Ternary algebras with braided statistics

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Abstract
Algebraic relations that characterize quantum statistics (Bose-Einstein statistic, Fermi-Dirac statistic, supersymmetry, parastatistic, anyonic statistic,...) are reformulated herein in terms of a new algebraic structure, which we call para-algebra.

1 Introduction
There have been suggestions for quantization procedures that lead to particle statistics different from the well known Bose-Einstein and Fermi-Dirac Statistics. Elaborate mathematical developments are needed to justify such quantization procedures. An interesting example is parastatistics which was introduced by H.S Green [1] in 1953. It is well known that the idea of supersymmetry [2] concerns bosons and fermions and the mathematics that describe this field is Lie superalgebras [3], or equivalently $\mathbb{Z}_2$-graded algebras, which is based on binary relations. In an earlier paper [5], we have shown that there are general mathematical structures that represent different statistics, and we have established the connection between them. In practice, braided groups are simply an analog of supergroups with ±1 Bose-Fermi statistics replaced by braid statistics. There are indications that particles of braid statistics arise in low dimension quantum field theory.
On the other hand the parasupersymmetric [4] quantum mechanics, is related to para-bosons and parafermions and the mathematical theory need to study this field is based on ternary operations which we will call para-algebra.
In this paper, we attempt to construct the theory of para-algebras, this terminology comes from physics in the context of parastatistics. Parasuperalgebras or equivalently $\mathbb{Z}_2$-graded para-algebras are para-algebras where the bilinear map $\sigma$ defined in paper [5] takes its values in $\mathbb{Z}_2$; from this formalism we derive the algebraic structures which lead to quantum
statistics, specially to generalized parastatistics \[6\], which we will call Lie parasuperalgebra. In section 2 we give the basic definitions concerning braided tensor algebra \[7, 8, 9\]. In section 3 we illustrate our method, by defining maps on a braided tensor algebra, then we construct ideals and enveloping algebras to define ternary maps which give the para-algebras. In section 4, we derive the formalism of the generalized parastatistics \[6\]. In section 5, we show that these ternary maps are related to Schur functor or Weyl module.

2 Preliminaries

2.1 definitions

Let \( \Gamma \) be an \( n \) dimensional vector space over a field \( K \) of characteristic 0, and let \((A, \otimes, I)\) be a tensorial category \[7, 8, 9\]; the objects of \( A \) are denoted by \( A_{v_i}, i = 0, 1, \ldots; A_{v_0} = I; v_i \in \Gamma \) and \( I \) is the unit element for the operation \( \otimes \), for simplicity \( A_{v_i} \otimes I = I \otimes A_{v_i} = A_{v_i} \) (this notation will be clear later; subsequently the \( A_{v_i} \) will denote vector spaces). \((A, \otimes, I)\) is a braided tensorial category if there exist natural isomorphisms called braiding defined between any two objects of \( A \) such that

\[
\Psi_{A_{v_i}, A_{v_j}} : A_{v_i} \otimes A_{v_j} \to A_{v_j} \otimes A_{v_i}
\]  

Let \( \{e_i\} \) and \( \{f_j\} \) be, respectively, the bases of the vector spaces \( A_{v_i} \) and \( A_{v_j} \). Then :

\[
\Psi_{A_{v_i}, A_{v_j}}(e_i \otimes f_j) = \Psi_{ij}^{mn} f_m \otimes e_n
\]  

(The summation on the repeated indices is to be understood). Therefore

\[
\Psi_{A_{v_i}, I} = \Psi_{I, A_{v_i}} = id_{A_{v_i}}
\]  

\[
\Psi_{A_{v_i}, A_{v_j} \otimes A_{v_k}} = \left( id_{A_{v_j}} \otimes \Psi_{A_{v_i}, A_{v_k}} \right) \circ \left( \Psi_{A_{v_i}, A_{v_j}} \otimes id_{A_{v_k}} \right)
\]  

\[
\Psi_{A_{v_i}, A_{v_j} \otimes A_{v_k}} = \left( \Psi_{A_{v_i}, A_{v_k}} \otimes id_{A_{v_j}} \right) \circ \left( id_{A_{v_i}} \otimes \Psi_{A_{v_j}, A_{v_k}} \right)
\]  

\[
\Psi_{ij}^{mn} \Psi_{rq}^{mn} = \delta_i^j \delta_r^q \text{ unitarity condition}
\]  

The property (4) is equivalent to the following triangular diagram :
The property (5) is equivalent to the following triangular diagram:

\[ A_{v_i} \otimes A_{v_j} \otimes A_{v_k} \xrightarrow{\Psi_{A_{v_i} \otimes A_{v_j} \otimes A_{v_k}}} A_{v_k} \otimes A_{v_i} \otimes A_{v_j} \]

In a braided tensorial category, we have the following identity:

\[
\begin{align*}
& \left( \Psi_{A_{v_j}, A_{v_k}} \otimes id_{A_{v_i}} \right) \left( id_{A_{v_j}} \otimes \Psi_{A_{v_i}, A_{v_k}} \right) \left( \Psi_{A_{v_i}, A_{v_j} \otimes A_{v_k}} \right) \\
= & \left( id_{A_{v_k}} \otimes \Psi_{A_{v_i}, A_{v_j}} \right) \left( \Psi_{A_{v_i}, A_{v_k} \otimes A_{v_j}} \right) \left( id_{A_{v_i}} \otimes \Psi_{A_{v_i}, A_{v_k}} \right)
\end{align*}
\]

which is the generalized Yang-Baxter identity:

\[ \text{(7)} \]

3 Ternary maps

3.1 Definitions

Let \( A \) be a \( \Gamma \)-graded vector space \( A = \oplus A_{v_i}; v_i \in \Gamma, i = 1 \ldots n \), \( A_{v_i} \) is a vector subspace of \( A \). Let \( T(A) = \oplus_{P \geq 0} A^\otimes P \) be the tensor algebra constructed from the vector space \( A \);
since $A$ is braided, $T(A)$ is also braided. From the properties a) and b) in section 2 one can see that there are only two ways for braiding a 3-fold tensor product. It is useful to write some notations; this consists of writing all isomorphisms pointing downwards with $\Psi = \begin{array}{c|c|c}
 \end{array} \begin{array}{c|c|c}
 \end{array}$, $\Psi^{-1} = \begin{array}{c|c|c}
 \end{array} \begin{array}{c|c|c}
 \end{array}$ as braids.

