Simulation of Dislocation in Cosserat Elastic Plates

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Abstract

In this article we present the numerical simulation of a dislocation incorporated into a Cosserat plate. The simulation is based on the mathematical model for bending of Cosserat elastic plates recently developed by the authors. The dislocation is modeled by a sequence of domains that converge to the point of the dislocation and by a residual force distributed around that point. The resulted plate deformation is calculated using the Finite Element method. We also discuss a possible effect of the dislocation on a hole incorporated into the plate.

Key words: dislocation, Cosserat plate, hoop stress, Dirac delta function.

1 Introduction

The classical theory of elasticity establishes that the movement of the particles within the body is described only by the displacement vector \(\mathbf{u}\). This implies that there are only three degrees of freedom for each particle and no rotations considered. Since only the force vector describes the surface loads the stress tensor is symmetric.
Such materials as foams, composites, concrete and human bones represent materials with microstructure and can not always be described by the classical elasticity [1], [2], [3], [4], [5], [6], [7].

The theory that takes into account the size effect of the particles and their rotations is Cosserat Elasticity. It employs three additional degrees of freedom for material particles, which represent their microrotations. The description of the surface loads is done by the force and moment vectors. The stress tensor is asymmetric and an additional couple stress tensor is incorporated. Six elastic constants are used in the description of the Cosserat solid: $\lambda$, $\mu$, $\alpha$, $\beta$, $\gamma$, $\epsilon$.

The Theory of Cosserat Elasticity starts with a pioneering work published by Eugene and Francois Cosserat in the beginning of the 20th century [8]. Since 1960s the number of publications on Cosserat Elasticity started to grow and has not stopped since. The complete theory of three-dimensional asymmetric elasticity gave rise to a variety of Cosserat plate theories. Naghdi obtained the linear theory of Cosserat plate [9] and Eringen proposed a complete theory of plates in the framework of Cosserat elasticity [10]. Steinberg proposed to use the Reissner plate theory as a basis for the theory of Cosserat plates in [11]. The finite element modeling for this theory is provided in [12].

The parametric theory of Cosserat plate, presented by the authors in [13], includes some additional assumptions leading to the introduction of the splitting parameter. The parametric theory produces the equilibrium equations, constitutive relations, and the optimal value of the minimization of the elastic energy of the Cosserat plate. The paper [13] also provides the analytical solutions of the presented plate theory and the three-dimensional Cosserat elasticity for simply supported rectangular plate. The comparison of these solutions showed that the precision of the developed Cosserat plate theory is similar to the precision of the classical plate theory developed by Reissner [14], [15].

The numerical modeling of bending of simply supported rectangular plates is given in [7]. We developed the Cosserat plate field equations and a rigorous formula for the optimal value of the splitting parameter. The solution of the Cosserat plate was shown to converge to the Reissner plate as the elastic asymmetric parameters tend to zero. The Cosserat plate theory demonstrates the agreement with the size effect, confirming that the plates of smaller thickness are more rigid than is expected from the Reissner model. The modeling of Cosserat plates with simply supported rectangular holes is also provided.

The extension of the static model of Cosserat elastic plates to the dynamic problems is presented in [18]. The computations predict a new kind of natural frequencies associated with the material microstructure and were shown to be compatible with
the size effect principle reported in [7] for the Cosserat plate bending.

The numerical study of Cosserat elastic plate deformation based on the parametric theory of Cosserat plates using the Finite Element Method is presented in [19]. The paper discusses the existence and uniqueness of the weak solution, convergence of the proposed FEM and its numerical validation by estimating the order of convergence. The Finite Element analysis of clamped Cosserat plates of different shapes under different loads is also provided. The numerical analysis of plates with circular holes shows that the stress concentration factor around the hole is less than the classical value, and smaller holes exhibit less stress concentration as would be expected on the basis of the classical elasticity.

The current article represents an extension of the paper [18] for different shapes and orientations of micro-elements incorporated into the Cosserat plates. It is based on the generalized variational principle for elastodynamics and includes a non-diagonal rotatory inertia tensor. The numerical computations of the plate free vibration showed the existence of some additional high frequencies of micro-vibrations depending on the orientation of micro-elements. The comparison with three-dimensional Cosserat elastodynamics shows a high agreement with the exact values of the eigenvalue frequencies.

2 Parametric Cosserat Plate Theory

The Cosserat linear elasticity balance laws are

\[
\begin{align*}
\text{div}\sigma & = 0, \\
\varepsilon \cdot \sigma + \text{div}\mu & = 0,
\end{align*}
\]  

(1) \hspace{2cm} (2)

where the \( \sigma \) is the stress tensor, \( \mu \) the couple stress tensor, \( p = \rho \frac{\partial u}{\partial t} \) and \( q = J \frac{\partial \phi}{\partial t} \) are the linear and angular momenta, \( \rho \) is the material density, \( \varepsilon \) is the Levi-Civita tensor.

