ANOMALY CANCELLATION AND MODULARITY II: THE $E_8 \times E_8$ CASE

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Abstract. In this paper we show that both of the Green-Schwarz anomaly factorization formula for the gauge group $E_8 \times E_8$ and the Hořava-Witten anomaly factorization formula for the gauge group $E_8$ can be derived through modular forms of weight 14. This answers a question of J. H. Schwarz. We also establish generalizations of these factorization formulas and obtain a new Hořava-Witten type factorization formula.

Introduction

In [[15], [8] and [9]], it has been shown that both of the Alvarez-Gaumé-Witten miraculous anomaly cancellation formula [2] and the Green-Schwarz anomaly factorization formula [7] for the gauge group $SO(32)$ can be derived (and extended) through a pair of modularly related modular forms, which are over the modular subgroup $\Gamma_0(2)$ and $\Gamma^0(2)$ respectively. In answering a question of J. H. Schwarz [16], we deal with the remaining case of gauge group $E_8 \times E_8$ in this article.

Let $Z \rightarrow X \rightarrow B$ be a fiber bundle with fiber $Z$ being 10 dimensional. Let $TZ$ be the vertical tangent bundle equipped with a metric $g^{TZ}$ and an associated Levi-Civita connection $\nabla^{TZ}$ (cf. [[3] Proposition 10.2]). Let $R^{TZ} = (\nabla^{TZ})^2$ be the curvature of $\nabla^{TZ}$, which we also for simplicity denote by $R$. Let $T_C Z$ be the complexification of $TZ$ with the induced Hermitian connection $\nabla^{T_C Z}$.

Let $(P_1, \vartheta_1), (P_2, \vartheta_2)$ be two principal $E_8$ bundles with connections over $X$. Let $\rho$ be the adjoint representation of $E_8$. Let $W_i = P_i \times_{\rho} \mathbb{C}^{248}$, $i = 1, 2$ be the associated vector bundles, which are of rank 248. We equip both $W_1, W_2$ with Hermitian metrics and Hermitian connections respectively. Let $F_i$ denote the curvature of the bundle $W_i$. Let “Tr” denote the trace in the adjoint representation. Then one has $\text{Tr} F_i^{2n+1} = 0$ (cf. the proof of Theorem 2.1 in this article), $\text{Tr} F_i^4 = \frac{1}{150} (\text{Tr} F_i^2)^2$, $\text{Tr} F_i^6 = \frac{1}{720} (\text{Tr} F_i^2)^3$ (cf. [[1]]). It’s easy to see that $c_2(W_i) = -\frac{1}{4} \text{Tr} F_i^2$. Simply denote $\text{Tr} F_1^n + \text{Tr} F_2^n$ by $\text{Tr} F^n$.

The Green-Schwarz anomaly formula [7] asserts that the following factorization for the 12 forms holds:

(0.1)

$I_{12}$

\[
\left\{ \hat{A}(TZ) \text{ch}(W_1 + W_2) + \hat{A}(TZ) \text{ch}(T_C Z) - 2 \hat{A}(TZ) \right\}^{(12)} \\
= \frac{-1}{64 \pi^6 \cdot 720} \left( - \frac{15}{8} \text{tr} R^2 \text{tr} R^4 - \frac{15}{32} (\text{tr} R^2)^3 + \text{Tr} F^6 + \text{Tr} F^2 \left( \frac{1}{16} \text{tr} R^4 + \frac{5}{64} (\text{tr} R^2)^2 \right) - \frac{5}{8} \text{Tr} F^4 \text{tr} R^2 \right) \\
= \frac{-1}{4 \pi^2 \cdot 2} \left( \text{tr} R^2 - \frac{1}{30} \text{Tr} F^2 \right) \cdot \frac{1}{16 \pi^4 \cdot 180} \left( \frac{1}{960} (\text{Tr} F^2)^2 - \frac{5}{16} \text{Tr} F^4 + \frac{1}{32} \text{tr} R^2 \text{Tr} F^2 - \frac{15}{16} \text{tr} R^4 - \frac{15}{64} (\text{tr} R^2)^2 \right) \\
= \left( p_1(TZ) + \frac{1}{30} (c_2(W_1) + c_2(W_2)) \right) \cdot I_8.
\]

