Electromagnetic Force as consequence of the Geometry of Minkowskian Spacetime

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Abstract

By describing the dynamical evolution of a test charged particle in the presence of an electromagnetic field as a succession of infinitesimal Lorentz boosts and rotations it is possible to obtain the Lorentz Force of Electrodynamics. A consequence of this derivation at the classical level is that, given the existence of electric and magnetic fields, the form of the electromagnetic force acting on the particle can be regarded as arising from the geometry of Minkowskian spacetime.
1. Introduction

As is well known, the Lorentz Force acting on a charged test particle under the action of an electromagnetic field cannot be obtained as a consequence of Maxwell Equations, but must be postulated in an independent manner. In some text books, heuristic arguments about the form of the force are given, such as being the simplest four-force compatible with special relativity and linear in the components of the four-velocity (Barut 1964, Rindler 1982), while in other undergraduate texts it is attempted to derive the existence of magnetic fields (and even the specific form of the magnetic force) from Coulomb’s law of electrostatics and the Lorentz transformations of special relativity (Lorrain and Corson 1970). Another approach to the problem is to make Newton’s second law applied to a particle in an electrostatic field consistent with relativistic invariance (Kobe, 1986).

The problem of the relation between the dynamical behaviour of charged bodies in electromagnetic fields and special relativity has been the subject of some controversy which we consider out of the scope of this work. A usual procedure which can be found in many text books is to consider the equation of motion of a particle in an inertial frame, in which there is only an electric field, and then transform to another inertial frame. As a matter of fact, from this approach it is possible to obtain some partial features of electric and magnetic forces (see the remarks of Jackson, 1975) but not to obtain the electromagnetic force which, as we shall attempt to show in this article, is not related to the usual finite Lorentz transformations but to infinitesimal boosts and rotations in spacetime.

The motivation for this article is to present a derivation of the covariant equations of motion of a charged particle in an electromagnetic field that suggests
a new interpretation of the Lorentz Force as an unavoidable consequence of the geometric structure of Minkowskian spacetime and also gives new insight into the nature of electric and magnetic fields.

Central to our discussion will be a somewhat detailed examination of how the evolution of the four velocity vector of a test particle can be described by means of infinitesimal boosts and rotations and how these transformations are seen from the standpoint of two different inertial frames of reference, this is done in sections 2 and 3. In section 4 we shall relate the equations of motion obtained from a description in terms of parallel axis boosts to the motion of a charged test particle under the action of an electric field. However, the equation of motion, in this case, are not susceptible of being written in a covariant tensor form. We shall see that a more general description based in the combination of boosts and rotations leads in a natural way to the covariant form of the electromagnetic force.

2. Infinitesimal boosts and rotations

Suppose we have a particle of mass $m$ following an accelerated, not necessarily rectilinear, trajectory relative to a certain inertial frame $S$. At a generic proper time $\tau$ the four velocity in frame $S$ is $\bar{u}(\tau)$ and after a small lapse of proper time $\delta \tau$ the four velocity has evolved to $\bar{u}(\tau + \delta \tau)$ relative to the same frame. By the principle of relativity it is always possible to choose another inertial frame $S^*$ such that at proper time $\tau$ the four velocity of the particle is

$$\bar{u}^*(\tau) = \bar{u}(\tau + \delta \tau).$$  \hspace{1cm} (1)

Since $S$ and $S^*$ are inertial frames we know that there is a Lorentz transformation, a spatial rotation or a combination of both that relate $\bar{u}(\tau)$ and $\bar{u}^*(\tau)$. 

4
Furthermore, from the assumed smallness of $\delta \tau$ the transformation need only be infinitesimal. If $A(\tau)$ is such a transformation we can, accordingly, write

$$\bar{u}^*(\tau) = A(\tau)\bar{u}(\tau),$$

and therefore

$$\bar{u}(\tau + \delta \tau) = A(\tau)\bar{u}(\tau).$$

Departing here from the conventional use of Lorentz boost as transformations relating geometric objects in two different inertial frames, we shall, henceforth, regard $A(\tau)$ as a linear operator relating $\bar{u}(\tau + \delta \tau)$ and $\bar{u}(\tau)$ in the same frame of reference.

