Effect of Self-Interaction on Charged Black Hole Radiance

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Abstract

We extend our previous analysis of the modification of the spectrum of black hole radiance due to the simplest and probably most quantitatively important back-reaction effect, that is self-gravitational interaction, to the case of charged holes. As anticipated, the corrections are small for low-energy radiation when the hole is well away from extremality, but become qualitatively important near extremality. A notable result is that radiation which could leave the hole with mass and charge characteristic of a naked singularity, predicted in the usual approximation of fixed space-time geometry, is here suppressed. We discuss the nature of our approximations, and show how they work in a simpler electromagnetic analogue problem.
1. Introduction

Considerable interest attaches to possible deviations of black hole radiance from exact thermality. In a previous paper [1], we showed how inclusion of the effect of gravitational self-interaction modifies the spectrum, introducing a definite departure from a thermal distribution. We did this by considering the full Hamiltonian for a spherically symmetric, electrically neutral black hole interacting with a single particle in the s-wave – i.e. a “shell”. Upon solving the constraints, we found an effective particle Hamiltonian. We analyzed the quantum theory for this effective Hamiltonian in the WKB approximation, which we found to be both unambiguous and adequate to describe the late-time radiation.

In this paper, two additional things are done. First, we extend the calculations to include a charged black hole, and charged matter. Although this step does not present any significant formal difficulties, the physical results we obtain are considerably richer than what we found in our previous calculations involving neutral holes and shells. In the neutral case the final result could be summarized as a simple replacement of the nominal temperature governing the radiation by the Hawking temperature for the mass after radiation, so that the “Boltzmann factor” governing emission of energy \( \omega \) from a hole of mass \( M \) became

\[
e^{-\omega/T_{\text{eff}}} = e^{-\omega 8\pi (M-\omega)}. \tag{1.1}
\]

Note that the argument of the exponential is not simply proportional to the energy \( \omega \), so that the spectrum is not, strictly speaking, thermal. While the deviation from thermality is important in principle its structure, in this case, is rather trivial, and one is left wondering whether that is a general result. Fortunately we find that for charged holes the final results are much more complex. We say “fortunately”, not only because this relieves us of the nagging fear that we have done a simple calculation in a complicated way, but also for more physical reasons. For one knows on general grounds that the thermal description of black hole radiance breaks down completely for near-extremal holes [2]. One might anticipate, therefore, that something more drastic than a simple modification of the nominal temperature will occur – as indeed we find. A particularly gratifying consequence of the accurate formula is a form of “quantum cosmic censorship”. Whereas a literal application of the conventional thermal formulas for radiation yields a non-zero amplitude for radiation past extremality – that is, radiation leaving behind a hole with larger charge than mass – we find (within our approximations) vanishing amplitude for such processes.

Second, we discuss in a more detailed fashion the relationship between our method of calculation, which proceeds by reduction to an effective particle theory, and more familiar approximations. We show that it amounts to saturation of the functional integral of the underlying s-wave field theory with one-particle intermediate states, or alternatively to neglect of vacuum polarization. It is therefore closely related to conventional eikonal
approximations. We demonstrate the reduction of the field theory to a particle theory explicitly in the related problem of particle creation by a strong spherically symmetric charge source, which is a problem of independent interest.

Before entering the body of this work, for later reference we here briefly collect a few formulas describing the Reissner-Nordstrom geometry. In the rather unconventional gauge \[^{3}\] whose use we have found to be extremely convenient, the classical line element for hole of mass \(M\) and charge \(Q\) is

\[
ds^2 = -dt^2 + \left( dr \pm \sqrt{\frac{2M}{r} - \frac{Q^2}{r^2}} dt \right)^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2). \tag{1.2}
\]

The outer horizon radius is

\[
R_+(M, Q) = M + \sqrt{M^2 - Q^2} \tag{1.3}
\]

and the inner horizon radius is

\[
R_-(M, Q) = M - \sqrt{M^2 - Q^2}. \tag{1.4}
\]

