New error bounds for Boole’s rule

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Abstract
In recent years, a lot of research was devoted to Simpson’s rule for numerical integration. In the paper we study a natural successor of Simpson’s rule, namely the Boole’s rule. It is the Newton-Cotes formula in the case where the interval of integration is divided into four subintervals of equal length. With computer software assistance, we prove novel error bounds for Boole’s rule.

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1. Introduction

In recent years, a significant amount of research has been devoted to Simpson’s rule for numerical integration:

\[
\int_a^b f(t) \, dt \sim \frac{h}{3} \cdot \left( f(a) + 4 f \left( \frac{a + b}{2} \right) + f(b) \right), \quad \text{where} \quad h = \frac{a + b}{2}.
\]

Among many formidable sources, let us mention just a few: \[1, 3, 4, 5, 6, 7, 8\]. It is the author’s impression that while Simpson’s rule enjoys considerable popularity, Boole’s rule remains somewhat neglected. The following paper is a modest attempt to change this status quo.

The main part of the paper is Section \[2\] in which we suggest six novel estimates for the Boole’s rule

\[
\int_a^b f(t) \, dt \sim \frac{2h}{45} \cdot \left( 7f(a) + 32f \left( \frac{3a + b}{4} \right) + 12f \left( \frac{a + b}{2} \right) + 32f \left( \frac{a + 3b}{4} \right) + 7f(b) \right),
\]

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where \( h = \frac{a+b}{4} \). For convenience, let us denote the right hand side of the Boole’s rule by \( B(f) \).

Since the calculations become quite tedious very fast, we frequently resort to Maple computer software, which greatly facilitates computations. We took the liberty of enclosing some parts of the code in case the Readers wanted to verify the constants on their own.

Last but not least, in Section 3 we test the performance of the novel estimates. It turns out that for monomials \( t \mapsto t^k \), (apart from first few instances), the new error bounds are better than the classical, well-know result.

2. Main results

Before we proceed with the main results of the paper, let us denote

\[
I(g) = \frac{g(b) - g(a)}{b-a}.
\]

Furthermore, let \( M(g) \) denote the constant \( M \) such that

\[
\forall \text{ a.e. } t \in [a,b] \quad g(t) \leq M,
\]

and similarly, let \( m(g) \) denote the constant \( m \) such that

\[
\forall \text{ a.e. } t \in [a,b] \quad m \leq g(t).
\]

**Theorem 1.** Let \( f : [a, b] \to \mathbb{R} \) be an absolutely continuous function such that \( m(f') \) and \( M(f') \) are both finite. Then

\[
\left| \int_a^b f(t) \, dt - B(f) \right| \leq \frac{11}{60} \cdot \left( I(f) - m(f') \right) (b-a)^2 \quad (1)
\]

and

\[
\left| \int_a^b f(t) \, dt - B(f) \right| \leq \frac{11}{60} \cdot \left( M(f') - I(f) \right) (b-a)^2 \quad (2)
\]

**Proof.** Let us define

\[
K_i(t) = \alpha_i t + \beta_i, \quad \text{for} \quad i = 1, 2, 3, 4,
\]
where $\alpha_i$, $\beta_i$ are constants which will be determined in the course of the proof. We put

$$K(t) = \begin{cases} K_1(t) & \text{if } t \in [a, \frac{3a+b}{4}], \\ K_2(t) & \text{if } t \in \left(\frac{3a+b}{4}, \frac{a+b}{2}\right], \\ K_3(t) & \text{if } t \in \left(\frac{a+b}{2}, \frac{a+3b}{4}\right], \\ K_4(t) & \text{if } t \in \left(\frac{a+3b}{4}, \frac{a+3b}{4}\right]. \end{cases}$$

Integration by parts yields

$$\int_a^b K(t) f'(t) \, dt = -K_1(a) f(a) + \left(K_1 \left(\frac{3a+b}{4}\right) - K_2 \left(\frac{3a+b}{4}\right) \right) f \left(\frac{3a+b}{4}\right) + \left(K_2 \left(\frac{a+b}{2}\right) - K_3 \left(\frac{a+b}{2}\right) \right) f \left(\frac{a+b}{2}\right) + \left(K_3 \left(\frac{a+3b}{4}\right) - K_4 \left(\frac{a+3b}{4}\right) \right) f \left(\frac{a+3b}{4}\right) + K_4(b) f(b) - \int_a^b K'(t) f(t) \, dt.$$

