Conditional positive definiteness in operator theory

Zenon Jan Jabłoński, Il Bong Jung, and Jan Stochel

Abstract. In this paper we extensively investigate the class of conditionally positive definite operators, namely operators generating conditionally positive definite sequences. This class itself contains subnormal operators, 2- and 3-isometries, complete hypercontractions of order 2 and much more beyond them. Quite a large part of the paper is devoted to the study of conditionally positive definite sequences of exponential growth with emphasis put on finding criteria for their positive definiteness, where both notions are understood in the semigroup sense. As a consequence, we obtain semispectral and dilation type representations for conditionally positive definite operators. We also show that the class of conditionally positive definite operators is closed under the operation of taking powers. On the basis of Agler’s hereditary functional calculus, we build an $L^\infty(M)$-functional calculus for operators of this class, where $M$ is an associated semispectral measure. We provide a variety of applications of this calculus to inequalities involving polynomials and analytic functions. In addition, we derive new necessary and sufficient conditions for a conditionally positive definite operator to be a subnormal contraction (including a telescopic one).

CONTENTS

1. Introduction 2
    1.1. Motivation 2
    1.2. Intuition 3
    1.3. Ideas and concise description 6
    1.4. Notation and terminology 8

2. Conditionally positive definite sequences 10
    2.1. Basic facts 10
    2.2. Exponential growth 11
    2.3. Additional constraints 19

3. Representations of conditionally positive definite operators 23
    3.1. Semispectral integral representations 23
    3.2. A dilation representation 28

2020 Mathematics Subject Classification. Primary 47B20, 44A60; Secondary 47A20, 47A60.

Key words and phrases. Conditional positive definiteness, positive definiteness, subnormality, functional calculus.

The second author was supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education (NRF-2021R111A1A01043569).
1. Introduction

1.1. Motivation. The concepts of positive and conditional positive definiteness (at least in the group setting) have their origins in stochastic processes that are stationary or which have stationary increments \([45, 50, 53, 14, 60]\). It seems that conditional positive definiteness appeared in operator theory for the first time on the occasion of investigating subnormal operators (see \([70]\)). Later it appeared sporadically in the context of complete hyperexpansivity and complete hypercontractivity of finite order, both related to \(m\)-tuples of commuting operators \([9, 21, 22]\). The main goal of the present paper is to exploit conditional positive definiteness in the semigroup setting to study a class of operators which is large enough to subsume subnormal operators \([33, 25]\) (which are integrally tied to positive definiteness), 2- and 3-isometries \([3, 4, 5]\), complete hypercontractions of order 2 \([21]\), certain algebraic operators which are neither subnormal nor \(m\)-isometric, and much more. Below we give a more detailed discussion on this.

Throughout this paper \(\mathcal{H}\) stands for a complex Hilbert space and \(B(\mathcal{H})\) for the \(C^*\)-algebra of all bounded linear operators on \(\mathcal{H}\). An operator \(T \in B(\mathcal{H})\) is said to be subnormal if there exist a complex Hilbert space \(\mathcal{K}\) and a normal operator \(N \in B(\mathcal{K})\), called a normal extension of \(T\), such that \(\mathcal{H} \subseteq \mathcal{K}\) (isometric embedding) and \(Th = Nh\) for all \(h \in \mathcal{H}\).

A sequence \(\{\gamma_n\}_{n=0}^\infty\) of real numbers is said to be positive definite (PD for brevity) if

\[
\sum_{i,j=0}^k \gamma_{i+j} \lambda_i \lambda_j \geq 0
\]

(1.1.1)

for all finite sequences of complex numbers \(\lambda_0, \ldots, \lambda_k\). The celebrated Lambert’s characterization of subnormality \([48]\) can be adapted to the context of not necessarily injective operators as follows (for (i)\(\iff\)(ii) see \([74]\) Theorem 7, while for (ii)\(\iff\)(iii) apply Theorem 2.1.3 substituting \(Th\) in place of \(h\)).

**Theorem 1.1.1.** If \(T \in B(\mathcal{H})\), then the following conditions are equivalent:

(i) \(T\) is subnormal,
(ii) the sequence \(\{\|T^n h\|^2\}_{n=0}^\infty\) is a Stieltjes moment sequence for every \(h \in \mathcal{H}\),
(iii) the sequence \(\{\|T^n h\|^2\}_{n=0}^\infty\) is PD for every \(h \in \mathcal{H}\).

The above theorem, which fails for unbounded operators (see \([40, 18]\)), turned out to be very useful when studying the concrete classes of bounded operators (see \([49, 39, 17, 72, 19]\)). Some of them are associated with the set \(\mathcal{F}\) of nonconstant entire functions with nonnegative Taylor’s coefficients at 0. The question of characterizing the subnormality of composition operators with matrix symbols on \(L^2(\mathbb{R}^d, \rho(x)dx)\) with a density function \(\rho\) coming from \(\Phi \in \mathcal{F}\) (see \([69]\)) led to the
following problem, which for thirty years remains unsolved even for second-degree monomials (see [70] p. 237)).

**Problem 1.1.2.** Let $T \in B(\mathcal{H})$ be a contraction and $\Phi \in \mathcal{F}$. Is it true that if 
$\{\Phi([T^nh]^2)\}_{n=0}^{\infty}$ is a PD sequence for every $h \in \mathcal{H}$, then $T$ is subnormal?

The answer to the question in Problem 1.1.2 is in the affirmative as long as $\Phi'(0)$, the derivative of $\Phi$ at 0, is positive or $T$ is algebraic (see [70] Theorems 5.1 and 6.3). Problem 1.1.2 without the assumption that $T$ is contractive has a negative solution (see [70] Example 5.4). Note also that the converse implication in Problem 1.1.2 is true even if $T$ is not contractive (see the proof of [70] Theorem 5.1).

Before we continue the discussion, let us give a necessary definition. A sequence $\{\gamma_n\}_{n=0}^{\infty}$ of real numbers is said to be **conditionally positive definite** (CPD for brevity) if inequality (1.1.1) holds for all finite sequences of complex numbers $\lambda_0, \ldots, \lambda_k$ such that $\sum_{i=0}^{k} \lambda_i = 0$. Continuing our discussion, we note that if $T \in B(\mathcal{H})$ and $\Phi = \exp$ (which is a member of $\mathcal{F}$ with $\Phi'(0) > 0$), then, by the Schoenberg characterization of CPD sequences (see Lemma 2.1.1), the sequence $\{\exp(([T^nh]^2))_{n=0}^{\infty}\}$ is PD for every $h \in \mathcal{H}$ if and only if the sequence $\{\|T^nh\|^2\}_{n=0}^{\infty}$ is CPD for every $h \in \mathcal{H}$. The situation becomes more complex if the function $\exp$ is replaced by an arbitrary member $\Phi$ of $\mathcal{F}$; then the hypothesis that the sequence $\{\Phi([T^nh]^2))_{n=0}^{\infty}\}$ is PD for every $h \in \mathcal{H}$ implies that for some positive integer $j$ (depending only on $\Phi$) and for every $h \in \mathcal{H}$, the sequence $\{\|T^nh\|^2\}_{n=0}^{\infty}$ is CPD (see [70] Lemma 5.2). It was shown in [70] Theorem 4.1 that if $T$ is a contraction, then $T$ is subnormal if and only if the sequence $\{\|T^nh\|^2\}_{n=0}^{\infty}$ is CPD for every $h \in \mathcal{H}$. The contractivity hypothesis cannot be removed (see [70] Example 5.4).

The above-mentioned results of [70] were obtained by using ad hoc methods. The main goal of the present paper is to systematically and rigorously study operators $T \in B(\mathcal{H})$ having the property that for every $h \in \mathcal{H}$, the sequence $\{\|T^nh\|^2\}_{n=0}^{\infty}$ is CPD. Such operators are called here **conditionally positive definite** (CPD for brevity). In view of Theorem 1.1.1 and the fact that PD sequences are CPD, subnormal operators are CPD but not conversely (see [70] Example 5.4). Our investigations are preceded by developing harmonic analysis of CPD functions of (at most) exponential growth on the additive semigroup of nonnegative integers. As a consequence, we gain, among other things, a deeper insight into the subtle relationship between subnormality and conditional positive definiteness.

1.2. **Intuition.** To develop an intuition about CPD operators, we answer a few natural simple questions that are usually asked when considering new classes of operators.

1. Is the class of CPD operators on $\mathcal{H}$ closed\footnote{Using the fact that the multiplication in $B(\mathcal{H})$ is sequentially continuous in SOT (the strong operator topology), one can show that the class of CPD operators is sequentially SOT-closed. The question of whether it is SOT-closed remains open. This is related to the celebrated Bishop theorem stating that the class of subnormal operators is the SOT-closure of the set of normal operators (see [15]; see also [25] Theorem II.1.17]).} in the operator norm?

2. Is the orthogonal sum of CPD operators CPD?

3. Is the restriction of a CPD operator to its invariant subspace CPD?

4. Are positive integer powers of CPD operators still CPD?

5. Is the inverse of an invertible CPD operator CPD?

6. Is the tensor product of two CPD operators CPD?
(7) Is it true that if $T$ is CPD, then $T + \lambda I$ is CPD for any complex number $\lambda$, where $I$ stands for the identity operator?

(8) Is it true that if $T$ is CPD, then $\lambda T$ is CPD for any complex number $\lambda$?

The answers to questions (1)-(5) are in the affirmative. The rest of the questions are answered in the negative. The affirmative answers to questions (1) and (2) follow from the fact that the class of CPD sequences is closed in the topology of pointwise convergence. In turn, the affirmative answer to question (3) is a direct consequence of the definition. The affirmative answer to question (4) is given in Theorem 3.3.2. The negative answer to question (7) implies that the algebraic sum of two commuting CPD operators may not be CPD. The negative answer to question (8) implies that the product of two commuting CPD operators may not be CPD.

Now, we show that the answer to question (5) is in the affirmative.

**Proposition 1.2.1.** If $T \in B(H)$ is a CPD operator which is invertible in $B(H)$, then its inverse $T^{-1}$ is CPD.

**Proof.** Let $\lambda_0, \ldots, \lambda_k$ be a finite sequence of complex numbers such that $\sum_{j=0}^k \lambda_j = 0$ and let $h \in H$. Since $T$ is surjective, there exists $f \in H$ such that $h = T^k f$. Using the assumption that $T$ is CPD, we conclude that

$$\sum_{i,j=0}^k \| (T^{-1})^{i+j} h \|^2 \lambda_i \bar{\lambda}_j = \sum_{i,j=0}^k \| (T^{(k-i)+(k-j)} f ) \|^2 \lambda_i \bar{\lambda}_j \geq 0,$$

which completes the proof. \hfill \Box

The negative answer to question (6) is justified in Example 1.2.2 below, which is essentially based on the concept of a strict $m$-isometry. Following [2], we call an operator $T \in B(H)$ an $m$-isometry, where $m$ is a positive integer, if

$$\sum_{k=0}^m (-1)^k \binom{m}{k} T^k T^* = 0.$$

An $m$-isometry $T$ is strict if $m = 1$ and $H \neq \{0\}$, or $m \geq 2$ and $T$ is not an $(m - 1)$-isometry. It is worth noting that any non-isometric 3-isometry is CPD, but not subnormal. This fact can be deduced from Proposition 3.3.4 and Proposition 4.5. Since any $m$-isometry is $(m + 1)$-isometry (see [3] p. 389), we conclude that any strict 2-isometry is CPD, but not subnormal.

**Example 1.2.2.** In this example, we use the following fact, which can be deduced from [41 Corollary 3.5].

The tensor product of a strict $m_1$-isometry and a strict $m_2$-isometry is a strict $(m_1 + m_2 - 1)$-isometry.

Hence, the tensor product of a strict 2-isometry and a strict 3-isometry is a strict 4-isometry, which, according to Proposition 3.3.4 is not CPD. However, by the same proposition, 2- and 3-isometries are CPD. This gives the negative answer to question (6). A similar conclusion can be drawn considering the tensor product of two strict 3-isometries (the resulting tensor product is a strict 5-isometry). We refer the reader to [7 Proposition 8] for examples of strict 2- and 3-isometries, which are unilateral weighted shifts. \hfill \diamond
That the answers to questions (7) and (8) are in the negative is shown in the following example (see also Remark 1.2.4).

**Example 1.2.3.** Let $N \in B(H)$ be a nonzero operator such that $N^2 = 0$. Fix $\theta \in [0, 2\pi)$ and set $T_\theta = N - e^{i\theta} I$. Note that

$$T_\theta + \lambda I = N + (\lambda - e^{i\theta})I \quad \text{for any complex number } \lambda. \tag{1.2.1}$$

It follows from [11, Theorem 2.2] that $T_\theta$ is a 3-isometry, so by Proposition 4.3.1, $T_\theta$ is CPD. Denote by $\Xi_{T_\theta}$ the set of all complex numbers $\lambda$ for which the operator $T_\theta + \lambda I$ is CPD. Since the class of CPD operators is closed in the operator norm, we see that $\Xi_{T_\theta}$ is a closed subset of the complex plane. If $\lambda$ is a complex number such that $|\lambda| < 1$, then by (1.2.1) and Corollary 3.4.8, $\lambda \not\in \Xi_{T_\theta}$. This gives the negative answer to question (7). Observe also that if $|\lambda - e^{i\theta}| = 1$, then by (1.2.1), [11, Theorem 2.2] and Proposition 4.3.1, $\lambda \not\in \Xi_{T_\theta}$.

Denote by $\tilde{\Xi}_{T_\theta}$ the set of all complex numbers $\lambda$ for which the operator $\lambda T_\theta$ is CPD. As above, we verify that $\tilde{\Xi}_{T_\theta}$ is a closed subset of the complex plane. Since $\lambda T_\theta = (\lambda N) - \lambda e^{i\theta} I$, we infer from Corollary 3.4.8 that $\lambda \not\in \tilde{\Xi}_{T_\theta}$ whenever $0 < |\lambda| < 1$. This answers question (8) in the negative. Let us also notice that if $|\lambda| \in \{0, 1\}$, then by the definition of a CPD operator, $\lambda \in \Xi_{T_\theta}$. ♦

**Remark 1.2.4.** Regarding question (8), note that by Corollary 3.4.7, for every non-subnormal CPD operator $T$, $r(T) > 1$ and $\lambda T$ is not CPD for any complex number $\lambda$ such that $0 < |\lambda| < \frac{1}{r(T)}$, where $r(T)$ stands for the spectral radius of $T$. Note that in Example 1.2.3, $r(T_\theta) = 1$. Using the description of CPD algebraic operators as in [42], one can show that

$$\Xi_{T_\theta} = \{ \lambda : \lambda \text{ is a complex number and } |\lambda - e^{i\theta}| = 1 \},$$

$$\tilde{\Xi}_{T_\theta} = \{ \lambda : \lambda \text{ is a complex number and } |\lambda| = 1 \} \cup \{0\},$$

where the sets $\Xi_{T_\theta}$ and $\tilde{\Xi}_{T_\theta}$ are as in Example 1.2.3. ♦

According to Proposition 4.3.1, the only $m$-isometries that are CPD are 3-isometries. Since subnormal operators are also CPD, this raises another natural question.

(9) Are there CPD operators that are not orthogonal sums of a subnormal operator and a 3-isometry?

As shown in Example 1.2.5, the answer to question (9) is in the affirmative.

**Example 1.2.5.** Let $a \in (1, \infty)$ and let $W_a$ be the unilateral weighted shift on $\ell^2$ as in Example 4.3.6. Then

$$W_a \text{ is CPD, but neither subnormal nor 3-isometric.} \tag{1.2.2}$$

Suppose to the contrary that $W_a$ is an orthogonal sum of a number of subnormal operators and a number of 3-isometries. This implies that there exists a nonzero closed subspace of $\ell^2$ which reduces $W_a$ either to a subnormal operator or to a 3-isometry. Since (injective) unilateral weighted shifts are irreducible (see [54, (3.0)]), $W_a$ is either subnormal or 3-isometric, which contradicts (1.2.2). ♦

As is well known, the class of unilateral weighted shifts is an important research area, providing useful tools for constructing examples and counterexamples (see [63]). This is also true for our paper, as seen in Examples 1.2.2, 1.2.5 and 4.3.6.
and Remark 3.3.3 (see also Propositions 3.4.9 and 3.4.10), where we make extensive use of weighted shifts. In particular, CPD operators that are neither subnormal nor 3-isometric can be implemented as unilateral weighted shifts (see (1.2.2)). A natural question then is to characterize unilateral weighted shifts that are CPD. An in-depth study of CPD unilateral weighted shifts based on a Lévy-Khinchin type formula (cf. [10] Theorem 4.3.19]) is carried out in the forthcoming paper [38]. It gives explicit methods to construct weighted shifts of this class and solves the flatness and the $n$-step backward extension problems in this class.

1.3. Ideas and concise description. In this paper we provide several characterizations of CPD operators. For the reader’s convenience, we make an excerpt from characterizations that are contained in Theorems 3.1.1, 3.2.5 and 3.3.1 (see also Theorem 3.1.0).

**Theorem 1.3.1.** Let $T \in B(H)$. Then the following conditions are equivalent:

(i) $T$ is CPD,

(ii) there exist $B, C \in B(H)$ and a $B(H)$-valued Borel semispectral measure $F$ on $[0, \infty)$ with compact support such that $B = B^*$, $C \geq 0$, $F(\{1\}) = 0$ and

\[
T^*T^n = I + nB + n^2C + \int_{[0, \infty)} Q_n(x)F(dx), \quad n = 0, 1, 2, \ldots, \tag{1.3.1}
\]

where $Q_n$ is the polynomial as in (2.2.1).

(iii) there exists a $B(H)$-valued Borel semispectral measure $M$ on $[0, \infty)$ with compact support such that

\[
T^*T^n = \int_{[0, \infty)} x^n M(dx), \quad n = 0, 1, 2, \ldots. \tag{1.3.2}
\]

Moreover, the triplet $(B, C, F)$ in (ii) and the measure $M$ in (iii) are unique, and

\[
B = T^*T - I - \frac{1}{2} M(\{1\}), \tag{1.3.3}
\]

\[
C = \frac{1}{2} M(\{1\}), \tag{1.3.4}
\]

\[
F(\Delta) = (1 - \chi_\Delta(1))M(\Delta), \quad \Delta \text{ is a Borel subset of } [0, \infty). \tag{1.3.5}
\]

Denote by $\text{CPD}_H$ the class of all CPD operators on $\mathcal{H}$ and by $\mathcal{F}_H$ the class of all triplets $(B, C, F)$, where $B, C \in B(H)$ are such that $B = B^*$ and $C \geq 0$, and $F$ is a $B(H)$-valued Borel semispectral measure on $[0, \infty)$ with compact support such that $F(\{1\}) = 0$. In view of Theorem 1.3.1, the mapping

\[
\Psi_\mathcal{H} : \text{CPD}_H \rightarrow \mathcal{F}_H \text{ given by } \Psi_\mathcal{H}(T) = (B, C, F), \tag{1.3.6}
\]

where $(B, C, F) \in \mathcal{F}_H$ satisfies 1.3.1, is well defined. The mapping $\Psi_\mathcal{H}$ is never injective (provided $\mathcal{H} \neq \{0\}$). Indeed, an operator $T \in B(H)$ is an isometry if and only if 1.3.1 holds with $(B, C, F) = (0, 0, 0)$, so

\[
\Psi_\mathcal{H}^{-1}(\{(0, 0, 0)\}) \text{ is the class of all isometries on } \mathcal{H}. \tag{1.3.7}
\]

More generally, if $T \in B(H)$ is a CPD operator and $V \in B(H)$ is any isometry that commutes with $T$, then the operator $VT$ is CPD and $\Psi_\mathcal{H}(T) = \Psi_\mathcal{H}(VT)$. Moreover, the mapping $\Psi_\mathcal{H}$ is not surjective in general. To have an example, consider the triplet $(B, C, 0) \in \mathcal{F}_H$. Then it can happen that $\Psi_\mathcal{H}^{-1}(\{(B, C, 0)\}) = \emptyset$ (e.g., when $\mathcal{H} = \mathbb{C}$ and $(B, C) \neq (0, 0)$), which means that in this case the expression on the right-hand side of the equality in 1.3.1 is a polynomial in $n$ with operator...
coefficients, but there is no \( T \in B(H) \) satisfying \((1.3.1)\). In other words, in view of [41 Corollary 3.5], the lack of surjectivity appears not only in the class of CPD operators but also in the class of \( m \)-isometries. What is more, this drawback also applies to other classes of operators, not to mention subnormal ones (cf. \((1.4.5)\)).

As for the second characterization of CPD operators given in Theorem \((1.3.1,iii)\), it reduces the number of parameters to one, but at the cost of increasing the complexity of the expression on the left-hand side of \((1.3.2)\). Denoting by \( \tilde{T}_H \) the class of all \( B(H) \)-valued Borel semispectral measures on \([0, \infty)\) with compact support, observe that according to Theorem \((1.3.1)\) the mapping

\[
\tilde{\Psi}_H: \text{CPD}_H \to \tilde{T}_H \text{ given by } \tilde{\Psi}_H(T) = M, \tag{1.3.8}
\]

where \( M \in \tilde{T}_H \) satisfies \((1.3.2)\), is well defined. Since

\[
\tilde{\Psi}_H^{-1}(\{0\}) \text{ is the class of all 2-isometries,} \tag{1.3.9}
\]

the mapping \( \tilde{\Psi}_H \) is never injective (provided \( H \neq \{0\} \)). Moreover, it is not surjective in general (e.g., if \( H = C \) and \( c \in (0, \infty) \), then \( \tilde{\Psi}_H^{-1}\{e_01\} = \emptyset \)). To compare the surjectivity of \( \Psi_H \) and \( \tilde{\Psi}_H \), note that if \( \tilde{\Psi}_H^{-1}(\{(B, C, F)\}) \neq \emptyset \), then by Theorem \((1.3.1)\) any \( T \in \Psi_H^{-1}(\{(B, C, F)\}) \) is CPD and \( T \in \tilde{\Psi}_H^{-1}(\{M\}) \), where \( M \) is defined by \((1.3.4)\) and \((1.3.5)\). And vice versa, if \( \tilde{\Psi}_H^{-1}(\{M\}) \neq \emptyset \), then any \( T \in \tilde{\Psi}_H^{-1}(\{M\}) \) is CPD and \( T \in \Psi_H^{-1}(\{(B, C, F)\}) \), where \( B, C \) and \( F \) are defined by \((1.3.3)\) and \((1.3.5)\). Note also that if \( (B, C, F) = (0, 0, 0) \), then \( M = 0 \) corresponds to \( (B, C, F) \) via \((1.3.4)\) and \((1.3.5)\) and by \((1.3.7)\) and \((1.3.9)\), we have

\[
\Psi_H^{-1}(\{(0, 0, 0)\}) \subseteq \tilde{\Psi}_H^{-1}(\{0\}) \text{ (provided } H \ni K_0).\]

The reader must be aware of the fact that there are operators \( T \in \tilde{\Psi}_H^{-1}(\{0\}) \) such that \( T \in \Psi_H^{-1}(\{(B, 0, 0)\}) \) with \( B = T^*T - I \neq 0 \), where \( (B, 0, 0) \) corresponds to \( M = 0 \) via \((1.3.3)\) and \((1.3.5)\). These are exactly non-isometric 2-isometries. It is worth mentioning that the ranges of the mappings \( \Psi_H \) and \( \tilde{\Psi}_H \) when restricted to operators of class \( Q \) can be described explicitly (see Remark \((3.3.7)\)). However, the problem of describing the ranges of the mappings \( \Psi_H \) and \( \tilde{\Psi}_H \) in full generality is highly non-trivial.

The organization of this paper is as follows. We begin by introducing notation and terminology in Subsection \((1.4)\) and collecting more or less known facts about PD and CPD (scalar) sequences in Subsection \((2.1)\). The remainder of Section \((2)\) is devoted to systematic study of CPD sequences. In Subsection \((2.2)\) we provide an integral representation for a CPD sequence of exponential growth and relate the rate of its growth to the “size” of the closed support of its representing measure (see Theorem \((2.2.5)\)). We also compare the integral representations for PD and CPD sequences (see Theorem \((2.2.12)\). Theorem \((2.2.13)\) which is the main result of this subsection, states that a sequence \( \{\gamma_n\}_{n=0}^\infty \) of exponential growth is PD if and only if \( 0 \) is an accumulation point of the set of all \( \theta \in (0, \infty) \) for which the sequence \( \{\theta^n\gamma_n\}_{n=0}^\infty \) is CPD. In Subsection \((2.3)\) we characterize CPD sequences of exponential growth for which the sequence of consecutive differences is either convergent or bounded from above plus some additional constraints (see Theorems \((2.3.2)\) and \((2.3.3)\)). As a consequence, we show that, subject to some mild constraints, convergent CPD sequences of exponential growth are PD (see Corollary \((2.3.4)\)).
Starting from Section 3, we begin the study of CPD operators. In Subsection 3.1, we give a semispectral integral representation for such operators and relate their spectral radii to the closed supports of representing semispectral measures (see Theorem 3.1.1). Certain semispectral integral representations for completely hypercontractive and completely hyperexpansive operators of finite order appeared in [37, 21, 22] with the representing semispectral measures concentrated on the closed interval \([0, 1]\). In our case there is no limitation on the size of the support (the reader should be aware of the fact that CPD operators are not scalable in general, see Corollary 3.4.6). Theorem 3.1.6 offers yet another semispectral integral representation for CPD operators satisfying a telescopic-like condition. Theorem 3.2.5, which is the main result of Subsection 3.2, provides a dilation representation for CPD operators based on Agler’s hereditary functional calculus and relates their spectral radii to the norms of positive operators appearing in the dilation representation. Subsection 3.3 contains simplified semispectral and dilation representations of CPD operators (see Theorem 3.3.1). As an application, we show that the class of CPD operators is closed under the operation of taking powers (see Theorem 3.3.2). We also completely characterize CPD operators of class \(Q\) (see Theorem 3.3.5). In both cases, we describe explicitly the corresponding semispectral integral and dilation representations. In Theorem 3.4.1 we give necessary and sufficient conditions for a CPD operator \(T\) to be subnormal written in terms of the semispectral integral representation of \(T\). Theorem 3.4.4, which is the main result of Subsection 3.4, provides several characterizations of subnormal contractions via conditional positive definiteness including the one appealing to the telescopic condition. This is a generalization of [70, Theorem 4.1]. On the basis of earlier results, we characterize conditional positive definiteness of a (bounded) operator \(T\) on \(H\) by subnormality of (in general unbounded) unilateral weighted shifts \(W_{T,h}\), \(h \in H\), canonically associated with \(T\) (see Proposition 3.4.9). If the sequence \(\{T^* (n+1) T^n \}^\infty_{n=0}\) is not convergent in the weak operator topology, then some (or even all except \(h = 0\)) weighted shifts \(W_{T,h}\) may be unbounded. If the limit exists and is nonzero, then all \(W_{T,h}\) are bounded, but \(T\) is not subnormal. Finally, if the limit exists and is equal to zero, then \(T\) is a subnormal contraction.

In Subsection 4.1, we construct an \(L^\infty(M)\)-functional calculus for CPD operators, where \(M\) is a semispectral measure on the closed half-line \([0, \infty)\) associated to \(T\) (see Theorem 4.1.2). As a consequence, we obtain a variety of estimates on norms of polynomial and analytic expressions coming from operators in question (see Corollary 4.1.3 and Subsection 4.2). The last subsection of this paper is devoted to characterizing CPD operators for which the closed support of the associated semispectral measure is one of the three sets \(\emptyset\), \(\{1\}\) and \(\{0\}\). It is shown that the first two cases completely characterize CPD \(m\)-isometries (see Proposition 4.3.1). The third case leads to CPD operators that are beyond the classes of subnormal and \(m\)-isometric operators (see Proposition 4.3.5 and Example 4.3.6).

