ON $n$-NORMED SPACES

HENDRA GUNAWAN and M. MASHADI

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ABSTRACT. Given an $n$-normed space with $n \geq 2$, we offer a simple way to derive an $(n-1)$-norm from the $n$-norm and realize that any $n$-normed space is an $(n-1)$-normed space. We also show that, in certain cases, the $(n-1)$-norm can be derived from the $n$-norm in such a way that the convergence and completeness in the $n$-norm is equivalent to those in the derived $(n-1)$-norm. Using this fact, we prove a fixed point theorem for some $n$-Banach spaces.

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1. Introduction. Let $n \in \mathbb{N}$ and $X$ be a real vector space of dimension $d \geq n$. (Here we allow $d$ to be infinite.) A real-valued function $\| \cdot,\cdots,\cdot \|$ on $X^n$ satisfying the following four properties

$\begin{align}
(1) \quad & \|x_1,\cdots,x_n\| = 0 \text{ if and only if } x_1,\cdots,x_n \text{ are linearly dependent;} \\
(2) \quad & \|x_1,\cdots,x_n\| \text{ is invariant under permutation;} \\
(3) \quad & \|x_1,\cdots,x_{n-1},\alpha x_n\| = |\alpha| \|x_1,\cdots,x_{n-1},x_n\| \text{ for any } \alpha \in \mathbb{R}; \\
(4) \quad & \|x_1,\cdots,x_{n-1},y+z\| \leq \|x_1,\cdots,x_{n-1},y\| + \|x_1,\cdots,x_{n-1},z\|,
\end{align}$

is called an $n$-norm on $X$ and the pair $(X,\| \cdot,\cdots,\cdot \|)$ is called an $n$-normed space.

A trivial example of an $n$-normed space is $X = \mathbb{R}^n$ equipped with the following $n$-norm:

\[
\|x_1,\cdots,x_n\|_E := \text{abs} \begin{pmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & \ddots & \vdots \\ x_{n1} & \cdots & x_{nn} \end{pmatrix},
\]

where $x_i = (x_{i1},\cdots,x_{in}) \in \mathbb{R}^n$ for each $i = 1,\ldots,n$. (The subscript $E$ is for Euclidean.)

Note that in an $n$-normed space $(X,\| \cdot,\cdots,\cdot \|)$, we have, for instance, $\|x_1,\ldots,x_n\| \geq 0$ and $\|x_1,\ldots,x_{n-1},x_n\| = \|x_1,\ldots,x_{n-1},x_n + \alpha_1 x_1 + \cdots + \alpha_{n-1} x_{n-1}\|$ for all $x_1,\ldots,x_n \in X$ and $\alpha_1,\ldots,\alpha_{n-1} \in \mathbb{R}$.

The theory of 2-normed spaces was first developed by Gähler [3] in the mid 1960’s, while that of $n$-normed spaces can be found in [11]. Recent results can be found, for example, in [9, 10]. Related works on $n$-metric spaces and $n$-inner product spaces may be found, for example, in [1, 2, 4, 5, 7, 6, 12].

In this note, we will show that every $n$-normed space with $n \geq 2$ is an $(n-1)$-normed space and hence, by induction, an $(n-r)$-normed space for all $r = 1,\ldots,n-1$. In particular, given an $n$-normed space, we offer a simple way to derive an $(n-1)$-norm from the $n$-norm, different from that in [5].
We will also apply our result to study convergence and completeness in $n$-normed spaces, which will be defined later. This enables us to prove a fixed point theorem for some $n$-normed spaces.

The case $n = 2$ was previously studied in [8].

2. Preliminary results. Suppose hereafter that $n \geq 2$ and $(X, \|\cdot, \ldots, \cdot\|)$ is an $n$-normed space of dimension $d \geq n$. Take a linearly independent set $\{a_1, \ldots, a_n\}$ in $X$. With respect to $\{a_1, \ldots, a_n\}$, define the following function $\|\cdot, \ldots, \cdot\|_\infty$ on $X^{n-1}$ by

$$\|x_1, \ldots, x_{n-1}\|_\infty := \max \{\|x_1, \ldots, x_{n-1}, a_i\| : i = 1, \ldots, n\}. \quad (2.1)$$

Then we have the following result.