We demand that $K'(t) \equiv 1$, so

$$\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 1.$$  
Furthermore, we impose the following conditions:

$$\begin{cases} K_1(a) = -\frac{7}{90} \cdot (b - a), \\ K_1 \left(\frac{3a+b}{4}\right) - K_2 \left(\frac{3a+b}{4}\right) = \frac{32}{90} \cdot (b - a), \\ K_2 \left(\frac{a+b}{2}\right) - K_3 \left(\frac{a+b}{2}\right) = \frac{12}{90} \cdot (b - a), \\ K_3 \left(\frac{a+3b}{4}\right) - K_4 \left(\frac{a+3b}{4}\right) = \frac{32}{90} \cdot (b - a), \\ K_4(b) = \frac{7}{90} \cdot (b - a). \end{cases}$$

The above system is equivalent to

$$\begin{cases} a + \beta_1 = -\frac{7}{90} \cdot (b - a), \\ \beta_1 - \beta_2 = \frac{32}{90} \cdot (b - a), \\ \beta_2 - \beta_3 = \frac{12}{90} \cdot (b - a), \\ \beta_3 - \beta_4 = \frac{32}{90} \cdot (b - a), \\ b + \beta_4 = \frac{7}{90} \cdot (b - a). \end{cases}$$

The following Maple code:
beta1 := simplify(-7/90*(b-a)-a);
beta2 := simplify(beta1 - 32/90*(b-a));
beta3 := simplify(beta2-12/90*(b-a));
beta4 := simplify(beta3 - 32/90*(b-a));

returns the solution:

\[
\begin{align*}
\beta_1 &= -\frac{83a+7b}{90}, \\
\beta_2 &= -\frac{17a+13b}{30}, \\
\beta_3 &= -\frac{30a+17b}{90}, \\
\beta_4 &= -\frac{7a+83b}{90}.
\end{align*}
\]

(4)

By the above choice of \(\alpha_i, \beta_i\) and the fact that \(K'(t) \equiv 1\), we have

\[
\int_a^b K(t)f'(t) \, dt = \mathcal{B}(f) - \int_a^b f(t) \, dt.
\]

With the help of Maple software we check that \(\int_a^b K(t) \, dt = 0\):

K1 := t-> alpha1*t + beta1;
K2 := t-> alpha2*t + beta2;
K3 := t-> alpha3*t + beta3;
K4 := t-> alpha4*t + beta4;
simplify(integrate(K1(t),t=a..(3*a+b)/4)
+ integrate(K2(t),t=(3*a+b)/4..(a+b)/2)
+ integrate(K3(t),t=(a+b)/2..(a+3*b)/4)
+ integrate(K4(t),t=(a+3*b)/4..b));

We have

\[
\left| \mathcal{B}(f) - \int_a^b f(t) \, dt \right| \leq \left| \int_a^b K(t)f'(t) \, dt \right| = \left| \int_a^b K(t)f'(t) - m(f') \, dt \right| \\
\leq \sup_{t \in [a,b]} |K(t)| \cdot \int_a^b |f'(t) - m(f')| \, dt = \sup_{t \in [a,b]} |K(t)| \cdot \left( I(f) - m(f') \right) (b - a).
\]

(5)

It remains to estimate \(\sup_{t \in [a,b]} |K(t)|\). Since every function \(|K_i|, i = 1, 2, 3, 4\) is convex, when searching for the maximal value it suffices to check
the endpoints of the subintervals. We obtain

\[
\begin{align*}
\sup_{t \in [a, \frac{3a+b}{4}]} |K_1(t)| &= \frac{31}{180} \cdot (b-a) = \sup_{t \in [\frac{3a+b}{4}, b]} |K_4(t)|, \\
\sup_{t \in [\frac{3a+b}{4}, \frac{a+b}{2}]} |K_2(t)| &= \frac{11}{60} \cdot (b-a) = \sup_{t \in [\frac{a+b}{2}, \frac{a+3b}{4}]} |K_3(t)|,
\end{align*}
\]

which may be verified with the Maple code:

\[
\begin{align*}
simplify(\max(abs(K_1(a)),abs(K_1((3a+b)/4)))); \\
simplify(\max(abs(K_2((3a+b)/4)),abs(K_2((a+b)/2)))); \\
simplify(\max(abs(K_3((a+b)/2)),abs(K_3((a+3b)/4)))); \\
simplify(\max(abs(K_4((a+3b)/4)),abs(K_4(b))));
\end{align*}
\]

