-1})(4\lambda^{(0)}(\mu^{(0)} - 1) + (c_\infty - c_0 + \eta^{-1} + 2\lambda^{(0)}^2 X))}{(2\lambda^{(0)}^2(\mu^{(0)} - 1) + (c_\infty - c_0 + \eta^{-1})\lambda^{(0)} + 2t)^2} \right).
\]
(4.16)
(ii) \(R(t, T_2(c), \eta) - R(t, c, \eta) = \frac{d}{dt} \log t + \frac{d}{dt} \log (-2(\mu^{(0)} - 1)^2 + (c_\infty - c_0 + \eta^{-1} X)
\]
\[
-2 \frac{d}{dt} \log (2\lambda^{(0)}(\mu^{(0)} - 1) + (c_\infty - c_0 + \eta^{-1})).
\]
(4.17)
Here $X = X(t, c, \eta)$ is a formal power series given by
\[
X(t, c, \eta) = \frac{\eta^{-1}t R}{2\lambda(0)^2} - \frac{\eta^{-1}t d\lambda(0)}{\lambda(0)^3} dt - \frac{c_0 - \eta^{-1}}{2\lambda(0)^2} + \frac{t}{\lambda(0)^3}. \tag{4.18}
\]

**Proof.** Here we only show the equality (4.17). Applying the Bäcklund transformation (4.8) to the transseries solution, we have the following transseries solution of the ($P_{III}$)$_{D_6}$ with the parameter $c$ is shifted by $T_2$ (the equation is denoted by $T_2(P_{III})_{D_6}$ in what follows):
\[
\Lambda_2(\lambda, \mu) = \Lambda^{(0)}(t, c, \eta) + \lambda_1^{-1/2} \Lambda^{(1)}(t, c, \eta)e^{\eta \phi} + (\lambda_1^{-1/2})^2 \Lambda^{(2)}(t, c, \eta)e^{2\eta \phi} + \cdots, \tag{4.19}
\]
where $\Lambda^{(k)}(t, c, \eta)$ is a formal power series in $\eta^{-1} (k \geq 0)$. Especially, $\Lambda^{(0)}(t, c, \eta) = \Lambda_2(\lambda(0), \mu(0))$ is a 0-parameter solution of $T_2(P_{III})_{D_6}$, and
\[
\Lambda^{(1)}(t, c, \eta) = \frac{-4t(\mu(0) - 1)^2 \lambda^{(1)} + 2t(c_{\infty} - c_0 + \eta^{-1})\mu^{(1)}}{(2\lambda(0)(\mu(0) - 1) + (c_{\infty} - c_0 + \eta^{-1}))^2}. \tag{4.20}
\]
Note that, since we can easily check that
\[
\frac{2t(\mu_0 - 1)}{2\lambda_0(\mu_0 - 1) + (c_{\infty} - c_0)} = \lambda_0 \tag{4.21}
\]
holds, we have $\Lambda^{(0)}(t, c, \eta) = \lambda^{(0)}(t, T_2(c), \eta)$ due to the uniqueness of the 0-parameter solution of $T_2(P_{III})_{D_6}$. Hence, as explained in Section 2.1, the formal power series (4.20) is expressed as
\[
\Lambda^{(1)}(t, c, \eta)e^{\eta \phi} = \mathcal{C}(\eta)\exp\left(\int^t R(t, T_2(c), \eta) dt\right) \tag{4.22}
\]
with a formal power series $\mathcal{C}(\eta)$ whose coefficients are independent of $t$. On the other hand, since $\mu^{(1)}$ can be written as
\[
\mu^{(1)}(t, c, \eta) = X(t, c, \eta)\Lambda^{(1)}(t, c, \eta) \tag{4.23}
\]
by (4.13) and (2.8), the formal power series (4.20) also has the following expression:
\[
\Lambda^{(1)}(t, c, \eta)e^{\eta \phi} = \frac{\mathcal{C}'(\eta)}{(2\lambda(0)(\mu(0) - 1) + (c_{\infty} - c_0 + \eta^{-1}))\lambda^{(1)}(t, c, \eta)} \exp\left(\int^t R(t, c, \eta) dt\right). \tag{4.24}
\]
Here $\mathcal{C}'(\eta)$ is a formal power series with constant (with respect to $t$) coefficients. Comparing (4.22) and (4.24), and taking logarithmic derivatives, we obtain (4.17). The equality (4.21) can be derived in the completely same manner.

Lemma 4.3 arrows us to compute integrals of $R(t, T_j(c), \eta) - R(t, c, \eta)$ ($j = 1, 2$) explicitly. Using this lemma, the difference equation satisfied by the Voros coefficient $W_T(c, \eta)$ can be derived. We can derive difference equations for all other Voros coefficients in the same manner.
Lemma 4.4. The Voros coefficient $W_T(c, \eta)$ defined by (4.1) satisfies the following difference equations:

(i) $W_T(T_1(c), \eta) - W_T(c, \eta) = 0, \quad (4.25)$

(ii) $W_T(T_2(c), \eta) - W_T(c, \eta) = 1 - \left( c_m \eta + 1 \right) \log \left( 1 + \frac{1}{c_m \eta} \right) + \log \left( 1 + \frac{1}{2 c_m \eta} \right), \quad (4.26)$

where $c_m$ is given by (3.21).

Proof. We only derive the difference equation (4.26) here. We introduce the following formal power series defined by

$I_\pm(t, c, \eta) = \int_{\Gamma_t} R_\pm(t, T_2(c), \eta) \, dt - \int_{\Gamma_t} R_\pm(t, c, \eta) \, dt , \quad (4.27)$

$I_k(t, c, \eta) = \int_{\Gamma_t} R_k(t, T_2(c)) \, dt - \int_{\Gamma_t} R_k(t, c) \, dt \quad (k \geq -1), \quad (4.28)$

where $\Gamma_t$ is a path shown in Figure 3.1 and 3.2. Then, it follows from the definition (2.11) of $R_{\text{odd}}$ that

$W_T(T_2(c), \eta) - W_T(c, \eta) = \frac{1}{2} \lim_{t \to \infty, +} \left( \frac{I_+(t, c, \eta) - I_-(t, c, \eta)}{2} - \eta I_{-1}(t, c, \eta) \right), \quad (4.29)$

where the limit is taken along the Stokes curve $\Gamma$ in Figure 2.4. Using Lemma 4.3, we have

\[
\frac{I_+(t, c, \eta) - I_-(t, c, \eta)}{2} = \log \left( \frac{-2(\mu(0) - 1)^2 + (c_\infty - c_0 + \eta^{-1})X_+}{-2(\mu(0) - 1)^2 + (c_\infty - c_0 - \eta^{-1})X_-} \right), \quad (4.30)
\]

where $X_\pm$ is a formal power series defined by taking $R = R_\pm$ in (4.18). Then, it follows from (C.15), (C.16) and (C.18) in Appendix C that

\[
\frac{I_+(t, c, \eta) - I_-(t, c, \eta)}{2} = 2 \log \left( 1 + \frac{1}{(c_\infty - c_0) \eta} \right) + \log \left( \frac{-(c_\infty - c_0)^2}{256 t} \right) + O(t^{-1/2}) \quad (4.31)
\]

as $t \to \infty_{3,+}$, with suitable branches of the logarithm. Moreover, we can compute the integral of $R_{-1}$ explicitly as

\[
\int_{\Gamma_t} R_{-1}(t, c) \, dt = 4 t R_{-1} - c_\infty \log \left( \frac{2 \lambda_0 - c_\infty + t R_{-1}}{2 \lambda_0 - c_\infty - t R_{-1}} \right) - c_0 \log \left( \frac{2 t^2 - c_0 t \lambda_0 + t^2 \lambda_0 R_{-1}}{2 t^2 - c_0 t \lambda_0 - t^2 \lambda_0 R_{-1}} \right). \quad (4.32)
\]

The equality (4.32) follows from the relationship between the Painlevé equations and associated isomonodromic deformation (cf. [JM]) of linear differential equations. Let $Q_0(x, t, c)$ be a rational function

\[
Q_0(x, t, c) = \frac{(x - \lambda_0)^2}{4 x^4} P(x, t, c), \quad P(x, t, c) = x^2 + 2(\lambda_0 - c_\infty)x + \frac{t^2}{\lambda_0^2},
\]

22
which is the leading term of the potential function of a Schrödinger equation relevant to $(P_{W})_{0}$ (cf. [KT3]), and $x = a(t, c)$ be a zero of $P(x, t, c)$. Then, it is shown in [KT3] that

$$\int_{a(t, c)}^{\lambda_{0}(t, c)} \sqrt{Q_{0}(x, t, c)} \, dx = \frac{1}{2} \int_{t}^{\tau} R_{-1}(t, c) \, dt.$$  \hspace{1cm} (4.33)

Therefore, the equality (4.32) follows from the equalities (4.33) and the explicit computation of the integral of $\sqrt{Q_{0}(x, t, c)}$. Thus we have

$$I_{-1}(t, c, \eta) = -2\eta^{-1} + (c_{\infty} - c_{0} + 2\eta^{-1}) \log \left( 1 + \frac{2}{(c_{\infty} - c_{0})\eta} \right)$$

$$+ \eta^{-1} \log \left( \frac{-(c_{\infty} - c_{0})^2}{256t} \right) + O(t^{-1/2}),$$  \hspace{1cm} (4.34)

as $t \to \infty$, with suitable branches of the logarithm. As a result, the difference equation (4.26) follows from (4.29), (4.31) and (4.34) directly. Here we note that, since any Voros coefficients has the form

$$\sum_{n \geq 1} \eta^{1-2n} W_{2n-1}(c), \quad W_{2n-1}(c) \in \mathbb{C}$$

(i.e., a formal power series in $\eta^{-1}$ without constant terms), the branches of logarithms in (4.31) and (4.34) must coincide, otherwise we have a contradiction; there appears a term of $\eta^{0}$ in the right-hand side of the difference equation (4.26). The difference equation (4.25) can be derived by the completely same manner.

