Rebounding Bandits for Modeling Satiation Effects

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Abstract

Psychological research shows that enjoyment of many goods is subject to satiation, with short-term satisfaction declining after repeated exposures to the same item. Nevertheless, proposed algorithms for powering recommender systems seldom model these dynamics, instead proceeding as though user preferences were fixed in time. In this work, we introduce rebounding bandits, a multi-armed bandit setup, where satiation dynamics are modeled as time-invariant linear dynamical systems. Expected rewards for each arm decline monotonically with consecutive exposures to it and rebound towards the initial reward whenever that arm is not pulled. Unlike classical bandit settings, methods for tackling rebounding bandits must plan ahead and model-based methods rely on estimating the parameters of the satiation dynamics. We characterize the planning problem, showing that the greedy policy is optimal when the arms exhibit identical deterministic dynamics. To address stochastic satiation dynamics with unknown parameters, we propose Explore-Estimate-Plan (EEP), an algorithm that pulls arms methodically, estimates the system dynamics, and then plans accordingly.

1 Introduction

Recommender systems suggest such diverse items as music, news, restaurants, and even job candidates. Practitioners hope that by leveraging historical interactions, they might provide services better aligned with their users’ preferences. However, despite their ubiquity in application, the dominant learning framework suffers several conceptual gaps that can result in misalignment between machine behavior and human preferences. For example, because human preferences are seldom directly observed, these systems are typically trained on the available observational data (e.g., purchases, ratings, or clicks) with the objective of predicting customer behavior \cite{4, 27}. Problematically, such observations tend to be confounded (reflecting exposure bias due to the current recommender system) and subject to censoring (e.g., users with strong opinions are more likely to write reviews) \cite{11, 10}.

Even if we could directly observe the utility experienced by each user, we might expect it to depend, in part, on the history of past items consumed. For example, consider the task of automated (music) playlisting. As a user is made to listen to the same song over and over again, we might expect that the utility derived from each consecutive listen would decline \cite{35}. However, after listening to other music for some time, we might expect the utility associated with that song to bounce back towards its baseline level. Similarly, a diner served pizza for lunch might feel diminished pleasure upon eating pizza again for dinner.

The psychology literature on satiation formalizes the idea that enjoyment depends not only on...
one’s intrinsic preference for a given product but also on the sequence of previous exposures and the time between them [3, 6]. Research on satiation dates to the 1960s (if not earlier) with early studies addressing brand loyalty [14, 25]. Interestingly, even after controlling for marketing variables like price, product design, promotion, etc., researchers still observe brand-switching behavior in consumers. Such behavior, referred as variety seeking, has often been explained as a consequence of utility associated with the change itself [25, 17]. For a comprehensive review on hedonic decline caused by repeated exposure to a stimulus, we refer the readers to [11].

In this paper, we introduce rebounding bandits, a multi-armed bandits (MABs) [37] framework that models satiation via linear dynamical systems. While traditional MABs draw rewards from fixed but unknown distributions, rebounding bandits allow each arm’s rewards to evolve as a function of both the per-arm characteristics (susceptibility to satiation and speed of rebounding) and the historical pulls (e.g., past recommendations). In rebounding bandits, even if the dynamics are known and deterministic, selecting the optimal sequence of $T$ arms to play requires planning in a Markov decision process (MDP) whose state space scales exponentially in the horizon $T$. When the satiation dynamics are known and stochastic, the states are only partially observable, since the satiation of each arm evolves with (unobserved) stochastic noises between pulls. And when the satiation dynamics are unknown, learning requires that we identify a stochastic dynamical system.

We propose Explore-Estimate-Plan (EEP) an algorithm that (i) collects data by pulling each arm repeatedly, (ii) estimates the dynamics using this dataset; and (iii) plans using the estimated parameters. We provide guarantees for our estimators in § 6.2 and bound EEP’s regret in § 6.3.

Our main contributions are: (i) the rebounding bandits problem (§3), (ii) analysis showing that when arms share rewards and (deterministic) dynamics, the optimal policy pulls arms cyclically, exhibiting variety-seeking behavior (§4.1); (iii) an estimator (for learning the satiation dynamics) along with a sample complexity bound for identifying an affine dynamical system using a single trajectory of data (§6.2); (iv) EEP, an algorithm for learning with unknown stochastic dynamics that achieves sublinear $w$-step lookahead regret [34] (§6); and (v) experiments demonstrating EEP’s efficacy (§7).

2 Related Work

Satiation effects have been addressed by such diverse disciplines as psychology, marketing, operations research, and recommendation systems. In the psychology and marketing literatures, satiation has been proposed as an explanation for variety-seeking consumer behavior [11, 25, 26]. In operations research, addressing continuous consumption decisions, [3] propose a deterministic linear dynamical system to model satiation effects. In the recommendation systems community, researchers have used semi-Markov models to explicitly model two states: (i) sensitization—where the user is highly interested in the product; and (ii) boredom—where the user is not engaged [18].

The bandits literature has proposed a variety of extensions where rewards depend on past exposures, both to address satiation and other phenomena. [14, 21, 39] tackle settings where each arm’s expected reward grows (or shrinks) monotonically in the number of pulls. By contrast, [19, 2, 7] propose models where rewards increase as a function of the time elapsed since the last pull. [34] model the expected reward as a function of the time since the last pull drawn from a Gaussian Process with known kernel. [43] propose a model where rewards are linear functions of the recent history of actions and [29] model the reward as a function of a context that evolves according to known deterministic dynamics. In rested bandits [12], an arm’s rewards changes only when it is played, and in restless bandits [44] rewards evolve independently from the play of
At time \( t \), we denote its pull history from 0 to \( T \) as the binary sequence \( u_{k,0:T} := (u_{k,0}, \ldots, u_{k,T}) \), where \( u_{k,0} = 0 \) and for \( t \in [T] \), \( u_{k,t} = 1 \) if \( \pi_t = k \) and \( u_{k,t} = 0 \) otherwise. The subsequence of \( u_{k,0:T} \) from \( t_1 \) to \( t_2 \) (including both endpoints) is denoted by \( u_{k,t_1:t_2} \).

At time \( t \), each arm \( k \) has a satiation level \( s_{k,t} \) that depends on a satiation retention factor \( \gamma_k \in [0,1) \), as follows

\[
s_{k,t} := \gamma_k (s_{k,t-1} + u_{k,t-1}) + z_{k,t-1}, \quad \forall t > t_0^k,
\]

where \( t_0^k := \min\{t : u_{k,t} = 1\} \) is the first time arm \( k \) is pulled and \( z_{k,t-1} \) is independent and identically distributed noise drawn from \( \mathcal{N}(0, \sigma_z^2) \), accounting for incidental (uncorrelated) factors in the satiation dynamics. Because satiation requires exposure, arms only begin to have nonzero satiation levels after their first pull, i.e., \( s_{k,0} = \ldots = s_{k,t_0^k} = 0 \).

At time \( t \in [T] \), if arm \( k \) is played with a current satiation level \( s_{k,t} \), the agent receives reward \( \mu_{k,t} := b_k - \lambda_k s_{k,t} \), where \( b_k \) is the base reward for arm \( k \) and \( \lambda_k \geq 0 \) is a bounded exposure influence factor. We use satiation influence to denote the product of the exposure influence factor \( \lambda_k \) and the satiation level \( s_{k,t} \). In Figure 3 we show how rewards evolve in response to both pulls and the stochastic dynamics under two sets of parameters. The expected reward of each arm.

**Key Differences** This may be the first bandits paper to model evolving rewards through continuous-state linear stochastic dynamical systems with unknown parameters. Our framework captures several important aspects of satiation: rewards decline by diminishing amounts with consecutive pulls and rebound towards the baseline with disuse. Unlike models that depend only on fixed windows or the time since the last pull, our model expresses satiation more organically as a quantity that evolves according to stochastic dynamics and is shocked (upward) by pulls. To estimate the reward dynamics, we leverage recent advances in the identification of linear dynamical systems [40, 38] that rely on the theory of self-normalized processes [33, 1] and block martingale conditions [40].

### 3 Rebounding Bandits Problem Setup

Consider the set of \( K \) arms \( [K] := \{1, \ldots, K\} \) with bounded base rewards \( b_1, \ldots, b_K \). Given a horizon \( T \), a policy \( \pi_{1:T} := (\pi_1, \ldots, \pi_T) \) is a sequence of actions, where \( \pi_t \in [K] \) depends on past actions and observed rewards. For any arm \( k \in [K] \), we denote its pull history from 0 to \( T \) as the binary sequence \( u_{k,0:T} := (u_{k,0}, \ldots, u_{k,T}) \), where \( u_{k,0} = 0 \) and for \( t \in [T] \), \( u_{k,t} = 1 \) if \( \pi_t = k \) and \( u_{k,t} = 0 \) otherwise. The subsequence of \( u_{k,0:T} \) from \( t_1 \) to \( t_2 \) (including both endpoints) is denoted by \( u_{k,t_1:t_2} \).

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Remark 1 (negative expected reward). We note that there exist choices of \( b_k, \gamma_k, \lambda_k \) for which the expected reward of arm \( k \) can be negative. In the traditional bandits setup, one must pull an arm at every time step. Thus, what matters are the relative rewards and the problem is mathematically identical, regardless of whether the expected rewards range from \(-10\) to \(0\) or \(0\) to \(10\). In addition, one might construct settings where negative expected rewards are reasonable. For example, when one of the arms corresponds to no recommendation with \( b_k = 0, \lambda_k = 0 \), then the interpretation of negative expected reward would be that the corresponding arm (item) is less preferred relative to not being recommended.

Given horizon \( T \geq 1 \), we seek an optimal pull sequence \( \pi_{1:T} \), where \( \pi_t \) depends on past rewards and actions \((\pi_1, \mu_{\pi_1,1}, \ldots, \pi_{t-1}, \mu_{\pi_{t-1},t-1})\) and maximizes the expected cumulative reward:

\[
G_T(\pi_{1:T}) := \mathbb{E} \left[ \sum_{t=1}^{T} \mu_{\pi_t, t} \right].
\] (2)

Additional Notation. Let \( \tau := \max_{k \in [K]} \gamma_k \) and \( \bar{\lambda} := \max_{k \in [K]} \lambda_k \). We use \( a \lesssim b \) when \( a \leq Cb \) for some positive constant \( C \).

4 Planning with Known Dynamics

Before we can hope to learn an optimal policy with unknown stochastic dynamics, we need to establish a procedure for planning when the satiation retention factors, exposure influence factors, and base rewards are known. We begin by presenting several planning strategies and analyzing them under deterministic dynamics, where the past pulls exactly determine each arm’s satiation level, i.e., \( s_{k,t} = \gamma_k(s_{k,t-1} + u_{k,t-1}) \), \( \forall t > t_t^0 \). With some abuse of notation, at time \( t \geq 2 \), given a pull sequence \( u_{k,0:t-1} \), we can express the satiation and the expected reward of each arm as

\[
s_{k,t}(u_{k,0:t-1}) = \gamma_k(s_{k,t-1} + u_{k,t-1}) = \gamma_k(\gamma_k(s_{k,t-2} + u_{k,t-2}) + \gamma_k u_{k,t-1}) = \sum_{i=1}^{t-1} \gamma_i^k u_{k,i},
\]

\[
\mu_{k,t}(u_{k,0:t-1}) = b_k - \lambda_k \left( \sum_{i=1}^{t-1} \gamma_i^k u_{k,i} \right).
\] (3)

At time \( t = 1 \), we have that \( s_{k,1}(u_{k,0:0}) = 0 \) and \( \mu_{k,1}(u_{k,0:0}) = b_k \) for all \( k \in [K] \). Since the arm parameters \( \{\lambda_k, \gamma_k, b_k\}_{k=1}^K \) are known, our goal simplifies to finding a pull sequence that solves the following bilinear integer program:

\[
\max_{u_{k,t}} \left\{ \sum_{k=1}^{K} \sum_{t=1}^{T} u_{k,t} \left( b_k - \lambda_k \sum_{i=0}^{t-1} \gamma_i^k u_{k,i} \right) : \sum_{k=1}^{K} u_{k,t} = 1, \forall t \in [T], u_{k,t} \in \{0,1\}, \ u_{k,0} = 0, \forall k \in [K], \forall t \in [T] \right\}
\] (4)

where the objective maximizes the expected cumulative reward associated with the pull sequence and the constraints ensure that at each time period we pull exactly one arm. Note that \( \sum_{i=0}^{t-1} \gamma_i^k u_{k,i} \) includes products of decision variables \( u_{k,t} \) leading to bilinear terms in the objective. In Appendix A we provide an equivalent integer linear program.

\footnote{We use “expected reward” to emphasize that all results in this section also apply to settings where the satiation dynamics are deterministic but the rewards are stochastic, i.e., \( \mu_{k,t} = b_k - \lambda_k s_{k,t} + \epsilon_k,t \) for independent mean-zero noises \( \epsilon_{k,t} \).}
4.1 The Greedy Policy

At each step, the greedy policy $\pi^g$ picks the arm with the highest instantaneous expected reward. Formally, at time $t$, given the pull history $\{u_{k,0:t-1}\}_{k=1}^K$, the greedy policy picks

$$\pi_t^g \in \arg \max_{k \in [K]} \mu_{k,t}(u_{k,0:t-1}).$$

In order to break ties, when all arms have the same expected reward, the greedy policy chooses the arm with the lowest index.

Note that the greedy policy is not, in general, optimal. Sometimes, we are better off allowing the current best arm to rebound even further, before pulling it again.

**Example 1.** Consider the case with two arms. Suppose that arm 1 has base reward $b_1$, satiation retention factor $\gamma_1 \in (0,1)$, and exposure influence factor $\lambda_1 = 1$. For any fixed time horizon $T > 2$, suppose that arm 2 has $b_2 = b_1 + \frac{\gamma_1^{-T}}{1-\gamma_1}$, where $\gamma_2 \in (0,1)$ and $\lambda_2 = 1$. The greedy policy $\pi_{1:T}^g$ will keep pulling arm 2 until time $T - 1$ and then play arm 1 (or arm 2) at time $T$. This is true because if we keep pulling arm 2 until $T - 1$, at time $T$, we have $\mu_{2,T}(u_{2,0:T-1}) = b_1 = \mu_{1,T}(u_{1,0:T-1})$. However, the policy $\pi_{1:T}^n$, where $\pi^n_t = 2$ if $t \leq T - 2$, $\pi^n_{T-1} = 1$, and $\pi^n_T = 2$, obtains a higher expected cumulative reward. In particular, the difference $G_T(\pi_{1:T}^n) - G_T(\pi_{1:T}^g)$ will be $\gamma_2 - \gamma_2^{T-1}$.

4.2 When is Greedy Optimal?

When the satiation retention factors $\gamma_k = 0$ for all $k \in [K]$, i.e., when the satiation effect is always 0, we know that the greedy policy (which always plays the arm with the highest instantaneous expected reward) is optimal. However, when satiation can be nonzero, it is less clear under what conditions the greedy policy performs optimally. This question is of special interest when we consider human decision-making, since we cannot expect people to solve large-scale bilinear integer programs every time they pick music to listen to.

In this section, we show that when all arms share the same properties ($\gamma_k, \lambda_k, b_k$ are identical for $k \in [K]$), the greedy policy is optimal. In this case, the greedy policy exhibits variety-seeking behavior as it plays the arms cyclically. Interestingly, this condition aligns with early research that has motivated studies on satiation \[28\]: when controlling for marketing variables (e.g., the arm parameters $\gamma_k, \lambda_k, b_k$), researchers still observe variety-seeking behaviors of consumers (e.g., playing arms in a cyclic order).

**Assumption 1.** $\gamma_1 = \ldots = \gamma_K = \gamma$, $\lambda_1 = \ldots = \lambda_K = \lambda$, and $b_1 = \ldots = b_K = b$.

We start with characterizing the greedy policy when Assumption 1 holds.

**Lemma 1** (Greedy Policy Characterization). Under Assumption 1 and the tie-breaking rule that when all arms have the same expected reward, the greedy policy chooses the one with the lowest arm index, the sequence of arms pulled by the greedy policy forms a periodic sequence: $\pi_1 = 1, \pi_2 = 2, \ldots, \pi_K = K$, and $\pi_{t+K} = \pi_t$, $\forall t \in \mathbb{N}_+$.

In this case, the greedy policy is equivalent to playing the arms in a cyclic order. All proofs for the paper are deferred to the Appendices.

**Theorem 1.** Under Assumption 1, given any horizon $T$, the greedy policy $\pi_{1:T}^g$ is optimal.

**Remark 2.** Theorem 1 suggests that when the (deterministic) satiation dynamics and base rewards are identical across arms, planning does not require knowledge of those parameters.
Lemma 4 and Theorem 4 lead us to conclude the following result: when recommending items that share the same properties, the best strategy is to show the users a variety of recommendations by following the greedy policy.

On a related note, Theorem 4 also gives an exact Max K-Cut of a complete graph $K_T$ on $T$ vertices, where the edge weight connecting vertices $i$ and $j$ is given by $e(i,j) = \lambda \gamma^{j-i}$ for $i \neq j$. The Max K-Cut problem partitions the vertices of a graph into $K$ subsets $P_1, \ldots, P_K$, such that the sum of the edge weights connecting the subsets are maximized \[10\]. Mapping the Max K-Cut problem back to our original setup, each vertex represents a time step. If vertex $i$ is assigned to subset $P_k$, it suggests that arm $k$ should be played at time $i$. The edge weights $e(i,j) = \lambda \gamma^{j-i}$ for $i \neq j$ can be seen as the reduction in satiation influence achieved by not playing the same arm at both time $i$ and time $j$. The goal \[4\] is to maximize the total satiation influence reduction.

**Proposition 2** (Connection to Max K-Cut). Under Assumption 7, an optimal solution to \[4\] is given by a Max K-Cut on $K_T$, where $K_T$ is a complete graph on $T$ vertices with edge weights $e(i,j) = \lambda \gamma^{j-i}$ for all $i \neq j$.

Using Lemma 4 and Theorem 4, we obtain that an exact Max K-Cut of $K_T$ is given by $\forall k \in [K], P_k = \{t \in [T] : t \equiv k \mod K\}$, which may be a result of separate interest.

### 4.3 The $w$-lookahead Policy

To model settings where the arms correspond to items with different characteristics (e.g., we can enjoy tacos on consecutive days but require time to recover from a trip to the steakhouse) we must allow the satiation parameters to vary across arms. Here, the greedy policy may not be optimal. Thus, we consider more general lookahead policies (the greedy policy is a special case). Given a window of size $w$ and the current satiation levels, the $w$-lookahead policy picks actions to maximize the total reward over the next $w$ time steps. Let $l$ denote $[T/w]$. Define $t_i = \min\{iw,T\}$ for $i \in [l]$ and $t_0 = 0$. More formally, the $w$-lookahead policy $\pi^w_{1:T}$ is defined as follows: for any $i \in [l]$, given the previously chosen arms' corresponding pull histories $\{u^w_{k,0:t_{i-1}}\}_{k=1}^K$ where $u^w_{k,0} = 0$ and $u^w_{k,t} = 1$ if (and only if) $\pi^w_{t_i} = k$, the next $w$ (or $T \mod w$) actions $\pi^w_{t_{i-1}+1:t_i}$ are given by

$$\max_{\pi^w_{t_{i-1}+1:t_i}} \left\{ \sum_{t=t_{i-1}+w}^{t_i} \mu_t(u_{\pi_t,0:t-1}) : \sum_{k=1}^K u_{k,t} = 1, \forall t \in [T], u_{k,t} \in \{0,1\}, \forall k \in [K], t \in [t_i] \right\} \tag{5}$$

In the case of a tie, one can pick any of the sequences that maximize \[5\]. We recover the greedy policy when the window size $w = 1$, and finding the $w$-lookahead policy for the window size $w = T$ is equivalent to solving \[4\].

**Remark 3.** Another reasonable lookahead policy, which requires planning ahead at every time step, would be the following: at every time $t$, plan for the next $w$ actions and follow them for a single time step. Studying the performance of such a policy is of future interest. To lighten the computational load, we adopt the current $w$-lookahead policy which only requires planning every $w$ time steps.

For the rest of the paper, we use Lookahead($\{\{\lambda_k, \gamma_k, b_k\}_{k=1}^K, \{u^w_{k,0:t_{i-1}}\}_{k=1}^K, t_i, t_{i-1}\}$) to refer to the solution of \[5\], where the arm parameters are $\{\lambda_k, \gamma_k, b_k\}_{k=1}^K$, the historical pull sequences of all arms till time $t_{i-1}$ are given by $\{u^w_{k,0:t_{i-1}}\}_{k=1}^K$, and the solution corresponds to the actions that should be taken for the next $t_i - t_{i-1}$ time steps.