We need the following maps:

### 3.2 Left ternary maps

By braiding from the left we construct the following maps

\[ id = id_{A_{v_i}} \otimes id_{A_{v_j}} \otimes id_{A_{v_k}} \]

\[ \Psi_{A_{v_i} A_{v_j}} \otimes id_{A_{v_k}} \]

\[
\left( \Psi_{A_{v_i} A_{v_j}} \otimes id_{A_{v_k}} \right) \left( id_{A_{v_j}} \otimes \Psi_{A_{v_i} A_{v_k}} \right)
\times \left( \Psi_{A_{v_j} A_{v_k}} \otimes id_{A_{v_i}} \right)
\]
3.3 Right ternary maps

By braiding from the right we construct the following maps

\[
\begin{align*}
\text{id}_{A_{v_i}} & \otimes \Psi_{A_{v_j},A_{v_k}} \quad \left( \Psi_{A_{v_j},A_{v_k}} \otimes \text{id}_{A_{v_i}} \right) \\
\text{id}_{A_{v_j}} & \otimes \Psi_{A_{v_i},A_{v_k}} \quad \left( \Psi_{A_{v_i},A_{v_j}} \otimes \text{id}_{A_{v_k}} \right) \\
\end{align*}
\times
\]

\[
\begin{align*}
\text{id}_{A_{v_i}} & \otimes \Psi_{A_{v_j},A_{v_k}} \\
\end{align*}
\]
Let \( \langle \ , \ , \rangle \) be a trilinear map defined on \( A \times A \times A \)

\[
\langle \ , \ , \rangle : A \times A \times A \rightarrow A
\]

(8)

Let \( I_1 \) be the two-sided ideal generated under the map

\[
id + \Psi_{A_{vi}, A_{vj}} \otimes id_{A_{vk}} - \left( id_{A_{vk}} \otimes \Psi_{A_{vj}, A_{vk}} \right) \left( \Psi_{A_{vj}, A_{vk}} \otimes id_{A_{vi}} \right) \left( id_{A_{vj}} \otimes \Psi_{A_{vi}, A_{vk}} \right) \left( \Psi_{A_{vi}, A_{vj}} \otimes id_{A_{vk}} \right)
\]
3.5 Definition of a para-algebra and Lie para-algebra:

Similarly for mapping of one mapping, we may identify of elements \( \Psi_{A,v_i, A,v_k} \),

\[
- \left( \Psi_{A,v_i, A,v_k} \otimes \text{id}_{A,v_i} \right) \left( \text{id}_{A,v_j} \otimes \Psi_{A,v_i, A,v_k} \right) \left( \Psi_{A,v_i, A,v_j} \otimes \text{id}_{A,v_k} \right) - \langle , , \rangle
\]  

(9)
on \( A,v_i \otimes A,v_j \otimes A,v_k \)

Let \( I_2 \) be the two-sided ideal generated under the map

\[
id + \text{id}_{A,v_i} \otimes \Psi_{A,v_i, A,v_k} - \left( \Psi_{A,v_i, A,v_j} \otimes \text{id}_{A,v_k} \right) \left( \Psi_{A,v_i, A,v_k} \otimes \text{id}_{A,v_j} \right) - \langle , , \rangle
\]  

(10)on \( A,v_i \otimes A,v_j \otimes A,v_k \)

where:

\( \text{id}_{A,v} \) is the identity on the vector space \( A,v \), that is

\[
id = \text{id}_{A,v_i} \otimes \text{id}_{A,v_j} \otimes \text{id}_{A,v_k}
\]  

(11)

3.4 Enveloping algebras

We define two enveloping algebras \( U_1(A) = T(A)/I_1 \) and \( U_2(A) = T(A)/I_2 \). We denote by \( \langle , , \rangle_1 \) the trilinear map for \( U_1(A) \), and by \( \langle , , \rangle_2 \) the trilinear map for \( U_2(A) \). Left \( \{e_i\} \), \( \{f_j\} \) and \( \{g_k\} \) be respectively the bases of the vector spaces \( A,v_i, A,v_j \) and \( A,v_k \), \( \Psi = \left( \Psi_{ij}^{kl} \right) \) the matrix of \( \Psi_{A,v_i, A,v_j} \) and denote the product in \( U_1(A) = T(A)/I_1 \) and \( U_2(A) = T(A)/I_2 \) of elements \( x \) and \( y \) in \( A \) by \( xy \), we have the following relations:

\[
U_1(A) = T(A)/I_1 \text{ is an associative algebra; since the composition of the canonical mapping of } T(A) \text{ onto } U_1(A) \text{ with the inclusion mapping of } A \text{ into } T(A) \text{ yields a one-to-one mapping, we may identify } A \text{ with its image in } U_1(A)
\]

The trilinear map \( \langle , , \rangle_1 \) reads:

\[
\langle e_i, f_j, g_k \rangle_1 = e_i \otimes f_j \otimes g_k + \Psi_{ij}^{mn} f_m \otimes e_n \otimes g_k \\
+ \Psi_{ij}^{mn} \Psi_{nk}^{pq} \Psi_{mp}^{rs} e_t \otimes f_u \otimes g_r \otimes f_s \otimes e_q
\]  

(12)

Similarly for \( U_2(A) \); we identity \( A \) with its image in \( U_2(A) \). The trilinear map \( \langle , , \rangle_2 \) is

\[
\langle e_i, f_j, g_k \rangle_2 = e_i \otimes f_j \otimes g_k + \Psi_{ij}^{mn} f_m \otimes e_n \otimes g_k \\
+ \Psi_{ij}^{mn} \Psi_{im}^{pq} \Psi_{qm}^{rs} e_t \otimes f_u \otimes g_r \otimes f_s \otimes e_q
\]  

(13)