1.4. Notation and terminology. We denote by \(\mathbb{R}\) and \(\mathbb{C}\) the fields of real and complex numbers, respectively. Since we consider suprema of subsets of \(\mathbb{R}\) which may be empty, we adhere to the often-used convention that
\[
\sup \emptyset = \sup_{x \in \emptyset} f(x) := -\infty \quad \text{whenever } f : \mathbb{R} \to \mathbb{R}.
\]
We write \(\mathbb{N}\), \(\mathbb{Z}_+\) and \(\mathbb{R}_+\) for the sets of positive integers, nonnegative integers and nonnegative real numbers, respectively. As usual, \(\mathbb{C}[X]\) stands for the ring
of all polynomials in indeterminate \( X \) with complex coefficients. We customarily identify members of \( \mathbb{C}[X] \) with polynomial functions of one real variable. The unique involution on \( \mathbb{C}[X] \) which sends \( X \) to itself is denoted by \( * \), that is, if \( p = \sum_{i \geq 0} \alpha_i X^i \in \mathbb{C}[X] \), then \( p^* = \sum_{i \geq 0} \overline{\alpha_i} X^i \), or in the language of polynomial functions \( p^*(x) = p(x) \) for all \( x \in \mathbb{R} \). If no ambiguity arises, the characteristic function of a subset \( \Omega_1 \) of a set \( \Omega \) is denoted by \( \chi_{\Omega_1} \). Given a compact topological Hausdorff space \( \Omega \), let \( C(\Omega) \) stand for the Banach space of all continuous complex functions on \( \Omega \) with the supremum norm
\[
\|f\|_{C(\Omega)} = \sup_{x \in \Omega} |f(x)|, \quad f \in C(\Omega).
\]

We write \( \mathcal{B}(\Omega) \) for the \( \sigma \)-algebra of all Borel subsets of a topological Hausdorff space \( \Omega \). If not stated otherwise, measures considered in this paper are assumed to be positive. The \emph{closed support} of a finite Borel measure \( \mu \) on \( \mathbb{R} \) (or \( \mathbb{C} \)) is denoted by \( \text{supp}(\mu) \) (recall that \( \text{supp}(\mu) \) exists because \( \mu \) is automatically regular, see [59, Theorem 2.18]). Given \( x \in \mathbb{R} \), we write \( \delta_x \) for the Borel probability measure on \( \mathbb{R} \) such that \( \text{supp}(\delta_x) = \{x\} \).

All Hilbert spaces considered in this paper are assumed to be complex. Given Hilbert spaces \( \mathcal{H} \) and \( \mathcal{K} \), we denote by \( \mathcal{B}(\mathcal{H}, \mathcal{K}) \) the Banach space of all bounded linear operators from \( \mathcal{H} \) to \( \mathcal{K} \). We abbreviate \( \mathcal{B}(\mathcal{H}, \mathcal{H}) \) to \( \mathcal{B}(\mathcal{H}) \) and denote by \( \mathcal{B}(\mathcal{H})_+ \) the convex cone \( \{T \in \mathcal{B}(\mathcal{H}); T \geq 0\} \) of nonnegative operators on \( \mathcal{H} \). We write \( I_\mathcal{H} \) (or simply \( I \) if no ambiguity arises) for the identity operator on \( \mathcal{H} \). Let \( T \in \mathcal{B}(\mathcal{H}) \). In what follows, \( \mathcal{N}(T) \), \( \mathcal{R}(T) \), \( \sigma(T) \), \( \sigma_p(T) \), \( r(T) \) and \( |T| \) stand for the kernel, the range, the spectrum, the point spectrum, the spectral radius and the modulus of \( T \), respectively. To comply with Gelfand’s formula for spectral radius, we adhere to the convention that \( r(T) = 0 \) if \( \mathcal{H} = \{0\} \). We say that \( T \) is \emph{normaloid} if \( r(T) = \|T\| \), or equivalently, by Gelfand’s formula for spectral radius, if and only if \( \|T^n\| = \|T\|^n \) for all \( n \in \mathbb{N} \). Let us recall the following basic fact (see [25, Proposition II.4.6], see also [31, p. 116]).

\[
\text{Any subnormal operator is normaloid.} \tag{1.4.2}
\]

This will be used several times in this article. Given an operator \( T \in \mathcal{B}(\mathcal{H}) \), we set
\[
\mathcal{B}_m(T) = \sum_{k=0}^{m} (-1)^k \binom{m}{k} T^{+k} T^{-k}, \quad m \in \mathbb{Z}_+.
\tag{1.4.3}
\]

Recall that if \( m \in \mathbb{N} \) and \( \mathcal{B}_m(T) = 0 \), then \( T \) is called an \emph{m-isometry} (see [2], p. 11] and [3, 4, 5]). An \( m \)-isometry \( T \) is said to be \emph{strict} if \( m = 1 \) and \( \mathcal{H} \neq \{0\} \), or \( m \geq 2 \) and \( T \) is not an \((m - 1)\)-isometry; in both cases \( \mathcal{H} \neq \{0\} \) (see [10]). Examples of strict \( m \)-isometries for each \( m \geq 2 \) are given in [7, Proposition 8]. We say that \( T \) is \emph{2-hyperexpansive} if \( \mathcal{B}_2(T) \leq 0 \) (see [56]). We call \( T \) \emph{completely hyperexpansive} if \( \mathcal{B}_m(T) \leq 0 \) for all \( m \in \mathbb{N} \) (see [8]).

Let \( F: \mathcal{A} \to \mathcal{B}(\mathcal{H}) \) be a semispectral measure on a \( \sigma \)-algebra \( \mathcal{A} \) of subsets of a set \( \Omega \), i.e., \( F \) is \( \sigma \)-additive in the weak operator topology (briefly, \( \text{wot} \)) and \( F(\Delta) \geq 0 \) for all \( \Delta \in \mathcal{A} \). Denote by \( L^1(F) \) the linear space of all complex \( \mathcal{A} \)-measurable functions \( \zeta \) on \( \Omega \) such that \( \int_{\Omega} |\zeta(x)| (F(dx)h, h) < \infty \) for all \( h \in \mathcal{H} \). Then for every \( \zeta \in L^1(F) \), there exists a unique operator \( \int_{\Omega} \zeta \, dF \in \mathcal{B}(\mathcal{H}) \) such that
If \( \Omega = \mathbb{R}, \mathbb{C} \) and \( F: \mathcal{B}(\Omega) \to \mathcal{B}(H) \) is a semispectral measure, then its closed support is denoted by \( \text{supp}(F) \) (recall that such \( F \) is automatically regular so \( \text{supp}(F) \) exists). By a semispectral measure of a subnormal operator \( T \in \mathcal{B}(H) \) we mean a normalized compactly supported semispectral measure \( \gamma: \mathcal{B}(\mathbb{C}) \to \mathcal{B}(H) \) defined by \( G(\Delta) = PE(\Delta)|_H \) for \( \Delta \in \mathcal{B}(\mathbb{C}) \), where \( E: \mathcal{B}(\mathbb{C}) \to \mathcal{B}(K) \) is the spectral measure of a minimal normal extension \( N \in \mathcal{B}(K) \) of \( T \) and \( P \in \mathcal{B}(K) \) is the orthogonal projection of \( K \) onto \( H \). (The minimality means that \( K \) has no proper closed vector subspace that reduces \( N \) and contains \( H \).) It follows from [44 Proposition 5] and [25 Proposition II.2.5] that a subnormal operator has exactly one semispectral measure. It is also easily seen that \( T^{*n}T^n = \int_\mathbb{C} |z|^{2n}G(dz) \) for all \( n \in \mathbb{Z}_+ \). Applying (1.4.4) and the measure transport theorem (cf. [6 Theorem 1.6.12]) yields

\[
T^{*n}T^n = \int_{\mathbb{R}_+} x^n G \circ \phi^{-1}(dx), \quad n \in \mathbb{Z}_+, \tag{1.4.5}
\]

where \( \phi: \mathbb{C} \to \mathbb{R}_+ \) is defined by \( \phi(z) = |z|^2 \) for \( z \in \mathbb{C} \) and \( G \circ \phi^{-1}: \mathcal{B}(\mathbb{R}_+) \to \mathcal{B}(H) \) is the semispectral measure defined by \( G \circ \phi^{-1}(\Delta) = G(\phi^{-1}(\Delta)) \) for \( \Delta \in \mathcal{B}(\mathbb{R}_+) \).

We refer the reader to [25] for the foundations of the theory of subnormal operators.

2. Conditionally positive definite sequences

2.1. Basic facts. Let \( \gamma = \{\gamma_n\}_{n=0}^\infty \) be a sequence of real numbers. Recall that \( \gamma \) is PD if (1.1.1) holds for all finite sequences \( \{\lambda_i\}_{i=0}^k \subseteq \mathbb{C} \); \( \gamma \) is CPD if (1.1.1) holds for all finite sequences \( \{\lambda_i\}_{i=0}^k \subseteq \mathbb{C} \) such that \( \sum_{i=0}^k \lambda_i = 0 \). It is a matter of routine to verify that \( \gamma \) is PD (resp., CPD) if and only if (1.1.1) holds for all finite sequences \( \{\lambda_i\}_{i=0}^k \subseteq \mathbb{R} \) (resp., for all finite sequences \( \{\lambda_i\}_{i=0}^k \subseteq \mathbb{R} \) such that \( \sum_{i=0}^k \lambda_i = 0 \)). Another important observation (cf. [10 Remark 3.1.2]) is that \( \gamma \) is PD (resp., CPD) if and only if

\[
\sum_{i,j=1}^k \gamma_{n_i+n_j} \lambda_i \bar{\lambda}_j \geq 0
\]

for all finite sequences \( \{n_i\}_{i=1}^k \subseteq \mathbb{Z}_+ \) and \( \{\lambda_i\}_{i=1}^k \subseteq \mathbb{C} \) (resp., for all finite sequences \( \{n_i\}_{i=1}^k \subseteq \mathbb{Z}_+ \) and \( \{\lambda_i\}_{i=1}^k \subseteq \mathbb{C} \) such that \( \sum_{i=1}^k \lambda_i = 0 \)). This shows that our definitions of positive definiteness and conditional positive definiteness are consistent with those in [10 Section 3.1]. Let us mention further that according to the terminology in [10], \( \gamma \) is CPD if and only if \( -\gamma \) is “negative definite”. It follows from the definition that if \( \gamma \) is PD (resp., CPD), then so is the sequence \( \{\gamma_{n+2k}\}_{n=0}^\infty \) for every \( k \in \mathbb{Z}_+ \). However, it may happen that \( \gamma \) is PD but \( \{\gamma_{n+1}\}_{n=0}^\infty \) is not (e.g., \( \gamma_n = (-1)^n \) for \( n \in \mathbb{Z}_+ \)).

The following fundamental characterization of conditional positive definiteness in terms of positive definiteness is essentially due to Schoenberg.

**Lemma 2.1.1** ([55 Lemma 1.7], [10 Theorem 3.2.2]). If \( \gamma = \{\gamma_n\}_{n=0}^\infty \) is a sequence of real numbers, then the following conditions are equivalent:

(i) \( \gamma \) is CPD,

(ii) \( \{e^{t\gamma_n}\}_{n=0}^\infty \) is PD for every positive real number \( t \).
A sequence \( \gamma = \{ \gamma_n \}_{n=0}^{\infty} \) of real numbers is said to be a Hamburger (resp., Stieltjes, Hausdorff) moment sequence if there exists a Borel measure \( \mu \) on \( \mathbb{R} \) (resp., \( \mathbb{R}_+, [0, 1] \)) such that
\[
\gamma_n = \int t^n \, d\mu(t), \quad n \in \mathbb{Z}_+.
\]
(2.1.1)

A Borel measure \( \mu \) on \( \mathbb{R} \) satisfying (2.1.1) is called a representing measure of \( \gamma \). If \( \gamma \) is a Hamburger moment sequence which has a unique representing measure on \( \mathbb{R} \), then we say that \( \gamma \) is determinate. Note that by [59 Ex. 4(e), p. 71], the Weierstrass theorem (see [58 Theorem 7.26]) and the Riesz representation theorem (see [69 Theorem 2.14]) the following holds.

**Lemma 2.1.2.** A Hamburger moment sequence \( \gamma = \{ \gamma_n \}_{n=0}^{\infty} \) of real numbers has a compactly supported representing measure if and only if \( \theta := \limsup_{n \to \infty} |\gamma_n|^{1/n} < \infty \). Moreover, if this is the case, then \( \gamma \) is determinate and \( \text{supp}(\mu) \subseteq [-\theta, \theta] \), where \( \mu \) is a unique representing measure of \( \gamma \).

In particular, a Hausdorff moment sequence is always determinate. For our later needs, we recall a theorem due to Stieltjes.

**Theorem 2.1.3 ([66,10 Theorem 6.2.5]).** A sequence \( \{ \gamma_n \}_{n=0}^{\infty} \subseteq \mathbb{R} \) is a Stieltjes moment sequence if and only if the sequences \( \{ \gamma_n \}_{n=0}^{\infty} \) and \( \{ \gamma_n+1 \}_{n=0}^{\infty} \) are PD.

We refer the reader to [10, 65] for the fundamentals of the theory of moment problems.

### 2.2. Exponential growth.

In this subsection we give an integral representation for CPD sequences of (at most) exponential growth (see Theorem 2.2.5). PD sequences of exponential growth are characterized by means of parameters appearing in the above-mentioned integral representation (see Theorem 2.2.12). Theorem 2.2.13 states that a sequence \( \{ \gamma_n \}_{n=0}^{\infty} \) of exponential growth is PD if and only if the sequences \( \{ \theta^n \gamma_n \}_{n=0}^{\infty}, \theta \in \mathbb{R} \), are CPD.

We begin by introducing the difference transformation \( \Delta \) which plays an important role in further considerations. Denote by \( \mathbb{C}^{\mathbb{Z}_+} \) the complex vector space of all complex sequences \( \{ \gamma_n \}_{n=0}^{\infty} \) with linear operations defined coordinatewise. The difference transformation \( \Delta : \mathbb{C}^{\mathbb{Z}_+} \to \mathbb{C}^{\mathbb{Z}_+} \) is given by
\[
(\Delta \gamma)_n = \gamma_{n+1} - \gamma_n, \quad n \in \mathbb{Z}_+, \quad \gamma = \{ \gamma_n \}_{n=0}^{\infty} \in \mathbb{C}^{\mathbb{Z}_+}.
\]
Clearly, \( \Delta \) is a linear. Denote by \( \Delta^k \) the \( k \)th composition power of \( \Delta \), i.e., \( \Delta^0 \) is the identity transformation of \( \mathbb{C}^{\mathbb{Z}_+} \) and \( \Delta^{k+1} \gamma = \Delta^k (\Delta \gamma) \) for \( \gamma = \{ \gamma_n \}_{n=0}^{\infty} \in \mathbb{C}^{\mathbb{Z}_+} \).

Given \( n \in \mathbb{Z}_+ \), we define the polynomial \( Q_n \in \mathbb{C}[X] \) by
\[
Q_n(x) = \begin{cases} 
0 & \text{if } x \in \mathbb{R} \text{ and } n = 0, 1, \\
\sum_{j=0}^{n-2} (n-j-1)x^j & \text{if } x \in \mathbb{R} \text{ and } n = 2, 3, 4, \ldots. 
\end{cases}
\]
(2.2.1)

Below, for fixed \( k \in \mathbb{N} \) and \( x \in \mathbb{R} \), we write \( \Delta^k Q_n(x) \) to denote the action of the transformation \( \Delta^k \) on the sequence \( \{ Q_n(x) \}_{n=0}^{\infty} \).

**Lemma 2.2.1.** The polynomials \( Q_n \) have the following properties:
\[
Q_n(x) = \frac{x^n - 1 - n(x - 1)}{(x - 1)^2}, \quad n \in \mathbb{Z}_+, \quad x \in \mathbb{R} \setminus \{1\},
\]
(2.2.2)
\[
Q_{n+1}(x) = xQ_n(x) + n, \quad n \in \mathbb{Z}_+, \; x \in \mathbb{R}, \quad (2.2.3)
\]
\[
\frac{Q_n(x)}{n} \leq \frac{Q_{n+1}(x)}{n+1}, \quad n \in \mathbb{N}, \; x \in [0, 1], \quad (2.2.4)
\]
\[
\lim_{n \to \infty} \frac{Q_n(x)}{n} = \frac{1}{1 - x}, \quad x \in [0, 1), \quad (2.2.5)
\]
\[
(\Delta Q)(x) = \begin{cases} 
0 & \text{if } n = 0, \; x \in \mathbb{R}, \\
\sum_{j=0}^{n-1} x^j & \text{if } n \in \mathbb{N}, \; x \in \mathbb{R}, 
\end{cases} \quad (2.2.6)
\]
\[
(\Delta^2 Q)(x) = x^n, \quad n \in \mathbb{Z}_+, \; x \in \mathbb{R}. \quad (2.2.7)
\]

**Proof.** Suppose \( n \geq 2 \). Then
\[
\frac{x^n - 1 - n(x - 1)}{(x - 1)^2} = \frac{\sum_{i=0}^{n-1} x^i - n}{x - 1} = \frac{\sum_{i=0}^{n-1} x^i - 1}{x - 1}
\]
\[
= \sum_{i=0}^{n-1} x^i = \sum_{j=0}^{n-2} (n - j - 1)x^j, \quad x \in \mathbb{R} \setminus \{1\}.
\]
This implies (2.2.2). Identities (2.2.3), (2.2.6) and (2.2.7) follow from (2.2.1) and the definition of \( \Delta \), while (2.2.4) and (2.2.5) can be deduced from (2.2.2). \( \square \)

Below, we denote by \( |\mu| \) the total variation measure of a complex Borel measure \( \mu \) on \( \mathbb{R} \). Recall that a complex Borel measure on \( \mathbb{R} \) is automatically regular, i.e., its total variation measure is regular (see [59, Theorem 2.18]).

**Lemma 2.2.2.** Suppose \( a, b, c \in \mathbb{C} \) and \( \mu \) is a complex Borel measure on \( \mathbb{R} \) such that \( \mu(\{1\}) = 0 \), the measure \( |\mu| \) is compactly supported and
\[
a + bn + cn^2 + \int_{\mathbb{R}} Q_n(x) d\mu(x) = 0, \quad n \in \mathbb{Z}_+.
\]
Then \( a = b = c = 0 \) and \( \mu = 0 \).

**Proof.** Define \( \gamma \in \mathbb{C}^{\mathbb{Z}_+} \) by \( \gamma_n = a + bn + cn^2 + \int_{\mathbb{R}} Q_n(x) d\mu(x) \) for \( n \in \mathbb{Z}_+ \). It follows from (2.2.7) that
\[
0 = (\Delta^2 \gamma)_n = \int_K x^n d(\mu + 2c\delta_1)(x), \quad n \in \mathbb{Z}_+,
\]
where \( K := \text{supp}(|\mu + 2c\delta_1|) \) is a compact subset of \( \mathbb{R} \). This implies that
\[
\int_K p(x) d(\mu + 2c\delta_1)(x) = 0, \quad p \in \mathbb{C}[X].
\]
Applying the Weierstrass theorem and the uniqueness part in the Riesz Representation Theorem (see [59, Theorem 6.19]), we deduce that \( (\mu + 2c\delta_1)(\Delta) = 0 \) for all \( \Delta \in \mathfrak{B}(\mathbb{R}) \). Substituting \( \Delta = \{1\} \), we get \( c = 0 \), and consequently \( \mu = 0 \). Clearly, \( a = \gamma_0 = 0 \). Putting all this together gives \( b = 0 \), completing the proof. \( \square \)

Now, for the reader’s convenience we state explicitly the fundamental characterization of CPD sequences. Recall that a Borel measure on a Hausdorff topological space is said to be Radon if it is finite on compact sets and inner regular with respect to compact sets.
Theorem 2.2.3 ([10] Theorem 6.2.6). A sequence \( \gamma = \{ \gamma_n \}_{n=0}^{\infty} \subseteq \mathbb{R} \) is CPD if and only if it has a representation of the form
\[
\gamma_n = \gamma_0 + bn + cn^2 + \int_{\mathbb{R} \setminus \{1\}} (x^n - 1 - n(x - 1))d\mu(x), \quad n \in \mathbb{Z}_+,
\]
where \( b \in \mathbb{R}, \ c \in \mathbb{R}_+ \) and \( \mu \) is a Radon measure on \( \mathbb{R} \setminus \{1\} \) such that
\[
\int_{0 < |x-1| < 1} (x - 1)^2d\mu(x) < \infty,
\]
\[
\int_{|x-1| \geq 1} |x|^n d\mu(x) < \infty, \quad n \in \mathbb{Z}_+.
\]

For our purpose, we need the following equivalent variant of Theorem 2.2.3.

Theorem 2.2.4. A sequence \( \gamma = \{ \gamma_n \}_{n=0}^{\infty} \) of real numbers is CPD if and only if it has a representation of the form
\[
\gamma_n = \gamma_0 + bn + cn^2 + \int_{\mathbb{R}} Q_n(x)d\nu(x), \quad n \in \mathbb{Z}_+, \tag{2.2.8}
\]
where \( b \in \mathbb{R}, \ c \in \mathbb{R}_+ \) and \( \nu \) is a Borel measure on \( \mathbb{R} \) such that \( \nu(\{1\}) = 0 \) and
\[
\int_{\mathbb{R}} |x|^n d\nu(x) < \infty, \quad n \in \mathbb{Z}_+. \tag{2.2.9}
\]

Proof. To prove the “only if” part apply Theorem 2.2.3 and define the finite Borel measure \( \nu \) on \( \mathbb{R} \) by
\[
\nu(\Delta) = \int_{\Delta \cap (\mathbb{R} \setminus \{1\})} (x - 1)^2d\mu(x), \quad \Delta \in \mathcal{B}(\mathbb{R}).
\]
Then, by Lemma 2.2.1, conditions (2.2.8) and (2.2.9) are satisfied (with the same \( b, c \)). The converse implication goes through by applying Theorem 2.2.3 to the Radon measure \( \mu \) defined by
\[
\mu(\Delta) = \int_{\Delta} (x - 1)^{-2}d\nu(x), \quad \Delta \in \mathcal{B}(\mathbb{R} \setminus \{1\}).
\]
That the so-defined \( \mu \) is a Radon measure follows from [59] Theorem 2.18. \( \square \)

CPD sequences of (at most) exponential growth can be characterized as follows (below we use the convention (1.4.1)).

Theorem 2.2.5. Let \( \gamma = \{ \gamma_n \}_{n=0}^{\infty} \) be a sequence of real numbers. Then the following conditions are equivalent:

(i) \( \gamma \) is CPD and there exist \( \alpha, \theta \in \mathbb{R}_+ \) such that
\[
|\gamma_n| \leq \alpha \theta^n, \quad n \in \mathbb{Z}_+,
\]

(ii) \( \gamma \) is CPD and \( \limsup_{n \to \infty} |\gamma_n|^{1/n} < \infty, \)

(iii) there exist \( b \in \mathbb{R}, \ c \in \mathbb{R}_+ \) and a finite compactly supported Borel measure \( \nu \) on \( \mathbb{R} \) such that \( \nu(\{1\}) = 0 \) and
\[
\gamma_n = \gamma_0 + bn + cn^2 + \int_{\mathbb{R}} Q_n(x)d\nu(x), \quad n \in \mathbb{Z}_+. \tag{2.2.10}
\]
Moreover, if (iii) holds, then the triplet \((b, c, \nu)\) is unique and
\[
\limsup_{n \to \infty} |\gamma_n|^{1/n} = \inf \left\{ \theta \in \mathbb{R}_+: \exists \alpha \in \mathbb{R}_+ \forall n \in \mathbb{Z}_+ \ |\gamma_n| \leq \alpha \theta^n \right\}, \tag{2.2.11}
\]
\[
c > 0 \implies \limsup_{n \to \infty} |\gamma_n|^{1/n} \geq 1, \tag{2.2.12}
\]
\[
supp(\nu) \subseteq \left[ -\limsup_{n \to \infty} |\gamma_n|^{1/n}, \limsup_{n \to \infty} |\gamma_n|^{1/n} \right], \tag{2.2.13}
\]
\[
\sup_{x \in supp(\nu)} |x| \geq 1 \implies \limsup_{n \to \infty} |\gamma_n|^{1/n} = \sup_{x \in supp(\nu)} |x|, \tag{2.2.14}
\]
\[
\limsup_{n \to \infty} |\gamma_n|^{1/n} \leq \max \left\{ 1, \sup_{x \in supp(\nu)} |x| \right\}. \tag{2.2.15}
\]
\[\]
**Proof.** It is a matter of routine to show that conditions (i) and (ii) are equivalent.

(ii)⇒(iii) By Theorem 2.2.4 there exist \(b \in \mathbb{R}, c \in \mathbb{R}_+\) and a finite Borel measure \(\nu\) on \(\mathbb{R}\) that satisfy conditions (2.2.8) and (2.2.9) and the equality \(\nu\{1\} = 0\). It follows from (2.2.1) and (2.2.10) that there exists a constant
\[
R(2.2.14), \text{ assume that } \gamma_n(2.2.15), \text{ it suffices to consider the case when } \limsup_{n \to \infty} |\gamma_n|^{1/n} < \infty \text{ can be deduced straightforwardly from (2.2.11) and (2.2.10).}
\]
\[\]
It remains to complete the proof of the “moreover” part. The uniqueness of the triplet \((b, c, \nu)\) in (iii) follows from Lemma 2.2.2. Identity (2.2.11) is a well-known fact in analysis. Condition (2.2.12) can be deduced from (2.2.18). To prove (2.2.14), assume that \(R := \sup_{x \in supp(\nu)} |x| \geq 1\) (then supp(\(\nu\)) = \(\emptyset\)). It is a matter of routine to deduce from (2.2.1) and (2.2.10) that there exists a constant \(\alpha \in \mathbb{R}_+\) such that
\[
|\gamma_n| \leq \alpha n^2 R^n, \quad n \in \mathbb{N}.
\]
This implies that \(\limsup_{n \to \infty} |\gamma_n|^{1/n} \leq R\). Combined with (2.2.13), this yields \(\limsup_{n \to \infty} |\gamma_n|^{1/n} = R\). Hence (2.2.14) holds. In view of (2.2.14), to prove (2.2.15), it suffices to consider the case when \(\nu(\mathbb{R} \setminus [-1, 1]) = 0\). Note that
\[
|Q_n(x)| \leq \sum_{j=0}^{n-2} (n - j - 1)|x|^j \leq n^2, \quad x \in [-1, 1], \ n \geq 2.
\]
Combined with (2.2.10), this implies that \(|\gamma_n| \leq \alpha \cdot n^2\) for all \(n \in \mathbb{N}\) with \(\alpha = |\gamma_0| + |b| + c + \nu(\mathbb{R})\), so \(\limsup_{n \to \infty} |\gamma_n|^{1/n} \leq 1\). This completes the proof. \(\square\)
**Definition 2.2.6.** If $\gamma = \{\gamma_n\}_{n=0}^\infty$ is a CPD sequence such that
\[ \limsup_{n \to \infty} |\gamma_n|^{1/n} < \infty \]
and $b$, $c$ and $\nu$ are as in statement (iii) of Theorem 2.2.5 we call $(b, c, \nu)$ the representing triplet of $\gamma$, or we simply say that $(b, c, \nu)$ represents $\gamma$.

The following example shows that the converse to the implication (2.2.14) in Theorem 2.2.3 is not true in general.

**Example 2.2.7.** For $\theta \in (0, 1)$, let $\gamma = \{\gamma_n\}_{n=0}^\infty$ be the sequence of real numbers defined by
\[ \gamma_n = \frac{\theta^n}{(\theta - 1)^2} = \frac{1}{(\theta - 1)^2} + \frac{n}{\theta - 1} + Q_n(\theta), \quad n \in \mathbb{Z}_+. \]

Then the sequence $\gamma$ is PD and thus CPD. Its representing triplet $(b, c, \nu)$ takes the form $b = \frac{1}{(\theta - 1)^2}$, $c = 0$ and $\nu = \delta_\theta$. Moreover, we have
\[ \limsup_{n \to \infty} |\gamma_n|^{1/n} = \sup_{x \in \text{supp}(\nu)} |x| = \theta < 1, \]
as required. ◊

Here are more examples of CPD sequences of exponential growth.

**Example 2.2.8.** It is a matter of direct computation to see that if $\mu$ is a finite Borel measure on $\mathbb{R}_+$, then the sequence $\left\{\int_{[0,1]} \frac{x^n - 1}{1 - x} \, d\mu(x)\right\}_{n=0}^\infty$ is CPD and
\[ \int_{[0,1]} \frac{x^n - 1}{1 - x} \, d\mu(x) = -n\mu([0,1]) + \int_{\mathbb{R}_+} Q_n(x) \, d\nu(x), \quad n \in \mathbb{Z}_+, \]
where
\[ \nu(\Delta) = \int_{\Delta \cap [0,1]} (1 - x) \, d\mu(x), \quad \Delta \in \mathfrak{B}(\mathbb{R}_+). \]

Similarly, if $\mu$ is a finite compactly supported Borel measure on $\mathbb{R}_+$, then the sequence $\left\{\int_{(1,\infty)} \frac{x^n - 1}{x - 1} \, d\mu(x)\right\}_{n=0}^\infty$ is CPD and
\[ \int_{(1,\infty)} \frac{x^n - 1}{x - 1} \, d\mu(x) = n\mu((1, \infty)) + \int_{\mathbb{R}_+} Q_n(x) \, d\nu(x), \quad n \in \mathbb{Z}_+, \]
where
\[ \nu(\Delta) = \int_{\Delta \cap (1,\infty)} (x - 1) \, d\mu(x), \quad \Delta \in \mathfrak{B}(\mathbb{R}_+). \] ◊

Yet another characterization of CPD sequences of exponential growth is given below. Let us mention that in view of [10] Theorem 4.6.11 sequences $\gamma = \{\gamma_n\}_{n=0}^\infty \subseteq \mathbb{R}$ for which $\Delta^2 \gamma$ is a Hausdorff moment sequence coincide with completely monotone sequences of order 2 introduced in [21].

