The Regularized Trace Formula Of A Second Order Differential Equation Given With Anti-Periodic Boundary Conditions

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Received: 09.11.2018; Accepted: 24.10.2019

http://dx.doi.org/10.17776/csj.480810

\begin{abstract}
In this study, we examined the formula of the regularized trace of the self-adjoint operator which is formed by
\[ \ell(y) = -y'' + p(x)y \]
differential expression and
\[ y(0) + y(\pi) = 0 \]
\[ y'(0) + y'(-\pi) = 0 \]
anti-periodic boundary condition.

\textbf{Keywords}: Regularized trace, Eigenvalues, Eigen functions.
\end{abstract}

\section{INTRODUCTION}

\( p(x) \) is a real valued, continuous function in \([0, \pi]\), \( L_0 \) and \( L \) get two self-adjoint operators generated by the following expressions

\[ \ell_0(y) = -y'' \]

and

\[ \ell(y) = -y'' + p(x)y \] \hfill (1)
with the same boundary conditions
\[
y(0) + y(\pi) = 0
\]
\[
y'(0) + y' (\pi) = 0
\] (2)
respectively, in the space \( L_2[0, \pi] \). The spectrum of operator \( L_0 \) coincides with the set \( \{(2n + 1)^2\}_{n=0}^{\infty} \). Every point of the spectrum is an eigenvalue with multiplicity two.

Let
\[
\mu_k = \begin{cases} 
k^2, & \text{if } k \text{ is odd} \\
(k-1)^2, & \text{if } k \text{ is even} 
\end{cases} \quad (k = 1, 2, \ldots)
\]
is the eigenvalues of operator \( L_0 \) and
\[
\psi_1 = \frac{2}{\sqrt{\pi}} \sin x, \psi_2 = \frac{2}{\sqrt{\pi}} \cos x, \psi_3 = \frac{2}{\sqrt{\pi}} \sin 3x, \psi_4 = \frac{2}{\sqrt{\pi}} \cos 3x, \ldots
\]
are the orthonormal eigenfunctions corresponding to this eigenvalues.

Also we showed the eigenvalues of operator \( L \) by \( \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots \leq \lambda_k \leq \cdots \) and corresponding orthonormal eigenfunctions by \( \varphi_0, \varphi_1, \varphi_2, \ldots, \varphi_k, \ldots \)

In this study, we obtained a formula for the sum of series by Dikii’s method,
\[
\sum_{n=1}^{\infty} (\lambda_n - \mu_n)
\]
which is called the formula of regularized trace of operator \( L \).

The regularized trace theory, which was first examined by Gelfand and Levitan and they derived the formula of regularized trace for the Sturm-Liouville operator [1], attracted the attention of many authors. Dikii [2] provided and developed Gelfand and Levitan’s formulas by their own method. Later, Levitan [6] suggested one more method for computing the traces of the Sturm–Liouville operator. There are numerous investigations on the calculation of the regularized trace of differential operator equations [3-17].

2. CALCULATION

Let us show the following equation
\[
\lim_{N \to \infty} \sum_{n=1}^{N} \left[ (\varphi_n, L\varphi_n) - (\psi_n, L\psi_n) \right] = 0 \quad (3)
\]
which will be used later. For this we consider the transfer matrix \( (u_{ik})_{i,k=1}^{\infty} \) from the orthonormal basis \{\varphi_k\} to orthonormal basis \{\psi_k\} as in [2]:
\[
\psi_k = \sum_{i=1}^{\infty} u_{ik} \varphi_i \quad (k = 1, 2, \ldots)
\]
where \( u_{ik} = (\varphi_i, \psi_k) \) and \((u_{ik})_{l,k=1}^{\infty}\) are the unitary matrix, that is
\[
\sum_{l=1}^{\infty} u_{lk}^2 = 1 \quad (k = 1, 2, \ldots)
\]