3.5 Definition of a para-algebra and Lie para-algebra:

A para-algebra is a \( \Gamma \)-graded vector space \( A = \oplus_{i} A,v_i \); \( v_i \in \Gamma \) (that is, if \( a \in A,v_i, b \in A,v_j \), \( v_i, v_j \in \Gamma \), then \( ab \in A,v_{i+j} \)); the braiding is taken to be \( \Psi_{A,v_i, A,v_j} = (-1)^{\sigma(v_i,v_j)} \), where
The relations (22)-(24) are the anticommutativity, and the Jacobi identity for the following relations:

\[ \langle a, b, c \rangle_1 = a \otimes b \otimes c + (-1)^{\sigma(v_j, v_k)} b \otimes a \otimes c \]
\[ + (-1)^{\sigma(v_i, v_k) + \sigma(v_j, v_k)} c \otimes a \otimes b + (-1)^{\sigma(v_i, v_j) + \sigma(v_j, v_k) + \sigma(v_i, v_k)} c \otimes b \otimes a \]

and

\[ \langle a, b, c \rangle_2 = a \otimes b \otimes c + (-1)^{\sigma(v_j, v_k)} a \otimes c \otimes b \]
\[ + (-1)^{\sigma(v_i, v_j) + \sigma(v_i, v_k)} b \otimes c \otimes a + (-1)^{\sigma(v_j, v_i) + \sigma(v_i, v_k) + \sigma(v_j, v_k)} c \otimes b \otimes a \]

A natural way of defining brackets \( \langle \ , \ , \ \rangle_1 \) or \( \langle \ , \ , \ \rangle_2 \) in a para-algebra \( A \) is through the following equalities,

\[ \langle a, b, c \rangle_1 = (-1)^{\sigma(v_j, v_k)} \langle b, a, c \rangle_1 a \in A_{v_i}, b \in A_{v_j}, c \in A_{v_k} \]  \hspace{1cm} (16)
\[ \langle a, b, c \rangle_2 = (-1)^{\sigma(v_j, v_k)} \langle a, c, b \rangle_2 a \in A_{v_i}, b \in A_{v_j}, c \in A_{v_k} \]  \hspace{1cm} (17)

When we take \( \langle \ , \ , \ \rangle_1 \), we have the left para-algebra, and when we take \( \langle \ , \ , \ \rangle_2 \), we have the right para-algebra.

For an associative para-algebra \( A \) the following identities hold:

\[ \langle a, b, cd \rangle_1 = \langle a, b, c \rangle_1 d + (-1)^{\sigma(v_i + v_j, v_k)} c \langle a, b, d \rangle_1 \]
\[ \langle a, b, cde \rangle_1 = \langle a, b, c \rangle_1 de + (-1)^{\sigma(v_i + v_j, v_k)} c \langle a, b, d \rangle_1 e \\
+ (-1)^{\sigma(v_i + v_j, v_k + v_l)} cd \langle a, b, e \rangle_1 \]
\[ \langle ab, c, d \rangle_2 = a \langle b, c, d \rangle_2 + (-1)^{\sigma(v_k + v_i, v_j)} \langle a, c, d \rangle_2 b \]
\[ \langle abc, d, e \rangle_2 = ab \langle c, d, e \rangle_2 + (-1)^{\sigma(v_k + v_i + v_m)} a \langle b, d, e \rangle_2 c \\
+ (-1)^{\sigma(v_j + v_k + v_m)} \langle a, d, e \rangle_2 bc \]

where \( a \in A_{v_i}, b \in A_{v_j}, c \in A_{v_k}, d \in A_{v_l}, e \in A_{v_m} \)

A Lie para-algebra is a para-algebra with a trilinear operation \( \langle \ , \ , \ \rangle_1 \) or \( \langle \ , \ , \ \rangle_2 \) satisfying the following relations:

\[ \langle a, b, c \rangle_1 = (-1)^{\sigma(v_i, v_j)} \langle b, a, c \rangle_1 a \in A_{v_i}, b \in A_{v_j}, c \in A_{v_k} \]  \hspace{1cm} (22)
\[ (-1)^{\sigma(v_i, v_k)} \langle a, b, c \rangle_1 + (-1)^{\sigma(v_j, v_k)} \langle b, c, a \rangle_1 + (-1)^{\sigma(v_i, v_k)} \langle c, a, b \rangle_1 = 0 \]  \hspace{1cm} (23)
\[ \langle a, b, \langle c, d, e \rangle_1 \rangle_1 = \langle \langle a, b, c \rangle_1, d, e \rangle_1 + (-1)^{\sigma(v_i + v_j, v_k)} \langle c, \langle a, b, d \rangle_1, e \rangle_1 \\
+ (-1)^{\sigma(v_i + v_j, v_k) + \sigma(v_i + v_j, v_l)} \langle e, d, \langle a, b, e \rangle_1 \rangle_1 \]  \hspace{1cm} (24)

where \( a \in A_{v_i}, b \in A_{v_j}, c \in A_{v_k}, d \in A_{v_l}, e \in A_{v_m} \)

The relations (22)-(24) are the anticommutativity, and the Jacobi identity for \( \langle \ , \ , \ \rangle_1 \)
respectively, while \((23)\) is a cyclic relation.

The relation \((22) - (24)\) define the left Lie para-algebra.

Similarly for \((\langle \ , \ , \ \rangle)_2\), we have

\[
\langle a, b, c \rangle_2 = (-1)^{\sigma(v_i,v_k)} \langle a, c, b \rangle_2, a \in A_{v_i}, b \in A_{v_j}, c \in A_{v_k}
\]

\[
(-1)^{\sigma(v_i,v_k)} \langle a, b, c \rangle_2 + (-1)^{\sigma(v_i,v_j)} \langle b, c, a \rangle_2 + (-1)^{\sigma(v_j,v_k)} \langle c, a, b \rangle_2 = 0
\]

\[
\langle (a, b, c)_2, d, e \rangle_2 = \langle a, b, \langle c, d, e \rangle_2 \rangle_2 + (-1)^{\sigma(v_k,v_i,v_m)} \langle a, (b, d, e) \rangle_2 \langle b, c \rangle_2 + (-1)^{\sigma(v_j+v_k,v_i+v_m)} \langle \langle a, d, e \rangle_2, b, c \rangle_2
\]

where \(a \in A_{v_i}, b \in A_{v_j}, c \in A_{v_k}, d \in A_{v_l}, e \in A_{v_m}\)

The relation \((25) - (27)\) define a right Lie para-algebra. The relations \((25)\), \((27)\) are respectively the anticommutativity, and the Jacobi identity for \((\langle \ , \ , \ \rangle)_2\), while \((26)\) is a cyclic relation. The Jacobi identities \((24)\) and \((27)\) follow respectively from \((18)\), \((19)\) and \((20)\), \((21)\) respectively.