We conclude that

\[
\sup_{t \in [a, b]} |K(t)| = \frac{11}{60} \cdot (b-a),
\]

which, due to (5), proves (1). Inequality (2) is proven analogously. \(\square\)

**Theorem 2.** Let \(f : [a, b] \to \mathbb{R}\) be such that \(f'\) is absolutely continuous and \(m(f'')\) and \(M(f'')\) are both finite. Then

\[
\begin{align*}
\left| \int_a^b f(t) \, dt - B(f) \right| &\leq \frac{17}{1440} \cdot \left( I(f') - m(f'') \right) (b-a)^3 \quad (6) \\
\text{and} \\
\left| \int_a^b f(t) \, dt - B(f) \right| &\leq \frac{17}{1440} \cdot \left( M(f'') - I(f') \right) (b-a)^3. \quad (7)
\end{align*}
\]

**Proof.** Let us define

\[
K_i(t) = \alpha_i t^2 + \beta_i t + \gamma_i, \quad \text{for } i = 1, 2, 3, 4,
\]

where \(\alpha_i, \beta_i, \gamma_i\) are constants which will be determined in the course of the proof. Again, we put

\[
K(t) = \begin{cases} 
K_1(t) & \text{if } t \in [a, \frac{3a+b}{4}], \\
K_2(t) & \text{if } t \in \left(\frac{3a+b}{4}, \frac{a+b}{2}\right), \\
K_3(t) & \text{if } t \in \left(\frac{a+b}{2}, \frac{a+3b}{4}\right), \\
K_4(t) & \text{if } t \in \left(\frac{a+3b}{4}, b\right). 
\end{cases}
\]

5
Integration by parts yields
\[ \int_a^b K(t)f''(t) \, dt = C_1 + C_2 + \int_a^b K''(t)f(t) \, dt, \]
where
\[ C_1 = -K_1(a)f'(a) + \left( K_1 \left( \frac{3a + b}{4} \right) - K_2 \left( \frac{3a + b}{4} \right) \right) f' \left( \frac{3a + b}{4} \right) \]
\[ + \left( K_2 \left( \frac{a + b}{2} \right) - K_3 \left( \frac{a + b}{2} \right) \right) f' \left( \frac{a + b}{2} \right) \]
\[ + \left( K_3 \left( \frac{a + 3b}{4} \right) - K_4 \left( \frac{a + 3b}{4} \right) \right) f' \left( \frac{a + 3b}{4} \right) + K_4(b)f'(b), \]
and
\[ C_2 = K_1'(a)f(a) - \left( K_1' \left( \frac{3a + b}{4} \right) - K_2' \left( \frac{3a + b}{4} \right) \right) f \left( \frac{3a + b}{4} \right) \]
\[ - \left( K_2' \left( \frac{a + b}{2} \right) - K_3' \left( \frac{a + b}{2} \right) \right) f \left( \frac{a + b}{2} \right) \]
\[ - \left( K_3' \left( \frac{a + 3b}{4} \right) - K_4' \left( \frac{a + 3b}{4} \right) \right) f \left( \frac{a + 3b}{4} \right) - K_4'(b)f(b). \]

We demand that \( K''(t) \equiv 1 \), so
\[ \alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = \frac{1}{2}.\]
Furthermore, we impose the following conditions
\[
\begin{align*}
K_1'(a) &= -\frac{7}{90} \cdot (b - a), \\
K_1' \left( \frac{3a + b}{4} \right) - K_2' \left( \frac{3a + b}{4} \right) &= \frac{32}{90} \cdot (b - a), \\
K_2' \left( \frac{a + b}{2} \right) - K_3' \left( \frac{a + b}{2} \right) &= \frac{12}{90} \cdot (b - a), \\
K_3' \left( \frac{a + 3b}{4} \right) - K_4' \left( \frac{a + 3b}{4} \right) &= \frac{32}{90} \cdot (b - a), \\
K_4'(b) &= \frac{7}{90} \cdot (b - a),
\end{align*}
\]
which are equivalent to the system \( \text{[3]} \). We already know that the solution to this system is given by \( \text{[4]} \).
Last but not least, we require that

\[
\begin{cases}
K_1(a) = 0, \\
K_1 \left( \frac{3a+b}{4} \right) - K_2 \left( \frac{3a+b}{4} \right) = 0, \\
K_2 \left( \frac{a+b}{2} \right) - K_3 \left( \frac{a+b}{2} \right) = 0, \\
K_3 \left( \frac{a+3b}{4} \right) - K_4 \left( \frac{a+3b}{4} \right) = 0, \\
K_4(b) = 0.
\end{cases}
\]