**Theorem 2.** Given any horizon $T$, let $\pi^*_{1:T}$ be a solution to \[4\]. For a fixed window size $w \leq T$, we have that

$$G_T(\pi^w_{1:T}) - G_T(\pi^w_{1:T}) \leq \frac{\bar{X}_T(1 - \pi^{T-w})}{(1-\pi)^2} [T/w].$$
Remark 4. Note that when \( w = T \), the \( w \)-lookahead policy by definition is the optimal policy and in such case, the upper bound for the optimality gap of \( w \)-lookahead established in Theorem 2 is also 0. In contrast to the optimal policy, the computational benefit of the \( w \)-lookahead policy becomes apparent when the horizon \( T \) is large since it requires solving for a much smaller program \( \text{[5]} \). In general, the \( w \)-lookahead policy is expected to perform much better than the greedy policy (which corresponds to the case of \( w = 1 \)) at the expense of a higher computational cost. Finally, we note that for the window size of \( w = \sqrt{T} \), we obtain \( G_T(\pi^*_1) - G_T(\pi^*_w) \leq O(\sqrt{T}) \).

5 Learning with Unknown Dynamics: Preliminaries

When the satiation dynamics are unknown and stochastic (\( \sigma_z > 0 \)), the learner faces a continuous-state partially observable MDP because the satiation levels are not observable. To set the stage, we first introduce our state representation (§ 5.1) and a regret-based performance measure (§ 5.2). In the next section, we will introduce EEP, our algorithm for rebounding bandits.

5.1 State Representation

Following \( \text{[32]} \), at any time \( t \in [T] \), we define a state vector \( x_t \) in the state space \( \mathcal{X} \) to be \( x_t = (x_{1,t}, n_{1,t}, x_{2,t}, n_{2,t}, \ldots, x_{K,t}, n_{K,t}) \), where \( n_{k,t} \in \mathbb{N} \) is the number of steps at time \( t \) since arm \( k \) was last selected and \( x_{k,t} \) is the satiation influence (product of \( \lambda_k \) and the satiation level) as of the most recent pull of arm \( k \). Since the most recent pull happens at \( t - n_{k,t} \), we have \( x_{k,t} = b_k - \mu_{k,t-n_{k,t}} = \lambda_k s_{k,t-n_{k,t}} \). Recall that \( \mu_{k,t-n_{k,t}} \) is the reward collected by pulling arm \( k \) at time \( t - n_{k,t} \). Note that \( b_k \) is directly observed when arm \( k \) is pulled for the first time because there is no satiation effect. The state at the first time step is \( x_1 = (0, \ldots, 0) \). Transitions between two states \( x_t \) and \( x_{t+1} \) are defined as follows: If arm \( k \) is chosen at time \( t \), and reward \( \mu_{k,t} \) is obtained, then the next state \( x_{t+1} \) will satisfy (i) for the pulled arm \( k \), \( n_{k,t+1} = 1 \) and \( x_{k,t+1} = b_k - \mu_{k,t} \); (ii) for other arms \( k' \neq k \), \( n_{k',t+1} = n_{k',t} + 1 \) if \( n_{k',t} \neq 0 \), \( n_{k',t+1} = 0 \) if \( n_{k',t} = 0 \), and the satiation influence remains the same \( x_{k',t+1} = x_{k',t} \).

Given \( \{\gamma_k, \lambda_k, b_k\}_{k=1}^K \), the reward function \( r : \mathcal{X} \times [K] \rightarrow \mathbb{R} \) represents the expected reward of pulling arm \( k \) under state \( x_t \):

If \( n_{k,t} = 0 \), then \( r(x_t,k) = b_k \). If \( n_{k,t} \geq 1 \), \( r(x_t,k) = b_k - \gamma_{n_{k,t}} x_{k,t} - \lambda_k \gamma_{n_{k,t}} \), which equals \( \mathbb{E}[\mu_{k,t}|x_t] \), where the expectation is taken over the noises in between the current pull and the last pull of arm \( k \). See Appendix C.1 for the full description of the MDP setup (including the transition kernel and value function definition) of rebounding bandits.

5.2 Evaluation Criteria: \( w \)-step Lookahead Regret

In reinforcement learning (RL), the performance of a learner is often measured through a regret that compares the expected cumulative reward obtained by the learner against that of an optimal policy in a competitor class \( \text{[20]} \). In most episodic (e.g., finite horizon) RL literature \( \text{[31]} \text{[13]} \), regrets are defined in terms of episodes. In such cases, the initial state is reset (e.g., to a fixed state) after each episode ends, independent of previous actions taken by the learner. Unlike these episodic RL setups, in rebounding bandits, we cannot restart from the initial state because the satiation level cannot be reset and user’s memory depends on past received recommendations. Instead, \( \text{[34]} \) proposed a version of \( w \)-step lookahead regret that divides the \( T \) time steps into \( [T/w] \) episodes where each episode (besides the last) consists of \( w \) time steps. At the beginning of each episode, the initial state is reset but depends on how the learner has interacted with the user previously. In particular, at the beginning of episode \( i + 1 \) (at time \( t = iw + 1 \)), given that the learner has played \( \pi_{1:iw} \) with corresponding pull sequence \( u_{k,0:iw} \) for \( k \in [K] \), we reset the initial state to be \( x_0' = (\mu_{1,iw+1}(u_{1,0:iw}), n_{1,iw+1}, \ldots, \mu_{K,iw+1}(u_{K,0:iw}), n_{K,iw+1}) \) where \( \mu_{k,t}(\cdot) \) is defined in \( \text{[3]} \) and \( n_{k,iw+1} \) is the number of steps since arm \( k \) is last pulled by the learner as of
time \( iw + 1 \). Then, given the learner’s policy \( \pi_{1:T} \), where \( \pi_1 : X \rightarrow [K] \), the \( w \)-step lookahead regret, against a competitor class \( C^w \) (which we define later), is defined as follows:

\[
\text{Reg}^w(T) = \sum_{i=0}^{\lfloor T/w \rfloor - 1} \max_{\bar{\pi}_{1:w} \in C^w} \mathbb{E} \left[ \sum_{j=1}^{\min\{w,T-iw\}} r(x_{iw+j}, \bar{\pi}_j(x_{iw+j})) \bigg| x_{iw+1} = x^1 \right] \\
- \mathbb{E} \left[ \sum_{j=1}^{\min\{w,T-iw\}} r(x_{iw+j}, \pi_{iw+j}(x_{iw+j})) \bigg| x_{iw+1} = x^1 \right],
\]

where the expectation is taken over \( x_{iw+2}, \ldots, x_{\min\{(i+1)w,T\}} \).

The competitor class \( C^w \) that we have chosen consists of policies that depend on time steps, i.e., \( C^w = \{ \bar{\pi}_{1:w} : \bar{\pi}_t = \bar{\pi}_t(x_t) = \bar{\pi}_t(x_t'), \bar{\pi}_t \in [K], \forall t \in [w], x_t, x_t' \in X \} \). We note that \( C^w \) subsumes many traditional competitor classes in bandits literature, including the class of fixed-action policies considered in adversarial bandits [20] and the class of periodic ranking policies [7].

In our paper, the \( w \)-lookahead policy (including the \( T \)-lookahead policy given by [3]) is a time-dependent policy that belongs to \( C^w \), since at time \( t \), it will play a fixed action by solving (5) using the true reward parameters \( \{\lambda_k, \gamma_k, b_k\}_{k=1}^K \). The time-dependent competitor class \( C^w \) differs from a state-dependent competitor class which includes all measurable functions \( \bar{\pi}_t \) that map from \( X \) to \([K]\). The state-dependent competitor class contains the optimal policy \( \pi^* \) where \( \pi_t^*(x_t) \) depends on not just the time step but also the exact state \( x_t \). Finding the optimal state-dependent policy requires optimal planning for a continuous-state MDP, which relies on state space discretization [31] or function approximation (e.g., approximate dynamic programming algorithms [30, 9, 36]). In Appendix C we provide discussion and analysis on an algorithm compared against the optimal state-dependent policy. We proceed the rest of the main paper with \( C^w \) defined above.

When \( w = 1 \), the 1-step lookahead regret is also known as the instantaneous regret, which is commonly used in restless bandits literature and some nonstationary bandits papers including [29]. Note that low instantaneous regret does not imply high expected cumulative reward in the long-term, i.e., one may benefit more by waiting for certain arms to rebound. When \( w = T \), we recover the full horizon regret. As we have noted earlier, finding the optimal competitor policy in this case is computationally intractable because the number of states, even when the satiation dynamics are deterministic, grows exponentially with the horizon \( T \). Finally, we note that the \( w \)-step lookahead regret can be obtained for not just policies designed to look \( w \) steps ahead but any given policy. For a more comprehensive discussion on these notions of regret, see [34, Section 4].

## 6 Explore-Estimate-Plan

We now present Explore-Estimate-Plan (EEP), an algorithm for learning in rebounding bandits with stochastic dynamics and unknown parameters, that (i) collects data by pulling each arm a fixed number of times; (ii) estimates the model’s parameters based on the logged data; and then (iii) plans according to the estimated model. Finally, we analyze EEP’s regret.

Because each arm’s base reward is known from the first pull, whenever arm \( k \) is pulled at time \( t \) and \( n_{k,t} \neq 0 \), we measure the satiation influence \( \lambda_k s_{k,t} \), which becomes the next state \( x_{k,t+1} \):

\[
x_{k,t+1} = \lambda_k s_{k,t} = \lambda_k \gamma_k^{n_{k,t}} s_{k,t-n_{k,t}} + \lambda_k \gamma_k^{n_{k,t}} + \lambda_k \sum_{i=0}^{n_{k,t}-1} \gamma_k i z_{k,t-i} \\
= \gamma_k^{n_{k,t}} x_{k,t+1-n_{k,t}} + \lambda_k \gamma_k^{n_{k,t}} + \lambda_k \sum_{i=0}^{n_{k,t}-1} \gamma_k i z_{k,t-i-1}.
\]

We note that the current state \( x_{k,t} \) equals \( x_{k,t+1-n_{k,t}} \), since \( x_{k,t+1-n_{k,t}} \) is the last observed satiation influence for arm \( k \) and \( n_{k,t} \) is the number of steps since arm \( k \) was last pulled.
6.1 The Exploration Phase: Repeated Pulls

We collect a dataset $P^n_k$ by consecutively pulling each arm $n+1$ times, in turn, where $n \geq \lceil T^{2/3}/K \rceil$ (Line 4-7 of Algorithm 1). Specifically, for each arm $k \in [K]$, the dataset $P^n_k$ contains a single trajectory of $n+1$ observed satiation influences $\tilde{x}_{k,1}, \ldots, \tilde{x}_{k,n+1}$, where $\tilde{x}_{k,1} = 0$ and $\tilde{x}_{k,j}$ ($j > 1$) is the difference between the first reward and the $j$-th reward from arm $k$. Thus, for $\tilde{x}_{k,j}, \tilde{x}_{k,j+1} \in P^n_k$, using (7) with $n_{k,t} = 1$ (because pulls are consecutive), it follows that

$$\tilde{x}_{k,j+1} = \gamma_k \tilde{x}_{k,j} + d_k + \tilde{z}_{k,j},$$

(8)

where $d_k = \lambda_k \gamma_k$ and $\tilde{z}_{k,j}$ are independent samples from $\mathcal{N}(0, \sigma^2_{\tilde{z},k})$ with $\sigma^2_{\tilde{z},k} = \lambda_k^2 \sigma^2_k$. In Appendix E.2, we discuss other exploration strategies (e.g., playing the arms cyclically) for EEP and their regret guarantees.

6.2 Estimating the Reward Model and Satiation Dynamics

For all $k \in [K]$, given the dataset $P^n_k$, we estimate $A_k = (\gamma_k, d_k)\top$ using the ordinary least squares estimator:

$$\hat{A}_k \in \arg \min_{A \in \mathbb{R}^2} \|Y_k - X_k A\|_2^2,$$

where $Y_k \in \mathbb{R}^n$ is an $n$-dimensional vector whose $j$-th entry is $\tilde{x}_{k,j+1}$ and $X_k \in \mathbb{R}^{n \times 2}$ takes as its $j$-th row the vector $x_{k,j} = (\tilde{x}_{k,j}, 1)\top$, i.e., $\tilde{x}_{k,j+1}$ is treated to be the response to the covariates $x_{k,j}$. This suggests that

$$\hat{A}_k = \begin{pmatrix} \hat{\gamma}_k \\ \hat{d}_k \end{pmatrix} = \left(\bar{X}_k \top \bar{X}_k \right)^{-1} \bar{X}_k \top Y_k,$$

(9)

and we take $\hat{\lambda}_k = |\hat{d}_k/\hat{\gamma}_k|$.

The difficulty in analyzing the ordinary least squares estimator for identifying an affine dynamical system using a single trajectory of data comes from the fact that the samples are not independent. Asymptotic guarantees of the ordinary least squares estimators in this case have been studied previously in the control theory and time series communities. Recent work on system identifications for linear dynamical systems focuses on the sample complexity. Adapting the proof of Theorem 2.4, we derive the following theorem for identifying our affine dynamical system.

**Theorem 3.** Fix $\delta \in (0, 1)$. For all $k \in [K]$, there exists a constant $n_0(\delta, k)$ such that if the dataset $P^n_k$ satisfies $n \geq n_0(\delta, k)$, then

$$\mathbb{P} \left( \|\hat{A}_k - A_k\|_2 \gtrsim \sqrt{1/(\psi n)} \right) \leq \delta,$$

where $\psi = \sqrt{\min \left\{ \frac{\sigma^2_{\tilde{z},k}(1-\gamma_k)^2}{16d^2_k(1-\gamma_k)^2 + (1-\gamma_k)^2 \sigma^2_{\tilde{z},k}}, \frac{\sigma^2_{\tilde{z},k}}{4(1-\gamma_k)^2} \right\}}$.

As shown in Theorem 3 when $d_k = \lambda_k \gamma_k$ gets larger, the convergence rate for $\hat{A}_k$ gets slower. Given a single trajectory of sufficient length, we obtain $|\hat{\gamma}_k - \gamma_k| \leq O(1/\sqrt{n})$ and $|\hat{d}_k - d_k| \leq O(1/\sqrt{n})$. In Corollary 4 we show that the estimator of $\lambda_k$ also achieves $O(1/\sqrt{n})$ estimation error.

**Corollary 4.** Fix $\delta \in (0, 1)$. Suppose that for all $k \in [K]$, we have $\mathbb{P}(\|\hat{A}_k - A_k\|_2 \gtrsim 1/\sqrt{n}) \leq \delta$ and $\hat{\gamma}_k > 0$. Then, with probability $1 - \delta$, we have that for all $k \in [K]$,

$$|\hat{\gamma}_k - \gamma_k| \leq O \left( \frac{1}{\sqrt{n}} \right), \quad |\hat{\lambda}_k - \lambda_k| \leq O \left( \frac{1}{\sqrt{n}} \right).$$
Algorithm 1: $w$-lookahead Explore-Estimate-Plan

**Input:** Lookahead window size $w$, Number of arms $K$, Horizon $T$

1. Initialize $t = 1$, $\pi_{1:T}$ to be an empty array of length $T$ and $\tilde{T} = T^{2/3} + w - (T^{2/3} \mod w)$.

2. for $k = 1, \ldots, K$ do
   3. Set $t' = t$ and initialize an empty array $\mathcal{P}_k^n$.
   4. for $c = 0, \ldots, \lfloor \tilde{T}/K \rfloor$ do
      5. Play arm $k$ to obtain reward $\mu_{k,t'+c}$ and add $\mu_{k,t'+c}$ to $\mathcal{P}_k^n$.
      6. Set $\pi_t = k$ and increase $t$ by 1.
   7. end
   8. Obtain $\gamma_k, \hat{d}_k$ using the estimator (9), set $\hat{\lambda}_k = |\hat{d}_k/\gamma_k|$ and $\hat{b}_k = \mu_{k,t'}$.

9. end

10. Let $t_0 = \tilde{T}$, set $\pi_{t_0} = (1, \ldots, \tilde{T} - t + 1)$, and play $\pi_{t:0}$.

11. for $i = 1, \ldots, \lfloor \tilde{T}/w \rfloor$ do
   12. Set $t_i = \min\{t_{i-1} + w, T\}$.
   13. Obtain $\pi_{t_i-1+t_i} = \text{Lookahead}(\hat{\lambda}_k, \hat{d}_k, \hat{b}_k)_{k=1}^K, \{u_{k,0:t_i-1}\}_{k=1}^K, t_i-1, t_i)$ where $\{u_{k,0:t_i-1}\}_{k=1}^K$ are the arm pull histories correspond to $\pi_{t_i-1}$.
   14. Play $\pi_{t_i-1+t_i}$.

15. end

6.3 Planning and Regret Bound

In the planning stage of Algorithm 1 (Line 11-15), at time $t_{i-1} + 1$, the next $w$ arms to play are obtained through the Lookahead function defined in (5) based on the estimated parameters from the estimation stage (Line 8). Using the results in Corollary 4 we obtain the following sublinear regret bound for $w$-lookahead EEP.

**Theorem 5.** There exists a constant $T_0$ such that for all $T > T_0$ and $w \leq T^{2/3}$, the $w$-step lookahead regret of $w$-lookahead Explore-Estimate-Plan satisfies

$$\text{Reg}^w(T) \leq O(K^{1/2}T^{2/3} \log T).$$

Remark 5. The fact that EEP incurs a regret of order $O(T^{2/3})$ is expected for two reasons: First, EEP can be viewed as an explore-then-commit (ETC) algorithm that first explores then exploits. The regret of EEP resembles the $O(T^{2/3})$ regret of the ETC algorithm in the classical $K$-armed bandits setting [20]. In rebounding bandits, the fundamental obstacle to mixing the exploration and exploitation stages is the need to estimate the satiation dynamics. When the rewards of each arm are not observed periodically, the obtained satiation influences can no longer be viewed as samples from the same time-invariant affine dynamical system, since the parameters of the system depend on the duration between pulls. In practice, one may utilize the maximum likelihood estimator to obtain estimates of the reward parameters but obtaining the sample complexity of such an estimator with dependent data is difficult. Second, it has been shown in [3] that when the rewards of the arms have temporal variation that depends on the horizon $T$, the worst case instantaneous regret has a lower bound $\Omega(T^{2/3})$. On the other hand, in $K$-armed bandits, the regret (following the classical definition [20]) is lower bounded by $\Omega(T^{1/2})$, and can be attained by methods like the upper confidence bound algorithm [20]. Precisely characterizing the regret lower bound for rebounding bandits is of future interest.

7 Experiments

We now evaluate the performance of EEP experimentally, separately investigating the sample efficiency of our proposed estimators (9) for learning the satiation and reward models (Figure 2 and
Figure 2: Figure 2a and 2b are the log-log plots of absolute errors of $\hat{\gamma}_k$ and $\hat{\lambda}_k$ with respect to the number of samples $n$ in a single trajectory. The results are averaged over 30 random runs, where the shaded area represents one standard deviation.

Figure 3: Figure 3a shows the expected cumulative reward collected by the $T$-lookahead policy (red line) and $w$- lookahead policy (blue dots) when $T = 30$. Figure 3b shows the log-log plot of the $w$-step lookahead regret of $w$-lookahead EEP under different $T$ averaged over 20 random runs.

the computational performance of the $w$-lookahead policies (Figure 3a). For the experimental setup, we have 5 arms with satiation retention factors $\gamma_1 = \gamma_2 = .5, \gamma_3 = .6, \gamma_4 = .7, \gamma_5 = .8$, exposure influence factors $\lambda_1 = 1, \lambda_2 = \lambda_3 = 3, \lambda_4 = \lambda_5 = 2$, base rewards $b_1 = 2, b_2 = 3, b_3 = 4, b_4 = 2, b_5 = 10$, and noise with variance $\sigma_z = 0.1$.

Parameter Estimation We first evaluate our proposed estimator for using a single trajectory per arm to estimate the arm parameters $\gamma_k, \lambda_k$. In Figure 2 we show the absolute error (averaged over 30 random runs) between the estimated parameters and the true parameters for each arm. Aligning with our theoretical guarantees (Corollary 4), the log-log plots show that the convergence rate of the absolute error is on the scale of $O(n^{-1/2})$.

$w$-lookahead Performance To evaluate $w$-lookahead policies, we solve (5) using the true reward parameters and report expected cumulative rewards of the obtained $w$-lookahead policies (Figure 3a). Recall that the greedy policy is precisely the 1-lookahead policy. In order to solve the resulting integer programs, we use Gurobi 9.1 [23] and set the number of threads for solving the problem to 10. When $T = 30$, the $T$-lookahead policy (expected cumulative rewards given by the red line in Figure 3a) solved through (4) is obtained in 1610s. On the other hand, all $w$-lookahead policies (expected cumulative rewards given by the blue dots in Figure 3a) for $w$ in between 1 and 15 are solved within 2s. We provide the results when $T = 100$ in Appendix G. Despite using significantly lower computational time, $w$-lookahead policies achieve a similar expected cumulative reward to the $T$-lookahead policy.