In what follows, we will write characteristic forms without specifying the connections when there is no confusion (cf. [[17]]).
In [11] and [12], Hořava and Witten observed, on the other hand, that the following anomaly factorization formula holds for each $i = 1, 2$,

\begin{equation}
\hat{I}_{12} = \left\{ \hat{A}(TZ) \text{ch}(W_i) + \frac{1}{2} \hat{A}(TZ) \text{ch}(T \text{c} \text{Z}) - \hat{A}(TZ) \right\}^{(12)}
\end{equation}

\begin{align*}
&= -\frac{1}{64\pi^6} \frac{1}{1440} \left( -\frac{15}{8} \text{tr} R^2 \text{tr} R^4 - \frac{15}{32} (\text{tr} R^2)^3 + 2 \text{Tr} F_i^6 + \text{Tr} F_i^2 \left( \frac{1}{8} \text{tr} R^4 + \frac{5}{32} (\text{tr} R^2)^2 \right) - \frac{5}{4} \text{Tr} F_i^4 \text{tr} R^2 \right) \\
&= -\frac{1}{4\pi^2} \left( \text{tr} R^2 - \frac{1}{15} \text{Tr} F_i^2 \right) \cdot \hat{I}_8 \\
&= \left( \frac{1}{2} p_1(TZ) + \frac{1}{30} c_2(W_i) \right) \cdot \hat{I}_8,
\end{align*}

where $\hat{I}_8$ can be written explicitly as

$$\hat{I}_8 = \frac{1}{16\pi^4} \frac{1}{24} \left( -\frac{1}{4} \left( \frac{1}{2} \text{tr} R^2 - \frac{1}{30} \text{Tr} F_i^2 \right)^2 - \frac{1}{8} \text{tr} R^4 + \frac{1}{32} (\text{tr} R^2)^2 \right),$$

and therefore

$$I_{12} = \hat{I}_{12} + \hat{I}_{12} = \left( \frac{1}{2} p_1(TZ) + \frac{1}{30} c_2(W_1) \right) \cdot \hat{I}_8 + \left( \frac{1}{2} p_1(TZ) + \frac{1}{30} c_2(W_2) \right) \cdot \hat{I}_8.$$

The purpose of this article is to show that the above anomaly factorization formulas can also be derived naturally from modularity as in the orthogonal group case dealt with in [9]. This provides a positive answer to a question of J. H. Schwarz mentioned at the beginning of the article.

To be more precise, we will construct in Section 2 a modular form $Q(P_i, P_j, \tau)$ of weight 14 over $SL(2, \mathbb{Z})$, for any $i, j \in \{1, 2\}$, such that when $i = 1, j = 2$, the modularity of $Q(P_i, P_2, \tau)$ gives the Green-Schwarz factorization formula (0.1), while when $i = j$, the modularity of $Q(P_i, P_i, \tau)$ gives the Hořava-Witten factorization formula (0.2). Actually what we construct is a more general modular form $Q(P_i, P_j, \xi, \tau)$, which involves a complex line bundle (or equivalently a rank two real oriented bundle) and we are able to obtain generalizations of the Green-Schwarz formula and the Hořava-Witten formula by using the associated modularity. Our construction of the modular form $Q(P_i, P_j, \xi, \tau)$ involves the basic representation of the affine Kac-Moody algebra of $E_8$.

Inspired by our modular method of deriving the Green-Schwarz and Hořava-Witten factorization formulas, we also construct a modular form $R(P_i, \xi, \tau)$ of weight 10 over $SL(2, \mathbb{Z})$, the modularity of which will give us a new factorization formula of Hořava-Witten type. See Theorem 0.2 for details. It would be interesting to compare (0.8), (0.9) with the Hořava-Witten factorization (0.2) or (0.6). Actually another interesting question of J.H. Schwarz is to construct quantum field theories associated to the generalized anomaly factorization formulas in this paper and [9].

In the rest of this section, we will present our generalized Green-Schwarz and Hořava-Witten formula, as well as the new formulas of Hořava-Witten type obtained from $R(P_i, \xi, \tau)$. They will be proved in Section 2 by using modularity after briefly reviewing some knowledge of the affine Kac-Moody algebra of $E_8$ in Section 1.

Let $\xi$ be a rank two real oriented Euclidean vector bundle over $X$ carrying a Euclidean connection $\nabla^\xi$. Let $c = e(\xi, \nabla^\xi)$ be the Euler form canonically associated to $\nabla^\xi$ (cf. [17] Section 3.4).
Theorem 0.1. The following identities hold,
\begin{equation}
(0.3)
\left\{ \hat{A}(TZ)e^\frac{2}{3}\text{ch}(W_1 + W_2) + \hat{A}(TZ)e^\frac{2}{3}\text{ch}(TCZ) - 2\hat{A}(TZ)e^\frac{2}{3}\text{ch}(\xi_C + 3\xi_C \otimes \xi_C) \right\}^{(12)}
= \left( p_1(TZ) - 3c^2 + \frac{1}{30}(c_2(W_1) + c_2(W_2)) \right)
\cdot \left\{ -e^{\frac{2}{3}p_1(TZ)}(p_1(TZ) + \frac{1}{30}(c_2(W_1) + c_2(W_2))) - 1 \hat{A}(TZ)e^\frac{2}{3}\text{ch}(\mathfrak{A}) + e^{\frac{4}{3}p_1(TZ)}(p_1(TZ) + \frac{1}{30}(c_2(W_1) + c_2(W_2))) \hat{A}(TZ)e^\frac{2}{3}\right\}^{(8)},
\end{equation}
where \( \mathfrak{A} = W_1 + W_2 + TCZ - 2 + \xi_C + 3\xi_C \otimes \xi_C; \)
and for each \( i, \)
\begin{equation}
(0.4)
\left\{ \hat{A}(TZ)e^\frac{2}{3}\text{ch}(W_i) + \frac{1}{2}\hat{A}(TZ)e^\frac{2}{3}\text{ch}(TCZ) - \frac{1}{2}\hat{A}(TZ)e^\frac{2}{3}\text{ch}(\xi_C + 3\xi_C \otimes \xi_C) \right\}^{(12)}
= \left( \frac{1}{2}p_1(TZ) - 3c^2 + \frac{1}{30}c_2(W_i) \right)
\cdot \left\{ -e^{\frac{2}{3}p_1(TZ)}(p_1(TZ) + \frac{1}{30}c_2(W_i)) - 1 \hat{A}(TZ)e^\frac{2}{3}\text{ch}(\mathfrak{B}_i) + e^{\frac{4}{3}p_1(TZ)}(p_1(TZ) + \frac{1}{30}c_2(W_i)) \hat{A}(TZ)e^\frac{2}{3}\right\}^{(8)},
\end{equation}
where \( \mathfrak{B}_i = 2W_i + TCZ - 2 + \xi_C + 3\xi_C \otimes \xi_C. \)

