OPTIMAL BOUNDS ON THE SPEED OF SUBSPACE EVOLUTION \(^*\)\(^†\)

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ABSTRACT. By a quantum speed limit one usually understands an estimate on how fast a quantum system can evolve between two distinguishable states. The most known quantum speed limit is given in the form of the celebrated Mandelstam-Tamm inequality that bounds the speed of the evolution of a state in terms of its energy dispersion. In contrast to the basic Mandelstam-Tamm inequality, we are concerned not with a single state but with a (possibly infinite-dimensional) subspace which is subject to the Schrödinger evolution. By using the concept of maximal angle between subspaces we derive optimal bounds on the speed of such a subspace evolution. These bounds may be viewed as further generalizations of the Mandelstam-Tamm inequality. Our study includes the case of unbounded Hamiltonians.

1. INTRODUCTION

By a quantum speed limit one usually calls a lower bound on the time that is needed for a quantum system to evolve from a given state to a target state or a target subspace. The history of the subject is already long, being traced back to the 1945’s pioneering work by Mandelstam and Tamm [1]. The volume of literature on quantum speed limits and their applications in a variety of areas is large and by no means we make here an attempt to present a more or less complete review of all relevant results. Instead, we only inform the interested reader that comprehensive surveys of the literature on various quantum speed limits may be found in the recent review articles [2] and [3] (see also the introductory part of the very recent paper [4]).

We begin with recalling how the main known quantum speed limits look. To this end, we consider an isolated quantum system described by a Hamiltonian \(H\), which is assumed to be a time-independent self-adjoint operator acting in the complex Hilbert space \(\mathcal{H}\). Any vector \(\phi\) from the unit sphere in \(\mathcal{H}\) represents a possible pure state of this system. Strictly speaking, a pure state \(\mathcal{S}\) is rather a class of equivalence of norm-one vectors in \(\mathcal{H}\): the vectors \(\phi, \psi \in \mathcal{H}\) with \(\|\phi\| = \|\psi\| = 1\) represent the same pure state if \(\psi = u\phi\) for some \(u \in \mathbb{C}\) such that \(|u| = 1\). In an obvious way, the state \(\mathcal{S}\) may be identified with a one-dimensional subspace \(\mathfrak{P}_\mathcal{S}\) which is the span of an arbitrarily chosen vector \(\psi\) in \(\mathcal{S}\), \(\mathfrak{P}_\mathcal{S} := \{f = \lambda \psi \mid \lambda \in \mathbb{C}\}\).

In what follows we will always suppose that units of measurement are chosen such that \(\hbar = 1\). The time evolution of a state vector \(\psi(t) \in \mathcal{H}, t \in \mathbb{R}\), is assumed to be governed by the Schrödinger equation

\[
\begin{align*}
\frac{i}{\hbar} \frac{d}{dt} \psi &= H \psi, \\
\psi(t)\big|_{t=0} &= \psi_0.
\end{align*}
\]

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where the initial-state vector $\psi_0$, $\|\psi_0\| = 1$, along with all other vectors on the path $\psi(t)$, $t \in \mathbb{R}$, should belong to the domain $\text{Dom}(H)$ of the Hamiltonian $H$.

Starting point in the study of quantum speed limits was the following natural question: How fast can a quantum system with the Hamiltonian $H$ arrive at a state orthogonal to its initial state $\psi_0$?

It is obvious that the answer to this question may be important in various respects. Perhaps, the very latest motivation stems from quantum information theory and quantum computing (see, e.g., [2, 3]).

Available answers to the above question have been given in the form of lower bounds for the orthogonalization time $T_\perp$ which is the time necessary for the system to evolve from the initial state $\psi_0$ to a state $\psi(T_\perp)$ such that $\langle \psi_0, \psi(T_\perp) \rangle = 0$. (Here and in what follows by $\langle \cdot, \cdot \rangle$ we denote the inner product in the Hilbert space $\mathcal{H}$ assuming that it is linear in the first entry.)

The oldest among these bounds is the celebrated Mandelstam–Tamm inequality discovered in the 1945’s paper [1]:

$$T_\perp \geq \frac{\pi}{2 \Delta E}, \quad (1.3)$$

where $\Delta E$ is the energy dispersion for the initial state $\psi_0$,

$$\Delta E = \left( \|H\psi_0\|^2 - \langle H\psi_0, \psi_0 \rangle^2 \right)^{1/2}, \quad \psi_0 \in \text{Dom}(H). \quad (1.4)$$

The second celebrated lower bound for the orthogonalization time, the Margolus–Levitin inequality [5] has been discovered half a century later, in 1998. This bound reads as

$$T_\perp \geq \frac{\pi}{2 \delta E}, \quad (1.5)$$

where the quantity

$$\delta E = \langle H\psi_0, \psi_0 \rangle - \min(\text{spec}(H)) \quad (1.6)$$

represents the difference between the average energy for the state $\psi_0$ and the lower edge of the spectrum of the Hamiltonian $H$ (which is assumed to be semibounded from below in this case).

The lower bounds (1.3) and (1.5) are not equivalent to each other but both of them have been proven to be optimal (see, e.g., [2, p. 7] and [3, p. 3923]).

Of course, one notices that by their form the bounds (1.3) and (1.5) resemble the uncertainty relation for energy and time. These bounds, however, are related not to the standard deviation in the measuring of the quantity $t$ but to the well-established time needed for a state of the system to evolve into an orthogonal state. Thus, in their essence the inequalities (1.3) and (1.5) are very different from the uncertainty relation.

Next, there is a version of the Mandelstam–Tamm inequality that works for intermediate time moments $t \in (0, T_\perp)$. This is the lower estimate found for the first time in 1973 by Fleming [6]:

$$T_\theta \geq \frac{\theta}{\Delta E}, \quad (1.7)$$

where $\Delta E$ is again given by (1.4) and $T_\theta$ denotes the first time moment when the acute angle

$$\angle(\psi_0, \psi(t)) := \arccos|\langle \psi_0, \psi(t) \rangle| \quad (1.8)$$

between the vectors $\psi_0$ and $\psi(t)$ reaches a certain value $\theta \in (0, \pi/2]$.

It is worth to remark that, through the years, the Mandelstam-Tamm bound (1.3)/(1.7) has been rediscovered several times (for related discussion and references, see, e.g., [2, p. 5]). The Mandelstam-Tamm bound has also been extended to the evolution of mixed states [7]. Furthermore, more detailed estimates for the evolution speed have been established for particular classes
of evolutionary problems (see [2, 3]). Probably, the latest among them is a speed limit for evolution of thermal states derived in [4]. Mandelstam-Tamm-type bounds for the orthogonalization time exist even for some non-self-adjoint (so-called pseudo-Hermitian and, in particular, PT-symmetric) Hamiltonians (see [8, 9] and references therein).

In our recent work [10] we have generalized the Mandelstam-Tamm-Fleming bound (1.7) to the Schrödinger evolution of a subspace. Like in the vast majority of publications on quantum speed limits, in [10] we restrict ourselves to the exclusive consideration of bounded Hamiltonians. However, typical quantum-mechanical Hamiltonians are unbounded operators. In the present work we drop the requirement of boundedness of $H$ and extend most of the results of [10] to the subspace evolution governed by arbitrary self-adjoint Hamiltonians. Assuming that $P_0$ is an orthogonal projection in $H$ such that the domain of a (possibly unbounded) self-adjoint operator (Hamiltonian) $H$ is invariant under $P_0$, that is, $P_0 \text{ Dom}(H) \subset \text{ Dom}(H)$, we study the subspace path $P_t = \text{ Ran}(P_t), t \in \mathbb{R}$, formed in the set of all the subspaces of $H$ by the ranges of the orthogonal projections $P_t = e^{-iHt}P_0e^{iHt}, t \in \mathbb{R}$.