We will also consider the constitutive equations in the following form [20]:

\[
\begin{align*}
\sigma & = (\mu + \alpha)\gamma + (\mu - \alpha)\gamma^T + \lambda(\text{tr}\gamma)1, \\
\mu & = (\gamma + \epsilon)\chi + (\gamma - \epsilon)\chi^T + \beta(\text{tr}\chi)1,
\end{align*}
\]  

(3) \hspace{2cm} (4)

and the kinematic relations in the form

\[
\gamma = (\nabla u)^T + \varepsilon \cdot \phi \text{ and } \chi = (\nabla \phi)^T,
\]

(5)
Here \( \mathbf{u} \) and \( \phi \) represent the displacement and rotation vectors, \( \gamma \) and \( \chi \) represent the strain and torsion tensors, \( \mu, \lambda \) are the Lamé parameters and \( \alpha, \beta, \gamma, \epsilon \) are the Cosserat elasticity parameters.

The constitutive equations (3) - (4) can be written in the reverse form [11]:

\[
\begin{align*}
\gamma &= (\mu' + \alpha')\sigma + (\mu' - \alpha')\sigma^T + \lambda'(\text{tr}\sigma)\mathbf{1}, \\
\chi &= (\gamma' + \epsilon')\mu + (\gamma' - \epsilon')\mu^T + \beta'(\text{tr}\mu)\mathbf{1},
\end{align*}
\]

where \( \mu' = \frac{1}{4\mu}, \alpha' = \frac{1}{4\alpha}, \gamma' = \frac{1}{4\gamma}, \epsilon' = \frac{1}{4\epsilon}, \lambda' = \frac{-\lambda}{6\mu(\lambda + \frac{4\mu}{3})} \) and \( \beta' = \frac{-\beta}{6\mu(\beta + \frac{2\mu}{3})} \).

We will consider the boundary conditions provided in [13]:

\[
\begin{align*}
\mathbf{u} &= \mathbf{u}_0, \quad \phi = \phi_0 \text{ on } G_1^t = \partial B_0 \setminus \partial B_0 \times [t_0, t], \\
\sigma_n &= \sigma \cdot \mathbf{n} = \sigma_0, \text{ on } G_2^t = \partial B_0 \times [t_0, t], \\
\mu_n &= \mu \cdot \mathbf{n} = \mu_0 \text{ on } G_2^t = \partial B_0 \times [t_0, t],
\end{align*}
\]

where and \( \phi_0 \) are prescribed on \( G_1, \sigma_0 \) and \( \mu_0 \) on \( G_2, \) and \( \mathbf{n} \) is the unit vector normal to the boundary \( \partial B_0 \) of the elastic body \( B_0 \).

In this section we review our stress, couple stress and kinematic assumptions of the Cosserat plate [13]. We consider the thin plate \( P \), where \( h \) is the thickness of the plate and \( x_3 = 0 \) represent its middle plane. The sets \( T \) and \( B \) are the top and bottom surfaces contained in the planes \( x_3 = h/2, x_3 = -h/2 \) respectively and the curve \( \Gamma \) is the boundary of the middle plane of the plate.

The set of points \( P = (\Gamma \times [-\frac{h}{2}, \frac{h}{2}]) \cup T \cup B \) forms the entire surface of the plate and \( \Gamma_u \times [-\frac{h}{2}, \frac{h}{2}] \) is the lateral part of the boundary where displacements and microrotations are prescribed. The notation \( \Gamma_\sigma = \Gamma \setminus \Gamma_u \) of the remainder we use to describe the lateral part of the boundary edge \( \Gamma_\sigma \times [-\frac{h}{2}, \frac{h}{2}] \) where stress and couple stress are prescribed. We also use notation \( P_0 \) for the middle plane internal domain of the plate.

In our case we consider the vertical load and pure twisting momentum boundary conditions at the top and bottom of the plate, which can be written in the form:

\[
\begin{align*}
\sigma_{33}(x_1, x_2, h/2, t) &= \sigma^t(x_1, x_2, t), \\
\sigma_{33}(x_1, x_2, -h/2, t) &= \sigma^b(x_1, x_2, t), \\
\sigma_{3\beta}(x_1, x_2, \pm h/2, t) &= 0, \\
\mu_{33}(x_1, x_2, h/2, t) &= \mu^t(x_1, x_2, t), \\
\mu_{33}(x_1, x_2, -h/2, t) &= \mu^b(x_1, x_2, t), \\
\mu_{3\beta}(x_1, x_2, \pm h/2, t) &= 0,
\end{align*}
\]
where \((x_1, x_2) \in P_0\).