If \( \xi \) is trivial, we obtain the Green-Schwarz formula (0.1) for \( E_8 \times E_8 \) and the Hořava-Witten formula (0.2) for \( E_8 \) in the following corollary.

Corollary 0.1. One has
\begin{equation}
(0.5)
\left\{ \hat{A}(TZ)\text{ch}(W_1 + W_2) + \hat{A}(TZ)\text{ch}(TCZ) - 2\hat{A}(TZ) \right\}^{(12)}
= \left( p_1(TZ) + \frac{1}{30}(c_2(W_1) + c_2(W_2)) \right)
\cdot \left\{ -e^{\frac{2}{3}p_1(TZ)}(p_1(TZ) + \frac{1}{30}(c_2(W_1) + c_2(W_2))) - 1 \hat{A}(TZ)\text{ch}(\mathfrak{C}) + e^{\frac{4}{3}p_1(TZ)}(p_1(TZ) + \frac{1}{30}(c_2(W_1) + c_2(W_2))) \hat{A}(TZ) \right\}^{(8)},
\end{equation}
where \( \mathfrak{C} = W_1 + W_2 + TCZ - 2; \)
and for each \( i, \)
\begin{equation}
(0.6)
\left\{ \hat{A}(TZ)\text{ch}(W_i) + \frac{1}{2}\hat{A}(TZ)\text{ch}(TCZ) - \hat{A}(TZ) \right\}^{(12)}
= \left( \frac{1}{2}p_1(TZ) + \frac{1}{30}c_2(W_i) \right)
\cdot \left\{ -e^{\frac{2}{3}p_1(TZ)}(p_1(TZ) + \frac{1}{30}c_2(W_i)) - 1 \hat{A}(TZ)\text{ch}(\mathfrak{D}_i) + e^{\frac{4}{3}p_1(TZ)}(p_1(TZ) + \frac{1}{30}c_2(W_i)) \hat{A}(TZ) \right\}^{(8)},
\end{equation}
where \( \mathfrak{D}_i = 2W_i + TCZ - 2. \)

Remark 0.1. It can be checked by direct computation that the second factors in the right hand sides of (0.5) and (0.6) are equal to \( I_8 \) and \( \tilde{I}_8^i \) respectively.
We now state a new factorization formula, which is of the Hořava-Witten type.

**Theorem 0.2.** For each $i$, the following identity holds,

$$
\left\{ \hat{A}(TZ)e^{\frac{i}{2\pi}(p_1(TZ) - 3c^2 + \frac{1}{30}c_2(W_i))} - \frac{1}{p_1(TZ) - 3c^2 + \frac{1}{30}c_2(W_i)} \hat{A}(TZ)e^{\frac{i}{2\pi}(p_1(TZ) - 3c^2 + \frac{1}{30}c_2(W_i))} \right\} (8)
$$

where $\mathfrak{E}_i = W_i + T_C Z + 246 + \xi_C + 3\xi_C \otimes \xi_C$.

If $\xi$ is trivial, we have

$$
\left\{ \hat{A}(TZ)ch(W_i) + \hat{A}(TZ)ch(T_C Z) + 246\hat{A}(TZ) \right\} (12)
$$

where $\mathfrak{g}_i = W_i + T_C Z + 246$.