Our studies of the subspace evolution path $P_t$, $t \in \mathbb{R}$, are essentially based on the concept of maximal angle between two subspaces of a Hilbert space. Recall, that the maximal angle between (arbitrary) subspaces $\mathcal{Q}$ and $\mathcal{R}$ of the Hilbert space $H$ is introduced as follows:

$$\theta(\mathcal{Q}, \mathcal{R}) = \arcsin \|Q - R\|,$$  

(1.9)

where $Q$ and $R$ are the orthogonal projections in $H$ onto $\mathcal{Q}$ and $\mathcal{R}$, respectively. The maximal angle (1.9) possesses all the properties of a distance, and thus it generates a metric on the set of all subspaces of $H$. By using this metric we establish, in particular, the following result (see Theorem 3.12 below).

Assume that $T_\theta$ is the time moment at which the maximal angle between the initial subspace $\mathcal{Q}_0$ and a subspace in the path $P_t$, $t \geq 0$, reaches a certain value $\theta \in (0, \pi/2]$. Then necessarily

$$T_\theta \geq \frac{\theta}{\Delta E_{\mathcal{Q}_0}},$$  

(1.10)

where

$$\Delta E_{\mathcal{Q}_0} = \sup_{\psi \in \mathcal{Q}_0 \cap \text{ Dom}(H)} \left( \|H\psi\|^2 - \langle H\psi, \psi \rangle^2 \right)^{1/2}.$$  

(1.11)

Clearly, the quantity $\Delta E_{\mathcal{Q}_0}$ is nothing but the least upper bound for the energy dispersion on the states belonging to the initial subspace $\mathcal{Q}_0$.

The Mandelstam-Tamm-Fleming bound (1.7) turns out to be a special case of the estimate (1.10) for a one-dimensional subspace $\mathcal{Q}_0$ spanned by a particular state $\psi_0$.

The plan of the paper is as follows. In Section 2 we collect some facts on the projection path $P_t = e^{-iHt}P_0e^{iHt}, t \in \mathbb{R}$. In particular, we notice that this path is strongly continuous on the whole Hilbert space $H$. Under the assumption $P_0 \text{ Dom}(H) \subset \text{ Dom}(H)$, the path $P_t, t \in \mathbb{R}$, is, in addition, strongly differentiable in $t \in \mathbb{R}$ on the domain of $H$. In such a case the strong derivative $P_t$ is expressed through the commutator of $H$ and $P_t$ (see Theorem 2.1). In Section 3 we work under the additional hypothesis that the commutator of $H$ and $P_0$ is a bounded operator on $\text{ Dom}(H)$ and, hence, its closure is a bounded operator on the whole space $H$. The main result of the section is Theorem 3.6. It presents the upper bound (3.16) for the maximal angle between the subspaces $\mathcal{Q}_s$ and $\mathcal{Q}_t$, $s, t \in \mathbb{R}$, in the subspace path $P_t = \text{ Ran}(P_t), \tau \in \mathbb{R}$, through the product of the times difference $|t - s|$ and the norm of the commutator of $H$ and $P_0$. This section

\[\text{For discussion of the concept of maximal angle and references see page 9} \]
also contains the proof of Theorem 3.12 that we already mentioned. The section is concluded with a new consideration of the case where \( H \) is bounded operator. In such a case the quantity \((1.11)\) is bounded by half the distance between the upper and lower edges of the spectrum of \( H \). Combining this with \((1.10)\) we obtain our last lower bound for \( T_\theta \) (see Theorem 3.18) in this paper. Finally, in Section 4 we present a summary of the work and point out some open problems.

Let us add a few words about notations used throughout the paper. By a subspace we always understand a closed linear subset of a Hilbert space. The identity operator is denoted by \( I \). For a linear operator \( L \), by \( \text{Dom}(L) \) we denote its domain and by \( \text{Ran}(L) \), its range. If \( Q \) is an orthogonal projection, the notation \( Q^\perp \) is always used for the complementary projection, \( Q^\perp = I - Q \). By \( \mathcal{M} \oplus \mathcal{N} \) we understand the orthogonal sum of two Hilbert spaces (or orthogonal subspaces or simply orthogonal linear subsets) \( \mathcal{M} \) and \( \mathcal{N} \). By \( E_T(\sigma) \) we always denote the spectral projection of a self-adjoint operator \( T \) associated with a Borel set \( \sigma \subset \mathbb{R} \). Notation \([A,B]\) is used for the commutator of linear operators \( A \) and \( B \) on \( \mathfrak{H} \). It is assumed that \( \text{Dom}\left([A,B]\right) := \{ x \in \text{Dom}(A) \cap \text{Dom}(B) \mid Ax \in \text{Dom}(B), Bx \in \text{Dom}(A) \} \) and \([A,B]x := ABx - BAx\) for any \( x \in \text{Dom}\left([A,B]\right) \).

2. **Projection path generated by the Schrödinger evolution of a subspace**

As it was already underlined, we are concerned not with a single state but with a multi-dimensional (possibly, even infinite-dimensional) subspace spanned by the states of the system that are subject to the Schrödinger evolution. In general, we allow the Hamiltonian \( H \) of the system to be an unbounded self-adjoint operator on the Hilbert space \( \mathfrak{H} \) with domain \( \text{Dom}(H) \). In this work we restrict ourselves to the consideration of a nontrivial subspace \( \mathfrak{P}_0 \subset \mathfrak{H}, \mathfrak{P}_0 \neq \{0\} \), such that

\[
P_0 \text{ Dom}(H) \subset \text{Dom}(H),
\]

where \( P_0 \) stands for the orthogonal projection in \( \mathfrak{H} \) onto \( \mathfrak{P}_0 \). That is, we assume that the linear set \( \text{Dom}(H) \) is invariant under \( P_0 \) in the sense that \( P_0 f \in \text{Dom}(H) \) for any \( f \in \text{Dom}(H) \). From \((2.1)\) it follows that the set \( \text{Dom}(H) \) is also invariant under the complementary orthogonal projection \( P_0^\perp \),

\[
P_0^\perp \text{ Dom}(H) \subset \text{Dom}(H),
\]

Moreover, any of the hypotheses \((2.1)\) and \((2.2)\) is equivalent to any of the equalities

\[
\text{Ran}\left(P_0\mid_{\text{Dom}(H)}\right) = \mathfrak{P}_0 \cap \text{Dom}(H) \quad \text{and} \quad \text{Ran}\left(P_0^\perp\mid_{\text{Dom}(H)}\right) = \mathfrak{P}_0^\perp \cap \text{Dom}(H)
\]

as well as to the combination of them,

\[
\text{Dom}(H) = (\mathfrak{P}_0 \cap \text{Dom}(H)) \oplus (\mathfrak{P}_0^\perp \cap \text{Dom}(H)).
\]