**Proposition 2.2.9.** Let $\gamma = \{\gamma_n\}_{n=0}^\infty$ be a sequence of real numbers such that
\[ \limsup_{n \to \infty} |\gamma_n|^{1/n} < \infty. \]
Then the following conditions are equivalent:

(i) $\gamma$ is CPD (resp., $\gamma$ is CPD with the representing triplet $(b, c, \nu)$ such that \text{supp}(\nu) \subseteq \mathbb{R}_+$),

(ii) $\Delta^2 \gamma$ is PD (resp., $\Delta^2 \gamma$ and $\{(\Delta^2 \gamma)_{n+1}\}_{n=0}^\infty$ are PD),
(iii) $\Delta^2 \gamma$ is a Hamburger moment sequence (resp., $\Delta^2 \gamma$ is a Stieltjes moment sequence).

Moreover, if $\gamma$ is CPD and has a representing triplet $(b, c, \nu)$ such that $\text{supp}(\nu) \subseteq \mathbb{R}_+$, then the sequence $\Delta \gamma$ is monotonically increasing.

**Proof.** (i)$\Rightarrow$(iii) Apply (2.2.16) to both versions.

(iii)$\Rightarrow$(i) Suppose that $\Delta^2 \gamma$ is a Hamburger moment sequence. Let $\mu$ be a representing measure of $\Delta^2 \gamma$. Using Lemma [2.1.2] and [2.2.17] we deduce that $\mu$ is compactly supported. Note that

$$
(\Delta^k \gamma)_n = (\Delta^{k-2}(\Delta^2 \gamma))_n = \int_{\mathbb{R}} (x - 1)^{k-2} x^n d\mu(x), \quad n \in \mathbb{Z}_+, k \geq 2.
$$

Let $\nu$ be the finite compactly supported Borel measure on $\mathbb{R}$ given by

$$
\nu(\Delta) = \mu(\Delta \setminus \{1\}), \quad \Delta \in \mathcal{B}(\mathbb{R}).
$$

(2.2.20)

Applying Newton’s binomial formula to $\Delta$ (see [41] (2.2)), we obtain

$$
\gamma_n = \sum_{k=0}^{n} \binom{n}{k} (\Delta^k \gamma)_0
$$

(2.2.19)

$$
= \gamma_0 + n(\gamma_1 - \gamma_0) + \int_{\mathbb{R}} \sum_{k=2}^{n} \binom{n}{k} (x - 1)^{k-2} d\mu(x)
$$

$$
= \gamma_0 + n(\gamma_1 - \gamma_0) + \frac{n(n-1)}{2} \mu(\{1\}) + \int_{\mathbb{R}} \sum_{k=2}^{n} \binom{n}{k} (x - 1)^{k} (x - 1)^{2} d\nu(x)
$$

(2.2.21)

$$
= \gamma_0 + n(\gamma_1 - \gamma_0 - \frac{1}{2} \mu(\{1\})) + \frac{n^2}{2} \mu(\{1\}) + \int_{\mathbb{R}} Q_n(x) d\nu(x), \quad n \geq 2.
$$

This implies that condition (iii) of Theorem 2.2.5 holds with $\nu$ as in (2.2.20) and the parameters $b$ and $c$ defined by

$$
b = \gamma_1 - \gamma_0 - \frac{1}{2} \mu(\{1\}) \quad \text{and} \quad c = \frac{1}{2} \mu(\{1\}).
$$

Thus $\gamma$ is CPD with the representing triplet $(b, c, \nu)$. Clearly, by (2.2.20), $\text{supp}(\mu) \subseteq \mathbb{R}_+$ if and only if $\text{supp}(\nu) \subseteq \mathbb{R}_+$. All this together proves both versions of the implication (iii)$\Rightarrow$(i).

(ii)$\Rightarrow$(iii) Use [10] Theorem 6.2.2 (resp., Theorem 2.1.3).

Since $\Delta \gamma$ is monotonically increasing if and only if $\Delta^2 \gamma \geq 0$, the “moreover” part follows from (2.2.19) applied to $k = 2$ and (2.2.20). \qed

Now we give necessary and sufficient conditions for a CPD sequence to have a polynomial growth of degree at most 2.

**Proposition 2.2.10.** Let $\gamma = \{\gamma_n\}_{n=0}^{\infty}$ be a CPD sequence with the representing triplet $(b, c, \nu)$. Then the following conditions are equivalent:

(i) there exists $\alpha \in \mathbb{R}_+$ such that

$$
|\gamma_n| \leq \alpha \cdot n^2, \quad n \in \mathbb{N},
$$

(2.2.21)

(ii) $\limsup_{n \to \infty} |\gamma_n|^{1/n} \leq 1$,

(iii) $\text{supp}(\nu) \subseteq [-1, 1]$.

Moreover, (iii) implies (2.2.21) with $\alpha = |\gamma_0| + |b| + c + \nu(\mathbb{R})$. 

Proof. (i)⇒(ii) This implication is obvious.
(ii)⇒(iii) It suffices to apply \((2.2.13)\).
(iii)⇒(i) Arguing as in the proof of \((2.2.13)\), one can verify that inequality \((2.2.21)\) holds with \(\alpha := |\gamma_0| + |b| + c + \nu(R)\), which completes the proof. □

The above lemma enables us to prove the following.

Proposition 2.2.11. Let \(p\) be a polynomial in one indeterminate with real coefficients. Then the following conditions are equivalent:

(i) the sequence \(\{p(n)\}_{n=0}^\infty\) is CPD,
(ii) either \(\deg p \leq 1\) or \(\deg p = 2\) and the leading coefficient of \(p\) is positive.

Proof. Suppose that the sequence \(\{p(n)\}_{n=0}^\infty\) is CPD. Since

\[
\limsup_{n \to \infty} \frac{|p(n)|}{1/n} \leq 1,
\]
we infer from Proposition 2.2.10 that there exists \(\alpha \in \mathbb{R}^+\) such that

\[
|p(n)| \leq \alpha \cdot n^2
\]
for all \(n \in \mathbb{N}\). As a consequence, \(\deg p \leq 2\). Straightforward computations complete the proof. □

PD sequences of exponential growth can be characterized by means of parameters describing conditional positive definiteness given in Theorem 2.2.5(iii).

Theorem 2.2.12. Let \(\gamma = \{\gamma_n\}_{n=0}^\infty\) be a sequence of real numbers such that

\[
\limsup_{n \to \infty} |\gamma_n|^{1/n} < \infty.
\]
Then the following conditions are equivalent.

(i) \(\gamma\) is PD,
(ii) \(\gamma\) is a Hamburger moment sequence,
(iii) \(\gamma\) is CPD, \(\int_{\mathbb{R}} \frac{1}{(x-1)^2} d\nu(x) \leq \gamma_0\), \(b = \int_{\mathbb{R}} \frac{1}{x-1} d\nu(x)\) and \(c = 0\), where \((b,c,\nu)\) is a representing triplet of \(\gamma\).

Moreover, if (iii) holds, then \(\gamma\) is a determinate Hamburger moment sequence, its unique representing measure \(\mu\) is compactly supported, and the following identities hold:

\[
\mu(\Delta) = \int_{\Delta} \frac{1}{(x-1)^2} d\nu(x) + \left(\gamma_0 - \int_{\mathbb{R}} \frac{1}{(x-1)^2} d\nu(x)\right)\delta_1(\Delta), \quad \Delta \in \mathfrak{B}(\mathbb{R}),
\]

\[
b = \int_{\mathbb{R}} (x-1) d\mu(x),
\]

\[
\nu(\Delta) = \int_{\Delta} (x-1)^2 d\mu(x), \quad \Delta \in \mathfrak{B}(\mathbb{R}).
\]

Proof. The equivalence (i)⇔(ii) follows from \([10]\) Theorem 6.2.2].
(ii)⇒(iii) Clearly, \(\gamma\) is CPD. Denote by \((b,c,\nu)\) the representing triplet of \(\gamma\).

Let \(\mu\) be a representing measure of \(\gamma\), that is

\[
\gamma_n = \int_{\mathbb{R}} x^n d\mu(x), \quad n \in \mathbb{Z}_+.
\]

By Lemma 2.1.2, \(\gamma\) is determinate and \(\mu\) is compactly supported. Note that

\[
\int_{\mathbb{R}} x^n (x-1)^2 d\mu(x) = \int_{\mathbb{R}} x^n d(\nu + 2c\delta_1)(x), \quad n \in \mathbb{Z}_+.
\]

If the inequality in (iii) holds, then by the Cauchy-Schwarz inequality, \(\frac{1}{x-1} \in L^1(\nu)\).
Since the measure $\nu + 2c\delta_1$ is compactly supported, we infer from Lemma 2.1.2 that
\[
\int_{\Delta} (x - 1)^2 \, d\mu(x) = (\nu + 2c\delta_1)(\Delta), \quad \Delta \in \mathcal{B}(\mathbb{R}).
\] (2.2.26)
Substituting $\Delta = \{1\}$ into (2.2.26), we deduce that $c = 0$. Combined with (2.2.24), this implies (2.2.25). As a consequence of (2.2.24) and $\nu(\{1\}) = 0$, we have
\[
\mu(\Delta) = \int_{\Delta} \frac{1}{(x - 1)^2} \, d\nu(x) + \mu(\{1\})\delta_1(\Delta), \quad \Delta \in \mathcal{B}(\mathbb{R}).
\] (2.2.27)
Since 1 is a common root of the polynomials $X^n - 1 - n(X - 1)$, where $n \in \mathbb{Z}_+$, and $\nu(\{1\}) = 0$, it follows from Lemma 2.2.1 that
\[
\gamma_n = \gamma_0 + b n + \int_{\mathbb{R}} \frac{x^n - 1 - n(x - 1)}{(x - 1)^2} \, d\nu(x)
\] (2.2.28)
\[
= \gamma_0 + b n + \int_{\mathbb{R}} (x^n - 1 - n(x - 1)) \, d\mu(x)
\] (2.2.29)
\[
= (\gamma_0 - \mu(\mathbb{R})) + \left( b - \int_{\mathbb{R}} (x - 1) \, d\mu(x) \right) n + \int_{\mathbb{R}} x^n \, d\mu(x)
\] (2.2.30)
\[
= (\gamma_0 - \mu(\mathbb{R})) + \left( b - \int_{\mathbb{R}} (x - 1) \, d\mu(x) \right) n + \gamma_n, \quad n \in \mathbb{Z}_+.
\] (2.2.31)
Hence, we have
\[
\gamma_0 = \mu(\mathbb{R}) = \int_{\mathbb{R}} \frac{1}{(x - 1)^2} \, d\nu(x) + \mu(\{1\}) \geq \int_{\mathbb{R}} \frac{1}{(x - 1)^2} \, d\nu(x),
\] (2.2.29)
and $b = \int_{\mathbb{R}} (x - 1) \, d\mu(x)$, which yields (2.2.28) and the inequality in (iii). Using (2.2.28), (2.2.29) and [59, Theorem 1.29], we deduce that $b = \int_{\mathbb{R}} \frac{1}{x - 1} \, d\nu(x)$. Summarizing, we have proved that (iii) holds. It follows from (2.2.29) that
\[
\mu(\{1\}) = \gamma_0 - \int_{\mathbb{R}} \frac{1}{(x - 1)^2} \, d\nu(x).
\]
Combined with (2.2.27), this implies (2.2.22). This also justifies the “moreover” part.

(iii) $\Rightarrow$ (ii) It follows from the inequality in (iii) that the formula
\[
\mu(\Delta) = \int_{\Delta} \frac{1}{(x - 1)^2} \, d\nu(x) + \left( \gamma_0 - \int_{\mathbb{R}} \frac{1}{(x - 1)^2} \, d\nu(x) \right) \delta_1(\Delta), \quad \Delta \in \mathcal{B}(\mathbb{R}),
\] (2.2.30)
defines a finite compactly supported Borel measure $\mu$ on $\mathbb{R}$. Arguing as in the first three lines of (2.2.28) and using (2.2.30) instead of (2.2.24), we verify that (2.2.25) is satisfied. This completes the proof. \qed

In view of the Schur product theorem (see [62, p. 14] or [35, Theorem 7.5.3]), the product of two PD sequences is PD; this is no longer true for CPD sequences, e.g., the powers $\{n^{2k}\}_{n=0}^{\infty}$, $k = 2, 3, \ldots$, of the CPD sequence $\{n^2\}_{n=0}^{\infty}$ are not CPD (see Proposition 2.2.11). As a consequence, if $\gamma = \{\gamma_n\}_{n=0}^{\infty}$ is a PD sequence, then the product sequence $\{\xi_n\gamma_n\}_{n=0}^{\infty}$ is CPD for every PD sequence $\{\xi_n\}_{n=0}^{\infty}$. Below, we show that the converse implication is true for sequences $\gamma$ of exponential growth. What is more, the above equivalence remains true if the class of all PD sequences $\{\xi_n\}_{n=0}^{\infty}$ is reduced drastically to the class of the sequences of the form $\{\theta^n\}_{n=0}^{\infty}$, where $\theta \in \mathbb{R}$. 

18  Z. J. Jabłoński, I. B. Jung, and J. Stochel
Theorem 2.2.13. Suppose that \( \{\gamma_n\}_{n=0}^{\infty} \) is a sequence of real numbers such that \( \limsup_{n \to \infty} |\gamma_n|^{1/n} < \infty \). Then the following conditions are equivalent:

(i) the sequence \( \{\gamma_n\}_{n=0}^{\infty} \) is PD,

(ii) the sequence \( \{\theta^n \gamma_n\}_{n=0}^{\infty} \) is CPD for all \( \theta \in \mathbb{R} \),

(iii) zero is an accumulation point of the set of all \( \theta \in \mathbb{R} \setminus \{0\} \) for which the sequence \( \{\theta^n \gamma_n\}_{n=0}^{\infty} \) is CPD,

(iv) there exists \( \theta \in \mathbb{R} \setminus \{0\} \) such that \( |\theta| \cdot \limsup_{n \to \infty} |\gamma_n|^{1/n} < 1 \) and the sequence \( \{\theta^n \gamma_n\}_{n=0}^{\infty} \) is CPD.

Proof. The implication (i) \( \Rightarrow \) (ii) is a direct consequence of the Schur product theorem. The implications (ii) \( \Rightarrow \) (iii) and (iii) \( \Rightarrow \) (iv) are obvious.

(iv) \( \Rightarrow \) (i) Replacing \( \{\gamma_n\}_{n=0}^{\infty} \) by \( \{\theta^n \gamma_n\}_{n=0}^{\infty} \) if necessary, we can assume that \( \{\gamma_n\}_{n=0}^{\infty} \) is CPD and

\[
r := \limsup_{n \to \infty} |\gamma_n|^{1/n} < 1.
\]

Then \( \lim_{n \to \infty} \gamma_n = 0 \) and consequently

\[
\lim_{n \to \infty} (\Delta^j \gamma)_n = 0, \quad j \in \mathbb{Z}_+,
\]

where \( \gamma := \{\gamma_n\}_{n=0}^{\infty} \). Let \( (b,c,\nu) \) be the representing triplet of \( \gamma \). It follows from (2.2.13) that \( \text{supp}(\nu) \subseteq [-r,r] \). Thus, by (2.2.10), we have

\[
(\Delta^2 \gamma)_n = 2c + \int_{[-r,r]} x^n \, d\nu(x), \quad n \in \mathbb{Z}_+.
\]

Using (2.2.31) for \( j = 2 \) and Lebesgue’s dominated convergence theorem, we deduce that \( c = 0 \). In view of (2.2.6) and (2.2.10), we get

\[
(\Delta \gamma)_n = b + \int_{[-r,r]} \frac{1 - x^n}{1 - x} \, d\nu(x), \quad n \in \mathbb{Z}_+.
\]

Since \( r < 1 \), we see that \( \frac{1 - x^n}{1 - x} \) is an accumulation point of \( \{\gamma_n\}_{n=0}^{\infty} \) for all \( j \in \mathbb{Z}_+ \). Hence, it follows from (2.2.31) for \( j = 1 \) and Lebesgue’s dominated convergence theorem that \( b = \int_{[-r,r]} \frac{1 - x^n}{1 - x} \, d\nu(x) \). According to (2.2.2) and (2.2.10), we have

\[
\gamma_n = \gamma_0 + \int_{[-r,r]} \left( \frac{n}{x - 1} + \frac{x^n - 1 - n(x - 1)}{(x - 1)^2} \right) \, d\nu(x)
\]

\[
= \gamma_0 + \int_{[-r,r]} \frac{x^n - 1}{(x - 1)^2} \, d\nu(x), \quad n \in \mathbb{Z}_+.
\]

Using (2.2.31) for \( j = 0 \) and Lebesgue’s dominated convergence theorem, we conclude that \( \int_{[-r,r]} \frac{1}{(x - 1)^2} \, d\nu(x) = \gamma_0 \). Applying Theorem 2.2.12 shows that \( \gamma \) is PD. This completes the proof. \( \square \)

2.3. Additional constraints. In this subsection we characterize CPD sequences \( \gamma = \{\gamma_n\}_{n=0}^{\infty} \) of exponential growth for which the sequence of consecutive differences \( \Delta \gamma \) is either convergent (see Theorem 2.3.2) or bounded from above plus some additional constraints (see Theorem 2.3.3). As a consequence, under slightly stronger hypotheses than those of Theorem 2.3.3 we show that CPD sequences \( \gamma \) of exponential growth with \( \lim_{n \to \infty} (\Delta \gamma)_n = 0 \) are PD (see Corollary 2.3.4).

We begin by proving a simple lemma on backward growth estimates for powers of the difference transformation \( \Delta \).
Lemma 2.3.1. Let \( \gamma = \{ \gamma_n \}_{n=0}^{\infty} \) be a sequence of real (resp., complex) numbers and \( k \in \mathbb{N} \) be such that
\[
\sup_{n \in \mathbb{N}} (\Delta^k \gamma)_n < \infty \quad \text{(resp., } \sup_{n \in \mathbb{N}} |(\Delta^k \gamma)_n| < \infty \text{)}.
\]
Then
\[
\sup_{n \in \mathbb{N}} \frac{(\Delta^j \gamma)_n}{n^{k-j}} < \infty \quad \text{(resp., } \sup_{n \in \mathbb{N}} \frac{|(\Delta^j \gamma)_n|}{n^{k-j}} < \infty \text{), } j = 0, \ldots, k.
\]

Proof. Because of the similarity of proofs, we concentrate on the real case. We use the backward induction on \( j \). By the first inequality in (2.3.4), the first inequality holds for a fixed \( j \in \{1, \ldots, k\} \); there exists \( \eta \in \mathbb{R}_+ \) such that
\[
(\Delta^{j-1} \gamma)_n = (\Delta^{j-1} \gamma)_0 + \sum_{m=0}^{n-1} (\Delta^j \gamma)_m \leq (\Delta^{j-1} \gamma)_0 + \eta \sum_{m=0}^{n-1} (m+1)^{k-j} \\
\leq (\Delta^{j-1} \gamma)_0 + \eta n^{k-j+1}, \quad n \in \mathbb{N}.
\]
Hence the first inequality in (2.3.2) holds for \( j - 1 \) in place of \( j \).

Next, we characterize CPD sequences \( \gamma \) for which the sequence \( \Delta \gamma \) is convergent.

Theorem 2.3.2. Let \( \gamma = \{ \gamma_n \}_{n=0}^{\infty} \) be a sequence of real numbers. Then the following statements are equivalent.

(i) \( \gamma \) is CPD and the sequence \( \Delta \gamma \) is convergent in \( \mathbb{R} \),
(ii) there exist a finite Borel measure \( \nu \) on \( \mathbb{R} \) and \( d \in \mathbb{R} \) such that
   (ii-a) \( \nu(\mathbb{R} \setminus (-1, 1)) = 0 \),
   (ii-b) \( \frac{1}{1-x} \in L^1(\nu) \),
   (ii-c) \( \gamma_n = \gamma_0 + nd - \int_{(-1,1)} \frac{1-x^n}{1-x} d\nu(x) \) for all \( n \in \mathbb{Z}_+ \).

Moreover, the following statements are satisfied:
(iii) if (i) holds and \( (b, c, \nu) \) represents \( \gamma \), then \( c = 0 \), \( \frac{1}{1-x} \in L^1(\nu) \) and the pair \( (d, \nu) \) with \( d = b + \int_{(-1,1)} \frac{1-x^n}{1-x} d\nu(x) \) is a unique pair satisfying (ii),
(iv) if (ii) holds, then \( d = \lim_{n \to \infty} (\Delta \gamma)_n \) and \( (b, 0, \nu) \) represents \( \gamma \) with \( b = d - \int_{(-1,1)} \frac{1}{1-x} d\nu(x) \).

Proof. (i)⇒(ii) It follows from footnote 3 and (2.2.13) that \( \operatorname{supp}(\nu) \subseteq [-1, 1] \), where \( (b, c, \nu) \) represents \( \gamma \). By using (2.2.13), we get
\[
(\Delta^2 \gamma)_n = 2c + (-1)^n \nu(\{-1\}) + \int_{(-1,1)} x^n d\nu(x), \quad n \in \mathbb{Z}_+.
\]
It follows from (2.3.5) that \( \lim_{n \to \infty} \Delta^2 \gamma = 0 \). By Lebesgue’s dominated convergence theorem, the third term on the right-hand side of the equality in (2.3.5)
converges to 0. This together with (2.3.5) implies that \( c = 0 \) and \( \nu(\{-1\}) = 0 \), which gives (ii-a). Now, using (2.2.6) and (2.2.10), we obtain
\[
(\triangle \gamma)_n = b + \int_{(-1,0)} \frac{1 - x^n}{1 - x} d\nu(x) + \int_{[0,1)} \frac{1 - x^n}{1 - x} d\nu(x), \quad n \in \mathbb{Z}_+.
\] (2.3.6)

Applying Lebesgue’s dominated and monotone convergence theorems to the second and the third terms on the right-hand side of the equality in (2.3.6) respectively, we infer from (2.3.4) that (ii-b) holds and
\[
b = d - \int_{(-1,1)} \frac{1}{1 - x} d\nu(x),
\] (2.3.7)
where \( d := \lim_{n \to \infty} (\triangle \gamma)_n \). Using again (2.2.10), we get
\[
\gamma_n = \gamma_0 + bn + \int_{(-1,1)} Q_n(x) d\nu(x)
\]
\[\quad \overset{(*)}{=} \gamma_0 + nd + \int_{(-1,1)} \left( \frac{n}{x - 1} + \frac{x^n - 1 - n(x - 1)}{(x - 1)^2} \right) d\nu(x)
\]
\[\quad = \gamma_0 + nd - \int_{(-1,1)} \frac{1 - x^n}{(1 - x)^2} d\nu(x), \quad n \in \mathbb{Z}_+.
\] (2.3.8)

where \((*)\) follows from (2.3.7) and (2.2.2). This implies (ii) and (iii) except for the uniqueness of \((d, \nu)\).

(ii) \(\Rightarrow\) (i) Using (ii-b) and (ii-c) and arguing as in (2.3.8), we see that
\[
\gamma_n = \gamma_0 + bn + \int_{(-1,1)} Q_n(x) d\nu(x), \quad n \in \mathbb{Z}_+.
\] (2.3.9)
where \( b \) is as in (2.3.7). Hence, by Theorem 2.2.5 the sequence \( \gamma \) is CPD and \( \limsup_{n \to \infty} |\gamma_n|^{1/n} < \infty \). By (ii-a), \((b, 0, \nu)\) represents \( \gamma \). It follows from (2.2.6) and (2.3.9) that
\[
(\triangle \gamma)_n = b + \int_{(-1,1)} \frac{1 - x^n}{1 - x} d\nu(x), \quad n \in \mathbb{Z}_+.
\] (2.3.10)

Using (ii-b) and applying Lebesgue’s dominated convergence theorem to (2.3.10), we see that (2.3.4) holds and \( d = \lim_{n \to \infty} (\triangle \gamma)_n \). Summarizing, we have proved that (i) and (iv) hold. As a consequence, this yields the uniqueness of \((d, \nu)\) in (iii), which completes the proof. \( \square \)

Under some additional constraints, CPD sequences \( \gamma \) for which the sequence \( \triangle \gamma \) is bounded from above can be characterized as follows.

**Theorem 2.3.3.** Let \( \gamma = \{\gamma_n\}_{n=0}^{\infty} \) be a sequence of real numbers such that
\[
\inf_{n \in \mathbb{Z}_+} \gamma_n > -\infty.
\] (2.3.11)

Then the following statements are equivalent.

---

4 Applying Lemma 2.3.1 to \( k = 1 \) and \( j = 0 \), we verify that (2.3.11) and (2.3.12) imply that \( \limsup_{n \to \infty} |\gamma_n|^{1/n} \leq 1 \).
Applying (2.2.6) and (2.2.10), we obtain

\[
\limsup_{n \to \infty} R_n \text{ increasing and convergent in } c.
\]

Moreover, the following statements are satisfied:

- (ii-a) \( \nu([0,1]) = 0 \),
- (ii-b) \( \frac{1}{1-x} \in L^1(\nu) \),
- (ii-c) \( \gamma_n = \gamma_0 + nd - \int_{[0,1]} \frac{1}{1-x} \, d\nu(x) \) for all \( n \in \mathbb{Z}_+ \).

Moreover, the following statements are satisfied:

- (iii) if (i) holds and \((b,c,\nu)\) represents \( \gamma \), then \( c = 0 \), \( \frac{1}{1-x} \in L^1(\nu) \) and the pair \( (d,\nu) \) with \( d = b + \int_{[0,1]} \frac{1}{1-x} \, d\nu(x) \) is a unique pair satisfying (ii),
- (iv) if (ii) holds, then \( d \geq 0 \), the sequence \( \triangle \gamma \) is monotonically increasing to \( d \) and \((b,0,\nu)\) represents \( \gamma \) with \( b = d - \int_{[0,1]} \frac{1}{1-x} \, d\nu(x) \).

**Proof.** We begin by proving the implication (i)\( \Rightarrow \) (ii). Suppose (i) holds. By footnote \( \footnote{\ref{footnote:4}} \limsup_{n \to \infty} |\gamma_n|^{1/n} < 1 \). Hence by \ref{2.2.13} and \ref{2.3.13}, (ii-a) holds. Applying \ref{2.2.16} and \ref{2.2.11}, we obtain

\[
(\triangle \gamma)_n = b + c(n + 1) + \int_{[0,1]} \frac{1-x^n}{1-x} \, d\nu(x), \quad n \in \mathbb{Z}_+,
\]

where \((b,c,\nu)\) represents \( \gamma \). By Lebesgue’s monotone convergence theorem, the third term on the right-hand side of the equality in \ref{2.3.14} is monotonically increasing to \( \int_{[0,1]} \frac{1}{1-x} \, d\nu(x) \). Since \( c \geq 0 \), we deduce from \ref{2.3.12} and \ref{2.3.14} that \( c = 0 \), \( \frac{1}{1-x} \in L^1(\nu) \) (which yields (ii-b)), the sequence \( \triangle \gamma \) is monotonically increasing and convergent in \( \mathbb{R} \) and

\[
b = d - \int_{[0,1]} \frac{1}{1-x} \, d\nu(x),
\]

where \( d = \lim_{n \to \infty} (\triangle \gamma)_n \). Using \ref{2.2.10} and \ref{2.3.15} and arguing as in \ref{2.3.8}, we deduce that (ii-c) holds. Since \( d = \sup_{n \in \mathbb{Z}_+} (\triangle \gamma)_n \), the telescopic argument (cf. \ref{2.3.3}) shows that

\[
\gamma_n \leq \gamma_0 + nd, \quad n \in \mathbb{N}.
\]

Applying \ref{2.3.11}, we conclude that \( d \geq 0 \). This proves (ii) and (iii) except for the uniqueness of \((d,\nu)\).

A close inspection of the proof of the implication (ii)\( \Rightarrow \) (i) of Theorem \ref{2.3.2} shows that (ii) implies (i) and that \((d,\nu)\) in (iii) is unique. By this uniqueness, statement (iv) follows from the proof of the implication (i)\( \Rightarrow \) (ii). \( \square \)

**Corollary 2.3.4.** Let \( \gamma = \{\gamma_n\}_{n=0}^{\infty} \) be a CPD sequence such that

\[
\limsup_{n \to \infty} |\gamma_n|^{1/n} < \infty
\]

and let \((b,c,\nu)\) be the representing triplet of \( \gamma \). Suppose \( \supp(\nu) \subseteq \mathbb{R}_+ \) and \( \gamma_n \geq 0 \) for \( n \) large enough. Then the following conditions are equivalent:

- (i) \( \lim_{n \to \infty} (\triangle \gamma)_n = 0 \),
- (ii) the sequence \( \gamma \) is monotonically decreasing,
- (iii) the sequence \( \gamma \) is convergent in \( \mathbb{R} \).
Moreover, if (i) holds, then $\gamma$ is PD and $\gamma_n \geq 0$ for all $n \in \mathbb{Z}_+$.