The plate stress and kinetic energy density are defined by the formulas

\[
U^S_K = \int_{P_0} \Phi(S) da,
\]

where \(P_0\) is the internal domain of the middle plane of the plate and \(\Phi(S)\) and \(\Upsilon_C(\frac{\partial U}{\partial t})\) are given as follows:

\[
\Phi(S) = \frac{3\lambda(M_{\alpha\alpha}) (M_{\beta\beta})}{h^3 \mu (3\lambda + 2\mu)} + \frac{3(\alpha + \mu) M^2_{\alpha\beta}}{2h^3 \alpha \mu} + \frac{3(\alpha + \mu)}{160h^3 \alpha \mu} \left(8\hat{Q}_\alpha \hat{Q}_\alpha + 15Q_\alpha \hat{Q}_\alpha\right) + \frac{3(\alpha + \mu)}{160h^3 \alpha \mu} \left(20\hat{Q}_\alpha Q^*_\alpha + 8Q_\alpha Q^*_\alpha\right) + \frac{3(\gamma + \epsilon) S^*_\alpha S^*_\alpha}{2h^3 \gamma \epsilon} + \frac{3(\alpha - \mu) M^2_{\alpha\beta}}{2h^3 \alpha \mu} + \frac{\alpha - \mu}{280h^3 \alpha \mu} \left[21Q_\alpha \left(5\hat{Q}_\alpha + 4Q^*_\alpha\right)\right] - \frac{\gamma - \epsilon}{160h^3 \gamma \epsilon} \left[24R^2_{\alpha\alpha} + 45R^*_\alpha R_{\alpha\alpha} + 60R_{\alpha\beta} R^*_\alpha R_{\alpha\beta}\right] + \frac{\gamma + \epsilon}{160h^3 \gamma \epsilon} \left[8R^2_{\alpha\beta} + 15R^*_{\alpha\beta} R_{\alpha\beta} + 20R_{\alpha\beta} R^*_{\alpha\beta}\right] + \frac{3\beta}{80h^3 \gamma (3\beta + 2\gamma)} \left[8(R_{\alpha\alpha}) (R_{\beta\beta}) + 15(R^*_{\alpha\alpha}) (R^*_{\beta\beta})\right] + \frac{3\beta}{80h^3 \gamma (3\beta + 2\gamma)} \left[20(R_{\alpha\alpha}) (R^*_{\alpha\alpha})\right] - \frac{\beta}{4\gamma (3\beta + 2\gamma)} [(2R_{\alpha\alpha} + 3R^*_{\alpha\alpha}) t - h(V^2 + T^2)] + \frac{\lambda}{560h\mu (3\lambda + 2\mu)} \left[\frac{5 + 3\eta}{(1 + \eta)} p M_{\alpha\alpha}\right] + \frac{(\lambda + \mu) h}{840\mu (3\lambda + 2\mu)} \left(\frac{140 + 168\eta + 51\eta^2}{4(1 + \eta)^2}\right) p^2 - \frac{(\lambda + \mu) h}{2\mu (3\lambda + 2\mu)} \sigma_0^2 + \frac{eh}{12h \gamma \epsilon} [(3T^2 + V^2)]
\]
\( S, u \) and \( E \) are the Cosserat plate stress, displacement and strain sets

\[
S = \left( M_{\alpha\beta}, Q_\alpha, Q_\ast^\alpha, \hat{Q}_\alpha, R_{\alpha\beta}, R_{\ast\alpha\beta}, S_{\ast\beta}^\ast \right),
\]

\[
S_n = \left( \hat{M}_\alpha, \hat{Q}_\ast^\alpha, \hat{Q}_\ast^\hat{\beta}, \hat{R}_{\alpha\beta}, \hat{R}_{\ast\alpha\beta}, \hat{S}_{\ast\beta}^\ast \right),
\]

\[
S_o = \left( \Pi_{\alpha\alpha}, \Pi_{\alpha\beta}, \Pi_{\beta\beta}^\ast, M_{\alpha\alpha}, M_{\alpha\beta}^\ast, M_{\beta\beta}^\ast \right),
\]

\[
u = \left( \Psi_\alpha, W, \Omega_3, \Omega_0^\ast, W^\ast, \Omega_0^0 \right),
\]

\[
E = \left( \epsilon_{\alpha\beta}, \omega_\beta, \hat{\omega}_\alpha, \tau_{3\alpha}, \tau_{3\beta}, \tau_{\alpha\beta}^\ast \right),
\]

where

\[
M_{\alpha\beta} n_\beta = \Pi_{\alpha\alpha}, R_{\alpha\beta} n_\beta = M_{\alpha\alpha},
\]

\[
Q_\ast^\alpha n_\alpha = \Pi_{\alpha\beta}, S_{\alpha}^\ast n_\alpha = M_{\alpha\beta}^\ast,
\]

\[
\hat{Q}_\alpha n_\alpha = \Pi_{\alpha\beta}^\ast, R_{\alpha\beta}^\ast n_\beta = M_{\alpha\beta}^\ast,
\]

\[
\hat{M}_\alpha = M_{\alpha\beta} n_\beta, \hat{Q}_\ast^\beta = Q_\ast^\beta n_\beta,
\]

\[
\hat{R}_{\alpha\beta} = R_{\alpha\beta} n_\beta, \hat{S}_{\ast\beta}^\ast = S_{\ast\beta}^\ast n_\beta,
\]

\[
\hat{Q}_\ast^\ast = \hat{Q}_\ast^\beta n_\beta, \hat{R}_{\alpha}^\ast = \hat{R}_{\alpha\beta} n_\beta.
\]

In the above \( n_\beta \) is the outward unit normal vector to \( \Gamma_u \).