Since \( H \) is a self-adjoint operator, its domain \( \text{Dom}(H) \) is dense in \( \mathfrak{H} \). By \((2.4)\), this implies that the sets \( \mathfrak{P}_0 \cap \text{Dom}(H) \) and \( \mathfrak{P}_0^\perp \cap \text{Dom}(H) \) are dense in the subspaces \( \mathfrak{P}_0 \) and \( \mathfrak{P}_0^\perp \), respectively. In particular,

\[
\mathfrak{P}_0 \cap \text{Dom}(H) \subset \mathfrak{P}_0.
\]

From \((2.3)\) it follows that the Hamiltonian \( H \) admits the following \( 2 \times 2 \) block matrix representation with respect to the decomposition \( \mathfrak{H} = \mathfrak{P}_0 \oplus \mathfrak{P}_0^\perp \):

\[
H = H_{\text{diag}} + H_{\text{off}}, \quad H_{\text{diag}} = \begin{pmatrix} H_{\mathfrak{P}_0} & 0 \\ 0 & H_{\mathfrak{P}_0^\perp} \end{pmatrix}, \quad H_{\text{off}} = \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix},
\]
where
\[ H_{\psi_0} = P_0 H |_{\psi_0}, \quad \text{Dom}(H_{\psi_0}) = \mathcal{P}_0 \cap \text{Dom}(H), \quad (2.7) \]
\[ H_{\psi_0}^* = P_0^* H |_{\psi_0^*}, \quad \text{Dom}(H_{\psi_0}^*) = \mathcal{P}_0^* \cap \text{Dom}(H), \quad (2.8) \]
\[ B = P_0 H |_{\psi_0}, \quad \text{Dom}(B) = \text{Dom}(H_{\psi_0}), \quad (2.9) \]
\[ C = P_0^* H |_{\psi_0^*}, \quad \text{Dom}(C) = \text{Dom}(H_{\psi_0}). \quad (2.10) \]

Notice that, in general, \( C \subset B^* \) and \( B \subset C^* \).

Every vector \( \psi_0 \subset \mathcal{P}_0 \cap \text{Dom}(H) \) is assumed to evolve into a vector \( \psi(t) \in \text{Dom}(H), t > 0 \), according to (1.1), (1.2). It is well known that for any \( \psi_0 \in \text{Dom}(H) \) the Cauchy problem (1.1), (1.2) has a unique solution \( \psi(t) \in \text{Dom}(H) \) for all \( t \geq 0 \). For convenience of the reader, we remark that the existence of the solution to the problem (1.1), (1.2) in the form
\[ \psi(t) = U(t) \psi_0, \quad t > 0, \quad (2.11) \]
where
\[ U(t) = e^{-i H t}, \quad t \in \mathbb{R}, \quad (2.12) \]
follows, e.g., from [11, Theorem VIII.7]. Surely, the exponential (2.12) is defined by the spectral theorem (see, e.g., [11, Theorem VIII.6]) as
\[ e^{-i H t} := \int_{\mathbb{R}} e^{-i \lambda t} E(d\lambda), \quad t \in \mathbb{R}, \quad (2.13) \]
where \( E \) stands for the spectral measure on \( \mathbb{R} \) associated with the self-adjoint operator \( H \). It is worth to notice that the domain of \( H \),
\[ \text{Dom}(H) = \left\{ f \in \mathcal{S} \left| \int_{\mathbb{R}} \lambda^2 \langle E(d\lambda) f, f \rangle \right. \right\}, \quad (2.14) \]
is an invariant of \( U(t) \),
\[ \text{Ran} \left( U(t) |_{\text{Dom}(H)} \right) = \text{Dom}(H), \quad \text{for any } t \in \mathbb{R}. \quad (2.15) \]

Given \( t \in \mathbb{R} \), by \( \mathcal{P}_t \), we denote the range \( \text{Ran} \left( U(t) |_{\mathcal{P}_0} \right) \) of the product of the unitary operator (2.12) and the orthogonal projection \( P_0 \) onto the subspace \( \mathcal{P}_0 \), that is,
\[ \mathcal{P}_t := \text{Ran} \left( U(t) |_{\mathcal{P}_0} \right), \quad t \in \mathbb{R}. \quad (2.16) \]

By (2.11), the subspace \( \mathcal{P}_t, t > 0 \), is nothing but the closure of the span of the vectors \( \psi(t) \subset \text{Dom}(H) \) representing the values, at the time moment \( t \), of the vector-valued functions \( \psi : \mathbb{R}^+ \to \mathcal{S} \) that solve (1.1), (1.2) for various \( \psi_0 \in \mathcal{P}_0 \cap \text{Dom}(H) \). So that we deal with the path \( \mathcal{P}_t, t \geq 0 \), in the set of all subspaces of the Hilbert space \( \mathcal{S} \). Or (and this is the same) with the path
\[ P_t, \quad t \geq 0, \quad \text{Ran}(P_t) = \mathcal{P}_t, \quad (2.17) \]
of the orthogonal projections \( P_t \) in \( \mathcal{S} \) onto the respective subspaces \( \mathcal{P}_t \). Clearly, the orthogonal projections \( P_t \) onto the subspaces \( \mathcal{P}_t \), defined in (2.16) are explicitly given by
\[ P_t = U(t) P_0 U(t)^* = e^{-i H t} P_0 e^{i H t}, \quad \text{for any } t \in \mathbb{R}. \quad (2.18) \]

It is almost obvious that the strong continuity of the unitary group \( e^{-i H t}, t \in \mathbb{R} \), implies the strong continuity of the path \( P_t, t \in \mathbb{R} \), on the whole Hilbert space \( \mathcal{S} \). Under the assumption (2.1) from (2.15) it immediately follows that the domain of \( H \) is mapped by \( P_t \) back into the domain of \( H \),
\[ \text{Ran} \left( P_t |_{\text{Dom}(H)} \right) \subset \text{Dom}(H), \quad t \in \mathbb{R}. \quad (2.19) \]
For convenience of the reader we present a proof of the strong differentiability of the projection family (2.18) under the hypothesis (2.1).

**Theorem 2.1.** Let $H$ be a (possibly unbounded) self-adjoint operator in the Hilbert space $S$. Assume that $P_0, P_0 \neq 0$, is an orthogonal projection in $S$ and that the domain of $H$ is invariant under $P_0$, i.e., $\text{Ran}(P_0|_{\text{Dom}(H)}) \subset \text{Dom}(H)$. Then the projection path $P_t = e^{-iHt}P_0e^{iHt}$, $t \in \mathbb{R}$, is strongly differentiable on $\text{Dom}(H)$ for any $t \in \mathbb{R}$, that is, the following limit exists

$$
P_t f := \lim_{\tau \to 0} \left( \frac{P_{t+\tau} - P_t}{\tau} \right), \quad t \in \mathbb{R},
$$

for any $f \in \text{Dom}(H)$.

Moreover, the inclusion (2.19) holds and the following equality takes place:

$$
iP_t f = H P_0 f - P_0 H f, \quad t \in \mathbb{R},
$$

for any $f \in \text{Dom}(H)$.