In the case that the change in the particle’s four velocity can be described by an infinitesimal Lorentz boost alone, the explicit form of $A(\tau)$ is a $4 \times 4$ matrix

$$A(\tau) = e^{\delta \vec{v} . \vec{K}} \simeq I + \delta \vec{v} . \vec{K},$$

$\delta \vec{v}$ being the change in the velocity of the body during the lapse of time $\delta \tau$ as measured in $S$ (we are using units in which $c = 1$ and signature of the special relativistic metric tensor $+---$), $I$ the unit matrix, and the components of $\vec{K}$ the generators of Lorentz boosts (see Jackson, 1975)

$$K_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \ K_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \ K_3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}. \quad (5)$$

When the accelerated motion follows a curved trajectory one should, in general, include spatial rotations as well as boosts which means that we need all
the six generators of the Lorentz Group (as infinitesimal boosts and rotations commute, see equation 6 below, the order in which the transformations are carried out is irrelevant). Equation (4) should be replaced by

\[ A = e^{-(\delta \vec{S} \cdot \vec{S} - \delta \vec{v} \cdot \vec{K})}, \]  

where the components of \( \vec{S} \) are the generators of spatial rotations, explicitly:

\[
S_1 = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0
\end{pmatrix},
S_2 = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix},
S_3 = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}.
\]

3. Transformation properties of the parameters of infinitesimal boosts and rotations

Consider two inertial systems \( S \) and \( S' \) with coordinates \((x_0, x_1, x_2, x_3)\) and \((x'_0, x'_1, x'_2, x'_3)\) respectively. The coordinate axes in the two frames are parallel and oriented so that the system \( S' \) is moving in the positive \( x_1 \) direction with velocity \( v \), as viewed from \( S \). In \( S' \) we have a test particle moving, at proper time \( \tau \), in the direction of the \( x'_2 \) axis with velocity \( v' \). Its four velocity being

\[
\vec{u}'(\tau) = \begin{pmatrix}
\frac{u'^0}{u'^2}
\end{pmatrix},
\]

with \( u'^2 = \gamma' v' \), \( \gamma' = (1 - v'^2)^{-1/2} \).

At proper time \( \tau \), and always in \( S' \), the particle accelerates in the \( x'_2 \) direction during a lapse of proper time \( \delta \tau \) thereby suffering a change in its velocity

\[
\delta \vec{v}' = \begin{pmatrix}
0 \\
\delta v'_2 \\
0
\end{pmatrix}.
\]
According to the starting discussion in this section, we can describe the evolution of the four velocity in frame $S'$ as

$$\ddot{u}'(\tau + \delta\tau) = A'(\tau)\dot{u}'(\tau),$$  \hspace{1cm} (10)

where $A'(\tau)$ is the operator for an infinitesimal Lorentz boost in the $x_2'$ direction.

From (4) and (5), $A'$ in matrix form is given by

$$A' = \begin{pmatrix} 1 & 0 & \delta v_2' & 0 \\ 0 & 1 & 0 & 0 \\ \delta v_2' & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$  \hspace{1cm} (11)

Applying $A'$ to $\ddot{u}'(\tau)$ given by (8) we get

$$\ddot{u}'(\tau + \delta\tau) = \begin{pmatrix} u'^0 + u'^2\delta v_2' \\ 0 \\ u'^0\delta v_2' + u'^2 \\ 0 \end{pmatrix}.$$  \hspace{1cm} (12)