We can define the trilinear maps \((\langle \ , \ , \ \rangle)_1\) and \((\langle \ , \ , \ \rangle)_2\) by introducing respectively the left multiplication operator and the right multiplication operator which we denote by \(L_{a,b}\) and \(R_{g,z}\) such that:

\[
L_{a,b} : A \rightarrow A
\]

\[
L_{a,b} : c \mapsto L_{a,b} c = \langle a, b, c \rangle_1, a, b \text{ and } c \in A
\]

and,

\[
R_{g,z} : A \rightarrow A
\]

\[
R_{g,z} : x \mapsto R_{g,z} (x) = xR_{g,z} = \langle x, y, z \rangle_2, x, y \text{ and } z \in A
\]

The operator \(L_{a,b}\) behave just like first order differential operator in that it obeys a product rule

\[
L_{a,b} \langle c, d, e \rangle_1 = \langle L_{a,b} c, d, e \rangle_1 + (-1)^{\sigma(v_i+v_j,v_k)} \langle c, L_{a,b} d, e \rangle_1 + (-1)^{\sigma(v_i+v_j+v_k)} \langle c, d, L_{a,b} e \rangle_1
\]

\[(28)\]

\(a \in A_{v_i}, b \in A_{v_j}, c \in A_{v_k}, d \in A_{v_l}, e \in A_{v_m}\)

This product rule is simply another way of writing the Jacobi identity \((24)\) for \((\langle \ , \ , \ \rangle)_1\).

The operator \(R_{g,z}\) has the property

\[
\langle a, b, c \rangle_2 R_{d,e} = \langle a, b, cR_{d,e} \rangle_2 + (-1)^{\sigma(v_k,v_l+v_m)} \langle a, bR_{d,e}, c \rangle_2 + (-1)^{\sigma(v_j+v_k,v_l+v_m)} \langle aR_{d,e}, b, c \rangle_2
\]

\[(31)\]

for \(a \in A_{v_i}, b \in A_{v_j}, c \in A_{v_k}, d \in A_{v_l}, e \in A_{v_m}\). This is also another way of writing the Jacobi identity \((27)\) for \((\langle \ , \ , \ \rangle)_2\).
4 Application to the formalism of parastatistics.

Since $A$ is a vector space, we define a quadratic form $Q$ on $A$, which takes its values in the field $K$, such that:

$$Q(a, b) = (-1)^{\sigma(v_i, v_j)} Q(b, a), a \in A_{v_i}, b \in A_{v_j}. \quad (32)$$

Let $J_1$ be the two sided ideal in $T(A)$ generated by the elements:

$$a \otimes b \otimes c + (-1)^{\sigma(v_i, v_j)} b \otimes a \otimes c - (-1)^{\sigma(v_i, v_k)+\sigma(v_j, v_k)} c \otimes a \otimes b$$

$$- (-1)^{\sigma(v_i, v_j)+\sigma(v_i, v_k)+\sigma(v_j, v_k)} c \otimes b \otimes a$$

$$- \left\{ aQ(b, c) + (-1)^{\sigma(v_i, v_j)} bQ(a, c) - (-1)^{\sigma(v_i, v_k)+\sigma(v_j, v_k)} Q(c, a)b \right\}.$$

(33)

where $a \in A_{v_i}, b \in A_{v_j}, c \in A_{v_k}$. Let $J_2$ be the two sided ideal in $T(A)$ generated by the elements:

$$a \otimes b \otimes c + (-1)^{\sigma(v_j, v_k)} a \otimes c \otimes b - (-1)^{\sigma(v_i, v_j)+\sigma(v_i, v_k)} b \otimes c \otimes a$$

$$- (-1)^{\sigma(v_i, v_k)+\sigma(v_i, v_j)+\sigma(v_j, v_k)} c \otimes b \otimes a$$

$$- \left\{ Q(a, b)c + (-1)^{\sigma(v_i, v_j)} Q(a, c)b - (-1)^{\sigma(v_i, v_k)+\sigma(v_j, v_k)} bQ(c, a) \right\}.$$

(34)

where $a \in A_{v_i}, b \in A_{v_j}, c \in A_{v_k}$.

We define the envelopping algebra $V_1(A) = T(A)/J_1$ and $V_2(A) = T(A)/J_2$; $V_1(A)$ and $V_2(A)$ are associative algebras, we denote the product of two elements $x$ and $y$ in $V_1(A)$ or in $V_2(A)$ by $xy$.

Since $A$ is embeded in $V_1(A)$ then in $A$ we have

$$abc + (-1)^{\sigma(v_i, v_j)}bac - (-1)^{\sigma(v_i, v_k)+\sigma(v_j, v_k)} cab = (-1)^{\sigma(v_i, v_j)+\sigma(v_i, v_k)+\sigma(v_j, v_k)} cba =$$

$$aQ(b, c) + (-1)^{\sigma(v_i, v_j)} bQ(a, c) - (-1)^{\sigma(v_i, v_k)+\sigma(v_j, v_k)} Q(c, a)b$$

$$- (-1)^{\sigma(v_i, v_j)+\sigma(v_i, v_k)+\sigma(v_j, v_k)} Q(c, b)a.$$

(35)

Similarly in $A$ we have

$$abc + (-1)^{\sigma(v_i, v_j)}bac - (-1)^{\sigma(v_i, v_k)+\sigma(v_j, v_k)} cab =$$

$$Q(a, b)c + (-1)^{\sigma(v_i, v_j)} Q(a, c)b - (-1)^{\sigma(v_i, v_j)+\sigma(v_j, v_k)} bQ(c, a)b$$

$$- (-1)^{\sigma(v_i, v_k)+\sigma(v_i, v_j)+\sigma(v_j, v_k)} cQ(b, a).$$

(36)

If we compare (14) and (35), then the trilinear map \langle a, b, c \rangle_1 reads in this case

$$\langle a, b, c \rangle_1 = aQ(b, c) + (-1)^{\sigma(v_i, v_j)} bQ(q, c) - (-1)^{\sigma(v_i, v_k)+\sigma(v_j, v_k)} Q(c, a)b$$

$$- (-1)^{\sigma(v_i, v_j)+\sigma(v_i, v_k)+\sigma(v_j, v_k)} Q(c, b)a.$$

(37)
Comparing (15) and (36) shows in this case the trilinear map $\langle \cdot, \cdot, \cdot \rangle_2$ reads

$$\langle a, b, c \rangle_2 = Q(a, b) + (-1)^{\sigma(v_j, v_k)}Q(a, c) - (-1)^{\sigma(v_i, v_j) + \sigma(v_i, v_k)}bQ(c, a)$$

$$-(-1)^{\sigma(v_j, v_k) + \sigma(v_i, v_j) + \sigma(v_i, v_k)}cQ(b, a)$$

(38)