**Remark 0.2.** We can express (0.8) by direct computations as follows,

$$
\frac{-1}{64\pi^6} \frac{1}{1440} \left( -\frac{15}{4}\text{tr}R^2\text{tr}R^4 - \frac{15}{16}(\text{tr}R^2)^3 + 2\text{Tr}F_i^6 + \text{Tr}F_i^2 \left( \frac{1}{8}\text{tr}R^4 + \frac{5}{32}(\text{tr}R^2)^2 \right) - \frac{5}{4}\text{Tr}F_i^4\text{tr}R^2 \right)
$$

$$
= \left( p_1(TZ) + \frac{1}{30}c_2(W_i) \right) \cdot \hat{J}_8
$$

**Remark 0.3.** As in [13], one may ask whether there is a physics model corresponding to (0.8) and (0.9).

1. **The Basic Representation of Affine $E_8$**

In this section we briefly review the basic representation theory for the affine $E_8$ by following [13] (see also [14]).

Let $\mathfrak{g}$ be the Lie algebra of $E_8$. Let $\langle , \rangle$ be the Killing form on $\mathfrak{g}$. Let $\tilde{\mathfrak{g}}$ be the affine Lie algebra corresponding to $\mathfrak{g}$ defined by

$$
\tilde{\mathfrak{g}} = \mathbb{C}[t, t^{-1}] \otimes \mathfrak{g} \oplus \mathbb{C}c,
$$

with bracket

$$
[P(t) \otimes x + \lambda c, Q(t) \otimes y + \mu c] = P(t)Q(t) \otimes [x, y] + \langle x, y \rangle \text{Res}_{t=0} \left( \frac{dP(t)}{dt}Q(t) \right) c.
$$

Let $\hat{\mathfrak{g}}$ be the affine Kac-Moody algebra obtained from $\tilde{\mathfrak{g}}$ by adding a derivation $t \frac{d}{dt}$ which operates on $\mathbb{C}[t, t^{-1}] \otimes \mathfrak{g}$ in an obvious way and sends $c$ to 0.
The basic representation $V(\Lambda_0)$ is the $\mathfrak{g}$-module defined by the property that there is a nonzero vector $v_0$ (highest weight vector) in $V(\Lambda_0)$ such that $cv_0 = v_0, (C[t] \oplus C(\frac{d}{dt}))v_0 = 0$. Setting $V_k := \{ v \in V(\Lambda_0) | t \frac{d}{dt} = -kv \}$ gives a $Z_+$-gradation by finite spaces. Since $[g, d] = 0$, each $V_k$ is a representation of $\mathfrak{g}$. Moreover, $V_1$ is the adjoint representation of $E_8$.

Let $q = e^{2\pi \frac{-1}{T_0}}$. Fix a basis $\{ z_i \}_{i=1}^8$ for the Cartan subalgebra. The character of the basic representation is given by

$$\text{ch}(z_1, z_2, \cdots, z_8, \tau) := \sum_{k=0}^{\infty} (\text{ch}V_k)(z_1, z_2, \cdots, z_8)q^k = \varphi(\tau)^{-1} \Theta(z_1, z_2, \cdots, z_8, \tau),$$

where $\varphi(\tau) = \prod_{n=1}^{\infty} (1 - q^n)$ so that $\eta(\tau) = q^{1/24} \varphi(\tau)$ is the Dedekind $\eta$ function; $\Theta(z_1, z_2, \cdots, z_8, \tau)$ is the theta function defined on the root lattice $Q$ by

$$\Theta(z_1, z_2, \cdots, z_8, \tau) = \sum_{\gamma \in Q} q^{\gamma^2/2} e^{2\pi \sqrt{-1} (\gamma, \tau)}.$$

It is proved in [6] (cf. [10]) that there is a basis for the $E_8$ root lattice such that

$$\Theta(z_1, \cdots, z_8, \tau) = \frac{1}{2} \left( \prod_{l=1}^{8} \theta(z_l, \tau) + \sum_{l=1}^{8} \theta_1(z_l, \tau) + \sum_{l=1}^{8} \theta_2(z_l, \tau) + \sum_{l=1}^{8} \theta_3(z_l, \tau) \right),$$

where $\theta$ and $\theta_i (i = 1, 2, 3)$ are the Jacobi theta functions (cf. [4] and [8]).

2. Derivation of Green-Schwarz and Horava-Witten type anomaly factorization via modularity

In this section, we will derive the Green-Schwarz and Hořava-Witten type factorization formulas in Theorems 0.1 and 0.2 via modularity.

For the principal $E_8$ bundles $P_i, i = 1, 2$, consider the associated bundles

$$V_i = \sum_{k=0}^{\infty} (P_i \times_{\rho_i} V_k) q^k \in K(X)[[q]].$$

Since $\rho_i$ is the adjoint representation of $E_8$, we have $W_i = P_i \times_{\rho_i} V_1$.

Following [5], set

$$\Theta(TCZ, \xi C) := \left( \bigotimes_{m=1}^{\infty} S_{q^m}(\tilde{TC}Z) \right) \otimes \left( \bigotimes_{n=1}^{\infty} \Lambda_{q^n}(\tilde{\xi} C) \otimes \left( \bigotimes_{u=1}^{\infty} \Lambda_{q^{-u^{-1/2}}}(\tilde{\xi} C) \right) \right) \in K(X)[[q]],$$

where $\xi C$ is the complexification of $\xi$, and for a complex vector bundle $E$, $\tilde{E} := E - C^{\text{rk}(E)}$.