Now if we Lorentz transform $\ddot{u}'(\tau)$ and $\ddot{u}'(\tau + \delta\tau)$ to $S$, we find

$$\ddot{u}(\tau) = \begin{pmatrix} \gamma u'^0 \\ \gamma v u'^0 \\ u'^2 \\ 0 \end{pmatrix},$$  \hspace{1cm} (13)

and

$$\ddot{u}(\tau + \delta\tau) = \begin{pmatrix} \gamma(u'^0 + u'^2\delta v_2') \\ \gamma v(u'^0 + u'^2\delta v_2') \\ u'^2 + u'^0\delta v_2' \\ 0 \end{pmatrix},$$  \hspace{1cm} (14)

where $\gamma = (1-v^2)^{-1/2}$ and the $4 \times 4$ matrix for the finite Lorentz transformation to $S$ is

$$\begin{pmatrix} \gamma & \gamma v & 0 & 0 \\ \gamma v & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The question which now arises is: how should the change in the four velocity of the test particle be regarded by an observer in $S$? Since the $x_1$ component of
\( \bar{u} \) has also changed during \( \delta \tau \), it is clear that something more involved than a single infinitesimal Lorentz boost along the \( x_2 \) axis would be necessary in order to produce the observed change. The answer to this question can be readily found in an intuitive way if we note that an accelerated straight path in \( S' \), along \( x_2' \), would be seen from the point of view of frame \( S \) as an accelerated but curved trajectory in the \((x_1, x_2)\) plane which could be described as a succession of infinitesimal boosts along the \( x_2 \) direction and rotations around \( x_3 \). The operator performing this task is

\[
A = e^{-(\delta \phi_3 S_3 - \delta v_2 K_2)} \simeq I - (\delta \phi_3 S_3 - \delta v_2 K_2) = \begin{pmatrix}
1 & 0 & \delta v_2 & 0 \\
0 & 1 & \delta \phi_3 & 0 \\
\delta v_2 & -\delta \phi_3 & 1 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}.
\]

(15)

Solving the linear system which comes out from \( \bar{u}(\tau + \delta \tau) = A\bar{u}(\tau) \), with \( \bar{u}(\tau + \delta \tau) \) and \( \bar{u}(\tau) \) given by (13) and (14) respectively, and \( A \) given by (15), we find

\[
\delta v_2 = \gamma \delta v_2'
\]

and

\[
\delta \phi_3 = \gamma v \delta v_2'.
\]

(16) (17)

Let us now write the familiar transformations equations between electric and magnetic fields in the two inertial frames \( S \) and \( S' \) (Jackson, 1975)

\[
E_1 = E_1', \quad E_2 = \gamma(E_2' + vB_3'), \quad E_3 = \gamma(E_3' - vB_2'). \\
B_1 = B_1', \quad B_2 = \gamma(B_2' - vE_3'), \quad B_3 = \gamma(B_3' + vE_2').
\]

(18)

From the way in which we have set up our experiment, it is clear that the rectilinear, accelerated, motion of the test particle in \( S' \), if charged, could be attributed to a pure electric field in the \( x_2' \) direction which would transform to \( S \) according to

\[
E_2 = \gamma E_2'.
\]
\[ B_3 = \gamma v E'_2. \]

By comparison of these two equations with (16) and (17) we see that, at least for the particular case under consideration, an electric field in the \( x'_2 \) direction has the same transformation law that an infinitesimal Lorentz boost in that direction and that magnetic fields might, perhaps, have some relation with spatial rotations.