This system is equivalent to:

\[
\begin{cases}
\alpha_1 a^2 + \beta_1 a + \gamma_1 = 0, \\
(\beta_1 - \beta_2) \cdot \frac{3a+b}{4} + \gamma_1 - \gamma_2 = 0, \\
(\beta_2 - \beta_3) \cdot \frac{a+b}{2} + \gamma_2 - \gamma_3 = 0, \\
(\beta_3 - \beta_4) \cdot \frac{a+3b}{4} + \gamma_3 - \gamma_4 = 0, \\
\alpha_4 b^2 + \beta_4 b + \gamma_4 = 0.
\end{cases}
\]

The following Maple code:

\[
\begin{align*}
\text{gamma1} & := -\text{simplify(}\alpha_1 a^2+\beta_1 a)\text{;} \\
\text{gamma2} & := \text{simplify(} (\beta_1-\beta_2) (3a+b)/4+\gamma_1)\text{;} \\
\text{gamma3} & := \text{simplify(} (\beta_2-\beta_3) (a+b)/2+\gamma_2)\text{;} \\
\text{gamma4} & := \text{simplify(} (\beta_3-\beta_4) (a+3b)/4+\gamma_3)\text{;}
\end{align*}
\]

returns the solution:

\[
\begin{cases}
\gamma_1 = \frac{19}{45} \cdot a^2 + \frac{7}{90} \cdot ab, \\
\gamma_2 = \frac{7}{45} \cdot a^2 + \frac{23}{90} \cdot ab + \frac{4}{45} \cdot b^2, \\
\gamma_3 = \frac{b(7a+38b)}{90}, \\
\gamma_4 = \frac{b(7a+38b)}{90}.
\end{cases}
\]

By the above choice of \(\alpha_i, \beta_i, \gamma_i\) and the fact that \(K''(t) \equiv 1\), we have

\[
\int_a^b K(t)f''(t) \, dt = B(f) - \int_a^b f(t) \, dt.
\]

With the help of Maple software we check that \(\int_a^b K(t) \, dt = 0\):

\[
\begin{align*}
K_1 & := t\rightarrow \alpha_1 t^2 + \beta_1 t + \gamma_1; \\
K_2 & := t\rightarrow \alpha_2 t^2 + \beta_2 t + \gamma_2; \\
K_3 & := t\rightarrow \alpha_3 t^2 + \beta_3 t + \gamma_3; \\
K_4 & := t\rightarrow \alpha_4 t^2 + \beta_4 t + \gamma_4;
\end{align*}
\]
\begin{align*}
simplify(&\int_a^b f(t) \, dt) \\
&\leq \left| \int_a^b K(t) f''(t) \, dt \right| = \left| \int_a^b K(t) \left( f''(t) - m(f'') \right) \, dt \right| \\
&\leq \sup_{t \in [a,b]} |K(t)| \cdot \int_a^b |f''(t) - m(f'')| \, dt = \sup_{t \in [a,b]} |K(t)| \cdot \left( I(f') - m(f'') \right) (b - a). \\
\end{align*}

(10)

It remains to estimate \( \sup_{t \in [a,b]} |K(t)| \). It is easy to see that the critical points of \( K_i \) are \( -\beta_i \) respectively. Furthermore, we have

\[
a \leq -\beta_1 \leq \frac{3a + b}{4},
\]

\[
\frac{3a + b}{4} \leq -\beta_2 \leq \frac{a + b}{2},
\]

\[
\frac{a + b}{2} \leq -\beta_3 \leq \frac{a + 3b}{4},
\]

\[
\frac{a + 3b}{4} \leq -\beta_4 \leq b.
\]

