An Automated Theorem Proving Framework for Information-Theoretic Results

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Abstract

We present a versatile automated theorem proving framework capable of automated discovery, simplification and proofs of inner and outer bounds in network information theory, deduction of properties of information-theoretic quantities (e.g. Wyner and Gács-Körner common information), and discovery of non-Shannon-type inequalities, under a unified framework. Our implementation successfully generated proofs for 32 out of 56 theorems in Chapters 1-14 of the book Network Information Theory by El Gamal and Kim. Our framework is based on the concept of existential information inequalities, which provides an axiomatic framework for a wide range of problems in information theory.

Index Terms

Automated theorem proving, non-Shannon-type inequalities, network information theory, capacity region.

I. INTRODUCTION

In information theory, information is commonly represented as random variables. Proofs of theorems in information theory often rely on properties of random variables, e.g. implications between conditional independence statements \cite{11, 3}, and inequalities among entropy and mutual information terms. Computer programs for automated deduction of these properties are therefore useful for proving information-theoretic results. For example, the linear programming approach to proving linear information inequalities studied by Yeung \cite{3} and Zhang and Yeung \cite{41, 4} (implemented in the Information Theoretic Inequality Prover \cite{6}) has found success in proving results on network coding \cite{7, 8, 9, 10, 11} and secret sharing \cite{12, 13, 14}.

In this paper, which is the complete version of \cite{15}, we develop a versatile automated theorem proving framework capable of automated discovery, simplification and proofs of inner and outer bounds in network information theory involving auxiliary random variables, deduction of properties of information-theoretic quantities, and discovery of non-Shannon-type inequalities. To this end, we study the inference problem of existential information inequalities, which is a generalization of the inference problem of linear information inequality. Existential information inequalities concern the problem of deciding whether there exist some random variables satisfying certain constraints on their entropy and mutual information terms. For example, the copy lemma \cite{3, 17}, which states that for any random variables $X, Y$, there exists a random variable $U$ that is conditionally independent of $Y$ given $X$, such that the joint distribution of $(X, U)$ is the same as that of $(X, Y)$, can be regarded as an existential information inequality. Therefore, the technique for proving non-Shannon-type inequalities using the copy lemma can be included in the framework of existential information inequalities.

Existential information inequalities are especially suitable for network information theory, where the problem of finding auxiliary random variables in proving converse results can be expressed as existential information inequalities. Refer to \cite{18} for various examples. For achievability results, while general coding theorems have been studied, for example, in \cite{19, 20, 21, 22}, the inner bounds produced by these methods are often more complicated than necessary, and manual manipulations are needed to simplify these bounds to more familiar forms. The simplification of rate regions can also be performed using existential information inequalities. The algorithm in this paper is capable of automatically computing inner and outer bounds for a multiuser setting, requiring only a simple description of the communication network as the input to the program. Examples of results provable by this framework include channels with state (Gelfand-Pinsker theorem \cite{23}, achievability and converse), lossy source coding with side information (Wyner-Ziv theorem \cite{24}, achievability and converse), multiple access channel \cite{25, 26, 27} (achievability and converse), broadcast channel (Marton’s inner bound \cite{28, 29, 30}) and distributed lossy source coding (Berger-Tung inner bound \cite{31, 32}). Achievability results are proved using the coding theorem by Lee and Chung \cite{21, 22}, together with a novel simplification algorithm to reduce the number of auxiliary random variables via existential information inequalities.

This paper was presented in part at the IEEE International Symposium on Information Theory 2021. A brief description of the contents of the conference version was also included in one subsection in an article in IEEE BITS the Information Theory Magazine.\footnote{The short version \cite{15} only includes the definition of existential information inequalities and the auxiliary searching algorithm. It does not include other results in this paper (e.g. inner and outer bounds in network information theory, simplification of regions, etc.). A very brief description of existential information inequalities and the PSITIP program was also given in one subsection in an article in IEEE BITS the Information Theory Magazine \cite{16}. Note that \cite{16} does not include any of the technical results in this paper.}
An open-source Python implementation of the framework described in this paper, called Python Symbolic Information Theoretic Inequality Prover (PSITIP), is available online.\footnote{Source code is available at \url{https://github.com/cheuktingli/psitip}} PSITIP is capable of proving 57.1% (32 out of 56) of the theorems in Chapters 1-14 of \cite{18} (if we exclude 16 theorems on Gaussian settings that are out of the scope of our framework, then PSITIP can prove 80% of the theorems on discrete settings).\footnote{For the list of theorems and proofs generated by the program, visit \url{https://github.com/cheuktingli/psitip}}

This paper is organized as follows. In Section \textbf{II} we review the linear programming method for proving information inequalities in \cite{6,8}. In Section \textbf{III} we introduce the notion of existential information inequalities (EIIs). In Section \textbf{IV} we describe the auxiliary searching algorithm, a method for proving EIIs. In Section \textbf{V} we describe some inference rules for EIIs. In Section \textbf{VI} we study some EIIs that cannot be proved by the algorithm (called nontrivial EIIs). In Section \textbf{VII} we explain how to incorporate these nontrivial EIIs into the algorithm to allow it to prove more nontrivial EIIs. In Section \textbf{VIII} we introduce the notion of existential information predicates (EIPs), which is useful for representing rate regions in network information theory, and describe how we can compare and simplify EIPs. In Section \textbf{XI-A} we demonstrate the algorithm by showing how it can automatically derive the capacity region of the degraded broadcast channel \cite{33}, \cite{34}. In Section \textbf{XI} we give a list of results that can be proved by the algorithm.

\section{Related Works}

The inference problem of conditional independence \cite{1,2,35} is to decide whether a statement on the conditional independence among some random variables follows from a list of other such statements. Pearl and Paz \cite{36} introduced an axiomatic system, called the semi-graphoid axioms, which is useful for characterizing conditional independence structures among random variables. It was shown by Studeny \cite{37} that the semi-graphoid axioms are incomplete (i.e., unable to prove all valid conditional independence inferences). Algorithms for conditional independence inference has been studied, for example, in \cite{38,39,40}. It was shown recently in \cite{41} that the inference problem of conditional independence is undecidable. Also refer to \cite{42} for another proof posted slightly later.

The problem of characterizing linear inequalities among entropy and mutual information of random variables (called linear information inequalities) was studied by Yeung \cite{13} and Zhang and Yeung \cite{4,5}. The Information Theoretic Inequality Prover (ITIP) by Yeung and Yan \cite{6} is a program that is capable of performing inference on linear information inequalities (i.e., deducing whether an inequality follows from a list of other inequalities) via linear programming. For other programs based on this linear programming method, see \cite{43,44,45,46}. Refer to \cite{47,46} for recent advances on the linear programming algorithm, and \cite{16} for an overview. A symbolic approach to proving Shannon-type linear inequalities was studied in \cite{48}. Note that the inference problem of linear information inequality is a generalization of the inference problem of conditional independence, since conditional independence can be expressed using conditional mutual information.

While ITIP is strictly more powerful than the semi-graphoid axioms (it can also prove the example given in \cite{37}), it is still incomplete. It is only capable of proving Shannon-type inequalities, whereas non-Shannon-type inequalities were discovered in \cite{41,5,49,50,51,52,17}. Proofs of non-Shannon-type inequalities often invoke the copy lemma \cite{5,17}, and specialized algorithms employing the copy lemma has been used in \cite{52,17} to discover non-Shannon-type inequalities, and in \cite{13,14} for other purposes.

Programs for performing inference on information inequalities have found success in proving results on several settings in information theory (e.g. network coding \cite{2,8,9,10,11} and secret sharing \cite{12,13,14}). For information theory in general, ITIP is a convenient tool for manipulating expressions involving entropy and mutual information terms. While it is useful for intermediate steps, it is often incapable of performing the whole proof. For example, in network information theory \cite{18}, capacity regions are often stated in terms of auxiliary random variables, and the existence of auxiliary random variables cannot be stated in terms of information inequalities. The role of ITIP (and related programs) in proving these results is akin to a calculator for simple computations (e.g. for Fourier-Motzkin elimination in \cite{44}), but not a complete package.

While the algorithm in this paper is capable of proving a wide range of problems in information theory, for many of these classes of problems, it is fundamentally impossible to have a algorithm that can solve all problems within the class. For example, it was shown recently in \cite{41} that the problem of conditional information inequalities is undecidable, i.e., cannot be solved by an algorithm. Also see \cite{53,54} for earlier partial undecidability results. The decidability of unconditional information inequalities is still open \cite{55,56,57,16}.

\section*{Notations}

The set of nonnegative real numbers is denoted as $\mathbb{R}_{\geq 0}$. The set of positive integers is denoted as $\mathbb{N}$. Given propositions $P, Q$, the logical conjunction (i.e., AND) is denoted as $P \land Q$, and the material implication is denoted as $P \Rightarrow Q := (NOT P) \lor Q$. We write $[a..b] := \{ x \in \mathbb{Z} : a \leq x \leq b \}$, $[n] := \{1..n\}$. Given vectors $\mathbf{v}, \mathbf{w} \in \mathbb{R}^k$, $\mathbf{v} \succeq \mathbf{w}$ means $v_i \geq w_i$ for $i \in [k]$. We use the same notation for entrywise comparison for matrices. We write $X^\circ := (X_{a}, X_{a+1}, \ldots, X_{b})$, $X^\triangledown := X^\circ \circ$. For finite
set $\mathcal{S} \subseteq \mathbb{N}$, write $X_S := (X_{a_1}, \ldots, X_{a_k})$, where $a_1, \ldots, a_k$ are the elements of $\mathcal{S}$ in ascending order. For a statement $s$, its indicator function is written as $1(s)$ (which is 1 if $s$ holds, 0 otherwise).

We usually use $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}$ for matrices, $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$ for column vectors, $X, Y, Z, U, V, W$ for random variables, $R$ for real (non-random) variables, and $\mathcal{S}, \mathcal{T}$ for sets. We write $\mathbf{1}^{n \times k}$ for the $n \times k$ matrix with ones, and $0^{n \times k}$ for the $n \times k$ matrix with zeros. Write $\mathbf{I}_n$ for the $n \times n$ identity matrix. For a more compact notation, we sometimes write $[\mathbf{A}; \mathbf{B}] = \begin{bmatrix} \mathbf{A} \\ \mathbf{B} \end{bmatrix}$ for the vertical stacking of two matrices.

II. LINEAR PROGRAM

In this section, we review the linear programming approach in [6], [3]. For a sequence of random variables (or random sequence in short) $X^n = (X_1, \ldots, X_n)$, its entropic vector [4] is defined as $\mathbf{h}(X^n) = \mathbf{h} \in \mathbb{R}^{2^n-1}$, where the entries of $\mathbf{h}$ are indexed by nonempty subsets of $[n]$ (there are $2^n - 1$ such nonempty subsets$^4$, and $\mathbf{h}_S := H(X_S)$ (where $\mathcal{S} \subseteq [n]$) is the joint entropy of $\{X_i\}_{i \in \mathcal{S}}$. The entropic region [4] is defined as the region of entropic vectors

$$\Gamma^n := \bigcup_{p_{X^n}} \{ \mathbf{h}(X^n) \}$$

over all discrete joint distributions $p_{X^n}$. The entropic region is hard to characterize (the problem of characterizing $\Gamma^n$ for $n \geq 4$ is open). Therefore, for the purpose of automated verification of information-theoretic inequalities, we often focus on Shannon-type inequalities in the form $I(X; Y|Z) \geq 0$, or equivalently $H(X, Z) + H(Y, Z) - H(X, Y, Z) - H(Z) \geq 0$. The set of vectors satisfying Shannon-type inequalities is given by [5]

$$\Gamma^n := \left\{ \mathbf{k} \in \mathbb{R}^{2^n-1} : \forall \mathcal{S} \subseteq \mathcal{T} : \mathbf{k}_S \leq \mathbf{k}_T \\
\wedge \forall \mathcal{S}, \mathcal{T} : \mathbf{k}_S + \mathbf{k}_T \geq \mathbf{k}_{S \cup T} + \mathbf{k}_{S \cap T} \right\},$$

where we let $\mathbf{k}_0 = 0$. Note that $\Gamma^n$ is a convex polyhedral cone. The number of constraints in $\Gamma^n$ can be reduced by considering only the elemental inequalities [3]. While we have $\Gamma^n \subseteq \Gamma_n$, the inclusion is strict for $n \geq 3$ [4], [8].