**Proof.** (i)$\Rightarrow$(ii) It follows from Theorem 2.3.3 that $\nu(\mathbb{R} \setminus [0,1)) = 0$, $b = \int_{\mathbb{R}} \frac{1}{x^2} \, d\nu(x)$, $c = 0$ and

$$
\gamma_n = \gamma_0 - \int_{[0,1)} \frac{1-x^n}{(1-x)^2} \, d\nu(x), \quad n \in \mathbb{Z}_+, 
$$

which yields (ii) and consequently implies that $\gamma_n \geq 0$ for all $n \in \mathbb{Z}_+$. Lebesgue’s monotone convergence theorem gives $\int_{\mathbb{R}} \frac{1}{x^2} \, d\nu(x) \leq \gamma_0$, so by Theorem 2.2.12 $\gamma$ is PD. This proves the “moreover” part.

The implications (ii)$\Rightarrow$(iii) and (iii)$\Rightarrow$(i) are obvious. $\square$

3. Representations of conditionally positive definite operators

3.1. Semispectral integral representations. Recall that an operator $T \in B(\mathcal{H})$ is said to be CPD if the sequence $\{\|T^n h\|^2\}_{n=0}^{\infty}$ is PD for every $h \in \mathcal{H}$. Occasionally, we will use a concise notation:

$$(\gamma_{T,h})_n := \|T^n h\|^2, \quad n \in \mathbb{Z}_+, \ h \in \mathcal{H}. \quad (3.1.1)$$

The class of CPD operators contains the class of complete hypercontractions of order 2 introduced by Chavan and Sholapurkar in [21] (see the paragraph preceding Proposition 3.1.4 for a more detailed discussion). The main difference between these two concepts is that the representing semispectral measures of complete hypercontractions of order 2 are concentrated on the closed interval $[0,1]$ (see [21], Theorem 4.11), while the representing semispectral measures of CPD operators can be concentrated on an arbitrary finite subinterval of $\mathbb{R}_+$ (see Theorem 3.1.1). Let us point out that CPD operators are not scalable in general (see Corollary 3.4.1). We also refer the reader to [37] for semispectral integral representations and the corresponding dilations for completely hypercontractive and completely hyperexpansive operators (still on $[0,1]$). The article [37] was an inspiration for the research carried out in [21, 22]. It is also worth mentioning that in view of [9, Theorem 2], an operator $T \in B(\mathcal{H})$ is completely hyperexpansive if and only if the sequence $\{-\|T^n h\|^2\}_{n=0}^{\infty}$ is CPD for every $h \in \mathcal{H}$.

**Theorem 3.1.1.** Let $T \in B(\mathcal{H})$. Then the following statements are equivalent:

(i) $T$ is CPD,

(ii) there exist operators $B, C \in B(\mathcal{H})$ and a compactly supported semispectral measure $F: \mathfrak{B}(\mathbb{R}_+) \to B(\mathcal{H})$ such that $B = B^*$, $C \geq 0$, $F(\{1\}) = 0$ and

$$
T^* T^n = I + n B + n^2 C + \int_{\mathbb{R}_+} Q_n(x) F(dx), \quad n \in \mathbb{Z}_+. \quad (3.1.2)
$$

Moreover, if (ii) holds, then the triplet $(B,C,F)$ is unique and

$$
supp(F) \subseteq [0, r(T)^2], \quad (3.1.3)
$$

$$
C \neq 0 \implies r(T) > 1, \quad (3.1.4)
$$

$$
\sup \sup \sup \sup (F) \geq 1 \implies r(T)^2 = \sup \sup \sup (F). \quad (3.1.5)
$$

Furthermore, $(\langle Bh, h \rangle, \langle Ch, h \rangle, \langle F(\cdot)h, h \rangle)$ is the representing triplet of the CPD sequence $\{\|T^n h\|^2\}_{n=0}^{\infty}$ for every $h \in \mathcal{H}$. 

Proof. (i)⇒(ii) By Theorem 2.2.5 for every \( h \in \mathcal{H} \) there exists a unique triplet \((b_h, c_h, \nu_h)\) consisting of a real number \(b_h\), nonnegative real number \(c_h\) and a finite compactly supported Borel measure \(\nu_h\) on \(\mathbb{R}\) such that \(\nu_h(\{1\}) = 0\) and

\[
(\gamma_{T,h})_n = \|T^n h\|^2 = \|h\|^2 + b_n + c_n n^2 + \int_{\mathbb{R}} Q_n(x) d\nu_h(x), \quad n \in \mathbb{Z}_+. \quad (3.1.6)
\]

First we show that

\[
\text{supp}(\nu_h) \subseteq \mathbb{R}_+, \quad h \in \mathcal{H}. \quad (3.1.7)
\]

For this, note that by (3.1.6) and (2.2.16) we have

\[
(\Delta^2 \gamma_{T,h})_n = \int_{\mathbb{R}} x^n d(\nu_h + 2c_n \delta_1)(x), \quad n \in \mathbb{Z}_+, \quad h \in \mathcal{H}. \quad (3.1.8)
\]

It is a simple matter to verify that the following identity holds

\[
(\Delta^2 \gamma_{T,h})_{n+1} = (\Delta^2 \gamma_{T,T,h})_n, \quad n \in \mathbb{Z}_+, \quad h \in \mathcal{H}. \quad (3.1.9)
\]

It follows from (3.1.8) and (3.1.9) that the sequences \(\Delta^2 \gamma_{T,h}\) and \(\{\Delta^2 \gamma_{T,T,h}\}_{n=0}^{\infty}\) are PD. Hence, by Theorem 2.1.3, \(\Delta^2 \gamma_{T,h}\) is a Stieltjes moment sequence. Since the measure \(\nu_h + 2c_n \delta_1\) is compactly supported, we infer from (3.1.8) and Lemma 2.1.2 that the Stieltjes moment sequence \(\Delta^2 \gamma_{T,h}\) is determinate (as a Hamburger moment sequence). Therefore, \(\text{supp}(\nu_h + 2c_n \delta_1) \subseteq \mathbb{R}_+\) for every \(h \in \mathcal{H}\), which implies (3.1.7).

Define the functions \(\hat{b}, \hat{c}: \mathcal{H} \times \mathcal{H} \to \mathbb{C}\) and \(\hat{\nu}: \mathcal{B}(\mathbb{R}_+) \times \mathcal{H} \times \mathcal{H} \to \mathbb{C}\) by

\[
\hat{b}(f, g) = \frac{1}{4} \sum_{k=0}^{3} i^k b_{f+i^k g}, \quad \hat{c}(f, g) = \frac{1}{4} \sum_{k=0}^{3} i^k c_{f+i^k g}, \quad \hat{\nu}(\Delta; f, g) = \frac{1}{4} \sum_{k=0}^{3} i^k \nu_{f+i^k g}(\Delta),
\]

where \(f, g \in \mathcal{H}\) and \(\Delta \in \mathcal{B}(\mathbb{R}_+)\). Clearly, \(\hat{\nu}(\cdot; f, g)\) is a complex measure for all \(f, g \in \mathcal{H}\). It follows from (3.1.6), (3.1.7) and the polarization formula that

\[
\langle T^n f, T^n g \rangle = \langle f, g \rangle + \hat{b}(f, g)n + \hat{c}(f, g)n^2 + \int_{\mathbb{R}_+} Q_n(x) d\hat{\nu}(dx, f, g), \quad n \in \mathbb{Z}_+, \quad f, g \in \mathcal{H}. \quad (3.1.10)
\]

Using (3.1.10) and Lemma 2.2.2 one can verify that \(\hat{b}\) is a Hermitian symmetric sesquilinear form and the functions \(\hat{c}\) and \(\hat{\nu}(\Delta; \cdot, \cdot)\), where \(\Delta \in \mathcal{B}(\mathbb{R}_+)\), are semi-inner products such that for all \(h \in \mathcal{H}\) and \(\Delta \in \mathcal{B}(\mathbb{R}_+)\),

\[
\hat{b}(h, h) = b_h, \quad \hat{c}(h, h) = c_h, \quad \hat{\nu}(\Delta; h, h) = \nu_h(\Delta). \quad (3.1.11)
\]

(cf. the proofs of [68, Proposition 1] and [37, Theorem 4.2]). By (3.1.6), we have

\[
\hat{\nu}(\Delta; h, h) + 2\hat{c}(h, h) \leq \nu_h(\mathbb{R}) + 2c_h \leq (\Delta^2 \gamma_{T,h})_0 \\ = \langle \mathcal{B}_2(T) h, h \rangle \\ \leq \|\mathcal{B}_2(T)\| \|h\|^2, \quad h \in \mathcal{H}, \quad \Delta \in \mathcal{B}(\mathbb{R}_+).
\]
This implies that the sesquilinear forms $\hat{c}$ and $\hat{\nu}(\Delta; \cdot, \cdot)$, where $\Delta \in \mathcal{B}(\mathbb{R}_+)$, are bounded. Hence, there exist $C, F(\Delta) \in \mathcal{B}(\mathcal{H})_+$, where $\Delta \in \mathcal{B}(\mathbb{R}_+)$, such that

$$
\langle Ch, h \rangle = \hat{c}(h, h) \quad (3.1.11)
$$

$$
\langle F(\Delta)h, h \rangle = \hat{\nu}(\Delta; h, h) \quad (3.1.12)
$$

In view of (3.1.13), $F$ is a Borel semispectral measure on $\mathbb{R}_+$.

Now we show that the so-constructed $F$ satisfies (3.1.3) and (3.1.5). It follows from Gelfand’s formula for spectral radius that

$$
\limsup_{n \to \infty} \|T^n h\|^{1/n} \leq r(T), \quad h \in \mathcal{H}.
$$

(3.1.14)

This together with (3.1.6), (3.1.13) and Theorem 2.2.5 applied to $\gamma_{T,h}$ yields

$$
\left\langle F\left(\left(r(T)^2, \infty\right)\right)h, h \right\rangle 
\leq \left(\left(\limsup_{n \to \infty} \|T^n h\|^{2/n}, \infty\right)\right)h, h \right\rangle 
= 0,
$$

$\quad h \in \mathcal{H},$

which, when combined with (3.1.7), implies (3.1.3). Hence, we have

$$
\sup \text{supp}(F) \leq r(T)^2.
$$

(3.1.15)

Observing that

$$
\text{supp}(\langle F(\cdot)h, h \rangle) \subseteq \text{supp}(F), \quad h \in \mathcal{H},
$$

we obtain

$$
\limsup_{n \to \infty} \|T^n h\|^{2/n} \leq \max \left\{1, \sup \text{supp}(F)\right\}, \quad h \in \mathcal{H}.
$$

(3.1.16)

It follows from [28, Corollary 3] that $r(T)^2 \leq \max \left\{1, \sup \text{supp}(F)\right\}$, which together with (3.1.15) gives (3.1.5).

Our next goal is to construct the operator $B$. By (3.1.6) and (3.1.12), we have

$$
\|Th\|^2 - \|h\|^2 = \langle \Delta \gamma_{T,h} \rangle_0 \hat{b}(h, h) + \langle Ch, h \rangle, \quad h \in \mathcal{H}.
$$

As a consequence, $\hat{b}$ is a bounded Hermitian symmetric sesquilinear form. This implies that there exists a self-adjoint operator $B \in \mathcal{B}(\mathcal{H})$ such that

$$
\langle Bh, h \rangle = \hat{b}(h, h) \quad (3.1.11)
$$

Combining (3.1.6) with (3.1.12), (3.1.13) and (3.1.15) gives (ii).

(ii)$\Rightarrow$(i) This implication is a direct consequence of Theorem 2.2.5 applied to the sequences $\gamma_{T,h}$, $h \in \mathcal{H}$.

It remains to justify the “moreover” part. Suppose (ii) holds. The uniqueness of the triplet $(B, C, F)$ follows from Theorem 2.2.5. Assertions (3.1.3) and (3.1.5) were proved above. To show (3.1.4), assume that $C \neq 0$. Then the set $U := \{h \in \mathcal{H} : \langle Ch, h \rangle > 0\}$ is nonempty. By the “moreover” part of Theorem 2.2.5 and (3.1.14), we have

$$
r(T) \geq \limsup_{n \to \infty} \|T^n h\|^{1/n} \geq 1, \quad h \in U,
$$

which implies (3.1.4). The last statement of the theorem is easily seen to be true. This completes the proof. □
The following definition is an operator counterpart of Definition 2.2.6.

**Definition 3.1.2.** If \( T \in \mathcal{B}(\mathcal{H}) \) is a CPD operator and \( B, C, F \) are as in statement (ii) of Theorem 3.1.1, we call \((B, C, F)\) the representing triplet of \( T \), or we simply say that \((B, C, F)\) represents \( T \).

**Remark 3.1.3.** Note that if \((B, C, F)\) represents a CPD operator \( T \) on \( \mathcal{H} \neq \{0\} \) and \( B \geq 0 \), then by 3.1.2, \( T^nT^n \geq I \) for every \( n \in \mathbb{N} \) which together with Gelfand’s formula for spectral radius yields \( r(T) \geq 1 \).

Proposition 3.1.4 below which gives characterizations of CPD operators is closely related to Proposition 2.2.9 (see also Theorem 3.3.1 for an alternative approach). The most important fact we need in its proof is that a sequence \( \{\gamma_n\}_{n=0}^\infty \subseteq \mathcal{B}(\mathcal{H}) \) of exponential growth is a Hamburger moment sequence (that is, \( \mathcal{B}(\mathcal{H}) \) holds for some semispectral measure \( \mu : \mathcal{B}(\mathbb{R}) \to \mathcal{B}(\mathcal{H}) \)) if and only if \( \{\langle \gamma_n h, h \rangle\}_{n=0}^\infty \) is a Hamburger moment sequence for all \( h \in \mathcal{H} \) (see [13] Theorem 2). Similar assertions are true for Stieltjes and Hausdorff operator moment sequences. In view of [76], operators \( T \in \mathcal{B}(\mathcal{H}) \) for which the sequence \( \{T^n \mathcal{B}_2(T)T^n\}_{n=0}^\infty\) is a Hausdorff moment sequence coincide with complete hypercontractions of order 2 introduced in [21]. On the other hand, by [29] Corollary (see also Theorem 1.1.1), an operator \( T \in \mathcal{B}(\mathcal{H}) \) is subnormal if and only if the sequence \( \{T^nT^n\}_{n=0}^\infty \) is a Stieltjes moment sequence. We refer the reader to [76] [13] for necessary definitions and facts related to the aforesaid operator moment problems.

**Proposition 3.1.4.** For \( T \in \mathcal{B}(\mathcal{H}) \), the following conditions are equivalent:

(i) \( T \) is CPD,

(ii) \( \{T^n \mathcal{B}_2(T)T^n\}_{n=0}^\infty \) is PD,

(iii) \( \{T^n \mathcal{B}_2(T)T^n\}_{n=0}^\infty \) is a Stieltjes moment sequence.

Moreover, if \( T \) is CPD, then \( \{T^{n+1} \mathcal{B}_2(T)T^n - T^nT^n\}_{n=0}^\infty \) is monotonically increasing and

\[
\inf_{n \in \mathbb{Z}_+} (\|T^{n+1}h\|^2 - \|T^n h\|^2) = -\langle \mathcal{B}_1(T)h, h \rangle = (Bh, h) + (Ch, h), \quad h \in \mathcal{H},
\]

where \( B \) and \( C \) are as in Theorem 3.1.1(ii).

**Proof.** Clearly, the sequence \( \{T^n \mathcal{B}_2(T)T^n\}_{n=0}^\infty \) is of exponential growth and \( (\Delta^2 \gamma_{T,h})_n = (T^n \mathcal{B}_2(T)T^n h, h) \) for all \( n \in \mathbb{Z}_+ \) and \( h \in \mathcal{H} \). This implies that \( \{T^n \mathcal{B}_2(T)T^n\}_{n=0}^\infty \) is PD (resp., a Stieltjes moment sequence) if and only if \( \Delta^2 \gamma_{T,h} \) is PD (resp., a Stieltjes moment sequence) for every \( h \in \mathcal{H} \). Applying Proposition 2.2.9 to the sequences \( \gamma_{T,h} \) and using Theorem 3.1.1 we deduce that conditions (i)-(iii) are equivalent. The “moreover” part is a direct consequence of the corresponding part of Proposition 2.2.9 Theorem 3.1.1 and 3.1.2 applied to \( n = 1 \).

Theorem 3.1.6 below can be thought of as an operator counterpart of Theorem 2.3.3. Before stating it, we will discuss the role played by condition 3.1.17, which is an operator counterpart of 2.3.12.

**Proposition 3.1.5.** Let \( T \in \mathcal{B}(\mathcal{H}) \) be a CPD operator. Then the following conditions are equivalent:

(i) the sequence \( \{T^{n+1} \mathcal{B}_2(T)T^n - T^nT^n\}_{n=0}^\infty \) is convergent in wot,

(ii) \( \sup_{n \in \mathbb{Z}_+} \|T^{n+1} \mathcal{B}_2(T)T^n - T^nT^n\| < \infty \),
In turn, there are CPD operators $T$ is the zero vector (see Remark 4.3.3c)). It is worth mentioning that in view of Remark 4.3.3a) and the sequence $\{T_n\}_{n=0}^\infty$ is monotonically increasing. Hence, by (3.1.17) and the polarization formula, the sequence $\{\langle D_n f, g \rangle\}_{n=0}^\infty$ is convergent in $\mathbb{C}$ for all $f, g \in \mathcal{H}$. Using the uniform boundedness principle again and the Riesz representation theorem, we deduce that (i) is valid.

Regarding Proposition 3.1.5, note that if an operator $T \in \mathcal{B}(\mathcal{H})$ is CPD, then by Proposition 3.1.4, “sup$_n$” can be replaced by “lim$_{n \to \infty}$” (in the extended real line). If $T$ is CPD and satisfies (3.1.17), then by Proposition 3.1.5, the sequence $(T^{(n+1)}T^{n+1} - T^nT^n)_{n=0}^\infty$ is convergent in wot, say to $D \in \mathcal{B}(\mathcal{H})$. It is worth mentioning that in view of Remark 3.3.3a) and [7, Proposition 8] (see also Lemma 6.1(ii)), there are CPD unilateral weighted shifts $T$ such that

$\langle Dh, h \rangle = \sup_{n \in \mathbb{Z}_+} (\|T^{n+1}h\|^2 - \|T^n h\|^2) = -\langle P(T)h, h \rangle > 0$, $h \in \mathcal{H} \setminus \{0\}$.

In turn, there are CPD operators $T$ for which the only vector $h$ satisfying (3.1.17) is the zero vector (see Remark 3.1.4).

**Theorem 3.1.6.** Let $T \in \mathcal{B}(\mathcal{H})$. Then the following are equivalent:

(i) $T$ is CPD and satisfies (3.1.17),

(ii) there exist a semispectral measure $F: \mathcal{B}(\mathbb{R}_+) \to \mathcal{B}(\mathcal{H})$ and a selfadjoint operator $D \in \mathcal{B}(\mathcal{H})$ such that

(ii-a) $F([1, \infty)) = 0$,

(ii-b) $\frac{1}{1-x} \in L^1(F)$,

(ii-c) $T^{n} T^n = I + nD - \int_{(0,1)} \frac{1-x^n}{(1-x)^2} F(dx)$ for all $n \in \mathbb{Z}_+$.

Moreover, the following statements are satisfied:

(iii) if (i) holds and $(B, C, F)$ represents $T$, then $C = 0$, $\frac{1}{1-x} \in L^1(F)$ and the pair $(D, F)$ with $D = B + \int_{(0,1)} \frac{1-x^n}{(1-x)^2} F(dx)$ is a unique pair satisfying (ii),

(iv) if (ii) holds, then $D \geq 0$. $(T^{(n+1)}T^{n+1} - T^nT^n)_{n=0}^\infty$ converges in wot to $D$ and $(B, 0, F)$ represents $T$ with $B = D - \int_{(0,1)} \frac{1-x^n}{(1-x)^2} F(dx)$.

**Proof.** Using Theorem 2.3.3 (together with its “moreover” part) and Theorem 3.1.1 (together with its “furthermore” part), we deduce that statements (i) and (ii) are equivalent and statements (iii) and (iv) are valid.

The simple argument given below shows that condition (3.1.17) is strong enough to guarantee that $r(T) \leq 1$.

**Proposition 3.1.7.** If an operator $T \in \mathcal{B}(\mathcal{H})$ satisfies condition (3.1.17), then $\alpha_T(h) := \sup_{n \in \mathbb{Z}_+} (\|T^{n+1}h\|^2 - \|T^n h\|^2) \geq 0$ for all $h \in \mathcal{H}$ and $r(T) \leq 1$.

**Proof.** Using the telescopic argument (cf. (2.3.13)) yields

$\|T^n h\|^2 \leq \|h\|^2 + n\alpha_T(h), \quad n \in \mathbb{N}, h \in \mathcal{H}$.

Hence, $\alpha_T(h) \geq 0$ and $\limsup_{n \to \infty} \|T^n h\|^{1/n} \leq 1$ for all $h \in \mathcal{H}$. Applying [28, Corollary 3], we conclude that $r(T) \leq 1$. 


We show below that if the operator $D$ in Theorem 3.1.6 is nonzero, then the spectral radius of $T$ is equal to 1. The case $D = 0$ is discussed in Theorem 3.4.4.

**Theorem 3.1.8.** Let $T \in B(\mathcal{H})$ be a CPD operator satisfying (3.1.17). Suppose that $\frac{1}{1-x^2} \in L^1(F)$ and $D \neq 0$, where $F$ as is in Theorem 3.1.1(ii) and $D := (\text{wot}) \lim_{n \to \infty} (T^{*(n+1)}T^{n+1} - T^*T^n)$. Then $r(T) = 1$.

**Proof.** By Proposition 3.1.5 the limit (wot) $\lim_{n \to \infty} (T^{*(n+1)}T^{n+1} - T^*T^n)$ exists. It follows from statements (iii) and (iv) of Theorem 3.1.6 that the pair $(D, F)$ satisfies condition (ii) of this theorem and $D \geq 0$. Hence $(Dh_0, h_0) > 0$ for some $h_0 \in \mathcal{H}$. In view of Lebesgue’s monotone convergence theorem, the sequence $\{\int_{(0,1)} \frac{1-x^n}{1-x^2} (F(dx)h_0, h_0)\}_{n=0}^\infty$ converges to $\int_{(0,1)} \frac{1-x^n}{1-x^2} (F(dx)h_0, h_0)$. Because the last integral is finite, we infer from equality (ii-c) of Theorem 3.1.6 that there exists $n_0 \in \mathbb{N}$ such that

$$\frac{\|T^nh_0\|^2}{n} = \|h_0\|^2 - \int_{(0,1)} \frac{1-x^n}{1-x^2} (F(dx)h_0, h_0) + \langle Dh_0, h_0 \rangle \geq \frac{1}{2} \langle Dh_0, h_0 \rangle, \quad n \geq n_0.$$

Combined with Gelfand’s formula for spectral radius, this implies that

$$r(T) \geq \limsup_{n \to \infty} \frac{\|T^nh_0\|^{1/n}}{n^{1/2n}} \geq 1.$$

Therefore applying Proposition 3.1.7 yields $r(T) = 1$. □

It follows from Theorem 3.1.6(ii-c) that if $T \in B(\mathcal{H})$ is a CPD operator satisfying (3.1.17), then there exists $\alpha \in \mathbb{R}_+$ such that $\|T^n\| \leq \alpha \sqrt{n}$ for all $n \in \mathbb{N}$. The next proposition shows that the powers of a CPD operator with spectral radius less than or equal to 1 have polynomial growth of degree at most 1.

**Proposition 3.1.9.** Let $T \in B(\mathcal{H})$ be a CPD operator with the representing triplet $(B, C, F)$. Then the following conditions are equivalent:

(i) there exists $\alpha \in \mathbb{R}_+$ such that

$$\|T^n\| \leq \alpha \cdot n, \quad n \in \mathbb{N},$$

(ii) $r(T) \leq 1$,

(iii) $\text{supp}(F) \subseteq [0, 1]$.

Moreover, (iii) implies (3.1.18) with $\alpha = \sqrt{1 + \|B\| + \|C\| + \|F([0,1])\|}$.

Proposition 3.1.9 can be deduced from Proposition 3.2.10 and Gelfand’s formula for spectral radius. Its proof is omitted. According to Proposition 3.3.1 and Remark 3.3.3 below, there are CPD operators having exactly polynomial growth of degree 1. Observe that a subnormal operator of polynomial growth of arbitrary degree, being normaloid (see (1.1.20)) is a contraction, that is, it has polynomial growth of degree zero.

### 3.2. A dilation representation.

First, we adapt Agler’s hereditary functional calculus [1, 51, 27] to our needs. For $T \in B(\mathcal{H})$, we set

$$p(T) = \sum_{i \geq 0} \alpha_i T^{*i} T^i \quad \text{for} \quad p = \sum_{i \geq 0} \alpha_i X^i \in \mathbb{C}[X].$$

(3.2.1)
In particular, we have (see (1.4.3))
\[ SN_m(T) = (1 - X)^m(T), \quad m \in \mathbb{Z}_+. \]  
(3.2.2)

The map \( \mathbb{C}[X] \ni p \mapsto p(T) \in B(H) \) is linear but in general not multiplicative (e.g., if \( T \in B(H) \) is a nilpotent operator with index of nilpotency 2 and \( p = X - 1 \), then \( p(T)^2 \neq (p^2)(T) \)). However, it has the following property.

The map \( p \mapsto p(T) \) is a unique linear map from \( \mathbb{C}[X] \) to \( B(H) \) such that \( X^0(T) = 1 \) and \( (Xp)(T) = T^* p(T) T \) for all \( p \in \mathbb{C}[X] \).

There is another way of defining \( p(T) \). Namely, let us consider the elementary operator \( \nabla_T : B(H) \to B(H) \) defined by
\[ \nabla_T(A) = T^* A T, \quad A \in B(H). \]  
(3.2.4)

It is then easily seen that \( p(T) = p(\nabla_T)(I) \) for any \( p \in \mathbb{C}[X] \) and by (3.2.3),
\[ p(\nabla_T)(q(T)) = ((pq)(\nabla_T))(I) = (pq)(T), \quad p, q \in \mathbb{C}[X]. \]  
(3.2.5)

Although the map \( \mathbb{C}[X] \ni p \mapsto p(T) \in B(H) \) is not multiplicative, it does have a property that resembles multiplicativity.

**Lemma 3.2.1.** Let \( T \in B(H) \) and \( q_0 \in \mathbb{C}[X] \). Then the set
\[ \mathcal{I} = \{ q \in \mathbb{C}[X] : (q_0 q)(T) = 0 \} \]
is a principal ideal in \( \mathbb{C}[X] \), that is, \( \mathcal{I} = \{ pu : p \in \mathbb{C}[X] \} \) for some \( w \in \mathbb{C}[X] \). Moreover, if \( q_0 = q_0^* \), then \( w \) can be chosen to satisfy \( w^* = w \).

**Proof.** First note that \( \mathcal{I} \) is an ideal. Indeed, if \( q \in \mathcal{I} \) and \( p \in \mathbb{C}[X] \), then
\[ 0 = p(\nabla_T) \left( (q_0 q)(T) \right) = (q_0 pq)(T). \]

By [36, Theorem III.3.9], \( \mathcal{I} \) is a principal ideal in \( \mathbb{C}[X] \). That \( w \) can be chosen to satisfy \( w^* = w \), follows from the fact that \( (p(T))^* = (p^*(T)) \) for all \( p \in \mathbb{C}[X] \). \( \square \)

**Remark 3.2.2.** It follows from (3.2.2) and Lemma 3.2.1 that if \( T \in B(H) \) is an \( m \)-isometry, that is \( (1 - X)^m(T) = SN_m(T) = 0 \), then \( ((1 - X)^m q)(T) = 0 \) for every \( q \in \mathbb{C}[X] \); in particular, \( SN_k(T) = (1 - X)^k(T) = 0 \) for all \( k \geq m \), which means that \( T \) is a \( k \)-isometry for all \( k \geq m \) (see [36] p. 389, line 6).

Given \( a \in \mathbb{C} \), we define the linear transformation \( \mathcal{D}_a : \mathbb{C}[X] \to \mathbb{C}[X] \) by
\[ \mathcal{D}_a p = \frac{p - p(a)}{X - a}, \quad p \in \mathbb{C}[X]. \]

Using the Taylor series expansion about the point \( a \), it is easily seen that the transformation \( \mathcal{D}_a \) is well defined and \((\mathcal{D}_a)^*(a)\) stands for the derivative of \( p \) at \( a \)
\[ \mathcal{D}_a^2 p = \frac{p - p(a) - p'(a)(X - a)}{(X - a)^2}, \quad a \in \mathbb{C}, \quad p \in \mathbb{C}[X]. \]  
(3.2.6)

It is a simple matter to verify that for each \( p \in \mathbb{C}[X], \mathcal{D}_a^p p = 0 \) whenever \( n > \deg p \).