Finally, the nominal Hawking temperature \[^{4}\] is

\[
T(M, Q) = \frac{1}{2\pi} \frac{\sqrt{M^2 - Q^2}}{(M + \sqrt{M^2 - Q^2})^2}. \tag{1.5}
\]

2. Self-Interaction Correction

Our system consists of a matter shell of rest mass \(m\) and charge \(q\) interacting with the electromagnetic and gravitational fields. The corresponding action is

\[
S = \int \left[ -m\sqrt{-\hat{g}_{\mu\nu}d\hat{x}^\mu d\hat{x}^\nu} + q\hat{A}_\mu d\hat{x}^\mu \right] + \frac{1}{16\pi} \int d^4x\sqrt{-g} \left[ R - F_{\mu\nu}F^{\mu\nu} \right] \tag{2.1}
\]

\(\hat{x}^\mu\) is the position of the shell, and a caret over a function means that it is to be evaluated at the shell \(\hat{f} \equiv f(\hat{x}^\mu)\). To pass to the Hamiltonian formulation we first write the general spherically symmetric metric in ADM form:

\[
ds^2 = -N^t(t, r)^2 dt^2 + L(t, r)^2 [dr + N^r(t, r)dt]^2 + R(t, r)^2 [d\theta^2 + \sin^2 \theta d\phi^2]. \tag{2.2}
\]
The action can then be written in canonical form as:

$$S = \int dt \ p \dot{\hat{r}} + \int dr \ [\pi_R \dot{R} + \pi_L \dot{L} - N^t (H^*_t + H^G_t + H^EM_t) - N^r (H^*_r + H^G_r)] - \int dt \ M_{ADM}$$

with

$$H^*_t = \left( \sqrt{\left(\frac{p}{L}\right)^2 + m^2} - q \hat{A}_t \right) \delta(r - \hat{r}) \quad ; \quad H^*_r = -p \delta(r - \hat{r})$$

$$H^G_t = \frac{L \pi L^2}{2R^2} - \frac{\pi L \pi R}{R} + \left( \frac{R R'}{L} \right)' - \frac{R^2}{2L} - \frac{L}{2} \quad ; \quad H^G_r = R' \pi_R - L \pi'_L$$

$$H^EM_t = \frac{N^t L \pi_A r^2}{2R^2} - A_t \pi_A'$$

where $'$ represents $d/dr$ and $\cdot$ represents $d/dt$. To arrive at this form we have chosen a gauge such that $A_t$ is the only nonvanishing component of $A_\mu$. Of course, we set $A_r = 0$ only after computing the canonical momentum $\pi_A$.

Constraints are found by varying the action with respect to $N^t$, $N^r$, and $A_t$,

$$H_t \equiv H^*_t + H^G_t + H^EM_t = 0 \quad ; \quad H_r \equiv H^*_r + H^G_r = 0$$

$$\pi'_{A_r} + q \delta(r - \hat{r}) = 0.$$ (2.7)

$\pi_R$ can be eliminated by forming the linear combination of constraints

$$0 = \frac{R'}{L} H_t + \frac{\pi L}{RL} H_r = -\mathcal{M}' + \frac{R'}{L} (H^*_t + H^EM_t) + \frac{\pi L}{RL} H^*_r$$

where

$$\mathcal{M} = \frac{\pi L^2}{2R^2} + \frac{R}{2} - \frac{RR'^2}{2L^2}.$$ (2.9)

$\mathcal{M}(r)$ is interpreted as being the mass contained within a sphere of coordinate size $r$; its value at infinity is the ADM mass. Similarly, we see from the Gauss’ law constraint that $-\pi_A(r)$ is the charge contained within a sphere of size $r$, so we define: $Q(r) \equiv -\pi_A(r)$

Now, if the shell was absent $\mathcal{M}(r)$ and $Q(r)$ would be given by

$$\mathcal{M}(r) = M - \int_r^\infty dr \frac{R'(r) H^EM(r)}{L(r)} \quad ; \quad Q(r) = Q$$ (2.10)

with $M$ and $Q$ being the mass and charge of the black hole as seen from infinity. In the gauge $L = 1, R = r$ these become

$$\mathcal{M}(r) = M - Q^2/2r \quad ; \quad Q(r) = Q.$$ (2.11)

With the shell present we retain the expression (2.11) for the region inside the shell, $r < \hat{r}$, whereas outside the shell we write (with $L = 1, \hat{R} = r$),

$$\mathcal{M}(r) = M_+ - (Q + q)^2/2r \quad ; \quad Q(r) = Q + q$$ (2.12)
where \( M_+ \) and \( Q + q \) are the mass and charge of the hole-shell system as measured at infinity.