The linear program in ITIP [6], [3] attempts to prove results in the form “for any $X^n$ such that $\mathbf{A}\mathbf{h}(X^n) \succeq 0$ holds, $\mathbf{B}\mathbf{h}(X^n) \succeq 0$ must hold”, i.e.,

$$\forall p_{X^n} : (\mathbf{A}\mathbf{h}(X^n) \succeq 0 \rightarrow \mathbf{B}\mathbf{h}(X^n) \succeq 0)$$

(1)

for $\mathbf{A} \in \mathbb{R}^{mA \times (2^n-1)}$, $\mathbf{B} \in \mathbb{R}^{mB \times (2^n-1)}$. Note that inequalities between (linear combinations of) entropy and mutual information terms involving $X_1, \ldots, X_n$ can always be expressed in the form $\mathbf{A}\mathbf{h}(X^n) \succeq 0$, since (conditional) mutual information can be written as a linear combination of entropy terms. We call (1) a conditional information inequality (CII). If $\mathbf{A}$ is an empty matrix (i.e., $m_A = 0$), then we also call (1) an unconditional information inequality (UII). ITIP tries to prove (1) by solving the linear program

minimize $\mathbf{b}^T \mathbf{k}$ subject to $\mathbf{k} \in \Gamma_n$, $\mathbf{A}\mathbf{k} \succeq 0$

(2)

for each row $\mathbf{b}^T$ of $\mathbf{B}$, and check whether the optimal values are non-negative for every row. If this holds, then the implication in (1) is true. A limitation of ITIP is that it can only verify Shannon-type inequalities (since it uses $\Gamma_n$ rather than $\Gamma^n$), i.e., it can only find the truth value of

$$\forall \mathbf{k} \in \Gamma_n : (\mathbf{A}\mathbf{k} \succeq 0 \rightarrow \mathbf{B}\mathbf{k} \succeq 0),$$

(3)

which is a sufficient, but not necessary condition for (1) to hold (i.e., it cannot prove that (1) does not hold). Examples of $\mathbf{A}, \mathbf{B}$ where (1) holds but (3) does not are called non-Shannon-type CIIs. A non-Shannon-type CII is given in [4], whereas non-Shannon-type UIIs are given in [5], [49], [50], [51], [52], [17].

Remark 1. Conventionally, in an information inequality (UII or CII), there is only one inequality in the consequence in (1), i.e., $\mathbf{B}$ has one row. It is indeed sufficient to consider the case where $\mathbf{B}$ has one row, since the case for multiple rows is equivalent to the conjunction of the CIIs for all rows (similar to (3)). However, in this paper, we consider the general case where $\mathbf{B}$ can have multiple rows, which is crucial to existential information inequalities defined in the next section.

$^4$The index $i \in \{1, \ldots, 2^n - 1\}$ in $\mathbf{h} \in \mathbb{R}^{2^n-1}$ corresponds to the subset $\mathcal{S} \subseteq [n]$ that is the binary representation of $i$, i.e., $i = \sum_{k \in \mathcal{S}} 2^{k-1}$. 

III. EXISTENTIAL INFORMATION INEQUALITIES

The random sequence $X^n$ in (1) is quantified with universal quantification (i.e., “∀”). In this section, we will extend (1) to incorporate existential quantification. Existential information inequalities (EIIs) are statements in the form:

$$\forall p_{X^n} : (A h(X^n) \geq 0 \rightarrow \exists p_{U^l | X^n} : B h(X^n, U^l) \geq 0),$$

(4)

where $U^l$ are called the auxiliary random variables (or “auxiliary” in short), and $p_{U^l | X^n}$ is their conditional distribution given $X^n$. Note that $h(X^n, U^l) \in \mathbb{R}^{2^n+1-1}$ is the entropic vector of the sequence of random variables $(X_1, \ldots, X_n, U_1, \ldots, U_l)$. Also note that $A \in \mathbb{R}^{m_A \times (2^n-1)}$, $B \in \mathbb{R}^{m_B \times (2^n+1-1)}$. For brevity, we will write (4) as

$$\forall X^n : (A h(X^n) \geq 0 \rightarrow \exists U^l : B h(X^n, U^l) \geq 0),$$

(5)

which reads as “for all $X^n$ satisfying $A h(X^n) \geq 0$, there exists $U^l$ satisfying $B h(X^n, U^l) \geq 0$”. When a new random variable is declared (e.g. “∀ $X^n$” and “∃ $U^l$”), it is assumed to be dependent on all previously declared random variables (e.g. they are shorthands for “∀ $p_{X^n}$” and “∃ $p_{U^l | X^n}$” respectively). The sets of UIIs, CIIs and EIIs satisfy the following inclusion: UIIs ⊆ CIIs ⊆ EIIs (EII is the most general since it reduces to CII when $l = 0$). As a corollary of the undecidability of CII shown recently in [41], EII (when $A$, $B$ have rational entries) is undecidable as well. Nevertheless, we will discuss several algorithms that can verify EIIs that arise from a wide range of problems.

There are several situations where EIIs arise. For example:

1) In network information theory, capacity regions are often stated in terms of auxiliary random variables (see various examples in [18]). We will show that a capacity region can often be expressed as an existential information predicate (defined in Section X). Both converse and achievability proofs can be performed under the framework of EII:

   a) (Converse). The most common technique for proving outer bounds and converse results pioneered by Gallager [34] involves writing the $n$-letter operational region of the coding setting, and identifying the auxiliary random variables (among the variables appearing in the operational region) so that the proposed outer bound is satisfied. The problem of finding auxiliaries satisfying some conditions can often be represented as an EII. Refer to Remark 4 for a discussion, Section XI-A for a demonstration on the degraded broadcast channel, and Section XI-B for an non-exhaustive list of results provable by the framework.

   b) (Achievability). Several general achievability results combining techniques such as superposition coding, simultaneous nonunique decoding and binning have been studied (e.g. [19], [20], [21]). A downside of these results is that the inner bounds they give are often complicated (contain more auxiliaries than needed), and need to be simplified manually to obtain the final inner bound. Moreover, these results include some parameters that are often chosen manually (e.g. which messages should a decoder decode uniquely and nonuniquely), and we often have to compare inner bounds obtained using different parameters in order to select the largest one. The framework in this paper provides a systematic method to simplify and compare rate regions. The PSITIP implementation allows automated discovery of inner bounds, using only the graphical representation of the network as input, via the inner bound in [21], [22] together with the simplification procedures in Section X.

2) Auxiliary random variables are used in the definitions of some information-theoretic quantities such as Wyner’s common information [58] (the $U$ is an auxiliary random variable)

$$J(X; Y) := \inf_{p_{U|X,Y} : I(U; X, Y) = 0} I(U; X, Y),$$

and Gács-Körner common information [59]

$$K(X; Y) := \sup_{p_{U|X,Y} : H(U|X,Y) = H(U|Y) = 0} H(U).$$

Other examples include common entropy [60], necessary conditional entropy [62], information bottleneck [63], privacy funnel [64], excess functional information [65], and other quantities studied in [66]. Properties of these quantities can often be stated as EIIs (see Example 3). In fact, for information quantities in the form

$$F(X^n) := \inf_{U^l : B h(X^n, U^l) \geq 0} b^l h(X^n, U^l),$$

(6)
where \( \mathbf{B} \in \mathbb{R}^{mn \times (2^n+1)} \) and \( \mathbf{b} \in \mathbb{R}^{2^n+1} \) (which includes all aforementioned information quantities; \( \sup \) can be expressed as \( \inf \) by flipping the sign), any linear inequality among such information quantities can be expressed as an EII. This can be shown by observing

\[
\forall X^n : \left( \mathbf{A}h(X^n) \geq 0 \Rightarrow \inf_{U : \mathbf{B}h(X^n, U^n) \geq 0} \mathbf{b}^T \mathbf{h}(X^n, U^n) \geq \inf_{V : \mathbf{C}h(X^n, V^n) \geq 0} \mathbf{c}^T \mathbf{h}(X^n, V^n) \right) \]

\[
\Leftrightarrow \forall X^n U^n : \left( \mathbf{A}h(X^n) \geq 0 \land \mathbf{B}h(X^n, U^n) \geq 0 \Rightarrow \exists V^k : \mathbf{C}h(X^n, V^k) \geq 0 \land \mathbf{b}^T \mathbf{h}(X^n, U^n) \geq \mathbf{c}^T \mathbf{h}(X^n, V^k) \right) \]

Note that linear inequalities among more than two information quantities (in the form (3)) can also be stated in this form, since we can combine all positive terms into one, and all negative terms into one, resulting in at most two information quantities. Therefore, by employing EII, we can handle linear inequalities not only among entropy and mutual information terms, but also on more general information quantities.

3) In the study of the entropic region \( \Gamma_n^* \), outer bounds (e.g., \( \Gamma_n \) and various non-Shannon-type inequalities) can be stated as UIIs. Nevertheless, inner bounds of \( \Gamma_n^* \) (e.g., [67], [68], [69]) cannot be stated as UIIs, though they can be stated as EIs. For example, the fact that \( \Gamma_2 \) is an inner bound of \( \Gamma_2^* \) (actually \( \Gamma_2 = \Gamma_2^* \) as shown in [3]) can be expressed as

\[
\forall R_1, R_2, R_3 \geq 0 : \left( R_1 \leq R_3 \land R_2 \leq R_3 \land R_3 \leq R_1 + R_2 \Rightarrow \exists U^2 : H(U_1) = R_1 \land H(U_2) = R_2 \land H(U_1, U_2) = R_3 \right). \]

Note that \( R_1, R_2, R_3 \) are real-valued non-random variables. Refer to Remark 3 for how to represent them in an EII. Combine this with the fact that the outer bound \( \Gamma_2^* \) can be expressed as a UII, the statement \( \Gamma_2 = \Gamma_2^* \) can also be expressed as an EII.

4) The copy lemma [5], [17], a useful tool for proving non-Shannon-type inequalities, can be stated as an EII. Refer to Section VII-A for details.

The following example demonstrates the usage of EIs.

**Example 2.** Consider the tensorization property of Wyner’s common information, i.e., \( (X_1, Y_1) \perp (X_2, Y_2) \) implies \( J(X_1, X_2; Y_1, Y_2) = J(X_1; Y_1) + J(X_2; Y_2) \). We can prove inequalities in both directions:

1) \( J(X_1, X_2; Y_1, Y_2) \geq J(X_1; Y_1) + J(X_2; Y_2) \) is equivalent to the EII

\[
\forall X^2, Y^2, V : \left( I(X_1, Y_1; X_2, Y_2) = I(X_1, X_2; Y_1, Y_2 | V) = 0 \Rightarrow \exists U^2 : I(X_1; Y_1 | U_1) = I(X_2; Y_2 | U_2) = 0 \land I(V; X^2, Y^2) \geq I(U_1; X_1, Y_1) + I(U_2; X_2, Y_2) \right). \]

This can be proved by identifying \( U_1 = U_2 = V \).

2) \( J(X_1, X_2; Y_1, Y_2) \leq J(X_1; Y_1) + J(X_2; Y_2) \) is equivalent to the EII

\[
\forall X^2, Y^2, U^2 : \left( I(X_1, Y_1; U_1; X_2, Y_2, U_2) = I(X_1; Y_1 | U_1) = I(X_2; Y_2 | U_2) = 0 \Rightarrow \exists V : I(X^2, Y^2 | V) = 0 \land I(V; X^2, Y^2) \leq I(U_1; X_1, Y_1) + I(U_2; X_2, Y_2) \right). \]

This can be proved by identifying \( V = (U_1, U_2) \) (the joint random variable). We can assume \( I(X_1, Y_1; U_1; X_2, Y_2, U_2) = 0 \) since \( J(X_1; Y_1) \) and \( J(X_2; Y_2) \) can be optimized separately. If we do not make this assumption (i.e., we only have \( I(X_1, Y_1; X_2, Y_2) = 0 \)), then the EII can still be proved using the conditional independence rule in Section VII-A.

**Remark 3.** In (5), the variables \( X_i, U_i \) are random variables. We may also want to introduce real-valued non-random variables (real variables in short), e.g., rates in a coding setting. For example, we have \( \forall R \geq 0 : \exists U : H(U) = R/2 \) (i.e., for any real number \( R \geq 0 \), there exists random variable \( U \) with \( H(U) = R/2 \)). To represent this statement in the form (5), we replace each nonnegative real variable \( R \) by \( H(X) \) for a new random variable \( X \), i.e., it becomes \( \forall X : \exists U : H(U) = H(X)/2 \). In
case $R$ is not constrained to be nonnegative, replace $R$ by $H(X_1) - H(X_2)$ for two new random variables $X_1, X_2$. This way we can transform a statement involving real variables into an equivalent EII. Nevertheless, since the dimension of the linear program is exponential in the number of random variables, this method is inefficient in practice. In the PSITIP implementation, real variables are represented separately.

Remark 4. We discuss a method to state a converse result in network information theory as an EII, which is basically Gallager’s approach [33]. For each sequence of random variables $X^n$ in the $n$-letter operational setting, define a group of three random variables $X^n Q^Q Q^{-1}, X^n Q^Q_{t+1}$, representing the present, past and future respectively, where $Q \sim \text{Unif}[n]$ is the time sharing random variable. The causal relations between the random variables and the decoding requirements are included in the EII.