Let us give some limitations for \( u_{ik} \). It is clear that
\[
L\psi_k = \mu_k \psi_k + p \psi_k \tag{4}
\]
If we multiply both side of equality (4) by \( \varphi_i \) we obtain
\[
(L\psi_k, \varphi_i) = (\mu_k \psi_k, \varphi_i) + (p \psi_k, \varphi_i)
\]
Or
\[
\lambda_i (\psi_k, \varphi_i) = \mu_k (\psi_k, \varphi_i) + (p \psi_k, \varphi_i)
\]
and
\[
(\lambda_i - \mu_k)(\psi_k, \varphi_i) = (p \psi_k, \varphi_i)
\]

With respect to [2] taking the square of both sides of the last equality and summing from 1 to \( \infty \) respect to \( i \) we obtain
\[
\sum_{i=1}^{\infty} (\lambda_i - \mu_k)^2 (\psi_k, \varphi_i)^2 = \sum_{i=1}^{\infty} (p \psi_k, \varphi_i)^2 = \|p \psi_k\|^2 = \int_0^\pi [p(x) \psi_k(x)]^2 dx \leq p_0^2 \tag{5}
\]
where \( p_0 = \max_{0 \leq x \leq \pi} |p(x)| \).

Suppose that the following conditions hold:

1. For the eigenvalues and the eigenfunctions of the \( L \) operator holds the asymptotic formulas
   \[
   \lambda_k = \mu_k + O \left( \frac{1}{k} \right), \quad \varphi_k = \psi_k + O \left( \frac{1}{k} \right) \quad [10].
   \]

2. \( \int_0^\pi p(x) \, dx = 0 \).

Hence
\[
\sum_{i=N+1}^{\infty} (\lambda_i - \mu_k)^2 u_{ik}^2 < C \quad (C = \text{const.}) \quad (k < N) \tag{6}
\]

We will use condition 1 in the inequalities we will obtain.

Obviously,
\[
\sum_{i=N+1}^{\infty} (\lambda_i - \mu_k) u_{ik}^2 < C \Rightarrow \sum_{i=N+1}^{\infty} (\lambda_i - \mu_k)(\lambda_i - \lambda_k) u_{ik}^2 < C
\]
\[
\Rightarrow \sum_{i=N+1}^{\infty} (\lambda_i - \lambda_k)^2 u_{ik}^2 < C
\]
is obtained for all integer $N$ from equation (6)

And we obtain

$$\sum_{i=N+1}^{\infty} (\lambda_i - \lambda_k) u_{ik}^2 \leq \frac{C}{\lambda_{N+1} - \mu_k} \quad (k < N). \quad (7)$$

Now let us prove the equation (3).

$$\sum_{i=1}^{N} (\psi_k, L \psi_k) = \left( \sum_{i=1}^{\infty} u_{ik} \phi_i, \sum_{i=1}^{\infty} \lambda_i u_{ik} \phi_i \right) = \sum_{i=1}^{\infty} \lambda_i u_{ik}^2$$

If we take the sum on $k$ from 1 to $N$ on both sides of this equation we get

$$\sum_{k=1}^{N} (\psi_k, L \psi_k) = \sum_{k=1}^{N} \sum_{i=1}^{\infty} \lambda_i u_{ik}^2.$$

Since $\sum_{i=1}^{\infty} u_{kl}^2 = 1$ we get

$$\sum_{k=1}^{N} (\varphi_k, L \varphi_k) = \sum_{k=1}^{N} \lambda_k = \sum_{k=1}^{N} \sum_{i=1}^{\infty} \lambda_k u_{kl}^2.$$

So now we need to prove

$$\lim_{N \to \infty} \left( \sum_{k=1}^{N} \sum_{i=1}^{\infty} \lambda_i u_{ik}^2 - \sum_{k=1}^{N} \sum_{i=1}^{\infty} \lambda_k u_{kl}^2 \right) = 0. \quad (8)$$

$$\sum_{k=1}^{N} \sum_{i=1}^{\infty} \lambda_i u_{ik}^2 - \sum_{k=1}^{N} \sum_{i=1}^{\infty} \lambda_k u_{kl}^2 = \sum_{k=1}^{N} \sum_{i=N+1}^{\infty} (\lambda_i - \lambda_k) u_{ik}^2 + \sum_{k=1}^{N} \sum_{i=N+1}^{\infty} \lambda_k (u_{ik}^2 - u_{kl}^2). \quad (9)$$

