The Regularity of the Solutions to the Cauchy Problem for the Quasilinear Second-Order Parabolic Partial Differential Equations

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Abstract: This article is dedicated to expanding our comprehension of the regularity of the solutions to the Cauchy problem for the quasilinear second-order parabolic partial differential equations under fair general conditions on the nonlinear perturbations. In this paper have been obtained that the sequence of the weak solutions \( u^\tau \in V_1, \tau = 1,2, \ldots \) to the Cauchy problems for the Equations (15) under the initial conditions \( u^\tau (0,x) = \varphi_0 \) converges to the weak solution to the Cauchy problem for the Equation (1) under the initial condition \( u(0, x) = u_0 \) in \( V_1 \).

Keywords: Quasi-Linear Partial Differential Equations, Nonlinear Partial Differential Equations, Parabolic, Nonlinear Operator, Weak Solution, A Priori Estimations

Introduction

Let us consider the quasilinear second-order parabolic partial differential equations:

\[
\begin{align*}
\frac{\partial}{\partial t} u + \lambda u &= \sum_{i,j=1 \atop i \neq j}^{l} \frac{\partial}{\partial x_i} \left( a_{ij}(t,x,u) \frac{\partial}{\partial x_j} u \right), \\
+ b(t,x,u,\nabla u) &= f(t,x),
\end{align*}
\]

under the initiation conditions:

\( u(0,x) = u_0(x) \),

where the \( u(t,x) \) is the unknown function, \( \lambda > 0 \) is a real number and \( f(t,x) = f \) is a given function. The term \( b(t,x,u,\nabla u) \) is a measurable function of four arguments.

The matrix \( a_{ij}(t,x,u) \) is a measurable elliptical matrix \( l \times l \) size such that there is a number \( \nu: 0 < \nu < \infty \) and:

\[
\nu \sum_{i=1}^{l} \xi_i^2 \leq \sum_{i,j=1 \atop i \neq j}^{l} a_{ij}(t,x,u) \xi_i \xi_j, \quad \forall \xi \in \mathbb{R}^l
\]

for almost every \( t \in [0,T] \) and \( x \in \mathbb{R}^l \). Or we will consider a more restrictive condition:

\[
\nu \sum_{i=1}^{l} \xi_i^2 \leq \sum_{i,j=1 \atop i \neq j}^{l} a_{ij}(t,x,u) \xi_i \xi_j \leq \mu \sum_{i=1}^{l} \xi_i^2, \quad \forall \xi \in \mathbb{R}^l
\]

Definition

A real-valued function \( u(t,x) \) is called a weak solution to the parabolical partial differential Equation (1) if the integral identity:

\[
\begin{align*}
\langle u(\tau), v(\tau) \rangle_V + \\
+ \int_0^\tau \left( \nu \frac{\partial u}{\partial \tau} + a_{ij}(t,x,u) \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} \right) dt + \\
+ \int_0^\tau \left( \nu b(t,x,u,\nabla u) v \right) dt = \\
+ \int_0^\tau \left( \nu \int_0^\tau f(\tau) \right) dt
\end{align*}
\]

holds for almost every \( t \in [0,T] \), \( x \in \mathbb{R}^l \) and for all \( v \in W^{1,\nu}_V \).

The main object of this paper is the regularity properties of the solutions to the quasilinear parabolical partial differential Equation (1) under the conditions that its coefficients belong to the certain functional classes and functional spaces.

The conditions of linear growth:

1. \( b(t,x,y,z) \) is a measurable function of its arguments and \( b \in L^\infty_1 \left( \mathbb{R}^l \right) \).
2. Function \( b(t,x,y,z) t \in [0,T] \) satisfies inequality:

\[
|b(t,x,y,z)| \leq \mu_1(x)|\nabla u| + \mu_2(x)|u| + \mu_3(x)
\]
for almost everywhere and almost every $t \in [0, T]$, where the functions $\mu_1 \in PK_1(A), \mu_2 \in PK_2(A)$ and $\mu_3 \in L^r(R^r)$.

3. The increase of function $b(t, x, y, z)$ satisfies the inequality:

$$|b(t, x, u, \nabla u)| \leq \mu_6(x)|\nabla(u - v)| + \mu_7(x)|u - v|^2,$$

almost everywhere and almost every $t \in [0, T]$, where the functions $\mu_6 \in PK_6(A), \mu_7 \in PK_7(A)$.

Here we introduce the class of form-bounded functions $PK_\beta$ according to formula-definition:

$$PK_\beta(A) = \left\{ g \in L^1_{loc}(R^r, d^r) : \left\| g|b|^2 \right\|_{L^\infty} \leq \beta \left( \frac{1}{A^2 h^2} + c(\beta)\|h\|^2 \right) \right\}.$$

where $h \in D\left( \frac{1}{A^2} \right)$ and $\beta > 0$ is a form-boundary and $c(\beta) \in R^r$.

The general information on the partial differential equations and the existence of their solutions can be found in the extensive literature on the conditions on their coefficients under which there are the solutions of these equations in a specific functional space (Adams and Hedberg, 1996; Gilbarg and Trudinger, 1983; Ladyzenskaja et al., 1968; Nirenberg, 1994; Veron, 1996; Yaremko, 2017a; 2017b). O. Ladyzhenskaya, N. Ural'tseva, O.A. Solonnikov developed the Ennio de Giorgi's method (DeGiorgi, 1968) for establishing a priory estimation of the solution of such equations. 1960 J. Moser enhance the maximum principle and created a new method of studying the regularity of the solutions of elliptic differential equations and Harnack’s inequality under the assumption that the coefficients are bounded measurable and satisfy a uniform ellipticity condition, these results were summarized in the work of Ladyzenskaja et al. (1968).

A Lebesgue space $L^p (R^r, d^r x)$ for $1 < p < \infty$ can be defined as a set of all real-valued measurable functions defined almost everywhere such that the Lebesgue integral of its absolute value raised to the $p$-th power is a finite number with its natural norm:

$$\|u\|_p = \left( \int |u(x)|^p d^r x \right)^{1/p} = \left( \int |u(x)|^p d^r x \right)^{1/p} = \left\langle |u|^p \right\rangle^{1/p}.$$

The dual or adjoint space of $L^p (R^r, d^r x)$ for $1 < p < \infty$ has a natural isomorphism with $L^q (R^r, d^r x)$, where

$$\frac{1}{p} + \frac{1}{q} = 1 \text{ or } q = \frac{p}{p - 1}.$$

We will use the inequality:

$$\langle f, g \rangle \leq \|f\|_p \|g\|_q^p + \frac{1}{p} \|f\|_p^p \|g\|_q^q,$$

where $f \in L^p (R^r), g \in L^q (R^r), \varepsilon > 0$ and its consequence:

$$\langle f, f \rangle^{1/2} = \|f\|_{L^p} \|f\|_{L^q}^{1/2} = \frac{1}{p} \|f\|_{L^p}^p \|f\|_{L^q}^{1/2}.$$

The $f \in L^p$ yields $f \|f\|_{L^p}^p \|f\|_{L^q}^{1/2} \in L^q$ that justify the last equation (Gilbarg and Trudinger, 1983; Ladyzenskaja et al., 1968).