Let us now see how a pure spatial rotation of angle \( \delta \phi'_3 \) around the \( x'_3 \) axis performed on the four velocity, in \( S' \), is seen by an observer in \( S \). In \( S' \), the four velocity evolves according to

\[
\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & \delta \phi'_3 & 0 \\ 0 & -\delta \phi'_3 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} w^0 \\ 0 \\ w^2 \delta \phi'_3 \\ 0 \end{pmatrix} = \begin{pmatrix} w^0 \\ w^2 \delta \phi'_3 \\ 0 \end{pmatrix}. \tag{19}
\]

Following a similar procedure as the one used in the precedent case, one finds, in \( S \), that a spatial rotation around \( x'_3 \) of angle \( \delta \phi'_3 \) in \( S' \), is seen from the standpoint of an observer in \( S \) as a boost

\[ \delta v_2 = \gamma v \delta \phi'_3 \tag{20} \]

in the \( x_2 \) direction followed by a rotation of angle

\[ \delta \phi_3 = \gamma \delta \phi'_3 \tag{21} \]

around \( x_3 \). On the other hand, from equations (18), it is readily seen that a pure magnetic field along \( x'_3 \), in \( S' \), would transform to \( S \) as

\[ E_2 = \gamma v B'_3, \]

9
\[ B_3 = \gamma B'_3. \]

We wish to show now that every relation between electric and magnetic fields in (18) has a geometric counterpart of whom equations (16), (17) and (20), (21) are particular cases. Following the same somewhat pedestrian, but clarifying approach, we consider first a \( x'_1 \) boost in \( S' \) acting on a generic four-velocity vector \( \vec{u}'(\tau) = (u'^0, u'^1, u'^2, u'^3) \), compute the change in \( \vec{u}'(\tau) \), Lorentz transform this change to \( S \) and find the relation between the operators responsible for the change in both inertial frames. As one should expect for longitudinal accelerations, the result is

\[ \delta v_1 = \delta v'_1 \] (22)

(as geometric counterpart of \( E_1 = E'_1 \)). Although somewhat laborious, it is also possible to show that no other combination of boosts and rotations in \( S' \) yields a \( \delta v_1 \neq 0 \) in \( S \). To prove that

\[ \delta \phi_1 = \delta \phi'_1, \] (23)

as counterpart of \( B_1 = B'_1 \), it is only necessary to note that in a rotation of the spatial part of the four velocity around the preferred axis \( x'_1 \) only the components orthogonal to \( u'_1 \) are changed and these components remain without alteration under a Lorentz transformation in the \( x_1 \) direction.

In order to find geometric expressions similar to the second and last relations in (18) we shall consider a boost in the \( x'_2 \) direction followed by a rotation around \( x'_3 \). The evolution equations are in \( S \)

\[
\begin{bmatrix}
1 & 0 & \delta v'_2 & 0 \\
0 & 1 & \delta \phi'_3 & 0 \\
\delta v'_2 & -\delta \phi'_3 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
u'^0 \\
u'^1 \\
u'^2 \\
u'^3
\end{bmatrix}
= 
\begin{bmatrix}
u'^0 + u'^2 \delta v'_2 \\
u'^1 + u'^2 \delta \phi'_3 \\
u'^0 \delta v'_2 - u'^1 \delta \phi'_3 + u'^2 \\
u'^3
\end{bmatrix}.
\] (24)
On the other hand, the Lorentz transformed vectors to $S \bar{u}(\tau + \delta \tau)$ and $\bar{u}(\tau)$ are

$$\bar{u}(\tau + \delta \tau) = \begin{pmatrix} \gamma(u^0 + u^2 \delta v^2) + \gamma v(u^1 + u^2 \delta \phi_3') \\ \gamma v(u^0 + u^2 \delta v^2) + \gamma(u^1 + u^2 \delta \phi_3') \\ u^0 \delta v'_2 - u^1 \delta \phi_3' + u^2 \\ u^3 \end{pmatrix}, \quad (25)$$

and

$$\bar{u}(\tau) = \begin{pmatrix} \gamma(u^0 + vu'^1) \\ \gamma(u'^1 + vu^0) \\ u'^2 \\ u^3 \end{pmatrix}. \quad (26)$$

The observed change in $S$ is thus

$$\delta \bar{u}(\tau) = \begin{pmatrix} \gamma(u^0 \delta v'_2 + vu'^2 \delta \phi_3') \\ \gamma(vu'^2 \delta v'_2 + u'^2 \delta \phi_3') \\ u^0 \delta v'_2 - u^1 \delta \phi_3' \end{pmatrix}. \quad (27)$$

