Global strong solution for 3D compressible heat-conducting magnetohydrodynamic equations revisited

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Abstract

We revisit the 3D Cauchy problem of compressible heat-conducting magnetohydrodynamic equations with vacuum as far field density. By delicate energy method, we derive global existence and uniqueness of strong solutions provided that

\[
\|\rho_0\|_{L^\infty} + 1 + \|\nabla u_0\|_{L^2}^2 + \|\nabla b_0\|_{L^2}^2
\]

is properly small. In particular, the smallness condition is independent of any norms of the initial data. This work improves our previous results [1, 2].

Key words and phrases. Compressible heat-conducting magnetohydrodynamic equations; global strong solution; Cauchy problem; vacuum.

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1 Introduction

Let $\Omega \subseteq \mathbb{R}^3$ be a domain, the motion of a viscous, compressible, and heat conducting magnetohydrodynamic (MHD) flow in $\Omega$ can be described by full compressible MHD equations (see [13, Chapter 3]):

\[
\begin{align*}
\rho_t + \text{div}(\rho u) &= 0, \\
\rho u_t + \rho u \cdot \nabla u - \mu \Delta u - (\lambda + \mu) \nabla \text{div} u + \nabla p &= \text{curl} b \times b, \\
c_v \rho (\theta_t + u \cdot \nabla \theta) + p \text{div} u - \kappa \Delta \theta &= \mathcal{Q}(|\nabla u|) + \nu |\text{curl} b|^2, \\
b_t - b \cdot \nabla u + u \cdot \nabla b + b \text{div} u &= \nu \Delta b, \\
\text{div} b &= 0,
\end{align*}
\]  

(1.1)

where the unknowns $\rho \geq 0$, $u \in \mathbb{R}^3$, $\theta \geq 0$, and $b \in \mathbb{R}^3$ are the density, velocity, pressure, absolute temperature, and magnetic field, respectively; $p = R \rho \theta$, with positive constant $R$, is the pressure, and

\[
\mathcal{Q}(\nabla u) = \frac{\mu}{2} |\nabla u + (\nabla u)^T|^2 + \lambda (\text{div} u)^2,
\]

(1.2)

with $(\nabla u)^T$ being the transpose of $\nabla u$. The constant viscosity coefficients $\mu$ and $\lambda$ satisfy the physical restrictions

\[
\mu > 0, \quad 2\mu + 3\lambda \geq 0.
\]

(1.3)

Positive constants $c_v$, $\kappa$, and $\nu$ are the heat capacity, the ratio of the heat conductivity coefficient over the heat capacity, and the magnetic diffusive coefficient, respectively.

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In this paper, we will consider the Cauchy problem for (1.1) in $\mathbb{R}^3 \times (0, T)$ with the initial condition
\[(\rho, \rho u, \rho \theta, b)(x, 0) = (\rho_0, \rho_0 u_0, \rho_0 \theta_0, b_0)(x), \quad x \in \mathbb{R}^3, \quad (1.4)\]
and the far field behavior
\[(\rho, u, \theta, b)(x, t) \to (0, 0, 0, 0) \text{ as } |x| \to \infty. \quad (1.5)\]

It should be noted that the system (1.1) becomes the compressible non-isentropic Navier-Stokes equations when there is no electromagnetic field, which is one of the most important systems in fluid dynamics. With the assumption that the initial data differ only slightly from the equilibrium values (constants), Matsumura and Nishida [20, 21] first proved the global existence of smooth solutions to initial boundary value problems and the Cauchy problem. Feireisl [4] obtained the global existence of the so-called “variational solutions” in the sense that the energy equation is replaced by an energy inequality. Recently, Huang and Li [10] derived global well-posedness of strong solutions to the full compressible Navier-Stokes equations in $\mathbb{R}^3$ with no-vacuum at infinity which are of small energy but possibly large oscillations. Wen and Zhu [23] showed global existence of strong solutions with far-field vacuum under the condition that the initial mass is properly small in certain sense. Meanwhile, Li [12] obtained a new type of global strong solutions under some smallness condition on the scaling invariant quantity. Very recently, Liang [16] established global strong solutions when the initial energy is small. Moreover, decay rates were also determined.

Let’s turn our attention to the full compressible MHD equations (1.1). As a couple system, (1.1) contains much richer structures than the full compressible Navier-Stokes equations. It is not merely a combination of fluid equations and magnetic field equations but an interactive system. Their distinctive features make analytical studies a great challenge but offer new opportunities. For the initial data satisfying some compatibility condition, Fan and Yu [3] established the local existence and uniqueness of strong solutions to the problem (1.1)–(1.5). Later, Liu and Zhong [17] extended this local existence result to be a global one provided that $\|\rho_0\|_{L^\infty} + \|b_0\|_{L^3}$ is suitably small and the viscosity coefficients satisfy $3\mu > \lambda$. This result was improved in [18] where the authors proved the global existence and uniqueness of strong solutions, which may be of possibly large oscillations, provided that the initial data are of small total energy. At the same time, Hou-Jiang-Peng [7] obtained global strong solutions under the condition that $\|\rho_0\|_{L^1} + \|b_0\|_{L^2}$ is suitably small. For the global existence of weak solutions, we refer to [2, 8, 14, 15] and references therein. There are also some interesting mathematical results concerning the global existence of (weak, strong or classical) solutions to the compressible isentropic MHD equations, please refer to [5, 6, 9, 11, 19, 22, 24].

The aim of the present paper is to extend the result in [12] to the compressible heat-conductive magneto-hydrodynamic flows. This is a nontrivial generalization, since one has to control the strong nonlinear terms involved with the magnetic field. Furthermore, the restriction on the viscosity coefficients is relaxed. These are exactly the new points of this paper.

For $1 \leq p < \infty$ and integer $k \geq 0$, we denote the standard homogeneous and inhomogeneous Sobolev spaces as follows:
\[
\begin{align*}
L^p &= L^p(\mathbb{R}^3), \quad W^{k,p} = L^p \cap D^{k,p}, \quad H^k = W^{k,2}, \\
D^{k,p} &= \{u \in L^1_{\text{loc}}(\mathbb{R}^3) : \|\nabla^k u\|_{L^p} < \infty\}, \quad D^k = D^{k,2}, \\
D_0^1 &= \{u \in L^3(\mathbb{R}^3) : \|\nabla u\|_{L^2} < \infty\}.
\end{align*}
\]

We can now state our main result.

**Theorem 1.1.** Assume that $3\mu > \lambda$ and let $q \in (3, 6]$ be a fixed constant. Let the initial data $(\rho_0 \geq 0, u_0, \theta_0 \geq 0, b_0)$ satisfy
\[
\rho_0 \in H^1 \cap W^{-1,q}, \quad (\sqrt{\rho_0} u_0, \sqrt{\rho_0} \theta_0) \in L^2, \quad (u_0, \theta_0) \in D_0^1 \cap D^2, \quad b_0 \in H^2, \quad (1.6)
\]
and the compatibility condition
\[
\begin{align*}
-\mu \Delta u_0 - (\lambda + \mu) \nabla \text{div} u_0 + \nabla (R \rho_0 \theta_0) - \text{curl} b_0 \times b_0 &= \sqrt{\rho_0} g_1, \\
-\kappa \Delta \theta_0 - Q(\nabla u_0) - \eta \text{curl} b_0^2 &= \sqrt{\rho_0} g_2,
\end{align*}
\]
with \( g_1, g_2 \in L^2(\mathbb{R}^3) \). There exists a small positive constant \( \varepsilon_0 \) depending only on \( R, \mu, \lambda, \nu, \kappa, \) and \( c_v \) such that if
\[
N_0 \triangleq \rho \left[ \| \rho_0 \|_{L^1} + \rho^2(\| \sqrt{\rho_0} u_0 \|_{L^2}^2 + \| b_0 \|_{L^2}^2) \right] \left[ \| \nabla u_0 \|_{L^2}^2 + \rho(\| \sqrt{\rho_0} E_0 \|_{L^2}^2 + \| \nabla b_0 \|_{L^2}^2) \right] \leq \varepsilon_0,
\] (1.8)
where \( E_0 = \frac{|u_0|^2}{2} + c_v \theta_0 \) and \( \bar{\rho} = \| \rho_0 \|_{L^\infty} + 1 \), then the problem (1.1)–(1.5) has a unique global strong solution \((\rho, u, \theta, b)\) satisfying
\[
\rho \in C([0, T]; H^1 \cap W^{1,q}), \quad \rho_t \in C([0, T]; L^2 \cap L^q), \\
(u, \theta) \in C([0, T]; D^1 \cap D^2) \cap L^2(0, T; D^2), \quad (u_t, \theta_t) \in L^2(0, T; D^2), \\
b \in C([0, T]; H^2) \cap L^2(0, T; H^3), \quad b_t \in C([0, T]; L^2) \cap L^2(0, T; H^1), \\
\left( \sqrt{\rho} u_t, \sqrt{\rho} \theta_t \right) \in L^\infty(0, T; L^2).
\]

**Remark 1.1.** It should be noted that our smallness assumption is independent of any norms of the initial data, which is in sharp contrast to [7, 17, 18] where they established global strong solution under some smallness conditions depending on the initial data.