Let us denote $W^p_{l^0} (R^r, d^r x)$ given Sobolev space for $1 < p \infty$ with a natural norm:

$$\|u\|_{W^p_{l^0}} = \left( \sum_{\alpha \in \mathbb{N}^r \atop \sum_{\alpha} \|D^\alpha u\|^p \right)^{1/p} = \left( \sum_{\alpha \in \mathbb{N}^r \atop \sum_{\alpha} \|D^\alpha u\|^p \right)^{1/p}.$$

The dual space of $W^p_{l^0} (R^r, d^r x)$ for $1 < p \infty$ is $W^q_{l^0} (R^r, d^r x)$ and the dual space of $W^p_{l^0} (R^r, d^r x)$ for $1 < p \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$ is $W^q_{l^0} (R^r, d^r x)$, Sobolev spaces are reflexive (Fijavž et al., 2007).

Let us consider a linear parabolic equation as an exemplar:

$$L u = \left[ \frac{\partial}{\partial t} - \sum_{i,k=1} a_{ik}(t, x)\nabla_i \nabla_k - \sum_{i,j=1} b_j(t, x) \nabla_j \right] u(t, x) = 0$$

under the conditions $\exists \nu, \mu: 0 < \nu \leq \mu < \infty$ such that:

$$\sum_{i=1}^q \xi_i^2 \leq \sum_{q=1}^r a_{ij}(t, x) \xi_j \leq \sum_{j=1}^r \xi_j^2$$

and linear perturbation-potential $b_j(t, x): R^r \rightarrow R^r$.

In traducing the notations:
and assuming $b = a^{1 - b} \in PK_{\varepsilon}(A)$ for some $\beta < 1$ we obtain:

$$\|h_b^h\| \leq \sqrt{\beta}\|Ah_h\| + c(\beta) \frac{1}{2\sqrt{\beta}} \|h\|^2$$

according to the KLMN-theorem, there is a preserving $C_p$-semigroups of $L^p$- contraction $e^{-\gamma t}$, $\frac{2}{2 - \sqrt{\beta}} \leq n \leq \infty$ such that $\Lambda_n = A + b \cdot \nabla$. Assuming $A$ is Laplace operator $A = \Delta$ we are obtaining an estimation:

$$\|h_b^h\| \leq \sqrt{\beta}\|h\|^2 + \frac{c(\beta)}{2\sqrt{\beta}} \|h\|^2 \quad \forall h \in D(\Delta).$$

The operator $B_h^h = \nabla \cdot b$ of the domain $D(B_h^h) = \{u \in L^1; \|\nabla u\| \in L^1; \nabla \cdot b \cdot \nabla u \in L^1\}$ is $A_1$-bounded with relative bound zero namely $D(B_h^h) \supset D(A_1)$ and:

$$\|B_h^h\| \leq \alpha \|A_h^h\| + k(\alpha) \|h\|, \quad h \in D(\Delta)$$

holds for all $\alpha > 0$ and $k(\alpha) < \infty$. There are $s > 0$ and $\beta(s) < 1$ such that $\int_0^1 B_h e^{-\alpha s} \|h\|_s \leq \beta(s) \|h\|_s$, $h \in D(A_1)$. The operator $A_1 + B_1$ of the domain $D(A_1)$ generates $C_p$-semigroup $T''$ consistent with $T' = \exp(-t(A + b \cdot \nabla))$ such that $\|T''\| \leq 1 - \exp\left(-t(1 - \beta(s))\right), t > 0$.

### The Estimation of the Solutions to the Equation (1)

For almost every $t \in [0, T]$, let us consider the integral identity:

$$\langle u(\tau), v(\tau) \rangle_{B^b} + \int_0^t \left( -\langle u(\tau), \partial_t\partial_t v(\tau) \rangle + \lambda \langle u(\tau), v(\tau) \rangle \right) d\tau$$

$$+ \int_0^t \left( \sum_{i,j=1}^n a_{ij} \frac{\partial}{\partial x_j} u(\tau) \frac{\partial}{\partial x_i} v(\tau) \right) d\tau + \int_0^t \langle b, v \rangle d\tau = \int_0^t \langle f, v \rangle d\tau,$$

where functions $u(t,x) \in W^s_{\lambda \beta}$ and $v \in W^s_{\lambda \beta}$.

For $t \in [0, T]$ identity (6) can be rewritten as:

$$\langle u(\tau), v(\tau) \rangle_{B^b} + \int_0^t \left( -\langle u(\tau), \partial_t\partial_t v(\tau) \rangle + \lambda \langle u(\tau), v(\tau) \rangle \right) d\tau$$

$$+ \int_0^t \left( \sum_{i,j=1}^n a_{ij} \frac{\partial}{\partial x_j} u(\tau) \frac{\partial}{\partial x_i} v(\tau) \right) d\tau + \int_0^t \langle b, v \rangle d\tau = \int_0^t \langle f, v \rangle d\tau.$$
In case of \( p = 2 \) there is the next estimation:
\[
\int_0^\infty \left[ \frac{\beta}{2} \left( \| \nabla u \|_w^2 + c(\beta) \| u \|_w^2 \right) \right] \, dt \\
\leq \left( \frac{1}{2} \| \nabla u \|_w^2 + c(\beta) \| u \|_w^2 \right) \, dt \\
+ \frac{1}{2} \left( \frac{1}{c^2} + \beta \frac{1}{c} \right) \int_0^\infty \left( \| u \|_w^2 \right) \, dt.
\]