By using the constraints we can determine \( \pi_R, \pi_L \), and an expression for \( M_+ \), in terms of the shell variables. These relations can then be inserted in the action \( (2.3) \) to give an effective action depending only on the shell variables. This program was carried out in ref. [1] for the uncharged case. The calculation for the present case runs precisely parallel, resulting in

\[
S = \int dt \left[ \dot{r} \left( \sqrt{2M\dot{r} - Q^2} - \sqrt{2M_+\dot{r} - (Q + q)^2} \right) - \eta \dot{r} \dot{r} \log \left| \sqrt{\dot{r} - \eta\sqrt{M_+ - (Q + q)^2/2\dot{r}}} - M_+ \right| \right]
\]

where \( \eta \equiv \text{sgn}(p) \), and we have now specialized to a massless shell \( (m = 0) \). The canonical momentum is then

\[
p_c = \sqrt{2M\dot{r} - Q^2} - \sqrt{2M_+\dot{r} - (Q + q)^2} - \eta \dot{r} \log \left| \sqrt{\dot{r} - \eta\sqrt{M_+ - (Q + q)^2/2\dot{r}}} \right|. \tag{2.14}
\]

At this point we would like to obtain a quantum mechanical wave equation by making the substitutions \( p_c \rightarrow -i\partial/\partial r \), \( M_+ - M \rightarrow -i\partial/\partial t \). However, as discussed in ref. [1], it is rather difficult to implement this because of the nonlocal form of \( (2.14) \). Fortunately, for our present purposes we need only compute a class of short wavelength solutions which are accurately described by the WKB approximation. Writing these solutions as \( v(t, r) = e^{iS(t, r)} \) with \( S \) rapidly varying, we can make the replacements

\[
p_c \rightarrow \frac{\partial S}{\partial r}; \quad M_+ - M \rightarrow \frac{\partial S}{\partial t}.
\]

\( S(t, r) \) satisfies the Hamilton-Jacobi equation, and so is found by computing classical action along classical trajectories. We first choose the initial conditions for \( S(t, r) \) at \( t = 0 \):

\[
S^{q_k}_t(0, r) = kr. \tag{2.15}
\]

We have appended a subscript and a superscript to denote the initial condition and charge of the solution. The corresponding classical trajectory has the initial condition \( p_c = k \) at \( t = 0 \). \( S^{q_k}_t(t, r) \) is then given by

\[
S^{q_k}_t(t, r) = k\dot{r}(0) + \int_{\dot{r}(0)}^{\dot{r}} d\dot{r} p_c(\dot{r}) - (M_+ - M)t. \tag{2.16}
\]

To determine the radianee from the hole we will will only need to consider the behaviour of the solutions near the horizon. Furthermore, only the most rapidly varying part of the
solutions will contribute to the late-time radiation. With this in mind, we can write the momentum as (choosing $\eta = 1$ for an outgoing solution)

$$p_c(\hat{r}) \approx -\hat{r} \log \left| \frac{\hat{r} - R_+(M_+, Q + q)}{(\hat{r} - R_+(M, Q))(\hat{r} - R_-(M, Q))} \right|$$

(2.17)

so that the initial condition becomes

$$k = -\hat{r}(0) \log \left| \frac{(\hat{r}(0) - R_+(M_+, Q + q))(\hat{r}(0) - R_-(M_+, Q + q))}{(\hat{r}(0) - R_+(M, Q))(\hat{r}(0) - R_-(M, Q))} \right|. \quad (2.18)$$