**Remark 1.2.** It is not hard to check that the quantity \( N_0 \) in (1.8) is scaling invariant under the transform
\[
\rho_\lambda(x, t) = \rho(\lambda x, \lambda^2 t), \quad u_\lambda(x, t) = \lambda u(\lambda x, \lambda^2 t), \quad \theta_\lambda(x, t) = \lambda^2 (\lambda x, \lambda^2 t), \quad b_\lambda(x, t) = \lambda b(\lambda x, \lambda^2 t).
\]
Thus we generalize the result in [12] to the compressible non-isentropic MHD equations. Nevertheless, we should point out that the magnetic field acts some significant roles since \( N_0 \) differs slightly from that of in [12] when \( b = b_0 \equiv 0 \). Moreover, the restriction \( 2\mu > \lambda \) in [12] on the viscosity coefficients is relaxed to \( 3\mu > \lambda \).

It seems that the methods used in [7, 17, 18] are not available here because our smallness condition depends only on the parameters in the system. We mainly apply some techniques developed by Li [12] to give the proof of Theorem 1.1. However, compared with compressible heat-conducting Navier-Stokes equations considered in [12], due to the strong coupling and interplay between the fluid motion and the magnetic field, the crucial techniques of proofs in [12] cannot be adapted directly. To overcome these difficulties, we first obtain \( L^\infty(0, T; L^4) \) bound of \( b \) in terms of \( L^1_tL^2_x \)-norm of \( b, \nabla b \), and \( \nabla u \) (see (3.3)). Then some necessary lower order time-independent estimates are obtained (see (3.7) and (3.13)). Next, with the help of the effect viscous flux, we derive \( L^\infty(0, T; L^2) \) estimate of \( \nabla u \) under a priori hypothesis (3.19) (see Lemma 3.5). So, the next key step is to complete the proof of the a priori hypothesis, that is, to show that \( \| \rho(t) \|_{L^\infty} \) is in fact strictly less than \( 4\bar{\rho} \). Inspired by [12, Proposition 2.6], we find that \( \| \rho \|_{L^\infty(0, T; L^\infty)} \) is indeed bounded by other norms of the solution and could be as small as desired under smallness condition on \( N_0 \) (see Proposition 3.1). This in particular completes the proof of a priori hypothesis. Having the time-independent estimates at hand, we can show Theorem 1.1 by continuity arguments as those in [17].

The rest of the paper is organized as follows. In Section 2, we recall some known facts and elementary inequalities which will be used later. Section 3 is devoted to the proof of Theorem 1.1.

## 2 Preliminaries

In this section, we recall some known facts and elementary inequalities which will be used frequently later.

We begin with the local existence of a unique strong solution with vacuum to the problem (1.1)–(1.5), whose proof can be obtained by similar ways as those in [3].

**Lemma 2.1.** Assume that the initial data \((\rho_0 \geq 0, u_0, \theta_0 \geq 0, b_0)\) satisfies the conditions in Theorem 1.1. Then, there exists a positive time \( T_0 > 0 \) depending only on \( R, \mu, \lambda, \) and \( \psi_0 \), such that the problem (1.1)–(1.5) admits a unique strong solution in \( \mathbb{R}^3 \times (0, T_0) \), where \( \psi_0 \) is a positive constant such that
\[
\| \rho_0 \|_{H^1 \cap W^{1,q}} + \| (u_0, \theta_0) \|_{D^1_0 \cap D^2} + \| b_0 \|_{H^2} + \| (\sqrt{\rho_0} u_0, \sqrt{\rho_0} \theta_0, g_1, g_2) \|_{L^2} \leq \psi_0.
\]

Next, the following well-known Gagliardo-Nirenberg inequality (see [1, Chapter 10, Theorem 1.1]) will be used frequently later.
Lemma 2.2. Assume that \( f \in D^{1,m} \cap L^r \) with \( m, r \geq 1 \), then there exists a constant \( C \) depending only on \( q, m \), and \( r \) such that
\[
\|f\|_{L^q} \leq C\|\nabla f\|_{L^m}^\vartheta \|f\|_{L^r}^{1-\vartheta},
\]
where \( \vartheta = \left( \frac{1}{r} - \frac{1}{q} \right) / \left( \frac{1}{r} - \frac{1}{m} + \frac{1}{3} \right) \) and the admissible range of \( q \) is the following:
- if \( m < 3 \), then \( q \) is between \( r \) and \( \frac{3m}{3-m} \);
- if \( m = 3 \), then \( q \in [r, \infty) \);
- if \( m > 3 \), then \( q \in [r, \infty] \).

3 Proof of Theorem 1.1

3.1 A priori estimates

In this subsection, we will establish some necessary a priori bounds for smooth solutions to the Cauchy problem (1.1)–(1.5). In what follows, we always assume that \((\rho, u, \theta, b)\) is a strong solution to the problem (1.1)–(1.5) in \(\mathbb{R}^3 \times (0, T)\) for some positive time \(T\). Meanwhile, \(C, C_i\), and \(c_i\) \((i = 1, 2, \ldots)\) denote generic positive constants which rely only on \(R, \mu, \lambda, \nu, \kappa\), and \(c_v\). For simplicity, in what follows, we write
\[
\int \cdot dx = \int_{\mathbb{R}^3} \cdot dx, \quad \nu = \kappa = c_v = 1.
\]

Lemma 3.1. It holds that
\[
\sup_{0 \leq t \leq T} \left( \|\sqrt{\rho}u\|_{L^2}^2 + \|b\|_{L^2}^2 \right) + \int_0^T \left( \|\nabla u\|_{L^2}^2 + \|\nabla b\|_{L^2}^2 \right) dt \leq \|\sqrt{\rho_0}u_0\|_{L^2}^2 + \|b_0\|_{L^2}^2 + C \int_0^T \|\rho\|_{L^3}^2 \|\nabla \theta\|_{L^2}^2 dt. \tag{3.1}
\]

Proof. Multiplying (1.1)\(_2\) by \(u\) and (1.1)\(_4\) by \(b\), respectively, then adding the two resulting equations together and integrating over \(\mathbb{R}^3\), we obtain from \(\mu + \lambda > 0\)\(^1\), Hölder’s inequality, and Sobolev’s inequality that
\[
\frac{1}{2} \frac{d}{dt} \left( \|\sqrt{\rho}u\|_{L^2}^2 + \|b\|_{L^2}^2 \right) + \mu \|\nabla u\|_{L^2}^2 + (\mu + \lambda) \|\nabla b\|_{L^2}^2 = R \int \rho \theta \Div u dx \leq R \|\rho\|_{L^3} \|\theta\|_{L^6} \|\nabla u\|_{L^2} \leq \frac{\mu + \lambda}{2} \|\nabla u\|_{L^2}^2 + C \|\rho\|_{L^3} \|\nabla \theta\|_{L^2}^2,
\]
which implies that
\[
\frac{d}{dt} \left( \|\sqrt{\rho}u\|_{L^2}^2 + \|b\|_{L^2}^2 \right) + \|\nabla u\|_{L^2}^2 + \|\nabla b\|_{L^2}^2 \leq C \|\rho\|_{L^3} \|\nabla \theta\|_{L^2}^2. \tag{3.2}
\]
Hence, integrating (3.2) over \([0, T]\) leads to (3.1).

Lemma 3.2. It holds that
\[
\sup_{0 \leq t \leq T} \left( \|\nabla b\|_{L^2}^2 + \|b\|_{L^4}^2 \right) + \int_0^T \left( \|b_t\|_{L^2}^2 + \|\nabla^2 b\|_{L^2}^2 + \|b_t\|_{L^2} \|\nabla b\|_{L^2}^2 \right) dt \leq \|\nabla b_0\|_{L^2}^2 + \|b_0\|_{L^4}^2 + C \sup_{0 \leq t \leq T} \left( \|b_t\|_{L^2} \|\nabla b\|_{L^2} \right)^2 \int_0^T \|\nabla^2 b\|_{L^2}^2 dt + C \sup_{0 \leq t \leq T} \left( \|b_t\|_{L^2} \|\nabla u\|_{L^2}^2 \right) \int_0^T \|\nabla u\|_{L^2}^2 dt. \tag{3.3}
\]

\(^1\)From (1.3), we have \(3\mu + 3\lambda > 0\). Thus the result follows.
Proof. 1. We deduce from (1.1) that

\[
\frac{d}{dt} \|\nabla b\|_{L^2}^2 + \|b_t\|_{L^2}^2 + \|\nabla^2 b\|_{L^2}^2 = \int |b_t - \Delta b|^2 \, dx = \int |b \cdot \nabla u - u \cdot \nabla b - b \, \text{div} \, u|^2 \, dx \\
\leq C \|\nabla u\|_{L^6}^2 \|b\|_{L^6}^2 + C \|u\|_{L^6}^2 \|\nabla b\|_{L^6}^2 \\
\leq C \|b\|_{L^2}^2 \|\nabla^2 b\|_{L^2}^2 \|\nabla u\|_{L^2}^2 \\
\leq \frac{1}{8} \|\nabla^2 b\|_{L^2}^2 + C(\|b\|_{L^2}^2 \|\nabla u\|_{L^2}^2) \|\nabla u\|_{L^2}^2, \tag{3.4}
\]

where we have used the following Gagliardo-Nirenberg inequality

\[
\|\nabla v\|_{L^2} \leq C \|v\|_{L^6}^{\frac{1}{3}} \|\nabla^2 v\|_{L^2}^{\frac{2}{3}}, \quad \|v\|_{L^\infty} \leq C \|v\|_{L^6}^{\frac{1}{3}} \|\nabla^2 v\|_{L^2}^{\frac{2}{3}}. \tag{3.5}
\]