Assuming that \( \epsilon^2 = \frac{1}{\sqrt{\beta}} \) then \( \frac{1}{2} \left( \frac{1}{c^2} + \beta \frac{1}{c} \right) + \beta = \sqrt{\beta} + \beta = \sqrt{\beta} + \sqrt{\beta} \) and we are obtaining:
\[
\frac{1}{2} \| u \|_w^2 + \int_0^\infty \left( \| \nabla u \|_w^2 + \lambda \right) \| u \|_w^2 \, dt \\
\leq \left( \frac{1}{2} \| \nabla u \|_w^2 + c(\beta) \| u \|_w^2 \right) \, dt \\
+ \frac{1}{2} \left( \frac{1}{c^2} + \beta \frac{1}{c} \right) \int_0^\infty \left( \| u \|_w^2 \right) \, dt.
\]

The Smoothness of the Weak Solutions to the Quasilinear Second-Order Parabolic Partial Differential Equation (1)

**Definition**

A real-valued function \( u(t,x) \in W_{1,1}^2 \) such that \( \text{vrai} \max |u(t,x)| < \infty \) is called a weak bound solution to the quasilinear second-order parabolic partial differential Equation (1) if the identity:
\[
\left\langle u(t), v(t) \right\rangle + \int_0^T \left( -\left\langle u(t), \frac{\partial}{\partial x} v(t) \right\rangle + \lambda \left( u(t), v(t) \right) \right) \, dt \\
+ \int_0^T \left\langle \sum_{p=1}^P a_{p,ij} \frac{\partial}{\partial x_i} u(t), \frac{\partial}{\partial x_j} v(t) \right\rangle \, dt \\
+ \int_0^T \left\langle h(t), v(t) \right\rangle \, dt \\
= \left\langle f(t), v(t) \right\rangle
\]

holds for all functions \( v \in W_{1,1}^2 \) such that \( \text{vrai} \max |v(t,x)| < \infty, \ t \in [0,T] \).

For arbitrary function \( v \in W_{1,1}^2 \) such that \( \text{vrai} \max |v(t,x)| < \infty, \ t \in [0,T] \) from that definition of the weak solution we are obtaining
\begin{equation}
\langle u(\tau), v(\tau) \rangle + \int_0^T \left[ \left( \sum_{i,j=1}^n a_{ij} \frac{\partial}{\partial x_j} \left( v_i \frac{\partial}{\partial x_i} v \right) \right) + \lambda (u_n, v) \right] d\tau
\leq \int_0^T \langle f, v \rangle d\tau - \int_0^T \langle \lambda (u(\tau), v(\tau)) \rangle d\tau
+ \int_0^T \left( \mu_1(t,x) |\nabla u| + \mu_2(t,x) |u| + \mu_3(t,x), v(\tau) \right) d\tau.
\tag{9}
\end{equation}

Let \( u(t,x) \) be a weak solution. We denote \( v_h(t,x) \) the average of function \( v(t,x) \) at \( t \) by formulae:
\begin{equation}
v_h(t,x) = \frac{1}{h} \int_{t-h}^t v(\tau,x) d\tau, \quad u_h(t,x) = \frac{1}{h} \int_{t-h}^t u(\tau,x) d\tau
\tag{10}
\end{equation}

we transform:
\begin{equation}
-\frac{\partial}{\partial t} \langle u(t,x) \rangle = -\frac{\partial}{\partial t} \langle u_h(t,x) \rangle = \int_0^T \langle \partial_t u_h(t,x) \rangle dt.
\end{equation}

since:
\begin{equation}
\frac{1}{h} \int_{t-h}^t u(x) v(t) dt \int_0^T u_h(x) v(t) dt
\end{equation}

where the function \( v(t,x) \) is tautological equals zero over \( t \leq 0 \) and \( T \geq t \geq T-h \).

\textbf{Remark}

The order of averaging and differentiation by \( x \) are interchangeable.

Let us rewrite (6) as:
\begin{equation}
\int_0^T \left[ \left( \sum_{i,j=1}^n a_{ij} \frac{\partial}{\partial x_j} \left( v_i \frac{\partial}{\partial x_i} v \right) \right) + \lambda (u_n, v) \right] d\tau
+ \int_0^T \left[ \left( \sum_{i,j=1}^n a_{ij} \frac{\partial}{\partial x_j} \left( v_i \frac{\partial}{\partial x_i} v \right) \right) + \lambda (u_n, v) \right] d\tau
= \int_0^T \langle f, v \rangle d\tau.
\end{equation}

Since in the last equality the function \( v \in W_{1,0}^2 \) is arbitrary, we can assume that \( v = u_n \) next integrating with respect to \( t \), we are passing to the limit as \( h \to 0 \) and are obtaining:
\begin{equation}
\frac{1}{2} \int_0^T \langle u, u \rangle d\tau + \frac{1}{2} \int_0^T \langle \nabla u \cdot a \nabla u \rangle d\tau
= \int_0^T \langle f, u \rangle d\tau.
\end{equation}

For an arbitrary function \( v \in V_{1,0}^2 \) the integrals:
\begin{equation}
\int_0^T \left[ \left( \sum_{i,j=1}^n a_{ij} \frac{\partial}{\partial x_j} \left( v_i \frac{\partial}{\partial x_i} v \right) \right) + \lambda (u_n, v) \right] d\tau
\end{equation}

and:
\begin{equation}
\int_0^T \langle f, v \rangle d\tau
\end{equation}

converge to:
\begin{equation}
\int_0^T \left( \sum_{i,j=1}^n a_{ij} \frac{\partial}{\partial x_j} \left( v_i \frac{\partial}{\partial x_i} v \right) + \lambda (u_n, v) \right) d\tau
\end{equation}

and:
\begin{equation}
\int_0^T \langle f, v \rangle d\tau
\end{equation}

as \( h \to 0 \) so it is true for \( v = u \).

