Global injectivity of differentiable maps via W-condition in $\mathbb{R}^2$ *

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Abstract

In this paper, we study the intrinsic relation between the global injectivity of differentiable local homeomorphisms $F$ and the rate that tends to zero of $Spec(F)$ in $\mathbb{R}^2$, where $Spec(F)$ denotes the set of all (complex) eigenvalues of $DF(x)$, for all $x \in \mathbb{R}^2$. This depends on the $W$-condition deeply, which extends the $*$-condition and $B$-condition. The $W$-condition reveals the rate that tends to zero of real eigenvalues of $DF$ can not exceed $O\left(x \log x \left(\frac{\log x}{\log \log x}\right)^2\right)^{-1}$ by the half-Reeb component method. This improves the theorems of Gutiérrez-Nguyen [14] and various results in the articles ([8], [9], [13], [15], [22], [23]). The $W$-condition is optimal for the half-Reeb component method in this paper setting.

Key words: $W$-condition; Half-Reeb component; Jacobian conjecture.

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1 Introduction

On the long-standing Jacobian conjecture, it is still open even in the case $n = 2$. There are many results on it, see for example [1] and [7].

A very important step, for example in $\mathbb{R}^2$, is the following result, due to A. Fernandes, C. Gutiérrez, and R. Rabanal:

**Theorem 1.1.** ([8]) Let $X = (f, g) : \mathbb{R}^2 \to \mathbb{R}^2$ be a differentiable map. For some $\varepsilon > 0$, if
\[ \text{Spec}(X) \cap [0, \varepsilon) = \emptyset, \] (1.1)
then $X$ is injective.

Theorem 1.1 is deep. If the assumptions (1.1) replaced by $0 \notin \text{Spec}(F)$, then the conclusion is false, even for polynomial map $X$, as the Pinchuck’s counterexample [21]. Pmyth and Xavier [26] proved that there exists $n > 2$ and non-injective polynomial map such that $\text{Spec}(X) \cap [0, +\infty) = \emptyset$.

Theorem 1.1 added to a long sequence of results on Markus-Yamabe conjecture [20] and the eigenvalue conditions of some map for injectivity in dimension two. The Markus-Yamabe Conjecture has been solved independently in 1993 by C. Gutiérrez [11] and R. Fessler [10]. It is false in dimension $n \geq 3$ even for polynomial vector field [5]. Theorem 1.1 also implies that Chamberland conjecture [4] is true in dimension $n = 2$.

The essential tool to prove Theorem 1.1 is making use of the concept of the half-Reeb component that we recall in Definition 2.1.

C. Gutiérrez and V. Ch. Nguyen [14] study the geometrical behavior of differentiable maps in $\mathbb{R}^2$ and the following $\ast$-condition on the real eigenvalues of $DF$ under the half-Reeb component technique.

For each $\theta \in \mathbb{R}$, we denote by $R_\theta$ the linear rotation
\[ R_\theta = R_\theta \circ F \circ R_{-\theta}. \]

and define the map $F_\theta = R_\theta \circ F \circ R_{-\theta}$.

**Definition 1.1.** ($\ast$-condition) A differentiable $F$ satisfies the $\ast$-condition if for each $\theta \in \mathbb{R}$, there does not exist a sequence $\mathbb{R}^2 \ni z_k \to \infty$ such that, $F_\theta(z_k) \to T \in \mathbb{R}^2$ and $DF_\theta(z_k)$ has a real eigenvalue $\lambda_k \to 0$.

**Theorem 1.2.** ([14]) Suppose that $X : \mathbb{R}^2 \to \mathbb{R}^2$ is a differentiable local homeomorphism. Then:

(i) If $X$ satisfies $\ast$-condition, then $X$ is injective and its image is a convex set.

(ii) $X$ is a global homeomorphism of $\mathbb{R}^2$ if and only if $X$ satisfies $\ast$-condition and its image $X(\mathbb{R}^2)$ is dense in $\mathbb{R}^2$.

$\ast$-condition is somewhat weaker than condition (1.1), thus one can obtain the Theorem 1.1 from Theorem 1.2 (i) by a standard procedure.