**Proof.** As we already noticed, under the hypothesis $\text{Ran}(P_0|_{\text{Dom}(H)}) \subset \text{Dom}(H)$ the inclusion (2.19) follows immediately from (2.15). Assume that $f \in \text{Dom}(H)$ and $t \in \mathbb{R}$. By (2.19) we have $P_t f \in \text{Dom}(H)$. Now take $0 \neq \tau \in \mathbb{R}$ and write

$$
\frac{P_{t+\tau} - P_t}{\tau} = \frac{e^{-iH(t+\tau)}P_0e^{iH(t+\tau)} - e^{-iHt}P_0e^{iHt}}{\tau}
$$

$$
= \frac{e^{-iH(t+\tau)}P_0e^{iHt}e^{-iH\tau} - e^{-iHt}P_0e^{iHt}}{\tau}
$$

$$
= \frac{e^{-iH(t+\tau)}P_0e^{iH}(t+\tau) - e^{-iHt}P_0e^{iHt}}{\tau} = g_1(t, \tau, f) + g_2(t, \tau, f),
$$

where the vectors $g_1$ and $g_2$ are given by

$$
g_1(t, \tau, f) = e^{-iH(t+\tau)}P_0e^{iH}(t+\tau) - I f,
$$

$$
g_2(t, \tau, f) = e^{-iHt}P_0e^{iHt}f.
$$

For $g_1(t, \tau, f)$ we have

$$
g_1(t, \tau, f) = g_{11}(t, \tau, f) + g_{12}(t, \tau, f),
$$

where

$$
g_{11}(t, \tau, f) = e^{-iH(t+\tau)}P_0e^{iH}(t+\tau) \left( \frac{e^{iH\tau} - I}{\tau} f - iH f \right),
$$

$$
g_{12}(t, \tau, f) = e^{-iH(t+\tau)}P_0e^{iH}(iH f).
$$

The unitarity of $e^{-iH(t+\tau)}$ and $e^{iHt}$ jointly with the fact that the orthogonal projection $P_0$ has unit norm, $\|P_0\| = 1$, yields

$$
\|g_{11}(t, \tau, f)\| \leq \left\| e^{-iH(t+\tau)} \right\| \|P_0\| \left\| e^{iHt} \right\| \left\| \frac{e^{iH\tau} - I}{\tau} f - iH f \right\| \leq \left\| \frac{e^{iH\tau} - I}{\tau} f - iH f \right\|. \quad (2.24)
$$

Taking into account the strong differentiability of the group $e^{iHt}$ on $\text{Dom}(H)$, from (2.24) one concludes that

$$
g_{11}(t, \tau, f) \to 0 \quad \text{as} \quad \tau \to 0 \quad (\text{for any} \ t \in \mathbb{R} \ \text{and any} \ f \in \text{Dom}(H)). \quad (2.25)
$$
At the same time, in view of the strong continuity of the group $e^{iHt}$ on $\mathcal{F}$,

$$g_{12}(t, \tau, f) \xrightarrow[t \to 0]{\tau} e^{-iHt} P_0 e^{iHt} (iH f) = iP_t H f \quad \text{(for any } t \in \mathbb{R} \text{ and any } f \in \operatorname{Dom}(H)). \quad (2.26)$$

As for the term $g_2(t, \tau, f)$, it is easy to observe that

$$g_2(t, \tau, f) \xrightarrow[t \to 0]{\tau} e^{-iHt} (-iH) P_0 e^{iHt} f \quad \text{(for any } t \in \mathbb{R} \text{ and any } f \in \operatorname{Dom}(H)). \quad (2.27)$$

This follows again from the strong differentiability of the group $e^{iHt}$ on $\operatorname{Dom}(H)$, taking into account that $e^{iHt} f \in \operatorname{Dom}(H)$ whenever $f \in \operatorname{Dom}(H)$ and then $P_0 e^{iHt} f \in \operatorname{Dom}(H)$ by the hypothesis. Now it only remains to recollect that $e^{-iHt} (-iH) P_0 e^{iHt} f = -iHe^{-iHt} P_0 e^{iHt} f = -iHP_t f$, which means that

$$g_2(t, \tau, f) \xrightarrow[t \to 0]{\tau} -iHP_t f \quad \text{(for any } t \in \mathbb{R} \text{ and any } f \in \operatorname{Dom}(H)). \quad (2.27)$$

Combining this result with (2.22), (2.23), (2.25), and (2.26) completes the proof. \hfill \square

**Remark 2.2.** Equality (2.21) implies that, under the hypothesis of Theorem 2.1, the projection path (2.18) is a strong solution (on $\operatorname{Dom}(H)$) to the Cauchy problem

$$i \frac{d}{dt} P_t = [H, P_t], \quad (2.28)$$

$$P_t|_{t=0} = P_0, \quad (2.29)$$

where $\frac{d}{dt} P_t = \dot{P}_t$ stands for the strong derivative (2.20) and

$$[H, P_t] := HP_t - P_t H, \quad \operatorname{Dom}([H, P_t]) = \operatorname{Dom}(H), \quad (2.30)$$

denotes the commutator of $H$ and $P_t$.

The following observation may be helpful in the study of variation of the subspaces $\mathcal{F}_t$ defined in (2.16).

**Remark 2.3.** The inclusion (2.19) implies that also $\operatorname{Ran} (P_t^\perp|_{\operatorname{Dom}(H)}) \subset \operatorname{Dom}(H)$. This means that, at any moment $t \in \mathbb{R}$, the commutator (2.30) may be written in the form

$$[H, P_t] = P_t^\perp HP_t - P_t HP_t^\perp. \quad (2.31)$$

From (2.31) it follows that, with respect to the orthogonal decomposition $\mathcal{H} = \mathcal{F}_t \oplus \mathcal{F}_t^\perp$, the operator $[H, P_t]$ admits representation in the form of the following $2 \times 2$ block off-diagonal skew-symmetric operator matrix:

$$[H, P_t] = \begin{pmatrix} 0 & -P_t \sigma \mathbb{I} \\ P_t^\perp H |_{\mathcal{F}_t^\perp} & 0 \end{pmatrix}, \quad t \in \mathbb{R}, \quad (2.32)$$

which is considered on the domain $(\operatorname{Dom}(H) \cap \mathcal{F}_t) \oplus (\operatorname{Dom}(H) \cap \mathcal{F}_t^\perp) = \operatorname{Dom}(H)$.

**Remark 2.4.** One more observation is that for any $t \in \mathbb{R}$ the commutator (2.30) of $H$ and $P_t$ remains unitary equivalent to its value $[H, P_0]$ at $t = 0$, namely,

$$[H, P_t] = e^{-iHt} [H, P_0] e^{iHt}, \quad t \in \mathbb{R}. \quad (2.33)$$
3. Bounds for the speed of the subspace evolution

In this section we again assume that $H$ is a self-adjoint operator on the Hilbert space $\mathcal{H}$ and that $P_0$ is an orthogonal projection on $\mathcal{H}$ with $\text{Ran}(P_0) = P_0 \neq \{0\}$ satisfying the condition (2.1).

It is well known that the set of all orthogonal projections in the Hilbert space $\mathcal{H}$ (and hence the set of all subspaces of $\mathcal{H}$) is a metric space with the distance given by the operator norm,

$$\rho(Q,R) := \|Q - R\|, \quad \rho(\Omega, \mathfrak{R}) := \rho(Q,R),$$

(3.1)

where $Q$, $R$ are arbitrary orthogonal projections and $\Omega$, $\mathfrak{R}$, their respective ranges. It is worth mentioning that

$$\|Q - R\| \leq 1$$

for any two orthogonal projections $Q$ and $R$ in $\mathcal{H}$ (see, e.g., [12, Section 34]) and, thus, we always have $\rho(\Omega, \mathfrak{R}) \leq 1$ for any subspaces $\Omega$ and $\mathfrak{R}$ in $\mathcal{H}$.