Clearly, $\Theta(TCZ, \xi C)$ admits a formal Fourier expansion in $q$ as

$$\Theta(TCZ, \xi C) = C + B_1q + B_2q^2 \cdots,$$

where the $B_j$’s are elements in the semi-group formally generated by complex vector bundles over $X$. Moreover, they carry canonically induced connections denoted by $\nabla^{B_j}$. Let $\nabla^{\Theta}$ be the induced connection with $q$-coefficients on $\Theta$.

For $1 \leq i, j \leq 2$, set

$$Q(P_i, P_j, \xi, \tau) := \left\{ e^{\frac{1}{c_2} S_2(\tau)}(p_1(TZ) - 3c_2 + \frac{1}{48}(c_2(W_i) + c_2(W_j))) \tilde{A}(TZ) \cosh \left( \frac{c}{2} \right) \text{ch} (\Theta(TCZ, \xi C)) \varphi(\tau)^{16} \text{ch}(V_i) \text{ch}(V_j) \right\}.$$

**Theorem 2.1.** $Q(P_i, P_j, \xi, \tau)$ is a modular form of weight 14 over $SL(2, \mathbb{Z})$. 
Proof: By the knowledge reviewed in Section 2, we see that there are formal two forms $y_i^l, 1 \leq l \leq 8, i = 1, 2$ such that

\begin{equation}
\varphi(\tau)^8 \text{ch}(V_i) = \frac{1}{2} \left( \prod_{l=1}^{8} \theta(y_i^l, \tau) + \prod_{l=1}^{8} \theta_1(y_i^l, \tau) + \prod_{l=1}^{8} \theta_2(y_i^l, \tau) + \prod_{l=1}^{8} \theta_3(y_i^l, \tau) \right).
\end{equation}

Since $\theta(z, \tau)$ is an odd function about $z$ and we only take forms of degrees not greater than 12, one has

\begin{equation}
\varphi(\tau)^8 \text{ch}(V_i) = \frac{1}{2} \left( \prod_{l=1}^{8} \theta_1(y_i^l, \tau) + \prod_{l=1}^{8} \theta_2(y_i^l, \tau) + \prod_{l=1}^{8} \theta_3(y_i^l, \tau) \right).
\end{equation}

Since $\theta_1(z, \tau), \theta_2(z, \tau)$ and $\theta_3(z, \tau)$ are all even functions about $z$, the right hand side of the above equality only contains even powers of $y_i^l$'s. Therefore $\text{ch}(W_i)$ only consists of forms of degrees divisible by 4. So

\begin{equation}
\text{ch}(V_i) = 1 + \text{ch}(W_i)q + \cdots = 1 + (248 - c_2(W_i) + \cdots)q + \cdots.
\end{equation}

On the other hand,

\begin{equation}
\frac{1}{2} \left( \prod_{l=1}^{8} \theta_1(y_i^l, \tau) + \prod_{l=1}^{8} \theta_2(y_i^l, \tau) + \prod_{l=1}^{8} \theta_3(y_i^l, \tau) \right) = 1 + \left( 240 + 30 \sum_{l=1}^{8} (y_i^l)^2 + \cdots \right)q + O(q^2).
\end{equation}

From (2.4), (2.5) and (2.6), we have

\begin{equation}
\sum_{l=1}^{8} (y_i^l)^2 = -\frac{1}{30} c_2(W_i).
\end{equation}

Let $\{\pm 2\pi \sqrt{-1} x_l\}$ be the formal Chern roots for $(T_Z \mathbb{C}, V^T_Z \mathbb{C})$. Let $c = 2\pi \sqrt{-1} u$. One has

\begin{equation}
Q(P_i, P_j, \xi, \tau) = \left\{ e^{\frac{1}{3\pi} E_2(\tau)(p_1(TZ) - 3c^2 + \frac{1}{30}(c_2(W_i) + c_2(W_j)))} \tilde{A}(TZ) \cosh \left( \frac{c}{2} \right) \text{ch}(\Theta(T_Z \mathbb{C}, \xi \mathbb{C})) \varphi(\tau)^{16} \text{ch}(V_i) \text{ch}(V_j) \right\}^{(12)}.
\end{equation}