The following lemma will be used in some proofs of subsequent results.

**Lemma 3.2.3.** The following assertions hold:
(a) if \( K \) is a Hilbert space, \( S \in B(K)_+ \), \( E \) is the spectral measure of \( S \) and \( \mathbb{M} \) is a vector subspace of \( K \), then
\[ \bigvee \left\{ E(\Delta) : \Delta \in \mathcal{B}(\mathbb{R}_+) \right\} = \bigvee \{ S^n \mathbb{M} : n \in \mathbb{Z}_+ \}, \]  
(3.2.7)
(b) if for $i = 1, 2$, $(\mathcal{K}_i, R_i, S_i)$ consists of a Hilbert space $\mathcal{K}_i$ and operators $R_i \in \mathcal{B}(\mathcal{H}, \mathcal{K}_i)$ and $S_i \in \mathcal{B}(\mathcal{K}_i)_+$ such that $\mathcal{K}_i = \sqrt{\{ S_i^* R_i : n \in \mathbb{Z}_+ \}}$ and $R_1^* S_1 R_1 = R_2^* S_2 R_2$ for all $n \in \mathbb{Z}_+$, then there exists a (unique) unitary isomorphism $U \in \mathcal{B}(\mathcal{K}_1, \mathcal{K}_2)$ such that $UR_1 = R_2$ and $US_1 = S_2U$; in particular, $\sigma(S_1) = \sigma(S_2)$ and $\| S_1 \| = \| S_2 \|$. 

**Proof.** (a) Since $S$ is bounded, $E(\mathbb{R}_+ \setminus [0, r]) = 0$, where $r := \| S \|$. Take a vector $g \in \mathcal{K}$. Then $g$ is orthogonal to the right-hand side of (3.2.7) if and only if

$$0 = \langle S^n h, g \rangle = \int_{[0,r]} x^n (E(dx)h, g), \quad n \in \mathbb{Z}_+, \quad h \in \mathcal{M}. \quad (3.2.8)$$

It follows from the Weierstrass approximation theorem and the uniqueness part of the Riesz representation theorem (see [59 Theorem 6.19]) that (3.2.8) holds if and only if $\langle E(\Delta)h, g \rangle = 0$ for all $\Delta \in \mathcal{B}(\mathbb{R}_+)$ and $h \in \mathcal{M}$, or equivalently if and only if $g$ is orthogonal to the left-hand side of (3.2.7). This implies (3.2.7).

(b) It is easily seen that there exists a unique unitary isomorphism $U \in \mathcal{B}(\mathcal{K}_1, \mathcal{K}_2)$ such that $US_1^* R_1 h = S_2^* R_2 h$ for all $h \in \mathcal{H}$ and $n \in \mathbb{Z}_+$. It is a matter of routine to verify that $U$ has the desired properties. This completes the proof. \qed

For the reader's convenience, we recall a version of the Naimark dilation theorem needed in this paper.

**Theorem 3.2.4 ([52 Theorem 6.4]).** If $M : \mathcal{B}(\mathbb{R}_+) \to \mathcal{B}(\mathcal{H})$ is a semispectral measure, then there exist a Hilbert space $\mathcal{K}$, an operator $R \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ and a spectral measure $E : \mathcal{B}(\mathbb{R}_+) \to \mathcal{B}(\mathcal{K})$ such that

$$M(\Delta) = R^* E(\Delta) R, \quad \Delta \in \mathcal{B}(\mathbb{R}_+), \quad (3.2.9)$$

$$\mathcal{K} = \sqrt{\{ E(\Delta) \mathcal{A}(R) : \Delta \in \mathcal{B}(\mathbb{R}_+) \}}, \quad (3.2.10)$$

We are now ready to give a dilation representation for CPD operators and relate their spectral radii to the norms of positive operators appearing in this representation. Dilation representations for complete hypercontractions and complete hyperexpansions were given in [37] and afterwards generalized to the case of complete hypercontractions of finite order in [21]. All aforesaid representations were built over the closed interval $[0, 1]$. What is more, the dilation representation for complete hypercontractions of order 2 (which are very particular instances of CPD operators) was proved under a restrictive assumption on the representing semispectral measure (see [21, Theorem 4.20]). Below we use the convention (1.4.1).

**Theorem 3.2.5.** Let $T \in \mathcal{B}(\mathcal{H})$. Then the following conditions are equivalent:

(i) $T$ is CPD,

(ii) there exists a semispectral measure $M : \mathcal{B}(\mathbb{R}_+) \to \mathcal{B}(\mathcal{H})$ with compact support such that

$$p(T) = p(1) I - p'(1) \mathcal{A}_1(T) + \int_{\mathbb{R}_+} (\mathcal{A}_1^2 p)(x) M(dx), \quad p \in \mathbb{C}[X], \quad (3.2.11)$$

(iii) there exist a Hilbert space $\mathcal{K}$, $R \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ and $S \in \mathcal{B}(\mathcal{K})_+$ such that

$$p(T) = p(1) I - p'(1) \mathcal{A}_1(T) + R^* (\mathcal{A}_1^2 p)(S) R, \quad p \in \mathbb{C}[X], \quad (3.2.12)$$
(iv) there exist a Hilbert space $K$, $R \in B(H,K)$ and $S \in B(K)_+$ such that (3.2.12) holds and
\[ K = \bigvee \{ S^n \mathcal{A}(R) : n \in \mathbb{Z}_+ \}. \] (3.2.13)

Moreover, if any of conditions (i)-(iv) holds, then

(a) the semispectral measure $M$ in (ii) is unique,

(b) if $(B,C,F)$ represents $T$, then $B + C = -\mathcal{B}_1(T)$, $C = \frac{1}{2}M(\{1\})$ and
\[ F(\Delta) = (1 - \chi(1))M(\Delta), \quad \Delta \in \mathcal{B}(\mathbb{R}_+), \] (3.2.14)

(c) if $(K,R,S)$ is as in (iv), then
\[ \sigma(S) = \text{supp}(M) \quad \text{and} \quad ||S|| = \max \{ 0, \text{sup}\text{sup}(M) \} \leq r(T)^2. \] (3.2.15)

Proof. (i)⇒(ii) By Theorem 3.1.1, $T$ has a representing triplet $(B,C,F)$. Define the compactly supported semispectral measure $M : \mathcal{B}(\mathbb{R}_+) \rightarrow B(H)$ by
\[ M(\Delta) = F(\Delta) + 2\chi(1)C, \quad \Delta \in \mathcal{B}(\mathbb{R}_+). \]

Using (3.1.2) and the fact that $Q_n(1) = \frac{n(n-1)}{2}$ (see (2.2.1)), we deduce that
\[ T^{*n}T^n = I + n(B + C) + \int_{\mathbb{R}_+} Q_n(x)M(dx), \quad n \in \mathbb{Z}_+. \] (3.2.16)

Since $Q_1 = 0$, substituting $n = 1$ into (3.2.16) yields $B + C = -\mathcal{B}_1(T)$. Hence
\[ T^{*n}T^n = I - n\mathcal{B}_1(T) + \int_{\mathbb{R}_+} Q_n(x)M(dx), \quad n \in \mathbb{Z}_+. \] (3.2.17)

Suppose $p \in \mathbb{C}[X]$ is of the form $p = \sum_{n \geq 1} \alpha_n X^n$, where $\alpha_n \in \mathbb{C}$. Multiplying (3.2.17) by $\alpha_n$ and summing with respect to $n$, gives
\[ p(T) = p(1)I - p'(1)\mathcal{B}_1(T) + \int_{\mathbb{R}_+} \sum_{n \geq 1} \alpha_n Q_n(x)M(dx). \] (3.2.18)

Notice that
\[ \sum_{n \geq 0} \alpha_n Q_n(x) = \sum_{n \geq 0} \alpha_n \frac{x^n - 1 - n(x - 1)}{(x - 1)^2} = (\Omega_p^n)(x), \quad x \in \mathbb{R} \setminus \{1\}. \] (3.2.19)

Combining (3.2.18) with (3.2.19) gives (3.2.11).

(iii)⇒(iv) Substituting the polynomial $p = X^n$ into (3.2.11) and using (2.2.2), (3.2.6), (3.2.11) and the fact that $Q_n(1) = \frac{n(n-1)}{2}$, we get
\[ T^{*n}T^n = I - n\mathcal{B}_1(T) + \int_{\mathbb{R}_+} Q_n(x)M(dx) \]
\[ = I - n\mathcal{B}_1(T) + Q_n(1)M(\{1\}) + \int_{\mathbb{R}_+} Q_n(x)F(dx) \]
\[ = I - n \left( \mathcal{B}_1(T) + \frac{1}{2}M(\{1\}) \right) + \frac{n^2}{2} M(\{1\}) + \int_{\mathbb{R}_+} Q_n(x)F(dx), \quad n \in \mathbb{Z}_+, \] (3.2.20)
where \( F : \mathcal{B}(\mathbb{R}_+) \to \mathcal{B}(\mathcal{H}) \) is the compactly supported semispectral measure given by (3.2.14). Hence, by Theorem 3.1.1, \( T \) is CPD. What is more, using (3.2.14), (3.2.20) and the uniqueness of representing triplets, we easily verify that (a) and (b) hold.

(ii)⇒(iv) By Theorem 3.2.4, there exists a triplet \((\mathcal{K}, R, E)\) satisfying (3.2.9) and (3.2.10). Notice that the measure \( E \) is compactly supported. This is a direct consequence of the identity \( \text{supp}(M) = \text{supp}(E) \), which follows from (3.2.9) and (3.2.10) (see the proof of [37, Theorem 4.4]). Set \( S = \int_{\mathbb{R}_+} x E(dx) \). Since \( E \) is compactly supported in \( \mathbb{R}_+ \), the operator \( S \) is bounded and positive (see [61, Theorem 5.9]). Applying the Stone-von Neumann functional calculus (cf. [12, 61] and [71]), we deduce from (3.2.11) that the triplet \((\mathcal{K}, R, S)\) satisfies (3.2.12). Using (3.2.10) and Lemma 3.2.3(a), we get (3.2.13), which yields (iv).

(iv)⇒(iii) This is obvious.

(iii)⇒(ii) Applying the Stone-von Neumann functional calculus to (3.2.12) yields (ii) with the semispectral measure \( M \) defined by (3.2.9), where \( E \) is the spectral measure of \( S \).

It remains to prove (c). Suppose that \((\mathcal{K}, R, S)\) is as in (iv). If \( \mathcal{K} = \{0\} \), then by (3.2.11), (3.2.12) and (a), we deduce that \( M = 0 \) which gives (3.2.15). Therefore we can assume that \( \mathcal{K} \neq \{0\} \). According to the proof of the implication (iv)⇒(ii), we see that the semispectral measure \( M \) is given by (3.2.9), where \( E \) is the spectral measure of \( S \). In view of (3.2.10) and Lemma 3.2.3(a), \((\mathcal{K}, R, E)\) satisfies (3.2.10), and so \( M \neq 0 \). As mentioned above, \( \text{supp}(M) = \text{supp}(E) \). Combined with [61, Theorem 5.9, Proposition 5.10], this implies that

\[ \sigma(S) = \text{supp}(M) \text{ and } \|S\| = \sup \sigma(S) = \sup \text{supp}(M). \]

Thus it suffices to show that \( \text{supp}(M) \leq r(T)^2 \). Let \((B, C, F)\) be the representing triplet of \( T \). Set \( \vartheta = \sup \text{supp}(F) \). Since \( \text{supp}(F) \) is compact, we see that \( \vartheta \in \{-\infty\} \cup \mathbb{R}_+ \). We now consider three possible cases that are logically disjoint.

**Case 1.** \( \vartheta \geq 1 \).

Then the following equalities hold

\[ r(T)^2 \stackrel{(a)}{=} \sup \text{supp}(F) \stackrel{(b)}{=} \sup \text{supp}(M). \]

**Case 2.** \( \vartheta < 1 \) and \( C \neq 0 \).

According to (3.1.4), \( r(T) \geq 1 \). Since \( \vartheta < 1 \), we obtain

\[ \sup \text{supp}(M) \stackrel{(b)}{=} 1 \leq r(T)^2. \]

**Case 3.** \( \vartheta < 1 \) and \( C = 0 \).

First observe that by (b), \( \vartheta = \sup \text{supp}(M) \). It follows from (3.1.3) that \( \vartheta \leq r(T)^2 \), hence \( \sup \text{supp}(M) \leq r(T)^2 \). This completes the proof. \( \square \)

**Corollary 3.2.6.** Suppose \( T \in \mathcal{B}(\mathcal{H}) \) is a CPD operator and \((\mathcal{K}, R, S)\) is as in Theorem 3.2.5(iv). If 1 is an accumulation point of \( \sigma(S) \cap (0,1) \) or if \( \sigma(S) \cap (1, \infty) \neq \emptyset \), then \( r(T)^2 = \|S\| \).

**Proof.** Let \((B, C, F)\) be the representing triplet of \( T \) and let \( M \) be as in Theorem 3.2.5(ii). Suppose first that 1 is an accumulation point of \( \sigma(S) \cap (0,1) \).
It follows from the first equality in (3.2.15) and (3.2.14) that \( \text{supp}(M) = \text{supp}(F) \) and \( 1 \in \text{supp}(F) \), so by the second equality in (3.2.15), we have
\[
\|S\| = \sup \text{supp}(F) \geq 1.
\]
In turn, if \( \sigma(S) \cap (1, \infty) \neq \emptyset \), then again by the first equality in (3.2.15) and (3.2.14),
\[
1 < \sup \sigma(S) = \sup \text{supp}(M) = \sup \text{supp}(F).
\]
In both cases, an application of (3.1.3) and \( \int \text{supp}(\cdot) \), assertion (i) is immediate from (3.2.21), while assertions (ii) and (iii) can be deduced from (3.1.4) and (3.2.14), and Lebesgue's dominated convergence theorem.

The next corollary enables us to determine the mass of the measure \( M \) at the point 0 provided the CPD operator has the spectral radius less than or equal to 1.

**Corollary 3.2.7.** Suppose \( T \in B(H) \) is a CPD operator and \( M \) is as in Theorem 3.2.5 (ii). Then
\[
\mathcal{B}_m(T) = \int_{\mathbb{R}^+} (1 - x)^{m-2} M(dx), \quad m \geq 2.
\]
In particular, the following assertions hold:

1. \( \mathcal{B}_k(T) \geq 0 \) for all \( k \in \mathbb{Z}_+ \).
2. If \( r(T) \leq 1 \), then \( \mathcal{B}_m(T) \geq 0 \) for all \( m \in \mathbb{Z}_+ \setminus \{1\} \).
3. If \( r(T) \leq 1 \), then the sequence \( \{\mathcal{B}_m(T)\}_{m=2}^{\infty} \) is monotonically decreasing and convergent to \( M(\{0\}) \) in the strong operator topology.

**Proof.** Fix an integer \( m \geq 2 \) and set \( p = (1 - X)^{m-2} \). Then by (3.2.6), we have \( D_+^2\rho = (1 - X)^{m-2} \). Applying (3.2.20) and Theorem 3.2.5 (ii), we get (3.2.21). Assertion (i) is immediate from (3.2.21), while assertions (ii) and (iii) can be deduced from (3.1.3), (3.2.14) and Lebesgue’s dominated convergence theorem.

Concluding this subsection, we make a few remarks related to Theorem 3.2.5 and Corollary 3.2.7.

**Remark 3.2.8.** a) Let us begin by discussing in more detail the relationship between \( r(T) \) and \( \vartheta = \sup \text{supp}(F) \), where \( T \in B(H) \) is a CPD operator and \((B, C, F)\) represents \( T \). As in the proof of Theorem 3.2.5 (c), we consider three cases. If \( \vartheta \geq 1 \), then by (3.1.5), \( 1 \leq \vartheta = r(T)^2 \). If \( \vartheta < 1 \) and \( C \neq 0 \), then by (3.1.4), \( \vartheta < 1 \leq r(T)^2 \).

Suppose now that \( \vartheta < 1 \) and \( C = 0 \). First, we consider the subcase when \( D := B + \int_{\mathbb{R}^+} F(dx) \neq 0 \). Then, there exists \( h_0 \in H \) such that \( \eta(h_0) := \langle Bh_0, h_0 \rangle \neq 0 \). According to (3.1.2), we have
\[
\|T^n h_0\|^2 = n \left( \frac{\|h_0\|^2}{n} + \langle Bh_0, h_0 \rangle + \int_{\mathbb{R}^+} \frac{Q_n(x)}{n} (F(dx)h_0, h_0) \right), \quad n \in \mathbb{N}.
\]
By assumption that \( \vartheta < 1 \), we infer from (3.2.21), (3.2.20) and Lebesgue’s monotone convergence theorem that
\[
\langle Bh_0, h_0 \rangle + \int_{\mathbb{R}^+} \frac{Q_n(x)}{n} (F(dx)h_0, h_0) \rightarrow \eta(h_0) \text{ as } n \rightarrow \infty.
\]
Since \( \eta(h_0) \neq 0 \), we deduce from (3.2.22) that \( \eta(h_0) > 0 \) and so by Gelfand’s formula for spectral radius we obtain
\[
r(T)^2 \geq \limsup_{n \rightarrow \infty} \|T^n h_0\|^{2/n} \geq 1 > \vartheta.
\]
It remains to consider the subcase when \( D = 0 \). Then by (3.2.4) and Theorem 3.4.3, \( T \) is subnormal and \( \vartheta \leq r(T)^2 = \|T\|^2 \leq 1 \) (see Example 4.3.3 and Remark 4.3.7 for the continuation of this discussion).

b) It follows from assertions (a) and (c) of Theorem 3.2.5 that \( \sigma(S) \) does not depend on a triplet \( (K, R, S) \) satisfying (3.2.12) and (3.2.13). This fact can be also deduced from Lemma 3.2.3 by applying (3.2.12) and (3.2.6) to the polynomials \( p = (X - 1)^2X^n \), where \( n \in \mathbb{Z}_+ \).

c) Concerning (3.2.15), observe that if \( T \in B(\mathcal{H}) \) is a 2-isometry and \( \mathcal{H} \neq \{0\} \), then \( T \) is CPD and \( 1 = r(T)^2 > \|S\| = 0 \) (use Proposition 4.3.1 and Lemma 1.21). It turns out that there are non-subnormal CPD operators \( T \) with \( r(T) = 1 \), so by Corollary 3.2.7(ii) for such \( T \)'s, \( \mathcal{B}_m(T) \geq 0 \) if and only if \( m \in \mathbb{Z}_+ \setminus \{1\} \) (see e.g., Example 1.3.6 cf. also 20 Section 9).

\[ \text{(3.3.1)} \]

3.3. A simplified representation with applications. First, following Proposition 3.1.4, we simplify the previous representations of CPD operators.

Theorem 3.3.1. For \( T \in B(\mathcal{H}) \), the following conditions are equivalent:

(i) \( T \) is CPD,

(ii) there exists a semispectral measure \( M : \mathcal{B}(\mathbb{R}_+) \to B(\mathcal{H}) \) with compact support such that

\[
\langle (X - 1)^2q(T) \rangle = \int_{\mathbb{R}_+} q(x)M(dx), \quad q \in \mathbb{C}[X],
\]

(ii') there exist a Hilbert space \( K \), \( R \in B(\mathcal{H}, K) \) and \( S \in B(K)_+ \) such that

\[
\langle (X - 1)^2q(T) \rangle = R^*q(S)R, \quad q \in \mathbb{C}[X],
\]

(iii) there exists a semispectral measure \( M : \mathcal{B}(\mathbb{R}_+) \to B(\mathcal{H}) \) with compact support such that

\[
T^n\mathcal{B}_2(T)T^n = \int_{\mathbb{R}_+} x^nM(dx), \quad n \in \mathbb{Z}_+,
\]

(iii') there exist a Hilbert space \( K \), \( R \in B(\mathcal{H}, K) \) and \( S \in B(K)_+ \) such that

\[
T^n\mathcal{B}_2(T)T^n = R^*S^nR, \quad n \in \mathbb{Z}_+.
\]

Moreover, the measures in (ii) and (iii) are unique and coincide with that in Theorem 3.2.5(ii); the triplets \( (K, R, S) \) in (ii') and (iii') can be chosen to satisfy 3.2.13.

Proof. (i)\(\Rightarrow\)(ii') Applying the implication (i)\(\Rightarrow\)(iv) of Theorem 3.2.5 to the polynomial \( p = (X - 1)^2q \) and using (3.2.6), we get a triplet \( (K, R, S) \) satisfying (3.2.2) and (3.2.13).

(ii')\(\Rightarrow\)(i) It follows from (3.2.6) that

\[
p = p(1) + p'(1)(X - 1) + (X - 1)^2\mathcal{B}_1p, \quad p \in \mathbb{C}[X].
\]

Since the mapping \( p \mapsto p(T) \) is linear, we obtain

\[
p(T) = p(1)I - p'(1)\mathcal{B}_1(T) + ((X - 1)^2\mathcal{B}_1p)(T).
\]
which means that \(3.2.12\) holds. Applying Theorem \(3.2.5\) gives (i).

(ii')\(\Leftrightarrow\)(iii') One can easily check that these two conditions are equivalent with the same triplet \((K, R, S)\) (use \((3.2.2)\) and \((3.2.3)\)). This together with the first paragraph of this proof justifies the second statement of the “moreover” part.

Arguing as in the proof of the equivalence (ii)\(\Leftrightarrow\)(iv) of Theorem \(3.2.5\), we deduce that the equivalences (ii)\(\Leftrightarrow\)(ii') and (iii)\(\Leftrightarrow\)(iii') hold. The first statement of the “moreover” part can be inferred from Theorem \(3.2.5(a)\) by observing that conditions \((3.2.11)\), \((3.3.1)\) and \((3.3.3)\) are equivalent (cf. the proof of the equivalence (i)\(\Leftrightarrow\)(ii')).

This completes the proof. 

\[\□\]

Many classes of operators are closed under the operation of taking powers. Among them are the classes of normaloid, subnormal, \(k\)-isometric, \(k\)-expansive, completely hyperexpansive and alternatingly hyperexpansive operators (see \([31]\, p. 99], [37]\) Theorem 2.3) and \([30]\) Theorem 2.3). On the other hand, the class of hyponormal operators does not share this property (see \([34]\), Problem 209). As the first application of Theorem \(3.3.1\), we show that the class of CPD operators does share this property. We also describe the semispectral and the dilation representations for powers of CPD operators.

**Theorem 3.3.2.** Suppose that \(T \in B(H)\) is a CPD operator and \(i \in \mathbb{N} \setminus \{1\}\). Then

(i) \(T^i\) is CPD,

(ii) if \(M\) and \(M_i\) are semispectral measures that correspond respectively to \(T\) and \(T^i\) via Theorem \(3.2.5(ii)\), then

\[M_i(\Delta) = \tilde{M}_i(\psi^{-1}_i(\Delta)), \quad \Delta \in \mathcal{B}(\mathbb{R}_+),\]  

where \(\tilde{M}_i: \mathcal{B}(\mathbb{R}_+) \rightarrow B(H)\) is the semispectral measure defined by

\[\tilde{M}_i(\Delta) = \int_{\Delta} (1 + x + \ldots + x^{i-1})^2 M(dx), \quad \Delta \in \mathcal{B}(\mathbb{R}_+),\]  

and \(\psi_i: \mathbb{R}_+ \rightarrow \mathbb{R}_+\) is given by \(\psi_i(x) = x^i\) for \(x \in \mathbb{R}_+\),

(iii) the representing triplet \((B_i, C_i, F_i)\) of \(T^i\) can be described by applying Theorem \(3.2.5(b)\) to \(M_i\) in place of \(M\),

(iv) if \((K, R, S)\) is as in Theorem \(3.2.5(iv)\), then the triplet \((K, R_i, S^i)\) with

\[R_i := (I + S + \ldots + S^{i-1})R,\]  

corresponds to \(T^i\) via Theorem \(3.2.5(iv)\).

**Proof.** (i)&(ii) First, it is easily seen that

\[\mathcal{F}_2(T^i) = (1 - X^i)^2(T).\]  

Let \(M\) be as in Theorem \(3.2.5(ii)\). By the “moreover” part of Theorem \(3.3.1\) \(M\) satisfies \((3.3.1)\). Clearly, the set functions \(\tilde{M}_i\) and \(M_i\) defined by \((3.3.6)\) and \((3.3.5)\), respectively, are semispectral measures that are compactly supported. Applying \((1.4.4)\) and the measure transport theorem, we get (for the definition of \(\nabla_T\), see \((3.2.3)\))

\[(T^i)^n \mathcal{F}_2(T^i)(T^i)^n \overset{\text{(3.3.8)}}{=} (\nabla_T)^n (1 - X^i)^2(T)\]
Using Theorem 3.3.1(iii) and the “moreover” part of this theorem, we see that (i) and (ii) hold.

(iii) Obvious.

(iv) Let \((K, R, S)\) be as in Theorem 3.2.5(iv). Denote by \(E_S\) and \(E_S^\dagger\) the spectral measures of \(S\) and \(S^\dagger\), respectively. In view of \[12\] Theorem 6.6.4, we have
\[
E_S^\dagger(\Delta) = E_S(\psi^{-1}_i(\Delta)), \quad \Delta \in \mathcal{B}(\mathbb{R}_+). \tag{3.3.9}
\]
According to the proof of the implication (iii)⇒(ii) of Theorem 3.2.5
\[
M(\Delta) = R^* E_S(\Delta) R, \quad \Delta \in \mathcal{B}(\mathbb{R}_+). \tag{3.3.10}
\]
It follows from (3.3.10) and Lemma 3.2.3(a) that
\[
\mathcal{K} = \bigvee \{ E_S(\Delta) \mathcal{B}(R) : \Delta \in \mathcal{B}(\mathbb{R}_+) \}. \tag{3.3.11}
\]
Define the function \(\zeta_i : \mathbb{R}_+ \to \mathbb{C}\) by \(\zeta_i(x) = 1 + x + \ldots + x^{i-1}\) for \(x \in \mathbb{R}_+.\) Using (3.3.5) and (3.3.10) and applying the Stone-von Neumann functional calculus, we get
\[
\langle M_i(\Delta) h, h \rangle = \langle \int_{\psi^{-1}_i(\Delta)} \zeta_i(x)^2 M(dx) h, h \rangle \tag{3.3.12}
\]
Since the operator \(I + S + \ldots + S^{i-1}\) commutes with \(E_S\) and is invertible in \(B(\mathcal{K})\), we obtain
\[
\bigvee \{ E_S(\Delta) \mathcal{B}(R) : \Delta \in \mathcal{B}(\mathbb{R}_+) \} = (I + S + \ldots + S^{i-1}) \bigvee \{ E_S(\psi^{-1}_i(\Delta)) \mathcal{B}(R) : \Delta \in \mathcal{B}(\mathbb{R}_+) \}
\]
Hence, by Lemma 3.2.3(a), \( \forall \{ (S^n)R_n : n \in \mathbb{Z}_+ \} = \mathcal{K}. \) Using (ii) and (3.3.12) and applying the Stone-von Neumann functional calculus to the operator \( S^i, \) we verify that equalities (3.2.12) and (3.2.13) hold with \( (T, R_i, S^i) \) in place of \( (T, R, S). \) This shows (iv) and completes the proof. \( \square \)

The following corollary extends the formula (3.2.21) of Corollary 3.2.7 to the case of powers of CPD operators.

