Approximate Atkin-Serre Conjecture

N. A. Carella

Abstract: Let $\lambda(n)$ be the $n$th coefficient of a modular form $f(z) = \sum_{n \geq 1} \lambda(n)q^n$ of weight $k \geq 4$, let $p^m$ be a prime power, and let $\varepsilon > 0$ be a small number. A pair of completely different approximations of the Atkin-Serre conjecture are presented in this note. The first approximation of the lower bound $p^{(k-1)m/2 - 2k + 2\varepsilon} \leq |\lambda(p^m)|$ is true for all sufficiently large prime powers, and the second approximation of the lower bound $n^{(k-3)/2 + \log \log \log n / \log \log n} \leq |\lambda(n)|$ is true on a subset of integers of density 1.

Contents

1 Introduction 1
2 Notation 4
3 Lower Bound For All Large Integers 4
4 Lower Bound For Almost Every Integer 5
5 Basic Results In Diophantine Approximations 7
6 Coefficients Characteristic Function 10
7 Wirsing Formula 10
8 Harmonic Sums And Products And Positive Densities 11
9 References 12

1 Introduction

The properties of the Fourier coefficients $\lambda : \mathbb{N} \rightarrow \mathbb{C}$ of modular forms

$$f(s) = \sum_{n \geq 1} \lambda(n)q^n = \lambda(1)q + \lambda(2)q^2 + \lambda(3)q^3 + \lambda(4)q^4 + \cdots,$$

(1)

where $s \in \mathbb{C}$ is a complex number in the upper half plane, and $q = e^{i2\pi s}$, are the topics of many studies. Basic information on modular forms, classified by various parameters such as level $N \geq 1$, weight $k \geq 1$, et cetera, and other data are archived in [11]. The
corresponding $L$-function $L(s, f) = \sum_{n \geq 1} \lambda(n)n^{-s}$ is analytic on the complex half plane $\mathcal{H}_f = \{ s \in \mathbb{C} : \Re(s) > (k + 1)/2 \}$, and its functional equation

$$\xi(s) = (2\pi)^{-s} \Gamma(s)L(s, f), \quad \xi(s) = \xi(k - s)$$

facilitates an analytic continuation to the entire complex plane, see [8, p. 376]. One of the basic properties of the Fourier coefficients is the dynamic range of its magnitude

$$-L \leq \lambda(n) \leq U,$$

where $L > 0$ and $U > 0$ are the lower bounds and upper bounds respectively. The earliest results for the upper bounds seem to be the Hecke estimate

$$|\lambda(p^m)| \leq cp^{(k-1)m},$$

where $p^m$ is a prime power, $c > 0$ is a constant, and $k \geq 1$ is the weight, see [19, Theorem 4], [3, Proposition 5.4], et cetera. After many partial results by many authors, this line of research culminated with the effective upper bound (known as Deligne theorem)

$$|\lambda(n)| \leq d(n)n^{(k-1)/2+\varepsilon},$$

where $d(n)$ is the divisors function, and $\varepsilon > 0$ is a small number. Furthermore, the partial upper bound

$$|\lambda(n)| \leq 2n^{(k-1)/2} (\log n)^{-1/2+o(1)},$$

on a subset of integers of density 1, was proved in [12, Theorem 1.1].

On the other hand, the lower bounds for nonvanishing Fourier coefficients have no effective results. However, there is a claims that specifies an effective lower bound.

**Conjecture 1.1.** (Atkin-Serre Conjecture) Let $f$ be a non-CM modular form of weight $k \geq 4$. If $p$ is sufficiently large prime, then, for each $\varepsilon > 0$, there exist constants $c(\varepsilon, f) > 0$, such that

$$|\lambda(p)| \geq c(\varepsilon, f)p^{(k-3)/2-\varepsilon}. $$

Various new partial results on this conjecture appear in [15], [7], et alii. Furthermore, the effective lower bound

$$2p^{(k-1)n/2} \log \log p^n \left(\log p^n\right)^{1/2} < |\lambda(p^n)|,$$

on a subset of primes of density 1, was proved in [7, Theorem 1]. A much weaker, almost trivial, but unconditional lower bound proved in [13, Theorem 1], has the form

$$|\lambda(n)| \geq (\log n)^c,$$

where $n$ is an integer for which $\lambda(n)$ is odd, and $c > 0$ is an effectively computable absolute constant. The authors commented, [op. cit., page 393], that an application of Roth theorem for the approximations of algebraic integers, see [33], to the Ramanujan tau function $\tau(p^n)$ yields

$$|\tau(p^n)| \gg p^{11(n-4)/2-\varepsilon},$$

where the implied constant depends on the prime power $p^n \geq 1$, and $\varepsilon > 0$. But the implied constant is not computable.