We start with proving our first quantum speed limit for the subspace variation based on the equation (2.21) under an additional assumption that the commutator of the operators $H$ and $P_0$ is a bounded operator on its domain $\text{Dom}([H,P_0]) = \text{Dom}(H)$.

**Lemma 3.1.** Assume the hypothesis of Theorem [2.1] Assume, in addition, that the commutator $[H,P_0]$ considered on $\text{Dom}([H,P_0]) = \text{Dom}(H)$ is a bounded operator, that is,

$$V_{H,P_0} := \sup_{f \in \text{Dom}(H)} \|HP_0f - P_0Hf\| < \infty,$$

(3.2)

and let $P_t = e^{-itH}P_0e^{itH}$, $t \in \mathbb{R}$. Then the closure $\bar{[H,P_t]}$ of the commutator $[H,P_t]$, $t \in \mathbb{R}$, is a bounded operator on the whole Hilbert space $\mathcal{H}$ and

$$\|\bar{[H,P_t]}\| = V_{H,P_0} \quad \text{for any } t \in \mathbb{R}.$$  

(3.3)

Furthermore, the following inequality holds

$$\|P_t - P_s\| \leq V_{H,P_0} |t - s|, \quad \text{for any } t, s \in \mathbb{R}.$$  

(3.4)

**Proof.** Since $H$ is a self-adjoint operator, its domain is dense in $\mathcal{H}$. Then the boundedness (3.2) of the commutator $[H,P_0]$ on $\text{Dom}(H)$ implies that its closure $\bar{[H,P_0]}$ is a bounded operator defined on the whole Hilbert space $\mathcal{H}$. Continuity of the norm implies $\|\bar{[H,P_0]}\| = \|[H,P_0]\| = V_{H,P_0}$. By Remark 2.4, for any $t \in \mathbb{R}$ the commutator $[H,P_t]$ is unitary equivalent to the commutator $[H,P_0]$ and the same concerns their closures. Hence, $\bar{[H,P_t]}$ is a bounded operator on $\mathcal{H}$ with the norm coinciding with that of $\bar{[H,P_0]}$, and this proves (3.3).

Now assume that $s, t \in \mathbb{R}$ and, for definiteness, let $s < t$. Since $\text{Dom}(H) = \mathcal{H}$, for the norm of $P_t - P_s$ we have

$$\|P_t - P_s\| = \sup_{f \in \mathcal{H}} \|P_t f - P_s f\| = \sup_{\|f\| = 1} \|P_t f - P_s f\|.$$  

(3.5)

Then by Theorem [2.1] (in particular, by the strong differentiability (2.20) of the path $P_t$) one concludes that

$$\|P_t - P_s\| = \sup_{\|f\| = 1} \|f^t \dot{P}_t f d\tau\| \leq \int_s^t \sup_{\|f\| = 1} \|\dot{P}_\tau f\| d\tau.$$  

(3.6)
Now by taking into account (2.21) and (3.3) one arrives at
\[
\|P_t - P_s\| \leq \int_s^t \sup_{f \in \text{Dom}(H)} \|HP_\tau f - P_\tau Hf\| \, d\tau = \int_s^t \|H, P_\tau\| \, d\tau = V_{H, P_0} (t - s),
\] (3.7)
which completes the proof. \[\square\]

Soon we will see that there is a stronger estimate of \(\|P_t - P_s\|\) through \(|t - s|\) than the one presented in (3.4). The stronger estimate (see inequality (3.16) in Theorem 3.6 below) is associated with another natural but much less known metric on the set of all subspaces of \(\mathcal{F}\). This metric is associated with the quantity
\[
\vartheta(\Omega, \mathcal{R}) := \arcsin(\|Q - R\|)
\] (3.8)
called the maximal angle between the subspaces \(\Omega\) and \(\mathcal{R}\). The fact that the maximal angle (3.8) is a metric has been proven in 1993 by L. Brown [13]. An alternative proof of this fact may be found in [14]. Let us also refer to the discussion of the metric (3.8) in [15].

Notice that, since the maximal angle is a metric, the triangle inequality
\[
\vartheta(\Omega, \mathcal{R}) \leq \vartheta(\Omega, \mathcal{S}) + \vartheta(\mathcal{S}, \mathcal{R})
\] (3.9)
holds for any subspaces \(\Omega, \mathcal{R}, \mathcal{S} \subset \mathcal{F}\).

**Remark 3.2.** Inequality \(x < \arcsin x, x \in (0, 1]\), implies that always \(\rho(\Omega, \mathcal{R}) < \vartheta(\Omega, \mathcal{R})\) if \(\Omega \neq \mathcal{R}\). Thus, the metric \(\vartheta\) is stronger than the metric \(\rho\) in the sense that the bound \(\vartheta(\Omega, \mathcal{R}) < c\) for some \(c > 0\) automatically requires that also \(\rho(\Omega, \mathcal{R}) < c\). The converse is not true in general.

**Remark 3.3.** One verifies by inspection that the non-negative operator \((Q - R)^2\) is block diagonal with respect to the decomposition \(\mathcal{F} = \mathcal{R} \oplus \mathcal{R}^\perp\), more precisely,
\[
(Q - R)^2 = RQS + R^\perp QR^\perp, \tag{3.10}
\]
which means that
\[
\|Q - R\| = \max\{\|RQS\|^\frac{1}{2}, \|R^\perp QR^\perp\|^\frac{1}{2}\} = \max\{\|Q\|^\frac{1}{2}, \|R\|^\frac{1}{2}\}. \tag{3.11}
\]

**Remark 3.4.** The concept of maximal angle between subspaces can be traced back at least to Krein, Krasnoselsky, and Milman [16]. Under the assumption that \((\Omega, \mathcal{R})\) is an ordered pair of subspaces and \(\Omega \neq \{0\}\), the notion of the (relative) maximal angle between \(\Omega\) and \(\mathcal{R}\) is applied in [16] to the number \(\varphi(\Omega, \mathcal{R}) \in [0, \pi/2]\) defined by
\[
\sin \varphi(\Omega, \mathcal{R}) = \sup_{x \in \Omega, \|x\|=1} \text{dist}(x, \mathcal{R}). \tag{3.12}
\]
Obviously, equality (3.12) is equivalent to
\[
\sin \varphi(\Omega, \mathcal{R}) = \|Q^\perp R\|. \tag{3.13}
\]
If both \(\Omega \neq \{0\}\) and \(\mathcal{R} \neq \{0\}\) then (3.11) implies
\[
\vartheta(\Omega, \mathcal{R}) = \arcsin\left(\max\{\|Q^\perp R\|, \|R^\perp Q\|\}\right) = \max\{\varphi(\Omega, \mathcal{R}), \varphi(\mathcal{R}, \Omega)\}. \tag{3.14}
\]
In contrast to \(\varphi(\Omega, \mathcal{R})\), the maximal angle \(\vartheta(\Omega, \mathcal{R})\) is always symmetric with respect to the interchange of the arguments \(\Omega\) and \(\mathcal{R}\). Moreover,
\[
\varphi(\mathcal{R}, \Omega) = \varphi(\Omega, \mathcal{R}) = \vartheta(\Omega, \mathcal{R}) \quad \text{whenever } \|Q - R\| < 1. \tag{3.15}
\]
For \(\|Q - R\| < 1\), this is simply a consequence of the equality \(\|Q^\perp R\| = \|R^\perp Q\|\), which is easily deduced, e.g., from [17] Corollary 3.4 (i) and Remark 3.6.
Remark 3.5. The maximal angle between subspaces admits a quantum-mechanical interpretation. To this end, one may apply the concept of a subspace-state of a quantum system. Namely, given a subspace $\mathcal{P} \subset \mathcal{H}$, one says that the system is in the $\mathcal{P}$-state if it is in a pure state described by a (non-specified) normalized vector $x \in \mathcal{P}$. By (3.12) and (3.14) the quantity $\cos^2 \vartheta(\mathcal{P}, \mathcal{R})$ is then treated as a minimum probability for a quantum system which is in a $\mathcal{P}$-state to be found also in an $\mathcal{R}$-state.