EEP Performance We evaluate the performance of EEP when $T$ ranges from 60 to 400. For each horizon $T$, we examine the $w$-step lookahead regret of $w$-lookahead EEP where $w = 2, 5, 8, 10$. All results are averaged over 20 random runs. As $T$ increases, the exploration stage of EEP becomes longer, which results in collecting more data for estimating the reward parameters and lower variance of the parameter estimators. We fit a line for the regrets with the same
lookahead size $w$ to examine the order of the regret with respect to the horizon $T$. The slopes of the lines (see Figure 3b’s legend) are close to $2/3$, which aligns with our theoretical guarantees (Theorem 5), i.e., the regrets are on the order of $O(T^{2/3})$. In Appendix G, we present additional experimental setups and results.

8 Conclusions

While our work has taken strides towards modeling the exposure-dependent evolution of preferences through dynamical systems, there are many avenues for future work. First, while our satiation dynamics are independent across arms, a natural extension might allow interactions among the arms. For example, a diner sick of pizza after too many trips to Di Fara’s, likely would also avoid Grimaldi’s until the satiation effect wore off. On the system identification side, we might overcome our reliance on evenly spaced pulls, producing more adaptive algorithms (e.g., optimism-based algorithms) that can refine their estimates, improving the agent’s policy even past the pure exploration period. Finally, our satiation model captures just one plausible dynamic according to which preferences might evolve in response to past recommendations. Characterizing other such dynamics (e.g., the formation of brand loyalty where the rewards of an arm increase with more pulls) in bandits setups is of future interest.

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## Contents (Appendix)

A Integer Linear Programming Formulation .................................................. 16
B Proofs and Discussion of Section 4 ......................................................... 17
  B.1 Proof of Lemma 1 .............................................................................. 17
  B.2 Proof of Theorem 1 ........................................................................... 17
  B.3 Proof of Proposition 2 ....................................................................... 21
  B.4 Proof of Theorem 2 ........................................................................... 21
C More Discussion on Learning with Unknown Dynamics ......................... 23
  C.1 MDP Setup ....................................................................................... 23
  C.2 Exploration and Estimation of the Reward Model ............................. 24
    C.2.1 Estimation using Multiple Trajectories ....................................... 24
    C.2.2 Estimation using a Single Trajectory ......................................... 25
  C.3 Planning ......................................................................................... 27
    C.3.1 Time-dependent Policy ............................................................... 27
    C.3.2 State-dependent Policy .............................................................. 28
D Proofs of Section 6.2 and Appendix C.2.2 ............................................. 29
  D.1 Proof of Theorem 3 and Theorem 8 .................................................. 29
  D.2 Proof of Corollary 4 and Corollary 9 ................................................ 34
E Additional Proofs and Discussion of Section 6 ...................................... 35
  E.1 Proof of Theorem 5 .......................................................................... 35
  E.2 Exploration Strategies ....................................................................... 38
F Additional Proofs of Appendix C ............................................................. 39
  F.1 Proof of Corollary 7 .......................................................................... 39
  F.2 Proof of Lemma 4 ............................................................................. 39
  F.3 Proof of Proposition 5 ....................................................................... 40
  F.4 Proof of Proposition 6 ....................................................................... 40
G Additional Experimental Details and Results ......................................... 43
A Integer Linear Programming Formulation

The bilinear integer program of (1) admits the following equivalent linear integer programming formulation:

$$\max_{u_{k,t}, z_{k,t,i}} \sum_{k \in [K]} \sum_{t \in [T]} b_k u_{k,t} - \lambda_k \sum_{i=0}^{t-1} \gamma_k^{t-i} z_{k,t,i}$$

s.t. \( \sum_{k \in [K]} u_{k,t} = 1, \quad \forall t \in [T], \)

\( z_{k,t,i} \leq u_{k,i}, \quad z_{k,t,i} \leq u_{k,t}, \quad u_{k,i} + u_{k,t} - 1 \leq z_{k,t,i}, \quad \forall k \in [K], t \in [T], i \in \{0, \ldots, t-1\}, \)

\( u_{k,t} \in \{0, 1\}, \quad u_{k,0} = 0, \quad \forall k \in [K], t \in [T], \)

\( z_{k,t,i} \in \{0, 1\}, \quad \forall k \in [K], t \in [T], i \in \{0, \ldots, t-1\}. \)
B Proofs and Discussion of Section 4

B.1 Proof of Lemma 1

Proof. When the expected rewards of all arms are the same, we know that the arm with the lowest index will be chosen and thus the first $K$ pulls will be $\pi_1 = 1, \ldots, \pi_K = K$. We will complete the proof through induction. Suppose that the greedy pull sequence is periodic with $\pi_1 = 1, \ldots, \pi_K = K$ and $\pi_{h+1} = \pi_h$ until time $h > K$. We define $k'$ to be $h \mod K$ and $n$ to be $(h - k')/K$. We will show that $\pi_{h+1} = 1$ if $\pi_h = K$ and $\pi_{h+1} = h + 1$ otherwise. When $k' = 0$ (i.e., $\pi_h = K$), all arms have been pulled exactly $n$ times as of time $h$. By the induction assumption, we know that $u_{1,1:h} = u_{2,2:h-K+1} = \ldots = u_{K,K:h}$, which implies that last time when each arm is pulled, all of them have the same expected rewards, i.e.,

$$
\mu_{1, h-K+1}(u_{1,0:h-K}) = \mu_{2, h-K+2}(u_{2,0:h-K+1}) = \ldots = \mu_{K,h}(u_{K,0:h-1}).
$$

Moreover, $u_{1, h-K+1:h} = (1, 0, \ldots, 0)$, $u_{2, h-K+1:h} = (1, 0, \ldots, 0)$, \ldots, $u_{K, h:h} = (1)$.

Therefore, by (3), at time $h + 1$, arm 1 has the highest expected reward and will be chosen. In the case where $k' > 0$ (i.e., $n_h = k'$), we let $h' := h - k'$. We have that $\mu_{1, h'-K+1}(u_{1,0:h'-K}) = \ldots = \mu_{K,h}(u_{K,0:h'-1})$ and $s = s_{1,h'-K+1}(u_{1,0:h'-K}) = \ldots = s_{K,h}(u_{K,0:h'-1}) \leq \frac{\gamma^K}{1-\gamma}$. Then, at time $h + 1$, the satiation level for the arms will be $s_{k,h+1}(u_{k,0:h}) = \gamma^{k'-k+1}(1 + \gamma^K s)$ for all $k \leq k'$ and $s_{k,h+1}(u_{k,0:h}) = \gamma^{K-k'+k'+1} s$ for all $k > k'$. Thus, the arm with the lowest satiation level will be $\pi_{h+1} = k' + 1 = \pi_h + 1$, since $s_{k'+1,h+1}(u_{k'+1,0:h}) < s_{1,h+1}(u_{1,0:h})$. Consequently, the greedy policy will select arm $\pi_h + 1$ at time $h + 1$. \hfill \Box

B.2 Proof of Theorem 1

Proof. First, when $T \leq K$, greedy policy is optimal since its cumulative expected reward is $Tb$. So, we consider the case of $T > K$. Assume for contradiction that there exists another policy $\pi_{1:T}$ that is optimal and is not greedy, i.e., $\exists t \in [T], \pi_{t} \notin \arg\max_{k \in [K]} b - \lambda s_{k,t}$. We will construct a new policy $\pi_{1:T}$ that obtains a higher cumulative expected reward than $\pi_{1:T}$. Throughout the proof, we use $s_{k,t}^n$ to denote the satiation levels for the new policy.

We first note two illustrative facts to give the intuition of the proof.

Fact 1: Any policy $\pi_{1:T}$ that does not pick the arm with the lowest satiation level (i.e., highest expected reward) at the last step is not optimal.

Proof of Fact 1: In this case, the policy $\pi_{1:T} = (\pi_{1:T}^0, \ldots, \pi_{T-1:T}^0, \pi_{T})$ where $\pi_T \in \arg\max_{k \in [K]} b - \lambda s_{k,T}$ will obtain a higher cumulative expected reward.

Fact 2: If a policy $\pi_{1:T}$ picks the lowest satiation level for the final pull $\pi_T^0$ but does not pick the arm with the lowest satiation level at time $T - 1$, we claim that $\pi_{1:T} = (\pi_{1:T}^0, \ldots, \pi_{T-2:T}^0, \pi_T^0, \pi_{T-1:T}^0) \neq \pi_{1:T}^0$ obtains a higher cumulative expected reward.

Proof of Fact 2: First, note that $\pi_{T-1:T}^0 \neq \pi_T^0$ because otherwise $\pi_{T-1:T}^0$ is the arm with the lowest satiation level at $T - 1$. Moreover, at time $T - 1$, $\pi_{T} \in \arg\min_{k \in [K]} s_{k,T-1}$ has the smallest satiation, since if not, then there exists another arm $k \neq \pi_T^0$ and $k \neq \pi_{T-1:T}^0$ that has a smaller satiation level than $\pi_T^0$ at time $T - 1$. In that case, $\pi_T^0$ will not be the arm with the lowest satiation at time $T$, which is a contradiction. Then, we deduce $s_{\pi_{T-1:T}^0, T-1}^0 > s_{\pi_T^0, T-1}^0$. Combining this with $\pi_{T-1:T}^0 \neq \pi_T^0$, we arrive at

$$
G_T(\pi_{1:T}^0) - G_T(\pi_{1:T}^0) = \lambda(1-\gamma)
\left(s_{\pi_{T-1:T}^0, T-1}^0 - s_{\pi_T^0, T-1}^0\right) > 0.
$$
For the general case, given any policy \( \pi_{1:T}^o \) that is not a greedy policy, we construct the new policy \( \pi_{1:T}^n \) that has a higher cumulative expected reward through the following procedure:

1. Find \( t^* \in [T] \) such that for all \( t > t^* \), \( \pi_{t}^o \in \arg\max_{k \in |K|} b - \lambda s_{k,t}^o \) and \( \pi_{t}^o \notin \arg\max_{k \in |K|} b - \lambda s_{k,t}^o \). Further, we know that \( \pi_{t+1}^o \in \arg\max_{k \in |K|} b - \lambda s_{k,t^o+1}^o \), using the same reasoning as the above example, i.e., otherwise \( \pi_{t+1}^o \notin \arg\max_{k \in |K|} b - \lambda s_{k,t^o+1}^o \). To ease the notation, we use \( k_1 \) to denote \( \pi_{t}^o \) and \( k_2 \) to denote \( \pi_{t+1}^o \).

2. For the new policy, we choose \( \pi_{1:t+1}^n = (\pi_{1}, \ldots, \pi_{t-1}, k_2, k_1, k_2) \). Let \( A_{t_1, t_2}^o \) denote the set \( \{t^o’ : t^o’ + 2 \leq t^o’ \leq t_2, \pi_{t}^o = \pi_{t}^o \} \). \( A_{t_1, t_2}^o \) contains a set of time indices between \( t^o’ + 2 \) and \( t_2 \) when arm \( \pi_{t}^o \) is played under policy \( \pi_{1:T}^o \). We construct the following three sets:

   \( T_A := \{t : t^o + 2 \leq t \leq T, |A_{t^o,t^o+1}^o| < |A_{t^o+1,t^o+1}^o|\}, T_B := \{t : t^o + 2 \leq t \leq T, |A_{t^o,t^o}^o > |A_{t^o+1,t^o+1}^o|\} \) and \( T_C := \{t : t^o + 2 \leq t \leq T, |A_{t^o,t^o}^o = |A_{t^o+1,t^o+1}^o|\} \).

   For time \( t^o \geq t^o + 2 \), we consider the following three cases:

   - **Case I**. \( T_B = \emptyset \), which means that at any time \( t \) in between \( t^o + 2 \) and \( T \), arm \( k_1 \) is played more than arm \( k_2 \) from \( t^o + 2 \) to \( t \). In this case, the new policy follows \( \pi_{1:T}^n = \pi_{1:T}^o + 2:T \).

   - **Case II**. \( T_A = \emptyset \), which means that at any time \( t \) in between \( t^o + 2 \) and \( T \), arm \( k_2 \) is played more than arm \( k_1 \) from \( t^o + 2 \) to \( t \). In this case, the new policy satisfies: for all \( t \geq t^o + 2 \), 1) \( \pi_{t}^o = \pi_{t}^o \) if \( \pi_{t}^o \neq k_1 \) and \( \pi_{t}^o \neq k_2 \); 2) \( \pi_{t}^o = k_2 \) if \( \pi_{t}^o = k_1 \); and 3) \( \pi_{t}^o = k_1 \) if \( \pi_{t}^o = k_2 \).

   - **Case III**. \( T_A \neq \emptyset \) and \( T_B \neq \emptyset \). Then, starting from \( t^o + 2 \), if \( t \in T_A \), \( \pi_{t}^n \) follows the new policy construction in Case I, i.e., \( \pi_{t}^n = \pi_{t}^o \). If \( t \in T_B \), \( \pi_{t}^n \) follows the new policy construction in Case II. Finally, for all \( t \in T_C \), define \( t_{A,t}^o = \max_{t^o \in T_A, t^o \leq t} t^o \) and \( t_{B,t}^o = \max_{t^o \in T_B, t^o \leq t} t^o \).

   If \( t_{A,t}^o > t_{B,t}^o \), then \( \pi_{t}^n \) follows the new policy construction as Case I. If \( t_{A,t}^o < t_{B,t}^o \), \( \pi_{t}^n \) follows the new policy construction as Case II. We note that \( t_{A,t}^o \neq t_{B,t}^o \) since \( T_A \cap T_B = \emptyset \).

When \( T_A = \emptyset \) and \( T_B = \emptyset \), we know that \( k_1 \) and \( k_2 \) are not played in \( \pi_{1:T}^o + 2:T \). In this case, the new policy construction can follow either Case I or Case II. To complete the proof, we state some facts first:

- From \( t^o \), the expected rewards collected by the policies \( \pi_{1:T}^o \) and \( \pi_{1:T}^n \) only differ at times when arm \( k_1 \) or arm \( k_2 \) is played.

- \( \pi_{1:t+1}^n \) obtains a higher cumulative expected reward than \( \pi_{1:t+1}^o \).

- At time \( t^o + 2 \), the new policy follows that \( s_{k_1,t^o+2}^n = \gamma + \gamma^2 s_{k_1,t^o}^o \), \( s_{k_2,t^o+2}^n = \gamma + \gamma^2 s_{k_2,t^o}^o \). On the other hand, the old policy has \( s_{k_1,t^o+2}^o = \gamma^2 + \gamma^2 s_{k_1,t^o}^o \) and \( s_{k_2,t^o+2}^o = \gamma + \gamma^2 s_{k_2,t^o}^o \).

Let \( N_k := \{t : t^o + 2 \leq t \leq T, \pi_{t}^o = k_1\} \) and \( N_k := \{t : t^o + 2 \leq t \leq T, \pi_{t}^o = k_2\} \) denote the sets of time steps when \( k_1 \) and \( k_2 \) are played in \( \pi_{1:T}^o \). For a given satisfaction level \( x \) at time \( t^o \) together with the time steps the arm is pulled \( N_k \), we have that at time \( t \geq t^o \), the arm has satisfaction level \( g_{N_k}(x, t, t^o) = \gamma^{t-t^o} x + \sum_{N_k,t^o < t} \gamma^{t-N_k} \), where \( N_k,t^o \) is the \( i \)-th smallest element in \( N_k \).

In Case I, the difference of the cumulative expected rewards between the two policies satisfies:

\[
G_T(\pi_{1:T}^n) - G_T(\pi_{1:T}^o) > \sum_{i=1}^{\lceil N_k\rceil} -\lambda g_{N_k}(s_{N_k,t^o+2}, N_{k_1,i}, t^o + 2) + \lambda g_{N_k}(s_{N_k,t^o+2}, N_{k_1,i}, t^o + 2) + \sum_{j=1}^{\lceil N_k\rceil} -\lambda g_{N_k}(s_{N_k,t^o+2}, N_{k_1,j}, t^o + 2) + \lambda g_{N_k}(s_{N_k,t^o+2}, N_{k_1,j}, t^o + 2)
\]
\[
\lambda \left( s_{k_2,t^*+2}^o - s_{k_2,t^*+2}^n \right) \sum_{i=1}^{\lfloor N_{k_2} \rfloor} \gamma^{N_{k_2} - (t^*+2)} + \lambda \left( s_{k_1,t^*+2}^o - s_{k_1,t^*+2}^n \right) \sum_{j=1}^{\lfloor N_{k_1} \rfloor} \gamma^{N_{k_1} - (t^*+2)} > 0,
\]
where we have used the fact that \( s_{k_2,t^*+2}^o - s_{k_2,t^*+2}^n = - \left( s_{k_1,t^*+2}^o - s_{k_1,t^*+2}^n \right) > 0 \), \( |N_{k_2}| \geq |N_{k_1}| \)
and for all \( j \in [|N_{k_1}|], N_{k_2,j} < N_{k_1,j} \). In Case II, similarly, we have that
\[
G_T(\pi_{1:T}^o) - G_T(\pi_{1:T}^n) > \sum_{j=1}^{\lfloor N_{k_1} \rfloor} -\lambda g_{N_{k_1}} \left( s_{k_1,t^*+2}^o, N_{k_1,j}, t^* + 2 \right) + \lambda g_{N_{k_1}} \left( s_{k_1,t^*+2}^n, N_{k_1,j}, t^* + 2 \right)
+ \sum_{i=1}^{\lfloor N_{k_2} \rfloor} -\lambda g_{N_{k_2}} \left( s_{k_2,i,t^*+2}, N_{k_2,i}, t^* + 2 \right) + \lambda g_{N_{k_2}} \left( s_{k_2,i,t^*+2}, N_{k_2,i}, t^* + 2 \right)
= \lambda \left( s_{k_1,t^*+2}^o - s_{k_1,t^*+2}^n \right) \sum_{j=1}^{\lfloor N_{k_1} \rfloor} \gamma^{N_{k_1} - (t^*+2)} + \lambda \left( s_{k_2,t^*+2}^o - s_{k_2,t^*+2}^n \right) \sum_{i=1}^{\lfloor N_{k_2} \rfloor} \gamma^{N_{k_2,i} - (t^*+2)} > 0,
\]
since \( s_{k_2,t^*+2}^o - s_{k_2,t^*+2}^n = - \left( s_{k_1,t^*+2}^o - s_{k_1,t^*+2}^n \right) > 0 \), \( |N_{k_2}| \leq |N_{k_1}| \) and for all \( i \in [|N_{k_2}|], N_{k_2,i} < N_{k_1,i} \).

Finally, for Case III, the new policy construction is a mix of Case I and Case II. We represent the time interval \([t^* + 2, T]\) to be \([t^* + 2, T] = [t_{i_1,s_1}, t_{i_1,e_1}] \cup [t_{i_2,s_2}, t_{i_2,e_2}] \cup \cdots \cup [t_{i_M,s_M}, t_{i_M,e_M}]\) where \( t^* + 2 = t_{i_1,s_1} \leq \cdots \leq t_{i_M,s_M} = T \), \( \cap_{m=1}^{M} [t_{i_m,s_m}, t_{i_m,e_m}] = \emptyset \) and \( M - 1 \) is the number of new policy construction switches happen in between \( t^* + 2 \) and \( T \). We say that a new policy construction switch happens at time \( t \) if the policy construction follows Case I at time \( t - 1 \) but follows Case II at time \( t \) or vice versa. Each \( i_m \neq i_{m-1} \) can take values I or II, representing which policy construction rule is used between the time period \( t_{i_m,s_m} \) and \( t_{i_m,e_m} \). For any time index set \( V \), we use the notation \( V[t_{i_m,s_m}, t_{i_m,e_m}] := \{ t \in V : t_{i_m,s_m} \leq t \leq t_{i_m,e_m} \} \).

We notice that at any switching time \( t_{i_m,s_m} \), the number of previous pulls of arm \( k_1 \) and \( k_2 \) from time \( t_{i_{m-1},s_{m-1}} \) to \( t_{i_{m-1},e_{m-1}} \) are equivalent, which is denoted by \( l = |N_{k_1} | = |N_{k_2} | = \lfloor N_{k_1} \rfloor \) for all \( m < M \). From our analysis of Case I and Case II, we know that to show that \( \pi_{1:T}^o \) obtains a higher cumulative expected reward, it suffices to prove: for all \( m < M \) such that
\[
s_{k_2,t_{i_m,s_m}^o} - s_{k_2,t_{i_m,s_m}^n} = - \left( s_{k_1,t_{i_m,s_m}^o} - s_{k_1,t_{i_m,s_m}^n} \right) > 0,
\]
we have
\[
\begin{align*}
s_{k_2,t_{i_m,s_m+1}^o} - s_{k_2,t_{i_m,s_m+1}^n} &= - \left( s_{k_1,t_{i_m,s_m+1}^o} - s_{k_1,t_{i_m,s_m+1}^n} \right) > 0, \\
s_{k_1,t_{i_m,s_m+1}^o} - s_{k_1,t_{i_m,s_m+1}^n} &= - \left( s_{k_2,t_{i_m,s_m+1}^o} - s_{k_2,t_{i_m,s_m+1}^n} \right) > 0.
\end{align*}
\]
We will establish these facts in Lemma 3. Finally, we note that the above required conditions are held at time \( t_{i_1,e_1} = t^* + 2 \).

