SIMPSON’S CONSTRUCTION OF VARIETIES WITH MANY LOCAL SYSTEMS

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One of the goals of this note is to say something about the fundamental group of a smooth complex projective variety in terms of the quantity of local systems on it. Given a finitely generated group \( \Gamma \), let \( d_N(\Gamma) \) be the dimension of the space of irreducible rank \( N \) representations. The number \( d_1(\Gamma) \) coincides with the first Betti number, so one may think of \( d_N(\Gamma) \) as a nonabelian generalization. The basic problem is to see how these numbers behave when \( \Gamma \) is the fundamental group of a smooth projective variety \( X \). In this case, these numbers are always even [A]. If \( X \) is a curve of genus at least two, or even if it maps onto such a curve, then \( d_N(\Gamma) > 0 \) for all \( N \). If \( X \) is an abelian variety, then \( d_1(\Gamma) > 0 \) but \( d_N(\Gamma) = 0 \) for all \( N > 0 \).

I want to consider examples which have the opposite behaviour, in that \( d_1 = 0 \) but some higher \( d_N > 0 \). Some cheap examples are given in the first section. However, they are not very interesting in the sense that they are very close to the examples we already know. In the second section I will turn to a beautiful construction due to Carlos Simpson [S], which also produces smooth projective varieties such that \( d_N(\pi_1(X)) > 0 \) for some \( N > 1 \). In fact, the real purpose of this article is to make Simpson’s construction a bit more accessible and explicit, with the hope that these examples will be studied more thoroughly in the future. Some specific problems are suggested in the last section.

1. Representation varieties

For \( \Gamma \) a group with generators \( g_1, \ldots, g_n \), an element of \( \text{Hom}(\Gamma, GL_N(\mathbb{C})) \) is given by \( n \) matrices subject to the relations of the group. In this way, the set becomes an affine scheme of finite type, called the representation “variety”. (For the present purposes, a scheme will be identified with the set of its closed points.) The algebraic group \( GL_N(\mathbb{C}) \) acts on the representation variety by conjugation, and the GIT quotient

\[
M(\Gamma, N) = \text{Hom}(\Gamma, GL_N(\mathbb{C}))/\!/GL_N(\mathbb{C})
\]

\[
:= \text{Spec} \mathcal{O}(\text{Hom}(\Gamma, GL_N(\mathbb{C})))^{GL_N(\mathbb{C})}
\]

can be identified with the set of isomorphism classes of semisimple representations of rank \( N \) [LM]. This is often called the character variety. Let

\[
M(\Gamma, N)_{irred} \subset M(\Gamma, N)
\]

denote the possibly empty open subset of irreducible representations. We have quasifinite (i.e. set theoretically finite to one) morphisms

\[
M(\Gamma, N_1) \times M(\Gamma, N_2) \to M(\Gamma, N_1 + N_2)
\]

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As noted above, the group on the right vanishes. Since $G \times H$ left also vanishes. Therefore

\( (2) \quad \text{Im} \, M(\Gamma, N)_{irred} \times \ldots \times M(\Gamma, N)_{irred} \)

we take it to be zero if it is empty. From (1), we obtain:

**Lemma 1.1.** We have

\[ \dim M(\Gamma, N) = \max_{N_1 + \ldots + N_r = N} d_{N_1}(\Gamma) + \ldots + d_{N_r}(\Gamma) \]

Therefore $\dim M(\Gamma, N) > 0$ if and only if $d_M(\Gamma) > 0$ for some $M \leq N$.

We have $M(\Gamma, 1) = \dim \text{Hom}(\Gamma, \mathbb{C}^*)$, therefore $d_1(\Gamma) = \text{rank } \Gamma/[\Gamma, \Gamma]$. For higher $N$, these numbers are usually very difficult to calculate, although there are some easy cases. We have $d_N(\Gamma) = 0$ when $N > 1$ and $\Gamma$ is abelian simply because in this case there are no irreducible representations of higher rank. If $\Gamma$ surjects onto a nonabelian free group then a bit of thought shows that $d_N(\Gamma) > 0$ for all $N$. This remark applies to the fundamental group of a smooth projective curve of genus at least two.

