ON ASYMMETRIC CHERN-SIMONS NUMBER DIFFUSION

BERT-JAN NAUTA
Institute for Theoretical Physics, University of Amsterdam
Valckenierstraat 65, 1018XE, Amsterdam, The Netherlands,
E-mail: nauta@science.uva.nl

We show that CP-violation can lead to an asymmetric diffusion of the Chern-Simons number in thermal equilibrium. This asymmetry leads to a linearly growing expectation value of the third power of the Chern-Simons number. In the long-time limit all expectation values of powers of Chern-Simon numbers are determined by their appropriate disconnected parts.

1. Chern-Simons number diffusion

At high temperatures the sphaleron rate determines the rate of Chern-Simons number changing transitions. Since a change in the Chern-Simons number is accompanied by a (3 times larger) change in the baryon number (in the standard model), the sphaleron rate is an important ingredient of baryogenesis scenarios. The sphaleron rate may be obtained by ignoring the back reaction of the generated baryons and leptons. This back reaction may then later be included through a linear response analysis. Without (the back reaction of) baryons and leptons the potential is periodic (as for instance in pure gauge-Higgs theory). When subsequent transitions are uncorrelated or the correlation dies away sufficiently fast, the process is Markovian and the Chern-Simons number diffuses. The diffusion process may be specified by

\[ \langle N_{CS}(t) - N_{CS}(0) \rangle = 0, \]
\[ \langle [N_{CS}(t) - N_{CS}(0)]^2 \rangle = 2V\Gamma_{sph}t, \]

with \( \Gamma_{sph} \) the sphaleron rate and \( V \) the volume. See e.g. the review 1.

CP-violation is one of three of Sakharov’s requirements for baryogenesis. This is our reason to study the effect of CP-violation on Chern-Simons number diffusion. Since the Chern-Simons number itself is CP-odd, CP-violation may cause odd powers of the Chern-Simons number to take non-zero values. The question that we want to answer is: what is the possible
behavior of the expectation values $\langle [N_{CS}(t) - N_{CS}(0)]^{2n+1} \rangle$ for $n = 0, 1, \ldots$ in thermal equilibrium when CP is violated.

2. CPT invariance and thermal equilibrium

In general, the expectation values of CPT-odd quantities, such as the Chern-Simons number, vanish in thermal equilibrium. Before we will show that this does not apply to odd powers of differences of Chern-Simons numbers, we review the argument.

Consider the expectation value of a CPT-odd operator, $B$, for instance the baryon number. Since in thermal equilibrium the (thermal) distribution function is time-independent, the thermal average of $B(t)$ may be taken at time $t$

$$\langle B(t) \rangle = \langle B \rangle. \quad (3)$$

The thermal average is defined by $\langle B \rangle = \text{Tr} \{ Be^{-\beta H} \}$, where $\beta$ is the inverse temperature and $H$ is the Hamiltonian.

Since the operator $B$ is CPT-odd and the Hamiltonian is CPT-even

$$\langle B \rangle^{\text{CPT}} = -\langle B \rangle \Rightarrow \langle B \rangle = 0. \quad (4)$$

see e.g. \cite{2}. In general, the thermal expectation value of CPT-odd operators vanishes.

Basically there are two conditions for this argument to work: the CPT-odd operator must be local in time, and the thermal average must exist.

3. Why $\langle [N_{CS}(t) - N_{CS}(0)]^{2n+1} \rangle$ can be non-zero

If we try to repeat the reasoning that leads to the conclusion that CPT-odd quantities vanish in thermal equilibrium (from (3) to (4)) for odd powers of differences in Chern-Simons numbers at different times, we run into the following problem. If we directly apply a CPT-transformation we arrive at

$$\langle [N_{CS}(t) - N_{CS}(0)]^{2n+1} \rangle^{\text{CPT}} = -\langle [N_{CS}(-t) - N_{CS}(0)]^{2n+1} \rangle \quad (5)$$