Let us calculate first sum on the right side of equality (9). For convenience while let $N + 1$ be even number then we have

$$\sum_{k=1}^{N} \sum_{i=N+1}^{\infty} (\lambda_i - \lambda_k) u_{ik}^2 = \sum_{k=1}^{N-1} \sum_{i=N+1}^{\infty} (\lambda_i - \lambda_k) u_{ik}^2 + (\lambda_{N+1} - \lambda_N) u_{(N+1)N}^2 + \sum_{i=N+2}^{\infty} (\lambda_i - \lambda_N) u_{iN}^2 \quad (10)$$

Let us calculate first and third sum on the right side of equality (10) by inequality (7), for $N \to \infty$

$$\sum_{k=1}^{N} \sum_{i=N+1}^{\infty} (\lambda_i - \lambda_k) u_{ik}^2 < \frac{1}{4N} + \frac{1}{2(N+1)} \left[ \ln \frac{N^2 + N}{N - 1} \right] \to 0 \quad (11)$$

and
\[
\sum_{i=1}^{\infty} (\lambda_i - \lambda_N) u_{iN}^2 \leq \frac{C}{\lambda_{N+2} - \mu_N} \leq \frac{C}{4N + 4} \to 0
\]  
(12)

Now we shall calculate the second term on the right side of equality (10) when \( N \to \infty \). Suppose that \( N + 1 \) is even, we have

\[
(\lambda_{N+1} - \lambda_N) u_{(N+1)N}^2 \leq N^2 + O\left(\frac{1}{N+1}\right) - N^2 - O\left(\frac{1}{N}\right) \to 0 \quad (N \to \infty)
\]  
(13)

In this way, for even number \( N + 1 \) from the expressions (10), (11), (12) and (13) we have

\[
\lim_{N \to \infty} \sum_{k=1}^{N} \sum_{i=N+1}^{\infty} (\lambda_i - \lambda_k) u_{ik}^2 = 0.
\]  
(14)

Formula (14) can also calculated for odd number \( N + 1 \).

Now we shall calculate second sum on the right side of equality (9).

\[
u_{ik} + u_{ki} = (\varphi_i, \psi_k) + (\varphi_k, \psi_i) = -(\varphi_i - \psi_i, \varphi_k - \psi_k)
\]  
(15)

By equality (15) and condition 1., we have

\[
|u_{ik} + u_{ki}| \leq \|\varphi_i - \psi_i\| \|\varphi_k - \psi_k\| < \frac{C}{ik}.
\]  
(16)

According to Cauchy-Schwarz inequality we have

\[
\sum_{i=N+1}^{\infty} (\lambda_i - \mu_k) |u_{ik}^2 - u_{ki}^2| = \sum_{i=N+1}^{\infty} (\lambda_i - \mu_k) |u_{ik} - u_{ki}| |u_{ik} + u_{ki}|
\]

\[
\leq \sqrt{\sum_{i=N+1}^{\infty} |u_{ik} - u_{ki}|^2} \sqrt{\sum_{i=N+1}^{\infty} (\lambda_i - \mu_k)^2 |u_{ik} - u_{ki}|^2}
\]

\[
< \frac{C}{(k-1)\sqrt{N+1}}.
\]  
(17)

Hence

\[
\sum_{i=N+1}^{\infty} |u_{ik}^2 - u_{ki}^2| < \frac{C}{(k-1)\sqrt{N+1}[N^2 - (k-1)^2]}
\]  
(18)

Now we shall evaluate the second sum on the right side of equality (9),

\[
\sum_{k=1}^{N} \lambda_k \sum_{i=N+1}^{\infty} |u_{ik}^2 - u_{ki}^2| = \lambda_N \sum_{i=N+1}^{\infty} |u_{iN}^2 - u_{Ni}^2| + \sum_{k=1}^{N-1} \lambda_k \sum_{i=N+1}^{\infty} |u_{ik}^2 - u_{ki}^2|
\]
\begin{align*}
&= \lambda_N \left| u_{N+1N}^2 - u_{NN+1}^2 \right| + \lambda_N \sum_{i=N+2}^{\infty} \left| u_i^2 - u_{Ni}^2 \right| + \sum_{k=1}^{N-1} \lambda_k \sum_{i=N+1}^{\infty} \left| u_{ik}^2 - u_{ki}^2 \right| \\
&\text{(19)}
\end{align*}