The proof of the first approximation in Section 3 applies an explicit version of Liouville theorem for the approximations of algebraic integers to obtain the following result.
Theorem 1.1. Let $\lambda(m) \neq 0$ be the $m$th coefficient of a modular form $f(z) = \sum_{m \geq 1} \lambda(m)q^m$ of weight $k \geq 4$. If the integer $m = p^n$ is a prime power, and $\varepsilon > 0$, then

$$|\lambda(p^n)| \geq \frac{1}{8} p^{(k-3)n/2-2k+2-\varepsilon},$$

(11)
as $p^{n/2} \to \infty$.

The parameters $p^{n/2} > p^5$, and $k \geq 4$ produce a nontrivial lower bound.

Example 1.1. For $k = 4$, consider the trace of the Frobenius $a(n)$ of the modular form $f(z) = \sum_{n \geq 1} a(n)q^n$ attached to a nonsingular elliptic curve. Suppose that $a(p) \neq 0$. Then, at the prime power $p^n$, it satisfies the new explicit lower bound

$$|a(p^n)| \geq \frac{1}{8} p^{n/2-6-\varepsilon},$$

(12)
as $p^5 \leq p^{n/2} \to \infty$.

Example 1.2. For $k = 12$, consider the Ramanujan tau function $\tau(n)$ associated with the modular form $f(z) = \sum_{n \geq 1} \tau(n)q^n$. Suppose that $\tau(p) \neq 0$. Then, at the prime power $p^n$, it satisfies the new explicit lower bound

$$|\tau(p^n)| \geq \frac{1}{8} p^{n/2-22-\varepsilon},$$

(13)
as $p^5 \leq p^{n/2} \to \infty$.

The proof of the second approximation in Section 4 employs completely different method to derive an explicit lower bound for almost all the coefficients. This result has the following claim.

Theorem 1.2. The $n$th Fourier coefficient of a modular form $f(z) = \sum_{n \geq 1} \lambda(n)q^n$ of weight $k \geq 4$ satisfies the followings lower bound and upper bound

$$\frac{k-3}{n} \frac{\log \log n}{\log n} \leq |\lambda(n)| \leq \frac{k-1}{n} \frac{\log \log n}{\log n} + \varepsilon,$$

where $\varepsilon > 0$ is a small number, on a subset of integers $\mathcal{N}_f = \{n \in \mathbb{N} : \lambda(n) \neq 0\}$ of density $\delta(\mathcal{N}_f) = 1$.

Consequently, it confirms the Atkin-Serre conjecture for almost every integer. A similar result, in [10] Theorem 1.6, proves an asymptotic formula for the number of exceptions.

Example 1.3. For $k = 4$, consider the trace of the Frobenius $a(n)$ of the modular form $f(z) = \sum_{n \geq 1} a(n)q^n$ attached to a nonsingular elliptic curve. Let $r = \text{rad}(n)$ be the radical of $n \geq 1$, and suppose that $a(r) \neq 0$. Then, it satisfies the new explicit lower bound

$$|a(n)| \geq n^{\frac{k-3}{2} \frac{\log \log n}{\log n}} ,$$

(14)
on a subset of integers $\mathcal{N}_f = \{n \in \mathbb{N} : \lambda(n) \neq 0\}$ of density $\delta(\mathcal{N}_f) = 1$.

Example 1.4. For $k = 12$, consider the Ramanujan tau function $\tau(n)$ associated with the modular form $f(z) = \sum_{n \geq 1} \tau(n)q^n$. Let $r = \text{rad}(n)$ be the radical of $n \geq 1$, and suppose that $\tau(r) \neq 0$. Then, it satisfies the new explicit lower bound

$$|\tau(p^n)| \geq n^{\frac{k-3}{2} \frac{\log \log n}{\log n}} ,$$

(15)
on a subset of integers $\mathcal{N}_f = \{n \in \mathbb{N} : \lambda(n) \neq 0\}$ of density $\delta(\mathcal{N}_f) = 1$. 
2 Notation

The followings sets of numbers are used within.