**Corollary 3.3.3.** Suppose \( T \in B(H) \) is a CPD operator and \( M \) is as in Theorem 3.3.5(ii). Then

\[
\mathcal{B}_m(T^i) = \int_{\mathbb{R}_+} (1 - x^i)^{m-2}(1 + x + \ldots + x^{i-1})^2 M(dx), \quad m \geq 2, i \geq 1.
\]

**Proof.** In view of Corollary 3.2.7, it suffices to consider the case \( i \geq 2. \) By assertions (i) and (ii) of Theorem 3.3.5(ii), \( T^i \) is CPD and the semispectral measure \( M_i \) corresponding to \( T^i \) via Theorem 3.2.7(ii) is given by (3.3.5) and (3.3.6). Using (3.3.7) and the measure transport theorem, we obtain

\[
\int_{\mathbb{R}_+} f(x)\int_{\mathbb{R}_+} f(x^i)(1 + x + \ldots + x^{i-1})^2 M(dx), \quad f \in L^1(M_i).
\]

Applying (3.2.21) to \( T^i \) and (3.3.14) to \( f(x) = (1 - x)^{m-2}, \) we get (3.3.13). \( \square \)

As the second application of Theorem 3.3.5, we give a characterization of CPD operators of class \( Q \) (a class of operators having upper triangular \( 2 \times 2 \) block matrix form) by using the Taylor spectrum approach developed in [20]. We also describe the semispectral and the dilation representations for such operators. According to Corollary 3.2.7(i), \( \mathcal{B}_{2k}(T) \geq 0 \) for all \( k \in \mathbb{Z}_+ \) whenever \( T \) is CPD. We will show in Theorem 3.3.3 below that the single inequality \( \mathcal{B}_{2k}(T) \geq 0 \) with \( k \geq 1 \) completely characterizes CPD operators of class \( Q. \) Following [20], we say that \( T \in B(H) \) is of class \( Q \) if it has a block matrix form

\[
T = \begin{bmatrix} V & E \\ 0 & Q \end{bmatrix}
\]

with respect to an orthogonal decomposition \( H = H_1 \oplus H_2, \) where \( H_1 \) and \( H_2 \) are nonzero Hilbert spaces and \( V \in B(H_1), E \in B(H_2, H_1), Q \in B(H_2) \) satisfy

\[
V^*V = I, \quad V^*E = 0, \quad QE^*E = E^*EQ \quad \text{and} \quad QQ^*Q = Q^*QQ.
\]

(In particular, \( T = \begin{bmatrix} V & E \\ 0 & Q \end{bmatrix} \) in \( Q_{H_1, H_2} \).) If this is the case, we denote \( \mathcal{B}_{2k}(T) \) by \( \sigma(T), \) and the Taylor spectrum of a pair \( (T_1, T_2) \) of commuting operators \( T_1, T_2 \in B(H) \) is denoted by \( \sigma(T_1, T_2). \) It is worth pointing out that in view of [20], Theorem 3.3.3 for any nonempty compact subset \( \Gamma \) of \( \mathbb{R}_+ \) and any separable infinite dimensional Hilbert space \( H_2, \) there exist a nonzero Hilbert space \( H_1 \) and \( T = \begin{bmatrix} V & E \\ 0 & Q \end{bmatrix} \in Q_{H_1, H_2} \) such that \( \sigma([Q, [E]]) = \Gamma. \) This important fact enables us to find the spectral region for conditional positive definiteness of operators of class \( Q \) (see Theorem 3.3.5 and Figure [1]). For a more thorough discussion of these topics the reader is referred to [20]. Before stating Theorem 3.3.5, we prove an auxiliary lemma which is of some independent interest.
LEMMA 3.3.4. Let $A, B \in \mathcal{B}(\mathcal{H})$ be two commuting normal operators. Then

$$\mathcal{N}(AB) = \mathcal{N}(A) + \mathcal{N}(B),$$  

(3.3.16)

$$\mathcal{R}(AB) = \mathcal{R}(A) \cap \mathcal{R}(B).$$  

(3.3.17)

PROOF. Let $G : \mathfrak{B}(\mathbb{C}^2) \to \mathcal{B}(\mathcal{H})$ be the joint spectral measure of $(A, B)$ (see Theorem 5.21). Since $G([\{0, 0\}]) \subseteq G(\mathbb{C} \times \{0\})$ and thus $\mathcal{R}(G([\{0, 0\}])) \subseteq \mathcal{R}(G(\mathbb{C} \times \{0\}))$, we obtain

$$\mathcal{R}(G(\mathbb{C} \times \{0\})) = \mathcal{R}(G([\{0, 0\}])) + \mathcal{R}(G(\mathbb{C} \times \{0\})).$$  

(3.3.18)

Applying the Stone-von Neumann functional calculus and (3.3.15) yields

$$\mathcal{N}(AB) = \mathcal{N} \left( \int_{\mathbb{C}^2} z_1 z_2 dG(z_1, z_2) \right)$$

$$= \mathcal{N} \left( G([\{z_1, z_2\} \in \mathbb{C}^2 : z_1 z_2 \neq 0]) \right)$$

$$= \mathcal{R} \left( G([\{z_1, z_2\} \in \mathbb{C}^2 : z_1 z_2 = 0]) \right)$$

$$= \mathcal{R}(G(\{0\} \times \mathbb{C})) + \mathcal{R}(G(\mathbb{C} \times \{0\}))$$

$$= \mathcal{R}(G([0]) + \mathcal{R}(G([0, 0])) + \mathcal{R}(G(\mathbb{C} \times \{0\}))$$

$$= \mathcal{R}(G([0] \times \mathbb{C})) + \mathcal{R}(G(\mathbb{C} \times \{0\})),$$  

(3.3.19)

where $\mathbb{C}_* := \mathbb{C} \setminus \{0\}$. Similarly,

$$\mathcal{N}(A) = \mathcal{N} \left( \int_{\mathbb{C}^2} z_1 dG(z_1, z_2) \right) = \mathcal{R}(G(\{0\} \times \mathbb{C})),$$  

(3.3.20)

$$\mathcal{N}(B) = \mathcal{N} \left( \int_{\mathbb{C}^2} z_2 dG(z_1, z_2) \right) = \mathcal{R}(G(\mathbb{C} \times \{0\})).$$  

(3.3.21)

Combining (3.3.19) with (3.3.20) and (3.3.21), we get (3.3.16). Finally, applying (3.3.16) to the adjoints of $A$ and $B$ and taking orthocomplements gives (3.3.17). □

THEOREM 3.3.5. Suppose that $T = \begin{bmatrix} V & F \\ 0 & Q \end{bmatrix} \in \mathcal{Q}_{\mathcal{H}_1, \mathcal{H}_2}$. Then the following conditions are equivalent:

(i) $T$ is CPD,

(ii) $\sigma([Q], |E|) \subseteq \{(s, t) \in \mathbb{R}_+^2 : s^2 + t^2 \leq 1\} \cup (1, \infty) \times \mathbb{R}_+),$

(iii) $\mathcal{B}_{2k}(T) \geq 0$ for every (equivalently, for some) $k \in \mathbb{N}$.

Moreover, if $T$ is CPD, then the following assertions hold:

(a) $A := (I - |Q|^2 - |E|^2)(I - |Q|^2) \in \mathcal{B}(\mathcal{H}_2)_+$, the operators $Q$, $|Q|$, $|E|$ and $A$ commute and

$$M(\Delta) = 0 \oplus \sqrt{A}P_{|Q|^2}(\Delta)\sqrt{A}, \quad \Delta \in \mathfrak{B}(\mathbb{R}_+),$$  

(3.3.22)

where $M$ is as in Theorem 3.2.5 ii) and $P_{|Q|^2}$ is the spectral measure of $|Q|^2$,

(b) the representing triplet $(B, C, F)$ of $T$ is described by Theorem 3.2.5 b),

(c) the triplet $(K, R, S)$ defined below corresponds to $T$ via Theorem 3.2.5 iv):

$$K := \mathcal{F}(A) = \mathcal{F}(I - |Q|^2 - |E|^2) \cap \mathcal{F}(I - |Q|),$$  

(3.3.23)

$$R(h_1 \oplus h_2) := \sqrt{A}h_2, \quad h_1 \in \mathcal{H}_1, h_2 \in \mathcal{H}_2,$$  

(3.3.24)

$$S := (|Q|_K)^2 \quad (K \text{ reduces } |Q|).$$
Proof. (i)⇔(ii) Using [20] Proposition 3.10 and Lemma 9.1, one can check that
\[ T^n \mathcal{B}_2(T)T^n = \begin{bmatrix} 0 & 0 \\ 0 & Q^nA^nQ^n \end{bmatrix}, \quad n \in \mathbb{Z}_+. \quad (3.3.25) \]
By the square root theorem and (3.3.15), the operators \( Q, |Q|, |E| \) and \( A \) commute. Combined with [20] (19), this implies that \( A = A^* \) and
\[ Q^nA^nQ^n = Q^nQ^nA = |Q|^{2n}A = \int_{\mathbb{R}_+^2} \tau_n dG, \quad n \in \mathbb{Z}_+, \quad (3.3.26) \]
where \( G \) is the joint spectral measure of \( (|Q|, |E|) \) and \( \tau_n : \mathbb{R}_+^2 \to \mathbb{R} \) is given by
\[ \tau_n(s, t) = (1 - s^2 - t^2)(1 - s^2)s^{2n}, \quad s, t \in \mathbb{R}_+, n \in \mathbb{Z}_+. \quad (3.3.27) \]
It follows from Proposition 3.1.4 [13] Theorem 2], (3.3.25) and (3.3.26) that \( T \) is CPD if and only if \( \{ \tau_n(s, t)(G(ds, dt)h, h) \}_{n=0}^{\infty} \) is a Stieltjes moment sequence for every \( h \in \mathcal{H}_2 \). By [20] Theorem 2.1(i) & Lemma 4.10, the latter holds if and only if \( \sigma(|Q|, |E|) \subseteq \Xi \), where
\[ \Xi := \{ (s, t) \in \mathbb{R}_+^2 : \{ \tau_n(s, t) \}_{n=0}^{\infty} \text{ is a Stieltjes moment sequence} \}. \]
In view of (3.3.27), it is easily seen that
\[ \Xi = \{ (s, t) \in \mathbb{R}_+^2 : (1 - s^2 - t^2)(1 - s^2) \geq 0 \}
= \{ (s, t) \in \mathbb{R}_+^2 : s^2 + t^2 \leq 1 \} \cup ([1, \infty) \times \mathbb{R}_+), \]
which shows that (i) and (ii) are equivalent.

(ii)⇔(iii) This equivalence is a direct consequence of [20] Theorem 9.2(i)].

We now prove the “moreover” part. Assume that (i) holds.
(a) Applying the spectral mapping theorem (see e.g., [20] Theorem 2.1]), we get
\[ \sigma(A) = \sigma(|Q|, |E|) = \sigma_0(\sigma(|Q|, |E|)) \subseteq \mathbb{R}_+, \]
which together with \( A = A^* \) implies that \( A \in B(\mathcal{H}_2)_+ \). Now, it is clear that the set function \( M : \mathcal{B}(\mathbb{R}_+) \to B(\mathcal{H}) \) defined by (3.3.31) is a semispectral measure with compact support. Recall that \( A \) commutes with \( |Q| \). Using this fact and applying the Stone-von Neumann functional calculus, we deduce from (3.3.25), (3.3.26) and the square root theorem that
\[ \langle T^n \mathcal{B}_2(T)T^n h, h \rangle = \langle Q^nA^n h, h \rangle 
= \| (|Q|^{2n/2}) A h \|^2 
= \int_{\mathbb{R}_+} x^n (\sqrt{A}P_{Q}x) (dx) \sqrt{A}h, h \rangle 
= \int_{\mathbb{R}_+} x^n (M(dx)h, h), \]
\[ \overset{1.4.4}{=} \left\langle \int_{\mathbb{R}_+} x^n M(dx) h, h \right\rangle, \quad h = h_1 \oplus h_2 \in \mathcal{H}, n \in \mathbb{Z}_+. \]
This shows that condition (iii) of Theorem 3.11 holds. Applying the “moreover” part of this theorem completes the proof of (a).

(b) Obvious.
Corollary 3.2. Let $X, Y \in \mathcal{B}(\mathcal{H})$ be commuting positive selfadjoint operators such that $\sigma(X, Y) \subseteq \{(s, t) \in \mathbb{R}^2: s^2 + t^2 \leq 1\} \cup ([1, \infty) \times \mathbb{R}_+).$

Then there exist $V \in \mathcal{B}(\mathcal{H}_1), E \in \mathcal{B}(\mathcal{H}_2, \mathcal{H}_1)$ and $Q \in \mathcal{B}(\mathcal{H}_2)$ such that $|Q| = X$, $|E| = Y$, $\begin{bmatrix} V & E \\ 0 & Q \end{bmatrix} \in \mathcal{Q}_{\mathcal{H}_1, \mathcal{H}_2}$ and $T := \begin{bmatrix} V & E \\ 0 & Q \end{bmatrix}$ is a CPD operator which satisfies assertion (a) of Theorem 3.2.4.

**Remark 3.3.7.** Assuming that (3.3.29) holds, the above results allow to describe the ranges of the mappings $\Psi_M$ and $\tilde{\Psi}_M$ restricted to operators of class $Q$ (for the definitions of $\Psi_M$ and $\tilde{\Psi}_M$, see (3.3.16) and (1.3.3)). First, we consider the case of the mapping $\tilde{\Psi}_M$. Take any $T = \begin{bmatrix} V & E \\ 0 & Q \end{bmatrix} \in \mathcal{Q}_{\mathcal{H}_1, \mathcal{H}_2}$ which is CPD with $M$ as in Theorem 3.2.5(ii) and set $X = |Q|$ and $Y = |E|$. Then by Theorem 3.3.5, $X$ and $Y$ are commuting positive selfadjoint operators, they satisfy (3.3.30) and

$$M(\Delta) = 0 \oplus \sqrt{A} P_{X^2}(\Delta) \sqrt{A}, \quad \Delta \in \mathcal{B}(\mathbb{R}_+),$$

where

$$A = (I - X^2 - Y^2)(I - X^2).$$

Conversely, if $X, Y \in \mathcal{B}(\mathcal{H}_2)$ are commuting positive selfadjoint operators which satisfy (3.3.30), then by Corollary 3.3.6, there is $T = \begin{bmatrix} V & E \\ 0 & Q \end{bmatrix} \in \mathcal{Q}_{\mathcal{H}_1, \mathcal{H}_2}$ which is CPD and satisfies (3.3.31) and (3.3.32), where $M$ is as in Theorem 3.2.5(ii).

The case of the mapping $\Psi_M$ can be deduced from the above description of $\tilde{\Psi}_M$, Subsection 1.3 and [20, Proposition 3.10(ii)].

**Corollary 3.3.8.** Suppose that $T = \begin{bmatrix} V & E \\ 0 & Q \end{bmatrix} \in \mathcal{Q}_{\mathcal{H}_1, \mathcal{H}_2}$ is CPD and $S$ is as in Theorem 3.3.5(c). Then the following conditions are equivalent:

(i) $S = |Q|^2$,

(ii) $1 \notin \sigma_p(|Q|^2 + |E|^2)$ and $1 \notin \sigma_p(|Q|)$.
Regarding Theorem 3.3.5, it is worth mentioning that in view of Theorem 1.2, the operator \( T = \begin{bmatrix} 1 & E \\ Q \end{bmatrix} \in \mathcal{Q}_H \) is subnormal if and only if
\[
\sigma(|Q|, |E|) \subseteq \{(s, t) \in \mathbb{R}^2_+: s^2 + t^2 \leq 1\} \cup ([1, \infty) \times \{0\}).
\]
For the reader’s convenience, the spectral regions for subnormality and conditional positive definiteness of operators of class \( \mathcal{Q} \) are illustrated in Figure 1.

3.4. Subnormality. In view of Theorem 1.1.1, any subnormal operator \( T \in B(H) \) has the property that the sequence \( \{\|T^n h\|_2\}_{n=0}^\infty \) is PD for every \( h \in H \). As a consequence, any subnormal operator is CPD. The converse implication is not true in general (see [70], Example 5.4). In this subsection, we deal with the problem of finding necessary and sufficient conditions for subnormality written in terms of conditional positive definiteness. Theorem 4.1 in [70], which is the first result in this direction formulated for \( d \)-tuples of operators, shows that a contraction is subnormal if and only if it is CPD. The main result of this subsection, namely Theorem 3.4.4, generalizes [70] Theorem 4.1. In particular, it covers the case of strongly stable operators (see Corollary 3.4.5).

Our first goal is to characterize those CPD operators that are subnormal in terms of the parameters \( B, C, F \) appearing in statement (ii) of Theorem 3.1.1.

**Theorem 3.4.1.** Let \( T \in B(H) \). Then the following statements are equivalent:

(i) \( T \) is subnormal,
(ii) \( T \) is CPD and its representing triplet \((B, C, F)\) satisfies the following conditions:

(ii-a) \( \frac{1}{(x-1)^2} \in L^1(F) \) and \( \int_{\mathbb{R}_+} \frac{1}{(x-1)^2} F(dx) \leq I \),
(ii-b) \( \frac{1}{x-1} \in L^1(F) \) and \( B = \int_{\mathbb{R}_+} \frac{1}{x-1} F(dx) \),
(ii-c) \( C = 0 \).

Moreover, if (ii) holds and \( G \) is the semispectral measure of \( T \) (see (1.4.5)), then
\[
F = M, \quad \text{where } M \text{ is as in Theorem 3.2.5(ii)},
\]
\[
B = \int_{\mathbb{R}_+} (x-1) G \circ \phi^{-1}(dx),
\]
\[
F(\Delta) = \int_{\Delta} (x-1)^2 G \circ \phi^{-1}(dx), \quad \Delta \in \mathfrak{B}(\mathbb{R}_+), \tag{3.4.1}
\]
\[
G \circ \phi^{-1}(\Delta) = \int_{\Delta} \frac{1}{(x-1)^2} F(dx) + \delta_1(\Delta) \left( I - \int_{\mathbb{R}_+} \frac{1}{(x-1)^2} F(dx) \right), \quad \Delta \in \mathfrak{B}(\mathbb{R}_+).
\]
Proof. (i)⇒(ii) It follows from Theorem 1.1.1 that \( \{\|T^n h\|^2\}_{n=0}^{\infty} \) is a Stieltjes moment sequence for every \( h \in \mathcal{H} \). Hence, by Theorem 2.2.12 we have
\[
\int_{\mathbb{R}} \frac{1}{(x-1)^2} \langle F(dx)h, h \rangle \leq \|h\|^2, \quad h \in \mathcal{H},
\]
(3.4.2)
\[
(Bh, h) = \int_{\mathbb{R}} \frac{1}{x-1} \langle F(dx)h, h \rangle, \quad h \in \mathcal{H},
\]
(3.4.3)
\[
(CH, h) = 0, \quad h \in \mathcal{H}.
\]
(3.4.4)

It follows from (1.4.4), (3.4.2) and (3.4.4) that conditions (ii-a) and (ii-c) are satisfied.

(ii)⇒(i) Applying (1.4.4) and Theorem 2.2.12 again, we deduce that the sequence \( \{\|T^n h\|^2\}_{n=0}^{\infty} \) is PD for all \( h \in \mathcal{H} \). Hence, by Theorem 1.1.1, \( T \) is subnormal.

The “moreover” part can be deduced straightforwardly from (1.4.5) and the corresponding part of Theorem 2.2.12 (that \( F = M \) follows from (ii-c) and Theorem 3.2.3(b)). This completes the proof. \( \square \)

Corollary 3.4.2. Let \( T \in \mathcal{B}(\mathcal{H}) \) be a subnormal operator, \( G \) be the semispectral measure of \( T \), \( N \) be the minimal normal extension of \( T \) and \( F \) be as in Theorem 3.1.1(ii). Then

(i) \( r(T) = \|T\| = \sup \{|z| : z \in \text{supp}(G)\} \),

(ii) \( \sigma(N) = \text{supp}(G) \) and \( \sigma(N^*N) = \{|z|^2 : z \in \text{supp}(G)\} \),

(iii) if \( G(T) = 0 \), where \( T = \{z \in \mathbb{C} : |z| = 1\} \), then

(iii-a) the measures \( F \) and \( G \circ \phi^{-1} \) are mutually absolutely continuous,

(iii-b) \( \sigma(N^*N) = \text{supp}(F) \),

(iii-c) \( \|T\|^2 = \text{sup}\text{sup}(\text{supp}(F)) \).

Proof. The first equality in (i) is a consequence of (1.4.2). It follows from Proposition 4 that
\[
\sigma(N) = \text{supp}(G), \tag{3.4.5}
\]
which gives the first equality in (ii). Using [25 Corollary II.2.17], we obtain
\[
\|T\| = \|N\| \quad \Leftrightarrow \quad r(N) = \text{sup}\{\|z\| : z \in \text{supp}(G)\}.
\]

This yields the second equality in (i). The second equality in (ii) follows from (3.4.5) and [12 eq. (14), p. 158]. It remains to prove (iii). According to (3.4.1), \( F \) is absolutely continuous with respect to \( G \circ \phi^{-1} \). In turn, if \( \Delta \in \mathcal{B}(\mathbb{R}_+) \) is such that \( F(\Delta) = 0 \), then (3.4.1) implies that \( G \circ \phi^{-1}(\Delta \setminus \{1\}) = 0 \). Since by assumption \( G \circ \phi^{-1}(\{1\}) = 0 \), we see that \( G \circ \phi^{-1}(\Delta) = 0 \). This means that the measures \( F \) and \( G \circ \phi^{-1} \) are mutually absolutely continuous, therefore (iii-a) holds. As a consequence, \( \text{supp}(F) = \text{supp}(G \circ \phi^{-1}) \). Combined with [23 Lemma 3(5)], this implies (iii-b). Finally, (iii-c) is a direct consequence of (i), (ii) and (iii-b). \( \square \)

Corollary 3.4.3. Let \( T \in \mathcal{B}(\mathcal{H}) \) be a subnormal operator and \( M \) be as in Theorem 3.2.5(ii). Then \( M = 0 \) if and only if \( T \) is an isometry.

Proof. If \( M = 0 \), then by Theorem 3.4.1 \( B = C = 0 \), so by (3.1.2), \( T \) is an isometry. Conversely, if \( T \) is an isometry, an application of the identity \( p(T) = p(1)I, p \in \mathbb{C}[X] \), gives (3.2.11) with \( M = 0 \). \( \square \)
Theorem 3.4.4 below gives new necessary and sufficient conditions for subnormality. Condition (v) of this theorem comprises the case $D = 0$ which is not covered by Theorem 3.1.8.

**Theorem 3.4.4.** Let $T \in B(\mathcal{H})$. Then the following conditions are equivalent:

(i) $T$ is a subnormal contraction,

(ii) $T$ is a CPD contraction,

(iii) $T$ is CPD and the telescopic series

$$\sum_{n=0}^{\infty} (\|T^{n+1}h\|^2 - \|T^nh\|^2)$$

is convergent in $\mathbb{R}$ for every $h \in \mathcal{H}$,

(iv) $T$ is CPD and

$$\lim_{n \to \infty} (\|T^{n+1}h\|^2 - \|T^nh\|^2) = 0, \quad h \in \mathcal{H},$$

(v) condition (ii) of Theorem 3.1.1 holds with $C = 0$, $D = 0$, $F([1, \infty)) = 0$ and $\frac{1}{1-x} \in L^1(F)$, where $D := B + \int_{(0,1)} \frac{1}{1-x} F(dx)$ (or equivalently if all of this holds with \(\frac{1}{1-x} \in L^1(F)\) in place of \(\frac{1}{1-x} \in L^1(F)\)),

(vi) condition (ii) of Theorem 3.1.6 holds with $D = 0$.

**Proof.** The implications (i)$\Rightarrow$(ii)$\Rightarrow$(iii)$\Rightarrow$(iv) are obvious because if $T$ is a contraction, then the sequence $\{\|T^n h\|^2\}_{n=0}^{\infty}$, being monotonically decreasing, is convergent in $\mathbb{R}_+$ for all $h \in \mathcal{H}$.

(iv)$\Rightarrow$(i) This implication can be deduced from Corollary 2.3.4 (applied to $\gamma_{T,h}$) and Theorems 3.1.4 and 3.1.7.

(iv)$\Rightarrow$(v) It follows from Theorem 3.1.6 that (iv) implies the variant of (v) with $\frac{1}{1-x} \in L^1(F)$\(^5\). That $\frac{1}{1-x} \in L^1(F)$ is a consequence of Theorem 3.4.1 and the fact that (iv) implies (i).

(v)$\Rightarrow$(vi) Assume that the variant of (v) with $\frac{1}{1-x} \in L^1(F)$ holds. Then, by the Cauchy-Schwarz inequality, $\frac{1}{1-x} \in L^1(F)$. Observe that (cf. (2.3.8))

$$T^{*n}T^n = I + n \left( B + \int_{(0,1)} \frac{Q_n(x)}{n} F(dx) \right)$$

and

$$= I + \int_{(0,1)} \frac{1-x^n}{1-x} F(dx), \quad n \in \mathbb{N}.$$  

This implies that the pair $(D,F)$ with $D = 0$ satisfies condition (ii) of Theorem 3.1.6.

(vi)$\Rightarrow$(iv) One can apply Theorem 3.1.6.

There are other ways to prove some implications of Theorem 3.4.4. Namely, one can show the implication (iv)$\Rightarrow$(ii) by using the “moreover” part of Proposition 3.1.4. In turn, the implication (ii)$\Rightarrow$(i) can be deduced from [11, Theorem 3.1] and Corollary 3.2.7(ii). The implication (ii)$\Rightarrow$(i) (with a different proof) is a part

\(^5\)It follows from Propositions 3.1.4 and 3.1.7 that, under the assumption that $T$ is CPD, (3.4.6) is equivalent to

$$\sup_{n \in \mathbb{N}_+} (\|T^{n+1}h\|^2 - \|T^nh\|^2) \leq 0, \quad h \in \mathcal{H}.$$
of the conclusion of [70] Theorem 4.1]. Observe also that by Theorem 3.4.4 an operator T ∈ B(H) is subnormal if and only if there exists α ∈ ℂ \ {0} such that the operator αT satisfies any of the equivalent conditions (ii)-(vi) of Theorem 3.4.4.

**Corollary 3.4.5.** Let T ∈ B(H) obey any of the following conditions:
(i) the sequence \( \{\|T^n h\|^2\}_{n=0}^\infty \) is convergent in \( \mathbb{R}_+ \) for every \( h \in H \),
(ii) T is strongly stable, i.e., \( \lim_{n \to \infty} \|T^n h\| = 0 \) for every \( h \in H \) (36, 47),
(iii) \( r(T) < 1 \).
Then T is CPD if and only if T is subnormal.

**Corollary 3.4.6.** Let T ∈ B(H). Then the following are equivalent:
(i) T is subnormal,
(ii) αT is CPD for all \( \alpha \in \mathbb{C} \),
(iii) zero is an accumulation point of the set of all \( \alpha \in \mathbb{C} \setminus \{0\} \) for which αT is CPD,
(iv) there exists \( \alpha \in \mathbb{C} \setminus \{0\} \) such that \( |\alpha| r(T) < 1 \) and αT is CPD.

**Corollary 3.4.7.** Suppose \( T \in B(H) \) is a non-subnormal CPD operator. Then \( r(T) \geq 1 \) and αT is not CPD for any complex number α such that
\[
0 < |\alpha| < \frac{1}{r(T)}.
\]

Regarding Corollary 3.4.7, we refer the reader to Example 1.3.6 for an example of a non-subnormal CPD operator with \( r(T) = 1 \). Below we apply the above to certain translations of quasinilpotent operators (cf. 34).

**Corollary 3.4.8.** Let \( N \in B(H) \) and \( \alpha \in \mathbb{C} \) be such that \( r(N) = 0 \) and \( |\alpha| < 1 \). Then \( \alpha I + N \) is CPD if and only if \( N = 0 \).

**Proof.** If \( \alpha I + N \) is CPD, then, since \( r(\alpha I + N) = |\alpha| < 1 \), we infer from Corollary 3.4.5 and (1.4.2) that \( \|N\| = r(N) = 0 \), which shows that \( N = 0 \).

Concerning Corollary 3.4.8 note that if \( N \) is a nilpotent operator with index of nilpotency 2 and \( \alpha \in \mathbb{C} \) is such that \( |\alpha| = 1 \), then by 11 Theorem 2.2, \( \alpha I + N \) is a strict 3-isometry, so by Proposition 1.3.1 \( \alpha I + N \) is CPD. It is an open question as to whether there exists a quasinilpotent operator \( N \) which is not nilpotent and such that \( I + N \) is CPD.

According to the above discussion, the class of CPD operators is not scalable, i.e., it is not closed under the operation of multiplying by nonzero complex scalars. Among non-scalable classes of operators are those which consist of \( m \)-isometric and 2-hyperexpansive operators (see 3 Lemma 1.21 and 50 Lemma 1, respectively). On the other hand, the classes of normaloid, hyponormal and subnormal operators are scalable (see 31 for more examples).

Condition (3.4.9) of Theorem 3.4.4 gives rise to a link between the conditional positive definiteness of a (bounded) operator \( T \) and the subnormality of (in general unbounded) unilateral weighted shift operators \( W_{T,h} \), \( h \in H \), defined below. Given an operator \( T \in B(H) \) and a vector \( h \in H \), we denote by \( W_{T,h} \) the unilateral weighted shift in \( \ell^2 \) with weights \( \{e^{\frac{r(T)}{2}(\|T^n h\|^2 - \|T^n h\|^2)}\}_{n=0}^\infty \), that is \( W_{T,h} = UD_{T,h} \), where \( U \in B(\ell^2) \) is the unilateral shift and \( D_{T,h} \) is the diagonal (normal) operator in \( \ell^2 \) with the diagonal \( \{e^{\frac{r(T)}{2}(\|T^n h\|^2 - \|T^n h\|^2)}\}_{n=0}^\infty \) (with respect to the standard
orthonormal basis of $\ell^2$. Then for every $h \in \mathcal{H}$,
\[ W_{T,h} \in B(\ell^2) \text{ if and only if } \sup_{n \in \mathbb{Z}_+} (\|T^{n+1}h\|^2 - \|T^n h\|^2) < \infty; \]
if this is the case, then $\|W_{T,h}\|^2 = e^{\sup_{n \in \mathbb{Z}_+} (\|T^{n+1}h\|^2 - \|T^n h\|^2)}$. (3.4.7)

In view of (3.4.7), the weighted shift $W_{T,h}$ is bounded for all $h \in \mathcal{H}$ if and only if $T$ satisfies condition (3.4.7) and Propositions 3.1.4 and 3.4.9 (row $\bullet\bullet\bullet$). To get the equivalence (ii-b) and (ii-c) follow from the equivalence (i) of Theorem 3.4.4. Noting first that the sequence $\{\|T^n h\|^2\}^\infty_{n=0}$ is convergent in $\mathbb{R}_+$ for all $h \in \mathcal{H}$ whenever $T$ is a contraction and then using (i) and (3.4.7), we get the equivalence (ii-b)$\Leftrightarrow$(ii-c). This completes the proof.