**Theorem 2.1.** The function $\|\cdot, \ldots, \cdot\|_\infty$ defines an $(n-1)$-norm on $X$.

**Proof.** We will verify that $\|\cdot, \ldots, \cdot\|_\infty$ satisfies the four properties of an $(n-1)$-norm.

1. If $x_1, \ldots, x_{n-1}$ are linearly dependent, then $\|x_1, \ldots, x_{n-1}\| = 0$ for each $i = 1, \ldots, n$, and hence $\|x_1, \ldots, x_{n-1}\|_\infty = 0$. Conversely, if $\|x_1, \ldots, x_{n-1}\|_\infty = 0$, then $\|x_1, \ldots, x_{n-1}, a_i\| = 0$ and accordingly $x_1, \ldots, x_{n-1}, a_i$ are linearly dependent for each $i = 1, \ldots, n$. But this can only happen when $x_1, \ldots, x_{n-1}$ are linearly dependent.

2. Since $\|x_1, \ldots, x_{n-1}, a_i\|$ is invariant under any permutation of $\{x_1, \ldots, x_{n-1}\}$, we find that $\|x_1, \ldots, x_{n-1}\|_\infty$ is also invariant under any permutation.

3. Observe that

$$\|x_1, \ldots, x_{n-2}, \alpha x_{n-1}\|_\infty = \max \{\|x_1, \ldots, x_{n-2}, \alpha x_{n-1}, a_i\| : i = 1, \ldots, n\}$$

$$= |\alpha| \max \{\|x_1, \ldots, x_{n-2}, x_{n-1}, a_i\| : i = 1, \ldots, n\}$$

$$= |\alpha| \|x_1, \ldots, x_{n-2}, x_{n-1}\|_\infty. \quad (2.2)$$

4. Observe that

$$\|x_1, \ldots, x_{n-2}, y + z\|_\infty = \max \{\|x_1, \ldots, x_{n-2}, y + z, a_i\| : i = 1, \ldots, n\}$$

$$\leq \max \{\|x_1, \ldots, x_{n-2}, y, a_i\| : i = 1, \ldots, n\}$$

$$+ \max \{\|x_1, \ldots, x_{n-2}, z, a_i\| : i = 1, \ldots, n\}$$

$$= \|x_1, \ldots, x_{n-2}, y\|_\infty + \|x_1, \ldots, x_{n-2}, z\|_\infty. \quad (2.3)$$

Therefore $\|\cdot, \ldots, \cdot\|_\infty$ defines an $(n-1)$-norm on $X$. \qed

**Corollary 2.2.** Every $n$-normed space is an $(n-r)$-normed space for all $r = 1, \ldots, n-1$. In particular, every $n$-normed space is a normed space.

**Remark 2.3.** Note that in general the function $\|x_1, \ldots, x_{n-1}\|_p := \left(\sum_{i=1}^{n-1} \|x_i, \ldots, x_{n-1}, a_i\|^p\right)^{1/p}$, where $1 \leq p \leq \infty$, also defines an $(n-1)$-norm on $X$. These $(n-1)$-norms, however, are equivalent to $\|\cdot, \ldots, \cdot\|_\infty$, as long as we use the same set of $n$ vectors $a_1, \ldots, a_n$. In certain cases, it is possible to get equivalent $(n-1)$-norms even if we use different sets of $n$ vectors.
2.1. The standard case. Take a look at a standard example. Let $X$ be a real inner product space of dimension $d \geq n$. Equip $X$ with the standard $n$-norm

$$
\|x_1,\ldots,x_n\|_S := \left| \begin{array}{c}
\langle x_1,x_1 \rangle & \cdots & \langle x_1,x_n \rangle \\
\vdots & & \ddots & \vdots \\
\langle x_n,x_1 \rangle & \cdots & \langle x_n,x_n \rangle
\end{array} \right|^{1/2},
$$

(2.4)

where $\langle \cdot,\cdot \rangle$ denotes the inner product on $X$. (If $X = \mathbb{R}^n$, then this $n$-norm is exactly the same as the Euclidean $n$-norm $\|\cdot,\ldots,\cdot\|_E$ mentioned earlier.)

