RATIONAL PERIODIC POINTS FOR DEGREE TWO POLYNOMIAL MORPHISMS ON PROJECTIVE SPACE

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Abstract. This article addresses the existence of \( \mathbb{Q} \)-rational periodic points for morphisms of projective space. In particular, we construct an infinitely family of morphisms on \( \mathbb{P}^N \) where each component is a degree 2 homogeneous form in \( N + 1 \) variables which has a \( \mathbb{Q} \)-periodic point of primitive period \( \frac{(N+1)(N+2)}{2} + \lfloor \frac{N-1}{2} \rfloor \). This result is then used to show that for \( N \) large enough there exists morphisms of \( \mathbb{P}^N \) with \( \mathbb{Q} \)-rational periodic points with primitive period larger that \( c(k)N^k \) for any \( k \) and some constant \( c(k) \).

1. Introduction and Statement of Results

Let \( \phi : \mathbb{P}^1 \to \mathbb{P}^1 \) be a morphism of degree 2 which has a totally ramified fixed point at infinity, in other words, a polynomial morphism. We will denote \( \phi^n \) as the \( n \)th iterate of \( \phi \). A point \( P \in K \) is called periodic of period \( n \) for \( \phi \) if there is a positive integer \( n \) such that \( \phi^n(P) = P \). If \( n \) is the smallest such integer, it is called the primitive period of \( P \) for \( \phi \). Northcott’s theorem [5] tells us that \( \phi \) can have only finitely many rational periodic points defined over a number field and, hence, the primitive periods of rational periodic points must be bounded. For \( n = 1 \), 2, and 3 there are infinitely many examples of degree 2 polynomial maps defined over \( \mathbb{Q} \) with \( \mathbb{Q} \)-rational primitive periodic points with primitive period \( n \). Morton [4] showed that there are no such maps with \( \mathbb{Q} \)-rational primitive 4-periodic points. Flynn, Poonen, and Schaefer [1] showed that there are no such maps with \( \mathbb{Q} \)-rational primitive 5-periodic points and made the following conjecture:

Conjecture 1. For \( n \geq 4 \) there is no quadratic polynomial \( f \in \mathbb{Q}[x] \) with a rational periodic point with primitive period \( n \).

More recently, Stoll [6] has shown conditionally that there are no degree 2 polynomial maps with \( \mathbb{Q} \)-rational primitive 6-periodic points. For degree two rational maps, Manes [3, Theorem 4] shows the existence of maps with \( \mathbb{Q} \)-rational periodic points of primitive period 4 and provides evidence for no maps with \( \mathbb{Q} \)-rational points of primitive period 5 or 6. This article examines the possible primitive period of a \( \mathbb{Q} \)-rational periodic point for a degree two polynomial morphism on \( \mathbb{P}^N \) defined over \( \mathbb{Q} \).

Definition. We define a polynomial map \( \phi : \mathbb{P}^N \to \mathbb{P}^N \) of degree \( d \) with coordinates \([x_0, \ldots, x_N]\) to be a map defined as

\[
\phi(x_0, \ldots, x_N) = [\phi_0(x_0, \ldots, x_N), \ldots, \phi_{N-1}(x_0, \ldots, x_N), x_N^d],
\]

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where each $\phi_i$ is a homogeneous form of degree $d$ in the variables $x_0, \ldots, x_N$. Such a map is a morphism if $\phi_1, \ldots, \phi_{N-1}$ have no nontrivial common zeroes when $x_N = 0$.

1.1. A First Example. Consider a degree two polynomial map $\phi : \mathbb{P}^1 \to \mathbb{P}^1$ given by

$$\phi(x, y) = [ax^2 + bxy + cy^2, y^2].$$

We wish to find constants $a, b,$ and $c$ such that $P = [0, 1]$ is a periodic point of primitive period 3 for $\phi$. To do so we choose any two distinct other points $P_1$ and $P_2$ and solve the three linear equations $\phi(P_1) = P_1$, $\phi(P_1) = P_2$, and $\phi(P_2) = P$ in the three unknowns $a$, $b$, and $c$ to find a suitable map $\phi$. Since we will use a similar, albeit more complicated, construction in Theorem 1, we explore this construction in detail for this simple example.

We see that $\phi([0, 1]) = [c, 1]$, so we choose $c \neq 0$, say $c = 1$. Then we have

$$\phi([1, 1]) = [a + b + 1, 1].$$

We now choose $b$ so that $\phi([1, 1]) \neq [0, 1]$ or $[1, 1]$, say $b = 1 - a$. Then, we have $\phi([1, 1]) = [2, 1]$ and so

$$\phi([2, 1]) = [2a + 3, 1].$$

Finally, we choose $a = -3/2$ to have $\phi([2, 1]) = [0, 1]$, making the degree two polynomial map

$$\phi([x, y]) = [-3/2x^2 + 5/2x + 1, y^2]$$

have $[0, 1]$ as a primitive 3-periodic point.

Trying to construct a $\mathbb{Q}$-rational primitive 4-periodic point in the same manner, at a minimum, requires more care. The obstruction lies in having to solve four equations in three unknowns. Conjecture 1 states that for a degree 2 polynomial morphism on $\mathbb{P}^1$ it is never possible to solve these larger systems of equations and the number of coefficients $(\frac{2 + 2}{2}) = 3$ is an upper bound on the primitive $\mathbb{Q}$-rational periods.