Csiszár sum identity [70], [71] applied on each pair of those groups is added to the condition in the EII. This way, the number of variables in the EII is only $3 \times$ the number of variables in the 1-letter operational setting, and does not grow with $n$. This strategy suffices for most of the converse and outer bound proofs in [18] (some results that have been verified by PSITIP are listed in Section XI).

We will not only discuss an algorithm that proves a given outer bound, but also an algorithm that automatically produces an outer bound given a description of the coding setting. The aforementioned method, which considers the present, past and future versions of each random sequence (where the past and future are regarded as auxiliaries), creates too many auxiliaries, making the resultant outer bound complicated and hard to interpret. To produce a satisfactory outer bound, we describe algorithms that reduce the number of auxiliaries. The algorithms basically automate the process of identifying new auxiliaries as combinations of existing auxiliaries (e.g. past and future variables), which was performed manually in conventional converse proofs. They are discussed in Section X.

Remark 5. The PSITIP implementation is capable of handling arbitrary first-order statement on entropy, i.e., composition of linear inequalities on entropy and mutual information, existential and universal quantification of random variables, AND, OR and NOT operators (it can handle statements with an arbitrary sequence of “$\lor$” and “$\exists$”, not only that in the form of (4)), though we will not discuss this here for simplicity.

IV. Substitution Operation

As demonstrated in Example 2 to prove an EII, we can identify the auxiliary random variables using the non-auxiliary random variables. To prove 4, a sufficient condition is to identify $S_1, S_2, \ldots, S_l \subseteq [n]$ such that when we substitute $U_l = X S_i$, we have

$$A h(X^n) \geq 0 \rightarrow B h(X^n, U_l) \geq 0,$$ (7)

which can be checked using linear programming after we fix $S_1, \ldots, S_l$.

A way of substituting the auxiliaries $U_l$ by the variables $X^n$ can be specified by a sequence of sets $S_1, S_2, \ldots, S_l \subseteq [n]$, where we substitute $U_l = X S_i$. We call $(S_1, S_2, \ldots, S_l)$ a substitution combination. We introduce a more compact way to represent a substitution combination using a matrix $S \in \mathbb{R}_{\geq 0}^{l \times n}$ of nonnegative entries, where

$$S_{i,j} = \begin{cases} 1 & \text{if } j \in S_i \\ 0 & \text{if } j \not\in S_i. \end{cases}$$

We call $S$ the substitution matrix corresponding to $(S_1, S_2, \ldots, S_l)$. Given a random sequence $X^n$ and a matrix $S \in \mathbb{R}_{\geq 0}^{l \times n}$, we write $S \circ X^n$ for the random sequence with length $l$, where the $i$-th entry is

$$(S \circ X^n)_i := X_{\{j \in [n] : S_{i,j} \neq 0 \}}.$$

We call “$\circ$” the substitution operation. If $S$ is the substitution matrix corresponding to $(S_1, S_2, \ldots, S_l)$, then $S \circ X^n$ is precisely the sequence $U_l$ where $U_l = X S_i$. Note that the substitution operation does not require the matrix to have $\{0,1\}$ entries (it only requires nonnegative entries), and it only depends on the positions of the nonzero entries in $S$, not their precise values. We can show that the substitution operation satisfies the following associativity law.

Proposition 6 (Associativity of substitution operation). For any random sequence $X^n$, and matrices $S \in \mathbb{R}_{\geq 0}^{l \times n}, T \in \mathbb{R}_{\geq 0}^{m \times l}$, we have

$$(T S) \circ X^n \triangleq T \circ (S \circ X^n),$$

where $Y^m \triangleq Z^m$ means $H(Y_i|Z_i) = H(Z_i|Y_i) = 0$ for $i \in [m]$, i.e., the two random sequences are informationally equivalent.

Note that even though $U_l$ can be joint random variables, it is considered as one random variable in $h(X^n, U^l)$, and we still have $h(X^n, U^l) \in 2^{n+l} - 1$. Nevertheless, since $X^n$ is the only randomness $(U_l$ are functions of $X^n$), each entry of $h(X^n, U^l)$ is also an entry of $h(X^n)$. 
Proof: This can be shown by
\[(TS) \circ X^n = X_{j \in [n]: (TS)_{i,j} \neq 0}\]
\[= X_{j \in [n]: \sum_{k \in [l]} T_{i,k} S_{k,j} \neq 0}\]
\[= X_{j \in [n]: \exists k \in [l]: T_{i,k} S_{k,j} \neq 0}\]
\[= U_{k \in [l]: T_{i,k} \neq 0},\]
where \(U_k := X_{j \in [n]: s_{k,j} \neq 0}\).

We now describe how the substitution operation affects the entropic vector. Given a matrix \(S \in \mathbb{R}^{n \times n}_{\geq 0}\), our goal is to find a matrix \(\text{sub}(S) \in \{0, 1\}^{(2^k-1) \times (2^n-1)}\), called the entropic substitution matrix of \(S\), such that for any random sequence \(X^n\), we have
\[h(S \circ X^n) = \text{sub}(S)h(X^n).\]

This is possible since each entry of \(h(S \circ X^n)\) is also an entry of \(h(X^n)\). More precisely, the \(T\)-th entry of \(h(S \circ X^n)\) (where \(T \subseteq [l]\)) is
\[H((S \circ X^n)_T) = H(X_{j \in T \{j \in [n]: s_{i,j} \neq 0\}}) = H(X_{j \in [n]: \sum_{i \in T} s_{i,j} \neq 0}).\]

Hence, we can take
\[\text{sub}(S)_{T, U} := \begin{cases} 1 & \text{if } U = \{j \in [n] : \sum_{i \in T} S_{i,j} \neq 0\} \\ 0 & \text{if } U \neq \{j \in [n] : \sum_{i \in T} S_{i,j} \neq 0\} \end{cases}\]
for nonempty \(T \subseteq [l]\), \(U \subseteq [n]\). Note that \(\text{sub}(S)_{T, U}\) denotes the entry of \(\text{sub}(S)\) in the \(T\)-th row and the \(U\)-th column (like entropic vectors, the rows and columns of this matrix are also indexed by nonempty subsets).

By the associativity of the substitution operation, the function \(\text{sub}\) is multiplicativce.

**Proposition 7** (Multiplicativity of entropic substitution matrix). For any matrices \(S \in \mathbb{R}^{n \times n}_{\geq 0}, T \in \mathbb{R}^{m \times l}_{\geq 0}\), we have
\[\text{sub}(T)\text{sub}(S) = \text{sub}(TS).\]

Proof: This can be shown by noting that for any \(X^n\),
\[\text{sub}(TS)h(X^n) = h((TS) \circ X^n) = h(T \circ (S \circ X^n)) = \text{sub}(T)h(S \circ X^n) = \text{sub}(T)\text{sub}(S)h(X^n),\]
and that \(\text{span}\left(\bigcup_{p \in [n]} \{h(X^n)\}\right) = \mathbb{R}^{2^n-1}\) by considering \(U \sim \text{Bern}(1/2), X_i = U\) if \(i \in S, X_i = 0\) if \(i \notin S\), where each choice of \(S \subseteq [n], S \neq \emptyset\) gives a linearly independent \(h(X^n)\).

For the substitution performed in (7), letting \(S \in \mathbb{R}^{l \times n}_{\geq 0}\) be the substitution matrix corresponding to \((S_1, S_2, \ldots, S_l)\), we take \(C = \text{Bsub}(I_n; S)\), where we write
\[\text{sub}(I_n; S) = \text{sub}([I_n; S]) = \text{sub}\left(\begin{bmatrix} I_n \\ S \end{bmatrix}\right)\]
for a more compact notation. When \(U_i = X_S, i\), we have \([I_n; S \circ X^n = (X^n, U^l)\) where \((X^n, U^l)\) is a random sequence with length \(n + l\) and hence \(Bh(X^n, U^l) = Ch(X^n)\). We can then verify (7) by checking \(Ah(X^n) \geq 0 \Rightarrow Ch(X^n) \geq 0\) using the linear program in (2). Hence, a method to verify an EII is to find \(S \in \{0, 1\}^{l \times n}\) such that the linear program succeeds in checking \(Ah(X^n) \geq 0 \Rightarrow Ch(X^n) \geq 0\). The class of EIs that can be proved this way can be defined formally as follows.

**Definition 8.** An EII \(X^n: (Ah(X^n) \geq 0) \Rightarrow \exists U^l: Bh(X^n, U^l) \geq 0\) is called a **trivial EII** if there exists a substitution matrix \(S \in \{0, 1\}^{l \times n}\) such that the following implication holds:
\[\forall k \in \Gamma_n: (Ak \geq 0 \Rightarrow \text{Bsub}(I_n; S)k \geq 0),\]
which can be checked using the linear program in (2).

Even though trivial EIs are trivial in a sense, searching for \(S \in \{0, 1\}^{l \times n}\) (with \(2^{nl}\) different choices) is still a computationally intensive task. An algorithm for finding \(S \in \{0, 1\}^{l \times n}\) using some pruning techniques will be discussed in Section (V).
V. Verification of EII via Auxiliary Searching

In this section, we present an algorithm for verifying an EII in the form (5), that is capable of verifying any trivial EII (9). A simple brute-force algorithm is to identify $U_i = X_{S_i}$, and exhaust all possible choices of $S_1, S_2, \ldots, S_i \subseteq [n]$, or equivalently exhaust all possible substitution matrices $S \in \{0, 1\}^{l \times n}$, which can be computationally prohibitive. We discuss methods to reduce the search space. The algorithm is divided into two steps: the sandwich procedure, and exhaustion with cached linear program optimization.

A. Sandwich Procedure

For $b \in \mathbb{R}^{2^n-1}$, we say that $b^T h(X^n)$ is increasing in $X_i$ ($i \in [n]$) if

$$\forall X^n, Y : (b^T h(X^n) \leq b^T h(X^{i-1}, (X_i, Y), X^{n+1})) .$$

The case for decreasing is defined similarly (with “$\leq$” replaced with “$\geq$”). There are several ways of checking whether $b^T h(X^n)$ is increasing/decreasing in $X_i$. One way is to check the above condition (which is an UII in the form of (1)) using the linear program (4). We call $b^T h(X^n)$ trivially increasing in $X_i$, if it can be verified this way. In case the expression $b^T h(X^n)$ is given in the form of a linear combination of entropy and mutual information terms, we can also make use of simple rules like $H(X|Z)$ is increasing in $X$ and decreasing in $Z$, and $I(X; Y|Z)$ is increasing in $X$ and $Y$ (but neither increasing nor decreasing in $Z$), to deduce whether the expression is increasing or decreasing.

If $b^T h(X^n, U^l)$ is trivially increasing in $U_i$, then in order for the linear program (2) to be able to verify the CII $Ah(X^n) \succeq 0 \rightarrow b^T h(X^n, U^l) \succeq 0$ for any choice of $\{S_i\}_i$ (recall that we try to identify $U_i = X_{S_i}$), it is necessary that (2) can verify the CII when $S_i = [n]$, which gives the largest $b^T h(X^n, U^l)$. By extension, in order for (2) to be able to verify the CII for any choice of $\{S_i\}_i$, satisfying $1 \not\in S_i$, it is necessary that (2) can verify the CII when $S_i = [n]\{1\}$ and $S_i = [n]$ for $i \neq 1$. Taking the contrapositive, if (2) fails to verify the CII for this choice of $\{S_i\}_i$, then we can assume $1 \in S_1$, giving a lower bound $\{1\}$ on $S_1$ (with respect to set inclusion, i.e., $\{1\} \subseteq S_1$).