For an arbitrary \( t_1, t_2 \in [h, T-h] \) applying (6) we can write:
\begin{equation}
\int_{t_1}^{t_2} \langle \partial_t u(t,x), v \rangle + \lambda \langle u_n, v \rangle d\tau
+ \int_{t_1}^{t_2} \left[ \left( \sum_{i,j=1}^n a_{ij} \frac{\partial}{\partial x_j} \left( v_i \frac{\partial}{\partial x_i} v \right) \right) + \lambda (u_n, v) \right] d\tau
= \int_{t_1}^{t_2} \langle f, v \rangle d\tau,
\end{equation}

assume \( v = u_n \), where \( u_n(t,x) = \max \{u(t,x), k, 0\} \) and we denote the set of points \( P(t) = \{x \in R^l: u(t,x) > k, t \in [0, T]\} \), \( R^l, l > 2 \) and \( P(t) = \{t(x) \in [0, \bar{T}] \times R^l: u(t,x) > k, \ t \in [0, T], l > 2\} \), we have:
\begin{equation}
\frac{1}{2} \int_0^T \langle u, u \rangle d\tau + \frac{1}{2} \int_0^T \langle \nabla u \cdot a \nabla u \rangle d\tau
+ \lambda \int_{\Omega} u^2 d\tau
\leq \left( \frac{1}{\sqrt{\beta}} + \frac{c(\beta)}{2\sqrt{\beta}} + c(\beta) \right) \int_0^T \langle u, u \rangle d\tau
+ \sqrt{\beta} \int_{\Omega} \left( \nabla u \cdot a \nabla u \right) d\tau
+ \sqrt{\beta} \int_{\Omega} \left( \nabla u \cdot a \nabla u \right) d\tau
+ \frac{\sqrt{\beta}}{2} \int_0^T \| u \|^2_{L^2(\Omega)} d\tau + \frac{\sqrt{\beta}}{2} \int_0^T \| u \|^2_{L^2(\Omega)} d\tau.
\end{equation}

From \( (a+b)^2 \leq 2(a^2+b^2) \), we obtain:
Lemma 1

Let element \( u \in V^2_\varphi \) satisfies following tautology:

\[
\int_0^\infty \left( \|u - k\|_{t_\varphi}^2 + k^2 \right) \varphi \left( \tau \right) \, d\tau \leq 2 \left( \|u - k\|_{t_\varphi}^2 + k^2 \right) \int_0^\infty \varphi \left( \tau \right) \, d\tau .
\]

where the \( \varphi \) is an arbitrary element of functional space \( W^2_\varphi \left( [0,T] \times \mathbb{R}^l \right) \) then element \( u \in V^2_\varphi \) belongs \( V^2_\varphi \left( [0,T] \times \mathbb{R}^l \right) \).

Space \( V^2_\varphi \left( [0,T] \times \mathbb{R}^l \right) \) is a subspace of \( W^2_\varphi \left( [0,T] \times \mathbb{R}^l \right) \) that consists of all continuous at \( t \) in \( L^2(R^l) \) norm elements with the norm \( \|u\|_{\varphi} = \max_{t\in[0,T]}|u(t)| + \|\nabla u\|_{\varphi} \) and the following condition

\[
\int_0^\infty \left( \frac{1}{2} |u(t+h,.) - u(t,.)|^2 \right) dt \to 0
\]

is satisfied.

Proof of Lemma 1

For arbitrary \( \varphi \in W^2_\varphi \left( [0,T] \times \mathbb{R}^l \right) \) we denote \( \varphi_h(t,x) = \frac{1}{h} \int_{t-h}^t \varphi(t,x) \, d\tau \) then:

\[
\int_0^\infty \left( \langle u, \partial_t \varphi \rangle + \lambda \langle u, \varphi \rangle \right) d\tau
\]

\[
+ \int_0^\infty \left( \sum_{i,j=1}^n a_{ij} \frac{\partial}{\partial x_j} u \right) \frac{\partial}{\partial x_i} \varphi \, d\tau + \langle b, \varphi \rangle \, d\tau
\]

\[
= \int_0^\infty \langle f, \varphi \rangle \, d\tau .
\]

Put \( \varphi(t,x) = \chi(t) \psi(x) \), where \( \chi(t) \) is a smooth function of time and \( \psi \in W^2_\varphi \left( \mathbb{R}^l \right) \). We have:

\[
\int_0^\infty \left( \langle u, \partial_t \chi(t) \psi \rangle + \lambda \chi(t) \langle u, \psi \rangle \right) d\tau
\]

\[
+ \int_0^\infty \chi(t) \left( \sum_{i,j=1}^n a_{ij} \frac{\partial}{\partial x_j} u \right) \frac{\partial}{\partial x_i} \psi \, d\tau + \langle b, \psi \rangle \, d\tau
\]

\[
= \int_0^\infty \chi(t) \langle f, \psi \rangle \, d\tau .
\]

By integrating with respect to time, we have:

\[
\frac{1}{2} \sum_{i,j=1}^n a_{ij} \int_0^\infty \frac{\partial}{\partial x_j} u \, d\tau + \lambda \int_0^\infty u \, d\tau
\]

\[
+ \int_0^\infty \left( \sum_{i,j=1}^n a_{ij} \frac{\partial}{\partial x_j} u \right) \frac{\partial}{\partial x_i} \psi \, d\tau + \langle b, \psi \rangle \, d\tau
\]

\[
= \int_0^\infty \chi(t) \langle f, \psi \rangle \, d\tau .
\]

Let pass to limit as \( h_1 \to 0, h_2 \to 0 \) we obtain:
\[ \left\| u_n - u_i \right\| + \left\| \frac{\partial}{\partial x_i} (u_n - u_i) \right\| \\
+ \left( \sum_{i,j=1}^{n} a_{ij} \frac{\partial}{\partial x_i} u_n \right)_n - \left( \sum_{i,j=1}^{n} a_{ij} \frac{\partial}{\partial x_j} u_i \right)_n \\
+ \| b_n - b_i \| + \left\| f_n - f_i \right\| \xrightarrow{h_n \to 0} 0. \]

We denote \( \psi(x) = \Delta u = u(t+h,x) - u(t,x) \) then:

\[ \int_{\Omega} \left( \frac{\partial}{\partial x_1} u(t+h,x) - \frac{\partial}{\partial x_2} u(t,x) \right) \, dt \]
\[ + \int_{\Omega} \left( \sum_{i,j=1}^{n} a_{ij} \frac{\partial}{\partial x_i} u(t+h,x) - \sum_{i,j=1}^{n} a_{ij} \frac{\partial}{\partial x_j} u(t,x) \right) \, dt \\
+ \int_{\Omega} \left( f_n, u(t+h,x) - u(t,x) \right) \, dt \\
= \int_{\Omega} \left( f, u(t+h,x) - u(t,x) \right) \, dt, \]

and we have:

\[ \int_{\Omega} \left( \Delta u \right) \, dt + \int_{\Omega} \left( u_n, \Delta u \right) \, dt \\
+ \int_{\Omega} \left( \sum_{i,j=1}^{n} a_{ij} \frac{\partial}{\partial x_i} u_n \right) \, dt + \int_{\Omega} \left( b_n, u_n \right) \, dt \\
= \int_{\Omega} \left( f, u_n \right) \, dt. \]

Applying Holder inequality and previous considerations, we have obtained:

\[ \int_{\Omega} \left( \Delta u \right) \, dt \leq c(h) \xrightarrow{h \to 0} 0, \]

that proves the lemma.