In other new case, the essential difficulty is that the eigenvalues of $DF$ which may be tending to zero implies $F$ is globally injective. R. Rabanal [23] extended the $\ast$ condition to the so called $B$-condition.
Definition 1.2. (B-condition) The differentiable map $F : \mathbb{R}^2 \to \mathbb{R}^2$ satisfies the B-condition if for each $\theta \in \mathbb{R}$, there does not exist a sequence $(x_k, y_k) \in \mathbb{R}^2$ with $x_k \to +\infty$ such that $F_0(x_k, y_k) \to T \in \mathbb{R}^2$ and $DF_0(x_k, y_k)$ has a real eigenvalue $\lambda_k$ satisfying $\lambda_k x_k \to 0$.

He obtains the following theorem where Theorem 1.4 holds if one replaced $*$-condition by B-condition.

Theorem 1.3. (12) Suppose that the differentiable map $F : \mathbb{R}^2 \to \mathbb{R}^2$ satisfies the B-condition and $\det DF(z) \neq 0$, $\forall z \in \mathbb{R}^2$, then $F$ is a topological embedding.

In fact, Theorem 1.3 improves the main results of (14). (see (25), (22)).

In 2014, F. Braun and V. S. Jean (3) considered the relation between the half-Reeb component and Palais-Smale condition for global injectivity.

Many references on other aspects of half Reeb component including in higher dimensional situations see (24), (17), (18), (19), (12).

For example, C. Gutiérrez and C. Maquera considered half-Reeb components for the global injectivity in dimension 3.

Theorem 1.4. (12) Let $Y = (f, g, h) : \mathbb{R}^3 \to \mathbb{R}^3$ be a polynomial map such that $\text{Spec}(Y) \cap [0, \varepsilon) = \emptyset$, for some $\varepsilon > 0$. If $\text{codim}(SY) \geq 2$, then $Y$ is a bijection.

Recently, W. Liu prove the following theorem by the Minimax method.

Theorem 1.5. (18) Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a $C^1$ map, $n \geq 2$. If for some $\varepsilon > 0$,

$$0 \notin \text{Spec}(F) \quad \text{and} \quad \text{Spec}(F + F^T) \subseteq (-\infty, -\varepsilon) \cup (\varepsilon, +\infty),$$

then $F$ is globally injective.

Let us return to study approaching to zero of the eigenvalues of $DF$ by the half-Reeb component method in $\mathbb{R}^2$.

For the convenience of our statement, let us set

$$\mathcal{P} = \left\{ P \mid \mathbb{R}^+ \to \mathbb{R}^+, P \text{ is nondecreasing and } \forall M > 0, \text{ there exists large constant } N \right\},$$

which depends on $M$ and $P$, such that

$$\int_2^N \frac{1}{P(x)} dx > M.$$
tends to zero of eigenvalues of $DF$ must be in the interval \( \left( O(x \log^2 x)^{-1}, \forall \beta > 1, O\left( x \log x \left( \log \left( \log x \right) \right)^2 \right)^{-1} \right) \) by the half-Reeb component method.

**Remark 1.2.** If \( x_k \) exchanges \( y_k \) in definition 1.3 then it is also applied.

We use the \( W \)-condition and obtain the following results.

**Theorem 1.6.** Let \( F : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) be a differentiable local homeomorphism. If \( F \) satisfies \( W \)-condition, then \( F \) is injective and \( F(\mathbb{R}^2) \) is convex.

Obviously, Theorem 1.6 implies Theorem 1.2 and Theorem 1.3(i).

Next, we have the following results.

**Theorem 1.7.** Let \( F : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) be a differentiable Jacobian map. If \( F \) satisfies \( W \)-condition, then \( F \) is a measure-preserving map and globally injective.

**Corollary 1.1.** If \( F \) is as in Theorem 1.7 and \( \text{Spec}(F) \subseteq \{ z \in \mathbb{C} | |z| < 1 \} \), then \( F \) has at most one fixed point.

**Theorem 1.8.** Let \( F = (f, g) : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) be a local homeomorphism such that for some \( s > 0 \), \( F|_{\mathbb{R}^2 \setminus D_s} \) is differentiable. If \( F \) satisfies the \( W \)-condition, then it is a globally injective and \( F(\mathbb{R}^2) \) is a convex set.