Hence, we have

\[
\sup_{t \in \left[ a, \frac{3a + b}{4} \right]} |K_1(t)| = \sup_{t \in \left[ \frac{3a + b}{4}, \frac{a + b}{2} \right]} |K_2(t)| = \sup_{t \in \left[ \frac{a + b}{2}, \frac{a + 3b}{4} \right]} |K_3(t)| = \sup_{t \in \left[ \frac{a + 3b}{4}, b \right]} |K_4(t)| = \frac{17}{1440} \cdot (b - a)^2,
\]

which can be verified with the following Maple code:

\begin{verbatim}
simplify(max(abs(K1(a)),abs(K1(-beta1)),abs(K1((3*a+b)/4))));
simplify(max(abs(K2((3*a+b)/4)),abs(K2(-beta2)),abs(K2((a+b)/2))));
simplify(max(abs(K3((a+b)/2)),abs(K3(-beta3)),abs(K3((a+3*b)/4))));
simplify(max(abs(K4((a+3*b)/4)),abs(K4(-beta4)),abs(K4(b))));
\end{verbatim}

We conclude that

\[
\sup_{t \in [a,b]} |K(t)| = \frac{17}{1440} \cdot (b - a)^2,
\]

which, due to (10) proves (6). Inequality (7) is proven analogously. \( \square \)
Theorem 3. Let \( f : [a, b] \rightarrow \mathbb{R} \) be such that \( f'' \) is absolutely continuous and \( m(f''') \) and \( M(f''') \) are both finite. Then

\[
\left| \int_a^b f(t) \, dt - B(f) \right| \leq \frac{1}{1620} \cdot \left( I(f'') - m(f''') \right) (b-a)^4 \quad (11)
\]

and

\[
\left| \int_a^b f(t) \, dt - B(f) \right| \leq \frac{1}{1620} \cdot \left( M(f''') - I(f'') \right) (b-a)^4. \quad (12)
\]

Proof. Let us define

\[
K_i(t) = \alpha_i t^3 + \beta_i t^2 + \gamma_i t + \delta_i, \quad \text{for } i = 1, 2, 3, 4,
\]

where \( \alpha_i, \beta_i, \gamma_i, \delta_i \) are constants which will be determined in the course of the proof. Again, we put

\[
K(t) = \begin{cases} 
K_1(t) & \text{if } t \in [a, \frac{3a+b}{4}], \\
K_2(t) & \text{if } t \in \left(\frac{3a+b}{4}, \frac{a+b}{2}\right], \\
K_3(t) & \text{if } t \in \left(\frac{a+b}{2}, \frac{a+3b}{4}\right], \\
K_4(t) & \text{if } t \in \left(\frac{a+3b}{4}, b\right].
\end{cases}
\]

Integration by parts yields

\[
\int_a^b K(t) f'''(t) \, dt = C_1 + C_2 + C_3 - \int_a^b K'''(t) f(t) \, dt,
\]

where

\[
C_1 = -K_1(a) f''(a) + \left( K_1 \left( \frac{3a+b}{4} \right) - K_2 \left( \frac{3a+b}{4} \right) \right) f'' \left( \frac{3a+b}{4} \right) \\
\quad + \left( K_2 \left( \frac{a+b}{2} \right) - K_3 \left( \frac{a+b}{2} \right) \right) f'' \left( \frac{a+b}{2} \right) \\
\quad + \left( K_3 \left( \frac{a+3b}{4} \right) - K_4 \left( \frac{a+3b}{4} \right) \right) f'' \left( \frac{a+3b}{4} \right) + K_4(b) f''(b),
\]

\[
C_2 = K_1'(a) f'(a) - \left( K_1' \left( \frac{3a+b}{4} \right) - K_2' \left( \frac{3a+b}{4} \right) \right) f' \left( \frac{3a+b}{4} \right) \\
\quad - \left( K_2' \left( \frac{a+b}{2} \right) - K_3' \left( \frac{a+b}{2} \right) \right) f' \left( \frac{a+b}{2} \right) \\
\quad - \left( K_3' \left( \frac{a+3b}{4} \right) - K_4' \left( \frac{a+3b}{4} \right) \right) f' \left( \frac{a+3b}{4} \right) - K_4'(b) f'(b),
\]

and

\[
C_3 = \int_a^b K(t) f'''(t) \, dt.
\]
and

\[ C_3 = -K''_1(a)f(a) + \left( K''_1 \left( \frac{3a+b}{4} \right) - K''_2 \left( \frac{3a+b}{4} \right) \right) f \left( \frac{3a+b}{4} \right) \]
\[ + \left( K''_2 \left( \frac{a+b}{2} \right) - K''_3 \left( \frac{a+b}{2} \right) \right) f \left( \frac{a+b}{2} \right) \]
\[ + \left( K''_3 \left( \frac{a+3b}{4} \right) - K''_4 \left( \frac{a+3b}{4} \right) \right) f \left( \frac{a+3b}{4} \right) + K''_4(b)f(b). \]