Similarly, the classical trajectory emanating from $\hat{r}(0)$ is given by approximately,

$$t \approx \frac{2}{R_+(M_+, Q + q) - R_-(M_+, Q + q)} \left[ R_+(M_+, Q + q)^2 \log \left| \frac{\hat{r} - R_+(M_+, Q + q)}{\hat{r}(0) - R_+(M_+, Q + q)} \right| - R_-(M_+, Q + q)^2 \log \left| \frac{\hat{r} - R_-(M_+, Q + q)}{\hat{r}(0) - R_-(M_+, Q + q)} \right| \right]. \quad (2.19)$$

These trajectories are in fact null geodesics of the metric

$$ds^2 = -dt^2 + (dr + \sqrt{2M_+/r - Q^2} dt)^2. \quad (2.20)$$

The relations (2.18) and (2.19) allow us to determine $M_+$ and $\hat{r}(0)$ in terms of the other variables, so that after integrating (2.16) we can obtain an expression for $S^q_k(t, r)$ as a function of $k$, $t$, and $r$.

We can now write down an expression for the field operator:

$$\hat{\phi}(t, r) = \int dk \left[ \hat{a}_k v^q_k(t, r) + \hat{b}^\dagger_k v^{-q}_k(t, r)^* \right]. \quad (2.21)$$

The modes $v^q_k(t, r)$ are nonsingular at the horizon, and so the state of the field is taken to be the vacuum with respect to these modes:

$$\hat{a}_k |0_v\rangle = \hat{b}_k |0_v\rangle = 0.$$  

Alternatively, we can consider modes which are positive frequency with respect to the Killing time $t$. We write these modes as $u^q_k(r)e^{-i\omega_k t}$ where the $u^q_k(r)$ are singular at the horizon, $r = R_+(M + \omega_k, Q + q)$. Then

$$\hat{\phi}(t, r) = \int dk \left[ \hat{c}_k u^q_k(r)e^{-i\omega_k t} + \hat{d}^\dagger_k u^{-q}_k(r)^*e^{i\omega_k t} \right]. \quad (2.22)$$

The two sets of operators are related by Bogoliubov coefficients,

$$\hat{c}_k = \int dk \left[ \alpha_{kk'} \hat{a}_{k'} + \beta_{kk'} \hat{b}_{k'} \right]. \quad (2.23)$$
The flux of outgoing particles of charge \( q \) with energy between \( \omega_k \) and \( \omega_k + d\omega_k \) is given by

\[
F(\omega_k) = \frac{d\omega_k}{2\pi} \frac{\Gamma(\omega_k)}{|\alpha_{kk'}/\beta_{kk'}|^2 - 1}
\]  

(2.24)

where \( \Gamma(\omega_k) \) is a grey-body factor. This identifies \( |\beta_{kk'}/\alpha_{kk'}|^2 \) as the effective Boltzmann factor. From (2.21, 2.22) \( \alpha_{kk'} \) and \( \beta_{kk'} \) are found to be

\[
\alpha_{kk'} = \frac{1}{2\pi u_k^q(r)} \int_{-\infty}^{\infty} dt e^{i\omega_k t} v_k^q(t, r)
\]

\[
\beta_{kk'} = \frac{1}{2\pi u_k^q(r)} \int_{-\infty}^{\infty} dt e^{i\omega_k t} v_{k'}^{-q}(t, r)^*.
\]  

(2.25)

Here, \( r \) is taken to be slightly outside the horizon, \( r = R_+(M + \omega_k, Q + q) + \epsilon \). These coefficients can be evaluated in the saddle point approximation. Recalling that \( v_k^q(t, r) = e^{iS_k^q(t, r)} \), the saddle point equation for \( \alpha_{kk'} \) becomes

\[
\omega_k = -\frac{\partial S^q_k}{\partial t} = M_+ + M.
\]  

(2.26)

This leads to a purely real value of \( t \) for the saddle point. For \( \beta_{kk'} \) we have

\[
\omega_k = \frac{\partial S_{k'}^{-q}}{\partial t} = M - M_+^{-q}.
\]  

(2.27)