The plate characteristics, being the functions of \( x_1, x_2 \) and \( t \), provide the approximation of the components of the three-dimensional tensors \( \sigma \) and \( \mu \)

\[
\sigma_{\alpha\beta} = \frac{6}{h^2} \zeta M_{\alpha\beta},
\]

\[
\sigma_{3\beta} = \frac{3}{2h} \left( 1 - \zeta^2 \right) Q_\beta,
\]

\[
\sigma_{\beta3} = \frac{3}{2h} \left( 1 - \zeta^2 \right) Q_\ast^\beta + \frac{3}{2h} \hat{Q}_\beta,
\]

\[
\sigma_{33} = -\frac{3}{4} \left( \frac{1}{3} \zeta^3 - \zeta \right) p_1 + \zeta p_2 + \sigma_0,
\]

\[
\mu_{\alpha\beta} = \frac{3}{2h} \left( 1 - \zeta^2 \right) R_{\alpha\beta} + \frac{3}{2h} R_{\ast\alpha\beta},
\]

\[
\mu_{3\beta} = \frac{6}{h^2} \zeta S_{\ast\beta}^\ast,
\]

\[
\mu_{33} = \zeta V + T,
\]
where

\[ p = \sigma^t(x_1, x_2, t) - \sigma^b(x_1, x_2, t), \tag{24} \]

\[ \sigma_0 = \frac{1}{2} \left( \sigma^t(x_1, x_2, t) + \sigma^b(x_1, x_2, t) \right), \tag{25} \]

\[ V = \frac{1}{2} \left( \mu^t(x_1, x_2, t) - \mu^b(x_1, x_2, t) \right), \tag{26} \]

\[ T = \frac{1}{2} \left( \mu^t(x_1, x_2, t) + \mu^b(x_1, x_2, t) \right), \tag{27} \]

\[ p_1 = \eta p(x_1, x_2, t), \tag{28} \]

\[ p_2 = \frac{(1 - \eta)}{2} p(x_1, x_2, t), \tag{29} \]

three-dimensional displacements \( u \) and microrotations \( \phi \)

\[ u_\alpha = \frac{h}{2} \zeta \Psi_\alpha, \tag{30} \]

\[ u_3 = W + (1 - \zeta^2) W^*, \tag{31} \]

\[ \phi_\alpha = \Omega_\alpha^0 (1 - \zeta^2) + \hat{\Omega}_\alpha, \tag{32} \]

\[ \phi_3 = \zeta \Omega_3, \tag{33} \]

and the three-dimensional strain and torsion tensors \( \gamma \) and \( \chi \)

\[ \gamma_{\alpha\beta} = \frac{6}{h^2} \zeta e_{\alpha\beta}, \tag{34} \]

\[ \gamma_{3\beta} = \frac{3}{2h} (1 - \zeta^2) \omega_\beta, \tag{35} \]

\[ \gamma_{\beta3} = \frac{3}{2h} (1 - \zeta^2) \omega^*_\beta + \frac{3}{2h} \hat{\omega}_\beta, \tag{36} \]

\[ \chi_{\alpha\beta} = \frac{3}{2h} (1 - \zeta^2) \tau_{\alpha\beta} + \frac{3}{2h} \tau^*_{\alpha\beta}, \tag{37} \]

\[ \chi_{3\beta} = \frac{6}{h^2} \zeta \tau^*_\beta, \tag{38} \]

where \( \zeta = \frac{2x_3}{h} \).

Then zero variation of the functional

\[ \delta \Theta(s, \eta) = 0 \]

is equivalent to the plate bending system of equations (A) and constitutive formulas (B) mixed problems.
(A). The bending equilibrium system of equations:

\[
M_{\alpha\beta,\alpha} - Q_\beta = 0, \quad (39) \\
Q_{\alpha,\alpha}^* + \hat{p}_1 = 0, \quad (40) \\
R_{\alpha\beta,\alpha} + \varepsilon_{3\gamma\gamma}(Q_*^\gamma - Q_\gamma) = 0, \quad (41) \\
\varepsilon_{3\gamma\gamma} M_{\beta\gamma} + S_{\alpha,\alpha}^* = 0, \quad (42) \\
\hat{Q}_{\alpha,\alpha} + \hat{\rho}_2 = 0, \quad (43) \\
R_{\alpha\beta,\alpha}^* + \varepsilon_{3\gamma\gamma} \hat{Q}_\gamma = 0, \quad (44)
\]

with the resultant traction boundary conditions :

\[
M_{\alpha\beta} n_\beta = \Pi_{\alpha\alpha}, \quad R_{\alpha\beta} n_\beta = M_{\alpha\alpha}, \quad (45) \\
Q_{\alpha}^* n_\alpha = \Pi_{\alpha3}, \quad S_{\alpha}^* n_\alpha = \Upsilon_{\alpha3}, \quad (46)
\]

at the part \(\Gamma_\sigma\) and the resultant displacement boundary conditions and \(\hat{\rho}_1 = \eta_{opt} p\) and \(\hat{\rho}_2 = \frac{2}{3} (1 - \eta_{opt}) p\).

\[
\Psi_{\alpha} = \Psi_{\alpha\alpha}, \quad W = W_\alpha, \quad \Omega^0_{\alpha} = \Omega^0_{\alpha\alpha}, \quad \Omega_3 = \Omega_{\alpha3}, \quad (47)
\]

at the part \(\Gamma_u\).