We demand that \( K''(t) \equiv 1 \), so

\[ \alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = \frac{1}{6}. \]

Furthermore, we impose the following conditions

\[
\begin{align*}
K''_1(a) &= -\frac{7}{90} \cdot (b - a), \\
K''_1 \left( \frac{3a+b}{4} \right) - K''_2 \left( \frac{3a+b}{4} \right) &= \frac{32}{90} \cdot (b - a), \\
K''_2 \left( \frac{a+b}{2} \right) - K''_3 \left( \frac{a+b}{2} \right) &= \frac{12}{90} \cdot (b - a), \\
K''_3 \left( \frac{a+3b}{4} \right) - K''_4 \left( \frac{a+3b}{4} \right) &= \frac{32}{90} \cdot (b - a), \\
K''_4(b) &= \frac{7}{90} \cdot (b - a),
\end{align*}
\]

which are equivalent to

\[
\begin{align*}
a + 2\beta_1 &= -\frac{7}{90} \cdot (b - a), \\
2(\beta_1 - \beta_2) &= \frac{32}{90} \cdot (b - a), \\
2(\beta_2 - \beta_3) &= \frac{12}{90} \cdot (b - a), \\
2(\beta_3 - \beta_4) &= \frac{32}{90} \cdot (b - a), \\
b + 2\beta_4 &= \frac{7}{90} \cdot (b - a).
\end{align*}
\]

The following Maple code:

\[
\begin{align*}
\text{beta1} &:= \text{simplify((-a-7/90*(b-a))/2)}; \\
\text{beta2} &:= \text{simplify(beta1-16/90*(b-a))}; \\
\text{beta3} &:= \text{simplify(beta2-6/90*(b-a))}; \\
\text{beta4} &:= \text{simplify(beta3-16/90*(b-a))};
\end{align*}
\]

returns the solution:

\[
\begin{align*}
\beta_1 &= \frac{-83a+7b}{180}, \\
\beta_2 &= \frac{-13a+17b}{180}, \\
\beta_3 &= \frac{69}{60}, \\
\beta_4 &= \frac{-33a+3b}{180}.
\end{align*}
\]
What is more, we demand that

\[
\begin{align*}
K_1'(a) &= 0, \\
K_1' \left(\frac{3a+b}{4}\right) - K_2' \left(\frac{3a+b}{4}\right) &= 0, \\
K_2' \left(\frac{a+b}{2}\right) - K_3' \left(\frac{a+b}{2}\right) &= 0, \\
K_3' \left(\frac{a+3b}{4}\right) - K_4' \left(\frac{a+3b}{4}\right) &= 0, \\
K_4'(b) &= 0.
\end{align*}
\]

This system is equivalent to:

\[
\begin{align*}
3\alpha_1 a^2 + 2\beta_1 a + \gamma_1 &= 0, \\
(\beta_1 - \beta_2) \cdot \frac{3a+b}{2} + \gamma_1 - \gamma_2 &= 0, \\
(\beta_2 - \beta_3) \cdot (a + b) + \gamma_2 - \gamma_3 &= 0, \\
(\beta_3 - \beta_4) \cdot \frac{a+3b}{2} + \gamma_3 - \gamma_4 &= 0, \\
3\alpha_4 b^2 + 2\beta_4 b + \gamma_4 &= 0.
\end{align*}
\]

The following Maple code:

\[
\begin{align*}
gamma_1 &= \text{simplify}(-3*\alpha_1*a^2-2*\beta_1*a); \\
gamma_2 &= \text{simplify}((3*a+b)/2*(\beta_1-\beta_2) + \gamma_1); \\
gamma_3 &= \text{simplify}((a+b)*(\beta_2-\beta_3) + \gamma_2); \\
gamma_4 &= \text{simplify}((a+3*b)/2*(\beta_3-\beta_4) + \gamma_3); \\
gamma_1 &= \text{simplify}((a+3*b)/2*(\beta_3-\beta_4) + \gamma_3);
\end{align*}
\]

returns the solution:

\[
\begin{align*}
\gamma_1 &= \frac{19}{45} \cdot a^2 + \frac{7}{15} \cdot ab, \\
\gamma_2 &= \frac{4}{15} \cdot a^2 + \frac{9}{90} \cdot ab + \frac{4}{45} \cdot b^2, \\
\gamma_3 &= \frac{b(7a+35b)}{90}, \\
\gamma_4 &= \frac{90}{90}.
\end{align*}
\]