From (2.18, 2.19) we find that the saddle point value for \( t \) has an imaginary part given by

\[
\text{Im}(t_s) = \frac{2 R_+(M - \omega_k, Q - q)^2}{R_+(M - \omega_k, Q - q) - R_-(M - \omega_k, Q - q)} \frac{1}{2 T(M - \omega_k, Q - q)}.
\]  

(2.28)

Therefore,

\[
|\beta_{kk'}/\alpha_{kk'}| = \frac{1}{2\pi u_k(r)} \exp \left( \omega_k/T(M - \omega_k, Q - q) + \text{Im}[S_{k'}^{-q}(t_s)] \right).
\]  

(2.29)

The terms in \( S_{k'}^{-q} \) which contribute to the second term in the exponent are

\[
\int_{\hat{r}(0)}^{\hat{r}} d\hat{r} p_c(\hat{r}) + \omega_k \text{Im}(t_s).
\]

Using (2.17, 2.19) this can be evaluated to give

\[
\text{Im}[S_{k'}^{-q}(t_s)] = \frac{M \omega + \sqrt{M^2 - Q^2} \left( \sqrt{(M - \omega)^2 - (Q - q)^2} - \sqrt{M^2 - Q^2} \right)}{2 T(M - \omega, Q - q) R_+(M, Q)}.
\]  

(2.30)
resulting in

\[ \frac{\beta_{kk'}}{\alpha_{kk'}}^2 = \exp\left( -\frac{\sqrt{M^2 - Q^2 \left[ \omega - \sqrt{(M - \omega)^2 - (Q - q)^2} + \sqrt{M^2 - Q^2} \right]}}{T(M - \omega, Q - q) R_+(M, Q)} \right). \]  

(2.31)

This is the effective Boltzmann factor governing emission. Sufficiently far from extremality, when \( \omega, q \ll \sqrt{M^2 - Q^2} \), we can expand (2.31) to give

\[ \frac{\beta_{kk'}}{\alpha_{kk'}}^2 \approx \exp\left( -\omega - \frac{Qq}{R_+(M, Q)} + \frac{M^2 q^2 + Q^2 \omega^2 - 2MQ\omega q}{2(M^2 - Q^2)R_+(M, Q)} \right) \]  

(2.32)

as compared to the free field theory result \( [4] \),

\[ \frac{\beta_{kk'}}{\alpha_{kk'}}^2 = \exp\left( -\frac{\omega}{T(M, Q)} \right). \]  

(2.33)

Near extremality, the self-interaction corrections cause the emission to differ substantially from (2.33).

We might ask whether it is possible to reach extremality after a finite number of emissions. Since \( T(M - \omega, Q - q) \) appears in the denominator of the exponent of (2.31), the transition probability to the extremal state is in fact zero. We can also ask whether there are transitions to a meta-extremal (\( Q > M \)) hole. This would have rather dramatic implications as the meta-extremal hole is a naked singularity. To address this question we return to the saddle point equation (2.27). When \( Q > M \), \( R_+ \) and \( R_- \) become complex. From (2.18) we see that a saddle point solution would require that \( k \) be complex, but we do not allow this since a complete family of initial conditions \( S_k(0, r) = kr \) was defined with \( k \) real. Therefore, in the saddle point approximation the extremal hole is stable.

Modes with \( |\beta/\alpha| > 1 \) formally require larger amplitudes for higher occupation numbers, and thus require special interpretation. Considering for simplicity the free field form of these coefficients, (2.33), we see that such modes occur when \( \omega < qQ/R_+ \), that is when the incremental energy gain from discharging the Coulomb field overbalances the cost of creating the charged particle. Under these conditions one has dielectric breakdown of the vacuum, just as for a uniform electric field in empty space. Since this physics is not our primary concern in the present note, we shall restrict ourselves to a few remarks. The occupation factor appearing in the formula for radiation in these “superradiant” modes is negative, but the reflection probability exceeds unity, so the radiation flux is positive as it should be. And in general the formulas for physical quantities will appear sensible, although Fock space occupation numbers are not. We can avoid superradiance altogether by considering a model with only massive charged fundamental particles, and holes with a charge/mass ratio small compared to the minimal value for fundamental quanta.
Another interesting variant is to consider a \textit{magnetically} charged hole interacting with neutral matter. In that case, one simply puts $q = 0$ in the formulae above (but $Q \neq 0$). One could also consider the interaction of dyonic holes with charged matter, and other variants (\textit{e.g.} dilaton black holes) but we shall not do that here.