**Theorem 1.9.** Let \( F = (f, g) : \mathbb{R}^2 \setminus D_{\sigma} \rightarrow \mathbb{R}^2 \) be a differential map which satisfies the \( W \)-condition. If \( \text{Spec}(F) \cap [0, +\infty) = \emptyset \) or \( \text{Spec}(F) \cap (-\infty, 0] = \emptyset \), then there exists \( s \geq \sigma \) such that \( F|_{\mathbb{R}^2 \setminus D_s} \) can be extended to an injective local homeomorphism \( \tilde{F} = (\tilde{f}, \tilde{g}) : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \).

In a word, we extend the theorems in the articles (8, 9, 15, 13, 16, 22, 25, 23).

These works are related to the Jacobian conjecture which can be reduce to that for all dimension \( n \geq 2 \), a polynomial map \( F : \mathbb{C}^n \rightarrow \mathbb{C}^n \) of the form \( F = x + H \), where \( H \) is cube-homogeneous and \( JH \) is symmetry, is injective if \( \text{Spec}(F) = \{ 1 \} \). (see 2).

In order to prove our theorems, we need to use the definition and propositions of the half-Reeb component.

## 2 Half-Reeb component

In this section, we will introduce some preparation on the eigenvalue conditions of \( \text{Spec}(F) \).

Let \( h_0(x, y) = xy \) and consider the set
\[
B = \{ (x, y) \in [0, 2] \times [0, 2] | 0 < x + y \leq 2 \}.
\]
Definition 2.1. (half-Reeb component) Let $X$ be a differentiable map from $\mathbb{R}^2 \to \mathbb{R}^2$. Let $D\mathcal{X}_p \neq \emptyset, \forall p \in \mathbb{R}^2$. Given $h \in \{f, g\}$, we will say that $A \subseteq \mathbb{R}^2$ is a half-Reeb component for $\mathcal{F}(h)$ (or simply a hRc for $\mathcal{F}(h)$) if there exists a homeomorphism $H : B \to A$ which is a topological equivalence between $\mathcal{F}(h)|_A$ and $\mathcal{F}(h_0)|_B$ and such that:

1. The segment $\{(x, y) \in B : x + y = 2\}$ is sent by $H$ onto a transversal section for the foliation $\mathcal{F}(h)$ in the complement of $H(1, 1)$; this section is called the compact edge of $A$.

2. Both segments $\{(x, y) \in B : x = 0\}$ and $\{(x, y) \in B : y = 0\}$ are sent by $H$ onto full half-trajectories of $\mathcal{F}(h)$. These two semi-trajectories of $\mathcal{F}(h)$ are called the noncompact edges of $A$.

Proposition 2.1. Suppose that $X = (f, g) : \mathbb{R}^2 \to \mathbb{R}^2$ is a differentiable map such that $0 \notin \text{Spec}(X)$. If $X$ is not injective, then both $\mathcal{F}(f)$ and $\mathcal{F}(g)$ have half-Reeb components.

Proposition 2.2. Let $X = (f, g) : \mathbb{R}^2 \to \mathbb{R}^2$ be a non-injective, differentiable map such that $0 \notin \text{Spec}(X)$: Let $\mathcal{A}$ be a hRc of $\mathcal{F}(f)$ and let $(f_0, g_0) = R_\theta \circ X \circ R_{-\theta}$, where $\theta \in \mathbb{R}$ and $R_\theta$ is in (1.3). If $\Pi(x, y) = x$ is bounded, where $\Pi : \mathbb{R}^2 \to \mathbb{R}$ is given by $\Pi(x, y) = x$, then there is an $\varepsilon > 0$ such that, for all $\theta \in (-\varepsilon, 0) \cup (0, \varepsilon)$; $\mathcal{F}(f_\theta)$ has a hRc $\mathcal{A}_\theta$ such that $\Pi(\mathcal{A}_\theta)$ is an interval of infinite length.

3 Half-Reeb component and $W$-condition

In this section, we will establish the essential fact that the $W$-condition ensures non-existence of half-Reeb component.