Theorem 1 demonstrates an infinite family of polynomial maps on $\mathbb{P}^N$ with periodic points with primitive period larger than $(\frac{N+1}{2})^2$. Theorem 2 shows that these families contain infinitely many maps which are in fact morphisms of $\mathbb{P}^N$. Theorem 3 uses Theorem 1 and Theorem 2 to show that the primitive period of $\mathbb{Q}$-rational periodic points for polynomial morphisms of $\mathbb{P}^N$ grows faster than $c(k)N^k$ for any $k$ and some constant $c(k)$.

1.2. The Main Results. In general, we can construct a degree 2 polynomial map with a $\mathbb{Q}$-rational periodic point with primitive period equal to the number of coefficients of a quadratic form in $N + 1$ variables $(\frac{N+2}{2})$, by choosing one coefficient with each successive iterate as we did above for $\mathbb{P}^1$. We show that for $N \geq 2$ we can construct polynomial maps on $\mathbb{P}^N$ with a periodic point with primitive period larger than this value.

**Theorem 1.** Let $N \geq 2$. There is an infinite family of degree two polynomial maps $\phi : \mathbb{P}^N \to \mathbb{P}^N$ with a $\mathbb{Q}$-rational periodic point with primitive period

$$\left\{ \begin{array}{ll} \leq 7 = \frac{(N+1)(N+2)}{2} + 1 & \text{for } N = 2 \\ \leq \frac{(N+1)(N+2)}{2} + \left\lfloor \frac{N-1}{2} \right\rfloor & \text{for } N \geq 3, \end{array} \right.$$
where \( |x| \) denotes the greatest integer less than or equal to \( x \). Moreover, the dimension of the family is at least \( N \).

**Theorem 2.** The infinite family of maps constructed in Theorem 1 contains infinitely many morphisms.

**Theorem 3.** For \( N \) large enough, there exists a degree two polynomial morphism of \( \mathbb{P}^N \) with a \( \mathbb{Q} \)-rational periodic point with primitive period larger than \( c(k)N^k \) for any \( k \) and some constant \( c(k) \) depending on \( k \).

In general, the bounds in Theorem 1 are not upper bounds on the primitive period. Several examples of polynomial morphisms with \( \mathbb{Q} \)-rational points with larger primitive period are included at the end of the article.

2. Proof of Theorem 1

We denote the \( i \)th coordinate of a point \( P \in \mathbb{P}^N \) as \( x_i(P) \) and denote the polynomial map \( \phi : \mathbb{P}^N \to \mathbb{P}^N \) as

\[
x_i(\phi(x_0, \ldots, x_N)) = \begin{cases} 
    \sum_{j=0}^{N-1} \sum_{k=j}^{N} c_i(j, k)x_jx_k & \text{for } i = 0, \ldots, N - 1 \\
    c_N(N,N)x_N^2 & \text{for } i = N.
\end{cases}
\]

We denote the \( n \)th image of \( P \) by \( \phi \) as \( \phi^n(P) = P_n \).

The method of construction is to choose appropriate values of the constants \( c_i(j, k) \) so that the coordinates of each iterate are linear in at most two of the \( c_i(j, k) \). When we have chosen all of the \( c_i(j, k) \) with \( j \neq k \), we will then be able to choose \( c_i(j, j) \) so that \( \phi(P) \) is determined and \( \phi(\phi(P)) \) is linear in one of the \( c_i(j, j) \), allowing the primitive period to increase beyond the trivial value \( \binom{N+2}{2} \) determined by the number of coefficients. We treat the case of \( N = 2 \) separately.

In Lemma 1 we choose the initial sequence of images by specifying \( c_i(j, k) \) \( 0 \leq i \leq N \) for one pair \( (j, k) \) for each image. Then we proceed with the construction to increase the primitive period beyond the trivial lower bound.

**Lemma 1.** Let \( \phi : \mathbb{P}^N \to \mathbb{P}^N \) be a degree two polynomial map. We may choose the first \( (N^2 + N)/2 - 1 \) images of \( [0, \ldots, 0, 1] \) as

\[
\begin{align*}
&[0, \ldots, 0, 1] \xrightarrow{\phi} [1, 0, \ldots, 0, 1] \xrightarrow{\phi} [0, 1, 0, \ldots, 0, 1] \xrightarrow{\phi} \cdots \xrightarrow{\phi} [0, \ldots, 0, 1, 1, 1] \\
&\xrightarrow{\phi} [1, 1, 0, \ldots, 0, 1] \xrightarrow{\phi} [0, 1, 1, 0, \ldots, 0, 1] \xrightarrow{\phi} \cdots \xrightarrow{\phi} [0, \ldots, 0, 1, 1, 1] \\
&\xrightarrow{\phi} [1, 1, 1, 0, \ldots, 0, 1] \xrightarrow{\phi} [0, 1, 1, 1, 0, \ldots, 0, 1] \xrightarrow{\phi} \cdots \xrightarrow{\phi} [0, \ldots, 0, 1, 1, 1, 1] \\
&\vdots
\end{align*}
\]

by choosing all of the \( c_i(j, k) \) except \( c_i(0, N - 1) \) and \( c_i(k, k) \) for each \( 0 \leq k \leq N - 1 \) and each \( 0 \leq i \leq N - 1 \). Furthermore, \( x_i(\phi([1, \ldots, 1])) \) is of the form

\[
a_0c_0(0, N - 1) + b_i
\]

for some constants \( a_i, b_i \) for all \( 0 \leq i \leq N - 1 \).
Proof. Let $P = [0, \ldots, 0, 1]$. Then $x_i(\phi(P))$ is linear in $c_i(N, N)$, so we can choose 

$$\phi(P) = [1, 0, \ldots, 0, 1]$$

by choosing 

$$c_i(N, N) = \begin{cases} 
1 & \text{for } i = 0 \text{ and } i = N \\
0 & \text{for } 1 \leq i \leq N - 1.
\end{cases}$$