3. Discussion

We have arrived at the result (2.31) by what may appear to be a somewhat circuitous route. Inspired by a field theory question, we calculated the solutions of a single self-gravitating particle at the horizon, and then passed back to field theory by interpreting the solutions as the modes of a second quantized field operator. In this section we hope to clarify the logic of this procedure, and show that it is both correct and efficient, by demonstrating how a single particle action emerges from the truncation of a complete field theory.

We can illustrate this explicitly if we consider the simpler model of spherically symmetric electromagnetic and charged scalar fields interacting in flat space. Our goal is to show that the propagator for the scalar field can be expressed as a Hamiltonian path integral for a single charged shell. To achieve this, two important approximations will be made. The first is that the effects of vacuum polarization will be assumed to be small, so we can ignore scalar loop diagrams. The second is to assume that the dominant interactions involve soft photons, so that the difference in the scalar particle’s energy before and after emission or absorption of a photon is small compared to the energy itself. Thus we expect that our expression will be valid for cases where the scalar particle has a large energy, so that the energy transfer per photon is relatively small, and is far from the origin, so that the classical electromagnetic self energy of the particle is a slowly varying function of the radial coordinate. Field theory in this domain is in fact well described by the eikonal approximation, which implements the same approximations we have just outlined. What follows is then essentially a Hamiltonian version of the eikonal method.

We start from the action

$$S = -\frac{1}{4\pi} \int d^4x \left[ (\partial_\mu - i q A_\mu) \phi^* (\partial^\mu + i q A^\mu) \phi + m^2 \phi^* \phi + \frac{1}{4} F_{\mu \nu} F^{\mu \nu} \right]$$

$$= \int dt dr \left[ \pi_{\phi^*} \dot{\phi}^* + \pi_\phi \dot{\phi} - \left( \frac{\pi_{\phi^*} \pi_\phi}{r^2} + r^2 \phi^* \phi' + m^2 r^2 \phi^* \phi + \frac{\pi_{A^r}^2}{2r^2} \right) - A_t \left( i q [\pi_{\phi^*} \phi^* - \pi_\phi \phi] - \pi_{A^r} \right) \right].$$

Defining the charge density

$$\rho(r) \equiv i q [\pi_{\phi^*} (r) \phi^* (r) - \pi_\phi (r) \phi (r)] \quad (3.1)$$

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the solution of the Gauss’ law constraint is

$$Q(r) \equiv -\pi_A = \int_0^r dr \rho(r)$$  \hspace{1cm} (3.3)$$

and so the scalar field Hamiltonian becomes

$$H = \int_0^\infty dr \left[ \frac{\pi_{\phi^*}\pi_{\phi}}{r^2} + r^2 \phi^{*}\phi' + m^2 \phi^{*}\phi + \frac{Q(r)^2}{2r^2} \right].$$  \hspace{1cm} (3.4)$$

The fields are now written as second quantized operators:

$$\hat{\phi} = \int \frac{dk}{\sqrt{2\pi}} \sqrt{\omega_k} \left[ \hat{a}_k e^{ikr} + \hat{b}_k^\dagger e^{-ikr} \right]$$

$$\hat{\pi}_\phi = i \int \frac{dk}{\sqrt{2\pi}} \sqrt{\frac{\omega_k}{2}} \left[ \hat{a}_k e^{-ikr} - \hat{b}_k e^{ikr} \right]$$  \hspace{1cm} (3.5)$$

where $\omega_k = \sqrt{k^2 + m^2}$, and we also have $\hat{\phi}^* = \hat{\phi}^\dagger$, $\hat{\pi}_{\phi^*} = \hat{\pi}_{\phi}^\dagger$. To ensure that the field is nonsingular at the origin we impose the conditions $\hat{a}_{-k} = -\hat{a}_k$, $\hat{b}_{-k} = -\hat{b}_k$, and take the limits of all $k$ integrals to be from $-\infty$ to $\infty$.