4.1 Exemple

when $\Gamma$ is a two-dimensional vector space on $Z_2$, $\Gamma = \{(0, 0), (1, 0), (0, 1), (1, 1)\}$, the algebra $A$ is

$$A = A\begin{pmatrix} 0 \\ 0 \end{pmatrix} \oplus A\begin{pmatrix} 1 \\ 0 \end{pmatrix} \oplus A\begin{pmatrix} 0 \\ 1 \end{pmatrix} \oplus A\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

In the paper $^5$ we have shown that the color superalgebra $C(2, s)$, the Lie superalgebra $C(1, s)$, the color algebra $C(2, a)$ and the Lie algebra are characterized, respectively, by the following four equivalence classes,

$$\{(0, 1), (1, 1), (0, 1)\}.$$ (39)

$$\{(0, 1), (1, 0), (1, 1)\}.$$ (40)

$$\{(0, 1), (0, 1), (1, 1)\}.$$ (41)

$$\{(0, 0), (0, 0)\}.$$ (42)

While $\sigma$ is represented by $M \in Sbil(Z_2)^5$, such that:

$\sigma(v_i, v_j) = v_i^tMv_j$, $v_i, v_j \in \Gamma$ and $v_i^t$ is the transpose of $v_i$.

Let $M = \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix}$, $v_i = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}$ and $v_j = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}$ where $\alpha_{11}, \alpha_{12}, \alpha_{21}, \alpha_{22}, p_1, q_1, p_2$ and $q_2 \in Z_2$

$$\sigma(v_i, v_j) = v_i^tMv_j = p_1\alpha_{11}q_1 + p_1\alpha_{12}q_2 + p_2\alpha_{21}q_1 + p_2\alpha_{22}q_2$$

(43)

Let:

$\langle E \rangle$ ($E$ is the identity) be the generator of $A\begin{pmatrix} 0 \\ 0 \end{pmatrix}$;

$\langle a_i, a_j^+ \rangle$, $i, j = 1, 2, \ldots, n$ be the generator of $A\begin{pmatrix} 1 \\ 0 \end{pmatrix}$.
\[ \langle b_i, b_j^+, i, j = 1, 2, \ldots, m \rangle \] be the generator of \( A \begin{pmatrix} 0 \\ 1 \end{pmatrix} \):

\[ \langle c_i, c_j^+, i, j = 1, 2, \ldots, p \rangle \] be the generator of \( A \begin{pmatrix} 1 \\ 1 \end{pmatrix} \).

and \( M = \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \).

\( A \begin{pmatrix} 0 \\ 0 \end{pmatrix} \) and \( A \begin{pmatrix} 1 \\ 1 \end{pmatrix} \) are even spaces (symmetric spaces or equivalently bosonic spaces)

with respect to \( \sigma \) since \( \sigma \left( \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right) = 0 \) and \( \sigma \left( \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right) = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} = 2 = 0 \), with implies that if \( a \) and \( b \in A \begin{pmatrix} 0 \\ 0 \end{pmatrix} \) or \( a \) and \( b \in A \begin{pmatrix} 1 \\ 1 \end{pmatrix} \) then \( Q(a, b) = -Q(b, a) \).

\( A \begin{pmatrix} 1 \\ 0 \end{pmatrix} \) and \( A \begin{pmatrix} 0 \\ 1 \end{pmatrix} \) are odd spaces (antisymmetric spaces or equivalently fermionic spaces) with respect to \( \sigma \) since \( \sigma \left( \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right) = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = 1 \)

and \( \sigma \left( \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} = 1 \), then \( Q(a, b) = Q(b, a) \) for \( a \) and \( b \in A \begin{pmatrix} 1 \\ 0 \end{pmatrix} \) or \( a \) and \( b \in A \begin{pmatrix} 0 \\ 1 \end{pmatrix} \).

First we consider the case where \( A = A \begin{pmatrix} 0 \\ 0 \end{pmatrix} \oplus A \begin{pmatrix} 1 \\ 0 \end{pmatrix}, A \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \emptyset \) and \( A \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \emptyset \).

\( a, b \) and \( c \) can be taken as annihilation or creation operators, \( a_i \) and \( a_j^+ \), \( i, j = 1, 2, \ldots, n \).

The quadratic form is symmetric \( Q(a, b) = Q(b, a) \); we denote the usual commutator by \( [a, b] = ab - ba \) and the usual anticommutator by \( \{a, b\} = ab + ba \).

\[ \langle a, b, c \rangle_1 = aQ(b, c) - bQ(a, c) - Q(c, a)b + Q(c, b)a = 2Q(b, c)a - 2Q(a, c)b \]

\[ = abc - bac - cab + cba \]

\[ = [a, b], c \] (44)

if we set, \( a = a_i \), \( b = a_j^+ \) and \( c = a_k \), and choose the bilinear form \( Q(a, b) \) to satisfy the following relations: \( Q(a_i, a_j^+) = Q(a_j^+, a_i) = \delta_{ij}, Q(a_i, a_j^+) = Q(a_j^+, a_i) = \delta_{ij}, Q(a_i^+, a_j^+) = Q(a_j^+, a_i^+) = 0, Q(a_i, a_j) = Q(a_j, a_i) = 0 \), then

\[ \langle a_i, a_j^+, a_k \rangle_1 = a_iQ(a_j^+, a_k) - Q(a_i, a_k)a_j^+ + a_j^+Q(a_k, a_i) + Q(a_k, a_j^+)a_i \]
If we set 

\[ a_i \delta_{jk} + \delta_{jk} a_i = 2 \delta_{jk} a_i = \left[ a_i, a_j^+ \right], a_k \]  

(45)

when \( a = a_i^+ \), \( b = a_j^+ \) and \( c = a_k \), it readily follows that

\[ \langle a_i^+, a_j^+, a_k \rangle_1 = 2 \delta_{jk} a_i^+ - 2 \delta_{ik} a_j^+ = \left[ a_i^+, a_j^+, a_k \right] \]

(46)

Eq. (46) and Eq. (47) are two of the relations that characterize the parafermion statistic. The other relations are obtained in a similar way.