(B). Constitutive formulas in the reverse form:

\[
M_{\alpha\alpha} = \frac{\mu (\lambda + \mu)}{3(\lambda + 2\mu)} h^3 \Psi_{\alpha,\alpha} + \frac{\lambda \mu h^3}{6(\lambda + 2\mu)} \Psi_{\beta,\beta} + \frac{(3p_1 + 5p_2) \lambda h^2}{30(\lambda + 2\mu)}, \\
M_{\beta\alpha} = \frac{(\mu - \alpha) h^3}{12} \Psi_{\alpha,\beta} + \frac{(\mu + \alpha) h^3}{12} \Psi_{\beta,\alpha} + (-1)^{\alpha'} \frac{\alpha h^3}{6} \Omega_3,
\]

\[
R_{\beta\alpha} = \frac{5 (\gamma - \epsilon) h}{6} \Omega^0_{\beta,\alpha} + \frac{5 (\gamma + \epsilon) h}{6} \Omega^0_{\alpha,\beta}, \\
R_{\alpha\alpha} = \frac{10h \gamma}{3(\beta + 2\gamma)} \Omega^0_{\alpha,\alpha} + \frac{5h\beta \gamma}{3(\beta + 2\gamma)} \Omega^0_{\beta,\beta}, \\
R_{\alpha\alpha}^* = \frac{2 (\gamma - \epsilon) h}{3} \hat{\Omega}_{\beta,\alpha} + \frac{2 (\gamma + \epsilon) h}{3} \hat{\Omega}_{\alpha,\beta}, \\
R_{\alpha\alpha}^* = \frac{8 \gamma (\gamma + \beta) h}{3(\beta + 2\gamma)} \hat{\Omega}_{\alpha,\alpha} + \frac{4 \gamma \beta h}{3(\beta + 2\gamma)} \hat{\Omega}_{\beta,\beta},
\]

\[1\]In the following formulas a subindex \(\beta = 1\) if \(\alpha = 2\) and \(\beta = 2\) if \(\alpha = 1\)
\[ Q_\alpha = \frac{5(\mu + \alpha)h}{6} \Psi_\alpha + \frac{5(\mu - \alpha)h}{6} W_{\alpha} + \frac{2(\mu - \alpha)h}{3} W^*_\alpha + (-1)^\beta \frac{5h\alpha}{3} \Omega^0_\beta + (-1)^\beta \frac{5h\alpha}{3} \hat{\Omega}_\beta, \]
\[ Q^*_\alpha = \frac{5(\mu - \alpha)h}{6} \Psi_\alpha + \frac{5(\mu - \alpha)^2h}{6(\mu + \alpha)} W_{\alpha} + \frac{2(\mu + \alpha)h}{3} W^*_\alpha + (-1)^\beta \frac{5h\alpha}{3} \left( \frac{\Omega^0_\beta}{(\mu + \alpha)\hat{\Omega}_\beta} \right), \]
\[ \hat{Q}_\alpha = \frac{8\alpha\mu h}{3(\mu + \alpha)} W_{\alpha} + (-1)^\alpha \frac{8\alpha\mu h}{3(\mu + \alpha)} \hat{\Omega}_\beta, \]
\[ S^*_\alpha = \frac{5\gamma \epsilon h^3}{3(\gamma + \epsilon)} \Omega_{3,\alpha}, \] (48)
and the optimal value \( \eta_{opt} \) of the splitting parameter is given as in [7]:
\[ \eta_{opt} = \frac{2\mathcal{W}^{(00)} - \mathcal{W}^{(10)} - \mathcal{W}^{(01)} - \mathcal{W}^{(01)}}{2\left(\mathcal{W}^{(11)} + \mathcal{W}^{(00)} - \mathcal{W}^{(10)} - \mathcal{W}^{(01)}\right)}. \] (49)

where \( \mathcal{W}^{(ij)} = S|_{\eta = i} \cdot \mathcal{E}|_{\eta = j} \).

The Cosserat plate field equations are obtained by substituting the relations into the system of equations (39) – (44) similar to [7]:
\[ L v = f(\eta) \] (50)