**Table 1**. When does conditional positive definiteness imply subnormality?

- **T is CPD and satisfies $\circ$, $\circ\circ$, or $\circ\circ\circ$**
  - **$\Rightarrow$ T subnormal**
  - **$r(T)$**

|   |   |   |
|---|---|---|
| 1 | $\exists h: W_{T,h}$ is not bounded | NO | $\geq 1$
| 2 | $\forall h: W_{T,h}$ is bounded and $D \neq 0$ | NEVER | $\leq 1$
| 3 | $\forall h: W_{T,h}$ is bounded and $D = 0$ | YES | $\leq 1$

We now recapitulate our considerations in Table 1. Note that if $T \in B(\mathcal{H})$ is CPD and $W_{T,h}$ is bounded for all $h \in \mathcal{H}$, then by (3.4.7) and Proposition 3.1.5 the limit $D := (\text{wot}) \lim_{n \to \infty} \{T^*(n+1)T^{n+1} - T^*nT^n\}_{n=0}^\infty$ exists. This is especially true in cases $\circ\circ$ and $\circ\circ\circ$. To get row $\circ\circ$ apply the Gelfand’s formula for spectral radius and (3.4.7); row $\circ\circ$ follows from (3.4.2), (3.4.7) and Proposition 3.1.7; row $\circ\circ\circ$ is a consequence of (3.4.7) and Propositions 3.1.4 and 3.4.9 (row $\circ\circ\circ$ also follows from (3.4.7) and Theorems 3.1.6 and 3.4.4).
We close this subsection with a new characterization of completely hyperexpansive operators. It can be deduced from [9, Theorem 2] and Lemma 2.1.1 by arguing as in the proof of Proposition 3.4.9(i). Despite the formal similarity, the characterizations given in Propositions 3.4.9 and 3.4.10 are radically different, because all unilateral weighted shifts appearing in Proposition 3.4.10 are contractive.

**Proposition 3.4.10.** An operator \( T \in B(\mathcal{H}) \) is completely hyperexpansive if and only if the unilateral weighted shift on \( \ell^2 \) with weights \( \{ e^{\frac{1}{2} \| T^n h \|^2 - \| T^{n+1} h \|^2} \} \) is subnormal for all \( h \in \mathcal{H} \).

### 4. A functional calculus and related matters

#### 4.1. A functional calculus.
We begin by discussing the space \( L^\infty(M) \). Suppose \( M : \mathfrak{B}(\mathbb{R}_+) \to B(\mathcal{H}) \) is a semispectral measure. We denote by \( L^\infty(M) \) the Banach space of all equivalence classes of \( M \)-essentially bounded complex Borel functions on \( \mathbb{R}_+ \) equipped with the \( M \)-essential supremum norm (see [7], Appendix; see also [57, Section 12.20]). We customarily regard elements of \( L^\infty(M) \) as functions that are identified by the equality a.e. \( M \), the latter meaning “almost everywhere with respect to \( M \).” In particular, the norm on \( L^\infty(M) \) takes the form

\[
\| f \|_{L^\infty(M)} = \min \{ \alpha \in \mathbb{R}_+ : M(\{ x \in \mathbb{R}_+ : |f(x)| > \alpha \}) = 0 \}, \quad f \in L^\infty(M).
\]

The relationship between \( L^\infty(M) \) and the classical \( L^\infty(\mu) \) is explained below.

If \( \mu \) is a Borel measure on \( \mathbb{R}_+ \), then \( L^\infty(M) = L^\infty(\mu) \) if and only if \( M \) and \( \mu \) are mutually absolutely continuous; if this is the case, then

\[
\| f \|_{L^\infty(M)} = \| f \|_{L^\infty(\mu)} \quad \text{for every} \quad f \in L^\infty(M).
\]

As shown in Example 4.1.1 below, it may not be possible to find a Borel probability measure on \( \mathbb{R}_+ \) with respect to which a given semispectral measure is absolutely continuous.

**Example 4.1.1.** Let \( \Omega \) be any uncountable bounded subset of \( \mathbb{R}_+ \) and let \( E : \mathfrak{B}(\mathbb{R}_+) \to B(\mathcal{H}) \) be the spectral measure given by

\[
E(\Delta) = \bigoplus_{x \in \Omega} \chi_\Delta(x) I_{\mathcal{H}_x}, \quad \Delta \in \mathfrak{B}(\mathbb{R}_+),
\]

where each \( \mathcal{H}_x \) is a nonzero Hilbert space. Clearly, the following holds.

If \( \Delta \in \mathfrak{B}(\mathbb{R}_+) \), then \( E(\Delta) = 0 \) if and only if \( \Delta \cap \Omega = \emptyset \).

Suppose to the contrary that \( E \) is absolutely continuous with respect to a finite Borel measure \( \mu \) on \( \mathbb{R}_+ \). Then by (4.1.2), \( \mu(\{ x \}) > 0 \) for every \( x \in \Omega \), which is impossible because \( \mu \) is finite and \( \Omega \) is uncountable (see [6, Problem 12, p. 12]). Plainly, \( E \) is compactly supported and \( \text{supp}(E) = \bar{\Omega} \).

The situation described in Example 4.1.1 cannot happen when \( \mathcal{H} \) is separable. What is more, the following statement holds.

**Suppose \( \mathcal{H} \) is separable and \( M : \mathfrak{B}(\mathbb{R}_+) \to B(\mathcal{H}) \) is a nonzero semispectral measure. Then there exists a Borel probability measure \( \mu \) on \( \mathbb{R}_+ \) such that \( M \) and \( \mu \) are mutually absolutely continuous.**

(4.1.3)
To see this, take an orthonormal basis \( \{ e_j \}_{j \in J} \) of \( \mathcal{H} \), where \( J \) is a countable index set. Let \( \{ a_j \}_{j \in J} \) be any system of positive real numbers such that
\[
\sum_{j \in J} a_j (M(\mathbb{R}_+) e_j, e_j) = 1. \tag{4.1.4}
\]
(This is possible because \( M \neq 0 \).) Define the Borel measure \( \mu \) on \( \mathbb{R}_+ \) by
\[
\mu(\Delta) = \sum_{j \in J} a_j (M(\Delta) e_j, e_j), \quad \Delta \in \mathcal{B}(\mathbb{R}_+).
\]
By (4.1.4), \( \mu \) is a probability measure. If \( \Delta \in \mathcal{B}(\mathbb{R}_+) \) is such that \( \mu(\Delta) = 0 \), then
\[
0 = (M(\Delta) e_j, e_j) = \| M(\Delta) \|_{1/2}^2, \quad j \in J,
\]
which implies that \( M(\Delta) = 0 \). Thus \( E \) is absolutely continuous with respect to \( \mu \). That \( \mu \) is absolutely continuous with respect to \( M \) is immediate.

We now prove the following fact.

If \( M : \mathcal{B}(\mathbb{R}_+) \rightarrow \mathcal{B}(\mathcal{H}) \) is a nonzero compactly supported semispectral measure, then \( \| f \|_{L_\infty(M)} = \| f \|_{C(\Omega)} \) for every \( f \in L_\infty(M) \) such that \( f|_\Omega \in C(\Omega) \), where \( \Omega := \text{supp}(M) \).

Indeed, the inequality “\( \leq \)” is obvious. If \( \alpha \in \mathbb{R}_+ \) is such that
\[
M(\{ x \in \mathbb{R}_+ : |f(x)| > \alpha \}) = 0,
\]
then \( M(\{ x \in \Omega : |f(x)| > \alpha \}) = 0 \) and, because the set \( \{ x \in \Omega : |f(x)| > \alpha \} \) is open in \( \Omega \), we deduce that \( |f(x)| \leq \alpha \) for all \( x \in \Omega \), which after taking infimum over such \( \alpha \)’s yields the inequality “\( \geq \)”.

As a consequence of (4.1.5), we have
\[
\text{if } f, g \in L_\infty(M) \text{ are such that } f|_\Omega, g|_\Omega \in C(\Omega) \text{ and } f = g \text{ a.e. } [M], \text{ then } f|_\Omega = g|_\Omega.
\]

The above discussion shows that (still under the assumptions of (4.1.5)) the map which sends a function \( g \in C(\Omega) \) to the equivalence class of any of extensions of \( g \) to a complex Borel function on \( \mathbb{R}_+ \) is an isometry from \( C(\Omega) \) to \( L_\infty(M) \).

Therefore, \( C(\Omega) \) can be regarded as a closed vector subspace of \( L_\infty(M) \); this fact plays an important role in Theorem 4.1.2(v) below. As shown in (4.1.3) and (4.1.1), if \( \mathcal{H} \) is separable and \( M \neq 0 \), then \( L_\infty(M) = L_\infty(\mu) \) for some Borel probability measure on \( \mathbb{R}_+ \), so \( C(\Omega) \) is a separable closed vector subspace of \( L_\infty(\mu) \) (see (24 Theorem V.6.6)), while, in general, \( L_\infty(\mu) \) is not separable (see [77 Problem 2, p. 62]). As is easily seen, the above facts (except for separability of \( C(\Omega) \)) are true for regular Borel semispectral measures on topological Hausdorff spaces.

We are now ready to construct an \( L_\infty(M) \)-functional calculus that is built up on the basis of Agler’s hereditary functional calculus.

**Theorem 4.1.2.** Suppose that \( T \in \mathcal{B}(\mathcal{H}) \) is a CPD operator. Let \( M : \mathcal{B}(\mathbb{R}_+) \rightarrow \mathcal{B}(\mathcal{H}) \) be a compactly supported semispectral measure satisfying (5.2.11). Then the map \( \Lambda_T : L_\infty(M) \rightarrow \mathcal{B}(\mathcal{H}) \) given by
\[
\Lambda_T(f) = \int_{\mathbb{R}_+} f dM, \quad f \in L_\infty(M), \tag{4.1.6}
\]
is continuous and linear. It has the following properties:

(i) \( \Lambda_T(q) = ((X - 1)^2 q)(T) \) for every \( q \in \mathbb{C}[X] \),
(ii) $A_T$ is positive, i.e., $A_T(f) \geq 0$ whenever $f \in L^\infty(M)$ and $f \geq 0$ a.e. $[M]$,
(iii) there exist a Hilbert space $K$, $R \in \mathcal{B}(\mathcal{H}, K)$ and $S \in \mathcal{B}(K)_+$ such that
\(8.2.13\) holds and
\[ A_T(f) = R^*f(S)R, \quad f \in L^\infty(M), \] (4.1.7)
(iv) $\|A_T\| = \|\mathcal{B}_2(T)\|$
(v) if $M$ is nonzero and $\Omega := \text{supp}(M)$, then $\mathbb{C}[X]$ is dense in $C(\Omega)$ in the
$L^\infty(M)$-norm, $A_T|_{C(\Omega)}: C(\Omega) \to \mathcal{B}(\mathcal{H})$ is a unique continuous linear
map satisfying (i), $\|A_T|_{C(\Omega)}\| = \|A_T\|$ and
\[
\left\| \sum_{j=0}^n \alpha_j T^{*j}\mathcal{B}_2(T)T^j \right\| \leq \|\mathcal{B}_2(T)\| \sup_{x \in \Omega} \left\| \sum_{j=0}^n \alpha_j x^j \right\|, \quad \{\alpha_j\}_{j=0}^n \subseteq \mathbb{C}, \ n \in \mathbb{Z}_+ . \quad (4.1.8)
\]

**Proof.** Let $(\mathcal{K}, R, E)$ be as in Theorem 3.2.4. Since for every $\Delta \in \mathcal{B}(\mathbb{R}_+)$,
$E(\Delta) = 0$ if and only if $M(\Delta) = 0$ (see the proof of 37 Theorem 4.4), we get
$L^\infty(M) = L^\infty(E)$ and $\|f\|_{L^\infty(M)} = \|f\|_{L^\infty(E)}$ for every $f \in L^\infty(M)$. (4.1.9)

Set $S = \int_{\mathbb{R}_+} xE(dx)$. Applying (4.1.3) and the Stone-von Neumann functional
calculations, we deduce from (4.1.9) that (4.1.7) is valid and consequently
\[
\|A_T(f)\| \leq \|R\|^2 \|f(S)\| = \|R\|^2 \|f\|_{L^\infty(E)} = \|R\|^2 \|f\|_{L^\infty(M)}, \quad f \in L^\infty(M).
\]
Hence, $A_T$ is a continuous positive linear map such that
\[
\|A_T\| \leq \|R\|^2 . \quad (4.1.10)
\]
Applying (3.2.11) to $p = (X - 1)^2q$, we deduce that $A_T$ satisfies (i). Substituting
$q = f = X^0$ into (i) and (4.1.7), we infer from (3.2.2) that
\[
\mathcal{B}_2(T) = A_T(X^0) = R^*R, \quad (4.1.11)
\]
which together with (4.1.9) yields $\|A_T\| = \|R\|^2 = \|\mathcal{B}_2(T)\|$. Thus, in view of
Lemma 3.2.4 (i), (i)-(iv) hold.

It remains to prove (v). Assume that $M$ is nonzero. It follows from (4.1.5) and the Stone-Weierstrass theorem (or the classical Weierstrass theorem combined with Tietze extension theorem) that $\mathbb{C}[X]$ is dense in $C(\Omega)$ in the $L^\infty(M)$-norm and so
$A_T|_{C(\Omega)}: C(\Omega) \to \mathcal{B}(\mathcal{H})$ is a unique continuous linear map satisfying condition (i).
Since by (3.2.2) and (3.2.3),
\[
\sum_{j=0}^n \alpha_j T^{*j}\mathcal{B}_2(T)T^j = ((X - 1)^2q)(T) \overset{(i)}{=} A_T(q), \quad (4.1.12)
\]
where $q = \sum_{j=0}^n \alpha_j X^j$, we can easily deduce (4.1.3) from (iv) and (4.1.5). Using
(4.1.11) and (iv) again, we conclude that $\|A_T|_{C(\Omega)}\| = \|A_T\|$. This proves (v) and
thus completes the proof. \(\square\)

Before stating a corollary to Theorem 4.1.2, we recall that a monic polynomial
$p \in \mathbb{C}[X]$ of degree at least one takes the form (see 36 p. 252)
\[
p = (X - z_1) \cdots (X - z_n), \quad (4.1.13)
\]

\(\text{Since } A_T \text{ is a positive map on a commutative } C^*\text{-algebra } L^\infty(M), \text{ the Stinespring theorem}\)
implies that $A_T$ is completely positive (see 67 Theorem 4).
where \( z_1, \ldots, z_n \in \mathbb{C} \). What is more, \( p \) can be written as

\[
p = \sum_{j=0}^{n} (-1)^{n-j} s_{n-j}(z_1, \ldots, z_n) X^j,
\]

where \( s_0 = 1 \) and \( s_1, \ldots, s_n \) are the elementary symmetric functions in complex variables \( z_1, \ldots, z_n \) given by

\[
s_j(z_1, \ldots, z_n) = \sum_{1 \leq i_1 < \cdots < i_j \leq n} z_{i_1} \cdots z_{i_j} \text{ for } z_1, \ldots, z_n \in \mathbb{C} \text{ and } j = 1, \ldots, n.
\]

**Corollary 4.1.3.** Assume that \( T \in B(\mathcal{H}) \) is CPD and \( M \) and \( \Omega \) are as in Theorem 4.1.2(v). Then for every \( n \in \mathbb{N} \),

\[
\left\| \sum_{j=0}^{n} (-1)^{j} s_{n-j}(z) T^{*j} B_2(T) T^j \right\| \leq \| B_2(T) \| \sup_{x \in \Omega} \prod_{j=1}^{n} |x - z_j|,
\]

\[
z = (z_1, \ldots, z_n) \in \mathbb{C}^n,
\]

\[
\left\| \sum_{j=0}^{n} \binom{n}{j} (-1)^{j} z^{n-j} T^{*j} B_2(T) T^j \right\| \leq \| B_2(T) \| \left( \sup_{x \in \Omega} |x - z| \right)^n, \quad z \in \mathbb{C},
\]

\[
\| B_{n+2}(T) \| \leq \| B_2(T) \| \left( \sup_{x \in \Omega} |x - 1| \right)^n.
\]

**Proof.** Applying (4.1.8) to the polynomial (4.1.14) and using (4.1.13), we get (4.1.15). The estimate (4.1.16) is a direct consequence of (4.1.15). Finally, the estimate (4.1.17) follows from (4.1.16) applied to \( z = 1 \) and the identity

\[
B_{n+2}(T) = \sum_{j=0}^{n} \binom{n}{j} (-1)^{j} T^{*j} B_2(T) T^j, \quad n \in \mathbb{Z}_+.
\]

which can be proved straightforwardly by using Agler’s hereditary functional calculus (see (3.2.2) and (4.1.12)). The estimate (4.1.17) can also be inferred from identity (3.2.21) and statements (viii) and (ix) of [71 Theorem A.1]. \( \square \)

In the case of CPD operators, Lemma 3.2.1 takes the following form for \( q_0 = (X - 1)^2 \).

**Proposition 4.1.4.** Let \( T \in B(\mathcal{H}) \) be a CPD operator and let \( M \) be as in Theorem 4.1.2. Then the set

\[
\mathcal{J}_T = \{ q \in \mathbb{C}[X] : ((X - 1)^2 q)(T) = 0 \},
\]

is the ideal in \( \mathbb{C}[X] \) generated by the polynomial \( w_T \in \mathbb{C}[X] \) defined by

\[
w_T = \begin{cases} 0 & \text{if } \Omega \text{ is infinite,} \\
\prod_{u \in \Omega} (X - u) & \text{if } \Omega \text{ is finite and nonempty,} \\
X^0 & \text{if } \Omega = \emptyset, \text{ or equivalently if } T \text{ is a 2-isometry,} 
\end{cases}
\]

where \( \Omega := \text{supp}(M) \). Moreover, if \( \Omega = \{ u_1, \ldots, u_n \} \), where \( n \in \mathbb{N} \) and \( u_1, \ldots, u_n \) are distinct, then the following identity holds

\[
\sum_{j=0}^{n} (-1)^{j} s_{n-j}(u_1, \ldots, u_n) T^{*j} B_2(T) T^j = 0.
\]
\textbf{Proof.} First note that by (4.1.6) and Theorem 4.1.2(i),
\[
\int_{\Omega} p(x)M(dx) = A_T(p) = ((X - 1)^2 p)(T), \quad p \in \mathbb{C}[X].
\] (4.1.18)
It follows from Lemma 3.2.1 that \( \mathcal{I}_T \) is an ideal in \( \mathbb{C}[X] \) generated by some polynomial \( w \in \mathbb{C}[X] \). Applying (4.1.18) to \( p = w^* w \), we see that \( \int_{\Omega} |w(x)|^2 M(dx) = 0 \), and so, by (4.1.18), we have
\[
w|_{\Omega} = 0.
\] (4.1.19)
We now consider three cases.

\textbf{Case 1} The set \( \Omega \) is infinite.
Since nonzero polynomials may have only finite number of roots, we deduce from (4.1.19) that \( w = 0 \).

\textbf{Case 2} The set \( \Omega \) is empty (or equivalently, by Proposition 4.3.1, \( T \) is a 2-isometry).
Then, in view of (4.1.18), \( \mathcal{I}_T = \mathbb{C}[X] \) and so \( X^0 \) generates the ideal \( \mathcal{I}_T \).

\textbf{Case 3} The set \( \Omega \) is finite and nonempty.
Set \( w_T = \prod_{u \in \Omega} (X - u) \). Clearly, by (4.1.18), \( w_T \in \mathcal{I}_T \). It follows from the fundamental theorem of algebra (see [36, Theorem V.3.19]) and (4.1.19) that the polynomial \( w_T \) divides \( w \). Since \( w \) generates the ideal \( \mathcal{I}_T \), \( w \) divides \( w_T \) and so \( w_T = \alpha w \), where \( \alpha \in \mathbb{C} \setminus \{0\} \). This means that \( w_T \) generates \( \mathcal{I}_T \).

The “moreover” part is a direct consequence of (4.1.15). \( \square \)

Theorem 4.1.2(i), Proposition 4.1.4 and (4.1.5) lead to the following corollary.

\textbf{Corollary 4.1.5.} Let \( T \in B(\mathcal{H}) \) be a CPD operator and let \( M \) and \( \Lambda_T \) be as in Theorem 4.1.2. Then the following assertions hold:

\begin{enumerate}
\item the map \( \mathbb{C}[X] \ni q \mapsto ((X - 1)^2 q)(T) \in B(\mathcal{H}) \) is injective if and only if \( \text{supp}(M) \) is infinite,
\item if \( q \in \mathbb{C}[X] \) is such that \( \Lambda_T(q) = 0 \), then \( q = 0 \) a.e. \( [M] \), which means that the restriction of \( \Lambda_T \) to equivalence classes of polynomials is injective.
\end{enumerate}

\section{4.2. Analytic implementations.}

In Subsection 4.1 we were discussing the action of the functional calculus established in Theorem 4.1.2 on polynomials. In this subsection we concentrate on showing how this functional calculus may work in the case of real analytic functions.

Let \( T, M \) and \( \Lambda_T \) be as in Theorem 4.1.2. Assume that \( M \) is nonzero. Set \( \Omega = \text{supp}(M) \). Suppose that \( \sum_{n=0}^{\infty} a_n x^n \) is a power series in the real variable \( x \) with complex coefficients \( a_n \) such that
\[
\limsup_{n \to \infty} |a_n|^{1/n} < \frac{1}{\sup \Omega} \quad \text{(with } \frac{1}{0} = \infty \text{)}.
\] (4.2.1)
Then the series \( \sum_{n=0}^{\infty} a_n x^n \) is uniformly convergent on \( [0, \sup \Omega] \) to a continuous function on \( \Omega \), say \( f \). Hence by Theorem 4.1.2 we have
\[
\Lambda_T(f) = \sum_{n=0}^{\infty} a_n \Lambda_T(X^n) = \sum_{n=0}^{\infty} a_n T^{*n} \mathcal{B}(T) T^n.
\] (4.2.2)
Let \( (\mathcal{K}, R, S) \) be as Theorem 4.1.2(iii). In particular, (3.2.13) holds and
\[
\Lambda_T(f) = R^* f(S) R.
\]
Combined with (4.2.2), this implies that
\[ \sum_{n=0}^{\infty} a_n T^n B_2(T) T^n = R^* f(S) R. \]  
(4.2.3)

It follows from Theorem 4.1.2(i) and (4.1.7) that (3.3.2) holds. According to the proof of the implication (ii')\(\Rightarrow\)(i) of Theorem 3.3.1, (3.2.12) holds, so by Theorem 3.2.5(c), \(\sigma(S) = \Omega\) and \(\|S\| = \sup \Omega\). Since the map \(C(\sigma(S)) \ni g \mapsto g(S) \in B(\mathcal{H})\) is a unital isometric \(\ast\)-homomorphism (see [24, Theorem VIII.2.6]), we get (see also (4.1.5) and (4.1.9))
\[ f(S) = \sum_{n=0}^{\infty} a_n S^n. \]  
(4.2.4)

Concerning (4.2.2), note that
\[ \sum_{n=0}^{\infty} a_n T^n B_2(T) T^n = \sum_{n=0}^{\infty} a_n \nabla^n T(\mathcal{B}_{2}(T)), \]
where \(\nabla_T\) is as in (3.2.4). Since \(r(\nabla_T) = r(T)^2\) (a general fact which follows from Gelfand’s formula for spectral radius), we deduce that the series \(\sum_{n=0}^{\infty} a_n \nabla^n_T\) converges in \(B(B(\mathcal{H}))\) if \(\limsup_{n \to \infty} |a_n|^{1/n} < \frac{1}{r(T)^2}\). The last inequality is in general stronger than (4.2.1) because by Theorem 3.2.5(c),
\[ \frac{1}{r(T)^2} \leq \frac{1}{\sup \Omega}. \]

Let us now discuss two important cases. We begin with \(a_n = z^n\) for every \(n \in \mathbb{Z}_+\), where \(z \in \mathbb{C}\). Then the above considerations lead to
\[ \sum_{n=0}^{\infty} z^n T^n B_2(T) T^n \overset{(i)}{=} R^* (I - zS)^{-1} R, \quad z \in \mathbb{C}, \ |z| < \frac{1}{\sup \Omega}. \]  
(4.2.5)

where \((i)\) follows from (4.2.1), (4.2.2), (4.2.3) and the Carl Neumann theorem (see [57, Theorem 10.7]). In particular, the following estimate holds (see (4.1.11))
\[ \left\| \sum_{n=0}^{\infty} z^n T^n B_2(T) T^n \right\| \leq |z|^{-1} \frac{\|B_2(T)\|}{\text{dist}(z^{-1}, \Omega)}, \quad z \in \mathbb{C}, \ 0 < |z| < \frac{1}{\sup \Omega}. \]

In view of the previous paragraph and (4.2.5), we have
\[ (I - z \nabla_T)^{-1}(B_2(T)) = R^* (I - zS)^{-1} R, \quad z \in \mathbb{C}, \ |z| < \frac{1}{r(T)^2}, \]
where \(I\) is the identity map on \(B(\mathcal{H})\). Note also that if (4.2.5) holds, then by differentiating the operator valued functions appearing on both sides of the equality in (4.2.5) \(n\) times at 0, we obtain (3.3.2), which by Theorem 3.3.1 implies that \(T\) is CPD.

It is a matter of routine to show that for an operator \(T \in B(\mathcal{H})\) the operator valued function appearing on the left-hand side of (4.2.5), call it \(\Psi\), is uniquely determined by the requirement that it be an analytic \(B(\mathcal{H})\)-valued function defined on an open disk \(D_r = \{ z \in \mathbb{C} : |z| < r \}\) for some \(r \in (0, \infty)\) such that
\[ \Psi(z) = B_2(T) + zT^* \Psi(z) T, \quad z \in D_r. \]  
(4.2.6)
In other words, we have proved that $T$ is CPD if and only if there exists $r \in (0, \infty)$ such that the analytic function $\Psi$ associated with $T$ via (4.2.6) satisfies the following equation

$$\Psi(z) = R^* (I - zS)^{-1} R, \quad z \in \mathbb{D}_r,$$

for some triplet $(\mathcal{K}, R, S)$ consisting of a Hilbert space $\mathcal{K}$, an operator $R \in B(\mathcal{H}, \mathcal{K})$ and a positive operator $S \in B(\mathcal{K})$ such that $r\|S\| \leq 1$.

In turn, if $a_n = z^n$ for every $n \in \mathbb{Z}^+$, where $z \in \mathbb{C}$, then

$$\sum_{n=0}^{\infty} \frac{z^n}{n!} t^n \mathcal{B}_2(T) T^n \overset{(1)}{=} R^* e^{iS} R, \quad z \in \mathbb{C},$$

where (1) is a consequence of (4.2.1), (4.2.3) and (4.2.4), or equivalently that $e^{i\nabla T} (\mathcal{B}_2(T)) = R^* e^{iS} R, \quad z \in \mathbb{C}$.

In particular, we have

$$\sum_{n=0}^{\infty} \frac{i^n z^n}{n!} T^n \mathcal{B}_2(T) T^n = R^* e^{iz} R, \quad x \in \mathbb{R}. \tag{4.2.7}$$

Since $\{e^{ixS}\}_{x \in \mathbb{R}}$ is a uniformly continuous group of unitary operators, we obtain

$$\left\| \sum_{n=0}^{\infty} \frac{i^n z^n}{n!} T^n \mathcal{B}_2(T) T^n \right\| \overset{(4.2.7)}{\leq} \|R\|^2 \overset{(4.1.11)}{=} \|\mathcal{B}_2(T)\|, \quad x \in \mathbb{R},$$

or equivalently

$$\|e^{ix\nabla T} (\mathcal{B}_2(T))\| \leq \|\mathcal{B}_2(T)\|, \quad x \in \mathbb{R}.$$  

As in the previous case, we observe that if (4.2.7) holds, then by differentiating the operator valued functions appearing on both sides of the equality in (4.2.7) $n$ times at 0, we obtain (3.3.4), which as we know implies that $T$ is CPD.