**A Priori Estimation of the Solution to (1)**

Let us assume that ellipticity condition and (4), (5) are satisfied and all weak solutions \( u(t,x) \) of the \( V_{1,h}^2 \) are bounded, we will show that \( u \in H^{\alpha,\alpha} \) for certain \( \alpha > 0 \) and estimate the norm \( \| u \|_{H^{\alpha,\alpha}} \).

Assume \( u \in V_{1,h}^2 \) for arbitrary element \( \varphi \in W_{1,h}^2 \), we have tautology (6) and we obtain an estimation:

\[ \left\| u(\tau), \varphi(\tau) \right\|_{L^2}^2 + \int_{\Omega} \left( \Delta u \right) \, dt \leq \int_{\Omega} \left( f, u \right) \, dt, \]

since for arbitrary element \( \varphi \in W_{1,h}^2 \), the following condition is executed:

\[ \int_{\Omega} \left( \Delta u, \varphi \right) \, dt \leq \int_{\Omega} \left( \Delta u, \varphi \right) \, dt \]

so:

\[ \int_{\Omega} \left( u(\tau), \varphi(\tau) \right) \, dt + \int_{\Omega} \left( u(\tau), \varphi(\tau) \right) \, dt + \int_{\Omega} \left( u(\tau), \varphi(\tau) \right) \, dt \]

\[ \leq \int_{\Omega} \left( \mu \| \nabla u \| + \mu_2 |u| + \mu_3 |\varphi| \right) \, dt \]

\[ + \int_{\Omega} \left( \mu_2 |u| \right) \, dt, \]

let us put \( \varphi(t,x) = (\xi(t,x))^2 u(t,x) = \xi^2 u \) and integrate by parts, we are obtaining

\[ \frac{1}{2} \int_{\Omega} \left| \xi(t,x) \right|^2 \, dt + \int_{\Omega} \left( \mu_2 u \right) \, dt \]

\[ + \int_{\Omega} \left( \mu_3 |\xi| \right) \, dt \]

\[ \leq \int_{\Omega} \left( \mu_2 |u| \right) \, dt + \int_{\Omega} \left( \mu_3 |\xi| \right) \, dt \]

\[ + \int_{\Omega} \left( f, \xi^2 u \right) \, dt, \]

where the \( K(\delta) \) is a cube in \( R^n \) with an edge length of \( \delta \). Next, we estimate:

\[ \left( \mu_2 u \right) \, dt \leq \left( \mu_2 u \right) \, dt \]

\[ \leq \frac{1}{2} \left( \int_{\Omega} \left| \xi \right|^2 \, dt \right) + \frac{1}{2} \left( \mu_2 \right) \, dt, \]
\[
\|u\|_2 \leq (\mu \|v\|^2 + c(\beta)\|\xi\|^2)^\frac{1}{2}.
\]

Similarly:
\[
\langle \mu \xi^2(\tau), u \rangle \leq \|\mu \xi^2(\tau)\|_2 \|u\| \leq \beta(\|\xi\|^2 + c(\beta)\|\xi\|^2)^\frac{1}{2} \|u\|.
\]

And:
\[
\|\nabla u\|_2 \leq \frac{1}{2} \left( \frac{1}{\epsilon^2} \|\nabla u\|^2 + \epsilon \|u\|^2 \right).
\]

These we have had the following inequality:
\[
\|u(t)\|_{1(\ell)}^2 + \int_{K(\ell)} \left( K_1 \|\nabla \xi + K_2 \|\xi\| + K_3 \|\xi\| \right) d\tau
\leq \|u(t)\|_{1(\ell)}^2 + \int_{K(\ell)} \left( K_4 \|f(\xi, \xi)\|_2 \right) d\tau,
\]

where \( K, K_1, K_2, K_3 \) are positive constants depend on the initial conditions and constants \( \varepsilon, \varepsilon_1, \varepsilon_2 \) are arbitrary constants, such that:
\[
\frac{1}{2} \left( \frac{1}{\epsilon^2} \|\mu \xi^2(\tau)\|^2 + \epsilon \|\nabla u\|^2 \right)
\leq \frac{1}{2} \left( \frac{1}{\epsilon^2} \beta(\|\xi\|^2 + c(\beta)\|\xi\|^2)^\frac{1}{2} + \epsilon \right)
\leq \frac{1}{2} \left( \frac{1}{\epsilon^2} \|\nabla u\|^2 + \epsilon \|u\|^2 \right).
\]

It is possible to presume \( \varepsilon^2 = c\beta \), where \( C \) is a constant. Thus we have obtained a prior estimation for the solution to the equation (1).

Let us assume the function \( u \in V_{1h}^2 \) is a solution to the equation (1) then for an arbitrary element \( v \in W_{1h}^2(R^2, dT) \) such that \( \text{var} \max |v(t, x)| < \infty \), \( t \in [0, T] \), we have an integral equality:
\[
\langle u(\tau), v(\tau) \rangle_h + \int_0^\tau \left[ -\langle u(\tau), \xi(\tau) \rangle + \lambda \langle u(\tau), v(\tau) \rangle \right] d\tau
\]
\[
+ \int_0^\tau \left( \frac{\partial}{\partial \xi_{ij}} \sum a_{ij} \frac{\partial}{\partial \xi_j} u \right) v \right) d\tau
\]
\[
+ \int_0^\tau \langle b(v) \rangle d\tau = \int_0^\tau \langle f(\xi) \rangle d\tau.
\]

We put \( v = u \) and obtain:
\[
\frac{1}{2} \|u(\tau)\|^2 + \lambda \int_0^\tau \|u(\tau)\|^2 d\tau
\]
\[
+ \int_0^\tau \left( \frac{\partial}{\partial \xi_{ij}} \sum a_{ij} \frac{\partial}{\partial \xi_j} u \right) v \right) d\tau
\]
\[
+ \int_0^\tau \langle b(u) \rangle d\tau = \int_0^\tau \langle f(\xi) \rangle d\tau.
\]

Thus we have obtained a prior estimation for the solution to the equation (1).
under the form-bounded of \( b \) and \( \max_{\Omega} \frac{\partial a_{ij}}{\partial x_j} < \infty \) conditions has a solution \( u \in W_{2,1}^2 \), then the solution belongs \( W_{2,1}^2 \).