### 4.2 Remark

One can take the trilinear application \( \langle , , \rangle_2 \)

\[ \langle a_i, a_j^+, a_k \rangle_2 = + \delta_{ij} a_k + a_k \delta_{ij} = 2 \delta_{ij} a_k = \left[ a_i, [a_j^+, a_k] \right] \]

(47)

For \( a = a_i^+ \), \( b = a_j^+ \) and \( c = a_k \)

\[ \langle a_i^+, a_j^+, a_k \rangle_2 = -2 \delta_{ik} a_j^+ = \left[ a_i, [a_j^+, a_k] \right] \]

(48)

the others relations follow directly.

Therefore, the present example reproduces the parafermion statistics.

Now let \( A = A \left( \begin{array}{cc} 0 & 1 \\ 0 & 1 \end{array} \right) \oplus A \sigma \left( \left( \frac{1}{1}, \frac{1}{1} \right) \right) = 0. \)

\( a, b \) and \( c \in A \left( \begin{array}{c} 1 \\ 1 \end{array} \right) \) may be identified with the annihilation and creation operators \( c_i \)

and \( c_j^+ i, j = 1, 2, ..., n \), the quadratic form satisfies \( Q(a, b) = -Q(b, a) \) such that:

\[ Q(c_i^+, c_i) = -Q(c_i, c_i^+) = \delta_{ij} \quad \text{and} \quad Q(c_i, c_j) = -Q(c_i^+, c_j^+) = 0 \]

(49)

If we set \( a = c_i \), \( b = c_j^+ \) and \( c = c_k \) then:

\[ \langle c_i, c_j^+, c_k \rangle_1 = c_i Q(c_j^+, c_k) + c_j^+ Q(c_i, c_k) - Q(c_k, c_i) c_j^+ - Q(c_k, c_j^+) c_i \]

\[ = c_i \delta_{jk} + \delta_{jk} c_i = 2 \delta_{jk} c_i \]

\[ = c_i c_j^+ c_k + c_j^+ c_k c_i - c_k c_i c_j^+ - c_k c_j^+ c_i \]

(50)

In term of the usual commutator and anticommutator the expression (48) is \( \left[ \left\{ c_i, c_j^+ \right\}, c_k \right] \)

If we set \( a = c_i \), \( b = c_j^+ \) and \( c = c_k^+ \)

\[ \langle c_i, c_j^+, c_k^+ \rangle_1 = c_i Q(c_j^+, c_k^+) + c_j^+ Q(c_i, c_k^+) - Q(c_k^+, c_i) c_j^+ - Q(c_k^+, c_j^+) c_i \]

\[ = -c_j^+ \delta_{ik} - \delta_{ik} c_j^+ = -2 \delta_{ik} c_j^+ \]

\[ = c_i c_j^+ c_k^+ + c_j^+ c_i c_k^+ - c_k^+ c_i c_j^+ - c_k^+ c_j^+ c_i \]

\[ = \left[ \left\{ c_i, c_j^+ \right\}, c_k^+ \right] \]

(51)
If we set \( a = c_i, b = c_j^+ \) and \( c = c_k^+ \) a direct calculation gives,
\[
\langle c_i, c_j, c_k^+ \rangle_1 = -2\delta_{jk}c_i - 2\delta_{ik}c_j = \{\{c_i, c_j\}, c_k^+\}
\]
(52)

If we set \( a = c_i^+, b = c_j^+ \) and \( c = c_k^+ \) then,
\[
\langle c_i^+, c_j^+, c_k^+ \rangle_1 = 2\delta_{jk}c_i^+ + 2\delta_{ik}c_j^+ = \{\{c_i^+, c_j^+\}, c_k^+\}
\]
(53)
\[
\langle c_i^+, c_j, c_k^+ \rangle_1 = 0 = \{\{c_i, c_j\}, c_k^+\}
\]
(54)

The case \( a = c_i^+, b = c_j^+ \) and \( c = c_k^+ \)
\[
\langle c_i^+, c_j^+, c_k^+ \rangle_1 = 0 = \{\{c_i^+, c_j^+\}, c_k^+\}
\]
(55)

therefore reproduces the paraboson statistics.

One can use the map \((, , )_2\), we found that:
\[
\langle c_i, c_j^+, c_k \rangle_2 = Q(c_i, c_j^+)c_k + Q(c_i, c_k)c_j^+ - c_j^+Q(c_k, c_i) - c_kQ(c_j^+, c_i)
\]
\[
= -\delta_{ij}c_k - c_k\delta_{ij} - 2\delta_{ij}c_k
\]
\[
= c_i c_j^+ c_k + c_i c_k c_j^+ - c_j^+ c_k c_i - c_k c_j^+ c_i
\]
\[
= \{\{c_i, c_j^+\}, c_k\}
\]
(56)

\[
\langle c_i, c_j^+, c_k^+ \rangle_2 = Q(c_i, c_j^+)c_k^+ + Q(c_i, c_k^+)c_j^+ - c_j^+Q(c_k^+, c_i) - c_k^+Q(c_j^+, c_i)
\]
\[
= -\delta_{ij}c_k^+ - \delta_{ik}c_j^+ - c_j^+\delta_{ik} - c_k^+\delta_{ij} - 2\delta_{ij}c_k^+
\]
\[
= c_i c_j^+ c_k^+ + c_i c_k c_j^+ - c_j^+ c_k c_i - c_k^+ c_j^+ c_i
\]
\[
= \{\{c_i, c_j^+\}, c_k^+\}
\]
(57)

A similar calculation gives:
\[
\langle c_i, c_j, c_k^+ \rangle_2 = -2\delta_{ik}c_j = \{\{c_i, c_j\}, c_k^+\}
\]
(58)
\[
\langle c_i^+, c_j^+, c_k \rangle_2 = 2\delta_{ik}c_j^+ = \{\{c_i^+, c_j^+\}, c_k\}
\]
(59)
\[
\langle c_i^+, c_j, c_k \rangle_2 = 0 = \{\{c_i^+, c_j\}, c_k\}
\]
(60)
\[
\langle c_i^+, c_j^+, c_k^+ \rangle_2 = 0 = \{\{c_i^+, c_j^+\}, c_k^+\}
\]
(61)

which reproduce the paraboson statistics.