2. Multiplying (1.1) by $4|b|^2 b$ and integration by parts, we get from (3.5) and Sobolev’s inequality that

\[
\frac{d}{dt} \int |b|^4 \, dx + 4 \int (|\nabla b|^2 |b|^2 + 2|\nabla |b|^2 |b|^2) \, dx \\
= 4 \int (b \cdot \nabla u - u \cdot \nabla b - b \, \text{div} \, u)|b|^2 \, dx \\
= 4 \int b \cdot \nabla u \cdot b |b|^2 \, dx - 3 \int |b|^4 \, \text{div} \, u \, dx \\
\leq C \|b\|_{L^\infty}^2 \|\nabla u\|_{L^2}^2 + C \|b\|_{L^6}^2 \|b\|_{L^6}^2 \|\nabla u\|_{L^2}^2 + C \|b\|_{L^2}^2 \|\nabla b\|_{L^2} \|b\|_{L^2} \|\nabla u\|_{L^2}^2 \\
\leq C \|b\|_{L^2}^2 \|\nabla^2 b\|_{L^2}^2 \|\nabla u\|_{L^2}^2 + C \|b\|_{L^2}^2 \|\nabla b\|_{L^2} \|b\|_{L^2} \|\nabla u\|_{L^2}^2 + C \|b\|_{L^2}^2 \|\nabla b\|_{L^2} \|b\|_{L^2} \|\nabla u\|_{L^2}^2 \|\nabla^2 b\|_{L^2}^2 \\
\leq \frac{1}{8} \|\nabla^2 b\|_{L^2}^2 + C(\|b\|_{L^2}^2 \|\nabla u\|_{L^2}^2) \|\nabla u\|_{L^2}^2 + C \|b\|_{L^2}^2 \|\nabla b\|_{L^2} \|b\|_{L^2} \|\nabla u\|_{L^2}^2 \|\nabla^2 b\|_{L^2}^2 \\
\leq \frac{1}{8} \|\nabla^2 b\|_{L^2}^2 + \frac{1}{2} \|b\|_{L^2}^2 \|\nabla b\|_{L^2}^2 + C \|b\|_{L^2}^2 \|\nabla b\|_{L^2} \|\nabla^2 b\|_{L^2}^2 + C \|b\|_{L^2}^2 \|\nabla u\|_{L^2}^2 \|\nabla u\|_{L^2}^2. \tag{3.6}
\]

This along with (3.4) yields that

\[
\frac{d}{dt} \left( \|\nabla b\|_{L^2}^2 + \|b_t\|_{L^2}^2 + \|\nabla^2 b\|_{L^2}^2 + \|b|\nabla b\|_{L^2}^2 \right) \\
\leq C \left( \|b\|_{L^2}^2 \|\nabla b\|_{L^2}^2 \|\nabla b\|_{L^2}^2 \right)^\frac{1}{2} \|\nabla^2 b\|_{L^2}^2 + C \|b\|_{L^2}^2 \|\nabla u\|_{L^2}^2 \|\nabla u\|_{L^2}^2. \tag{3.6}
\]

Thus, (3.3) follows from (3.6) integrated in $t$ over $[0, T]$.

\[\square\]

**Lemma 3.3.** Assume that $3\mu > \lambda$, it holds that

\[
\sup_{0 \leq t \leq T} \|\sqrt{\rho} E\|_{L^2}^2 + \int_0^T \left( \|u\|_{L^2} \|\nabla u\|_{L^2}^2 + \|\nabla \theta\|_{L^2}^2 \right) \, dt \\
\leq C \|\sqrt{\rho_0} E_0\|_{L^2}^2 + C \sup_{0 \leq t \leq T} (\|b\|_{L^2} \|\nabla b\|_{L^2}) \int_0^T \|\nabla^2 b\|_{L^2}^2 \, dt \\
+ C \sup_{0 \leq t \leq T} (\|b\|_{L^2} \|\nabla b\|_{L^2}) \int_0^T \|\nabla u\|_{L^2}^6 \, dt + C \sup_{0 \leq t \leq T} \|\nabla b\|_{L^2}^4 \int_0^T \|\nabla u\|_{L^2}^2 \, dt \\
+ C \int_0^T \|\sqrt{\rho} \theta\|_{L^2} \|\nabla \theta\|_{L^2} \|u\|_{L^2} \|\nabla u\|_{L^2} \|\rho\|_{L^1}^\frac{1}{2} \, dt. \tag{3.7}
\]

**Proof.** 1. For $E = \frac{|u|^2}{2} + c_\theta \theta$, we infer from (1.1) that

\[
\rho (E_t + u \cdot \nabla E) + \text{div} (u p) - \Delta \theta = \text{div} (S \cdot u) + \text{curl} \, b \times b + |\text{curl} \, b|^2, \tag{3.8}
\]
where $S = \mu(\nabla u + (\nabla u)^T) + \lambda \text{div} u I_3$ with $I_3$ being the identity matrix of order 3. Multiplying (3.8) by $E$ and integrating the resultant over $\mathbb{R}^3$, it follows from integration by parts and Young’s inequality that

$$
\frac{1}{2} \frac{d}{dt} \|\sqrt{\rho} E\|_{L^2}^2 + \|\nabla \theta\|_{L^2}^2 \leq -\frac{1}{2} \int \nabla \theta \cdot \nabla |u|^2 dx + \int (up - S \cdot u) \cdot \nabla E dx
$$

$$+ C \int (|u||b|^2|\nabla E| + |\nabla u||b|^2 E) dx + \int |\text{curl} b|^2 E dx
$$

$$\leq \frac{1}{6} \|\nabla \theta\|_{L^2}^2 + \frac{3}{8} \|u||\nabla u\|_{L^2}^2 + C \int \rho^2 \theta^2 |u|^2 dx
$$

$$+ C \int (|u||b|^2|\nabla E| + |\nabla u||b|^2 E) dx + C \int |\nabla E||\nabla b||b| dx
$$

Applying Hölder’s inequality and Sobolev’s inequality, we have that

$$I_3 \leq C\|\sqrt{\rho} \theta\|_{L^2}\|\theta\|_{L^6}\|\nabla u\|_{L^6}\|\rho\|_{L^3}^{\frac{3}{2}}$$

$$\leq C\|\rho\|_{L^\infty}\|\sqrt{\rho} \theta\|_{L^2}\|\nabla \theta\|_{L^2}\|u||\nabla u\|_{L^2}\|\rho\|_{L^3}^{\frac{1}{2}},$$

$$I_4 \leq C\|u\|_{L^6}\|b\|_{L^6}\|\nabla E\|_{L^2} + C\|\nabla u\|_{L^2}\|b\|_{L^3}\|E\|_{L^6}$$

$$\leq C\|\nabla u\|_{L^2}\|b\|_{L^6}\|\nabla E\|_{L^2}$$

$$\leq \frac{1}{8} \|\nabla \theta\|_{L^2}^2 + C\|u||\nabla u\|_{L^2}^2 + C\|\nabla b\|_{L^2}\|\nabla u\|_{L^2}^2,$$

$$I_5 + I_6 \leq C\|E\|_{L^6}\|\nabla b\|_{L^2}\|b\|_{L^3} + C\|\nabla E\|_{L^2}\|\nabla b\|_{L^6}\|b\|_{L^3}$$

$$\leq C\|\nabla E\|_{L^2}\|b\|_{L^3}\|\nabla b\|_{L^3}$$

$$\leq \frac{1}{6} \|\sqrt{\rho} \theta\|_{L^2}^2 + C\|u||\nabla u\|_{L^2}^2 + C\|\nabla E\|_{L^2}\|\nabla b\|_{L^2}\|b\|_{L^2}.$$

Substituting the above inequalities into (3.9) yields that

$$\frac{d}{dt} \|\sqrt{\rho} E\|_{L^2}^2 + \|\nabla \theta\|_{L^2}^2 \leq c_1\|u||\nabla u\|_{L^2}^2 + C\|\nabla b\|_{L^2}\|\nabla b\|_{L^2}\|\nabla b\|_{L^2}^2 + C\|\nabla b\|_{L^2}\|\nabla u\|_{L^2}^2$$

$$+ C\|\sqrt{\rho} \theta\|_{L^2}\|\nabla \theta\|_{L^2}\|u||\nabla u\|_{L^2}\|\rho\|_{L^3}^{\frac{1}{2}}. \quad (3.10)$$

2. Exactly in the same way as that in [17, Lemma 3.4], we find that

$$\frac{d}{dt} \|\rho^2 u^4\|_{L^4}^2 + \|u||\nabla u\|_{L^2}^2 \leq C \int \rho^2 \theta^2 |u|^2 dx + C\|b\|_{L^2}\|\nabla b\|_{L^2}\|\nabla b\|_{L^2}^2 + C\|b\|_{L^3}\|\nabla u\|_{L^2}^2$$

$$\leq C\|b\|_{L^2}\|\nabla b\|_{L^2}\|\nabla b\|_{L^2}^2 + C\|b\|_{L^2}\|\nabla b\|_{L^2}\|\nabla u\|_{L^2}^2$$

$$+ C\|\sqrt{\rho} \theta\|_{L^2}\|\nabla \theta\|_{L^2}\|u||\nabla u\|_{L^2}\|\rho\|_{L^3}^{\frac{1}{2}},$$

$$\leq C\|b\|_{L^2}\|\nabla b\|_{L^2}\|\nabla b\|_{L^2}^2 + C\|b\|_{L^2}\|\nabla b\|_{L^2}\|\nabla u\|_{L^2}^2$$

$$+ C\|\sqrt{\rho} \theta\|_{L^2}\|\nabla \theta\|_{L^2}\|u||\nabla u\|_{L^2}\|\rho\|_{L^3}^{\frac{1}{2}}, \quad (3.11)$$

due to Hölder’s inequality and Sobolev’s inequality. Adding (3.10) to (3.11) multiplied by $2c_1$, one gets that

$$\frac{d}{dt} \left(\|\sqrt{\rho} E\|_{L^2}^2 + 2c_1\|\rho^2 u^4\|_{L^4}^2\right) + c_1\|u||\nabla u\|_{L^2}^2 + \|\nabla \theta\|_{L^2}^2$$

$$\leq C\|b\|_{L^2}\|\nabla b\|_{L^2}\|\nabla b\|_{L^2}^2 + C\|b\|_{L^2}\|\nabla b\|_{L^2}\|\nabla u\|_{L^2}^2 + C\|\nabla b\|_{L^2}\|\nabla u\|_{L^2}^2$$

$$+ C\|\sqrt{\rho} \theta\|_{L^2}\|\nabla \theta\|_{L^2}\|u||\nabla u\|_{L^2}\|\rho\|_{L^3}^{\frac{1}{2}}. \quad (3.12)$$