Then we can preform the transformation formulas for the theta functions and $E_2(\tau)$ (c.f. [3] and [8]) to show that $Q(P_i, P_j, \xi, \tau)$ is a modular form of weight 14 over $SL(2, \mathbb{Z})$. Q.E.D.
Proof of Theorem 0.1: Expanding the \( q \)-series, we have

\[
(2.9) \quad e^{\frac{2\pi i}{	au}} E_2(p_1TZ) - 3c^2 + \frac{1}{30}(c_2(W_i) + c_2(W_j)) \tilde{A}(TZ) \cosh \left( \frac{c}{2} \right) \chi(T(CZ, \xi_C)) \phi(\tau)^{16} \chi(V_i) \chi(V_j)
\]

\[
e\left( e^{\frac{2\pi i}{	au}} (p_1TZ) - 3c^2 + \frac{1}{30}(c_2(W_i) + c_2(W_j)) \right) 
\]

\[
- e^{\frac{2\pi i}{	au}} (p_1TZ) - 3c^2 + \frac{1}{30}(c_2(W_i) + c_2(W_j)) \left( p_1(TZ) - 3c^2 + \frac{1}{30}(c_2(W_i) + c_2(W_j)) \right) q + O(q^2)
\]

\[
\cdot \tilde{A}(TZ) \cosh \left( \frac{c}{2} \right) \chi(\mathbf{C} + B_1q + O(q^2))(1 - 16q + O(q^2))(1 + \chi(W_i)q + O(q^2))(1 + \chi(W_j)q + O(q^2))
\]

\[
= e^{\frac{2\pi i}{	au}} (p_1TZ) - 3c^2 + \frac{1}{30}(c_2(W_i) + c_2(W_j)) \tilde{A}(TZ) \cosh \left( \frac{c}{2} \right) 
\]

\[
+ q \left( e^{\frac{2\pi i}{	au}} (p_1TZ) - 3c^2 + \frac{1}{30}(c_2(W_i) + c_2(W_j)) \right) \tilde{A}(TZ) \cosh \left( \frac{c}{2} \right) \chi(B_1 - 16 + W_i + W_j)
\]

\[
- e^{\frac{2\pi i}{	au}} (p_1TZ) - 3c^2 + \frac{1}{30}(c_2(W_i) + c_2(W_j)) \left( p_1(TZ) - 3c^2 + \frac{1}{30}(c_2(W_i) + c_2(W_j)) \right) \tilde{A}(TZ) \cosh \left( \frac{c}{2} \right) 
\]

\[
+ O(q^2).
\]

It is well known that modular forms over \( SL(2, \mathbb{Z}) \) can be expressed as polynomials of the Eisenstein series \( E_4(\tau), E_6(\tau) \), where

\[
(2.10) \quad E_4(\tau) = 1 + 240q + 2160q^2 + 6720q^3 + \cdots,
\]

\[
(2.11) \quad E_6(\tau) = 1 - 504q - 16632q^2 - 122976q^3 + \cdots.
\]

Their weights are 4 and 6 respectively.

Since the weight of the modular form \( Q(P, P_j, \xi, \tau) \) is 14, it must be a multiple of

\[
(2.12) \quad E_4(\tau)^2 E_6(\tau) = 1 - 24q + \cdots.
\]

So from (2.9) and (2.12), we have

\[
(2.13) \quad \left\{ e^{\frac{2\pi i}{	au}} (p_1TZ) - 3c^2 + \frac{1}{30}(c_2(W_i) + c_2(W_j)) \right\} \tilde{A}(TZ) \cosh \left( \frac{c}{2} \right) \chi(B_1 - 16 + W_i + W_j)
\]

\[
- \left\{ e^{\frac{2\pi i}{	au}} (p_1TZ) - 3c^2 + \frac{1}{30}(c_2(W_i) + c_2(W_j)) \right\} \left( p_1(TZ) - 3c^2 + \frac{1}{30}(c_2(W_i) + c_2(W_j)) \right) \tilde{A}(TZ) \cosh \left( \frac{c}{2} \right)
\]

\[
= -24 \left\{ e^{\frac{2\pi i}{	au}} (p_1TZ) - 3c^2 + \frac{1}{30}(c_2(W_i) + c_2(W_j)) \right\} \tilde{A}(TZ) \cosh \left( \frac{c}{2} \right) \}
\]

Therefore

\[
(2.14) \quad \left\{ \tilde{A}(TZ) \cosh \left( \frac{c}{2} \right) \chi(W_i + W_j + B_1 + 8) \right\}^{(12)}
\]

\[
= \left( p_1(TZ) - 3c^2 + \frac{1}{30}(c_2(W_i) + c_2(W_j)) \right)
\]

\[
\cdot \left\{ - \frac{1}{p_1(TZ) - 3c^2 + \frac{1}{30}(c_2(W_i) + c_2(W_j))} - 1 \right\} \tilde{A}(TZ) \cosh \left( \frac{c}{2} \right) \chi(W_i + W_j + B_1 + 8)
\]

\[
+ e^{\frac{2\pi i}{	au}} (p_1TZ) - 3c^2 + \frac{1}{30}(c_2(W_i) + c_2(W_j)) \tilde{A}(TZ) \cosh \left( \frac{c}{2} \right) \}
\]
To find $B_1$, we have

$$\Theta(TCZ, \xi C) = \left( \bigotimes_{m=1}^{\infty} S_{q^m}(TCZ) \right) \otimes \left( \bigotimes_{n=1}^{\infty} \Lambda_{q^n}(\tilde{\xi} C) \right) \otimes \left( \bigotimes_{u=1}^{\infty} \Lambda_{q^{u-1/2}}(\tilde{\xi} C) \right)$$