Notice that for $n = 1$, the above $n$-norm is the usual norm $\|x_1\|_S = \langle x_1,x_1 \rangle^{1/2}$, which gives the length of $x_1$, while for $n = 2$, it defines the standard 2-norm $\|x_1,x_2\|_S = \sqrt{\langle x_1,x_1 \rangle \langle x_2,x_2 \rangle - \langle x_1,x_2 \rangle^2}$, which represents the area of the parallelogram spanned by $x_1$ and $x_2$. Further, if $X = \mathbb{R}^3$, then $\|x_1,x_2,x_3\|_S = \sqrt{\langle x_1,x_2 \rangle \langle x_2,x_3 \rangle - \langle x_1,x_3 \rangle^2}$ is nothing but the volume of the parallelograms spanned by $x_1$, $x_2$, and $x_3$. In general, $\|x_1,\ldots,x_n\|_S$ represents the volume of the $n$-dimensional parallelepiped spanned by $x_1,\ldots,x_n$ in $X$.

Now let $\{e_1,\ldots,e_n\}$ be an orthonormal set in $X$. Then, by Theorem 2.1, the following function

$$
\|x_1,\ldots,x_n\| := \max \{ \|x_1,\ldots,x_{n-1},e_i\|_S : i = 1,\ldots,n \}
$$

(2.5)

defines an $(n-1)$-norm on $X$. Further, we have the following fact.

**Fact 2.4.** On a standard $n$-normed space $X$, the derived $(n-1)$-norm $\|\cdot,\ldots,\cdot\|_\infty$, defined with respect to $\{e_1,\ldots,e_n\}$, is equivalent to the standard $(n-1)$-norm $\|\cdot,\ldots,\cdot\|_S$. Precisely, we have

$$
\|x_1,\ldots,x_{n-1}\|_\infty \leq \|x_1,\ldots,x_{n-1}\|_S \leq \sqrt{n} \|x_1,\ldots,x_{n-1}\|_\infty
$$

(2.6)

for all $x_1,\ldots,x_{n-1} \in X$.

**Proof.** Assume that $x_1,\ldots,x_{n-1}$ are linearly independent. For each $i = 1,\ldots,n$, write $e_i = e_i^t + e_i^\perp$ where $e_i^t \in \text{span}\{x_1,\ldots,x_{n-1}\}$ and $e_i^\perp \perp \text{span}\{x_1,\ldots,x_{n-1}\}$. Then we have

$$
\|x_1,\ldots,x_{n-1},e_i\|_S = \|x_1,\ldots,x_{n-1},e_i^t\|_S

= \left| \begin{array}{c}
\langle x_1,x_1 \rangle & \cdots & \langle x_1,x_{n-1} \rangle & 0 \\
\vdots & & \ddots & \vdots \\
\langle x_{n-1},x_1 \rangle & \cdots & \langle x_{n-1},x_{n-1} \rangle & 0 \\
0 & \cdots & 0 & \langle e_i^t,e_i^t \rangle
\end{array} \right|^{1/2}

\leq \left| \begin{array}{c}
\langle x_1,x_1 \rangle & \cdots & \langle x_1,x_{n-1} \rangle \\
\vdots & & \ddots \\vdots \\
\langle x_{n-1},x_1 \rangle & \cdots & \langle x_{n-1},x_{n-1} \rangle
\end{array} \right|^{1/2}

= \|x_1,\ldots,x_{n-1}\|_S.
$$

Hence we get $\|x_1,\ldots,x_{n-1}\|_\infty \leq \|x_1,\ldots,x_{n-1}\|_S$. 


Next, take a unit vector \( e = \alpha_1 e_1 + \cdots + \alpha_n e_n \) such that \( e \perp \text{span}\{x_1, \ldots, x_{n-1}\} \). (Here we are still assuming that \( x_1, \ldots, x_{n-1} \) are linearly independent.) Then, by properties (3) and (4) of the \( n \)-norm, we have

\[
\|x_1, \ldots, x_{n-1}, e\|_S \leq |\alpha_1| \|x_1, \ldots, x_{n-1}, e_1\|_S + \cdots + |\alpha_n| \|x_1, \ldots, x_{n-1}, e_n\|_S \quad (2.8)
\]

But, by the Cauchy-Schwarz inequality, we have

\[
\sum_{i=1}^n |\alpha_i| \leq \left\{ \sum_{i=1}^n 1^2 \right\}^{1/2} \left\{ \sum_{i=1}^n |\alpha_i|^2 \right\}^{1/2} = \sqrt{n}. \quad (2.9)
\]