From which, the conclusion follows by integrating (3.12) over $[0, T]$.
By virtue of Hölder’s inequality and Sobolev’s inequality, we derive that
\[ \sup_{0 \leq t \leq T} \| \rho \|_{L^3}^3 + \int_0^T \rho^3 p dx dt \leq \| \rho_0 \|_{L^3}^3 + C \sup_{0 \leq t \leq T} \left( \| \rho \|_{L^\infty} \| \rho u \|_{L^2} \| \sqrt{\rho} u \|_{L^2} \right) + C \int_0^T \| \rho \|_{L^\infty} \| \rho \|_{L^3}^2 \| \nabla u \|_{L^2} dt + C \int_0^T \| \rho \|_{L^\infty} \| \rho \|_{L^3}^2 \| \nabla b \|_{L^2} dt. \] (3.13)

**Proof.** Applying the operator $\Delta^{-1} \text{div}$ to (1.1) leads to
\[ \Delta^{-1} \text{div}(\rho u)_t + \Delta^{-1} \text{div}(\rho u \otimes u) - (2 \mu + \lambda) \text{div} u + p + \frac{|b|^2}{2} = \Delta^{-1} \text{div}(b \otimes b). \] (3.14)

In view of (1.1), one obtains that
\[ \partial_t \rho^3 + \text{div}(u \rho^3) + 2 \text{div} u \rho^3 = 0. \] (3.15)

Then, multiplying (3.14) by $\rho^3$ and using (3.15), we have
\[ \frac{2 \mu + \lambda}{2} (\partial_t \rho^3 + \text{div}(u \rho^3)) + \rho^3 p + \rho^3 \Delta^{-1} \text{div}(\rho u)_t + \rho^3 \Delta^{-1} \text{div}(\rho u \otimes u) = \rho^3 \Delta^{-1} \text{div}(b \otimes b). \] (3.16)

Using (3.15) again, we deduce that
\[ \int \rho^3 \Delta^{-1} \text{div}(\rho u)_t dx = \frac{d}{dt} \int \rho^3 \Delta^{-1} \text{div}(\rho u) dx + \int [\text{div}(\rho^3 u) + 2 \text{div} u \rho^3] \Delta^{-1} \text{div}(\rho u) dx \leq \int [2 \text{div} u \rho^3 \Delta^{-1} \text{div}(\rho u) - \rho^3 u \cdot \nabla \Delta^{-1} \text{div}(\rho u)] dx + \frac{d}{dt} \int \rho^3 \Delta^{-1} \text{div}(\rho u) dx, \]
which combined with (3.16) yields that
\[ \frac{d}{dt} \int \left( \frac{2 \mu + \lambda}{2} + \Delta^{-1} \text{div}(\rho u) \right) \rho^3 dx + \int \rho^3 p dx \leq \int [\rho^3 (u \cdot \nabla \Delta^{-1} \text{div}(\rho u)) - \Delta^{-1} \text{div}(\rho u \otimes u) - 2 \text{div} u \rho^3 \Delta^{-1} (\rho u)] dx + \int \rho^3 \Delta^{-1} \text{div}(b \otimes b) dx \equiv J_1 + J_2. \] (3.17)

By virtue of Hölder’s inequality and Sobolev’s inequality, we derive that
\[ |J_1| \leq C \| \rho \|_{L^\infty} \| \rho \|_{L^3}^2 \| \nabla u \|_{L^2}^2, \]
\[ |J_2| \leq \| \rho \|_{L^\infty} \| \rho \|_{L^3}^2 \| \nabla u \|_{L^2}^2 \Delta^{-1} \text{div}(b \otimes b) \|_{L^3} \leq C \| \rho \|_{L^\infty} \| \rho \|_{L^3}^2 \| \nabla b \|_{L^2}^2, \]
which together with (3.17) implies that
\[ \frac{d}{dt} \int \left( \frac{2 \mu + \lambda}{2} + \Delta^{-1} \text{div}(\rho u) \right) \rho^3 dx + \int \rho^3 p dx \leq C \| \rho \|_{L^\infty} \| \rho \|_{L^3}^2 \| \nabla u \|_{L^2}^2 + C \| \rho \|_{L^\infty} \| \rho \|_{L^3}^2 \| \nabla b \|_{L^2}^2. \] (3.18)

Noticing that
\[ \| \Delta^{-1} \text{div}(\rho u) \|_{L^\infty} \leq C \| \Delta^{-1} \text{div}(\rho u) \|_{L^2}^\frac{1}{2} \| \nabla \Delta^{-1} \text{div}(\rho u) \|_{L^2}^\frac{1}{2} \]
\[ \leq C \| \rho u \|_{L^6} \| \rho u \|_{L^4}^\frac{1}{2} \leq C \| \rho \|_{L^\infty} \| \sqrt{\rho} u \|_{L^2} \| \sqrt{\rho} u \|_{L^2} \]

This along with (3.18) integrated in $t$ over $[0, T]$ leads to (3.13). \(\square\)
Lemma 3.5. Assume that

$$\sup_{0 \leq t \leq T} \|\rho\|_{L^\infty} \leq 4\bar{\rho},$$

then it holds that

$$\sup_{0 \leq t \leq T} \left\| \nabla u \right\|^2_{L^2} + \int_0^T \left\| \frac{\sqrt{p} u_t, \nabla F}{\sqrt{\rho}} \right\|^2_{L^2} dt \leq C \|\nabla u_0\|^2_{L^2} + C\bar{\rho} \sup_{0 \leq t \leq T} \|\sqrt{\rho} \theta\|^2_{L^2} + C \sup_{0 \leq t \leq T} \|b\|^2_{L^1} + \frac{\eta_1}{2} \int_0^T \|b\|_{L^2} dt + C\bar{\rho} \int_0^T \|\nabla b\|^2_{L^2} dt + C \int_0^T \|\int \nabla b\|^2_{L^2} dt + C\bar{\rho} \int_0^T \|\nabla b\|_{L^2} \|b\|^2_{L^2} dt,$$

where $F = (2\mu + \lambda) \text{div} u - p - \frac{|b|^2}{2}$ and $w = \text{curl} u$.

Proof. Multiplying (1.1) by $u_t$ and integration by parts, we get that

$$\int p \text{div} u_t dx = \frac{1}{2} \frac{d}{dt} \left( \mu \|\nabla u\|^2_{L^2} + (\mu + \lambda) \|\text{div} u\|^2_{L^2} \right) - \int p \text{div} u_t dx + \|\sqrt{\rho} u_t\|^2_{L^2} = - \int \rho u \cdot \nabla u \cdot u_t dx - \int b \cdot \nabla b \cdot u_t dx - \frac{1}{2} \int \nabla |b|^2 \cdot u_t dx. \tag{3.21}$$

By the definition of effective viscous flux $F$, we use $u = \frac{F + p + \frac{1}{2}|b|^2}{2\mu + \lambda}$ to obtain that

$$\int p \text{div} u_t dx = \frac{1}{2} \frac{d}{dt} \int p \text{div} u dx - \int p_t \text{div} u dx = \frac{1}{2} \frac{d}{dt} \|p\|^2_{L^2} + \frac{1}{2} \frac{d}{dt} \int p |F|^2 + \frac{1}{2} \frac{d}{dt} \int p |b|^2 + \frac{1}{2} \frac{d}{dt} \int p F dx - \frac{1}{2} \frac{d}{dt} \int p |b|^2 dx.$$

$$= \frac{1}{2} \frac{d}{dt} \|p\|^2_{L^2} + \frac{1}{2} \frac{d}{dt} \int p |F|^2 + \frac{1}{2} \frac{d}{dt} \int p |b|^2 + \frac{1}{2} \frac{d}{dt} \int p F dx - \frac{1}{2} \frac{d}{dt} \int p |b|^2 dx.$$

It follows (1.1)$_1$, (1.1)$_3$, and the state equation that

$$p_t = - \text{div}(pu) - (\gamma - 1) (p \text{div} u - \Delta \theta - Q(\nabla u) - |\text{curl} b|^2) \tag{3.23}$$

with $\gamma - 1 = \frac{K}{c_v}$, which leads to

$$\int p_t \left( F + \frac{1}{2} |b|^2 \right) dx = \int \left[ (\gamma - 1) (Q(\nabla u) - p \text{div} u + |\text{curl} b|^2) F + (up - (\gamma - 1) \nabla \theta) \cdot \nabla F \right] dx + \frac{1}{2} \int \left[ (\gamma - 1) (Q(\nabla u) - p \text{div} u + |\text{curl} b|^2) |b|^2 + (up - (\gamma - 1) \nabla \theta) \cdot \nabla |b|^2 \right] dx. \tag{3.24}$$