By using the maximal-angle metric (3.8) we obtain the following result.

Theorem 3.6. Assume the hypothesis of Lemma 3.7 and let $\mathcal{P}_\tau = \text{Ran}(P_\tau)$, $\tau \in \mathbb{R}$. Then the following inequality holds:

$$\vartheta(\mathcal{P}_s, \mathcal{P}_t) \leq V_{H,P_0} |t-s| \quad \text{for any } s, t \in \mathbb{R}. \quad (3.16)$$

Proof. Assume, for definiteness, that $s \neq t$ and set

$$\tau_j = s + j \frac{t-s}{n}, \quad j = 0, 1, \ldots, n, \quad (3.17)$$

where $n$ is a natural number, $n \in \mathbb{N}$. Notice that $\tau_0 = s$ and $\tau_n = t$. Under the assumption that $n \geq 2$, by applying the triangle inequality (3.9) to the subspaces $\mathcal{P}_s$, $\mathcal{P}_t$, and intermediate subspaces $\mathcal{P}_{\tau_j}$, $j = 1, 2, \ldots, n-1$, one arrives at the following bound for the maximal angle $\vartheta(\mathcal{P}_s, \mathcal{P}_t)$ between the subspaces $\mathcal{P}_s$ and $\mathcal{P}_t$:

$$\vartheta(\mathcal{P}_0, \mathcal{P}_t) \leq \sum_{j=1}^n \arcsin \|P_{\tau_j} - P_{\tau_{j-1}}\|. \quad (3.18)$$

By (3.17) we have

$$|\tau_j - \tau_{j-1}| = \frac{|t-s|}{n}, \quad j = 1, 2, \ldots, n. \quad (3.19)$$

Now take $n$ such that $V_{H,P_0} \frac{|t-s|}{n} \leq 1$. Then combining (3.18) and (3.19) with the estimate (3.4) in Lemma 3.1 yields

$$\vartheta(\mathcal{P}_0, \mathcal{P}_t) \leq \sum_{j=1}^n \arcsin \left( V_{H,P_0} |\tau_j - \tau_{j-1}| \right) = n \arcsin \frac{V_{H,P_0} |t-s|}{n}. \quad (3.20)$$

Passing in (3.20) to the limit $n \to \infty$ one arrives at (3.16), completing the proof.

Remark 3.7. Given $s \neq t$ and $V_{P_0,H} > 0$, the bound (3.16) is more tight for $\|P_s - P_t\|$ than the bound (3.4) (cf. Remark 3.2).

Remark 3.8. Under the hypothesis of Lemma 3.1 by Remark 2.3 from (3.2) it follows that

$$V_{H,P_t} = \|P_t H P_t^\perp\| = \|P_t^\perp H P_t\| \quad (= \|P_t H P_t^\perp\| = \|P_t^\perp H P_t\|) \quad \text{for any } t \in \mathbb{R} \quad (3.21)$$

and, in particular,

$$V_{H,P_0} = \|P_0 H P_0^\perp\| = \|P_0^\perp H P_0\|. \quad (3.22)$$

Therefore, the bound (3.16) may be interpreted in the sense that only the off-diagonal entries $P_0 H |_{\mathcal{P}_0}$ and $P_0^\perp H |_{\mathcal{P}_0}$ in the block matrix representation (2.6) of the Hamiltonian $H$ contribute into the variation of the subspace $\mathcal{P}_0$. If $H$ is block diagonal with respect to the decomposition $\mathcal{H} = \mathcal{P}_0 \oplus \mathcal{P}_0^\perp$ and, hence, the subspace $\mathcal{P}_0$ is reducing for $H$, it does not change with time at all. In particular, none of the spectral subspaces of $H$ can be a subject of the time evolution.
Corollary 3.9. Under the hypothesis of Lemma 3.1, suppose that $T_\theta$ is the time when the maximal angle between the initial subspace $\mathcal{V}_0$ and a subspace in the path $\mathcal{V}_t$, $t \geq 0$, reaches the value of $\theta$, $0 < \theta \leq \frac{\pi}{2}$, that is,

$$\vartheta(\mathcal{V}_0, \mathcal{V}_{T_\theta}) = \theta.$$  \hfill (3.23)

Then

$$T_\theta \geq \frac{\theta}{V_{H,P_0}}.$$  \hfill (3.24)

Example 3.10. Let the Hamiltonian $H$ describe a two-level quantum system with bound states $e_1$ and $e_2$, that is, $\|e_1\| = \|e_2\| = 1$, $\langle e_1, e_2 \rangle = 0$, the Hilbert space $\mathcal{H}$ is the span of the vectors $e_1$ and $e_2$, and

$$H = E_1 \langle \cdot, e_1 \rangle e_1 + E_2 \langle \cdot, e_2 \rangle e_2,$$

where the binding energies $E_1$ and $E_2$ are supposed to be different, $E_1 \neq E_2$. Assume that $P_0$ is the orthogonal projection on the one-dimensional subspace $\mathcal{V}_0$ spanned by the vector $e = \frac{1}{\sqrt{2}}(e_1 + e_2)$. One immediately observes that

$$[H, P_0] = \frac{E_2 - E_1}{2} \left( \langle \cdot, e_1 \rangle e_2 - \langle \cdot, e_2 \rangle e_1 \right).$$  \hfill (3.25)

and then for the norm of the commutator of $H$ and $P_0$ we get

$$V_{H,P_0} = \| [H, P_0] \| = \frac{|E_2 - E_1|}{2}.$$  \hfill (3.26)

One also verifies by inspection that this norm may be written as

$$V_{H,P_0} = \left( \|He\|^2 - \langle He, e \rangle^2 \right)^{1/2}.$$  \hfill (3.27)

Furthermore, an elementary computation shows that for any $\tau \in \mathbb{R}$ the orthogonal projection $P_\tau = e^{-iH\tau}P_0e^{iH\tau}$ is given by

$$P_\tau = \frac{1}{2} \left( \langle \cdot, e_1 \rangle e_1 + e^{-i(E_2-E_1)\tau} \langle \cdot, e_1 \rangle e_2 + e^{i(E_2-E_1)\tau} \langle \cdot, e_2 \rangle e_1 + \langle \cdot, e_2 \rangle e_2 \right).$$  \hfill (3.28)