If $b^T h(X^n, U^l)$ is trivially increasing for some $U_i$, and trivially decreasing for some other $U_i$, then the most conservative choice will be $S_i = [n]$ for $U_i$ increasing, $S_i = \emptyset$ for $U_i$ decreasing. If we know beforehand that $S_i \subseteq S_i \subseteq \overline{S}_i$ for $i \in [l]$, where $\underline{S}_i \subseteq \overline{S}_i \subseteq \overline{S}_i$ are lower and upper bounds for $S_i$ respectively, then the most conservative choice will be $S_i = \overline{S}_i$ for $U_i$ increasing, $S_i = \underline{S}_i$ for $U_i$ decreasing. In terms of the substitution matrix $S \in \{0, 1\}^{l \times n}$, if we know beforehand that $\underline{S} \leq S \leq \overline{S}$, where $\underline{S}, \overline{S} \in \{0, 1\}^{l \times n}$, then the most conservative choice for the substitution matrix is

$$\text{diag}\left(\frac{1-g}{2}\right)\underline{S} + \text{diag}\left(\frac{1+g}{2}\right)\overline{S},$$

where $g \in \mathbb{R}^l$ with $g_i = 1$ if $b^T h(X^n, U^l)$ is trivially increasing for $U_i$, and $g_i = -1$ if trivially decreasing (if it is not trivially increasing or decreasing for some $i$, we ignore this row $b^T$ in the sandwich procedure). Therefore, we can repeat this procedure for each row $b^T$ of $B$ to refine the lower and upper bounds on $S$, reducing the search space for $S$. The description of the algorithm for the EII (5) is given in Algorithm 1.
Algorithm 1 **SANDWICH(A, B, S, S)**

Input: \( A \in \mathbb{R}^{m_A \times (2^n-1)}, B \in \mathbb{R}^{m_B \times (2^{n+1}-1)} \), initial values of \( S, \overline{S} \in \{0,1\}^{1 \times n} \) (lower and upper bound of \( S \) respectively, default values are \( S = 0^{1 \times n}, \overline{S} = 1^{1 \times n} \))

Output: final values of \( S, \overline{S} \in \{0,1\}^{1 \times n} \) or failure

1: repeat
2: for row \( b^T \) of \( B \) do
3: for \( i \in [n] \) do
4: \( g_i \leftarrow \begin{cases} 
1 & \text{if } b^T h(X^n, U^l) \text{ triv. increasing in } U_i \\
-1 & \text{else if triv. decreasing in } U_i \\
0 & \text{otherwise}
\end{cases} \)
5: if \( g_i = 0 \), then skip row \( b^T \) and continue to next row
6: end for
7: \( S \leftarrow \text{diag} \left( \frac{1-g_i}{g_i} \right) S + \text{diag} \left( \frac{1-g_i}{g_i} \overline{S} \right) \)
8: \( c^T \leftarrow b^T_{\text{sub}}(I_i; S) \)
9: if CI \( \forall X^n : (Ah(X^n)) \geq 0 \rightarrow c^T h(X^n) \geq 0 \)
10: cannot be verified by (\text{2}), then return failure
11: (note that \( c^T h(X^n) = b^T h(X^n, U^l) \) when \( U^l = S \circ X^n \); see (\text{3}))
12: for \( i \in [l], j \in [n] \) where \( S_{i,j} < \overline{S}_{i,j} \) do
13: \( S \leftarrow S \)
14: \( S_{i,j} \leftarrow 1 - S_{i,j} \)
15: \( c^T \leftarrow b^T_{\text{sub}}(I_i; S) \)
16: if CI \( \forall X^n : (Ah(X^n)) \geq 0 \rightarrow c^T h(X^n) \geq 0 \)
17: cannot be verified by (\text{3}), then
18: set \( S_{i,j} \leftarrow S_{i,j}, \overline{S}_{i,j} \leftarrow S_{i,j} \)
19: end for
20: end for
21: until no further changes are made to \( S, \overline{S} \)
22: return \( S, \overline{S} \)

Step [7] sets each auxiliary to the most conservative choice (set it to the largest if \( b^T h(X^n, U^l) \) is increasing, or smallest if decreasing). If \( b^T h(X^n, U^l) \geq 0 \) cannot be verified even for the most conservative choices, then it cannot possibly be verified by this algorithm. In Step [12] we set the auxiliaries to the most conservative choice except for one position. If this change causes \( b^T h(X^n, U^l) \geq 0 \) to fail to be verified, then we know that position must be included if the change is to exclude it, or that position must be excluded if the change is to include it, i.e., the choice of whether the position is included must be fixed to its most conservative choice.

**Remark 9.** We say that \( b^T h(X^n) \) is *conditionally increasing* in \( X_i \ (i \in [n]) \) given \( Ah(X^n) \geq 0 \) if
\[
\forall X^n, Y : (Ah(X^n)) \geq 0 \wedge Ah(X^{i-1}, (X_i, Y), X^n_i) \geq 0 \\
\rightarrow b^T h(X^n) \leq b^T h(\hat{X}^{i-1}, (X_i, Y), X^n_i)
\]
We may use conditional increasing/decreasing given \( Ah(X^n) \geq 0 \) instead in the sandwich procedure to slightly strengthen the result.

**B. Cached Linear Program**

After the sandwich procedure, we have to identify \( S \in \{0,1\}^{1 \times n} \) with \( S \preceq S \preceq \overline{S} \), such that \( Ah(X^n) \geq 0 \rightarrow Bh(X^n, U^l) \geq 0 \) can be verified when \( U^l = S \circ X^n \). One method is to exhaust all \( 2^{\sum_{i,j} (S_{i,j} - \overline{S}_{i,j})} \) possible choices of \( S \), and solve the linear program (2) for each combination. We describe a way to skip the linear program for some of the combinations in Algorithm [2]. The main idea is that for \( S \in \{0,1\}^{1 \times n} \), if the linear program (2) fails to verify \( Ah(X^n) \geq 0 \rightarrow Bh(X^n, U^l) \geq 0 \) after substituting \( U^l = S \circ X^n \) (i.e., there exists a row \( b \) and a vector \( k \in \Gamma_n \) that gives a negative \( b^T k \) in (2), we call \( k \) a *witness of failure* of \( S \), then we not only know that \( U^l = S \circ X^n \) is not a solution to the EII, but we also guess that the same \( k \) may also be a witness of failure of another substitution matrix.
In Algorithm 2 if the linear program gives a negative optimal value, then the witness of failure $k$ is stored, and can be used to quickly check the linear program of another substitution matrix before running a full linear programming algorithm. In Step 10 the constraint $k_{2^n-1} \leq 1$ (meaning $H(X^n) \leq 1$) is to ensure that the linear program is bounded, so we can obtain the optimal value of $k$. This caching optimization gives a significant speedup in the PSITIP implementation. Note that this optimization can also be applied to the linear programming steps in the sandwich procedure.

Another optimization is to use the increasing/decreasing information (computed in the sandwich procedure) to skip some substitution matrices if a better combination has failed. We will not discuss this optimization in this paper.

VI. Inference Rules

For brevity, we will write the EI in (5) as $A \xrightarrow{EI} B$. Note that $n, l$ can be deduced from the widths of $A$ and $B$. In the following sections, we will discuss several inference rules, which are rules that allow us to deduce new EIs from existing EIs. We first discuss some simple rules, which we call elementary rules. We switch between the notation in (5) and the “$A \xrightarrow{EI} B$” notation depending on which is simpler. Also recall that the index $i \in \{1, \ldots, 2^n-1\}$ in $h \in \mathbb{R}^{2^n-1}$ corresponds to the subset $S \subseteq [n]$ that is the binary representation of $i$, i.e., $i = \sum_{k \in S} 2^{k-1}$.

Proposition 10 (Elementary rules). For $A \in \mathbb{R}^{m_A \times (2^n-1)}$, $B \in \mathbb{R}^{m_B \times (2^{n+l}-1)}$, $C \in \mathbb{R}^{m_C \times (2^{n+i+k}-1)}$, $n, l, k \geq 0$, we have

1) Shannon-type inequality (Sha). $0^{0 \times (2^l-1)} \xrightarrow{EI} [0, 0, 0, -1, 1, 1, -1]$. Equivalently, $\forall X, Y, Z : \Gamma(X; Y|Z) \geq 0$.

2) Substitution of variables (SubV). For any $n' \in \mathbb{N}$, $S \in \{0,1\}^{n \times n}$,

$$A \xrightarrow{EI} B \Rightarrow A_{\text{sub}(S)} \xrightarrow{EI} B_{\text{sub}} \left( \begin{bmatrix} S & 0^{n \times l} \\ 0^{l \times n} & I_l \end{bmatrix} \right).$$

Equivalently,

$$\forall X^n : (Ah(X^n) \geq 0 \Rightarrow \exists U^l : Bh(X^n, U^l) \geq 0)$$

$$\Rightarrow \forall Y^n : \left( Ah(S \circ Y^n) \geq 0 \Rightarrow \exists U^l : Bh(S \circ Y^n, U^l) \geq 0 \right).$$

This corresponds to substituting the variables $X^n = S \circ Y^n$.

3) Substitution of auxiliaries (SubA). For any $l' \in \mathbb{N}$, $S \in \{0,1\}^{l \times (n+l')}$,

$$A \xrightarrow{EI} B_{\text{sub}} \left( \begin{bmatrix} I_n & 0^{l \times n} \\ S & 0^{n \times l'} \end{bmatrix} \right) \Rightarrow A \xrightarrow{EI} B.$$
where \((X^n, V^l) = (X_1, \ldots, X_n, V_1, \ldots, V_l)\) is a random sequence with length \(n + l'\). This corresponds to substituting the auxiliaries \(U^l = S \circ (X^n, V^l)\).

4) Conic combination (Cone). Let \(D \in \mathbb{R}_{\geq 0}^{m_A \times m_A}\) (\(D\) has nonnegative entries). We have

\[
A^{EII} \Rightarrow DA.
\]

5) Transitivity (Tran).

\[
(A \Rightarrow B \land B \Rightarrow C) \Rightarrow A \Rightarrow C.
\]

Equivalently,

\[
\forall X^n : (Ah(X^n) \geq 0 \Rightarrow \exists U^l : Bh(X^n, U^l) \geq 0)
\]

\[
\land \forall Y^{n+l} : (Bh(Y^{n+l}) \geq 0 \Rightarrow \exists V^k : Ch(Y^{n+l}, V^k) \geq 0)
\]

\[
\Rightarrow \forall Z^n : (Ah(Z^n) \geq 0 \Rightarrow \exists W^{l+k} : Ch(X^n, U^l, W^{l+k}) \geq 0),
\]

which can be seen simply by substituting \(X^n = Z^n, Y^{n+l} = (X^n, U^l)\) and \(W^{l+k} = (U^l, V^k)\). Note that transitivity together with the conic combination rule implies that \(a_{EII}^{EII}\) is a preorder.

6) Absorption (Abs).

\[
A \Rightarrow B \Rightarrow A \Rightarrow \begin{bmatrix} B \\ A \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix},
\]

where the block matrix on the right hand side is formed by vertically stacking \(B\) and \(A\) (padded with zeros on the right to make the width the same as the width of \(B\)). Note that \([A \mid 0]h(X^n, U^l) = Ah(X^n)\) since \(h(X^n)\) is the first \(2^n - 1\) entries of \(h(X^n, U^l)\). Equivalently,

\[
\forall X^n : (Ah(X^n) \geq 0 \Rightarrow \exists U^l : Bh(X^n, U^l) \geq 0)
\]

\[
\Rightarrow \forall X^n : (Ah(X^n) \geq 0 \Rightarrow \exists U^l : Bh(X^n, U^l) \geq 0 \land Ah(X^n) \geq 0).
\]

We will refer the rules by the abbreviations in parentheses. We remark that these rules are meant to be computer verifiable, so it is necessary to specify the rules (even the logically trivial ones) precisely. We also want to find a minimal set of rules that is sufficient to prove the results in this paper, so redundant rules (that can be deduced from other rules) are not included.

For example, the following rule can be deduced from the elementary rules: For \(A \in \mathbb{R}^{m_A \times (2^n - 1)}, B \in \mathbb{R}^{m_B \times (2^n - 1)}, C \in \mathbb{R}^{m_C \times (2^n - 1)},\) we have

\[
(A \Rightarrow B \land A \Rightarrow C) \Rightarrow A \Rightarrow \begin{bmatrix} B \\ C \end{bmatrix}.
\]

To show this,

\[
A \Rightarrow B \land A \Rightarrow C
\]

\[
\Rightarrow A \Rightarrow B \land \begin{bmatrix} B \\ A \end{bmatrix} \Rightarrow A \land A \Rightarrow C \quad \text{(Cone)}
\]

\[
\Rightarrow A \Rightarrow B \land \begin{bmatrix} B \\ A \end{bmatrix} \Rightarrow C \quad \text{(Tran)}
\]

\[
\Rightarrow A \Rightarrow \begin{bmatrix} B \\ A \end{bmatrix} \land \begin{bmatrix} B \\ A \end{bmatrix} \Rightarrow \begin{bmatrix} C \\ B \end{bmatrix} \quad \text{(Abs)}
\]

\[
\Rightarrow A \Rightarrow \begin{bmatrix} C \\ B \end{bmatrix} \quad \text{(Tran)}
\]

\[
\Rightarrow A \Rightarrow \begin{bmatrix} C \\ B \end{bmatrix} \land \begin{bmatrix} C \\ B \end{bmatrix} \Rightarrow \begin{bmatrix} B \\ C \end{bmatrix} \quad \text{(Cone)}
\]

\[
\Rightarrow A \Rightarrow \begin{bmatrix} B \\ C \end{bmatrix} \quad \text{(Tran)}.
\]

We may regard the elementary rules in Proposition[10] as the axioms of EII. Nevertheless, this set of axioms is not complete. We will show that the EII that can be verified by Algorithm[2] are precisely the EII that can be deduced using the elementary rules, i.e., an EII is trivial if and only if it can be deduced using the elementary rules. The proof of the following theorem is in Appendix[1]

**Theorem 11.** For an EII \(A \Rightarrow B\), the following are equivalent:
1) It can be deduced using the elementary rules.
2) \( A \xrightarrow{EI} B \) is trivial \(^3\) (it can be verified by Algorithm \(^2\)).
3) There exists \( S \in \{0, 1\}^{n \times n} \) and a matrix \( D \) with nonnegative entries such that
   \[
   B_{\text{sub}}(I_n; S) = D \begin{bmatrix} A \\ \Gamma_n \end{bmatrix},
   \]
   where \( \Gamma_n \) is a matrix such that \( \Gamma_n = \{k \in \mathbb{R}^{2n-1} : \Gamma_n k \geq 0 \} \) (i.e., the rows of \( \Gamma_n \) correspond to the elementary inequalities for Shannon-type inequalities).

