Stability of black holes in Hořava gravity: Gravitational quasinormal modes

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Abstract

Gravitational perturbations in the geometry of asymptotically flat black hole solution in the deformed Hořava-Lifshitz (HL) gravity are considered and the associated quasinormal mode (QNM) frequencies are evaluated using WKB method. Black holes are found to be stable against small perturbations. We also find a significant deviation in the behavior of QNMs in HL theory for low values of HL parameter \( \omega \), from that in the standard General Relativity (GR). QNMs are long lived in Hořava theory compared with the usual Schwarzschild black hole case.

Keywords: Hořava-Lifshitz gravity, Black holes, Linear stability, Quasinormal modes.

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1. Introduction

One of the greatest goals in theoretical physics is to realize a quantum version of gravity. Attempts are being made in the past to explain gravity in the framework of quantum field theory but none are satisfactory. Recently a renormalizable theory of gravity in 3+1 dimensions was proposed by Hořava[1], inspired from the Lifshitz theory in solid state physics, now known as Hořava-Lifshitz theory. The theory is a potential candidate of

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quantum field theory of gravity. It assumes a Lifshitz-like anisotropic scaling between space and time at short distances, characterized by a dynamical critical exponent $z = 3$ and thus breaking the Lorentz invariance. While in the IR limit it flows to $z = 1$, retrieving the Einstein’s GR. Even though the discussions on some issues with the theory are going on [2, 3, 4, 5], as a new theory, it is interesting to investigate its various aspects in parallel. Cosmological implications [6, 7, 8, 9, 10, 11] and astrophysically viable tests of the theory [12, 13, 14, 15] had been studied. Since the theory has the same Newtonian and post Newtonian corrections as those of GR, systems of strong gravity, like black holes, are needed to get observable deviation from the standard GR. The black hole solutions in HL theory were studied in [16, 17, 18, 19, 20, 21] and some of its properties were investigated in [22, 23, 24, 25, 26, 27, 28].

Stability of black holes in HL theory is the topic of this paper. There are strong indirect evidences for the existence of black holes, and if the black holes are found to be unstable in the new theory it may question the reliability of the theory itself.

Gravitational perturbation and stability studies of black holes were originally formulated by Regge and Wheeler [29, 30]. Considering small perturbations in the space time outside the horizon and assuming harmonic time dependence to this perturbation, seek the solutions of the perturbation equation by imposing the boundary conditions that the perturbations will vanish at the horizon and at spatial infinity. If it admits solutions with damping amplitudes (frequencies with negative imaginary part) then the black hole is stable, otherwise the perturbations will grow in time and the gravitational waves will carry away energy of the system leading to an unstable black hole. This complex frequency with negative imaginary part is called quasinormal modes and its existence proves the stability of the black hole. Also features of the black holes in the new theory would be imprinted on the QNMs and if the gravitational detectors can observe them it can test the validity of the theory.

The paper is organized as follows. In the following section we briefly review asymptotically flat black hole solution in modified HL theory and derive the radial equation for perturbation in that black holes spacetime. In section 2 we evaluate the QNMs. The main conclusions are discussed in section 4.
2. Gravitational perturbation around black hole in Hořava theory

The IR vacuum of pure HL gravity is found to be anti-de Sitter\cite{16, 18}. Even though HL gravity could recover GR in IR at the action level for a particular value of the parameter $\lambda = 1$, there found a significant difference between these black hole solutions and the usual Schwarzschild AdS. The asymptotic fall-off of the metric for these black hole solutions is much slower than that of usual Schwarzschild AdS black holes in GR. Meanwhile Kehagias and Sfetsos\cite{17} could find a black hole solution in asymptotically flat Minkowski spacetimes applying deformation in HL theory by adding a term proportional to the Ricci scalar of three-geometry, $\mu^4R^{(3)}$ and then taking the limit $\Lambda_W \rightarrow 0$. This will not alter the UV properties of the theory but it does the IR ones leading to Minkowski vacuum analogous to Schwarzschild spacetime in GR. The deformed action in this particular limit is given as\cite{17},