1. $\mathbb{P} = \{2, 3, 5, 7, 11, 13, \ldots\}$ denotes the set of prime integers,
2. $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$ denotes the set of nonnegative integers,
3. $\mathbb{Z} = \{3, -2, -1, 0, 1, 2, 3, \ldots\}$ denotes the set of integers,
4. $\mathbb{Q} = \{a/b : a, b \in \mathbb{Z}\}$ denotes the set of rational numbers,
5. $\mathbb{R}$ denotes the set of real numbers,
6. $\mathbb{C} = \{s = a + ib : a, b \in \mathbb{R}\}$ denotes the set of complex numbers.

For a pair of real valued functions $f : \mathbb{R} \to \mathbb{C}$ and $g : \mathbb{R} \to \mathbb{C}$, the followings symbols are used without explanations.

1. $f \ll g$ implies that $a f(x) \leq g(x) \leq b f(x)$, where $a > 0$ and $b > 0$ are constants, for all large real numbers $x$,
2. $f \gg g$ implies that $a f(x) \geq g(x) \geq b f(x)$, where $a > 0$ and $b > 0$ are constants, for all large real numbers $x$,
3. $f = O(g)$ implies that $|f(x)| \leq |g(x)|$, for all large real numbers $x$,
4. $f = o(g)$ implies that $|f(x)/g(x)| \to 0$ as $x \to \infty$,

The followings arithmetic functions are used within.

1. $d(n) = \{d \mid n\}$, is the divisors counting function,
2. $\omega(n) = \{p \mid n\}$, is the prime divisors counting function,
3. $\pi(x) = \{p \leq x\}$, is the prime counting function,
4. $v_p(n)$, is the $p$-adic valuation, or e maximal prime power divisor of the integer $n$,
5. $rad(n) = \prod_{p | n} p$, is the radical of the integer $n$,

3 Lower Bound For All Large Integers

This result is based on an application of effective Diophantine approximation for algebraic numbers.
Proof. (Theorem 1.1) Let \( \alpha_p = p^{(k-1)/2} e^{i\theta_p} \), where \( 0 \leq \theta_p \leq \pi \), be the root of the polynomial
\[
f(T) = a_2 T + a_1 T + a_0 = T^2 - \lambda(p)T + p^{k-1}
\] of weight \( k \geq 4 \). Modify the Binet formula (for integers sequences of the second order defined by a quadratic polynomial) into the following form
\[
\lambda(p^n) = \frac{\alpha_p^{n+1} + \overline{\alpha_p}^{n+1}}{\alpha_p + \overline{\alpha_p}} \frac{\alpha_p^{n+1} - \overline{\alpha_p}^{n+1}}{\alpha_p - \overline{\alpha_p}}
\]
where \( \alpha_p \) is the conjugate of \( \alpha_p \), and \( p^n \) is a prime power. Replace the algebraic integer \( \alpha_p = p^{(k-1)/2} e^{i\theta_p} \), where \( 0 < \theta_p < \pi \). Taking absolute value and simplifying yield
\[
|\lambda(p^n)| = \left| p^{(k-1)/2} \cos(\theta_p) \right| \frac{p^{(k-1)(n+1)/2} e^{i\theta_p}}{2p^{(k-1)/2} \sin(2\theta_p)} \left| \alpha_p^{n+1} \overline{\alpha_p}^{-(n+1)} - 1 \right|
\]
\[
\geq p^{(k-1)n/2} \left| \alpha_p^{n+1} \overline{\alpha_p}^{-(n+1)} - 1 \right|
\]
\[
\geq p^{(k-1)n/2} \cdot \frac{1}{8p^{n+2k-2}}
\]
\[
\geq \frac{1}{8} p^{(k-3)n/2 - 2k + 2 - \varepsilon},
\]
where \( \varepsilon > 0 \).

4 Lower Bound For Almost Every Integer

This result is based on an application of the current effective version of the Sato-Tate conjecture for modular forms, see [16] for recent developments. Define the quantities
\[
L_1 = p^{\frac{k-1}{2} + \frac{\log \log p}{(\log p)^{1/2}}} \quad \text{and} \quad U_1 = p^{\frac{k-1}{2} + \varepsilon}.
\]
Given a modular form \( f(z) = \sum_{n \geq 1} \lambda(n) q^n \) of weight \( k \geq 1 \), let
\[
\mathcal{P}_f = \{ p \in \mathbb{P} : \lambda(p) \geq L_1 \} \subset \mathbb{P}
\]
be a subset of primes of density \( \delta(\mathcal{P}_f) \geq 0 \), and let
\[
\mathcal{N}_f = \{ n = \prod_{p|n} p^{v_p} \in \mathbb{N} : p \in \mathcal{P}_f \} \subset \mathbb{N},
\]
where \( v_p(n) \mid n \) is the maximal prime power divisor, be the generated multiplicative subset of integers of density \( \delta(\mathcal{N}_f) \geq 0 \).
Lemma 4.1. The multiplicative subset of integers $N_f \subset \mathbb{N}$ has density $\delta(N_f) = 1$.