Next we choose the sequence of points 

$$[1, 0, \ldots, 0, 1] \xrightarrow{\phi} [0, 1, 0, \ldots, 0, 1] \xrightarrow{\phi} \cdots$$

$$\xrightarrow{\phi} [0, 0, \ldots, 0, 1, 1] \xrightarrow{\phi} [1, 1, 0, \ldots, 0, 1],$$

where 

$$[1, 1, 0, \ldots, 0, 1] = P_{N+1}.$$ 

We can do this because $x_i(\phi(P))$ is of the form 

$$x_i(\phi(P)) = \begin{cases} 
c_i(j - 1, j - 1) + c_i(j - 1, N) + 1 & \text{for } i = 0 \\
c_i(j - 1, j - 1) + c_i(j - 1, N) & \text{for } 1 \leq i \leq N - 1 \\
1 & \text{for } i = N.
\end{cases}$$

We choose 

$$c_i(j - 1, N) = \begin{cases} 
-1 - c_i(j - 1, j - 1) & \text{for } i = 0 \\
1 - c_i(j - 1, j - 1) & \text{for } i = j - 1 \\
-c_i(j - 1, j - 1) & \text{otherwise}.
\end{cases}$$

Remark. With these choices of $c_i(N, N)$ and $c_i(k, N)$ for all $0 \leq k \leq N - 1$, the image $x_k(\phi(x_0, x_1, \ldots, x_{N-1}))$ contains terms of the form $c_k(i, i)x_i^2 - c_k(i, i)x_i$ for all $0 \leq i \leq N$ and $0 \leq k < N - 1$. Consequently, if $x_i = 1$ or $x_i = 0$, then $c_k(i, i)$ does not appear in the $k$th coordinate of the image.

Next we choose the sequence of images 

$$[1, 1, 0, \ldots, 0, 0, 1] \xrightarrow{\phi} [0, 1, 1, 0, \ldots, 0, 1] \xrightarrow{\phi} \cdots$$

$$\xrightarrow{\phi} [0, 0, \ldots, 0, 1, 1, 1] \xrightarrow{\phi} [1, 1, 1, 0, \ldots, 0, 1]$$

until 

$$P_{2N-1} = [1, 1, 1, 0, \ldots, 0, 1].$$

We can do this because we have already chosen $c_i(k, N)$ for all $0 \leq k \leq N - 1$ and $c_i(N, N)$, causing the $i$th coordinate for all $0 \leq i \leq N - 1$ of each image in this sequence to be linear only in the single coefficient $c_i(k, k + 1)$ for some $0 \leq k \leq N - 2$.

Next we choose the sequence of images 

$$[1, 1, 1, 0, \ldots, 0, 1] \xrightarrow{\phi} [0, 1, 1, 1, 0, \ldots, 0, 1] \xrightarrow{\phi} \cdots$$

$$\xrightarrow{\phi} [0, 0, \ldots, 0, 1, 1, 1] \xrightarrow{\phi} [1, 1, 1, 1, 0, \ldots, 0, 1].$$
Since we have already chosen \( c_i(N, N) \), \( c_i(k, N) \) for all \( 0 \leq k \leq N - 1 \), and \( c_i(k, k + 1) \) for all \( 0 \leq k \leq N - 2 \) and \( 0 \leq i \leq N - 1 \), the \( i \)th coordinate of these iterates is linear in \( c_i(k, k + 2) \) for some \( 0 \leq k \leq N - 3 \).

We repeat this process until we have
\[
\begin{align*}
\phi(P_{(N^2+N)/2-3}) &= [1, \ldots, 1, 0, 1, 1] \quad \text{(linear in } c_i(0, N - 2)) \\
\phi(P_{(N^2+N)/2-2}) &= [0, 1, \ldots, 1] \quad \text{(linear in } c_i(1, N - 1)) \\
\phi(P_{(N^2+N)/2-1}) &= [1, \ldots, 1] \quad \text{(linear in } c_i(0, N - 1)).
\end{align*}
\]
The only coefficients not yet chosen are \( c_j(i, i) \) for all \( 0 \leq i \leq N - 1 \) and \( c_j(0, N - 1) \). We also know that \( x_i(\phi(P)) \) is linear in \( c_i(0, N - 1) \) for all \( 0 \leq i \leq N - 1 \). \( \square \)

We are now ready to increase the primitive period beyond the trivial lower bound.

**Proof of Theorem 4**

**Case 1.** \((N = 2)\)

We begin where Lemma 1 finished. We have chosen \( c_i(2, 2) \) for all \( 0 \leq i \leq 2 \) and \( c_i(0, 2) \) and \( c_i(1, 2) \) for all \( 0 \leq i \leq 1 \) to have the sequence of points
\[
[0, 0, 1] \xrightarrow{\phi} [1, 0, 1] \xrightarrow{\phi} [0, 1, 1] \xrightarrow{\phi} [1, 1, 1].
\]

Now we choose \( c_i(0, N - 1) = c_i(0, 1) \) so that
\[
\phi([1, 1, 1]) = [0, K(0, 1), 1]
\]
for some constant \( K(0, 1) \notin \{0, 1\} \). Each coordinate \( x_i(\phi([0, K(0, 1), 1])) \) is linear in \( c_i(1, 1) \). We then choose the \( c_i(1, 1) \) so that
\[
\phi([0, K(0, 1), 1])) = [0, K(1, 1), 1],
\]
where \( K(1, 1) \) is a constant with \( K(1, 1) \notin \{0, 1, K(0, 1)\} \). So we have three points chosen of the form \([0, x_1, 1]\) whose images are given by
\[
\begin{align*}
[0, 0, 1] &\xrightarrow{\phi} [1, 0, 1] \\
[0, 1, 1] &\xrightarrow{\phi} [1, 1, 1] \\
[0, K(0, 1), 1] &\xrightarrow{\phi} [0, K(1, 1), 1].
\end{align*}
\]