Hence we obtain

\[
\|x_1, \ldots, x_{n-1}\|_S \leq \sqrt{n} \|x_1, \ldots, x_{n-1}\|_\infty, \quad (2.10)
\]

and this completes the proof.

\[\square\]

### 2.2. The finite-dimensional case.

For finite-dimensional \( n \)-normed space \((X, \|\cdot, \ldots, \cdot\|)\), we can in general derive an \((n-1)\)-norm from the \( n \)-norm in the following way. Take a linearly independent set \( \{a_1, \ldots, a_m\} \) in \( X \), with \( n \leq m \leq d \). With respect to \( \{a_1, \ldots, a_m\} \), define the following function \( \|\cdot, \ldots, \cdot\|_\infty \) on \( X^{n-1} \) by

\[
\|x_1, \ldots, x_{n-1}\|_\infty := \max \{ \|x_1, \ldots, x_{n-1}, a_i\| : i = 1, \ldots, m \}. \quad (2.11)
\]

Then, as in Theorem 2.1, the function \( \|\cdot, \ldots, \cdot\|_\infty \) defines an \((n-1)\)-norm on \( X \).

As we will see later, we can obtain a better \((n-1)\)-norm by using a set of \( d \), rather than just \( n \), linearly independent vectors in \( X \) (that is, by using a basis for \( X \)).

### 3. Applications and further results.

Recall that a sequence \( x(k) \) in an \( n \)-normed space \((X, \|\cdot, \ldots, \cdot\|)\) is said to converge to an \( x \in X \) (in the \( n \)-norm) whenever

\[
\lim_{k \to \infty} \|x_1, \ldots, x_{n-1}, x(k) - x\| = 0 \quad (3.1)
\]

for every \( x_1, \ldots, x_{n-2} \in X \).

The following proposition says that the convergence in the \( n \)-norm implies the convergence in the derived \((n-1)\)-norm \( \|\cdot, \ldots, \cdot\|_\infty \), defined with respect to an arbitrary linearly independent set \( \{a_1, \ldots, a_n\} \) in \( X \).

**Proposition 3.1.** If \( x(k) \) converges to an \( x \in X \) in the \( n \)-norm, then \( x(k) \) also converges to \( x \) in the derived \((n-1)\)-norm \( \|\cdot, \ldots, \cdot\|_\infty \), that is,

\[
\lim_{k \to \infty} \|x_1, \ldots, x_{n-2}, x(k) - x\|_\infty = 0 \quad (3.2)
\]

for every \( x_1, \ldots, x_{n-2} \in X \).

**Proof.** If \( x(k) \) converges to an \( x \in X \) in the \( n \)-norm, then

\[
\lim_{k \to \infty} \|x_1, \ldots, x_{n-2}, x(k) - x, a_i\| = 0 \quad (3.3)
\]
Then, as mentioned before, the function $\|\cdot\|$ for every $x \in X$ and $i = 1, \ldots, n$, and hence
\[
\lim_{k \to \infty} \|x_1, \ldots, x_{n-2}, x(k) - x\|_\infty = 0
\] (3.4)
for every $x_1, \ldots, x_{n-2} \in X$, that is, $x(k)$ converges to $x$ in the derived $(n-1)$-norm $\|\cdot, \cdot\|_\infty$.

3.1. The standard case. In a standard $n$-normed space $(X, \|\cdot, \cdot\|_S)$, the converse of Proposition 3.1 is also true, especially when the derived $(n-1)$-norm $\|\cdot, \cdot\|_\infty$ is defined with respect to an orthonormal set $\{e_1, \ldots, e_n\}$ in $X$ as in Section 2.1.

**FACT 3.2.** A sequence in a standard $n$-normed space $X$ is convergent in the $n$-norm if and only if it is convergent in the derived $(n-1)$-norm $\|\cdot, \cdot\|_\infty$.