In view of $\|\nabla u\|^2_{L^2} = \|w\|^2_{L^2} + \|\text{div} u\|^2_{L^2}$, then one obtains from (3.21)–(3.24) that

$$\frac{1}{2} \frac{d}{dt} \left( \mu \|w\|^2_{L^2} + \frac{|F|^2_{L^2}}{2(2\mu + \lambda)} \right) + \frac{1}{2(2\mu + \lambda)} \int |b|^4 dx + \frac{1}{2(2\mu + \lambda)} \int |b|^4 dx + \|\sqrt{\rho} u_t\|^2_{L^2}.$$
\[
= - \int \rho u \cdot \nabla u \cdot u_t dx + \int b \cdot \nabla b \cdot u_t dx - \frac{1}{2} \int \nabla |b|^2 \cdot u_t dx \\
- \frac{\gamma - 1}{2\mu + \lambda} \int \left[ Q(\nabla u) - p \text{div} \, u + |\text{curl} \, b|^2 \right] \left( F + \frac{1}{2} |b|^2 \right) dx \\
+ \frac{1}{2\mu + \lambda} \int \left( (\gamma - 1) \nabla \theta - up \right) \cdot \nabla \left( F + \frac{1}{2} |b|^2 \right) dx.
\] (3.25)

Using \( \Delta u = \nabla \text{div} \, u - \text{curl} \, w \) to rewrite (1.1) as follows
\[
\rho u_t + \rho u \cdot \nabla u = \nabla F - \mu \text{curl} \, w + b \cdot \nabla b.
\] (3.26)

Multiplying (3.26) by \( \nabla F \) and using \( \int \nabla F \cdot \text{curl} \, w \, dx = 0 \) and (3.19), we have
\[
\|\nabla F\|_{L^2}^2 = \int \left[ \rho(u_t + u \cdot \nabla u) - b \cdot \nabla b \right] \cdot \nabla F \, dx \\
\leq \int \left( \frac{\|\nabla F\|^2}{2} + 2\rho|u_t|^2 \right) dx + \int \rho u \cdot \nabla u \cdot \nabla F \, dx - \int b \cdot \nabla b \cdot \nabla F \, dx,
\]
which yields that
\[
\frac{\|\nabla F\|_{L^2}^2}{16\rho} \leq \frac{1}{4} \|\nabla u_t\|_{L^2}^2 + \frac{1}{8\rho} \int \rho u \cdot \nabla u \cdot \nabla F \, dx - \frac{1}{8\rho} \int b \cdot \nabla b \cdot \nabla F \, dx.
\] (3.27)

Similarly, one deduces that
\[
\frac{\mu \|\nabla w\|_{L^2}^2}{16\rho} \leq \frac{1}{4} \|\nabla u_t\|_{L^2}^2 + \frac{1}{8\rho} \int \rho u \cdot \nabla u \cdot \text{curl} \, w \, dx - \frac{1}{8\rho} \int b \cdot \nabla b \cdot \text{curl} \, w \, dx.
\] (3.28)

Putting (3.27) and (3.28) into (3.25) gives that
\[
\frac{1}{2} \frac{d}{dt} \left( \mu \|w\|_{L^2}^2 + \|F\|_{L^2}^2 + \frac{1}{2\mu + \lambda} \int |b|^2 \, F \, dx + \frac{1}{2\mu + \lambda} \int |b|^4 \, dx \right) \\
+ \frac{1}{2} \sqrt{\rho} u_t \left( \|w\|_{L^2}^2 + \frac{\|F\|_{L^2}^2}{2\mu + \lambda} + \frac{1}{2\mu + \lambda} \int |b|^2 \, F \, dx + \frac{1}{2\mu + \lambda} \int |b|^4 \, dx \right) \\
\leq C \int \rho u \|\nabla u_t\| \left[ |u_t| + \frac{1}{\rho} (|\nabla F| + |\nabla w|) \right] \, dx + C \int \left( \|\nabla \theta + \rho \theta |u| \|\nabla F\| + |b| |\nabla b| \right) \, dx \\
+ C \int (\|\nabla u\|^2 + \rho \theta |\nabla u|)(\|F\| - \frac{1}{2} |b|^2) + \frac{C}{\rho} \int |b| |\nabla b| (|\nabla F| + |\nabla w|) \, dx \\
+ C \int |\nabla b|^2 \, F \, dx + \int b \cdot \nabla b \cdot u_t \, dx - \frac{1}{2} \int \nabla |b|^2 \cdot u_t \, dx \leq \sum_{i=1}^{7} Q_i.
\] (3.29)

By (3.19), Hölder’s inequality, and Young’s inequality, we get that
\[
Q_1 \leq C \sqrt{\rho} \|u\|_{L^6} \|\nabla u\|_{L^2} \sqrt{\rho} u_t \|L^2\| + C \|u\|_{L^2} \|\nabla u\|_{L^2} (\|\nabla F\|_{L^2} + \|\nabla w\|_{L^2}) \\
\leq \frac{1}{4} \|\nabla u_t\|_{L^2}^2 + \frac{1}{2\rho} \|\nabla F\|_{L^2}^2 + \mu^2 \|\nabla w\|_{L^2}^2 + C \rho \|u\|_{L^2} \|\nabla u\|_{L^2}^2,
\]
\[
Q_2 \leq C \rho \theta |u|_{L^6} (\|\nabla F\|_{L^2} + \|\nabla b\|_{L^2}) + C \|\nabla \theta\|_{L^2} (\|\nabla F\|_{L^2} + \|\nabla b\|_{L^2}) \\
\leq C \sqrt{\rho} \|F\|_{L^2} \sqrt{\theta} \|\nabla \theta\|_{L^2} \|u\|_{L^2} \|F\|_{L^2} + \|u\|_{L^2} \|\nabla b\|_{L^2} \\
+ C \|\nabla \theta\|_{L^2} (\|\nabla F\|_{L^2} + \|\nabla b\|_{L^2}) \\
\leq \frac{1}{2\rho} \|F\|_{L^2}^2 + \varepsilon_1 \rho \|\nabla b\|_{L^2}^2 + C (\rho^2 \rho) \frac{1}{L^2} \|\nabla \theta\|_{L^2}^2 + \rho \|\nabla \theta\|_{L^2}^2 + \|u\|_{L^2} \|\nabla u\|_{L^2}^2.
\]

Noticing that
\[
\|\nabla u\|_{L^6} \leq C (\|w\|_{L^6} + \|\text{div} \, u\|_{L^6})
\]
Then it follows from Hölder’s, Young’s, and Gagliardo-Nirenberg inequalities that
\[
Q_3 \leq C \left( \| \nabla u \|_{L^2} \| \nabla u \|_{L^6} (\| F \|_{L^3} + \| b \|_{L^6}^2) + C \left( \| \nabla u \|_{L^2} \| \rho \theta \|_{L^6} (\| F \|_{L^3} + \| b \|_{L^6}^2) \right) \right)
\leq C \left( \| \nabla u \|_{L^2} + \| \nabla F \|_{L^2} + \| \rho \theta \|_{L^2} + \| b \|_{L^2} \| \nabla b \|_{L^2} \right).
\] (3.30)

By virtue of Hölder’s, Young’s, and Gagliardo-Nirenberg inequalities, one obtains that
\[
Q_4 \leq C \| b \|_{L^3} \| \nabla b \|_{L^6} (\| F \|_{L^2} + \| \nabla w \|_{L^2})
\leq \frac{1}{224 \rho} (\| \nabla F \|_{L^2}^2 + \mu^2 \| \nabla w \|_{L^2}^2) + C \| b \|_{L^2} \| \nabla b \|_{L^2} \| \nabla b \|_{L^2}^2,
\]
\[
Q_5 \leq C \int (|b| \| \nabla^2 b \| F) + |b| \| \nabla b \| | \nabla F | \) dx \leq C \| b \|_{L^3} \| \nabla^2 b \|_{L^2} \| F \|_{L^6}
\leq \frac{1}{224 \rho} \| \nabla F \|_{L^2}^2 + C \| b \|_{L^2} \| \nabla b \|_{L^2} \| \nabla^2 b \|_{L^2}^2.
\]

Using Hölder’s inequality and (3.30), we arrive at
\[
Q_6 = -\frac{d}{dt} \int b \cdot \nabla u \cdot b dx + \int b \cdot \nabla u \cdot b dt + \int b \cdot \nabla u \cdot b_t dx
\leq -\frac{d}{dt} \int b \cdot \nabla u \cdot b dx + C \| b \|_{L^3} \| b_t \|_{L^2} \| \nabla u \|_{L^6}
\leq -\frac{d}{dt} \int b \cdot \nabla u \cdot b dx + C \| b \|_{L^2} \| \nabla b \|_{L^2} \| b_t \|_{L^2}^2 + \| b \|_{L^2} \| \nabla u \|_{L^6}^2
\leq -\frac{d}{dt} \int b \cdot \nabla u \cdot b dx + \frac{1}{224 \rho} (\| \nabla F \|_{L^2}^2 + \mu^2 \| \nabla w \|_{L^2}^2) + \frac{\varepsilon_1 \rho}{4} \| \nabla \theta \|_{L^2}^2
\leq \frac{\varepsilon_1 \rho}{2 \rho} \| b \|_{L^2} \| \nabla b \|_{L^2}^2 + C \| b \|_{L^2} \| \nabla b \|_{L^2} \| b_t \|_{L^2}^2.
\]