Since \( x_i(\phi[0, x_1, 1]) \) is a quadratic polynomial in \( x_1 \) for \( 0 \leq i \leq 1 \), the image \( \phi([0, K(1, 1), 1]) \) is determined by these three known points and is of the form
\[
\phi([0, K(1, 1), 1]) = [k_0, k_1, 1]
\]
for some constants \( k_0 \) and \( k_1 \). Note that we may need to exclude finitely many choices of \( K(0, 1) \) and \( K(1, 1) \) (and hence of \( c_i(0, 1) \) and \( c_i(1, 1) \)) so that \( k_0 \notin \{0, 1\} \). So we have that each \( x_i(\phi([k_0, k_1, 1])) \) is linear in \( c_i(0, 0) \) and we choose the \( c_i(0, 0) \) so that
\[
\phi(\phi([0, K(1, 1), 1]) = [0, 0, 1].
\]

This is a primitive 7-periodic point, and the family is dimension 2 since we can choose \( c_i(0, 1) \) and \( c_i(1, 1) \) arbitrarily (with finitely many exceptions).

It is easy to modify this construction to get points with primitive periods 1, \ldots, 6 because at each stage we are linear in at most two variables. So we simply choose the constant so that
\( \phi^3(P) = [0,0,1] \) at the appropriate iterate. The dimension of these families is larger since there are more free coefficients.

**Case 2. \((N \geq 3)\)**

We begin where Lemma \([1]\) finished and choose the \( c_i(0,N-1) \) so that we have

\[ \phi(P_{(N^2+N)/2}) = [0,K(0,N-1),1,\ldots,1] \]

where \( K(0,N-1) \notin \{0,1\} \) is a constant. Each coordinate \( x_i(\phi([0,K(0,N-1),1,\ldots,1])) \) is linear in \( c_i(1,1) \). Choose the \( c_i(1,1) \) to have

\[ \phi([0,K(0,N-1),1,\ldots,1]) = [K(1,1),1,\ldots,1,0,1], \]

where \( K(1,1) \notin \{0,1\} \) is some constant. Now we have that the \( i^{\text{th}} \) coordinate of the image of \([K(1,1),1,\ldots,1,0,1]\) is linear in \( c_i(0,0) \) for all \( 0 \leq i \leq N-1 \). So we choose the \( c_i(0,0) \) so that

\[ \phi([K(1,1),1,\ldots,1,0,1]) = [0,K(0,0),1,\ldots,1] \]

for some constant \( K(0,0) \notin \{0,1,K(0,N-1)\} \). Note that there are three points of the form \([0,x_1,1,\ldots,1]\) whose images are given by

\[
\begin{align*}
[0,1,1,\ldots,1] &\xmapsto{\phi} [1,\ldots,1] \\
[0,K(0,N-2),1,\ldots,1] &\xmapsto{\phi} [K(1,1),1,\ldots,1,0,1] \\
[0,0,1,\ldots,1] &\xmapsto{\phi} [1,1,\ldots,1,0,1].
\end{align*}
\]

Since each \( x_i(\phi([0,x_1,1,\ldots,1])) \) is a degree 2 polynomial in \( x_1 \), the three known points and their images completely determine the image of any point of the form \([0,x_1,1,\ldots,1]\). The \( 0^{\text{th}} \) coordinate is a non-constant function of \( x_1 \) and the \((N-1)^{\text{st}}\) coordinate is a non-constant function of \( x_1 \). The remaining coordinates take on a constant value of 1. Therefore, we have

\[ \phi([0,K(0,0),1,\ldots,1]) = [k_0,1,1,\ldots,1,k_{N-1},1] \]

for some constants \( k_0 \) and \( k_{N-1} \) with \( k_{N-1} \notin \{0,1\} \) (again we may need to exclude finitely many choices of \( K(1,1) \) and \( K(0,N-2) \) so that \( k_{N-1} \notin \{0,1\} \)). In particular, since the \( c_i(0,0) \) are already chosen, each \( x_i(\phi([k_0,1,1,\ldots,1,k_{N-1},1])) \) is linear in \( c_i(N-1,N-1) \). From our choice of \([0,K(0,0),1,\ldots,1]\) we have determined \( \phi([0,K(0,0),1,\ldots,1]) \) and have \( \phi([0,K(0,0),1,\ldots,1]) \) as our next iterate to consider. We have thus increased the primitive period of \([0,\ldots,0,1]\) by two with the choice of the \( c_i(0,0) \).

If \( N = 3 \) we are done since we are linear in the \( c_i(N-1,N-1) \) and they are the only unchosen coefficients; so we choose the \( c_i(N-1,N-1) \) to make the point periodic.