Consider now the case:
If we set
\[ A = A \begin{pmatrix} 0 \\ 0 \end{pmatrix} \oplus A \begin{pmatrix} 1 \\ 0 \end{pmatrix} \oplus A \begin{pmatrix} 1 \\ 1 \end{pmatrix} \]
and choose the quadratic form \( Q \) such that
\[
Q(a_i^+, a_j) = Q(a_j, a_i^+) = \delta_{ij}
\]
\[
Q(a_i, a_j) = Q(a_i^+, a_j^+) = 0
\]
\[
Q(c_i^+, c_j) = -Q(c_j, c_i^+) = \delta_{ij}
\]
\[
Q(c_i, c_j) = Q(c_i^+, c_j^+) = 0
\]
\[
Q(a_i^+, c_j) = Q(a_j, c_i^+) = Q(a_i, c_j) = Q(a_i^+, c_j^+) = 0
\]
\[
M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}
\]
like in the precedent examples; we have:
\[
\sigma \left( \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = 1
\]
\[
\sigma \left( \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right) = \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = 2 = 0
\]
\[
\sigma \left( \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = 1
\]
\[
\langle a_i, c_j, a_k^+ \rangle_1 = -2\delta_{ik} c_j = [a_i, c_j, a_k^+] = 0
\]
\[
\langle a_i, c_j, a_k^+ \rangle_1 = -2\delta_{ik} c_j = [a_i, [c_j, a_k^+]]
\]
Note that the bosonic operator \( c_j \) does not commute with the fermionic operator \( a_i \) in Eq. (66) or with the fermionic operator \( a_k^+ \) in Eq. (67).
The relations below follow easily if
\[
A = A \begin{pmatrix} 0 \\ 0 \end{pmatrix} \oplus A \begin{pmatrix} 1 \\ 0 \end{pmatrix} \oplus A \begin{pmatrix} 1 \\ 1 \end{pmatrix}, M = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, b \in A
\]
\[ c \in A \begin{pmatrix} 0 \\ 1 \end{pmatrix} \]
and
\[
Q(a_i^+, a_j) = Q(a_j, a_i^+) = \delta_{ij}
\]
\[
Q(a_i, a_j) = Q(a_i^+, a_j^+) = 0
\]
\[
Q(b_i^+, b_j) = -Q(b_j, b_i^+) = \delta_{ij}
\]
\[
Q(b_i, b_j) = Q(b_i^+, b_j^+) = 0
\]
\[
Q(a_i^+, b_j) = Q(a_j, b_i^+) = Q(a_i, b_j) = Q(a_i^+, b_j^+) = 0
\]
If we set \( a = a_i^+, b = b_j^+ \) and \( c = b_k \), we find that,
\[
\langle a_i^+, b_j^+, b_k \rangle_1 = 2\delta_{jk} a_i^+ = \left[ [a_i^+, b_j^+], b_k \right]
\]
Since $A \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ and $A \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$ are odd spaces (antisymmetric spaces i.e. fermionic spaces) with respect to $\sigma$, we can say that different fermion species do not anticommute.

If $A = A \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \oplus A \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \oplus A \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$ let $a \in A \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$, $b \in A \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}$ and $c \in A \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}$

$$\langle a, b, c \rangle_1 = aQ(b, c) - bQ(a, c) + Q(c, a)b - Q(c, b)a$$
$$= abc - bac + cab - cba$$
$$= \{[a, b], c\}$$ (69)

we have $Q(b, c) = -Q(c, b) = 0$ and $Q(a, c) = Q(c, a)$ then

$$\langle a, b, c \rangle_1 = 2Q(b, c)a = \{[a, b], c\}$$ (70)

If we set $a = a_i^+, b = b_j$ or $b_j^+$, $c = a_k$ and $Q(a_i^+, a_k) = \delta_{ik}$ we have

$$\langle a_i^+, b_j, a_k \rangle_1 = \{[a_i^+, b_j], a_k\}$$ (71)
$$\langle a_i^+, b_j^+, a_k \rangle_1 = \{[a_i^+, b_j^+], a_k\}$$ (72)

$$\langle a, b, c \rangle_2 = Q(a, b)c - Q(a, c)b + bQ(c, a) - cQ(b, a) = \{a, [b, c]\}$$ (73)

we have $Q(a, b) = -Q(b, a)$ and $Q(a, c) = Q(c, a)$ then

$$\langle a, b, c \rangle_2 = 2Q(a, b)c = \{a, [b, c]\}$$ (74)

If we set $a = a_i^+, b = b_j$ or $b_j^+$, $c = a_k$ and $Q(a_i^+, a_k) = \delta_{ik}$ we have

$$\langle a_i^+, b_j, a_k \rangle_2 = \{a_i^+, [b_j, a_k]\} = -2\delta_{ik}b_j$$ (75)
$$\langle a_i^+, b_j^+, a_k \rangle_2 = \{a_i^+, [b_j^+, a_k]\} = -2\delta_{ik}b_j^+$$ (76)

Note that in this case $\langle a, b, c \rangle_1 \neq \langle a, b, c \rangle_2$

If $a \in A \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $b \in A \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $c \in A \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$$\langle a, b, c \rangle_1 = aQ(b, c) + bQ(a, c) - Q(c, a)b - Q(c, b)a$$
$$= abc + bac - cab - cba$$
$$= \{[a, b], c\}$$ (77)
we have

\[ Q(b, c) = Q(c, b) \quad \text{and} \quad Q(a, c) = Q(c, a) \quad \text{then} \] (78)

\[ \langle a, b, c \rangle_1 = 0 = \{ [a, b], c \} \] (79)

\[ \langle a, b, c \rangle_2 = Q(a, b)c - Q(a, c)b + bQ(c, a)b - cQ(b, a) \]

\[ = abc - acb + bca - cba \]

\[ = \{ a, [b, c] \} \] (80)

we have

\[ Q(a, b) = -Q(b, a) \quad \text{and} \quad Q(a, c) = Q(c, a) \quad \text{then} \] (81)

\[ \langle a, b, c \rangle_2 = 2Q(a, b)c = \{ a, [b, c] \} \] (82)

Note that \( a, b \) and \( c \) belong to different spaces, by definition \( Q(a, b) = 0 \), \( Q(a, c) = 0 \) and \( Q(b, c) = 0 \), hence:

\[ \langle a, b, c \rangle_2 = 2Q(a, b)c = \{ a, [b, c] \} = 0 \] (83)

5 Application to the bilinear case

5.1 Anyonic Vector Spaces

We consider braided categories associated to \( \mathbb{Z}_n \), the finite group of order \( n \). Let \( g \) be the generator of \( \mathbb{Z}_n \) with \( g^n = 1 \). As a category of objects and morphisms we take the category \( \text{Rep}(\mathbb{Z}_n) \) of finite dimensional representations of \( \mathbb{Z}_n \).