### 4.3. Small supports.

In view of Subsection 4.2, the natural question arises of when the closed support of the semispectral measure $M$ associated with a given CPD operator $T$ via Theorem 3.2.5(ii) is equal to $\emptyset$, $\{0\}$ or $\{1\}$. Surprisingly, the answers to this seemingly simple question that are given in Propositions 4.3.1 and 4.3.5 (see also Corollary 3.4.3) lead to three relatively broad classes of operators, including 2- and 3-isometries. The fact that 3-isometries are CPD was already proved in [21] Proposition 2.7. In Proposition 4.3.1 below, $F$ and $M$ denote the semispectral measures appearing in Theorems 3.1.1(ii) and 3.2.5(ii), respectively.

**Proposition 4.3.1.** Let $T \in B(\mathcal{H})$. Then

(i) if $T$ is CPD, then $M = 0$ if and only if $A_T = 0$, or equivalently if and only if $T$ is a 2-isometry,

(ii) the following conditions are equivalent:

(a) $T$ is CPD and $F = 0$,
(b) $T$ is CPD and $\text{supp}(M) \subseteq \{1\}$,
(c) $T^n T^n = I - n \mathcal{B}_1(T) + \frac{n(n-1)}{2} \mathcal{B}_2(T)$ for all $n \in \mathbb{Z}_+$,
(d) $T$ is a 3-isometry.

Moreover, if an $m$-isometry is CPD, then it is a 3-isometry.
Proof. (i) The first equivalence in (i) follows from (4.3.1) by considering characteristic functions, while the second is a direct consequence of Theorem 4.1.2(iv).

(ii) The implication (a)⇒(c) follows from 3.1.2. Straightforward computations shows that the implication (c)⇒(d) holds. If (d) holds, then for all \( n \geq 2, \)

\[
T^{*n}T^n = ((X - 1) + 1)^n(T) = \sum_{j=0}^{n} \binom{n}{j}(X - 1)^j(T) = \sum_{j=0}^{2} \binom{n}{j}(-1)^jB_j(T),
\]

where \((*)\) follows from Remark 3.2.2. This yields (c). If (c) holds, then the right-hand side of the equality in (c) is nonnegative for all \( n \in \mathbb{Z}_+, \) which implies that \( B_2(T) \geq 0. \) Clearly (3.1.2) holds with \( B_2(T) = 0, \) so by Theorem 3.1.1, (a) holds. By (3.2.14), (a) and (b) are equivalent.

Remark 4.3.3. a) First, note that each 2-isometry \( T \in B(H) \) satisfies condition (i) of Theorem 3.1.6. Indeed, by Proposition 4.3.1(ii), any 2-hyperexpansive operator \( T \) is expansive, i.e., \( \mathcal{B}_1(T) \leq 0. \) This and Remark 3.2.2 implies that any 2-isometry is completely hyperexpansive. In view of Proposition 4.3.1(ii), (a) is valid.

(b) Suppose that \( T \in B(H) \) is a strict 3-isometry. Then, by Proposition 4.3.1, \( T \) is CPD. However, \( T \) does not satisfy condition 3.1.17. In fact, we can show more. By 3.2.21 and Proposition 4.3.1

\[
\mathcal{B}_2(T) \geq 0 \text{ and } T^{*n+1}T^{n+1} - T^{*n}T^n = -\mathcal{B}_1(T) + n\mathcal{B}_2(T) \text{ for all } n \in \mathbb{Z}_+.
\]

This yields

\[
\sup_{n \in \mathbb{Z}_+} (\|T^{n+1}h\|^2 - \|T^nh\|^2) = \begin{cases} -\langle \mathcal{B}_1(T)h, h \rangle & \text{if } h \in \mathcal{N}(\mathcal{B}_2(T)), \\ \infty & \text{if } h \in H \setminus \mathcal{N}(\mathcal{B}_2(T)). \end{cases}
\]

(4.3.1)

Since \( T \) is not a 2-isometry, \( \mathcal{N}(\mathcal{B}_2(T)) \neq H, \) so \( T \) does not satisfy 3.1.17.

c) It turns out that there are strict 3-isometries \( T \) such that \( \mathcal{N}(\mathcal{B}_2(T)) = \{0\}. \) Indeed, let \( W \) be the unilateral weighted shift on \( \ell^2 \) with weights \( \left\{ \frac{n+3}{\sqrt{n+1}} \right\}_{n=0}^{\infty}. \) It
follows from [7] Proposition 8] and [3] Lemma 1.21] that \( W \) is a strict 3-isometry for which \( r(W) = 1 \). We claim that

\[
\mathcal{N}(\mathcal{B}_2(W)) = \{0\}. \tag{4.3.2}
\]

Indeed, it is a matter of routine to verify that \( \mathcal{B}_2(W) \) is the diagonal operator (with respect to the the standard orthonormal basis of \( \ell^2 \)) with the diagonal \( \{\|W^n\|\}_{n=0}^{\infty} \), which yields \( (4.3.2) \). In particular, \( (4.3.2) \) implies that \( W \) is a strict 3-isometry and, by \( (4.3.1) \),

\[
\sup_{n \in \mathbb{Z}_+} (\|W^n+1\|^2 - \|W^n\|^2) = \infty, \quad h \in \ell^2 \setminus \{0\}.
\]

d) Let \( W \) be the unilateral weighted shift as in c). Then \( W \) is a 3-isometry and, by Proposition \( (4.3.1) \) we have

\[
W^{+2}W^n = I + nB + n^2C, \quad n \in \mathbb{Z}_+ \tag{4.3.3}
\]

where \( B = -\langle \mathcal{B}_1(W) + \frac{1}{2} \mathcal{B}_2(W) \rangle \) and \( C = \frac{1}{2} \mathcal{B}_2(W) \). We easily check that \( B \) and \( C \) are diagonal operators with diagonals \( \{\frac{\alpha}{(n+1)(n+2)}\}_{n=0}^{\infty} \) and \( \{\frac{1}{(n+1)(n+2)}\}_{n=0}^{\infty} \), respectively, so \( B \geq 0, \ C \geq 0 \) and \( \mathcal{N}(B) = \mathcal{N}(C) = \{0\} \). By \( (4.3.3) \), \( \|W^n\| = \alpha \cdot n \) for all \( n \in \mathbb{N} \), where \( \alpha = \sqrt{1 + \|B\|^2 + \|C\|^2} \). We show that there are no \( \varepsilon \in (0, \infty) \) and \( \beta \in \mathbb{R}_+ \) such that \( \|W^n\| \leq \beta \cdot n^{1-\varepsilon} \) for all \( n \in \mathbb{N} \). Indeed, otherwise we have

\[
\langle Ch, h \rangle \leq \frac{(I + nB + n^2C)h, h}{n^2} \leq \frac{\beta^2 \|h\|^2}{n^{2\varepsilon}}, \quad n \in \mathbb{N}, \ h \in \ell^2,
\]

which contradicts \( \mathcal{N}(C) = \{0\} \).

We now turn to the case when \( \text{supp}(M) = \{0\} \). We first prove a result that is of some independent interest (see [26] Proposition 8] for the case of unilateral weighted shifts).

**Lemma 4.3.4.** Suppose that the restriction of an operator \( T \in \mathcal{B}(\mathcal{H}) \) to \( \mathcal{N}(T) \) is subnormal. Then \( T \) is subnormal if and only if

\[
\int_{\mathbb{R}_+} \frac{1}{t} d\mu_h(t) \leq 1 \text{ for all } h \in \mathcal{H} \text{ such that } \|h\| = 1, \tag{4.3.4}
\]

where \( \mu_h \) stands for the (unique) representing measure of the Stieltjes moment sequence \( \{\|T^{n+1}h\|^2\}_{n=0}^{\infty} \).

**Proof.** Applying Theorem \( (1.1.1) \) to \( T|_{\mathcal{N}(T)} \) and using Lemma \( (2.1.2) \) we see that the sequence \( \{\|T^{n+1}h\|^2\}_{n=0}^{\infty} \) is a determinate Stieltjes moment sequence for every \( h \in \mathcal{H} \). By Theorem \( (1.1.1) \) \( T \) is subnormal if and only if for every \( h \in \mathcal{H} \) for which \( \|h\| = 1 \), the sequence \( \{\|T^n h\|^2\}_{n=0}^{\infty} \) is a Stieltjes moment sequence, or equivalently, by \( (39) \) Lemma 6.1.2], if and only if condition \( (4.3.4) \) holds. \( \Box \)

**Proposition 4.3.5.** For \( T \in \mathcal{B}(\mathcal{H}) \), the following conditions are equivalent:

(i) \( T \) is CPD and \( \text{supp}(M) = \{0\} \), where \( M \) is as in Theorem \( (4.1.2) \).

(ii) \( \mathcal{B}_2(T) \neq 0, \mathcal{B}_2(T) \geq 0 \) and \( \mathcal{B}_2(T) \neq 0 \),

(iii) \( T^{+n}T^n = I - \mathcal{B}_2(T) + n(\mathcal{B}_2(T) - \mathcal{B}_1(T)) \) for all \( n \in \mathbb{N} \), \( \mathcal{B}_2(T) \geq 0 \) and \( \mathcal{B}_2(T) \neq 0 \).

(iv) \( T \) satisfies Theorem \( (3.1.0) \) ii] with \( \text{supp}(F) = \{0\} \).

Moreover, if (i) holds, then

(a) \( r(T) = 1 \) whenever \( T \neq 0 \),
(b) \( T \) is subnormal if and only if \( \mathcal{B}_1(T)T = 0 \) and \( \|T\| \leq 1 \); if this is the case, then \( \|T\| = 1 \) provided \( T \neq 0 \).

**Proof.** (i)⇒(ii) Substituting \( q = X \) into (4.1.12) yields \( T^*\mathcal{B}_2(T)T = 0 \). By Corollary 3.2.7, \( \mathcal{B}_2(T) = M(\mathbb{R}_+) \geq 0 \). Putting this all together implies (ii).

(ii)⇒(i) Note that the set function \( M: \mathfrak{B}(\mathbb{R}_+) \rightarrow \mathcal{B}(\mathcal{H}) \) defined by \( M(\Delta) = \chi_\Delta(0)\mathcal{B}_2(T) \) for \( \Delta \in \mathfrak{B}(\mathbb{R}_+) \) is a semispectral measure such that \( \text{supp}(M) = \{0\} \). Clearly (3.3.3) holds, so by Theorem 3.3.1, \( T \) is CPD and (3.2.11) is valid.

(i)⇒(iii) Let \( (B, C, F) \) be the representing triplet of \( T \). According to Theorem 3.2.5(b), \( F = M, C = 0 \) and \( B = -\mathcal{B}_1(T) \), so by 3.1.2, and Corollary 3.2.7,

\[
T^*n^n = I - n\mathcal{B}_1(T) + Q_n(0)\mathcal{B}_2(T)
\]

\[
= I - n\mathcal{B}_1(T) + (n - 1)\mathcal{B}_2(T), \quad n \in \mathbb{N}.
\]

This together with the implication (i)⇒(ii) gives (iii).

(iii)⇒(iv) As above, the set function \( F: \mathfrak{S}(\mathbb{R}_+) \rightarrow \mathcal{B}(\mathcal{H}) \) defined by \( F(\Delta) = \chi_\Delta(0)\mathcal{B}_2(T) \) for \( \Delta \in \mathfrak{S}(\mathbb{R}_+) \) is a semispectral measure for which \( \text{supp}(F) = \{0\} \).

Set \( D = \mathcal{B}_2(T) - \mathcal{B}_1(T) \). It is easily seen that \( D \) and \( F \) satisfy Theorem 3.1.6(ii).

(iv)⇒(i) Apply Theorems 3.1.6 and 3.2.5(b).

We now prove the “moreover” part.

(a) If \( D \neq 0 \), then by Theorems 3.1.6 and 3.1.8, \( r(T) = 1 \). Suppose that \( D = 0 \).

Then by (iii), \( T^n = I - n\mathcal{B}_1(T) \) for all \( n \in \mathbb{N} \). This together with \( T \neq 0 \) implies that \( I - \mathcal{B}_2(T) \neq 0 \), so by Gelfand’s formula for spectral radius \( r(T) = 1 \).

(b) Suppose first that \( T \) is subnormal. It follows from (iii) that for every \( h \in H \),

\[
\|T^n h\|^2 = \langle (I - \mathcal{B}_2(T))h, h \rangle + n(\langle \mathcal{B}_2(T) - \mathcal{B}_1(T) h, h \rangle), \quad n \in \mathbb{N}.
\]

By Theorem 4.3.5, \( \{\|T^{n+1} h\|^2\}_{n=0}^\infty \) is a Stieltjes moment sequence for every \( h \in H \). Combined with 4.3.5 and 20, Lemma 4.7, this implies that

\[
\langle (\mathcal{B}_2(T) - \mathcal{B}_1(T))h, h \rangle = 0, \quad h \in H,
\]

or equivalently that \( T^*\mathcal{B}_1(T)T = 0 \). By (a) and (4.3.2), \( T \) is a contraction (in fact, \( \|T\| = 1 \) if \( T \neq 0 \), so \( \mathcal{B}_1(T) \geq 0 \) and consequently \( \mathcal{B}_1(T)T = 0 \).

In turn, if \( \|T\| \leq 1 \) and \( \mathcal{B}_1(T)T = 0 \), then \( T \) is a contraction whose restriction to \( \mathcal{H}(T) \) is an isometry, so an application of Lemma 4.3.4 with \( \mu_n := \|T h\|^2\delta_1 \) shows that \( T \) is subnormal. This completes the proof.

Now we give an example of an operator satisfying condition (i) of Proposition 4.3.3. In particular, we show that the class of operators satisfying this condition can contain both (non-isometric) subnormal and non-subnormal operators.

**Example 4.3.6.** Fix real numbers \( a \in (0, \infty) \) and \( b \in [1, \infty) \) such that

\[
\theta := 1 - 2a + ab > 0.
\]

Define the sequence \( \{\lambda_n\}_{n=0}^\infty \subseteq (0, \infty) \) by

\[
\lambda_n = \begin{cases} \sqrt{a} & \text{if } n = 0, \\
\sqrt{1+n(b-1)} & \text{if } n \geq 1.
\end{cases}
\]

Let \( W_{a,b} \) be the unilateral weighted shift on \( \ell^2 \) with weights \( \{\lambda_n\}_{n=0}^\infty \). It follows from [43] Lemma 6.1 & Proposition 6.2(iii) that \( W_{a,b} \in \mathcal{B}(\ell^2) \) and

\[
\|W_{a,b}\|^2 = \max \{a, b\}.
\]
One can also verify that \( \mathcal{B}_2(W_{a,b}) \) is the diagonal operator (with respect to the standard orthonormal basis of \( l^2 \)) with the diagonal \((\theta, 0, 0, \ldots)\). This together with \((4.3.6)\) implies that \( \mathcal{B}_2(W_{a,b})W_{a,b} = 0 \), \( \mathcal{B}_2(W_{a,b}) \geq 0 \) and \( \| \mathcal{B}_2(W_{a,b}) \| = 0 > 0 \). In view of Proposition \(4.3.5\), the operator \( W_{a,b} \) satisfies condition (i) of this proposition. From Propositions \(4.3.1\) and \(4.3.5\) it follows that \( W_{a,b} \) is a CPD operator which is not \( m \)-isometric for any \( m \in \mathbb{N} \). If \( a > 1 \), we see that \( \| W_a \| = \sqrt{a} > 1 \) and \( \| \mathcal{B}_2(W_a) \| = (a - 1)^2 \), where \( W_a := W_{a,a} \). Since, by Proposition \(4.3.5(a)\), \( r(W_a) = 1 \) for every \( a \in (1, \infty) \), we deduce that

\[
W_a \text{ is not normaloid for all } a > 1 \text{ and } \lim_{a \to \infty} \| W_a \| = \lim_{a \to \infty} \| \mathcal{B}_2(W_a) \| = \infty.
\]

In turn, if \( a \in (0, 1) \) and \( b = 1 \), then one can verify that \( \mathcal{B}_1(W_{a,1})W_{a,1} = 0 \) and by \((4.3.7)\), \( \| W_{a,1} \| = 1 \), so by Proposition \(4.3.5\) the operator \( W_{a,1} \) is subnormal and \( r(W_{a,1}) = 1 \). \( \diamondsuit \)

We conclude this subsection with a remark related to Proposition \(4.3.5\) and Example \(4.3.6\).

**Remark 4.3.7.** Suppose that \( T \in \mathcal{B}(\mathcal{H}) \) is nonzero and satisfies condition (i) of Proposition \(4.3.5\) (the zero operator on nonzero \( \mathcal{H} \) does satisfy (i)). By Proposition \(4.3.1\) \( T \) is not an \( m \)-isometry for any \( m \in \mathbb{N} \). According to condition (iii) of Proposition \(4.3.5\), \( T^*T^n \) is a polynomial in \( n \) if \( n \) varies over \( \mathbb{N} \) however not when \( n \) varies over \( \mathbb{Z}_+ \). Indeed, otherwise, since a nonzero polynomial may have only finite number of roots, we deduce from (iii) that \( I = I - \mathcal{B}_2(T) \), which contradicts \( \mathcal{B}_2(T) \neq 0 \). In other words, in view of \(3\) p. 389 (see also \(11\) Corollary 3.5), the requirement that \( T^*T^n \) be a polynomial in \( n \) if \( n \) varies over \( \mathbb{N} \) is not enough for \( T \) to be an \( m \)-isometry no matter what is \( m \). Finally note that \( T \) falls under Case 3 of the proof of Theorem \(3.2.5(c)\) and the discussion performed in Remark \(3.2.8\)). Indeed, by Theorem \(3.2.5(b)\), Proposition \(4.3.5(a)\) and Corollary \(3.2.7\) we see that \( B = -\mathcal{B}_1(T), \ C = 0, \ F = M, \vartheta := \sup \text{supp}(F) = 0, \ r(T) = 1 \) and

\[
D := B + \int_{\mathbb{R}_+} \frac{1}{1 - x} F(dx) = \mathcal{B}_2(T) - \mathcal{B}_1(T).
\]

Moreover, in view of Example \(4.3.6\) both cases \( D = 0 \) and \( D \neq 0 \) can appear. \( \diamondsuit \)

**Acknowledgement.** The authors would like to express their deepest thanks to the anonymous reviewer for reading the article carefully and catching any ambiguities, as well as for suggestions and questions that made the article more readable and reader friendly. A part of this paper was written while the first and the third author visited Kyungpook National University during the autumn of 2019. They wish to thank the faculty and the administration of this unit for their warm hospitality.

**References**

[1] J. Agler, Hypercontractions and subnormality, *J. Operator Theory* **13** (1985), 203-217.

[2] J. Agler, A disconjugacy theorem for Toeplitz operators, *Amer. J. Math.* **112** (1990), 1-14.

[3] J. Agler, M. Stankus, \( m \)-isometric transformations of Hilbert spaces, I, *Integr. Equ. Oper. Theory* **21** (1995), 383-429.

[4] J. Agler, M. Stankus, \( m \)-isometric transformations of Hilbert spaces, II, *Integr. Equ. Oper. Theory* **23** (1995), 1-48.

[5] J. Agler, M. Stankus, \( m \)-isometric transformations of Hilbert spaces, III, *Integr. Equ. Oper. Theory* **24** (1996), 379-421.

[6] R. B. Ash, *Probability and measure theory*, Harcourt/Academic Press, Burlington, 2000.
[7] A. Athavale, Some operator theoretic calculus for positive definite kernels, *Proc. Amer. Math. Soc.* **112** (1991), 701-708.
[8] A. Athavale, On completely hyperexpansive operators, *Proc. Amer. Math. Soc.* **124** (1996), 3745-3752.
[9] A. Athavale, The complete hyperexpansivity analog of the Embry conditions, *Studia Math.* **154** (2003), 233-242.
[10] C. Berg, J. P. R. Christensen, P. Ressel, *Harmonic Analysis on Semigroups*, Springer-Verlag, Berlin 1984.
[11] T. Bermúdez, A. Martinón, J. A. Noda, An isometry plus a nilpotent operator is an m-isometry. Applications, *J. Math. Anal. Appl.* **407** (2013), 505-512.
[12] M. Sh. Birman, M. Z. Solomjak, *Spectral theory of selfadjoint operators in Hilbert space*, D. Reidel Publishing Co., Dordrecht, 1987.
[13] T. M. Bisgaard, Positive definite operator sequences, *Proc. Amer. Math. Soc.* **121** (1994), 1185-1191.
[14] T. M. Bisgaard, Z. Sasvári, *Characteristic functions and moment sequences. Positive definiteness in probability*, Nova Science Publishers, Inc., Huntington, NY, 2000.
[15] E. Bishop, Spectral theory for operators on a Banach space, *Trans. Amer. Math. Soc.* **86** (1957), 414-445.
[16] F. Botelho, J. Jamison, Isometric properties of elementary operators, *Linear Algebra Appl.* **432** (2010), 357-365.
[17] P. Budzyński, Z. J. Jabłoński, I. B. Jung, J. Stochel, Unbounded subnormal composition operators in $L^2$-spaces, *J. Funct. Anal.* **269** (2015), 2110-2164.
[18] P. Budzyński, Z. J. Jabłoński, I. B. Jung, J. Stochel, Subnormality of unbounded composition operators over one-circuit directed graph: exotic examples, *Adv. Math.* **310** (2017), 484-556.
[19] P. Budzyński, Z. J. Jabłoński, I. B. Jung, J. Stochel, Unbounded weighted composition operators in $L^2$-spaces, *Lect. Notes Math.*, Volume 2209, Springer 2018.
[20] S. Chavan, Z. J. Jabłoński, I. B. Jung, J. Stochel, Taylor spectrum approach to Brownian-type operators with quasinormal entry, *Ann. Mat. Pur. Appl.* **200** (2021), 881-922.
[21] S. Chavan, V. M. Sholapurkar, Complete monotone functions of finite order and Agler’s conditions, *Studia Math.* **226** (2015), 229-258.
[22] S. Chavan, V. M. Sholapurkar, Completely hyperexpansive tuples of finite order, *J. Math. Anal. Appl.* **447** (2017), 1009-1026.
[23] D. Cichoń, Jan Stochel, Subnormality, analyticity and perturbations, *Rocky Mountain J. Math.* **37** (2007), 1831-1869.
[24] J. B. Conway, *A course in functional analysis*, Graduate Texts in Mathematics **96**, Springer-Verlag, New York, 1990.
[25] J. B. Conway, *The theory of subnormal operators*, Mathematical Surveys and Monographs, **36**, American Mathematical Society, Providence, RI, 1991.
[26] R. E. Curto, Quadratically hyponormal weighted shifts, *Integr. Equ. Oper. Theory* **13** (1990), 49-66.
[27] R. Curto and M. Putinar, Nearly subnormal operators and moment problems, *J. Funct. Anal.* **115** (1993), 480-497.
[28] J. Daneš, On local spectral radius, *Časopis Pěst. Mat.* **112** (1987), 177-187.
[29] M. R. Embry, A generalization of the Halmos-Bram criterion for subnormality, *Acta Sci. Math. (Szeged)* **35** (1973), 61-64.
[30] G. Exner, I. B. Jung, C. Li, On $k$-hyperexpansive operators, *J. Math. Anal. Appl.* **323** (2006), 569-582.
[31] T. Furuta, Invitation to linear operators, Taylor & Francis, Ltd., London, 2001.
[32] C. Gu, On $(m, p)$-expansive and $(m, p)$-contractive operators on Hilbert and Banach spaces, *J. Math. Anal. Appl.* **426** (2015), 893-916.
[33] P. Halmos, Normal dilations and extensions of operators, *Summa Bras. Math.* **2** (1950), 124-134.
[34] P. R. Halmos, *A Hilbert space problem book*, Springer-Verlag, New York Inc. 1982.
[35] R. A. Horn, C. R. Johnson, *Matrix analysis*, Cambridge University Press, Cambridge, 1985.
[36] T. W. Hungerford, *Algebra*, Graduate Texts in Mathematics 73, Springer-Verlag, New York, 1974.
[37] Z. Jabłoński, Complete hyperexpansivity, subnormality and inverted boundedness conditions, *Integr. Equ. Oper. Theory* **44** (2002), 316-336.
[38] Z. J. Jabłoński, I. B. Jung, E. Y. Lee, J. Stochel, Conditionally positive definite weighted shifts, preprint, 28 pp., arXiv:2106.03222.
[39] Z. J. Jabłoński, I. B. Jung, J. Stochel, Weighted shifts on directed trees, Memoirs of the AMS 216 (2012).
[40] Z. J. Jabłoński, I. B. Jung, J. Stochel, A non-hyponormal operator generating Stieltjes moment sequences, J. Funct. Anal. 262 (2012), 3946-3980.
[41] Z. J. Jabłoński, I. B. Jung, J. Stochel, m-Isometric operators and their local properties, Linear Algebra Appl. 596 (2020), 49-70.
[42] Z. J. Jabłoński, I. B. Jung, J. Stochel, Conditionally positive definite algebraic operators, in preparation.
[43] Z. Jabłoński, J. Stochel, Unbounded 2-hyperexpansive operators, Proc. Edin. Math. Soc. 44 (1991), 613-629.
[44] I. Jung, J. Stochel, Subnormal operators whose adjoints have rich point spectrum, J. Funct. Anal. 255 (2008), 1797-1816.
[45] K. Parthasarathy, K. Schmidt, Positive definite kernels, continuous tensor products, and central limit theorems of probability theory, Lecture Notes in Mathematics, Vol. 272, Springer-Verlag, Berlin-New York, 1972.
[46] W. Rudin, Functional analysis, McGraw-Hill Series in Higher Mathematics, McGraw-Hill Book Co., New York, 1973.
[47] W. Rudin, Principles of mathematical analysis, International Series in Pure and Applied Mathematics, McGraw-Hill Book Co., New York, 1976.
[48] W. Rudin, Real and Complex Analysis, McGraw-Hill Book Co., New York, 1987.
[49] Z. Sasvári, Multivariate characteristic and correlation functions, De Gruyter Studies in Mathematics, 50, Walter de Gruyter & Co., Berlin, 2013.
[50] J. Schur, Bemerkungen zur Theorie der beschränkten Bilinearformen mit unendlich vielen Veränderlichen, J. Reine Angew. Math. 140 (1911), 1-29.
[51] A. L. Shields, Weighted shift operators and analytic function theory, Topics in operator theory, pp. 49-128. Math. Surveys, No. 13, Amer. Math. Soc., Providence, R.I., 1974.
[52] V. M. Sholapurkar, A. Athavale, Completely and alternatingly hyperexpansive operators, J. Operator Theory 43 (2000), 43-68.
[53] W. Stinespring, Positive functions on C*-algebras, Proc. Amer. Math. Soc. 6 (1955), 211-216.
[68] J. Stochel, The Fubini theorem for semi-spectral integrals and semi-spectral representations of some families of operators, *Univ. Iagel. Acta Math.* **26** (1987), 17-27.

[69] J. Stochel, Seminormal composition operators on $L^2$ spaces induced by matrices, *Hokkaido Math. J.* **19** (1990), 307-324.

[70] J. Stochel, Characterizations of subnormal operators, *Studia Math.* **97** (1991), 227-238.

[71] J. Stochel, Decomposition and disintegration of positive definite kernels on convex *-semigroups, *Ann. Polon. Math.* **56** (1992), 243-294.

[72] J. Stochel, J. B. Stochel, Composition operators on Hilbert spaces of entire functions with analytic symbols, *J. Math. Anal. Appl.* **454** (2017), 1019-1066.

[73] J. Stochel, F. H. Szafraniec, On normal extensions of unbounded operators. I, *J. Operator Theory* **14** (1985), 31-55.

[74] J. Stochel, F. H. Szafraniec, On normal extensions of unbounded operators. II, *Acta. Sci. Math. (Szeged)* **53** (1989), 153-177.

[75] J. Stochel, F. H. Szafraniec, On normal extensions of unbounded operators. III. Spectral properties, *Publ. RIMS, Kyoto Univ.* **25** (1989), 105-139.

[76] B. Sz.-Nagy, A moment problem for self-adjoint operators, *Acta Math. Acad. Sci. Hungar.* (1953), 285-293.

[77] A. E. Taylor, D. C. Lay, *Introduction to functional analysis*, Second edition, John Wiley & Sons, New York-Chichester-Brisbane, 1980.

**Instytut Matematyki, Uniwersytet Jagielloński, ul. Łojasiewicza 6, PL-30348 Kraków, Poland**

*Email address: Zenon.Jablonski@im.uj.edu.pl*

**Department of Mathematics, Kyungpook National University, Daegu 41566, Korea**

*Email address: ibjung@knu.ac.kr*

**Instytut Matematyki, Uniwersytet Jagielloński, ul. Łojasiewicza 6, PL-30348 Kraków, Poland**

*Email address: Jan.Stochel@im.uj.edu.pl*