Note that the only rule that introduces new auxiliary random variables is the substitution of auxiliaries rule, and hence these rules only allow us to identify auxiliary random variables that are subsets of \( X^n \). The elementary rules alone cannot infer any of the nontrivial EII (e.g. non-Shannon inequalities and the EII in the next section), and hence they are incomplete if viewed as axioms. The reason we discuss the elementary rules is that, while they are not powerful enough to derive nontrivial EII on their own, if we assume some nontrivial EII as premises (i.e., introduce more axioms), then the elementary rules can help us derive more nontrivial EII. This will be discussed in the following sections.

VII. NONTRIVIAL EXISTENTIAL INFORMATION INEQUALITIES

A. Copy Lemma

We state the copy lemma \(^5\), \(^17\) as an EII.

**Proposition 12** (Copy lemma \(^5\), \(^17\)). For any \( n, l \geq 0 \),

\[
\forall X^n, Y^l : \left( \exists U^l : I(U^l; Y^l|X^n) = 0 \land h(X^n, U^l) = h(X^n, Y^l) \right).
\]

Note that we only enforce equalities of entropy terms here instead of equality in distribution in the original copy lemma, resulting in a weaker statement.

Even though there are non-Shannon-type inequalities that cannot be proved using the linear program in \(^5\), all known unconditional non-Shannon-type inequalities (e.g. \(^30\), \(^32\), \(^17\)) can be proved using the elementary rules and the copy lemma. We remark that automated application of the copy lemma has been used in \(^32\), \(^17\) to discover non-Shannon-type inequalities. The framework in this paper is more general in the sense that it can incorporate any nontrivial EII, not only the copy lemma. Refer to Section VIII for the algorithm which incorporates known EII in the verification of new EII.

The copy lemma can be equivalently stated as the following rule, which we call the conditional independence rule. Intuitively, the rule states that if the auxiliaries \( U^l \) only interact with \( X^n \), then \( U^l \) can be assumed to be conditionally independent of all other random variables given \( X^n \). We remark that a similar observation was made in \(^72\).

**Proposition 13** (Conditional independence rule). The copy lemma is equivalent to the following rule:

\[
\forall X^n, Y^k : \left( Ah(X^n, Y^k) \geq 0 \rightarrow \exists U^l : Bh(X^n, U^l) + Ch(X^n, Y^k) \geq 0 \right) \Rightarrow \forall X^n, Y^k : \left( Ah(X^n, Y^k) \geq 0 \rightarrow \exists U^l : Bh(X^n, U^l) + Ch(X^n, Y^k) \geq 0 \land I(U^l; Y^k|X^n) = 0 \right).
\]

**Proof:** Assume the copy lemma is true. Substituting \( X^n \leftarrow X^n, Y^l \leftarrow (U^l, Y) \) to the copy lemma,

\[
\forall X^n, U^l, Y : \left( \exists V^l, Z : I(V^l, Z; U^l, Y|X^n) = 0 \land h(X^n, V^l, Z) = h(X^n, U^l, Y) \right) \Rightarrow \forall X^n, U^l, Y : \left( Bh(X^n, U^l) + Ch(X^n, Y^k) \geq 0 \rightarrow \exists V^l, Z : Bh(X^n, U^l) + Ch(X^n, Y^k) \geq 0 \land I(V^l, Z; U^l, Y|X^n) = 0 \land h(X^n, V^l, Z) = h(X^n, U^l, Y) \right). \quad \text{(11)}
\]
where the implication is by applying transitivity with $\mathbf{B}h(X^n, U^l) + \mathbf{C}h(X^n, Y^k) \geq 0 \rightarrow \text{True}$ (by conic combination), and then by absorption. We have

$$
\forall X^n, Y^k : \left( \mathbf{A}h(X^n, Y^k) \geq 0 \rightarrow \\
\exists U^l : \mathbf{B}h(X^n, U^l) + \mathbf{C}h(X^n, Y^k) \geq 0 \right)
$$

by (11) and transitivity.

For the other direction, assume the conditional independence rule is true. We have

$$
\forall Y^l : \left( \exists U^l : \left\{ \text{Join, Tran} \right\} \right)
$$

$$
\Rightarrow \forall X^n, Y^l : \left( \exists U^l : \left\{ \text{Elim} \right\} \right)
$$

where the last implication is by the conditional independence rule.

\[ \blacksquare \]

### B. Functional Representation Lemma

We state the functional representation lemma \[18\] as an EII.

**Proposition 14** (Functional representation lemma \[18\]).

$$
\forall X, Y : \left( \exists U : I(X; U) = H(Y | X, U) = 0 \right).
$$

The strong functional representation lemma \[65\] states that, in addition to $I(X; U) = H(Y | X, U) = 0$, we can also have $H(Y | U) \leq I(X; Y) + \log(I(X; Y) + 1) + 4$. Technically, the strong functional representation lemma is not an EII due to the non-linear logarithmic term. Nevertheless, it can still be handled by the FSTIP implementation using a gap term, and can be used to prove the achievability of Gelfand-Pinsker theorem \[23\] automatically (see Section VIII).

### C. Other Examples

The double Markov property \[73\] can be stated as an EII:

$$
\forall X, Y, Z : \left( I(X; Z | Y) = I(Y; Z | X) = 0 \rightarrow \\
\exists U : H(U | X) = H(U | Y) = I(X, Y; Z | U) = 0 \right).
$$

The existence of the Gács-Körner common part \[59\] can be stated as

$$
\forall X, Y : \left( \exists U : H(U | X) = H(U | Y) = 0 \\
\and \forall V : \left( H(V | X) = H(V | Y) = 0 \rightarrow H(V | U) = 0 \right) \right).
$$
This can be considered as a “nested” EII due to the extra nested layer of the universally quantified variable $V^{'}$. Similarly, the existence of the minimal sufficient statistic can be stated as

$$\forall X, Y : \left( \exists U : H(U|X) = I(X;Y|U) = 0 \right)$$
$$\land \forall V : \left( H(V|X) = I(X;Y|V) = 0 \rightarrow H(U|V) = 0 \right) \right).$$

We remark that the PSITIP implementation can handle arbitrarily nested expressions such as the two aforementioned examples.

The infinite divisibility of information [74] states that for any random variable $X$ and positive integer $n$, there exists i.i.d. $U_1, \ldots, U_n$ such that $H(X|U^n) = 0$ and $H(U_1) \leq e/(n(e-1))H(X) + 2.43$. A weaker form (which does not require $U_1, \ldots, U_n$ to have the same distribution) can be stated as

$$\forall X : \left( \exists U^n : H(U_1) = \cdots = H(U_n) = \frac{1}{n}H(U^n) \right)$$
$$\land H(X|U^n) = 0 \land H(U_1) \leq \frac{e}{n(e-1)}H(X) + 2.43.$$ 

While this is technically not an EII due to the constant term, it can still be handled by the PSITIP implementation (which can handle affine expressions).

VIII. Auxiliary Searching with EII Premises

Suppose we want to prove that an EII follows from a collection of EII premises assumed to be true (a premise can be one of the nontrivial EII in Section VII that are always true). We describe how to incorporate the premises using the transitivity rule and the conditional independence rule (Proposition 13). Suppose we want to verify the EII $\forall X^n : (Ah(X^n) \geq 0 \rightarrow \exists U^l : Bh(X^n, U^l) \geq 0)$, using the premise $\forall Y^n : (Ch(Y^n) \geq 0 \rightarrow \exists V^l : Dh(Y^n, V^l) \geq 0)$. The idea is to apply the premise on $Y^n = S \circ X^n$ where $S \in \{0, 1\}^n \times n$. If we can verify $Ch(Y^n) \geq 0$ for this choice of $Y^n$, then we can use the premise to assume that $\exists V^l : Dh(Y^n, V^l) \geq 0$ is true. We now state this precisely using the elementary rules. Using the substitution of variables rule to substitute $Y^n = S \circ X^n$ to the premise, and use the substitution of variables rule to introduce $X^n_{n+1}$, and use the conditional independence rule to obtain

$$\forall X^n : (Ch(S \circ X^n) \geq 0 \rightarrow \exists V^l : Dh(S \circ X^n, V^l) \geq 0)$$
$$\land I(V^l; X^n | S \circ X^n) = 0.$$  

Therefore, if we can verify the CII:

$$\forall X^n : (Ah(X^n) \geq 0 \rightarrow Ch(S \circ X^n) \geq 0),$$

and the EII:

$$\forall X^n, V^l : (Ah(X^n) \geq 0 \land Ch(S \circ X^n) \geq 0)$$
$$\land Dh(S \circ X^n, V^l) \geq 0 \land I(V^l; X^n | S \circ X^n) = 0$$
$$\rightarrow \exists U^l : Bh(X^n, U^l) \geq 0,$$

then we can verify the desired EII $\forall X^n : (Ah(X^n) \geq 0 \rightarrow Ch(S \circ X^n) \geq 0)$, by applying the transitivity rule on 13, 12, 14. Therefore, to verify the desired EII $\forall X^n$, we can use the linear program 2 to verify 13, and then the auxiliary searching algorithm (Algorithm 2) to verify 14.

We can exhaust all choices of $S \in \{0, 1\}^n \times n$ and repeat the above procedures. To speed up the search, note that Algorithm 2 returns one valid substitution matrix $S$ for 13, and hence can be modified to enumerate all substitution matrices satisfying 13 more efficiently. We define a variant of Algorithm 2 called VERIFY_EII_LIST(A, B, S, S), by making the following change to Algorithm 2:

- Change the line “return S” to “add S to a list $T$ (initialized to $T = \emptyset$).”
- Change the last line “return failure” to “return $T$.”

This will allow the modified algorithm to output a list $T$ of valid substitution matrices $S$ that proves the EII.

The algorithm for verifying $A \rightarrow B$ under the premise $C \rightarrow D$ is given in Algorithm 3. It returns two matrices $S \in \{0, 1\}^n \times n$, $S \in \{0, 1\}^{n \times (n+l)}$, which means that the EII $A \rightarrow B$ can be proved by substituting $X^n = S \circ X^n$ and $U^n = S \circ (X^n, U^l)$ (note that $U_i$ is allowed to be a combination of the entries in $X^n$ and $V^l$). If the program outputs $\emptyset$, $S$, it means the premise $C \rightarrow D$ is not needed. We remark that this algorithm is useful not only when the desired consequence is an

A subsequent work by the author [53] studies the first-order theory of random variables, which allows arbitrary nesting of existential and universal quantifiers.

A more efficient implementation is to use a generator or coroutine, e.g. using “yield S” in Python, so the function can stop as soon as one $S$ gives the desired result.
EII, but also when it is an UII or CII (which are special cases of EII). For example, this algorithm can prove non-Shannon-type UIIs and CIs using the copy lemma (Proposition 12) as premise.

Algorithm 3 VerifyEII_Premise($A, B, C, D$)

- **Input:** $A \in \mathbb{R}^{m_A \times (2^n-1)}$, $B \in \mathbb{R}^{m_B \times (2^{n+1}-1)}$, $C \in \mathbb{R}^{m_C \times (2^n-1)}$, $D \in \mathbb{R}^{m_D \times (2^{n+1}-1)}$
- **Output:** $\bar{S} \in \{0, 1\}^{n \times n}$, $S \in \{0, 1\}^{(n+1) \times l}$, or failure

1. $S \leftarrow$ VerifyEII($A, B$)
2. if not failure then return $\emptyset, S$
3. for $\bar{S} \in$ VerifyEII_List($A, C$) do
4. $S \leftarrow$ VerifyEII($E, B$), where $E \xrightarrow{EII} B$
5. if not failure then return $S, S$
6. end for
7. return failure

**Remark 15.** If there are multiple premises, we can replace line 4 in Algorithm 3 by a recursive call to the algorithm itself, using another premise. Note that even when we only have one premise, it may be useful to repeat the premise multiple times, so it can be applied multiple times.