The shift operators $T_{1}$ and $T_{2}$ induce $T_{1}(c_{p}, c_{m}) = (c_{p} + \eta^{-1}, c_{m})$ and $T_{2}(c_{p}, c_{m}) = (c_{p}, c_{m} + \eta^{-1})$. Therefore, the first difference equation (4.25) implies that $W_{\Gamma}$ does not depend on $c_{p}$; that is, $W_{\Gamma}$ is a formal power series of $\eta^{-1}$ whose coefficients depend only on $c_{m}$. Moreover, according to the homogeneity property (B.11), $W_{\Gamma}$ can be expressed as

$$W_{\Gamma}(c, \eta) = \sum_{n \geq 1} w_{2n-1}(c_{m} \eta)^{1-2n},$$  \hspace{1cm} (4.35)

with some $w_{2n-1} \in \mathbb{C}$ which is independent of both $\eta$ and $c_{m}$ ($n \geq 1$). Then, (4.26) and (4.35) implies that $W_{\Gamma}$ has the explicit representation (4.2) by (ii) of Lemma 4.1. Thus we have proved Theorem 4.1.

As well as the above proof, we can obtain an explicit representations of Voros coefficients from difference equations satisfied by them, and these difference equations can be derived in the same manner. The list of asymptotic behaviors of $\lambda^{(0)}$, $\mu^{(0)}$ and $R_{\pm}$ when $t$ tends to infinity, which are necessary for derivations of the difference equations, are summarized in Appendix C. Here we show the difference equations satisfied by the Voros coefficients $W_{\infty}$ for any choice of $\infty$.  \hspace{1cm} $\square$
Lemma 4.5. (i) The Voros coefficient \( W_{\pm}(c, \eta) = W_{\pm,\pm}(c, \eta) \) (for \( j = 1, 2 \)) satisfies the following difference equations:

\[
W_{\pm}(T_1(c), \eta) - W_{\pm}(c, \eta) = \pm \left\{ 1 - (c_p \eta + 1) \log \left( 1 + \frac{1}{c_p \eta} \right) + \log \left( 1 + \frac{1}{2c_p \eta} \right) \right\},
\]

(4.36)

\[
W_{\pm}(T_2(c), \eta) - W_{\pm}(c, \eta) = 0.
\]

(4.37)

(ii) The Voros coefficient \( W_{\pm}(c, \eta) = W_{\pm,\pm}(c, \eta) \) (for \( j = 3, 4 \)) satisfies the following difference equations:

\[
W_{\pm}(T_1(c), \eta) - W_{\pm}(c, \eta) = 0,
\]

(4.38)

\[
W_{\pm}(T_2(c), \eta) - W_{\pm}(c, \eta) = \pm \left\{ 1 - (c_m \eta + 1) \log \left( 1 + \frac{1}{c_m \eta} \right) + \log \left( 1 + \frac{1}{2c_m \eta} \right) \right\}.
\]

(4.39)

Here \( c_p \) and \( c_m \) are given by (3.21).

The explicit representations of Voros coefficients in Theorem 3.1 are obtained from these difference equations and Lemma 4.1 directly. Thus we have proved Theorem 3.1. \( \square \)

4.2 Proof of Theorem 3.2

Next we show Theorem 3.2 about the explicit representations of the Voros coefficients \( W_{0,\pm}(c, \eta) \) and \( W_{0,\pm}(c, \eta) \) for double-poles defined by (3.22). Using Lemma 4.3 and the list of asymptotic behaviors when \( t \) tends to double-poles which are summarized in Appendix C, we can derive the following difference equations in the completely same method in Section 4.1.

Lemma 4.6. (i) The Voros coefficient \( W_{c,\pm}(c, \eta) = W_{c,\pm}(c, \eta) \) satisfies the following difference equations:

\[
W_{c,\pm}(T_1(c), \eta) - W_{c,\pm}(c, \eta) = \pm \left\{ -2 - (c_p \eta + 1) \log \left( 1 + \frac{1}{c_p \eta} \right) + \log \left( 1 + \frac{1}{2c_p \eta} \right) \right\} + 3 \left( c_{\infty} \eta + \frac{1}{2} \right) \log \left( 1 + \frac{1}{c_{\infty} \eta} \right),
\]

(4.40)

\[
W_{c,\pm}(T_2(c), \eta) - W_{c,\pm}(c, \eta) = \pm \left\{ -2 - (c_m \eta + 1) \log \left( 1 + \frac{1}{c_m \eta} \right) + \log \left( 1 + \frac{1}{2c_m \eta} \right) \right\} + 3 \left( c_{\infty} \eta + \frac{1}{2} \right) \log \left( 1 + \frac{1}{c_{\infty} \eta} \right).
\]

(4.41)

(ii) The Voros coefficient \( W_{c,\pm}(c, \eta) = W_{c,\pm}(c, \eta) \) satisfies the following difference
observed in Section 2.2, and derive connection formulas in some cases.

In this section we analyze parametric Stokes phenomena relevant to the degeneration due to (ii) of Lemma 4.1 and (B.11). Thus we have proved Theorem 3.2. □

by combining the equations (4.44) + 3 (c_0 \eta - \frac{1}{2}) \log \left( 1 + \frac{1}{c_0 \eta} \right), \quad (4.43)

W_{c_0, \pm}(T_2(c), \eta) - W_{c_0, \pm}(c, \eta) = \pm \left\{ -2 - \left( c_p \eta + 1 \right) \log \left( 1 + \frac{1}{c_p \eta} \right) + \log \left( 1 + \frac{1}{2 c_p \eta} \right) \right\},

Here \( c_p \) and \( c_m \) are given by \((3.21)\).

It follows from Lemma 4.1 and Lemma 4.6 that, the formal power series

\[ \tilde{W}_{c_{\infty}, \pm}(c, \eta) = W_{c_{\infty}, \pm}(c, \eta) \mp \left( \mathcal{F}(c_p, \eta) + \mathcal{F}(c_m, \eta) \right), \]

\[ \tilde{W}_{c_0, \pm}(c, \eta) = W_{c_0, \pm}(c, \eta) \mp \left( \mathcal{F}(c_p, \eta) - \mathcal{F}(c_m, \eta) \right), \]

satisfies the following equations:

\[ \tilde{W}_{c_{\infty}, \pm}(T_1(c), \eta) - \tilde{W}_{c_{\infty}, \pm}(c, \eta) = \pm \left\{ -3 + \left( c_{\infty} \eta + \frac{1}{2} \right) \log \left( 1 + \frac{1}{c_{\infty} \eta} \right) \right\}, \quad (4.44) \]

\[ \tilde{W}_{c_{\infty}, \pm}(T_2(c), \eta) - \tilde{W}_{c_{\infty}, \pm}(c, \eta) = \pm \left\{ -3 + \left( c_{\infty} \eta + \frac{1}{2} \right) \log \left( 1 + \frac{1}{c_{\infty} \eta} \right) \right\}, \quad (4.45) \]

\[ \tilde{W}_{c_0, \pm}(T_1(c), \eta) - \tilde{W}_{c_0, \pm}(c, \eta) = \pm \left\{ -3 + \left( c_0 \eta + \frac{1}{2} \right) \log \left( 1 + \frac{1}{c_0 \eta} \right) \right\}, \quad (4.46) \]

\[ \tilde{W}_{c_0, \pm}(T_2(c), \eta) - \tilde{W}_{c_0, \pm}(c, \eta) = \pm \left\{ 3 + \left( c_0 \eta - \frac{1}{2} \right) \log \left( 1 - \frac{1}{c_0 \eta} \right) \right\}. \quad (4.47) \]