Finally, we require that

\[
\begin{align*}
K_1'(a) &= 0, \\
K_1' \left(\frac{3a+b}{4}\right) - K_2' \left(\frac{3a+b}{4}\right) &= 0, \\
K_2' \left(\frac{a+b}{2}\right) - K_3' \left(\frac{a+b}{2}\right) &= 0, \\
K_3' \left(\frac{a+3b}{4}\right) - K_4' \left(\frac{a+3b}{4}\right) &= 0, \\
K_4'(b) &= 0.
\end{align*}
\]
This system is equivalent to:

\[
\begin{align*}
\alpha_1 a^3 + \beta_1 a^2 + \gamma_1 a + \delta_1 &= 0, \\
(\beta_1 - \beta_2) \cdot \left( \frac{3a+b}{4} \right)^2 + (\gamma_1 - \gamma_2) \cdot \frac{3a+b}{4} + \delta_1 - \delta_2 &= 0, \\
(\beta_2 - \beta_3) \cdot \left( \frac{a+b}{2} \right)^2 + (\gamma_2 - \gamma_3) \cdot \frac{a+b}{2} + \delta_2 - \delta_3 &= 0, \\
(\beta_3 - \beta_4) \cdot \left( \frac{a+3b}{4} \right)^2 + (\gamma_3 - \gamma_4) \cdot \frac{a+3b}{4} + \delta_3 - \delta_4 &= 0, \\
\alpha_4 b^3 + \beta_4 b^2 + \gamma_4 b + \delta_4 &= 0.
\end{align*}
\]

The following Maple code:

```maple
delta1:= -simplify(alpha1*a^3+beta1*a^2+gamma1*a);
delta2:= simplify((beta1-beta2)*((3*a+b)/4)^2 + (gamma1-gamma2)*((3*a+b)/4) + delta1);
delta3:= simplify((beta2-beta3)*((a+b)/2)^2 + (gamma2-gamma3)*((a+b)/2) + delta2);
delta4:= simplify((beta3-beta4)*((a+3*b)/4)^2 + (gamma3-gamma4)*((a+3*b)/4) + delta3);
```

returns the solution:

\[
\begin{align*}
\delta_1 &= -\frac{a^2(23a+7b)}{180}, \\
\delta_2 &= -\left( \frac{1}{36} \cdot a^3 + \frac{13}{180} \cdot a^2 b + \frac{1}{18} \cdot ab^2 + \frac{1}{90} \cdot b^3 \right), \\
\delta_3 &= -\left( \frac{2}{90} \cdot a^3 + \frac{7}{18} \cdot a^2 b + \frac{1}{36} \cdot ab^2 + \frac{1}{90} \cdot b^3 \right), \\
\delta_4 &= -\frac{b^2(7a+23b)}{180}.
\end{align*}
\]

By the above choice of \( \alpha_i, \beta_i, \gamma_i, \delta_i \) and the fact that \( K'''(t) \equiv 1 \), we have

\[
\int_a^b K(t)f'''(t) \, dt = B(f) - \int_a^b f(t) \, dt.
\]

With the help of Maple software we check that \( \int_a^b K(t) \, dt = 0 \):

```maple
K1 := t-> alpha1*t^3 + beta1*t^2 + gamma1*t + delta1;
K2 := t-> alpha2*t^3 + beta2*t^2 + gamma2*t + delta2;
K3 := t-> alpha3*t^3 + beta3*t^2 + gamma3*t + delta3;
K4 := t-> alpha4*t^3 + beta4*t^2 + gamma4*t + delta4;
simplify(integrate(K1(t),t=a..(3*a+b)/4) + integrate(K2(t),t=(3*a+b)/4..(a+b)/2) + integrate(K3(t),t=(a+b)/2..(a+3*b)/4) + integrate(K4(t),t=(a+3*b)/4..b));
```
We have
\[
|B(f) - \int_a^b f(t) \, dt| \leq \left| \int_a^b K(t) f'''(t) \, dt \right| = \left| \int_a^b K(t) \left( f'''(t) - m f''' \right) \, dt \right|
\]
\[
\leq \sup_{t \in [a,b]} |K(t)| \cdot \left| \int_a^b f'''(t) - m(f''') \, dt \right| = \sup_{t \in [a,b]} |K(t)| \cdot \left( I(f''' - m f''') \right) (b - a).
\]