We take now into account that, as we have previously seen, $\delta v'_1 = 0 \Rightarrow \delta v_1 = 0$ and $\delta \phi'_1 = 0 \Rightarrow \delta \phi_1 = 0$. Also, since the $u'^3$ component have remained unchanged in both inertial frames: $\delta v_3 = 0$, $\delta \phi_2 = 0$. The evolution equations (written now as $(A - I)\bar{u}(\tau) = \delta \bar{u}(\tau)$) in $S$ are then

$$\begin{pmatrix} 0 & 0 & \delta v_2 & 0 \\ 0 & 0 & \delta \phi_3 & 0 \\ \delta v_2 & -\delta \phi_3 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \gamma(u^0 + vu'^1) \\ \gamma(u'^1 + vu^0) \\ u'^2 \\ u^3 \end{pmatrix} = \begin{pmatrix} \gamma(u^0 \delta v'_2 + vu'^2 \delta \phi_3') \\ \gamma(vu'^2 \delta v'_2 + u'^2 \delta \phi_3') \\ u^0 \delta v'_2 - u^1 \delta \phi_3' \end{pmatrix}, \quad (28)$$

and solving for $\delta v_2$ and $\delta \phi_3$:

$$\delta v_2 = \gamma(\delta v'_2 + v\delta \phi'_3), \quad (29)$$

$$\delta \phi_3 = \gamma(v\delta v'_2 + \delta \phi'_3). \quad (30)$$

In the same way, equations similar to the third and fifth of (18) can be obtained.
4. Lorentz Force

In the last two sections, we have gained some understanding of how the evolution of a particle’s four velocity can be related to infinitesimal boosts and rotations in spacetime and with the way these geometric operations, considered here as linear operators, transform between inertial reference frames. We shall now attempt to relate the Lorentz force to these infinitesimal transformations. To this end, it is convenient to consider first a general infinitesimal Lorentz boost, the explicit form of $A(\tau)$ is

$$A(\tau) = e^{(\delta \vec{v}, \vec{K})} \simeq I + \delta \vec{v}, \vec{K} =$$

$$= I + \begin{pmatrix}
0 & \delta v_1 & \delta v_2 & \delta v_3 \\
\delta v_1 & 0 & 0 & 0 \\
\delta v_2 & 0 & 0 & 0 \\
\delta v_3 & 0 & 0 & 0
\end{pmatrix}.$$  \(31\)

Expanding (3) with $A(\tau)$ given by (31), and $\bar{u}(\tau) = (u^0, \vec{u}) = (\gamma, \gamma \vec{v})$, we get

$$u^0(\tau + \delta \tau) = u^0(\tau) + \delta \vec{v}, \vec{u}$$

$$u^i(\tau + \delta \tau) = u^i(\tau) + \delta v^i u^0 = u^i(\tau) + \gamma \delta v^i$$ \(32\)

$i = 1, 2, 3$.

From these equations, the change in the four velocity $\delta \bar{u}$ is defined as

$$\delta \bar{u} \equiv (\delta \vec{v}, \vec{u}, \gamma \delta \vec{v}).$$ \(33\)

We note that $\delta \bar{u}$ is orthogonal to the four velocity $\bar{u} = (\gamma, \gamma \vec{v})$:

$$\delta \bar{u}, \bar{u} = \gamma (\delta \vec{v}, \vec{u}) - \gamma^2 \delta \vec{v}, \vec{v} = 0.$$
Equations (32) can also be written as

\[
\frac{u^0(\tau + \delta \tau) - u^0(\tau)}{\delta \tau} = \frac{\delta \vec{v}}{\delta \tau} \cdot \vec{u},
\]

and

\[
\frac{u^i(\tau + \delta \tau) - u^i(\tau)}{\delta \tau} = \frac{\delta v^i}{\delta \tau} u^0
\]

(i = 1, 2, 3).