We can confirm that \( \tilde{W}_{c_{\infty}, \pm} \) and \( \tilde{W}_{c_0, \pm} \) satisfy

\[ \tilde{W}_{c_{\infty}, \pm}(c_{\infty}, c_0 + \eta^{-1}) - \tilde{W}_{c_{\infty}, \pm}(c_{\infty}, c_0 - \eta^{-1}) = 0 \]

\[ \tilde{W}_{c_0, \pm}(c_{\infty} + \eta^{-1}, c_0) - \tilde{W}_{c_0, \pm}(c_{\infty} - \eta^{-1}, c_0) = 0 \]

by combining the equations \((4.44) \sim (4.47)\). This implies that \( \tilde{W}_{c_{\infty}, \pm} \) (resp., \( \tilde{W}_{c_0, \pm} \)) does not depend on \( c_0 \) (resp., \( c_{\infty} \)). As a result, we have

\[ \tilde{W}_{c_{\infty}, \pm}(c, \eta) = \mp 3 \mathcal{G}(c_{\infty}, \eta), \quad (4.48) \]

\[ \tilde{W}_{c_0, \pm}(c, \eta) = \mp 3 \mathcal{G}(c_0, \eta), \quad (4.49) \]

due to (ii) of Lemma 4.1 and \((B.11)\). Thus we have proved Theorem 3.2. □

5 Connection formulas for parametric Stokes phenomena

In this section we analyze parametric Stokes phenomena relevant to the degeneration observed in Section 2.2 and derive connection formulas in some cases.
5.1 Borel sum of Voros coefficients and their jump property

First we note that the Borel sum (as a formal power series in $\eta^{-1}$) of Voros coefficients can be computed explicitly. (See [KT3 §1] for the definition of Borel sums of formal power series.)

**Proposition 5.1** (cf. [T2 §2], [KoT]). Let $\mathcal{F}(c, \eta)$ and $\mathcal{G}(c, \eta)$ be the formal power series in $\eta^{-1}$ given by (3.18) and (3.29). They are not Borel summable when $c \in i\mathbb{R}$, and Borel summable otherwise. Moreover, the Borel sum $S_{\pm}[\mathcal{F}(c, \eta)]$ and $S_{\pm}[\mathcal{G}(c, \eta)]$ of the formal power series $\mathcal{F}(c, \eta)$ and $\mathcal{G}(c, \eta)$ when $\arg c = \pi/2 \pm \delta$ for a sufficiently small positive number $\delta$ are given explicitly by the followings:

\[
S_{-}[\mathcal{F}(c, \eta)] = \log \frac{\Gamma(c\eta + 1/2)}{\sqrt{2\pi}} - c\eta \left( \log(c\eta) - 1 \right) \quad (5.1)
\]

\[
S_{+}[\mathcal{F}(c, \eta)] = -\log \frac{\Gamma(-c\eta + 1/2)}{\sqrt{2\pi}} - c\eta \left( \log(c\eta) - 1 \right) + \pi i c\eta. \quad (5.2)
\]

\[
S_{-}[\mathcal{G}(c, \eta)] = \log \frac{\Gamma(c\eta)}{\sqrt{2\pi}} - c\eta \left( \log(c\eta) - 1 \right) + \frac{1}{2} \log(c\eta). \quad (5.3)
\]

\[
S_{+}[\mathcal{G}(c, \eta)] = -\log \frac{\Gamma(-c\eta)}{\sqrt{2\pi}} - c\eta \left( \log(c\eta) - 1 \right) - \frac{1}{2} \log(c\eta) + \pi i (c\eta + 1/2). \quad (5.4)
\]

As a corollary of Proposition 5.1, we have the following equalities.

**Corollary 5.1.** After the analytic continuation across the imaginary axis $\{\arg c = \pi/2\}$, the following holds:

\[
S_{+}[e^{\mathcal{F}(c, \eta)}] = (1 + e^{2\pi i c\eta}) S_{-}[e^{\mathcal{F}(c, \eta)}]. \quad (5.5)
\]

\[
S_{+}[e^{\mathcal{G}(c, \eta)}] = (1 - e^{2\pi i c\eta}) S_{-}[e^{\mathcal{G}(c, \eta)}]. \quad (5.6)
\]

These jump properties of the Voros coefficients are essential in the derivation of connection formulas describing the parametric Stokes phenomena.

It follows from Proposition 5.1 and the explicit representations of Voros coefficients obtained by Theorem 3.1 and Theorem 3.2 that Voros coefficients are not Borel summable when the parameter $c = (c_{\infty}, c_0)$ lies on the “walls” $\mathcal{W}_1, \ldots, \mathcal{W}_8$ in the parameter space defined by

\[
\mathcal{W}_1 = \{ c \in \mathbb{S}; \text{Re} c_0 = 0, \text{Re} c_{\infty} > 0 \}, \quad \mathcal{W}_2 = \{ c \in \mathbb{S}; \text{Re} c_m = 0, \text{Re} c_p > 0 \},
\]

\[
\mathcal{W}_3 = \{ c \in \mathbb{S}; \text{Re} c_{\infty} = 0, \text{Re} c_0 > 0 \}, \quad \mathcal{W}_4 = \{ c \in \mathbb{S}; \text{Re} c_p = 0, \text{Re} c_m < 0 \},
\]

\[
\mathcal{W}_5 = \{ c \in \mathbb{S}; \text{Re} c_0 = 0, \text{Re} c_{\infty} < 0 \}, \quad \mathcal{W}_6 = \{ c \in \mathbb{S}; \text{Re} c_m = 0, \text{Re} c_p < 0 \},
\]

\[
\mathcal{W}_7 = \{ c \in \mathbb{S}; \text{Re} c_0 = 0, \text{Re} c_{\infty} < 0 \}, \quad \mathcal{W}_8 = \{ c \in \mathbb{S}; \text{Re} c_p = 0, \text{Re} c_m > 0 \},
\]

where $\mathbb{S}$ is given by (2.2), $c_p$ and $c_m$ are given by (3.21). These walls divide the parameter space into eight chambers $I, \ldots, VIII$. Figure 5.1 describes the projections of these walls and chambers to $(\text{Re} c_{\infty}, \text{Re} c_0)$-plane. The Voros coefficients are
Figure 5.1: Projection of the walls and chambers in the parameter space of $c = (c_\infty, c_0)$.

Borel summable in each chambers, and jump when the parameter cross these walls. This jump property causes parametric Stokes phenomena. We show some examples of connection formulas on these walls in subsequent discussions.

5.2 Connection problem for parametric Stokes phenomena relevant to the triangle-type degeneration

Here we discuss the connection problem for the parametric Stokes phenomenon relevant to the triangle-type degeneration of Stokes geometry observed when $c = (2, 2 - i)$; that is, the connection problem on the wall $W_2$ in Figure 5.1. To be more specific, we discuss the connection problem for the transseries solution $\lambda_{\tau_1}(t, c, \eta; \alpha)$ normalized at the turning point $\tau_1$, when the independent variable $t$ is in a sufficiently small neighborhood of the point $t_0$ in Figure 5.2 and $t_1$ in Figure 5.3. These figures describe only Sheet 2 of the Riemann surface of $\lambda_0$; see Figure 3.1 ~ 3.4. Note that $t_0$ (or the corresponding point on the $u$-plane) lies inside of the “triangle” in Figure 2.11 formed by three bounded Stokes curves, and $t_1$ lies outside of the triangle. Here we mean the triangle by “the triangle on the $u$-plane described in in Figure 2.11”, which is not the triangle on the $t$-plane in Figure 2.4. (The inside of the triangle in Figure 2.4 are not mapped to the inside of the triangle in Figure 2.11 by (2.28).)

Before the discussion of connection problems, we impose the following assumptions (A-1) and (A-2) for a neighborhood $U_{t_*}$ of the point $t = t_*$ ($* = 0$ or 1) and $\varepsilon > 0$:

(A-1) There exists a neighborhood $U_{t_*}$ of the point $t_*$ such that

$$t \in U_{t_*} \Rightarrow \Re \phi(t) < 0.$$  

(A-2) The small number $\varepsilon > 0$ satisfies that, for any $0 \leq \varepsilon' \leq \varepsilon$, any Stokes curves never touch with any points in $U_{t_*}$ when $c = (2 \pm \varepsilon', 2 - i)$. 

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These assumptions are expected to be essential for the Borel summability of transseries solutions. According to a recent result obtained by Kamimoto (for the Borel summability of transseries solutions), we can expect the following.

**Conjecture 5.1.** Assume that the integral of $R_{\text{odd}}$ in (2.15) is taken along a path which never touches with any turning points, the simple-pole and Stokes curves, and the real part of $\phi$ is negative. Then, the corresponding transseries solution $\lambda(t, c, \eta; \alpha)$ is Borel summable (in general sense of \([C]\)); that is, the $k$-th formal power series $\lambda^{(k)}(t, c, \eta)$ in $\lambda(t, c, \eta; \alpha)$ is Borel summable for each $k$, and the (generalized) Borel sum of $\lambda(t, c, \eta; \alpha)$ defined by the infinite sum

$$
S[\lambda(t, c, \eta; \alpha)] = \sum_{k \geq 0} (\alpha \eta^{-1/2})^k S[\lambda^{(k)}(t, c, \eta)] e^{k \eta \phi}
$$

converges for sufficiently large $\eta > 0$ and represents an analytic solution of $(P_{\text{III}'})_{D_6}$. Here $S[\lambda^{(k)}(t, c, \eta)]$ is the Borel sum of the formal power series $\lambda^{(k)}(t, c, \eta)$.