\[
S = \int dtd^3x\sqrt{g}N\left\{ \frac{2}{k^2}(K_{ij}K^{ij} - \lambda K^2) - \frac{k^2}{2\omega^4}C_{ij}C^{ij} + \frac{k^2\mu^2}{2\omega^2}\epsilon^{ijk}R^{(3)}_{il}\nabla_j R^{(3)}_{kl} - \frac{k^2\mu^2}{8}R^{(3)}_{ij}R^{(3)ij} + \frac{k^2\mu^2}{8(1 - 3\lambda)}\frac{1 - 4\lambda}{4}\left(R^{(3)}\right)^2 + \mu^4R^{(3)} \right\}, \tag{1}
\]

with second fundamental form $K_{ij}$ defined as:

\[
K_{ij} = \frac{1}{2N}(\dot{g}_{ij} - \nabla_i N_j - \nabla_j N_i), \tag{2}
\]

and Cotton tensor $C^{ij}$,

\[
C^{ij} = \epsilon^{ikl}\nabla_k \left( R^{(3)lj} - \frac{1}{4}R^{(3)}\delta^{lj} \right). \tag{3}
\]

Considering the metric ansatz for a static, spherically symmetric solution,

\[
ds^2 = -N(r)^2dt^2 + f(r)^{-1}dr^2 + r^2d\Omega^2, \tag{4}
\]

the Lagrangian after angular integration reduces to

\[
\mathcal{L} = \frac{k^2\mu^2}{8(1 - 3\lambda)}\frac{N}{\sqrt{f}}\left[ (2\lambda - 1)\frac{(f - 1)^2}{r^2} - 2\lambda\frac{f - 1}{r}f' + \frac{\lambda - 1}{2}f^2 - 2\omega(1 - f - rf') \right]. \tag{5}
\]

In the IR limit $\lambda = 1$ of the theory, one gets asymptotic flat solution.
\[ N(r)^2 = f(r) = 1 + \omega r^2 - \sqrt{r(\omega^2 r^3 + 4\omega M)}, \] (6)

with horizons at,
\[ r_{\pm} = M \left(1 \pm \sqrt{1 - \frac{1}{2\omega M^2}}\right), \] (7)

where \( M \) is the integration constant related to ADM mass. In order that the singularity at \( r = 0 \) not to be naked it requires \( \omega M^2 \geq \frac{1}{2} \). In the limit \( \omega M^2 \gg 1 \) this solution recovers Schwarzschild solution in conventional General Relativity. Metric \( g_{\mu\nu} \) with coefficients given in Eq.(6) is our initial time-independent equilibrium system up on which we consider small perturbation \( h_{\mu\nu} \) with harmonic time dependence \( e^{-ikt} \). Where \( k \) is the frequency. Now at late time, we want to see whether this frequency has an imaginary part which makes the perturbations to grow exponentially in time(unstable) or leads to its damping(stable).

Equation governing the perturbation is the Einstein’s equation for this perturbed spacetime,
\[ R_{\mu\nu}(g + h) = 0, \] (8)

where \( R_{\mu\nu}(g + h) \) is the Ricci tensor computed from the total metric \( g_{\mu\nu} + h_{\mu\nu} \). Since the perturbation is small, taking terms up to linear in \( h_{\mu\nu} \), above equation can be expanded as,
\[ R_{\mu\nu}(g) + \delta R_{\mu\nu}(h) = 0, \] (9)

where \( R_{\mu\nu}(g) \) is the Ricci tensor computed from the unperturbed metric \( g_{\mu\nu} \) which will vanish and the equation becomes,
\[ \delta R_{\mu\nu}(h) = 0. \] (10)

Variation of Ricci tensor can be found from the expression
\[ \delta R_{\mu\nu} = -\delta \Gamma^\beta_{\mu\nu;\beta} + \delta \Gamma^\beta_{\mu;\beta\nu}, \] (11)

where the variation of affine connections is
\[ \delta \Gamma^\alpha_{\beta\gamma} = \frac{1}{2} g^{\alpha\nu} (h_{\beta\nu;\gamma} + h_{\gamma\nu;\beta} - h_{\beta\gamma;\nu}), \] (12)

Any arbitrary perturbations can be decomposed in to normal modes, since the background we are considering is spherically symmetric. For any given
value of the angular momentum $L$, associated with these modes, there are two classes of perturbations. Even, $(-1)^L$ and odd, $(-1)^{L+1}$ parity perturbations. Here we consider odd parity case. The canonical form of odd wave perturbations in Regge-Wheeler gauge\cite{20} is

$$h_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & h_0(r) \\ 0 & 0 & 0 & h_1(r) \\ 0 & 0 & 0 & 0 \\ h_0(r) & h_1(r) & 0 & 0 \end{pmatrix} e^{-ikt} \sin(\theta) \frac{\partial}{\partial\theta} P_L(\cos(\theta)). \quad (13)$$