Proof. By Theorem 1 in [7], the inequality
\[
p^{(k-1)/2} \frac{\log \log p}{(\log p)^{1/2}} < |\lambda(p)| < 2p^{(k-1)/2},
\]
is valid on a subset of primes $P_f = \{p \in \mathbb{P} : \lambda(p) \geq L_1\}$ of density 1. Therefore, the number of integers generated by the subset of primes $P_f$ has the asymptotic counting function
\[
N_f(x) = \sum_{n \leq x} \chi_f(n)
\]
(24)
\[
= C_f \cdot \frac{x}{\log x} \prod_{p \leq x, p \in P_f} \left(1 + \frac{\chi_f(p)}{p} + \frac{\chi_f(p^2)}{p^2} + \cdots \right),
\]
where the constant
\[
C_f = \frac{1}{e^{\gamma} \Gamma(\tau)} + o(1) = \frac{1}{e^{\gamma} + o(1)}
\]
(25)
where $\tau = 1$ is the density of the subset of primes $P_f$, $\gamma$ is Euler constant, and $\Gamma(s)$ is the gamma function, see Lemma 6.1 and Lemma 7.1. Now, applying Mertens theorem, see Lemma 8.2 returns
\[
N_f(x) = \left(\frac{1}{e^{\gamma} + o(1)}\right) \cdot \frac{x}{\log x} \prod_{p \leq x, p \in P_f} \left(1 + \frac{1}{p} + \frac{1}{p^2} + \cdots \right)
\]
(26)
\[
= \left(\frac{1}{e^{\gamma} + o(1)}\right) \cdot \frac{x}{\log x} \prod_{p \leq x, p \in P_f} \left(1 - \frac{1}{p}\right)^{-1}
\]
\[
= \left(\frac{1}{e^{\gamma} + o(1)}\right) \cdot \frac{x}{\log x} \left(\left(e^{\gamma} \log x\right)^\tau + O\left(\frac{1}{\log x}\right)\right)
\]
\[
= x + O\left(\frac{x}{\log^2 x}\right),
\]
as $x \to \infty$. ■

For sufficiently large integers $n \geq 1$, the estimates
\[
d(n) = O(n^\varepsilon),
\]
\[
\omega(n) \leq (1 + \varepsilon) \log n/\log \log n,
\]
where $\varepsilon > 0$ is a small number, are used in the proof below, see [1, Theorem 13.12] for more details.

Proof. (Theorem 1.2) The density 1 claim for the subset of integers
\[
N_f = \{n \in \mathbb{N} : p \mid n \Rightarrow p \in P_f\}
\]
(28)
is proved in Lemma 4.1. Next, to estimate the lower bound on the subset of integers $N_f$, consider the prime decomposition

$$|\lambda(n)| = \prod_{p^r || n} |\lambda(p^r)| = \prod_{p^r || n, p \notin P_f} |\lambda(p^r)| \prod_{p^r || n, p \in P_f} |\lambda(p^r)|. \quad (29)$$

Replacing (23) into (29) returns

$$|\lambda(n)| \geq \prod_{p^r || n, p \notin P_f} |\lambda(p^r)| \prod_{p^r || n, p \in P_f} p^{-k-1} \cdot \log \log p \cdot \left(\frac{\log n}{\log \log n}\right)^{1/2} \geq \prod_{p^r || n, p \in P_f} \left(\frac{\log n}{\log \log n}\right)^{1/2} \geq \prod_{p^r || n, p \in P_f} \left(\frac{\log n}{\log \log n}\right)^{1/2} \geq \prod_{p^r || n, p \in P_f} \left(\frac{\log n}{\log \log n}\right)^{1/2} \geq \prod_{p^r || n, p \in P_f} \left(\frac{\log n}{\log \log n}\right)^{1/2}$$

where $\omega(n) \leq (1 + \varepsilon) \log n / \log \log n$ is an upper bound for the number of prime factors from the subset of primes $P_f = \{p \in \mathbb{P} : |\lambda(p)| \geq L_1\}$ of density 1. The lower bound (30) is valid for all large numbers $n \geq 1$. Furthermore, the upper bound follows from the first relation in (27).