Kamimoto proved the same statement for the first and second Painlevé equation \([Kam]\). In this paper we assume that the Conjecture 5.1 is true, and discuss connection problems for parametric Stokes phenomena.

![Figure 5.2: The path from $\infty_{3,+}$ for $t \in U_{t_0}$.](image)

![Figure 5.3: A path from $\infty_{1,+}$ for $t \in U_{t_1}$.](image)

First we discuss the connection problem for the parametric Stokes phenomenon in the case $t \in U_{t_0}$. Let

$$
\lambda_\infty(t, c, \eta; \alpha) = \sum_{k \geq 0} (\alpha \eta^{-1/2})^k \lambda_\infty^{(k)}(t, c, \eta) e^{k \eta \phi}
$$

(5.7)

(resp., $\lambda_{\tau_1}(t, c, \eta; \alpha) = \sum_{k \geq 0} (\alpha \eta^{-1/2})^k \lambda_{\tau_1}^{(k)}(t, c, \eta) e^{k \eta \phi}$)

be the transseries solution whose 1-instanton part is normalized at infinity along the path shown in Figure 5.2 (resp., normalized at the turning point $\tau_1$ along the path.

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If Conjecture 5.1 holds, for any $0 \leq \varepsilon' \leq \varepsilon$ the transseries solution $\lambda_\infty(t, c, \eta; \alpha)$ is Borel summable when $t \in U_t$ and $c = (2 \pm \varepsilon', 2 - i)$ since the real part of

$$\phi(t, c) = \int_{t_1}^t R_{-1}(t, c) dt = \int_{t_1}^t \sqrt{\Delta(t, c)} dt$$

is negative by the assumption (A-1) and the normalization path in Figure 5.2 can be deformed homotopically such that it does not touch with any turning points and Stokes curves by the assumptions (A-2). On the other hand, using the relations

$$\lambda^{(k)}_{\tau_1}(t, c, \eta) = e^{kW_\Gamma(c, \eta)} \lambda^{(k)}_\infty(t, c, \eta) \quad (k \geq 0)$$

and

$$W_{\Gamma}(c, \eta) = F(c_m, \eta)$$

(cf. Theorem 4.1 and Proposition 5.1) we can conclude that $\lambda_{\tau_1}(t, c, \eta; \alpha)$ is Borel summable when $c = (2 \pm \varepsilon', 2 - i)$ for any $0 < \varepsilon' \leq \varepsilon$, and the Borel sum has the analytic continuation with respect to the parameter $c$ across the wall $W_2$.

Now we derive the connection formula for the transseries solution $\lambda_{\tau_1}$. Let $S_{II}[\lambda_{\tau_1}(t, c, \eta; \alpha)]$ (resp., $S_I[\lambda_{\tau_1}(t, c, \eta; \alpha)])$ be the Borel sum of $\lambda_{\tau_1}(t, c, \eta; \alpha)$ when $c = (2 - \varepsilon, 2 - i)$ (resp., of $\lambda_{\tau_1}(t, c, \eta; \alpha)$ when $c = (2 + \varepsilon, 2 - i)$) for a sufficiently small $\varepsilon > 0$. Moreover, we assume that, after the analytic continuation with respect to the parameter $c$ across the wall $W_2$, $S_{II}[\lambda_{\tau_1}(t, c, \eta; \alpha)]$ and $S_I[\lambda_{\tau_1}(t, c, \eta; \alpha)]$ represent the same analytic solution of $(P_{III})_{D_6}$ defined on $U_t$. That is, we assume

$$S_{II}[\lambda_{\tau_1}(t, c, \eta; \alpha)] = S_I[\lambda_{\tau_1}(t, c, \eta; \alpha)].$$

(5.11)

Taking into account the relation (5.9) and comparing the coefficients of $\exp(k\eta\phi)$ ($k \geq 0$) in (5.11), we have

$$\alpha^k S_{II}[e^{kW_\Gamma(c, \eta)} \lambda^{(k)}_\infty(t, c, \eta)] = \tilde{\alpha}^k S_I[e^{kW_\Gamma(c, \eta)} \lambda^{(k)}_\infty(t, c, \eta)].$$

(5.12)

The Borel summability of $\lambda^{(k)}_\infty(t, c, \eta)$ when $c = (2, 2 - i)$ implies that

$$S_{II}[\lambda^{(k)}_\infty(t, c, \eta)] = S_I[\lambda^{(k)}_\infty(t, c, \eta)]$$

(5.13)

holds for all $k \geq 0$ after the analytic continuation across $W_2$. Therefore it follows from (5.12) and (5.13) that

$$\alpha^k S_{II}[e^{kW_\Gamma(c, \eta)}] = \tilde{\alpha}^k S_I[e^{kW_\Gamma(c, \eta)}]$$

(5.14)
holds for all $k \geq 0$. Moreover, (5.10) and Corollary 5.1 implies that

$$
S_I \left[ e^{W(t,c)} \right] = (1 + e^{2\pi i c \eta}) S_I \left[ e^{W(t,c)} \right] = (1 + e^{\pi i (c_\infty - c_0) \eta}) S_I \left[ e^{W(t,c)} \right].
$$

Thus, (5.14) is true if and only if the parameters satisfy the relation

$$
\bar{\alpha} = (1 + e^{\pi i (c_\infty - c_0) \eta}) \alpha.
$$

Thus we obtain the following connection formula.

**Connection formula on $W_2$ near $t = t_0$.** Assume that a neighborhood $U_{t_0}$ of the point $t_0$ and a small number $\varepsilon > 0$ satisfies the assumptions (A-1) and (A-2). If the Borel sum of $\lambda_{t_1}(t,c,\eta;\alpha)$ when $c = (2 - \varepsilon, 2 - i)$ and that of $\lambda_{t_1}(t,c,\eta;\bar{\alpha})$ when $c = (2 + \varepsilon, 2 - i)$ represent the same analytic solution of $(P_{W_2})_{c_0}$ defined on on $t \in U_{t_0}$ after the analytic continuation with respect to the parameter $c$ across $W_2$, then the parameters $\alpha$ and $\bar{\alpha}$ satisfy (5.16). That is, we have the following connection formula:

$$
S_I \left[ \lambda_{t_1}(t,c,\eta;\alpha) \right] = S_I \left[ \lambda_{t_1}(t,c,\eta;\bar{\alpha}) \right] \bigg|_{\bar{\alpha} = (1 + e^{\pi i (c_\infty - c_0) \eta}) \alpha}.
$$

The formula (5.17) describes the parametric Stokes phenomenon; that is, it gives an explicit relationship between the Borel sums of the transseries solution in different regions in the parameter space of $c$. Note that the difference $\exp(\pi i (c_\infty - c_0) \eta)$ of the Borel sums is exponentially small for large $\eta$ near $c = (2, 2 - i)$.

**Remark 5.1.** As we see, the formula (5.17) is derived by comparing $\lambda_{t_1}$ with $\lambda_\infty$, and the point is that the latter one is (conjectured to be) Borel summable even if the Stokes geometry degenerates. We note that, in deriving the formula for $\lambda_{t_1}$ we may compare it with another transseries solution $\lambda_{0,c_0,-}$ which is normalized at the double-pole $t = 0_{c_0,-}$ in Figure 5.2 instead of $\lambda_\infty$, since $\lambda_{0,c_0,-}$ is also Borel summable when the Stokes geometry degenerates under the assumption that Conjecture 5.1 holds, as well as $\lambda_\infty$. These transseries solutions are related as

$$
\lambda_{t_1}(t,c,\eta;\alpha) = \lambda_{0,c_0,-}(t,c,\eta;\alpha e^{W_{0,c_0,-}})
$$

with

$$
W_{0,c_0,-} = W_{0,c_0,-}(c,\eta) = -\mathcal{F}(c_r,\eta) + \mathcal{F}(c_m,\eta) + 3\mathcal{G}(c_0,\eta).
$$

due to Theorem 3.2. Since the formal power series $\mathcal{F}(c_r,\eta)$ and $\mathcal{G}(c_0,\eta)$ in (5.19) are Borel summable when $c = (2, 2 - i)$ by Proposition 5.1, they never jump at the wall $W_2$. Thus we have

$$
S_I \left[ e^{W_{0,c_0,-}(c,\eta)} \right] = (1 + e^{\pi i (c_\infty - c_0) \eta}) S_I \left[ e^{W_{0,c_0,-}(c,\eta)} \right].
$$

Therefore, we obtain the same conclusion as (5.17).
Next we discuss the connection problem when \( t \in U_{t_1} \). In this case we should compare \( \lambda_{t_1} \) with \( \lambda_{\infty} \) which is normalized along the path from \( \infty_{1,-} \) in Figure 5.3 so that \( \lambda_{\infty} \) is Borel summable when \( c = (2, 2 - i) \). Then, we have the following relation between these two transseries instead of (5.9):

\[
\lambda^{(k)}(t, c, \eta) = e^{kW} \lambda^{(k)}_{\infty}(t, c, \eta) \quad (k \geq 0),
\]

with \( W = W(c, \eta) = -F(c, \eta) \) because of Theorem 3.2. Since the formal power series \( F(c, \eta) \) is Borel summable when \( c = (2 - \varepsilon, 2 - i) \) by Proposition 5.1, we have

\[
S_{II}[e^{W(c, \eta)}] = S_I[e^{W(c, \eta)}]
\]

instead of (5.15). As a result, we can conclude the following.