If we define now

\[
\chi_i(\tau) = \lim_{\delta \tau \to 0} \frac{\delta v^i}{\delta \tau},
\]

we can write instead of (34):

\[
\frac{du^0}{d\tau} = \bar{\chi}(\tau) \cdot \vec{u}
\]

and

\[
\frac{du^i}{d\tau} = \chi_i(\tau) u^0.
\]

For the case under consideration (i.e. infinitesimal Lorentz boosts), the last equations are valid irrespective of the external field. (they do not describe the more general kind of change as we have not included spatial rotations). In the context of flat spacetime, we know that any change in a particle’s four velocity should arise from the interaction with an external driving field. Due to this fact, we can attempt to relate the quantities \(\chi_i(\tau)\) with the components of the electric field.

To elucidate further this point let us now examine the part of the Lorentz force that only depends on the electric field:

\[
\frac{du^0}{d\tau} = \frac{e}{m} \vec{E} \cdot \vec{u}
\]
\[
\frac{du^i}{d\tau} = \frac{e}{m} E_i u^0.
\]  

(37)

It is apparent that the last equations are completely similar to equations (36).

Furthermore, experimentally we know that:

a) The electric force acting on a particle is proportional to its charge.

b) The electric force depends on the point of spacetime where a charge is located and not of \( \vec{u} \).

It seems thus that no conflict with previous notions about the electric field arises if we substitute in (36), up to a proportionality constant, the quantities \( \chi_i \) by the components of the electric field and consider this field as responsible for the succession of infinitesimal Lorentz boosts that change the four velocity of the particle. (Note that we could write now instead of (3)

\[
\bar{u}(\tau + \delta\tau) = e^{\frac{\vec{E} \cdot \vec{K}}{m}\delta\tau} \bar{u}(\tau)
\]

(38)

and obtain the electric force).

To get the full electromagnetic force one should only notice that equations (36) cannot be completely general because if the evolution in the four velocity vector can be described in terms of an infinitesimal Lorentz boost in a certain frame, from the point of view of another inertial frame we have seen in the preceding section that it is necessary, in general, to use the combination of Lorentz boosts and infinitesimal spatial rotations. As tensor equations express physical laws in a form independent of the choice of inertial frame one should expect to find the covariant form of the equation of motion from a more general transformation on the four velocity. As a matter of fact, although equations (36) are compatible with the condition \((du^\alpha/d\tau).u_\alpha = 0\), the real limitation is that it does not seem possible to write them in the form of a tensor expression valid in
any coordinate frame. One is thus lead to conclude that equations of the kind of (36) and (37), should only be valid in a preferred frame of reference (see section 4 below). By considering a more general kind of motion, we shall now see that the form of the Lorentz Force, and the number of non null components of the field, is dictated by the specific form of the 6 matrices ($S_i$ and $K_i$) generators of the Lorentz Group.

As previously noted, a general description of the evolution of the four velocity vector, valid in any frame of reference, should include spatial rotations as well as boosts which means that we need all the six generators of the Lorentz Group. The operator (31) should be replaced by

$$A = e^{-(\delta \vec{\phi} \cdot \vec{S} - \delta \vec{v} \cdot \vec{K})},$$

(39)

We now relate the change in the velocity of the particle $\delta \vec{v}$ with an external force field which we call $\vec{\epsilon}(x)$ and the rotation $\delta \vec{\phi}$ with another field $\vec{b}(x)$. The change in the velocity vector $\delta \vec{v}$ should be proportional to $\delta \tau$ and depend on the particle’s spacetime location. Accordingly, we write

$$\delta \vec{v} = k \vec{\epsilon}(x) \delta \tau,$$

(40)

and, since from a geometric standpoint a spatial rotation is quite similar to a space-time rotation (boost), we also write

$$\delta \vec{\phi} = k \vec{b}(x) \delta \tau,$$

(41)