Hence, almost every integer $n \geq 1$, the $n$th Fourier coefficient satisfies the inequalities

$$n^{k/2} \cdot \frac{\log \log n}{\log \log n} < |\lambda(n)| < n^{k/2 + \varepsilon}, \quad (31)$$

as $n \to \infty$. \hfill \blacksquare

Example 4.1. For $k = 12$, the $n$th Fourier coefficient is given by the Ramanujan tau function $\tau(n)$. At $n = 1000000$, the values specified by Theorem 1.2 are the followings.

1. $n^{k/2} \cdot \frac{\log \log \log n}{\log \log n} = 1.60 \times 10^{29},$

2. $\tau(10^6) = 26219141861258689102548992000000 = 2.62 \times 10^{32},$

3. $d(n)n^{-k/2} = 4.90 \times 10^{34}.$

5 Basic Results In Diophantine Approximations

The concept of measures of irrationality of real numbers is discussed in [18, p. 556], [2, Chapter 11], et alii. This concept can be approached from several points of views.
Definition 5.1. The irrationality measure \( \mu(\alpha) \) of a real number \( \alpha \in \mathbb{R} \) is the infimum of the subset of real numbers \( \mu(\alpha) \geq 1 \) for which the Diophantine inequality

\[
|\alpha - \frac{p}{q}| \gg \frac{1}{q^{\mu(\alpha)+\varepsilon}}
\]

where \( \varepsilon > 0 \) is an arbitrary small number, holds for all large \( q \geq 1 \).

Let \( \alpha \) be a root of an irreducible polynomial \( f(x) \in \mathbb{Z}[x] \) of degree \( \deg f = d \). The Liouville inequality

\[
|\alpha - \frac{p}{q}| > \frac{c}{q^d},
\]

where \( c > 0 \) is a constant, is the oldest result for irrationality measure of an irrational number as \( \alpha \). The Liouville result was superseded by Thue-Roth theorem for the approximations of algebraic integers

\[
|\alpha - \frac{p}{q}| > \frac{c(\alpha, \varepsilon)}{q^{d+\varepsilon}},
\]

where the constant \( c(\alpha, \varepsilon) > 0 \) is not computable.

A completely explicit version of the Liouville theorem is considered here. The proof given below is relatively recent, an older version appears in [14, Theorem 7.8]. The preliminary definitions are a few concepts used in the proof.

Definition 5.2. Let \( r = a/b \in \mathbb{Q}^\times \) with \( \gcd(a, b) = 1 \), be a rational number. The \textit{height} is defined by

\[
H(r) = \max\{|a|, |b|\}.
\]

Definition 5.3. Let \( \alpha \) be a root of the irreducible polynomial

\[
f(x) = a_d(x - \alpha_1)(x - \alpha_2)\cdots(x - \alpha_d).
\]

The \textit{Mahler height} is defined by

\[
H(\alpha) = |a_d| \prod_{1 \leq i \leq d} \max\{1, |\alpha_i|\}.
\]

Theorem 5.1. (Liouville) Let \( \alpha \) be a real algebraic number of degree \( d \geq 1 \). There is a constant \( c(\alpha) > 0 \) depending only on \( \alpha \) such that

\[
|\alpha - \frac{p}{q}| \geq \frac{c(\alpha)}{H(p/q)^d},
\]

where \( c(\alpha) \geq 2^{1-d}H(\alpha)^{-1} \), for every rational number \( p/q \) if \( d \geq 2 \).

Proof. (Same as [5, Theorem 8.1.1]) Let \( f(x) = a_dx^d + a_{d-1}x^{d-1} + \cdots + a_1x + a_0 \in \mathbb{Z}[x] \) be the minimal polynomial of \( \alpha \). Take \( p/q \in \mathbb{Q} \), where \( p, q \in \mathbb{Z} \), and consider the number

\[
F(p, q) = q^df(p/q) = a_dx^d + a_{d-1}x^{d-1}q + \cdots + a_1qx^{d-1} + a_0q^d.
\]

By assumption, \( f(p/q) \neq 0 \). Hence, \( F(p, q) \) is a nonzero integer. So \( |F(p, q)| \geq 1 \). To obtain an upper bound for \( |F(p, q)| \), rewrite it in factored form

\[
F(p, q) = a_d(p - \alpha_1)(p - \alpha_2)\cdots(p - \alpha_d),
\]

and estimate each factor as follows. Let \( \alpha = \alpha_1 \), then
1. \(|p - \alpha q| = \left| \frac{p}{q} - \alpha \right| |q| \leq \left| \frac{p}{q} - \alpha \right| H(\alpha)|