Then for any $s, t \in \mathbb{R}$ we have

$$P_t - P_s = \frac{1}{2} \left( e^{-i(E_2-E_1)s} - e^{-i(E_2-E_1)t} \right) \langle \cdot, e_1 \rangle e_2 + \frac{1}{2} \left( e^{i(E_2-E_1)t} - e^{i(E_2-E_1)s} \right) \langle \cdot, e_2 \rangle e_1.$$  \hfill (3.29)

The eigenvalues of the rank-two operator (3.29) are

$$\lambda_{\pm} = \pm \sin \left( \frac{|E_2 - E_1|}{2} |t - s| \right),$$  \hfill (3.30)

which means in particular that

$$\vartheta(\mathcal{V}_s, \mathcal{V}_t) = \arcsin \|P_t - P_s\| = \frac{|E_2 - E_1|}{2} |t - s| \quad \text{whenever} \quad \frac{|E_2 - E_1|}{2} |t - s| \leq \frac{\pi}{2},$$  \hfill (3.31)

where, as usually, $\mathcal{V}_\tau = \text{Ran}(P_\tau)$, $\tau \in \mathbb{R}$.

Remark 3.11. Example 3.10 is used in many papers on quantum speed limits (see, e.g., \cite{2} \cite{3} \cite{8} \cite{9}). In particular, this example proves the sharpness of both the Mandelstam-Tamm and Margolus-Levitin inequalities (see, e.g., \cite{2} Section 2.4). Example 3.10 also works for the bounds (3.16) and (3.24). In this example, due to (3.26) and (3.31) both of these bounds transform into equalities, which proves that both the bounds (3.16) and (3.24) are optimal.

\footnote{For the case where $V_{H,P_0} = 0$, which implies that $\mathcal{V}_0$ is a reducing subspace of $H$, one adopts the convention that $T_0 = \infty$.}
Theorem 3.12. Assume the hypothesis of Theorem 2.7 and let \( \mathcal{P}_\tau = \text{Ran}(e^{-i\tau P_0}e^{i\tau}) \), \( \tau \in \mathbb{R} \). Assume, in addition, that

\[
\Delta E_{\mathcal{P}_0} := \sup_{f \in \mathcal{P}_0 \cap \text{Dom}(H)} \left( \|Hf\|^2 - \langle Hf, f \rangle \right)^{1/2} < \infty, \tag{3.32}
\]

Then

\[
\vartheta(\mathcal{P}_s, \mathcal{P}_t) \leq \Delta E_{\mathcal{P}_0} |t - s| \quad \text{for any } t, s \in \mathbb{R}, \tag{3.33}
\]

and

\[
T_\theta \geq \frac{\theta}{\Delta E_{\mathcal{P}_0}}, \tag{3.34}
\]

where \( \theta, T_\theta \) are the same as in Corollary 3.9.

Proof. We start with the elementary observation that

\[
\|P_0^\perp Hf\| = \left( \|Hf\|^2 - \|P_0Hf\|^2 \right)^{1/2} \quad \text{for any } f \in \text{Dom}(H). \tag{3.35}
\]

If, in addition, \( f \in \mathcal{P}_0 \) and \( \|f\| = 1 \) then automatically \( \|P_0y\| \geq |\langle y, f \rangle| \) for any \( y \in \mathcal{S}_f \). In particular,

\[
\|P_0Hf\| \geq \langle Hf, f \rangle \quad \text{for any } f \in \mathcal{P}_0 \cap \text{Dom}(H), \|f\| = 1. \tag{3.36}
\]

Hence, from (3.35) it follows that

\[
\|P_0^\perp Hf\| \leq \left( \|Hf\|^2 - \langle Hf, f \rangle \right)^{1/2} \quad \text{for any } f \in \mathcal{P}_0 \cap \text{Dom}(H), \|f\| = 1. \tag{3.37}
\]

Meanwhile, by Remark 3.8 (see second equality in (3.22)) we have

\[
V_{H,P_0} = \sup_{f \in \text{Dom}(H)} \|P_0^\perp Hf\| = \sup_{f \in \mathcal{P}_0 \cap \text{Dom}(H)} \|P_0^\perp Hf\|, \tag{3.38}
\]

by taking into account that, by the hypothesis, the linear set \( \text{Dom}(H) \) is invariant under \( P_0 \) and then \( P_0 \text{Dom}(H) = \mathcal{P}_0 \cap \text{Dom}(H) \) (see equalities (2.1)–(2.3)). By combining (3.38) with (3.32) and (3.37), one obtains

\[
V_{H,P_0} \leq \Delta E_{\mathcal{P}_0}. \tag{3.39}
\]

Then (3.33) follows from the estimate (3.16) in Theorem 3.6, while (3.34) is implied by the bound (3.24) in Corollary 3.9. The proof is complete. \( \square \)

Remark 3.13. We underline that the above proof is based on the inequality (3.39). Due to this inequality, the hypothesis of Theorem 3.12 implies the one of Theorem 3.6 and the bound (3.24) implies the bound (3.34). In this sense, Theorem 3.12 is a weaker version of Theorem 3.6 but the bound (3.34) much closer resembles a Mandelstam-Tamm-Fleming bound (1.7). (Also notice that in the case of a one-dimensional subspace \( \mathcal{P}_0 \) inequality (3.39) turns into equality and both the bounds (3.24) and (3.34) reduce to the Mandelstam-Tamm-Fleming bound (1.7) with the energy dispersion (1.4) involving a state vector \( \psi_0 \) spanning \( \mathcal{P}_0 \).

Remark 3.14. Combining equality (3.31) with equalities (3.26) and (3.27) shows that in Example 3.10 both bounds (3.33) and (3.34) turn into precise equalities. Thus, Example 3.10 proves that these bounds are sharp.

The following proposition is verified by direct inspection.

\footnote{For the case where \( \Delta E_{\mathcal{P}_0} = 0 \), which implies that \( \mathcal{P}_0 \) is the eigenspace associated with an eigenvalue of \( H \), one adopts the convention that \( T_\theta = \infty \) (cf. footnote on page 11).}
Proposition 3.15. Assume that T is a linear operator on a Hilbert space $\mathcal{H}$ with domain $\text{Dom}(T)$. Then the following identity holds
\[ \|Tx\|^2 - |\langle Tx, x \rangle|^2 = \| (T - cI)x \|^2 - |\langle (T - cI)x, x \rangle|^2 \] (3.40)
for any $c \in \mathbb{C}$ and any $x \in \text{Dom}(T)$, $\|x\| = 1$.

The statement below is rather well known. We present and prove it only for convenience of the reader.

Lemma 3.16. Let T be a bounded self-adjoint operator on the Hilbert space $\mathcal{H}$. Then the following inequalities hold:
\[ 0 \leq \|Tx\|^2 - \langle Tx, x \rangle^2 \leq \frac{1}{4}(M - m)^2, \quad \text{for any } x \in \mathcal{H}, \|x\| = 1, \] (3.41)
where $m = \min(\text{spec}(T))$ and $M = \max(\text{spec}(T))$ are the upper and lower bounds of the spectrum of T, respectively.