For \( N > 3 \) we repeat the process. Choose the \( c_i(N-1,N-1) \) to get

\[ [0,0,K(N-1,N-1),1,\ldots,1,0,1] \]

with \( K(N-1,N-1) \notin \{0,1\} \). The coordinates of the image are linear in the \( c_i(2,2) \) so we choose

\[ \phi([0,0,K(N-1,N-1),1,\ldots,1,0,1]) = [0,0,K(2,2),1,\ldots,1,0,1], \]
for $K(2,2) \not\in \{0,1,K(N-1,N-1)\}$. The image
\[
\phi([0,0, K(2,2), 1, \ldots, 1, 0, 1])
\]
is completely determined since we have three points of the form
\[
[0,0, x, 1, \ldots, 1, 0, 1]
\]
whose images are known. These images are
\[
[0,0, K(N-1,N-1), 1, \ldots, 1, 0, 1] \overset{\phi}{\rightarrow} [0,0, K(2,2), 1, \ldots, 1, 0, 1]
\]
\[
[0,0, 1, \ldots, 1, 0, 1] \overset{\phi}{\rightarrow} [0,0, 0, 1, \ldots, 1, 1, 1]
\]
\[
[0,0, 0, 1, \ldots, 1, 0, 1] \overset{\phi}{\rightarrow} [0,0, 0, 0, 1, \ldots, 1, 1, 1].
\]
We have
\[
\phi([0,0, K(2,2), 1, \ldots, 1, 0, 1]) = [0,0, x, y, 1, \ldots, 1, z, 1]
\]
for some constants $x$, $y$, and $z$. Note that we may need to exclude finitely many choices of $K(N-1,N-1)$ and $K(2,2)$ so that $y \not\in \{0,1\}$. We have already chosen the $c_i(2,2)$ and the $c_i(N-1,N-1)$, so each
\[
x_i(\phi([0,0, K(2,2), 1, \ldots, 1, 0, 1]))
\]
is linear in $c_i(3,3)$, again increasing the primitive period by two with the choice of a single set of coefficients $c_i(2,2)$.

Continuing in this manner, we choose the $c_i(k,k)$ to get
\[
[0, \ldots, 0, K(k,k), 1, \ldots, 1, 0, 1],
\]
where $K(k,k) \not\in \{0,1\}$ and is the $(k+1)^{\text{st}}$ coordinate. The $i^{\text{th}}$ coordinate of the image is linear in $c_i(k+1,k+1)$. We choose
\[
\phi([0, \ldots, 0, K(k,k), 1, \ldots, 1, 0, 1]) = [0, \ldots, 0, K(k+1,k+1), 1, \ldots, 1, 0, 1],
\]
where $K(k+1,k+1)$ is the $(k+1)^{\text{st}}$ coordinate and $K(k+1,k+1) \not\in \{0,1,K(k,k)\}$. The image $\phi([0, \ldots, 0, K(k+1,k+1), 1, \ldots, 1, 0, 1])$ is completely determined since we have three points of the form
\[
[0, \ldots, 0, x_{k+1}, 1, \ldots, 1, 0, 1]
\]
whose images are known; they are
\[
[0, \ldots, 0, K(k,k), 1, \ldots, 1, 0, 1] \overset{\phi}{\rightarrow} [0, \ldots, 0, K(k+1,k+1), 1, \ldots, 1, 0, 1]
\]
\[
[0, \ldots, 0, 1, \ldots, 1, 0, 1] \overset{\phi}{\rightarrow} [0, \ldots, 0, 0, 1, \ldots, 1, 1, 1]
\]
\[
[0, \ldots, 0, 0, 1, \ldots, 1, 0, 1] \overset{\phi}{\rightarrow} [0, \ldots, 0, 0, 0, 1, \ldots, 1, 1, 1].
\]
We have
\[
\phi([0, \ldots, 0, K(k+1,k+1), 1, \ldots, 1, 0, 1]) = [0, \ldots, 0, x, y, 1, \ldots, 1, z, 1]
\]
for some constants $x$, $y$, and $z$. Note that we may need to exclude finitely many choices of $K(k,k)$ and $K(k+1,k+1)$ so that $y \not\in \{0,1\}$. We have already chosen the $c_i(k+1,k+1)$ and the $c_i(N-1,N-1)$ so each $x_i(\phi([0, \ldots, 0, K(k+1,k+1), 1, \ldots, 1, 0, 1]))$ is linear in $c_i(k+2,k+2)$, again increasing the primitive period by 2.

We continue this process until the only non-chosen coefficients are either $\{c_i(N-2,N-2),c_i(N-3,N-3)\}$ or $\{c_i(N-2,N-2)\}$. In this first case, we do not have enough unchosen coefficients remaining to increase the primitive period further beyond the trivial value, so we simply choose the $c_i(N-3,N-3)$ to have the point
\[
[0, \ldots, 0, K(N-3,N-3), 1, 1]
\]}
with $K(N - 3, N - 3) \not\in \{0, 1\}$. Each $x_i(\phi([0, \ldots, 0, K(N - 3, N - 3), 1, 1]))$ is linear in $c_i(N - 2, N - 2)$. We have now reduced to the second case and choose the $c_i(N - 2, N - 2)$ to have
\[
\phi([0, \ldots, 0, K(N - 3, N - 3), 1, 1]) = [0, 0, \ldots, 0, 1],
\]
making the point periodic of primitive period $\frac{(N + 1)(N + 2)}{2} + \lfloor \frac{N - 1}{2} \rfloor$.

Note, that along the way we were able to choose
\[
\{c_1(0, N - 1), c_1(0, 0), c_0(1, 1), \ldots, c_{N-2}(N - 3, N - 3), c_2(N - 1, N - 1)\}
\]
arbitrarily, except for excluding a finite set of values, making this an infinite family of dimension $N$.