The Existence of the Solution to the Parabolic Partial Differential Equation (1)

**Theorem 2**

The quasi-linear parabolic partial differential Equation (1) under the conditions (4), (5) has the solution from \( W_{2,1}^2 ([0,T] \times \mathbb{R}^n) \).

**Proof**

To prove the existence of the solution to (1) we construct the sequence of approximate solutions \( \{u_m(t,x)\}, m = 1, 2, \ldots \) to the equation:

\[
\frac{\partial u}{\partial t} + \lambda u - \sum_{i,j} a_{ij}(t,x,u) \frac{\partial^2 u}{\partial x_i \partial x_j} + b(t,x,u,\nabla u) = f.
\]

as \( \{u_m(t,x)\} = \left\{ \sum_{i=1}^m c_{ij}^m(t) \phi_i(x) \right\} \), where the elements \( \{ \phi_i(x) \} \) form the basis of \( W_{2,1}^2 (\mathbb{R}^n) \) with the properties \( \phi_i(x) = \delta_i \) and \( \max_i |\phi_i(x)| \leq c, \quad \lambda < \infty \). The functional coefficients \( c_{ij}^m(t) \) of \( \{u_m(t,x)\} = \left\{ \sum_{i=1}^m c_{ij}^m(t) \phi_i(x) \right\} \) are determined by:

\[
\begin{align*}
(c_{ij}^m(t), \phi_j) &+ \sum_{i,j=1}^m a_{ij}(t,x,u) (\phi_i, \phi_j) + (b, \phi_m) = (f, \phi_m), m = 1, 2, \ldots, m \end{align*}
\]

and initial conditions:

\[
c_{ij}^m(0) = (u_{ij}(x), n = 1, 2, \ldots, m.
\]

From the initial conditions for \( t \in [0, T] \) we are obtaining \( |c_{ij}^m| \leq \text{const}, m = 1, 2, \ldots, m \), from ellipticity follows uniformly boundedness of the solutions over \( t \in [0, T] \), to show this we multiply the Equation (1) by \( c_{ij}^m \) and a sum of \( n \) up to \( m \) then we obtain the inequality:

\[
\frac{1}{2} \|u_m(t)\|^2 + \int_0^t (\nabla u_m + a \nabla u_m)d\tau + \lambda \int_0^t \|u_m\|^2 d\tau \\
\leq \left( 1 + \frac{c(\beta)}{2\sqrt{\beta}} + c(\beta) \right) \|u_m\|^2 d\tau \\
+ \frac{\sqrt{\beta}}{2} \left\| \psi \right\|^2 d\tau + \frac{\sqrt{\beta}}{2} \left\| \mu \right\|^2 d\tau.
\]

We will apply the following lemma.

**Lemma 2**

Let \( \psi(t) \) be a positive absolute continuous function such that \( \psi(0) = 0 \) and for almost all \( t \in [0, T] \) holds the inequality:

\[
\frac{d}{dt} \psi(t) \leq c(t) \psi(t) + F(t)
\]

where the \( c(t) \) and \( F(t) \) are positive integrable on \( [0, T] \) functions. Then:

\[
\psi(t) \leq \exp \left( \int_0^t c(t) d\tau \right) \int_0^t F(t) d\tau,
\]

and:

\[
\frac{d}{dt} \psi(t) \leq c(t) \exp \left( \int_0^t c(t) d\tau \right) \int_0^t F(t) d\tau + F(t).
\]

Since \( u_m \in L^2(R^n) \) there is an estimation:

\[
\max_{t \in [0,T]} \left\| \sum_{i=1}^m c_{ij}^m(t) \phi_i(x) \right\| \leq \text{const}.
\]

Functions \( c_{ij}^m(t) = (u_{ij}(t,x), \phi_i(x)), m = 1, 2, \ldots \) are continuous on \( [0, T] \). On the interval \( [t, t + \Delta t] \), we can estimate:

\[
\left\| u_m(t + \Delta t) - u_m(t, x) \right\| \phi_m(x) \\
\leq \int_t^{t+\Delta t} \left( \sum_{ij} a_{ij} \frac{\partial u_m}{\partial x_i} \phi_j \right) d\tau \\
+ \int_t^{t+\Delta t} (f, \phi_m) d\tau + \lambda \int_t^{t+\Delta t} (u_m, \phi_m) d\tau \\
+ \int_t^{t+\Delta t} \left( \mu(t,x) |\nabla u_m| + \mu_2(t,x) |u_m|^2 \right) \phi_m d\tau \\
\leq c_m \int_t^{t+\Delta t} \left( \sum_{ij} a_{ij} \frac{\partial u_m}{\partial x_i} \phi_j \right) d\tau \\
- \lambda c_m \int_t^{t+\Delta t} |u_m| d\tau.
continuous functions with bounded weak derivatives, we consider the equality:

\[
\langle u_n(t), v(t) \rangle + \int \langle \nabla u_n, \partial_t v(t) \rangle + \lambda \langle u_n(t), v(t) \rangle dt + \int \langle \sum_{j=1}^{m} a_j \frac{\partial}{\partial x_j} u_n, \frac{\partial}{\partial x_j} v \rangle dt + \int \langle b, v \rangle dt = \int \langle f, v \rangle dt.
\]

The \( \varphi_n \) is the set of functions \( u_n \) and \( \varphi = \bigcup_m \varphi_n \), the set \( \varphi \) is dense in \( W_2^1 \). Passing to the limit as \( m \to \infty \) we obtain:

\[
\int \langle (-u(t), \partial_t v(t)) + \lambda \langle u(t), v(t) \rangle \rangle dt = \int \langle f, v \rangle dt
\]

for any function \( v \in \varphi \).