Similarly, we get that
\[
Q_7 \leq -\frac{d}{dt} \int \| b \|^4 h dx + \frac{1}{224 \rho} (\| \nabla F \|_{L^2}^2 + \mu^2 \| \nabla w \|_{L^2}^2) + \frac{\varepsilon_1 \rho}{4} \| \nabla \theta \|_{L^2}^2
\leq \frac{\varepsilon_1 \rho}{2 \rho} \| b \|_{L^2} \| \nabla b \|_{L^2}^2 + C \| b \|_{L^2} \| \nabla b \|_{L^2} \| b_t \|_{L^2}^2.
\]

Substituting all those estimates for \( Q_i \) (\( i = 1, 2, \cdots, 7 \)) into (3.29) leads to
\[
\frac{d}{dt} \left( \mu \| w \|_{L^2}^2 + \| F \|_{L^2}^2 \right) + \frac{1}{2 \mu + \lambda} \int \| b \|^2 F dx + \frac{1}{2 \mu + \lambda} \int \| b \|^4 dx
\leq \frac{1}{4 \mu + \lambda} \left( \| \nabla \theta \|_{L^2}^2 \right) + \frac{1}{16 \rho} (\| \nabla F \|_{L^2}^2 + \mu^2 \| \nabla w \|_{L^2}^2)
\leq -\frac{d}{dt} \int (b \cdot \nabla u \cdot b + \| b \|^2 \text{div} \ u) dx + \epsilon_1 \rho \| b \|_{L^2} \| \nabla b \|_{L^2}^2 + C \rho^3 \| b \|_{L^2} \| \nabla u \|_{L^2} \| F \|_{L^2}^2
\leq -\frac{d}{dt} \int (b \cdot \nabla u \cdot b + \| b \|^2 \text{div} \ u) dx + \epsilon_1 \rho \| b \|_{L^2} \| \nabla b \|_{L^2}^2 + C \rho^3 \| b \|_{L^2} \| \nabla u \|_{L^2} \| F \|_{L^2}^2
\leq C \| \nabla u \|_{L^2} \| b \|_{L^2} \| \nabla b \|_{L^2}^2 + \epsilon_1 \rho \| \nabla \theta \|_{L^2}^2,
which integrating in $t$ over $[0,T]$, together with the following facts
\[
\|\nabla u\|_{L^2} \leq C(\|w\|_{L^2} + \|F\|_{L^2} + \|\rho\|_{L^2} + |b|^2_{L^2}),
\]
\[
\left| - \int (b \cdot \nabla u \cdot b + |b|^2 \|u\|_{L^2} dx \right| \leq C\|\nabla u\|_{L^2} |b|^2_{L^2} \leq \varepsilon_2 \|\nabla u\|_{L^2}^2 + C|b|^4_{L^2},
\]
\[
\int |b|^2 F dx \leq \varepsilon_3 \|F\|_{L^2}^2 + C|b|^4_{L^2},
\]
with sufficiently small constants $\varepsilon_2$ and $\varepsilon_3$, yields (3.20).

**Lemma 3.6.** Let (3.19) be satisfied, then it holds that
\[
\sup_{0 \leq t \leq T} \|\rho\|_{L^\infty} \leq \rho_0 \exp \left( \int_0^T \left( C \rho \frac{u^2}{\rho} \sup_{0 \leq s \leq T} \left( \|\nabla u\|_{L^2}^2 \right)^\frac{1}{2} \right) \right),
\]
\[
\|\rho\|_{L^\infty} \leq \rho_0 \left( \sup_{0 \leq t \leq T} \left( \|\nabla u\|_{L^2}^2 \right)^\frac{1}{2} \right).
\]

**Proof.** According to (3.14), one obtains that
\[
\Delta^{-1} \text{div}(\rho u) + u \cdot \nabla \Delta^{-1} \text{div}(\rho u) - (2\mu + \lambda) \text{div} u + p + \frac{|b|^2}{2} \Delta^{-1} \text{div} (b \otimes b) = u \cdot \nabla \Delta^{-1} \text{div}(\rho u) - \Delta^{-1} \text{div}(\rho u \otimes u).
\]
Similarly to [12, Proposition 2.6], we can deduce that
\[
\sup_{0 \leq t \leq T} \|\rho\|_{L^\infty} \leq \rho_0 \left( \sup_{0 \leq t \leq T} \left( \|\nabla u\|_{L^2}^2 \right)^\frac{1}{2} \right). \tag{3.31}
\]
Employing Gagliardo-Nirenberg and Hölder’s inequalities, one gets that
\[
\int_0^T |b|^2_{L^2} dt \leq C \int_0^T \|\nabla b\|_{L^2} \|\nabla^2 b\|_{L^2} dt \leq C \left( \int_0^T \|\nabla b\|_{L^2}^2 dt \right)^\frac{1}{2} \left( \int_0^T \|\nabla^2 b\|_{L^2}^2 dt \right)^\frac{1}{2}.
\]
This together with (3.31) implies the conclusion immediately.

**Lemma 3.7.** Let
\[
N_T \triangleq \bar{\rho} \left[ \|\rho\|_{L^3} + \bar{\rho}^2 (\|\nabla u\|_{L^2}^2 + |b|^2_{L^2}) \right] \left[ \|\nabla u\|_{L^2}^2 + \bar{\rho} (\|\nabla E\|_{L^2}^2 + |b|^2_{L^2}) \right].
\]
There exists a positive constant $\eta_0$ depending only on $R$, $\mu$, and $\lambda$ such that if
\[
\sup_{0 \leq t \leq T} \|\rho\|_{L^\infty} \leq 4\bar{\rho}, \quad N_T \leq \sqrt{\eta}, \quad \eta \leq \eta_0, \tag{3.32}
\]
then it holds that
\[
\sup_{0 \leq t \leq T} \|\rho\|_{L^3} + \left( \int_0^T \rho^3 p dx dt \right)^\frac{1}{4} \leq C \left[ \|\rho_0\|_{L^3} + \bar{\rho}^2 (\|\nabla u_0\|_{L^2}^2 + |b_0|^2_{L^2}) \right], \tag{3.33}
\]
\[
\bar{\rho}^2 \left( \sup_{0 \leq t \leq T} \left( \|\nabla u\|_{L^2}^2 + |b|^2_{L^2} \right) \right)^\frac{1}{4} \int_0^T \|\nabla u, \nabla b\|_{L^2}^2 dt \leq C \left[ \|\rho_0\|_{L^3} + \bar{\rho}^2 (\|\nabla u_0\|_{L^2}^2 + |b_0|^2_{L^2}) \right], \tag{3.34}
\]
\[
\sup_{0 \leq t \leq T} \left( \|\nabla b\|_{L^2}^2 + |b|^2_{L^2} + \|\nabla E\|_{L^2}^2 + |\nabla u|^2_{L^2} \right) \leq C (\bar{\rho} \|\nabla u_0\|^2_{L^2} + \|\nabla E_0\|^2_{L^2} + \|\nabla u_0\|^2_{L^2}), \tag{3.35}
\]
\[
\sup_{0 \leq t \leq T} \|\rho\|_{L^\infty} \leq \bar{\rho} e^{C N_T^\frac{1}{2} + C N_T^\frac{1}{4}}. \tag{3.36}
\]

Proof. 1. By virtue of (3.1), (3.32), and \( \bar{\rho} = \| \rho_0 \|_{L^\infty} + 1 \), we deduce that

\[
\bar{\rho} \int_0^T \| \nabla u \|_{L^2}^6 \leq C \bar{\rho} \sup_{0 \leq t \leq T} \| \nabla u \|_{L^2}^4 \int_0^T \| \nabla u \|_{L^2}^2 dt
\]

\[
\leq C \bar{\rho} \sup_{0 \leq t \leq T} \| \nabla u \|_{L^2}^4 \left( \frac{1}{\sqrt{\rho_0}} \| u_0 \|_{L^2}^2 + \| b_0 \|_{L^2}^2 + \sup_{0 \leq t \leq T} \| \rho \|_{L^3}^2 \int_0^T \| \nabla \theta \|_{L^2}^2 dt \right)
\]

\[
\leq C \bar{\rho} \sup_{0 \leq t \leq T} \| \nabla u \|_{L^2}^4 \left( \sup_{0 \leq t \leq T} ( \| \sqrt{\rho} u \|_{L^2}^2 + \| b \|_{L^2}^2 ) + \sup_{0 \leq t \leq T} \| \rho \|_{L^3}^2 \int_0^T \| \nabla \theta \|_{L^2}^2 dt \right)
\]

\[
\leq C \eta^3 \sup_{0 \leq t \leq T} \| \nabla u \|_{L^2}^2 + C \eta \int_0^T \| \nabla \theta \|_{L^2}^2 dt.
\]

(3.37)

It follows from (3.3) and (3.32) that

\[
\bar{\rho} \sup_{0 \leq t \leq T} ( \| \nabla b \|_{L^2}^2 + \| b \|_{L^2}^4 ) + \bar{\rho} \int_0^T ( \| b \|_{L^2}^2 + \| \nabla b \|_{L^2}^2 + \| \nabla^2 b \|_{L^2}^2 ) dt
\]

\[
\leq C \bar{\rho} ( \| \nabla b_0 \|_{L^2}^2 + \| b_0 \|_{L^2}^4 ) + C \eta \sup_{0 \leq t \leq T} \| \nabla u \|_{L^2}^2 + C \eta^2 \int_0^T \| \nabla b \|_{L^2}^2 dt.
\]