**Connection formula on \( \mathcal{W}_2 \) near \( t = t_1 \).** Assume that a neighborhood \( U_{t_1} \) of the point \( t_1 \) and a small number \( \varepsilon > 0 \) satisfies the assumptions (A-1) and (A-2). If the Borel sum of \( \lambda_{t_1}(t, c, \eta; \alpha) \) when \( c = (2 - \varepsilon, 2 - i) \) and that of \( \lambda_{t_1}(t, c, \eta; \tilde{\alpha}) \) when \( c = (2 + \varepsilon, 2 - i) \) represent the same analytic solution of \( (P_{III})_{D_6} \) defined on \( t \in U_{t_1} \) after the analytic continuation with respect to the parameter \( c \) across \( \mathcal{W}_2 \), then the parameters \( \alpha \) and \( \tilde{\alpha} \) satisfy \( \tilde{\alpha} = \alpha \). That is, no parametric Stokes phenomenon occurs on \( \mathcal{W}_2 \) when \( t \in U_{t_1} \):

\[
S_{II}[\lambda_{t_1}(t, c, \eta; \alpha)] = S_I[\lambda_{t_1}(t, c, \eta; \alpha)].
\]

**Remark 5.2.** From the comparison of two formulas (5.17) and (5.22), we can see that the connection formulas describing parametric Stokes phenomena are different depending on the location of the independent variable \( t \). More precisely, the connection formula when \( t \) lies inside of the triangle formed by three bounded Stokes curves in Figure 2.11 is different from that when \( t \) lies outside of the triangle. This is also observed for the second Painlevé equation in [12]. We expect that the same phenomena also happen to other Painlevé equations and higher order analogues of them.

Since we have computed all Voros coefficients of \( (P_{III})_{D_6} \), we can derive connection formulas for parametric Stokes phenomenon at any point \( t \in \Omega_{D_6} \) if the
assumptions (A-1) and (A-2) are satisfied and Conjecture 5.1 holds. Here we give the statement of connection formulas for parametric Stokes phenomena relevant to the other triangle-type degeneration in Figure 5.5 which describes the Stokes geometry on the $u$-plane when $c = (-2 + i, 2 + i/2)$; that is $c$ lies on the wall $W_4$. In this case we impose the following assumption $(A-2)'$ for $\varepsilon > 0$ instead of $(A-2)$:

(A-2)' The small number $\varepsilon > 0$ satisfies that, for any $0 \leq \varepsilon' \leq \varepsilon$, any Stokes curves never touch with any points in $U_t$, when $c = (-2 \pm \varepsilon' + i, 2 - i)$.

**Connection formula on $W_4$.** Let $\lambda(t, c, \eta; \alpha)$ be a transseries solution normalized at a turning point or the simple-pole, and $S_{IV}[\lambda(t, c, \eta; \alpha)]$ (resp., $S_{III}[\lambda(t, c, \eta; \tilde{\alpha})]$) be the Borel sum of $\lambda(t, c, \eta; \alpha)$ when $c = (-2 - \varepsilon + i, 2 + i/2)$ (resp., of $\lambda(t, c, \eta; \tilde{\alpha})$ when $c = (-2 + \varepsilon + i, 2 + i/2)$) for a sufficiently small $\varepsilon > 0$.

(i) Fix $u_*$ outside of the triangle formed by bounded Stokes curves in Figure 5.5. Assume that a neighborhood $U_t*$ of the point $t_*$ which corresponds to $u_*$ and a small number $\varepsilon > 0$ satisfies the assumptions (A-1) and (A-2)'. If $S_{IV}[\lambda(t, c, \eta; \alpha)]$ and $S_{III}[\lambda(t, c, \eta; \tilde{\alpha})]$ represent the same analytic solution of $(P_{IV})D_6$ defined on $t \in U_t*$ after the analytic continuation with respect to the parameter $c$ across $W_4$, then the parameters $\alpha$ and $\tilde{\alpha}$ satisfy

$$\tilde{\alpha} = (1 + e^{2\pi i(c_{\infty} + c_0)\eta})^{\pm 1} \alpha,$$

where the sign $\pm 1$ depends on the choice of square root $R_{-1}(t, c) = \sqrt{\Delta(t, c)}$. That is, we have the following connection formula:

$$S_{IV}[\lambda(t, c, \eta; \alpha)] = S_{III}[\lambda(t, c, \eta; \tilde{\alpha})]
\bigg|_{\tilde{\alpha} = (1+e^{2\pi i(c_{\infty} + c_0)\eta})^{\pm 1} \alpha}.$$  

(ii) Fix $u_*$ inside of the triangle formed by bounded Stokes curves in Figure 5.5. Assume that a neighborhood $U_t*$ of the point $t_*$ which corresponds to $u_*$ and a small number $\varepsilon > 0$ satisfies the assumptions (A-1) and (A-2)'. If $S_{IV}[\lambda(t, c, \eta; \alpha)]$ and $S_{III}[\lambda(t, c, \eta; \tilde{\alpha})]$ represent the same analytic solution of $(P_{IV})D_6$ defined on $t \in U_t*$ after the analytic continuation with respect to the parameter $c$ across $W_4$, then the parameters $\alpha$ and $\tilde{\alpha}$ satisfy $\tilde{\alpha} = \alpha$. That is, no parametric Stokes phenomenon occurs on the wall $W_4$ when $t \in U_t*$:

$$S_{IV}[\lambda(t, c, \eta; \alpha)] = S_{III}[\lambda(t, c, \eta; \alpha)].$$

**5.3 Connection problem for parametric Stokes phenomena relevant to the loop-type degeneration**

Next we discuss the connection problem relevant to the loop-type degeneration of Stokes geometry observed when $c = (i, 3 + i/2)$; that is, the connection problem on the wall $W_3$. Figure 5.7 ~ 5.9 describes the Stokes geometry on the $u$-plane near $c = (i, 3 + i/2)$.

First we consider the case that the independent variable $t$ (or $u$) lies outside of the “loop”. Here we impose the following assumption (A-2") instead of (A-2):

$$S_{IV}[\lambda(t, c, \eta; \alpha)] = S_{III}[\lambda(t, c, \eta; \alpha)].$$
The small number $\varepsilon > 0$ satisfies that, for any $0 \leq \varepsilon' \leq \varepsilon$, any Stokes curves never touch with any points in $U_t$, when $c = (\pm \varepsilon' + i, 3 + i/2)$. In this case the discussion given in Section 5.2 can be applicable, and the relevant Voros coefficients to be considered to derive the connection form are one of $W_{\infty_j}(c, \eta) \ (1 \leq j \leq 4)$ or $W_{0, \pm}(c, \eta)$. These Voros coefficients are all Borel summable when $c$ lies on the wall $W_3$, and hence they never jump on the wall. Therefore, we have the following conclusion:

**Connection formula on $W_3$.** Let $\lambda(t, c, \eta; \alpha)$ be a transseries solution normalized at a turning point or the simple-pole, and $S_{\text{III}}[\lambda(t, c, \eta; \alpha)]$ (resp., $S_{\text{II}}[\lambda(t, c, \eta; \tilde{\alpha})]$) be the Borel sum of $\lambda(t, c, \eta; \alpha)$ when $c = (-\varepsilon + i, 3 + i/2)$ (resp., of $\lambda(t, c, \eta; \tilde{\alpha})$ when $c = (+\varepsilon + i, 3 + i/2)$) for a sufficiently small $\varepsilon > 0$. Fix $u_*$ outside of the loop formed by a bounded Stokes curve in Figure 5.8. Assume that a neighborhood $U_{t_*}$ of the point $t_*$ which corresponds to $u_*$ and a small number $\varepsilon > 0$ satisfies the assumptions (A-1) and (A-2)”. If $S_{\text{III}}[\lambda(t, c, \eta; \alpha)]$ and $S_{\text{II}}[\lambda(t, c, \eta; \tilde{\alpha})]$ represent the same analytic solution of $(P_{\text{III}}')_D$ defined on $t \in U_{t_*}$ after the analytic continuation with respect to the parameter $c$ across $W_3$, then the parameters $\alpha$ and $\tilde{\alpha}$ satisfy $\tilde{\alpha} = \alpha$. That is, no parametric Stokes phenomenon occurs on the wall $W_3$ when $t \in U_{t_*}$:

$$S_{\text{III}}[\lambda(t, c, \eta; \alpha)] = S_{\text{II}}[\lambda(t, c, \eta; \alpha)].$$  \hspace{1cm} (5.26)