A time translation on the r.h.s. shows that this is a trivial identity. The argument leading to (4) only works for CPT-odd operators that are local in time, so that the thermal average can be taken at the time the operator is evaluated, as in (3). With this in mind, we might try to apply the reasoning to $\langle N_{CS}^{2n+1}(t) \rangle$ and we would find it vanishes. However only differences of Chern-Simons numbers are gauge-invariant. The thermal average $\langle N_{CS}^{2n+1}(t) \rangle$ does not exist, and the argument leading to (4) cannot be
applied in this case. It is important to realize that there is no way around this: only differences $N_{\text{CS}}(t_1) - N_{\text{CS}}(t_2)$ are gauge invariant and physical, but a thermal average of powers of these differences can be taken only at a single time $t_1$, $t_2$, or some other time. As a result these quantities may depend on $t_1 - t_2$ and do not need to vanish.

Hence, we conclude that the standard argument that the thermal average of CPT-odd quantities vanishes based on CPT-invariance is not applicable to odd powers of the difference of Chern-Simons numbers.

The combination of CPT-invariance and thermal equilibrium still make gauge-invariant CPT-odd quantities that depend on a single time vanish, such as

$$\langle \{N_{\text{CS}}(t) - \text{round}[N_{\text{CS}}(t)]\}^{2n+1} \rangle = 0,$$

with round the function that maps a real number to its nearest integer. Equations as (6) pose restrictions on the distribution function of the Chern-Simons number. But it poses no restrictions on the expectation values we are interested in, expectation values of powers of differences of the Chern-Simons number.

But, there is an additional important restriction, that follows from

$$\langle \frac{d}{dt}[N_{\text{CS}}(t) - N_{\text{CS}}(0)] \rangle = 0,$$

since the thermal average of a time derivative of a quantity that is local in time vanishes. This gives

$$\langle [N_{\text{CS}}(t) - N_{\text{CS}}(0)] \rangle = 0. (7)$$

For expectation values of higher powers there is not such a restriction

$$\langle \frac{d}{dt}[N_{\text{CS}}(t) - N_{\text{CS}}(0)]^m \rangle \neq 0, \ m = 2, 3, 4, \ldots \text{ as is well-known for } m = 2.$$

Hence, we may summarize that on general grounds only the expectation value of a single power of a difference in Chern-Simons numbers vanishes in thermal equilibrium (7), but that generically

$$\langle [N_{\text{CS}}(t) - N_{\text{CS}}(0)]^m \rangle \neq 0, (8)$$

for all, including odd, $m \geq 2$. Here it should be understood that (8) means that it is possible that these expectation values are non-zero. Of course for the expectation values of odd powers to actually obtain a non-zero value, at least CP has to violated. (Formally, this can be shown by the same argument as leading to (4), but with a CP-transformation instead of a CPT-transformation.)

4. Asymmetric diffusion

We have argued that when CP is violated there is no reason why

$$\langle [N_{\text{CS}}(t) - N_{\text{CS}}(0)]^{2n+1} \rangle \text{ with } n = 1, 2, 3, \ldots \text{ should be equal to zero in ther-}$$
mal equilibrium.

For Chern-Simons number diffusion to develop asymmetrically, an asymmetry must exist in sphaleron transitions. We have investigated such an asymmetry by considering the effect of CP-violation on the motion of configurations at the sphaleron barrier. The CP-violation was added by including effective CP-odd operators to the action. The lowest order operator, that is mostly considered, is the dimension-six operator, $\bar{\phi}^a \phi^a \tilde{F}_{\mu\nu} F^{\mu\nu}$. In out-of-equilibrium situation this operator can generate an effective chemical potential for baryons, and therefore it introduces a bias in the sphaleron transitions. However, we found that, within the chosen approach, in thermal equilibrium the dimension-six operator does not lead to any asymmetry. But we also found that CP-odd dimension-eight operators induce an asymmetry in the average velocity with which a configuration crosses the sphaleron barrier. Unfortunately, we have not succeeded to relate this asymmetry in the velocity to an asymmetry in the distribution of the Chern-Simons number after several sphaleron transitions. Nevertheless, one may assume the the distribution to develop an asymmetry, at least, there is absolutely no reason why it should not.