2. \(|p - \alpha_i q| \leq 2 \max \{1, |\alpha_i|\} \cdot \max \{|p|, |q|\} \leq 2 \max \{1, |\alpha_i|\} H(p/q)|

where \(i = 2, 3, \ldots, d\). Thus,

\[
|F(p, q)| = |a_d| \prod_{1 \leq i \leq d} |p - \alpha_i q| \leq |a_d| \left| \frac{p}{q} - \alpha \right| \cdot H(p/q)^d \cdot 2^{d-1} \cdot \prod_{2 \leq i \leq d} \max \{1, |\alpha_i|\}
\]

\[
\leq |a_d| \left| \frac{p}{q} - \alpha \right| \cdot H(p/q)^d \cdot 2^{d-1} \cdot H(\alpha).
\]

Combined the lower bound \(|F(p, q)| \geq 1\) and the inequality \((41)\) to complete the proof. 

**Corollary 5.1.** Let \(\beta\) be an algebraic number of degree \(d = 2\) and height \(H(\beta) \leq 4p^{2(k-1)}\). Let \(\{p_m/q_m : m \geq 1\}\) be a subsequence of convergents of height \(H(p_m/q_m) \leq p^n/2\), where \(p^n\) is a large prime power. Then,

\[
\left| \beta - \frac{p_m}{q_m} \right| \geq \frac{1}{8p^{n+2k-2}},
\]

as \(q_m \leq p^{n/2} \to \infty\).

**Proof.** Let \(\beta\) be a root of the polynomial \(f(T) = a_2T^2 + a_1T + a_0\) of degree \(\deg f = 2\), and coefficients \(|a_0|, |a_1| \leq p^{k-1}\), \(a_2 = 1\). These data imply that \(|\beta| \leq 2p^{k-1}\), and the height is at most

\[
H(\beta) = |a_d| \prod_{1 \leq i \leq d} \max \{1, |\beta_i|\} \leq 4p^{2(k-1)},
\]

see Definition 5.3. Hence, the constant in inequality \((36)\) has at least the value

\[
c(\beta) \geq 2^{1-d} H(\beta)^{-1} = 2^{1-2} \cdot (4p^{2(k-1)})^{-1} = \frac{1}{8p^{2(k-1)}}
\]

and the height of the rational approximation is

\[
H(p_m/q_m) \leq p^{n/2}.
\]

An application of Theorem 5.1 yields

\[
\left| \beta - \frac{p_m}{q_m} \right| \geq \frac{c(\beta)}{H(p_m/q_m)^d} \geq \frac{1}{8p^{2(k-1)} (p^{n/2})^2} \geq \frac{1}{8p^{n+2k-2}}.
\]

**Corollary 5.2.** Let \(\beta = \alpha_p^{n+1} \alpha_p^{-(n+1)}\) be an algebraic number of degree \(d = 2\) and height \(H(\beta) \leq 4p^{2(k-1)}\). Let \(\{p_m/q_m : m \geq 1\}\) be a subsequence of convergents of height \(H(p_m/q_m) \leq p^n/2\), where \(p^n\) is a large prime power. Then,

\[
|\beta - 1| > \frac{1}{8p^{n+2k-2}},
\]

as \(q_m \leq p^{n/2} \to \infty\).
Proof. The inverse triangle inequality, and Corollary 5.1 lead to
\[
|\beta - 1| = \left| \beta - 1 + \frac{p_m}{q_m} - \frac{p_m}{q_m} \right| \\
\geq \left| \beta - \frac{p_m}{q_m} \right| - \left| 1 - \frac{p_m}{q_m} \right| \\
\geq \frac{1}{8p^{n+2k-2}},
\]
since
\[
|1 - \frac{p_m}{q_m}| = \left| \frac{q_m - p_m}{q_m} \right| \geq \frac{1}{p^{n/2}},
\]
as \(q_m \leq p^{n/2} \to \infty\).

6 Coefficients Characteristic Function

The characteristic function of the nonzero Fourier coefficients of a modular form \(f(z) = \sum_{n \geq 1} \lambda(n)q^n\) is defined by
\[
\chi_f(p) = \begin{cases} 
1 & \text{if } \lambda(p) \neq 0; \\
0 & \text{if } \lambda(p) = 0.
\end{cases}
\]

Lemma 6.1. The function \(\chi_f : \mathbb{Z} \to \{0, 1\}\) is completely multiplicative.