On the other hand, the cases when $t$ (or $u$) lies inside the loop are quite different from the above case. When the loop-type degeneration appears and $t_*$ lies inside the loop, the assumption (A-2)” is not satisfied for any $\varepsilon > 0$ because of “infinitely many spirals”. Actually, if $t_*$ is fixed inside of the loop, then Stokes curves touch with $t_*$ infinitely many times as $c$ tends to $\varepsilon'$ varies $0 \leq \varepsilon' \leq \varepsilon$, and hence usual Stokes phenomena occur to transseries solutions infinitely many times. These loop-type degenerations have not been analyzed even in the case of linear differential equations, due to the same difficulty. In order to describe connection formulas, we need some modification of the Borel resummation method, but we do not have an appropriate way at this time. This is a future issue to be discussed.
A Examples of Stokes geometry of \((P_{III})_{D_6}\)

Here we show the examples of Stokes geometry of \((P_{III})_{D_6}\) on the \(u\)-plane \((u\) is defined by (2.28)) when the parameter \(c\) is in the chambers \(I \sim \text{VIII}\) and on the walls \(W_1 \sim W_8\) in Figure 5.1. Note that, since the quadratic differential (2.29) is invariant under the exchange of the parameters \(c_\infty \leftrightarrow c_0\), so is the Stokes geometry. Therefore it is enough to show the Stokes geometries when \(c\) is in the chambers \(II, III, IV, V\) and on the boundaries of them. We conjecture that, as long as the parameter \(c\) does not across the walls, the topological type of configuration of the Stokes curves never change, and degeneration of Stokes geometry occur only on these walls. This is true as far as we checked by numerical experiments, but we can not confirm it analytically.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{example1}
\caption{On \(W_2\) : \(c = (2 + i, 2 + i/2)\).}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{example2}
\caption{In \(II\) : \(c = (1 + i, 3 + i/2)\).}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{example3}
\caption{On \(W_3\) : \(c = (0 + i, 3 + i/2)\).}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{example4}
\caption{In \(III\) : \(c = (-1 + i, 3 + i/2)\).}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{example5}
\caption{On \(W_4\) : \(c = (-2 + i, 2 + i/2)\).}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{example6}
\caption{In \(IV\) : \(c = (-3 + i, 1 + i/2)\).}
\end{figure}
**B Homogeneity**

As is explained in the beginning of Section 2, quantities appearing in this paper have a homogeneity with respect to the following scaling operation:

\[(t, c_\infty, c_0, \eta) \mapsto (r^{-2} t, r^{-1} c_\infty, r^{-1} c_0, r \eta) \quad (r > 0).\]

For example, the homogenous degree of \(\lambda_0(t, c) = \lambda_0(t, c_\infty, c_0)\) which is the algebraic function defined by \(F(\lambda_0, t, c) = 0\) is \(-1\); that is,

\[\lambda_0(r^{-2} t, r^{-1} c_\infty, r^{-1} c_0) = r^{-1} \lambda_0(t, c_\infty, c_0).\]

We list the homogenous degrees of quantities below. We regard the free parameter \(\alpha\) as a constant with homogenous degree 0.

---

Figure A.7: On \(W_5\): \(c = (-3 + i, 0 + i/2)\).

Figure A.8: In \(V\): \(c = (-3 + i, -1 + i/2)\).

Figure A.9: On \(W_6\): \(c = (-2 + i, -2 + i/2)\).

Figure A.10: \(c = (0 + i, 0 + i/2)\).
Asymptotics of coefficients

As is noted in Section 4, asymptotic behaviors of $\lambda^{(0)}(t, c, \eta)$, $\mu^{(0)}(t, c, \eta)$ and $R_{\pm}(t, c, \eta)$ when $t$ tends to singular points $\infty$ and 0 is important in derivations of difference equations satisfied by Voros coefficients. Here we summarize all asymptotic behaviors of them. (We only show the case $t \to \infty$ for $1 \leq j \leq 4$ and $t \to 0_{c_{*},+}$ ($* = \infty$ or 0). Behaviors when $t \to \infty_{j,-}$ and $t \to 0_{c_{*},-}$ can be obtained from these list by replacing $R_{-1} \mapsto -R_{-1}$ and $R_{\pm} \mapsto R_{\mp}$.)

C. Asymptotics of coefficients

$$\lambda_0(t, c) : -1. \quad \text{(B.1)}$$
$$\mu_0(t, c) : 0. \quad \text{(B.2)}$$
$$\Delta(t, c) : +2. \quad \text{(B.3)}$$

A turning point $\tau(c) : -2. \quad \text{(B.4)}$

$$R(t, c, \eta) : +2. \quad \text{(B.5)}$$
$$R_{\text{odd}}(t, c, \eta) : +2. \quad \text{(B.6)}$$
$$\phi(t, c) : -1. \quad \text{(B.7)}$$
$$\lambda(t, c, \eta; \alpha) : -1. \quad \text{(B.8)}$$
$$\mu(t, c, \eta; \alpha) : 0. \quad \text{(B.9)}$$
$$W_{\tau,\infty}(c, \eta) : 0. \quad \text{(B.10)}$$
$$W_{\tau,\infty_{c_*}}(c, \eta) : 0 \quad (* = \infty, 0). \quad \text{(B.11)}$$
• The case $t \to \infty_{1,+}$:

$$\lambda_0(t, c) = t^{1/2} + \frac{c_\infty - c_0}{4} - \frac{(c_\infty - c_0)(3c_\infty + c_0)}{32}t^{-1/2} + O(t^{-1}), \quad \text{(C.1)}$$

$$\mu_0(t, c) = \frac{c_\infty + c_0}{4}t^{-1/2} + O(t^{-3/2}), \quad \text{(C.2)}$$

$$\lambda^{(0)}(t, c, \eta) = t^{1/2} + \frac{c_\infty - c_0}{4} + \frac{(c_\infty - c_0)(3c_\infty + c_0)}{32}t^{-1/2} + O(t^{-1}), \quad \text{(C.3)}$$

$$\mu^{(0)}(t, c, \eta) = \frac{c_\infty + c_0 - \eta^{-1}}{4}t^{-1/2} + O(t^{-3/2}), \quad \text{(C.4)}$$

$$R_{-1}(t, c) = 2t^{-1/2} - \frac{c_\infty + c_0}{4}t^{-1} + \frac{(3c_\infty - c_0)(c_\infty - 3c_0)}{64}t^{-3/2} + O(t^{-2}), \quad \text{(C.5)}$$

$$R_{\pm}(t, c, \eta) = \pm 2\eta^{-1/2} - \frac{(\pm c_\infty \eta \pm c_0 \eta - 1)}{4}t^{-1}$$

$$+ \frac{\pm 3c_\infty^2 \eta^2 \mp 10c_\infty c_0 \eta^2 \pm 3c_0^2 \eta^2 - 10c_\infty \eta + 6c_0 \eta \mp 1}{64 \eta}t^{-3/2} + O(t^{-2}). \quad \text{(C.6)}$$

• The case $t \to \infty_{2,+}$:

$$\lambda_0(t, c) = -t^{1/2} + \frac{c_\infty - c_0}{4} - \frac{(c_\infty - c_0)(3c_\infty + c_0)}{32}t^{-1/2} + O(t^{-1}), \quad \text{(C.7)}$$

$$\mu_0(t, c) = -\frac{c_\infty + c_0}{4}t^{-1/2} + O(t^{-3/2}), \quad \text{(C.8)}$$

$$\lambda^{(0)}(t, c, \eta) = -t^{1/2} + \frac{c_\infty - c_0}{4} - \frac{(c_\infty - c_0)(3c_\infty + c_0)}{32}t^{-1/2} + O(t^{-1}), \quad \text{(C.9)}$$

$$\mu^{(0)}(t, c, \eta) = -\frac{c_\infty + c_0 - \eta^{-1}}{4}t^{-1/2} + O(t^{-3/2}), \quad \text{(C.10)}$$

$$R_{-1}(t, c) = -2t^{-1/2} - \frac{c_\infty + c_0}{4}t^{-1} - \frac{(3c_\infty - c_0)(c_\infty - 3c_0)}{64}t^{-3/2} + O(t^{-2}), \quad \text{(C.11)}$$

$$R_{\pm}(t, c, \eta) = \mp 2\eta^{-1/2} - \frac{(\pm c_\infty \eta \mp c_0 \eta - 1)}{4}t^{-1}$$