Next we determine the large-time behavior of the expectation values of odd powers of Chern-Simons number differences. We start with the third power. It can be shown that in the long-time limit $\langle [N_{CS}(t) - N_{CS}(0)]^3 \rangle$ in thermal equilibrium either goes to a constant or grows linearly in time (for a one-dimensional system this was shown in 5). When the sphaleron barrier is large, subsequent transitions are practically uncorrelated. In that case, an asymmetry in the sphaleron transitions will lead to a linearly growing value in time for $\langle [N_{CS}(t) - N_{CS}(0)]^3 \rangle$. Hence in a theory with CP-violation, an additional equation is required to describe the (asymmetric) diffusion of the Chern-Simons number

$$\langle [N_{CS}(t) - N_{CS}(0)]^3 \rangle = \delta_{CP} V \Gamma_{sph} t,$$

with $\delta_{CP}$ a dimensionless parameter proportional to the coefficients of the CP-odd operators in the theory. Note that the assumption that the sphaleron barrier is large implies that we have only argued (9) to hold in the broken phase. We would be surprised if in the symmetric phase (9) is incorrect, but we do not have any physical arguments for this.

Equation (9) can be tested in a one-dimensional model. All arguments leading to (9) apply also to a model of a particle with (CP-odd) coordinate $x$ with a Lagrangian that is not invariant under $x \rightarrow -x$, but that is invariant under the combined reflections of $t \rightarrow -t$ and $x \rightarrow -x$ (T and CP), and
has a periodic potential (with a barrier between different vacua). Such a one-dimensional model was coupled to a thermal bath and the diffusion was studied, the result showed indeed a linearly growing value for $\langle x^3 \rangle$ proportional to the diffusion constant of $\langle x^2 \rangle$.

To relate the constant $\delta_{CP}$ in (9) to the coefficient of certain CP-odd operator seems to be very difficult analytically, since sphaleron transitions require non-perturbatively large fields. Also numerically this may be a problem, since in 3+1D the classical theory is divergent and this situation may worsen when non-renormalizable (effective) CP-odd operators are added to a theory. We are performing a numerical study in 1+1D. Our preliminary results confirm (9).

Expectation values of higher odd powers are expected to be dominated by their disconnected parts in the large-time limit

$$\langle [N_{CS}(t) - N_{CS}(0)]^{2n+1} \rangle \approx \frac{(2n+1)!}{2^{n-1}3!(n-1)!} \langle [N_{CS}(t) - N_{CS}(0)]^3 \rangle \times \langle [N_{CS}(t) - N_{CS}(0)]^2 \rangle^{n-1}. \quad (10)$$

In this way all correlation function are determined by the set of equations (1), (2), and (9).

5. Conclusion and outlook

We have shown that CP-violation can induce an asymmetry in the distribution function of the Chern-Simons number. We have argued that in the broken phase in the presence of CP-violation Chern-Simons number diffusion is determined by the set of equations (1), (2), and (9).

The study of asymmetric Chern-Simons number diffusion is motivated by the the question how the baryon asymmetry in the universe was generated. It has been argued that asymmetric diffusion may result in a non-zero expectation value of the baryon number in a time-dependent effective potential.

References

1. V. A. Rubakov and M. E. Shaposhnikov, Phys. Usp. 39 461 (1996).
2. W. Bernreuther, Lect. Notes Phys. 591 237 (2002).
3. B. J. Nauta, Phys. Lett. B478 275 (2000).
4. J. Garcia-Bellido, D. Y. Grigoriev, A. Kusenko, and M.E. Shaposhnikov, Phys. Rev. D60 123504 (1999); J. Smit and A. Tranberg, hep-ph/0211243, and references therein.
5. B. J. Nauta and A. Arrizabalaga, Nucl. Phys. B635, 255 (2002).