(3.38)

which, after choosing \( \eta_0 \) suitably small, together with (3.37) yields that

\[
\bar{\rho} \sup_{0 \leq t \leq T} ( \| \nabla b \|_{L^2}^2 + \| b \|_{L^2}^4 ) + \bar{\rho} \int_0^T ( \| b \|_{L^2}^2 + \| \nabla b \|_{L^2}^2 + \| \nabla^2 b \|_{L^2}^2 ) dt
\]

\[
\leq C \bar{\rho} ( \| \nabla b_0 \|_{L^2}^2 + \| b_0 \|_{L^2}^4 ) + C \eta \sup_{0 \leq t \leq T} \| \nabla u \|_{L^2}^2 + C \eta^2 \int_0^T \| \nabla \theta \|_{L^2}^2 dt.
\]

(3.39)

2. We infer from Gagliardo-Nirenberg and Hölder’s inequalities, (3.7), (3.32), and (3.38) that

\[
\sup_{0 \leq t \leq T} \bar{\rho} \| \sqrt{\rho} E \|_{L^2}^2 + \bar{\rho} \int_0^T ( \| \sqrt{\rho} u \|_{L^2}^2 + \| \nabla \theta \|_{L^2}^2 ) dt
\]

\[
\leq C \bar{\rho} \| \sqrt{\rho_0} E_0 \|_{L^2}^2 + C \eta \bar{\rho} \int_0^T \| \nabla^2 b \|_{L^2}^2 dt
\]

\[
+ C \eta \bar{\rho} \int_0^T \| \nabla u \|_{L^2}^2 dt + C \eta \sup_{0 \leq t \leq T} \| \nabla b \|_{L^2}^2 \int_0^T \| \nabla u \|_{L^2}^2 dt
\]

\[
+ C \bar{\rho} \int_0^T \| \sqrt{\rho} \|_{L^2} \| \nabla \theta \|_{L^2} \| u \|_{L^2} \| \nabla u \|_{L^2} \| \rho \|_{L^2}^4 dt
\]

\[
\leq C \bar{\rho} ( \| \sqrt{\rho_0} E_0 \|_{L^2}^2 + \| \nabla b_0 \|_{L^2}^2 + ( \| b_0 \|_{L^2} \| \nabla b_0 \|_{L^2} ) \| \nabla b_0 \|_{L^2}^2 ) + C \eta \bar{\rho} \int_0^T \| \nabla^2 b \|_{L^2}^2 dt
\]

\[
+ C \bar{\rho} \left( \sup_{0 \leq t \leq T} \| \nabla b \|_{L^2}^2 ( \sup_{0 \leq t \leq T} ( \| \sqrt{\rho} u \|_{L^2}^2 + \| b \|_{L^2}^2 ) + \sup_{0 \leq t \leq T} \| \rho \|_{L^3}^2 \int_0^T \| \nabla \theta \|_{L^2}^2 dt \right)
\]

\[
+ C \eta \bar{\rho} \int_0^T \| \nabla \theta \|_{L^2}^2 dt + C \eta \bar{\rho} \int_0^T ( \| \nabla \theta \|_{L^2}^2 + \| u \|_{L^2} \| \nabla u \|_{L^2} )^2 dt
\]

\[
\leq C \bar{\rho} ( \| \sqrt{\rho_0} E_0 \|_{L^2}^2 + \| \nabla b_0 \|_{L^2}^2 ) + C \rho \bar{\rho} \sup_{0 \leq t \leq T} \| \nabla b \|_{L^2}^2 + C \eta \bar{\rho} \int_0^T ( \| \nabla \theta \|_{L^2}^2 + \| u \|_{L^2} \| \nabla u \|_{L^2} )^2 dt
\]

\[
+ C \eta \bar{\rho} \int_0^T \| \nabla b \|_{L^2}^2 dt + C \eta \bar{\rho} \int_0^T ( \| \nabla \theta \|_{L^2}^2 + \| u \|_{L^2} \| \nabla u \|_{L^2} )^2 dt.
\]

This, combined with the fact \( \eta_0 \) is suitably small, implies that

\[
\sup_{0 \leq t \leq T} \bar{\rho} \| \sqrt{\rho} E \|_{L^2}^2 + \bar{\rho} \int_0^T ( \| \sqrt{\rho} u \|_{L^2}^2 + \| \nabla \theta \|_{L^2}^2 ) dt
\]

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\[
\leq C\rho(\|\sqrt{\rho}E_0\|_{L^2}^2 + \|\nabla b_0\|_{L^2}^2) + C\rho\eta^2 \sup_{0 \leq t \leq T} \|\nabla b\|_{L^2}^2 + C\eta^2 \sup_{0 \leq t \leq T} \|\nabla u\|_{L^2}^2 \\
+ C\eta^2 \rho \int_0^T \|\nabla^2 b\|_{L^2}^2 dt.
\] (3.40)

3. Using (3.20), (3.39), and (3.40), and choosing \(\varepsilon_1 \leq \eta\), we deduce from (3.32) that

\[
\sup_{0 \leq t \leq T} \|\nabla u\|_{L^2}^2 + \int_0^T \left(\|\sqrt{\rho} u_t, \nabla F, \nabla w\|_{L^2}^2\right) dt \\
\leq C\|\nabla u_0\|_{L^2}^2 + C\rho(\sup_{0 \leq t \leq T} \|\sqrt{\rho}E_0\|_{L^2}^2 + \|\nabla b_0\|_{L^2}^2) + \eta\rho \int_0^T \|\nabla \|\nabla b\|_{L^2}^2 dt \\
+ C\eta^2 \int_0^T \|\nabla u\|_{L^2}^2 \|\nabla b\|_{L^2}^2 dt \\
+ C \sup_{0 \leq t \leq T} (\rho^2 \|\|\nabla \|\nabla E\|_{L^2}^2 + \|\nabla b\|_{L^2}^2) \int_0^T (\|\nabla \|\nabla\|_{L^2}^2 + \|\nabla \|\nabla u\|_{L^2}^2) dt + \eta\rho \int_0^T \|\nabla \|\nabla \|_{L^2}^2 dt \\
+ C\rho \int_0^T \|\nabla\|_{L^2}^2 \|\nabla b\|_{L^2}^2 \|\nabla b\|_{L^2}^2 dt + C\rho \eta^2 \sup_{0 \leq t \leq T} \|\nabla b\|_{L^2}^2 + C\eta^2 \sup_{0 \leq t \leq T} \|\nabla u\|_{L^2}^2 \\
+ C\eta^2 \rho \int_0^T \|\nabla^2 b\|_{L^2}^2 dt + C\rho \eta^2 \int_0^T \|\nabla u\|_{L^2}^2 dt.
\] (3.41)

In view of (3.32) and (3.1), we obtain after choosing \(\eta_0 < 1\) that

\[
\rho^3 \int_0^T \|\nabla u\|_{L^2}^2 (\|\nabla u\|_{L^2}^2 + \rho \|\sqrt{\rho}E\|_{L^2}^2 + \|b\|_{L^2}^2) dt \\
\leq C\rho^3 \sup_{0 \leq t \leq T} (\|\nabla u\|_{L^2}^2 + \rho \|\sqrt{\rho}E\|_{L^2}^2 + \eta^2 \|\nabla b\|_{L^2}^2) \sup_{0 \leq t \leq T} \|\nabla u\|_{L^2}^2 \int_0^T \|\nabla u\|_{L^2}^2 dt \\
\leq C\rho^3 (\|\nabla u\|_{L^2}^2 + \rho \|\sqrt{\rho}E\|_{L^2}^2 + \|\nabla b\|_{L^2}^2) \sup_{0 \leq t \leq T} \|\nabla u\|_{L^2}^2 \\
\times \left(\sup_{0 \leq t \leq T} (\rho \|\nabla u\|_{L^2}^2 + \|b\|_{L^2}^2) + C \sup_{0 \leq t \leq T} \|\nabla \|\nabla\|_{L^2}^2 dt \right) \\
\leq C\eta^2 \sup_{0 \leq t \leq T} \|\nabla u\|_{L^2}^2 + C\eta \int_0^T \|\nabla \|\nabla\|_{L^2}^2 dt,
\] (3.42)

and

\[
\rho \int_0^T \|\nabla b\|_{L^2}^2 \|\nabla u\|_{L^2}^2 dt \leq C\rho \sup_{0 \leq t \leq T} \|\nabla b\|_{L^2}^2 \left(\sup_{0 \leq t \leq T} (\rho \|\nabla u\|_{L^2}^2 + \|b\|_{L^2}^2) + \rho \|\nabla \|\nabla\|_{L^2}^2 dt \right) \\
\leq C\eta^2 \sup_{0 \leq t \leq T} \|\nabla b\|_{L^2}^2 + C\eta \int_0^T \|\nabla \|\nabla\|_{L^2}^2 dt.
\] (3.43)