Let $F = (f, g) : \mathbb{R}^2 \to \mathbb{R}^2$ be a local homeomorphism of $\mathbb{R}^2$. For each $\theta \in \mathbb{R}$, we denoted by $R_\theta$ the linear rotation (see (1.2)):

$$(x, y) \to (x \cos \theta - y \sin \theta, x \sin \theta + y \cos \theta),$$

and

$$F_\theta := (f_\theta, g_\theta) = R_\theta \circ F \circ R_{-\theta}.$$ 

In other words, $F_\theta$ represents the linear rotation $R_\theta$ in the linear coordinates of $\mathbb{R}^2$.

Proposition 3.1. A differentiable local homeomorphism $F : \mathbb{R}^2 \to \mathbb{R}^2$ which satisfies $W$-condition has no half-Reeb components.

Proof. Suppose by contradiction that $F$ has a half-Reeb component. In order to obtain this result, we consider the map $(f_\theta, g_\theta) = F_\theta$. From Proposition 2.2, there exists some $\theta \in \mathbb{R}$, such that $\mathcal{F}(A_\theta)$ has a half-Reeb component which $\Pi(A)$ is unbounded interval, where $\Pi(A)$ denote orthogonal projection onto the first coordinate in $A$. Therefore $\exists b$ and a half-Reeb component $A$, such that $[b, +\infty) \subseteq \Pi(A)$. Then, for large enough $a > b$ and any $x \geq a$, the vertical line $\Pi^{-1}(x)$ intersects exactly one trajectory $\alpha_x \cap [x, +\infty) = x$, i.e. $x$ is maximum of the the trajectory $\Pi_x$. If $x \geq a$, the intersection $\alpha_x \cap \Pi^{-1}(x)$ is compact subset in $A$. 

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Thus, we can define functions $H : (a, +\infty) \to \mathbb{R}$ by
\[
H(x) = \sup \{ y : (x, y) \in \Pi^{-1}(x) \cap \alpha_x \}.
\]
As $\mathcal{F}(f_\theta)$ is a foliation and we can obtain
\[
\Phi : (a, +\infty) \quad \text{by} \quad \Phi(x) = f_\theta(x, H(x)).
\]
We can know that $\Phi$ is a bounded, strictly monotone function such that, for some full measure subset $M \subseteq (a, +\infty)$.

Since the image of $\Phi$ is contained in $f_\theta(\Gamma)$ where $\Gamma$ is compact edge of $hRc \mathcal{A}$, the function $\Phi$ is bounded in $(a, +\infty)$. Furthermore, $\Phi$ is continuous because $\mathcal{F}(f_\theta)$ is a $C^0$ foliation. And since $\mathcal{F}(f_\theta)$ is transversal to $\Gamma$, $\Phi$ is monotone strictly.

For the measure subset $M \subseteq (a, +\infty)$, such that $\Phi(x)$ is differentiable on $M$ and the Jacobian matrix of $F_\theta(x, y)$ at $(x, H(x))$ is
\[
DF_\theta(x, H(x)) = \begin{pmatrix}
\Phi'(x) & 0 \\
\partial_x g_\theta(x, H(x)) & \partial_y g_\theta(x, H(x))
\end{pmatrix}.
\]
Therefore, $\forall x \in M$, $\Phi'(x) = \partial_x f_\theta(x, H(x))$ is a real eigenvalue of $DF_\theta(x, H(x))$ and we denote it by $\lambda(x) := \Phi'(x)$.

Since $F$ is local homeomorphism, without loss of generality, we assume $\Phi$ is strictly monotone increasing, $\Phi'(x) > 0, \forall x \in M$. Let any function $P \in \mathcal{P}$, where $\mathcal{P} = \{ P : \mathbb{R}^+ \to \mathbb{R}^+, P \text{ is nondecreasing and } \forall M > 0, \text{ there exists large constant } N \text{ which depends on } M \text{ and } P, \text{ such that } \int_2^N \frac{1}{P(x)} \, dx > M \}$.