It is easy to modify this construction to get points with periods $< \frac{(N+1)(N+2)}{2} + \lfloor \frac{N-1}{2} \rfloor$ since at each stage we are linear in at most two variables. So we simply choose the coefficients so that $\phi^n(P) = [0, \ldots, 0, 1]$ at the appropriate iterate. The dimension of these families is larger since there are more free coefficients.

\[\square\]

3. Proof of Theorem 2

We will use the theory of Macaulay resultants to show that we can choose the coefficients of the maps in Theorem 1 so that they are morphisms; in other words, so that $\phi_0, \ldots, \phi_N$ have no nontrivial common zeroes. Following [2], given $N + 1$ homogeneous forms $F_0, \ldots, F_N$ of degree $d_i$ in $N + 1$ variables $x_0, \ldots, x_N$, construct a matrix denoted $M_d(F_0, \ldots, F_N)$ where $d = 1 + \sum_i (d_i - 1)$. The columns of $M_d$ correspond to the monomials of degree $d$ in the variables $x_0, \ldots, x_N$, and the rows correspond to polynomials of the form $rF_i$ where $r$ is a monomial such that $\deg(rF_i) = d$. The entries of $M_d$ are the coefficients of the column monomials in the row polynomials. The matrix has $\binom{N+d}{d}$ columns and the number of rows corresponding to each $F_i$ is $\binom{N+d-d_i}{d-d_i}$. It is the transpose of the matrix of the linear map
\[
(P_0, \ldots, P_N) \mapsto P_0F_0 + \cdots + P_NF_N,
\]
where $P_i$ is homogenous of degree $d - d_i$. Consider the maximal minors of $M_d(F_0, \ldots, F_N)$. The determinants of these minors are polynomials in the coefficients of $F_0, \ldots, F_N$. Let $R$ be the greatest common divisor of these determinants (as polynomials in the coefficients). Then $R$ is called the resultant of $F_0, \ldots, F_N$ and (among other properties) it satisfies $R = 0$ if and only if the forms $F_0, \ldots, F_N$ have a common nontrivial zero.

**Proof of Theorem 2.** We are in the case of $N + 1$ homogeneous forms $\phi_i$ in $N + 1$ variables $x_0, \ldots, x_N$. We have each $\phi_i$ is degree 2 and hence $d = N + 2$. We will show that the Macaulay matrix has a maximal minor that has nonzero determinant and hence that the resultant is nonzero for infinitely many maps in the family. In the matrix there are $\binom{2N+2}{N+2}$ columns corresponding to all of the monomials with degree $N + 2$ and $(N + 1)\binom{2N}{N}$ rows corresponding to the $\binom{2N}{N}$ monomials of degree $d - 2$ for each of the $N + 1$ forms $\phi_i$. We need to extract a $\binom{2N+2}{N+2} \times \binom{2N+2}{N+2}$ minor with nonzero determinant. We first consider the case of largest possible period from Theorem 1.

For $N = 2$ we can write down the matrix (but do not do so here) and explicitly check that it has a maximal minor with nonzero determinant.
For \( N \geq 3 \) define (with \( N = 2 \) replaced with \( N - 1 \) for \( N = 3 \))

\[
S_N = \{ F : \deg(F) = d, x_N^2 \mid F \}
\]
\[
S_{N-2} = \{ F : \deg(F) = d, x_N^2 \mid F, \text{ and } x_{N-2}^2 \mid F \}
\]
\[
S_{N-1} = \{ F : \deg(F) = d, x_N^2 \mid F, x_{N-2}^2 \mid F, \text{ and } x_{N-1}^2 \mid F \}
\]
\[
S_{N-3} = \{ F : \deg(F) = d, x_N^2 \mid F, x_{N-1}^2 \mid F, x_{N-3}^2 \mid F, \text{ and } x_{N-2}^2 \mid F \}
\]
\[
S_{N-4} = \{ F : \deg(F) = d, x_N^2 \mid F, \ldots, x_{N-3}^2 \mid F, \text{ and } x_{N-4}^2 \mid F \}
\]
\[
\vdots
\]
\[
S_0 = \{ F : \deg(F) = d, x_N^2 \mid F, \ldots, x_1^2 \mid F, \text{ and } x_0^2 \mid F \}.
\]

Order the columns in reverse lexicographic order, \( x_N > x_{N-1} > \cdots > x_0 \), with the largest to the left. For the columns corresponding to a monomial in \( S_N \), choose the row with a 1 on the diagonal (the row contains all 0’s except one entry which is 1 since \( \phi_N(x_0, \ldots, x_N) = x_N^2 \)). For columns corresponding to monomials in \( S_{2k} \) with \( k \neq 0 \), choose the row with \( c_{2k}(2k, 2k) \) on the diagonal. For columns corresponding to monomials in \( S_{2k-1} \) with \( k \neq 1 \), choose the row with \( c_{2k}(2k - 1, 2k - 1) \) on the diagonal. For \( S_1 \) we choose the row with \( c_0(1, 1) \) on the diagonal, and for \( S_0 \) we choose the row with \( c_1(0, 0) \) on the diagonal. Finally, columns corresponding to monomials in \( S_{N-2} \) we fix \( i > 1 \) odd and choose the row with \( c_i(N - 2, N - 2) \) on the diagonal (use \( S_{N-1} \) and \( c_i(N - 1, N - 1) \) for \( N = 3 \)).

We have two facts to verify:

(1) These choices contain no duplicate rows.

(2) The resulting minor has nonzero determinant.

The first is clear since \( S_i \) and \( S_j \) are disjoint for \( i \neq j \) and each row associated to an element of \( S_k \) has at most one entry containing a \( c_i(k, k) \).