$$- \frac{(\pm 3c_\infty^2 \eta^2 \mp 10c_\infty c_0 \eta^2 \pm 3c_0^2 \eta^2 - 10c_\infty \eta + 6c_0 \eta \mp 1)}{64 \eta}t^{-3/2} + O(t^{-2}). \quad \text{(C.12)}$$
• The case $t \to \infty_{3,+}$:

\[
\lambda_0(t, c) = it^{1/2} + \frac{c_\infty + c_0}{4} - \frac{i(c_\infty + c_0)(3c_\infty - c_0)}{32} t^{-1/2} + O(t^{-1}), \quad (C.13)
\]

\[
\mu_0(t, c) = 1 + \frac{i(c_\infty - c_0)}{4} t^{-1/2} + O(t^{-3/2}), \quad (C.14)
\]

\[
\lambda^{(0)}(t, c, \eta) = it^{1/2} + \frac{c_\infty + c_0}{4} - \frac{i(c_\infty + c_0)(3c_\infty - c_0)}{32} t^{-1/2} + O(t^{-1}), \quad (C.15)
\]

\[
\mu^{(0)}(t, c, \eta) = 1 + \frac{i(c_\infty - c_0 + \eta^{-1})}{4} t^{-1/2} + O(t^{-3/2}), \quad (C.16)
\]

\[
R_{-1}(t, c) = 2it^{-1/2} - \frac{c_\infty - c_0 t^{-1} - i(3c_\infty + c_0)(c_\infty + 3c_0)}{64} t^{-3/2} + O(t^{-2}), \quad (C.17)
\]

\[
R_{\pm}(t, c, \eta) = \mp 2i \eta t^{-1/2} + \frac{(\mp c_\infty \eta \pm c_0 \eta + 1)}{4} t^{-1}
\]

\[
\pm i(\mp 3c_\infty^2 \eta^2 \mp 10c_\infty c_0 \eta^2 \mp 3c_0^2 \eta^2 \mp 10c_\infty \eta - 6c_0 \eta + 1)}{64 \eta} t^{-3/2} + O(t^{-2}). \quad (C.18)
\]

• The case $t \to \infty_{4,+}$:

\[
\lambda_0(t, c) = -it^{1/2} + \frac{c_\infty + c_0}{4} + \frac{i(c_\infty + c_0)(3c_\infty - c_0)}{32} t^{-1/2} + O(t^{-1}), \quad (C.19)
\]

\[
\mu_0(t, c) = 1 - \frac{i(c_\infty - c_0)}{4} t^{-1/2} + O(t^{-3/2}), \quad (C.20)
\]

\[
\lambda^{(0)}(t, c, \eta) = -it^{1/2} + \frac{c_\infty + c_0}{4} + \frac{i(c_\infty + c_0)(3c_\infty - c_0)}{32} t^{-1/2} + O(t^{-1}), \quad (C.21)
\]

\[
\mu^{(0)}(t, c, \eta) = 1 - \frac{i(c_\infty - c_0 + \eta^{-1})}{4} t^{-1/2} + O(t^{-3/2}), \quad (C.22)
\]

\[
R_{-1}(t, c) = -2it^{-1/2} - \frac{c_\infty - c_0 t^{-1} + i(3c_\infty + c_0)(c_\infty + 3c_0)}{64} t^{-3/2} + O(t^{-2}), \quad (C.23)
\]

\[
R_{\pm}(t, c, \eta) = \mp 2i \eta t^{-1/2} + \frac{(\mp c_\infty \eta \pm c_0 \eta + 1)}{4} t^{-1}
\]

\[
\mp i(\mp 3c_\infty^2 \eta^2 \mp 10c_\infty c_0 \eta^2 \mp 3c_0^2 \eta^2 \mp 10c_\infty \eta + 6c_0 \eta + 1)}{64 \eta} t^{-3/2} + O(t^{-2}). \quad (C.24)
\]

Next we summarize asymptotic behaviors when $t$ tends to double-poles.
The case \( t \to 0_{c_0,+} \):

\[
\lambda_0(t, c) = c_\infty - \frac{c_0}{c_\infty} t + \frac{c_\infty^2 - c_0^2}{c_\infty^2} c_\infty^2 t^2 + O(t^3), \tag{C.25}
\]

\[
\mu_0(t, c) = \frac{c_\infty + c_0}{2c_\infty} - \frac{c_\infty^2 - c_0^2}{2c_\infty^4} t - \frac{3c_\infty(c_\infty^2 - c_0^2)}{2c_\infty^7} t^2 + O(t^3), \tag{C.26}
\]

\[
\lambda^{(0)}(t, c, \eta) = c_\infty - \frac{c_0}{c_\infty - \eta^{-2}} t + \frac{c_\infty^4 - 2c_\infty^2 c_0^2 - 2c_\infty^2 \eta^{-2} + c_0^2 \eta^{-2} + \eta^{-4}}{c_\infty(c_\infty^2 - 4\eta^{-2})(c_\infty^2 - \eta^{-2})} t^2 + O(t^3), \tag{C.27}
\]

\[
\mu^{(0)}(t, c, \eta) = \frac{c_\infty + c_0 - \eta^{-1}}{2c_\infty} - \frac{c_\infty^2 - (c_0 - \eta^{-1})^2}{2c_\infty^2(c_\infty^2 - \eta^{-2})} t - \frac{3(c_\infty^2 c_0 - c_0^3 c_\infty \eta^{-1} + 3c_0^2 \eta^{-1} - 3c_0 \eta^{-2} + \eta^{-3})}{2c_\infty^3(c_\infty^2 - 4\eta^{-2})(c_\infty^2 - \eta^{-2})} t^2 + O(t^3), \tag{C.28}
\]

\[
R_{-1}(t, c) = c_\infty t^{-1} - \frac{2c_0}{c_\infty^3} + \frac{5c_\infty^2 - 9c_0^2}{2c_\infty^5} t + O(t^2), \tag{C.29}
\]

\[
R_{\pm}(t, c, \eta) = \pm c_\infty \eta t^{-1} + \frac{2c_0 \eta}{c_\infty^2 - \eta^{-2}} + \frac{r_{\pm}(c, \eta)}{2c_\infty^2(c_\infty - \eta^{-1})^3(c_\infty + \eta^{-1})^2(c_\infty^2 - 4\eta^{-2})} t + O(t^2). \tag{C.30}
\]

\[
r_{\pm}(c, \eta) = \pm \eta(-5c_\infty^6 + 9c_\infty^4 c_0^2) + (4c_\infty^5 - 6c_\infty^3 c_0^2) \pm \eta^{-1}(14c_\infty^4 + c_\infty^2 c_0^2) + \eta^{-2}(-8c_\infty^3 - 12c_\infty c_0^2) \pm \eta^{-3}(-13c_\infty^2 - 4c_0^2) + 4\eta^{-4} c_\infty \pm 4\eta^{-5}. \tag{C.31}
\]

The case \( t \to 0_{c_0,+} \):

\[
\lambda_0(t, c) = \frac{1}{c_0} t + \frac{c_\infty}{c_0^2} t^2 + \frac{3c_\infty^2 - c_0^2}{c_0^2} t^3 + O(t^4), \tag{C.31}
\]

\[
\mu_0(t, c) = \frac{c_\infty + c_0}{2c_0} + \frac{c_\infty^2 - c_0^2}{2c_0^4} t + O(t^2), \tag{C.32}
\]

\[
\lambda^{(0)}(t, c, \eta) = \frac{1}{c_0} t + \frac{c_\infty}{c_0^2(c_0^2 - \eta^{-2})} t^2 + \frac{3c_\infty^2 - c_0^2 + \eta^{-2}}{c_0^3(c_\infty^2 - 4\eta^{-2})(c_0^2 - \eta^{-2})} t^3 + O(t^4), \tag{C.33}
\]

\[
\mu^{(0)}(t, c, \eta) = \frac{c_\infty + c_0 - \eta^{-1}}{2(c_0 - \eta^{-1})} + \frac{c_\infty^2 - (c_0 - \eta^{-1})^2}{2c_0(c_0 - 2\eta^{-1})(c_0 - \eta^{-1})^2} t + O(t^2), \tag{C.34}
\]

\[
R_{-1}(t, c) = c_0 t^{-1} - \frac{2c_\infty}{c_0^2} + \frac{5c_\infty^2 - 9c_0^2}{2c_\infty^5} t + O(t^2), \tag{C.35}
\]

\[
R_{\pm}(t, c, \eta) = (\pm c_0 \eta + 1) t^{-1} + \frac{2c_\infty \eta}{c_0(c_0 + \eta^{-1})} + \frac{r_{\pm}(c, \eta)}{2c_\infty^2(c_\infty - \eta^{-1})^3(c_\infty + \eta^{-1})(c_\infty^2 - 2\eta^{-2})} t + O(t^2). \tag{C.36}
\]

\[
r_{\pm}(c, \eta) = \pm \eta(5c_\infty^4 - 9c_\infty^2 c_0^2) + (11c_\infty^3 - 13c_\infty c_0^2) \pm \eta^{-1}(-2c_\infty^2 + c_0^2) - 11\eta^{-2} c_\infty \pm 6\eta^{-3}. \tag{39.36}
\]
D The Voros coefficients of the third Painlevé equation of the type $D_7$