Putting (3.42) and (3.43) into (3.41), we derive that

\[
\sup_{0 \leq t \leq T} \|\nabla u\|_{L^2}^2 + \int_0^T \left(\|\sqrt{\rho} u_t, \nabla F, \nabla w\|_{L^2}^2\right) dt \\
\leq C\|\nabla u_0\|_{L^2}^2 + C\rho(\sup_{0 \leq t \leq T} \|\sqrt{\rho}E_0\|_{L^2}^2 + \|\nabla b_0\|_{L^2}^2) + \eta\rho \int_0^T \|\nabla \|\nabla b\|_{L^2}^2 dt \\
+ C\eta^2 \sup_{0 \leq t \leq T} \|\nabla u\|_{L^2}^2 + C\eta \int_0^T \|\nabla \|\nabla\|_{L^2}^2 dt + C\eta^2 \rho \int_0^T \|b\|_{L^2}^2 dt \\
+ C\rho \eta^2 \sup_{0 \leq t \leq T} \|\nabla b\|_{L^2}^2 + C\eta^2 \sup_{0 \leq t \leq T} \|\nabla u\|_{L^2}^2 + C\eta^2 \rho \int_0^T \|\nabla^2 b\|_{L^2}^2 dt.
\]
\[ + C \eta^4 \bar{\rho} \int_0^T (\|\nabla \theta\|^2_{L^2} + ||u||^2_{\nabla u}) dt, \]

which, after choosing \( \eta_0 \) suitably small, together with (3.39) and (3.40) yields that

\[
\begin{align*}
\sup_{0 \leq t \leq T} \left[ \bar{\rho}(\|\nabla b\|^2_{L^2} + \|b\|^2_{L^2} + \|\sqrt{\rho} E\|^2_{L^2}) + \|\nabla u\|^2_{L^2} \right] & + \int_0^T \left( \|\nabla \rho u_{1, \bar{\rho}} \cdot \nabla \bar{w} \|_{L^2} \right)^2 dt \\
& + C \bar{\rho} \int_0^T (\|b\|_{L^2}^2 + \|\nabla b\|_{L^2}^2 + \|b\|_{L^2}^2 + \|u\|_{\nabla u}^2 + \|\nabla \theta\|^2_{L^2}) dt \\
& \leq C(\|\nabla u\|_{L^2}^2 + C \bar{\rho}(\|\sqrt{\rho_0} E\|_{L^2}^2 + \|\nabla b\|_{L^2}^2). \\
(3.44)
\end{align*}
\]

This implies (3.35).

4. We deduce from (3.1) and (3.44) that

\[
\sup_{0 \leq t \leq T} (\|\sqrt{\rho_0} u\|^2_{L^2} + \|b\|^2_{L^2}) + \int_0^T (\|\nabla u\|^2_{L^2} + \|\nabla b\|^2_{L^2}) dt
\leq \|\sqrt{\rho_0} u_0\|^2_{L^2} + \|b_0\|^2_{L^2} + C \sup_{0 \leq t \leq T} \|\rho\|^2_{L^2} \int_0^T \|\nabla \theta\|^2_{L^2} dt
\leq \|\sqrt{\rho_0} u_0\|^2_{L^2} + \|b_0\|^2_{L^2} + C \frac{1}{\bar{\rho}} \sup_{0 \leq t \leq T} \|\rho\|^2_{L^2} \left[ \bar{\rho}(\|\sqrt{\rho_0} E\|_{L^2}^2 + \|\nabla b\|_{L^2}^2 + \|\nabla u\|_{L^2}^2) \right]
\leq \|\sqrt{\rho_0} u_0\|^2_{L^2} + \|b_0\|^2_{L^2} + C \eta^2 \frac{1}{\bar{\rho}} \sup_{0 \leq t \leq T} \|\rho\|_{L^2}. \quad (3.45)
\]

Indeed, it follows from (3.13), (3.32), (3.45), and Young’s inequality that

\[
\sup_{0 \leq t \leq T} \|\rho\|^3_{L^3} + \int_0^T \rho^3 \|dx dt
\leq C \|\rho_0\|^3_{L^3} + C \sup_{0 \leq t \leq T} \left( \|\rho\|^2_{L^\infty} \|\sqrt{\rho_0} u\|^2_{L^2} \|\sqrt{\rho} E\|^2_{L^2} \right) \|\rho\|^3_{L^1} + C \bar{\rho}^2 \sup_{0 \leq t \leq T} \|\rho\|^2_{L^3} \int_0^T (\|\nabla u\|^2_{L^2} + \|\nabla b\|^2_{L^2}) dt
\leq C \|\rho_0\|^3_{L^2} + C \left( \eta^2 + \eta^2 \right) \sup_{0 \leq t \leq T} \|\rho\|^3_{L^3} + C \bar{\rho}^2 \sup_{0 \leq t \leq T} \|\rho\|^2_{L^3} \left( \|\sqrt{\rho_0} u_0\|^2_{L^2} + \|b_0\|^2_{L^2} \right)
\leq C \|\rho_0\|^3_{L^3} + C \left( \eta^2 + \frac{1}{4} \right) \sup_{0 \leq t \leq T} \|\rho\|^3_{L^3} + C \bar{\rho}^2 (\|\sqrt{\rho_0} u_0\|^2_{L^2} + \|b_0\|^2_{L^2})^3, \quad (3.46)
\]

which implies (3.33) by choosing \( \eta_0 \) suitably small.

Combining (3.45) and (3.46), we get that

\[
\sup_{0 \leq t \leq T} \bar{\rho}^2 (\|\sqrt{\rho_0} u\|^2_{L^2} + \|b\|^2_{L^2}) + \bar{\rho}^2 \int_0^T (\|\nabla u\|^2_{L^2} + \|\nabla b\|^2_{L^2}) dt
\leq C \bar{\rho}^2 (\|\sqrt{\rho_0} u_0\|^2_{L^2} + \|b_0\|^2_{L^2}) + C \eta^2 \sup_{0 \leq t \leq T} \|\rho\|_{L^3}
\leq C(\|\rho_0\|_{L^3} + \bar{\rho}^2 (\|\nabla u_0\|^2_{L^2} + \|b_0\|^2_{L^2})),
\]

which gives (3.34).

Finally, (3.36) follows from Lemma 3.6, (3.34), and (3.35). \( \square \)

**Proposition 3.1.** Assume that \( 3\mu > \lambda \). Let \( \eta_0, N_T \), and \( N_0 \) be as in Lemma 3.7. Then, the followings hold true.

(i) There exists a number \( \varepsilon_0 \in (0, \eta_0) \) depending only on \( R \), \( \mu \), and \( \lambda \) such that if

\[
\sup_{0 \leq t \leq T} \|\rho\|_{L^\infty} \leq 4\bar{\rho}, \quad N_T \leq \sqrt{\varepsilon_0}, \quad N_0 \leq \varepsilon_0,
\]

(3.47)
then
\[ \sup_{0 \leq t \leq T} \| \rho \|_{L^\infty} \leq 2 \bar{\rho}, \quad N_T \leq \frac{\sqrt{\varepsilon_0}}{2}. \]  

(ii) As a consequence of (i), the following estimates hold
\[ \sup_{0 \leq t \leq T} \| \rho \|_{L^\infty} \leq 2 \bar{\rho}, \quad N_T \leq \frac{\sqrt{\varepsilon_0}}{2}, \]  
provided that \( N_0 \leq \varepsilon_0 \) is sufficiently small.

Proof. (i) By (3.47), all the conditions in Lemma 3.7 hold true when \( \varepsilon_0 \leq \eta_0 \) is small enough. Thus, we have
\[ N_T \leq \bar{\rho}(\| \rho_0 \|_{L^3} + \bar{\rho}^2(\| \sqrt{\rho_0} u_0 \|_{L^2}^2 + \| b_0 \|_{L^2}^2))(\| \nabla u_0 \|_{L^2}^2 + \bar{\rho}(\| \sqrt{\rho_0} E_0 \|_{L^2}^2 + \| \nabla b_0 \|_{L^2}^2)) \leq C \varepsilon_0 \leq \frac{\sqrt{\varepsilon_0}}{2}, \]
and
\[ \sup_{0 \leq t \leq T} \| \rho \|_{L^\infty} \leq C \bar{\rho} e^{CN_0^{1/2} + CN_0^{1/2}} \leq \rho e^{CN_0^{1/2} + CN_0^{1/2}} \leq 2 \bar{\rho}, \]
provided that \( \varepsilon_0 \) is sufficiently small.

(ii) Define
\[ T^\# \triangleq \max \left\{ T' \in (0, T] \left| \sup_{0 \leq t \leq T'} \| \rho \|_{L^\infty} \leq 2 \bar{\rho}, \quad N_{T'} \leq \sqrt{\varepsilon_0} \right. \right\}. \]
Then, by (i), we have
\[ \sup_{0 \leq t \leq T} \| \rho \|_{L^\infty} \leq 2 \bar{\rho}, \quad N_{T'} \leq \frac{\sqrt{\varepsilon_0}}{2}, \quad \forall T' \in (0, T^\#). \]  

If \( T^\# < T \), noticing that \( N_{T'} \), and \( \sup_{0 \leq t \leq T'} \| \rho \|_{L^\infty} \) are continuous on \([0, T]\), there is another time \( T^{\#\#} \in (T^\#, T] \) such that
\[ \sup_{0 \leq t \leq T'} \| \rho \|_{L^\infty} \leq 2 \bar{\rho}, \quad N_{T'} \leq \sqrt{\varepsilon_0}, \]
which contradicts to the definition of \( T^\# \). Thus, we have \( T^\# = T \), and (3.48) follows from (3.49) and the continuity of \( N_{T'} \) and \( \sup_{0 \leq t \leq T'} \| \rho \|_{L^\infty} \) on \([0, T]\).

3.2 Proof of Theorem 1.1
With all the \textit{a priori} estimates established in Section 2, we can immediately obtain the existence result of Theorem 1.1 by standard arguments as those in [17]. Here we omit the details for simplicity.

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