Given an object \( V \) of \( \text{Rep}(\mathbb{Z}_n) \) we can decompose it under the action of \( \mathbb{Z}_n \) as

\[ V = \bigoplus_{p=0}^{n-1} V_p, \quad p = 0, 1 \ldots, n - 1. \] (84)

Here \( p \) runs over the set of irreducible representation \( \rho_p \). We have the action,

\[ g \triangleright v = e^{2\pi ip/n} v, \quad \forall \ v \in V_p \]

where the action of \( \mathbb{Z}_n \) is denoted by \( \triangleright \). If \( v \in V_p \), we say that \( v \) is homogeneous of degree \( |v| = p \)

On this category \( \text{Rep}(\mathbb{Z}_n) \), we can now define the braiding

\[ \Psi_{V,W}(v \otimes w) = e^{2\pi i |v||w|/n} w \otimes v \] (85)

In physics the quantities \( e^{2\pi i |v||w|/n} \) are called fractional or anyonic statistics.
5.2 Classical case

\( A = \oplus_i A_v \) is a \( \Gamma \)-graded algebra\(^5\), we consider (anti)symmetric bilinear maps that we have defined in an earlier paper\(^5\) \( \sigma : \Gamma \times \Gamma \longrightarrow Z_2; Z_2 = \{0,1\}; \) and \( T(A) \) the tensor algebra of \( A \). We define isomorphisms \( \Psi_{A_v,A_v} : A_v \otimes A_v \longrightarrow A_v \otimes A_v \) which satisfy the properties (2). Let \([\ ,\ ]_\sigma\) be a bilinear map, \([\ ,\ ]_\sigma : A \times A \longrightarrow A; \) and \( J \) be the two sided ideal generated by all expressions of the following form, \( x \otimes y + \Psi_{A_v,A_v} (x \otimes y) - [x,y]_\sigma \) where \( x \in A_v, y \in A_v, \ i \) and \( j \in Z_2 \). The envelopping in algebra is defined as \( U(A) = T(A)/J \).

We take the braiding as:

\[
\Psi_{A_v,A_v}(x \otimes y) = -(-1)^{\sigma(v_i,v_j)} y \otimes x
\]

In \( U(A) \), and also in \( A \) we have:

\[
[x,y]_\sigma = x.y - (-1)^{\sigma(v_i,v_j)} y.x
\]

this is the case that we have developped in the paper\(^5\), which gives the color superalgebra, the Lie superalgebra, the color algebra, etc...

Note that the anyonic case is obtained by taking the braiding as:

\[
\Psi_{A_v,A_v}(x \otimes y) = \exp\left(\frac{2\pi i \sigma(v_i,v_j)}{n}\right)y \otimes x
\]

now the bilinear map \( \sigma \) is:

\[
\sigma : \Gamma \times \Gamma \longrightarrow Z_n; Z_n = \{0,1,\ldots,n-1\}
\]

Then in \( U(A) \), and also in \( A \), we have:

\[
[x,y]_\sigma = x.y + \exp\left(\frac{2\pi i \sigma(v_i,v_j)}{n}\right)y.x
\]

6 Schur functors

It is well known that the functors generalizing the symmetric powers an exterior powers are defined in terms of the Young symmetrizers \( c_\lambda \). For any finite complex vector space \( A \), we consider the dth tensor power of \( A \), on which the symmetric group \( S_d \) acts, say on the right, by permuting the factors \((v_1 \otimes \ldots \otimes v_d).\sigma = v_{\sigma(1)} \otimes \ldots \otimes v_{\sigma(d)}\). This action commute with the left action of \( GL(A) \). For any partition \( \lambda \) of \( d \) we have a Young Symmetrizer \( c_\lambda \) in the group algebra \( \mathbb{C}S_d \). We denote the image of \( c_\lambda \) on \( A^{\otimes d} \) by \( S_\lambda A \).

( The functoriality means that a linear map \( \varphi : V \rightarrow W \) of vector spaces determines a linear map \( S_\lambda(\varphi) : S_\lambda(V) \rightarrow S_\lambda(W) \), with \( S_\lambda(\varphi \circ \psi) = S_\lambda(\varphi) \circ S_\lambda(\psi) \) and \( S_\lambda(Id_V) = Id_{S_\lambda(V)} \)
\[ S_\lambda A = \text{Im}(c_\lambda |_A \otimes \epsilon) \] (91)

Which is a representation of \( GL(A) \). The functor \( A \hookrightarrow S_\lambda A \) is called the Schur functor or Weyl module corresponding to \( \lambda \). We have the canonical decomposition

\[ A \otimes A = \text{Sym}^2 A \oplus \Lambda^2 A \] (92)

The group \( GL(A) \) acts on \( A \otimes A \), and decompose it into a direct sum of irreducible \( GL(A) \)-representations. We will be interested by the next tensor power

\[ A \otimes A \otimes A = \text{Sym}^3 A \oplus \Lambda^3 A \oplus (S(2,1)A)^{\otimes 2} \] (93)

The partition \( d = 3 \) corresponds to the functor \( A \hookrightarrow \text{Sym}^3 A \) and the partition \( d = 1 + 1 + 1 \) to the functor \( A \hookrightarrow \Lambda^3 A \). We found new and more generalized things; for example the partition \( 3 = 2 + 1 \), the corresponding “symmetrizer” \( c_\lambda, \Psi \) is

\[ c_{(2,1), \Psi} = e_1 + \Psi_{ij}^{mn} e_{12} + \Psi_{ij}^{pq} \Psi_{mp}^{rs} e_{13} + \Psi_{ij}^{mn} \Psi_{nk}^{pq} \Psi_{mp}^{rs} \Psi_{sq}^{lu} e_{132} \] (94)

\[ c_{(2,1), \Psi}^* = e_1 + \Psi_{jk}^{mn} e_{23} + \Psi_{jk}^{pq} \Psi_{im}^{rs} e_{13} + \Psi_{jk}^{mn} \Psi_{im}^{pq} \Psi_{qn}^{rs} \Psi_{pr}^{tu} e_{123} \] (95)

7 Conclusion

We can consider the algebra \( A \otimes \mathbb{C}^\infty(\mathbb{R}^n) \) and interested by \( S_3 \) irreducible representations of such algebra ; this algebra may be viewed as an algebra of operators, and also to understand the unitarity condition (6) in this case.

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