Proof. The left inequality in (3.41) is obvious since $|\langle Tx, x \rangle| \leq \|Tx\|\|x\| = \|Tx\|$ because of $\|x\| = 1$. Now choose $c = \frac{1}{2}(m + M)$. By Proposition 3.15 one infers that
\[ \|Tx\|^2 - \langle Tx, x \rangle^2 \leq \|T_c x\|^2 - |\langle T_c x, x \rangle|^2 \quad \text{for any } x \in \mathcal{H}, \|x\| = 1, \] (3.42)
where $T_c = T - cI$. Notice that $\|T_c\| = \frac{1}{2}(M - m)$. Then from (3.42) it follows that
\[ \|Tx\|^2 - \langle Tx, x \rangle^2 \leq \|T_c x\|^2 \leq \frac{1}{4}(M - m)^2, \] (3.43)
proving the right inequality in (3.41) and, thus, completing the whole proof. □

Remark 3.17. Combining equalities (3.26) and (3.27) in Example 3.10 proves that the upper bound in (3.41) is sharp.

Our final bound only concerns the special case of bounded Hamiltonians.

Theorem 3.18. Let H be a bounded self-adjoint operator on the Hilbert space $\mathcal{H}$ and let
\[ E_{\min}(H) = \min(\text{spec}(H)), \quad E_{\max}(H) = \max(\text{spec}(H)) \] (3.44)
be the upper and lower bounds of the spectrum of H, respectively. Let $P_t$, $t \geq 0$, be the projection path (2.18) where $P_0$ is an orthogonal projection in $\mathcal{H}$. Then
\[ \vartheta (\mathcal{P}_s, \mathcal{P}_t) \leq \frac{E_{\max}(H) - E_{\min}(H)}{2} |t - s|, \quad \text{for any } s, t \in \mathbb{R}, \] (3.45)
where $\mathcal{P}_\tau = \text{Ran}(P_\tau)$, $\tau \in \mathbb{R}$. Furthermore, the following lower bound holds\(^4\):
\[ T_\vartheta \geq \frac{2\vartheta}{E_{\max}(H) - E_{\min}(H)}, \] (3.46)
where $\vartheta$ and $T_\vartheta$ are the same as in Corollary 3.9.

Proof. In the case under consideration, the quantity $E_{3\vartheta}$ from (3.32) may be rewritten as
\[ \Delta E_{3\vartheta} = \sup_{f \in \mathcal{F}_\vartheta} (\|Hf\|^2 - \langle Hf, f \rangle^2)^{1/2} \leq \sup_{f \in \mathcal{F}} (\|Hf\|^2 - \langle Hf, f \rangle^2)^{1/2}, \] (3.47)
\(^4\)For the case where $E_{\max}(H) - E_{\min}(H) = 0$, which implies that H is a multiple of unity, one adopts the convention that $T_\vartheta = \infty$ (cf. footnotes on pages 11 and 12).
since the self-adjoint operator $H$ is bounded and $\text{Dom}(H) = \mathcal{H}$. Then by Lemma 3.16 one concludes that

$$\Delta E_{\mathcal{P}_0} \leq \frac{E_{\max}(H) - E_{\min}(H)}{2}, \quad \text{for any subspace } \mathcal{P}_0 \subset \mathcal{H}. \quad (3.48)$$

Now inequality (3.33) in Theorem 3.12 implies the estimate (3.45) while the lower bound (3.46) follows from the lower bound (3.34), and this completes the proof. \hfill $\square$

4. Summary and Future Perspectives

This paper is aimed in establishing sharp lower bounds on the time required for an initial state subspace $\mathcal{P}_0$ of a quantum system described by a Hamiltonian $H$ to evolve into another subspace particularly positioned with respect to the subspace $\mathcal{P}_0$. The operator $H$ is assumed to be time-independent and self-adjoint; it is allowed to be unbounded. In the latter case, we require, in addition, that the domain $\text{Dom}(H)$ of $H$ is invariant under the orthogonal projection $P_0$ on $\mathcal{P}_0$, that is, $P_0 \text{Dom}(H) \subset \text{Dom}(H)$. As a measure of the difference between the subspaces $\mathcal{P}_s$ and $\mathcal{P}_t$, $s,t \in \mathbb{R}$, in the subspace evolution path generated by $H$ out of $\mathcal{P}_0$ we use the maximal angle $\theta(\mathcal{P}_s, \mathcal{P}_t)$ between them, defined as in (1.9).

Our first principal result (see Theorem 3.6) is as follows. Assume that the commutator of $H$ and $P_0$ is a bounded operator on $\text{Dom}(H)$ and let $V_{H,P_0} = \|[H,P_0]\|$. Then for any $s,t \in \mathbb{R}$ the maximal angle between the subspaces $\mathcal{P}_s$ and $\mathcal{P}_t$ satisfies inequality $\theta(\mathcal{P}_s, \mathcal{P}_t) \leq V_{H,P_0} |s-t|$. Furthermore, we show that this inequality is optimal. Thus, it is the quantity $V_{H,P_0}$ that bounds the maximal possible “angular” speed of variation of the subspaces in the path $\mathcal{P}_t$, $t \in \mathbb{R}$. In particular, we make the conclusion that $T_\theta \geq \theta / V_{H,P_0}$ where $T_\theta$ is the first time moment when the maximal angle between $\mathcal{P}_0$ and $\mathcal{P}_t$ reaches the value of $\theta$, $0 < \theta \leq \frac{\pi}{2}$ (see Corollary 3.9).

The second principal result of the paper is based on the bound (3.39), $V_{H,P_0} \leq \Delta E_{\mathcal{P}_0}$, where $\Delta E_{\mathcal{P}_0}$ denotes the maximum energy dispersion (3.32) over the subspace $\mathcal{P}_0$. By using this bound we derive for $T_\theta$ another sharp lower estimate $T_\theta \geq \theta / \Delta E_{\mathcal{P}_0}$ (see Theorem 3.12 and Remark 3.14) which is, in general, weaker than our previous lower limit $T_\theta \geq \theta / V_{H,P_0}$. However the estimate $T_\theta \geq \theta / \Delta E_{\mathcal{P}_0}$ is featuring much closer resemblance to a Mandelstam-Tamm-Fleming bound (1.7), which is a particular case of it.

Surely, it is also interesting to have for $T_\theta$ a lower bound written directly in terms of the spectrum of the Hamiltonian $H$. We have found a bound of such a kind, namely the bound (3.46) in Theorem 3.18. It should be underlined, however, that this bound only works in the case where $H$ is a bounded operator.

We believe that the established bounds on the speed of evolution of a quantum subspace admit an extension to the case of some unbounded time-dependent self-adjoint Hamiltonians, and we work on this. It is worth noting that this extension is not straightforward since, unlike the one-parameter evolution semigroup entries (2.12), the corresponding propagators do not commute with a time-dependent Hamiltonian.

Another challenging problem consists in establishing bounds on the evolution speed for a subspace whose variation is governed by a non-Hermitian Hamiltonian. Of course, the general non-Hermitian case should be extremely difficult. Furthermore, this case is not of direct interest for quantum physics. However, special cases where a non-Hermitian Hamiltonian is similar to a self-adjoint operator are definitely of interest. This concerns, in particular, the case of $PT$-symmetric Hamiltonians with real spectrum where some quantum speed limits have already been established for the evolution of pure states (see [8], [9], and references therein). One may expect that in this case quantum speed limits exist for the evolution of subspaces as well.
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