When $\Gamma = \pi_1(X)$ is the fundamental group of a smooth projective variety $X$, Hodge theory tells us that $d_1(\Gamma) = \dim H^1(X)$ is even. Moreover, nonabelian Hodge theory implies that $M(\Gamma, N)_{irred}$ carries a quaternionic or hyperkähler structure, therefore every $d_N(\Gamma)$ is even \[\text{[A]} \text{ thm 3.1}\]. Here is the example promised in the introduction.

**Theorem 1.2.** There exists a smooth projective variety $X$ with $d_1(\pi_1(X)) = 0$ and $d_N(\pi_1(X)) \geq 2d$ for any given $N > 1$ and $d > 0$.

**Proof.** Let $C \to \mathbb{P}^1$ be a cyclic cover of the form $y^N = f(x)$, where $f$ has distinct roots. Let $x_0$ denote one of the roots. By choosing $\deg f$ sufficiently large, we can assume that the genus $g$ of $C$ is greater than or equal to $d$. The group $G = \mathbb{Z}/N\mathbb{Z}$ will act on $C$ with $C/G \cong \mathbb{P}^1$. If follows that $H^1(C, \mathbb{Q})^G = 0$. Consequently, if $\gamma \in G$ denotes a generator, it will act nontrivially on $H_1(C, \mathbb{Z})$. By Serre [Se prop 15], there exists a simply connected variety $Y$ on which $G$ acts freely. Let $X = (C \times Y)/G$, where $G$ acts diagonally. The projection $X \to Y/G$ is a fibration with fibre $C$ and section given by $y \mapsto (x_0, y)$. Therefore we have split exact sequence

\[ 1 \to \pi_1(C) \to \pi_1(X) \to G \to 1 \]

Using the Hochschild-Serre spectral sequence, we obtain an exact sequence

\[ H^1(G, H^0(\pi_1(C), \mathbb{Q})) \to H^1(\pi_1(X), \mathbb{Q}) \to H^0(G, H^1(\pi_1(C), \mathbb{Q})) \]

As noted above, the group on the right vanishes. Since $G$ is finite, the group on the left also vanishes. Therefore $H^1(\pi_1(X), \mathbb{Q}) = 0$, which means that $d_1(\pi_1(X)) = 0$.

Let $\rho \in \text{Hom}(\pi_1(C), \mathbb{C}^*) = (\mathbb{C}^*)^{2g}$ be a one dimensional character. For a generic choice of $\rho$, the characters $\rho, \rho \circ \gamma, \ldots, \rho \circ \gamma^{N-1}$ are all distinct. Let $\mathbb{C}_\rho$ denote the $\mathbb{C}[\pi_1(C)]$-module associated to $\rho$. The induced representation $V_\rho = \text{Ind } \mathbb{C}_\rho$ gives a rank $N \mathbb{C}[\pi_1(X)]$-module. As an $\mathbb{C}[\pi_1(C)]$-module

\[ (2) \quad V_\rho = \mathbb{C}_\rho \oplus \mathbb{C}_{\rho_0\gamma} \oplus \ldots \]
and \( \gamma \) acts by cyclically permuting the factors. It follows easily that \( V_\rho \) is an irreducible \( \pi_1(X) \)-module for generic \( \rho \). Also by computing characters, using (2), we see that \( V_\rho \cong V_\rho' \) only if \( \rho' \in \{ \rho, \gamma \rho, \ldots \} \). Therefore the map \( \rho \mapsto V_\rho \) is a quasifinite morphism from an open subset of \((\mathbb{C}^*)^{2g}\) to \( M(\pi_1(X), N)^{irred} \). Thus 
\[
d_N(\pi_1(X)) \geq 2g \geq 2d.
\]

The drawback of this method is that it does not produce any really new examples of fundamental groups of smooth projective varieties. I will describe a more subtle construct in the next section, but first I want to record the following useful fact which was used implicitly above.