Proof. Suppose that \(\lambda(p) \neq 0\), equivalently \(\chi_f(p) = 1\). It is sufficient to verify the multiplicative property at the prime power \(p^k\), where \(k \geq 2\). Recursively eliminating the term \(\chi_f(p) = 1\) yields
\[
\chi_f(p^k) = \chi_f(p^{k-1}) \chi_f(p) \\
= \chi_f(p^{k-2}) \chi_f(p) \\
\vdots \\
= \chi_f(p) \\
= 1.
\]
Therefore, \(1 = \chi_f(p) = \chi_f(p^2) = \cdots = \chi_f(p^k)\), as claimed.

7 Wirsing Formula

This formula provides decompositions of some summatory multiplicative functions as products over the primes supports of the functions. This technique works well with certain multiplicative functions, which have supports on subsets of primes numbers of nonzero densities.

Lemma 7.1. ([17] p. 71) Suppose that \(f : \mathbb{N} \to \mathbb{C}\) is a multiplicative function with the following properties.
(i) $f(n) \geq 0$ for all integers $n \in \mathbb{N}$.

(ii) $f(p^k) \leq c^k$ for all integers $k \in \mathbb{N}$, and $c < 2$ constant.

(iii) There is a constant $\tau > 0$ such

$$\sum_{p \leq x} f(p) = (\tau + o(1)) \frac{x}{\log x}$$

as $x \to \infty$.

Then

$$\sum_{n \leq x} f(n) = \left(\frac{1}{e^{\tau} \Gamma(\tau)} + o(1)\right) \frac{x}{\log x} \prod_{p \leq x} \left(1 + \frac{f(p)}{p} + \frac{f(p^2)}{p^2} + \cdots\right).$$

The gamma function appearing in the above formula is defined by $\Gamma(s) = \int_0^\infty t^{s-1} e^{-st} dt, s \in \mathbb{C}$.

8 Harmonic Sums And Products And Positive Densities

Let $f(z) = \sum_{n \geq 1} \lambda(n)q^n$ be a modular form of weight $k \geq 1$, let $L_1 = L_1(p) > 0$ be a decreasing function of the primes $p$, and let $P_f = \{p \in \mathbb{P} : |\lambda(n)| \geq L_1\} \subset \mathbb{P}$ be a subset of primes of nonzero density $\delta(P_f) > 0$. The proofs of these results are based on standard analytic number theory methods in the literature.

**Lemma 8.1.** If $x \geq 1$ is a large number, and $\tau = \delta(P_f) > 0$ is the density of the subset of primes, then,

$$\sum_{p \leq x} \frac{1}{p} = (\log \log x + B) \tau + O\left(\frac{1}{\log x}\right),$$

where $B > 0$ is a constant

**Proof.** Let $\pi_f(x) = \#\{p \leq x : |\lambda(n)| \geq L_1\} = \tau \pi(x)$ be the counting measure of the corresponding subset of primes $P_f$. To estimate the asymptotic order of the prime harmonic sum, use the Stieltjes integral representation:

$$\sum_{p \leq x} \frac{1}{p} = \int_1^x \frac{1}{t} d\pi_f(t) = \frac{\pi_f(x)}{x} + \int_1^x \frac{\pi_f(t)}{t^2} dt.$$  (55)

Applying the prime number theorem $\pi(x) = x/\log x + O(x/\log^2 x)$, see [4, Section 27.12], [6], etc., yields

$$\int_1^x \frac{1}{t} d\pi_f(t) = \frac{\tau}{\log x} + O\left(\frac{1}{\log^2 x}\right) + \tau \int_1^x \frac{1}{t \log t} + O\left(\frac{1}{t \log^2 t}\right) dt$$

$$= \tau (\log \log x + B) + O\left(\frac{1}{\log x}\right),$$

where $B > 0$ is Mertens constant.  ■
The Euler constant and the Mertens constant are defined by the limits
\[
\gamma = \lim_{x \to \infty} \left( \sum_{p \leq x} \frac{\log p}{p} - \log x \right) \quad \text{and} \quad B = \lim_{x \to \infty} \left( \sum_{p \leq x} \frac{1}{p} - \log \log x \right),
\]
(57)
or some other equivalent definitions, respectively.