**Proof.** Suppose that $x(k)$ converges to an $x \in X$ in the derived $(n-1)$-norm $\|\cdot, \cdot\|_\infty$. We want to show that $x(k)$ also converges to $x$ in the $n$-norm. Take $x_1, \ldots, x_{n-1} \in X$. Then one may observe that
\[
\|x_1, \ldots, x_{n-2}, x_{n-1}, x(k) - x\|_S \leq \|x_1, \ldots, x_{n-2}, x(k) - x\|_S \|x_{n-1}\|_S,
\] (3.5)
where $\|\cdot, \cdot\|_S$ and $\|\cdot\|_S$ on the right-hand side denote the standard $(n-1)$-norm and the usual norm on $X$, respectively. By Fact 2.4, we have
\[
\|x_1, \ldots, x_{n-2}, x_{n-1}, x(k) - x\|_S \leq \sqrt{\|x_1, \ldots, x_{n-2}, x(k) - x\|_\infty \|x_{n-1}\|_S}.
\] (3.6)
But $\lim_{k \to \infty} \|x_1, \ldots, x_{n-2}, x(k) - x\|_\infty = 0$, and so we conclude that
\[
\lim_{k \to \infty} \|x_1, \ldots, x_{n-2}, x(k) - x\|_S = 0,
\] (3.7)
that is, $x(k)$ converges to $x$ in the $n$-norm.

**Corollary 3.3.** A sequence in a standard $n$-normed space is convergent in the $n$-norm if and only if it is convergent in the standard $(n-1)$-norm and, by induction, in the standard $(n-r)$-norm for all $r = 1, \ldots, n-1$. In particular, a sequence in a standard $n$-normed space is convergent in the $n$-norm if and only if it is convergent in the usual norm $\|\cdot\|_S := \langle \cdot, \cdot \rangle^{1/2}$.

3.2. The finite-dimensional case. We also have a similar result for finite-dimensional $n$-normed space $(X, \|\cdot, \cdot\|)$. Let $\{b_1, \ldots, b_d\}$ be a basis for $X$. With respect to $\{b_1, \ldots, b_d\}$, define the following function $\|\cdot, \cdot\|_\infty$ on $X^{n-1}$ by
\[
\|x_1, \ldots, x_{n-1}\|_\infty := \max \{|x_1, \ldots, x_{n-1}, b_i| : i = 1, \ldots, d\}.
\] (3.8)
Then, as mentioned before, the function $\|\cdot, \cdot\|_\infty$ defines an $(n-1)$-norm on $X$.

With this derived $(n-1)$-norm, we have the following result.

**Proposition 3.4.** A sequence in the finite-dimensional $n$-normed space $X$ is convergent in the $n$-norm if and only if it is convergent in the derived $(n-1)$-norm $\|\cdot, \cdot\|_\infty$. 
**Proof.** If a sequence in $X$ is convergent in the $n$-norm, then it will certainly be convergent in the $(n-1)$-norm $\|\cdot,\cdot\|_{(n-1)}$. Conversely, suppose that $x(k)$ converges to an $x \in X$ in $\|\cdot,\cdot\|_{(n-1)}$. Take $x_1,\ldots,x_{n-1} \in X$. Writing $x_{n-1} = \alpha_1 b_1 + \cdots + \alpha_d b_d$, we get

$$\begin{align*}
\|x_1,\ldots,x_{n-2},x_{n-1},x(k) - x\| &\leq \|x_1,\ldots,x_{n-2},x(k) - x, b_1\| \\
& \quad + \cdots + \|x_{n-2},x(k) - x, b_d\| \\
& \leq \left( \|x_1\| + \cdots + \|x_{n-2}\| \right) \|x(k) - x\|_{(n-1)}.
\end{align*}$$

But $\lim_{k \to \infty} \|x_1,\ldots,x_{n-2},x(k) - x\|_{(n-1)} = 0$, and so we obtain

$$\lim_{k \to \infty} \|x_1,\ldots,x_{n-1},x(k) - x\| = 0,$$  

that is, $x(k)$ converges to $x$ in the $n$-norm. \hfill $\square$

### 3.3. The standard, separable case

We go back to the standard case, where $X$ is a real inner product space of dimension $d \geq n$ equipped with the standard $n$-norm $\|\cdot,\cdot\|_n$ as in Section 2.1. But suppose now that $X$ is separable and that $\{e_i : i \in I_d\}$, where $I_d := \{1,\ldots,d\}$ (if $d < \infty$) or $\mathbb{N}$ (if $d = \infty$), is an orthonormal basis for $X$. For every $x_1,\ldots,x_{n-1} \in X$ and every basis vector $e_i$ ($i \in I_d$), we have