**Lemma 1.3.** Suppose that \( \Gamma_1 \subset \Gamma \) is a subgroup of index \( r < \infty \).

(a) If \( W_\rho \) is a nontrivial (i.e. nonconstant) family of representations in \( M(\Gamma, N) \), then the restrictions \( \text{Res} W_\rho \) give a nontrivial family in \( M(\Gamma_1, N) \).

(b) Conversely if \( \text{Res} W_\rho \) is a nontrivial family, then so is \( W_\rho \).

(c) If \( V_\rho \) is a nontrivial family of representations in \( M(\Gamma_1, N) \), then \( \text{Ind} V_\rho \) is a nontrivial family in \( M(\Gamma, rN) \).

**Proof.** The first two items are the content of lemma 1.5 of [S]. For (c), we have that \( \text{Res} (\text{Ind} V_\rho) = V_\rho \oplus \ldots \) is nontrivial, so \( \text{Ind} V_\rho \) is nontrivial by (b).

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2. **Simpson’s construction**

Let \( Z \) be a smooth projective variety with dimension \( 2n+1 \geq 3 \) and positive first Betti number. Fix an embedding \( Z \subset \mathbb{P}^K \) such that \( \mathcal{O}_Z(1) \) is sufficiently ample in the sense that it is a high enough power of a given ample bundle. Sufficient ampleness is needed for the proofs of proposition 2.1 and theorem 2.4. Let \( P \subset \mathbb{P}^K \) be a general linear subspace of the dual space of dimension \( d \geq 2 \). Then we can form the incidence variety

\[
Y = \{(z, H) \in Z \times P \mid z \in H\}
\]

with projections and inclusions labelled as follows

\[
\begin{array}{ccc}
Y & \xrightarrow{f} & Z \\
\downarrow \pi & & \downarrow \iota \\
P & \xleftarrow{\Pi} & Z \times P
\end{array}
\]

Denote the fibre of \( \pi \) over \( t \) by \( X_t \). Let \( D_1 = \{ t \in P \mid X_t \text{ is singular} \} \) be the (reduced) discriminant. The following standard fact is stated in [DL] and various other places. A proof, assuming sufficient ampleness, can be found in [S] prop 6.1.

**Proposition 2.1.** The discriminant \( D_1 \) is an irreducible hypersurface and for a generic 2 dimensional plane \( Q \subset P \), the singularities of \( D_1 \cap Q \) are nodes and cusps.

The next step is to form a double cover branched over \( D_1 \). If \( g(x_1, \ldots, x_d) = 0 \) is an affine equation of \( D_1 \), then the cover \( y^2 = g \) may acquire additional ramification at infinity. It is better to control this in advance by defining

\[
D = \begin{cases} 
D_1 & \text{if } \deg D_1 \text{ is even} \\
D_1 + D_2 & \text{otherwise, where } D_2 \text{ is a hyperplane in general position}
\end{cases}
\]
Let $U = P - D$. Let $p' : X' \to P$ be the double cover branched along $D$. As a scheme

$$X' = \text{Spec} \left( \mathcal{O}_P \oplus \mathcal{O}_P \left( \frac{\deg D}{2} \right) \right)$$

where the sheaf in parantheses is made into an algebra in the standard way (cf [EV, p 22]). This will usually be singular but the singularities are normal local complete intersections. The singular set $X'_\text{sing} \subseteq \Sigma = p^{−1}D_{\text{sing}}$. Let $q : X \to X'$ be a desingularization which is an isomorphism on the complement of $\Sigma$. This variety is what we are after. It is very similar to, although not identical to, Simpson's construction in [S, lemma 6.3]. The difference is that Simpson's variety is a branched cover of $P$ of indeterminate degree, on which, by design, the local systems $V_\rho$ constructed below extend. This makes it simpler for the purpose of constructing local systems. However, the lack of explicitness makes it harder to do precise computations.