**Lemma 3.** Let \( N_k[t_s,t_e] \) denote the set of time steps when arm \( k \) is pulled in between (and including) time \( t_s \) and \( t_e \) under policy \( \pi_{1:T}^o \). Let \( s_{k,t}^o \) and \( s_{k,t}^n \) represent the satiation level of arm \( k \) at time \( t \) when following the policy \( \pi_{1:T}^o \) and \( \pi_{1:T}^n \), respectively. For two different arms \( k_1 \) and \( k_2 \), suppose that at time \( t_s \) we have
\[
s_{k_2,t_s}^o - s_{k_2,t_s}^n = - \left( s_{k_1,t_s}^o - s_{k_1,t_s}^n \right) > 0,
\]
Further, suppose that from time $t_s$ to $t_e$, $\pi^n_{1:T}$ follows either Case I (or Case II) of new policy construction (see proof of Theorem 1 for their definitions); and at time $t'_s = t_e + 1$, the new policy construction for $\pi^n_{1:T}$ has switched to Case II (or Case I if Case II is used from $t_s$ to $t_e$). Then at time $t'_s$, we have that

$$s_{k_1,t'_s}^n - s_{k_2,t'_s}^n = - (s_{k_1,t'_s}^n - s_{k_1,t_s}^n) > 0.$$

Proof of Lemma 3. Following the definition in the proof of Theorem 1, given that at time $t_s$, arm $k$ has satiation $s$, let $g_{N_k}[t_s,t_e](s, t'_s, t_s)$ denote the satiation level of arm $k$ at time $t'_s$ after being pulled at the time steps in the set $N_k[t_s, t_e]$. Let $N_{k,i}[t_s, t_e]$ be the $i$-th smallest element in the set $N_k[t_s, t_e]$. From the definition of the new policy construction given in the proof of Theorem 1 we also know that (1) $N := |N_k[t_s, t_e]| = |N_{k_2}[t_s, t_e]|$; (2) if Case I is applied in between $t_s$ and $t_e$, we have that for all $i \in [N], N_{k_2,i}[t_s, t_e] < N_{k_1,i}[t_s, t_e]$; and (3) if Case II is applied in between $t_s$ and $t_e$, we have that for all $i \in [N], N_{k_2,i}[t_s, t_e] > N_{k_1,i}[t_s, t_e]$.

We first consider the setting when Case I new policy construction is applied, then at time $t'_s$, we can show that

$$s_{k_1,t'_s}^n = g_{N_k}[t_s,t_e](s_{k_1,t_s}, t'_s, t_s) - g_{N_{k_2}[t_s,t_e]}(s_{k_2,t_s}, t'_s, t_s)$$

$$= \gamma t'_s - t_s (s_{k_1,t_s}^n - s_{k_2,t_s}^n) + \sum_{i=1}^k \gamma t'_s - N_{k_1,i}[t_s,t_e] - \gamma t'_s - N_{k_2,i}[t_s,t_e]$$

$$= \gamma t'_s - t_s (s_{k_1,t_s}^n - s_{k_2,t_s}^n) + \sum_{i=1}^k \gamma t'_s - N_{k_1,i}[t_s,t_e] - \gamma t'_s - N_{k_2,i}[t_s,t_e]$$

$$s_{k_1,t'_s}^n - s_{k_2,t'_s}^n > 0,$$

where the last inequality has used the fact that when we use Case I construction, we have $N_{k_2,i}[t_s, t_e] < N_{k_1,i}[t_s, t_e]$. Meanwhile, we also have that

$$s_{k_2,t'_s}^n = g_{N_k}[t_s,t_e](s_{k_2,t_s}, t'_s, t_s) - g_{N_{k_2}[t_s,t_e]}(s_{k_2,t_s}, t'_s, t_s)$$

$$= \gamma t'_s - t_s (s_{k_2,t_s}^n - s_{k_2,t_s}^n) = - \gamma t'_s - t_s (s_{k_1,t_s}^n - s_{k_1,t_s}^n)$$

$$= - (s_{k_1,t'_s}^n - s_{k_1,t_s}^n) > 0.$$

When Case II new policy construction is applied, then at time $t'_s$, we get

$$s_{k_1,t'_s}^n - s_{k_2,t'_s}^n = g_{N_k}[t_s,t_e](s_{k_1,t_s}, t'_s, t_s) - g_{N_{k_2}[t_s,t_e]}(s_{k_2,t_s}, t'_s, t_s)$$

$$= \gamma t'_s - t_s (s_{k_1,t_s}^n - s_{k_2,t_s}^n) = - \gamma t'_s - t_s (s_{k_2,t_s}^n - s_{k_1,t_s}^n)$$

$$= - (s_{k_2,t'_s}^n - s_{k_1,t'_s}^n) > 0,$$

since $s_{k_1,t'_s}^n - s_{k_2,t'_s}^n > 0$. On the other hand, we have that

$$s_{k_2,t'_s}^n - s_{k_2,t'_s}^n = g_{N_k}[t_s,t_e](s_{k_2,t_s}, t'_s, t_s) - g_{N_{k_2}[t_s,t_e]}(s_{k_2,t_s}, t'_s, t_s)$$

$$= \gamma t'_s - t_s (s_{k_2,t_s}^n - s_{k_2,t_s}^n) + \sum_{i=1}^k \gamma t'_s - N_{k_2,i}[t_s,t_e] - \gamma t'_s - N_{k_1,i}[t_s,t_e]$$

$$= \gamma t'_s - t_s (s_{k_2,t_s}^n - s_{k_2,t_s}^n) + \sum_{i=1}^k \gamma t'_s - N_{k_2,i}[t_s,t_e] - \gamma t'_s - N_{k_1,i}[t_s,t_e]$$

$$s_{k_2,t'_s}^n - s_{k_2,t'_s}^n > 0.$$
\[= \gamma_{t_s}^t - t_s \left( s_{k_1,t_s}^n - s_{k_1,t_s}^o \right) + \sum_{i=1}^{l} \gamma_{t_s}^i - N_{k_2,i} \left( t_s, t_e \right) - \gamma_{t_s}^i - N_{k_1,i} \left( t_s, t_e \right) \]
\[= s_{k_1,t_s}^n - s_{k_1,t_s}^o > 0, \]
where the last inequality is true because when Case II new policy construction is applied, we have \( N_{k_1,i} \left( t_s, t_e \right) < N_{k_2,i} \left( t_s, t_e \right). \]

\[\square\]

B.3 Proof of Proposition 2

**Proof.** If \( T \leq K \), a Max K-Cut of \( \mathcal{K}_T \) is \( \forall k \in [T], P_k = \{k\} \), which is the same as an optimal solution to (4). Let \( 1_{\{\cdot\} } \) denote the indicator function. When \( T > K \), the integer program in (4) is equivalent to

\[ \max_{\forall t \in [T]; \sum u_{k,t} = 1} \sum_{k=1}^{K} bu_{k,1} + \sum_{k=1}^{K} \sum_{t=2}^{T} \left( bu_{k,t} - \lambda \sum_{i=1}^{l-1} \gamma_{t-i} u_{k,i}u_{k,t} \right) \]
\[= \max_{P_1, \ldots, P_K \subseteq [T]: \cup_k P_k = [T], \forall k \neq k', P_k \cap P_{k'} = \emptyset} T b - \sum_{k=1}^{K} \sum_{t \in P_k} \lambda \gamma_{t-i} + \max_{P_1, \ldots, P_K \subseteq [T]: \cup_k P_k = [T], \forall k \neq k', P_k \cap P_{k'} = \emptyset} \sum_{k=1}^{K-1} \sum_{k'=k+1}^{K} \sum_{t \in P_k \cap P_{k'}} \lambda \gamma_{t-i}, \]

where the second equality uses the fact \( \sum_{k=1}^{K} 1_{\{t \in P_k\}} = 1 \) for all \( t \in [T] \) and the third equality is true because for any \( P_1, \ldots P_K \) such that \( \forall k \neq k', P_k \cap P_{k'} = \emptyset \), we have

\[ \text{Total Edge Weights of } \mathcal{K}_T = \sum_{t=2}^{T-l} \sum_{i=1}^{l-1} e(t,i) = \sum_{t \in [T]; i < t, \exists k \in [K], i \notin P_k} e(t,i) + \sum_{t \notin [T]; i < t, \forall k \in [K], i \notin P_k} e(t,i). \]

\[\square\]

B.4 Proof of Theorem 2

**Proof.** Given \( \pi_{1:T}^w \) and \( \pi_{1:T}^w \), define a set of new policies \( \{\pi_{1:T}^i\}_{i=1}^{l-1} \) such that for all \( i, \pi_{1:T}^i = (\pi_{1:i-1}^w, \pi_{i+1:T}^i) \). Based on this, we have the following decomposition

\[ G_T(\pi_{1:T}^w) - G_T(\pi_{1:T}^w) = \left( G_T(\bar{\pi}_{1:T}^1) - G_T(\bar{\pi}_{1:T}^1) \right) + \left( \sum_{i=1}^{l-2} G_T(\bar{\pi}_{1:T}^1) - G_T(\bar{\pi}_{1:T}^1) \right) + G_T(\bar{\pi}_{1:T}^1) - G_T(\bar{\pi}_{1:T}^1). \]

To distinguish the past pull sequences of each arm under different policies, we use the following notations: \( \mu_{k,t}(u_{k,0:t-1}; \pi^w) \) gives the expected reward of arm \( k \) at time \( t \) by following pull sequence \( \pi_{1:t-1}^w \). By the definition of \( \pi_{1:T}^w \), we have that

\[ A_0 = \sum_{t=1}^{w} \mu_{\pi_{1:T}^i}(u_{\pi_{1:T}^i,0:t-1}; 0^w) - \mu_{\pi_{1:T}^i}(u_{\pi_{1:T}^i,0:t-1}; 0^w) + \sum_{t=w+1}^{T} \mu_{\pi_{1:T}^i}(u_{\pi_{1:T}^i,0:t-1}; 0^w) - \mu_{\pi_{1:T}^i}(u_{\pi_{1:T}^i,0:t-1}; 0^w) \]

21
≤ \sum_{t=w+1}^{T} \mu_{\pi_t^w,t}(u_{\pi_t^w,0:t-1}^w; \pi^w) - \mu_{\pi_t^w,t}(u_{\pi_t^w,0:t-1}^w; \tilde{\pi}^w),

where the inequality follows from the fact that \( \pi_{1:w}^w \) is optimal for (4) when \( T = w \). Similarly, we obtain that for all \( i \in [l - 2] \),

\[ A_i = \sum_{t=1}^{iw} \mu_{\pi_t^w,t}(u_{\pi_t^w,0:t-1}^w) - \mu_{\pi_t^w,t}(u_{\pi_t^w,0:t-1}^w) + \sum_{t= iw+1}^{(i+1)w} \mu_{\pi_t^w,t}(u_{\pi_t^w,0:t-1}^w) - \mu_{\pi_t^w,t}(u_{\pi_t^w,0:t-1}^w) \leq 0 \]

\[ + \sum_{t=(i+1)w+1}^{T} \mu_{\pi_t^w,t}(u_{\pi_t^w,0:t-1}^w) - \mu_{\pi_t^w,t}(u_{\pi_t^w,0:t-1}^w) \leq \sum_{t=(i+1)w+1}^{T} \mu_{\pi_t^w,t}(u_{\pi_t^w,0:t-1}^w) - \mu_{\pi_t^w,t}(u_{\pi_t^w,0:t-1}^w) \]

Finally, we have \( A_{l-1} = \sum_{t=(l-1)w+1}^{T} \mu_{\pi_t^w,t}(u_{\pi_t^w,0:t-1}^w) - \mu_{\pi_t^w,t}(u_{\pi_t^w,0:t-1}^w) \leq 0 \). To complete the proof, it suffices to use the fact that for all \( i \in \{1, \ldots, l - 1\} \),

\[ \max_{\pi_{iw+1:T}^w, \pi_{iw+1:T}} \sum_{t= iw+1}^{T} \mu_{\pi_t^w,t}(u_{\pi_t^w,0:t-1}^w) - \mu_{\pi_t^w,t}(u_{\pi_t^w,0:t-1}^w) \leq \sum_{t=0}^{T-iw-1} \frac{T-iw-1}{1-\gamma} \frac{\lambda_{\gamma}(1-\gamma^{T-iw})}{(1-\gamma)^2} \leq \frac{\lambda_{\gamma}(1-\gamma^{T-iw})}{(1-\gamma)^2} \]

where the first inequality holds because for any arm, the maximum satiation level discrepancy under two pull sequences (after \( iw \) time steps) is \( \frac{\gamma}{1-\gamma} \) and from time \( iw + 1 \) till time \( T \), the objective will be maximized when the arm with the maximum satiation discrepancy is played all the time.
C  More Discussion on Learning with Unknown Dynamics

As we have noted in Section 3, when the learner makes a decision on which arm to pull, the learner does not observe the hidden satiation level the user has for the arms. The POMDP the learner faces can be cast as a fully observable MDP (Appendix C.1) where the estimated reward model (Appendix C.2) can be used for planning (Appendix C.3). In addition to policies that are time-dependent (actions taken by time-dependent policies only depend on the time steps at which they are taken) considered in Section 6, we also consider state-dependent policies where the states are continuous.

C.1 MDP Setup

We begin with describing the full MDP setup of rebounding bandits, including the state representation and reward function defined in Section 5.1. Following [32], at any time \( t \in [T] \), we define our state vector to be \( x_t = (x_{1,t}, n_{1,t}, x_{2,t}, n_{2,t}, \ldots, x_{K,t}, n_{K,t}) \), where \( n_{k,t} \in \mathbb{N} \) is the number of steps since arm \( k \) is last selected and \( x_{k,t} \) is the satiation influence of the most recent pull of arm \( k \). Since the most recent pull happens at \( t - n_{k,t} \), we have \( x_{k,t} = b_k - \mu_{k,t - n_{k,t}} = \lambda_k s_{k,t - n_{k,t}} \).

We note that
\[
\text{At time } t, \text{ the learner follows an action } \pi_t \text{, and reward } \mu_{k,t} \text{ is obtained when arm } k \text{ is pulled for the first time since the satiation effect is 0 if an arm has not been pulled before. The initial state is } x_{init} = (0, \ldots, 0). \text{ Transitions between two states } x_t \text{ and } x_{t+1} \text{ are defined as follows: If arm } k \text{ is chosen at time } t, \text{ i.e., } \pi_t = k, \text{ and reward } \mu_{k,t} \text{ is obtained, then the next state } x_{t+1} \text{ will be:}
\]

A.1 For the pulled arm \( k \), \( n_{k,t+1} = 1 \) and \( x_{k,t+1} = b_k - \mu_{k,t} \).

A.2 For other arms \( k' \neq k \), \( n_{k',t+1} = n_{k',t} + 1 \) if \( n_{k',t} \neq 0 \) and \( n_{k',t+1} = 0 \) if \( n_{k',t} = 0 \). The satiation influence remains the same, i.e., \( x_{k',t+1} = x_{k',t} \).

For all \( x_t \in X \) and \( k \in [K] \), we have that
\[
\mathbb{E}[x_{k,t}] \leq \overline{\lambda} \tau/(1 - \tau) \quad \text{and} \quad \text{Var}[x_{k,t}] \leq \overline{\lambda}^2 \sigma_z^2/(1 - \tau^2).
\]

Hence, for any \( \delta \in (0, 1) \), \( \mathbb{P} \left( \max_{k,t} |x_{k,t}| \geq B(\delta) \right) \leq \delta \), where
\[
B(\delta) := \frac{\overline{\lambda} \tau}{1 - \tau} + \overline{\lambda} \sigma_z \sqrt{\frac{2 \log(2KT/\delta)}{1 - \tau^2}}. \tag{10}
\]

The MDP the learner faces can be described as a tuple \( \mathcal{M} := (x_{init}, [K], \{\gamma_k, \lambda_k, b_k\}_{k=1}^K, T) \) of the initial state \( x_{init} \), actions (arms) \( [K] \), the horizon \( T \) and parameters \( \{\gamma_k, \lambda_k, b_k\}_{k=1}^K \). Let \( \Delta(\cdot) \) denote the probability simplex. Given \( \{\gamma_k, \lambda_k, b_k\}_{k=1}^K \), the expected reward \( r: X \times [K] \rightarrow \mathbb{R} \) and transition functions \( p: X \times [K] \times [T] \rightarrow \Delta(X) \) are defined as follows:

1. \( r: X \times [K] \rightarrow \mathbb{R} \) gives the expected reward of pulling arm \( k \) conditioned on \( x_t \), i.e., \( r(x_t, k) = \mathbb{E}[\mu_{k,t}|x_t] \). If \( n_{k,t} = 0 \), then \( r(x_t, k) = b_k \). If \( n_{k,t} \geq 1 \), \( r(x_t, k) = b_k - \gamma_k^{n_{k,t}} x_{k,t} - \lambda_k \gamma_k^{n_{k,t}} \).

2. When pulling arm \( k \) at time \( t \) and state \( x_t \), \( p(x_{t+1}|x_t, k, t) = 0 \) if \( x_{t+1} \) does not satisfy eq. A.1 or A.2. When \( x_{t+1} \) fulfills both eq. A.1 and A.2, we consider two cases of \( x_t \). If \( n_{k,t} \neq 0 \), then the transition function \( p(x_{t+1}|x_t, k, t) \) is given by the Gaussian density with mean \( \gamma_k^{n_{k,t}} (x_{k,t} + \lambda_k) \) and variance \( \lambda^2_k \sigma^2_z \sum_{i=0}^{n_{k,t}} \gamma_i^2 \lambda_i \), as illustrated in [11]. If \( n_{k,t} = 0 \), then \( p(x_{t+1}|x_t, k, t) = 1 \) since for the first pull of arm \( k \), the obtained reward \( \mu_{k,t} = b_k \).

At time \( t \), the learner follows an action \( \pi_t: X \rightarrow [K] \) that depends on the state. We use \( V_{t,M}^\pi: X \rightarrow \mathbb{R} \) to denote the value function of policy \( \pi_{1:T} \) at time \( t \) under MDP \( \mathcal{M} \). The value function \( V_{t,M}^\pi(x_t) = r(x_t, \pi_t(x_t)) + \mathbb{E}[x_{t+1} \sim p(\cdot|x_t, \pi_t(x_t), t)] [V_{t+1,M}^\pi(x_{t+1})] \) and \( V_{T+1,M}^\pi(x) = 0 \) for all \( x \in X \). To restate our goal (2) in terms of the value function: for an MDP \( \mathcal{M} \), we would like to find a policy \( \pi_{1:T} \).

\(^2\)By conditioning on \( x_t \), we mean conditioning on the \( \sigma \)-algebra generated by past actions and observed rewards.
that maximizes
\[ V^*_t(x_{\text{init}}) = \mathbb{E}\left[ \sum_{t=1}^{T} r(x_t, \pi_t(x_t)) \middle| x_1 = x_{\text{init}} \right]. \]

To simplify the notation, we use \( \pi \) to refer to a policy \( \pi_{1:T} \). Given an MDP \( \mathcal{M} \), we denote its optimal policy by \( \pi^*_\mathcal{M} \) and the value function for the optimal policy by \( V^*_t \), i.e., \( V^*_t(x) := V^*_{t,\mathcal{M}}(x) \).

### C.2 Exploration and Estimation of the Reward Model

As we have discussed in § 6.1, based on our satiation and reward models, the satiation influence \( x_{k,t} \) of arm \( k \) forms a dynamical system where we only observe the value of the system when arm \( k \) is pulled. When arm \( k \) is pulled at time \( t \) and \( n_{k,t} \neq 0 \), we observe the satiation influence \( \lambda_k s_{k,t} \) which becomes the next state \( x_{k,t+1} \), i.e.,

\[
x_{k,t+1} = \lambda_k s_{k,t} = \lambda_k n_{k,t} s_{k,t-n_{k,t}} + \lambda_k n_{k,t} + \lambda_k \sum_{i=0}^{n_{k,t}-1} \gamma_k^{i} z_{k,t-1-i}
\]

We note that the current state \( x_{k,t+1} \) equals to \( x_{k,t+1-n_{k,t}} \) since \( x_{k,t+1-n_{k,t}} \) is the last observed satiation influence for arm \( k \) and \( n_{k,t} \) is the number of steps since arm \( k \) is last pulled.

#### Exploration Settings

Depending on the nature of the recommendation domain, we consider two types of exploration settings: one where the users only interact with the recommendation systems for a short time after they log in to the service (Appendix C.2.1) and the other where the users tend to interact with the system for a much longer time, e.g., automated music playlisting (Appendix C.2.2). In the first case, the learner collects multiple \( n \) short trajectories of user utilities, while in the second case, similar to § 6.2, the learner obtains a single trajectory of user utilities that has length \( n \). In both settings, we obtain that under some mild conditions, the estimation errors of our estimators for \( \gamma_k \) and \( \lambda_k \) are \( O(1/\sqrt{n}) \).