**Lemma 8.2.** If \(x \geq 1\) is a large number, and \(\mathcal{P}_f \subset \mathbb{P}\) is a subset of primes of nonzero density \(\tau = \delta(\mathcal{P}_f) > 0\), then,
\[
\prod_{p \leq x, p \in \mathcal{P}_f} \frac{1}{1 - \frac{1}{p}} = (e^\gamma \log x)^\tau + O\left(\frac{1}{\log x}\right).
\]
(58)

**Proof.** Express the logarithm of the product as
\[
\sum_{p \leq x, p \in \mathcal{P}_f} \log \left(1 - \frac{1}{p}\right)^{-1} = \sum_{p \leq x} \sum_{k \geq 1} \frac{1}{kp^k} = \sum_{p \leq x} \frac{1}{p} + \sum_{p \leq x} \sum_{k \geq 2} \frac{1}{kp^k}.
\]
(59)

Apply Lemma 8.1 and the linear independence relation \(B = \gamma - \sum_{p \geq 2} \sum_{k \geq 2} (kp^k)^{-1}\), see [9, Theorem 427], to complete the verification.

9 References

References

[1] Apostol, Tom M. *Introduction to analytic number theory*. Undergraduate Texts in Mathematics. Springer-Verlag, New York-Heidelberg, 1976.

[2] Borwein, J. M. and Borwein, P. B. *AGM: A Study in Analytic Number Theory and Computational Complexity*. New York: Wiley, pp. 362-386, 1987.

[3] H. Cohen. *An Introduction to Modular Forms*. Ilker Inam; Engin Büyükaşık, Editors. Birkhäuser, pp.3-62, 2019, Workshops in the Mathematical Sciences. ffhal-01883058.

[4] F. Olver, M. McClain, et al, Editors. *Digital Library of mathematical Functions*. http://dlmf.nist.gov/5.4.el.

[5] J. H. Evertse. *Notes on Diophantine Approximation*. Lecture Notes, 2007.

[6] Ellison, William; Ellison, Fern. *Prime numbers*. A Wiley-Interscience Publication. John Wiley and Sons, Inc., New York; Hermann, Paris, 1985.

[7] Ayla Gafni, Jesse Thorner, And Peng-Jie Wong. *Applications Of The Sato-Tate Conjecture*. https://arxiv.org/pdf/2003.09026.

[8] Hafner, James. Ivi, Aleksandar. *On Sums Of Fourier Coefficients Of Cusp Forms*. L’Enseignement Mathematique, Vol.35 (1989).

[9] Hardy, G. H.; Wright, E. M. *An introduction to the theory of numbers*. Sixth edition. Oxford University Press, 2008.
[10] Alexandra Hoey, Jonas Iskander, Steven Jin, Fernando Trejos Suarez. *An unconditional explicit bound on the error term in the Sato-Tate conjecture*. [http://arxiv.org/abs/2108.03520](http://arxiv.org/abs/2108.03520).

[11] LMFDB. *L-Functions, Modular Forms Data Base*, lmfdb.org, 2019.

[12] F. Luca, M. Radziwill, I. E. Shparlinski. *On the typical size and cancellations among the coefficients of some modular forms*. Mathematical Proceedings of the Cambridge Philosophical Society, Volume 166, Issue 1, pp. 173-189, 2019. [http://arxiv.org/abs/1308.6606](http://arxiv.org/abs/1308.6606).

[13] V. K. Murty, R. Murty, N. Saradha. *Odd prime values of the Ramanujan tau function*. Bull. Soc. Math. France 115 (1987), 391-395.

[14] Niven, I. *Irrational Numbers*, Am. Math. Assotiation, Weley and Sons Inc., 1956.

[15] Jeremy Rouse. *Atkin-Serre Type Conjectures For Automorphic Representations On GL(2)*. Math. Res. Lett. 14 (2007), no. 2, 189-204.

[16] Thorner, Jesse. *Effective forms of the Sato–Tate conjecture*. [http://arXiv.org/abs/2002.10450](http://arXiv.org/abs/2002.10450) doi 10.1007/s40687-020-00234-3.

[17] Wirsing, E. *Das asymptotische Verhalten von Summen über multiplikative Funktionen*, Math. Ann. 143 (1961) 75-102.

[18] Waldschmidt, M. *Diophantine approximation on linear algebraic groups. Transcendence properties of the exponential function in several variables*. Grundlehren der Mathematischen Wissenschaften, 326. Springer-Verlag, Berlin, 2000.

[19] Zagier, D.. *Modular Forms of One variable*, [https://people.mpim-bonn.mpg.de/zagier/files/tex/UtrechtLectures/UtBook.pdf](https://people.mpim-bonn.mpg.de/zagier/files/tex/UtrechtLectures/UtBook.pdf), 2013.

AtkinSerreConjectureApproximate-08-30-21-17-arxiv.tex.