We now write the Hamiltonian in terms of the creation and annihilation operators. In doing so we shall normal order the operators, which corresponds to omitting vacuum polarization since we do not allow particle-antiparticle pairs to be created out of the vacuum. Also when evaluating $\phi'$ we shall use the geometrical optics approximation, $(e^{ikr}/r)' \approx ike^{ikr}/r$, valid for $k \gg 1/r$. Then the quadratic part of the Hamiltonian becomes,

$$\int_0^\infty dr \left[ \frac{\pi_{\phi^*}\pi_{\phi}}{r^2} + r^2 \phi^{*}\phi' + m^2 \phi^{*}\phi \right] = \frac{1}{2} \int dk \omega_k \left[ \hat{a}_k^\dagger \hat{a}_k + \hat{b}_k^\dagger \hat{b}_k \right].$$  \hspace{1cm} (3.6)$$

Next we consider the interaction term. When evaluating this there will arise factors of $\sqrt{\omega_k}/\omega_k$. The essence of the soft photon approximation is that we replace these factors by 1, since we are assuming that $\Delta \omega/\omega \ll 1$ for the emission or absorption of a single photon. Then, after normal ordering, we can evaluate the charge density to be:

$$\hat{\rho}(r) = q \int \frac{dk dk'}{2\pi} \left[ \hat{a}_k^\dagger \hat{a}_{k'} - \hat{b}_k^\dagger \hat{b}_{k'} \right] e^{i(k-k')r}.$$  \hspace{1cm} (3.7)$$

We now wish to calculate matrix elements of the Hamiltonian between one particle states. A basis of one particle states labelled by position is given by

$$|r\rangle = \int \frac{dk}{\sqrt{2\pi}} e^{-ikr} \hat{a}_k^\dagger |0\rangle.$$  \hspace{1cm} (3.8)$$

The free part of the Hamiltonian then has matrix elements

$$\langle r_2 | \frac{1}{2} \int dk \omega_k \left[ \hat{a}_k^\dagger \hat{a}_k + \hat{b}_k^\dagger \hat{b}_k \right] | r_1 \rangle = \int \frac{dk}{2\pi} \omega_k \left[ e^{ik(r_2-r_1)} - e^{ik(r_2+r_1)} \right].$$  \hspace{1cm} (3.9)$$
The second term in the brackets corresponds to the path from $r_1$ to $r_2$ which passes through the origin. These paths will not contribute to local processes far from the origin, so we drop this term. The matrix elements of the interaction term for closely spaced points $r_1$ and $r_2$ are:

$$\langle r_2 | \int_0^\infty dr \frac{\hat{Q}(r)^2}{2r^2} | r_1 \rangle = \frac{q^2}{2r_1} \int \frac{dk}{2\pi} e^{ik(r_2-r_1)} .$$

(3.10)

Putting these together, we find the matrix elements of the Hamiltonian,

$$\langle r_2 | \hat{H} | r_1 \rangle = \int \frac{dk}{2\pi} e^{ik(r_2-r_1)} (\sqrt{k^2 + m^2} + q^2/2r_1) .$$

(3.11)

Now we can follow the standard route which leads from matrix elements of the Hamiltonian to a path integral expression for the time evolution operator, with the result

$$\langle r_f | e^{-i\hat{H}t} | r_i \rangle = \int_{r(0)=r_i}^{r(t)=r_f} Dp \ D\!r \ e^{i\int_0^t (p\dot{r} - \sqrt{p^2 + m^2} - q^2/2r)} .$$

(3.12)

The action in the exponent is precisely that of a charged shell, with $q^2/2r$ being the electromagnetic self energy.