#### Exploration Strategies

Generalizing from the case where arms are pulled repeatedly, we explore by pulling the same arm at a fixed interval \( m \). In particular, when \( m = 1 \), the exploration strategy is the same as repeatedly pulling the same arm for multiple times, which is the exploration strategy used in § 6.1. When \( m = K \), the exploration strategy is to pull the arms in a cyclic order. We present the estimator for \( \gamma_k, \lambda_k \) using the dataset collected by this exploration strategy in both the multiple trajectory and single trajectory settings.

#### C.2.1 Estimation using Multiple Trajectories

For each arm \( k \in [K] \), we use \( D^{n,m}_k \) to denote a dataset containing \( n \) trajectories of evenly spaced observed satiation influences that are collected by our exploration phase. The time interval between two pulls of an arm is denoted by \( m \). Each trajectory is of length at least \( T_{\text{min}} + 1 \) for \( T_{\text{min}} > 1 \). For trajectory \( i \in [n] \), the observed satiation influences are denoted by \( \tilde{x}_{k,1}^{(i)}, \ldots, \tilde{x}_{k,T_{\text{min}}+1}^{(i)} \), where \( \tilde{x}_{k,1}^{(i)} = 0 \) is the initial satiation influence and the rest of the satiation influences \( \tilde{x}_{k,j}^{(i)} \) \( (j > 1) \) is the difference between the first received reward, i.e., the base reward \( b_k \), and the reward from the \( j \)-th pull of arm \( k \). In other words, for \( \tilde{x}_{k,j}^{(i)}, \tilde{x}_{k,j+1}^{(i)} \in D^{n,m}_k \), it follows that

\[
\tilde{x}_{k,j+1}^{(i)} = a_k \tilde{x}_{k,j}^{(i)} + d_k + \tilde{z}_{k,j}^{(i)}.
\]
where \( a_k = \gamma_k^m \), \( d_k = \lambda_k \gamma_k^m \) and \( \tilde{z}^{(i)} \) are the independent samples from \( \mathcal{N}(0, \sigma_{z,k}^2) \) with \( \sigma_{z,k}^2 = \lambda_k^2 \sigma_z^2(1 - \gamma_k^2)/\left(1 - \gamma_k^2\right) \).

To estimate \( d_k \), we use the estimator \( \hat{d}_k = \frac{1}{n} \sum_{i=1}^n \tilde{z}^{(i)}_{k,2} = d_k + \frac{1}{n} \sum_{i=1}^n \tilde{z}^{(i)}_{k,1} \). By the standard Gaussian tail bound, we obtain that for \( \delta \in (0, 1) \), with probability \( 1 - \delta \),

\[
|\hat{d}_k - d_k| \leq \sqrt{\frac{2\sigma_{z,k}^2 \log(2/\delta)}{n}} =: \epsilon_d(n, \delta, k).
\] (13)

When estimating \( a_k \), we first take the difference between the first \( T_{\min} + 1 \) entries of two trajectories \( i \) and \( 2i \) for \( i \in [n/2] \) and obtain a new trajectory \( \tilde{y}^{(i)}_{k,1}, \ldots, \tilde{y}^{(i)}_{k,T_{\min}+1} \) where \( \tilde{y}^{(i)}_{k,j} = \tilde{x}^{(i)}_{k,j} - \tilde{x}^{(2i)}_{k,j} \) for \( j \in [T_{\min} + 1] \). We note that the new trajectory forms a linear dynamical system without the bias term \( d_k \), i.e.,

\[
\tilde{y}^{(i)}_{k,j+1} = a_k \tilde{y}^{(i)}_{k,j} + \tilde{w}^{(i)}_{k,j},
\]

where \( \tilde{w}^{(i)}_{k,j} \) are samples from \( \mathcal{N}(0, 2\sigma_{z,k}^2) \). We use the ordinary least squares estimator to estimate \( a_k \):

\[
\hat{a}_k = \arg \min_a \sum_{i=1}^{[n/2]} \left( \tilde{y}^{(i)}_{k,T_{\min}+1} - a \tilde{y}^{(i)}_{k,T_{\min}} \right)^2 = \frac{\sum_{i=1}^{[n/2]} \tilde{y}^{(i)}_{k,T_{\min}+1} \tilde{y}^{(i)}_{k,T_{\min}}}{\sum_{i=1}^{[n/2]} \left( \tilde{y}^{(i)}_{k,T_{\min}} \right)^2}.
\] (14)

**Theorem 6.** [24] Theorem II.4] Fix \( \delta \in (0, 1) \). Given \( n \geq 64 \log(2/\delta) \), with probability \( 1 - \delta \), we have that

\[
|\hat{a}_k - a_k| \leq 4 \sqrt{\frac{2 \log(4/\delta)}{n \sum_{t=0}^{T_{\min}+1} a_t^2 k}} =: \epsilon_a(n, \delta, k).
\] (15)

We notice that as the minimum length of the trajectory gets greater, the upper bound of the estimation error of \( a_k \) gets smaller. Using our estimators for \( a_k \) and \( d_k \), we estimate \( \gamma_k \) and \( \lambda_k \) through \( \hat{\gamma}_k = |\hat{a}_k|^{1/m} \) and \( \hat{\lambda}_k = |\hat{d}_k/\hat{a}_k| \).

**Corollary 7.** Fix \( \delta \in (0, 1) \). Suppose that for all \( k \in [K] \), we are given \( P_k^{n,m} \) where \( n \geq 64 \log(2/\delta) \) and \( \hat{a}_k > 0 \) where \( \hat{a}_k \) is defined in (14). Then, with probability \( 1 - \delta \), we have that for all \( k \in [K] \),

\[
|\hat{\gamma}_k - \gamma_k| \leq \frac{\epsilon_a(n, \delta/K, k)}{\gamma_k^{m-1}} = O\left(\frac{1}{\sqrt{n}}\right) \quad \text{and} \quad |\hat{\lambda}_k - \lambda_k| \leq O\left(\frac{1}{\sqrt{n}}\right).
\]

The proof of Corollary 7 can be found in Appendix F.1. In the case where we are have collected \( n \) trajectories of evenly spaced user utilities for each arm, when the sample size \( n \) is sufficient large, the estimation errors of \( \hat{\gamma}_k \) and \( \hat{\lambda}_k \) are \( O(1/\sqrt{n}) \).

### C.2.2 Estimation using a Single Trajectory

In the case where the learner gets to interact with the user for a long period of time (which is the setting considered in § 5 and § 6), we collect a single trajectory of evenly spaced arm pulls for each arm: for each arm \( k \in [K] \), we use \( P_k^{n,m} \) to denote a dataset containing a single
A trajectory of $n+1$ observed satiation influences $\tilde{x}_{k,1}, \ldots, \tilde{x}_{k,n+1}$, where similar to the multiple trajectories case, $\tilde{x}_{k,1} = 0$, $\tilde{x}_{k,j}$ ($j > 1$) is the difference between the first received reward and the $j$-th received reward and the time interval between two consecutive pulls is $m$. Thus, for $\tilde{x}_{k,j}, \tilde{x}_{k,j+1} \in \mathcal{P}_{k}^{n,m}$, it follows that

$$\tilde{x}_{k,j+1} = a_k \tilde{x}_{k,j} + d_k + \tilde{z}_{k,j}, \quad (16)$$

where $a_k, d_k$ and $\tilde{z}_{k,j}$ are defined the same as the ones in (12). For all $k \in [K]$, given $\mathcal{P}_{k}^{n,m}$, we use the following estimators to estimate $A_k = (a_k, d_k)^\top$,

$$\hat{A}_k = \begin{pmatrix} \hat{a}_k \\ \hat{d}_k \end{pmatrix} = (X_k^\top X_k)^{-1} X_k^\top Y_k, \quad (17)$$

where $Y_k \in \mathbb{R}^n$ is an $n$-dimensional vector whose $j$-th entry is $\tilde{x}_{k,j+1}$ and $X_k \in \mathbb{R}^{n \times 2}$ has its $j$-th row to be the vector $x_{k,j} \in \mathbb{R}$.

Finally, we take $\hat{\gamma}_k = |\hat{a}_k|^{1/m}$ and $\lambda_k = |\hat{d}_k/\hat{a}_k|$. We note that $\hat{A}_k = \arg\min_{A_k \in \mathbb{R}^2} \|Y_k - X_k A_k\|_2^2$, i.e., it is the ordinary least squares estimator for $A_k$ given the dataset that treats $\tilde{x}_{k,j+1}$ to be the response of the covariates $x_{k,j}$.

As we have noted earlier (§ 6.2), unlike the multiple trajectories setting, in the single trajectory case, the difficulty in analyzing the ordinary least squares estimator (17) comes from the fact that the samples are not independent. Asymptotic guarantees of the ordinary least squares estimators in this case have been studied previously in control theory and time series community [13, 22]. The recent work on system identifications for linear dynamical systems focuses on studying the sample complexity of the problem [40, 38]. Adapting the proof of [40, Theorem 2.4], we derive the following theorem for identifying our affine dynamical system (16).

**Theorem 8.** Fix $\delta \in (0, 1)$. For all $k \in [K]$, there exists a constant $n_0(\delta, k)$ such that if the dataset $\mathcal{P}_{k}^{n,m}$ satisfies $n \geq n_0(\delta, k)$, then

$$\mathbb{P} \left( \|\hat{A}_k - A_k\|_2 \geq \sqrt{1/(\psi n)} \right) \leq \delta,$$

where $\psi = \sqrt{\min \left\{ \frac{\sigma^2 z_k (1 - a_k)^2}{16d_k^2 (1 - a_k^2) + (1 - a_k)}; \frac{\sigma^2 z_k}{4(1 - a_k)} \right\}}$.

As shown in Theorem 8 when $d_k = \lambda_k \gamma_k^m$ gets larger, the rates of convergence for $\hat{A}_k$ gets slower. Given that we have a single trajectory of sufficient length, $|\hat{a}_k - a_k| \leq O(1/\sqrt{n})$ and $|\hat{d}_k - d_k| \leq O(1/\sqrt{n})$. Similar to the multiple trajectories case, as shown in Corollary 9 the estimators of $\gamma_k$ and $\lambda_k$ also achieve $O(1/\sqrt{n})$ estimation error.

**Corollary 9.** Fix $\delta \in (0, 1)$. Suppose that for all $k \in [K]$, we have $\mathbb{P}(\|\hat{A}_k - A_k\|_2 \geq 1/\sqrt{n}) \leq \delta$ and $\hat{a}_k > 0$ where $\hat{A}_k$ and $\hat{a}_k$ are defined in (17). Then, with probability $1 - \delta$, we have that for all $k \in [K]$,

$$|\hat{\gamma}_k - \gamma_k| \leq O \left( \frac{1}{\sqrt{n}} \right) \quad \text{and} \quad |\hat{\lambda}_k - \lambda_k| \leq O \left( \frac{1}{\sqrt{n}} \right).$$

In the next section, we assume that the satiation and reward models are estimated using the dataset collected by the proposed exploration strategies and estimators for multiple trajectories or a single trajectory of user utilities. We will show that performing planning based on these estimated models will give us policies that perform well for the true MDP.
C.3 Planning

For a continuous-state MDP, planning can be done through either dynamic programming with a discretized state space or approximate dynamic programming that uses function approximations. In Appendix C.3.2 we consider the case where we are given a continuous-state MDP planning oracle and provide guarantees of the optimal state-dependent policy planned under the estimated satiation dynamics and reward model. Within the state-dependent policies, we also consider a set of policies that only depend on time (Appendix C.3.1), i.e., the time-dependent competitor class defined in §5.2. In addition to not requiring discretization of the state space to solve the planning problem, such policies can be deployed to settings where user utilities are hard to attain after the exploration stage. We will show that using the dataset (collected by our exploration strategy in Appendix C.2) with sufficient trajectories (or a sufficient long trajectory) to estimate \( \{ \gamma_k, \lambda_k \}_{k=1}^{K} \), the optimal policy \( \pi^*_\hat{M} \) for \( \hat{M} = (x_1, [K], \{ \hat{\gamma}_k, \hat{\lambda}_k, b_k \}_{k=1}^{K}, T) \) also performs well in the original MDP \( M \). We note that \( b_k \) is known exactly since it is the same as the first observed reward for arm \( k \), as discussed in Appendix C.2.

C.3.1 Time-dependent Policy

We first show that finding the optimal time-dependent policy is equivalent to solving the bilinear program \([4]\). 

**Lemma 4.** Consider a policy \( \pi \) that depends only on the time step \( t \) but not the state \( x_t \), i.e., \( \pi \) satisfies \( \pi_t = \pi_t(x_t) = \pi_t(x'_t) \) for all \( t \in [T] \) and \( x_t, x'_t \in X \). Then, we have

\[
V_{t,\mathcal{M}}(x_{\text{init}}) = \sum_{t=1}^{T} \mu_{\pi_t,t}(u_{\pi_t,0:t-1}),
\]

where \( u_{\pi_t,0:t-1} \) is the corresponding pull sequence of arm \( \pi_t \) under policy \( \pi \) and \( \mu_{k,t} \) is defined in \([3]\).

**Remark 6.** We denote the policy obtained by solving \([4]\) using model parameters in \( M \) by \( \pi^T_M \). Because solving \([4]\) is equivalent to maximizing \( \sum_{t=1}^{T} \mu_{\pi_t,t}(u_{\pi_t,0:t-1}) \), Lemma 4 suggests that, for MDP \( M \), the best policy \( \pi \) that depends only on the time step \( t \) but not the exact state \( x_t \) (which we refer as time-dependent policies), is \( \pi^T_M \).

**Proposition 5.** Fix \( \delta \in (0, 1) \). Suppose that for all \( k \in [K] \), we are given \( \mathcal{D}^n_{\gamma_k} \) such that \( n \geq 64 \log(2/\delta) \) and \( \hat{\alpha}_k \in (\alpha, \overline{\alpha}) \) for some \( 0 < \alpha < \overline{\alpha} < 1 \) almost surely where \( \hat{\alpha}_k \) is defined in \([4]\). Consider a policy \( \pi \) that depends on only the time step \( t \) but not the state \( x_t \). Then, with probability \( 1 - \delta \), we have that

\[
|V_{1,\mathcal{M}}^\pi(x_{\text{init}}) - V_{1,\mathcal{M}}^{\pi^*_\hat{M}}(x_{\text{init}})| \leq O \left( \frac{T}{\sqrt{n}} \right).
\]

**Remark 7.** Proposition 5 applies to time-dependent policies. Such policies can be constructed from an optimal solution to \([4]\) or the \( \sqrt{T} \)-lookahead policy \([3]\). From these results, we deduce that when the historical trajectory is of size \( n = O(T) \), the \( \sqrt{T} \)-lookahead policy \( \pi^w_M \) obtained from solving \([5]\) with the parameters from the estimated MDP \( \hat{M} \) will be \( O(\sqrt{T}) \)-separated from the optimal time-dependent policy \( \pi^T_M \) obtained by solving \([4]\) with the true parameters of \( M \). That is,

\[
0 \leq V_{1,\mathcal{M}}^{\pi^T_M}(x_{\text{init}}) - V_{1,\mathcal{M}}^{\pi^w_M}(x_{\text{init}}) = V_{1,\mathcal{M}}^{\pi^T_M}(x_{\text{init}}) - V_{1,\mathcal{M}}^{\pi^T_M}(x_{\text{init}}) + V_{1,\mathcal{M}}^{\pi^w_M}(x_{\text{init}}) - V_{1,\mathcal{M}}^{\pi^w_M}(x_{\text{init}}) \approx V_{1,\mathcal{M}}^{\pi^*_{\hat{M}}}(x_{\text{init}}) - V_{1,\mathcal{M}}^{\pi^*_{\hat{M}}}(x_{\text{init}}).
\]
\[
\leq |V_{1,\hat{M}}^{\pi}(x_{\text{init}})| + |V_{1,\hat{M}}^{\pi}(x_{\text{init}})| + |V_{1,\hat{M}}^{\pi}(x_{\text{init}})| + |V_{1,\hat{M}}^{\pi}(x_{\text{init}})|
\leq O(\sqrt{T}),
\]

where the second inequality follows from the fact that \( V_{1,\hat{M}}^{\pi}(x_{\text{init}}) - V_{1,\hat{M}}^{\pi}(x_{\text{init}}) \leq 0 \) (since for the MDP \( \hat{M}, \pi_{1,\hat{M}}^{T} \) is the optimal time-dependent policy), and the third (last) inequality is derived by applying Proposition 5 twice and using Remark 4.

### C.3.2 State-dependent Policy

In Proposition 6, we show that the difference between the value of the optimal state-dependent policy \( \pi_{\hat{M}}^{*} \), and the value of the optimal state-dependent policy \( \pi_{\hat{M}}^{*} \) planned under the estimated \( \hat{M} \) is of order \( O\left(\frac{T^2}{\sqrt{n}}\right) \) where \( n \) is the number of historical trajectories if we use multiple trajectories to estimate \( \gamma_k \) and \( \lambda_k \).

**Proposition 6.** Fix \( \delta \in (0,1) \). Suppose that for all \( k \in [K] \), we are given \( D_{k}^{n,m} \) such that \( n \geq 64 \log(2/\delta) \) and \( \hat{a}_k \in (\underline{a}, \bar{a}) \) for some \( 0 < \underline{a} < \bar{a} < 1 \) almost surely where \( \hat{a}_k \) is defined in (14). Then, with probability \( 1 - \delta \),

\[
|V_{1,M}^{\pi}(x_{\text{init}}) - V_{1,M}^{\pi}(x_{\text{init}})| \leq O\left(\frac{T^2}{\sqrt{n}}\right).
\]

**Remark 8.** The assumptions in Proposition 5 and 6 correspond to the case where we use multiple trajectories to estimate the satiation dynamics and reward model. They can be replaced by conditions on single trajectory datasets when one uses a single trajectory to estimate the parameters.

In summary, as Proposition 6 suggests, when given a continuous-state MDP planning oracle, our algorithm obtain a policy \( \pi_{\hat{M}}^{*} \) that is \( O\left(\frac{T^2}{\sqrt{n}}\right) \) away from the optimal policy \( \pi_{\hat{M}}^{*} \) under the true MDP \( M \) where the size of the exploration stage for our algorithm (EEP) is \( O(Kn) \) and the horizon of the exploitation/planning stage is \( T \). We also note that the optimal state-dependent policy \( \pi_{M}^{*} \) is the optimal competitor policy when the competitor class (§ 5.2) contains all measurable functions from \( \mathcal{X} \) to \( [K] \).
D Proofs of Section 6.2 and Appendix C.2.2

D.1 Proof of Theorem 3 and Theorem 8

We notice that Theorem 3 is a consequence of Theorem 8 when \( m = 1 \). More specifically, the dataset \( P_k^n \) and the parameter \( A_k = (\gamma_k, \lambda_k \gamma_k)^T \) in Theorem 3 is a special case of the dataset \( P_k^{n,m} \) and parameter \( A_k = (\gamma_k^m, \lambda_k \gamma_k^m)^T \) considered in Theorem 8 by taking \( m = 1 \). Thus, below we directly present the proof of Theorem 8 where we use the notation from Theorem 8 (and Appendix C.2.2), i.e., \( a_k = \gamma_k^m \) and \( d_k = \lambda_k \gamma_k^m \).

We begin with presenting some key results from [40]; we utilize these results in establishing the sample complexity of our estimator for identifying an affine dynamical system in Appendix C.2.2.

**Definition 2.1** [40] Let \( \{ \phi_t \}_{t \geq 1} \) be an \( \{ \mathcal{F}_t \}_{t \geq 1} \)-adapted random process taking values in \( \mathbb{R} \). We say \( \{ \phi_t \}_{t \geq 1} \) satisfies the \((k, \nu, p)\)-block martingale small-ball (BMSB) condition if, for any \( j \geq 1 \), one has \( \frac{1}{k} \sum_{i=1}^k \mathbb{P}(|\phi_{j+i}| \geq \nu \mathcal{F}_j) \geq p \) almost surely. Given a process \( (X_t)_{t \geq 1} \) taking values in \( \mathbb{R}^d \), we say that it satisfies the \((k, \Gamma_{sb}, p)\)-BMSB condition for \( \Gamma_{sb} > 0 \) if for any fixed \( w \) in the unit sphere of \( \mathbb{R}^d \), the process \( \phi_t := \langle w, X_t \rangle \) satisfies \((k, \sqrt{w^T \Gamma_{sb} w}, p)\)-BMSB.