Let us assume \( v = u_n - \varphi \) then we have:

\[
\int \langle u_n(t), u_n - \varphi \rangle + \int \langle -u_n(t), \partial_t (u_n - \varphi) \rangle + \lambda \int \langle u_n(t), (u_n - \varphi) \rangle dt + \int \langle \sum_{j=1}^{m} a_j \frac{\partial}{\partial x_j} u_n, \frac{\partial}{\partial x_j} (u_n - \varphi) \rangle dt + \int \langle b, (u_n - \varphi) \rangle dt = \int \langle f, (u_n - \varphi) \rangle dt
\]

so:

\[
\int \langle \sum_{j=1}^{m} a_j \frac{\partial}{\partial x_j} u_n, \frac{\partial}{\partial x_j} (u_n - \varphi) \rangle dt = \int \langle f, (u_n - \varphi) \rangle dt
\]

and:

\[
c(t+\Delta t) - c(t) \leq \varepsilon(\Delta t)\|\varphi\|_{W_2^1} \rightarrow 0.
\]
\[
\begin{align*}
\int_0^1 & \left( -\{u_n(\tau), \partial_t (u_n - \phi)(\tau) + \lambda \{u_n(\tau), (u_n - \phi)(\tau) \} \right) d\tau \\
+ & \int_0^1 \left( \sum_{i,j,l} a_{i,j} \frac{\partial}{\partial x_i} u_n \frac{\partial}{\partial x_j} (u_n - \phi) \right) d\tau \\
+ & \int_0^1 \left( h(u_n(\phi)) d\tau - \int_0^1 f(u_n(\phi)) d\tau \right) \\
- & \frac{1}{2} \|u_n\|_{W^{1,2}_0}^2 + \|u_n, \phi\|_{W^{1,2}_0} + \text{function} \left( \|u_n - \phi\| \right) \geq 0,
\end{align*}
\]

we fix the function \( \phi \) and pass to the limit as \( m \to \infty \) obtain:
\[
\begin{align*}
\int_0^1 & \left( -\{u(\tau), \partial_t (u_n - \phi)(\tau) + \lambda \{u(\tau), (u_n - \phi)(\tau) \} \right) d\tau \\
+ & \int_0^1 \left( \sum_{i,j,l} a_{i,j} \frac{\partial}{\partial x_i} u_n \frac{\partial}{\partial x_j} (u_n - \phi) \right) d\tau \\
+ & \int_0^1 \left( h(u_n(\phi)) d\tau - \int_0^1 f(u_n(\phi)) d\tau \right) \\
- & \frac{1}{2} \|u_n\|_{W^{1,2}_0}^2 + \|u_n, \phi\|_{W^{1,2}_0} + \text{function} \left( \|u_n - \phi\| \right) \geq 0.
\end{align*}
\]

In the last inequality, we put \( v = u \) and have:
\[
\begin{align*}
\frac{1}{2} \|u\|_{W^{1,2}_0}^2 + & \int_0^1 \{u(\tau), (u_n - \phi)(\tau) \} d\tau \\
+ & \int_0^1 \left( \sum_{i,j,l} a_{i,j} \frac{\partial}{\partial x_i} u_n \frac{\partial}{\partial x_j} u_n \right) d\tau + \int_0^1 h(u_n) d\tau = \int_0^1 \{f, u\} d\tau.
\end{align*}
\]

Since \( v \in \varphi_n \) for arbitrary \( m \) therefore for arbitrary function \( v \in \varphi = \bigcup_1^\infty \varphi_n \), we have:
\[
\begin{align*}
\int_0^1 & \left( -\{u(\tau), \partial_t (u_n - v)(\tau) + \lambda \{u(\tau), (u_n - v)(\tau) \} \right) d\tau \\
+ & \int_0^1 \left( \sum_{i,j,l} a_{i,j} \frac{\partial}{\partial x_i} u_n \frac{\partial}{\partial x_j} (u_n - v) \right) d\tau \\
+ & \int_0^1 \left( h(u_n(\phi)) d\tau - \int_0^1 f(u_n(\phi)) d\tau \right) \\
- & \frac{1}{2} \|u_n\|_{W^{1,2}_0}^2 + \|u_n, \phi\|_{W^{1,2}_0} + \text{function} \left( \|u_n - v\| \right) \geq 0.
\end{align*}
\]

Since the set \( \varphi \) is dense in \( W^{1,2}_0 \) therefore for any \( \varepsilon > 0 \) and any function \( \phi \in \varphi \), we can put \( v = u - \varepsilon \phi \) and estimate:
\[
\begin{align*}
\varepsilon \int_0^1 & \left( -\{u(\tau), \partial_t (u_n - \phi)(\tau) + \lambda \{u(\tau), (u_n - \phi)(\tau) \} \right) d\tau \\
+ & \varepsilon \int_0^1 \left( \sum_{i,j,l} a_{i,j} \frac{\partial}{\partial x_i} u_n \frac{\partial}{\partial x_j} (u_n - \phi) \right) d\tau + \varepsilon \int_0^1 \{h, \phi\} d\tau \\
- & \varepsilon \int_0^1 \left( f, \phi \right) d\tau + \text{function} \left( \varepsilon \|\phi\| \right) \geq 0.
\end{align*}
\]

We pass to the limit as \( \varepsilon \to 0 \) have:
\[
\begin{align*}
\int_0^1 & \left( -\{u(\tau), \partial_t (u_n - \phi)(\tau) + \lambda \{u(\tau), (u_n - \phi)(\tau) \} \right) d\tau \\
+ & \int_0^1 \left( \sum_{i,j,l} a_{i,j} \frac{\partial}{\partial x_i} u_n \frac{\partial}{\partial x_j} (u_n - \phi) \right) d\tau \\
+ & \int_0^1 \left( h(\phi) d\tau - \int_0^1 \{f, \phi\} d\tau \right) \\
- & \frac{1}{2} \|u_n\|_{W^{1,2}_0}^2 + \|u_n, \phi\|_{W^{1,2}_0} + \text{function} \left( \|u_n - \phi\| \right) \geq 0.
\end{align*}
\]

Since the set \( \varphi \) is dense in \( W^{1,2}_0 \), from the last inequality, the estimation:
\[
\begin{align*}
\int_0^1 & \left( -\{u(\tau), \partial_t (u_n - \phi)(\tau) + \lambda \{u(\tau), (u_n - \phi)(\tau) \} \right) d\tau \\
+ & \int_0^1 \left( \sum_{i,j,l} a_{i,j} \frac{\partial}{\partial x_i} u_n \frac{\partial}{\partial x_j} (u_n - \phi) \right) d\tau \\
+ & \int_0^1 \left( h(\phi) d\tau - \int_0^1 \{f, \phi\} d\tau \right) = 0.
\end{align*}
\]

is true for arbitrary \( \phi \in W^{1,2}_0 \), which means that function \( u \in W^{1,2}_0 \) is a solution to (1).