Substituting Eq.(13) in Eq.(10) and separating the angular and radial parts we get the following radial equations:

$$\frac{ikh_0(r)}{f} + \frac{d}{dr} [fh_1(r)] = 0, \text{ from } \delta R_{23}, \quad (14)$$

$$\frac{ik}{f(r)} \left( \frac{dh_0}{dr} - \frac{2h_0}{r} \right) + h_1 \left\{ \frac{L(L+1)}{r^2} - \frac{k^2}{f} + \frac{f}{r} \left[ \frac{2}{r} + \frac{2\omega}{r - \frac{m + r^3\omega}{\sqrt{r\omega(4M + r^3\omega)}}} \right] \right\} = 0, \text{ from } \delta R_{13}. \quad (15)$$

Defining $dr_* = \frac{1}{f} dr$ and $Q = \frac{fh_1}{r}$, the above two equations can be combined to get a second order equation by eliminating $h_0(r)$:

$$\frac{d^2Q}{dr_*^2} + (k^2 - V_{eff})Q = 0. \quad (16)$$

where

$$V_{eff} = f \left[ \frac{L(L+1)}{r^2} - 6\omega \left( 1 - \frac{m + r^3\omega}{r\sqrt{r\omega(4M + r^3\omega)}} \right) \right] \quad (17)$$

Effective potential is plotted in Figure\ref{fig1} for $L = 3$. The height of the potential decreases with the increase of the parameter $\omega$ and finally merges with the Schwarzschild potential as $\omega$ tends to $\infty$.

3. Evaluation of quasinormal modes

Third order WKB approximation method is used here to evaluate the complex normal mode frequencies of black hole, a semi analytic method
originally developed by Schutz and Will \cite{32} for the lowest order. Later Iyer and Will \cite{33} carried out this approach to third WKB order and Konoplya \cite{34} to sixth order to get more accurate results. The method is found to be accurate for low lying modes and can be used to explore the QNM behavior of black holes efficiently, without going to the complicated numerical methods.

WKB method gives a simple condition which will lead to discrete, complex values for the normal mode frequencies:

\[
k^2 = [V_0 + (-2V_0')^2 Λ] - i(n + \frac{1}{2})(-2V_0')^{\frac{3}{2}}(1 + Ω), \tag{18}
\]

where Λ and Ω are second and third order WKB correction terms,

\[
Λ = \frac{1}{(-2V_0''')^{1/2}} \left\{ \frac{1}{8} \left( \frac{V_0'''}{V_0''} \right)^2 \left( \frac{1}{4} + α^2 \right) - \frac{1}{288} \left( \frac{V_0'''}{V_0''} \right)^2 \left( 7 + 60α^2 \right) \right\}, \tag{19}
\]

\[
Ω = \frac{1}{(-2V_0''')^{1/2}} \left\{ \frac{5}{6912} \left( \frac{V_0'''}{V_0''} \right)^4 \left( 77 + 188α^2 \right) - \frac{1}{384} \left( \frac{V_0'''}{V_0''} \right)^2 \left( 51 + 100α^2 \right) \right\}
\]

\[
+ \frac{1}{2304} \left( \frac{V_0'''}{V_0''} \right)^2 \left( 67 + 68α^2 \right) + \frac{1}{288} \left( \frac{V_0'''}{V_0''} \right)^2 \left( 19 + 28α^2 \right)
\]

\[
+ \frac{1}{288} \left( \frac{V_0'''}{V_0''} \right)^2 \left( 5 + 4α^2 \right) \left\}, \right. \tag{20}
\]
\[ \alpha = n + \frac{1}{2} \text{ and } n \text{ is the mode number,} \]

\[
n = \{ 0, 1, 2, \ldots \ldots \ldots \text{Re}(E) > 0 \}
\]

\[
n = \{-1, -2, -3, \ldots \ldots \ldots \text{Im}(E) < 0. \}
\]

\[ V^{(n)}_0 \] denotes the \( n^{th} \) derivative of \( V \) evaluated at \( r_0 \), the value of \( r \) at which \( V \) attains maximum. Table 1 and Table 2 list the QNMs evaluated.