**Proposition 7.** [40] Proposition 2.5 Fix a unit vector \( w \in \mathbb{R}^d \), define \( \phi_t = w^T X_t \). If the scalar process \( \{ \phi_t \}_{t \geq 1} \) satisfies the \((l, \sqrt{w^T \Gamma_{sb} w}, p)\)-BMSB condition for some \( \Gamma_{sb} \in \mathbb{R}^{d \times d} \), then

\[
\mathbb{P} \left( \sum_{t=1}^n \phi_t^2 \leq \frac{w^T \Gamma_{sb} w p^2}{8} |T/l| \right) \leq \exp \left( -\frac{|T/l| p^2}{8} \right).
\]

**Theorem 10.** [40] Theorem 2.4 Fix \( \delta \in (0, 1) \), \( T \in \mathbb{N} \) and \( 0 < \Gamma_{sb} \preceq \Gamma \). Then if \( (X_t, Y_t)_{t \geq 1} \in (\mathbb{R}^d \times \mathbb{R}^n)^n \) is a random sequence such that \((a) Y_t = AX_t + \eta_t \), where \( \mathcal{F}_t = \sigma(\eta_1, \ldots, \eta_t) \) and \( \eta_t \mid \mathcal{F}_{t-1} \) is \( \sigma^2 \)-sub-Gaussian and mean zero, \((b) X_1, \ldots, X_T \) satisfies the \((l, \Gamma_{sb}, p)\)-BMSB condition, and \((c) \mathbb{P}(\sum_{t=1}^n X_tX_t^T \preceq T\Gamma) \geq \delta \). Then if

\[
T \geq \frac{10l}{p^2} \left( \log \left( \frac{1}{\delta} \right) + 2d \log(10/p) + \log \det(\Gamma_{sb}) \right),
\]

we have that for \( \hat{A} = \arg\min_{A \in \mathbb{R}^{d \times d}} \sum_{t=1}^n \| Y_t - AX_t \|_2^2 \),

\[
\mathbb{P} \left( \| \hat{A} - A \|_{op} > \frac{9\sigma}{p} \sqrt{\frac{n + d \log(10/p) + \log \det(\Gamma_{sb}^{-1}) + \log(1/\delta)}{T \lambda_{\min}(\Gamma_{sb})}} \right) \leq 3\delta.
\]

We note that in the proof of Theorem 10 in [40], condition (b) is used through applying Proposition 7 to ensure that for any unit vector \( w \in \mathbb{R}^d \),

\[
\mathbb{P} \left( \sum_{t=1}^T \langle w, X_t \rangle^2 \leq \frac{(w^T \Gamma_{sb} w)p^2}{8} |T/l| \right) \leq \exp \left( -\frac{|T/l| p^2}{8} \right).
\]

To apply Theorem 10 in our setting to obtain Theorem 8, we verify condition (a) and (c). For condition (b), we show a result similar to (18). The below technical lemmas are used in our proof of Theorem 8.

**Lemma 8.** Let \( a, b \) be scalars with \( b > 0 \). Suppose that \( X \sim N(a, b) \). Then for any \( \theta \in [0, 1] \),

\[
\mathbb{P}( |X| \geq \sqrt{\theta(a^2 + b)} ) \geq \frac{(1 - \theta)^2}{9}.
\]
Proof. By the Paley-Zygmund inequality,
\[ \mathbb{P}(|X| \geq \sqrt{\theta E[X^2]}) = \mathbb{P}(X^2 \geq \theta E[X^2]) \geq (1 - \theta)^{E[X^2]} \frac{E[X^4]}{E[X^2]}. \]

Using the mean and variance of non-central chi-squared distributions, we obtain that
\[ E[X^2] = a^2 + b, \quad E[X^4] = a^4 + 6a^2b + 3b^2 = (a^2 + 3b)^2 - 6b^2. \]

Plugging them back to the Paley-Zygmund inequality, we have that
\[ \mathbb{P}(|X| \geq \sqrt{\theta (a^2 + b)}) \geq (1 - \theta)^{2 E[X^2]} \frac{9}{9}, \]

where the last inequality uses the fact that \( E[X^4] \leq (a^2 + 3b)^2 \leq 9(a^2 + b)^2 = 9E[X^2]^2 \).

Lemma 9. Let \( \{\phi_t\}_{t \geq 1} \) be a scalar process satisfying that
\[ \mathbb{P}(\sum_{t=1}^{l} \phi_{t+i} \geq \nu_i | \mathcal{F}_t) \geq p, \]
for \( \nu_t \) depending on \( \mathcal{F}_t \). If \( \mathbb{P}(\min_t \nu_t \geq \nu) \geq 1 - \delta \) for \( \nu > 0 \) that depends on \( \delta \), then
\[ \mathbb{P}\left( \sum_{t=1}^{T} \phi_t^2 \leq \frac{\nu_t \theta^2}{8} l[T/l] \right) \leq \exp\left( -\frac{3[T/l] p}{4} \right) + \delta. \]

Proof. We begin with partitioning \( Z_1, \ldots, Z_T \) into \( S := \lfloor T/l \rfloor \) blocks of size \( l \). Consider the random variables
\[ B_j = 1 \left( \sum_{i=1}^{l} \phi_{jl+i}^2 \geq \frac{\nu_j \theta^2}{2} \right) \quad \text{for} \ 0 \leq j \leq S - 1. \]

We observe that
\[ \mathbb{P}\left( \sum_{t=1}^{T} \phi_t^2 \leq \frac{\nu_t \theta^2}{8} l[T/l] \right) = \mathbb{P}\left( \sum_{t=1}^{T} \phi_t^2 \leq \frac{\nu_t \theta^2}{8} l[T/l] \right) \cap \{ \min_t \nu_t \geq \nu \} \]
\[ + \mathbb{P}\left( \sum_{t=1}^{T} \phi_t^2 \leq \frac{\nu_t \theta^2}{8} l[T/l] \right) \cap \{ \min_t \nu_t < \nu \} \]
\[ \leq \mathbb{P}\left( \sum_{t=1}^{T} \phi_t^2 \leq \frac{\nu_t \theta^2}{8} l[lS] \right) \cap \{ \min_t \nu_t \geq \nu \} + \mathbb{P}(\min_t \nu_t < \nu) \]
\[ \leq \mathbb{P}\left( \sum_{t=1}^{T} \phi_t^2 \leq \frac{\nu_t \theta^2}{8} kS \right) + \delta. \]

Using Chernoff bound, we obtain that
\[ \mathbb{P}\left( \sum_{t=1}^{T} \phi_t^2 \leq \frac{\nu_t \theta^2}{8} kS \right) \leq \mathbb{P}\left( \sum_{j=0}^{S-1} \sum_{i=1}^{l} \phi_{jl+i}^2 \leq \frac{\nu_j \theta^2}{8} lS \right) = \mathbb{P}\left( \sum_{j=0}^{S-1} \sum_{i=1}^{l} \phi_{jl+i}^2 \leq \frac{\nu_j \theta^2}{8} lS \right). \]
\[
\leq \mathbb{P}\left( \sum_{j=0}^{S-1} B_j \leq \frac{P}{2} \right) \leq \inf_{\lambda \leq 0} e^{-\frac{pS}{2}} \mathbb{E}[e^{\lambda \sum_{j=0}^{S-1} B_j}],
\]

where the second to the last inequality uses the fact that \( \frac{\nu^2 P}{2} B_j \leq \sum_{t=1}^{l} \phi_{jl+i}^2 \). Further, we have that
\[
\mathbb{E}[B_j | \mathcal{F}_{jl}] = \mathbb{P}\left( \sum_{i=1}^{l} \phi_{jl+i}^2 \geq \frac{\nu^2 P}{2} | \mathcal{F}_{jl} \right) \geq \mathbb{P}\left( \frac{1}{l} \sum_{i=1}^{l} \left( \phi_{jl+i} \geq \nu_j \right) \geq \frac{P}{2} | \mathcal{F}_{jl} \right)
\]
\[
\geq \frac{P}{2},
\]

where the first inequality uses the fact that \( \frac{1}{\nu_j^2} \phi_{jl+i}^2 \geq 1 \{ \phi_{jl+i} \geq \nu_j \} \) and the last inequality uses the fact that for a random variable \( X \) supported on \([0, 1]\) almost surely such that \( \mathbb{E}[X] \geq p \) for some \( p \in (0, 1) \), then for all \( t \in [0, p] \), \( \mathbb{P}(X \geq t) \geq \frac{p}{t} \). This is true because
\[
\mathbb{P}(X \geq t) = \int_t^1 d\mathbb{P}(x) = \int_t^1 x d\mathbb{P}(x) = \int_0^1 x d\mathbb{P}(x) - \int_0^t x d\mathbb{P}(x) = p - t (1 - \mathbb{P}(X \geq t)).
\]

In our case, \( \mathbb{E}\left[ \frac{1}{l} \sum_{i=1}^{l} \left( \phi_{jl+i} \geq \nu_j \right) \mathcal{F}_{jl} \right] = \frac{1}{l} \sum_{i=1}^{l} \mathbb{P}\left( \phi_{jl+i} \geq \nu_j | \mathcal{F}_{jl} \right) \geq p \). Thus, we obtain that for \( \lambda \leq 0 \), i.e., \( e^\lambda \leq 1 \),
\[
\mathbb{E}[e^{\lambda B_j} | \mathcal{F}_{jl}] = e^{\lambda \mathbb{P}(B_j = 1 | \mathcal{F}_{jl})} + \mathbb{P}(B_j = 0) = (e^\lambda - 1) \mathbb{E}[B_j | \mathcal{F}_{jl}] + 1 \leq (e^\lambda - 1)^{\frac{P}{2} + 1}.
\]

By law of iterated expectation, we obtain that
\[
\mathbb{E}[e^{\lambda \sum_{j=0}^{S-1} B_j}] = \mathbb{E}\left[ e^{\lambda \sum_{j=0}^{S-2} B_j} \mathbb{E}[e^{\lambda B_j} | \mathcal{F}_{(S-1)k}] \right] \leq \left( (e^\lambda - 1)^{\frac{P}{2} + 1} \right) \mathbb{E}\left[ e^{\lambda \sum_{j=0}^{S-2} B_j} \right] \leq \left( (e^\lambda - 1)^{\frac{P}{2} + 1} \right)^{S}.
\]

Finally, we need to find
\[
\inf_{\lambda \leq 0} e^{-\frac{pS}{2}} \left( (e^\lambda - 1)^{\frac{P}{2} + 1} \right)^{S}.
\]

We can see that \( \lambda^* = -\infty \), which gives that
\[
\inf_{\lambda \leq 0} e^{-\frac{pS}{2}} \left( (e^\lambda - 1)^{\frac{P}{2} + 1} \right)^{S} = e^{-\frac{pS}{2}} \left( 1 - \frac{P}{2} \right)^{S} \leq e^{-\frac{pS}{2}} e^{-\frac{pS}{2}} = e^{-3pS/4},
\]

where we have used the fact that \( 1 + x \leq e^x \) for all real-valued \( x \). \( \square \)

To apply Theorem 10, we first recall that the affine dynamical system we aim to identify is as follows:
\[
\tilde{x}_{k,j+1} = a_k \tilde{x}_{k,j} + d_k + \tilde{z}_{k,j},
\]
where \( \tilde{x}_{k,1} = 0 \), \( a_k \in (0, 1) \) and \( \tilde{z}_{k,j} \sim \mathcal{N}(0, \sigma^2 z_{k,j}) \). We define the following quantities
\[
\Gamma_{k,j} := \sigma^2 z_{k,j} \sum_{i=0}^{j-1} a_k^{2i}, \quad d_{k,j} := \sum_{i=0}^{j-1} a_k^i d_k,
\]
and \( \Gamma_{k,\infty} = \sigma^2 z_{k,\infty} \sum_{i=0}^{\infty} a_k^{2i} = \frac{\sigma^2 z_{k,j}}{1 - a_k^2} \). We notice that for all \( t \in [T], j \geq 1 \),
\[
\tilde{x}_{k,t+j} | \tilde{x}_{k,t} \sim \mathcal{N}\left( a_k^j \tilde{x}_{k,t} + d_{k,j}, \Gamma_{k,j} \right).
\]

31
Lemma 10. Fix \( t \geq 0 \) and \( j \geq 1 \). Recall that \( \mathbf{x}_{k,t} := (\tilde{x}_{k,t}, 1) \in \mathbb{R}^2 \). Fix a unit vector \( w \in \mathbb{R}^2 \). For any \( \epsilon \in (0, 1) \), we have
\[
P \left( |\langle w, \mathbf{x}_{k,t+j} \rangle| \geq \frac{1}{\sqrt{2}} \sqrt{\min \left\{ 1 - \epsilon, \Gamma_{k,j} - \left( \frac{1}{\epsilon} - 1 \right) (a_k^j \tilde{x}_{k,t} + d_{k,j})^2 \right\}} \right) \geq \frac{1}{36}
\]

Proof. By Lemma 8, we have that for any unit vector \( w \in \mathbb{R}^2 \),
\[
P \left( |\langle w, \mathbf{x}_{k,t+j} \rangle| \geq \frac{1}{\sqrt{2}} \sqrt{(w_1 \Gamma_{k,j} (a_k^j \tilde{x}_{k,t} + d_{k,j})^2 + w_2^2 \Gamma_{k,j})} \right) \geq \frac{1}{36}.
\]
For all \( \epsilon \in (0, 1) \), we have
\[
(w_1 (a_k^j \tilde{x}_{k,t} + d_{k,j}) + w_2^2 \Gamma_{k,j})^2 + w_2^2 \Gamma_{k,j} = \left( w_1 \left( a_k^j \tilde{x}_{k,t} + d_{k,j} \right) \right)^2 + w_2^2 + 2w_2 w_1 \left( a_k^j \tilde{x}_{k,t} + d_{k,j} \right) + w_2^2 \Gamma_{k,j}
\]
\[
\geq (1 - \epsilon) w_2^2 - \left( \frac{1}{\epsilon} - 1 \right) \left( w_1 \left( a_k^j \tilde{x}_{k,t} + d_{k,j} \right) \right)^2 + w_2^2 \Gamma_{k,j}
\]
\[
\geq \min \left\{ 1 - \epsilon, \Gamma_{k,j} - \left( \frac{1}{\epsilon} - 1 \right) (a_k^j \tilde{x}_{k,t} + d_{k,j})^2 \right\}.
\]

\( \square \)

Lemma 11. Fix \( \delta \in (0, 1) \). \( \{\mathbf{x}_{k,t}\}_{t=1}^{n} \) satisfy that for any unit vector \( w \in \mathbb{R}^2 \),
\[
P \left( \sum_{t=1}^{n} |\langle w, \mathbf{x}_{k,t} \rangle|^2 \leq \frac{\psi^2 n^2}{16} j_* \frac{n}{j_*} \right) \leq \exp \left( -\frac{3 |n/j_*|^p}{4} \right) + \delta
\]
with \( p = 1/72 \),
\[
j_* := \max \left\{ -\log_{a_k} \left( 1 + (1 - a_k) \sqrt{2 \Gamma_{k,\infty} \log(n/\delta)} \right), -\log_{a_k} \sqrt{2} \right\},
\]
\[
\psi := \min \left\{ \frac{\Gamma_{k,\infty}}{16d_k^2 (1-a_k)^2} + \frac{\Gamma_{k,\infty}}{4}, \frac{\Gamma_{k,\infty}}{4} \right\}.
\]

Proof. Fix \( \delta \in (0, 1) \). Recall that from Lemma 9 we have shown that for all \( t \geq 0 \) and \( k \geq 1 \), given a unit vector \( w \in \mathbb{R}^2 \), for any \( \epsilon \in (0, 1) \), we have
\[
P \left( |\langle w, \mathbf{x}_{k,t+j} \rangle| \geq \frac{1}{\sqrt{2}} \sqrt{\min \left\{ 1 - \epsilon, \Gamma_{k,j} - \left( \frac{1}{\epsilon} - 1 \right) (a_k^j \tilde{x}_{k,t} + d_{k,j})^2 \right\}} \right) \geq \frac{1}{36}.
\]
Denote \( q_{t,j} = a_k^j \tilde{x}_{k,t} + d_{k,j} \) where \( \tilde{x}_{k,t} \sim \mathcal{N}(d_{k,t}, \Gamma_{k,t}) \). Fix \( \delta \in (0, 1) \). Using the standard Gaussian tail bound and the union bound, we have that with probability \( 1 - \delta \),
\[
\max_{t \in [T]} q_{t,j} \leq a_k^j \left( \frac{d_k}{1-a_k} + \sqrt{2 \Gamma_{\infty} \log(n/\delta)} \right) + \frac{d_k}{1-a_k}.
\]

32
When \( j \geq j^* \), \( \Gamma_{k,j} \geq \Gamma_{k,\infty}/2 \), and with probability \( 1 - \delta \), \( \max_{t \in [T]} q_{t,j} \leq \frac{2d_k}{1-a_k} \). Thus, for \( j \geq j^* \), and

\[
\epsilon = \frac{4d_k^2}{(1-a_k)^2} + \Gamma_{k,\infty}/4,
\]

we have

\[
\nu_{t,j}^2 := \min \left\{ 1 - \epsilon, \Gamma_{k,j} - \left( \frac{1}{\epsilon} - 1 \right) q_{t,j}^2 \right\} \\
\geq \min \left\{ 1 - \epsilon, \Gamma_{k,\infty}/2 - \left( \frac{1}{\epsilon} - 1 \right) \frac{4d_k^2}{(1-a_k)^2} \right\} \\
\geq \min \left\{ \frac{\Gamma_{k,\infty}}{16d_k^2 + \Gamma_{k,\infty}/4}, \Gamma_{k,\infty} \right\} = \psi^2.
\]

Putting it altogether, we have

\[
\frac{1}{2j^*} \sum_{j=1}^{2j^*} P\left( |\langle w, \pi_{k,t+j} \rangle| \geq \nu_{t,j} / \sqrt{2} |F_t| \right) \geq \frac{1}{2j^*} \sum_{j=j^*}^{2j^*} \text{Pr}( |\langle w, \pi_{k,t+j} \rangle| \geq \nu_{t,j} / \sqrt{2} |F_t| ) \geq \frac{1}{72}.
\]

Further, we have

\[
P\left( \min_{t \in [T]} \nu_{t,j}^2, \psi^2 \right) \geq 1 - \delta.
\]

Applying Lemma 9, we have that for \( p = \frac{1}{72} \),

\[
P\left( \sum_{t=1}^{n} \langle w, \pi_{k,t} \rangle^2 \leq \frac{\psi^2 p^2}{16} j^* [n/j^*] \right) \leq \exp\left( -\frac{3[n/j^*]p}{4} \right) + \delta.
\]

**Proof of Theorem 8** Based on our setup, condition (a) of Theorem 10 is satisfied. For any \( n \), using Lemma 11 with \( \delta = \exp(-n) \), we have that

\[
\forall w \in \mathbb{R}^2, \quad P\left( \sum_{t=1}^{n} \langle w, \pi_{k,t} \rangle^2 \leq \frac{\psi^2 p^2}{16} j^* [n/j^*] \right) \leq \exp\left( -\frac{3[n/j^*]p}{4} \right) + \delta \leq 2 \exp\left( -\frac{3[n/j^*]p}{4} \right),
\]

with \( p = 1/72 \),

\[
\psi := \min \left\{ \frac{\Gamma_{k,\infty}}{16d_k^2 + \Gamma_{k,\infty}/4}, \frac{\Gamma_{k,\infty} \sqrt{\log(n) + n}}{d_k} \right\},
\]

Thus, we have provided a similar result to (18), which is what condition (b) of Theorem 10 is used for. In this case, we have \( \Gamma_{ab} = \psi I \) where \( I \) is a 2 \( \times \) 2 identity matrix. Finally, to verify condition (c), we notice that we have

\[
\Gamma_{k,j} := E[\pi_{k,j} \pi_{k,j}^\top] = \left( \frac{6d_k^2}{(1-a_k)^2} + \frac{a_k^2(1-a_k^2-2)}{1-a_k} \frac{(1-a_k^{-1})b_k}{1-a_k} \right).
\]

33
and we denote
\[ \Gamma := \Gamma_{k,n} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + \Gamma_{sb}, \]
which gives that \( 0 \prec \Gamma_{sb} \prec \Gamma \) and for all \( j \geq 1 \), \( 0 \preceq \Gamma_{k,j} \prec \Gamma \). Then, we have that
\[
P \left( \mathbf{X}_k^\top \mathbf{X}_k \not\preceq \frac{2n}{\delta} \Gamma \right) = P \left( \lambda_{\text{max}} \left( \left( n \Gamma \right)^{-1/2} \mathbf{X}_k^\top \mathbf{X}_k \left( n \Gamma \right)^{-1/2} \right) \geq \frac{2}{\delta} \right)
\[
\leq \frac{\delta}{2} \mathbb{E} \left[ \lambda_{\text{max}} \left( \left( n \Gamma \right)^{-1/2} \mathbf{X}_k^\top \mathbf{X}_k \left( n \Gamma \right)^{-1/2} \right) \right]
\[
\leq \frac{\delta}{2} \mathbb{E} \left[ \text{tr} \left( \left( n \Gamma \right)^{-1/2} \mathbf{X}_k^\top \mathbf{X}_k \left( n \Gamma \right)^{-1/2} \right) \right] \leq \delta,
\]
where the last inequality is true since \( \mathbb{E} \left[ \mathbf{X}_k^\top \mathbf{X}_k \right] = \sum_{j=1}^n \Gamma_{k,j} \preceq n \Gamma \) (for all \( j \in [n] \), trace(\( \Gamma - \Gamma_{k,j} \)) > 0 and det(\( \Gamma - \Gamma_{k,j} \)) > 0). Following Theorem 10, for \( \delta \in (0,1) \), when the number of samples satisfy that
\[
\frac{n}{J^*} \geq \frac{10}{\delta^2} \left( \log (1/\delta) + 4 \log(10/p) + \log \text{det}(\Gamma_{sb}^{-1}) \right),
\]
we have that
\[
P \left( \| \hat{A}_k - A_k \|_2 > \frac{9\sigma_{z,k}}{p} \sqrt{1 + 2 \log(10/p) + \log \text{det}(\Gamma_{sb}^{-1}) + \log(1/\delta)} \right) \leq 3\delta.
\]

\[\square\]

**D.2 Proof of Corollary 4 and Corollary 9**

Similar to Appendix D.1, Corollary 4 is a special case of Corollary 9 when \( m = 1 \). Hence, we directly present the proof of Corollary 9 below.