**Remark**

The monotonousness can be proven as:
\[
\begin{align*}
\int_0^1 & \left( -\{u_n(\tau), \partial_t (u_n - v)(\tau), \partial_t (u_n - v)(\tau) \} \right) d\tau \\
+ & \lambda \int_0^1 \left( \sum_{i,j,l} a_{i,j} \frac{\partial}{\partial x_i} u_n \frac{\partial}{\partial x_j} (u_n - v) \right) d\tau \\
+ & \int_0^1 \left( h(u_n) - h(v) \right) d\tau \\
- & \int_0^1 \left( f, v \right) d\tau + \text{function} \left( \|u_n - v\| \right) \geq 0.
\end{align*}
\]

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are satisfied, these equations mean that the coefficient of (15) converge to the coefficients (1) and additional condition:

$$b^*(-\tau, u, \nabla u) - b^*(\tau, u', \nabla u') \leq \mu_4(\tau)|u - u'|^2 + \mu_5(\tau)|u - u'|$$

is executed.

Then the sequence of the weak solution $u^* \in V_{1, b}^2$, $z = 1, 2, \ldots, \tau$ to the Cauchy problems for the equations (15) under the initial conditions $u^*(0, x) = \phi_b^*$ converges to the weak solution to the Cauchy problem for the equation (1) under the initial condition $u(0, x) = \phi_b$ in $V_{1, b}^2$.

**Proof**

The proving will be accomplished according to the schema:

- compose the integral identity for the solution $u(t, x)$ to the Cauchy problem for the equation (1) under the initial condition $u(0, x) = u_0$ and for the sequence of the weak solutions $u^* \in V_{1, b}^2$, $z = 1, 2, \ldots, \tau$ to the Cauchy problems for the equations (15) under the initial conditions $u^*(0, x) = \phi_b^*$
- subtract integral identity for the solution $u^* = u-u^*$
- obtain the priory estimations for the differences $v^* = u-u^*$
- apply the priory estimations to substantiate the passing to the limit $\lim_{z \to \infty} v^* = 0$ in $V_{1, b}^2$ topology.

Let us compose the integral identity for the (1):

$$\langle u(t), \eta(t) \rangle + \int_0^t \left(-\langle u(t, \tau), \epsilon \eta(\tau) \rangle + \lambda \langle u(t, \tau), \eta(\tau) \rangle \right) d\tau$$

$$+ \int_0^t \left( - \sum_{i,j} a_{ij} \frac{\partial u}{\partial x_j} \frac{\partial \eta}{\partial x_i} \right) d\tau + \int_0^t \left( b, \eta \right) d\tau = \int_0^t \left( f, \eta \right) d\tau$$

for an arbitrary $\eta \in W_{1, b}^2$ and the integral identities for the Equations (15):

$$\langle u^*(t), \eta(t) \rangle + \int_0^t \left(-\langle u^*(t, \tau), \epsilon \eta(\tau) \rangle + \lambda \langle u^*(t, \tau), \eta(\tau) \rangle \right) d\tau$$

$$+ \int_0^t \left( - \sum_{i,j} a_{ij} \frac{\partial u^*}{\partial x_j} \frac{\partial \eta}{\partial x_i} \right) d\tau + \int_0^t \left( b^*(t, \tau, u^*, \nabla u^*), \eta \right) d\tau = \int_0^t \left( f^*(t, \tau), \eta \right) d\tau$$
for an arbitrary \( \eta \in W_{1+}^2 \), after the subtraction, we are obtaining the equation:

\[
\begin{align*}
\langle v'(\tau), \eta(\tau) \rangle & \\
+ \int_0^\tau \left[ -\langle v'(\tau), \partial_t \eta(\tau) \rangle + \lambda \langle v'(\tau), \eta(\tau) \rangle \right] d\tau \\
+ \int_0^\tau \left[ \sum_{j=1}^n a_{ij}(\tau, x) \frac{\partial v_j}{\partial x_j} u_i \frac{\partial \eta}{\partial x_i} \right] d\tau \\
+ \int_0^\tau \left[ \sum_{j=1}^n a_j(\tau, x) \frac{\partial v_j}{\partial x_j} \right] d\tau \\
+ \int_0^\tau \left[ b(\tau, x, \nabla u) - \frac{\partial b}{\partial x} \right] d\tau \\
= \int_0^\tau \left[ f(\tau) - f(\tau, \eta) \right] d\tau.
\end{align*}
\]

Let us estimate the term

\[
\lim_{\tau \to \infty} \left| \sum_{j=1}^n a_j(\tau, x) \frac{\partial v_j}{\partial x_j} \right| d\tau,
\]

since

\[
\lim_{\tau \to \infty} \left| \sum_{j=1}^n a_j(\tau, x) \right|^2 d\tau = 0,
\]

therefore:

\[
\lim_{\tau \to \infty} \int_0^\tau \left| \sum_{j=1}^n a_j(\tau, x) \frac{\partial v_j}{\partial x_j} \right| d\tau = 0.
\]

applying the notation \( v' = u' \cdot u' \) and fact \( v' \in W_{1+}^2 \), we have had:

\[
\begin{align*}
\left| \sum_{j=1}^n a_j(\tau, x) \frac{\partial v_j}{\partial x_j} \right| & \\
& \leq \sum_{j=1}^n a_j(\tau, x) \frac{\partial v_j}{\partial x_j} \left| \frac{\partial v_j}{\partial x_j} \right|
\end{align*}
\]

From the conditions we have:

\[
\lim_{\tau \to \infty} \int_0^\tau \left[ f(\tau) - f(\tau, \eta) \right] d\tau = 0.
\]

since:

\[
\lim_{\tau \to \infty} \int_0^\tau \left[ b(\tau, x, \nabla u) - \frac{\partial b}{\partial x} \right] d\tau = 0,
\]

and \( \eta = v' \), we obtain:

\[
\langle b(\tau, x, \nabla u) - \frac{\partial b}{\partial x}, v' \rangle
\]

\[
\leq \left\| v' \right\| \left\| \nabla v' \right\| + \left\| \frac{\partial b}{\partial x} \right\| \left\| v' \right\|
\]

\[
\leq \frac{1}{2} \left\| \frac{\partial b}{\partial x} \right\|^2 + \sigma^2 \left\| v' \right\|^2 + c(\beta) \left\| v' \right\|^2
\]

so:

\[
\left\| \mu v' \right\|^2 = \langle v' \rangle \left\| v' \right\|^2 \leq \beta \left\| \nabla v' \right\|^2 + c(\beta) \left\| v' \right\|^2
\]

similarly, the term containing \( \mu \) can be estimated. After reducing similar terms, we obtain the statement of the theorem.

**Ethics**

This article is original and contains unpublished material. The corresponding author confirms that all of the other authors have read and approved the manuscript and no ethical issues involved.

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