Claim:
\[
\lim_{x_k \to +\infty} \Phi'(x_k)P(x_k) > 0.
\]
Because $P(x)$ and $\Phi'(x)$ are both positive, we can suppose by contradiction that \( \lim_{x_k \to +\infty} \Phi'(x_k)P(x_k) = 0 \). There exists a subsequence denoted still $\{ x_k \}$ with $x_k \to +\infty$ such that $\Phi'(x_k)P(x_k) \to 0$. That is $\lambda(x_k)P(x_k) \to 0$. Since $F_\theta(\mathcal{A})$ is bounded, $F_\theta(x_k, H(x_k))$ converges to a finite value $T$ on compact set $\overline{\mathcal{F}_\theta(\mathcal{A})}$. This contradicts the $W$-condition.

Therefore, there exist constant $a_0$ ($a_0 > 2$) and small $\varepsilon_0 > 0$, such that
\[
\Phi'(x)P(x) > \varepsilon_0, \quad \forall x \geq a_0.
\]
Since $\Phi(x)$ is bounded, there exists $L > 0$, such that
\[
\Phi(x) - \Phi(a_0) \leq L, \quad \forall x \geq a_0.
\]
By the definiton of $\mathcal{P}$, we can choose $C$ large enough, such that
\[
\int_{a_0}^{C} \frac{1}{P(x)} \, dx > \frac{L}{\varepsilon_0}.
\]
Thus,

\[ L \geq \Phi(C) - \Phi(a_0) = \int_{a_0}^{C} \Phi'(x)dx \geq \int_{a_0}^{C} \frac{\varepsilon_0}{F(x)}dx > L. \]

It is contradiction.

4 The Proof of Theorem 1.6

Proof. Suppose by contradiction that \( F \) is not injective. By Proposition 2.1, \( F \) has a half-Reeb component, this contradicts Proposition 3.1 that \( F \) has no half-Reeb component if \( F \) satisfies the \( W \)-condition.

5 The Proof of Theorem 1.7

Proof. Firstly, we prove the equivalence of the Jacobian map and measure-preserving in any dimension \( n \).

For any nonempty measurable set \( \Omega \subset \mathbb{R}^n \). Since \( F : \mathbb{R}^n \to \mathbb{R}^n \), denote \( V = \{ F(x) \mid x \in \Omega \} \). Let the components of \( F(x) \) be \( v_i(i = 1, 2...n) \), i.e. \( F(x_1,...x_n) = (v_1(x_1,...x_n),...v_n(x_1...x_n)) \). So \( dv = det F'(x)dx \). Since \( det F'(x) \equiv 1 \), we get \( dv = dx \).

Therefore, \( \int_V dv = \int_\Omega dx \). It implies \( F \) preserves measure.

Inversely, let \( v = F(x) \), \( \forall x \in \Omega \). We still denote \( V = \{ F(x) \mid x \in \Omega \} \).

Since \( F \) preserves measure, one gets \( \int_V dv = \int_\Omega dx \).

Combining with \( dv = det F'(x)dx \), we obtain \( \int_V dv = \int_\Omega det F'(x)dx \).

Thus, we have \( \int_\Omega dx = \int_\Omega det F'(x)dx \). That is

\[ \int_\Omega (1 - det F'(x))dx = 0, \forall \Omega \subset \mathbb{R}^n. \]

Claim: \( det F'(x) \equiv 1, \forall x \in \mathbb{R}^n \). It’s proof by contradiction. Suppose \( \exists x_0 \in \mathbb{R}^n \), \( det F'(x_0) \neq 1 \). Without loss of generality, we suppose \( det F'(x_0) > 1 \), denote \( C = det F'(x_0) - 1 > 0 \). Since \( F \in C^1 \), \( det F'(x) \in C \). \( \exists \delta > 0 \), such that \( det F'(x) - 1 \geq \frac{C}{2}, \forall x \in U(x_0, \delta) \).

Choosing \( \Omega = U(x_0, \delta) \), thus

\[ \int_{U(x_0, \delta)} (1 - det F'(x))dx \leq \int_{U(x_0, \delta)} -\frac{C}{2} dx = -\frac{C}{2}m(U(x_0, \delta)) < 0, \]

it contradicts.

Thus, Theorem 1.7 is proved by Theorem 1.6.

By similar methods, we can prove the Theorem 1.8 and Theorem 1.9 by the \( W \) condition and half Reeb component.

Remark 5.1. It is very important and meaningful to study the relation between half-Reeb component in higher dimensions and the rate of tending to zero of eigenvalues of \( DF \).
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