where
\[
L = \begin{pmatrix}
L_{11} & L_{12} & L_{13} & L_{14} & 0 & L_{16} & kL_{13} & 0 & L_{16} \\
L_{12} & L_{22} & L_{23} & L_{24} & L_{16} & 0 & kL_{23} & L_{16} & 0 \\
L_{13} & -L_{23} & L_{33} & 0 & L_{35} & L_{36} & L_{77} & L_{38} & L_{39} \\
L_{14} & L_{42} & 0 & L_{44} & 0 & 0 & 0 & 0 & 0 \\
0 & -L_{16} & -L_{38} & 0 & L_{55} & L_{56} & -kL_{35} & L_{58} & 0 \\
L_{16} & 0 & -L_{39} & 0 & L_{56} & L_{66} & -kL_{36} & L_{58} & 0 \\
L_{13} & -L_{14} & L_{73} & 0 & L_{35} & L_{36} & L_{77} & L_{78} & L_{79} \\
0 & -L_{16} & -L_{78} & 0 & L_{85} & L_{56} & -kL_{35} & L_{88} & kL_{56} \\
L_{16} & 0 & -L_{79} & 0 & L_{56} & L_{55} & -kL_{36} & kL_{56} & L_{99}
\end{pmatrix}
\]
\[ f(\eta) = \begin{pmatrix} -3h^2\lambda(3p_{1,1}+5p_{2,1}) \\ \frac{30(\lambda+2\mu)}{30(\lambda+2\mu)} \\ -3h^2\lambda(3p_{1,2}+5p_{2,2}) \\ -\frac{30(\lambda+2\mu)}{30(\lambda+2\mu)} \\ -p_1 \\ 0 \\ 0 \\ 0 \\ \frac{h^2(3p_1+4p_2)}{24} \\ 0 \\ 0 \end{pmatrix} \]

Here \( p_1 = \eta p, p_2 = \frac{1-\eta}{2} p \) and \( \mathcal{U} \) is given as before

\[ v = (\Psi_1, \Psi_2, W, \Omega_3, \Omega_1^0, \Omega_2^0, W^*, \Omega_1^0, \Omega_2^0)^T \]
The operators $L_{ij}$ are given as follows

\[ L_{11} = c_1 \frac{\partial^2}{\partial x_1^2} + c_2 \frac{\partial^2}{\partial x_2^2} - c_3, \]
\[ L_{12} = (c_1 - c_2) \frac{\partial}{\partial x_1 x_2}, \]
\[ L_{13} = c_{11} \frac{\partial}{\partial x_1}, \]
\[ L_{14} = c_{12} \frac{\partial}{\partial x_2}, \]
\[ L_{16} = c_{13}, \]
\[ L_{17} = k_1 c_{11} \frac{\partial}{\partial x_1}, \]
\[ L_{22} = c_2 \frac{\partial^2}{\partial x_1^2} + c_1 \frac{\partial^2}{\partial x_2^2} - c_3, \]
\[ L_{23} = c_{11} \frac{\partial}{\partial x_1}, \]
\[ L_{24} = c_{12} \frac{\partial}{\partial x_2}, \]
\[ L_{26} = c_{13}, \]
\[ L_{27} = c_{11} \frac{\partial}{\partial x_1}, \]
\[ L_{28} = c_{12} \frac{\partial}{\partial x_2}, \]
\[ L_{29} = c_{13}, \]
\[ L_{33} = c_3 (\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}), \]
\[ L_{35} = c_{11} \frac{\partial}{\partial x_1}, \]
\[ L_{36} = c_{12} \frac{\partial}{\partial x_2}, \]
\[ L_{38} = c_{13}, \]
\[ L_{39} = c_{11} \frac{\partial}{\partial x_1}, \]
\[ L_{41} = c_{12} \frac{\partial}{\partial x_2}, \]
\[ L_{42} = c_{13}, \]
\[ L_{44} = c_6 \left( \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} \right) - 2c_{12}, \]
\[ L_{45} = (c_7 - c_8) \frac{\partial^2}{\partial x_1 x_2}, \]
\[ L_{47} = c_6 \left( \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} \right) - 2c_{13}, \]
\[ L_{48} = c_7 \frac{\partial}{\partial x_2}, \]
\[ L_{49} = c_8 \frac{\partial}{\partial x_1}, \]
\[ L_{55} = c_7 \frac{\partial^2}{\partial x_1^2} + c_8 \frac{\partial^2}{\partial x_2^2} - 2c_{13}, \]
\[ L_{56} = c_8 \frac{\partial}{\partial x_2}, \]
\[ L_{58} = -c_9, \]
\[ L_{59} = c_7 \frac{\partial}{\partial x_1}, \]
\[ L_{66} = c_8 \frac{\partial^2}{\partial x_1^2} + c_7 \frac{\partial^2}{\partial x_2^2} - 2c_{13}, \]
\[ L_{73} = c_5 \left( \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} \right), \]
\[ L_{77} = c_4 \left( \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} \right), \]
\[ L_{78} = -c_{14} \frac{\partial}{\partial x_2}, \]
\[ L_{79} = c_{14} \frac{\partial}{\partial x_1}, \]
\[ L_{85} = c_7 \frac{\partial^2}{\partial x_1^2} + c_8 \frac{\partial^2}{\partial x_2^2} - 2c_{13}, \]
\[ L_{88} = c_7 \frac{\partial^2}{\partial x_1^2} + c_8 \frac{\partial^2}{\partial x_2^2} - c_{15}, \]
\[ L_{99} = c_8 \frac{\partial^2}{\partial x_1^2} + c_7 \frac{\partial^2}{\partial x_2^2} - c_{15}. \]
The coefficients \( c_i \) are given as