$k$ being a constant.
Since $A$ is infinitesimal:

$$A(x) \simeq I - k[\vec{b}(x) \cdot \vec{S} - \vec{e}(x) \cdot \vec{K}] \delta \tau. \quad (42)$$

Once expanded with $A(x)$ given by (42), equation (3) now reads

$$\begin{pmatrix}
  u^0(\tau + \delta \tau) \\
u^1(\tau + \delta \tau) \\
u^2(\tau + \delta \tau) \\
u^3(\tau + \delta \tau)
\end{pmatrix} \simeq
\begin{pmatrix}
  u^0(\tau) \\
u^1(\tau) \\
u^2(\tau) \\
u^3(\tau)
\end{pmatrix} +
k \delta \tau
\begin{pmatrix}
  0 & \epsilon_1 & \epsilon_2 & \epsilon_3 \\
\epsilon_1 & 0 & b_3 & -b_2 \\
\epsilon_2 & -b_3 & 0 & b_1 \\
\epsilon_3 & b_2 & -b_1 & 0
\end{pmatrix}
\begin{pmatrix}
  u^0(\tau) \\
u^1(\tau) \\
u^2(\tau) \\
u^3(\tau)
\end{pmatrix}. \quad (43)$$

In the limit $\delta \tau \to 0$ the former matrix equation can be expressed as the following two equations

$$\frac{d u^0}{d \tau} = k \vec{\bar{u}} \cdot \vec{e} \quad (44)$$

and

$$\frac{d \vec{\bar{u}}}{d \tau} = k(\vec{\bar{e}} u^0 + \vec{\bar{u}} \times \vec{b}), \quad (45)$$

which for $k \equiv e/m$ are identical to the equations describing the Lorentz force.

From (43) and recalling that contravariant vectors are represented by column matrices, the last two equations can be joined into a single tensor expression:

$$\frac{d u^\alpha}{d \tau} = k F^\alpha_\beta u^\beta, \quad (46)$$

16
where $F^\alpha_\beta$ is associated with the \textit{antisymmetric} electromagnetic tensor by

$$F^\alpha_\beta = \eta^{\alpha\delta} F_{\delta\beta},$$

(47)

$\eta^{\alpha\delta}$ being the metric tensor.

5. Final Remarks

A very special situation which deserves a comment is the situation corresponding to a charged test particle moving in circular motion around a charged object. As the energy remain constant in this case, an observer in the rest frame of the source might conclude that the test particle is performing a succession of rotations. However notice that the equations of motion in this case are, from (44) and (45)

$$\frac{du^0}{d\tau} = k\vec{u}.\vec{e} = 0$$

(48)

and

$$\frac{d\vec{u}}{d\tau} = k\vec{e}u^0.$$  

(49)

From (43), it is readily seen that the last equations are related to boosts not to rotations (The reader can convince himself that an infinitesimal boost orthogonal to $\vec{u}$ is both compatible, to first order, with the normalization condition $u^\alpha u_\alpha = 1$ and with the conservation of $|\vec{u}|$). This kind of motion, as well as any other taking place under a central electrostatic field, should be understood, as a sort of continuous fall, in terms of infinitesimal boosts when described in the preferred frame in which the source is at rest. The opposite situation would be a particle undergoing circular or helicoidal motion in a constant magnetic field which should be understood in terms of infinitesimal rotations.
Finally note that the interpretation of magnetic fields as performers of infinitesimal spatial rotations give also a geometric explanation of why there is no magnetic force in the rest system of a charged particle and in the direction of the field. Being the spatial components of the four velocity null, there is nothing to rotate and, also, the longitudinal component of the velocity remain unchanged in a rotation. However, there is no way of avoiding the existence of magnetic fields as it is impossible, for two or more source charges in relative motion, to set up a preferred reference system in which all the sources are at rest.
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