We now discuss how this analysis can be applied to the case where we include gravitational interactions. The resulting field Hamiltonian is much more complex, and so we will not be able to explicitly calculate the effective shell action. However, the preceding derivation allows us to argue that were we to do so, we would simply derive the effective action obtained in section 2. The nature of the black hole radiance calculation makes us believe that the approximations used to arrive at a shell action are justified. This is so because for a large ($M \gg M_{pl}$) hole the relevant field configurations are short wavelength solutions moving in a region of relatively low curvature, and these are the conditions which we argued make the eikonal approximation valid.

For simplicity, we will consider an uncharged self-gravitating scalar field. If we truncate to the s-wave we arrive at what is known as the BCMN model, originally considered in [3] and corrected in [4]. The action is

$$S = \frac{1}{4\pi} \int d^4x \sqrt{-g} \left[ \frac{1}{4} R - \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi \right]$$

$$= \int dt \ dr \left[ \pi_\phi \dot{\phi} + \pi_R \dot{R} + \pi_L \dot{L} - N^t (\mathcal{H}_t^\phi + \mathcal{H}_t^\phi') - N^r (\mathcal{H}_r^\phi + \mathcal{H}_r^\phi') \right] - \int dt M_{ADM}$$

(3.13)

with

$$\mathcal{H}_t^\phi = \frac{1}{2} \left( \pi_\phi^2 + \frac{R^2}{L} \phi'^2 \right) ; \quad \mathcal{H}_r^\phi = \pi_\phi \phi'.$$

(3.14)
The analog of (2.8) is now
\[ M' = \frac{R'}{L} \mathcal{H}_t^\phi + \frac{\pi_L}{RL} \mathcal{H}_r^\phi = \frac{R'}{2L^2} \left( \frac{\pi_\phi^2}{R^2} + R^2 \phi'^2 \right) + \frac{\pi_L \pi_\phi \phi'}{RL} \] (3.15)

The Hamiltonian is
\[ H = M_{ADM} = \mathcal{M}(\infty) = M + \int_0^\infty dr \left[ \frac{R'}{L} \mathcal{H}_t^\phi + \frac{\pi_L}{RL} \mathcal{H}_r^\phi \right]. \] (3.16)

To obtain an expression for \( H \) which depends only on \( \phi \) and \( \pi_\phi \) we must choose a gauge and solve the constraints. We can obtain an explicit result if we choose the gauge \( R = r, \pi_L = 0 \). Then, defining
\[ h(r) \equiv \frac{1}{2} \left( \frac{\pi_\phi^2}{r^2} + r^2 \phi'^2 \right), \] (3.17)

\( L \) is determined from (3.15),
\[ \mathcal{M}'(r) = \left( \frac{r}{2} - \frac{r}{2L^2} \right)' = \frac{h(r)}{L^2} \] (3.18)
so
\[ \frac{1}{L^2} = -\frac{2M}{r} e^{-2 \int_0^r dr' h(r'/r')} + \frac{1}{r} e^{-2 \int_0^r dr' h(r'/r')} \int_0^r dr' e^2 \int_0^{r'} dr'' h(r''/r') \] (3.19)
which then leads to
\[ H = Me^{-2 \int_0^\infty dr h(r)/r} + \int_0^\infty dr h(r) e^{-2 \int_r^\infty dr' h(r')/r'}. \] (3.20)

This generalizes the result of [6] to include a nonzero mass \( M \) for the pure gravity solution. To make a direct comparison with our work in the previous section, it would be preferable to obtain the Hamiltonian in \( L = 1, R = r \) gauge. This is more difficult and we do not know the explicit expression. For the moment, though, we are mainly interested in the qualitative structure of the Hamiltonian, and (3.20) will be sufficient for our purposes.

The various nonlocal terms contained in the Hamiltonian (3.20) correspond to gravitons attaching onto the particle’s worldline. If we expand the exponentials in (3.20), we see that there arise an infinite series of bi-local, tri-local, . . . , terms resulting from the non-linearity of gravity. Now we could, in principle, repeat the analysis which led to an effective shell action for the charged field in flat space. In that case the calculation could be done with only modest effort because there was only a single quartic interaction term. In the present case we would have to sum the infinite series of terms that arise; our point is that handling all of these terms is cumbersome, to say the least, and that it is much simpler to proceed as in section 2.
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