$$\|x_1,\ldots,x_{n-1},e_i\|_S \leq \|x_1,\ldots,x_{n-1}\|_S,$$  

where $\|\cdot,\cdot\|_S$ on the right-hand side denotes the standard $(n-1)$-norm on $X$. Hence, with respect to $\{e_i : i \in I_d\}$, we may define the function $\|\cdot,\cdot\|_{(n-1)}$ on $X^{n-1}$ by

$$\|x_1,\ldots,x_{n-1}\|_{(n-1)} := \sup \{ \|x_1,\ldots,x_{n-1},e_i\|_S : i \in I_d\},$$

and check that it also defines an $(n-1)$-norm on $X$. Moreover, we have the following relation between the two derived $(n-1)$-norms $\|\cdot,\cdot\|_{(n-1)}$ and $\|\cdot,\cdot\|_{(n-1)}$ and the standard $(n-1)$-norm $\|\cdot,\cdot\|_S$ (the latter being defined with respect to $\{e_1,\ldots,e_n\}$ only):

$$\|x_1,\ldots,x_{n-1}\|_{(n-1)} \leq \|x_1,\ldots,x_{n-1}\|_{(n-1)} \leq \|x_1,\ldots,x_{n-1}\|_S \leq \sqrt{n} \|x_1,\ldots,x_{n-1}\|_{(n-1)}$$

for every $x_1,\ldots,x_{n-1} \in X$. Hence we conclude the following fact.

**Fact 3.5.** On a standard $n$-normed space $X$, the two derived $(n-1)$-norms $\|\cdot,\cdot\|_{(n-1)}$ and $\|\cdot,\cdot\|_{(n-1)}$ and the standard $(n-1)$-norm $\|\cdot,\cdot\|_S$ are equivalent. Accordingly, a sequence in a standard $n$-normed space $X$ is convergent in the $n$-norm if and only if it is convergent in one of the three $(n-1)$-norms.

### 3.4. Cauchy sequences, completeness and fixed point theorem

Recall that a sequence $x(k)$ in an $n$-normed space $(X,\|\cdot,\cdot\|)$ is called **Cauchy** (with respect to the $n$-norm) if

$$\lim_{k,l \to \infty} \|x_1,\ldots,x_{n-2},x_{n-1},x(k) - x(l)\| = 0$$

#### References
for every \(x_1, \ldots, x_{n-1} \in X\). If every Cauchy sequence in \(X\) converges to an \(x \in X\), then \(X\) is said to be complete (with respect to the \(n\)-norm). A complete \(n\)-normed space is then called an \(n\)-Banach space.

By replacing the phrases “\(x(k)\) converges to \(x\)” with “\(x(k)\) is Cauchy” and “\(x(k) − x(l)\)” with “\(x(k) − x(l)\),” we see that the analogues of Proposition 3.1, Fact 3.2, Corollary 3.3, Proposition 3.4, and Fact 3.5 hold for Cauchy sequences.

Hence, for the standard or finite-dimensional case, we have the following result.

**Proposition 3.6.** (a) A standard \(n\)-normed space is complete if and only if it is complete with respect to one of the three \((n − 1)\)-norms \(\|\cdot, \ldots, \cdot\|_\infty, \|\cdot, \ldots, \cdot\|_\bowtie, \text{or} \|\cdot, \ldots, \cdot\|_S\).

By induction, a standard \(n\)-normed space is complete if and only if it is complete with respect to the usual norm \(\|\cdot\|_S := (\cdot, \cdot)^{1/2}\).

(b) A finite-dimensional \(n\)-normed space is complete if and only if it is complete with respect to the derived \((n − 1)\)-norm \(\|\cdot, \ldots, \cdot\|_\bowtie\).

Consequently, we have the following result.

**Corollary 3.7** (fixed point theorem). Let \((X, \|\cdot, \ldots, \cdot\|)\) be a standard or finite-dimensional \(n\)-Banach space, and \(T\) a contractive mapping of \(X\) into itself, that is, there exists a constant \(C \in (0, 1)\) such that

\[
\|x_1, \ldots, x_{n-1}, Ty − Tz\| \leq C\|x_1, \ldots, x_{n-1}, y − z\| \tag{3.15}
\]

for all \(x_1, \ldots, x_{n-1}, y, z \in X\). Then \(T\) has a unique fixed point in \(X\).