For the second, we start by examining the entries in each row. Each row associated to \( \phi_i \) for \( i \neq N \) contains a \( c_i(N - 2, N - 2) \) (or \( c_i(N - 1, N - 1) \) for \( N = 3 \)) whose value depends on at least \( c_i(N - 3, N - 3) \) (or \( c_i(1, 1) \) for \( N = 3 \)) so is not identically 0. In addition, each of these rows contains a corresponding

\[
c_i(N - 2, N) = \begin{cases} 
-1 - c_i(N - 2, N - 2) & \text{for } i = 0 \\
1 - c_i(N - 2, N - 2) & \text{for } i = N - 2 \\
-c_i(N - 2, N - 2) & \text{otherwise}.
\end{cases}
\]

For \( x_1^2 \) we have \( c_0(1, 1) \) and \( c_0(1, N) = -1 - c_0(1, 1) \). For \( x_0^2 \), we have \( c_1(0, 0) \) and \( c_1(0, N) = -c_0(1, 1) \). For \( x_{2k}^2 \) with \( k > 0 \), there is a \( c_{2k}(2k, 2k) \) and a \( c_{2k}(2k, N) = 1 - c_{2k}(2k, 2k) \). For \( k > 1 \) there is a \( c_{2k}(2k - 1, 2k - 1) \) and a \( c_{2k}(2k - 1, N) = 1 - c_{2k}(2k - 1, 2k - 1) \). The rest of the entries are either constants or depend on \( c_1(0, N - 1) \). Also note that each row contains each \( c_i(k, k) \) at most once (in addition to the corresponding \( c_i(k, N) \)).

Note that \( c_i(k, k) \) and \( c_i(k, N) \) are possibly linearly dependent and that our choice of ordering has \( c_i(k, k) \) appearing farther right in the matrix than \( c_i(k, N) \). For the other entries, we are choosing one coefficient per iteration, so they are either constant, independent, or the next depends on the previous in a quadratic (or higher) fashion (since each \( \phi_i \) is degree 2).
Assume that we have some linear combination of the rows that produces a row identically 0. Each row contains a $c_i(k, k)$ on the diagonal for some $i$ and $k$ and a $c_i(k, N)$ in some other entry. For the linear combination to result in 0, there are three cases to consider.

**Case 1.** Assume two rows in the combination contain $c_i(k, k)$ and $c_i(k, N)$ in the same column. By our choice of ordering, the respective $c_i(k, N)$ and $c_i(k, k)$ in those rows would not be in the same column. Hence, we must also include rows in the combination that contain $c_i(k, N)$ and $c_i(k, k)$ in the corresponding columns. Again by our choice of ordering, we need to include at least two rows to do this and then we still have unpaired $c_i(k, k)$ and $c_i(k, N)$ as before. Therefore, we cannot choose any number of rows so that all of the $c_i(k, k)$ and $c_i(k, N)$ are paired by column.

**Case 2.** Notice that by our choice of the $S_i$ we have guaranteed that we cannot have $c_j(N - 2, N - 2)$ and $c_i(k, k)$ in the same column for any $k \neq N - 2$. Let $j$ be such that $c_j(N - 2, N - 2)$ is on the diagonal of the minor. Assume two rows in the combination have $c_i(N - 2, N - 2)$ and $c_j(N - 2, N - 2)$ in the same column for $i \neq j$. But with $c_j(N - 2, N - 2)$ used for $S_{N - 2}$, we must have that the row containing $c_i(N - 2, N - 2)$ also contains $c_i(k, k)$ for some $k \neq N - 2$. As in Case 1, we are unable to find a combination of rows that pairs all of the $c_i(k, k)$ and $c_i(k, N)$.

**Case 3.** Assume we have $c_i(k, k)$ and $c_i(k, N)$ paired with constants to get a combination of rows identically 0. However, every row containing a $c_i(k, k)$ with $k \neq N - 2$ also contains $c_j(N - 2, N - 2)$ for some $j$. These $c_j(N - 2, N - 2)$ must be paired either with constants or with other $c_l(N - 2, N - 2)$. However, they cannot be paired with constants since the $c_i(k, k)$ are already paired with constants in a combination that results in 0, and $c_i(k, k)$ and $c_j(N - 2, N - 2)$ are not related in a linear fashion. Case 2 eliminates the possibility of pairing with another $c_t(N - 2, N - 2)$ for some $t$. So we must have $k = N - 2$. Then all of the rows in the combination are associated to the same $\phi_j$ and, hence, entries in columns cannot be paired appropriately to result in a combination of 0.

Therefore, no linear combination can have all entries as 0 and the determinant of this minor is not identically 0. Therefore, there are infinitely many choices of the coefficients that produce a map that is a morphism.

For the families with a periodic point with smaller primitive period, the matrix is similar but with more free constants, so similar choices of rows will also produce a minor with nonzero determinant.  

\[ \Box \]

**4. Proof of Theorem 3**

**Lemma 2.** Given $\phi_1 : \mathbb{P}^N \rightarrow \mathbb{P}^N$ a polynomial morphism with $P_1$ of primitive period $n$ and $\phi_2 : \mathbb{P}^M \rightarrow \mathbb{P}^M$ a polynomial morphism with $P_2$ of primitive period $m$, then there exists a polynomial morphism $\psi : \mathbb{P}^{N+M} \rightarrow \mathbb{P}^{N+M}$ and a point $P$ with primitive period $\text{lcm}(n, m)$.