IX. Unioi Rule and Leave-One-Out Procedure

We present another inference rule, called the union rule.

**Proposition 16 (Union rule).** For $A \in \mathbb{R}^{m_A \times (2^n-1)}$, $B \in \mathbb{R}^{m_B \times (2^{n+1}-1)}$, $c \in \mathbb{R}^{2^n-1}$,

$$
\left[ \begin{array}{c} A \\ c^T \end{array} \right] \xrightarrow{EII} B \land \left[ \begin{array}{c} A \\ -c^T \end{array} \right] \xrightarrow{EII} B \Rightarrow A \xrightarrow{EII} B.
$$

We can see that the union rule is true simply by considering the cases whether $c^T h(X^n) \geq 0$ (where the result follows from the first EII $Ah(X^n) \geq 0 \land c^T h(X^n) \geq 0 \Rightarrow Bh(X^n) \geq 0$ above) or $c^T h(X^n) < 0$ (where the result follows from the second EII). While the union rule is logically trivial, it is useful since the choices of $U^l$ in the two cases do not need to be the same. We describe how to modify the algorithms in Sections VII and VIII to automatically apply the union rule. Suppose we are trying to prove the EII in (5):

$$
\forall X^n : (Ah(X^n) \geq 0 \to \exists U^l : Bh(X^n, U^l) \geq 0),
$$

and we find a substitution $U^l = S \circ X^n$ (where $S \in \{0, 1\}^{(n+1) \times l}$ such that $Ah(X^n) \geq 0 \to b^T h(X^n, S \circ X^n) \geq 0$ holds for all rows $b^T$ of $B$ except one, i.e., it is “almost correct”). We can let that one row be $b^T$, $b \in \mathbb{R}^{2^n+1}$, and let $c^T = b^T \text{sub}(I_n; S) \in \mathbb{R}^{2^n+1}$ (such that $c^T h(X^n) = b^T h(X^n, U^l)$ for all $X^n$ when $U^l = S \circ X^n$; see (5)). Adding $c^T h(X^n) \geq 0$ as an assumption, we know this EII holds:

$$
\left[ \begin{array}{c} A \\ c^T \end{array} \right] h(X^n) \geq 0 \to \exists U^l : Bh(X^n, U^l) \geq 0.
$$

By the union rule, it is left to prove that

$$
\left[ \begin{array}{c} A \\ -c^T \end{array} \right] h(X^n) \geq 0 \to \exists U^l : Bh(X^n, U^l) \geq 0,
$$

which is easier to prove because of the additional assumption. Therefore, we can utilize “almost correct” choices of $U^l$ to strengthen the assumption. This procedure can be repeated until we find the correct choice of $U^l$.

Consider the toy example of proving the EII

$$
\forall X, Y : \exists U : I(X; Y | U) \leq 0 \land 2H(U) \leq H(X) + H(Y).
$$

Note that there are two rows in $B$ corresponding to $I(X; Y | U) \leq 0$ and $2H(U) \leq H(X) + H(Y)$. We try different choices of $U$ and note which of these two rows are satisfied. First we try $U = X$, which satisfies $I(X; Y | U) \leq 0$, but does not satisfy $2H(U) \leq H(X) + H(Y)$, which becomes $H(X) \leq H(Y)$ after substituting $U = X$. Therefore, we know that if we add an
assumption $H(X) \leq H(Y)$, then $U = X$ would be a valid choice. It is left to consider the case $H(X) \geq H(Y)$, i.e., it is left to prove the EII

$$\forall X, Y : (H(X) \geq H(Y) \rightarrow \exists U : I(X; Y|U) \leq 0 \wedge 2H(U) \leq H(X) + H(Y)).$$

We then try $U = Y$. It satisfies both $I(X; Y|U) \leq 0$ and $2H(U) \leq H(X) + H(Y)$ (due to the new assumption $H(X) \geq H(Y)$). Hence $U = Y$ is a valid choice. The algorithm will conclude that the original EII can be proved by considering either $U = X$ or $U = Y$.

We call this the leave-one-out procedure. Refer to Algorithm 4 for details. Algorithm 4 combines the cached linear program in Section V-B with the leave-one-out procedure. For each substitution matrix $S$, it stores a list $C$ of rows $c$ (refer to the previous discussion for the meaning of $c$) where $A_h(X^n) \geq 0 \rightarrow c^T h(X^n) \geq 0$ fails. If $C$ is empty, then we successfully find a valid substitution matrix $S$. If $|C| = 1$, then we can apply the leave-one-out procedure to add the row $-c^T$ to $A$ (note that the cache $K$ must be cleared since $A_k \geq 0$ for $k \in K$ may fail under the new $A$). Algorithm 4 outputs a list $T$ of substitution matrices $S$ such that the original EII can be proved by considering the cases $U^i = S \circ X^n$ for each $S \in T$ (note that this is different from VERIFY_EII_LIST in Section VIII where each $S$ in the list $T$ is sufficient to prove the EII by itself).

Using this procedure, the program can automatically generate proofs that are impossible using only the elementary rules, e.g., the converse proof of the capacity region of the state-dependent semideterministic broadcast channel [75].

---

**Algorithm 4 VERIFY_EII_LEAVE_ONE_OUT($A, B, S, \overline{S}$)**

**Input:** $A \in \mathbb{R}^{m_A \times (2^n - 1)}$, $B \in \mathbb{R}^{m_B \times (2^n + l - 1)}$, lower and upper bound $S, \overline{S} \in \{0, 1\}^{l \times n}$

**Output:** list $T$ of substitution matrices or failure

1: $T \leftarrow \emptyset$ (empty list)
2: $(S, \overline{S}) \leftarrow$ SANDWICH($A, B, S, \overline{S}$) (if failure, return failure)
3: $K \leftarrow \emptyset$ (empty list)
4: for each matrix $S \in \{0, 1\}^{l \times n}$ with $S \leq S \leq \overline{S}$ do
5:   $C \leftarrow \emptyset$ (empty list)
6:   for row $b^T$ of $B$ do
7:     $c^T \leftarrow b^T_{\text{sub}(I_n; S)}$
8:     (so $c^T h(X^n) = b^T h(X^n, U^i)$ when $U^i = S \circ X^n$; see 3)
9:     if $c^T k < 0$ for any $k \in K$ then
10:        Add $c$ to list $C$
11:    else
12:       Solve the linear program:
13:       minimize $c^T k$ s.t. $k \in \Gamma_n$, $A_k \geq 0$, $k_{2^n-1} \leq 1$
14:       if optimal value < 0 then
15:          Add $k$ attaining optimum to list $K$
16:       Add $c$ to list $C$
17:    end if
18:   end if
19: if $|C| \geq 2$ then
20:   Skip $S$ and continue to the next matrix
21: end if
22: if $C = \emptyset$ then
23:   Add $S$ to list $T$
24: return $T$
25: else if $|C| = 1$ (let $C = \{c\}$) then
26:   $A \leftarrow \begin{bmatrix} A \\ -c^T \end{bmatrix}$
27:   $K \leftarrow \emptyset$
28:   Add $S$ to list $T$
29: end if
30: end for
31: return failure
X. EXISTENTIAL INFORMATION PREDICATE AND SIMPLIFICATION OF RATE REGIONS

An EII is a proposition that is either true of false. We may be interested in predicates with truth value depending on the distribution of some random variables. Define the existential information predicate (EIP) on the random sequence $X^n$ and inequality matrix $A \in \mathbb{R}^{m_A \times (2^n+1)}$ to be the predicate

$$\exists U^l: A h(X^n, U^l) \geq 0.$$ 

We denote the above predicate as $EIP_A(X^n)$. Note that $l$ can often be deduced from $n$ and the width of $A$.

Rate regions and bounds in network information theory can often be stated as EIP. For example, the superposition coding inner bound \[33\], \[34\] for the broadcast channel $p(y_1, y_2|x)$ can be stated as the following EIP on $X, Y_1, Y_2, R_1, R_2$:

$$\exists U : (R_1 \leq I(X; Y_1|U) \land R_2 \leq I(U; Y_2)$$

$$\land R_1 + R_2 \leq I(X; Y_1)$$

$$\land I(U; Y_1, Y_2|X) = 0).$$

Note that $R_1, R_2$ are real variables (refer to Remark 5 for how to represent them).

A. EIP Implication Problem

An EII can be stated using EIPs:

$$(A \xrightarrow{EII} B) \iff (\forall X^n : EIP_A(X^n) \Rightarrow EIP_B(X^n)).$$

Note that the $A$ in the above expression has width $2^n - 1$, and hence $EIP_A(X^n) : A h(X^n) \geq 0$ does not involve any auxiliary. In general, implication between EIPs in the form $\forall X^n : EIP_A(X^n) \Rightarrow EIP_B(X^n)$ (where $A$ can have width larger than $2^n - 1$) may not have an equivalent EII, though we have the following sufficient EII: for any $n, l, k \geq 0, A \in \mathbb{R}^{m_A \times (2^n+1)}, B \in \mathbb{R}^{m_B \times (2^n+1)}$, we have

$$\forall X^n, U^l : (A h(X^n, U^l) \geq 0 \rightarrow \exists V^k : B h(X^n, V^k) \geq 0)$$

$$\Rightarrow (\forall X^n : EIP_A(X^n) \Rightarrow EIP_B(X^n)).$$

Therefore, implications between EIPs may be checked using algorithms for proving EIIIs (applied on the EII in the line \[15\]) discussed in the previous sections. This is useful for deciding whether a rate region is included in (i.e., is an inner bound of) another rate region.

B. Subtractive EIP Simplification and Inner Bounds

We now discuss methods for simplifying an EIP $EIP_A(X^n)$. Common methods for simplifying a rate region in network information theory include removing redundant inequalities (i.e., if there is a row $a^T$ of $A$ such that $A h(X^n, U^l) \geq 0 \rightarrow a^T h(X^n, U^l) \geq 0$, where $A$ is $A$ with row $a^T$ removed, then we can remove row $a^T$ from $A$) and Fourier-Motzkin elimination on existentially quantified real variables, e.g. see \[18\]. Here we discuss an algorithm, called the subtractive simplification procedure, for reducing the number of auxiliaries in an EIP.

Suppose we want to know whether $U_1$ in the EIP $EIP_A(X^n) : \exists U^l : A h(X^n, U^l) \geq 0$ can be removed. If we can find $S \in \{0, 1\}^{l \times (n+1)}$ with $S_{i,n+1} = 0$ for $i \in [l]$ such that

$$\forall X^n, U^l : (A h(X^n, U^l) \geq 0 \rightarrow$$

$$A h(X^n, S \circ (X^n, U^l)) \geq 0),$$

then we know that substituting $U^l \leftarrow S \circ (X^n, U^l)$ to $EIP_A(X^n)$ does not strengthen the EIP. We also know a priori that substituting a particular choice of auxiliaries cannot weaken an EIP. Hence if \[17\] holds, then substituting $U^l \leftarrow S \circ (X^n, U^l)$ results in an equivalent EIP. Since $S_{i,n+1} = 0$ for $i \in [l]$, the resultant EIP does not contain $U_1$, and hence we can remove the auxiliary $U_1$.

To find $S \in \{0, 1\}^{l \times (n+1)}$, we apply the auxiliary searching algorithm (Algorithm \[2\]) on the EII

$$\forall X^n, U^l : (A h(X^n, U^l) \geq 0 \rightarrow$$

$$\exists V^l : A h(X^n, V^l) \geq 0)$$

(18)

to search for $S \in \{0, 1\}^{l \times (n+1)}$ with $S_{i,n+1} = 0$ for $i \in [l]$ such that $V^l = S \circ (X^n, U^l)$ satisfies the above EII. Since we want to eliminate $U_1$ after substituting $V_i = (X_{S_i}, U_{T_i})$, we must have $S_{i,n+1} = 0$ for $i \in [l]$. This can be enforced by removing the index of $U_1$ from the initial upper bounds $S$ passed to Algorithm \[2\].