**Proof of Corollary 9**

Fix \( \delta \in (0,1) \). We have that with probability \( 1 - \delta \), \( \epsilon(n, \delta, k) := \| \hat{A}_k - A_k \|_2 \leq O(1/\sqrt{n}) \). With probability at least \( 1 - \frac{\delta}{\kappa} \), \( \epsilon_{ak} := |\hat{a}_k - a_k| \leq \| \hat{A}_k - A_k \|_2 = \epsilon(n, \delta/K, k) = O(1/\sqrt{n}) \) and \( \epsilon_{dk} := |\hat{d}_k - b_k| \leq \| \hat{A}_k - A_k \|_2 = \epsilon(n, \delta/K, k) = O(1/\sqrt{n}) \). When \( m = 1 \), then \( |\hat{\gamma}_k - \gamma_k| = |\hat{a}_k - a_k| \leq |\hat{a}_k - a_k| = \epsilon_{ak} \leq \epsilon(n, \delta/K, k) \). When \( m \geq 2 \), since \( \gamma_k \neq 0 \), we have that
\[
|\hat{\gamma}_k - \gamma_k| = \left| \frac{|\hat{a}_k - a_k|}{\gamma_k} \right| < \left| \frac{|\hat{a}_k - a_k|}{\gamma_k^{(m-1)/m} |\hat{a}_k|^{(m-2)/m} \gamma_k^{(m-3)/m} \cdots + \gamma_k^{m-1}} \right| \leq \frac{|\hat{a}_k - a_k|}{\gamma_k^{m-1}}.
\]
On the other hand, we obtain that
\[
|\hat{\lambda}_k - \lambda_k| = \left| \frac{\hat{d}_k}{\hat{a}_k} - \frac{d_k}{a_k} \right| \leq \frac{\hat{d}_k}{\hat{a}_k} - \frac{d_k}{a_k} + \frac{d_k}{a_k} - \frac{d_k}{a_k} \leq \frac{\epsilon_{dk}}{\hat{a}_k} + \frac{\lambda_k \epsilon_{ak}}{a_k} \leq O \left( \frac{1}{\sqrt{n}} \right).
\]
The proof completes as follows:
\[
P \left( \forall k \in [K], |\hat{\gamma}_k - \gamma_k| \leq O(1/\sqrt{n}), |\hat{\lambda}_k - \lambda_k| \leq O(1/\sqrt{n}) \right) \geq \prod_{k=1}^K \left( 1 - \frac{\delta}{\kappa} \right) \geq 1 - \delta,
\]
where the last inequality follows from Bernoulli’s inequality. \(\square\)
E Additional Proofs and Discussion of Section 6

E.1 Proof of Theorem 5

Lemma 12. Consider any episode $i + 1$ (from time $t_i + 1$ to $t_{i+1}$) where the initial state $x^t = (\mu_{1:t_i+1}(u_{1:t_i}), n_{1:t_i}, \ldots, \mu_{K:t_i+1}(u_{K:t_i}), n_{K:t_i})$ and $\{u_{k,0,t_i}\}_{k=1}^K$ are the past pull sequences of the proposed policy $\pi_{1:t_i}$. For all $\pi_{t_i+1:t_{i+1}}$ such that $\pi_t = \pi_t(x_t) = \pi_t(x')$, $\pi_t \in [K], \forall t \in [t_i + 1, t_{i+1}], x_t, x'_t \in \mathcal{X}$, we have that

$$\sum_{t=t_i+1}^{t_{i+1}} \mathbb{E}_{x_{t_i+2}, \ldots, x_t} \left[ r(x_t, \pi_t(x_t)) | x_{t_i+1} = x^t \right] = \sum_{t=t_i+1}^{t_{i+1}} \mu_{k,t}(u_{k,0,t-1}),$$

where $\{u_{k,0,t+1:t_{i+1}}\}_{k=1}^K$ is the arm pull sequence of $\tilde{\pi}_{t_i+1:t_{i+1}}$.

Proof. Let $k$ denote $\tilde{\pi}_t$, where $t \in \{t_i + 1, \ldots, t_{i+1}\}$. Recall that we use $u_{k,0,t-1}$ to denote the pull sequence of arm $k$ under policy $\tilde{\pi}_{t_i+1:t_{i+1}} = (\pi_{1:t_i}, \pi_{t_i+1:t_{i+1}})$. If $k$ has not been pulled before time $t$ by $\tilde{\pi}_{1:t_i}$, then $\mathbb{E}_{x_{t_i+2}, \ldots, x_t} \left[ r(x_t, \tilde{\pi}_t) | x_{t_i+1} = x^t \right] = b_{\pi_t} = \mu_{\pi_t}(\pi_{t_i+1:t_{i+1}})$. If $k$ has been pulled before, then let $q_1, \ldots, q_n$ denote the time steps that arm $k$ has been pulled before time $t$ by $\tilde{\pi}_{1:t_i}$, i.e., $u_{k,q_i} = 1$ for $i \in [n]$ and $u_{k,t'} = 0$ for $t' \notin \{q_1, \ldots, q_n\}$. We have that for $t \in \{t_i + 1, \ldots, t_{i+1}\}$,

$$\mathbb{E}_{x_{t_i+2}, \ldots, x_{t_i+1}} \left[ r(x_t, \tilde{\pi}_t) | x_{t_i+1} = x^t \right] = b_k - \left( \mathbb{E}_{x_{t_i+2}, \ldots, x_{t_i+1}-1} \left[ \mathbb{E}_{x_{t_i+1}} \left[ \gamma_k n_{k,t_{i+1}} x_{k,t_{i+1}} + \lambda_k n_{k,t_{i+1}} | x_{t_i+1} = x^t \right] \right] \right)$$

$$= b_k - \left( \mathbb{E}_{x_{t_i+2}, \ldots, x_{t_i+n}} \left[ \mathbb{E}_{x_{t_i+n+1}} \left[ \gamma_k n_{k,t_{i+1}} x_{k,q_{n+1}} + \lambda_k n_{k,t_{i+1}} | x_{t_{i+1}} = x^t \right] \right] \right)$$

$$= b_k - \left( \mathbb{E}_{x_{t_i+2}, \ldots, x_{t_i+n}} \gamma_k n_{k,t_{i+1}} (\gamma_k n_{k,q_n} x_{k,q_n} + \lambda_k n_{k,q_n}) + \lambda_k \gamma_k n_{k,t_{i+1}} | x_{t_{i+1}} = x^t \right)$$

$$= \ldots = b_k - \lambda_k (\gamma_k n_{k,t_{i+1}} + \gamma_k n_{k,t_{i+1}} + \gamma_k n_{k,q_n} + \ldots + \gamma_k n_{k,t_{i+1}} + \gamma_k n_{k,t_{i+1}} + \gamma_k n_{k,q_n})$$

$$= \mu_{k,t}(u_{k,0,t-1}),$$

where the second equality is true because when arm $k$ is not pulled for example at time $t_{i+1} - 1$, the state for arm $k$ at time $t_{i+1} - 1$ will satisfy that $x_{k,t_{i+1}} = x_{k,t_{i+1}-1}$ and $n_{k,t_{i+1}} = n_{k,t_{i+1}-1} + 1$ with probability 1. In this case, we have that

$$\mathbb{E}_{x_{t_i+1}} \left[ \gamma_k n_{k,t_{i+1}} x_{k,t_{i+1}} + \lambda_k n_{k,t_{i+1}} | x_{t_i+1} \right] = \gamma_k n_{k,t_{i+1}} + \lambda_k n_{k,t_{i+1}} + 1$$

$$= \gamma_k n_{k,t_{i+1}} + \lambda_k n_{k,t_{i+1}}.$$
Proof. By Lemma 12, we have that the optimal time-dependent competitor policy \( \tilde{\pi}_{t+1} \) maximizes \( \sum_{t=0}^{t_{t+1}} \mu_{k,t}(u_{k,0:t-1}) \), by choosing \( u_{k,t+1:t_{t+1}} \). Thus, by the definition of Lookahead [3], given our proposed policy \( \pi_{t+1} \), the optimal time-dependent competitor policy is given by \( \text{Lookahead}(\{\lambda_k, \gamma_k, b_k\}_{k=1}^K, \{u_{k,0:t}\}_{k=1}^K, t, t_{t+1}) \). □

Proof of Theorem [4] Using Lemma 13, we have that given our policy \( \pi_{t+1} \) and its corresponding pull sequence \( u_{k,0:t} \) for \( k \in [K] \), \( t \in [T] \), the optimal competitor policy for episode \( i+1 \) where \( i \in \{0, \ldots, [T/w]\} \) (episode \( i+1 \) ranges from time \( t_{i+1} = iw + 1 \) to \( t_{i+2} = \min\{iw + w, T\} \)) is given by \( \text{Lookahead}(\{\lambda_k, \gamma_k, b_k\}_{k=1}^K, \{u_{k,0:t} \}_{k=1}^K, t, t_{i+1}) \). We use \( M(\{\lambda_k, \gamma_k, b_k\}_{k=1}^K, \{u_{k,0:t} \}_{k=1}^K, t, t_{i+1}) \) to denote the (optimal) objective value of (5) given by \( \text{Lookahead}(\{\lambda_k, \gamma_k, b_k\}_{k=1}^K, \{u_{k,0:t} \}_{k=1}^K, t, t_{i+1}) \). Denote \( \bar{b} = \max_k b_k \) and \( \underline{b} = \min_k b_k \).

Exploration Stage Recall that in Algorithm 1, we have defined \( \hat{T} = T^{2/3} + w - (T^{2/3} \mod w) \) which is a multiple of \( w \). For the first \( \hat{T} \) time steps, as defined in Algorithm 1, our policy \( \pi_{i,T} \) is a time-dependent policy, i.e., it satisfies that \( \pi_t = \pi_t(x_t) = \pi_t(x_t') \), \( \forall t \in [K], \forall x_t, x_t' \in X \). Using 12, we obtain that the regret for the first \( \hat{T}/w \) episodes is given by

\[
\begin{align*}
\hat{T}/w - 1 & \sum_{i=0}^{\hat{T}/w-1} \max_{\pi_{1,w} \in C_w} \mathbb{E} \left[ \sum_{j=1}^{w} r(x_{iw+j}, \tilde{\pi}_j(x_{iw+j})) \bigg| x_{iw+1} = x^i \right] \\
& - \sum_{i=0}^{\hat{T}/w-1} \mathbb{E} \left[ r(x_{iw+j}, \pi_{iw+j}(x_{iw+j})) \bigg| x_{iw+1} = x^i \right] \\
& \leq \sum_{i=0}^{\hat{T}/w-1} M(\{\lambda_k, \gamma_k, b_k\}_{k=1}^K, \{u_{k,0:iw} \}_{k=1}^K, iw, iw + w) - \hat{T} (\underline{b} + \frac{\lambda_T}{\frac{1}{1-\pi}}) \\
\end{align*}
\]

since \( \hat{T} \leq T^{2/3} + w \) and by assumption, \( w \leq T^{2/3} \).

Estimation Stage By Theorem [3] and Corollary [4], we have that for any \( \delta \in (0,1) \) and \( n \geq n_0(\delta, k) \) where \( n_0(\delta, k) \) depends on \( \delta \) logarithmically, with probability \( 1 - \delta \), for all \( k \in [K] \)

\[ |\hat{\gamma}_k - \gamma_k| \leq \frac{C_{\text{log}(1/\delta)}}{\sqrt{n}} \quad \text{and} \quad |\hat{\lambda}_k - \lambda_k| \leq \frac{C_{\text{log}(1/\delta)}}{\sqrt{n}} \text{ when } \gamma_k > 0. \]

We define two numbers \( T_0' := \min_{\tau} \left\{ \tau : (\sum_{k=1}^K n_0(k, \tau^{-1/3}))^{3/2} = C_1 K (\log T)^{3/2} < T \right\} \) and \( T_0'' := \min_{\tau} \left\{ \tau : \max_k \gamma_k + \frac{C_{\text{log}(1/\delta)}}{\sqrt{T^{1/3}}} < 1 \right\} \). These two numbers exist as \( T \) can be chosen to be arbitrarily large. Take \( T_0 = \max\{T_0', T_0''\} \). Then for all \( T \geq T_0 \), with probability \( 1 - \delta \) where \( \delta = T^{-1/3} \), we have that \( \forall k \in [K], |\hat{\gamma}_k - \gamma_k| \leq \epsilon_\gamma = O(\sqrt{K}T^{-1/3} \log T), |\hat{\lambda}_k - \lambda_k| \leq \epsilon_\lambda = O(\sqrt{K}T^{-1/3} \log T) \) and \( (\epsilon_\lambda + \hat{\gamma}_k - \gamma_k + \frac{C_{\text{log}(1/\delta)}}{\sqrt{T_0^{1/3}}} < 1 \text{ and } \gamma_k \leq \gamma_k < 1 \).

For any pull sequence \( u_{k,0:t-1} \), using our obtained estimated parameters \( \{\hat{\gamma}_k, \hat{\lambda}_k, b_k\}_{k=1}^K \), we define the estimated reward function: for \( t \geq 2, \hat{\mu}_{k,t}(u_{k,0:t-1}) = b_k - \lambda_k \left( \sum_{i=1}^{t-1} \hat{\gamma}_k^{t-i} u_{k,i} \right), \) and for \( t = 1, \hat{\mu}_{k,1}(u_{k,0:1}) = b_k = \mu_{k,1}(u_{k,0:1}) \), where we note that \( \hat{b} = b_k \) since it is the reward of the first pull of arm \( k \). Given \( t \geq 2 \), we have that

\[ |\mu_{k,t}(u_{k,0:t-1}) - \hat{\mu}_{k,t}(u_{k,0:t-1})| \]
where

\[ \begin{align*}
\lambda_k &= \left( \sum_{i=1}^{t-1} \gamma_k^{t-i} u_{k,i} \right) - \lambda_k \left( \sum_{i=1}^{t-1} \gamma_k^{t-i} u_{k,i} \right) \\
&= |\tilde{\lambda}_k - \lambda_k| \\
&\leq \epsilon \left| \frac{\tilde{\gamma}_k}{1 - \tilde{\gamma}_k} \right| + \epsilon \gamma \left| \frac{\lambda}{(1 - \tilde{\gamma}_k)(1 - \gamma_k)} \right|.
\end{align*} \]

(19)

**Planning Stage** Given our policy \( \pi_{1:T} \) (along with its pull sequence \( \{u_{k,0:T}\}_{k=1}^K \)), starting from time \( T+1 \), for any episode \( i+1 \geq T/w \), we denote the optimal competitor policy to be \( \pi_{t_i+1:t_i+1} = \text{Lookahead}(\{\lambda_k, \gamma_k, b_k\}_{k=1}^K, \{u_{k,0:t_i}\}_{k=1}^K, t_i, t_i+1) \) where \( t_i = iw \) and \( t_i+1 = \min\{iw+w, T\} \). The cumulative expected reward collected by \( \pi_{t_i+1:t_i+1} \) and \( \pi_{1:t_i+1} \) has the difference

\[ M(\{\pi_{1:T}\}_{k=1}^K, \{u_{k,0:t_i}\}_{k=1}^K, t_i, t_i+1) = \sum_{t=t_i+1}^{t_i+1} \mu_{\pi_i,t}(u_{\pi_i,0:t_i-1}) - \sum_{t=t_i+1}^{t_i+1} \mu_{\pi_i,t}(u_{\pi_i,0:t_i-1}) \]

\[ = \sum_{t=t_i+1}^{t_i+1} \mu_{\pi_i,t}(u_{\pi_i,0:t_i-1}) - \sum_{t=t_i+1}^{t_i+1} \mu_{\pi_i,t}(u_{\pi_i,0:t_i-1}) \]

\[ + \sum_{t=t_i+1}^{t_i+1} \tilde{\mu}_{\pi_i,t}(u_{\pi_i,0:t_i-1}) - \sum_{t=t_i+1}^{t_i+1} \mu_{\pi_i,t}(u_{\pi_i,0:t_i-1}) \]

\[ + \sum_{t=t_i+1}^{t_i+1} \tilde{\mu}_{\pi_i,t}(u_{\pi_i,0:t_i-1}) - \sum_{t=t_i+1}^{t_i+1} \mu_{\pi_i,t}(u_{\pi_i,0:t_i-1}). \]

where \( u_{\pi_i,0:t_i-1} \) is the corresponding pull sequence of arm \( \pi_i \) under policy \( \pi_{1:t_i} = (\pi_{1:t_i}, \pi_{t_i+1:t_i+1}) \), and the last inequality holds because \( \pi_{t_i+1:t_i+1} = \text{Lookahead}(\{\tilde{\lambda}_k, \tilde{\gamma}_k, \tilde{b}_k\}_{k=1}^K, \{u_{k,0:t_i}\}_{k=1}^K, t_i, t_i+1) \) is the optimal solution under the estimated parameters \( \{\tilde{\lambda}_k, \tilde{\gamma}_k, \tilde{b}_k\}_{k=1}^K \) and \( \pi \)'s previous past pull sequence \( \{u_{k,0:t_i}\}_{k=1}^K \). Further, using (19) and the fact that \( t_i - t_i-1 \leq w \), we obtain that

\[ \sum_{t=t_i+1}^{t_i+1} \mu_{\pi_i,t}(u_{\pi_i,0:t_i-1}) - \sum_{t=t_i+1}^{t_i+1} \tilde{\mu}_{\pi_i,t}(u_{\pi_i,0:t_i-1}) \]

\[ + \sum_{t=t_i+1}^{t_i+1} \tilde{\mu}_{\pi_i,t}(u_{\pi_i,0:t_i-1}) - \sum_{t=t_i+1}^{t_i+1} \mu_{\pi_i,t}(u_{\pi_i,0:t_i-1}) \]

\[ \leq 2w \max_k \left( \epsilon \lambda \left| \frac{\tilde{\gamma}_k}{1 - \tilde{\gamma}_k} \right| + \epsilon \gamma \left| \frac{\lambda}{(1 - \tilde{\gamma}_k)(1 - \gamma_k)} \right| \right). \]

Finally, putting it altogether, we have obtained that for all \( T \geq T_0 \),

\[ \text{Reg}^w(T) = \sum_{i=0}^{T/w-1} \max_{\bar{\pi}_{1:w} \in \mathcal{C}^w} \mathbb{E} \left[ \sum_{j=1}^{\min\{w,T-w\}} r(x_{iw+j}, \bar{\pi}_j(x_{iw+j})) \right] x_{iw+1} = x_i \]
− \mathbb{E} \left[ \sum_{j=1}^{\min\{w, T-iw\}} r(x_{iw+j}, \pi_{iw+j}(x_{iw+j})) \bigg| x_{iw+1} = x_i \right] \\
\leq O(T^{2/3}) + (1 - T^{-1/3}) \left( \sum_{i=T/w}^{[T/w]-1} 2wO(\sqrt{KT}T^{-1/3} \log T) \right) + T^{-1/3} \left( T \left( \bar{b} - \bar{b} + \frac{\lambda}{1-\gamma} \right) \right) \\
\leq O(T^{2/3}) + (T - T^{2/3})O(\sqrt{KT}T^{-1/3} \log T) + O(T^{2/3}) \\
\leq O(\sqrt{KT}T^{2/3} \log T),

which we notice that with probability \( \delta = T^{-1/3} \), the cumulative expected reward from time \( \tilde{T} \) to \( T \) between the optimal competitor policy and our policy \( \pi \) is at most \( T \left( \bar{b} - \bar{b} + \frac{\lambda}{1-\gamma} \right) \). This completes the proof. \( \square \)

E.2 Exploration Strategies

In the exploration phase of Algorithm 1 (from time 1 to \( \tilde{T} \)), in addition to playing each arm repeatedly for \( \tilde{T}/K \) times, in general, we could explore by playing each arm at a fixed interval, i.e., the time interval between two consecutive pulls of arm \( k \) should be a constant \( m_k \). For example, this includes playing the arms cyclically with the cycle being 1, 2, \ldots, \( K \) or playing the first two arms in an alternating fashion from time 1 to 2\( \tilde{T}/K \), then the next two arms, etc. As shown in Theorem 8 and Corollary 9, using the datasets (of size \( n \)) collected by these exploration strategies, we can obtain estimators \( \hat{\gamma}_k \) and \( \hat{\lambda}_k \) with the estimation error being on the order of \( O(1/\sqrt{n}) \). Using these results (in replacement of Theorem 3 and Corollary 4 in the estimation stage of the proof of Theorem 5), we can obtain that there exists \( T_0 \) such that for all \( T \geq T_0 \), the regret upper bound of EEP under these exploration strategies are of order \( O(\sqrt{KT}T^{2/3} \log T) \).
F Additional Proofs of Appendix C

F.1 Proof of Corollary 7

Proof. Fix \( \delta \in (0, 1) \). By Theorem 4 for all \( k \in [K] \), with probability \( 1 - \frac{\delta}{2K} \), we have the following: When \( m = 1 \), then \( |\gamma_k - \gamma_k| = ||a_k| - a_k| \leq |a_k - a_k| \leq \epsilon_a(n, \frac{\delta}{2K}, k) \). When \( m \geq 2 \), we have that

\[
|\gamma_k - \gamma_k| = \frac{|a_k| - a_k}{|\gamma_k| (m-1)/m + |a_k| (m-2)/m \gamma_k + \ldots + \gamma_k^{m-1}} \leq \frac{|a_k - a_k|}{\gamma_k^{m-1}}.
\]

On the other hand, given that \( |a_k - a_k| \leq \epsilon_a(n, \frac{\delta}{2K}, k) \), we have that with probability \( 1 - \frac{\delta}{2K} \),

\[
|\gamma_k - \gamma_k| = \left| \frac{d_k}{a_k} - \frac{d_k}{a_k} \right| \leq \frac{d_k}{a_k} - \frac{d_k}{a_k} + \frac{d_k}{a_k} - \frac{d_k}{a_k} \leq \epsilon_d(n, \frac{\delta}{2K}, k) + \frac{\lambda_k \epsilon_a(n, \frac{\delta}{2K}, k)}{a_k} \leq O \left( \frac{1}{\sqrt{n}} \right).
\]

The proof completes as follows:

\[
P \left( \forall k, |\gamma_k - \gamma_k| \leq \frac{|a_k - a_k|}{\gamma_k^{m-1}}, |\gamma_k - \gamma_k| \leq \epsilon_a(n, \frac{\delta}{2K}, k) + \frac{\lambda_k \epsilon_a(n, \frac{\delta}{2K}, k)}{a_k} \leq O \left( \frac{1}{\sqrt{n}} \right). \right)
\]

where the last inequality follows from Bernoulli’s inequality.