(2.15)

$$= (1 + (TCZ - 10)q + O(q^2)) \otimes (1 + \tilde{\xi} C q + O(q^2))$$

$$\otimes (1 - \tilde{\xi} C q^{1/2} - 2\tilde{\xi} C q + O(q^{3/2})) \otimes (1 + \tilde{\xi} C q^{1/2} - 2\tilde{\xi} C q + O(q^{3/2}))$$

$$= 1 + (TCZ - 10 + \tilde{\xi} C + 3\tilde{\xi} C \otimes \tilde{\xi} C) q + O(q^2).$$

So $B_1 = TCZ - 10 + \tilde{\xi} C + 3\tilde{\xi} C \otimes \tilde{\xi} C$.

Plugging $B_1$ into (2.14), we have

$$\left\{ \hat{A}(TZ) \cosh \left( \frac{c}{2} \right) \text{ch}(W_i + W_j + TCZ - 2 + \tilde{\xi} C + 3\tilde{\xi} C \otimes \tilde{\xi} C) \right\}^{(12)}$$

(2.16)

$$= \left( p_1(TZ) - 3c^2 + \frac{1}{30} (c_2(W_i) + c_2(W_j)) \right)$$

$$\cdot \left\{ -e^{\frac{c}{2}}(p_1(TZ) - 3c^2 + \frac{1}{30} (c_2(W_i) + c_2(W_j))) - 1 \right\} \hat{A}(TZ) \cosh \left( \frac{c}{2} \right) \text{ch}(W_i + W_j + TCZ - 2 + \tilde{\xi} C + 3\tilde{\xi} C \otimes \tilde{\xi} C)$$

$$+ e^{\frac{c}{2}}(p_1(TZ) - 3c^2 + \frac{1}{30} (c_2(W_i) + c_2(W_j))) \hat{A}(TZ) \cosh \left( \frac{c}{2} \right) \right\}^{(8)}.$$

Since $\text{ch}(W_i), \text{ch}(W_j)$ only contribute degree 4l forms, we can replace $\cosh \left( \frac{c}{2} \right)$ by $e^{\frac{c}{2}}$. Then in (2.16), putting $i = 1, j = 2$ gives (0.4) and putting $i = j$ gives (0.5). Q.E.D.

To prove Theorem 0.2, for each $i$, set

$$R(P, \xi, \tau)$$

(2.17)

$$:= \left\{ e^{\frac{c}{2}}E_2(\tau)(p_1(TZ) - 3c^2 + \frac{1}{30} c_2(W_i)) \hat{A}(TZ) \cosh \left( \frac{c}{2} \right) \text{ch} (\Theta(TCZ, \xi C)) \varphi(\tau)^8 \text{ch}(V_i) \right\}^{(12)}.$$

**Theorem 2.2.** $R(P, \xi, \tau)$ is a modular form of weight 10 over $SL(2, \mathbb{Z})$.

**Proof:** This can be similarly proved as Theorem 2.1 by seeing that

$$R(P, \xi, \tau)$$

(2.18)

$$= \left\{ e^{\frac{c}{2}}E_2(\tau)(p_1(TZ) - 3c^2 + \frac{1}{30} c_2(W_i)) \hat{A}(TZ) \cosh \left( \frac{c}{2} \right) \text{ch} (\Theta(TCZ, \xi C)) \varphi(\tau)^8 \text{ch}(V_i) \right\}^{(12)}$$

$$= \left\{ e^{\frac{c}{2}}E_2(\tau)(p_1(TZ) - 3c^2 + \frac{1}{30} c_2(W_i)) \left( \prod_{l=1}^{5} \left( x_l^l \right)^{\theta_1'(0, \tau)} \right) \right\}^{(12)}$$

$$\cdot \left( \prod_{l=1}^{8} \theta_1(y_i^l, \tau) + \prod_{l=1}^{8} \theta_2(y_i^l, \tau) + \prod_{l=1}^{8} \theta_3(y_i^l, \tau) \right)$$

and then apply the transformation laws of theta functions. Q.E.D.
Proof of Theorem 0.2: Similar as in the proof of Theorem 0.1, expanding the \(q\)-series, we have

\[
e^{\frac{\pi}{24} (p_1(TZ) - 3c^2 + \frac{1}{30}c_2(W_i)) \tilde{A}(TZ) \cosh \left( \frac{c}{2} \right) \operatorname{ch} \left( \Theta(TCZ, \xi C) \right) \varphi(\tau)^8 \operatorname{ch}(V_i)}
\]