The \textit{full subtractive simplification procedure} repeats this step to remove auxiliaries until no more auxiliary can be removed. Refer to Algorithm \[5\] for details.
Algorithm 5 SIMPLIFY_SUB_FULL(n, A)

Input: n ∈ N, A ∈ R^{m_A × (2^{n+l}-1)}
Output: A after simplification

1: repeat
2:   for i = 1, . . . , l (we try to remove U_i) do
3:     Let A_{EII} be the EII:
4:     ∀X^n, U^l : (Ah(X^n, U^l) ≥ 0 → ∃V^l : Ah(X^n, V^l) ≥ 0).
5:     S ← VERIFYEII(A, C, O^l × (n+l), S)
6:     where S_{j,k} = 0 if k = n + i (the index of U_i), S_{j,k} = 1 otherwise
7:     if not failure then
8:       Compute A ∈ R^{m_A × (2^{n+l}-1)} such that for any X^n, U^l,
9:       Ah(X^n, S ◦ (X^n, U^l)) = Ah(X^n, U[\{i\}] \cup \{1\})
10:      A ← A
11:      l ← l − 1
12:   end if
13: end for
14: until no more auxiliaries are removed
15: return A

Algorithm 5 can be quite slow since it has to search for the choices of l auxiliaries V^l in the VERIFYEII step. To improve the efficiency of the algorithm, we use the observation that if we are trying to remove the auxiliary U_1, then we usually only need to consider the choice of U_1, that is, we find S ∈ {0, 1}^{1×(n+l)} with S_{1,n+1} = 0 such that it suffices to consider the choice U_1 ← S ◦ (X^n, U^l), i.e.,

∀X^n, U^l : (Ah(X^n, U^l) ≥ 0 → Ah(X^n, S ◦ (X^n, U^l), U^l_j) ≥ 0).

For the other auxiliaries U_j, we simply leave them unchanged. If the above EII is true, then we can substitute U_1 ← S ◦ (X^n, U^l) and remove the auxiliary U_1. To find S, we apply Algorithm 3 on the EII

∀X^n, U^l : (Ah(X^n, U^l) ≥ 0 → ∃V : Ah(X^n, V, U^l_j) ≥ 0)

(19)
to search for S ∈ {0, 1}^{1×(n+l)} with S_{1,n+1} = 0 such that V = S ◦ (X^n, U^l) satisfies the above EII. We call this the partial subtractive simplification procedure, which is less powerful than the full version, but is significantly faster since we only search for the choice of one auxiliary V. Refer to Algorithm 5 for details.

Algorithm 6 SIMPLIFY_SUB_PARTIAL(n, A)

Input: n ∈ N, A ∈ R^{m_A × (2^{n+l}-1)}
Output: A after simplification

1: repeat
2:   for i = 1, . . . , l (we try to remove U_i) do
3:     Let A_{EII} be the EII:
4:     ∀X^n, U^l : (Ah(X^n, U^l) ≥ 0 → ∃V : Ah(X^n, U^l−1, V, U^l_i) ≥ 0),
5:     S ← VERIFYEII(A, C, O^l × (n+l), S)
6:     where S_{j,k} = 0 if k = n + i (the index of U_i), S_{j,k} = 1 otherwise
7:     if not failure then
8:       Compute A ∈ R^{m_A × (2^{n+l}-1)} such that for any X^n, U^l,
9:       Ah(X^n, U^l−1, S ◦ (X^n, U^l), U^l_{i+1}) = Ah(X^n, U[\{i\}] \cup \{1\})
10:      A ← A
11:      l ← l − 1
12:   end if
13: end for
14: until no more auxiliaries are removed
15: return A
Subtractive simplification is best suited for EIPs with few auxiliaries, e.g., inner bounds for multiuser coding settings (see Section XI-A). It can be inefficient for a large number of auxiliaries due to the semi-exhaustive search for each auxiliary.

Remark 17. Note that even if the EII (19) fails for the choice \( V = S \circ (X^n, U^l) \), we can still substitute \( U_1 \leftarrow S \circ (X^n, U^l) \) to the original EIP \( \exists U^l: \text{Ah}(X^n, U^l) \succeq 0 \) in order to obtain a new EIP that implies (i.e., is an inner bound of) the original EIP.

C. Additive EIP Simplification and Outer Bounds

Next, we discuss another algorithm, called the \textit{additive simplification procedure}, for reducing the number of auxiliaries in an EIP. Consider an EIP \( \exists U^l: \text{Ah}(X^n, U^l) \succeq 0 \). We want to use it to deduce a new equivalent EIP with auxiliaries \( V^k, k < l \), where the choice of auxiliaries is \( V^k = S \circ (X^n, U^l) \) (where \( S \in \{0, 1\}^{k \times (n+l)} \)). Let

\[
B = \text{sub} \left( \begin{bmatrix} I_n & 0^{n \times l} \\ S \end{bmatrix} \right) \in \mathbb{R}^{(2^n+k-1) \times (2^n+l-1)}.
\]

Note that \( B \) satisfies \( h(X^n, V^k) = Bh(X^n, U^l) \). Therefore, the original EIP implies the new EIP

\[
\exists V^k : h(X^n, V^k) \in P
\]

(20)

where

\[ P := \{Bk : k \in \Gamma_{n+l}, Ak \succeq 0\} \]

Note that \( P \) is the polyhedral cone obtained by projecting the polyhedral cone \( \{k \in \Gamma_{n+l} : Ak \succeq 0\} \) by the matrix \( B \). It can be computed using any algorithm for polyhedron projection, for example, Fourier-Motzkin elimination, or the convex hull method [76] (which is used in the PSITIP implementation). We note that the convex hull method is also used in [52] for the discovery of non-Shannon-type inequalities, though our framework is more general since it is capable of incorporating any EII, not only the copy lemma.

To check whether (20) is equivalent to the original EIP, it remains to check the other direction of the implication, which can be checked using the method in Section XI-A. If (20) is equivalent to the original EIP, then we can simplify the original EIP to (20), which contains fewer auxiliaries since \( k < l \). The \textit{additive simplification procedure} simply perform this checking for every substitution matrix \( S \in \{0, 1\}^{k \times (n+l)} \). Refer to Algorithm 7 for details.

Algorithm 7 SIMPLIFY_ADD(n, A)

\begin{algorithm}
\begin{aligned}
\textbf{Input:} & \ n \in \mathbb{N}, \ A \in \mathbb{R}^{m \times (2^n+l-1)} \\
\textbf{Output:} & \ A \text{ after simplification} \\
1: \ & \text{for } k = 0, 1, \ldots, l-1 \text{ do} \\
2: \ & \quad \text{for each matrix } S \in \{0, 1\}^{k \times (n+l)} \text{ do} \\
3: \ & \quad \quad B \leftarrow \text{sub} \left( \begin{bmatrix} I_n & 0^{n \times l} \\ S \end{bmatrix} \right) \\
4: \ & \quad \quad \text{Compute the facets of the projected cone:} \\
5: \ & \quad \quad \quad P \leftarrow \{Bk : k \in \Gamma_{n+l}, Ak \succeq 0\} \\
6: \ & \quad \quad \quad \text{Compute } \tilde{A} \text{ such that } h(X^n, V^k) \in P \Leftrightarrow \tilde{A}h(X^n, V^k) \succeq 0 \\
7: \ & \quad \quad \quad \text{Apply VERIFYEII } \text{(or VERIFYEII_LEAVEONEOUT) to check the EII:} \\
8: \ & \quad \quad \quad \quad \forall X^n, V^k : (\tilde{A}h(X^n, V^k) \succeq 0 \rightarrow \exists U^l : \text{Ah}(X^n, U^l) \succeq 0) \\
9: \ & \quad \quad \text{if not failure then} \\
10: \ & \quad \quad \quad \text{return } \tilde{A} \\
11: \ & \quad \text{end if} \\
12: \ & \text{end for} \\
13: \ & \text{end for} \\
14: \ & \text{return } A
\end{aligned}
\end{algorithm}

If we are only interested in obtaining an outer bound of the EIP, then it is unnecessary to check that the new EIP implies the original EIP (step 7 in Algorithm 5). Each \( S \) produces an outer bound \( \exists V^k : h(X^n, V^k) \in P \) of the original EIP. We then choose the tightest outer bound by comparing these outer bounds using the method in Section XI-A. Given the desired number of auxiliaries \( k \), Algorithm 8 which we call the \textit{additive outer bound procedure}, finds an outer bound of the given EIP with \( k \) auxiliaries.
The additive outer bound procedure is suitable when the number of auxiliaries is large, and only an outer bound is desired. This is useful for automated discovery of outer bounds for multiuser coding settings, where we may encounter an outer bound with too many auxiliaries, and we are interested in a (possibly weaker) outer bound with fewer auxiliaries (see Section XI-A for an example). This is also useful for discovering non-Shannon inequalities (see Section XI-B for an example).

All the aforementioned methods (removing redundant inequalities, Fourier-Motzkin elimination, removing redundant auxiliaries) are implemented in the PSITIP implementation. The PSITIP implementation allows automated discovery of inner and outer bounds, using only the graphical representation of the network as input, via the inner bound in [21], Gallager’s approach [34] for outer bounds, together with the aforementioned simplification procedures.

**Remark 18.** The simplification methods can sometimes eliminate the need of auxiliary random variables and convert an EIP to a non-existence information predicate (e.g. \( \exists U : I(X : U) = I(X ; Y | U) = 0 \) can be simplified to \( I(X ; Y) = 0 \) by identifying \( U = Y \)). This is known as quantifier elimination in mathematical logic. Examples of quantifier elimination includes Fourier-Motzkin elimination and the Tarski-Seidenberg theorem [77], [78] for real numbers. While the Tarski-Seidenberg theorem establishes the decidability of the theory of real closed fields by showing that quantifier elimination is always possible, the same is not true for information inequalities. For example, the EIP \( \exists U : (H(U ; X) = 0 \land H(U) = H(X) / 2) \) cannot be reduced to any statement that only concerns \( H(X) \) (i.e., for any \( t > 0 \), we can construct \( X \) where \( H(X) = t \) and the EIP holds, and also construct \( X \) where \( H(X) = t \) and the EIP does not hold), and hence the auxiliary \( U \) cannot be eliminated. Note that CIIs, and hence EIIIs, are undecidable [41].

**XI. Examples**

### A. Degraded Broadcast Channel

In this section, we demonstrate the use of the algorithm in finding and proving the capacity region of 2-receiver degraded broadcast channel [33], [34]. The Python code for finding the capacity region (which uses the PSITIP package that implements the algorithms in this paper [28] and Pyomo [79] and GLPK [80] for linear programming) is given below:

```python
from psitip import PsiOpt
PsiOpt.solve(solver = "pyomo.glpk")
PsiOpt.solve(style = "latex")

X, Y, Z = rv("X, Y, Z")
M1, M2 = rv_array("M", 1, 3)
R1, R2 = real_array("R", 1, 3)
model = CodingModel()
```

[Source code of Python Symbolic Information Theoretic Inequality Prover (PSITIP) is available at https://github.com/cheuktingli/psitip][1]

---

**Algorithm 8 OUTER_ADD(n, A, k)**

| Input: | \( n \in \mathbb{N}, A \in \mathbb{R}^{mA \times (2^n+1)} \), target number of auxiliaries \( k \) |
| Output: | \( C \) such that EIP \( C(X^n) \) is an outer bound of EIP \( A(X^n) \) |
|---|---|
| 1: | \( C \leftarrow \emptyset \) |
| 2: | for each matrix \( S \in \{0, 1\}^{k \times (n+1)} \) do |
| 3: | \( B \leftarrow \text{sub} \left( \begin{bmatrix} I_n & 0^{n \times l} \end{bmatrix} \right) \) |
| 4: | Compute the facets of the projected cone: |
| 5: | \( P \leftarrow \{ Bk : k \in \Gamma_{n+l}, Ak \succeq 0 \} \) |
| 6: | Compute \( A \) such that \( h(X^n, V^k) \in P \Leftrightarrow \tilde{A}h(X^n, V^k) \succeq 0 \) |
| 7: | if \( C = \emptyset \) then |
| 8: | \( C \leftarrow A \) |
| 9: | else |
| 10: | Apply VERIFY_EII (or VERIFY_EII_LEAVEONEOUT) to check the EII: |
| 11: | \( \forall X^n, U^k : (Ch(X^n, U^k) \succeq 0 \rightarrow \exists V^k : \tilde{A}h(X^n, V^k) \succeq 0) \) |
| 12: | if failure then |
| 13: | \( C \leftarrow A \) |
| 14: | end if |
| 15: | end if |
| 16: | end for |
| 17: | return \( C \) |
This is the union of 4 EIPs (separated by "\lor" which means "or"). Note that $A_{M_1}$ and $A_{M_2}$ are auxiliary random variables, and "\leftrightarrow" denotes Markov chain. Each individual EIP is obtained via a different choice of simultaneous nonunique decoding sets in \[21\]. While the inner bound in \[21\] is general, it requires many manual choices of parameters in the coding scheme (e.g. the simultaneous nonunique decoding sets), and hence requires effort to compare the regions obtained with different choices of parameters in order to find the largest.
The second output of the program (inner bound with simplification turned on) is given below:

\[ \exists A_{M_2} : \begin{cases} R_1 \geq 0, \\
R_2 \geq 0, \\
R_2 \leq I(A_{M_2}; Z), \\
R_1 + R_2 \leq I(A_{M_2}; Z) + I(X; Y|A_{M_2}), \\
A_{M_2} \leftrightarrow X \leftrightarrow Y \leftrightarrow Z \end{cases} \]  

(22)

This EIP is the same as the superposition region \[33, 34\]. The algorithm simplifies the union in \[21\] into the EIP in \[22\] by comparing the 4 EIPs in \[21\] using \[16\] and the auxiliary searching algorithm (Algorithm \[2\]) to find the largest, and using the subtractive simplification procedure described in Section X-B to reduce the number of auxiliaries (the auxiliary \(A_{M_1}\) is removed). We can see how the algorithm eliminates the need of manual efforts to compare and simplify rate regions.