| \( \omega \) | \( L=2 \) | \( L=3 \) | \( L=4 \) | \( L=5 \) |
|---|---|---|---|---|
| 0.5 | 0.4322 - 0.05945 i | 0.68305 - 0.06558 i | 0.91050 - 0.06764 i | 1.13252 - 0.06863 i |
| 1 | 0.4001 - 0.07875 i | 0.63226 - 0.08339 i | 0.84942 - 0.08506 i | 1.06027 - 0.08586 i |
| 1.5 | 0.38949 - 0.08221 i | 0.61995 - 0.08694 i | 0.83446 - 0.08857 i | 1.04250 - 0.08933 i |
| 2 | 0.38488 - 0.08397 i | 0.61434 - 0.08853 i | 0.82762 - 0.09011 i | 1.03435 - 0.09087 i |
| 3 | 0.38065 - 0.08573 i | 0.60904 - 0.09001 i | 0.82113 - 0.09156 i | 1.02662 - 0.09229 i |
| 4 | 0.37866 - 0.08661 i | 0.60650 - 0.09072 i | 0.81801 - 0.09224 i | 1.02290 - 0.09297 i |
| 6 | 0.37676 - 0.08749 i | 0.60403 - 0.09140 i | 0.81497 - 0.09290 i | 1.01927 - 0.09362 i |
| 8 | 0.37583 - 0.08792 i | 0.60281 - 0.09174 i | 0.81348 - 0.09323 i | 1.01748 - 0.09394 i |
| 10 | 0.37528 - 0.08819 i | 0.60209 - 0.09194 i | 0.81259 - 0.09342 i | 1.01642 - 0.09413 i |
| \( \infty \) | 0.37316 - 0.08922 i | 0.59927 - 0.09273 i | 0.80910 - 0.09417 i | 1.02568 - 0.09067 i |

Table 1: Fundamental \( (n = 0) \) QNMs of gravitational perturbations for various values of \( \omega \) and \( L \). \( \omega = \infty \) represents the Schwarzschild limit.

| \( n \) | \( \omega = 1 \) | \( \omega = 2 \) | \( \omega = 3 \) | \( \omega = 4 \) | \( \omega = \infty \) |
|---|---|---|---|---|---|
| 0 | 0.63220 - 0.08339 i | 0.61434 - 0.08853 i | 0.60904 - 0.09001 i | 0.60650 - 0.09072 i | 0.59927 - 0.09273 i |
| 1 | 0.62029 - 0.09228 i | 0.59996 - 0.26827 i | 0.59381 - 0.27290 i | 0.59085 - 0.27511 i | 0.58235 - 0.28141 i |
| 2 | 0.59924 - 0.45383 i | 0.57504 - 0.45373 i | 0.56747 - 0.46182 i | 0.56380 - 0.46568 i | 0.55320 - 0.47668 i |
| 3 | 0.57207 - 0.60378 i | 0.54310 - 0.64417 i | 0.53372 - 0.65589 i | 0.52913 - 0.66149 i | 0.51575 - 0.67743 i |
| 4 | 0.54009 - 0.78467 i | 0.50538 - 0.83773 i | 0.49372 - 0.85315 i | 0.48797 - 0.86053 i | 0.47197 - 0.88154 i |

Table 2: Higher mode frequencies for different values of \( \omega \) with \( L=3 \)

The variation of real and imaginary parts of QNMs with the parameter \( \omega \) are plotted in Figure 2. The oscillation frequency, \( \text{Re}(\omega) \) decreases with the increase of \( \omega \) while the damping time \( |\text{Im}(\omega)| \), increases with the increase of \( \omega \). For values near \( \omega = 0.5 \) there is a sudden decrease in damping time. Finally comparing with the Schwarzschild black hole, oscillation frequencies of QNMs in HL gravity is larger and has a lower damping time.

4. Conclusion

We have investigated the evolution of gravitational perturbations in black hole spacetime in HL theory and associated QNMs are evaluated. The negative imaginary part of QNMs show that the perturbation will decay in time
and hence we can conclude that black holes are stable against small perturbations in the spacetime. The present calculations show that the gravitational QNMs are long lived in Hořava theory compared with the Schwarzschild black hole.

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