**Theorem 2.2.** The first Betti number of $X$ is zero. For some $M > 1$, $d_M(\pi_1(X)) > 0$.

The rest of this section will be devoted to the proof of this theorem.

**Proposition 2.3.** The first Betti number of $X$ is zero.

**Proof.** By Hodge theory, the proposition is equivalent to $H^1(X, \mathcal{O}_X) = 0$. We prove the last equation by induction on $d$ starting with $d = 2$. In this case, $\Sigma$ consists of a finite set of singular points. The local analytic germ of $X'$ at $p \in \Sigma$ is either of the form $y^2 = x_1x_2$ or $y^2 = x_1^2 - x_3^2$. These are the well known singularities of type $A_n$ for $n = 1, 2$ [D]. These are rational singularities which implies that $H^1(X, \mathcal{O}_X) = H^1(X', \mathcal{O}_{X'})$. The last group $H^1(X', \mathcal{O}_{X'}) \cong H^1(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}) \oplus H^1(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(\deg D/2)) = 0$

For $d > 2$, choose a general hyperplane $H \subset P$. By the Bertini, $G = p^{−1}H$ is smooth. By induction, we can assume that $H^1(G, \mathcal{O}_G) = 0$. We have an exact sequence

$$H^1(X, \mathcal{O}_X(-G)) \to H^1(X, \mathcal{O}_X) \to H^1(X, \mathcal{O}_G) = 0$$

The first group $H^1(X, \mathcal{O}_X(-G)) = H^{d-1}(X, \omega_X(G))$ is zero by the Kawamata-Viehweg vanishing theorem [EV, p 49].

We turn to the second part of theorem. By assumption $Z$ carries a positive dimensional family of rank one local systems. Fix a generic such system $\mathcal{C}_\rho$, and consider the sheaf

$$V_\rho = \text{coker}(R^n\pi_* (F^* \mathcal{C}_\rho) \to R^n\pi_* (f^* \mathcal{C}_\rho))|_U$$

This is a local system of some rank $N > 1$. The stalk of $V_\rho$ over $t$ is the primitive $n$th cohomology of $X_t$ with coefficients in $\mathcal{C}_\rho$. The rank $N$ is just the dimension of this space. Let $R_\rho : \pi_1(U) \to GL_N \mathbb{C}$ denote the representation corresponding to $V_\rho$.

**Theorem 2.4** (Simpson [S, thm 5.1]). As $\rho$ varies, $V_\rho$ gives a nontrivial family in $M(\pi_1(U), N)$. 
The proof is rather involved, so we will be content to make a few brief comments about it. The key ingredient is nonabelian Hodge theory, which sets up a correspondence between semisimple local systems and certain Higgs bundles, which for our purposes can be viewed as sheaves on the cotangent bundle. Simpson then checks that as the $\rho$ vary, the supports of the Higgs bundles corresponding to $V_\rho$, called spectral varieties, also vary nontrivially. When $Z$ is an abelian variety, there is a more elementary argument which avoids Higgs bundles [S, p 358], and this already suffices for constructing nontrivial examples.

Let $\gamma_1$ be a loop going once around a smooth point $D_1$. This involves a choice, but any two choices are conjugate because $D_1$ is irreducible. We have

$$R_\rho(\gamma_1)^2 = I$$

by the Picard-Lefschetz formula or see [S, lemma 6.5]. Let $\gamma_2$ be a loop around $D_2$ when it exists. Then

$$R_\rho(\gamma_2) = I$$

because $V_\rho$ extends to a local system on $P - D_1$. Let $p = p' \circ q$ and $\hat{U} = p^{-1}U$. We can identify $\hat{U} = p'^{-1}U \subset X'$. This is an étale double cover of $U$ corresponding to an index two subgroup $\pi_1(\hat{U}) \subset \pi_1(U)$. This subgroup contains $\gamma_2^2$. We can identify $\pi_1(X' - \Sigma)$ with the quotient of $\pi_1(\hat{U})$ by the normal subgroup generated by the $\gamma_2^2$. Combining this with (3) and (4) yields

**Lemma 2.5.** The pullback of the local system $p'^*V_\rho$ extends to $X' - \Sigma$.

Let $X'_Q = X' \cap p'^*Q$ where $Q \subset P$ is a general 2-plane.