**Proof.** We restrict $\phi_1$ to the affine chart $\mathbb{A}^N$ with $x_N \neq 0$ and $\phi_2$ to the affine chart $\mathbb{A}^M$ with $x_M \neq 0$. The restricted points $\tilde{P}_1$ and $\tilde{P}_2$ still have period $n$ and $m$, and the product map $\tilde{\phi}_1 \times \tilde{\phi}_2 : \mathbb{A}^{N+M} \rightarrow \mathbb{A}^{N+M}$ has the product of the dehomogenizations $\tilde{P} = (\tilde{P}_1, \tilde{P}_2)$ as a periodic point of primitive period $\text{lcm}(n, m)$. This fact is simply the statement that the product of a cyclic group of order $n$ with a cyclic group of order $m$ has order $\text{lcm}(n, m)$.  

\[ \Box \]
Now, homogenizing $\tilde{\phi}_1 \times \tilde{\phi}_2$ to a map $\psi : \mathbb{P}^{N+M} \to \mathbb{P}^{N+M}$ we know that the first $N$ forms and $x_{N+M}^2$ have no common nontrivial zeros in $x_0, \ldots, x_{N-1}, x_{N+M}$ and the next $M$ forms and $x_{N+M}^2$ have no common nontrivial zeros in $x_N, \ldots, x_{N+M}$. Since the only variable shared between the two sets of forms is $x_{N+M}$, the map $\psi$ is also a morphism and the homogenization of $\tilde{P}$ has primitive period $\text{lcm}(n, m)$ for $\psi$. □

Proof of Theorem 3. From Theorem 1 and Theorem 2 we can find morphisms $\phi : \mathbb{P}^N \to \mathbb{P}^N$ with $\mathbb{Q}$-rational periodic points with primitive period $1, 2, \ldots, (N+1)(N+2)/2$. Fix $s$ a positive integer. Let $M = \lfloor N/s \rfloor$. Then $\frac{(M+1)(M+2)}{2} > \frac{(N/s)(N/s)}{2} = \frac{N^2}{2s^2}$ and for every prime $p \leq \frac{N^2}{2s^2}$ there is a point with primitive period $p$ for some polynomial morphism of $\mathbb{P}^M$. Fix $\epsilon > 0$ and choose $N$ large enough that the interval $((1-\epsilon)N^2/2s^2, N^2/2s^2)$ has at least $s$ primes $p_1, \ldots, p_s$. We apply Lemma 2 to combine these points and associated morphisms to get a point $P \in \mathbb{P}^s = \mathbb{P}^N$, which has primitive period

$$p_1 \cdots p_s \geq \frac{(1-\epsilon)^s}{2^{2s^2}} N^{2s}$$

for a polynomial morphism $\psi : \mathbb{P}^N \to \mathbb{P}^N$. □

5. SOME EXAMPLES WITH LARGER PRIMITIVE PERIODS

With slightly different choices of coefficients, it is occasionally possible to increase the primitive period by more than 2 with a choice of a single set of coefficients. While a general method to ensure this occurrence was not discovered, in practice it is possible to construct a polynomial map for a specific $N$ with a periodic point with primitive period that exceeds the bound presented in Theorem 1. These can then be combined as in Lemma 2 to produce morphisms of $\mathbb{P}^N$ with $\mathbb{Q}$-rational periodic points of large primitive period. The following examples present such maps for $N = 2, 3, 4$. For the reader’s convenience, the following table shows the trivial lower bound $\binom{N+2}{2}$, the lower bound from Theorem 1 and the primitive period exhibited in the example of a polynomial morphism $\phi : \mathbb{P}^N \to \mathbb{P}^N$. Note that since we are dealing with maps on $\mathbb{P}^N$ outside of the scope of Theorem 2 the maps were verified explicitly to be morphisms, but the details are omitted here.

| N  | trivial bound | Theorem 1 bound | Example period |
|----|---------------|-----------------|---------------|
| 2  | 6             | 7               | 9             |
| 3  | 10            | 11              | 24            |
| 4  | 15            | 16              | 72            |

Example 1. The point $[0, 0, 1] \in \mathbb{P}^2$ is a periodic point of primitive period 9 for the morphism

$$\phi([x_0, x_1, x_2]) =$$

$$[-38/45x_0^2 + (2x_1 - 7/45x_2)x_0 + (-1/2)x_1^2 - 1/2x_2 x_1 + x_2^2],$$

$$-67/90x_0^2 + (2x_1 + 157/90x_2)x_0 - x_2 x_1, x_2^2].$$
Example 2. The point \([0,0,0,1] \in \mathbb{P}^3\) is a periodic point of primitive period 24 for the morphism

\[
\phi([x_0,x_1,x_2,x_3]) = \\
((-x_1 - x_3)x_0 + (-13/30 x_1^2 + 13/30 x_3 x_1 + x_3^2), \\
-1/2 x_0^2 + (-x_1 + 3/2 x_3)x_0 + (-1/3 x_1^2 + 4/3 x_3 x_1), \\
-3/2 x_2^2 + 5/2 x_2 x_3 + x_3^2, x_2^3, x_3^3]
\]

created by combining a periodic point of primitive period 8 in \(\mathbb{P}^2\) and a periodic point of primitive period 3 in \(\mathbb{P}^1\).

Example 3. The point \([0,0,0,0,1] \in \mathbb{P}^4\) is a periodic point of primitive period 72 for the morphism

\[
\phi([x_0,x_1,x_2,x_3,x_4]) = \\
(-38/45 x_0^2 + (2 x_1 - 7/45 x_4)x_0 + (-1/2 x_1^2 - 1/2 x_4 x_1 + x_4^2), \\
-67/90 x_0^2 + (2 x_1 + 157/90 x_4)x_0 - x_4, x_1, \\
(-x_3 - x_4)x_2 + (-13/30 x_3^2 + 13/30 x_4 x_3 + x_4^2), \\
-1/2 x_2^2 + (-x_3 + 3/2 x_4)x_2 + (-1/3 x_2^3 + 4/3 x_4 x_3), x_2^4, x_3^2, x_4^3]
\]

created by combining periodic points of primitive period 8 and 9 in \(\mathbb{P}^2\).

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