(13)

It remains to estimate \( \sup_{t \in [a,b]} |K(t)| \). At first we calculate critical points for \( K_i \):

| Function | Critical points |
|----------|-----------------|
| \( K_1 \) | \( a, \frac{38a+7b}{45} \) |
| \( K_2 \) | \( \frac{a+3b}{4}, \frac{2a+b}{4} \) |
| \( K_3 \) | \( \frac{a+2b}{3}, \frac{8a+7b}{15} \) |
| \( K_4 \) | \( \frac{a+3b}{4}, b \) |

The above values are obtained with the Maple code:

```maple
solve(D(K1)(t)=0,t);
solve(D(K2)(t)=0,t);
solve(D(K3)(t)=0,t);
solve(D(K4)(t)=0,t);
```

However, we observe that
\[
\frac{7a + 8b}{15} \notin \left[ \frac{3a + b}{4}, \frac{a + b}{2} \right],
\]
\[
\frac{8a + 7b}{15} \notin \left[ \frac{a + b}{2}, \frac{a + 3b}{4} \right].
\]

The Maple code

```maple
t1:=(38*a+7*b)/45;
t2:=(2*a+b)/3;
t3:=(a+2*b)/3;
t4:=(7*a+38*b)/45;
simplify(max(abs(K1(a)),abs(K1(t1)),abs(K1((3*a+b)/4))));
simplify(max(abs(K2((3*a+b)/4)),abs(K2(t2)),abs(K2((a+b)/2))));
simplify(max(abs(K3((a+b)/2)),abs(K3(t3)),abs(K3((a+3*b)/4))));
simplify(max(abs(K4((a+3*b)/4)),abs(K4(t4)),abs(K4(b))));
```

13
returns
\[
\sup_{t \in [a, \frac{3a+b}{4}]} |K_1(t)| = \frac{343}{1093500} \cdot (b-a)^3 = \sup_{t \in [\frac{a+b}{2}, b]} |K_4(t)|,
\]
\[
\sup_{t \in [\frac{3a+b}{4}, a+b]} |K_2(t)| = \frac{1}{1620} \cdot (b-a)^3 = \sup_{t \in [\frac{a+b}{2}, \frac{3a+b}{4}]} |K_3(t)|,
\]
We conclude that
\[
\sup_{t \in [a, b]} |K(t)| = \frac{1}{1620} \cdot (b-a)^3,
\]
which, due to (13) proves (11). Inequality (12) is proven analogously.

3. Examples

Let us consider a monomial \( f(t) = t^k \), where the power \( k \) is treated as a parameter. Let \( a = 0 \) and \( b > 0 \). The classical error estimate (comp. [2], p. 266) for Boole’s rule is
\[
8 \cdot 945 \cdot h^7 \| f^{(6)} \|,
\]
which in our case turns out to be
\[
\frac{b^{k+1}}{1935360} \cdot k(k-1)(k-2)(k-3)(k-4)(k-5).
\]
(14)

Furthermore, we have
\[
I(f) = 1, \quad I(f') = k, \quad I(f'') = k(k-1), \quad m(f') = m(f'') = m(f'''') = 0,
\]
and
\[
M(f') = kb^{k-1}, \quad M(f'') = k(k-1)b^{k-2}, \quad M(f'''') = k(k-1)(k-2)b^{k-3}.
\]
In the table below, we summarize the results. The second column is the error bound of the estimate, i.e. the term on the right-hand side of the inequality. The last column indicates for which \( k \) does the novel estimate perform better than (14).
Estimate | Error bound | Better than (14) for $k \geq$  
--- | --- | ---  
(1) | $\frac{11}{6} \cdot b^{k+1}$ | 15  
(2) | $\frac{11}{6} \cdot (k-1)b^{k+1}$ | 24  
(3) | $\frac{11}{140} \cdot kb^{k+1}$ | 11  
(4) | $\frac{14}{1440} \cdot k(k-2)b^{k+1}$ | 16  
(5) | $\frac{1}{1620} \cdot k(k-1)b^{k+1}$ | 10  
(6) | $\frac{1}{1620} \cdot k(k-1)(k-3)b^{k+1}$ | 15

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