**Lemma 2.6.** $\pi_1(X') \cong \pi_1(X'_Q)$

**Proof.** Since $X$ has local complete intersection singularities, we can apply the Lefschetz theorem of [FL, p 28] to deduce the above isomorphism. $\square$

To simplify notation, replace $\Sigma$ by its restriction to $X'_Q$. Then $\Sigma$ consists of a finite set of points. For each $p \in \Sigma$, let $L_p$ denote the link which is the boundary of a small contractible neighbourhood of $p$. The group $\pi_1(X') = \pi_1(X'_Q)$ is the quotient of $\pi = \pi_1(X'_Q - \Sigma)$ by the normal subgroup $N$ generated by $\bigcup_p \pi_1(L_p)$. For any group $\Gamma$, let

$$K(\Gamma) = \ker[\Gamma \rightarrow \hat{\Gamma}]$$

where $\hat{\Gamma}$ is the profinite completion. This can also be characterized as the intersection of all finite index subgroups, or as the smallest normal subgroup for which $\Gamma/K(\Gamma)$ is residually finite.

**Lemma 2.7.** There exists a normal subgroup of finite index $\Gamma \subseteq \pi$ such that $\pi_1(L_p) \cap \Gamma \subseteq K(\pi)$ for each $p \in \Sigma$.

**Proof.** As noted above, $\Sigma$ consists of a finite set of singular points of type $A_1$ or $A_2$. These singularities can also be described as quotients of $(\mathbb{C}^2, 0)$ by an action of $\mathbb{Z}/2\mathbb{Z}$ or $\mathbb{Z}/3\mathbb{Z}$ [D]. Therefore $\pi_1(L_p)$ must either be $\mathbb{Z}/2\mathbb{Z}$ or $\mathbb{Z}/3\mathbb{Z}$ and in particular finite. Since $\pi/K(\pi)$ is residually finite, we can find a finite index subgroup $\bar{\Gamma}$ of it avoiding the nonzero elements $\text{im}(\pi_1(L_p))$. Let $\Gamma$ be the preimage. $\square$

Let $\Psi \subseteq \pi_1(X')$ denote the image of $\Gamma$.

**Lemma 2.8.**
(a) If $\bar{\Gamma}$ and $\bar{N}$ denote the images of $\Gamma$ and $N$ in $\pi/K(\pi)$, then $\bar{\Gamma} \cap \bar{N} = 1$.

(b) $\Gamma/K(\pi) \cong \Psi/(\pi_1(X'))$.

**Proof.** Item (a) follows immediately from lemma. The canonical map $\Gamma/K(\pi) \to \Psi/(\pi_1(X'))$ is clearly surjective. The kernel is $\bar{\Gamma} \cap \bar{N}$. So (b) follows from (a). \qed

**Lemma 2.9.** The restriction of $R_\rho$ to $\Gamma$ is the pull back of a representation of $\Psi$.

**Proof.** By a theorem of Mal’cev [M, p 309], any finitely generated linear group is residually finite. Therefore the restriction $\text{Res} V_\rho = R_\rho|_\Gamma$ factors through $\Gamma/K(\pi) \cong \Psi/(\pi_1(X'))$. \qed

To finish the proof of theorem 2.2 observe that by the above results, the restriction $\text{Res} V_\rho$ comes from a $\Psi$-module $W_\rho$. We can form the induced $\pi_1(X')$-module $\text{Ind} W_\rho$. This corresponds to a nontrivial family of semisimple local systems on $X'$ by lemma 1.3, which pulls back to a nontrivial family on $X$. Therefore by lemma 1.1 $d_M(\pi_1(X)) > 0$ for some $M$. By proposition 2.3, $M > 1$, and this concludes the proof.