\[
\begin{align*}
  c_1 &= \frac{h^3 \mu (\lambda + \mu)}{3(\lambda + 2\mu)}, \\
  c_2 &= \frac{h^3(\alpha + \mu)}{12}, \\
  c_3 &= \frac{5h(\alpha + \mu)}{6}, \\
  c_4 &= \frac{5h(\alpha - \mu)^2}{6(\alpha + \mu)}, \\
  c_5 &= \frac{h(5\alpha^2 + 6\alpha \mu + 5\mu^2)}{6(\alpha + \mu)}, \\
  c_6 &= \frac{h^3 \gamma \epsilon}{3(\gamma + \epsilon)}, \\
  c_7 &= \frac{10h \gamma (\beta + \gamma)}{3(\beta + 2\gamma)}, \\
  c_8 &= \frac{5h(\gamma + \epsilon)}{6}, \\
  c_9 &= \frac{10h \alpha^2}{3(\alpha + \mu)}, \\
  c_{10} &= \frac{5h \alpha(\alpha - \mu)}{3(\alpha + \mu)}, \\
  c_{11} &= \frac{5h(\alpha - \mu)}{6}, \\
  c_{12} &= \frac{h^3 \alpha}{6}, \\
  c_{13} &= \frac{5h \alpha}{3}, \\
  c_{14} &= \frac{h \alpha(5\alpha + 3\mu)}{3(\alpha + \mu)}, \\
  c_{15} &= \frac{2h \alpha(5\alpha + 4\mu)}{3(\alpha + \mu)}.
\end{align*}
\]

3 Numerical Simulation

Even though a lot of research has been done based on the Classical Elasticity, there have not been much progress for the case of Cosserat materials. The appearance of defects, cracks, dislocations or other inhomogeneities can affect the performance of the material. This creates a stress field around the dislocation and might affect the body as a whole and also act on its cavities and displace them. Therefore correctly assessing the effect of the dislocation is essential for use of the material in applications.

For the numerical simulation we will consider a two-dimensional body \( B_0 \) (plate) with a dislocation (inclusion, defect, inhomogeneity) and follow \([23]\).
Let $b_i$ be the Burger’s vector, denoting an additional displacement of the lattice points. The distortion tensor is

$$w_{ik} = u_{k,i}$$
and
\[ \oint_L w_{ik} dx_i = -b_k \]
or equivalently
\[ \int_{S_L} e_{ilm} w_{mk,j} df_i = -b_k \]

The two-dimensional delta-function \( \delta (\xi) \) satisfies
\[ \int \delta (\xi) \tau \cdot df = \tau_i \int_{S_L} \delta (\xi) \cdot df_i = 1 \]

Therefore
\[ e_{ilm} w_{mk,j} df_i = -\tau_i b_k \delta (\xi) \]

Let \( (a, b) \) be the position of the dislocation. We will simulate the effect of the two-dimensional delta function \( \delta (\xi) \) by assuming the boundary condition on some circular neighbourhood represented by the function \( g(x_1, x_2) \):
\[ g(x_1, x_2) = \frac{1}{2\pi} \arctan \left( \frac{x_2 - b}{x_1 - a} \right) \]  \( (51) \)

The function \( g(x_1, x_2) \) gives a constant for any line integral along the circle \( L(r) \) of radius \( r \) centred at the dislocation:
\[ \oint_{L(r)} g_i dx_i = 1 \]

Let us define
\[ J_j = \int_S b_{ij} n_i dA \]
where \( b_{ij} \) is the material stress tensor.

The values of \( J_j \) can be found from [23]:
\[ J_1 = \int_0^{2\pi} \sigma_{\phi\phi} (\phi) \cos \phi d\phi \]
\[ J_2 = \int_0^{2\pi} \sigma_{\phi\phi} (\phi) \sin \phi d\phi \]

where
\[ \sigma_{\phi\phi} (\phi) = \sigma_{11}^C + \sigma_{22}^C + 2 \left( \sigma_{\phi\phi}^L (r, \phi) - \sigma_{rr}^L (r, \phi) \right) \]
\( \sigma_{\phi\phi}(\phi) \) is the hoop stress, \( \sigma_{11}^C, \sigma_{22}^C \) are the stresses in Cartesian coordinates at the center of the hole, \( \sigma_{\phi\phi}^L(r, \phi), \sigma_{rr}^L(r, \phi) \) are the stresses in polar coordinates along the boundary of the hole (functions of \( \phi \)).

The stresses in Cartesian coordinates at the center of the hole \( \sigma_{11}^C, \sigma_{22}^C \) can be found from the solution set of kinematic variables \( v = [\Psi_1, \Psi_2, W, \Omega_3, \Omega_1^0, \Omega_2^0, W^*, \Omega_1^0, \Omega_2^0] \) and the constitutive formulas.