By inequality (16) we have

\begin{align*}
\lambda_N \left| u_{N+1N}^2 - u_{NN+1}^2 \right| &= \lambda_N \left| u_{N+1N} - u_{NN+1} \right| \left| u_{N+1N} + u_{NN+1} \right| \\
&\leq \frac{CN^2}{N^2(N + 1)^2} \left| u_{N+1N} - u_{NN+1} \right| \to 0 \quad (N \to \infty) \\
&\text{(20)}
\end{align*}

By the expression (18) we evaluate the second and third sum on the right side of equality (19)

\begin{align*}
\lambda_N \sum_{i=N+2}^{\infty} \left| u_i^2 - u_{Ni}^2 \right| &< \frac{CN^2}{(N - 1)\sqrt{N} + 2[(N + 2)^2 - (N + 1)^2]} \to \infty \quad (N \to \infty) \\
&\text{(21)}
\end{align*}

and

\begin{align*}
\sum_{k=1}^{N-1} \lambda_k \sum_{i=N+1}^{\infty} \left| u_{ik}^2 - u_{ki}^2 \right| &< \frac{CN}{\sqrt{N} + 1} \sum_{k=2}^{N} \frac{1}{N^2 - (k - 1)^2} \sim \frac{\ln N}{\sqrt{N}} \to 0 \quad (N \to \infty). \\
&\text{(22)}
\end{align*}

From the expressions (19), (20),(21) and (22) we have

\begin{align*}
\lim_{N \to \infty} \sum_{k=1}^{N} \sum_{i=N+1}^{\infty} \lambda_k (u_{ik}^2 - u_{ki}^2) = 0 \\
&\text{(23)}
\end{align*}

Thus from the expressions (9), (14), and (23) we obtain formula (8). Therefore formula (3) have proved.

3. CONCLUSION

\begin{align*}
(\varphi_k, L\varphi_k) = \lambda_k \quad \text{and} \quad (\psi_k, L\psi_k) = \mu_k + (\psi_k, p\psi_k).
\end{align*}

If we use these into formula (3) then we obtain

\begin{align*}
\sum_{k=1}^{N} [(\psi_k, L\psi_k) - (\varphi_k, L\varphi_k)] = \sum_{k=1}^{N} (\mu_k - \lambda_k) + \sum_{k=1}^{N} (\psi_k, p\psi_k) \to 0, \quad (N \to \infty). \\
&\text{(24)}
\end{align*}

Now we shall calculate

\begin{align*}
\lim_{N \to \infty} \sum_{k=1}^{N} (\psi_k, p\psi_k).
\end{align*}

According to condition 2. we have for even number \( N \)
\[
\sum_{k=1}^{N} (\psi_k, p\psi_k) = \frac{1}{\pi} \int_0^{\pi} p(x) \, dx + \frac{N}{\pi} \int_0^{\pi} p(x) \, dx = 0
\] (25)

Similarly we have for odd number \( N \)
\[
\sum_{k=1}^{N} (\psi_k, p\psi_k) = -\frac{1}{\pi} \int_0^{\pi} p(x) \cos 2Nx \, dx \to 0 \quad (N \to \infty).
\] (26)

From the expressions (25) and (26) we have
\[
\lim_{N \to \infty} \sum_{k=1}^{N} (\psi_k, p\psi_k) = 0
\]

Hence from the expressions (24) and (26) we have
\[
\lim_{N \to \infty} \sum_{k=1}^{N} (\lambda_k - \mu_k) = 0.
\]

So we have proved the following theorem.

**THEOREM** : The following formula is true when we considered \( p(x) \) is a continuous function and conditions 1.,2. are fulfilled
\[
\sum_{n=1}^{\infty} (\lambda_n - \mu_n) = 0.
\] (27)

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