\[
e^{\frac{1}{24} (p_1(TZ) - 3c^2 + \frac{1}{30}c_2(W_i))}
\]

\[
- e^{\frac{1}{24} (p_1(TZ) - 3c^2 + \frac{1}{30}c_2(W_i))} \left( p_1(TZ) - 3c^2 + \frac{1}{30}c_2(W_i) \right) q + O(q^3)
\]

\[
\times \tilde{A}(TZ) \cosh \left( \frac{c}{2} \right) \operatorname{ch}(C + B_1 q + O(q^3))(1 - 8q + O(q^2))(1 + \operatorname{ch}(W_i)q + O(q^3))
\]

\[
= e^{\frac{1}{24} (p_1(TZ) - 3c^2 + \frac{1}{30}c_2(W_i)) \tilde{A}(TZ) \cosh \left( \frac{c}{2} \right)}
\]

\[
+ q \left( e^{\frac{1}{24} (p_1(TZ) - 3c^2 + \frac{1}{30}c_2(W_i)) \tilde{A}(TZ) \cosh \left( \frac{c}{2} \right)} \operatorname{ch}(B_1 - 8 + W_i) \right.
\]

\[
- e^{\frac{1}{24} (p_1(TZ) - 3c^2 + \frac{1}{30}c_2(W_i))} \left( p_1(TZ) - 3c^2 + \frac{1}{30}c_2(W_i) \right) \tilde{A}(TZ) \cosh \left( \frac{c}{2} \right)
\]

\[
+ O(q^2).
\]

However modular form of weight 10 must be a multiple of \(E_4(\tau)E_6(\tau) = 1 - 264q + \cdots\), so we have

\[
\left\{ e^{\frac{1}{24} (p_1(TZ) - 3c^2 + \frac{1}{30}c_2(W_i)) \tilde{A}(TZ) \cosh \left( \frac{c}{2} \right)} \operatorname{ch}(B_1 - 8 + W_i) \right\}^{(12)}
\]

\[
- \left\{ e^{\frac{1}{24} (p_1(TZ) - 3c^2 + \frac{1}{30}c_2(W_i)) \tilde{A}(TZ) \cosh \left( \frac{c}{2} \right)} \operatorname{ch}(B_1 - 8 + W_i) \right\}^{(12)}
\]

\[
= -264 \left\{ e^{\frac{1}{24} (p_1(TZ) - 3c^2 + \frac{1}{30}c_2(W_i)) \tilde{A}(TZ) \cosh \left( \frac{c}{2} \right)} \right\}^{(12)}.
\]

Therefore

\[
\left\{ \tilde{A}(TZ) \cosh \left( \frac{c}{2} \right) \operatorname{ch}(W_i + B_1 + 256) \right\}^{(12)}
\]

\[
= \left( p_1(TZ) - 3c^2 + \frac{1}{30}c_2(W_i) \right)
\]

\[
- \left\{ - e^{\frac{1}{24} (p_1(TZ) - 3c^2 + \frac{1}{30}c_2(W_i))} \right. \left( p_1(TZ) - 3c^2 + \frac{1}{30}c_2(W_i) \right) \tilde{A}(TZ) \cosh \left( \frac{c}{2} \right) \operatorname{ch}(W_i + B_1 + 256)
\]

\[
+ e^{\frac{1}{24} (p_1(TZ) - 3c^2 + \frac{1}{30}c_2(W_i))} \tilde{A}(TZ) \cosh \left( \frac{c}{2} \right) \right\}^{(8)}.
\]

Plugging in \(B_1\), we have

\[
\left\{ \tilde{A}(TZ) \cosh \left( \frac{c}{2} \right) \operatorname{ch}(W_i + TCZ + 246 + \tilde{\xi} C + 3\tilde{\xi} C \otimes \tilde{\xi} C) \right\}^{(12)}
\]

\[
= \left( p_1(TZ) - 3c^2 + \frac{1}{30}c_2(W_i) \right)
\]

\[
- \left\{ - e^{\frac{1}{24} (p_1(TZ) - 3c^2 + \frac{1}{30}c_2(W_i))} \right. \left( p_1(TZ) - 3c^2 + \frac{1}{30}c_2(W_i) \right) \tilde{A}(TZ) \cosh \left( \frac{c}{2} \right) \operatorname{ch}(W_i + TCZ + 246 + \tilde{\xi} C + 3\tilde{\xi} C \otimes \tilde{\xi} C)
\]

\[
+ e^{\frac{1}{24} (p_1(TZ) - 3c^2 + \frac{1}{30}c_2(W_i))} \tilde{A}(TZ) \cosh \left( \frac{c}{2} \right) \right\}^{(8)}.
\]

Since \(\operatorname{ch}(W_i)\) only contribute degree \(4l\) forms, we can replace \(\cosh \left( \frac{c}{2} \right)\) by \(e^\frac{c}{2}\), (2.22) gives us (0.7). Q.E.D.
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