**Proof.** First consider the case \(n = 2\) (see [8]). By Proposition 3.6, we know that \(X\) is a Banach space with respect to the derived norm \(\|\cdot\|_\infty\) (for standard case) or \(\|\cdot\|_\bowtie\) (for finite-dimensional case). Since the mapping \(T\) is also contractive with respect to \(\|\cdot\|_\infty\) or \(\|\cdot\|_\bowtie\), we conclude by the fixed point theorem for Banach spaces that \(T\) has a unique fixed point in \(X\). For \(n > 2\), the result follows by induction.

**Remark 3.8.** In the finite-dimensional case, it is actually enough to assume that \(X\) is an \(n\)-normed space because we know that all finite-dimensional normed spaces are complete and, by Proposition 3.6(b), so are all finite-dimensional \(n\)-normed spaces.

4. Concluding remark. We have shown that an \(n\)-normed space with \(n \geq 2\) is an \((n − 1)\)-normed space and that, for the standard or finite-dimensional case, the \((n − 1)\)-norm can be derived from the \(n\)-norm in such a way that the convergence and completeness in the \(n\)-norm is equivalent to those in the derived \((n − 1)\)-norm.

Below is an example of a non-standard, infinite-dimensional \(2\)-normed space for which we can derive a norm from the \(2\)-norm such that the convergence and completeness in the \(2\)-norm is equivalent to those in the derived norm.

Let \(X = l^\infty\), the space of bounded sequences of real numbers. Equip \(X\) with the following \(2\)-norm

\[
\|x, y\| := \sup_{i \in \mathbb{N}} \sup_{j \in \mathbb{N}} |x_i y_j − x_j y_i|, \tag{4.1}
\]

where \(x = (x_1, x_2, x_3, \ldots)\) and \(y = (y_1, y_2, y_3, \ldots)\). Let \(\alpha_1 = (1, 0, 0, \ldots)\) and \(\alpha_2 = (0, 1, 0, \ldots)\).
With respect to \(\{a_1, a_2\}\), we derive the norm \(\|\cdot\|_\infty\) via
\[
\|x\|_\infty := \max \{\|x, a_1\|, \|x, a_2\|\}. \tag{4.2}
\]
But \(\|x, a_1\| = \sup_{i \in \mathbb{N}\setminus\{1\}} |x_i|\) and \(\|x, a_2\| = \sup_{i \in \mathbb{N}\setminus\{2\}} |x_i|,\) and so we obtain
\[
\|x\|_\infty = \sup_{i \in \mathbb{N}} |x_i|, \tag{4.3}
\]
the usual norm on \(l^\infty\).

Now suppose that \(x^{(k)}\) is a sequence in \(X\) that converges to \(x\) in the derived norm \(\|\cdot\|_\infty\). For every \(y \in X\), we have
\[
\|x^{(k)} - x, y\| = \sup_{i \in \mathbb{N}} \sup_{j \in \mathbb{N}} |(x_i^{(k)} - x_i)y_j - (x_j^{(k)} - x_j)y_i|
\leq \sup_{i \in \mathbb{N}} \sup_{j \in \mathbb{N}} |x_i^{(k)} - x_i| |y_j| + |x_j^{(k)} - x_j| |y_i|
\leq 2 \|x^{(k)} - x\|_\infty \|y\|_\infty, \tag{4.4}
\]
whence \(\lim_{k \to \infty} \|x^{(k)} - x, y\| = 0\). Hence \(x^{(k)}\) converges to \(x\) in the 2-norm \(\|\cdot, \cdot\|\).

Thus, for this particular example, we see that the convergence in the 2-norm is equivalent to that in the derived norm. By similar arguments, we can also verify that the completeness in the 2-norm is equivalent to that in the derived norm.

For general non-standard, infinite-dimensional \(n\)-normed spaces, however, it is unknown whether we can always derive an \((n - 1)\)-norm from the \(n\)-norm such that the convergence and completeness in the \(n\)-norm is equivalent to those in the derived \((n - 1)\)-norm.

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HENDRA GUNAWAN: DEPARTMENT OF MATHEMATICS, BANDUNG INSTITUTE OF TECHNOLOGY, BANDUNG 40132, INDONESIA

E-mail address: hgunawan@dns.math.itb.ac.id

MASHADI: DEPARTMENT OF MATHEMATICS, UNIVERSITY OF RIAU, PEKANBARU 28293, INDONESIA