3. **Problems**

I will end by discussing a few follow up problems.

**Problem 3.1.** Determine (a presentation for) the fundamental group of $X$, constructed in section two, for some explicit choice of $Z \subset \mathbb{P}^K$, such as when it is an abelian variety.

My hope is that this will give a genuinely new and interesting example of a group in the class of fundamental groups of smooth projective varieties. It is clear that it would differ from most of the standard known examples which either have positive first Betti number or are rigid in the sense that all $d_N = 0$. Furthermore $\pi_1(X)$ would be different from the examples constructed in section one. Simpson’s arguments [S] show that in his terminology that $X$, with the local system $\mathcal{L} = q^* \text{Ind} W_\rho$ above, has the nonfactorization property $NF_1$. This means that $\mathcal{L}$ is not the pull back of a local system on a curve even if we allow $X$ to be replaced by another variety mapping surjectively to it. This will imply that $\pi_1(X)$ cannot contain the fundamental group of a curve as a subgroup of finite index.

**Problem 3.2.** Find an example of a smooth projective variety with an infinite family of irreducible unitary representations which do not come from curves, i.e. that satisfy $NF_1$

This is equivalent to asking for a variety with an infinite family of stable vector bundles, with vanishing Chern classes, which do not come from curves. This can be rephrased as asking for an infinite family of stable Higgs bundles of the above type with zero Higgs fields. Simpson’s construction described above yields Higgs bundles with nonzero Higgs fields. This is clear from his proof of theorem 2.4.

For applications to the fundamental group, it suffices to stick with dimension $d = 2$. One reason for allowing $d > 2$ is that I feel that these varieties should be interesting from other points of view.

**Problem 3.3.** Study the birational geometry of these varieties.
For instance, although they have zero first Betti number, I suspect that they behave like varieties with large Albanese. One way to try to make this precise is by using the notion of Shafarevich maps in the sense of Campana and Kollár [K]. In most cases, I suspect that this map should be birational. This would be an analogue of the Albanese map being generically finite.

REFERENCES

[A] D. Arapura Higgs bundles, integrability, and holomorphic forms. Motives, polylogarithms and Hodge theory, Part II (Irvine, CA, 1998), 605-624, Int. Press Lect. Ser., 3, II, Int. Press, Somerville, MA, (2002)

[DL] I. Dolgachev, A. Libgober, On the fundamental group of the complement to a discriminant variety. Algebraic geometry (Chicago, Ill., 1980), pp. 1-25, Lecture Notes in Math., 862, Springer (1981)

[D] A. Durfee, Fifteen characterizations of rational double points, Enseign. Math. (2) 25 (1979), no. 1-2, 131-163.

[EV] H. Esnault, E. Viehweg, Lectures on vanishing theorems, DMV Seminar, 20. Birkhäuser Verlag, (1992)

[FL] W. Fulton, R. Lazarsfeld, Connectivity and its applications in algebraic geometry, Algebraic geometry (Chicago, Ill., 1980), pp. 26-92, Lecture Notes in Math., 862, Springer (1981)

[K] J. Kollár, Shafarevich maps and automorphic forms. Princeton University Press, (1995)

[LM] A. Lubotzky, A. Magid, Varieties of representations of finitely generated groups, Mem. Amer. Math. Soc. 58 (1985), no. 336

[M] W. Magnus, Residually finite groups, Bull. Amer. Math. Soc. 75 (1969) 305-316

[Se] J.P. Serre, Sur la topologie des variétés algébriques en caractéristique p, (1958) Symp. Int. Top. Alg, Mexico, pp. 24-53

[S] C. Simpson, Some families of local systems over smooth projective varieties, Annals of Math 138 (1993), no. 2, 337-425.