F.2 Proof of Lemma 4

Proof. Let \( \pi_{t,T} \) denote the sequence that policy \( \pi \) will take from time 1 to \( T \). By the definition of the value function, we have that

\[
V_{t, A}^\pi(x_{init}) = b_{\pi_1} + \sum_{t=2}^{T} \mathbb{E}_{x_{t-1}, x_1}[r(x_t, \pi_t)],
\]

where \( x_1 \sim \pi_{t,T}(x_{t-1}, \pi_{t-1}, t-1) \) is a state vector drawn from the transition distribution defined in Section 4.1. Let \( k \) denote \( \pi_t \) and \( u_{k,0:t-1} \) denote the past pull sequence for arm \( k \) under policy \( \pi \). If \( k \) has not been pulled before time \( t \), then \( \mathbb{E}_{x_{t-1}, x_1}[r(x_t, \pi_t)] = b_{\pi_t} = \mu_{\pi_t, 1}(u_{k,0:t-1}, 1) \). If \( k \) has been pulled before, then let \( t_1, \ldots, t_n \) denote the time steps at which arm \( k \) has been pulled before time \( t \). We have that

\[
\mathbb{E}_{x_{t-1}, x_1}[r(x_t, k)] = b_k - \left( \mathbb{E}_{x_{t-1}, x_1}[r(x_t, k, t-1)] \right) = b_k - \left( \mathbb{E}_{x_{t-1}, x_1}[r(x_t, k, t-1)] \right);
\]

where we note that the second equality is true because when arm \( k \) is not pulled for example at time \( t-1 \), the state for arm \( k \) at time \( t-1 \) will satisfy that \( x_{k,t} = x_{k,t-1} \) and \( n_{k,t} = n_{k,t-1} + 1 \) with probability 1. In this case, we have that \( \mathbb{E}_{x_t \sim \pi_{t,T}(x_{t-1}, k, t-1)}[r_{k,t} x_{k,t} + \lambda_k \gamma_k^{n_{k,t}}] = \gamma_k^{n_{k,t}+1} + \lambda_k \gamma_k^{n_{k,t}+1} = \gamma_k^{n_{k,t}+1} + \lambda_k \gamma_k^{n_{k,t}+1} \). The third equality is true since when arm \( k \) is pulled for example at time \( t-1 \), then we have that \( \mathbb{E}_{x_t \sim \pi_{t,T}(x_{t-1}, k, t-1)}[r_{k,t} x_{k,t} + \lambda_k \gamma_k^{n_{k,t}}] = \gamma_k^{n_{k,t}+1} + \lambda_k \gamma_k^{n_{k,t}+1} + \lambda_k \gamma_k^{n_{k,t}} \). The proof completes by summing over \( \mathbb{E}_{x_{t-1}, x_1}[r(x_t, \pi_t)] \) for all \( t \geq 2 \).
F.3 Proof of Proposition 5

Proof. Fix δ ∈ (0, 1). Let E1 be the event that

\[ \forall k \in [K], |\hat{\gamma}_k - \gamma_k| = \epsilon_{\gamma_k} \leq O \left( \frac{1}{\sqrt{n}} \right), \ |\hat{\lambda}_k - \lambda_k| = \epsilon_{\lambda_k} \leq O (1/\sqrt{n}) . \]

From Corollary 7 we have that \( \mathbb{P}(E_1) \geq 1 - \delta \). Let \( \pi_{1:T} \) denote the sequence that policy \( \pi \) will take from time 1 to \( T \). From Lemma 4 we have that

\[ |V_{1, M}(x_{\text{init}}) - V_{1, \hat{M}}(x_{\text{init}})| = \left| \sum_{t=1}^{T} \mu_{\pi_t,t}(u_{\pi_t,t-1}) - \hat{\mu}_{\pi_t,t}(u_{\pi_t,t-1}) \right| , \]

where \( u_{\pi_t,t-1} \) is the past pull sequence for arm \( \pi_t \) under policy \( \pi \) before time \( t \) and \( \hat{\mu}_{k,t}(u_{k,0:t-1}) = b_k - \hat{\lambda}_k \sum_{i=1}^{t-1} \hat{\gamma}_k^{t-i} u_{k,i} \) for \( t \geq 2 \) and \( \hat{\mu}_{k,1}(u_{k,0:1}) = b_k = \mu_{k,1}(u_{k,0:1}) \). Given \( t \geq 2 \), let \( k \) denote \( \pi_t \), we have that

\[ |\mu_{k,t}(u_{k,0:t-1}) - \hat{\mu}_{k,t}(u_{k,0:t-1})| \]

\[ = |\hat{\lambda}_k \sum_{i=1}^{t-1} \hat{\gamma}_k^{t-i} u_{k,i} - \lambda_k \sum_{i=1}^{t-1} \gamma_k^{t-i} u_{k,i}| \]

\[ = |\hat{\lambda}_k \sum_{i=1}^{t-1} \hat{\gamma}_k^{t-i} u_{k,i} - \lambda_k \sum_{i=1}^{t-1} \gamma_k^{t-i} u_{k,i} + \lambda_k \sum_{i=1}^{t-1} \gamma_k^{t-i} u_{k,i} - \lambda_k \sum_{i=1}^{t-1} \gamma_k^{t-i} u_{k,i}| \]

\[ \leq |\hat{\lambda}_k - \lambda_k| \left( \hat{\gamma}_k - \gamma_k \right) + \lambda_k \left( \frac{\hat{\gamma}_k - \gamma_k}{1 - \gamma_k} \right) \]

\[ \leq \epsilon_{\gamma_k} \epsilon_{\lambda_k} + \frac{\lambda_k \epsilon_{\gamma_k}}{1 - \gamma_k} (1 - \gamma_k) \]

Since \( \hat{\gamma}_k < 1 (\hat{\gamma}_k \in (a, \pi)) \) almost surely and with probability \( 1 - \delta \), for all \( k \in [K] \), \( \epsilon_{\gamma_k} \leq O (1/\sqrt{n}) \) and \( \epsilon_{\lambda_k} \leq O (1/\sqrt{n}) \). We have that with probability \( 1 - \delta \),

\[ \left| \sum_{t=1}^{T} \mu_{\pi_t,t}(u_{\pi_t,t-1}) - \hat{\mu}_{\pi_t,t}(u_{\pi_t,t-1}) \right| \leq \sum_{t=1}^{T} |\mu_{\pi_t,t}(u_{\pi_t,t-1}) - \hat{\mu}_{\pi_t,t}(u_{\pi_t,t-1})| \leq \left( \frac{T}{\sqrt{n}} \right) . \]

\( \square \)

F.4 Proof of Proposition 6

Proof. Fix δ ∈ (0, 1). Let E1 be the event that

\[ \forall k \in [K], |\hat{\gamma}_k - \gamma_k| = \epsilon_{\gamma_k} \leq O \left( \frac{1}{\sqrt{n}} \right), \ |\hat{\lambda}_k - \lambda_k| = \epsilon_{\lambda_k} \leq O (1/\sqrt{n}) . \]

From Corollary 9 we have that \( \mathbb{P}(E_1) \geq 1 - \delta/2 \). Let \( \epsilon_{\lambda} := \max_{k} \epsilon_{\lambda_k} \). Let \( E_2 \) denote the event that \( \forall t \in [T], k \in [K], |x_{k,t}| \leq B(\delta/2) \). We know that \( \mathbb{P}(E_2) \geq 1 - \delta/2 \). When \( E_1 \) and \( E_2 \) happen, we first observe that for all positive integer \( n \) and \( k \in [K] \),

\[ |\hat{\gamma}_k^n - \gamma_k^n| \leq |\hat{\gamma}_k - \gamma_k| \left( n \max(\gamma_k^{n-1}, \hat{\gamma}_k^{n-1}) \right) \leq \frac{|\hat{\gamma}_k - \gamma_k|}{\max(\gamma_k, \hat{\gamma}_k) \ln (1/ \max(\gamma_k, \hat{\gamma}_k))} = O(1/\sqrt{n}) , \]

where and the second inequality uses the assumption that \( \hat{\gamma}_k, \gamma_k \) are bounded away from 0 and 1.

To continue, we first bound the distance between the transition function in \( \hat{M} \) and \( M \). At any time \( t \) and state \( x_t = (x_{1,t}, n_{1,t}, \ldots, x_{K,t}, n_{K,t}) \), when we pull arm \( \pi_t = k \), the next state \( x_{t+1} \)
is updated by: (i) for arm $k$, $n_{k,t+1} = 1$ and (ii) for all other arms $k' \neq k$, $n_{k',t+1} = n_{k',t} + 1$ if $n_k \neq 0$, $n_{k,t+1} = 0$ if $n_k = 0$, and $x_{k',t+1} = x_{k',t}$. Then, by [8, Theorem 1.3], we have that when $n_{\pi_t,t} \neq 0$,

\[
\|p_{M} (x_{t+1}|x_t, \pi_t, t) - p_{M} (x_{t+1}|x_t, \pi_t, t)\|_1 \\
\leq \frac{3\lambda_k^2 \sum_{i=0}^{n_{k,t}-1} \gamma_{k,i}^2 \lambda_k^2 \sum_{i=0}^{n_{k,t}-1} \gamma_{k,i}^2}{\lambda_k^2 \sum_{i=0}^{n_{k,t}-1} \gamma_{k,i}^2} + |\gamma_{k,t} x_{k,t} + \lambda_k \gamma_{k,t} - \hat{\gamma}_{k,t} x_{k,t} - \hat{\lambda}_{k,t} \gamma_{k,t}| \\
+ \frac{|\hat{\gamma}_{k,t} - \gamma_{k,t}| |B(\delta/2) + \lambda_k \gamma_{k,t} - \hat{\lambda}_{k,t} \gamma_{k,t}|}{\lambda_k \sqrt{\sum_{i=0}^{n_{k,t}-1} \gamma_{k,i}^2}} \\
\leq 3 \left( \lambda_k^2 \left( \sum_{i=0}^{n_{k,t}-1} \gamma_{k,i}^2 - \sum_{i=0}^{n_{k,t}-1} \gamma_{k,i}^2 \right) + (\hat{\lambda}_k^2 - \lambda_k^2) \sum_{i=0}^{n_{k,t}-1} \gamma_{k,i}^2 \right) + |\hat{\gamma}_{k,t} - \gamma_{k,t}| |B(\delta/2) + \lambda_k| + |\lambda_k - \hat{\lambda}_k| \\
= \epsilon_P = O \left( \frac{1}{\sqrt{n}} \right),
\]

where (*) holds since $p_{M} (x_{t+1}|x_t, \pi_t, t)$ is a Gaussian density with mean $\gamma_{k,t} x_{k,t} + \lambda_k \gamma_{k,t}$ and variance $\lambda_k^2 \sum_{i=0}^{n_{k,t}-1} \gamma_{k,i}^2$ and (**) uses the fact that $\lambda_k^2 \sum_{i=0}^{n_{k,t}-1} \gamma_{k,i}^2 \geq \lambda_k^2 \geq 1$. When $n_{\pi_t,t} = 0$ and condition (i) and (ii) are satisfied, we have that $\|p_{\hat{M}} (x_{t+1}|x_t, \pi_t, t) - p_{M} (x_{t+1}|x_t, \pi_t, t)\|_1 = 0$. Otherwise, that is, if condition (i) or (ii) is not satisfied, we also have that $\|p_{\hat{M}} (x_{t+1}|x_t, \pi_t, t) - p_{M} (x_{t+1}|x_t, \pi_t, t)\|_1 = 0$ since $p_{\hat{M}} (x_{t+1}|x_t, \pi_t, t) = p_{M} (x_{t+1}|x_t, \pi_t, t) = 0$. Next, we examine the difference of the expected reward obtained by pulling arm $k$ at state $x_t$ at time $t$ in MDP $M$ and $\hat{M}$; when $n_{k,t} \neq 0$, this is given by

\[
|\hat{r}(x_t, k)| - r(x_t, k)| = |\gamma_{k,t} x_{k,t} + \lambda_k \gamma_{k,t} - \hat{\gamma}_{k,t} x_{k,t} - \hat{\lambda}_{k,t} \gamma_{k,t}| \\
\leq |x_{k,t}| \cdot |\gamma_{k,t} - \hat{\gamma}_{k,t}| + |\lambda_k \gamma_{k,t} - \hat{\lambda}_{k,t} \gamma_{k,t} + \lambda_k \gamma_{k,t} - \hat{\lambda}_{k,t} \gamma_{k,t}| \\
\leq (B(\delta/2) + \lambda_k) |\gamma_{k,t} - \gamma_{k,t}| + |\lambda_k - \hat{\lambda}_k| =: \epsilon_R = O \left( \frac{1}{\sqrt{n}} \right),
\]

where $\hat{r}(x_t, k)$ is the expected reward of pulling arm $k$ at state $x_t$ in MDP $\hat{M}$. Putting it altogether, we have that for any deterministic policy $\pi$,

\[
V_{1,M}^\pi (x_{init}) - V_{1,\hat{M}}^\pi (x_{init}) = r(x_{init}, \pi_1 (x_{init})) - \hat{r}(x_{init}, \pi_1 (x_{init})) + \mathbb{E}_{x_{2} \sim p_M (\cdot|x_{1}, \pi_1)} [V_{2,M}^\pi (x_{2})] \\
- \mathbb{E}_{x_{2} \sim p_{\hat{M}} (\cdot|x_{1}, \pi_1)} [V_{2,\hat{M}}^\pi (x_{2})] \\
\leq \epsilon_R + \mathbb{E}_{x_{2} \sim p_M (\cdot|x_{1}, \pi_1)} [V_{2,M}^\pi (x_{2})] - \mathbb{E}_{x_{2} \sim p_{\hat{M}} (\cdot|x_{1}, \pi_1)} [V_{2,\hat{M}}^\pi (x_{2})] \\
+ \mathbb{E}_{x_{2} \sim p_{\hat{M}} (\cdot|x_{1}, \pi_1)} [V_{2,\hat{M}}^\pi (x_{2})] - \mathbb{E}_{x_{2} \sim p_{\hat{M}} (\cdot|x_{1}, \pi_1)} [V_{2,\hat{M}}^\pi (x_{2})] \\
\leq T \epsilon_R + \sum_{t=1}^{T} \mathbb{E}_{M,\pi} \left\{ \mathbb{E}_{x_{t+1} \sim p_M (\cdot|x_{t}, \pi)} [V_{t+1,\hat{M}}^\pi (x_{t+1})] \right\}
\]
\[- \mathbb{E}_{x_{t+1} \sim p_{\tilde{M}}(\cdot|x_t, \pi, t)}[V_{t+1,M}(x_{t+1})]\]
\[\leq T \varepsilon_R + T^2 \varepsilon_P \max_k b_k,\]

where \(p_M(\cdot|x_t, \pi, t)\) denotes \(p(\cdot|x_t, \pi_t(x_t), t)\) in MDP \(M\) and the last inequality uses the fact that \(\langle p_M(\cdot|x_t, \pi, t) - p_{\tilde{M}}(\cdot|x_t, \pi, t), V_\pi^{x,t} \rangle \leq \|p_M(\cdot|x_t, \pi, t) - p_{\tilde{M}}(\cdot|x_t, \pi, t)\|_1 \|V_\pi^{x,t}\|_\infty \leq \varepsilon_P T \max_k b_k\). Finally, we have that

\[V_1^{\pi_*}(x_{\text{init}}) - V_1^{\pi_*}(x_{\text{init}}) = V_1^{\pi_*}(x_{\text{init}}) - V_1^{\pi_*}(x_{\text{init}}) + V_1^{\pi_*}(x_{\text{init}}) - V_1^{\pi_*}(x_{\text{init}}) + V_1^{\pi_*}(x_{\text{init}}) - V_1^{\pi_*}(x_{\text{init}}) \leq 2T \varepsilon_R + 2T^2 \varepsilon_P \max_k b_k,\]

where the equation follows from the fact that \(V_1^{\pi_*}(x_{\text{init}}) = V_1^{\pi_*}(x_{\text{init}})\) and rearranging the terms, and the inequality follows from applying the bound of \(V_1^{\pi_*}(x_{\text{init}}) - V_1^{\pi_*}(x_{\text{init}}) \leq T \varepsilon_R + T^2 \varepsilon_P \max_k b_k\) that was derived above for \(\pi = \pi_*\) and \(\pi = \pi_*\) and using the fact that the policy \(\pi_*\) is optimal for MDP \(\tilde{M}\). Let \(E_3\) denote the event that \(V_1^{\pi_*}(x_{\text{init}}) - V_1^{\pi_*}(x_{\text{init}}) \leq O(T^2/\sqrt{n})\). Putting it altogether, we have that \(\mathbb{P}(E_3) \geq \mathbb{P}(E_2, E_1) = 1 - \mathbb{P}(E_2^c \cup E_1^c) \geq 1 - \delta. \)
Figure 4: Figure 4a shows the cumulative expected reward collected by and $w$-lookahead policy (blue dots) when $T = 100$. When solving for the $T$-lookahead policy (with $T = 100$), after 24 hours, Gurobi 9.1 obtains an objective value of 491.3 (red solid line) with an upper bound 555.3 (red dotted line) and an absolute optimality gap 64.0 (13.0%). The true cumulative expected reward for $T$-lookahead policy for this problem lies in between the solid and dotted red lines. Figure 4b shows the log-log plot of the $w$-step lookahead regret of $w$-lookahead EEP (averaged over 5 random runs) under different $T$.

**G Additional Experimental Details and Results**

In this appendix, we present additional experimental details and results.

**$w$-lookahead Performance** When evaluating the performance of $w$-lookahead policies, in addition to the case where $T = 30$ (Figure 3a), we have also run the experiments with $T = 100$ (Figure 4a). When solving for the 100-lookahead policy, we have increased the number of threads to 50 to solve for (4) and stopped the program at a time limit of 24 hours. In such settings, we obtain an upper bound on the absolute optimality gap of 64.0 (percentage optimality gap of 13.0%). When solved for $w$-lookahead policies with $w$ in between 1 and 15 using 10 threads, Gurobi ends up solving (5) within 40s for all different $w$ values. Thus, despite using significantly lower computational time, $w$-lookahead policies achieve a similar cumulative expected reward to the $T$-lookahead policies (see Figures 3a and 4a).

**EEP Performance** Figure 3b is the log-log plot of the $w$-step lookahead regret of $w$-lookahead EEP against the horizon $T$ when $T = 60, 80, 100, 150, 200, 300, 400$ (averaged over 20 random runs) and Figure 4b is the log-log plot when $T = 60, 80, 100, 150, 200, 300, 400, 600, 800$ (averaged over 5 random runs), under the experimental setup provided in § 7.