The third output of the program (automatic outer bound) is an EIP obtained via the Bayesian network of the past, present and future random variables (e.g. for \(Y^n\), the past is \(Y_p := Y_1^{Q-1}\), the present is \(Y_Q\), and the future is \(Y_F := Y_{Q+1}^n\), where \(Q \sim \text{Unif}[n]\)), and the Csiszár sum identity \[70, 71\] applied on every triple of random variables:

\[ \exists Y_p, Y_F, Z_p, M_1, M_2 : \begin{cases} I(M_1; Y; M_2|Y_p) \leq 0, \\
R_1 \leq I(M_1; Y|Y_p), \\
I(M_2; Z; M_1|Z_p) \leq 0, \\
R_2 \leq I(M_2; Z|Z_p), \\
R_1 \geq 0, \\
R_2 \geq 0, \\
I(Y_F; Y|Y_p) = I(Y_F; Y|Y_F), \\
I(Y_F; Y|Y_p, M_1) = I(Y_F; Y|Y_F, M_1), \\
I(Y_F; Y|Y_p, M_2) = I(Y_F; Y|Y_F, M_2), \\
I(Y_F; Y|Y_p, M_1, M_2) = I(Y_F; Y|Y_F, M_1, M_2), \\
I(Y_F; Z|Z_p, M_1) = I(Z_p; Y|Y_F, M_1), \\
I(Y_F; Z|Z_p, M_2) = I(Z_p; Y|Y_F, M_2), \\
I(Y_F; Z|Z_p, M_1, M_2) = I(Z_p; Y|Y_F, M_1, M_2), \\
H(X|M_1, M_2, Y_p) = 0, \\
H(X|M_1, M_2, Y_F) = 0, \\
H(X|M_1, M_2, Z_p) = 0, \\
(M_1, M_2, Y_F) \leftrightarrow X \leftrightarrow Y \leftrightarrow Z, \\
(Y, Z) \leftrightarrow (M_1, M_2, Y_F, X) \leftrightarrow Y_p \leftrightarrow Z_p \end{cases} \]  

(23)

We write \(I(X; Y; Z) = I(X; Y) - I(X; Y|Z)\). Note that the time-sharing random variable \(Q\) is absorbed into the auxiliaries and is not present. Also, the program omits some past/future random variables (e.g. \(X_p, X_F, Z_p\)) that are determined to be unnecessary.\(^9\) Next, the program checks the optimality of the inner bound by showing that the outer bound implies the inner bound. The program uses \[16\] and Algorithm \[2\] to show that \[23\] implies \[22\]. It outputs the choice of auxiliary \(A_{M_2} = (Y_F, M_2)\).

Finally, the program uses the additive outer bound procedure in Section X-C on the bound in \[23\] to find an outer bound with one auxiliary. It outputs the superposition region \[33, 34\]:

\[ \exists A : \begin{cases} R_1 \geq 0, \\
R_2 \geq 0, \\
R_2 \leq I(A; Z), \\
R_1 \leq I(X; Y|A), \\
A \leftrightarrow X \leftrightarrow Y \leftrightarrow Z \end{cases} \]

We can see that for both automated inner bound and outer bound, the program correctly outputs the superposition region.

\(^9\)Technically, instead of \(R_1 \leq I(M_1; Y|Y_p)\), we should have \(R_1 \leq I(M_1; Y|Y_p) + \epsilon\) where \(\epsilon \to 0\) as the error probability tends to 0, though this is unnecessary since \(\epsilon\) can be absorbed into \(R_1\).
B. Non-Shannon Inequalities

To demonstrate the additive outer bound procedure in Section X-C we use it to discover non-Shannon-type inequalities in a way similar to [52]. Python code:

```python
from psitip import *
PsiOpt.solver = "pyomo.glpk" # Set linear programming solver
X, Y, Z, W, U = rv("X, Y, Z, W, U") # Declare random variables

# State the copy lemma as an EII
r = eqdist([X, Y, U], [X, Y, Z]).exists(U)

# Automatically discover non-Shannon-type inequalities using copy lemma
print(r.discover([X, Y, Z, W]).simplified().latex())
```

Output of the program, which contains the Zhang-Yeung inequality [5] (note that \( I(X;Y;Z) = I(X;Y) - I(X;Y|Z) \)):

\[
\begin{align*}
I(X;W;Z) &\leq I(Y;W) + I(Z;Y|X) + 2I(Z;X|Y) + I(X;Y|Z), \\
I(Z;Y,X) &\leq I(Z;W) + I(Y;Z|X) + I(X;Z|Y) + I(Z;X|Y) + I(Y;X|W), \\
I(Y;W;Z) &\leq I(X;W) + I(Y;Z|X) + I(Z;X|Y) + I(Y;X;Z)
\end{align*}
\]

XII. List of Results Provable by the Framework

The following is a non-exhaustive list of results that can be proved by the PSITIP implementation of the framework [3]. Achievability results are obtained via the inner bound in [21] together with the simplification procedures in this paper. In some cases marked with “(simplest)”, the framework can automatically derive the inner bound in its simplest form using the graphical representation of the network as input. For inner and outer bound results not marked with “(simplest)”, the framework can only derive a more complicated bound, and verify that it is at least as good as the known bound (and hence give an automated proof of the known bound).

1) Gelfand-Pinsker theorem [23] for channels with state, achievability (simplest) and converse.
2) Shannon’s result [81] on the capacity of channels with state available causally at the encoder, achievability (simplest) and converse.
3) 2-receiver degraded broadcast channel [33], [34], achievability (simplest) and converse.
4) 2-receiver broadcast channel: Marton’s inner bound [28], [29], [30] (simplest) and Nair-El Gamal outer bound [82].
5) 2-receiver broadcast channel with less noisy and more capable [83], [84] receivers, achievability (simplest) and converse.
6) State-dependent semideterministic broadcast channel [75], achievability (simplest) and converse of capacity region.
7) Multiple access channel [25], [26], [27], achievability (simplest) and converse of capacity region.
8) Interference channel, Han-Kobayashi inner bound [65].
9) Wyner-Ahlswede-Körner network [86], [87], achievability (simplest) and converse of capacity region.
10) Wyner-Ziv theorem [24] for lossy source coding with side information, achievability (simplest) and converse.
11) Gray-Wyner network [88], achievability and converse.
12) Slepian-Wolf theorem [89] for distributed lossless source coding, achievability (simplest) and converse.
13) Distributed lossy source coding: Berger-Tung inner bound [31], [32] (simplest) and Berger-Tung outer bound [31], [32].
14) Multiple description coding: Zhang-Berger inner bound [90] (simplest), Venkataramani-Kramer-Goyal inner bound [91], [92].
15) Vámos network [93]: upper bound 10/11 of capacity in [93] via Zhang-Yeung inequality [5], upper bound 5/6 of linear coding capacity in [93] via Ingleton inequality [94].
16) Wyner’s common information [58]: data processing property, lower-bounded by \( I(X;Y) \), tensorization.
17) Gács-Körner common information [59]: data processing property, upper-bounded by \( I(X;Y) \), tensorization.
18) Common entropy [60]: data processing property, lower-bounded by Wyner’s common information, subadditivity.
19) Excess functional information [65], upper bounds in [65] Proposition 3.5].
20) Non-Shannon-type inequalities such as the Zhang-Yeung inequality [5] and [50].

---

10This code is also given as an example in [https://github.com/cheuktingli/psitip](https://github.com/cheuktingli/psitip)
11Source code is available at [https://github.com/cheuktingli/psitip](https://github.com/cheuktingli/psitip). Jupyter notebooks of the proofs are available at [https://nbviewer.jupyter.org/github/cheuktingli/psitip/tree/master/examples/](https://nbviewer.jupyter.org/github/cheuktingli/psitip/tree/master/examples/).
A. Proof of Theorem 11

The equivalence between statement 2 and statement 3 follows directly from the duality of linear programming.

We then prove if \(A \rightarrow B\) satisfies statement 3, then it can be deduced using the elementary rules. Assume statement 3 is satisfied. By \((\text{Sha})\), \(\forall X, Y, Z : I(X; Y|Z) \geq 0\). By \((\text{SubV})\), \(\forall X^n : I(X_A; X_B|X_C) \geq 0\) for any \(A, B, C \subseteq [n]\). Applying \((10)\) repeatedly, we have \(\forall X^n : \Gamma_n h(X^n) \geq 0\), or equivalently, \(0^{0 \times (2^n - 1)} \rightarrow \Gamma_n\). We have

\[
A^{EII} \rightarrow B
\]

Finally prove that if \(A \rightarrow B\) can be deduced using the elementary rules, then it satisfies statement 3 (or equivalently, it is trivial). We only need to show that, for any elementary rule, if we start with trivial EIIs, then the new EIIs derived are still trivial. It is clear that \((\text{Sha})\) gives a trivial EII, and \((\text{Cone})\) and \((\text{Abs})\) give trivial EIIs if we start with trivial EIIs.

For \((\text{SubV})\), if \(A \rightarrow B\) is trivial, then there exists \(T \in \{0, 1\}^{l \times n}\) and a matrix \(D\) with nonnegative entries such that \(B_{\text{sub}}(I_n; T) = D\). Fix any \(S \in \{0, 1\}^{n \times n'}\). By the multiplicativity of sub (Proposition 7), we have

\[
B_{\text{sub}}(S; TS) = B_{\text{sub}}(I_n; T_{\text{sub}}(S))
\]

Note that each row of \(\Gamma_{n_{\text{sub}}(S)}\) corresponds to a Shannon-type inequality, and hence is a conic combination of rows of \(\Gamma_{n'}\). Therefore, the EII

\[
A_{\text{sub}}(S) \rightarrow B_{\text{sub}}(I_n; T)
\]

also satisfies statement 3.

For \((\text{SubA})\), let \(S \in \{0, 1\}^{l \times (n+l')}\), and assume \(A \rightarrow B_{\text{sub}}(I_n; T)\) is trivial. There exists \(T \in \{0, 1\}^{l' \times n}\) and a matrix \(D\) with nonnegative entries such that

\[
B_{\text{sub}}(I_n; T) = D\Gamma_n
\]
By the multiplicativity of $\text{sub}$,
\[
B_{\text{sub}} \left( \left[ \frac{L_n - 1}{S} \right] \right) \text{sub}(I_n; T) = B_{\text{sub}}(I_n; U).
\]

where $U = S \left[ \begin{array}{c} I_n \\ T \end{array} \right] \in \mathbb{R}^{k \times n}_{\geq 0}$. Therefore, $A \overset{E,I}{\rightarrow} B$ also satisfies statement 3.

For (Tran), assume $A \overset{E,I}{\rightarrow} B$ and $B \overset{E,I}{\rightarrow} C$ are trivial for $A \in \mathbb{R}^{m_A \times (2^n - 1)}$, $B \in \mathbb{R}^{m_B \times (2^n+1-1)}$, $C \in \mathbb{R}^{m_C \times (2^n+k-1)}$. There exist $S \in \{0,1\}^{k \times n}$, $T \in \{0,1\}^{k \times (n+l)}$ and matrices $D, E$ with nonnegative entries such that
\[
B_{\text{sub}}(I_n; S) = D \left[ \begin{array}{c} A \\ \Gamma_n \end{array} \right],
\]
\[
C_{\text{sub}}(I_{n+l}; T) = E \left[ \begin{array}{c} B \\ \Gamma_{n+l} \end{array} \right].
\]

Let $U = T \left[ \begin{array}{c} I_n \\ S \end{array} \right] \in \mathbb{R}^{k \times n}_{\geq 0}$. We have
\[
C_{\text{sub}}(I_n; S; U) = C_{\text{sub}}(I_{n+l}; T)_{\text{sub}}(I_n; S)
\]
\[
= E \left[ \begin{array}{c} B_{\text{sub}}(I_n; S) \\ \Gamma_n_{\text{sub}}(I_n; S) \end{array} \right]
\]
\[
= E \left[ \begin{array}{c} DA \\ D \Gamma_n \end{array} \right]_{\text{sub}}(I_n; S).
\]

Note that each row of $\Gamma_{n+l,\text{sub}}(I_n; S)$ corresponds to a Shannon-type inequality, and hence is a conic combination of rows of $\Gamma_n$. Therefore, the EII $A \overset{E,I}{\rightarrow} C$ also satisfies statement 3.

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