The stresses in polar coordinates \( \sigma_{\phi\phi}^L(r, \phi), \sigma_{rr}^L(r, \phi) \) can be found from the stress tensor \( \sigma_{ij} \) in Cartesian coordinates by the following transformation:

\[
\begin{align*}
\sigma_{\phi\phi} &= \sigma_{11} \cos^2(\phi) + \sigma_{22} \sin^2(\phi) + (\sigma_{12} + \sigma_{21}) \sin(\phi) \cos(\phi) \\
\sigma_{rr} &= \sigma_{22} \cos^2(\phi) + \sigma_{11} \sin^2(\phi) - (\sigma_{12} + \sigma_{21}) \sin(\phi) \cos(\phi)
\end{align*}
\]

Once we find \( J_1 \) and \( J_2 \) the direction of the force acting on the cavity induced by the dislocation can be calculated as:

\[
\frac{dx_1}{dx_2} = \frac{J_1}{J_2}
\]

We will model the simulation using the two-dimensional Dirac delta function \( \delta(\xi) \) being non-zero at the point of dislocation. The numerical simulation of the Dirac delta function is proposed to be done by the function (51).

By shrinking the hole around the dislocation and applying the boundary conditions that are consistent with (51) we will simulate the dislocation as a limiting case of these domains. In the presence of an additional cavity this will result in a residual force, which can be calculated from the vector of solutions for kinematic variables \( v \).
Figure 2: Distribution of the large values of $\sigma_{11}$ stress component (top 10% of the magnitude).
Figure 3: Distribution of the $\sigma_{11}$ stress component.

The vector of solutions for kinematic variables $v$ can be obtained by solving the elliptic system of partial differential equations (50) applying the Finite Element method developed in [19].

In our computations we consider the plates made of polyurethane foam – a material reported in the literature to behave Cosserat like – and the values of the technical
elastic parameters presented in [25]:

\[ E = 299.5 \text{ MPa}, \]
\[ \nu = 0.44, \]
\[ l_t = 0.62 \text{ mm}, \]
\[ l_b = 0.327 \text{ mm}, \]
\[ N^2 = 0.04. \]

Taking into account that the ratio \( \beta/\gamma \) is equal to 1 for bending [25], these values of the technical constants correspond to the following values of Lamé and Cosserat parameters:

\[ \lambda = 762.616 \text{ MPa}, \]
\[ \mu = 103.993 \text{ MPa}, \]
\[ \alpha = 4.333 \text{ MPa}, \]
\[ \beta = 39.975 \text{ MPa}, \]
\[ \gamma = 39.975 \text{ MPa}, \]
\[ \epsilon = 4.505 \text{ MPa}. \]

We consider a plate \( 10 \times 10 \) with its points represented on the coordinate plane by the Descartes product of the segments: \([-7, 3] \times [-5, 5]\). Let \( h = 0.3 \) be the thickness of the plate. The dislocation is located at the point \((-3, 2)\) and the cavity located in the origin \((0, 0)\) inside the plate (see Figure 1).

Let \( G = G_1 \cup G_2 \) be the external boundary of the plate:

\[ G_1 = \{(x_1, x_2) : x_1 \in \{0, a\}, x_2 \in [0, a]\} \]
\[ G_2 = \{(x_1, x_2) : x_2 \in \{0, a\}, x_1 \in [0, a]\} \]

and the following hard simply supported boundary conditions [13] similar to [24]:

\[ G_1 : \quad W = 0, \quad W^* = 0, \quad \Psi_2 = 0, \quad \Omega_2 = 0, \quad \hat{\Omega}_2 = 0; \quad \Omega_3 = 0, \quad \hat{\Omega}_3 = 0, \quad (52) \]
\[ G_2 : \quad \partial \Psi_2 / \partial n = 0, \quad \partial \Omega_2 / \partial n = 0, \quad \partial \hat{\Omega}_2 / \partial n = 0; \quad (53) \]
\[ G_1 : \quad \partial \Psi_1 / \partial n = 0, \quad \partial \Omega_1 / \partial n = 0, \quad \partial \hat{\Omega}_1 / \partial n = 0; \quad (54) \]
\[ G_2 : \quad \partial \Psi_2 / \partial n = 0, \quad \partial \Omega_2 / \partial n = 0, \quad \partial \hat{\Omega}_2 / \partial n = 0; \quad \Omega_3 = 0, \quad \hat{\Omega}_3 = 0; \quad (55) \]
\[ G_2 : \quad \partial \Psi_2 / \partial n = 0, \quad \partial \Omega_2 / \partial n = 0, \quad \partial \hat{\Omega}_2 / \partial n = 0. \quad (56) \]

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The results of the stress fields around the dislocation are given on the Figures 2–3. These stress fields induce the force that acts on the cavity. The direction of the force can be obtained by the comparison of the values $J_1$ and $J_2$. The direction of the force acting on the hole is given in the Figure 4. In case of a crack, the direction of the residual force will show the trajectory of the propagation of the crack.
4 Conclusion

In this article we presented the numerical simulation of a dislocation incorporated into a Cosserat plate. The simulation is based on the mathematical model for bending
of Cosserat elastic plates recently developed by the authors. The dislocation is modeled by a sequence of domains that converge to the point of the dislocation and by a residual force distributed around that point. The resulted plate deformation is calculated using the Finite Element method. We also discuss a possible effect of the dislocation on a hole incorporated into the plate.

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