We also compute the Voros coefficients of the degenerate third Painlevé equation of the type $D_7$:

$$(P_{III})_{D_7}: \frac{d^2 \lambda}{dt^2} = \frac{1}{\lambda} \left( \frac{d \lambda}{dt} \right)^2 - \frac{1}{t} \frac{d \lambda}{dt} + \eta^2 \left( -\frac{2 \lambda^2}{t^2} + \frac{c}{t} - \frac{1}{\lambda} \right).$$

(D.1)

Here we assume that

$$c \neq 0.$$  

(D.2)

Let $F_{D_7}(\lambda, t, c)$ be the coefficient of $\eta^2$ in the right-hand side of $(P_{III})_{D_7}$, and $\lambda_0 = \lambda_0(t, c)$ be an algebraic function defined by

$$F_{D_7}(\lambda_0, t, c) = -\frac{2 \lambda_0^2}{t^2} + \frac{c}{t} - \frac{1}{\lambda_0} = 0.$$  

(D.3)

Turning points, simple-poles and Stokes curves of $(P_{III})_{D_7}$ are defined in the same way in Section 2.2 in terms of $\lambda_0$. Note that $\lambda_0$ has the following asymptotic behaviors

$$\lambda_0(t, c) = (-2)^{-1/3} \omega_j t^{2/3}(1 + O(t^{-1/3})) \quad \text{as} \quad t \to \infty \quad (j = 1, 2, 3),$$

(D.4)

where $\omega = e^{2\pi i/3}$, when $t$ tends to infinity, and

$$\lambda_0(t, c) = \pm (c/2)^{1/2} t^{1/2}(1 + t^{1/2}) \quad \text{as} \quad t \to \tau_{sp},$$

(D.5)

$$\lambda_0(t, c) = t/c + O(t^2) \quad \text{as} \quad t \to 0,$$  

(D.6)

when $t$ tends to 0. Here we used the symbol $\infty_j$ ($j = 1, 2, 3$), $\tau_{sp}$ (which corresponds to the simple-pole $u = 0$ of (D.8)) and $0_c$ (which corresponds to the double-pole $u = c$ of (D.8)) to distinguish the branch of $\lambda_0$. We consider the lift of them onto the Riemann surface of $\lambda_0$ by taking a new variable $u$ given by

$$u = 1/\mu_0, \quad \mu_0 = \frac{c \lambda_0 - t}{2 \lambda_0^2},$$

(D.7)

and the quadratic differential determining the Stokes geometry is written as

$$\partial_\lambda F_{D_7}(\lambda_0, t, c) dt^2 = \frac{(3u - 2c)^3}{u(u - c)^2} du^2.$$  

(D.8)

On the $u$-plane we have one turning point at $u = 2c/3$, and one simple-pole at $u = 0$ (which corresponds to (D.5)), and a double pole at $u = c$ (which corresponds to (D.6)). The following figures (Figure D.1 $\sim$ D.3) describe the $P$-Stokes geometries lifted on the $u$-plane near arg $c = \pi/2$. We can observe that Stokes geometry admit a loop-type degeneration when arg $c = \pi/2$. The loop turn around the double-pole $u = c$. 

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Here we define Voros coefficients for \((P_{III})_{D_7}\) as follows. Let
\[
\lambda^{(0)}(t, c, \eta) = \sum_{\ell \geq 0} \eta^{-\ell} \lambda^{(0)}_{\ell}(t, c)
\]
be a 0-parameter solution of \((P_{III})_{D_7}\), and
\[
R_{\text{odd}}(t, c, \eta) = \sum_{\ell \geq 0} \eta^{1-2\ell} R_{2\ell-1}(t, c)
\]
be the odd part of the formal power series solution \(R = R(t, c, \eta)\) of the following Riccati equation:
\[
R^2 + \frac{dR}{dt} = \left( \frac{2}{\lambda^{(0)}} \frac{d\lambda^{(0)}}{dt} - \frac{1}{t} \right) R + \eta^2 \left\{ \frac{\partial F_{D_7}}{\partial \lambda}(\lambda^{(0)}, t, c) - \eta^{-2} \left( \frac{1}{\lambda^{(0)}} \frac{d\lambda^{(0)}}{dt} \right)^2 \right\}. \tag{D.9}
\]
In order to fix the square root \(R^{-1}(t, c) = \sqrt{\partial_\lambda F_{D_7}(\lambda_0, t, c)}\) near the infinity and the double-pole, we use further symbol \(\infty_j, \pm (j = 1, 2, 3)\) and \(0_{c, \pm}\) such that the following holds:
\[
\lambda_0 = (-2)^{-1/3} \omega^2 t^{2/3} (1 + O(t^{-1/3})) , \quad R_{-1} = \pm \left\{ 6^{1/2}(-2)^{-1/6} \omega^{j/2} t^{1/3} (1 + O(t^{-1/3})) \right\} \quad \text{as } t \to \infty_{j, \pm} (j = 1, 2, 3),
\]
and
\[
\lambda_0 = t/c + O(t^2), \quad R_{-1} = \pm \left\{ ct^{-1}(1 + O(t)) \right\} \quad \text{as } t \to 0_{c, \pm}. \tag{D.11}
\]

**Definition D.1.** Let \(\tau\) be the turning point or the simple-pole of \((P_{III})_{D_7}\), and \(\ast = \infty_{j, \pm}\) or \(0_{c, \pm}\). For a path \(\Gamma(\tau, \ast)\) from \(\tau\) to \(\ast\) on the \(t\)-plane, the Voros coefficient of \((P_{III})_{D_7}\) for the path \(\Gamma(\tau, \ast)\) is defined by
\[
W_{\ast}(c, \eta) = \int_{\Gamma(\tau, \ast)} (R_{\text{odd}}(t, c, \eta) - \eta R_{-1}(t, c)) dt. \tag{D.12}
\]

Then we have the following list of Voros coefficients.

**Theorem D.1.** The Voros coefficients are represented explicitly as follows:
\[
W_{\infty_{j, \pm}}(c, \eta) = 0 \quad (j = 1, 2, 3), \tag{D.13}
\]
\[
W_{0_{c, \pm}}(c, \eta) = \mp 3 \mathcal{G}(c, \eta). \tag{D.14}
\]
Here \(\mathcal{G}(c, \eta)\) is the formal power series given by \(3.29\).
This theorem can be proved by the completely same manner used in Section 4. Here we only show the Bäcklund transformation for the Hamiltonian system \((H_{III})_{D_7}\) which is equivalent to \((P_{III})_{D_7}\).

\[
(H_{III})_{D_7} : \quad \frac{d\lambda}{dt} = \eta \frac{\partial H}{\partial \mu}, \quad \frac{d\mu}{dt} = -\eta \frac{\partial H}{\partial \lambda},
\]

\[
tH = \lambda^2 \mu^2 - (c - \eta^{-1}) \lambda \mu + t \mu + \lambda.
\]  

**Proposition D.1** (e.g., [OKSO]). Let \((\lambda, \mu)\) be a solution of \((H_{III})_{D_7}\). Then, 

\[(\Lambda, M) = (\Lambda(\lambda, \mu), M(\lambda, \mu))\]

defined by

\[
\begin{cases}
  \Lambda = -t\mu + \frac{ct}{\lambda} - \frac{t^2}{\lambda^2}, \\
  M = \frac{\lambda}{t},
\end{cases}
\]  

is a solution of the following equation:

\[
\frac{d\Lambda}{dt} = \eta \frac{\partial H}{\partial M}, \quad \frac{dM}{dt} = -\eta \frac{\partial H}{\partial \Lambda},
\]

where the Hamiltonian \(H\) is given by

\[
tH = \Lambda^2 M^2 - c\Lambda M + tM + \Lambda.
\]  

Using the Bäcklund transformation, we can confirm that \(W_{\infty j, \pm}(c, \eta)\) satisfies

\[
W_{\infty j, \pm}(c + \eta^{-1}, \eta) - W_{\infty j, \pm}(c, \eta) = 0,
\]

and \(\mp W_{0c, \pm}(c, \eta)/3\) satisfies \((4.4)\). Thus we can prove the Theorem D.1.

**Remark D.1.** In parallel with the discussion presented in Section 5.3, we can conclude that, if the independent variable \(t\) lies outside of the loop in Figure D.2, then the parametric Stokes phenomena never occur to transseries solutions of \((P_{III})_{D_7}\) because the Voros coefficients for \(\infty\) are trivial in this case. However, due to the same reason as the case of \((P_{III})_{D_6}\), connection formula when \(t\) lies inside the loop is remains to be analyzed.

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